Topological Phase Transition of Non-Hermitian Crosslinked Chain

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The contents of topological classification of matter are enriched by non-Hermiticity, such as exceptional points, bulk-edge correspondence, and skin effects. Physically, gain and loss can be introduced by imaginary on-site potentials of lattice Hamiltonians, and the topological phase transition for a cross-linked chain in the presence of such non-Hermiticity is investigated. The topological phase diagram in terms of a winding number is obtained analytically with phase boundaries coinciding with the surfaces of exceptional points. The topologically original edge states with distribution mainly at the joints between domains of different phases are protected even for long chains. The non-Hermitian topological feature can also be reflected by vortex structures in the vector fields of complex eigenenergies, expected values of Pauli matrices, and trajectories of these quantities. This model may be implemented in coupled photonic crystals, fermions trapped in optical lattice, or non-Hermitian electrical-circuit lattices, and the edge states are immune to various kinds of disorders until topological phase transition occurs. This work gives insight into the influence of non-Hermiticity on topological phase of matter.

1. Introduction

In the Hermitian realm, topological classification of matter benefits our understanding of condensed matter physics based on topological invariants in a global manner. Non-Hermiticity can manifest as gain and loss or asymmetric hopping in lattice systems. Extending topological classification theory for non-Hermitian systems may bring intriguing contents associated with some peculiar properties related to topological invariants, complex energy spectra and edge states, unusual bulk-edge correspondence, and skin phenomenon. In the presence of non-Hermiticity, proper topological invariants may be discussed in a new manner. For 1D topological systems with chiral symmetry, winding numbers are usually employed to characterize the topological properties. Besides topological invariants defined in terms of eigenstates, one may conjecture that complex eigenenergy spectra and expectation values of operators may reveal topological features from a new perspective.

Non-Hermitian degenerate points with coalesced eigenenergies and eigenvectors may manifest themselves as exceptional points, lines, and surfaces. Non-Hermitian systems usually perform intriguing and extraordinarily at such points and nearby. For example, unidirectional invisibility has been observed in parity-time symmetric fiber networks near exceptional points. The configurations of isolated exceptional points can characterize the symmetry-protected topological phases. Exceptional lines in nodal-line semimetals have been investigated in 3D dissipative cold atom gas. Exceptional rings protected by chiral symmetry for correlated 2D systems have been found robust against perturbations. Non-Hermitian nodal semimetals are promoted to be symmetry-protected, where the surfaces of exceptional points form the boundaries of open Fermi volumes. Besides, the exceptional surface has been studied by symmetry-preserving non-Hermitian deformations of topological nodal line. It is intriguing to investigate exceptional points in this work.

The bulk-boundary correspondence bridges the appearance of the edge states to the bulk topological invariants in Hermitian realm. It is intriguing to examine such correspondence in non-Hermitian situations. Topological invariants characterizing the anomalous helical edge states of non-Hermitian Chern insulator has been discussed in ref. [38]. Skin modes and topology in non-Hermitian systems have been anatomized in ref. [39]. With the breakdown of conventional bulk-boundary correspondence, non-Bloch Chern numbers predicting the numbers of chiral edge modes were introduced in ref. [17]. Based on the notion of biorthogonal quantum mechanics, generalized bulk-boundary correspondence was studied for the non-Hermitian SSH model and Chern insulators. It was found that non-Hermitian Aharonov–Bohm effect is essential for the...
validation of conventional bulk-boundary correspondence in non-Hermitian realm, and the inversion or combined inversion symmetry is critical for restoring conventional bulk-boundary correspondence. The topology of real-energy gapless phase resulting from exceptional point has been discussed in ref. [43]. It has been found that exotic nodal manifolds can be bounded by exceptional points in non-Hermitian systems.

There are currently platforms which can be employed to investigate non-Hermitian topological properties, for example, coupled photonic systems, ultracold fermions trapped in optical lattice, and non-Hermitian electrical-circuit lattices with synthetic magnetic field introduced. The immunity against imperfections in such systems makes topological edge states appropriate for topological quantum computation. Such topological imperfections in such systems makes topological edge states appropriate for topological quantum computation. Such topological stability has been investigated for non-Hermitian systems in refs. [49–51].

