A virtual Kawasaki formula

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Abstract

Kawasaki’s formula is a tool to compute holomorphic Euler characteristics of vector bundles on a compact orbifold $\mathcal{X}$. Let $\mathcal{X}$ be an orbispace with perfect obstruction theory which admits an embedding in a smooth orbifold. One can then construct the virtual structure sheaf and the virtual fundamental class of $\mathcal{X}$. In this paper we prove that Kawasaki’s formula “behaves well” with working “virtually” on $\mathcal{X}$ in the following sense: if we replace the structure sheaves, tangent and normal bundles in the formula by their virtual counterparts then Kawasaki’s formula stays true. Our motivation comes from studying the quantum K-theory of a complex manifold $X$ (see [GT]), with the formula applied to Kontsevich’ moduli spaces of genus 0 stable maps to $X$.

1 Introduction

Given a manifold $\mathcal{X}$ and a vector bundle $V$ on $\mathcal{X}$ then Hirzebruch-Riemann-Roch formula states that:

$$\chi(\mathcal{X}, V) = \int_{\mathcal{X}} ch(V) Td(T\mathcal{X}).$$

In [Ka] Kawasaki generalized this formula to the case when $\mathcal{X}$ is an orbifold. He reduces the computation of Euler characteristics on $\mathcal{X}$ to computation of certain cohomological integrals on the inertia orbifold $I\mathcal{X}$:

$$\chi(\mathcal{X}, V) = \sum_{\mu} \frac{1}{m_{\mu}} \int_{X_{\mu}} Td(T_{X_{\mu}}) ch \left( \frac{Tr(V)}{Tr(A^*N^*_{\mu})} \right). \quad (1)$$

We explain below the ingredients in the formula:

$I\mathcal{X}$ is defined as follows: around any point $p \in \mathcal{X}$ there is a local chart $(\tilde{U}_p, G_p)$ such that locally $\mathcal{X}$ is represented as the quotient of $\tilde{U}_p$ by $G_p$. Consider the set of conjugacy classes $(1) = (h_1^p), (h_2^p), \ldots, (h_{n_p}^p)$ in $G_p$. Define:

$$I\mathcal{X} := \{(p, (h_i^p)) \mid i = 1, 2, \ldots, n_p\}.$$
Pick an element $h^i_p$ in each conjugacy class. Then a local chart on $I\mathcal{X}$ is given by:

$$\prod_{i=1}^{n_p} \tilde{U}^{(h^i_p)}/Z_{G_p}(h^i_p),$$

where $Z_{G_p}(h^i_p)$ is the centralizer of $h^i_p$ in $G_p$. Denote by $\mathcal{X}_\mu$ the connected components of the inertia orbifold (we’ll often refer to them as Kawasaki strata). The multiplicity $m_\mu$ associated to each $\mathcal{X}_\mu$ is given by:

$$m_\mu := \left| \ker \left( Z_{G_p}(g) \to \operatorname{Aut}(\tilde{U}^g) \right) \right|.$$

For a vector bundle $V$ we will denote by $V^*$ the dual bundle to $V$. The restriction of $V$ to $\mathcal{X}_\mu$ decomposes in characters of the $g$ action. Let $E^{(l)}_r$ be the subbundle of the restriction of $E$ to $\mathcal{X}_\mu$ on which $g$ acts with eigenvalue $e^{2\pi ir}$. Then the trace $Tr(V)$ is defined to be the orbibundle whose fiber over the point $(p, (g))$ of $\mathcal{X}_\mu$ is:

$$Tr(V) := \sum_l e^{2\pi ir} E^{(l)}_r.$$

Finally, $\Lambda^\bullet N^*_\mu$ is the K-theoretic Euler class of the normal bundle $N_\mu$ of $\mathcal{X}_\mu$ in $\mathcal{X}$. $Tr(\Lambda^\bullet N^*_\mu)$ is invertible because the symmetry $g$ acts with eigenvalues different from 1 on the normal bundle to the fixed point locus. We call the terms corresponding to the identity component in the formula fake Euler characteristics:

$$\chi_f(\mathcal{X}, V) = \int_X \operatorname{ch}(V)\operatorname{Td}(T\mathcal{X}).$$

In the case where $\mathcal{X}$ is a global quotient formula (1) is the Lefschetz fixed point formula.

