M(embrane) theory on $T^9$.

P.K. Townsend$^a$

$^a$DAMTP, University of Cambridge
Silver St., Cambridge CB3 9EW, U.K.

An 'algebraic' approach to M-theory is briefly reviewed, and a proposal is made for a similar algebraic structure underlying the $T^9$ compactification of 'M(embrane) theory', i.e. the M(atrix) model with area-preserving diffeomorphism gauge group.

1. INTRODUCTION

This conference provided evidence of a consensus that the most promising candidate for a non-perturbative theory of quantum gravity is the conjectured 11-dimensional 'M-theory', despite the fact that it has yet to be adequately defined. That this is possible is due largely to the remarkable power of supersymmetry, which often allows otherwise unwarranted extrapolation of semi-classical results. For some time I have been advocating an approach to M-theory based on the M-theory superalgebra [1], i.e. the 'maximal central extension' of the $D=11$ supertranslation algebra (originally introduced in a mathematical context [2]). This algebra, which is a contraction of $osp(1|32;\mathbf{R})$, is spanned by the 32-component spinor charge $Q$, the 11-momentum $P$, a two-form charge $Z$ and a 5-form charge $Y$.

The only non-zero (anti)commutator is

\begin{equation}
\{Q_\alpha, Q_\beta\} = \frac{1}{2}(C_{\Gamma m})_{\alpha\beta}P_m + \frac{1}{5!}(C_{\Gamma m_1...m_5})_{\alpha\beta}Y_{m_1...m_5}.
\end{equation}

where $\Gamma^m$ are Dirac matrices, $\Gamma^{m_1...m_p}$ ($p = 2, 5$) their antisymmetrized products, and $C$ the charge conjugation matrix. In the Majorana representation spinors are real and $C = \Gamma^0$. Setting $m = (0, m)$ ($m = 1, \ldots, 10$), the anticommutator \((1)\) can be rewritten as

\begin{equation}
\{Q, Q\} = P^0 (1 + \bar{\Gamma})
\end{equation}

where

\begin{equation}
P^0 \bar{\Gamma} = \Gamma^0 P_m + \frac{1}{2} \Gamma^{0mn} Z_{mn}
\end{equation}

It is remarkable what one can learn about M-theory by studying this superalgebra. For example, the positivity of the anticommutator $\{Q, Q\}$ implies that $P^0 \geq 0$, and that no eigenvalue of $\bar{\Gamma}^2$ can exceed unity. If $P^0 = 0$ we have the $D=11$ vacuum, or toroidal compactifications of it, in which supersymmetry is unbroken. Otherwise, the fraction $\nu$ of supersymmetry preserved is $1/32$ times the number of eigenvectors of $\bar{\Gamma}$ with eigenvalue $-1$. Since $\bar{\Gamma}$ is tracefree, $\nu$ cannot exceed $1/2$. In fact, the possible values are

\begin{equation}
\nu = \frac{1}{2}, \frac{1}{4}, \frac{3}{16}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}.
\end{equation}

The $\nu = 1/2$ charge configurations correspond to the 'basic' objects of M-theory: $P_m$ is associated with massless quanta of the effective field theory (or M-waves of $D=11$ supergravity), $Z_{mn}$ is associated with the supermembrane (the M-2-brane) and $Y_{m_1...m_5}$ with the superfivebrane (the M-5-brane). The time components of $Z$ and $Y$ are associated respectively, with the D-6-branes and D-8-branes that arise on compactification to $D=10$.

1This is an oversimplification because there are additional $\nu = 1/2$ 'mixed brane' configurations corresponding to non-marginal bound states of M-2-branes with M-5-branes. However, these can alternatively be viewed as excited states of the M-5-brane so there is a sense in which the 'basic' branes are sufficient.

