A NEW IMPROVEMENT OF HÖLDER INEQUALITY VIA ISOTONIC LINEAR FUNCTIONALS

İMDAT İŞCAN

ABSTRACT. In this paper, new improvement of celebrated Hölder inequality by means of isotonic linear functionals is established. An important feature of the new inequality obtained in here is that many existing inequalities related to the Hölder inequality can be improved via new improvement of Hölder inequality. We also show this in an application.

1. Introduction

The famous Young’s inequality, as a classical result, state that: if $a, b > 0$ and $t \in [0, 1]$, then

\begin{equation}
    a^t b^{1-t} \leq ta + (1-t)b
\end{equation}

with equality if and only if $a = b$. Let $p, q > 1$ such that $1/p + 1/q = 1$. The inequality (1.1) can be written as

\begin{equation}
    ab \leq \frac{a^p}{p} + \frac{b^q}{q}
\end{equation}

for any $x, y \geq 0$. In this form, the inequality (1.2) was used to prove the celebrated Hölder inequality. One of the most important inequalities of analysis is Hölder’s inequality. It contributes wide area of pure and applied mathematics and plays a key role in resolving many problems in social science and cultural science as well as in natural science.

**Theorem 1 (Hölder Inequality for Integrals [8]).** Let $p > 1$ and $1/p + 1/q = 1$. If $f$ and $g$ are real functions defined on $[a, b]$ and if $|f|^p, |g|^q$ are integrable functions on $[a, b]$ then

\begin{equation}
    \int_a^b |f(x)g(x)| \, dx \leq \left( \int_a^b |f(x)|^p \, dx \right)^{1/p} \left( \int_a^b |g(x)|^q \, dx \right)^{1/q},
\end{equation}

with equality holding if and only if $A |f(x)|^p = B |g(x)|^q$ almost everywhere, where $A$ and $B$ are constants.

**Theorem 2 (Hölder Inequality for Sums [8]).** Let $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_n)$ be two positive $n$-tuples and $p, q > 1$ such that $1/p + 1/q = 1$. Then we have

\begin{equation}
    \sum_{k=1}^n a_k b_k \leq \left( \sum_{k=1}^n a_k^p \right)^{1/p} \left( \sum_{k=1}^n b_k^q \right)^{1/q}.
\end{equation}

2000 Mathematics Subject Classification. Primary 26D15; Secondary 26A51.

Key words and phrases. Hölder Inequality, Young Inequality, Integral Inequalities, Hermite-Hadamard Type Inequality.
Equality hold in (1.4) if and only if \(a^p\) and \(b^q\) are proportional.

In [7], İscan gave new improvements for integral ans sum forms of the Hölder inequality as follow:

**Theorem 3.** Let \(p > 1\) and \(\frac{1}{p} + \frac{1}{q} = 1\). If \(f\) and \(g\) are real functions defined on interval \([a, b]\) and if \(|f|^p\), \(|g|^q\) are integrable functions on \([a, b]\) then

\[
\int_a^b |f(x)g(x)| \, dx \leq \frac{1}{b-a} \left\{ \left( \int_a^b (b-x) |f(x)|^p \, dx \right)^{1/p} \left( \int_a^b (x-a) |g(x)|^q \, dx \right)^{1/q} \right\}.
\]

**Theorem 4.** Let \(a = (a_1, \ldots, a_n)\) and \(b = (b_1, \ldots, b_n)\) be two positive n-tuples and \(p, q > 1\) such that \(1/p + 1/q = 1\). Then

\[
\sum_{k=1}^n a_kb_k \leq \frac{1}{n} \left\{ \left( \sum_{k=1}^n k a_k^p \right)^{1/p} \left( \sum_{k=1}^n k b_k^q \right)^{1/q} \right\} + \left( \sum_{k=1}^n (n-k) a_k^p \right)^{1/p} \left( \sum_{k=1}^n (n-k) b_k^q \right)^{1/q}.
\]

