Estimation of pulsed driven qubit parameters via quantum Fisher information

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Abstract

We estimate the initial weight and phase parameters ($\theta, \phi$) of a single qubit system initially prepared in the coherent state $|\theta, \phi\rangle$ and interacts with three different shape of pulses; rectangular, exponential, and $\sin^2$-pulses. In general, we show that the estimation degree of the weight parameter depends on the pulse shape and the initial phase angle, ($\phi$). For the rectangular pulse case, increasing the estimating rate of the weight parameter via the Fisher information function ($F_\theta$) is possible with small values of the atomic detuning parameter and large values of the pulse strength. Fisher information ($F_{\phi}$) increases suddenly at resonant case to reach its maximum value if the initial phase $\phi = \pi/2$ and consequently one may estimate the phase parameter with high degree of precision. If the initial system is coded with classical information, the upper bounds of Fisher information for resonant and non-resonant cases are much larger and consequently one may estimate the phase parameter with high degree of estimation. Similarly as the detuning increases the Fisher information decreases and therefore the possibility of estimating the phase parameter decreases. For exponential, and $\sin^2$-pulses the Fisher information is maximum ($F_{\theta,\phi} = 1$) and consequently one can always estimate the weight and the phase parameters ($\theta, \phi$) with high degree of precision.

1 Introduction

The essence of conditional probability is that, learning about one event (or measurement) changes or affects the probability for a second event. Classically, this is well known termed as Fisher information (FI) function [1]. In the other words, the FI extracts information about an unknown parameter, $\beta$, from a previously measured result result, say, $x$.

For quantum systems, the corresponding quantum Fisher information (QFI) is nowadays an important physical quantity of estimation within the context of quantum metrology and quantum information theory [2]. It describes the sensitivity of a quantum state with respect to changes in its initial parameters [3] or gained system parameters during
the quantum information processing as teleportation [5]. Evaluation of QFI with respect to the desired estimation parameters have been given in [6, 7]. Dynamics of QFI were studied in many models, such as, the Ising model [8], mixed Hamiltonian model [9], Bell state pairs model under decoherence channel [10] and steady state open and noisy systems [11]. QFI in non-inertial frames has been also quantified for different systems. For example, Yao et. al [12] have investigated the dynamics of Fisher information of a pure-two qubit state in non-inertial frame. Metwally [5] has investigated the dynamics of the teleported QFI by using accelerated quantum channel. The Unruh acceleration effect on the precision of parameter estimation for a general two qubit system is discussed in [13].

Investigating QFI of a driven single qubit by a laser pulse is important within the context of quantum information theory. Previously, we have investigated the transfer and exchange information between a single qubit system and the driving rectangular pulse [13]. Also, the possibility of achieving long-lived entanglement between two entangled pulsed driven qubits is given in [15].

Here, we extend our investigation to the behavior of QFI of a single qubit driven by different shapes of pulses. Specifically, we investigate the problem of parameter estimation of the quantum channel parameter for a single qubit, initially prepared in a coherent state and driven by three different shapes of pulses, namely, rectangular, exponential and \( \sin^2 \) pulses.

The paper is organized as follows. In Sec.2, we present the model and its exact solution in terms of its Bloch vectors, together with the definition of the QFI. Computational results of the effect of the pulse shapes on the QFI are represented in Sec.3, followed by a summary in Sec.4.

## 2 The suggested Model

### 2.1 The Hamiltonian Model

Her, we consider a single qubit taken as 2-level atomic transition of frequency \( \omega_q \) and driven by a laser pulse of arbitrary shape and of circular frequency \( \omega_c \) in the absence of any dissipation process. The quantized Hamiltonian of the system (in units of \( \hbar = 1 \)) in the dipole and rotating wave approximation is given by [16, 17]

\[
\hat{H} = \omega_q \hat{S}_z + \frac{\Omega(t)}{2} (\hat{S}_+ e^{-i\omega_c t} + \hat{S}_- e^{i\omega_c t})
\]  

(1)
where, the spin-\(\frac{1}{2}\) operators \(\hat{S}_\pm, z\) obey the \(SU(2)\) algebra,

\[
[\hat{S}_+, \hat{S}_-] = 2\hat{S}_z, \quad [\hat{S}_z, \hat{S}_\pm] = \pm \hat{S}_\pm
\]  

