Solution and Hyers-Ulam-Rassias Stability of Generalized Mixed Type Additive-Quadratic Functional Equations in Fuzzy Banach Spaces

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By using fixed point methods and direct method, we establish the generalized Hyers-Ulam stability of the following additive-quadratic functional equation

\[ f(x + ky) + f(x - ky) = f(x + y) + f(x - y) + (2k+1)f(ky) - 2(k+1)f(y) \]

for fixed integers \( k \) with \( k \neq 0, \pm 1 \) in fuzzy Banach spaces.

1. Introduction and Preliminaries

The stability problem of functional equations was originated from a question of Ulam [1] in 1940, concerning the stability of group homomorphisms. Let \( (G_1, \cdot) \) be a group and let \( (G_2, *, d) \) be a metric group with the metric \( d(\cdot, \cdot) \). Given \( \epsilon > 0 \), does there exist a \( \delta > 0 \), such that if a mapping \( h : G_1 \to G_2 \) satisfies the inequality \( d(h(x \cdot y), h(x) * h(y)) < \delta \) for all \( x, y \in G_1 \), then there exists a homomorphism \( H : G_1 \to G_2 \) with \( d(h(x), H(x)) < \epsilon \) for all \( x \in G_1 \)? In other words, under what condition does there exist a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1941, Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. Let \( f : E \to E' \) be a mapping between Banach spaces such that

\[ \| f(x + y) - f(x) - f(y) \| \leq \delta, \] (1.1)
for all \( x, y \in E \), and for some \( \delta > 0 \). Then there exists a unique additive mapping \( T : E \rightarrow E' \) such that

\[
\| f(x) - T(x) \| \leq \delta,
\]

(1.2)

for all \( x \in E \). Moreover if \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in E \), then \( T \) is linear. In 1978, Rassias [3] provided a generalization of Hyers’ Theorem which allows the Cauchy difference to be unbounded. In 1991, Gajda [4] answered the question for the case \( p > 1 \), which was raised by Rassias. This new concept is known as Hyers-Ulam-Rassias stability of functional equations (see [5–17]).

The functional equation

\[
f(x + y) + f(x - y) = 2f(x) + 2f(y)
\]

(1.3)

is related to a symmetric biadditive function. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.3) is said to be a quadratic function. It is well known that a function \( f \) between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function \( B \) such that \( f(x) = B(x, x) \) for all \( x \) (see [6, 18]). The biadditive function \( B \) is given by

\[
B(x, y) = \frac{1}{4}(f(x + y) - f(x - y)).
\]

(1.4)

A Hyers-Ulam-Rassias stability problem for the quadratic functional equation (1.3) was proved by Skof for functions \( f : A \rightarrow B \), where \( A \) is normed space and \( B \) Banach space (see [19–22]). Borelli and Forti [23] generalized the stability result of quadratic functional equations as follows (cf. [24, 25]): let \( G \) be an Abelian group, and \( X \) a Banach space. Assume that a mapping \( f : G \rightarrow X \) satisfies the functional inequality:

\[
\| f(x + y) + f(x - y) - 2f(x) - 2f(y) \| \leq \varphi(x, y),
\]

(1.5)

for all \( x, y \in G \), and \( \varphi : G \times G \rightarrow [0, \infty) \) is a function such that

\[
\Phi(x, y) := \sum_{i=0}^{\infty} \frac{1}{4^{i+1}} \varphi(2^i x, 2^i y) < \infty,
\]

(1.6)

for all \( x, y \in G \). Then there exists a unique quadratic mapping \( Q : G \rightarrow X \) with the property

\[
\| f(x) - Q(x) \| \leq \Phi(x, x),
\]

(1.7)

for all \( x \in G \).

Now, we introduce the following functional equation for fixed integers \( k \) with \( k \neq 0, \pm 1 \):

\[
f(x + ky) + f(x - ky) = f(x + y) + f(x - y) + \frac{2(k + 1)}{k} f(ky) - 2(k + 1) f(y),
\]

(1.8)
with $f(0) = 0$ in a non-Archimedean space. It is easy to see that the function $f(x) = ax + bx^2$ is a solution of the functional equation (1.8), which explains why it is called additive-quadratic functional equation. For more detailed definitions of mixed type functional equations, we can refer to [26–47].

Definition 1.1 (see [48]). Let $X$ be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a fuzzy norm on $X$ if for all $x, y \in X$ and all $s, t \in \mathbb{R},$

\begin{align*}
(N1) & \ N(x, t) = 0 \text{ for } t \leq 0; \\
(N2) & \ x = 0 \text{ if and only if } N(x, t) = 1 \text{ for all } t > 0; \\
(N3) & \ N(cx, t) = N(x, t/|c|) \text{ if } c \neq 0; \\
(N4) & \ N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}; \\
(N5) & \ N(x, \cdot) \text{ is a nondecreasing function of } \mathbb{R} \text{ and } \lim_{t \to \infty} N(x, t) = 1; \\
(N6) & \text{ for } x \neq 0, \ N(x, \cdot) \text{ is continuous on } \mathbb{R}.
\end{align*}

The pair $(X, N)$ is called a fuzzy normed vector space.

Example 1.2. Let $(X, \| \cdot \|)$ be a normed linear space and $\alpha, \beta > 0.$ Then

$$N(x, t) = \begin{cases} 
\frac{\alpha t}{\alpha t + \beta \|x\|}, & t > 0, \ x \in X, \\
0, & t \leq 0, \ x \in X,
\end{cases} \quad (1.9)$$

is a fuzzy norm on $X.$

Definition 1.3. Let $(X, N)$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in $X$ is said to be convergent or converge if there exists an $x \in X$ such that $\lim_{n \to \infty} N(x_n - x, t) = 1$ for all $t > 0.$ In this case, $x$ is called the limit of the sequence $\{x_n\}$ in $X$ and one denotes it by $\text{lim}_{n \to \infty} x_n = x.$

Definition 1.4. Let $(X, N)$ be a fuzzy normed vector space. A sequence $\{x_n\}$ in $X$ is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0,$ one has $N(x_{n+p} - x_n, t) > 1 - \epsilon.$

It is well known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

Example 1.5. Let $N : \mathbb{R} \times \mathbb{R} \to [0,1]$ be a fuzzy norm on $\mathbb{R}$ defined by

$$N(x, t) = \begin{cases} 
\frac{t}{t + |x|}, & t > 0, \\
0, & t \leq 0.
\end{cases} \quad (1.10)$$
Let \((\mathbb{R}, N)\) be a fuzzy Banach space. Let \(\{x_n\}\) be a Cauchy sequence in \(\mathbb{R}\), \(\delta > 0\), and \(\epsilon = \delta / (1 + \delta)\). Then there exist \(m \in \mathbb{N}\) such that for all \(n \geq m\) and all \(p > 0\), one has

\[
\frac{1}{1 + |x_{n+p} - x_n|} \geq 1 - \epsilon.
\]

(1.11)

So \(|x_{n+p} - x_n| < \delta\) for all \(n \geq m\) and all \(p > 0\). Therefore \(\{x_n\}\) is a Cauchy sequence in \((\mathbb{R}, |\cdot|)\). Let \(x_n \to x_0 \in \mathbb{R}\) as \(n \to \infty\). Then \(\lim_{n \to \infty} N(x_n - x_0, t) = 1\) for all \(t > 0\).

We say that a mapping \(f : X \to Y\) between fuzzy normed vector spaces \(X\) and \(Y\) is continuous at a point \(x \in X\) if for each sequence \(\{x_n\}\) converging to \(x_0 \in X\), the sequence \(\{f(x_n)\}\) converges to \(f(x_0)\). If \(f : X \to Y\) is continuous at each \(x \in X\), then \(f : X \to Y\) is said to be continuous on \(X\) ([49]).