2. Hölder’s Inequality for Positive Functionals

Let \(E\) be a nonempty set and \(L\) be a linear class of real valued functions on \(E\) having the following properties
- **L1:** If \(f, g \in L\) then \((\alpha f + \beta g) \in L\) for all \(\alpha, \beta \in \mathbb{R}\);
- **L2:** \(1 \in L\), that is if \(f(t) = 1, t \in E\), then \(f \in L\);
- **L3:** If \(f \in L, E_1 \in L\) then \(f \chi_{E_1} \in L\),

where \(\chi_{E_1}\) is the indicator function of \(E_1\). It follows from L2 and L3 that \(\chi_{E_1} \in L\) for every \(E_1 \in L\).

We also consider positive isotonic linear functionals \(A : L \to \mathbb{R}\) is a functional satisfying the following properties:
- **A1:** \(A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)\) for \(f, g \in L\) and \(\alpha, \beta \in \mathbb{R}\);
- **A2:** If \(f \in L, f(t) \geq 0\) on \(E\) then \(A(f) \geq 0\).

Furthermore, It follows from L3 that for every \(E_1 \in L\) such that \(A(\chi_{E_1}) > 0\), the functional \(A_{E_1}\) is defined for all \(f \in L\) by \(A_{E_1}(f) = A(f \chi_{E_1}) / A(\chi_{E_1})\) is a fixed positive isotonic linear functional with \(A_{E_1}(1) = 1\). We observe that

\[
A(\chi_{E_1}) + A(\chi_{E \setminus E_1}) = 1,
\]

\[
A(f) = A(f \chi_{E_1}) + A(f \chi_{E \setminus E_1}).
\]

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. Functional versions of well-known inequalities and related results could be found in [1][2][3][4][5][6][7].
Example 1. \textbf{i.)} If $E = [a, b] \subseteq \mathbb{R}$ and $L = L[a, b]$, then

$$A(f) = \int_{a}^{b} f(t) dt$$

is an isotonic linear functional.

\textbf{ii.)} If $E = [a, b] \times [c, d] \subseteq \mathbb{R}^2$ and $L = L([a, b] \times [c, d])$, then

$$A(f) = \int_{a}^{b} \int_{c}^{d} f(x, y) dx dy$$

is an isotonic linear functional.

\textbf{iii.)} If $(E, \Sigma, \mu)$ is a measure space with $\mu$ positive measure on $E$ and $L = L(\mu)$ then

$$A(f) = \int_{E} f d\mu$$

is an isotonic linear functional.

\textbf{iv.)} If $E$ is a subset of the natural numbers $\mathbb{N}$ with all $p_k \geq 0$, then $A(f) = \sum_{k \in E} p_k f_k$ is an isotonic linear functional. For example; If $E = \{1, 2, \ldots, n\}$ and $f : E \to \mathbb{R}, f(k) = a_k$, then $A(f) = \sum_{k=1}^{n} a_k$ is an isotonic linear functional. If $E = \{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$ and $f : E \to \mathbb{R}, f(k, l) = a_{k,l}$, then $A(f) = \sum_{k=1}^{n} \sum_{l=1}^{m} a_{k,l}$ is an isotonic linear functional.

\textbf{Theorem 5} (Hölder’s inequality for isotonic functionals \cite{10}). Let $L$ satisfy conditions $L1$, $L2$, and $A$ satisfy conditions $A1$, $A2$ on a base set $E$. Let $p > 1$ and $p^{-1} + q^{-1} = 1$. If $w, f, g \geq 0$ on $E$ and $w f^p, w g^q, w f g \in L$ then we have

$$(2.1) \quad A(w f g) \leq A^{1/p} (w f^p) A^{1/q} (w g^q).$$

In the case $0 < p < 1$ and $A(w g^q) > 0$ (or $p < 0$ and $A(w f^p) > 0$), the inequality in (2.1) is reversed.

