Problems in field theoretical approach to gravitation

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Abstract

We consider gravitational self interaction in the lowest approximation and assume that graviton interacts with gravitational energy-momentum tensor in the same way as it interacts with particles. We show that, using gravitational vertex with a preferred gravitational energy-momentum tensor, it is possible to obtain a metric necessary for explaining perihelion precession. The preferred gravitational energy-momentum tensor gives positive gravitational energy density of Newtonian center. We show also that, employing "improvement" technique, any gravitational energy-momentum tensor can be made suitable for using in gravitational wave equation for obtaining metric which explains perihelion precession. Yet the "improvement" leads to negative gravitational energy density of the Newtonian center.

1 Introduction

The field theoretical approach to gravitation gives an additional insight into gravity both within general relativity and beyond. For example, this approach can help to choose the preferred coordinate system, which enable as to see directly from $g_{\mu\nu}$ how the gravitational field affects the Cartesian latticework of rigid rods, existed before switching on gravitational field. The field theoretical approach suggests that in a weak gravitational field of a Newtonian center the influence on rods is isotropic [Thirring (1961)]. The same conclusion follows from the dimensional analysis in [Dehnen, Hönl, Westpfahl (1960)]. On the other hand, the standard Schwarzschild coordinate system suggests that only radial lengths are affected by spherically symmetric gravitational field, cf. for example, [Cohn (1969)], see also §22 in [Einstein (1916)]. Yet, in the framework of general relativity we cannot get the answer to this question without performing first the complicated procedure of identification coordinates $x$ in $g_{\mu\nu}(x)$ with points in the laboratory, cf. §22 in [Misner, Thorne, Wheeler (1973)].

If the influence of gravitational field is indeed anisotropic, then the atom in gravitational field should lose its spherical symmetry and this must affect its spectrum, cf. [Thirring (1961)]. We note also that the ability to define the preferred system can also help us to

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resolve the problem of nonuniqueness of gravitational energy-momentum tensor [Nikishov (2003)].

It is commonly accepted that using field theoretical methods in building up gravity theory one almost inevitably end up with general relativity, see, for example, [Wald (1986)]. We show that at least in the lowest nonlinear approximation this is not the case. One appealing way to get a version of gravity theory is to use the $S$–matrix formalism with propagators and vertices without any recourse to gravitational wave equation. This way we are free from the assumption that there are purely gravitational Lagrangian and matter Lagrangian; they may be inextricable.

We consider the 3-graviton vertex build with the help of a preferred gravitational energy-momentum tensor which yields positive gravitational energy density of the Newtonian center. Using this vertex it is possible to obtain metric necessary for explanation of perihelion precession.

It seems natural to assume that gravitational field should have positive energy density. In this case the existence of black holes should be impossible. Some arguments in favor of this are given in [Baryshev (1999)]. It is likely that in the future the gravitational theory will be able to define uniquely the gravitational energy-momentum tensor.

In my previous paper [Nikishov (1999)] I have considered several gravitational energy-momentum tensors. Only one of them gives positive gravitational energy density of the Newtonian center. In this paper I use this tensor in 3-graviton vertex.

In Section 2 a notation is introduced for the building blocks of gravitational energy-momentum tensors. It facilitates the comparison of different tensors and different theories. In Section 3 we obtain the necessary $g_{\mu\nu}$ from our vertex. In Section 4 we show how to get different versions of gravitational wave equations by using ”improvement” technique for gravitational energy-momentum tensors. In Section 5 we compare the Feynman’s gravitational Lagrangian [Feynman Moringo, Wagner (1995)] with that of general relativity.

2 Notation and some preliminary relations

We define

\begin{equation}
    g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h, \quad h = h_\alpha^\alpha, \quad \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1).
\end{equation}

If not said otherwise Latin and Greek indices run from 0 to 3. As we are going to compare different gravitational energy-momentum tensors, it is helpful to introduce special notation for the building blocks of these tensors. We denote

\begin{align*}
    1\tau_{j^k} &= \eta^{jk} h_{\alpha\beta,\gamma} h^{\gamma\beta,\alpha}; \quad 2\tau_{j^k} = \eta^{jk} h_{\alpha\beta,\gamma} h^{\alpha\beta,\gamma}; \quad 3\tau_{j^k} = \eta^{jk h_{\rho\sigma,\sigma}} h; \quad 4\tau_{j^k} = \eta^{jk h_{\rho\sigma}} h; \\
    5\tau_{j^k} &= h_{\alpha\beta,\gamma} h^{\gamma\beta,\alpha}; \quad 6\tau_{j^k} = h_{\alpha\beta,\gamma} h^{\alpha\beta,\gamma}; \quad 7\tau_{j^k} = h_{\alpha\beta,\gamma} h^{\alpha\beta,\gamma}; \quad 8\tau_{j^k} = \frac{1}{2} (h_{\alpha\beta,\gamma} h^{\alpha\beta,\gamma}) h_{\alpha\beta,\gamma}; \\
    9\tau_{j^k} &= h_{\alpha\beta,\gamma} h^{\alpha\beta,\gamma}; \quad 10\tau_{j^k} = \frac{1}{2} (h_{\alpha\beta,\gamma} h^{\alpha\beta,\gamma}) h_{\alpha\beta,\gamma} h^{\alpha\beta,\gamma}; \quad 11\tau_{j^k} = h_{\alpha\beta,\gamma} h^{\alpha\beta,\gamma}; \quad 12\tau_{j^k} = h_{\alpha\beta,\gamma} h^{\alpha\beta,\gamma}; \\
    13\tau_{j^k} &= \frac{1}{2} (h_{\alpha\beta,\gamma} h^{\alpha\beta,\gamma}) h_{\alpha\beta,\gamma} h^{\alpha\beta,\gamma}; \quad 14\tau_{j^k} = h_{\alpha\beta,\gamma} h^{\alpha\beta,\gamma}; \quad 15\tau_{j^k} = \frac{1}{2} (h_{\alpha\beta,\gamma} h^{\alpha\beta,\gamma}) h_{\alpha\beta,\gamma} h^{\alpha\beta,\gamma}; \\
    16\tau_{j^k} &= \frac{1}{2} (h_{\alpha\beta,\gamma} h^{\alpha\beta,\gamma}) h_{\alpha\beta,\gamma} h^{\alpha\beta,\gamma}; \quad h_{i} = \frac{\partial h}{\partial x^i}.
\end{align*}

