THE $\theta$-CONGRUENT NUMBERS ELLIPTIC CURVES VIA A
FERMAT-TYPE THEOREM

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Abstract. A positive integer $N$ is called a $\theta$-congruent number if there is a $\theta$-triangle $(a, b, c)$ with rational sides for which the angle between $a$ and $b$ is equal to $\theta$ and its area is $N\sqrt{r^2 - s^2}$, where $\theta \in (0, \pi)$, $\cos(\theta) = s/r$, and $0 \leq |s| < r$ are coprime integers. It is attributed to Fujiwara [4] that $N$ is a $\theta$-congruent number if and only if the elliptic curve $E_N^\theta : y^2 = x(x + (r + s)N)(x - (r - s)N)$ has a point of order greater than 2 in its group of rational points. Moreover, a natural number $N \neq 1, 2, 3, 6$ is a $\theta$-congruent number if and only if rank of $E_N^\theta(\mathbb{Q})$ is greater than zero.

In this paper, we answer positively to a question concerning with the existence of methods to create new rational $\theta$-triangle for a $\theta$-congruent number $N$ from given ones by generalizing the Fermat’s algorithm, which produces new rational right triangles for congruent numbers from a given one, for any angle $\theta$ satisfying the above conditions. We show that this generalization is analogous to the duplication formula in $E_N^\theta(\mathbb{Q})$. Then, based on the addition of two distinct points in $E_N^\theta(\mathbb{Q})$, we provide a way to find new rational $\theta$-triangles for the $\theta$-congruent number $N$ using given two distinct ones. Finally, we give an alternative proof for the Fujiwara’s theorem 2.2 and one side of Theorem 2.3. In particular, we provide a list of all torsion points in $E_N^\theta(\mathbb{Q})$ with corresponding rational $\theta$-triangles.

Keywords: $\theta$-congruent number, elliptic curve, Fermat type theorem, Pythagorean $\theta$-triples

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1. Introduction

A positive integer $N$ is called a congruent number if it is equal to the area of a right triangle with rational sides; equivalently, if there exist positive rational numbers $a, b$ and $c$ such that $a < b < c$, and

$$a^2 + b^2 = c^2, \quad ab = 2N.$$  

Determining all congruent numbers is an old problem in the theory of numbers and there are different types of generalizations in the literature, see for example [1][5][10]. Specifically, in [4], Fujiwara introduced the notion of $\theta$-congruent numbers for an angle $\theta \in (0, \pi)$ with rational cosine as a generalization of the usual congruent numbers. To be precise, assume that $\theta \in (0, \pi)$ is an angle of a triangle given by a triple $(a, b, c)$ of positive rational numbers for which $\cos(\theta) = s/r$ with $r, s \in \mathbb{Z}$ satisfying $0 \leq |s| < r$ and gcd$(r, s) = 1$. The case $\theta = \pi/2$ refers to the usual congruent numbers. We call such a triple $(a, b, c)$ a rational $\theta$-triangle. A positive integer $N$ is called a $\theta$-congruent number if there exists a rational $\theta$-triangle with area $N\sqrt{r^2 - s^2}$, where $\theta$ is the angle between the sides $a$ and $b$; equivalently, if there is a triple $(a, b, c)$ of (positive) rational numbers satisfying

$$a^2 + b^2 - \frac{2s}{r}ab = c^2, \quad ab = 2rN. \tag{1}$$
A triple \((a, b, c)\) of rational numbers satisfying (1) it is called a \textit{rational \(\theta\)-triangle} for \(N\). It is also said that \((a, b, c)\) is a \textit{rational \(\theta\)-triangle} for \(N\), if \(a, b, c\) are all positive rational numbers. Note that if we have a \(\theta\)-triangle \((a, b, c)\) for \(N\), then \((|a|, |b|, |c|)\) is a rational \(\theta\)-triangle for \(N\). We identify the \(\theta\)-triples \((a, b, c)\) and \((b, a, c)\) since the equations (1) are symmetric in \(a\) and \(b\). It is clear that if a positive integer \(N\) is a \(\theta\)-congruent number with a \(\theta\)-triangle \((a, b, c)\), then \(N m^2\) is also a \(\theta\)-congruent number with the rational \(\theta\)-triangle \((ma, mb, mc)\). Hence, one may assume the square-free positive numbers in the study of \(\theta\)-congruent numbers.

There are some natural questions related to the rational \(\theta\)-triples and hence \(\theta\)-triangles for \(\theta\)-congruent numbers.

**Question 1.1.** Let \(\theta \in (0, \pi)\) be an angle with rational cosine and assume that \(N\) is a \(\theta\)-congruent number.

(i) Is it possible to introduce new rational \(\theta\)-triples for \(N\) using a given one? If yes, can one repeat this procedure infinitely many times?

(ii) Is it possible to find new rational \(\theta\)-triples for \(N\) using given two distinct ones?

The part (i) of Question 1.1 in the case \(\theta = \pi/2\), was answered around 360 years ago by Fermat [3, 8]. Indeed, without any proof, he provided an algorithm which is capable to produce infinitely many different right triangles for a given congruent number \(N\) from a given right triangle. In [7], Halbeisen and Hungerbühler provided a detailed proof for the Fermat’s algorithm. They used their results to give an elementary proof of the fact that if \(N\) is a congruent number and \(P_0 = (x_0, y_0)\) with \(y_0 \neq 0\) is a rational point on the congruent number elliptic curve \(E_N : y^2 = x^3 - N^2 x\), then \(P_0\) cannot be of finite order; in other words, it is a point of infinite order in the Mordell-Weil group \(E_N(\mathbb{Q})\) and so it does not belong to the set \(T_N(\mathbb{Q}) = \{\infty, (0, 0), (-N, 0), (N, 0)\}\), called the torsion subgroup of \(E_N(\mathbb{Q})\), where \(\infty = (0 : 1 : 0)\) denotes the point at infinity. In the recent work [1], for \(\theta = \pi/2\), Chan gave a positive answer to the part (ii) of Question 1.1 using geometric methods.

In [4, 5], Fujiwara proved a theorem (see Theorem 2.2) on the \(\theta\)-congruent number elliptic curves defined by

\[E^\theta_N : y^2 = x(x + (r + s)N)(x - (r - s)N),\]

where \(r\) and \(s\) are as mentioned earlier, relating the \(\theta\)-congruent number problem to the existence of a non-2-torsion point in \(E^\theta_N(\mathbb{Q})\). It is clear that the points \((0, 0), (-r + s)N, 0\) and \((r - s)N, 0\) are the only 2-torsion points in \(E^\theta_N(\mathbb{Q})\). This implies that \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\) is contained in \(T^\theta_N(\mathbb{Q})\), the torsion subgroup of \(E^\theta_N(\mathbb{Q})\). Thus, based on the Mazur’s theorem 2.1 on the classification of torsion subgroups of elliptic curves over \(\mathbb{Q}\), the only possibilities for \(T^\theta_N(\mathbb{Q})\) are \(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z}\) with \(1 \leq n \leq 4\). In [5], Fujiwara determined certain conditions on \(\theta\) and \(N\) so that \(T^\theta_N(\mathbb{Q})\) is isomorphic to one of the above possibilities, see Theorem 2.3.

