ASSOCIATIVITY AND THOMPSON’S GROUP
ROSS GEOGHEGAN AND FERNANDO GUZMÁN

Abstract. Given a set $S$ equipped with a binary operation (we call this a “bracket algebra”) one may ask to what extent the binary operation satisfies some of the consequences of the associative law even when it is not actually associative? We define a subgroup $\text{Assoc}(S)$ of Thompson’s Group $F$ for each bracket algebra $S$, and we interpret the size of $\text{Assoc}(S)$ as determining the amount of associativity in $S$ - the larger $\text{Assoc}(S)$ is, the more associativity holds in $S$. When $S$ is actually associative, $\text{Assoc}(S) = F$; that is the trivial case. In general, it turns out that only certain subgroups of $F$ can occur as $\text{Assoc}(S)$ for some $S$, and we describe those subgroups precisely. We then explain what happens in some familiar examples: Lie algebras with the Lie bracket as binary operation, groups with the commutator bracket as binary operation, the Cayley numbers with their usual multiplication, as well as some less familiar examples. In the case of a group $G$, with the commutator bracket as binary operation, it is better to think of the “virtual size of $G$”, determined by all the groups $\text{Assoc}(H)$ such that $H$ is a subgroup of finite index in $G$. This gives a way of partitioning groups into “small”, “intermediate” and “large” - a partition suggestive of, but different from, traditional measures of a group’s size such as growth, isoperimetric inequality and “amenable vs. non-amenable”

1. Introduction

1.1. Thompson’s Group. We begin by recalling the “pairs of binary trees” definition of Thompson’s group $F$ as we will need precise terminology.

By a binary tree we will mean either the trivial tree consisting of a single vertex and no edges, or a finite tree having one vertex of order 2 (called the root) and all other vertices having order either 1 (they are the leaves) or 3 (they are the interior vertices). The only vertex in the trivial tree is considered to be both the root and a leaf. Included in the structure of what we call a binary tree is a labeling of the edges as follows: each edge has a canonical orientation away from the root, and in non-trivial trees exactly two edges point away from each non-leaf vertex; one of those edges is labeled 0 and the other 1. This is most intuitively seen when the binary tree is embedded in the Euclidean plane as in the pictures below, with
all oriented edges pointing downward, 0 to the left and 1 to the right. Indeed, every binary
tree (in our sense) can be represented in this way. We will not distinguish between two
binary trees when there exists a label-preserving isomorphism between them.

The leaves of a binary tree have a canonical ordering as follows: there is a unique geodesic
dgepath from the root to each leaf specified by a word in the alphabet \( \{0, 1\} \): 0 means
follow the left edge, 1 means follow the right edge. For example, in the figure below, the leaf
marked \( b \) is specified by 010 and the leaf marked \( d \) is given by 10; The canonical ordering
of the leaves is then the corresponding lexicographic ordering of those geodesics. By the
\( i \)th leaf we will mean the \( i \)th term in this ordering; if there are \( n \) leaves they are numbered
1,\ldots,\( n \).

A \textit{caret} is a binary tree with exactly two edges. If \( p \) is a binary tree having \( n \) leaves, the
\( i \)-th \textit{elementary expansion} of \( p \) is the binary tree \( \beta^i(p) \) obtained from the disjoint union of
\( p \) and a single caret by identifying the root of the caret with the \( i \)th leaf of \( p \). This makes
sense when \( 1 \leq i \leq n \); it is convenient to define \( \beta^i(p) = p \) when \( i > n \).

Let \( T_n \) denote the set of (isomorphism classes of) binary trees which have \( n \) leaves. We
write \( T^{(2)} = \coprod_{n=1}^\infty T_n \times T_n \) and we define \( b^i : T^{(2)} \to T^{(2)} \) to agree with \( \beta^i \times \beta^i \) on \( T_n \times T_n \).
The relations \( (p, q) \sim b^i(p, q) \), one such relation for each \( i \), together generate an equivalence
relation on \( T^{(2)} \). The set of equivalence classes is denoted by \( F \). We write \( \langle p, q \rangle \) for the
equivalence class of \( (p, q) \).

To define a multiplication on \( F \) we note first that any two binary trees have a common
expansion, where an \textit{expansion} of \( p \) is a binary tree of the form \( \beta(p) = \beta^{i_k} \ldots \beta^{i_1}(p) \).
We define a \textit{simultaneous expansion} of the ordered pair \( (p, q) \) to be an ordered pair of the form
\( (\beta(p), \beta(q)) \). The \textit{product} \( \langle p, q \rangle \cdot \langle r, s \rangle \) is then defined to be \( \langle p', s' \rangle \), where \( (p', q') \) is an expansion
of \( (p, q) \), \( (r', s') \) is an expansion of \( (r, s) \), and \( q' = r' \). This is well-defined and associative.
With respect to this multiplication the element \( \langle p, p \rangle \) (where \( p \) is any binary tree - this is
independent of \( p \)) is a two-sided identity, and \( \langle q, p \rangle \) is a multiplicative inverse for \( \langle p, q \rangle \). Thus
we have a group - Thompson’s group \( F \).
The group $F$ has standard generators $x_0$ and $x_1$ represented by the following two pairs of binary trees:

\[
\left( \begin{array}{c}
\ \ \\
\ \ \\
\ \ \\
\ \ \\
\ \ \\
\ \ \\
\ \ \\
\ \ \\
\end{array} \right)
\]

\[
\left( \begin{array}{c}
\ \ \\
\ \ \\
\ \ \\
\ \ \\
\ \ \\
\ \ \\
\ \ \\
\ \ \\
\ \ \\
\ \ \\
\end{array} \right)
\]

For details, and for information about some of the many remarkable properties of this group, see [2] or the expository article [5].

1.2. Bracket Algebras. By a bracket algebra we will mean a set $S$ together with a binary operation $\alpha : S \to S$. A binary tree having $n$ leaves determines a rule for associating ordered $n$-tuples of members of $S$ using $\alpha$, i.e. determines an $n$-ary operation on $S$; see Section 2 for details. So an ordered pair $(p, q)$ of binary trees having the same number of leaves, can be interpreted as encoding a “law” in $S$ saying that the method of association defined by $p$ always gives the same result in $S$ as the method of association defined by $q$. In that case we can say that $S$ “satisfies” $(p, q)$. Similarly, an element $\langle p, q \rangle$ of $F$ can be interpreted as defining a “stable law” in $S$; i.e. $S$ satisfies some simultaneous expansion of $(p, q)$. Note that if $S$ satisfies $(p, q)$ then it satisfies every simultaneous expansion of $(p, q)$.

**Proposition 1.1.** The elements $\langle p, q \rangle$ of $F$ which define stable laws in the bracket algebra $S$ form a subgroup of $F$.