(2)

and \(\Omega(t) = \Omega_0 f(t)\) is the real laser Rabi frequency with \(f(t)\) is the pulse shape. Introducing the rotating frame operators,

\[
\hat{\sigma}_\pm(t) = \hat{S}_\pm(t) e^{\pm \omega_c t}, \quad \hat{\sigma}_z(t) = \hat{S}_z(t)
\]

(3)

where the \(\hat{\sigma}\) operators obey the same algebraic form of Eq.(2). Heisenberg equations of motion for the atomic operators \(\hat{\sigma}_\pm, z\) according to (1) are of the form,

\[
\dot{\hat{\sigma}}_+ = i\Delta \hat{\sigma}_+ - i\Omega(t) \hat{\sigma}_z = \left[\hat{\sigma}_-, \hat{\sigma}_+\right],
\]

\[
\dot{\hat{\sigma}}_z = -i\Omega(t) \left(\hat{\sigma}_+ - \hat{\sigma}_-\right)
\]

(4)

where \(\Delta = \omega_q - \omega_c\), is the atomic detuning. Eqs(4) are of variable coefficient and have exact solutions in the following two cases \cite{16,17}:

1. Arbitrary atomic detuning \((\Delta)\) and constant Rabi frequency \(\Omega(t) = \Omega_0\) with \(f(t) = 1\). This case corresponds to the rectangular pulse shape.

2. Exact atomic resonance \((\Delta = 0)\) and arbitrary Rabi frequency \(\Omega(t)\). This case corresponds to a pulse of arbitrary shape, \(f(t)\).

Initially, we assume the qubit is prepared in the coherent state,

\[
|\psi_q\rangle = \cos(\theta/2)|0\rangle + e^{-i\phi}\sin(\theta/2)|1\rangle,
\]

(5)

where \(0 \leq \phi \leq 2\pi\), \(0 \leq \theta \leq \pi\) and \(|0\rangle, |1\rangle\) are the lower and upper states, respectively. We use the notations,

\[
s_i(0) = \langle \theta, \phi | \hat{\sigma}_i | \theta, \phi \rangle
\]

(6)

for the initial Bloch vector components, where \((i = x, y, z)\) and \(\hat{\sigma}_x = \frac{1}{2}(\hat{\sigma}_+ + \hat{\sigma}_-), \quad \hat{\sigma}_y = \frac{1}{2i}(\hat{\sigma}_+ - \hat{\sigma}_-)\). Explicitly from (5) and (6), we have,

\[
s_x(0) = \sin \theta \cos \phi, \quad s_y(0) = \sin \theta \sin \phi, \quad s_z(0) = -\cos \theta
\]

(7)
2.2 Exact Solutions

In the light of the comments (i),(ii) after Heisenberg eq (4), we now present their exact solutions in the following three cases of the pulse shape:

1. Rectangular pulse:
   For a rectangular laser pulse of duration $T$, the Rabi frequency $\Omega(t) = \Omega_0 f(t)$, where $\Omega_0$ real and $f(t)$ is defined as,

   $$ f(t) = \begin{cases} 1 & \text{for } t \in [0, T] \\ 0 & \text{otherwise} \end{cases} $$

   where the pulse duration $T$ is much shorter than the lifetime of the qubit upper state, hence the damping can be discarded. The exact solutions of (4) in terms of the Bloch vector $s_{x,y,z}(t)$ are given in the form [14, 16],

   $$ s_x(t) = \left[ \left( \frac{1}{\eta} + \delta^2 \cos(\tau \sqrt{\eta}) \right) - \frac{\delta \sin(\tau \sqrt{\eta})}{\sqrt{\eta}} \right] s_x(0) $$

   $$ + \left[ \frac{1 + 2\delta^2}{2\eta} + \frac{\cos(\tau \sqrt{\eta})}{2\eta} + \frac{\delta \sin(\tau \sqrt{\eta})}{\sqrt{\eta}} \right] s_y(0) $$

   $$ + \left[ \frac{\delta}{\eta} \left( 1 - \cos(\tau \sqrt{\eta}) \right) + \frac{1}{\sqrt{\eta}} \sin(\tau \sqrt{\eta}) \right] s_z(0), $$

   $$ s_y(t) = \left[ \left( \frac{1}{\eta} + \frac{\delta^2 \cos(\tau \sqrt{\eta})}{2\eta} + \frac{\delta \sin(\tau \sqrt{\eta})}{\sqrt{\eta}} \right) s_x(0) $$

