For a Tychonoff space $X$ and a subspace $Y \subset \mathbb{R}$, we study Baire category properties of the space $C_{\downarrow F}(X, Y)$ of continuous functions from $X$ to $Y$, endowed with the Fell hypograph topology. We characterize pairs $X, Y$ for which the function space $C_{\downarrow F}(X, Y)$ is $\infty$-meager, meager, Baire, Choquet, strong Choquet, (almost) complete-metrizable or (almost) Polish.

\section{Introduction and Main Results}

In this paper we study Baire category properties of function spaces $C_{\downarrow F}(X, Y)$ and answer a problem, posed by McCoy and Ntantu in [11].

For a topological space $X$, the \textit{Fell topology} on the space $\text{Cl}(X)$ of all closed subsets of $X$ is generated by the subbase consisting of the sets

$$U^- = \{ F \in \text{Cl}(X) : F \cap U \neq \emptyset \} \text{ and } (X \setminus K)^+ = \{ F \in \text{Cl}(X) : F \subset X \setminus K \}$$

\textbf{Key words and phrases.} Fell hypograph topology, compact-open topology, Moving Off Property, Baire space, meager space.

\section{Function spaces $C_{\downarrow F}(X, Y)$ over $F$-spaces $X$}

\section{Separate continuity of the lattice operations on $C_{\downarrow F}(X, Y)$}

\section{Extension of functions defined on $Y$-separated spaces}

\section{The $\infty$-density of some subsets in $C_{\downarrow F}(X, Y)$}

\section{The subspace $C'_{\downarrow F}(X, Y)$}

\section{Recognizing $\infty$-meager function spaces $C_{\downarrow F}(X, Y)$}

\section{Recognizing Baire spaces among function spaces $C_{\downarrow F}(X, Y)$}

\section{Recognizing Choquet spaces among function spaces $C_{\downarrow F}(X, Y)$}

\section{Recognizing strong Choquet spaces among function spaces $C_{\downarrow F}(X, Y)$}

\section{Recognizing almost Polish spaces among function spaces $C_{\downarrow F}(X, Y)$}

\section{Recognizing Polish spaces among the function spaces $C_{\downarrow F}(X, Y)$}

\section{Recognizing function spaces $C_{\downarrow F}(X, Y)$ which are neither Baire nor meager}

\section{A dichotomy for analytic function spaces $C_{\downarrow F}(X, Y)$}

\section{References to proofs of the statements in Table 1}

\section{References}
where $U$ and $K$ run over open and compact sets in $X$, respectively. The space $Cl(X)$ endowed with the Fell topology is denoted by $Cl_F(X)$.

For a topological space $X$ and a subspace $Y \subset \mathbb{R}$ of the real line, let $C(X,Y)$ denote the set of continuous functions from $X$ to $Y$. Identifying each function $f \in C(X,Y)$ with its hypograph $\downarrow f := \{(x,y) \in X \times \mathbb{R} : y \leq f(x)\}$, we identify $C(X,Y)$ with the subset $\{\downarrow f : f \in C(X,Y)\}$ of the hyperspace $Cl_F(X \times \mathbb{R})$. The topology on the function space $C(X,Y)$, inherited from the hyperspace $Cl_F(X \times \mathbb{R})$, is called the Fell hypograph topology. Let $C_k(Y)$ denote the function space $C(X,Y)$ endowed with the Fell hypograph topology.

Repeating the argument of the proof of Lemma 2.1 in [11], it can be shown that for any Hausdorff space $X$ and any subspace $Y \subset \mathbb{R}$, the Fell hypograph topology on $C(X,Y)$ is given by the subbase consisting of the sets

\[ [U; y] := \{f \in C(X,Y) : \sup f(U) > y\} \quad \text{and} \quad [K; y] := \{f \in C(X,Y) : \max f(K) < y\} \]

where $U$ is a non-empty open set in $X$, $K$ is a non-empty compact subset of $X$, and $y \in \mathbb{R}$.

This description implies that the Fell hypograph topology on $C(X,Y)$ is weaker than the compact-open topology, which is generated by the subbase consisting of the sets

\[ [K; U] := \{f \in C(X,Y) : f(K) \subset U\} \]

where $K$ is a compact set in $X$ and $U$ is an open set in $Y$. The function space $C(X,Y)$ endowed with the compact-open topology will be denoted by $C_k(X,Y)$.

Topological properties of the function spaces $C_k(X,Y)$ were studied in [11], [22], [19], [20], [23], [15], [21].

In this paper we shall explore Baire category properties of the function spaces $C_k(X,Y)$. Let us recall that a topological space $X$ is

- **Baire** if the intersection $\bigcap_{n \in \omega} U_n$ of any sequence $(U_n)_{n \in \omega}$ of open dense subsets of $X$ is dense in $X$;
- **meager** if $X$ can be written as the countable union of (closed) nowhere dense subsets.

It is well-known [10, 8.1] that a topological space $X$ is Baire if and only if each non-empty open subspace of $X$ is not meager, and similarly a topological space $X$ is meager if and only if each non-empty open subspace of $X$ is not Baire.

By the classical theorem of Oxtoby [15], Baire spaces can be characterized as topological spaces $X$ in which the player $E$ does not have a winning strategy in the Choquet game $G_{EN}(X)$. The game $G_{EN}(X)$ is played by two players, $E$ and $N$ (abbreviated from **Empty** and **Non-Empty**). The player $E$ starts the game selecting a non-empty open set $U_1 \subset X$. Then the player $N$ responds selecting a non-empty open set $V_1 \subset U_1$. At the $n$-th inning the player $E$ selects a non-empty open set $U_n \subset V_{n-1}$ and player $N$ responds selecting a non-empty open set $V_n \subset U_n$. At the end of the game the player $E$ is declared the winner if the intersection $\bigcap_{n \in \mathbb{N}} U_n = \bigcap_{n \in \mathbb{N}} V_n$ is empty. Otherwise the player $N$ wins the game.

We shall also be interested in a variation $G_{EN}(X)$ of the Choquet game, called the strong Choquet game. This game is played by two players, $E$ and $N$. The player $E$ starts the game selecting an open set $U_1 \subset X$ and a point $x_1 \in U_1$. Then the player $N$ responds selecting an open neighborhood $V_1 \subset U_1$ of $x_1$. At the $n$-th inning the player $E$ selects an open set $U_n \subset V_{n-1}$ and a point $x_n \in U_n$ and player $N$ responds selecting a neighborhood $V_n \subset U_n$ of $x_n$. At the end of the game the player $E$ is declared the winner if the intersection $\bigcap_{n \in \mathbb{N}} U_n = \bigcap_{n \in \mathbb{N}} V_n$ is empty. Otherwise the player $N$ wins the game.

A topological space $X$ is called
For every topological space we have the implications

\[
\begin{align*}
\text{Polish} &\Rightarrow \text{complete-metrizable} \Rightarrow \text{strong Choquet} \Rightarrow \text{non-meager} \\
\text{almost Polish} &\Rightarrow \text{almost complete-metrizable} \Rightarrow \text{Choquet} \Rightarrow \text{Baire}
\end{align*}
\]

By [10, 8.16, 8.17], a metrizable separable space is

- complete-metrizable if and only if it is strong Choquet;
- almost complete-metrizable if and only if it is Choquet.

In [11, 5.2] McCoy and Ntantu proved that for a Tychonoff space \(X\) the function space \(C_{\downarrow}^F(X, \mathbb{R})\) is complete-metrizable if and only if \(C_{\downarrow}^F(X, \mathbb{R})\) is Polish if and only if \(X\) is countable and discrete.

In [11, 5.3] McCoy and Ntantu posed a problem of characterization of Tychonoff spaces \(X\) for which the function space \(C_{\downarrow}^F(X, \mathbb{R})\) is Baire. In Corollary 1.3 we shall prove that this happens if and only if the space \(X\) is discrete if and only if the space \(C_{\downarrow}^F(X, \mathbb{R})\) is (strong) Choquet. Then we shall consider a more difficult problem of detecting Baire and (strong) Choquet spaces among function spaces \(C_{\downarrow}^F(X, Y)\) where \(Y\) is a subset of the real line with \(\inf Y \in Y\) and \(X\) is a \(Y\)-separated space.

**Definition 1.1.** Let \(Y\) be a topological space. A topological space \(X\) is defined to be \(Y\)-separated if for any distinct points \(x_1, x_2 \in X\) and any points \(y_1, y_2 \in Y\) there exists a continuous map \(f : X \to Y\) such that \(f(x_i) = y_i\) for every \(i \in \{1, 2\}\).

A topological space \(X\) is called

- functionally Hausdorff if it is \([0, 1]\)-separated;
- totally disconnected if it is \([0, 1]\)-separated.

It is easy to see that

- for a connected subspace \(Y \subset \mathbb{R}\) containing more than one point, a topological space \(X\) is \(Y\)-separated if and only if \(X\) is functionally Hausdorff;
- for a disconnected subspace \(Y \subset \mathbb{R}\) a topological space \(X\) is \(Y\)-separated if and only if \(X\) is totally disconnected.

**Theorem 1.2.** Let \(Y \subset \mathbb{R}\) be a subset with \(\inf Y \notin Y\). For any \(Y\)-separated space \(X\), the function space \(C_{\downarrow}^F(X, Y)\) is

1. Baire if and only if \(X\) is discrete and the space \(Y\) is Baire;
2. Choquet if and only if the space \(X\) is discrete and the space \(Y\) is almost Polish;
3. strong Choquet if and only if the space \(X\) is discrete and the space \(Y\) is Polish;
4. almost complete-metrizable if and only if \(C_{\downarrow}^F(X, Y)\) is almost Polish if and only if \(X\) is countable and discrete and the space \(Y\) is almost Polish;
(5) complete-metrizable if and only if $C_{\downarrow F}(X, Y)$ is Polish if and only if $X$ is countable and discrete and the space $Y$ is Polish.

The statements (1)–(5) of Theorem 1.2 are proved in Lemmas 8.4, 9.4, 10.4, 11.2, 12.1, respectively. Taking into account that the real line is a Polish space with $\inf \mathbb{R} = -\infty \not\in \mathbb{R}$, we conclude that Theorem 1.2 implies the following characterization that answers Problem 5.3 [11] of McCoy and Ntantu.

**Corollary 1.3.** For a Tychonoff space $X$, the following conditions are equivalent:

1. $C_{\downarrow F}(X, \mathbb{R})$ is Baire;
2. $C_{\downarrow F}(X, \mathbb{R})$ is Choquet;
3. $C_{\downarrow F}(X, \mathbb{R})$ is strong Choquet;
4. $X$ is discrete.

Now, we present a characterization of Baire and Choquet spaces among function spaces $C_{\downarrow F}(X, Y)$ where $Y$ is a subset of the real line with $\inf Y \not\in Y$.

This characterization involves the Discrete Moving Off Property and Winning Discrete Moving Off Properties (abbreviated by DMOP and WDMOP), which were introduced and studied by the authors in [5]. The Discrete Moving Off Property is a modification of MOP, the Moving Off Property of Gruenhage and Ma [9].

A point $x$ of a topological space $X$ is called isolated if its singleton $\{x\}$ is clopen set in $X$ (which means that $\{x\}$ is closed-and-open in $X$).

**Notation 1.4.** For a topological space $X$ let

- $\hat{X}$ be the (open) set of all isolated points of $X$,
- $X'$ be the (closed) set of non-isolated points in $X$,
- $\overset{o}{X'}$ be the interior of the set $X'$ in $X$,
- $\overline{\overset{o}{X'}}$ be the closure of the set $\overset{o}{X'}$ in $X$.

A family $\mathcal{F}$ of subsets of a topological space $X$ is called

- discrete if each point $x \in X$ has a neighborhood $O_x \subset X$ that meets at most one set $F \in \mathcal{F}$;
- a moving off family if for any compact subset $K \subset X$ there is a non-empty set $F \in \mathcal{F}$ with $F \cap K = \emptyset$.

It is clear that each discrete infinite family is moving off.

**Definition 1.5.** A topological space $X$ is defined to have the Discrete Moving Off Property (abbreviated DMOP) if any moving off family $\mathcal{F}$ of finite subsets in $\hat{X}$ contains an infinite subfamily $\mathcal{D} \subset \mathcal{F}$, which is discrete in $X$.

By [5], a topological space $X$ has DMOP if and only if the player $F$ does not have the winning strategy in the infinite game $G_{KF}(X)$, played by two players, $K$ and $F$ according to the following rules. The player $K$ starts the game. At the $n$-th inning the player $K$ chooses a compact subset $K_n \subset X$ and the player $F$ responds by choosing a finite subset $F_n \subset \hat{X} \setminus K_n$. At the end of the game, the player $K$ is declared the winner if the indexed family $(F_n)_{n \in \mathbb{N}}$ is discrete in $X$ (which means that each point $x \in X$ has a neighborhood $O_x \subset X$ that meets at most one set $F_n$); otherwise the player $F$ wins the game.

**Definition 1.6.** A topological space $X$ is defined
to have the Winning Discrete Moving Off Property (abbreviated WDMOP) if the player $K$ has a winning strategy in the game $G_{KF}(X)$;

- to be a $k_\omega$-space if a subset $D \subseteq \hat{X}$ is closed in $X$ if and only if for every compact set $K \subseteq X$ the intersection $D \cap K$ is finite;

- to be a $k_{\omega_1}$-space if there exists a countable family $\mathcal{K}$ of compact subsets of $X$ such that $\hat{X} \subseteq \bigcup \mathcal{K}$ and a subset $D \subseteq \hat{X}$ is closed in $X$ if and only if for every $K \in \mathcal{K}$ the intersection $D \cap K$ is finite.

By [5] 6.2, every $k_\omega$-space has WDMOP, and WDMOP implies DMOP. These properties have nice characterizations in terms of the Baire category properties of the function space

$$C_k^0(X, 2) = \{ f \in C_k(X, \{0, 1\}) : f(X') \subset \{0\} \}.$$ The following theorem was proved in [5].

**Theorem 1.7.** For a topological space $X$ the function space $C^0_k(X, 2)$ is

- discrete iff $\hat{X} \subset K$ for some compact set $K \subset X$;

- complete-metrizable iff $X$ is a $k_\omega$-space;

- Polish iff $X$ is a $k_{\omega_1}$-space and the set $X$ is countable;

- Choquet iff $X$ has WDMOP;

- Baire iff $X$ has DMOP;

- meager iff $X$ does not have DMOP.

Also we need the notion of a $Y$-separable space, which is defined with the help of the $Y$-topology.

For topological spaces $X, Y$, the $Y$-topology on $X$ is the weakest topology in which all maps $f \in C(X, Y)$ remain continuous. This topology is generated by the subbase consisting of the sets $f^{-1}(U)$ where $f \in C(X, Y)$ and $U$ is an open set in $Y$. Observe that a topological space $X$ is Tychonoff (and zero-dimensional) if and only if its topology coincides with the $\mathbb{R}$-topology (and with the $\{0, 1\}$-topology).

For a subset $A$ of a topological space $X$ by $\overline{A}^Y$ we denote the closure of $A$ in the $Y$-topology of $X$ and call $\overline{A}^Y$ the $Y$-closure of $A$. It is clear that the closure $\overline{A}$ of any set $A \subset X$ is contained in its $Y$-closure $\overline{A}^Y$.

**Definition 1.8.** A topological space $X$ is defined to be $Y$-separable if $X$ contains a meager $\sigma$-compact subset $M \subset X$ such that $X' = \overline{M}^Y$.

Observe that a topological space $X$ is $Y$-separable if its set $X'$ of non-isolated points is separable (in the standard sense). In Lemma 12.22, we shall prove that a $Y$-separated topological space $X$ is $Y$-separable if the function space $C_{1F}(X, Y)$ has a countable network.

A topological space $X$ is called Polish+meager if it contains a Polish subspace $P \subset X$ whose complement $X \setminus P$ is meager in $X$. It is easy to see that a Polish+meager space is Baire if and only if it is almost Polish. It is known [10], 8.23, that each Borel subset of a Polish space is Polish+meager. A subset $A$ of a topological space $X$ is sequentially closed if $A$ contains the limit point of any sequence $\{a_n\}_{n \in \omega} \subset A$ that converges in $X$.

**Theorem 1.9.** Let $Y \subset \mathbb{R}$ be a Polish+meager subspace with $\inf Y \in Y \neq \{\inf Y\}$. For a $Y$-separable $Y$-separated space $X$, the function space $C_{1F}(X, Y)$ is

1. Baire if and only if the space $Y$ is Baire, the set $X$ is dense in $X$, and the space $X$ has DMOP.
(2) Choquet if and only if the space $Y$ is almost Polish, the set $\hat{X}$ is dense in $X$, and the space $X$ has WDMOP;

(3) strong Choquet if (and only if) the space $Y$ is Polish and the set $\hat{X}$ is (sequentially) closed in $X$;

(4) almost complete-metrizable if and only if the space $Y$ is almost Polish and $X$ is a $\kappa_\omega$-space with dense set $\hat{X}$ of isolated points;

(5) almost Polish if and only if the space $Y$ is almost Polish and $X$ is a $\kappa_\omega$-space with countable dense set $\hat{X}$ of isolated points.

(6) complete-metrizable if and only if $C_{\mathcal{F}}(X,Y)$ is Polish if and only if the space $Y$ is Polish and $X$ is countable and discrete.

The statements (1)–(6) of Theorem 1.9 are proved in Propositions 8.8, 9.7, 10.6, 11.5, 11.6 respectively. The $Y$-separability of the space $X$ is essential and cannot be removed as shown by the following theorem treating Baire category properties of function spaces $C_{\mathcal{F}}(X,Y)$ on zero-dimensional compact $F$-spaces.

Let us recall that a topological space $X$ is called an $F$-space if the closures of any disjoint open $F_\sigma$-sets are disjoint. By Theorem 1.2.5 [12], for any locally compact $\sigma$-compact non-compact space $X$ the remainder $\beta X \setminus X$ of the Stone-Čech compactification of $X$ is an $F$-space. In particular, the remainder $\beta \mathbb{N} \setminus \mathbb{N}$ of the Stone-Čech compactification of $\mathbb{N}$ is an $F$-space.

