Skew-symmetric forms: On integrability of equations of mathematical physics.

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Abstract

The study of integrability of the mathematical physics equations showed that the differential equations describing real processes are not integrable without additional conditions. This follows from the functional relation that is derived from these equations. Such a relation connects the differential of state functional and the skew-symmetric form. This relation proves to be nonidentical, and this fact points to the nonintegrability of the equations. In this case a solution to the equations is a functional, which depends on the commutator of skew-symmetric form that appears to be unclosed. However, under realization of the conditions of degenerate transformations, from the nonidentical relation it follows the identical one on some structure. This points out to the local integrability and realization of a generalized solution.

In doing so, in addition to the exterior forms, the skew-symmetric forms, which, in contrast to exterior forms, are defined on nonintegrable manifolds (such as tangent manifolds of differential equations, Lagrangian manifolds and so on), were used.

In the present paper, the partial differential equations, which describe any processes, the systems of differential equations of mechanics and physics of continuous medium and field theory equations are analyzed.

1 Analysis of partial differential equations that describe real processes

Let us take the simplest case: the first-order partial differential equation

\[ F(x^i, u, p_i) = 0, \quad p_i = \frac{\partial u}{\partial x^i} \]  

We consider the functional relation

\[ du = \theta \]  

where \( \theta = p_i \, dx^i \) is a skew-symmetric differential form of the first degree (the summation over repeated indices is implied).

In the general case, when differential equation (1) describes any physical processes, the functional relation (2) is nonidentical one. If to take the differential of this relation, we will have \( ddu = 0 \) in the left-hand side, whereas in the right-hand side \( d\theta \) is not equal to zero. The differential \( d\theta \) is equal to \( K_{ij} \, dx^i \, dx^j \), where \( K_{ij} = \frac{\partial p_j}{\partial x^i} - \frac{\partial p_i}{\partial x^j} \) is components of differential form commutator constructed of the mixed derivatives. From equation (1) it does not follow (explicitly) that the derivatives \( p_i = \frac{\partial u}{\partial x^i} \), which obey to the equation (and given boundary or initial conditions) are consistent, and that mixed derivatives are commutative. Components of commutator \( K_{ij} \) is nonzero. Therefore the
differential form commutator and the differential of form $\theta$ are nonzero. Thus, $d\theta$ is not equal to zero.

The nonidentity of functional relation (2) means that the equation (1) is nonintegrable: the derivatives $p_i$ of equation do not make up a differential. The solution $u$ of the equation (1), obtained from such derivatives, is not a function of only variables $x^i$. This solution will depend on the commutator $K_{ij}dx^i dx^j$, that is, it is a functional.

To obtain a solution that is a function (i.e., the derivatives of this solution make up a differential), it is necessary to add the closure condition for the form $\theta = p_i dx^i$ and for the relevant dual form (in the present case the functional $F$ plays a role of a form dual to $\theta$) [1]:

$$\begin{cases} 
    dF(x^i, u, p_i) = 0 \\
    d(p_i dx^i) = 0
\end{cases} \tag{3}$$

If we expand the differentials, we get a set of homogeneous equations with respect to $dx^i$ and $dp_i$ (in the $2n$-dimensional tangent space):

$$\begin{cases} 
    \left( \frac{\partial F}{\partial x^i} + \frac{\partial F}{\partial u} p_i \right) dx^i + \frac{\partial F}{\partial p_i} dp_i = 0 \\
    dp_i dx^i - dx^i dp_i = 0
\end{cases} \tag{4}$$

It is well known that vanishing the determinant composed of coefficients at $dx^i$, $dp_i$ is a solvability condition of the system of homogeneous differential equations. This leads to relations:

$$\frac{dx^i}{\partial F/\partial p_i} = -\frac{dp_i}{\partial F/\partial x^i + p_i \partial F/\partial u} \tag{5}$$

Relations (5) specify the integrating direction, namely, a pseudostructure, on which the form $\theta = p_i dx^i$ turns out to be closed one, i.e. it becomes a differential, and from relation (2) the identical relation is produced. One the pseudostructure, which is defined by relation(5), the derivatives of differential equation (1) constitute a differential $\delta u = p_i dx^i = du$ (on the pseudostructure), and this means that the solution to equation (1) becomes a function.

Solutions, namely, functions on the pseudostructures formed by the integrating directions, are the so-called generalized solutions.

It is evident that this solution is obtained only under degenerate transformation, that is, when the determinant vanishes.

[It is evident that the degenerate transformation is a transition from tangent space to cotangent space (the Legendre transformations). The coordinates in relations (5) are not identical to the independent coordinates of the initial space on which equation (1) is defined.]

The first-order partial differential equation has been analyzed, and the functional relation with the form of the first degree has been considered.