   $$ + \left[ \frac{\cos(\tau \sqrt{\eta})}{\eta} - \frac{\delta}{\sqrt{\eta}} \sin(\tau \sqrt{\eta}) \right] s_y(0) $$

   $$ + \left[ \frac{\delta}{\eta} \left( 1 - \cos(\tau \sqrt{\eta}) \right) - \frac{\delta}{\sqrt{\eta}} \sin(\tau \sqrt{\eta}) \right] s_z(0), $$

   $$ s_z(t) = \frac{\delta}{\eta} \left( 1 - \cos(\tau \sqrt{\eta}) \right) s_x(0) + \frac{1}{\sqrt{\eta}} \sin(\tau \sqrt{\eta}) s_y(0) $$

   $$ + \left( \frac{\delta^2}{\eta} + \frac{1}{\eta} \cos(\tau \sqrt{\eta}) \right) s_z(0) $$

   where $\delta = \frac{\Delta}{\Omega_0}$, $\eta = 1 + \delta^2$, $\tau = \Omega_0 t$ is the scaled time and $s_{x,y,z}(0)$ are given in (7).

2. Exponential pulse:
   In this case $\Omega(t) = \Omega_0 f(t)$, where $\Omega_0$ is the Rabi frequency associated with the laser pulse and the pulse shape $f(t)$ is defined as,

   $$ f(t) = \begin{cases} e^{-\gamma t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} $$

   (10)
where $\gamma_p^{-1}$ is the (short) time scale of the pulse. At exact resonance ($\Delta = 0$), the exact solution for the Bloch vector components are given by [18, 19],

$$
\begin{align*}
    s_x(t) &= s_x(0), \\
    s_y(t) &= \cos \omega_c(t)s_y(0) - \sin \omega_c(t)s_z(0) \\
    s_z(t) &= \cos \omega_c(t)s_z(0) + \sin \omega_c(t)s_y(0)
\end{align*}
$$

(11)

where

$$
\omega_c(t) = \frac{\Omega_0}{\gamma_p}(1 - e^{-\gamma_p t})
$$

(12)

3. $\sin^2$-pulse:

In this case $\Omega(t) = \Omega_0 f(t)$ and the laser pulse shape $f(t)$ is given by

$$
f(t) = \sin^2(n\omega_q(t)); n = 1, 2, ...
$$

(13)

with $(n\omega_q)$ is the beating frequency of the pulse. The pulse shape (13) represents $n$-sequential pulses, each of duration $\pi$ over the interval $[0, n\pi]$. At exact resonance ($\Delta = 0$), the Bloch vector components have the same form as in (11), but with the time-dependent frequency $\omega_c(t)$ is replaced by $\omega_s(t)$ [20, 21], where

$$
\omega_s(t) = \frac{\Omega'}{2}\left(\tau - \frac{1}{2n}\sin(2n\tau)\right), \quad n = 1, 2, 3, ...
$$

(14)

where $\tau = \omega_q t$ is the normalized time and $\Omega'_0 = \frac{\Omega_0}{\omega_q}$ is the normalized pulse strength.

### 2.3 Quantum Fisher Information

The density operator for 2-level atomic system is given by,

$$
\rho_q = \frac{1}{2}(I + \vec{s} \cdot \hat{\sigma})
$$

(15)

where, $\vec{s} = (s_x(0), s_y(0), s_z(0))$ is the Bloch vector and $\hat{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ are the spin Pauli operators. In terms of Bloch vector $\vec{s}(\beta)$, the QFI with respect to the parameter $\beta$ is defined as [6, 22],

$$
\mathcal{F}_\beta = \begin{cases} 
\frac{1}{1-|\vec{s}(\beta)|^2} \left[ \vec{s}(\beta) \cdot \frac{\partial \vec{s}(\beta)}{\partial \beta} \right] + \left( \frac{\partial \vec{s}(\beta)}{\partial \beta} \right)^2 & \text{for mixed state, } |\vec{s}(\beta)| < 1, \\
|\frac{\partial \vec{s}(\beta)}{\partial \beta}|^2 & \text{for pure state, } |\vec{s}(\beta)| = 1
\end{cases}
$$

(16)
where $\beta$ is the parameter to be estimated. From Eq.(7), it is clear that the final solution depends on the initial parameters $(\theta, \phi)$ in addition to the system parameters $\delta, \Omega'_0$. In the following subsections, we shall estimate these parameters by calculating their corresponding QFI, $F_\beta$. The larger QFI is the higher degree of estimation for the parameter $\beta$.

