INTEGRAL TOPOLOGICAL HOCHSCHILD HOMOLOGY OF CONNECTIVE COMPLEX K-THEORY

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Abstract. We compute the homotopy groups of \( \text{THH}(\text{ku}) \) as a \( \text{ku}_* \)-module using the descent spectral sequence for the map \( \text{THH}(\text{ku}) \to \text{THH}(\text{ku}/\text{MU}) \), which is the motivic spectral sequence for \( \text{THH}(\text{ku}) \) in the sense of Hahn-Raksit-Wilson. We compute the \( E_2 \)-page using an algebraic Bockstein spectral sequence, which is an approximation to the topological Bockstein spectral sequence computing \( \text{THH}(\text{ku}) \) from \( \text{THH}(\text{ku})/(p, \beta) \), where \( \beta \) is the Bott element. Then, we show that the descent spectral sequence degenerates at the \( E_2 \)-page.

1. Introduction

1.1. Motivation. Topological Hochschild homology of a ring spectrum \( R \), defined as \( \text{THH}(R) = R \otimes_{R \otimes R^{op}} R \) is an object of interest largely due to its connection to the algebraic K-theory \( K(R) \). The Dundas-Goodwillie-McCarthy theorem [DGM13, Thm. 7.2.2.1] states that the difference between topological cyclic homology \( \text{TC}(R) \) and \( K \)-theory \( K(R) \) is locally constant, therefore providing a way to compute \( K(R) \). Computing \( \text{THH}(R) \) is the very first step towards computing \( \text{TC}(R) \). See [NST18] for a modern construction of \( \text{TC} \) from \( \text{THH} \).

In this paper, we compute the homotopy groups of \( \text{THH}(\text{ku}) \), where \( \text{ku} = \tau_{\geq 0} \text{KU} \) is the connective cover of the topological K-theory spectrum, in the hope that this would eventually lead to the computation of \( K(\text{ku}) \).

The interest in \( K(\text{ku}) \) comes from the chromatic redshift philosophy which roughly suggests that the algebraic \( K \) theory of a “chromatic height \( n \) theory” is of “chromatic height \( n + 1 \)”. In this perspective, Ausoni proved that \( K(\text{ku}) \) is of chromatic height 2 in a suitable sense [Aus10]. Also, according to [BDRR13], the cohomology theory \( K(\text{ku}) \) has a geometric meaning in terms of 2-vector bundles.

While our work is the first to completely compute \( \text{THH}_*(\text{ku}) \), several previous works make important, related computations that motivate our work. In [Aus05], Ausoni computes homotopy groups of various quotients of \( \text{THH}(\text{ku}) \) including

\[
\text{THH}(\text{ku})/p, \text{THH}(\text{ku})/(p, v_1), \text{ and } \text{THH}(\text{ku})/\beta
\]

where \( \beta \) is the Bott element. Another very relevant work is the computation of \( \text{THH}_*(\ell)^p/\ell \) by Angeltveit, Hill, and Lawson in [AHL10], where \( \ell \) is the Adams summand of ku localized at a prime \( p \). These works compute various Bockstein spectral sequences. For example, for each arrow in the diagram

\[
\text{THH}(\ell)/(p, v_1) \Rightarrow \text{THH}(\ell)^p/v_1 \Rightarrow \text{THH}(\ell)^p
\]

there is a Bockstein spectral sequence computing the homotopy groups of the next spectrum from the homotopy groups of the previous spectrum. However, the computation of these Bockstein differentials in [AHL10] is highly nontrivial. Their work depends on the knowledge of the homotopy groups of \( \text{THH}(\ell)/p \), which is rather complicated even compared to those of \( \text{THH}(\ell) \). We note that the homotopy groups of \( \text{THH}(\ell)/p \) were computed in [MS93] when \( p \) is odd and later extended to the case \( p = 2 \) in [AR05].
In this work, we simplify the computation of Bockstein differentials (for $\text{ku}$ instead of $\ell$) by first making an algebraic approximation of homotopy groups and computing Bockstein differentials in algebra. More precisely, for each $M = \mathbb{F}_p, \mathbb{Z}_p, \text{ku}_p^\wedge$, we shall construct a spectral sequence

$$E_2(\text{THH}(\text{ku}; M)) \Rightarrow \text{THH}_*(\text{ku}; M).$$

Here, $\text{THH}(\text{ku}; M)$ denotes the THH with coefficients in $M$ \[2.1\]. Then, we shall construct Bockstein spectral sequences computing each group in the order of arrows in the diagram

$$E_2(\text{THH}(\text{ku}; \mathbb{F}_p)) \Rightarrow E_2(\text{THH}(\text{ku}; \mathbb{Z}_p)) \Rightarrow E_2(\text{THH}(\text{ku})_p^\wedge).$$

This task is arguably simpler because we get to work with chain complexes instead of spectra, allowing us to use explicit elements. Also, the computation can be done without any knowledge of $\text{THH}(\text{ku})/p$. Although we claim that our method of computing Bockstein differentials is simpler than previous works, figuring out what the differentials should be in this paper (Theorem 4.2.1) was heavily inspired by [AHL10, Theorem 6.4].

The spectral sequence $E_2(\text{THH}(\text{ku}; M))$ to be used in the paper is the descent spectral sequence for $\text{THH}(\text{ku}; M)$ along $\text{THH}(\text{ku}) \to \text{THH}(\text{ku}/\text{MU})$. This idea is from the work [HW] of Hahn and Wilson, in which they descend along $\text{THH}(\text{BP}(n)) \to \text{THH}(\text{BP}(n)/\text{MU})$ to analyze $\text{TC}(\text{BP}(n))$. We shall see that our descent cover $\text{THH}(\text{ku}/\text{MU})$ has homotopy groups concentrated in even degrees (Prop. 2.2.2). Computations of $\text{THH}$ by descending from even rings have successfully been carried out by many authors for $\text{THH}$ of ring of integers of $p$-adic number fields and their quotient rings ([BMS19], [KN], [LW]). The relation between these works and our work can be explained by the notion of even/motivic filtration in the work [HRW] of Jeremy Hahn, Arpon Raksit, and Dylan Wilson. See also Remark 3.2.1.

Further pursuing in this direction, we expect that similar descent spectral sequences will be useful for computing the $\text{THH}$ of other complex oriented ring spectra such as the truncated Brown-Peterson spectra $\text{BP}(n)$. We discuss this in Section 6. We would also like to note that several computations related to $\text{THH}(\text{BP}(n))$ have recently been made in the work [AKCH] of Angelini-Knoll, Culver, and Höning.

### 1.2. Main Results.

To describe $\text{THH}_*(\text{ku})$ as a $\text{ku}_* = \mathbb{Z}[\beta]$-module with $|\beta| = 2$, we need to define two graded $\mathbb{Z}[\beta]$-modules $F$ and $T$. The graded $\mathbb{Z}[\beta]$-module $F$ is defined to be

$$F := \Sigma^3 \left( \mathbb{Z} \left\{ \frac{\beta^k}{f(k)} \mid k \geq 0 \right\} \right) \subseteq \Sigma^3(\mathbb{Q}[\beta])$$

where $f(k)$ is a sequence defined as

$$f(0) = 1$$
$$f(k) = \begin{cases} pf(k-1) & \text{if } k + 2 = p^m \text{ for some prime } p \\ f(k-1) & \text{otherwise.} \end{cases}$$

The graded module $T$ is defined to be the direct sum

$$T := \bigoplus_{p: \text{prime}} T(p)$$

where $T(p)$ is a torsion $\mathbb{Z}_p[\beta]$-module described below.

**Theorem 1.2.1.** There is an isomorphism

$$\text{THH}_*(\text{ku}) = \mathbb{Z}[\beta] \oplus F \oplus T$$

as $\mathbb{Z}[\beta]$-modules.
Let us describe the torsion $\mathbb{Z}_p[\beta]$-module $T(p)$ for each prime $p$. We first define a $\mathbb{Z}_p[\beta]$-module $T_1(p)$ using generators $h_{m,j}$ of degree $2pm + 2$ where $m$ varies over positive integers and $j$ is a nonnegative integer such that $0 \leq j \leq \text{val}_p(m)$. If we write $d = \text{val}_p(m), d' = \text{val}_p(m - p^d(p - 1))$, the relations are

\[ \beta^{p^d-p^d+2}h_{m,j} = 0 \]

and

\[ ph_{m,j} = \begin{cases} h_{m,j+1} + \beta^{p^{d'}-p^{d'+1}}h_{m-p^d(p-1),d'-d-1} & \text{if } j = 0, \ m > p^d(p-1), \text{ and } d' > d \\ h_{m,j+1} & \text{otherwise.} \end{cases} \]

If $j = \text{val}_p(m)$, then the $h_{m,j+1}$'s on the right-hand side should read zero. Then, when $p$ is odd, we define $T(p)$ as $T_1(p)$, and if $p = 2$, then $T(2)$ is defined to be the subquotient of $T_1(2)$ generated by the elements of the form $h_{2m,j}$ with additional relations $h_{2m,\text{val}_2(2m)} = 0$ for all positive integers $m$.

Figure 1 shows the associated graded group of $T(3)$ with respect to the $(3, \beta)$-adic filtration in a range of degrees. The generators $h_{m,0}$ for $6 \leq m \leq 17$ are detected by the dots in the bottom row. Compare this with [AHL10, Figure 1].

We shall prove Theorem 1.2.1 by first computing the $p$-completed homotopy groups of $\text{THH}(\text{ku})$ for each prime $p$ and then applying the arithmetic fracture square.

