LONG TIME DECAY OF 3D-NSE IN LEI-LIN-GEVREY SPACES

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ABSTRACT. In this paper, we prove a global well-posedness of the three-dimensional incompressible Navier-Stokes equation under initial data, which belongs to the Lei-Lin-Gevrey space $Z_{-\frac{1}{a},\sigma}^1(\mathbb{R}^3)$ and if the norm of the initial data in the Lei-Lin space $X_{-1}^1$ is controlled by the viscosity. Moreover, we will show that the norm of this global solution in the Lei-Lin-Gevrey space decays to zero as time approaches to infinity.

1. Introduction

The 3D incompressible Navier-Stokes equations are given by:
\[
\begin{aligned}
\partial_t u - \nu \Delta u + u \cdot \nabla u &= -\nabla p \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\
\text{div } u &= 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\
u(0, x) &= u^0(x) \quad \text{in } \mathbb{R}^3,
\end{aligned}
\]
(NSE)

where $\nu > 0$ is the viscosity of the fluid, $u = u(t, x) = (u_1, u_2, u_3)$ and $p = p(t, x)$ denote respectively the unknown velocity and the unknown pressure of the fluid at the point $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3$, and $(u \cdot \nabla u) := u_1 \partial_1 u + u_2 \partial_2 u + u_3 \partial_3 u$, while $u^0 = (u^0_1(x), u^0_2(x), u^0_3(x))$ is an initial given velocity. If $u^0$ is quite regular, the divergence free condition determines the pressure $p$.

The local existence of solutions to the system (NSE) was studied by several researchers (for example [9,13,14]). The global existence of weak solutions goes back to Leray [14] and Hopf [8]. The global well-posedness of strong solutions for small initial data in the critical Sobolev space $\dot{H}^{\frac{1}{2}}$ is due to Fujita and Kato [6]. In [4], Chemin considered initial data that belongs to the space $\dot{H}^s$ for $s > \frac{1}{4}$. Kato [10] obtained a well-posedness result for initial data in the Lebesgue space $L^3$. In [11], Koch and Tataru considered the space $\text{BMO}^{-1}$ (see, also [3,5,15]). In all these works, the norms in the corresponding spaces of the initial data are assumed to be very small. More precisely, the norm was supposed to be bounded by the viscosity $\nu$ multiplied by some positive constant. More results and details in this direction can be found in the book by Cannone [2].

In [12], the authors consider a new critical space that is contained in $\text{BMO}^{-1}$ and prove that it is sufficient to assume that the norms of initial data are less than exactly the viscosity coefficient $\nu$. Then, the used space in [12] is the following
\[
\mathcal{X}^{-1}(\mathbb{R}^3) = \left\{ f \in \mathcal{D}'(\mathbb{R}^3); \int_{\mathbb{R}^3} \frac{\|\hat{u}(\xi)\|}{|\xi|} d\xi < \infty \right\}
\]

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which is equipped with the norm
\[ \|f\|_{X^{-1}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \frac{|\hat{f}(\xi)|}{|\xi|} d\xi. \]

For the small initial data, the global existence is proved in [12].

**Theorem 1.1 (See [12]).** Let \( u^0 \in X^{-1}(\mathbb{R}^3) \), such that \( \|u^0\|_{X^{-1}(\mathbb{R}^3)} < \nu \). Then, there is a unique
\( u \in C(\mathbb{R}^+, X^{-1}(\mathbb{R}^3)) \) such that \( \Delta u \in L^1(\mathbb{R}^+, X^{-1}(\mathbb{R}^3)) \). Moreover, for all \( t \geq 0 \)
\[ \sup_{0 \leq t < \infty} \left( \|u(t)\|_{X^{-1}} + (\nu - \|u^0\|_{X^{-1}}) \int_0^t \|\nabla u\|_{L^\infty} d\tau \right) \leq \|u^0\|_{X^{-1}}. \]

On the other hand, in [16] the authors proved the local existence for the initial data and blow-up criteria if the maximal time is finite, precisely:

**Theorem 1.2 (See [16]).** Let \( u^0 \in X^{-1}(\mathbb{R}^3) \). There exists time \( T \) such that the system \( \text{NSE} \) has a unique solution \( u \in L^2([0,T], X^0(\mathbb{R}^3)) \) which also belongs to
\[ C([0,T], X^{-1}(\mathbb{R}^3)) \cap L^1([0,T], X^1(\mathbb{R}^3)) \cap L^\infty([0,T], X^{-1}(\mathbb{R}^3)) \]
Let \( T^* \) denote the maximal time of existence of such solution. Hence if \( \|u\|_{X^{-1}} < \nu \), then
\[ T^* = \infty; \]
if \( T^* \) is finite, then
\[ \int_0^{T^*} \|u(t)\|^2_{X^0} = \infty. \]

Also, the long time decay for the global solution was studied in [1], precisely:

**Theorem 1.3 (See [1]).** Let \( u \in C(\mathbb{R}^+, X^{-1}(\mathbb{R}^3)) \) be a global solution of \( \text{NSE} \), then
\[ \lim_{t \to \infty} \sup \|u(t)\|_{X^{-1}} = 0. \]

To prepare for announcing our main results, we need to introduce the Lei-Lin-Gevrey spaces:
For \( a > 0 \),\( \sigma > 1 \) and \( \rho \in \mathbb{R} \), the following spaces are defined
\[ Z^\rho_{a,\sigma}(\mathbb{R}^3) = \left\{ f \in S'(\mathbb{R}^3); \hat{f} \in L^1_{\text{loc}}(\mathbb{R}^3), \int_{\mathbb{R}^3} |\xi|^\rho e^{a|\xi|^{1/\sigma}} |\hat{f}(\xi)| d\xi < \infty \right\} \]
which are equipped with the norm
\[ \|f\|_{Z^\rho_{a,\sigma}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |\xi|^\rho e^{a|\xi|^{1/\sigma}} |\hat{f}(\xi)| d\xi. \]

Our first result is the following:

**Theorem 1.4.** Let \( u^0 \in Z^{-1}_{a,\sigma}(\mathbb{R}^3) \), such that \( \|u\|_{X^{-1}} < \nu \). Then, there exists a unique global solution \( u \in C(\mathbb{R}^+, Z^{-1}_{a,\sigma}(\mathbb{R}^3)) \cap L^1(\mathbb{R}^+, Z^1_{a,\sigma}(\mathbb{R}^3)) \) of \( \text{NSE} \).

Our second result is as follows:

**Theorem 1.5.** Let \( u \in C(\mathbb{R}^+, Z^{-1}_{a,\sigma}(\mathbb{R}^3)) \) be the global solution of \( \text{NSE} \). Then
\[ \limsup_{t \to \infty} \|u(t)\|_{Z^{-1}_{a,\sigma}} = 0. \]
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The paper is organized in the following way: In Section 2, we give some notations and important preliminary results. Section 3 is devoted to prove that (NSE) is well posed in \( Z^{-1}_{\alpha,\sigma}(\mathbb{R}^3) \). In Section 4, we prove the global existence under the condition \( \|u\|_{X^{-1}} < \nu \). Finally, in the Section 5, we state that the norm of global solution in \( Z^{-1}_{\alpha,\sigma}(\mathbb{R}^3) \) goes to zero at infinity.

