MODULI OF QUADRILATERALS AND QUASICONFORMAL REFLECTION

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Abstract. We study the interior and exterior moduli of polygonal quadrilaterals. The main result is a formula for a conformal mapping of the upper half plane onto the exterior of a convex polygonal quadrilateral. We prove this by a careful analysis of the Schwarz-Christoffel transformation and obtain the so-called accessory parameters and then the result in terms of the Lauricella hypergeometric function. This result enables us to understand the dissimilarities of the exterior and interior of a convex polygonal quadrilateral. We also give a Mathematica algorithm for the computation. In particular, we study the special case of an isosceles trapezoidal polygon $L$ and obtain some estimates for the coefficient of quasiconformal reflection over $L$ in terms of special functions and geometric parameters of $L$.

1. Introduction

A quadrilateral $Q = (Q; z_1, z_2, z_3, z_4)$ is a Jordan domain $Q$ on the Riemann sphere with four fixed points $z_1, z_2, z_3, z_4$ on its boundary. We label $z_j$ in such an order that increasing of the index $j$ corresponds to the positive traverse of the boundary $\partial Q$; we will name such quadruples $(z_1, z_2, z_3, z_4)$ admissible.

One of the main geometric characteristics of a quadrilateral $Q$ is its conformal modulus. There are many equivalent definitions of this concept. If $f$ is a conformal mapping of $Q$ onto a rectangle $[0,1] \times [0,h], h > 0$, such that the points $z_1, z_2, z_3, z_4$ correspond to $0, 1, 1 + ih, ih$, then $h$ is uniquely defined and is called the conformal modulus of $Q$ \cite{PS} p.52, Def. 2.1.3. In this case, we write

$$h = \text{Mod}(Q).$$

Another way to define the modulus is to use the concept of the extremal length $\lambda(\Gamma)$ of a curve-family $\Gamma$ (see, e.g. \cite{A1}). If $\Gamma$ is the family of curves in the domain $Q$, connecting the sides $z_1z_2$ and $z_3z_4$ of the quadrilateral $Q$, then $\text{Mod}(Q) = \lambda(\Gamma)$. If we consider the family $\Gamma_1$ of curves connecting the sides $z_2z_3$ and $z_1z_4$ in $Q$, then $\text{Mod}(Q) = 1/\lambda(\Gamma_1)$. At last,

$$\text{Mod}(Q) = \left( \inf_u \int_Q |\nabla u|^2 dxdy \right)^{-1}$$

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where the infimum is taken over all smooth functions \( u \) in \( Q \), continuous on \( \overline{Q} \), which are equal to 0 on the boundary arc \( z_1z_2 \) and equal to 1 on the boundary arc \( z_3z_4 \). Moreover, it is known that the infimum here is attained by a harmonic function [AI, p. 65].

Assume that \( Q \) is a bounded Jordan domain in \( \mathbb{C} \), \( L = \partial Q \), and \( z_1, z_2, z_3, z_4 \) are some points on \( L \) satisfying the above requirements. Consider the quadrilateral \( Q = (Q; z_1, z_2, z_3, z_4) \). Then we will say that the value of \( \text{Mod}(Q; z_1, z_2, z_3, z_4) \) is the interior modulus. We can also consider the quadrilateral \( Q^c := (Q^c; z_4, z_3, z_2, z_1) \) where \( Q^c \) is the complement of \( Q \) with respect to the Riemann sphere. In that case, we will name \( \text{Mod}(Q^c) \) the exterior modulus.

The main topic of this paper is to analyze the interior and exterior moduli of polygonal quadrilaterals and to apply the results to geometric function theory. Because the upper half-plane can be conformally mapped onto a polygonal domain in terms of the Schwarz-Christoffel transformation, both the interior and exterior moduli of polygonal quadrilaterals can be computed. This transformation is semi-explicit, there are so called accessory parameters that have to be determined separately for each case [AF, KY]. There are no universal recipes for finding the accessory parameters, which itself leads to numerically ill-conditioned problems. The best methods for numerical computation of Schwarz-Christoffel type conformal mappings are those developed and implemented by T.A. Driscoll and L.N. Trefethen [DT]. For a survey of the available methods, see N. Papamichael and N. Stylianopoulos [PS].

We study the conformal mapping from upper half plane onto the exterior of a convex polygonal quadrilateral with vertices \( 0, 1, a, b \) with \( \text{Im} \, a > 0, \text{Im} \, b > 0 \) from analytic point of view and our goal is to explicitly find the Schwarz-Christoffel mapping and its parameters. Recall first that by classical theory of elliptic functions it is known that the upper half plane is conformally mapped onto a rectangle under the inverse of the elliptic function \( sn \) and the ratio of the side lengths is given by a quotient of complete elliptic integrals [AF, KY, PS]. This fact extends to the case of conformal mapping of the upper half plane onto parallelograms and the mapping is given by generalized elliptic functions, now depending on the least angle of the parallelogram, and the interior modulus is given by a quotient of generalized complete elliptic integrals as shown in [AQVV, Section 2]. These generalized complete elliptic integrals were introduced in [BB, p.158] and we mention in passing that during the past decade these integrals have been studied intensively, cf. e.g. [CQW, QMC, QMB] and the bibliographies therein. A further extension of the above conformal mapping problem is to map the upper half plane onto a convex polygonal quadrilateral and such a mapping is given by the Schwarz-Christoffel transformation expressed in terms of the Gaussian hypergeometric function \( _2F_1(a, b; c; z) \) with parameters depending on the angles; the previous case is a special case of this one as shown in [HVV, Section 2]. The conformal mapping problem onto the exterior of a polygonal quadrilateral that we study here is much more difficult. In one of our main results, Theorem 3.22, we prove that the mapping can be expressed in terms of the Lauricella hypergeometric function \( F^{(3)}_D \). A particular case of this mapping was studied by P. Duren and J. Pfaltzgraff [DP], namely a conformal mapping of the upper half plane onto the exterior of a rectangle. As pointed out
in [HV], this mapping already appeared in the works of W. Burnside [BU]. We also give a Mathematica function for the computation of the exterior modulus based on Theorem 3.22 and compare its numerical precision by a comparison to the recent numerical computation results reported in [NRRV] and observe very good agreement of the results.

Conformal moduli play an important role in geometric function theory and applications, in particular, they are valuable tools in the study of quasiconformal mappings (see, e.g. [A1, AVV, D, GH, K1]). One of the definitions of quasiconformal mappings (the so-called geometric definition) uses the moduli. A sense-preserving homeomorphism of the Riemann sphere $\mathbb{C}$ is called $K$-quasiconformal ($K \geq 1$) if it satisfies the following condition: conformal moduli are quasiinvariant under the mapping, i.e. if $Q = (Q; z_1, z_2, z_3, z_4)$ is an arbitrary quadrilateral and $f(Q) = (f(Q); f(z_1), f(z_2), f(z_3), f(z_4))$, then

\[(1.1) \quad K^{-1}\text{Mod}(Q) \leq \text{Mod}(f(Q)) \leq K\text{Mod}(Q)\]

A closed Jordan curve $L$ on the Riemann sphere is called a quasicircle if it is the image of the unit circle under a quasiconformal mapping defined in the whole plane. If we know $K$ such that (1.1) is valid, then $L$ is called a $K$-quasicircle. An important problem is to determine, for a given Jordan curve, whether it is a quasicircle or not, and, if the answer is positive, to either find or estimate the minimal possible value of $K$, denoted by $K_L$. The problem of finding $K_L$ is open also for the case of curves $L$ as simple as the boundaries of long rectangles and we will discuss this below [HKV, p. 455 Probl. (20)].

Ahlfors [A2] gave the following geometric characterization for quasicircles. If $L$ is a closed Jordan curve in the plane and there is a constant $C$ such that for every three points $z_1, z_2,$ and $z_3$ on $L$ such that $z_3$ lies on the subarc of $L$ with smaller diameter and with endpoints $z_1$ and $z_2$, the inequality

\[(1.2) \quad |z_1 - z_3| + |z_2 - z_3| \leq C|z_1 - z_2|\]

holds, then $L$ is a $K$-quasicircle where $K$ depends only on $C$ [GH, p.23, Def. 2.2.2, Thm 2.2.5]. Conversely, if $L$ is a $K$-quasicircle, then for every appropriate triple on $L$ the inequality (1.2) holds with $C$ depending only on $K$. The monograph of Gehring and Hag [GH] gives many more characterizations of quasicircles and provides a survey of their many applications.

If $L$ is a quasicircle, then there is a quasiconformal reflection with respect to $L$, i.e. a sense-reversing quasiconformal automorphism $g$ of the Riemann sphere which keeps every point of $L$ fixed and maps the bounded complementary component of $L$ onto the unbounded one and vice versa. Another important problem is to find or estimate the minimal coefficient of quasiconformality for such a mapping $g$; further we will denote this coefficient of quasiconformal reflection by $QR_L$. This is a very difficult problem even for polygonal curves in $\mathbb{C}$ studied by R. Kühnau in a series of papers, e.g. [K2]. A nice survey is given in [Kr, pp.525-531]. Here we give some of these results. Since every such a curve $L$ in $\mathbb{C}$ determines its interior in a unique way, we will also say that $QR_L$ is the coefficient for $Q := \text{int}(L)$. If there exists a circle tangent to every side of a closed polygon $L$, then $QR_L = 2/\alpha - 1$ where $\pi\alpha$ is the least interior angle of $L$. In particular, for triangles the
problem is solved. For quadrilaterals, the problem is open even for the case of rectangles. The value of $QR_L$ is known only for rectangles $[0, a] \times [0, 1]$ close to a square $[W]$: if $1 \leq a < 1.037$, then $QR_L = 3$. For sufficiently long rectangles with $a > 2.76$ it is proven $[W]$ that $QR_L > 3$. Moreover, for any $a > 1$ the estimate

$$\frac{\pi}{4} a < QR_L < \pi a$$

holds $[W]$, see also $[K1, Kr]$.

The value $QR_L$ is closely connected with $M_L := \sup \frac{\text{Mod}(Q)}{\text{Mod}(Q^c)}$ where $Q = (Q; z_1, z_2, z_3, z_4)$, $\partial Q = L$ and the supremum is taken over all admissible quadruples $(z_1, z_2, z_3, z_4)$ on $L$. As Kühnau noted in $[K3]$, (1.4) $QR_L \geq M_L$, therefore, every lower estimation for $M_L$ also gives a lower estimation for $QR_L$. For related results, see Shen $[Sh]$.

In the present paper, we investigate the problem of estimations of $M_L$ and $QR_L$ for isosceles trapezoidal curves. With the help of the Schwarz-Christoffel formula, we construct conformal mappings of the upper half-plane onto the interior and exterior of $L$. Comparison of the interior and exterior moduli for some quadruples of points $z_1, z_2, z_3, z_4 \in L$ allows us to obtain lower estimates for $M_L$, and, therefore, for $QR_L$.

