Persistence of Some Delayed Complex Balanced Systems\

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\section*{Abstract}

Time-delay is very common phenomenon in real biological or industrial systems. This paper study persistence of complex balanced systems with any constant time-delayed. We give several of sufficient conditions for this kind of systems to be persistent. Besides, persistence implies global asymptotic stability for complex balanced systems which means that all trajectories starting at a positive concentration trend to the unique positive constant equilibrium in its stoichiometric compatibility class. These results are analogous to the theorems which have proven by D.F.Anderson in \cite{1} and \cite{3} for the non-delayed mass action system. And from the proof of these conditions, we can see lots of relations between delayed system and non-delayed system. We can apply these results to help judge the long-term property of delayed complex balanced system.

\textbf{Keywords:} Mass action kinetics, Time delay system, Complex balanced, Persistence.

\section{1. Introduction}

A complex process with many components in biochemistry, population system or even transportation system can be described by a network assigned with kinetics. The mathematical model of a chemical reaction network is often based on a system of ordinary differential equations(ODE). But in actual system the reaction rate is often difficult to measure. So we need mathematical method to derive some properties that have nothing to do with reaction rate. Given the high nonlinearity of dynamical equation and complexity of system, chemical reaction network theory(CRNT) is established to find the deep relationship between the dynamical property

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and network structure[9]. CRNT has a quick development in the past 40 years and it is originate from work by F. Horn, R. Jackson, M. Feinberg in [14], [11], [9], [10]. CRNT has made a great contribution to researching the existence and uniqueness of equilibrium, stability analysis, persistence, condition of oscillation phenomenon and so on. The most important two theorems in CRNT are zero deficiency theorem and one deficiency theorem. The two theorems are derived based on the mass action kinetics. A reaction network with mass action kinetics means that the reaction rate is determined by a constant and the concentration of the species. And it is a realistic assumption in chemical engineering and has a widely research in CRNT.

Complex balanced system is system we are interested in which means that for each complex in the network the creating and the consuming of this complex are the same per unit time. Complex balance goes back to the semidetailed balance which is introduced in [6] by L. Boltzmann. It is all known each positive stoichiometric compatibility only have one positive equilibrium and it is local asymptotically stable by Lyapunov function[9]. As for global asymptotically stable, there exist a Global Attractor Conjecture in [14] which said that each trajectory starts from a positive point in phase space trends to the unique positive equilibrium in its stoichiometric compatibility class. And it is equivalent to the persistence of this kind of systems. Persistence is an important property of system that originate from ecological system. A system is said to be persistence if any trajectory remains in positive stoichiometric compatibility class as long as the initial point in \( \mathbb{R}^n_+ \). Persistence of chemical reaction network first presented in [10]. Some subsequent researches can be found in [4], [5], [1], [3], [2], [13], [16], [8], [7], [15] by D. Angeli, D. F. Anderson, G. Craciun, A. Shiu ect.

Now time-delay systems have attracted lots of attention as it is found that bringing time-delay into system can describe the phenomena in a variety of biological systems and technical systems better. Time-delay is common and inevitable and it can cause stable system unstable or stabilize the unstable system, even generate oscillation[12]. So CRNT has been extended to time-delayed form in [19]. [19] reveals the semistability of complex balanced system with time delay. The first research on persistence of time-delayed system is in [17] by Hirokazu Komatsu and Hiroyuki Nakajima which is a nature extension of the results in [4], [5]. This paper we focus on the persistence property of two kinds of time-delayed complex balanced systems. The two analogous kinds of systems without time-delay have been proved by D. F. Anderson in [1] and [3]. [1], [3] discuss the persistence property in the view of the dimension of the face \( L_W \cap \mathcal{P} \)(the
intersection of the stoichiometric compatibility class and the face of phase space). The purpose of this paper is to generalize some results presented in [1],[3] so that we can use these in the system with time-delay. Time delayed system is defined on the functional space $C([-\tau,0];\mathbb{R}^n_{\geq 0})$, so firstly we get the $\omega$-limit set theorem [21] in this functional space, namely, each trajectory in complex balanced system with time-delay trends to either the unique positive constant equilibrium or the constant boundary equilibriums. Then we give the definition of ”face” and ”facet” in this system and prove the delayed form of theorems in [1],[3].

Now we give an outline of this paper. In section 2, we first recall some basic concepts of CRNT without time-delay, then introduce CRNT in delayed form[19]. In section 3, we give $\omega$-limit set theorem in time-delay system. In section 4, we state two main results about the persistence of complex balanced system and some related definitions. In section 5, we apply our results in section 4 to two examples.

Mathematical Notation:

\[
\mathbb{R}^n, \mathbb{R}^n_{\geq 0}, \mathbb{R}^n_* : n\text{-dimensional real space, non-negative and positive real space, respectively.}
\]

\[
\mathcal{G}_+, \mathcal{G}_+^* = C([-\tau,0];\mathbb{R}^n_{\geq 0}), \mathcal{G}_+^* = C([-\tau,0];\mathbb{R}^n_{\leq 0}) : \text{the non-negative, positive continuous functions defined on the interval } [-\tau,0], \text{respectively.}
\]

\[
x^{\nu_i} : x^{\nu_i} \triangleq \prod_{i=1}^{d} x_i^{\nu_i}, \text{where } x, \nu_i \in \mathbb{R}^n.
\]

\[
\text{Ln}(x) : \text{Ln}(x) \triangleq (\ln x_1, \cdots, \ln x_n)^T, \text{where } x \in \mathbb{R}^n_{\geq 0}.
\]

\[
\text{supp } x : \text{Support set, defined by } \text{supp } x = \{i | x_i \neq 0\}, \forall x \in \mathbb{R}^n_{\geq 0}.
\]

\[
0_n : \text{An } n\text{-dimensional vector with each element to be zero.}
\]

\[0^0 : \text{The result is defined by 1.}\]

2. Preliminaries

This section contributes to introducing the chemical reaction network with and without time delays.
2.1. Chemical reaction systems without delay

Now we consider the chemical reaction network which is composed of \( n \) species. The \( i \)-th reaction can be written as the following form:

\[
v_{i1}X_1 + \cdots + v_{in}X_n \rightarrow v'_{i1}X_1 + \cdots + v'_{in}X_n
\]

where \( v_{ji} \) and \( v'_{ji} \) are the reaction coefficient and result coefficient of \( X_j \) in the \( i \)-th reaction. So we can use the following three finite sets to express the chemical reaction network.

**Definition 2.1 (Chemical Reaction Network).**

- \( S = \bigcup_{j=1}^{n} \{ X_j \} \) is a set of all the species that participate in the reactions.
- \( C = \bigcup_{i=1}^{r} \{ v_i, v'_i \} \) is a set of complexes that appear on the left and right side of the reaction arrow satisfying \( \text{Card}(C) = c \) and

\[
\bigcup_{i=1,\ldots,r} \{ \text{supp } v_i \cup \text{supp } v'_i \} = S,
\]

where the \( j \)-th entry of \( v_i, v'_i \in \mathbb{Z}_{\geq 0}^S \) express the stoichiometric coefficient of species \( X_j \) in complex \( v_i \).
- \( R = \bigcup_{i=1}^{r} \{ v_i \rightarrow v'_i \} \) is the set of all the reactions of this reaction network. We often use the triple \( (S, C, R) \) to denote a chemical reaction network \( N \).

Each reaction network \( (S, C, R) \) can be expressed by a unique directed reaction graph \( \{ V, E \} \). \( V \) is composed of all the complexes in \( C \). And \( (v_i, v'_i) \) is an edge in \( E \) if and only if \( v_i \rightarrow v'_i \in R \).