2This point has been made independently in [3].
Charge configurations with $\nu < 1/2$ are associated with intersections of the ‘basic’ branes. They may also be associated with compactifications on manifolds of reduced holonomy but, at least in certain limits of moduli space, these may be viewed locally as intersecting brane configurations. Since there are some orbifold compactifications that can only be interpreted in this way (e.g. \( \mathbb{R}^4 \)), it seems that the intersecting brane perspective is the more powerful one. In my talk at the conference I described some aspects of work with Jerome Gauntlett, Gary Gibbons and George Papadopoulos on intersecting brane configurations with $\nu = 3/16$, but I intend to review this work, and intersecting brane configurations in general, elsewhere. I also discussed some aspects of how configurations preserving a given fraction of supersymmetry are related by dualities, but a rather more complete treatment of this has been provided by others (e.g. \[7\]). Here, I shall take the opportunity to present some investigations into the algebraic structure underlying the M(atrix) model conjecture \[6\] for the microscopic degrees of M-theory. The conclusion will be that it is remarkably similar to the M-theory superalgebra just described, which can be viewed as the algebraic structure underlying the macroscopic semi-classical aspects of M-theory.

2. FROM M(ATRIX) to M(EMBRANE)

The M(atrix) models of M-theory are $SU(k)$ supersymmetric gauge quantum mechanics (SGQM) models with 16 supersymmetries, of the type first investigated in \[1\]. It was shown in \[10\] that these models can be viewed as regularizations of the light-cone gauge-fixed D=11 supermembrane \[11\], which is an SGQM model with the area-preserving diffeomorphism (APD) group of the membrane as its gauge group. It was observed in \[12\] that the SGQM model is also the effective action for a condensate of IIA D=0-branes, and that the continuity of the supermembrane spectrum \[13\] can therefore be re-interpreted as a ‘no force’ condition between constituent D=0-branes. This feature was used in \[8\] to interpret the first quantized Hilbert space of the SGQM model as an interacting multi-particle Hilbert space. The existence of a novel $k \to \infty$ limit was also proposed in which D=11 Lorentz covariance would (hopefully) be recovered without the usual problems associated with removal of a regulator. There is now evidence (see e.g. \[14\]) that scattering amplitudes of D=11 supergravity can be successfully recovered in this approach, with the virtue that the divergences of quantum D=11 supergravity are now under control, at least in principle.

For finite $k$ the M(atrix) model Hamiltonian is indeed one in which the variables are matrices, but this is not true of the $k \to \infty$ limit. In addition, the restoration of D=11 Lorentz invariance in this limit appears to be linked to the recovery of the APD group and hence to the supermembrane interpretation \[14\]. I therefore propose to call the SGQM model with area-preserving diffeomorphism gauge group the ‘M(embrane) model’. It differs from the (first quantized) supermembrane only in the insight derived from D-branes that M-theory compactified on $T^n$ is described by the ‘M(embrane) model on $\tilde{T}^n$’-meter, where $\tilde{T}^n$ is the dual n-torus to $T^n$ \[16\]. That is, the SGQM with APD gauge group must be replaced by an $(n+1)$-dimensional gauge field theory with APD gauge group, where the n-dimensional space is $\tilde{T}^n$. According to this prescription, M-theory on $S^1$, alias (second quantized) IIA superstring theory, is described by a $(1+1)$-dimensional APD gauge theory. This has recently been verified in considerable detail \[15\].

For $n > 1$ we should find the appropriate U-duality group of toroidally-compactified M-theory \[20\]. For $n = 2$ this group is $Sl(2; \mathbb{Z})$, which indeed arises in the M(atrix) models as the modular group of $\tilde{T}^2$. More generally, a gauge theory on $\tilde{T}^n$ has a manifest invariance under the $Sl(n; \mathbb{Z})$ modular group of $\tilde{T}^n$, but this is a proper subgroup of the U-duality group when $n > 2$. To recover the full U-duality group for $n \geq 3$ one must include the magnetic degrees of freedom of M(atrix) models, corresponding to 5-branes in M-theory. At this point we can see that the M(atrix) model approach is an ‘optimally democratic’ formulation of M-theory, in the sense of \[1\]. That is, it incorporates into a new type of perturbation theory all of the electric degrees of freedom of M-theory (as it was argued in \[8\]) that a su-
permembrane theory might do). Since the supermembrane is the strong coupling limit of the IIA superstring, the M(atrix) model approach is non-perturbative with respect to superstring theory, but it is still perturbative as a formulation of M-theory. A truly non-perturbative, and fully democratic, formulation of M-theory must incorporate the magnetic degrees of freedom.

