UNIFORM CONTINUITY AND $\varphi$-UNIFORM DOMAINS

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Abstract. We discuss the modulus of continuity of the identity mapping $id : (G, d_1) \rightarrow (G, d_2)$, where $(G, d_i)$, $i = 1, 2$, are metric spaces in $\mathbb{R}^n$. In particular, we find a sharp bound for the modulus of continuity when $d_2$ is the Euclidean and $d_1$ the quasihyperbolic metric. We also characterize domains $G$ in which the identity mapping is uniformly continuous when $d_2$ is the quasihyperbolic metric and $d_1$ is the distance ratio metric. In addition, we discuss some properties of $\varphi$-uniform domains.

1. Introduction

Let $(X_j, d_j)$, $j = 1, 2$, be metric spaces. A function $f : X_1 \rightarrow X_2$ is said to be uniformly continuous if there exists a function, Lipschitzian modulus of continuity $\omega : [0, r_1) \rightarrow [0, r_2)$ such that $\omega(0) = 0$ and $\omega(t) \rightarrow 0$, as $t \rightarrow 0$, and for all $x, y \in X_1$ with $d_1(x, y) < r_1$ we have $d_2(f(x), f(y)) < \omega(d_1(x, y))$. In this case we also say that $f : (X_1, d_1) \rightarrow (X_2, d_2)$ is $\omega$-uniformly continuous. To simplify matters we always assume that $\omega : [0, r_1) \rightarrow [0, r_2)$ is an increasing homeomorphism. In this general setup uniform continuity occurs in many areas of mathematical analysis from real analysis, functional analysis, measure theory, topology to geometric function theory.

Here we will mainly be motivated by geometric function theory and therefore give a few related examples. If $X_1 = B^n = X_2$ and $f : B^n \rightarrow B^n$ is $K$-quasiconformal, then the quasiconformal counterpart of the Schwarz lemma says that $f : (B^n, \rho) \rightarrow (B^n, \rho)$ is uniformly continuous where $\rho$ is the hyperbolic metric of $B^n$. If $X_1 = B^2$, $X_2 = \mathbb{R}^2 \setminus \{0, 1\}$, the Schottky theorem gives, in an explicit form, a growth estimate for $|f(z)|$ in terms of $|z|$ when $f : B^2 \rightarrow \mathbb{R}^2 \setminus \{0, 1\}$ is an analytic function [8, p. 685, 702]. In fact, Nevanlinna’s principle of the hyperbolic metric [8, p. 683] yields an estimate for the modulus of continuity of $f : (B^2, \rho) \rightarrow (X_2, d_2)$ where $d_2$ is the hyperbolic metric of the twice punctured plane $X_2$. If $q$ is the chordal metric and $f : (B^2, \rho) \rightarrow (\mathbb{R}^2, q)$ is a meromorphic function, then $f$ is normal (in the sense of Lehto and Virtanen) if and only if it is uniformly continuous. In the context of quasiregular mappings, uniform continuity has been discussed in [19, 21].

We will consider two problems which naturally fit in this framework for some other metric spaces. For a subdomain $G \subset \mathbb{R}^n$ and $x, y \in G$ the distance ratio metric $j_G$ is defined [19] by

$$j_G(x, y) = \log \left( 1 + \frac{|x - y|}{\min\{\delta(x), \delta(y)\}} \right).$$
In a slightly different form of this metric was studied in \[5\]. The \textit{quasihyperbolic metric} of \(G\) is defined by the quasihyperbolic length minimizing property
\[
k_G(x, y) = \inf_{\gamma \in \Gamma(x, y)} \ell_k(\gamma), \quad \ell_k(\gamma) = \int_{\gamma} |dz|/\delta(z),
\]
where \(\ell_k(\gamma)\) is the quasihyperbolic length of \(\gamma\) (cf. \[6\]). For a given pair of points \(x, y \in G\), the infimum is always attained \[5\], i.e., there always exists a quasihyperbolic geodesic \(J_G[x, y]\) which minimizes the above integral, \(k_G(x, y) = \ell_k(J_G[x, y])\) and furthermore with the property that the distance is additive on the geodesic: \(k_G(x, y) = k_G(x, z) + k_G(z, y)\) for all \(z \in J_G[x, y]\).

If the domain \(G\) is emphasized we call \(J_G[x, y]\) a \(k_G\)-geodesic.

The first problem is to find a bound, as sharp as possible, for the modulus of continuity of the identity mapping
\[
id : (G, m_G) \to (G, |\cdot|)
\]
where \(G \subseteq \mathbb{R}^n\) is a domain, \(m_G \in \{k_G, j_G\}\) and \(|\cdot|\) is the Euclidean norm. In the case \(G = B^n\), our first main result gives a sharp bound in Lemma \[2.7\]. This problem is motivated by a recent work of Earle and Harris \[3\] dealing with the case of the hyperbolic metric \(\rho_{B^n}\) of the unit ball. The well-known inequality in Lemma \[2.5\] served in their work as a standpoint for further generalizations to other metrics. Another motivation comes from \[21\] pp. 322–323 where it was pointed out that finding the modulus of continuity is an open territory of research for numerous combinations of metric spaces, classes of domains and categories of maps. This simple question leads to a very large class of open questions because it allows a great number of variations: one may vary the classes of metrics, domains and mappings independently. The present paper will provide some answers to the questions raised in \[21\] pp. 322–323.

Another question we study is to explore the connection between \(k_G\) and the distance ratio metric \(j_G\). We characterize domains \(G\) such that the identity mapping
\[
id : (G, j_G) \to (G, k_G)
\]
is uniformly continuous. These domains are so called \(\varphi\)-uniform domains (see Theorem \[3.12\]) and were first introduced in \[19\]. We also prove several properties of \(\varphi\)-uniform domains, in particular we prove that a domain preserves the \(\varphi\)-uniform property even if we remove a finite number of points from the domain.

The \(\varphi\)-uniform domains generalize the notion of uniform domains introduced by Martio and Sarvas in \[15\] and extensively studied thereafter. It is a natural question whether these results also have a counterpart in the case of \(\varphi\)-uniform domains. In some particular instances this very general question is answered here.

\section{Preliminary results}

We shall now specify some necessary notations, definitions and facts that we frequently use below. The standard unit vectors in the Euclidean \(n\)-space \(\mathbb{R}^n\) \((n \geq 2)\) are represented by \(e_1, e_2, \ldots, e_n\). We write \(x \in \mathbb{R}^n\) as a vector \((x_1, x_2, \ldots, x_n)\). The Euclidean length (Euclidean norm) of \(x \in \mathbb{R}^n\) is denoted by \(|x|\). We denote \([x, y]\) for the Euclidean line segment joining \(x\) and \(y\). The one point compactification of \(\mathbb{R}^n\) (the Möbius \(n\)-space \(\mathbb{R}^n\)) is defined by \(\mathbb{R}^n = \mathbb{R}^n \cup \{\infty\}\). We denote by \(B^n(x, r)\) and \(S^{n-1}(x, r)\), the Euclidean ball and sphere with radius \(r\) centered at \(x\) respectively. We set \(B^n(0, r) := \mathbb{R}^n \cap B^n(0, r)\) and \(S^{n-1}(0, r) := S^{n-1}(0, r)\). Let \(G\) be a domain (open connected non-empty set) in \(\mathbb{R}^n\). The boundary, closure and diameter...
of $G$ are denoted by $\partial G$, $\overline{G}$ and $\text{diam} \, G$ respectively. For $x \in G$ we write $\delta(x) := d(x, \partial G)$, the Euclidean distance from $x$ to $\partial G$. In what follows, all paths $\gamma \subset G$ are required to be rectifiable, i.e. $\ell(\gamma) < \infty$ where $\ell(\gamma)$ stands for the Euclidean length of $\gamma$. Given $x, y \in G$ $\Gamma(x, y)$ stands for the collection of all rectifiable paths $\gamma \subset G$ joining $x$ and $y$.

Some basic properties of the quasihyperbolic and hyperbolic metrics will be frequently used (cf. [20, Sections 2 and 3]). First recall the monotonicity with respect to the domain: if $G_1$ and $G_2$ are domains with $G_2 \subset G_1$ then for all $x, y \in G_2$ we have $k_{G_1}(x, y) \leq k_{G_2}(x, y)$. It is obvious that this property holds for the metric $j_G$ as well.