A topological space $X$ is called countably base-compact if it has a base $\mathcal{B}$ of the topology such that for any decreasing sequence $\{B_n\}_{n \in \omega} \subseteq \mathcal{B}$ the intersection $\bigcap_{n \in \omega} B_n$ is not empty. It is easy to see that each countably base-compact regular space is strong Choquet. The countable base-compactness is one of Amsterdam properties, discussed by Aarts and Lutzer in [1, 2.1.4].

**Theorem 1.10.** For any compact zero-dimensional $F$-space $X$ and any closed subset $Y \subset \mathbb{R}$ with $\inf Y \in Y$ the function space $C_{\mathcal{F}}(X,Y)$ is countably base-compact and strong Choquet.

Theorem 1.10 will be proved in Section 2.

Now we turn to the problem of classification of meager spaces among function spaces $C_{\mathcal{F}}(X,Y)$. This classification is rather complicated and depends on the interplay between the following 6+7 properties of the spaces $Y$ and $X$.

For a non-empty subset $Y \subset \mathbb{R}$ we consider the following 6 properties:

- $(Y_M)$ $Y$ is meager;
- $(Y_B)$ $Y$ is Baire;
- $(Y_N)$ $Y$ is neither meager nor Baire;
- $(Y_0)$ $\inf Y \notin Y$;
- $(Y_1)$ $\inf Y \in Y \setminus \hat{Y}$;
- $(Y_2)$ $\inf Y \in \hat{Y} \subset Y$.

For two symbols $L \in \{B, M, N\}$ and $n \in \{0, 1, 2\}$ we say that the space $Y$ has property $Y_{Ln}$ if $Y$ has the properties $Y_L$ and $Y_n$. So, for example, $Y_{M2}$ means that the space $Y$ is meager and has the smallest element $\inf Y$, which is an isolated point of $Y$. In fact, among all possible 9 combinations of the properties $Y_M, Y_B, Y_N, Y_0, Y_1, Y_2$ we shall need only 6: $Y_{00}, Y_{01}, Y_{02}, Y_{B0}, Y_{B1}, Y_{B2}$.

Next, we introduce 7 properties $X_0, X_1, X_2, X_3, X_C, X_B, X_M$ of a topological space $X$ and 12 combinations of these properties (of which we shall need only 6).

For any topological space $X$ consider the following 7 properties:
(X₀) \( X' = \emptyset \);
(X₁) \( X' \neq \emptyset = X^{\circ} \);
(X₂) \( X^{\circ} \) is not empty and compact;
(X₃) \( X^{\circ} \) is not compact;
(X₅) \( \bar{X} \subset K \) for some compact set \( K \subset X \);
(X₆) \( X \) does not have DMOP.

For two symbols \( L \in \{ C, B, M \} \) and \( n \in \{ 0, 1, 2, 3 \} \) we say that the space \( X \) has property \( X_{Ln} \) if \( X \) has the properties \( X_{L} \) and \( X_{n} \). So, for example, \( X_{M1} \) means that the space \( X \) does not have DMOP and the set \( X^{\circ} \neq \emptyset \) is nowhere dense in \( X \). In fact, among all possible 12 combinations of the properties \( X₀, X₁, X₂, X₃, X₅, X₆ \) we shall be interested only in 6: \( X₅₀, X₅₁, X₅₂, X₆₀, X₆₁, X₆₂ \).

Finally, let us consider the following three Baire category properties of the function space \( C_{1F}(X, Y) \):

(M) \( C_{1F}(X, Y) \) is meager;
(B) \( C_{1F}(X, Y) \) is Baire;
(N) \( C_{1F}(X, Y) \) is neither meager nor Baire.

The following table describes the Baire category properties \( M, B, N \) of the function space \( C_{1F}(X, Y) \), where \( Y \subset \mathbb{R} \) is a Polish+meager space containing more than one point and \( X \) is a \( Y \)-separable \( Y \)-separated topological space.

|        | \( Y_{M} \) | \( Y_{N0} \) | \( Y_{N1} \) | \( Y_{N2} \) | \( Y_{B0} \) | \( Y_{B1} \) | \( Y_{B2} \) |
|--------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| \( X_{C0} \) | M           | N           | N           | N           | B           | B           | B           |
| \( X_{C1} \) | M           | N           | N           | N           | M           | B           | B           |
| \( X_{C2} \) | M           | M           | M           | N           | M           | M           | N           |
| \( X_{B0} \) | M           | M           | M           | M           | B           | B           | B           |
| \( X_{B1} \) | M           | M           | M           | M           | M           | B           | B           |
| \( X_{B2} \) | M           | M           | M           | M           | M           | M           | N           |
| \( X_{M} \)  | M           | M           | M           | M           | M           | M           | M           |
| \( X₃ \)    | M           | M           | M           | M           | M           | M           | M           |

This table consists of \( 8 \times 7 \) statements on the Baire Category properties of the function spaces \( C_{1F}(X, Y) \). The references to lemmas proving these 56 statements will be given in Section 15. In fact, the meagerness of the function spaces \( C_{1F}(X, Y) \) will be proved in a stronger form of \( \infty \)-meagerness, defined as follows.

**Definition 1.11.** A subset \( A \) of a topological space \( X \) is called

- \( \infty \)-dense in \( X \) if for any compact Hausdorff space \( K \), the subset \( C_{k}(K, A) = \{ f \in C_{k}(K, X) : f(K) \subset A \} \) is dense in \( C_{k}(K, X) \);
- \( \infty \)-codense if its complement \( X \setminus A \) is \( \infty \)-dense in \( X \);
- \( \infty \)-meager if \( A \) is contained in a countable union of closed \( \infty \)-codense subsets of \( X \).

A topological space \( X \) is called

- \( \infty \)-meager if it is \( \infty \)-meager in itself;
- \( \infty \)-comeager if \( X \) contains an \( \infty \)-dense Polish subspace.
It is easy to see that each closed $\infty$-codense set is nowhere dense, so each $\infty$-meager set is meager. On the other hand, the singleton $\{0\}$ in the real line is nowhere dense but not $\infty$-codense in $\mathbb{R}$.

It should be mentioned that $\infty$-meager and $\infty$-comeager spaces play an important role in Infinite-Dimensional Topology and enter as key ingredients in many characterization theorems of model infinite-dimensional spaces, see [2], [4], [6], [13], [14], [16].

For any topological space we have the implications

$$\infty\text{-meager} \Rightarrow \text{meager} \Rightarrow \text{not Baire} \Rightarrow \text{not } \infty\text{-comeager.}$$

By [3], the linear hull of the Erdős set $E = \{(x_i)_{i \in \omega} \in \ell_2 : (x_i)_{i \in \omega} \in \mathbb{Q}^\omega\}$ in the separable Hilbert space $\ell_2$ is an example of a meager (pre-Hilbert) space, which is not $\infty$-meager.

**Theorem 1.12.** Let $Y$ be a Polish+$\infty$-meager subset $Y \subset \mathbb{R}$ and $X$ be a $Y$-separable $Y$-separated topological space $X$. The function space $C_{\downarrow F}(X,Y)$ is meager if and only if $C_{\downarrow F}(X,Y)$ is $\infty$-meager.

This theorem will be proved in Section 15. In Section 14 we prove an interesting dichotomy for analytic function spaces $C_{\downarrow F}(X,Y)$. A topological space is called analytic if it is a continuous image of a Polish space.

**Theorem 1.13.** Let $Y \subset \mathbb{R}$ be a non-empty Polish subspace with $\inf Y \notin \dot{Y}$. If for a $Y$-separated topological space $X$ the function space $C_{\downarrow F}(X,Y)$ is analytic, then $C_{\downarrow F}(X,Y)$ is either $\infty$-meager or $\infty$-comeager.

Typical examples of sets $Y \subset \mathbb{R}$ with properties $Y_{B0}$, $Y_{B1}$ and $Y_{B2}$ are the real line $\mathbb{R}$, the closed interval $[0,1]$ and the doubleton $\{0,1\}$, respectively.

For these spaces the classification given in Table 1 implies the following characterizations.

**Corollary 1.14.** For an $\mathbb{R}$-separable functionally Hausdorff space $X$, the following statements are equivalent:

1. $C_{\downarrow F}(X,\mathbb{R})$ is Baire;
2. $C_{\downarrow F}(X,\mathbb{R})$ is not meager;
3. $C_{\downarrow F}(X,\mathbb{R})$ is not $\infty$-meager;
4. the space $X$ is discrete.

**Corollary 1.15.** For an $\mathbb{R}$-separable functionally Hausdorff space $X$, the following statements are equivalent:

1. $C_{\downarrow F}(X,[0,1])$ is Baire;
2. $C_{\downarrow F}(X,[0,1])$ is not meager;
3. $C_{\downarrow F}(X,[0,1])$ is not $\infty$-meager;
4. $X$ has DMOP and the set $\dot{X}$ is dense in $X$;
5. $C_{\downarrow F}(X,[0,1])$ is Baire;
6. $C_{\downarrow F}(X,[0,1])$ is not meager;
7. $C_{\downarrow F}(X,[0,1])$ is not $\infty$-meager.

On the other hand, the function space $C_{\downarrow F}(X,\{0,1\})$ behaves differently.

**Corollary 1.16.** For a $\{0,1\}$-separable totally disconnected space $X$, the following characterizations hold:

1. $C_{\downarrow F}(X,\{0,1\})$ is Baire if and only if $X$ has DMOP and $X^{\circ\circ} = \emptyset$;
(2) \( C_{\text{IF}}(X, \{0, 1\}) \) is meager if and only if \( X \) does not have DMOP or \( \overline{X^0} \) is not compact;
(3) \( C_{\text{IF}}(X, \{0, 1\}) \) is neither Baire nor meager if and only if \( X \) has DMOP and the set \( \overline{X^0} \) is compact and not empty.

**Remark 1.17.** Theorem [10] shows that the \( \{0, 1\} \)-separability of space \( X \) cannot be removed from the assumptions of Corollary [10](3): for the compact \( F \)-space \( X = \beta N \setminus N \) the function space \( C_{\text{IF}}(X, 2) \) is Baire but \( X \) has DMOP and \( X^0 = X' = X \).

2. **Function spaces \( C_{\text{IF}}(X, Y) \) over \( F \)-spaces \( X \)**

In this section we prove Theorem [10]. Given a compact zero-dimensional \( F \)-space \( X \) and a closed subset \( Y \subset \mathbb{R} \) with \( \inf Y \subset Y \), we need to show that the function space \( C_{\text{IF}}(X, Y) \) is countably base-compact and strong Choquet.

In the space \( C_{\text{IF}}(X, Y) \) consider the family \( \mathcal{B} \) of all non-empty open sets of the form

\[
[\mathcal{U}; a, b] := \bigcap_{U \in \mathcal{U}} [U; a(U)] \cap [U; b(U)],
\]

where \( \mathcal{U} \) is a finite cover of \( X \) by pairwise disjoint clopen sets and \( a, b : \mathcal{U} \to \mathbb{R} \) are two functions. It follows from \( [\mathcal{U}; a, b] \neq \emptyset \) that for every \( U \in \mathcal{U} \) the order interval

\[
\langle a(U), b(U) \rangle_Y := \{ y \in Y : a(U) < y < b(U) \}
\]

is not empty and its closure \( [a(U), b(U)]_Y \) in \( Y \) is compact. It is clear that \( [a(U), b(U)]_Y = [\bar{a}(U), \bar{b}(U)] \cap Y \) for some real numbers \( \bar{a}(U), \bar{b}(U) \) such that \( a(U) \leq \bar{a}(U) < \bar{b}(U) < b(U) \).

It can be shown that \( \mathcal{B} \) is a base of the Fell hypograph topology of \( C_{\text{IF}}(X, Y) \). We claim that this base witnesses that the function space \( C_{\text{IF}}(X, Y) \) is countably base-compact.

Fix a decreasing sequence \( \{ [\mathcal{U}_n; a_n, b_n] \}_{n \in \omega} \subset \mathcal{B} \) of basic open sets. Replacing each cover \( \mathcal{U}_n \) by a finer disjoint open cover, we can assume that for every \( n \in \omega \) each set \( U \in \mathcal{U}_{n+1} \) is contained in some set \( V \in \mathcal{U}_n \). Also we lose no generality assuming that \( \mathcal{U}_0 = \{ X \} \).

For any \( n \in \omega \) and \( U \in \mathcal{U}_n \), fix a point \( y_n(U) \in \langle a_n(U), b_n(U) \rangle_Y \) and let \( \bar{a}_n(U) \) and \( \bar{b}_n(U) \) be two real numbers such that \( [a_n(U), b_n(U)]_Y = Y \cap [\bar{a}_n(U), \bar{b}_n(U)] \).

For any \( n \leq m \) we can use the inclusion \( [\mathcal{U}_m; a_m, b_m] \subset [\mathcal{U}_n; a_n, b_n] \) to show that the following two conditions are satisfied:

(a) for any \( U \in \mathcal{U}_0 \) there exists \( V \in \mathcal{U}_n \) such that \( V \subset U \) and \( y_m(V) \geq \bar{a}_m(V) \geq \bar{a}_n(U) \);
(b) for any \( U \in \mathcal{U}_n \) and \( V \in \mathcal{U}_m \) with \( V \subset U \) we have \( y_m(V) \leq \bar{b}_n(V) \leq \bar{b}_m(U) \).

The condition (b) implies that the set \( \bigcup_{n \in \omega} \{ y_n(V) : V \in \mathcal{U}_n \} \) is contained in the compact set \( \inf Y, \bar{b}_0(X) \rangle_Y := Y \cap \inf Y, \bar{b}_0(X) \rangle \).

For every point \( x \in X \) let \( \text{Lim}(x) \) be the set of all points \( y \in Y \) such that for any neighborhoods \( O_x \subset X \) and \( O_y \subset \mathbb{R} \) of \( y \) the set \( \bigcup_{n \in \omega} \{ U \in \mathcal{U}_n : U \cap O_x \neq \emptyset, y_n(U) \in O_y \} \) is infinite.

The compactness of the set \( \inf Y, \bar{b}_0(X) \rangle_Y \supset \{ y_n(U) : n \in \omega, U \in \mathcal{U}_n \} \) implies that the set \( \text{Lim}(x) \) is not empty. We claim that \( \text{Lim}(x) \) is a singleton. To derive a contradiction, assume that \( \text{Lim}(x) \) contains two points \( y, z \) with \( y < z \). Then

\[
W_- = \bigcup_{n \in \omega} \{ U \in \mathcal{U}_n : y_n(U) < \frac{1}{2}(y + z) \} \quad \text{and} \quad W_+ = \bigcup_{n \in \omega} \{ U \in \mathcal{U}_n : y_n(U) > \frac{1}{2}(y + z) \}
\]

are two disjoint open \( F_\sigma \)-sets with \( x \in \overline{W_-} \cap \overline{W_+} \), which is not possible in \( F \)-spaces. This contradiction shows that the set \( \text{Lim}(x) \) contains a single point \( \lambda(x) \in Y \).
Using the equality \( \text{Lim}(x) = \{\lambda(x)\} \) and the compactness of the set \([\inf Y, b_0(X)]_Y\), it is possible to prove that the function \( \lambda : X \to [\inf Y, b_0(X)]_Y \subset Y \) is continuous.

It remains to show that \( \lambda \) belongs to the closure \([\mathcal{U}_n; a_n, b_n]\) of each basic set \([\mathcal{U}_n; a_n, b_n]\) in \(C_{\mathcal{I}_F}(X, Y)\). It is easy to see that
\[
[\mathcal{U}_n; a_n, b_n] = \bigcap_{U \in \mathcal{U}_n} \{f \in C_{\mathcal{I}_F}(X, Y) : \bar{a}_n(U) \leq \max f(U) \leq \bar{b}_n(U)\}.
\]

So, we need to check that \(\bar{a}_n(U) \leq \max \lambda(U) \leq \bar{b}_n(U)\). By the condition (a), for any \(m \geq n\) there exists a set \(V_m \in \mathcal{U}_m\) such that \(y_m(V_m) \geq \bar{a}_m(V_m) \geq \bar{a}_n(U)\). By the compactness of \(U\), there exists a point \(x \in U\) whose any neighborhood \(O_x\) intersects infinitely many sets \(V_m\). For this point \(x\) the value \(\lambda(x)\) is contained in the closure of the set \([y_m(V_m)]_{m \geq n} \subset [\bar{a}_n(U), \bar{b}_n(U)]\). So, \(\max \lambda(U) \geq \lambda(x) \geq \bar{a}_n(U)\).

On the other hand, the condition (b) guarantees that
\[
\bigcup_{m \geq n} \{y_m(V) : V \in \mathcal{U}_m, V \subset U\} \subset [\inf Y, \bar{b}_n(U)],
\]
which implies that \(\lambda(U) \subset [\inf Y, \bar{b}_n(U)]\) and finally
\[\bar{a}_n(U) \leq \max \lambda(U) \leq \bar{b}_n(U)\].

This completes the proof of the countable base-compactness of \(C_{\mathcal{I}_F}(X, Y)\).

By \(\text{[11, Theorem 3.7]}\), the function space \(C_{\mathcal{I}_F}(X, Y)\) is regular (since \(X\) is compact). Being countably base-compact, the regular space \(C_{\mathcal{I}_F}(X, Y)\) is strong Choquet.

3. **Separate continuity of the lattice operations on \(C_{\mathcal{I}_F}(X, Y)\)**

Observe that any subset \(Y \subset \mathbb{R}\) is closed under the operations of \(\min\) and \(\max\). For any topological space \(X\), these two operations induce two lattice operations on the function space \(C_{\mathcal{I}_F}(X, Y)\):
\[
\min : C_{\mathcal{I}_F}(X, Y) \times C_{\mathcal{I}_F}(X, Y) \to C_{\mathcal{I}_F}(X, Y), \quad \min : (f, g) \mapsto \min\{f, g\}
\]
and
\[
\max : C_{\mathcal{I}_F}(X, Y) \times C_{\mathcal{I}_F}(X, Y) \to C_{\mathcal{I}_F}(X, Y), \quad \max : (f, g) \mapsto \max\{f, g\},
\]
where \(\min\{f, g\} : x \mapsto \min\{f(x), g(x)\}\) and \(\max\{f, g\} : x \mapsto \max\{f(x), g(x)\}\) for \(x \in X\).

