NON UNIFORM DECAY OF THE ENERGY OF SOME DISSIPATIVE EVOLUTION SYSTEMS

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ABSTRACT. In this paper we consider second order evolution equations with bounded damping. We give a characterization of a non uniform decay for the damped problem using a kind of observability estimate for the associated undamped problem.

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1. Introduction and main results

Let $X$ be a complex Hilbert space with norm and inner product denoted respectively by $||\cdot||_X$ and $\langle\cdot,\cdot\rangle_X$. Let $A$ be a linear unbounded self-adjoint and strictly positive operator in $X$ and $V = \mathcal{D}(A^{1/2})$ be the domain of $A^{1/2}$, with

$$\|x\|_V = \|A^{1/2}x\|_X, \forall x \in V.$$ Denote by $(\mathcal{D}(A^{1/2}))'$ the dual space of $\mathcal{D}(A^{1/2})$ obtained by means of the inner product in $X$. Further, let $U$ be a complex Hilbert space (identified to its dual) and $B \in \mathcal{L}(U, X)$.

Most of the linear control problems coming from elasticity can be written as

\begin{equation}
\begin{cases}
   w''(t) + Aw(t) + Bu(t) = 0, \\
   w(0) = w_0, \ w'(0) = w_1,
\end{cases}
\end{equation}

where $w : [0, T] \rightarrow X$ is the state of the system, $u \in L^2(0, T; U)$ is the input function and denote the differentiation with respect to time by “$'$”.

2010 Mathematics Subject Classification. 35B30, 35B40.

Key words and phrases. bounded feedback, kind of observability estimate, non uniform decay.
We define the energy of \( w(t) \) at instant \( t \) by
\[
E(w(t)) = \frac{1}{2} \left\{ ||w'(t)||_X^2 + ||A^\frac{1}{2}w(t)||_X^2 \right\}.
\]
Simple formal calculations give
\[
E(w(0)) - E(w(t)) = \int_0^t \langle Bu(s), w'(s) \rangle_X ds, \quad \forall \ t \geq 0.
\]
This is why, in many problems coming in particular from elasticity, the input \( u \) is given in the feedback form \( u(t) = B^*w'(t) \), which obviously gives a nonincreasing energy and which corresponds to collocated actuators and sensors.

The aim of this paper is to give sufficient and necessary conditions on the conservative system (1.5) making the corresponding closed loop system (1.3)
\[
\begin{cases}
  w''(t) + Aw(t) + BB^*w'(t) = 0, \\
  w(0) = w_0, w'(0) = w_1,
\end{cases}
\]
non uniformly stable. The strategy to get such a decay rate will consist to generalize a kind of observability estimate given in [6].

Any sufficiently smooth solution of (1.3) satisfies the energy estimate
\[
E(w(0)) - E(w(t)) = \int_0^t ||B^*w'(s)||_U^2 ds, \quad \forall \ t \geq 0.
\]
In particular (1.4) implies that
\[
E(w(t)) \leq E(w(0)), \quad \forall \ t \geq 0.
\]
In the natural well-posedness space \( V \times X \), the existence and uniqueness of finite energy solutions of (1.3) can be obtained by standard semi-group methods.

Denote by \( \phi \) the solution of the associated undamped problem
\[
\begin{cases}
  \phi''(t) + A\phi(t) = 0, \\
  \phi(0) = w_0, \ \phi'(0) = w_1.
\end{cases}
\]
It is well known that (1.5) is well-posed in \( D(A) \times V \) and in \( V \times X \).

Our main result is stated as follows:

Let \( \mathcal{G} \) be a continuous positive increasing real function on \( [0, +\infty) \) and define the function \( F \) by \( F(x) = x (\mathcal{G}(x))^2 \).

