Uniqueness of differential polynomials sharing one value

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Abstract

We prove some uniqueness results which improve and generalize results of Jiang-Tao Li and Ping Li\cite{Li15}\textit{Uniqueness of entire functions concerning differential polynomials. Commun. Korean Math. Soc. 30 (2015), No. 2, pp. 93-101}.

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1 Introduction

Let $f$ be a non-constant meromorphic function in the complex plane $\mathbb{C}$. We assume that the reader is familiar with the standard notions of the Nevanlinna value distribution theory such as $T(r,f)$, $m(r,f)$, $N(r,f)$ (see e.g., [3]).

For $a \in \mathbb{C} \cup \{\infty\}$, we say that two meromorphic functions $f$ and $g$ share a CM, if $f - a$ and $g - a$ have the same set of zeros with same multiplicities, and if we do not consider the multiplicities then $f$ and $g$ are said to share a IM.

In [11], C.C. Yang posed the following question:

**Question:** What can be said about two entire functions $f$ and $g$, when they share 0 CM and their derivatives share 1 CM?

In 1990, Yi [4, 5], answered the above question by proving: Let $f$ and $g$ be two non-constant entire functions such that $f$ and $g$ share the value 1 CM and $\delta(0,f) > 1/2$, where $k$ is non-negative integer, then $f \equiv g$ unless $f(k), g(k) = 1$; and for meromorphic functions he proved: Let $f$ and $g$ be two non-constant meromorphic functions such that $f$ and $g$ share 0 and $\infty$ CM. If $f(k)$ and $g(k)$ share the value 1 CM and $2\delta(0,f) + (k+2)\Theta(\infty,f) > k + 3$, where $k$ is non-negative integer, then $f \equiv g$ unless $f(k), g(k) = 1$.

For a non-constant meromorphic function $h$, we denote by

$$L(h) = h^{(k)} + a_1 h^{(k-1)} + a_2 h^{(k-2)} + \ldots + a_{k-1} h' + a_k h,$$

the differential polynomial of $h$, where $a_1, a_2, \ldots, a_k$ are finite complex numbers and $k$ is a positive integer. We denote the order and lower order of $h$ by $\lambda(h)$ and $\mu(h)$, respectively. Also by $\sigma(h)$ and $\sigma(1/h)$, we denote the exponent of convergence of zeros and poles of $h$ respectively.

Recently, Jiang-Tao Li and Ping Li [2] generalized first result of Yi (as stated above) for entire functions as

**Theorem A.** Let $f$ and $g$ be two non-constant entire functions such that $f$ and $g$ share 0 CM. Suppose $L(f)$ and $L(g)$ share 1 CM and $\delta(0,f) > 1/2$. If $\lambda(f) \neq 1$, then $f \equiv g$ unless $L(f), L(g) \equiv 1$.

**Theorem B.** Let $f$ and $g$ be two non-constant entire functions such that $f$ and $g$ share 0 CM. Suppose $L(f)$ and $L(g)$ share 1 IM and $\delta(0,f) > 4/5$. If $\lambda(f) \neq 1$, then $f \equiv g$ unless $L(f), L(g) \equiv 1$.

We recall the following definition of weighted sharing:

**Definition 1.1.** Let $f$ and $g$ be two non constant meromorphic functions and $k$ be a non-negative integer or $\infty$. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a,f)$ the set of all $a$-points of $f$, where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a,f) = E_k(a,g)$, we say that $f$ and $g$ share the value $a$ with weight $k$. 

We write “$f$ and $g$ share $(a,k)$” to mean that “$f$ and $g$ share the value $a$ with weight $k$”. Clearly if $f$ and $g$ share $(a,k)$, then $f$ and $g$ share $(a,p)$, $0 \leq p < k$. Also we note that $f$ and $g$ share the value $a$ IM(ignoring multiplicity) or CM(counting multiplicity) if and only if $f$ and $g$ share $(a,0)$ or $(a,\infty)$, respectively.