In addition, using fairly simple methods, we get two-sided estimates of $QR_L$ and $M_L$ for isosceles trapezoidal polygons $L$ of height 1 in terms of lengths of its sides and angles. In particular, our main result about quasiconformal reflection, Theorem 6.4, states that for such $L$ with acute angle $\pi \alpha$ and bases $c$ and $d$, $d = c + \cot(\pi \alpha)$, the following estimates hold:

$$M_L \geq \begin{cases} g(\lambda_0)(1 + C(\alpha))d, & \text{if } \frac{\pi}{4} \geq \lambda_0, \\ g(\lambda)(1 + C(\alpha))d, & \text{if } \frac{\pi}{4} < \lambda_0. \end{cases}$$

Here $g(\lambda) = \lambda \kappa (\sqrt{1 - \lambda^2}) / \kappa(\lambda)$, $\kappa(\lambda)$ is the complete elliptic integral of the first kind, $\lambda_0 = 0.7373921\ldots$ is the unique point of maximum of $g$ on $(0, 1)$, $g(\lambda_0) = 0.708434\ldots$ and

$$C(\alpha) = \left( \sqrt{1 + \tan^2(\pi \alpha)/4} - \tan(\pi \alpha)/2 \right)^2, \quad 0 < \alpha \leq 1/2.$$ 

At last, in Section 7 with the help of the concept of strongly starlike curve, we give upper bounds for $QR_L$ for isosceles trapezoidal polygons.

2. HYPERGEOMETRIC FUNCTIONS AND THEIR GENERALIZATIONS

To find some explicit formulas for the interior and exterior conformal moduli for a trapezoidal curve we need special functions. In this section we recall the Gaussian and Appell hypergeometric functions and some of their generalizations $[B] [BF] [AVV]$.
First we recall that, for $|z| < 1$, the Gaussian hypergeometric function is defined by the equality

$$2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!},$$

for $|z| < 1$, where $(q)_n$ denotes the Pochhammer symbol, i.e. $(q)_n = q(q+1)\ldots(q+(n-1))$ for every natural $n$ and $(q)_0 = 1$. It can be extended analytically to the domain $|z| > 1$ along any path avoiding 1 and $\infty$. Moreover, we have

$$\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-tz)^{-b}dt = B(a, c-a)\ _2F_1(a, b; c; z)$$

where $B(\cdot, \cdot)$ is the Euler beta function and the integral in (2.1) converges if $\text{Re } c > \text{Re } a > 0$. We also recall that the beta function can be expressed via the Euler gamma function:

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$

The complete elliptic integrals $\mathcal{K}, \mathcal{E}$ of the first and second kinds

$$\mathcal{K}(\lambda) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\lambda^2t^2)}}, \quad \mathcal{E}(\lambda) = \int_0^1 \sqrt{\frac{1-\lambda^2t^2}{1-t^2}}dt$$

are, in fact, special cases of the Gaussian hypergeometric function; we have

$$\mathcal{K}(\lambda) = \frac{\pi}{2} _2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \lambda^2\right), \quad \mathcal{E}(\lambda) = \frac{\pi}{2} _2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; \lambda^2\right).$$

For the decreasing homeomorphism $\mu : (0, 1) \to (0, \infty)$

$$\mu(r) = \frac{\pi}{2} \frac{\mathcal{K}(\sqrt{1-r^2})}{\mathcal{K}(r)}, \quad 0 < r < 1,$$

the following differentiation formulas hold [AVV] p.475

$$\mu'(r) = -\frac{\pi^2}{4r(1-r^2)\mathcal{K}^2(r)}, \quad \mu''(r) = \frac{\pi^2(2\mathcal{E}(r) - (1 + r^2)\mathcal{K}(r))}{4r^2(1-r^2)^2\mathcal{K}^3(r)}.$$

Let $\mathbb{H}^2 = \{z: \text{Im } z > 0\}$ be the upper half-plane and $0 < r < 1$. Then the modulus of the quadrilateral $\mathcal{H}^2_r = (\mathbb{H}^2; 0, 1, 1/r^2, \infty)$ can be written as follows

$$\text{Mod}(\mathcal{H}^2_r) = \frac{\mathcal{K}(\sqrt{1-r^2})}{\mathcal{K}(r)}.$$

This formula follows from the definition, if we use of a canonical conformal mapping of the the upper half plane onto a rectangle. It also follows from [HKV] 7.12, 7.33.

The Appell hypergeometric function $F_1(a, b_1, b_2; c; z, w)$ is defined as

$$F_1(a; b_1, b_2; c; z, w) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}} \frac{z^m w^n}{m! n!},$$

see, e.g. [B]. The series converges in the bidisk $B^2 := \{|z|, |w| < 1\}$ and similar to the case of hypergeometric function, can be continued analytically outside the bidisk $B^2$ along
any path not containing the points with \( z = 1 \) and \( w = 1 \). The following formula is due to Picard:

\[
\int_{0}^{1} t^{a-1}(1-t)^{\mu-1}(1-tz)^{-b_1}(1-tw)^{-b_2} dt = B(a, \mu)F_1(a; b_1, b_2; a + \mu; z, w).
\]

The integral converges for \( \Re a > 0, \Re \mu > 0 \), if \( z, w \neq 1 \). It is evident that if either \( b_2 = 0 \) or \( w = 0 \), then \( F_1(a; b_1, b_2; c; z, w) = 2F_1(a, b_1; c; z) \).

The Lauricella hypergeometric function \([L]\), see also \([S]\) generalizes the Appell hypergeometric function for the case of arbitrary number \( n \) of variables \( z_1, \ldots, z_n \):

\[
F_D^{(n)}(a; b_1, \ldots, b_n; c; z_1, \ldots, z_n) = \sum_{m_1, \ldots, m_n \geq 0} \frac{(a)_{m_1+\ldots+m_n}(b_1)_{m_1} \ldots (b_n)_{m_n}}{(c)_{m_1+\ldots+m_n}m_1! \ldots m_n!} z_1^{m_1} \ldots z_n^{m_n},
\]

\(|z_1|, \ldots, |z_n| < 1\). It has the integral representation

\[
F_D^{(n)}(a; b_1, \ldots, b_n; c; z_1, \ldots, z_n) = (B(a, c-a))^{-1} \int_{0}^{1} u^{a-1}(1-u)^{c-a-1}(1-uz_1)^{-b_1} \ldots (1-uz_n)^{-b_n} du,
\]

\( \Re c > \Re a > 0 \), which gives an analytic continuation of the Lauricella hypergeometric function outside the polydisk \(|z_k| < 1, 1 \leq k \leq n\), except for the hyperplanes \( z_k = 1 \).

3. Conformal mappings of the half-plane onto the exterior of a polygonal quadrilateral

Heikkala, Vamanamurthy, and Vuorinen [HVV] pp.78-82 studied the Schwarz-Christoffel mapping of the upper half-plane onto a convex polygonal quadrilateral. In this section we consider the conformal mapping of the upper half-plane onto the exterior of a convex polygonal quadrilateral.

To specify the geometry, we suppose that the bounded polygonal quadrilateral has four segments as its sides which form the exterior angles \( \pi(1+\alpha), \pi(1+\beta), \pi(1+\gamma), \) and \( \pi(1+\delta) \) at the vertices, with the angle parameters satisfying the constraints

\[
\alpha + \beta + \gamma + \delta = 2, \quad 0 < \alpha, \beta, \gamma, \delta < 1.
\]

The conformal mapping \( f \) of the upper half plane onto the complement of this quadrilateral is given by the generalized Schwarz-Christoffel formula [AF]. Section 5.6, formula (5.6.3b)]

\[
f(z) = C_1 \int_{0}^{z} \frac{\zeta^\alpha (\zeta - 1)^\beta (\zeta - t)^\gamma}{(\zeta - z_0)^2 (\zeta - \infty)^2} d\zeta + C_2.
\]

The preimages of the vertices are the points \( 0, 1, t \ (t > 1) \), and \( \infty; \) the point \( z_0 \) is an unknown pole. We note that from \([B1]\) it follows, in particular, that the bounded complementary polygonal domain is convex, since the interior angles at the vertices are \( (1-\alpha)\pi, (1-\beta)\pi, (1-\gamma)\pi, \) and \( (1-\delta)\pi \) and \( \alpha, \beta, \gamma, \delta > 0 \).
To find $z_0 = x_0 + iy_0$ we derive an equation which is obtained from the fact that the residue of the integrand at the point $z_0$ vanishes. The integrand is equal to
\[
\frac{g(\zeta)}{(\zeta - z_0)^2}, \quad \text{where} \quad g(\zeta) = \frac{\zeta^\alpha(\zeta - 1)^\beta(\zeta - t)^\gamma}{(\zeta - z_0)^2},
\]
and hence it has a pole of the second order at the point $z_0$. Computation yields
\[
\text{res}_{\zeta = z_0} \frac{g(\zeta)}{(\zeta - z_0)^2} = g'(z_0).
\]
Consequently, $z_0$ must satisfy the equality $g'(z_0) = 0$ which is equivalent to $(\log g(z_0))' = 0$. At last,
\[
(\log g(\zeta))' = \frac{\alpha}{\zeta} + \frac{\beta}{\zeta - 1} + \frac{\gamma}{\zeta - t} - \frac{2}{\zeta - z_0},
\]
and from this we obtain the desired equation for $z_0$:
\[
\frac{\alpha}{z_0} + \frac{\beta}{z_0 - 1} + \frac{\gamma}{z_0 - t} = \frac{1}{iy_0}, \quad y_0 = \text{Im } z_0.
\]

**Lemma 3.3.** In the upper half-plane $\{z : \text{Im } z > 0\}$ the equation
\begin{equation}
(3.4) \quad \frac{\alpha}{z} + \frac{\beta}{z - 1} + \frac{\gamma}{z - t} = \frac{1}{iy},
\end{equation}
z = x + iy, has a unique solution.

**Proof.** We can represent (3.4) in the form
\[
z = \frac{2z(z - 1)(z - t)}{Q(z)}
\]
where
\[
Q(z) = \alpha(z - 1)(z - t) + \beta z(z - t) + \gamma z(z - 1) = (\alpha + \beta + \gamma)z^2 - (\alpha(1 + t) + \beta t + \gamma)z + \alpha t.
\]
Therefore, $\overline{z} = R(z)$ where
\begin{equation}
(3.5) \quad R(z) = z - \frac{2z(z - 1)(z - t)}{Q(z)} = \frac{(\alpha + \beta + \gamma - 2)z^3 - ((\alpha - 2)(1 + t) + \beta t + \gamma)z^2 + (\alpha - 2)t z}{Q(z)}.
\end{equation}
Since $R(z)$ is a rational function with real coefficients, we have $\overline{R(z)} = R(\overline{z})$ and, therefore, all solutions of (3.4) are also solutions of the equation
\begin{equation}
(3.6) \quad z = R(R(z)) = R(z) - \frac{2R(z)(R(z) - 1)(R(z) - t)}{Q(R(z))}.
\end{equation}
The function $R(R(z))$ is a rational function of degree 9; for real $x$ it has poles at the points where $Q(x) = 0$ and $Q(R(x)) = 0$. For the quadratic function $Q(x)$ we have $Q(0), Q(t) > 0$ and $Q(1) < 1$, therefore, it has two real zeroes $x_1 \in (0, 1)$ and $x_2 \in (1, t)$. Moreover,
\[
\lim_{x \to x_k^-} R(x) = -\infty, \quad \lim_{x \to x_k^+} R(x) = +\infty, \quad k = 1, 2,
\]
and, because of the inequality $\alpha + \beta + \gamma < 2$, we have

$$\lim_{x \to -\infty} R(x) = +\infty, \quad \lim_{x \to +\infty} R(x) = -\infty.$$ 

This implies that on each of the intervals $(-\infty, x_1)$, $(x_1, x_2)$ and $(x_2, +\infty)$ there are exactly two points where either $R(x) = x_1$ or $R(x) = x_2$. Therefore, in addition to $x_1$ and $x_2$, we have six points on the real axis where $R$ has poles. It is evident that all these eight points, which are poles of $R(R(x))$, are different. Denote them by $\tau_j$, $1 \leq j \leq 8$; the points are labelled so that $\tau_1 < \tau_2 < \ldots < \tau_8$.