Similar functional properties have all differential equations describing real processes. And, if the order of the differential equation is $k$, the functional relation with the $k$-degree form corresponds to this equation.
Here the following should be emphasized. Under degenerate transformation from an initial nonidentical functional relation the integrable identical relation is obtained. As a result of integrating, one obtains the relation that contains skew-symmetric forms of degree less by one and which in turn proves to be a nonidentical (without additional conditions). By integrating the functional relations sequentially obtained (it is possible only under realization of the degenerate transformations), from the initial functional relation of degree $k$ one can obtain $(k+1)$ functional relations each involving exterior forms of one of degrees: $k, k-1, \ldots, 0$. In particular for the first-order partial differential equation it is also necessary to analyze the functional relation of zero degree.

Thus one can see that the nonintegrability of differential equations describing real processes is due to the nonconjugacy (noncommutativity) of the derivatives with respect to different variables: the commutator made up of relevant mixed derivatives is nonzero. Without the realization of additional conditions, the solution will depend on this commutator, that is, it will be a functional. For ordinary differential equations the relevant commutator is generated due to the conjugacy of the derivatives of the functions desired and those of initial data.

It should be emphasized once more that all differential equations describing real physical processes allow solutions of two types, namely, generalized solutions that depend on variables only, and solutions that are functionals since they depend on the commutator made up by mixed derivatives.

[The dependence of the solution on the commutator may lead to instability of the solution. Equations that do not satisfy the integrability conditions may have unstable solutions. Unstable solutions appear in the case when the additional conditions are not realized and no exact solutions (their derivatives make up a differential) are formed. Thus, the solutions to the equations of the elliptic type may be unstable.

Investigation of nonidentical functional relations lies at the basis of the qualitative theory of differential equations. It is well known that the qualitative theory of differential equations is based on the analysis of unstable solutions and the integrability conditions. From the functional relation it follows that the dependence of the solution on the commutator leads to instability, and the closure conditions of skew-symmetric forms constructed by derivatives are integrability conditions. That is, the qualitative theory of differential equations that solves the problem of unstable solutions and integrability bases on the properties of nonidentical functional relation.]

Thus one can see that the solutions to equations of mathematical physics, on which no additional conditions are imposed, are functionals. The solutions prove to be exact (the generalized solution) only under realization of additional requirements, namely, the conditions of degenerate transformations. Mathematically, this corresponds to some functional expressions that become equal to zero. Such functional expressions are Jacobians, determinants, the Poisson brackets, residues, and others. These conditions define integral surfaces. The characteristic manifolds, the envelopes of characteristics, singular points, potentials of simple and double layers, residues and others are the examples of such surfaces.

Since generalized solutions are possible only under realization of the conditions of degenerate transforms, they are discrete solutions (defined only on pseu-
dostructures) and have discontinuities in the direction normal to pseudostructures.

While studying the integrability of differential equation, which was carried out by using the skew-symmetric differential forms, a nontraditional mathematical apparatus made its appearance, namely, the nonidentical relation and degenerate transformation. The skew-symmetric differential forms, which, in contrast to interior forms, are defined on nonintegrable manifolds (such, for example, as the tangent manifolds of differential equations) and possess the evolutionary properties, are provided with such mathematical apparatus.

Below, the analysis of integrability of the equations of mechanics and physics of continuous medium will be carried out by using the exterior and evolutionary forms. In so doing, the mechanism of realization the integrability of these equations will be disclosed in more detail.

2 Integrability of the equations of mechanics and physics of continuous medium

While studying the integrability of partial differential equations, the conjugacy of derivatives with respect to different variables was analyzed by using the non-identical functional relation. When describing physical processes in continuous media (in material systems) one obtains not one differential equation but a set of differential equations. And in this case it is necessary to investigate the conjugacy of not only derivatives with respect to different variables but also the conjugacy (consistency) of the equations of this set. In this case, from this set of equations one also obtains a nonidentical relation that enables one to investigate the integrability of equations and the specific features of their solutions.

The equations of mechanics and physics of continuous media (of material systems) is a set of equations that describe the conservation laws for energy, linear momentum, angular momentum and mass. The Euler and Navier-Stokes equations are examples of such a set of equations [1].

[It should be noted that the conservation laws for material systems are described with differential equations since they are balance ones (they specify the balance between the variations of physical quantities and external actions). In contrast to this, the conservation laws for physical fields are described by closed exterior forms since they are exact (point out to the availability of conserved quantities).]

Let us analyze the equations of energy and linear momentum.