For \( n = 3 \) it is clear how this is to be done \[27\]. The U-duality group of \( T^3 \)-compactified M-theory is \( SL(3;\mathbb{Z}) \times SL(2;\mathbb{Z}) \), the \( SL(2;\mathbb{Z}) \) factor being the ‘electromagnetic’ duality group acting on dyonic membranes \[22\] (equivalently, wrapped 2-brane/5-brane bound states in \( D=11 \)). The M(atrix) models on \( T^3 \) have an obvious \( SL(3;\mathbb{Z}) \) invariance but they are also conjectured to be invariant under a non-perturbative \( SL(2;\mathbb{Z}) \) electromagnetic S-duality \[23\]. For \( n \geq 3 \) the situation is further complicated by the non-renormalizability of \((n+1)\)-dimensional gauge theories. The M-theory dynamics should be governed by a superconformal fixed point theory to which the gauge theory dynamics should be governed by a superconformal UV-fixed point theory having some non-local theory, which is presumably admissible in a non-local theory.

These works point to a non-local \( D=10 \) superconformal UV-fixed point theory having some connection to area-preserving diffeomorphisms as the true ‘master’ theory underlying the M(atrix) model approach. In any case, the existence of some such theory, which I will call ‘Membra’ theory’, is clearly required by the conjecture of \[8\]. The problem is that it is apparently excluded by the absence of a superconformal group; according to the standard classification \[30\], superconformal groups exist for \( D \leq 6 \) but not for \( D > 6 \). Here I will suggest a resolution of this puzzle.

3. M(EMBRANE) SUPERALGEBRA

We first note that the classification of superconformal groups in \[28,21\] is based on the Coleman-Mandula theorem, which requires the bosonic symmetry group to be the direct product of the spacetime symmetry group (the \( D=10 \) conformal group \( SO(10, 2) \) in the case of interest here) and some internal symmetry group. The Coleman-Mandula theorem holds for local QFT. Once locality is abandoned there is no obstacle to superconformal invariance for \( D=10 \). In fact, there is then a natural candidate for the \( D=10 \) superconformal group.

As motivation, let us first recall that \( D=10 \)
gauge theories have instantonic fivebranes \footnote{I thank Paul Howe for reminding me of this.} which (as in D=4 \footnote{See footnote.}) lead to a central extension of the D=10 N=1 supertranslation algebra spanned by the 16 (independent) component chiral spinor charges $Q$, the 10-momentum $P$ and a self-dual 5-form charge $Z^+$. The only non-zero (anti)commutator is

$$\{Q_\alpha, Q_\beta\} = (CT^m P^+)^{\alpha\beta} P_m + \frac{1}{5!} (CT^{mnpqr})^{\alpha\beta} Z^+_{mnpqr}$$

where $P^+$ is the positive chirality projection operator on spinors. As noted in \footnote{See footnote.}, this is the ‘maximal central extension’ of the $N=1$ D=10 supertranslation algebra. On the basis of this algebra one would expect the stress-tensor supermultiplet to include a 6-form current associated to the 5-form charge. In fact\footnote{See footnote.}, there is a non-local 256-component superconformal current multiplet in D=10 for which the 128 bosonic components consist of the traceless stress tensor $T_{mn}$ and a 6-form current $J_{mnpqrs}$ \footnote{See footnote.}.

This confirms both that we must abandon locality and that, in doing so, we should seek a superconformal extension of \footnote{See footnote.}, which must involve an antichiral conformal supersymmetry charge $S^\alpha$ (we use the position of indices to keep track of chirality). It must also contain both the D=10 conformal algebra $so(10,2)$ and the supertranslation algebra \footnote{See footnote.} as sub-(super)algebras. It is not clear to me whether the solution to these requirements is unique but there is an obvious solution: the superalgebra $osp(1|32;\mathbf{R})$. The additional anticommutation relations are

$$\{S^\alpha, S^\beta\} = (CT^m P^-)^{\alpha\beta} K_m + \frac{1}{5!} (CT^{mnpqr})^{\alpha\beta} Z^-_{mnpqr}$$

$$\{S^\alpha, Q_\beta\} = \delta^\alpha_\beta D + \frac{1}{2} (\Gamma^{mn})^{\alpha\beta} M_{mn} + \frac{1}{4!} (\Gamma^{mnpq})^{\alpha\beta} Y_{mnpq}$$

where $K, D, M$ are the generators of conformal boosts, dilations and Lorentz transformations, respectively. The anti-self-dual 5-form charge $Z^-$ is the conformal analogue of $Z^+$ while $Y$ is a new 4-form charge.