Definition 1.6. Let \(X\) be a set. A function \(d : X \times X \to [0, \infty)\) is called a generalized metric on \(X\) if \(d\) satisfies the following conditions:

1. \(d(x, y) = 0\) if and only if \(x = y\) for all \(x, y \in X\);
2. \(d(x, y) = d(y, x)\) for all \(x, y \in X\);
3. \(d(x, z) \leq d(x, y) + d(y, z)\) for all \(x, y, z \in X\).

Theorem 1.7. Let \((X, d)\) be a complete generalized metric space and let \(J : X \to X\) be a strictly contractive mapping with Lipschitz constant \(L < 1\). Then, for all \(x \in X\), either

\[
d\left(J^n x, J^{n+1} x\right) = \infty,
\]

for all nonnegative integers \(n\), or there exists a positive integer \(n_0\) such that

1. \(d(J^n x, J^{n+1} x) < \infty\) for all \(n \geq n_0\);
2. the sequence \(\{J^n x\}\) converges to a fixed point \(y^*\) of \(J\);
3. \(y^*\) is the unique fixed point of \(J\) in the set \(Y = \{y \in X : d(J^n x, y) < \infty\}\);
4. \(d(y, y^*) \leq 1/(1 - L)d(y, Jy)\) for all \(y \in Y\).

We have the following theorem from [42], which investigates the solution of (1.8).

Theorem 1.8. A function \(f : X \to Y\) with \(f(0) = 0\) satisfies (1.8) for all \(x, y \in X\) if and only if there exist functions \(A : X \to Y\) and \(Q : X \times X \to Y\), such that \(f(x) = A(x) + Q(x, x)\) for all \(x \in X\), where the function \(Q\) is symmetric biadditive and \(A\) is additive.

2. A Fixed Point Method

Using the fixed point methods, we prove the Hyers-Ulam stability of the additive-quadratic functional equation (1.8) in fuzzy Banach spaces. Throughout this paper, assume that \(X\) is a vector space and that \((Y, N)\) is a fuzzy Banach space.
Theorem 2.1. Let \( \varphi : X^2 \to [0, \infty) \) be a mapping such that there exists an \( \alpha < 1 \) with
\[
\varphi(x, y) \leq |k|\alpha \varphi \left( \frac{x}{k}, \frac{y}{k} \right),
\]
for all \( x, y \in X \). Let \( f : X \to Y \) be an odd function satisfying \( f(0) = 0 \) and
\[
N \left( f(k(x + y)) + f(k(x - y)) - f(kx + y) - f(kx - y) - \frac{2(k+1)}{k} f(ky) + 2(k+1)f(y), t \right) \\
\geq \frac{t}{t + \varphi(x, y)},
\]
for all \( x, y \in X \) and all \( t > 0 \). Then \( A(x) := N - \lim_{n \to \infty} (f(k^n x) / k^n) \) exists for all \( x \in X \) and defines a unique additive mapping \( A : X \to Y \) such that
\[
N(f(x) - A(x), t) \geq \frac{(|2k + 2| - |2k + 2|\alpha)t}{(|2k + 2| - |2k + 2|\alpha)t + \varphi(0, x)},
\]
for all \( x \in X \) and \( t > 0 \).

Proof. Note that \( f(0) = 0 \) and \( f(-x) = -f(x) \) for all \( x \in X \) since \( f \) is an odd function. Putting \( x = 0 \) in (2.2), we get
\[
N \left( f(ky) - f(y), \frac{t}{|2k + 2|} \right) \geq \frac{t}{t + \varphi(0, y)},
\]
for all \( y \in X \) and all \( t > 0 \). Replacing \( y \) by \( x \) in (2.4), we have
\[
N \left( f(kx) - f(x), \frac{t}{|2k + 2|} \right) \geq \frac{t}{t + \varphi(0, x)},
\]
for all \( x \in X \) and all \( t > 0 \). Consider the set \( S := \{ h : X \to Y; h(0) = 0 \} \) and introduce the generalized metric on \( S \):
\[
d(g, h) = \inf_{\mu \in (0, +\infty)} \left\{ N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(0, x)}, \forall x \in X \right\},
\]
where, as usual, \( \inf \varphi = +\infty \). It is easy to show that \( (S, d) \) is complete (see [50]). We consider the mapping \( J : (S, d) \to (S, d) \) as follows:
\[
Jg(x) := \frac{1}{k}g(kx),
\]
for all $x \in X$. Let $g, h \in S$ be given such that $d(g, h) = \beta$. Then

$$N(g(x) - h(x), \beta t) \geq \frac{t}{t + \varphi(0, x)},$$

(2.8)

for all $x \in X$ and all $t > 0$. Hence

$$N(Jg(x) - Jh(x), a\beta t) = N\left(\frac{1}{k}g(kx) - \frac{1}{k}h(kx), a\beta t\right)$$

$$= N(g(kx) - h(kx), |k|a\beta t)$$

$$\geq \frac{|k|at}{|k|at + \varphi(0, x)}$$

$$\geq \frac{|k|at}{|k|at + |k|\alpha\varphi(0, x)}$$

$$= \frac{t}{t + \varphi(0, x)},$$

(2.9)

for all $x \in X$ and all $t > 0$. So $d(g, h) = \beta$ implies that $d(Jg, Jh) \leq a\beta$. This means that $d(Jg, Jh) \leq ad(g, h)$ for all $g, h \in S$. It follows from (2.5) that

$$d(f, Jf) \leq \frac{1}{|2k + 2|},$$

(2.10)

By Theorem 1.7, there exists a mapping $A : X \to Y$ satisfying the following.

(1) $A$ is a fixed point of $J$, that is,

$$kA(x) = A(kx),$$

(2.11)

for all $x \in X$. The mapping $A$ is a unique fixed point of $J$ in the set $M = \{g \in S : d(h, g) < \infty\}$. This implies that $A$ is a unique mapping satisfying (2.11) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - A(x), \mu t) \geq \frac{t}{t + \varphi(0, x)},$$

(2.12)

for all $x \in X$.

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality $\lim_{n \to \infty} (f(k^n x)/k^n) = A(x)$, for all $x \in X$.

(3) $d(f, A) \leq (1/(1 - a))d(f, Jf)$, which implies the inequality

$$d(f, A) \leq \frac{1}{|2k + 2| - |2k + 2|a},$$

(2.13)

This implies that the inequality (2.3) holds.
Corollary 2.2. Let \( \theta \geq 0 \) and let \( r \) be a real positive number with \( r < 1 \). Let \( X \) be a normed vector space with norm \( \| \cdot \| \). Let \( f : X \to Y \) be an odd mapping satisfying

\[
N \left( f(k(x + y)) + f(k(x - y)) - f(kx + y) - f(kx - y) - \frac{2(k + 1)}{k} f(ky) + 2(k + 1) f(y), t \right) \geq \frac{t}{t + \theta(\|x\|^r + \|y\|^r)},
\]

for all \( x, y \in X \) and all \( t > 0 \). Then the limit \( A(x) := N - \lim_{n \to \infty} (f(k^n x)/k^n) \) exists for each \( x \in X \) and defines a unique additive mapping \( A : X \to Y \) such that

\[
N(f(x) - A(x), t) \geq \frac{|2k + 2||k| - |k|^r|t}{|2k + 2||k| - |k|^r|t + |k|\theta\|x\|^r},
\]

for all \( x \in X \) and all \( t > 0 \).
Proof. The proof follows from Theorem 2.1 by taking \( \varphi(x, y) := \theta(\|x\|^r + \|y\|^r) \) for all \( x, y \in X \). Then we can choose \( a = |k|^{-1} \) and we get the desired result. \( \square \)

**Theorem 2.3.** Let \( \varphi : X^2 \to [0, \infty) \) be a mapping such that there exists an \( \alpha < 1 \) with

\[
\varphi\left(\frac{x}{k}, \frac{y}{k}\right) \leq \frac{\alpha}{|k|} \varphi(x, y),
\]

for all \( x, y \in X \). Let \( f : X \to Y \) be an odd mapping satisfying \( f(0) = 0 \) and (2.2). Then the limit \( A(x) := N - \lim_{n \to \infty} k^n f(x/k^n) \) exists for all \( x \in X \) and defines a unique additive mapping \( A : X \to Y \) such that

\[
N(f(x) - A(x), t) \geq \frac{(|2k + 2| - |2k + 2| \alpha) + \alpha \varphi(0, x)}{|2k + 2| - |2k + 2| \alpha + \alpha \varphi(0, x)},
\]

for all \( x \in X \) and all \( t > 0 \).