In the accompanying reference system, which is tied to the manifold built by the trajectories of particles (elements of material system), the equation of energy is written in the form

\[
\frac{\partial \psi}{\partial \xi^1} = A_1
\] (6)

Here \(\xi^1\) are the coordinates along the trajectory, \(\psi\) is the functional of the state that specifies material system, \(A_1\) is the quantity that depends on specific...
features of the system and on external energy actions onto the system. The action functional, entropy, wave function can be regarded as examples of the functional $\psi$. Thus, the equation for energy expressed in terms of the action functional $S$ has a similar form: $DS/ Dt = L$, where $\psi = S$ and $A_1 = L$ is the Lagrange function. The equation for energy of an ideal gas can be presented in the form [1]: $D s/ Dt = 0, \text{ where } s \text{ is entropy.}$

In a similar manner, in the accompanying reference system the equation for linear momentum appears to be reduced to the equation of the form

$$\frac{\partial \psi}{\partial \xi^\nu} = A_\nu, \quad \nu = 2, ...$$

(7)

where $\xi^\nu$ are the coordinates in the direction normal to the trajectory, $A_\nu$ are the quantities that depend on the specific features of material system and on external force actions.

Eqs. (6) and (7) can be convoluted into the relation

$$d \psi = A_\mu d \xi^\mu, \quad (\mu = 1, \nu)$$

(8)

where $d \psi$ is the differential expression $d \psi = (\partial \psi / \partial \xi^\mu) d \xi^\mu$.

Relation (8) can be written as

$$d \psi = \omega$$

(9)

here $\omega = A_\mu d \xi^\mu$ is the skew-symmetrical differential form of the first degree.

Relation (9) has been obtained from the equation of the balance conservation laws for energy and linear momentum. In this relation the form $\omega$ is that of the first degree. If the equations of the balance conservation laws for angular momentum be added to the equations for energy and linear momentum, this form will be a form of the second degree. And, in combination with the equation of the balance conservation law for mass, this form will be a form of degree 3.

In general case the evolutionary relation can be written as

$$d \psi = \omega^p$$

(10)

where the form degree $p$ takes the values $p = 0, 1, 2, 3$. (The relation for $p = 0$ is an analog to that in the differential forms, and it has been obtained from the interaction of energy and time.)

The relation obtained from the equation of the balance conservation laws possess the properties that enable one to investigate the integrability of the original set of equations.

This relation is, firstly, an evolutionary one since the original equations are evolutionary.

Secondly, it, as well as functional relation (2), turns out to be nonidentical. To justify this we shall analyze the relation (9).

The evolutionary relation $d \psi = \omega$ is a nonidentical relation as it involves the unclosed differential form $\omega = A_\mu d \xi^\mu$. The commutator of the form $\omega$ is nonzero. The components of the commutator of such a form can be written as follows:

$$K_{\alpha \beta} = \left( \frac{\partial A_\beta}{\partial \xi^\alpha} - \frac{\partial A_\alpha}{\partial \xi^\beta} \right)$$
(here the term connected with the manifold metric form has not yet been taken into account).

The coefficients $A_{\mu}$ of the form $\omega$ have been obtained either from the equation of the balance conservation law for energy or from that for linear momentum. This means that in the first case the coefficients depend on the energetic action and in the second case they depend on the force action. In actual processes the energetic and force actions have different nature and appear to be inconsistent. The commutator of the form $\omega$ constructed of the derivatives of such coefficients is nonzero. This means that the differential of the form $\omega$ is nonzero as well. Thus, the form $\omega$ proves to be unclosed and cannot be a differential.

The nonidentity of the evolutionary relation, as well as the nonidentity of the functional relation (2), means that the initial equations of conservation laws are not conjugated, and hence, they are not integrable. The solutions to these equations can be functional or generalized solutions. In this case, the generalized solutions are obtained only under degenerated transformations.

The questions arise of how the conditions of degenerate transformation are realized and how the degenerate transformation proceeds.

The evolutionary relation has a peculiarity that enables one to answer these questions. Since this relation is evolutionary and nonidentical, it turns out to be a selfvarying one (it is an evolutionary relation and it contains two objects one of which appears to be unmeasurable and cannot be compared with another one, and therefore the process of mutual variation cannot terminate). The self-varying evolutionary relation leads to realization of the conditions of degenerate transformation. Under degenerate transformation, from nonidentical relation the relation that is identical on pseudostructure is obtained. If the conditions of degenerate transformation are realized, from the unclosed evolutionary form with nonvanishing differential $d\omega^p \neq 0$, one can obtain a differential form closed on pseudostructure. The differential of this form equals zero. That is, it is realized the transition

$$d\omega^p \neq 0 \rightarrow \text{ (degenerate transformation) } \rightarrow d_\pi \omega^p = 0, \ d_\pi^* \omega^p = 0$$

The condition $d_\pi^* \omega^p = 0$ is an equation of a certain pseudostructure $\pi$ on which the differential of evolutionary form vanishes: $d_\pi \omega^p = 0$. That is, the closed (inexact) exterior form $\omega^p_\pi$ is obtained on pseudostructure. On the pseudostructure, from evolutionary relation $d\psi = \omega^p$ it is obtained an identical relation $d_\pi \psi = \omega^p_\pi$, since the closed exterior form $\omega^p_\pi$ is a differential of some differential form (this relation will be an identical one as the left and right sides of the relation contain differentials). The identity of the relation obtained from the evolutionary relation means that on pseudostructures the original equations for material systems (the equations of conservation laws) become consistent and integrable.