The proof is in Section 2. We call the subgroup in Proposition 1.1 $\text{Assoc}(S)$, the group of stable associativities of $S$.

1.3. Characterization of the groups $\text{Assoc}(S)$. A natural first question is: which subgroups of $F$ can occur as $\text{Assoc}(S)$ for some bracket algebra $S$? To answer this we need some further vocabulary concerning binary trees.

The right shift of $p$ is the binary tree $\sigma_1(p)$ obtained from the disjoint union of $p$ and a single caret by identifying the root of $p$ with the second (i.e. right) leaf of the caret. The left shift of $p$, $\sigma_0(p)$, is defined similarly.
There is an involution on the set of binary trees taking a binary tree \( p \) to its \textit{reflection} \( \rho(p) \). In terms of planar pictures, \( \rho(p) \) is the mirror image of \( p \) in the \( y \)-axis.

The two shifts induce endomorphisms \( s_i : F \to F \) \((i = 0 \text{ or } 1)\) taking \( \langle p, q \rangle \) to \( \langle \sigma_i(p), \sigma_i(q) \rangle \). These are the \textit{left and right shift endomorphisms}. The \textit{reflection} automorphism \( R : F \to F \) is defined by \( R(\langle p, q \rangle) = \langle \rho(p), \rho(q) \rangle \). Note that \( s_0 = R.s_1.R \). The right shift \( s_1 \) is, of course, well-known, and is defined by the formula \( s_1(x_0) = x_1 \) and \( s_1(x_1) = x_2 \) where \( x_2 := x_1^{x_0} \). The formula for the left shift in terms of the standard generators is less pleasant: \( s_0(x_0) = (x_0 x_1^{-1})^{x_0^{-1}} \) and \( s_0(x_1) = (x_1 x_2^{-1})^{(x_0 x_1)^{-1}} \).

The shift endomorphisms can be much better understood in terms of the “dyadic piecewise linear” model of \( F \). This is a well-known faithful representation of \( F \) in the group of increasing self-homeomorphisms of the closed unit interval \( I \). Given \( \langle p, q \rangle \in F \), two copies of \( I \) are to be dyadically subdivided according to the instructions of the trees \( p \) and \( q \), and then \( \langle p, q \rangle \) is identified with the dyadic piecewise linear increasing homeomorphism of \( I \) which maps each segment of the \( p \)-subdivision affinely onto the corresponding segment of the \( q \)-subdivision. In these terms, the two shift endomorphisms can be easily understood from the following picture:

Now we can answer the question about what subgroups of \( F \) occur as \( \text{Assoc}(S) \):

**Theorem 1.2.** A subgroup \( H \) of \( F \) is \( \text{Assoc}(S) \) for some bracket algebra \( S \) if and only if both shift endomorphisms, \( s_0 \) and \( s_1 \), map \( H \) into itself.

The proof is at the end of Section 2. Examples 2.1 and 7.1 illustrate how a non-normal subgroup of \( F \) can arise as \( \text{Assoc}(S) \). There are infinitely many such subgroups.
A companion question is: which normal subgroups of $F$ occur as $\text{Assoc}(S)$ for some bracket algebra $S$? Recall that $F/F'$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ where the standard generators $x_0$ and $x_1$ map respectively to $(1,0)$ and $(0,1)$. Since the non-trivial normal subgroups of $F$ are the subgroups containing $F'$, they are in bijective correspondence with the subgroups of $\mathbb{Z} \times \mathbb{Z}$. Both shifts preserve $F'$ and so induce endomorphisms $\bar{s}_i$ of $\mathbb{Z} \times \mathbb{Z}$; the formulas are: $\bar{s}_0(m,n) = (m,-m)$ and $\bar{s}_1(m,n) = (0,m+n)$. Thus we have:

**Corollary 1.3.** A normal subgroup of $F$ is $\text{Assoc}(S)$ for some $S$ if and only if it is the preimage in $F$ of a subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by $(m,-m)$ and $(0,n)$ for some integers $m$ and $n$.

In Proposition 2.4 we give a bracket algebra criterion for $\text{Assoc}(S)$ to be normal.

1.4. **Strongly Regular Laws.** Given a bracket algebra $S$, an element $(p,q)$ of $T^{(2)}$ may or may not hold as a law in $S$. For example, the associative law in $S$

$$\alpha(\alpha(x_1,x_2),x_3) \approx \alpha(x_1,\alpha(x_2,x_3))$$

corresponds to the pair of trees

$$\left( \begin{array}{c} \wedge, \\
\wedge \end{array} \right)$$

We write $S \models p \approx q$ when that law holds in $S$.

Such laws are of a special type: each variable occurs exactly once on each side, and variables occur in the same order on both sides. We will call such laws (represented by pairs $(p,q) \in T^{(2)}$) strongly regular laws. Similarly, we say that the law $p \approx q$ eventually holds in $S$, and write $S \models_e p \approx q$, if there is a simultaneous expansion $(p',q')$ of $(p,q)$ such that $S \models p' \approx q'$. In these terms we obviously have:

**Proposition 1.4.** $\text{Assoc}(S)$ is trivial if and only if no strongly regular law eventually holds in $S$.

1.5. **Example: Lie algebras.** For Lie algebras with the Lie bracket as the binary operation we have:

**Theorem 1.5.** Let $L$ be a finite-dimensional complex Lie algebra. $\text{Assoc}(L)$ is either trivial or is $F$. It is $F$ if and only if $L$ is a solvable Lie algebra.
The proof is in Section 4.

Kac-Moody algebras are infinite-dimensional, but the proof of Theorem 4.6 will show:

**Addendum 1.1.** If $L$ is a Kac-Moody algebra then $\text{Assoc}(L)$ is trivial.

1.6. **Example: Groups.** A group $G$ with the commutator bracket $[x, y] = xyx^{-1}y^{-1}$ as the binary operation is an important example of a bracket algebra. When we discuss $G$ as a bracket algebra this binary operation is always understood.

**Proposition 1.6.** Let $G$ be a group. $\text{Assoc}(G) = F$ if and only if $G$ is a solvable group.

**Proposition 1.7.** $\text{Assoc}(A_5)$ is trivial, where $A_5$ denotes the finite alternating group on five letters.

Proposition 1.6 is proved in Section 3, and Proposition 1.7 is proved in Section 5. Proposition 1.7 is of interest because $A_5$ has a subgroup of finite index, the trivial group $\{1\}$, such that $\text{Assoc}(\{1\})$ is $F$. Such instability suggests that in order to use $\text{Assoc}(G)$ as a good measure of the size of a group $G$ we should consider things virtually.

We define $G$ to be **large** if for every subgroup $H$ of finite index $\text{Assoc}(H)$ is trivial. Equivalently, by Proposition 1.4, $G$ is large if and only if no strongly regular law holds eventually in any subgroup $H$ of finite index in $G$.