For $p = 2$, we have $\text{ku}^\wedge_2 = \ell^\wedge_2$ so that

\[ \text{THH}_*(\text{ku})^\wedge_2 = \mathbb{Z}_2[\beta] \oplus F^\wedge_2 \oplus T(2) \]

by [AHL10, Thm. 2.6]. More precisely, for a positive integer $m$, let

\[ m = a_02^n + \cdots + a_k2^{n-k} \]

be the 2-adic representation of $m$ with $a_1, \ldots, a_{k-1} \in \{0, 1\}$ and $a_0 = a_k = 1$. Then, the corresponding element to $h_{2m,j} \in T(2)$ in the notation of [AHL10, Thm. 2.8] is $g_w \in \Sigma^{24op^{n+2}+2(p-1)}T_n$ where $w$ is the string $(a_1, \ldots, a_k, 0, \ldots, 0)$ with $j$ trailing zeros.

When $p$ is odd, we shall prove the following theorem in this paper.
Theorem 1.2.2. Let \( p \) be an odd prime. The \( E_2 \)-page of the descent spectral sequence for \( \text{THH}(\text{ku})_p \rightarrow \text{THH}(\text{ku}/\text{MU})_p \) has the following description as graded \( (\text{ku}_p)_* = \mathbb{Z}_p[\beta] \)-modules:

\[
\begin{align*}
E_0^{*,*} &= \mathbb{Z}_p[\beta] \\
E_1^{*,*} &= \Sigma F_p \oplus T(p) \\
E_2^{*,*} &= \Sigma^2 T(p) \\
E_r^{*,*} &= 0 \quad \text{if } r \neq 0, 1, 2.
\end{align*}
\]

The spectral sequence degenerates at \( E_2 \) and we have an isomorphism

\[
\text{THH}_*(\text{ku})^\wedge = \mathbb{Z}_p[\beta] \oplus F_p \oplus T(p)
\]

as \( \mathbb{Z}_p[\beta] \)-modules.

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1.4. Notations and Conventions.

- Given a nonnegative integer \( m \) and a prime number \( p \), \( \text{val}_p(m) \) is the largest integer such that \( p^{\text{val}_p(m)} \) divides \( m \). We define \( \text{val}_p(0) = \infty \).
- The word category shall always mean \( \infty \)-category as in [Lur09]. We assume that we are working in the category of spectra \( \text{Sp} \) as in [Lur17]. Following its notation, we use \( \otimes \) to denote the smash product of spectra.
- If \( R \) is a commutative ring, we shall use the same symbol \( R \) (instead of \( HR \)) to denote the Eilenberg-Maclane ring spectra of \( R \), which has a canonical structure of an \( \mathbb{E}_\infty \)-ring spectra.

2. Preliminaries

2.1. Topological Hochschild Homology. Suppose that \( S \) is an \( \mathbb{E}_\infty \)-ring spectrum and \( R \) is an \( \mathbb{E}_1 \)-\( S \)-algebra. Then, we define the relative topological Hochschild homology as

\[
\text{THH}(R/S) := R \otimes_{R \otimes_S R^{op}} R.
\]

More generally, if \( M \) is a \( R \)-bimodule, or equivalently a \( (R \otimes_S R^{op}) \)-module, then the relative THH with coefficients in \( M \) is defined as

\[
\text{THH}(R/S; M) := M \otimes_{R \otimes_S R^{op}} R.
\]

In this paper, we shall only consider \( \mathbb{E}_\infty \)-ring spectra. More precisely, we shall only consider \( \text{THH}(R/S; M) \) when \( R \) is an \( \mathbb{E}_\infty \)-\( S \)-algebra and \( M \) is an \( \mathbb{E}_\infty \)-\( R \)-algebra. In this case, \( \text{THH}(R/S; M) \) naturally has a structure of an \( \mathbb{E}_\infty \)-\( R \)-algebra and we have an equivalence

\[
\text{THH}(R/S; M) = \text{THH}(R/S) \otimes_R M
\]

of \( \mathbb{E}_\infty \)-\( R \)-algebras.

Suppose that the following is a commutative diagram in the category of \( \mathbb{E}_\infty \)-ring spectra:

\[
\begin{array}{ccc}
R_2 & \leftrightarrow & R_1 \\
\uparrow & & \uparrow \\
S_2 & \leftrightarrow & S_1 \\
& & \\
& & \uparrow \\
& & \uparrow \\
& & \uparrow \\
& & \\
R_3 & \leftrightarrow & S_3
\end{array}
\]

Then, we have an equivalence

\[
\text{THH}(R_2 \otimes_{R_1} R_3/S_2 \otimes_{S_1} S_3) = \text{THH}(R_2/S_2) \otimes_{\text{THH}(R_1/S_1)} \text{THH}(R_3/S_3).
\]

This equivalence will be used several times without reference.
2.2. Suspension Elements. Suppose that $S$ is a connective $E_{\infty}$ ring spectrum and $R$ is a connective $E_{\infty}$-$S$-algebra. Let $F = \text{fib}(S \to R)$. We shall also assume that $S_\ast \to R_\ast$ is surjective, so that $F_\ast$ is an ideal of $S_\ast$.

Then, we define the suspension map $\sigma$ to be the composite

$$\Sigma F = 0 \coprod_{F} 0 \to R \coprod_{S} R \to R \otimes_{S} R.$$  \hspace{1cm} (2.2)

We shall also use $\sigma$ to denote the induced map on the homotopy groups $\pi_\ast(F) \to \pi_{\ast+1}(R \otimes_{S} R)$.

**Lemma 2.2.1.** We have $\sigma(xy) = x\sigma(y)$ for any $x \in S_\ast$ and $y \in F_\ast$. In particular, $\sigma$ annihilates the ideal $(F_\ast)^2$.

**Proof.** This follows from the fact that every map in (2.2) is a map of $S$-modules. \hfill $\square$

The suspension elements will play an important role as primitive elements in Hopf algebras.

**Proposition 2.2.2.** Let $R$ and $S$ be as above. In addition, assume that the odd homotopy groups of $S$ are zero and that the unit map $S_\ast \to R_\ast$ is a quotient by a regular sequence $R_\ast = S_\ast/(x_1, x_2, \ldots)$.

Then, there is a $R_\ast$-Hopf algebra structure on $\text{THH}_\ast(R/S)$ such that

(a) it is free as a $R_\ast$-module,
(b) the submodule of primitive elements is the free module generated by the elements $\sigma^2 x_i$’s,
(c) and for each $k$ and a prime $p$, $(\sigma^2 x_k)^p$ is a $R_\ast$-linear combination of $\sigma^2 x_i$’s modulo $p$.

**Proof.** Let $T = R \otimes_{S} R$. By [Ang08, Prop. 3.6], the homotopy group of $T$ is an exterior algebra $T_\ast = \Lambda_{R_\ast}(\sigma x_1, \sigma x_2, \ldots)$. Then, the $E_2$-page of the Künneth spectral sequence for $\text{THH}_\ast(R/S)$

$$E_2 = \text{Tor}^{T_\ast}(R_\ast, R_\ast) \Rightarrow \text{THH}_\ast(R/S)$$

can be computed as a divided power algebra

$$E_2 = \Gamma_{R_\ast}[\sigma^2 x_1, \sigma^2 x_2, \ldots].$$

The spectral sequence degenerates at the $E_2$-page since everything is concentrated in even degrees. From the $E_2$-page, we can see that $\text{THH}_\ast(R/S)$ is free as a $R_\ast$-module and that it is a Hopf algebra rather than just a Hopf algebroid.

Taking Whitehead tower of a spectrum defines a lax symmetric monoidal functor $\tau_{\geq \ast} : \text{Sp} \to \text{Fun}(\mathbb{Z}, \text{Sp})$, so that

$$\tau_{\geq \ast} R \otimes_{\tau_{\geq \ast} T} \tau_{\geq \ast} R$$

filters $\text{THH}(R/S)$. Taking the associated graded group, which is a symmetric monoidal process, we get

$$\pi_\ast R \otimes_{\pi_\ast T} \pi_\ast R$$

whose homotopy group is

$$\text{Tor}^{T_\ast}(R_\ast, R_\ast).$$

Therefore, this is the filtration that constructs the Künneth spectral sequence. We then see that the Künneth filtration is compatible with the coalgebra structure since it is induced by the map

$$\tau_{\geq \ast} R \otimes_{\tau_{\geq \ast} T} \tau_{\geq \ast} R \xrightarrow{\text{id} \otimes 1 \otimes \text{id}} \tau_{\geq \ast} R \otimes_{\tau_{\geq \ast} T} \tau_{\geq \ast} R \otimes_{\tau_{\geq \ast} T} \tau_{\geq \ast} R$$

of filtered spectra.

It follows that the element $\sigma^2 x_k$ is primitive, since $\Delta(\sigma^2 x_k)$ must have Künneth filtration 1, and the only possibility is $\sigma^2 x_k \otimes 1 + 1 \otimes \sigma^2 x_k$. Conversely, any primitive element in $\text{THH}_\ast(R/S)$ must be detected by a primitive element in the associated graded group. But in the associated graded
group, the primitives are linear combinations of $\sigma^2 x_i$'s, since it is a divided power coalgebra. This proves (b).

Lastly, the (underived) quotient $(\pi_*(R \otimes S R))/p$ is a $(R_*/p)$-Hopf algebra and has a filtration such that the associated graded is a divided power Hopf algebra on $\sigma^2 x_i$'s. Therefore, as in the previous paragraph, the primitive elements form a free $(R_*/p)$-module generated by $\sigma^2 x_i$'s. Then, (c) follows since $(\sigma^2 x_k)^p$ is primitive.

\[\square\]

Example 2.2.3. In this paper, we consider $\text{THH}(ku/MU)$ and $\text{MU} \otimes \text{THH}(MU) \otimes MU$. In the notation of Definition 3.2.4, $\text{THH}_*(MU/MU \otimes MU)$ is a $\text{MU}_*$-Hopf algebra with primitive elements $\sigma^2 b_i$'s for $i \geq 1$, and these elements will play an important role. For $\text{THH}_*(ku/MU)$, we will need only (c) of the previous proposition and not its Hopf algebra structure.

