Critical Heat Engines in Massive Gravity

Pavan Kumar Yerra\footnote{pk1@iitbbs.ac.in} and Chandrasekhar Bhamidipati\footnote{chandrasekhar@iitbbs.ac.in}

School of Basic Sciences
Indian Institute of Technology Bhubaneswar
Bhubaneswar, Odisha, 752050, India

Abstract
With in the extended thermodynamics, we study the efficiency $\eta_k$ of critical heat engines for charged black holes in massive gravity for spherical ($k = 1$), flat ($k = 0$) and hyperbolic ($k = -1$) topologies. Although, $\eta_k$ is in general higher (lower) for hyperbolic (spherical) topology, we show that this order can be reversed in critical heat engines with efficiency higher for spherical topology, following in particular the order: $\eta_{-1} < \eta_0 < \eta_{+1}$. Furthermore, the study of the near horizon region of the critical hole shows that, apart from the known $q \to \infty$ condition, additional scalings of massive gravity parameters, based on the topology of the geometry are required, to reveal the presence of a fully decoupled Rindler space-time with vanishing cosmological constant.
1 Introduction

Investigations of the critical region of black holes in AdS [1,8], particularly, in the context of extended phase space approach [4-7] has been an exciting area of research. In the neighborhood of a second order phase transition, the thermodynamic quantities of charged black holes in AdS, turn out to show scaling behavior, with respect to charge $q$, i.e., Entropy $S \sim q^2$, Pressure $p \sim q^{-2}$, and Temperature $T \sim q^{-1}$. More intriguingly, the black hole geometry turns out to be a fully decoupled Rindler space-time in the double scaling limit of: nearing the horizon while also taking the charge to be large [8, 9]. In the past, the emergence of such decoupled space-times in the near horizon limit of charged black holes leading to extremal black holes have generated enormous interest, such as, microscopic counting of black hole entropy, among other issues involving AdS/CFT duality. The appearance of fully decoupled Rindler geometries in the context of critical charged black holes in AdS is quite non-trivial and should lead to novel results from CFT point of view in this scenario. It is important to know if the above mentioned features are generic to black holes in AdS or not. The scaling behavior of thermodynamic quantities mentioned above, of course varies depending on the critical behavior of black holes in question and can also be mildly broken if one of the thermodynamic quantities depends on additional parameters (as will be the case in the model considered in this paper) and the near horizon limit in this case needs to be reexamined. We will see that the setting of considering black holes in massive gravity theories gives interesting results on the aforementioned issues, which are motivated at the end of this section.

Novel studies of the critical region were in fact possible only due to active developments in extended thermodynamic description of charged black holes in AdS, which reveal a phase structure consisting of line of first order phase transitions terminating in a second order transition point [1, 6, 7, 10, 11]. Here, apart from continued research on Hawking-Page transition in the bulk (holographically dual to confinement-deconfinement transition in gauge theories), van der Waals transition has also attracted wide spread attention, with a holographic interpretation being actively pursued. For instance, in [12] (see also [4, 13]), this transition is thought to be in the space of dual field theories (labeled by $N$, the number of colors in the gauge theory). Varying the cosmological constant in the bulk may correspond to perturbing the dual CFT, triggering a field theory renormalization group flow. This flow in the bulk is expected to be captured by Holographic heat engines with black holes playing the role of working substances [12]. Specifically, for hyperbolic space-times, the efficiency of heat engines may have a nontrivial connection with central charges and degrees of freedom of dual CFT [14]. We should mention, that this is currently a very interesting topic with various aspects being studied both from the gravity as well as the dual gauge theory side [3-5, 11, 12, 15-33].

There is another interesting context in which the critical region of black holes in AdS plays a central role, namely, for improving the efficiency of heat engines. It has been shown recently, that the efficiency of heat engines when the black hole is on the verge of a second order phase transition, leads to the interesting possibility of reaching Carnot efficiency. With this hope, following the works of Johnson, efficiency of critical black hole heat engines has been computed for several systems, involving Gauss-Bonnet and non-linearly charged black holes. In all the cases, it was noted that the engine efficiency reaches the Carnot efficiency, however only in the
limit that the engine runs for an infinite time\(^1\) as certain parameters of the engine are taken to be large \(8\, 43\, 44\). Following the earlier works, in this paper, we study the efficiency of critical heat engines with black holes in massive gravity as working substances. In the next paragraph, we present a general motivation for studying massive gravity theories and following that, we give a specific reasons for choosing this system for studying critical heat engines.