\textbf{Remark 1. \textbf{i.)}} If we choose $E = [a, b] \subseteq \mathbb{R}$, $L = L[a, b]$, $w = 1$ on $E$ and $A(f) = \int_{a}^{b} |f(t)| dt$ in the Theorem 2, then the inequality (2.1) reduce the inequality \cite{13}.

\textbf{ii.)} If we choose $E = \{1, 2, \ldots, n\}$, $w = 1$ on $E$, $f : E \to [0, \infty), f(k) = a_k$, and $A(f) = \sum_{k=1}^{n} a_k$ in the Theorem 2, then the inequality (2.1) reduce the inequality (1.4).

\textbf{iii.)} If we choose $E = [a, b] \times [c, d], L = L(E), w = 1$ on $E$ and $A(f) = \int_{a}^{b} \int_{c}^{d} |f(x, y)| dx dy$ in the Theorem 2, then the inequality (2.1) reduce the following inequality for double integrals:

$$\int_{a}^{b} \int_{c}^{d} |f(x, y)||g(x, y)| dx dy \leq \left( \int_{a}^{b} \int_{c}^{d} |f(x, y)|^p dx \right)^{1/p} \left( \int_{a}^{b} \int_{c}^{d} |g(x, y)|^q dx \right)^{1/q}.$$

The aim of this paper is to give a new general improvement of Hölder inequality for isotonic linear functional. As applications, this new inequality will be rewritten for several important particular cases of isotonic linear functionals. Also, we give an application to show that improvement is hold for double integrals.
3. Main results

**Theorem 6.** Let \( L \) satisfy conditions \( L_1, L_2 \), and \( A \) satisfy conditions \( A_1, A_2 \) on a base set \( E \). Let \( p > 1 \) and \( p^{-1} + q^{-1} = 1 \). If \( \alpha, \beta, w, f, g \geq 0 \) on \( E \) and \( \alpha wfg, \beta wfg, \alpha wfg^p, \alpha wg^p, \beta wfg^p, \beta wg^p, wfg \in L \) then we have

\[ \text{(3.1)} \quad A(wfg) \leq A^{1/p}(\alpha wfg^p) A^{1/q}(\alpha wg^p) + A^{1/p}(\beta wfg^p) A^{1/q}(\beta wg^p) \]

\[ \text{ii.)} \quad A^{1/p}(\alpha wfg^p) A^{1/q}(\alpha wg^p) + A^{1/p}(\beta wfg^p) A^{1/q}(\beta wg^p) \leq A^{1/p}(wfg) A^{1/q}(wg^p). \]

**Proof.** i.) By using of Hölder inequality for isotonic functionals in (2.1) and linearity of \( A \), it is easily seen that

\[ A(wfg) = A(\alpha wfg + \beta wfg) = A(\alpha wfg) + A(\beta wfg) \leq A^{1/p}(\alpha wfg^p) A^{1/q}(\alpha wg^p) + A^{1/p}(\beta wfg^p) A^{1/q}(\beta wg^p). \]

ii.) Firstly, we assume that \( A^{1/p}(wfg) A^{1/q}(wg^p) \neq 0 \). Then

\[ A^{1/p}(\alpha wfg^p) A^{1/q}(\alpha wg^p) + A^{1/p}(\beta wfg^p) A^{1/q}(\beta wg^p) \]

\[ = \left( \frac{A(\alpha wfg^p)}{A(wfg)} \right)^{1/p} \left( \frac{A(\alpha wg^p)}{A(wg^p)} \right)^{1/q} + \left( \frac{A(\beta wfg^p)}{A(wfg)} \right)^{1/p} \left( \frac{A(\beta wg^p)}{A(wg^p)} \right)^{1/q}, \]

