WEIGHTED ELLIPTIC ESTIMATES FOR A MIXED BOUNDARY SYSTEM RELATED TO THE DIRICHLET-NEUMANN OPERATOR ON A CORNER DOMAIN

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ABSTRACT. Based on the $H^2$ existence of the solution, we investigate weighted estimates for a mixed boundary elliptic system in a two-dimensional corner domain, when the contact angle $\omega \in (0, \pi/2)$. This system is closely related to the Dirichlet-Neumann operator in the water-waves problem, and the weight we choose is decided by singularities of the mixed boundary system. Meanwhile, we also prove similar weighted estimates with a different weight for the Dirichlet boundary problem as well as the Neumann boundary problem when $\omega \in (0, \pi)$.

1. Introduction. Based on the classical non-smooth theory in [14, 17], we consider weighted estimates for a mixed boundary elliptic system in a two-dimensional corner domain $\Omega$. This domain is bounded by a top surface $\Gamma_t = \{(x,z)\mid z = \eta(x)\}$ and a smooth bottom $\Gamma_b = \{(x,z)\mid z = l(x)\}$, that is

$$\Omega = \{(x,z)\mid l(x) < z < \eta(x), x \geq 0\}$$

and the bottom satisfies

$$l(x) = -\gamma x, \quad \text{when } x \leq x_0$$

for some fixed constant $x_0$ and slope $-\gamma < 0$. Without loss of generality, we place the contact point $X_c$ intersected by $\Gamma_t$, $\Gamma_b$ to be at the origin $O = (0,0)$. The free surface $z = \eta(x)$ satisfies

$$\eta(0) = 0, \quad \text{and } 0 < \eta(x) - l(x) \leq H, \quad \forall x > 0$$

for some constant $H > 0$. Moreover, the notation $X = (x,z)$ is used for the space coordinates, and we have $\nabla_X = (\partial_x, \partial_z)$ and $dX = dx dz$.

Closely related to the Dirichlet-Neumann operator in the water-waves problem, we will focus on the following mixed boundary elliptic problem for $u$ when proper conditions $h$, $f$, $g$ are given:

$$\begin{align*}
\Delta u &= h, & \text{in } \Omega \\
u |_{\Gamma_1} &= f, & \partial_{\nu_b} u |_{\Gamma_b} &= g.
\end{align*}$$

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When the domain changes with time $t$, the top surface becomes a free surface with a fixed bottom, and the contact point also varies. This kind of corner domains are related to a scene of sea waves moving near the beach in the real world, which is already used when we investigate the water-waves problem and related elliptic systems in [24, 25]. For the moment, we only consider a fixed surface $z = \eta(x)$ independent of the time in this paper.

To prove the estimates for the mixed boundary problem (MBVP), one needs to notice firstly that this problem contains some singularity on the boundary, which requires naturally the non-smooth elliptic theory. Therefore, before stating our main results, we shall recall some previous works on the non-smooth elliptic theory.

To start with, here *non-smooth* is generally referred to Lipschitz. When the boundary is Lipschitz, the classical elliptic theory for a smooth boundary doesn’t apply any more. The non-smooth elliptic theory has been fully developed in recent decades, and fundamental works are done by Kondrat’ev [11, 12]. One can find some other early works by Birman and Skvortsov [2], Eskin [8], Lopatinskiy [16], Maz’ya [18, 19], Kondrat’ev and Oleinik [13], Maz’ya and Plamenevskiy [20], Maz’ya and Rossmann [23], Grisvard [9], Dauge [6] etc.. These works analyze singularities near the corner and provide regularity results in Sobolev space or weighted Sobolev space for general linear elliptic problems on Lipschitz domains.

In fact, the existence of a variation solution in $H^1$ can be proved most of the time for a Lipschitz domain, see for example [9]. Compared to the smooth elliptic theory, when a higher regularity is considered, the key for the non-smooth theory lies in singularities, which can be expressed by a summation of singular functions like $r^\lambda \log^q r \phi(\theta)$ near the corner point, where $r$ is the radius to the corner point, $\lambda$ is an eigenvalue of the corresponding problem, $q$ is some constant, and $\phi(\theta)$ is a bounded trigonometric function. Compared to $H^1$ solutions, it is well known that singularities arise when a higher-order regularity is referred to. At that time, the solution $u$ to an elliptic problem can be decomposed into

$$u = u_r + \sum_i c_i S_i$$

where $u_r$ is the regular part, $c_i$ the singular coefficient, and $S_i$ some singular function with an explicit formula as mentioned above. Moreover, it is also well known that, the number in the summation of singular functions are finite and can be decided explicitly by the elliptic operator, the contact angle and the regularity, see for example [9]. In fact, when one considers higher-order regularities or larger contact angles, the number of singular functions usually increases. The decompositions and estimates for the regular part and the singular coefficients in Sobolev spaces can be found in Kondrat’ev [12], Maz’ya and Plamenevskiy [21, 22], Dauge, Nicaise, Bourlard and Lubuma [7], Grisvard [9], Costabel and Dauge [4, 5], Ming and Wang [24] etc.

Based on the study for singular functions, a smart way to obtain a clean elliptic estimate as in the classical case is to use weighted Sobolev spaces, for example, space $V^{\beta}_l$ defined in Section 2 with some weight number $\beta$ and order $l$. Due to the expressions of singular functions, the weight is naturally in a form of $r^\beta$, where $r$ is the radius to the corner point. We refer to general weighted estimates in Kozlov, Mazya and Rossmann [14], Dauge [6], Mazya and Rossmann [17] etc.. These works provide some general weighted estimates assuming that the right side of the elliptic system also lies in corresponding weighted spaces. Meanwhile, to obtain the
weighted estimates, there are usually conditions between $\beta, l$ and the eigenvalues of the corresponding eigenvalue problem: One requires that no eigenvalues $\lambda$ lie on the line $Re\lambda = -\beta + l - 1$, see Theorem 6.1.1 [14].

Using the weighted spaces introduced in these works and starting with the $H^2$ existence (which is already proved in [24]), we prove proper weighted estimates for the mixed boundary problem and trace the dependence of the upper boundary in the coefficients.

Firstly, “proper” means that we identify the power $\beta$ of the weight $r^\beta$ very specifically, which is based on our analysis for the same mixed boundary problem in [24]. On one hand, the weight $\beta$ we choose is decided by the order of singularities which appear in our problem. Thanks to Proposition 5.19 [24], when one considers $H^1(\Omega)$ solution $u$ ($l \geq 3$), one needs at least the weight $r^{l-2}$ to eliminate the singular part

$$r^{-(m+1/2)\pi}$$

for $m \in \mathbb{Z}$

such that $r^{l-2}\nabla l u \in L^2$ near the corner. On the other hand, we obtain the weighted elliptic estimates without extra condition between eigenvalues $\lambda$ and $\beta, l$ as mentioned above (which is an important ingredient in our results). These two points result in the weighted space $V_{l-2+\beta}(\Omega)$ with $\beta \in [0, 2]$ in our main theorem, which is defined in Section 2.

On the other hand, one can see that the dependence of the upper boundary is not clearly proved in previous works. We provide detailed estimates for tracing this dependence in this paper.

The main theorem is presented below.

\textbf{Theorem 1.1.} (Mixed boundary) Let the contact angle $\omega \in (0, \pi/2)$ and $u \in H^2(\Omega)$ be the solution to (MBVP) for given $h \in L^2(\Omega)$, $f \in H^{3/2}(\Gamma_t)$ and $g \in H^{1/2}(\Gamma_b)$. Moreover, for a real $\beta \in [0, 2]$ and an integer $l \geq 2$, one assumes that

$$h \in V_{l-2}^{l-2+\beta}(\Omega), \quad f \in V_{l-2+\beta}^{l-1/2}(\Gamma_t) \quad \text{and} \quad g \in V_{l-2+\beta}^{l-3/2}(\Gamma_b).$$

Then, one has $u \in V_{l-2+\beta}^{l-1}(\Omega)$, and

(i) If $\eta \in W^{l, \infty}(\mathbb{R}^+)$, there holds

$$\|u\|_{V_{l-2+\beta}^{l-1}(\Omega)} \leq C(\|\eta\|_{W^{l, \infty}(\mathbb{R}^+)})(\|h\|_{V_{l-2+\beta}^{l-1}(\Omega)} + \|f\|_{V_{l-2+\beta}^{l-1/2}(\Gamma_t)} + \|g\|_{V_{l-2+\beta}^{l-3/2}(\Gamma_b)}).$$

(ii) If $\eta \in H^{1/2-l}(\mathbb{R}^+)$ and $l \geq 3$, there holds

$$\|u\|_{V_{l-2+\beta}^{l-1}(\Omega)} \leq C(\|\eta\|_{H^{1/2-l}(\mathbb{R}^+)})(\|h\|_{V_{l-2+\beta}^{l-1}(\Omega)} + \|f\|_{V_{l-2+\beta}^{l-1/2}(\Gamma_t)} + \|g\|_{V_{l-2+\beta}^{l-3/2}(\Gamma_b)}).$$

The coefficient $C$ is a positive polynomial of $\|\eta\|_{W^{l, 1}(\mathbb{R}^+)}$ or $\|\eta\|_{H^{1/2-l}(\mathbb{R}^+)}$.

\textbf{Remark 1.} In fact, this result can be adjusted immediately to the case of a bounded corner domain, where there are two contact points between the upper surface and the bottom.

On the other hand, we also consider about weighted estimates for the Dirichlet boundary problem

$$\text{(DVP)} \left\{ \begin{array}{ll} \Delta u = h, & \text{in} \quad \Omega \\ u|_{\Gamma_t} = f, & \text{in} \quad \Gamma_t \\ u|_{\Gamma_b} = g & \text{on} \quad \Gamma_b \end{array} \right.$$

as well as for the Neumann boundary problem

$$\text{(NVP)} \left\{ \begin{array}{ll} \Delta u = h, & \text{in} \quad \Omega \\ \partial_n u|_{\Gamma_t} = f, & \text{on} \quad \Gamma_t \\ \partial_n u|_{\Gamma_b} = g & \text{on} \quad \Gamma_b \end{array} \right.$$
with the compatibility condition
\[ \int_{\Omega} h dX = \int_{\Gamma_t} f ds + \int_{\Gamma_b} g ds. \]

Similar weighted estimates are proved in this paper for both (DVP) and (NVP) when the contact angle varies in a much larger interval, and meanwhile the weight is slightly different from the mixed boundary case.

**Theorem 1.2.** Assume that the contact angle \( \omega \in (0, \pi) \), \( \beta \in (0, 1] \) be a real number and the integer \( l \geq 2 \).

(i) (Dirichlet boundary) Let \( u \in H^2(\Omega) \) be the solution to (DVP) for given \( h \in L^2(\Omega), f \in H^{3/2}(\Gamma_t) \) and \( g \in H^{3/2}(\Gamma_b) \) satisfying
\[ f|\Gamma_c = g|\Gamma_c. \]
Moreover, one assumes that
\[ h \in V^{l-2}_{l-1,1+\beta}(\Omega), \quad f \in V^{l-1/2}_{l-1,1+\beta}(\Gamma_t) \quad \text{and} \quad g \in V^{l-1/2}_{l-1,1+\beta}(\Gamma_b). \]
Then, one has \( u \in V^{l}_{l-1,1+\beta}(\Omega) \). When \( \eta \in H^{l-1/2}(\mathbb{R}^+) \) and \( l \geq 3 \), there holds
\[ \|u\|_{V^{l}_{l-1,1+\beta}(\Omega)} \leq C(\|\eta\|_{H^{l-1/2}(\mathbb{R}^+)})(\|h\|_{V^{l-2}_{l-1,1+\beta}(\Omega)} + \|f\|_{V^{l-1/2}_{l-1,1+\beta}(\Gamma_t)} + \|g\|_{V^{l-1/2}_{l-1,1+\beta}(\Gamma_b)}). \]

(ii) (Neumann boundary) Let \( u \in H^2(\Omega) \) be the solution to (NVP) for given \( h \in L^2(\Omega), f \in H^{1/2}(\Gamma_t) \) and \( g \in H^{1/2}(\Gamma_b) \). Moreover, one assumes that
\[ h \in V^{l-2}_{l-1,1+\beta}(\Omega), \quad f \in V^{l-3/2}_{l-1,1+\beta}(\Gamma_t) \quad \text{and} \quad g \in V^{l-3/2}_{l-1,1+\beta}(\Gamma_b). \]
Then, one has \( u \in V^{l}_{l-1,1+\beta}(\Omega) \). When \( \eta \in H^{l-1/2}(\mathbb{R}^+) \) and \( l \geq 3 \), there holds
\[ \|u\|_{V^{l}_{l-1,1+\beta}(\Omega)} \leq C(\|\eta\|_{H^{l-1/2}(\mathbb{R}^+)})(\|h\|_{V^{l-2}_{l-1,1+\beta}(\Omega)} + \|f\|_{V^{l-3/2}_{l-1,1+\beta}(\Gamma_t)} + \|g\|_{V^{l-3/2}_{l-1,1+\beta}(\Gamma_b)}). \]

The coefficient \( C \) above is a positive polynomial of \( \|\eta\|_{H^{l-1/2}(\mathbb{R}^+)} \).

### 1.1. Organization of the paper.
In Section 2 we introduce the weighted spaces on \( \Omega \) and its boundaries with some useful lemmas. Section 3 proves the main theorem for the mixed boundary problem. In Section 4, some other boundary problems are considered, while Section 5 provides the application of our theory on the Dirichlet-Neumann operator.

### 1.2. Notations.
- \( X_c \) denotes the contact point. We simply set \( X_c = O(0, 0) \) here;
- \( K \) is the cone \( \{(x, z) \mid -\gamma x \leq z \leq \eta'(0)x\} \);
- \( C \) is the strip \( \{(t, \theta) \mid t \in \mathbb{R}, -\omega_2 \leq \theta \leq \omega_1\} \);
- The angular interval \( I = [-\omega_2, \omega_1] \). The contact angle \( \omega = \omega_1 + \omega_2 \);
- \( \Gamma_t, \Gamma_b \) denote the upper boundary and the lower boundary respectively for the domain \( \Omega, K \) or \( C \), when no confusion will be made;
- Recalling from [24] that, the function \( d(\cdot) \) introduced in the transformation of the domain is
\[ d(\cdot) = 1 - \frac{1}{\eta'(\eta^{-1}(\cdot)) + \gamma} \]
with \( \eta(x) = \eta(x) + \gamma x \) invertible near \( X_c \).
2. Weighted Sobolev spaces on corner domains.