3. Descent Spectral Sequence

3.1. Descent Spectral Sequence.

Definition 3.1.1. Let $B \to C$ be a map of connective $E_\infty$-ring spectra. Then, we can form the augmented cosimplicial diagram

$$B \to C \Rightarrow C \otimes_B C \Rightarrow \cdots$$

of $E_\infty$-ring spectra, called the \textit{descent diagram}. Then, we obtain a spectral sequence

$$E_1^{s,t} = \pi_t(C \otimes_B (s+1)) \Rightarrow \pi_{t-s}(B).$$

More generally, if $M$ is a connective $B$-module, we have a descent diagram

$$M \to M \otimes_B C \Rightarrow M \otimes_B C \otimes_B C \Rightarrow \cdots$$

and a spectral sequence

$$E_1^{s,t} = \pi_t(M \otimes_B C \otimes_B (s+1)) \Rightarrow \pi_{t-s}(M).$$

This will be called the \textit{descent spectral sequence} for $M \to M \otimes_B C$ (along $B \to C$).

Definition 3.1.2. Suppose that $(k, \Gamma)$ is a graded Hopf algebroid such that $\Gamma$ is flat as a $k$-module and $A$ is a right $\Gamma$-comodule. By the cobar complex $\text{CB}_{\Gamma}(A)$ we mean the chain complex

$$A \to A \otimes_k \Gamma \to A \otimes_k \Gamma \otimes_k \Gamma \to \cdots$$

where the differentials are the alternating sum of comultiplication maps and the coaction map. This is bigraded so that $\text{CB}_{\Gamma_{t,s}}(A)$ is the degree $t$ part of $A \otimes_k \Gamma^\otimes_{k^s}$. The cohomology of $\text{CB}_{\Gamma}(A)$ is denoted by $\text{Ext}_{\Gamma}(A)$, which is again bigraded. This agrees with the usual definition of $\text{Ext}$. See, for example, [Rav86, A.1.2.12].

Proposition 3.1.3. In Definition 3.1.1 if $\pi_*(C \otimes_B C)$ is flat over $\pi_* C$, then $\pi_*(C \otimes_B C)$ is a Hopf algebroid over $\pi_* C$, $\pi_*(M \otimes_B C)$ is a comodule, and we have

$$E_1 = \text{CB}_{\pi_*(C \otimes_B C)}(\pi_*(M \otimes_B C))$$

$$E_2 = \text{Ext}_{\pi_*(C \otimes_B C)}(\pi_*(M \otimes_B C)).$$

Proof. The identification of $E_1$-page follows from the flatness assumptions and $E_2$-page follows from the definition. \[\square\]

Remark 3.1.4. In the previous definition, if $M$ is an $E_\infty$-$B$-algebra, then the descent spectral sequence for $M \to M \otimes_B C$ along $B \to C$ is isomorphic to the descent spectral sequence for $M \to M \otimes_B C$ along $M \to M \otimes_B C$. So in this case, the definition is less ambiguous without the phrase “along $B \to C$”.

Lemma 3.1.5. Let $f : B \to C$ be a 1-connective map of connective $E_\infty$-ring spectra. Then, for any connective $B$-module $M$, the descent spectral sequence for $M \to M \otimes_B C$ converges strongly to $\pi_* M$. 

Proof. Since the descent spectral sequence is constructed from the coskeletal filtration, the lemma follows from the fact that the fiber of the map
\[ M \to \cosk^s(M \otimes_B C^{\otimes_B(s+1)}) \]
is \( M \otimes_B I^{\otimes_B(s+1)} \) where \( I \) is the fiber of \( B \to C \). See, for example, [MNN17, Prop. 2.14]. □

**Lemma 3.1.6.** Suppose that a map of connective \( E_\infty \)-ring spectra \( f : B \to C \) induces a surjection on homotopy groups. Let \( x \in F_* \) where \( F = \text{fib}(f) \). Then, \( \sigma x \in \pi_* (C \otimes_B C) \), considered as an element in the \( E_1 \)-page of the descent spectral sequence for \( f \), is a permanent cycle and detects \( x \in B \).

Proof. It is enough to show that \( x \) is detected by \( \sigma x \) in the equalizer
\[ \text{Eq}(C \Rightarrow C \otimes_B C). \]
This follows from chasing the following diagram

\[
\begin{array}{ccc}
F & \Rightarrow & \text{Eq}(0 \Rightarrow 0 \sqcup F 0) \\
\downarrow & & \downarrow \\
B & \Rightarrow & \text{Eq}(C \Rightarrow C \sqcup_B C) \Rightarrow \text{Eq}(C \Rightarrow C \otimes_B C)
\end{array}
\]
from \( x \in F_* \). □

### 3.2. Complex K-theory spectrum

Until the end of section 4, we shall assume that \( p \) is a fixed odd prime number and that every spectrum is \( p \)-complete. For example, we shall write \( \text{THH}(\text{ku}) \) instead of \( \text{THH}(\text{ku})_p \).

We write \( \text{ku} \) for the connective cover of the complex K-theory spectrum \( \text{KU} \) equipped with the standard complex orientation \( \text{MU} \to \text{ku} \), which can be lifted to be an \( E_\infty \) orientation according to [Joa04].

We shall compute the descent spectral sequence for
\[ \text{THH}(\text{ku}) \to \text{THH}(\text{ku}/\text{MU}) = \text{THH}(\text{ku}) \otimes_{\text{THH}(\text{MU})} \text{MU} \]
along \( \text{THH}(\text{MU}) \to \text{MU} \). More generally, \( M = \mathbb{F}_p \) or \( \mathbb{Z}_p \) with a canonical \( \text{ku} \)-algebra structure, we shall compute the descent spectral sequence for
\[ \text{THH}(\text{ku}; M) \to \text{THH}(\text{ku}/\text{MU}; M). \]

Let \( E_{r,*}^*(\text{THH}(\text{ku}; M)) \) denote the \( E_r \)-page of this descent spectral sequence. Until the end of Section 4, let us write
\[ \Gamma := \text{MU} \otimes_{\text{THH}(\text{MU})} \text{MU} \]
\[ A := \text{THH}(\text{ku}/\text{MU}). \]

Using Propositions 2.2.2 and 3.1.3 we note a few things about the homotopy groups of these ring spectra.

- \( A_* \) and \( \Gamma_* \) are commutative rings with no odd homotopy groups.
- \( \Gamma_* \) is a \( \text{MU} \)-Hopf algebra and \( A_* \) is a right \( \Gamma_* \)-comodule algebra. We shall write \( \eta_R : A_* \to A_* \otimes \Gamma_* \) for the coaction map for reasons to be explained in Proposition 3.2.3.
- \( E_{1,*}^*(\text{THH}(\text{ku})) \) can be identified with the cobar complex

\[
\text{CB}_{\Gamma_*}(A_*) = (A_* D^0 \to A_* \otimes_{\text{MU}} \Gamma_* D^1 \to A_* \otimes_{\text{MU}} \Gamma_* \otimes_{\text{MU}} \Gamma_* D^2 \cdots)
\]
where we write \( D^0, D^1, \ldots \) for the differentials in \( \text{CB}_{\Gamma_*}(A_*) \). This is the \( d_1 \) differential in the descent spectral sequence.
Remark 3.2.1. Suppose $B \to C$ is a map of $E_\infty$-ring spectra. Instead of the coskeletal filtration of the descent diagram for $B \to C$, there is an alternative filtration using Whitehead covers. The $k$'th filtration of $B$ is given by

$$\text{Tot}(\tau_{\geq 2k} C^\otimes_B(s+1)).$$

If we further assume that $C^\otimes_B(s+1)$ has no odd homotopy groups for all $s \geq 0$, then this filtration gives us a shearing of the descent spectral sequence in the sense that we have

$$E^{k,3k-s}_{r} = E^{s,2k}_{2r+1}$$

where the left-hand side is the spectral sequence associated with the new filtration and the right-hand side is the descent spectral sequence.

In our case of $B = \text{THH}(ku)$ and $C = \text{THH}(ku/MU)$, the new filtration using Whitehead covers is an example of the even/motivic filtration of Hahn, Raksit, and Wilson \cite[Def. 4.1.2]{HRW} since $C$ is evenly free over $B$ in their sense.

**Lemma 3.2.2.** $\text{CB}_{\Gamma_*}(A_*)$ is a chain complex of free $ku_*$-modules. Therefore, we have identifications

$$E_1(\text{THH}(ku; Z_p)) = \text{CB}_{\Gamma_*}(A_*/\beta)$$

$$E_2(\text{THH}(ku; Z_p)) = \text{Ext}_{\Gamma_*}(A_*/\beta)$$

$$E_1(\text{THH}(ku; F_p)) = \text{CB}_{\Gamma_*}(A_*/(p, \beta))$$

$$E_2(\text{THH}(ku; F_p)) = \text{Ext}_{\Gamma_*}(A_*/(p, \beta)).$$

**Proof.** From Example 2.2.3, $A_*$ is free as a $ku_*$-module and $\Gamma_*$ is free as a $MU_*$-module. \hfill $\square$

**Notation.** From this point, all ordinary modules or ordinary rings will naturally be modules or algebras over $MU_*$, and all tensor product $\otimes$ will be over $MU_*$ unless the base ring is explicitly written. We shall continue to write the base for the tensor product of spectra unless it is over the sphere spectrum.

There is an alternative description of the descent spectral sequence. The augmented cosimplicial diagram

$$S^0 \to MU \Rightarrow MU^\otimes 2 \Rightarrow \cdots$$

induces an augmented cosimplicial diagram

$$\text{THH}(ku) \to \text{THH}(ku/MU) \Rightarrow \text{THH}(ku/MU^\otimes 2) \Rightarrow \cdots$$

(3.1)

of $E_\infty$-ring spectra.