Each connected component of this graph is termed a linkage class of this reaction network. There exist two kinds of special structure which have lots of good properties.

**Definition 2.2 (Reversible and Weakly Reversible CRN).** A CRN \( (S, C, R) \) is called weakly reversible if its directed graph is strong connected, namely, for any reaction \( v_i \rightarrow v'_i \in R \) there exists a chain of reactions starting from \( v'_i \) and ending with \( v_i \), i.e., \( v'_i \rightarrow v_{i_1} \in R \), \( \ldots \), \( v_{i_m} \rightarrow v_i \in R \), \( m < r \). In particular, a reaction \( v_i \rightarrow v'_i \in R \) is reversible if \( v'_i \rightarrow v_i \) also in \( R \).

For each reaction \( v_i \rightarrow v'_i \), its effect on the species concentration is exactly the reaction vector: \( v'_i - v_i \). The stoichiometric subspace is defined as:

\[
\mathcal{S} = \text{span}\{ v'_i - v_i, i = 1, \cdots, r \}
\] (1)
The above properties is only determined by the network structure, and have nothing to do with the dynamics. Now we transfer our attention to the dynamics without time delay.

The dynamics of a chemical reaction network reflects on the continuous rate function \( R_k(x) \) of each reaction \( v_k \rightarrow v'_k \). With this kinetics, the dynamical equation of this system can be written as the following coupled set of nonlinear ordinary differential equation:

\[
\dot{x}(t) = \sum_{k=1}^{r} R_k(x(t))(y'_k - y_k).
\]

And the integral form of above equation is:

\[
x(t) = x_0 + \sum_{k=1}^{r} \left( \int_{0}^{t} R_k(x(s))ds \right) (y'_k - y_k).
\]

From above equation we can see that the transition of any trajectory is due to the reactions, thus any two concentration vectors \( x_1 \) and \( x_2 \) on the trajectory, \( x_1 - x_2 \) must be the linear combination of the reaction vectors. In this case, if the trajectory \( x(t) \) starting at \( x_0 \) remains in the \( (x_0 + \mathcal{S}) \cap \mathbb{R}^n_{\geq 0} \) where \( x_0 + \mathcal{S} := \{ z \in \mathbb{R}^n | z = x_0 + v, \text{for some } v \in \mathcal{S} \} \). And this closed set denoted by \( \mathcal{P}_{x_0} \) is called the non-negative stoichiometric compatibility class of \( x_0 \). And the interior of \( \mathcal{P} : (x_0 + \mathcal{S}) \cap \mathbb{R}^n_{>0} \) is termed as positive stoichiometric compatibility class. So for a given system, \( \mathcal{P}_{x_0} \) is only determined by the initial point \( x_0 \). Stiochiometric compatibility class is positively invariant sets of dynamical equation \( [2] \). From \( [3] \) we know \( \mathcal{P}_{x_0} \) is indeed a polyhedron and \( \dim \mathcal{P}_{x_0} = \dim \mathcal{S} \) as the dimension of a polyhedron in \( \mathbb{R}^n \) is the dimension of the span of the translate of this polyhedron which contains the origin. For the polyhedron \( \mathcal{P}_{x_0} \), each face of polyhedron is also a polyhedron with lower dimension.

**Definition 2.3 (facet).** For a polyhedron \( \mathcal{Q} \), a facet of \( \mathcal{Q} \) is the face of this polyhedron with dimension one less than it.

It can easily see that the phase space of this system is \( \mathbb{R}^n \) and we usually use \( \mathcal{P}_W \) where \( W \) is a subset of species set \( S \) denote the boundary of \( \mathbb{R}^n_{\geq 0} \).

\[
\partial \mathbb{R}^n = \bigcup_{W \subset S} \mathcal{Q}_W
\]

where

\[
\mathcal{Q}_W = \{ x \in \mathbb{R}^n | x_i = 0, \ x_i \in W, \ x_i \neq 0, \ x_i \notin W \}.
\]

But what we are really interested in is \( N_W : \mathcal{Q}_W \cap \mathcal{P}_{x_0} \).
This paper we consider the network with the most common kinetics—mass action kinetics. And this kind of system are called mass action systems with all the reaction rates have the following form

\[ R_k(x) = k_k x_1^{y_{i1}} x_2^{y_{i2}} \cdots x_n^{y_{in}} =: k_k x_i \]  

(5)

for some positive reaction rate constant \( k_k \). Combining 2 and 5, we can express the dynamical equation as following:

\[ \dot{x}(t) = \sum_{i=1}^{r} k_k x(t)^{y_i} (y'_i - y_i) =: f(x(t)). \]  

(6)

As for above dynamics, a concentration vector \( \bar{x} \in \mathbb{R}_+^n \) is an equilibrium if \( f(\bar{x}) = 0 \). This paper we focus on the complex balanced. This is originally from the research in thermodynamic compatibility of reaction networks. Now we give the definition of complex balanced equilibrium:

**Definition 2.4 (Complex balanced).** A positive equilibrium of the dynamical system 2 is complex balanced if for any complex \( y \in \mathbb{C} \), there exists:

\[ \sum_{v_i = y} k_v \bar{x} \bar{y} = \sum_{v'_i = y} k_{v'} \bar{x} \bar{y} \]  

(7)

If all the positive equilibria are complex balanced, we call this system is complex balanced.

From equation 7, complex balanced system means that at each positive equilibrium, the reaction fluxes producing and consuming of one complex are equal for any complex \( y \). [9] also reveals that if one equilibrium in a system is complex balanced, all the equilibrium are complex balanced.

It is well-known that each complex balanced system has a Lyapunov function independent of rate coefficients which called pseudo-Helmholtz function in [14] as follows:

\[ H(x) = \sum_{i=1}^{n} (\bar{x}_i - x_i) \ln \left( \frac{x_i}{\bar{x}_i} \right) = \sum_{i=1}^{n} \hat{s}_i G \left( \frac{x_i}{\bar{x}_i} \right) \]  

(8)

where \( G(s) = s \ln s + 1 - s \). And it is already proved that each positive equilibrium is locally asymptotically stable relative to its stoichiometric compatibility class from above Lyapunov function. Above is the dynamics for non-delay system. But as for delayed mass action system, there exist another model.

### 2.2. Dynamics of mass action system with time-delay

Time-delay often exist in biology system. Let \((S, C, R, k, \tau)\) denote a mass action system where \(i\)-th element of vector \(\tau \in \mathbb{R}^r\) is the time-delay of reaction \(R_i\). It does not affect the property
only concerning network structure. So the stoichiometric subspace, equilibrium, linkage class are all the same. So only the dynamics should be introduced. The dynamical equation can be expressed in the form of:

\[ \dot{x}(t) = \sum_{i=1}^{r} k_i [(x(t - \tau_k))^\nu y_k' - (x(t))^\nu y_k], \quad t \geq 0 \] (9)

where \( \tau_k \geq 0 \) is the time-delay of the reaction \( y_k \rightarrow y_k' \). Specially, some \( \tau_k = 0 \). And if all \( \tau_k = 0 \), \( k = 1, \cdots, r \), (9) reduces to (2). Analogous to non-delayed system, solutions of (9) is only determined by initial function \( x(s) = \theta(s) \) where \( s \in [-\tau, 0] \) and \( \tau = \max_{1 \leq k \leq r} \tau_k \) is the maximum delay. So it is different to the non-delayed system whose solution is defined on \( \mathbb{R}_{\geq 0}^n \), the solution space for delayed system defined in [19] and [18] is \( \mathcal{C}_+ \) which denotes the functional space \( C([-\tau, 0]; \mathbb{R}_{\geq 0}^n) \). Besides, we denote \( C([-\tau, 0]; \mathbb{R}^n) \) as \( \mathcal{C}_+ \). And the norm of the space \( \mathcal{C}([-\tau, 0]; \mathbb{R}^n) \) is:

\[ \|\phi\| := \sup_{-\tau \leq s \leq 0} |\phi(s)| \] (10)

for any \( \phi \) in this space. Now we will give some definitions that are different to that in non-delayed system.