The hyperbolic metrics $\rho_{B^n}$ of the unit ball and $\rho_{\mathbb{H}^n}$ of the upper half space $\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n > 0\}$ are defined in terms of the hyperbolic length minimizing property, in the same way as the quasihyperbolic metric (see [20, Section 2]). The density functions are $2/(1 - |x|^2)$ and $1/x_n$ for the unit ball and the half space, resp. This leads to the observations that $k_{\mathbb{H}^n} = \rho_{\mathbb{H}^n}$ and

$$k_{B^n}(x, y)/2 \leq k_{B^n}(x, y) \leq \rho_{B^n}(x, y)$$

for all $x, y \in B^n$. For the case of $B^n$ we make use of an explicit formula [20, (2.18)] to the effect that for $x, y \in B^n$

$$(2.1) \quad \sinh \frac{\rho_{B^n}(x, y)}{2} = \frac{|x - y|}{t}, \quad t = \sqrt{(1 - |x|^2)(1 - |y|^2)}.$$ 

The following proposition gathers together several basic properties of the metrics $k_G$ and $j_G$, see for instance [6, 20].

**Proposition 2.2.**

1. For a domain $G \subset \mathbb{R}^n$, $x, y \in G$, we have

$$k_G(x, y) \geq \log \left(1 + \frac{L}{\min\{\delta(x), \delta(y)\}}\right) \geq j_G(x, y),$$

where $L = \inf\{\ell(\gamma) : \gamma \in \Gamma(x, y)\}$.

2. For $x \in B^n$ we have

$$k_{B^n}(0, x) = j_{B^n}(0, x) = \log \frac{1}{1 - |x|}.$$ 

3. Moreover, for $b \in S^{n-1}$ and $0 < r < s < 1$ we have

$$k_{B^n}(br, bs) = j_{B^n}(br, bs) = \log \frac{1 - r}{1 - s}.$$ 

4. Let $G \subset \mathbb{R}^n$ be any domain and $z_0 \in G$. Let $z \in \partial G$ be such that $\delta(z_0) = |z_0 - z|$. Then for any $u, v \in [z_0, z]$ we have

$$k_G(u, v) = j_G(u, v) = \left| \log \frac{\delta(z_0) - |z_0 - u|}{\delta(z_0) - |z_0 - v|} \right| = \left| \log \frac{\delta(u)}{\delta(v)} \right|.$$ 

5. For $x, y \in B^n$ we have

$$j_{B^n}(x, y) \leq \rho_{B^n}(x, y) \leq 2j_{B^n}(x, y)$$

with equality on the right hand side when $x = -y$. 
Remark 2.4. \( x, y \) for 

Therefore by Proposition 2.2(5) 

\[ \ell_k(\gamma) = \int_0^u \frac{\gamma'(t)|\,dt}{d(\gamma(t), \partial G)} \geq \int_0^u \frac{dt}{\delta(x) + t} = \log \frac{\delta(x) + u}{\delta(x)} \geq \log \left(1 + \frac{|x - y|}{\delta(x)} \right) = j_G(x, y). \]

(2) We see by (1) that 

\[ j_{B^n}(0, x) = \log \frac{1}{1 - |x|} \leq k_{B^n}(0, x) \leq \int_{[0, x]} \frac{|dz|}{\delta(z)} = \log \frac{1}{1 - |z|} \]

and hence \([0, x]\) is the \(k_{B^n}\)-geodesic between 0 and \(x\) and the equality in (2) holds.

The proof of (3) follows from (2) because the quasihyperbolic length is additive along a geodesic

\[ k_{B^n}(0, bs) = k_{B^n}(0, br) + k_{B^n}(br, bs). \]

The proof of (4) follows from (3).

The proof of (5) is given in [1, Lemma 7.56].

\[ \Box \]

Lemma 2.3. (1) For \(0 < s < 1\) and \(x, y \in B^n(s)\) we have

\[ j_{B^n}(x, y) \leq k_{B^n}(x, y) \leq (1 + s) j_{B^n}(x, y). \]

(2) Let \(G \subseteq \mathbb{R}^n\) be a domain, \(w \in G\), and \(w_0 \in (\partial G) \cap S^{n-1}(w, \delta(w))\). If \(s \in (0, 1)\) and \(x, y \in B^n(w, s\delta(w))\) are such that \(\delta(x) = |x - w_0| \leq \delta(y)\), then we have 

\[ k_G(x, y) \leq (1 + s) j_G(x, y). \]

(3) Let \(s \in (0, 1), G = \mathbb{R}^n \setminus \{0\}\), \(x, y, w \in G\) with \(|x| \leq |y|\) and \(|x - w| < s\delta(w), |y - w| < s\delta(w)\). Then we have

\[ k_G(x, y) \leq (1 + s) j_G(x, y). \]

Proof. (1) Fix \(x, y \in B^n(s)\) and the geodesic \(\gamma\) of the hyperbolic metric joining them. Then \(\gamma \subset B^n(s)\) and for all \(w \in B^n(s)\) we have

\[ \frac{1}{1 - |w|} < \frac{1 + s}{2} \frac{2}{1 - |w|^2}. \]

Therefore by Proposition 2.2(5)

\[ k_{B^n}(x, y) \leq \int_{\gamma} \frac{|dw|}{1 - |w|} \leq \frac{1 + s}{2} \int_{\gamma} \frac{2|dw|}{1 - |w|^2} \leq \frac{1 + s}{2} \rho_{B^n}(x, y) \leq (1 + s) j_{B^n}(x, y). \]

for \(x, y \in B^n(s)\). The first inequality follows from Proposition 2.2(1).

For the proof of (2) set \(B = B^n(w, \delta(w))\). Then by part (1)

\[ k_G(x, y) \leq k_B(x, y) \leq (1 + s) j_B(x, y) = (1 + s) j_G(x, y). \]

The proof of (3) follows from the proof of (2). \[ \Box \]

Remark 2.4. (1) Lemma 2.3(1) and (3) improve [20, Lemma 3.7(2)] for the cases of \(B^n\) and \(\mathbb{R}^n \setminus \{0\}\). We have been unable to prove a similar statement for a general domain.

(2) The proof of Proposition 2.2 shows that the diameter \((-e, e), e \in S^{n-1}\), of \(B^n\) is a geodesic of \(k_{B^n}\) and hence the quasihyperbolic distance is additive on a diameter. At the
At the same time we see that the $j$ metric is additive on a radius of the unit ball but not on the full diameter because for $x \in B^n \setminus \{0\}$
\[ j_{B^n}(-x, x) < j_{B^n}(-x, 0) + j_{B^n}(0, x). \]

Our next goal is to compare the Euclidean and the quasihyperbolic metric in a domain and we recall in the next lemma a sharp inequality for the hyperbolic metric of the unit ball proved in [20 (2.27)].

**Lemma 2.5.** For $x, y \in B^n$ let $t$ be as in (2.1). Then
\[ \tanh^2 \frac{\rho_{B^n}(x, y)}{2} = \frac{|x - y|^2}{|x - y|^2 + t^2}, \]
\[ |x - y| \leq 2 \tanh \frac{\rho_{B^n}(x, y)}{4} = \frac{2|x - y|}{\sqrt{|x - y|^2 + t^2 + t}}, \]
where equality holds for $x = -y$.

Earle and Harris [3] provided several applications of this inequality and extended this inequality to other metrics such as the Carathéodory metric. Several remarks about Lemma 2.5 are in order. Notice that Lemma 2.5 gives a sharp bound for the modulus of continuity $id : (B^n, \rho_{B^n}) \to (B^n, |\cdot|)$.

For a $K$-quasiconformal homeomorphism
\[ f : (B^n, \rho_{B^n}) \to (B^n, \rho_{B^n}) \]
an upper bound for the modulus of continuity is well-known, see [20 Theorem 11.2]. For $n = 2$ the result is sharp for each $K \geq 1$, see [11, p. 65 (3.6)]. The particular case $K = 1$ yields a classical Schwarz lemma.

As a preliminary step we record Jung’s Theorem [2 Theorem 11.5.8] which gives a sharp bound for the radius of a Euclidean ball containing a given bounded domain.

**Lemma 2.6.** Let $G \subset \mathbb{R}^n$ be a domain with $\operatorname{diam} G < \infty$. Then there exists $z \in \mathbb{R}^n$ such that $G \subset B^n(z, r)$, where $r \leq \sqrt{n/(2n + 2)} \operatorname{diam} G$.

**Lemma 2.7.**
1. If $x, y$ are on a diameter of $B^n$ and $w = |x - y| e_1/2$, then we have
\[ k_{B^n}(x, y) \geq k_{B^n}(-w, w) = 2 k_{B^n}(0, w) = 2 \log \frac{2}{2 - |x - y|} \geq |x - y|, \]
where the first inequality becomes equality when $y = -x$.
2. If $x, y \in B^n$ are arbitrary and $w = |x - y| e_1/2$, then
\[ k_{B^n}(x, y) \geq k_{B^n}(-w, w) = 2 k_{B^n}(0, w) = 2 \log \frac{2}{2 - |x - y|} \geq |x - y|, \]
where the first inequality becomes equality when $y = -x$.
3. Let $G \subset \mathbb{R}^n$ be a domain with $\operatorname{diam} G < \infty$ and $r = \sqrt{n/(2n + 2)} \operatorname{diam} G$. Then we have
\[ k_G(x, y) \geq 2 \log \frac{2}{2 - t} \geq t = |x - y|/r, \]
for all distinct $x, y \in G$ with equality in the first step when $G = B^n(z, r)$ and $z = (x + y)/2$. 
Proof. In the proof of \((1)\) and \((2)\), without loss of generality, we assume that \(|x| \geq |y|\).

