Averages over Ginibre’s Ensemble of Random Real Matrices

CHRISTOPHER D. SINCLAIR

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Abstract

We give a method for computing the ensemble average of multiplicative class functions over the Gaussian ensemble of real asymmetric matrices. These averages are expressed in terms of the Pfaffian of Gram-like antisymmetric matrices formed with respect to a skew-symmetric inner product related to the class function.

1 Introduction

In the 1960’s Ginibre introduced three statistical ensembles of matrices whose entries are chosen independently with Gaussian probability from (resp.) \( \mathbb{R} \), \( \mathbb{C} \), and Hamilton’s Quaternions [4]. These ensembles are respectively labeled GinOE, GinUE and GinSE in analogy with their Hermitian counterparts. Ginibre introduced a physical analogy between GinUE and two-dimensional electrostatics, but gave no applications for the other two ensembles. Since their introduction, many applications have been found for GinOE and GinSE.

Here we report a method for determining ensemble averages over GinOE of certain functions which are constant on similarity (conjugacy) classes. GinOE is the space of \( N \times N \) real matrices \( \mathbb{R}^{N \times N} \) together with the probability measure \( \nu \) given by

\[
\nu(S) := \mathcal{B}_N^{-1} \int_S \exp\left(-\frac{\text{Tr}(X^T X)}{2}\right) d\mu(X),
\]

where \( \mu \) is Lebesgue measure on \( \mathbb{R}^{N \times N} \), and \( \mathcal{B}_N = (2\pi)^{N^2/2} \). Our goal is to find

\[
\langle \Psi \rangle := \mathcal{B}_N^{-1} \int_{\mathbb{R}^{N \times N}} \Psi(X) \exp\left(-\frac{\text{Tr}(X^T X)}{2}\right) d\mu(X),
\]

where \( \Psi : \mathbb{R}^{N \times N} \to \mathbb{R} \) is (i) constant on similarity classes: \( \Psi(A X A^{-1}) = \Psi(X) \) for all invertible \( A \in \mathbb{R}^{N \times N} \), and (ii) there exists a function \( \phi : \mathbb{C} \to \mathbb{R} \) such that if \( D \) is a diagonal matrix with entries \( \gamma_1, \gamma_2, \ldots, \gamma_N \) then \( \Psi(D) = \phi(\gamma_1)\phi(\gamma_2)\cdots\phi(\gamma_N) \). We remark that \( \mu \)-almost every matrix in \( \mathbb{R}^{N \times N} \) is similar to a diagonal matrix and consequently \( \langle \Psi \rangle \) is uniquely determined by \( \psi \).

As an example, when \( \psi(\gamma) = \gamma^n \), we are in the well-studied situation \( \Psi(X) = (\det X)^n \) [8][1].

Ginibre’s original interest was the joint eigenvalue probability density function (JPDF) and the \( n \)-point correlation functions of GinUE, GinSE and GinOE. In the case of GinOE,
Ginibre was only able to report the JPDF in the restrictive case where all eigenvalues are real: One difficulty in determining the full JPDF being that the space of eigenvalues is naturally represented as a disjoint union indexed over the possible numbers of real and complex conjugate pairs of eigenvalues. The full JPDF for GinOE was finally computed in the 1990’s by Lehmann and Sommers [6], and later independently by Edelman [2].

Surprisingly, the formulation for the ensemble average of a multiplicative class functions we present here is seemingly independent of this decomposition of space. Nonetheless, since we will use the JPDF (which is dependent on this decomposition) we introduce the details here.

Throughout the discussion $N$ will be fixed, and $(L, M)$ will represent pairs of non-negative integers such that $L + 2M = N$. $L$ will represent the number of real eigenvalues and $M$ the number of complex conjugate pairs of eigenvalues. The symbols $\alpha$ and $\beta$ will represent real and (non-real) complex eigenvalues respectively. We also use $\lambda_L$ and $\lambda_{2M}$ to represent Lebesgue measure on $\mathbb{R}^L$ and $\mathbb{C}^M$ respectively. The space of all possible eigenvalues of $N \times N$ matrices can then be identified with the disjoint union

$$\bigcup_{(L, M)} \mathbb{R}^L \times (\mathbb{C} \setminus \mathbb{R})^M.$$  

It shall be convenient to write the partial JPDF’s in terms of the Vandermonde determinant. Given $\gamma \in \mathbb{C}^N$ we define $V^\gamma$ to be the $N \times N$ Vandermonde matrix in the coordinates of $\gamma$. (Superscripts on matrices will indicate variables on which the entries are dependent). The Vandermonde determinant is then given by

\begin{equation}
\Delta(\gamma) := \det V^\gamma = \prod_{m<n} \gamma_n - \gamma_m,
\end{equation}

and given $\alpha \in \mathbb{R}^L$ and $\beta \in \mathbb{C}^M$ we define

\begin{equation}
\Delta(\alpha, \beta) := \det V^\gamma \quad \text{where} \quad \gamma := (\overline{\beta_1}, \beta_1, \ldots, \overline{\beta_M}, \beta_M, \alpha_1, \ldots, \alpha_L)
\end{equation}

