Analytic two-loop results for selfenergy- and vertex-type diagrams with one non-zero mass

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Abstract

For a large class of two-loop selfenergy- and vertex-type diagrams with only one non-zero mass ($M$) and the vertices also with only one non-zero external momentum squared ($q^2$) the first few expansion coefficients are calculated by the large mass expansion. This allows to 'guess' the general structure of these coefficients and to verify them in terms of certain classes of 'basis elements', which are essentially harmonic sums. Since for this case with only one non-zero mass the large mass expansion and the Taylor series in terms of $q^2$ are identical, this approach yields analytic expressions of the Taylor coefficients, from which the diagram can be easily evaluated numerically in a large domain of the complex $q^2$–plane by well known methods. It is also possible to sum the Taylor series and present the results in terms of polylogarithms.

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1 Introduction

Higher loop calculations present great difficulties especially in the presence of masses. When there is a large number of parameters (masses and external momenta) in the problem, analytic expressions can hardly be obtained beyond one-loop. In such cases one has to resort to numerical or approximative methods. In particular the rather large spectrum of masses in the Standard Model requires numerous and different approximation schemes (e.g. small fermion masses \(m_f^2/m_Z^2 \ll 1\), close vector boson masses \((m_Z^2 - m_W^2)/m_Z^2 \ll 1\), large top mass \(m_Z^2/m_t^2 \ll 1\) etc.) and often a problem is reduced to diagrams with one mass.

In this work we consider the evaluation of 2-loop diagrams of self-energy and vertex type with one non-zero mass and all others zero.

For the analytic evaluation of ‘master integrals’ with masses various techniques have been developed. Such calculations in QED are known since [1]. Using dispersion relations 2-loop 2-point integrals relevant for QED and QCD were obtained in [2]. Results for a class of 2-loop self energies are given in [4]-[6]. The method of differential equations was developed in [7]. Some recent results, applying this method for 2-loop vertex functions, are presented in [8]. At the 3-loop order analytic results are known only for vacuum bubbles so far [9].

Very powerful approaches for the evaluation of diagrams, which are in fact our starting point, are the asymptotic expansion [10] and/or Taylor expansion [11, 12] in an external momentum squared, which are identical if only one non-zero mass is involved. These methods allow to expand a diagram in a series with coefficients, which in many cases can be represented in a relatively simple analytic form. Analytic continuation by means of a mapping in the plane of momentum squared allows to calculate the diagrams on their cut. For propagator type diagrams such expansion was successfully applied in [13, 14] while for vertex functions it is discussed in [11].

So far the method of expansion is considered as semianalytic in the sense that only a limited number of coefficients can be obtained explicitly. In this paper, however, we want to go one step further. We calculate the first few coefficients of the expansion of a diagram with only one non-zero mass by means of the asymptotic expansion [10] method. The method of differential equations [7] then yields an idea, like in [8], what the general analytic form of these coefficients might be, providing some ‘basis’ in terms of which they might be expressed. The Ansatz of equating the explicit coefficients obtained from the asymptotic expansion to a linear combination in terms of the basis elements, yields a system of linear equations, which can be solved to yield the desired representation of the coefficients.

The main problem in this approach is the choice of the basis elements. We start from so-called harmonic sums which are particularly relevant for moments of structure functions in QCD (e.g. [15]). These functions are directly related to (generalized) polylogarithms [16] and therefore it is not surprising that they appear in the analysis of massive diagrams (see also recent research about these sums [17]). However, not every massive diagram even of self-energy type can be expressed in terms of these sums. In particular, if a diagram possesses a cut with three massive particles in the intermediate state then it is rather a subject of elliptic integrals than polylogarithmic ones [2, 13]. The method should work in this case as well provided one finds the relevant class of basis elements. However, in this paper we do not consider diagrams of such type and concentrate ourselves on those with one- and/or two-massive-particle \((m^2\) and \(2m^2\)) cuts. We give the results for all selfenergies and a certain number of vertices with \(m^2\) and \(2m^2\)-cuts. In particular we include diagrams relevant for \(Z\)-decay.
The paper is organized as follows: after introducing some notation in Sect. 2, we explain in Sect. 3 the main idea of our approach in terms of some simple examples. Sect. 4 contains a ‘straightforward’ prescription for the generalization of our basis elements, while in Sect. 5 we demonstrate how to find more complicated ones occurring in $2m$-cuts. The differential equation method (DEM) in Sect. 6 demonstrates a further method to explore the form of basis elements. Sect. 7 contains a proposal of how to perform the large $q^2$-expansion by means of the Sommerfeld-Watson transformation, providing an explicit example. The Appendices contain further notations and results.

2 Notation and definitions

The generic topologies considered in this work are displayed in Fig. 1. Lines of the diagrams represent scalar propagators $(p^2 - m^2 + i0)^{-1}$ with either mass $m$ or zero mass. For both ultraviolet and infrared divergences we use dimensional regularization in the Minkowski space with dimension $D = 4 - 2\varepsilon$. Each loop integration is normalized as follows

$$\int d\tilde{k} = (m^2 e^{\gamma_E})^\varepsilon \int \frac{d^D k}{\pi^{D/2}}$$

with $\gamma_E$ the Euler constant.

Our starting point is the large mass expansion $[10]$ of the diagrams. Here we just note that the result of a large mass expansion for these diagrams reads ($z = q^2/m^2$)

$$J = \frac{1}{(q^2)^a} \sum_{n \geq 1} z^n \sum_{j=0}^{\omega} \sum_{k=0}^{\nu} \log^k (-z) A_{n,j,k},$$

where $a$ is the dimensionality of the diagram, $\omega$ and $\nu$ independent of $n$ are the highest degree of divergence and the highest power of $\log(-q^2/m^2)$, respectively (in our cases $\omega, \nu \leq 4$). The coefficients $A_{n,j,k}$ are of the form $r_1 + \zeta_2 r_2 + \ldots + \zeta_\nu r_\nu$ with $r_c$ being rational numbers and $\zeta_c = \zeta(c)$ is the Riemann $\zeta$-function.

Series (2) always has a nonzero radius of convergence, which is defined by the position of the nearest nonzero threshold in the $q^2$-channel. For brevity we shall call $m$-cuts (2$m$-cuts, etc.) possible cuts of a diagram in $q^2$ corresponding to 1 (2, etc.) massive particles in the intermediate state.

3 Diagrams with $m$-cuts

We start from diagrams with the simplest threshold structure, i.e. with $m$-cuts and (possible) 0-cuts. Such diagrams are particularly easy to handle and results for all of them within the considered generic topologies of Fig. 1 can be found in the appendices. It will be shown that all are expressible in terms of harmonic sums.

As a first example consider the 2-loop 2-point function of Fig. 2, $I_{13}$, which was considered in [2]. Using the standard large mass expansion technique one can get the first few coefficients of the expansion of this diagram in powers of $z = q^2/m^2$

$$\frac{1}{q^2} \sum_{n=1}^{\infty} a_n z^n = \frac{1}{q^2} \left(2 \zeta_2 z + \left(\zeta_2 + \frac{1}{2}\right) z^2 + \left(\frac{2}{3} \zeta_2 + \frac{1}{2}\right) z^3 + \left(\frac{1}{2} \zeta_2 + \frac{65}{144}\right) z^4 + \ldots \right),$$

(3)
where $\zeta_2 = \zeta(2)$ is the Riemann $\zeta$-function.

It takes seconds to evaluate the first 10 terms (we use a package written in FORM) but the evaluation time increases exponentially for the higher order terms. Although 10–20 coefficients are sufficient to get a very precise numerical value for the diagram, even at $|z| \sim 10^2$ (via conformal mapping and accelerated convergence), this cannot be called a satisfying result.

Our key observation is that the information contained in the first 10 (or even less!) coefficients of the series is enough to restore the whole series. Namely, each coefficient $a_n$ can be expressed as some combination of harmonic sums and their extensions. In this case the series can be continued numerically or analytically to any point of the complex $z$-plane.

Thus our first step is to find an expression for the higher order terms of the series as a functions of $n$ i.e. $a_n = a(n)$.

To achieve this let us search for $a_n$ as a linear combination of elements of a certain kind. For these we take $1/n$ and $S_k(n-1) = \sum_{j=1}^{n-1} 1/j^k$ (harmonic sum). As a matter of observation, the rule is that one has to take into account all possible products of the type $\zeta_2 S_b_1 \ldots S_b_k/n^c$ with the 'transcendentality level' (TL) $a + b_1 + \ldots + b_k + c = 3$. It is obvious that one can exclude $\zeta_3$ beforehand since it never appears on the r.h.s. of (3). Thus we have the following Ansatz for $a_n$

$$a_n = \frac{\zeta_2}{n} x_1 + \zeta_2 S_1 x_2 + S_3 x_3 + S_2 S_1 x_4 + S_1^2 x_5 + \frac{S_2}{n} x_6 + \frac{S_1^2}{n} x_7 + \frac{S_1}{n^2} x_8 + \frac{1}{n^3} x_9,$$  \tag{4}

where $x_1, \ldots, x_9$ are rational numbers independent on $n$ and we omitted the arguments in the $S$-functions. These can be taken with argument $n$ or $n-1$. We choose the latter option. However, either choice gives a solution since the difference is only in rearranging $1/n$ terms. Later on we refer to the structures in (3) as 'basis elements' or just 'basis'. Indeed, the functions $\zeta_2 S_1(n-1) \ldots S_b_k(n-1)/n^c$ are algebraically independent.

Inserting the expression for $a_n$ from (3) into the l.h.s. of (3) and equating equal powers of $z$, we obtain a system of linear equations for the $x_i$. One needs at least 9 first coefficients on the r.h.s. of (3) to solve the equations for 9 variables and any system of more than 9 equations should be consistent and have the same solution if the Ansatz (4) is correct. An explicit computation ensures that the system can be solved in terms of rational numbers for the $x_i$ and this latter consistency property holds. The solution is $x_1 = 2, x_6 = 2, x_8 = -2, x_2,3,4,5,7,8 = 0$ i.e. the answer for the diagram at hand is

$$I_{13} = \frac{1}{q^2} \sum_{n=1}^{\infty} z^n \left( 2\zeta_2/n + 2 S_2(n-1)/n^2 - 2 S_1(n-1)/n^3 \right).$$ \tag{5}

This is the desired result. One notices that in (3) and Ansatz (4) the terms proportional to $\zeta_2$ are linearly independent and can be treated separately, i.e one could solve a system of linear equations for $x_1, x_2$ and another one for $x_3, \ldots, x_9$. In this sense only the 7 first coefficients of the large mass expansion are required.