Now we will show that $R'(x) < 0$ for real $x$ different from the points $x_1$ and $x_2$. We have

$$R(x) = x - \frac{2}{f(x)}, \quad f(x) = \frac{\alpha}{x} + \frac{\beta}{x-1} + \frac{\gamma}{x-t}.$$ 

Consequently,

$$R'(x) = 1 + \frac{2f'(x)}{f^2(x)}, \quad f'(x) = -\frac{\alpha}{x^2} - \frac{\beta}{(x-1)^2} - \frac{\gamma}{(x-t)^2} < 0.$$ 

From the Cauchy-Schwarz inequality we obtain

$$f^2(x) = \left(\frac{\alpha}{x} + \frac{\beta}{x-1} + \frac{\gamma}{x-t}\right)^2 \leq (\alpha + \beta + \gamma) \left(\frac{\alpha}{x^2} + \frac{\beta}{(x-1)^2} + \frac{\gamma}{(x-t)^2}\right) < 2|f'(x)|,$$ 

and hence, $R'(x) < 0$. It is evident that on every one of the intervals $I_j := (\tau_j, \tau_{j+1})$, $1 \leq j \leq 7$, $R(R(x))$ is strictly increasing. Then

$$\lim_{x \to +\infty} (x - R(R(x))) = +\infty, \quad \lim_{x \to -\infty} (x - R(R(x))) = -\infty.$$ 

Therefore, $x - R(R(x))$ takes all real values on $I_j$ and there exists a point $s_j \in I_j$ such that $s_j = R(R(s_j))$. From this we conclude that every $s_j$, $1 \leq j \leq 7$, satisfies the equality (3.6). From (3.5) we conclude that $R(x)$ is a rational function of degree 3 which is the ratio of two polynomials of degrees 3 and 2. Moreover, $R(x) \sim (1 - 2(\alpha + \beta + \gamma)^{-1})x$, $x \to \infty$. Therefore, $R(R(x))$ is a rational function of degree 9 and $R(R(x)) \sim (1 - 2(\alpha + \beta + \gamma)^{-1})^2x$, $x \to \infty$. This implies that the rational function $x - R(R(x))$ has degree 9. Since at least 7 of its zeros are real, we conclude that it has at most one complex (i.e. non-real) root in the upper half-plane. Since all solutions of (3.4) satisfy (3.6), we see that (3.4) has at most one complex root in the upper half-plane.

To prove that such a root exists, we note that if we fix one side of the boundary polygon $A_1A_2A_3A_4$, say $A_1A_2$, and assume that it coincides with the segment $[0, 1]$ of the real axis and change the length of the other side, $A_2A_3$ from 0 to $\infty$ (with fixed values of angles $\alpha$, $\beta$, and $\gamma$), then the exterior conformal moduli of the obtained polygons continuously increase from 0 to $+\infty$ and, therefore, there exists such a polygon $P$ that its modulus coincides with the modulus of the upper half-plane with vertices at the points 0, 1, $t$, and $\infty$. Then there exists a conformal mapping $f$ of the upper half-plane onto the exterior of $P$ such that the points 0, 1, $t$, and $\infty$ are mapped to the vertices of $P$. Denote $z_0 = f^{-1}(\infty)$. Then $f$ has the form (3.2) and $z_0$ satisfies (3.4). \(\square\)
Now we will give a formula for the unique solution of (3.4). Denote
\[ E = \alpha + \beta + \gamma - 1, \]
\[ A = 2(E - 1)^2, \]
\[ B = (E - 1)[4 - 3(\alpha + \gamma) + (4 - 3(\alpha + \beta))t], \]
\[ C = 2 - 3(\alpha + \gamma) + (\alpha + \gamma)^2 + 2(3 - 5\alpha - 2\beta - 2\gamma + 2\alpha^2 + 2\alpha\beta + 2\alpha\gamma + \beta\gamma)t \]
\[ + (2 - 3(\alpha + \beta) + (\alpha + \beta)^2)t^2, \]
\[ D = (1 - \alpha)(\alpha + \gamma - 1 + (\alpha + \beta - 1)t)t, \]

(3.7)
\[ \rho(x) = \frac{\alpha x}{(1 - E)x + E(t + 1) - \gamma t - \beta}. \]

**Lemma 3.8.** The unique solution of (3.4), lying in the upper half-plane, has the form
\[ z_0 = x_0 + iy_0 \] where \( x_0 \) is the unique solution of the cubic equation

(3.9)
\[ Ax^3 + Bx^2 + Cx + D = 0, \]

satisfying the inequality \( x^2 < \rho(x) \) and \( y_0 = \sqrt{\rho(x_0) - x_0^2} \).

**Proof.** Multiplying both sides of (3.4) by \( z - t \) we obtain
\[ \alpha(z - t) + \beta + \gamma(z - t) = 1 + \frac{x - t}{iy}, \]

consequently,
\[ \text{Re} \left[ \frac{\alpha t}{z} + \frac{\beta(t - 1)}{z - 1} \right] = E, \]

and
\[ \frac{\alpha x}{|z|^2} + \frac{\beta(t - 1)(x - 1)}{|z - 1|^2} = E. \]

If we denote \( \rho = x^2 + y^2 \), we have
\[ \frac{\alpha x}{\rho} + \frac{\beta(t - 1)(x - 1)}{\rho + 1 - 2x} = E \]

and
\[ \alpha x(\rho + 1 - 2x) + \rho \beta(t - 1)(x - 1) = E \rho (\rho + 1 - 2x). \]

Similarly, multiplying both sides of (3.4) by \( z - 1 \) we obtain
\[ \frac{\alpha(z - 1)}{z} + \beta + \gamma(z - 1) = 1 + \frac{x - 1}{iy}. \]

\[ \text{Re} \left[ \frac{\alpha}{z} + \frac{\gamma(1 - t)}{z - t} \right] = E, \]

\[ \frac{\alpha x}{\rho} + \frac{\gamma(1 - t)(x - t)}{\rho + t^2 - 2tx} = E, \]

\[ \alpha x(\rho + t^2 - 2tx) + \rho \gamma(1 - t)(x - t) = E \rho (\rho + t^2 - 2tx). \]
Thus, we have a system of two equations with respect to \( x \) and \( \rho \):

\[
\alpha tx(\rho + 1 - 2x) + \rho \beta (t - 1)(x - 1) = E\rho (\rho + 1 - 2x). 
\]

(3.10)

\[
\alpha x(\rho + t^2 - 2tx) + \rho \gamma (1 - t)(x - t) = E\rho (\rho + t^2 - 2tx). 
\]

(3.11)

Subtracting (3.10) from (3.11) we obtain

\[
E\rho (t^2 - 1 - 2(t - 1)x) = \alpha x(1 - t)\rho + \alpha x(t^2 - t) + \rho \gamma (\gamma + \beta)(x - \gamma t - \beta) 
\]

This is a linear equation with respect to \( \rho \). Solving it, we obtain (3.7). If we substitute this expression into any of the equations (3.10), (3.11), we find a cubic equation (3.9) for \( x_0 \).

By Lemma 3.3, there exists only one root \( x_0 \) of (3.9) satisfying the inequality \( x^2 < \rho(x) \).

Then we find \( y_0 = \sqrt{\rho^2(x_0) - x_0} \). □

After finding the value of \( z_0 \) we can simply express the coordinates of the vertices in terms of \( \alpha, \beta, \gamma, \) and \( t \). Further, we will assume that the vertices of the boundary polygon, corresponding to the points 0, 1, \( t \) and \( \infty \), are located at the points \( A_1 = 1, A_2 = 0, A_3 \) and \( A_4 \). Then the conformal mapping of the upper half-plane onto the exterior of the polygonal region is given by the formula

\[
f(z) = 1 - \frac{h(z)}{h(1)},
\]

where

\[
h(z) = \int_0^z x^\alpha (1-x)^\beta (1-x/t)^\gamma dx 
\]

(3.13)

\[
(1-x/z_0)^2(1-x/z_0^2)^2.
\]

(The branch of the integrand is fixed such that it takes positive values on \((0,1)\)). Here \( z_0 = z_0(t) \) is described in Lemma 3.3. With the help of the Lauricella function \( F_D^{(n)} \), we can write

\[
h(1) = B(1 + \alpha, 1 + \beta)F_D^{(3)}(1 + \alpha; -\gamma, 2, 2; 2 + \alpha + \beta; 1/t, 1/z_0, 1/z_0^2),
\]

\[
h(z) = \frac{z^{1+\alpha}}{1+\alpha} F_D^{(4)}(1 + \alpha; -\beta, -\gamma, 2, 2; 2 + \alpha; z, z/t, z/z_0, z/z_0).
\]

The length of the side \( A_2A_3 \) is given by the formula

\[
l_2 := |A_2A_3| = \frac{I}{h(1)}, \quad I = \int_1^t \frac{x^\alpha(x - 1)^\beta (1-x/t)^\gamma dx}{(1-x/z_0)^2(1-x/z_0^2)^2}.
\]

(3.15)
After the change of variables $x = 1 + (t - 1)\tau$ we find

\[(3.16)\]

\[
I = \int_1^t x^\alpha(x - 1)^\beta (1 - x/t)^\gamma dx = \frac{(t - 1)^{1+\beta+\gamma}|z_0|^4}{t^\gamma|z_0 - 1|^4} \int_0^1 \tau^\beta(1 - \tau)^\gamma(1 + (t - 1)\tau)^\alpha d\tau
\]

\[
= \frac{(t - 1)^{1+\beta+\gamma}|z_0|^4}{t^\gamma|z_0 - 1|^4} B(1 + \beta, 1 + \gamma) F_D^{(3)}(1 + \beta; -\alpha, 2, 2 + \beta + \gamma; -(t - 1), \frac{t-1}{z_0-1}, \frac{t-1}{z_0-1}).
\]

Denote $r = 1/\sqrt{t} \in (0, 1)$. From the conformal invariance of the modulus we obtain that the desired exterior conformal modulus $\text{Mod}(Q)$ coincides with the conformal modulus of the quadrilateral $H_r^2$ which is the upper half-plane with vertices $z_1 = 0$, $z_2 = 1$, $z_3 = t = 1/r^2$, and $z_4 = \infty$. Applying (2.4), we have

\[(3.17)\]

\[\text{Mod}(Q) = \mathcal{K}(r')/\mathcal{K}(r), \quad r' = \sqrt{1-r^2},\]

where $\mathcal{K}(r)$ is the complete elliptic integral of the first kind.