2. Notations and preliminary results

2.1. Notations

In this section, we collect some notations and definitions that will be used later. First, the Fourier transformation is normalized as

\[
\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^3} \exp(-ix \cdot \xi)f(x)dx, \quad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3,
\]

the inverse Fourier formula is

\[
\mathcal{F}^{-1}(g)(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} \exp(i\xi \cdot x)g(\xi)d\xi, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3,
\]

and the convolution product of a suitable pair of functions \( f \) and \( g \) on \( \mathbb{R}^3 \) is given by

\[
(f * g)(x) := \int_{\mathbb{R}^3} f(y)g(x-y)dy.
\]

If \( f = (f_1, f_2, f_3) \) and \( g = (g_1, g_2, g_3) \) are two vector fields, then we set

\[
f \otimes g := (g_1 f, g_2 f, g_3 f),
\]

and

\[
\text{div}(f \otimes g) := (\text{div}(g_1 f), \text{div}(g_2 f), \text{div}(g_3 f)).
\]

We denote by \( P \) the Leray projection operator defined by the formula:

\[
\mathcal{F}(Pf)(\xi) = \hat{f}(\xi) - \frac{(f(\xi) \cdot \xi)}{||\xi||^2} \xi.
\]

Finally, let \( \langle B, \| \cdot \| \rangle \), be a Banach space, \( 1 \leq p \leq \infty \) and \( T > 0 \). We define \( L^p_T(B) \) to be the space of all measurable functions

\[
[0, T] \ni t \mapsto f(t) \in B \text{ such that } t \mapsto \|f(t)\| \in L^p([0, T]).
\]

2.2. Preliminary results

In this section, we recall some classical results and we give new technical lemmas.

**Lemma 2.1.** Let \( u, v \in Z^{-1}_{\alpha,\sigma}(\mathbb{R}^3) \cap Z^1_{\alpha,\sigma}(\mathbb{R}^3) \). Then

\[
\|uv\|_{Z^0_{\alpha,\sigma}} \leq \|u\|_{Z^{-1}_{\alpha,\sigma}}^{2/3} \|u\|_{Z^1_{\alpha,\sigma}}^{2/3} \|v\|_{Z^{-1}_{\alpha,\sigma}}^{2/3} \|v\|_{Z^1_{\alpha,\sigma}}^{2/3}.
\]

**Proof.** We have

\[
\|uv\|_{Z^0_{\alpha,\sigma}} = \int_{\mathbb{R}^3} e^{a|\xi|^{1/\sigma}} |\hat{u}v(\xi)|d\xi \leq \int_{\mathbb{R}^3} e^{a|\xi|^{1/\sigma}} \left( \int_{\mathbb{R}^3} |\hat{u}(\xi - \eta)||\hat{v}(\eta)|d\eta \right)d\xi.
\]
Using the inequality \(e^{a|\xi|^{1/\sigma}} \leq e^{a|\xi-n|^{1/\sigma}} e^{a|\eta|^{1/\sigma}}\) and taking \(F(\xi) = e^{a|\xi|^{1/\sigma}} |\hat{u}(\xi)|,\)
\(G(\xi) = e^{a|\xi|^{1/\sigma}} |\hat{v}(\xi)|\), we obtain
\[
\|uv\|_{Z_{a,\sigma}^0} \leq \int \left( \int e^{a|\xi-n|^{1/\sigma}} |\hat{u}(\xi-\eta)| e^{a|\eta|^{1/\sigma}} |\hat{v}(\eta)| d\eta \right) d\xi
\leq \|F \ast G\|_{L^1} \leq \|F\|_{L^1} \|G\|_{L^1} \leq \|u\|_{Z_{a,\sigma}^0} \|v\|_{Z_{a,\sigma}^0}.
\]
Then, from Cauchy Schwartz inequality we obtain
\[
\|uv\|_{Z_{a,\sigma}^0} \leq \|u\|_{Z_{a,\sigma}^{-1}} \|v\|_{Z_{a,\sigma}^1} \|v\|_{Z_{a,\sigma}^1}.
\]

**Lemma 2.2.** Let \(u, v \in L^\infty_T(Z_{a,\sigma}^{-1}(\mathbb{R}^3)) \cap L^1_T(Z_{a,\sigma}^1(\mathbb{R}^3))\). Then
\[
\left\| \int_0^t e^{\nu(t-\tau)} \Delta \mathbb{P} (\text{div}(u \otimes v)) d\tau \right\|_{Z_{a,\sigma}^0} \leq \nu \left( \left\| \int_0^t e^{\nu(t-\tau)} \Delta \mathbb{P} (\text{div}(u \otimes v)) \right\|_{Z_{a,\sigma}^{-1}} d\tau \right)
\leq \nu \left( \int_0^t \left\| e^{\nu(t-\tau)} \Delta \text{div}(u \otimes v) \right\|_{Z_{a,\sigma}^{-1}} d\tau \right)
\leq \nu \int_0^t \left\| e^{\nu(t-\tau)} \Delta (u \otimes v) \right\|_{Z_{a,\sigma}^0} d\tau \leq \nu \int_0^t \left\| u \otimes v \right\|_{Z_{a,\sigma}^0} d\tau.
\]

**Proof.** We have
\[
\left\| \int_0^t e^{\nu(t-\tau)} \Delta \mathbb{P} (\text{div}(u \otimes v)) d\tau \right\|_{Z_{a,\sigma}^0} \leq \nu \left( \left\| \int_0^t e^{\nu(t-\tau)} \Delta \mathbb{P} (\text{div}(u \otimes v)) \right\|_{Z_{a,\sigma}^{-1}} d\tau \right)
\leq \nu \left( \int_0^t \left\| e^{\nu(t-\tau)} \Delta \text{div}(u \otimes v) \right\|_{Z_{a,\sigma}^{-1}} d\tau \right)
\leq \nu \int_0^t \left\| e^{\nu(t-\tau)} \Delta (u \otimes v) \right\|_{Z_{a,\sigma}^0} d\tau \leq \nu \int_0^t \left\| u \otimes v \right\|_{Z_{a,\sigma}^0} d\tau.
\]
Using the Lemma 2.1, we obtain
\[
\left\| \int_0^t e^{\nu(t-\tau)} \Delta \mathbb{P} (\text{div}(u \otimes v)) d\tau \right\|_{Z_{a,\sigma}^0} \leq \nu \left( \left\| \int_0^t e^{\nu(t-\tau)} \Delta \mathbb{P} (\text{div}(u \otimes v)) \right\|_{Z_{a,\sigma}^{-1}} d\tau \right)
\leq \nu \left( \int_0^t \left\| e^{\nu(t-\tau)} \Delta \text{div}(u \otimes v) \right\|_{Z_{a,\sigma}^{-1}} d\tau \right)
\leq \nu \int_0^t \left\| e^{\nu(t-\tau)} \Delta (u \otimes v) \right\|_{Z_{a,\sigma}^0} d\tau \leq \nu \int_0^t \left\| u \otimes v \right\|_{Z_{a,\sigma}^0} d\tau.
\]

**Lemma 2.3.** Let \(u, v \in L^\infty_T(Z_{a,\sigma}^{-1}(\mathbb{R}^3)) \cap L^1_T(Z_{a,\sigma}^1(\mathbb{R}^3))\). Then
\[
\int_0^T \left\| e^{\nu(t-\tau)} \Delta \mathbb{P} (\text{div}(u \otimes v)) \right\|_{Z_{a,\sigma}^0} d\tau \leq \nu^{-1} \left\| e^{\nu(t-\tau)} \right\|_{L^\infty_T(Z_{a,\sigma}^{-1})} \left\| e^{\nu(t-\tau)} \right\|_{L^1_T(Z_{a,\sigma}^1)} \left\| e^{\nu(t-\tau)} \right\|_{L^1_T(Z_{a,\sigma}^1)}.
\]
Then, from Lemma 2.1, we have