Series (3) converges for $|z| < 1$ and represents in the domain of convergence an analytic function. It can be analytically continued into the whole $z$-plane except for the cut on the real axis starting at $z = 1$ where $I_{13}$ has a branch point (threshold). In particular for the given diagram, summing the series, we find the representation

$$I_{13} = \frac{1}{q^2} \left( -2\zeta_2 \log(1-z) - 2 \log(1-z) Li_2(z) - 6 S_{1,2}(z) \right),$$ \tag{6}
where $\text{Li}_\nu(z)$ is a polylogarithm and $S_{p,q}(z)$ are the generalized Nielsen polylogarithms \cite{24} (see also Appendix C).

Both representations (3) and (4) are equivalent in the sense that they both represent the same analytic function and can be unambiguously converted into each other. However, we find it more convenient to use the series representation and refer for the continuation to App. C and E. Moreover a series representation is usually shorter than a representation in terms of known functions and/or an integral representation.

Equation (4) can be considered as expansion of the general coefficient $a_n$ in terms of a ‘basis’ with (unknown, rational) coefficients $x_i$. Now the question arises: what are the basis elements in general? Definitely there must be a certain connection between the topology and the structure of thresholds of a diagram on the one hand and the structure of the basis elements on the other hand. So far we are lacking rules for predicting a basis of a given diagram. The power of the method, however, is that given a set of basis elements for one diagram, it can be used to find the solution for other diagrams. Often, though, one has to ‘generalize’ already known (harmonic) sums in a basis. In Sect. 4 we explain how to do this.

In particular it will be shown that many 2-point and 3-point functions have similar bases and thus solving some 2-point integrals (which are simpler) we can find solutions for 3-point integrals. As an illustration consider the diagram shown in Fig. 3, $P_5$. Its large mass expansion looks like

$$P_5 = \frac{1}{(q^2)^2} \sum_{n=1}^{\infty} z^n \left( r_n^{(2)} \log^2(-z) + r_n^{(1)} \log(-z) + r_n^{(0,3)} \zeta_3 + r_n^{(0,2)} \zeta_2 + r_n^{(0,0)} \right),$$

with the $r$’s being rational numbers. It is obvious that one can search for a solution for each of the $r$’s independently. Again we use the same set of functions ($1/n^a$ and $S_b$) as above but now with different transcendentality level(s). The system of equations has a solution only if we add the factor $(-)^n$ which can be seen easily by inspection of the series. At the end we arrive at

$$P_5 = \frac{1}{(q^2)^2} \sum_{n=1}^{\infty} z^n (-)^n \left( \frac{S_1}{n} \log^2(-z) + \left( -4 \frac{S_2}{n} + \frac{S_1^2}{n^2} - 2 \frac{S_1}{n^2} \right) \log(-z) 
- \frac{\zeta_3}{n} + \frac{\zeta_2 S_1}{n} + \frac{S_3}{n} - 2 \frac{S_2 S_1}{n^2} + 4 \frac{S_2}{n^2} \frac{S_1^2}{n^4} + 2 \frac{S_1}{n^4} \right),$$

where as above we take all harmonic sums ($S'_i$) with the argument $n - 1$, which is omitted.

The similarity of (3) and (8) is spectacular. There are two comments in order.

At first we point out that in (8) the part without $\log(-z)$ has basis elements $\zeta_a S_b/n^c$ obeying $a + b + c = 4$ (i.e. it has a basis with TL= 4). Terms proportional to $\log(-z)$ are of 3rd level while those proportional to $\log^2(-z)$ of the 2nd. One can say that $\log(-z)$ itself is of 1st level and each $\log^a(-z)$ reduces the level of basis elements by $a$ units. This is the general behaviour for all diagrams we consider. The same rule applies to UV and IR poles $1/\varepsilon$ if they are present in a diagram i.e. each $1/\varepsilon^a$ reduces the level of basis elements by $a$ units.

Secondly we wish to underline the presence of the factor $(-)^n$. Strictly speaking the basis now is not $\zeta_a S_b/n^c$ any more but rather $(-)^n \zeta_a S_b/n^c$. In principle one can expect that a mixture of both bases occurs and this is indeed the case for some diagrams. Moreover, the $(-)$ may
stand not only in front of $S_a$ but also inside the harmonic summation. Thus we are led to the alternating harmonic sum $K_a(n-1) = \sum_{j=1}^{n-1} (-1)^{j+1}/j^a$.

It will turn out that the results obtained above indicate already a general property, i.e. in 2-loops all the selfenergy functions have TL=3 and all vertices have TL=4 (at least for the diagrams under consideration).

4 Generalization of basis elements

It was mentioned above that we cannot directly predict the basis of a diagram. Therefore we experiment with basis elements of diagrams calculated before to match the expansion coefficients of other ones. Another extremely useful trick is that we first establish the structure of the lowest level in a diagram (i.e. the structure of the coefficients of the highest pole in $\varepsilon$ or the highest power of $\log(-z)$) and then try to reconstruct terms of higher level. If, say, the $1/\varepsilon^k$ coefficients are expressed symbolically in terms of basis elements $f_a$ with TL=a then the $1/\varepsilon^{k-1}$ part is expressed in terms of some $f_{(a+1)}$ etc. Therefore we need rules of generalization, i.e. to find higher level elements from lower level ones. These are discussed in this Section.

We start from the following observation. Let $J$ be a $D$-dimensional Feynman integral (properly normalized to be dimensionless) depending on $z = q^2/m^2$. It can be formally written as multiple series of the hypergeometric type

$$J(z) = \sum_{j_1,j_2,\ldots,j_s} \frac{\Gamma(\alpha_1\{j,D\}) \ldots \Gamma(\alpha_\nu\{j,D\})}{\Gamma(\beta_1\{j,D\}) \ldots \Gamma(\beta_\rho\{j,D\})} z^{\gamma\{j,D\}},$$

where $\Gamma$ is the Euler $\Gamma$-function. Here the symbols $\alpha_i\{j,D\}$, $\beta_i\{j,D\}$ and $\gamma\{j,D\}$ stand for some linear combinations of the summation indices $j_k$ and the space-time dimension $D = 4-2\varepsilon$.

The form (9), though rather obvious by itself, can be deduced e.g. from the Feynman parameter representation or $\alpha$-representation. From this point of view, ‘to evaluate’ diagram $J$ means to bring multiple sums in (9) to some ‘known’ hypergeometric form of series or at least to remove as many summations as possible.

Usually, however, one is interested in the expansion of a diagram around $\varepsilon = 0$. On expansion, (9) develops some singularities in the $z$-plane and possible poles $1/\varepsilon$. The expansion of $\Gamma$-functions produces $\psi$-functions or equivalently $S$-sums. Namely for $j$ integer

$$\frac{\Gamma(j+a\varepsilon)}{\Gamma(1+a\varepsilon)} = (j-1)! \exp \left\{ \sum_{k=1}^{\infty} a^k \varepsilon^k S_k(j-1) \right\},$$

with $S_k(n) = \sum_{s=1}^{n} 1/s^k$. This shows that harmonic sums naturally play a special role in our subject.

Terms of lowest order are obtained by replacing the exponent in (10) by unity and (9) becomes a multiple sum of factorials. Then the coefficient of $z^n$ is a finite sum over factorials. Terms of higher level are obtained by expanding the exponent in (10) in $\varepsilon$, which produces $S$-functions in (9).

This suggests the following recipe of generalization: if $f_{(a)}$ is found to be a multiple (finite) sum $f_{(a)}(n) = \sum_{\{j\}} c_{(j)}$, then higher level objects are obtained either by multiplying $f_{(a)}$ by

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These sums were used in [19, 20].
S-functions or by inserting $S$-functions under the summation sign of $f(a)$ to give

$$f_{(a+1)}(n) = \sum_{\{j\}}^n c_{\{j\}} S_1(\{j\}),$$
$$f_{(a+2)}(n) = \sum_{\{j\}}^n c_{\{j\}} S_2(\{j\}),$$
$$f_{(a+1+1)}(n) = \sum_{\{j\}}^n c_{\{j\}} S_1(\{j\}) S_1(\{j\}),$$
$$\ldots \text{ etc.,}$$

(11)

where the arguments of $S_k$ are some linear combinations of summation indices. The same applies of course to the function $S_a$ itself giving $S_{a,b}$ etc.

From (11) it is seen that any $f(a)$ produces in general many functions $f_{(a+1)}$ and higher ones. In the case of a multiple summation the number of possible combinations of $S_k(\{j\})$ may get too large and this general rule is getting rather formal than practical. However, in practice we always met the situation that $f_a$ is a simple (one-fold) sum. Then, inserting $S_k(j-1)$ and $S_k(j)$ or equivalently $S_k(j-1)$ and $1/j$, (11) becomes

$$f_{(a+1)}(n) = \sum_{j=1}^n c_j \left\{ \frac{1}{j}, S_1(j-1) \right\},$$
$$f_{(a+2)}(n) = \sum_{j=1}^n c_j \left\{ \frac{1}{j^2}, \frac{S_1(j-1)}{j}, S_1^2(j-1), S_1(j-1), S_2(j-1) \right\},$$
$$\ldots \text{ etc.}$$

(12)

The choice (12) is justified a posteriori.

This prescription, however, must be modified if a diagram possesses $2m$-cuts (with a threshold at $q^2 = 4m^2$). Such diagrams produce in the sum (11) functions like $\Gamma(2j + a\varepsilon)$ and according to (10) $S_k(2j-1)$. In such cases, along with $S_k(j-1)$, we insert also $S_k(2j-1)$.

In conclusion of this Section we want to stress that the recipes formulated above do not necessarily work in every case for any diagram. But in many cases they do. A possible reason of a failure of the prescription (11) may be e.g. the following: some new basis element begins to contribute only, say, at the order $1/\varepsilon$ while at the order $1/\varepsilon^2$ it happens to drop out. In such cases it will be missed in the analysis and one has to use other more powerful and direct methods of determination of low level terms.

## 5 Diagrams having both $m$- and $2m$-cuts and $W$-functions

We now establish the structure of the lowest level for the set of diagrams having both $m$- and $2m$-cuts. While for diagrams with only $m$-cuts the complete basis consists of elements $1/n^a, S_b, K_c$ and their generalization due to (12), in the presence of $2m$-cuts the basis changes drastically. In this case the behaviour of the coefficients is governed by a new class of elements which we call $W$. The generic structure of the lowest level is

$$W_1(n) = \sum_{j=1}^n \binom{2j}{j} \frac{1}{j}$$

(13)

and higher level functions are obtained following the rules of the previous Section.
The $W$-structure can be deduced from different diagrams with $2m$-cuts but most easily we find it from the nonplanar graph shown in Fig. 3, $N_{12}$. Namely this diagram has a double collinear pole and we can easily perform both integrations in calculating the $1/\varepsilon^2$ contribution.