**Definition 2.5.** For any \( \theta \in \mathcal{C}_+ \), the positive stoichiometric compatibility class of \( \theta \) is as follows:

\[ D_\theta := \{ \psi \in \mathcal{C}_+ | c_a(\theta) = c_a(\psi), \ \forall a \in \mathcal{S}^\perp \} \]

where the functional \( c_a : \mathcal{C}_+ \rightarrow \mathbb{R} \) is defined as:

\[ c_a(\theta) := a^\top \left[ \theta(0) + \sum_{i \rightarrow i'} \int_{s = -\tau}^{0} k_i \theta(s)^{\nu_i} \cdot i_{i'} \right] \] (11)

Equilibrium is same for both system with or without delay. And the Laypunov-function \( V \) of delayed system \( V : \mathcal{C}_+ \rightarrow \mathcal{R}_+ \) in [19] as follows:

\[ V(\psi) = \sum_{i=1}^{N} (\phi_i(0)(\ln(\phi_i(0)) - \ln(\bar{x}_i)) - 1 + \bar{x}_i) \]
\[ + \sum_{k=1}^{M} k_i \int_{s = -\tau_k}^{0} [(\psi(s))^{\nu_k} [\ln(\psi(s))]^{\nu_k} - \ln(\bar{x}_k) - 1] + \bar{x}_k] \ ds \] (12)

reveals that the unique positive equilibrium is local asymptotic stability. As for global asymptotic stability, we need further research the persistence of this system. So let us recall some concepts relative to persistence. In CRNT, persistence means that none of the concentrations of species
can tend to zero if they are not zero at the beginning of reactions. Mathematically, it is defined as follows.

**Definition 2.6 (Persistence).** A MAS $(S, C, R, k)$ which is described by (2) is called persistence if any forward trajectory $x^\psi(t) \in \mathbb{R}^S_{\geq 0}$ with positive initial condition $\psi \in \mathbb{R}^S_{> 0}$ satisfies

$$\liminf_{t \to \infty} x^\psi_j(t) > 0 \quad \text{for all } j \in \{1, \cdots, n\}.$$  

And we can use the following definition to help us judge the persistence of bounded system.

Before introducing this concept, we should first recall $\omega$-limit set.

**Definition 2.7 ($\omega$-limit Set).** The set of $\omega$-limit set for the trajectory $x^\psi(t)$ with positive initial condition $\psi \in C^+$ is

$$\omega(\psi) := \{ \phi \in C^+ | x^\psi_{t_N} \to \phi, \text{ for some sequence } t_N \to \infty \text{ with } t_N \in \mathbb{R} \}.$$  

And each element in this $\omega$-limit set is an $\omega$-limit point of this trajectory.

**Definition 2.8 (Semilocking Set).** A nonempty subset $W$ is called a semilocking set if it satisfies:

- $W$ is a subset of $S$.
- $W \cap \text{supp } v_i \neq \emptyset$ if $W \cap \text{supp } v'_i \neq \emptyset$.

**Definition 2.9 (Persistence for Bounded Trajectory).** For a MAS $(S, C, R, k, \tau)$ with bounded trajectory, it is persistent if

$$\omega(\psi) \cap (\bigcup_{W: \text{semilocking set}} L_W) = \emptyset, \quad \forall \psi \in C^+.$$  

where

$$L_W = \left\{ w \in C([-\tau, 0]; \mathbb{R}^n) | x_i(\tau) = 0, x_i(W), x_i \neq 0, \forall s \in [-\tau, 0] \right\}.$$  

We will use above definition to help us judge one system is persistence or not.

### 3. $\omega$-limit set theorem for delayed complex balanced system

Given a complex balanced system with arbitrary time delay, it is known that the unique positive equilibrium $\bar{x}$ in each stoichiometric compatibility classes is locally asymptotic stability. Using this property we can only judge the trajectory whose initial point lies in the sufficiently small neighborhood of $\bar{x}$. So we want know the dynamics of other solutions with initial condition out of this neighborhood.
Theorem 3.1 (ω-limit set theorem for delayed complex balanced system). If the system is a complex balanced delayed system, the solution beginning at $x_0$ either trends to the boundary equilibria or the single positive point of complex balanced equilibrium of $C_{x_0}$.

Above theorem is analogous to ω-limit set theorem in [21]. Before we prove it we should first claim that the following lemma is also right for delayed system.

Lemma 3.2. For a trajectory $\psi$ of delayed differential equation with an initial function $\phi \in E$, the ω-limit set of $\Gamma = \omega(\psi)$ is a closed set. Besides, if $\psi$ is contained in a compact subset of $C([-\tau, 0]; \mathbb{R}^n_{>0})$, then $\omega(\Gamma)$ is a non-empty, connected, compact subset of $E$.

Proof. Let $\omega_m$ be a sequence of functions in $\omega(\Gamma)$ and converges to $\omega \in C([-\tau, 0]; \mathbb{R}^n_{>0})$. There exist a sequence $t_m$ trends to infinity such that:

$$\lim_{k \to \infty} \psi(t_m, \phi) = \omega_m.$$ (16)

In other words, for each $m$, there exists $K(m)$ such that when $k \geq K(m)$,

$$\sup_{-\tau \leq s \leq 0} |\psi(t_m, \phi)(s) - \omega_m(s)| < \frac{1}{m}.$$ (17)

Let $t_m = t_{\omega_m}$, we have:

$$\sup_{-\tau \leq s \leq 0} |\psi(t_m, \phi)(s) - \omega_m(s)| \leq \sup_{-\tau \leq s \leq 0} |\psi(t_m, \phi)(s) - \omega_m(s)| + \sup_{-\tau \leq s \leq 0} |\omega_m(s) - \omega_m(s) + \omega_m(s) - \omega_m|$$ (18)

$$\leq 1/m + |\omega_m(s) - \omega_m|.$$ (19)

Above equation trends to zero when $n$ trends to infinity. We can easily conclude that $\omega \in \omega(\phi)$. Therefore, $\omega(\psi)$ is a closed set. If $\psi$ is contained in a compact subset $K$ and we can choose a sequence $t_m$ such that $\psi(t_m, \phi) \to \omega \in \omega(\phi)$. $\omega(\phi) \subseteq K$ and hence compact.

