Warped product spaces and geodesic motion in the neighbourhood of hypersurfaces

F. Dahia\textsuperscript{a}, C. Romero\textsuperscript{b}, L. F. P. da Silva\textsuperscript{b}, and R. Tavakol\textsuperscript{c}

\textsuperscript{a}Departamento de Física, Universidade Federal de Campina Grande, 58109-970
Campina Grande, Pb, Brazil
\textsuperscript{b}Departamento de Física,
Universidade Federal da Paraíba,
Caixa Postal 5008, 58059-979 João Pessoa, Pb, Brazil
\textsuperscript{c}School of Mathematical Sciences,
Queen Mary, University of London,
London E1 4NS United Kingdom

We study the classical geodesic motions of nonzero rest mass test particles and photons in five-dimensional warped product spaces. We show that it is possible to obtain a general picture of these motions, using the natural decoupling that occurs in such spaces between the motions in the fifth dimension and the motion in the hypersurfaces. This splitting allows the use of phase space analysis in order to investigate the possible confinement of particles and photons to hypersurfaces in five-dimensional warped product spaces. Using such analysis, we find a novel form of quasi-confinement which is oscillatory and neutrally stable. We also find that this class of warped product spaces locally satisfy the $Z_2$ symmetry by default. The importance of such a confinement is that it is purely due to the classical gravitational effects, without requiring the presence of brane–type confinement mechanisms.

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I. INTRODUCTION

Motivated by string inspired theories \cite{1, 2}, a great deal of effort has, over the recent years, gone into the study of higher dimensional general relativity (GR) theories. Examples of such theories include noncompactified $(n+1)$-dimensional GR \cite{3} (where the observable universe is assumed to lie on an embedded 3 + 1 (4D) hypersurface), Kaluza–Klein theory \cite{4} (where the extra dimensions are compactified) and the braneworld models inspired by String/M-theory \cite{5, 6, 7}.

A central issue in such theories has been how to explain the fact that the observable universe is (very nearly) confined to a particular hypersurface. In most braneworld models, stringy effects are invoked to argue that in low energy regimes particles are restricted to a special 3 + 1 \textit{brane} hypersurface, which is embedded in a higher (usually five)-dimensional \textit{bulk}, while the gravitational field is free to propagate in the bulk \cite{1}.

In the case of other higher-dimensional GR theories, however, such confinement needs to be explained. This has provided strong motivation for the consideration of the so-called warped product spaces \cite{8} and the investigation of their geometrical properties. We recall that the geometrical considerations of warped product spacetimes predate their applications in these higher-dimensional contexts and were first considered by Bishop and O’Neill \cite{7} in the context of 3 + 1 general relativity. Geodesic motion in connection with higher-dimensional theories has been considered by a number of authors in recent years \cite{8, 9, 10}. Our interest in studying such motions specifically in the context of warped product spaces arises from the realization of the fact that there is a strong connection between the 5D geodesics and the analytical form of the warping function. Indeed, as we shall see in this paper, it is possible to make a general qualitative analysis of the behaviour of massive particles and photons in the fifth dimension from the knowledge of the warping function. This can in turn allow questions concerning the possibility of hypersurface confinement as well as the stability of the confined motions with respect to small perturbations to be treated with great generality.

An assumption usually made in the braneworld scenarios is that the branes possess an infinitesimal thickness. In this sense one may talk about $\delta$-\textit{confinement} of massive particles and light rays. There are also smooth versions of this scenario which consider the possibility that branes may have a finite thickness \cite{12, 13, 14}. In that case one may talk about \textit{quasi-confinement}.

Here we shall demonstrate a novel example of the latter in the context of 4 + 1-dimensional warped product spaces which we shall refer to as \textit{oscillatory confinement}, whereby the particles in the four-dimensional hypersurface can oscillate about the hypersurface while remaining close to it. As we shall show below such behaviour can indeed occur for large classes of bulks which possess warped product geometries. Our analysis is fairly general and the results may be applied to different examples of warped product spacetimes considered in the literature.

The paper is organized as follows. In Section 2 we write the geodesic equations for warped product spaces and consider the parts due to 4D and the fifth dimension separately. We then show that the equation that describes the motion in 5D decouples from the rest. We proceed...
in Section 3 to rewrite the geodesic equation in the fifth dimension as an autonomous planar dynamical system. We then employ phase plane analysis to study the motion of particles with nonzero rest mass (timelike geodesics) and photons (null geodesics) respectively. Such qualitative analysis allows very general conclusions to be drawn about the possible existence of confined motions and their stability in the neighbourhood of hypersurfaces. We conclude with a discussion of further applications in Section 4.

