Precise asymptotics for Fisher–KPP fronts

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Abstract
We consider the one-dimensional Fisher–KPP equation with step-like initial data. Nolen et al showed that the solution $u$ converges at long time to a travelling wave $\phi$ at a position $\tilde{\sigma}(t) = 2t - (3/2) \log t + \alpha_0 - 3\sqrt{\pi}/\sqrt{t}$, with error $O(t^{\gamma-1})$ for any $\gamma > 0$. With their methods, we find a refined shift $\sigma(t) = \tilde{\sigma}(t) + \mu^*(\log t)/t + \alpha_1/t$ such that in a frame moving with $\sigma$, the solution $u$ satisfies $u(t,x) = \phi(x) + \psi(x)/t + O(t^{\gamma-3/2})$ for a certain profile $\psi$ independent of the initial data. The coefficient $\alpha_1$ depends on the initial data, but $\mu^* = 9(5 - 6\log 2)/8$ is universal, and agrees with a finding of Berestycki et al. Furthermore, we predict the asymptotic forms of $\sigma$ and $u$ to arbitrarily high order.

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1. Introduction
We study solutions to the Fisher–KPP equation:

$$u_t = u_{xx} + u(1-u) \quad \text{with } (t,x) \in \mathbb{R}_+ \times \mathbb{R}. \quad (1.1)$$

For initial data we take $u(0,x) = u_0(x)$ for $x \in \mathbb{R}$, where $u_0$ is a compact perturbation of a step function. That is, there exists $L \geq 0$ such that $u_0(x) = 1$ when $x \leq -L$ and $u_0(x) = 0$ when $x \geq L$. We further assume $0 \leq u_0 \leq 1$ on $\mathbb{R}$, so that $0 < u < 1$ on $\mathbb{R}_+ \times \mathbb{R}$. We study the long-time behaviour of $u$.

This matter has a rich history, beginning with Fisher’s introduction of equation (1.1) in [6]. Fisher studied travelling front solutions to (1.1), which have the form $u(t,x) = \phi_c(x-ct)$, where $\phi_c: \mathbb{R} \to (0,1)$ satisfies
\[ -c\phi''_c = \phi''_c + \phi_c - \phi_c^2, \quad \phi_c(-\infty) = 1, \quad \phi_c(+\infty) = 0. \]

Such solutions model steady-speed invasions of the unstable state 0 by the stable state 1. Fisher used heuristic and numerical arguments to identify the minimal speed \( c_\ast = 2 \) of traveling fronts. At the minimal speed there exists a front \( \phi_{c_\ast} \) unique up to translation. We let \( \phi \) denote the unique minimal-speed front satisfying

\[ \phi(s) = A_0 se^{-s} + \mathcal{O}(e^{-(1+\omega)s}) \quad \text{as} \quad s \to +\infty \quad (1.2) \]

for universal constants \( A_0, \omega > 0 \).

Kolmogorov et al published their groundbreaking work [9] in the same year as Fisher. The authors showed that if \( u_0 \) is a step function, the solution \( u \) converges to the minimal-speed front \( \phi \), in the sense that

\[ \lim_{t \to \infty} u(t, x + \sigma(t)) = \phi(x) \quad (1.3) \]

uniformly in \( x \), for some function \( \sigma \) satisfying \( \sigma(t) = 2t + \mathcal{O}(t) \) as \( t \to \infty \). The precise nature of this convergence has since been well-studied, and is the subject of this work.

In a striking series of papers [3, 4], Bramson proved that \( \sigma(t) - 2t \) is not asymptotically constant. Rather:

**Theorem 1 (Bramson).** There exists \( \alpha_0 \in \mathbb{R} \) such that

\[ \sigma(t) = 2t - \frac{3}{2} \log t + \alpha_0 + \mathcal{O}(1) \quad \text{as} \quad t \to +\infty. \quad (1.4) \]

Significantly, the constant shift \( \alpha_0 \) depends on the initial data, but the coefficient of the logarithmic delay does not. In this sense the logarithmic term is ‘universal’. Bramson used elaborate probabilistic methods to prove theorem 1, drawing on intimate connections between the Fisher–KPP equation (1.1) and branching Brownian motion. Soon after, Lau [10] provided a different proof of the results of [3, 4] for more general nonlinearities, using the intersection properties of solutions to parabolic Cauchy problems.

Recent years have seen substantial progress through purely PDE methods. In [7], Hamel et al related the Cauchy problem for (1.1) to a moving linear Dirichlet boundary problem, and established

\[ \sigma(t) = 2t - \frac{3}{2} \log t + \mathcal{O}(1). \]

In a subsequent work [12], Nolen et al used the same approach to recover (1.4) for the initial data of the form studied here: compact perturbations of a step function.

To further analyse \( \sigma \), we consider the rate of convergence in (1.3). We wish to find \( \mathcal{O}(1) \) adjustments to \( \sigma \) that accelerate the convergence of \( u(t, x + \sigma(t)) \) to \( \phi(x) \). In [5], Ebert and van Saarloos performed formal calculations suggesting:

\[ \sigma(t) = 2t - \frac{3}{2} \log t + \alpha_0 - \frac{3\sqrt{\pi}}{\sqrt{t}} + \mathcal{O}\left( \frac{1}{\sqrt{t}} \right). \]

That is, they predicted that for such \( \sigma \),

\[ u(t, x + \sigma(t)) = \phi(x) + \sigma(t^{-1/2}) \quad \text{as} \quad t \to \infty \quad (1.5) \]

locally uniformly in \( x \). Notably, the coefficient of the \( t^{-1/2} \) correction is again universal, in that it is independent of the initial data. This is particularly striking given that a larger term, \( \alpha_0 \), does depend on \( u_0 \).
In [13], Nolen et al proved the $t^{-1/2}$ refinement derived by Ebert and van Saarloos. Precisely, the authors constructed an approximate travelling solution $\tilde{U}_{app}$ incorporating both the travelling wave $\phi$ and the linear behaviour of the ‘pulled front’ at $x \gg 2t$. Let

$$\tilde{\sigma}(t) := 2t - \frac{3}{2} \log t + \alpha_0 - \frac{3\sqrt{\pi}}{\sqrt{t}}$$

denote their front shift. Then:

**Theorem 2 (Nolen et al).** There exists $\alpha_0 \in \mathbb{R}$ depending on the initial data $u_0$ such that for any $\gamma > 0$ there exists $C_\gamma > 0$ also depending on $u_0$ such that

$$|u(t, x + \tilde{\sigma}(t)) - \tilde{U}_{app}(t, x)| \leq \frac{C_\gamma (1 + |x|) e^{-x}}{t^{1-\gamma}} \text{ for all } (t, x) \in [3, \infty) \times \mathbb{R}.$$

The approximate travelling solution satisfies $\tilde{U}_{app}(t, x) = \phi(x) + \mathcal{O}(t^{-1})$ as $t \to +\infty$ locally uniformly in $x$. Hence theorem 2 proves (1.5). In [8], Henderson established the same $t^{-1/2}$ correction for a related moving-boundary problem.

In recent works [1, 2], Berestycki et al have discovered remarkable formulae relating initial data and front position in several evolution equations, including (1.1). Their formulae predict a universal $\frac{\log t}{t}$ correction of the form:

$$\sigma(t) = 2t - \frac{3}{2} \log t + \alpha_0 - \frac{3\sqrt{\pi}}{\sqrt{t}} + \frac{9}{8} (5 - 6 \log 2) \frac{\log t}{t} + \mathcal{O} \left( \frac{1}{t} \right). \tag{1.6a}$$

For concision, we let $\mu_* = \frac{9}{8} (5 - 6 \log 2)$ denote this universal coefficient. In the present work, we prove (1.6a). Furthermore, we characterize $u$ to order $t^{-1}$, and find that it cannot be represented as a simple shift of the travelling front $\phi$.

Our main theorem makes these observations precise. For our front shift, we include the $\frac{\log t}{t}$ correction predicted in [1] and an undetermined order $t^{-1}$ term:

$$\sigma(t) := 2t + \alpha_0 - \frac{3}{2} \log t - \frac{3\sqrt{\pi}}{\sqrt{t}} + \mu_* \frac{\log t}{t} + \frac{\alpha_1}{t}. \tag{1.6b}$$

The constants $\alpha_0$ and $\alpha_1$ will depend on the initial data $u_0$. There is a second correction at order $t^{-1}$, however. For any $\gamma > 0$, we construct an approximate travelling solution $U_{app}$ satisfying

$$U_{app}(t, x) = \phi(x) + \frac{1}{t} \psi(x) + \mathcal{O} \left( t^{-\frac{3}{2}} \right) \text{ as } t \to \infty$$

locally uniformly in $x$. The profile $\psi$ solves

$$\psi'' + 2\psi' + (1 - 2\phi) \psi = \frac{3}{2} \phi',$$

and is independent of $u_0$. This $\frac{1}{t} \log t$ term is a consequence of the $\frac{1}{2} \log t$ delay in the front position.

We will show:

**Theorem 3.** There exist $\alpha_0$ and $\alpha_1$ in $\mathbb{R}$ depending on the initial data $u_0$ such that the following holds. For any $\gamma > 0$, there exists $C_\gamma > 0$ also depending on $u_0$ such that for all $(t, x) \in [3, \infty) \times \mathbb{R}$,

$$|u(t, x + \sigma(t)) - U_{app}(t, x)| \leq \frac{C_\gamma (1 + |x|) e^{-x}}{t^{1-\gamma}}, \tag{1.7}$$

with $\sigma$ defined in (1.6b).
Remark 1. Because $\alpha_1$ depends on $u_0$, we find that the asymptotic behaviour of $u$ at order $\frac{1}{t}$ is not universal.

Remark 2. The $\frac{1}{t}$ correction $\psi$ varies in space, so at this order and lower, the asymptotics of $u$ cannot be described as simple shifts of the travelling front $\phi$.

Our main theorem implies:

Corollary 4. For each $s \in (0, 1)$, let $\sigma_s(t) := \max \{x \in \mathbb{R}; u(t, x) = s\}$ denote the leading edge of $u$ at value $s$. Then

$$\sigma_s(t) = 2t - \frac{3}{2} \log t + \alpha_0 + \phi^{-1}(s) - \frac{3\sqrt{\pi}}{\sqrt{t}} + \mu_* \log t + O\left(\frac{1}{t}\right).$$

The proofs of these results closely follow the methods of Nolen et al in [12, 13].

Theorem 3 raises the question of the general behaviour of $\sigma$ and $u$. We informally argue the existence of an asymptotic series

$$\sigma(t) \sim 2t - \frac{3}{2} \log t + \sum_{a \geq 0} \sum_{0 \leq b \leq \lfloor \frac{a}{2} \rfloor} \sigma_{a,b} t^{-a/2} \log^b t,$$

such that

$$u(t, x + \sigma(t)) \sim \phi(x) + \sum_{a \geq 2} \sum_{0 \leq b \leq \lfloor \frac{a}{2} \rfloor - 1} t^{-a/2} \log^b t u_{a,b}(x).$$

Furthermore, for any fixed value of $a$, the corresponding terms in $u$ and $\sigma$ with maximal degree in $\log t$ are independent of $u_0$. In this sense, the ‘leading logarithmic’ terms are universal.

Our paper is structured as follows. We outline the proof of theorem 3 in section 2, and intuitively motivate the result and methods. In section 3, we perform a matched asymptotic expansion for $U_{app}$, and derive an implicit equation for the coefficient $\mu_*$. In section 4, we explicitly compute $\mu_*$, to show agreement with [1, 2]. We extend our asymptotic analysis to all orders in section 5, and thereby describe the Fisher–KPP front shift to arbitrarily high order. In section 6, we use the approach of [13] to prove theorem 3. We close with an appendix detailing an ODE lemma.

2. Proof outline

Recall our main equation, which we begin from $t = 1$ for convenience:

$$\begin{cases}
  \frac{\partial u}{\partial t} = u_{xx} + u - u^2 & \text{on } (1, \infty) \times \mathbb{R}, \\
  u(1, \cdot) = u_0, & \text{on } \mathbb{R}.
\end{cases}$$

As in the introduction, we assume the initial data $0 \leq u_0 \leq 1$ are step-like. We expect $u$ to converge to a travelling front at position $\sigma$ of the form

$$\sigma(t) = 2t - \frac{3}{2} \log t + \alpha_0 - \frac{3\sqrt{\pi}}{\sqrt{t}} + \mu_1 \log t + \frac{\alpha_1}{t}.$$ 

It is therefore natural to change coordinates to the moving frame given by

$$x_{\text{new}} = x_{\text{old}} - \sigma(t).$$
Now, $v$ is a ‘pulled front’, meaning its dynamics are determined by its behaviour at $x \gg 1$. In this regime, $v$ is very close to $\phi$, which decays exponentially as $x \to \infty$. To detect fine effects in the tail, it is helpful to remove this exponential decay. With this motivation, we study

$$v(t, x) := A_0^{-1} e^{t} u(t, x),$$

where $A_0$ is the prefactor in (1.2).

Incorporating the shift and the exponential multiplier, (1.1) becomes

$$v_t - v_{xx} - \left(\frac{3}{2t} - \frac{3\sqrt{\tau}}{2t^2} + \mu \frac{\log t}{t^2} + \frac{\alpha_1 - \mu}{t^2}\right)(v - v_0) + A_0 e^{-\tau} v^2 = 0 \quad \text{on } (1, \infty) \times \mathbb{R}. \quad (2.1)$$

In particular, when $t$ and $x$ are large, (2.1) resembles the heat equation $v_t - v_{xx} = 0$. Crucially, at any fixed $\eta > 0$, the prefactor $e^{t} e^{-\eta t^{1/2}}$ of the nonlinear term decays rapidly. Thus on $\mathbb{R}_+$, the nonlinear nature of the problem only manifests in a boundary layer near $\eta = 0$. Furthermore, since $v \leq e^{\eta t^{1/2}}$, we have $v \leq e^{\eta t^{1/2}}$. Hence when $\tau < 0$, $v$ approaches 0 rapidly. We therefore expect $v$ to approximately solve a linear Dirichlet boundary value problem on $\mathbb{R}_+$.