We define $G$ to be **small** if it has a subgroup $H$ of finite index such that $\text{Assoc}(H)$ is a non-trivial normal subgroup of $F$. (Recall that a subgroup of $F$ is normal if and only if it is either trivial or contains the commutator subgroup $F'$.)

If $G$ satisfies the third possibility, neither large nor small, we say that $G$ is **of intermediate size**. We do not know the answer to the following:

**Question 1.1.** Does there exist a group of intermediate size?

Indeed, we do not know a group $G$ for which $\text{Assoc}(G)$ is neither $F$ nor the trivial group. Let $F_k$ denote the free group on $k$ generators $\{X_1, \ldots, X_k\}$ and let $w(X_1, \ldots, X_k)$ be a non-empty word in those generators and their inverses. This word $w$ defines a **law** in a group $G$ if the statement $w(g_1, \ldots, g_k) = 1$ is true for all choices of $g_i \in G$. A group **satisfies no law stably** if each subgroup of finite index satisfies no law. Every strongly regular law in the commutator algebra of $G$ can be recast as a law in this sense, so by Proposition 1.4...
every group which satisfies no law stably is large. This and some related observations are summarized in:

**Proposition 1.8.** Groups satisfying no law stably are large. If a group $G$ has a quotient containing a large subgroup then $G$ is also large. Virtually solvable groups are small.

This is proved in Section 3.

Many interesting groups are known to satisfy no law stably, for example non-abelian free groups, non-elementary hyperbolic groups, and Thompson’s Group $F$, and are therefore large. (Proof of this for $F$: that $F$ itself satisfies no law was proved in [4]; every subgroup of finite index in $F$ contains a normal subgroup of finite index, which in turn contains the commutator subgroup, and hence contains a copy of $F$.) Moreover, by Proposition 1.8 the Tits Alternative separates the finitely generated linear groups neatly into the large and the small (i.e. every such group either contains a free non-abelian subgroup or is virtually solvable).

The relationship between “stably large” and “non-amenable” is not so clean. Among the large groups are the non-abelian free groups (which are non-amenable) and Thompson’s group $F$ (which is conjectured to be non-amenable). However, it is not true that “non-amenable” is equivalent to “stably large”. An example is the “first Grigorchuk group” $\Gamma$ ([6] or [3]), which is finitely generated, has subexponential growth (and is therefore amenable) but is large since it satisfies no law stably ([1] or [7]). Now this group $\Gamma$ is not elementary amenable, but even in the elementary amenable case there are distinctions to be made. We ask:

**Question 1.2.** Is there a finitely generated large elementary amenable group?

The words “finitely generated” are included in this question because of:

**Proposition 1.9.** Let $S_n$ denote the group of permutations of $\{1, \ldots, n\}$ and let $S_\infty$ denote the union of the groups $S_n$. Then $S_\infty$ is elementary amenable and is large.

This is proved in Section 5.

1.7. **The Five Variable Law.** The commutator of the two standard generators $[x_0, x_1] = x_0 x_1 x_0^{-1} x_1^{-1}$ is most efficiently represented by the following pair of binary trees.
This pair of trees defines a strongly regular law which we call the Five Variable Law.

**Theorem 1.10.** The commutator $[x_0, x_1]$ lies in $\text{Assoc}(S)$ if and only if $\text{Assoc}(S)$ is a non-trivial normal subgroup of $F$. In other words, $\text{Assoc}(S)$ is a non-trivial normal subgroup of $F$ if and only if the Five Variable Law holds eventually in $S$.

This is proved in Section 6.

An elegant application is that the Five Variable Law holds eventually in a finite-dimensional complex Lie algebra $L$ if and only if $L$ is solvable. Perhaps this is well-known?

We call the bracket algebra $(S, \alpha)$ **simply perfect** if $\alpha$ is surjective.

**Remark 1.11.** It is clear that if $S$ is simply perfect and if some simultaneous expansion of the Five Variable Law holds in $S$ then the Five Variable Law itself holds in $S$. Indeed, this holds for any strongly regular law. This immediately implies the next Corollary.

**Corollary 1.12.** Let $S$ be a simply perfect bracket algebra. $\text{Assoc}(S)$ is a non-trivial normal subgroup of $F$ if and only if the Five Variable Law holds (“on the nose”) in $S$.

1.8. **Example: The Cayley numbers; bracket algebras having identity elements.** A classical example of a non-associative bracket algebra is the algebra of Cayley numbers, also known as the octonion algebra; its underlying real vector space is eight-dimensional. Since it possesses a two-sided identity element 1, it is covered by the following general theorem:

**Theorem 1.13.** Let $S$ be a bracket algebra which has a two-sided identity element 1. If $S$ is non-associative then it satisfies no non-trivial strongly regular identity. Thus $\text{Assoc}(S)$ is either $F$ or is trivial depending on whether $S$ is associative or not.

In particular, this theorem (which is proved in Section 7) applies to all loops. (A **loop** is a bracket algebra with two-sided identity element such that each element has a unique right inverse and a unique left inverse.)
By contrast, we exhibit a four-element non-associative bracket algebra $S$ having a right identity element but no left identity element such that $\text{Assoc}(S)$ is a non-normal subgroup of $F$; see Example 7.1.

1.9. **Acknowledgments.** An early version of some of these ideas was presented by the first-named author in the symposium “Thompson’s Group at Forty Years” at the American Institute of Mathematics in January 2004. He thanks that institution for their support. He also takes this opportunity to warmly thank his colleagues for their dedication of this volume, and for attending the accompanying special session held in his honor at the October 2004 meeting of the American Mathematical Society at Vanderbilt University. And he expresses appreciation to Michael Mihalik and Mark Sapir who organized that session.

## 2. Bracket algebras and Thompson’s Group

Let $T$ denote the set of all binary trees, made into a bracket algebra as follows. The trivial binary tree will be denoted by $\cdot$, and the binary tree having $p$ as the left subtree and $q$ as the right subtree,

$$\begin{array}{c}
\bigwedge \\
p & q
\end{array}$$

will be denoted by $[p, q]$. This is the free bracket algebra generated by $\cdot$, the trivial binary tree. Recall that we extended the function $\beta^i : T_n \to T_{n+1}$ to a function $T \to T$ by setting $\beta^i(p) = p$ for $p \in T_n$ when $i > n$. Let $B$ denote the monoid generated by $\{\beta^i | i \in \mathbb{N}\}$ with composition as multiplication (the *monoid of expansions*). Thus we have an action of $B$ on the set $T$. Observe that the elementary expansions $\beta^i$, for $i \in \mathbb{N}$ satisfy:

$$\beta^i \beta^j = \beta^{i+1} \beta^i,$$

when $i < j$.