Proof. Let \((S, d)\) be the generalized metric space defined as in the proof of Theorem 2.1.

Consider the mapping \( J : S \to S \) by

\[
Jg(x) := kg\left(\frac{x}{k}\right),
\]

for all \( g \in S \). Let \( g, h \in S \) be given such that \( d(g, h) = \beta \). Then

\[
N(g(x) - h(x), \beta t) \geq \frac{t}{t + \varphi(0, x)},
\]

for all \( x \in X \) and all \( t > 0 \). Hence

\[
N(Jg(x) - Jh(x), \alpha \beta t) = N\left(kg\left(\frac{x}{k}\right) - kh\left(\frac{x}{k}\right), \alpha \beta t\right)
\]

\[
= N\left(g\left(\frac{x}{k}\right) - h\left(\frac{x}{k}\right), \frac{\alpha \beta t}{|k|}\right)
\]

\[
\geq \frac{t}{\alpha \beta t / |k| + \varphi(0, x/k)} \geq \frac{t}{t + \varphi(0, x)}
\]

for all \( x \in X \) and all \( t > 0 \). So \( d(g, h) = \beta \) implies that \( d(Jg, Jh) \leq \alpha \beta \). This means that \( d(Jg, Jh) \leq \alpha d(g, h) \) for all \( g, h \in S \). It follows from (2.5) that

\[
N\left(kf\left(\frac{x}{k}\right) - f(x), \frac{kt}{|2k + 2|}\right) \geq \frac{t}{t + \varphi(0, x/k)} \geq \frac{t}{t + (\alpha / |k|) \varphi(0, x)},
\]
This implies that the inequality
\[ N\left( k f\left( \frac{x}{k}\right) - f(x)^r, \frac{at}{|2k+2|} \right) \geq \frac{t}{t + \varphi(0,x)} . \]  
(2.25)

So \( d(f, Jf) \leq a \). By Theorem 1.7, there exists a mapping \( A : X \to Y \) satisfying the following.

(1) \( A \) is a fixed point of \( J \), that is,
\[ A\left( \frac{x}{k}\right) = \frac{1}{k} A(x), \]  
(2.26)

for all \( x \in X \). The mapping \( A \) is a unique fixed point of \( f \) in the set \( \Omega = \{ h \in S : d(g, h) < \infty \} \).

This implies that \( A \) is a unique mapping satisfying (2.26) such that there exists \( \mu \in (0, \infty) \) satisfying
\[ N\left( f(x) - A(x), \mu t \right) \geq \frac{t}{t + \varphi(0,x)}, \]  
(2.27)

for all \( x \in X \) and \( t > 0 \).

(2) \( d(J^n f, A) \to 0 \) as \( n \to \infty \). This implies the equality \( N - \lim_{n \to \infty} k^n f(x/k^n) = A(x) \) for all \( x \in X \).

(3) \( d(f, A) \leq d(f, Jf)/(1 - L) \) with \( f \in \Omega \), which implies the inequality
\[ d(f, A) \leq \frac{\alpha}{|2k+2| - |2k+2|\alpha}. \]  
(2.28)

This implies that the inequality (2.20) holds.

The rest of proof is similar to the proof of Theorem 2.1.

\[ \square \]

**Corollary 2.4.** Let \( \theta \geq 0 \) and let \( r \) be a real number with \( r > 1 \). Let \( X \) be a normed vector space with norm \( \| \cdot \| \). Let \( f : X \to Y \) be an odd mapping satisfying (2.17). Then \( A(x) := N - \lim_{n \to \infty} k^n f(x/k^n) \) exists for each \( x \in X \) and defines a unique additive mapping \( A : X \to Y \) such that
\[ N\left( f(x) - A(x), t \right) \geq \frac{|2k+2|(|k|^r - |k|)t}{|2k+2|(|k|^r - |k|)t + |k|\theta \|x\|^r}, \]  
(2.29)

for all \( x \in X \) and all \( t > 0 \).

**Proof.** The proof follows from Theorem 2.3 by taking \( \varphi(x, y) := \theta(\|x\|^r + \|y\|^r) \) for all \( x, y \in X \). Then we can choose \( \alpha = |k|^{r-1} \) and we get the desired result.

\[ \square \]

**Theorem 2.5.** Let \( \varphi : X^2 \to [0, \infty) \) be a function such that there exists an \( \alpha < 1 \) with
\[ \varphi(x, y) \leq \kappa^2 \alpha \varphi\left( \frac{x}{k}, \frac{y}{k} \right), \]  
(2.30)
for all \( x, y \in X \). Let \( f : X \to Y \) be an even mapping with \( f(0) = 0 \) and satisfying (2.2). Then \( Q(x) := N - \lim_{n \to \infty} (f(k^nx)/k^{2n}) \) exists for all \( x \in X \) and defines a unique quadratic mapping \( Q : X \to Y \) such that

\[
N(f(x) - Q(x), t) \geq \frac{(2|k| - 2|k|\alpha)t}{(2|k| - 2|k|\alpha)t + \varphi(0, x)},
\]

for all \( x \in X \) and all \( t > 0 \).

**Proof.** Replacing \( x \) by \( kx \) in (2.2), we get

\[
N\left(f(k(x + y)) + f(k(x - y)) - f(kx + y) - f(kx - y) - \frac{2(k + 1)}{k} f(ky) + 2(k + 1)f(y), t \right) \\
\geq \frac{t}{t + \varphi(kx, y)},
\]

for all \( x, y \in X \) and all \( t > 0 \). Putting \( x = 0 \) and replacing \( y \) by \( x \) in (2.32), we have

\[
N\left(\frac{f(kx)}{k} - k f(x), t / 2 \right) \geq \frac{t}{t + \varphi(0, x)},
\]

for all \( x \in X \) and all \( t > 0 \). By (2.33), (N3), and (N4), we get

\[
N\left(\frac{f(kx)}{k^2} - f(x), t / 2|k| \right) \geq \frac{t}{t + \varphi(0, x)},
\]

for all \( x \in X \) and all \( t > 0 \). Consider the set \( S^* := \{ h : X \to Y ; h(0) = 0 \} \) and introduce the generalized metric on \( S^* \):

\[
d(g, h) = \inf_{\mu \in (0, +\infty)} \left\{ N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(0, x)}, \forall x \in X \right\},
\]

where, as usual, \( \inf \phi = +\infty \). It is easy to show that \( (S^*, d) \) is complete (see [50]). Now we consider the linear mapping \( J : (S^*, d) \to (S^*, d) \) such that

\[
Jg(x) := \frac{1}{k^2} g(kx),
\]

for all \( x \in X \). Proceeding as in the proof of Theorem 2.1, we obtain that \( d(g, h) = \beta \) implies that \( d(Jg, Jh) \leq a\beta \). This means that \( d(Jg, Jh) \leq ad(g, h) \) for all \( g, h \in S \). It follows from
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(2.34) that

\[ d(f, Jf) \leq \frac{1}{2|k|}. \]  

(2.37)

By Theorem 1.7, there exists a mapping \( Q : X \rightarrow Y \) such that one has the following.

