The Evolution of Travelling Waves in a KPP Reaction-Diffusion Model with Cut-off Reaction Rate. I. Permanent Form Travelling Waves.

Alex D O Tisbury, David J Needham & Alexandra Tzella
School of Mathematics, University of Birmingham, Edgbaston, Birmingham, B15 2TT, UK
E-mail: d.j.needham@bham.ac.uk and a.tzella@bham.ac.uk

May 2018

Abstract. We consider Kolmogorov–Petrovskii–Piscounov (KPP) type models in the presence of an arbitrary cut-off in reaction rate at concentration $u = u_c$. In Part I we examine permanent form travelling wave solutions (a companion paper, Part II, is devoted to their evolution in the large time limit). For each fixed cut-off value $0 < u_c < 1$, we prove the existence of a unique permanent form travelling wave with a continuous and monotone decreasing propagation speed $v^*(u_c)$. We extend previous asymptotic results in the limit of small $u_c$ and present new asymptotic results in the limit of large $u_c$ which are respectively obtained via the systematic use of matched and regular asymptotic expansions. The asymptotic results are confirmed against numerical results obtained for the particular case of a cut-off Fisher reaction function.

Keywords: reaction-diffusion equations, nonlinear boundary-value problems, asymptotic expansions, singular perturbations
1. Introduction

Travelling waves arise in a wide range of applications in mathematical chemistry, biology, ecology and genetics [1, 2]. They describe the invasion of chemical or biological reactions and are usually established as a result of the interaction between molecular diffusion, local growth and saturation. The simplest model that encapsulates this interaction is the Kolmogorov–Petrovskii–Piscounov (KPP) reaction–diffusion equation (also called Fisher-KPP [3, 4] equation). In one spatial dimension this describes the evolution of the concentration \( u(x,t) \) as

\[
\begin{align*}
    u_t &= u_{xx} + f(u), \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+ , \\
    u(x,0) &= u_0(x),
\end{align*}
\]

(1a)

where \( u_0 : \mathbb{R} \rightarrow \mathbb{R} \) is piecewise continuous and smooth with limits 0 and 1 as \( x \to \infty \) and \( x \to -\infty \), respectively. This is typically supplemented with boundary conditions

\[
    u(x,t) \rightarrow \begin{cases} 
        1, & \text{as } x \to -\infty \\
        0, & \text{as } x \to \infty 
    \end{cases}
\]

(1c)

with these limits being uniform for \( t \in [0,T] \) and any \( T > 0 \). The function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is a normalised KPP-type reaction function which satisfies conditions that \( f \in C^1(\mathbb{R}) \) with

\[
    f(0) = f(1) = 0, \quad f'(0) = 1, \quad f'(1) < 0
\]

(2a)

and in addition

\[
    0 < f(u) \leq u \quad \forall u \in (0,1), \quad f(u) < 0 \quad \forall u \in (1,\infty).
\]

(2b)

A prototypical example of such a KPP reaction is the Fisher reaction function [4] given by

\[
    f(u) = u(1-u).
\]

(3)

An illustration of \( f(u) \) against \( u \) is given in Figure 1(a).

It is well-known [1, 3, 5, 6] that the initial-boundary value problem [1] for the KPP equation supports a one-parameter family of non-negative permanent form travelling wave solutions of the form

\[
    u(x,t) = U(y) = U(x - vt) \quad \forall (x,t) \in \mathbb{R} \times \mathbb{R}^+.
\]

(4)

These remain steady in time in a reference frame moving in the positive \( x \) direction with speed \( v \geq 0 \) to be determined. Their existence and uniqueness (up to translation in origin) is established for

\[
    v \geq v_m = 2,
\]

(5)
where $v_m$ denotes the minimum speed of propagation. This is achieved by analysing the following nonlinear boundary value problem, namely,

$$U'' + v U' + f(U) = 0, \quad -\infty < y < \infty, \quad (6a)$$

$$U(y) \geq 0, \quad -\infty < y < \infty, \quad (6b)$$

$$U(y) \to \begin{cases} 1, & \text{as } y \to -\infty \\ 0, & \text{as } y \to \infty \end{cases} \quad (6c)$$

obtained by inserting (4) into equation (1a) and using (1c). The analysis is based on examining the existence of a unique heteroclinic orbit connecting the stable fixed point $(U, U') = (0, 0)$ to the unstable fixed point $(U, U') = (1, 0)$ in the $(U, U')$ phase plane of the equivalent two-dimensional dynamical system obtained from (6). It is also used to establish that $U(y)$ is monotone decreasing in $y \in (-\infty, \infty)$, giving explicit expressions of the behaviour of the permanent form travelling wave near the two fixed points as

$$U(y) \sim \begin{cases} (A_\infty y + B_\infty) e^{-y}, & \text{as } y \to \infty, v = v_m = 2 \\ C_\infty e^{\alpha(v)y}, & \text{as } y \to \infty, v > v_m = 2 \end{cases} \quad (7a)$$

and for all $v \geq v_m = 2$,

$$U(y) \sim 1 - A_{-\infty} e^{-\gamma y}, \quad \text{as } y \to -\infty, \quad (7b)$$

where

$$\alpha(v) = \frac{1}{2}(-v + \sqrt{v^2 - 4}) < 0, \quad \gamma = (-1 + \sqrt{1 + |f'(1)|}) > 0, \quad (7c)$$

with $A_\infty > 0$, $B_\infty$, $C_\infty > 0$ and $A_{-\infty} > 0$ being globally determined constants, dependent on the nonlinearity of the boundary value problem (6) (see, for example, [1, 7]).
A key result is that the initial condition in (1b) determines the permanent form travelling wave solution that emerges at large times. When $u_0(x)$ is sufficiently close to a Heaviside function, specifically,

$$u_0(x) \leq O(e^{-x}), \quad \text{as } x \to \infty$$

(8)

the solution to the KPP initial-boundary value problem (1) converges at large times to the permanent form travelling solution with minimum speed $v_m = 2$ at an algebraic rate determined in [10, 11, 12]. The mechanism which selects the speed of propagation of the emerging permanent form travelling wave solution (as well as the rate of convergence) is based on the linearisation of the KPP equation (1a) at the leading edge of the travelling wave. There, the concentration $u$ is small and the dynamics are unstable. As a result, any modification of the dynamics near the leading edge of the travelling wave would invalidate this speed selection mechanism.

This is precisely the case for the cut-off KPP model that Brunet and Derrida [13] proposed and considered. Motivated by the discrete nature of chemical and biological phenomena at the microscopic level, they took a reaction function that is effectively deactivated at points where the concentration $u$ lies at or below a threshold value $u_c \in (0, 1)$. This case corresponds to the cut-off KPP equation given by

$$u_t = u_{xx} + f_c(u), \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+,$$

(9a)

$$u(x, 0) = u_0(x), \quad (9b)$$

which is once more supplemented with the boundary conditions

$$u(x, t) \rightarrow \begin{cases} 1, & \text{as } x \to -\infty \\ 0, & \text{as } x \to \infty \end{cases}$$

(9c)

uniformly for $t \in [0, T]$ for all $T > 0$. The main difference is that the reaction function $f : \mathbb{R} \rightarrow \mathbb{R}$ in the KPP equation (1) is replaced with a cut-off reaction function $f_c : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f_c(u) = \begin{cases} f(u), & u \in (u_c, \infty) \\ 0, & u \in (-\infty, u_c] \end{cases}$$

(9d)

where $f(u)$ satisfies the KPP conditions (2). An illustration of $f_c(u)$ against $u$ is given in Figure 1b, with $f_c^+ = f_c(u_c^+)$. Focussing on the initial conditions

$$u_0(x) = \begin{cases} 1, & \text{as } x < 0 \\ 0, & \text{as } x \geq 0 \end{cases}$$

(9e)

we henceforth refer to this initial-boundary value problem as IVP. Examining the specific example which has

$$f(u) = u(1 - u^2), \quad (10)$$

Brunet and Derrida [13] considered the behaviour of permanent form travelling wave solutions for (very) small values of $u_c$ corresponding to a single particle cut-off. Their
The main result is a prediction for the propagation speed \( v^*(u_c) \) of the unique permanent form travelling wave given by

\[
v^*(u_c) \approx 2 - \frac{\pi^2}{(\ln u_c)^2}, \quad \text{as } u_c \to 0^+,
\]

which they obtained using a point patching procedure. This significant result demonstrates the strong influence of a cut-off on the value of \( v^*(u_c) \) for small values of \( u_c \). Subsequently, a more rigorous approach was employed by Dumortier, Popovic and Kaper [14] who proved the existence and uniqueness of a permanent form travelling wave solution when \( u_c \) is small and used matched asymptotics in the phase plane to obtain that

\[
v^*(u_c) \sim 2 - \frac{\pi^2}{(\ln u_c)^2} + O\left(\frac{1}{|\log u_c|^3}\right), \quad \text{as } u_c \to 0^+.
\]

All these results have focused in the small \( u_c \) limit and with particular focus on the reaction function \( f(u) \).

There are a number of fundamental questions that remain. The first one concerns the existence and uniqueness of a permanent form travelling wave solution for arbitrary threshold values \( u_c \) and KPP reaction functions \( f(u) \). The second one concerns the propagation speed of such permanent form travelling wave solutions for arbitrary threshold values \( u_c \). The third one is with regard to the shape of the permanent form travelling wave solution. A final question concerns the evolution in time to the permanent form travelling wave solution via the initial boundary value problem IVP.

In this series of papers we address all of these questions. In particular, we study classical solutions \( u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \) to the IVP for the cut-off KPP equation (9). In this paper we proceed as follows. In Section 2, we re-formulate the IVP as a moving boundary problem. We then make a simple coordinate transformation to consider an equivalent initial-boundary value problem that we refer to as QIVP. In section 3, we examine the possibility that QIVP supports permanent form travelling wave solutions \( U_T(y) = U_T(x - vt) \) where \( U_T \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\}) \) satisfy the nonlinear boundary value problem,

\[
\begin{align*}
U''_T + vU'_T + f_c(U_T) &= 0, \quad y \in \mathbb{R} \setminus \{0\}, \\
U_T &\geq u_c \quad \forall y < 0, \quad 0 \leq U_T \leq u_c \quad \forall y > 0, \\
U_T(0) &= u_c, \\
U_T(y) &\to \begin{cases} 1, & \text{as } y \to -\infty \\ 0, & \text{as } y \to \infty \end{cases}
\end{align*}
\]

We establish the following theorem.