The partial JPDF is then given by $P_{L, M} : \mathbb{R}^L \times (\mathbb{C} \setminus \mathbb{R})^M \to \mathbb{R}$ where

\begin{equation}
P_{L, M}(\alpha, \beta) = \mathcal{C}_N^{-1} \frac{\Delta(\alpha, \beta)}{L! M!} \prod_{\ell=1}^L e^{-\alpha_\ell^2/2} \prod_{m=1}^M \text{erfc} \left( \sqrt{2} |\text{Im} \beta_m| \right) e^{-(\beta_m^2 + \beta_{-m}^2)/2},
\end{equation}

and

\begin{equation}
\mathcal{C}_N := 2^{N(N+1)/4} \prod_{n=1}^N \Gamma(n/2).
\end{equation}

This defines the full JPDF since the domains of the partial JPDF’s are disjoint. The only essential difference between the formulation for $P_{L, M}(\alpha, \beta)$ presented here and that presented by Lehmann and Sommers (and Edelman) is that they use the right hand side of (1.1).

From (1.3) we can see the two main difficulties in the computation of $\langle \Psi \rangle$: (i) the decomposition of the space of eigenvalues, and (ii) the complicated nature of $|\Delta(\alpha, \beta)|$. 
2 Statement of Results

In order to state our results it shall be convenient to define
\[ \phi(\gamma) := \exp(-\gamma^2/2) \left\{ \text{erfc}(\sqrt{2}\text{Im}(\gamma)) \right\}^{1/2} \psi(\gamma). \]

We then define two skew-symmetric inner products,
\[ \langle P, Q \rangle_R := \int_{R^2} \phi(\alpha_1) \phi(\alpha_2) P(\alpha_1)Q(\alpha_2) \text{sgn}(\alpha_2 - \alpha_1) \, d\alpha_1 \, d\alpha_2, \]
and
\[ \langle P, Q \rangle_C := -2i \int_C \phi(\beta) \phi(\overline{\beta}) P(\beta)Q(\overline{\beta}) \text{sgn}(\text{Im}(\beta)) \, d\lambda_2(\beta) \]
where \( \text{sgn}(0) = 0 \) and \( \text{sgn}(x) = x/|x| \). These skew-symmetric inner products also appear in another paper on the statistics of the eigenvalues of matrices in GinOE by Kanzieper and Akemann [5].

**Theorem 2.1.** Let \( J \) be the integer part of \((N + 1)/2\). If \( P = \{P_1(\gamma), P_2(\gamma), \ldots, P_N(\gamma)\} \subset \mathbb{C}[\gamma] \) is a set of monic polynomials with \( \deg P_n = n - 1 \), then, assuming \( \langle \Psi \rangle \) exists,
\[ \langle \Psi \rangle = C_N^{-1} \text{Pf} U_P, \]
the Pfaffian of \( U_P \), where \( U_P \) is the \( 2J \times 2J \) antisymmetric matrix whose \( j,k \) entry is given by
\[ U_P[j,k] := \begin{cases} \langle P_j, P_k \rangle_R + \langle P_j, P_k \rangle_C & \text{if } j, k \leq N, \\ \text{sgn}(k - j) \int_R \phi(\alpha) P_{\min\{j,k\}}(\alpha) \, d\alpha & \text{otherwise}, \end{cases} \]
and \( C_N \) is given as in (1.4).

Notice that when \( N \) is even the first condition in Equation (2.1) always holds. The asymmetry between even and odd cases is due to the fact that the Pfaffian is not defined for odd by odd matrices.

If \( P \) satisfies the conditions of Theorem 2.1, we will say that \( P \) is a complete set of monic polynomials. When \( N \) is even it is sensible to use a complete family of monic polynomials which are skew-orthogonal.

**Corollary 2.2.** Suppose \( N = 2J \), and let \( Q = \{Q_1, Q_2, \ldots, Q_N\} \) be any complete family of monic polynomials specified by
\[ \langle Q_{2k-1}, Q_{2j} \rangle = -\langle Q_{2j}, Q_{2k-1} \rangle = \delta_{kj} M_j \quad \text{and} \quad \langle Q_{2j}, Q_{2k} \rangle = \langle Q_{2j-1}, Q_{2k-1} \rangle = 0, \]
where \( \langle P, Q \rangle = \langle P, Q \rangle_R + \langle P, Q \rangle_C \). Then,
\[ \langle \Psi \rangle = C_N^{-1} \prod_{j=1}^J M_j. \]

The quantities \( M_j \) are referred to as the normalization(s) of \( Q \). See [7, Ch. 5] or [3, Ch. 5] for more about skew-orthogonal polynomials.