Proof. We have

\[
\int_0^T \int_0^t e^{\nu(t-\tau)} \Delta \mathbb{P}(\text{div}(u \otimes v)) \, d\tau \, dt \leq \int_0^T \int_0^t e^{\nu(t-\tau)} \Delta \mathbb{P}(\text{div}(u \otimes v)) \, d\tau \, dt
\]

\[
\leq \int_0^T \int_0^t e^{\nu(t-\tau)} \mathbb{P}(\text{div}(u \otimes v)) \, d\tau \, dt
\]

\[
\leq \int_0^T \int_0^t e^{-\nu(t-\tau)} |\xi|^2 |\xi|^{1/\sigma} |u \otimes v(\tau, \xi)| \, d\tau \, d\xi
\]

\[
\leq \int_0^T \int_0^t e^{-\nu(T-\tau)} |\xi|^2 |\xi|^{1/\sigma} \left( \int_0^t e^{-\nu(t-\tau)} |u \otimes v(\tau, \xi)| \, d\tau \right) \, d\xi.
\]

Integrating the function \(e^{-\nu(t-\tau)} |\xi|^2\) twice with respect to \(\tau \in [0, t]\) and \(t \in [0, T]\), we get

\[
\int_0^T \int_0^t e^{-\nu(t-\tau)} |\xi|^2 |u \otimes v(\tau, \xi)| \, d\tau \, dt = \int_0^T |u \otimes v(\tau, \xi)| \left( \frac{1 - e^{-\nu(T-\tau)} |\xi|^2}{\nu |\xi|^2} \right) \, d\tau
\]

Then, from Lemma 2.1 we have

\[
\int_0^T \int_0^t e^{\nu(t-\tau)} \mathbb{P}(\text{div}(u \otimes v)) \, d\tau \, dt \leq \int_0^T \int_0^t \frac{e^{-\nu(t-\tau)} |\xi|^2}{\nu |\xi|^2} |u \otimes v(\tau, \xi)| \, d\tau \, d\xi
\]

\[
\leq \nu^{-1} \int_0^T \int_0^t \left\| u \otimes v \right\| Z_{a,\sigma}^1 \, d\tau \leq \nu^{-1} \int_0^T \int_0^t \left\| u \right\| Z_{a,\sigma}^0 \left\| v \right\| Z_{a,\sigma}^0 \left\| u \right\| Z_{a,\sigma}^1 \left\| v \right\| Z_{a,\sigma}^1 \left\| u \right\| Z_{a,\sigma}^1 \left\| v \right\| Z_{a,\sigma}^1
\]

\[
\leq \nu^{-1} \left\| u \right\| _{L_T^\infty(Z_{a,\sigma}^{-1}(\mathbb{R}^3))} \left\| v \right\| _{L_T^\infty(Z_{a,\sigma}^{-1}(\mathbb{R}^3))} \left\| u \right\| _{L_T^1(Z_{a,\sigma}^1(\mathbb{R}^3))} \left\| v \right\| _{L_T^1(Z_{a,\sigma}^1(\mathbb{R}^3))}
\]

□

Lemma 2.4. Let \(u \in L_T^\infty(Z_{a,\sigma}^{-1}(\mathbb{R}^3)) \cap L_T^1(Z_{a,\sigma}^1(\mathbb{R}^3))\). Then

\[
\left\| u \otimes u \right\| Z_{a,\sigma}^2 \leq 2c_{a,\sigma} \left\| u \right\| Z_{a,\sigma}^{-1} \left\| u \right\| _{Z_{a,\sigma}^{-1}} \left\| u \right\| _{Z_{a,\sigma}^1}.
\]

Proof. We have

\[
\left\| u \otimes u \right\| Z_{a,\sigma}^2 = \int_{\mathbb{R}^3} e^{\alpha |\xi|^{1/\sigma}} |u \otimes u(\xi)| \, d\xi = \int_{\mathbb{R}^3} e^{\alpha |\xi|^{1/\sigma}} \left( \int_{\mathbb{R}^3} |\hat{u}(\xi - \eta)| \left\| \hat{u}(\eta) \right\| d\eta \right) \, d\xi
\]

\[
= \int_{\mathbb{R}^3} e^{\alpha |\xi|^{1/\sigma}} \left( \int_{|\eta| < |\xi - \eta|} |\hat{u}(\xi - \eta)| \left\| \hat{u}(\eta) \right\| d\eta + \int_{|\eta| > |\xi - \eta|} |\hat{u}(\xi - \eta)| \left\| \hat{u}(\eta) \right\| d\eta \right) \, d\xi.
\]

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Using the inequality $e^{a|\xi|^{1/\sigma}} \leq e^{a \max(|\xi-\eta|,|\eta|)^{1/\sigma}} e^{a \min(|\xi-\eta|,|\eta|)^{1/\sigma}}$ and taking $F_1 = e^{a|\xi|^{1/\sigma}} |\hat{u}(\xi)|$, $F_2 = e^{a|\xi|^{1/\sigma}} |\hat{v}(\xi)|$, we get

$$\|u \otimes u\|_{Z_{a,\sigma}^{-1}} \leq \|F_1 \ast F_2\|_{L^1} + \|F_1 \ast F_2\|_{L^1} \leq 2\|F_1\|_{L^1}\|F_2\|_{L^1} \leq 2\|u\|_{Z_{a,\sigma}^{-1}} \|u\|_{Z_{a,\sigma}^{-1}}.$$  

Hence, from Cauchy Schwartz we have

$$\|u\|_{Z_{a,\sigma}^{-1}} \leq \|u\|_{Z_{a,\sigma}^{-1}}^{1/2} \|u\|_{Z_{a,\sigma}^{-1}}^{1/2}, \quad (2.1)$$

and

$$\|u\|_{Z_{a,\sigma}^{-1}} = \int_{\mathbb{R}^3} e^{a|\xi|^{1/\sigma}} |\hat{u}(\xi)| d\xi \leq \int_{\mathbb{R}^3} |\xi| e^{a(\frac{\xi}{\sigma} - \frac{(\xi-\eta)}{\sigma})} |\xi| e^{a|\xi|^{1/\sigma}} |\hat{u}(\xi)| d\xi \leq c_{a,\sigma} \int_{\mathbb{R}^3} |\hat{u}(\xi)| d\xi \leq c_{a,\sigma} \|u\|_{Z_{a,\sigma}^{-1}}, \quad (2.2)$$

where $c_{a,\sigma} = \sup_{x \geq 0} e^{a(\frac{x}{\sigma} - \frac{x}{\sigma})} x^{-1/\sigma}$, $a > 0$, $\sigma > 1$. Using the inequalities $(2.1)$ and $(2.2)$, we obtain the desired results. 

\[\square]\]

3. Well-posedness of $\text{(NSE)}$ in $Z_{a,\sigma}^{-1}(\mathbb{R}^3)$

In the following Theorem, we study the existence and uniqueness of the solution.

