DEFORMATION THEORY OF ORTHOGONAL AND SYMPLECTIC SHEAVES

EMILIO FRANCO

Abstract. We show that the space of first-order deformations of an orthogonal (resp. symplectic) sheaf over a smooth projective scheme is the first hypercohomology space of a complex which is naturally constructed out of the orthogonal (resp. symplectic) sheaf. We also provide an obstruction theory of these objects whose target is the second hypercohomology space of this complex.

1. Introduction

Moduli spaces of principal bundles usually carry interesting geometric structures, being a powerful, and often unique, source of examples of varieties with prescribed properties and characteristics. Nevertheless, these spaces might be non-compact whenever the base (smooth) scheme has dimension higher than 1. Principal sheaves or singular principal bundles [GS1, GS2, Sch1, Sch2, GLSS1, GLSS2] provide a natural compactification of the moduli space of principal bundles for a connected complex reductive structure group. Therefore, moduli spaces of principal sheaves are projective varieties equipped with an interesting geometry, at least, on a dense subset. In order to check whether or not these properties extend to the compactification, we need a local description of the moduli spaces, precisely over the locus where the principal sheaves fail to be principal bundles. Such description would naturally derive from deformation theory of principal sheaves, which is still missing at present date.

Moduli spaces of sheaves have been very useful in defining invariants. For instance, Donaldson polynomials or, more recently, Donaldson-Thomas invariants. It is natural to consider also $G$-principal objects for a reductive group $G$ but, again, we first need to study the deformation theory of these objects.

In this article we consider orthogonal and symplectic sheaves (see Section 2.3 for the definition). We show that the deformation and obstruction theory of these objects is controlled by a deformation complex naturally built out of our starting orthogonal (resp. symplectic) sheaf. A close version of this deformation complex appears in [S1] where a preliminary study of the deformation theory of quadratic sheaves is presented, along with a beautiful study of framed symplectic sheaves and their moduli spaces (see also [S2]).

Let us briefly sketch the structure of our paper. After recalling the basic definitions of deformation and obstruction theory in §1, we review in Section 2.2 the classical case of coherent sheaves achieved by Grothendieck. We finish the preliminaries by presenting orthogonal and symplectic sheaves in Section 2.3. In Section 3 we introduce the deformation complex, providing a description of its zero, first and second hypercohomology spaces. We prove in Section 4 that the space of first order deformations of orthogonal (resp. symplectic) sheaves coincide with the first hypercohomology space and we construct, in Section 5, an obstruction theory for these objects with the second hypercohomology space as a target.

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2. Preliminaries

2.1. Deformation and obstruction theory. See [Ni] for an introduction to deformation theory. Let \( k \) be an algebraically closed field, \((\text{Art})\) the category of all finite Artin local \( k \)-algebras with residue field \( k \) and denote by \((\text{Sets})\) the category of all sets. Denote by \((\text{FinVect})\) the category of finite dimensional \( k \)-vector spaces. We construct the functor \( k(\bullet) : (\text{FinVect}) \to (\text{Art}) \) by setting \( k(V) = k \oplus V \) as \( k \)-vector spaces, and ring structure given by \( (k,v) \cdot (k',v') = (kk', k'v + kv') \). Note that \( k(V) \) is the Artin local algebra whose maximal ideal is the vector space \( m = V \), satisfying \( m^2 = 0 \), and its residue field is \( k \). Note that one naturally has that \( k(k) \cong k[e]/(e^2) \). We say that \( 0 \to H \to B \xrightarrow{\tau} A \to 0 \) is a small extension in \((\text{Art})\) if \( m_B H = 0 \).

Given a deformation functor \( F : (\text{Art}) \to (\text{Sets}) \), we list below the so-called Schlessinger conditions for \( F \).

**S1:** For any homomorphism \( C \to A \) and any small extension \( 0 \to H \to B \xrightarrow{\tau} A \to 0 \) in \((\text{Art})\), the induced morphism

\[
F(B \times_A C) \to F(B) \times_{F(A)} F(C)
\]

is surjective.

**S2:** For any \( B \in (\text{Art}) \) and any \( V \in (\text{FinVect}) \), the induced morphism

\[
F(B \times_k k(V)) \to F(B) \times F(k(V))
\]

is bijective.

**S3:** The space of first-order deformations \( F(k[e]/(e^2)) \) is a finite dimensional \( k \)-vector space.

**S4:** For any \( 0 \to H \to B \xrightarrow{\tau} A \to 0 \) small extension in \((\text{Art})\), the induced morphism

\[
F(B \times_A B) \to F(B) \times_{F(A)} F(B)
\]

is bijective.

A pro-family is a family \( r \) parametrized by a complete local \( k \)-algebra \( R \) with residue field \( k \). A pro-family is versal if any family parametrized by \( A \in (\text{Art}) \) is the pull-back of \( r \) by a morphism \( f : R \to A \). It is a miniversal pro-family if furthermore the induced map on first order infinitesimal deformations

\[
\text{Hom}_{k-\text{alg}}(R, k[e]/(e^2)) \to F(k[e]/(e^2))
\]

is an isomorphism. A functor is pro-representable if there is a universal pro-family (i.e, the morphism \( f : R \to A \) above is unique).

Schlessinger [Sc] proved that a deformation functor admits a miniversal pro-family if and only if it satisfies **S1**, **S2** and **S3**. Moreover, it is pro-representable if and only if it satisfies **S4** along with the previous conditions.

An obstruction theory for the deformation functor \( F \) consists on a \( k \)-vector space \( \text{Obs}(F) \) and, for any small extension \( 0 \to H \to B \xrightarrow{\tau} A \to 0 \) in \((\text{Art})\), a morphism

\[
\Omega_\tau : F(A) \to H \otimes_k \text{Obs}(F)
\]

satisfying the conditions listed below:

**O1:** The sequence of sets

\[
F(B) \xrightarrow{F(\tau)} F(A) \xrightarrow{\Omega_\tau} H \otimes_k \text{Obs}(F)
\]

is exact in the middle.
O2: For any morphism of small extensions, that is, a commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & H \\
\downarrow{h} & & \downarrow{\beta} \\
0 & \rightarrow & B \\
\downarrow{\alpha} & & \downarrow{\tau} \\
0 & \rightarrow & A \\
& & \downarrow{\tau'} \\
& & 0,
\end{array}
\]

the induced diagram,

\[
\begin{array}{c}
\mathbf{F}(A) \xrightarrow{\Omega} H \otimes_k \mathbb{O}(\mathbf{F}) \\
\mathbf{F}(\alpha) \downarrow \quad \quad \quad \quad \downarrow 1_{\mathbb{O}(\mathbf{F})} \\
\mathbf{F}(A') \xrightarrow{\Omega'} H' \otimes_k \mathbb{O}(\mathbf{F}),
\end{array}
\]

commutes.

2.2. **Coherent sheaves.** Let \( X \) be a projective scheme over \( k \). Denote by \( \text{Coh}(X) \) the category of coherent sheaves on \( X \), and write \( \mathcal{D}^b(X) \) for the bounded derived category of quasi-coherent sheaves with coherent cohomology. Given \( E \in \text{Coh}(E) \), by abuse of notation, we denote by \( E \in \mathcal{D}^b(X) \) the complex supported on 0-degree given by \( E \).

For any coherent sheaf \( E \) one defines its **dual sheaf** by setting \( E^\vee := \text{Hom}_{\text{Coh}(X)}(E, \mathcal{O}_X) \) and, for any complex \( F^\bullet \in \mathcal{D}^b(X) \), its associated **dual complex** is \( D F^\bullet := \text{Hom}_{\mathcal{D}^b(X)}(F^\bullet, \mathcal{O}_X) \). Every torsion free coherent sheaf injects naturally into its double dual, \( E \hookrightarrow E^{\vee\vee} \), but unless \( E \) is reflexive, \( E^{\vee\vee} \) is not isomorphic to \( E \). On the other hand \( D \circ D = 1 \), so \( D \) is an autoequivalence of \( \mathcal{D}^b(X) \). If \( E \) is locally free, \( D E \cong E^\vee \), but this does not hold for a general coherent sheaf.

For any coherent sheaf \( E \) over a projective scheme \( X \), we define its deformation functor

\[
\text{Def}_E : (\text{Art}) \rightarrow (\text{Sets})
\]

by associating to any \( A \in (\text{Art}) \) the set of isomorphism classes of pairs \((\mathcal{E}, \gamma)\), where \( \mathcal{E} \) is a coherent sheaf on \( X_A := X \times \text{Spec}(A) \), flat over \( \text{Spec}(A) \), and \( \gamma : \mathcal{E}|_X \rightarrow E \) is an isomorphism. We say that two pairs \((\mathcal{E}, \gamma)\) and \((\mathcal{E}', \gamma')\) are isomorphic if there exists an isomorphism \( f : \mathcal{E} \rightarrow \mathcal{E}' \) such that \( \gamma' \circ (f|_X) = \gamma \). Every morphism of Artin algebras \( a : A \rightarrow A' \), induces naturally a morphism \( p_a : X_A \rightarrow X_{A'} \). Functoriality of \( \text{Def}_E \) follows from applying pull-backs under \( p_a \).

Grothendieck showed that the cohomology of the complex \( DE \otimes L E \) rules the deformation and obstruction theory of \( E \). In particular \( \text{Def}_E \) admits a universal pro-family and its space of first-order deformations is

\[
\text{Def}_E(k[\epsilon]/(\epsilon^2)) \cong \text{Ext}^1_X(E, E) \cong H^1(DE \otimes L E).
\]

If further, \( E \) is simple, then \( \text{Def}_E \) is pro-representable. Also, \( \text{Def}_E \) admits a deformation theory with vector space

\[
\text{Obs}(\text{Def}_E) \cong \text{Ext}^2_X(E, E) \cong H^2(DE \otimes L E).
\]

In particular, when \( \text{Ext}^2_X(E, E) = 0 \), the deformation functor \( \text{Def}_E \) is formally smooth.

2.3. **Orthogonal and symplectic sheaves.** An **orthogonal sheaf** (resp. a symplectic sheaf) on the projective scheme \( X \) is a pair \((\mathcal{E}, \phi)\), where \( E \) is a torsion-free coherent sheaf on \( X \) and \( \phi : E \otimes E \rightarrow \mathcal{O}_X \) is a homomorphism which is symmetric, \( \phi \circ \theta_E = \phi \) (resp. anti-symmetric, \( \phi \circ \theta_E = \phi \)), under the permutation \( \theta_E : E \otimes E \rightarrow E \otimes E \), and such that its restriction \( \phi|_{U_E} \) to the open subset \( U_E \) where \( E \) is locally free is non-degenerate. Recalling that the centre of \( \text{O}(n, k) \) (resp. \( \text{Sp}(2m, k) \)) is \( \{1, -1\} \), we say that an orthogonal (resp. symplectic) sheaf \((\mathcal{E}, \phi)\) is **simple** if its automorphism group is \( \text{Aut}(E, \phi) = \{1_E, -1_E\} \).

A **family of orthogonal** (resp. symplectic) sheaves parametrized by \( S \) is a pair \((\mathcal{E}, \Phi)\) such that \( \mathcal{E} \rightarrow X \times S \) is a torsion-free coherent sheaf, flat over \( S \) and such that \( \mathcal{E}_s := \mathcal{E}|_{X \times \{s\}} \) is torsion-free for each closed point \( s \in S \), and \( \Phi : \mathcal{E} \otimes \mathcal{E} \rightarrow \mathcal{O}_{X \times S} \) is a symmetric (resp. anti-symmetric) homomorphism whose restriction \( \Phi|_{U_E} \) to the open set where \( \mathcal{E} \) is locally free is non-degenerate. Two families \((\mathcal{E}, \Phi)\) and \((\mathcal{E}', \Phi')\) of orthogonal (resp. symplectic) sheaves are **isomorphic** if there exists an isomorphism \( f : \mathcal{E} \rightarrow \mathcal{E}' \) such that \( \Phi|_{U_E} = \Phi'|_{U_E} \circ (f|_{U_E} \otimes f|_{U_E}) \).
For any coherent sheaf \( E \), adjunction gives the identification
\[
\hom_{\coh(X)}(E \otimes E, \mathcal{O}_X) \cong \hom_{\coh(X)}(E, E^\vee).
\]
(2.2)

Permutation composed with the adjunction is the same as restricting to \( E \to E^\vee \)
the dual to the adjoint morphism,
\[
(\phi \circ \theta_E)_{\text{ad}} = \phi_{\text{ad}}|E : E \to E^\vee \to \phi_{\text{ad}}^{|E}.
\]
(2.3)

Remark 2.1. It follows from (2.2) and (2.3) that there exists a 1:1 correspondence between orthogonal (resp. symplectic) sheaves and pairs \((E, \phi_{\text{ad}})\), where \( E \) is a torsion-free sheaf, and \( \phi_{\text{ad}} : E \to E^\vee \) such that \( \phi_{\text{ad}} = \phi_{\text{ad}}^{|E} \) (resp. \( \phi_{\text{ad}} = -\phi_{\text{ad}}^{|E} \)), and \( \phi_{\text{ad}}^{|U_E} \) is an isomorphism.