To make these heuristics precise, we construct an approximate solution $V_{\text{app}}$ through a matched pair of asymptotic expansions. When $x \sim 1$, we solve the nonlinear equation (2.1) by expanding in successively smaller orders of $t$. For $x \sim \sqrt{t}$, we solve the linear part of (2.2) on $\mathbb{R}_+$ with Dirichlet boundary data, again expanding in orders of $t = e^\tau$. To link the inner expansion at $x \sim 1$ with the outer expansion at $x \sim \sqrt{t}$, we match them at an intermediate scale $x = \epsilon t^{2}$. In this matching, the inner expansion $V^-$ sets additional boundary conditions on the outer expansion $V^+$, through the Neumann data $\partial_x V^+|_{x=1}$. To solve the resulting overdetermined boundary problem, we use degrees of freedom in the shift $\sigma$. The universal coefficients of $\sigma$ are uniquely chosen to admit a solution $V^+$ satisfying the boundary conditions prescribed by $V^-$.

This method determines the universal terms $-\frac{3}{2} \log t - \frac{3\sqrt{\tau}}{t}$, and $\frac{\alpha_1 - \mu}{t}$, which depend on the initial data $v_0$. In general, the spectral properties of the Dirichlet problem make the matched expansion insensitive to shift terms of order $t^{-a}$ with $a \in \mathbb{Z}_{>0}$. Rather, these terms are chosen to eliminate components of the difference $v - V_{\text{app}}$.

For instance, the principal eigenfunction of the Dirichlet problem on $\mathbb{R}_+$ is $\eta e^{-\eta t^{1/2}}$, and the leading term of $V^+$ will be $e^{\tau/2} \eta e^{-\eta t^{1/2}}$. On the other hand, Nolen et al [12] showed the existence of $q_0 \in \mathbb{R}$ such that $v(t, \eta) \sim q_0 e^{\tau/2} \eta e^{-\eta t^{1/2}}$ when $\tau \gg 1$. By adjusting $\alpha_0$, we can force $q_0 = 1$, so that $V$ and $V^+$ agree to the leading order. In other words, we choose $\alpha_0$ to eliminate the principal component of $v - V_{\text{app}}$. Similarly, $\alpha_1$ will be chosen to kill the component of $v - V_{\text{app}}$ corresponding to the second eigenfunction of the Dirichlet problem.

In summary, we wish to construct an approximate solution $V_{\text{app}}$ to (2.1) which closely models the exact solution $v$. To do so, we perform a matched asymptotic expansion at the scales...
\( x \sim 1 \) and \( x \sim \sqrt{t} \). The universal terms of \( \sigma \) are uniquely chosen to ensure the existence of such an expansion. The remaining terms \( \alpha_0 \) and \( \frac{\log t}{t} \) are then chosen so that \( V_{\text{app}} \) and \( v \) agree up to a certain order in the eigenbasis of the linear Dirichlet problem. In all these steps, we closely follow Nolen et al [13], who developed this method to the first order.

3. Matched asymptotics for the approximate solution

As described above, we transform (1.1) by translating to a moving frame and removing the exponential decay of \( u \):

\[
x \mapsto x - \frac{3}{2} \log t + \frac{1}{2} \sqrt{\pi} - \frac{\mu t}{t} - \frac{\alpha_1}{t}, \quad v(t, x) = A_0^{-1} e^t u(t, x).
\]

Here we use an undetermined coefficient \( \mu \in \mathbb{R} \) for the \( \frac{\log t}{t} \) term in the shift. We will show that only the special value \( \mu = \mu_* \) will allow us to approximate \( u \) with \( O(\frac{\log t}{t}) \) accuracy.

We construct asymptotic solutions to (2.1) at the scales \( x \sim 1 \) and \( x \sim \sqrt{t} \). We denote these as \( V^- \) and \( V^+ \) respectively, and match them at the intermediate position \( x = \xi \) to construct \( V_{\text{app}} \). Our choice of \( 0 < \varepsilon \ll 1 \) will depend on the parameter \( \gamma \) in theorem 3.

3.1. The inner approximation

We first take \( x \sim 1 \), and expand (2.1) in orders of \( t \). Since we expect \( O(t^{-3/2}) \) error in theorem 3, we may discard terms of this order and lower. Two terms in (2.1) remain, of order 1 and \( t^{-1} \). We thus use the ansatz

\[
V^-(t, x) = V_0^-(x) + t^{-1} V_1^-(x).
\]

Considering only order 1 terms, we find the equation for \( V_0^- \):

\[
-(V_0^-)'' + A_0 e^{-x} (V_0^-)^2 = 0.
\]

The travelling front \( \phi \) provides a natural solution:

\[
V_0^-(x) = A_0^{-1} e^x \phi(x).
\]

By (1.2) and the standard theory of travelling fronts, \( V_0^- \) satisfies

\[
V_0^-(x) = x + O(e^{-\omega x}) \quad \text{and} \quad (V_0^-)'(x) = 1 + O(e^{-\omega x}) \quad \text{as } x \rightarrow +\infty
\]

for some \( \omega \in (0, 1) \). In the other direction,

\[
V_0^+(x) = A_0^+ e^x + O(e^{(1+\omega)x}) \quad \text{and} \quad (V_0^+)'(x) = A_0^+ e^x + O(e^{(1+\omega)x}) \quad \text{as } x \rightarrow -\infty.
\]

We now collect the terms of order \( t^{-1} \) in (2.1):

\[
-(V_1^-)'' + 2A_0 e^{-x} V_1^- = \frac{3}{2} [V_0^- - (V_0^-)'].
\]

From the asymptotics of \( V_0^- \), (3.1) is an exponentially small perturbation of \( -(V_1^-)'' = \frac{3}{2} (x - 1) \) on \( \mathbb{R}^+ \). We therefore expect

\[
V_1^-(x) = -\frac{1}{4} x^3 + \frac{3}{4} x^2 + C_1^- x + C_0^- + O(e^{-\omega x/2}) \quad \text{as } x \rightarrow \infty,
\]

for some \( C_1^-, C_0^- \in \mathbb{R} \).
To uniquely specify $V_1^-$, we must impose boundary conditions. One condition is straightforward: $V_1^-$ must be a perturbation of $V_0^-$, so it must decay as $x \to -\infty$. Furthermore, we shall find that an accurate matching between the inner and outer approximations requires $C_0 = 0$ in (3.2). In the appendix, we prove:

**Lemma 5.** There exist $C_1^+ \in \mathbb{R}$ and a solution $V_1^+$ to (3.1) satisfying

$$V_1^+(x) = -\frac{1}{4}x^3 + \frac{3}{4}x^2 + C_1^+ x + \mathcal{O}(e^{-\omega x/2})$$

and

$$(V_1^+)'(x) = -\frac{3}{4}x^2 + \frac{3}{2}x + C_1^+ + \mathcal{O}(e^{-\omega x/2})$$

as $x \to +\infty$ and $(V_1^-)' = \mathcal{O}(e^x)$ as $x \to -\infty$.

For the remainder of the paper, $V_1^-$ denotes this solution.

Finally, we note that $V_1^-$ will be spatially shifted by a time-dependent quantity $\zeta(t)$ to ensure the continuity of $V_{\text{app}}$ at the matching point $x = r^t$. We defer this technicality to section 6.

### 3.2. The outer approximation

The outer layer $V^+$ requires a more elaborate analysis, and involves several more terms. To emphasize the diffusive nature of the problem, we switch to the self-similar variables

$$\tau := \log t, \quad \eta := \frac{x}{\sqrt{t}}.$$

Recall that in these variables, $\tau$ satisfies (2.2). As noted in section 2, we will neglect the nonlinear term $A_0 e^{\tau} e^{-\eta e^{-\tau/2}} \partial^2 \tau$ on $\mathbb{R}_+$. Furthermore, $\partial \tau$ decays rapidly on $\mathbb{R}_-$, so we approximate (2.2) with the linear Dirichlet problem

$$V_\tau - V_{\eta \eta} - \frac{\eta}{2} V_\eta - \left( \frac{3}{2} - \frac{3}{2} e^{-\tau/2} + \mu \tau e^{-\tau} + (\alpha_1 - \mu) e^{-\tau} \right) \left( V - e^{-\tau/2} V_\eta \right) = 0$$

with $V(0, \tau) = 0$ for all $\tau \geq 0$.

Consider $V^+$ near $\eta = 0$, where

$$V^+(\tau, \eta) \sim \partial_\eta V^+(\tau, 0) \eta.$$ We will match this behaviour with $V^-(x) \sim x = e^{\tau/2} \eta$. We therefore require $\partial_\eta V^+(\tau, 0) \sim e^{\tau/2}$.

This motivates our asymptotics for $V^+$: we expand in orders of $\tau$, and assume the leading order is $e^{\tau/2}$. At fixed $x$, we are only interested in behaviour of order $t^{-1}$ or higher. Since $V^+$ satisfies the Dirichlet condition, this corresponds to terms of order $e^{-\tau/2}$ in $V^+$. We therefore neglect all smaller terms in (3.3). Performing this expansion, we find:

$$V^+(\tau, \eta) = e^{\tau/2} V_0^+(\eta) + V_1^+(\eta) + \tau e^{-\tau/2} V_2^+(\eta) + e^{-\tau/2} V_3^+(\eta).$$

We impose the boundary conditions independently on each term, so $V_i^+(0) = V_i^+(\infty) = 0$ for all $i = 0, \ldots, 3$.

Before writing the equations for $V_i^+$, we introduce

$$\mathcal{L} := -\partial_\eta^2 - \frac{\eta}{2} \partial_\eta - 1,$$

a differential operator closely related to (3.3). We are interested in the Dirichlet problem for $\mathcal{L}$ on the half-line. The discrete spectrum of $\mathcal{L}$ is $\mathbb{Z}_{\geq 0}$ without multiplicity. The functions defined by

$$\phi_0(\eta) := e^{-\eta^2/4}, \quad \phi_{k+1} := \phi_k'' \quad \text{for } k \in \mathbb{Z}_{\geq 0}$$

are eigenfunctions of $\mathcal{L}$ with eigenvalues $k^2$.
are eigenfunctions of $\mathcal{L}$ satisfying $\mathcal{L}\phi_k = k\phi_k$. The adjoint operator is given by

$$\mathcal{L}^* = -\partial_n^2 + \frac{\eta}{2}\partial_n - \frac{1}{2}. $$

Its eigenfunctions $\psi_k$ are polynomials; we choose their normalization so that $\langle \phi_i, \psi_j \rangle |_{\mathcal{L}^* (\mathbb{R}_+)} = \delta_{ij}$. We defer a more detailed study of these eigenfunctions to section 4.

Now consider the asymptotic expansion of (3.3). We substitute the ansatz (3.4) in place of $V$, and group terms by order in $n$. The first two terms proceed as in [13]. At order $\epsilon^{\tau/2}$, we find

$$\mathcal{L}V_0^+ = 0. $$

It follows that $V_0^+ = q_0\phi_0$ for some $q_0 \in \mathbb{R}$.

To find $q_0$, we introduce the matching between $V^-$ and $V^+$. We need these two functions to agree to order $t^{-1}$ at $x = t^2$. For the sake of concision, we use the self-similar variables for the matching at $\eta = m(\tau) := e^{(\epsilon-1/2)\tau}$. From the form of $V^- = V_0^- + t^{-1}V_1^-$,

$$V^- (\tau, m(\tau)) = e^{\epsilon\tau} + \left( -\frac{1}{4}e^{3\epsilon\tau} + \frac{3}{4}e^{2\epsilon\tau} + C_1 e^{\epsilon\tau} \right) e^{-\tau} + \mathcal{O} \left( e^{-\omega\epsilon^{\tau/2}} \right). \ (3.5) $$

With its double-exponential decay, the error term is negligible. To compare (3.5) with $V^+(\tau, m(\tau))$, we Taylor-expand $V^+$ in $\eta$, evaluate at $\eta = m(\tau)$, and group the resulting terms by order in $\tau$. To simplify the resulting expression, we compute its terms sequentially. Using the explicit form of $V_0^+$,

$$V^+(\tau, m(\tau)) = q_0 e^{\epsilon\tau} + (V_1^+)'(0) e^{(\epsilon-1/2)\tau} + \mathcal{O}(e^{(3\epsilon-1)\tau}). $$

Comparing this with (3.5), we see that necessarily $q_0 = 1$ and $(V_1^+)'(0) = 0$.

Having determined $V_0^+$, we turn to $V_1^+$. The expansion of (3.3) implies:

$$\left( \mathcal{L} - \frac{1}{2} \right) V_1^+ + \frac{3}{2} (V_1^+)' + \frac{3\sqrt{\pi}}{2} V_0^+ = 0. \ (3.6) $$

This equation has a unique solution, since $\frac{1}{2}$ is not in the spectrum of $\mathcal{L}$. Furthermore, in [13] it is shown that $V_1^+$ satisfies $(V_1^+)'(0) = 0$. Indeed, this condition determines the universal coefficient $3\sqrt{\pi}$ for $\tau^{-\frac{1}{2}}$ in the time shift $\sigma$.

To compute further terms in $V^+(\tau, m(\tau))$, we require the values

$$(V_0^+)^{(')}(0) = -\frac{3}{2}, \ (V_1^+)'(0) = \frac{3}{2}. $$

The latter follows from (3.6) and $V_1^+(0) = 0$. Then:

$$V^+(\tau, m(\tau)) = e^{\epsilon\tau} + \left( -\frac{1}{4}e^{3\epsilon\tau} + \frac{3}{4}e^{2\epsilon\tau} \right) e^{-\tau} + (V_1^+)'(0) e^{(\epsilon-1)\tau} + \mathcal{O}(e^{(\epsilon-1)\tau}). $$

Again comparing with (3.5), we find $(V_2^+)'(0) = 0$.