Now we assume that $\omega(\phi)$ is not connected. In this case two nonempty, closed sets $A, B$ can be found such that $\omega(\phi) = A \cup B$. $\delta$ is a positive finite distance between $A$ and $B$ where

$$d(A, B) = \inf_{\varphi_1 \in K, \varphi_2 \in B} \sup_{s \in [-\tau, 0]} |\varphi_1(s) - \varphi_2(s)|$$ (20)

Since $A$ and $B$ are both the subsets of $\omega(\phi)$, we can find a big enough $t_m$ such that $\psi_{t_m}$ within the $\delta/2$-neighbourhood of $A$, and a big enough $t_m$, such that $\psi_{t_m}$ out of $\delta/2$-neighbourhood of $A$. From the continuity of $d(\psi_{t_m}, A)$, we can get there exist a $t_m$ lead to $d(\psi_{t_m}, A) = \delta/2$. Thus
\( \omega \notin A \cup B \) is also an omega-limit point of \( \psi^\varphi_i \). This contradicts to \( \omega(\psi) = A \cup B \). So the connectivity of \( \omega(\psi) \) is proved. \( \square \)

Now we use the Lyapunov function for delayed complex balanced system to prove

**Proof.**[Proof of 3.1] From the result in [19], we know that \( V : \mathcal{C}_+ \to \mathcal{R} \) in [12] is a convex function and decrease along the trajectory. So the trajectory must be bounded as the same reason with the zero-delayed system. [19] tells us that only equilibrium can be \( \omega\)-limit point in positive stoichiometric compatibility class. Now we consider the points on the boundary in two cases:

Firstly, if \( W \) is a locking set, according to the definition of the boundary \( L_W \), all the functions in \( L_W \) are all equilibria. And it is obviously that \( \dot{V}(\psi) = 0 \) for any \( \psi \in L_W \). Now we prove that \( \omega\)-limit set can only be constant equilibria. If \( \psi \in \omega(\phi) \cap L_W \), we can find a \( \psi_1 \in \omega(\phi) \cap L_W \) such that \( x^{\psi_1}(t) = \psi(t - \tau) \). Now we consider the dynamics of the \( x^{\psi_1}(t) \) for any \( t \in [0, \tau] \)

\[
\dot{x}^{\psi_1}(t) = \sum_{i=1}^{r} k_i \left( x(t - \tau - y_i) \right)^{y_i} - \sum_{i=1}^{r} k_i (x(t))^{y_i} y_i = 0, \quad \forall \ t \in [0, \tau].
\]

Above equation holds as \( x(t - \tau) \) and \( x(t) \) are all in \( L_W \) and all the \( y_i \) satisfy \( \supp y_i \cap W \neq \emptyset \). Therefore, if \( \psi \in \omega(\phi) \cap L_W, \psi \) must be a constant equilibrium.

The second part contributes to prove the case that \( W \) is a semilocking set but not a locking set. In this case, we can find a subnetwork \( N_W = \{S', C', \mathcal{R}'\} \) where

\[
\mathcal{R}' = \{y \to y'|\supp y \cap W = \emptyset, \supp y' \cap W = \emptyset\}
\]

with \( |\mathcal{R}'| = r' \). \( S', C' \) are the set of species and complex appear in the reactions in \( \mathcal{R}' \) respectively. Without loss of generality, we can assume \( S' = \{X_1, ..., X_N\} \). This \( N_W \) is obviously a complex balanced network and its dynamical function is [2] From the properties of complex balanced delayed system, each stoichiometric compatibility class only have one positive constant equilibrium, denoted by \( \bar{x} \). From the Lyapunov function of \( N_W \) -- \( V' \), defined as:

\[
V'(\psi') = \sum_{i=1}^{N'} (\psi'_i(0)(\ln(\psi'_i(0)) - \ln(\bar{x}_i) - 1) + \bar{x}_i)
+ \sum_{k=1}^{r'} k_k \int_{-\tau_k}^{0} \{e^x(s)^{y_i}(\ln(\psi'(s))^{y_i}) - \ln(\bar{x}^{y_i})] - 1 + \bar{x}^{y_i}]ds \leq 0
\]

form [19] we know above equation with equality if and only if for each \( k = 1, ..., r' \),

\[
\left( \frac{\psi'(0)}{\psi'} \right)^{\frac{1}{y_i}} = \left( \frac{\psi'(-\tau_k)}{\psi'} \right)^{\frac{1}{y_i}}
\]

(21)
Then by using above equation and rewrite into following form we can obtain that:
\[
\dot{x}(t) = \sum_{c \in C} \left[ \sum_{c = c'} \left( \frac{x(t - \tau_k)}{x} \right)^{y_k} c \right] = \sum_{c \in C} \left( \frac{x(t)}{x} \right)^c \left[ \sum_{c = c'} \left( \sum_{c = c} y_k \right) k \left( x(t - \tau_k) \right) y_k \left( x(t) - \tau_k \right) \right] c = 0
\]

The last equation equal to zero can be derived from the definition of complex balanced equilibrium. From above discussion, we can obtain that only constant equilibrium can be the omega-limit function in \( L_W \) as \( V'(\psi) \) is a constant if and only if \( \psi = \bar{\psi} \) is a positive equilibrium of \( N_W \).

We can easily see \( V'(\varphi') = \lim_{\gamma \to \varphi} V(\gamma) \) where \( \gamma \in \mathcal{C}, \varphi \in \mathcal{C} \cap L_W \) and \( \varphi' \) is the function restricted on \( W^c \). Hence, from the continuity of \( V \), we obtain that \( \dot{V}(\bar{x}) = \lim_{\gamma \to \bar{x}} \dot{V}(\gamma) = \dot{V}(\bar{x}') = 0 \). And the nonequilibrium function \( \varphi_1 \) in \( L_W \) can not be the omega-limit function of any trajectories due to \( \dot{V}(\varphi_1) < 0 \). Thus we complete our proof. \( \square \)

4. Persistence in some kinds of complex balanced system with time delay.

Now we consider which network structure have persistent property with arbitrary finite delayed mass action kinetics. Now we will verify some kinds of networks that are persistent under delayed system. We will mainly focus on the two classes of delayed systems which have a close connection with the systems whose persistence have been proved by D.F. Anderson in [1] and [3].

4.1. Systems with \( \dim Z_W = \dim \mathcal{F} - 1 \) for all semilocking set \( W \)

Before we prove the persistent property, first let us define a facet in delayed system.

**Definition 4.1.** Define \( Z_W \) as:
\[
Z_W = \{ v | v = g(x_1(s)) - g(x_2(s)), x_1(s), x_2(s) \in L_W \}
\tag{22}
\]

where
\[
g(x) = x(0) + \sum_{k=1}^{K} k \int_{-\tau_k}^{0} (x(s))^{y_k} dsy_k.
\]

If \( \dim(Z_W) = s - 1 \), then \( Z_W \) called the facet of stoichiometric subspace \( \mathcal{F} \).

**Lemma 4.2.** \( N = (S, C, R) \) is a complex balanced chemical reaction network with delayed mass action dynamics. There does not exist \( \omega \)-limit function in \( L_W \) if \( Z_W \) is a facet.
Proof. As we have proved in [3, 4] that \( \omega \)-limit set can only contain constant equilibria for complex balanced system. Now we are in a position to prove that all the boundary equilibria can not be \( \omega \)-limit function in \( L_W \).