**Theorem 1.1.** (1) Assume that there exists \( C > 0 \) such that for all non-identically zero initial data \((w_0, w_1) \in V \times X\) and for all \( t > 0 \), the solution \( w \) of (1.3) satisfies:
\[
E(w(t)) \leq C ||(w_0, w_1)||_{V \times X}^2 \mathcal{G}^{-1} \left( \frac{1}{t} \right),
\]
then there exists \( C > 0 \) such that the solution \( \phi \) of (1.5) satisfies:
\[
|| (w_0, w_1) ||_{V \times X}^2 \leq 16 \int_0^t \mathcal{G}(\frac{1}{\lambda t}) \ ||B^*\phi'(t)||_U^2 dt,
\]
where
\[
\Lambda = \frac{|| (w_0, w_1) ||_{D(A) \times V}^2 \mathcal{G}^{-1} \left( || (w_0, w_1) ||_{V \times X}^2 \right)}{|| (w_0, w_1) ||_{V \times X}^2}.
\]
(2) Assume that $x \mapsto x F^{-1}(\frac{1}{x})$ is an increasing function and there exists $C > 0$ such that for all non-identically zero initial data $(w_0, w_1) \in D(A) \times V$, the solution $\phi$ of (1.5) satisfies:

$$|| (w_0, w_1) ||^2_{V \times X} \leq C \int_{0}^{G(H(s))} ||B^* \phi'(t)||^2_{U} dt. $$

Then there exists $C > 0$ such that for all $t > 0$, the solution $w$ of (1.3) satisfies:

$$E(w(t)) \leq C || (w_0, w_1) ||^2_{D(A) \times V} F^{-1}(\frac{1}{\sqrt{t}}).$$

**Corollary 1.2.** The weak observability i.e. there exist $T, C > 0$ such that for all $(w_0, w_1) \in V \times X$ the solution $\phi$ of (1.5) satisfies

$$\int_{0}^{T} ||B^* \phi'(t)||^2_{U} dt \geq C || (w_0, w_1) ||^2_{V \times X} G \left( \frac{|| (w_0, w_1) ||^2_{X \times (D(A)^{1/2})'}}{|| (w_0, w_1) ||^2_{V \times X}} \right),$$

implies in particular (1.7).

The paper is organized as follows: In section 2 we prove our main result and in the last section we give some applications both in the linear and the nonlinear case. Decay rates for nonlinear dissipations were obtained under our generalized observability estimate. Here we mention that the literature is less provide. We cite essentially [1, 7].

### 2. Proof of Theorem 1.1

The following lemma will be very useful.

**Lemma 2.1.** Let $H$ (resp. $G$) be a continuous positive decreasing (resp. increasing) real function on $[0, +\infty)$. Suppose that $H$ is bounded by one and there exists a positive constant $c$ such that

$$H(s) \leq \frac{c}{(G(H(s))^2} \left( H(s) - H\left( \frac{1}{G(H(s))} + s \right) \right), \forall s > 0.$$  

Suppose that $x \mapsto x F^{-1}(\frac{1}{x})$ is an increasing function, then there exists $C > 0$ such that for any $t > 0$,

$$H(t) \leq CF^{-1}\left( \frac{1}{\sqrt{t}} \right).$$

The proof is similar to that of Lemma B in [6]. Since our result is more general, we give it for the reader’s convenience.