The two spinor charges $Q$ and $S$ can be assembled into a 32-component real spinor of $SO(10,2)$ with components $Q_A = (Q_\alpha, S^\beta)$. Introducing the $SO(10,2)$ Dirac matrices $\Gamma^M$ and their antisymmetrized products, $\Gamma^{MN}$ etc., we can rewrite the anticommutators of (5), (6) and (7) in the manifestly $SO(10,2)$-covariant form

$$\{Q_A, Q_B\} = (CT^{MN} P^+)_{AB} L_{MN} + \frac{1}{6!} (CT^{MNPQRS})_{AB} T^+_{MNPQRS}$$

where $C$ is now the $SO(10,2)$ ‘charge conjugation’ matrix, and $P^+$ the chiral projector on $SO(10,2)$ spinors. $L$ comprises the generators $(K, D, M, P)$ of $SO(10,2)$, while $T^+$ (composed of $Z^+$ and $Y$) is a self-dual antisymmetric 6th rank tensor of $SO(10,2)$. In fact, the algebra is $Sp(32)$ covariant with $(L, T^+)$ in the $\mathbf{528}$ adjoint representation and $Q$ in the $\mathbf{32}$ representation. Similar algebras have appeared elsewhere in the context of speculations concerning a 12-dimensional extension of M-theory \footnote{See footnote.}, but here there is an important difference; the charge $T^+$ does not commute with $L$. The commutation relations of $(L, T^+)$ are those of $sp(32)$ for which the matrices $\Gamma^{MN} P^+$ and $\Gamma^{MNPQRS} P^+$ form the 32-dimensional representation. The full set of (anti)commutation relations are those of $osp(1|32;\mathbf{R})$.

4. COMMENTS

I began this article by explaining how the ‘M-theory superalgebra’, a contraction of $osp(1|32;\mathbf{R})$, encapsulates much of what we can learn from a semi-classical analysis of M-theory. I then turned to the consideration of the M(atrix), or M(embrane), model approach to the microscopic theory and asked whether there were some comparable algebraic structure underlying it. My conclusion was that there is and that it is $osp(1|32;\mathbf{R})$. As far as I can see this is just a coincidence; the algebras are actually quite different. The M-theory algebra does not include the Lorentz group (this has to be considered separately as its automorphism group) and the M(embrane) algebra is superconformal. Nevertheless, the possibility of a deeper connection.
hoefully provides a justification for the joint discussion here of both algebraic structures.

Of course, identifying the superconformal group associated with M(embrane) theory is only a small step towards its construction. Other clues are the expected $E_9$ invariance, and the fact that it should involve 5-branes. Also, it should reduce, on $T^4$ compactification to the (2,0) D=6 superconformal theory underlying M(atrix) theory on $\tilde{T}_5$; the strings of the latter model are presumably $T^4$-wrapped 5-branes of M(embrane) theory. Similarly, the membranes of the theory proposed to govern the $T^6$ compactification of M-theory are presumably $T_3$-wrapped 5-branes of M(embrane) theory.

REFERENCES

1. P.K. Townsend, $p$-brane democracy, in Particles, Strings and Cosmology, eds. J. Bagger, G. Domokos, A Falk and S. Kovacs-Domokos (World Scientific 1996), pp.271-285, hep-th/9507048; Four Lectures on M-theory, to appear in proceedings of the Summer School on High Energy Physics and Cosmology, ICTP, Trieste, June 1996, hep-th/9612121.

2. J.W. van Holten and A. Van Proeyen, J. Phys. A: Math Gen. 15 (1982) 3763.

3. D. Sorokin and P.K. Townsend, M-theory superalgebra from the M-5-brane, hep-th/9708007.

4. C.M. Hull, Gravitational duality, branes and charges, hep-th/9705162.

5. K. Dasgupta and S. Mukhi, Nucl. Phys. B465 (1996) 399; E. Witten, Nucl. Phys. B463 (1996) 383.

6. J.P. Gauntlett, G.W. Gibbons, G. Papadopoulos and P.K. Townsend, Hyper-Kähler manifolds and multiply intersecting branes, hep-th/9702202.