**Proposition 3.2.3.** The augmented cosimplicial diagram \([3.1]\) is equivalent to the descent diagram for $\text{THH}(ku) \to \text{THH}(ku/MU)$. Furthermore, under the identification $\text{THH}_*(ku/MU^\otimes 2) = A_* \otimes \Gamma_*$ the two maps $\text{THH}_*(ku/MU) \to \text{THH}_*(ku/MU^\otimes 2)$ induced by the left and right units $\eta_L, \eta_R : MU \to MU^\otimes 2$ can be identified with id $\otimes 1$ and the coaction map $A_* \to A_* \otimes \Gamma_*$, respectively.

**Definition 3.2.4.** Following the classical notation, We write $x_1, x_2, \ldots$ with $|x_i| = 2i$ for the polynomial generators of the Lazard ring $MU_*$, and we write $b_1, b_2, \ldots$ for the generators of $MU_*MU = MU_*[b_1, b_2, \ldots]$ as $MU_*$ algebras where $MU_*MU$ is given the algebra structure by the left unit $MU_* \to MU_*MU$ and $b_i$'s vanish under the multiplication map $MU_*MU \to MU_*$. There are many choices for the generators and for now, we only require that $x_1$ maps to $\beta$ under

$$MU_* \to ku_* = Z_p[\beta]$$

and that $x_i$ maps to zero for $i \geq 2$. We shall give more specific choices of generators in Lemma 3.2.8.

For the lightness of notations, we shall often write $v_k$ instead of $x_{p^k-1}$ when $k \geq 2$ and $t_k$ instead of $b_{p^k-1}$ when $k \geq 1$. 

**Notation.** The elements $x_i \in \text{MU}_*$ for $i \geq 2$ and $b_i \in \text{MU}_* \text{MU}$ for $i \geq 1$ admit double suspensions $\sigma^2 x_i \in A_*$ and $\sigma^2 b_i \in \Gamma_*$. In the cobar complex $\text{CB}_*(A_*)$, we shall write $\sigma^2 x_i$ (or $\sigma^2 v_i$) for the corresponding element in either $A_*$ or $A_* \otimes \Gamma_*$. Which element the notation is referring to will be clear from the context. Similarly, we shall write $\sigma^2 b_i$ (or $\sigma^2 t_i$) for the corresponding element in $A_* \otimes \Gamma_*$. We will not need any notation for elements in $A_* \otimes \Gamma_*^s$ for $s \geq 2$.

The following two remarks hold for any choice of generators.

**Remark 3.2.5.** Consider the descent spectral sequence for $\text{THH}(\text{MU}) \to \text{THH}(\text{MU}/\text{MU})$, whose $E_1$-page is $\text{CB}_*(\text{MU}_*)$. Since

$$\text{THH}_*(\text{MU}) = \Lambda_{\text{MU}_*}(\sigma b_1, \sigma b_2, \ldots),$$
we can see, by Lemma 3.1.6, that the element $\sigma^2 b_i \in \text{CB}_*(\text{MU}_*)$ is a permanent cycle in the descent spectral sequence for any $i$ and that it detects $\sigma b_i \in \text{THH}_*(\text{MU})$.

Mapping to the descent spectral sequence for $\text{THH}(\text{ku}) \to \text{THH}(\text{ku}/\text{MU})$, we can see that $\sigma^2 b_i \in \text{CB}_*(\text{A}_*)$ is a permanent cycle in this descent spectral sequence and detects $\sigma b_i \in \text{THH}_*(\text{ku})$. Here, $\sigma b_i$ is the suspension of the class $b_i \in \text{ku}_* \text{ku}$, which is defined to be the image of $b_i \in \text{MU}_* \text{MU}$.

**Remark 3.2.6.** There is a multiplicative structure on the descent spectral sequence, which is represented in $E_1 = \text{CB}_*(\text{A}_*)$ by the standard formula for cup product of cocycles. We shall only be interested in the multiplication by $\sigma^2 b_i \in A_* \otimes \Gamma_*$, which is a permanent cycle by the previous remark. In this case, we can check that the cup product formula in the $E_1$-page for $x \in A_* \otimes \Gamma_*^s$ and $\sigma^2 b_i$ equals $x \otimes \sigma^2 b_i \in A_* \otimes \Gamma_*^s(s+1)$.

**Lemma 3.2.7.** For any choice of generators in Definition 3.2.4, we have

$$(\sigma^2 v_k)^p \equiv \sigma^2 v_{k+1} \quad (\text{mod } p, \beta)$$

for $k \geq 2$ and

$$(\sigma^2 x_{p-1})^p \equiv \sigma^2 v_2 \quad (\text{mod } p, \beta)$$
in $A_*$ up to a $p$-adic unit. Similarly, we have

$$(\sigma^2 b_i)^p \equiv \sigma^2 b_{p+i-1} \quad (\text{mod } p, x_1, x_2, \ldots)$$

for $i \geq 1$ in $\Gamma_*$ up to a $p$-adic unit.

**Proof.** For $A_*$, the proof is the same as the proof of [HW Prop. 2.5.3]. For $\Gamma_*$, it is similar and we shall sketch the proof. We wish to show that $(\sigma^2 v_k)^p = \sigma^2 v_{k+1}$ in

$$\Gamma_* \otimes \text{MU}_* \mathbb{F}_p = \pi_*(\Gamma \otimes \text{MU} \mathbb{F}_p).$$

Since base changing along $\text{MU} \to \mathbb{F}_p$ is a symmetric monoidal functor, we have

$$\Gamma \otimes \text{MU} \mathbb{F}_p = \mathbb{F}_p \otimes \text{THH}(\text{MU}; \mathbb{F}_p) \mathbb{F}_p$$

where $\text{THH}(\text{MU}; \mathbb{F}_p) = \mathbb{F}_p \otimes \mathbb{F}_p \otimes \text{MU} \mathbb{F}_p$ again by base change. By the stability of Dyer-Lashof operations, we have

$$(\sigma^2 b_i)^p = Q_0(\sigma^2 b_i) = \sigma^2(Q_2 b_i).$$

Then, the statement follows from the computation of the operation $Q_2$ in $(\mathbb{F}_p)_* \text{MU} = H_*(BU; \mathbb{F}_p)$, done in [Koc71 Thm. 6].

**Lemma 3.2.8.** We can choose the generators $x_1, x_2, \ldots, b_1, b_2, \ldots$ so that the following properties hold.

(a) There is a sequence of $p$-adic units $\delta_0 = 1, \delta_1, \delta_2, \ldots$ such that $(\sigma^2 v_2)^p \equiv \delta_k \sigma^2 v_{k+2} \quad (\text{mod } p)$ for $k \geq 0$ in $A_*$. 

(b) We have $(\sigma^2 t_1)^p \equiv \delta_k \sigma^2 t_{k+1} \quad (\text{mod } p)$ for $k \geq 0$ in $\Gamma_*$ with the same sequence $\delta_1, \delta_2, \ldots$ as in (a).

(c) We have $(\sigma^2 b_1)^p \equiv \sigma^2 b_{2p-1} \quad (\text{mod } p)$. 
(d) The coaction of the element $\sigma^2 v_k \in A_*$ is given as
$$\eta_R \sigma^2 v_k = \sigma^2 v_k + \sigma^2 t_k + \beta \sigma^2 \sigma^2 t_{k-1}$$
and the coaction of the element $\sigma^2 x_{p-1}$ is given as
$$\eta_R \sigma^2 x_{p-1} = \sigma^2 x_{p-1} + \sigma^2 t_1 + \beta \sigma^2 \sigma^2 b_1.$$

(e) There is a constant $\delta'$ such that the coaction of the element $\sigma^2 x_{2p-1}$ is given as
$$\eta_R \sigma^2 x_{2p-1} = \sigma^2 x_{2p-1} + \sigma^2 b_{2p-1} + \delta' \sigma^2 t_1.$$

(f) Assuming all of the above, the constant $\delta'$ in (e) is a $p$-adic unit.

Proof. By the naturality of $\sigma$, we have $\eta_R(\sigma^2 \alpha) = \sigma^2 \eta_R(\alpha)$ for any $\alpha \in \pi_* \text{fib}(\text{MU} \to \text{ku})$. Therefore, the lemma will be proved using properties of the right unit $\text{MU}_* \to \text{MU}_* \text{MU}$. In this proof, the elements of the ideal $(x_2, x_3, \ldots)^2 \subseteq A_*$ or $(x_2, x_3, \ldots, b_1, b_2, \ldots)^2 \subseteq A_* \otimes \Gamma_*$ will be called decomposables. Decomposables are annihilated by $\sigma^2$.

Note that we do not need to require that the sequence $\delta_1, \delta_2, \ldots$ of $p$-adic units appearing in (a) and (b) are the same sequences because it follows automatically from the first equation of (d) by taking $p$-th powers.

