Quantum-Fluctuation-Driven Coherent Spin Dynamics in Small Condensates

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We have studied quantum spin dynamics of small condensates of cold sodium atoms. For a condensate initially prepared in a mean field ground state, we show that coherent spin dynamics are purely driven by quantum fluctuations of collective spin coordinates and can be tuned by quadratic Zeeman coupling and magnetization. These dynamics in small condensates can be probed in a high-finesse optical cavity where temporal behaviors of excitation spectra of a coupled condensate-photon system reveal the time evolution of populations of atoms at different hyperfine spin states.

Recently, single-atom detection in optical cavities has been realized in experiments by having atoms and cavity photons in a strongly coupling regime\textsuperscript{[3]}. This remarkable achievement has been applied to study optically transported atoms in cavities\textsuperscript{[3]}; furthermore the controllable coupling has been realized in experiments by having atoms and cavity photons and resultant collective excitations have also been successfully investigated\textsuperscript{[4]}. The sensitivity that a cavity-based atom detector has, together with a translating optical lattice which can effectively transport ultra cold atoms from a magnetic-optical trap to a cavity make it possible to study the physics of small BECs. Especially, this potentially opens the door to explore coherent dynamics of ultra-cold atoms in relatively small condensates. The physics of BECs of small numbers of atoms can be qualitatively different from the physics of big condensates and represents a new domain of cold-atom research. In small condensates, various intrinsic beyond-mean-field dynamics can be relevant within an experimentally accessible time scale. These new physical phenomena however have been quite difficult to study using the standard absorption-imaging approach to cold atoms because of relatively fewer atoms are involved in small condensates. Cavity electrodynamics in a strong coupling regime and high sensitivities to intra-cavity atoms on the other hand are ideal for investigating small condensates where the beyond-mean-field dynamics are mostly visible. In this letter, we focus on the basic concepts of beyond-mean-field coherent spin dynamics in BECs with typically a few tens to a few hundreds of atoms and detailed analysis of detecting these fascinating properties of small condensates in optical cavities with high-finesse. Research on this subject could substantially advance our understanding of the nature of quantum-fluctuation dynamics\textsuperscript{[3]}, in this particular case, dynamics purely driven by fluctuations with wavelengths of the size of condensates. Secondly, results obtained can help to better recognize limitations of precise measurements of various interaction constants based on mean-field coherent dynamics\textsuperscript{[6]}. Thirdly, our results should shed some light on the feasibility of investigating fluctuation dynamics of small condensates using optical cavities and also pave the way for future studies of dynamics of coupled small condensates.

To understand spin dynamics of a small condensate, we first study the evolution of a condensate of \( N \) hyperfine spin-one sodium atoms which is initially prepared in a mean field ground state,

\[
|\mathbf{n}| = \frac{(\mathbf{n} \cdot \psi_{\alpha})^N}{\sqrt{N!}} | \mathbf{0} \rangle.
\] (1)

Here \( \mathbf{n} \) is a unit director and three components of \( \psi_{\alpha} \), \( \psi_{\alpha}^\dagger \), \( \alpha = x, y, z \) are creation operators for three spin-one states, \( | \mathbf{x} \rangle = (|1\rangle - | -1\rangle)/\sqrt{2} \), \( | \mathbf{y} \rangle = (|1\rangle + | -1\rangle)/\sqrt{2} \) and \( | \mathbf{z} \rangle = | 0 \rangle \) respectively. And in this representation, \( S_\alpha = -i\epsilon_{\alpha\beta\gamma} \psi_{\beta}^\dagger \psi_{\gamma} \) is the total spin operator. States in Eq.(1) with \( \mathbf{n} = \mathbf{e}_z \) minimize the interaction energy of the following Hamiltonian for spin-one atoms in the presence of a quadratic Zeeman coupling along the \( z \)-direction,

\[
H = \frac{c_2}{N} S^2 + q(\psi_{\alpha}^\dagger \psi_{\alpha} + \psi_{\beta}^\dagger \psi_{\beta}).
\] (2)