**Proof.** Let $t > 0$. We distinguish two cases:

- If

  $$G(H(s)) < \frac{1}{t},$$

  then

  $$H(s) \leq G^{-1}\left( \frac{1}{t} \right).$$

*see [2, 3, 4] for more details
• If \( G(H(s)) \geq \frac{1}{t} \), then
\[
\frac{1}{G(H(s))} + s \leq t + s,
\]
therefore
\[
H(t + s) \leq H \left( \frac{1}{G(H(s))} + s \right).
\]
and we get
\[
F(H(s)) \leq H(s) - H(t + s).
\]
The inequalities (2.13) and (2.15) give
\[
H(s) \leq F^{-1}(H(s) - H(t + s)) + G^{-1} \left( \frac{1}{t} \right), \forall s, t > 0.
\]
We introduce the function \( \Psi_t \) defined on \([0, +\infty[\) by:
\[
\Psi_t(s) = \frac{1}{F^{-1} \left( \frac{s}{t} \right) + G^{-1} \left( \frac{1}{t} \right)}.
\]
We distinguish two cases:
• If \( H(s) - H(t + s) < \frac{1}{t+s} \) then \( H(s) \leq \Psi_t(t + s) \) and we deduce
\[
\Psi_t(t + s)H(t + s) \leq 1.
\]
• If \( H(s) - H(t + s) > \frac{1}{t+s} \). Taking into account that \( H(s) \leq 1 \) we obtain
\[
\frac{t}{t + s}H(s) \leq \frac{t}{t + s},
\]
so
\[
\frac{t}{t + s}H(s) \leq \frac{t}{t + s} < H(s) - H(t + s),
\]
and we deduce
\[
H(t + s) < H(s) \frac{s}{t + s}.
\]
Consequently,
\[
\Psi_t(t + s)H(t + s) < \Psi_t(t + s)H(s) \frac{s}{t + s} = \frac{\Psi_t(t + s)}{t + s}H(s)\Psi_t(s) \frac{s}{\Psi_t(s)}
\]
\[
= H(s)\Psi_t(s) \frac{\Psi_t(t + s)}{t + s} \frac{1}{\Psi_t(s)}.
\]
Using the increasing property of \( x \mapsto xF^{-1}(\frac{1}{x}) \), we obtain
\[
\Psi_t(t + s)H(t + s) < \Psi_t(s)H(s).
\]
We have proved that for all \( s, t > 0 \), we have either
\[
\Psi_t(t + s)H(t + s) \leq 1 \text{ or } \Psi_t(t + s)H(t + s) < \Psi_t(s)H(s).
\]
In particular, we deduce that for any \( t > 0 \) and \( n \in \mathbb{N}^* \), either
\[
\Psi_t((n + 1)t)H((n + 1)t) \leq 1 \text{ or } \Psi_t((n + 1)t)H((n + 1)t) < \Psi_{nt}(t)H(nt).
\]
Hence, we have
\begin{equation}
\Psi_t((n + 1)t) \mathcal{H}((n + 1)t) \leq \max(1, \Psi_t(t)\mathcal{H}(t)) = 1.
\end{equation}
Therefore, for all \( t > 0 \) and \( n \in \mathbb{N}^* \),
\begin{equation}
\mathcal{H}((n + 1)t) \leq F^{-1}\left(\frac{1}{n + 1}\right) + G^{-1}\left(\frac{1}{t}\right).
\end{equation}
Choose \( n \) such that \( n + 1 \leq t < n + 2 \) and make use again of the increasing property of \( x \mapsto xF^{-1}(\frac{x}{2}) \), we get for all \( t \geq 2 \):
\begin{equation}
\mathcal{H}(t^2) \leq F^{-1}\left(\frac{1}{t}\right) + G^{-1}\left(\frac{1}{t}\right).
\end{equation}
Since \( |G^{-1}(x)| \leq |F^{-1}(x)| \) close to zero, the desired result follows immediately. \( \square \)

After, we give the proof of the main result.

**Proof of (1).** We combine \( \text{(1.0)} \) and the following formula:
\begin{equation}
E(w(t)) = E(\phi(0)) - 2 \int_0^t \|B^* \phi'(t)\|^2_U \, ds \quad \forall t > 0,
\end{equation}
to get:
\begin{equation}
E(\phi(0)) - 2 \int_0^t \|B^* \phi'(t)\|^2_U \, ds \leq C \mathcal{G}^{-1}\left(\frac{1}{t}\right) \|(w_0, w_1)\|^2_{\mathcal{B}(A) \times \mathcal{V}}.
\end{equation}
Take \( t = \frac{1}{v(\frac{1}{2})} \), we obtain
\begin{equation}
E(\phi(0)) - 2 \int_0^{\frac{1}{v(\frac{1}{2})}} \|B^* \phi'(t)\|^2_U \, ds \leq \frac{1}{2} \|(w_0, w_1)\|^2_{\mathcal{V} \times \mathcal{X}}.
\end{equation}
We deduce that
\begin{equation}
\|(w_0, w_1)\|^2_{\mathcal{V} \times \mathcal{X}} \leq 4 \int_0^{\frac{1}{v(\frac{1}{2})}} \|B^* \phi'(t)\|^2_U \, ds.
\end{equation}
Now, let us consider \( v = \phi - w \), then \( v \) satisfies the following system:
\begin{equation}
\begin{cases}
v''(t) + Av(t) + BB^*v' = BB^*\phi'(t), & t > 0, \\
(v(0), v'(0)) = (0, 0).
\end{cases}
\end{equation}
Multiply the first equation of \( \text{(3.31)} \) by \( v' \), and integrate by parts to get
\begin{equation}
E(v(t)) + 2 \int_0^t \|B^* v'(s)\|^2_U \, ds = 2 \int_0^t \langle B^* \phi'(s), B^* v'(s) \rangle_U \, ds.
\end{equation}
Make use of Young inequality,
\begin{equation}
E(v(t)) + 2 \int_0^t \|B^* v'(s)\|^2_U \, ds \leq \int_0^t \left( \|B^* \phi'(s)\|^2_U + \|B^* v'(t)\|^2_U \right) \, ds.
\end{equation}
Hence,
\begin{equation}
E(v(t)) + \int_0^t \|B^* v'(s)\|^2_U \, ds \leq \int_0^t \|B^* \phi'(s)\|^2_U \, ds.
\end{equation}
Since
\begin{equation}
\|B^* w'\|^2_U = \|B^* \phi' - B^* v'\|^2_U \leq 2 \left( \|B^* \phi'\|^2_U + \|B^* v'\|^2_U \right),
\end{equation}
we obtain
\begin{equation}
\|B^* w'\|^2_U \leq 2 \int_0^t \|B^* \phi'(s)\|^2_U \, ds.
\end{equation}