At order $\epsilon^{\tau/2}$ in (3.3), we have

$$(\mathcal{L} - 1)V_2^+ - \mu V_0^+ = 0. $$

Expanding $V_2^+$ in the eigenbasis of $\mathcal{L}$, we explicitly find $V_2^+ = -\mu \phi_0 + q_2 \phi_1$ for some $q_2 \in \mathbb{R}$. Using the condition derived above,

$$0 = (V_2^+)'(0) = -\mu - \frac{3}{2} q_2. $$
So $q_2 = -\frac{7}{2} \mu$ and

$$V_2^+ = -\mu \left( \phi_0 + \frac{2}{3} \phi_1 \right).$$

Finally, at order $e^{-\tau/2}$ we have

$$(\mathcal{L} - 1) V_3^+ + V_2^+ + \frac{3}{2} (V_1)' + \frac{3 \sqrt{\pi}}{2} V_1^+ - \frac{3 \sqrt{\pi}}{2} (V_0^+)' + (\mu - \alpha_1) V_0^+ = 0.$$  

Using the explicit forms for $V_0^+$ and $V_2^+$, we write this as

$$(\mathcal{L} - 1) V_3^+ = \frac{2}{3} \mu \phi_1 - \frac{3}{2} (V_1^+)' - \frac{3 \sqrt{\pi}}{2} V_1^+ + \frac{3 \sqrt{\pi}}{2} \phi_0' + \alpha_1 \phi_0. \quad (3.7)$$

Now, by the definition of the adjoint eigenfunctions, $\psi_1$ is $L^2(\mathbb{R}^+)$-orthogonal to the range of $\mathcal{L} - 1$. In fact, (3.7) has a solution if and only if $\psi_1$ is orthogonal to the right-hand side—that is, if and only if

$$\langle \frac{2}{3} \mu \phi_1 - \frac{3}{2} (V_1^+)' - \frac{3 \sqrt{\pi}}{2} V_1^+ + \frac{3 \sqrt{\pi}}{2} \phi_0', \psi_1 \rangle_{L^2(\mathbb{R}^+)} = 0. \quad (3.8)$$

Here we use $\langle \phi_0, \psi_1 \rangle = 0$, so the $\alpha_1$-term drops out. This equation determines the unique value $\mu_*$ that permits us to match $V^-$ and $V^+$ with sufficiently high accuracy. We explicitly compute $\mu_*$ in section 4, where we show:

**Lemma 6.** Equation (3.8) implies $\mu_* = \frac{2}{8} (5 - 6 \log 2)$.

This is the value found by Berestycki et al in [1, 2].

Having determined $\mu_*$, at least implicitly, we return to the equation for $V_3^+$. Although we have guaranteed the existence of a solution to (3.7), we do not have uniqueness. Indeed, $\mathcal{L} - 1$ has nullspace spanned by $\phi_1$, so we have only determined $V_3^+$ up to a multiple of $\phi_1$. More precisely, let $V_3^+$ denote a particular solution to (3.7) when $\alpha_1 = 0$. Then a general solution to (3.7) has the form

$$V_3^+ = \tau V_3^+ - \alpha_1 \phi_0 + q_3 \phi_1 \quad (3.9)$$

for some $q_3 \in \mathbb{R}$. For the moment, we leave $q_3$ undetermined. In the proof of theorem 3, we will use this free constant to push the accuracy of (1.7) below $O(\tau^{-1})$. We will see that $q_3$ depends on the initial data $u_0$.

For the moment, fix $q_3 \in \mathbb{R}$, and consider $V^+(\tau, m(\tau))$. We have now defined all terms in $V^+$, so

$$V^+(\tau, m(\tau)) = e^{\tau} + \left( \frac{1}{4} e^{3 \epsilon \tau} + \frac{3}{4} e^{2 \epsilon \tau} + (V_3^+)'(0) e^{\tau} \right) e^{-\tau} + O(e^{(4 \epsilon - 3/2)\tau}). \quad (3.10)$$

Comparing this expansion with (3.5), we require $(V_3^+)'(0) = C_1^-$. We therefore choose $\alpha_1$ so that

$$C_1^- = (V_3^+)'(0) = (V_3^+)'(0) - \alpha_1 - \frac{3}{2} q_3. \quad (3.11)$$

Thus $\alpha_1$ depends on $u_0$ through $q_3$. Note also that the absence of a pure $e^{-\tau}$ term in (3.10) forces $C_0^- = 0$ in (3.2). This condition motivates the form of $V_1^-$ given by lemma 5.
4. Computation of $\mu_*$

We now offer an explicit computation of the coefficient $\mu_*$ determined by (3.8). We ultimately recover the value found by Berestycki et al in [1, 2].

Recalling that $\langle \phi_1, \psi_1 \rangle = 1$, we rewrite (3.8) as

$$\mu_* = \frac{3}{2} \left( \frac{3}{2} V_1^+ + \frac{3\sqrt{\pi}}{2} \phi_0', \psi_1 \right). \quad (4.1)$$

From the explicit form of $\phi_0$, we can compute $\langle \phi_0', \psi_1 \rangle = -\frac{1}{\sqrt{\pi}}$. Also, from (3.6) we have

$$(\mathcal{L} - \frac{1}{2}) V_1^+ + \frac{3}{2} \phi_0' + \frac{3\sqrt{\pi}}{2} \phi_0 = -\frac{1}{2} V_1^+.$$

Since $\psi_1$ is orthogonal to the range of $\mathcal{L} - 1$, we have

$$\langle V_1^+, \psi_1 \rangle = -\langle 3\phi_0' + 3\sqrt{\pi} \phi_0, \psi_1 \rangle = -3 \langle \phi_0', \psi_1 \rangle = \frac{3}{\sqrt{\pi}}.$$

Now let $\theta$ denote the unique Dirichlet solution to $(\mathcal{L} - \frac{1}{2}) \theta = \phi_0'$. Then (3.6) implies

$$V_1^+ = -\frac{3}{2} \theta + 3\sqrt{\pi} \phi_0.$$ 

Hence

$$\langle (V_1^+)', \psi_1 \rangle = -\frac{3}{2} \langle \theta', \psi_1 \rangle + 3\sqrt{\pi} \langle \phi_0', \psi_1 \rangle = -\frac{3}{2} \langle \theta', \psi_1 \rangle - 3.$$

Combining these calculations, (4.1) yields

$$\mu_* = \frac{9}{4} - \frac{27}{8} \langle \theta', \psi_1 \rangle. \quad (4.2)$$

Before examining $\theta$, we first relate $\phi_k$ and $\psi_k$ to the well-known Hermite polynomials. For $n \in \mathbb{Z}_{\geq 0}$, let

$$H_n(\eta) := (\eta - 2\partial_\eta)^n 1.$$

Then $H_n$ is a scaled variant of the $n$th Hermite polynomial. From the definition of $\phi_k$ and well-known properties of the Hermite polynomials, it is straightforward to check that

$$\phi_k(\eta) = 4^{-k} H_{2k+1}(\eta)e^{-\eta^2/4}, \quad \psi_k(\eta) = \frac{1}{2\sqrt{\pi}(2k+1)!} H_{2k+1}(\eta).$$

We now express $\theta$ in the $\{\phi_k\}$ basis:

$$\theta = \sum_{k \geq 0} c_k \phi_k$$

for $c_k \in \mathbb{R}$. By the defining equation for $\theta$,

$$(\mathcal{L} - \frac{1}{2}) \theta = \sum_k c_k \left( k - \frac{1}{2} \right) \phi_k = \phi_0'.$$

Taking the inner product with the dual basis, orthogonality implies

$$c_k = \frac{1}{k - 1/2} \langle \phi_0', \psi_k \rangle.$$

Integrating by parts, we have

$$\langle \theta', \psi_1 \rangle = -\langle \theta, \psi_1' \rangle = -\sum_k c_k \langle \phi_k, \psi_1' \rangle.$$
But
\[
\langle \varphi_k, \psi_k \rangle = \int_{\mathbb{R}^+} 4^{k+1} H_{2k+1}(\eta) e^{-\eta^2/4} \frac{1}{4\sqrt{\pi}}(\eta^2 - 2) \, d\eta
\]
\[
= -(2k+1)!4^{-k} \int_{\mathbb{R}^+} \left(1 - \frac{\eta^2}{2}\right) e^{-\eta^2/4} \frac{1}{2\sqrt{\pi}(2k+1)!} H_{2k+1}(\eta) \, d\eta = -(2k+1)!4^{-k}
\]
\[
\langle \varphi_k', \psi_k \rangle.
\]
Hence
\[
\langle \varphi_k', \psi_k \rangle = - \sum_k c_k \langle \varphi_k, \psi_k \rangle = \sum_{k \geq 0} \frac{(2k+1)!}{4^k(k-1/2)} \langle \varphi_k', \psi_k \rangle^2.
\] (4.3)
Next, we claim that
\[
\langle \varphi_k', \psi_k \rangle = \frac{(-1)^k}{\sqrt{\pi}(2k-1)k!} \quad \text{for all } k \geq 0.
\] (4.4)

**Proof of claim.** First note that \( \phi_0' = -\frac{1}{2} H_2 \), so
\[
\langle \phi_0', \psi_k \rangle = - \frac{1}{4\sqrt{\pi}(2k+1)!} \int_{\mathbb{R}^+} H_2 H_{2k+1}e^{-\eta^2/4} \, d\eta.
\] (4.5)
Using the definition of \( H_n \), and integrating by parts, we find
\[
\int_{\mathbb{R}^+} H_2 H_{2k+1}e^{-\eta^2/4} \, d\eta = -2 \int_{\mathbb{R}^+} H_2 H_{2k} \partial_\eta (e^{-\eta^2/4}) \, d\eta - 2 \int_{\mathbb{R}^+} H_2 H_{2k} e^{-\eta^2/4} \, d\eta
\]
\[
= 2 \int_{\mathbb{R}^+} H_2' H_{2k} e^{-\eta^2/4} \, d\eta + 2H_2(0)H_{2k}(0).
\]
Repeating this procedure, we further find
\[
2 \int_{\mathbb{R}^+} H_2' H_{2k} e^{-\eta^2/4} \, d\eta = 4 \int_{\mathbb{R}^+} H_2'' H_{2k-1} e^{-\eta^2/4} \, d\eta = 8H_2''(0)H_{2k-2}(0).
\]
Using the explicit form for \( H_2 \), this work yields
\[
\int_{\mathbb{R}^+} H_2 H_{2k+1} e^{-\eta^2/4} \, d\eta = -4H_{2k}(0) + 16H_{2k-2}(0).
\]
From the standard formulae for the Hermite polynomials,
\[
H_{2k}(0) = (-1)^k 2^k (2k - 1)!!, \quad H_{2k-2}(0) = (-1)^{k-1} 2^{k-1} (2k - 3)!!. \]
So
\[
\int_{\mathbb{R}^+} H_2 H_{2k+1} e^{-\eta^2/4} \, d\eta = -(-1)^k 2^k [4(2k - 1) + 8](2k - 3)!! = -4(-1)^k 2^k (2k + 1)(2k - 3)!!.
\]
By (4.5), we obtain (4.4):
\[
\langle \varphi_k', \psi_k \rangle = \frac{(-1)^k 2^k (2k - 3)!!}{\sqrt{\pi}(2k)!!(2k - 1)} = \frac{(-1)^k 2^k}{\sqrt{\pi}(2k)!!(2k - 1)} = \frac{(-1)^k}{\sqrt{\pi}(2k - 1)k!}.
\]
Combining (4.3) and (4.4), we obtain the series representation

$$\langle \theta', \psi_1 \rangle = \frac{2}{\pi} \sum_{k \geq 0} \frac{(2k + 1)!}{4^k (k!)^2 (2k - 1)^3}. \quad (4.6)$$

**Lemma 7.** We have

$$\sum_{k \geq 0} \frac{(2k + 1)!}{4^k (k!)^2 (2k - 1)^3} = \frac{\pi}{2} (2 \log 2 - 1). \quad (4.7)$$

Before proving lemma 7, we use it to conclude the computation of $\mu_\star$.

**Proof of lemma 6.** From (4.6) and (4.7), $\langle \theta', \psi_1 \rangle = 2 \log 2 - 1$. Therefore (4.2) implies

$$\mu_\star = \frac{9}{4} - \frac{27}{8} (2 \log 2 - 1) = \frac{9}{8} (5 - 6 \log 2). \quad \square$$

We have thus reduced the problem to computing a sum in closed form.

**Proof of lemma 7.** Let $S$ denote the sum in (4.7). We first note that the sum converges by Stirling’s formula. Using $(2k + 1)! = (2k + 1) \cdot (2k)!$ and $(2k)! (k!)^{-2} = \binom{2k}{k}$, we have

$$S = \sum_{k \geq 0} \binom{2k}{k} \frac{2k + 1}{4^k (2k - 1)^3} = \sum_{k \geq 0} \binom{2k}{k} \frac{1}{2^{2k} (2k - 1)^3} \left[ \frac{1}{(2k - 1)^2} + \frac{2}{(2k - 1)^3} \right].$$

We view this sum as a power series evaluated at $x = \frac{1}{2}$. As noted in [11], the binomial theorem implies

$$\sum_{k \geq 1} \binom{2k}{k} x^{2k} = \frac{1}{\sqrt{1 - 4x^2}} \quad \text{for} \quad x \in \left[ -\frac{1}{2}, \frac{1}{2} \right].$$

To introduce negative powers of $2k - 1$, we repeatedly divide by powers of $x$ and integrate, so that we always integrate terms of the form $x^{2k-2}$. We move the constant term in the sum to the right-hand side, to ensure integrability. So:

$$\sum_{k \geq 1} \binom{2k}{k} \frac{x^{2k-1}}{2k - 1} = \int_0^1 \frac{(1 - 4y^2)^{-1/2} - 1}{y^2} \, dy = \frac{4x^2 + \sqrt{1 - 4x^2} - 1}{x \sqrt{1 - 4x^2}}.$$  

Repeatedly dividing by $x$ and integrating, we find:

$$\sum_{k \geq 1} \binom{2k}{k} \frac{x^{2k}}{(2k - 1)^2} = \sqrt{1 - 4x^2} - 1 + 2x \arcsin(2x),$$

$$\sum_{k \geq 1} \binom{2k}{k} \frac{x^{2k}}{(2k - 1)^3} = 1 - \sqrt{1 - 4x^2} - 2x \arcsin(2x) + 2x \int_0^\pi \arcsin(2y) \, dy.$$  

The integrations induce convergence at the right endpoint $x = \frac{1}{2}$. Evaluating there and restoring the constant terms, we obtain
\[ S = 1 - 1 + \frac{\pi}{2} + 2 \left( -1 + 1 - \frac{\pi}{2} + \int_0^1 \frac{\arcsin y}{y} \, dy \right) = -\frac{\pi}{2} + 2 \int_0^1 \frac{\arcsin y}{y} \, dy. \]