(1) If \(0 \in [x, y]\), by Proposition 2.2(2) we have
\[
k_{B^n}(x, y) = k_{B^n}(x, 0) + k_{B^n}(0, y) = \log \frac{1}{(1 - |x|)(1 - |y|)},
\]
and hence
\[
k_{B^n}(-w, w) = 2 \log \frac{1}{1 - |w|}.
\]
We need to prove that
\[
(1 - |w|)^2 \geq (1 - |x|)(1 - |y|),
\]
It suffices to show that
\[
\frac{|x| + |y|}{2} = \frac{|x - y|}{2} = |w| \leq 1 - \sqrt{(1 - |x|)(1 - |y|)},
\]
which is equivalent to \((|x| - |y|)^2 \geq 0\).
If \(y \in [x, 0]\), then the proof goes in a similar way. Indeed, we note that \(|x| - |y| = |x - y| = 2|w|\). Then by Proposition 2.2(3) we have
\[
k_{B^n}(x, y) = \log \frac{1 - |y|}{1 - |x|}.
\]
It is enough to show that
\[
(1 - |w|)^2 \geq \frac{1 - |x|}{1 - |y|} = 1 + \frac{|y| - |x|}{1 - |y|}.
\]
Substituting the value of \(|w|\) and then squaring we see that
\[
(|x| - |y|) \left(1 - \frac{1}{1 - |y|}\right) \leq \left(\frac{|x| - |y|}{2}\right)^2,
\]
which is trivial as the left hand term is \(\leq 0\). Equality holds if \(y = -x\).

(2) Choose \(y' \in B^n\) such that \(|x - y| = |x - y'| = 2|w|\) with \(x\) and \(y'\) on a diameter of \(B^n\).

Then
\[
k_{B^n}(x, y) \geq k_{B^n}(x, y') \geq k_{B^n}(-w, w),
\]
where the first inequality holds trivially and the second one holds by \((1)\).

(3) Since \(G\) is a bounded domain, by Lemma 2.6 there exists \(z \in \mathbb{R}^n\) such that \(G \subset B^n(z, r)\). Denote \(B := B^n(z, r)\). Then the domain monotonicity property gives
\[
k_G(x, y) \geq k_B(x, y).
\]
Without loss of generality we may now assume that \(z = 0\). Choose \(u, v \in B\) in such a way that \(u = -v\) and \(|u - v| = 2|u| = |x - y|\). Hence by \((2)\) we have
\[
k_G(x, y) \geq k_B(x, y) \geq k_B(-u, u) = 2 \log \frac{r}{r - |u|}.
\]
This completes the proof. \(\square\)

**Corollary 2.8.** (1) For every \(x, y \in B^n\) we have
\[
|x - y| \leq 2(1 - \exp(-k_{B^n}(x, y)/2)) \leq k_{B^n}(x, y),
\]
where the first inequality becomes equality when \(y = -x\).
(2) If $G \subseteq \mathbb{R}^n$ is a domain with diam $G < \infty$ and $r = \sqrt{n/(2n + 2)} \text{diam } G$, then we have

$$|x - y|/r \leq 2(1 - \exp(-k_G(x,y)/2)) \leq k_G(x,y)$$

for all distinct $x, y \in G$ with equality in the first step when $G = B^n(z, r)$ and $z = (x + y)/2$.

A counterpart of Lemma 2.7 for the metric $j_G$ is discussed below.

**Lemma 2.9.**

(1) If $x, y$ are on a diameter of $B^n$ and $w = |x - y|e_1/2$, then we have

$$j_B^n(x, y) \geq j_B^n(-w, w) = \log \frac{2 + t}{2 - t} \geq t = |x - y|,$$

where the first inequality becomes equality when $y = -x$.

(2) If $x, y \in B^n$ are arbitrary and $w = |x - y|e_1/2$, then

$$j_B^n(x, y) \geq j_B^n(-w, w) = \log \frac{2 + t}{2 - t} = 2 \tanh(t/2) \geq t = |x - y|,$$

where the first inequality becomes equality when $y = -x$.

(3) Let $G \subseteq \mathbb{R}^n$ be a domain with diam $G < \infty$ and $r = \sqrt{n/(2n + 2)} \text{diam } G$. Then we have

$$j_G(x, y) \geq j_B^n(-w, w) = \log \frac{2 + t}{2 - t} \geq t = |x - y|/r,$$

for all distinct $x, y \in G$ with equality in the first step when $G = B^n(z, r)$ and $z = (x + y)/2$.

**Proof.** In the proof of (1) and (2), without loss of generality, we may assume that $|x| \geq |y|$.

(1) If $0 \in [x, y]$, we have

$$j_B^n(x, y) = \log \left(1 + \frac{|x - y|}{1 - |x|}\right) = \log \frac{1 + |y|}{1 - |x|}$$

and hence

$$j_B^n(-w, w) = \log \frac{1 + |w|}{1 - |w|} = \log \frac{2 + |x - y|}{2 - |x - y|} \geq |x - y|.$$  

The inequality $j_B^n(x, y) \geq j_B^n(-w, w)$ is clear due the fact that $2|w| = |x| + |y|$ and $|x| \geq |y|$. If $y \in [x, 0]$, a similar reasoning gives the conclusion.

(2) Choose $y' \in B^n$ such that $|x - y| = |x - y'| = 2|w|$ with $x$ and $y'$ on a diameter of $B^n$. Then

$$j_B^n(x, y) = j_B^n(x, y') \geq j_B^n(-w, w),$$

where the lower bound holds by (1).

(3) The proof is very similar to the proof of Lemma 2.7(3).

This completes the proof.

**Corollary 2.10.**

(1) For every $x, y \in B^n$ we have

$$|x - y| \leq 2 \tanh(j_B^n(x, y)/2) \leq j_B^n(x, y),$$

where the first inequality becomes equality when $y = -x$. 
Remark 2.11. Let us denote the spherical chordal metric in $\mathbb{R}^n$ by $q(x, y)$. Starting with the sharp inequality $[11]$ 7.17 (3), p. 378

$$|x - y| \geq \frac{2q(x, y)}{1 + \sqrt{1 - q(x, y)^2}}$$

we deduce that

$$q(x, y) \leq \frac{|x - y|}{1 + (|x - y|/2)^2}$$

with equality for $y = -x$. Therefore, we see that the identity mapping

$$id : (\mathbb{R}^n, | \cdot |) \to (\mathbb{R}^n, q)$$

has the sharp modulus of continuity $\omega(t) = t/(1 + (t/2)^2)$ for $t \in (0, 2)$.

We conclude this section by formulating some basic results on quasihyperbolic distances which are indeed used latter in Section 3. The following lemma is established in $[19]$ Lemma 2.53.

Lemma 2.12. If $\theta \in (0, 1)$, then there exists a positive number $a(\theta)$ such that the following holds. If $x, y, z \in G$ with $x, y \in G \setminus B^n(z, \theta d(z, \partial G))$ then

$$k_{G \setminus \{z\}}(x, y) \leq a(\theta) k_G(x, y),$$

where $a(\theta) = 1 + (2/\theta) + \pi/(2\log(2 + 2e))$.

Lemma 2.13. If $r > 0$ and $x, y \in G = \mathbb{R}^n \setminus \overline{B^n}(r)$ with $|x| = |y|$, then

$$k_G(x, y) \leq \frac{|x|}{|x| - r} k_{\mathbb{R}^n \setminus \{0\}}(x, y) \leq \frac{|x - y| \pi}{2(|x| - r)}.$$

Proof. The first inequality follows from $[10]$ Theorem 5.20. For the second inequality we see that if $\theta = \angle x0y$, then we have the identity

$$|x - y|^2 = |x|^2 + |y|^2 - 2 |x||y| \cos \theta.$$

Since $|x| = |y|$ it follows that $\sin(\theta/2) = |x - y|/(2|x|)$. For $0 \leq \theta \leq \pi$, the well-known inequality $\theta \leq \pi \sin(\theta/2)$ gives that $\theta \leq \pi |x - y|/(2|x|)$. Since $k_{\mathbb{R}^n \setminus \{0\}}(x, y) \leq \theta$, when $|x| = |y|$, we conclude the second inequality. \hfill $\square$

We also need a generalization of Lemma 2.12.