First, let us choose the generators $x_{p-1}, t_1$ and $b_1$ satisfying the second equation of (d). Let $i_Q : (\text{BP}_*, \text{BP}_* \text{BP}) \to (\text{MU}_*, \text{MU}_* \text{MU})$ be the map of Hopf algebroids induced by the Quillen idempotent and let $v'_1, v'_2, \ldots, t'_1, t'_2, \ldots$ be the Hazewinkel generators ([Rav86, A2.2]) for $\text{BP}_* = \mathbb{Z}[v'_1, \ldots]$ and $\text{BP} \text{BP} = \text{BP}_*[t'_1, \ldots]$. Note that $i_Q(v'_1) \in \text{MU}_*$ is well-defined modulo $p$ since it is the coefficient of $X^p$ in the $p$-series of the universal formal group law mod $p$. This means that the image of $i_Q(v'_1)$ in $\text{ku}_*$ is $c\beta p^{-1}$ for some $p$-adic unit $c$. We define $x_{p-1} \in \text{MU}_*$ to be
$$x_{p-1} := i_Q(v'_1) - cx_1^{p-1}$$
and define $t_1 \in \text{MU}_* \text{MU}$ to be $i_Q(t'_1)$. If we choose $b_1$ so that $\eta_R(x_1) = x_1 + b_1$ at least for now, then we have
$$\eta_R(x_{p-1}) = i_Q(\eta_R v_1) - c\eta_R(x_1^{p-1})$$
$$= i_Q(v'_1 + pt'_1) - c(x_1 + b_1)^{p-1}$$
$$= x_{p-1} + pt_1 - cx_1^{p-2}b_1 + \text{(decomposables)}.$$

Therefore, after scaling $b_1$ by a $p$-adic unit, we have
$$\eta_R(\sigma^2 x_{p-1}) = \sigma^2 x_{p-1} + \sigma^2 t_1 + \beta \sigma^2 \sigma^2 b_1.$$

Next, make arbitrary choices for all $b_i$'s that have not already been defined. We shall redefine $t_k$'s inductively in $k \geq 2$ so that (b) holds. Suppose that we have chosen $t_{k-1}$. By Proposition 2.2.2, we have
$$(\sigma^2 t_{k-1})^p \equiv \alpha_0 \sigma^2 t_k + \sum_{i=1}^{p^k-2} \alpha_i \sigma^2 b_{k-1-i} \pmod{p}$$
for some $\text{MU}_*$-coefficients $\alpha_i \cdots \in \text{MU}_*$. By Lemma 3.2.7, $\alpha_0$ is a $p$-adic unit. Therefore, we may redefine $t_k$ to be
$$\alpha_0 t_k + \sum_{i=1}^{p^k-2} \alpha_i b_{k-1-i}$$
then $(\sigma^2 t_{k-1})^p$ would be a unit multiple of $\sigma^2 t_k$ modulo $p$, and (b) follows from this. Any future changes on the $t_k$'s will not violate (b).

By the same argument, we can choose $b_{2p-1}$ satisfying (c).
We shall now choose \( v_2 \). We start with an arbitrary choice, not even requiring that \( v_2 \) maps to zero in \( ku \). Then, the right unit formula says

\[
\eta_R v_2 = v_2 + pt_2 + \sum_{i=1}^{p^2-2} c_i x_1^i b_{p^2-1-i} + \text{ (decomposables)}.
\]

Since \( \eta_R x_i = x_i + b_i + \cdots \) for \( 2 \leq i < p^2 - 1, i \neq p - 1 \), we can redefine \( v_2 \) by subtracting linear combinations of \( x_1^{p^2-1-i} x_i \) from \( v_2 \) to get

\[
\eta_R v_2 = v_2 + pt_2 + c x_1^{p^2-p} t_1 + c' x_1^{p^2-2} b_1 + \text{ (decomposables)}.
\]

We can further subtract multiples of \( x_1^{p^2-p} v_1 \) and get

\[
\eta_R v_2 = v_2 + pt_2 + c x_1^{p^2-p} t_1 + \text{ (decomposables)}.
\]

The question is whether the constant \( c \) is a \( p \)-adic unit or not. One can check that the answer to this question does not depend on the initial arbitrary choice of \( v_2 \) and even \( t_2 \). Therefore, we can test the question on \( i_Q(v_2') \) and \( i_Q(t_2') \), and we can check that \( c \) must be a \( p \)-adic unit using the formula

\[
\eta_R(v_2') = v_2' + pt_2' - p + 1)(v_1')^{p+1}t_1' \quad (\text{mod } (t_1')^2)
\]

for Hazewinkel generators. Scaling \( v_2 \) and \( t_2 \) by \( p \)-adic units if necessary, we obtain a generator \( v_2 \) such that (d) holds with \( k = 2 \).

We can deduce (e) with the same argument on \( x_{2p-1} \). Note that even if \( \delta' \) turns out to be a \( p \)-adic unit, we cannot remove it because \( b_{2p-1} \) cannot be scaled to be consistent with (c) and \( x_{2p-1} \) cannot be scaled to fix the coefficient of \( \sigma^2 b_{2p-1} \) in \( \eta_R(\sigma^2 x_{2p-1}) \).

By the same argument as in (b), we can choose \( v_3, v_4, \ldots \) satisfying (a).

Then, we shall modify \( v_k \)'s inductively for \( k \geq 3 \) so that (d) is true. Suppose that we have chosen \( v_{k-1} \) so that (d) is true. Then, taking the \( p \)-th power of

\[
\eta_R \sigma^2 v_{k-1} = \sigma^2 v_{k-1} + p \sigma^2 t_{k-1} + \beta p^{k-1-p^{k-2}} \sigma^2 t_{k-2}
\]

and using (a) and (b), we see that

\[
\eta_R v_k = v_k + pt_k + \sum_{i=1}^{p^k-2} c_i x_1^i b_{p^k-1-i} + \text{ (decomposables)}
\]

with \( c_i \)'s being multiples of \( p \) except when \( i = p^k - p^{k-1} \). Subtracting multiples of \( x_1^i x_{p^k-1-i} \) for \( 1 \leq i < p^k - p^{k-1} \), we can assume that \( c_i = 0 \) for all \( 1 \leq i < p^k - p^{k-1} \). Then, we can redefine \( t_{k-1} \) to be

\[
\sum_{i=p^k-p^{k-1}}^{p^k-2} c_i x_1^i b_{p^k-1-i}
\]

and we are done.

Finally, let us prove (f). By Proposition 2.2.2 and Lemma 3.2.7, we have

\[
((\sigma^2 x_{p-1})^p) = 2 \sigma^2 v_2 + \sum_{i=1}^{p^2-2} \alpha_i \beta^i \sigma^2 x_{p^2-1-i} \quad (\text{mod } p)
\]

for some constants \( \alpha_0, \ldots, \alpha_{p^2-2} \) with \( \alpha_0 \) a unit. We shall compare both sides after applying \( \eta_R \).

Since

\[
\eta_R((\sigma^2 x_{p-1})^p) = (\sigma^2 x_{p-1})^p + \beta (p-2) \sigma^2 b_{2p-1} \quad (\text{mod } p)
\]
we have $\alpha_i \equiv 0 \pmod{p}$ for $i \neq 0, p^2 - 2p, p^2 - p$ since it would otherwise introduce a nonzero term $\beta \sigma^2 b_{p^2 - i}$ when $\eta_R$ is applied. If $\delta' \equiv 0 \pmod{p}$, then on the right hand side, we would not be able to cancel out $\beta p^2 - \sigma^2 t_1$ appearing in $\eta_R(\alpha_0 \sigma^2 v_2)$. □

Remark 3.2.9. Using that $\sigma^2 b_1$ can be identified with the image of Bott map in [Ada74 Prop. II.12.6], it could be possible to determine the sequence $(d_i)$ or even the generators.

4. Bockstein Spectral Sequences

As before, until the end of this section, $p$ is a fixed odd prime and every spectrum is $p$-complete. We also fix a set of generators $x_1, x_2, \ldots, b_1, b_2, \ldots$ of $M_u$ and $M_u, M_u$ satisfying Lemma 3.2.8.

Recall that $\text{CB}_{\gamma}(A_*)$ is a cobar complex of free $k_u$-modules. Then, filtering this cobar complex by powers of $\beta$, we obtain the $\beta$-Bockstein spectral sequence

$$E_1 = \text{Ext}_{\gamma}(A/\beta)[\beta] \Rightarrow \text{Ext}_{\gamma}(A), \quad (4.1)$$

and filtering $\text{CB}_{\gamma}(A_*/\beta)$ by powers of $p$, we obtain the $v_0$-Bockstein spectral sequence

$$E_1 = \text{Ext}_{\gamma}(A/(p, \beta))[v_0] \Rightarrow \text{Ext}_{\gamma}(A/\beta). \quad (4.2)$$

The convergence of the $v_0$-Bockstein spectral sequence follows from $p$-completeness and the convergence of the $\beta$-Bockstein spectral sequence follows from the fact that $A \otimes \Gamma^{\otimes s}$ is bounded below for any $s$. We shall compute these spectral sequences in this section.

Note that a $v_0$-Bockstein differential $d_*(x) = v_0^r y$ is equivalent to finding a class $\bar{x} \in \text{CB}_{\gamma}(A_*/\beta)$ such that $\bar{x}$ represents $x$ modulo $p$ and $D^i(\bar{x}) = p^i \bar{y}$ for some $\bar{y} \in \text{CB}_{\gamma}(A_*/\beta)$ representing $y$ modulo $p$. There is a similar description for $\beta$-Bockstein differentials.

4.1. $v_0$-Bockstein.

Proposition 4.1.1. We have

$$\text{Ext}_{\gamma}(A_*/(p, \beta)) = F_p[\mu] \otimes F_p A_{\gamma}(\lambda_1, \lambda_2)$$

as $F_p$-algebras, where $\mu, \lambda_1, \lambda_2$ are represented by $\sigma^2 x_{p-1}, \sigma^2 t_1, \sigma^2 b_1$, respectively, in $\text{CB}_{\gamma}(A_*/(p, \beta))$.

Proof. We can mimic the proof of [HIW Prop. 6.1.6]. The necessary ingredient is [Aus05 Thm. 6.8] which states that

$$\text{THH}_*(ku; F_p) \simeq F_p[\sigma^2 x_{p-1}] \otimes A_{\gamma}(\sigma t_1, \sigma b_1).$$

Corollary 4.1.2. The descent spectral sequence for $\text{THH}(ku) \to \text{THH}(ku/MU)$ degenerates at $E_2$.

Proof. Since $\text{Ext}_{\gamma}(A_*/(p, \beta))$ is concentrated in $0 \leq s \leq 2$, so is $\text{Ext}_{\gamma}(A_*)$. This follows from the convergence of Bockstein spectral sequences and will become clearer as we compute Bockstein differentials. Furthermore, the Ext groups are nonzero only if $t \in 2Z$ because this is already true for cobar complexes. Therefore, there is no room for differentials and the descent spectral sequence degenerates. □

Lemma 4.1.3. (a) For any nonnegative integers $m, k$, we have

$$\text{val}_p \left( \binom{m}{k} \right) \geq \text{val}_p(m) - \text{val}_p(k),$$

and if $k$ is a power of $p$, then it is an equality.