(Indeed, it can be shown that $B$ is isomorphic to the negative monoid of Thompson’s group $F$.)

Now, let $\alpha : S \times S \to S$ be a bracket algebra. (We will sometimes denote the binary operation in such an algebra by $[,]$ rather than by $\alpha$, and we will often omit explicit reference to the operation altogether.) We define a bracket algebra $X(S)$, a quotient $B$-set of the $B$-set $T$, which makes precise the notion of “expansion” in the context of any bracket algebra.

First we define $X_n(S)$ and a bracket on $\coprod_{n=1}^{\infty} X_n(S)$ recursively by:
• The identity map, $I_S : S \to S$, is the only member of $X_1(S)$

• if $f \in X_n(S)$, $g \in X_m(S)$, and $\bar{s} = (s_1, \ldots, s_n, s_{n+1}, \ldots, s_{n+m})$, then

$$[f,g] : S^{n+m} \to S$$

$$\bar{s} \mapsto \alpha(f(s_1, \ldots, s_n), g(s_{n+1}, \ldots, s_{n+m}))$$

belongs to $X_{n+m}(S)$.

The promised $X(S)$ is $\coprod_{n=1}^{\infty} X_n(S)$ with this bracket. It is the bracket algebra of “$\alpha$-operations” on $S$, generated by one element, namely, $I_S$. Hence, it is a quotient of $T$, with quotient epimorphism $\hat{\cdot} : T \to X(S)$ given recursively by

• $\hat{\cdot} = I_S$ where $\cdot$ denotes the trivial binary tree, and

• $\hat{[p,q]}(s_1, \ldots, s_n, s_{n+1}, \ldots, s_{n+m}) = \alpha(\hat{p}(s_1, \ldots, s_n), \hat{q}(s_{n+1}, \ldots, s_{n+m}))$

Proposition 2.1. The action of $B$ on $T$ descends via the epimorphism $\hat{\cdot} : T \to X(S)$ to an action of $B$ on $X(S)$, so that $\hat{\cdot}$ is also a morphism of $B$-sets.

Proof. Clearly the function $\hat{\cdot}$ is a surjective morphism of bracket algebras. If $p, q \in T$ are such that $\hat{p} = \hat{q}$ then $p$ and $q$ have the same number $n$ of leaves. For $i \leq n$

$$\hat{\beta^i}(p)(s_1, \ldots, s_i, s_{i+1}, \ldots, s_n) = \hat{p}(s_1, \ldots, \alpha(s_i, s_{i+1}), \ldots, s_n)$$

$$= \hat{q}(s_1, \ldots, \alpha(s_i, s_{i+1}), \ldots, s_n)$$

$$= \hat{\beta^i}(q)(s_1, \ldots, s_i, s_{i+1}, \ldots, s_n)$$

so, $\hat{\beta^i}(p) = \hat{\beta^i}(q)$.

For $\beta \in B$ we call $\beta(f)$ an expansion of $f \in X(S)$.

The action of $B$ on $T$ induces a diagonal action of $B$ on $T^{(2)}$ and the set of orbits, $T^{(2)}/B$, is precisely Thompson’s Group $F$.

We write $X^{(2)}_n(S) = X_n(S) \times X_n(S)$ and $X^{(2)} = \coprod_{n=1}^{\infty} X^{(2)}_n$. We call $(\beta(f), \beta(g))$ a simultaneous expansion of the pair $(f, g)$. Again using the diagonal action of $B$, we consider the set of orbits $X^{(2)}/B$.

Clearly we have:
Proposition 2.2.  
(1) The function

\[ \hat{\circ} : \mathcal{T}^{(2)} \to X^{(2)} \]
\[ (p, q) \mapsto (\hat{p}, \hat{q}) \]

is a surjective \(B\)-map.

(2) The pre-image under the induced function

\[ \hat{\circ} : \mathcal{T}^{(2)}/B \to X^{(2)}/B \]
\[ (p, q) \mapsto (\hat{p}, \hat{q}) \]

of the trivial element of \(X^{(2)}/B\) (i.e. of the image of \(1 \in F = \mathcal{T}^{(2)}/B\)) is the set \(\text{Assoc}(S)\).

When we consider strongly regular identities which eventually hold in \(S\), we look at pairs \((p, q) \in \mathcal{T}_n \times \mathcal{T}_n\), such that \(S \models_e p \approx q\), i.e. such that \(\hat{p'} = \hat{q'}\) where \((p', q') = (\beta(p), \beta(q))\) for some simultaneous expansion \(\beta \times \beta\). Of course, this is the same as looking at classes \(\langle p, q \rangle \in \mathcal{T}^{(2)}/B(= F)\).

Consideration of binary trees leads immediately to the observation that \(B\) has the “common left multiples property”, i.e. given \(\beta_1, \beta_2 \in B\) there are \(\beta_3, \beta_4 \in B\) such that \(\beta_3\beta_1 = \beta_4\beta_2\).

Proposition 1.1 can be restated as:

Proposition 2.3. Let \(S\) be a bracket algebra. The set of stable associativities \(\text{Assoc}(S) = \{ \langle p, q \rangle \in F \mid S \models_e p \approx q \}\) is a subgroup of Thompson’s Group \(F\).

Proof. Clearly \(\text{Assoc}(S)\) contains the identity element of \(F\) and is closed under inverses. For closure under composition, let \(\langle p, q \rangle, \langle s, r \rangle \in F\). WLOG we may assume \(q = s\), so we have \(S \models_e p \approx q\) and \(S \models_e q \approx r\). Let \(\beta_1, \beta_2\) be expansions such that \(\beta_1(\hat{p}) = \beta_1(\hat{q})\) and \(\beta_2(\hat{q}) = \beta_2(\hat{r})\). By “common left multiples” there are expansions \(\beta_3, \beta_4\) such that \(\beta_3\beta_1 = \beta_4\beta_2\). Let \(\beta = \beta_3\beta_1 = \beta_4\beta_2\). Then, \(\beta(\hat{p}) = \beta_3\beta_1(\hat{p}) = \beta_3\beta_1(\hat{q}) = \beta_4\beta_2(\hat{q}) = \beta_4\beta_2(\hat{r}) = \beta(\hat{r})\). So, \(S \models_e p \approx r\). \(\square\)

The next proposition gives us necessary and sufficient conditions for \(\text{Assoc}(S)\) to be a normal subgroup of \(F\). First a definition.
Definition 2.1. Let $S$ be a bracket algebra. We say that $S$ is normal provided that: whenever $\beta_1, \beta_2 \in \mathcal{B}$ and $g \in X(S)$ are such that $g \neq \beta_1(g) = \beta_2(g)$, then for any $f \in X(S)$ there is $\beta \in \mathcal{B}$ such that $\beta \beta_1(f) = \beta \beta_2(f)$.