We can also find the lengths of sides $l_3 = |A_3A_4|$ and $l_4 = |A_4A_1|$ and the vertices $A_3$ and $A_4$. It is easy to verify that

\[(3.18)\]

\[l_3 = \frac{\sin \pi \alpha + l_2 \sin \pi(\alpha + \beta)}{\sin \pi \delta}, \quad l_4 = \frac{\sin \pi(\beta + \gamma) + l_2 \sin \pi \gamma}{\sin \pi \delta},\]

\[(3.19)\]

\[A_3 = -l_2 e^{-\pi \beta i}, \quad A_4 = A_3 - l_3 e^{-\pi (\beta + \gamma) i} = 1 + l_4 e^{\pi \alpha i}.\]

Therefore, we have

**Theorem 3.20.** Let $f(z) = 1 - \frac{h(z)}{z^{\alpha (1)}}$ where $h$ is given by (3.13). Then $f$ maps conformally the upper half-plane onto the exterior of the polygonal line $A_1A_2A_3A_4$ where $A_1 = 1$, $A_2 = 0$, $A_3$ and $A_4$ are given by (3.19) where $l_3$ and $l_4$ are given by (3.18) and $l_2$; here $l_2$ is defined by (3.15), $h(1)$ and $I$ are given by (3.14) and (3.16), taking into account (3.14) and (3.16). The length of sides of $A_1A_2A_3A_4$ are $l_1 = 1$, $l_2$, $l_3$, and $l_4$.

It is evident that, under the above assumptions, the exterior of the quadrilateral is defined uniquely by the value of the length $l_2$ (or $l_4$). From Lemma 3.8 it follows

**Theorem 3.21.** For fixed angles $(1 + \alpha)\pi$, $(1 + \beta)\pi$, $(1 + \gamma)\pi$, and $(1 + \delta)\pi$, the exterior conformal modulus is strictly increasing as a function of $l_2$.

Actually, if it is not the case, then we can find two different (exterior) quadrilaterals with vertices $A_1$, $A_2$, $A_3$, $A_4$ and $A_1$, $A_2$, $A_3$, $A_4$ such that $A_1 = 1$, $A_2 = 0$, and their conformal moduli coincide. Then the conformal mappings of the upper half-plane onto these exterior quadrilaterals is defined by the formula

\[f_k(z) = C_k \int_0^z \frac{\zeta^\alpha(\zeta - 1)^\beta(\zeta - t_k)^\gamma}{(\zeta - z_{0k})^2(\zeta - z_{0k})^2} d\zeta + 1, \quad k = 1, 2.\]

Since the moduli of the quadrilaterals are uniquely defined by $t_k$ and the dependence is strictly monotone, we see that $t_1 = t_2$. Then, by Lemma 3.8 we obtain that $z_{01} = z_{02}$.
From the normalization $f_k(1) = 0$, $k = 1, 2$ it follows that $C_{11} = C_{12}$. Thus, $f_1 \equiv f_2$ and, therefore, the exterior quadrilaterals coincides, which contradicts our assumptions.

Now we will describe how to determine the conformal modulus of the given exterior polygonal quadrilateral with angles $(1 + \alpha)\pi$, $(1 + \beta)\pi$, $(1 + \gamma)\pi$, and $(1 + \delta)\pi$, satisfying (3.1).

**Theorem 3.22.** For a given exterior quadrilateral, the conformal modulus is given by the formula (3.17) where $r = 1/\sqrt{t}$ and $t$ is a unique solution to the equation (3.15) where $h(1)$ and $I$ are defined by (3.14) and (3.16), keeping in mind that $z_0 = z_0(t) = x_0(t) + i\sqrt{r(x_0(t)) - x_0^2(t)}$, $x_0(t)$ is a solution to the equation (3.9) satisfying the inequality $x^2 < \rho(x)$ and $\rho(x)$ is given by (3.7).

**Remark 3.23.** If we consider the general (non-convex) case, where $\alpha$, $\beta$, $\gamma$, and $\delta$ satisfy the conditions

$$\alpha + \beta + \gamma + \delta = 2, \quad -1 < \alpha, \beta, \gamma, \delta < 1,$$

instead of (3.1), then we obtain that, by Lemma 3.8, for every $M$ there are at most three different exterior quadrilaterals of the form $A_1A_2A_3A_4$, $A_1 = 1$, $A_2 = 0$, with angles $(1 + \alpha)\pi$, $(1 + \beta)\pi$, $(1 + \gamma)\pi$, and $(1 + \delta)\pi$, such that their conformal moduli equal $M$.

In connection with Remark 3.22, we can suggest

**Conjecture 3.24.** For every $M > 0$ there is only one exterior quadrilateral with given angles and vertices $A_1 = 1$, $A_2 = 0$, $A_3, A_4$, conformal modulus of which equals $M$.

With the help of Theorem 3.22 we can calculate the exterior modulus of a sufficiently arbitrary convex polygonal line $A_1A_2A_3A_4$ with vertices $A_1 = 1$, $A_2 = 0$, $A_3 = A$, and $A_4 = B$. With the help of Wolfram Mathematica package, we created a function ExtMod[$A, B, n, wp$] which calculated the values of the exterior modulus and also gives the values of the parameters $\alpha$, $\beta$, $\gamma$, $\delta$, $t$, and $z_0$; here $n$ and $wp$ are some additional parameters; their meaning will be explained below. The code is contained in Appendix A.

Now we briefly describe its structure.

To find the exterior modulus with the help of (3.17), we determine the value of $t = 1/r^2$. For this, we use the bisection method on the segment $[1, T_2]$, $T_2 = 10^n$, $n > 0$; the number of iterations is $S = [5(n + 15)]$ where $[x]$ denotes the integer part of $x$. The input consists of coordinates of the vertices $A_3$ and $A_4$, the value of $n$ and the parameter $wp$ that specifies how many digits of precision should be maintained in internal computations (the Wolfram Mathematica option "WorkingPrecision"). The program calculates the values of angles $\alpha$, $\beta$, $\gamma$, and $\delta$, which are denoted for short by $a$, $b$, $c$, and $d$. Then, on every iteration, we solve the cubic equation (3.9) with coefficients corresponding to the current value of $t$ and determine its unique root $x_0$ satisfying $x^2 < \rho(x)$. After this, we find $y_0$, $z_0 = x_0 + iy_0$ and calculate the integrals

\[
J_1 = \int_0^1 \frac{x^\alpha(1 - x)^\beta(1 - x/t)^\gamma dx}{|1 - x/z_0|^4}, \quad J_2 = \int_1^t \frac{x^\alpha(x - 1)^\beta(1 - x/t)^\gamma dx}{|1 - x/z_0|^4},
\]

corresponding to the current value of the parameter $t$, and compare its ratio $J_2/J_1$ with $L$ which is the length of the side $|A_2A_3|$. (In fact, $J_2/J_1$ coincides with $l_2$ given by (5.3).)
Now we give a suggestion how to fix the value of \( n \) if we can obtain an a priori estimate from above for the value of the desired exterior modulus \( M \). If \( M \) is less than \( 2.3 \), we can take \( n = 2 \). For \( 2.3 \leq M \leq 11.8 \) the value \( n = (10M - 6)/7 \) is suitable. We note that for \( M > 11.8 \) the program does not work properly because of degeneration of the elliptic integrals.

Now we will give results of some numeric examples.

**Example 3.26.** In \([\text{NRRV} \text{, subsect. 5.3}]\) exterior polygonal quadrilaterals with the following vertices are considered:

1. \( A_1 = 1, A_2 = 0, A_3 = -19/25 + i21/25, A_4 = 28/25 + i69/50; \)
2. \( A_1 = 1, A_2 = 0, A_3 = -3/25 + i21/25, A_4 = 42/25 + 4i. \)

With the use of the boundary integral equations method, approximate values of their exterior moduli \( M_1 \) and \( M_2 \) were found.

We calculated the moduli \( M_j \) by our method and the results are given in Table 1 (see also Appendix A). To estimate the accuracy of our calculations we also find the moduli of the conjugate quadrilaterals, \( M_j^* \) and \( M_j \). Theoretically, the values of \( M_j \) coincide with \((M_j^*)^{-1}\) but difference in approximate values shows how the obtained values are distinct from the exact ones. We see that the values of our calculations coincide with those from \([\text{NRRV}]\) with accuracy \(10^{-9}\). The differences between \( M_j \) and \((M_j^*)^{-1}\) do not exceed \(5 \times 10^{-15}\); this gives a reason to hope that in the values of moduli we received 14 correct digits after the decimal dot.

On Fig. we give the exterior quadrilateral and the image of a grid under the mapping (3.12) for the case (1).

**Table 1.** The values of exterior moduli for two quadrilaterals.

|                  | \( j = 1 \)          | \( j = 2 \)          |
|------------------|----------------------|----------------------|
| Approx. values of \( M_j \)       | 0.992341633097863    | 0.959257179199002    |
| Approx. values of \((M_j^*)^{-1}\) | 0.992341633097868    | 0.959257179199007    |
| Approx. values of \( M_j \) from \([\text{NRRV}]\) | 0.9923416331         | 0.9592571729         |

**Example 3.27.** Now consider the case of the exterior of a rectangle with vertices \( A_1 = 1, A_2 = 0, A_3 = iH, A_4 = 1 + iH, H > 0 \). The function

\[
(3.28) \quad w = \frac{1}{k} \frac{z - \sqrt{t}}{z + \sqrt{t}}, \quad k = \frac{\sqrt{t} - 1}{\sqrt{t} + 1},
\]

maps the upper half-plane onto itself with the correspondence of points \( 0 \mapsto -1/k, 1 \mapsto -1, t \mapsto 1, \infty \mapsto 1/k \). According to the Duren-Pfaltzgraff formula \([\text{DP} \text{, sect. (iv)}]\),

\[
(3.29) \quad H = \frac{2 \varepsilon(k) - (1 - k) \kappa(k)}{\varepsilon'(k) - k \kappa'(k)}.
\]
Here $\mathcal{K}(k)$ and $\mathcal{E}(k)$ are the complete elliptic integrals given by (2.2), $\mathcal{K}'(k) = \mathcal{K}(k')$, $\mathcal{E}'(k) = \mathcal{E}(k')$, where $k' = \sqrt{1 - k^2}$. Using (3.29), we can check the accuracy of our calculation. For a given $H$, we find the approximate value of $t$, then, with the help of (3.28), determine $k$ and by (3.29) find the value of $H_{\text{app}}$ corresponding to the found approximate values of parameters. Comparing $H_{\text{app}}$ with the initial value of $H$, we can estimate the accuracy of the approximate method.

In Table 2, for some $H$ we give the values of the exterior moduli $M$ and the corresponding $H_{\text{app}}$. It is interesting that the method gives very good results even for very large $H$. Comparing $H_{\text{app}}$ with $H$ shows that the accuracy of results for large $H$ is much better than those obtained by considering the conjugate modulus $M^*$ (we can simply put $H^{-1}$ instead of $H$) and funding, after this, the reciprocal value $(M^*)^{-1}$.

