COUNTABLE MARKOV PARTITIONS SUITABLE FOR THERMODYNAMIC FORMALISM

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Abstract. We study hyperbolic attractors of some dynamical systems with a-priori given countable Markov partitions. Assuming that contraction is stronger than expansion, we construct new Markov rectangles such that their cross-sections by unstable manifolds are Cantor sets of positive Lebesgue measure. Using new Markov partitions we develop thermodynamical formalism and prove exponential decay of correlations and related properties for certain Hölder functions. The results are based on the methods developed by Sarig [26]–[28].

1. Introduction

Examples of one-dimensional maps with countable Markov partitions go back to the Gauss transformation, and further developments appeared in particular in [24, 3, 31]. Beginning in [4], theorems about ergodic properties of such maps are often referred to as Folklore.

More recently an interest in such maps was motivated by works on ergodic and statistical properties of quadratic-like and Hénon-like maps. The study of such maps is typically based on various tower constructions, see in particular [16, 17, 7, 8, 32]. Power maps defined on the tower satisfy hyperbolicity and distortion estimates.

Although the following remark is not directly related to the results of our paper, it might be useful for the further work in the area under discussion.

Remark 1.1. Since numerical evidence for the existence of a strange attractor was presented in the original Hénon paper [13], rigorous results were only obtained in unspecified small neighborhoods of one-dimensional maps.

One possible approach to that problem is to prove that in a sufficiently small neighborhood of the classical Hénon values, $a_H = 1.4$, $b_H = .3$, there is a positive Lebesgue measure set $M$, such that for $(a, b) \in M$, Hénon maps $f_{a,b}$ have SRB measures with strong mixing properties.

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The main difficulty in that direction is to design a set of checkable numeric estimates which can be maintained through the induction. In the one-dimensional case such estimates were used in [22, 15, 11].

We study apriori given two-dimensional systems with countable Markov partitions satisfying hyperbolicity and distortion conditions. In [18] we proved strong mixing properties of such systems assuming distortion condition (D1), requiring boundedness of the quotient of the second derivatives over the first derivative. This condition is too strong for our purposes since it does not hold for power maps induced by quadratic and Hénon maps.

Here we return to the more general setting of [20, 21] where only boundedness of the quotient of the second derivatives over the square of first derivatives is assumed. In order to study the decay of correlations we require additionally that contraction in our models grows faster than expansion, see condition (H5) below. Condition (H5) naturally holds for power maps generated by Hénon-like maps with Jacobian less than 1. We also require distortions of our initial maps to be uniformly bounded, see condition BIV below. That is a standard requirement for maps defined on a tower.

The main new idea of the paper is to develop thermodynamic formalism by using special Markov rectangles such that their intersections with unstable manifolds are Cantor sets of positive Lebesgue measure.

2. Description of models and statement of results

2.1. Description of the model. The setting for our model is the same as in [18]–[21]. To summarize, let \( Q \) be the unit square. Let \( \xi = \{E_1, E_2, \ldots\} \) be a countable collection of full-height, closed, curvilinear rectangles in \( Q \). Hyperbolicity conditions that we will recall below imply that the left and right boundaries of \( E_i \) are graphs of smooth functions \( x^{(i)}(y) \) with \( \left| \frac{dx^{(i)}}{dy} \right| \leq \alpha \) for \( 0 < \alpha < 1 \).

Assume that each \( E_i \) lies inside a domain of definition of a \( C^2 \) diffeomorphism \( F \) which maps \( E_i \) onto its image \( S_i \subset Q \). The images \( F|_{E_i}(E_i) = f_i(E_i) = S_i \) are disjoint, full-width strips of \( Q \) which are bounded from above and below by the graphs of smooth functions \( y^i(x) \), \( \left| \frac{dy^i}{dx} \right| \leq \alpha \).

We recall geometric and hyperbolicity conditions from [21].

2.2. Geometric conditions. For \( z \in Q \), let \( \ell_z \) be the horizontal line through \( z \). We define \( \delta_z(E_i) = \text{diam}(\ell_z \cap E_i) \), \( \delta_{i,\text{max}} = \max_{z \in Q} \delta_z(E_i) \), \( \delta_{i,\text{min}} = \min_{z \in Q} \delta_z(E_i) \).

- (G1) For \( i \neq j \), \( \text{int } E_i \cap \text{int } E_j = \emptyset \), \( \text{int } S_i \cap \text{int } S_j = \emptyset \).
- (G2) \( \mu \text{mes}(Q \sim \cup_i \text{int } E_i) = 0 \), where \( \mu \text{mes} \) stands for Lebesgue measure.
- (G3) for some \( 0 < a \leq b < 1 \) and some \( \tilde{C} \geq 1 \) it holds that \( \tilde{C}^{-1} a^i \leq \delta_{i,\text{min}} \leq \delta_{i,\text{max}} \leq \tilde{C} b^i \).

2.3. Hyperbolicity conditions. Let \( J_F(z) \) be the absolute value of the Jacobian determinant of \( F \) at \( z \). There exist constants \( 0 < \alpha < 1 \) and \( K_0 > 1 \) such that for each \( i \) the map \( F(z) = f_i(z) \) for \( z \in E_i \).
satisfies

\( \text{(H1) } |F_{2x}(z)| + \alpha |F_{2y}(z)| + \alpha^2 |F_{1y}(z)| \leq \alpha |F_{1x}(z)|. \)

\( \text{(H2) } |F_{1x}(z)| - \alpha |F_{1y}(z)| \geq K_0. \)

\( \text{(H3) } |F_{1y}(z)| + \alpha |F_{2y}(z)| + \alpha^2 |F_{2x}(z)| \leq \alpha |F_{1x}(z)|. \)

\( \text{(H4) } |F_{1x}(z)| - \alpha |F_{2x}(z)| \geq J_F(z)K_0. \)

We recall some notation. Given a finite string \( i_0 \ldots i_{n-1}, \ n \geq 1, \) we define inductively

\[ E_{i_0 \ldots i_{n-1}} = E_{i_0} \cap f_{i_0}^{-1} E_{i_1 \ldots i_{n-1}}. \]

Then, each set \( E_{i_0 \ldots i_{n-1}} \) is a full height subrectangle of \( E_{i_0}. \)

Analogously, for a string \( i_{-m} \ldots i_{1} \) we define

\[ S_{i_{-m} \ldots i_1} = f_{i_1} \left(S_{i_{-m} \ldots i_2} \cap E_{i_0}\right) \]

and get that \( S_{i_{-m} \ldots i_1} \) is a full width strip in \( Q. \) It is easy to see that

\[ S_{i_{1-m} \ldots i_1} = f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_{-m}} \left(E_{i_{-m} \ldots i_1}\right) \]

and that \( f_{i_0}^{-1}(S_{i_{-m} \ldots i_1} E_{i_0}) \) is a full-width substrip of \( E_{i_0}. \)

We also define curvilinear rectangles \( R_{i_{-m} \ldots i_1,i_0 \ldots i_{n-1}} \) by

\[ R_{i_{-m} \ldots i_1,i_0 \ldots i_{n-1}} = S_{i_{-m} \ldots i_1} \cap E_{i_0 \ldots i_{n-1}} \]

If there are no negative indices then respective rectangle is full height in \( Q. \)

The following is a well known fact in hyperbolic theory, see [21].

**Proposition 2.1.** Any \( C^1 \) map \( F \) satisfying the above geometric conditions (G1)–(G3) and hyperbolicity conditions (H1)–(H4) has a "topological attractor"

\[ \Lambda = \bigcup_{\ldots i_{n},i_{1}=1}^{\infty} S_{i_{-n} \ldots i_1}. \]

The infinite intersections

\[ \bigcap_{k=1}^{\infty} S_{i_{-k} \ldots i_1} \]

define \( C^1 \) curves \( y(x), \ |dy/dx| \leq \alpha, \) which are the unstable manifolds for the points of the attractor.