Here \( c_2 \) is a spin interaction constant and \( q \) is the quadratic Zeeman coupling\textsuperscript{[7, 8, 9, 10]}. Mean field ground states are stationary solutions to the multi-component Gross-Pitaevskii equations for spin-one atoms and dynamics of these initial states demonstrated below are therefore a beyond-mean-field phenomenon. When deriving Eq.(2) for a trapped condensate, we assume that spin dynamics are described by a single mode, i.e. \( \psi_{\alpha}(\mathbf{r},t) = \sqrt{\rho(\mathbf{r})} \psi_{\alpha}(t) \); for a small condensate of less than one thousand weakly interacting atoms, this approximation is always valid. \( c_2 \) is typically a few nano kelvin for sodium atoms; \( q = (\mu_B B)^2/(4\Delta_{hf}) \) and the hyperfine splitting is \( \Delta_{hf} = (2\pi)1.77\text{GHz} \) (\( \mu_B \) is the Bohr magneton and \( \hbar \) is set to be unity).

To illustrate the nature of non-mean-field dynamics and crucial role played by quantum fluctuations, we expand the full Hamiltonian in Eq.(2) around a mean field ground state. In the lowest order expansion, we approximate \( \psi_{\alpha}^\dagger \approx \sqrt{N} \mathbf{e}_x + \psi_{\alpha}^\dagger \mathbf{e}_x + \psi_{\beta}^\dagger \mathbf{e}_y \), and \( \psi_{x,y}^\dagger \) are much

\[
H = \frac{c_2}{N} S^2 + q(\psi_{\alpha}^\dagger \psi_{\alpha} + \psi_{\beta}^\dagger \psi_{\beta}).
\] (2)
corresponds to a mean field ground state. The inset is Eq. 3,

\[ H_B = \sum_{\alpha=x,y} \frac{q + 4c_2}{2N} p_\alpha^2 + \frac{qN}{2} \theta_\alpha^2 + \ldots \]  

(3)

where for \( \alpha = x, y \), \( \theta_\alpha = \frac{1}{\sqrt{2N}}(\psi_\alpha^\dagger + \psi_\alpha) \) and \( P_\alpha = i\sqrt{\frac{N}{2}}(\psi_\alpha^\dagger - \psi_\alpha) \) are pairs of conjugate operators which satisfy the usual commutation relations \[ [\theta_\alpha, P_\beta] = i\delta_{\alpha,\beta} \].

Semiclassically, collective coordinates \( \theta_\alpha, \alpha = x, y \) represent projections of \( \psi^\dagger \) or order parameter \( n \) in the \( xy \)-plane, and \( P_{x(y)} = S_{y(x)} \) is the spin projection along the \( y(x) \)-direction. The bilinear Hamiltonian is equivalent to the bilinear Hamiltonian can be solved exactly using

\[ \langle \theta_\alpha \theta_\beta \rangle = \frac{1}{2N} \sqrt{q + 4c_2} \]. This is a measure of how strongly \( n \) fluctuates in the \( xy \)-plane. As expected, these quantum fluctuations are substantial only when \( q \) is small and are suppressed by a quadratic Zeeman field which effectively pins the order parameter along the \( z \)-direction. A direct calculation also shows that the amplitude of quantum fluctuations \( \langle \theta_\alpha^2 \rangle_{MF} \) in the mean field ground state defined in Eq. 1 is \( 2N \). This indicates that the mean field ground state is a good approximation only when \( q \gg 4c_2 \).

On the other hand, as \( q \) decreases and the effective spring constant gets smaller, the deviation becomes more and more severe. When \( q \) approaches zero, quantum fluctuations \( \theta_\alpha \) in the harmonic oscillator ground state become divergent implying that the mean field ground state is no longer a good approximation.