From Table 2 we see that for $H < 100$ the $|H_{\text{app}} - H| < 10^{-13}$. For large $H$, the relative error grows but even for $H = 10^6$ it less than $10^{-10}$ what can be considered a very good result.
### Table 2. The values of exterior moduli $M$ and $k$ for some rectangles.

| $H$ | $H_{app}$   | $M$          | $k$            |
|-----|-------------|--------------|----------------|
| 1   | 0.9999999999999784 | 0.999999999999997 | 0.171528752538083 |
| 2   | 1.9999999999999971  | 1.15424858699707 | 0.2589511664373517 |
| 3   | 2.9999999999999959  | 1.25423186704834 | 0.3183618249446048 |
| 4   | 3.9999999999999940  | 1.328560829309608 | 0.3630445515606185 |
| 5   | 4.9999999999999927  | 1.387897041604210 | 0.3985903936736862 |
| 10  | 9.99999999999999874 | 1.50900257847724 | 0.5096661128249422 |
| 50  | 49.9999999999999927 | 2.062779488244626 | 0.7306010544314864 |
| 100 | 99.9999999999999962 | 2.27819583070594 | 0.7996714751224258 |
| $10^3$ | 999.9999999999126  | 3.005361525457626 | 0.9312093496761309 |
| $10^4$ | 10000.00000000519 | 3.7355617586474 | 0.9776888723313666 |
| $10^5$ | 100000.0000021733 | 4.47031757015527 | 0.9928890530750033 |
| $10^6$ | 1000000.000038778  | 5.203265238854191 | 0.998045791670292 |

4. **Conformal mappings of the interior and exterior of an isosceles trapezoidal polygon onto the half-plane**

4.1. **Interior of a trapezoidal polygon.**

Further, for convenience, in investigation of the interior and exterior moduli for considered trapezoidal lines, we will assume that one of its parallel sides is on the real axis and, therefore, $A_1 A_2$ does not coincide with the segment $[0, 1]$. This assumption does not play a significant role because it is evident that the formulas for the corresponding conformal mappings can be simply obtained from each other by applying conformal automorphisms of the complex plane of the form $z \mapsto a_0 z + b_0$, $a_0, b_0 \in \mathbb{C}$. So, let $L$ be the boundary of a trapezoid with vertices $A_1(−d−i), A_2(−c), A_3(c)$ and $A_4(−d+i)$. Here $d > c > 0$ (Fig. 2).

Denote by $T^+ = T^+(c, \alpha)$ and $T^- = T^-(c, \alpha)$ the interior and the exterior of $L$. Let $\alpha$ be the value of the angle of $T^+$ at $A_4$. Then the angles of $T^+$ at $A_1$, $A_2$, $A_3$, and $A_4$ are equal $\pi \alpha$, $\pi(1-\alpha)$, $\pi(1-\alpha)$, and $\pi \alpha$. The corresponding angles of $T^-$ are equal $\pi(2-\alpha)$, $\pi(1+\alpha)$, $\pi(1+\alpha)$, and $\pi(2-\alpha)$. Moreover, $d - c = \cot(\pi \alpha)$ and the both non-horizontal sides are of length equal to $1/\sin(\pi \alpha)$.

Let us map the lower half-plane conformally onto $T^+$ such that $-1/(\lambda) \mapsto A_1$, $-1 \mapsto A_2$, $1 \mapsto A_3$, and $1/\lambda \mapsto A_4$. Here $0 < \lambda < 1$ is some number depending on the modulus of $T^+$. According to the Schwarz-Christoffel formula, the mapping is given by the formula

$$f^+(z) = C \int_0^z (t^2 - 1)^{-\alpha}(\lambda^2 t^2 - 1)^{\alpha-1} dt$$

with the constant

$$C = c/I, \quad I = \int_0^1 (1 - t^2)^{-\alpha}(1 - \lambda^2 t^2)^{\alpha-1} dt.$$
Comparing the lengths of the bases, we obtain

\[
\frac{\int_0^1 (1 - t^2)^{-\alpha} (1 - \lambda^2 t^2)^{-\alpha - 1} dt}{\int_{1/\lambda}^{\infty} (t^2 - 1)^{-\alpha} (\lambda^2 t^2 - 1)^{-\alpha - 1} dt} = \frac{c}{d}.
\]

From \((4.2)\) we can find the parameter \(\lambda\) and the conformal modulus of \(T^+\):

\[
\text{mod}(T^+) = \frac{2\mathcal{K}(\lambda)}{\mathcal{K}(\lambda')}, \quad \lambda' = \sqrt{1 - \lambda^2},
\]

where \(\mathcal{K}(\lambda)\) is the complete elliptic integral of the first kind defined in \((2.2)\).

Now we will express the above integrals \((4.2)\) via the Gaussian hypergeometric function. The change of variables \(\tau = t^2\) and \((2.1)\) yield

\[
\int_0^1 (1 - t^2)^{-\alpha} (1 - \lambda^2 t^2)^{-\alpha - 1} dt = \frac{1}{2} B(1 - \alpha, \frac{1}{2}) {}_2F_1\left(\frac{1}{2}, 1 - \alpha; \frac{3}{2} - \alpha; \lambda^2\right).
\]

Another change of variables \(s = 1/(\lambda t)\) and the substitution \(\tau = s^2\) lead to

\[
\int_{1/\lambda}^{\infty} (t^2 - 1)^{-\alpha} (\lambda^2 t^2 - 1)^{-\alpha - 1} dt = \lambda^{2\alpha - 1} \int_0^1 (1 - s^2)^{\alpha - 1} (1 - \lambda^2 s^2)^{-\alpha} ds
\]

\[= \frac{1}{2} B(\alpha, \frac{1}{2}) \lambda^{2\alpha - 2} {}_2F_1\left(\frac{1}{2}, \alpha; \alpha + \frac{1}{2}; \lambda^2\right).
\]

The relations \((4.3)\) and \((4.4)\) allow us to write \((4.2)\) in the form

\[
\frac{B(\alpha, \frac{1}{2})}{B(1 - \alpha, \frac{1}{2})} \frac{\lambda^{2\alpha - 1} {}_2F_1\left(\frac{1}{2}, \alpha; \alpha + \frac{1}{2}; \lambda^2\right)}{\lambda^{2\alpha - 2} {}_2F_1\left(\frac{1}{2}, 1 - \alpha; \frac{3}{2} - \alpha; \lambda^2\right)} = \frac{d}{c}.
\]

From the geometric reasoning we conclude that \((4.5)\) has a unique solution \(\lambda\) on \((0, 1)\).
We will also need the boundary correspondence between points of the real axis and points on the bases of the trapezoid. For \( x \in [0, 1] \), with the help of the change of variables, \( t = xs \), \( s^2 = \tau \) and \((2.6)\), we have

\[
\int_{0}^{x} (1 - t^2)^{-\alpha} (1 - \lambda^2 t^2)^{\alpha-1} dt = x \int_{0}^{1} (1 - x^2 s^2)^{-\alpha} (1 - \lambda^2 x^2 s^2)^{\alpha-1} ds
\]

\[
= x F_1 \left( \frac{1}{2}; \alpha, 1 - \alpha; \frac{3}{2}; x^2, x^2 \lambda^2 \right).
\]

where \( F_1 \) is the Appell hypergeometric function \((2.5)\). Similarly, for \( x > 1/\lambda \) we obtain

\[
\int_{x}^{\infty} (t^2 - 1)^{-\alpha} (\lambda^2 t^2 - 1)^{\alpha-1} dt = x \int_{0}^{1} (x^2 - s^2)^{-\alpha} (\lambda^2 x^2 - s^2)^{\alpha-1} ds
\]

\[
= \lambda^{2(\alpha-1)} x^{-1} F_1 \left( \frac{1}{2}; \alpha, 1 - \alpha; \frac{3}{2}; x^{-2}, x^{-2} \lambda^{-2} \right).
\]

4.6. Exterior of a trapezoidal polygon.

Now we describe the conformal mapping of the upper half-plane onto the exterior \( T^- \) of a trapezoidal polygon. We will assume that the pole of the mapping function is at the point \( i \). Let \( A_1, A_2, A_3, \) and \( A_4 \) correspond to the points \(-b, -a, a, \) and \( b \) for some \( 0 < a < b \).

According to the generalized Schwarz-Christoffel formula \((3.2)\), we have

\[
f^{-1}(z) = \tilde{C} \int_{0}^{z} \frac{(t^2 - a^2)^{\alpha} (t^2 - b^2)^{1-\alpha}}{(1 + t^2)^2} dt
\]

with the constant

\[
\tilde{C} = c/\bar{I}, \quad \bar{I} = \int_{0}^{\alpha} \frac{(t^2 - a^2)^{\alpha} (b^2 - t^2)^{1-\alpha}}{(1 + t^2)^2} dt.
\]

Because the residue of the integrand vanishes at the point \( t = i \), we deduce that the values of \( a \) and \( b \) are connected by the equality

\[
(4.7) \quad \frac{\alpha}{1 + a^2} + \frac{1 - \alpha}{1 + b^2} = \frac{1}{2}.
\]

Denote \( k = a/b, \) \( 0 < k < 1 \).

Comparing the lengths of the sides we obtain

\[
\int_{0}^{a} \frac{(t^2 - a^2)^{\alpha} (b^2 - t^2)^{1-\alpha}}{(1 + t^2)^2} dt = c \quad \frac{d}{d}
\]

\[
\int_{b}^{\infty} \frac{(t^2 - a^2)^{\alpha} (t^2 - b^2)^{1-\alpha}}{(1 + t^2)^2} dt = \frac{c}{d}.
\]

After the change of variables \( t = as \), we have

\[
(4.8) \quad \int_{0}^{1} (1 - s^2)^{\alpha} (1 - k^2 s^2)^{1-\alpha} \frac{ds}{(1 + a^2 s^2)^2}
\]

\[
= \frac{c}{d}.
\]
From (4.7) we find that, for a fixed \( \alpha \),
\[
a^2(a(k)) = \sqrt{A^2 + k^2} - A, \quad A = (\frac{1}{2} - \alpha)(1 - k^2).
\]
Solving (4.8), we find the value of \( k \) and then
\[
\mod(T^-) = \frac{2\kappa(k)}{\kappa(k')}, \quad k' = \sqrt{1 - k^2}.
\]

Now we will write the integrals from (4.8) through special functions. After the change of variable \( s^2 = \tau \), with the help of (2.6), we obtain:
\[
\int_0^1 \frac{(1 - s^2)^\alpha (1 - k^2 s^2)^{1-\alpha}}{(1 + a^2 s^2)^2} \, dt = \frac{1}{2} B(\frac{1}{2}, 1 + \alpha) F_1(\frac{1}{2}; \alpha - 1, 2; \frac{3}{2} + \alpha; k^2, -a^2).
\]
Similarly,
\[
\int_{1/k}^{\infty} \frac{(s^2 - 1)^\alpha (k^2 s^2 - 1)^{1-\alpha}}{(1 + a^2 s^2)^2} \, ds = k^{3-2\alpha} \int_0^1 \frac{(1 - k^2 \tau^2)^\alpha (1 - \tau^2)^{1-\alpha}}{(a^2 + k^2 \tau^2)^2} \, d\tau
\]
\[
= \frac{k^{3-2\alpha}}{2a^4} B(\frac{1}{2}, 2 - \alpha) F_1(\frac{1}{2}; -\alpha, 2; \frac{5}{2} - \alpha; k^2, -a^2 k^2).
\]
Therefore, we have the equation to determine \( k \):
\[
(4.10) \quad \frac{k^{3-2\alpha} B(\frac{1}{2}, 2 - \alpha) F_1(\frac{1}{2}; -\alpha, 2; \frac{5}{2} - \alpha; k^2, -a^2 k^2)}{2a^4 B(\frac{1}{2}, 1 + \alpha) F_1(\frac{1}{2}; \alpha - 1, 2; \frac{3}{2} + \alpha; k^2, -a^2)} = \frac{d}{c}
\]
where \( a = a(k) \) (see (4.9)).