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The infinite intersections

\[ \bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} R_{i_{-m} \ldots i_1,i_0 \ldots i_n} \]

define points of the attractor.
2.4. Distortion condition. We formulate certain assumptions on the second derivatives. We use the distance function
\[ d((x, y), (x_1, y_1)) = \max(|x - x_1|, |y - y_1|) \]
associated with the norm \(|v| = \max(|v_1|, |v_2|)\) on vectors \(v = (v_1, v_2)\).

Our first condition is the same as in [21]; we recall it below.
Let \(f_i(x, y) = (f_{i1}(x, y), f_{i2}(x, y))\). We use \(f_{ij}, f_{ijkl}\) to denote partial derivatives of \(f_{ij}, j = 1, 2\). We define
\[ |D^2 f_i(z)| = \max_{j=1,2,(k,l)=(x,x),(x,y),(y,y)} |f_{ijkl}(z)|. \]

We assume that there exists a constant \(C_0 > 0\) such that the following distortion condition holds:
\[ D1. \sup_{z \in E_i, i \geq 1} |D^2 f_i(z)| \delta_z(E_i) < C_0. \]

2.5. A result about systems satisfying geometric, hyperbolicity and distortion conditions. Our conditions imply the following theorem proved in [20, 21].

**Theorem 2.2.** Let \(F\) be a piecewise smooth mapping as above satisfying the geometric conditions (G1)–(G3), the hyperbolicity conditions (H1)–(H4), and the distortion condition (D1). Then, \(F\) has an SRB measure \(\mu\) supported on \(\Lambda\) whose basin has full Lebesgue measure in \(Q\). The dynamical system \((F, \mu)\) satisfies the following properties.

(a) \((F, \mu)\) is measure-theoretically isomorphic to a Bernoulli shift.

(b) \(F\) has finite entropy with respect to the measure \(\mu\), and the entropy formula holds
\[ h_\mu(F) = \int \log |D^u F|d\mu, \]
where \(D^u F(z)\) is the norm of the derivative of \(F\) in the unstable direction at \(z\).

(c) \[ h_\mu(F) = \lim_{n \to \infty} \frac{1}{n} \log |DF^n(z)|, \]
where the latter limit exists for Lebesgue almost all \(z\) and is independent of such \(z\).

2.6. Additional distortion and hyperbolicity conditions and statement of the main theorems. Properties of the function \(\phi(z) = -\log(D^u F(z))\) are important when applying thermodynamic formalism to hyperbolic attractors. We consider systems satisfying conditions of Theorem 2.2 and some extra hyperbolicity conditions, which can be used for power maps arising from Hénon-type diffeomorphisms.

We explore a general principle that can be stated as: contraction increases faster than expansion, see hyperbolicity condition (H5) below. For such systems we construct new Markov partitions such that the pullback of \(\phi(z)\) into a respective symbolic space is a locally Hölder function.
New Markov rectangles are Cantor sets, such that their one-dimensional cross-sections by \( W^u(z) \) have positive Lebesgue measure.

We consider a class of system which satisfy conditions of the Theorem 2.2 as well as the following additional assumptions.

(i) **Bounded Initial Variation** (BIV). There exists \( B_0 > 0 \) such that for all \( i \) and all \( \{z_1 = (x_1, y_1), z_2 = (x_2, y_2)\} \in E_i \), it holds

\[
| \log f_{i1x}(z_1) - \log f_{i1x}(z_2) | < B_0. \tag{5}
\]

BIV does not allow unbounded oscillations of widths for initial rectangles.

(ii) **Contraction grows faster than expansion**. We assume that there is a constant \( a_1 \) satisfying

\[
0 < a_1 < a, \tag{6}
\]

where \( a \) is from G3, such that for each \( j \), for each \( z \in E_j \), and for any vector \( v \) in the stable cone \( K^s_{\alpha_j}(z) \).

(H5) \( | Df^{-1}j \cdot v | \geq a^{-j} \cdot | v | \).

Condition (H5) means that up to a uniform factor, contractions of \( f_j \) grow faster than expansions. In particular it implies that up to a uniform factor, heights of \( S_i \cap E_k \) are smaller than widths of \( E_k \) for all \( k \leq i \).

**Remark 2.3.** Uniform hyperbolicity and distortion conditions were used in [20], [21] to extend the classical approach of [6] and to study ergodic properties of systems with countable Markov partitions. By combining (D1) and (H5) we can add methods of thermodynamic formalism.

Let \( \mathcal{H}_\gamma \) be the space of functions on \( Q \) satisfying Hölder property with exponent \( \gamma \)

\[
| \phi(x) - \phi(y) | \leq c | x - y |^\gamma.
\]

We state our main theorems.

**Theorem 2.4 (Exponential Decay of Correlations).** Let \( F \) be a piecewise smooth mapping as above, satisfying geometric conditions (G1)–(G3), hyperbolicity conditions (H1)–(H5), distortion condition (D1), and the BIV condition. Then the system \((F, \mu)\) has exponential decay of correlations for \( f \in \mathcal{H}_\gamma \) and \( g \in L^\infty(\mu) \). Namely there exists \( 0 < \eta < 1 \), \( \eta = \eta(\gamma) \), such that

\[
\left| \int f( g \circ F^n) \, d\mu - \int f \, d\mu \int g \, d\mu \right| < C(f, g)\eta^n. \tag{7}
\]

**Theorem 2.5 (Central Limit Theorem).** Let \((F, \mu)\) satisfy the assumptions of Theorem 2.4 and suppose that \( f \in \mathcal{H}_\gamma \). If \( \int f \, d\mu = 0 \) and \( f \) cannot be expressed as \( g - g \circ F \) for \( g \) continuous, then there is a positive constant \( \sigma = \sigma(f) \), such that for every \( t \in \mathbb{R} \),

\[
\lim_{n \to \infty} \mu \left\{ x \colon \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f \circ F^k(x) < t \right\} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} \, ds.
\]
3. Hölder properties of $\log(D^u F(z))$ and Markov partitions in the phase space

The key step toward the proof of Theorem 2.4 is to establish that the pullback of the function $\log D^u F$ into the respective symbolic space is Hölder continuous. Then one can follow the Ruelle-Bowen approach ([25, 9]), in particular using results of Sarig [26]–[28] to develop thermodynamic formalism for the systems under consideration. Hölder properties of the pullback of $\log D^u F$ into symbolic space follow from Hölder properties of $\log D^u F$ in the phase space.

In order to get an appropriate symbolic space, we construct a partition of a subset of positive measure $\mathcal{C} \subset \Lambda$, such that the first return map to $\mathcal{C}$ is Markov. Elements of the Markov partition of $\Lambda$ are elements of the Markov partition of $\mathcal{C}$ and their orbits before the first return.

Elements $C_i$ of the Markov partition $\mathcal{C}$ have the following property. For $z \in C_i$ the cross-section $C_i^u(z)$ of $C_i$ by $W^u(z)$ is a Cantor set of positive linear Lebesgue measure.

The Cantor sets that we construct are inscribed in curvilinear rectangles

$$R_{i_1, \ldots, i_{m-1}}.$$

Recall that $R_{i_1, \ldots, i_{m-1}, b_0 \ldots b_{n-1}}$ are bounded from above and below by arcs of unstable curves $\Gamma^u$, which are images of some pieces of the top and bottom of $Q$, and from the left and right by arcs of stable curves $\Gamma^s$ which are preimages of some pieces of the left and right boundaries of $Q$.

For $x \in R_{i_1, \ldots, i_{m-1}, b_0 \ldots b_{n-1}} \cap \Lambda$ let

$$\Gamma^s(x, R_{i_1, \ldots, i_{m-1}, b_0 \ldots b_{n-1}}) = W^s(x) \cap R_{i_1, \ldots, i_{m-1}, b_0 \ldots b_{n-1}},$$

$$\Gamma^u(x, R_{i_1, \ldots, i_{m-1}, b_0 \ldots b_{n-1}}) = W^u(x) \cap R_{i_1, \ldots, i_{m-1}, b_0 \ldots b_{n-1}}.$$

We define the height of $R_{i_1, \ldots, i_{m-1}, b_0 \ldots b_{n-1}}$ as

$$H(R_{i_1, \ldots, i_{m-1}, b_0 \ldots b_{n-1}}) = \sup_x |\Gamma^s(x, R_{i_1, \ldots, i_{m-1}, b_0 \ldots b_{n-1}})|.$$

The width $W(R_{i_1, \ldots, i_{m-1}, b_0 \ldots b_{n-1}})$ is defined similarly.