We note that Fujiwara did not provide explicitly the points of \(T^\theta_N(\mathbb{Q})\) in his proof.

In this paper, we give a positive answer to the both parts of Question 1.1 for any \(\theta \in (0, \pi)\) with rational cosine and any \(\theta\)-congruent number \(N\). To do this, in Section 2, we briefly review the arithmetic of \(\theta\)-congruent numbers and the Fujiwara’s results. Then, in Section 3, we answer to the part (i) of Question 1.1 by generalizing the Fermat’s algorithm for the rational \(\theta\)-triangles. The Section 4 is devoted to show the analogy between our generalization of the Fermat’s algorithm and the duplication formula in \(E^\theta_N(\mathbb{Q})\). We also show how to use the addition of two distinct points in \(E^\theta_N(\mathbb{Q})\) to give a positive answer to the part (ii) of Question 1.1. In the last section, we provide an alternative proof for the Fujiwara’s theorem 2.2 and the right-to-left side of Theorem 2.3 using the rational \(\theta\)-triples. We also list all possible torsion points in \(E^\theta_N(\mathbb{Q})\) with corresponding rational \(\theta\)-triples.
2. Elliptic curves and \( \theta \)-congruent numbers

An elliptic curve over \( \mathbb{Q} \) is a genus one smooth algebraic (plane) curve defined by the affine short Weierstrass form

\[
E : \quad y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Z}, \quad 4a^3 + 27b^2 \neq 0,
\]

together with the point \( \infty \). The set of rational points on \( E \), denoted by \( E(\mathbb{Q}) \), forms an Abelian group with respect to the chord-tangent addition law. To see more on the geometry of elliptic curves, we cite the reader to \([12]\). By a celebrated theorem of Mordell and Weil \([12]\), the set \( E(\mathbb{Q}) \) is a finitely generated Abelian group with the identity \( \infty \); in other words, we have

\[
E(\mathbb{Q}) \cong T(E, \mathbb{Q}) \times \mathbb{Z}^{r(E)},
\]

where \( T(E, \mathbb{Q}) \) is a finite group called the torsion subgroup of \( E(\mathbb{Q}) \) and \( r(E) \) is a non-negative integer called the Mordell-Weil rank of \( E(\mathbb{Q}) \). Determining the Mordell-Weil rank of elliptic curves over \( \mathbb{Q} \) is a very challenging problem since there is no algorithm to determine it in general. But, in contrast, Mazur classified all possibilities for torsion subgroups of elliptic curves over \( \mathbb{Q} \) by studying modular curves in a groundbreaking work \([11]\). Here is the resume of Mazur’s results.

**Theorem 2.1** (Mazur). Let \( E \) be an elliptic curve over \( \mathbb{Q} \). Then, the torsion subgroup \( T(E, \mathbb{Q}) \) is one of the following finite groups:

\begin{itemize}
  \item[(i)] \( \mathbb{Z}/n\mathbb{Z} \) with \( 1 \leq n \leq 10 \) or \( n = 12 \),
  \item[(ii)] \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \) with \( 1 \leq n \leq 4 \).
\end{itemize}

Let us recall the algebraic version of the addition law and the duplication formula in \( E_N^\theta(\mathbb{Q}) \). The sum \( P_0 + P_1 = (x_2, y_2) \), of any two points \( P_0 = (x_0, y_0) \) and \( P_1 = (x_1, y_1) \) in \( E_N^\theta(\mathbb{Q}) \), is given by the following rules:

\begin{itemize}
  \item[(1)] If \( x_0 \neq x_1 \), then \( x_2 = \lambda^2 - 2sN - x_0 - x_1 \) and \( y_2 = -(\lambda x_2 + \nu) \), where

\[
\lambda = \frac{y_1 - y_0}{x_1 - x_0}, \quad \text{and} \quad \nu = y_0 - \lambda x_0,
\]

\item[(2)] If \( P_0 = P_1 \), then \( x_2 = \lambda^2 - 2sN - 2x_0 \) and \( y_2 = -\lambda^3 + (3x_0 + 2sN)\lambda - y_0 \), where

\[
\lambda = \frac{3x_0^2 + 4sNx_0 - (r^2 - s^2)N^2}{2y_0}.
\]
\end{itemize}

For the \( \theta \)-congruent number elliptic curve \( E_N^\theta \), we denote by \( T_N^\theta(\mathbb{Q}) \) the torsion subgroup and by \( r_N^\theta(\mathbb{Q}) \) the Mordell-Weil rank of \( E_N^\theta(\mathbb{Q}) \). In the recent decades, the notion of \( \theta \)-congruent numbers has attracted the interests of some mathematicians, consult \([2, 4, 5, 6, 9]\). Among other results, Fujiwara \([4]\) showed the following relation between \( \theta \)-congruent numbers and \( r_N^\theta(\mathbb{Q}) \).

**Theorem 2.2** (Fujiwara). Let \( \theta \in (0, \pi) \) be an angle with rational cosine.

\begin{itemize}
  \item[(1)] A positive integer \( N \) is a \( \theta \)-congruent number if and only if \( E_N^\theta(\mathbb{Q}) \) has a point of order greater than 2,
  \item[(2)] If \( N \nmid 6 \), then \( N \) is a \( \theta \)-congruent number if and only if \( r_N^\theta(\mathbb{Q}) > 0 \) for the group \( E_N^\theta(\mathbb{Q}) \).
\end{itemize}

Moreover, in \([5]\), the torsion subgroups of \( \theta \)-congruent elliptic curves are classified by Fujiwara as follows.

