FAST TRACK COMMUNICATION

On the structure of critical energy levels for the cubic focusing NLS on star graphs

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Abstract

We provide information on a non-trivial structure of phase space of the cubic nonlinear Schrödinger (NLS) on a three-edge star graph. We prove that, in contrast to the case of the standard NLS on the line, the energy associated with the cubic focusing Schrödinger equation on the three-edge star graph with a free (Kirchhoff) vertex does not attain a minimum value on any sphere of constant $L^2$-norm. We moreover show that the only stationary state with prescribed $L^2$-norm is indeed a saddle point.

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1. Introduction

A major issue in nonlinear dynamics consists in the search for stationary solutions and the study of their stability properties. In the case of Hamiltonian systems, a first picture of the phase space can be drawn by identifying critical points of the energy and their nature. In particular, it is important to know if a ground state exists, where a ground state is defined as the minimizer of the energy functional, possibly restricted to suitable submanifolds. In the context of the nonlinear Schrödinger (NLS) equation, one typically restricts the energy functional to a manifold on which the second conserved quantity (sometimes called mass, or charge) is constant. From a physical point of view, such a restriction is meaningful, as the extra conserved quantity often represents a physical characteristic of the system...
(e.g., the mass, or the number of particles). In the one-dimensional case, the ground states of the NLS with power nonlinearity on the line are well known and completely described in the classical paper [3], where more general nonlinearities are also treated. It turns out that on the line, the NLS energy constrained to the manifold of the states of constant mass attains its minimum value in correspondence with a unique (up to translation) positive, symmetric and decreasing (for $x > 0$) function. No other critical points of the energy exist.

In this communication, we are interested in the case of the focusing nonlinear cubic Schrödinger equation on a three-edge star graph, sometimes called in the physical literature a Y-junction. To put the issue in a physical context, we begin by recalling that the linear Schrödinger equation on a graph is a well-developed subject, as an effective description of dynamics of many mesoscopic systems such as, for example, quantum nanowires (see [5, 8, 9] and references therein). It is of interest to extend the analysis to the nonlinear wave propagation on networks. In particular, it is well known that the NLS appears as an effective equation in several different areas: the description of Bose condensates, the propagation of electromagnetic pulses in nonlinear (Kerr) media and Langmuir waves in plasma physics. In many situations, it seems of interest to treat the propagation of NLS solutions associated with such phenomena in one-dimensional ramified structures, the prototype of which is a three-edge star graph, or Y-junction. The subject is at its beginning, and for some preliminary experimental, numerical and analytical works, see [6, 7, 10–13]. A first rigorous analysis of nonlinear stationary (or bound) states for NLS on a star graphs is outlined in [2], where solitary waves for a star graph with delta vertex are constructed, including the case of a free or Kirchhoff vertex. The graph with a Kirchhoff vertex is the closest analog to the free particle on the line, to which it reduces when the line is considered as a graph with two edges. In the linear case, the Kirchhoff Laplacian on a graph has only an absolutely continuous spectrum, and so only scattering states are possible for the Schrödinger dynamics. However, in the presence of a nonlinearity, a three-edge star graph with Kirchhoff conditions at the vertex admits a unique stationary state, which is quite simply described: it is the state on the graph which coincides on every edge with half a soliton. One could suspect that this stationary state is the ground state. In contrast, we show that, perhaps unexpectedly, this is not the case.

This fact highlights that the NLS on graphs, even on the simplest one, exhibits remarkable differences w.r.t. the same evolution equation on the line.

In this communication, we consider the case of the cubic focusing NLS and analyze the energy constrained to a fixed sphere in $L^2$, i.e. the set of states of prescribed mass. We show that the constrained energy is bounded from below (a fact that can be shown similarly to the case of $\mathbb{R}^n$ and subcritical nonlinearity, see [4] and [2]), but it approaches the infimum without attaining a minimum value. Moreover, the only nonlinear stationary state is a saddle point of the constrained energy functional. The existence of a saddle point of the energy is a remarkable feature, inducing on the phase space of the system stable and unstable manifolds that, in the absence of other critical points, are the main structural properties of the dynamics. The consequences of this fact will be investigated in a subsequent paper.