**Theorem 1.1.** For each fixed \( u_c \in (0, 1) \), QIVP has a unique permanent form travelling wave solution \( U_T : \mathbb{R} \to \mathbb{R} \), with the propagation speed given by \( v^*(u_c) \). Here \( v^* : (0, 1) \to \mathbb{R}^+ \) is continuous and monotone decreasing, with

\[
v^*(u_c), \to \begin{cases} 0, & \text{as } u_c \to 1^- \\ 2, & \text{as } u_c \to 0^+ \end{cases}
\]
where 2 is the minimum propagation speed of the permanent form travelling wave solution in the absence of cut-off \( (u_c = 0) \). In addition, \( U_T(y) \) is strictly monotone decreasing for \( y \in \mathbb{R} \), with \( U_T(0) = u_c \), and

\[
U_T''(0^+) - U_T''(0^-) = -f'_c,
\]

\[
U_T(y) = u_c e^{-v^*(u_c)y} \quad \forall y \in \mathbb{R}^+,
\]

\[
U_T(y) \sim 1 - A_{-\infty} e^{\lambda_+(v^*(u_c))y} \quad \text{as} \quad y \to -\infty,
\]

for some global constant \( A_{-\infty} > 0 \) (which depends upon \( u_c \)), and

\[
\lambda_+(v) = \frac{1}{2} \left( -v + \sqrt{v^2 + 4|f'_c(1)|} \right) > 0.
\]

Furthermore,

\[
v^*(u_c) \sim |f'_c(1)|^{\frac{1}{2}}(1 - u_c) \quad \text{as} \quad u_c \to 1^-.
\]

In sections 4 and 5 we use matched asymptotic expansions to develop the detailed asymptotic structure to the permanent form travelling wave solutions as \( u_c \to 0^+ \) and as \( u_c \to 1^- \) respectively. These are used to obtain higher-order corrections to \( (12) \) and \( (15) \) in a systematic manner. In the first limit, the analysis is carried out on the direct problem (rather than the phase plane). It highlights that higher-order corrections are controlled by two global constants \( A_\infty \) and \( B_\infty \) associated with the minimum speed of permanent form travelling wave solution to the non cut-off KPP problem \( (1) \). These global constants represent the nonlinearity in the problem when \( u_c \) is small. The analysis is readily generalised to degenerate and singular KPP conditions, obtained for example when \( f'(0) = 0 \) or \( f(u) \sim u^{1/2} \) as \( u \to 0^+ \), respectively. Section 6 presents numerical examples for the specific Fisher cut-off reaction function. The paper concludes with a discussion in Section 7.

2. Formulation of Evolution Problem QIVP

Due to the discontinuity in \( f_c(u) \) at \( u = u_c \), it is convenient to re-structure IVP as a moving boundary problem. To this end, we introduce the domains:

\[
D^L = \{(x,t) \in \mathbb{R} \times \mathbb{R}^+ : x < s(t)\},
\]

\[
D^R = \{(x,t) \in \mathbb{R} \times \mathbb{R}^+ : x > s(t)\},
\]

and the curve

\[
\mathcal{L} = \{(x,t) \in \mathbb{R} \times \mathbb{R}^+ : x = s(t)\},
\]

that describes the moving boundary between the two domains. The boundary is expressed in terms of \( s(t) \) which satisfies \( u(s(t),t) = u_c \), with \( u \geq u_c \) in \( D^L \) and \( u \leq u_c \) in \( D^R \). In this context, a classical solution will have \( u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \) and \( s : \mathbb{R}^+ \to \mathbb{R} \) such that,

\[
u \in C(\mathbb{R} \times \mathbb{R}^+ \setminus \{(0,0)\}) \cap C^{1,1}(\mathbb{R} \times \mathbb{R}^+) \cap C^{2,1}(D^L \cup D^R),
\]

\[
s \in C^1(\mathbb{R}^+),
\]

\[
s(0^+) = 0.
\]
The Evolution of Travelling Waves in a Cut-off Reaction-Diffusion Model.

The moving boundary problem is then formulated as follows,

\[
\begin{align*}
  u_t & = u_{xx} + f_c(u), \quad (x, t) \in D_L \cup D_R, \\
  u & \geq u_c \text{ in } D_L, \quad u \leq u_c \text{ in } D_R, \\
  u(x, 0) & = \begin{cases} 
    1, & \text{for } x < 0 \\
    0, & \text{for } x \geq 0
  \end{cases}, \\
  u(x, t) & = \begin{cases} 
    1, & \text{for } x \to -\infty \\
    0, & \text{for } x \to \infty
  \end{cases}
\end{align*}
\]

uniformly for \( t \in [0, T] \) for all \( T > 0 \) and

\[
\begin{align*}
  u(s(t), t) & = u_c, \quad t \in \mathbb{R}^+, \\
  u_x(s(t)^+, t) & = u_x(s(t)^-, t), \quad t \in \mathbb{R}^+, \\
  s(0^+) & = 0.
\end{align*}
\]

The situation is illustrated in Figure 2. It is now convenient to make the simple coordinate transformation \((x, t) \to (y, t)\) with \( y = x - s(t)\). We then introduce the following domains:

\[
Q^L = \mathbb{R}^- \times \mathbb{R}^+, \quad Q^R = \mathbb{R}^+ \times \mathbb{R}^+.
\]

with \( u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \) and \( s : \mathbb{R}^+ \to \mathbb{R} \) such that

\[
\begin{align*}
  u & \in C(\mathbb{R} \times \mathbb{R}^+ \setminus \{(0, 0)\}) \cap C^{1,1}(\mathbb{R} \times \mathbb{R}^+) \cap C^{2,1}(Q^L \cup Q^R), \\
  s & \in C^1(\mathbb{R}^+).
\end{align*}
\]

The equivalent problem to (18) is then given by

\[
\begin{align*}
  u_t - \dot{s}(t)u_y = u_{yy} + f_c(u), \quad (y, t) \in Q^L \cup Q^R, \\
  u & \geq u_c \text{ in } Q^L, \quad u \leq u_c \text{ in } Q^R, \\
  u(y, 0) & = \begin{cases} 
    1, & y < 0 \\
    0, & y \geq 0
  \end{cases}, \\
  u(y, t) & \to \begin{cases} 
    1, & \text{as } y \to -\infty \\
    0, & \text{as } y \to \infty
  \end{cases}
\end{align*}
\]

uniformly for \( t \in [0, T] \) for all \( T > 0 \) and

\[
\begin{align*}
  u(0, t) & = u_c, \quad t \in \mathbb{R}^+, \\
  u_y(0^+, t) & = u_y(0^-, t), \quad t \in \mathbb{R}^+, \\
  s(0^+) & = 0,
\end{align*}
\]

where the dot denotes differentiation with respect to time, \( t \). This initial-boundary value problem will henceforth be referred to as QIVP. On using the classical maximum principle and comparison theorem (see, for example, [1] and [15]), together with translational invariance in \( y \), and the regularity in (20), we can readily establish the following basic qualitative properties concerning QIVP, namely,
The Evolution of Travelling Waves in a Cut-off Reaction-Diffusion Model.

\[ x = s(t) \]

Figure 2. A sketch of the moving boundary problem.

\[ 0 < u(y, t) < u_c \quad \forall (y, t) \in Q^R, \] (22a)
\[ u_c < u(y, t) < 1 \quad \forall (y, t) \in Q^L, \] (22b)
\[ u(y, t) \text{ is strictly monotone decreasing in } y \in \mathbb{R} \quad \forall t \in \mathbb{R}^+. \] (22c)

In addition, via the partial differential equation (21a) and the regularity conditions (20), we have

\[ \lim_{y \to 0^+} u_{yy}(y, t) = \lim_{y \to 0^+} (u_t(y, t) - \dot{s}(t)u_y(y, t)) \]
\[ = -\dot{s}(t)u_y(0, t) \quad \forall t \in \mathbb{R}^+, \] (22d)
\[ \lim_{y \to 0^-} u_{yy}(y, t) = \lim_{y \to 0^-} (u_t(y, t) - \dot{s}(t)u_y(y, t) - f(u(y, t))) \]
\[ = -\dot{s}(t)u_y(0, t) - f_c^+ \quad \forall t \in \mathbb{R}^+, \] (22e)

with the limits in (22d) and (22e) being uniform for \( t \in [t_0, t_1] \) (for any \( 0 < t_0 < t_1 \)). It follows from (22d) and (22e) that

\[ [u_{yy}(y, t)]_{y=0^+} = f_c^+ \quad \forall t \in \mathbb{R}^+, \] (22f)

whilst, using (22c), (22e) and the regularity condition (20) establish that

\[ u_y(y, t) < 0 \quad \forall (y, t) \in \mathbb{R} \times \mathbb{R}^+. \] (22g)

The remainder of this paper and its companion (part II) concentrates on the analysis of QIVP. Specifically, in this paper we consider the existence and uniqueness of permanent form travelling wave solutions to QIVP including their asymptotic behaviour in the limits of \( u_c \to 0^+ \) and \( u_c \to 1^- \) via the method of matched and regular asymptotic expansions.

3. Permanent Form Travelling Waves in QIVP

We anticipate that as \( t \to \infty \), a permanent form travelling wave solution will develop in the solution to QIVP, advancing with a (non-negative) propagation speed, allowing for the transition between the fully reacted state, \( u = 1 \) as \( y \to -\infty \), to the unreacted state, \( u = 0 \) as \( y \to \infty \). Therefore, in this section we focus attention on the possibility of QIVP supporting permanent form travelling wave solutions (henceforth referred to as PTW solutions). We begin by establishing the existence and uniqueness of a PTW to QIVP.
The Evolution of Travelling Waves in a Cut-off Reaction-Diffusion Model.

for each fixed $u_c \in (0, 1)$, denoting the unique propagation speed by $v = v^*(u_c)$. We then consider limiting values of $v^*(u_c)$ as $u_c \to 0^+$ and $u_c \to 1^-$. The results established in this section provide proof of Theorem 1.1 as stated in the introduction.