1. \( Q \) is a fixed point of \( J \), that is,

\[ k^2 Q(x) = Q(kx), \]  

(2.38)

for all \( x \in X \). The mapping \( Q \) is a unique fixed point of \( J \) in the set \( M = \{ g \in S^* : d(h, g) < \infty \} \).

This implies that \( Q \) is a unique mapping satisfying (2.38) such that there exists a \( \mu \in (0, \infty) \) satisfying \( N(f(x) - Q(x), t\mu) \geq t((t + \varphi(0, x)) \) for all \( x \in X \).

2. \( d(J^n f, Q) \rightarrow 0 \) as \( n \rightarrow \infty \). This implies the equality \( \lim_{n \rightarrow \infty} (f(k^n x)/k^{2n}) = Q(x) \) for all \( x \in X \).

3. \( d(f, Q) \leq (1/(1 - \alpha))d(f, Jf) \), which implies the inequality \( d(f, Q) \leq 1/(2|k| - 2|k|\alpha) \). This implies that the inequality (2.31) holds.

The rest of the proof is similar to the proof of Theorem 2.1.

**Corollary 2.6.** Let \( \theta \geq 0 \) and let \( r \) be a real positive number with \( r < 1 \). Let \( X \) be a normed vector space with norm \( \| \cdot \| \). Let \( f : X \rightarrow Y \) be an even mapping with \( f(0) = 0 \) and satisfying (2.17). Then the limit \( Q(x) := N - \lim_{n \rightarrow \infty} (f(k^n x)/k^{2n}) \) exists for each \( x \in X \) and defines a unique quadratic mapping \( Q : X \rightarrow Y \) such that

\[ N(f(x) - Q(x), t) \geq \frac{(2k^2 - 2k^2\alpha)t}{(2k^2 - 2k^2\alpha)t + |k\|\|x\|^r}, \]  

(2.39)

for all \( x \in X \) and all \( t > 0 \).

**Proof.** The proof follows from Theorem 2.5 by taking \( \varphi(x, y) := \theta(\|x\|^r + \|y\|^r) \) for all \( x, y \in X \). Then we can choose \( \alpha = k^{2r-2} \) and we get the desired result.

**Theorem 2.7.** Let \( \varphi : X^2 \rightarrow [0, \infty) \) be a function such that there exists an \( \alpha < 1 \) with

\[ \varphi\left(\frac{x}{k}, \frac{y}{k}\right) \leq \frac{\alpha}{k^r} \varphi(x, y), \]  

(2.40)

for all \( x, y \in X \). Let \( f : X \rightarrow Y \) be an even mapping with \( f(0) = 0 \) and satisfying (2.2). Then the limit \( Q(x) := N - \lim_{n \rightarrow \infty} k^n f(x)/k^n \) exists for all \( x \in X \) and defines a unique quadratic mapping \( Q : X \rightarrow Y \) such that

\[ N(f(x) - Q(x), t) \geq \frac{(2|k| - 2|k|\alpha)t}{(2|k| - 2|k|\alpha)t + \alpha \varphi(0, x)}, \]  

(2.41)

for all \( x \in X \) and \( t > 0 \).
Proof. Let \((S^*, d)\) be the generalized metric space defined as in the proof of Theorem 2.5. It follows from (2.34) that
\[
N \left( k^2 f \left( \frac{x}{k} \right) - f(x), \frac{|k| t}{2} \right) \geq \frac{t}{t + \varphi(0, x/k)} \geq \frac{t}{t + (\alpha/k^2)\varphi(0, x)},
\]
for all \(x \in X\) and \(t > 0\). So
\[
N \left( f(x) - k^2 f \left( \frac{x}{k} \right), \frac{a t}{2|k|} \right) \geq \frac{t}{t + \varphi(0, x)}.
\]
The rest of the proof is similar to the proofs of Theorems 2.1 and 2.3. \(\square\)

**Corollary 2.8.** Let \(\theta \geq 0\) and let \(r\) be a real number with \(r > 1\). Let \(X\) be a normed vector space with norm \(|| \cdot ||\). Let \(f : X \rightarrow Y\) be an even mapping with \(f(0) = 0\) and satisfying (2.17). Then \(Q(x) := N - \lim_{n \rightarrow \infty} k^{2n} f(x/k^n)\) exists for each \(x \in X\) and defines a unique quadratic mapping \(Q : X \rightarrow Y\) such that
\[
N(f(x) - Q(x), t) \geq \frac{\left(2|k|^{2r+1} - 2|k|^3\right) t}{2|k|^{2r+1} - 2|k|^3 t + k^2 \theta ||x||^r},
\]
for all \(x \in X\) and all \(t > 0\).

**Proof.** It follows from Theorem 2.7 by taking \(\varphi(x, y) = \theta(||x||^r + ||y||^r)\) for all \(x, y \in X\). Then we can choose \(\alpha = k^{2-2r}\) and we get the desired result. \(\square\)

### 3. Direct Method

In this section, using direct method, we prove the Hyers-Ulam stability of functional equation (1.8) in fuzzy Banach spaces. Throughout this section, we assume that \(X\) is a linear space, \((Y, N)\) is a fuzzy Banach space, and \((Z, N')\) is a fuzzy normed space. Moreover, we assume that \(N(x, \cdot)\) is a left continuous function on \(\mathbb{R}\).

**Theorem 3.1.** Assume that a mapping \(f : X \rightarrow Y\) is an odd mapping with \(f(0) = 0\) satisfying the inequality
\[
N \left( f(k(x + y)) + f(k(x - y)) - f(kx + y) - f(kx - y) - \frac{2(k + 1)}{k} f(ky) + 2(k + 1) f(y), t \right) \\
\geq N' \left( \varphi(x, y), t \right),
\]
for all \(x, y \in X\), \(t > 0\), and \(\varphi : X^2 \rightarrow Z\) is a mapping for which there is a constant \(r \in \mathbb{R}\) satisfying \(0 < |r| < 1/|k|\) such that
\[
N' \left( \varphi \left( \frac{x}{k}, \frac{y}{k} \right), t \right) \geq N' \left( \varphi(x, y), \frac{t}{|r|} \right),
\]
(3.2)
for all $x, y \in X$ and all $t > 0$. Then there exists a unique additive mapping $A : X \rightarrow Y$ satisfying (1.8) and the inequality

$$N\left(f(x) - A(x), t\right) \geq N'\left(\varphi(0, x), \frac{|2k + 2|(1 - |kr|)t}{|r|}\right),$$  \hfill (3.3)

for all $x \in X$ and all $t > 0$.