Now let $\mathcal{X}$ be a compact, complex orbispace (Deligne-Mumford stack) with a perfect obstruction theory $E^{-1} \to E^0$. This gives rise to the intrinsic normal cone, which is embedded in $E_1$ - the dual bundle to $E^{-1}$ (see [L], also [BF]). The virtual structure sheaf $\mathcal{O}^\text{vir}_\mathcal{X}$ was defined in [L] as the K-theoretic pull-back by the zero section of the structure sheaf of this cone. Let $I\mathcal{X} = \bigsqcup_\mu \mathcal{X}_\mu$ be the inertia orbifold of $\mathcal{X}$. We denote by $i_\mu$ the inclusion of a stratum $\mathcal{X}_\mu$ in $\mathcal{X}$. For a bundle $V$ on $\mathcal{X}$ we write $i_\mu^* V = V^f_\mu \oplus V^m_\mu$ for its decomposition as the direct sum of the fixed part and the moving part under the action of the symmetry associated to $\mathcal{X}_\mu$. To avoid ugly notation we will often simply write $V^m, V^f$. The virtual normal bundle to $\mathcal{X}_\mu$ in $\mathcal{X}$ is defined
as \([E^m_0] - [E^1_1]\). We will in addition assume that \(\mathcal{X}\) admits an embedding \(j\) in a smooth compact orbifold \(\mathcal{Y}\). This is always true for the moduli spaces of genus 0 stable maps \(X_{0,n,d}\) because an embedding \(X \hookrightarrow \mathbb{P}^N\) induces an embedding \(X_{0,n,d} \hookrightarrow (\mathbb{P}^N)_{0,n,d}\).

**Theorem 1.1.** Denote by \(N^\text{vir}_\mu\) the virtual normal bundle of \(\mathcal{X}_\mu\) in \(\mathcal{X}\). Then

\[
\chi\left(\mathcal{X}, j^*(V) \otimes O^\text{vir}_\mathcal{X}\right) = \sum_{\mu} \frac{1}{m_\mu} \chi^f\left(\mathcal{X}_\mu, \frac{Tr(V_\mu \otimes O^\text{vir}_\mathcal{X}_\mu)}{Tr\left(\Lambda^\bullet(N^\text{vir}_\mathcal{X}_\mu)^*\right)}\right) .
\]

(2)

**Remark 1.2.** A perfect obstruction theory \(E^{-1} \to E^0\) on \(\mathcal{X}\) induces canonically a perfect obstruction theory on \(\mathcal{X}_\mu\) by taking the fixed part of the complex \(E^{-1}_\mu \to E^0_\mu\). The proof is the same as that of Proposition 1 in [GP]. This is then used to define the sheaf \(O^\text{vir}_{\mathcal{X}_\mu}\).

**Remark 1.3.** It is proved in [FG] that if \(\mathcal{X}\) is a scheme, the Grothendieck-Riemann-Roch theorem is compatible with virtual fundamental classes and virtual fundamental sheaves i.e.:

\[
\chi^f(\mathcal{X}, V \otimes O^\text{vir}_\mathcal{X}) = \int_{[\mathcal{X}]} \text{ch}(V \otimes O^\text{vir}_\mathcal{X}) \cdot Td(T^\text{vir})
\]

where \([\mathcal{X}]\) is the virtual fundamental class of \(\mathcal{X}\) and \(T^\text{vir}\) is its virtual tangent bundle. Their arguments carry over to the case when \(\mathcal{X}\) is a stack.

**Remark 1.4.** The bundles \(V\) to which we apply Theorem 1.1 in [GT] are (sums and products of) cotangent line bundles \(L_i\) and evaluation classes \(ev^*_i(a_i)\). They are pull-backs of the corresponding bundles on \((\mathbb{P}^N)_{0,n,d}\).

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## 2 Proof of Theorem 1.1

Before proving Theorem 1.1 we recall a couple of background facts and lemmata on K-theory which we will use.

Let \(K_0(X)\) be the Grothendieck group of coherent sheaves on \(X\). Given a map \(f : X \to Y\), the K-theoretic pullback \(f^*(\mathcal{F}) : K_0(Y) \to K_0(X)\) is defined as the alternating sum of derived functors \(Tor^i_{O_Y}(\mathcal{F}, O_X)\), provided
that the sum is finite. This is always true for instance if \( f \) is flat or if it is a regular embedding.