3.1. The Existence and Uniqueness of a PTW Solution to QIVP

A PTW solution to QIVP, with constant speed of propagation $v \geq 0$, is a steady state solution to QIVP with $u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ and $s : \mathbb{R}^+ \to \mathbb{R}$ such that

\[
u(y, t) = U_T(y) \quad \forall (y, t) \in \mathbb{R} \times \mathbb{R}^+, \tag{23}
\]

\[
\dot{s}(t) = v \quad \forall t \in \mathbb{R}^+, \tag{24}
\]

where $U_T \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$ and $v \geq 0$ satisfy the nonlinear boundary value problem,

\[
U''_T + vU'_T + f_c(U_T) = 0, \quad y \in \mathbb{R} \setminus \{0\}, \tag{25a}
\]

\[
U_T \geq u_c \quad \forall y < 0, \quad 0 \leq U_T \leq u_c \quad \forall y > 0, \tag{25b}
\]

\[
U_T(0) = u_c, \tag{25c}
\]

\[
U_T(y) \to \begin{cases} 
1, & \text{for } y \to -\infty \\
0, & \text{for } y \to \infty 
\end{cases} \tag{25d}
\]

where the dash denotes differentiation with respect to $y$. The nonlinear boundary value problem (25) can be thought of as a nonlinear eigenvalue problem with the eigenvalue being the propagation speed $v \geq 0$.

It is convenient to consider the ordinary differential equation (25) as the following equivalent autonomous first-order two-dimensional dynamical system, with $\alpha = U_T$ and $\beta = U'_T$, namely,

\[
\alpha' = \beta \tag{26}
\]

\[
\beta' = -v\beta - f_c(\alpha). \quad \forall y \in \mathbb{R}.
\]

We will analyse this dynamical system in the $(\alpha, \beta)$ phase plane for $v \geq 0$. In particular, it is straightforward to establish that the existence of a solution to (25) is equivalent to the existence of a heteroclinic connection in the $(\alpha, \beta)$ phase plane, for the dynamical system (26), which connects the equilibrium point $(1, 0)$, as $y \to -\infty$, to the equilibrium point $(0, 0)$, as $y \to \infty$ (the translational invariance is then fixed by condition (25c) which requires that $\alpha(0) = u_c$). From (25b), this heteroclinic connection must remain in the $\alpha \geq 0$ half plane of the $(\alpha, \beta)$ phase plane, which we denote by $R^+ = \{ (\alpha, \beta) : (\alpha, \beta) \in \mathbb{R}^+ \times \mathbb{R} \}$. We henceforth focus on this region of the $(\alpha, \beta)$ phase plane.

However, before we proceed further, it is first worth considering the effect of introducing the cut-off into the reaction function on the dynamical system (26). To that end, we introduce the function $\bar{Q} : \mathbb{R}^2 \to \mathbb{R}^2$ where $\bar{Q}(\alpha, \beta)$ is given by

\[
\bar{Q}(\alpha, \beta) = (\beta, -v\beta - f_c(\alpha)), \tag{27}
\]
to represent the vector field generating the dynamical system \( \text{(26)} \). We observe that, in the \((\alpha, \beta)\) phase plane, the effect of the discontinuity in \( f_c(\alpha) \) across the line \( \alpha = u_c \) is simply to \textit{refract} the phase paths passing through this line. In particular, for each \( \beta \in \mathbb{R} \), there is exactly one phase path passing through \((u_c, \beta)\), which has tangent vectors, \( \vec{Q}(u_c^-, \beta) = (\beta, -v \beta) \) and \( \vec{Q}(u_c^+, \beta) = (\beta, -v \beta + f_c^+) \). Thus, the refraction vector for the phase paths which cross the line \( \alpha = u_c \) is
\[
\vec{R}(u_c, \beta) = \vec{Q}(u_c^+, \beta) - \vec{Q}(u_c^-, \beta) = (0, -f_c^+).
\] (28)

We observe that the refraction vector \( \text{(28)} \) is independent of \((\beta, v) \in \mathbb{R} \times \mathbb{R}^+ \) and depends continuously on \( u_c \in (0, 1) \). It follows that
\[
\vec{R}(u_c, \beta) \to \vec{0} \quad \text{as} \quad u_c \to 0,
\] (29)
uniformly in \((\beta, v) \in \mathbb{R} \times \mathbb{R}^+ \). After determining the effect of the discontinuity on the phase paths of the dynamical system \( \text{(26)} \) in \( \mathbb{R}^+ \), we next consider the equilibrium points of \( \text{(26)} \) in \( \mathbb{R}^+ \). These are readily found to be at locations
\[
\begin{align*}
\vec{e}_a &= (a, 0) \quad \text{for each} \quad a \in [0, u_c], \quad (30a) \\
\vec{e}_1 &= (1, 0). \quad (30b)
\end{align*}
\]

We begin by examining the local phase portrait in the neighbourhood of the equilibrium point \( \vec{e}_1 \). We find that \( \vec{e}_1 \) is a hyperbolic equilibrium point. Moreover, \( \vec{e}_1 \) is a saddle point with eigenvalues
\[
\lambda_{\pm}(v) = \frac{1}{2} \left( -v \pm \sqrt{v^2 + 4|f_c'(1)|} \right). 
\] (31)

The associated local straight line paths of \( \vec{e}_1 \) are given by
\[
\beta(\alpha) = -\lambda_{\pm}(v)(1 - \alpha), 
\] (32)
where the negative (positive) eigenvalue corresponds to the local stable (unstable) manifold. We denote the phase path which forms the part of the unstable manifold entering \( D_+ = \{ (\alpha, \beta) : 0 < \alpha < 1, \beta < 0 \} \) as \( S_+^1 \). Similarly, we denote \( S_-^1 \) as the phase path which forms part of the unstable manifold entering \( D_- = \{ (\alpha, \beta) : \alpha > 1, \beta > 0 \} \). The situation is illustrated in Figure 3.

We next determine the local phase portrait of the equilibrium points \( \vec{e}_a \) for each \( a \in [0, u_c] \). For \( a \in (0, u_c) \) and \( v > 0 \), each of the equilibrium points \( \vec{e}_a \) is non-hyperbolic with a single stable manifold in \( \mathbb{R}^+ \) given by \( \{ (\alpha, \beta) : \beta = -v(\alpha - a); 0 \leq \alpha \leq u_c \} \). Also, the equilibrium point \( \vec{e}_0 \) is non-hyperbolic with a single stable manifold in \( \mathbb{R}^+ \) which we will denote by
\[
S_0 = \{ (\alpha, \beta) : \beta = -v\alpha; 0 \leq \alpha \leq u_c \}. 
\] (33)

Finally, the equilibrium point \( \vec{e}_{u_c} \) is again non-hyperbolic, and, for \( 0 \leq \alpha \leq u_c \), has a single stable manifold in \( \mathbb{R}^+ \) given by \( \{ (\alpha, \beta) : \beta = -v(\alpha - u_c); 0 \leq \alpha \leq u_c \} \).
The Evolution of Travelling Waves in a Cut-off Reaction-Diffusion Model.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3}

\caption{The local phase portrait for the equilibrium points of the dynamical system (26). The thick blue arrows denote the direction of the vector field $\vec{Q}(\alpha, \beta)$ along the line segments $L_0$, $L_1$ and on the boundary of $D_-$.}
\end{figure}

In fact, the collection of phase paths of the dynamical system (26) in the region $\{(\alpha, \beta) : 0 \leq \alpha \leq u_c, \beta \leq 0\}$ is given by the family of curves $\beta = c - v\alpha$, for each $c \in \mathbb{R}$. This is illustrated in Figure 3.

Next, for the line segment $\{(\alpha, \beta) : \alpha = 1, \beta > 0\}$, we observe the following,

$$\vec{Q}(\alpha, \beta) \cdot (1, 0) = \beta > 0.$$  (34)

Similarly, for the line segment $\{(\alpha, \beta) : \alpha > 1, \beta = 0\}$, we observe that

$$\vec{Q}(\alpha, \beta) \cdot (0, 1) = -f_c(\alpha) > 0.$$  (35)

Together with the local structure at the equilibrium point $\vec{e}_1$, we conclude from (34) and (35) that the region $D_-$ is a strictly positively invariant region for the dynamical system (26). We now examine the line segments $L_0 = \{(\alpha, \beta) : \alpha = 1, \beta < 0\}$ and $L_1 = \{(\alpha, \beta) : u_c < \alpha < 1, \beta = 0\}$, we observe that

$$\vec{Q}(\alpha, \beta) \cdot (-1, 0) = -\beta > 0 \quad \forall (\alpha, \beta) \in L_0,$$

$$\vec{Q}(\alpha, \beta) \cdot (0, -1) = f_c(\alpha) > 0 \quad \forall (\alpha, \beta) \in L_1.$$  (36)  (37)

In addition, for $v > 0$, we observe that for all $$(\alpha, \beta) \in R^+$$

$$\nabla \cdot \vec{Q}(\alpha, \beta) = -v < 0.$$  (38)

Thus, for any $v > 0$, it follows from the Bendixson negative criterion (see, for example, [16]) that (26) has no periodic orbits, homoclinic orbits or heteroclinic cycles in $R^+$. Finally, we observe that at each $$(\alpha, \beta) \in R^+ \setminus \{\{\vec{e}_1\} \cup \{\vec{e}_a : 0 \leq a \leq u_c\}\}$$. the vector field $\vec{Q}(\alpha, \beta)$ rotates continuously clockwise for increasing $v \geq 0$. At the equilibrium point $\vec{e}_1$, the unstable manifold $S_1^-$ rotates clockwise for increasing $v \geq 0$, as does the stable manifold $S_0$ at the equilibrium point $\vec{e}_0$.