By the inequality (1.1) and linearity of \( A \), we have

\[ A^{1/p}(wfg) A^{1/q}(wg^p) \leq \frac{1}{p} \left[ A(\alpha wfg^p) + A(\beta wfg^p) \right] + \frac{1}{q} \left[ A(\alpha wg^p) + A(\beta wg^p) \right] \]

\[ = 1. \]

Finally, suppose that \( A^{1/p}(wfg) A^{1/q}(wg^p) = 0 \). Then \( A^{1/p}(wfg) = 0 \) or \( A^{1/q}(wg^p) = 0 \), i.e. \( A(wfg) = 0 \) or \( A(wg^p) = 0 \). We assume that \( A(wfg) = 0 \). Then by using linearity of \( A \) we have,

\[ 0 = A(wfg) = A(\alpha wfg + \beta wfg) = A(\alpha wfg) + A(\beta wfg). \]

Since \( A(\alpha wfg) + A(\beta wfg) \geq 0 \), we get \( A(\alpha wfg) = 0 \) and \( A(\beta wfg) = 0 \). From here, it follows that

\[ A^{1/p}(\alpha wfg^p) A^{1/q}(\alpha wg^p) + A^{1/p}(\beta wfg^p) A^{1/q}(\beta wg^p) \leq 0 = A^{1/p}(wfg) A^{1/q}(wg^p). \]

In case of \( A(wg^p) = 0 \), the proof is done similarly. This completes the proof. \( \square \)

**Remark 2.** The inequality (3.2) shows that the inequality (3.1) is better than the inequality (2.1).

If we take \( w = 1 \) on \( E \) in the Theorem then we can give the following corollary:

**Corollary 1.** Let \( L \) satisfy conditions \( L_1, L_2 \), and \( A \) satisfy conditions \( A_1, A_2 \) on a base set \( E \). Let \( p > 1 \) and \( p^{-1} + q^{-1} = 1 \). If \( \alpha, \beta, f, g \geq 0 \) on \( E \) and \( \alpha f, \beta fg, \alpha f^p, \alpha g^p, \beta f^p, \beta g^p, fg \in L \) then we have
\[ A(fg) \leq A^{1/p}(\alpha f^p) A^{1/q}(\alpha g^q) + A^{1/p}(\beta f^p) A^{1/q}(\beta g^q) \]

Remark 3. i.) If we choose \( E = [a, b] \subseteq \mathbb{R}, L = L[a, b], \alpha(t) = \frac{b-t}{b-a}, \beta(t) = \frac{t-a}{b-a} \) on \( E \) and \( A(f) = \int_a^b |f(t)| \, dt \) in the Corollary, then the inequality (3.3) reduce the inequality (3.2).

ii.) If we choose \( E = \{1, 2, ..., n\}, \alpha(k) = \frac{k}{n}, \beta(k) = \frac{n-k}{n} \) on \( E, f : E \to [0, \infty], f(k) = a_k \), and \( A(f) = \sum_{k=1}^n a_k \) in the Theorem, then the inequality (3.3) reduce the inequality (3.2).

We can give more general form of the Theorem as follows:

**Theorem 7.** Let \( L \) satisfy conditions L1, L2, and \( A \) satisfy conditions A1, A2 on a base set \( E \). Let \( p > 1 \) and \( p^{-1} + q^{-1} = 1 \). If \( \alpha_i, w, f, g \geq 0 \) on \( E, \alpha_i w f g, \alpha_i w f^p, \alpha_i w g^q, w f g \in L, i = 1, 2, ..., m, \) and \( \sum_{i=1}^m \alpha_i = 1 \), then we have

i.)
\[
A(wfg) \leq \sum_{i=1}^m A^{1/p}(\alpha_i w f^p) A^{1/q}(\alpha_i w g^q)
\]

ii.)
\[
\sum_{i=1}^m A^{1/p}(\alpha_i w f^p) A^{1/q}(\alpha_i w g^q) \leq A^{1/p}(w f^p) A^{1/q}(w g^q).
\]