Now, the integrand \( \frac{\arcsin y}{y} \) has no elementary antiderivative, so we use contour integration to compute the definite integral. We first change variables and integrate by parts:

\[ \int_0^1 \frac{\arcsin y}{y} \, dy = \int_0^{\frac{\pi}{2}} u \cot u \, du = u \log(\sin u) \bigg|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \log(\sin u) \, du = -\int_0^{\frac{\pi}{2}} \log(\sin u) \, du. \]

By trigonometric symmetries,

\[ \int_0^{\frac{\pi}{2}} \log(\sin u) \, du = \int_0^{\frac{\pi}{2}} \log(\cos u) \, du = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log(\cos u) \, du. \]

Now consider the open half-strip \( D \subset \mathbb{C} \) in the upper half-plane bounded by the lines \( \text{Re}\, z = \pm \frac{\pi}{2} \). Within \( D \), the function \( f(z) := \log(e^{2iz} + 1) \) is analytic (using the standard branch of the logarithm). Furthermore, since the complex arguments of \( e^{iz} \) and \( e^{iz} - e^{-iz} \) stay within \((-\frac{\pi}{2}, \frac{\pi}{2})\) in \( D \), we have

\[ f(z) = \log \left( e^{iz} (e^{iz} + e^{-iz}) \right) = \log(e^{iz}) + \log(e^{iz} + e^{-iz}) = iz + \log 2 + \log(\cos z). \]

By a standard limiting argument,

\[ \int_{\partial D} f(z) \, dz = 0. \]

On the other hand, \( f \left( -\frac{\pi}{2} + it \right) = f \left( \frac{\pi}{2} + it \right) \) for \( t > 0 \), so the contributions from the vertical rays in \( \partial D \) are cancelled in \( \int_{\partial D} f \). Thus

\[ 0 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(z) \, dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [iz + \log 2 + \log(\cos z)] \, dz = \pi \log 2 + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log(\cos u) \, du. \]

Hence \( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log(\cos u) \, du = -\pi \log 2 \), and

\[ \int_0^1 \frac{\arcsin y}{y} \, dy = \frac{\pi}{2} \log 2. \]

Finally, this implies

\[ S = -\frac{\pi}{2} + 2 \int_0^1 \frac{\arcsin y}{y} \, dy = \frac{\pi}{2} (2 \log 2 - 1) \]

as claimed. \( \square \)

5. Complete front asymptotics

We now generalize the asymptotic methods in section 3 to describe the behaviour of \( u \) to all orders in \( t \). We make (1.9) and (1.10) precise, and present the method for their derivation.
However, we do not rigorously prove the full expansion. Nevertheless, we expect that the proof in section 6 can be generalized to verify our proposed asymptotic form.

In this section, we let $V^+, V^\pm$, and $\sigma$ denote complete asymptotic developments of the inner and outer expansions and front shift. As in section 3, we describe $V^\pm$ and $\sigma$ in successively smaller orders of $t$. All orders will have the form $t^{-a/2} \log^b t$ for integers $a$ and $b$. To facilitate our discussion, we introduce notation adapted to this structure. We let the subscript $(a, b)$ denote the coefficient of order $t^{-a/2} \log^b t$ in an asymptotic expansion in $t$. Of course, not all terms of the form $t^{-a/2} \log^b t$ appear: only finitely many factors of $\log t$ accompany any fixed $t^{-a/2}$. As we shall see, $V^-$, $V^+$, and $\sigma$ have closely related but distinct expansions in $t$. To be precise, define:

$$\Omega^- := \{(0, 0)\} \cup \{(a, b) \in \mathbb{Z} \times \mathbb{Z}; a \geq 2, 0 \leq b \leq \frac{a-2}{2}\},$$

$$\Omega^+ := \{(a, b) \in \mathbb{Z} \times \mathbb{Z}; a \geq 0, 0 \leq b \leq \frac{a}{2}\},$$

$$\Omega^\sigma := \{(-2, 0), (0, 1)\} \cup \Omega^+.$$

We will argue that

$$V^-(t, x) \sim \sum_{(a, b) \in \Omega^-} t^{-a/2} \log^b t V^-_{a,b}(x),$$

$$V^+(\tau, \eta) \sim \sum_{(a, b) \in \Omega^+} \tau^b e^{-(a-1)\tau/2} V_+^{a, \eta}(\eta),$$

$$\sigma(t) \sim \sum_{(a, b) \in \Omega^\sigma} \sigma_{a, b} t^{-a/2} \log^b t. \tag{5.1}$$

We have graphically organized this structure in figure 1. We emphasize $\sigma'$ rather than $\sigma$, since the shift always enters equations through its time derivative.

In (5.1), $\sim$ denotes an asymptotic expansion in powers of $t$. That is, for any $A > 0$ we may truncate the series by omitting terms with $a > A$. Then each series will equal its left-hand side up to an error $O(t^{-A/2})$ in the physical variables $(t, x)$. We let $\sigma_{(A)}$ and $V_{(A)}^\pm$ denote these truncations. For instance,

$$\sigma_{(A)}(t) := \sum_{(a, b) \in \Omega^{A}} \sigma_{a, b} t^{-a/2} \log^b t. \tag{5.2}$$

We propose the following generalization of theorem 3:

**Proposition 8.** There exists an asymptotic series of the form (5.1) depending on $u_0$ such that the following holds. For any $A > 0$, let $\sigma_{(A)}$ and $V_{(A)}^\pm$ be as in (5.2). Then for any $\gamma > 0$, there exist $\varepsilon > 0$ and $C_\gamma > 0$ depending also on $u_0$ such that

$$\left|u(t, x + \sigma_{(A)}(t)) - Ae^{-x} \left[V_{(A)}^- (t, x) 1_{x < r} + V_{(A)}^+ (t, x) 1_{x \geq r}\right]\right| \leq \frac{C_\gamma (1 + |x|) e^{-x}}{t^{(A+1)/2 - \gamma}} \text{ on } [3, \infty) \times \mathbb{R}.$$

Furthermore, for each fixed power of $t$, the terms in $\sigma$ and $V^\pm$ of the highest order in $\log t$ are independent of $u_0$.

**Remark 3.** This proposition justifies (1.9) and (1.10) in the introduction.
In the remainder of this section, we outline the derivation of the expansion (5.1). We proceed inductively on orders in \( t \). Suppose we have determined \( V^\pm \) and \( \sigma \) to order \( t^{-(A-1)/2} \) for some \( A \geq 1 \), and they have the form in (5.1). We wish to show that (5.1) continues to hold to order \( t^{-A/2} \).

5.1. The inner expansion

First consider the inner expansion \( V^- \). Recall that \( V^- \) is an approximate solution to

\[
v_t - v_{xx} - (\sigma' - 2)(v - v_x) + A_0 e^{-\tau} v^2 = 0. \tag{5.3}
\]

We choose \( V^- \) to cancel all terms of order \( t^A \) or higher in (5.3). Note that the time derivative on \( \sigma \) lowers the order of its terms by a factor of \( t^{-1} \). Since \( V^- \) has the leading order \( \mathcal{O}(1) \), terms of the form \( \sigma_{a,b} \) with \( a \geq A - 1 \) make \( \mathcal{O}(t^{-A}) \) contributions to (5.3). They thus have no influence on the equations for \( V_\pm^{(A-1)} \). Rather, these equations depend only on \( V_\pm^{(A-1)} \) and \( \sigma^{(A-2)} \).

To find the highest power of \( \log t \) paired with \( t^{-A} \) in \( V_\pm^{(A)} \), we substitute \( V_\pm^{(A-1)} \) into (5.3). Since \( V_\pm^{(A-1)} \) is chosen to eliminate all terms of order \( t^{-(A-1)/2} \) or higher, we are left with terms of the form \( t^{-a/2} \log^b t \) with \( a \geq A \). By the inductive hypothesis, \( V^- \) and \( \sigma \) obey (5.1) up to order \( t^{-A/2} \). Using figure 1, we can visually track the contributions from \( (\sigma' - 2)(v - v_x) \) and \( e^{-\tau} v^2 \) by combining appropriate columns of \( \sigma' \) and \( V^- \).

For instance, suppose we wish to compute the \( \log t \) factors paired with \( t^{-2} \). To do so, we substitute \( \sigma^{(2)} \) and \( V_\pm^{(3)} \) for \( \sigma \) and \( v \) in (5.3). We examine (5.3) term by term, to find the factors of \( \log t \) at order \( a = 4 \).

The time derivative \( \partial_t V^- \) will generate no logarithmic factors, since \( V^- \) has none at order \( t^{-1} \). The spatial derivative \( \partial_x V^- \) can be ignored, as it does not generate any term of order \( t^{-2} \) when we plug in \( V_\pm^{(3)} \). To handle the product \((\sigma' - 2)(V^- - (V^-)',)\), we combine columns of \( \sigma' - 2 \) and \( V^- \) whose \( a \)-values sum to 4. Of these, only the product

\[-\sigma_2^t t^{-2} \log t \left[ V_-^{(0,0)} - (V_-^{(0,0)})' \right] \]

generates a factor of \( \log t \). Applying an identical approach to \( e^{-\tau}(V^-)^2 \), we see that it contributes no logarithmic factors, since \( V_\pm^{(3)} \) has no such factors. Therefore

\[V_-^{(4)}(t,x) = V_-^{(3)}(t,x) + t^{-2} \log t V_-^{(4)}(x) + t^{-2} V_-^{(0)}(x).\]

In general, let \( B := \left\lceil \frac{3}{2} \right\rceil \). The above argument shows that the leading \( \log t \) term at order \( t^{-A/2} \) is due to
− \frac{d}{dt}\left(\sigma_{A-2,\beta}t^{-(A-2)/2}\log^b t\right)\left[V_{0,0}^- - \{V_{0,0}^-\}\right] = \frac{A - 2}{2}\sigma_{A-2,\beta}t^{-A/2}\log^{b-1} t\left[V_{0,0}^- - \{V_{0,0}^-\}\right] + O(t^{-A/2}\log^{b-2} t).

We must therefore include a term of the form \(V_{A,B}^-\) in \(V_{(A)}^-\). Naturally, all lower powers of \(\log t\) appear as well, so

\[V_{(A)}^-(t,x) = V_{(A-1)}^-(t,x) + \sum_{b=0}^{B-1} t^{-A/2}\log^b t\ V_{A,b}^-(x),\]

as predicted by (5.1). If we substitute \(V_{(A)}^-\) in (5.3), only \(c(t^{-A/2})\) terms remain. That is, \(NL[V_{(A)}^-] = c(t^{-A/2})\), where \(NL\) is the nonlinear operator in (5.3).

Recall, however, that further constraints on \(V^-\) are necessary. In particular, \(V^-\) must decay as \(x \to -\infty\), and must match well with \(V^+\). Now, (5.3) implies that \(V_{A,b}^-\) solves an inhomogeneous linear ODE of the form

\[-V'' + 2e^{-x}V_{0,0}^- V = F,\]

where \(F\) is some combination of the known functions \(e^{-x}, V_{a,b}\), and \(V_{a,b}^- = (V_{a,b})\) with \(a < A\). We can easily verify that \(F = O(e^x)\) as \(x \to -\infty\) and that \(F\) grows polynomially as \(x \to +\infty\). We now desire a solution to (5.4) decaying at \(-\infty\) and lacking a constant term in its polynomial expansion at \(+\infty\) (in order to match \(V^+\)). The proof of lemma 5 can be adapted to show the existence of a unique solution to (5.4) satisfying these boundary conditions. We have thus uniquely determined the inner expansion \(V_{(A)}^-\) and it conforms to (5.1).

5.2. The outer expansion and shift

We now determine the next terms of \(V_{(A)}^+\) and \(\sigma_{(A)}\). Recall that by the Dirichlet condition, order \(t^{-A/2}\) terms in physical variables correspond to order \(e^{-\left(\frac{\sigma}{2}\right)}\) terms in the self-similar variables \((\tau, \eta)\). We therefore assume \(V^+\) obeys (5.1) up to order \(e^{-\left(\frac{\sigma}{2}\right)}\), and wish to continue the pattern to order \(e^{-\left(\frac{\sigma}{2}\right)\tau/2}\). Likewise, we assume \(\sigma\) obeys (5.1) to order \(t^{-\left(\frac{A}{2}\right)\tau/2}\), and seek to continue its pattern to order \(t^{-\left(\frac{A}{2}\right)\tau/2}\).

In the self-similar variables, \(V^+\) is an approximate Dirichlet solution to

\[v_\tau - v_\eta - \frac{\eta}{2}v_\eta + e^{-\frac{\sigma}{2}}(\sigma - 2)\left(e^{-\tau/2}v_\eta - v\right) + A_0\exp\left(\frac{\sigma - \eta}{2}\right)v^2 = 0.\]

Furthermore, \(V^+\) must agree with \(V^-\) at the matching point \(x = \tau^2\) to high order. Because we have determined \(V_{(A)}^+\) the matching criteria for \(V_{(A)}^+\) are fixed. Let \(\mathcal{N}\mathcal{L}\) denote the nonlinear operator in (5.5). We choose \(\sigma_{(A)}\) and \(V_{(A)}^+\) to ensure the existence of an approximate solution to (5.5) such that \(\mathcal{N}\mathcal{L}[V^+] = o(e^{-\left(\frac{\sigma}{2}\right)\tau/2})\) and \(|V^+ - V^-| = o(t^{-A/2})\) at \(x = \tau^2\).

Given \(V_{(A-1)}^+\), let us consider the equations for \(V_{A,B}^+\). As in section 3, we may neglect the nonlinear term in (5.5), as it decays super-exponentially as \(\tau \to \infty\). The \(e^\cdot\) prefactor before \(\sigma^\cdot\) means \(\sigma_{A,B}\) affects \(V_{A,B}^+\). In particular, \(V_{A,B}^+\) must solve an inhomogeneous linear equation of the form

\[
\left(\mathcal{L} - \frac{A}{2}\right) V - \frac{A}{2}\sigma_{A,B} V_{0,0} = G,
\]
where $G$ depends on the ‘larger’ terms: $V_{a,b'}^+$ and $\sigma_{a,b'}$ with $a < A$ or $a = A$ and $b' > b$. We therefore iteratively determine $V_{A,b}^+$ and $\sigma_{A,b}$, beginning with the largest value $b$. We divide our analysis into two cases, determined by the Dirichlet invertibility of $L - \frac{A}{2}$.