Now, as in the case of the interior modulus, we find the relations between boundary points of the half-plane and points of the sides of \( T \). We have for \( x \in (0, 1) \):
\[
\int_0^x \frac{(1 - s^2)^\alpha (1 - k^2 s^2)^{1-\alpha}}{(1 + a^2 s^2)^2} \, ds = x \int_0^1 \frac{(1 - x^2 s^2)^\alpha (1 - k^2 x^2 s^2)^{1-\alpha}}{(1 + a^2 x^2 s^2)^2} \, ds
\]
\[
= x F_D^{(2)}(1; -\alpha, -1, 2; \frac{1}{2}, k^2, a^2; x^2; 1)
\]
where \( F_D^{(3)} \) is the Lauricella hypergeometric function (see (2.7) and (2.8)).

For \( x > 1/k \) we have
\[
\int_x^{\infty} \frac{(s^2 - 1)^\alpha (k^2 s^2 - 1)^{1-\alpha}}{(1 + a^2 s^2)^2} \, ds = x \int_0^1 \frac{(1 - x^2 s^2)^\alpha (1 - k^2 x^2 s^2)^{1-\alpha}}{(1 + a^2 x^2 s^2)^2} \, ds
\]
\[
= \frac{k^{2(1-\alpha)}}{xa^4} \int_0^1 \frac{(1 - x^2 s^2)^\alpha (1 - k^2 x^2 s^2)^{1-\alpha}}{(1 + a^2 x^2 s^2)^2} \, ds
\]
\[
= \frac{k^{2(1-\alpha)}}{2xa^4} F_D^{(3)}(1; -\alpha, -1, 2; x^{-2}, k^2, a^{-2}; x^{-2}).
\]
Comparing the interior and exterior moduli we immediately obtain the following statement.
Theorem 4.11. Let $L$ be the isosceles trapezoidal curve with acute angle $\pi \alpha$ and bases $c$ and $d$. Let $\lambda$ and $k$ be solutions of (4.5) and (4.10), (4.9). Then the coefficient $M_L$ satisfies the inequality

$$M_L \geq \max \left[ \frac{\mathcal{K}(\lambda) \mathcal{K}(k')}{\mathcal{K}(\lambda')(k)}, \frac{\mathcal{K}(\lambda') \mathcal{K}(k)}{\mathcal{K}(\lambda)(k')}, \frac{\mathcal{K}(\lambda) \mathcal{K}(k')}{\mathcal{K}(\lambda')(k)} \right]$$

where $\lambda' = \sqrt{1 - \lambda^2}$, $k' = \sqrt{1 - k^2}$.

It is evident that the estimation from Theorem 4.11 is not sharp. Numerical experiments with sufficiently long rectangles show that the ratio of moduli of two quadrilaterals, external and internal, with the same vertices $z_1$, $z_2$, $z_3$, and $z_4$ on the line is not maximal if $z_k$ coincides with the 'natural' vertices of the rectangle. The best result is for the case where $z_k$ are on the bigger sides and are symmetric with respect to the axes of symmetry of the rectangle. In the next section we will try to explain this fact theoretically.

5. THE BOUNDARY OF A RECTANGLE

Consider the case $\alpha = 1/2$. Let $\Pi_d = [-d, d] \times [-1, 0]$ be a rectangle. Denote by $\Pi_d^c$ its exterior. Now we fix a number $\delta \in (0, d)$. Let $\Pi_\delta$ be the quadrilateral $(\Pi_d, \delta - i, \delta, -\delta, -\delta - i)$ and $\Pi_\delta^c = (\Pi_d^c, -\delta - i, -\delta, \delta, \delta - i)$. We will compare their conformal moduli. Let $\mathcal{K}(\lambda)$ be the complete elliptic integral of the first kind defined by (2.2); we will write for short $\mathcal{K}'(\lambda) = \mathcal{K}(\lambda')$ where $\lambda' = \sqrt{1 - \lambda^2}$.

**Theorem 5.1.** We have

$$\frac{\text{Mod}(\Pi_\delta)}{\text{Mod}(\Pi_\delta^c)} \geq \frac{2\lambda \mathcal{K}'(\lambda)}{\mathcal{K}(\lambda)} d,$$

where $\lambda = \delta/d$.

**Proof.** It is obvious that

$$\text{Mod}(\Pi_\delta) = 2\delta.$$

Now we will estimate $\text{Mod}(\Pi_\delta^c)$.

![Figure 3. Rectangle with shifted vertices.](image)
Let $\Pi_d^+\delta$ be the part of $\Pi_d^\delta$, lying in the quarter of the plane $\{z: \text{Re} \ z \geq 0, \text{Im} \ z \geq -1/2\}$. Consider the quadrilateral $\Pi_d^+\delta := (\Pi_d^c+: 0, \delta, d-i/2, \infty)$. By the symmetry principle,

\begin{equation}
\text{Mod}(\Pi_d^+\delta) = \text{Mod}^{-1}(\Pi_d^\delta).
\end{equation}

On the other side, the modulus is equal to the extremal length of the family of curves, $\Gamma$, connecting in $\Pi_d^c+\delta$ the sides $[0, \delta]$ and $[d-i/2, \infty]$. Now consider the subdomain $G$ of $\Pi_d^c+\delta$ which is the first quarter of the plane. The modulus of the quadrilateral $G := (G; 0, \delta, d, \infty)$ is equal to the extremal length of the family of curves, $\Gamma_1$, connecting in $G$ the sides $[0, \delta]$ and $[d, \infty]$. Since $\Gamma_1 < \Gamma$, we obtain

\begin{equation}
\text{Mod}(\Pi_d^+\delta) \geq \text{Mod}(G).
\end{equation}

Under the conformal automorphism $w = (1/\delta)z$ of $G$, the vertices of the quadrilateral $G$ are mapped to the points $0, 1, 1/\lambda, \infty$. The modulus of the obtained quadrilateral is well-known; it is expressed via elliptic integrals. Since conformal modulus is invariant under similarity mappings, we find

\begin{equation}
\text{Mod}(G) = \frac{\chi'(\lambda)}{\chi(\lambda)}.
\end{equation}

From (5.3)–(5.6) we obtain (5.2). \hfill \Box

Consider the function

$$g(\lambda) := \frac{\lambda \chi'(\lambda)}{\chi(\lambda)}.$$

The graph of $g(\lambda)$ is given on the Fig. 4. It is evident that $\lim_{\lambda \to 0^+} g(\lambda) = 0$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{The graph of the function $g(\lambda)$.}
\end{figure}
Lemma 5.7. The function $g$ is concave on $(0, 1)$ and has a unique maximum point $\lambda_0 = 0.7373921\ldots$ which is a unique root of the equation

$$\pi = (\lambda')^2 \kappa(\lambda) \kappa'(\lambda),$$

on the interval $(0, 1)$. The maximal value of $g$ is equal to $g(\lambda_0) = 0.708434\ldots$

Proof. In terms of the function $\mu$ we see that $g(\lambda) = 2\lambda \mu(\lambda)/\pi$ and hence

$$\frac{\pi}{2} g'(\lambda) = \mu(\lambda) + \lambda \mu'(\lambda), \quad \frac{\pi}{2} g''(\lambda) = 2 \mu'(\lambda) + \lambda \mu''(\lambda).$$

By (2.3) we obtain

$$g'(\lambda) = g(\lambda) - \frac{\pi/2}{(\lambda')^2 \kappa^2(\lambda)} = \frac{(\lambda')^2 \kappa(\lambda) \kappa'(\lambda) - \pi/2}{(\lambda')^2 \kappa^2(\lambda)},$$

and also, after simplification,

$$g''(\lambda) = -\frac{\pi/2}{\lambda(1 - \lambda^2)^2 \kappa^3(\lambda)}((3 - \lambda^2) \kappa(\lambda) - 2 \varepsilon(\lambda)) < 0.$$

For the last inequality note that $(3 - \lambda^2) \kappa(\lambda) > 2 \kappa(\lambda) > 2 \varepsilon(\lambda), 0 < \lambda < 1$. Therefore, $g$ is concave on $(0, 1)$ and the statement on the maximum follows from (5.9). \qed

Corollary 5.10. For $\lambda \in (0, \lambda_0)$ we have $g(\lambda) > g(\lambda_0)\lambda$.

Corollary 5.11. For the coefficient of quasiconformal reflection $M_{\partial \Pi_d}$ we have the estimation $M_{\partial \Pi_d} \geq \gamma d$ where $\gamma = 2g(\lambda_0) = 1.4168687\ldots$

Actually, it is easy to show that the maximal value of the function $g(\lambda)$ is attained at a unique point $\lambda_0 \in (0, 1)$ which is the unique root of the equation (5.8) on $(0, 1)$.

Remark 5.12. From Corollary 5.11 and (1.4) it follows that $QR_{\partial \Pi_d} \geq \gamma d$. A sharper estimate for $QR_{\Pi_d}$ follows from (1.3): $QR_{\partial \Pi_d} \geq (\pi/2)d, \quad \pi/2 = 1.5707963\ldots$

But the method of the proof of Theorem 5.1, which is rather simple, can be used to obtain a similar estimate for the case of isosceles trapezoids.

6. ESTIMATION OF THE COEFFICIENT $M_L$ FOR ISOSCELES TRAPEZOIDAL POLYGON $L$

Now we apply the same method to obtain a lower estimate for the coefficients $M_L$ and $QR_L$ of arbitrary isosceles trapezoidal polygon $L$.

First we will prove

**Lemma 6.1.** Let $D^+$ be the part of $T^-$ lying in the half-plane $\{y \geq -1\}$ and let $0 < \delta < c$. Denote by $M_1$ the conformal modulus of $(D; -\delta, \delta, d - i, -d - i)$ and by $M_2$ the conformal modulus of the quadrilateral which is the upper half-plane with vertices $-\delta, \delta, d, -d$. Then

$$M_1 \geq C(\alpha)M_2$$
where
\[ C(\alpha) = \left( \sqrt{1 + \tan^2(2\alpha)} / 4 - \tan(\pi \alpha) / 2 \right)^2, \quad 0 < \alpha \leq 1/2. \]

**Proof.** Consider the piecewise-linear mapping \( F(x, y) = x + iv(x, y) \) where
\[
v(x, y) = \begin{cases} 
  y, & |x| \leq c, \\
  y + (|x| - c) \tan(\pi \alpha), & c \leq |x| \leq d, \\
  y + 1, & |x| \geq d.
\end{cases}
\]

It is easy to verify that \( F \) is a homeomorphism of \( \mathbb{C} \), mapping \((D; -\delta, \delta, d - i, -d - i)\) onto \((H; -\delta, \delta, d, -d)\). We will show that \( F \) is a \( K \)-quasiconformal mapping with
\[
(6.3) \quad K = \left( \sqrt{4 + \tan^2(\pi \alpha)} + \tan(\pi \alpha) \right)^2 / 4.
\]
Actually, \( F \) is conformal in \( \{ |x| \leq c \} \) and \( \{ |x| \geq d \} \). Since \( F(x, y) \) is even with respect to \( x \), we only need to show that \( F \) is \( K \)-quasiconformal in the strip \( \{ c \leq x \leq d \} \) where it has the form
\[
F(z) = z + i(x - c) \tan(\pi \alpha) = \frac{1}{2} \left( (2 + i \tan(\pi \alpha))z - i \tan(\pi \alpha) \overline{z} \right) - ic \tan(\pi \alpha),
\]
therefore,
\[
\left| \frac{F_{\overline{z}}}{F_z} \right| \leq k := \frac{\tan(\pi \alpha)}{\sqrt{4 + \tan^2(\pi \alpha)}},
\]
and this implies that \( F \) is \( \frac{1 + k}{1 - k} \)-quasiconformal mapping. But \( \frac{1 + k}{1 - k} = K \) where \( K \) is given by (6.3).