**Definition 1.2.** Let $f$ and $g$ share 1 IM, and let $z_0$ be a zero of $f − 1$ with multiplicity $p$ and a zero of $g − 1$ with multiplicity $q$. We denote by $N_1^{(1)} (r,1/(f − 1))$, the counting function of the zeros of $f − 1$ when $p = q = 1$. By $\overline{N}_E (r,1/(f − 1))$, we denote the counting function of the zeros of $f − 1$ when $p = q \geq 2$ and by $\overline{N}_L (r,1/(f − 1))$, we denote the counting function of the zeros of $f − 1$ when $p > q \geq 1$, each point in these counting functions is counted only once; similarly, the terms $N_1^{(1)} (r,1/(g − 1))$, $\overline{N}_E^{(1)} (r,1/(g − 1))$ and $\overline{N}_L (r,1/(g − 1))$. Also, we denote by $\overline{N}_{f > k} (r,1/(g − 1))$, the reduced counting function of those zeros of $f − 1$ and $g − 1$ such that $p > q = k$, and similarly the term $\overline{N}_{g > k} (r,1/(f − 1))$.

With the help of weighted sharing, we generalize Theorem A and Theorem B as

**Theorem 1.3.** Let $f$ and $g$ be two non-constant entire functions such that $f$ and $g$ share 0 CM. Suppose $L(f)$ and $L(g)$ share $(1,l)$, $l \geq 0$ with one of the following conditions:

- (i) $l \geq 2$ and $\delta(0,f) > 1/2$
- (ii) $l = 1$ and $\delta(0,f) > 3/5$
- (iii) $l = 0$ and $\delta(0,f) > 4/5$.

If $\lambda(f) \neq 1$, then $f \equiv g$ unless $L(f).L(g) \equiv 1$.

For meromorphic functions, we prove the following result:

**Theorem 1.4.** Let $f$ and $g$ be two non-constant meromorphic functions of finite order such that $f$ and $g$ share 0 and $\infty$ CM. Suppose $L(f)$ and $L(g)$ share $(1,l)$, $l \geq 0$ with one of the following conditions:

- (i) $l \geq 2$ and

$$ (k + 2)\Theta(\infty,f) + 2\delta(0,f) > k + 3 \quad (1.1) $$

- (ii) $l = 1$ and

$$ (3k + 5)\Theta(\infty,f) + 5\delta(0,f) > 3k + 9 \quad (1.2) $$

- (iii) $l = 0$ and

$$ (4k + 5)\Theta(\infty,f) + 5\delta(0,f) > 4k + 9 \quad (1.3) $$

If $\lambda(f) \neq 1$ and $\sigma(1/f) \leq \sigma(f)$, then $f \equiv g$ unless $L(f).L(g) \equiv 1$.

The main tool of our investigations in this paper is Nevanlinna value distribution theory of meromorphic functions(see [3]).
2 Proof of the Main Result

We shall use the following results in the proof of our main result:

Lemma 2.1. [2] Let \( f \) be a non-constant meromorphic function and \( k \) be a non-negative integer. Then

\[
T(r, L(f)) \leq T(r, f) + kN(r, f) + S(r, f). \tag{2.1}
\]

Lemma 2.2. [2] Let \( f \) be a non-constant meromorphic function and \( a \) be a meromorphic function such that \( T(r, a) = o(T(r, f)) \) as \( r \to \infty \). If \( f \) is not a polynomial, then

\[
N \left( r, \frac{1}{L(f) - L(a)} \right) \leq T(r, L(f)) - T(r, f) + N \left( r, \frac{1}{f - a} \right) + S(r, f) \tag{2.2}
\]

and

\[
N \left( r, \frac{1}{L(f) - L(a)} \right) \leq N \left( r, \frac{1}{f - a} \right) + kN(r, f) + S(r, f). \tag{2.3}
\]

Lemma 2.3. [1] Let \( f \) and \( g \) be two non-constant meromorphic functions.

(i) If \( f \) and \( g \) share \((1,0)\), then

\[
N_L \left( r, \frac{1}{f - 1} \right) \leq N \left( r, \frac{1}{f} \right) + N(r, f) + S(r), \tag{2.4}
\]

where \( S(r) = o(T(r)) \) as \( r \to \infty \) with \( T(r) = \max \{ T(r, f); T(r, g) \} \).

