Emergence of cosmic space and horizon entropy maximization from Tsallis and Cirto entropy

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Abstract In this note by use of the idea of the emergence of cosmic space suggested by Padmanabhan (Emergence and expansion of cosmic space as due to the quest for holographic equipartition; Res Astro Astrophys 12:891, 2012), we derive the modified Friedmann equation of a Friedmann-Robertson-Walker (FRW) Universe from Tsallis and Cirto entropy in \((n + 1)\) dimension with any spatial curvature. We investigate further the consistency of the law of emergence with the maximization of horizon entropy in the context of Tsallis and Cirto horizon entropy. Our results reveal the deep connection between the law of emergence and horizon thermodynamics.

1 Introduction

Ever since the discovery of thermodynamic properties of black holes in the 1970s [1–4], the deep connection between gravity and thermodynamics has drawn more and more attention. In the year of 1995, Jacobson first derived Einstein’s field equations by employing the Clausius relation \(\delta Q = T dS\) on a local Rindler causal horizon, where \(\delta Q\) and \(T\) are energy flux and Unruh temperature respectively, observed by an accelerating observer inside the horizon [5]. In 2011, Verlinde proposed a viewpoint about gravity that gravity should be interpreted as an entropic force caused by changes in the information associated with the positions of material bodies [6], which has got a lot of attention [7–14]. These studies support the point of view that gravitational field equations should have the same status as the equations of emergent phenomena like fluid mechanics or elasticity.

However, in the most work on the emergent paradigm, the gravitational field equations are treated as an emergent phenomenon leaving the space-time background as pre-existing. In 2012, Padmanabhan found that in the cosmological context, when one chooses the time variable as the proper time of the geodesic observer to whom CMBR (cosmic microwave background radiation) appears homogeneous and isotropic, the spatial expansion of our universe can be regarded as the consequence of emergence space with the progress of cosmic time, that is, cosmic space is emergent as cosmic time progresses [15,16]. He proposed that in an infinitesimal interval \(dt\) of cosmic time, the increase \(dV\) of the cosmic volume is given by

\[
\frac{dV}{dt} = L_p^2 (N_{\text{surf}} - N_{\text{bulk}}),
\]

where \(N_{\text{surf}}\) denotes the number of degrees of freedom on the holographic surface of Hubble radius \(H^{-1}\), and \(N_{\text{bulk}}\) is the number of degrees of freedom in the bulk. Based on this assumption, the Friedmann equation of a flat FRW Universe in general relativity is successfully derived. By properly modifying the volume increase and the number of the surface degrees of freedom on the holographic surface, Cai obtained the Friedmann equations of a \((n + 1)\)-dimensional FRW Universe in Einstein, Gauss-Bonnet and Lovelock gravities for a spatially flat universe [17]. An interesting improvement on Cai’s proposal, which works in a non-flat universe, was suggested by Sheykhi [18], and one can obtain the corresponding Friedmann equations of a FRW Universe with any spatial curvature not only in Einstein gravity, but also in Gauss-Bonnet and more general Lovelock gravities. Then Sheykhi successfully applied his proposal to derive the modified Friedmann equations in different horizon entropy scenarios such as Tsallis and Cirto entropy [19], and Barrow entropy [20]. Another generalization to Padmanabhan’s proposal was suggested in [21,22]. Instead of modifying the degrees of freedom, the change in Hubble volume is assumed to be proportional to a more general function \(f(\Delta N, N_{\text{surf}})\), where \(\Delta N = N_{\text{surf}} - N_{\text{bulk}}\), and different functions of \(f(\Delta N, N_{\text{surf}})\) could be interpreted as Friedmann equations corresponding to different gravity theories. The Friedmann equations of a \((n + 1)\)-dimensional FRW Universe with any spatial curvature in Einstein, Gauss-Bonnet and Lovelock theories are also obtained. The idea of treating the cosmic
space as an emergent process has been applied to the contexts of horizon entropy scenarios, modified gravities, Bionic systems and brane scenarios, and the corresponding cosmological equations in these theories have been successfully derived [23–37]. For more investigations on Padmanabhan’s idea, see [38–54].