**Theorem 2.3** (Fujiwara). For \( T_N^\theta(\mathbb{Q}) \), one of the following cases happens.
(1) \( T^\theta_N(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \) if and only if there exist integers \( u,v > 0 \) such that \( \gcd(u,v) = 1 \), \( u \) and \( v \) have opposite parity and satisfy either of the following:

(i) \( N = 1, r = 8u^4v^4, r - s = (u^2 - v^2)^4, (1 + \sqrt{2})v > u > v \),

(ii) \( N = 2, r = (u^2 - v^2)^4, r - s = 32u^4v^4, u > (1 + \sqrt{2})v \),

(2) \( T^\theta_N(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \) if and only if there exist integers \( u,v > 0 \) such that \( \gcd(u,v) = 1 \), \( u > 2v \) and satisfy either of the following:

(i) \( N = 1, r = (u - v)^3(u + v)/2, r + s = u^3(u - 2v) \),

(ii) \( N = 2, r = (u - v)^3(u + v), r + s = 2u^3(u - 2v) \),

(iii) \( N = 3, r = (u - v)^3(u + v)/6, r + s = u^3(u - 2v)/3 \),

(iv) \( N = 6, r = (u - v)^3(u + v)/3, r + s = 2u^3(u - 2v)/3 \),

(3) \( T^\theta_N(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \) if and only if

(i) \( N = 1, 2r \) and \( r - s \) are squares but not satisfy (1)(i),

(ii) \( N = 2, r \) and \( 2(r - s) \) are squares but not satisfy (1)(ii).

(4) Otherwise, \( T^\theta_N(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \).

Here, we recall the following one-to-one correspondence between two sets with different objects which has crucial role in the proof of the above theorems and the results of the current paper. For \( \theta \in (0, \pi) \) with rational cosine and a \( \theta \)-congruent number \( N \), we consider the sets:

\[
\mathcal{T}_N^\theta = \left\{ (a,b,c) \in \mathbb{Q}^3 \setminus (0,0,0) : c^2 = a^2 + b^2 - \frac{2s}{r}ab, ab = 2rN \right\},
\]

\[
\mathcal{P}_N^\theta = \left\{ (x,y) \in \mathbb{Q}^2 : (x,y) \in E_N^\theta(\mathbb{Q}) \setminus \{\infty\}, y \neq 0 \right\}.
\]

Then, by a straightforward computation, one can easily check that the sets \( \mathcal{T}_N^\theta \) and \( \mathcal{P}_N^\theta \) are bijective via the following maps:

\[
\phi : (x,y) \mapsto \left( \frac{2rxN}{x}, \frac{x^2 + (r^2 - s^2)N^2}{y} \right),
\]

\[
\psi : (a,b,c) \mapsto \left( \frac{rN(a + c - sb/r)}{b}, \frac{2r^2N^2(a + c - sb/r)}{b^2} \right).
\]

Using the fact \( a/2 = rN/b \), we can rewrite

\[
\psi(a,b,c) = \left( \frac{a(a + c - sb/r)}{2}, \frac{a^2(a + c - sb/r)}{2} \right).
\]

Note that in the set \( \mathcal{P}_N^\theta \), it is allowed \( a,b,c \) to be negative rational numbers and that \( a,b \) have the same sign since their product is a positive integer. Hence, all the four \( \theta \)-triples

\[(a,b,c), (a,b,-c), (-a,-b,c), (-a,-b,-c) \in \mathcal{P}_N^\theta \]

correspond to the rational \( \theta \)-triangle \((|a|,|b|,|c|)\) for the \( \theta \)-congruent number \( N \). It is an easy fact that \( a + c - sb/r > 0 \) for a given rational \( \theta \)-triangle \((a,b,c)\) for \( N \). This means that any rational \( \theta \)-triangle \((a,b,c)\) is mapped by \( \psi \) on a point \((x,y)\) with positive coordinates.

We end this section by the following remark on \( \theta \)-triples.

**Remark 2.4.** Using the equation (1) it is easy to see that there exists a rational \( \theta \)-triple \((a,b,c)\) with \(|a| = |b|\) for a \( \theta \)-congruent number \( N \) if and only if the rational numbers \( 2r(r-s) \) and \( 2r(r+s) \) are squares. Indeed, we have

\[
a = b \iff 2rN = a^2, \quad 2r(r-s) = (rc/a)^2,
\]

\[
a = -b \iff 2rN = a^2, \quad 2r(r+s) = (rc/a)^2.
\]
In this case, the $\theta$-triple is given by

$$(a, \mp a, c) = (\sqrt{2rN}, \sqrt{2rN}, 2\sqrt{(r \pm s)N}).$$

In particular, there exists a rational $\theta$-triangle $(a, b, c)$ with $a = b = c$ for a $\theta$-congruent number $N$ if and only if $\theta = \pi/3$, $a = 2$ and $N = 1$.

3. Generalizing the Fermat’s algorithm for rational $\theta$-triples

Let us recall the Fermat’s algorithm [3, 8] which gives infinitely many distinct right triangles for the congruent numbers if only one of them exists.

**Theorem 3.1.** Suppose that $N$ is a congruent number and $(a_0, b_0, c_0)$ is a rational right triangle for $N$. Then,

$$a_1 = \frac{2c_0a_0b_0}{b_0^2 - a_0^2} = \frac{4c_0N}{\sqrt{c_0^4 - 16N^2}},$$

$$b_1 = \frac{c_0^4 - 4a_0^2b_0^2}{2c_0(b_0^2 - a_0^2)} = \frac{\sqrt{c_0^4 - 16N^2}}{2c_0},$$

$$c_1 = \frac{c_0^4 + 4a_0^2b_0^2}{2c_0(b_0^2 - a_0^2)} = \frac{c_0^4 + 16N^2}{2c_0\sqrt{c_0^4 - 16N^2}},$$

give another rational right triangle for $N$. Furthermore, this procedure leads to infinitely many different rational right triangles for $N$.

First, we adapt the Fermat’s algorithm to the rational $\theta$-triples associated to $\theta$-congruent numbers for any $\theta \in (0, \pi)$. The following theorem gives a naive algorithm for constructing new rational $\theta$-triples for the $\theta$-congruent numbers from given ones.

**Theorem 3.2.** Suppose that $\theta \in (0, \pi)$ has a rational cosine and $(a_0, b_0, c_0)$ is a rational $\theta$-triple for a $\theta$-congruent number $N$. If $b_0^2 - a_0^2 \neq 0$, then

$$a_1 = \frac{c_0^4 - 4(r^2 - s^2)a_0^2b_0^2/r^2 + 4sa_0b_0c_0^2/r}{2c_0(b_0^2 - a_0^2)},$$

$$b_1 = \frac{4c_0^3a_0b_0}{2c_0(b_0^2 - a_0^2)},$$

$$c_1 = \frac{c_0^4 + 4(r^2 - s^2)a_0^2b_0^2/r^2}{2c_0(b_0^2 - a_0^2)},$$

is another rational $\theta$-triple for $N$. It is different from $(a_0, b_0, c_0)$ provided that $b_0^2 - a_0^2 \neq 2a_0c_0$. In particular, $(|a_1|, |b_1|, |c_1|)$ is a rational $\theta$-triangle for $N$.