(b) Let $R$ be any commutative ring and $x, y \in R$ be any elements. Then

$$(x + p^e y)^m \equiv x^m + mp^e x^{m-1} y \pmod{p^{e + \text{val}_p(m) + 1}}$$

for any positive integers $m, e$. 


(c) If \( m \) is a multiple of \( p^k \) for some \( k \geq 0 \), then we have
\[
\binom{m}{p^k} \equiv \frac{m}{p^k} \quad \text{(mod } p).\]

**Proof.** (1) follows from the fact that \( \text{val}_p \left( \binom{m}{k} \right) \) is equal to the number of carries in the \( p \)-adic addition of \( k \) and \( m - k \). (2) easily follows from (1). (3) is a special case of Lucas’s theorem. \( \square \)

**Proposition 4.1.4.** We have
\[
\text{Ext}_{\Gamma_*}(A_*; \beta) \simeq \left[ Z_p \{1\} \oplus \bigoplus_{k=1}^{\infty} \mathbb{Z}/p^{\text{val}_p(k)+1} \{a_k\} \right] \otimes_{\mathbb{Z}_p} \Lambda_{\mathbb{Z}_p}(\lambda_2)
\]
as \( \mathbb{Z} \)-modules. The generators are represented in the \( \text{CB}_{\Gamma_*}(A_*; \beta) \) as the following:

- \( Z_p \) is generated by 1,
- The generator \( a_k \) of \( \mathbb{Z}/p^{\text{val}_p(k)+1} \) is represented by the bidegree \( (1, 2pk) \) cycle (also denoted \( a_k \) by abuse of notation)
\[
a_k := \frac{D^0((\sigma^2 x_{p-1} - 1)^k)}{p^{\text{val}_p(k)+1}} = \frac{(\sigma^2 x_{p-1} + p\sigma^2 b_{p-1})^k - (\sigma^2 x_{p-1})^k}{p^{\text{val}_p(k)+1}},
\]
- \( \lambda_2 \) is represented by \( \sigma^2 b_1 \).

**Multiplication by \( \lambda_2 \) should be interpreted as in Remark 4.2.6.**

**Proof.** We work in \( \text{CB}_{\Gamma_*}(A_*; \beta) \). Then, by Lemma 4.1.3(b), we have
\[
D^0((\sigma^2 x_{p-1} - 1)^k) = (\sigma^2 x_{p-1} + p\sigma^2 t_1)^k - (\sigma^2 x_{p-1})^k \equiv kp(\sigma^2 x_{p-1})^{k-1}\sigma^2 t_1 \quad \text{(mod } p^{\text{val}_p(k)+2}),
\]
which gives us the differentials
\[
d_{\text{val}_p(k)+1}(\mu^k) = v_0^{\text{val}_p(k)+1}\mu^{k-1}\lambda_1
\]
up to a \( p \)-adic unit in the \( v_0 \)-Bockstein spectral sequence 4.2. Since \( \lambda_2 \) is a permanent cycle represented by \( \sigma^2 b_1 \), we have the differentials
\[
d_{\text{val}_p(k)+1}(\mu^k\lambda_2) = v_0^{\text{val}_p(k)+1}\mu^{k-1}\lambda_1\lambda_2.
\]
The generators in the statement can be derived from this computation. \( \square \)

**Corollary 4.1.5.** We have
\[
\text{THH}_{\ast}(\text{ku}; \mathbb{Z}_p) = \left[ Z_p \oplus \bigoplus_{k=1}^{\infty} \mathbb{Z}\text{}/p^{2pk-1}\mathbb{Z}/p^{\text{val}_p(k)+1} \right] \otimes_{\mathbb{Z}_p} \Lambda_{\mathbb{Z}_p}(\lambda_2)
\]
as graded \( \mathbb{Z}_p \)-modules with \( |\lambda_2| = 3 \).

**Proof.** Since
\[
E_2^{s,t}(\text{THH}(\text{ku}; \mathbb{Z}_p)) = \text{Ext}_{\Gamma_*}^{s,t}(A_*; \beta)
\]
is concentrated in \( 0 \leq s \leq 2 \) and \( t \in 2\mathbb{Z} \), there is no room for any further differential in the descent spectral sequence computing \( \text{THH}(\text{ku}; \mathbb{Z}_p) \). There is no extension problem since \( E_2^{0,s} \) is free as a \( \mathbb{Z}_p \)-module. \( \square \)

**Remark 4.1.6.** The previous corollary agrees with the known result from [Aus05, Cor. 6.9].
4.2. β-Bockstein. The differentials of the β-Bockstein spectral sequence is given as the following.

**Theorem 4.2.1.** (a) For all \( e \geq 0, p^e a_p \in \text{Ext}_1^{\mathcal{F}_2}(A_*/\beta) \) can be lifted to a class in \( \text{Ext}_1^{\mathcal{F}_2}(A_*) \) represented by the cycle \( \sigma^2 t_{e+1} \in \text{CB}^{\mathcal{F}_2}_1(A_*) \).

(b) For each \( 0 \leq e \leq \text{val}_p(m) \), there is a differential

\[
d_{p^{e+1}-2}(p^e a_m) = \frac{m - p^e}{p^e} \beta^{p^{e+1}-2} a_{m-p^e} \lambda_2
\]

up to a \( p \)-adic unit in the β-Bockstein spectral sequence (4.1).

To prove the theorem, let us first construct some elements in the cobar complex. Define

\[
y = (\delta')^{-1}(-\sigma^2 x_{p-1})^p + \beta^{p(p-2)} \sigma^2 x_{2p-1} \in A_*.
\]

**Lemma 4.2.2.** (a) The coaction modulo \( p^2 \) of \( y \) is given as

\[
\eta_R(y) \equiv y + \beta^{p(p-1)} \sigma^2 t_1 \pmod{p^2}.
\]

(b) We have \( y \equiv \sigma^2 v_2 \pmod{p} \).

**Proof.** (1) follows from Lemma [3.2.8](#) and Lemma [4.1.3](#) (b).

By Proposition [2.2.2](#) \( y \) is a \( ku_* \)-linear combination of \( \sigma^2 x_i \)'s. However, using Lemma [3.2.8](#) we can see that \( \eta_R(y) \pmod{p} \) completely determines \( y \pmod{p} \) if \( y \) is a linear combination of \( \sigma^2 x_i \)'s. □

Next, fix a pair of positive integers \((m, e)\) such that \( 1 \leq e \leq \text{val}_p(m) \). Then, we define constants \( \varepsilon_1, \ldots, \varepsilon_e \) inductively as

\[
\varepsilon_1 = -\frac{\delta_{e-1}}{p^{\text{val}_p(m)-e}} \left( \frac{m/p}{p^{e-1}} \right), \quad \varepsilon_{k+1} = -\varepsilon_k \left( 1 + \left( \frac{m-p^e}{p^{e-k-1}} \right) \right).
\]

Then, define

\[
f_{m,e} := \frac{1}{p^{\text{val}_p(m)-e+1}} D^0\left(y^{m/p}\right) + \sum_{k=1}^{e} \frac{\varepsilon_k}{p^{k+1}} \beta^{p^{e+1}-p^{e-k+1}} D^0\left(y^{(m-p^e)/p^{e-k}} \sigma^2 x_{p^{e-k+1}-1} \right)
\]

which is an element in \((A_* \otimes \Gamma_*)[p^{-1}]\). We also define

\[
f_{m,0} := \frac{1}{p^{\text{val}_p(m)+1}} D^0\left((\sigma^2 x_{p-1})^m\right).
\]

**Lemma 4.2.3.** (a) \( \varepsilon_1, \ldots, \varepsilon_e \) are \( p \)-adic units.

(b) For \( 1 \leq k < e \), we have \( \varepsilon_{k+1} \equiv -\varepsilon_k \pmod{p^k} \). If \( \text{val}_p(m-p^e) > e \), then we have \( \varepsilon_{k+1} \equiv -\varepsilon_k \pmod{p^{k+1}} \).

**Proof.** Both statements follow from Lemma [4.1.3](#). □

**Lemma 4.2.4.** The image of \( f_{m,e} \) in \((A_* \otimes MU_*, \Gamma_*))/(p^\infty, \beta^{p^{e+1}-1})\) is

\[
\frac{1}{p^{e+1}} \beta^{p^{e+1}-2}(\sigma^2 x_{p-1})^{m-p^e} \sigma^2 b_1.
\]

up to a \( p \)-adic unit.