It is well known that an ordinary macroscopic system evolves to an equilibrium state of maximum entropy [55]. Pavon and Radicella recently showed that our universe with a Hubble expansion history behaves as an ordinary macroscopic system that proceeds to a state of maximum entropy [56]. Subsequently, Krishna and Mathew investigated the relationship of the emergence of cosmic space and the maximization of horizon entropy in the context of Einstein, Gauss-Bonnet and Lovelock gravities, and they found that both the law of emergence and the horizon entropy maximization lead to the same constraints, so the emergence of cosmic space can be viewed as a tendency for maximizing the horizon entropy [57–59].

It is important to note that in order to derive the Friedmann equation and discuss the relationship between the emergence of cosmic space and the maximization of horizon entropy, the expression of entropy of the black hole is important. In Ref. [57], Tsallis and Cirto showed that thermodynamical entropy of a gravitational system such as a black hole must be generalized to the non-additive entropy, which can be modified as

\[ S = \gamma A^\beta, \]  

(2)

where \( \gamma \) is an unknown constant, \( A \) is the black hole horizon area, and \( \beta \) is known as Tsallis and Cirto parameter or nonextensive parameter which quantifies the degree of non-extensivity. The area law of entropy is restored for \( \gamma = 1/(4L_p^2) \) and \( \beta = 1 \). Assuming the entropy associated with the apparent horizon is in the form of Tsallis and Cirto entropy, Sheykhi derived the corresponding modified Friedmann equation in \((3+1)\) dimension by employing his modified law of emergence and the first law of thermodynamics at the apparent horizon of a FRW Universe respectively, and both of methods result in the same Friedmann equation [19].

In recent years, ideas in brane-world cosmology, string theory, and gauge/gravity duality have all motivated a study of space-time in more than four dimensions, with surprising results. Brane-world ideas suggest that our familiar three spatial dimensions might just be a surface in a higher-dimensional space. In these theories, non-gravitational forces are confined to the brane but gravity is higher dimensional. String theory, one of the most promising approaches to quantum gravity, predicts that space-time has more than four dimensions. This incorporates older ideas of unification based on the idea that extra dimensions are curled up into a small ball. Furthermore gauge/gravity duality, which has emerged from string theory, relates certain strongly coupled non-gravitational theories to higher-dimensional theories with gravity [61]. Thus considering the emergence of cosmic space in higher space-time dimensions is important. In this note, we will derive the modified Friedmann equation of a FRW Universe from Tsallis and Cirto entropy in the higher \((n + 1)\)-dimensional case with non-zero spatial curvature, and investigate the consistency of the law of emergence with the maximization of Tsallis and Cirto horizon entropy.

The remainder of this paper is organized as follows. In Sect. 2, we derive the modified Friedmann equation of a FRW Universe from Tsallis and Cirto entropy in \((n + 1)\) dimension with non-zero spatial curvature. In Sect. 3, we obtain the constraints for the maximization of Tsallis and Cirto horizon entropy. In Sect. 4, we check the consistency of the law of emergence with the horizon entropy maximization in the context of Tsallis and Cirto entropy. Finally, some discussions and conclusions are given in Sect. 5.

In this paper, we take the natural units of \( k_B = \hbar = c = 1 \) for simplicity.

2 Modified Friedmann equation in \((n + 1)\) dimension from Tsallis and Cirto entropy

In this section, we briefly review the modification of Padmanabhan’s proposal made by Sheykhi [19], which successfully gets the modified Friedmann equation from Tsallis and Cirto entropy in \((3+1)\) dimension with any spacial curvature. We then generalize Sheykhi’s derivation process to obtain the modified Friedmann equation from Tsallis and Cirto entropy in the \((n + 1)\)-dimensional case with \( n > 3 \). To confirm our results, we also derive the modified Friedmann equation by using the first law of thermodynamics at the apparent horizon of a FRW Universe [62].

2.1 Review of Sheykhi’s derivation of the modified Friedmann equation from emergence of cosmic space

The background of space-time is spatially homogeneous and isotropic, which is described by the line element

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu + \bar{r}^2(d\theta^2 + \sin^2 \theta d\phi^2), \]  

(3)

where \( \bar{r} = a(t)r \), \( x^0 = t \), \( x^1 = r \), and \( g_{\mu\nu} = \text{diag}(-1, a^2/(1 - kr^2)) \) represents the two dimensional metric. The spatial curvature constant \( k \) corresponds to a closed, flat, and open universe for \( k = 1, 0, \) and \( -1 \), respectively. The physical boundary of the universe, which is consistent with laws of thermodynamics, is the apparent horizon with radius

\[ \bar{r}_A = \frac{1}{\sqrt{H^2 + k/a^2}}, \]  

(4)
where \( H = \dot{a}/a \) is the Hubble parameter.