3 Computational results of QFI

3.1 Rectangular Pulse

1. Estimating the weight parameter $(\theta)$

Fig.(1) shows the behavior of $F_\theta$ against the detuning parameter where we assume that the single qubit system is initially prepared in the state $|\psi_q\rangle = \cos(\theta/2)|0\rangle - \sin(\theta/2)|1\rangle$.

Figure 1: Fisher information ($F_\theta$) in the rectangular pulse case with respect to the parameter $\theta$ and the detuning parameter $\delta$ for fixed $\phi = \pi$, and $\Omega'_0 = 0.3, 0.9$ for the figures (a,b), (c,d), respectively.
\[ i \sin(\theta/2)|1\rangle, \] namely, we set the phase angle \( \phi = \pi \). The general behavior shows that \( F_\theta \) decays as the detuning parameter increases. Moreover, the decay is less by increasing the pulse strength \( \Omega_0' \). Fig1.(a,b) describe the behavior of the Fisher information \( F_\theta \) at \( \Omega_0' = 0.3 \). It is shown that at \( \delta = 0 \) and \( \theta = 0, \) i.e, \( |\psi_q\rangle = |0\rangle \), \( F_\theta \) is maximum. However as one increases \( \theta \) at zero detuning (\( \delta = 0 \)) the Fisher information \( F_\theta \) decays gradually to reach its minimum value at \( \theta = \pi/2 \). In this case the initial single qubit system reduces to be \( |\psi_q\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) \). For further values of \( \theta \in [\pi/2, \pi] \), \( F_\theta \) increases gradually to reach its maximum value at \( \theta = \pi \). This maximum value is reached for a qubit system initially prepared in \( |\psi_q\rangle = -i|1\rangle \). However, the Fisher information \( F_\theta \) decays gradually as \( \delta \) increases and starts to disappear for \( \delta > 0.6 \).

The effect of the larger value of the pulse strength (\( \Omega_0' = 0.9 \)) is displayed in Figs.(1c&1d). It is clear that by increasing \( \Omega_0' \), the Fisher information increases even within larger values of the detuning. Also, \( F_\theta \) decays gradually for \( \theta \in [0, \pi/2] \)
but its upper bounds are larger than those predicted at $\Omega'_0 = 0.3$. The Fisher information increases as the initial weight parameter increases, namely for $\theta \in [\pi/2, \pi]$. Figs.(1c&1d) display explicitly the upper and lower bounds of the Fisher information. The most bright regions indicate that one can estimate the weight parameter $\theta$ with high precision.

From Fig.(1) one may conclude the following: one may estimate the weight parameter ($\theta$) with high degree of precision at small values of the detuning parameter. This precision decreases as one increases the value of the detuning parameter. However, one may improve the degree of estimation by increasing the pulse strength. If the initial qubit system encodes only classical information (i.e. $\theta = 0$ or $\pi$), the possibility of estimating the weight parameter ($\theta$) is larger than that predicted if the initial qubit encodes quantum information (i.e. $0 < \theta < \pi$).

Fig.(2), shows the effect of different values of the phase parameter $\phi = \pi/4, \pi/2$ on the precision of estimating the weight parameter ($\theta$). The general behavior is similar to that displayed in Fig.(1c,d), but with smaller values. It is seen that the upper bounds of $F_\theta$ at $\phi = \pi/4$ are larger than that for $\phi = \pi/2$.

From Figs.(1&2), one observes that the possibility of increasing the estimating rate of the weight parameter ($\theta$) depends on the structure of the initial state, small values of the detuning parameter and larger values of the pulse strength.

2. Estimating the phase parameter ($\phi$):

Fig.(3) displays the behavior of Fisher information ($F_\phi$) with respect to the phase parameter $\phi$ against the detuning parameter $\delta$ for fixed values of the pulse strength, $\Omega'_0$. It is assumed that the qubit system is initially prepared in the state $\psi(0) = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle)$, namely, the weight parameter $\theta = \pi/4$. The behavior of $F_\phi$ shows that for small values of the detuning parameter, Fisher information increases as the phase parameter increases to reach its maximum values at $\phi = \pi/2$ and decreases gradually to vanish completely at $\phi = \pi$. This behavior is repeated in the interval of $\phi \in [\pi, 2\pi]$, where the maximum value is reached at $\phi = 3\pi/2$ as shown in Fig.(3a). However, as $\delta$ increases Fisher information decreases gradually and completely vanishes at $\delta > 0.6$. From Fig.(3b) it is clear that the upper bounds of $F_\phi$ are shifted as $\delta$ increases.