Proof. The proof can be easily done similarly to the proof of Theorem then we can give the following corollary:

**Corollary 2.** Let \( L \) satisfy conditions L1, L2, and \( A \) satisfy conditions A1, A2 on a base set \( E \). Let \( p > 1 \) and \( p^{-1} + q^{-1} = 1 \). If \( \alpha_i, f, g \geq 0 \) on \( E, \alpha_i f g, \alpha_i f^p, \alpha_i g^q, f, g \in L, i = 1, 2, ..., m, \) and \( \sum_{i=1}^m \alpha_i = 1 \), then we have

i.)
\[
A(fg) \leq \sum_{i=1}^m A^{1/p}(\alpha_i f^p) A^{1/q}(\alpha_i g^q)
\]

ii.)
\[
\sum_{i=1}^m A^{1/p}(\alpha_i f^p) A^{1/q}(\alpha_i g^q) \leq A^{1/p}(f^p) A^{1/q}(g^q).
\]

**Corollary 3** (Improvement of Hölder inequality for double integrals). Let \( p, q > 1 \) and \( 1/p + 1/q = 1 \). If \( f \) and \( g \) are real functions defined on \( E = [a, b] \times [c, d] \) and if \( |f|^p, |g|^q \in L(E) \) then

\[
\int_a^b \int_c^d |f(x, y)| |g(x, y)| \, dx \, dy \leq \sum_{i=1}^4 \left( \int_a^b \int_c^d \alpha_1(x, y) |f(x, y)|^p \, dx \, dy \right)^{1/p} \left( \int_a^b \int_c^d \alpha_2(x, y) |g(x, y)|^q \, dx \, dy \right)^{1/q},
\]

where \( \alpha_1(x, y) = \frac{(b-x)(d-y)}{(b-a)(d-c)}, \alpha_2(x, y) = \frac{(b-x)(y-c)}{(b-a)(d-c)}, \alpha_3(x, y) = \frac{(x-a)(y-c)}{(b-a)(d-c)}, \alpha_4(x, y) = \frac{(x-a)(d-y)}{(b-a)(d-c)} \) on \( E \)
Proof. If we choose $E = [a, b] \times [c, d] \subseteq \mathbb{R}^2$, $L = L(E)$, $\alpha_1(x, y) = \frac{(b-x)(d-y)}{(b-a)(d-c)}$, $\alpha_2(x, y) = \frac{(b-x)(y-c)}{(b-a)(d-c)}$, $\alpha_3(x, y) = \frac{(x-a)(y-c)}{(b-a)(d-c)}$, $\alpha_4(x, y) = \frac{(x-a)(d-y)}{(b-a)(d-c)}$ on $E$ and $A(f) = \int_a^b \int_c^d |f(x, y)| \, dx \, dy$ in the Corollary [1] then we get the inequality (3.5).

Corollary 4. Let $(a_{k,l})$ and $(b_{k,l})$ be two tuples of positive numbers and $p, q > 1$ such that $1/p + 1/q = 1$. Then we have

\[
\sum_{k=1}^n \sum_{l=1}^m a_{k,l} b_{k,l} \leq 4 \left( \sum_{k=1}^n \sum_{l=1}^m \alpha_1(k, l)a_{k,l}^p \right)^{1/p} \left( \sum_{k=1}^n \sum_{l=1}^m \alpha_2(k, l)b_{k,l}^q \right)^{1/q},
\]

where $\alpha_1(k, l) = \frac{kt}{nm}$, $\alpha_2(k, l) = \frac{(n-k)t}{nm}$, $\alpha_3(k, l) = \frac{(n-k)(m-l)}{nm}$, $\alpha_4(k, l) = \frac{k(m-l)}{nm}$ on $E$.