Lemma 2.14. If $\alpha, \theta \in (0, 1)$, then there exists a positive number $a(\alpha, \theta)$ such that the following holds. If $x, y, z \in G$ with $x, y \in G \setminus B^n(z, \theta d(z, \partial G))$ then

$$k_{G'}(x, y) \leq a(\alpha, \theta) k_G(x, y),$$

where $G' = G \setminus \overline{B^n}(z, \alpha \theta d(z, \partial G))$ and

$$a(\alpha, \theta) = \frac{2 + \theta + \alpha \theta}{\theta(1 - \alpha^2)} + \frac{(1 + \alpha)\pi}{2(1 - \alpha) \log((2 + 2\theta)/(2 + \theta + \alpha \theta))}.$$
Proof. Denote by \( \delta(z) = d(z, \partial G) \). Fix \( \beta \in (0, 1) \) and \( w \in G \setminus \overline{B^n}(z, \beta \delta(z)) \). Choose \( q \in S^{n-1}(z, \alpha \beta \delta(z)) \) and \( p \in \partial G \) such that \( |w - q| = d(w, S^{n-1}(z, \alpha \beta \delta(z))) \) and \( |p - z| = \delta(z) \). Then we have \( |w - q| \geq (1 - \alpha) \beta \delta(z) \) and hence

\[
|p - q| \leq (1 + \alpha \beta) \delta(z) \leq \frac{1 + \alpha \beta}{\beta(1 - \alpha)} |w - q|.
\]

It follows by triangle inequality that

\[
(2.15) \quad d(w, \partial G) \leq \frac{1 + 1/\beta}{1 - \alpha} d(w, \partial G \cup S^{n-1}(z, \alpha \beta \delta(z))).
\]

Let \( J \) be a geodesic joining \( x \) and \( y \) in \( G \) (i.e. \( J = J_G[x, y] \)) and \( U = J \cap \overline{B}(z, (1 + \alpha) \theta \delta(z)/2) \).

If \( U \neq \emptyset \) then we denote by \( y' \) the first point in \( U \), when we traverse along \( J \) from \( x \) to \( y \).

We similarly denote \( y' \) in \( U \) along the reverse way. By (2.15) and Lemma 2.13

\[
k_G(x, y) \leq k_G(x, x') + k_G(x', y') + k_G(y', y)
\]

\[
\leq \frac{2 + \theta + \alpha \theta}{\theta(1 - \alpha^2)} k_G(x, x') + \frac{1 + \alpha}{1 - \alpha} \frac{2 + \theta + \alpha \theta}{\theta(1 - \alpha^2)} k_G(y', y)
\]

\[
\leq \frac{2 + \theta + \alpha \theta}{\theta(1 - \alpha^2)} k_G(x, y) + \frac{1 + \alpha}{1 - \alpha} \frac{2 + \theta + \alpha \theta}{\theta(1 - \alpha^2)}.
\]

Since \( k \geq j \) we have

\[
k_G(x, y) \geq k_G(x, x') + k_G(y', y) \geq 2 \log \left(1 + \frac{\theta - (1 + \alpha) \theta/2}{1 + (1 + \alpha) \theta/2}\right) = 2 \log \frac{2 + 2 \theta}{2 + \theta + \alpha \theta}
\]

and therefore

\[
k_G(x, y) \leq a(\alpha, \theta) k_G(x, y)
\]

holds for

\[
a(\alpha, \theta) = \frac{2 + \theta + \alpha \theta}{\theta(1 - \alpha^2)} + \frac{(1 + \alpha) \pi}{2(1 - \alpha) \log((2 + 2 \theta)/(2 + \theta + \alpha \theta))}.
\]

If \( U = \emptyset \), then by (2.15)

\[
k_G(x, y) \leq \frac{2 + \theta + \alpha \theta}{\theta(1 - \alpha^2)} k_G(x, y) \leq a(\alpha, \theta) k_G(x, y).
\]

The assertion follows. \( \square \)

Clearly Lemma 2.14 implies Lemma 2.12 as \( \alpha \to 0 \).

3. \( \varphi \)-UNIFORM DOMAINS

In 1979, the class of uniform domains was introduced by Martio and Sarvas [15]. In the same year, Gehring and Osgood [5] characterized uniform domains in terms of an upper bound for the quasihyperbolic metric as follows: a domain \( G \) is uniform if and only if there exists a constant \( C \geq 1 \) such that

\[
(3.1) \quad k_G(x, y) \leq C j_G(x, y)
\]

for all \( x, y \in G \). As a matter of fact, the above inequality appeared in [5] in a form with an additive constant on the right hand side: it was shown by Vuorinen [19, 2.50] that the additive constant can be chosen to be 0. This observation lead in [19] to the definition of \( \varphi \)-uniform domains.
Definition 3.2. Let \( \varphi : [0, \infty) \to [0, \infty) \) be a continuous strictly increasing function with \( \varphi(0) = 0 \). A domain \( G \subset \mathbb{R}^n \) is said to be \( \varphi \)-uniform if
\[
(3.3) \quad k_G(x, y) \leq \varphi(|x - y|/\min\{\delta(x), \delta(y)\})
\]
for all \( x, y \in G \).

In order to give a simple criterion for \( \varphi \)-uniform domains, consider domains \( G \) satisfying the following property [19]: there exists a constant \( C \geq 1 \) such that each pair of points \( x, y \in G \) can be joined by a rectifiable path \( \gamma \in G \) with \( \ell(\gamma) \leq C|x - y| \) and \( \min\{\delta(x), \delta(y)\} \leq C d(\gamma, \partial G) \). Then \( G \) is \( \varphi \)-uniform with \( \varphi(t) = C^2t \). In particular, every convex domain is \( \varphi \)-uniform with \( \varphi(t) = t \). However, in general, convex domains need not be uniform. The above examples of \( \varphi \)-uniform domains are studied in [19, Examples 2.50 (1)]. More complicated nontrivial examples of \( \varphi \)-uniform domains can be seen by considering that of uniform domains which are studied by several researchers. For instance, it is noted in [13, 16] that complementary components of quasimöbius (and hence bi-Lipschitz) spheres are uniform.

Because simply connected uniform domains in plane are quasidisks [15] (see also [4]), it follows that the complement of such a uniform domain also is uniform. A motivation to this observation of uniform domains leads to investigate the complementary domains in the case of \( \varphi \)-uniform domains. In fact we see from the following examples that complementary domains of \( \varphi_1 \)-uniform domains are not always \( \varphi_2 \)-uniform. The first example investigates the matter in the case of half-strips.

Example 3.4. Since the half-strip defined by \( S = \{(x, y) \in \mathbb{R}^2 : x > 0, -1 < y < 1\} \) is convex, by the above discussion we observe that it is \( \varphi \)-uniform with \( \varphi(t) = t \). On the other hand, by considering the points \( z_n = (n, -2) \) and \( w_n = (n, 2) \) we see that \( G := \mathbb{R}^2 \setminus S \) is not a \( \varphi \)-uniform domain. Indeed, we have \( j_G(z_n, w_n) = \log 5 \) and for some \( m \in \mathbb{R} \cap J_G[z_n, w_n] \)
\[
k_G(z_n, w_n) \geq k_G(m, w_n) \geq \log \left( 1 + \frac{|m - w_n|}{\delta(w_n)} \right) \geq \log(1 + n) \to \infty \quad \text{as} \quad n \to \infty.
\]
This shows that \( G \) is not \( \varphi \)-uniform for any \( \varphi \).

Example 3.5. Consider the domain
\[
D = \{(x, y) \in \mathbb{R}^2 : -\exp(-1 - x) < y < \exp(-1 - x), \quad x > 0\}.
\]
It is clear by [19, 2.50] that \( D \) is \( \varphi \)-uniform with \( \varphi(t) = 4t \). We next show that its complementary domain \( G := \mathbb{R}^2 \setminus D \) is not \( \varphi \)-uniform. We see that the points \( z_n = (n, -e^{-n}) \) and \( w_n = (n, e^{-n}) \) are in \( G \), and \( j_G(z_n, w_n) = \log 3 \). On the other hand, let \( m \in J_G[z_n, w_n] \cap \mathbb{R} \). Then
\[
k_G(z_n, w_n) \geq k_G(z_n, m) \geq \log \left( 1 + \frac{|z_n - m|}{e^{-n}} \right) \geq \log(1 + ne^n) > n \to \infty \quad \text{as} \quad n \to \infty.
\]
This shows that \( G \) is not \( \varphi \)-uniform for any \( \varphi \).