**Lemma 3.1.** For any non-empty set \(Y \subset \mathbb{R}\) and any continuous function \(h : X \to Y\) defined on a topological space \(X\), the functions
\[
\land_h : C_{\mathcal{I}_F}(X, Y) \to C_{\mathcal{I}_F}(X, Y), \quad \land_h : f \mapsto \min\{f, h\}, \quad \text{and}
\]
\[
\lor_h : C_{\mathcal{I}_F}(X, Y) \to C_{\mathcal{I}_F}(X, Y), \quad \lor_h : f \mapsto \max\{f, h\},
\]
are continuous.

**Proof.** For the continuity of the function \(\land_h\), it suffices to prove that for any open set \(U \subset X\), compact set \(K \subset X\) and real number \(r\) the preimages
\[
\land_h^{-1}([U; r]) \quad \text{and} \quad \land_h^{-1}([K; r])
\]
are open in \(C_{\mathcal{I}_F}(X, Y)\).

To show that \(\land_h^{-1}([U; r])\) is open, fix any function \(f \in \land_h^{-1}([U; r])\). It follows that \(\min\{f, h\} \in [U; r]\) and hence \(\min\{f(x), h(x)\} > r\) for some \(x \in U\). By the continuity of
the functions $f$ and $h$, the point $x$ has an open neighborhood $O_x \subset U$ such that $\inf f(O_x) > r$ and $\inf h(O_x) > r$. Then $[O_x; r]$ is an open neighborhood of $f$ in $C_{\delta}(X, Y)$ such that $[O_x; r] \subset \Lambda^{-1}(U; r)$.

To show that $\Lambda^{-1}(K; r)$ is open, fix any function $f \in \Lambda^{-1}(K; r)$. It follows that $\min \{f, h\} \in [K; r]$. Consider the closed (and thus compact) subset $K = \{x \in K : h(x) > r\}$ of $K$ and observe that $[K, r]$ is an open neighborhood of $f$, contained in the set $\Lambda^{-1}(K, r)$.

Next, we check that the map $\Lambda$ is continuous. Fix an open set $U \subset X$, a compact set $K \subset X$, and a real number $r$.

To show that $\Lambda^{-1}(U; r)$ is open, fix any function $f \in \Lambda^{-1}(U; r)$. It follows that $\max \{f, h\} \in [U; r]$ and hence $\max \{f(x), h(x)\} > r$ for some $x \in U$. If $h(x) > r$, then $\Lambda^{-1}(U; r) = C_{\delta}(X, Y)$ is trivially open in $C_{\delta}(X, Y)$. If $h(x) \leq r$, then $f(x) > r$ and then $f \in [U; r] \subset \Lambda^{-1}(U; r)$ and $f$ is an interior point of $[U; r]$.

To show that $\Lambda^{-1}(K; r)$ is open, fix any function $f \in \Lambda^{-1}(K; r)$. It follows that $\max \{f, h\} \in [K; r]$ and hence $\max f(K) < r$ and $\max h(K) < r$. Then $f \in [K; r] \subset \Lambda^{-1}(K; r)$. □

4. Extension of functions defined on $Y$-separated spaces

In this section we establish one helpful extension property of $Y$-separated spaces.

**Lemma 4.1.** Let $Y \subset \mathbb{R}$ be a non-empty subspace and $X$ be a $Y$-separated topological space. Any continuous function $f : K \to Y$ defined on a compact subset $K \subset X$ admits a continuous extension $\tilde{f} : X \to Y$.

**Proof.** The conclusion of the lemma is trivially true if $Y$ is a singleton. So, assume that $Y$ contains more than one point. Let $Z = [0, 1]$ if $Y$ is connected and $Z = \{0, 1\}$ if $Y$ is disconnected. The $Y$-separated property of $X$ implies that the space $X$ is $Z$-separated.

Then for any distinct points $a, b \in K$ we can choose a continuous function $\delta_{a,b} : X \to Z$ such that $\delta_{a,b}(a) \neq \delta_{a,b}(b)$. Let $D = \{(a, b) \in K \times K : a \neq b\}$ and observe that the map

$$
\delta : X \to Z^D, \quad \delta : x \mapsto (\delta_{a,b}(x))_{(a, b) \in D}
$$

is continuous and its restriction $h = \delta|K : K \to \delta(K) \subset Z^D$ is injective and hence is a homeomorphism of the compact space $K$ onto $\delta(K)$. The set $\delta(K) \subset Z^D$ is compact and hence closed in the compact Hausdorff space $Z^D$. We claim that the continuous map

$$
g : \delta(K) \to Y, \quad g : z \mapsto f \circ h^{-1}(z),
$$

admits a continuous extension $\tilde{g} : Z^D \to Y$.

If $Y$ is connected, then this follows from the normality of the compact Hausdorff space $Z^D$ and the Tietze-Urysohn Theorem 2.1.8 in [8].

If $Y$ is disconnected, then the space $Z^D = \{0, 1\}^D$ is zero-dimensional, and the continuous map $g : \delta(K) \to g(K) \subset Y$ has a continuous extension $\tilde{g} : Z^D \to g(K) \subset Y$ by Proposition 6.1.10 in [2].

Then $\tilde{f} := \tilde{g} \circ \delta : X \to Y$ is a required continuous extension of the map $f : K \to Y$. □
5. The \( \infty \)-density of some subsets in \( C_{\mathcal{F}}(X, Y) \)

**Lemma 5.1.** Let \( Y \subset \mathbb{R} \) be a subset and \( X \) be a \( Y \)-separated space. For any non-empty compact nowhere dense set \( K \subset X \) and any real numbers \( y < u \) with \( y \in Y \) the basic open set \( [K; u] \) is \( \infty \)-dense in \( C_{\mathcal{F}}(X, Y) \).

**Proof.** If \( u \) is greater than any element of \( Y \), then \( [K; u] = C_{\mathcal{F}}(X, Y) \) and there is nothing to prove. So, we assume that \( u \leq \bar{y} \) for some \( \bar{y} \in Y \).

Given any compact Hausdorff space \( Z \), we need to prove that the subset
\[
C_k(Z, [K; u]) = \{ f \in C_k(Z, C_{\mathcal{F}}(X, Y)) : f(Z) \subset [K; u] \}
\]
is dense in \( C_k(Z, C_{\mathcal{F}}(X, Y)) \). Fix any function \( \mu \in C_k(Z, C_{\mathcal{F}}(X, Y)) \) and a neighborhood \( O_\mu \) of \( \mu \) in \( C_k(Z, C_{\mathcal{F}}(X, Y)) \). Given a point \( z \in Z \), it will be convenient to denote the function \( \mu(z) \in C_{\mathcal{F}}(X, Y) \) by \( \mu_z \).

By definition, the Fell-hypograph topology \( \mathcal{B} \) on \( C_{\mathcal{F}}(X, Y) \) has a base \( \mathcal{B} \) consisting of the sets
\[
[U_1; a_1] \cap \cdots \cap [U_n; a_n] \cap [K_1; b_1] \cap \cdots \cap [K_m; b_m]
\]
where \( U_1, \ldots, U_n \subset X \) are non-empty open sets, \( K_1, \ldots, K_m \subset X \) are non-empty compact sets, and \( a_1, \ldots, a_n, b_1, \ldots, b_m \in \mathbb{R} \).

On the other hand, the compact-open topology on the space \( C_k(K, C_{\mathcal{F}}(X, Y)) \) is generated by the subbase consisting of the sets
\[
[Z; B] := \{ f \in C_{\mathcal{F}}(X, Y) : f(Z) \subset B \},
\]
where \( Z \) is a non-empty compact set in \( K \) and \( B \in \mathcal{B} \).

So, without loss of generality, we can assume that the neighborhood \( O_\mu \) is of basic form
\[
O_\mu = \bigcap_{i=1}^m [Z_i; B_i]
\]
for some non-empty compact sets \( Z_1, \ldots, Z_m \subset Z \) and some basic open sets \( B_1, \ldots, B_m \in \mathcal{B} \).

For every \( i \leq m \) find a non-empty finite family \( \mathcal{U}_i \) of non-empty open sets in \( X \), a finite family \( \mathcal{K}_i \) of non-empty compact sets in \( X \), and two functions \( a_i : \mathcal{U}_i \to \mathbb{R} \) and \( b_i : \mathcal{K}_i \to \mathbb{R} \) such that
\[
B_i = \bigcap_{U \in \mathcal{U}_i} [U_i; a_i(U_i)] \cap \bigcap_{\kappa \in \mathcal{K}_i} [\kappa; b_i(\kappa)].
\]

Let
\[
a := \max \max_{i \leq m} a_i(\mathcal{U}_i).
\]

**Claim 5.2.** There exists a point \( s \in Y \) such that \( s > \max\{a, y\} \).

**Proof.** Consider a point \( z \in Z \) such that \( z \in Z_1 \) and consider the continuous function \( \mu_{z_1} = \mu(z_1) : X \to Y \). It follows from \( z_i \in Z_i \) and \( \mu_{z_i} \in [Z_i, B_i] \) that \( \mu_{z_i} \in B_i \subset [U; a_i(U)] \) and hence \( \sup \mu_{z_1}(U) > a_i(U) = a \). Then there exists an element \( t \in \mu_{z_1}(U) \subset Y \) such that \( t > a \). Then the element \( s = \max\{t, \bar{y}\} \) belongs to \( Y \) and \( s > \max\{a, \bar{y}\} \).

For every \( i \leq m \) and \( z \in Z_i \) consider the function \( \mu_z = \mu(z) \in C_{\mathcal{F}}(X, Y) \) and observe that \( \mu \in O_\mu \subset [Z_i; B_i] \) implies \( \mu_z \in B_i \subset [U; a_i(U)] \). Then for every \( U \in \mathcal{U}_i \) we can choose a point \( x_{z, U} \in U \) such that \( \mu_z(x_{z, U}) > a_i(U) \). Since the compact set \( K \) is nowhere dense in \( X \), we can additionally assume that \( x_{z, U} \notin K \). Using Lemma 5.1, we construct a continuous
function $h_{z;U} : X \to Y$ such that $h_{z;U}(K) = \{y\}$ and $h_{z;U}(x_{z;U}) = s > a \geq a_i(U)$. Then the open set $W_{z;U} := \{x \in U : h_{z;U}(x) > a_i(U)\}$ is an open neighborhood of the point $x_{z;U}$ and $O_{z;U} := \mu^{-1}([W_{z;U}; a_i(U)])$ is an open neighborhood of $z$ in $Z$. By the compactness of $Z_i$, there exists a finite set $F_{i;U} \subset Z_i$ such that $Z_i \subset \bigcup_{z \in F_{i;U}} O_{z;U}$. Consider the continuous function $h : X \to Y$, defined by

$$h = \max\{h_{z;U} : i \leq m, U \in \mathcal{U}_i, z \in F_{i;U}\}$$

and observe that $h(K) = \{y\}$.

Lemma 3.1 implies that the map $\mu' : Z \to C_{\beta\mathcal{F}}(X, Y)$ assigning to each $z \in Z$ the function $\mu'_z = \min\{\mu_z, h\}$ is continuous. Taking into account that $\max\mu'_z(K) \leq \max h(K) = y < u$, we conclude that $\mu'_z \in [K; u]$ and hence $\mu'(Z) \subset [K; u]$.

It remains to check that $\mu' \in O_{u}$. Since $O_{u} = \bigcap_{i=1}^m [Z_i; B_i]$, we should prove that for any $i \leq n$ and point $z_i \in Z_i$, the function $\mu'_z = \min\{\mu_z, h\}$ belongs to the set $B_i = \bigcap_{U \in \mathcal{U}_i} [U; a_i(U)] \cap \bigcap_{n \in \mathcal{K}_i} [\kappa; b_i(\kappa)]$.

Observe that for every $\kappa \in \mathcal{K}_i$ we have $\max\mu'_z(\kappa) \leq \max\mu_z(\kappa) < b_i(\kappa)$. This implies that $\mu'_z \in \bigcap_{n \in \mathcal{K}_i} [\kappa; b_i(\kappa)]$.

To show that $\mu'_z \in \bigcap_{U \in \mathcal{U}_i} [U; a_i(U)]$, take any $U \in \mathcal{U}$ and find $z \in F_{i;U}$ such that $z_i \in O_{z;U}$. By the definition of $O_{z;U} = \mu^{-1}([W_{z;U}; a_i(U)])$, we get $\sup\mu(z) > a_i(U)$. Consequently, there exists a point $w \in W_{z;U} \subset U$ such that $\mu(w) > a_i(U)$. By the definition of the set $W_{z;U} = \{x \in U : h_{z;U}(x) > a_i(U)\}$, we get $h(w) \geq h_{z;U}(w) > a_i(U)$. Then

$$\sup\mu'_z(U) \geq \mu'_z(w) = \min\{\mu_z(w), h(w)\} > a_i(U)$$

and $\mu'_z \in [U; a_i(U)]$.

Therefore,

$$\mu'_z \in \bigcap_{U \in \mathcal{U}_i} [U; a_i(U)] \cap \bigcap_{n \in \mathcal{K}_i} [\kappa; b_i(\kappa)] = B_i$$

and we are done. □

Lemma 5.3. Let $Y \subset \mathbb{R}$ be a subspace such that $\inf Y \notin \bar{Y}$. For any topological space $X$ and any non-empty open subset $V \subset X$ the basic open set $[V; \inf Y]$ is $\infty$-dense in $C_{\beta\mathcal{F}}(X, Y)$.

Proof. Given any compact Hausdorff space $Z$, we need to prove that the subset

$$C_k(Z, [V; \inf Y]) = \{f \in C_k(Z, C_{\beta\mathcal{F}}(X, Y)) : f(Z) \subset [V; \inf Y]\}$$

is dense in $C_k(Z, C_{\beta\mathcal{F}}(X, Y))$. Fix any function $\mu \in C_k(Z, C_{\beta\mathcal{F}}(X, Y))$ and a neighborhood $O_{\mu}$ of $\mu$ in $C_k(Z, C_{\beta\mathcal{F}}(X, Y))$.

We lose no generality assuming that $O_{\mu}$ is of the basic form $O_{\mu} = \bigcap_{i=1}^m [Z_i; B_i]$ for some non-empty compact sets $Z_1, \ldots, Z_m \subset K$ and some sets

$$B_i = \bigcap_{U \in \mathcal{U}_i} [U; a_i(U)] \cap \bigcap_{n \in \mathcal{K}_i} [\kappa; b_i(\kappa)],$$

where $\mathcal{U}_i$ is a finite family of non-empty open sets in $X$, $\mathcal{K}_i$ is a finite non-empty family of non-empty compact sets in $X$, and $a_i : \mathcal{U}_i \to \mathbb{R}, b_i : \mathcal{K}_i \to \mathbb{R}$ are functions.

Let $b = \min_{i \leq m} \min_{n \in \mathcal{K}_i} b_i(\kappa)$. Find $i \leq m$ and $\kappa \in \mathcal{K}_i$ such that $b = b_i(\kappa)$. The inclusion $\mu \in O_{\mu} \subset [Z_i; B_i]$ implies that the set $B_i$ is not empty and hence contains some function $\beta : X \to Y$. For this function we get $\beta \in B_i \subset [\kappa; b_i(\kappa)]$ and hence $\inf Y \leq \max\beta(\kappa) < b_i(\kappa) = b$. So, $b > \inf Y$. Since the point $\inf Y$ is not isolated in $Y$, there exists an element $y \in Y$ such
that $\inf Y < y < b$. Let $h : X \to \{y\} \subset Y$ be the constant function. By Lemma 3.1, the function

$$\mu' : K \to C_{\lambda F}(X, Y), \quad \mu' : z \mapsto \mu'_z := \max\{\mu_z, h\},$$

is continuous. It is easy to see that $\mu'(K) \subset \{U; \inf Y\}$ and $\mu' \in O_\mu$. \hfill $\square$

**Lemma 5.4.** Let $Y \subset \mathbb{R}$ be a subspace such that $\inf Y \in Y$. For any $Y$-separated space $X$, any open subset $V \subset X$ with non-compact closure $\overline{V}$, and any real number $u$ with $\{y \in Y : y > u\} \neq \emptyset$, the basic open set $[V, u]$ is $\infty$-dense in $C_{\lambda F}(X, Y)$.

**Proof.** Given any compact Hausdorff space $Z$, we need to prove that the subset

$$C_k(Z, [V; u]) = \{f \in C_k(Z, C_{\lambda F}(X, Y)) : f(Z) \subset [V; u]\}$$

is dense in $C_k(Z, C_{\lambda F}(X, Y))$. Fix any function $\mu \in C_k(Z, C_{\lambda F}(X, Y))$ and a neighborhood $O_\mu$ of $\mu$ in $C_k(Z, C_{\lambda F}(X, Y))$.

We lose no generality assuming that $O_\mu$ is of the basic form $O_\mu = \bigcap_{i=1}^m [Z_i; B_i]$ for some non-empty compact sets $Z_1, \ldots, Z_m \subset K$ and some sets

$$B_i = \bigcap_{\mathclap{U \in U_i}} [U; a_i(U)] \cap \bigcap_{\mathclap{\kappa \in K_i}} [\kappa; b_i(\kappa)],$$

where $U_i$ is a finite non-empty family of non-empty open sets in $X$, $K_i$ is a finite non-empty family of non-empty compact sets in $X$, and $a_i : U_i \to \mathbb{R}$, $b_i : K_i \to \mathbb{R}$ are functions.

Let $a = \max_{i \leq m} \max_{U \in U_i} a_i(U_i)$.

**Claim 5.5.** There exists an element $s \in Y$ such that $s > \max\{a, u\}$.

**Proof.** If $a \leq u$, then take any element $s \in Y$ with $s > u$ and conclude that $s > u = \max\{a, u\}$.

So, we assume that $a > u$. Find $i \leq m$ and $U \in U_i$ with $a = a_i(U)$. Since $\mu \in O_\mu \subset [Z_i; B_i]$, the set $B_i$ is not empty and hence contains some function $\beta \in B_i \subset [U, a_i(U)]$. Then $\sup \beta(U) > a_i(U) = a$ and hence there exists a point $s \in \beta(U) \subset Y$ with $s > a = \max\{a, u\}$.