(2)
Similarly for terms with second derivative:

\[
\begin{align*}
\frac{\alpha}{\tau} jk &= \eta^{jk} h_{\sigma} \sigma h; \quad \frac{b}{\tau} jk = \eta^{jk} h_{\alpha \beta} \alpha \beta h; \quad \frac{c}{\tau} jk = \eta^{jk} h_{\alpha \beta} \sigma h; \quad \frac{d}{\tau} jk = \eta^{jk} h_{\alpha \beta} \alpha \beta h; \\
\frac{e}{\tau} jk &= \eta^{jk} h_{\alpha \beta} \alpha \beta h; \quad \frac{f}{\tau} jk = h_{jk} h; \quad \frac{9}{\tau} jk = h_{jk} \sigma h; \quad \frac{h}{\tau} jk = \frac{1}{2} (h_{jk} \sigma + h_{k \alpha \beta}) h; \\
\frac{i}{\tau} jk &= h_{jk} \alpha \beta h; \quad \frac{j}{\tau} jk = \frac{1}{2} (h_{jk} \alpha \beta + h_{k \alpha \beta}) h; \quad \frac{k}{\tau} jk = h_{\alpha \beta} \alpha \beta h; \quad \frac{l}{\tau} jk = h_{jk} \sigma h; \\
\frac{m}{\tau} jk &= h_{jk} h_{\alpha \beta} \alpha \beta h; \quad \frac{\nu}{\tau} jk = \frac{1}{2} (h_{jk} \alpha \beta + h_{k \alpha \beta}) h; \quad \frac{\omega}{\tau} jk = \frac{1}{2} (h_{jk} \alpha \beta + h_{k \alpha \beta}) h; \\
\frac{p}{\tau} jk &= \frac{1}{2} (h_{jk} \alpha \beta + h_{k \alpha \beta}) h; \quad \frac{q}{\tau} jk = \frac{1}{2} (h_{jk} \alpha \beta + h_{k \alpha \beta}) h.
\end{align*}
\]

(3)

Tensors $\mathcal{T} jk, \ldots, \mathcal{T} 16 jk$, and $\mathcal{A} jk, \ldots, \mathcal{A} 9 jk$ are obtained from (1) and (2) by substitutions $h_{\mu \nu} \rightarrow \tilde{h}_{\mu \nu}$. There are relations

\[
\begin{align*}
\mathcal{A} jk &= \frac{a}{\tau} jk, \quad \mathcal{B} jk = \frac{b}{\tau} jk, \quad \mathcal{C} jk = \frac{c}{\tau} jk, \quad \mathcal{D} jk = \frac{d}{\tau} jk, \\
\mathcal{E} jk &= \frac{e}{\tau} jk, \quad \mathcal{F} jk = \frac{f}{\tau} jk, \quad \mathcal{G} jk = \frac{g}{\tau} jk, \quad \mathcal{H} jk = \frac{h}{\tau} jk, \\
\mathcal{I} jk &= \frac{i}{\tau} jk, \quad \mathcal{J} jk = \frac{j}{\tau} jk, \quad \mathcal{K} jk = \frac{k}{\tau} jk, \quad \mathcal{L} jk = \frac{l}{\tau} jk, \\
\mathcal{M} jk &= \frac{m}{\tau} jk, \quad \mathcal{N} jk = \frac{n}{\tau} jk, \quad \mathcal{O} jk = \frac{o}{\tau} jk, \\
\mathcal{P} jk &= \frac{p}{\tau} jk, \quad \mathcal{Q} jk = \frac{q}{\tau} jk.
\end{align*}
\]

(4)