Throughout Latin indices take values in the range (0,1,...,4) while Greek indices run over (0,1,2,3).

II. WARPED PRODUCT SPACES AND FIVE-DIMENSIONAL GEODESICS

Warped geometries are at the core of the Randall-Sundrum models. Technically we can define a warped product space in the following way. Let \((M^m, h)\) and \((M^n, k)\) be two Riemannian (or pseudo-Riemannian) manifolds of dimension \(m\) and \(n\), with metrics \(h\) and \(k\), respectively. Suppose we are given a smooth function \(f : M^n \to \mathbb{R}\) (which is referred to as the warping function). Then we can construct a new Riemannian (pseudo-Riemannian) manifold by setting \(M = M^n \times M^m\) and defining a metric \(g = e^{2f}h \otimes k\). In this paper we shall take \(M = M^4 \times \mathbb{R}\), where \(M^4\) is a four-dimensional Lorentzian manifold with signature \(+−−−\) (referred to as the \((3+1)\)-dimensional \(\text{spacetime}\)) . In local coordinates \(\{y^\alpha\}\) the line element corresponding to this metric will be denoted by

\[
dS^2 = g_{\alpha\beta}dy^\alpha dy^\beta.
\]

Let us now consider the equations of geodesics in the five-dimensional space

\[
d^2y^\alpha/dx^2 + (5)\Gamma^\alpha_{\beta\gamma}dy^\beta dy^\gamma = 0,
\]

where \(\lambda\) is an affine parameter and \((5)\Gamma^\alpha_{\beta\gamma}\) are the 5D Christoffel symbols of the second kind defined by \((5)\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}(g_{\delta\beta,c} + g_{\delta\gamma,b} - g_{\delta\gamma,c}).\) Denoting the fifth coordinate \(y^5\) by \(l\) and the remaining coordinates \(y^\alpha\) (the "\(\text{spacetime}\)" coordinates) by \(x^\mu\), it is not difficult to show that the "\(\text{4D part}\)" of the geodesic equations

\[
d^2x^\mu/d\lambda^2 + (4)\Gamma^\mu_{\alpha\beta}dx^\alpha/d\lambda dx^\beta/d\lambda = \phi^\mu,
\]

where \(\phi^\mu = -\frac{1}{2}(5)\Gamma^\mu_{\alpha\beta} \left(\frac{dl}{d\lambda}\right)^2 - 2(5)\Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dl}{d\lambda} - \frac{1}{2}g^{\alpha\delta}(g_{\delta\alpha,\beta} + g_{\delta\beta,\alpha} - g_{\delta\alpha,\beta}) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda},\)

and \(\frac{4}{2} \Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\nu\alpha,\beta}).\) We shall assume that the five-dimensional manifold \(M\) can be foliated by a family of hypersurfaces \(\Sigma\) defined by the equation \(l = \text{constant}.\) Then the geometry of each hypersurface, say \(l = l_0\), will be determined by the induced metric

\[
ds^2 = g_{\alpha\beta}(x,l_0)dx^\alpha dx^\beta.
\]

Therefore the quantities \(\frac{4}{2} \Gamma^\mu_{\alpha\beta}\) which appear on the left-hand side of Eq. (2) may be identified with the Christoffel symbols associated with the induced metric in the leaves of the foliation defined above.

We shall consider the class of warped geometries given by the following line element

\[
ds^2 = e^{2f}h_{\alpha\beta}dx^\alpha dx^\beta - dl^2,
\]

where \(f = f(l)\) and \(h_{\alpha\beta} = h_{\alpha\beta}(x).\) For this metric it is easy to see that \((5)\Gamma^\mu_{\alpha\beta} = 0\) and \((5)\Gamma^\mu_{\alpha\beta} = f'\delta^\mu_\nu\), where a prime denotes a derivative with respect to \(l\). Thus in the case of the warped product space (1), the right-hand side of Eq. (2) reduces to \(\phi^\mu = -2f' \frac{dx^\mu}{d\lambda} \frac{dl}{d\lambda}\) and the 4D part of the geodesic equations becomes

\[
\frac{d^2x^\mu}{d\lambda^2} + (4)\Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = -2f' \frac{dx^\mu}{d\lambda} \frac{dl}{d\lambda}.
\]

