High frequency Transition Radiation from uniaxial crystal

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Abstract. Transition radiation from an ultrarelativistic particle flying out of the crystal surface at an arbitrary angle is considered. The expression for the spectral-angular density of radiation in the high frequencies has been obtained. The dependence of the resulting radiation distribution depending on the directions of the crystal main axis relative to the surface and the angle of the emission particle velocity is investigated.

1. Introduction
Transition radiation arises when a charged particle is passing through an interface between two media. Polarization currents that are produced in the material by moving charged particle field are the transition radiation sources. Usually transition radiation is considered in the frames of macroscopic electrodynamics when boundary conditions at the interface between two media is used [1]. If we use boundary conditions when surface shape is complex, it leads to the calculating difficulties. However, in the high frequencies area
\[ \left( \frac{e^2}{hc} \right)^2 mc^2 << h\omega << \left( \frac{e^2}{hc} \right) mc^2, \] (1)
when the radiation field frequency is much more higher than atomic frequencies and wavelength more than the atom size, so the permittivity \( \varepsilon(\mathbf{r}, \omega) \) is close to unity. This gives us an opportunity to use the method of successive approximation by degree of the difference \( 1 - \varepsilon(\mathbf{r}, \omega) \) for transition radiation in the high frequencies area. We can neglect the difference \( \varepsilon(\mathbf{r}, \omega) \) from unity in the expression for the field in the calculation of the polarization current in linear approximation, because the Fourier transform of the polarization current is proportional to the difference \( 1 - \varepsilon(\mathbf{r}, \omega) \), i.e. we can replace the field \( \mathbf{E}(\mathbf{r}, \omega) \), that is produced by a charged particle in medium by the field \( \mathbf{E}^0(\mathbf{r}, \omega) \), that is created by the same charged particle motion in a vacuum. In this approximation, medium boundaries are taken into account as boundaries when polarization current existence and at the same time boundaries conditions for the fields are automatically satisfied. If linear approximation by \( 1 - \varepsilon(\mathbf{r}, \omega) \) is used, it makes consideration of the transition radiation from complex surface much simpler. In general, transition radiation from interface between anisotropic medium and vacuum depends on the crystal surface orientation. It is of interest to evaluate the high frequencies transition radiation intensity when a fast charged particle shoot through from a uniaxial crystal.
2. Polarization currents in uniaxial crystal

It is known [7] that considering monocrystal to a crystal optics approximation one should take into account orientational ordering of crystals molecules and neglect the positional molecules ordering. So, in the crystal optics approximation the crystal is considered as an anisotropic homogeneous medium. The unit vector \( \mathbf{e} \) is the vector that shows the direction of a uniaxial crystal principal axis. Then we can write its permittivity in the crystal optics approximation as:

\[
\varepsilon_0 = \varepsilon_0(\omega)((\delta_\omega - \varepsilon_{\omega})(\omega) + \varepsilon_0(\omega)\mathbf{e}_x \cdot \mathbf{e}_y.
\]

(2)

In frequencies region equation (1), permittivity distinction from unity is small, i.e.

\[
(\delta_\omega - \varepsilon_0(\omega))(\omega) - 1 \ll 1.
\]

(3)

Inequality equation (3) allows one to find an approximate solution by using the method of successive approximation in powers of the small difference \( \delta_\omega - \varepsilon_0(\omega) \). If the medium is at \( z < Z(x,y) \), the Fourier transforms of polarization current density \( J(r,\omega) \) and electric field in medium \( E(r,\omega) \) are connected with permittivity by the ratio:

\[
J_\omega(r,\omega) = \frac{i\omega}{4\pi} \cdot (\delta_\omega - \varepsilon_0(\omega)) \cdot E_\omega(r,\omega) \theta(Z(x,y) - z),
\]

(4)

where \( \theta(u) = (u + |u|)/2|u| \).

