Correct Solvability of Integrodifferential Equations in Spaces of Vector Functions Holomorphic in an Angular Domain

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Abstract—Integrodifferential equations with unbounded operator coefficients in a Hilbert space are studied. The main part of an equation of this kind is an abstract parabolic equation perturbed by a Volterra integral operator. The fundamental difference between this work and the other ones is that integrodifferential equations are considered and studied in this paper for vector functions the arguments of which take values in an angular domain on the complex plane.

Keywords: Volterra integrodifferential equations, vector function holomorphic in an angular domain, Hardy space

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In [1, 2], the class \( \mathcal{H}_2(S_0) \) of functions holomorphic in an angular domain of the form

\[ S_0 = \{ \tau \in \mathbb{C}, \arg \tau < \theta \} \]

such that

\[ \sup_{|\theta| < \theta} \left( \int_0^\infty |f(\tau e^{i\theta})|^2 dt \right) < \infty \]

is studied. In [1, 2], it is established that \( \mathcal{H}_2(S_0) \) equipped with the corresponding norm is a Hilbert space and a theorem of Paley–Wiener type is proved for it.

Let \( H \) be a separable Hilbert space, and let \( A \) be a self-adjoint positive operator \( A^* = A \geq \kappa I \) \( (\kappa > 0) \) acting in \( H \) with a compact inverse. Let \( (\cdot, \cdot) \) and \( \| \cdot \| \) denote the scalar product and the norm in \( H \), respectively.

This paper studies the classes \( \mathcal{H}_2(S_0, H) \) and \( W^2_2(S_0, A^\beta) \) of functions with values in \( H \) that are holomorphic in the domain \( S_0 \). The class \( \mathcal{H}_2(S_0, H) \) consists of vector functions such that

\[ \sup_{|\theta| < \theta} \left( \int_0^\infty |f(\tau e^{i\theta})|^2 dt \right) < \infty, \]

while the class \( W^2_2(S_0, A^\beta) \) consists of vector functions such that

\[ \sup_{|\theta| < \theta} \left( \int_0^\infty \left( \|\nabla^\beta u(\tau e^{i\theta})\|^2 + \|A^\beta u(\tau e^{i\theta})\|^2 \right) dt \right) < \infty. \]

We turn the domain \( D(A^\beta) \) of the operator \( A^\beta \), \( \beta > 0 \), into the Hilbert space \( H_\beta \) by introducing the norm \( \| \cdot \|_\beta \), which is equivalent to the norm of the graph of the operator \( A^\beta \). In what follows, a function with values in \( H \) is called a function (without the word vector), while a function with values in \( \mathbb{C} \) is called a scalar (or numeric) function.

Let \( L^2(\mathbb{R}_+, H) \) denote the space of (classes of) functions with values in \( H \) measurable with respect to the Lebesgue measure \( dt \) on the half-line \( \mathbb{R}_+ = (0, +\infty) \) such that

\[ \|f\|_{L^2(\mathbb{R}_+, H)} = \left( \int_0^\infty \|f(t)\|^2 dt \right)^{1/2} < +\infty. \]

Let \( W^2_\beta(\mathbb{R}_+, A^\beta) \) denote the Sobolev space of functions on the half-line \( \mathbb{R}_+ = (0, +\infty) \) with values in \( H \) equipped with the norm

\[ \|f\|_{W^2_\beta(\mathbb{R}_+, A^\beta)} = \left( \int_0^\infty \left( \|u(t)\|^2 + \|A^\beta u(t)\|^2 \right) dt \right)^{1/2}. \]
See [3, Ch. 1] for more detail about the spaces $W^n_2(\mathbb{R}_+, A^n)$. For $n = 0$, we assume that $W^0_2(\mathbb{R}_+, A^0) \equiv L_2(\mathbb{R}_+, H)$. We assume in what follows that $\mathfrak{K}_2(S_0, H) = L_2(\mathbb{R}_+, H)$ and $W^n_2(S_0, A^n) = W^n_2(\mathbb{R}_+, A^n)$.

1. BASIC PROPERTIES OF THE SPACES $\mathfrak{K}_2(S_0, H)$ AND $W^n_2(S_0, A^n)$

Below are the main properties of the space $\mathfrak{K}_2(S_0, H)$.