Proposition 2.4. Let $S$ be a bracket algebra. $\text{Assoc}(S)$ is a normal subgroup of $F$ if and only if $S$ is a normal bracket algebra.

Proof. Assume first that the bracket algebra $S$ is normal. Let $\langle p, q \rangle \in \text{Assoc}(S)$. WLOG we may assume that $\hat{p} = \hat{q}$. For any $\langle r, s \rangle$ we have

$$\langle r, s \rangle \cdot \langle p, q \rangle \cdot \langle s, r \rangle = \langle \beta_1(r), \beta_1(s) \rangle \cdot \langle \beta_2(p), \beta_2(q) \rangle \cdot \langle \beta_3(s), \beta_3(r) \rangle = \langle \beta_1(r), \beta_3(r) \rangle$$

where the $\beta_i$ are chosen so that $\beta_1(s) = \beta_2(p)$ and $\beta_2(q) = \beta_3(s)$.

So $\hat{\beta_1} = \hat{\beta_2} = \hat{\beta_3}$. Thus, either $s \neq \beta_1(s) = \beta_3(s)$ in which case, by normality, there exists $\beta$ such that $\hat{\beta \beta_1} = \hat{\beta \beta_3}$, implying that our conjugate of $\langle p, q \rangle$ also lies in $\text{Assoc}(S)$; or $s = \beta_1(s) = \beta_3(s)$ in which case $\beta_1(r) = \beta_3(r) = r$, which would mean that $\langle p, q \rangle$ is the trivial element of $F$.

Conversely, if $\text{Assoc}(S)$ is normal then the multiplication on $F = \hat{T}^{(2)}/\mathcal{B}$ descends to $X^{(2)}/\mathcal{B}$ and the condition for normality of $S$ follows easily.

We now describe an example where the group $\text{Assoc}(S)$ is not normal in $F$. For this, we need further notation concerning binary trees.

If $p$ is a binary tree, with each vertex of $p$ is associated a unique word $w$ in the alphabet $\{0, 1\}$. This was explained for leaves in the Introduction but applies equally to all vertices. The leaves of $p$ are numbered 1 through $n$ from left to right, and the word of the $i$-th leaf is denoted by $l_i$ or $l_i(p)$. We say that $p$ has a free caret at the interior vertex $w$ if both $w0$ and $w1$ denote leaves of $p$. The depth of the $i$-th leaf is the length of the word that denotes it, $d_i = |l_i|$. If $w$ denotes a vertex of $p$, the subtree of $p$ at that vertex is denoted by $p_w$.

Example 2.1. Let
Let $S$ be a free algebra in the variety of bracket algebras that satisfy the identity: $p \approx q$. Consider the expansions $\beta^4$, $\beta^2$ and the function $g(X, Y, Z, W) = \hat{p}(X, Y, Z, W) = \hat{q}(X, Y, Z, W)$. Notice that $\beta^4(p) = \beta^2(q)$, so $\beta^4(g) = \beta^2(g)$. Let

$$r = \begin{array}{c}
\text{Diagram 1}
\end{array}$$

$$r_1 = \beta^4(r) = \begin{array}{c}
\text{Diagram 2}
\end{array}$$

$$r_2 = \beta^2(r) = \begin{array}{c}
\text{Diagram 3}
\end{array}$$

We claim that $S$ does not satisfy any simultaneous expansion of $r_1 \approx r_2$. Let $\beta \in B$ and let $r'_1 = \beta(r_1)$, and $r'_2 = \beta(r_2)$. There are $t_1, \ldots, t_5 \in \mathcal{T}$ such that

$$r'_1 = r_1(t_1, \ldots, t_5) = \begin{array}{c}
\text{Diagram 4}
\end{array}$$

$$r'_2 = r_2(t_1, \ldots, t_5) = \begin{array}{c}
\text{Diagram 5}
\end{array}$$

If it were true that $S \models r'_1 \approx r'_2$ then it would be possible to rewrite $r'_1$ to $r'_2$ by a finite number of applications of $p \approx q$ on different subtrees. (Strictly, this informal statement is a consequence of the Completeness Theorem of Equational Logic.) If $t$ is any one of the intermediate trees, then applying $p \approx q$ at the root of $t$ does not move any leaves from the left to the right subtree of $t$; and applying $p \approx q$ at an interior vertex of $t$ does not either. But any leaf of $r'_1$ under $t_3$ is in the left subtree of $r'_1$, and any leaf of $r'_2$ under $t_3$ is in the right subtree of $r'_2$.

The pair $(p, q)$ used in this Example has the property that $\langle p, q \rangle = x_1 \in F$. It will follow from the proof of Theorem 1.2 below, that the group $\text{Assoc}(S)$ in Example 2.1 is the smallest subgroup of $F$ containing $x_1$ which is invariant under both shifts. Viewed in the “dyadic piecewise linear” model this subgroup is the stabilizer of $\{\frac{1}{2^n} | n \in \mathbb{N}\}$. It is abstractly isomorphic to a countably infinite weak direct product of copies of $F$.

The proof of Theorem 1.2 can now be given. The “only if” part is clear: i.e. $\text{Assoc}(S)$ is invariant under both shifts. The “if” part follows from the following more precise Lemma:
Lemma 2.5. Let $K := \{(p_m, q_m) \mid m \in \mathcal{M}\}$ be a subset of $F$ and let $S$ be a free algebra in the variety defined by $\overline{K} := \{(p_m, q_m) \mid m \in \mathcal{M}\}$; i.e., the defining set for $\overline{K}$ contains one representative of each member of $K$. Then $\text{Assoc}(S)$ is the smallest subgroup $H$ of $F$ containing $K$ and invariant under both shifts.