In the high-frequency region Fourier transform of the polarization current density is proportional to the small difference equation (3). Therefore, in the first approximation it is possible to replace in equation (4) exact value of electric field in medium \( E(r,\omega) \) the electric field \( E_0(r,\omega) \) of a charged particle in a vacuum:

\[
J_\omega(r,\omega) = \frac{i\omega}{4\pi} \cdot (\delta_\omega - \varepsilon_0(\omega)) \cdot E_\omega(r,\omega) \theta(Z(x,y) - z).
\]

(5)

In an isotropic medium the Fourier transform of polarization currents density is:

\[
J_\omega(r,\omega) = \frac{i\omega}{4\pi} \cdot (1 - \varepsilon_0(\omega)) \cdot E_\omega(r,\omega) \theta(Z(x,y) - z).
\]

(6)

Comparing equations (5) and (6) it can be seen that for the same medium geometry and the same particle motion Fourier transform for polarization current density in a uniaxial crystal \( J(q,\omega) \) in the linear on \( \delta_\omega - \varepsilon_0(\omega) \) is connected to Fourier transform approximation of polarization current density in an isotropic medium with permittivity \( \varepsilon_0(\omega) \), \( J_\omega(r,\omega) \) by ratio:

\[
J_\omega(r,\omega) = \frac{\delta_\omega - \varepsilon_0(\omega)}{[1 - \varepsilon_0]} J_\omega(r,\omega).
\]

(7)

In the linear approximation by \( \delta_\omega - \varepsilon_0(\omega) \) polarization current density equation (5) can be assumed to be known, so, the problem reduces to finding the polarization current equation (6). The angular and frequency radiation distribution in vacuum is created by a current \( J(k,\omega) \) can be represented as:

\[
\frac{d^3W}{d\omega dk} = \frac{2\pi}{c} |k J(k,\omega)|^2 = \frac{2\pi}{c} \left| k^2 |J(k,\omega)|^2 - |k J(k,\omega)|^2 \right|.
\]

(8)

where

\[
J(q,\omega) = \int d^3r J(r,\omega) \exp(-iqr).
\]

(9)

It followed from equations (3) and (5) that the Fourier transform of the polarization current density can be represented as the sum of the part \( J'_\omega(q,\omega) \) that is perpendicular to a principal crystal optical axis and the part \( J''(q,\omega) \) that is parallel to this axis:
The contributions of each of these parts depends on the monocrystal surface orientation relative to the crystal main axis. Using equation (7), it can be written:

\[ \mathbf{k} \left[ J_0 (\mathbf{k}, \omega) \right] = \mathbf{k} \left[ J_{\omega} (\mathbf{k}, \omega) \right] + \mathbf{k} \left[ J'_{\omega} (\mathbf{k}, \omega) \right] + 2 \mathbf{k} \mathrm{Re} \left[ \left[ \mathbf{k}, J' (\mathbf{k}, \omega) \right] \right]. \]  

(10)

Considering the radiation along the crystals main axis direction \( \mathbf{k} \parallel \mathbf{e} \), so that \( \left[ \mathbf{k}, J' (\mathbf{k}, \omega) \right] = 0 \), it is easy to see that \( \left[ \mathbf{k}, J (\mathbf{k}, \omega) \right] = \mathbf{k} \left[ J_{\omega} (\mathbf{k}, \omega) \right] \). This implies that there is only radiation to the parallel to the main axis part of polarization current with any crystal surface orientation. Using equation (11), one can express the distribution of the transition radiation in a uniaxial crystal though the polarization current density in an isotropic medium \( \mathbf{J}_0 (\mathbf{r}, \omega) \). This will determine us the transition radiation intensity ratio in a crystal and in isotropic medium:

\[
\begin{align*}
\frac{d^2 W}{d \omega d \Omega} &= \frac{(2\pi)^6}{c} \left[ k^2 \left| \mathbf{J}_0 (\mathbf{k}, \omega) \right|^2 - \left| \mathbf{k}, J_0 (\mathbf{k}, \omega) \right|^2 + k^2 \xi^2 (\omega) \left| \mathbf{e}, J_0 (\mathbf{k}, \omega) \right|^2 \right] \\
&= -2 \xi^2 (\omega) \left| \mathbf{k}, \mathbf{e} \right| \left| \mathbf{e}, J_0 (\mathbf{k}, \omega) \right|^2 + 2 \xi (\omega) \mathrm{Re} \left[ \left[ \left[ \mathbf{k}, J_0 (\mathbf{k}, \omega) \right]^2 \right] \cdot \mathbf{k}, \mathbf{e} \right].
\end{align*}
\]

(12)

Polarization current in an isotropic medium \( \mathbf{J}_0 (\mathbf{k}, \omega) \) is not dependent on the uniaxial crystal axis direction. That’s why, some of the orientational dependences features of transition radiation that is created by charged particle, flying out from an unaxial crystal, can be seeing without using the explicit form of \( \mathbf{J}_0 (\mathbf{k}, \omega) \), which will be demonstrated below.