(ii) If \( f \) and \( g \) share \((1,1)\), then

\[
2N_L \left( r, \frac{1}{f - 1} \right) + 2N_L \left( r, \frac{1}{g - 1} \right) + N_E^2 \left( r, \frac{1}{f - 1} \right) - N_{f>2} \left( r, \frac{1}{g - 1} \right)
\leq N \left( r, \frac{1}{f - 1} \right) - N \left( r, \frac{1}{g - 1} \right). \tag{2.5}
\]

Lemma 2.4. [10] Suppose \( f_j \ (j = 1, 2, \ldots, n) \) and \( g_j \ (j = 1, 2, \ldots, n) \) \((n \geq 1)\) are entire functions satisfying the following conditions:

(i) \( \sum_{j=1}^{n} f_j(z)e^{g_j(z)} = f_{n+1}(z) \),

(ii) The order of \( f_j(z) \) is less than the order of \( e^{g_k(z)} \) for \( 1 \leq j \leq n+1 \), \( 1 \leq k \leq n \). And furthermore, the order of \( f_j(z) \) is less than the order of \( e^{g_h(z)} \) for \( n \geq 2 \) and \( 1 \leq j \leq n+1 \), \( 1 \leq h < k \leq m \).

Then \( f_j \equiv 0 \ (j = 1, 2, \ldots, n+1) \).

Lemma 2.5. [10] Suppose \( f_j \ (j = 1, 2, \ldots, n) \) are meromorphic functions and \( g_j \ (j = 1, 2, \ldots, n) \) \((n \geq 2)\) are entire functions satisfying the following conditions:
(i) \( \sum_{j=1}^{n} f_j(z)e^{g_j(z)} = 0. \)

(ii) \( g_j(z) - g_k(z) \) are non-constants for \( 1 \leq j < k \leq n. \)

(iii) For \( 1 \leq j \leq n, \ 1 \leq h < k \leq n, \)

\[ T(r, f_j) = o(T(r, e^{g_h-g_k})), \]

as \( r \to \infty. \) Then \( f_j(z) \equiv 0 \ (j = 1, 2, ..., n). \)

Lemma 2.6. [10] If \( h(z) \) be a polynomial of degree \( p \) and \( f(z) = e^{h(z)} \), then \( \lambda(f) = \mu(f) = p. \)

Lemma 2.7. [10] Let \( f(z) \) and \( g(z) \) be two non-constant meromorphic functions in the complex plane. If \( \lambda(f) < \mu(g) \), then \( T(r, f) = o(T(r, g)) \) as \( r \to \infty. \)

We only prove Theorem 1.4 as the proof of Theorem 1.3 follows on the similar lines.

Proof of Theorem 1.4: First we assume that \( L(f) \equiv c, \) a finite constant. Then \( f \) has to be entire and

\[ f \equiv c_1 + \sum_{i=1}^{m} p_i(z)e^{\alpha_i z}, \]

where \( c_1 \) is finite constant, \( m(\leq k) \) is a positive integer, \( \alpha_i \) are distinct complex numbers and \( p_i(z) \) are polynomials \( (i = 1, 2, ..., m). \)

Since \( \lambda(f) \neq 1, \) we get \( \lambda(f) < 1 \) and so \( e^{\alpha_i z} \) is constant. Thus \( f \) is a polynomial and so \( \delta(0, f) = 0, \) which contradicts (1.1), (1.2) and (1.3).

Assume that both \( L(f) \) and \( L(g) \) are non-constant. Since \( f \) and \( g \) share 0 and \( \infty \) CM, and \( L(f) \) and \( L(g) \) share \( (1, l) \), it follows from Milloux’s inequality and (2.3)

\[ T(r, f) \leq \overline{N}(r, f) + N(r, \frac{1}{f}) + \overline{N} \left( r, \frac{1}{L(f)-1} \right) + S(r, f) \]

\[ = \overline{N}(r, g) + N(r, \frac{1}{g}) + \overline{N} \left( r, \frac{1}{L(g)-1} \right) + S(r, f) \]

\[ \leq 2T(r, g) + k\overline{N}(r, g) + N \left( r, \frac{1}{g-1} \right) + S(r, f) + S(r, g) \]

\[ \leq (k + 3)T(r, g) + S(r, f) + S(r, g). \]

Similarly

\[ T(r, g) \leq (k + 3)T(r, f) + S(r, f) + S(r, g). \]

Thus \( S(r, f) = S(r, g) \) and \( \lambda(f) = \lambda(g). \)
Let \( F = L(f) \) and \( G = L(g) \). Then \( F \) and \( G \) share \((1, l)\), \( l \geq 0 \). Define
\[
H = \left( \frac{F''}{F'} - \frac{2F'}{F - 1} \right) - \left( \frac{G''}{G'} - \frac{2G'}{G - 1} \right).
\] (2.6)