If \( \psi \) satisfies an additional symmetry, then we may write \( \langle \Psi \rangle \) as a determinant
Corollary 2.3. Let $J$ be the integer part of $(N + 1)/2$. If $\psi(-\beta) = \psi(\beta)$ for every $\beta \in \mathbb{C}$, and $P$ is a complete family of monic polynomials in $\mathbb{C}[\gamma]$ such that $P_n$ is even when $n - 1$ is even and odd when $n - 1$ is odd, then

$$\langle \Psi \rangle = \mathbb{C}_N^{-1} \det A_P$$

where $A_P$ is the $J \times J$ matrix whose $j, k$ entry is given by

$$A_P[j, k] := U_P[2j - 1, 2k].$$

We reiterate the striking fact that these formulations of $\langle \Psi \rangle$ are seemingly independent of the decomposition of the space of eigenvalues into all the different possible numbers of real and complex conjugate pairs of eigenvalues. In fact, the decomposition of the space of eigenvalues does enter into the statement of Theorem 2.1 — it is the reason that the inner product in the entries of $U_P$ are sums of skew inner products. One of these inner products introduces to $\langle \Psi \rangle$ contributions from $\psi$ on real eigenvalues, while the other introduces contributions from pairs of complex conjugate eigenvalues.

3 Averages over GinUE

The simplicity of Theorem 2.1 and its corollaries can (perhaps) be most appreciated when compared with the analogous results for GinUE. GinUE is the set $\mathbb{C}^{N \times N}$ together with the probability measure whose density is proportional to $\exp(\text{Tr}(Z^*Z)/2)$. The joint eigenvalue probability density function was given by Ginibre as

$$P_N(\gamma) := D_N^{-1} |\Delta(\gamma)|^2 \prod_{n=1}^{N} e^{-|\gamma_n|^2/2},$$

where $D_N := (2\pi)^N \prod_{n=1}^{N} \Gamma(n + 1)$.

We will denote the ensemble average of $\Psi$ over GinUE by $\{\Psi\}$, and we define the inner product,

$$\langle P|Q \rangle := \int_{\mathbb{C}} e^{-|\gamma|^2/2} \psi(\gamma) \overline{P(\gamma)}Q(\gamma) d\lambda_2(\gamma).$$

Theorem 3.1. Let $P$ be any complete set of monic polynomials. Then, assuming $\{\Psi\}$ exists,

$$\{\Psi\} = D_N^{-1} \det W_P,$$

where $W_P$ is the $N \times N$ symmetric matrix whose $j, k$ entry is given by $W_P[j, k] = \langle P_j|P_k \rangle$, and $D_N$ is given as in (3.1).

Corollary 3.2. Let $Q$ be the complete family of monic polynomials which are orthogonal with respect to $\langle \cdot | \cdot \rangle$, then

$$\{\Psi\} = D_N^{-1} \prod_{n=1}^{N} \langle Q_n|Q_n \rangle.$$
4 The Proof of Theorem 2.1

The proof of Theorem 2.1 is based on, indeed almost identical to, another computation by the author in the study of heights of polynomials with integer coefficients [11]. The connection between that computation and the one presented here is that the Jacobian of the change of variables from the roots to the coefficients of a monic degree $N$ real polynomial with $L$ real roots and $M$ pairs of complex conjugate roots is a constant times $|\Delta(\alpha, \beta)| / L! M!$. Since the audiences of these two results are likely disjoint many of the details are presented here, though the proof of many formulas which are purely combinatorial will only be referenced.

From (1.3) we see,

\[(4.1) \quad \langle \Psi \rangle = C_N^{-1} \sum_{(L,M)} \frac{1}{L!M!} \int_{\mathbb{R}^L \times \mathbb{C}^M} \left\{ \prod_{i=1}^L \phi(\alpha) \prod_{m=1}^M \phi(\beta) \phi(\overline{\beta}) \right\} |\Delta(\alpha, \beta)| d\lambda_L(\alpha) d\lambda_2M(\beta),\]

Next, for each pair $(L, M)$ we use the Laplace expansion in order to expand the Vandermonde determinant. Since the first $2M$ columns of $V^{\alpha, \beta}$ depend only on $\beta$, while the remaining columns depend only on $\alpha$, we will expand $\Delta(\alpha, \beta)$ via $2M \times 2M$ and $L \times L$ minors.