Broad motivations for studying massive gravity theories follow. Einstein’s General relativity has met with lot of success, with important predictions having received experimental confirmation, more recently in agreement with recent observational data of LIGO collaboration \(45\, 46\) on gravitational waves. However, there are also phenomena which, such as, accelerated expansion of the universe and the cosmological constant problem, to name a few, which warrant extensions of the Einstein’s theory. In this context, an important extension involves massive graviton theories, motivated by hierarchy problems and their usefulness in quantum gravity \(47\, 48\), which binds well with recent data \(49\), putting lower limits on the mass of gravitons. Massive gravity theories have long history, starting from the models introduced by Fierz and Paullo in 1939 \(50\), which underwent several modifications and inclusion of novel ideas, such as, New massive gravites \(51–54\), which has been actively studied in current literature \(55–61\). Black hole solutions, their thermodynamical properties \(62\, 63\) and applications in cosmology/astrophysics with motivations to see deviations from Einstein’s General relativity are being actively pursued too \(66\, 72\). One class of massive gravity theories with possible applications to holographic duality was considered in \(73\), involving the use of a singular metric, showing that the massive gravity might be stable and free of ghosts \(74\), including the presence of black hole solutions \(75–79\). Massive gravity theories are expected to play important role in solving problems discussed above in Einstein’s gravity \(80\, 91\), such as, the ability to explain the current observations related to dark matter \(92\) and also the accelerating expansion of universe without requiring any dark energy component \(93\, 94\). Attempts to embed massive gravities in string theory are being pursued too \(95\). More importantly, Van der Waals type liquid gas phase transitions in the extended phase space have been shown to exist and studied in a number of works \(96\, 101\).

Now we describe specific motivation for studying critical heat engines in massive gravity theories. First, the massive gravity theories considered in this paper can be regarded as the minimal modification of general relativity which takes in to account a massive graviton. The effect of introducing graviton mass is phenomenal as it leads to the existence of van der Waals phase transitions for non-spherical topologies, which are forbidden in Einstein as well as higher curvature Lovelock gravity. It is then interesting to see how the efficiency of heat engines varies with topology of the black hole. Preliminary studies in this direction were undertaken in \(102\), showing in particular that black holes with hyperbolic horizons as heat engines turn out to have maximum efficiency, following by flat and spherical horizon cases. Our aim in this work is to check whether the above dependence of efficiency on horizon topology, continues to hold when the heat engine is operated close to the critical point in thermodynamic phase space. The reason for checking this, is that efficiency of heat engines is very sensitive to the scheme chosen for computations\(^2\) and can show an increase in one scheme and a decrease in another \(19\);

\(^1\)see e.g. \(34\, 42\), for ongoing work on approaching Carnot efficiency in statistical mechanics literature.
\(^2\)There are actually various schemes possible, obtained by choosing a thermodynamic cycle in which certain parameters are held fixed and others varied. For instance, one possible scheme involves picking operating pressures
and also show certain universal features when the engine runs close to criticality in the large charge limit \cite{8, 9}. We thus, analyze properties of heat engines in massive gravity at the critical point and compare the efficiencies for horizons of various topologies. Interestingly, we find that efficiencies are highest for spherical topologies, followed by flat and hyperbolic horizon cases, a result, quite opposite to the situation when the engine runs far from criticality \cite{102}. A second motivation is that, the study of the near horizon limit of all charged or neutral critical black holes in AdS, in the large charge limit, thus far has shown the emergence of a fully decoupled Rindler geometry, which is because of a perfect scaling of all thermodynamic quantities with charge $q$ at the critical point (as mentioned in the first paragraph of this section). However, as we see below, one of the effects of massive graviton is to spoil the perfect of scaling of temperature with respect to charge (see eqn. (2.17) to be discussed later). We show that the appearance of a fully decoupled Rindler space-time may or may not appear in the near horizon limit, depending on the values chosen for massive gravity parameters.

Rest of the paper is organized as follows. In section-(2), we give brief details of charged black holes in AdS in massive gravity theories and collect results on various thermodynamic quantities, including the equation of state and the PV critical behavior. In section-(3), we set up the computation of efficiency of heat engines at the critical point and bring out the role played by the massive gravity parameters. The results on efficiency at the critical point are compared for various topologies. We also analyze the critical region of black hole and present the conditions under which a fully decoupled Rindler space-time appears in the near horizon limit. We summarize our findings in section-(4).