Example 3.6. Define
\[
D_m = \left\{ (x, y) \in \mathbb{R}^2 : |x| < \frac{1}{1 + \log m}, \quad 0 < y < \frac{me}{10} \right\}.
\]
Then, by a similar reasoning to Example 3.5 we conclude that \( D = \bigcup_{m=1}^{\infty} D_m \) is a \( \varphi \)-uniform domain whereas its complement \( G' = \mathbb{R}^2 \setminus D \) is not (see Figure 1).

A domain \( G \subset \mathbb{R}^n \) is said to be *quasiconvex* if there exists a constant \( c > 0 \) such that any pair of points \( x, y \in G \) can be joined by a rectifiable path \( \gamma \subset G \) satisfying \( \ell(\gamma) \leq c |x - y| \). The domains \( G \) and \( G' \) in the above examples are not quasiconvex and are not bounded as well. This naturally leads to the following question.

**Problem 3.7.** Are there any bounded \( \varphi \)-uniform domains whose complementary domains are not \( \varphi \)-uniform?

**Problem 3.8.** Is it true that quasiconvex domains are \( \varphi \)-uniform and viceversa?

In the dimension \( n = 3 \), we have solutions to Problem 3.7 as follows:

**Example 3.9.** Let \( T \) be the triangle with vertices \((1, -1), (0,0)\) and \((1,1)\). Consider the domain \( D \) bounded by the surface of revolution generated by revolving \( T \) about the vertical axis (see Figure 2).

Then we see that \( D \) is \( \varphi \)-uniform. Indeed, let \( x, y \in D \) be arbitrary. Without loss of generality we assume that \( |x| \geq |y| \). Consider the path \( \gamma = [x, x'] \cup C \) joining \( x \) and \( y \), where \( x' \in S^1(|y|) \) is chosen so that \( |x' - x| = d(x, S^1(|y|)) \); and \( C \) is the smaller circular arc of \( S^1(|y|) \) joining \( x' \) to \( y \). For \( z \in D \), we write \( \delta(z) := d(z, \partial D) \). Then for all \( x, y \in D \) we have

\[
k_D(x, y) \leq \int_{\gamma} \frac{|dz|}{\delta(z)} = \int_{[x,x']} \frac{|dz|}{\delta(z)} + \int_C \frac{|dz|}{\delta(z)} \leq \frac{|x - y|}{\min\{\delta(x), \delta(y)\}} + \int_C \frac{|dz|}{\delta(z)} \leq \left( 1 + \frac{\pi}{2} \right) \frac{|x - y|}{\min\{\delta(x), \delta(y)\}},
\]

where the last inequality follows by the fact that \( \ell(C) \leq \pi|x - y|/2 \) (see the proof of Lemma 2.13).

On the other hand, its complement \( G = \mathbb{R}^3 \setminus D \) is not \( \varphi \)-uniform. Because for the point \( z_t = te_2 \in G, 0 < t < 1 \), we have \( j_G(-z_t, z_t) = \log(1 + 2\sqrt{2}) \); and by a similar argument as in
Figure 2. A bounded $\varphi$-uniform domain in $\mathbb{R}^3$ whose complementary domain is not $\varphi$-uniform (the right hand side figure shows the revolution).

Example 3.5 we have

$$k_G(-z_t, z_t) \geq \log \left(1 + \frac{\sqrt{2}}{t}\right) \to \infty \quad \text{as } t \to 0.$$  

This shows that $G$ is not $\varphi$-uniform for any $\varphi$. △

Example 3.9 gives an bounded $\varphi$-uniform domain in $\mathbb{R}^3$ which is not simply connected. In the following, we construct a bounded simply connected domain in $\mathbb{R}^3$ which is $\varphi$-uniform but its complement is not.

Example 3.10. Fix $h = 1/3$. For the sake of convenience, we denote the coordinate axes in $\mathbb{R}^3$ by $x$-, $y$- and $z$-axes. Let $D$ be a domain obtained by rotating the triangle with vertices $(0, 0, h)$, $(1, 0, 0)$ and $(0, 0, -h)$ around the $z$-axis. For each $k \geq 1$, we let

$$x_k = 1 - 4^{-k} \quad \text{and} \quad h_k = (1 - x_k)/(10h).$$

We now modify the boundary of $D$ as follows: let us drill the cavity of $D$ from two opposite directions of $z$-axis such that the drilling axis, parallel to $z$-axis, goes through the point $(x_k, 0, 0)$. From the positive direction we drill until the tip of the drill is at the height $h_k$ and from the opposite direction we drill up to the height $-h_k$. The cross section of the upper conical surface will have its tip at $(x_k, y_k)$ described by

$$y - h_k = A(x - x_k), \quad A = \pm 1.$$  

This intersects the boundary of the cavity represented by $z = h(1 - x)$ at

$$x = \frac{h - h_k + Ax_k}{A + h}.$$  

This gives

$$u_k = x|_{A=-1} = \frac{h - h_k - x_k}{-1 + h} \quad \text{and} \quad v_k = x|_{A=1} = \frac{h - h_k + x_k}{1 + h}.$$  

Obviously, $u_k \leq x_k \leq v_k$. Since $v_k \leq u_{k+1}$, we see that two successive drilling do not interfere.
Figure 3. A double cone domain with twosided drillings. This picture provides a schematic view of the simply-connected domain \(D \subset \mathbb{R}^3\) constructed in Example 3.10. The domain is uniform but its complement is not \(\varphi\)-uniform for any \(\varphi\).

Induction on \(k\) gives us a new domain \(G \subset \mathbb{R}^3\) which is simply connected and uniform, but its complement is not \(\varphi\)-uniform for any \(\varphi\). Indeed, the choice of points \(z_k = (x_k, 0, 2h_k)\) and \(w_k = (x_k, 0, -2h_k)\) in \(U = \mathbb{R}^3 \setminus G\) gives that

\[d(z_k, \partial U) = \min\{x_k - u_k, v_k - x_k\} = h_k f(h) .\]

It follows that

\[j_U(z_k, w_k) = \log \left(1 + \frac{4h_k}{h_k f(h)}\right) < \infty .\]

On the other hand,

\[k_U(z_k, w_k) \geq \log \left(1 + \frac{\sqrt{1 + h^2}}{d(z_k, \partial U)}\right) = \log \left(1 + \frac{\sqrt{1 + h^2}}{h_k f(h)}\right) \to \infty \quad \text{as} \quad k \to \infty .\]

This shows that \(U\) is not \(\varphi\)-uniform for any \(\varphi\). \(\triangle\)

We now discuss solution to Problem 3.8. The following example says that there exist quasiconvex domains which are not \(\varphi\)-uniform.

**Example 3.11.** Consider the unit square \(D = \{(x, y) \in \mathbb{R}^2 : |x| < 1, |y| < 1\}\). For \(n \geq 1\) we define the set

\[P^n_0 = \left\{\left(0, \pm \frac{1}{2^n}\right), \left(\pm \frac{1}{2^n}, 0\right)\right\},\]

and for \(1 \leq m \leq n - 1\) we define

\[P^n_m = \left\{\left(\sum_{k=1}^{m} \frac{1}{2^k}, \sum_{k=1}^{m} \frac{1}{2^k} \pm \frac{1}{2^n}\right), \left(\sum_{k=1}^{m} \frac{1}{2^k} \pm \frac{1}{2^n}, \sum_{k=1}^{m} \frac{1}{2^k}\right)\right\} .\]

Then consider the domain defined by

\[G := D \setminus \bigcup_{m=0}^{k} P^n_m, \quad \text{for} \quad n \geq m + 1 .\]

When \(k\) is large enough we observe that \(G\) is not \(\varphi\)-uniform but it is quasiconvex. \(\triangle\)
Next, we characterize domains $G$ such that the identity mapping defined by (1.2) is uniformly continuous with Lipschitzian modulus of continuity. The uniform continuity of the identity mapping defined on certain metric spaces is studied in [10] as well. Due to Proposition 2.2(1) we see that for any domain $G \not\subseteq \mathbb{R}^n$ the identity mapping
\[ \text{id} : (G, j_G) \rightarrow (G, k_G) \]
is uniformly continuous with Lipschitzian modulus of continuity $\omega(t) = t$. On the other hand, the mapping (1.2) is not uniformly continuous with Lipschitzian modulus of continuity in general. One may ask, is it true that $G$ is uniform if and only if the identity mapping $\text{id} : (G, j_G) \rightarrow (G, k_G)$ is uniformly continuous with Lipschitzian modulus of continuity. The answer is provided by the next theorem.

**Theorem 3.12.** The identity mapping $\text{id} : (G, j_G) \rightarrow (G, k_G)$ is uniformly continuous with Lipschitzian modulus of continuity if and only if $G$ is $\varphi$-uniform.