Consider at first the massless lines 3 and 5 (see (1) for the normalization)

\[
N_{12} = \int \frac{d\vec{k}_1}{k_1^2(k_1 - p_1)^2} \phi(k_1, k_2, p_1, p_2),
\]

where $\phi$ stands for the rest of the diagram. Introducing the Feynman parameter $\alpha_1$ this can be written as

\[
N_{12} = \int_0^1 d\alpha_1 \int \frac{d\vec{k}_1}{(k_1^2 - 2\alpha_1 k_1 p_1)^2} \phi = \int_0^1 d\alpha_1 \int \frac{d\vec{k}_1}{(k_1 - \alpha_1 p_1)^4} \phi,
\]

where we have used $p_1^2 = 0$. This integral diverges in 4-dimensions when $(k_1 - \alpha_1 p_1) \rightarrow 0$, i.e. there is a collinear divergence. For the calculation of the most singular term in $\varepsilon$ one can effectively replace

\[
\frac{1}{(k_1 - \alpha_1 p_1)^4} \rightarrow i\frac{\pi^2}{\varepsilon} \delta^{(D)}(k_1 - \alpha_1 p_1),
\]

and the same replacement applies to the pair of lines 4 and 6 and integration $d\vec{k}_2$. Therefore

\[
N_{12} \overset{1/\varepsilon^2}{=} -\frac{\pi^4}{\varepsilon^2} \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int \frac{d\vec{k}_1 d\vec{k}_2 \delta^{(D)}(k_1 - \alpha_1 p_1) \delta^{(D)}(k_2 - \alpha_2 p_2)}{[(k_1 - k_2 - p_1)^2 - m^2][(k_2 - k_1 - p_2)^2 - m^2]}.
\]

Integrating out $k_1, k_2$ with the help of the $\delta$-functions and using $p_1^2 = p_2^2 = 0, p_1 p_2 = q^2/2$, we arrive at ($z = q^2/m^2$)

\[
N_{12} \overset{1/\varepsilon^2}{=} -\frac{1}{\varepsilon^2(q^2)^2} \int_0^1 \frac{d\alpha_1 d\alpha_2}{(1 - \alpha_1(1 - \alpha_2)z)(1 - (1 - \alpha_1)\alpha_2 z)}
\]

\[
= -\frac{1}{\varepsilon^2(q^2)^2} \sum_{n=1}^{\infty} z^n \frac{(n!)^2}{2(n-1)!^2} \sum_{j=1}^{n-1} \frac{(2j)!}{(j!)^2} \frac{1}{j}
\]

\[
= -\frac{1}{\varepsilon^2(q^2)^2} \sum_{n=1}^{\infty} z^n \left(2n\right)^{-1} \frac{W_1(n-1)}{n}.
\]

If we assign to the factor $\binom{2n}{n}^{-1}$ the 0th level, then the expression on the r.h.s of (18) has the correct 2nd level as it should be for the double pole part of a vertex function. Using the results of Sect. 4 we obtain functions of higher levels $W_2, W_{1,1}, W_3$ etc. (see Appendix A).

### 6 Differential equation method

It has been demonstrated above that the most important point in our approach to set up a Taylor series for the diagrams under consideration is to find the proper basis elements. In the last Section we have seen a special calculation to find one more basis element. There is no general way of how to find further such elements but another effective approach is that of the differential equation method (DEM) [9]. It was used in [8] to find the expansion of diagrams $P_{126}$ and $P_{56}$ in the notation of Fig.3. Therefore we give in this Section a short review of
the DEM and demonstrate the finding of another basis element by calculating the two-point diagram \( I_{15} \) of Fig. 2.

The DEM allows one to calculate massive diagrams by reducing them to others having an essentially simpler structure. More precisely, one writes a linear differential equation (w.r.t. masses) for the diagram under consideration. The inhomogeneous term of the equation is a sum of diagrams with smaller number of lines. These in turn can be reduced via another differential equation to even simpler diagrams and so on. Solving the so obtained equations iteratively one gets for the initial diagram an \( s \)-fold integral representation which is in general simpler than the usual parametric or dispersion integrals. Often some of the integrations can be performed analytically.

For completeness we give some formulae that will be useful in the further discussion. We introduce the following graphical representation for the propagators

\[
\frac{1}{(q^2 + i0)^\alpha} = \cdots \frac{1}{(q^2 - m^2 + i0)^\alpha} = \frac{\alpha}{m^2}
\]

\( \alpha \) and \( m \) are called the index and mass of this line. Further (except when mentioned otherwise) all solid lines have the same mass \( m \). Lines with index 1 and mass \( m \) are not marked.

The basic tool is the integration by part relation \[25\] which is obtained by multiplying the integrand of a diagram with \( (d/dk_\mu) k_\mu \) and using

\[ \int dD \text{div}(\ldots) = 0. \]

As a result for a triangle \[9\] with arbitrary masses one has (here lines with index \( \alpha_i \) have mass \( m_i \))

\[
(D - 2\alpha_1 - \alpha_2 - \alpha_3) = 2m_1^2 \alpha_1
\]

\[
+ \left\{ \alpha_2 \left( \frac{\alpha_2 + 1}{\alpha_1} \right)^\alpha_3 + \frac{\alpha_2 + 1}{\alpha_1} \right\} + \left( m_1^2 + m_2^2 \right) \left( \frac{\alpha_2 + 1}{\alpha_1} \right) + (\alpha_2 \leftrightarrow \alpha_3) \right\}
\]

We stress the fact that the basic line of the triangle plays a special role (in the following we call it 'distinctive'). This relation is used in the following manner. All triangles on the r.h.s. with 'shifted' indices \( \alpha_i + 1 \) can be written as derivatives of the initial triangle w.r.t. \( m_i^2 \). Generally speaking, one has to consider all masses \( m_i \) to be different and put them equal at the end of the calculation. In practice one can often eliminate some of the derivatives by combining several differential equations.

Whenever a self-energy subdiagram occurs the following relation is used

\[
q \left\langle \frac{\alpha_1}{\alpha_2}, \frac{m_1}{m_2} \right\rangle = i^{1+D} \frac{\Gamma(\alpha_1 + \alpha_2 - D/2)}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \int_0^1 ds \frac{d^D}{(1-s)^{\alpha_1+1-D/2} s^{\alpha_2 + 1 - D/2}} \frac{q^{\alpha_1 + \alpha_2 - D/2}}{(m_1^2 + m_2^2)^2}
\]

This allows one to reduce an \( l \)-loop to an \( (l - 1) \)-loop diagram with one propagator having 'mass' \( m_1^2/(1-s) + m_2^2/s \).

We now apply this technique to the evaluation of diagrams with one mass. Consider first the symmetric planar diagram \( P_{126} \) (Fig. 3). It possesses only \( 2m \)-cuts and zero-cuts but no \( m \)-cuts. Therefore the nearest singularity in the complex \( z = q^2/m^2 \) plane is placed at \( z = 4 \)

---

8A similar technique has been introduced recently by Remiddi [21].

9Similar relations may easily be obtained for arbitrary \( n \)-point diagrams (see e.g. [7] and discussion in [22]).
and the diagram obviously has some new structures different from those already discussed. Applying (20) with distinctive central rung we have

\[
(D - 4) \begin{array}{c}
\text{\textbullet}\hline
\text{\textbullet}\hline
\text{\textbullet}
\end{array} = 2 \begin{array}{c}
\text{\textbullet}\hline
\text{\textbullet}\hline
\text{\textbullet}
\end{array} - 2 \begin{array}{c}
\text{\textbullet}\hline
\text{\textbullet}\hline
\text{\textbullet}
\end{array} - 4m^2 \begin{array}{c}
\text{\textbullet}\hline
\text{\textbullet}\hline
\text{\textbullet}
\end{array} - 2m^2 \begin{array}{c}
\text{\textbullet}\hline
\text{\textbullet}\hline
\text{\textbullet}
\end{array} \tag{22}
\]

It is easy to see that the last two terms on the r.h.s. can be combined into the total derivative of the initial diagram w.r.t. \(m^2\). Therefore we immediately obtain the differential equation for our diagram. The inhomogeneous term now is the sum of the first and the second diagram on the r.h.s. of (22). The former is trivial while for the latter we repeat the procedure with the left vertical line being distinctive:

\[
(D - 4) \begin{array}{c}
\text{\textbullet}\hline
\text{\textbullet}\hline
\text{\textbullet}
\end{array} = \begin{array}{c}
\text{\textbullet}\hline
\text{\textbullet}\hline
\text{\textbullet}
\end{array} + \begin{array}{c}
\text{\textbullet}\hline
\text{\textbullet}\hline
\text{\textbullet}
\end{array} \tag{23}
\]

The resulting diagrams on the r.h.s. all have self-energy insertions which can be replaced according to (21) and the remaining triangles are solved by usual Feynman parameters.

After some rather long but trivial transformations we obtain

\[
P_{126} = -\frac{1}{2(q^2)^2} \sum_{n=1}^{\infty} \frac{z^n}{(1-s)^n} \left( \frac{2n}{n} \right)^{n-1} \frac{1}{\varepsilon^2} \log(1 - \xi) + \frac{1}{\varepsilon} \left( \frac{1}{2} \log^2(1 - \xi) - \log(-z) \log(1 - \xi) \right) + \log[s(1 - s)] \log^2(1 - \xi) - \frac{7}{6} \log^3(1 - \xi) - 12 S_{1,2}(\xi) - 5 \log(1 - \xi) \text{Li}_2(\xi) + \frac{1}{2} \log(-z) \log^2(1 - \xi) + \frac{1}{2} \log^2(-z) \log(1 - \xi) \right], \tag{24}
\]

were \(\xi = zs(1-s)\) and \(S_{1,2}(z)\) is the Nielsen polylogarithm (see Appendix C). The above remaining integral is very difficult to evaluate analytically. However the expansion in \(z\) is welcome

\[
P_{126} = \frac{1}{(q^2)^2} \sum_{n=1}^{\infty} z^n \binom{2n}{n}^{-1} \frac{1}{n^2} \left( \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \left[ -S_1 - \log(-z) \right] \right)
- \frac{3}{2} S_2 - \frac{15}{2} S_1^2 + 4 S_1 \mathcal{S}_1 + 2 \frac{S_1}{n} - S_1 \log(-z) + \frac{1}{2} \log^2(-z) \right), \tag{25}
\]

with \(S_a = S_a(n-1)\) and \(\mathcal{S}_1 = S_1(2n-1)\). This expression has the correct threshold at \(q^2 = 4m^2\) due to the factor \(\binom{2n}{n}^{-1}\). The appearance of \(\mathcal{S}\) is somehow expected as was mentioned at the end of Sect. 5, but the factor \(\binom{2n}{n}^{-1}\) is difficult to guess without additional information (compare however with \([13]\)).