Likewise the geodesic equation for the fifth coordinate \(l\) in the warped product space becomes

\[
\frac{d^2 l}{d\lambda^2} + f' \left(1 + \left(\frac{dl}{d\lambda}\right)^2\right) = 0.
\]

By restricting ourselves to 5D timelike geodesics \((g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 1)\) we can readily decouple the above equation from the 4D spacetime coordinates to obtain

\[
\frac{d^2 l}{d\lambda^2} + f' \left(1 + \left(\frac{dl}{d\lambda}\right)^2\right) = 0.
\]

Similarly, to study the motion of photons in 5D, we must consider the null geodesics \((g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0)\), in which case Eq. (6) becomes

\[
\frac{d^2 l}{d\lambda^2} + f' \left(\frac{dl}{d\lambda}\right)^2 = 0.
\]

Equations (7) and (8) are ordinary differential equations of second-order which, in principle, can be solved if the function \(f' = f'(l)\) is given. Conversely, if \(l = l(\lambda)\) is known then \(f = f(l)\) can be determined (up to a constant of integration) provided that we can write \(\lambda = \lambda(l)\).
III. QUALITATIVE ANALYSIS OF MOTION IN THE FIFTH DIMENSION

We recall that the hypersurface sets the boundary conditions for the warping function. If the warping function \( f = f(l) \) is given a priori, then we can obtain a great deal of information about the motion in the fifth dimension by performing a qualitative analysis of Eqs. (7) and (8), without the need to solve these equations analytically. To do this we define the variable \( q = \frac{dl}{d\lambda} \) and consider instead of (7) or (8) the autonomous dynamical system

\[
\begin{align*}
\frac{dl}{d\lambda} &= q \\
\frac{dq}{d\lambda} &= P(q, l)
\end{align*}
\]

with \( P(q, l) = -f'(\epsilon + q^2) \), where \( \epsilon = 1 \) in the case of (7) (corresponding to the motion of particles with nonzero rest mass) and \( \epsilon = 0 \) in the case of (8) (corresponding to the motion of photons). In the investigation of dynamical systems a crucial role is played by their equilibrium points, which in the case of system (9) are given by \( \frac{dl}{d\lambda} = 0 = \frac{dq}{d\lambda} \). The knowledge of these points together with their stability properties allows a great deal of information to be gained regarding the types of behaviour allowed by the system. Thus if for a particular geodesic we know \( t \) as a function of the parameter \( \lambda \), i.e. \( t = t(\lambda) \), then since \( \frac{d\lambda}{dl} \neq 0 \), knowing the behaviour of \( l = l(\lambda) \) from qualitative analysis will enable us to deduce the time evolution of the motion of a particle (or light ray) in the fifth dimension.

A. The 5D motion of particles with nonzero rest mass near the hypersurface

We begin by considering the case of nonzero rest mass particles, whose motion in the fifth dimension is governed by the dynamical system

\[
\begin{align*}
\frac{dl}{d\lambda} &= q \\
\frac{dq}{d\lambda} &= -f'(1 + q^2)
\end{align*}
\]

The equilibrium points of (10) are given by \( q = 0 \) and the zeros of the function \( f'(l) \) (if they exist) which we generically denote by \( l_0 \). These solutions, pictured as fixed points in the phase plane, correspond to curves which lie entirely in a hypersurface \( \Sigma \) of our foliation (since they have \( l = \text{constant} \)). It turns out that these curves are timelike geodesics with respect to the hypersurface induced geometry. Indeed, the existence of equilibrium points of the dynamical system (10) has the important consequence that the geodesics of the bulk and that of the hypersurface \( (l_0 = \text{constant}) \) coincide. This can be seen directly from the equations (2) and (6), and is perfectly consistent with a well-known theorem of differential geometry according to which the geodesics of a Riemannian space and that of a particular hypersurface coincide (i.e. confinement occurs) if and only if the extrinsic curvature of that hypersurface vanishes (see e.g. (16) where such hypersurfaces are referred to as totally geodesic). Note that for the metric (1) the extrinsic curvature of the hypersurfaces \( \Sigma \) is given by \( \Omega_{\alpha\beta} = -f''e^{2\lambda}h_{\alpha\beta}(x) \), which is clearly zero at the equilibrium points, where \( f' = 0 \). Thus in the absence of equilibrium points (i.e. \( f' \neq 0 \)) the extrinsic curvature does not vanish and according to the above theorem the bulk and hypersurface geodesics cannot be identical.