For any fiber square:

\[
\begin{array}{ccc}
V' & \longrightarrow & V \\
\downarrow & & \downarrow \\
B' & \longrightarrow & B
\end{array}
\]

with \( i \) a regular embedding one can define K-theoretic refined Gysin homomorphisms \( i^!: K_0(V) \rightarrow K_0(V') \) (see [L]). One way to define the map \( i^! \) is the following: the class \( i_*(\mathcal{O}_{B'}) \in K^0(B) \) has a finite resolution of vector bundles, which is exact off \( B' \). We pull it back to \( V \) and then cap (i.e. tensor product) with classes in \( K_0(V) \), to get a class on \( K_0(V') \) with homology supported on \( V' \), which we can regard as an element of \( K_0(V') \), because there is a canonical isomorphism between complexes on \( V \) with homology supported on \( V' \) and \( K_0(V') \).

In the following two theorems \( X, Y, Y' \) are assumed DM stacks. We will use the following result:

**Lemma 2.1.** Consider the diagram:

\[
\begin{array}{ccc}
t^*C_{X/Y} & \longrightarrow & C_{X/Y} \\
\downarrow & & \downarrow \\
X' & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y' & \longrightarrow & Y
\end{array}
\]

with \( i \) a regular embedding and \( j \) an embedding, \( C_{X/Y} \) is the normal cone of \( X \) in \( Y \) and both squares are fiber diagrams. Then:

\[
i^![\mathcal{O}_{C_{X/Y}}] = [\mathcal{O}_{C_{X'/Y'}}] \in K_0(t^*C_{X/Y}). \tag{3}
\]

This is stated and proved in [L] (Lemma 2). The proof is based on a more general statement (Lemma 1 of [L]), which has been worked out in [Kr] on the level of Chow rings. Since K-theoretic statements are stronger, we give below the key-ingredient which allows one to carry over Kresch’s proof to K-theory:

**Lemma 2.2.** Let \( f : X \rightarrow Y \) be a closed embedding let \( g : Y \rightarrow \mathbb{P}^1 \) be a surjection such that \( g \circ f \) is flat. Denote by \( X_0 \) and \( Y_0 \) the fibers over 0 of \( g \circ f \) and \( g \) respectively. Moreover assume that the restriction of \( f \) to \( X \setminus X_0 \) is an isomorphism. Then if \( i \) is the inclusion of \( \{0\} \) in \( \mathbb{P}^1 \), \( i^!(\mathcal{O}_Y) = \mathcal{O}_{X_0} \in K_0(Y_0) \).
Proof: the skyscraper sheaves at all points of \( P^1 \) represent the same element in \( K_0(P^1) \), hence if we pull-back a resolution of any point \( P \in P^1 \) by \( g \) we get the same elements of \( K_0(Y) \). On the other hand since \( f \) is an isomorphism above \( P^1 \setminus \{0\} \), pulling-back by \( g \) of the structure sheaf of a point \( P \neq 0 \) is the same as pulling back by \( g \circ f \) followed by \( f^* \). By what we said above we can replace \( P \) with 0. Now from the flatness of \( g \circ f \) above 0 the pull-back of the structure sheaf of 0 by \( g \circ f \) followed by \( f^* \). By what we said above we can replace \( P \) with 0. Now from the flatness of \( g \circ f \) above 0 the pull-back of the structure sheaf of 0 by \( g \circ f \) is the structure sheaf of the fiber \( X_0 \). The result then follows from the definition of \( i^! \).

Remark 2.3. Lemma 2.2 allows one to show Lemma 2.1: intermediately one shows, following [Kr], (notation is as in Lemma 2.1) that \( [O_{C_1}] = [O_{C_2}] \) in \( K_0(C_{X'}Y \times_Y C_XY) \) where \( C_1 := C \cap C_{X'}(C_XY) \) and \( C_2 := C_j \cap C_{X'}(C_{Y'}Y) \).