We now turn to the 2-point asymmetric diagram \(I_{15}\) (Fig. 2) which contains another new basis element. Using (20) with distinctive central rung for the left triangle we obtain the following differential equation

\[
\left[ D - 4 - 2m^2 \frac{d}{dm^2} \right] \begin{array}{c}
\text{\textbullet}\hline
\text{\textbullet}\hline
\text{\textbullet}
\end{array} = \begin{array}{c}
\text{\textbullet}\hline
\text{\textbullet}\hline
\text{\textbullet}
\end{array} + \begin{array}{c}
\text{\textbullet}\hline
\text{\textbullet}\hline
\text{\textbullet}
\end{array} \tag{26}
\]

with \(\xi = zs(1-s)\) and \(S_{1,2}(z)\) is the Nielsen polylogarithm (see Appendix C). The above remaining integral is very difficult to evaluate analytically. However the expansion in \(z\) is welcome

\[
P_{126} = \frac{1}{(q^2)^2} \sum_{n=1}^{\infty} z^n \binom{2n}{n}^{-1} \frac{1}{n^2} \left( \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \left[ -S_1 - \log(-z) \right] \right)
- \frac{3}{2} S_2 - \frac{15}{2} S_1^2 + 4 S_1 \mathcal{S}_1 + 2 \frac{S_1}{n} - S_1 \log(-z) + \frac{1}{2} \log^2(-z) \right), \tag{25}
\]

with \(S_a = S_a(n-1)\) and \(\mathcal{S}_1 = S_1(2n-1)\). This expression has the correct threshold at \(q^2 = 4m^2\) due to the factor \(\binom{2n}{n}^{-1}\). The appearance of \(\mathcal{S}\) is somehow expected as was mentioned at the end of Sect. 5, but the factor \(\binom{2n}{n}^{-1}\) is difficult to guess without additional information (compare however with \([13]\)).

We now turn to the 2-point asymmetric diagram \(I_{15}\) (Fig. 2) which contains another new basis element. Using (20) with distinctive central rung for the left triangle we obtain the following differential equation

\[
\left[ D - 4 - 2m^2 \frac{d}{dm^2} \right] \begin{array}{c}
\text{\textbullet}\hline
\text{\textbullet}\hline
\text{\textbullet}
\end{array} = \begin{array}{c}
\text{\textbullet}\hline
\text{\textbullet}\hline
\text{\textbullet}
\end{array} + \begin{array}{c}
\text{\textbullet}\hline
\text{\textbullet}\hline
\text{\textbullet}
\end{array} \tag{26}
\]
One gains little from this because of the last term on the r.h.s. This has the structure of the initial diagram but with one massless propagator squared. To get rid of this term we can write another equation, using the second massive line as distinctive of the triangle:

\[
(D - 4 - 2m^2 \frac{d}{dm^2}) + m^2 - 2 \\rightarrow + 2 \\rightarrow + 2 \\rightarrow \]

Taking a proper linear combination of these equations, we can exclude the bad last term. In the resulting equation the inhomogeneous term consists only of diagrams with self-energy insertions which can be solved immediately. As a result for \( I_{15} \) we write its expansion in \( z \)

\[
I_{15} = \frac{1}{q^2} \sum_{n=1}^{\infty} \frac{z^n}{n} \left[ -\frac{1}{n} \log(-z) - \zeta_2 + 3 \sum_{j=1}^{n-1} \left( \frac{2}{j} \right)^{-1} \frac{1}{j^2} + 2 \frac{1}{n^2} + 4 \left( \frac{2n}{n} \right)^{-1} \frac{1}{n^2} \right]. \tag{26}
\]

The inner sum in (26) is the new type of basis elements which we discovered from this calculation. In a somewhat more general way of writing it is

\[
V_a(n - 1) = \sum_{j=1}^{n-1} \left( \frac{2}{j} \right)^{-1} \frac{1}{j^a}. \tag{27}
\]

Further basis elements were obtained in a similar fashion. They are listed in the appendix. We do not give all the details of their derivation, which was done in a way analogous to the above.

### 7 Analytical continuation

Taking care of the zero-thresholds by factorizing \( \log(-z), (z = q^2/M^2) \) as described above, all series under consideration have finite radius of convergence \( |z| < |z_0| \) where \( z_0 \) is the lowest non-zero (pseudo)-threshold. Therefore these series need to be analytically continued on their cut since this is in general the region of physical interest. The most straightforward way is the continuation via conformal mapping and special summation techniques (usually Padé approximants) \[11\]. In principle this allows one to continue the diagrams to the whole complex \( z \)-plane including the cut; at least this method works perfectly for low and moderate \( |z/z_0| \). For the diagrams we consider here, the situation is even better than in cases with several masses since our analytic results for the Taylor coefficients allow to calculate arbitrary many of them easily with any desired precision.

Nevertheless it is of interest to investigate the domain \( |z/z_0| \gg 1 \) and have series in powers of \( 1/z \). The first few coefficients of such series can of course be obtained by expanding any diagram in the ‘standard’ manner \[10], [13], [26\] in large \( q^2 \). This expansion is known to be
much more difficult than the corresponding $q^2/M^2$-expansion. One of the reasons is that the number of contributing subgraphs in this case is usually larger. In the language of the present paper the ‘basis’ is more extended (see examples below).

Here we present a different approach for the diagrams with one non-zero mass, starting from the series in $z = q^2/M^2$. The idea is to apply the Sommerfeld–Watson transformation to the $z$-series: given $J = \sum_{n=1}^{\infty} c_n z^n$, it can be written as a Sommerfeld–Watson contour integral

$$J = \frac{1}{2\pi i} \int_{C_1+C_2} \frac{e^{i\pi n} c_n z^n}{\sin(\pi n)} \, dn.$$

The integral in (28) is to be performed along a closed contour in the $n$-plane consisting of two pieces: $C_1$ is a line parallel to the imaginary axis $(\gamma - i\infty, \gamma + i\infty)$ with $0 < \gamma < 1$ and $C_2$ is a semicircle closed in the right-half-plane at infinity.

The representation (28) is valid of course only if the coefficients $c_n$ (considered as analytic function of $n$) decrease fast enough at infinity and do not have singularities to the right of $C_1$. Moving $C_1$ to the left in the negative direction, one has to add the corresponding residues.

We now suppose that $c_n$ is regular in the whole $n$-plane except (possible) poles at nonpositive integers $n = -k$, $k = 0, 1, 2, \ldots$ and has the required behaviour at infinity. Then

$$J = -\sum_{k=0}^{\infty} \text{res}_{n=-k} \frac{e^{i\pi n} c_n z^n}{\sin(\pi n)}.$$

(29)

The crucial point now is to show how to continue analytically each of the structures occurring in this paper. First let us note without proof that all singularities of these functions are placed on the real axis at nonpositive integers. Therefore we need to continue them through the neighborhood of these points i.e $n = -k + \delta$ with $k = 0, 1, 2, \ldots$ and $|\delta| < 1$.

To start with, let us consider one-fold sums. First of all we note that $S_1$ is continued by

$$S_1(n-1) = \psi(n) + \gamma_E$$

(30)

where $\psi(n)$ is the logarithmic derivative of the $\Gamma$-function. All sums $S_a$ with $a > 1$ are continued via the $\psi$-function and its derivatives. All $K_a$ are simply expressed through $S_a$

$$K_a(n) = S_a(n) - 2^{1-a} S_a([n/2]).$$

(31)

To continue the other functions we use the following general method. Let $f(n-1) = \sum_{j=1}^{n-1} \phi(j)$. Then, formally

$$f(n-1) = \sum_{j=1}^{\infty} \phi(j) - \sum_{j=n}^{\infty} \phi(j) = f(\infty) - \sum_{j=0}^{\infty} \phi(j + n).$$

(32)

By inspection of the elements under consideration we see that $\phi(n)$ is always an analytic function in the whole $n$-plane apart from (possible) poles at nonpositive integers. Then, with $n = -k + \delta$, we obtain for the last term in (32)

$$\sum_{j=0}^{\infty} \phi(j - k + \delta) = \sum_{j=0}^{k-1} \phi(j - k + \delta) + \phi(\delta) + \sum_{j=k+1}^{\infty} \phi(j - k + \delta) = \sum_{j=1}^{k} \phi(-j + \delta) + \phi(\delta) + \sum_{j=1}^{\infty} \phi(j + \delta).$$

(33)
It is convenient to introduce the notation \( f^{(\delta)}(n - 1) = \sum_{j=1}^{n-1} \phi(j + \delta) \) for the sum with 'shifted' summation index. Combining (32) and (33) we finally get

\[
f(-k - 1 + \delta) = f(\infty) - f^{(\delta)}(\infty) - \sum_{j=1}^{k} \phi(-j + \delta) - \phi(\delta),
\]

with \( k = 0, 1, 2, \ldots \). The remaining finite sum can be easily transformed in our cases to a 'known' function and (possible) poles in \( \delta \) can be explicitly separated. In this way we obtain the following formulæ

\[
S_a(-k - 1 + \delta) = S_a(\infty) - S_a^{(\delta)}(\infty) - (-)^a S_a^{(-\delta)}(k) - \frac{1}{\delta^a},
\]

\[
K_a(-k - 1 + \delta) = K_a(\infty) - K_a^{(\delta)}(\infty) - (-)^{a+2\delta} K_a^{(-\delta)}(k) + \frac{(-1)^\delta}{\delta^a},
\]

\[
V_a(-k - 1 + \delta) = V_a(\infty) - V_a^{(\delta)}(\infty) + (-)^a \pi \cot(\pi \delta) W_a^{(-\delta)}(k) - \left(\frac{2\delta}{\delta}\right)^{-1} \frac{1}{\delta^a},
\]

\[
W_a(-k - 1 + \delta) = W_a(\infty) - W_a^{(\delta)}(\infty) + (-)^a \frac{1}{\pi \cot(\pi \delta)} V_a^{(-\delta)}(k) - \left(\frac{2\delta}{\delta}\right)^{-1} \frac{\psi(\delta) + \gamma_E}{\delta^a}.
\]

The continuation of double etc. sums is done in a similar manner. In the inner summation one uses the continuation of \( S_a \) via \( \psi \)-function or (33). The results are similar to (33)-(38) e.g.

\[
V_{a,1}(-k - 1 + \delta) = V_{a,1}(\infty) - V_{a,1}^{(\delta)}(\infty)
\]

\[
+ (-)^a \pi \cot(\pi \delta) \left( W_{a,1}^{(-\delta)}(k) + W_{a,1}^{(-\delta)}(k) - \pi \cot(\pi \delta) W_{a,1}^{(-\delta)}(k) \right) - \left(\frac{2\delta}{\delta}\right)^{-1} \frac{\psi(\delta) + \gamma_E}{\delta^a}.
\]

In the same manner functions with 2 arguments are continued. There is one shortcoming, however, in such a procedure—some of the infinite sums are divergent, e.g. \( W_a(\infty) \). This may cause problems in some cases. One way out may be that these functions are evaluated as hypergeometric series with the corresponding argument, like e.g.

\[
W_1^{(\delta)}(\infty) = \frac{\Gamma(3 + 2\delta)}{\Gamma(1 + \delta) \Gamma(2 + \delta)} {}_3F_2(1, \frac{3}{2} + \delta, 1 + \delta; 2 + \delta, 2 + \delta; 4).
\]

Moreover, some of them (e.g. \( S_1 \)) cancel after applying the SW-transformation.