2.1. Definitions for weighted spaces and transformations of domains. We will introduce definitions of weighted Sobolev spaces firstly on the cone $\mathcal{K}$ and then on the corner domain $\Omega$, which can be found in [14, 17].

For an integer $l \geq 0$ and a real $\beta$, the space $V^l_\beta(\mathcal{K})$ can be defined as the closure of $C^\infty_0(\mathcal{K}\setminus X_c)$ with the norm

$$
\|w\|_{V^l_\beta(\mathcal{K})} = \left( \int_{\mathcal{K}} \sum_{|\alpha| \leq l} r^{2(\beta-l+|\alpha|)}|\nabla^\alpha_X w|^2 dX \right)^{\frac{1}{2}}
$$

with $r$ the radius with respect to $X_c$.

Next, we recall straightening transformations $T_S$ and $T_R$ from [24]. To begin with, Let $S = \{(\bar{x}, \bar{z})| \bar{x} \geq 0, 0 \leq \bar{z} \leq \bar{x}\}$. $T_S$ is the local transformation near the point $X_c$ which maps $S \cap U_{\delta}S$ into $\Omega \cap U_{\delta}$:

$$
T_S: (\bar{x}, \bar{z}) \in S \cap U_{\delta}S \mapsto (\bar{x}, \bar{z}) \in \Omega \cap U_{\delta}
$$

with

$$
\bar{x} = \bar{x} + \bar{\eta}^{-1}(\bar{z}) - \bar{z}, \quad \bar{z} = \bar{z} - \gamma(\bar{x} + \bar{\eta}^{-1}(\bar{z}) - \bar{z}),
$$

where $\bar{\eta}^{-1}(\bar{z})$ is the inverse of $\bar{\eta}(\bar{x}) = \eta(\bar{x}) + \gamma \bar{x}$. $U_{\delta}S$ and $U_{\delta}$ are two corresponding neighborhoods of $X_c$. We know that $T_S$ is invertible:

$$
T^{-1}_S: (\bar{x}, \bar{z}) \in \Omega \cap U_{\delta} \mapsto (\bar{x}, \bar{z}) \in S \cap U_{\delta}S
$$

where

$$
\bar{x} = \bar{x} - \bar{\eta}^{-1}(\gamma \bar{x} + \bar{z}) + \gamma \bar{x} + \bar{z}, \quad \bar{z} = \gamma \bar{x} + \bar{z}.
$$

Moreover, we introduce the linear transform

$$
T_0: X = (x, z) \in \mathcal{K} \mapsto \bar{X} = (\bar{x}, \bar{z}) = XP_0 \in S
$$

with

$$
P_0 = \begin{pmatrix} 1 + \gamma d(0) & \gamma \\ d(0) & 1 \end{pmatrix} \quad \text{where} \quad d(0) = 1 - \frac{1}{\gamma + \eta'(0)}.
$$

Together with $T_S$, we set

$$
T_c = T_S \circ T_0
$$

which maps the cone $\mathcal{K}$ to the domain $\Omega$ near the corner.

Besides, we also have the transform $T_R$ which maps a flat strip $R$ to the rest part of $\Omega$:

$$
T_R: (x, z) \in R \mapsto (\bar{x}, \bar{z}) \in \Omega
$$

with

$$
\bar{x} = x, \quad \bar{z} = \eta(x)z + l(x)(1 - z),
$$

where $R = \{(x, z)| x \geq x_{\delta}, 0 \leq z \leq 1\}$ is a flat strip, and $x_{\delta} > 0$ is a constant fixed by $U_{\delta}$. The inverse transform $T_R$ is

$$
T^{-1}_R: (\bar{x}, \bar{z}) \in \Omega \cap \{\bar{x} \geq x_{\delta}\} \mapsto (x, z) \in R
$$

where

$$
x = \bar{x}, \quad z = \frac{\bar{z} - l(\bar{x})}{\eta(\bar{x}) - l(\bar{x})}.
$$

Now it’s the time to define the weighted space $V^l_\beta(\Omega)$ on $\Omega$. We firstly set $\chi_c \in C^\infty_0(\Omega)$ supported near $X_c$ with some diameter $\delta > 0$ small enough. Since
Moreover, the weighted space $V^j_\Omega$ is equipped with the norm
\[ \|u\|_{V^j_\Omega} = \|v_c\|_{V^j(K)} + \|v_R\|_{H^j(\Gamma)}, \]
where
\[ v_c = u_c \circ T_c \quad \text{with} \quad u_c = \chi_c u \]
and
\[ v_R = (1 - \chi_c)u \circ T_R. \]
Obviously, the space doesn’t depend on the choices of the cut-off function $\chi_c$.

On the other hand, one also needs to use another type of weighted space $W_{2,\beta}(C)$ on the infinite strip $C = \mathbb{R} \times [-\omega_2, \omega_1]$, which can be found in [14]. In fact, for a function $w(t, \theta)$ on $C$, the norm for $W_{2,\beta}(C)$ is defined as
\[ \|w\|_{W_{2,\beta}(C)} = \|e^{\beta t}w\|_{H^j(\Gamma)}. \]
Similarly, the corresponding weighted space $W_{2,\beta}^{1-1/2}(\Gamma_t), W_{2,\beta}^{1-3/2}(\Gamma_b)$ on the upper and lower boundaries are defined with norms
\[ \|w\|_{W_{2,\beta}^{1-1/2}(\Gamma_t)} = \|e^{\beta t}w\|_{H^{1-1/2}(\Gamma_t)}, \quad \|w\|_{W_{2,\beta}^{1-3/2}(\Gamma_b)} = \|e^{\beta t}w\|_{H^{1-3/2}(\Gamma_b)}. \]
Moreover, $W_{2,\beta}(\mathbb{R})$ used in Section 2.3 is defined in a similar way.

In the end, we recall a regularizing diffeomorphism near $X_c$ from [24] which is a variation based on the transformations $T_S$ and $T_c$.

To begin with, we define $\tilde{s}(\tilde{x}, \tilde{z})$ on $S$ satisfying the Dirichlet boundary condition:
\[ \tilde{s}(\tilde{x}, \tilde{z})|_{\Gamma_t: \tilde{z} = \tilde{x}} = \beta(\tilde{x})\tilde{\eta}^{-1}(\tilde{x}), \]
where $\beta$ is a cut-off function defined on $[0, +\infty)$ and vanish away from 0. Consequently, one has from Remark 4.8 [24] that if $\beta\tilde{\eta}^{-1} \in H^{1-1/2}(\mathbb{R}^+)$, then $\tilde{s}(x, z) \in H^j(S)$ with the estimate
\[ \|\tilde{s}\|_{H^j(S)} \leq C|\beta\tilde{\eta}^{-1}|_{H^{1-1/2}(\mathbb{R}^+)} \leq C(\eta|H^{m_0}(\mathbb{R}^+))|\eta|_{H^{1-1/2}(\mathbb{R}^+)}, \]
where the constant $m_0 > 3/2$.

As a result, we define the regularized transformation $\tilde{T}_S$ as
\[ \tilde{T}_S : (\tilde{x}, \tilde{z}) \in S \cap U_{\delta} \mapsto (\tilde{x}, \tilde{z}) \in \Omega \cap U_{\delta} \]
with
\[ \tilde{x} = \tilde{x} + \tilde{s}(\varepsilon\tilde{x} + (1 - \varepsilon)\tilde{z}, \tilde{z}), \quad \tilde{z} = \tilde{z} - \gamma(\tilde{x} + \tilde{s}(\varepsilon\tilde{x} + (1 - \varepsilon)\tilde{z}, \tilde{z}) - \tilde{z})) \]
where $\varepsilon$ is a small constant to be explained. A direct computation shows that
\[ \text{Det}(\nabla \tilde{T}_S) = 1 + \varepsilon \partial_z \tilde{s}, \]
so $\tilde{T}_S$ is invertible as long as the constant $\varepsilon$ is small enough such that
\[ \varepsilon \leq \frac{1}{2\|\partial_z \tilde{s}\|_{\infty}}. \]

Some more computations lead to the associated coefficient matrix related to (MBVP)
\[ \tilde{P}_S = ((\nabla \tilde{T}_S^{-1}) \circ \tilde{T}_S) \]
\[ = \frac{1}{1 + \varepsilon \partial_z \tilde{s}} \begin{pmatrix} 1 + \gamma(1 - (1 - \varepsilon)\partial_{\tilde{x}} \tilde{s} - \partial_{\tilde{z}} \tilde{s}) & \gamma(1 + \varepsilon \partial_z \tilde{s}) \\ 1 - (1 - \varepsilon)\partial_{\tilde{x}} \tilde{s} - \partial_{\tilde{z}} \tilde{s} & 1 + \varepsilon \partial_z \tilde{s} \end{pmatrix} |(\varepsilon\tilde{x} + (1 - \varepsilon)\tilde{z}, \tilde{z})| \]
and we denote

\[ \tilde{P}_0 = \tilde{P}_S|_{\chi_r} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \]

Similarly as before, the transformation \( \tilde{T}_0 \) from \( K \) to \( S \) is defined as

\[ \tilde{T}_0 : \quad X = (x, z) \in K \mapsto \tilde{X} = (\tilde{x}, \tilde{z}) = XP_0 \in S, \]

and we also define on \( K \) that

\[ s(x, z) = \tilde{s}(XP_0P_\varepsilon) = \tilde{s}(\varepsilon \tilde{x} + (1 - \varepsilon) \tilde{z}, \tilde{z}) \quad \text{with} \quad P_\varepsilon = \begin{pmatrix} \varepsilon & 0 \\ 1 - \varepsilon & 1 \end{pmatrix}. \quad (5) \]

So one can replace \( P_S \) in system 9 by

\[ \tilde{P}_S \circ \tilde{T}_0 = \frac{1}{1 + a_2 \partial_z s + c_2 \partial_z s} \begin{pmatrix} 1 + \gamma(1 - b_1 \partial_z s - d_1 \partial_z s) & \gamma(1 + a_1 \partial_z s + c_1 \partial_z s) \\ 1 - b_1 \partial_z s - d_1 \partial_z s & 1 + a_1 \partial_z s + c_1 \partial_z s \end{pmatrix} \quad (6) \]

when we need it.

Similarly as before, we set

\[ \tilde{T}_c = \tilde{T}_S \circ \tilde{T}_0 \]

which maps the cone \( K \) to the domain \( \Omega \) near the corner.

2.2. Traces on the boundary. The weighted spaces on the boundary are also needed in our theory. We introduce the definitions of the trace spaces (see [17]), and some trace theorems are discussed, too.

Firstly, we define \( V_{\beta}^{l-1/2}(\Gamma_t) \) (and \( V_{\beta}^{l-1/2}(\Gamma_b) \)) for \( l \geq 1 \) as the spaces for traces of functions from \( V_{\beta}^l(\Omega) \) on \( \Gamma_t \) (and \( \Gamma_b \)) respectively. \( V_{\beta}^{l-1/2}(\Gamma_t) \) is equipped with the norm

\[ \| u \|_{V_{\beta}^{l-1/2}(\Gamma_t)} = \inf \left\{ \| u_{\text{ex}} \|_{V_{\beta}^l(\Omega)} \mid u_{\text{ex}} \in V_{\beta}^l(\Omega), u_{\text{ex}}|_{\Gamma_t} = u \right\}, \]

and the norm of \( V_{\beta}^{l-1/2}(\Gamma_b) \) is defined similarly.

Consequently, one concludes the following lemma immediately.

**Lemma 2.1.** Let \( u \in V_{\beta}^l(\Omega) \) and set \( f = u|_{\Gamma_t} \), then one has \( f \in V_{\beta}^{l-1/2}(\Gamma_t) \) and the estimate

\[ \| f \|_{V_{\beta}^{l-1/2}(\Gamma_t)} \leq \| u \|_{V_{\beta}^l(\Omega)}. \]

Similar conclusion holds for the trace on \( \Gamma_b \) and for the case \( V_{\beta}^l(\Gamma) \).

Since the traces related to the cone \( K \) will be used frequently, one needs to go further with the norms defined above. Notice that the angle \( \theta \equiv \omega_1 \) on the upper boundary \( \Gamma_t \) of \( K \), and \( \theta \equiv -\omega_2 \) for \( \Gamma_b \). Lemma 6.1.2 from [14] gives an equivalent norm for \( V_{\beta}^{l-1/2}(\Gamma_t) \) as

\[ \| u \|_{V_{\beta}^{l-1/2}(\Gamma_t)}^2 = \sum_{j \leq l-1} \int_{\mathbb{R}^+} r^{2(\beta-j)+1} \left| (r\partial_r)^j u(r, \omega_1) \right|^2 \, dr \]

\[ + \sum_{j \leq l-1} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} r^{2(\beta-j)+2} \left| (r\partial_r)^j u(r, \omega_1) - (\rho\partial_\rho)^j u(\rho, \omega_1) \right|^2 \, d\rho dr, \]

which will be used frequently in our paper. The equivalent norm for \( V_{\beta}^{l-1/2}(\Gamma_b) \) is defined similarly.

The following lemma concerns the trace theorem with Dirichlet boundary conditions, which is modified from Lemma 2.2.1 [17].
Lemma 2.2. (Dirichlet boundary) Let \( f \in V_{\beta}^{l-1/2}(\Gamma_t) \) and \( g \in V_{\beta}^{l-1/2}(\Gamma_b) \) with integer \( l \geq 1 \). Then there exists a function \( w \in V_{\beta}^l(K) \) such that
\[
  w|_{\Gamma_t} = f, \quad w|_{\Gamma_b} = g
\]
with the estimate
\[
  \|w\|_{V_{\beta}^l(K)} \leq C \left( \|f\|_{V_{\beta}^{l-1/2}(\Gamma_t)} + \|g\|_{V_{\beta}^{l-1/2}(\Gamma_b)} \right),
\]
where the constant \( C \) depends only on \( \beta, l, K \).