Now we just consider \( L_W \) where \( W \) is a semilocking set. Besides, without loss of generality, let \( W = \{X_{d+1}, \ldots, X_n\} \). We will first prove that if \( Z_W \) is a facet, then for zero-delayed system the boundary \( M_W \) is also a facet of the stoichiometric compatibly class \( P \). We assume that the set \( \{w_1, w_2, \ldots, w_{r-1}\} \) where any \( w_i \) and \( w_j \) are independent can denote the elements of \( Z_W \) by linear combination. For any \( x_1(s) \) and \( x_2(s) \) in \( L_W \), we have:

\[
x_1(0) - x_2(0) + \sum_{k=1}^{r} k \int_{-\tau_1}^{0} (x_1(s))^{\hat{y}_k} ds - \sum_{k=1}^{r} k \int_{-\tau_2}^{0} (x_2(s))^{\hat{y}_k} ds = \left( \frac{\hat{x}_1(0)}{0_{n-d}} \right) - \left( \frac{\hat{x}_2(0)}{0_{n-d}} \right) + \sum_{\text{supp } y_l \cap W = 0} k \int_{-\tau_2}^{0} (\hat{x}_1(s))^{\hat{y}_k} - (\hat{x}_2(s))^{\hat{y}_k} ds \left( \frac{\hat{y}_k}{0_{n-d}} \right)
\]

Let \( (\hat{x}_1(s))^{\hat{y}_k} \) smaller than \( (\hat{x}_2(s))^{\hat{y}_k} \), otherwise we can exchange \( x_1 \) and \( x_2 \). Let

\[
x_0 = \left( \frac{\hat{x}_2(0)}{0_{n-d}} \right) - \sum_{\text{supp } y_l \cap W = 0} k \int_{-\tau_2}^{0} (\hat{x}_1(s))^{\hat{y}_k} - (\hat{x}_2(s))^{\hat{y}_k} ds \left( \frac{\hat{y}_k}{0_{n-d}} \right)
\]

So we can easily verify that \( x_0 \) is a point in \( M_W(x_1) \). Because all the stoichiometric compatibly classes are all translated from subspace \( \mathcal{F} \), they share the same dimension. Any two functions \( x_1(s) \) and \( x_2(s) \) in \( L_W \) we can find corresponding points \( x_1(0) \) and \( x_0 \) in zero-delayed system and the subtraction of these point are all in the \( P_W \cap \mathcal{F} \), thus \( \dim(P_W \cap \mathcal{F}) = s - 1 \). Therefore, \( M_W \) is a facet in stoichiometric compatibly class for mass action system without delay. Especially, when \( W \) is a locking set, all the reactions are "locked" and all the points in the \( Z_W \) are all equilibria. Then:

\[
g(x_1(s)) - g(x_2(s)) = x_1(0) - x_2(0) \quad (23)
\]

Hence, if \( x_1(s), x_2(s) \in L_W \) are in the same stoichiometric compatibly class of delayed system, we have \( x_1(0), x_2(0) \) also in the same compatibly class of system without delay and vice versa. In this case, it is obvious that \( \dim(P_W \cap \mathcal{F}) = s - 1 \), namely, \( M_W \) is a facet for system without delay.

For delayed system, if the trajectory \( x_t^\phi \) trends to the interior of \( L_W \), the \( \omega \)-limit set can only be a connected equilibria set—\( \omega(\phi) \). Then for any \( \epsilon > 0 \) we can find an \( \epsilon \)-neighbourhood and a large enough \( t_0 \) such that the trajectory will always stay in this \( \epsilon \)-neighbourhood when \( t > t_0 \). If this is not true, the trajectory will go out of the neighbourhood infinitely. Then a
infinite sequence $\psi_n(s)$, $t_n \to \infty$ can be choosen with $\text{dist}(x(t_n), \omega(\psi)) > \epsilon$ for all $t_n$. And all the trajectories in complex balanced system is bounded, so from Bolzano-Weierstrass theorem, there must exist an $\omega$-limit point that out of the $\epsilon$-neighbourhood. But this is contradiction to the connectivity of $\omega$-limit set. Thus when $t > t_0$, the trajectory $x(t)$ stays in $\epsilon$-neighbourhood. As we know all the $\omega$-limit functions are constant equilibria, in this case for any $t_k$ large enough, there exists $w_k \in \omega(\psi)$ and an $\epsilon_1$ such that:

$$\text{dist}(x_{t_n}(s), w_k) = \sup_{s \in [-\tau,0]} |x_{t_n}^\psi(s) - w_k| < \epsilon_1.$$  

In this case, we can obtain that:

$$\text{dist}(x_{t_n}^\psi(0), x_{t_n}^\psi(-\tau)) < \epsilon_1. \quad (24)$$

From [3] we know for zero-delayed system we can find a minimal complex $y_l$ (namely, $y_{il} < y'_{il}$ for all $i \in W$ and all $l' \neq l$) in each linkage class $L_l$. $y_l$ is the dominate complex in all complexes in $L_l$. In other words, for any linkage class $L_l$ the relation between 0 and the following equation:

$$\sum_{y_k \to y_{il}' \in R} k_k(x(t)) y_k (y_{il}' - y_{il}) \quad (25)$$

is determined by $(y_{l}' - y_l)$. For complex balanced system, each result complex is also reactant complex. So $y_{l}' - y_{il} > 0$ for each $i \in W$ and all $l$. So the dynamical equation for non-delayed system is:

$$\dot{x}_l = \sum_{k \in L_l} k_k(x(t)) y_k (y_{l}' - y_{il}) > 0 \quad (26)$$

for all $i \in W$. Then we consider the dynamics system with delay.

$$\dot{x}_{l}^\psi(t) = \sum_{y_l \to y_l' \in R} k_k(x(t - \tau_k)) y_{l}' - \sum_{y_l \to y_l' \in R} k_k(x(t)) y_k, \quad t \geq 0$$

The same as the non-delayed system, $y_l$ is the minimal complex in each $L_l$. Thus

$$(x(t))^{y_l} \gg (x(t))^{y_l'}, \quad t \text{ large enough} \quad (27)$$

for all $l'$. The rate constant $k$ and the stoichiometric coefficient $y_l$ are all bounded. As proved above when $t$ large enough, $x(t - \tau)$ and $x(t)$ almost same. So we can also obtain the following equation by [27] and [24]:

$$x(t - \tau)^{y_l} \gg x(t)^{y_l'}, \quad t \text{ large enough}.$$
Proof. When \( W \) is a semilocking set \( W = \{ X_1, ..., X_d \} \), from condition we know \( Z_W \) is a facet or empty. \( Z_W \) is a facet, there can not exists \( \omega \)-limit function in \( L_{W_i} \) with \( W \subseteq W_1 \). Without loss of generality, let \( W_1 = \{ X_1, ..., X_{d+m} \} \) If there exists an \( \omega \)-limit function \( y_0 \in L_{W_1} \). Thus the corresponding \( Z_{W_1} \) must also be a facet. We assume \( \{ w_1, ..., w_{s-1}, v \} \) spans the stoichiometric subspace \( \mathcal{S} \). Then for any two \( \psi_1 \) and \( \psi_2 \),

\[
(h(\psi_1) - h(\psi_2))|_{W} = qv|_{W_1}.
\]

\( v|_{W_1} \) is a basis spans \( \mathcal{S}|_{W_1} \) and \( q \) is a real number. \( v|_{W_1} = (v_1, ..., v_{d+m}) \) and there does not exist species whose concentration never change. So all \( v_1, ..., v_{d+m} \) is not zero. But consider \( \psi_1 \in L_{W_1} \) and \( \psi_2 \in L_{W_1} \), we obtain:

\[
(g(\psi_1) - g(\psi_2))|_{W_1} = g = (0, ..., 0, g_{m+1}, ..., g_{m+d})
\]

with \( g_{m+j}, j = 1, ..., d \) is nonzero. This obviously can not be expressed by \( v|_{W_1} \). So \( Z_{W_1} \) can not be a facet. Then \( Z_{W_1} \) does not exist \( \omega \)-limit point. So we obtain if a trajectory trends to a facet, its \( \omega \)-limit set can only contain in the interior of \( L_{W_1} \). Then combining with \[4.2\] we can obtain the network is persistent when \( Z_W \) is a facet or empty for all semilocking set \( W \). □
4.2. Systems with $\dim Z_W = 0$ with all semilocking set $W$.