Consider the compact set $\kappa = \bigcup_{i \leq m} \bigcup K_i$. Since the set $V$ has non-compact closure, $V \not\subseteq \kappa$, so we can choose a point $v \in V \setminus \kappa$. Applying Lemma 4.1, find a continuous function $h : X \to Y$ such that $h(\kappa) \subset \{\inf Y\}$ and $h(v) = s$. By Lemma 3.1, the map

$$\mu' : K \to C_{\lambda F}(X, Y), \quad \mu' : z \mapsto \max\{\mu(z), h\},$$

is continuous. It is easy to see that $\mu'(K) \subset [V, u]$ and $\mu' \in O_\mu$. \hfill $\square$

For sets $A \subset X$ and $B \subset Y \subset \mathbb{R}$ let

$$[A; B] := \{f \in C_{\lambda F}(X, Y) : f(A) \subset B\}.$$ 

For a point $y \in Y \subset \mathbb{R}$ we put $\downarrow y := \{u \in Y : u \leq y\}$. The following lemma is a modification of Lemma 5.1.

**Lemma 5.6.** For any subset $Y \subset \mathbb{R}$, real numbers $y < \bar{y}$ in the set $Y$, and a topological space $X$, the set $[X'; \downarrow y]$ is $\infty$-dense in the subspace $[X'^0; \downarrow y]$ of $C_{\lambda F}(X, Y)$.

**Proof.** Given any compact Hausdorff space $Z$, we need to prove that the subset

$$C_k(Z, [X'; \downarrow y]) = \{\mu \in C_k(Z, C_{\lambda F}(X, Y)) : \mu(Z) \subset [X'; \downarrow y]\}$$

is dense in $C_k(Z, [X'^0; \downarrow y])$. Fix any function $\mu \in C_k(Z, [X'^0; \downarrow y])$ and a neighborhood $O_\mu$ of $\mu$ in $C_k(Z, [X'^0; \downarrow y])$. The following lemma is a modification of Lemma 5.1.
Proof. Being Lemma 6.2. C the set Lemma 6.1. Following lemma is a partial case of Lemma 5.6. Where
\( U \) family of non-empty compact sets in \( X \), \( K \) is a finite non-empty family of non-empty compact sets in \( X \), and \( a_i : U_i \to \mathbb{R} \), \( b_i : K \to \mathbb{R} \) are functions.

Let \( a = \max_{i \leq m} \max_{U \in U_i} a_i(U_i) \) and \( b = \min_{i \leq m} \min_{\kappa \in K_i} b_i(\kappa) \). Repeating the argument of Claim 5.2 we can find a real number \( s \in Y \) such that \( s > a \) and \( s \geq y > y \).

For every \( i \leq m \) and \( z \in Z \), consider the function \( \mu_z = \mu(z) \in [X^\infty; \downarrow y] \) and observe that \( \mu \in O_\mu \subset [X^\infty; \downarrow y] \cap [Z_i; B_i] \) implies \( \mu_z \in B_i \subset \bigcap_{U \in \mathcal{U}_i} [U; a_i(U)] \). Then for every \( U \in \mathcal{U}_i \) with \( \mu_z(\mu) \geq y \), we can choose a point \( x_{z.U} \in U \) such that \( \mu_z(x_{z.U}) > a_i(U) \). Since \( a_i(U) \geq y \), the inclusion \( \mu_z \in [X^\infty; \downarrow y] = [X^\infty; \downarrow y] \) implies that \( x_{z.U} \notin \bar{X}^\infty \). Since the set \( X \setminus X^\infty \supset X \setminus \bar{X}^\infty \) is nowhere dense in \( X \), we can replace \( x_{z.U} \) by a near point in the point \( U \setminus X \) and additionally assume that \( x_{z.U} \in \bar{X} \).

It follows that \( O_{z,U} := \mu^{-1}([\{x_{z.U}\}; a_i(U)]) \) is an open neighborhood of \( z \) in \( Z \). By the compactness of \( Z_i \), there exists a finite set \( F_{i,U} \subset Z_i \) such that \( Z_i \subset \bigcup_{z \in F_{i,U}} O_{z,U} \). Consider the finite set

\[
E = \bigcup_{i=1}^{m} \bigcup_{U \in U_i} \{x_{z,U}: a_i(U) \geq y, \ z \in F_{i,U}\} \subset \bar{X}
\]

and define a continuous function \( h : X \to Y \) by the formula

\[
h(x) = \begin{cases} s & \text{if } x \in E; \\ y & \text{if } x \in X \setminus E. \end{cases}
\]

Lemma 3.1 implies that the map \( \mu' : Z \to [X^\infty; \downarrow y] \) assigning to each \( z \in Z \) the function \( \mu'_z = \min\{\mu_z, h\} \) is continuous. Taking into account that \( \max \mu'_z(\mu) \leq \max h(X') = y \), we conclude that \( \mu'_z \in [X'; \downarrow y] \) and hence \( \mu'(Z) \subset [X'; \downarrow y] \).

By analogy with the proof of Lemma 5.1, we can show that \( \mu' \in O_\mu \). \( \square \)

6. The subspace \( C'_\mathcal{F}(X,Y) \)

Given a topological space \( X \) and a subset \( Y \subset \mathbb{R} \) with \( \inf Y \in Y \), consider the subset

\[
C'_\mathcal{F}(X,Y) := \{f \in C'_\mathcal{F}(X,Y) : f(X') \subset \{\inf Y\}\} = [X'; \{\inf Y\}]
\]

in the function space \( C'_\mathcal{F}(X,Y) \).

In this section we establish some properties of the subspace \( C'_\mathcal{F}(X,Y) \) of \( C_\mathcal{F}(X,Y) \). The following lemma is a partial case of Lemma 5.6.

Lemma 6.1. For any subset \( Y \subset \mathbb{R} \) with \( \inf Y \in Y \neq \{\inf Y\} \) and any topological space \( X \) the set \( C'_\mathcal{F}(X,Y) \) is \( \infty \)-dense in the subspace \( [X^\infty; \{\inf Y\}] \) (which is equal to \( C'_\mathcal{F}(X,Y) \) if \( X^\infty = \emptyset \)).

Lemma 6.2. If \( X \) is \( Y \)-separable, then \( C'_\mathcal{F}(X,Y) \) is a \( G_\delta \)-set in \( C_\mathcal{F}(X,Y) \).

Proof. Being \( Y \)-separable, the space \( X \) contains a meager \( \sigma \)-compact set \( M \) such that \( X' = \overline{M}' \). Write \( M \) as the countable union \( M = \bigcup_{n \in \omega} M_n \) of compact nowhere dense sets \( M_n \subset M_{n+1} \) in \( X \). Fix a strictly decreasing sequence \( (y_n)_{n \in \omega} \) of real numbers such that \( \inf_{n \in \omega} y_n = \inf Y \).
The equality $X' = \overline{M}'$ implies the equality $C'_f(X, Y) = \bigcap_{n<\omega} [M_n; y_n]$, which means that $C'_f(X, Y)$ is a $G_\delta$-set in $C_f(X, Y)$. \hfill \Box$

**Lemma 6.3.** The Fell hypograph topology on $C'_f(X, Y)$ coincides with the compact-open topology.

**Proof.** Since the Fell hypograph topology is weaker than the compact open topology, it suffices to show that each subbasic set $W$ of the compact-open topology of $C'_f(X, Y)$ is contained in the Fell hypograph topology of the space $C'_f(X, Y)$.

First assume that $W = [K; a] \cap C'_f(X, Y) := \{ f \in C'_f(X, Y) : \min(K) > a \}$ for some non-empty compact set $K \subset X$ and some $a \in \mathbb{R}$. Fix any function $f \in W$ and observe that $f \in C'_f(X, Y)$ and $\min(f(K)) > a$ imply that $a < \inf Y$ or $K \cap X' = \emptyset$. If $a < \inf Y$, then $W = C'_f(X, Y)$ is (trivially) open in the Fell hypograph topology of the space $C'_f(X, Y)$.

If $K \cap X' = \emptyset$, then $K$ is finite and open in $X$. Then $f \in \bigcap_{x \in K} \{ x \} ; a \cap C'_f(X, Y) \subset W$, which means that $W$ is open in the Fell hypograph topology of $C'_f(X, Y)$.

If $W = [K; b] \cap C'_f(X, Y)$ for some non-empty compact set $K \subset X$ and some $b \in \mathbb{R}$, then by definition, $W$ is open in the Fell hypograph topology on $C'_f(X, Y)$. \hfill \Box$

Lemma 6.3 allows us to identify the subspace $C'_k(X, Y)$ of $C_f(X, Y)$ with the subspace

$$C'_k(X, Y) := \{ f \in C_k(X, Y) : f(X') \subset \{ \inf Y \} \} = [X'; \{ \inf Y \}]$$

of the function space $C_k(X, Y)$ endowed with the compact-open topology.

The Baire category properties of the function spaces $C'_k(X, Y)$ are described in the following theorem, proved in [5].

**Theorem 6.4.** Let $X$ be a topological space containing an isolated point, and $Y \subset \mathbb{R}$ be a set with $\inf Y \in Y \neq \{ \inf Y \}$.

1. If $X$ does not have DMOP, then the function space $C'_k(X, Y)$ is $\infty$-meager.
2. If $X$ has DMOP and the space $Y$ is almost Polish, then $C'_k(X, Y)$ is Baire.
3. $C'_k(X, Y)$ is Choquet if and only if $Y$ is almost Polish and $X$ has WDMOP.
4. $C'_k(X, Y)$ is (almost) complete-metrizable if and only if $Y$ is (almost) Polish and $X$ is a $\kappa$-$\omega$-space.
5. The function space $C'_k(X, Y)$ is (almost) Polish if and only if $Y$ is (almost) Polish and $X$ is a $\kappa$-$\omega$-space with countable set $X$ of isolated points.
6. The function space $C'_k(X, Y)$ is separable if $Y$ is separable and $X$ is countable.

In [5] we also proved the following dichotomy for analytic spaces $C'_k(X, Y)$.

**Theorem 6.5.** Let $Y \subset \mathbb{R}$ be a Polish subspace with $\inf Y \in Y$. If for a topological space $X$ the function space $C'_k(X, Y)$ is analytic, then $C'_k(X, Y)$ is either Polish or $\infty$-meager.

In fact, under some assumptions, the analyticity of the function space $C'_k(X, Y)$ is equivalent to the analyticity of the function space $C'_p(X, Y) = \{ f \in C_p(X, Y) : f(X') \subset \{ \inf Y \} \} \subset Y^X$, endowed with the topology of pointwise convergence. The following characterization was proved in [5].

**Proposition 6.6.** For any non-empty subspace $Y \subset \mathbb{R}$ and a topological space $X$, the function space $C'_p(X, Y)$ is analytic if and only if $C'_k(X, Y)$ has a countable network and the function space $C'_p(X, Y)$ is analytic.
7. Recognizing $\infty$-meager function spaces $C_{\infty}(X, Y)$

In this section we find some conditions on spaces $Y \subset \mathbb{R}$ and $X$ under which the function space $C_{\infty}(X, Y)$ is $\infty$-meager.

**Lemma 7.1.** Let $Y \subset \mathbb{R}$ be a non-empty subset with $\inf Y \notin Y$. For any non-discrete $Y$-separated topological space $X$, the function space $C_{\infty}(X, Y)$ is $\infty$-meager.

**Proof.** Since $\inf Y \notin Y$, we can fix a strictly decreasing sequence $\{y_n\}_{n \in \omega} \subset Y$ such that $\inf_{n \in \omega} y_n = \inf Y$. Take any non-isolated point $x \in X$. By Lemma 5.1 for every $n \in \omega$ the basic open set $\{\{x\}; y_n\}$ is $\infty$-dense in $C_{\infty}(X, Y)$. Then its complement $C_{\infty}(X, Y) \setminus \{\{x\}; y_n\}$ is a closed $\infty$-codense set in $C_{\infty}(X, Y)$. Since

$$C_{\infty}(X, Y) = \bigcup_{n \in \omega} (C_{\infty}(X, Y) \setminus \{\{x\}; y_n\}),$$

the space $C_{\infty}(X, Y)$ is $\infty$-meager. \(\square\)

A topological space $X$ is called a $T_{-1}$-space if for each point $x \in X$ the singleton $\{x\}$ is closed in $X$. A point $x$ of a $T_{-1}$-space is isolated if and only if the singleton $\{x\}$ is open in $X$ if and only if $\{x\}$ is not nowhere dense in $X$.

**Lemma 7.2.** Let $Y \subset \mathbb{R}$ be a non-empty subset with $\inf Y \notin Y$. For any non-discrete $T_{-1}$-space $X$ with dense set $\overline{X}$ of isolated points, the function space $C_{\infty}(X, Y)$ is $\infty$-meager.

**Proof.** Fix a strictly decreasing sequence of real numbers $\{y_n\}_{n \in \omega} \subset Y$ such that $\inf_{n \in \omega} y_n = \inf Y$. Take any non-isolated point $x \in X$. By Lemma 5.6 for every $n \in \omega$ the basic open set $\{\{x\}; y_n\}$ is $\infty$-dense in $C_{\infty}(X, Y)$. Then its complement $C_{\infty}(X, Y) \setminus \{\{x\}; y_n\}$ is a closed $\infty$-codense set in $C_{\infty}(X, Y)$, and the space $C_{\infty}(X, Y) = \bigcup_{n \in \omega} (C_{\infty}(X, Y) \setminus \{\{x\}; y_n\})$ is $\infty$-meager. \(\square\)

**Lemma 7.3.** Let $Y \subset \mathbb{R}$ be a subset with $\inf Y \notin Y \neq \{\inf Y\}$. Let $X$ be a $Y$-separated topological space containing a meager $\sigma$-compact set $M$ such that $\emptyset \neq X^{\infty} \subset \overline{M}^{Y}$. Then the basic open set $[X^{\infty}, \inf Y]$ is $\infty$-meager in $C_{\infty}(X, Y)$.

**Proof.** Fix a strictly decreasing sequence of real numbers $(y_n)_{n \in \omega}$ with $\lim_{n \to \infty} y_n = \inf Y$. Write the meager $\sigma$-compact set $M$ as the union $M = \bigcup_{n \in \omega} M_n$ of compact nowhere dense sets $M_n \subset M_{n+1}$ in $X$. By Lemma 5.11 for every $n \in \omega$ the basic open set $[M_n, y_n]$ is $\infty$-dense in $C_{\infty}(X, Y)$. Then the complement $C_{\infty}(X, Y) \setminus [M_n, y_n]$ is a closed $\infty$-codense set in $C_{\infty}(X, Y)$. It follows from $X^{\infty} \subset \overline{M}^{Y}$ that

$$[X^{\infty}, \inf Y] \subset \bigcup_{n \in \omega} (C_{\infty}(X, Y) \setminus [M_n, y_n]),$$

which means that the set $[X^{\infty}, \inf Y]$ is $\infty$-meager in $C_{\infty}(X, Y)$. \(\square\)

**Lemma 7.4.** Let $Y \subset \mathbb{R}$ be a subset with $\inf Y \notin Y \neq \{\inf Y\}$. Let $X$ be a $Y$-separated topological space containing a meager $\sigma$-compact set $M$ such that $\emptyset \neq X^{\infty} \subset \overline{M}^{Y}$. If $\inf Y \notin \hat{Y}$ or $X^{\infty}$ is not compact, then the function space $C_{\infty}(X, Y)$ is $\infty$-meager.

**Proof.** By Lemma 7.3 the basic open set $[X^{\infty}, \inf Y]$ is $\infty$-meager in $C_{\infty}(X, Y)$ and by Lemmas 5.3 and 5.4 the set $[X^{\infty}, \inf Y]$ is $\infty$-dense in $C_{\infty}(X, Y)$. Then its complement $C_{\infty}(X, Y) \setminus [X^{\infty}, \inf Y]$ is closed and $\infty$-codense in $C_{\infty}(X, Y)$. Consequently, the set

$$C_{\infty}(X, Y) = (C_{\infty}(X, Y) \setminus [X^{\infty}, \inf Y]) \cup [X^{\infty}, \inf Y]$$

is $\infty$-meager in $C_{\infty}(X, Y)$. \(\square\)
is $\infty$-meager (being the countable union of two $\infty$-meager sets in $C_{1_\mathbb{F}}(X, Y)$).

\[\□\]

**Lemma 7.5.** Let $Y \subseteq \mathbb{R}$ be a subset containing more than one point and $X$ be a $Y$-separable $T_1$-space with dense set $X$ of isolated points. If the space $X$ does not have $\text{DMOP}$, then the function space $C_{1_\mathbb{F}}(X, Y)$ is $\infty$-meager.

**Proof.** Since the space $X$ does not have $\text{DMOP}$, it is not discrete. If $\inf Y \notin Y$, then the space $C_{1_\mathbb{F}}(X, Y)$ is $\infty$-meager according to Lemma 7.2. So, we assume that $\inf Y \in Y$. Choose a strictly decreasing sequence of real numbers $(y_n)_{n \in \omega}$ with $\inf_{n \in \omega} y_n = \inf Y$.

Being $Y$-separable, the space $X$ contains a meager $\sigma$-compact subset $M$ with $X' \subseteq \mathcal{M}^Y$. Write $M$ as the union $M = \bigcup_{n \in \omega} M_n$ of compact nowhere dense sets $M_n \subset M_{n+1}$ in $X$.

By Lemma 6.6 for every $n \in \omega$ the open basic set $[M_n, y_n]$ is $\infty$-dense in $C_{1_\mathbb{F}}(X, Y)$. Then the complement $C_{1_\mathbb{F}}(X, Y) \setminus [M_n, y_n]$ is a closed $\infty$-codense set in $C_{1_\mathbb{F}}(X, Y)$.

By Lemma 6.1 the subspace $C'_{1_\mathbb{F}}(X, Y)$ is $\infty$-dense in $C_{1_\mathbb{F}}(X, Y)$ and by Lemma 6.3 and Theorem 6.4(1), the space $C'_{1_\mathbb{F}}(X, Y) = C'_{1_\mathbb{F}}(X, Y)$ is $\infty$-meager. So, $C'_{1_\mathbb{F}}(X, Y)$ can be written as the countable union $C'_{1_\mathbb{F}}(X, Y) = \bigcup_{n \in \omega} F_n$ of closed $\infty$-codense sets in $C'_{1_\mathbb{F}}(X, Y)$. For every $n \in \omega$ let $\bar{F}_n$ be the closures of the set $F_n$ in $C_{1_\mathbb{F}}(X, Y)$.