Assume that \( H \not\equiv 0 \). Then from (2.6), we have
\[
m(r, H) = S(r, F) + S(r, G).
\]

By the Second fundamental theorem of Nevanlinna, we have
\[
T(r, F) + T(r, G) \leq N(r, F) + N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{F - 1} \right) + N(r, G) + N \left( r, \frac{1}{G} \right)
\]
\[
+ N \left( r, \frac{1}{G - 1} \right) - N_0 \left( r, \frac{1}{F'} \right) - N_0 \left( r, \frac{1}{G'} \right) + S(r, F) + S(r, G)
\]
\[
= 2N(r, F) + N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{F - 1} \right) + N \left( r, \frac{1}{G} \right)
\]
\[
+ N \left( r, \frac{1}{G - 1} \right) - N_0 \left( r, \frac{1}{F'} \right) - N_0 \left( r, \frac{1}{G'} \right) + S(r, F) + S(r, G),
\] (2.7)

where \( N_0(r, 1/F') \) denotes the counting function of the zeros of \( F' \) which are not the zeros of \( F(F - 1) \) and \( N_0(r, 1/G') \) denotes the counting function of the zeros of \( G' \) which are not the zeros of \( G(G - 1) \).

We consider the following cases:

**Case (i).** If \( l \geq 1 \), then from (2.6), we have
\[
N_E^{(1)} \left( r, \frac{1}{F - 1} \right) \leq N \left( r, \frac{1}{H} \right) + S(r, F) + S(r, G)
\]
\[
\leq T(r, H) + S(r, F) + S(r, G)
\]
\[
= N(r, H) + S(r, F) + S(r, G)
\]
\[
\leq N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{G} \right) + N \left( r, \frac{1}{F - 1} \right)
\]
\[
+ N \left( r, \frac{1}{G - 1} \right) - N_0 \left( r, \frac{1}{F'} \right) - N_0 \left( r, \frac{1}{G'} \right) + S(r, F) + S(r, G)
\]
and so

\[
N \left( r, \frac{1}{F-1} \right) + N \left( r, \frac{1}{G-1} \right) = N^1 \left( r, \frac{1}{F-1} \right) + N^2 \left( r, \frac{1}{F-1} \right) + N \left( r, \frac{1}{F-1} \right) \\
+ N \left( r, \frac{1}{G-1} \right) + N \left( r, \frac{1}{G-1} \right) + S(r, F) + S(r, G) \\
\leq N(2 \left( r, \frac{1}{F} \right) + N(2 \left( r, \frac{1}{G} \right) + 2N \left( r, \frac{1}{F-1} \right) \\
+ 2N \left( r, \frac{1}{G-1} \right) + N \left( r, \frac{1}{F-1} \right) + N \left( r, \frac{1}{G-1} \right) \\
+ N_0 \left( r, \frac{1}{F} \right) + N_0 \left( r, \frac{1}{G} \right) + S(r, F) + S(r, G). \\
(2.8)
\]

Subcase 1.1: When \( l = 1 \). Then we have

\[
N \left( r, \frac{1}{F-1} \right) \leq \frac{1}{2} N \left( r, \frac{1}{F-1} | F \neq 0 \right) \leq \frac{1}{2} N(r, F) + \frac{1}{2} N \left( r, \frac{1}{F} \right), \\
(2.9)
\]

where \( N \left( r, \frac{1}{F} | F \neq 0 \right) \) denotes the zeros of \( F' \), that are not the zeros of \( F \).