**Theorem 3.1.** Let $u^0 \in Z_{a,\sigma}^{-1}$. Then, there are a time $T > 0$ and a unique solution $u \in \mathcal{C}([0,T], Z_{a,\sigma}^{-1}(\mathbb{R}^3))$ of $\text{(NSE)}$ such that $u \in L^1([0,T], Z_{a,\sigma}^{-1}(\mathbb{R}^3))$.

**Proof.** Firstly, let us prove the existence. The idea of the proof is to write the initial condition as a sum of higher and lower frequencies. For small frequencies, we will give a regular solution of the associated linear system to $\text{(NSE)}$. For the higher frequencies, we consider a partial differential equation that is very small to $\text{(NSE)}$ and with small initial data in $Z_{a,\sigma}^{-1}(\mathbb{R}^3)$ for which we can solve it by the Fixed Point Theorem.

Let $r_* = \frac{1}{1+r^*}$, $r = \inf \left\{1, \frac{r_1}{10}, \frac{r_2}{10\|\hat{u}\|_{Z_{a,\sigma}^{-1}}} \right\}$, and $N \in \mathbb{N}$ be such that

$$\int_{|\xi| < N} \frac{e^{a|\xi|^{1/\sigma}}}{|\xi|} |\hat{u}(\xi)| d\xi < \frac{r}{5}.$$  

Set

$$u^0 = F^{-1}(1_{(|\xi| < N}) \hat{u}(\xi))$$

and

$$w^0 = F^{-1}(1_{(|\xi| > N}) \hat{u}(\xi)).$$

Clearly,

$$\|w^0\|_{Z_{a,\sigma}^{-1}} < \frac{r}{5}. \quad (3.1)$$
Let \( v = e^{\nu t} \Delta v^0 \) be the unique solution to
\[
\begin{cases}
\partial_t v - \nu \Delta v = 0 \\
v(0, x) = v^0(x).
\end{cases}
\]

We have
\[
\|v\|_{Z^{-1}_{a, \sigma}} \leq \|u^0\|_{Z^{-1}_{a, \sigma}} \quad \text{for all } t \geq 0,
\]
and
\[
\|v\|_{L_1^1(Z_{a, \sigma}^1)} = \int_0^T \int_{\mathbb{R}^3} |\xi| e^{|\xi|^{1/\sigma}} |\hat{v}(\xi)| d\xi dt
\]
\[
\leq \int_0^T \int_{\mathbb{R}^3} e^{-\nu t |\xi|^2} |\xi| e^{|\xi|^{1/\sigma}} |\hat{u^0}(\xi)| d\xi dt
\]
\[
\leq \int \left( \int_0^T e^{-\nu t |\xi|^2} dt \right) |\xi| e^{|\xi|^{1/\sigma}} |\hat{u^0}(\xi)| d\xi
\]
\[
\leq \frac{1}{\nu} \int_{\mathbb{R}^3} (1 - e^{-\nu T |\xi|^2}) |\xi|^{-1} e^{|\xi|^{1/\sigma}} |\hat{u^0}(\xi)| d\xi.
\]

Using the Dominated Convergence Theorem, we get
\[
\lim_{T \to 0^+} \|v\|_{L_1^1(Z_{a, \sigma}^1)} = 0. \quad (3.2)
\]
Let \( 0 < \varepsilon < \inf \left\{ r, \frac{r^2}{5\|u^0\|_{Z^{-1}_{a, \sigma}}} , \frac{r^2}{25\|u^0\|_{Z^{-1}_{a, \sigma}}} \right\} \). Using the aforementioned choice of \( r^* \) and \( r \), it’s easy to deduce that
\[
(1 + \nu^{-1})\varepsilon \|u^0\|_{Z^{-1}_{a, \sigma}} < \frac{r}{5}, \quad (3.3)
\]
\[
(1 + \nu^{-1})\varepsilon \frac{1}{2} \|u^0\|_{Z^{-1}_{a, \sigma}} < \frac{1}{5}, \quad (3.4)
\]
\[
(1 + \nu^{-1})r < \frac{1}{5}. \quad (3.5)
\]
We have also
\[
2(1 + \nu^{-1})(\varepsilon^2 + r^2) (\|u^0\|_{Z^{-1}_{a, \sigma}}^2 + r^2) \leq \frac{2}{r^*} \left( 2r^2 (\|u^0\|_{Z^{-1}_{a, \sigma}}^2 + r^2) \right) \leq \frac{1}{2}. \quad (3.6)
\]
By (3.2), there is a time \( T = T(\varepsilon) > 0 \) such that
\[
\|v\|_{L_1^1(Z_{a, \sigma}^1)} < \varepsilon.
\]
Put \( w = u - v \), clearly \( w \) is the solution of the following system
\[
\begin{cases}
\partial_t w - \nu \Delta w + (v + w) \cdot \nabla (v + w) = -\nabla p \\
w(0, x) = w^0(x),
\end{cases}
\]
The integral representation of \( w \) is given by
\[
w = e^{\nu t \Delta} w^0 - \int_0^t e^{\nu (t - \tau) \Delta} P((v + w) \cdot \nabla (v + w)) d\tau.
\]
In order to prove the existence of $w$, we consider the following operator
\[
\psi(w) = e^{\nu t} \Delta w^0 - \int_0^t e^{\nu (t-\tau)} \Delta P((v + w) \cdot \nabla (v + w)) \, d\tau.
\]

Now, we introduce the space
\[
Z_T = C([0, T], Z_{a, \sigma}^{-1}(\mathbb{R}^3)) \cap L^1([0, T], Z_{a, \sigma}^1(\mathbb{R}^3))
\]
equipped with the norm
\[
\|f\|_{Z_T} = \|f\|_{L^\infty_T(Z_{a, \sigma}^{-1})} + \|f\|_{L^1_T(Z_{a, \sigma}^1)}.
\]

Using Lemma 2.2, we can prove $\psi(Z_T) \subset Z_T$. Let $B_r$ the subset of $Z_T$ defined by
\[
B_r = \left\{ u \in Z_T; \|u\|_{L^\infty_T(Z_{a, \sigma}^{-1})} \leq r; \|u\|_{L^1_T(Z_{a, \sigma}^1)} \leq r \right\}.
\]
For $w \in B_r$, we prove that $\psi(w) \subset B_r$. In fact,
\[
\|\psi(w)(t)\|_{Z_{a, \sigma}^{-1}} \leq \sum_{k=0}^{4} I_k,
\]
where
\[
I_0 = \|e^{\nu t} \Delta w^0\|_{Z_{a, \sigma}^{-1}},
\]
\[
I_1 = \left\| \int_0^t e^{\nu (t-\tau)} \Delta P(v \cdot \nabla v) \, d\tau \right\|_{Z_{a, \sigma}^{-1}},
\]
\[
I_2 = \left\| \int_0^t e^{\nu (t-\tau)} \Delta P(v \cdot \nabla w) \, d\tau \right\|_{Z_{a, \sigma}^{-1}},
\]
\[
I_3 = \left\| \int_0^t e^{\nu (t-\tau)} \Delta P(w \cdot \nabla v) \, d\tau \right\|_{Z_{a, \sigma}^{-1}},
\]
\[
I_4 = \left\| \int_0^t e^{\nu (t-\tau)} \Delta P(w \cdot \nabla w) \, d\tau \right\|_{Z_{a, \sigma}^{-1}}.
\]