First suppose $A$ is odd, so $L - \frac{A}{2}$ is Dirichlet invertible. We substitute $V_{(A-1),b}^+$ and $\sigma_{(A-1),b}$ into (5.5), and use figure 1 as before to find the size of the nontrivial nullspace of $\tilde{G}$ and $\sigma_0 = \sigma_0(a, b)$. Recalling that $b = \sigma_0(a, b)$, $\sigma_0$ satisfies $\sigma_0(a, b) = \sigma_{A,b}$. Matching with $\sigma_0$ rate matching of $B_0$, we need terms $V_{A,b}^+$ with $0 \leq b \leq \frac{A-1}{2} = \left\lfloor \frac{A}{2} \right\rfloor$, which agrees with (5.1). Let $B := \frac{A-1}{2}$. Recalling that $V_{0,0} = \phi_0$, the equation for $V_{A,b}^+$ has the form

$$\left( L - \frac{A}{2} \right) V - \frac{A}{2} \sigma_{A,b} V = G,$$

(5.6)

where $G$ depends only on already determined terms. This equation has a unique solution for any $\sigma_{A,b}$ and changing $\sigma_{A,b'}$ changes $V_{A,b}^+$ by a multiple of $\phi_0$. As in section 3, an accurate matching of $V^+$ with $V^-$ requires a prescribed value of $\partial_\eta V_{A,b}(0)$. In fact, since $V^-$ has no term of order $t^{-A/2} \log^B t$, we need $\partial_\eta V_{A,b}(0) = 0$. We therefore choose $\sigma_{A,b}$ so that $\partial_\eta V_{A,b}(0) = 0$.

Now suppose we have uniquely determined $\sigma_{A,b'}$ and $V_{A,b'}^+$ for $b' > b$. Then $V_{A,b}^+$ satisfies an equation of the form $\left( L - \frac{A}{2} \right) V - \frac{A}{2} \sigma_{A,b} V = G$ for some already determined $G$. As above, we uniquely choose $\sigma_{A,b}$ so that $\partial_\eta V_{A,b}(0)$ has the value required by matching (which is not necessarily 0). Iterating in $b$, we thus uniquely determine $\sigma_{(A)}$ and $V_{(A)}^+$ when $A$ is odd. At each stage, we use the degree of freedom afforded by $\sigma$ to impose a second boundary condition on $V^+$, which permits an accurate matching with $V^-$.

Next suppose $A$ is even, and let $B := \frac{A}{2}$. Now $L - \frac{A}{2} = L - B$ is not invertible, but rather has a one-dimensional kernel and cokernel. Thus at each stage we have an additional constraint: the inhomogeneity $G$ in (5.6) must be orthogonal to $\psi_B$, the $B$-eigenfunction of the adjoint operator $L^\ast$. However, if this constraint is satisfied, we obtain a new degree of freedom: $L - B$ has a nontrivial kernel, so (5.6) only determines $V$ up to a multiple of the eigenfunction $\phi_B$. We therefore typically have two constraints and two degrees of freedom, which result in a unique solution.

To be more precise, consider the leading-order term $V_{A,b}^+$. For this term,

$$G = -\sum_{b=1}^{B-1} b \sigma_{2b,b} V_{A,2b,b-b}^+.$$

Also, we can inductively show that $V_{A,2b,b-b}^+$ lies in the span of $\{\phi_0, \ldots, \phi_b\}$ for all $0 < b < B$. But $\langle \phi_b, \psi_B \rangle = 0$ when $b < B$, so automatically $\langle G, \psi_B \rangle = 0$. In fact, in this case we have an explicit solution:

$$V_{A,b}^+ = -\sigma_{A,b} \phi_0 + q_{A,b} \phi_B + \sum_{b=0}^{B-1} G_b \phi_b,$$

where $G_b \in \mathbb{R}$ are already determined and $q_{A,b} \in \mathbb{R}$ is a free parameter corresponding to the nontrivial nullspace of $L - B$. Matching with $V^-$ forces $\partial_\eta V_{A,b}(0) = 0$, as $V^-$ has no term of order $t^{-A/2} \log^B t$. We therefore have one constraint on the parameters $\sigma_{A,b}$ and $q_{A,b}$.

Now consider the equation for $V_{A,B-1}^+$. It will have the form

$$(L - B) V - B \sigma_{A,B-1} \phi_0 = \tilde{G} - \sigma_{A,B} \phi_0 - B V_{A,B}^+ =: G,$$

(5.7)
for already determined \( \tilde{c} \). Indeed, the \( \sigma_{A,B} \) term is due to \( \partial_t \) acting on the logarithmic factor in \( \sigma_{A,B} t^{-A/2} \log^B t \). Likewise, \( BV_{A,B}^+ \) arises when \( \partial_\tau \) acts on the polynomial factor in \( \tau^B e^{-(A-1)/2} \). For (5.7) to have a solution, we must have \( \langle G, \psi_B \rangle = 0 \). Recalling that \( \langle \phi_B, \psi_B \rangle = 0 \), we have

\[
\langle G, \psi_B \rangle = \langle \tilde{G}, \psi_B \rangle - Bq_{A,B}.
\]

We may therefore choose \( q_{A,B} \) to ensure \( \langle G, \psi_B \rangle = 0 \). In turn, \( q_{A,B} \) determines \( \sigma_{A,B} \), through the requirement that \( \partial_\tau V_{A,B}^+(0) = 0 \).

We may repeat this procedure for successively smaller values of \( b \). At each stage, the free coefficient \( q_{A,B} \) of \( \phi_B \) in \( V_{A,B}^+ \) is chosen to ensure existence for \( V_{A,B}^+ \). The matching condition then determines \( \sigma_{A,B} \). This procedure continues until \( b = 0 \). Then there are no lower-order equations, so \( q_{A,0} \) seems undetermined. This parameter mimics \( q_1 \) in section 3. It is undetermined by matching, and effectivly controls the \( \phi_B \)-component of the difference between the approximate solution \( V_{app} \) and the true solution \( V \). As in the proof of theorem 9 presented below, we can uniquely choose \( q_{A,0} \) to kill this component. Through this choice, \( q_{A,0} \) depends on the initial data \( u_0 \). Having fixed \( q_{A,0} \), the shift \( \sigma_{A,0} \) is determined, and likewise depends on \( u_0 \). With these choices, we have completely determined \( V_{(A)}^+ \) and \( \sigma_{(A)} \), which have the claimed forms.

### 5.3. Universality

We now consider the universality of terms in (5.1).

We claim that shift coefficients of the form \( \sigma_{A,B} \) with \( A \geq 1 \) and \( B = \lfloor \frac{A}{2} \rfloor \) are independent of the initial data \( u_0 \), as are \( \sigma_{-2,0} = 2 \) and \( \sigma_{0,1} = -\frac{1}{2} \). As a direct result, the outer expansion terms \( V_{A,B}^+ \) are universal for all \( A \geq 0 \). We again argue inductively, so suppose this universality holds up to order \( \tau^{(A+1)^2} \) for some \( A \geq 1 \).

In the equation for \( V_{A,B}^+ \), \( G \) is a linear combination of universal terms in \( \sigma \) and \( V^+ \). Furthermore, since \( V^- \) has no matching term, the boundary data for \( V_{A,B}^- \) are \( V_{A,B}^- = 0 \). When \( A \) is odd, the shift \( \sigma_{A,B} \) is determined solely by the equation for \( V_{A,B}^+ \) and its boundary data. Hence in this case \( \sigma_{A,B}^+ \) and \( V_{A,B}^+ \) are universal.

When \( A \) is even, \( \sigma_{A,B} \) also depends on the equation for \( V_{A,B}^- \). The only non-universal term in the equation for \( V_{A,B}^- \) is \( -B\sigma_{A,B-1} \phi_0 \). However, \( \langle \phi_0, \psi_B \rangle = 0 \), so this term has no effect on the solvability of the equation. Since \( q_{A,B} \) is chosen solely to ensure this solvability, it is independent of \( u_0 \). Thus so are \( \sigma_{A,B} \) and \( V_{A,B}^+ \). It follows that the claimed terms in \( \sigma \) and \( V^+ \) are independent of \( u_0 \).

We next argue that inner expansion terms of the form \( V_{A,B}^+ \) are universal. The equation for such a term is

\[
-V'' + 2e^{-t}V_{0,0} = -\left(\frac{A-2}{2}\right)\sigma_{A-2,B-1}[V_{0,0} - (V_{0,0}^\tau)^\tau].
\]

Comparing with (3.1), we see that \( V_{A,B}^- \) is a multiple of \( V_{2B}^- \). The scaling factor is propotional to \( \sigma_{A-2,B-1} \). Since this shift term is universal, so is \( V_{A,B}^- \).

Finally, we note that the shift terms \( \sigma_{A,B} \) are universal in a broader sense. Suppose we change the form of the nonlinearity in (1.1) so the equation becomes

\[
u_t = u_{xx} + f(u)
\]
with a more general KPP reaction $f$. Assume $f'(0) = 1$, $f(u) \leq u$, and $f'(1) < 0$. Then the form of the travelling front $\phi$ will change, but its speed will not, since $f'(0) = 1$. Our preceding arguments hold for the nonlinearity $f$, and the associated linear operator $L$ is unchanged. It follows that the equations for $V_{A, b}^+$ are likewise unchanged. The inner expansion will change with the front $\phi$, and will affect $V^-$ through the boundary data for $V_{A, b}^-$. However, terms of the form $V_{A, b}^+$ have no matching $V^-$ term, and are thus independent of the changes to the inner expansion. It follows that these terms, and their shifts $\sigma_{A, b}$, are independent of the precise form of the nonlinearity. In effect, they only ‘see’ the linear behaviour of (1.1). This strong universality suggests that the special coefficients $\sigma_{A, b}$ may arise in more general pulled front settings.

6. Proof of main theorem

We now proceed with the proof of theorem 3. Following section 3, we work in the shifted frame $x \mapsto x - \sigma(t)$. We know from [3, 12] that there exists $\alpha_0 \in \mathbb{R}$ depending on $v_0$ such that $u(t, x) \to \phi(x)$ as $t \to \infty$. Without loss of generality, we shift the initial data $u_0$ so that $\alpha_0 = 0$.

As in section 3, we primarily study $v = A_0^{-1}e^u$. We will construct $V_{\text{app}}$ and prove:

**Theorem 9.** There exists a choice of $q_3$ in (3.9) depending on the initial data $u_0$ such that the following holds. For all $\gamma > 0$, there exists $C_\gamma > 0$ also depending on $u_0$ such that for all $t \geq 3$ and $x \geq 2 - t^{1/6}$,

$$|\partial(t, x) - V_{\text{app}}(t, x)| \leq C_\gamma \frac{(1 + |x|)}{t^{2-\gamma}},$$

Our main results follow from theorem 9:

**Proof of theorem 3.** Let $U_{\text{app}} := A_0 e^{-x} V_{\text{app}}$ and $\psi := A_0 e^{-x} V^-$. To extend our bound from $x \geq 2 - t^{1/6}$ to all $x \in \mathbb{R}$, note that $u$ and $U_{\text{app}}$ are uniformly bounded, say by $C$. Now $t^{1/6} \geq C_\gamma > 0$ when $x \leq 2 - t^{1/6}$ and $t \geq 1$. Hence (1.7) is trivial when $x \leq 2 - t^{1/6}$, provided we take $C_\gamma \geq 2C_\gamma^{-1}$. \hfill \Box

**Proof of corollary 4.** We wish to track the rightmost edge of the level set $\{x: u(t, x) = s\}$. Theorem 3 shows that $u(t, x + \sigma(t))$ is close to $U_{\text{app}}(t, x)$, uniformly on rays $x \geq C$. Recall that $U_{\text{app}}(t, x) = \phi(x) + O(t^{-1})$ uniformly on the same rays. Hence if we apply theorem 3 on a ray $x \geq \phi^{-1}(s) - 1$, we find $\sigma_{A, b}(t) - \phi^{-1}(s) - \sigma(t) = O(t^{-1})$. Equation (1.8) follows. \hfill \Box

Before proving theorem 9, we must first construct the approximate solution $V_{\text{app}}$. Roughly, we use $V_{\text{app}} = V^-$ when $x < t^\epsilon$ and $V_{\text{app}} = V^+$ when $x > t^\epsilon$, with $\epsilon = \epsilon(\gamma)$ to be determined. However, we must join $V^\pm$ near $t^\epsilon$ so that $V_{\text{app}}$ is $C^1$ in space. Our work in section 3 shows

$$|V^+(t, r^\epsilon) - V^-(t, r^\epsilon)| = O(t^{2\epsilon - 3/2}). \tag{6.1}$$

To make $V_{\text{app}}$ continuous, we change the spatial argument of $V^-$ by a time-dependent shift $\zeta$ so that

$$V_0^-(t^\epsilon + \zeta(t)) + t^{-1}V^-_1(t^\epsilon + \zeta(t)) = V^+(t, r^\epsilon).$$

Since $(V^-)' \to 1$ near $x = t^\epsilon$ as $t \to \infty$, (6.1) and the construction of $V^\pm$ imply

$$\zeta(t) = O(t^{2\epsilon - 3/2}), \quad \zeta(t) = O(t^{2\epsilon - 5/2}).$$
For the remainder of the paper,

\[ V^{-}(t, x) := V_{0}^{-}(x + \zeta(t)) + t^{-1}V_{1}^{-}(x + \zeta(t)), \]

so that \( V^{-}(t, t') = V^{+}(t, t') \).

We further require \( \partial_{t}V_{\text{app}} \) to be continuous. To enforce this, we add a term to \( V_{\text{app}} \) whose derivative has a discontinuity precisely cancelling that between \( \partial_{t}V^{-} \) and \( \partial_{t}V^{+} \). Let

\[ K(t) := \partial_{t}V^{+}(t, t') - \partial_{t}V^{-}(t, t'), \]

so

\[ K(t) = \mathcal{O}(t^{\varepsilon-\frac{1}{2}}), \quad \dot{K}(t) = \mathcal{O}(t^{\varepsilon-\frac{1}{2}}). \]

Now define \( \varphi \geq 0 \) satisfying

\[ -\varphi_{xx} + \varphi = \delta(x - t'), \quad \varphi(0) = \varphi(\infty) = 0. \]

Explicitly,

\[ \varphi(t, x) = \begin{cases} e^{-t'} \sinh x & \text{for } 0 \leq x \leq t', \\ \sinh(x/t')e^{-x} & \text{for } x > t'. \end{cases} \]

A term \( K\varphi \) would fix the discontinuity. However, it will be convenient for this perturbation to be compactly supported in space. Therefore let \( \vartheta \in C_{c}^\infty(\mathbb{R}) \) satisfy \( \vartheta(1) = 1 \) and \( \vartheta|_{(0,2)} \equiv 0 \). Then let

\[ V_{\text{app}}(t, x) := 1_{|x|<t'}V^{-}(t, x) + 1_{|x|\geq t'}V^{+}(t, x) + K(t)\vartheta(t-x)\varphi(t, x). \]

By the construction of \( K, \varphi, \) and \( \vartheta, V_{\text{app}} \) is \( C^{1} \) in space.