**Proof.** Given any $m, n \in \mathbb{Q}^*$, defining $X, Y$ and $Z$ as

$$X = m^2 - (\frac{r^2 - s^2}{r^2})n^2 + \frac{2smn}{r}, \quad Y = 2mn, \quad Z = m^2 + (\frac{r^2 - s^2}{r^2})n^2,$$
one can get a rational \( \theta \)-triple for \( N_0 := \langle XY/(2r) \rangle \). It is an easy task to check out the relation
\[
X^2 + Y^2 - 2sXY/r = Z^2.
\]
Setting \( m = c_0^2 \) and \( n = 2a_0b_0 \) implies that
\[
X = c_0^4 - 4 \left( \frac{r^2 - s^2}{r^2} \right) a_0^2 b_0^2 + 4 \frac{s}{r} c_0^2 a_0 b_0 ,
\]
\[
Y = 4c_0^2 a_0 b_0 ,
\]
\[
Z = c_0^4 + 4 \left( \frac{r^2 - s^2}{r^2} \right) a_0^2 b_0^2 .
\]
Thus, we have
\[
N_0 = \left| \frac{XY}{2r} \right| = \left| \frac{2c_0^2 a_0 b_0 }{r} \left( \frac{2}{c_0} - \frac{2(r-s)a_0 b_0}{r} \right) \left( \frac{2}{c_0} + \frac{2(r+s)a_0 b_0}{r} \right) \right| .
\]
Since \( c_0^2 = a_0^2 + b_0^2 - 2s a_0 b_0/r \), it follows
\[
N_0 = \left| 2a_0 b_0 c_0^2 (a_0^2 - b_0^2) ^2 /r \right| ,
\]
which is nonzero by the assumption. Now, we define \( (a_1, b_1, c_1) := \left( X/d_0, Y/d_0, Z/d_0 \right) \), where \( d_0 = 2c_0 (b_0^2 - a_0^2) \neq 0 \), again by the same assumption. Clearly, we have \( a_1^2 + b_1^2 - 2s a_1 b_1/r = c_1^2 \) and
\[
a_1 b_1 = \frac{XY}{2r} \frac{2r N_0}{2r} = \frac{4a_0 b_0 c_0^2 (b_0^2 - a_0^2)^2}{2r} = \frac{a_0 b_0}{2 r} .
\]
Thus, both the \( \theta \)-triangles \( |a_0|, |b_0|, |c_0| \) and \( |a_1|, |b_1|, |c_1| \) have the same area \( N \sqrt{r^2 - s^2} \).

We note that if \( 2a_0 c_0 = b_0^2 - a_0^2 \), then \( b_1 = b_0 \) and using \( c_0^2 = a_0^2 + b_0^2 - 2s a_0 b_0/r \) one can get \( 3a_0 - 8s a_0^3 b_0/r + 6a_0^2 b_0^2 - b_0^2 = 0 \) which divides both of the equations \( a_1 - a_0 = 0 \) and \( c_1 - c_0 = 0 \). Thus, by the assumption we have \( (a_1, b_1, c_1) \neq (a_0, b_0, c_0) \). Therefore, the proof of Theorem \( 3.2 \) is completed.

Note that in the above theorem it is possible to have \( a_1 = b_1 \). Indeed, substituting \( c_0^2 = a_0^2 + b_0^2 - 2s a_0 b_0/r \) in the equations \( 3.2 \) we have \( a_1 = b_1 \) if and only if \( (a_0 - b_0)^4 = 32r(r-s)N^2 \). We refer the reader to Table \( 1 \) in the end of this paper to see examples satisfying these conditions.

The following theorem shows that, under mild conditions, repeating the procedure given by the theorem \( 3.2 \) leads to infinitely many distinct rational \( \theta \)-triangles for a given \( \theta \)-congruent number. These conditions will be checked and used in the second half of Section 5 when Fujiiwara’s Theorem \( 2.2 \) is proved.

**Theorem 3.3.** For any integer \( n \geq 0 \), suppose that \( (a_{n+1}, b_{n+1}, c_{n+1}) \) is the rational \( \theta \)-triple for \( N \) obtained by Theorem \( 3.2 \) from \( (a_n, b_n, c_n) \) satisfying the conditions \( b_n^2 - a_n^2 \neq 0 \) and \( b_n^2 - a_n^2 \neq 2a_n c_n \). Then, \( |c_n| \neq |c_n'| \) for any two distinct non-negative integers \( n, n' \).

**Proof.** Fix an arbitrary number \( n \geq 0 \). Since \( N = |\frac{2a_n b_n}{2r}| \), we have
\[
a_n^2 b_n^2 = 4r^2 N^2 .
\]
On the other hand, the equality \( a_n^2 + b_n^2 - 2s a_n b_n/r = c_n^2 \) implies that
\[
c_n^4 - 8r N \left( a_n - \frac{sb_n}{r} \right) \left( b_n - \frac{sa_n}{r} \right) = (b_n^2 - a_n^2)^2 > 0.
\]
Therefore,
\[
d_n := b_n^2 - a_n^2 = \sqrt{c_n^4 - 8r N \left( a_n - \frac{sb_n}{r} \right) \left( b_n - \frac{sa_n}{r} \right)}.
\]
Using the equations (11), (5), and doing a little bit of algebraic simplifications one can get that
\[ d_n = \sqrt{c_n^4 + 8s^Nc_n^2 - 16N^2r^2 - s^2}, \quad |c_{n+1}| = \frac{c_n^4 + 16N^2r^2 - s^2}{2d_n|c_n|}. \] (7)

Note that \( d_n \) is a rational number because \( c_{n+1} \in \mathbb{Q}^* \). Assume that \( |c_n| = u/v \), where \( u, v \) are in lowest term. To prove the assertion of the theorem, we consider the following two cases.

**Case 1: \( u \) is even.** Let us write \( u = 2^t \cdot \tilde{u} \), where \( t \geq 1 \) and \( \tilde{u} \) is an odd number, hence \( c_n = 2^t \cdot \tilde{u}/v \) and \( N = 2^t \cdot \tilde{N} \), where \( t \geq 1 \) and \( \tilde{N} \) is an odd number. Thus, we have
\[ d_n^2 = \frac{2^{4t} \cdot \tilde{u}^4 + 2^{2t+\ell+3} \cdot \tilde{N}sv^2 \tilde{u}^2 - 2^{2\ell+4} \cdot \tilde{N}^2 (r^2 - s^2)v^4}{v^4} = \left( \frac{2^m u_0}{v^2} \right)^2, \]
\[ c_n^4 + 16N^2(r^2 - s^2) = \frac{2^{4t} \cdot \tilde{u}^4 + 2^{2t+4} \cdot \tilde{N}^2 (r^2 - s^2)v^4}{v^4} = \left( \frac{2^m u_1}{v^2} \right)^2, \] (8)
where the both numbers \( u_0, u_1 \) are odd and \( 2 \leq m \leq 2t \). Therefore, from the equations (7) and (8), we can write
\[ |c_{n+1}| = \frac{2^{m-t-1} \cdot u'}{v'}, \]
where \( u_0, u_1, u', v' \) are odd numbers. Since \( m < 2t + 1 \), we have \( m - t - 1 < t \), which implies that: If \( |c_n| = 2^t \cdot \tilde{u}/v \) with odd \( \tilde{u}, v \) and \( t \geq 0 \), then we have \( |c_{n+1}| = 2^{t'} \cdot u'/v' \) with odd \( u', v' \) and \( 0 \leq t' < t \).