2 Charged Black Holes in Massive Gravity

Consider the action for 4-dimensional Einstein-Maxwell theory with a negative cosmological constant $\Lambda$ in massive gravity as \cite{97, 102}:

\[
I = \frac{-1}{16\pi} \int d^4x \sqrt{-g} \left( R - 2\Lambda - F + m^2 \sum_{i} c_i U_i(g, f) \right),
\]

(2.1)

where $R$ is the Ricci scalar, $F = F_{\mu\nu}F^{\mu\nu}$ is the Maxwell invariant, $F_{\mu\nu} = \partial_{\mu} A_\nu - \partial_{\nu} A_\mu$ is the electromagnetic field tensor with gauge potential $A_\mu$, and $m$ is massive parameter. The $c_i$’s are the constants, $f$ is a reference metric and the $U_i$’s are symmetric polynomials of the eigenvalues of the $4 \times 4$ matrix $\mathcal{K}^\mu_\nu = \sqrt{g^{\mu\nu}} f_{\alpha\nu}$, which can be written as

\[
\begin{align*}
U_1 &= |\mathcal{K}|, \\
U_2 &= |\mathcal{K}|^2 - |\mathcal{K}^2|, \\
U_3 &= |\mathcal{K}|^3 - 3 |\mathcal{K}| |\mathcal{K}^2| + 2 |\mathcal{K}^3|, \\
U_4 &= |\mathcal{K}|^4 - 6 |\mathcal{K}^2| |\mathcal{K}|^2 + 8 |\mathcal{K}^3| |\mathcal{K}| + 3 |\mathcal{K}^2|^2 - 6 |\mathcal{K}^4|. 
\end{align*}
\]

(2.2)

The above action admits the static topological black hole solution with the metric \cite{97, 102}:

\[
ds^2 = -Y(r) dt^2 + \frac{dr^2}{Y(r)} + r^2 h_{ij} dx_i dx_j, 
\]

(2.3)

and temperatures, with volume being left is unfixed (to be determined from equation of state) and so on.
and together with a reference metric $f_{\mu\nu}$:

$$f_{\mu\nu} = \text{diag}(0, 0, c_0^2 h_{ij}) ,$$

(2.4)

where $c_0$ is a positive constant, $i,j = 1, 2$ and $h_{ij} dx_i dx_j$ is a spatial metric of constant curvature $2k$ with volume $4\pi$. Here, one can take $k = +1$, $0$, or $-1$, for a spherical, Ricci flat, or hyperbolic topology of the black hole horizon, respectively. Using the reference metric $f_{\mu\nu}$, the $U_i$’s are read as

$$U_1 = \frac{2c_0}{r}, \quad U_2 = \frac{2c_0^2}{r^2}, \quad U_3 = 0, \quad U_4 = 0 ,$$

(2.5)

one can set $c_3 = c_4 = 0$, since $U_3 = U_4 = 0$. The metric function $Y(r)$, using the gauge potential ansatz $A_\mu = h(r) \delta_\mu^0$, is given by

$$Y(r) = k - \frac{m_0}{r} - \frac{\Lambda r^2}{3} + \frac{q^2}{r^2} + m^2 \left( \frac{c_0 c_1}{2} r + c_0^2 c_2 \right) ,$$

(2.6)

where the integration constants $m_0$ and $q$ are correspond to the mass $M$ and the electric charge $Q$ of the black hole, respectively. The solution (2.6), is asymptotically AdS and in the absence of graviton mass $(m = 0)$, it can be reduced to the Reissner-Nordstrom black hole [97]. We note that the choice of the reference metric makes the graviton mass terms to have a Lorentz-breaking property [104].

The horizon radius $r_+$ of the black hole is the largest positive root of $Y(r_+) = 0$, in terms of which the temperature $T$, mass $M$, entropy $S$, charge $Q$, and the electric potential $\Phi$ of the black hole can be expressed as

$$T = \frac{k}{4\pi r_+} - \frac{r_+^2}{4\pi} - \frac{\Lambda r_+^2}{4\pi} + \frac{m^2}{4\pi r_+} \left( c_0 c_1 r_+ + c_0^2 c_2 \right) ,$$

(2.7)

$$M = \frac{m_0}{2} = \frac{r_+}{2} \left( k - \frac{\Lambda r_+^2}{3} + \frac{q^2}{r_+^2} + m^2 \left( \frac{c_0 c_1}{2} r_+ + c_0^2 c_2 \right) \right) ,$$

(2.8)

$$S = \pi r_+^2 ,$$

(2.9)

$$Q = q ,$$

(2.10)

$$\Phi = A_\mu \chi^\mu |_{r \to \infty} - A_\mu \chi^\mu |_{r \to r_+} = \frac{q}{r_+} .$$

(2.11)