From (2.8) and (2.10), we have

\[
2N \left( r, \frac{1}{F-1} \right) + 2N \left( r, \frac{1}{G-1} \right) + N \left( r, \frac{1}{F-1} \right) + N \left( r, \frac{1}{G-1} \right) \\
\leq N \left( r, \frac{1}{G-1} \right) + N \left( r, \frac{1}{F-1} \right) + S(r, F) + S(r, G) \\
\leq N \left( r, \frac{1}{F-1} \right) + \frac{1}{2} N(r, F) + \frac{1}{2} N \left( r, \frac{1}{F} \right) + S(r, F) + S(r, G). \\
(2.10)
\]

Thus, from (2.8) and (2.10), we have

\[
N \left( r, \frac{1}{F-1} \right) + N \left( r, \frac{1}{G-1} \right) \leq \frac{1}{2} N(r, F) + N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{G} \right) \\
+ \frac{1}{2} N \left( r, \frac{1}{F-1} \right) + N \left( r, \frac{1}{G-1} \right) + N_0 \left( r, \frac{1}{F} \right) + N_0 \left( r, \frac{1}{G} \right) + S(r, F) + S(r, G) \\
\leq \frac{1}{2} N(r, F) + N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{G} \right) \\
+ \frac{1}{2} N \left( r, \frac{1}{F-1} \right) + N \left( r, \frac{1}{G-1} \right) + T(r, G) \\
+ N_0 \left( r, \frac{1}{F} \right) + N_0 \left( r, \frac{1}{G} \right) + S(r, F) + S(r, G). \\
(2.11)
\]
From (2.2), (2.3), (2.7) and (2.11), we obtain

$$\begin{align*}
T(r, F) & \leq \frac{5}{2} N(r, F) + N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{G} \right) + N \left( r, \frac{1}{H} \right) \\
& \quad + \frac{1}{2} N \left( r, \frac{1}{F} \right) + S(r, F) + S(r, G) \\
& \leq \frac{5}{2} N(r, F) + N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{G} \right) + \frac{1}{2} N \left( r, \frac{1}{F} \right) + S(r, F) + S(r, G) \\
& = \frac{5}{2} N(r, F) + N \left( r, \frac{1}{L(f)} \right) + \frac{1}{2} N \left( r, \frac{1}{L(g)} \right) + \frac{1}{2} N \left( r, \frac{1}{L(h)} \right) + S(r, F) + S(r, G) \\
& \leq \frac{5}{2} N(r, f) + T(r, L(f)) - T(r, f) + N \left( r, \frac{1}{f} \right) + \frac{1}{2} N \left( r, \frac{1}{f} \right) \\
& \quad + \frac{k}{2} N(r, f) + N \left( r, \frac{1}{g} \right) + k N(r, g) + S(r, f) + S(r, g) \\
& = T(r, L(f)) - T(r, f) + \left( \frac{3k + 5}{2} \right) N(r, f) + \frac{5}{2} N \left( r, \frac{1}{f} \right) + S(r, f).
\end{align*}$$

That is,

$$2T(r, f) \leq (3k + 5\Theta(\infty, f) + 5\delta(0, f) \leq 3k + 8, a contradiction to (1.2).

Subcase 1.2: When $l \geq 2$.

In this case, we have

$$\begin{align*}
2 N_L \left( r, \frac{1}{R-1} \right) & + 2 N_L \left( r, \frac{1}{R-1} \right) + N_L^2 \left( r, \frac{1}{R-1} \right) + N \left( r, \frac{1}{R-1} \right) \\
& \leq N \left( r, \frac{1}{R-1} \right) + S(r, F) + S(r, G).
\end{align*}$$

Thus from (2.8), we get

$$\begin{align*}
N \left( r, \frac{1}{F-1} \right) + N \left( r, \frac{1}{G-1} \right) & \leq N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{G} \right) + N \left( r, \frac{1}{H} \right) \\
& \quad + N_0 \left( r, \frac{1}{F} \right) + N_0 \left( r, \frac{1}{G} \right) + S(r, F) + S(r, G) \\
& \leq N \left( r, \frac{1}{F} \right) + N \left( r, \frac{1}{G} \right) + S(r, F) + S(r, G).
\end{align*}$$

(2.12)
Since \( f \) and \( g \) share 0 and \( \infty \) CM, from (2.2), (2.3), (2.7) and (2.12), we obtain

\[
T(r, F) \leq 2N(r, F) + N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{G}\right) + S(r, F) + S(r, G)
\]

\[
\leq 2N(r, f) + N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{G}\right) + S(r, F) + S(r, G)
\]

\[
= 2N(r, f) + N\left(r, \frac{1}{L(f)}\right) + N\left(r, \frac{1}{L(g)}\right) + S(r, f) + S(r, g)
\]

\[
\leq 2N(r, f) + T(r, L(f)) - T(r, f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right) + kN(r, g) + S(r, f) + S(r, g)
\]

\[
= T(r, L(f)) - T(r, f) + (k + 2)\overline{N}(r, f) + 2N\left(r, \frac{1}{f}\right) + S(r, f).
\]

That is,

\[
T(r, f) \leq (k + 2)\overline{N}(r, f) + 2N\left(r, \frac{1}{f}\right) + S(r, f),
\]

and so \((k + 2)\Theta(\infty, f) + 2\delta(0, f) \leq k + 3\), a contradiction to (1.1).