Proof. If we choose $E = \{1, 2, ..., n\} \times \{1, 2, ..., m\}$, $\alpha_1(k, l) = \frac{kt}{nm}$, $\alpha_2(k, l) = \frac{(n-k)t}{nm}$, $\alpha_3(k, l) = \frac{(n-k)(m-l)}{nm}$, $\alpha_4(k, l) = \frac{k(m-l)}{nm}$ on $E$, $f : E \rightarrow [0, \infty)$, $f(k, l) = a_{k,l}$, and $A(f) = \sum_{k=1}^n \sum_{l=1}^m a_{k,l}$ in the Theorem [1] then we get the inequality (3.6). □

4. An Application for Double Integrals

In [1], Sarıkaya et al. gave the following lemma for obtain main results.

Lemma 1. Let $f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\frac{\partial^2 f}{\partial t \partial s} \in L(\Delta)$, then the following equality holds:

\[
\frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dx \, dy
\]

\[
- \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b [f(x, c) + f(x, d)] \, dx + \frac{1}{d-c} \int_c^d [f(a, y) + f(b, y)] \, dy \right]
\]

\[
= \frac{(b-a)(d-c)}{4} \int_0^1 \int_0^1 (1 - 2t)(1 - 2s) \frac{\partial^2 f}{\partial t \partial s} (ta + (1-t)b, sc + (1-s)d) \, dt \, ds.
\]

By using this equality and Hölder integral inequality for double integrals, Sarıkaya et al. obtained the following inequality:

Theorem 8. Let $f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\left| \frac{\partial^2 f}{\partial t \partial s} \right|^q \in L(\Delta)$, $q > 1$, is convex function on the co-ordinates on $\Delta$, then one has the inequalities:

\[
\frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dx \, dy - A
\]

\[
\leq \frac{(b-a)(d-c)}{4(p+1)^{2/p}} \left[ |f_{st}(a, c)|^q + |f_{st}(a, d)|^q + |f_{st}(b, c)|^q + |f_{st}(b, d)|^q \right]^{1/q},
\]

where

\[
A = \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b |f(x, c) + f(x, d)| \, dx + \frac{1}{d-c} \int_c^d |f(a, y) + f(b, y)| \, dy \right],
\]

$1/p + 1/q = 1$ and $f_{st} = \frac{\partial^2 f}{\partial t \partial s}$. 

If Theorem 8 are resulted again by using the inequality (3.5), then we get the following result:

**Theorem 9.** Let $f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\frac{\partial^2 f}{\partial t \partial s}, q > 1$, is convex function on the co-ordinates on $\Delta$, then one has the inequalities:

\[
\begin{align*}
\left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} - \frac{1}{(b - a)(d - c)} \int_c^d \int_a^b f(x, y) dxdy - A \right| \\
\leq \frac{(b - a)(d - c)}{4^{1+1/p}(p + 1)^2/p} \left\{ \frac{1}{36} \left[ 4|f_{st}(a, c)|^q + 2|f_{st}(a, d)|^q + 2|f_{st}(b, c)|^q + |f_{st}(b, d)|^q \right]^{1/q} \\
+ \frac{2|f_{st}(a, c)|^q + |f_{st}(a, d)|^q + 2|f_{st}(b, c)|^q + 2|f_{st}(b, d)|^q}{36}^{1/q} \\
+ \frac{2|f_{st}(a, c)|^q + 4|f_{st}(a, d)|^q + |f_{st}(b, c)|^q + 2|f_{st}(b, d)|^q}{36}^{1/q} \\
+ \frac{|f_{st}(a, c)|^q + 2|f_{st}(a, d)|^q + 2|f_{st}(b, c)|^q + 4|f_{st}(b, d)|^q}{36}^{1/q} \right\},
\end{align*}
\]

where

\[ A = \frac{1}{2} \left[ \frac{1}{b - a} \int_a^b [f(x, c) + f(x, d)] dx + \frac{1}{d - c} \int_c^d [f(a, y) + f(b, y)] dy \right], \]

$1/p + 1/q = 1$ and $f_{st} = \frac{\partial^2 f}{\partial t \partial s}$.