Hyperbolicity conditions imply that stable boundaries of rectangles belong to stable cones. Since standard horizontal lines belong to unstable cones, and stable and unstable cones are separated, we get the following.

For every $\epsilon_1$ there is an $\epsilon_0$ such that, if

$$H(R_{i_1, \ldots, i_{m-1}, b_0 \ldots b_{n-1}}) < \epsilon_0 w_{\min}(R_{i_1, \ldots, i_{m-1}, b_0 \ldots b_{n-1}}),$$

then, for all $m \geq 1$, $n \geq 1$, the ratio of lengths of any two unstable curves

$$\Gamma^u(x, R_{i_1, \ldots, i_{m-1}, b_0 \ldots b_{n-1}})$$

is bounded by $1 \pm \epsilon_1$. Similarly it follows from hyperbolicity conditions that, if $l$ is the length of a standard horizontal cross-section of $E_{b_0 \ldots b_{n-1}}$ through a point $x \in R_{i_1, \ldots, i_{m-1}, b_0 \ldots b_{n-1}}$, then for some $c_0$,

$$|\Gamma^u(x, R_{i_1, \ldots, i_{m-1}, b_0 \ldots b_{n-1}})| < c_0 l.$$
3.1. Admissible objects. We define the following strings of indices as admissible:

(A1) A string $\bar{a} = [i_1 \ldots i_k]$ is admissible if for each $l = 1, 2, \ldots, k-1$ it holds that

$$\sum_{m=l}^{k} i_m \geq i_{l+1}. \quad \text{(10)}$$

A rectangle $R_{i_m \ldots i_l, i_0 \ldots i_{n-1}}$ is admissible if the string $[i_m \ldots i_l i_0 \ldots i_{n-1}]$ is admissible. It follows from definition that if $R_{i_m \ldots i_l, i_0 \ldots i_{n-1}}$, $m \geq 0, n \geq 1$ is admissible, then all rectangles obtained by moving the comma in the index to the left or to the right are admissible. A one-sided sequence $i_1 i_2 \ldots i_n$ is admissible if all strings $[i_1 \ldots i_n]$ are admissible.

In particular, all strings $[i j], i \geq j$ are admissible, and thus, the respective rectangles are admissible.

Note that distortion estimates may be satisfied on non-admissible rectangles if their heights are small enough, but we ignore that possibility, and organize our construction based on condition (A1).

We estimate the variation of $\log D^u F$ on admissible two-dimensional curvilinear rectangles $R_{i_m \ldots i_l, i_0 \ldots i_{n-1}}$. For any function $a(x, y)$ the variation of $a(x, y)$ over a rectangle $R$ is defined as

$$\text{var}(a(x, y))|R = \sup_{(x_1, y_1) \in R, (x_2, y_2) \in R} |a(x_1, y_1) - a(x_2, y_2)|. \quad \text{(11)}$$

The function $\log D^u F$ is locally Hölder on admissible rectangles $R_{i_m \ldots i_l, i_0 \ldots i_{n-1}}$ if for $m \geq 0, n \geq 1$, the variation of $\log D^u F$ on $R_{i_m \ldots i_l, i_0 \ldots i_{n-1}}$ satisfies

$$\text{var}(\log D^u F)|R_{i_m \ldots i_l, i_0 \ldots i_{n-1}} < C\theta_{m, n}^{\min(n, m)} \quad \text{(12)}$$

for some $C > 0, \theta_0 < 1$.

Note that on initial rectangles $E_i$, the estimate (12) is satisfied because of BIV. The proof of the following proposition is similar to the proof of Proposition 5.1 in [18].

**Proposition 3.1.** For any admissible string $[i_m \ldots i_l i_0 \ldots i_{n-1}]$ the variation of $\log D^u F$ on $R_{i_m \ldots i_l, i_0 \ldots i_{n-1}}$ satisfies (12) with some $C$ and $\theta_0$ independent of $m, n$ and determined by hyperbolicity and distortion conditions.

Note that in [18] and [19] we proved the Hölder property of $\log D^u F$ on arbitrary rectangles $R_{i_m \ldots i_l, i_0 \ldots i_{n-1}}$. Here we prove it for admissible rectangles.

**Proof.** (a) Admissible rectangles $R_{i_m \ldots i_l, i_0 \ldots i_{n-1}}$ are bounded from above and below by some arcs of unstable curves $\Gamma^u_{i_m \ldots i_l}$ which are images of some pieces of the top and bottom of $Q$ and from left and right by some arcs of two stable curves $\Gamma^s_{i_0 \ldots i_{n-1}}$ which are preimages of some pieces of the left and right boundaries of $Q$.

Let $z_1, z_2 \in R_{i_m \ldots i_l, i_0 \ldots i_{n-1}} \cap \Lambda$. We consider two points $z_3, z_4$ such that $W^s(z_3) = W^s(z_4)$ and for which we can connect $z_1$ to $z_3$ and $z_2$ to $z_4$ along their respective
unstable manifolds. We define the following curves inside $R_{k,m-1,\ldots,i_0\ldots i_n}$,

$$\begin{align*}
\gamma_1 &= \gamma(z_1, z_3) \subset W^u(z_1), \\
\gamma_2 &= \gamma(z_2, z_4) \subset W^u(z_2), \\
\gamma_3 &= \gamma(z_3, z_4) \subset W^s(z_3).
\end{align*}$$

Now we bound each term on the right hand side of the inequality

$$|\log D^uF(z_1) - \log D^uF(z_2)| \leq |\log D^uF(z_1) - \log D^uF(z_3)|$$

(13)

$$+ |\log D^uF(z_3) - \log D^uF(z_4)|$$

$$+ |\log D^uF(z_4) - \log D^uF(z_2)|.$$ Estimates of $|\log D^uF(z_1) - \log D^uF(z_2)|$ and $|\log D^uF(z_4) - \log D^uF(z_2)|$ are the same as estimates (15) – (28) in the proof of Proposition 5.1 in [18]. Then we get

(14) $$|\log D^uF(z_1) - \log D^uF(z_3)| < C_2 \frac{1}{K_0^n}.$$ Similarly,

(15) $$|\log D^uF(z_2) - \log D^uF(z_4)| < C_2 \frac{1}{K_0^n}.$$ (b) The second part of the proof, depending on $m$, follows again the ideas in [18] and [19] but also utilizes condition (H5). We are left with estimating the difference

(16) $$|\log D^uF(z_3) - \log D^uF(z_4)|.$$ Note that the BIV condition implies that the above difference is uniformly bounded on full-height rectangles. From [21] we get that the hyperbolicity conditions imply that any unit vector in $K^u_i$ at a point $z \in E_i$, in particular a tangent vector to $W^u(z)$, has coordinates $(1, a_z)$ with $|a_z| < \alpha$. Thus we need to estimate

(17) $$\log |F_{1x}(z_3) + a_{z_3}, F_{1y}(z_3)| - \log |F_{1x}(z_4) - a_{z_1}, F_{1y}(z_4)|.$$ Now we are moving along $\gamma_3 \subset W^s(z_3)$ connecting $z_3$ and $z_4$. We cover $\gamma_3$ by rectangles $\tilde{R}_k$ for which the widths $\Delta x$ and lengths $\Delta y$ satisfy $|\Delta x| < \alpha |\Delta y|$. As in [18], we get (17) by estimating differences

(18) $$|\log F_{1x}(\tilde{z}) - \log F_{1x}(\tilde{z}')|$$

for $\tilde{z}, \tilde{z}' \in \tilde{R}_k \cap W^s(z_3)$, and

(19) $$|a_{z_3} - a_{z_1}|.$$ To estimate (18) we use the mean value theorem and get on each rectangle $\tilde{R}_k$ an estimate not exceeding

(20) $$\text{Const} \sup_{z \in \tilde{R}_k} \frac{|f_{ij}(z)|}{|f_{1x}(z)|} \Delta y.$$
Let $\Gamma_3 = W^s(z_3) \cap R_{i_1, i_0}$. The sum of the contributions from (20) is bounded by

$$\text{Const} \sup_{z \in R_{i_1, i_0}} \frac{|f_{ij}(z)|}{|f_{1x}(z)|} |\gamma_3|.$$  

(21)

Since the distortion condition (D1) is expressed using the width of $E_{i_0}$, we use condition (H5).