To obtain information about the possible modes of behaviour of particles and light rays in such hypersurfaces, it is important to study the nature and stability of the corresponding equilibrium points. This can be done by linearising equations (10) and studying the eigenvalues of the corresponding Jacobian matrix about the equilibrium points. Assuming that the function \( f'(l) \) vanishes, at least at one point \( l_0 \), it can readily be shown that the corresponding eigenvalues are determined by the sign of the second derivative \( f''(l_0) \), at the equilibrium point, and the following possibilities can arise for the equilibrium points of the dynamical system (10):

Case I. If \( f''(l_0) > 0 \), then the equilibrium point \( (q = 0, l = l_0) \) is a center. This represents a case in which the solutions near the equilibrium point have the topology of a circle. In that case the phase portrait consists of closed curves describing the motions of particles oscillating about the hypersurface \( \Sigma (l = l_0) \) indefinitely (see the top panel in Fig. 1). The amplitudes of the oscillations will depend on initial conditions. We note that the existence of such cyclic motions is independent of the ordinary 4D spacetime dimensions, and, except for the conditions \( f'(l_0) = 0 \) and \( f''(l_0) > 0 \), the warping function \( f(l) \) remains completely arbitrary. The presence of such center equilibrium points with the consequent quasi-confinement of particles can be viewed as providing examples of almost totally geodesic hypersurfaces (see the lower panel in Fig. 1).

Case II. If \( f''(l_0) < 0 \), then the point \( (q = 0, l = l_0) \) is a saddle point. In this case the solution corresponding to the equilibrium point \( E \) is highly unstable and the smallest perturbations will lead to exponential divergence of the solutions and hence the particles - from the hypersurface (See Fig. 2). The only (measure zero) exception is provided by the stable manifold of the point \( E \), represented by the lines \( AE \) and \( BE \), along which particles are attracted towards the brane. An example of this highly unstable ”confinement” is provided by the warping function (16):

\[
f(l) = -b \ln \cosh(c l), \quad (11)
\]

where \( b \) and \( c \) are positive constants. Here we have a unique equilibrium point at \( l = 0 \) and it is easy to
FIG. 1: Depicted in the top panel is the center equilibrium point E in the case where $f''(l_0) > 0$. In this case the particles oscillate about the hypersurface $l = l_0$. This provides a quasi-confinement mechanism whereby massive particles enter and leave the hypersurface $\Sigma$ indefinitely (lower panel).

verify that $f''(0) < 0$ in this case. In this connection we note that for large values of $l$ the warping function approaches that of the Randall-Sundrum metric.

$$ds^2 = e^{-2k|l|}\eta_{\alpha\beta}dx^\alpha dx^\beta - dl^2,$$

where $k$ is a constant. In this case $f'(l) = \mp k$ according to whether $l$ is positive or negative. Hence for $l \neq 0$ there exist no equilibrium points, and therefore no confinement of particles purely due to gravitational effects. However, in this thin wall limit if we take the extrinsic curvature of the brane $l = 0$ as being zero, then implies confinement. Nevertheless, as was pointed out in ref. [11] this is a highly unstable confinement in the sense that any small transversal perturbations in the motion of massive particles along the brane will cause them to be expelled into the extra dimension. However, this case lies somewhat outside the scope of the metrics that we are considering here as in this case the warping function is not smooth (the first derivative of $f(l)$ with respect to the extra coordinate is not continuous at $l = 0$). In the braneworld scenario, as is well known, stable confinement of matter fields is possible at the quantum level if we take into account interaction with a scalar field [17]. In a purely classical picture, however, one would require mechanisms other than geodesic confinement in order to constrain massive particles to move on hypersurfaces in a stable way.

Case III. If $f''(l_0) = 0$, then both eigenvalues are zero, which corresponds to a degenerate case. To carry out a qualitative analysis of the solutions near the equilibrium point, in this case, would require the knowledge of the third (or higher) derivative of the function $f(l)$. We shall not consider this case in its generality. However, if $f(l)$ is constant (case IV), then we can easily draw a global phase portrait of the system.

Case IV. If $f''(l_0) = 0$, then both eigenvalues are zero, which corresponds to a degenerate case. To carry out a qualitative analysis of the solutions near the equilibrium point, in this case, would require the knowledge of the third (or higher) derivative of the function $f(l)$. We shall not consider this case in its generality. However, if $f(l)$ is constant (case IV), then we can easily draw a global phase portrait of the system.

FIG. 2: When $f''(l_0) < 0$ the equilibrium point E is a saddle. In this case confinement is highly unstable.