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\[ \int_0^t \|B^* \phi'(s)\|^2_U \, ds \leq 4 \int_0^{\frac{1}{v(\frac{1}{2})}} \|B^* \phi'(t)\|^2_U \, ds. \]
therefore
\begin{equation}
\int_0^1 \left( \frac{1}{2} C \Lambda \right) \left( \left\| B^* w'(t) \right\|^2_U + \left\| B^* v'(t) \right\|^2_U \right) dt \leq 2 \int_0^1 \left( \frac{1}{2} C \Lambda \right) dt.
\end{equation}

Thanks to (2.34) and (2.35), we obtain
\begin{equation}
\int_0^1 \left( \frac{1}{2} C \Lambda \right) \left( \left\| B^* w'(t) \right\|^2_U + \left\| B^* v'(t) \right\|^2_U \right) dt \leq 4 \int_0^1 \left( \frac{1}{2} C \Lambda \right) \left( \left\| B^* w'(t) \right\|^2_U + \left\| B^* v'(t) \right\|^2_U \right) dt.
\end{equation}

Finally, by virtue of (2.30), we conclude the following estimate
\begin{equation}
\|(w_0, w_1)\|^2_{V \times X} \leq 16 \int_0^1 \left( \frac{1}{2} C \Lambda \right) \left( \left\| B^* w'(t) \right\|^2_U + \left\| B^* v'(t) \right\|^2_U \right) dt.
\end{equation}

\[ \square \]

Proof of (2). Using similar arguments as previously, the inequality (1.8) becomes
\begin{equation}
E(w(0)) \leq 2C \int_0^1 \left( \left\| B^* w'(t) \right\|^2_U + \left\| B^* v'(t) \right\|^2_U \right) dt.
\end{equation}

Multiply the first equation of (2.31) by $v'$ and integer by parts. It follows that
\begin{equation}
E(v(t)) \leq \int_0^t \left( \frac{\left\| B^* w'(s) \right\|^2_U}{\varepsilon} + \varepsilon \left\| B^* v'(s) \right\|^2_U \right) ds, \forall \varepsilon > 0.
\end{equation}

Let $T > 0$ be fixed, for all $0 \leq t \leq T$, we have
\[
\sup_{0 \leq t \leq T} E(v(t)) \leq \int_0^T \left( \frac{\left\| B^* w'(t) \right\|^2_U}{\varepsilon} + \varepsilon \left\| B^* v'(t) \right\|^2_U \right) dt
\leq \int_0^T \left\| B^* w'(t) \right\|^2_U dt + \varepsilon \int_0^T E(v(t)) dt
\leq \int_0^T \left\| B^* w'(t) \right\|^2_U dt + \varepsilon CT \sup_{0 \leq t \leq T} E(v(t)).
\]

Choose $\varepsilon = \frac{1}{2CT}$, so we have
\begin{equation}
\sup_{0 \leq t \leq T} E(v(t)) \leq 4CT \int_0^T \left\| B^* w' \right\|^2_U dt.
\end{equation}