On an admissible rectangle $R_{i_{-m} \ldots i_{-1}, i_0 \ldots i_{n-1}}$, the string $[i_{-m} \ldots i_{-1} i_0]$ is admissible. Let $h_{\text{max}}$ be the maximal height of $S_{i_{-m} \ldots i_{-1}}$ and let $w_{\text{min}}$ be the minimal width of $E_{i_0}$. From condition (H5) we know that contraction of $f_i$ is stronger than $a_1^{i_0}$. Since the rectangle is admissible, the sum of the indices satisfies $i_{-m} + \ldots + i_{-1} \geq i_0$. Since contraction of the composition is stronger than $a_1^{i_0}$, it follows that

$$\frac{h_{\text{max}}}{w_{\text{min}}} < C \left( \frac{a_1}{a} \right)^{i_0}.$$  

(22)

As each index is at least 1 we get that $i_0 \geq m$ and thus

$$h_{\text{max}} < \left( \frac{a_1}{a} \right)^m C^{-1} w_{\text{min}}.$$  

(23)

Therefore, the heights of rectangles $R_{i_{-m} \ldots i_{-1}, i_0 \ldots i_{n-1}}$ decay exponentially comparatively to the width of $E_{i_0}$. Since $|\gamma_3| < h_{\text{max}}$ and $w_{\text{min}} < \delta_z(E_{i_0})$ for any $z \in E_{i_0}$, we can apply (D1) and obtain the following bound for the sum of the contributions from (18),

$$C_3 \left( \frac{a_1}{a} \right)^m.$$  

(24)

We estimate (19) as in [18], [19]. We assume by induction

$$|a_{z_3} - a_{z_4}| < c_1 \theta_1^m.$$  

(25)

As in [19] one can assume by taking, if needed, instead of $F$ some power of $F$,

$$\frac{1}{K_0^2} + a^2 < 1.$$  

(26)

Note that differently from [19] the variation of $\log D^u F$ along stable manifolds inside admissible rectangles is controlled not by using bounded distortions of inverse maps, but from (23) and (26).

With that modification we prove like in Lemma 5.2 from [18]

$$|a_{F(z_3)} - a_{F(z_4)}| < c_1 \theta_1^{m+1},$$  

(27)

where

$$\max \left\{ \frac{1}{K_0^2} + a^2 \cdot \frac{a_1}{a} \right\} < \theta_1 < 1.$$  

(28)

Combining (24) and (27) gives us

$$|\log D^u F(z_3) - \log D^u F(z_4)| < C_4 \theta_1^m.$$  

(29)
Finally, combining (14), (15), and (29) concludes the proof of Proposition 3.1, if we take \( \theta_0 < 1 \) satisfying

\[
\theta_0 > \max \left\{ \frac{1}{K_0}, \theta_1 \right\}.
\]

3.2. **Construction of full height Cantor sets.** We define full height Cantor sets \( C_n \) inside full height rectangles \( E_n, n \geq 1 \) by

\[
C_n = \bigcap_{m=1}^{\infty} \bigcup_{k=1}^{m} E_{i_0...i_{k-1}},
\]

where \( i_0 = n, k \geq 1 \) and all \([i_0...i_{k-1}]\) are admissible strings.

As an example let us consider several rectangles with indices starting from 1. It follows from the definition that \([11]\) is admissible and \([1i], i > 1\) are not. Then \( R_{[11]} \) is the only defining rectangle of order two for the Cantor set \( C_1 \) and that the other \( R_{[1i]} \) are gaps. The next defining rectangles of order three for \( C_1 \) are \( R_{[111]}, R_{[112]} \), of order four \( R_{[1111]}, R_{[1112]}, R_{[1113]}, R_{[1121]}, R_{[1122]}, R_{[1123]}, R_{[1124]} \) and so on.

As each index is at least 1 we get from the definition of admissible rectangles that inadmissible indices satisfy \( i_N > N \). Geometric condition (G3) implies that on any unstable manifold, the relative measure of the union of rectangles with indices greater than \( N \) decays exponentially. Then uniformly bounded distortion implies that the total relative linear measure of gaps in an unstable manifold of any defining rectangle of order \( N \) has uniform exponential decay.

This implies the following corollary.

**Corollary 3.2.** There is a \( c_0 > 0 \) such that for any initial full height rectangle \( E_n \) and respective full height Cantor set \( C_n \) constructed inside \( E_n \) and for any \( z \in C_n \cap \Lambda \) the relative linear measure of \( C_n \) in \( W^u(z, E_n) \) is greater than \( c_0 \). Moreover that relative measure tends to one when \( n \to \infty \).

Also as in [18, Remark 5.10], we get the following corollary from distortion estimates.

**Corollary 3.3.** Let \( E_{i_0...i_{n-1}} \) be a full height admissible rectangle of order \( n \). Then for any two points \( z_1, z_2 \in E_{i_0...i_{n-1}} \cap \Lambda \), it holds that

\[
\frac{|W^u(z_1, E_{i_0...i_{n-1}})|}{|W^u(z_2, E_{i_0...i_{n-1}})|} < c.
\]

As Corollary 3.3 is valid for all defining rectangles we get:

**Corollary 3.4.** Let \( C_n \) be the full height Cantor set constructed inside \( E_n \), let \( z \in C_n \cap \Lambda \), and let \( |W^u(z, C_n)| \) be the linear Lebesgue measure of \( C_n \) in \( W^u(z, E_n) \). Then for any two points \( z_1, z_2 \in C_n \cap \Lambda \), it holds that

\[
\frac{|W^u(z_1, C_n)|}{|W^u(z_2, C_n)|} < c.
\]
3.3. **Markov properties of** $C_n$. Every Cantor set $C_i$ is determined by its defining rectangles and equivalently by its gaps. Defining rectangles are labeled by admissible strings $[i_1 i_2 \ldots i_n]$ satisfying (A1). Gaps are labeled by nonadmissible strings $[i_1 i_2 \ldots i_n]$], where $[i_1 i_2 \ldots i_n]$ is admissible and $j > i_1 + i_2 + \ldots + i_n$.

For example gaps of $C_3$ are:

- $E_{3k}$, $k > 3$ - gaps of order 1,
- $E_{31k}$, $k > 4$, $E_{32k}$, $k > 5$, $E_{33k}$, $k > 6$ - gaps of order 2,
- $E_{311k}$, $k > 5$ - gaps of order 3, and so on.

The following example illustrates Markov relations between Cantor sets $C_i$.

Consider an admissible rectangle $E_{112}$. Then $F(E_{112}) = R_{11,2}$ is a subrectangle of a non-admissible rectangle $E_{12}$. However, because of (H5), $F^2(E_{112}) = R_{11,2}$ has height less than the width of $E_2$.

We will use the notation

$$C_{112} = C_1 \cap E_{112}$$

and in general,

$$C_{i_1 \ldots i_n} = C_{i_1} \cap E_{i_1 \ldots i_n},$$

$$C_{i_1 \ldots i_k, i_{k+1} \ldots i_l} = C_{i_{k+1} \ldots i_l} \cap R_{i_1 \ldots i_k, i_{k+1} \ldots i_l}.$$  

We have that

$$F^2(C_{112}) \supset C_{11,2}.$$  

As the sum of indices in $[112]$ is greater than 2, the inclusion in (34) is not an equality. Namely strings [23] and [24] are not admissible, but [1123] and [1124] are admissible. The image $F^2(C_{112})$ covers the respective slice of $C_2$ and also covers some parts of gaps $C_{23}$ and $C_{24}$ used in the construction of the Cantor set $C_2$.