Now we prove that for complex balanced delayed mass action system, the results in [1] also hold.

**Lemma 4.4.** For complex balanced delayed system, if $W$ is a semilocking set but not a locking set, $L_W \cap D_0$ can not only have one element, namely, $Z_W = 0$.

**Proof.** Assume $L_W \cap D_0$ have exactly one element. Namely, $\dim(Z_W \cap \mathcal{S}) = 0$. Because $W$ is a semilocking set but not a locking set. Without loss of generality, let $W = \{X_{a-1}, \ldots, X_a\}$. So we consider the subnetwork $\mathcal{N}_W = \{S_W, C_W, R_W\}$ with $R_W = \{y_i | W^c \rightarrow y_i' \mid W^c, \text{supp}_y \cap W = \emptyset, \text{supp}_{y'} \cap W = \emptyset\}$. The set $S_W$ and set $C_W$ are the species and complexes appear in the $R_W$. The definition of network tells us that reactions $y_i \rightarrow y_i' \in R_W$ must satisfy $(y_i' - y_i)|W = 0$. So we can choose two functions $\hat{\psi}_1(s)$ and $\hat{\psi}_2(s)$, such that:

$$g(\hat{\psi}_1) - g(\hat{\psi}_2) \neq 0.\quad (31)$$

Then let $\psi_1 = (\hat{\psi}_1, 0_\nu^\top)$ and $\psi_2 = (\hat{\psi}_2, 0_\nu^\top)$, we can see $\psi_1$ and $\psi_2$ are all in $L_W$. Complex balanced system has the property that in one linkage either all the complexes have the element in $W$ or neither of the complexes have the intersection with $W$. So we have the following equation:

$$g(\psi_1) - g(\psi_2) = (g(\hat{\psi}_1) - g(\hat{\psi}_2), 0_\nu^\top) \in Z_W \cap \mathcal{S}.\quad (30)$$

Above equation holds as that $g(\psi_1) - g(\psi_2)$ can be expressed by the linear combination of $\{y' - y \mid \text{supp}_y \cap W = 0, \text{supp}_{y'} \cap W = 0\} \subset \mathcal{S}$. So it is obviously contradict to the assumption. Thus $L_W \cap D_0$ can not have only one element.

\[\Box\]

Now we prove the main result by using chain method and above lemma. So before prove the main theorem in this subsection, we first recall the chain method in the chemical reaction kinetics.

In delayed system, if the system model is as following equation:

$$\dot{x}(t) = f(x(t)) + f_1(x(t - \tau))H, \forall t \geq 0,\quad (31)$$
where \( x(t) \in \mathbb{R}^n \) is the state vector, \( \tau > 0 \), \( f : \mathbb{R}^n \to \mathbb{R}^n \), \( h : \mathbb{R}^n \to \mathbb{R} \) are continuous functions. \( H \) is a constant vector, \( x(t) = \theta(t) \) for \(-\tau \leq t \leq 0\) is a continuous initial function. Then we can use a set of ODEs with a "N" new state variables chain to approximate the original delayed system. The ODEs have the following form:

\[
\begin{align*}
\dot{z}(t) &= f(z(t)) + \frac{N}{\tau} v_N(t) H \\
\dot{v}_1(t) &= f_1(z(t)) - \frac{N}{\tau} v_1(t) \\
\dot{v}_i(t) &= \frac{N}{\tau} v_{i-1}(t) - \frac{N}{\tau} v_i(t), \quad 2 \leq i \leq N
\end{align*}
\]  

(32)

with \( z(0) = \theta(0) \), and \( v_i(0) = \int_{-\tau}^{-(i-1)\tau} h(\theta(s))ds, \quad 1 \leq i \leq N \). \( z(t) \in \mathbb{R}^n \approx x(t) \) and \( v_i, \quad i = 1, \ldots, N \) is the chain we add. And [20] reveals that if the initial function of delayed system is sufficiently smooth, the solution of above ODE converges uniformly to the solution of the original delayed model on any finite time interval \([0, T]\) when \( N \) goes to infinity. As for chemical reaction network with delayed mass action kinetics. We assume that the last \( r' \) reactions is delayed, namely, \( \tau_k = 0 \) for \( k = 1, \ldots, r - r' \) and \( \tau_k > 0 \) for the rest of \( k \). In this case, the delayed dynamical equation can be written as:

\[
\dot{z}(t) = \sum_{k=1}^{r-r'} k_k(x(t))^y_k [y'_k - y_k] + \sum_{k=r-r'+1}^{r} k_k \left[ (x(t) - \tau_k)^y_k - (x(t))^y_k \right].
\]  

(33)

We can write out the approximating system of above delayed system by applying the [32]

\[
\begin{align*}
\dot{z}(t) &= \sum_{k=1}^{r-r'} k_k(z(t))^y_k [y'_k - y_k] + \sum_{k=r-r'+1}^{r} \frac{N}{\tau_k} v_{kN}(t)y'_k - \sum_{k=r-r'+1}^{r} k_k(x(t))^y_k \\
\dot{v}_{k1}(t) &= k_k(z(t))^y_k - \frac{N}{\tau_k} v_{k1}(t) \\
\dot{v}_{ki}(t) &= \frac{N}{\tau_k} v_{k,i-1}(t) - \frac{N}{\tau_k} v_{ki}(t), \quad 2 \leq i \leq N,
\end{align*}
\]  

(34)

with the initial point:

\[
z(0) = \theta(0), \quad \text{and} \quad v_{kN}(0) = k_k \int_{-\tau}^{-\tau_{kN}} (\theta(s))^{y_k} ds
\]  

(35)

In above dynamical equation, we take each time-delay \( \tau_k \) as a chain of first-order intermediate reactions with each reaction rate \( \frac{N}{\tau_k} \). Let \( \tilde{y}_k = [y_k^T, 0^T]^T \) and \( \tilde{y}'_k = [y'_k^T, 0^T]^T \) for \( k = 1, \ldots, r \). Then the stoichiometric subspace denoted by \( \mathcal{S}'' \) can be expressed by:

\[
\mathcal{S}'' = \mathcal{S} \oplus \mathcal{S}_{1k}' \oplus \mathcal{S}_{2k}' \quad k = r - r' + 1, \ldots, r.
\]
where:

\[ \mathcal{F} = \text{span} \left( \{ \hat{y}_k - \tilde{y}_k \mid k = 1, \ldots, r \} \right) \]

\[ \mathcal{F}'_{1,k} = \text{span} \left( \{ e_{2,i+1} - e_{2,1} \mid i = (n + 1), \ldots, (n + N - 1) \} \right), \tag{36} \]

\[ \mathcal{F}'_{2,k} = \text{span} \left( \{ e_{2,n+1} - \tilde{y}_k \} \right) \]

Then each point \( \tilde{x}(t) \) in approximating system can be written as:

\[ \tilde{x}(t) = (z(t)^T, v_1(t) = (z(t)^T, v_{r'-1+1}(t), \ldots, v_{r'-e+1,N}(t), \ldots, v_{r+1}(t), \ldots, v_{r,N}(t)). \]

Theorem 4.5. For a complex balanced delayed mass action system \( \{ S, C, R, k, \tau \} \), if \( L_0 \cap D_0 \) only have one element for each semilocking set \( W \), this system is persistent.

\( \square \) Lemma 4.4 reveals that if \( L_0 \) only have one element in each stoichiometric compatibly class \( D_0 \), then \( W \) must be a locking set. Without loss of generality, we assume \( W = \{ X_1, \ldots, X_n \} \). And the only element \( \tilde{y}(s) \) must be a constant equilibrium. Instead of considering the original system \( N = \{ S, C, R, k, \tau \} \), we study the approximating system \( N' = \{ S', C', R', k' \} \) derived from adding a chain of first-order reactions to each delayed reactions in original system.