As in the case of coherent sheaves, the deformation theory of an orthogonal (resp. symplectic) sheaf is related to a complex in the derived category naturally built out of it. In the remaining of the section we will see how orthogonal and symplectic sheaves provide a well defined geometrical object living in the derived category.

Given any complex \( F^* \in D^b(X) \), denote by \( \theta_{F^*} \) the derived permutation of \( F^* \otimes^L F^* \).

Adjoining gives
\[
\hom_{D^b(X)}(F^* \otimes F^*, \mathcal{O}_X) \cong \hom_{D^b(X)}(F^*, DF^*),
\]
(2.4)

As in the case of sheaves, permutation composed with adjunction dualizes morphisms of the form \( \psi^* : F^* \to DF^* \),
\[
(\psi^* \circ \theta_{F^*})_{\text{ad}} = D\psi^*_{\text{ad}}.
\]
(2.5)

One can give a description of orthogonal and symplectic sheaves in terms of the derived category.

**Proposition 2.2.** There exists a 1:1 correspondence between orthogonal (resp. symplectic) sheaves and pairs \((E, \phi_{\text{ad}})\), where \( E \) is a complex supported on 0 determined by a torsion-free sheaf, and \( \phi_{\text{ad}} \in \hom_{D^b(X)}(E, \mathcal{D} E) \) such that \( \phi_{\text{ad}} = \mathcal{D} \phi_{\text{ad}} \) (resp. \( \phi_{\text{ad}} = -\mathcal{D} \phi_{\text{ad}} \)) and \( \mathcal{H}^0(\phi_{\text{ad}})|_{U_E} : E|_{U_E} \to E^\vee|_{U_E} \) is an isomorphism.

**Proof.** This follows immediately after (2.2), (2.4), (2.5) and the identification
\[
\hom_{D^b(X)}(E, \mathcal{D} E) \cong \hom_{\coh(X)}(E, E^\vee),
\]
which can be proved by applying the standard truncations on \( \mathcal{D} E \).

\[\square\]

3. The deformation complex

Consider a coherent sheaf \( E \) over \( X \) projective and a morphism \( \phi : E \otimes E \to \mathcal{O}_X \) such that \( \phi \circ \theta_E = \phi \).

Inspired by J. Scalise [S1], we define
\[
\Delta^{+}_{(E, \phi)} := (1_{DE} \otimes^L \mathcal{D} E + \theta_{DE}) \circ (1_{DE} \otimes \phi_{\text{ad}}) : DE \otimes^L E \to (DE \otimes^L DE)^+.
\]

Analogously, when \( \phi \circ \theta_E = -\phi \), set
\[
\Delta^{--}_{(E, \phi)} := (1_{DE} \otimes^L \mathcal{D} E - \theta_{DE}) \circ (1_{DE} \otimes \phi_{\text{ad}}) : DE \otimes^L E \to (DE \otimes^L DE)^-.
\]

Let us also consider their associated mapping cone shifted by \(-1\),
\[
\Delta^{+\bullet}_{(E, \phi)} := C^\bullet(\Delta^{+}_{(E, \phi)})[-1],
\]
and
\[
\Delta^{--\bullet}_{(E, \phi)} := C^\bullet(\Delta^{--}_{(E, \phi)})[-1].
\]

These complexes fit in the distinguished triangle
\[
\Delta^{+\bullet}_{(E, \phi)} \to DE \otimes^L E \xrightarrow{\Delta^{+}_{(E, \phi)}} (DE \otimes^L DE)^+ \to \Delta^{--\bullet}_{(E, \phi)}[1],
\]
giving a long-exact sequence in hypercohomology.
Proposition 3.1. Let $E$ be a coherent sheaf over $X$ projective and consider $\phi : E \otimes E \to O_X$ such that $\phi \circ \theta_E = \pm \Phi$. Consider a finite dimensional $k$-vector space $V$. Then,

$$V \otimes_k \mathbb{H}^0 \left( \Delta^\pm_{(E,\phi)} \right) = \{ \lambda \in \text{Hom}_{\text{Coh}}(E, E) \text{ such that } \phi \circ (1_E \otimes \lambda) + \phi \circ (\lambda \otimes 1_E) = 0 \},$$

Proof. By definition $\mathbb{H}^{i+1} \left( \Delta^\pm_{(E,\phi)} \right) = \mathbb{H}^i \left( C^* \left( \Delta^\pm_{(E,\phi)} \right) \right)$, so we focus on the description of the complex $C^* \left( \Delta^\pm_{(E,\phi)} \right)$. Picking the locally free resolution $W^\bullet \to E \to 0$, one can see that this complex is 0 for $H < -1$, and in degrees $-1$ and 0 amounts to

$$0 \to \text{Hom}_X(W_0, E) \xrightarrow{\delta_{-1}} \frac{\text{Hom}_X(W_{-1}, E)}{\text{Hom}_X(W_0 \otimes W_0, O_X)^\pm},$$

with

$$\delta_{-1} = \left( \phi \circ (\pi \otimes (\bullet)) \circ (1_{W_0} \otimes \theta_W) \right),$$

where $\pi$ denotes the projection $W_0 \to E$ and its dual, $\pi^\vee$, the inclusion $0 \to E^\vee \to W_0^\vee$. The first statement follows from the fact that $\mathbb{H}^{-1} \left( C^* \left( \Delta^\pm_{(E,\phi)} \right) \right) = \ker(\delta_{-1})$ and

$$\phi \circ (\pi \otimes \lambda) \pm \phi \circ (\pi \otimes \lambda) \circ \theta_W = \phi \circ (\pi \otimes \lambda) \pm \phi \circ \theta_E \circ (\lambda \otimes \pi)$$

$$= \phi \circ (\pi \otimes \lambda) \pm \phi \circ (\lambda \otimes \pi).$$

And the result follows. \qed

Proposition 3.2. Let $E$ be a coherent sheaf over $X$ projective and consider $\phi : E \otimes E \to O_X$ such that $\phi \circ \theta_E = \pm \Phi$. Consider a finite dimensional $k$-vector space $V$. Then, $V \otimes_k \mathbb{H}^1 \left( \Delta^\pm_{(E,\phi)} \right)$ is the finite dimensional vector space classifying isomorphism classes of $1$-extensions

$$0 \to V \otimes_k E \xrightarrow{i} F \xrightarrow{j} E \to 0,$$

equipped with

$$\Phi : F \otimes F / I \to V \otimes_k O_X,$$

where $I \subset F \otimes F$ is the subsheaf generated by $(f_1, (i \circ j)(f_2)) - ((i \circ j)(f_1), f_2)$ for all $f \in F$, and $\Phi$ is such that

$$\Phi \circ \theta_F = \pm \Phi$$

and

$$\Phi \circ (1_F \otimes i) = (1_V \otimes_k \phi) \circ (\pi \otimes 1_{(V \otimes_k E)}).$$

Proof. We have to describe $V \otimes_k \mathbb{H}^0 \left( C^* \left( \Delta^\pm_{(E,\phi)} \right) \right)$, which is finite dimensional since $V$ is and so are all the cohomology spaces of $DE \otimes^L E$ and $DE \otimes^L DE$. Taking the locally free resolution $W^\bullet \to E \to 0$, the complex $C^* \left( \Delta^\pm_{(E,\phi)} \right)$ can be described in degrees 0 and 1 as

$$\left( \frac{\text{Hom}_X(W_{-1}, E)}{\text{Hom}_X(W_0 \otimes W_0, O_X)^\pm} \right) \xrightarrow{\delta_0} \left( \frac{\text{Hom}_X(W_{-2}, E)}{\text{Hom}_X(W_0 \otimes W_{-1}, O)} \right),$$

with

$$\delta_0 = \left( \begin{array}{c} -(\bullet) \circ \partial_{-2} \\ \phi \circ (\pi \otimes (\bullet)) \circ (1_{W_0} \otimes \partial_{-1}) \end{array} \right).$$

We have

$$\ker(\delta_0) = \left\{ \begin{array}{c} (\eta, \Psi) \in \text{Hom}_X(W_{-1}, E) \oplus \text{Hom}_X(W_0 \otimes W_0, O_X)^\pm \\ \eta \circ \partial_{-2} = 0 \\ \phi \circ (\pi \otimes \eta) + \Psi \circ (1_{W_0} \otimes \partial_{-1}) = 0 \end{array} \right\}$$

and $\mathbb{H}^0 \left( C^* \left( \Delta^\pm_{(E,\phi)} \right) \right) = H^0(\ker(\delta_0))$. For any pair $(\eta, \Psi) \in V \otimes_k H^0(\ker(\delta_0))$, with $\eta \in \text{Hom}_X(W_{-1}, V \otimes_k E) \otimes_k \text{Hom}_X(W_0 \otimes W_0, V \otimes_k O_X)^\pm$, one can naturally construct an extension [3.1] setting

$$F := (V \otimes_k E) \oplus W_0/(\eta \oplus \partial_{-1})(W_{-1}).$$
with the injection
\[
i : \quad E \longrightarrow F = (V \otimes_k E) \oplus W_0/([\pi \oplus \partial_{-1}](W_{-1})
\]
and the projection
\[
j : \quad F = (V \otimes_k E) \oplus W_0/([\pi \oplus \partial_{-1}](W_{-1}) \longrightarrow \quad W_0/\partial_{-1}(W_{-1}) \cong E
\]

Let $\phi$ denote $1_V \otimes_k \phi$. We have that $\phi \circ (1_V \otimes_k \theta_E) = \pm \phi$ by hypothesis on $\phi$. Observe also that we pick $\Psi$ in $\text{Hom}_X(W_0 \otimes W_0, V \otimes \mathcal{O}_X)^\pm$, hence $\Psi \circ \theta_{W_0} = \pm \Psi$. Therefore,
\[
\phi + \Psi : (V \otimes E \otimes E) \oplus (W_0 \otimes W_0) \otimes (E \oplus W_0) \rightarrow V \otimes_k \mathcal{O}_X
\]
satisfies
\[
\phi + \Psi = \pm \Psi.(3.4)
\]
Recalling that $(\eta, \Psi) \in V \otimes_k \ker(\delta_0)$, we have $\phi \circ (\pi \otimes \eta) + \Psi \circ (1_{W_0} \otimes \partial_{-1}) = 0$, we observe that
\[
\phi + \Psi \mid_{(E \otimes W_0) \otimes ([\pi \oplus \partial_{-1}](W_{-1}))} = 0.
\]
As a direct consequence of (3.4), one has
\[
\phi + \Psi \mid_{([\pi \oplus \partial_{-1}](W_{-1}) \otimes (E \oplus W_0))} = 0.
\]
Then, $\phi + \Psi$ defines $\Phi : F \otimes F \rightarrow \mathcal{O}_X$. Since $\phi + \Psi$ satisfies (3.4), it follows that $\Phi$ satisfies (3.2). Obviously, $\phi + \Psi$ restricted to $E \oplus 0$ coincides with $\phi$, hence $\Phi$ satisfies (3.3) as well.