Proof. It follows from (3.2) that

$$N'\left(\varphi\left(\frac{x}{k^j}, \frac{y}{k^j}\right), t\right) \geq N'\left(\varphi(x, y), \frac{t}{|r|}\right),$$  \hfill (3.4)

for all $x, y \in X$ and all $t > 0$. Putting $x = 0$ in (3.1) and then replacing $y$ by $x/k$, we get

$$N\left(kf\left(\frac{x}{k}\right) - f(x), \frac{|k|t}{2k + 2}\right) \geq N'\left(\varphi\left(0, \frac{x}{k}\right), t\right),$$  \hfill (3.5)

for all $x \in X$ and all $t > 0$. Replacing $x$ by $x/k^j$ in (3.5), we have

$$N\left(k^{j+1}f\left(\frac{x}{k^{j+1}}\right) - k^j f\left(\frac{x}{k^j}\right), \frac{|k|^{j+1}t}{2k + 2}\right) \geq N'\left(\varphi\left(0, \frac{x}{k^{j+1}}\right), t\right) \geq N'\left(\varphi(0, x), \frac{t}{|r|^{j+1}}\right),$$  \hfill (3.6)

for all $x \in X$, all $t > 0$, and all integer $j \geq 0$. So

$$N\left(f(x) - k^n f\left(\frac{x}{k^n}\right), \sum_{j=0}^{n-1} \frac{|k|^{j+1}t}{2k + 2}\right)$$

$$= N\left(\sum_{j=0}^{n-1} k^{j+1} f\left(\frac{x}{k^{j+1}}\right) - k^j f\left(\frac{x}{k^j}\right), \sum_{j=0}^{n-1} \frac{|k|^{j+1}t}{2k + 2}\right)$$

$$\geq \min_{0 \leq j \leq n-1} \left\{ N\left(k^{j+1} f\left(\frac{x}{k^{j+1}}\right) - k^j f\left(\frac{x}{k^j}\right), \frac{|k|^{j+1}t}{2k + 2}\right) \right\}$$

$$\geq \min_{0 \leq j \leq n-1} \{ N'\left(\varphi(0, x), t\right) \}$$

$$= N'\left(\varphi(0, x), t\right),$$

which yields

$$N\left(k^{n+p} f\left(\frac{x}{k^{n+p}}\right) - k^n f\left(\frac{x}{k^p}\right), \sum_{j=0}^{n-1} \frac{|k|^{j+p+1}t}{2k + 2}\right) \geq N'\left(\varphi\left(0, \frac{x}{2^p}\right), t\right) \geq N'\left(\varphi(0, x), \frac{t}{|r|^p}\right),$$  \hfill (3.8)
for all $x \in X$, $t > 0$, and all integers $n > 0$, $p \geq 0$. So

$$N \left( k^{n+p} f \left( \frac{x}{k^{n+p}} \right) - k^p f \left( \frac{x}{k^p} \right), \sum_{j=0}^{n-1} \frac{|k|^{j+p+1} |r|^{j+p+1}}{|2k + 2|} \right) \geq N' \left( \varphi(0, x), t \right), \tag{3.9}$$

for all $x \in X$, $t > 0$, and any integers $n > 0$, $p \geq 0$. Hence one can obtain

$$N \left( k^{n+p} f \left( \frac{x}{k^{n+p}} \right) - k^p f \left( \frac{x}{k^p} \right), t \right) \geq N' \left( \varphi(0, x), \frac{t}{\sum_{j=0}^{n-1} \left( |k|^{j+p+1} |r|^{j+p+1} / |2k + 2| \right)} \right), \tag{3.10}$$

for all $x \in X$, $t > 0$, and any integers $n > 0$, $p \geq 0$. Since the series $\sum_{j=0}^{\infty} k^j |r|^j$ is a convergent series, we see by taking the limit $p \to \infty$ in the last inequality that the sequence $\{k^n f(x/k^n)\}$ is a Cauchy sequence in the fuzzy Banach space $(Y, N)$ and so it converges in $Y$. Therefore a mapping $A : X \to Y$ defined by $A(x) := N - \lim_{n \to \infty} k^n f(x/k^n)$ is well defined for all $x \in X$. This means that

$$\lim_{n \to \infty} N \left( A(x) - k^n f \left( \frac{x}{k^n} \right), t \right) = 1, \tag{3.11}$$

for all $x \in X$ and all $t > 0$. In addition, it follows from (3.10) that

$$N \left( f(x) - k^n f \left( \frac{x}{k^n} \right), t \right) \geq N' \left( \varphi(0, x), \frac{t}{\sum_{j=0}^{n-1} \left( |k|^{j+p+1} |r|^{j+p+1} / |2k + 2| \right)} \right), \tag{3.12}$$

for all $x \in X$ and all $t > 0$. So

$$N \left( f(x) - A(x), t \right) \geq \min \left\{ N \left( f(x) - k^n f \left( \frac{x}{k^n} \right), (1 - \epsilon) t \right), N \left( A(x) - k^n f \left( \frac{x}{k^n} \right), \epsilon t \right) \right\}$$

$$\geq N' \left( \varphi(0, x), \frac{\epsilon t}{\sum_{j=0}^{n-1} \left( |k|^{j+p+1} |r|^{j+p+1} / |2k + 2| \right)} \right)$$

$$\geq N' \left( \varphi(0, x), \frac{|2k + 2|(1 - |k||r|) \epsilon t}{|kr|} \right), \tag{3.13}$$

for sufficiently large $n$ and for all $x \in X$, $t > 0$, and $\epsilon$ with $0 < \epsilon < 1$. Since $\epsilon$ is arbitrary and $N'$ is left continuous, we obtain

$$N \left( f(x) - A(x), t \right) \geq N' \left( \varphi(0, x), \frac{|2k + 2|(1 - |k||r|) t}{|kr|} \right), \tag{3.14}$$
for all $x, y \in X$ and $t > 0$. Therefore, we obtain in view of (3.11)

$$
N\left( \frac{f(k^n (x + ky))}{k^n} + \frac{f(k^n (x - ky))}{k^n} - \frac{f(k^n (x + y))}{k^n} - \frac{f(k^n (x - y))}{k^n} - \frac{2(k + 1)}{k} f(k^{n+1}y), t \right) 
\geq \min \left\{ N\left( \frac{f(k^n (x + ky))}{k^n} + \frac{f(k^n (x - ky))}{k^n} - \frac{f(k^n (x + y))}{k^n} - \frac{f(k^n (x - y))}{k^n} - \frac{2(k + 1)}{k} f(k^{n+1}y) - 2(k + 1) \frac{f(k^n y)}{k^n}, \frac{t}{2} \right), 
N\left( \frac{f(k^n (x + ky))}{k^n} + \frac{f(k^n (x - ky))}{k^n} - \frac{f(k^n (x + y))}{k^n} - \frac{f(k^n (x - y))}{k^n} - \frac{2(k + 1)}{k} f(k^{n+1}y) + 2(k + 1) \frac{f(k^n y)}{k^n}, \frac{t}{2} \right) \right\}
= N\left( \frac{f(k^n (x + ky))}{k^n} + \frac{f(k^n (x - ky))}{k^n} - \frac{f(k^n (x + y))}{k^n} - \frac{f(k^n (x - y))}{k^n} - \frac{2(k + 1)}{k} f(k^{n+1}y) + 2(k + 1) \frac{f(k^n y)}{k^n}, \frac{t}{2} \right) \right\}
\geq N'\left( \frac{f(x,y)}{2|k|^n |r|^n}, \frac{t}{2 |k|^n |r|^n} \right) \rightarrow 1 \quad \text{as } n \rightarrow +\infty,
$$

(3.16)

for all $x, y \in X$ and all $t > 0$, which implies that

$$
A(k(x + y)) + A(k(x - y)) = A(kx + y) + A(kx - y) + \frac{2(k + 1)}{k} A(ky) - 2(k + 1) A(y).
$$

(3.17)

Hence the mapping $A : X \rightarrow Y$ is additive, as desired.
To prove the uniqueness, let there be another mapping \( L : X \to Y \) which satisfies the inequality (3.3). Since \( L(k^n x) = k^n L(x) \) for all \( x \in X \), we have

\[
N(A(x) - L(x), t) = N\left( k^n A\left( \frac{x}{k^n} \right) - k^n L\left( \frac{x}{k^n} \right), t \right)
\]
\[
\geq \min \left\{ N\left( k^n A\left( \frac{x}{k^n} \right) - k^n f\left( \frac{x}{k^n} \right), \frac{t}{2} \right), N\left( f\left( \frac{x}{k^n} \right) - k^n L\left( \frac{x}{k^n} \right), \frac{t}{2} \right) \right\}
\]
\[
\geq N'\left( \frac{2k + 2|1 - |k|||t|}{2|k||^n| \cdot |r|} \right) \rightarrow 1 \quad \text{as} \quad n \to \infty,
\]