As an example of this technique we transform the series \( z^n V_3/n \) into a \( 1/z^n \) series. Utilizing formula (37) we get

\[
\sum_{n=1}^{\infty} \frac{V_3(n - 1)}{n} z^n = \frac{1}{24} \log^4(-z) + 2 \zeta_3 \log(-z) - 3 \zeta_4 - 2 V_4(\infty) + 2 V_{3,1}(\infty) - 2 \tilde{V}_{3,1}(\infty)
\]

\[
- \sum_{n=1}^{\infty} \frac{1}{z^n} \frac{1}{n} \left( \frac{1}{6} \log^3(-z) + \frac{1}{2n} \log^2(-z) + \left( \frac{1}{n^2} + W_2(n) \right) \log(-z) + 2 \zeta_3 + 3 W_3(n) + 2 W_{2,1}(n) - 2 \tilde{W}_{2,1}(n) + \frac{W_2(n)}{n} + \frac{1}{n^3} \right).
\]
8 Conclusion

For a certain class of selfenergy and vertex functions with only one non-zero mass we have developed a new technique of obtaining analytic results. This method uses information from the large mass expansion of the diagrams by reproducing the obtained expansion coefficients in terms of a certain set of ‘basis elements’, which are harmonic sums in many cases but often more complicated (‘W- and V-sums’). However, we did not investigate systematically all \( 2m \)-vertex functions. In some cases we failed to find the solution. Supposedly some of them include new exceptional basis elements.

Given the expansion coefficients analytically, it is possible to sum most of the series in closed form to yield a representation of the diagrams under consideration in terms of polylogarithms. Furthermore the analytic form of the expansion coefficients allows by means of the Sommerfeld-Watson transformation a direct transition to the large \( q^2 \)-expansion. For diagrams involving elliptic functions (essentially those with \( 3m \)-cuts) the basis elements of the corresponding expansions were not found.

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Appendix

A Basic functions

Here we list all basic functions we used and point their ‘transcendentality level’ (TL).

A.1 Zero-fold sums

\[ \zeta_a, \quad \frac{1}{n^a}, \quad \binom{2j}{j}, \quad \binom{2j}{j}^{-1}, \quad \text{where } \zeta_a = \zeta(a) \text{ is the Riemann } \zeta \text{-function. (42) and (43) have TL= } a. \quad (44) \text{ and (45) have TL= 0.} \]

A.2 One-fold sums

\[ S_a(n) = \sum_{j=1}^{n} \frac{1}{j^a}, \quad (46) \]
\[ K_a(n) = - \sum_{j=1}^{n} \frac{(-)^j}{j^a} , \quad (47) \]
\[ W_a(n) = \sum_{j=1}^{n} \binom{2j}{j} \frac{1}{j^a} , \quad (48) \]
\[ V_a(n) = \sum_{j=1}^{n} \frac{(2j)^{-1}}{j^a} . \quad (49) \]

All have TL= \( a \). An exceptional sum occurs in the diagram \( I_{123} \) with TL= 2

\[ \hat{V}_2(n) = \sum_{j=1}^{n} \binom{2j}{j}^{-1} \frac{1}{j(n+1-j)} . \quad (50) \]

### A.3 Two-fold sums

\[ S_{a,b}(n) = \sum_{j=1}^{n} \frac{1}{j^a} S_b(j-1) , \quad (51) \]
\[ K_{a,b}(n) = - \sum_{j=1}^{n} \frac{(-)^j}{j^a} S_b(j-1) , \quad (52) \]
\[ W_{a,b}(n) = \sum_{j=1}^{n} \binom{2j}{j} \frac{1}{j^a} S_b(j-1) , \quad (53) \]
\[ V_{a,b}(n) = \sum_{j=1}^{n} \frac{(2j)^{-1}}{j^a} S_b(j-1) . \quad (54) \]

If instead of \( S_a(j-1) \) there is an insertion of \( S_a(2j-1) \) we use the tilde, e.g.

\[ \hat{V}_{a,b}(n) = \sum_{j=1}^{n} \binom{2j}{j}^{-1} \frac{1}{j^a} S_b(2j-1) , \quad (55) \]

etc. All have TL= \( a + b \).

### A.4 Three-fold sums

\[ S_{a,(b+c)}(n) = \sum_{j=1}^{n} \frac{1}{j^a} S_b(j-1) S_c(j-1) , \quad (56) \]
\[ W_{a,(b+c)}(n) = \sum_{j=1}^{n} \frac{(2j)^{-1}}{j^a} S_b(j-1) S_c(j-1) . \quad (57) \]

All have TL= \( a + b + c \).
A.5 Functions with two arguments

Functions $S$ and $V$ may occur with two arguments

\[
S_a(n; p) = \sum_{j=1}^{n} \frac{p_j^{j-n}}{j^a},
\]

(58)

\[
V_a(n; p) = \sum_{j=1}^{n} \left(\frac{2j}{j}\right)^{-1} \frac{p_j^{j-n}}{j^a},
\]

(59)

All have TL= $a$.

B The 2-point functions

Below we present results for the diagrams shown in Fig. 2 All functions have argument $n – 1$ which is omitted. Functions with two arguments have their first argument $n – 1$ which is omitted. $\mathcal{S}_1 = S_1(2n - 1)$ and $\hat{V}_2$ in $I_{123}$ is given by (50). $z = q^2/m^2$.

\[
I_1 = \frac{1}{q^2} \sum_{n=1}^{\infty} z^n \frac{1}{n} \left[ \frac{1}{2} \log^2(-z) - \frac{2}{n} \log(-z) + \zeta_2 + 2S_2 - 2S_1/n + 3/n^2 \right]
\]

(60)

\[
I_5 = \frac{1}{q^2} \sum_{n=1}^{\infty} z^n \frac{(-)^n}{n} \left[ -\log^2(-z) + \frac{2}{n} \log(-z) - 2\zeta_2 + 4K_2 - 2\frac{1}{n^2} - 2(\frac{-)^n}{n^2} \right]
\]

(61)

\[
I_{12} = \frac{1}{q^2} \sum_{n=1}^{\infty} z^n \frac{1}{n^2} \left[ \frac{1}{n} + \left(\frac{2n}{n}\right)^{-1} \left(-2\log(-z) - 3W_1 + \frac{2}{n}\right) \right]
\]

(62)

\[
I_{13} = \frac{1}{q^2} \sum_{n=1}^{\infty} z^n \frac{1}{n} \left[ 2\zeta_2 + 2S_2 - \frac{S_1}{n} \right]
\]

(63)

\[
I_{14} = \frac{1}{q^2} \sum_{n=1}^{\infty} z^n \frac{1}{n} \left[ -S_1(\;2) \log(-z) + 2\zeta_2 - 6V_2 + 2V_2(\;2) + S_2(\;2) - S_1,1(\;2) + 2\frac{S_1}{n} + \frac{S_1(\;2)}{n^2} \right]
\]

(64)

\[
I_{15} = \frac{1}{q^2} \sum_{n=1}^{\infty} z^n \frac{1}{n} \left[ -\frac{1}{n} \log(-z) - \zeta_2 + 3V_2 + \frac{2}{n^2} + 4\left(\frac{2n}{n}\right)^{-1} \frac{1}{n^2} \right]
\]

(65)

\[
I_{123} = \frac{1}{q^2} \sum_{n=1}^{\infty} z^n \frac{1}{n} \left[ \zeta_2 - 3V_2 + 3\hat{V}_2 \right] - \left(\frac{2n}{n}\right)^{-1} \frac{1}{n^2} \left(3W_1/n\right)
\]

(66)

\[
I_{125} = \frac{1}{q^2} \sum_{n=1}^{\infty} z^n \frac{1}{n^2} \left[ \left(\frac{2n}{n}\right)^{-1} - \log(-z) + \frac{3}{n} \right]
\]

(67)

\[
I_{1234} = \frac{1}{q^2} \sum_{n=1}^{\infty} z^n \frac{1}{n} \left[ V_2(\;4) + \left(\frac{2n}{n}\right)^{-1} \frac{1}{n} \left(-8S_1 + 4\mathcal{S}_1 - \frac{2}{n}\right) \right]
\]

(68)

First seven coefficients of the expansion for $I_1$, $I_5$, $I_{12}$, $I_{14}$, $I_{15}$ and $I_{125}$ were compared with [14].
C Integral representations for the 2-point functions

Here we give the representations for the 2-point functions. Following [24] we introduce the notation

\[ S_{a+1,b}(z) = \frac{(-1)^{a+b}}{a! b!} \int_0^1 \frac{\log^a(t) \log^b(1 - zt)}{t} dt. \]  \tag{69}