**Case (ii).** If \( l = 0 \), then we have

\[
N_E^{(1)}\left(r, \frac{1}{r-1}\right) = N_E^{(1)}\left(r, \frac{1}{G-1}\right) + S(r, F) + S(r, G),
\]

\[
\overline{N}_E^{(2)}\left(r, \frac{1}{r-1}\right) = \overline{N}_E^{(2)}\left(r, \frac{1}{G-1}\right) + S(r, F) + S(r, G),
\]

and also from (2.6), we have

\[
\overline{N}\left(r, \frac{1}{F-1}\right) + N\left(r, \frac{1}{G-1}\right) \leq N_E^{(1)}\left(r, \frac{1}{F-1}\right) + \overline{N}_E^{(2)}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{F-1}\right)
\]

\[
+ N\left(r, \frac{1}{G-1}\right) + N\left(r, \frac{1}{G-1}\right) + S(r, F) + S(r, G)
\]

\[
\leq N_E^{(1)}\left(r, \frac{1}{F-1}\right) + \overline{N}_E^{(2)}\left(r, \frac{1}{F-1}\right) + N\left(r, \frac{1}{F-1}\right) + N\left(r, \frac{1}{G-1}\right)
\]

\[
+ S(r, F) + S(r, G)
\]

\[
\leq \overline{N}(2)\left(r, \frac{1}{F}\right) + \overline{N}(2)\left(r, \frac{1}{G}\right) + 2\overline{N}\left(r, \frac{1}{F-1}\right)
\]

\[
+ \overline{N}\left(r, \frac{1}{G-1}\right) + N\left(r, \frac{1}{G-1}\right) + N_0\left(r, \frac{1}{F}\right)
\]

\[
+ N_0\left(r, \frac{1}{G}\right) + S(r, F) + S(r, G).
\]

(2.13)
From (2.2), (2.3), (2.4), (2.7) and (2.13), we obtain

\[
T(r, F) \leq 2N(r, F) + N\left(r, \frac{1}{F}\right) + N(2, r, F) + N(2, r, G) + 2N(r, F) + 2N(r, F)
\]

\[
+ 2N_L\left(r, \frac{1}{F-1}\right) + N_L\left(r, \frac{1}{G-1}\right) + S(r, F) + S(r, G)
\]

\[
\leq 2N(r, F) + N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{G}\right) + 2N\left(r, \frac{1}{F}\right) + 2N(r, F)
\]

\[
+ N\left(r, \frac{1}{G}\right) + N(r, G) + S(r, F) + S(r, G)
\]

\[
\leq 5N(r, f) + N\left(r, \frac{1}{L(f)}\right) + 2N\left(r, \frac{1}{L(f)}\right) + 2N\left(r, \frac{1}{L(g)}\right) + S(r, F) + S(r, G)
\]

\[
\leq 5N(r, f) + T(r, L(f)) - T(r, f) + N\left(r, \frac{1}{f}\right) + 2N\left(r, \frac{1}{f}\right) + 2kN(r, f)
\]

\[
+ 2N\left(r, \frac{1}{g}\right) + 2kN(r, g) + S(r, f) + S(r, g)
\]

\[
\leq T(r, L(f)) - T(r, f) + (4k + 5)N(r, f) + 5N\left(r, \frac{1}{f}\right) + S(r, f).
\]

That is,

\[
T(r, f) \leq (4k + 5)N(r, f) + 5N\left(r, \frac{1}{f}\right) + S(r, f),
\]

and so \((4k + 5)\Theta(\infty, f) + 5\delta(0, f) \leq 4k + 9\), a contradiction to (1.3).

Thus our supposition is wrong and hence \(H \equiv 0\). So (2.6) implies that

\[
\frac{F''}{F'} - \frac{2F'}{F-1} = \frac{G''}{G'} - \frac{2G'}{G-1},
\]

and so we obtain

\[
\frac{1}{F - 1} = \frac{C}{G - 1} + D,
\]

(2.14)

where \(C \neq 0\) and \(D\) are constants.