On the other hand, as
\[ \left\| v'(t) \right\|^2_X \leq E(v(t)), \]
we integer on $[0, T]$, to get
\[
\int_0^T \left\| v'(t) \right\|^2_X dt \leq \int_0^T E(v(t)) dt
\leq T \sup_{0 \leq t \leq T} E(v(t))
\leq 4CT^2 \int_0^T \left\| B^* w' \right\|^2_U dt,
\]
so
\begin{equation}
\int_0^T \left\| B^* v' \right\|^2_U dt \leq 4C^2T^2 \int_0^T \left\| B^* w' \right\|^2_U dt.
\end{equation}
Hence, for \( T = \frac{1}{G(\frac{1}{2C})} \) we conclude that

\[
E(w(0)) \leq 2C \left( 1 + 4C^2 \left( \frac{1}{G(\frac{1}{2C})} \right)^2 \right) \int_0^{\frac{1}{G(\frac{1}{2C})}} \| B^* w' \|_{L^1_U}^2 dt.
\]

One can easily verify that

\[
\Lambda \leq \frac{E(w(0)) + E(w'(0))}{E(w(0))} := \tilde{\Lambda},
\]

and consequently

\[
E(w(0)) \leq 2C \left( 1 + 4C^2 \left( \frac{1}{G(\frac{1}{2C})} \right)^2 \right) \int_0^{\frac{1}{G(\frac{1}{2C})}} \| B^* w'(t) \|_{L^1_U}^2 dt.
\]

Since \( \tilde{\Lambda} \) is minimized, we get

\[
E(w(0)) \leq C \left( \frac{1}{G(\frac{1}{2C})} \right)^2 \int_0^{\frac{1}{G(\frac{1}{2C})}} \| B^* w'(t) \|_{L^1_U}^2 dt.
\]

By translating the time variable and using the formula

\[
E(w(t_1)) - E(w(t_2)) + \int_{t_2}^{t_1} \| B^* w'(t) \|_{L^1_U}^2 dt = 0,
\]

we obtain:

\[
\frac{E(w(0))}{E(w(0)) + E(w'(0))} \leq C \left( \frac{1}{G(\frac{1}{2C})} \right)^2 \int_s^{\frac{1}{G(\frac{1}{2C})}} \| B^* w'(t) \|_{L^1_U}^2 dt
\]

\[
\leq C \left( \frac{1}{G(\frac{1}{2C})} \right)^2 \left( \frac{E(w(s))}{E(w(0)) + E(w'(0))} - \frac{E(w(\frac{1}{G(\frac{1}{2C})} + s))}{E(w(0)) + E(w'(0))} \right).
\]

Put \( \mathcal{H}(s) = \frac{E(w(s))}{2C(E(w(0)) + E(w'(0)))} \).

Make use of the previous inequality and the decay of \( \mathcal{H} \), it follows that

\[
\mathcal{H}(s) \leq C \left( \frac{1}{G(\mathcal{H}(s))} \right)^2 \left( \mathcal{H}(s) - \mathcal{H} \left( \frac{1}{G(\mathcal{H}(s))} + s \right) \right), \forall s > 0.
\]

Thanks to \( \mathcal{H}(t) \) and Lemma 2.1 there exists \( C \) such that

\[
\frac{E(w(t))}{E(w(0)) + E(\partial_t w(0))} \leq CF^{-1} \left( \frac{1}{\sqrt{t}} \right).
\]

We conclude the desired result. \( \square \)

3. Some applications

We give some applications of Theorem 1.1.
3.1. The linear case.

3.1.1. Example 1. Let \( G(x) = x^p \) on \((0, r_0), r_0 > 0\) and \( p \in \mathbb{R} \setminus \left[-\frac{1}{2}, 0\right] \).
Then the following two statements are equivalent.

i) There exists \( C > 0 \) such that for all non-identically zero initial data \((w_0, w_1) \in V \times X\), the solution \( \phi \) of (1.5) satisfies:
\[
\|(w_0, w_1)\|_{V \times X}^2 \leq 16 \int_0^{C \Lambda} \|B^\ast \phi'(t)\|_U^2 \, dt.
\]

ii) There exists \( C > 0 \) such that for all non-identically zero initial data \((w_0, w_1) \in D(A) \times V\) and for all \( t > 0 \), the solution \( w \) of (1.3) satisfies:
\[
E(w(t)) \leq \frac{C}{t^p} \|(w_0, w_1)\|_{D(A) \times V}^2.
\]

Remark 3.1. In [6], the author constructs a geometry with a trapped ray for the linear dissipative wave equation (the geometric control condition is then not fulfilled) and establishes a polynomial decay rate when \((w_0, w_1) \in [H^2(\Omega) \cap H^1_0(\Omega)] \times H^1_0(\Omega)\), the estimate (1.7) is satisfied for \( G(x) = x^\delta, \delta > 0 \).