The matter and energy content of the universe is assumed as the form of perfect fluid with stress-energy tensor

\[
T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu},
\]

where \( \rho \) and \( p \) are the energy density and pressure respectively. The conservation of the energy-stress tensor in a FRW background, \( \nabla_\mu T^{\mu\nu} = 0 \), leads to the continuity equation

\[
\dot{\rho} + 3H(\rho + p) = 0.
\]

Let us begin with Sheykhi’s generalization of Padmanabhan’s conjecture. The effective area of the holographic surface corresponding to the entropy (2) is defined as

\[
\widetilde{A} = A^\beta = (4\pi \tilde{r}_A^3)^\beta.
\]

The increasing in the effective volume is

\[
\frac{d\tilde{V}}{dt} = \frac{\tilde{r}_A}{2} \frac{d\tilde{A}}{dt} = \beta \frac{(4\pi \tilde{r}_A^3)^\beta \tilde{r}_A}{2\beta - 4} \frac{d}{dt} (\tilde{r}_A^{2\beta - 4}).
\]

Inspired by (8), the number of degrees of freedom on the apparent horizon with Tsallis and Cirto entropy is given by

\[
N_{\text{surf}} = \frac{4\pi \beta}{2 - \beta} (4\pi \tilde{r}_A^3)^\beta.
\]

The temperature associated with the apparent horizon is the Hawking temperature

\[
T = \frac{1}{2\pi \tilde{r}_A}.
\]

The energy contained inside the sphere with volume \( V = 4\pi \tilde{r}_A^3/3 \) is the Komar energy

\[
E_{\text{Komar}} = |(\rho + 3p)|V.
\]

Based on the equipartition law of energy, the number of degrees of freedom in the bulk is given by

\[
N_{\text{bulk}} = \frac{2E_{\text{Komar}}}{\dot{r}_A} = -\frac{16\pi^2}{3} \tilde{r}_A^4 (\rho + 3p).
\]

In the accelerating phase with \( \rho + 3p < 0 \), in order to have \( N_{\text{bulk}} > 0 \), a minus sign is contained in the above equation. The law of emergence from Tsallis and Cirto entropy takes the form

\[
\frac{d\tilde{V}}{dt} = \frac{1}{4\beta} H^{-1} (N_{\text{surf}} - N_{\text{bulk}}).
\]

Substituting relations (8), (9) and (12) in Eq. (13), using the continuity Eq. (6), and multiplying the hand both side of the equation by factor \( \dot{a}/a \), we reach

\[
\frac{d}{dt} (\tilde{r}_A^{2\beta - 4}) = \frac{2\pi(2 - \beta)}{3\beta} (4\beta)^{1-\beta} \frac{d}{dt} (\rho a^2).
\]

Integrating, yields

\[
\left( H^2 + \frac{k}{a^2} \right)^{2-\beta} = \frac{2\pi(2 - \beta)}{3\beta} (4\pi)^{1-\beta} \rho,
\]

where the integration constant is set to zero. This is the modified Friedmann equation from Tsallis and Cirto entropy (2). By defining

\[
\gamma \equiv \frac{2 - \beta}{4\beta} (4\pi)^{1-\beta},
\]

one obtain

\[
\left( H^2 + \frac{k}{a^2} \right)^{2-\beta} = \frac{8\pi L_p^2}{3} \rho.
\]

Sheyki also derived the modified Friedmann equation from the first law of thermodynamics in Ref. [19], which coincides with Eq. (15) from emergence of cosmic space in this section.

2.2 Modified Friedman equation from emergence of cosmic space in \((n + 1)\)-dimensional case

Based on the fact that ideas in brane-world cosmology, string theory, and gauge/gravity duality have all motivated a study of space-time in more than four dimensions, with surprising results, we now investigate the emergence of cosmic space from Tsallis and Cirto entropy in higher space-time dimensions. We generalize Sheyki’s derivation process to obtain the modified Friedmann equation from Tsallis and Cirto entropy in \((n + 1)\)-dimensional case with any spatial curvature. The effective area of the holographic surface is

\[
\widetilde{A} = A^\beta = (n\Omega_n \tilde{r}_A^{n-1})^\beta,
\]

where \( A = n\Omega_n \tilde{r}_A^{n-1} \), and \( \Omega_n \) is the volume of a unit \( n \)-sphere. The effective volume increase satisfies

\[
\frac{d\tilde{V}}{dt} = \frac{\tilde{r}_A}{n - 1} \frac{d\tilde{A}}{dt} = \frac{n\Omega_n^\beta}{(n-1)\beta - n - 1} \tilde{r}_A^{2\beta - 4} \frac{d}{dt} (\tilde{r}_A^{n-1}).
\]