**Case 2: \( u \) is odd.** We write \( v = 2^t \cdot \tilde{v} \), where \( t \geq 0 \) and \( \tilde{v} \) is odd, hence \( c_n = u/(2^t \cdot \tilde{v}) \) from which we obtain
\[ d_n^2 = \left( \frac{\tilde{u}}{2^{2t+\ell+2}} \right)^2, \quad c_n^4 + 16N^2(r^2 - s^2) = \frac{\tilde{u}}{2^{4t} \cdot \tilde{v}^4}. \]
for some odd positive numbers \( \tilde{u} \) and \( \tilde{v} \). After some simple computations, we have
\[ |c_{n+1}| = \frac{u'}{2^{t+1} \cdot v'}, \]
where \( u', v' \) are odd integers and \( \gcd(u', v') = 1 \). Thus, we conclude that:
If \( |c_n| = u/2^t \cdot \tilde{v} \) with odd \( u, \tilde{v} \), and \( t \geq 0 \), then \( |c_{n+1}| = u'2^{t+1} \cdot v' \) with odd \( u', v' \) and \( 0 \leq t' < t \).

Therefore, we have finished the proof of Theorem 3.3.

Considering the equations (6) and (7) in the proof of Theorem 3.3, we can reformulate Theorem 3.2 as follows.

**Corollary 3.4.** Suppose that \( \theta \in (0, \pi) \) has a rational cosine and \((a_0, b_0, c_0)\) is a rational \( \theta \)-triple with \( d_0 \neq 0 \) and \( d_0 \neq 2a_0c_0 \) for a \( \theta \)-congruent number \( N \). Then,
\[ (a_1, b_1, c_1) = \left( \frac{d_0}{2c_0}, \frac{4rc_0N}{d_0}, \frac{c_n^4 + 16N^2(r^2 - s^2)}{2c_0d_0} \right) \]
is a distinct rational \( \theta \)-triple for \( N \).

4. **Rational \( \theta \)-Triples and the Addition in \( E_N^\theta(\mathbb{Q}) \)**

In this section, we show that our Fermat-type algorithm, given in Theorem 3.2, is essentially doubling of points in \( E_N^\theta(\mathbb{Q}) \).
Theorem 4.1. Let \( \theta \in (0, \pi) \) be an angle with rational cosine. Given a rational \( \theta \)-triple \((a_0, b_0, c_0)\) with \(d_0 \neq 0\) and \(d_0 \neq 2apc_0\) for a \(\theta\)-congruent number \(N\), we assume that \((a_1, b_1, c_1)\) is a \(\theta\)-triple for \(N\) obtained by Theorem 3.2. Moreover, we let \((x_i, y_i)\) be the rational points in \(E_N^\theta(\mathbb{Q})\) corresponding to \((a_i, b_i, c_i)\) by the map \((2)\) for \(i = 0, 1\). Then,

\[
[2](x_0, y_0) = (x_1, -y_1).
\]

Proof. By Corollary 3.4 the rational \(\theta\)-triple \((a_1, b_1, c_1)\) is given by

\[
(a_1, b_1, c_1) = \left( \frac{d_0}{2c_0}, \frac{4rc_0N}{d_0}, \frac{c_0^4 + 16N^2(r^2 - s^2)}{2c_0d_0} \right),
\]

which is corresponded by (3) to the point \((x_1, y_1) = (c_0^2/4, c_0d_0/8)\) in \(E_N^\theta(\mathbb{Q})\). Indeed, replacing \(N = a_0b_0/2r\) and the above values for \(a_1, b_1\) and \(c_1\), and doing some simplifications lead to

\[
x_1 = \frac{rN(a_1 + c_1 - sb_1/r)}{b_1} = \frac{c_0^2}{4},
\]

\[
y_1 = \frac{2r^2N^2(a_1 + c_1 - sb_1/r)}{b_1^2} = \frac{c_0d_0}{8}.
\]

On the other side, replacing \(N = a_0b_0/2r\) and using (3), the triple \((a_0, b_0, c_0)\) corresponds to \((x_0, y_0)\) in \(E_N^\theta(\mathbb{Q})\) with coordinates,

\[
x_0 = \frac{a_0(a_0 + c_0 - sb_0/r)}{2}, \quad y_0 = \frac{a_0^2(a_0 + c_0 - sb_0/r)}{2}.
\]

Using the last equations and applying the duplication formula described in Section 2 one can show that for \((x_2, y_2) := [2](x_0, y_0)\) we have,

\[
x_2 = \lambda^2 - 2Ns - 2x_0 = c_0^2/4 = x_1,
\]

\[
y_2 = -\lambda^3 + (3x_0 + 2Ns)\lambda - y_0 = -\frac{c_0d_0}{8} = -y_1,
\]

where \(\lambda\) is given by

\[
\lambda = \frac{3x_0^4 + 4NsN^2x_0^2 - 4N^2(r^2 - s^2)}{2y_0} = \frac{2a_0 + c_0}{2}.
\]

Thus, we have completed the proof of Theorem 4.1.

In the following, using the addition law in \(E_N^\theta(\mathbb{Q})\), we provide a way to find new rational \(\theta\)-triples for \(N\) from given two distinct ones.

Let \((a_0, b_0, c_0)\) and \((a_1, b_1, c_1)\) be two distinct \(\theta\)-triples for a \(\theta\)-congruent number \(N\), where \(\theta \in (0, \pi)\) has rational cosine as before. Then, applying the transformation (4), we obtain the following two points \(P_i = (x_i, y_i)\) in \(E_N^\theta(\mathbb{Q})\) for \(i = 0, 1\), where

\[
x_i = \frac{a_i(a_i + c_i)}{2} - sN, \quad y_i = \frac{a_i^2(a_i + c_i)}{2} - sNa_i.
\]

Defining \(t_i = a_i(a_i + c_i)\) for \(i = 0, 1\) and

\[
\lambda = \frac{y_1 - y_0}{x_1 - x_0} = \frac{a_0t_0 - a_1t_1 - 2sN(a_0 - a_1)}{t_0 - t_1}
\]

The next step is to determine the parameters \(a_0, b_0, c_0, a_1, b_1, c_1, d_0, d_1\) from a given \(\theta\)-triple \((a, b, c)\) together with \(N\). This can be done by using the duplication formula and the addition law as described in the following section.
we obtain that the point \( P_2 = (x_2, y_2) := P_0 + P_1 \) has the coordinates

\[
x_2 = \frac{(a_0 - a_1)^2 (t_0 - 2sN)(t_1 - 2sN)}{(t_0 - t_1)^2},
\]

\[
y_2 = \frac{(a_0 - a_1)(t_0 - 2sN)(t_1 - 2sN)T}{(t_0 - t_1)^3},
\]

where \( T = 4r^2N^2 + t_0t_1 + a_0a_1 (t_0 + t_1 - 4sN) \). (9)

Therefore, using the transformation (2) and some algebraic computations, we conclude the following theorem.