**Proof.** For \( e = 0 \), we have

\[
D^0\left((\sigma^2 x_{p-1})^m\right) \equiv (\sigma^2 x_{p-1} + \beta^{p-2} \sigma^2 b_1)^m - (\sigma^2 x_{p-1})^m \pmod{p^{\text{val}_p(m)+1}}
\]

\[
\equiv m \beta^{p-2}(\sigma^2 x_{p-1})^{m-1} \sigma^2 b_1 \pmod{\beta^{p-1}, p^{\text{val}_p(m)+1}}
\]

by Lemma [4.1.3](#) and Lemma [3.2.8](#) and the statement follows.
For $e > 0$, we similarly have
\[
D^0(y^{m/p}) \equiv (y + \beta^{(p-1)}\sigma^2 t_1)^{m/p} - y^{m/p} \quad \text{(mod } p^{\text{val}_p(m+1)})
\]
\[
\equiv \left(\frac{m/p}{p^{e-1}}\right) \beta^{p^{p-1} (p-1) y^{(m-p)/p} \sigma^2 t_1^{p^{e-1}} - \beta^{p^{p-1} (p-1) y^{(m-p)/p} \sigma^2 t_1}} \quad \text{(mod } p^{\text{val}_p(m+1)})
\]
\[
\equiv \delta_{e-1} \left(\frac{m/p}{p^{e-1}}\right) \beta^{p^{p-1} (p-1) y^{(m-p)/p} \sigma^2 t_e} \quad \text{(mod } p^{\text{val}_p(m+1)})
\]
Also, using Lemma 4.1.3 we can check that the fractions in the equation above are actually elements of $D^0(y^{(m-p)/p} \sigma^2 t_{e-k+1})$
\[
\equiv py^{(m-p)/p} \sigma^2 t_{e-k+1} + \left(1 + \left(\frac{(m-p)/p}{p^{e-k-1}}\right)\right) \beta^{p^{p-2} (p-1) y^{(m-p)/p} \sigma^2 t_{e-k}} \quad \text{(mod } p^{e+1}, \beta^{p-1}).
\]
and similarly
\[
D^0(y^{(m-p)/p} \sigma^2 x_{p-1}) = py^{(m-p)/p} \sigma^2 x_{p-1} + \beta^{p-2} y^{(m-p)/p} \sigma^2 b_1 \quad \text{(mod } p^{e+1}, \beta^{p-1}).
\]
Combining these calculations, we obtain the result. \qed

**Proof of Theorem 4.2.1.** (a) If $e = 0$,
\[
D^0(\sigma^2 x_{p-1}) - \beta^{p-2} \sigma^2 b_1 = \sigma^2 t_1.
\]
is a cycle representing $a_1$ modulo $\beta$, and if $e > 0$,
\[
D^0(y^{p^{e-1}} - \delta_{e-1} \beta^{p^{e+1} - p^{e} \sigma^2 t_e}) \equiv D^0\left(\frac{y^{p^{e-1}} - \delta_{e-1} \sigma^2 v_{e+1}}{p}\right) + \delta_{e-1} \sigma^2 t_{e+1}
\]
is a cycle representing $p^e a_{p^e}$ modulo $\beta$ which is homologous to $\delta_{e-1} \sigma^2 t_{e+1}$. Using Lemmas 3.2.8 and 4.2.2, we can check that the fractions in the equation above are actually elements of $A_\ast$ or $A_\ast \otimes \Gamma_\ast$.

(b) By Lemma 4.2.4 we have $f_{m,e}^+ \in A_\ast \otimes \Gamma_\ast$ and $g_{m,e} \in (A_\ast \otimes \Gamma_\ast)[p^{-1}]$ such that
\[
f_{m,e}^+ = f_{m,e}^+ - \beta^{p^{e+1} - 2} g_{m,e},
\]
and
\[
g_{m,e} = \frac{1}{p^{e+1}} (\sigma^2 x_{p-1})^{m-p} \sigma^2 b_1 \quad \text{(mod } \beta)
\]
up to a $p$-adic unit, so that $D^1(g_{m,e})$ represents
\[
\frac{m-p^e}{p^e} a_{m-p^e} \lambda_2
\]
modulo $\beta$.

Also, we have
\[
f_{m,e}^+ \equiv f_{m,e} \equiv a_m \quad \text{(mod } \beta),
\]
and since $D^1(f_{m,e}) = 0$ by construction, we have
\[
D^1(f_{m,e}^+) = \beta^{p^{e+1} - 2} D^1(g_{m,e})
\]
which implies the Bockstein differentials. \qed

Using Theorem 4.2.1, we can compute the associated graded group $\text{gr}_\beta \text{Ext}_1(A_\ast)$ with respect to the $\beta$-adic filtration. We shall discuss these groups and extension problems in 4.3 but we have a rough description as follows.

- $\text{gr}_\beta \text{Ext}_0$ is $\text{ku}_\ast$ and is generated by an element represented by the cycle 1.
• \( \text{gr}_\beta \text{Ext}^1 \) is generated by the elements detected by \( p^e a_{p^e} \) for \( e \geq 0 \) and \( \lambda_2 \). These generators are \( p \)-torsions and support infinite \( \beta \)-tower.

• \( \text{gr}_\beta \text{Ext}^2 \) is generated by elements detected by \( a_m \lambda_2 \) in \( \text{Ext}^1_{\Gamma_s}(A_s) \), which is \( p^j \beta p^{\text{val}_p(m)-j+1-2} \) torsion for each \( 0 \leq j \leq \text{val}_p(m) \) and \( p^{\text{val}_p(m)+1} \)-torsion.

4.3. Extension Problems. We shall resolve the extension problems with respect to the \( \beta \)-adic filtration thereby proving Theorem 1.2.1.

Proof of Theorem 1.2.1. By Theorem 4.2.1, \( \text{gr}_\beta \text{Ext}^1_{\Gamma_s}(A_s) \) is a free \( \mathbb{Z}/p[\beta] \)-modules generated by classes represented by the cycles \( \sigma^2 b_1, \sigma^2 t_1, \sigma^2 t_2, \ldots \). From the following differentials in \( \text{CB}_{\Gamma_s}(A_s) \)

\[
D^0(\sigma^2 c_{e+1}) = \sigma^2 c_{e+1} + \beta \sigma^2 c_e,
\]

we can see that \( \text{Ext}^1_{\Gamma_s}(A_s) \) is isomorphic to \( F_p^\wedge \), defined in Section 1.2, by identifying the class in \( \text{Ext}^1 \) represented by \( \sigma^2 c_e \) with \( (\beta \sigma^2 f/p^e) \) in \( F_p^\wedge \) up to a \( p \)-adic unit.

Next, let us describe the extension problems in \( \text{Ext}^2_{\Gamma_s}(A_s) \). From Theorem 4.2.1, we see that the class \( p^j a_m \lambda_2 \in \text{Ext}^1_{\Gamma_s}(A_s/\beta) \) is hit by the Bockstein differential

\[
d^0_{\text{val}_p(m)-j+1-2} \left( \text{Ext}^1_{\Gamma_s}(A_s/\beta) \right) = \text{Ext}^2_{\Gamma_s}(A_s/\beta).
\]

From the proof of that theorem, we see that the class \( p^j a_m \lambda_2 \) can be lifted to a \( \beta \text{Ext}^1_{\Gamma_s}(A_s/\beta) \)-torsion class \( h_{m,j} \in \text{Ext}^1_{\Gamma_s}(A_s) \) represented by the cycle

\[
D^1 \left( g_{m+p^{\text{val}_p(m)-j}, \text{val}_p(m)-j} \right).
\]

Let \( d = \text{val}_p(m) \) and \( j \) be an integer such that \( 0 \leq j < d \). Then, let \( \varepsilon_1, \ldots, \varepsilon_{d-j-1} \) be the sequence defined in (4.3) for the pair \( (m + p^{d-j}, d - j) \). If in addition \( j < d - 1 \), then let \( \varepsilon'_1, \ldots, \varepsilon'_{d-j-1} \) be the sequence for the pair \( (m + p^{d-j-1}, d - j - 1) \).

Case 1: If \( j < d - 1 \), we have

\[
p f_{m+p^{d-j},d-j} - \frac{\varepsilon_d}{\varepsilon_{d-j-1}} p^{d-j}(p-1) f_{m+p^{d-j-1},d-j-1} = \frac{1}{p^{\text{val}_p(m+p^{d-j})-d-j}} D^0(y_{(m+p^{d-j})/p})
\]

\[
+ \frac{1}{p} \beta^{d-j}(p-1) \left( \varepsilon_1 D^0(y^{m/p} \sigma^2 v_{d-j}) - \frac{\varepsilon_d}{\varepsilon_{d-j-1}} D^0(y^{(m+p^{d-j-1})/p}) \right)
\]

\[
+ \sum_{k=1}^{d-j-1} \frac{1}{p^{k+1}} \beta^{d-j+1-p^d-j-k} \left( \varepsilon_{k+1} - \frac{\varepsilon_d}{\varepsilon_{d-j-1}} \varepsilon'_k \right) D^0(y^{m/p} \sigma^2 x_{p^{d-j-k}-1}).
\]

We shall show that this is integral, i.e. an element of \( A_s \otimes \Gamma_s \), possibly except for the first term. For the second term, we have

\[
D(y^{(m+p^{d-j-1})/p}) \equiv \delta_{d-j-2} D(y^{m/p} \sigma^2 v_{d-j}) \pmod{p},
\]

so that we need to show

\[
\varepsilon_1 - \frac{\varepsilon_d}{\varepsilon_{d-j-1}} \delta_{d-j-2} \equiv 0 \pmod{p}.
\]