Pseudostructures constitute the integral surfaces (such as characteristics, singular points, potentials of simple and double layers and others) on which the quantities of material system desired (such as the temperature, pressure, density) become functions of only independent variables and do not depend on the commutator (and on the path of integrating). This are generalized solutions.
One can see that the integral surfaces are obtained from the condition of degenerate transformation of the evolutionary relation. As it was already mentioned, the conditions of degenerate transformation are a vanishing of such functional expressions as determinants, Jacobians, Poisson’s brackets, residues and others. They are connected with the symmetries, which can be due to the degrees of freedom (for example, the translational, rotational and oscillatory freedom of the material system).

The degenerate transformation is realized as the transition from the noninertial frame of reference to the locally inertial system, i.e. the transition from nonintegrable manifold (for example, tangent one) to the integrable structures and surfaces (such as the characteristics, potential surfaces, eikonal surfaces, singular points).

Thus, one can see that the problem of integrability is based on the properties of nonidentical evolutionary relation.

Nonidentical evolutionary relation, which is obtained from the equations for material media (and from which identical relations with closed forms describing physical fields are obtained), allows to understand the specific features of the field-theory equations as well.

3 Analysis of the field-theory equations

The field-theory equations are equations for physical fields. As the physical fields are described by the closed exterior (inexact) forms, it is obvious that solutions to the field-theory equations must be closed exterior forms, i.e. to be differentials. And such differentials, which are closed exterior forms, can be obtained from the nonidentical evolutionary relation. This means that in field theory the nonidentical evolutionary relation can play a role of the field-theory equation.

One can see that there exists the correspondence between the field-theory equations and the nonidentical evolutionary relation.

The nonidentical evolutionary relation is a relation for functionals such as wave-function, action functional, entropy, and others. The field-theory equations are those for such functionals.

Another correspondence relates to the peculiarity of field-theory equations.

The peculiarity of field-theory equations consists in the fact that all these equations have the form of relations. They can be relations in differential forms or in the forms of their tensor or differential (i.e. expressed in terms of derivatives) analogs.

The Einstein equation is a relation in differential forms. This equation relates the differential of the first degree form (Einstein’s tensor) and the differential form of second degree, namely, the energy-momentum tensor. (It should be noted that Einstein’s equation follows from the differential form of third degree).

The Dirac equation relates Dirac’s bra- and ket-vectors, which made up the differential form of zero degree.
The Maxwell equations have the form of tensor relations.

The Schrödinger’s equations have the form of relations expressed in terms of derivatives and their analogs.

From the field-theory equations, as well as from the nonidentical evolutionary relation, the identical relation, which contains the closed exterior form, is obtained.

The closed exterior forms or their tensor or differential analogs, which are obtained from identical relations, are solutions to the field-theory equations.

As one can see, from the field-theory equations it follows such identical relation as

1) The Dirac relations made up of Dirac’s bra- and ket-vectors, which connect a closed exterior form of zero degree;

2) The Poincare invariant, which connects closed exterior forms of first degree;

3) The relations $d\theta^2 = 0$, $d^*\theta^2 = 0$ are those for closed exterior forms of second degree obtained from Maxwell equations;

4) The Bianchi identity for gravitational field.

From the Einstein equation it is obtained the identical relation in the case when the covariant derivative of the energy-momentum tensor vanishes.

It is evident that all equations of existing field theories are in essence the relations that connect skew-symmetric forms or their analogs. It may be emphasized once more that the equations of field theories have the form of relations for functionals such as wave function (the relation corresponding to differential form of zero degree), action functional (the relation corresponding to differential form of first degree), the Pointing vector (the relation corresponding to differential form of second degree). The tensor functionals that correspond to Einstein’s equation are obtained from the relation connecting the differential forms of third degree.

The nonidentical evolutionary relations derived from the equations for material media, as it was already mentioned, unites the relations for all these functionals. This is, all equations of field theories are analogous to the nonidentical evolutionary relation.

From this it follows that the nonidentical evolutionary relation can play a role of the equation of general field theory that discloses common properties and peculiarities of existing equations of field theory.

Can see that investigation of integrability of the field-theory equations also is based on the properties of nonidentical evolutionary relation.

References

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