Are waves all localized in two dimensional random media?

Zhen Ye

Department of Physics, National Central University, Chungli, Taiwan 32054

(Dated: October 16, 2002)

PACS numbers: 43.25.Fx, 71.55.Jv

The concept of localization was originally introduced by Anderson[1] for electrons in a crystal. In the case of a perfectly periodic lattice, except in the gaps all the electronic states are extended and are represented by Bloch states. When a sufficient amount of disorders is added to the lattice, for example in the form of random potentials, the electrons may become spatially localized due to the multiple scattering by the disorders. In such a case, the eigenstates are exponentially confined in the space. By a scaling analysis[2], Abraham et al. suggested that there can be no metallic state or metal-insulator transition in two dimensions in zero magnetic field. In other words, all electrons are always localized in two dimensions (2D), as reviewed in [3, 4].

The fact that the electronic localization is due to the wave nature of electrons has led to the conjecture that the localization phenomenon also exists for classical waves in random media. All predictions for the electronic localization are believed to hold for classical waves. Following the scaling analysis of the electronic localization, it was widely accepted that all waves are localized in 2D random media. This has been the prevailing view for the past twenty years (e.g. Ref. [5, 6, 7]). Hereafter I will refer to this view as ‘2D Conjecture’.

In this Letter, I propose that the popular view on 2D localization may not be valid. While the ‘2D Conjecture’ has been challenged and shown to be likely incorrect for electronic systems (Reviewed in [3, 4]), here I will focus on classical waves. For the purpose, I will first review the origin of the ‘2D Conjecture’ and the current theory on 2D wave localization. Then point out the ambiguities in the theory and discuss the evidence that is in conflict with the ‘2D Conjecture’. We take the following steps.

(1) Re-inspect the scaling analysis, checking for ambiguities. (2) Examine the predictions from the current theory, checking for its validity. (3) Examine the previous experimental and numerical results that claim to support the ‘2D Conjecture’, checking for their appropriateness. (4) Find the self-conflicting points in the current theory, and discuss an apparent mechanism which is in conflict with the theory. The idea is that if there are conflicts, the 2D conjecture should be at least skeptical. The confidence is: Wave scattering in many 2D systems is exactly calculable following Twersky[8]; thus previous predictions can be put under a close scrutiny. We note that the electronic system is more complicated because effects such as the Coulomb interaction makes data interpretation difficult. In this sense, the classical wave systems are advantageous in studying localization effects.

The scaling analysis According to [2], an hypercubic geometry is used for the scaling analysis. In the metallic state, the resistance follows the Ohmic behavior

\[ R \sim L^{2-d}, \]

where \( d \) is the dimension. For a localized state, i.e. large \( R \), the resistance grows exponentially

\[ R \sim e^{L/L_1}, \]

where \( L_1 \) is the localization length which may differ for different dimensions. A scaling function is defined as \( \beta = \frac{2\ln R}{L} \). Taking Eqs. (1) and (2) into this, we obtain the asymptotic behavior

\[ \beta \sim \begin{cases} \ln R, & \text{as } R \to \infty \text{ (Localized)} \\ 2-d, & \text{as } R \to 0 \text{ (Ohmic)} \end{cases} \]

From the asymptotic behavior in Eq. (3), one can sketch the universal curves in \( d = 1, 2, 3 \) dimensions. The central assumptions in [2] are (1) \( \beta \) is continuous; (2) \( \beta \) is a function of \( R \) and depends on other parameters such as disorders and length scale only through \( R \); and (3) once wave is localized, the increasing sample size would always mean more localization.