Let \( \mathcal{NL} \) denote the nonlinear operator in (2.1). We are interested in controlling the size of \( \mathcal{NL}[V_{\text{app}}] \), which measures how badly \( V_{\text{app}} \) fails to be a true solution of (2.1). We consider the contributions from \( V^{-}, V^{+}, \) and \( K\vartheta\varphi \) separately.

Recall that \( V^{-}(t, x) = V_{0}^{-}(x + \zeta) + t^{-1}V_{1}^{-}(x + \zeta) \). Before the shift by \( \zeta \), we constructed \( V_{0}^{-} \) and \( V_{1}^{-} \) to eliminate terms up to order \( t^{-1} \) in \( \mathcal{NL}[V_{0}^{-} + t^{-1}V_{1}^{-}] \). By the decay of \( V_{1}^{-} \) on \( \mathbb{R}_{-} \),

\[ \mathcal{NL}[V_{0}^{-} + t^{-1}V_{1}^{-}] = \mathcal{O}(t^{-\frac{3}{2}}e^{\varepsilon}). \]

Spatially shifting by \( \zeta \) introduces new terms in \( \mathcal{NL}[V^{-}] \). Of these, the most significant is

\[ (e^{-x} - e^{-x-\zeta})(V_{0}^{-})^{2}(x + \zeta), \]

which is due to the mismatch in argument between \( e^{-x}(V_{0}^{-})^{2} \) and \( (V_{0}^{-})''(x + \zeta) = e^{-x-\zeta}(V_{0}^{-})^{2} \).

Nonetheless, the decay of \( \zeta \) and \( V_{0}^{-} \) implies

\[ |(e^{-x} - e^{-x-\zeta})(V_{0}^{-})^{2}(x + \zeta)| = \mathcal{O}(\varepsilon^{3}) = \mathcal{O}(t^{\varepsilon-\frac{3}{2}}e^{\varepsilon}). \]

Therefore

\[ \mathcal{NL}[V^{-}] = \mathcal{O}(t^{\varepsilon-\frac{3}{2}}e^{\varepsilon}) \quad \text{on } (-\infty, 0]. \]

An identical analysis shows that \( \mathcal{NL}[V^{-}] = \mathcal{O}(t^{\varepsilon-\frac{3}{2}}) \) on \( [0, t'] \).

Now consider \( V^{+} \). If \( \mathcal{NL} \) denotes the nonlinear operator in (3.3), we have constructed \( V^{+} \) so that

\[ \mathcal{NL}(V^{+}) \leq \mathcal{O}(\tau e^{-\eta^{2}/5}) \quad \text{for } \eta \in \mathbb{R}_{+}. \]
However, when we derived (3.3) from (2.1), we cleared a common factor of $e^{-\tau}$. Thus informally, $NL = e^{-\tau}N\mathcal{L}$. Changing to the physical variables $(t, x)$, this observation implies

$$
NL[V^\tau] = O\left(\log t \cdot t^{-2} \exp\left[-\frac{x^2}{5t}\right]\right) \quad \text{on} \quad [t^\epsilon, \infty).
$$

Finally, the bounds on $V^\pm$ and $K$ show that the correction $K\vartheta/\varphi$ perturbs $NL[V_{\text{app}}]$ by order $O(t^{3\epsilon - 3/2})$ solely on $[0, 2t^\epsilon]$. Thus there exists $C_\epsilon > 0$ depending also on $u_0$ (through $\alpha_1$) such that

$$
|NL[V_{\text{app}}]| \leq C_\epsilon \left[t^{\epsilon \pm \frac{3}{2}} e^\epsilon 1_{(-\infty, 0]}(x) + t^\epsilon e^{-\frac{2}{3}} 1_{[0, 2t^\epsilon]}(x) + t^{-2} 1_{[t^\epsilon, \infty]}(x) \exp\left(-\frac{x^2}{5t}\right)\right]. \quad (6.2)
$$

With the estimate (6.2) in hand, we are ready to prove theorem 9. We will transform our equation into a Dirichlet problem on the half-line, switch to the self-similar variables

$$
\tau = \log t, \quad \eta = \frac{x}{\sqrt{t}},
$$

and show that our problem is still dominated by linear theory related to the operator $\mathcal{L}$ introduced in section 3.

**Proof of theorem 9.** Let $W := v - V_{\text{app}}$. Then $W$ satisfies an equation of the form

$$
W_t - W_{xx} = \left(-\frac{3}{2t} - \frac{3\sqrt{\pi}}{2t^{3/2}} + \mu_\ast - \frac{\mu_\ast}{t^2} + \frac{\alpha_1 - \mu_\ast}{t^2}\right) (W - W_x) + A_\ast e^{-\frac{x}{\sqrt{t}}} (v + V_{\text{app}})W = F,
$$

where by (6.2),

$$
|F(t, x)| \leq C_\epsilon \left[t^{\epsilon \pm \frac{3}{2}} e^\epsilon 1_{(-\infty, 0]}(x) + t^{\epsilon - \frac{2}{3}} 1_{[0, 2t^\epsilon]}(x) + t^{-2} 1_{[t^\epsilon, \infty]}(x) \exp\left(-\frac{x^2}{5t}\right)\right]. \quad (6.3)
$$

Recall that the constant $C_\epsilon$ depends on $\epsilon$ and the initial data $u_0$. For the remainder of the proof we suppress such constants with the notation $\lesssim$, which denotes inequality up to a multiplicative constant depending on $\epsilon$ and $u_0$. Similarly, we frequently use larger-than-necessary multiples of $\epsilon$ in exponents, to simplify presentation. Under these conventions, (6.3) may be written as

$$
|F(t, x)| \lesssim t^{\epsilon \pm \frac{3}{2}} e^\epsilon 1_{(-\infty, 0]}(x) + t^{\epsilon - \frac{2}{3}} 1_{[0, 2t^\epsilon]}(x) + t^{-2} 1_{[t^\epsilon, \infty]}(x) \exp\left(-\frac{x^2}{5t}\right).
$$

We now enforce a Dirichlet condition at $x = -t^\epsilon$ by subtracting the boundary value from $W$. To simplify notation, we then shift $x$ by $t^\epsilon$, so the Dirichlet condition occurs at $x = 0$. Therefore define

$$
W(t, x) := W(t, x - t^\epsilon) - W(t, -t^\epsilon)\vartheta(x + 1),
$$

recalling that $\vartheta(1) = 1$ and $\vartheta(0, 2t^\epsilon) \equiv 0$. To control $W(t, -t^\epsilon)$, we use the exponential decay of $v$ and $V_{\text{app}}$. Indeed, $v, V_{\text{app}}^- \leq C e^\epsilon$ on $\mathbb{R}_-$. Thus

$$
|W(t, -t^\epsilon)| \lesssim e^{-t} \lesssim t^{-2}.
$$
It follows that \( \dot{W} \) satisfies
\[
\dot{W} - \dot{\omega}_0 + \varepsilon t^{-1} \dot{\omega}_0 - \left( \frac{3}{2 \mu} - \frac{3 \sqrt{\pi}}{2 \alpha_1} + \mu_\ast \frac{\log t}{t^2} + \frac{\alpha_1 - \mu_\ast}{t^2} \right) (\dot{W} - \dot{\omega}_0) + A_0 e^{-\varepsilon t} (\dot{\omega} + \dot{V}_{\text{app}}) W = G_1 + G_2,
\]
where \( \dot{v}(t,x) := v(t,x-t') \), \( \dot{V}_{\text{app}} \) is analogous, and
\[
\begin{align*}
|G_1(t,x)| & \lesssim e^{-\varepsilon t} I_{10.3|t|}(x) + e^{-2 \varepsilon t} I_{12|t|,\infty}(x) \exp \left[ -\frac{(x-t')^2}{5 \mu} \right], \\
|G_2(t,x)| & \lesssim t^{-2} I_{10.2}(x).
\end{align*}
\]

Switching to the self-similar variables, we find:
\[
\dot{W}_s + \left( \mathcal{L} - \frac{1}{2} \right) \dot{W} + A_0 \exp \left( \tau - \frac{\eta - m(\tau)}{2 \tau} \right) (\dot{v} + \dot{V}_{\text{app}}) W = G_1 + G_2 + g(\tau) \dot{W}_{\tau} + h(\tau) W
\]
on the half-line \( \eta \in \mathbb{R}_+ \) with \( W(\tau, 0) = 0 \) for all \( \tau \geq 0 \). We use the notation
\[
\begin{align*}
m(\tau) &= e^{(\varepsilon - 1)\tau}, \\
g(\tau) &= e^{(\varepsilon - 1)\tau} - \frac{3}{2} e^{-\tau/2} + \frac{3 \sqrt{\pi}}{2} e^{-\tau} - \mu_\ast \tau e^{-3\tau/2} + (\mu_\ast - \alpha_1) e^{-3\tau/2}, \\
h(\tau) &= -\frac{3 \sqrt{\pi}}{2} e^{-\tau/2} + \mu_\ast \tau e^{-\tau} + (\alpha_1 - \mu_\ast) e^{-3\tau/2}.
\end{align*}
\]

Finally, we symmetrize the operator \( \mathcal{L} \) by multiplying through by \( e^{\eta/3} \). This transforms \( \mathcal{L} \) to
\[
\mathcal{M} := -\partial_\eta^2 + \frac{\eta^2}{16} - \frac{5}{4}.
\]

Then
\[
w(\tau, \eta) := e^{\eta/3} \dot{W}(\tau, \eta)
\]
satisfies
\[
w_{\tau} - \mathcal{M} w + A_0 \exp \left( \tau - \frac{\eta - m(\tau)}{2 \tau} \right) (\dot{v} + \dot{V}_{\text{app}}) w = \sum_{i=1}^{3} E_i,
\]
where the errors \( E_i \) satisfy
\[
\begin{align*}
|E_1| & \lesssim e^{(\varepsilon - 1/2)\tau} I_{10.3|\tau|}(\eta) + e^{(\varepsilon - 1)\tau} e^{-2 \eta/3} I_{12|\tau|,\infty}(\eta) =: E_{11} + E_{12}, \\
|E_2| & \lesssim e^{-\tau} I_{10.2|\tau|}(\eta), \\
E_3 & = g(\tau) \left( w_0 - \frac{\eta}{4} w \right) + h(\tau) w.
\end{align*}
\]

Furthermore, the convergence of \( v \) to \( A_0^{-1} e^\phi(x) \) and the definition of \( V_{\text{app}} \) imply
\[
\left| (\dot{v} + \dot{V}_{\text{app}})(\tau, \eta) \right| \lesssim e^{(\eta - m(\tau))/3} I_{10,\infty}(\eta) + \left( 1 + \eta^3 e^{3\tau/2} \right) I_{10,\infty}(\eta).
\]
To control the behaviour of \( w \), we bootstrap from the bounds obtained in [13]. The main result in [13] does not directly apply, as it uses a different shift and approximate solution. However, the proof in [13] works in our situation with trivial modifications. Thus, as in (4.70) and (4.71) in [13], we have

\[
\|w\|_{L^2(\mathbb{R}^+)} + \|w\|_{L^\infty(\mathbb{R}^+)} \lesssim e^{(\epsilon-1/2)\tau}, \quad \|w(\tau, \eta)\| \lesssim \eta e^{(\epsilon-1/2)\tau} \text{ for all } (\tau, \eta) \in [0, \infty) \times [0, \infty).
\]

(6.7)

Here we have replaced the exponent 100\( \gamma \) in [13] with our small parameter \( \epsilon \).

We use the method of [13] to improve this bound to

\[
\|w\|_{L^2(\mathbb{R}^+)} + \|w\|_{L^\infty(\mathbb{R}^+)} \lesssim e^{(5\epsilon-1)\tau}, \quad \|w(\tau, \eta)\| \lesssim \eta e^{(5\epsilon-1)\tau},
\]

(6.8)

provided \( q_1 \) in \( V_1^+ \) is chosen appropriately. As we shall see, this control implies theorem 9. For the initial stage of the proof, take \( q_3 = 0 \).

In the following, let \( \{e_k\}_{k \in \mathbb{Z}_{\geq 0}} \) denote orthonormal eigenfunctions of \( \mathcal{M} \). Since \( \mathcal{M} \) has the same spectrum as \( \mathcal{L} = \frac{1}{\tau} \), we have

\[
\mathcal{M}e_k = \left( k - \frac{1}{2} \right) e_k \quad \text{for } k \in \mathbb{Z}_{\geq 0}.
\]

There exist unique \( c_k > 0 \) such that \( e_k = c_k \phi_k e^{\eta^2/8} \) and \( \|e_k\|_{L^2(\mathbb{R}^+)} = 1 \). In particular,

\[
e_0(\eta) = c_0 e^{\eta^2/8} \quad \text{and} \quad e_1(\eta) = \frac{c_1}{4} (\eta^3 - 6\eta) e^{-\eta^2/8}.
\]

We begin by proving:

**Lemma 10.** There exists \( r \in \mathbb{R} \) such that

\[
\left\| e^{\tau/2} w(\tau, \cdot) - re_1(\cdot) \right\|_{L^2(\mathbb{R}^+)} \lesssim e^{(2\epsilon-1/4)\tau} \quad \text{as } \tau \to \infty.
\]

(6.9)

We will use this lemma to choose the final value of \( q_3 \) in \( V_3^+ \).