In this work, inspired by the works in refs. [16], [52], we study the topological features for a cross-linked chain in the presence of balanced on-site gain and loss. The topological phase diagram is given in terms of a winding number versus parameter systems including the hopping phases and the non-Hermiticity, and the relation between the phase transition and appearance of exceptional points is examined. Topological edge states locating mainly at the joints between domains of different topological phases are protected even if the chain is long. With the strengthening of the balanced gain and loss, the chain will be pushed into the topologically trivial phase according to the winding number with most of the eigenstates tending to pile up at the boundaries. Such a topological feature can also be reflected by the vortex structures in the vector fields of complex eigenenergies or expectations of Pauli matrices and their trajectories. The robustness of the edge states against various disorders is discussed in terms of the experimental implementations for this model.

This work is organized as follows: In Section 2, we put forward the cross-linked chain in the presence of on-site balanced gain and loss. In Section 3, the phase diagram in terms of a winding number versus parameters is given with exceptional points, edge states, and bulk-boundary correspondence discussed. In Section 4, we show the influence of non-Hermiticity on the energy spectra. Complex eigenenergies and expectation values of Pauli matrices are employed to reflect the topological feature. In Section 5, the experimental implementations are proposed and the immunity of the edge states against several kinds of disorders are examined. At last, we conclude in Section 6.

2. The Non-Hermitian Cross-Linked Chain

We consider the lattice model composed of two-component fermions and without considering the spin freedom (or spin polarized). The Hamiltonian in real space reads

\[
H = \sum_{n} \left( w_\parallel (\epsilon_0 - \epsilon_0) \frac{1}{4} (a_n^\dagger a_{n+1}^\dagger - b_n b_{n+1}) + \frac{\nu}{2} \epsilon_0 \langle a_n^\dagger b_{n+1}^\dagger + b_n a_{n+1}^\dagger \rangle + h.c. + H_{\text{int}} \right)
\]

where \( a_n \) and \( b_n \) are the annihilation operators for \( a \) and \( b \) elements on cell \( n \) as Figure 1 shows, \( w_\parallel \) is the coupling strength, and \( \nu \) would be used as the energy unit in this work (\( \nu = 1 \) hereafter). While this model is regarded as modified Creutz ladder, the hopping phases result from integral of gauge potential for electrons on the lattice along a path connecting adjacent sites [52]. Since the magnetic field \( B = V \times (\mathbf{A} + \nabla \Phi) \) with \( V \Phi \) freely added, the freedom of gauge choice makes the phases adjustable. Without considering the spin freedom, such a structure may be simulated by coupled photonic systems, ultracold fermions trapped in optical lattice, and non-Hermitian electrical-circuit lattices which will be discussed at last.

\[
H_{\text{int}} = h_z \sigma_z + h_x \sigma_x
\]

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\]

where \( \sigma_x \) and \( \sigma_z \) are Pauli matrices and \( h_z = \nu \sin(k + \frac{\theta_0 - \theta_\parallel}{2}) \sin(\theta_0 - \theta_\parallel) + i \Gamma \), \( h_x = \cos(k + \theta_\parallel \), \( \theta_\parallel \) and \( \theta_0 \) are the coupling strength, and \( \nu \) would be used as the energy unit in this work (\( \nu = 1 \) hereafter). While this model is regarded as modified Creutz ladder, the hopping phases result from integral of gauge potential for electrons on the lattice along a path connecting adjacent sites [52]. Since the magnetic field \( B = V \times (\mathbf{A} + \nabla \Phi) \) with \( V \Phi \) freely added, the freedom of gauge choice makes the phases adjustable. Without considering the spin freedom, such a structure may be simulated by coupled photonic systems, ultracold fermions trapped in optical lattice, and non-Hermitian electrical-circuit lattices which will be discussed at last.

Under periodic boundary condition, through the Fourier transformation \( c_n = \frac{1}{N} \sum_k c_k e^{i k n} \), \( c_k \) represents the operators \( a_{n0} \) or \( b_{n0} \), in momentum space, the Hamiltonian can be written as

\[
H_k = h_z \sigma_z + h_x \sigma_x
\]

where \( \sigma_x \) and \( \sigma_z \) are Pauli matrices and \( h_z = \nu \sin(k + \frac{\theta_0 - \theta_\parallel}{2}) \sin(\theta_0 - \theta_\parallel) + i \Gamma \), \( h_x = \cos(k + \theta_\parallel \), \( \theta_\parallel \) and \( \theta_0 \) are the coupling strength, and \( \nu \) would be used as the energy unit in this work (\( \nu = 1 \) hereafter). While this model is regarded as modified Creutz ladder, the hopping phases result from integral of gauge potential for electrons on the lattice along a path connecting adjacent sites [52]. Since the magnetic field \( B = V \times (\mathbf{A} + \nabla \Phi) \) with \( V \Phi \) freely added, the freedom of gauge choice makes the phases adjustable. Without considering the spin freedom, such a structure may be simulated by coupled photonic systems, ultracold fermions trapped in optical lattice, and non-Hermitian electrical-circuit lattices which will be discussed at last.