**Proof.** Sufficiency part is trivial. Indeed, for $x, y \in G$ we have
\[ k_G(x, y) \leq \varphi(\exp(j_G(x, y)) - 1) = \omega(j_G(x, y)) \]
where $\omega(t) = \varphi(e^t - 1)$.

For the necessary part, we define $\varphi : [0, \infty) \rightarrow [0, \infty)$ by
\[ \varphi(t) = \sup\{k_G(x, y) : j_G(x, y) \leq t\}, \quad t \geq 0. \]
Our assumption is that the identity mapping $\text{id} : (G, j_G) \rightarrow (G, k_G)$ is uniformly continuous with Lipschitzian modulus of continuity. This is equivalent to the condition that (see [20, P. 134]) the Lipschitzian modulus of continuity $\varphi : (0, \infty) \rightarrow (0, \infty)$ is increasing and that $\varphi(t) \rightarrow 0$ as $t \rightarrow 0$. For $G$ to be a $\varphi$-uniform domain we remain to check (3.3). Note that $k_G(x, y) \leq \varphi(t)$ for all $t$ with $j_G(x, y) \leq t$. Since $\log(1 + t) \leq t$ for all $t \geq 0$, in particular, for $t = |x - y|/ \min\{\delta(x), \delta(y)\}$ we see that $G$ satisfies (3.3) and hence $G$ is $\varphi$-uniform. □

It is well-known that uniform domains are preserved under bilipschitz mappings (see for instance [20 p. 37]). We now present the bilipschitz properties of $\varphi$-uniform domains.

**Proposition 3.13.** Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an $L$-bilipschitz mapping, that is
\[ |x - y|/L \leq |f(x) - f(y)| \leq L|x - y| \]
for all $x, y \in \mathbb{R}^n$. If $G \not\subseteq \mathbb{R}^n$ is $\varphi$-uniform, then $f(G)$ is $\varphi_1$-uniform with $\varphi_1(t) = L^2\varphi(L^2t)$.

**Proof.** We denote $\delta(z) := d(z, \partial G)$ and $\delta'(w) := d(w, \partial f(G))$. Since $f$ is $L$-bilipschitz, it follows that
\[ \delta(z)/L \leq \delta'(f(z)) \leq L \delta(z) \]
for all $z \in G$. Also, we have the following well-know relation (see for instance [20 p. 37])
\[ k_G(x, y)/L^2 \leq k_{f(G)}(f(x), f(y)) \leq L^2k_G(x, y) \]
for all $x, y \in G$. Hence, $\varphi$-uniformity of $G$ yields
\[ k_{f(G)}(f(x), f(y)) \leq L^2k_G(x, y) \]
\[ \leq L^2\varphi(|x - y|/ \min\{\delta(x), \delta(y)\}) \]
\[ \leq L^2\varphi(L^2|x - y|/ \min\{\delta'(f(x)), \delta'(f(y))\}). \]
This concludes our claim. □
A mapping $h : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by

$$h(x) = a + \frac{r^2(x-a)}{|x-a|^2}, \quad h(\infty) = a, \quad h(a) = \infty$$

is called an inversion in the sphere $S^{n-1}(a, r)$ for $x, a \in \mathbb{R}^n$ and $r > 0$. We recall the following well-known identity from [20, (1.5)]

$$|h(x) - h(y)| = \frac{r^2|x - y|}{|x - a||y - a|}, \quad x, y \in \mathbb{R}^n \setminus \{a\}.$$  \hspace{1cm} (3.14)

We next show that $\varphi$-uniform domains are preserved under inversion in a sphere.

**Corollary 3.15.** Let $z_0 \in \mathbb{R}^n$ and $R > 0$ be arbitrary. Denote by $h$ an inversion in $S^{n-1}(z_0, R)$. For $0 < m < M$, if $G \subset B^n(z_0, M) \setminus \overline{B^n}(z_0, m)$ is a $\varphi_1$-uniform domain, then $h(G)$ is $\varphi_1$-uniform with $\varphi_1(t) = (M/m)^2 \varphi(M^2 t/m^2)$.

**Proof.** We denote $\delta(z) := d(z, \partial G)$ and $\delta'(w) := d(w, \partial h(G))$. Without loss of generality we can assume that $z_0 = 0$. By the assumption on $G$ we see that $m \leq |z| \leq M$ for all $z \in G$. Hence, by the identity (3.14) we have

$$R^2 |x - y|/M^2 \leq |h(x) - h(y)| \leq R^2 |x - y|/m^2$$

which implies

$$R^2 \min\{\delta(x), \delta(y)\}/M^2 \leq \min\{\delta'(h(x), \delta'(y))\} \leq R^2 \min\{\delta(x), \delta(y)\}/m^2.$$

It follows that

$$(m/M)^2 k_G(x, y) \leq k_{h(G)}(h(x), h(y)) \leq (M/m)^2 k_G(x, y)$$

for all $x, y \in G$. Since $G$ is $\varphi$-uniform, by a similar argument as in the proof of Proposition 3.13 we conclude our assertion. \hspace{1cm} \Box

There are a number of classes of domains that have been investigated in analysis and related topics (e.g. see [7]). In general, all such classes of domains do not behave same if we remove a finite number of points from the domains [7], but some do behave (for example, see [7, Theorem 5.1] and [17, Theorem 5.4]). The following result is one such investigation.

**Theorem 3.16.** Let $G \subset \mathbb{R}^n$ be a $\varphi_1$-uniform domain and $z_0 \in G$. Then $G \setminus \{z_0\}$ is $\varphi$-uniform for some $\varphi$ depending on $\varphi_1$ only.

**Proof.** In this proof we denote by $\delta_1$ the Euclidean distance to the boundary of $G$ and $\delta_2$ the Euclidean distance to that of $G \setminus \{z_0\}$. Fix $\theta \in (0, 1)$ and let $x, y \in G \setminus \{z_0\}$ be arbitrary. We prove the theorem by considering three cases.

**Case I:** $x, y \in B^n(z_0, \theta \delta_1(z_0)) \setminus \{z_0\}$. We see that

$$k_{G \setminus \{z_0\}}(x, y) = k_{\mathbb{R}^n \setminus \{z_0\}}(x, y) \leq \frac{\pi}{\log 3} j_{\mathbb{R}^n \setminus \{z_0\}}(x, y) \leq \frac{\pi}{\log 3} j_{G \setminus \{z_0\}}(x, y),$$
where the equality follows from \cite[Lemma 4.4]{18} and the first inequality is due to Lindén \cite[Theorem 1.6]{12}. It follows that

\begin{equation}
(3.17) \quad k_{G\setminus\{z_0\}}(x, y) \leq \varphi_2(|x - y|/ \min\{\delta_2(x), \delta_2(y)\})
\end{equation}

for $\varphi_2(t) = \frac{\pi}{\log 3} \log(1 + t)$.

**Case II:** $x, y \in G \setminus B^n(z_0, \theta_\delta_1(z_0)/2)$.

Since $G$ is $\varphi_1$-uniform, using Lemma \ref{lem:2.12 we obtain}

\[
k_{G\setminus\{z_0\}}(x, y) \leq a(\theta/2)k_G(x, y) = a(\theta/2)\varphi_1(|x - y|/ \min\{\delta_1(x), \delta_1(y)\}) \leq a(\theta/2)\varphi_1(|x - y|/ \min\{\delta_2(x), \delta_2(y)\}),
\]

where the last inequality holds because $\delta_1 \geq \delta_2$. This gives that

\begin{equation}
(3.18) \quad k_{G\setminus\{z_0\}}(x, y) \leq \varphi_3(|x - y|/ \min\{\delta_2(x), \delta_2(y)\})
\end{equation}

with $\varphi_3(t) = a(\theta/2)\varphi_1(t)$.

**Case III:** $x \in B^n(z_0, \theta_\delta_1(z_0)/2) \setminus \{z_0\}$ and $y \in G \setminus B^n(z_0, \theta_\delta_1(z_0))$.