All polylogarithmic functions are particular cases of \( S \) functions, namely

\[ \text{Li}_a(z) = S_{a-1,1}(z). \]  \tag{70}

We introduce the following new variable \( (z = q^2/m^2) \)

\[ y = \frac{1 - \sqrt{z/(z - 4)}}{1 + \sqrt{z/(z - 4)}}. \]  \tag{71}

Then

\[ q^2 \cdot I_1 = -\frac{1}{2} \log^2(-z) \log(1 - z) - 2 \log(-z) \text{Li}_2(z) + 3 \text{Li}_3(z) - 6 S_{1,2}(z) \]
\[ - \log(1 - z) \left( \zeta_2 + 2 \text{Li}_2(z) \right), \]  \tag{72}

\[ q^2 \cdot I_5 = 2 \zeta_2 \log(1 + z) + 2 \log(-z) \text{Li}_2(-z) + \log^2(-z) \log(1 + z) + 4 \log(1 + z) \text{Li}_2(z) \]
\[ - 2 \text{Li}_3(-z) - 2 \text{Li}_3(z) + 2 S_{1,2}(z^2) - 4 S_{1,2}(z) - 4 S_{1,2}(-z), \]  \tag{73}

\[ q^2 \cdot I_{12} = \text{Li}_3(z) - 6 \zeta_3 - \zeta_2 \log y - \frac{1}{6} \log^3 y - 4 \log y \text{Li}_2(y) \]
\[ + 4 \text{Li}_3(y) - 3 \text{Li}_3(-y) + \frac{1}{3} \text{Li}_3(-y^3), \]  \tag{74}

\[ q^2 \cdot I_{13} = -6 S_{1,2}(z) - 2 \log(1 - z) \left( \zeta_2 + \text{Li}_2(z) \right), \]  \tag{75}

\[ q^2 \cdot I_{14} = \log(2 - z) \left( \log^2(1 - z) - 2 \log(-z) \log(1 - z) - 2 \text{Li}_2(z) \right) \]
\[ - \frac{2}{3} \log^3(1 - z) + \log(-z) \log^2(1 - z) - 2 \zeta_2 \log(1 - z) \]
\[ - S_{1,2}(1/(1 - z)^2) + 2 S_{1,2}(1/(1 - z)) + 2 S_{1,2}(-1/(1 - z)) \]
\[ + \frac{1}{3} \log^3 y + \log^2 y \left( 2 \log(1 + y^2) - 3 \log(1 - y + y^2) \right) \]
\[ - 6 \zeta_3 - \text{Li}_3(-y^2) + \frac{2}{3} \text{Li}_3(-y^3) - 6 \text{Li}_3(-y) \]
\[ + 2 \log y \left( \text{Li}_2(-y^2) - \text{Li}_2(-y^3) + 3 \text{Li}_2(-y) \right), \]  \tag{76}

\[ q^2 \cdot I_{15} = 2 \text{Li}_3(z) - \log(-z) \text{Li}_2(z) + \zeta_2 \log(1 - z) \]
\[ + \frac{1}{6} \log^3 y - \frac{1}{2} \log^2 y \left( 8 \log(1 - y) - 3 \log(1 - y + y^2) \right). \]
\[-6\zeta_3 - \frac{1}{3} \text{Li}_3(-y^3) + 3\text{Li}_3(-y) + 8\text{Li}_3(y)
+ \log y \left( \text{Li}_2(-y^3) - 3\text{Li}_2(-y) - 8\text{Li}_2(y) \right), \quad (77)\]

\[q^2 \cdot I_{123} = -\zeta_2 \left( \log(1 - z) + \log y \right) - 6\zeta_3 - \frac{3}{2} \log(1 - y + y^2) \log^2 y
+ \text{Li}_3(-y^3) - 9\text{Li}_3(-y) - 2 \log y \left( \text{Li}_2(-y^3) - 3\text{Li}_2(-y) \right), \quad (78)\]

\[q^2 \cdot I_{125} = -2 \log^2 y \log(1 - y) - 6\zeta_3 + 6\text{Li}_3(y) - 6 \log y \text{Li}_2(y), \quad (79)\]

\[q^2 \cdot I_{1234} = -6\zeta_3 - 12\text{Li}_3(y) - 24\text{Li}_3(-y)
+ 8 \log y \left( \text{Li}_2(y) + 2\text{Li}_2(-y) \right) + 2 \log^2 y \left( \log(1 - y) + 2 \log(1 + y) \right). \quad (80)\]

Different analytic representations for \(I_1, I_{13}, I_{125}\) and \(I_{1234}\) were obtained in [2]. In [3] the same four integrals were calculated for arbitrary dimension \(d\) in terms of hypergeometric functions. \(I_5\) and \(I_{12}\) were calculated also in [4]. The numerical check shows agreement for these integrals.

### D 3-point functions

Below we present results for the diagrams shown in Fig.3. All functions have argument \(n - 1\) which is omitted. Overlined functions have argument \(2n - 1\) which is omitted (\(\overline{S}_1 = S_1(2n - 1)\) etc.). \(\overline{V}_{2,1}\) in \(P_{56}\) is given by [5] and \(z = q^2/m^2\).

\[P_1 = \frac{1}{(q^2)^2} \sum_{n=1}^{\infty} z^n \frac{1}{n} \left\{ -\frac{1}{2\varepsilon} s_1 + \frac{1}{\varepsilon} \left[ -\frac{1}{2} \zeta_2 + \frac{5}{2} S_2 - \frac{1}{2} S_1^2 + \frac{1}{n^2} - \frac{1}{n} \log(-z) + \frac{1}{2} \log^2(-z) \right] \right. \]
\[-\frac{8}{3} \zeta_3 - \zeta_2 S_1 - \frac{\zeta_2}{n} + \frac{8}{3} S_3 + \frac{9}{2} S_1 S_2 + \frac{5}{6} S_1^3 + 4 \frac{S_2}{n} + 2 \frac{S_1}{n^2} + \frac{3}{n^3} \]
\[+ \left( \zeta_2 - 4 S_2 - 2 \frac{S_1}{n} - \frac{3}{n^2} \right) \log(-z) + \left( S_1 + \frac{3}{2} \frac{n}{S_2} \right) \log^2(-z) - \frac{1}{2} \log^3(-z) \} \quad (81)\]

\[P_3 = \frac{1}{(q^2)^2} \sum_{n=1}^{\infty} z^n \left\{ \frac{1}{\varepsilon} \left[ \frac{1}{2} \zeta_2 - \frac{1}{2} S_2 \right] + \frac{1}{\varepsilon} \left[ \zeta_3 + \zeta_2 S_1 - \frac{5}{2} S_3 - S_{1,2} + (-\zeta_2 + S_2) \log(-z) \right] \right. \]
\[+ \zeta_4 + \zeta_3 S_1 + \frac{7}{2} \zeta_2 S_2 + \zeta_2 S_1^2 + \frac{3}{2} S_4 + 3 S_{1,3} - 2 S_{1,(1+2)} - 8 S_1 S_3 - 2 S_1 S_{1,2} + 2 S_2 + S_1^2 S_2 \]
\[+ \left( -\zeta_3 - 2 \zeta_2 S_1 + S_3 + 2 S_1 S_2 \right) \log(-z) + \left( \zeta_2 - \frac{1}{2} S_2 \right) \log^2(-z) \right\} \quad (82)\]

\[P_5 = \frac{1}{(q^2)^2} \sum_{n=1}^{\infty} z^n \frac{(-)^n}{n} \]
\[
\begin{align*}
\left\{-6\zeta_3 + 2\zeta_2 S_1 + 6S_3 - 2S_1 S_2 + 4\frac{S_2}{n} - \frac{S_1^2}{n^2} + 2\frac{S_1}{n^2} \right. \\
\left. + \left(-4S_2 + S_1^2 - 2\frac{S_1}{n}\right) \log(-z) + S_1 \log^2(-z) \right\} \\
\end{align*}
\]

\[
P_6 = \frac{1}{(q^2)^2} \sum_{n=1}^{\infty} z^n \left(\frac{-n}{n}\right)
\left\{ \frac{1}{\varepsilon^2} \left[-\frac{1}{n} \log(-z) \right] \\
+ \frac{1}{\varepsilon} \left[ -\frac{\zeta}{3} - 3S_2 + 4K_2 - 3\frac{S_1}{n} - \frac{3}{n^2} + \left(3S_1 + \frac{3}{n}\right) \log(-z) - \frac{3}{2} \log^2(-z) \right] \\
+ 2\zeta_3 + 7\zeta_2 S_1 + 2\frac{\zeta_2}{n} - 2S_3 - 9S_1 S_2 + 2K_3 + 12K_{2,1} + 4S_1 K_2 - \frac{7}{2} S_2 \\
- \frac{9}{2} S_1^2 - \frac{5}{2} S_1 - \frac{7}{3} S_1 - 7 \frac{S_1}{n^2} + \left( -2\zeta_2 + \frac{7}{2} S_2 + \frac{9}{2} S_1^2 + 5 \frac{S_1}{n} + \frac{7}{n^2} \right) \log(-z) \\
+ \left( -\frac{5}{2} \frac{S_1}{n} - \frac{7}{2} \right) \log^2(-z) + \frac{7}{6} \log^3(-z) \right\}
\]

\[
P_{13} = \frac{1}{(q^2)^2} \sum_{n=1}^{\infty} z^n
\left\{ \frac{1}{\varepsilon^2} \left[ -\frac{1}{2} S_2 + \frac{1}{\varepsilon} \left(-\frac{1}{2} S_3 - 2S_{1,2} \right) \right] \\
+ \frac{1}{2} \zeta_2 S_1 - \frac{5}{2} S_4 - S_{1,3} + 3S_{1,(1+2)} - S_1 S_3 - 8S_1 S_{1,2} + \frac{1}{2} S_2^2 + \frac{5}{2} S_1 S_2 \right\}
\]

\[
P_{12} = \frac{1}{(q^2)^2} \sum_{n=1}^{\infty} z^n \left(\frac{2n}{n}\right)^{-1} \frac{1}{n^2}
\left\{ \frac{2}{\varepsilon^2} + \frac{1}{\varepsilon} \left[ -6W_1 + 2S_1 + \frac{2}{n} - 2 \log(-z) \right] - 6W_2 - 18W_{1,1} - 13S_2 \\
+ S_1^2 - 6S_1 W_1 + \frac{2S_1}{n} + \frac{2}{n^2} + \left( -2S_1 - \frac{2}{n} \right) \log(-z) + \log^2(-z) \right\}
\]

\[
P_{56} = \frac{1}{(q^2)^2} \sum_{n=1}^{\infty} z^n \left(\frac{-n}{n}\right)
\left\{ -6\zeta_2 S_1 + 6V_3 - 2S_3 - 10K_3 - 12V_{2,1} + 12V_{2,1} - 12K_{2,1} + 12S_1 K_2 + 6 \frac{V_2}{n} \\
- \frac{3S_2}{n} - 4S_1^2 \log(-z) - 2S_1 \log^2(-z) \right\}
\]