3. Topological Phase Diagram and Edge States

3.1. Topological Phase Diagram

First, we check the topological phase diagram of this non-Hermitian cross-linked chain in terms of a winding number

\[
H_{\text{int}} = i \Gamma (a_n^\dagger a_n - b_n^\dagger b_n)
\]
versus hopping phases and strength of the balanced gain and loss in the Hamiltonian (2). Besides the relation between topological phase transition and appearance of exceptional points, bulk-boundary correspondence correlating topological invariants to edge states under open boundary condition is going to be examined especially in the presence of gain and loss in this model.

Considering the Hamiltonian (2), inspired by (31), the winding number defined as

\[ n = \frac{1}{2\pi} \oint_{BZ} \text{tr} \phi \text{dk} \]

in the Brillouin zone (subscript BZ) where \( \phi = \text{atan} \frac{h_{xy}}{h_{yz}} \) is employed to indicate the topological phase of this model. According to (3), \( n \) has the meaning of accumulation of the angle \( \phi \) as \( k \) sweeps the Brillouin zone. The complex angle \( \phi = \text{Re}(\phi) + i\text{Im}(\phi) = \phi_1 + i\phi_2 \).

Then, \( n = \frac{1}{\pi} \oint \phi_1 \text{dk} + \frac{1}{\pi} \oint \phi_2 \text{dk} \). Since \( e^{i\phi_1} = \frac{h_{xy} + i h_{yz}}{h_{yz} - ih_{xy}} \), it can be calculated that

\[ \phi_1 = \frac{1}{2} \text{atan} \frac{2(h_{xy}h_{zx} + h_{xw}h_{zy})}{|h_x|^2 - |h_z|^2} \]

and

\[ \phi_2 = \frac{1}{4} \log \left| \frac{|h_x|^2 + |h_z|^2 - 2(h_{xy}h_{zx} - h_{xw}h_{zy})}{|h_x|^2 + |h_z|^2 + 2(h_{xy}h_{zx} - h_{xw}h_{zy})} \right| \]

where \( (h_{x,y,z}) \) denotes real (imaginary) parts of \( h_{x,y,z} \) and \( x,y,z, i=1,2 \). After some calculations (34), we obtain \( \frac{1}{\pi} \oint_{BZ} \phi_{1,2} \text{dk} = 0 \) and

\[ n = \frac{1}{2\pi} \oint_{BZ} \phi_{1,2} \text{dk} \]

\[ = \frac{1}{2} \text{sgn}(\phi_{1,2} - \frac{\theta_x + \theta_y}{2}) \sin \left( \frac{\theta_x - \theta_y}{2} \right) \]

\[ \text{sgn}(\Gamma^2 - \cos^2(\phi_{1,2} - \frac{\theta_x + \theta_y}{2})) - n \]

where \( n = \text{sgn}(\Gamma^2 + w^2 \cos^2(\phi_{1,2} - \frac{\theta_x + \theta_y}{2})) \text{sgn}(\sin(\phi_{1,2} - \frac{\theta_x + \theta_y}{2})) \geq 0 \) and \( \text{sgn}(x) \) outputs the sign of \( x \). Thus when \( \Gamma^2 \geq \cos^2(\phi_{1,2} - \frac{\theta_x + \theta_y}{2}) \), the chain is in the topological trivial phase. The topological phase diagram versus the hopping parameters and non-Hermiticity is shown in Figure 2.

In the topological classification of matter, degeneration in energy dispersion usually accompanies with topological phase transition. According to the winding number in (5), the phase boundaries occur at surfaces \( \Gamma^2 = \cos^2(\phi_{1,2} - \frac{\theta_x + \theta_y}{2}) \) or \( \theta_x - \theta_y = 2\pi n \) (\( n \) is an integer). This topological invariant also applies to the Hermitian case as shown in Figure 2b. Namely, the phase boundaries versus the parameters are consistent with the trajectories of degenerate points of the Hamiltonian (2) when \( \Gamma = 0 \).