At last, because of quasiinvariance of conformal modulus under quasiconformal mapping, we obtain that \( M_1 \geq K^{-1}M_2 \) where \( K^{-1} = (\sqrt{4 + \tan^2(\pi \alpha)} - \tan(\pi \alpha))^2 / 4 = C(\alpha) \). \( \square \)

Now, with the help of Lemma 6.1 we will estimate \( M_L \).

**Theorem 6.4.** Let \( L \) be the isosceles trapezoidal polygon with acute angle \( \pi \alpha \) and bases \( c \) and \( d \), \( c < d \).

1) If \( \frac{\sqrt{c}}{d} \geq \lambda_0 \), where \( \lambda_0 \) is described in Lemma 5.7, then
\[
(6.5) \quad M_L \geq g(\lambda_0)(1 + C(\alpha))d.
\]

2) If \( \frac{\sqrt{c}}{d} < \lambda_0 \), then
\[
M_L \geq g(\lambda)(1 + C(\alpha))d, \quad \lambda = c/d.
\]

**Proof.** 1) Let \( \frac{\sqrt{c}}{d} \geq \lambda_0 \). Then we put \( \delta = \lambda_0d, \quad 0 < \delta < c \). Denote
\[
T^+_{\delta} = (T^+; \delta - i, \delta, -\delta, -\delta - i), \quad T^-_{\delta} = (T^-; -\delta - i, -\delta, \delta, \delta - i),
\]
then
\[
(6.6) \quad \text{Mod}(T^+_{\delta}) \geq 2\delta = 2\lambda_0d.
\]
The line \( y = -1 \) separates \( T^- \) into two domains. One of them is \( D^+ \), denote by \( D^- \) the other one. Let \( D^+ = (D^+; -\delta, \delta, d - i, -d - i) \), \( D^- = (D^-; -\delta - i, \delta - i, d - i, -d - i) \). Then, with the use of Lemma 6.1, we obtain

\[
\text{Mod}(D^-) = \frac{3'(\lambda_0)}{2\lambda(\lambda_0)}, \quad \text{Mod}(D^+) \geq C(\alpha)\text{Mod}(D^-)
\]

Now we note that \( D^+ \), \( D^- \) and the quadrilateral \((T^+)^* = (T^-; -\delta, \delta, d - i, -d - i)\), conjugate to \( T^+ \), are symmetric with respect to the imaginary axis. Applying the Grötzsch’ lemma \( [V A, \text{pp.13-15}] \) to the right halves of the considered quadrilaterals and taking into mind that, by the symmetry principle, their moduli are half as small as the initial moduli, we obtain

\[
(6.7) \quad (\text{Mod}(T^+))^* = \text{Mod}((T^-)^*) \geq \text{Mod}(D^+) + \text{Mod}(D^-)
\]

Then, multiplying (6.6) and (6.7), we obtain

\[
\frac{\text{Mod}(T^+)}{\text{Mod}(T^-)} \geq g(\lambda_0)(1 + C(\alpha))d.
\]

and this implies (6.5).

2) Now let \( \frac{c}{d} < \lambda_0 \). Then, by similar reasoning as above, we obtain

\[
\frac{\text{Mod}(T^+)}{\text{Mod}(T^-)} \geq g(\lambda)(1 + C(\alpha))d
\]

where \( \lambda = \frac{c}{d} \).

\[ \square \]

**Remark 6.8.** The estimation obtained in Theorem 6.4 is good for small \( \alpha \) because then \( C(\alpha) \) is close to 1. For \( \alpha \), close to \( \pi/2 \), the value of \( C(\alpha) \) is close to zero and probably can be essentially improved.

Using (1.4), we can estimate \( QRL \) for trapezoidal curves.

**Corollary 6.9.** Under the assumption of Theorem 6.4, if \( \frac{c}{d} \geq \lambda_0 \), then \( QRL \geq g(\lambda_0)(1 + C(\alpha))d \). If \( \frac{c}{d} < \lambda_0 \), then \( QRL \geq g(\lambda)(1 + C(\alpha))d \) where \( \lambda = c/d \).

We can also enhance the estimations given in Theorem 6.4 and Corollary 6.9, if we calculate \( \text{Mod}(T^+_\delta) \) and \( \text{Mod}(T^-_\delta) \) with the help of the formulas given in Section 4.

Now we will describe the algorithm in more detail.

1) Let \( \frac{c}{d} \geq \lambda_0 \). First we find the preimages of \( \delta \) and \( \delta - i, \delta = \lambda_0d \), under the conformal mapping of the lower half-plane onto \( T^+ \) described in Subsection 4.1.

For this, we find a unique \( x_* \in [0, 1] \) from the equation

\[
x_*F_1\left(\frac{1}{2}; \alpha, 1 - \alpha; \frac{3}{2}; x_*^2, \lambda^2 x_*^2\right) = \frac{\lambda_0d}{c} B(1 - \alpha, \frac{1}{2}) \int F_1\left(\frac{1}{2}; 1 - \alpha, \frac{3}{2}; x_*^2, \lambda^2 x_*^2\right).
\]
and a unique $x_{ss} \in (1/\lambda, +\infty)$, satisfying

$$x_{ss}^{-1} \lambda^{2(1-\alpha)} F_1(\frac{1}{2}; \alpha, 1 - \alpha, \frac{3}{2}; x_{ss}^{-2}, \lambda^{-2} x_{ss}^{-2}) = \frac{\lambda_0 d}{c} B(1 - \alpha, \frac{1}{2}) F_1(\frac{1}{2}; 1 - \alpha, \frac{3}{2} - \alpha, \lambda^2).$$

Now we find $\tilde{\lambda} = x_* / x_{ss}$ and the modulus of $T^+$:

$$(6.10) \quad \text{Mod}(T^+) = \frac{2 \kappa(\tilde{\lambda})}{\kappa(\tilde{\lambda}')}, \quad \tilde{\lambda}' = \sqrt{1 - \tilde{\lambda}^2}.$$

Then we find the preimages of $\delta$ and $\delta - i$ under the conformal mapping of the upper half-plane onto $T^-$ described in Subsection 4.6.

We find $y_* \in (0, 1)$ from the equation

$$y_* F_D^{(3)}(\frac{1}{2}; -\alpha, \alpha - 1, 2; \frac{3}{2}; y_*^2, y_*^2 k^2, -y_*^2 a^2) = \frac{\lambda_0 d}{c} F_1(\frac{1}{2}; \alpha - 1, \frac{3}{2} + \alpha; k^2 m - a^2),$$

and $y_{ss} \in (1/k, +\infty)$ from the equation

$$y_{ss}^{-1} k^{2(1-\alpha)} a^{-4} F_D^{(3)}(\frac{1}{2}; -\alpha, \alpha - 1, 2; \frac{3}{2}; y_{ss}^2, y_{ss}^2 k^2, -y_{ss}^2 a^2).$$

Then we put $\tilde{k} = y_* / y_{ss}$ and obtain

$$(6.11) \quad \text{Mod}(T^-) = \frac{2 \kappa(\tilde{k})}{\kappa(\tilde{k}')}, \quad \tilde{k}' = \sqrt{1 - \tilde{k}^2}.$$

From (6.10) and (6.11) we deduce that

$$M_L \geq \frac{\kappa(\tilde{\lambda}) \kappa(\tilde{k})}{\kappa(\tilde{\lambda}') \kappa(\tilde{k})}.$$

2) If $\frac{c}{d} < \lambda_0$, then we replace $x_*$ and $y_*$ with 1 and in the second equation we put $c/d$ instead of $\lambda_0$.

**Remark 6.12.** If the base $c$ is sufficiently large, then the choice of quadrilateral with vertices $\pm \delta$, $\pm \delta - i$ for estimation of $M_L$ and $QR_L$ is rather good because of a result by W. Hayman (see, e.g. [PS, thrm. 2.3.8]). It states that if two sides of a quadrilateral are segments on the vertical lines $\{ x = 0 \}$ and $\{ x = 1 \}$ and the other two are graphs of two continuous on $[0, 1]$ functions $y = \varphi(x)$ and $y = \psi(x)$, and $h := \min \psi - \max \varphi > 0$, then the modulus $M$ of the quadrilateral satisfies the inequality

$$h \leq M \leq h + 1.$$
Let \( \alpha \) be a real number with \( 0 < \alpha < 1 \). An analytic function \( f \) on the unit disk \( \mathbb{D} \) is said to be strongly starlike of order \( \alpha \) if \( f'(0) \neq 0 \) and if \( f \) satisfies the inequality

\[
\left| \arg \frac{zf'(z)}{f(z) - f(0)} \right| < \frac{\pi \alpha}{2}
\]

for \( 0 < |z| < 1 \). In particular, \( \text{Re} \left[ \frac{zf'(z)}{(f(z) - f(0))} \right] > 0 \) and thus \( f \) is a starlike univalent function on the unit disk. A simply connected domain \( \Omega \) in \( \mathbb{C} \) is said to be strongly starlike of order \( \alpha \) with respect to \( w_0 \) if the conformal homeomorphism \( f : \mathbb{D} \to \Omega \) with \( f(0) = w_0 \) and \( f'(0) > 0 \) is strongly starlike of order \( \alpha \). The following result is due to Fait, Krzyż and Zygmunt [FKZ]. Here and hereafter, we set

\[
(7.1) \quad K(\alpha) = \frac{1 + \sin(\pi \alpha/2)}{1 - \sin(\pi \alpha/2)}
\]

for \( 0 < \alpha < 1 \).

**Lemma 7.2.** Let \( 0 < \alpha < 1 \). A strongly starlike function \( f \) of order \( \alpha \) on \( \mathbb{D} \) extends to a \( K(\alpha) \)-quasiconformal mapping of \( \mathbb{C} \).

In particular, we see that the boundary of a strongly starlike domain is a Jordan curve in \( \mathbb{C} \). As a consequence, we obtain the following result.

**Corollary 7.3.** Let \( L \) be the boundary of a strongly starlike domain \( \Omega \) of order \( \alpha \). Then \( QR_L \leq K(\alpha) \), where \( K(\alpha) \) is given in (7.1).

**Proof.** We follow Ahlfors’ construction [A2]. Let \( j \) be the inversion in the unit circle, \( j(z) = 1/\bar{z} \). By Lemma 7.2, a conformal mapping \( f : \mathbb{D} \to \Omega \) extends to a \( K(\alpha) \)-quasiconformal mapping of the Riemann sphere \( \mathbb{C} \cup \{\infty\} \), which is denoted by the same symbol \( f \). Then \( F = f \circ j \circ f^{-1} \) is a \( K(\alpha) \)-quasiconformal reflection across \( L \). \( \square \)

To check strong starlikeness, it is convenient to look at the following quantity. For a domain \( \Omega \) and \( w_0 \in \Omega \), we define

\[
R_{\Omega,w_0}(\theta) = \sup \{ r > 0 : w_0 + te^{i\theta} \in \Omega \text{ for all } t \in [0,r) \}
\]

for \( \theta \in \mathbb{R} \). The following result is contained in [Su].