Proof. Taking \( m = 1 \) in Lemma 2.2.1 [17], one obtains the desired result immediately. \( \square \)

2.3. Some preparations. Some preparations are done in this part. Firstly, embeddings between different weighted spaces are discussed. Moreover, one considers the relationships between different weighted spaces and ordinary spaces. In the end, the Laplace transform is introduced with some basic properties, and an equivalent norm for a weighted space is defined based on this transform.

The functions considered here are always compactly supported near \( X_c \) with a size \( \delta \), and we focus on the cone \( K \) most of the time.

Lemma 2.3. Assume that integers \( l_2 \geq l_1 \geq 0 \) and real \( \beta_1, \beta_2 \) satisfy
\[
  l_2 - \beta_2 \geq l_1 - \beta_1.
\]
For any \( v \in V_{\beta_2}^{l_2}(K) \) with a compact support of size \( \delta \) near \( X_c \), one can have \( v \in V_{\beta_1}^{l_1}(K) \) such that
\[
  \|v\|_{V_{\beta_1}^{l_1}(K)} \leq \delta^{|l_2 - \beta_2| - (l_1 - \beta_1)} \|v\|_{V_{\beta_2}^{l_2}(K)}.
\]
Moreover, similar results hold for \( V_{\beta}^{l-1/2}(\Gamma_t) \) and \( V_{\beta}^{l-3/2}(\Gamma_b) \) with constants \( C = C(l_1, l_2, \beta_1, \beta_2, \delta) \).

Proof. One only needs to check from the definitions to prove this lemma. In fact, for any \( v \in V_{\beta_2}^{l_2}(K) \) with a compact support of size \( \delta \) near \( X_c \), a simple computation shows that
\[
  \|v(x, \beta_1 - l_1, \alpha) \partial^\alpha v\|_{L^2(K)} = \|v(x, l_2 - \beta_2, \beta_2 - l_1 + |\alpha|) \partial^\alpha v\|_{L^2(K)} \leq \delta^{|l_2 - \beta_2| - (l_1 - \beta_1)} \|v(x, \beta_2 - l_2, -|\alpha|) \partial^\alpha v\|_{L^2(K)}
\]
where \( \partial^\alpha = \partial^{\alpha_1}_x \partial^{\alpha_2}_z \) satisfying \( |\alpha| = \alpha_1 + \alpha_2 \leq l_1 \). Therefore, the case for \( V_{\beta_2}^{l_2}(K) \) is proved, and the other cases can be done similarly. \( \square \)

Lemma 2.4. Let \( v \) and \( f \) be two functions on \( K \) and \( \Gamma_t \) (or \( \Gamma_b \)) respectively with a compact support of size \( \delta \) near \( X_c \).
(i) When \( v \in H^2(K) \), one has \( v \in V_2^2(K) \) satisfying
\[
  \|v\|_{V_2^2(K)} \leq C \|v\|_{H^2(K)};
\]
(ii) When \( v \in L^2(K) \), one has \( v \in V_0^2(K) \) satisfying
\[
  \|v\|_{V_0^2(K)} \leq C \|v\|_{L^2(K)};
\]
(iii) When \( f \in H^{3/2}(\Gamma_t) \), one has \( f \in V_2^{3/2}(\Gamma_t) \) satisfying
\[
  \|f\|_{V_2^{3/2}(\Gamma_t)} \leq C \|f\|_{H^{3/2}(\Gamma_t)}.
\]

Similar inequality holds also for the case from \( H^{3/2}(\Gamma_b) \) to \( V_2^{1/2}(\Gamma_b) \). Moreover, the constant \( C \) above depends on \( \delta, K \).
Proof. The first two cases can be proved in a similar way as in Lemma 2.3, and it
only remains to check (iii).

In fact, when \( f \in V^{3/2}_2(\Gamma_t) \), one knows directly from the definition that
\[
\|f\|_{V^{3/2}_2(\Gamma_t)}^2 = \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} r^2 f(r)^2 dr + \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} r^3 |f'(r)|^2 dr + \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} r^2 \frac{|f(r) - f(\rho)|^2}{|r - \rho|^2} dr d\rho
\]
\[
\quad + \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} r^2 |f'(r) - \rho f'(\rho)|^2 dr d\rho
\]
\[
:= A_1 + A_2 + A_3 + A_4
\]
where the first two terms can be handled easily since \( f \) is compactly supported near \( X_c \):
\[
A_1 + A_2 \leq (\delta + \delta^3)\|f\|^2_{H^{1/2}(\Gamma_t)}.
\]
Now it remains to take care of the last two terms. To begin with, one has
\[
A_3 \leq \int_{\mathbb{R}^+} \int_{\rho/2}^{2\rho} \ldots dr d\rho + \int_{\mathbb{R}^+} \int_{0}^{\rho/2} \ldots dr d\rho + \int_{\mathbb{R}^+} \int_{2\rho}^{+\infty} \ldots dr d\rho
\]
\[
:= A_{31} + A_{32} + A_{33},
\]
where a direct analysis shows that
\[
A_{31} \leq \delta^2 C \int_{\mathbb{R}^+} \int_{\rho/2}^{2\rho} \frac{|f(r) - f(\rho)|^2}{|r - \rho|^2} dr d\rho \leq \delta^2 C\|f\|^2_{H^{1/2}(\Gamma_t)}
\]
since one has \( r \sim \rho \) in this case and remember that \( f \) is compactly supported near \( X_c \). Moreover, one can also have
\[
A_{32} \leq C \left( \int_{\mathbb{R}^+} \int_{2\rho}^{\infty} r^2 |f(r)|^2 dr + \int_{\mathbb{R}^+} \int_{\rho/2}^{\rho} \frac{1}{|r - \rho|^2} d\rho dr + \int_{\mathbb{R}^+} \int_{0}^{\rho/2} |f(\rho)|^2 \frac{r^2}{|r - \rho|^2} dr d\rho \right)
\]
\[
\leq C \int_{\mathbb{R}^+} |f(r)|^2 dr \leq \delta C\|f\|^2_{H^{1/2}(\Gamma_t)},
\]
and a similar inequality holds for \( A_{33} \). Consequently, we arrive at
\[
A_3 \leq \delta C\|f\|^2_{H^{3/2}(\Gamma_t)}.
\]
On the other hand, similar computations can be done for the term \( A_4 \). Therefore, the proof for the case \( H^{3/2}(\Gamma_t) \) is finished.

Lemma 2.5. Let \( f \in V^{3/2}_0(\Gamma_t) \) and \( g \in V^{1/2}_0(\Gamma_b) \) be functions compactly supported near \( X_c \) of \( K \) with a size \( \delta \). Then one has \( f \in H^{3/2}(\Gamma_t) \) and \( g \in H^{1/2}(\Gamma_b) \) satisfying
\[
\|f\|_{H^{3/2}(\Gamma_t)} \leq C\|f\|_{V^{3/2}_0(\Gamma_t)}, \quad \|g\|_{H^{1/2}(\Gamma_b)} \leq C\|g\|_{V^{1/2}_0(\Gamma_b)},
\]
where the constant \( C \) depends on \( \delta, K \).

Proof. The proof can be done similarly as in the previous lemma. In fact, using the definition of \( H^{3/2}(\Gamma_t) \) and \( V^{3/2}_0(\Gamma_t) \), one writes directly that
\[
\|f\|_{H^{3/2}(\Gamma_t)}^2 = \|f\|^2_{H^{1}(\Gamma_t)} + \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{|f(r) - f(\rho)|^2}{|r - \rho|^2} dr d\rho
\]
and
\[
|f|_{V^{3/2}_0(\Gamma_t)}^2 = \|r^{-3/2}f\|^2_{L^2(\Gamma_t)} + \|r^{-1/2}f'\|^2_{L^2(\Gamma_t)} + \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} r^{-2} \frac{|f(r) - f(\rho)|^2}{|r - \rho|^2} dr d\rho
\]
\[
+ \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} r^{-2} |r| f'(r) - \rho f'(\rho)|^2 dr d\rho.
\]
Since \( f \) is supported near \( X \) with a size \( \delta \), one can easily see that
\[
\|f\|_{H^1(\Gamma_1)}^2 = \|r^{3/2}r^{-3/2}f\|_{L^2(\Gamma_1)}^2 + \|r^{1/2}r^{-1/2}f\|_{L^2(\Gamma_1)}^2 \leq \delta C\|f\|_{V_0^{3/2}(\Gamma_1)}^2,
\]
so it remains to check the last term in \( H^{3/2} \) norm.

Similarly as before, one can write
\[
\begin{align*}
\int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \frac{|f'(r) - f'(\rho)|^2}{|r - \rho|^2} dr d\rho & = \int_{\mathbb{R}^+} \int_{\rho/2}^{2\rho} \frac{r f'(r) - r f'(\rho)}{|r - \rho|^2} dr d\rho + \int_{\mathbb{R}^+} \int_{0}^{\rho/2} \frac{|r f'(r)|}{|r - \rho|^2} dr d\rho + \int_{\mathbb{R}^+} \int_{2\rho}^{+\infty} \frac{|f'(\rho)|}{|r - \rho|^2} dr d\rho + \int_{\mathbb{R}^+} \int_{\rho/2}^{+\infty} \frac{|r f'(r)|}{|r - \rho|^2} dr d\rho \\
& := A_1 + A_2 + A_3.
\end{align*}
\]
Direct computations show that
\[
A_1 = \int_{\mathbb{R}^+} \int_{\rho/2}^{2\rho} \frac{r f'(r) - r f'(\rho)}{|r - \rho|^2} dr d\rho 
\]
\[
\leq C \left( \int_{\mathbb{R}^+} \int_{\rho/2}^{2\rho} \frac{|r f'(r) - \rho f'(\rho)|^2}{|r - \rho|^2} dr d\rho + \int_{\mathbb{R}^+} \int_{\rho/2}^{2\rho} \frac{r^2 |f'(\rho)|^2 dr d\rho}{} \right) 
\]
\[
\leq C\|f\|_{V_0^{3/2}(\Gamma_1)},
\]
and \( A_2, A_3 \) can be handled similarly as before. Consequently, the case of \( H^{3/2}(\Gamma_1) \) is proved. Moreover, the case of \( H^{1/2}(\Gamma_1) \) can also be proved similarly. \( \square \)

The following lemma deals with the relationship between \( V_{\beta}^1(\mathcal{K}) \) and \( W_{2, \beta}^l(\mathcal{C}) \), which is quoted directly from (6.1.6) and (6.1.7) [14].

**Lemma 2.6.** Let \( r = e^t \) with \( (r, \theta) \) polar coordinates and denote \( w(t, \theta) = v(r, \theta) \), where \( v(r, \theta) \) is defined on \( \mathcal{K} \). Then \( w(t, \theta) \) is defined on \( \mathcal{C} \) and there exist constants \( C_1, C_2 \) depending on \( l, \beta \) and \( \mathcal{K} \) such that
\[
C_1 \|w\|_{V_{l, \beta}^{1-1}(\Gamma_1)} \leq \|v\|_{V_{\beta}^{1}(\mathcal{K})} \leq C_2 \|w\|_{W_{l, \beta}^{1-1}(\Gamma_1)} \text{ i.e. } \|v\|_{V_{\beta}^{1}(\mathcal{K})} \simeq \|w\|_{W_{l, \beta}^{1-1}(\Gamma_1)}.
\]
Moreover, similar results hold on the boundary:
\[
\|v\|_{V_{l, \beta}^{1/2}(\Gamma_1)} \simeq \|w\|_{W_{l, \beta}^{1/2}(\Gamma_1)} \text{ and } \|v\|_{V_{l, \beta}^{1/2}(\Gamma_1)} \simeq \|w\|_{W_{l, \beta}^{1/2}(\Gamma_1)}.
\]

In the end of this section, we introduce the Laplace transform \( \mathcal{L} \) acting on any \( w \in C_0^\infty(\mathbb{R}) \):
\[
\tilde{w}(\lambda) = \mathcal{L}w(\lambda) = \int_\mathbb{R} e^{-\lambda t}w(t)dt, \quad \forall \lambda \in \mathbb{C}.
\]
Some well-known properties of this transform are recalled below, quoted directly from Lemma 5.2.3 [14].

**Lemma 2.7.** (i) The transform defines a linear and continuous mapping from \( C_0^\infty(\mathbb{R}) \) into the space of analytic functions on the complex plane \( \mathbb{C} \). Further more, one has
\[
\mathcal{L}(\partial_t w) = \lambda \mathcal{L}w
\]
(ii) For all \( u, v \in C_0^\infty(\mathbb{R}) \), the Parseval equality
\[
\int_{-\infty}^{+\infty} e^{2\beta t}u(t)v(t)dt = \frac{1}{2\pi i} \int_{Re\lambda = -\beta} \tilde{u}(\lambda) \tilde{v}(\lambda)d\lambda
\]
where $u$ denotes the conjugation of $z$. Then there exists a unique solution $u$

A word, we focus on the weighted estimates near the corner in the following text. Sobolev spaces are used, so standard elliptic estimates can be applied directly. In our paper). When it is away from the corner, ordinary weighted spaces near the corner, and the elliptic estimates need to be proved (which corner and away from the corner are treated in completely different ways. We use weighted estimates is proved by an induction argument, and the

Lemma 3.1. Suppose that $V^l(\Omega)$ and $W^{l-1/2}(\Gamma_\ell)$, we recall from \cite{14} the following lemma. The norm $2$ with an integer $l \geq 0$ is equivalent to the norm

\[ \|w\| = \left( \frac{1}{2\pi i} \int_{\Re\lambda = -\beta} \|\tilde{w}\|^2_{H^1(I,\lambda)} d\lambda \right)^{1/2}, \]

where

\[ \|\tilde{w}\|_{H^1(I,\lambda)} = \left( \|\tilde{w}(\lambda,\cdot)\|^2_{H^1(I)} + |\lambda|^{2l} \|\tilde{w}(\lambda,\cdot)\|^2_{L^2(I)} \right)^{1/2}. \]

Analogously, an equivalent norm to 3 for $W^{l-1/2}(\Gamma_\ell)$ ($l \geq 1$) on $\Gamma_\ell$ is

\[ \|w\| = \left( \frac{1}{2\pi i} \int_{\Re\lambda = -\beta} (1 + |\lambda|^{2l-1}) |\tilde{w}(\lambda,\omega_1)|^2 d\lambda \right)^{1/2}, \]

and there is a similar norm for the case of $W^{l-3/2}(\Gamma_b)$.