2. Preliminaries

Before giving the main results, we start by fixing the framework, the notation and recalling some basic results.
A three-edge star graph $G$ can be thought of as composed by three half-lines with a common origin, called vertex. A state or wavefunction on the graph is an element of the Hilbert space $L^2(G) = \bigoplus_{i=1}^{3} L^2(\mathbb{R}^+; dx_i)$ and can be represented as a column vector:

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix}, \quad \Psi_i \in L^2(\mathbb{R}^+).$$

The space $L^2(G)$ is naturally endowed with the Hermitian product

$$\langle \Phi, \Psi \rangle_{L^2(G)} = \sum_{i=1}^{3} \int_{0}^{\infty} \overline{\Phi_i(x_i)} \Psi_i(x_i) \, dx_i.$$

Sobolev and $L^p$-spaces on $G$ are defined analogously:

$$H^s(G) = \bigoplus_{i=1}^{3} H^s(\mathbb{R}^+), \quad L^p(G) = \bigoplus_{i=1}^{3} L^p(\mathbb{R}^+).$$

We denote by $H^1_G$ the set of the functions in $H^1(G)$ that fulfil the continuity condition at the vertex $\Psi_1(0) = \Psi_2(0) = \Psi_3(0)$.

In the following, we denote by $\|\Psi\|_p$ the norm of the function $\Psi$ in the space $L^p(G)$. When $p = 2$, we shall simply write $\|\Psi\|$.

The dynamics of the system is described by the Schrödinger equation

$$i \partial_t \Psi(t) = -\Delta \Psi(t) - |\Psi(t)|^2 \Psi(t). \quad (2.1)$$

Here

- The operator $-\Delta$ acts on the domain

$$D(-\Delta) := \{ \Psi \in H^2(G), \ \Psi_1(0) = \Psi_2(0) = \Psi_3(0), \ \Psi_1'(0) + \Psi_2'(0) + \Psi_3'(0) = 0 \}$$

and its action reads

$$-(\Delta \Psi)_i = -\Psi_i'' , \quad i = 1, 2, 3.$$

The condition at the vertex is usually referred to as Kirchhoff’s boundary condition.

The operator $-\Delta$ is self-adjoint on $L^2(G)$. Note that, on the graph, the Laplacian with Kirchhoff boundary conditions is the natural generalization of the free Laplacian on the line, as it is easily seen by considering the line as a two-edge star graph, and noting that the boundary conditions reduce to continuity of the wavefunction and continuity of the derivative at the vertex, i.e. $\Psi \in H^2(\mathbb{R})$.

- The nonlinear term in (2.1) is defined componentwise:

$$\langle |\Psi|^2 \Psi_i \rangle := |\Psi_i|^2 \Psi_i.$$

The resulting equation (2.1) has to be interpreted as a system of NLS equations on the half-line, in which the coupling is assigned through the boundary condition at the vertex.

- We call standing waves the solutions to equation (2.1) of the form

$$\Psi(t) = e^{i\omega t} \Phi, \quad \omega > 0.$$

The amplitude $\Phi$, for which we use the common expression stationary state, satisfies the equation

$$-\Delta \Phi - |\Phi|^2 \Phi = -\omega \Phi.$$
(3) The problem \((2.1)\) is globally well-posed in \(H^1_{\epsilon}(\mathcal{G})\) (see [1]). The \(L^2\)-norm and the energy
\[
E(\Psi) = \frac{1}{2} \| \Psi \|_2^2 - \frac{1}{4} \| \Psi \|_4^4 = \sum_{i=1}^{3} \left( \frac{1}{2} \| \Psi_i \|_{L^2(\mathbb{R}^+)}^2 - \frac{1}{4} \| \Psi_i \|_{L^4(\mathbb{R}^+)}^4 \right)
\]
are conserved by time evolution.