We now study the action of $\text{Im}(\delta_{-1})$ on $\ker(\delta_0)$. For any $(\eta, \Psi) \in \text{Hom}_X(W_{-1}, E) \oplus \text{Hom}_X(W_0 \otimes W_0, \mathcal{O}_X)$ and any $\lambda \in \text{Hom}_X(W_0, E)$, set
\[
(3.7) \quad (\eta, \Psi) = (\eta, \Psi) = (\eta - \lambda \circ \partial_{-1}, \Psi \circ (\pi \otimes \lambda) + \phi \circ (\lambda \otimes \pi)),
\]
Let $F'$ be $E \oplus W_0/((\pi - \partial_{-1})W_{-1})$. Consider the isomorphism
\[
(3.8) \quad \begin{pmatrix} 1_E & \lambda \\ 0 & 1_{W_0} \end{pmatrix} : E \oplus W_0 \cong E \oplus W_0.
\]
Since the image under (3.8) of $(\eta + \partial_{-1})(W_{-1})$ is precisely $(\eta + \partial_{-1})(W_{-1})$, this descends to an isomorphism
\[
\lambda_1 : F \cong F'.
\]
Let $\Phi' \in \text{Hom}_X(F' \otimes F', \mathcal{O}_X)^\pm$ defined by $\phi + \Psi$. We can check that
\[
\Phi = \Phi' \circ (\lambda_1 \otimes \lambda_1), \quad (3.9)
\]
so $(\eta, \Psi)$ and $(\eta', \Psi')$ define isomorphic extensions, with this isomorphism relating the corresponding quadratic form.

Conversely, suppose we are given an extension of the form (3.1) and $\Phi : F \otimes F \rightarrow \mathcal{O}_X$ satisfying (3.2). Picking a locally free resolution $W^* \longrightarrow E \rightarrow 0$, the extension (3.1) determines $\pi \in \text{Hom}_X(W_{-1}, E)$ such that $\pi(\partial_{-2}(W_{-2})) = 0$. Taking the pull-back of $\Phi$ under $E \oplus W_0 \rightarrow F$ and restricting to $W_0$, we obtain $\Psi \in \text{Hom}_X(W_0 \otimes W_0, \mathcal{O}_X)^\pm$. Since it comes from $\Phi$ defined over $F$, it follows that $\Psi$ satisfies (3.5) and (3.6). Therefore $(\eta, \Psi)$ lies in $\ker(\delta_0)$. Suppose further that we are given two isomorphic extensions
\[
0 \longrightarrow E \longrightarrow F' \longrightarrow E \longrightarrow 0
\]
and $\Phi \in \text{Hom}_X(F \otimes F, \mathcal{O}_X)^\pm$ and $\Phi' \in \text{Hom}_X(F' \otimes F', \mathcal{O}_X)^\pm$ satisfying (3.9). Let $(\eta, \Psi)$ be the element of $\ker(\delta_0)$ associated to the first extension equipped with $\Phi$, and let $(\eta', \Psi') \in \ker(\delta_0)$ be the pair associated to the second extension and $\Phi'$. Since $\lambda_1$ defines an isomorphism of extensions, it then comes from some isomorphism $E \oplus W_0 \rightarrow E \oplus W_0$ of the form (3.8) for some
Proposition 3.3. Let $E$ be a coherent sheaf over $X$ smooth and projective and consider $\phi : E \otimes E \to \mathcal{O}_X$ such that $\phi \circ \theta_E = \pm \phi$. Consider a finite dimensional $k$-vector space $V$. Then, $V \otimes_k \mathbb{H}^2(\Delta_{(E,\phi)}^\pm)$ is the space classifying equivalence classes of 2-extensions

\[(3.10) \quad 0 \to V \otimes_k E \xrightarrow{i} F \xrightarrow{f} G \xrightarrow{j} E \to 0,\]

together with a class

\[\mu \in \text{Hom}_X(G \otimes F, V \otimes_k \mathcal{O}_X) / (\text{Hom}_X(G \otimes G, V \otimes_k \mathcal{O}_X) \circ (1_G \otimes f))\]

whose elements satisfy

\[\mu \circ (1_G \otimes i) = 1_V \otimes_k \phi(j \otimes 1_E),\]

and

\[\mu \circ (f \otimes 1_F) = \pm \mu \circ (f \otimes 1_F) \circ \theta_F.\]

The zero element in $V \otimes_k \mathbb{H}^2(\Delta_{(E,\phi)}^\pm)$ corresponds to a 2-extension that splits

\[(3.13) \quad 0 \to V \otimes_k E \xrightarrow{i} F = (V \otimes_k E) \oplus \ker j \xrightarrow{f} G \xrightarrow{j} E \to 0,\]

and $[\mu]$ such that

\[(3.14) \quad [\mu] \circ (1_G \otimes q) = 0\]

in $\text{Ext}^1(E \otimes E, V \otimes_k \mathcal{O}_X)^\pm$, where $q$ denotes the projection $F \to \ker j$.

Proof. We study $V \otimes_k \mathbb{H}^1\left(C^\bullet\left(\Delta_{E,\phi}^\pm\right)\right)$. Using the locally free resolution $W^\bullet \to E \to 0$, the mapping cone of $\Delta_{(E,\phi)}^\pm$ in degrees 1 and 2 is given by

\[
\begin{pmatrix}
\text{Hom}_X(W_{-2}, E) \\ \text{Hom}_X(W_0 \otimes W_{-1}, \mathcal{O}_X)
\end{pmatrix}
\xrightarrow{\delta_1}
\begin{pmatrix}
\text{Hom}_X(W_{-3}, E) \\ \text{Hom}_X(W_{-1} \otimes W_{-1}, \mathcal{O}_X)^+ \\ \text{Hom}_X(W_0 \otimes W_{-2}, \mathcal{O}_X)
\end{pmatrix},
\]

where

\[\delta_1 = \begin{pmatrix}
-(\bullet) \circ \partial_{-3} & 0 \\
0 & (\bullet) \circ (\partial_{-1} \otimes 1_{W_{-1}}) \circ (1_{W_{-1}} \oplus \theta_{W_{-1}}) \\
\phi \circ (\pi \otimes (\bullet)) & -\phi \circ (1_{W_0} \otimes \partial_{-2})
\end{pmatrix}.
\]

Then,

\[\ker(\delta_1) =\]

\[
\begin{cases}
(\chi, \Xi) \in \text{Hom}_X(W_{-2}, E) \oplus \text{Hom}_X(W_0 \otimes W_{-1}, \mathcal{O}_X) & \\
\chi(\partial_{-3}(W_{-3})) = 0 \\
\Xi \circ (\partial_{-1} \otimes 1_{W_{-1}}) = \pm \Xi \circ (\partial_{-1} \otimes 1_{W_{-1}}) \circ \theta_{W_{-1}} \\
\phi \circ (\pi \otimes \chi) - \Xi \circ (1_{W_0} \otimes \partial_{-2}) = 0
\end{cases}.
\]

Using $(\chi, \Xi) \in V \otimes_k \ker(\delta_1)$, where $\chi \in \text{Hom}_X(W_{-1}, V \otimes_k E)$ and $\Xi \in \text{Hom}_X(W_0 \otimes W_{-1}, V \otimes_k \mathcal{O}_X)$ consider the projection $j : G \to E$ to be $\pi : W_0 \to E$ and let us construct

\[(3.15) \quad F := (V \otimes_k E) \oplus W_{-1} / (\chi \oplus \partial_{-2}) (W_{-2})\]

and consider the injection

\[i : E \to F = (V \otimes_k E) \oplus W_{-1} / (\chi \oplus \partial_{-2}) (W_{-2}) \]

and the morphism

\[f : F = (V \otimes_k E) \oplus W_{-1} / (\chi \oplus \partial_{-2}) (W_{-2}) \to G = W_0 \]

which is well defined since $\partial_{-1} \circ \partial_{-2} = 0$. Note that we have obtained a 2-extension of the form of $[\delta_1]$. 

\[\square\]
Consider now
\[ 1_V \otimes_k \phi \circ (\pi \otimes 1_E) - \Xi : W_0 \otimes ((V \otimes_k E) \oplus W_{-1}) \rightarrow V \otimes_k \mathcal{O}_X. \]
Since \((\chi, \Xi) \in V \otimes_k \ker(\delta_1)\) satisfy \(1_V \otimes_k \phi \circ (\pi \otimes \chi) - \Xi \circ (1_{W_0} \otimes \partial_{-2}) = 0\), it follows that \(1_V \otimes_k \phi \circ (\pi \otimes 1_E) - \Xi\) vanishes at \(W_0 \otimes ((\chi \oplus \partial_{-2})(W_{-2}))\), hence it descends to \(\mu : G \otimes F \rightarrow V \otimes_k \mathcal{O}_X\),
where we recall \((3.15)\) and that \(G = W_0\). Since \(1_V \otimes_k \phi \circ (\pi \otimes 1_E) - \Xi\) restricted to \(G \otimes (V \otimes_k E \oplus 0)\) amounts to \(1_V \otimes_k \phi \circ (\pi \otimes 1_E)\), \((3.11)\) follows naturally. Note also that
\[ (1_V \otimes_k \phi \circ (\pi \otimes 1_E) - \Xi) \circ (\partial_{-1} \otimes 1_{(V \otimes_k E) \oplus W_{-1}}) = \Xi \circ (\partial_{-1} \otimes 1_{W_{-1}}). \]
Then \((3.12)\) follows from the identity \(\Xi \circ (\partial_{-1} \otimes 1_{W_{-1}}) = \Xi \circ (\partial_{-1} \otimes 1_{W_{-1}}) \circ \theta_{W_{-1}}\) that any \((\chi, \Xi) \in V \otimes_k \ker(\delta_1)\) satisfies.

For any \(\eta \in V \otimes_k \Hom_X(W_{-2}, E)\), we set \((\chi', \Xi') = (\chi, \Xi) + (1_V \otimes_k \delta_0) \cdot (\eta, 0) = (\chi - \eta \circ \partial_{-2}, \Xi + 1_V \otimes_k \phi \circ (\pi \otimes \eta))\), and define
\[ F' = (V \otimes_k E) \oplus W_{-1}/(\chi' \oplus \partial_{-2})(W_{-2}). \]
Observe that the isomorphism
\[ (3.16) \quad \left( \begin{array}{cc} 1_{V \otimes_k E} & -\Xi' \\ 0 & 1_{W_0} \end{array} \right) : (V \otimes_k E) \oplus W_{-1} \xrightarrow{\cong} (V \otimes_k E) \oplus W_{-1}. \]

sends \((\chi \oplus \partial_{-2})(W_{-2})\) to \((\chi' \oplus \partial_{-2})(W_{-2})\), hence \((3.10)\) provides an isomorphism \(\eta_F : F \xrightarrow{\cong} F'\).

One obtains a commutative diagram
\[
\begin{array}{cccccc}
0 & \rightarrow & V \otimes_k E & \rightarrow & F & \rightarrow & G = W_0 & \rightarrow & E & \rightarrow & 0 \\
\downarrow & & \downarrow \cong & & \downarrow \eta_F & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & V \otimes_k E & \rightarrow & F' & \rightarrow & G' = W_0 & \rightarrow & E & \rightarrow & 0
\end{array}
\]
so both \(\chi\) and \(\chi'\) define the same class of extensions. Note also that \(1_V \otimes_k \phi \circ (\pi \otimes 1_E) - \Xi'\) vanishes in \((\chi' \oplus \partial_{-2})(W_{-2})\), defining \(\mu' : G' \otimes F' \rightarrow V \otimes_k \mathcal{O}_X\) satisfying \((3.11)\) and \((3.12)\). Note also that
\[ 1_V \otimes_k \phi \circ (\pi \otimes 1_E) - \Xi' = 1_V \otimes_k \phi \circ (\pi \otimes 1_E) \circ \left( \begin{array}{cc} 1_{V \otimes_k E} & -\Xi' \\ 0 & 1_{W_0} \end{array} \right), \]
so one gets
\[ \mu' \circ (1_G \otimes \eta_F) = \mu. \]

Therefore, the 2-extension and the morphism that we obtain from \((\chi, \Xi)\) are equivalent to the 2-extension and the morphism that we obtain from \((\chi', \Xi')\).