As the phase path $S_1^-$ enters $D_-$ on leaving $\vec{e}_1$, and we have established that $D_-$ is a strictly positively invariant region for the dynamical system (26), we conclude that
this cannot correspond to a heteroclinic connection between $\vec{e}_1$ and $\vec{e}_0$. Thus, at any $v \geq 0$, the existence of a heteroclinic connection in $R^+$ connecting $\vec{e}_1$, as $y \rightarrow -\infty$, to $\vec{e}_0$, as $y \rightarrow \infty$, is equivalent to the phase path $S_1^+$, leaving $\vec{e}_1$, being coincident with the phase path $S_0$, entering $\vec{e}_0$. It also follows that, at those $v \geq 0$ when such a heteroclinic connection exists, then it is unique.

We are now in a position to investigate for which values of $v \geq 0$, if any, the required heteroclinic connection exists in $R^+$. When $v = 0$, it follows directly from (26) that the phase path $S_1^+$ has graph $(\alpha, \beta_0(\alpha))$ where

$$\beta_0(\alpha) = -\left(2 \int_\alpha^1 f_{c}(\gamma) d\gamma\right)^{\frac{1}{2}},$$

for $\alpha \in [0, 1]$. Thus, $\beta_0(\alpha)$ is (non-positive) non-decreasing for $\alpha \in [0, 1]$ with

$$\beta_0(0) = -\left(2 \int_{u_c}^1 f_{c}(\gamma) d\gamma\right)^{\frac{1}{2}} < 0,$$

and, via the vector field (27) and local straight line paths of $\vec{e}_1$ (32),

$$\beta'_0(1) = (-f'_{c}(1))^\frac{1}{2}.\tag{40b}$$

We denote the phase path $S_1^+|_{v=0}$ as $C_0$, and note from (39) that $C_0 \subset D^+$ as illustrated in Figure 4. We conclude from (40a) that when $v = 0$ no heteroclinic connection exists from $\vec{e}_1$ to $\vec{e}_0$. Moreover, it follows from the rotational properties of the vector field $\vec{Q}(\alpha, \beta)$ with increasing $v \geq 0$, as discussed earlier, that, for each $v > 0$, we have

$$\vec{Q}(\alpha, \beta_0(\alpha)) \cdot \vec{n}_0(\alpha) < 0,$$

for all $\alpha \in [0, 1)$, where $\vec{n}_0(\alpha)$ is the unit normal to $C_0$ as shown in Figure 4. We define the line segments $L_2 = \{(\alpha, \beta) : \alpha = 0, \beta_0(0) < \beta < 0\}$ and $L_3 = \{(\alpha, \beta) : 0 \leq \alpha \leq 1, \beta = 0\}$ and denote the region $\Omega_0 \subset D_+$ as that region bounded by $\partial \Omega_0 = L_2 \cup L_3 \cup C_0$.

We observe, via the rotational properties of $S_1^+$ at $\vec{e}_1$ with increasing $v \geq 0$, that for any $v > 0$, then $S_1^+|_v$ enters $\Omega_0$ on leaving $\vec{e}_1$. Moreover, from (38), $\Omega_0$ contains no periodic orbits, homoclinic orbits or heteroclinic cycles. It then follows from (36), (37), (41) and the Poincaré-Bendixson Theorem (see, for example, [16]), that (recalling that $\Omega_0$ contains no periodic or homoclinic orbits, or heteroclinic cycles) $S_1^+|_v$ must leave $\Omega_0$ through $L_2$ (at finite $y$) or connect with $\vec{e}_a$ for some $a \in [0, u_c]$ (as $y \rightarrow \infty$). For each $v \geq 0$, this observation allows us to classify the behaviour of $S_1^+|_v$, by introducing the following function $s : \mathbb{R}^+ \rightarrow \mathbb{R}$, such that,

$$s(v) = \begin{cases} \text{The distance, measured from the origin of the (} \alpha, \beta \text{) plane,} & \\
\text{to the point of intersection of } S_1^+|_v \text{ with } L_2 \text{ (measuring negative distance) or } L_3 \text{ (measuring positive distance).} & 
\end{cases}$$

We have immediately that

$$s(0) = \beta_0(0) < 0.$$

(42)
The Evolution of Travelling Waves in a Cut-off Reaction-Diffusion Model.

Figure 4. The phase path $c_0 = S_1^+|_{v=0}$.

and

$$\beta_0(0) < \bar{s}(v) \leq u_c,$$  \hspace{1cm} (43)

for all $v > 0$. Moreover, since $\bar{Q}(\alpha, \beta)$ depends continuously on $(\alpha, \beta, v) \in D_+ \times \mathbb{R}^+ \setminus \{ (\beta, u_c) : \beta \leq 0 \} \times \mathbb{R}^+$, the refraction vector $[28]$ for phase paths crossing the line $\alpha = u_c$ in $D_+$ is independent of $(\beta, v) \in \mathbb{R}^- \times \mathbb{R}^+$, and $\Omega_0$ is compact, we may conclude that

$$\bar{s} \in C(\mathbb{R}^+).$$  \hspace{1cm} (44)

In addition, from the rotational properties of the vector field $\bar{Q}(\alpha, \beta)$ in $\mathbb{R}^+$ with increasing $v \geq 0$, we deduce that

$$\bar{s}(v_2) > \bar{s}(v_1) \hspace{1cm} \forall v_2 > v_1 \geq 0.$$  \hspace{1cm} (45)

Therefore, $\bar{s} : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a continuous and strictly monotone increasing function. Next, take

$$v > v_c(u_c) = \left( \frac{1}{u_c} \sup_{\gamma \in (u_c, 1)} f_c(\gamma) \right)^{\frac{1}{2}}.$$  \hspace{1cm} (46)

Then, with $\beta_c = -vu_c$, we have

$$\bar{Q}(\alpha, \beta_c) \cdot (0, 1) = v^2 u_c - f_c(\alpha)$$
$$> \sup_{\gamma \in (u_c, 1)} f_c(\gamma) - f_c(\alpha) \geq 0,$$  \hspace{1cm} (47)

for all $\alpha \in (u_c, 1]$, and recall that $S_0|_v$ is given by $\beta = -v\alpha$ for $\alpha \in [0, u_c]$. It then follows, from (47), that

$$\bar{s}(v) > 0 \hspace{1cm} \forall v > v_c(u_c).$$  \hspace{1cm} (48)

We now observe that, at any $v \geq 0$, the dynamical system $[26]$ has a heteroclinic connection between $\vec{e}_1$ and $\vec{e}_0$, in $\mathbb{R}^+$ (which is unique, and is, in fact, contained in $\Omega_0 \subset \mathbb{R}^+$) if and only if $\bar{s}(v) = 0$. It follows that since $\bar{s} : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a continuous and strictly monotone increasing function, which satisfies (42) and (48), then, for each $u_c \in (0, 1)$, there exists a unique $v^*(u_c) > 0$ such that

$$\bar{s}(v^*(u_c)) = 0,$$  \hspace{1cm} (49)
whilst,
\[ s(v) < 0 \quad \forall v \in [0, v^*(u_c)), \]  
\[ s(v) > 0 \quad \forall v \in (v^*(u_c), \infty). \]  
(50a)  
(50b)

We conclude that, for each \( u_c \in (0, 1) \), QIVP has a PTW solution if and only if \( v = v^*(u_c)(>0) \) which we write as \( u = U_T(y), y \in \mathbb{R} \). Moreover, this PTW solution is unique. In addition, since the associated heteroclinic connection between \( \vec{e}_1 \) and \( \vec{e}_0 \) is contained in \( \Omega_0 \), then we conclude that \( U_T : \mathbb{R} \to \mathbb{R} \) satisfies:
\[ 0 < U_T(y) < 1, \quad U_T'(y) < 0 \quad \forall y \in \mathbb{R}, \]  
(51a)

with \( U_T(0) = u_c \), and
\[ U_T''(0^+) - U_T''(0^-) = -f_+^c, \]  
(51b)
\[ U_T(y) = u_c e^{-v^*(u_c)y} \quad \forall y \in \mathbb{R}^+, \]  
(51c)
\[ U_T(y) \sim 1 - A_{-\infty} e^{\lambda_+(v^*(u_c))y} \quad \text{as} \quad y \to -\infty, \]  
(51d)

for some constant \( A_{-\infty} > 0 \) (depending upon \( u_c \in (0, 1) \)), and with the eigenvalue \( \lambda_+(v) \) given in (31).

We next consider \( u_c \in (0, 1) \) as a parameter, regarding \( v^* \) as a function of \( u_c \), with \( v^* : (0, 1) \to \mathbb{R}^+ \) such that \( v^* = v^*(u_c) \), and associated PTW solution \( u = U_T(y, u_c) \) for \( (y, u_c) \in \mathbb{R} \times (0, 1) \). We recall that the vector field \( \vec{Q}(\alpha, \beta, v) \) is continuously differentiable on \( (\alpha, \beta, v) \in \left( ([0, u_c) \times \mathbb{R}) \cup ((u_c, 1] \times \mathbb{R}) \right) \times \mathbb{R}^+, \) whilst the refraction vector \( [28] \) depends on \( u_c \in (0, 1) \) and is continuous. It follows that on fixing \( u^0_c \in (0, 1) \), and taking \( \varepsilon > 0 \), then with \( u_c = u^0_c \) and \( v = v^*(u^0_c) - \varepsilon \), we have that \( s(v^*(u^0_c) - \varepsilon)|_{u_c = u^0_c} < 0 \), where we have used equation \([49]\). Hence, there exists \( \delta^-_\varepsilon > 0 \), which depends on \( \varepsilon > 0 \), such that for all \( u_c \in (u^0_c - \delta^-_\varepsilon, u^0_c + \delta^-_\varepsilon) = I^-_\varepsilon \), then
\[ s(v^*(u^0_c) - \varepsilon)|_{u_c \in I^-_\varepsilon} < 0. \]  
(52)

It follows that \( v^*(u_c) > v^*(u^0_c) - \varepsilon \) for all \( u_c \in I^-_\varepsilon \). Similarly, we establish that there exists \( \delta^+_\varepsilon > 0 \), which depends on \( \varepsilon > 0 \), such that for all \( u_c \in (u^0_c - \delta^+_\varepsilon, u^0_c + \delta^+_\varepsilon) = I^+_\varepsilon \), then
\[ s(v^*(u^0_c) + \varepsilon)|_{u_c \in I^+_\varepsilon} > 0. \]  
(53)