In Fig.3(c,d), we increase the pulse strength, where we set $\Omega'_0 = 0.6$. The general behavior of the Fisher information is similar to that depicted in Fig.3(a,b), but the
upper bounds of \( \langle F_\phi \rangle \) is larger. On the other hand, larger pulse strength protects the vanishing of the Fisher information with larger values of the detuning parameter.

In Fig.(4) we investigate the behavior of the Fisher information \( \langle F_\phi \rangle \) with respect to the phase parameter for a system is initially prepared in the state \( |\psi_0\rangle = ie^{i\phi}|1\rangle \),

Figure 3: Fisher information \( \langle F_\phi \rangle \) in the rectangular pulse case with respect to the parameter \( \phi \) and the detuning parameter \( \delta \) for fixed \( \theta = \pi/4 \), and \( \Omega'_\phi = 0.3, 0.6, 0.9 \) for the figures (a,b), (c,d), (e,f), respectively.
namely, $\theta = \pi/2$. This behavior shows that as soon as the system is pulsed at zero detuning, the Fisher information increases suddenly to reach its maximum values ($F_\phi = 1$). As $\phi$ increases the upper values of $F_\phi$ slightly decrease. However as $\phi \rightarrow \pi$, the Fisher information decreases suddenly to reach its minimum value at $\phi = \pi$. Similar to the behavior in Fig.(4), $F_\phi$ decreases as the detuning $\delta$ increases, but the upper bounds are larger than that displayed in Fig.3.

From Figs.(3,4), one may conclude that the possibility of estimating the phase pa-
rameter \( \phi \) decreases as detuning parameter increases. This possibility can be improved by increasing the pulse strength. By increasing the weight parameter \( \theta \) the possibility of estimating the phase parameter \( \phi \) increases.

### 3.2 Exponential and \( \sin^2 \)-pulses

With the initial qubit system is prepared in the state (15), where the initial Bloch vector in the coherent state \(|\theta, \phi\rangle\), i.e. \( \vec{s}(0) = (\cos \phi \sin \theta, \sin \phi \sin \theta, -\cos \theta) \), and hence \(|\vec{s}(0)| = 1\). Using this initial value for \( \vec{s}(0) \) in Eq.(11), one obtains the Bloch vector of the final state in a pure state, i.e. \(|\vec{s}(t)| = 1\). One can show that \( F_{\theta,\phi} = 1 \) and hence it is independent of the initial parameters \( \theta, \phi \). Therefore, in the case of the Exponential and \( \sin^2 \)-pulses, one can estimate these initial parameters with high degree precision.

### 4 conclusion

The dynamics of a single qubit initially prepared in a coherent state \(|\theta, \phi\rangle\) interacts with three different pulse shapes, namely, rectangular, exponential and \( \sin^2 \)-pulses is discussed, utilizing the analytical solutions in terms of the corresponding final Bloch vectors.

We investigate the possibility of estimating the weight parameter \( \theta \) and the phase parameter \( \phi \) of the initial single qubit coherent state \(|\theta, \phi\rangle\) under the effect of these pulses. It is shown that the strength of the pulses have different effect, while the initial phase angle \( \phi \) has a similar effect in the three different cases as follows.

- For the rectangular pulse, the estimation degree of the weight parameter \( \theta \) decays as the atomic deuning parameter increases. This decay can be recovered by increasing the value of the pulse strength. The upper bounds of estimation degree decrease if the initial qubit prepared with a suitable phase angle \( \phi \), encodes quantum information.

- The amount of Fisher information with respect to the phase parameter \( \phi \) is quantified for a system which is initially coded with quantum and classical informations. It is shown that, Fisher information \( F_\phi \) increases suddenly at resonant case to reach its maximum value if the initial phase \( \phi = \pi/2 \) and consequently one may estimate the phase parameter with high degree of precision. However, as the detuning increases the Fisher information decreases and therefore the possibility of estimating
the phase parameter decreases. If the initial system is coded with classical information, the upper bounds of Fisher information for resonant and non-resonant cases are much larger.

- For resonant exponential and $\sin^2$ pulses, the Fisher information is maximum and consequently one can always estimate the weight and the phase parameters $(\theta, \phi)$ with high degree of precision.