Take now \(\Xi' \in V \otimes_k \Hom_X(W_0 \otimes W_0, O_X)\) and define
\[ (\chi, \Xi') = (\chi, \Xi) + (1_V \otimes_k \delta_0) \cdot (0, \Xi') = (\chi, \Xi + \Xi' \circ (1_{W_0} \otimes \partial_{-1})). \]

Since \(\chi\) does not change, we get \(F\) and \(G\) as before. We observe that \(1_V \otimes_k \phi \circ (\pi \otimes 1_E) - \Xi'\) descends to \(\mu'' = \mu + \Xi' \circ (1_G \otimes f), \) so \([\mu''] = [\mu]\). Observe that \(\mu''\) also satisfies \((3.11)\) and \((3.12)\).

Conversely, choose representants \((3.10)\) and \(\mu : G \otimes F \rightarrow V \otimes_k \mathcal{O}_X\) of a given 2-extension class of \(E\) by \(E\), and of a certain class in \(\Hom_X(G \otimes F, V \otimes_k \mathcal{O}_X)/\Hom_X(G \otimes G, V \otimes_k \mathcal{O}_X)\) of \((1_{W_0} \otimes f)\).
satisfying $\Xi$ and $\Theta$. Using the universal property of projective modules (recall that locally free sheaves are projective) one can always complete to a commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & W_{-2}/\partial_{-3}(W_{-3}) & \rightarrow & W_{-1} & \rightarrow & W_{0} & \rightarrow & E & \rightarrow & 0 \\
& & \downarrow{\tau} & & \downarrow{\chi_{F}} & & \downarrow{\chi_{G}} & & \downarrow{1} & & 0. \\
0 & \rightarrow & V \otimes_{k} E & \rightarrow & F & \rightarrow & G & \rightarrow & E & \rightarrow & 0.
\end{array}
\]

This defines a morphism $\chi : W_{-2} \rightarrow V \otimes_{k} E$ with $\chi(\partial_{-3}(W_{-3})) = 0$. Let us denote the composition of $\Xi := \mu \circ (\chi_{G}, \chi_{F})$. Note that the commutativity of (3.17) together with (3.11) and (3.12) imply, respectively, that

\[
1_{V} \otimes_{k} \phi \circ (\pi \otimes \chi) - \Xi \circ (1_{W_{0}} \otimes \partial_{-2}) = 0
\]

and

\[
\Xi \circ (\partial_{-1} \otimes 1_{W_{-1}}) = \pm \Xi \circ (\partial_{-1} \otimes 1_{W_{-1}}) \circ \theta_{W_{-1}}.
\]

This completes the proof of the first statement.

To describe the 0 element in $V \otimes_{k} \mathbb{H}^{1}(C^{\ast} \left(\Delta_{E,\phi}^{+}\right))$, we first note that any $(0, \Xi) \in V \otimes_{k} \ker(\delta_{1})$ gives

\[
F = (V \otimes_{k} E) \oplus W_{-1}/(0 \oplus \delta_{-2}(W_{-2}) = (V \otimes_{k} E) \oplus \ker(\pi),
\]

with $\Xi$ satisfying

\[
\Xi \circ (\partial_{-1} \otimes 1_{W_{-1}}) = \pm \Xi \circ (\partial_{-1} \otimes 1_{W_{-1}}) \circ \theta_{W_{-1}}
\]

and

\[
\Xi \circ (1_{W_{0}} \otimes \partial_{-2}) = 0.
\]

Therefore, $(0, \Xi) \in \ker(\delta_{1})$ determines a short exact sequence of the form $\Xi$ together with an element $[\Xi] \in \operatorname{Ext}^{1}_{X}(E \otimes V, V \otimes \mathcal{O}_{X})^{\pm}$. If $\mu$ is the descent of $1_{V} \otimes \phi \circ (\pi \otimes 1_{E}) - \Xi : W_{0} \otimes (V \otimes E \oplus W_{-1}) \rightarrow V \otimes \mathcal{O}_{X}$ to a morphism in $\operatorname{Hom}_{X}(W_{0} \otimes \ker(\pi), V \otimes \mathcal{O}_{X})^{\pm}$, observe that $\mu \circ (1_{G} \otimes q)$ coincides with $\Xi$. This concludes the proof. \(\square\)

4. DEFORMATION THEORY OF ORTHOGONAL AND SYMPLECTIC SHEAVES

Given an orthogonal sheaf $(E, \phi)$ over the projective scheme $X$, we define its deformation functor

\[
\operatorname{Def}_{(E,\phi)}^{+} : (\operatorname{Art}) \rightarrow (\operatorname{Sets})
\]

by associating to any $A \in (\operatorname{Art})$ the set of isomorphism classes of triples $(E, \Phi, \gamma)$, where $(E, \Phi)$ is a family of orthogonal sheaves on $X_{A}$, (so $E$ is a torsion-free coherent sheaf on $X_{A}$ flat over $\operatorname{Spec}(A)$), and $\gamma : (E, \Phi)|_{X} \rightarrow (E, \phi)$ is an isomorphism of orthogonal (resp. symplectic) sheaves. Two triples $(E, \Phi, \gamma)$ and $(E', \Phi', \gamma')$ are isomorphic if there exists an isomorphism $f : (E, \Phi) \rightarrow (E', \Phi')$ such that $\gamma' \circ (f|_{X}) = \gamma$. As in the case of the deformation functor of coherent sheaves, functoriality of $\operatorname{Def}_{(E,\phi)}^{+}$ follows from applying pull-backs under the morphisms $p_{a} : X_{A'} \rightarrow X_{A}$ for any $a : A \rightarrow A'$.

Analogously, associated to any symplectic sheaf $(E, \phi)$ over $X$, its deformation functor

\[
\operatorname{Def}_{(E,\phi)}^{-} : (\operatorname{Art}) \rightarrow (\operatorname{Sets})
\]

is constructed by associating to every $A \in (\operatorname{Art})$ the set of isomorphism classes of triples $(E, \Phi, \gamma)$, where $(E, \Phi)$ is a family of symplectic sheaves on $X_{A}$, and $\gamma : (E, \Phi)|_{X} \rightarrow (E, \phi)$ is an isomorphism of symplectic sheaves. The notion of isomorphism of triples is analogous to the case of orthogonal sheaves. As before, functoriality under pull-backs holds in this case as well.

We see that the complex $\Delta_{E,\phi}^{\ast, \bullet}$ governs the deformation theory of $\operatorname{Def}_{(E,\phi)}^{\pm}$.

**Theorem 4.1.** Let $(E, \phi)$ be an orthogonal (resp. symplectic) sheaf over the smooth projective scheme $X$. Then the deformation functor $\operatorname{Def}_{(E,\phi)}^{+}$ (resp. $\operatorname{Def}_{(E,\phi)}^{-}$) admits a universal pro-family and the associated space of first-order deformations is

\[
\operatorname{Def}_{(E,\phi)}^{\pm}(k[\epsilon]/(\epsilon^{2})) \cong \mathbb{H}^{1}(\Delta_{(E,\phi)}^{\pm, \bullet}).
\]
If, further, \((E, \phi)\) is simple, \(\text{Def}^\pm_{(E, \phi)}\) is pro-representable.

**Proof.** Given a torsion-free sheaf \(E\) with \(\phi : E \otimes E \to \mathcal{O}_X\) satisfying \(\phi = \pm \phi \circ \theta_E\), we have to check whether \(\text{Def}^\pm_{(E, \phi)}\) satisfies the Schlessinger conditions \(\text{S1, S2 and S3}\). Let us denote by \(b : B \times_A C \to B\) the projection to the first factor and by \(c : B \times_A C \to C\) the projection to the second. Consider the associated morphisms \(p_b : X_B \to X_{B \times_A C}\) and \(p_c : X_{B \times_A C} \to X_C\). Condition \(\text{S1}\) holds if for any homomorphism \(B \to A\) and any small extension \(0 \to H \to C \to A \to 0\) in \((\text{Art})\), the morphism induced by taking pull-backs under \(p_b\) and \(p_c\),

\[
\text{Def}^\pm_{(E, \phi)}(B \times_A C) \to \text{Def}^\pm_{(E, \phi)}(B) \times \text{Def}^\pm_{(E, \phi)}(A) \text{Def}^\pm_{(E, \phi)}(C),
\]

is surjective. Consider \((\mathcal{E}_B, \Phi_B, \gamma_B) \in \text{Def}^\pm_{(E, \phi)}(B)\) and \((\mathcal{E}_C, \Phi_C, \gamma_C) \in \text{Def}^\pm_{(E, \phi)}(C)\) for which there exists an isomorphism \(h : \mathcal{E}_B|_{X_A} \to \mathcal{E}_C|_{X_A}\) satisfying \(\gamma_B = \gamma_C \circ f|_X\) and \(\Phi_B|_{X_A} = \Phi_C|_{X_A} \circ (f \circ f)\). Thanks to deformation theory of sheaves we know that \(\text{Def}_E\) verifies \(\text{S1}\), so

\[
\text{Def}_E(B \times_A C) \to \text{Def}_E(B) \times \text{Def}_E(A) \text{Def}_E(C),
\]

is surjective and there exists \((\mathcal{E}', \gamma) \in \text{Def}_E(B \times_A C)\) such that \(g_B : (\mathcal{E}', \gamma)|_{X_B} \to (\mathcal{E}_B, \gamma_B)\) and \(g_C : (\mathcal{E}', \gamma)|_{X_C} \to (\mathcal{E}_C, \gamma_C)\) satisfy that \(g_C|_{X_A} = f \circ g_B|_{X_A}\). It remains to construct a quadratic form on \(\mathcal{E}'\) compatible with \(\Phi_B\) and \(\Phi_C\) under pull-backs.

Write \(\mathcal{E}'_B, \mathcal{E}'_C\) and \(\mathcal{E}'_A\) for \(\mathcal{E}'|_{X_B}, \mathcal{E}'|_{X_C}\) and \(\mathcal{E}'|_{X_A}\) respectively. One trivially has that

\[
(4.2) \quad \mathcal{E}' = \mathcal{E}'_B \times_{\mathcal{E}'_A} \mathcal{E}'_C,
\]

so

\[
\mathcal{E}' \otimes \mathcal{E}' = (\mathcal{E}'_B \otimes \mathcal{E}'_B) \times (\mathcal{E}'_C \otimes \mathcal{E}'_C).
\]

Since \(g_C|_{X_A} = f \circ g_B|_{X_A}\) and \(\Phi_B|_{X_A} = \Phi_C|_{X_A} \circ (f \circ f)\), it follows that

\[
(g_B \circ g_B)^* \Phi_B|_{X_A} = (g_C \circ g_C)^* \Phi_C|_{X_A}.
\]

Then, they define

\[
\Phi : \mathcal{E}' \otimes \mathcal{E}' \to \mathcal{O}_{X_B \times A C},
\]

which naturally satisfies \(p_b^* \Phi = \Phi_B\) and \(p_c^* \Phi = \Phi_C\). By hypothesis, one has that \(p_b^* \Phi_B = \pm p_b^* \Phi_B \circ \theta_{\mathcal{E}_B}\) and similarly for \(C\) and \(A\). It then follows that

\[
\Phi = \pm \Phi \circ \theta_{\mathcal{E}'_B}.
\]

Recall that \(U_{\mathcal{E}}\) the open subset of \(X_{(B \times A C)}\) where \(\mathcal{E}'\) is locally free. Thanks to \(\text{S2}\), one has that

\[
U_{\mathcal{E}'_B} = U_{\mathcal{E}'_B} \times_{U_{\mathcal{E}'_A}} U_{\mathcal{E}'_C}.
\]

By hypothesis, \((g_B \circ g_B)^* \Phi_B\) and \((g_C \circ g_C)^* \Phi_C\) are non-degenerate over \(U_{\mathcal{E}'_B} = U_{\mathcal{E}'_B}\) and \(U_{\mathcal{E}'_C} = U_{\mathcal{E}'_C}\). Since \(\Phi\) is constructed by gluing \((g_B \circ g_B)^* \Phi_B\) and \((g_C \circ g_C)^* \Phi_C\), it then follows that \(\Phi\) is non-degenerate over \(U_{\mathcal{E}'_B}\). This proves that \(\text{Def}^\pm_{(E, \phi)}\) satisfies \(\text{S1}\).