It follows that \( v^*(u_c) < v^*(u^0_c) + \varepsilon \) for all \( u_c \in I^+_\varepsilon \). We now set \( \delta_\varepsilon = \min(\delta^-_\varepsilon, \delta^+_\varepsilon) \). Thus, for all \( u_c \in (u^0_c - \delta_\varepsilon, u^0_c + \delta_\varepsilon) = I_\varepsilon \), then
\[ |v^*(u_c) - v^*(u^0_c)| < \varepsilon. \]  
(54)

We conclude that \( v^* : (0, 1) \to \mathbb{R} \) is continuous. In addition, we recall that
\[ v^*(u_c) > 0 \quad \forall u_c \in (0, 1). \]  
(55)

Next, let \( u^0_c \in (0, 1) \) and consider \( S^1_{\varepsilon}|_{((u^0_c, v^*(u^0_c))} \). It follows from the refraction vector \([28]\) that there exists \( \delta > 0 \), such that on fixing \( v = v^*(u^0_c) \), then for any \( u_c \in (u^0_c, u^0_c + \delta) = P_\delta \),
the intersection point of $S^+_1|\{(u_c,v^*(u_c^0))\}$ with the line $\alpha = u_c$ lies above the intersection point of the line $\beta = -v^*(u_c^0)\alpha$ with the line $\alpha = u_c$. Consequently, $s(v^*(u_c^0))|_{u_c \in P_3} > 0$, from which we conclude that $v^*(u_c) < v^*(u_0^c)$ for all $u_c \in P_3$. Thus, $v^* : (0,1) \to \mathbb{R}$ is locally decreasing, and continuous, and so $v^* : (0,1) \to \mathbb{R}$ is strictly monotone decreasing. It then also follows from (55) that $v^*(u_c)$ has a finite non-negative limit as $u_c \to 1^-$. Hence, $v^*(u_c) \to v_1^*$ as $u_c \to 1^-$, for some $v_1^* \geq 0$. When $(1-u_c)$ is sufficiently small, $S^+_1$ can be approximated in the region $(\alpha, \beta) \in [u_c, 1] \times \mathbb{R}^-$ by its linearised form at the equilibrium point $e_1$; it is then readily established that $v_1^* = 0$, and, moreover, that

$$v^*(u_c) \sim |f'(e_1)|\frac{1}{2}(1-u_c) \quad \text{as} \quad u_c \to 1^-.$$  

We now investigate $v^*(u_c)$ as $u_c \to 0^+$. To begin with we consider the dynamical system (26) when $u_c = 0$. In this case, the dynamical system (26) has a (unique) heteroclinic connection which connects $e_1^+$, as $y \to -\infty$, to $e_0^-$, as $y \to \infty$, if and only if $v \in [2, \infty)$, see for example [3, 5, 8, 9]. Moreover, $s(v)|_{u_c=0} < 0$ for all $v \in [0,2)$. From (28) and (29), it follows that $S^+_1$ depends continuously on $u_c \geq 0$. Thus, for $\varepsilon > 0$, there exists $\sigma_\varepsilon > 0$ such that for $u_c \in (0, \sigma_\varepsilon)$, then $s(2-\varepsilon)|_{u_c} < 0$. Therefore, from (49), we deduce that $v^*(u_c) > 2-\varepsilon$ for all $u_c \in (0, \sigma_\varepsilon)$. However, it also follows from (28) and (29) that $s(2)|_{u_c} > 0$ for all $u_c \in (0,1)$. Thus, $v^*(u_c) < 2$ for all $u_c \in (0,1)$. We conclude that,

$$2-\varepsilon < v^*(u_c) < 2 \quad \forall \ u_c \in (0, \sigma_\varepsilon).$$  

(57)

Since (57) holds for all $\varepsilon > 0$, we conclude immediately that $v^*(u_c)$ has limit 2 as $u_c \to 0^+$. We conclude that $v^* : (0,1) \to \mathbb{R}$ is continuous and monotone decreasing, with

$$\lim_{u_c \to 1^-} v^*(u_c) = 0, \quad \lim_{u_c \to 0^+} v^*(u_c) = 2.$$  

(58)

This completes the proof of Theorem 1.1. In the next two sections we consider the structure of the PTW solutions in the limits $u_c \to 0^+$ and $u_c \to 1^-$ respectively.

4. Asymptotic Structure of the PTW Solution when $u_c \to 0^+$

In this section we investigate the detailed asymptotic form of $v^*(u_c)$ as $u_c \to 0^+$, in the small cut-off limit, via the method of matched asymptotic expansions. To that end, we write $u_c = \varepsilon$ with $0 < \varepsilon \ll 1$. It then follows from Theorem 1.1 that we may write,

$$v^*(\varepsilon) = 2 - \bar{v}(\varepsilon),$$  

(59)

where now,

$$\bar{v}(\varepsilon) > 0 \quad \forall \ \varepsilon \in (0,1),$$  

(60)

and

$$\bar{v}(\varepsilon) = o(1) \quad \text{as} \quad \varepsilon \to 0^+.$$  

(61)
With \( U_T : \mathbb{R} \to \mathbb{R} \) being the associated PTW solution, then from (25),

\[
\begin{align*}
U_{Ty} + (2 - \bar{v}(\varepsilon))U_T + f(U_T) &= 0, \quad y < 0, \\
U_T(y) &> \varepsilon \quad \forall \ y < 0, \\
U_T(0) &= \varepsilon, \\
U_Ty(0) &= -(2 - \bar{v}(\varepsilon))\varepsilon, \\
U_T(y) &\to 1 \quad \text{as} \quad y \to -\infty.
\end{align*}
\]

(62a) to (62e)

It is convenient, in what follows, to make a shift of origin by introducing the coordinate \( \bar{y} \) via

\[
\bar{y} = \bar{y}_c(\varepsilon) + y,
\]

where \( \bar{y}_c(\varepsilon) \) is chosen so that (62) becomes,

\[
\begin{align*}
U_{\bar{y}\bar{y}} + (2 - \bar{v}(\varepsilon))U_{\bar{y}} + f(U_T(\bar{y})) &= 0, \quad \bar{y} < \bar{y}_c(\varepsilon), \\
U_T(\bar{y}) &> \varepsilon \quad \forall \ \bar{y} < \bar{y}_c(\varepsilon), \\
U_T(\bar{y}_c(\varepsilon)) &= \varepsilon, \\
U_T(\bar{y}_c(\varepsilon)) &= -(2 - \bar{v}(\varepsilon))\varepsilon, \\
U_T(\bar{y}) &\to 1 \quad \text{as} \quad \bar{y} \to -\infty,
\end{align*}
\]

(63a) to (63e)

with now the shift of origin fixing

\[
U_T(0) = \frac{1}{2}.
\]

(64)

It follows from (63) and (64) that

\[
\bar{y}_c(\varepsilon) \to +\infty \quad \text{as} \quad \varepsilon \to 0^+.
\]

(65)

Our objective is now to examine the boundary value problem (63) and (64) as \( \varepsilon \to 0^+ \), and, in particular, to determine the asymptotic structure of \( \bar{v}(\varepsilon) \) as \( \varepsilon \to 0^+ \). Anticipating the requirement of outer regions, we begin in an inner region when \( \bar{y} = O(1) \) and \( U_T = O(1) \) as \( \varepsilon \to 0^+ \), and we label this as region I. In region I we thus expand as

\[
U_T(\bar{y}; \varepsilon) = U_m(\bar{y}) + O(\bar{v}(\varepsilon)) \quad \text{as} \quad \varepsilon \to 0^+,
\]

(66)

with \( \bar{y} = O(1) \). On substitution from (66) into (63) and (64), and using (65), we obtain the leading order problem as

\[
\begin{align*}
U_{m\bar{y}\bar{y}} + 2U_{m\bar{y}} + f(U_m) &= 0, \quad -\infty < \bar{y} < \infty, \\
U_m(\bar{y}) &> 0, \quad -\infty < \bar{y} < \infty, \\
U_m(\bar{y}) &\to \begin{cases} 
1, & \text{as} \quad \bar{y} \to -\infty \\
0, & \text{as} \quad \bar{y} \to \infty
\end{cases}
\end{align*}
\]

(67a) to (67c)

with \( \bar{y} = O(1) \). On substitution from (66) into (63) and (64), and using (65), we obtain the leading order problem as

\[
U_{m\bar{y}\bar{y}} + 2U_{m\bar{y}} + f(U_m) = 0, \quad -\infty < \bar{y} < \infty,
\]

(67a)

\[
U_m(\bar{y}) > 0, \quad -\infty < \bar{y} < \infty,
\]

(67b)

\[
U_m(\bar{y}) \to \begin{cases} 
1, & \text{as} \quad \bar{y} \to -\infty \\
0, & \text{as} \quad \bar{y} \to \infty
\end{cases}
\]

(67c)

\[
U_m(0) = \frac{1}{2}.
\]

(67d)

The leading order problem is immediately recognised as the boundary value problem (25) for the permanent form travelling wave solution to the corresponding KPP problem.
The Evolution of Travelling Waves in a Cut-off Reaction-Diffusion Model.

without cut-off \((\varepsilon = 0)\). Let \(U_m : \mathbb{R} \rightarrow \mathbb{R}\) be the unique solution to \((67)\). For use in what follows, we recall \((7)\) with higher order corrections given by

\[
U_m(y) = \begin{cases} 
(A_\infty y + B_\infty) e^{-\gamma y} + O(y^2 e^{-2y}), & \text{as } y \to \infty \\
1 - A_\infty e^{\gamma y} + O(e^{2\gamma y}), & \text{as } y \to -\infty
\end{cases}
\]  

(68)

where \(\gamma = -1 + \sqrt{1 + |f'(1)|} \) \((> 0)\). On proceeding to \(O(\bar{\varepsilon}^2)\) in region we observe that the inner region expansion \((66)\) becomes non-uniform when \(|\bar{y}| \gg 1\), and in particular when \((-\bar{y}) = O(\bar{\varepsilon}^{-\frac{1}{2}})\) and \(\bar{y} = O(\bar{\varepsilon}^{-\frac{1}{2}})\). Therefore, to complete the asymptotic structure of the solution to \((63)\) as \(\varepsilon \to 0^+\), we must introduce two outer regions, namely region \(\Pi^+\) when \(\hat{y} = O(\bar{\varepsilon}^{-\frac{1}{2}})\) and region \(\Pi^-\) when \((-\hat{y}) = O(\bar{\varepsilon}^{-\frac{1}{2}})\). We begin in region \(\Pi^-\). To formalize region \(\Pi^-\), we introduce the scaled variable,

\[
\hat{y} = \bar{\varepsilon}^{\frac{1}{2}} \bar{y},
\]