Consider the union of all full height Cantor sets $C_n$,

$$\mathcal{C} = \bigcup_{n=1}^{\infty} C_n.$$  

For $z \in C_{i_1 \ldots i_n}$ let us denote $W^u(z, C_{i_1 \ldots i_n}) = C_{i_1 \ldots i_n} \cap W^u(z, E_{i_1 \ldots i_n})$.

Let $T$ be the first return map to $\mathcal{C}$ generated by $F$. Similarly to (34) above we get the following Markov properties.

**Proposition 3.5.** If $[i_1 \ldots i_n]$ is admissible, $z \in C_{i_1 \ldots i_n}$, $T(z) \in C_{i_n}$, then

$$T(W^u(z, C_{i_1 \ldots i_n})) \supset W^u(T(z), C_{i_n}),$$  

$$T(W^s(z, C_{i_1 \ldots i_n})) \subset W^s(T(z), C_{i_{n-1}, i_n}).$$
3.4. Estimates of measure. Let $T_1$ be the first return map to $C_1 \subset \mathcal{E}$. Next we estimate the measure of points in $C_1$ which return to $C_1$ after at least $n$ iterates of $F$.

**Proposition 3.6.** Let $B_n$ be the set of points in the domain of $T_1$ such that the return time for $z \in B_n$ is greater than $n$. For some $C > 0$ and $0 < \beta < 1$,

$$\mu(B_n) < C\beta^n. \quad (38)$$

*Proof.* We begin by proving (38) for the first return map $T$ onto $\mathcal{E}$. Suppose $x = (x_0, x_1, x_2, \ldots) \in \mathcal{E}$, $y = Fx = (x_1, x_2, x_3, \ldots) \notin \mathcal{E}$, and $F^n x \in \mathcal{E}$ is the first return. As $y \notin \mathcal{E}$, $y_k$ such that $y_k = x_{k+1} > y_0 + \ldots + y_{k-1} = x_1 + \ldots + x_k$. If $k \geq n$, then

$$x_{k+1} > x_1 + \ldots + x_k \geq x_n + \ldots + x_k$$

which contradicts that $F^n(x) \in \mathcal{E}$.

Suppose $k < n$ satisfies

$$x_1 + \ldots + x_k < x_{k+1}.$$ Then $x_{k+1} \geq k + 1$, because each coordinate is at least 1. As all images $F^k x$, $1 \leq k < n$, do not belong to $\mathcal{E}$, there is a coordinate $x_N$, $N \geq k$ such that

$$x_k + \ldots + x_N < x_{N+1}.$$ If $N \geq n$, then

$$x_n + \ldots + x_N < x_{N+1}$$

and we get a contradiction as above to $F^n x \in \mathcal{E}$. So $N < n$ and we get

$$N + 1 = 1 + (k + 1) + (N - k - 1) \leq x_k + x_{k+1} + \ldots + x_N < x_{N+1}. \quad (39)$$

Proceeding as above we get

$$x_n \geq n. \quad (40)$$

As widths of $E_i$ decay exponentially, the measure of the collection of $x$ satisfying (40) decays exponentially. Since the measure of $\mathcal{E}$ is positive, we get that the measure of points which do not return to $\mathcal{E}$ after $n$ iterates is less than

$$C_1 \beta^n \quad (41)$$

for some $C_1 > 0$ and $0 < \beta_1 < 1$.

Next we note that if $z \in C_1$ is mapped by $T$ into $C_i$, then, because all transitions from $C_i$ to $C_1$ are admissible, $z$ will be mapped into $C_1$ by the next iterate of $F$. Therefore points which do not return into $C_1$ after $n$ iterates are subdivided into two subsets: points which did not return into $\mathcal{E}$ after $n$ iterates and points which returned into $C_i$, $i > 1$, after $n - 1$ iterates and at the next iterate were not mapped into $C_1$. Because of uniformly bounded distortion the measure of the second set is less than

$$C_1 \beta^{n-1}(1 - \gamma_0), \quad (42)$$

where $0 < \gamma_0 < 1$. Then (38) follows from (41) and (42) if we take $C = 2C_1$ and $\beta = \max(\beta_1, 1 - \gamma_0)$.

This concludes the proof of Proposition 3.6. \qed
3.5. First return maps and respective transition matrices. Note that on every unstable leaf, relative measures of $C_i$ inside $E_i$ are uniformly bounded away from 0. Together with uniformly bounded distortion it implies that in the orbits of the first return map each gap is substituted by a union of new gaps and a Cantor set of relative measure greater than some uniform $c_0 > 0$. Then at the end of that construction we get, up to a set of measure zero,

\[ C_i = \bigcup_k \left( T^{-1}(TC_i \cap C_k) = C_{ik} \right). \tag{43} \]

Note that $C_{ik}$ can be unions of several Cantor sets which belong to disjoint full height rectangles. For example, consider $C_{1113} \subset E_{1113}$. Admissible rectangle $E_{1113}$ is mapped as follows:

\[ E_{1113} \rightarrow E_{113} \rightarrow E_{13} \rightarrow E_3. \]

As $E_{113}, E_{13}$ are inadmissible, we get that $T = F^3$ maps $C_{1113}$ onto $C_3$ in a Markov way. Similarly $T = F^3$ maps $C_{1123}$ onto $C_3$ in a Markov way. The correct labeling is provided by respective strings of the original alphabet starting width $i$ and ending with $k$.

To get an authentic Markov partition which generates a transition matrix of 0’s and 1’s we partition each $C_{j_i}$ into subsets

\[ C_{i_1i_2...i_{n-1}}, \tag{44} \]

where

\[ i_0 = j_0, i_1, \ldots, i_{n-1} = j_1 \tag{45} \]

is admissible, and for all $k > 0$

\[ i_k, \ldots, i_{n-1} \tag{46} \]

are not admissible. In other words the first return map $T$ maps $C_{i_1i_2...i_{n-1}}$ onto a full width substrip of $C_{i_{n-1}}$ and all intermediate images of $C_{i_1i_2...i_{n-1}}$ belong to various gaps.

As in the proof of (38) we get that the length of the above strings starting from $j_0$ and ending with $j_1$ is at most $j_1 - j_0 + 2$. The union of $C_{j_1i_1...i_{n-2}j_1}$ forms a Markov partition

\[ \mathcal{MP} = \{C_{j_0i_1...i_{n-2}j_1}\}, \tag{47} \]

of ‘C’. Using Proposition 3.5 we get

**Lemma 3.7.** To any one-sided $T$-admissible sequence of transitions

\[ C_{j_0i_1...i_{n-2}j_1} \rightarrow C_{j_{i_1}i_2...i_{n-1}j_2} \rightarrow \ldots \]

corresponds a unique one-sided sequence of the original alphabet

\[ j_0, i_1^1, \ldots, i_{n-1}^1 = j_1, i_1^2, \ldots, i_{n-1}^2 = \ldots \tag{48} \]

such that

\[ C_{j_{i_1}i_2...i_{n-1}j_1} \cap T^{-1}(C_{j_{i_2}i_3...i_{n-1}j_2}) \cap \ldots \]

coincides with the stable manifold labeled by (48).
The union of elements of $\mathcal{MP}$ and all intermediate iterates of $C_{j_{0}i_{1}...i_{n-2}j_{1}}$ form a tower over $\mathcal{C}$. Elements of this tower form a Markov partition of the attractor $\Lambda$.

Up to a set of $\mu$ measure zero, any point of the attractor is uniquely labeled by a two-sided sequence of admissible transitions

$$... \rightarrow C_{j_{-1}i_{-1}...i_{n-2}j_{0}} \rightarrow C_{j_{0}i_{1}...i_{n-2}j_{1}} \rightarrow C_{j_{1}i_{2}...i_{n-2}j_{2}} \rightarrow ...$$

We consider a new alphabet $\Omega$ corresponding to the elements of the tower from Lemma 3.7 and get a subshift

$$\left(\Omega, X, \sigma\right).$$

Recall that a subshift is topologically mixing if for any states $a$ and $b$ there is $n(a, b)$ such that for $n \geq n(a, b)$ there is an admissible word of length $n$ starting from $a$ ending with $b$.

We will need the following proposition.