Indeed, the energy of mean field ground state is \( E_{MF} = \frac{1}{2}N \omega^2 c_2 \), which is much higher than \( \frac{1}{2}N \omega^2 \) when \( q \ll c_2 \); such a state corresponds to a highly excited wave packet, because of a relatively narrow spread along \( \theta_\alpha \)-directions and consequently an enormous kinetic energy associated with momenta \( P_\alpha \). We therefore expect that dynamics in this limit could dramatically differ from a stationary solution. Since the total number of atoms \( N \) is equal to \( \sum_\alpha \psi_\alpha^\dagger \psi_\alpha \), the population of atoms at \( \ket{z} \) (or \( \ket{1,0} \)) state \( N_0 = \langle \psi_1^\dagger \psi_2 \rangle \) is directly related to quantum fluctuations of \( \theta_\alpha \) and \( P_\alpha \),

\[ N_0 = N + 1 - \sum_\alpha \left( \frac{N}{2} \langle \theta_\alpha^2 \rangle + \frac{1}{2N} \langle P_\alpha^2 \rangle \right). \]  

(4)

Eq. 4 shows that the time evolution of \( N_0(t) \) is effectively driven by quantum fluctuations in \( \theta_\alpha \) and \( P_\alpha \): a study of \( N_0(t) \) probes underlying quantum-fluctuation dynamics.

For an initial state prepared in a mean field ground state with \( n = e_z \) where all atoms occupy \( \ket{1,0} \) state, one finds that \( \langle \theta_\alpha^2 \rangle = \frac{1}{2N} \) and \( \langle P_\alpha^2 \rangle = \frac{N}{2} \). The evolution of such a symmetric Gaussian wave packet subject to the bilinear Hamiltonian can be solved exactly using the standard theory for harmonic oscillators. The wave packet will remain to be a Gaussian one with the width oscillating as a function of time. Qualitatively, because of the symmetry, only harmonic states with even-parity are involved in dynamics and therefore the oscillation frequency is \( 2\omega \). Furthermore during oscillations, the kinetic energy stored in initial wave packets is converted into the potential one and vice versa. Especially when \( q \ll c_2 \), oscillations are driven by the enormous initial kinetic energy associated with \( P_\alpha \); the oscillation amplitude can be estimated by equaling the total energy \( E_{MF} \) to the potential energy which leads to \( \langle \theta_\alpha^2 \rangle \sim c_2/(Nq) \).

A straightforward calculation yields the time dependence of \( \langle \theta_\alpha^2 \rangle \) and \( \langle P_\alpha^2 \rangle \) that leads to

\[ \frac{\theta_{x,y}}{N} \text{coordinates in the ground state can be estimated as} \]

\[ \langle \theta_\alpha \theta_\alpha \rangle = \frac{1}{2N} \sqrt{q + 4c_2}. \]
\[
\frac{N_0}{N} = 1 - \frac{8c^2}{q(q + 4c^2)}N \sin^2 wt. \tag{5}
\]

The oscillating term in Eq.\(\text{5}\) shows the deviation from the stationary behavior due to quantum fluctuations in \(\theta_o\)-coordinates. The deviation is insignificant when \(q\) is not too small; however when \(q\) is of the order of \(c_2/N\), we expect that the non-mean field dynamics becomes very visible. Note that the approach outlined here neglects all higher order anharmonic interactions and therefore is only valid when the relative amplitude of fluctuations is small; that is when \(q \gg c_2/N\).

When \(q\) approaches zero, the short time dynamics following the bilinear Hamiltonian is equivalent to a particle of a mass \(m_{\text{eff}} = N/4c_2\) that is initially localized within a spread \((\theta_o)^2 = 1/N\) having a ballistic expansion with a typical velocity given as \(v = 8c^2/N\). The time dependence of spread \((\theta_o)^2\) therefore is \(1/N + (8c^2/N)^t\). So at \(t \sim N/c_2\), the number of atoms not occupying the initially prepared \(|1, 0\rangle\) state becomes of order of \(N\). This limit was first addressed by Law et al.\cite{11} and also in early works\cite{12, 13}; to describe the physics after this characteristic time scale requires analysis of full quantum dynamics. This time scale however becomes quite long for a few million atoms which makes it difficult to observe quantum dynamics in large condensates.