3. Estimates for the mixed boundary problem. We start with the existence of the solution to (MBVP) in certain weighted space, and then the regularity is considered. The weighted estimates is proved by an induction argument, and the dependence of the upper boundary is traced at the same time.

To begin with, one must consider about the existence of the solution in proper weighted space, which we wish to be built on the existence result in ordinary Sobolev spaces from \cite{1, 24}. In fact, recalling Theorem 5.2 and Remark 5.3 \cite{24}, we state the following lemma for the unique existence of the solution in $H^2(\Omega)$.

Lemma 3.1. Suppose that $\Omega$ has a $C^{2,0}$ upper boundary $\Gamma_\ell$. Let functions $h \in L^2(\Omega)$, $f \in H^{3/2}(\Gamma_\ell)$ and $g \in H^{1/2}(\Gamma_b)$ be given and the contact angle $\omega \in (0, \pi/2)$. Then there exists a unique solution $u \in H^2(\Omega)$ to (MBVP).

One can see from the definition of $V^l(\Omega)$ that, the parts of the norm near the corner and away from the corner are treated in completely different ways. We use weighted spaces near the corner, and the elliptic estimates need to be proved (which is the key ingredient in our paper). When it is away from the corner, ordinary Sobolev spaces are used, so standard elliptic estimates can be applied directly. In a word, we focus on the weighted estimates near the corner in the following text.

Recalling from 1 that, we defined on $K$ the function

\[ v_c = u_c \circ T_c \quad \text{with} \quad u_c = \chi_c u \]

where $u \in H^2(\Omega)$ the solution to (MBVP).

Some computations as in \cite{24} show that $v_c$ satisfies the system

\[
\begin{align*}
\nabla \cdot P_c \nabla v_c &= h_c \quad \text{on} \quad K \\
v_c|_{\Gamma_\ell} &= f_c, \quad \partial_{\nu_{\gamma_c}} v_c|_{\Gamma_b} = (1 + \gamma^2)^{1/2} g_c.
\end{align*}
\]

(9)
where \(-\gamma\) is the constant slope of \(\Gamma_t\) near \(X_c\) and
\[
h_c = (\chi_c \cdot h - [\chi_c, \Delta] u) \circ T_c, \quad f_c = (\chi_c \cdot f) \circ T_c, \quad g_c = (\chi_c \mid_{\Gamma_t} g - (\partial_{n_b} \chi_c) u \mid_{\Gamma_b}) \circ T_c.
\]
Besides, the coefficient matrix is
\[
P_c = (P_0^{-1})^t A P_0^{-1},
\]
where
\[
P_S = (\nabla T_S^{-1}) \circ T_S = \begin{pmatrix} 1 + \gamma d(\gamma x + z) & \gamma \\ d(\gamma x + z) & 1 \end{pmatrix}, \quad P_0 = P_S |_{X_c} = \begin{pmatrix} 1 + \gamma d(0) & \gamma \\ d(0) & 1 \end{pmatrix}.
\]

To prove the main theorem, we need to focus on system 9 for \(v_c\). First of all, under the assumptions of Theorem 1.1 and combining Lemma 3.1, one finds immediately that
\[
v_c \in H^2(\mathcal{K}) \quad \text{with} \quad h_c \in L^2(\mathcal{K}), \quad f_c \in H^{3/2}(\Gamma_t), \quad g_c \in H^{1/2}(\Gamma_b), \tag{10}
\]
while notice that all functions are compactly supported near \(X_c\).

Now we are in a position to introduce proper weighted spaces for the system of \(v_c\). In fact, combining Lemma 2.4, one has immediately
\[
v_c \in V^2_2(\mathcal{K}), \quad h_c \in V^0_2(\mathcal{K}), \quad f_c \in V^{3/2}_2(\Gamma_t), \quad g_c \in V^{1/2}_2(\Gamma_b) \tag{11}
\]
where the weight \(\beta = 2\). Based on these spaces, we improve the regularity of \(v_c\) in the following two subsections. One will see that, when the contact angle \(\omega \in (0, \pi/2)\) and a proper weight \(\beta\) is chosen, there is no extra singularity when higher regularity is considered.

3.1. \textbf{Lower-order regularity near the corner.} So far, \(v_c\) belongs to \(V^2_2(\mathcal{K})\) with the weight \(\beta = 2\). The aim of this subsection is to show that \(v_c\) also belongs to \(V^0_2(\mathcal{K})\) with a lower weight \(\beta = 0\), which is a very important step and leads us to the proper weighted space \(V^{l-2}_l(\mathcal{K})\).

\textbf{Proposition 1.} Let \(v_c\) be the solution to 9 and 10 holds. Moreover, for a real \(\beta \in [0, 2]\) one assumes that
\[
h_c \in V^0_{\beta}(\mathcal{K}), \quad f_c \in V^{3/2}_{\beta}(\Gamma_t), \quad g_c \in V^{1/2}_{\beta}(\Gamma_b),
\]
and
\[
\|\eta\|_{W^{2,\infty}} \leq C_0
\]
for some constant \(C_0\).

Then one has \(v_c \in V^2_2(\mathcal{K})\) and the weighted estimate holds
\[
\|v_c\|_{V^2_2(\mathcal{K})} \leq C \left( \|h_c\|_{V^0_{\beta}(\mathcal{K})} + \|f_c\|_{V^{3/2}_{\beta}(\Gamma_t)} + \|g_c\|_{V^{1/2}_{\beta}(\Gamma_b)} \right), \tag{12}
\]
where the constant \(C = C(\mathcal{K}, \beta)\).

Before we prove this proposition, some preparations are needed. Firstly, let
\[
\mathcal{B} = (\nabla \cdot P_c \nabla, \cdot |_{\Gamma_t}, \partial_{n_b} \cdot |_{\Gamma_b})
\]
be the elliptic operator for system 9. In particular, since direct computations show that
\[
P_c |_{X_c} = Id,
\]
we denote
\[
\mathcal{B}_0 = \mathcal{B} |_{X_c} = (\Delta, \cdot |_{\Gamma_t}, \partial_{n_b} \cdot |_{\Gamma_b})
\]
as the restriction of coefficients in \(\mathcal{B}\) on the contact point \(X_c\).
System 9 can be rewritten into a perturbation form of operator $B_0$ near the contact point $X_c$:

$$B_0 v_c = (h_c, f_c, g_c) - (B - B_0)v_c$$

or equivalently the following system

$$\begin{cases}
\Delta v_c = h_v, & \text{on } K \\
v_c|_{\Gamma} = f_v, & \partial_{n_b}v_c|_{\Gamma_b} = g_v
\end{cases}$$

(13)

where

$$h_v = h_c - \nabla \cdot (P_c - Id)\nabla v_c, \ f_v = f_c, \text{ and } g_v = (1 + \gamma^2)^{1/2} g_c - \partial^{P_c-Id}_{n_b} v_c|_{\Gamma_b}.$$

**Lemma 3.2.** Under the assumptions of Proposition 1 with $\beta \in [0, 2]$, there exists a number $\varepsilon \in (0, 1)$ depending on $\beta$ such that

$$h_v \in V^1_{1+\varepsilon}(K), \ f_v \in V^3_{1+\varepsilon}(K), \text{ and } g_v \in V^{1/2}_{1+\varepsilon}(\Gamma_b)$$

in system 13.

**Proof.** Since one assumes that $h_c \in V^0_{\beta}(K)$, applying Lemma 2.3 with $l_2 = l_1 = 0$, $\beta_2 = \beta$ and $\beta_1 = 1 + \varepsilon$ on $h_c$ leads to

$$h_c \in V^0_{1+\varepsilon}(K).$$

Here one requires that $\beta \leq 1 + \varepsilon$. Similarly, applying Lemma 2.3 with $l_2 = l_1 = 2$, $\beta_2 = \beta$ and $\beta_1 = 1 + \varepsilon$ on $f_c$ and $g_c$ leads to

$$f_c \in V^{3/2}_{1+\varepsilon}(\Gamma_b), \ g_c \in V^{1/2}_{1+\varepsilon}(\Gamma_b).$$

It remains to deal with perturbation terms

$$(B - B_0)v_c = (\nabla \cdot (P_c - Id)\nabla v_c, 0, \partial^{P_c-Id}_{n_b} v_c|_{\Gamma_b}).$$

Firstly, one can show directly that

$$r^{1+\varepsilon} \nabla \cdot (P_c - Id)\nabla v_c \in L^2(K)$$

since one has $v_c \in H^2(K)$ with a compact support near $X_c$ and the assumption for $\eta$ in Proposition 1. This infers that $\nabla \cdot (P_c - Id)\nabla v_c \in V^0_{1+\varepsilon}(K)$.

On the other hand, the boundary term can be written as

$$\partial^{P_c-Id}_{n_b} v_c|_{\Gamma_b} = n_b \cdot (P_c - Id)\nabla v_c|_{\Gamma_b},$$

so one can show in a similar way as above that

$$(P_c - Id)\nabla v_c \in V^1_{1+\varepsilon}(K).$$

Applying Lemma 2.1 on $\Gamma_b$, one has immediately that $\partial^{P_c-Id}_{n_b} v_c|_{\Gamma_b} \in V^{1/2}_{1+\varepsilon}(\Gamma_b)$.

Summing up these results above, one can finish the proof. $\square$

For the moment, we are ready to change the weight for $v_c$, that is, from $V^2_{\beta}(K)$ to $V^{2}_{1+\varepsilon}(K)$. Concerning elliptic systems on corner domains, it is well known that one will meet with singularities most of the time when one wants to consider about two different spaces, see for example [9, 14]. The key lemma below tells us that in our settings with the contact angle $\omega \in (0, \pi/2)$, no singularity happens when we choose the space carefully. Moreover, we only investigate about the proper space without establishing any estimate at this time.
Proposition 2. Let the contact angle \( \omega \in (0, \pi/2) \). Assume that system 13 admits a solution \( v_c \in V^2_2(\mathcal{K}) \) with
\[ h_v \in V^0_{1+\varepsilon}(\mathcal{K}) \cap V^0_2(\mathcal{K}), \quad f_v \in V^{3/2}_2(\Gamma_1) \cap V^{3/2}_2(\Gamma_t) \quad \text{and} \quad g_v \in V^{1/2}_2(\Gamma_b) \cap V^{1/2}_2(\Gamma_b), \]
then one has \( v_c \in V^2_{0+\varepsilon}(\mathcal{K}) \) without any singularity decomposition.

Proof. The idea of this proof follows the proofs for Theorem 5.4.1 and Theorem 6.1.4 [14]. In fact, we convert system 9 on the cone \( \mathcal{K} \) equivalently to a system on a horizontal strip, and then the Laplace transform is applied to derive the related eigenvalue problem. As a result, the solution \( v_c \) under Laplace transform could be expressed through an ODE. Based on some analysis on eigenvalues, we are able to use Cauchy’s Formula to show that \( v_c \) eventually lies in the desired weighted space.

Step 1. Change of variable. First of all, system 13 can be rewritten under polar coordinates:
\[
\begin{align*}
\{(r^2 \partial_r^2 + \partial_\theta^2) v_c &= r^2 h_v \quad \text{on} \quad \mathcal{K} \\
v_c |_{\theta = \omega_1} &= f_v, \quad -\partial_\theta v_c |_{\theta = -\omega_2} = r g_v.
\end{align*}
\]
Secondly, introducing the following change of variable
\[ t = \ln r, \quad \text{i.e.} \quad r = e^t \quad \text{for} \quad \forall t \in \mathbb{R} \]
and denoting
\[ w(t, \theta) = v_c(r, \theta), \]
the system above for \( v_c \) can be changed equivalently into the system for \( w(t, \theta) \) on an infinite strip \( \mathcal{C} = \mathbb{R} \times [-\omega_2, \omega_1] \):
\[
\begin{align*}
\{(r^2 \partial_r^2 + \partial_\theta^2) w &= e^{2t} h_w \quad \text{on} \quad \mathcal{C} \\
w |_{\theta = \omega_1} &= f_w, \quad -\partial_\theta w |_{\theta = -\omega_2} = e^t g_w
\end{align*}
\]
with the notations
\[ h_w(t, \theta) = h_v(r, \theta), \quad f_w(t, \theta) = f_v(r, \theta), \quad \text{and} \quad g_w(t, \theta) = g_v(r, \theta). \]
As a result, applying the assumptions of this proposition and Lemma 2.6 on \( v_c, h_v, f_v \) and \( g_v \), one derives immediately that
\[ w \in W^{2, 1}_{2, 1}(\mathcal{C}), \]
and the right side of 14 satisfies
\[
e^{2t} h_w \in W^0_{2, 1}(\mathcal{C}) \cap W^0_{2, 1}(\mathcal{C}), \quad f_w \in W^{3/2}_{2, 1}(\Gamma_t) \cap W^{3/2}_{2, 1}(\Gamma_t), \quad e^t g_w \in W^{1/2}_{2, 1}(\Gamma_b) \cap W^{1/2}_{2, 1}(\Gamma_b).
\]

Step 2. Laplace transform to an ODE. One performs the Laplace transform on \( w(t, \theta) \) with respect to \( t \) and denote
\[ \tilde{w}(\lambda, \cdot) = (\mathcal{L} w)(\lambda, \cdot) = \int_{\mathbb{R}} e^{-\lambda t} w(t, \cdot) dt, \quad \forall \lambda \in \mathbb{C} \]
Applying Lemma 2.7 (i), one arrives at the system for \( \tilde{w}(\lambda, \cdot) \) from 14:
\[
\begin{align*}
\lambda^2 \tilde{w} + \partial_\theta^2 \tilde{w} &= \mathcal{L}(e^{2t} h_w), \quad \lambda \in I \\
\tilde{w} |_{\theta = \omega_1} &= \mathcal{L}(f_w), \quad -\partial_\theta \tilde{w} |_{\theta = -\omega_2} = \mathcal{L}(e^t g_w),
\end{align*}
\]
and one knows from 15 and Lemma 2.8 that
\[ \tilde{w} \in H^2(I, \lambda), \quad \mathcal{L}(e^{2t} h_w) \in L^2(I, \lambda). \]
One can see that our system 9 turns into an ordinary differential system with parameter \( \lambda \), which becomes more handy.
We denote by
\[ U(\lambda) = \left( -\partial^2_\theta - \lambda^2, \mid \psi = \omega_1, -\partial_\theta \mid \psi = -\omega_2 \right) \]
the operator of system 16 with parameter \( \lambda \in \mathbb{C} \). For each fixed \( \lambda \), \( U(\lambda) \) continuously maps \( H^l(I) \) into \( H^{l-2}(I) \) for any \( l \geq 2 \) with corresponding boundary values.