**Lemma 7.4.** Let \( \Omega \) be a domain in \( \mathbb{C} \) containing a point \( w_0 \) and let \( 0 < \alpha < 1 \). The domain \( \Omega \) is strongly starlike of order \( \alpha \) with respect to \( w_0 \) if and only if \( R(\theta) = R_{\Omega,w_0}(\theta) \) is absolutely continuous and satisfies the inequality \( |R'(\theta)|/R(\theta) \leq \tan(\alpha \pi/2) \) for almost all \( \theta \in \mathbb{R} \).

We consider the isosceles trapezoid \( L \) described in Section 4. Let \( \Omega \) be the domain bounded by \( L \). We now show the following for \( \Omega \).

**Lemma 7.5.** Let \( 0 < s < 1 \). Then \( \Omega \) is strongly starlike of order \( \alpha(s) \) with respect \( -is \), where \( \alpha(s) \) is determined by

\[
\tan \frac{\pi \alpha(s)}{2} = \max \left\{ c \cdot \frac{d}{1-s} + (d-c) \frac{1-s + (d-c)d}{c + (d-c)s} \cdot s - (d-c) \frac{1-s + (d-c)d}{c + (d-c)s} \right\}
\]
Thus the required assertion follows with the help of the formula

\[ \Omega \]  

apply the previous lemma to show that

We recall that Werner’s estimation (1.3) gives us

\[ QR \]

where

\[ R_c \]

Proof. Since \( \Omega \) is symmetric in the imaginary axis, it is enough to consider the function

\[ R(\theta) = R_{\Omega,-s}(\theta) \]

for \(-\pi/2 < \theta < \pi/2\). We define \( \theta_1 \) and \( \theta_2 \) in \((0, \pi/2)\) by requiring

\[
\tan \theta_1 = \frac{s}{c}, \quad \tan \theta_2 = \frac{1-s}{d}.
\]

Then the function \( R(\theta) \) is described by

\[
R(\theta) = \begin{cases} 
\frac{s}{\sin \theta}, & \theta_1 < \theta < \pi/2, \\
\frac{(1-s)c+sd}{\cos \theta + (d-c)\sin \theta}, & -\theta_2 \leq \theta \leq \theta_1, \\
\frac{1-s}{-\sin \theta}, & -\pi/2 < \theta < -\theta_2.
\end{cases}
\]

In the first case \( \theta_1 < \theta < \pi/2 \), we have \(|R'(\theta)|/R(\theta) = 1/\tan \theta \leq 1/\tan \theta_1 = c/s\). Similarly, we have \(|R'(\theta)|/R(\theta) \leq 1/\tan \theta_2 = d/(1-s)\) in the third case. When \(-\theta_2 < \theta < \theta_1\), we have

\[
\frac{R'(\theta)}{R(\theta)} = \sin \theta - (d-c)\cos \theta \quad \text{and} \quad \left( \frac{R'(\theta)}{R(\theta)} \right)' = \frac{1+(d-c)^2}{(\cos \theta + (d-c)\sin \theta)^2} > 0.
\]

Hence, \( R'/R \) is increasing in this interval and, in particular,

\[
\frac{1-s+(d-c)d}{d-(d-c)(1-s)} = \frac{1-s+(d-c)d}{c+(d-c)s} \leq \frac{R'(\theta)}{R(\theta)} \leq \frac{s-(d-c)c}{c+(d-c)s}
\]

for \( \theta \in (-\theta_2, \theta_1) \). Therefore, by Lemma 7.3, the required formula follows. \( \square \)

We are now able to show the following result.

**Theorem 7.6.** Let \( L \) be an isosceles trapezoidal polygon of height 1 and bases \( c \) and \( d \) with \( c \leq d \). Then

\[
QR_L \leq (\sqrt{1+\tau^2} + \tau)^2,
\]

where

\[
\tau = \max \left\{ c + d, \frac{1-c^2+d^2}{2c} \right\}.
\]

Proof. Let \( \Omega \) be the domain bounded by \( L \). If \( s = c/(c+d) \), then \( c/s = d/(1-s) \) and we apply the previous lemma to show that \( \Omega \) is strongly starlike of order \( \alpha \) with \( \tan(\alpha \pi/2) = \tau \). Thus the required assertion follows with the help of the formula \( \sin \theta = \tan \theta/\sqrt{1+\tan^2 \theta} \).

**Remark 7.7.** When \( c = d \), we have a rectangle \( L \) of height 1 and width \( a = 2d \). We may assume that \( a \geq 1 \). Then we have \( \tau = a \) and thus

\[
QR_L \leq (\sqrt{1+a^2} + a)^2 \leq (3+2\sqrt{2})a^2.
\]

We recall that Werner’s estimation (1.3) gives us \( QR_L \leq \pi a \). Therefore, the last theorem yields only a poor estimate.
8. Appendix A. The Wolfram Mathematica Code for Calculation of Exterior Modulus for Polygonal Quadrilateral

Here we give a code which defines the function ExtMod(A, B, n, wp) described in Section 3.

```mathematica
ExtMod[A_?NumberQ, B_?NumberQ, n_, wp_] := Module[{m = n, A1 = 1., A2 = 0., A3 = A, A4 = B, a, b, c, d, i, r1, r2, s, t, t1, t2, x, x0, x1, x2, x3, x4, y0, z0, A0, B0, C0, D0, sol, J1, J2, K, L, L2, M, S, T2},
T2 = 10^-m; S = IntegerPart[5 (m + 15)]; a = Arg[A4 - A1]/Pi;
b = 1 - Arg[A3 - A2]/Pi; c = 1 - b - Arg[A4 - A3]/Pi; d = 2 - a - b - c;
L = Abs[A3 - A2]; t1 = 1; t2 = T2;
Do[t = (t1 + t2)/2; K = a + b + c - 1;
A0 = 2 (K - 1)^-2; B0 = (K - 1) (4 - 3 (a + c) + (4 - 3 (a + b)) t);
C0 = 2 - 3 (a + c) + (a + c)^-2 + 2 (3 - 5*a - 2*b - 2*c + 2*a^-2 + 2*a*b + 2*a*c + b*c) t + (2 - 3 (a + b) + (a + b)^-2) t^-2;
D0 = (1 - a) (a + c - 1 + (a + b - 1) t) t;
sol = Solve[A0*x^-3 + B0*x^-2 + C0*x + D0 == 0, x];
x1 = x /. sol[[1]]; x2 = x /. sol[[2]]; x3 = x /. sol[[3]];
r1[x_] = a*t*x; r2[x_] = ((1 - K) x + K (t + 1) - c*t - b);
x4 = If[r2[x1]^2 x1^2 < r1[x1] r2[x1], x1, x2];
x0 = If[r2[x3]^2 x3^2 < r1[x3] r2[x3], x3, x4];
y0 = Sqrt[r1[x0]/r2[x0] - x0^2]; z0 = x0 + I*y0;
Quiet[J1 = Re[NIntegrate[s^a (1 - s)^b (1 - s/t)^c/Abs[1 - s/z0]^4, {s, 0, 1}, WorkingPrecision -> wp]];
Quiet[J2 = Re[NIntegrate[s^a (s - 1)^-b (1 - s/t)^c/Abs[1 - s/z0]^4, {s, 1, t}, WorkingPrecision -> wp]];
L2 = J2/J1; If[L2 < L, t1 = t, t2 = t1, i, S]];
t = N[t]; M = EllipticK[(1. - 1/t)]/EllipticK[1/t]; {M, a, b, c, d, t, z0}
]
```

Now we will give the results of calculating the exterior moduli of the polygons from Example 3.26. The working precision is equal 16 and the values of n is equal 2 because the values of the moduli are sufficiently small.

**Polygon 1.**

A = 28/25 + I*69/50; B = -19/25 + I*21/25; sol = ExtMod[B, A, 2, 16];
Print["ExtMod(" + A + ", " + B + ", 16) = " ", NumberForm[sol[[1]], 16], "];
Print["alpha = " ", NumberForm[sol[[2]], 16], "; ", beta = " ", NumberForm[sol[[3]], 16], "; ", gamma = " ", NumberForm[sol[[4]], 16], "; " + delta = " ", NumberForm[sol[[5]], 16], "; "];
Print["t = " ", NumberForm[sol[[6]], 16], "; ", z0 = " ", NumberForm[sol[[7]], 16], "."]

ExtMod[28/25+(69 I)/50, -19/25+(21 I)/25,16] = 0.992341633097864,
alpha = 0.4723903292882761, beta = 0.2659022512561763,
gamma = 0.645065158079917, delta = 0.6166422676475559,
t = 1.96691045621464, z0 = (1.215406699779183+1.315084271771535 I).
Polygon 2.

\[ A = \frac{42}{25} + i \cdot 4; \quad B = -\frac{3}{25} + i \cdot \frac{21}{25}; \quad \text{sol} = \text{ExtMod}[B, A, 2, 16]; \]
\[
\text{Print}["\text{ExtMod}(, A, , B, , 16, , ) = " \text{NumberForm}[\text{sol}[[1]], 16], ","; \text{Print}["alpha = " \text{NumberForm}[\text{sol}[[3]], 16], "," " gamma = " \text{NumberForm}[\text{sol}[[5]], 16], "," " delta = " \text{NumberForm}[\text{sol}[[7]], 16], ","; \text{Print}["t = " \text{NumberForm}[\text{sol}[[6]], 16], "," " z0 = " \text{NumberForm}[\text{sol}[[7]], 16], "."]
\]

\[
\text{ExtMod}(\frac{42}{25}+4 \, i, -(\frac{3}{25})+(21 \, i)/25, 16) = 0.959257171919002, \]
\[
\text{alpha} = 0.4463997482438991, \quad \text{beta} = 0.4548327646991334, \]
\[
\text{gamma} = 0.2099823197839025, \quad \text{delta} = 0.888785167273065, \]
\[
\text{t} = 1.83346758954612, \quad \text{z0} = (0.7429152683728336+1.983082728044083 \, i). \]

| A         | B         | ExtMod[B, A] |
|-----------|-----------|--------------|
| 7 + 5 I   | -1 + 2 I  | 1.158095606321043 |
| 8 + 3 I   | -1 + I    | 1.13041008465672 |
| 5 + 5 I   | -3 + I    | 1.233703270301942 |
| 7 + 4 I   | -3 + 3 I  | 1.274708414007269 |
| 5 + 5 I   | -1 + 2 I  | 1.14057649170462 |
| 7 + 5 I   | - I       | 1.015468479689712 |
| 7 + 3 I   | 1 + 2 I   | 1.135151674872884 |
| 4 + 5 I   | -2 + I    | 1.157883901548636 |
| 1 + I     | - I       | 0.999999999999995 |

Table 3 was computed with the function ExtMod.

Remark 8.1. There is also another method to validate the results of the function ExtMod. We can compare the results in the case of a rectangle with vertices \(0, 1, 1 + i \cdot h, i \cdot h, h > 0\), to the analytic formula given by Duren and Pfaltzgraff \([DP]\) see also \([HRV1]\) for further bibliographic references. By this formula, defining

\[
\psi(r) = \frac{2(\varepsilon(r) - (1 - r)\kappa(r))}{\varepsilon(\sqrt{1 - r^2}) - r\kappa(\sqrt{1 - r^2})},
\]

\[
\text{DP}(h) = \mu(\psi^{-1}(h))/\pi,
\]

we have

\[
\text{ExtMod}[I \ast h, 1 + I \ast h, 2, 16][[1]] = \text{DP}(1/h).
\]

For the range \(h \in [0.5, 10]\) this last identity holds with approximate error \(10^{-14}\).
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