4.1 Notation for Minors

For each $K \leq N$ we define $\mathcal{J}_K^N$ to be the set of increasing functions from $\{1, 2, \ldots, K\}$ to $\{1, 2, \ldots, N\}$. That is,

$$\mathcal{J}_K^N := \{\{1, 2, \ldots, K\} \rightarrow \{1, 2, \ldots, N\} : t(1) < t(2) < \cdots < t(K)\}.$$ 

Associated to each $t \in \mathcal{J}_K^N$ there exists a unique $t' \in \mathcal{J}_{N-K}^N$ such that the images of $t$ and $t'$ are disjoint. Each $t \in \mathcal{J}_K^N$ induces a unique permutation in $\iota_t \in S_N$ by specifying that

$$\iota_t(n) := \begin{cases} 
  t(n) & \text{if } 1 \leq n \leq K, \\
  t'(n-K) & \text{if } K < n \leq N.
\end{cases}$$

We define the sign of $t$ by setting $\operatorname{sgn}(t) := \operatorname{sgn}(\iota_t)$. The identity map in $\mathcal{J}_K^N$ is denoted by $i$.

Given an $N \times N$ matrix $W$ and $u, t \in \mathcal{J}_K^N$, define $W_{u,t}$ to be the $K \times K$ minor whose $j,k$ entry is given by $W_{u,t}(j,k) = W[u(j), t(k)]$. The complimentary minor is given by $W_{u',t'}$. As an example of the utility of this notation, the Laplace expansion of the determinant can be written as

\[(4.2) \quad \det W = \sum_{t \in \mathcal{J}_K^N} \operatorname{sgn}(t) \det W_{u,t} \cdot \det W_{u',t'},\]

where $u$ is any fixed element of $\mathcal{J}_K^N$. We will also use the abbreviated notation $W_u$ for $W_{u,u}$; this is useful notation for working with Pfaffians since if $W$ is an antisymmetric matrix then minors of the form $W_u$ are also antisymmetric.

We recall the definition of the Pfaffian here. If $N = 2J$ and $U$ is an $N \times N$ antisymmetric matrix, then the Pfaffian of $U$ is given by

\[(4.3) \quad \operatorname{Pf} U = \frac{1}{2^J J!} \sum_{\tau \in S_N} \operatorname{sgn}(\tau) \prod_{j=1}^J U[\tau(2j-1), \tau(2j)],\]
where $S_N$ is the $N$th symmetric group. We will also use the formula

$$\text{Pf } U = \frac{1}{J!} \sum_{\sigma \in \Pi_{2J}} \text{sgn}(\sigma) \prod_{j=1}^{J} U[\tau(2j-1), \tau(2j)],$$

where $\Pi_{2J}$ is the subset of $S_{2J}$ composed of those $\sigma$ with $\sigma(2j) > \sigma(2j-1)$ for $j = 1, \ldots, J$. The Pfaffian is related to the determinant by the formula $\det U = (\text{Pf } U)^2$ (see for instance [9, Appendix: Pfaffians]).

If $U = R + C$ where $R$ and $C$ are antisymmetric $2J \times 2J$ matrices, then $\text{Pf } U$ has an expression in terms of the minors of $R$ and $C$ [11, Lemma 8.7].

$$\text{Pf } U = \sum_{M=0}^{J} \sum_{u \in S_{2J}} \text{sgn}(u) \text{Pf } R_{u} \cdot \text{Pf } C_{u}. \leqno{(4.5)}$$

This will be useful since ultimately we intend to express $\langle \Psi \rangle$ as the Pfaffian of a sum of matrices.

### 4.2 Steps in the Proof

**Lemma 4.1.** Let $\gamma \in \mathbb{C}^N$ be given as in (1.2) and let $W^{\alpha,\beta}$ be the $N \times N$ matrix whose $j, k$ entry is given by

$$W^{\alpha,\beta}[j, k] := P_k(\gamma_j). \leqno{(4.6)}$$

Then, if $i \in S_{2M}$ is the identity map on $\{1, 2, \ldots, 2M\}$,

$$|\Delta(\alpha, \beta)| = \sum_{t \in S_{2M}} \text{sgn}(t) \left\{ \begin{array}{c} \det W^{\beta}_{i,t}(-i)^M \prod_{m=1}^{M} \text{sgn} \Im(\beta_m) \end{array} \right\} \left\{ \begin{array}{c} \det W^{\alpha}_{i,t'} \prod_{j<k} \text{sgn}(\alpha_k - \alpha_j) \end{array} \right\},$$

where as suggested by the notation, the minors $W^{\beta}_{i,t}$ and $W^{\alpha}_{i,t'}$ of $W^{\alpha,\beta}$ are dependent only on $\beta$ and $\alpha$ respectively.