and these

\[
\begin{align*}
\mathcal{A} jk &= \frac{a}{\tau} jk - \frac{3}{\tau} jk + \frac{4}{\tau} jk; \quad \mathcal{B} jk = \frac{b}{\tau} jk; \quad \mathcal{C} jk = - \frac{3}{\tau} jk + \frac{4}{\tau} jk; \quad \mathcal{D} jk = \frac{4}{\tau} jk; \\
\mathcal{E} jk &= \frac{5}{\tau} jk; \quad \mathcal{F} jk = \frac{6}{\tau} jk - \frac{8}{\tau} jk + \frac{1}{\tau} jk; \quad \mathcal{G} jk = \frac{1}{\tau} jk + \frac{11}{\tau} jk; \quad \mathcal{H} jk = - \frac{8}{\tau} jk + \frac{11}{\tau} jk; \\
\mathcal{I} jk &= \frac{9}{\tau} jk; \quad \mathcal{J} jk = \frac{10}{\tau} jk = - \frac{10}{\tau} jk + \frac{11}{\tau} jk; \quad \mathcal{K} jk = \frac{11}{\tau} jk; \quad \mathcal{L} jk = \frac{11}{\tau} jk; \\
\mathcal{M} jk &= \frac{12}{\tau} jk = - \frac{1}{2} \frac{3}{\tau} jk + \frac{1}{2} \frac{4}{\tau} jk - \frac{1}{2} \frac{9}{\tau} jk + \frac{12}{\tau} jk; \quad \mathcal{N} jk = - \frac{1}{2} \frac{8}{\tau} jk - \frac{1}{2} \frac{10}{\tau} jk + \frac{1}{2} \frac{11}{\tau} jk + \frac{13}{\tau} jk; \\
\mathcal{O} jk &= \frac{14}{\tau} jk = - \frac{10}{\tau} jk + \frac{1}{2} \frac{11}{\tau} jk + \frac{14}{\tau} jk; \quad \mathcal{P} jk = - \frac{1}{2} \frac{8}{\tau} jk - \frac{1}{2} \frac{10}{\tau} jk + \frac{1}{2} \frac{11}{\tau} jk + \frac{15}{\tau} jk; \quad \mathcal{Q} jk = \frac{16}{\tau} jk = - \frac{3}{\tau} jk + \frac{1}{2} \frac{4}{\tau} jk + \frac{16}{\tau} jk.
\end{align*}
\]

(5)

The reversed relations are obtained from (4) and (5) by substitutions $\mathcal{T} \leftrightarrow \tau$. 
Next we introduce the Newtonian potential $\phi$. For one Newtonian center $\phi = -\frac{GM}{r}$, for several centers

$$\phi = -G \sum_a \frac{m_a}{|\vec{r} - \vec{r}_a|}. \tag{6}$$

For these centers the field theoretical approach gives in linear approximation [Thirring (1961)]

$$\bar{h}_{\mu\nu}^{(1)} = -4\phi \delta_{\mu0} \delta_{\nu0}, \quad h_{\mu\nu}^{(1)} = -2\phi \delta_{\mu\nu}, \quad h^{(1)} = \eta_{\sigma\alpha}^{(1)} = -4\phi = -\bar{h}^{(1)}, \quad \eta_{\mu\nu} = \text{diag}(-1, 1, 1). \tag{7}$$

The Schwarzschild metric in harmonic and isotropic (but not in standard) frames gives (7) in linear approximation. The same expressions is obtained by a heuristic method [Dehnen, Hönl, and Westpfahl (1960)]. Using (7), for the Newtonian centers we find for $T_{jk}$

$$\tau_{jk} = \tau_{jk} = \frac{3}{4} \tau_{jk} = \frac{6}{4} \tau_{jk} = \frac{8}{4} \tau_{jk} = \frac{10}{4} \tau_{jk} = \frac{12}{4} \tau_{jk} = \frac{14}{4} \tau_{jk} = \frac{16}{4} \tau_{jk} = 0;$$

$$\tau_{jk} = \frac{2}{4} \tau_{jk} = \frac{4}{4} \tau_{jk} = \frac{5}{4} \tau_{jk} = \frac{7}{4} \tau_{jk} = \frac{7}{4} \tau_{jk} = \frac{9}{4} \tau_{jk} = -16(\nabla \phi)^2 \delta_{j0} \delta_{k0};$$

$$\tau_{jk} = \frac{a}{4} \tau_{jk} = \frac{d}{4} \tau_{jk} = \frac{4}{4} \tau_{jk} = \frac{p}{4} \tau_{jk} = -16\phi \delta_{j0} \delta_{k0}; \quad \tau_{jk} = \frac{f}{4} \tau_{jk} = \frac{k}{4} \tau_{jk} = 16\phi \delta_{j0} \delta_{k0};$$

$$\tau_{jk} = \frac{h}{4} \tau_{jk} = \frac{i}{4} \tau_{jk} = \frac{j}{4} \tau_{jk} = \frac{l}{4} \tau_{jk} = \frac{m}{4} \tau_{jk} = \frac{n}{4} \tau_{jk} = \frac{o}{4} \tau_{jk} = \frac{q}{4} \tau_{jk} = 0, \tag{8}$$

and for $\tau^{jk}$

$$\frac{1}{4} \tau^{jk} = \frac{1}{4} \tau^{jk} = \frac{1}{4} \tau^{jk} = \frac{1}{4} \tau^{jk} = \frac{1}{4} \tau^{jk} = \frac{1}{4} \tau^{jk} = \frac{1}{4} \tau^{jk} = \frac{1}{4} \tau^{jk} = 4\eta^{jk}(\nabla \phi)^2; \quad \frac{5}{4} \tau^{jk} = \frac{6}{4} \tau^{jk} =$$

$$\frac{2}{4} \tau^{jk} = \frac{10}{4} \tau^{jk} = \frac{11}{4} \tau^{jk} = \frac{13}{4} \tau^{jk} = \frac{14}{4} \tau^{jk} = \frac{15}{4} \tau^{jk} = \frac{9}{4} \tau^{jk} = \frac{2}{4} \tau^{jk} = \frac{12}{4} \tau^{jk} = 8\delta_{jk}(\nabla \phi)^2;$$

$$\frac{1}{2} \tau^{jk} = \frac{b}{2} \tau^{jk} = \frac{c}{2} \tau^{jk} = \frac{d}{2} \tau^{jk} = \frac{2}{2} \tau^{jk} = \frac{2}{2} \tau^{jk} = \frac{2}{2} \tau^{jk} = \frac{2}{2} \tau^{jk} = \frac{2}{2} \tau^{jk} = \frac{2}{2} \tau^{jk} = \frac{2}{2} \tau^{jk} = \frac{2}{2} \tau^{jk} = \frac{2}{2} \tau^{jk} = 8\phi \phi_{jk}. \tag{9}$$

$j, k = 0, 1, 2, 3$. It is assumed in (8) and (9) that $\eta_{jk} = \text{diag}(-1, 1, 1, 1)$. Otherwise one must replace $\eta_{jk}$ by $-\eta_{jk}$.