Since \(\text{Def}_E\) satisfies the condition \(\text{S2}\), to prove that \(\text{Def}^\pm_{(E, \phi)}\) also full-fills this condition it is enough to see that, when \(A = k\), given a quadratic form \(\Phi'\) on \(\mathcal{E}'\) compatible with \(\gamma\) and such that \(\Phi'|_{\mathcal{E}'_B \otimes \mathcal{E}'_B} = (h_B \otimes h_B)^* \Phi'|_{\mathcal{E}'_B \otimes \mathcal{E}'_B}\) for an automorphism \(h_B\) of \(\mathcal{E}'_B, \gamma|_{\mathcal{E}'_B}\) and \(\Phi'|_{\mathcal{E}'_C \otimes \mathcal{E}'_C} = (h_C \otimes h_C)^* \Phi'|_{\mathcal{E}'_C \otimes \mathcal{E}'_C}\) under an automorphism \(h_C\) of \(\mathcal{E}'_C, \gamma|_{\mathcal{E}'_C}\), one can construct an automorphism \(h\) of \((\mathcal{E}', \gamma)\) sending \(\Phi'\) to \(\Phi\). Since \(h_B|_X = \gamma|_X = h_C|_X\), it follows from \(\text{S2}\) that one can construct \(h : \mathcal{E}' \to \mathcal{E}'\) by gluing \(h_B\) and \(h_C\) along \(X\). It is straight-forward that \(h\) satisfies the required condition so \(\text{Def}^\pm_{(E, \phi)}\) satisfies \(\text{S2}\).

We address \(\text{S3}\) now. First, note that one can endow the space of first-order deformations \(\text{Def}^\pm_{(E, \phi)}(k\langle k \rangle)\) with a \(k\)-vector space structure. Using the inverse of the bijective map of \(\text{S2}\) when \(B = k\langle k \rangle\), and the morphism \(\langle + \rangle : k\langle k \otimes k \rangle \to k\langle k \rangle\) induced by the sum of the elements in the maximal ideal, one can define the sum within the space of first-order deformations,

\[
\text{Def}^\pm_{(E, \phi)}(k\langle k \rangle) \times \text{Def}^\pm_{(E, \phi)}(k\langle k \rangle) \xrightarrow{1:1} \text{Def}^\pm_{(E, \phi)}(k\langle k \otimes k \rangle) \xrightarrow{\text{Def}^\pm_{(E, \phi)}(\langle + \rangle)} \text{Def}^\pm_{(E, \phi)}(k\langle k \rangle).
\]
Lemma 4.2. For any □, Then consider the push-forward under the natural projection \( k \).

Proof. Otherwise, as \( b \) is naturally determined by \( \Phi \) by the usual \( \{ \text{class of morphisms } [\bullet] \} \), one can then consider the map

\[
(4.3) \quad \mathbb{H}^1(\Delta^{\pm}_{(E,\phi)}) \rightarrow \text{Def}^{\pm}_{(E,\phi)}(k[e]/(e^2))
\]

that sends the short exact sequence of \( \mathcal{O}_X \)-modules \( 0 \rightarrow E \overset{j}{\rightarrow} F \overset{j'}{\rightarrow} E \rightarrow 0 \) and \( \Phi_1 : F \otimes F \rightarrow \mathcal{O}_X \) to the triple \( (\mathcal{E}, \Phi, \gamma) \) where \( \mathcal{E} \) is \( F \) endowed with a \( \mathcal{O}_X \times_k k[e]/(e^2) \)-module structure determined by the usual \( \mathcal{O}_X \)-module structure on \( F \) and the action of \( e \) on \( \mathcal{E} \) defined by the composition \( j \circ i : E \rightarrow E \). The isomorphism \( \gamma : \mathcal{E}|_X \rightarrow E \) is determined by the projection \( j : F \rightarrow E \) and \( \Phi \) is naturally determined by \( \Phi_1 \).

Conversely, given the isomorphism class of \( (\mathcal{E}, \Phi, \gamma) \) in \( \text{Def}^{\pm}_{(E,\phi)}(k[e]/(e^2)) \), we construct an exact sequence \( 0 \rightarrow E \overset{j}{\rightarrow} F \overset{j'}{\rightarrow} E \rightarrow 0 \). Tensorize \( \mathcal{E} \) with \( 0 \rightarrow k \rightarrow k[e]/(e^2) \rightarrow k \rightarrow 0 \) to obtain

\[
0 \rightarrow k \otimes k[e]/(e^2) \mathcal{E} \cong E \overset{j}{\rightarrow} \mathcal{E} \overset{j'}{\rightarrow} k \otimes k[e]/(e^2) \mathcal{E} \cong E \rightarrow 0.
\]

Then consider the push-forward under the natural projection \( \pi : X \times_k \text{Spec}(k[e]/(e^2)) \rightarrow X \) given by \( F = \pi_* \mathcal{E} \), the projection \( j : F \rightarrow E \) is determined by the composition \( \gamma \circ \pi_* j' \) and \( \Phi_1 = (\pi \circ \pi)^* \Phi \). This provides an inverse for \( (4.3) \) so \( \text{S3} \) is satisfied and \( (4.1.1) \) holds.

Finally, we address \( \text{S4} \) for \((E, \phi)\) simple. Observe that it follows naturally from \( \text{S1} \) and Lemma 3.7.2. \( \square \)

The following result provides a characterization of simple orthogonal and symplectic sheaves on \( X_A \) in terms of their restriction to the the closed subset \( X \subset X_A \).

Lemma 4.2. For any \( A \in \text{(Art)} \) and any orthogonal (resp. symplectic) sheaf \((\mathcal{E}_A, \Phi_A)\) over \( X_A \), we have that \((\mathcal{E}_A, \Phi_A)\) is simple if and only if \((E, \phi) := (\mathcal{E}_A|_X, \Phi_A|_X)\) is simple.

Proof. The \( A \)-module \( H^0(X_A, \text{End}(\mathcal{E}_A)) \) is finitely generated. Note that \( H^0(X_A, \text{End}(\mathcal{E}_A)) \otimes_A k = H^0(X, \text{End}(E)) \) and \( 1_{\mathcal{E}_A} \otimes_A 1 = 1_E \).

By Nakayama’s lemma, if \( \{1_{\mathcal{E}_A}, b_1, \ldots, b_n\} \subset H^0(X_A, \text{End}(\mathcal{E}_A)) \) are such that \( \{1_E, (b_2 \otimes 1), \ldots, (b_n \otimes A_1)\} \) is a basis of \( H^0(X, \text{End}(E)) \), then \( \{1_{\mathcal{E}_A}, b_1, \ldots, b_n\} \) generate \( H^0(X_A, \text{End}(\mathcal{E}_A)) \). Then, there is no \( b' \in H^0(X_A, \text{End}(\mathcal{E}_A)) \) such that \( b' \neq 1_{\mathcal{E}_A} \) with restriction \( b' \otimes A_1 = 1_E \).

Otherwise, as \( b' = a_1 \otimes A_1 + a_2 \otimes A_2 b_2 + \cdots + a_n \otimes A_n b_n \), we would obtain a contradiction with the linear independence of \( \{1_E, (b_2 \otimes 1), \ldots, (b_n \otimes A_1)\} \).

Then, the only element in \( H^0(X_A, \text{End}(\mathcal{E}_A)) \) that restrict to \( 1_E \) is \( 1_{\mathcal{E}_A} \) itself. If \( \text{Aut}(E, \phi) = \{1_E, -1_E\} \), it then follows that \( \text{Aut}(\mathcal{E}_A, \Phi_A) = \{1_{\mathcal{E}_A}, -1_{\mathcal{E}_A}\} \). \( \square \)

5. Obstruction theory for orthogonal and symplectic sheaves

In this section we will see that the second cohomology space of the deformation complex defined in Section 3 provides an obstruction theory for the deformation functors of orthogonal and symplectic sheaves. We begin by the construction of the morphism \((2.1)\) in this case.

Let \((E, \phi)\) be an orthogonal (resp. symplectic) sheaf over \( X \) projective and let \( 0 \rightarrow H \rightarrow B \rightarrow A \rightarrow 0 \) be a small extension of Artin algebras with residue field \( k \). We want to construct a morphism from \( \text{Def}^{\pm}_{(E,\phi)}(A) \) to \( H \otimes_k \mathbb{H}^2(\Delta^{\pm}_{(E,\phi)}) \). Then, after Proposition 3.3, given \((\mathcal{E}_A, \Phi_A, \gamma_A) \in \text{Def}^{\pm}_{(E,\phi)}(A) \), we should construct a 2-extension of the form \((5.1)\) equipped with a class of morphisms \([\mu] \in \text{Hom}_X(G \otimes F, \mathcal{O}_X)/(\text{Hom}_X(G \otimes G, \mathcal{O}_X))^\pm \circ (1_G \otimes f) \) satisfying \((5.11)\) and \((5.12)\).

From the small extension \( 0 \rightarrow H \rightarrow B \rightarrow A \rightarrow 0 \), one naturally obtains the short exact sequence of \( \mathcal{O}_{X_B} \)-modules

\[
(5.1) \quad 0 \rightarrow H \otimes_k \mathcal{O}_X \overset{\alpha_1}{\rightarrow} \mathcal{O}_{X_B} \overset{\alpha_2}{\rightarrow} p_* \mathcal{O}_{X_A} \rightarrow 0,
\]
where we denote by $p : X_A \rightarrow X_B$ the morphism associated to $B \rightarrow A$.

Let us consider a locally free resolution $\mathcal{W}_A^* \xrightarrow{\pi_A} \mathcal{E}_A \rightarrow 0$ such that, for $i > 0$,

$$H^i(X_A, \mathcal{W}_{A,0}) = 0.$$ 

Along with the above locally free resolution, consider a locally free sheaf $\mathcal{W}_{B,0}$ satisfying $\mathcal{W}_{B,0}|_{X_A} \cong \mathcal{W}_{A,0}$ and

$$(5.2) \quad H^i(X_B, \mathcal{W}_{B,0}) = 0,$$

for $i > 0$. It can be easily verified that such a choice exists. Set $W_i := \mathcal{W}_{A,i}|_X$, with differentials $\partial_i := \partial_{A,i}|_X$ and $\pi := \gamma_A \circ \pi_{A,i} : W_0 \rightarrow E$. Note that $W^* \rightarrow E \rightarrow 0$ is a locally free resolution.

Inspired by the classical approach to obstruction theory of coherent sheaves, we consider the short exact sequences of $\mathcal{O}_{X_B}$-modules

$$0 \rightarrow H \otimes_k \partial_{-1}(W_{-1}) \rightarrow \mathcal{W}_{B,0} \rightarrow \pi_\ast \mathcal{W}_{A,0} \rightarrow 0,$$

where $\pi$ is induced by the restriction $X_B \rightarrow X_A$ and $\sigma$ is the inclusion of the kernel of $\rho$. Denote

$$(5.4) \quad \mathcal{F} := \rho^{-1}(\pi_\ast \partial_{-1}(\mathcal{W}_{A,-1})) \big/ ((H \otimes_k \partial_{-1}(W_{-1})).$$

We then see that, out of (5.3), one can naturally construct an extension

$$(5.5) \quad 0 \rightarrow H \otimes_k E \xrightarrow{\iota_0} \mathcal{F} \xrightarrow{f_0} \pi_\ast \mathcal{W}_{A,-1} \rightarrow 0,$$

which defines naturally $\zeta \in \text{Ext}^1_{X_A}(\pi_\ast \partial_{-1}(\mathcal{W}_{A,-1}), H \otimes_k E)$.