Following the similar reasoning method in Refs. [17–19], we propose from Eq. (19) that the number of degrees of freedom on the holographic surface is given by

\[
N_{\text{surf}} = \frac{\beta(n\Omega_n^\beta)}{(n - 1)\beta - n - 1} \times \tilde{r}_A^{n+1} \times \tilde{r}_A^{(n-1)\beta - n - 1}
\]

\times (-2) \times (4\beta) \times \alpha

\[
= \frac{8\alpha \beta \gamma}{n + 1 - (n - 1)\beta} \tilde{r}_A^{n-1},
\]

where \( \alpha = \frac{n-1}{2(n-2)} \). The bulk Komar energy in the case of \((n + 1)\) dimension is given by [7,63]

\[
E_{\text{Komar}} = \frac{(n - 2)\rho + np}{n - 2} V = \frac{(n - 2)\rho + np}{n - 2} \Omega_n \tilde{r}_A^n.
\]
where the volume $V = \Omega_n \tilde{r}_A^n$. Then the bulk degrees of freedom is obtained as
\begin{equation}
N_{\text{bulk}} = -\frac{2E_{\text{komar}}}{T} = -4\pi \Omega_n \tilde{r}_A^{n+1}(n-2)\rho + np
\frac{n}{n-2}, \tag{22}
\end{equation}
where $T = \frac{1}{2\pi \tilde{r}_A}$ is the Hawking temperature of the apparent horizon, and we have added a minus sign in front of $E_{\text{komar}}$ in order to have $N_{\text{bulk}} > 0$, which makes sense only in the accelerating phase with $(n-2)\rho + np < 0$. In the $(n+1)$-dimensional case, the conservation of the energy-stress tensor of the perfect fluid, $\nabla_\mu T^{\mu\nu} = 0$, leads to the continuity equation as
\[ \dot{\rho} + nH(\rho + p) = 0. \tag{23} \]

For the case of $(n+1)$ dimension, we modify the proposal in Eq. (13) as
\[ \frac{d\tilde{V}}{dt} = \frac{1}{4\gamma \alpha} \tilde{r}_A \left( N_{\text{surf}} - N_{\text{bulk}} \right). \tag{24} \]

This is the law of emergence proposed from Tsallis and Cirto entropy in the $(n+1)$-dimensional case. When $n = 3, \alpha = 1$, and the Eq. (24) is simplified to Eq. (13).

By substituting relations (19), (20) and (22) in Eq. (24), using the continuity Eq. (23), and multiplying the both hand side of the equation by factor $a\dot{a}$, we get
\[ \frac{4\pi \Omega_n}{n-2} \frac{d}{dt} \left[ a^2 \rho \right] = 4\gamma \beta \alpha \frac{(n\Omega_n)^\beta}{n+1-(n-1)\beta} \frac{d}{dt} \left[ a^2 \tilde{r}_A^{(n-1)\beta-(n-1)} \right]. \tag{25} \]

Integrating, yields
\[ \tilde{r}_A^{(n-1)\beta-(n-1)} = \frac{2\pi [n+1-(n-1)\beta][n\Omega_n]^{1-\beta}}{n(n-1)\gamma \beta} \rho. \tag{26} \]

Substituting $\tilde{r}_A$ into the above equation, we obtain
\[ \left( H^2 + \frac{k}{a^2} \right) \frac{1}{2} \left[ \frac{1}{n(n-1)} \right] = \frac{16\pi L_p^{n-1}}{n(n-1)} \rho, \tag{27} \]
where we define
\[ \gamma = \frac{[n+1-(n-1)\beta][n\Omega_n]^{1-\beta}}{8\beta L_p^{n-1}}. \tag{28} \]

This is the modified Friedmann equation of a FRW Universe from the non-additive Tsallis and Cirto entropy in the case of $(n+1)$ dimension with non-zero spatial curvature. From Eq. (28), we must have
\[ n+1-(n-1)\beta > 0, \tag{29} \]
that is,
\[ \beta < \frac{n+1}{n-1}. \tag{30} \]
This is the upper bound on the non-additive parameter of Tsallis and Cirto entropy.