There exists a quasihyperbolic geodesic joining $x$ and $y$ that intersects the boundary of $B^n(z_0, \theta_\delta_1(z_0)/2)$. Let an intersecting point be $m$. Along this geodesic we have the following equality

\begin{equation}
(3.19) \quad k_{G\setminus\{z_0\}}(x, y) = k_{G\setminus\{z_0\}}(x, m) + k_{G\setminus\{z_0\}}(m, y).
\end{equation}

Now, **Case I** and **Case II** respectively give

\[
k_{G\setminus\{z_0\}}(x, m) \leq \varphi_2(|x - m|/ \min\{\delta_2(x), \delta_2(m)\})
\]

and

\[
k_{G\setminus\{z_0\}}(m, y) \leq \varphi_3(|m - y|/ \min\{\delta_2(m), \delta_2(y)\}).
\]

We note that $\max\{|x - m|, |m - y|\} \leq 3|x - y|$ and $\delta_2(m) \geq \delta_2(x)$. Also, $\varphi_2$ and $\varphi_3$ being monotone, from (3.19) we obtain

\[
k_{G\setminus\{z_0\}}(x, y) \leq \varphi_2(3|x - y|/ \min\{\delta_2(x), \delta_2(y)\}) + \varphi_3(3|x - y|/ \min\{\delta_2(x), \delta_2(y)\}) \leq 2 \max\{\varphi_2(3|x - y|/ \min\{\delta_2(x), \delta_2(y)\}), \varphi_3(3|x - y|/ \min\{\delta_2(x), \delta_2(y)\})\} = \varphi_4(|x - y|/ \min\{\delta_2(x), \delta_2(y)\})
\]

where $\varphi_4(t) = 2 \max\{\varphi_2(3t), \varphi_3(3t)\}$.

We verified all the cases, and hence our conclusion with $\varphi = \varphi_4$ holds. \hfill \Box

**Corollary 3.20.** Suppose that $(z_i)_{i=1}^m$ is a finite non-empty sequence of points in a domain $G \subset \mathbb{R}^n$. If $G$ is $\varphi_0$-uniform, then $G \setminus \{z_1, z_2, \ldots, z_m\}$ is $\varphi$-uniform for some $\varphi$ depending on $\varphi_0$ only.

**Proof.** As a consequence of Theorem 3.16 proof follows by induction on $m$. Indeed, we obtain

\[
\varphi(t) = 2^m a(\theta/2)^{m-1} \max\{\pi(1 + 3t)/ \log 3, a(\theta/2)\varphi_0(3t)\},
\]

where $\theta = \min\{d(z_i, \partial G), |z_i - z_j|\}$ with $i \neq j$ and $i, j = 1, 2, \ldots, m$. \hfill \Box

The following property of uniform domains, first noticed by Väisälä (see \cite[Theorem 5.4]{17}) in a different approach, is a straightforward consequence of Theorem 3.16. For convenient reference we record the following Bernoulli inequality:

\begin{equation}
(3.21) \quad \log(1 + at) \leq a \log(1 + t); \quad a \geq 1, \ t \geq 0.
\end{equation}
Corollary 3.22. Suppose that \((z_i)_{i=1}^m\) is a finite non-empty sequence of points in a uniform domain \(G \subsetneq \mathbb{R}^n\). Then \(G' = G \setminus \{z_1, z_2, \ldots, z_m\}\) also is uniform. More precisely if (3.1) holds for \(G\) with some constant \(c\), then it also holds for \(G'\) with a constant \(c'\) depending only on \(c\).

Proof. It is enough to consider the domain \(G \setminus \{z_1\}\) when (3.1) holds for \(G\) with some constant \(c\). We follow the proof of Theorem 3.16. Our aim is to find a constant \(c'\) such that

\[ k_{G \setminus \{z_1\}}(x, y) \leq c' j_{G \setminus \{z_1\}}(x, y). \]

\(\varphi\) Let \(n\)(3.24)\(\delta\). From Case I we have \(c' = \pi/\log 3\). Since (3.1) holds for \(G\) with the constant \(c\), from Case II we get \(c' = c a(\theta/2)\).

By Case I and Case II, we see that

\[ k_{G \setminus \{z_1\}}(x, y) = k_{G \setminus \{z_1\}}(x, m) + k_{G \setminus \{z_1\}}(m, y) \]

\[ \leq \max\{\pi/\log 3, c a(\theta/2)\} \left[j_{G \setminus \{z_1\}}(x, m) + j_{G \setminus \{z_1\}}(m, y)\right] \]

\[ \leq c' j_{G \setminus \{z_1\}}(x, y), \]

where \(c' = 6 \max\{\pi/\log 3, c a(\theta/2)\}\). Note that the last inequality follows by similar reasoning as in Case III and by the Bernoulli inequality (3.21).

Inductively, we notice that uniformity constant for \(G'\) is

\[ 6^m a(\theta/2)^{m-1} \max\{\pi/\log 3, c a(\theta/2)\} = 6^m a(\theta/2)^m c, \]

where \(a(\theta)\) is defined in Lemma 2.12.

Theorem 3.23. Let \(\theta \in (0, 1)\). Assume that \(G \subsetneq \mathbb{R}^n\) is \(\varphi_1\)-uniform and \(z_0 \in G\). If \(E \subset B^n(z_0, \theta d(z_0, \partial G)/5)\) is a non-empty closed set such that \(\mathbb{R}^n \setminus E\) is \(\varphi_2\)-uniform, then \(G \setminus E\) is \(\varphi\)-uniform for \(\varphi\) depending on \(\varphi_1\) and \(\varphi_2\).

Proof. In this proof we denote by \(\delta_1\), \(\delta_2\) and \(\delta_3\) the Euclidean distances to the boundary of \(G\), \(G \setminus E\) and \(\mathbb{R}^n \setminus E\) respectively. Let \(\theta \in (0, 1)\) and \(x, y \in G \setminus E\) be arbitrary. We undertake the proof into several cases.

Case A: \(x, y \in G \setminus B^n(z_0, \theta \delta_1(z_0)/4)\).

Denote \(G'\) as in Lemma 2.14 but with \(\alpha = 1/5\). Then \(\varphi_1\)-uniformity of \(G\) gives

\[ k_{G \setminus E}(x, y) \leq k_{G'}(x, y) \]

\[ \leq a(1/5, \theta/4) k_G(x, y) \]

\[ \leq a(1/5, \theta/4) \varphi_1(|x - y|/\min\{\delta_1(x), \delta_1(y)\}) \]

\[ \leq a(1/5, \theta/4) \varphi_1(|x - y|/\min\{\delta_2(x), \delta_2(y)\}), \]

where the first inequality holds by the monotonicity property, second inequality follows by Lemma 2.14 and last follows trivially.

Case B: \(x, y \in B^n(z_0, \theta \delta_1(z_0)/2) \setminus E\).

If \(x, y \in B^n(z_0, \theta \delta_1(z_0)/4) \setminus E\), then the quasihyperbolic geodesic \(J := J_{G \setminus E}[x, y]\) may entirely lie in \(B^n(z_0, \theta \delta_1(z_0)/3)\) or may intersect the sphere \(S^{n-1}(z_0, \theta \delta_1(z_0)/3)\). This means that the shape of \(J\) will depend on the shape of \(E\). So, we divide the case into two parts.

Case B1: \(J \cap S^{n-1}(z_0, \delta_1(z_0)/3) = \emptyset\).

Since \(\mathbb{R}^n \setminus E\) is \(\varphi_2\)-uniform and \(\delta_2(\varphi) = \delta_3(\varphi)\) for any \(\varphi \in J\) we have

\[ k_{G \setminus E}(x, y) = k_{\mathbb{R}^n \setminus E}(x, y) \leq \varphi_2(|x - y|/\min\{\delta_2(x), \delta_2(y)\}) \]

\[ \leq \varphi_2(|x - y|/\min\{\delta_2(x), \delta_2(y)\}). \]

Case B2: \(J \cap S^{n-1}(z_0, \delta_1(z_0)/3) \neq \emptyset\).
To get a conclusion like in (3.24) it is enough to show that
\[(3.25) \quad k_{G \setminus E}(x, y) \leq C k_{\mathbb{R}^n \setminus E}(x, y)\]
for some constant $C > 0$.

**Case B2a:** $x, y \in B^n(z_0, \theta \delta_1(z_0)/4) \setminus E$ and $k_{\mathbb{R}^n \setminus E}(x, y) > \log \frac{3}{2}$.