Case IV. If $f''(l_0) = \text{constant}$, then $f'(l)$ vanishes for all values of $l$, which implies there is an infinite number of non-isolated equilibrium points, that is, we have a line of equilibrium points ($q = 0$). Perturbations along this line are neutrally stable, which implies that particles placed on any of the hypersurfaces of the foliation $l = \text{constant}$, will remain there so long as they do not receive a transversal velocity to the hypersurface.
Case V. If there are no equilibrium points, that is, the warping function \( f(l) \) does not have any turning points for any value of \( l \), then we cannot have confinement of classical particles to hypersurfaces solely due to gravitational effects. An example of this situation is illustrated by the warping function \( f(l) = \frac{1}{2} \ln \left( \Lambda^2/3 \right) \) considered in Ref. [18].

We note that even when there are no equilibrium points, the picture of dynamics can still be obtained if \( f(l) \) is known, by obtaining a first integral of the system (9). This can easily be done by writing this system as the first-order differential equation

\[
\frac{dl}{dq} = -\frac{q}{f'(\epsilon + q^2)},
\]

which can be readily integrated to yield

\[
f(l) = -\ln \sqrt{\epsilon + q^2} + K,
\]

where \( K \) is a constant of integration.

### B. The 5D motion of photons near the hypersurface

Finally we examine the motion of photons (null geodesics) and ask whether they can be confined to or a neighbourhood of a hypersurface. In this case the dynamical system (9) becomes

\[
\begin{align*}
\frac{dl}{d\lambda} &= q \\
\frac{dq}{d\lambda} &= -f'q^2.
\end{align*}
\]

The equilibrium points now are given by \( q = 0 \), which consists of a line of equilibrium points along the \( l \)-axis, with eigenvalues both equal to zero. As a result there exist 5D null geodesics in any hypersurface \( l = \text{constant} \). This demonstrates that confinement of photons to hypersurfaces does not depend upon the warping factor. There are also regions of stability and instability with respect to small perturbations along the \( l \)-axis. As was pointed out in the previous section we can easily obtain a first integral of the system (13) which is given by (12). In the case of the motion of photons (\( \epsilon = 0 \)) this gives

\[
q = Ae^{-f(l)}
\]

where \( A \) is a constant of integration. Therefore we can obtain a global picture of the solutions (13) provided that we have some qualitative knowledge of the function \( f(l) \).

Many different cases can arise, depending upon the nature of \( f(l) \). Here for definiteness and simplicity we impose \( Z_2 \) symmetry on the geometry of the bulk. The following are some possible examples:

(i) \( f(l) \) is an increasing monotonic function for \( l \geq 0 \) which approaches the limit \( f(l) \to \infty \) as \( l \to \infty \) (see the top panel of Fig. 3). This includes the case of the warping function considered in (18).

(ii) \( f(l) \) is a decreasing monotonic function for \( l \geq 0 \) that approaches the limit \( f(l) \to -\infty \) as \( l \to \infty \), which is the case of the warping function given by (11) (see the lower panel of Fig. 3). Note the interesting behaviour of photons that are not confined: if we set them in motion towards the hypersurface \( \Sigma(l = 0) \) then after reaching a minimal distance from \( \Sigma \) they bounce back to infinity, showing clearly that they are repelled by \( \Sigma \).

(iii) \( f(l) \) is constant, in which case we have the same phase diagram as in the case of massive particles.

Finally it is worth noting that in the case of null geodesics, the equilibrium points of the Eqs. (13) do not require \( f' = 0 \), which is a consequence of the fact that the above mentioned theorem [16] does not apply to null geodesics.

### IV. CONCLUSIONS

We have studied some aspects of the motion of massive particles and photons in five-dimensional warped product spaces. Spaces of this type have received a great deal of attention over the recent years mainly in connection with the braneworld scenarios. Our treatment has been geometrical and classical in nature and we have considered confinement to hypersurfaces purely due to classical gravitational effects.

Employing the splitting that naturally occurs in such spaces between the motion in the fifth dimension and the remaining dimensions, we have found, using phase plane analysis, a novel form of quasi-confinement which is neutrally stable. In a sense this amounts to a generalisation of the \( \delta \)-confinement of particles in thin branes. An important reason to consider models outside the framework of thin branes is that there is a characteristic minimum length given by the string scale [10, 12, 13, 14].

The importance of the type of confinement discussed here is that it is due purely to the classical gravitational effects, without requiring the presence of brane type confinement mechanisms.