3.1.2. Example 2. Let \( G(x) = \frac{e^{p(-1)x^2}}{\sqrt{x}} \) on \((0, r_0), p \in \mathbb{R}_+\). The following statements hold.

i) The existence of a constant \( C > 0 \) such that the solution \( \phi \) of (1.5) satisfies:
\[
\|(w_0, w_1)\|_{H^1_0(\Omega) \times L^2(\Omega)}^2 \leq 16 \int_0^{\Lambda^p} \|B^\ast \phi'(t)\|_U^2 \, dt,
\]
implies the existence of a constant \( C_1 > 0 \) such that for all non-identically zero initial data \((w_0, w_1) \in D(A) \times V\) and for all \( t > 0 \), the solution \( w \) of (1.3) satisfies:
\[
E(w(t)) \leq \frac{C_1}{(\ln t)^p} \|(w_0, w_1)\|_{D(A) \times V}^2.
\]

ii) The existence of a constant \( C_1 > 0 \) such that for all non-identically \((w_0, w_1) \in D(A) \times V\) and for all \( t > 0 \), the solution \( w \) of (1.3) satisfies:
\[
E(w(t)) \leq \frac{C_1}{(\ln t)^p} \|(w_0, w_1)\|_{D(A) \times V}^2.
\]
implies the existence of a constant \( C > 0 \) such that the solution \( \phi \) of (1.5) satisfies:
\[
\|(w_0, w_1)\|_{V \times X}^2 \leq 16 \int_0^{\Lambda^p} \|B^\ast \phi'(t)\|_U^2 \, dt.
\]
3.2. The nonlinear case. Let \( \Omega \) be a bounded connected open set of \( \mathbb{R}^n, n > 1 \) with a \( C^2 \) boundary \( \partial \Omega \). Let also \( M = (\alpha^{ij})_{1 \leq i,j \leq n} \in C^\infty(\overline{\Omega};\mathbb{R}^{n \times n}) \) be a symmetric and uniformly positive definite matrix. Denote by \( \nabla = (\sum_{j=1}^{n} \beta^{ij} \partial_{x_j}, \ldots, \sum_{j=1}^{n} \beta^{ij} \partial_{x_j}) \) and \( \Delta = \sum_{i,j=1}^{n} \partial_{x_i} (\alpha^{ij} \partial_{x_j}) \). We deal with the following second order differential equation:

\[
\begin{align*}
\partial_t^2 u - \Delta u + a(x)g(\partial_t u) &= 0, \text{ in } \Omega \times (0, +\infty), \\
u &= 0, \text{ on } \partial \Omega \times (0, +\infty), \\
(u, \partial_t u)(., 0) &= (u_0, u_1), \text{ in } \Omega,
\end{align*}
\]

where \( a = a(x) \in L^\infty(\Omega) \) is a bounded function with \( a(x) \geq 0 \) for all \( x \in \Omega \) and \( g : \mathbb{R} \to \mathbb{R} \) is a continuous strictly increasing function with \( g(0) = 0, s g(s) \geq 0 \). We assume the additional conditions:

(i) \( 2r \in [1, \infty), \exists c_1, c_2 > 0, |s| \leq 1 \Rightarrow c_1 |s|^r \leq |g(s)| \leq c_2 |s|^{1/r}. \)

(ii) \( 3k \in [0, 1], \exists p \in [1, \infty), \exists c_3, c_4 > 0, |s| > 1 \Rightarrow c_3 |s|^k \leq |g(s)| \leq c_4 |s|^p. \)

(iii) \((n - 2)(1 - k) \leq 4r \) and \((n - 2)(p - 1) \leq 1. \)