A direct computation shows that the corresponding eigenvalue problem for \( U(\lambda) \) reads
\[
\begin{cases}
-\phi''(\theta) - \lambda^2 \phi(\theta) = 0, & \theta \in I \\
\phi(\omega_1) = 0, & -\phi'(-\omega_2) = 0,
\end{cases}
\]
where the eigenvalues are countable and real with the explicit expressions
\[ \lambda_m = \frac{(m + 1/2)\pi}{\omega} \quad \text{for } \forall m \in \mathbb{Z}. \]

By the way, the eigenfunctions are \( \phi_m(\theta) = \cos(\lambda_m(\theta + \omega_2)) \). In fact, these eigenvalues and eigenfunctions coincide with those in [24], which is characteristic for the mixed-type elliptic problem.

Since the contact angle \( \omega \) is assumed to be in \( (0, \pi/2) \) in this paper, one finds immediately that
\[ \lambda_m \notin [-1, 1] \quad \forall m \in \mathbb{Z}, \]
which implies
\[ U(\lambda) \text{ is invertible when } \lambda \in [-1, 1]. \]

**Step 3.** Singularity decomposition without singularity. For this moment, we plan to show that \( w \in W^2_{2,0}(\mathcal{C}) \) by solving system 16. First of all, we will start from expressing \( \tilde{w} \in H^2(I, \lambda) \) in terms of the right hand side.

In fact, when we take \( \text{Re}\lambda = -1 \), system 16 is uniquely solvable in \( H^2(I) \) (which is already known since system 14 admits a solution \( w \in W^2_{2,1}(\mathcal{C}) \)).

Moreover, one can express the solution as below
\[ \tilde{w}(\lambda, \theta) = U(\lambda)^{-1}(\mathcal{L}(\text{e}^{2t} h_w), \mathcal{L}(f_w), \mathcal{L}(\text{e}^t g_w)) \in H^2(I, \lambda), \]
where \( \mathcal{L}(\text{e}^{2t} h_w) \) satisfies 17 with \( \text{Re}\lambda = -1 \).

Applying the inverse Laplace transform and Lemma 2.7 (iii), one obtains
\[ w(t, \theta) = \frac{1}{2\pi i} \int_{\text{Re}\lambda=-1} e^{\lambda t} \tilde{w}(\lambda, \theta)d\lambda \in W^2_{2,1}(\mathcal{C}). \quad (18) \]

From Lemma 2.7 (iv), one can see that for each \( \theta \in I \), \( \mathcal{L}(\text{e}^{2t} h_w), \mathcal{L}(f_w) \) and \( \mathcal{L}(\text{e}^t g_w) \) are holomorphic in the strip \( -1 < \text{Re}\lambda < -\varepsilon \). Therefore, the only singularities of the function
\[ e^{\lambda t} \tilde{w}(\lambda, \theta) = e^{\lambda t} U(\lambda)^{-1}(\mathcal{L}(\text{e}^{2t} h_w), \mathcal{L}(f_w), \mathcal{L}(\text{e}^t g_w)) \]
from 18 in the strip \( -1 < \text{Re}\lambda < -\varepsilon \) are the poles of \( U(\lambda)^{-1} \), i.e. the eigenvalues of \( U(\lambda) \). Combining previous analysis on \( U(\lambda) \), this implies immediately that no singularity takes place in the strip \( -1 < \text{Re}\lambda < -\varepsilon \).

Now we are in a position to show that \( w \in W^2_{2,0}(\mathcal{C}) \). In fact, let \( \rho > 0 \) to be a constant, then the complex domain
\[ D_\rho = \{ \lambda \in \mathbb{C} \mid -1 < \text{Re}\lambda < -\varepsilon, \ |Im\lambda| > \rho \} \]
doesn’t contain any eigenvalue of \( U(\lambda) \).
Lemma 3.3. Let \( w(t, \theta) \) be the operator of system 16 for \( \omega(\lambda, \theta) \) with \( (h_w, f_w, g_w) \) satisfying 15. Then there holds

\[
\lim_{\rho \to +\infty} \int_{-1-i\rho}^{-1+i\rho} e^{\lambda t} \tilde{w}(\lambda, \theta) d\lambda = 0.
\]

Consequently, with the help of this lemma we arrive at

\[
w(t, \theta) = \frac{1}{2\pi i} \lim_{\rho \to +\infty} \int_{-\varepsilon-i\rho}^{-\varepsilon+i\rho} e^{\lambda t} \tilde{w}(\lambda, \theta) d\lambda = \frac{1}{2\pi i} \int_{\Re \lambda = -\varepsilon} e^{\lambda t} \tilde{w}(\lambda, \theta) d\lambda.
\]

Recalling that \( \tilde{w} \in H^2(I, \lambda) \), we can finally conclude with Lemma 2.7 (ii) (iii) that \( w \in W^2_{2,\varepsilon}(\mathcal{C}) \).

Therefore, we apply Lemma 2.6 to find

\[
v_c \in V^2_{1+\varepsilon}(\mathcal{K}) \cap V^2_2(\mathcal{K})
\]

and the proof is finished. \( \square \)

Proof of Lemma 3.3. Since this proof is adapted from the proof of Lemma 5.4.1 [14], we only sketch the main idea here to be self-contained.

Firstly, let

\[
w_\rho(t, \theta) = \int_{-1+i\rho}^{-\varepsilon+i\rho} e^{\lambda t} \tilde{w}(\lambda, \theta) d\lambda
\]

and

\[
w_\rho(t, \theta) = \frac{1}{2\pi i} \int_{-1+i\rho}^{-\varepsilon+i\rho} e^{\lambda t} \tilde{w}(\lambda, \theta) d\lambda.
\]

Taking \( L^2 \) norm on \( C_N = [-N, N] \times I \) for a constant \( N > 0 \), one has

\[
\| w_\rho \|^2_{L^2(C_N)} = \int_{C_N} |w_\rho(t, \theta)|^2 dt d\theta \leq C \int_I \int_{-\varepsilon+i\rho}^{-\varepsilon+i\rho} |\tilde{w}(\lambda, \theta)|^2 d\lambda d\theta.
\]

On the other hand, checking from Theorem 3.6.1 [14] one finds the elliptic estimate for system 16 of \( \omega(\lambda, \theta) \):

\[
\| \tilde{w} \|_{H^2(I, \lambda)} \leq C \left( \| \mathcal{L}(e^{2t} h_w) \|_{H^2(I, \lambda)} + (1 + |\lambda|^{1/2}) |\mathcal{L}(f_w)| + (1 + |\lambda|^{1/2}) |\mathcal{L}(e^t g_w)| \right),
\]

which implies immediately

\[
\int_I |\tilde{w}(\lambda, \theta)|^2 d\theta = \| \bar{w}(\lambda, \cdot) \|^2_{L^2(I)}
\]

\[
\leq C|\lambda|^{-2} \left( \| \mathcal{L}(e^{2t} h_w) \|^2_{H^{-2}(I, \lambda)} + (1 + |\lambda|) |\mathcal{L}(f_w)|^2 + (1 + |\lambda|^{1/2}) |\mathcal{L}(e^t g_w)|^2 \right).
\]
As a result, one can show that
\[
\int_{c_1}^{c_2} \|w_\rho\|_{L^2(CN)}^2 \, d\rho \leq C \int_{c_1}^{c_2} \int_{-1+i\rho}^{c+1+i\rho} \left( \|\mathcal{L}(e^{2t} h_w)\|_{H^{l-2}(I,\lambda)}^2 + (1 + |\lambda|^{2l-1})|\mathcal{L}(f_w)|^2 \right) \, d\lambda \, d\rho
\]
with the constant \( C = C(N, c_1) \). Rewriting this double integral by changing the order of the integration, one derives
\[
\int_{c_1}^{c_2} \|w_\rho\|_{L^2(CN)}^2 \, d\rho \leq C \int_{c_1}^{c_2} \int_{-\lambda-\beta}^{1} \left( \|\mathcal{L}(e^{2t} h_w)\|_{H^{l-2}(I,\lambda)}^2 + (1 + |\lambda|^{2l-1})|\mathcal{L}(f_w)|^2 \right) \, d\lambda \, d\beta,
\]
which together with Lemma 2.8 leads to
\[
\int_{c_1}^{c_2} \|w_\rho\|_{L^2(CN)}^2 \, d\rho \leq C \int_{c_1}^{c_2} \left( \|e^{2t} h_w\|_{W^{l-2}_{2,\beta}(\Omega)}^2 + \|f_w\|_{W^{l-1/2}_{2,\beta}(\Gamma_{\theta})}^2 + \|g_w\|_{W^{1/2}_{2,\beta}(\Gamma_{\theta})}^2 \right) \, d\beta.
\]
Therefore, combining 15, one knows that \( \|w_\rho\|_{L^2(CN)} \) is also square integrable over the interval \((c_1, \infty)\) and the proof can be finished.

In order to prove the estimate in Proposition 1, we quote the weighted elliptic estimate for system 13 in the following lemma, which can be found in Theorem 6.1.1 [14]. Notice that this lemma holds due to the previous analysis on \( \mathcal{U}(\lambda) \): No eigenvalues of \( \mathcal{U}(\lambda) \) lie on the line \( Re\lambda = -\beta + l - 1 \), where we take \( \beta \in [0, 2] \) and \( l = 2 \) here.

**Lemma 3.4.** Let \( \beta \in [0, 2] \). Assume that there exists a solution \( v_c \in V^\beta_{2}(\mathcal{K}) \) for system 13 with \( h_v \in V^0_{\beta}(\mathcal{K}) \), \( f_v \in V^{3/2}_{\beta}(\Gamma_{\theta}) \) and \( g_v \in V^{1/2}_{\beta}(\Gamma_{\theta}) \). Then there holds
\[
\|v_c\|_{V^\beta_{2}(\mathcal{K})} \leq C \left( \|h_v\|_{V^0_{\beta}(\mathcal{K})} + \|f_v\|_{V^{3/2}_{\beta}(\Gamma_{\theta})} + \|g_v\|_{V^{1/2}_{\beta}(\Gamma_{\theta})} \right),
\]
where the constant \( C = C(\mathcal{K}, \beta) \).

Now we are ready to prove Proposition 1.

**Proof for Proposition 1.** Firstly, one needs to show that \( v_c \in V^\beta_{2}(\mathcal{K}) \). The case when \( \beta = 2 \) has been proved in 11. For the case when \( \beta \in (0, 2) \), applying Lemma 3.2, one can see that the assumptions of Proposition 2 are satisfied for some \( \varepsilon \in (0, 1) \) depending on \( \beta \). As a result, one knows from Proposition 2 that \( v_c \in V^{\varepsilon}_{2}(\mathcal{K}) \). Repeating this procedure finite times to reach a lower weight at each time, one can finally show that \( v_c \in V^\beta_{2}(\mathcal{K}) \).

Secondly, to prove the weighted estimate 12, one applies Lemma 3.4 on system 13 to derive
\[
\|v_c\|_{V^\beta_{2}(\mathcal{K})} \leq C \left( \|h_v\|_{V^0_{\beta}(\mathcal{K})} + \|f_v\|_{V^{3/2}_{\beta}(\Gamma_{\theta})} + \|g_v\|_{V^{1/2}_{\beta}(\Gamma_{\theta})} \right)
\]
with the constant \( C = C(\mathcal{K}, \beta) \). Substituting the expressions of \( h_v, f_v, g_v \) from system 13, one has
\[
\|v_c\|_{V^\beta_{2}(\mathcal{K})} \leq C \left( \|h_v\|_{V^0_{\beta}(\mathcal{K})} + \|f_v\|_{V^{3/2}_{\beta}(\Gamma_{\theta})} + \|g_v\|_{V^{1/2}_{\beta}(\Gamma_{\theta})} \right)
+ \|\nabla \cdot (P_c - Id) \nabla v_c\|_{V^0_{\beta}(\mathcal{K})} + \|\partial^{P_c}_{n_b} Id v_c\|_{V^{1/2}_{\beta}(\Gamma_{\theta})},
\]
where the last two terms need to be handled.
In fact, recall from system 9 that $P_c = (P_0^{-1})^t P_S^t P_S P_0^{-1}$, which implies
$$P_c - Id = (P_0^{-1})^t (P_S - P_0)^t P_S P_0^{-1} + (P_0^{-1})^t P_0 (P_s - P_0)^t P_0^{-1}$$
where
$$P_S - P_0 = (d(z) - d(0)) \begin{pmatrix} \gamma & 0 \\ 1 & 0 \end{pmatrix} \text{ with } d(z) - d(0) = \frac{\eta'(\bar{\eta}^{-1}(z)) - \eta'(0)}{(\eta'(0) + \gamma)(\eta'(\bar{\eta}^{-1}(z)) + \gamma)}.$$

Here, recall from [24] that we have $\bar{\eta}^{-1}(0) = 0$ since we set $\eta(0) = 0$.