Using the inequality (3.1), Lemma 2.2 and the fact that $w \in B_r$, we get
\[
I_0 \leq \frac{r}{5},
\]
\[
I_1 \leq \|v\|_{L^\infty_T(Z_{a, \sigma}^{-1})} \|v\|_{L^1_T(Z_{a, \sigma}^1)} \leq \varepsilon \|u^0\|_{Z_{a, \sigma}^{-1}} < \frac{r}{5} \quad \text{(by (3.3))},
\]
\[
I_2, I_3 \leq \|v\|_{L^\infty_T(Z_{a, \sigma}^{-1})} \|v\|_{L^1_T(Z_{a, \sigma}^1)} \|w\|_{\frac{1}{2} L^\infty_T(Z_{a, \sigma}^{-1})} \|w\|_{\frac{1}{2} L^1_T(Z_{a, \sigma}^1)} \leq \varepsilon \|u\|_{Z_{a, \sigma}^{-1}} \frac{1}{2} < \frac{r}{5} \quad \text{(by (3.4))}
\]
and
\[
I_4 \leq \|w\|_{L^\infty_T(Z_{a, \sigma}^{-1})} \|w\|_{L^1_T(Z_{a, \sigma}^1)} \leq r^2 < \frac{r}{5} \quad \text{(by (3.5))}.
\]

Then
\[
\|\psi(w)(t)\|_{Z_{a, \sigma}^{-1}} \leq r. \quad (3.7)
\]

Similarly,
\[
\|\psi(w)(t)\|_{L^1(Z_{a, \sigma}^1)} \leq \sum_{k=0}^{4} J_k,
\]

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where

\[ J_0 = \int_0^T \|e^{\nu\tau}\Delta w_0\|_{Z_{1,\sigma}^1}^2 dt, \]

\[ J_1 = \int_0^T \left\| \int_0^t e^{\nu(t-\tau)}\Delta F(v \cdot \nabla v) d\tau \right\|_{Z_{1,\sigma}^1}^2 dt, \]

\[ J_2 = \int_0^T \left\| \int_0^t e^{\nu(t-\tau)}\Delta F(v \cdot \nabla w) d\tau \right\|_{Z_{1,\sigma}^1}^2 dt, \]

\[ J_3 = \int_0^T \left\| \int_0^t e^{\nu(t-\tau)}\Delta F(w \cdot \nabla v) d\tau \right\|_{Z_{1,\sigma}^1}^2 dt, \]

\[ J_4 = \int_0^T \left\| \int_0^t e^{\nu(t-\tau)}\Delta F(w \cdot \nabla w) d\tau \right\|_{Z_{1,\sigma}^1}^2 dt. \]

Using Lemma 2.3 and the fact that \( w \in B_r \), we get

\[ J_0 \leq \frac{r}{5} \]

\[ J_1 \leq \nu^{-1}\|v\|_{L^\infty_t(Z_{a,1}^1)}\|v\|_{L^1_t(Z_{a,1}^1)} \leq \nu^{-1}\|w\|_{Z_{a,1}^1} < \frac{r}{5} \quad \text{(by (3.3))}, \]

\[ J_2, J_3 \leq \nu^{-1}\|v\|_{L^\infty_t(Z_{a,1}^1)}\|v\|_{L^1_t(Z_{a,1}^1)} \leq \frac{1}{\nu} \|a\|_{Z_{a,1}^1} < \frac{r}{5} \quad \text{(by (3.4))} \]

and

\[ J_4 \leq \nu^{-1}\|w\|_{L^\infty_t(Z_{a,1}^1)}\|w\|_{L^1_t(Z_{a,1}^1)} \leq \frac{r}{5} \quad \text{(by (3.5))}. \]

Then

\[ \|\psi(w)(t)\|_{L^1(Z_{a,1}^1)} \leq r. \tag{3.8} \]

Combining (3.7) and (3.8), we get \( \psi(w) \subset B_r \), and we can deduce

\[ \psi(B_r) \subset B_r. \tag{3.9} \]

Now, we shall prove the following estimate

\[ \|\psi(w_2) - \psi(w_1)\|_{Z_T} \leq \frac{1}{2}\|w_2 - w_1\|_{Z_T}, \quad w_1, w_2 \in B_r. \]

We have

\[ \psi(w_2) - \psi(w_1) = -\int_0^t e^{\nu(t-\tau)}\Delta F((v + w_2) \cdot \nabla (v + w_2) - (v + w_1) \cdot \nabla (v + w_1)) d\tau \]

\[ = -\int_0^t e^{\nu(t-\tau)}\Delta F((v + w_2) \cdot \nabla (w_2 - w_1) + (w_2 - w_1) \cdot \nabla (v + w_1)) d\tau. \]

Then

\[ \|\psi(w_2) - \psi(w_1)\|_{Z_{a,1}^1} \leq K_1 + K_2, \]

with

\[ K_1 = \left\| \int_0^t e^{\nu(t-\tau)}\Delta F((v + w_2) \cdot \nabla (w_2 - w_1)) d\tau \right\|_{Z_{a,1}^1}, \]

\[ K_2 = \left\| \int_0^t e^{\nu(t-\tau)}\Delta F((w_2 - w_1) \cdot \nabla (v + w_1)) d\tau \right\|_{Z_{a,1}^1}. \]
Using Lemma 2.2 we get

\[ K_1 \leq \int_0^t \|v + w_2\|_{Z_{u,\sigma}}^{1/2} \|v + w_2\|_{Z_{u,\sigma}}^{1/2} \|w_2 - w_1\|_{Z_{u,\sigma}}^{1/2} \|w_2 - w_1\|_{Z_{u,\sigma}}^{1/2} \, d\tau \]

\[ \leq (\|v\|_{L^p(Z_{u,\sigma})}^{1/2} + \|w_2\|_{L^p(Z_{u,\sigma})}^{1/2})(\|v\|_{L^p(Z_{u,\sigma})}^{1/2} + \|w_2\|_{L^p(Z_{u,\sigma})}^{1/2}) \|w_2 - w_1\|_{Z_T} \]

\[ \leq (\|u^0\|_{Z_{u,\sigma}}^{1/2} + r^{1/2})(\varepsilon^{1/2} + r^{1/2}) \|w_2 - w_1\|_{Z_T}. \]

Similarly, we get

\[ K_2 \leq (\|u^0\|_{Z_{u,\sigma}}^{1/2} + r^{1/2})(\varepsilon^{1/2} + r^{1/2}) \|w_2 - w_1\|_{Z_T}. \]

Then

\[ \|\psi(w_2) - \psi(w_1)\|_{L^p(Z_{u,\sigma})} \leq 2(\|u^0\|_{Z_{u,\sigma}}^{1/2} + r^{1/2})(\varepsilon^{1/2} + r^{1/2}) \|w_2 - w_1\|_{Z_T}. \]

Therefore, we have

\[ \|\psi(w_2) - \psi(w_1)\|_{L^1(Z_{u,\sigma})} \leq K_3 + K_4, \]

with

\[ K_3 = \int_0^t \left\| \int_0^t e^{(t-\tau)\Delta} \mathbb{P}((v + w_2) \cdot \nabla (w_2 - w_1)) \, d\tau \right\|_{Z_{u,\sigma}} \, dt, \]

\[ K_4 = \int_0^t \left\| \int_0^t e^{(t-\tau)\Delta} \mathbb{P}((w_2 - w_1) \cdot \nabla (v + w_1)) \, d\tau \right\|_{Z_{u,\sigma}} \, dt. \]