(69)

so that \(\hat{y} = O(1)^-\) in region \(\Pi^-\) as \(\varepsilon \to 0^+\). It then follows from \((66)\) and \((68)\) that

\[
U_T(\hat{y}; \varepsilon) = 1 - O \left( e^{-\bar{\varepsilon}^{-\frac{1}{2}}} \right),
\]

(70)

as \(\varepsilon \to 0^+\) in region \(\Pi^-\). It is then straightforward to develop an exponential expansion in region \(\Pi^-\), which, after matching (following the Van Dyke matching principle, [17]) with region I, via \((66)\) and \((68)\), gives the outer expansion in region \(\Pi^-\) as,

\[
U_T(\hat{y}; \varepsilon) = 1 - A_\infty \exp \left[ \gamma \bar{\varepsilon}^{-\frac{1}{2}} \left( 1 + O(\bar{\varepsilon}) \right) \hat{y} \right] \\
+ O \left( \exp \left[ 2\gamma \bar{\varepsilon}^{-\frac{1}{2}} \left( 1 + O(\bar{\varepsilon}) \right) \hat{y} \right] \right),
\]

(71)

as \(\varepsilon \to 0^+\) with \(\hat{y} = O(1)^-\). Thus, the solution in region \(\Pi^-\) is at this order unaffected by the cut-off. We now proceed to region \(\Pi^+\), where \(\hat{y} = O(1)^+\) as \(\varepsilon \to 0^+\). It is within this region that the conditions at \(\bar{y} = \bar{y}_c(\varepsilon)\) must be satisfied, which then requires \(\bar{y}_c(\varepsilon) = O(\bar{\varepsilon}^{-\frac{1}{2}})\) as \(\varepsilon \to 0^+\), which is consistent with \((65)\). Thus, we write

\[
\bar{y}_c(\varepsilon) = \bar{\varepsilon}^{-\frac{1}{2}} \hat{y}_c(\varepsilon),
\]

(72)

so that now,

\[
\hat{y}_c(\varepsilon) = O(1)^+ \quad \text{as} \quad \varepsilon \to 0^+.
\]

(73)

In region \(\Pi^+\) it follows from \((66)\) and \((68)\) that

\[
U_T(\hat{y}; \varepsilon) = O \left( \bar{\varepsilon}^{-\frac{1}{2}} e^{-\bar{\varepsilon}^{-\frac{1}{2}}} \right),
\]

as \(\varepsilon \to 0^+\). Again, it is then straightforward to develop an exponential expansion in region \(\Pi^+\), which, after matching with region I, via \((66)\) and \((68)\), gives the outer expansion in region \(\Pi^+\) as,

\[
U_T(\hat{y}; \varepsilon) = \left( A_\infty \bar{\varepsilon}^{-\frac{1}{2}} \sin \left( \hat{y} \left( 1 + O(\bar{\varepsilon}) \right) \right) + B_\infty \cos \left( \hat{y} \left( 1 + O(\bar{\varepsilon}) \right) \right) \right) \times \exp \left( -\bar{\varepsilon}^{-\frac{1}{2}} \left( 1 + O(\bar{\varepsilon}) \right) \hat{y} \right) \\
+ O \left( \exp \left[ -2\bar{\varepsilon}^{-\frac{1}{2}} \left( 1 + O(\bar{\varepsilon}) \right) \hat{y} \right] \right)
\]

(74)
as \( \varepsilon \to 0^+ \) with \( \hat{y} = O(1)^+ \). It now remains to apply conditions (63b), (63c) and (63d) to (74). In the outer region \( \Pi^+ \), these conditions become,

\[
U_T(\hat{y}; \varepsilon) > \varepsilon \quad \forall \ O \left( \bar{v}(\varepsilon)^{-\frac{1}{2}} \right)^+ < \hat{y} < \hat{y}_c(\varepsilon), \tag{75a}
\]

\[
U_T(\hat{y}_c(\varepsilon); \varepsilon) = \varepsilon, \tag{75b}
\]

\[
\left. u_{T y}(\hat{y}_c(\varepsilon); \varepsilon) = -\varepsilon \bar{v}(\varepsilon)^{-\frac{1}{2}} (2 - \bar{v}(\varepsilon)). \right) \tag{75c}
\]

We now turn to conditions (75b) and (75c). It is convenient to first eliminate \( \varepsilon \) explicitly between (75b) and (75c) to give,

\[
\left. u_{T y}(\hat{y}_c(\varepsilon); \varepsilon) = -\bar{v}(\varepsilon)^{-\frac{1}{2}} (2 - \bar{v}(\varepsilon)) U_T(\hat{y}_c(\varepsilon); \varepsilon), \right) \tag{76}
\]

which replaces (75c). On substitution from (74) into (76) and expanding, using (60), (61) and (73), we obtain,

\[
A_\infty \sin \omega = -\bar{v}(\varepsilon)^{\frac{1}{2}} (A_\infty + B_\infty) \cos \omega, \quad \omega = \hat{y}_c(\varepsilon) (1 + O(\bar{v}(\varepsilon))) \tag{77}
\]

as \( \varepsilon \to 0^+ \). Following (73) and (77), we now expand,

\[
\hat{y}_c(\varepsilon) = \hat{y}_c^0 + \hat{y}_c^1 \bar{v}(\varepsilon)^{\frac{1}{2}} + O(\bar{v}(\varepsilon)), \tag{78}
\]

as \( \varepsilon \to 0^+ \), with the constants \( \hat{y}_c^0 > 0 \) and \( \hat{y}_c^1 \) to be determined. On substitution from (78) into (77), we obtain, at \( O(1) \),

\[
A_\infty \sin \hat{y}_c^0 = 0.
\]

Since \( A_\infty > 0 \), then we must have (recalling \( \hat{y}_c^0 > 0 \) \( \hat{y}_c^0 = k\pi \), for some \( k \in \mathbb{N} \). However, condition (75a), with (74), then requires \( k = 1 \), and so

\[
\hat{y}_c^0 = \pi. \tag{79}
\]

Proceeding to \( O(\bar{v}(\varepsilon)^{\frac{1}{2}}) \), we find that, on using (79),

\[
\hat{y}_c^1 = -\frac{(A_\infty + B_\infty)}{A_\infty}. \tag{80}
\]

Thus, via (78), (79) and (80) we have,

\[
\hat{y}_c(\varepsilon) = \pi - \frac{(A_\infty + B_\infty)}{A_\infty} \bar{v}(\varepsilon)^{\frac{1}{2}} + O(\bar{v}(\varepsilon)), \tag{81}
\]

as \( \varepsilon \to 0^+ \). It remains to apply condition (75b). On using (74) and (81), condition (75b) becomes

\[
\ln \varepsilon = -\frac{\pi}{\bar{v}(\varepsilon)^{\frac{1}{2}}} + \left( \frac{(A_\infty + B_\infty)}{A_\infty} \ln A_\infty \right) + o(1), \tag{82}
\]

as \( \varepsilon \to 0^+ \). A re-arrangement of (82) then gives,

\[
\bar{v}(\varepsilon) = \frac{\pi^2}{(\ln \varepsilon)^2} + \frac{2\pi^2 ((A_\infty + B_\infty) A_\infty^{-1} + \ln A_\infty)}{(\ln \varepsilon)^3} + o \left( \frac{1}{(\ln \varepsilon)^3} \right), \tag{83}
\]
The Evolution of Travelling Waves in a Cut-off Reaction-Diffusion Model.

as $\varepsilon \to 0^+$. It then follows from (81) and (83) that,

$$\hat{y}_c(\varepsilon) = \pi + \frac{(A_\infty + B_\infty)\pi}{A_\infty} \frac{1}{\ln \varepsilon} + O\left(\frac{1}{(\ln \varepsilon)^2}\right),$$  \hspace{1cm} \text{(84)}

as $\varepsilon \to 0^+$. Finally, via (59) and (83), we can construct $v^*(\varepsilon)$ as,

$$v^*(\varepsilon) = 2 - \frac{\pi^2}{(\ln \varepsilon)^2} - \frac{2\pi^2 ((A_\infty + B_\infty)A_\infty^{-1} + \ln A_\infty)}{(\ln \varepsilon)^3} + o\left(\frac{1}{(\ln \varepsilon)^3}\right),$$  \hspace{1cm} \text{(85)}

as $\varepsilon \to 0^+$. We observe that (85) is decreasing in $\varepsilon$ as $\varepsilon \to 0^+$, and is in full accord with the rigorous results established in Theorem 1.1. We see immediately that the result derived here, via a rational application of the method of matched asymptotic expansions, agrees in the first two terms with the results that Brunet and Derrida [13] and Dumortier, Popovic and Kaper [14] obtained. However, the method of matched asymptotic expansions has enabled us to obtain the next correction term in (85), and higher order terms could be obtained by systematically following this approach. We now consider the asymptotic structure of the PTW solution to QIVP as $u_c \to 1^-$.