**Theorem 4.2.** Suppose that \( \theta \in (0, \pi) \) has a rational cosine, \((a_0, b_0, c_0)\) and \((a_1, b_1, c_1)\) are two distinct rational \( \theta \)-triples for a \( \theta \)-congruent number \( N \). Then, the \( \theta \)-triple \((a_2, b_2, c_2)\) given by

\[
a_2 = \frac{4r^2N^2 + t_0t_1 + a_0a_1 (t_0 + t_1 - 4sN)}{(a_0 - a_1)(t_0 - t_1)}, \quad b_2 = \frac{2rN}{a_2},
\]

\[
c_2 = \frac{(a_0 - a_1)^4(t_0 - 2sN)^2(t_1 - 2sN)^2 + (r^2 - s^2)(t_0 - t_1)^2N^2}{(a_0 - a_1)(t_0 - t_1)(t_0 - 2sN)(t_1 - 2sN)T},
\]

is another rational \( \theta \)-triple for \( N \), where \( T \) is given by (9). In particular, \((|a_2|, |b_2|, |c_2|)\) is a rational \( \theta \)-triangle for \( N \).

**Proof.** We may assume that \( a_0 \neq a_1 \) to have two distinct triples. Then, the relations \( a_i b_i = 2rN \) for \( i = 0, 1 \) give us \( a_0 \neq b_1 \) and hence \( c_0 \neq c_1 \). Thus, we have \( t_0 \neq t_1 \), hence both \( x_2 \) and \( y_2 \) are well-defined. Moreover, none of the factors in the defining equations of \( x_2 \) and \( y_2 \) can be zero, since the distinctness of \( \theta \)-triples implies \( x_i \neq 0 \) and \( t_i \neq 2sN \) for \( i = 0, 1 \).

In order to show that \((a_2, b_2, c_2)\) is a rational \( \theta \)-triangle for \( N \), it is enough to check that \( a_2b_2 = 2rN \) and \( a_2^2 + b_2^2 - 2abs/r = c_2^2 \). The first equality is trivial and the second one is equivalent to \( a_2^2 + 4r^2N^2/a_2^2 - 4sN - c_2^2 = 0 \) by considering the first one. It is straightforward to check the latter equality, so we leave it to the reader. Therefore, we have a new rational \( \theta \)-triangle for \( N \).

5. **Proof of Fujiwara’s theorems**

First, using the equations (1), (3) and the duplication formula, we prove the right-to-left side of Theorem (2) case by case as follows. For the other side, we refer the reader to Fujiwara’s original proof in [5].

**Case (1)(i).** \( N = 1 \), \( r = 8u^4v^4 \), \( r - s = (u^2 - v^2)^4 \), \( (1 + \sqrt{2})v > u > v > 0 \), and \( \text{gcd}(u, v) = 1 \).

From \( ab = 2rN \), letting \( a = 4uv^3 \) and \( b = 4u^3v \), we get \( c = 2(u^4 - v^4) \) satisfying \( c^2 = a^2 + b^2 - 2abs/r \). By (3), the \( \theta \)-triple \((a, b, c)\) transforms to the point \( P_1 = (x_1, y_1) \) or \(-P_1\) with

\[
x_1 = (u + v)(u^2 + v^2)(-v^2 + 2uv + u^2)(u - v)^3, \quad y_1 = 4uv^3x_1,
\]

where...
which is of order 8, carried out by SAGE. By the duplication formula, we get the other order 8 points $P_j = (x_j, y_j)$ and $-P_j$, $j = 2, 3, 4$, where

$$x_2 = (u-v)(u^2 + v^2)(-v^2 - 2uv + u^2)(u + v)^3, \quad y_2 = 4uv^2x_2,$$

$$x_3 = (u-v)(u^2 + v^2)(-v^2 + 2uv + u^2)(u + v)^3, \quad y_3 = 4u^3v^3x_3,$$

$$x_4 = (u+v)(u^2 + v^2)(-v^2 - 2uv + u^2)(u - v)^3, \quad y_4 = 4u^3v^4x_4,$$

so that $[2](±P_k) = ((u^4 - v^4)^2, ±4u^2v^2(u^4 - v^4)^2)$ and $[4](±P_k) = ((u^2 - v^2)^4, 0)$ for $k = 1, 2, 3, 4$. Hence, $T^6_N(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$.

Case (1)(ii). $N = 2$, $r = (u^2 - v^2)^4$, $r - s = 32u^4v^4$, $u > (1 + \sqrt{2})v > 0$, and gcd$(u, v) = 1$.

The $\theta$-triple $(a, b, c) = (2(u + v)(u - v)^3, 2(u + v)^3(u - v)3, 8uv(u^2 + v^2))$ satisfies $\mathbf{(1)}$ and hence, by $\mathbf{(3)}$, transforms to the order 8 point $P_1 = (x_1, y_1)$ or $-P_1$ with

$$x_1 = 16uv^3(u^2 + v^2)(u^2 + 2uv - v^2), \quad y_1 = 2(u - v)(u - v)^3x_1.$$

By the duplication formula, we get the other order 8 points $P_j = (x_j, y_j)$ and $-P_j$, $j = 2, 3, 4$, having the coordinates

$$x_2 = 16u^3v(u^2 + v^2)(v^2 + 2uv - u^2), \quad y_2 = 2(u + v)(u - v)^3x_2,$$

$$x_3 = 16u^3v(u^2 + v^2)(u^2 + 2uv - v^2), \quad y_3 = 2(u - v)(u + v)^3x_3,$$

$$x_4 = 16uv^3(u^2 + v^2)(v^2 + 2uv - u^2), \quad y_4 = 2(u - v)(u + v)^3x_4.$$  

It is easy to check that $[2](±P_k) = (16u^2v^2(u^2 + v^2)^2, ±32u^2v^2(u^4 - v^4)^2)$ and $[4](±P_k) = (64u^4v^4, 0)$ for $k = 1, 2, 3, 4$. Hence, $T^6_N(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$.