Consequently, one can show directly that
$$\|\nabla \cdot (P_c - Id) \nabla v_c\|_{V^0_2(\mathcal{K})} = \|r^{\beta} \nabla \cdot (P_c - Id) \nabla v_c\|_{L^2(\mathcal{K})} \leq \delta C(\eta\|W_{2,\infty}\|) + C(\|\nabla v_c\|_{L^2(\mathcal{K})})$$
where $\delta$ comes from $d(z) - d(0)$ and remember that $v_c$ is compactly supported near $X_c$ with radius $\delta$. Moreover, the last step is proved using the inequality
$$\|r^{\beta} \nabla v_c\|_{L^2(\mathcal{K})} = \|r^{\beta-1} \nabla v_c\|_{L^2(\mathcal{K})} \leq \delta \|v_c\|_{V^2_\beta(\mathcal{K})}.$$ 

Secondly, one has for the term $\|\partial_{n_k} P_c - Id\|_{V^1_{\beta,2}(\Gamma_k)}$ the following estimate
$$\|\partial^{P_c - Id}_{n_k} v_c\|_{V^1_{\beta,2}(\Gamma_k)} \leq C\|\partial^{P_c - Id}_{n_k} v_c\|_{V^2_{\beta}(\mathcal{K})} \leq C(\|\eta\|_{W_{2,\infty}}) \|v_c\|_{V^2_{\beta}(\mathcal{K})}$$
where the constant vector $n_k$ is extended on $\mathcal{K}$ and Lemma 2.1 is applied. Besides, one can show similarly as before that
$$\|r^{\beta-1} \partial^{P_c - Id}_{n_k} v_c\|_{L^2(\mathcal{K})} = \|r^{\beta-1} n_k \cdot (P_c - Id) \nabla v_c\|_{L^2(\mathcal{K})} \leq \delta C(\eta\|W_{2,\infty}\|) \|v_c\|_{V^2_{\beta}(\mathcal{K})}$$
and
$$\|r^{\beta} \nabla \partial^{P_c - Id}_{n_k} v_c\|_{L^2(\mathcal{K})} \leq \delta C(\eta\|W_{2,\infty}\|) \|v_c\|_{V^2_{\beta}(\mathcal{K})}.$$ 

Summing these up, one arrives at
$$\|\partial^{P_c - Id}_{n_k} v_c\|_{V^1_{\beta,2}(\Gamma_k)} \leq \delta C(\eta\|W_{2,\infty}\|) \|v_c\|_{V^2_{\beta}(\mathcal{K})}.$$ 

As a result, substituting the estimates above into 19, we conclude that
$$\|v_c\|_{V^2_{\beta}(\mathcal{K})} \leq C \left( \|h_c\|_{V^0_{\beta}(\mathcal{K})} + \|f_c\|_{V^{1/2}_{\beta}(\Gamma_t)} + \|g_c\|_{V^{3/2}_{\beta}(\Gamma_t)} + \delta C(\eta\|W_{2,\infty}\|) \|v_c\|_{V^2_{\beta}(\mathcal{K})} \right).$$ 

In the end, using the assumption that $\|\eta\|_{W_{2,\infty}} \leq C_0$, the proof can be finished if one choose $\delta$ small enough depending on $C_0$. \hfill \Box

3.2. higher-order regularity near the corner. In this part, we continue to improve the regularity of $v_c$. At this time, the target space is $V^{l-2+\beta}_{l-2+\beta}(\mathcal{K})$ for integer $l \geq 2$. The case when $l = 2$ is already considered in last subsection. Compared to previous analysis, we don’t meet with singularity here, and standard elliptic theory can be applied locally for the regularity.

**Proposition 3.** Let $l \geq 3$, $\beta \in [0, 2]$ and the contact angle $\omega \in (0, \pi/2)$. Assume that system 13 admits a solution $v_c \in V^{l-1}_{l-3+\beta}(\mathcal{K})$ with
$$h_c \in V^{l-2}_{l-2+\beta}(\mathcal{K}), \quad f_c \in V^{l-1/2}_{l-2+\beta}(\Gamma_t) \text{ and } g_c \in V^{l-3/2}_{l-2+\beta}(\Gamma_k),$$
then one has \( v_c \in V^i_{l-2+\beta}(\mathcal{K}) \) with the estimate

\[
\|v_c\|_{V^i_{l-2+\beta}(\mathcal{K})} \leq C(\|\eta\|_{W^{i-1,\infty}}) \left( \|h_c\|_{V^{i-2}_{l-2+\beta}(\mathcal{K})} + \|f_c\|_{V^{i-1/2}_{l-2+\beta}(\Gamma_1)} + \|g_c\|_{V^{i-1/2}_{l-2+\beta}(\Gamma_b)} \right).
\]

**Proof.** Firstly, we will prove the case when \( l = 3 \). To begin with, we use again the change of variable \( r = e^t \) to convert system 9 of \( v_c \) on \( \mathcal{K} \) to the system of \( w \) on \( \mathcal{C} \), where we denote

\[
w(t, \theta) = v_c(r, \theta).
\]

Direct computations lead to the system for \( w \) as below

\[
U(e^t, \partial_t)w = (e^{2t} h_c(e^t, \theta), f_c(e^t), e^t g_c(e^t)),
\]

where the operator

\[
U(e^t, \partial_t) = (\nabla \cdot P_w \nabla, \mid \Gamma_1, \partial_{\nu_{\mathcal{C}}} P \cdot \mid \Gamma_b).
\]

Here the coefficient matrix reads

\[
P_w(t, \theta) = P^t_{\theta} P_{e^t}(e^t, \theta) P_\theta \quad \text{with} \quad P_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}
\]

and notice that

\[
\nabla = \nabla_{t,\theta}
\]

in the strip domain \( \mathcal{C} \).

Applying Lemma 2.6 and recalling the assumptions of this proposition, we have \( w \in W^2_{2,\beta - 1}(\mathcal{C}) \) and the right side of 20 satisfies

\[
U(e^t, \partial_t)w \in W^2_{2,\beta - 1}(\mathcal{C}) \times W^{5/2}_{2,\beta - 1}(\Gamma_1) \times W^{3/2}_{2,\beta - 1}(\Gamma_b).
\]

We want to show that \( w \in W^3_{2,\beta - 1}(\mathcal{C}) \), which can be done in two steps.

**Step 1.** Localization in \( t \) and standard elliptic estimates for \( \zeta_k w \). Similarly as in [14], let \( \{\zeta_k\}_{k \in \mathbb{Z}} \subset C^\infty_0(\mathbb{R}) \) be a partition of unity with \( \zeta_k \) supported on \( (k - 1, k + 1) \) and satisfying

\[
|\zeta_k^{(j)}(t)| < c_j, \quad \forall t \in \mathbb{R}, \quad j \in \{0, 1, 2, \ldots\}.
\]

Here the constant \( c_j \) doesn’t depend on \( k, t \). Meanwhile, take

\[
\eta_k = \zeta_{k-1} + \zeta_k + \zeta_{k+1},
\]

so one has \( \eta_k \zeta_k = \zeta_k \), i.e. \( \eta_k = 1 \) on the support of \( \zeta_k \).

Recalling that \( w \in W^2_{2,\beta - 1}(\mathcal{C}) \), which implies

\[
\zeta_k w \in H^2(\mathcal{C})
\]

satisfying the system

\[
U(t, \partial_t)\zeta_k w = \zeta_k U(t, \partial_t)w + [U(t, \partial_t), \zeta_k]w,
\]

or equivalently

\[
\left\{ \begin{array}{l}
\nabla \cdot P_w \nabla (\zeta_k w) = \zeta_k e^{2t} h_c(e^t, \theta) + [\nabla \cdot P_w \nabla, \zeta_k]w, \\
\zeta_k w|_{\Gamma_1} = \zeta_k f_c(e^t), \quad \partial_{\nu_{\mathcal{C}}} (\zeta_k w)|_{\Gamma_b} = \zeta_k e^t g_c(e^t) + [\partial_{\nu_{\mathcal{C}}}, \zeta_k]w|_{\Gamma_b}.
\end{array} \right.
\]

To estimate the right side of the system above, one knows firstly from 21 that

\[
\zeta_k U(t, \partial_t)w \in H^1(\mathcal{C}) \times H^{5/2}(\Gamma_1) \times H^{3/2}(\Gamma_b).
\]

On the other hand, a direct computation from 20 shows

\[
[\nabla \cdot P_w \nabla, \zeta_k]w = [\nabla \cdot P_w \nabla, \zeta_k] \eta_k w = \nabla \cdot P_w \begin{pmatrix} \zeta'_k \\
0
\end{pmatrix} (\eta_k w) + \begin{pmatrix} \zeta'_k \\
0
\end{pmatrix} \cdot P_w \nabla(\eta_k w)
\]

satisfies
and
\[ [\partial_{\nu_b}^P \cdot \zeta_k] w |_{\Gamma_b} = n_b \cdot P_w \left( \begin{array}{c} \zeta_k' \\ 0 \end{array} \right) (\eta_k w) \bigg|_{\theta = -\omega_2}. \]

Consequently, one has
\[ [\mathcal{U}(t, \partial_t), \zeta_k] w = [\mathcal{U}(t, \partial_t), \zeta_k] \eta_k w \in H^1(C) \times H^{5/2}(\Gamma_t) \times H^{3/2}(\Gamma_b) \]
satisfying the estimate
\[
\begin{align*}
\left\| [\mathcal{U}(t, \partial_t), \zeta_k] w \right\|_{H^1(C) \times H^{5/2}(\Gamma_t) \times H^{3/2}(\Gamma_b)} & \leq \left\| \nabla \cdot P_w \left( \begin{array}{c} \zeta_k' \\ 0 \end{array} \right) (\eta_k w) \right\|_{H^1(C)} + \left\| P_w \nabla (\eta_k w) \right\|_{H^1(C)} \\
& + \left\| n_b \cdot P_w \left( \begin{array}{c} \zeta_k' \\ 0 \end{array} \right) (\eta_k w) \right\|_{H^{3/2}(\Gamma_b)} \\
& \leq C \left( \| \eta' \|_{W^{2, \infty}} \right) \| \eta_k w \|_{H^2(C)}
\end{align*}
\]

where Lemma 2.1 is applied on the boundary. Besides, one notices that \( e^t = r \) appears in \( \nabla P_{\xi}(e^t, \theta) \), which can be bounded by \( \delta \).

As a result, summing up the estimates above and applying standard elliptic theories (for example Theorem 2.9 [15]) leads to \( \zeta_k w \in H^3(C) \) with the estimate
\[
\begin{align*}
\| \zeta_k w \|_{H^3(C)} & \leq C \left( \| P_w \|_{W^{2, \infty}} \right) \left( \| \zeta_k \mathcal{U}(t, \partial_t) w \|_{H^1(C) \times H^{5/2}(\Gamma_t) \times H^{3/2}(\Gamma_b)} \\
& + \| [\mathcal{U}(t, \partial_t), \zeta_k] w \|_{H^1(C) \times H^{5/2}(\Gamma_t) \times H^{3/2}(\Gamma_b)} \right) \\
& \leq C \left( \| \eta' \|_{W^{2, \infty}} \right) \left( \| \zeta_k \mathcal{U}(t, \partial_t) w \|_{H^1(C) \times H^{5/2}(\Gamma_t) \times H^{3/2}(\Gamma_b)} + \| \eta_k w \|_{H^2(C)} \right).
\end{align*}
\]

Notice that the coefficient \( C \left( \| \eta' \|_{W^{2, \infty}} \right) \) above doesn’t depend on \( k \), which is the key to go back to the weighted norm for \( w \).

**Step 2.** The estimate for \( w \). To begin with, we will convert the estimate above for \( \zeta_k w \) to the estimate for \( w \). In fact, one has for each \( k \in \mathbb{Z} \) that
\[ \zeta_k w \in W^{3, 2}_{2, \beta - 1}(C) \]
from the definition of \( W^{3, 2}_{2, \beta - 1}(C) \) and \( \zeta_k \). Moreover, it’s straightforward to see that
\[ c_1 \| \zeta_k w \|_{H^3(C)} \leq e^{(1 - \beta)k} \| \zeta_k w \|_{W^{2, \beta - 1}_{2, \beta - 1}(C)} \leq c_2 \| \zeta_k w \|_{H^3(C)} \]
where \( c_1, c_2 \) are two constants independent of \( k \).

Consequently, multiplying \( e^{(\beta - 1)k} \) on both sides of 23, one derives
\[
\begin{align*}
\| \zeta_k w \|_{W^{3, 2}_{2, \beta - 1}(C)} & \leq C \left( \| \eta' \|_{W^{2, \infty}} \right) \left( \| \zeta_k \mathcal{U}(t, \partial_t) w \|_{W^1_{2, \beta - 1}(C) \times W^{5/2}_{2, \beta - 1}(\Gamma_t) \times W^{3/2}_{2, \beta - 1}(\Gamma_b)} \\
& + \| \eta_k w \|_{W^{2, \beta - 1}_{2, \beta - 1}(C)} \right).
\end{align*}
\]

The following lemma from [14] tells us the relationship between the norms of \( w \) and \( \zeta_k w \).

**Lemma 3.5.** Let \( \{ \zeta_k \} \) be the partition of unity in \( \mathbb{R} \) defined above and \( \beta_0 \in \mathbb{R} \).
Then there exist positive real constants \( c_1, c_2 \) depending only on \( l \geq 1 \) such that
\[
\begin{align*}
c_1 \| w \|_{W^{l, \beta}_2(C)} \leq \left( \sum_{k = -\infty}^{+\infty} \| \zeta_k w \|_{W^{l, 2}_{2, \beta_0}(C)}^2 \right)^{1/2} & \leq c_2 \| w \|_{W^{l, \beta}_2(C)}
\end{align*}
\]
for each \( w \in W^3_{2,\beta_1}(C) \).

Consequently, applying this lemma, we have immediately \( w \in W^3_{2,\beta_1}(C) \) with the estimate

\[
\|w\|_{W^3_{2,\beta_1}(C)} \leq C(\|\eta\|_{W^{2,\infty}}) \left( \|u(t, \partial_t)\|_{W^2_{2,\beta_1}(C)} \times W^2_{2,\beta_1}(\Gamma_t) \times W^2_{2,\beta_1}(\Gamma_b) \right)
\]

Combining Lemma 2.6 and Proposition 1, we finish the proof for the case \( l = 3 \).