As an example, we have found that in the case of the Randall-Sundrum metric gravity alone is not capable of confining massive particles, although photons may be confined. As is well known, the particle confinement in this scenario is achieved by means of a scalar field [2].

An assumption usually made in braneworld scenarios, which plays an important role in connection with confinement in these models, is that of \( Z_2 \) symmetry. It is therefore important to take a closer look at the models we have found, which are capable of providing successful confinement, from this perspective.
FIG. 3: Depicted in the top panel is the phase portrait representing the motion of photons in presence of $Z_2$ symmetry with an increasing monotonic function $f(l)$ for $l \geq 0$ and $f(l) \to \infty$ as $l \to \infty$. There is confinement of photons at any hypersurface $\Sigma$ of the bulk foliation $l = \text{constant}$. The bottom panel shows the corresponding phase portrait for the cases where $f(l)$ is a decreasing monotonic function for $l \geq 0$ that approaches the limit $f(l) \to -\infty$ as $l \to \infty$. Photons that are not confined are repelled by the hypersurface $\Sigma$ ($l = 0$).

Recalling that $Z_2$ symmetry implies

\[
\Omega^\alpha_{\alpha\beta} + \Omega^-_{\alpha\beta} = 0,
\]

(14)
it is easy to check that this condition is satisfied in cases where the hypersurfaces are located at the turning points $l = l_0$ of the functions $f$. This follows from the fact that in such cases $f'$, and hence $\Omega^\alpha_{\alpha\beta} = -f'e^{2f} h_{\alpha\beta}(x)$, have different signs on either sides of the hypersurface and thus $Z_2$ symmetry is satisfied by default. Importantly this includes all the equilibrium points we have found in the massive particle case.

Finally it is interesting to compare our results with the classical theorem, according to which the geodesics of a Riemannian space coincide with the geodesics of an embedded (totally geodesic) hypersurface if and only if the extrinsic curvature of that hypersurface vanishes. The analysis involved in proving this theorem [16], though general, does not address the case of null geodesics, as it confines itself to positive definite metrics. Nor does it provide any information regarding the stability of the this coincidence. Our analysis on the other hand, although restricted to the case of warped product spaces, provides information regarding the stability of confinement to totally geodesic hypersurfaces, through the use of dynamical system analysis.

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[20] In the above calculation we have used the fact that the matrix $h_{\alpha \beta}$ has an inverse $h^{\alpha \beta}$, that is, $h^{\alpha \beta} h_{\beta \nu} = \delta^{\alpha}_{\nu}$. This may be easily seen since, by definition, $\det h = - \det g \neq 0$. 

[21] In the above calculation we have used the fact that the matrix $h_{\alpha \beta}$ has an inverse $h^{\alpha \beta}$, that is, $h^{\alpha \beta} h_{\beta \nu} = \delta^{\alpha}_{\nu}$. This may be easily seen since, by definition, $\det h = - \det g \neq 0$. 

[22] In the above calculation we have used the fact that the matrix $h_{\alpha \beta}$ has an inverse $h^{\alpha \beta}$, that is, $h^{\alpha \beta} h_{\beta \nu} = \delta^{\alpha}_{\nu}$. This may be easily seen since, by definition, $\det h = - \det g \neq 0$. 

[23] In the above calculation we have used the fact that the matrix $h_{\alpha \beta}$ has an inverse $h^{\alpha \beta}$, that is, $h^{\alpha \beta} h_{\beta \nu} = \delta^{\alpha}_{\nu}$. This may be easily seen since, by definition, $\det h = - \det g \neq 0$. 

[24] In the above calculation we have used the fact that the matrix $h_{\alpha \beta}$ has an inverse $h^{\alpha \beta}$, that is, $h^{\alpha \beta} h_{\beta \nu} = \delta^{\alpha}_{\nu}$. This may be easily seen since, by definition, $\det h = - \det g \neq 0$. 

[25] In the above calculation we have used the fact that the matrix $h_{\alpha \beta}$ has an inverse $h^{\alpha \beta}$, that is, $h^{\alpha \beta} h_{\beta \nu} = \delta^{\alpha}_{\nu}$. This may be easily seen since, by definition, $\det h = - \det g \neq 0$. 

[26] In the above calculation we have used the fact that the matrix $h_{\alpha \beta}$ has an inverse $h^{\alpha \beta}$, that is, $h^{\alpha \beta} h_{\beta \nu} = \delta^{\alpha}_{\nu}$. This may be easily seen since, by definition, $\det h = - \det g \neq 0$.