Let $x_1$ be the first intersection point of $J$ with $S^{n-1}(z_0, \delta_1(z_0)/3)$ when we traverse along $J$ from $x$ to $y$. Similarly, we let $x_2$ when we traverse from $y$ to $x$ (see Figure 4). In a similar fashion, let us denote $y_1$ and $y_2$ the first intersection points of $J_{\mathbb{R}^n \setminus E}[x, y]$ with $S^{n-1}(z_0, \delta_1(z_0)/3)$ along both the directions respectively. We observe that $\delta_2(z) = \delta_3(z)$ for all $z \in J[x, x_1]$, where $J[x, x_1]$ denotes part of $J$ from $x$ to $x_1$. Hence, along the geodesic $J$ we have
\[(3.26) \quad k_{G \setminus E}(x, y) = k_{\mathbb{R}^n \setminus E}(x, x_1) + k_{G \setminus E}(x_1, x_2) + k_{\mathbb{R}^n \setminus E}(x_2, y).\]

Now, by the triangle inequality we see that
\[(3.27) \quad k_{\mathbb{R}^n \setminus E}(x, x_1) \leq k_{\mathbb{R}^n \setminus E}(x, y_1) + k_{\mathbb{R}^n \setminus E}(y_1, x_1).\]

By comparing the quasihyperbolic distance along the circular path joining $y_1$ and $x_1$, we obtain
\[(3.28) \quad k_{\mathbb{R}^n \setminus E}(y_1, x_1) \leq 4\pi.\]

On the other hand, we see that
\[(3.29) \quad k_{\mathbb{R}^n \setminus E}(x, y_1) \geq j_{\mathbb{R}^n \setminus E}(x, y_1) \geq \log \frac{8}{7},\]
because $|x - y_1| \geq \theta \delta_1(z_0)/12$ and $\delta_3(y_1) \leq 7\theta \delta_1(z_0)/12$. Combining (3.28) and (3.29), from (3.27) we obtain
\[k_{\mathbb{R}^n \setminus E}(x, x_1) \leq \left(1 + \frac{4\pi}{\log \frac{8}{7}}\right) k_{\mathbb{R}^n \setminus E}(x, y_1).\]
Similarly we get

\[ k_{\mathbb{R}^n \setminus E}(x_2, y) \leq \left( 1 + \frac{4\pi}{\log \frac{\delta}{\pi}} \right) k_{\mathbb{R}^n \setminus E}(y_2, y). \]

A similar argument as in (3.28) and the last two inequalities together with (3.26) give

\[ k_{G \setminus E}(x, y) \leq 4\pi + \left( 1 + \frac{4\pi}{\log \frac{\delta}{\pi}} \right) k_{\mathbb{R}^n \setminus E}(x, y). \]

By our assumption in this case, (3.25) follows from the last inequality with \( C = 1 + (4\pi/\log \delta) + (4\pi/\log \frac{\delta}{2}) \).

Case B2b: \( x, y \in B^n(z_0, \theta \delta_1(z_0)/4) \setminus E \) and \( k_{\mathbb{R}^n \setminus E}(x, y) \leq \log \frac{3}{2} \).

The well-known inequality \( j_{\mathbb{R}^n \setminus E}(x, y) \leq k_{\mathbb{R}^n \setminus E}(x, y) \) reduces to

\[ (3.30) \quad R := \frac{1}{2} \min\{\delta_3(x), \delta_3(y)\} \geq |x - y|. \]

Without loss of generality we assume that \( \min\{\delta_3(x), \delta_3(y)\} = \delta_3(x) \). Then there exists a point \( x_0 \in S^{n-1}(x, 2R) \cap \partial E \) such that \( \delta_3(x) = |x - x_0| = 2R \). For the proof of (3.25), we proceed as follows

\[
k_{G \setminus E}(x, y) \leq k_{B^n(x, 2R)}(x, y) \\
\leq 2 j_{B^n(x, 2R)}(x, y) \\
= 2 \log \left( 1 + \frac{|x - y|}{\delta_3(x) - |x - y|} \right) \\
\leq 2 \log \left( 1 + \frac{2|x - y|}{\delta_3(x)} \right) \\
\leq 4 \log \left( 1 + \frac{|x - y|}{\delta_3(x)} \right) \\
= 4 j_{\mathbb{R}^n \setminus \{x_0\}}(x, y) \\
\leq 4 j_{\mathbb{R}^n \setminus E}(x, y),
\]

where the second, third and fourth inequalities follow from [1] Lemma 7.56], (3.30) and (3.21) respectively. Hence we proved Case B when \( x, y \in B^n(z_0, \theta \delta_1(z_0)/4) \setminus E \).

If \( x \in B^n(z_0, \theta \delta_1(z_0)/4) \setminus E \) and \( y \in B^n(z_0, \theta \delta_1(z_0)/3) \setminus \overline{B^n(z_0, \theta \delta_1(z_0)/4)} \), by considering a sphere \( S^{n-1}(z_0, \theta \delta_1(z_0)r) \) with \( r \in (1/4, 1/3) \) we proceed like before.

If \( x \in B^n(z_0, \theta \delta_1(z_0)/4) \setminus E \) and \( y \in B^n(z_0, \theta \delta_1(z_0)/2) \setminus \overline{B^n(z_0, \theta \delta_1(z_0)/3)} \), then the geodesic \( J_{G \setminus E}[x, y] \) will intersect \( S^{n-1}(z_0, \theta \delta_1(z_0)/3) \). Let \( m \) be the first intersection point when we traverse along the geodesic from \( x \) to \( y \). Then along the geodesic we have

\[
k_{G \setminus E}(x, y) = k_{G \setminus E}(x, m) + k_{G \setminus E}(m, y) \\
\leq k_{\mathbb{R}^n \setminus E}(x, m) + a(1/4, \theta/3) \varphi_1(|m - y|/\min\{\delta_2(m), \delta_2(y)\}) \\
\leq \varphi_2(|x - m|/\min\{\delta_3(x), \delta_3(m)\}) + a(1/4, \theta/3) \varphi_1(|m - y|/\min\{\delta_2(m), \delta_2(y)\}) \\
\leq \varphi_2(4|x - y|/\min\{\delta_2(x), \delta_2(y)\}) + a(1/4, \theta/3) \varphi_1(10|x - y|/\min\{\delta_2(x), \delta_2(y)\}),
\]

where the first and second inequalities follow by Case A and the assumption on \( E \) respectively. Thus, we conclude that if \( x, y \in B^n(z_0, \theta \delta_1(z_0)/2) \setminus E \), then

\[ k_{G \setminus E}(x, y) \leq 2a(1/4, \theta/3) \varphi_3(|x - y|/\min\{\delta_2(x), \delta_2(y)\}), \]
where \( \varphi_3(t) = \max\{\varphi_2(10t), \varphi_1(10t)\} \).

Case C: \( x \in B^n(z_0, \theta \delta_1(z_0)/4) \setminus E \) and \( y \in G \setminus B^n(z_0, \theta \delta_1(z_0)/2) \).

Let \( p \in J_{G \setminus E}[x, y] \cap S^{n-1}(z_0, \theta \delta_1(z_0)/4) \). Then we see that

\[
k_{G \setminus E}(x, y) = k_{G \setminus E}(x, p) + k_{G \setminus E}(p, y)
\leq 2a(1/4, \theta/3)\varphi_3(|x - p|/\min\{\delta_2(x), \delta_2(p)\})
+ a(1/5, \theta/4)\varphi_1(|p - y|/\min\{\delta_2(p), \delta_2(y)\})
\leq 2a(1/4, \theta/3)\varphi_3(|x - p|/\min\{\delta_2(x), \delta_2(y)\})
+ a(1/5, \theta/4)\varphi_1(|p - y|/\min\{\delta_2(x), \delta_2(y)\}),
\]

where the first inequality holds by Case B and Case A, and last holds by a similar argument as above (or as in the proof of Case III in Theorem 3.16). It is easy to see that

\[
\max\{|x - p|, |p - y|\} \leq 3|x - y|.
\]

In the same way, as in Theorem 3.16, the monotonicity property of \( \varphi_3 \) and \( \varphi_1 \) gives

\[
k_{G \setminus E}(x, y) \leq 4a(1/4, \theta/3) \max\{\varphi_2(30|x - y|/\min\{\delta_2(x), \delta_2(y)\}),
\varphi_1(30|x - y|/\min\{\delta_2(x), \delta_2(y)\})\}.
\]

By combining all the above cases, a simple computation concludes that the domain \( G \setminus E \) is \( \varphi \)-uniform for \( \varphi(t) = 4a(1/4, \theta/3) \max\{\varphi_1(30t), \varphi_2(30t)\} \), where \( a(1/4, \theta/3) \) is obtained from Lemma 2.14. \( \Box \)

**Corollary 3.31.** Fix \( \theta \in (0, 1) \). Assume that \( G \varsubsetneq \mathbb{R}^n \) is \( \varphi_0 \)-uniform and \((z_i)_{i=1}^m\) are non-empty finite sequence of points in \( G \) such that \( \delta(z_1) = \min\{\delta(z_i)\}_{i=1}^m \). Denote

\[
d := \min_{i \neq j}\{|z_i - z_j|/2\} \text{ and } \delta := \min\{\delta(z_i), d\}.
\]

For all \( i = 1, 2, \ldots, m \) if \( E_i \) are non-empty closed sets in \( B^n(z_i, \theta \delta/5) \) such that \( \mathbb{R}^n \setminus \bigcup_{i=1}^m E_i \) is \( \varphi_1 \)-uniform for some \( \varphi_1 \), then the domain \( G \setminus \bigcup_{i=1}^m E_i \) is \( \varphi \)-uniform for some \( \varphi \).

**Proof.** As a consequence of Theorem 3.23 the proof follows by induction. \( \Box \)

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