Proof. Let $\langle r, s \rangle \in \text{Assoc}(S)$. WLOG we may assume that $S \models r \approx s$. Then we can pass from $r$ to $s$ in finitely many steps $r = t^0, t^1, \ldots, t^n = s$ so that each $(t^i, t^{i+1})$ consists of a pair of trees which are identical except that at a certain vertex $v$ and for some $m \in \mathcal{M}$, $t^i_v = p'_m$ and $t^{i+1}_v = q'_m$ (or vice versa) for a simultaneous expansion $(p'_m, q'_m)$ of $(p_m, q_m) \in \overline{K}$. Let $\epsilon_1 \cdots \epsilon_k$ be the word in the alphabet $\{0, 1\}$ which labels $v$. For any tree $p$ define $\sigma_v(p) := \sigma_{\epsilon_1} \sigma_{\epsilon_2} \cdots \sigma_{\epsilon_k}(p)$. Then $\langle t^i, t^{i+1} \rangle = \langle \sigma_v(p'_m), \sigma_v(q'_m) \rangle$ (or $\langle \sigma_v(q'_m), \sigma_v(p'_m) \rangle$) in the “vice versa” case) in $F$. Thus $\langle r, s \rangle = \langle t^0, t^1 \rangle, \langle t^1, t^2 \rangle, \ldots, \langle t^{n-1}, t^n \rangle$ is a product of members of $F$ of the form $s_{\epsilon_1} \cdots s_{\epsilon_k}(\langle p_m, q_m \rangle)$ (or its inverse in the “vice versa” case) and hence $\langle r, s \rangle$, which is an arbitrary element of $\text{Assoc}(S)$, lies in $H$; i.e., $\text{Assoc}(S) \leq H$. Since $\text{Assoc}(S)$ contains $K$ and (as we have already said) is invariant under both shifts, we also have $H \leq \text{Assoc}(S)$. □

In Example 2.1 we saw a non-normal $\text{Assoc}(S)$, expressed as the stabilizer of a subset of $I$. There are of course many other such non-normal subgroups, for example many expressible as stabilizers.

3. Solvable bracket algebras

We say that the bracket algebra $S$ is solvable if there exist $0 \in S$, $n \geq 1$, and $u \in T_n$ such that $\widehat{u}(S^n) = \{0\}$. Note that if $v \in T_m$ is an expansion of $u$ then $\widehat{v}(S^m) = \{0\}$.

Theorem 3.1. Let $S$ be a solvable bracket algebra. Any strongly regular law $p \approx q$ eventually holds in $S$.

Proof. We write $r \geq s$ if the binary tree $r$ is an expansion of the binary tree $s$. This is a partial ordering. Find expansions $\beta_1, \beta_2$ such that $\beta_1(p) \geq u$, and $\beta_2(\beta_1(q)) \geq u$. Let $\beta = \beta_2 \circ \beta_1$. Then $\beta(p) = \beta_2(\beta_1(p)) \geq \beta_1(p) \geq u$, and $\beta(q) = \beta_2(\beta_1(q)) \geq u$. Since $\beta(p), \beta(q) \geq u$ we have $\widehat{\beta}(p) = 0 = \widehat{\beta}(q)$ and $S \models_e p \approx q$. □

This theorem implies that if $S$ is solvable then $\text{Assoc}(S) = F$. In the case of groups or Lie algebras our definition of “solvable” is easily seen to be equivalent to the usual definitions.
of “solvable” in those contexts. For solvable Lie algebras and solvable groups the group of stable associativities is \( F \).

The proofs of Propositions 1.6 and 1.8 are now complete.

4. Lie Algebras

In this section we prove Theorem 1.5 and Addendum 1.1.

Lemma 4.1. The four element bracket algebra \( S = \{0, a, b, c\} \) with operation table

|   | 0  | a  | b  | c  |
|---|----|----|----|----|
| 0 | 0  | 0  | 0  | 0  |
| a | 0  | 0  | a  | b  |
| b | 0  | a  | 0  | c  |
| c | 0  | b  | c  | 0  |

satisfies no non-trivial strongly regular law, so \( \text{Assoc}(S) \) is trivial.

Proof. We note two things from this table. First, the function \( \hat{} : S \times S \to S \) is surjective; i.e. \( S \) is simply perfect. Therefore for any \( p \in T \) the \( n \)-ary function \( \hat{p} \) is surjective. Secondly, for any two non-zero \( u \neq v \in S \), the centralizer of \( \{u, v\} \), \( \text{C}_S(\{u, v\}) = \{x \in S | [x, u] = [x, v] = 0\} \) is trivial, i.e. \( \{0\} \).

Fix \( p \in T_n \). Given \( u \in S \) we denote by \( \hat{p}|_{x_i = u} \) the restriction of \( \hat{p} \) to the subset \( S^{i-1} \times \{u\} \times S^{n-i} \subseteq S^n \); we think of it as fixing the value of one of the arguments of \( \hat{p} \). Similarly for \( \hat{p}|_{x_i = u, x_j = v} \) where \( 1 \leq i \neq j \leq n \) and \( u, v \in S \). We will denote the image of \( \hat{p}|_{x_i = u} \) by \( \tilde{p}|_{x_i = u} \). Let \( d_i \) denote the depth of the \( i \)-th leaf of \( p \).

Claim 1: If \( d_i = 1 \) then for any \( u \in S \setminus \{0\} \) the set \( \tilde{p}|_{x_i = u} \) is non-trivial (contains 2 of the values in \( S \setminus \{0\} \)) and has trivial centralizer. It follows that if \( d_i \geq 1 \) then \( \tilde{p}|_{x_i = u} \) is non-trivial and has trivial centralizer.

Claim 2: If the \( i \)-th and \((i+1)\)-th leaves do not form a free caret, i.e. \( l_i \) and \( l_{i+1} \) differ at more than the last bit, then for any \( u, v \in S \setminus \{0\} \) the function \( \hat{p}|_{x_i = u, x_{i+1} = v} \) is not trivial. To see this, WLOG assume that the \( i \)-th leaf is in the left subtree \( p_0 \) of \( p \), and the \((i+1)\)-th leaf is in the right subtree \( p_1 \) of \( p \), i.e. that \( l_i \) and \( l_{i+1} \) have no non-empty common prefix. (Otherwise, apply the result to \( p_w \), the subtree of \( p \) at the vertex \( w \), where \( w \) is the largest common prefix of \( l_i \) and \( l_{i+1} \) i.e. to the smallest subtree of \( p \) that contains both the \( i \)-th
and the $(i + 1)$-th leaves, and use Claim 1.) Also WLOG assume $d_i \geq d_{i+1}$. Since $l_i$ and $l_{i+1}$ differ at more than the last bit it cannot be the case that $d_i = d_{i+1} = 1$, so $d_i \geq 2$. Note that the depths of leaves in $p_0$ and $p_1$ are one less than the corresponding depths in $p$. From Claim 1 we know that $\tilde{p}_0|x_i=u$ has trivial centralizer, and $\tilde{p}_1|x_i=v$ is non-trivial. So $\hat{p}|x_i=u,x_{i+1}=v$ is non-trivial.

Now fix $p, q \in T_n$. We want to show that if $S \models p \approx q$ then $p = q$. Since $S$ is simply perfect, by Remark 1.11 we may assume that the pair $(p, q)$ is the reduced representative of $\langle p, q \rangle$ in $F$. Suppose $p \neq q$. The non-trivial binary tree $q$ has at least one free caret, say with the $i$-th and $(i + 1)$-th leaves. Then the $i$-th and $(i + 1)$-th leaves of $p$ do not form a free caret. Select $u \in S - \{0\}$. Then $\hat{q}|x_i=u,x_{i+1}=u = 0$ but $\hat{p}|x_i=u,x_{i+1}=u \neq 0$. So, $S \not\models p \approx q$.