We claim that the set $\bar{F}$ is $\infty$-codense in $C_{1_\mathbb{F}}(X, Y)$. Given any compact Hausdorff space $K$ and a non-empty open set $W \subset C_k(K, C_{1_\mathbb{F}}(X, Y))$, we need to find a map $\mu \in W$ with $\mu(K) \cap \bar{F} = \emptyset$. Since the space $C_{1_\mathbb{F}}(X, Y)$ is $\infty$-dense in $C_{1_\mathbb{F}}(X, Y)$, the intersection $W \cap C_k(K, C'_{1_\mathbb{F}}(X, Y))$ is a non-empty open set in the function space $C_k(K, C'_{1_\mathbb{F}}(X, Y))$. Since the set $\bar{F}$ is $\infty$-codense in $C'_{1_\mathbb{F}}(X, Y)$, there exists a map $\mu \in \bar{F} \cap C_k(K, C'_{1_\mathbb{F}}(X, Y))$ such that $\mu(K) \cap \bar{F} = \emptyset$. Then

$$\mu(K) \cap \bar{F} = (\mu(K) \cap C'_{1_\mathbb{F}}(X, Y)) \cap \bar{F} = \mu(K) \cap (C'_{1_\mathbb{F}}(X, Y) \cap \bar{F}) = \mu(K) \cap F = \emptyset.$$  

Now we see that the space

$$C_{1_\mathbb{F}}(X, Y) = C'_{1_\mathbb{F}}(X, Y) \cup (C_{1_\mathbb{F}}(X, Y) \setminus C'_{1_\mathbb{F}}(X, Y)) \subset \bigcup_{n \in \omega} \bar{F}_n \cup \bigcup_{n \in \omega} (C_{1_\mathbb{F}}(X, Y) \setminus [M_n; y_n])$$

is $\infty$-meager, being the countable union of closed $\infty$-codense sets.  

\[\□\]

**Lemma 7.6.** Let $Y \subseteq \mathbb{R}$ be a set containing more than one point and $X$ be a $Y$-separable $Y$-separable topological space. If the space $X$ does not have $\text{DMOP}$, then the function space $C_{1_\mathbb{F}}(X, Y)$ is $\infty$-meager.

**Proof.** Since the space $X$ does not have $\text{DMOP}$, it is not discrete. If $\inf Y \notin Y$, then the function space $C_{1_\mathbb{F}}(X, Y)$ is $\infty$-meager by Lemma 7.5. So, we assume that $\inf Y \in Y$.

If the set $X'$ is nowhere dense in $X$, then the space $C_{1_\mathbb{F}}(X, Y)$ is $\infty$-meager by Lemma 7.5. So, we assume that the interior $X'^\circ$ is not empty. If the closure $X'^\circ$ is not compact or $\inf Y \notin Y'$, then the space $C_{1_\mathbb{F}}(X, Y)$ is $\infty$-meager by Lemma 7.4. So, we assume that $X'^\circ$ is compact and $\inf Y' \in Y'$.

In this case we can choose a real number $\varepsilon > 0$ such that $\{\inf Y \} = \{y \in Y : y < \varepsilon\}$ and conclude that the set $[X'^\circ; \{\inf Y\}] = [X'^\circ, \varepsilon] = C_{1_\mathbb{F}}(X, Y) \setminus [X'^\circ, \inf Y]$ is clopen in $C_{1_\mathbb{F}}(X, Y)$. By Lemma 6.1, the space $C'_{1_\mathbb{F}}(X, Y)$ is $\infty$-dense in the clopen subspace $[X'^\circ; \{\inf Y\}]$ of $C_{1_\mathbb{F}}(X, Y)$.

By Theorem 6.4(1), the space $C'_{1_\mathbb{F}}(X, Y)$ is $\infty$-meager. So, $C_{1_\mathbb{F}}(X, Y) = \bigcup_{n \in \omega} F_n$ for some closed $\infty$-codense sets $F_n$ in $C_{1_\mathbb{F}}(X, Y)$. Let $\bar{F}_n$ be the closure of the set $F_n$ in the space $X'$.

\[\□\]
$C_{iF}(X,Y)$. Taking into account that $C_{iF}(X,Y)$ is $\infty$-dense in $[X^\infty;\{\inf Y\}]$, we can show that each set $\bar{F}_n$ is $\infty$-codense in $[X^\infty;\{\inf Y\}]$ by analogy with the proof of Lemma 7.5. Since $[X^\infty;\{\inf Y\}]$ is clopen in $C_{iF}(X,Y)$, the set $\bar{F}_n$ is $\infty$-codense in $C_{iF}(X,Y)$.

Fix a strictly decreasing sequence of real numbers $(y_n)_{n \in \omega}$ with $\lim_{n \to \infty} y_n = \inf Y$. Being $Y$-separable, the space $X$ contains a meager $\sigma$-compact $M$ such that $X' \subset \overline{M}$. Write $M$ as the union $M = \bigcup_{n \in \omega} M_n$ of compact nowhere dense sets $M_n \subset M_{n+1}$ in $X$. By Lemma 7.1 for every $n \in \omega$ the open basic set $[M_n, y_n]$ is $\infty$-dense in $C_{iF}(X,Y)$. Then the complement $C_{iF}(X,Y) \setminus [M_n, y_n]$ is a closed $\infty$-codense set in $C_{iF}(X,Y)$.

Now we see that the space

$$C_{iF}(X,Y) = C_{iF}(X,Y) \cup (C_{iF}(X,Y) \setminus C_{iF}(X,Y)) \subset \bigcup_{n \in \omega} \bar{F}_n \cup \bigcup_{n \in \omega} (C_{iF}(X,Y) \setminus [M_n, y_n])$$

is $\infty$-meager (being the countable union of closed $\infty$-codense sets). \hfill \Box

**Lemma 7.7.** Let $Y \subset \mathbb{R}$ be a subset, $Z \subset Y$ be an open zero-dimensional subspace in $Y$ and $F \subset Z$ be a closed nowhere dense subset in $Y$. Then the set $F$ is $\infty$-codense in $Y$.

**Proof.** Given any compact Hausdorff space $K$, a continuous map $f : K \to Y$ and a neighborhood $O_f \subset C_k(K,Y)$, we need to find a continuous map $g \in O_f$ such that $g(K) \cap F = \emptyset$. By the normality of the space $Y$, there exists an open set $W \subset Y$ such that $F \subset W \subset \overline{W} \subset Z$. The zero-dimensional space $Z$, being metrizable and separable, has large inductive dimension zero, see [3, 7.3.3]. Consequently, there exists a clopen set $V \subset Z$ such that $F \subset V \subset W$. Since $V \subset W \subset \overline{W} \subset Z$, the clopen subset $V$ of $Z$ remains clopen in the space $Y$.

By [3, 8.2.7], the compact-open topology on the function space $C_k(K,Y)$ is generated by the metric $\rho(g,g') = \sup_{x \in K} |g(x) - g'(x)|$, where $g, g' \in C_k(K,Y)$. Consequently, we can find $\varepsilon > 0$ such that any map $g : K \to Y$ with $\rho(f,g) < \varepsilon$ belongs to the neighborhood $O_f$ of $f$. The space $f(K) \cap V$ is compact and zero-dimensional. So, admits a finite cover $\{V_1, \ldots, V_n\}$ by pairwise disjoint clopen sets in $V$ of diameter $< \varepsilon$. For each $i \leq n$ choose a point $v_i \in V_i \setminus F$. Consider the continuous map $g : K \to Y$ defined by the formula:

$$g(x) = \begin{cases} v_i & \text{if } f(x) \in V_i \text{ for some } i \leq n; \\ f(x) & \text{otherwise.} \end{cases}$$

Then $\rho(f,g) < \varepsilon$ and hence $g \in O_f$. Also

$$g(K) = g(f^{-1}(f(K) \cap V)) \cup g(f^{-1}(f(K) \setminus V)) \subset \{v_1, \ldots, v_n\} \cup (f(K) \setminus V) \subset Y \setminus F.$$ \hfill \Box

**Lemma 7.8.** For any meager space $Y \subset \mathbb{R}$ and any topological space $X$ containing an isolated point $x$, the function space $C_{iF}(X,Y)$ is $\infty$-meager.

**Proof.** Assume that the space $Y \subset \mathbb{R}$ is meager. Then $Y$ can be written as the countable union $Y = \bigcup_{n \in \omega} Y_n$ of closed nowhere dense sets $Y_n$. Being a meager subset of the real line, the space $Y$ is zero-dimensional. By Lemma 7.7 the set $Y_n$ is $\infty$-codense in $Y$. Since the point $x$ of $X$ is isolated, the map

$$H : C_{iF}(X,Y) \to Y \times C_{iF}(X \setminus \{x\},Y), \quad H : f \mapsto (f(x),f|_{X \setminus \{x\}}),$$

is a homeomorphism. The $\infty$-codensity of the closed set $Y_n$ in $Y$ implies the $\infty$-codensity of the closed set $Y_n \times C_{iF}(X \setminus \{x\},Y)$ in $Y \times C_{iF}(X \setminus \{x\},Y) = H(C_{iF}(X,Y))$. Since $H$ is a
homeomorphism, the closed set
\[ F_n = \{ f \in C_{I\mathcal{F}}(X,Y) : f(x) \in Y_n \} = H^{-1}(Y_n \times C_{I\mathcal{F}}(X \setminus \{x\}, Y)) \]
is \( \infty \)-codense in \( C_{I\mathcal{F}}(X,Y) \). Then the space \( C_{I\mathcal{F}}(X,Y) = \bigcup_{n \in \omega} F_n \) is \( \infty \)-meager, being a countable union of closed \( \infty \)-codense sets \( F_n, n \in \omega \).

**Lemma 7.9.** Let \( Y \subset \mathbb{R} \) be a subspace containing more than one point and \( X \) be a non-empty \( Y \)-separable \( Y \)-separated space. If the space \( Y \) is meager, then the function space \( C_{I\mathcal{F}}(X,Y) \) is \( \infty \)-meager.

**Proof.** If the space \( X \) contains an isolated point, then the function space \( C_{I\mathcal{F}}(X,Y) \) is \( \infty \)-meager by Lemma 7.8. So, we assume that the space \( X \) contains no isolated points. If \( \inf Y \notin Y \), then the space \( C_{I\mathcal{F}}(X,Y) \) is \( \infty \)-meager by Lemma 7.4. So, we assume that \( \inf Y \in Y \). Since the space \( Y \) is meager, the point \( \inf Y \) is not isolated in \( Y \). In this case the space \( C_{I\mathcal{F}}(X,Y) \) is \( \infty \)-meager by Lemma 7.4. \( \square \)

**Lemma 7.10.** Let \( Y \subset \mathbb{R} \) be a non-empty subspace, \( X \) be a topological space, and \( D \subset X \) be an infinite closed set in \( X \). If the space \( Y \) is not Baire, then the function space \( C_{I\mathcal{F}}(X,Y) \) is \( \infty \)-meager.

**Proof.** Replacing \( D \) by a smaller infinite subset, we can assume that \( D \) is countable.

The space \( Y \) is not Baire and hence contains a non-empty open meager subset \( Z \subset Y \). Being a meager subset of the real line, the space \( Z \) is zero-dimensional. Being a meager \( F_\sigma \)-set in \( Y \), the space \( Z \) can be written as the countable union \( Z = \bigcup_{n \in \omega} Z_n \) of nowhere dense closed subsets \( Z_n \) of \( Y \).

Observe that
\[ C_{I\mathcal{F}}(X,Y) = \{ f \in C_{I\mathcal{F}}(X,Y) : f(x) \in Z \} \cup \bigcup_{x \in D} \bigcup_{n \in \omega} \{ f \in C_{I\mathcal{F}}(X,Y) : f(x) \in Z_n \}. \]

Using Lemma 7.7, it can be shown that for every \( x \in D \) and \( n \in \omega \) the closed set \( \{ f \in C_{I\mathcal{F}}(X,Y) : f(x) \in Z_n \} \) is \( \infty \)-codense in \( C_{I\mathcal{F}}(X,Y) \). It remains to prove that the closed set \( \{ f \in C_{I\mathcal{F}}(X,Y) : f(x) \in Z \} \) is \( \infty \)-codense in \( C_{I\mathcal{F}}(X,Y) \).

Given a continuous map \( \mu : K \to C_{I\mathcal{F}}(X,Y) \), defined on a compact Hausdorff space \( K \) and a neighborhood \( O_{\mu} \subset C_b(K,C_{I\mathcal{F}}(X,Y)) \) of \( \mu \), we need to find a continuous map \( \mu' \in O_{\mu} \) such that \( \mu'(K) \subset C_{I\mathcal{F}}(X,Y) \setminus [D; Y \setminus Z] \). We can assume that the neighborhood \( O_{\mu} \) is of basic form
\[ O_{\mu} = \bigcap_{i=1}^m [K_i; B_i] \]where \( K_1, \ldots, K_m \) are compact sets in \( K \) and each set \( B_i \subset C_{I\mathcal{F}}(X,Y) \) is of basic form
\[ B_i = \bigcap_{U \in \mathcal{U}_i} \{ a_i(U) \} \cap \bigcap_{\kappa \in \mathcal{K}_i} [\kappa, b_i(\kappa)], \]
where \( \mathcal{U}_i \) is a non-empty open family of non-empty open sets in \( X \), \( \mathcal{K}_i \) is a non-empty finite family of non-empty compact sets in \( X \) and \( a_i : \mathcal{U}_i \to \mathbb{R}, b_i : \mathcal{K}_i \to \mathbb{R} \) are functions. We can also assume that for every \( i \leq m \) and \( U \in \mathcal{U}_i \) the open set is either singleton \( \{ x \} \subset X \) or \( U \subset X^i \). In this case the set \( E = \bigcup_{i \leq m} (X \cap \bigcup_{U \in \mathcal{U}_i}) \) is finite.

Since the infinite set \( D \) is discrete and closed in \( X \), there exists a point \( d \in D \setminus (E \cup \bigcup_{i=1}^m K_i) \). Fix any point \( z \in Z \) and consider the continuous map \( s : C_{I\mathcal{F}}(X,Y) \to C_{I\mathcal{F}}(X,Y) \setminus [D; Y \setminus Z] \) assigning to each function \( f \in C_{I\mathcal{F}}(X,Y) \) the function \( f' : X \to Y \) such that \( f'(d) = z \) and \( f'(x) = f(x) \setminus \{ d \} \). Then the continuous map \( \mu' = s \circ \mu : K \to C_{I\mathcal{F}}(X,Y) \setminus [D; Y \setminus Z] \) belongs to the neighborhood \( O_{\mu} \), witnessing that the closed set \( [D; Y \setminus Z] \) is \( \infty \)-codense in \( C_{I\mathcal{F}}(X,Y) \). \( \square \)
Lemma 7.11. Let $Y \subset \mathbb{R}$ be a non-empty subspace and $X$ be a $Y$-separable $Y$-separated space such that the set $\hat{X}$ is not contained in a compact subset of $X$. If the space $Y$ is not Baire, then the function space $C_{1}\mathcal{F}(X,Y)$ is $\infty$-meager.

Proof. If the topological space $X$ does not have DMOP, then the function space $C_{1}\mathcal{F}(X,Y)$ is $\infty$-meager by Lemma 7.6. So, we assume that the space $X$ has DMOP. Since the set $\hat{X}$ is not contained in a compact subset of $X$, the family of singletons $\{\{x\} : x \in \hat{X}\}$ is moving off. Since $X$ has DMOP, this family has an infinite discrete subfamily, which implies that $X$ contains an infinite subset $D \subset X$, which is closed in $X$. Now we can apply Lemma 7.10 to conclude that the space $C_{1}\mathcal{F}(X,Y)$ is $\infty$-meager. □

8. Recognizing Baire spaces among function spaces $C_{1}\mathcal{F}(X,Y)$

Lemma 8.1. Let $Y$ be a Baire subspace of the real line. For any discrete topological space $X$, the function space $C_{1}\mathcal{F}(X,Y)$ is Baire.

Proof. Taking into account that $Y$ is second countable and applying [15, Theorem 3], we conclude that the Tychonoff power $Y^{X}$ is Baire. Since $X$ is discrete, the Fell hypograph topology on $C(Y,X) = Y^{X}$ coincides with the Tychonoff product topology on $Y^{X}$, which implies that the function space $C_{1}\mathcal{F}(X,Y)$ is Baire.

Lemma 8.2. Let $Y \subset \mathbb{R}$ be a non-empty space with inf $Y \notin Y$ and $X$ be a $T_{1}$-space with dense set $\hat{X}$ of isolated points. If the function space $C_{1}\mathcal{F}(X,Y)$ is Baire, then $X$ is discrete and $Y$ is Baire.

Proof. Being Baire, the space $C_{1}\mathcal{F}(X,Y)$ is not meager and by Lemma 7.2, the space $X$ is discrete. In this case the Fell hypograph topology on $C(X,Y)$ coincides with the topology of pointwise convergence on $C(X,Y) = Y^{X}$, which implies that the Tychonoff power $Y^{X}$ is Baire and so is the space $Y$. □

Lemma 8.3. Let $Y \subset \mathbb{R}$ be a non-empty space with inf $Y \notin Y$ and $X$ be a $Y$-separated space. If the function space $C_{1}\mathcal{F}(X,Y)$ is Baire, then $X$ is discrete and $Y$ is Baire.

Proof. Being Baire, the space $C_{1}\mathcal{F}(X,Y)$ is not meager and by Lemma 7.1, the space $X$ is discrete. In this case the Fell hypograph topology on $C(X,Y)$ coincides with the topology of pointwise convergence on $C(X,Y) = Y^{X}$, which implies that the Tychonoff power $Y^{X}$ is Baire and so is the space $Y$. □

These three lemmas imply the following characterization.

Lemma 8.4. Let $X$ be a non-empty $T_{1}$-space and $Y \subset \mathbb{R}$ be a non-empty space with inf $Y \notin Y$. Assume that $X$ is $Y$-separated or $\hat{X}$ is dense in $X$. The function space $C_{1}\mathcal{F}(X,Y)$ is Baire if and only if $X$ is discrete and $Y$ is Baire.