5. Asymptotic Structure of the PTW Solution when $u_c \to 1^-$

In this section we investigate the asymptotic form of $v^*(u_c)$ in the large cut-off limit $u_c \to 1^-$. To this end, we write $u_c = 1 - \delta$ with $0 < \delta \ll 1$. Theorem 1.1 guarantees the existence and uniqueness of a PTW solution, whose speed $v^*(\delta) = o(1)$ as $\delta \to 0^+$. In this case, it is most convenient to consider the problem in the $(\alpha, \beta)$ phase plane corresponding to the phase path representing the PTW when $u_c = 1 - \delta$ and $v = v^*(\delta)$. Via (26), (31), (32) and (33), this is given by the phase path $\beta = \beta(\alpha; \delta)$, which satisfies the boundary value problem

$$\frac{d\beta}{d\alpha} = -v^*(\delta) - \frac{f(\alpha)}{\beta}, \quad \alpha \in (1 - \delta, 1),$$  \hspace{1cm} \text{(86a)}

$$\beta(\alpha; \delta) \sim -\lambda_+(v^*(\delta))(1 - \alpha) \quad \text{as} \quad \alpha \to 1^-,$$  \hspace{1cm} \text{(86b)}

$$\beta(1 - \delta; \delta) = -v^*(\delta)(1 - \delta).$$  \hspace{1cm} \text{(86c)}

We now examine the boundary value problem (86) as $\delta \to 0^+$. Since $v^*(\delta) = o(1)$ as $\delta \to 0^+$, we expand $\lambda_+(v^*(\delta))$, via (31), which determines that $\lambda_+(v^*(\delta)) = O(1)$ as $\delta \to 0^+$. It follows from the boundary condition (86b), that $\beta = O(\delta)$ as $\delta \to 0^+$. We therefore introduce the following re-scalings

$$\beta = \delta Y, \quad \alpha = 1 - \delta X,$$  \hspace{1cm} \text{(87)}

with $Y, X = O(1)$ as $\delta \to 0^+$. The form of the boundary condition (86c) then necessitates that $v^*(\delta) = O(\delta)$ as $\delta \to 0^+$. Thus, we write

$$v^*(\delta) = \delta V(\delta),$$  \hspace{1cm} \text{(88)}
The Evolution of Travelling Waves in a Cut-off Reaction-Diffusion Model.

where $V(\delta) = O(1)$ as $\delta \to 0^+$. These re-scalings transform the boundary value problem (86) into,

$$
\frac{dY}{dX} = \delta V(\delta) + \frac{f(1 - \delta X)}{\delta Y}, \quad X \in (0, 1),
$$

(89a)

$$
Y(X; \delta) \sim -\lambda_+ (\delta V(\delta))X \quad \text{as} \quad X \to 0^+,
$$

(89b)

$$
Y(1; \delta) = -V(\delta)(1 - \delta).
$$

(89c)

We now expand $Y(X; \delta)$ and $V(\delta)$ according to,

$$
Y(X; \delta) = Y_0(X) + \delta Y_1(X) + o(\delta), \quad X \in [0, 1],
$$

(90a)

$$
V(\delta) = V_0 + \delta V_1 + o(\delta),
$$

(90b)

as $\delta \to 0^+$. Substituting the expansions from (90) into the boundary value problem (89) and expanding, at $O(1)$, we obtain the following boundary value problem for $Y_0(X)$, namely,

$$
\frac{dY_0}{dX} = -f'(1) X \frac{Y_1}{Y_0}, \quad X \in (0, 1),
$$

(91a)

$$
Y_0(X) \sim -|f'(1)|^{\frac{1}{2}} X \quad \text{as} \quad X \to 0^+,
$$

(91b)

$$
Y_0(1) = -V_0.
$$

(91c)

The general solution to (91a) is $Y_0^2(X) = c_1 - f'(1)X^2$, for $X \in [0, 1]$, where $c_1$ is an arbitrary constant of integration. Applying the boundary condition (91b) determines $c_1 = 0$. Therefore,

$$
Y_0(X) = -|f'(1)|^{\frac{1}{2}} X, \quad X \in [0, 1].
$$

(92)

Application of the boundary condition (91c) then determines

$$
V_0 = |f'(1)|^{\frac{1}{2}}.
$$

(93)

At $O(\delta)$, we obtain the following boundary value problem for $Y_1(X)$, namely,

$$
\frac{dY_1}{dX} - \frac{Y_1}{Y_0(X)^2} f'(1)X = V_0 + \frac{1}{2} f''(1) \frac{X^2}{Y_0(X)}, \quad X \in (0, 1),
$$

(94a)

$$
Y_1(X) \sim \frac{1}{2} V_0 X \quad \text{as} \quad X \to 0^+,
$$

(94b)

$$
Y_1(1) = V_0 - V_1.
$$

(94c)

On substituting $Y_0(X)$, given by (92), into equation (94a) and solving, we find that the general solution is

$$
Y_1(X) = \frac{1}{2} V_0 X - \frac{1}{6} f''(1) \frac{X^2}{|f'(1)|^{\frac{1}{2}}} + \frac{c_2}{X}, \quad X \in (0, 1],
$$

(95)

where $c_2$ is an arbitrary constant of integration. From the boundary condition (94b), $Y_1(X)$ remains bounded as $X \to 0^+$. Therefore, we require $c_2 = 0$. Thus, we obtain the solution for $Y_1(X)$ as

$$
Y_1(X) = \frac{1}{6} |f'(1)|^{\frac{1}{2}} X \left(3 - \frac{f''(1)}{|f'(1)|} X\right), \quad X \in [0, 1].
$$

(96)
Finally, an application of the boundary condition (97) determines

$$V_1 = \frac{1}{6} |f'(1)|^{1/2} \left( 3 + \frac{f''(1)}{|f'(1)|} \right).$$  \hfill (97)

On collecting expressions (90a), (92) and (96), we have established that

$$Y(X; \delta) = -|f'(1)|^{1/2} X + \frac{1}{6} \delta |f'(1)|^{1/2} X \left( 3 - \frac{f''(1)}{|f'(1)|} X \right) + o(\delta) \quad \text{as} \quad \delta \to 0^+, \hfill (98)$$

uniformly for $X \in [0, 1]$. Similarly, on collecting expressions (90b), (93) and (97), we obtain,

$$V(\delta) = |f'(1)|^{1/2} + \frac{1}{6} \delta |f'(1)|^{1/2} \left( 3 + \frac{f''(1)}{|f'(1)|} \right) + o(\delta) \quad \text{as} \quad \delta \to 0^+. \hfill (99)$$

Using (88), the propagation speed of the PTW solution to QIVP is given by

$$v^*(\delta) = \delta |f'(1)|^{1/2} + \frac{1}{6} \delta^2 |f'(1)|^{1/2} \left( 3 + \frac{f''(1)}{|f'(1)|} \right) + o(\delta^2) \quad \text{as} \quad \delta \to 0^+. \hfill (100)$$

We use (87) to express the PTW solution to QIVP in terms of the cut-off $u_c$ as

$$\beta(\alpha) = -\frac{1}{2} |f'(1)|^{1/2} (1 + u_c)(1 - \alpha) - \frac{1}{6} \frac{f''(1)}{|f'(1)|^{1/2}} (1 - \alpha)^2$$

$$+ o((1 - u_c)^2) \quad \text{as} \quad u_c \to 1^-. \hfill (101)$$

Its speed of propagation is given by

$$v^*(u_c) = (1 - u_c) |f'(1)|^{1/2} + \frac{1}{6} (1 - u_c)^2 |f'(1)|^{1/2} \left( 3 + \frac{f''(1)}{|f'(1)|} \right)$$

$$+ o((1 - u_c)^2) \quad \text{as} \quad u_c \to 1-. \hfill (102)$$

In the next section we consider the specific case of a cut-off Fisher reaction, determining $v^*: (0, 1) \to \mathbb{R}$ via numerical integration.

6. Numerical Example

We here compare our predictions for the PTW solutions $U_T(y)$ and the speed $v^*(u_c)$ derived in the limits of small and large cut-off $u_c$, with the corresponding values obtained from the numerical evaluation of (25) (or (63) when $u_c$ is small) carried out for the particular case of the cut-off Fisher reaction function, namely,

$$f_c(u) = \begin{cases} 
    u(1 - u), & u \in (u_c, \infty) \\
    0, & u \in (-\infty, u_c]
\end{cases} \hfill (103)$$

where $u_c \in (0, 1)$. For our numerical calculations we adopt a shooting method that combines a standard fourth order Runge-Kutta discretisation scheme with a bisection method. We use (32) to approximate the unstable manifold near the unstable fixed
Figure 5. (a) Numerical solutions of $U_T(\bar{y})$ obtained from (63) for $u_c = 0.001, 0.01$ and 0.1, with the arrow pointing in the direction of increasing $u_c$ (thick black lines), and exact solution derived from (14b) (thin black lines). These are plotted against the numerical solution of $U_m(\bar{y})$ (in red) obtained from (67). (b) Comparison between numerical and asymptotic results for $U_T(\bar{y})$ obtained for $\bar{y} \leq \bar{y}_c(u_c)$ for $u_c = 10^{-7}$ (in black) and $u_c = 10^{-5}$ (in blue). The numerical results are juxtaposed against the asymptotic expression (74) valid for $u_c \to 0^+$ and $\bar{y} \gg 1$ (dashed black and blue lines, respectively). These are plotted against the numerical solution of $U_m(\bar{y})$ (in solid red) and the large-$\bar{y}$ asymptotic expression (68) (dashed red line).

The asymptotic predictions for the PTW solutions $U_T(\bar{y})$ and the speed $v^*(u_c)$ obtained for small values of $u_c$ rely on the global constants $A_\infty$ and $B_\infty$ (see equations (74) and (85)) associated with the leading edge behaviour of $U_m(\bar{y})$ (see expression (68)). We determine the values of $A_\infty$ and $B_\infty$ from the numerical evaluation of (67) carried out for the particular case of the Fisher reaction function (3). We again adopt a fourth order Runge-Kutta discretisation scheme and use (68) to approximate the unstable manifold near the unstable fixed point $(U_m, U'_m) = (1, 0)$, taking $U_m = 1 - \epsilon$ and $U'_m = -\lambda_+(v)\epsilon$ where $\epsilon = 10^{-10}$. We choose $\Delta\alpha = 10^{-12}$ to ensure that the values of $U_T(y)$ (or $U_T(\bar{y})$ when $u_c$ is small) and $v^*(u_c)$ are obtained to ten decimal places of accuracy.