Using Lemma 2.3 we get

\[ K_3 \leq \nu^{-1} \int_0^t \|v + w_2\|_{Z_{u,\sigma}}^{1/2} \|v + w_2\|_{Z_{u,\sigma}}^{1/2} \|w_2 - w_1\|_{Z_{u,\sigma}}^{1/2} \|w_2 - w_1\|_{Z_{u,\sigma}}^{1/2} \, d\tau \]

\[ \leq \nu^{-1}(\|v\|_{L^\infty(Z_{u,\sigma})}^{1/2} + \|w_2\|_{L^\infty(Z_{u,\sigma})}^{1/2})(\|v\|_{L^1(Z_{u,\sigma})}^{1/2} + \|w_2\|_{L^1(Z_{u,\sigma})}^{1/2}) \|w_2 - w_1\|_{Z_T} \]

\[ \leq \nu^{-1}(\|u^0\|_{Z_{u,\sigma}}^{1/2} + r^{1/2})(\varepsilon^{1/2} + r^{1/2}) \|w_2 - w_1\|_{Z_T}. \]

Similarly, we get

\[ K_4 \leq \nu^{-1}(\|u^0\|_{Z_{u,\sigma}}^{1/2} + r^{1/2})(\varepsilon^{1/2} + r^{1/2}) \|w_2 - w_1\|_{Z_T}. \]

Then

\[ \|\psi(w_2) - \psi(w_1)\|_{L^1(Z_{u,\sigma})} \leq 2\nu^{-1}(\|u^0\|_{Z_{u,\sigma}}^{1/2} + r^{1/2})(\varepsilon^{1/2} + r^{1/2}) \|w_2 - w_1\|_{Z_T}. \]

By (3.10) and (3.11), we obtain

\[ \|\psi(w_2) - \psi(w_1)\|_{Z_T} \leq 2(1 + \nu^{-1})(\|u^0\|_{Z_{u,\sigma}}^{1/2} + r^{1/2})(\varepsilon^{1/2} + r^{1/2}) \|w_2 - w_1\|_{Z_T}. \]

Using the inequality (3.6), we get

\[ \|\psi(w_2) - \psi(w_1)\|_{Z_T} \leq \frac{1}{2} \|w_2 - w_1\|_{Z_T}. \]

Combining (3.9) and (3.12) and the Fixed Point Theorem, so there is a unique \( w \in B_r \) such that \( u = v + w \) is the solution of \( \text{NSE} \) with \( u \in Z_T(\mathbb{R}^3) \).

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Now, we have to prove the uniqueness. Let \( u_1, u_2 \in C([0, T], Z_{a,\sigma}^{-1}(\mathbb{R}^3)) \cap L^1([0, T], Z_{a,\sigma}^1(\mathbb{R}^3)) \) of NSE provided that \( u_1(0) = u_2(0) \). Put \( \delta = U_1 - U_2 \). We have
\[
\partial_t \delta - \nu \Delta \delta + u_1 \cdot \nabla \delta + \delta \cdot \nabla u_2 = -\nabla (p_1 - p_2). \tag{3.13}
\]
Then
\[
\partial_t \hat{\delta} + \nu |\xi|^2 \hat{\delta} + (u_1 \cdot \nabla \hat{\delta}) + (\hat{\delta} \cdot \nabla u_2) = 0.
\]
Multiplying the previous equation by \( \hat{\delta} \), we get
\[
\partial_t \hat{\delta} + \nu |\xi|^2 \hat{\delta} + (u_1 \cdot \nabla \hat{\delta}) + (\hat{\delta} \cdot \nabla u_2) = 0. \tag{3.14}
\]
From Eq. (3.13), we have
\[
\partial_t \hat{\delta} + \nu |\xi|^2 \hat{\delta} + (u_1 \cdot \nabla \hat{\delta}) + (\hat{\delta} \cdot \nabla u_2) = 0.
\]
Multiplying this equation by \( \hat{\delta} \), we get
\[
\partial_t \hat{\delta}^2 + \nu |\xi|^2 \hat{\delta}^2 + (u_1 \cdot \nabla \hat{\delta}) \cdot \hat{\delta} + (\hat{\delta} \cdot \nabla u_2) \cdot \hat{\delta} = 0. \tag{3.15}
\]
By summing (3.14) and (3.15), we get
\[
\partial_t |\hat{\delta}|^2 + 2\nu |\xi|^2 |\hat{\delta}|^2 + 2 \text{Re}((u_1 \cdot \nabla \hat{\delta}) \cdot \hat{\delta}) + 2 \text{Re}(\hat{\delta} \cdot \nabla u_2) = 0,
\]
and
\[
\partial_t |\hat{\delta}|^2 + 2\nu |\xi|^2 |\hat{\delta}|^2 \leq 2((u_1 \cdot \nabla \hat{\delta}) |\hat{\delta}| + 2|\hat{\delta} \cdot \nabla u_2||\hat{\delta}|.
\]
Let \( \varepsilon > 0 \), thereby we have
\[
\partial_t |\hat{\delta}|^2 = \partial_t (|\hat{\delta}|^2 + \varepsilon^2) = 2\sqrt{|\hat{\delta}|^2 + \varepsilon^2} \cdot \partial_t \sqrt{|\hat{\delta}|^2 + \varepsilon^2}
\]
then
\[
2\partial_t \sqrt{|\hat{\delta}|^2 + \varepsilon^2} + 2\nu |\xi|^2 \sum_{t=0}^{t} \frac{|\hat{\delta}|^2}{|\hat{\delta}|^2 + \varepsilon^2} \leq 2((u_1 \cdot \nabla \hat{\delta}) |\hat{\delta}| + 2|\hat{\delta} \cdot \nabla u_2||\hat{\delta}|}
\]
By integrating with respect to time
\[
\sqrt{|\hat{\delta}|^2 + \varepsilon^2} + \nu \int_{0}^{t} \left| \xi \right|^2 |\hat{\delta}| |d\tau| \leq \int_{0}^{t} |(u_1 \cdot \nabla \hat{\delta})| d\tau + \int_{0}^{t} |\hat{\delta} \cdot \nabla u_2| d\tau.
\]
Letting \( \varepsilon \to 0 \), we get
\[
|\hat{\delta}| + \nu \int_{0}^{t} \left| \xi \right|^2 |\hat{\delta}| d\tau \leq \int_{0}^{t} |(u_1 \cdot \nabla \hat{\delta})| d\tau + \int_{0}^{t} |\hat{\delta} \cdot \nabla u_2| d\tau.
\]
Multiplying by \( \frac{e^{\xi|\xi|^\frac{1}{2}}}{|\xi|} \) and integrating with respect to \( \xi \), we get
\[
\|\hat{\delta}\|_{Z_{a,\sigma}^{-1}} + \nu \int_{0}^{t} \|\Delta \hat{\delta}\|_{Z_{a,\sigma}^{-1}} d\tau \leq \int_{0}^{t} \|u_1 \cdot \nabla \hat{\delta}\|_{Z_{a,\sigma}^{-1}} d\tau + \int_{0}^{t} \|\hat{\delta} \cdot \nabla u_2\|_{Z_{a,\sigma}^{-1}} d\tau
\]
\[
\leq \int_{0}^{t} \|\delta u_1\|_{Z_{a,\sigma}^{-1}} d\tau + \int_{0}^{t} \|u_2\|_{Z_{a,\sigma}^{-1}} d\tau.
\]
Using the elementary inequality $xy \leq \frac{x^2}{2} + \frac{y^2}{2}$, we get
\[
\|\delta u_1\|_{Z^{-1}_{a,\sigma}} \leq \|\delta\|_{Z^{-1}_{a,\sigma}} \|u_1\|_{Z^{-1}_{a,\sigma}} \\
\leq \|\delta\|_{Z^{-1}_{a,\sigma}} \|\Delta \delta\|_{Z^{-1}_{a,\sigma}} \|u_1\|_{Z^{-1}_{a,\sigma}} \|\Delta u_1\|_{Z^{-1}_{a,\sigma}} \\
\leq \frac{2}{\nu} \|\delta\|_{Z^{-1}_{a,\sigma}} \|u_1\|_{Z^{-1}_{a,\sigma}} \|\Delta u_1\|_{Z^{-1}_{a,\sigma}} + \frac{\nu}{2} \|\Delta \delta\|_{Z^{-1}_{a,\sigma}}.
\]
Similarly,
\[
\|u_2\|_{Z^{-1}_{a,\sigma}} \leq \frac{2}{\nu} \|\delta\|_{Z^{-1}_{a,\sigma}} \|u_2\|_{Z^{-1}_{a,\sigma}} \|\Delta u_2\|_{Z^{-1}_{a,\sigma}} + \frac{\nu}{2} \|\Delta \delta\|_{Z^{-1}_{a,\sigma}}.
\]
Then
\[
\|\delta\|_{Z^{-1}_{a,\sigma}} \leq \frac{2}{\nu} \int_0^t \|\delta\|_{Z^{-1}_{a,\sigma}} \|u_1\|_{Z^{-1}_{a,\sigma}} \|\Delta u_1\|_{Z^{-1}_{a,\sigma}} d\tau + \frac{2}{\nu} \int_0^t \|\delta\|_{Z^{-1}_{a,\sigma}} \|u_2\|_{Z^{-1}_{a,\sigma}} \|\Delta u_2\|_{Z^{-1}_{a,\sigma}} d\tau.
\]
Using Gronwall Lemma and the fact $(t \mapsto \|u_1\|_{Z^{-1}_{a,\sigma}} \|\Delta u_1\|_{Z^{-1}_{a,\sigma}}) \in L^1([0,T])$, $(t \mapsto \|u_2\|_{Z^{-1}_{a,\sigma}} \|\Delta u_2\|_{Z^{-1}_{a,\sigma}}) \in L^1([0,T])$, we can deduce that $\delta = 0$ in $[0,T]$ which gives the uniqueness.