Lemma 8.5. Let $Y \subset \mathbb{R}$ be a subspace with inf $Y \notin \{\inf Y\}$, and $X$ be a $Y$-separable $Y$-separated space. If the function space $C_{1}\mathcal{F}(X,Y)$ is Baire, then the set $\hat{X}$ of isolated points is dense in $X$.

Proof. To derive a contradiction, assume that the set $X'$ of isolated points of $X$ has non-empty interior $X'^{\circ}$. Being $Y$-separable, the space $X$ contains meager $\sigma$-compact subset $M$ with $X' \subset \overline{M}'$. Write $M$ as the countable union $M = \bigcup_{n \in \omega} M_{n}$ of nowhere dense compact sets $M_{n} \subset M_{n+1}$ in $X$. 
Choose a strictly decreasing sequence of real numbers \((y_n)_{n \in \omega}\) such that \(\lim_{n \to \infty} y_n = \inf Y\). By Lemma 5.1, for any \(n \in \omega\) the basic open set \([M_n; y_n]\) is dense in \(C_\delta(X, Y)\) and hence its complement \(C_\delta(X, Y) \setminus [M_n; y_n]\) is closed and nowhere dense in \(C_\delta(X, Y)\).

Since \([X^o; \inf Y] \subset \bigcup_{n \in \omega} [C_\delta(X, Y) \setminus [M_n; y_n]]\), the non-empty basic open set \([X^o; \inf Y]\) is meager. So, \(C_\delta(X, Y)\) cannot be Baire. \(\square\)

**Lemma 8.6.** Let \(Y \subset \mathbb{R}\) be a subset with \(\inf Y \in Y\) and \(X\) be a \(T_1\)-space with dense set \(\check{X}\) of isolated points.

1. If \(X\) has DMOP and \(Y\) is almost Polish, then \(C_\delta(X, Y)\) is Baire;
2. If \(C_\delta(X, Y)\) is not meager and \(X\) is \(Y\)-separable, then the space \(X\) has DMOP;
3. If \(C_\delta(X, Y)\) is Baire, then so is the space \(Y\).

**Proof.** 1. Assume that the space \(X\) has DMOP and the space \(Y\) is almost Polish. By Lemma 6.1, the space \(C_\delta'(X, Y)\) is dense in \(C_\delta(X, Y)\). By Lemma 6.3 and Theorem 6.4(2), the function space \(C_\delta'(X, Y) = C_\delta'(X, Y)\) is Baire. Then the space \(C_\delta(X, Y)\) is Baire, too (because it contains a dense Baire subspace).

2. Assume that the function space \(C_\delta'(X, Y)\) is not meager and the space \(X\) is \(Y\)-separable. By Lemmas 6.1 and 6.2, \(C_\delta'(X, Y)\) is a dense \(G_\delta\)-set in \(C_\delta(X, Y)\), which implies that the complement \(C_\delta(X, Y) \setminus C_\delta'(X, Y)\) is meager in \(C_\delta(X, Y)\). Assuming that the space \(C_\delta'(X, Y)\) is meager, we would conclude that the space \(C_\delta(X, Y) = C_\delta'(X, Y) \cup (C_\delta(X, Y) \setminus C_\delta'(X, Y))\) is meager, which contradicts our assumption. This contradiction shows that the space \(C_\delta'(X, Y)\) is not meager. By Lemma 6.3, \(C_\delta'(X, Y) = C_\delta'(X, Y)\) and by Theorem 6.4(1), the space \(X\) has DMOP.

3. Assuming that the space \(C_\delta'(X, Y)\) is Baire, we shall prove that the space \(Y\) is Baire. Take any isolated point \(x \in X\) and consider the subspace \(Z := X \setminus \{x\}\) of \(X\). It is easy to see that the map

\[H : C_\delta(X, Y) \to C_\delta(Z, Y) \times Y, \quad H : f \mapsto (f|Z, f(x)),\]

is a homeomorphism. Then the product \(C_\delta(Z, Y) \times Y\) is Baire and so is the space \(Y\). \(\square\)

**Lemma 8.7.** Let \(Y \subset \mathbb{R}\) be a Polish+meager space with \(\inf Y \in Y \neq \{\inf Y\}\). For any \(Y\)-separable space \(X\) with dense set \(\check{X}\) of isolated points, the following conditions are equivalent:

1. the function space \(C_\delta(X, Y)\) is Baire;
2. \(Y\) is Baire and \(X\) has DMOP.

**Proof.** (1) \(\Rightarrow\) (2) If the function space \(C_\delta(X, Y)\) is Baire, then by Lemma 8.6(2,3), the space \(X\) has DMOP and the space \(Y\) is Baire.

(2) \(\Rightarrow\) (1) Assume that the space \(Y\) is Baire and the space \(X\) has DMOP. By definition, the Polish+meager space \(Y\) contains a Polish subspace \(P \subset Y\) whose complement \(Y \setminus P\) is meager in \(Y\). We claim that the Polish space \(P\) is dense in \(Y\). In the opposite case the non-empty open subset \(Y \setminus P\) of \(Y\) is meager and \(Y\) cannot be Baire. Now Lemma 8.6(1) implies that the function space \(C_\delta(X, Y)\) is Baire. \(\square\)

Combining Lemmas 8.5 and 8.7, we obtain the following proposition which implies Theorem 1.9(1) announced in the introduction.

**Proposition 8.8.** Let \(Y \subset \mathbb{R}\) be a Polish+meager subspace such that \(\inf Y \in Y \neq \{\inf Y\}\). For any \(Y\)-separable \(Y\)-separated space, the following conditions are equivalent:
(1) the function space $C_{\downarrow}^\dagger(X,Y)$ is Baire;
(2) the space $Y$ is Baire, the space $X$ has DMOP and the set $\hat{X}$ is dense in $X$.

9. Recognizing Choquet spaces among function spaces $C_{\downarrow}^\dagger(X,Y)$

We shall use the following known properties of Choquet spaces, see [10, 8.13] and [17].

Lemma 9.1. (1) The Tychonoff product of Choquet spaces is Choquet.
(2) Each dense $G_\delta$-set of a Choquet space $X$ is Choquet.
(3) A topological space is Choquet if it contains a dense Choquet subspace.
(4) An open continuous image of a Choquet space is Choquet.
(5) A metrizable space is Choquet if and only if it is almost complete-metrizable.
(6) A metrizable separable space is Choquet if and only if it is almost Polish.

Lemma 9.2. Let $Y$ be a non-empty almost Polish subspace of the real line. Then for any discrete topological space $X$ the function space $C_{\downarrow}^\dagger(X,Y)$ is Choquet.

Proof. Since $X$ is discrete, the Fell hypograph topology on $C_{\downarrow}^\dagger(X,Y)$ coincides with the topology of pointwise convergence. So, $C_{\downarrow}^\dagger(X,Y)$ can be identified with the Tychonoff power $Y^X$, which is Choquet by Lemma 9.1(1,6). □

Lemma 9.3. Let $Y \subset \mathbb{R}$ be a non-empty space and $X$ be a topological space containing an isolated point $x$. If the function space $C_{\downarrow}^\dagger(X,Y)$ is Choquet, then $Y$ is Choquet and almost Polish.

Proof. Since the point $x$ is isolated in $X$, the map $\delta_x : C_{\downarrow}^\dagger(X,Y) \to Y$, $\delta_x : f \mapsto f(x)$, is surjective, continuous and open. If $C_{\downarrow}^\dagger(X,Y)$ is Choquet, then its open continuous image $Y$ is Choquet and almost Polish by Lemma 9.1(4,6). □

Lemma 9.4. Let $X$ be a non-empty $T_1$-space and $Y \subset \mathbb{R}$ be a non-empty subspace with $\inf Y \notin Y$. Assume that $X$ is $Y$-separated or $\hat{X}$ is dense in $X$. The function space $C_{\downarrow}^\dagger(X,Y)$ is Choquet if and only if $X$ is discrete and $Y$ is almost Polish.

Proof. The “if” part follows from Lemma 9.2. To prove the “only if” part, assume that the function space $C_{\downarrow}^\dagger(X,Y)$ is Choquet. Then it is Baire and by Lemma 9.4, the space $X$ is discrete and hence has an isolated point. By Lemma 9.3, the space $Y$ is almost Polish. □

Lemma 9.5. Let $Y \subset \mathbb{R}$ be a subspace with $\inf Y \in Y \neq \{\inf Y\}$ and let $X$ be a $T_1$-space with dense set $\hat{X}$ of isolated points. The function space $C_{\downarrow}^\dagger(X,Y)$ is Choquet if the space $Y$ is almost Polish and the space $X$ has WDMOP.

Proof. By Lemma 6.1, the set $C_{\downarrow}^\prime(X,Y)$ is dense in $C_{\downarrow}^\dagger(X,Y)$ and by Lemma 6.3, $C_{\downarrow}^\prime(X,Y)$ is homeomorphic to the function space $C_k^\prime(X,Y)$.

If $Y$ is almost Polish and $X$ has WDMOP, then by Theorem 6.4(3), the function space $C_k^\prime(X,Y)$ is Choquet and so is its topological copy $C_{\downarrow}^\dagger(X,Y)$. Then the space $C_{\downarrow}^\dagger(X,Y)$ is Choquet since it contains a dense Choquet subspace $C_{\downarrow}^\dagger(X,Y)$. □

Lemma 9.6. Let $Y \subset \mathbb{R}$ be a subspace with $\inf Y \in Y \neq \{\inf Y\}$ and let $X$ be a $Y$-separable $T_1$-space with dense set $\hat{X}$ of isolated points. The function space $C_{\downarrow}^\dagger(X,Y)$ is Choquet if and only if the space $Y$ is almost Polish and $X$ has WDMOP.
Proof. The “if” part follows from Lemma 9.5. To prove the “only if” part, assume that the function space \( C^\prime_k(X,Y) \) is Choquet. By Lemmas 6.1 and 6.2, \( C^\prime_k(X,Y) \) is a dense \( G_\delta \)-set in \( C^\prime_k(X,Y) \). By Lemma 9.1(2), the space \( C^\prime_k(X,Y) \) is Choquet and so is its topological copy \( C^\prime_k(X,Y) \). Applying Theorem 6.4(3), we conclude that the space \( Y \) is Choquet and the space \( X \) has WDMOP. \( \square \)

The following proposition implies Theorem 10.9(2) announced in the introduction.

**Proposition 9.7.** Let \( Y \subset \mathbb{R} \) be a subspace with \( \inf Y \in Y \neq \{\inf Y\} \). For a \( Y \)-separable \( Y \)-separated space \( X \), the function space \( C^\prime_k(X,Y) \) is Choquet if and only if the space \( Y \) is almost Polish, the set \( \hat{X} \) is dense in \( X \), and the space \( X \) has WDMOP.

**Proof.** The “if” part is proved in Lemma 9.6. To prove the “only” if part, assume that the function space \( C^\prime_k(X,Y) \) is Choquet. Then it is Baire and by Lemma 8.5 the set \( \hat{X} \) is dense in \( X \). By Lemma 9.6 the space \( Y \) is almost Polish and \( X \) has WDMOP. \( \square \)

10. Recognizing strong Choquet spaces among function spaces \( C^\prime_k(X,Y) \)

We shall use the following known properties of strong Choquet spaces, see [10, 8.16, 8.17].

**Lemma 10.1.**

1. The Tychonoff product of strong Choquet spaces is strong Choquet.
2. An open continuous image of a strong Choquet space is strong Choquet.
3. A metric separable space is strong Choquet if and only if it is Polish.

**Lemma 10.2.** Let \( Y \subset \mathbb{R} \) be a non-empty subspace. If \( Y \) is Polish, then for any discrete topological space \( X \) the function space \( C^\prime_k(X,Y) \) is strong Choquet.

**Proof.** Since \( X \) is discrete, the Fell hypograph topology on \( C^\prime_k(X,Y) \) coincides with the topology of pointwise convergence. So, \( C^\prime_k(X,Y) \) can be identified with the Tychonoff power \( Y^X \), which is strong Choquet by Lemma 10.1(1,3). \( \square \)

**Lemma 10.3.** Let \( Y \subset \mathbb{R} \) be a non-empty space and \( X \) be a topological space containing an isolated point \( x \). If the function space \( C^\prime_k(X,Y) \) is strong Choquet, then \( Y \) is Polish.

**Proof.** Since the point \( x \) is isolated in \( X \), the map \( \delta_x: C^\prime_k(X,Y) \to Y \), \( \delta_x: f \mapsto f(x) \), is surjective, continuous and open. If \( C^\prime_k(X,Y) \) is strong Choquet, then its open continuous image \( Y \) is strong Choquet and Polish by Lemma 10.1(2,3). \( \square \)

**Lemma 10.4.** Let \( X \) be a topological space and \( Y \subset \mathbb{R} \) be a non-empty subspace such that \( \inf Y \notin Y \). Assume that \( X \) is \( Y \)-separated or the set \( \hat{X} \) is dense in \( X \). The function space \( C^\prime_k(X,Y) \) is strong Choquet if and only if \( X \) is discrete and \( Y \) is Polish.

**Proof.** The “if” part follows from Lemma 10.2. To prove the “only if” part, assume that the function space \( C^\prime_k(X,Y) \) is strong Choquet. Then it is Baire and by Lemma 8.4 the space \( X \) is discrete. Then \( x \) has an isolated point and by Lemma 10.3 the space \( Y \) is Polish. \( \square \)

The case of \( Y \) with \( \inf Y \in Y \) is more complicated and requires playing the strong Choquet game on the function space \( C^\prime_k(X,Y) \).

**Lemma 10.5.** Let \( Y \subset \mathbb{R} \) be a subset containing more than one point. If a topological space \( X \) contains a metrizable compact subset \( K \subset X \) with infinite intersection \( K \cap X \), then the player \( E \) has a winning strategy in the strong Choquet game \( G_{EN}(C^\prime_k(X,Y)) \).
Proof. Since the intersection $X \cap K$ contains a non-trivial convergent sequence, we can replace $K$ by a smaller compact space and assume that $K \cap X$ is dense in $K$ and $K$ has a unique non-isolated point $x' \in K \cap X'$. Write the countable infinite set $K \cap X$ as the union $K \cap X = \bigcup_{n \in \omega} F_n$ of an increasing sequence $(F_n)_{n \in \omega}$ of finite sets.

By our assumption, the set $Y$ contains two real numbers $\bar{u} < \bar{v}$.

Let $\tau$ be the family of all non-empty open sets in $C_{\mathcal{F}}(X, Y)$ and let

$$[X', \{\bar{u}\}] := \{f \in C_{\mathcal{F}}(X, Y) : f(X') \subset \{\bar{u}\}\}.$$ 

For every sequence $s = (W_0, \ldots, W_n) \in \tau^{< \omega}$ of non-empty open sets we shall define a function $f_s \in W_n$, a neighborhood $V_s \subset W_n$ of $f_s$, and two points $v_s, w_s \in X \cap K \setminus F_n$ such that if $W_n \cap [X'; \{\bar{u}\}] \neq \emptyset$, then

$$f_s \in [X', \{\bar{u}\}] \quad \text{and} \quad f_s \in V_s \subset W_n \cap \left[\left\{w_s\right\}, \frac{1}{3}u + \frac{2}{3} \bar{u}\right] \cap \left[\left\{v_s\right\}, \frac{2}{3}u + \frac{1}{3} \bar{u}\right].$$

Now we shall explain how to construct $f_s, V_s, v_s$ and $w_s$.

If $W_n \cap [X'; \{\bar{u}\}] = \emptyset$, then put $V_s = W_n, f_s$ be any element of $V_s$, and $v_s, w_s \in X \cap K \setminus F_n$ be any distinct points.

If $W_n \cap [X'; \{\bar{u}\}] \neq \emptyset$, then choose any function $g_s \in W_n \cap [X', \{\bar{u}\}]$. Next, using the definition of the Fell hypograph topology, we can find a finite family $\mathcal{U}_s$ of non-empty open sets in $X$, a non-empty finite family $\mathcal{K}_s$ of non-empty compact sets in $X$ and two functions $a_s : \mathcal{U}_s \to \mathbb{R}, b_s : \mathcal{K}_s \to \mathbb{R}$ such that the basic open set

$$[\mathcal{U}_s, \mathcal{K}_s; a_s, b_s] := \bigcap_{U \in \mathcal{U}_s} [U; a(U)] \cap \bigcap_{\kappa \in \mathcal{K}_s} [\kappa; b_s(\kappa)]$$

is a neighborhood of $g_s$, contained in $W_n$. Replacing the sets $U \in \mathcal{U}_s$ by smaller sets, we can assume that each set $U \in \mathcal{U}_s$ intersecting the set $X$ is a singleton. In this case the union $\bigcup \mathcal{U}_s$ of the family $\mathcal{U}_s = \{U \in \mathcal{U}_s : U \cap X \neq \emptyset\}$ is finite. Since each compact subset of $X$ is finite, the union $\bigcup \mathcal{K}_s$ of the family $\mathcal{K}_s = \{\kappa \in \mathcal{K}_s : \kappa \cap X' = \emptyset\}$ also is finite.

Let $b_s = \min\{b_s(\kappa) : \kappa \in \mathcal{K}_s \setminus \mathcal{K}_s\}$. We claim that $\bar{u} < b_s$. Indeed, find $\kappa \in \mathcal{K}_s \setminus \mathcal{K}_s$ with $b_s = b_s(\kappa)$ and observe that $g_s \in [\kappa; b_s(\kappa)]$ implies that $\bar{u} = \max g_s(\kappa \cap X') \leq \max g_s(\kappa) < b_s(\kappa) = b_s$.

Using the continuity of the function $g_s$, at the unique accumulation point $x'$ of the compact set $K$, we can find a point $w_s \in X \cap K \setminus (F_n \cup \bigcup (\mathcal{U} \cup \mathcal{K}))$ such that

$$\frac{1}{3}u + \frac{2}{3} \bar{u} < g_s(w_s) < b_s.$$

Next, choose any point $v_s \in X \cap K \setminus \left(\left\{w_s\right\} \cup F_n \cup \bigcup (\mathcal{U} \cup \mathcal{K})\right)$. Put

$$V_s := [\mathcal{U}_s, \mathcal{K}_s; a_s, b_s] \cap \left[\left\{w_s\right\}, \frac{1}{3}u + \frac{2}{3} \bar{u}\right] \cap \left[\left\{v_s\right\}, \frac{2}{3}u + \frac{1}{3} \bar{u}\right].$$

Finally, define a function $f_s \in C_{\mathcal{F}}(X, Y)$ letting $f_s(v_s) = \bar{u}$ and $f_s(x) = g_s(x)$ for any $x \in X \setminus \{v_s\}$. It is easy to see that $f_s, V_s, v_s, w_s$ have the required properties.