**Proposition 4.2.** If $L$ is the simple Lie algebra $sl_2$ then $\text{Assoc}(L)$ is trivial.

*Proof.* The bracket operation of $sl_2$ is given by

|        | $e_{-1}$ | $e_0$ | $e_1$ |
|--------|----------|-------|-------|
| $e_{-1}$ | $0$      | $-e_{-1}$ | $-2e_0$ |
| $e_0$    | $e_{-1}$ | $0$    | $-e_1$    |
| $e_1$    | $2e_0$  | $e_1$  | $0$    |

It is clear that the algebra $S$ in Lemma 4.1 is a quotient of a bracket subalgebra of $sl_2$. Since $S$ satisfies no strongly regular law, neither does $sl_2$. \[\square\]

**Proposition 4.3.** A finite dimensional complex Lie algebra is either solvable or it has a subquotient isomorphic to $sl_2$.

*Proof.* If $L$ is not solvable, the solvable radical $R(L)$ is a proper ideal of $L$ and $L/R(L)$ is a non-trivial semisimple Lie algebra. By Serre’s theorem on semisimple Lie algebras, $L/R(L)$ contains a copy of $sl_2$. \[\square\]

We can now prove Theorem 1.5.

*Proof.* If $L$ is solvable, use Theorem 3.1. If $L$ is not solvable Proposition 4.3 implies that $L$ has a subquotient isomorphic to $sl_2$. But Lemma 4.2 says that the bracket algebra $sl_2$ satisfies no strongly regular law; so $L$ doesn’t either. \[\square\]
Addendum follows from the proof since every infinite dimensional Kac-Moody algebra contains a copy of sl₂.

5. The groups \( A_5 \) and \( S_\infty \)

We first prove Proposition \( \tag{1.7} \)

\textbf{Proof.} Note that \([,] : A_5 \times A_5 \to A_5\) is surjective. Fix \( p \in T_n \). We will use the same notation as in the proof of Lemma \( \tag{4.1} \)

Claim 1 of that lemma is replaced by: If \( d_i = 1 \) then for any \( 1 \neq a \in A_5 \), \( \tilde{p}|_{x_i=a} \) is a set with 12, 15, or 30 elements, depending on whether \( v \) is a 5-cycle, a 3-cycle, or a product of two disjoint 2-cycles. It follows that if \( d_i \geq 1 \) then \( \tilde{p}|_{x_i=a} \) has trivial centralizer. This follows from direct calculation. The rest of the argument is the same as in Lemma \( \tag{4.1} \)

\( \square \)

Next we prove Proposition \( \tag{1.9} \). Recall that \( S_\infty \) denotes the group of permutations of \( \mathbb{N} \) with finite support. This group is not finitely generated but it is obviously elementary amenable, being the union of finite groups.

\textbf{Proof.} It is obvious that \( S_\infty \) has trivial group of associativities, since it contains a copy of \( A_5 \). We will show that any \( H \leq G \) of finite index also contains a copy of \( A_5 \).

Since there are infinitely many pairwise disjoint 5-cycles in \( S_\infty \), there are two in the same coset of \( H \), so \( H \) contains a product of two disjoint 5-cycles. In fact, \( H \) contains infinitely many such products, and these infinitely many can be chosen to be pairwise disjoint. Now for each of these products of two disjoint 5-cycles in \( H \) pick two symbols in one of the 5-cycles and consider the transposition of those two symbols. There are infinitely many of these transpositions, so two of them are in the same coset of \( H \), and \( H \) contains a product of two such transpositions. So we have elements of \( H \) of the form

\[
\begin{align*}
x &= (a_1a_2a_3a_4a_5)(b_1b_2b_3b_4b_5) \\
y &= (c_1c_2c_3c_4c_5)(d_1d_2d_3d_4d_5) \\
z &= (a_1a_2)(c_1c_2)
\end{align*}
\]

Considering the action of the subgroup \( \langle xy, z \rangle \) on the set \( \{a_1, a_2, a_3, a_4, a_5\} \) we see that \( S_5 \) is a quotient of this subgroup. By Propositions \( \tag{1.8} \) and \( \tag{1.7} \) it follows that \( S_\infty \) is large. \( \square \)
6. The Five Variable Law

Recall that the Five Variable Law is represented by the following picture, and in $F$ by the element $c_0 = [x_0, x_1]$

\[
\begin{pmatrix}
\begin{array}{c}
\text{\text{\text{\text{}}}}
\end{array}
\end{pmatrix}
\]

We are to prove Theorem 1.10, namely that $\text{Assoc}(S)$ is a non-trivial normal subgroup of $F$ if and only if this law holds eventually. (Recall that by Corollary 1.12 the word “eventually” is unnecessary in the case of a simply perfect bracket algebra.) The “only if” part is clear since every non-trivial normal subgroup of $F$ contains the commutator subgroup. We prove “if”.

Again we use the PL homeomorphism model of $F$. It is well known that the commutator subgroup $F'$ consists of those homeomorphisms in $F$ which agree with the identity map near 0 and near 1. Let $F_k$ denote the subgroup of $F$ consisting of homeomorphisms supported on the closed interval $[\frac{1}{2k}, 1 - \frac{1}{2k}]$ and let $F_k^+$ denote the subgroup generated by $F_k \cup s_0(F_k) \cup s_1(F_k)$.

**Lemma 6.1.** For each $k \geq 2$ we have $F_k^+ = F_{k+1}$.

Postponing the proof, this lemma implies that the smallest subgroup of $F$ containing $F_2$ and invariant under the shifts $s_0$ and $s_1$ is the commutator subgroup $F'$. Now, inspection of pictures shows that $F_2$ is a copy of $F$ generated by $c_0$ and $c_1$, the latter being defined by the following picture (it is just $x_1$ concentrated on the interval $[\frac{1}{4}, \frac{3}{4}]$):

\[
c_1 = \begin{pmatrix}
\begin{array}{c}
\text{\text{\text{\text{}}}}
\end{array}
\end{pmatrix}
\]

As self-homeomorphisms of the closed unit interval $I$ we have:
But $c_1$ is just the commutator $[c_0, s_1(c_0)]$, so the smallest subgroup containing $c_0$ and invariant under both shifts is $F'$. The “if” part of Theorem 1.10 follows.

It remains to prove the Lemma:

**Proof.** Let $F_{k+1}(\frac{1}{2})$ denote the subgroup of $F_{k+1}$ which fixes pointwise the closed interval of length $\frac{1}{2^k}$ whose center point is $\frac{1}{2} \in I$. This is the subgroup generated by $s_0(F_k) \cup s_1(F_k)$. Starting with $h \in F_{k+1}$, we wish to “work on” $h$, i.e. to compose $h$ with some members of $F_k^+$ until we get a homeomorphism in $F_{k+1}(\frac{1}{2})$; that will be enough. (We sketch the idea since a detailed write-up only obscures it; this will prove one of the inclusions and the other is obvious.)