In the following, we prove a global existence if the initial condition is small in the Lei-Lin-Gevrey spaces.

**Theorem 3.2.** Let $u^0 \in Z^{-1}_{a,\sigma}(\mathbb{R}^3)$ such that $\|u^0\|_{Z^{-1}_{a,\sigma}} < \nu$. Then, there exists a unique global solution $u \in C(\mathbb{R}^+, Z^{-1}_{a,\sigma}(\mathbb{R}^3)) \cap L^1(\mathbb{R}^+, Z^1_{a,\sigma}(\mathbb{R}^3))$ of (NSE) such that
\[
\|u(t)\|_{Z^{-1}_{a,\sigma}} + \left(\frac{\nu - \|u^0\|_{Z^{-1}_{a,\sigma}}}{2}\right) \int_0^t \|\Delta u\|_{Z^{-1}_{a,\sigma}} d\tau \leq \|u^0\|_{Z^{-1}_{a,\sigma}}.
\]

**Proof.** From Theorem 3.1 if $u^0 \in Z^{-1}_{a,\sigma}(\mathbb{R}^3)$, we have a local existence
\[
u \in L^\infty_T(Z^{-1}_{a,\sigma}(\mathbb{R}^3)) \cap L^1_T(Z^1_{a,\sigma}(\mathbb{R}^3)).
\]
Assume that $\|u^0\|_{Z^{-1}_{a,\sigma}} < \nu$ and $u \in C([0,T^*), Z^{-1}_{a,\sigma}(\mathbb{R}^3)) \cap L^1_{loc}([0,T^*), Z^1_{a,\sigma}(\mathbb{R}^3))$ is the maximal solution of (NSE). We have
\[
\frac{\partial}{\partial t}\|u(t)\|_{Z^{-1}_{a,\sigma}} + \nu \|\Delta u\|_{Z^{-1}_{a,\sigma}} \leq \|\text{div}(u \otimes u)\|_{Z^{-1}_{a,\sigma}}.
\]
Integrating over $(0,t)$, we get
\[
\|u(t)\|_{Z^{-1}_{a,\sigma}} + \nu \int_0^t \|\Delta u\|_{Z^{-1}_{a,\sigma}} d\tau \leq \|u^0\|_{Z^{-1}_{a,\sigma}} + \int_0^t \|u \otimes u\|_{Z^1_{a,\sigma}} d\tau
\]
\[
\leq \|u^0\|_{Z^{-1}_{a,\sigma}} + \int_0^t \|u\|_{Z^{-1}_{a,\sigma}} \|\Delta u\|_{Z^{-1}_{a,\sigma}} d\tau.
\]
Therefore, for $T_\alpha = \sup\{t \in [0,T^*) / \|u(t)\|_{Z^{-1}_{a,\sigma}} < \alpha\}$, where $\alpha = \frac{\nu + \|u^0\|_{Z^{-1}_{a,\sigma}}}{2}$.
Take $t \in [0,T_\alpha)$. Then we have
\[
\|u(t)\|_{Z^{-1}_{a,\sigma}} + \nu \int_0^t \|\Delta u\|_{Z^{-1}_{a,\sigma}} d\tau \leq \|u^0\|_{Z^{-1}_{a,\sigma}} + \alpha \int_0^t \|\Delta u\|_{Z^{-1}_{a,\sigma}} d\tau.
\]
This implies
\[ \|u(t)\|_{Z_{a,\sigma}^{-1}} + (\nu - \alpha) \int_0^t \|\Delta u\|_{Z_{a,\sigma}^{-1}} \, d\tau \leq \|u^0\|_{Z_{a,\sigma}^{-1}} < \alpha. \]

Then \( T_* = T^* \). Particularly if \( T < T^* \), we have
\[ \|u(T)\|_{Z_{a,\sigma}^{-1}} + (\nu - \alpha) \int_0^T \|\Delta u\|_{Z_{a,\sigma}^{-1}} \, d\tau \leq \|u^0\|_{Z_{a,\sigma}^{-1}}. \]

Therefore, \( T^* = \infty \).

\[ \square \]

4. Global solution

In this section, we prove the first main Theorem \[1.3\]

Let \( u \in C([0, T_{a,\sigma}^*), Z_{a,\sigma}^{-1}(\mathbb{R}^3)) \cap L^2_{\text{loc}}([0, T_{a,\sigma}^*), Z_{a,\sigma}^{1}(\mathbb{R}^3)) \) be the maximal solution of \( \text{(NSE)} \), such that \( \|u^0\|_{\mathcal{X}^{-1}} < \nu \).