2.3 Modified Friedman equation from the first law of thermodynamics at the apparent horizon of a FRW Universe in $(n+1)$ dimension

In this subsection, to confirm the correctness of the modified Friedman equation from the emergence of cosmic space, we derive the modified Friedman equation from Tsallis and Cirto entropy in $(n+1)$ dimension by employing the first law of thermodynamics, $dE = TdS + WdV$, at the apparent horizon of a FRW Universe [62].

The associated temperature at the apparent horizon can be defined as [64,65]
\[ T = \frac{k}{2\pi} = -\frac{1}{2\pi \tilde{r}_A} \left( 1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A} \right), \tag{31} \]
where $k$ is the surface gravity, and the over-dot denotes the derivative with respect to the cosmic time $t$. The work density is defined as [62]
\[ W = -\frac{1}{2} T^{\mu\nu} h_{\mu\nu}. \tag{32} \]

A simple calculation obtains
\[ W = \frac{1}{2}(\rho - p). \tag{33} \]

The work density is the work done by the volume change of the universe due to the change of the apparent horizon radius. The first law of thermodynamics at the apparent horizon is assumed to be satisfied, and has the form
\[ dE = TdS + WdV. \tag{34} \]

The energy $E = \rho V$ is the total energy of matter inside the apparent horizon, which is actually the direct $(n+1)$-dimensional generalization of the $(3+1)$-dimensional one, given by Misner and Sharp [62,66]. $E$ is different from the Komar energy, $E_{\text{Kamor}}$, used in the emergence of cosmic space. $V = \Omega_n \tilde{r}_A^n$ is the volume enveloped by $n$-dimensional sphere with the area of $A = n\Omega_n \tilde{r}_A^{n-1}$. This law reduces to the conventional law, $dE = TdS - pdV$, for a pure de Sitter space where $\rho = -p$.

Taking the differential of $E$, we get
\[ dE = \rho \Omega_n n \tilde{r}_A^{n-1} d\tilde{r}_A + \Omega_n \tilde{r}_A^n \dot{\rho} dt. \tag{35} \]

Using the continuity Eq. (23), we obtain
\[ dE = \rho \Omega_n n \tilde{r}_A^{n-1} d\tilde{r}_A - \Omega_n \tilde{r}_A^n nH(\rho + p) dt. \tag{36} \]

The entropy associated with the apparent horizon is assumed to be in the form of Tsallis and Cirto entropy (2)
\[ S = \gamma A^\beta = \gamma (n\Omega_n \tilde{r}_A^{n-1})^\beta. \tag{37} \]

After differentiation, we have
\[ dS = \gamma \beta (n\Omega_n \tilde{r}_A^{n-1})^{\beta-1} n\Omega_n (n-1) \tilde{r}_A^{n-2} d\tilde{r}_A. \tag{38} \]
Substituting Eqs. (31), (33), (36) and (38) in Eq. (34), we get
\[(\rho + p)Hdt = \frac{\gamma \beta (n\Omega_n r_A^n - 1) - 1}{2n r_A^2} d\tilde{r}_A. \tag{39}\]

Using the continuity Eq. (23), we can get
\[\frac{2\pi}{n(n - 1)\gamma \beta (n\Omega_n)^{\beta - 1}} d\tilde{r}_A = \tilde{r}_A^{(n - 1)(\beta - 1) - 3} d\tilde{r}_A. \tag{40}\]

After integration of Eq. (40), we obtain
\[\tilde{r}_A^{(n - 1)\beta - n - 1} = \frac{2\pi [n - 1 - (n - 1)\beta](n\Omega_n)^{1 - \beta}}{n(n - 1)\gamma \beta} \rho. \tag{41}\]

where we have set the integration constant to zero. Substituting \(\tilde{r}_A\), we get
\[\left(H^2 + \frac{k}{a^2}\right)^{1 - \frac{(n - 1)(\beta - 1)}{2}} = \frac{2\pi [n - 1 - (n - 1)\beta](n\Omega_n)^{1 - \beta}}{n(n - 1)\gamma \beta} \rho. \tag{42}\]