### 3 Gravitational energy-momentum tensor

In my previous paper [Nikishov (1999)] I have considered several gravitational energy-momentum tensors in the lowest (e.i. $h^2$) approximation. One of them (corresponding to general relativity) is obtainable from the Lagrangian

$$\frac{T^{\text{hir}}}{L} = \frac{1}{32\pi G} \left[ -\frac{1}{2} \bar{h}_{\alpha\beta,\gamma} \bar{h}^{\alpha\beta,\gamma} + \frac{1}{4} \bar{h}_{\alpha} \bar{h}^{\alpha} + \bar{h}_{\alpha\beta,\lambda} \bar{h}^{\lambda\beta,\alpha} \right],$$

the other one from

$$\frac{M^{\text{TW}}}{L} = \frac{1}{32\pi G} \left[ -\frac{1}{2} \bar{h}_{\alpha\beta,\gamma} \bar{h}^{\alpha\beta,\gamma} + \frac{1}{4} \bar{h}_{\alpha} \bar{h}^{\alpha} + \bar{h}_{\alpha\beta,\lambda} \bar{h}^{\lambda\beta,\alpha} \right]. \tag{10}$$
The Lagrangian (10) is eq. (7.8b) in [Misner, Thorne, Wheeler (1973)]; earlier it was used by Feynman, see eqs. (3.6.1) and (3.6.5) in [Feynman, Moringo, Wagner (1995)]. These Lagrangians are related as follows

\[
T_{\text{biv}} = T_{\text{MTW}} + \frac{1}{32\pi G} [(\tilde{h}_{\alpha\beta}\tilde{h}^{\alpha\beta},)_{,\lambda} - (\tilde{h}_{\alpha\beta}\tilde{h}^{\alpha\beta},)_{,\lambda}]^\alpha.
\]

Symmetrizing the canonical energy-momentum tensor, obtained from (10) by Belinfante method and adding the interaction tensor

\[
t_{jk} = M T_{jn} h_{nk}^j,
\]

we obtain [Nikishov (1999)]

\[
t_{jk} = \frac{1}{32\pi G} [\frac{1}{2} T_{jk} + \frac{1}{4} T_{jk} + \frac{5}{2} T_{jk} - \frac{11}{2} T_{jk} - 2 T_{jk} - 2 T_{jk} + \frac{1}{2} (T_{jn} h_{nk}^j + T_{kn} h_{nj}^k)].
\]

Here \(M T_{jk}\) is energy-momentum tensor of particles

\[
M T_{jk} = \sum_{a} m_a u^j u^k \frac{d}{dt} \delta(\vec{x} - \vec{x}_a(t)), \quad u^\mu = dx^\mu / ds.
\]

The remarkable feature of tensor (11), called in [Nikishov (1999)] MTW-tensor, is that it gives positive energy density of gravitational field of the Newtonian center. Moreover, the expression for gravitational energy density has the same form as electromagnetic energy density of a Coulomb center:

\[
t^{00} = 2 M T^{00} \phi + \frac{1}{8\pi G} (\nabla \phi)^2.
\]

Together with (12) we find for slowly moving particles

\[
M T^{00} + t^{00} = \sum_{a} m_a \delta(\vec{x} - \vec{x}_a(t)) + M T^{00} \phi + \frac{1}{8\pi G} (\nabla \phi)^2.
\]

Here we have used eq.(29) below and eq.(7) according to which \(h_{00} = -2\phi\). Now we consider only two particles \(m_a\) and \(m_b\) and drop the self interaction terms in \(M T^{00}\)\):

\[
-\frac{G m_a}{|\vec{r} - \vec{r}_a|} \delta(\vec{r} - \vec{r}_a) + \frac{G m_b}{|\vec{r} - \vec{r}_b|} \delta(\vec{r} - \vec{r}_b) + \frac{G m_a m_b}{|\vec{r}_a - \vec{r}_b|} \delta(\vec{r} - \vec{r}_b).
\]

Then, the probing particle \(m_a\), slowly moving in the gravitational field of \(m_b\), has the energy \(m_a \left(1 - \frac{G m_b}{|\vec{r}_a - \vec{r}_b|}\right)\), see terms proportional to \(\delta(\vec{r} - \vec{r}_a)\) in (14) and (15). This is what is needed for explaining the red shift according to §4 in Ch 2 in [Schwinger (1970)]. Indeed, let \(w_\infty\) be the frequency of an atom outside the gravitational field. Then \(w_\infty (1 + \phi(r_a))\) is the frequency in the field in terms of (coordinate) time \(t\). In terms of observable time \(t^{\text{obs}} = \sqrt{|g_{00}|} t\) at point \(r\) the observable frequency is \(w_\infty (1 + \phi(r_a))(1 - \phi(r))\). In particular, near the point
of emission \( r = r_a \) the observable frequency is \( w_\infty \), i.e. the same as without gravitational field. This is in agreement with Thirring (1961), who considered electron-proton system in gravitational field, using observable length and \( t^\text{obs} \).