**Proof.** We first consider the \( e_0 \)-component of \( w \). By (6.4),

\[
\frac{d}{d\tau} \langle e_0, w \rangle = \frac{1}{2} \langle e_0, w \rangle + \langle e_0, A_0 \exp \left( \tau - [\eta - m(\tau)] e^{\tau/2} \right) (\dot{\nu} + \tilde{V}_{\text{app}}) w \rangle = \sum_{i=1}^{3} \langle e_0, E_i \rangle.
\]

The bound (6.5) implies

\[
|\langle e_0, E_1 \rangle| \lesssim e^{(4\epsilon-1)\tau}, \quad |\langle e_0, E_2 \rangle| \lesssim e^{-2\tau}.
\]

By (6.7), integration by parts, and Cauchy–Schwarz,

\[
|\langle e_0, E_3 \rangle| \lesssim e^{(\epsilon-1/2)\tau} \big( |\langle e_0, w \rangle| + |\langle \eta e_0, w \rangle| \big) \lesssim e^{(2\epsilon-1)\tau}.
\]

Now consider the term \( \langle e_0, A_0 \exp \left( \tau - [\eta - m(\tau)] e^{\tau/2} \right) (\dot{\nu} + \tilde{V}_{\text{app}}) w \rangle \). On the interval \([0, m(\tau)]\), (6.6) and (6.7) imply
\[ A_0 \int_0^{m(\tau)} e_0 \exp \left( \tau - [\eta - m(\tau)]e^{\tau/2} \right) \left| (\dot{v} + V_{\text{app}})w \right| \leq e^{(4\varepsilon-1)\tau} \int_0^{m(\tau)} \eta^2 \, d\eta \leq e^{(4\varepsilon-1)\tau}. \]

Similarly,
\[ A_0 \int_{m(\tau)}^{\infty} e_0 \exp \left( \tau - [\eta - m(\tau)]e^{\tau/2} \right) \left| (\dot{v} + V_{\text{app}})w \right| \leq e^{(2\varepsilon+1)\tau} \int_0^{\infty} \eta^2 (1 + \eta^2) e^{3\varepsilon/2} \, d\eta \leq e^{(2\varepsilon-1)\tau} \int_0^{\infty} x^2 (1 + x^3) e^{-x} \, dx \leq e^{(2\varepsilon-1)\tau}. \]

Therefore
\[ \left| \langle e_0, A_0 \exp \left( \tau - [\eta - m(\tau)]e^{\tau/2} \right) (\dot{v} + V_{\text{app}})w \rangle \right| \leq e^{(4\varepsilon-1)\tau}, \]

and
\[ \frac{d}{d\tau} \langle e_0, w \rangle - \frac{1}{2} \langle e_0, w \rangle = \nu_0(\tau) \quad (6.10) \]

with \( |\nu_0(\tau)| \leq e^{(4\varepsilon-1)\tau} \).

Now \( \lim_{\tau \to \infty} \langle e_0, w \rangle = 0 \) by Cauchy–Schwarz (and ultimately by our choice of \( a_0 \)). Hence we may integrate (6.10) back from \( \tau = +\infty \) (with the integrating factor \( e^{-\tau/2} \)) to obtain
\[ \left| \langle e_0, w \rangle \right| \leq e^{\tau/2} \int_{\tau}^{\infty} e^{-\tau'/2} |\nu_0(\tau')| \, d\tau' \leq e^{(4\varepsilon-1)\tau}. \quad (6.11) \]

Thus the \( e_0 \)-component of \( w \) is as small as desired.

We next consider the \( e_1 \)-component, which satisfies
\[ \frac{d}{d\tau} \langle e_1, w \rangle + \frac{1}{2} \langle e_1, w \rangle + \langle e_1, A_0 \exp \left( \tau - [\eta - m(\tau)]e^{\tau/2} \right) (\dot{v} + V_{\text{app}})w \rangle = \sum_{i=1}^3 \langle e_1, E_i \rangle. \]

An identical argument shows
\[ \frac{d}{d\tau} \langle e_1, w \rangle + \frac{1}{2} \langle e_1, w \rangle = \nu_1(\tau) \quad (6.12) \]

with \( |\nu_1(\tau)| \leq e^{(4\varepsilon-1)\tau} \). We rewrite (6.12) as
\[ \frac{d}{d\tau} \left( e^{\tau/2} \langle e_0, w \rangle \right) = e^{\tau/2} \nu_1(\tau). \]

Integrating from \( \tau = 0 \), we obtain
\[ \langle e_1, w(\tau, \cdot) \rangle = e^{-\tau/2} \left[ \langle e_0, w(0, \cdot) \rangle + \int_0^{\infty} e^{\tau'/2} \nu_1(\tau') \, d\tau' \right] - e^{-\tau/2} \int_0^{\infty} e^{\tau'/2} \nu_1(\tau') \, d\tau'. \]

We therefore choose
\[ r = \langle e_0, w(0, \cdot) \rangle + \int_0^{\infty} e^{\tau'/2} \nu_1(\tau') \, d\tau'. \]
It follows that

$$\langle e_1, w(\tau, \cdot) \rangle = re^{-\tau/2} + O(e^{(4\varepsilon-1)\tau}).$$

(6.13)

We must now control the remaining terms in $w$, namely

$$w^\perp := w - \langle e_0, w \rangle e_0 - \langle e_1, w \rangle e_1.$$

From (6.4),

$$\frac{1}{2} \frac{d}{d\tau} \|w^\perp\|^2 + \langle M w^\perp, w^\perp \rangle + A_0 e^\tau \int_{\mathbb{R}^+} \exp \left( [m(\tau) - \eta]e^{\tau/2} \right) (\partial + \bar{V}_{\text{app}}) w^\perp \, d\eta = \sum_{j=1}^3 \langle E_j, w^\perp \rangle. \tag{6.14}$$

Note that (6.11) and (6.13) imply the bounds in (6.7) hold for $w^\perp$ as well. So

$$\left| A_0 e^\tau \int_{\mathbb{R}^+} \exp \left( [m(\tau) - \eta]e^{\tau/2} \right) (\partial + \bar{V}_{\text{app}}) w^\perp \, d\eta \right| \lesssim e^{3\varepsilon \tau} \int_{\mathbb{R}^+} \eta^2 (1 + \eta^3 e^{3\varepsilon \tau}) \exp \left( -\eta e^{\tau/2} \right) \, d\eta$$

$$\lesssim e^{(3\varepsilon-3/2)\tau}.$$

Next,

$$\left| \langle E_{11}, w^\perp \rangle \right| \lesssim e^{(4\varepsilon-1)\tau} \int_0^{2m(\tau)} \eta \, d\eta \lesssim e^{(6\varepsilon-2)\tau}.$$

Similarly $|\langle E_2, w^\perp \rangle| \lesssim e^{(3\varepsilon-5)/2\tau}$. For the $E_{12}$ term, we use a Peter–Paul inequality and keep track of constants:

$$\left| \langle E_{12}, w^\perp \rangle \right| \lesssim \varepsilon \|w^\perp\|^2 + C_\varepsilon \|E_{12}\|^2 \lesssim \varepsilon \|w^\perp\|^2 + C_\varepsilon e^{(8\varepsilon-2)\tau}.$$

The $E_3$ term requires a more elaborate analysis. First, we easily have

$$\left| \langle \partial(\tau)w, w^\perp \rangle \right| \lesssim \varepsilon^{-\tau/2} \|w^\perp\|^2 \lesssim e^{(2\varepsilon-3/2)}.$$

Now turn to $g(\tau)(w_0 - \eta w/4)$. Integrating by parts,

$$\int_{\mathbb{R}^+} w_0 w^\perp = \int_{\mathbb{R}^+} \left[ \frac{1}{2} \partial_\eta(w^2) + \langle e_0, w \rangle \partial_\eta(e_0)w + \langle e_1, w \rangle \partial_\eta(e_1)w \right].$$

Now $w$ satisfies Dirichlet boundary conditions and $(e_0)_\eta, (e_1)_\eta \in L^2(\mathbb{R}^+)$, so by Cauchy–Schwarz

$$\left| \int_{\mathbb{R}^+} \eta w^\perp \right| \lesssim \left( \|w_0\| + |\langle e_1, w \rangle| \|w\| \right) \lesssim e^{(\varepsilon-1)\tau}.$$

Next, consider

$$\int_{\mathbb{R}^+} \eta w^\perp = \int_{\mathbb{R}^+} \eta(w^\perp)^2 + \langle e_0, w \rangle \int_{\mathbb{R}^+} \eta w_0 w^\perp + \langle e_1, w \rangle \int_{\mathbb{R}^+} \eta e_1 w^\perp.$$
Since $\eta_0, \eta_1 \in L^2(\mathbb{R}_+)$, the last two terms are $O(e^{(\varepsilon-1)\tau})$. For the first term, 
\[ \int_0^T (w^\perp)^2 \lesssim e^{(2\varepsilon-1)\tau}, \]
so we have 
\[ \int_{\mathbb{R}_+} \eta(w^\perp)^2 \lesssim \int_{\mathbb{R}_+} \eta^2(w^\perp)^2 + O(e^{(2\varepsilon-1)\tau}). \]

Finally, 
\[ \int_{\mathbb{R}_+} \eta^2(w^\perp)^2 = 16 \langle Mw^\perp, w^\perp \rangle + 20\|w^\perp\|^2 - 16\|w^\perp\|^2_q \lesssim 16 \langle Mw^\perp, w^\perp \rangle + C \varepsilon e^{(2\varepsilon-1)\tau}. \]

The prefactor $g(\tau)$ in $E_3$ is eventually positive and of order $e^{(\varepsilon-1/2)\tau}$. So for large $\tau$,
\[ \langle E_3, w^\perp \rangle \leq C e^{(\varepsilon-1/2)\tau} \langle Mw^\perp, w^\perp \rangle + C \varepsilon e^{(3\varepsilon-3/2)\tau}. \]
Combining these bounds and using $\langle Mw^\perp, w^\perp \rangle \geq \frac{3}{2} \|w^\perp\|^2$, we obtain for large $\tau$:
\[ \frac{1}{2} \frac{d}{d\tau} \|w^\perp\|^2 + \left( \frac{3}{2} - \varepsilon - C e^{(\varepsilon-1/2)\tau} \right) \|w^\perp\|^2 \lesssim C e^{(4\varepsilon-3/2)\tau}, \]
where $C$ depends on $\gamma$ and $\alpha_0$. We can absorb $C e^{(\varepsilon-1/2)\tau/2} \|w^\perp\|^2$ into the right-hand side and integrate to obtain 
\[ \|w^\perp\|^2 \lesssim e^{(4\varepsilon-3/2)\tau}. \] (6.15)

Together with (6.11) and (6.13), this bound implies (6.9).

We are now able to set the final value of $q_3$ in (3.9), and thus to fully specify $V_{\text{app}}$. We let $q := c_1^{-1} r$, and set $q_3 = q$. We claim that with this choice,
\[ \lim_{\tau \to \infty} e^{\tau/2} |\langle e_1, w \rangle| = 0. \] (6.16)

Thus $q_3$ is chosen to kill the $e_1$-component of $w$, just as $\alpha_0$ was chosen in [13] to kill the $e_0$-component.

To see (6.16), we consider how the change in $q_3$ affects $w$. We use superscript $o$ and $n$ to denote the old and new definitions, respectively. So $w$ changes from $w^o$ to $w^n$. By the calculations in the proof of lemma 10, the changes to $w$ on the interval $[0, 2m(\tau)]$ are negligible in $L^2(\mathbb{R}_+)$. We therefore focus on the changes to $w$ on $[2m(\tau), \infty)$.

When we increase $q_3$ from 0 to $q$, we must decrease $\alpha_1$ by $\frac{3}{2q}$ to satisfy (3.11). So 
\[ \sigma^n(t) = \sigma^o(t) - \frac{3q}{2t}. \]

Evaluating $u$ in the unshifted frame, we have:
\[ u(t, x) := e^\tau u(t, x + \sigma^o(t)). \]

Thus 
\[ v(t, x) = A_0^{-1} e^\tau u(t, x + \sigma^n(t)) = A_0^{-1} e^\tau u \left( t, x + \sigma^n(t) - \frac{3}{2} qt^{-1} \right) = e^{i\tau v^-} v^n \left( t, x - \frac{3}{2} qt^{-1} \right). \]
We shift $x$ by $r^*$, and switch to the self-similar variables. By the decay of $w$ and the form of $V^+$, we know that $v(\tau, \cdot) = e^{\sigma/2} \phi_0(\cdot) + O(1)$ in $L^2$. Using $e^{3q \sigma^{-1}} = 1 + \frac{3}{2} q r^{-1} + O(r^{-2})$, we have:

$$\hat{v}^0(\tau, \eta) = \hat{v}^0(\tau, \eta) - \frac{3}{2} q e^{-3\tau/2} + \frac{3}{2} q e^{-\tau/2} \phi_0(\eta - m(\tau)) + O(e^{-\tau/2})$$

in $L^2(\mathbb{R}^+)$.

The approximate solution $V_{\text{app}}$ is changed through $V^+_3$ by $q e^{-\tau/2} (\frac{3}{2} \phi_0 + \phi_1)$. So

$$\hat{V}^n_{\text{app}}(\tau, \eta) = \hat{v}^0(\tau, \eta) - \frac{3}{2} q e^{-\tau/2} \phi_0(\eta - m(\tau)) + q e^{-\tau/2} \phi_1(\eta - m(\tau))$$

Recall that on $[2m(\tau), \infty)$, $w = e^{\sigma/8}(\hat{v} - \hat{V}_{\text{app}})$. Thus the above observations imply

$$w^0(\tau, \eta) - w^0(\tau, \eta) = -q e^{\sigma/8} e^{-\tau/2} \phi_1(\eta - m(\tau)) + O(e^{-\tau}) = -q e^{\sigma/8} e^{-\tau/2} \phi_1(\eta - m(\tau)) + O(e^{(1-\epsilon)\tau})$$

Hence by lemma 10,

$$\lim_{\tau \to \infty} e^{\tau/2} |\langle e_1, w^0 \rangle| = 0.$$  

For the remainder of the proof we use the new forms of all functions defined with $q_3 = q$, and drop the superscript $n$. The calculations in the proof of lemma 10 continue to hold for $w$, but now (6.16) implies

$$\langle e_1, w \rangle = -e^{-\tau/2} \int^\infty_\tau e^{-\tau/2} \nu_1(\tau') d\tau'$$

with $|\nu_1(\tau)| \lesssim e^{(4\epsilon-1)\tau}$, so

$$|\langle e_1, w \rangle| \lesssim e^{(4\epsilon-1)\tau}.$$  

By (6.15),

$$\|w\|_{L^2(\mathbb{R}^+)} \lesssim e^{(2\epsilon - 3/4)\tau}.$$  

We now wish to obtain uniform bounds on $w$ as well.