In the extended phase space, we define the pressure from the cosmological constant, using $p = -\frac{\Lambda}{8\pi}$, and its conjugate quantity is the thermodynamic volume $V$, then one should identify the mass $M$ of the black hole as the enthalpy $H$ [4], which satisfies the first law of black hole thermodynamics [97]:

$$dM = TdS + \Phi dQ + V dp + C_1 dc_1 ,$$

(2.12)

where

$$V = \left( \frac{\partial M}{\partial p} \right)_{S,Q,c_1} = \frac{4\pi}{3} r_+^3 ,$$

(2.13)

$$C_1 = \left( \frac{\partial M}{\partial c_1} \right)_{S,Q,p} = \frac{c_0 m^2 r_+^2}{4} .$$

(2.14)
Now, using \( p = -\frac{\Lambda}{8\pi} \), in equation (2.7) and equation (2.8), we obtain the expressions for equation of state \( p(V, T) \) and enthalpy \( H(S, p) \) as:

\[
P = \frac{1}{8\pi} \left\{ \frac{(4\pi T - m^2c_0c_1)}{(3V/4)^{3/4}} - \frac{(k + m^2c_2c_0^2)}{(3V/4)^{3/4}} + \frac{q^2}{(3V/4)^{3/4}} \right\}, \tag{2.15}
\]

\[
H \equiv M = \frac{1}{6\sqrt{\pi}S} \left\{ 8pS^2 + 3S(k + m^2c_2c_0^2) + 3\pi q^2 + \frac{3m^2c_0c_1}{2} \sqrt{\frac{S^3}{\pi}} \right\}. \tag{2.16}
\]

The presence of massive graviton could admit the critical behavior for the black holes with topology flat \( (k = 0) \) and hyperbolic \( (k = -1) \) as well, unlike the case of massless graviton, where only the black holes with spherical topology \( (k = +1) \) can exhibit the critical behavior \( [6, 97] \).

The equation of state (2.15), facilitates to study the critical behavior of topological black holes

![Figure 1: Sample isotherms in \( p - V \) plane for the equation of state (2.15). The central (black) isotherm is for critical temperature \( T_{cr} \), the temperature of the isotherms decreases from top to bottom and the critical point is highlighted with green colored dot where the corner 3 of the engine cycle is placed (see the inset for labeling of the cycle). Here, the parameters \( k = -1, q = 2, m = c_0 = 1, c_1 = 0.01, \) and \( c_2 = 3 \), are used and the similar phase structure exist for other topologies.](image)

on plotting different isotherms as shown in figure [1]. For fixed parameters \( (k, q, m, c_0, c_1, c_2) \), there exist a critical temperature \( T_{cr} \), corresponds to critical isotherm. The isotherms above the critical isotherm, behave like a ideal gas and indicate a unique phase of the black holes, while, those below the critical isotherm, show oscillatory behavior that indicate the small and large black holes phase. These small and large black holes undergo a first order phase transition that terminates at the critical point, from where the phase transition is of second order. This phase structure of the topological black holes in massive gravity is a reminiscent of the liquid/gas phase transition of van der Waals fluid \( [97, 105] \). The critical point can be obtained from the condition of stationary point of inflection \( (i.e., \partial p/\partial V = \partial^2 p/\partial V^2 = 0) \), given by \( 97 \):

\[
p_{cr} = \frac{\epsilon^2}{96\pi q^2}, \quad V_{cr} = \frac{8\sqrt{6\pi q^3}}{\epsilon^2}, \quad T_{cr} = \frac{\epsilon^3}{3\sqrt{6}\pi q} + \frac{m^2c_1c_0}{4\pi}, \tag{2.17}
\]

where \( r_{cr} = \sqrt{\frac{6}{\epsilon}q} \) and \( \epsilon = (k + m^2c_2c_0^2) > 0 \). Furthermore, the specific heats of the black holes
at constant volume $C_V$, and at constant pressure $C_p$ are given by [102, 103]:

$$C_V = 0 ; \quad C_p = 2S \left( \frac{8pS^2 + S(k + m^2c_0^2c_2) - \pi q^2 + m^2c_0c_2S^{3/2}}{8pS^2 - S(k + m^2c_0^2c_2) + 3\pi q^2} \right). \quad (2.18)$$