The behavior of \( \beta \) is plotted in Fig. 1. It is clear that in the 3D case, the curve crosses the horizontal axis, yielding an unstable fixed point (B). Above this point, the
waves become more and more localized as the sample size increases. Below the critical point, the system tends to follow the Ohmic behavior as the sample size is enlarged. This fixed point separates the localized and non-localized states. For the two dimensional case, in the Ohmic regime $\beta$ approaches zero as $\ln(R) \to 0$. But the perturbation calculation including the wave interference effect shows that $\beta$ is always greater than zero. Therefore for both one and two dimensions, the curves do not cross the horizontal axis, and there is thus no fixed point. As the sample size increases, all states move towards the localization regime. This has been the main reason that led previously to the conclusion that all waves are localized in one and two dimensions.

The current localization theory Now I briefly review the existing theory for localization. As wave propagates in random media, it experiences multiple scattering, and as a result, the wave loses its phase, leading to the gradual decreases of the coherence of the wave in the absence of absorption. Meanwhile, diffusive wave is built up as more and more scattering takes place. The procedure to obtain the localization state can be briefly summarized as follows.

The quantity, $D^{(B)}$ which is a measure of diffusion of classical waves is called the classical Boltzmann diffusion constant and it may be derived under the coherent potential approximation [3], and is given as

$$D^{(B)} \sim \frac{v_l l}{d} \quad (4)$$

where $v_l$ is the transport velocity, $l$ is the mean free path and $d$ is the dimensionality.

As waves scattered along any two reversed paths in the backward direction interfere constructively, leading to the enhanced backscattering effect, which will add corrections to the diffusion coefficient. In the field theory approach, such an enhanced backscattering effect is represented by a set of maximally crossed ladder diagrams[4]. In the two dimension case, the evaluation of these diagrams leads to an integration for which two cut-off limits have to be introduced to avoid the divergence. The correction to the diffusion constant for two dimensional systems is thus found as

$$\delta D \sim -\ln(L_M/l_m) \quad (5)$$

where $L_M$ and $l_m$ are the two cut-off limits. It is then interpreted in the previous theory that the cut-off limit $l_m$ is a measure of the minimum scaling for the waves and is thought to be related to (for example) the mean free path, whereas $L_M$ is a measure of the effective size of the sample. It is rather important to note that the correction in Eq. (5) is not only negative but diverges as $L_M$ approach infinity. This is obviously unphysical, since the corrected diffusion constant cannot be negative. To avoid the problem, it was suggested that $L_M$ is related to the localization range, or simply the localization length, in such way that when $L_M$ is equal to the localization length denoted by $\xi$ say, the corrected diffusion coefficient becomes zero:

$$D_R(\xi) = D^{(B)} + \delta D(\xi) = 0. \quad (6)$$

The localization length $\xi$ is subsequently solved for from this equation. It is obvious that this equation always allows a solution. Therefore a localization length can always be found in two dimensions. Such a backscattering induced absence-of-diffusion mechanism is the core of the current theory of localization in two dimensions, and is considered a strong support of the '2D Conjecture' from the scaling analysis[2].

On the scaling analysis While it is simple and straightforward, the above scaling analysis is not without ambiguities in investigating the localization effect. The reasons follow. Whether a system has non-localized or only localized states is an intrinsic property of the system, and should not rely on neither the boundary nor the source. As long as the analysis cannot exclude the possibility that the boundary or the source is playing a role, the consequence from the analysis is deemed to be questionable. In order to isolate the localization or non-localization effect, therefore, a genuine analysis should not be plagued by boundary effects not only in the localization region but also in the non-localization region. Of course, if the system has indeed only localized states, the boundary is not an issue, as the dependence on the boundary is exponentially vanishing. However, the care must be taken for the non-localized regime. It is not difficult to see that the above scaling theory may work for situations when both probing contacts, used to measure the resistance or conductance from which the localization is inferred, are located outside the system. In this case, the Ohmic behavior given by Eq. (1) is valid under the condition that the current flows uniformly in one direction. This is possible only with properly scaled sources and with the presence of confining boundaries, obviously in conflict with the proclamation that whether it is a localization or non-localized state is the intrinsic property of the system and should not rely on a boundary nor a source. Thus the above analysis is more appropriate for studying transport phenomena. It is our opinion that the reduction in the conductance does not necessarily mean that all waves are actually localized. In other words, it is necessary to differentiate the situation that the electrons are prohibited from transmission through a random medium from the situation that the system has only localized states.