Now we define a strategy $S_{\mathcal{E}}$ of the player $\mathcal{E}$ in the strong Choquet game $G_{\mathcal{E}}(C_{\mathcal{F}}(X, Y))$ assigning to each $s = (W_0, \ldots, W_n) \in \tau^{< \omega}$ of non-empty open sets of $C_{\mathcal{F}}(X, Y)$ the pair $(f_s, V_s)$. For the empty sequence, we assume that $V_{\emptyset} = C_{\mathcal{F}}(X, Y)$ and $f_{\emptyset} : X \to \{\bar{u}\}$ is the constant function. We claim that this strategy of the player $\mathcal{E}$ is winning. Given any sequence $s = (W_n)_{n \in \omega} \in \tau^{< \omega}$ with $f_{s[n]} \in W_n \subset V_{s[n]}$ for every $n \in \omega$, we need to show that the intersection $\bigcap_{n \in \omega} W_n = \bigcap_{n \in \omega} V_{s[n]}$ is empty. To derive a contradiction, assume that this intersection contains some function $f \in C_{\mathcal{F}}(X, Y)$. 


By induction it can be shown that \( f_{s/n} \in [X', \{ \tilde{u} \}] \) and hence

\[
f \in V_{s/n} \subset \left[ \{ w_{s/n} \}, \frac{1}{3}\tilde{u} + \frac{2}{3}\tilde{u} \right] \cap \left[ \{ v_{s/n} \}, \frac{2}{3}\tilde{u} + \frac{1}{3}\tilde{u} \right],
\]

which contradicts the continuity of \( f \) as the sequences \((v_{s/n})_{n \in \omega}\) and \((w_{s/n})_{n \in \omega}\) both accumulate at the unique non-isolated point \( x' \) of the compact set \( K \). \( \square \)

The following proposition implies Theorem 1.9(3) announced in the introduction.

**Proposition 10.6.** Let \( Y \subset \mathbb{R} \) be a subspace containing more than one point and \( X \) be a non-empty \( Y \)-separable \( Y \)-separated space. The function space \( C_{\mathcal{F}}(X, Y) \) is strong Choquet if (and only if) the space \( Y \) is Polish, \( \hat{X} \) is dense in \( X \) and the set \( \hat{X} \) is (sequentially) closed in \( X \).

**Proof.** To prove the “if” part, assume that \( Y \) is Polish, \( \hat{X} \) is dense in \( X \) and \( \hat{X} \) is closed in \( X \). In this case \( \hat{X} = X \) and the space \( X \) is discrete. By Lemma 10.2, the function space \( C_{\mathcal{F}}(X, Y) \) is strong Choquet.

To prove the “only if” part, assume that the function space \( C_{\mathcal{F}}(X, Y) \) is strong Choquet. If \( \inf Y \notin Y \), then we can apply Lemma 10.4 to conclude that \( \hat{X} \) is discrete and \( Y \) is Polish. So, assume that \( \inf Y \in Y \). By Proposition 9.7, the set \( \hat{X} \) is dense in \( X \) and hence \( X \) contains an isolated point \( x \). By Lemma 10.3, the space \( Y \) is Polish. Lemma 10.3 implies that the set \( \hat{X} \) is sequentially closed in \( X \).

**Example 10.7.** For the Stone-Čech compactification \( X = \beta \mathbb{N} \) of the countable discrete space \( \mathbb{N} \) and any closed subset \( Y \subset \mathbb{R} \) with \( \inf Y \in Y \neq \{ \inf Y \} \) the function space \( C_{\mathcal{F}}(X, Y) \) is strong Choquet (by Theorem 1.10). The set \( \hat{X} = \mathbb{N} \) is sequentially closed but not closed in \( X = \beta \mathbb{N} \).

11. Recognizing almost Polish spaces among function spaces \( C_{\mathcal{F}}(X, Y) \)

Since the countable Tychonoff product of (almost) complete-metrizable spaces is (almost) complete-metrizable, we have the following simple lemma.

**Lemma 11.1.** If \( Y \subset \mathbb{R} \) is an (almost) Polish space, then for any countable discrete space \( X \) the function space \( C_{\mathcal{F}}(X, Y) \) is (almost) Polish.

**Lemma 11.2.** Let \( X \) be a non-empty \( T_1 \)-space and \( Y \subset \mathbb{R} \) be a non-empty subspace with \( \inf Y \notin Y \). Assume that \( X \) is \( Y \)-separated or \( \hat{X} \) is dense in \( X \). The following conditions are equivalent:

1. \( C_{\mathcal{F}}(X, Y) \) is almost complete-metrizable;
2. \( C_{\mathcal{F}}(X, Y) \) is almost Polish;
3. \( Y \) is almost Polish and \( X \) is countable and discrete.

**Proof.** The implication (3) \( \Rightarrow \) (2) was proved in Lemma 11.1 and (2) \( \Rightarrow \) (1) is trivial.

(1) \( \Rightarrow \) (3) Assume that \( C_{\mathcal{F}}(X, Y) \) is almost complete-metrizable. Then it is Choquet and by Lemma 9.4, the space \( Y \) is almost Polish and the space \( X \) is discrete. In this case the Fell hypograph topology coincides with the topology of pointwise convergence and the function space \( C_{\mathcal{F}}(X, Y) \) can be identified with the Tychonoff power \( Y^X \). Being almost complete-metrizable, the space \( Y^X \) contains a dense first-countable subspace \( D \). Being regular, \( Y^X \) is first-countable at each point of the set \( D \). This implies that the set \( X \) is countable (otherwise singletons in \( Y^X \) are not \( G_\delta \)). \( \square \)
Lemma 11.3. Let \( Y \subset \mathbb{R} \) be a subspace with \( \inf Y \in Y \neq \{ \inf Y \} \) and let \( X \) be a \( T_1 \)-space with dense set \( \hat{X} \) of isolated points. The function space \( C_{\downarrow F}(X, Y) \) is almost complete-metrizable (resp. almost Polish) if the space \( Y \) is almost Polish and \( X \) is a \( \kappa_\omega \)-space (with countable set \( \hat{X} \) of isolated points).

Proof. By Lemma 6.1, the set \( C'_{\downarrow F}(X, Y) \) is dense in \( C_{\downarrow F}(X, Y) \) and by Lemma 6.3, \( C'_{\downarrow F}(X, Y) \) is homeomorphic to the function space \( C'_{\hat{k}}(X, Y) \).

If \( Y \) is almost Polish and \( X \) is a \( \kappa_\omega \)-space (with countable set \( \hat{X} \) of isolated points), then by Theorem 6.4(4,5), the function space \( C'_{\downarrow F}(X, Y) \) is almost complete-metrizable (resp. almost Polish) and so is its topological copy \( C'_{\downarrow F}(X, Y) \). Then the space \( C_{\downarrow F}(X, Y) \) is almost complete-metrizable (resp. almost Polish) since it contains a dense almost complete-metrizable (resp. almost Polish) subspace \( C'_{\downarrow F}(X, Y) \).

Lemma 11.4. Let \( Y \subset \mathbb{R} \) be a subspace with \( \inf Y \in Y \neq \{ \inf Y \} \) and let \( X \) be a \( Y \)-separable \( T_1 \)-space with dense set \( \hat{X} \) of isolated points. The function space \( C_{\downarrow F}(X, Y) \) is almost complete-metrizable (resp. almost Polish) if and only if the space \( Y \) is almost Polish and \( X \) is a \( \kappa_\omega \)-space (with countable set \( \hat{X} \) of isolated points).

Proof. The “if” part follows from Lemma 11.3. To prove the “only if” part, assume that the function space \( C_{\downarrow F}(X, Y) \) is almost complete-metrizable. So, \( C_{\downarrow F}(X, Y) \) contains a dense complete-metrizable space \( D \). By Lemmas 6.1 and 6.2, \( C'_{\downarrow F}(X, Y) \) is a dense \( G_\delta \)-set in \( C_{\downarrow F}(X, Y) \). Then \( D \cap C'_{\downarrow F}(X, Y) \) is a \( G_\delta \)-set in the completely-metrizable space \( D \). Since the complement \( C_{\downarrow F}(X, Y) \setminus C'_{\downarrow F}(X, Y) \) is meager in \( C_{\downarrow F}(X, Y) \), the complement \( D \setminus C'_{\downarrow F}(X, Y) \) is meager in \( D \) by the density of \( D \) in \( C_{\downarrow F}(X, Y) \). By the Baire Theorem, the intersection \( D \cap C'_{\downarrow F}(X, Y) \) is a dense \( G_\delta \)-set in \( D \) and also in \( C'_{\downarrow F}(X, Y) \). By 10.3.11, the \( G_\delta \)-subset space \( D \cap C'_{\downarrow F}(X, Y) \) of the completely-metrizable space \( D \) is complete-metrizable, so \( C'_{\downarrow F}(X, Y) \) is almost complete-metrizable. By Theorem 6.4(4), the space \( Y \) is almost Polish and \( X \) is a \( \kappa_\omega \)-space.

If the space \( C_{\downarrow F}(X, Y) \) is almost Polish, then we can assume that the complete-metrizable space \( D \) is Polish. Then \( C'_{\downarrow F}(X, Y) \) is almost Polish and by Theorem 6.4(5), the set \( \hat{X} \) is at most countable.

The following two propositions imply Theorem 11.9(4,5).

Proposition 11.5. Let \( Y \subset \mathbb{R} \) be a subspace with \( \inf Y \in Y \neq \{ \inf Y \} \) and \( X \) be a \( Y \)-separable \( Y \)-separated space \( X \). The function space \( C_{\downarrow F}(X, Y) \) is almost complete-metrizable if and only if the space \( Y \) is almost Polish and \( X \) is a \( \kappa_\omega \)-space with dense set \( \hat{X} \) of isolated points.

Proof. The “if” part follows from Lemma 11.3. To prove the “only if” part, assume that the function space \( C_{\downarrow F}(X, Y) \) is almost complete-metrizable. So, \( C_{\downarrow F}(X, Y) \) contains a dense complete-metrizable space \( D \). Then it is Baire and by Lemma 8.3, the set \( \hat{X} \) is dense in \( X \). By Lemma 11.4 the space \( Y \) is almost complete-metrizable and \( X \) is a \( \kappa_\omega \)-space.

By analogy we can prove

Proposition 11.6. Let \( Y \subset \mathbb{R} \) be a subspace with \( \inf Y \in Y \neq \{ \inf Y \} \), and let \( X \) be a \( Y \)-separable \( Y \)-separated space. The function space \( C_{\downarrow F}(X, Y) \) is almost Polish if and only if the space \( Y \) is almost Polish and \( X \) is a \( \kappa_\omega \)-space with dense and countable set \( \hat{X} \) of isolated points.
12. Recognizing Polish spaces among the function spaces $C_{\downarrow F}(X,Y)$

In this section we recognize complete-metrizable and Polish spaces among function spaces $C_{\downarrow F}(X,Y)$. The case $\inf Y \notin Y$ is simple.

**Lemma 12.1.** Let $X$ be a non-empty $T_1$-space and $Y \subset \mathbb{R}$ be a non-empty subspace with $\inf Y \notin Y$. Assume that $X$ is $Y$-separated or $\check{X}$ is dense in $X$. Then the following conditions are equivalent:

1. $C_{\downarrow F}(X,Y)$ is complete-metrizable;
2. $C_{\downarrow F}(X,Y)$ is Polish;
3. $Y$ is Polish and $X$ is countable and discrete.

**Proof.** The implication (3) $\Rightarrow$ (2) trivially follows from the preservation of Polish spaces by countable Tychonoff products and (2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (3) Assume that the space $C_{\downarrow F}(X,Y)$ complete-metrizable. By Lemma 11.2, the space $Y$ is almost Polish and the space $X$ is countable and discrete. Since $X$ is discrete, the Fell-hypograph topology on $C_{\downarrow F}(X,Y)$ coincides with the Tychonoff product topology on $Y^X$. The complete-metrizability of the function space $C_{\downarrow F}(X,Y) = Y^X$ implies the complete-metrizability of $Y$. Being almost Polish, the complete-metrizable space $Y$ is separable and hence Polish. 

The case $\inf Y \in Y$ is more complicated and requires some preliminary work. We start with reminding two known definitions.

A topological space $X$

- is *hemicompact* if there exists a countable family $K$ of compact subsets of $X$ such that each compact set $C \subset X$ is contained in some set $K \in K$;
- has a *countable network* if there exists a countable family $\mathcal{N}$ of subsets of $X$ such that for any open set $U \subset X$ and point $x \in U$ there exists a set $N \in \mathcal{N}$ such that $x \in N \subset U$.

A partial case (for $Y = \mathbb{R}$ and Tychonoff $X$) the following lemma was proved by McCoy and Ntantu [11].

**Lemma 12.2.** Let $Y \subset \mathbb{R}$ be a subspace containing more than one point and $X$ be a $Y$-separated space.

1. If $C_{\downarrow F}(X,Y)$ is first-countable, then $X$ is hemicompact and $\check{X}$ is countable.
2. If $C_{\downarrow F}(X,Y)$ has a countable network, then the $Y$-topology of $X$ has countable network; consequently, $\check{X}$ is at most countable and $X$ is $Y$-separable.
3. If $C_{\downarrow F}(X,Y)$ has a countable network, then each compact subset of $X$ is metrizable.
4. If $C_{\downarrow F}(X,Y)$ is first-countable and has a countable network, then the space $X$ has a countable network.

**Proof.** Fix two real numbers $u < \bar{u}$ in $Y$.

1h. Assume that the space $C_{\downarrow F}(X,Y)$ is first-countable at the constant function $e : X \to \{u\} \subset Y$ and fix a countable neighborhood base $\{O_n\}_{n \in \omega}$ at $e$. By the definition of the Fell hypograph topology, for every $n \in \omega$ there exits a finite family $\mathcal{U}_n$ of non-empty open sets in $X$, a non-empty compact subset $K_n \subset X$, a function $a_n : \mathcal{U}_n \to \mathbb{R}$, and a real number $b_n > u$ such that

$$e \in [K_n; b_n] \cap \bigcap_{U \in \mathcal{U}_n} \left[ U_n; a_n(U) \right] \subset O_n.$$
We claim that \( \dot{\mathcal{C}} \)-topology of \( Y \) is larger compact set, we can assume that \( K \) has non-empty intersection with each set \( U \subseteq \mathcal{U}_n \).

We claim that the countable family \( \{K_n\}_{n \in \omega} \) witnesses that the space \( X \) is hemicompact. Given any compact subset \( K \subseteq X \), consider the open neighborhood \( [K; \bar{u}] \subseteq C_\mathcal{F}(X, Y) \) of \( \bar{u} \) and find \( n \in \omega \) such that \( O_n \subseteq [K; \bar{u}] \). We claim that \( K \subseteq K_n \). Assuming that \( K \notin K_n \), find a point \( x \in K \setminus K_n \). Using Lemma 4.1, construct a function \( f : X \to Y \) such that \( f(K_n) \subseteq \{\bar{u}\} \) and \( f(x) = \bar{u} \). Observe that for every \( U \subseteq \mathcal{U}_n \), we have \( \sup f(U) \geq \sup (f(U \cap K_n) \geq \bar{u} > a_n(U) \). Consequently,

\[
f \in [K_n; b_n] \cap \bigcap_{U \subseteq \mathcal{U}_n} [U; a_n(U)] \subseteq O_n \subseteq [K; \bar{u}],
\]

and hence \( f(x) \leq \max f(K) < \bar{u} \), which contradicts the choice of \( f \). This contradiction completes the proof of the hemicompactness of \( X \).

1c. Assume that the space \( C_\mathcal{F}(X, Y) \) is first-countable at the constant function \( \bar{c} : X \to \{\bar{u}\} \subseteq Y \) and fix a countable neighborhood base \( \{O_n\}_{n \in \omega} \) at \( \bar{c} \). By the definition of the Fell hypograph topology, for every \( n \in \omega \) there exits a finite family \( \mathcal{U}_n \) of non-empty open sets in \( X \), a non-empty compact subset \( K_n \subset X \), a function \( a_n : \mathcal{U}_n \to \mathbb{R} \), and a real number \( b_n > \bar{u} \) such that

\[
\bar{c} \in [K_n; b_n] \cap \bigcap_{U \subseteq \mathcal{U}_n} [U; a_n(U)] \subseteq O_n.
\]

Replacing each set \( U \subseteq \mathcal{U}_n \) by a suitable non-empty open subset of \( U \), we can assume that either \( U \subseteq X' \) or \( U = \{x_U\} \subset \hat{X} \) for some isolated point \( x_U \) of \( X \). Replacing \( K_n \) by a larger compact set, we can assume that \( K \) intersects each set \( U \subseteq \mathcal{U}_n \). It follows that \( a_n(U) < \sup \bar{c}(U) = \bar{u} \). Let

\[
\mathcal{U}_n := \{U \subseteq \mathcal{U}_n : U \cap \hat{X} \neq \emptyset\} = \{x \in \mathcal{U}_n : x \in \hat{X}\}.
\]

We claim that \( \hat{X} = \bigcup_{n \in \omega} \bigcup \mathcal{U}_n \).