This modified Friedmann equation from the first law of thermodynamics coincides with the equation from the emergence of cosmic space, which supports the viability of Padmanabhan’s perspective of emergent gravity and our modification given by Eq. (24). It’s worth noting that attempts of deriving the expansion law from the first law of thermodynamics were done in references [42,50]. The modified expansion laws of the universe proposed by Cai [17], Sheykhi [18], and Yang [21] could be derived starting from the first law of thermodynamics, \(dE = T dS + W dV\), in \((n + 1)\)-dimensional Einstein, Gauss-Bonnet and more general Lovelock gravities. These works show that the first law of thermodynamics could be considered as the origin of the expansion law of the universe. According to the derivation in this section, it is easy to derive the expansion law (Eq. (24)) from the first law of thermodynamics at the apparent horizon of a FRW Universe, so our results reveal the deep connection between the law of emergence and the first law of thermodynamics at the apparent horizon of a FRW universe.

3 Horizon entropy maximization of Tsallis and Cirto entropy

It is well known that an ordinary, isolated macroscopic system evolves to an equilibrium state of maximum entropy with the constraints [55],
\[\dot{S} \geq 0, \quad \text{always} \tag{43}\]
and
\[\dot{S} < 0, \quad \text{at least in long run}. \tag{44}\]

Recently, it was shown in Ref. [56] that in the context of Einstein gravity, our universe behaves as an ordinary macroscopic system that proceeds to a maximum entropy state consistent with the above constraints. In this case, \(S\) represents the total entropy of the universe which can be approximated as the horizon entropy, and dots denote the derivatives with respect to the cosmic time. Subsequently, Krishna and Mathew extended the procedure to Gauss Bonnet and Lovelock gravities for a spatially non-flat universe [57–59]. Our aim here is to study the maximization of Tsallis and Cirto horizon entropy in a \((n + 1)\)-dimensional FRW Universe with non zero spatial curvature. That is to say, we investigate whether the Tsallis and Cirto horizon entropy is getting extremized.

Motivated by the non-extensivity of Bekenstein entropy, and also the long-range nature of gravity, Tsallis and Cirto [60] introduced a new entropy for black holes as Eq. (2). The Tsallis and Cirto entropy is assumed to be hold for the apparent horizon of a FRW Universe, and is expressed as
\[S = \gamma A^\beta = \gamma (n\Omega_n r_A^{n - 1})^\beta = \gamma (n\Omega_n)^{\beta} \tilde{r}_A^{(n - 1)}, \tag{45}\]
where \(A = n\Omega_n r_A^{n - 1}\) for \(n \geq 3\) with \(\Omega_n\) being the volume of the unit \(n\)-sphere.

To check whether the horizon entropy is getting maximized in course of time, let us take the derivative of Eq. (45) with respect to the cosmic time
\[\dot{S} = \gamma \beta (n\Omega_n)^{\beta} (n - 1) \tilde{r}_A^{(n - 1) - 1} \dot{\tilde{r}}_A. \tag{46}\]

From this equation, the constraint \(\dot{S} \geq 0\) means that \(\dot{\tilde{r}}_A\) must be greater than or equal to zero. By using the modified Friedmann equation in \((n + 1)\) dimension from Tsallis and Cirto entropy (Eq. (41)),
\[\dot{\tilde{r}}_A^{(n - 1)(\beta - 1) - 2} = \frac{2\pi [n - 1 - (n - 1)\beta](n\Omega_n)^{1 - \beta}}{n(n - 1)\gamma \beta} \rho. \tag{47}\]

one can get
\[\dot{r}_A = \frac{\pi [(n - 1)\beta - n - 1](n\Omega_n)^{1 - \beta}}{n(n - 1)\gamma \beta [1 - (n - 1)(\beta - 1)]} \tilde{r}_A^{3 - (n - 1)(\beta - 1)} \dot{\rho}. \tag{48}\]

With the help of continuity Eq. (23), we get
\[\dot{r}_A = \frac{n}{2 - (n - 1)(\beta - 1)} H(1 + \omega) \tilde{r}_A, \tag{49}\]
where \(\omega\) is the parameter defined through the equation of state \(p = \omega \rho\). The recent observations show that our universe is evolving to a pure de Sitter state with \(\omega \geq -1\), so \(\dot{r}_A \geq 0\), which ensures the non negativity of \(\dot{S}\).