We now go on to prove Theorem 1.1. We have:

\[
\chi(\mathcal{X}, j^* V \otimes O_{\mathcal{X}}^{vir}) = \chi(\mathcal{Y}, V \otimes j_* O_{\mathcal{X}}^{vir}).
\]

Kawasaki’s formula applied to the sheaf \( V \otimes j_* O_{\mathcal{X}}^{vir} \) on \( Y \) gives:

\[
\chi(\mathcal{Y}, V \otimes j_* O_{\mathcal{X}}^{vir}) = \sum_{\mu} \frac{1}{m_{\mu}} \chi^f \left( \mathcal{Y}_{\mu}, \frac{Tr(V_{\mu} \otimes i_{\mu}^* j_* O_{\mathcal{X}}^{vir})}{Tr(\Lambda^* N_{\mu}^*)} \right). \tag{4}
\]

From the fiber diagram:

\[
\begin{array}{ccc}
\mathcal{X}_{\mu} & \xrightarrow{i_{\mu}^*} & \mathcal{X} \\
j' & & j \\
\mathcal{Y}_{\mu} & \xrightarrow{i_{\mu}^*} & \mathcal{Y}
\end{array}
\]

and Theorem 6.2 in [FL] (where this is proved for Chow rings) we have \( i_{\mu}^* j_* O_{\mathcal{X}}^{vir} = j'_* i_{\mu}^* O_{\mathcal{X}}^{vir} \). Plugging this in \((4)\) gives:

\[
\chi^f \left( \mathcal{Y}_{\mu}, \frac{Tr(V_{\mu} \otimes i_{\mu}^* j_* O_{\mathcal{X}}^{vir})}{Tr(\Lambda^* N_{\mu}^*)} \right) = \chi^f \left( \mathcal{Y}_{\mu}, \frac{Tr(V_{\mu} \otimes j'_* i_{\mu}^* O_{\mathcal{X}}^{vir})}{Tr(\Lambda^* N_{\mu}^*)} \right). \tag{5}
\]

Let \( G_{\mu} \) be the cyclic group generated by one element of the conjugacy class associated to \( \mathcal{X}_{\mu} \). Then we will show that:

\[
Tr \left( \frac{i_{\mu}^* O_{\mathcal{X}}^{vir}}{\Lambda^* (N_{\mu}^*)} \right) = Tr \left( \frac{O_{X_{\mu}^{vir}}}{\Lambda^* (N_{\mu}^{vir})} \right) \tag{6}
\]

in the \( G_{\mu} \)-equivariant K-ring of \( \mathcal{X}_{\mu} \). This is essentially the computation of Section 3 in [GP] carried out in \( C^* \)-equivariant K-theory. Relation \((6)\) then
follows by embedding the group $G_\mu$ in the torus and specializing the value of the variable $t$ in the ground ring of $\mathbb{C}^*$-equivariant $K$-theory to a $|G_\mu|$-root of unity.

If we define a cone $D := C_{X/Y} \times_X E_0$, then this is a $TY$ cone (see [BF]). The virtual normal cone $D^{vir}$ is defined as $D/TY$ and $\mathcal{O}_{D^{vir}}$ is the pull-back by the zero section of the structure sheaf of $D^{vir}$. Alternatively there is a fiber diagram:

$$
\begin{array}{ccc}
TY & \longrightarrow & D \\
\downarrow & & \downarrow \\
X & \longrightarrow & E_1
\end{array}
$$

where the bottom map is the zero section of $E_1$. Then one can define $\mathcal{O}_{X^{vir}}$ as $0_{TY}^*0_{E_1}^![\mathcal{O}_D]$. We’ll prove formula (9) following closely the calculation in [GP]. First by definition of $\mathcal{O}_{X^{vir}}$ and by commutativity of Gysin maps we have:

$$
i_\mu^!\mathcal{O}_X^{vir} = i_\mu^!0_{TY}^!0_{E_1}^![\mathcal{O}_D] = 0_{TY}^*0_{E_1}^!i_\mu^![\mathcal{O}_D].
$$

(7)

We pull-back relation (3) to $(i_\mu')^*D = (i_\mu')^*(C_{X/Y} \times E_0)$ to get:

$$
i_\mu^!\mathcal{O}_D = [\mathcal{O}_{D_\mu} \times (E_0^m)^*].
$$

(8)