Using Lemma 4.1, Equation (4.1) becomes

$$\langle \Psi \rangle = \mathcal{C}^{-1}_N \sum_{(L,M)} \frac{1}{M! L!} \sum_{t \in S_{2M}} \text{sgn}(t) \int_{L} \int_{C^M} \left\{ \prod_{\ell=1}^{L} \phi(\alpha_{\ell}) \right\} \left\{ \prod_{m=1}^{M} \phi(\beta_m) \phi(\overline{\beta_m}) \right\}$$

$$\times \left\{ \begin{array}{c} \det W^{\alpha}_{i,t'} \prod_{j<k} \text{sgn}(\alpha_k - \alpha_j) \end{array} \right\} \left\{ \begin{array}{c} \det W^{\beta}_{i,t}(-i)^M \prod_{m=1}^{M} \text{sgn} \Im(\beta_m) \end{array} \right\} d\lambda_L(\alpha) d\lambda_2(\beta),$$

and Fubini’s Theorem yields

$$\langle \Psi \rangle = \mathcal{C}^{-1}_N \sum_{(L,M)} \sum_{t \in S_{2M}} \text{sgn}(t) \frac{1}{L!} \int_{L} \left\{ \prod_{\ell=1}^{L} \phi(\alpha_{\ell}) \prod_{j<k} \text{sgn}(\alpha_k - \alpha_j) \right\} \frac{(-i)^M}{M!} \int_{C^M} \left\{ \prod_{m=1}^{M} \phi(\beta_m) \phi(\overline{\beta_m}) \text{sgn} \Im(\beta_m) \right\} d\lambda_L(\alpha) \leqno{(4.6)}$$

$$\times \left\{ \begin{array}{c} \det W^{\alpha}_{i,t'} \end{array} \right\} d\lambda_2(\beta).$$
We remark that by using Fubini’s Theorem we are assuming that the average $\langle \Psi \rangle$ actually exists.

Next, let $K$ be the integer part of $(L+1)/2$. Define $T^\alpha$ to be the $2K \times 2K$ antisymmetric matrix whose $j,k$ entry is given by

$$T^\alpha[j,k] := \begin{cases} \text{sgn}(\alpha_k - \alpha_j) & \text{if } j,k < L+1, \\ \text{sgn}(k-j) & \text{otherwise}. \end{cases}$$

Then, it is well known that,

$$\prod_{1 \leq j < k \leq L} \text{sgn}(\alpha_k - \alpha_j) = \text{Pf} T^\alpha,$$

(see [3, Ch.5], or [11, Lemma 8.4]). It is worth remarking that when $L$ is even then the first condition defining $T^\alpha$ is always in force. Since the Pfaffian is only defined for even rank antisymmetric matrices, the second condition is used when $L$ is odd to bootstrap a $2K \times 2K$ antisymmetric matrix from an $L \times L$ matrix.

Substituting (4.8) into (4.6) we see

$$\langle \Psi \rangle = \sum_{(L,M) \in \mathcal{H}_2} \left( \frac{1}{L!} \int_{\mathbb{R}^L} \left\{ \prod_{\ell=1}^L \phi(\alpha_\ell) \right\} \text{Pf} T^\alpha \cdot \det W_{\alpha_i}^{\alpha_{i'}} d\lambda_L(\alpha) \right) \times \left( \frac{(-i)^M}{M!} \int_{\mathbb{C}^M} \left\{ \prod_{m=1}^M \phi(\beta_m) \phi(\bar{\beta}_m) \text{sgn}(\beta_m) \right\} \det W_{\beta_i}^\beta d\lambda_2^\beta(\beta) \right)$$

(4.9)

Now, let $J$ be the integer part of $(N+1)/2$. It is necessary for our calculations to replace the $t \in \mathcal{H}_2^N$ with elements of $\mathcal{H}_2^J$. Each $t \in \mathcal{H}_2^N$ induces a unique $t_0 \in \mathcal{H}_2^J$ by setting $t = t_0$. Notice that $t'$ and $t_0'$ differ in the fact that if $N \neq 2J$ then $t_0'(2J-2M) = 2J$. Clearly, $\text{sgn}(t_0) = \text{sgn}(t)$.

**Lemma 4.2.** Let $R$ be the $2J \times 2J$ matrix whose $j,k$ entry is given by

$$R[j,k] := \begin{cases} \langle P_j, P_k \rangle_R & \text{if } j,k < N+1 \\ \text{sgn}(k-j) \int_{\mathbb{R}} \phi(\alpha) P_{\min(j,k)}(\alpha) d\alpha & \text{otherwise}, \end{cases}$$

and suppose $t \in \mathcal{H}_2^N$. Then,

$$\frac{1}{L!} \int_{\mathbb{R}^L} \left\{ \prod_{\ell=1}^L \phi(\alpha_\ell) \right\} \text{Pf} T^\alpha \cdot \det W_{\alpha_i}^{\alpha_{i'}} d\lambda_L(\alpha) = \text{Pf} R_{t'}.$$

When $N$ is odd and $t \in \mathcal{H}_2^N$ then $R_{t'}$ is an odd by odd matrix. The introduction of $t_0$ is useful since the Pfaffian of $R_{t_0'}$ is defined.