Figure 5 is devoted to the structure of $U_m(\bar{y})$ and that of $U_T(\bar{y})$ obtained for small cut-off $u_c$. For $\bar{y} \leq \bar{y}_c(u_c)$ this is numerically determined from (63). For $\bar{y} > \bar{y}_c(u_c)$ the solution is exact (see equation (14b)). Figure 5(a) contrasts the behaviour of $U_T(\bar{y})$ against that of $U_m(\bar{y})$. It is clear that when $\bar{y} = O(1)$, $U_T(\bar{y})$ remains close to $U_m(\bar{y})$ while when $|\bar{y}| \gg 1$, $U_T(\bar{y})$ approaches $1^-$ and $u_c$ exponentially fast. This behaviour is consistent with expressions (66), (71) and (74) (with $\epsilon = u_c$). Figure 5(b) focuses on the behaviour of $U_m(\bar{y})$ and $U_T(\bar{y})$ when the value of $\bar{y}$ is large. It shows that as long as $\bar{y} \gtrsim 5$, the numerical solution of $U_m(\bar{y})$ and the leading edge asymptotics given by (68) (with numerical values of $A_\infty$ and $B_\infty$ given above) are in excellent agreement. The same
The Evolution of Travelling Waves in a Cut-off Reaction-Diffusion Model.

Figure 6. (a) Numerical solutions of $U_T(y)$ obtained from (25) for $u_c = 0.4, 0.8, 0.9$ and $0.99$ (thick lines), with the arrow pointing in the direction of increasing $u_c$, and exact solution given by (51c) (thin lines). (b) Comparison between the numerical solutions of $U_T(y)$ and the asymptotic expression for $u_c \to 1^-$ derived from (101) (dashed black lines).

Figure 6 focuses on the structure of $U_T(y)$ obtained for larger cut-off $u_c$. For $y \leq 0$, this is numerically determined from (25); for $y > 0$ it is exactly given by (14b). It is clear that the asymptotic prediction associated with (101) is an excellent approximation of $U_T(y)$ for all values of $u_c$ considered (see Figure 6(b)).

We now examine the behaviour of the speed $v^*(u_c)$. Figure 7(a) focuses on speed values obtained for small values of $u_c$. It shows that expression (11) of Brunet and Derrida [13] is very good as long as $u_c \lesssim 0.02$. Higher order corrections are captured by the asymptotic expression (85) which remains good (though only marginally better than expression (11)) as long as $u_c \lesssim 0.05$ (when $u_c = 0.05$, $\delta \approx 1$ associated with expansion (66) is no longer small). Figure 7(b) shifts the focus to larger values of $u_c$. It shows that the asymptotic expression (102) accurately captures the speed $v^*(u_c)$ for a wide range of values given by $0.4 \lesssim u_c < 1$ (when $u_c = 0.4$, $\delta = 1 - u_c = 0.6$ associated with expansions (90) is no longer small).

7. Discussion and Conclusions

In this paper we have considered a canonical evolution problem for a reaction-diffusion process when the reaction function is of standard KPP-type, but experiences a cut-off in the reaction rate below the normalised cut-off concentration $u_c \in (0, 1)$. We have formulated this evolution problem in terms of the moving boundary initial-boundary
value problem QIVP. In Section 2 we have obtained some very general results concerning the solution to QIVP. In particular, these general results indicate that in the large-time, as \( t \to \infty \), the solution to QIVP will involve the propagation of an advancing non-negative permanent form travelling wave, effecting the transition from the unreacting state \( u = 0 \) (ahead of the wave-front) to the fully reacted state \( u = 1 \) (at the rear of the wave-front). With this in mind, this paper has concentrated on examining the existence of permanent form travelling wave solutions to QIVP with propagation speed \( v \geq 0 \), referred to as PTW solutions. In Section 3 we have used a phase plane analysis of the nonlinear boundary value problem (25) to establish that (i) for each \( u_c \in (0, 1) \), then QIVP has a unique PTW solution, with propagation speed \( v = v^*(u_c) > 0 \) and (ii) \( v^*: (0, 1) \to \mathbb{R}^+ \) is continuous and monotone decreasing, with \( v^*(u_c) \to 0^+ \) as \( u_c \to 1^- \), and \( v^*(u_c) \to 2^- \) as \( u_c \to 0^+ \). It should be noted that 2 is the minimum propagation speed of permanent form travelling wave solutions for the related KPP-type function in the absence of cut-off. In Section 4, we have developed asymptotic methods to determine the asymptotic forms of \( v^*(u_c) \) as \( u_c \to 0^+ \) and \( u_c \to 1^- \). The first limit was previously considered by Brunet and Derrida \[13\] and Dumortier, Popovic and Kaper \[14\]. The latter employed matched asymptotics expansions in the phase plane to determine the order of the error in \[13\]. We have here used matched asymptotics expansions on the direct problem (25) to obtain higher order corrections in a systematic manner. We show that these are controlled by the detailed structure ahead of the wave-front solution travelling with speed 2 for the related KPP problem obtained in the absence of a cut-off. For larger values of \( u_c \), the asymptotic behaviour is obtained via the use of regular asymptotic expansions in the phase plane.
We anticipate that the approach developed in this paper, for considering PTW solutions to QIVP, will be readily adaptable to corresponding problems, when the KPP-type cut-off reaction function is replaced by a broader class of cut-off reaction functions. In comparing the PTW theory for the cut-off KPP-type reaction function studied here, and its associated KPP-type reaction function without cut-off, we make the observation that, in the absence of cut-off, a PTW solution exists for each propagation speed \( v \in [2, \infty) \), whilst at each fixed cut-off value \( u_c \in (0, 1) \), a PTW solution exists only at the single propagation speed \( v = v^*(u_c) \), with \( 0 < v^*(u_c) < 2 \). This will have implications for the development of PTW solutions as large-\( t \)-structures in QIVP, with more general classes of initial data. In the companion paper we consider the evolution problem QIVP in more detail. Specifically we establish that, as \( t \to \infty \), the solution to QIVP does indeed involve the formation of the PTW solution considered in this paper, and we give the detailed asymptotic structure of the solution to QIVP as \( t \to \infty \).

Finally, it is interesting to contrast our results with results obtained from an alternative model whose purpose is also to account for microscopic discrete particles [18, 19]. This is given by the (stochastic) randomly perturbed KPP equation

\[
\begin{align*}
  u_t &= u_{xx} + f(u) + (\hat{u}_c f(u) \Theta(u - 1))^{1/2} W(x,t), \\
  u(x,0) &= u_0(x),
\end{align*}
\]

(104) (105)

where \( \Theta(u - 1) \) is 1 for \( u \leq 1 \) and 0 otherwise. \( W(x,t) \) is a Gaussian white noise, \( \delta \)-correlated in time and space and \( \hat{u}_c^{1/2} \) is the strength of the noise term. Mueller, Mytnik and Quastel [20] have recently proved that for this stochastic KPP model, the speed of propagation is given by

\[
\hat{v}(\hat{u}_c) = 2 - \frac{\pi^2}{(\ln \hat{u}_c)^2} + O \left( \frac{\log |\log \hat{u}_c|}{(\log \hat{u}_c)^3} \right), \quad \text{as} \quad \hat{u}_c \to 0^+.
\]

Thus, taking \( \hat{u}_c \sim u_c \), the difference between the speed obtained from this model and \( v^*(u_c) \) obtained from the deterministic cut-off model considered here only arises in the order of the error as both \( u_c \) and \( \hat{u}_c \to 0^+ \). The two models behave very differently when \( u_c \) and \( \hat{u}_c \) can no longer be regarded as small, as might be anticipated. In this case, [21] conjectured that the speed of propagation associated with the stochastic KPP model with \( f(u) = u(1 - u) \) is given by \( \hat{v}(\hat{u}_c) \sim 2/\hat{u}_c \), as \( \hat{u}_c \to \infty \). The behaviour in this limit should be contrasted against expression (102) obtained for \( u_c \to 1^- \). A comparison suggests that \( \hat{u}_c \) and \( u_c \) may in this case be related according to \( \hat{u}_c \sim 2/(1 - u_c) \) as \( u_c \to 1^- \).

Acknowledgments

The research of A D O Tisbury was supported by an EPRSC grant with reference number 1537790. A Tzella thanks C Doering for useful conversations.

[1] Fife P C 1979 Mathematical Aspects of Reacting and Diffusing Systems (Springer-Verlag, Berlin)
[2] Murray J 2002 Mathematical Biology I: An introduction (Springer-Verlag, 3rd edition)
The Evolution of Travelling Waves in a Cut-off Reaction-Diffusion Model.

[3] Kolmogorov A N, Petrovsky I G and Piskunov N S 1937 Bull. Univ. Moskov. Ser. Internat. Sect. 1 1–25
[4] Fisher R A 1937 Ann. Eugenics 7 355–369
[5] Aronson D G and Weinberger H F 1975 Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation vol 446 (Heidelberg: Springer)
[6] Smoller J 1989 Shock Waves and Reaction-Diffusion equations (Springer, Berlin)
[7] Hadeler K P and Rothe F 1975 Journal of Mathematical Biology 2 251–263
[8] Larson D A 1978 SIAM J. Appl. Math. 34 93–103
[9] Needham D J 1992 Quart. J. Mech. Appl. Math. 45(3) 469–498
[10] McKea H P 1975 Commun. Pur. Appl. Math. 28 323–331
[11] Bramson M 1983 Mem. Am. Math. Soc. 44
[12] Merkin J H and Needham D J 1993 J. Appl. Math. Phys. (ZAMP) 44(4) 707–721
[13] Brunet E and Derrida B 1997 Phys. Rev. E. 56 2597 – 2604
[14] Dumortier F, Popovic N and Kaper T J 2007 Nonlinearity 20 855–877
[15] Aronson D G and Serrin J 1967 Arch. Rational Mech. Anal. 25 81–122
[16] Verhulst F 1990 Nonlinear Differential Equations and Dynamical Systems (Springer)
[17] Van Dyke M 1975 Perturbation Methods in Fluid Mechanics (Parabolic Press)
[18] Conlon J G and Doering C R 2005 J. Stat. Phys. 120 421–477
[19] Tesser F and Doering C R 2014 Discrete and Continuum Dynamics of Reacting and Interacting Individuals. In: A. Muntean, F. Toschi (eds) Collective Dynamics from Bacteria to Crowds. CISM International Centre for Mechanical Sciences vol 553 (Springer, Vienna)
[20] Mueller C, Mytnik L and Quastel J 2011 J. Invent. Math. 184 405–453
[21] Doering C R, Mueller C and Smereka P 2003 AIP Conference Proceedings 665 523–530