Not to be neglected, the recent numerical results on acoustic waves[11] do support the above comments on the scaling theory. They point out that waves can be blocked from propagation through a medium by disorders, but such an inhibition is not necessarily an indication that waves can be localized in the medium when the transmitting source is put inside the medium. The same conclusion is also obtained from the simulations on propagation of electromagnetic waves (EM) in 2D random dielectric media[12]. More explicitly, it is shown that waves certainly cannot propagate through a random medium when it has only localized states. But the
observation that waves cannot propagate through does not necessarily imply that the system has only localized states.

To this end, it is proper to mention that a recent scaling analysis shows that the difference between 2D and 3D disordered systems is insignificant\(^{[13]}\), i.e. like 3D the localization-delocalization transition is also possible in 2D, which could explain the recently observed ‘unusual’ metallic behavior in 2D electronic systems (See references cited in \([4]\)). This scaling analysis has been pointed out to be absolutely in line with recent findings of the use of the Kubo formula (Y. Tarasov, private communication).

On the validity of the current theory Among many, there are two basic ways to check the validity of the aforementioned theory.

First, the key parameter obtained from the current theory is the localization length. This quantity can also be obtained exactly by numerical computation using the scheme detailed in \([8]\). Then we can make a comparison of the two results. We consider the model of acoustic scattering in water with air-cylinders detailed in \([14]\). The comparison is shown in Fig. 2. The comparison clearly shows that there is a significant difference between the numerically exact results and the results obtained from the theory. Further comparisons indicate that the difference between the two results is significant not only quantitatively but qualitatively\(^{[3]}\).

![FIG. 2: Localization length (ξ) is shown as a function of frequency (ka) for β=0.001. The dashed curve with circles represent the exact values obtained numerically while the solid curve is obtained from theory. Here k is the wavenumber, a is the radius of the cylinder, and β is the fraction of area occupied by the cylinders per unit area.](image)

Second, one can evaluate the localization length from the theory. Using an exactly solvable model, then inspect the spatial distribution of the wave energy density and check whether localization occurs when the system is larger than the obtained localization length. Again we consider the model of acoustic localization in air-filled cylinders in water. From Fig. 2, the theory predicts that the shortest localization length is around \(ka = 0.005\) which is the nature frequency of an air-cylinder in water\(^{[14]}\). The numerical results show, however, that no localization occurs around this frequency. Instead, localization appears at somewhat higher frequency ranges with the same system size, referring to Fig. 3 of \([4]\).

On the previous experimental and numerical evidence It was widely believed that the theory of 2D localization has been tested experimentally to be successful. We find that the claimed success is mainly based on two types of experiments. One is the indirect method which measures the effects of the enhanced backscattering (e.g. Ref. \([14]\)). In a rigorous simulation, it has been shown that the enhanced backscattering is not related to the localization\(^{[7]}\). Waves are not necessarily localized when a strong enhanced backscattering exists, and sometimes waves can be localized even when the enhanced backscattering is weak. Besides from few exceptions\(^{[3]}\), the other type of experiments is based on observations of the exponential decay of waves as they propagate through disordered media, as stated in Ref. \([8]\). According to the above and following discussions, this type of experiments is not sufficient to discern whether the medium really only has localized states. Unwanted effects of non-localization origin can also contribute to the wanted exponential decay, making data interpretation ambiguous. In fact, the authors in Ref. \([7]\) pointed out that there is no conclusive experimental evidence for localization of EM waves in 2D. We mention that there was a report of the observation of microwave localization in two dimensions when a transmitting source is inside disordered media\(^{[1]}\). However, the diffusion based theory has not been verified against this experimental result.