We see that tensor (11) may be the real gravitational energy-momentum tensor and all other tensors are only effective tensors, leading to correct expressions for special cases such as total energy of a system, or energy density of gravitational wave.

In view of this properties we try to use MTW-tensor in three graviton vertex. The immediate aim is to obtain \( g_{00} \) for the Newtonian center in the form

\[
g_{00} = -(1 + 2\phi + 2\phi^2)
\]

(16)

necessary for explaining the perihelion precession of Mercury, if the linear approximation for \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \) is given by (7). So we use the gravitation interaction Lagrangian in the form

\[
\int L = \frac{1}{2} h_{\mu\nu} t^{\mu\nu}
\]

(17)

with \( t^{\mu\nu} \) given in (11).

The corresponding general relativity Lagrangian in terms of \( \tau^{\mu\nu} \) is (see [Iwasaki (1971)])

\[
\frac{1}{64\pi G} h_{\mu\nu} \sum_{s=1}^{13} a_s \, \tau^s_{\mu\nu}.
\]

(18)

Here \( a_s \) are given in Table 1

| \( s \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( a_s \) | 1 | -\( \frac{1}{7} \) | -1 | \( \frac{1}{7} \) | 1 | -2 | 2 | -2 | 2 | -1 | 2 | -4 |

(In [Iwasaki (1971)] \( a_{11} = -2; \) this must be a mistake.)

If we rewrite (16) in the form

\[
g_{00} = -[1 + 2(\phi + c\phi^2)],
\]

(19)

then we can find from Iwasaki paper the contribution \( c_s = a_s R_s \) to the sum \( c = \sum_{s=1}^{13} c_s \) for each \( s = 1, 2, \cdots, 13 \). Hence we can find \( R_s \) and use them in different versions of theory. So we get \( R_s \) given in Table 2.

| \( s \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( R_s \) | -\( \frac{1}{7} \) | -2 | -1 | -2 | -1 | -\( \frac{1}{7} \) | -\( \frac{1}{7} \) | -2 | -\( \frac{1}{7} \) | -1 | -1 | -\( \frac{1}{7} \) | -\( \frac{1}{7} \) | -\( \frac{1}{7} \) | -\( \frac{1}{7} \) |
| \( e_s \) | 1 | -\( \frac{1}{7} \) | -1 | \( \frac{1}{7} \) | 1 | 2 | 0 | 1 | 1 | 3 | -2 | -2 | -2 | 0 | 0 |

The table is enlarged because in general \( s = 14, 15, 16 \) also occur. We note that \( R_s \) is determined by Newtonian value of \( \tau^s_{\mu\nu} \). For example, from \( \tau^1_{\mu\nu} = \tau^2_{\mu\nu} = \tau^{16}_{\mu\nu} \) in (9) it follows \( R_1 = R_2 = R_{16} \) and so on.

To use Table 2 for the case with three graviton vertex (17), we rewrite \( t^{jk} \) in (11) in terms of \( \tau \). With the help of (4) and (5) we find

\[
t^{jk} = \frac{1}{32\pi G} \left[ -\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \right]
\]
Next, we convert the terms with second derivatives into terms with first derivatives. For the term $\frac{a}{7} \mu \nu$ we have

$$ h_{\mu \nu} \frac{a}{7} \mu \nu \equiv h_{\mu \nu} \eta^{\mu \nu} h_{,\sigma}^\sigma h = (h^{\sigma} h h)_{,\sigma} - h^{\sigma} 2 h h_{,\sigma}. $$

(21)

Dropping divergence, we write $h_{\mu \nu} \frac{a}{7} \mu \nu \to -2 h_{\mu \nu} \frac{4}{7} \mu \nu$, or symbolically $\frac{a}{7} \mu \nu \to -2 \frac{4}{7} \mu \nu$ and similarly for other terms. Thus in cubic terms we may make the substitutions

$$ \frac{a}{7} \mu \nu \to -2 \frac{4}{7} \mu \nu; \quad b \mu \nu \to -2 \frac{3}{7} \mu \nu; \quad c \mu \nu \to -\frac{3}{7} \mu \nu - \frac{11}{7} \mu \nu; \quad d \mu \nu \to -\frac{2}{7} \mu \nu - \frac{9}{7} \mu \nu; \quad e \mu \nu \to -\frac{1}{7} \mu \nu - \frac{8}{7} \mu \nu; \quad f \mu \nu \to -\frac{3}{7} \mu \nu - \frac{11}{7} \mu \nu; \quad g \mu \nu \to -\frac{2}{7} \mu \nu - \frac{9}{7} \mu \nu; \quad h \mu \nu \to -\frac{1}{7} \mu \nu - \frac{8}{7} \mu \nu; \quad i \mu \nu \to -\frac{5}{7} \mu \nu - \frac{12}{7} \mu \nu; \quad j \mu \nu \to -\frac{13}{7} \mu \nu - \frac{19}{7} \mu \nu; \quad k \mu \nu \to -\frac{5}{7} \mu \nu - \frac{12}{7} \mu \nu; \quad l \mu \nu \to -\frac{2}{7} \mu \nu - \frac{9}{7} \mu \nu; \quad m \mu \nu \to -\frac{12}{7} \mu \nu; \quad n \mu \nu \to -\frac{8}{7} \mu \nu - \frac{10}{7} \mu \nu; \quad o \mu \nu \to -\frac{6}{7} \mu \nu - \frac{13}{7} \mu \nu; \quad p \mu \nu \to -\frac{2}{7} \mu \nu - \frac{9}{7} \mu \nu; \quad q \mu \nu \to -\frac{6}{7} \mu \nu - \frac{13}{7} \mu \nu, \quad \text{or} \quad q \mu \nu \to -\frac{14}{7} \mu \nu - \frac{15}{7} \mu \nu. $$