In what follows, we use the following notation:
\[
E_1(\psi) = \frac{1}{2} \| \psi' \|_{L^2(\mathbb{R}^+)}^2 - \frac{1}{4} \| \psi \|_{L^4(\mathbb{R}^+)}^4, \quad \psi \in H^1(\mathbb{R}^+),
\]
and
\[
E_2(\psi) = \frac{1}{2} \| \psi' \|_{L^2(\mathbb{R})}^2 - \frac{1}{4} \| \psi \|_{L^4(\mathbb{R})}^4, \quad \psi \in H^1(\mathbb{R}).
\]

(4) Let us recall a classical result on minimization for the cubic NLS on the line, i.e. the two-edge star graph (see, e.g., [4], chapter 8), and derive an immediate consequence for the cubic NLS on the half-line, i.e. the one-edge star graph.

(i) The minimum of the functional \(E_2\) on the functions in \(H^1(\mathbb{R})\) with squared \(L^2\)-norm equal to \(m > 0\) is achieved on the functions (up to a phase factor)
\[
\phi_\nu^m(x) = \frac{m}{2\sqrt{2}} \sech \left( \frac{m}{4} x \right), \quad x \in \mathbb{R},
\]
and gives
\[
E_2(\phi_\nu^m) = -\frac{m^3}{96}.
\]

(ii) The minimum of the functional \(E_1\) on the functions in \(H^1(\mathbb{R}^+)\) with squared \(L^2\)-norm equal to \(m > 0\) is achieved on the function (up to a phase factor)
\[
\phi_m(x) = \frac{m}{\sqrt{2}} \sech \left( \frac{m}{2} x \right), \quad x \geq 0,
\]
and gives
\[
E_1(\phi_m) = -\frac{m^3}{24}.
\]

Indeed, consider a function \(\eta \in H^1(\mathbb{R}^+)\), such that \(\| \eta \|_{L^2(\mathbb{R}^+)}^2 = m\) and \(E_1(\eta) \leq E_1(\phi_m)\), and define the function \(\mu(x) = \eta(|x|)\), \(x \in \mathbb{R}\). Then,
\[
E_2(\mu) = 2E_1(\eta) \leq 2E_1(\phi_m) = E_2(\phi_\nu^m),
\]
so, by (i), and due to the even character of \(\mu\), it must be \(\mu = \phi_\nu^m\), and thus \(\eta = \phi_m\).

3. Results

As recalled in the introduction, the energy functional \((2.2)\) at fixed \(L^2\)-norm is bounded from below for subcritical nonlinearity, which is our case.

Let us introduce the following family of states.

**Definition 3.1.** We call sesquisoliton (i.e. 'one and half' soliton) any function of the form
\[
\Phi_{m_1,m_2}^\epsilon(x_1, x_2, x_3) := \begin{pmatrix}
\frac{m_1}{\sqrt{2}} \sech \left( \frac{m_1}{2} x_1 \right) \\
\frac{m_2}{\sqrt{2}} \sech \left( \frac{m_2}{4} (x_2 - x) \right) \\
\frac{m_2}{\sqrt{2}} \sech \left( \frac{m_2}{4} (x_3 + x) \right)
\end{pmatrix},
\]
where \(0 < m_1 \leq m_2, x \geq 0\), and the following condition of 'continuity at the vertex' holds:
\[
m_1 = \frac{m_2}{2} \sech \left( \frac{m_2}{4} x \right).
\]
Note that the sesquisolitons (see figure 1) are obviously the elements of $H^1_c(G)$. As a matter of fact, they also belong to $D(-\Delta)$. Due to equations (3.1) and (3.2), sesquisolitons generate a two-parameter family of states in $D(-\Delta)$.

Moreover, in the case $x = 0$, one has $m_1 = \frac{m_2}{2}$, and one obtains a symmetric configuration with three half solitons concurring at the vertex (see figure 2). In [2], it was shown that this is a stationary state for the NLS equation (2.1). We remark that for $x \neq 0$ the sesquisoliton is not a stationary state, its components on each edge satisfy the stationary equation but for different values of $\omega$.