**Step 3.** The case \( l > 3 \). In fact, applying Theorem 2.9 [15] to \( \zeta_k w \) system 22, one obtains

\[
\|\zeta_k w\|_{H^1(C)} \leq C(\|P_w\|_{W^{1,\infty}}) \left( \|\zeta_k U(t, \partial_t) w\|_{H^1(\Gamma_t) \times H^1(\Gamma_b)} \right)
\]

The rest part can be proved similarly as well and the proof is finished.

\[\square\]

3.3. **Proof of Theorem 1.1.** Now we are ready to prove this main theorem. First of all, recalling the norm of the weighted space \( V^1_{\beta}(\Omega) \), one knows that \( u \) is divided into \( v_c \) and \( v_R \). Therefore, the proof deals with these two parts and an inductive method is applied here for \( l \geq 2 \).

**Step 1.** \( l = 2 \). For the key part concerning \( v_c \), we apply Proposition 3 directly if the assumptions there are satisfied. In fact, checking from system 9, one can see that

\[
h_c = (\chi_c \cdot h - [\chi_c, \Delta] u) \circ T_c \in V^0_{\beta}(\mathcal{K}).
\]

This holds since one has

\[
\chi_c \cdot h \circ T_c \in V^0_{\beta}(\mathcal{K})
\]

by the assumption \( h \in V^0_{\beta}(\Omega) \) and moreover \( u \in H^2(\Omega) \) leads to

\[
r^\beta \cdot [\chi_c, \Delta] u \circ T_c \in L^2(\mathcal{K})
\]

from definition 1.

On the other hand, \( f \in V^3_{\beta}(\Gamma_t) \) implies \( f_c \in V^3_{\beta}(\Gamma_t) \) immediately, so it remains to check \( g_c \). Recalling that

\[
g_c = (\chi_c \cdot \partial_n \chi_c) \cdot [\partial_n u] \circ T_c,
\]

one obtains \( \chi_c \cdot \partial_n \chi_c \circ T_c \in V^3_{\beta}(\Gamma_b) \) directly from the assumption of this theorem. Meanwhile, one also has \( u \in H^3(\Gamma_b) \), which infers

\[
(\partial_n \chi_c) \cdot [\partial_n u] \circ T_c \in H^3(\Gamma_b).
\]

Consequently, checking 8 for the norm of \( V^3_{\beta}(\Gamma_b) \) and noticing that

\[
(\partial_n \chi_c) \cdot [\partial_n u] \circ T_c = 0 \quad \text{near } X_c
\]

one derives \( (\partial_n \chi_c) \cdot [\partial_n u] \circ T_c \in V^1_{\beta}(\Gamma_b) \) immediately.
In a word, the assumptions of Proposition 1 are all satisfied indeed. Applying this proposition, we have from estimate 12 that
\[
\|v_c\|_{V^2_\beta(\Omega)} \leq C \left( \|h_c\|_{V^0_\beta(\Omega)} + \|f_c\|_{V^{3/2}_\beta(\Gamma)} + \|g_c\|_{V^{1/2}_\beta(\Gamma)} \right)
\]
where the constant \(C\) depends on \(\mathcal{K}, \chi_c\), and the following inequalities are applied:

\[
\|\chi_c, \Delta\| u \circ T_c \|_{V^0_\beta(\Omega)} \leq C \left( \|r^\beta(\Delta\chi_c)\chi_c u \circ T_c\|_{L^2(\Omega)} + ||r^\beta\nabla\chi_c \cdot \nabla(\chi_c u) \circ T_c\|_{L^2(\Omega)} \right) 
\]
and

\[
\left\| (\partial_{\eta_0}\chi_c)\|_{\Gamma_0} u \circ T_c \|_{V^{3/2}(\Gamma_0)} \right\| \leq C \left\| (\partial_{\eta_0}\chi_c)\|_{\Gamma_0} u \circ T_c \|_{V^{3/2}(\Omega)} \leq \delta C \|u\|_{V^2_\beta(\Omega)}
\]

with \(\tilde{\chi}_c\) another \(C^\infty_0(\Omega)\) function satisfying \(\chi_c = \tilde{\chi}_c\chi_c\).

Secondly, for the remainder \(v_R = (1 - \chi_c)u \circ T_R\), direct computations show that \(v_R\) satisfies the following system

\[
\left\{ \begin{array}{l}
\nabla \cdot P_R \nabla v_R = h_R & \text{on } R \\
v_R|_{\Gamma_0} = f_R, \quad \partial_{\eta_0} P_R v_R|_{\Gamma_0} = g_R
\end{array} \right.
\]

where

\[
h_R = \left( (1 - \chi_c) h - [1 - \chi_c, \Delta] u \right) \circ T_R, \quad f_R = \left( (1 - \chi_c)|_{\Gamma_0} f \right) \circ T_R,
\]

and

\[
g_R = (1 + (x')^2)^{1/2} \left( (1 - \chi_c)|_{\Gamma_0} g + (\partial_{\eta_0}\chi_c) u|_{\Gamma_0} \right) \circ T_R.
\]

Moreover, the coefficient matrix reads

\[
P_R = (P_r)^t P_r
\]

with

\[
P_r = (\nabla T_R^{-1}) \circ T_R = \begin{pmatrix} 1 & -\left(\eta(x) - l(x)\right)^{-1}(\eta'(x) - l'(x))z + l'(x)) \\
0 & \left(\eta(x) - l(x)\right)^{-1}
\end{pmatrix}
\]

Since this system for \(v_R\) is defined on the flat strip \(R\), standard elliptic theories apply directly (for example Theorem 2.9 [15]). Meanwhile, one has immediately that \(v_R \in H^2(\Omega)\) with the estimate

\[
\|v_R\|_{H^2(\Omega)} \leq C(\|\eta'\|_{W^{1,\infty} \left( (\eta')\right)} \left( \|h_R\|_{L^2(\Omega)} + \|f_R\|_{H^{3/2}(\Gamma)} + \|g_R\|_{H^{1/2}(\Gamma)} \right)
\]
\[
\leq C(\|\eta'\|_{W^{1,\infty} \left( (\eta')\right)} \left( \|h \circ T_R\|_{L^2(\Omega)} + \|f \circ T_R\|_{H^{3/2}(\Omega)} + \|g \circ T_R\|_{H^{1/2}(\Omega)} + \right)
\]

and notice that the term \(\|u\|_{H^1(\Omega)}\) can be handled using \(H^2\) estimate from [24] and Lemma 2.5:

\[
\|u\|_{H^1(\Omega)} \leq C(\|\eta'\|_{W^{1,\infty} \left( (\eta')\right)} \left( \|h\|_{L^2(\Omega)} + \|f\|_{H^{3/2}(\Gamma)} + \|g\|_{H^{1/2}(\Gamma)} \right)
\]
\[
\leq C(\|\eta'\|_{W^{1,\infty} \left( (\eta')\right)} \left( \|h\|_{V^0_\beta(\Omega)} + \|f\|_{V^{3/2}_\beta(\Gamma)} + \|g\|_{V^{1/2}_\beta(\Gamma)} \right).
\]

As a result, combining these estimates above, we have proved that \(u \in V^2_\beta(\Omega)\) satisfies the estimate

\[
\|u\|_{V^2_\beta(\Omega)} \leq C(\|\eta'\|_{W^{1,\infty} \left( (\eta')\right)} \left( \|h\|_{V^0_\beta(\Omega)} + \|f\|_{V^{3/2}_\beta(\Gamma)} + \|g\|_{V^{1/2}_\beta(\Gamma)} \right).
\]
Step 2. Case (i) \( l \geq 3 \) when \( \eta \in W^{l, \infty}(\mathbb{R}^+) \). An induction method is used in this part. To begin with, we know already from the assumptions of this theorem that

\[
h \in V_{l-2+\beta}^{l-1}(\Omega), \quad f \in V_{l-2+\beta}^{l-1/2}(\Gamma_t), \quad g \in V_{l-2+\beta}^{l-3/2}(\Gamma_b).
\]

Assuming \( u \in V_{l-3+\beta}^{l-1}(\Omega) \) and the following estimate holds

\[
\|u\|_{V_{l-3+\beta}^{l-1}(\Omega)} \leq C(\|\eta\|_{W^{l-2, \infty}}(\|h\|_{V_{l-3+\beta}^{l-1}(\Omega)} + |f|_{V_{l-3+\beta}^{l-1/2}(\Gamma_t)} + |g|_{V_{l-3+\beta}^{l-1/2}(\Gamma_b)})),
\]

we are going to show that \( u \in V_{l-2+\beta}^{l}(\Omega) \) with corresponding estimate.

Firstly, we deal with the part \( v_c \). In fact, it remains again to check the assumptions of Proposition 3. Since one has \( u \in V_{l-1+\beta}^{l}(\Omega) \cap H^{2}(\Omega) \), one deduces directly that

\[
[\chi_c, \Delta]u \circ T_c \in V_{l-2+\beta}^{l-2}(\mathcal{K})
\]

and direct computations lead to

\[
\|[\chi_c, \Delta]u \circ T_c\|_{V_{l-2+\beta}^{l-2}(\mathcal{K})} \leq C(\|\Delta\chi_c\|_{H^{l-1}(\Omega)} + \|\nabla \chi_c \cdot \nabla u\|_{L^2(\Omega)} + \|T_c \|_{L^2(\mathcal{K})})
\]

where one notices that \( \Delta\chi_c, \nabla \chi_c \) vanish near \( X_c \) and \( C(\|\eta\|_{W^{l-2, \infty}}(\|h\|_{V_{l-1+\beta}^{l-1}(\Omega)} + \|u\|_{V_{l-1+\beta}^{l-1}(\Omega)}) \).

On the other hand, the definition of \( V_{l-2+\beta}^{l-1/2}(\Gamma_t) \) infers immediately that \( f_c \in V_{l-2+\beta}^{l-1/2}(\Gamma_t) \) holds. Meanwhile, for the term \( g_c \), one can show directly from Lemma 2.1 and Lemma 2.3 that

\[
\|g_c\|_{V_{l-2+\beta}^{l-3/2}(\Gamma_b)} \leq C(\|g\|_{V_{l-2+\beta}^{l-3/2}(\Gamma_b)} + \|T_c \|_{V_{l-2+\beta}^{l-1}(\mathcal{K})})
\]

Summing these up, one can see that the assumptions of Proposition 3 are all satisfied. Applying Proposition 3 and Lemma 2.3 together with 25, we finally have \( v_c \in V_{l-2+\beta}^{l}(\mathcal{K}) \) with the estimate

\[
\|v_c\|_{V_{l-2+\beta}^{l}(\mathcal{K})} \leq C(\|\eta\|_{W^{l-1, \infty}}(\|h\|_{V_{l-2+\beta}^{l-2}(\Omega)} + |f|_{V_{l-2+\beta}^{l-1/2}(\Gamma_t)} + |g|_{V_{l-2+\beta}^{l-1/2}(\Gamma_b)} + \|u\|_{V_{l-1+\beta}^{l-1}(\Omega)})
\]

Secondly, applying standard elliptic theories, one derives an estimate for \( v_R \in H^l(R) \) similarly as before.
As a result, combining these two parts together, one concludes that \( u \in V_{l-2+\beta}^1(\Omega) \) satisfying the desired estimate.

**Step 3.** Case (ii) \( l \geq 3 \) when \( \eta \in H^{l-1/2}(\mathbb{R}^+) \). Here we need the regularizing transformation \( \tilde{T}_c \) defined in 7 in Section 2.1. Therefore, we change every \( T_c \) we meet into \( \tilde{T}_c \), and the corresponding coefficient matrix \( P_c \) in system 9 of \( v_c \) should be replaced by

\[
P_c^{\#} = (\tilde{P}_0^{-1})^t(\tilde{P}_S \circ \tilde{T}_0)^t \tilde{P}_S \circ \tilde{T}_0(\tilde{P}_0)^{-1},
\]

where \( \tilde{P}_S \circ \tilde{T}_0 \) is given by 6. So one can tell directly that

\[
\tilde{P}_c = \tilde{P}_c(\nabla s).
\]

Similarly as before, one can check the estimates from the beginning to find out that all we need is to focus on system 22 for \( \zeta_k w \) again. The corresponding coefficient matrix \( P_w \) should be replaced by

\[
P_w = P_0^t \tilde{P}_c(\nabla s(e^t, \theta)) P_0 = \tilde{P}_{w,1} + \tilde{P}_{w,2}
\]

with

\[
\tilde{P}_{w,1} = P_0^t \tilde{P}_c(0) P_0, \quad \tilde{P}_{w,2} = P_0^t (\tilde{P}_c(\nabla s(e^t, \theta)) - \tilde{P}_c(0)) P_0.
\]

Applying Theorem 2.9 ii)[15] and assuming that \( l \geq 3 \), one has

\[
\|\zeta_k w\|_{H^1(\mathcal{C})} \leq C(l, \|\tilde{P}_{w,1}\|_{W^{1,\infty}(\mathcal{C})}, \|\tilde{P}_{w,2}\|_{H^{l-1}(\mathcal{C})}) \times \left( \|\zeta_k \mathcal{U}(t, \partial_t)\|_{H^{l-2}(\mathcal{C})} \times H^{l-1/2}(\Gamma_1) \times H^{l-3/2}(\Gamma_2) \right), \quad (26)
\]

To handle the second term on the right side of the inequality above, the following three terms

\[
\left\| \nabla \cdot \tilde{P}_w \begin{pmatrix} \zeta_k \\ 0 \end{pmatrix} w \right\|_{H^{l-2}(\mathcal{C})}, \quad \left\| \tilde{P}_w \nabla (\eta k w) \right\|_{H^{l-2}(\mathcal{C})}
\]

and

\[
\left\| \mathbf{n}_b \cdot \tilde{P}_w \begin{pmatrix} \zeta_k \\ 0 \end{pmatrix} (\eta k w) \right\|_{H^{l-3/2}(\Gamma_k)}
\]

need to be taken care of according to the proof of Proposition 3.