This is true since we have

\[
\varepsilon_1 - \frac{\varepsilon_d}{\varepsilon_{d-j-1}} \delta_{d-j-2} \equiv \varepsilon_1 + \frac{\varepsilon_1}{\varepsilon_{d-j-2}} \delta_{d-j-2} \pmod{p}.
\]
by Lemma 4.2.3 and
\[ \varepsilon_1' + \delta_{d-j-2} = \delta_{d-j-2} \left( - \frac{m + p^{d-j-1}}{p^{d-j-1}} + 1 \right) \equiv 0 \pmod{p} \]
by Lemma 4.1.3 Next, for the summation part, we wish to show that
\[ \frac{\varepsilon_d}{\varepsilon_{d-j-1}} = \frac{\varepsilon_{d-j}}{\varepsilon_d} \pmod{p^{k+1}} \]
and this follows from Lemma 4.2.3 Therefore, we have proved that
\[
pf_{m+p^{d-j},d-j} - \frac{\varepsilon_{d-j}}{\varepsilon_d} \beta p^{d-j}(p-1) f_{m+p^{d-j-1},d-j-1} \]
\[ = \frac{1}{p^{\text{val}_p(m+p^{d-j})-d-j}} D^0(y^{(m+p^{d-j})/p}) + \text{(integral element)}. \] (4.4)
We divide into 3 cases.
Case 1-1: If \( \text{val}_p(m + p^{d-j}) = d - j \), then the whole right-hand side of (4.4) is integral. In this case, we see that
\[ p g_{m+p^{d-j},d-j} - \frac{\varepsilon_{d-j}}{\varepsilon_d} g_{m+p^{d-j-1},d-j-1} \]
is integral so that by taking \( D^1 \), we have
\[ ph_{m,j} = h_{m,j+1} \]
in \( \text{Ext}^2_{\Gamma}(A_*) \) up to a \( p \)-adic unit.
Case 1-2: If \( m = p^d(p-1) \) and \( j = 0 \), then from the proof of Theorem 4.2.1(a),
\[ \frac{D^0(y^{d^2})}{p} = \delta_d \beta p^{d+2} - p^{d+1} \frac{\sigma^2 t_{e+1}}{p} + \text{(integral element)}. \]
Therefore,
\[ pg_{p^{d+1},d} - \frac{\varepsilon_d}{\varepsilon_{d-1}} g_{m+p^{d-1},d-1} - \delta_d \beta p^{d+2} - p^{d+1} + 2 \frac{\sigma^2 t_{e+1}}{p} \]
is integral, and by taking \( D^1 \), we obtain
\[ ph_{p^d(p-1),0} = h_{p^d(p-1),1}. \]
Case 1-3: The remaining case is when \( \text{val}_p(m + p^{d-j}) > d - j \) but not \((m, j) \neq (p^d(p-1), 0)\).
This can happen only if \( j = 0 \). In this case, we have
\[ f_{m+p^d,d+1} = \frac{1}{p^{\text{val}_p(m+p^d)-d}} D^0(y^{(m+p^d)/p}) + \beta p^{d+2} \frac{t_{e+1}}{p} \]
where \((\text{cycle})\) means some cycle in \( A_* \otimes \Gamma_*[p^{-1}] \). Comparing with (4.4), we see that
\[ pg_{m+p^d,d} - \frac{\varepsilon_d}{\varepsilon_{d-1}} g_{m+p^{d-1},d-1} - \beta p^{d+2} - p^{d+1} g_{m+p^d,d+1} + \beta p^{d+2} \frac{t_{e+1}}{p} \]
is integral. Taking \( D^1 \), we have
\[ ph_{m,0} = h_{m,1} + \beta p^{d+2} - p^{d+1} h_{m-p^d(p-1),d'-d-1} \]
where \( d' = \text{val}_p(m - p^d(p-1)) \). This completes the proof when \( j < d - 1 \).
Case 2: If \( j = d - 1 \), then we have
\[
pf_{m+1} - c \beta p^{(p-1)} f_{m+1,0} = \frac{1}{p^{\text{val}_p(m+p)-1}} D^0(y^{(m+p)/p}) + \beta p^{(p-1)} \left( \varepsilon_1 D^0(y^{m/p} \sigma^2 x_{p-1}) - c D^0((\sigma^2 x_{p-1})^{m+1}) \right) \]
and we can choose a $p$-adic unit $c$ so that
\[
\frac{1}{p^{p(p-1)}} \left( \varepsilon_1 D^0 \left( y^{m/p} \sigma^2 x_{p-1} \right) - c D^0 \left( (\sigma^2 x_{p-1})^{m+1} \right) \right)
\]
is $\beta^{2p^2-3p}$ times a cycle by the definition of $y$. The rest of the argument is similar to the previous cases.

Case 3: For $j = d$, we have
\[
pf_{m+1,0} = \frac{1}{p^{\val_p(m+1)}} D^0 \left( (\sigma^2 x_{p-1})^{m+1} \right)
\]
and the rest of the argument is similar to the previous cases.

This completes the proof of the description of $\Ext_{\Gamma_*} \left( A_* \right)$. The rest of the statement immediately follows. \hfill \Box

5. INTEGRAL HOMOTOPY VIA FRACTURE SQUARE

In this section, we shall compute the homotopy groups of $\text{THH}(ku)$, thereby proving 1.2.1, by assembling the $p$-complete homotopy groups of $\text{THH}(ku)$. Recall that $\text{THH}(ku)$ splits into the direct sum
\[
\text{THH}(ku) = ku \oplus \text{THH}(ku).
\]
Let $X = \text{THH}(ku)$. Then, we can compute the homotopy groups of $X$ using the arithmetic fracture square

\[
\begin{array}{ccc}
X & \longrightarrow & \prod_p X_p^\wedge \\
\downarrow & & \downarrow \\
X \otimes \mathbb{Q} & \longrightarrow & \left( \prod_p X_p^\wedge \right) \otimes \mathbb{Q}.
\end{array}
\]

which is a pullback. Equivalently, there is a cofiber sequence
\[
X \to (X \otimes \mathbb{Q}) \oplus \left( \prod_p X_p^\wedge \right) \to \left( \prod_p X_p^\wedge \right) \otimes \mathbb{Q}.
\]

**Lemma 5.0.1.** The rational homotopy groups of $\text{THH}(ku)$ are
\[
\pi_*(\text{THH}(ku) \otimes \mathbb{Q}) = \mathbb{Q}[\beta] \oplus \Sigma^3 \mathbb{Q}[\beta]
\]
as $\mathbb{Q}[\beta]$-modules. The second summand is generated by $\sigma b_1$ where $b_1$ denotes the image of $b_1 \in \text{MU}_*\text{MU}$ in $ku_*ku$.

**Proof.** The $E_2$-page of the Bökstedt spectral sequence
\[
E_2 = \text{HH}_* (\pi_* (ku \otimes \mathbb{Q}) / \mathbb{Q}) \Rightarrow \pi_* (\text{THH}(ku) \otimes \mathbb{Q})
\]
is
\[
E_2 = \Lambda_{\mathbb{Q}[\beta]} (\sigma(\beta_1 - \beta_2))
\]
where $\beta_1, \beta_2$ are the two copies of $\beta$ in $\mathbb{Q}[\beta] \otimes \mathbb{Q}[\beta]$. The spectral sequence degenerates since there is no room for any differential, and the conclusion follows from the observation that $\sigma(\beta_1 - \beta_2)$ detects $\sigma b_1$ rationally. \hfill \Box

**Proof of Theorem 1.2.1.** We shall prove that the map
\[
(X \otimes \mathbb{Q}) \oplus \left( \prod_p X_p^\wedge \right) \to \left( \prod_p X_p^\wedge \right) \otimes \mathbb{Q}
\]
in the arithmetic fracture square is a surjection on homotopy groups and compute the kernel.
At even degrees, (5.1) is
\[ 0 \oplus \bigoplus_p T(p) \to 0 \]
so that the even homotopy group of \( X \) is \( \bigoplus_p T(p) \).

Next, we note that in the expression
\[ \text{THH}_*(\text{ku})_p^\beta = \mathbb{Z}_p[\beta] \oplus F_p^\beta \oplus T(p) \]
of Theorem 1.2.2, the generator of \( \text{THH}_3(\text{ku})_p^\beta \), i.e. the lowest degree generator of \( F_p^\beta \), is \( \sigma b_1 \). This follows from the proof of Theorem 1.2.2 in Section 1.3 combined with Remark 3.2.5.

Then, at an odd degree, say \( 2k + 3 \), (5.1) is
\[ \mathbb{Q} \oplus \prod_p \mathbb{Z}_p \to \left( \prod_p \mathbb{Z}_p \right) \otimes \mathbb{Q} \]
where \( \mathbb{Q} \) on the left-hand side is generated by \( \beta^k \sigma b_1 \) and the \( \mathbb{Z}_p \)'s on both sides are generated by \( (\beta^k/f(k))\sigma b_1 \). This is the fracture square for the ordinary ring \( \mathbb{Q} \). Therefore, it is surjective and the kernel is \( \mathbb{Z} \) generated by \( (\beta^k/f(k))\sigma b_1 \).

\[ \square \]

6. FURTHER QUESTIONS

It would be interesting if the descent spectral sequence for \( \text{THH}(R) \to \text{THH}(R/\text{MU}) \) degenerates for more general \( R \). When \( R = \text{BP}(n) \) is a \( \mathbb{E}_3 \)-MU-algebra, [HW] Prop. 6.1.6 implies that \( E_2^{s,t}(\text{THH}(R)) \) is concentrated in \( 0 \leq s \leq n + 1 \), so that the degeneracy is not immediate as in Corollary 4.1.2. However, since the descent spectral sequence for \( R = \text{MU} \) degenerates at the \( E_2 \)-page, the following conjecture would imply the degeneracy for \( R = \text{BP}(n) \). Note that the assumption that \( R \) is a \( \mathbb{E}_3 \)-MU-algebra is needed to ensure that \( \text{THH}_*(R/\text{MU}) \) is a commutative ring.

**Conjecture 6.1.** Suppose that we have \( \text{BP}(n) \) with an \( \mathbb{E}_3 \)-MU-algebra structure, which exists by [HW]. Let \( E_2(\text{THH}(\text{MU})) \) and \( E_2(\text{THH}(\text{BP}(n))) \) be the \( E_2 \)-page of the descent spectral sequence for \( \text{THH}(\text{MU}) \to \text{THH}(\text{MU}/\text{MU}) \) and \( \text{THH}(\text{BP}(n)) \to \text{THH}(\text{BP}(n)/\text{MU}) \). Then, \( E_2^{s,t}(\text{THH}(\text{MU})) \to E_2^{s,t}(\text{THH}(\text{BP}(n))) \) is surjective for \( 0 \leq s \leq n \).

Theorem 4.2.1(a) shows that the conjecture is true for \( \text{ku} \) instead of \( \text{BP}(1) \), i.e. the map \( E_2^{s,t}(\text{THH}(\text{MU})) \to E_2^{s,t}(\text{THH}(\text{ku})) \) is surjective for \( 0 \leq s \leq 1 \). Similar computations can be done to show that the conjecture is true for \( 0 \leq s \leq 1 \) and any \( n \).

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