**Proposition 1.** For a function $f(\tau) \in \mathfrak{K}_2(S_0, H)$, there are boundary values $f(\tau^\pm\theta) \in L_2(\mathbb{R}_+, H)$ such that

$$
\lim_{\theta \to \pm \theta} \|f(\tau^\theta) - f(\tau^0)||_{L_2(\mathbb{R}_+, H)} = 0.
$$

**Theorem 1.** $\mathfrak{K}_2(S_0, H)$ is a Banach space.

2. The class of functions $\mathfrak{K}_2(S_0, H)$ with the scalar product

$$
\langle f, g \rangle_{2,0} = \int_0^{+\infty} (f(\tau^0), g(\tau^0)) d\tau + \int_0^{+\infty} (f(\tau^0), g(\tau^0)) d\tau
$$

is a Hilbert space.

3. If $f(\tau)$ is an arbitrary function in the class $\mathfrak{K}_2(S_0, H)$ and

$$
\|f\|_{2,0} = \langle f, f \rangle_{2,0}^{1/2},
$$

then

$$
\|f\|_{2,0} \leq \sqrt{2} \|f\|_{2,0} \leq 2 \|f\|_{2,0}.
$$

We give a theorem that is an analogue of the Paley–Wiener theorem for the space $\mathfrak{K}_2(S_0, H)$.

**Theorem 2.** Let $\theta \in (0, \pi/2)$. The following assertions are true:

1. The class of functions $\mathfrak{K}_2(S_{0+\pi/2}, H)$ coincides with the set of functions admitting the representation

$$
F(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{-\lambda x} f(\tau^0) dt,
$$

$$
|\arg \lambda - \varphi| < \frac{\pi}{2}, \quad \varphi \in (-\theta, \theta),
$$

$$
f(\tau) \in \mathfrak{K}_2(S_0, H).
$$

2. The function $f(\tau) \in \mathfrak{K}_2(S_0, H)$ in representation (1) is unique for each fixed function $F(\lambda) \in \mathfrak{K}_2(S_{0+\pi/2}, H)$, and the following inversion formula is valid:

$$
f(\tau^0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\lambda y} F(\lambda) \left( e^{i\lambda\arg \tau} \right) d\lambda.
$$

3. If a function $F(\lambda) \in \mathfrak{K}_2(S_{0+\pi/2}, H)$ is representable in terms of $f(\tau) \in \mathfrak{K}_2(S_0, H)$ using (1), then

$$
\|F\|_{2,0} \leq 2 \|f\|_{2,0} \leq 2 \sqrt{2} \|F\|_{2,0}.
$$

Now we consider and study analogues of the Sobolev spaces $W^n_2(S_0, A^n)$ of functions holomorphic in the angular domain $S_\theta$.

In what follows, let $\frac{du}{d\tau}$ denote the derivative of a function $u(\tau)$ in the sense of functions of a complex variable. The class of functions $W^n_2(S_0, A^n)$ coincides with the class of functions holomorphic in the angular domain $S_\theta$ such that

$$
\sup_{\varphi \in [-\theta, \theta]} \int_0^{+\infty} \left( \left| \frac{d^n}{d\tau^n} u(\tau^0) \right|^2 + \left| A^n u(\tau^0) \right|^2 \right) d\tau < +\infty.
$$

The following lemma is an analogue of the intermediate derivative theorem, which is well known for $W^n_2(\mathbb{R}_+, A^n)$ (see [3]).

**Lemma 1.** Let $u(\tau) \in W^n_2(S_0, A^n)$. Then

$$
A^{-j} \frac{d^j}{d\tau^j} u(\tau) \in \mathfrak{K}_2(S_0, H), \quad j = 0, 1, \ldots, n.
$$

**Proposition 2.** For a function $u(\tau) \in W^n_2(S_0, A^n)$, there are boundary values $u_{\pm\theta}(t) = u(\tau^0 \pm \theta)$ in the class $W^n_2(\mathbb{R}_+, A^n)$ such that

$$
\lim_{\theta \to \pm \theta} \int_0^{+\infty} \left( \left| \frac{d^n}{d\tau^n} u(\tau^0) - \frac{d^n}{d\tau^n} u(\tau^0) \right|^2 + \left| A^n u(\tau^0) - u(\tau^0) \right|^2 \right) d\tau = 0.
$$

**Theorem 3.** $\mathfrak{K}_2(S_0, H)$ is a Banach space.
2. CORRECT SOLVABILITY OF THE INITIAL VALUE PROBLEM FOR AN INTEGRODIFFERENTIAL EQUATION IN THE SPACE $W^1_2(S_0, A)$