The first step is to follow $h$ by a member of $F_k^+$ to “move” $h(\frac{1}{2})$ to $\frac{1}{2}$. This can be done with a member of $F_k$ if $h(\frac{1}{2})$ lies in the open interval $(\frac{1}{2^k}, 1 - \frac{1}{2^k})$; if that is not so, use an element of $s_0(F_k) \cup s_1(F_k)$ to make it so. In summary, we replace $h$ by $h_1$ which fixes the point $\frac{1}{2}$. We then work further on $h_1$ without altering it at $\frac{1}{2}$ so as to replace it by $h_2$ whose slope is 1 on the closed interval of length $\frac{1}{2^k}$ lying immediately to the left of $\frac{1}{2}$; this may involve working on $h_1^{-1}$ instead, but that is just as good. Finally, similar further work gets us to $h_3$ (or its inverse) whose slope is also 1 on the closed interval of length $\frac{1}{2^k}$ immediately to the right of $\frac{1}{2}$. That is an element of $F_{k+1}(\frac{1}{2}).$  

\[\square\]

7. Bracket algebras having two-sided identity elements

Here we prove Theorem 1.13. For this a change of notation is convenient. Previously, we denoted the depth of the $i$-th leaf by $d_i$. In what follows only the depths of the first (leftmost) and last (rightmost) leaves of a tree $p$ are important; here we denote those depths by $l(p)$ and $r(p)$.

**Proof.** Let $p, q \in T_n$ be such that $S \models p \approx q$. We want to show that $p = q$. Define $p^l, p^r \in T_{n-1}$ to be such that $\tilde{p}^l(x_1, \ldots, x_{n-1}) = \tilde{p}(1, x_1, \ldots, x_{n-1})$ and $\tilde{p}^r(x_1, \ldots, x_{n-1}) =$
\(\widehat{p}(x_1, \ldots, x_{n-1}, 1)\). Similarly for \(q\). Then by induction on \(n\) we have \(p^l = q^l\) and \(p^r = q^r\).

Case 1: If \(l(p) = l(q) = 1\) then \(p^l = p_1\) and \(q^l = q_1\), so we have \(p_1 = q_1\) and \(p = q\). Similarly, if \(r(p) = r(q) = 1\) then \(p = q\).

Case 2: \(1 < l(p)\) and \(1 < l(q)\); then \(r(p) = r(p^l) = r(q^l) = r(q)\). Subcase 2a: if \(r(p) = r(q) = 1\) then \(p = q\) by Case 1. Subcase 2b: If \(r(p) = r(q) > 1\) and \(l(p) = l(q) = 1\) then \(p = q\) by Case 1. Subcase 2c: If \(r(p) = r(q) > 1\) and \(l(p) = l(q) > 1\) then \(p_0 = (p^r)_0 = (q^r)_0 = q_0\) and \(p_1 = (p^l)_1 = (q^l)_1 = q_1\), so \(p = q\). Similarly, if \(1 < r(p)\) and \(1 < r(q)\) then \(p = q\).

Case 3: If \(l(p) = 1 < l(q)\) [resp. \(r(q) = 1 < r(p)\)] then \(r(p) = 1 + r(p_1) = 1 + r(p^l) = 1 + r(q^l) = 1 + r(q)[\text{resp. } l(q) = 1 + l(p)\text{ by similar reasoning}]\). By Case 2, we cannot have \(r(p) > 1\), so \(r(p) = 1\) and \(r(q) = 2\)[resp. \(l(p) = 1\) and \(l(q) = 2\)]. Now, \(p_1 = p^l = q^l = [q_{01}, \cdot]\), so \(p_{10} = q_{01}\), and \(p \approx q\) is an expansion of the three variable associative law. But then, since \(S\) is simply perfect (because it has an identity element), by Remark 1.1.4 the assertion \(S \models p \approx q\) implies that \(S\) is associative, contrary to our hypothesis. So Case 3 does not arise.

\(\square\)

A one-sided identity element is not enough in Theorem 1.1.3

**Example 7.1.** Consider the bracket algebra \(S(4) = \{1, a, b, c\}\) with binary operation \([x, 1] = x\), \([x, a] = b\), \([x, b] = c\) and \([x, c] = c\), for any \(x \in S(4)\). It has a right identity element, 1. It is not associative since \([[x, [y, a]], a] = b\), but \([[x, [y, a]], [x, b]] = [x, b] = c\). However, it satisfies the law \([x, [[y, z], w]] = [x, [y, [z, w]]]\). To see this consider the cases \(w = 1\) and \(w \neq 1\). If \(w = 1\), then both sides reduce to \([x, [y, z]]\). If \(w \neq 1\), then both sides equal \(c\).

If \((p, q)\) is the usual representative pair of 4-leaf trees for the standard generator \(x_1\) then \(\widehat{p} = \widehat{q}\) represents the law \([x, [[y, z], w]] = [x, [y, [z, w]]]\) which holds in \(S(4)\). Thus \(S(4)\) is a quotient of the bracket algebra \(S\) in Example 2.4.1 We saw that Assoc(\(S\)) is a non-normal subgroup of \(F\). We remark that Assoc(\(S(4)\)) is that same group. We omit the proof of this.
REFERENCES

[1] M. Abert, *Group laws and free subgroups in topological groups* Bull. London Math. Soc. (to appear).

[2] K.S.Brown and R.Geoghegan, *An infinite-dimensional torsion free $FP_{\infty}$ group*, Inventiones Math. 77 (1984) 367-381.

[3] L. Bartholdi, R. Grigorchuk, and Z. Šunič, *Branch groups*. Handbook of algebra, Vol. 3, 989–1112, North-Holland, Amsterdam, 2003.

[4] M. Brin and C. C. Squier, *Groups of piecewise linear homeomorphisms of the real line*, Inventiones Math. 79 (1985) 485–498.

[5] J.W. Cannon, W.J. Floyd, and W.R. Parry, *Introductory notes on Richard Thompson's groups*, Enseign. Math. (2) 42 (1996), 215–256.

[6] R. I. Grigorchuk, *On Burnside's problem on periodic groups* Funktsional. Anal. i Prilozhen. 14 (1980), 53–54.

[7] Z. Sunik, private communication.

Ross Geoghegan, Binghamton University (SUNY), Binghamton N.Y., 13902, U.S.A.

Fernando Guzmán, Binghamton University (SUNY), Binghamton N.Y., 13902, U.S.A.