Furthermore, we compare this phase transition and the appearance of exceptional points in the non-Hermitian case. For this two-band model when \( H_\nu \neq 0 \), exceptional point occurs when \( H_\nu \) is defective when the eigenenergies and eigenstates both coalesce with orthogonality between the left and right eigenstates. Thus, after some algebra, one can find that the exceptional points locate on the surfaces of parameters: \( \Gamma^2 = \cos^2(\theta_{ab} - \frac{\theta_a + \theta_b}{2}) \) or \( \theta_b - \theta_a = 2\pi n \) (\( n \) is an integer). While \( k = \pi \frac{\theta_b - \theta_a}{2} \). Those parameters on the surfaces \( \theta_b - \theta_a = 2\pi n \) separate \( n \) with different sigh within the surfaces \( \Gamma^2 = \cos^2(\theta_{ab} - \frac{\theta_a + \theta_b}{2}) \) as shown in Figure 2a. \( k \) of real value is essential to reveal the exceptional points, but acts as a latent variable in the phase diagram in Figure 2. The results coincide with the phase boundaries determined by \( n \) in Figure 2. With the parameters on the exceptional surfaces, the Hamiltonian \( H_k \) turns to be proportional to the defective matrix \( i\sigma_x \pm \sigma_x \). Thus, similar to the degenerate points in the Hermitian realm, the exceptional points also separate topological phases in this case.

### 3.2. Topological Edge States

Topological edge states with distributions mainly near the joints between domains with different topological invariants comply the bulk-boundary correspondence. Different from localized states resulting from finite size effect or defects, such topological edge states would be protected with elongating the chain. In view of the winding number \( n \), the vacuum can be regarded as topological trivial domain which results in edge states at the boundaries of topological nontrivial systems under open boundary condition. By jointing chains with the same topological invariant but different parameters, there should be no edge states with distribution mainly at the joints on the long chain. This conjecture in non-Hermitian realm has been checked by jointing two chains with different \( \theta_a \) and \( \Gamma \) into one chain under open boundary condition as sketched in Figure 3a.
The energy spectra as a function of the difference between the two $\theta_i$ in the two parts are shown in Figure 3a). The spectra touch at the phase transition points $\theta_{a2}$ with zero energy edge states in the gap which coincides with the phase transition in Figure 2, and the symmetry of the spectra versus zero energy results from the symmetry of this model: $H_1$, $H_2$, and $H_0$ has an eigenvector $|u\rangle$ with eigenvalue $E$, then $\sigma_y|u\rangle$ is also an eigenvector with eigenvalue $-E$. In real space, this symmetry operator reads $\Theta_\delta \sigma_y$, applied to the counterpart of the spinless Hamiltonian obtained by Jordan–Wigner transformation. The edge states correspond to the $Z$ topological number of sublattice symmetry, class A in Table VI.

As shown in Figure 3a), it can be found that no edge states exist at the joints between domains with $\theta_1 = \theta_2$, although $\theta_{a1} \neq \theta_{a2}$ for the two halves of the long chain in the topological nontrivial realm. Yet we find that the localized states may appear with distributions mainly around the joints when the chain is short, but such states due to finite-size effect would submerge into the other extended eigenstates with elongating the chain. Around the joints between domains with $\theta_1 \neq \theta_2$, the edge states is stable, even the chain is very long which confirms their topological origin. It has been found that the non-Hermitian Aharonov–Bohm effect is essential for the validity of conventional bulk-boundary correspondence. The relation about bulk-boundary correspondence, symmetry, and periodic structure of the lattice in topological realm may need further argument and the relation between the edge localized states and the lattice size may also be an intriguing topic.

In the non-Hermitian case, with increasing $\Gamma$, the eigenstates tend to distribute locally on fractional sites instead of extensively on the chain. When $\Gamma$ pushes the chain to the topologically trivial phase, most of the states tend to distribute near the joints between domains with different parameters and the edge states tend to submerge into these localized states. This phenomenon has been named as skin effect in non-Hermitian topological researches.

4. Topological Features in Complex Energies and Expectation Values of Operators

4.1. Complex Spectra

To get insight into the influence of the balanced gain and loss on the topological feature more concretely, we focus on the chain with parameters $\Gamma = 0$, $\theta_1 = 0$, $\theta_2 = \frac{\pi}{4}$ as the dashed line shows in Figure 2c when the phase transition occur at $\Gamma = \pm 1$. It may benefit revealing the topological phase transition by checking the energy spectra versus $\Gamma$ both in momentum and real spaces. Such complex eigenenergy spectra versus $\Gamma$ are shown in Figure 4. ($\Gamma, k) = (\pm 1, \frac{2n+1}{2\pi})$ (n is an integer) are the points where the trajectories of the complex energies cross. Such points in momentum space are exceptional points with coalesced eigenstates and eigenergies of the defective Hamiltonian as mentioned in Section 3.1.