(3.18)

for all \( t > 0 \). Therefore \( A(x) = L(x) \) for all \( x \in X \). This completes the proof.

\[ \square \]

**Corollary 3.2.** Let \( X \) be a normed space and let \((\mathbb{R}, N')\) be a fuzzy Banach space. Assume that there exist real numbers \( \theta \geq 0 \) and \( p > 1 \) such that an odd mapping \( f : X \to Y \) with \( f(0) = 0 \) satisfies the following inequality:

\[
N\left( f(k(x + y)) + f(k(x - y)) - f(kx + y) - f(kx - y) - \frac{2(k + 1)}{k} f(ky) + 2(k + 1) f(y), t \right)
\]
\[
\geq N'\left( \theta(||x||^p + ||y||^p), t \right)
\]

(3.19)

for all \( x, y \in X \) and \( t > 0 \). Then there is a unique additive mapping \( A : X \to Y \) satisfying (1.8) and the inequality

\[
N(f(x) - A(x), t) \geq N'\left( \frac{\theta||x||^p}{2k + 2}, \left( \frac{|k|^p - |k|}{|k|} \right)t \right).
\]

(3.20)

**Proof.** Let \( \varphi(x, y) := \theta(||x||^p + ||y||^p) \) and \( |r| = |k|^p \). Applying Theorem 3.1, we get desired results.

\[ \square \]

**Theorem 3.3.** Let \( f : X \to Y \) be an odd mapping with \( f(0) = 0 \) satisfying the inequality (3.1) and let \( \varphi : X^2 \to Z \) be a mapping for which there exists a constant \( r \in \mathbb{R} \) satisfying \( 0 < |r| < |k| \) such that

\[
N'(\varphi(x, y), |r|t) \geq N'\left( \varphi\left( \frac{x}{k}, \frac{y}{k} \right), t \right).
\]

(3.21)
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Let

\[ \text{Corollary 3.4.} \]

Then there exists a unique additive mapping \( A : X \to Y \) satisfying (1.8) and the following inequality:

\[
N(f(x) - A(x), t) \geq N' \left( \varphi(0, x), \frac{|2k + 2|(|k| - |r|)t}{|k|} \right),
\]

(3.22)

for all \( x \in X \) and all \( t > 0 \).

**Proof.** It follows from (3.5) that

\[
N \left( \frac{f(kx) - f(x)}{k} - \frac{f(k^n x) - f(x)}{k^n}, \frac{t}{2k + 2} \right) \geq N' \left( \varphi(0, x), t \right),
\]

(3.23)

for all \( x \in X \) and all \( t > 0 \). Replacing \( x \) by \( k^n x \) in (3.41), we obtain

\[
N \left( \frac{f(k^{n+1} x) - f(k^n x)}{k^{n+1}} - \frac{f(k^n x) - f(x)}{k^n}, \frac{t}{2k + 2 k^n} \right) \geq N' \left( \varphi(0, k^n x), t \right) \geq N' \left( \varphi(0, x), \frac{t}{|r|^n} \right).
\]

(3.24)

So

\[
N \left( \frac{f(k^{n+1} x) - f(k^n x)}{k^{n+1}} - \frac{f(k^n x) - f(x)}{k^n}, \frac{|r|^n t}{2k + 2 k^n} \right) \geq N' \left( \varphi(0, x), t \right),
\]

(3.25)

for all \( x \in X \) and all \( t > 0 \). Proceeding as in the proof of Theorem 3.1, we obtain that

\[
N \left( f(x) - \frac{f(k^n x)}{k^n}, \sum_{j=0}^{n-1} \frac{|r|^j t}{2k + 2 |k|^j} \right) \geq N' \left( \varphi(0, x), t \right),
\]

(3.26)

for all \( x \in X, t > 0 \), and any integer \( n > 0 \). So

\[
N \left( f(x) - \frac{f(k^n x)}{k^n}, t \right) \geq N' \left( \varphi(0, x), \frac{t}{\sum_{j=0}^{n-1} |r|^j / |2k + 2 |k|^j} \right).
\]

(3.27)

The rest of the proof is similar to the proof of Theorem 3.1. \( \square \)

**Corollary 3.4.** Let \( X \) be a normed space and let \((\mathbb{R}, N')\) be a fuzzy Banach space. Assume that there exist real numbers \( \theta \geq 0 \) and \( 0 < p < 1 \) such that an odd mapping \( f : X \to Y \) with \( f(0) = 0 \) satisfies (3.19). Then there exists a unique additive mapping \( A : X \to Y \) satisfying (1.8) and the inequality

\[
N(f(x) - A(x), t) \geq N' \left( \varphi(0, x), \frac{|2k + 2|(|k| - |k|^p) t}{|k|} \right).
\]

(3.28)
Proof. Let \( \varphi(x, y) := \theta(\|x\|^p + \|y\|^p) \) and \( |r| = |k|^p \). Applying Theorem 3.3, we get the desired results.

\[ \square \]

**Theorem 3.5.** Let \( f : X \to Y \) be an even mapping with \( f(0) = 0 \) satisfying the inequality (3.1) and let \( \varphi : X^2 \to Z \) be a mapping for which there exists a constant \( r \in \mathbb{R} \) such that \( 0 < |r| < 1/k^2 \) and that

\[ N'(\varphi\left( \frac{x}{k} - \frac{y}{k} \right), t) \geq N'(\varphi(x, y), t), \]  

for all \( x, y \in X \) and all \( t > 0 \). Then there exists a unique quadratic mapping \( Q : X \to Y \) satisfying (1.8) and the inequality

\[ N(f(x) - Q(x), t) \geq N'(\varphi(0, x), \frac{2(1 - |k|^2)}{|k|^2} |t|), \]  

for all \( x \in X \) and all \( t > 0 \).

**Proof.** Replacing \( x \) by \( kx \) in (3.1), we get

\[ N\left( f(k(x + y)) + f(k(x - y)) - f(kx + y) - f(kx - y) - \frac{2(k + 1)}{k} f(ky) + 2(k + 1)f(y), t \right) \]

\[ \geq N'(\varphi(kx, y), t), \]  

for all \( x, y \in X \) and all \( t > 0 \). Putting \( x = 0 \) and replacing \( y \) by \( x \) in (3.31), we have

\[ N\left( \frac{f(kx)}{k^2} - f(x), \frac{t}{|k|^2} \right) \geq N'(\varphi(0, x), t), \]  

for all \( x \in X \) and all \( t > 0 \). Replacing \( x \) by \( x/k \) in (3.32), we find

\[ N\left( \frac{k^2 f\left( \frac{x}{k} \right) - f(x), \frac{|k|^2 t}{2} \right) \geq N'(\varphi\left( 0, \frac{x}{k} \right), t), \]  

for all \( x \in X \) and all \( t > 0 \). Also, replacing \( x \) by \( x/k^n \) in (3.33), we obtain

\[ N\left( k^{2n+2} f\left( \frac{x}{k^n} \right) - k^{2n} f\left( \frac{x}{k^n} \right), \frac{|k|^{2n+1} t}{2} \right) \geq N'(\varphi\left( 0, \frac{x}{k^{n+1}} \right), t) \geq N'(\varphi(0, x), \frac{t}{|r|^{n+1}}). \]