3. Angles and frequencies distribution of transition radiation

If charge moves with the constant velocity \( \mathbf{v} \parallel \mathbf{z} \) at an angle \( \xi \) to the crystal surface, where \( \xi \) is the angle between velocity direction and surface normal (see figure 1), the polarization current density in isotropic medium has the form:

\[ J_0 (\mathbf{r}, \omega) = J_0^\rho (\mathbf{r}, \omega). \]

Figure 1. Scheme of a charged particle emission.
\[
J_{0x}(k, \omega) = \frac{i \omega}{16\pi^4v} \cdot (1 - \epsilon_0(\omega)) \int_0^\infty \int dx \int dz \cdot e^{-i\frac{2\pi}{\epsilon_0}x} \cdot e^{-ikz} \int dq \cdot e^{iqy} E_0(q, k, \omega) \frac{q}{v}.
\]

The field is produced by such a particle can be represented as:

\[
E_0(r,t) = \int d^3q \int d\omega E_0(q, \omega) \exp(i\mathbf{q}\mathbf{r} - i\mathbf{q}\mathbf{v}t),
\]

\[
E_0(q, \omega) = -\frac{ie}{2\pi^2} \frac{v\omega/c^2-q}{q^2-\omega^2/c^2} \delta(\omega - q_v).
\]

Furthermore, a case when a charged particle moves through a crystal with an ultrarelativistic speed was considered. In ultrarelativistic case the radiation concentrates at small angles \( \theta \) near the particle velocity, so that one can write \( k_x = k\vartheta\cos\varphi \); \( k_y = k\vartheta\sin\varphi \); \( k_z = k\left(1 - \frac{\vartheta^2}{2}\right) \). Therefore

\[
\left(\frac{\omega}{v} - k_z\right) = \frac{k}{2} L; \quad \left(\frac{\omega}{v} - k_z\right) tg \xi + k_z = k\left(\frac{L}{2} tg \xi + n_i \sqrt{\epsilon_0}\right),
\]

where \( L = \gamma^2 + \vartheta^2 + \frac{\omega_T^2}{\omega^2} \),

\( \mathbf{n} = (n_x, n_y, n_z) = \left(\vartheta\cos\varphi, \vartheta\sin\varphi, 1 - \frac{\vartheta^2}{2}\right) \) and when \( \omega >> \omega_p \) one can assume that \( \sqrt{\epsilon} = 1 - \frac{\omega_T^2}{2\omega^2} \).

According to the relations given in [2],

\[
\int_0^\infty dx e^{-px} \sin(qx + \lambda) = \frac{1}{p^2 + q^2} (q \cos \lambda + p \sin \lambda),
\]

\[
\int_0^\infty dx e^{-px} \cos(qx + \lambda) = \frac{1}{p^2 + q^2} (p \cos \lambda - q \sin \lambda)
\]

and neglecting the amendments of the order of \( \gamma^2 \), the expression for the polarization current density components can be obtained:

\[
J_{0x}(k, \omega) = \frac{\omega e}{i8\pi^4vk^2} \frac{1}{L} \cdot \frac{1}{F^2 + \left(\frac{L}{2} \frac{tg \xi}{n_i \sqrt{\epsilon_0}}\right)^2} \left(\frac{L}{2} \frac{tg \xi}{n_i \sqrt{\epsilon_0}} + \frac{1}{\sqrt{\epsilon_0}}\right)(1 - \epsilon_0),
\]

\[
J_{0y}(k, \omega) = -\frac{n_i \frac{\sqrt{\epsilon_0}(1 - \epsilon_0)}{\omega e}}{i8\pi^4vk^2} \frac{1}{L} \cdot \frac{1}{F^2 + \left(\frac{L}{2} \frac{tg \xi}{n_i \sqrt{\epsilon_0}}\right)^2},
\]

\[
J_{0z}(k, \omega) = -\frac{\gamma^2 \omega e}{i8\pi^4vk^2} \frac{1}{L} \cdot \frac{1}{F^2 + \left(\frac{L}{2} \frac{tg \xi}{n_i \sqrt{\epsilon_0}}\right)^2} (1 - \epsilon_0).
\]

where \( F = \frac{n_i^2 \epsilon_0 + \gamma^2}{} \).