**Proposition 3.8.** Subshift $\left(\Omega, X, \sigma\right)$ is topologically mixing.

Note that although our original map is clearly topologically mixing, elements of the Markov partition are Cantor sets, so the statement is not obvious.

**Proof of Proposition 3.8.** By construction, any Cantor set $C$ which coincides with some element of the tower is mapped by some iterate of $F$ onto a full width substrip of some Cantor set $C_i$. As the image of any $C_i$ (including $C_1$) contains a full with substrip of $C_1$, we get that all consecutive images of $C_i$ have Markov intersections with $C_1$.

It remains to prove that for any element $\Delta$ of the tower there is an $n(\Delta)$ such that $F^n C_1$ has Markov intersection with $\Delta$ for $n > n(\Delta)$. By construction any $\Delta = F^{k(\Delta)} P$ where $P$ is a full height Cantor subrectangle of some $C_i$. So it is enough to prove that $F^n C_1$ intersects $C_i$ for $n > n(i)$. But $C_1$ contains a full height Cantor subset $C_{11...1i}$ and all images of $C_1$ have Markov intersections with $C_{11...1i}$. That proves Proposition 3.8.

Next we consider the first return map $T_1$ induced by $T$ on $C_1$. Consider the Markov partition $\mathcal{MP}_1$ of $C_1$

$$\mathcal{MP}_1 = \{C_{1i_1...i_{m1}}\}$$

generated on $C_1$ by $T_1$. By construction $T_1$ maps its domains (which are full height Cantor sets) onto full width substrips of $C_1$. Therefore the transition matrix corresponding to the map $T_1$ on $\mathcal{MP}_1$ consists of all 1’s.

**Remark.** [1] will correspond to our choice of state $[a]$ in the symbolic dynamics in later sections.

4. **Thermodynamic formalism, reduction of Theorems 2.4 and 2.5 to results for functions defined on one-sided sequences**

4.1. **Reduction arguments.** By first reducing to functions defined on one-sided sequences, we will show that our transfer operator (53) has the spectral gap property (56) on a particular Banach Space. This property implies exponential
decay of correlations and Central Limit Theorem for functions defined on one-sided sequences. Then we can extend these results to certain functions defined on two-sided sequences, Theorems 2.4 and 2.5, respectively. This exchange between the two settings is a consequence of the reduction from two-sided shifts to one-sided shifts following from the classical arguments of Ruelle and Bowen, see [25], [9]. In the case of an infinite alphabet, detailed reduction arguments can be found in sections 4 and 5 of [33].

If we restrict our consideration to Hölder functions on \( \mathcal{Q} \), then we are left with the proof of the spectral gap property (SGP) of the transfer operator acting on a suitable space \( \mathcal{L} \) of functions defined on one-sided sequences of the alphabet \( \Omega \). We do this in the next sections, following [10] and [28].

4.2. Thermodynamic formalism. Now by following [25], [9] we develop thermodynamic formalism on the space of one-sided sequences for the function \( \Phi(x, y) = -\log|D^u F| \).

Let \( \mathcal{E}_i \subset E_i \) be an element of the Markov partition \( \mathcal{M} \mathcal{P} \). For each \( \mathcal{E}_i \) we fix some unstable manifold \( W^u_0 \), and to any \((x, y) \in \mathcal{E}_i \) we let correspond \((x, y_0) = W^s(x, y) \cap W^u_0 \). We define

\[
    u(x, y) = \sum_{k=0}^{\infty} \Phi(F^k(x, y)) - \Phi(F^k(x, y_0)).
\]

Then we can construct a Hölder function on one-sided sequences cohomologous to \( \Phi(x, y) \) in the following way,

\[
    \phi(x) = \Phi(x, y) + u(F(x, y)) - u(x, y).
\]

We call \( \phi \) the potential.

The transfer operator \( L_\phi \) is defined as

\[
    (L_\phi f)(x) = \sum_{F(y) = x} e^{\phi(y)} f(y).
\]

In the next several sections we consider the space of functions on one-sided sequences for which we can prove the spectral gap property for \( L_\phi \). We denote by \( X \) the space of one-sided sequences. From this point forward, points \( x, y \) will be one-sided sequences belonging to \( X \).

4.3. Induced system. Just as in general case, we get SGP as a result of a particular induction procedure, see [10], [28], and references to earlier works in [28]. In our setting we induce on \( C_1 \); here our \([a]\) is \([1]\).

The induced system on \([1]\) is \( F_1 : X_1 \to X_1 \) where

\[
    X_1 := \{ x \in X \mid x_0 = 1, x_i = 1 \text{ infinitely often} \}
\]

and

\[
    F_1(x) := F^{\varphi_1(x)},
\]

for

\[
    \varphi_1(x) := [1][x_0 = 1 \mid n \geq 1 : x_n = 1].
\]
The resulting transformation can be given the structure of a Markov shift as follows. Let
\[ \mathcal{S} := \{[1, \xi_2, ..., \xi_{n-2}, 1] : 2 \leq i \leq n - 2, \xi_i \neq 1 \} \]
and let \( F^\phi : \mathcal{X} \to \mathcal{X} \) denote the left shift on \( \mathcal{X} = (\mathcal{S})^\mathbb{N} \). Then \( F^\phi \) is topologically conjugate to \( F_1 \). The conjugacy \( \pi : \mathcal{X} \to \mathcal{X}_1 \) is given by
\[
\pi([1, \xi^0_0, 1], [1, \xi^1_1, 1], ...):=(1, \xi^0_0, 1, \xi^1_1, ...).
\]
Let
\[ \phi : \mathcal{X} \to \mathbb{R} \]
the induced potential.

4.4. Gurevich pressure. We introduce a few preliminary definitions and results. Let \( \phi_n(x) = \sum_{k=0}^{n-1} \phi \circ F^k(x) \), where \( x = (x_0, x_1, ...) \). Let
\[ Z_n(\phi) = \sum_{\{x : T^n x = x, x_0 = 1\}} e^{\phi_n(x)} . \]
Then the limit, called the Gurevich pressure,
\[ P_G(\phi) = \lim_{n \to \infty} \frac{1}{n} \log Z_n(\phi) \]
exists, see [26], and we can calculate it explicitly in our setting.

Let \( T_1 \) be the first return map to \( C_1 \). Each periodic orbit of \( T_1 \) is contained in some admissible cylinder of \( F \), also periodic but which has, in general, a larger period. Moreover there are \( F \)-strings of arbitrary large periods which correspond to a given \( T_1 \) period. Admissible cylinders of the same \( T_1 \) periods but with different \( T_1 \) labels do not intersect.

Proposition 3.1 and uniformly bounded distortions of \( D^u F \) imply that the contribution to \( P_G(\phi) \) from each periodic \( T_1 \) orbit differs from the length of the horizontal cross-section of the respective two-dimensional cylinder by a uniformly bounded factor. That implies that the quantities \( Z_n(\phi) \) are uniformly bounded from above.

As the measure of the Cantor set \( C_1 \) is positive, we get that \( Z_n(\phi) \) are uniformly bounded from below. Thus,
\[ P_G(\phi) = 0. \]

REMARK. This calculation works for any \( C_i \), so as in the general case (see [28]) the Gurevich pressure is independent of the choice of partition set \( [a] \).

4.5. Spectral gap. Following [10], [28] we would like to show that \( L_\phi \) has spectral gap on the appropriate Banach space \( \mathcal{L} \), defined below in (64). The implications of such a property are as follows.

If \( L_\phi \) has spectral gap then it can be written as
\[ L_\phi = \lambda P + N, \]
where
\[ \lambda > e^{P_G(\phi)}, \quad PN = NP = 0, \quad P^2 = P \]
and the spectral radius, \( \rho \), of \( N \) is less than \( \lambda \). Since \( \rho < \lambda \),
\[
|| \lambda^{-n} L_{\phi}^n - \Pi ||_{\mathcal{L}} = \lambda^{-n} || N^n ||_{\mathcal{L}} \to 0
\]
exponentially fast as \( n \to \infty \). In our setting,
\[
\lambda = e^{P_G(\phi)} = 1
\]
and
\[
P f = h \int f \, d\nu,
\]
where \( h \) is the eigenfunction of \( L_{\phi} \) and \( \nu \) is the eigenmeasure of \( L_{\phi}^* \).