Having (5.5) in mind, observe that any $\mathcal{O}_{X_B}$-module $\mathcal{M}$ such that $(H \otimes_k \mathcal{O}_X) \otimes \mathcal{M} = 0$ is naturally equipped with the inherited $\mathcal{O}_{X_B}/(H \otimes_k \mathcal{O}_X) \cong p_*\mathcal{O}_{X_A}$-module structure. Note that there exists an equivalence of categories sending the $p_*\mathcal{O}_{X_A}$-module $\mathcal{M}$ to the $\mathcal{O}_{X_A}$-module $\mathcal{M}^A$. Under this equivalence, $p_*\partial_{-1}(\mathcal{W}_{A,-1})$ gives naturally the $\mathcal{O}_{X_A}$-module $\partial_{-1}(\mathcal{W}_{A,-1})$.

Since $H^2 = 0$, we see that $H \otimes_k E$ is annihilated by $H \otimes_k \mathcal{O}_X$ producing the $\mathcal{O}_{X_A}$-module $H \otimes_k p_*\mathcal{E}_A$, where $p_A : X \hookrightarrow X_A$ is the inclusion of the closed reduced subscheme associated to the structural projection to the residue field, $A \rightarrow k$. Hence, after (5.5) and right-exactness of tensor product, one has that $(H \otimes_k \mathcal{O}_X) \otimes \mathcal{F} = 0$ so $\mathcal{F}$ gives rise to the $\mathcal{O}_{X_A}$-module $\mathcal{F}^A$.

Therefore, from (5.5) we obtain the extension of $\mathcal{O}_{X_A}$-modules

$$(5.6) \quad 0 \rightarrow H \otimes_k p_*\mathcal{E}_A \xrightarrow{i_0^A} \mathcal{F}^A \xrightarrow{f_0^A} \partial_{-1}(\mathcal{W}_{A,-1}) \rightarrow 0,$$

associated to $\zeta^A \in \text{Ext}^1_{X_A}(\partial_{-1}(\mathcal{W}_{A,-1}), H \otimes_k p_*\mathcal{E}_A)$.

Composing (5.6) with the projection $\mathcal{W}_{A,0} \xrightarrow{\pi_A} \mathcal{E}_A$, one gets an element $\pi_A^\ast \zeta^A$ of $\text{Ext}^2_{X_A}(\mathcal{E}_A, H \otimes_k p_*\mathcal{E}_A)$ associated to the 2-extension

$$(5.7) \quad 0 \rightarrow H \otimes_k p_*\mathcal{E}_A \xrightarrow{i_A^A} \mathcal{F}^A \xrightarrow{f_A^A} \mathcal{W}_{A,0} \xrightarrow{\pi_A} \mathcal{E}_A \rightarrow 0.$$

One can prove that $\text{Ext}^2_{X_A}(\mathcal{E}_A, H \otimes_k p_*\mathcal{E}_A) \cong \text{Ext}^2_{X}(\mathcal{E}_A, H \otimes_k E)$, so (5.7) is completely determined by its restriction to $X$,

$$(5.8) \quad 0 \rightarrow H \otimes_k E \xrightarrow{i} F := \mathcal{F}^A|_X \xrightarrow{f} W_0 \xrightarrow{\pi} E \rightarrow 0.$$
Note that $i$ and $f$ are given respectively by $i^A|_X$, $f^A|_X$ and we recall that $\pi = \gamma_A \circ \pi_A|_X$. Setting $G := W_0$ and $j := \pi$, we see that (5.8) gives a 2-extension of the form (5.10).

Take now $\Phi_A : E_A \otimes E_A \to O_{X_A}$, and consider $(\pi_A \otimes \pi_A)^* \Phi_A \in \text{Hom}_{X_A}(W_{A,0} \otimes W_{A,0}, O_{X_A})^\pm$.

Recalling (5.2), one naturally has that $\text{Ext}_X^1(W_{B,0} \otimes W_{B,0}, O_{X_B}) = 0$, hence the functor $\text{Hom}_{X_B}(W_{B,0} \otimes W_{B,0}, \bullet)^\pm$ applied to the short exact sequence (5.11) returns the following short exact sequence in cohomology,

$$\text{(5.9)} \quad 0 \to \text{Hom}_{X_B}(W_{B,0} \otimes W_{B,0}, H \otimes_k O_X)^\pm \to \text{Hom}_{X_B}(W_{B,0} \otimes W_{B,0}, O_{X_B})^\pm \to \text{Hom}_{X_A}(W_{A,0} \otimes W_{A,0}, O_{X_A})^\pm \to 0.$$ 

It follows that $(\pi_A \otimes \pi_A)^* \Phi_A$ determines a class $[\Upsilon]$ in

$$\text{(5.10)} \quad \text{Hom}_{X_B}(W_{B,0} \otimes W_{B,0}, O_{X_B})^\pm / \sigma_0 \circ \text{Hom}_{X_B}(W_{B,0} \otimes W_{B,0}, H \otimes_k O_X)^\pm,$$

where

$$\text{(5.11)} \quad \text{Hom}_{X_B}(W_{B,0} \otimes W_{B,0}, H \otimes_k O_X)^\pm \cong \text{Hom}_{X_B}(W_{B,0} \otimes W_{B,0}, O_{X_B})^\pm \cong H \otimes_k \text{Hom}_X(W_0 \otimes W_0, O_X)^\pm.$$

Pick any representant $\Upsilon \in \text{Hom}_{X_B}(W_{B,0} \otimes W_{B,0}, O_{X_B})^\pm$ of the class $[\Upsilon]$ in (5.10), fixed by $(\pi_A \otimes \pi_A)^* \Phi_A$. We obviously have that $\Upsilon|_{X_A} = (\pi_A \otimes \pi_A)^* \Phi_A$. Therefore, the restriction to $X_A$ of the image under $\Upsilon$ of the subspace $W_{B,0} \otimes \rho^{-1}(p_*\partial_{A,-1}(W_{A,-1})) \subset W_B \otimes W_B$ vanishes,

$$\Upsilon(W_{B,0} \otimes \rho^{-1}(p_*\partial_{A,-1}(W_{A,-1})))|_{X_A} = (\pi_A \otimes \pi_A)^* \Phi_A(W_A \otimes \partial_{A,-1}(W_{A,-1})) = 0.$$

It then follows that,

$$\text{(5.12)} \quad \Upsilon(W_{B,0} \otimes \rho^{-1}(p_*\partial_{A,-1}(W_{A,-1}))) \subset \sigma(H \otimes_k O_X).$$

Since $H^2 = 0$ one has that the intersection of $(H \otimes_k W_0) \otimes W_B$ and $W_{B,0} \otimes (H \otimes_k W_0)$ is $0 \cong (H \otimes_k W_0) \otimes (H \otimes_k W_0)$. Thanks to this, and recalling that $H \cdot m_A = 0$, one can then consider the subspaces of $W_{B,0} \otimes W_{B,0}$

$$V_1 := (H \otimes_k W_0) \otimes W_{B,0} \cong (H \otimes_k W_0) \otimes W_{A,0} \cong H \otimes_k (W_0 \otimes W_0),$$

and

$$V_2 := W_{B,0} \otimes (H \otimes_k W_0) \cong W_{A,0} \otimes (H \otimes_k W_0) \cong H \otimes_k (W_0 \otimes W_0).$$

By construction $\Upsilon|_{X} = (\pi \otimes \pi)^* \phi$. Then, by continuity, we have that

$$\text{(5.13)} \quad \Upsilon|_{V_i} \cong 1_H \otimes (\pi \otimes \pi)^* \phi.$$ 

It is a consequence of (5.13) that

$$\text{(5.14)} \quad \Upsilon(W_{B,0} \otimes (H \otimes_k \partial_{A,-1}(W_{A,-1}))) = 0$$

since $W_B \otimes (H \otimes_k \partial_{A,-1}(W_{A,-1})) \cong W_0 \otimes (H \otimes_k \partial_{A,-1}(W_{A,-1}))$. Also, as $(H \otimes_k W_0) \otimes \rho^{-1}(p_*\partial_{A,-1}(W_{A,-1})) \cong (H \otimes_k W_0) \otimes \partial_{A,-1}(W_{A,-1})$, we have

$$\text{(5.15)} \quad \Upsilon((H \otimes_k W_0) \otimes \rho^{-1}(p_*\partial_{A,-1}(W_{A,-1}))) = 0.$$ 

It follows from (5.12), (5.14) and (5.15) that $\Upsilon$ applied to $W_{B,0} \otimes \rho^{-1}(p_*\partial_{A,-1}(W_{A,-1}))$ descends to the morphism of $O_{X_B}$-modules

$$\Upsilon_F : W_{A,0} \otimes F \to H \otimes_k O_X.$$

Consider $\Upsilon^A_F : W_{A,0} \otimes F^A \to H \otimes_k O_X$ to be the morphism of $O_{X_A}$-modules associated to $\Upsilon_F$ under the equivalence mentioned above. Since we defined $F_1 = F^A|_X$, let us set accordingly

$$\text{(5.16)} \quad \mu := \Upsilon^A_F|_X : W_0 \otimes F_1 \to H \otimes_k O_X.$$ 

Recall that $\Upsilon$ is defined up to the additive action of (5.11). We note that this action corresponds to the additive action of $\text{Hom}_X(W_0 \otimes W_0, H \otimes_k O_X)^\pm \circ (1_{W_0} \otimes f)$ over $\mu$. It follows from (5.13) that $\mu$ satisfies (5.11). Also, as $\Upsilon = \pm \Upsilon \circ \theta_{W_{R,0}}$ by construction, one naturally has that $\mu$ satisfies (5.12) as well. Note that for any other locally free resolution satisfying (5.2), we would obtain a 2-extension in the same equivalence class as (5.8) and the corresponding class of morphism as (5.16).
Thus, we have completed the construction of a morphism associated to the deformation functor $\text{Def}_{(E,\phi)}^+\!$ and an small extension of Artin algebras $0 \to H \to B \xrightarrow{\tau} A \to 0$,

\begin{equation}
\Omega^+_\tau : \text{Def}_{(E,\phi)}^+(A) \to H \otimes_k \mathbb{H}^2 \left( \Delta_{(E,\phi)}^+ \right),
\end{equation}

sending $(E_A, \Phi_A, \gamma_A) \in \text{Def}_{(E,\phi)}^+(A)$ to the point of $H \otimes_k \mathbb{H}^2 \left( \Delta_{(E,\phi)}^+ \right)$ given by the 2-extension \([5.8]\) and the class of morphisms given by \([5.10]\). Similarly, associated to $\text{Def}_{(E,\phi)}^-$, we construct

\begin{equation}
\Omega^-_\tau : \text{Def}_{(E,\phi)}^-(A) \to H \otimes_k \mathbb{H}^2 \left( \Delta_{(E,\phi)}^- \right).
\end{equation}

We now see that these maps provide an obstruction theory for orthogonal and symplectic sheaves. Some results of obstruction theory of sheaves are needed and, for the reader’s convenience, we include them instead of just cite them. We start by checking condition \textbf{O1}.

**Proposition 5.1.** Consider the small extension of Artin algebras $0 \to H \to B \xrightarrow{\tau} A \to 0$. Given $(E_A, \Phi_A, \gamma_A) \in \text{Def}_{(E,\phi)}^+(A)$, one has $\Omega^+_\tau(E_A, \Phi_A, \gamma_A) = 0$ if and only if there exists $(E_B, \Phi_B, \gamma_B) \in \text{Def}_{(E,\phi)}^+(B)$ such that $(E_B, \Phi_B, \gamma_B)|_{X_A} \cong (E_A, \Phi_A, \gamma_A)$.