Here, the following three cases can arise:

**Case (a):** When \(D \neq 0, -1\). We rewrite (2.14) as

\[
\frac{G - 1}{C} = \frac{F - 1}{D + 1 - DF},
\]

we have

\[
N(r, G) = N\left(r, \frac{1}{F - (D + 1)/D}\right).
\]
By Second fundamental theorem of Nevanlinna and (2.2), we have
\[ T(r, L(f)) = T(r, F) + S(r, f) \]
\[ \leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F-(D+1)/D}\right) + S(r, f) \]
\[ \leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}(r, G) + S(r, f) \]
\[ \leq N\left(r, \frac{1}{L(f)}\right) + 2\overline{N}(r, f) + S(r, f) \]
\[ \leq T(r, L(f)) - T(r, f) + 2\overline{N}(r, f) + N(r, \frac{1}{f}) + S(r, f). \]

Thus
\[ T(r, f) \leq 2\overline{N}(r, f) + N(r, \frac{1}{f}) + S(r, f), \]
and so \(2\Theta(\infty, f) + \delta(0, f) \leq 2\), which contradicts (1.1), (1.2) and (1.3).

**Case (b) :** When \( D = 0 \). Then from (2.14), we have
\[ G = CF - (C - 1). \]  
(2.15)

So if \( C \neq 1 \), then
\[ \overline{N}\left(r, \frac{1}{G}\right) = \overline{N}\left(r, \frac{1}{F - (C - 1)/C}\right). \]

Since \( f \) and \( g \) share \( 0 \) and \( \infty \) CM, by Second fundamental theorem of Nevanlinna, (2.2) and (2.3) gives
\[ T(r, L(f)) = T(r, F) + S(r, f) \]
\[ \leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F-(C-1)/C}\right) + S(r, f) \]
\[ = \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}(r, \frac{1}{G}) + S(r, f) \]
\[ \leq \overline{N}(r, f) + N\left(r, \frac{1}{L(f)}\right) + N\left(r, \frac{1}{L(g)}\right) + S(r, f) \]
\[ \leq \overline{N}(r, f) + T(r, L(f)) - T(r, f) + N(r, \frac{1}{f}) + N(r, \frac{1}{g}) + k\overline{N}(r, g) + S(r, f) \]
\[ = T(r, L(f)) - T(r, f) + (k + 1)\overline{N}(r, f) + 2N(r, \frac{1}{f}) + S(r, f). \]

Thus
\[ T(r, f) \leq (k + 1)\overline{N}(r, f) + 2N(r, \frac{1}{f}) + S(r, f), \]
and so \((k + 1)\Theta(\infty, f) + 2\delta(0, f) \leq k + 2\), which contradicts (1.1), (1.2) and (1.3).
Thus, $C = 1$ and so in this case from (2.15), we obtain $F \equiv G$ and so

$$L(f) \equiv L(g).$$

**Case (c) :** When $D = -1$. Then from (2.14) we have

$$\frac{1}{F - 1} = \frac{C}{G - 1} - 1. \quad (2.16)$$

So if $C \neq -1$, then

$$N\left( r, \frac{1}{G} \right) = N\left( r, \frac{1}{F - c/(C + 1)} \right).$$

Since $f$ and $g$ share 0 and $\infty$ CM, by Second fundamental theorem of Nevanlinna, (2.2) and (2.3), we have

$$T(r, L(f)) = T(r, F) + S(r, f)$$

\[
\leq N(r, F) + N\left( r, \frac{1}{F} \right) + N\left( r, \frac{1}{F - c/(C + 1)} \right) + S(r, f)
\]

\[
= N(r, f) + N\left( r, \frac{1}{F} \right) + N\left( r, \frac{1}{G} \right) + S(r, f)
\]

\[
\leq N(r, f) + N\left( r, \frac{1}{L(f)} \right) + N\left( r, \frac{1}{L(g)} \right) + S(r, f)
\]

\[
\leq N(r, f) + T(r, L(f)) - T(r, f) + N\left( r, \frac{1}{f} \right) + N\left( r, \frac{1}{g} \right) + kN(r, g) + S(r, f)
\]

\[
= T(r, L(f)) - T(r, f) + (k + 1)N(r, f) + 2N(r, \frac{1}{f}) + S(r, f).
\]

Thus

$$T(r, f) \leq (k + 1)N(r, f) + 2N(r, \frac{1}{f}) + S(r, f),$$

and so $(k + 1)\Theta(\infty, f) + 2\delta(0, f) \leq k + 2$, which contradicts (1.1), (1.2) and (1.3).