For \((u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)\), there exists a unique solution \( u \in C([0, +\infty), H^1_0(\Omega)) \cap C^1([0, +\infty), L^2(\Omega)) \). For more regular initial data \((u_0, u_1) \in H^2(\Omega) \cap H^1_0(\Omega) \) \times \( H^1_0(\Omega) \), the solution \( u \) has the following regularity \( u \in L^\infty(0, +\infty; H^2(\Omega) \cap H^1_0(\Omega)) \cap W^{1, \infty}(0, +\infty; H^1_0(\Omega)) \cap W^{2, \infty}(0, +\infty; L^2(\Omega)). \)

The energy of a solution is defined at instant \( t \geq 0 \) by

\[
E(u(t)) = \frac{1}{2} \int_\Omega \left( |\partial_t u(x, t)|^2 + |\nabla u(x, t)|^2 \right) dx.
\]

\( E(u(t)) \) is a non-increasing function of time and satisfies, for all \( t_2 > t_1 \geq 0 \) the identity

\[
E(u(t_2)) - E(u(t_1)) = - \int_{t_1}^{t_2} \int_\Omega a(x)g(\partial_t u(x, t))\partial_t u(x, t) dx \; dt \leq 0.
\]

Denote by

\[
X(u_0, u_1) = E(u(0)) + E_1(u(0)) + [E_1(u(0))]^{(2p-1)} + [E_1(u(0))]^{1+\frac{4}{n+4}},
\]

where

\[
E_1(u(0)) = \|(\Delta u_0 - a g(u_1), u_1)\|_{L^2(\Omega) \times H^1_0(\Omega)}^2.
\]

We introduce \( u_t \) the solution of the linear locally damped problem:

\[
\begin{align*}
\partial_t^2 u_t - \Delta u_t + a(x)\partial_t u_t &= 0, \text{ in } \Omega \times (0, +\infty), \\\nu &= 0, \text{ on } \partial \Omega \times (0, +\infty), \\
(u_t, \partial_t u_t)(., 0) &= (u_0, u_1), \text{ in } H^2(\Omega) \times H^1_0(\Omega) \cap H^1_0(\Omega),
\end{align*}
\]

and make the following assumption:
(A) Assume that \( x \mapsto x F^{-1}(\frac{1}{2}) \) is an increasing function and there exists \( C > 0 \) such that for all non-identically zero initial data \((u_0, u_1) \in [H^2(\Omega) \times H_0^1(\Omega)] \cap H_0^1(\Omega)\), the solution \( \phi \) of
\[
\begin{aligned}
\left\{ \begin{array}{l}
\partial_t^2 \phi(t) - \Delta \phi(t) = 0, \\
\phi(0) = u_0, \partial_t \phi(0) = u_1,
\end{array} \right.
\end{aligned}
\]
satisfies:
\[
\text{Theorem 3.3.}
\]
(3.49) \[ ||(u_0, u_1)||_{H_0^1(\Omega) \times L^2(\Omega)}^2 \leq C \int_0^{\frac{1}{2}(1-h^4)} \int_\Omega a(x) |\partial_t \phi(x, t)|^2 \, dx \, dt, \]
where
\[
\Lambda_r = \frac{(r-1) + X(u_0, u_1)}{E(u(0))}.
\]
The following result is deduced from Theorem 1.1 and Proposition 3.

\textbf{Proposition 3.2.} Let (A) holds. There exists \( c > 0 \) such that for any \((u_0, u_1) \in [H^2(\Omega) \times H_0^1(\Omega)] \cap H_0^1(\Omega)\), the solution \( u \) of (3.44) satisfies
\[
\begin{aligned}
E(u(s)) &\leq ch((r-1) + X(u_0, u_1)) \\
&\quad +c \int_s^{r \Lambda_r} \int_\Omega a(x) g(\partial_t u(x, t)) \partial_t u(x, t) \, dx \, dt,
\end{aligned}
\]
for any \( h > 0 \) and any \( s \geq 0 \) where
\[
G(h) := C h^{2r+1} F(h)^{4(r+1)}.
\]

We have the following stabilization result for the nonlinear damped wave equation.