\[
P_{126} = \frac{1}{(q^2)^2} \sum_{n=1}^{\infty} z^n \left(\frac{2n}{n}\right)^{-1} \frac{1}{n^2}
\left\{ \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \left[ -S_1 - \log(-z) \right] - \frac{3}{2} S_2 - \frac{15}{2} S_1^2 + 4S_1 S_1 \right. \\
\left. + 2\frac{S_1}{n} - S_1 \log(-z) + \frac{1}{2} \log^2(-z) \right\}
\]
\[ P_{3456} = \frac{1}{(q^2)^2} \sum_{n=1}^{\infty} z^n \left( \frac{2n}{n} \right)^{-1} \frac{1}{n} \\
= \left\{ 7S_3 - 4S_{1,2} - 8K_3 + 8S_1K_2 + 8S_2 + 8S_{1,2} + 8K_{2,1} + 4S_1S_2 - 8S_1K_2 - 8S_1S_2 \\
+ (-2S_2 + 4K_2) \log(-z) \right\} \]  
(89)\\\\
N_1 = \frac{1}{(q^2)^2} \sum_{n=1}^{\infty} z^n \\
= \left\{ \frac{1}{4\epsilon^4} + \frac{1}{\epsilon^3} \left[ S_1 - \frac{1}{2} \log(-z) \right] + \frac{1}{\epsilon^2} \left[ -\frac{1}{4} \zeta_2 + 3S_2 + 2S_1^2 - 2S_1 \log(-z) + \frac{1}{2} \log^2(-z) \right] \\
+ \frac{1}{\epsilon} \left[ -\frac{3}{3} \zeta_3 - \zeta_2S_1 + \frac{10}{3} S_3 + 2S_{1,2} + 10S_1S_2 + \frac{8}{3} S_3^2 - \frac{1}{2} \zeta_2 - 4S_2 - 4S_1^2 \right] \log(-z) + 2S_1 \log^2(-z) - \frac{1}{3} \log^3(-z) \right\} \\
\frac{57}{16} \zeta_4 - \frac{32}{3} \zeta_3 S_1 + \zeta_2 S_2 - 2 \zeta_2 S_1^2 + 4S_4 - 4S_{1,3} - 2S_{1,(i+2)} + 7S_2^2 + \frac{52}{3} S_1 S_3 \\
+ 8S_1 S_{1,2} + 17S_2 S_2 + \frac{8}{3} S_4^2 + \left( \frac{16}{3} \zeta_3 + 2\zeta_2 S_1 - \frac{20}{3} S_3 - 16S_1 S_2 - \frac{16}{3} S_3^2 \right) \log(-z) \\
+ \left( -\frac{1}{2} \zeta_2 + 4S_2 + 4S_1^2 \right) \log^2(-z) - \frac{4}{3} S_1 \log^3(-z) + \frac{1}{6} \log^4(-z) \right\} \\
(90)\\\\
N_3 = \frac{1}{(q^2)^2} \sum_{n=1}^{\infty} z^n (-n) \\
= \left\{ \frac{1}{\epsilon^2} \left[ -\frac{1}{2} \zeta_2 + K_2 \right] + \frac{1}{\epsilon} \left[ -\frac{1}{2} \zeta_3 - 2\zeta_2 S_1 + 2S_3 - 2K_3 + 4S_1 K_2 + (\zeta_2 - S_2) \log(-z) \right] \\
+ \zeta_4 - 2\zeta_3 S_1 - 7\zeta_2 S_2 - 4\zeta_2 S_1^2 + 7\zeta_2 K_2 \\
- \frac{7}{2} S_4 + \frac{7}{2} S_2^2 + 6S_1 S_3 + 2S_{1,3} + 8K_4 - 8S_1 K_3 + 8S_2^2 K_2 \\
+ (\zeta_3 + 4\zeta_2 S_1 - S_{1,2} - 3S_1 S_2 - 4K_3) \log(-z) + \left( -\zeta_2 + 4S_2 + 4S_1^2 \right) \log^2(-z) \right\} \\
(91)\\\\
N_{13} = \frac{1}{(q^2)^2} \sum_{n=1}^{\infty} z^n \frac{1}{n} \\
= \left\{ \frac{1}{\epsilon^2} S_1 + \frac{1}{\epsilon} \left[ -\frac{1}{2} S_2 - \frac{7}{2} S_2^2 + 2 \frac{S_1}{n} \right] \\
- 5\zeta_2 S_1 + \frac{14}{3} S_3 + 2S_{1,2} - \frac{1}{2} S_1 S_2 - \frac{37}{6} S_3^2 + S_2 + 7 \frac{S_1^2}{n} - 4 \frac{S_1}{n^2} \right\} \\
(92)\\\\
N_{12} = \frac{1}{(q^2)^2} \sum_{n=1}^{\infty} z^n \left( \frac{2n}{n} \right)^{-1} \frac{1}{n} \\
= \left\{ -\frac{6}{\epsilon^2} W_1 + \frac{1}{\epsilon} \left[ -4W_2 - 12W_{1,1} - 12S_1 W_1 - 16S_2 \right] \\
- 18\zeta_2 W_1 - 4W_3 - 8S_3 + 12W_{1,2} - 8W_{2,1} - 12W_{1,(i+1)} \right\}
\[-16S_{1,2} - 8S_1W_2 - 24S_1W_{1,1} - 16S_1S_2 - 12S_1^2W_1 \}

(93)

E Useful sums

In Appendix B we have presented our results for two-point functions in terms of the basis elements given in appendix A. Summing the series, we were able to obtain closed expressions in terms of polylogarithms. In appendix D we also gave the series representation for the vertex functions under consideration. The summation and representation in terms of integrals of polylogarithms is done for separate structures, occurring repeatedly in these series.

In the case when only harmonic sums enter the expression the formulae given in [24] are very useful. For convenience we introduce the notation

\[ \text{Li}_1(z) = -\log(1 - z). \]

(94)

E.1 Sums involving \( S \) and \( K \)

The following general relations hold true

\[
\sum_{n=1}^{\infty} \frac{f(n-1)}{n^a} z^n = \int_0^z \frac{dz'}{z'} \sum_{n=1}^{\infty} \frac{f(n-1)}{n^{a-1}} (z')^n,
\]

(95)

\[
\sum_{n=1}^{\infty} f(n-1) z^n = \frac{z}{1-z} \sum_{n=1}^{\infty} \left[ f(n) - f(n-1) \right] z^n,
\]

(96)

where \( f(n-1) \) is any function.

As a particular result we have

\[
\sum_{n=1}^{\infty} \frac{1}{n^a} z^n = \text{Li}_a(z),
\]

(97)

\[
\sum_{n=1}^{\infty} S_a(n-1) z^n = \frac{z}{1-z} \text{Li}_a(z),
\]

(98)

\[
\sum_{n=1}^{\infty} S_1(n-1) \frac{1}{n^a} z^n = S_{a-1,2}(z),
\]

(99)

\[
\sum_{n=1}^{\infty} K_a(n-1) z^n = \frac{z}{1-z} \text{Li}_a(-z).
\]

(100)

Only \( S \)-functions (for brevity: on the l.h.s. \( S_a = S_a(n-1) \) etc. and on the r.h.s. \( \text{Li}_a = \text{Li}_a(z) \) etc.)

\[
\sum_{n=1}^{\infty} \frac{S_2}{n} z^n = \text{Li}_1 \text{Li}_2 - 2S_{1,2},
\]

(101)

\[
\sum_{n=1}^{\infty} \frac{S_1^2}{n} z^n = \text{Li}_1 \text{Li}_2 + \frac{1}{3} \text{Li}_1^3 - 2S_{1,2},
\]

(102)

\[
\sum_{n=1}^{\infty} \frac{S_3}{n} z^n = \text{Li}_1 \text{Li}_3 - \frac{1}{2} \text{Li}_2^2,
\]

(103)
\[
\sum_{n=1}^{\infty} \frac{S_1 S_2}{n} z^n = \frac{1}{2} \text{Li}_1^2 \text{Li}_2 + \text{Li}_1 \text{Li}_3 - \frac{1}{2} \text{Li}_2^2 - \text{Li}_1 S_{1,2},
\]
(104)
\[
\sum_{n=1}^{\infty} \frac{S_1^3}{n} z^n = \frac{1}{4} \text{Li}_1^4 + \text{Li}_1 \text{Li}_3 - \frac{1}{2} \text{Li}_2^2 + \frac{3}{2} \text{Li}_1^2 \text{Li}_2 - 3\text{Li}_1 S_{1,2},
\]
(105)
\[
\sum_{n=1}^{\infty} \frac{S_{1,2}}{n} z^n = \frac{1}{2} \text{Li}_1^2 \text{Li}_2 - 2\text{Li}_1 S_{1,2} + 3 S_{1,3},
\]
(106)
\[
\sum_{n=1}^{\infty} \frac{S_2}{n^2} z^n = \frac{1}{2} \text{Li}_2^2 - 2 S_{2,2},
\]
(107)
\[
\sum_{n=1}^{\infty} \frac{S_1^2}{n^2} z^n = \frac{1}{2} \text{Li}_2^2 + 2 S_{1,3} - 2 S_{2,2},
\]
(108)
\[
\sum_{n=1}^{\infty} \frac{S_1^2}{n} z^n = \frac{1}{1-z} \left( \text{Li}_1^2 + \text{Li}_2 \right),
\]
(109)
\[
\sum_{n=1}^{\infty} S_1 S_2 z^n = \frac{1}{1-z} \left( \text{Li}_3 + \text{Li}_1 \text{Li}_2 - S_{1,2} \right),
\]
(110)
\[
\sum_{n=1}^{\infty} S_{1,2} z^n = \frac{1}{1-z} \left( \text{Li}_1 \text{Li}_2 - 2 S_{1,2} \right),
\]
(111)
\[
\sum_{n=1}^{\infty} S_1^2 z^n = \frac{1}{1-z} \left( \text{Li}_3 + 3 \text{Li}_1 \text{Li}_2 + \text{Li}_1^3 - 3 S_{1,2} \right),
\]
(112)
\[
\sum_{n=1}^{\infty} S_1 S_3 z^n = \frac{1}{1-z} \left( \text{Li}_4 + \text{Li}_1 \text{Li}_3 - \frac{1}{2} \text{Li}_2^2 + S_{2,2} \right),
\]
(113)
\[
\sum_{n=1}^{\infty} S_2^2 z^n = \frac{1}{1-z} \left( \text{Li}_2^2 + \text{Li}_4 - 4 S_{2,2} \right),
\]
(114)
\[
\sum_{n=1}^{\infty} S_1^2 S_2 z^n = \frac{1}{1-z} \left( \text{Li}_1^2 \text{Li}_2 + \text{Li}_4 + 2 \text{Li}_1 \text{Li}_3 - 2 \text{Li}_1 S_{1,2} + 2 S_{1,3} - 2 S_{2,2} \right),
\]
(115)
\[
\sum_{n=1}^{\infty} S_1^4 z^n = \frac{1}{1-z} \left( \text{Li}_2^2 + \text{Li}_4 + 4 \text{Li}_1 \text{Li}_3 + 6 \text{Li}_1^2 \text{Li}_2 + \text{Li}_1^4 - 12 \text{Li}_1 S_{1,2} + 12 S_{1,3} - 8 S_{2,2} \right),
\]
(116)
\[
\sum_{n=1}^{\infty} S_{1,3} z^n = \frac{1}{1-z} \left( \text{Li}_1 \text{Li}_3 - \frac{1}{2} \text{Li}_2^2 \right),
\]
(117)
\[
\sum_{n=1}^{\infty} S_{1,(1+2)} z^n = \frac{1}{1-z} \left( \text{Li}_1 \text{Li}_3 + \frac{1}{2} \text{Li}_1^2 \text{Li}_2 - \frac{1}{2} \text{Li}_2^2 - \text{Li}_1 S_{1,2} \right).
\]
(118)