**Lemma 4.3.** Let $C$ be the $2J \times 2J$ matrix whose $j,k$ entry is given by

$$C[j,k] := \begin{cases} \langle P_j, P_k \rangle_C & \text{if } j,k < N+1 \\ 0 & \text{otherwise}, \end{cases}$$
and suppose \( t \in \mathcal{J}_{2M}^N \). Then,
\[
\frac{(-i)^M}{M!} \int_{C^M} \left\{ \prod_{m=1}^{M} \phi(\beta_m) \phi(\beta_m) \operatorname{sgn} \mathfrak{I}(\beta_m) \right\} \det W^\beta_{t,t} d\lambda_{2M}(\beta) = Pf C_t.
\]

Using Lemma 4.2 and Lemma 4.3 we may rewrite (4.9) as
\[
\langle \Psi \rangle = C^{-1}_N \sum_{(L,M) \in \mathcal{J}_{2M}^J} \sum_{t \in \mathcal{J}_{2M}^N} \operatorname{sgn}(t) Pf R_{u_t} \cdot Pf C_t.
\]

If \( u \in \mathcal{J}_{2M}^J \) then either \( 2J \) is in the image of \( u \) or \( 2J \) is in the image of \( u' \). Notice that if \( 2J \) is in the image of \( u \) then \( Pf C_u = 0 \). If \( 2J \) is in the image of \( u' \) then \( u'(2J - 2M) = 2J \) and hence \( u = t \) for some \( t \in \mathcal{J}_{2M}^N \). Thus we may replace the sum over \( \mathcal{J}_{2M}^N \) in (4.10) with a sum over \( \mathcal{J}_{2M}^J \). Consequently,
\[
\langle \Psi \rangle = C^{-1}_N \sum_{(L,M) \in \mathcal{J}_{2M}^J} \sum_{u \in \mathcal{J}_{2M}^{2J}} \operatorname{sgn}(u) Pf R_{u} \cdot Pf C_u,
\]

where the second equation follows since the summand has been made to be independent of \( L \). From (4.5) we see that \( \langle \Psi \rangle = C^{-1}_N Pf (R + C) \). Consequently, by (2.1), \( \langle \Psi \rangle = C^{-1}_N Pf U \).

### 4.3 The Proof of Lemma 4.1

Applying (1.1) to (1.2),
\[
\Delta(\alpha, \beta) = \left\{ \prod_{j<k} (\alpha_k - \alpha_j) \right\} \prod_{m=1}^{M} \prod_{l=1}^{L} |\beta_m - \alpha_l|^2
\]
\[
\times \left\{ \prod_{m<n} |\beta_m - \beta_n|^2 |\beta_n - \beta_m|^2 \right\} \prod_{m=1}^{M} 2i\mathfrak{I}(\beta_m).
\]

And hence,
\[
|\Delta(\alpha, \beta)| = (-i)^M \left\{ \prod_{j<k} \operatorname{sgn}(\alpha_k - \alpha_j) \prod_{m=1}^{M} \operatorname{sgn} \mathfrak{I}(\beta_m) \right\} \Delta(\alpha, \beta).
\]

We may replace the monomials in the Vandermonde matrix with any complete family of monic polynomials without changing its determinant. That is, \( \Delta(\alpha, \beta) = \det W^{\alpha, \beta} \). Using the Laplace expansion of the determinant (4.2) with \( u = i \in \mathcal{J}_{2M}^N \), we see that
\[
\det W^{\alpha, \beta} = \sum_{t \in \mathcal{J}_{2M}^N} \operatorname{sgn}(t) \det W^{\alpha, \beta}_{t,t} \cdot \det W^{\alpha, \beta}_{t',t'}.
\]
Notice that the minors of the form \( W_{\alpha, \beta} \) consists of elements from the first \( 2M \) columns of \( W^{\alpha, \beta} \). These columns are not dependent on \( \alpha \) and thus we may write these minors as \( W_{i, t}^{\alpha, \beta} \). Similarly we may write the minors of the form \( W_{\alpha', \beta'}^{\alpha, \beta} \) as \( W_{i', t'}^{\alpha, \beta} \). It follows that

\[
\text{det} V^{\alpha, \beta} = \sum_{t \in \mathcal{N}_{2M}} \text{sgn}(t) \text{det} W_{i, t}^{\beta} \cdot \text{det} W_{i', t'}^{\alpha},
\]

and the lemma follows by substituting (4.13) into (4.12) and simplifying.

### 4.4 The Proof of Lemma 4.2

We start with

\[
\mathfrak{1} = \frac{1}{L!} \int_{\mathbb{R}^L} \det W_{i', t}^{\alpha} \cdot \text{Pf} T^{\alpha} \left\{ \prod_{\ell=1}^L \phi(\alpha_{\ell}) \right\} d\lambda_L(\alpha),
\]

where \( t \) is an element of \( \mathcal{N}_{2M} \). Expanding \( \det W_{i', t}^{\alpha} \) as a sum over \( S_L \) allows us to write \( \mathfrak{1} \) as

\[
\mathfrak{1} = \frac{1}{L!} \sum_{\sigma \in S_L} \text{sgn}(\sigma) \int_{\mathbb{R}^L} \left\{ \prod_{\ell=1}^L \phi(\alpha_{\ell}) \prod_{\ell=1}^L P_{t(\ell)}(\alpha_{\sigma(\ell)}) \right\} \text{Pf} T^{\alpha} d\lambda_L(\alpha).
\]