\textbf{Theorem 3.3.} Let (A) holds and suppose that there exists \( c_0 \) such that the function \( G \) satisfies \( G^{-1}(x) \geq \frac{c_0}{x+1} G^{-1}(x(c_0 + 1)) \) for all \( x \geq 0 \). Then the energy of the solution of (3.44) satisfies the estimate:
\[
E(u(t)) \leq CG^{-1} \left( \frac{c}{t} \right) ((r-1) + X(u_0, u_1)), \quad \text{for } t \text{ sufficiently large},
\]
and all non-identically zero initial data \((u^0, u^1) \in [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)\), the constant \( C \) depend on the initial data \((u^0, u^1)\).

\textbf{Proof.} Choosing
\[
h = \frac{1}{2CA_r},
\]
this implies the existence of a constant \( c > 0 \) such that
\[
E(u(s)) \leq c \int_s^{r \Lambda_r} \int_\Omega a(x) g(\partial_t u(x, t)) \partial_t u(x, t) \, dx \, dt.
\]
Denoting by \( \mathcal{H}(s) = \frac{E(u(s))}{(r-1) + X(u_0, u_1)} \), we deduce from (3.51) that
\[
\mathcal{H} \left( s + \frac{1}{G(\mathcal{H}(s))} \right) \leq \mathcal{H}(s) \leq c \left( \mathcal{H}(s) - \mathcal{H} \left( s + \frac{1}{G(\mathcal{H}(s))} \right) \right),
\]
which gives
\[
\mathcal{H} \left( s + \frac{1}{G(\mathcal{H}(s))} \right) \leq \frac{c}{c+1} \mathcal{H}(s).
\]
• If \( c_0 s \leq \frac{1}{G(H)} \), then \( H(s) \leq G^{-1}\left(\frac{1}{c_0 s}\right) \) and

\[
(3.52) \quad H((1 + c_0)s) \leq H(s) \leq G^{-1}\left(\frac{c}{c_0 s}\right).
\]

• If \( c_0 s > \frac{1}{G(H)} \), then

\[
(3.53) \quad H((1 + c_0)s) \leq H\left(s + \frac{1}{G(H)}\right) \leq \frac{c}{c + 1} H(s).
\]

By induction, we deduce from (3.52) and (3.53) that \( \forall s > 0 \) and \( \forall n \in \mathbb{N}^* \),

\[
H((1 + c_0)s) \leq \max\left[G^{-1}\left(\frac{c}{c_0 s}\right), \frac{c}{c + 1} G^{-1}\left(\frac{c(c_0 + 1)}{c_0 s}\right), \ldots, \left(\frac{c}{c + 1}\right)^n G^{-1}\left(\frac{c(c_0 + 1)^n}{c_0 s}\right), \left(\frac{c}{c + 1}\right)^{n+1} H\left(\frac{s}{c_0 + 1)^{n+1}}\right)\right].
\]

Now, remark that with the above hypothesis on the function \( G \),

\[
\frac{c}{c + 1} G^{-1}\left(\frac{c(c_0 + 1)}{c_0 s}\right) \leq G^{-1}\left(\frac{c}{c_0 s}\right).
\]

Consequently,

\[
H((1 + c_0)s) \leq \max\left[G^{-1}\left(\frac{c}{c_0 s}\right), \left(\frac{c}{c + 1}\right)^{n+1} H\left(\frac{s}{c_0 + 1)^{n+1}}\right)\right],
\]

(3.54)

\[
\leq \max\left[G^{-1}\left(\frac{c}{c_0 s}\right), \left(\frac{c}{c + 1}\right)^{n+1}\right], \quad \forall n \geq 1,
\]

and we conclude that

\[
H(s) \leq G^{-1}\left(\frac{c(1 + c_0)}{c_0 s}\right), \forall s > 0.
\]

\[\square\]

**Remarks 3.4.**

1. For \( G(x) = x^p \), we have \( F(x) = x^{2p+1} \) and \( G(x) = x^{(4p+3)(2r+1)-1} \).

   The energy of the solution of (3.44) satisfies the estimate:

\[
(3.55) \quad E(u(t)) \leq \frac{c}{t^{(4p+3)(2r+1)+1}} (r - 1 + X(u_0, u_1)), \text{ for } t \text{ sufficiently large.}
\]

2. For the wave equation with arbitrary localized nonlinear damping, we obtain in [4] a weak observability which implies in particular the estimate (3.47) and the logarithmic decay of the energy.

   At the same time, this gives a geometry where the observability estimate (3.47) is satisfied and simplify the proof of the decay result in [5].
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