Now, for any $M > 0$, we define the manifold of the sesquisolitons with fixed $L^2$-norm as follows:

$$S_M := \{ \Phi_{m_1,m_2}^x, \| \Phi_{m_1,m_2}^x \|^2 = M \}.$$ 

**Theorem 3.2.** For any $\Psi \in H^1_c(G)$ such that $\| \Psi \|^2 = M$, the following chain holds:

$$E(\Psi) > \inf_{\| \Psi \|^2 = M} E(\Psi) = \inf_{\Psi \in S_M} E(\Psi) = -\frac{M^3}{96}. \quad (3.3)$$

**Proof.** Given $\Psi \in L^2(G)$, it is possible to construct a sesquisoliton with the same $L^2$-norm but with lower energy. We proceed as follows. Let us suppose that

$$\| \Psi_1 \| \leq \min(\| \Psi_2 \|, \| \Psi_3 \|). \quad (3.4)$$
Then, consider the sesquisoliton (3.1) with $m_1 = \|\Psi_1\|^2$, $m_2 = \|\Psi_2\|^2 + \|\Psi_3\|^2$ and $x \geq 0$ chosen in order to satisfy condition (3.2). Note that such a choice is always possible since $2m_1 \leq m_2$ and is unique.

If condition (3.4) is not fulfilled, then one first relabels the edges in order to have the minimal mass on the first one, and thus proceeds as before.

It is immediately seen that $\|\Psi_1\|^2 = \|\phi_1\|^2 = m_1$ and $\|\Psi_2\|^2 + \|\Psi_3\|^2 = \|\phi_2\|^2 + \|\phi_3\|^2 = m_2$, and thus $\|\phi_1\|^2 + \|\phi_2\|^2 = M$. Once the total mass $M$ is fixed, sesquisolitons generate a one-parameter family of states. Let us define the following function on the real line:

$$
\psi(\xi) := \begin{cases} 
\Psi_2(-\xi), & \xi < 0, \\
\Psi_3(\xi), & \xi > 0,
\end{cases}
$$

(3.5)

and note that, by (2.5) and (3.1),

$$
\psi_{m_2}^{-1}(\xi) := \begin{cases} 
(\phi_{m_1,m_2}^x)_{2}(-\xi), & \xi < 0, \\
(\phi_{m_1,m_2}^x)_{1}(\xi), & \xi > 0.
\end{cases}
$$

(3.6)

Furthermore, by (2.7), (2.6), (2.2), (2.3), (2.4), (3.5) and (3.6), one immediately has the following chain of inequalities:

$$
E(\psi) = E_1(\psi_1) + E_2(\psi) \geq E_1(\phi_{m_1}) + E_2(\psi_{m_2}^{-1}) = E(\phi_{m_1,m_2})
$$

The inequality $\inf_{\|\psi\|=M} E(\psi) \leq \inf_{\psi \in S_M} E(\psi)$ is immediate, and then, the first of the two identities in (3.3) is proven. To prove the second, we use (2.7) and (2.6) and obtain

$$
E(\phi_{m_1,m_2}^x) = -\frac{m_1^3}{24} - \frac{m_2^3}{96}
$$

so, noting that $M = m_1 + m_2$, we obtain

$$
E(\phi_{m_1,m_2}^x) = \frac{m_1^3}{24} + \frac{(m_1 - M)^3}{96},
$$

(3.7)

where $m_1$ plays the role of a parameter. We stress that, due to the constraint of the mass of the soliton, $m_1$ can vary in the interval $(0, M/3)$. Differentiating (3.7), one immediately has that $E(\phi_{m_1,m_2}^x)$ is monotonically increasing in such interval, so

$$
\inf_{\psi \in S_M} E(\psi) = \lim_{m_1 \to 0^+} E(\phi_{m_1,m_2}^x) = \frac{M^3}{96}.
$$

To complete the proof, we must show that, for any $\Psi$ in $H^1_1(\mathcal{G})$, $E(\Psi)$ is strictly larger than $-M^3/96$. With this aim, we note that such an infimum cannot be achieved, as for $m_1 = 0$, the condition (3.2) does not correspond to an admissible sesquisoliton, so it cannot be fulfilled.