Following [10], [28] we introduce the \( a \)-discriminant

\[
\Delta_a[\phi] := \sup_{p \in \mathbb{R}} \{ P_G(\phi + p) \mid P_G(\phi + p) < \infty \}.
\]

The Discriminant Theorem 6.7 from [28] gives necessary and sufficient conditions for the spectral gap based on certain properties of the \( a \)-discriminant. Specifically, it links the strict positivity of the discriminant to the spectral gap property. It involves the Gurevich pressure evaluated with respect to the induced system. We state one of the properties relevant to our setting.

**Proposition 4.1.** Let \( X \) be a topologically mixing TMS and suppose \( \phi : X \to \mathbb{R} \) is a weakly Hölder continuous function such that \( P_G(\phi) < \infty \). If for some state \( a \),
\[
\Delta_a[\phi] > 0,
\]
then \( \phi \) has the SGP on the Banach space \( \mathcal{L} \) defined below in (64).

The following results prove \( \Delta_1[\phi] > 0 \) by showing that for sufficiently small \( p \),
\[
0 < P_G(\phi + p) < \infty.
\]

**Proposition 4.2.** For sufficiently small \( p > 0 \), \( P_G(\phi + p) < \infty \).

**Proof.** Recall that

\[
P_G(\phi + p) = \lim_{n \to \infty} \frac{1}{n} \log Z_n(\phi + p).
\]

We begin by calculating \( Z_1(\phi + p) \):

\[
Z_1(\phi + p) = \sum_{\{x \mid T x = x; \, x_0 = 1\}} e^{\phi + p} = \sum_{n=1}^{\infty} \sum_{\{x \mid T^n x = x; \, x_0 = 1\}} e^{\phi} e^{pn}
\]

\[
= \sum_{n=1}^{\infty} e^{pn} \sum_{\{x \mid T^n x = x; \, x_0 = 1\}} e^{\phi}
\]

\[
\leq C_\phi \sum_{n=1}^{\infty} e^{pn} \beta^n = C_\phi M_1 < \infty
\]

Here \( \beta \) comes from the estimate in Proposition 3.6 and we use \( p < \log \left( \frac{1}{\beta} \right) \). Constant \( C_\phi \) depends on constant \( C \) from the same proposition, on the uniform distortion bounds and on the uniform bound of measures of cross-sections of the Cantor set \( C_1 \) by unstable manifolds.
We define a sum similar to (54),
\[
Z'_n(\phi) = \sum_{\tilde{a}_n} e^{\tilde{a}_n \phi(x)},
\]
where $\tilde{a}_n = \lfloor i_1 \ldots i_{n-1} \rfloor$. The definition of $Z'_n(\phi)$ implies
\[
Z'_{n+m}(\phi) \leq Z'_n(\phi) Z'_m(\phi).
\]

Also from definitions of $Z_n(\phi)$ and of $Z'_n(\phi)$ and from uniformly bounded distortions we get
\[
Z_n(\phi) \leq Z'_n(\phi) \leq d Z_n(\phi).
\]
for some constant $d$. Combining (60), (62), and (63),
\[
Z_n(\phi + p) \leq Z'_n(\phi + p) \leq \left( Z'_1(\phi + p) \right)^n \leq C \left( Z_1(\phi + p) \right)^n \leq d^n C^n M^n_1,
\]
and thus,
\[
P_G(\phi + p) = \lim_{n \to \infty} \frac{1}{n} \log Z_n(\phi + p) \leq \lim_{n \to \infty} \frac{1}{n} \log(d^n C^n M^n_1) = \log d + \log C_\phi + \log M_1 < \infty.
\]

**Proposition 4.3.** For $p > 0$ as in Proposition 4.2, $P_G(\phi + p) > 0$.

**Proof.** We combine the following properties from [28] with the fact that in our setting $P_G(\phi) = 0$ for $\phi(x)$ cohomologous to $\Phi(x, y)$.

i. $\phi \geq \phi + p, \forall p \in \mathbb{R}^+.$

ii. If $\phi \leq \psi$, then $P_G(\phi) \leq P_G(\psi)$.

iii. $P_G(\phi + p) = P_G(\phi) + p, \forall p \in \mathbb{R}.$

iv. $P_G(\phi) = 0 \iff P_G(\phi + p) = 0.$

For our $p > 0$, $P_G(\phi + p) \geq P_G(\phi) + p = P_G(\phi) + p > 0.$

4.6. **Banach space.** For all $x, y \in X$, let
\[
t(x, y) = \min\{n : x_n \neq y_n\}
\]
\[
s_1(x, y) = \#\{0 \leq i \leq t(x, y) - 1 : x_i = y_i = 1\}.
\]
Let [1] be the collection of one-sided sequences, $x = .x_0 x_1 \ldots$, such that $x_0 = 1$.

As proved in [10] there is a positive function $h_0 : \mathbb{Z}_+ \to \mathbb{R}$ with the following properties.

Consider the set of continuous functions $\{ f : \|f\|_{\mathcal{L}} < \infty \}$, where
\[
\|f\|_{\mathcal{L}} = \sup_{b \in \mathbb{Z}_+} \frac{1}{h_0(b)} \left[ \sup_{x \in [b]} |f(x)| + \sup_{(x, y) \in [b] : x \neq y} \frac{|f(x) - f(y)|}{\theta_{s_1(x, y)}} \right].
\]

Then $\mathcal{L}$ is an $L_\phi$-invariant Banach space, and $L_\phi$ on $\mathcal{L}$ is a bounded operator with spectral gap. Additionally the eigenfunction $h$ of Ruelle operator belongs to $\mathcal{L}$ and for any bounded Hölder function $\psi$ it holds that
\[
\psi h \in \mathcal{L}.
\]
Note that bounded Hölder functions belong to \( \mathcal{L} \).

It follows from Propositions 4.2 and 4.3 that the discriminant is strictly positive, and thus, by Proposition 4.1, \( L_\phi \) has spectral gap on the Banach Space \( \mathcal{L} \). As in \([10], [28]\) this implies that \((\sigma, \mu_\phi)\) has exponential decay of correlations.

**Theorem 4.4.** For \( \sigma \) a one-sided full shift, consider \( \phi \) the potential defined in (52), and let \( \mu_\phi \) be the respective invariant measure. Then \((\sigma, \mu_\phi)\) has exponential decay of correlations for bounded Hölder functions \( f \) and \( g \in L^\infty(\mu_\phi) \). Namely there exists \( 0 < \eta_1 < 1 \) such that

\[
\int f(g \circ \sigma^n) \, d\mu_\phi - \int f \, d\mu_\phi \int g \, d\mu_\phi < C \| g \|_\infty \| fh \|_{\mathcal{L}} \eta^n_1.
\]  

Note that \( fh \in \mathcal{L} \) because of (65). As in \([10], [28]\), the subsequent result follows.

**Theorem 4.5** (Central Limit Theorem for one-sided shift). Let \((\sigma, \mu_\phi)\) satisfy the assumptions of Theorem 4.4 and suppose that \( f \in \mathcal{L} \). If \( \int f \, d\mu_\phi = 0 \) and \( f \) cannot be expressed as \( g - g \circ \sigma \) for \( g \) continuous, then there is a positive, finite constant \( d = d(f) \) such that for every \( t \in \mathbb{R} \),

\[
\lim_{n \to \infty} \mu_\phi \left\{ x : \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f \circ \sigma^k(x) < t \right\} = \frac{1}{\sqrt{2\pi d^2}} \int_{-t}^{\infty} e^{-\frac{x^2}{2d^2}} \, ds.
\]

**Remark.** It follows from Theorem 1.1 part \( (d) \) in \([10]\) that, for \( g \) bounded and Hölder continuous, \( P(\phi + \tau g) \) is real analytic in a neighborhood of 0.