\( S_L \) naturally acts on \( \mathbb{R}^L \) by permuting the coordinates (denote this action by \( \sigma \cdot \alpha \)), and it is easy to verify that for \( \sigma \in S_L, \) \( \text{Pf} T^{\sigma \cdot \alpha} = \text{sgn}(\sigma) \text{Pf} T^{\alpha} \). Using this fact, and the change of variables \( \alpha \mapsto -\sigma^{-1} \cdot \alpha \) we may write \( \mathfrak{1} \) as

\[
\mathfrak{1} = \sum_{\sigma \in S_L} \text{sgn}(\sigma^{-1}) \int_{\mathbb{R}^L} \left\{ \prod_{\ell=1}^L \phi(\alpha_{\ell}) \prod_{\ell=1}^L P_{t(\ell)}(\alpha_{\sigma(\ell)}) \right\} \text{Pf} T^{\alpha} d\lambda_L(\alpha).
\]

Substituting this into (4.15) we see that the sum over \( S_L \) exactly cancels \( 1/L! \). That is,

\[
\mathfrak{1} = \int_{\mathbb{R}^L} \prod_{\ell=1}^L \phi(\alpha_{\ell}) \prod_{\ell=1}^L P_{t(\ell)}(\alpha_{\ell}) \text{Pf} T^{\alpha} d\lambda_L(\alpha).
\]

Setting \( K \) to the integer part of \( (L + 1)/2 \), and using (4.4), we see

\[
\text{Pf} T^{\alpha} = \frac{1}{K!} \sum_{\tau \in \Pi_{2K}} \text{sgn}(\tau) \prod_{k=1}^K \text{sgn}(\alpha_{\tau(2k)} - \alpha_{\tau(2k-1)}).
\]

#### 4.4.1 L Odd

In the case of \( L \) odd, we view \( \alpha_{L+1} = +\infty \) so as to be consistent with (4.7). Substituting (4.17) into (4.16) we find

\[
\mathfrak{1} = \frac{1}{K!} \sum_{\tau \in \Pi_{2K}} \text{sgn}(\tau) \int_{\mathbb{R}^L} \prod_{\ell=1}^L \phi(\alpha_{\ell}) P_{t(\ell)}(\alpha_{\ell}) \prod_{k=1}^K \text{sgn}(\alpha_{\tau(2k)} - \alpha_{\tau(2k-1)}) d\lambda_L(\alpha).
\]
For each $\tau \in \Pi_{2K}$ there is a $k_o$ such that $\alpha_{\tau(2k_o)} = \alpha_{L+1}$. If we set $\ell_o = \tau(2k_o) - 1$ then we may write (4.20) as

$$\Phi = \phi(\alpha_{\ell_o}) P_{\ell_o}^{(\ell_o)}(\alpha_{\ell_o}) \left\{ \prod_{\ell \neq \ell_o}^{L} \phi(\alpha_{\ell}) P_{\ell}^{(\ell)}(\alpha_{\ell}) \prod_{k=1}^{K} \text{sgn}(\alpha_{\tau(2k)} - \alpha_{\tau(2k-1)}) \right\}$$

$$= \phi(\alpha_{\ell_o}) P_{\ell_o}^{(\ell_o)}(\alpha_{\ell_o}) \left\{ \prod_{k=1}^{K} \phi(\alpha_{\tau(2k)}) \phi(\alpha_{\tau(2k-1)}) \right\}$$

$$\times P_{(t'\circ\tau)(2k)}(\alpha_{\tau(2k)}) P_{(t'\circ\tau)(2k-1)}(\alpha_{\tau(2k-1)}) \text{sgn}(\alpha_{\tau(2k)} - \alpha_{\tau(2k-1)}) \right\},$$

where the second equation comes from reindexing the first product by $\ell \mapsto \tau^{-1}(\ell)$ together with the fact that $2(K - 1) = L - 1$. Substituting this into (4.18) and applying Fubini’s Theorem we find

$$\Phi = \frac{1}{K!} \sum_{\tau \in \Pi_{2K}} \text{sgn}(\tau) \left\{ \int_{\mathbb{R}} \phi(x) P_{(t'\circ\tau)(2k_o-1)}(x) dx \right\}$$

$$\times \left\{ \prod_{k=1}^{K} \int_{\mathbb{R}^2} \phi(x) \phi(y) P_{(t'\circ\tau)(2k)}(y) P_{(t'\circ\tau)(2k-1)}(x) \text{sgn}(y - x) dy \right\}$$

$$= \frac{1}{K!} \sum_{\tau \in \Pi_{2K}} \text{sgn}(\tau) \left\{ \int_{\mathbb{R}} \phi(x) P_{(t'\circ\tau)(2k_o-1)}(x) dx \right\} \prod_{k=1}^{K} \langle P_{(t'\circ\tau)(2k-1)}, P_{(t'\circ\tau)(2k)} \rangle_{\mathbb{R}}$$