Using equations (12) and (16), it is easy to obtain the following expression for angles and frequencies radiation distribution in vacuum:
\[
\frac{d^3W}{d\omega d\Omega} = \frac{e^2}{\pi^3v^2L^2} \frac{1}{\left(F^2 + \left(\frac{L}{2}tg \xi + n,\sqrt{E_0}\right)^2\right)^2(1-\varepsilon_0)^2} \\
\times \left[2\zeta(\omega)\left[\frac{L^2}{4}tg^2\xi \left(n^2_s - \vartheta^2\right)\sin \alpha + \sin \alpha - n_s \cos \alpha\right] + Ltg^2\zeta \left(n_s \sqrt{E_0} \sin \alpha - n_s \sqrt{E_0} \cos \alpha + \frac{1}{2} n_s^2 \sqrt{E_0} \cos \alpha + n_s^2 \left(1 - n_s^2\right)\right) \cdot \sin \alpha\right] \\
\times \left[2\zeta(\omega)\left[\frac{L^2}{4}tg^2\xi \left(1 - n_s^2\right) + Ltg^2\zeta \cdot n_s \sqrt{E_0} + \left(n_s^2 + n_s^2\right)\right]\right].
\] (17)

We would like to note that this distribution is obtained in the first approximation to the difference \(\Delta \varepsilon(\omega) = \varepsilon_1(\omega) - \varepsilon_0(\omega)\).

4. Discussion of the results

Let us see, how angles and frequencies radiation distribution in a vacuum, given by equation (17) depends on the angle \(\alpha\) between the direction of the charged particles velocity and the crystal main axis. Ad hoc, the following ratio was considered:

\[
\frac{d^3W}{d\omega d\Omega}(\alpha) = \frac{d^3W}{d\omega d\Omega} - \frac{d^3W}{d\omega d\Omega_{\text{extr}}} \\
= \frac{d^3W}{d\omega d\Omega_{\text{extr}}} \\
= 2\zeta(\omega)\left[\frac{L^2}{4}tg^2\xi \left(n^2_s - \vartheta^2\right)\sin \alpha + \sin \alpha - n_s \cos \alpha\right] + Ltg^2\zeta \left(n_s \sqrt{E_0} \sin \alpha - n_s \sqrt{E_0} \cos \alpha + \frac{1}{2} n_s^2 \sqrt{E_0} \cos \alpha + n_s^2 \left(1 - n_s^2\right)\right) \cdot \sin \alpha.] (18)
\]

In the special case when radiation quanta are emitted at an angle close to the surface normal one can find that:

\[
\cos \xi >> \gamma^2 + \frac{\alpha^2}{\vartheta^2}.
\]

Introducing polar and azimuthal emission angles by the relations \(n = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta)\) and taking into account that the radiation is emitted at small angles to the charged particle motion direction \(\vartheta << 1\), one gets from equation (17):

\[
\frac{d^3W}{d\omega d\Omega} = \frac{e^2}{\pi^3v^2} \frac{1}{\left(\gamma^2 + \vartheta^2 + \frac{\alpha^2}{\vartheta^2}\right)^2} \frac{1}{(1-\varepsilon_0)^2} \gamma^2 \left(2\zeta(\omega) \vartheta^2 \sin^2 \alpha + 1\right). (19)
\]

In a cubic crystal case, in which \(\Delta \varepsilon(\omega) = \varepsilon_1(\omega) - \varepsilon_0(\omega)\), the well-known result for particles that emitting perpendicular to crystal surface can be obtained:

\[
\frac{d^3W}{d\omega d\Omega} = \frac{ce^2}{\pi^3v^2} \frac{1}{(1-\varepsilon_0)^2} \vartheta^2 \left(\gamma^2 + \vartheta^2 + \frac{\alpha^2}{\vartheta^2}\right)^2 \left(\gamma^2 + \vartheta^2\right)^2. (20)
\]

The result (8) was obtained theoretically in [3] and experimentally confirmed in [4].

Thus, the dependence of the resulting distribution radiation on the crystal main axis direction and relative to the surface was considered. Also, the dependence of the resulting
distribution radiation on the angle of particle emission relative to the normal to the surface was investigated.

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