**Proof.** From obstruction theory of sheaves, $(E_A, \gamma_A)$ lifts to $(E_B, \gamma_B)$ if and only if one can give an exact filler of \([5.5]\). It is a standard result of abelian categories (see [Ni, Lemma 3.10] for instance) that exact fillers of \([5.5]\) exist if and only if the short exact sequence \([5.5]\) splits. In that case there exists a splitting morphism,

$s : p_* \partial_{A,-1}(W_{A,-1}) \to \mathcal{F},$

whose composition with $\rho$ is the identity. Hence

\begin{equation}
s \circ (p_* \partial_{A,-1}(W_{A,-1})) \cong \ker \pi_A.
\end{equation}

Fixing an splitting morphism, we can define $\mathcal{V}_B \subset W_{B,0}$ as the preimage of $s \circ (p_* \partial_{A,-1}(W_{A,-1}))$ under the projection

$\rho^{-1} (p_* \partial_{A,-1}(W_{A,-1})) \to \mathcal{F} = \rho^{-1} (p_* \partial_{A,-1}(W_{A,-1})) / H \otimes_k \partial_{A,-1}(W_{A,-1}).$

Then, set

$\mathcal{E}_B := W_{B,0} / \mathcal{V}_B.$

One can easily check that this construction provides a coherent sheaf $\mathcal{E}_B$ over $X_B$ satisfying $\mathcal{E}_B|_{X_A} \cong \mathcal{E}_A$ and, furthermore, $\mathcal{E}_B$ is flat over $B$ (see for instance [Ni, Lemma 3.14]). One trivially has that $\mathcal{E}_B|_{X} = \mathcal{E}_A|_{X}$, so if we pick $\gamma_B$ to be $\gamma_A$ : $\mathcal{E}_B|_{X} = \mathcal{E}_A|_{X} \cong \mathbb{E}$. We now see that this construction provides a coherent sheaf $\mathcal{E}_B$ over $X_B$ satisfying $\mathcal{E}_B|_{X_A} \cong \mathcal{E}_A$ and, furthermore, $\mathcal{E}_B$ is flat over $B$ (see for instance [Ni, Lemma 3.14]). One trivially has that $\mathcal{E}_B|_{X} = \mathcal{E}_A|_{X}$, so we pick $\gamma_B$ to be $\gamma_A : \mathcal{E}_B|_{X} = \mathcal{E}_A|_{X} \cong \mathbb{E}$. Up to this point, we have just reproduced the classical theory of sheaves, seeing $(E_A, \gamma_A)$ lifts to $(E_B, \gamma_B)$ if and only if \([5.5]\) splits. Pick $[\mathcal{T}]$ to be the class in \([5.10]\) given by $(\pi_A \otimes \pi_A)^* \Phi_A$. If \([5.5]\) splits, we have that a representant $\Upsilon$ of $[\mathcal{T}]$ defines $\Phi_B \in \text{Hom}_{X_B}(\mathcal{E}_B \otimes \mathcal{E}_B, \mathcal{O}_{X_B})^\pm$ if and only if

\begin{equation}
\Upsilon(\mathcal{V}_{B,0} \otimes \mathcal{V}_B) = 0.
\end{equation}

It is a consequence of \([5.12]\) and \([5.14]\) that \([5.20]\) holds whenever

\begin{equation}
\mathcal{Y}_F(\mathcal{V}_{B,0} \otimes s(p_* \partial_{A,-1}(W_{A,-1}))) = 0.
\end{equation}

It remains to show that \([5.5]\) splits and \([5.21]\) holds for some representant of $[\mathcal{T}]$ in \([5.10]\) if and only if the image of $\Omega^+_\tau(E_A, \Phi_A, \gamma_A) = 0$. By the second statement of Proposition \textbf{3.3}, the later is equivalent to the fact that the 2-extension given in \([5.8]\) has the form \([3.13]\) and the class $[\mu]$ in $\text{Hom}_{X}(W_0 \otimes F, H \otimes_k \mathcal{O}_X)^\pm / \text{Hom}_{X}(W_0 \otimes W_0, H \otimes_k \mathcal{O}_X)^\pm \circ (1_{W_0} \otimes f)$ defined in \([5.10]\) satisfies \([3.13]\). Note that \([5.5]\) splits if and only if \([5.8]\) splits, and further, \([5.6]\) splits if and only if the 2-extension \([5.4]\) splits. Since $\text{Ext}^2_{X_A}(\mathcal{E}_A, H \otimes_k p_{A,*} \mathcal{E}) \cong \text{Ext}^2_{X}(\mathcal{E}, H \otimes_k \mathcal{E})$, we have that \([5.7]\) splits if and only if \([5.8]\) splits giving rise to a 2-extension of the form \([5.13]\). In this case and recalling \([5.19]\), the equation \([5.21]\) holds if and only if

\begin{equation}
\mu(W_0 \otimes \ker \pi) = 0.
\end{equation}

This is the case whenever the class $[\mu]$ satisfies \([3.14]\), so $\Omega^+_\tau$ the proof is complete. □
We check now that the morphisms $\Omega^{\pm}_{\tau}$ satisfy condition $\mathbf{O2}$. Observe that any morphism of small extension decomposes into

$$
(5.23) \quad \begin{array}{cccccc}
0 & \longrightarrow & H & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \\
\tau & & \downarrow & & \downarrow & & \downarrow & & \tau & \\
0 & \longrightarrow & H := H/ \ker \tau & \longrightarrow & B := B/ \ker \tau & \longrightarrow & A & \longrightarrow & 0,
\end{array}
$$

and

$$
(5.24) \quad \begin{array}{cccccc}
0 & \longrightarrow & H & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \\
\tau & \downarrow & & \downarrow & & \downarrow & & \tau & \\
0 & \longrightarrow & H' := H/ \ker \tau & \longrightarrow & B' & \longrightarrow & A' & \longrightarrow & 0,
\end{array}
$$

where $h$ is an injection. Therefore, to check that $\mathbf{O2}$ holds, it is enough to check it for morphisms of small extensions of the form $(5.23)$ and $(5.24)$. We start by the first type.

**Proposition 5.2.** Consider the morphism of small extensions $(5.23)$. Given $(E_A, \Phi_A, \gamma_A) \in \text{Def}^{\pm}_{(E, \phi)}(A)$, one has that

$$
\Omega^{\pm}_{\tau}(E_A, \Phi_A, \gamma_A) = (\bar{h} \otimes 1_{\mathbb{R}^2}) \circ \Omega^{\pm}_{\tau}(E_A, \Phi_A, \gamma_A).
$$

**Proof.** Since one obviously has that $\bar{h}$ sends $\ker \bar{h}$ to 0, observe that $\bar{h} \otimes 1_{\mathbb{R}^2}$ applied to the 2-extension $(5.8)$ gives

$$
(5.25) \quad 0 \rightarrow \bar{H} \otimes_k E \rightarrow (F/ \ker \bar{h} \otimes_k E) \rightarrow W_0 \rightarrow E \rightarrow 0.
$$

Pick now $\mu : W_0 \otimes F \rightarrow H \otimes \mathcal{O}_X$ given in $(5.11)$ and note that

$$
(\bar{h} \otimes_k 1_{\mathcal{O}_X}) \circ \mu(W_0 \otimes (\ker \bar{h} \otimes_k E)) = 0.
$$

Then, $(\bar{h} \otimes_k 1_{\mathcal{O}_X}) \circ \mu$ descends to

$$
(5.26) \quad \bar{\mu} : W_0 \otimes (F/ (\ker \bar{h} \otimes_k E)) \rightarrow \bar{H} \otimes_k E,
$$

which is the image of $\mu$ under $\bar{h} \otimes_k 1_{\mathbb{R}^2}$. We have seen that $(\bar{h} \otimes 1_{\mathbb{R}^2}) \circ \Omega^{\pm}_{\tau}(E_A, \Phi_A, \gamma_A)$ is determined by $(5.30)$ and $(5.28)$.

We now study $\Omega^{\pm}_{\tau}(E_A, \Phi_A, \gamma_A)$. Since $X_A \leftarrow X_B$ is the composition $X_A \overset{\tau}{\rightarrow} X_B \overset{\gamma}{\leftarrow} X_B$ one can consider in this case $W_{\mathbb{R}, 0} := W_{B, 0}|_{X_A}$ and the morphism $\bar{\rho} : W_{B, 0} \rightarrow \bar{\rho}_0$ is being the restriction $\rho|_{X_A}$. Pick also $\sigma : \bar{H} \otimes_k W_0 \rightarrow W_{\mathbb{R}, 0}$ corresponding to $\sigma$ on $X_A$. Recall $\mathcal{F}$ from $(5.4)$ and construct $\mathcal{F}'$ accordingly. It follows from the previous discussion that

$$
\mathcal{F}|_{X_A} = \mathcal{F}.
$$

Observing that $\ker \bar{h} \otimes_k E$ is the kernel of $\mathcal{F} \rightarrow \mathcal{F} = \mathcal{F}|_{X_A}$, it then follows that

$$
(5.27) \quad \mathcal{F}'A \cong \mathcal{F}A/ \ker \bar{h} \otimes_k E.
$$

Up to here, we have been dealing with obstruction theory of sheaves. We now address the quadratic form. Observe that $(5.29)$ also holds for $X_B$. Therefore $(\pi_A \otimes \pi_A)^* \Phi_A$ defines a class in the space $(5.10)$ adapted to $X_B$ from which can pick a representant $\overline{\tau} \in \text{Hom}_{X_B}(W_{\mathbb{R}, 0} \otimes W_{\mathbb{R}, 0}, \mathcal{O}_B)^\pm$ that satisfies

$$
(5.28) \quad \overline{\tau}|_{X_B} = \overline{\tau}.
$$

It is a direct consequence of $(5.27)$ and $(5.28)$ that $\Omega^{\pm}_{\tau}(E_A, \Phi_A, \gamma_A)$ is determined by $(5.25)$ and $(5.26)$. This concludes the proof. \qed
Since $h$ on (5.24) is injective, $H$ defines naturally a subspace of $H'$ so one can give (non-canonically) a decomposition of the vector spaces
\[(5.29)\quad H' = H \oplus H''\]
and we can assume
\[(5.30)\quad h = 1_H \oplus 0.\]
Out of (5.24) and the (non-canonical) decomposition (5.29), one can always construct a small extension $\tau'$ and a morphism of small extensions making the following diagram commutative,
\[(5.31)\quad \begin{array}{ccccccccc}
0 & \rightarrow & H & \rightarrow & B & \rightarrow & A & \rightarrow & 0 \\
\downarrow h & & \downarrow \beta & & \downarrow \alpha & & & & \\
0 & \rightarrow & H \oplus H'' & \rightarrow & B' & \rightarrow & A' & \rightarrow & 0 \\
\downarrow \tilde{h} & & \downarrow \tilde{\beta} & & & & & & \\
0 & \rightarrow & H & \rightarrow & \tilde{B}' := B'/H'' & \rightarrow & A' & \rightarrow & 0,
\end{array}\]
where $\tilde{h}$ and $\tilde{\beta}$ are the obvious projections. If we set $\beta' := \tilde{\beta} \circ \beta$ and note that $\tilde{h} \circ h = 1_H$, we obtain the morphism of small extensions
\[(5.32)\quad \begin{array}{ccccccccc}
0 & \rightarrow & H & \rightarrow & B & \rightarrow & A & \rightarrow & 0 \\
\downarrow \beta' & & \downarrow \alpha & & & & & & \\
0 & \rightarrow & H & \rightarrow & B' & \rightarrow & A' & \rightarrow & 0.
\end{array}\]
Let us consider the morphisms of schemes associated to the morphism of algebras appearing in (5.31) and (5.32). Thanks to the commutativity of (5.31) one has the following commuting diagram of morphisms between schemes,
\[
\begin{array}{ccccccccc}
X_{A'} & \rightarrow & X_A & \rightarrow & X_B' \\
\downarrow p' & & \downarrow p & & \downarrow p_{\beta'} & & \\
X_{B'} & \rightarrow & p_{\beta} & & & & & & \\
\downarrow p & & \downarrow & & \downarrow & & \\
X_{A'} & \rightarrow & X_A & \rightarrow & X_B
\end{array}
\]
whose right-upper subdiagram
\[(5.33)\quad \begin{array}{ccccccccc}
X_{A'} & \rightarrow & X_A & \rightarrow & X_B' \\
\downarrow p_{\alpha} & & \downarrow p & & \downarrow p_{\beta'} & & \\
\end{array}\]
is Cartesian.

Before checking $O2$ restricted to morphisms of small extensions of the form (5.24), we will study its compatibility with those of the form (5.32).