We consider the initial value problem for an integro-differential equation of the form

$$
\frac{du}{d\tau} + Au(\tau) - \int_0^\tau K(\tau - s)Au(s)ds = f(\tau), \quad \tau \in S_0,
$$

subject to the condition

$$
u(0) = \phi_0,
$$

where $A$ is a self-adjoint positive operator acting in a separable Hilbert space $H$ with a compact inverse. The kernel $K(\tau)$ belongs to the Hardy space $H_1(S_0)$ (see [7]), $f(\tau) \in \mathfrak{H}_2(S_0, H)$, and $\phi_0 \in H_{1/2}$. Note that the integration in the integral term in (4) is carried out over the interval connecting the origin and the point $\tau \in S_0$. However, due to the regularity of the functions $K(\tau)$ and $u(\tau)$, the integral can be taken over any rectifiable piecewise smooth contour that connects 0 and $\tau$ and lies in the domain $S_0$.

The main result is the following solvability theorem for problem (4), (5) in $W^1_2(S_0, A)$.

**Theorem 5.** Assume that the kernel $K(\tau)$ belongs to the Hardy space $H_1(S_0)$ and its Laplace transform $\hat{K}(\lambda)$ satisfies the estimate

$$
\sup_{\varphi \in (\frac{-\pi}{2}, \frac{\pi}{2})} \left| \hat{K}(\tau)e^{\varphi(\lambda)} \right| \sup_{\alpha \geq 0} \frac{a}{(a^2 + r^2)^{\frac{1}{2} - \frac{1}{2} \sin \theta}} < 1, \quad (6)
$$

where $\alpha_0 = \inf_{(A, x)} f(\tau) \in \mathfrak{H}_2(S_0, H)$, and $\phi_0 \in H_{1/2}$. Then there exists a unique solution $u(\tau) \in W^1_2(S_0, A)$ of problem (4), (5) such that

$$
\|u\|_{W^1_2(S_0, A)} \leq d \left( \int_{S_0} f^2 + \|\phi_0\|^2 \right)^{1/2}, \quad (7)
$$

where the constant $d$ is independent of the vector function $f(\tau)$ and the vector $\phi_0$.

**Corollary 1.** Assume that the assumptions of Theorem 5 are satisfied. Then there exists a unique solution $u(\tau) \in W^1_2(S_0, A)$ of problem (4), (5) satisfying inequality (6).

The idea of proving Theorem 5 can be described as follows. We consider the Laplace transform $\hat{u}(\lambda)$ of the strong solution $u(\tau)$ of (4) with zero initial data $\phi_0 = 0$, which has the form

$$
\hat{u}(\lambda) = \mathcal{L}^{-1}(\lambda) \hat{f}(\lambda).
$$

Here, the operator function $L(\lambda)$ is the symbol of Eq. (4) and it can be represented as

$$
L(\lambda) = \lambda I + A - \hat{K}(\lambda) A,
$$

where $\hat{K}(\lambda)$ is the Laplace transform of the kernel $K(\tau)$ and $I$ is the identity operator in the space $H$.

To prove Theorem 5, due to Theorem 2 (Paley–Wiener theorem), it suffices to show that the vector functions $\lambda \hat{u}(\lambda)$ and $A \hat{u}(\lambda)$ are in the space $\mathfrak{H}_2(S_{0+\pi/2}, H)$ and to derive their estimates. With this aim in view, we establish that the operator function $L^{-1}(\lambda)$ is holomorphic and admits the following estimates in the domain $G_{0+\pi/2} = \{ \lambda \in \mathbb{C}: |\arg \lambda| < \theta + \frac{\pi}{2} \}$:

$$
\|AL^{-1}(\lambda)\| < +\infty, \quad \|\lambda L^{-1}(\lambda)\| < +\infty.
$$

In turn, to deduce (8), we use the representation

$$
L^{-1}(\lambda) = (\lambda I + A)^{-1}(I - \hat{K}(\lambda) A (\lambda I + A)^{-1})^{-1},
$$

the estimates

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\[ \left\| A (\lambda I + A)^{-1} \right\| \leq \sup_{\alpha > 0} \frac{a}{(a^2 + r^2)^{3/2}} \left(1 - \sin \theta \right)^{1/2}, \quad (10) \]

\[ \lambda \in G_{0+ \pi/2}, \quad \lambda = re^{i\theta}, \]

\[ \left\| \mathbf{K} (\lambda I + A)^{-1} \right\| < 1, \quad \lambda \in G_{0+ \pi/2}, \]

which is implied by (6), (10), (11), and (9).