(22)

Noting that together with $h \frac{a}{10} \mu \nu \to -\frac{1}{7} \mu \nu - \frac{8}{7} \mu \nu$ we also have $h \frac{a}{10} \mu \nu \to -\frac{10}{7} \mu \nu - \frac{16}{7} \mu \nu$, we find

$$ \frac{16}{7} \mu \nu \to \frac{1}{7} \mu \nu + \frac{8}{7} \mu \nu - \frac{10}{7} \mu \nu. $$

(23)

(We note by the way that substitutions $\tau \to T$ in (22) and (23) give corresponding relations for cubic terms expressed in terms of $\hat{h}$).

With the help of (22) and (23) we find that $t^{\mu \nu}$ in (17) can be substituted as follows

$$ t^{\mu \nu} \to \frac{1}{32 \pi G} \sum_{1}^{16} c_s \frac{s}{7} \mu \nu + \frac{1}{2} \left( T^{\mu \alpha \sigma} h_{\alpha}^{\nu} + T^{\nu \alpha \sigma} h_{\alpha}^{\mu} \right), $$

(24)

c_s are given in Table 2. Now it easy to verify that the contribution to $c$ in (19) from the sum in (24) is zero:

$$ \sum_{1}^{16} c_s R_s = 0. $$

(25)

Using the linearized Einstein equation

$$ -h^{jn,\sigma}_{,\sigma} + h^{j}_{,\sigma} h^n_{,\sigma} - h^{jn} + \eta^{jn} (h_{,\sigma}^\sigma - h_{,\sigma}^\sigma - h_{,\sigma}^\sigma) = 16 \pi G \frac{M}{T} j^n, $$

(26)

we rewrite the interaction term in (24) in the form

$$ \frac{1}{2} (T^{jn} h^{k} + T^{kn} h^{j}) = \frac{1}{16 \pi G} \left[ l_{jk} - m_{jk} - n_{jk} + o_{jk} - p_{jk} + q_{jk} \right]. $$

(27)

For the Newtonian centers this interaction term is equal $2 \phi^{M}_{T} 00$.

Now, using (22) we find

$$ h_{jk} \frac{1}{4} (T^{jn} h^{k} + T^{kn} h^{j}) \to \frac{1}{64 \pi G} h_{jk} \left[ -4 \frac{6}{7} j^{k} + 4 \frac{7}{7} j^{k} + 2 \frac{8}{7} j^{k} - 4 \frac{9}{7} j^{k} + \right]. $$
\[2 \tau^{10} jk + 4 \tau^{12} jk - 4 \tau^{13} jk]. \tag{28}\]

These terms contribute 2 to \(c\) in (19). Finally we have to take into account that \(T^{jk}\) consist of the observable energy-momentum and interaction part. We need here \(T^{00}\). For it we have

\[
T^{00} = \sum_{a} m_a \frac{dx^0}{ds} \delta(\vec{x} - \vec{x}_a(t)) \approx T^{00}_{\text{observable}} (1 + \frac{1}{2} h_{00}). \tag{29}\]

This is because in the presence of gravitational field

\[
ds^2 = -g_{00} dt^2 (1 - v^2). \tag{30}\]

Here \(v^2\) is physical velocity, see §88 in [Landau and Lifshitz (1973)]. Hence

\[
\frac{dx^0}{ds} = \frac{1}{\sqrt{-g_{00}(1 - v^2)}} \approx \frac{1}{\sqrt{1 - v^2}} (1 + \frac{1}{2} h_{00}). \tag{31}\]

The interaction term \(\frac{1}{2} M^{00} h_{00} = -\phi M^{00}\) in (29) together with (27) makes \(c = 1\) in (19). Thus, we can obtain the desired value of \(g_{00}\) using gravitational energy-momentum tensor (11) in three graviton vertex (17).

## 4 Gravitational wave equation

If we want to consider the wave equation approach, we can’t just add the gravitational energy-momentum tensor (11) to \(T^{jk}\) in the wave equation (26). This would lead to an incorrect \(g_{00}\). We must first ”improve” \(T^{jk}\) by adding some properly chosen expression, which do not affect the conservation laws. The simplest possibility is to use \((\partial_j \partial_k - \eta_{jk} \partial_\alpha \partial^\alpha) \tilde{h}^{\alpha \beta} \tilde{h}_{\alpha \beta}\) or \((\partial_j \partial_k - \eta_{jk} \partial_\alpha \partial^\alpha) \tilde{h}^2\). The first possibility will lead to an expression proportional to \(\tilde{T}^{5jk} - \tilde{T}^{jk} - \tilde{T}^{djk} + \tilde{T}^{jk}\). We discard this possibility because it would change the coefficient in front of \(\tilde{T}^{jk}\) in (11), which gives the correct energy-momentum tensor of gravitational plane wave, cf. §107 in [Landau and Lifshitz (1973)].