**Proposition 5.3.** Consider the morphism of small extensions (5.32). Given $(E_A, \Phi_A, \gamma_A) \in \text{Def}^\pm_{(E,\phi)}(A)$, one has that
\[
\Omega^\pm_{\tilde{\tau}}(p^\alpha, \tilde{E}_A, \Phi_A, \gamma_A) = \Omega^\pm_{\tau}(E_A, \Phi_A, \gamma_A).
\]

**Proof.** We study $\Omega^\pm_{\tilde{\tau}}(p^\alpha, \tilde{E}_A, \Phi_A, \gamma_A)$. Consider the locally free resolution $W_A^* \xrightarrow{\pi_A} E_A \rightarrow 0$ and take its pull-back $p_{\alpha}^* W_A^* \xrightarrow{p_{\alpha}^* \pi_A} p_{\alpha}^* E_A \rightarrow 0$ which is obviously a locally free resolution of $p_{\alpha}^* E_A$. \]
One can choose $\mathcal{W}_{B,0}$ and $\mathcal{W}_{B',0} := p_{\beta'}^* \mathcal{W}_{B,0}$ satisfying in both cases the cohomology vanishing $(5.2)$ for $i > 0$.

Recall that the morphism

$$\tilde{\rho} : \mathcal{W}_{B',0} = p_{\beta'}^* \mathcal{W}_{B,0} \rightarrow \mathcal{W}_{A',0} = \tilde{\rho}',$$

is given by the restriction to $X_{A'}$. Since $p_{\beta'}|_{X_{A'}} = \rho,$ $\mathcal{W}|_{X_{A'}} = 1_{X_{A'}}$ and $p|_{X_A} = 1_{X_A},$ one has that

$$\tilde{\rho}'|_{X_{A'}} = p_{\beta'}^* p_{\mu}^* \mathcal{W}_{A,0}|_{X_{A'}}.$$  

By the Cartesianness of $(5.33)$, it follows that $\tilde{\rho}'|_{X_{A'}}$ is supported over $X_{A'}$. By all of the above, one has that $\mathcal{W} \cong p_{\beta'}^* \rho$ and this implies that

$$(5.34) \quad \mathcal{W}' \cong p_{\beta'}^* \mathcal{W}.$$  

Hence $\mathcal{W}' := (\mathcal{W}')^\Lambda|_X$ is isomorphic to $\mathcal{W}$ and fits in the 2-extension $(5.8)$.

We now move forward from the obstruction theory of sheaves. Recall that we chose $\mathcal{W}_{B,0}$ and $\mathcal{W}_{B',0} := p_{\beta'}^* \mathcal{W}_{B,0}$ in such a way that both satisfy the cohomology vanishing $(5.2)$ for $i > 0$. Let $[\mathcal{T}]$ be the class in $(5.10)$ determined by $(\pi_A \otimes \pi_A)^* \Phi_A$ and note that $p_{\beta'}^* \mathcal{T}$ is a representant of the corresponding class determined to $(\pi_A \otimes \pi_A)^* p_{\mu}^* \Phi_A$. After this and $(5.34)$, it follows that

$$\mathcal{W} := (p_{\beta'}^* \mathcal{T})^\Lambda|_{X} \cong p_{\mu}^* (\mathcal{T}^\Lambda)|_{X} \cong \mu,$$

where $\mu$ is defined in $(5.10)$. This concludes the proof, as $\mu$ and the 2-extension $(5.8)$ determine $\Omega^\pm_i (\mathcal{E}_A, \Phi_A, \gamma_A)$ as well.

We now check that the set of maps defined in $(5.17)$ and $(5.18)$ satisfy condition O2 restricted to morphisms of small extensions of the form $(5.24)$.

**Proposition 5.4.** Consider the morphism of small extensions $(5.24)$. Given $(\mathcal{E}_A, \Phi_A, \gamma_A) \in \text{Def}^\pm (\mathcal{E}_A, \Phi_A, \gamma_A)(A)$, one has that

$$(5.35) \quad \Omega^\pm_i (p_{\mu}^* (\mathcal{E}_A, \Phi_A, \gamma_A)) = (h \otimes 1_{\mathbb{E}^2}) \circ \Omega^\pm_i (\mathcal{E}_A, \Phi_A, \gamma_A)).$$

**Proof.** We first describe $(h \otimes 1_{\mathbb{E}^2}) \circ \Omega^\pm_i (\mathcal{E}_A, \Phi_A, \gamma_A)$, where we recall that $\Omega^\pm_i (\mathcal{E}_A, \Phi_A, \gamma_A)$ is determined by the 2-extension $(5.8)$ and $\mu$ given in $(5.16)$. Recall as well that $\mathcal{H}'$ and $h$ decompose as indicated in $(5.27)$ and $(5.30)$. In that case, observe that $h \otimes 1_{\mathbb{E}^2}$ applied to $(5.8)$ gives

$$(5.36) \quad \mu \otimes (1_{\mathcal{H}'} \otimes \omega (\sqrt{\mathcal{H}' \otimes \mathcal{E}})) : W_0 \otimes ((\mathcal{F} \otimes \mathcal{E}) \otimes (\mathcal{H}' \otimes \mathcal{E})) \rightarrow (\mathcal{H} \otimes \mathcal{H}') \otimes \mathcal{O}_X.$$

Let us now study $\Omega^\pm_i (p_{\mu}^* (\mathcal{E}_A, \Phi_A, \gamma_A))$. We consider $\mathcal{W}_{A',i}$, $\mathcal{W}_{B',0}$ and the morphisms $\rho' : \mathcal{W}_{B',0} \rightarrow p_{\mu}^* \mathcal{W}_{A',0}$ as we did in the beginning of this section. Set as well

$$(5.37) \quad \mathcal{W}_{B',0}|_{X_{B'}} = \mathcal{W}_{B',0},$$

and

$$(5.38) \quad \rho'|_{X_{B'}} = \tilde{\rho}' : \mathcal{W}_{B',0} \rightarrow \mathcal{W}_{A',0} = p_{\mu}^* \mathcal{W}_{A',0}|_{X_{B'}}.$$  

Defining $\mathcal{F}'$ and $\mathcal{F}'$ as in $(5.4)$, it follows from $(5.37)$ and $(5.38)$ that

$$(5.39) \quad \mathcal{F}'|_{X_{B'}} = \mathcal{F}'.$$

From the description of $\mathcal{F}'$ that we obtain from $(5.5)$ one can obtain the following short exact sequence of $\mathcal{O}_{X_{B'}}$-modules

$$0 \rightarrow \mathcal{H}' \otimes \mathcal{E} \rightarrow \mathcal{F}' \rightarrow p_{\mu}^* \mathcal{F}' \rightarrow 0,$$
that gives rise to the short exact sequence of $\mathcal{O}_{X_A}$-modules

$$(5.40) \quad 0 \longrightarrow H'' \otimes_k p_{A',*}E \longrightarrow (\mathcal{F}')^A' \longrightarrow \pi_{B, *}(\overline{\mathcal{F}})^A' \longrightarrow 0.$$  

We see that $[6.39]$ provides naturally a splitting of $(5.40)$, so

$$(5.41) \quad (\mathcal{F}')^A' \equiv \pi_{B, *}(\overline{\mathcal{F}})^A' \oplus (H'' \otimes_k p_{A',*}E).$$

Hence, the 2-extension determined by $\Omega^\pm_\tau(p^*_\alpha(\mathcal{E}_A, \Phi_A, \gamma_A))$ is

$$(5.42) \quad 0 \rightarrow (H \otimes_k E) \oplus (H'' \otimes_k E) \rightarrow (\overline{\mathcal{F}}' \otimes_k E) \oplus (H'' \otimes_k E) \rightarrow W'_0 \rightarrow E \rightarrow 0,$$

where $\overline{\mathcal{F}}'$ denotes $(\mathcal{F})^A'|_X$ and $W'_0$ is the restriction to $X$ of $\mathcal{W}_A'.0$.

Only at this point, we find ourselves in a position to give a step forward from the classical case of sheaves. One has that $[5.9]$ also holds over $X_{B'}$ and $X_{\overline{B}'}$. Therefore $(\pi_A \otimes \pi_A)^*\Phi_A$ defines classes in the corresponding spaces $[5.10]$ defined over $X_{B'}$ and $X_{\overline{B}'}$. Note also that one can choose representatives $\Upsilon' \in \text{Hom}_{X_{B'}}(\mathcal{W}_{B',0} \otimes \mathcal{W}_{B',0}, \mathcal{O}_{B'})^\pm$ and $\overline{\Upsilon}' \in \text{Hom}_{X_{\overline{B}'}}(\mathcal{W}_{\overline{B}',0} \otimes \mathcal{W}_{\overline{B}',0}, \mathcal{O}_{\overline{B}'})^\pm$ satisfying

$$(5.43) \quad \Upsilon'|_{X_{\overline{B}'}} = \overline{\Upsilon}'.$$

Denote by $\mu'$ and $\overline{\mu}'$ the maps defined in $(5.16)$ out of $\Upsilon'$ and $\overline{\Upsilon}'$. It then follows from $(5.13)$, $(5.41)$ and $(5.43)$ that

$$(5.44) \quad \mu' = \overline{\mu}' \oplus (1_{H''} \otimes_k \phi(\pi \otimes 1_E)).$$

The result follows from Proposition 5.3 after comparing $[5.39]$ with $[5.42]$ and $[5.36]$ with $[5.41]$. \qed

The following summarizes all the previous results in this section.

**Theorem 5.5.** Let $(E, \phi)$ be an orthogonal (resp. symplectic) sheaf over the projective scheme $X$. Then the deformation functor $\text{Def}^+_{(E, \phi)}$ (resp. symplectic) admits an obstruction theory with vector space

$$\text{Obs}_{(E, \phi)}(\text{Def}^+_{(E, \phi)}) = \mathbb{H}^2\left(\Delta^\pm_{(E, \phi)}\right).$$

Therefore, $\text{Def}^+_{(E, \phi)}$ are formally smooth when $\mathbb{H}^2\left(\Delta^\pm_{(E, \phi)}\right) = 0$.

**Proof.** The theorem is a consequence of Propositions 5.1, 5.2 and 5.4. \qed

**References**

[GLSS1] T. Gómez, A. Langer, A. H. W. Schmitt, I. Sols. *Moduli spaces for principal bundles in arbitrary characteristic*. Adv. Math. 219 (2008), no. 4, 1177–1245.

[GLSS2] T. Gómez, A. Langer, A. H. W. Schmitt, I. Sols. *Moduli spaces for principal bundles in large characteristic*. Teichmüller theory and moduli problem, 281–371, Ramanujan Math. Soc. Lect. Notes Ser., 10, Ramanujan Math. Soc., Mysore, 2010.

[GS1] T. L. Gómez and I. Sols, *Stable tensors and moduli space of orthogonal sheaves*, [math.AG/0103150].

[GS2] T. L. Gómez and I. Sols, *Moduli space of principal sheaves over projective varieties*, Ann. of Math., 161 (2005), pp. 1033–1088.

[Ni] N. Nitsure, *Deformation theory for vector bundles*, in *Moduli spaces and vector bundles*, Lecture note series 359 (2009), Cambridge University Press (edited by L. Bambra, S. Bradlow, O. Garcia-Prada, and S. Ramanan), pp 128–164.

[Sc] M. Schlessinger, *Functors of Artin rings*, Trans. Amer. Math. Soc. 130 (1968), 208-222.

[Sch1] A. H. W. Schmitt, *Singular principal bundles over higher-dimensional manifolds and their moduli spaces*, Int. Math. Res. Not. 2002 (2002), no. 23, 1183–1209.

[Sch2] A. H. W. Schmitt, *A closer look at semistability for singular principal bundles*. Int. Math. Res. Not. 2004, no. 62, 3327–3366.

[S1] J. Scalise, *Framed symplectic sheaves on surfaces and their ADHM data*, PhD Thesis (2016), Scuola Internazionale Superiore di Studi Avanzati (SISSA), Trieste (Italia).

[S2] J. Scalise, *Framed symplectic sheaves on surfaces*, Int. J. Math. 29 (1) (2018).