Next, we check whether this entropy reaches a maximum value in the long run characterized by the constraint \(\dot{S} < 0\).
Differentiating Eq. (46), we obtain the second derivative of the horizon entropy which is expressed as
\[ \ddot{S} = \gamma \beta (n \Omega_4)^\beta (n - 1) \dot{r}_A (n - 1) (\beta - 1) (\beta - 1)^{-2} [\dot{r}_A^2 + \dot{r}_A^2 r_A + \dot{r}_A]. \]  
(50)

The constrain \( \dddot{S} < 0 \) means that
\[ \{\beta (n - 1) - 1\} \dddot{r}_A < -\dddot{r}_A. \]  
(51)

In the asymptotic limit, \( t \to \infty \), \( \dot{r}_A \) will tend to zero, so \( \dddot{r}_A \) should be less than zero in the long run. Taking the derivative of the Eq. (49), one gets
\[
\dddot{r}_A = \frac{n \dot{r}_A}{2 - (n - 1)(\beta - 1)} [\dot{H}(1 + \omega) + H \dot{\omega} + \frac{n H^2 (1 + \omega)^2}{2 - (n - 1)(\beta - 1)}].
\]  
(52)

In the final de Sitter epoch, \( \omega \) approaches \(-1\), so all terms containing \((1 + \omega)\) vanish. The negativity of \( \dot{\omega} \) guarantees the negativity of \( \dddot{r}_A \). So the constrain (51) will hold true in the final stage, that is to say, the Tsallis and Cirto horizon entropy of a non flat universe reaches a maximum value in the long run.

4 Relationship between the law of emergence and the horizon entropy maximization

In this section, we check whether the law of emergence leads to the maximization of horizon entropy in the context of Tsallis and Cirto entropy. Combining Eqs. (19) and (46), we can relate the rate of emergence to the rate of change of entropy as
\[
\frac{d \dot{V}}{dt} = \frac{1}{\gamma (n - 1)} \dddot{r}_A \ddot{S}.
\]  
(53)

Then the law of emergence in Eq. (24) could be expressed as
\[ \dddot{S} = -\frac{n - 1}{4 \alpha} H (N_{\text{surf}} - N_{\text{bulk}}). \]  
(54)

For the non negativity of \( \dddot{S} \), the holographic discrepancy, \( (N_{\text{surf}} - N_{\text{bulk}}) \), in the above equation should be greater than or equal to zero. Now let’s see whether the definitions of \( N_{\text{surf}} \) and \( N_{\text{bulk}} \) ensure the non negativity of \( \dddot{S} \). From Eq. (20), the degrees of freedom on the apparent horizon is
\[ N_{\text{surf}} = \beta (n \Omega_4)^\beta \frac{r_A^{(n-1)\beta}}{(n-1)\beta - n - 1} (-8) \gamma \alpha, \]  
(55)

and from Eq. (22), the bulk degrees of freedom is
\[ N_{\text{bulk}} = -4 \pi \Omega_n r_A^{n+1} \frac{(n - 2) \rho + np}{n - 2}. \]  
(56)

From the continuity Eq. (23) and the modified Friedmann Eq. (27), the discrepancy, \( N_{\text{surf}} - N_{\text{bulk}} \), can be expressed as
\[ N_{\text{surf}} - N_{\text{bulk}} = -\frac{n - 1}{n - 2} \frac{1}{H} \frac{\beta (n \Omega_4)^\beta}{\dot{r}_A} \frac{(n-1)\beta - 1}{r_A^2}. \]  
(57)

The universe is assumed to be asymptotically de Sitter, so \( \ddot{r}_A \) will be greater than or equal to zero, which ensures \( N_{\text{surf}} - N_{\text{bulk}} \geq 0 \).

The second derivative of horizon entropy is obtained from Eq. (54) as
\[ \dddot{S} = \frac{n - 1}{4 \alpha} H (N_{\text{surf}} - N_{\text{bulk}}) + \frac{H (n - 1)}{4 \alpha} \frac{d}{dt} (N_{\text{surf}} - N_{\text{bulk}}). \]  
(58)