We assume \( m \) reactions in system \( N \) with time delay. Without loss of generality, let \( \{ y_k \rightarrow y_k', k = r - r' + 1, \ldots, r \} \). Then we can derive the dynamical equation of \( N' \) as Eq. 36. And when \( \tilde{y}(s) \) is a constant equilibrium in delayed system, the point \( (\tilde{z}^T, \tilde{v}^T)^T \) with:

\[ \tilde{z}^T = \tilde{y}(s), \tilde{v}_k = k_t \tau_k \tilde{y}^{(k)} = k_t \tau_k \tilde{y}^{(k)} = 0 \]

Above equation is true from the approximation of Eq. 37 and \( \tilde{y}^{(k)} \equiv 0 \) for all the points in \( L_0 \) where \( W \) is a locking set. \( \tilde{y} \) is an \( \omega \)-limit point, \( (\tilde{y}^T, 0^T)^T \) is also a equilibrium in the approximating system. In this way, for arbitrary \( \epsilon, T_\epsilon \) is the time that the trajectory \( (\tilde{z}(t)^T, \tilde{v}^T)^T \) enter into \( \epsilon \)-neighbourhood of \( (\tilde{y}^T, 0^T)^T \). \( T_\epsilon \) is the time that trajectory enter into \( \xi \)-neighbourhood. The
trajectory at time $T_{\epsilon}$ and $T_{\bar{z}}$ are:

$$(\bar{z}^T, \bar{v}^T)^T = (z(T_{\epsilon})^T, v^T(T_{\epsilon}))^T, \quad (\bar{z}^T, \bar{v}^T)^T = (z(T_{\bar{z}})^T, v^T(T_{\bar{z}}))^T.$$ 

If the original system is complex balanced, then approximating system is also complex balanced. So the new system have the following Lyapunov function,

$$H(x) = \sum_{i=1}^{n} h_i(x) + \sum_{k=r+1}^{c} \sum_{m=1}^{N} h_{km}(x) = \xi V\left(\frac{\bar{z}}{\bar{v}}\right) + \sum_{k=r+1}^{c} \sum_{m=1}^{N} \nu_{km} V\left(\frac{v_{km}}{V_{km}}\right)$$

where $V(r) = r \ln r + 1 - r$. From the dissipation of function $H$, we have:

$$H\left((z(T_{\epsilon})^T, v(T_{\epsilon})^T)^T\right) - H\left((z(T_{\bar{z}})^T, v(T_{\bar{z}})^T)^T\right) \leq 0 \tag{38}$$

As $W$ is locking set, above equation can be rewritten as:

$$H\left((z(T_{\bar{z}})^T, v(T_{\bar{z}})^T)^T\right) - H\left((z(T_{\epsilon})^T, v(T_{\epsilon})^T)^T\right)$$

$$= \sum_{i=1}^{d} h_i(z(T_{\bar{z}})) - h_i(z_{\epsilon}(T_{\epsilon})) + \sum_{j=d+1}^{n} h_j(z_{\bar{z}}(T_{\bar{z}})) - h_j(z_{\epsilon}(T_{\epsilon}))$$

$$+ \sum_{k=r+1}^{c} \sum_{m=1}^{N} (h_{km}(v_{km}(T_{\bar{z}})) - h_{km}(v_{km}(T_{\epsilon})))$$

$$= \sum_{i=1}^{d} \left(\ln(\bar{z}_i) - \ln(\bar{z}_{\epsilon})\right)(z_{\bar{z}} - z_{\epsilon}) + \sum_{j=d+1}^{n} \left(\ln(\bar{z}_j) - \ln(\bar{z}_{\epsilon})\right)(z_{\bar{z}} - z_{\epsilon})$$

$$+ \sum_{\supp \chi_{(\bar{v})}} \sum_{m=1}^{N} (h_{km}(v_{km}(T_{\bar{z}})) - h_{km}(v_{km}(T_{\epsilon}))) \leq 0.$$ 

The last equation holds from differential mean value theorem and $\bar{z}_i \in [z(T_{\epsilon}), z(T_{\bar{z}})]$ so that $\bar{z}_i \leq \epsilon$. When we choose $\epsilon$ small enough such that all the $x_i$ with $X_i \in W$ are smaller than $\bar{z}_i$. And in this case, $a \geq 0$. Denote $z_{\bar{z}} - z_{\epsilon}$ as $\Delta z_i$ and we have:

$$|a| \leq \left|\sum_{j=d+1}^{n} \left(\ln(\bar{z}_j) - \ln(\bar{z}_{\epsilon})\right) \Delta z_j + \sum_{\supp \chi_{(\bar{v})}} \sum_{m=1}^{N} (h_{km}(v_{km}(T_{\bar{z}})) - h_{km}(v_{km}(T_{\epsilon})))\right|$$

18
For each \( i \leq d \), if let \( |\ln(z_i)| = M_i \), \( |\ln(\tilde{z}_i)| - |\ln(z_i)| \geq |\ln \epsilon| - M_i \). Because \( \epsilon \) is small enough, \( |\ln(\epsilon)| - M_i \gg 0 \), and \( h_{km}(v_{km}) \) trends to infinity only when \( v_{km} \) goes to infinity. But from the boundedness of complex balanced and \( W \) is a locking set, we can obtain:

\[
|\Delta z_i| \leq \frac{1}{|\ln(\epsilon)| - M_i} \left( \sum_{j=d+1}^{n} c_j |\Delta z_j| + M \right)
\]

(39)

where \( c_j = \ln(\tilde{z}_j)) - \ln(z_j) \) and \( M \) is the upper bound of part \( b \). Further we assume \( \Delta_{\text{max}} = \sup_{j=d+1, \ldots, n}(|\Delta x_j|) \), \( \delta(\epsilon) = \sup_{\epsilon \in [1, \ldots, d]}(|\ln(\epsilon)|) \) and \( C = \sum_{j=d+1}^{n} c_j \). Then for \( i = 1, \ldots, d \), Eq. (39) can be written as:

\[
|\Delta x_i| \leq \delta(\epsilon)(C_{\text{max}} + M).
\]

(40)

Now it is the time to consider the vector:

\[
(z^i(\epsilon), v^i(\epsilon)) = \frac{\Delta z_i, v}{\Delta_{\text{max}}} \left[ \frac{\epsilon(T_z)}{\Delta_{\text{max}}} - \frac{\epsilon(T_v)}{\Delta_{\text{max}}} \right] \in \mathcal{S}.
\]

For \( i = 1, \ldots, d \),

\[
|z^i(\epsilon)| \leq \delta(\epsilon)D
\]