Thus, $C = -1$ and so in this case from (2.16), we obtain $FG \equiv 1$ and so $L(f)L[f] = 1$.

If $L(f) \equiv L(g)$, then $L(f - g) \equiv 0$ and so $f - g$ has to be entire and we have (see (3))

$$f - g = \sum_{i=1}^{m} p_i(z)e^{\alpha_i z},$$

where $m \leq k$ is a positive integer, $\alpha_i$ are distinct complex numbers and $p_i(z)$ are polynomials ($i = 1, 2, ..., m$).

Thus

$$\lambda(f - g) = \lambda\left( \sum_{i=1}^{m} p_i(z)e^{\alpha_i z} \right) \leq 1.$$
We consider the following cases:

**Case (i).** When $\lambda(f) < 1$. Since $f$ and $g$ share $0$ and $\infty$ CM, we have $f / g = e^{h(z)}$, where $h(z)$ is an entire function. Also as $\lambda(f) = \lambda(g)$, we have

$$\lambda(e^{h(z)}) = \lambda(f / g) \leq \max \{\lambda(f), \lambda(1/g)\} < 1.$$ 

Thus $e^{h(z)}$ is a constant, say $c$ and so $f = cg$ which implies that $L(f) \equiv cL(g)$. But $L(f) \equiv L(g)$, so we get $c = 1$ and thus $f \equiv g$.

**Case (ii).** When $\lambda(f) > 1$. Since $f$ and $g$ are meromorphic functions of finite order, by Hadamard’s factorization theorem we have

$$f(z) = \frac{P(z)}{Q(z)} e^{l_1(z)} \quad \text{and} \quad g(z) = \frac{P(z)}{Q(z)} e^{l_2(z)},$$

where $P(z)$ is the canonical product formed with the common zeros of $f$ and $g$, $Q(z)$ is the canonical product formed with the common poles of $f$ and $g$, and $l_1$, $l_2$ are the polynomials of degree less than or equal to $\lambda(f)$, $\lambda(g)$ respectively. Thus

$$f - g = \frac{P(z)}{Q(z)} e^{l_1(z)} - \frac{P(z)}{Q(z)} e^{l_2(z)},$$

or we can write

$$\frac{P(z)}{Q(z)} e^{l_1(z)} - \frac{P(z)}{Q(z)} e^{l_2(z)} - (f - g)e^{l_3(z)} \equiv 0, \quad (2.17)$$

where $l_3(z) \equiv 0$.

Also

$$\lambda(P) = \sigma(f) \leq \sigma(f - g) \leq \lambda(f - g) \leq 1,$$

and since $\sigma(1/f) \leq \sigma(f)$, we have

$$\lambda(Q) = \sigma(1/f) \leq \sigma(f - g) \leq \lambda(f - g) \leq 1.$$ 

Thus

$$\lambda \left( \frac{P}{Q} \right) \leq \max \{\lambda(P), \lambda(Q)\} \leq 1.$$ 

Since $f - g = (e^{l_1 - l_2})g$ and $\lambda(f) = \lambda(g) > 1$, we have $\lambda(e^{l_1}) > 1$, $\lambda(e^{l_1}) > 1$ and $\lambda(e^{l_1 - l_2}) > 1$, so $\lambda(e^{l_i}) > 1$, where $1 \leq i < j \leq 3$. Thus $l_i - l_j$ is non-constant, where $1 \leq i < j \leq 3$ and by lemma 2.6 and 2.7, we get

$$T(r, f - g) = o(T(r, e^{l_1 - l_2})) \quad \text{and} \quad T(r, P/Q) = o(T(r, e^{l_i - l_j})),$$

as $r \to \infty$. Thus by lemma 2.5, we have $P/Q \equiv 0$ and $f - g \equiv 0$ which implies that $f(z) \equiv 0$, which is a contradiction. So $l_1 = l_2$ and hence $f \equiv g$. 

□
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