### 3 Critical Heat Engines in Massive Gravity

With the set up of extended thermodynamics given in last section, one can proceed to define heat engines for extracting mechanical work from heat energy via the $pdV$ term present in the First Law of extended black hole thermodynamics [12], where, the working substance is a black hole solution of the massive gravity system satisfying the equation of state given in eqn. (2.15). First step is to define a cycle in thermodynamic state space with input heat flow $Q_H$, output heat flow $Q_C$, and a net output work $W$, satisfying the relation $Q_H = W + Q_C$. The efficiency of heat engines can then be written in the well known way as $\eta = W/Q_H = 1 - Q_C/Q_H$. Actual computation of efficiency can be done by evaluating $\int C_p dT$ along the isobars, where $C_p$ is the specific heat at constant pressure or more efficiently via the exact formula given in [19, 20, 106]:

$$\eta = 1 - \frac{M_3 - M_4}{M_2 - M_1}, \quad (3.1)$$

which needs to be evaluated at all four corners of the cycle. We define a rectangle in $p - V$ plane as our engine cycle (which is a natural choice for static black holes with $C_V = 0$ [12]), and compute its efficiency using an exact formula (3.1).

#### 3.1 Efficiency at Criticality

It is noted in [38, 39] that, running the engine cycle in the vicinity of critical point leads to approach the Carnot’s efficiency with non zero power. This novel feature was also realized in the context of black holes on taking large parameter limit, such as charge [8] or other couplings of theories under consideration [43, 44]. Taking this advantage of critical region into consideration, to probe the behavior of efficiency with parameters of the black hole in critical region and put the corner 3 of the cycle at critical point (see fig. 1) and take the boundaries of the cycle in the following way:

$$p_3 = \frac{p_1 = p_{cr}}, \quad p_1 = \frac{p_2 = 3p_{cr}/2,}{V_2 = \frac{V_3 = V_{cr}}, \quad \text{and} \quad V_1 = \frac{V_4 = V_{cr}\left(1 - \frac{L}{q\sqrt{\epsilon}}\right)}{}} \quad (3.2)$$

where $L$ is a constant with dimensions of charge and $0 < \frac{L}{q\sqrt{\epsilon}} < 1$. This setup makes the work done $W$ (which is simply the area of the cycle) to be a constant and independent of charge [8], obtained to be:

$$W = \frac{L}{4\sqrt{6}}. \quad (3.3)$$

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Footnotes:

- Other choices are also permissible and equivalent.
In fact, the scalings in eqn. (3.2) have been chosen in such a way that $W$ is even independent of topology, which is quite useful as we will see later. The heat flow on the other hand is seen to be:

$$Q_H = M_2 - M_1$$

$$= \left\{ \frac{3L}{4\sqrt{6}} + \frac{3c_0c_1m^2q^2}{2\epsilon} \left[ 1 - \left(1 - \frac{L}{q\sqrt{\epsilon}} \right)^{\frac{3}{2}} \right] + \frac{q}{2\sqrt{6}} \left[ 1 - \left(1 - \frac{L}{q\sqrt{\epsilon}} \right)^{-\frac{1}{2}} \right] \right\}$$

$$+ 3q\sqrt{\frac{\epsilon}{6}} \left[ 1 - \left(1 - \frac{L}{q\sqrt{\epsilon}} \right)^{\frac{1}{2}} \right] \right\},$$

(3.4)

and explicitly depends on charge as well as topology. Engine efficiency is given as $\eta = W/Q_H$, while the Carnot efficiency $\eta_C$ is calculated from the highest ($T_H = T_2$) and lowest ($T_C = T_4$) temperatures using the equation of state (2.15) as:

$$\eta_C = 1 - \frac{T_C}{T_H}$$

$$= \frac{19 - 6\left(1 - \frac{L}{q\sqrt{\epsilon}}\right)^{\frac{3}{2}} - 12\left(1 - \frac{L}{q\sqrt{\epsilon}}\right)^{-\frac{1}{2}} + 2\left(1 - \frac{L}{q\sqrt{\epsilon}}\right)^{-1}}{19 + 12\sqrt{6}c_0c_1m^2q^{-3/2}}.$$

(3.5)

Now, we can examine the behavior of efficiency in two special cases, i.e., when the massive coefficient $c_1 = 0$ and $c_1 \neq 0$. We will also study efficiency as a function of graviton mass $m$. Without loss of generality, one can set the constants $c_0$ and $c_2$ to fixed values for the above study.