Recalling the definition of $t'_o$ gives $(t'_o \circ \tau)(2k_o) = 2J$, and hence

$$\Phi = \frac{1}{K!} \sum_{\tau \in \Pi_{2K}} \text{sgn}(\tau) R_{t'_o}[\tau(2k_o - 1), \tau(2k_o)] \prod_{k=1}^{K} R_{t'_o}[\tau(2k - 1), \tau(2k)]$$

(4.19)

4.4.2 \: L Even

When $L$ is even, substituting (4.17) into (4.16) and simplifying, $\Phi$ becomes

$$\Phi = \frac{1}{K!} \sum_{\tau \in \Pi_{2K}} \text{sgn}(\tau) \left\{ \prod_{k=1}^{K} \int_{\mathbb{R}^2} \phi(x) \phi(y) P_{(t'\circ\tau)(2k)}(y) P_{(t'\circ\tau)(2k-1)}(x) \text{sgn}(y - x) dy \right\}$$

(4.20)

$$= \frac{1}{K!} \sum_{\tau \in \Pi_{2K}} \text{sgn}(\tau) \prod_{k=1}^{K} \langle P_{(t'\circ\tau)(2k-1)}, P_{(t'\circ\tau)(2k)} \rangle_{\mathbb{R}}.$$
4.5 The Proof of Lemma 4.3

To prove Lemma 4.3 we set

\[ \Theta = \frac{(-i)^M}{M!} \int_{\mathbb{C}^M} \left\{ \prod_{m=1}^{M} \phi(\beta_m)\overline{\phi(\beta_m)} \operatorname{sgn} \Im(\beta_m) \right\} \det W_{i,t}^\beta \, d\lambda_{2M}(\beta). \]

From the definition of \( W_{i,t}^\beta \) we can write

\[ \det W_{i,t}^\beta = \sum_{\tau \in S_{2M}} \operatorname{sgn}(\tau) \prod_{m=1}^{M} P_{(t \circ \tau)(2m-1)}(\overline{\beta_m}) P_{(t \circ \tau)(2m)}(\beta_m). \]

Substituting this into \( \Theta \) we see

\[ \Theta = \frac{1}{M} \sum_{\tau \in S_{2M}} \operatorname{sgn}(\tau) (-i)^M \int_{\mathbb{C}^M} \left\{ \prod_{m=1}^{M} \phi(\beta_m)\overline{\phi(\beta_m)} \operatorname{sgn} \Im(\beta_m) \right\} \times \left\{ \prod_{m=1}^{M} P_{(t \circ \tau)(2m-1)}(\overline{\beta_m}) P_{(t \circ \tau)(2m)}(\beta_m) \right\} d\lambda_{2M}(\beta). \]

By Fubini’s Theorem,

\[ \Theta = \frac{1}{2M M!} \sum_{\tau \in S_{2M}} \operatorname{sgn}(\tau) \left\{ \prod_{m=1}^{M} (-2i)^{\overline{\beta}} \phi(\beta) \times P_{(t \circ \tau)(2m-1)}(\overline{\beta}) P_{(t \circ \tau)(2m)}(\beta) \right\} \operatorname{sgn} \Im(\beta) d\lambda_2(\beta) \]

\[ = \frac{1}{2M M!} \sum_{\tau \in S_{2M}} \operatorname{sgn}(\tau) \prod_{m=1}^{M} \langle P_{(t \circ \tau)(2m-1)}, P_{(t \circ \tau)(2m)} \rangle_{\mathbb{C}}, \]

which is \( \text{Pf} C_t \). But, by definition, \( t = t_\circ \), and hence \( \Theta = \text{Pf} C_{t_\circ} \) as desired.

4.6 The Proof of Corollary 2.3

Corollary 2.3 follows from the fact that if \( U \) is a \( 2J \times 2J \) antisymmetric matrix, and \( U[j,k] = 0 \) when \( j - k \equiv 0 \mod 2 \), then \( \text{Pf} U = \det A \) where \( A \) is the \( J \times J \) matrix given by \( A[j,k] = U[2j-1,2k] \). This is Lemma 8.10 of [11], or can be proven by conjugating \( U \) by an appropriate permutation matrix \( B \) so that

\[ BUB^{-1} = \begin{bmatrix} 0 & A \\ -A & 0 \end{bmatrix}. \]

Then, using well-known results about the Pfaffian, \( \text{Pf} U = \text{Pf}(BUB^{-1}) = \det A \).

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Pacific Institute for the Mathematical Sciences, Vancouver, British Columbia
email: sinclair@math.ubc.ca