The same situation can be said about the numerical simulations. Take Ref. \([1]\) as the example. The authors considered the EM propagation in a random array of dielectric cylinders. The localization length is computed from the reduction in the transmission across the random sample. Following their method, we first compute the transmission versus the sample size at various frequencies with the source and the receiver located at the opposite sides of the sample. We do observe exponential decays, and the decay rates depend on the frequency. According to \([7]\), the localization lengths can be estimated from this decay rates. Then as long as the sample size is bigger than this length, we would also expect to observe an exponential decay in the transmission when source is put inside the medium. But we found that the exponential decay disappears for some frequencies. As an example, the results for two frequencies are shown in Fig. 3. In line with \([8]\), the following parameters are used in the computation. The dielectric constants of the cylinders and the medium are 10 and 1 respectively. The fraction of area occupied by the cylinders per unit area, is 0.28. The radius a of the cylinders is 0.38 cm. The lattice constant \(d\) of the corresponding square lattice array is calculated as 1.28 cm. All lengths are scaled by the lattice constant \(d\). Here we see that the exponential decay at 14.70GHz, shown when the transmission is across the sample, disappears when the source is moved into the medium. The
results suggest that waves are not localized at this frequency. One may still argue that the non-localization is due to the fact that the localization length is long compared to the sample size in the ‘Inside’ case. Even if this were the case, the exponential decay shown for the ‘Outside’ case could not be due to the localization effect. Reiterating, it is not sufficient to extract the localization effect by merely computing the transmission reduction across the sample. Therefore the claim about the 2D localization like in [14] is not appropriate. At 11.75GHz, we observe that the exponential decay with nearly the same slope holds for both cases - the slight difference is due to the finite width in the ‘Outside’ case[12], and reveals the genuine localization behavior.

\[ \psi_j = \text{constant at least by domains while } |\psi|^2 = 0. \]

The physical picture of localization It seems that a general picture of localization may be obtained. For quantum mechanic or acoustic waves (the same argument also holds for EM systems[13]), the current can be written as \( \vec{J} \sim \text{Re}[\psi(-i) \nabla \psi] \), where \( \psi \) stands for the wave function for quantum mechanical systems and for the pressure in acoustic systems. Writing the field as \( \psi = |\psi|e^{i\theta} \), the current becomes \( \vec{J} \sim |\psi|^2 \nabla \theta \). It is clear that when \( \theta \) is constant at least by domains while \( |\psi| \neq 0 \), the flow stops, i.e., \( \vec{J} = 0 \), and the wave or the energy is localized in space, i.e., \( |\psi|^2 \neq 0 \). Obviously the constant phase \( \theta \) indicates the appearance of a coherence in the system. This coherent-phase picture has been demonstrated successfully not only for two dimensional media[20], but for one and three dimensions as well[21].

The current diffusion-based theory does not support the above picture. The physical picture of the theory is: Waves will undergo a diffusion process when the system is smaller than the localization size. As the system increases, the diffusion gradually diminishes and finally comes to a complete stop when the size exceeds the localization length. If this picture were valid, then one would expect a significant change in the spatial distribution of the energy density, from that of diffusion to the exponentially confined envelop. This is not evident in the theory and is not supported by numerical results. In contrast, the numerical results in Fig. 3 show that the exponential decay starts even when the sample size is smaller than the localization length; the distribution of the characters of a diffusion process does not appear.

In summary, concerns have been raised about the previous claim that all waves are localized in 2D. In fact, non-localized states have already been reported by numerical computation in, for instance, Ref. [14]. Even if this could be argued to be due to the finite sample size limited by computing facilities, there are still many other reasons for being doubtful about the ‘2D Conjecture’. Some important reasons are corroborated here.

\[ |\psi|^2 \neq 0 \]

\[ \vec{J} \sim |\psi|^2 \nabla \theta \]

\[ \text{Re}[\psi(-i) \nabla \psi] \]

\[ |\psi|e^{i\theta} \]

\[ \text{constant at least by domains while } |\psi|^2 = 0. \]