Based on (8), (10), and the fact that the vector function \( \hat{f}(\lambda) \) is in the space \( \mathfrak{H}_2 (S_{0+ \pi/2}, H) \), we find that the vector functions \( \hat{\lambda} \hat{u}(\lambda) \) and \( \hat{A} \hat{u}(\lambda) \) belong to the space \( \mathfrak{H}_2 (S_{0+ \pi/2}, H) \). Thus, using Theorem 2 (Paley–Wiener theorem), we conclude that the vector functions \( \frac{du}{d\tau}(\tau) \) and \( Au(\tau) \) belong to \( \mathfrak{H}_2 (S_0, H) \) and satisfy the inequalities

\[ \left\| \frac{du}{d\tau}(\tau) \right\|_{\mathfrak{H}_2 (S_0, H)} \leq d_1 \left\| f(\tau) \right\|_{\mathfrak{H}_{1,2} (S_0, H)}, \quad (12) \]

\[ \left\| Au(\tau) \right\|_{\mathfrak{H}_2 (S_0, H)} \leq d_2 \left\| f(\tau) \right\|_{\mathfrak{H}_{1,2} (S_0, H)}. \quad (13) \]

Finally, it follows from (12) and (13) that

\[ \left\| u(\tau) \right\|_{\mathfrak{H}_2 (S_0, A)} \leq d_0 \left\| f(\tau) \right\|_{\mathfrak{H}_{1,2} (S_0, H)}, \quad (14) \]

where \( d_0 = \sqrt{d_1^2 + d_2^2} \). Furthermore, the case of inhomogeneous initial data \( \varphi_0 \neq 0 \) is standardly reduced to a problem with homogeneous initial data and a new right-hand side of (4) of the form \( f_1(\tau) = f(\tau) + h(\tau) \), where

\[ h(\tau) = \int_0^\tau K (\tau - \zeta) A \exp (-\zeta A) \varphi_0 d\zeta. \]

and the vector function \( h(\tau) \) is then estimated.

**Remark 1.** It is well known that the solution of the initial boundary value problem for the homogeneous heat equation under natural assumptions on the initial data admits an analytic extension with respect to the time variable \( t \) to an angular domain on the complex plane. The theory of analytic semigroups of operators is closely related to this fact (see, for example, [6, 8–10]).

In Corollary 1 to Theorem 5, not only the analyticity of the solution of an abstract parabolic equation is established, but also estimate (7) is derived in the Hilbert spaces \( W_2^1 (S_0, A) \) and \( \mathfrak{H}_2 (S_0, H) \), which is a much more profound result than the analyticity (holomorphicity) of the solution.

Theorem 5 also immediately yields the corresponding assertion for \( \theta = 0 \), that is, for the case when problem (4), (5) is considered on the half-line \( \mathbb{R}_+ \), rather than in the angular domain \( S_0 \). In this case, the space \( \mathfrak{H}_2 (S_0, H) \) is replaced by \( L_2 (\mathbb{R}_+, H) \), while \( W_2^1 (S_0, A) \) is replaced by the Sobolev space \( W_2^1 (\mathbb{R}_+, A) \).

It is worth noting that Theorem 5 is closely related to Theorem 2.1 in [11] and Theorem 3.2.3 in [12]. These theorems establish the correct solvability of the initial value problem for an integrodifferential equation similar to (4) in the weighted Sobolev spaces \( W_2^1 (\mathbb{R}_+, A) \).

**Remark 2.** A significant difference of Theorem 5 from the solvability results deduced in [11, 12] for the traditional Sobolev spaces \( W_2^1 (\mathbb{R}_+, A) \) is that we give estimates of the Laplace transform \( \hat{u}(\lambda) \) of the solution in the domain \( S_{0+ \pi/2} \), rather than in the right half-plane. Our results rely heavily on Theorem 2 (an analogue of the Paley–Wiener theorem) for the angular domains \( S_0 \) and \( S_{0+ \pi/2} \), whereas the previous works use the conventional Paley–Wiener theorem.

Note that a number of solvability results for integrodifferential equations with operator coefficients in the Sobolev spaces \( W_2^\alpha (\mathbb{R}_+, A^\alpha) \) are given in [13].

Some profound results on the solvability of elliptic functional differential equations in Sobolev spaces are described in [14].

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**CONFLICT OF INTEREST**

The authors declare that they have no conflicts of interest.

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