For example, in the first term above, we consider the estimate for the term where all the derivatives are taken on one \( \nabla s(e^t, \theta) \). In fact, this term is like

\[
A := P_0^t (\partial \tilde{P}_c)(\nabla s(e^t \cos \theta, e^t \sin \theta)) P_0 (\partial^\alpha \partial s)(e^t \cos \theta, e^t \sin \theta) \times e^{(l-1)t} \phi(\cos \theta, \sin \theta) \begin{pmatrix} \zeta_k \\ 0 \end{pmatrix} w,
\]

which comes from \( \partial^\alpha \tilde{P}_w \) with \( |\alpha| = l - 1 \). Then the following estimate holds:

\[
\|A\|_{L^2(\mathcal{C})} \leq C(\|\partial^\alpha \partial s\|_{L^2(\mathcal{C})}) \|\eta k w\|_{L^\infty(\mathcal{C})} \leq C\|s\|_{H^{l}(\mathcal{K})} \|\eta k w\|_{H^{l-1}(\mathcal{C})},
\]

where we notice that \( e^{(l-1)t} \) is used to transform between different domains. Similarly, one can have estimates for the other terms in 26. Plugging all the estimates back into 26, one derives

\[
\|\zeta_k w\|_{H^1(\mathcal{C})} \leq C\|s\|_{H^{l}(\mathcal{K})} \|\eta k w\|_{H^{l-1}(\mathcal{C})}.
\]
As a result, remembering the definition of \( s \) and applying 4, one finds immediately that
\[
\|s\|_{H^1(K)} \leq C(\|\eta\|_{H^{1-\gamma}(\mathbb{R}^+)}).
\]
Therefore, the proof of Theorem 1.1 ii) can be finished.

4. Some other boundary-value problems.

4.1. Dirichlet boundary problem. To consider about the system (DVP), one needs to verify first of all the existence of the solution \( u \in H^2(\Omega) \). To begin with, it is well-known from Lemma 4.4.3.1 [9] that there exists a unique variational solution \( u \in H^1(\Omega) \) under the assumptions of Theorem 1.2.

Secondly, we recall an adjusted version of Theorem 3.2.5 [1] here.

**Theorem 4.1.** (Banasiak-Roach1989) Let
(i) \( \Omega_c \subset \mathbb{R}^2 \) be a bounded, open set with only one angle and a \( C^{2,0} \) curvilinear polygon as its boundary;
(ii) The boundary values \( f, g \) satisfy the following conditions:
For system (DVP), one assumes that \( f \in H^{3/2}(\Gamma_t), \ g \in H^{3/2}(\Gamma_b) \) and
\[
f|_{X_c} = g|_{X_c};
\]
For system (NVP), one assumes that \( f \in H^{1/2}(\Gamma_t), \ g \in H^{1/2}(\Gamma_b) \).

Then the variational solution \( u \in H^1(\Omega) \) of system (DVP) or system (NVP) can be uniquely represented in the form
\[
u = u_r + \sum_{m \in L} c_m S_m
\]
with \( u_r \in H^2(\Omega_c) \) and some \( c_m \in \mathbb{R} \). Moreover, the set \( L \) is defined as
\[
L = \{m | -1 < \lambda_m < 0\}, \quad \text{where} \quad \lambda_m = \frac{m\pi}{\omega},
\]
and \( \omega \) is the angle.

Therefore, localizing system (DVP) as in the mixed-boundary case and checking directly from this theorem, it is clear that when the contact angle \( \omega \in (0, \pi) \), one has
\[
\lambda_m \notin (-1, 0),
\]
which implies that there exists a unique solution \( u \in H^2(\Omega) \) to (DVP). By the way, the compatibility condition can also be found in [9, 24].

Consequently, the solution \( u \) lies in the space \( V_{2,1}^2(\Omega) \) as before. To prove the first part in Theorem 1.2, we only needs to follow the proof in Section 3 and check line by line. In this case, we will consider the Dirichlet boundary system for \( v_c \):
\[
\begin{cases}
\nabla \cdot P_c \nabla v_c = h_c & \text{on } K \\
v_c|_{\Gamma_t} = f_c, \quad v_c|_{\Gamma_b} = g_c,
\end{cases}
\]
and the system for \( w(t, \theta) = v_c(e^t, \theta) \) is slightly different as well.

In Proposition 2, one also has a different eigenvalue problem for \( \mathcal{U}(\lambda) \) here:
\[
\begin{cases}
-\phi''(\theta) - \lambda^2 \phi(\theta) = 0, & \theta \in I \\
\phi(\omega_1) = 0, \quad \phi(-\omega_2) = 0
\end{cases}
\]
with the eigenvalues and eigenfunctions
\[
\lambda_m = \frac{m\pi}{\omega}, \quad \phi_m(\theta) = \sin \left( \lambda_m (\theta + \omega_2) \right) \quad \text{for } \forall m \in \mathbb{Z}.
\]
Consequently, we can tell that the eigenvalues
\[ \lambda_m \notin [-1, 0) \quad \text{for} \quad \omega \in (0, \pi), \forall m \in \mathbb{Z}. \]
Therefore, following the proof of Proposition 2 and Proposition 1, we conclude that
\[ v_c \in V_{1+\beta}^2(\mathcal{K}) \quad \text{for any} \quad \beta \in (0, 1], \]
since we cannot cross over 0 for the eigenvalue \( \lambda_m \).
Moreover, the regularity of \( v_c \) can be improved in the same way as before, and the weighted estimates rely on standard elliptic estimates with Dirichlet boundaries (which can be proved similarly as in [15]).
As a result, the proof for the Dirichlet case in Theorem 1.2 is finished.

4.2. Neumann boundary problem. The case of Neumann boundaries can be proved similarly, and the eigenvalue value \( \lambda_m \) from the eigenvalue problem for the Tailor and~\( \Omega \) turns out to be the same as in the Dirichlet case.
Here, one needs to notice that we assume the existence of the solution \( u \) when \( \Omega \) is unbounded. When the domain is bounded, the existence can be proved under the compatibility condition for \( \text{NVP} \).
Therefore, the proof for the second part of Theorem 1.2 follows.

5. Application to the Dirichlet-Neumann operator. In the end, we show that the weighted elliptic theory above can be applied to the Dirichlet-Neumann operator, which is an important operator in the water-waves problem.
To begin with, recalling that for a proper function \( f \) on \( \Gamma_t \), the D-N operator \( \mathcal{N} \) is defined as
\[ \mathcal{N} f := \nabla_n f_h |_{\Gamma_t} \] (27)
where \( f_h \) is the harmonic extension of \( f \) satisfying the system
\[ \begin{cases} \Delta f_h = 0, & \text{on} \quad \Omega, \\ f_h |_{\Gamma_t} = f, \quad \nabla_n f_h |_{\Gamma_b} = 0. \end{cases} \]
When the function \( f \in V_{l-2+\beta}^1(\Gamma_t) \) for any \( \beta \in [0, 2] \), we know directly from Theorem 1.1 that
\[ \| f_h \|_{V_{l-2+\beta}^1(\Omega)} \leq C(\| g \|_{H^{l-1/2}(\mathbb{R}^2)}) \| f \|_{V_{l-2+\beta}^1(\Gamma_t)}. \] (28)
On the other hand, to consider the estimate for the D-N operator, we need the following lemma about the product estimate in the weighted space:

**Lemma 5.1.** For any two functions \( f \in V_{\beta}^{k+1/2}(\Gamma_t) \) and \( g \in H^{k+1/2}(\Gamma_t) \) with an integer \( k \geq 2 \) and a real \( \beta \), one has the estimate for the product of \( f, g \):
\[ \| f g \|_{V_{\beta}^{k+1/2}(\Gamma_t)} \leq C \| f \|_{V_{\beta}^{k+1/2}(\Gamma_t)} \| g \|_{H^{k+1/2}(\Gamma_t)} \]
where the constant \( C \) depends only on \( k \).

**Proof.** According to the definition of \( V_{\beta}^{k+1/2}(\Gamma_t) \), it suffices to show that the estimate holds for \( f_c, g_c \) defined on \( \Gamma_t \) of \( \mathcal{K} \) near the contact point.
In fact, checking 8, one knows immediately that
\[ \| f_c g_c \|_{V_{\beta}^{k+1/2}(\Gamma_t)} \leq C(k) \sum_{j \leq k} \int_{\mathbb{R}^+} r^{2(\beta-k)-1+2j} |\partial_r^j (f_c(r)g_c(r))|^2 dr. \]
where all the terms can be treated similarly as in the case of $A_j$.

Firstly, one has for the term $A_j$ that
\begin{equation}
A_j \leq \sum_{0 \leq \alpha \leq j} C_{\alpha} \int_{R^+} r^{2(\beta - k) - 1 + 2j} |\partial^\alpha_r f_c(r) \partial^{j-\alpha}_r g_c(r)|^2 \, dr,
\end{equation}
which is separated into two cases.

For the first case when $j - \alpha = k$, i.e. $j = k \geq 2$, $\alpha = 0$, all the derivatives are taken on $g$, so the corresponding term becomes
\begin{equation}
\int_{R^+} r^{2(\beta - k) - 1 + 2j} |f_c(r) \partial^k_r g_c(r)|^2 \, dr \leq \int_{R^+} r^{2(\beta - k) - 1} |f_c(r)|^2 |\partial^k_r g_c(r)|^2 \, dr \leq C_{\delta} \int_{R^+} r^{\beta - k - 1/2} |f_c(r)|^2 |\partial^k_r g_c(r)|^2 \, dr
\end{equation}

\begin{equation}
\leq C\|r^{\beta - k - 1/2} f_c\|_{L^\infty(R^+)} \|\partial^k_r g_c\|_{L^2(R^+)}^2
\end{equation}

\begin{equation}
\leq C\|r^{\beta - k - 1/2} f_c\|_{H^1(R^+)} \|g_c\|_{H^{k}(R^+)}^2
\leq C\|f_c\|_{V_{\beta}^{k+1/2}(\Gamma_1)}^2 \|g_c\|_{H^{k}(R^+)}^2
\end{equation}

where the imbedding theorem from $H^1(\mathbb{R}^+)$ to $L^\infty(\mathbb{R}^+)$ is applied.

For the other case when $j - \alpha \leq k - 1$, the corresponding terms are handled in a slightly different way:
\begin{equation}
\int_{R^+} r^{2(\beta - k) - 1 + 2j} |\partial^\alpha_r f_c(r) \partial^{j-\alpha}_r g_c(r)|^2 \, dr \leq \int_{R^+} r^{2(\beta - k) - 1 + 2j} |\partial^\alpha_r f_c(r)|^2 |\partial^{j-\alpha}_r g_c(r)| L^\infty(R^+)
\leq C_{\delta} \|f_c\|_{V_{\beta}^{k+1/2}(\Gamma_1)}^2 \|g_c\|_{H^{k}(R^+)}^2,
\end{equation}

where the imbedding theorem from $H^1(\mathbb{R}^+)$ to $L^\infty(\mathbb{R}^+)$ is applied again. Consequently, substituting the two cases above into 29, one arrives at
\begin{equation}
A_j \leq C(\delta) \|f_c\|_{V_{\beta}^{k+1/2}(\Gamma_1)}^2 \|g_c\|_{H^{k}(R^+)}^2.
\end{equation}

Secondly, we deal with $B_j$, one deduces that
\begin{equation}
B_j \leq \sum_{0 \leq \alpha \leq j} C_{\alpha} \int_0^\delta \int_{r/2}^{2r} r^{2(\beta - k)} \frac{|r^{\beta}\partial^\alpha \rho \partial^\alpha f_c(r) - \rho^{\beta}\partial^\alpha \rho f_c(\rho)|^2}{|r - \rho|^2} \frac{|r^{j-\alpha}\partial^{j-\alpha}_r g_c(r)|^2}{|r - \rho|^2} \, dr \, d\rho
+ \sum_{0 \leq \alpha \leq j} C_{\alpha} \int_0^{2\delta} \int_{r/2}^{2r} r^{2(\beta - k)} \rho^{\beta}\partial^\alpha \rho f_c(\rho)|^2 \frac{|r^{j-\alpha}\partial^{j-\alpha}_r g_c(r) - \rho^{j-\alpha}\partial^{j-\alpha}_\rho g_c(\rho)|^2}{|r - \rho|^2} \, dr \, d\rho.
\end{equation}

where all the terms can be treated similarly as in the case of $A_j$. 

\begin{equation}
C(k) \sum_{j \leq k} \int_{R^+} \int_{r/2}^{2r} r^{2(\beta - k)} \frac{|r^{\beta}\partial^\alpha \rho \partial^\alpha f_c(r) - \rho^{\beta}\partial^\alpha \rho f_c(\rho)|^2}{|r - \rho|^2} \, dr \, d\rho
\end{equation}

\begin{alignat}{2}
&= C \sum_{j \leq k} \left( A_j + B_j \right)
\end{alignat}
For example, when \( j = k \geq 2, \alpha = 0 \), there is a term from 31 satisfying
\[
\int_0^\delta \int_{r/2}^{2r} r^{2(\beta-k)} |\partial_r f_c(t)|^2 |r^k \partial^k_r g_c(r)|^2 d\rho dr \\
\leq C(\delta) \|t^{\beta-k-1/2} t^2 \partial_r f_c(t)\|_{L^\infty(R^+)}^2 \|g_c\|_{H^1(R^+)}^2 \\
\leq C \|f_c\|_{V^{k+1/2}_\beta(\Gamma_\delta^+)} \|g_c\|_{H^1(R^+)}^2,
\]
where \( t \) is some number between \( r \) and \( \rho \) in the integral, and the imbedding theorem from \( H^1(R^+) \) to \( L^\infty(R^+) \) is applied on \( f_c \) one more time. We omit the estimates for the remainder terms since one only needs to check from one term to another similarly as before. As a result, the proof is finished.

Now we conclude the weighted estimate for the D-N operator.

**Proposition 4.** Let \( k \geq 2 \) be an integer and \( \beta \in [k,k+2] \) be real. For any function \( f \in V^{k+3/2}_\beta(\Gamma_\delta) \), one has the following estimate for the D-N operator \( \mathcal{N} \):
\[
\|\mathcal{N}f\|_{V^{k+1/2}_\beta(\Gamma_\delta)} \leq C(\|f\|_{H^{k+3/2}(R^+)}) \|f\|_{V^{k+3/2}_\beta(\Gamma_\delta)}.
\]

**Proof.** The proof is a direct application of Theorem 1.1 and Lemma 5.1.

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