Given any point \( x \in \hat{X} \), consider the open neighborhood \( [\{x\}; \bar{u}] \) of \( \bar{c} \) in \( C_\mathcal{F}(X, Y) \) and find \( n \in \omega \) such that \( O_n \subseteq [\{x\}; \bar{u}] \). We claim that \( \{x\} \in \mathcal{U}_n \). Assuming that \( \{x\} \notin \mathcal{U}_n \), we conclude that \( x \notin \bigcup \mathcal{U}_n \). Consider the function \( \chi_x : X \to \{\bar{u}, \bar{u}\} \subseteq Y \), defined by \( \chi_x^{-1}(\bar{u}) = \{x\} \). It follows from \( \max_{U \subseteq \mathcal{U}_n} a_n(U) < \bar{u} < b_n \) that

\[
\chi_x \in [K_n; b_n] \cap \bigcap_{U \subseteq \mathcal{U}_n} [U; a_n(U)] \subseteq O_n \subseteq [\{x\}; \bar{u}],
\]

and hence \( \chi_x(x) > \bar{u} \), which contradicts the definition of the function \( \chi_x \). This contradiction shows that \( \{x\} \in \mathcal{U}_n \). Now we see that the set \( \hat{X} \subset \bigcup_{n \in \omega} \bigcup \mathcal{U}_n \) is countable (we recall that each set \( \mathcal{U}_n \), \( n \in \omega \), is finite).

2. Assume that the space \( C_\mathcal{F}(X, Y) \) has a countable network \( \mathcal{N} \). We shall show that the \( Y \)-topology of \( X \) has countable network. For every set \( N \in \mathcal{N} \) consider the set \( N^* := \{x \in X : N \subset [\{x\}; \bar{u}]\} \). We claim that the family \( \mathcal{N}^* = \{N^* : N \in \mathcal{N}\} \) is a countable network for the \( Y \)-topology of the space \( X \). Fix any point \( x \in X \) and its neighborhood \( O_x \) in the \( Y \)-topology of \( X \).

If the space \( Y \) is connected, then the \( Y \)-topology coincides with the \( \mathbb{R} \)-topology of \( X \). So, we can find a continuous function \( f : X \to [\bar{u}, \bar{u}] \subseteq Y \) such that \( f(x) = \bar{u} \) and \( f(X \setminus O_x) \subset \{\bar{u}\} \).
If the space $Y$ is disconnected, then the $Y$-topology on $X$ is generated by the base consisting of clopen subsets of $X$. In this case we can find a continuous function $f : X \to \{u, \bar{u}\} \subset Y$ such that $f(x) = u$ and $f(X \setminus O_x) \subset \{\bar{u}\}$.

In both cases we have a continuous function $f : X \to Y$ such that $f(x) = u$ and $f(X \setminus O_x) \subset \{\bar{u}\}$.

For the open neighborhood $[\{x\}; \bar{u}] \subset C_{\mathcal{F}}(X, Y)$ of $f$, find a set $N \in \mathcal{N}$ such that $f \in N \subset [\{x\}; \bar{u}]$. Then $x \in N^*$ by the definition of $N^*$. On the other hand, for any $z \in X \setminus O_x$ we get $f \notin [\{z\}; \bar{u}]$ and then $N \nsubseteq [\{z\}; \bar{u}]$ and $z \notin N^*$, which implies $x \in N^* \setminus O_x$. Therefore, $N^*$ is a countable network for $Y$-topology. Since spaces with countable network are hereditarily separable, the space $X'$ contains a countable set $M$ such that $X' = \mathcal{T}Y$, which means that the space $X$ is $Y$-separable.

Since each isolated point of $X$ remains isolated in the $Y$-topology (which has a countable network), the set $\hat{X}$ is at most countable.

3. Assume that the space $C_{\mathcal{F}}(X, Y)$ has a countable network $\mathcal{N}$. By the preceding statement, the $Y$-topology of $X$ has a countable network. Since the space $X$ is $Y$-separated, on any compact subset $K$ of $X$ the $Y$-topology induces the original subspace topology of $K$, which implies that $K$ has a countable network and hence is metrizable by [8, 3.1.19].

4. If the space $C_{\mathcal{F}}(X, Y)$ is first-countable and has countable network, then $X$ is hereditarily separable, and hence $\sigma$-compact (by the first statement). Consequently, $X$ contains a countable family $\{K_i\}_{i \in \omega}$ of compact subsets such that $X = \bigcup_{i \in \omega} K_i$. By the third statement, each compact set $K_i$ has a countable network $\mathcal{N}_i$. Then $\mathcal{N} = \bigcup_{i \in \omega} \mathcal{N}_i$ is a countable network for the space $X$.

**Lemma 12.3.** Let $Y \subset \mathbb{R}$ be a subspace containing more than one point. For any $Y$-separated space $X$, the function space $C_{\mathcal{F}}(X, Y)$ is Polish if and only if $Y$ is Polish and the space $X$ is countable and discrete.

**Proof.** The “if” follows from the preservation of Polish spaces by countable Tychonoff products.

To prove the “only if” part, assume that the space $C_{\mathcal{F}}(X, Y)$ is Polish. If $\inf Y \notin Y$, then by Lemma 11.12 the space $X$ is countable and discrete. Then $Y$ is Polish, being homeomorphic to a closed subset of the Polish space $C_{\mathcal{F}}(X, Y) = Y^X$.

Now assume that $\inf Y \in Y$. By Lemma 12.2(2), the space $X$ is $Y$-separable and the set $\hat{X}$ is at most countable. By Lemma 8.5 the set $\hat{X}$ is dense in $X$. Being Polish, the space $C_{\mathcal{F}}(X, Y)$ is strong Choquet. By Lemma 10.3 the space $Y$ is Polish and by Lemma 10.5 the set $\hat{X}$ is sequentially closed in $X$. By Lemma 12.2(4), the space $X$ has a countable network and hence it has countable tightness. Then the sequentially closed set $\hat{X}$ is closed in $X$, which implies that the space $X = \hat{X}$ is discrete and at most countable.

**Lemma 12.4.** Let $Y \subset \mathbb{R}$ be a subspace containing more than one point and $X$ be a $Y$-separable $Y$-separated $T_1$-space with dense set $\hat{X}$ of isolated points. The function space $C_{\mathcal{F}}(X, Y)$ is complete-metrizable if and only if $C_{\mathcal{F}}(X, Y)$ is Polish.

**Proof.** The “if” part is trivial. To prove the “only if” part, assume that the space $C_{\mathcal{F}}(X, Y)$ complete-metrizable. If $\inf Y \notin Y$, then by Lemma 11.2 the space $Y$ is almost Polish and the space $X$ is countable and discrete. Then $Y$ is complete-metrizable, being homeomorphic to a closed subset of the complete-metrizable space $C_{\mathcal{F}}(X, Y) = Y^X$. Being almost Polish, the complete-metrizable space $Y$ is separable and hence Polish.
Now assume that \( \inf Y \in Y \). In this case Lemma \([12.2](1)\) implies that \( X \) is a \( \kappa_{\omega} \)-space. By Lemma \([12.2](1)\), the set \( \hat{X} \) is countable. By Theorem \([6.4](6)\), the space \( C_{\mathcal{F}}^\prime(X,Y) \) is separable. By Lemmas \([6.1](1)\) and \([6.3](1)\), the set \( C_{\mathcal{F}}(X,Y) = C_{\mathcal{F}}^\prime(X,Y) \) is dense in \( C_{\mathcal{F}}(X,Y) \), which implies that the complete-metrizable space \( C_{\mathcal{F}}(X,Y) \) is separable and hence Polish.

The following proposition implies Theorem \([1.9](6)\) announced in the introduction.

**Proposition 12.5.** Let \( Y \subset \mathbb{R} \) be a subspace containing more than one point. For a non-empty \( Y \)-separable \( Y \)-separated space \( X \), the following conditions are equivalent:

1. \( C_{\mathcal{F}}(X,Y) \) is complete-metrizable;
2. \( C_{\mathcal{F}}(X,Y) \) is Polish;
3. \( Y \) is Polish and the space \( X \) is countable and discrete.

**Proof.** The implication (3) \( \Rightarrow \) (2) trivially follows from the preservation of Polish spaces by countable Tychonoff products, and (2) \( \Rightarrow \) (1) is trivial.

(1) \( \Rightarrow \) (3) Assume that the function space \( C_{\mathcal{F}}(X,Y) \) is complete-metrizable. If \( \inf Y \notin Y \), then we can apply Lemma \([12.1](1)\) and conclude that \( Y \) is Polish and \( X \) is countable and discrete.

Next, consider the case \( \inf Y \in Y \). Being complete-metrizable, the space \( C_{\mathcal{F}}(X,Y) \) is Baire. By Lemma \([5.5](1)\) the set \( \hat{X} \) is dense in \( X \). Now we can apply Lemmas \([12.3](1)\) and \([12.4](1)\) and conclude that the space \( Y \) is Polish and \( X \) is countable and discrete. \( \square \)

13. Recognizing function spaces \( C_{\mathcal{F}}(X,Y) \) which are neither Baire nor meager

**Theorem 13.1.** Let \( Y \subset \mathbb{R} \) be a Polish+meager subspace of the real line and \( X \) be a non-empty \( Y \)-separable \( Y \)-separated topological space. The function space \( C_{\mathcal{F}}(X,Y) \) is neither Baire nor meager if and only if one of the following conditions is satisfied:

1. \( X \) is finite and \( Y \) is neither Baire nor meager;
2. \( X \) is infinite compact, \( X^{\circ} = \emptyset \), \( \inf Y \in Y \) and \( Y \) is neither Baire nor meager;
3. \( X \) is compact, \( X^{\circ} \neq \emptyset \) and \( \inf Y \in \hat{Y} \);
4. \( X \) is not compact, \( X \) has DMOP, \( \inf Y \in \hat{Y} \), \( X^{\circ} \) is compact and not empty, and \( Y \) is Baire.

**Proof.** First we prove that each of the conditions (1)–(4) implies that the function space \( C_{\mathcal{F}}(X,Y) \) is neither meager nor Baire.

1. Assume that \( X \) is finite and \( Y \) is neither Baire nor meager. It follows that the Fell hypograph topology on \( C_{\mathcal{F}}(X,Y) \) coincides with the topology of Tychonoff product \( Y^X \). Taking into account that \( Y \) is neither meager nor Baire, we can find an non-empty open meager subspace \( M \subset Y \) and a non-empty open Baire subspace \( B \subset Y \). Then \( M^X \) is an open meager subspace in \( Y^X = C_{\mathcal{F}}(X,Y) \), which implies that the space \( C_{\mathcal{F}}(X,Y) \) is not Baire. Since the Baire space \( B \) is second-countable its power \( B^X \) is Baire according to \([15](1)\). Then the space \( Y^X = C_{\mathcal{F}}(X,Y) \) contains the non-empty open Baire subspace \( B^X \) and hence is not meager.

2. Assume that the space \( X \) is compact and infinite, \( X^{\circ} = \emptyset \), \( \inf Y \in Y \) and \( Y \) is neither Baire nor meager. The compactness of the space \( X \) implies that \( X \) has DMOP. By Proposition \([5.8](1)\), the space \( C_{\mathcal{F}}(X,Y) \) is not Baire (otherwise \( Y \) would be Baire). Since the space \( Y \) is not meager, it contains a non-empty open Baire subspace \( B \subset Y \). Replacing \( B \) by \( B \cup \{ \inf Y \} \), we can assume that \( \inf Y \in B \). The space \( B \) is Polish+meager (being an open subspace of the Polish+meager space \( Y \)) and hence \( B \) contains a dense Polish subspace (being
Lemma 7.4 implies that \( \inf Y \) is Baire by Lemma 7.11. Being Polish+meager, the Baire space considered two subcases. 

4. Assume that the space \( X \) is not compact, \( X^{\omega} \neq \emptyset \), and \( \inf Y \in \check{Y} \). Find a real number \( \varepsilon \) such that \( \{y \in Y : y < \varepsilon\} \) and observe that the constant function \( c : X \to \{\inf Y\} \subset Y \) is a unique point of the basic open set \( [X;\varepsilon] \). This implies that the function space \( C_{\downarrow F}(X,Y) \) is not meager. On the other hand, Lemma 7.3 ensures that its complement \( [X^{\omega}\setminus\{\inf Y\}] = [X^{\omega},\inf Y] \) is a meager open set in \( C_{\downarrow F}(X,Y) \), which implies that \( C_{\downarrow F}(X,Y) \) is not Baire. 

Now assume that function space \( C_{\downarrow F}(X,Y) \) is neither Baire or meager, we shall prove that one of the conditions (1)–(4) is satisfied. By Lemmas 7.9 and 7.10 the space \( X \) is not meager and the space \( X \) has DMOP. 

First assume that \( X \) is discrete. In this case the function space \( C_{\downarrow F}(X,Y) \) can be identified with the power \( Y^{X} \) of \( Y \). By Lemma 8.1 the space \( Y \) is not Baire (as \( C_{\downarrow F}(X,Y) = Y^{X} \) is not Baire). By Lemma 7.10 the space \( X \) is finite. So the condition (1) holds.

So, assume that \( X \) is not discrete. In this case Lemma 7.1 implies that \( \inf Y \in Y \). Now consider two subcases. 

First we assume that \( X \) is compact. If \( X^{\omega} = \emptyset \), then Proposition 8.8 implies that Polish-meager space \( Y \) is not Baire and hence the condition (2) holds. If \( X^{\omega} \neq \emptyset \), then Lemma 7.4 implies that \( \inf Y \in \check{Y} \), which yields the condition (3).

Next, assume that \( X \) is not compact. If \( \check{X} \) is not contained in a compact subset of \( X \), then \( Y \) is Baire by Lemma 7.11. Being Polish+meager, the Baire space \( \check{Y} \) is almost Polish. By Lemma 8.1, the set \( X' \) has non-empty interior in \( X \). By Lemma 7.4 \( \inf Y \in \check{Y} \) and \( X^{\omega} \) is compact. This means that the condition (4) is satisfied.

Finally, assume that the set \( \check{X} \) is contained in a compact subset \( K \) of \( X \). Since \( X \) is not compact, the set \( X' \) has non-empty interior. By Lemma 7.4 \( \inf Y \in \check{Y} \) and \( X^{\omega} \) is compact. Then the space \( X = \check{X} \cup \check{X}^{\omega} \) is compact, which contradicts our assumption. 

14. A dichotomy for analytic function spaces \( C_{\downarrow F}(X,Y) \)

In this section we prove Theorem 1.13 announced in the introduction.
Lemma 7.4 implies that the space $C$.

Table 1.

Proposition 14.2. Let $a$.

Applying Lemma 7.5, we conclude that the space $C$.

Proof. Assume that $C$.

By Lemmas 6.1, 6.2 and 6.3, the space $C$.

So, assume that $X$ is discrete, then $X = \hat{X}$ is at most countable and $C(X, Y) = Y^X$ is Polish and hence $\infty$-comeager.

So, we assume that $X$ is not discrete. If $\inf Y \notin \hat{Y}$, then the function space $C(X, Y)$ is $\infty$-meager by Lemma 7.1. So, we assume that $\inf Y \in Y$. If the set $X'$ has non-empty interior in $X$, then the space $C(X, Y)$ is $\infty$-meager by Lemma 7.4 (since $\inf Y \notin \hat{Y}$).

So, we assume that the set $X'$ is nowhere dense in $X$. By Lemmas 6.1 and 6.2, the subset $C'(X, Y)$ is an $\infty$-dense $G_\delta$-set in $C(X, Y)$. Being a $G_\delta$-subset of the analytic space $C(X, Y)$, the space $C'(X, Y)$ is analytic. By Lemma 6.3, the space $C(X, Y)$ can be identified with function space $C(X, Y)$. So, the space $C'(X, Y)$ is analytic and by Theorem 6.5, $C'(X, Y)$ is either Polish or $\infty$-meager. If $C'(X, Y)$ is Polish, then the space $C(X, Y)$ contains the $\infty$-dense Polish subspace $C'(X, Y)$ and hence is $\infty$-comeager.

If $C'(X, Y)$ is $\infty$-meager, then by Theorem 6.4(2), the space $X$ does not have DMOP. Applying Lemma 7.5, we conclude that the space $C(X, Y)$ is $\infty$-meager.

In fact, if $\inf Y \in Y$, then the analyticity of the space $C(X, Y)$ in Theorem 14.4 can be replaced by the analyticity of the space $C'(X, Y)$.

Proposition 14.2. Let $Y \subset \mathbb{R}$ be a non-empty Polish subspace with $\inf Y \in Y \setminus \hat{Y}$, and $X$ be a $Y$-separable $T_1$-space with dense set $\hat{X}$ of isolated points. If the function space $C'(X, Y)$ is analytic, then $C(X, Y)$ is either $\infty$-meager or $\infty$-comeager.

Proof. By Lemmas 6.1, 6.2, and 6.3, the space $C'(X, Y) = C(X, Y)$ is an $\infty$-dense $G_\delta$-set in $C(X, Y)$. By Theorem 6.5, the analytic space $C'(X, Y)$ is either Polish or $\infty$-meager. If $C'(X, Y)$ is Polish, then the space $C(X, Y)$ contains the $\infty$-dense Polish subspace $C'(X, Y)$ and hence is $\infty$-comeager.

If $C'(X, Y)$ is $\infty$-meager, then by Theorem 6.4(2), the space $X$ does not have DMOP. Applying Lemma 7.5, we conclude that the space $C(X, Y)$ is $\infty$-meager.

Proposition 14.3. Let $Y \subset \mathbb{R}$ be a non-empty Polish subspace with $\inf Y \in Y \setminus \hat{Y}$, and $X$ be a $Y$-separable $Y$-separated space. If the function space $C'(X, Y)$ is analytic, then $C(X, Y)$ is either $\infty$-meager or $\infty$-comeager.

Proof. Assume that the function space $C'(X, Y)$ is analytic. If the set $\hat{X}$ is dense in $X$, then by Proposition 14.2, the space $C(X, Y)$ is either $\infty$-meager or $\infty$-comeager.

So, assume that $\hat{X}$ is not dense in $X$ and hence the set $X^\infty$ is not empty. Since $\inf Y \notin \hat{Y}$, Lemma 7.4 implies that the space $C(X, Y)$ is $\infty$-meager.

15. References to Proofs of the Statements in Table 1

In this section we provide references to lemmas that prove the statements in $8 \times 7$ cells of Table 1.
The equivalence of the meagerness and $\infty$-meagerness (claimed in Theorem 1.12) follows from the facts that in Lemmas 7.1, 7.2, 7.4, 7.6, 7.9, 7.11 we establish the $\infty$-meagerness of the spaces $C_{\downarrow F}(X, Y)$ and the cells in the above table exhaust all possible cases of the interplay between the properties of the spaces $X$ and $Y$.

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