(41)

where \( D \) is a positive constant. Besides, as there must exist some \( j \in \{d+1, \ldots, n\} \) such that \( z_j = \Delta_{\text{max}} \), then \( 1 \leq ||(z^i(\epsilon), v^i(\epsilon))|| \leq M_i \) for some positive constant \( M_i \). Because \( \epsilon \) is small enough and arbitrary, we can choose an decreasing sequence \( \{\epsilon_n\} \) with \( \epsilon_n \to 0 \). Then from above analysis, we can also obtain a sequence of vectors \( \{(z^i(\epsilon_n), v^i(\epsilon_n))\} \) and each vector has one \( \delta(\epsilon_n) \) such that \( \delta(\epsilon_n) \to 0 \) when \( \epsilon_n \to 0 \). From Eq. (41) we can further derive that \( z^i(\epsilon_n), i \in \{1, \ldots, d\} \) and \( v^i_{km} \) where \( k \in \{r-r'+1, \ldots, r\}, m \in \{1, \ldots, N\} \) also trends to zero as \( \epsilon_n \to 0 \). \( v^i_{km} \to 0 \) follows that \( W \) is a locking set and each \( v^i_{km} \) can be expressed as: \( C_{0}z_i \) for some constant \( C_{0} \) and some \( i \in \{1, \ldots, d\} \). For each \( n \), vector \( (z^i(\epsilon_n), v^i(\epsilon_n)) \) is contained in compact space

\[
\mathcal{G}' = \{(z^i, v^i) : 1 \leq ||(z^i, v^i)|| \leq M_1 \} \cap \mathcal{S}
\]

So there exist a convergent sub-sequence \( \{(z^i(\epsilon_n), v^i(\epsilon_n))\} \) and it converges to the point \( (z^0, v^0) \) in \( \mathcal{G}' \cap \mathcal{S} \) when \( j \to +\infty \). From above analysis, \( z^0 > 1 \) and have the following form:

\[
(z^0, v^0) = (0, z_{d+1}, \ldots, z_n, 0_{r-N}) \in \mathcal{S}
\]

(42)

Because \( \mathcal{S}' = \mathcal{S} \oplus \mathcal{S}_{1,k} \oplus \mathcal{S}_{2,k} \), where \( \mathcal{S}_{1,k} \) have no impact on \( z^0 \) and because \( v_{k,r+1} = 0 \) for all \( k \in \{r-r'+1, \ldots, r\} \), \( \mathcal{S}_{2,k} \) also have no impact on \( z^0 \). Thus \( z^0 \) is in \( \mathcal{S} \). And then for any
\( \alpha > 0 \), we have \((z^2, v^2)^\top = (\tilde{\psi}, 0, r')^\top + \alpha(z^0, v^0)^\top \) in the same stoichiometric compatibility class with \((\tilde{\psi}, 0, r')^\top \). When \( N \to +\infty \), \((z^2, v^2)^\top\) corresponding to a constant function \( \tilde{\phi}(s) = z^2 \) for \( s \in [-\tau, 0] \). This function is also in \( L_W \). And we can compute

\[ g(\tilde{\psi}) - g(\tilde{\phi}) = z^0 \in \mathcal{S}. \]

In other words, \( \tilde{\phi} \) is also in \( L_W \cap D_\theta \). This is obviously contradict to the condition. So the unique point of intersection can not be \( \omega\)-limit point. Thus the system is also persistent. □

Then we can concluded the following result about delayed system:

**Theorem 4.6.** For the complex balanced mass action system with time delay \((S, C, R, k, \tau)\), if \( \bigcup \bigcup W \cap D_\theta \) where \( W \) is a semilocking set is discrete, then this system is persistent.

**Proof.** If \( \bigcup \bigcup W \cap D_\theta \) where \( W \) is a semilocking set is discrete, we can obtain that for each \( W \) is a semilocking set \( L_W \cap D_\theta \) must have an unique intersection. Then we prove this theorem by using above theorem. □

**Theorem 4.7.** \((S, C, R, k, \tau)\) denote a complex balanced network with delayed mass action kinetics. If \( L_W \cap D_\theta \) is empty, a vertex or a facet for each semilocking set \( W \), this system is persistent.

**Proof.** It can be proved by Theorem 4.5 and Lemma 4.2 directly. □

5. Some examples

This section we will give two examples to show the main two results in our paper.

**Example 1.** Consider the network \((S, C, R, k, \tau)\) as follows:

\[ 3X_1 \iff 2X_1 + X_2 \quad X_2 \iff X_3 \]

where \( k, \tau \) are the vector of reaction rate and time-delay of reactions respectively. It is easy checked that the deficiency of above network is zero and the stoichiometric subspace \( \mathcal{S} \):

\[ \mathcal{S} = \text{span}([(1, -1, 0)^\top, (0, 1, -1)^\top]. \]

And \((1, 1, 1)^\top\) is a mass conservation law of this network. Now we consider the long term property of this network. There only exist two semilocking set: \( W_1 = \{X_1\} \) and \( W_2 = \{X_1, X_2, X_3\} \) and

\[ L_{W_1} = \{\psi \in \mathcal{C}, \psi(s) = (0, \psi_2(s), \psi_3(s)), s \in [-\tau, 0], \psi_2(s), \psi_3(s) \neq 0\} \text{ and } L_{W_2} = \{(0, 0, 0)\}. \]
Figure 1: This graph shows the evolution of species as time goes on with different initial data and time delay: $x_1(t), x_2(t), x_3(t)$ of the network.
Figure 2: This graph shows the evolution of species as time goes on with different initial data and time delay: $x_1(t), x_2(t), x_3(t)$ of network 2.

But from the conservation of the network, we know $L_{W_2}$ is unattainable. So we just need to consider $L_{W_1}$. So we first to derive the dimension of $Z_{W_1}$. Any two functions $\psi, \phi$ in $L_{W_1}$,

$$v_1 = g_1(\psi) - g_1(\phi) = 0 - 0 = 0 \quad (43)$$

where $g_1$ is the first component of the vector $g$. From 22 and 43, we know that each element $v$ in $Z_{W_1}$ can be expressed by vector $(0, 1, -1)^T$. So the dimension is one. Thus $Z_{W_1}$ is the facet. Thus we use 4.3 to obtain that the above system is persistence.

We can also see in [7] when we choose different time delay, the trajectory starting from a positive data will trend to the unique positive constant equilibrium in its stoichiometric compatibility class(equilibrium in this network: $x_1(t) = x_2(t) = x_3(t)$).

Now we give another system which can apply 4.5 to judge the persistence property.
Example 2. The time-delayed complex balanced system $(S, C, R, k, \tau)$

\[2X_1 \iff X_1 + X_2 \iff X_2 + X_3 \quad (44)\]

and $k, \tau \in \mathbb{R}_{\geq 0}^r$ denote the vectors of reaction rate constant and time-delay of each reaction respectively. The stoichiometric subspace:

\[\mathcal{S} = \text{span} \{(-1, 1, 0)^\top, (-1, 0, 1)^\top\}\]

Thus time-delay system has three complexes, one linkage class and the dimension of stoichiometric subspace is two. In this case, this system is a zero-deficiency system. And all semilocking sets $W_1 = \{X_1, X_2\}, W_2 = \{X_1, X_3\}$ and $W_3 = \{X_1, X_2, X_3\}$ are locking sets. Then

\[L_{W_1} = \{\psi \in \mathcal{S}^+_T | \psi(s) = (0, 0, \psi_3(s)), s \in [-\tau, 0], \psi_3(s) > 0\}\]
\[L_{W_2} = \{\psi \in \mathcal{S}^+_T | \psi(s) = (0, \psi_2(s), 0), s \in [-\tau, 0], \psi_2(s) > 0\}\]
\[\text{and } L_{W_3} = \{(0, 0, 0)\}\]

Any trajectory starting from a positive initial function never goes to $L_{W_3}$ for the conservation of this network. From the definition of $Z_W$ and $Z_W \subset \mathcal{S}$, the dimension of $Z_{W_1}$ and $Z_{W_2}$ is zero. Thus from 4.5 we can obtain this system is persistence.

Besides, the four graphs in 2 also reveal that no matter what time-delay is, the trajectory converges to the positive equilibrium(equilibrium: $x_1 = x_2 = x_3$)

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