In conclusion, the maximization of the estimation degree of the weight parameter $(\theta)$ and the phase parameter $(\phi)$ of the single qubit depends on the structure of the initial state, small values of the detuning parameter and larger values of the pulse strength. The pulse strength is the effective control parameter to maximize the degree of estimation.

References

[1] S. M. Barnett," Quantum Information (Oxford Univ. Press, Oxford 2009).

[2] V. Giovannetti, S. Lloyd, and L. Maccone, " Quantum-Enhanced Measurements: Beating the Standard Quantum Limit", Science **306**, 1330-1336 (2004).

[3] J. Ma, Yi.-h Huang, X. Wang and C. P. Sun, " Quantum Fisher informationof the Greenbereger-Horne-Zeillinger state in decoherence channels", Phys. Rev. A **84** 022302 (2011).

[4] Q. Zheng, Y. Yao and Y. Li," Optimal quantum channel estimation of two interacting qubits subject to decoherence", Eur. Phys. J. D **68** 170 (2014).

[5] N. Metwally,"Estimation of teleported and gained parameters in a non-inertial frame", Laser Phys. Lett. **13** 105206 (2016)

[6] W. Zhong, Z. Sun, J. Ma, X. Wang, and F. Nori," Fisher information under decoherence in Bloch representation", Phys. Rev. A **87**, 022337 (2013).

[7] J. Liu, X. Jing and X. Wang ,"Phase-matching condition for enhancement of phase sensitivity in quantum metrology", Phys. Rev. A **88**, 042316 (2013).

[8] H. Na Xiong and X. Wang"Dynamics of quantum Fisher information in the Ising model", Physica A **390** 4719 (2011).
[9] G.-J. Hu and X.-X. Hu”, Spin squeezing and quantum Fisher information for a mixed Hamiltonian model”, Int. J. Theor. Phys. 53 533 (2014).

[10] F. Ozaydin, A. A. Altintas, S. Bugu and C. Yesilyurt,” Behavior of quantum Fisher information of Bell pairs under decoherence channels”, Acta Phys. Pol A125 606 (2014).

[11] A. A. Altinatas,” Quantum Fisher information of an open and noisy system of the steady state”, Annals of Physics 367 192 (2016).

[12] Y. Yao, X. Xiao, Li Ge, X.G. Wang and C. pu Sun,Quantum Fisher information in noninertial frames”, Phys. Rev. A. 89 042336 (2014).

[13] N. Metwally, ”Unruh acceleration effect on the precision of parameter estimation”, arXiv:1609.02092 (2016)

[14] N. Metwally and S. S. Hassan” Information Transfer and orthogonality speed via-pulsed driven qubit”, Nonlinear Optics and Quantum Optics,44 267 (2012).

[15] N. Metwally, H. A. Batarfi and S. S. Hassan,” Long-lived entanglement with pulsed-driven initially entangled qubit pair”, Int. J. Quantum Infor. 12 1450003 (2014).

[16] S. S. Hassan, A. Joshi and N. M. M. Al-Madhari,”Spectrum of a pulsed driven qubit”, J. Phys. B 41 145503 (2008); and corrigendum: J. Phys. B 42 089801 (2009).

[17] S. S. Hassan, A. Joshi and A. Batarfi” Spectrum of triagnular pulsed-driven atom”, Int. J. Therotical Phys., Group Theory and Nonlinear Opt. 13 371 (2010).

[18] H. A. Batarfi,” Specturm of spin-$\frac{1}{2}$ system driven by resonant exponential pulse” J. Nonlinear Opt. Phys. & Materials 21 120025 (2012).

[19] H. A. Batarfi and S. S. Hassan” Haar wavelet spectrum of an exponentially pulsed driven qubit”, Nonlinear Optics and Quantum Optics (2017), to appear.

[20] R. A. Alharabey,” Transient spectrum of sin$^2$ – pulsed driven qubit”, Int. J. Pure &App. Maths 110 193 (2015).

[21] R. A. Alharabey,” Haar wavelet spectrum of sin$^2$ – pulsed driven qubit”, Optik 127 9878 (2016).
[22] X. Xiao, Y. Yao, W.-J. Zhong, Y.-Ling and Y.-Mao Xie,” Enhancing teleportation of quantum Fisher information by measurements”, Phys. Rev. A 93012307 (2016).