**Proof.** Using Lemma 1 and the inequality (3.5), we find

\[
\begin{align*}
\left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} - \frac{1}{(b - a)(d - c)} \int_c^d \int_a^b f(x, y) dxdy - A \right| \\
\leq \frac{(b - a)(d - c)}{4} \int_0^1 \int_0^1 \left| 1 - 2t \right| \left| 1 - 2s \right| |f_{st} (ta + (1 - t)b, sc + (1 - s))| dt ds,
\end{align*}
\]
\[
\left( \int_0^1 \int_0^1 ts |1 - 2t|^p |1 - 2s|^p \, dt \, ds \right)^{1/p} \times \left( \int_0^1 \int_0^1 t(1-s) |1 - 2t|^p |1 - 2s|^p \, dt \, ds \right)^{1/q} \times \left( \int_0^1 \int_0^1 (1-t) |f_{st} (t \cdot a + (1-t)b, sc + (1-s))|^q \, dt \, ds \right)^{1/p} \times \left( \int_0^1 \int_0^1 (1-t)(1-s) |f_{st} (t \cdot a + (1-t)b, sc + (1-s))|^q \, dt \, ds \right)^{1/q}.
\]

Since \(|f_{st}|^q\) is convex function on the co-ordinates on \(\Delta\), we have for all \(t, s \in [0, 1]\)
\[
|f_{st}(ta + (1-t)b, sc + (1-s))|^q \leq t s |f_{st}(a, c)|^q + t(1-s) |f_{st}(a, d)|^q + (1-t)s |f_{st}(a, c)|^q + (1-t)(1-s) |f_{st}(a, c)|^q
\]
for all \(t, s \in [0, 1]\). Further since
\[
\int_0^1 \int_0^1 ts |1 - 2t|^p |1 - 2s|^p \, dt \, ds = \int_0^1 \int_0^1 t(1-s) |1 - 2t|^p |1 - 2s|^p \, dt \, ds
\]
(4.5)
\[
= \int_0^1 \int_0^1 (1-t) s |1 - 2t|^p |1 - 2s|^p \, dt \, ds
\]
\[
= \int_0^1 \int_0^1 (1-t)(1-s) |1 - 2t|^p |1 - 2s|^p \, dt \, ds
\]
(4.6)
\[
= \frac{1}{4(p+1)^4},
\]
a combination of (4.3) - (4.5) immediately gives the required inequality (4.2). \(\square\)

**Remark 4.** Since \(\eta : [0, \infty) \to \mathbb{R}, \eta(x) = x^s, 0 < s \leq 1\), is a concave function, for all \(u, v \geq 0\) we have
\[
\eta \left( \frac{u + v}{2} \right) = \left( \frac{u + v}{2} \right)^s \geq \frac{\eta(u) + \eta(v)}{2} = \frac{u^s + v^s}{2}.
\]
From here, we get
\[
I = \left\{ \frac{4 |f_{st}(a,c)|^q + 2 |f_{st}(a,d)|^q + 2 |f_{st}(b,c)|^q + |f_{st}(b,d)|^q}{36} \right\}^{1/q} \\
+ \left\{ \frac{2 |f_{st}(a,c)|^q + |f_{st}(a,d)|^q + 4 |f_{st}(b,c)|^q + 2 |f_{st}(b,d)|^q}{36} \right\}^{1/q} \\
+ \left\{ \frac{2 |f_{st}(a,c)|^q + 4 |f_{st}(a,d)|^q + |f_{st}(b,c)|^q + 2 |f_{st}(b,d)|^q}{36} \right\}^{1/q} \\
+ \left\{ \frac{|f_{st}(a,c)|^q + 2 |f_{st}(a,d)|^q + 2 |f_{st}(b,c)|^q + 4 |f_{st}(b,d)|^q}{36} \right\}^{1/q} \right\}
\leq 2 \left\{ \frac{6 |f_{st}(a,c)|^q + 3 |f_{st}(a,d)|^q + 6 |f_{st}(b,c)|^q + 3 |f_{st}(b,d)|^q}{72} \right\}^{1/q} \\
+ \left\{ \frac{3 |f_{st}(a,c)|^q + 6 |f_{st}(a,d)|^q + 3 |f_{st}(b,c)|^q + 6 |f_{st}(b,d)|^q}{72} \right\}^{1/q} \right\}
\leq 4 \left\{ \frac{|f_{st}(a,c)|^q + |f_{st}(a,d)|^q + |f_{st}(b,c)|^q + |f_{st}(b,d)|^q}{16} \right\}^{1/q}
\]