Therefore, we have
\[ \|u(t)\|_{Z_{a,\sigma}^{-1}} + \nu \int_0^t \|\Delta u\|_{Z_{a,\sigma}^{-1}} \, d\tau \leq \|u^0\|_{Z_{a,\sigma}^{-1}} + \int_0^t \|\text{div}(u)\|_{Z_{a,\sigma}^{-1}} \, d\tau \]
\[ \leq \|u^0\|_{Z_{a,\sigma}^{-1}} + \int_0^t \|u\|_{Z_{a,\sigma}^{2}} \, d\tau. \]

Using the Lemma \[2.4\] and the inequality \( xy \leq \frac{x^2}{2} + \frac{y^2}{2} \), thus we get
\[ \|u(t)\|_{Z_{a,\sigma}^{-1}} + \nu \int_0^t \|\Delta u\|_{Z_{a,\sigma}^{-1}} \, d\tau \leq \|u^0\|_{Z_{a,\sigma}^{-1}} + 2c_{a,\sigma} \int_0^t \|u\|_{Z_{a,\sigma}^{-1}} \|u\|_{Z_{a,\sigma}^{2}} \|\Delta u\|_{Z_{a,\sigma}^{-1}} \, d\tau \]
\[ \leq \|u^0\|_{Z_{a,\sigma}^{-1}} + \frac{4c^2_{a,\sigma}}{\nu} \int_0^t \|u\|_{Z_{a,\sigma}^{-1}} \|\Delta u\|_{Z_{a,\sigma}^{-1}} \, d\tau + \nu \int_0^t \|\Delta u\|_{Z_{a,\sigma}^{-1}} \, d\tau. \]

This implies that
\[ \|u(t)\|_{Z_{a,\sigma}^{-1}} + \frac{\nu}{2} \int_0^t \|\Delta u\|_{Z_{a,\sigma}^{-1}} \, d\tau \leq \|u^0\|_{Z_{a,\sigma}^{-1}} + \frac{4c^2_{a,\sigma}}{\nu} \int_0^t \|u\|_{Z_{a,\sigma}^{-1}} \, d\tau. \]

By the Gronwall Lemma, we get
\[ \|u(t)\|_{Z_{a,\sigma}^{-1}} \leq \|u^0\|_{Z_{a,\sigma}^{-1}} \exp \left( \frac{4c^2_{a,\sigma}}{\nu} \int_0^t \|u\|_{Z_{a,\sigma}^{-1}} \, d\tau \right). \quad (4.1) \]

As \( Z_{a,\sigma}^{-1}(\mathbb{R}^3) \hookrightarrow Z_{a,\sigma}^{1}(\mathbb{R}^3) \). Then \( T_{a,\sigma}^* = T_{\frac{a,\sigma}{\nu}}^* \). Thus
\[ T_{a,\sigma}^* = T_{\frac{a,\sigma}{\nu}}^* = \cdots = T_{\frac{a,\sigma}{\nu^n}}^* \quad \text{for all } n \in \mathbb{N}. \]
Therefore, from the Dominated Convergence Theorem
\[
\lim_{n \to \infty} \|u^0\|_{Z^{-1}} < \nu.
\]
Then, there exists \(n_0 \in \mathbb{N}\) such that
\[
\|u^0\|_{Z^{-1}} < \nu \quad \text{for all } n \geq n_0.
\]
Applying Theorem 3.1 so we have for all \(n \geq n_0\)
\[
\|u(t)\|_{Z^{-1}} + \left(\frac{\nu - \|u^0\|_{Z^{-1}}}{2}\right) \int_0^t \|\Delta u\|_{Z^{-1}} \, d\tau \leq \|u^0\|_{Z^{-1}}.
\] (4.2)

Now, if \(T_{a,\sigma}^* < \infty\). We prove \(T_{a,\sigma}^* \int_0^T \|\Delta u\|_{Z_{a,\sigma}} \, d\tau = \infty\). Suppose that \(T_{a,\sigma}^* \int_0^T \|\Delta u\|_{Z_{a,\sigma}} \, d\tau < \infty\). Let \(\varepsilon > 0\), there is \(0 < T < T_{a,\sigma}^*\) such that
\[
T_{a,\sigma}^* \int_0^T \|\Delta u\|_{Z_{a,\sigma}} \, d\tau < \varepsilon.
\]
For \(T < t < T_{a,\sigma}^*\), we have
\[
\|u(t)\|_{Z_{a,\sigma}} + \nu \int_T^t \|\Delta u\|_{Z_{a,\sigma}} \, d\tau \leq \|u(T)\|_{Z_{a,\sigma}} + \int_T^t \|u\|_{Z_{a,\sigma}} \, d\tau
\]
\[
\leq \|u(T)\|_{Z_{a,\sigma}} + \int_T^t \|\Delta u\|_{Z_{a,\sigma}} \, d\tau.
\]
Let’s take \(M(t) = \sup_{T \leq z \leq t} \|u(z)\|_{Z_{a,\sigma}}\), then
\[
M(t) \leq \|u(T)\|_{Z_{a,\sigma}} + \int_T^{T_{a,\sigma}^*} \|u\|_{Z_{a,\sigma}} \|\Delta u\|_{Z_{a,\sigma}}
\]
\[
\leq \|u(T)\|_{Z_{a,\sigma}} + \varepsilon \int_T^{T_{a,\sigma}^*} \|u\|_{Z_{a,\sigma}}
\]
\[
\leq \|u(T)\|_{Z_{a,\sigma}} + \varepsilon \|u(T)\|_{Z_{a,\sigma}} e^{4c_2\nu}, \quad \text{(by (4.1) - (4.2))}
\]
\[
\leq \|u(T)\|_{Z_{a,\sigma}} (1 + \varepsilon e^{4c_2\nu}) = M_T.
\]
We get \(M(t) \leq M_T\), which is absurd.
So, \(T_{a,\sigma}^* = \infty\).

5. Long time decay for the global solution

In this section, we prove the second main Theorem 1.5.
Let \(u \in C(\mathbb{R}^+, Z_{a,\sigma}^{-1}(\mathbb{R}^3))\). As \(Z_{a,\sigma}^{-1}(\mathbb{R}^3) \hookrightarrow X^{-1}(\mathbb{R}^3)\). Then \(u \in C(\mathbb{R}^+, X^{-1}(\mathbb{R}^3))\).
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For the results of Hantaek Bae (see [7]) and J. Benamou (see [1]). There exist $t_0 > 0$ and $\alpha > 0$ such that
\[ \|e^{\alpha|D|}u(t)\|_{X^{-1}(\mathbb{R}^3)} \leq c_0 \quad \text{for all } t \geq t_0, \tag{5.1} \]
where $\alpha = \varphi(t) = \sqrt{t - t_0}$.

Therefore, let $a > 0$ and $\beta > 0$. Then, there exists $c_1 > 0$ such that
\[ ax^{\frac{1}{2}} \leq c_1 + \beta x, \quad x \geq 0. \tag{5.2} \]

Taking $\beta = \frac{\alpha}{2}$ and using the inequalities (5.1)–(5.2) and the Cauchy-Schwartz inequality, so we obtain
\[ \|u(t)\|_{Z^{-1,a,\sigma}} \leq c_0 e^{c_1 \|u\|^\frac{1}{2}} \|u\|^{\frac{1}{2}}_{X^{-1}} \]
\[ \leq c_0 e^{c_1 \|u\|^\frac{1}{2}} \|u\|^\frac{1}{2} \]
\[ \leq c_0^2 e^{c_1} \|u\|^\frac{1}{2} \]
\[ \leq c_0^2 e^{c_1} \|u\|^\frac{1}{2}. \]

Using Theorem 1.3. So, $\lim_{t \rightarrow \infty} \|u(t)\|_{Z^{-1,a,\sigma}} = 0.$

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