First we concentrate on the case when the massive coefficient $c_1 = 0$. The heat inflow $Q_H$ can be obtained from eqn. (3.4) in the binomial expansion (Since $0 < \frac{L}{q\sqrt{\epsilon}} < 1$) to be:

$$Q_H \approx \frac{19L}{12\sqrt{6}} + \frac{2}{9\sqrt{6}} \frac{L^2}{q\sqrt{k + m^2c_2c_0^3}}.$$

(3.6)

In this case, one can see that as the topological parameter $k$ increases from $k = -1$ to $+1$, $Q_H$ decreases. The implication is that the efficiency $\eta$ increases (since, work is fixed) as $k$ takes higher values, i.e., one gets the order of efficiencies w.r.t. to topology as:

$$\eta_{-1} < \eta_0 < \eta_{+1}.$$

(3.7)

Hence, black holes with spherical topology are more efficient followed by flat topology and the black holes with hyperbolic topology are less efficient. This is opposite to the result noted for heat engines in this system, when the thermodynamic cycle is considered far from criticality [24,102]. One of course needs to check how the ratio $\eta/\eta_C$ behaves too. For this, the behavior of Carnot efficiency $\eta_C$ (3.5) with topology can be estimated from:

$$\eta_C \approx \frac{3}{19} + \frac{8}{513} \frac{L^3}{q^3(k + m^2c_2c_0^2)^{3/2}},$$

(3.8)

which shows, $\eta_{C-1} > \eta_{C_0} > \eta_{C+1}$. Moreover, one can go for large charge limit as done in [8], to approach the Carnot limit.
In the large charge limit, the expressions for $Q_H, \eta$ and $\eta_C$ are given by:

$$Q_H = \frac{19\sqrt{6}}{72} L + \frac{\sqrt{6}}{27} \frac{L^2}{q\sqrt{\epsilon}} + \frac{4\sqrt{6}}{243} \frac{L^3}{q^2\epsilon} + \frac{25\sqrt{6}}{2916} \frac{L^4}{q^3\epsilon^{3/2}} + O(q^{-4})$$

$$\eta = \frac{3}{19} - \frac{8}{361} \frac{L}{q\sqrt{\epsilon}} - \frac{416}{61731} \frac{L^2}{q^2\epsilon} - \frac{3286}{1172889} \frac{L^3}{q^3\epsilon^{3/2}} - \frac{764594}{601692057} \frac{L^4}{q^4\epsilon^2} + O(q^{-5})$$

$$\eta_C = \frac{3}{19} + \frac{8}{513} \frac{L^3}{q^3\epsilon^{3/2}} + \frac{14}{513} \frac{L^4}{q^4\epsilon^2} + \frac{166}{4617} \frac{L^5}{q^5\epsilon^{5/2}} + \frac{590}{13851} \frac{L^6}{q^6\epsilon^3} + O(q^{-7})$$

As a consistency check of our results, we note that taking the special values $m = 0$ and $k = 1$ (corresponding to charged black holes in AdS with corrections from massive gravity dropped) the above expressions in eqn. (3.9)-(3.11) for large $q$, yield exactly the large $q$ results of Johnson [8].

We can now see the effect of topology and graviton mass on efficiency of our engines. Figure 2 shows that, at fixed topology as the charge $q$ increases, the Carnot efficiency $\eta_C$ decreases, while the efficiency $\eta$ and the ratio $\eta/\eta_C$ increase. However, $\eta = \eta_C$ is possible only in the limit $q \to \infty$. At finite charge, the behavior of the quantities, $\eta, \eta_C$ and $\eta/\eta_C$, with topology is quite different, with the exception that in the limit $q \to \infty$, their topological dependence vanishes. Furthermore, as shown in figures 3 and 4, the presence of graviton mass $m$ improves the efficiency $\eta$ when $c_2$ is positive, while lowers the efficiency $\eta$ when $c_2$ is negative.
Figure 3: In the case of massive coefficients $c_1 = 0$, and positive $c_2$, the effect of graviton mass $m$ on (a) $\eta$ (b) $\eta_C$ (c) $\eta/\eta_C$. (Here the parameters $c_0 = L = 1$, $q = 5$, $c_2 = 3$ are used).

Figure 4: In the case of massive coefficients $c_1 = 0$, and negative $c_2$, the effect of graviton mass $m$ on (a) $\eta$ (b) $\eta_C$ (c) $\eta/\eta_C$. (Here the parameters $c_0 = L = 1$, $q = 5$, $c_2 = -3$ are used). Note here that when $c_2$ is negative, no critical behavior exist for $k = -1$ and 0.
For the case \( c_1 \neq 0 \), figures (5) and (6), show that the variation of efficiency for various topologies, follows the inequality presented in eqn. (3.7), when the thermodynamic cycle is placed closed to critical point.