Using the second possibility, we add to \(T^{jk}\) in (11) the expression

\[
\tilde{t}^{jk} = \frac{1}{32 \pi G} (\tilde{T}^{5jk} - \tilde{T}^{jk} - \tilde{T}^{djk} + \tilde{T}^{jk}). \tag{32}\]

In terms of \(\tau\) this expression has the same form, see (4) and (5). The constant factor on the r.h.s. in (32) is chosen so that \(g_{00}\) has the desired form (16). Thus, we obtain the ”improved” gravitational energy-momentum tensor

\[
t_{jk}^{\text{imp}} = t_{jk} + \tilde{t}_{jk}. \tag{33}\]

Now, for simplicity we consider a single Newtonian center. Outside the matter we find

\[
\tilde{t}_{00} = -\frac{GM^2}{2\pi r^4}, \quad \tilde{t}_{jk} = \frac{GM^2}{\pi} \left( \frac{\delta_{jk}}{r^4} - \frac{2x_j x_k}{r^6} \right), \tag{34}\]

\[
\tilde{\tau}^{10} jk + 4 \tau^{12} jk - 4 \tau^{13} jk].
\]
and for $t_{\mu\nu}$ in (11)

$$t_{00} = \frac{GM^2}{8\pi r^4}, \quad t_{jk} = \frac{GM^2}{8\pi} \left( -\frac{\delta_{jk}}{r^4} + \frac{2x_jx_k}{r^6} \right). \tag{35}$$

Beginning from equation (34) in all equations up to equation (46) $j, k = 1, 2, 3$. From (32)-(35) it follows

$$t_{00}^{\text{imp}} = -3GM^2 \frac{2}{8\pi r^4}, \quad t_{jk}^{\text{imp}} = 7GM^2 \frac{2}{8\pi} \left( \frac{\delta_{jk}}{r^4} - \frac{2x_jx_k}{r^6} \right). \tag{36}$$

We note that $t_{jk}$, $\tilde{t}_{jk}$, and $t_{jk}^{\text{imp}}$ differ only by constant factors and

$$\left( \frac{\delta_{jk}}{r^4} - \frac{2x_jx_k}{r^6} \right) = 0. \tag{37}$$

Thus, the conservation laws $t_{\mu\nu}^{\text{imp}} = 0$ are satisfied.

Comparison of $t_{\mu\nu}^{\text{imp}}$ in (36) with corresponding Weinberg tensor (see equations (7.6.3) and (7.6.4) in [Weinberg (1972)]) shows that for the considered special case they coincide: $t_{\mu\nu}^{\text{imp}} = t_{\mu\nu}^{W}$, if (7) is used for calculating $t_{\mu\nu}^{W}$. The expressions for $g_{jk}$ in Hilbert ($\tilde{h}_{jk} = 0$), harmonic and isotropic frames are

$$g_{Hilb}^{jk} = \delta_{jk}(1 - 2\phi) + \phi^2(5\delta_{jk} - \frac{7x_jx_k}{r^2}), \quad g_{\text{har}}^{jk} = \delta_{jk}(1 - 2\phi) + \phi^2(\delta_{jk} + \frac{x_jx_k}{r^2}),$$

$$g_{\text{iso}}^{jk} = \delta_{jk}(1 - 2\phi + \frac{3}{2}\phi^2). \tag{38}$$

In all these frames $g_{00}$ are the same as in (16). The differences in $g_{jk}$ are proportional to

$$\Lambda_{i,k} + \Lambda_{k,i} = 2\left( \frac{\delta_{jk}}{r^2} - \frac{2x_jx_k}{r^4} \right), \quad \Lambda_i = \frac{x_i}{r^2}. \tag{39}$$

i.e. these $g_{jk}$ are related by gauge transformations. This is in agreement with the fact that in considered approximation Weinberg gravitational energy-momentum tensors in harmonic and isotropic frames coincide [Nikishov (2003)].

Defining as usual $\bar{t}_{\mu\nu} = t_{\mu\nu} - \frac{1}{2}g_{\mu\nu}t$, we get from (36)

$$\bar{t}_{00}^{\text{imp}} = \frac{GM^2}{4\pi r^4}, \quad \bar{t}_{jk}^{\text{imp}} = \frac{GM^2}{4\pi} \left( \frac{\delta_{jk}}{r^4} - \frac{7x_jx_k}{r^6} \right). \tag{40}$$

Using improvement technique we can make any gravitational energy-momentum tensor suitable for using in the wave equation. In this way we obtain another version of theory in considered approximation. For example, for Thirring (1961) tensor we have instead of (35)

$$t_{00}^{Th} = -\frac{7GM^2}{8\pi r^4}, \quad t_{jk}^{Th} = \frac{GM^2}{8\pi} \left( \frac{2x_jx_k}{r^6} - \frac{\delta_{jk}}{r^4} \right). \tag{41}$$

The ”improved” tensor has the form

$$t_{\mu\nu}^{\text{imp}Th} = t_{\mu\nu}^{Th} + 3\bar{t}_{\mu\nu}, \tag{42}$$

where $\bar{t}_{\mu\nu}$ is given in (34). From (41), (42) and (34) we find

$$t_{00}^{\text{imp}Th} = -\frac{19GM^2}{8\pi r^4}, \quad t_{jk}^{\text{imp}Th} = \frac{23GM^2}{8\pi} \left( -\frac{2x_jx_k}{r^6} + \frac{\delta_{jk}}{r^4} \right). \tag{43}$$
From here for barred quantities we get
\[ \bar{p}_{00}^{\text{impTh}} = \frac{GM^2}{4\pi r^4}; \quad \bar{p}_{jk}^{\text{impTh}} = \frac{GM^2}{4\pi} \left( -\frac{23x_j x_k}{r^6} + \frac{\delta_{jk}}{r^4} \right). \] (44)