Case (2)(i). $N = 1$, $r = (u - v)^3(u + v)/2$, $r + s = u^3(u - 2v)$, $u > 2v > 0$, gcd$(u, v) = 1$, and

Case (2)(iii). $N = 3$, $r = (u - v)^3(u + v)/6$, $r + s = u^3(u - 2v)/3$, $u > 2v > 0$, gcd$(u, v) = 1$.

Considering $(a, b, c) = (u^2 - v^2, (u - v)^2, 2uv)$ which satisfies $\mathbf{(1)}$, we obtain, by $\mathbf{(3)}$, the order 6 point $P_1 = (x_1, y_1)$ or $-P_1$ with

$$x_1 = (2u - v)u^2v, \quad y_1 = (u^2 - v^2)x_1.$$  

By the duplication formula, we get the other order 6 points $P_j = (x_j, y_j)$ and $-P_j$ for $j = 2, 3$ with coordinates

$$x_2 = (2v - u)uv^2, \quad x_3 = (2u - v)(2v - u)uv,$$

and

$$y_2 = (u^2 - v^2)x_2, \quad y_3 = (u - v)^3x_3,$$

so that $[2](±P_k) = (u^2v^2, ±u^2v^2(u - v)^2)$ for $k = 1, 2, 3$. Hence, we have $T^6_N(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.

Case (2)(ii). $N = 2$, $r = (u - v)^3(u + v)$, $r + s = 2u^3(u - 2v)$, $u > 2v > 0$, gcd$(u, v) = 1$, and

Case (2)(iv). $N = 6$, $r = (u - v)^3(u + v)/3$, $r + s = 2u^3(u - 2v)/3$, $u > 2v > 0$, gcd$(u, v) = 1$. 
The $\theta$-triple $(a, b, c) = (2(u^2 - v^2), 2(u - v)^2, 4uv)$ satisfies (1) and, by (3), transforms to
the order 6 point $P_1 = (x_1, y_1)$ or $-P_1$ with $x_1 = 4(2u - v)u^2v, y_1 = 2(u^2 - v^2)x_1$. By
the duplication formula, we get the other order 6 points $P_j = (x_j, y_j)$ and $-P_j, j = 2, 3$, with
coordinates
\[ x_2 = 4(2v - u)uv^2, \quad x_3 = 4(2u - v)(2v - u)uv, \]
and
\[ y_2 = 2(u^2 - v^2)x_2, \quad y_3 = 2(u - v)^2x_3, \]
so that $[2](\pm P_k) = (4u^2v^2, \pm 8u^2v^2(u - v^2))$. Hence, $E_N^0(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.

Case 3(i). $N = 1, 2r$ and $r - s$ are squares but not satisfy (1)(i).

We may let $r = 2u^2$ and $s = 2u^2 - v^2$ and get the $\theta$-triple $(a, b, c) = (2u, 2u, 2v)$ which satisfies
(1) and, by (3), transforms to the order 4 point $P_1 = ((2u + v)4w(2u + v))$ or $-P_1$. By
the duplication formula, we get the other order 4 points $P_2 = ((v - 2u)4w(2v - 2u))$ and
$-P_2$, so that $[2](\pm P_j) = (v^2, 0)$ for $j = 1, 2$. Hence, $E_N^0(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

Case (3)(ii). $N = 2, r$ and $2(r - s)$ are squares but not satisfy (1)(ii).

We may let $r = u^2$ and $s = u^2 - 2v^2$ and get the $\theta$-triple $(a, b, c) = (2u, 2u, 4v)$ which satisfies
(1) and transforms to the order 4 point $P_1 = (4(u + v)4w(2u + v))$ or $-P_1$. By
the duplication formula, we get the other order 4 points $P_2 = (4(v - u)4w(2v - 2u))$ and
$-P_2$, so that $[2](\pm P_j) = (4v^2, 0)$ for $j = 1, 2$. Hence, $E_N^0(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

Case (4). Otherwise.

In this case, it is evident that $E_N^0(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ because the image of $\phi$ is the set
of all rational points on $E_N^0(\mathbb{Q})$, except for the three points on the $x$-axis, which are
the non-trivial points of order 2, and for the point at infinity.

In Table 1 we bring in the possible corresponding rational $\theta$-triples to the above torsion
points of orders 4, 6, 8, obtained by the transformation (2) and Theorem 3.2. Therefore,
we have completed the proof of the right-to-left side of Fujiwara’s Theorem 2.3.

Secondly, we note that given any $\theta$-congruent number $N$ and rational $\theta$-triangle $(a, b, c)$
one has $a + c - sb/r > 0$ which implies that its image $\psi(a, b, c)$, by the transformation (3), has
a nonzero $y$-coordinate. Hence, it is a non-2-torsion point in $E_N^0(\mathbb{Q})$ as desired. Conversely,
by the fact that the only 2-torsion points in $E_N^0(\mathbb{Q})$ are $(0, 0), (-r + s)N, 0), ((r - s)N, 0)$
and the point at the infinity, any non-2-torsion gives us a rational $\theta$-triangle for $N$ by
transformation (2). This shows the part (1) of the Fujiwaras’ theorem 2.2.

Finally, if $N \nmid 6$ is a $\theta$-congruent number, $(x_0, y_0) \in E_N^0(\mathbb{Q})$ with $y \neq 0$ and
\[ (a_0, b_0, c_0) = \phi(x_0, y_0) = \left( \frac{y_0}{x_0}, \frac{2rx_0N}{y_0}, \frac{x_0^2 + (r^2 - s^2)N^2}{y_0} \right) \]
is its corresponding rational $\theta$-triangle, then it satisfies the conditions of Theorems 3.2 and
3.3. Indeed, we have $b_0^2 = a_0^2$ if and only if $a_0 = \pm b_0$ if and only if
$y_0^2 = \pm 2r x_0^2 N$. On substituting $y_0^2 = x_0(x_0^2 + 2sN x_0 - (r^2 - s^2)N^2)$ and
some simplification we obtain that $a_0 = \pm b_0$ if and only if
\[ x_0^2 - 2N(r \pm s)x_0 - (r^2 - s^2)N^2 = 0, \]
which is soluble in $x_0 \in \mathbb{Q}$ if and only if $2(r - s)$ is a rational square. But, the proof of
the left-to-right side of Fujiwara’s Theorem 2.3 in [5] shows the condition holds only in the
cases $N = 1, 2, 3$ and 6, each of which gives us a contradiction and hence we have $a_0^2 \neq b_0^2$.