In the final de Sitter state, \( N_{\text{bulk}} \) will equal \( N_{\text{surf}} \), so the first term in the above equation vanishes. As the universe is trying to reduce the holographic discrepancy between \( N_{\text{surf}} \) and \( N_{\text{bulk}} \), we have the following condition
\[ \frac{d}{dt} (N_{\text{surf}} - N_{\text{bulk}}) < 0, \]  
(59)

which implies the non positivity of \( \dddot{S} \) in the long run. Substituting Eq. (57) in Eq. (58), we obtain
\[ \dddot{S} = \frac{(n - 1)^2}{2 \alpha (n - 2)} \gamma \beta (n \Omega_4)^\beta \times \{(n - 1)\beta - 1\} \dddot{r}_A \frac{r_A^{(n-1)\beta - 2} r_A^2}{\dot{r}_A^2} \left[ \frac{r_A^{(n-1)\beta - 1} \dddot{r}_A}{\dot{r}_A^2} \right]. \]  
(60)

It is easy to get the condition for the negativity of \( \dddot{S} \) as
\[ [(n - 1)\beta - 1] \dddot{r}_A^2 \leq -\dddot{r}_A \dddot{r}_A^2. \]  
(61)

This condition is the same as the constraint in Eq. (51) which we have got from the maximization of horizon entropy in the above section. The law of emergence therefore leads to the maximization of horizon entropy in the context of Tsallis and Cirto entropy.

5 Discussions and conclusions

Motivated by the non-extensivity of Bekenstein entropy, and also the long-range nature of gravity, Tsallis and Cirto showed that thermodynamical entropy of a gravitational system such as a black hole must be generalized to the non-additive entropy which is given by \( S = \gamma A^\beta \), where \( A \) is the horizon area, \( \gamma \) is an unknown constant, and \( \beta \) is the non-extensive parameter [60]. In this note, based on the idea of the emergence of cosmic space suggested by T. Padmanabhan [15, 16], we generalize Sheykhi’s derivation process to obtain the modified Friedmann equation from Tsallis and Cirto entropy in \((n + 1)\) dimension with non-zero spatial curvature. The upper bound on the non-additive parameter of Tsallis and Cirto entropy is obtained. To confirm our results, we also derive the modified Friedmann equation from Tsallis
and Cirto entropy by use of the first law of thermodynamics, \( dE = TdS + WdV \), at the apparent horizon of a FRW Universe, which coincides with the equation obtained from the emergence approach. Refs. [42, 50] studied the relationship between the expansion law and the first law of thermodynamics at the apparent horizon, and showed that the modified expansion laws of a FRW Universe proposed by Cai [17], Sheykhi [18], and Yang [21] could be derived starting from the first law of thermodynamics, \( dE = TdS + WdV \), in \((n + 1)\)-dimensional Einstein, Gauss-Bonnet and more general Lovelock gravities. These works show that the first law of thermodynamics could be considered as the origin of the emergence of cosmic space. According to our calculation, it is easy to derive the modified expansion law (Eq. (24)) from the first law of thermodynamics at the apparent horizon of a FRW Universe. So our results not only support the viability of Padmanabhan’s perspective of emergence gravity, but also point out the deep connection between the law of emergence and the first law of thermodynamics.

In Ref. [56], Pavon and Radicella showed that our universe with a Hubble expansion history behaves as an ordinary macroscopic system that proceeds to a state of maximum entropy. Subsequently, Krishna and Mathew proved that the laws of emergence in the context of Einstein, Gauss-Bonnet and Lovelock gravities, implies the maximization of horizon entropy [57–59]. In this note, we investigate the consistency of the law of emergence with the maximization of horizon entropy in the context of Tsallis and Cirto entropy. We find that both the law of emergence and the horizon entropy maximization lead to the same constraints, and the law of emergence can be viewed as a tendency for maximizing the horizon entropy. In the cosmological context, a system that obeys the law of the emergence behaves as an ordinary macroscopic system that proceeds to an equilibrium state of maximum entropy. Therefore, our results reveal the deep connection between the law of emergence and horizon thermodynamics. We would like to mention one point here. From Eq. (19), the number of degrees of freedom on the holographic surface is assumed as Eq. (20). This assumption is consistent with the first law of thermodynamics at the apparent horizon of a FRW Universe. According to Ref. [50], it is easy to derive the modified expansion law (Eq. (24)) from the first law of thermodynamics at the apparent horizon of a FRW Universe, and find the expression of \( N_{\text{surf}} \). The deeper physical meaning of \( N_{\text{surf}} \) is worthy of further studies, which is beyond the scope of our investigation.

It is interesting to consider other forms of entropy, e.g. Renyi entropy [67] and the generalized Kaniadakis entropy [68] to investigate the modified Friedmann equations and the consistency of the law of emergence with the maximization of horizon entropy. These issues are now under consideration.

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