### 3.2 Critical region of black holes

We now move on to study the critical region of the black hole, following the idea that a coupled system can drive the system to Carnot efficiency at criticality [38, 39]. In the context of black holes, this can be done by using a toy model of \( q \) interacting constituent objects [8] in the background of critical hole. In particular, we consider a particle of mass \( \mu \) moving in the background of this critical black hole in the probe approximation. Following the methods in [8, 107, 108], the effective potential is seen to be

\[
V_{\text{eff}}(r) = \frac{e q}{r} + \sqrt{Y_{\text{cr}}(r)\sqrt{\mu^2 + \frac{L^2}{r^2}}},
\]  

with \( L \) denoting the angular momentum of the particle. To take closer look at the critical region one studies the metric function of charged massive black hole geometry with critical
Figure 6: In the case of massive coefficient \( c_1 \neq 0 \), the effect of topology \( k \) on \( \eta \), \( \eta_C \) and \( \eta/\eta_C \), over a sample range of charge \( q \) and \( c_1 = -0.01 \) in (a), (b), (c). (Here, the parameters \( L = m = c_0 = 1 \), and \( c_2 = 3 \), are used.)

values inserted, i.e.,

\[
Y_{cr}(r) = k - \frac{2M_{cr}}{r^2} + \frac{r^2}{l_{cr}^2} + \frac{q^2}{r^2} + m^2 \left( \frac{c_0 c_1}{2} r + \frac{c_0^2 c_2}{2} r^2 \right),
\]  
(3.13)

where

\[
M_{cr} = \frac{q \left( 3\sqrt{6}c_0 c_1 m^2 q \sqrt{\left( \frac{1}{c_0^2 c_2 m^2 + k} \right)^3 + 8} \right)}{2\sqrt{6} \sqrt{\frac{1}{c_0^2 c_2 m^2 + k}}}
\]
(3.14)

\[
l_{cr}^2 = 36 \frac{q^2}{c_2^2}
\]

The critical values of mass \( M_{cr} \) and cosmological constant parameter \( l_{cr} \), are closely related to the RN-AdS case studied in [9] with corrections involving massive gravity parameters. The critical mass is plotted in figure-(7) for various values of two key parameters, namely, \( k \) and \( c_1 \), in comparison to the RN-AdS case (where \( M_{cr} \) is a linear function of charge \( q \)). In the present case, in the large charge limit, for any value of other parameters of the model (such as \( k, m, c_0, c_2 \)): \( M_{cr} \) is always higher than the RN-AdS case for positive \( c_1 \) and can be less than RN-AdS case, including vanishing at some \( q \), when \( c_1 \) is negative. This can be seen from the
expression for critical mass in eqn. (3.14), plotted in figure-(7), showing that $M_{cr}$ vanishes at two points, namely, at $q = 0$ and at

$$q = -\frac{4\sqrt{\frac{2}{3}} (c_0^2 c_2 m^2 + k)}{3c_0 c_1 m^2}.$$  \hspace{1cm} (3.15)

In contrast, in the RN-AdS case, $M_{cr}$ vanishes only when $q = 0$. Implications of this feature while considering near horizon limit of critical black holes will be discussed below. First, we study the effective potential in eqn. (3.12), which may generally have a minimum at some value of $r_{min}$ ($> r_{cr}$), depending on the values taken by $\mu, e$ and $L$. It was argued in [9], that the presence of such a local minimum for the critical hole would lead to a condensation and possibly an instability. The presence or absence of such a minimum can be studied numerically by looking for a possible $r_{min}$ with the mass to charge ratio taken to be identical to $M_{cr}/q$. As can be seen from figure (8), there is a local minimum for certain values of $\mu/e$, which quickly disappears once $\mu/e = M_{cr}/q$ and hence the potential is purely attractive type binding all the microsystems together, with no local minimum. Now, we can analyze another aspect of the critical black hole

![Figure 7: Variation of critical Mass w.r.t. charge q for various cases as compared to RN-AdS case.](image)

![Figure 8: Main plot: Effective Potential for $L = 0, k = q = 1, m = 1, c_0 = 1, c_2 = 2, c_3 = 3$. Inset: $\mu/e = 0.1745, M_{cr}/q = 4.015$.](image)

by taking a double scaling limit where the charge parameter $q$ is taken to be large while at
the same time nearing the horizon. The analysis can proceed parallel to the proposal in [9], by writing \( r = r_+ + \eta \sigma \) and \( t = \tau/\eta \), where, \( Y(r = r_+) = 0 \) and \( Y'(r = r_+) = 4\pi T_{cr} \). The near horizon limit was obtained in [9] by taking \( \eta \to 0 \), while at the same time taking the large \( q \) limit by holding \( \eta q \) fixed. In the present case, in addition to the above limits, looking at the form of critical quantities in eqn. (2.17) one also needs to take the limit \( c_1 \to 0 \) (or \( m \to 0 \) limit\(^4\)) due to the requirement of \( \epsilon > 0 \) for the existence of critical region) to get a consistent near horizon metric, for the case of general topology. Thus, the metric in (3.12) goes over to:

\[
ds^2 = -(4\pi \tilde{T}_{cr}) \sigma d\tau^2 + \frac{1}{(4\pi \tilde{T}_{cr})} \frac{d\sigma^2}{\sigma} + d\mathbb{R}^3. \tag{3.16}
\]