Bulk-boundary correspondence is an intriguing issue when exploring topological properties for Hermitian systems.
Non-Hermitian items in Hamiltonians usually raise complex energy spectrum which makes the problem intricate. In this model, the results in Figure 4 conform to the bulk-boundary correspondence. In Figure 4b, we show the real eigenenergy as the dashed vertical line shows in Figure 4c, and the insets (b1) and (b2) provide the profiles of the real and imaginary parts of the edge states \( |\psi_1\rangle \) projecting on the chain sites. The real (imaginary) part of the amplitude of one edge state is identical to the imaginary (real) part of the other one. This restoration of the conventional bulk-boundary correspondence results from the absence of non-Hermitian Aharonov–Bohm effect as mentioned in Section 3.2.

4.2. Topological Feature Reflected by Vortex

Topological classification of matter in terms of topological invariants is usually defined in a global manner by invariants \(^1,^2\) for non-Hermitian systems. Complex energy may result to specific vector patterns reflecting the topological properties, merely ambiguity of band-labeling occurs at the exceptional points. We set the eigenenergies with the same sign of imaginary part versus \( k \) in the Brillouin zone as in the same branch in Figure 4a. Since a complex number \( x \) can be visually represented by a pair of real numbers \( (\text{Re}(x), \text{Im}(x)) \), we employ such pairs of numbers to construct a vector field for the complex eigenenergies represented by arrows on the parameter grid as in Figure 5b. The vector field of complex energy \( (\text{Re}(E), \text{Im}(E)) \) has half-vortex structures at the exceptional points \( (k, \Gamma) \) in \( (\pm 1, \frac{2\pi + 1}{2} \pi) \) with the branch cut between \( (-1, \frac{2n+1}{2} \pi) \) and \( (1, \frac{2n+1}{2} \pi) \) on the grid of \( (k, \Gamma) \). It should be emphasized that the structures of the half-vortices and the position of branch cut are related to the choice of energy branches due to the ambiguity of band-labeling at the exceptional points.\(^28\) For example, while one sets the eigenenergies with the same sign of real part (opposite sign of imaginary part) versus \( k \) as in one branch, the location of the center of the half-vortices do not change but the branch cut would change to \((-\infty, \frac{2n+1}{2} \pi), (-1, \frac{2n+1}{2} \pi), (1, \frac{2n+1}{2} \pi), (+\infty, \frac{2n+1}{2} \pi)\) on the grid of \( (k, \Gamma) \).

4.3. Topological Feature Reflected by Trajectory

Considering the eigenenergies are complex values, similar to ref. [21], we conjecture that a topological character may be identified from the trajectories of the complex eigenenergies, with no counterparts in the Hermitian realm. In the video (CrossLinkavi.avi) in the Supporting Information, we show the loop of \( (\Gamma, k) \) and the corresponding trajectories of \( (\text{Re}(E_1), \text{Im}(E_1)) \). One can see that the loops of the complex eigenenergies do not connect into one loop unless the loop of \( (\Gamma, k) \) encircles one of the exceptional points \( (\Gamma, k) = (1, \frac{\pi}{4}) \) regardless of the local shape of \( (\Gamma, k) \)-loop. This is also a kind of topological character, but different to that one in terms of the topological invariant. It coincides with the argument in Section 3 that the exceptional points act as the topological critical points. This is similar but different in characterizing topological phase transition by defining a topological invariant on a closed curve surrounding the phase transition point in the parameter space in ref. [60].

Similarly, half-vortex structures also appear in the vector field of \( ((\sigma_x), (\sigma_y)) \) at the exceptional points where \( (\sigma_z) = \langle u_{k,s}|\sigma_3|u_{k,s}\rangle \) \((z = x, y, \text{or} \ z = 1 \text{ or} \ 2)\) as shown in Figure 5c. One gains the same conclusion when \( |u_{i,j}\rangle \) is employed.

Figure 4. a) The energy spectrum versus momentum \( k \) for \( \Gamma \in [0, 2] \) (darker color corresponds to less \(\Gamma\)) when \( (\theta_x, \theta_y, \theta_{ab}, \omega) = (0, \pi, 0.5\pi, 0.5) \). b) The real parts of eigenenergies as the dashed vertical line in (c) shows. The insets (b1) and (b2) provide the profiles of the real and imaginary parts of the edge states \( |\psi_1\rangle \) projecting on the chain. c, d) The real and imaginary parts of the eigenenergy spectrum under open boundary condition.