**Corollary 3.3.** The sesquisoliton $\phi_{M/3,2M/3}^0$ is a saddle point for the energy functional.

**Proof.** First, we note that $\phi_{M/3,2M/3}^0$ is a critical point. Indeed, it satisfies the Euler–Lagrange equation for the energy functional constrained on the manifold $\|\psi\|^2 = M$:

$$
-\Delta \psi - |\psi|^2 \psi + \omega \psi = 0,
$$

where $\omega$ is a Lagrange multiplier, coinciding with $M^2/36$.

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In order to prove that $\Phi^{0}_{M/3,2M/3}$ is a saddle point, it is sufficient to show that it maximizes the energy restricted to a submanifold to which it belongs in the constraint manifold, and minimizes the energy when restricted to a different submanifold.

By the proof of theorem (3.2) (see equation (3.7) and the comment below it) the stationary state $\Phi^{0}_{M/3,2M/3}$ has the higher energy among the one-parameter family of sesquisolitons at fixed mass. To state this fact precisely, we have a submanifold on which $\Phi^{0}_{M/3,2M/3}$ is a maximum of the energy, i.e. the curve made of sesquisolitons parametrized by $m_1$, namely

$$m_1 \mapsto \Phi^{(m_1)}_{m_1,M-m_1}, \quad m_1 \in (0, M/3],$$

where $x(m_1)$ is the unique positive solution to the equation

$$m_1 = M - m_1 \frac{2}{\text{sech} \left( \frac{M - m_1}{4} x(m_1) \right)}.$$

On the other hand, we know that $\Phi^{0}_{M/3,2M/3}$ restricted to any edge, minimizes the energy at fixed mass on any edge, recall the result on the NLS on the half-line stated in point 4 in section 2. So it is a minimum of the energy restricted to the submanifold of functions with $\Psi_1 = \Psi_2 = \Psi_3$.

The previous result can be extended to all star graphs with a similar construction, and a more systematic analysis of the character of stationary states on star graphs will be given in a future work.

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References

[1] Adami R, Cacciapuoti C, Finco D and Noja D 2011 Fast solitons on star graphs Rev. Math. Phys. 23 409–51
[2] Adami R, Cacciapuoti C, Finco D and Noja D 2011 Stationary states of NLS on star graphs arXiv:1104.3839v2
[3] Berestycki H and Lions P L 1983 Nonlinear scalar field equations I and II Arch. Ration. Mech. Anal. 82 (4) 313–75
[4] Cazenave T 2003 Semilinear Schrödinger Equations (Providence, RI: American Mathematical Society)
[5] Exner P, Keating J P, Kuchment P, Sunada T and Teplyaev A 2008 Analysis on Graphs and Its Applications (Providence, RI: American Mathematical Society)
[6] Gnutzmann S, Smilansky U and Derevyanko S 2011 Stationary scattering from a nonlinear network Phys. Rev. A 83 033831
[7] Kevrekidis P G, Frantzeskakis D J, Theocharis G and Kevrekidis I G 2003 Guidance of matter waves through Y-junctions Phys. Rev. Lett. A 317 51322
[8] Kostrykin V and Schrader R 1999 Kirchhoff’s rule for quantum wires J. Phys. A: Math. Gen. 32 595–630
[9] Kuchment P 2002 Graph models for waves in thin structures Waves Random Media 12 R1–24
[10] Sobirov Z, Matrasulov D, Sabirov K, Sawada S and Nakamura K 2010 Integrable nonlinear Schrödinger equation on simple networks: connection formula at vertices Phys. Rev. E 81 066602
[11] Tokuno A, Oshikawa M and Demler E 2008 Dynamics of one dimensional Bose liquids: Andreev-like reflection at Y-junctions and the absence of the Aharonov–Bohm effect Phys. Rev. Lett. 100 140402
[12] Miroshnichenko A E, Molina M I and Kivshar Y S 2007 Localized modes and bistable scattering in nonlinear network junctions Phys. Rev. E 75 046602
[13] Wan W, Muenzel S and Fleischer W 2010 Wave tunneling and hysteresis in nonlinear junctions Phys. Rev. Lett. 104 073903