Here, \( \tilde{T}_{cr} \) is \( T_{cr} \) in equation (2.17) with \( q \) replaced by \( \tilde{q} = \eta q \) and \( c_1 \) replaced by \( \tilde{c}_1 = c_1/\eta \), where both \( \tilde{q} \) and \( \tilde{c}_1 \) are held fixed. Also, since \( l_{cr} \) diverges in this limit, the effective cosmological constant zero and \( r_{cr} \) also diverges in the above scaling limit, from eqn. (2.17), the metric there is essentially flat. Thus, this new triple scaling limit results in a completely decoupled Rindler space-time with zero cosmological constant. We should mention here, that a large charge limit does not exist for the case when \( c_1 \) is negative, as the critical mass vanishes at the point noted in eqn. (3.15) and the critical temperature at this point takes the value \( T_{cr} = \frac{c_0 \sqrt{m^2}}{8\pi} \), which is unphysical, due to the bound that needs to be satisfied by thermodynamic quantities [78]. Thus, in the case of negative \( c_1 \), a suitable near horizon limit which gives a decoupled Rindler space as in [9], does not exist in general.

4 Conclusions

In this manuscript, we studied the efficiency of black hole heat engines in massive gravity theories for various topologies, when the system is close to the critical point. Earlier, heat engines in massive gravity theories were studied at generic points in the thermodynamic phase space (not necessarily at the critical point) [24, 102], where for a given fixed cycle with \( C_V = 0 \), efficiency \( \eta \) was shown to follow the order: \( \eta_{-1} > \eta_0 > \eta_{+1} \), which correspond to efficiencies of heat engines for black holes with hyperbolic, flat and spherical horizon topologies, respectively. However, here we showed that when the engine runs close to the critical point of a second order phase transition, and in particular, for case of massive coefficient \( c_1 = 0 \), efficiency \( \eta \) follows the reverse order, i.e., \( \eta_{-1} < \eta_0 < \eta_{+1} \). The reason for this is as follows. In the cases considered in [24, 102], as per the chosen scheme, the highest and lowest pressure, as well as the volumes in the engine are fixed\(^5\) so that the work done is fixed, while the heat inflow \( Q_H \) changes with topology only via an overall contribution from the term \( k \times (\text{positive quantity}) \). On the other hand, in the case of critical engines considered in this work, the highest and lowest pressure, as well as volumes in the engine depend on topology in a non-trivial way, and contribute every term in \( Q_H \) (see eq. 3.4), although, the work remains independent of topology. This explains the overall observed behavior of efficiency \( \eta \) with topology \( k \).

\(^4m \to 0 \) or \( c_0 \to 0 \) limit may not be smooth, particularly, for the cases \( k = 0, -1 \). For \( k = 1 \) the results are not very different from [9]. Further, a large value of \( m \) can also destabilize other thermodynamic quantities [78].

\(^5\) though the variation with topology is captured by equation of state, the change is compensated by changes in temperature
Moreover, for the critical engines, the Carnot efficiency $\eta_C$ and the ratio $\frac{n}{\eta_C}$ follow the order: 

$$\eta_{C_{-1}} > \eta_{C_0} > \eta_{C_{+1}}$$

and $(\frac{\eta}{\eta_C})_{-1} < (\frac{\eta}{\eta_C})_{0} < (\frac{\eta}{\eta_C})_{+1}$, which shows that the approach to Carnot efficiency is higher for the engine with higher $k$. However, $\eta$ and $\eta_C$ converge at large charge $q$ to $\frac{3}{4\pi}$, which is independent of topology of horizon and massive gravity parameters and in fact same as the charged black hole case [8].

We also studied the critical region of black holes in massive gravity by analyzing the behavior of charged particles in the probe approximation, moving in the background of the critical hole. It was noted that when there is an attractive potential binding the system together with no local minimum, when the mass to charge ratio of the particle is equal to the critical mass to charge ratio of the black hole. We further showed that a fully decoupled Rindler space-time appears in the near horizon limit in a new triple scaling limit. However, such a near horizon limit may not always exist and is not universal.

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