So

\[ N\left( k^{2n+2} f\left( \frac{x}{k^n} \right) - k^{2n} f\left( \frac{x}{k^n} \right), \frac{|k|^{2n+1} |t|}{2} \right) \geq N'(\varphi(0, x), t), \]

(3.35)
for all $x \in X$ and all $t > 0$. Proceeding as in the proof of Theorem 3.1, we obtain that

$$N\left( f(x) - k^{2n} f\left( \frac{x}{k^n} \right), \sum_{j=0}^{n-1} \frac{|k^{j+1}r|^{j+1}}{2^j}, t \right) \geq N'(\psi(0, x), t),$$

(3.36)

for all $x \in X$, all $t > 0$, and any integer $n > 0$. So

$$N\left( f(x) - k^{2n} f\left( \frac{x}{k^n} \right), t \right) \geq N'\left( \psi(0, x), \frac{t}{\sum_{j=0}^{n-1} \left( |k^{j+1}r|^{j+1}t/2 \right)} \right).$$

(3.37)

The rest of the proof is similar to the proof of Theorem 3.1.

\[ \Box \]

**Corollary 3.6.** Let $X$ be a normed space and let $(\mathbb{R}, N')$ be a fuzzy Banach space. Assume that there exist real numbers $\Theta \geq 0$ and $p > 1$ such that an even mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.19). Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (1.8) and the inequality

$$N\left( f(x) - Q(x), t \right) \geq N'\left( \theta \|x\|^p, \frac{2(k^2 - k^2)t}{\|x\|} \right).$$

(3.38)

**Proof.** Let $\psi(x, y) := \theta(\|x\|^p + \|y\|^p)$ and $|r| = |k|^{-2p}$. Applying Theorem 3.5, we get the desired results. \[ \Box \]

**Theorem 3.7.** Assume that an even mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality (3.1) and $\psi : X^2 \to Z$ is a mapping for which there is a constant $r \in \mathbb{R}$ satisfying $0 < |r| < k^2$ such that

$$N'(\psi(x, y), |r|t) \geq N'\left( \psi\left( \frac{x}{k^2}, \frac{y}{k^2} \right), t \right),$$

(3.39)

for all $x, y \in X$ and all $t > 0$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (1.8) and the following inequality

$$N\left( f(x) - Q(x), t \right) \geq N'\left( \psi(0, x), \frac{2(k^2 - |r|)t}{|r|} \right),$$

(3.40)

for all $x \in X$ and all $t > 0$.

**Proof.** It follows from (3.32) that

$$N\left( \frac{f(kx)}{k^2} - f(x), \frac{t}{2k} \right) \geq N'\left( \psi(0, x), t \right),$$

(3.41)
for all $x \in X$ and all $t > 0$. Replacing $x$ by $k^n x$ in (3.41), we obtain

$$N\left(\frac{f(k^{n+1}x)}{k^{2n+2}} - \frac{f(k^nx)}{k^{2n}}, \frac{t}{2|k|^{2n+1}}\right) \geq N'(\varphi(0, k^n x), t) \geq N'(\varphi(0, x), \frac{t}{|r|^n}), \quad \text{(3.42)}$$

for all $x \in X$ and all $t > 0$. So

$$N\left(\frac{f(k^{n+1}x)}{k^{2n+2}} - \frac{f(k^nx)}{k^{2n}}, \frac{|r|^nt}{2|k|^{2n+1}}\right) \geq N'(\varphi(0, x), t), \quad \text{(3.43)}$$

for all $x \in X$ and all $t > 0$. So

$$N\left(f(x) - \frac{f(k^n x)}{k^{2n}}, t\right) \geq N'(\varphi(0, x), \frac{t}{\sum_{j=0}^{n-1} (|r|^jt/2|k|^{2j+1})}). \quad \text{(3.44)}$$

The rest of the proof is similar to the proof of Theorem 3.1.

**Corollary 3.8.** Let $X$ be a normed space and let $(\mathbb{R}, N')$ be a fuzzy Banach space. Assume that there exist real numbers $\theta \geq 0$ and $0 < p < 1$ such that an even mapping $f : X \to Y$ with $f(0) = 0$ satisfies (3.19). Then there is a unique quadratic mapping $Q : X \to Y$ satisfying (1.8) and the inequality

$$N(f(x) - Q(x), t) \geq N'(\varphi(0, x), \frac{2(k^2 - k^{2p})t}{|k|}), \quad \text{(3.45)}$$

for all $x \in X$, all $t > 0$.

**Proof.** Let $\varphi(x, y) := \theta(||x||^p + ||y||^p)$ and $|r| = k^{2p}$. Applying Theorem 3.7, we get the desired results.

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**References**

[1] S. M. Ulam, *Problems in Modern Mathematics*, John Wiley & Sons, Inc, New York, NY, USA, 1964.
[2] D. H. Hyers, “On the stability of the linear functional equation,” *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.
[3] T. M. Rassias, “On the stability of the linear mapping in Banach spaces,” *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.
[4] Z. Gajda, “On stability of additive mappings,” *International Journal of Mathematics and Mathematical Sciences*, vol. 14, no. 3, pp. 431–434, 1991.
[5] S. Abbaszadeh, “Intuitionistic fuzzy stability of a quadratic and quartic functional equation,” International Journal of Nonlinear Analysis and Applications, vol. 1, no. 2, pp. 100–124, 2010.
[6] J. Aczél and J. Dhombres, Functional Equations in Several Variables, vol. 31, Cambridge University Press, Cambridge, UK, 1989.
[7] T. Aoki, “On the stability of the linear transformation in Banach spaces,” Journal of the Mathematical Society of Japan, vol. 2, pp. 64–66, 1950.
[8] D. G. Bourgin, “Classes of transformations and bordering transformations,” Bulletin of the American Mathematical Society, vol. 57, pp. 223–237, 1951.
[9] M. B. Savadkouhi, M. E. Gordji, J. M. Rassias, and N. Ghobadipour, “Approximate ternary Jordan derivations on Banach ternary algebras,” Journal of Mathematical Physics, vol. 50, no. 4, article 042303, p. 9, 2009.
[10] P. Gavruta, “A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings,” Journal of Mathematical Analysis and Applications, vol. 184, no. 3, pp. 431–436, 1994.
[11] P. Gavruta and L. Gavruta, “A new method for the generalized Hyers-Ulam-Rassias stability,” Journal of Mathematical Analysis and Applications, vol. 1, no. 2, pp. 11–18, 2010.
[12] D. H. Hyers, G. Isac, and T. M. Rassias, Stability of Functional Equations in Several Variables, Progress in Nonlinear Differential Equations and their Applications, 34, Birkhäuser Boston Inc., Boston, Mass, USA, 1998.
[13] G. Isac and T. M. Rassias, “On the Hyers-Ulam stability of g -additive mappings,” Journal of Approximation Theory, vol. 72, no. 2, pp. 131–137, 1993.
[14] C. Park and M. E. Gordji, “Comment on “Approximate ternary Jordan derivations on Banach ternary algebras” [B. Savadkouhi Journal of Mathematical Physics vol. 50, article 042303, 2009],” Journal of Mathematical Physics, vol. 51, no. 4, 2010.
[15] C. Park and A. Najati, “Generalized additive functional inequalities in Banach algebras,” International Journal of Nonlinear Analysis and Applications, vol. 1, no. 2, pp. 54–62, 2010.
[16] C. Park and T. M. Rassias, “Isomorphisms in unital C*-algebras,” International Journal of Nonlinear Analysis and Applications, vol. 1, no. 2, pp. 1–10, 2010.
[17] T. M. Rassias, “On the stability of functional equations and a problem of Ulam,” Acta Applicandae Mathematicae, vol. 62, no. 1, pp. 23–130, 2000.
[18] P. Kannappan, “Quadratic functional equation and inner product spaces,” Results in Mathematics, vol. 27, no. 3-4, pp. 368–372, 1995.
[19] F. Skof, “Local properties and approximation of operators,” Rendiconti del Seminario Matematico e Fisico di Milano, vol. 53, pp. 113–129, 1983.
[20] P. W. Cholewa, “Remarks on the stability of functional equations,” Aequationes Mathematicae, vol. 27, no. 1-2, pp. 76–86, 1984.
[21] S. Czerwik, “On the stability of the quadratic mapping in normed spaces,” Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, vol. 62, pp. 59–64, 1992.
[22] A. Grabiec, “The generalized Hyers-Ulam stability of a class of functional equations,” Publicationes Mathematicae Debrecen, vol. 48, no. 3-4, pp. 217–235, 1995.
[23] C. Borelli and G. L. Forti, “On a general Hyers-Ulam stability result,” International Journal of Mathematics and Mathematical Sciences, vol. 18, no. 2, pp. 229–236, 1995.
[24] C.-G. Park, “Generalized quadratic mappings in several variables,” Nonlinear Analysis. Theory, Methods & Applications, vol. 57, no. 5-6, pp. 713–722, 2004.
[25] C.-G. Park, “On the stability of the quadratic mapping in Banach modules,” Journal of Mathematical Analysis and Applications, vol. 276, no. 1, pp. 135–144, 2002.
[26] A. Ebadian, A. Najati, and M. Eshaghi Gordji, “On approximate additive-quartic and quadratic-cubic functional equations in two variables on abelian groups,” Results in Mathematics, vol. 58, no. 1-2, pp. 39–53, 2010.
[27] M. Eshaghi Gordji, “Stability of a functional equation deriving from quartic and additive functions,” Bulletin of the Korean Mathematical Society, vol. 47, no. 3, pp. 491–502, 2010.
[28] M. Eshaghi Gordji, “Stability of an additive-quadratic functional equation of two variables in F-spaces,” Journal of Nonlinear Science and its Applications, vol. 2, no. 4, pp. 251–259, 2009.
[29] M. Eshaghi Gordji and M. B. Savadkouhi, “Stability of cubic and quartic functional equations in non-Archimedean spaces,” Acta Applicandae Mathematicae, vol. 110, no. 3, pp. 1321–1329, 2010.
[30] M. Eshaghi Gordji and M. B. Savadkouhi, “Stability of a mixed type cubic-quartic functional equation in non-Archimedean spaces,” Applied Mathematics Letters, vol. 23, no. 10, pp. 1198–1202, 2010.
[31] M. Eshaghi Gordji and M. B. Savadkouhi, “Approximation of generalized homomorphisms in quasi-Banach algebras,” *Mathematical Journal of the Ovidius University of Constantza*, vol. 17, no. 2, pp. 203–213, 2009.