In the following, we are going to present our numerical results on dynamics and focus on its dependence on quadratic Zeeman coupling \(q\) and magnetization \(m\). For a condensate of \(N = 200\) atoms, we numerically integrate the time-dependent \(N\)-body Schrödinger equation of the quantum Hamiltonian in Eq.\(\text{2}\) The time evolution of \(N_0\) driven by quantum fluctuations is shown in Fig.\(\text{1}\). As \(q\) increases far beyond \(0.2c_2\), \(N_0\) oscillates as a function of time with frequency \(2\omega\) and the amplitude of oscillations decreases; the damping is not visible over tens of oscillations. When \(q\) is below \(0.2c_2\), anharmonic effects become substantial and oscillations are no longer perfect; when \(q = c_2/40\), oscillations are strongly damped after a few cycles and revived afterwards. For \(q = 0\), \(N_0\) drops to a minimum of about \(0.38N\) when \(t = t_c = 0.53N/c_2\) and remains to be a constant before reviving to be \(0.8N\) at about 10\(t_c\). For sodium atoms with a typical density \(2 \times 10^{14}\text{cm}^{-3}\), \(c_2 = (2\pi/50)\text{Hz}^2/\text{ms} = 23.8\text{ms}^2N = 200\) and increases to a few seconds when \(N\) reaches \(2 \times 10^{10}\).

We have also studied the quantum dynamics of a mean field condensate with a finite magnetization along the \(z\)-direction, \(\mathbf{m} = me_z\). States which minimize the mean field energy of the Hamiltonian in Eq.\(\text{2}\) with \(q = 0\) are

\[
|m\rangle = [(\cos \eta e_x + i \sin \eta e_y) \cdot \psi^\dagger]N \sqrt{N!} |0\rangle. \tag{6}
\]

where \(\sin 2\eta = m, m(\in [-1, 1])\) is the normalized magnetization. By expanding the Hamiltonian around these mean field states, one obtains a harmonic oscillator Hamiltonian defined in terms of conjugate operators, \(\theta_z = \frac{1}{\sqrt{2N}}(\psi^\dagger + \psi_z)\) and \(P_z = i\sqrt{2N}(\psi^\dagger - \psi_z)\). The effective mass is \(m_{\text{eff}} = \frac{N}{2c_2(1 + 1 - m^2)}\) and the harmonic oscillator frequency \(\omega = 2|m|c_2\). States shown in Fig.\(\text{2}\) have a narrow width along the direction of \(\theta_z\), \((\theta_z)^2 = 1/2N\) and therefore carry large conjugate momenta \(P_z\); the corresponding large kinetic energy drives a unique non-mean field quantum spin dynamics. The harmonic expansion again is only valid when \(m\) is large and fluctuations are weak. Simulations of the full Hamiltonian have been carried out in this case; in Fig.\(\text{1}\), we show the time dependence of \(N_0(t)\) for different magnetization. Only when \(m\) is close to unity, weakly damped oscillations are observed.

![FIG. 2: (a,b,c) (color online) Eigenfrequencies \(\Delta_p\) as a function of \(t\) for different detuning \(\Delta_c\) when the relative population at state \(|1, 0\rangle\), \(\rho_{10}(t)\) evolves. At \(t = 0\), all atoms occupy \(|1, 0\rangle\) state and \(N = 200\). d) \(\Delta_p\) as a function of time \(t\) for \(\Delta_c = 0\).](image)
populations at different states.

We restrict ourselves to excitations which involve a single cavity photon interacting with atoms in a BEC. We study atomic transitions from $3S_{1/2} \rightarrow 3P_{1/2}$ in sodium atoms. The Hamiltonian consists the following terms,

$$H_{\text{cavity}} = \sum_i \hbar w_{g_i} \hat{g}_i^\dagger \hat{g}_i + \sum_j \hbar w_{e_j} \hat{e}_j^\dagger \hat{e}_j + \sum_p \hbar w_{p} \hat{e}_p^\dagger \hat{e}_p + i\hbar \sum_{i,j} g_{ij}^p \hat{e}_j^\dagger \hat{g}_i + h.c.,$$