7. S. Ferrara and J. Maldacena, Branes, central charges and U-duality invariant BPS conditions, hep-th/9706097.

8. T. Banks, W. Fischler, S.H. Shenker and L. Susskind, Phys. Rev. D55 (1977) 112.

9. M. Claudson and M.B. Halpern, Nucl. Phys. B250 (1985) 689; R. Flume, Ann. Phys. (N.Y.) 164 (1985) 189; M. Baake, P. Reinicke and V. Rittenberg, J. Math. Phys. 26 (1985) 1070.

10. B. de Wit, J. Hoppe and H. Nicolai, Nucl. Phys. B305 [FS 23] (1988) 545.

11. E. Bergshoeff, E. Sezgin and P.K. Townsend, Phys. Lett. 189B (1987) 75; Ann. Phys. (N.Y.) 185 (1988) 330.

12. P.K. Townsend, Phys. Lett. 373B, (1996) 68.

13. B. de Wit, M. Lüscher and H. Nicolai, Nucl. Phys. B320 (1989) 135.

14. K. Becker, M. Becker, J. Polchinski and A. Tseytlin, Higher order graviton scattering in M(atrix) theory, hep-th/9706072.

15. K. Ezawa, Y. Matsuo and K. Muralid, Lorentz symmetry of supermembrane in light cone gauge formulation, hep-th/9705005.

16. W. Taylor IV, Phys. Lett. 394B (1997) 283.

17. L. Susskind, T-duality in M(atrix) theory and S-duality in field theory, hep-th/9611163; O.J. Ganor, S. Ramgoolam, W. Taylor IV, Nucl. Phys. B492 (1997) 191.

18. L. Motl, Proposals on non-perturbative superstring interactions, hep-th/9701025.

19. R. Dijkgraaf, E. Verlinde and H. Verlinde, Matrix String Theory, hep-th/9703030.

20. E. Cremmer and B. Julia, Nucl. Phys. B159 (1979) 141.

21. C.M. Hull and P.K. Townsend, Nucl. Phys. B438 (1995) 109.

22. J.M. Izquierdo, N.D. Lambert, G. Papadopoulos and P.K. Townsend, Nucl. Phys. B460, (1996) 560.

23. A. Sen, Phys. Lett. B329 (1994) 217.

24. M. Rozali, Phys. Lett. 400B (1997) 260; M. Berkooz, M. Rozali and N. Seiberg, Matrix description of M-theory on $T^4$ and $T^5$, hep-th/9704089.

25. N. Seiberg, New theories in six dimensions and Matrix description of M-theory on $T^5$ and $T^5/Z_2$, hep-th/9705221.

26. I. Brunner and A. Karch, Matrix description of M-theory on $T^6$, hep-th/9707059.
A. Losev, G. Moore and S.L. Shatashvili, $M\& m$’s, [hep-th/9707256].
A. Hanany and G. Liischytz, *Matrix Theory on $T^6$ and a m(atrix) description of KK monopoles*, [hep-th/9708037].

27. S. Elitzur, A. Giveon, D. Kutasov and E. Rabinovici, *Algebraic Aspects of Matrix Theory on $T^d$*, [hep-th/9707217].

28. B. Julia, *Group Disintegrations*, in *Super-space and Supergravity*, eds. S.W. Hawking and M. Roček, (C.U.P. 1981);
H. Nicolai, *Phys. Lett. B194* (1987) 402;
H. Nicolai and N.P. Warner, *Commun. Math. Phys. 125* (1989) 369.

29. E. Martinec, *Matrix theory and $N = (2,1)$ strings*, [hep-th/9706194].

30. W. Nahm, *Nucl. Phys. B135* (1978) 149.
31. P.K. Townsend, *Phys. Lett. 202B* (1988) 53.
32. E. Witten and D. Olive, *Phys. Lett. 78B* (1978) 97.
33. E. Bergshoeff and M. de Roo, *Phys. Lett. B112* (1982) 53;
P.S. Howe, H. Nicolai and A. Van Proeyen, *Phys. Lett. B112* (1982) 446.

34. I. Bars, *Phys. Rev. D54* (1996) 5203.