**4.7. Exponential decay of correlations and Central Limit Theorem for functions of two variables.** One can follow arguments of \([33, \text{Section 4}]\) to reduce the estimate of the \( n \)-th correlation for functions defined on the two-sided shift to the following estimate for functions defined on the one-sided shift,

\[
| L^{n-2k}_\phi(L^{2k}_\phi(f_k h)) - (\int L^{2k}_\phi(f_k h) \, dm) h |,
\]

where \( h \, dm = dv \) and \( v \) is the eigenmeasure of \( L^*_\phi \) as in (57). Here \( 2k < n \), and \( f_k \) is a piecewise constant approximation of \( f \) on cylinders of length \( k \). Thus \( f_k \) is a Hölder function bounded by \( \max |f| \). From (65) we get \( f_k h \in \mathcal{L} \).

As in \([33, \text{Section 4}]\) we get that norms \( \| L^{2k}_\phi(f_k h) \|_{\mathcal{L}} \) are uniformly bounded by a constant which only depends on \( \max |f| \). From (67) we get an estimate similar to (66) but with a different constant and a different \( 0 < \eta < 1 \). That proves Theorem 2.4.

Theorem 2.5 follows from arguments in \([33, \text{Section 5}]\) regarding a result referred to as Theorem \([G]\) from \([12]\). Using the spectral gap property, one obtains the following estimate,

\[
\int | L^i_\phi(f_0 h) | \, dm \leq C_0 \| t^i_\phi(f_0 h) \|_{\mathcal{L}} \leq C_0 \eta_0 \| f_0 h \|_{\mathcal{L}},
\]

where, as above, \( f_0 \) is a bounded Hölder approximation of \( f \) and \( 0 < \eta_0 < 1 \). Theorem 2.5 relies on showing that the key assumption in Theorem \([G]\) holds—finiteness of the sum of the \( L^2 \)-norms of the relevant conditional expectations.
Showing that this assumptions holds reduces to showing that the sum of estimate (68) is bounded, and thus, we just need that $f_0 h \in \mathcal{L}$. This again follows from (65).

4.8. **Concluding remarks.** The study of countable Markov partitions in the 1980’s originated in particular from the work of Roy Adler [4]. His work motivated the use of countable Markov partitions as a tool for studying one-dimensional dynamics with critical points, and subsequently, Hénon-like systems.

The first author keeps warmest memories of his visit in 1990 to IBM Thomas J Watson Research Center, when he worked within the wonderful group directed by Roy Adler.

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**REFERENCES**

[1] J. Aaronson and M. Denker, Ergodic local limit theorems for Gibbs-Markov maps, preprint, 1996.
[2] J. Aaronson, M. Denker and M. Urbański, Ergodic Theory for Markov fibered systems and parabolic rational maps, *Trans. Amer. Math. Soc.*, **337** (1993), 495–548.
[3] R. Adler, F-expansions revisited, in *Recent Advances in Topological Dynamics* (Proc. Conf., Yale Univ., New Haven, Conn., 1972; in honor of Gustav Arnold Hedlund), Lecture Notes in Math., **318**, Springer, Berlin, 1973, 1–5.
[4] R. Adler, Afterword to R. Bowen, Invariant measures for Markov maps of the interval, *Comm. Math. Phys.*, **69** (1979), 1–17.
[5] V. M. Alekseev, Quasirandom dynamical systems. I. Quasirandom diffeomorphisms, *Math. of the USSR, Sbornik*, **5** (1968), 73–128.
[6] D. V. Anosov and Ya. G. Sinai, Some smooth ergodic systems, *Russian Math. Surveys*, **22** (1967), 103–167.
[7] M. Benedicks and L. Carleson, The dynamics of the Hénon map, *Ann. of Math. (2)*, **133** (1991), 73–169.
[8] M. Benedicks and L.-S. Young, Markov extensions and decay of correlations for certain Hénon maps, *Astérisque*, **261** (2000), 13–56.
[9] R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Lecture Notes in Mathematics, **470**, Springer-Verlag, Berlin-New York, 1975.
[10] V. Cyr and O. Sarig, Spectral gap and transience for Ruelle operators on countable Markov shifts, *Comm. Math. Phys.*, **292** (2009), 637–666.
[11] A. Golmakani, S. Luzzatto and P. Pilarczyk, Uniform expansivity outside the critical neighborhood in the quadratic family, *Exp. Math.*, **25** (2016), 116–124.
[12] M. I. Gordin, The central limit theorem for stationary processes, *Soviet Math. Dokl.*, **10** (1969), 1174–1176.
[13] M. Hénon, A two-dimensional mapping with a strange attractor, *Comm. Math. Phys.*, **50** (1976), 69–77.
[14] M. Hirsch and C. Pugh, Stable manifolds and hyperbolic sets, in *1970 Global Analysis* (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968), Amer. Math. Soc., Providence, R.I., 133–163.
[15] Y.-R. Huang, Measure of Parameters With Acim Nonadjacent to the Chebyshev Value in the Quadratic Family, PhD Thesis, University of MD, 2011.
[16] M. Jakobson, Absolutely continuous invariant measures for one-parameter families of one-dimensional maps, *Comm. Math. Phys.*, **81** (1981), 39–88.
[17] M. Jakobson, Piecewise smooth maps with absolutely continuous invariant measures and uniformly scaled Markov partitions, in *Smooth Ergodic Theory and its Applications* (Seattle, WA, 1999), Proc. Sympos. Pure Math., 69, Amer. Math. Soc., Providence, RI, 2001, 825–881.
[18] M. Jakobson, Thermodynamic formalism for some systems with countable Markov structures, in Modern Theory of Dynamical Systems, Contemp. Math., 692, Amer. Math. Soc., Providence, RI, 2017, 177–193.

[19] M. Jakobson, Mixing properties of some maps with countable Markov partitions, in Dynamical Systems, Ergodic Theory, and Probability: In Memory of Kolya Chernov, Contemporary Mathematics, 698, Amer. Math. Soc., Providence, RI, 2017, 181–194.

[20] M. V. Jakobson and S. E. Newhouse, A two-dimensional version of the folklore theorem, in Sinai’s Moscow Seminar on Dynamical Systems, American Math. Soc. Translations, Series 2, 171, Adv. Math. Sci., 28, Amer. Math. Soc., Providence, RI, 89–105, 1996.

[21] M. V. Jakobson and S. E. Newhouse, Asymptotic measures for hyperbolic piecewise smooth mappings of a rectangle, Astérisque, 261 (2000), 103–160.

[22] S. Luzzatto and H. Takahashi, Computable conditions for the occurrence of non-uniform hyperbolicity in families of one-dimensional maps, Nonlinearity, 19 (2006), 1657–1695.

[23] C. Pugh and M. Shub, Ergodic attractors, Transactions AMS, 312 (1989), 1–54.

[24] A. Rényi, Representations for real numbers and their ergodic properties, Acta Math. Acad. Sci. Hungar, 8 (1957), 477–493.

[25] D. Ruelle, A measure associated with axiom-A attractors, Amer. J. Math., 98 (1976), 619–654.

[26] O. Sarig, Thermodynamic formalism for countable Markov shifts, Ergodic Theory Dynam. Systems, 19 (1999), 1565–1593.

[27] O. Sarig, Existence of Gibbs measures for countable Markov shifts, Proc. Amer. Math. Soc., 131 (2003), 1751–1758.

[28] O. Sarig, Thermodynamic formalism for countable Markov shifts, in Hyperbolic Dynamics, Fluctuations and Large Deviations, Proc. Symp. Pure Math., 89, Amer. Math. Soc., Providence, RI, 2015, 81–117.

[29] Ya. G. Sinai, Topics in Ergodic Theory, Princeton Mathematical Series, 44, Princeton University Press, Princeton, NJ, 1994.

[30] S. Smale, Diffeomorphisms with many periodic points, in Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse), Princeton University Press, Princeton, NJ, 1965, 63–80.

[31] P. Walters, Invariant measures and equilibrium states for some mappings which expand distances, Trans. AMS, 236 (1978), 121–153.

[32] Q. Wang and L.-S. Young, Toward a theory of rank one attractors, Annals of Math. (2), 167 (2008), 349–480.

[33] L.-S. Young, Statistical properties of dynamical systems with some hyperbolicity, Annals of Math. (2), 147 (1998), 585–650.

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