In the equality above we have used the fact that $D_\mu = C_{X/\mu} \times E_0^f$ and we identified the sheaf of sections of the bundle $E_0^m$ with the dual bundle $(E_0^m)^*$. Plugging (8) in (7) we get:

$$
i_\mu^!\mathcal{O}_X^{vir} = 0_{TY}^*0_{E_1}^![\mathcal{O}_{D_\mu} \times (E_0^m)^*].
$$

(9)

Notice that the action of $TY_\mu$ leaves $D_\mu \times (E_0^m)^*$ invariant (it acts trivially on $(E_0^m)^*$). Now we can write $0_{TY_\mu}^* = 0_{TY_\mu}^*0_{TY_\mu}^! \times 0_{TY_\mu}^!$ and since $D^{vir}_\mu = D_\mu/TY_\mu$ we rewrite (9) as:

$$
i_\mu^!\mathcal{O}_X^{vir} = 0_{TY_\mu}^*0_{E_1}^![\mathcal{O}_{D^{vir}_\mu} \times (E_0^m)^*].
$$

(10)

The proof of Lemma 1 in [GP] works in our set-up as well: it uses excess intersection formula which holds in $K$-theory. It shows that the following relation holds in the $\mathbb{C}^*$-equivariant $K$-ring of $X_\mu$:

$$
0_{TY_\mu}^*0_{E_1}^![\mathcal{O}_{D^{vir}_\mu} \times (E_0^m)^*] = 0_{E_0}^*0_{E_1}^![\mathcal{O}_{D^{vir}_\mu} \times (E_0^m)^*] \cdot \Lambda^*(TY_\mu)^* \cdot \Lambda^*(E_0)^*.
$$

(11)
The class $0^!_{E_1} [O_{Dvir} \times E_0^m]$ lives in the $\mathbb{C}^*$-equivariant K-ring of $E_0^m$. The class doesn’t depend on the bundle map $E_0^m \to E_1^m$ so we can assume this map to be 0. Then by excess intersection formula and the definition of $O_{X_{\mu}}^{vir}$ we get:

$$0^*_{E_0^m} \left( 0^!_{E_1} [O_{Dvir} \times (E_0^m)^*] \right) = O_{X_{\mu}}^{vir} \cdot \Lambda^*(E_1^m)^*.$$  \hfill (12)

Formula (12) holds because $D_{vir}^{m} \times (E_0^m) \subset E_0^f \times E_0^m$ and $0^!_{E_1} \times 0^!_{E_1}$ acts as $0^!_{E_1} \times 0^!_{E_1}$ on factors. $0^!_{E_1} [O_{Dvir}^{m}] = O_{X_{\mu}}^{vir}$ by definition of $O_{X_{\mu}}^{vir}$. By excess intersection formula applied to the fiber square:

$\begin{array}{ccc}
E_0^m & \longrightarrow & E_0^m \\
\downarrow & & \downarrow \\
X_{\mu} & \stackrel{0_{E_0^m}}{\longrightarrow} & E_1^m
\end{array}$

we have $0^*_{E_0^m} 0^!_{E_1} [(E_0^m)^*] = 0^*_{E_0^m} \pi^* \Lambda^*(E_1^m)^* = \Lambda^*(E_1^m)^*$. Plugging formula (12) in (11) (note that $N_{\mu} = T_{X_\mu}^m$ and $N_{vir}^{\mu} = [E_0^m] - [E_1^m]$) and taking traces proves (6). We now plug (6) in (5) and then pull-back to $X_{\mu}$ to get:

$$\chi^f \left( \mathcal{Y}_{\mu}^\prime, \frac{Tr(V_{\mu} \otimes j^*_{\mu} O_{X_{\mu}}^{vir})}{Tr(\Lambda^* N_{\mu}^*)} \right) = \chi^f \left( \mathcal{Y}_{\mu}^\prime, \frac{Tr(V_{\mu} \otimes j^*_{\mu} O_{X_{\mu}}^{vir})}{Tr(\Lambda^* (N_{vir}^{\mu})^*)} \right) = \chi^f \left( X_{\mu}^\prime, \frac{Tr(V_{\mu} \otimes O_{X_{\mu}}^{vir})}{Tr(\Lambda^* (N_{vir}^{\mu})^*)} \right).$$  \hfill (13)

This concludes the proof of the proposition.

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