On the other hand, a simple algebraic calculation shows that for the above $\theta$-triple
$(a_0, b_0, c_0)$, the relation $b_0^2 - a_0^2 = 2a_0c_0$ holds if and only if the following quartic equation

\[ 3x_0^4 + 8Nz x_0^3 - 6N^2(r^2 - s^2)x_0^2 = N^4(r^2 - s)^2 = 0 \]
has rational solution \(x_0\). This is equivalent to saying that the \(x\)-coordinate of \([2]P_0\) is equal to \(x_0\); in other words, \(P_0\) is a point of order 3. But, this is impossible by the Fujiwara’s theorem \([2,3]\) since \(N \geq 6\). Thus, we have \(b_0^2 - a_0^2 \neq 2a_0c_0\) as desired.

Now, let \(P_0 = (x_0, y_0)\) and for any \(n \geq 0\) suppose that \((a_{n+1}, b_{n+1}, c_{n+1})\) is the \(\theta\)-triple obtained by applying Theorem \(3.2\) for \((a_n, b_n, c_n)\) which are all distinct. Denote by \(P_n = (x_n, y_n)\) the rational point in \(E_{N}^0(\mathbb{Q})\) corresponding to \((a_n, b_n, c_n)\). Then, a similar argument as given in the proof of Theorem \(4.1\) shows that \(x_n = c_n^2/4\) and \(P_n = [2^n]P_0\) for any \(n \geq 0\). Since \(|c_n| \neq |c_{n'}|\), the points \(P_n, P_{n'}\) are distinct in \(E_{N}^0(\mathbb{Q})\) for \(n \neq n' \geq 0\) by Theorem \(3.3\). In other words, the point \(P_0\) is of infinite order so the rank of \(E_{N}^0(\mathbb{Q})\) is greater than or equal to one. The converse is trivial by considering the Fujiwara’s theorem \(2.3\) on the possible torsion points in \(E_{N}^0(\mathbb{Q})\). Therefore, we have completed the proof of Theorem \(2.2\).

### Table 1. Rational \(\theta\)-triples obtained by \([2]\) and Theorem \(3.2\) for the points of \(T_{N}^0(\mathbb{Q})\)

| Cases | Points | \((a_0, b_0, c_0)\) | \((a_1, b_1, c_1)\) | \cdots |
|-------|--------|----------------------|----------------------|--------|
| (1)(i) | \(\pm P_1, \pm P_2\) | \((4uv^3, 4u^3v, 2(u^4-v^4))\) | \((4u^2v^2, 4u^2v^2, 2(u^2-v^2)^2)\) | * |
|       | \(\pm P_3, \pm P_4\) | \((4u^2v, 4uv^3, 2(u^4-v^4))\) | \((4u^2v^2, 4u^2v^2, 2(u^2-v^2)^2)\) | * |
| \([2](\pm P_k)\) | \((4u^2v^2, 4u^2v^2, 2(u^2-v^2)^2)\) | | |
| (1)(ii) | \(\pm P_1, \pm P_2\) | \((a_0 = 2(u+v)(u-v)^3)\) | \((a_1 = 2(u^2-v^2)^2)\) | * |
|       | \(\pm P_3, \pm P_4\) | \((b_0 = 2(u+v)(u-v))\) | \((b_1 = 2(u^2-v^2)^2)\) | * |
| \([2](\pm P_k)\) | \((c_0 = 8uv(u^2+v^2))\) | \((c_1 = 16u^2v^2)\) | |
| (2)(ii) | \(\pm P_1, \pm P_2\) | \((u^2-v^2, (u-v)^2, 2uv)\) | \((u^2-v^2, (u-v)^2, 2uv)\) | ** |
| (2)(iii) | \(\pm P_3, \pm P_4\) | \(2(u^2-v^2, (u-v)^2, 4uv)\) | \((u^2-v^2, u^2-v^2, 2uv)\) | ** |
| (2)(iv) | \(\pm P_1, \pm P_2\) | \((2(u^2-v^2), 2(u-v)^2, 4uv)\) | \((2(u^2-v^2), 2(u-v)^2, 4uv)\) | ** |
| \([2](\pm P_k)\) | \((2(u^2-v^2), (u-v)^2, 4uv)\) | \((2(u^2-v^2), (u-v)^2, 4uv)\) | ** |
| (3)(i) | \(\pm P_1, \pm P_2\) | \((2u, 2u, 2v)\) | | *
| (3)(ii) | \(\pm P_1, \pm P_2\) | \((2u, 2u, 4v)\) | | *

The symbol “*” in the above table shows that the theorem \(3.2\) cannot be applied further because of the lack of the condition \(b_i^2 - a_i^2 \neq 0\) for some index \(i = 1, 2\) depending on the cases. By the symbol “**”, we mean that applying Theorem \(3.2\) leads to a rational \(\theta\)-triple which can be identified with the given one if we permute the \(a\) and \(b\) components. This happens in the cases (2)(i) to (2)(iv), since the condition \(b_0^2 - a_0^2 = 2a_0c_0\) does not hold.

### References

[1] S. Chan, *Rational right triangles of a given area*, The American Mathematical Monthly, **125**(8) pp. 689–703.

[2] A. Dujella, A. S. Janfada, C. J. Peral, and S. Salami, *On the high rank \(\pi/3\) and \(2\pi/3\)-congruent number elliptic curves*, Rocky Mountain J. Math. **44**(6) (2014) 1867–1880.

[3] Fermat, P.: *Fermat’s Diophanti Alex. Arith., 1670 in Oeuvres III*, (Ministère de l’instruction publique, ed.), Gauthier-Villars et. fils, Paris, 1896, 254–256.
[4] M. Fujiwara, θ-congruent numbers, in: Number Theory, K. Győry, A. Pethő and V. Sós (eds.), de Gruyter, 1997, pp. 235–241.
[5] M. Fujiwara, Some properties of θ-congruent numbers, Natural Science Report, Ochanomizu University, 118(2) (2001) 1–8.
[6] A. S. Janfada, S. Salami, On θ-congruent numbers on real quadratic number fields, Kodai Math. J. 38 352–364 (2015).
[7] Halbeisen, L. and Hungerbühler, N., A Theorem of Fermat and Congruent Numbers Curves, Hardy-Ramanujan Journal 41 (2018) 15-21.
[8] Hungerbühler, N., A proof of a conjecture of Lewis Carroll, Mathematics Magazine 69 (1996), 182–184.
[9] M. Kan, θ-congruent numbers and elliptic curves, Acta Arith. 94 (2000) 153–160.
[10] N. Koblitz, Introduction to Elliptic curves and Modular Forms, Springer-Verlag, Graduate text in Mathematics 97, 2nd ed, Berlin, 1993.
[11] B. Mazur, Modular curves and the Eisenstein ideal, Pub. Math. IHES 47 (1978), 33–186.
[12] J. H. Silverman, The Arithmetic of Elliptic Curves, second edition Graduate text in Mathematics, Springer-Verlag, Berlin, Vol. 106, (2009).

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