Fix $A > 0$ large enough that $\frac{\epsilon^2}{16} - \frac{3}{4} - 100\eta - 100 \geq 0$ for $\eta \geq A$. On the interval $[0, A]$, parabolic regularity implies

$$\|w\|_{L^\infty([0, A])} \lesssim C e^{(2\epsilon - 3/4)\tau}$$

for $\tau \geq 1$. Now consider a maximum of $|w|$ on $[A, \infty)$. There $w_\eta$ vanishes, so our previous bounds imply

$$w_\tau + \left[ M + \frac{1}{4} g(\tau) \eta - h(\tau) \right] w = E$$

with $\|E\|_{L^\infty(\mathbb{R}^+)} \lesssim e^{(4\epsilon-1)\tau}$ and $|g|, |h| \lesssim 100$. By (6.4), the form of $M$, and the definition of $A$, any maximum of $|w|$ on $[A, \infty)$ larger than $C e^{(4\epsilon-1)\tau}$ will decrease in magnitude as $e^{-3\tau/4}$. Since $w$ is initially bounded, this implies $\|w\|_{L^\infty([A, \infty))} \lesssim C e^{(2\epsilon - 3/4)\tau}$. Combining this with the bound on $[0, A]$, we obtain

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\[ \|w\|_{L^2(\mathbb{R}_+)} + \|w\|_{L^\infty(\mathbb{R}_+)} \lesssim e^{(2e-3/4)\tau} \quad \text{for } \tau \geq 1. \quad (6.17) \]

Next, we wish to use the Dirichlet condition on \( w \) to show that in fact
\[ |w(\tau, \eta)| \lesssim \eta e^{(2e-3/4)\tau} \quad \text{for } \tau \geq 1. \]

By the Kato inequality, on a sufficiently small interval \( \eta \in (0, a) \) with \( a > 0 \),
\[ \frac{\partial}{\partial \tau} |w| - \frac{\partial}{\partial \eta} |w| - 10 |w| - g(\tau) \frac{\partial}{\partial \eta} |w| \leq C \left[ e^{(4e-1/2)\tau} I_{[0,3m(\tau)]}(\eta) + e^{(4e+1)\tau} \right]. \]

By (6.17), we have boundary conditions \( |w(\tau, 0)| = 0 \) and \( |w(\tau, a)| \lesssim Ce^{(2e-3/4)\tau} \). Let \( \varphi_0 \) solve
\[-\frac{\partial}{\partial \eta} \varphi_0 = Ce^{(4e-1/2)\tau} I_{[0,3m(\tau)]}\] on \((0, a)\) with \( \varphi_0(0) = \varphi_0(a) = 0 \). Then \( \varphi_0 \) is explicitly given by:
\[ \varphi_0(\eta) = \begin{cases} C e^{(4e-1/2)\tau} \frac{\gamma(6m(\tau) - \frac{9m(\tau)^2}{a} - \eta)}{2a} & \text{for } \eta \in [0, 3m(\tau)], \\ C e^{(4e-1/2)\tau} \frac{9m(\tau)^2}{2a} (a - \eta) & \text{for } \eta \in [3m(\tau), a]. \end{cases} \]

From this form, we see that \( \varphi_0(\eta) \lesssim Ce^{(5e-1)\tau} \). We may then write \( |w| \leq \varphi_0 + Ce^{(2e-3/4)\tau} \varphi_1 \) with \( \varphi_1 \) satisfying
\[ \frac{\partial}{\partial \tau} \varphi_1 - \frac{\partial}{\partial \eta} \varphi_1 - 11 \varphi_1 - g(\tau) \frac{\partial}{\partial \eta} \varphi_1 = e^{(2e-1/4)\tau}. \]

By choosing a small \( a \), we may ensure the eigenvalue \( \lambda_a \) of the Dirichlet Laplacian on \((0, a)\) satisfies \( \lambda_a > 100 \). This forces \( \varphi_1 \leq C\eta \). Therefore \( |w(\tau, \eta)| \lesssim \eta e^{(2e-3/4)\tau} \) when \( \tau \geq 1 \), as desired.

In summary, we have bootstrapped (6.7) to
\[ \|w\|_{L^2(\mathbb{R}_+)} + \|w\|_{L^\infty(\mathbb{R}_+)} \lesssim e^{(e-3/4)\tau}, \quad |w(\tau, \eta)| \lesssim \eta e^{(2e-3/4)\tau} \quad \text{for } \tau \geq 1. \]

(6.18)

However, this bound is still weaker than (6.8). We improve it further by performing the computations in the proof of lemma 10 again, now using (6.18) and
\[ |\langle e_0, w \rangle| + |\langle e_1, w \rangle| \lesssim e^{(4e-1)\tau}. \]

The term of concern is thus \( \|w^\perp\| \).

Consider (6.14). We wish to control \( \|w^\perp\| \) with error \( O(e^{(6e-2)\tau}) \). Hence our earlier bounds on \( E_1 \) and \( E_2 \) suffice. Now note that (6.18) holds with \( w \) replaced by \( w^\perp \). So
\[ A e^{\tau} \int_{\mathbb{R}_+} \exp \left( \frac{( m(\tau) - \eta)e^{\tau/2} }{ 2 + V_{app} } w_{\perp} \right) d\eta \lesssim e^{(3e-1/2)\tau} \int_{\mathbb{R}_+} \eta^2 (1 + \eta^2 e^{3\tau/2}) \exp \left( -\eta e^{\tau/2} \right) d\eta \lesssim e^{(3e-2)\tau}. \]

Following the earlier analysis of the \( E_3 \) term, we find
\[ \int_{\mathbb{R}_+} w_{\eta} w_{\perp} \lesssim |\langle e_0, w \rangle| + |\langle e_1, w \rangle| \|w\| \lesssim e^{(6e-7)\tau} \]

and
\[ \int_{\mathbb{R}_+} \eta w_{\perp} \lesssim 16 \langle M w^\perp, w_{\perp} \rangle + 2 \|w^\perp\|^2 + C e^{(4e-3/2)\tau}. \]

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Thus
\[
\langle E_3, w^{-} \rangle \lesssim C e^{-\tau/2} \langle M w^{-}, w^{-} \rangle + C e^{(5\varepsilon - 2)\tau}.
\]
Arguing as before, these bounds and (6.14) imply
\[
\|w^{-}\|^2 \lesssim e^{(10\varepsilon - 2)\tau}.
\]
Therefore
\[
\|w\|_{L^2(\mathbb{R}^+)} \lesssim e^{(5\varepsilon - 1)\tau}.
\]
Repeating the $L^\infty$ arguments with this new control, we obtain (6.8). In particular,
\[
|w(\tau, \eta)| \lesssim \eta e^{(5\varepsilon - 1)\tau} \quad \text{for } \tau \geq 1.
\]
Finally, choose $\varepsilon = \frac{\gamma}{6}$. In the physical variables, we find
\[
|v(t, x) - V_{\text{app}}(t, x)| \leq C \gamma \left( \frac{x + t^\varepsilon}{\sqrt{t}} \right) t^{\varepsilon - 1} \leq \frac{C \gamma (1 + |x|)}{t^{\frac{1}{\varepsilon} - 1}}
\]
when $t \geq 3$ and $x \geq 2 - t^\varepsilon$. This concludes the proof of theorem 9.

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**Appendix**

In this appendix, we use standard ODE theory to establish lemma 5.

**Proof of lemma 5.** We first show that there exists a solution to (3.1) decaying as $x \to -\infty$.

Consider the travelling front $\phi$, which satisfies
\[
\phi'' + 2\phi' + \phi - \phi^2 = 0. \tag{A.1}
\]
Expanding this equation around $\phi = 1$, we find that
\[
\phi(x) = 1 - B_0 e^{(\sqrt{2} - 1)x} + O(e^{2(\sqrt{2} - 1)x}) \quad \text{as } x \to -\infty
\]
for some $B_0 > 0$. Recalling that $V_0^{-}(x) = A_0^{-1} e^x \phi(x)$, we have:
\[
\frac{3}{2} [V_0^{-} - (V_0^{-})'] = \frac{3}{2} A_0^{-1} B_0 (\sqrt{2} - 1) e^{\sqrt{2}x} + O(e^{2(\sqrt{2} - 1)x}).
\]
So (3.1) has the form
\[
-V'' + 2V = FV + \frac{3}{2} A_0^{-1} B_0 (\sqrt{2} - 1) e^{\sqrt{2}x} + G, \tag{A.2}
\]
where $F = O(e^{(\sqrt{2} - 1)x})$ and $G = O(e^{(2(\sqrt{2} - 1)x)}).$ We construct a series solution to (A.2). We first seek a decaying solution to
\[ -V''_0 + 2V_0 = \frac{3}{2}A_0^{-1}B_0(\sqrt{2} - 1)e^{\sqrt{2}x} + G. \] \tag{A.3}

The homogeneous solutions to \(-V'' + 2V = 0\) are \(e^{\pm \sqrt{2}x}\). By the theory of constant-coefficient ODEs, there exists a solution to (A.3) of the form
\[ V_0 = -\frac{3}{4\sqrt{2}}A_0^{-1}B_0(\sqrt{2} - 1)e^{\sqrt{2}x} + O(e^{(2\sqrt{2}-1)x}). \]

Thus for fixed small \(\delta > 0\), there exists \(C > 0\) such that
\[ |V_0(x)| \leq Ce^{(\sqrt{2}-\delta)x} \quad \text{for } x \leq 0. \]

Choose a \(C\) large enough that \(|F(x)| \leq Ce^{(\sqrt{2}-1)x}\). Then define a sequence of functions \((V_k)\) by
\[ -V''_{k+1} + 2V_{k+1} = FV_k, \quad \lim_{x \to -\infty} e^{\sqrt{2}x}V_{k+1}(x) = 0, \quad \text{for } k \in \mathbb{Z}_{\geq 0}. \]

We will show by induction that
\[ |V_k(x)| \leq \frac{C^{k+1}}{(\sqrt{2} - 1 - \delta)^k}e^{(k(\sqrt{2} - 1) + \sqrt{2} - \delta)x}. \] \tag{A.4}

This already holds for \(V_0\), so suppose it holds for \(V_k\). We can bound \(V_{k+1}\) by writing the second-order equation for \(V_{k+1}\) as a first-order system, which we solve with matrix exponentials. Taking norms, we obtain:
\[ |V_{k+1}(x)| \leq \int_{-\infty}^{x} e^{\sqrt{2}(x-y)} |FV_k(y)| \, dy \leq \int_{0}^{\infty} e^{\sqrt{2}z} |FV_k(x-z)| \, dz \]
\[ \leq \frac{C^{k+1}}{(\sqrt{2} - 1 - \delta)^k}e^{(k(\sqrt{2} - 1) + \sqrt{2} - \delta)x} \int_{0}^{\infty} e^{-(k+1)(\sqrt{2} - 1 - \delta)z} \, dz. \]

Bounding the final integral by \(|(k+1)(\sqrt{2} - 1 - \delta)|^{-1}\), we have (A.4). Similar bounds can be shown for \(V''_{k+1}\) and \(V''_k\). Thus
\[ V := \sum_{k \geq 0} V_k \]
converges in \(C^2(\mathbb{R})\), and solves
\[ -V'' + 2V = -V''_0 + 2V_0 + \sum_{k \geq 0} (-V''_{k+1} + 2V_{k+1}) = FV + \frac{3}{2}A_0^{-1}B_0(\sqrt{2} - 1)e^{\sqrt{2}x} + G. \]

Finally, \(V = O(e^{(\sqrt{2}-\delta)x})\) on \(\mathbb{R}_-\), so \(V\) is a decaying solution to (A.2), as desired.

Now let
\[ \hat{V}(x) := A_0^{-1}e^{-x}V(x). \]

Equation (A.1) implies
\[ -\hat{V}'' + 2A_0e^{-x}V_0^{-1}V = 0. \] \tag{A.5}
With the bounds noted previously, we have
\[ \dot{V}(x) = O\left(e^{x/2}\right) \] as \( x \to -\infty \) and \( \dot{V}(x) = 1 - x + O(e^{-\omega x}) \) as \( x \to +\infty \).

So \( \dot{V} \) is a solution of the homogeneous equation (A.5), which decays at \(-\infty\) and has known asymptotics at \(+\infty\).

Now consider the behaviour of \( V \) as \( x \to \infty \). We claim that \( V \) satisfies
\[ V(x) = -\frac{1}{4}x^3 + \frac{3}{4}x^2 + C_1x + C_0 + O(e^{-\omega x/2}) \]
for some \( C_1, C_0 \in \mathbb{R} \). Let
\[ Z(x) := V(x) + \frac{1}{4}x^3 - \frac{3}{4}x^2. \]
Then since \( V(x) = x - 1 + O(e^{-\omega x}) \) as \( x \to \infty \), we have
\[ Z'' = HZ + K \tag{A.6} \]
with \( H, K = O(e^{-\omega x}) \). We will argue that \( Z = C_1x + C_0 + O(e^{-\omega x/2}) \). Fix \( D \geq 0 \) such that \( |H(x)| \leq \frac{\omega}{4} \) when \( x \geq D \). We solve (A.6) using the matrix exponential again. Taking norms, we can show that \( |Z| \) is dominated on \([D, \infty)\) by solutions to \( \tilde{Z}'' = \frac{\omega}{4} \tilde{Z} \), namely linear combinations of \( e^{\pm \omega x/2} \). So \( |Z(x)| \leq Ce^{\omega x/2} \) on \([D, \infty)\). With this \textit{a priori} bound, we see that \( Z'' = O(e^{-\omega x/2}) \) on \([D, \infty)\). Integrating twice, we obtain
\[ Z(x) = C_1x + C_0 + O(e^{-\omega x/2}) \]
for some \( C_1, C_0 \in \mathbb{R} \), as desired.

Finally, let
\[ V_1^{-} := V - C_0\dot{V}. \]
Then \( V_1^{-} \in C^2(\mathbb{R}) \) solves (3.1), and satisfies the bounds in lemma 5 with \( C_1^{-} = C_1 + C_0 \). Although we have not explicitly discussed \((V_1^{-})'\), its bounds follow those for \( V_1^{-} \). \( \square \)

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