[32] M. Eshaghi Gordji and M. Bavand Savadkouhi, “On approximate cubic homomorphisms,” *Advances in Difference Equations*, vol. 2009, Article ID 618463, 11 pages, 2009.

[33] M. Eshaghi Gordji, S. Zolfaghari, J. M. Rassias, and M. B. Savadkouhi, “Solution and stability of a mixed type cubic and quartic functional equation in quasi-Banach spaces,” *Abstract and Applied Analysis*, vol. 2009, Article ID 41747, 14 pages, 2009.

[34] M. Eshaghi Gordji and M. B. Savadkouhi, “Stability of mixed type cubic and quartic functional equations in random normed spaces,” *Journal of Inequalities and Applications*, vol. 2009, Article ID 527462, 9 pages, 2009.

[35] M. Eshaghi Gordji, A. Ebadian, and S. Zolfaghari, “Stability of a functional equation deriving from cubic and quartic functions,” *Abstract and Applied Analysis*, vol. 2008, Article ID 801904, 17 pages, 2008.

[36] M. Eshaghi Gordji, S. Kaboli Gharehpayeh, J. M. Rassias, and S. Zolfaghari, “Solution and stability of a mixed type additive, quadratic, and cubic functional equation,” *Advances in Difference Equations*, vol. 2009, Article ID 826130, 17 pages, 2009.

[37] M. Eshaghi-Gordji, S. Kaboli-Gharetapeh, C. Park, and S. Zolfaghari, “Stability of an additive-cubic-quartic functional equation,” *Advances in Difference Equations*, vol. 2009, Article ID 395693, 20 pages, 2009.

[38] M. Eshaghi Gordji, T. Karimi, and S. Kaboli-Gharetapeh, “Approximately \( n \)-Jordan homomorphisms on Banach algebras,” *Journal of Inequalities and Applications*, vol. 2009, Article ID 870843, 8 pages, 2009.

[39] M. Eshaghi Gordji and H. Khodaei, “Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi-Banach spaces,” *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 11, pp. 5629–5643, 2009.

[40] M. Eshaghi Gordji, H. Khodaei, and R. Khodabakhsh, “General quartic-cubic-quadratic functional equation in non-Archimedean normed spaces,” *“Politehnica” University of Bucharest Scientific Bulletin. Series A*, vol. 72, no. 3, pp. 69–84, 2010.

[41] M. Eshaghi Gordji and A. Najati, “Approximately \( J \)-homomorphisms: a fixed point approach,” *Journal of Geometry and Physics*, vol. 60, no. 5, pp. 809–814, 2010.

[42] M. Eshaghi Gordji, S. Zolfaghari, S. Kaboli-Gharetapeh, A. Ebadian, and C. Park, “Solution and stability of generalized mixed type additive and quadratic functional equation in non-Archimedean spaces,” to appear in *Annali dell’Università di Ferrara*.

[43] R. Farokhzad and S. A. R. Hosseinioun, “Perturbations of Jordan higher derivations in Banach ternary algebras: an alternative fixed point approach,” *International Journal of Nonlinear Analysis and Applications*, vol. 1, no. 1, pp. 42–53, 2010.

[44] M. Eshaghi Gordji and H. Khodaei, “On the generalized Hyers-Ulam-Rassias stability of quadratic functional equations,” *Abstract and Applied Analysis*, vol. 2009, Article ID 923476, 11 pages, 2009.

[45] M. Eshaghi Gordji, S. Kaboli-Gharetapeh, J. M. Rassias, and S. Zolfaghari, “Solution and stability of a mixed type additive, quadratic, and cubic functional equation,” *Advances in Difference Equations*, vol. 2009, Article ID 826130, 17 pages, 2009.

[46] M. Eshaghi Gordji, S. Kaboli-Gharetapeh, E. Rashidi, T. Karimi, and M. Aghaei, “Ternary Jordan-derivations in \( C \)-ternary algebras,” *Journal of Computational Analysis and Applications*, vol. 12, no. 2, pp. 463–470, 2010.

[47] M. Eshaghi Gordji, J. M. Rassias, and N. Ghobadipour, “Generalized Hyers-Ulam stability of generalized \((N,K)\)-derivations,” *Abstract and Applied Analysis*, vol. 2009, Article ID 437931, 8 pages, 2009.

[48] T. Bag and S. K. Samanta, “Finite dimensional fuzzy normed linear spaces,” *Journal of Fuzzy Mathematics*, vol. 11, no. 3, pp. 687–705, 2003.

[49] T. Bag and S. K. Samanta, “Fuzzy bounded linear operators,” *Fuzzy Sets and Systems*, vol. 151, no. 3, pp. 513–547, 2005.

[50] D. Miheţ and V. Radu, “On the stability of the additive Cauchy functional equation in random normed spaces,” *Journal of Mathematical Analysis and Applications*, vol. 343, no. 1, pp. 567–572, 2008.