As seen from (40) and (44) \( \bar{p}_{00}^{\text{impTh}} = \bar{p}_{00}^{\text{imp}} \). Hence, \( g_{00} \) are the same in both versions, but \( g_{jk} \) are different. Using the wave equation in the form
\[ -h_{\mu\nu,\sigma} + \bar{h}_{\mu\sigma} \tau_{\nu} + \bar{h}_{\nu\sigma} \tau_{\mu} = 16\pi G \bar{T}_{\mu\nu}, \] (45)
we find in Hilbert gauge (\( \bar{h}_{\mu\nu,\nu} = 0 \))
\[ \bar{g}_{jk}^{\text{Th}} = \delta_{jk}(1-2\phi) + G^2 M^2 \left( -\frac{23x_j x_k}{r^6} + \frac{21\delta_{jk}}{r^4} \right). \] (46)

In conclusion of this Section we note that the nonuniqueness of gravitational energy-momentum tensor in general relativity is connected with the nonphysical degrees of freedom taking part in its formation [Nikishov (2003)]. If we could define the preferred frame as the one formed by only physical degrees of freedom, then the problem of uniqueness could be solved. It seems reasonable to assume that for the Newtonian center in linear approximation the preferred metric is given by (7) and in the second approximation it may be the harmonic one.

5 Remark on Feynman gravitational Lagrangian

The Feynman approach to building up gravity theory seems to me quite natural because he uses throughout \( h_{\mu\nu} \) as gravitational variable, not some function of it. His cubic term of the Lagrangian (see equation (6.1.13) in [Feynman, Moringo, Wagner (1995)]) in our notation has the form
\[ F^3 = -\frac{1}{64\pi G} h^{jk} \left\{ \frac{1}{8} \tau_{jk} + \frac{1}{4} \frac{a}{b} \tau_{jk} - \frac{1}{4} \frac{c}{f} \tau_{jk} - \frac{1}{4} \frac{g}{h} \tau_{jk} + \frac{i}{j} \tau_{jk} - \frac{3}{4} \frac{l}{m} \tau_{jk} + \frac{n}{o} \tau_{jk} - 2 \frac{p}{q} \tau_{jk} + \frac{r}{s} \tau_{jk} + \frac{t}{u} \tau_{jk} \right\}. \] (47)

This expression have to be compared with cubic term in the expansion of general relativity Lagrangian \(-\frac{1}{16\pi G} \sqrt{-g} R\). My calculation gives the following result for this term
\[ -\frac{1}{16\pi G} h^{jk} \left\{ \frac{1}{8} \tau_{jk} - \frac{1}{8} \frac{a}{b} \tau_{jk} - \frac{1}{4} \frac{c}{f} \tau_{jk} - \frac{1}{4} \frac{g}{h} \tau_{jk} + \frac{i}{j} \tau_{jk} - \frac{3}{4} \frac{l}{m} \tau_{jk} + \frac{n}{o} \tau_{jk} - 2 \frac{p}{q} \tau_{jk} + \frac{r}{s} \tau_{jk} + \frac{t}{u} \tau_{jk} \right\} \] (48)

Using reductions to first derivatives (22) and equation (23), we get for (48) the expression (18). For Feynman’s Lagrangian (47) the reduction gives (18) with substitution \( 4 \frac{13}{2} \tau_{jk} \rightarrow 2 \frac{13}{2} \tau_{jk} \). Thus, it seems that Feynman’s Lagrangian (47) differs essentially from (48). Evidently Feynman do not think so. In any case his Lagrangian, used as a vertex, explain perihelion precession. This is seen from the fact that for Newtonian centers \( \tau_{jk} = \frac{14}{2} \tau_{jk} \), see (9) and text below Table 2.
6 Conclusion

Our considerations show that the ways are open in search for gravitation theory with unique gravitational energy-momentum tensor, giving the positive energy density. It is still not clear which coordinate condition better exclude the nonphysical degrees of freedom; is it harmonic, Hilbert or some other condition? Will the future gravitational theory be the theory of gravitons (spin-2), or gravitons and spin-0 field (as in [Baryshev (1999)] and [Logunov (1998)])?

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References

Baryshev Yu., gr-qc/9912003.
Cohn J., Int. Jour. of Theor. Phys. 2, 267 (1969).
Dehnen H., Hönl H., and Westpfahl K., Ann. der Phys. 6, 7 Folge, Band 6, Heft 7-8, S.670 (1960).
Einstein A., Ann. Phys., 49, 769 (1916).
Feynman R.P., Moringo F.B., Wagner W.G., Feynman Lectures on Gravitation, Addison-Wesley Company (1995).
Iwasaki Y., Prog. Theor. Phys. 46, 1587 (1971).
Landau L.D. and Lifshitz E.M., The classical theory of fields, Moscow, (1973) (in Russian).
Logunov A.A., Part. and Nucl. 29 (1) Jan.-Feb 1998, p 1; The Theory of Gravitational Field, Moscow, Nauka (2000), (in Russian).
Misner C.W., Thorne K.S., Wheeler J.A., Gravitation. San Francisco(1973).
Nikishov A., gr-qc/0310072 Part. and Nucl. 32, 5 (2001).
Schwinger J. Particles, Sources, and Fields. V.1 Addison-Wesley (1970).
Thirring W.E., Ann. Phys. (N.Y.) 16, 96 (1961).
Wald R.M., Phys. Rev. 33, 3613,(1986).
Weinberg S., Gravitation and Cosmology, New York (1972).