Thus we obtain
\[
\frac{(b-a)(d-c)}{4^{1+1/p}(p+1)^{2/p}} I \\
\leq \frac{(b-a)(d-c)}{4^{1+1/p}(p+1)^{2/p}} \left\{ \frac{|f_{st}(a,c)|^q + |f_{st}(a,d)|^q + |f_{st}(b,c)|^q + |f_{st}(b,d)|^q}{16} \right\}^{1/q} \\
\leq \frac{(b-a)(d-c)}{4(p+1)^{2/p}} \left\{ \frac{|f_{st}(a,c)|^q + |f_{st}(a,d)|^q + |f_{st}(b,c)|^q + |f_{st}(b,d)|^q}{4} \right\}^{1/q}. 
\]

This show us that the inequality (4.2) is better than the inequality (4.1).

References

[1] S. Abramovich, J.E. Pečarić, and S. Varošanec, Sharpening Hölder’s and Popoviciu’s inequalities via functionals, The Rocky Mountain Journal of Mathematics, 34(3) (2004), 793-810.
[2] L. Ciurdariu, Some refinements of Hölder’s inequalities via isotonic linear functionals, Journal of Science and Arts 14(3) (2014), 221-228.
[3] L. Ciurdariu, Several Applications of Young-Type and Hölder’s Inequalities, Applied Mathematical Sciences 10(36) (2016), 1763-1774.
[4] S.S. Dragomir, A Grüss type inequality for isotonic linear functionals and applications, Demonstratio Mathematica 36(3) (2003), 551-562.
[5] S.S. Dragomir, Some results for isotonic functionals via an inequality due to Kittaneh and Manasrah, Fasciculi Mathematici 59(1) (2017), 29-42.
[6] S.S. Dragomir, M.A. Khan, and A. Abathun, Refinement of the Jensen integral inequality, Open Mathematics 14(1) (2016), 221-228.
[7] İ. İşcan, New Refinements for integral and sum forms of Hölder inequality, arXiv:1901.05841 [math.GM].
[8] D.S. Mitrinović, J.E. Pečarić, and A.M. Fink. Classical and new inequalities in analysis, Kluwer Academic Publishers, Dordrecht, Boston, London, 1993.
[9] J.E. Pečarić, Generalization of the power means and their inequalities, Journal of Mathematical Analysis and Applications 161(2) (1991), 395-404.
[10] J.E. Pečarić, F. Proschan and Y.L. Tong, Convex functions, partial orderings and statistical applications, Academic Press, New York, 1992.

[11] M.Z. Sarıkaya, E. Set, M.E. Özdemir and S.S. Dragomir, New Some Hadamard’s Type Inequalities for Co-ordinated Convex Functions, Tamsui Oxford Journal of Information and Mathematical Sciences 28(2) (2012) 137-152.

DEPARTMENT OF MATHEMATICS, FACULTY OF ARTS AND SCIENCES, GİRESUN UNIVERSITY, 28200, GİRESUN, TURKEY.

E-mail address: imdati@yahoo.com, imdat.iscan@giresun.edu.tr