where $i$ labels three states $|F = 1, m_F = 0, \pm 1\rangle$ in $3S_{1/2}$ orbital and $j$ eight states $|F' = 1, m_{F'}, |F' = 2, m_{F'}\rangle$ in $3P_{1/2}$ orbital. $\hat{g}_i$ and $\hat{e}_j$ create atoms in one of $3S_{1/2}$ and $3P_{1/2}$ states respectively with corresponding frequencies $w_{g_i}, w_{e_j}$. $\hat{e}_p^\dagger$ creates a photon with frequency $w_p$ and polarization $p$ in the cavity mode. $g_{ij}^p = (D_{ij}^p \sqrt{\hbar m_{c}/2\Omega V})$ is the coupling strength for a transition $i \rightarrow j$ driven by a cavity photon with polarization $p$, which depends on the dipole matrix element $D_{ij}^p$, the effective mode volume $V$.

$$\rho_{mF} = N_{mF}/N, \quad m_F = 0, \pm 1.$$ The latter three eigen frequencies are determined by the eigen value equation,

$$\left(\Delta_p - \Delta_\sigma\right)\left(\Delta_p^2 - \Delta^2\right) - \Delta_p N g_1^2 F_1 - \Delta N g_2^2 F_2 = 0. \quad (8)$$

Here $\Delta_p = w_p - w_a$, $m = 0 to \pm 1$ is the normalized magnetization, and $g_1$ is the coupling between $3S_{1/2}$ and $3P_{1/2}$ orbital. $F_1 = 1$, $m_F = 1$ and $3P_{1/2}$ and $3P_{1/2}$ in $3S_{1/2}$ orbital. $\hat{g}_i$ and $\hat{e}_j$ create atoms in one of $3S_{1/2}$ and $3P_{1/2}$ states respectively with corresponding frequencies $w_{g_i}, w_{e_j}$. $\hat{e}_p^\dagger$ creates a photon with frequency $w_p$ and polarization $p$ in the cavity mode. $g_{ij}^p = (D_{ij}^p \sqrt{\hbar m_{c}/2\Omega V})$ is the coupling strength for a transition $i \rightarrow j$ driven by a cavity photon with polarization $p$, which depends on the dipole matrix element $D_{ij}^p$, the effective mode volume $V$.

![Figure 3: Eigenfrequencies $\Delta_p$ as a function of time $t$ driven by dynamics of population $\rho(t)$ in the presence of quadratic Zeeman coupling $q = 0.05c_2$ or $B = 95mG$. Again at $t = 0$, all atoms occupy $|1, 0\rangle$ state and $N = 200$, $\Delta_\sigma = 0$.](image)

For simplicity, we set the energy of $3S_{1/2}$ states to be zero, i.e. $w_{g_i} = 0$; the energy of excited states is $w_{e_j}^2 = w_a + \Delta$ with $\Delta$ being the hyperfine splitting between $F' = 2$ and $F' = 1$ states. For atomic transitions induced by left-circularly($\sigma^+$) polarized photons, the selection rule is $\Delta F = 0, \pm 1, \Delta m_F = 1$. In a cavity, a state with a cavity photon (labeled as $1c$), $N_{mF}$ atoms at $3S_{1/2}|1, m_F\rangle$ states and no atoms at excited states (labeled as $0j$) is expressed as $|1c; N_1, N_0, N_{-1}; 0j\rangle$; it is coupled to the following states with one of atoms excited to $3P_{1/2}$ states (labeled $1F$) and no cavity photons (as $0c$), $|0c; N_1 - 1, N_0, N_{-1}; 1F\rangle$, $|0c; N_1, N_0 - 1, N_{-1}; 1F\rangle$, $|0c; N_1, N_0, N_{-1} - 1; 1F\rangle$, $|0c; N_1, N_0, N_{-1}; 1F\rangle$. We diagonalize the Hamiltonian matrix and obtain six eigenfrequencies $\omega_2$ for this coupled system. Three are $3P_{1/2}$ orbitals without mixing with $3S_{1/2}$ states, $w_p = w_a + \Delta$; the other three depend on relative populations of atoms at each spin state, $\rho_{mF} = N_{mF}/N, m_F = 0, \pm 1$. The latter three eigen frequencies are determined by the eigen value equation,

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