Figure 5. a) The absolute value of \( E_1 \) (or \( E_2 \)) versus \( k \) and \( \Gamma \). b, c) The vector fields of \( (\text{Re}(E), \text{Im}(E)) \) and \( ((\sigma_x), (\sigma_y)) \) on the grid of \( (\Gamma, k) \). d1–d3) The trajectories of the points with coordinates \( ((\sigma_x), (\sigma_y), (\sigma_z)) \) for \( k \in [0, 2\pi] \) when \( \Gamma = (0.5, 1, 1.5) \), respectively. Blue and red trajectories correspond to the trajectories obtained by different eigenstates. The eigenenergies with the same sign of imaginary part are set in the same branch. The other parameters are \( (\theta_x, \theta_y, \theta_{ab}) = (0, \pi, 0.5\pi) \) as the vertical dashed line shown in Figure 2c.
When $|\Gamma| < 1$, the two trajectories connect to each other with two cross points on the sphere, but when $|\Gamma| > 1$, the trajectories divide into two non-connected loops with two crosses. $|\Gamma| = 1$ are the topological critical points with four crosses on the sphere. This topological behavior is consistent with the phase diagram in Section 3 and the energy spectra in Figure 4.

These results indicate that the energy spectra and expectations of operators advocate the bulk-boundary correspondence in the parameter regime considered here, and these results hint that besides winding number $\nu$, other quantities can also reflect the topological character of matter. This may shed light on detecting topological phase of matter by more accessible avenues like those mentioned above.

5. Experimental Implementations and Robustness of Edge States Against Disorders

Experimentially, coupled optical micro resonators of photonic crystals, cold fermions in optical lattices, and non-Hermitian electrical-circuit lattices[44–48] may be candidates for fabricating simulators of this chain. In these proposals, the coupling between sites is the essential factor. The hopping strengths and phases can be tuned by varying the coupling conditions between the sites and the coupling phases result from the integral of synthetic magnetic fields. The gain can be obtained by coupling the simulators to gain medium or pumping, and the loss can be introduced naturally by scattering due to impurity or defects in the materials. Furthermore, non-Hermitian topological phase transition in the presence of non-linearity may be an intriguing issue to be investigated.[161]

In terms of the real implementations mentioned above, various kinds of disorders like defects or impurities in the materials are usually inevitable due to fabrication error. The protection for the topological edge modes against such disorders is an advantage in view of applications. Such states usually manifest themselves as gaped states, as the energy bands show in this model in Figure 4. The edge states should be protected until the disorders exceed the topological critical points. Considering the Hamiltonian and the implementations of this model mentioned above, we mainly focus on the disorders of $\theta_a$, $\theta_{ab}$, $w$, and $\Gamma$. The simulation results are shown in Figure 6. In each simulation, the disorders distribute on all the cells of the chain with strength randomly and uniformly within the range $[0, \delta x]$ ($\delta x$ represents the maximal amplitudes for the disorders in $\theta_a$, $\theta_{ab}$, $w$, and $\Gamma$). The parameters in the Hamiltonian (1) become $x + \delta x$ in each simulation. It can be seen that the edge states behave robustly to the disorders until topological phase transition occurs, namely, passing the exceptional points. Since the amplitude of $w$ does not change $\nu$ in (5), the edge states are stable with increasing $\delta w$. It should be noted that the amplitudes of the disorders should not be too large since there are perturbations here.

6. Conclusion

The topological phase transition of a cross-linked chain in the presence of balanced gain and loss has been investigated in this work. The phase transition is indicated by a winding number and the phase boundaries coincide with the surfaces of exceptional points. The edge states with distributions are mainly near the joints between domains of different phases. With increase of the balanced gain and loss, the eigenstates tend to locate on fractional sites. Besides the winding number, the topological feature of this model can also be reflected by the vortex structures in the vector fields of complex eigenenergies, expectation values of operators, and their trajectories. This means that several approaches can be employed to reveal topological phase of matter. This model may be demonstrated by instruments made of coupled photonic crystals, polarized cold fermions trapped in optical lattice, or non-Hermitian electrical-circuit lattices, and the edge states behave tenaciously against various disorders until the topological phase transition occurs. This work gives insight into the field of topological classification of matter in the non-Hermitian realm.

Supporting Information

Supporting Information is available from the Wiley Online Library or from the author.

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