\(S\) and \(K\)
\[
\sum_{n=1}^{\infty} \frac{K_2}{n} z^n = \phi(z),
\]
(119)
\[
\sum_{n=1}^{\infty} \frac{K_{2,1}}{n} z^n = -\int_0^z \frac{S_{1,2}(-t)}{1-t} dt,
\]
(120)
\[
\sum_{n=1}^{\infty} \frac{K_3}{n} z^n = -\int_0^z \frac{\text{Li}_3(-t)}{1-t} dt,
\]
(121)
\[
\sum_{n=1}^{\infty} \frac{S_1 K_2}{n} z^n = -\log(1-z)\phi(z) - \int_0^z \frac{\log(1-t)\text{Li}_2(-t) + \text{Li}_3(-t) + S_{1,2}(-t)}{1-t} dt,
\]
(122)
\[ \sum_{n=1}^{\infty} \frac{K_2}{n^2} z^n = \phi(z) \log z + \int_0^z \frac{\log t \text{Li}_2(-t)}{1-t} \, dt, \quad (123) \]
\[ \sum_{n=1}^{\infty} K_{2,1} z^n = \frac{z}{1-z} \left( -S_{1,2}(-z) \right), \quad (124) \]
\[ \sum_{n=1}^{\infty} S_1 K_2 z^n = \frac{z}{1-z} \left( \phi(z) - S_{1,2}(-z) - \text{Li}_3(-z) \right), \quad (125) \]
\[ \sum_{n=1}^{\infty} S_1 K_3 z^n = \frac{z}{1-z} \left( -\text{Li}_4(-z) - S_{2,2}(-z) - \int_0^z \frac{\text{Li}_3(-t)}{1-t} \, dt \right), \quad (126) \]
\[ \sum_{n=1}^{\infty} S_2 K_2 z^n = \frac{z}{1-z} \left( -\text{Li}_4(-z) - \frac{1}{2} \text{Li}_2^2(-z) + 2S_{2,2}(-z) + \phi(z) \log z \right. \]
\[ \left. + \int_0^z \log t \text{Li}_2(-t) \frac{1}{1-t} \, dt \right), \quad (127) \]
\[ \sum_{n=1}^{\infty} K_{3,1} z^n = \frac{z}{1-z} \left( -S_{2,2}(-z) \right), \quad (128) \]
\[ \sum_{n=1}^{\infty} S_1^2 K_2 z^n = \frac{z}{1-z} \left( -\text{Li}_4(-z) - \frac{1}{2} \text{Li}_2^2(-z) - 2S_{1,3}(-z) + \phi(z) \left( \log z + 2 \log(1-z) \right) \right. \]
\[ \left. - \int_0^z (2 \log(1-t) - \log t) \text{Li}_2(-t) + 2 \text{Li}_3(-t) + 2S_{1,3}(-t) \frac{dt}{1-t} \right), \quad (129) \]

where
\[ \phi(z) = \frac{1}{2} S_{1,2}(z^2) - S_{1,2}(z) - S_{1,2}(-z) + \log(1-z) \text{Li}_2(-z). \quad (130) \]

### E.2 Sums involving V and W

The factor \( \binom{2n}{n} \) can be written in the following useful form
\[ \binom{2n}{n}^{-1} = \frac{n}{2} \int_0^1 ds s^{n-1} (1-s)^{n-1} \]

Again we make use of some general formulae.

For any function \( f \) we have (hereafter \( p = s(1-s) \))
\[ \sum_{n=1}^{\infty} \binom{2n}{n}^{-1} f(n-1) \frac{n}{s} z^n = \int_0^1 ds \sum_{n=1}^{\infty} f(n-1) (zp)^n, \]
\[ \sum_{n=1}^{\infty} \binom{2n}{n}^{-1} f(n-1) \frac{n}{s} \left[ S_1 - S_1 \right] z^n = \frac{1}{2} \int_0^1 ds \log(p) \sum_{n=1}^{\infty} f(n-1) (zp)^n, \]
\[ \sum_{n=1}^{\infty} \binom{2n}{n}^{-1} f(2n-1) \frac{n}{s} z^n = \frac{1}{2} \int_0^1 ds \sum_{n=1}^{\infty} f(n-1) \left[ (\sqrt{zp})^n + (-\sqrt{zp})^n \right]. \quad (131) \]

When the function \( V_{a,b}(n-1) \) contribute we have
\[ \sum_{n=1}^{\infty} \frac{V_2}{n} z^n = \int_0^1 ds f_2(z, p) \]

22
\[
\sum_{n=1}^{\infty} \frac{1}{n} \left[ V_2 + \tilde{V}_2 \right] z^n = \int_0^1 \frac{ds}{s} \log(1 - z) \log(1 - pz)
\]
\[
\sum_{n=1}^{\infty} \frac{V_{2,1}}{n} z^n = \int_0^1 \frac{ds}{s} \left[ -I_2(z, p) \log(1 - p) + I_{2,1}(z, p) \right]
\]
\[
\sum_{n=1}^{\infty} \frac{1}{n} \left[ V_{2,1} - \tilde{V}_{2,1} \right] z^n = \frac{1}{2} \int_0^1 \frac{ds}{s} \log(p) I_2(z, p)
\]
\[
\sum_{n=1}^{\infty} \frac{V_3}{n} z^n = -\int_0^1 \frac{ds}{s} \left[ \log(1 - z) I_2(zp) + S_{1,2}(z) + S_{1,2}(zp) + I_3(z, p) \right]
\]
\[
\sum_{n=1}^{\infty} \frac{V_2}{n^2} z^n = \int_0^1 \frac{ds}{s} \left[ I_2(z, p) \log(z) + \log(1 - p) \left( \log(1 - z) - \zeta_2 \right) \right.
onumber
\]
\[
- \frac{1}{2} \log^2(1 - p) \log(1 - z) + I_{2,1}(z, p) + I_{1,2}(z, p) \right].
\]

where

\[
I_2(z, p) = \text{Li}_2 \left( \frac{p(1 - z)}{1 - zp} \right) - \text{Li}_2(p) - \log \left( \frac{1 - zp}{1 - z} \right) \log(1 - p) + \frac{1}{2} \log^2(1 - zp)
\]
\[
I_3(z, p) = \text{Li}_3(p) - \zeta_3 - \text{Li}_3 \left( \frac{p(1 - z)}{1 - zp} \right) + \text{Li}_3 \left( \frac{1 - z}{1 - zp} \right) + \log \left( \frac{1 - z}{1 - zp} \right) \left[ \text{Li}_2 \left( \frac{p(1 - z)}{1 - zp} \right) - \text{Li}_2 \left( \frac{1 - z}{1 - zp} \right) \right] - \frac{1}{2} \log(z) \log^2 \left( \frac{1 - z}{1 - zp} \right)
\]

When the functions \( W_{a,b}(n - 1) \) contribute we have

\[
3 \sum_{n=1}^{\infty} \binom{2n}{n}^{-1} \frac{1}{n} W_1 z^n = -\int_0^1 \frac{ds}{s} \frac{z}{1 - zp} \log(1 - zs)
\]
\[
3 \sum_{n=1}^{\infty} \binom{2n}{n}^{-1} \frac{1}{n} S_1 W_1 z^n = -\int_0^1 \frac{ds}{s} \frac{z}{1 - zp} \log(1 - zs) \log \left( \frac{s}{(1 - s)(1 - zp)} \right)
\]
\[
\sum_{n=1}^{\infty} \binom{2n}{n}^{-1} \frac{1}{n} \left[ 3W_{1,1} + W_2 \right] z^n = -\int_0^1 \frac{ds}{s} \frac{z}{1 - zp} \left[ 2\text{Li}_2(pz) + \log(1 - zs) \log \left( \frac{s}{1 - s} \right) \right]
\]
\[
3 \sum_{n=1}^{\infty} \binom{2n}{n}^{-1} \frac{1}{n^2} W_1 z^n = -\int_0^1 \frac{ds}{p} \left[ \text{Li}_2(s) - \text{Li}_2 \left( \frac{s}{1 - zp} \right) + \log(1 - s) \log(1 - zp) \right.
onumber
\]
\[
- \frac{1}{2} \log^2(1 - zp) \right]
\]

23
\[
3 \sum_{n=1}^{\infty} \left( \frac{2n}{n} \right)^{-1} \frac{1}{n^2} S_1 W_1 z^n = \int_0^1 ds \left[ \frac{\text{Li}_3(s) - \text{Li}_3\left( \frac{s}{1-zp} \right)}{p} \right] - \frac{1}{3} \log^3(1-zp) + \frac{1}{2} \log(1-s) \log^2(1-zp) - \log(1-zp) \log(1-s) \log\left( \frac{s}{1-s} \right) \]

\[
\sum_{n=1}^{\infty} \left( \frac{2n}{n} \right)^{-1} \frac{1}{n^2} \left[ 3W_{1,1} + W_2 \right] z^n = 3 \sum_{n=1}^{\infty} \left( \frac{2n}{n} \right)^{-1} \frac{1}{n^2} S_1 W_1 z^n = \int_0^1 ds \frac{z}{1-zp} \Phi_1(z, p) \]

\[
3 \sum_{n=1}^{\infty} \left( \frac{2n}{n} \right)^{-1} \frac{1}{n^2} S_2 W_1 z^n = \int_0^1 ds \frac{z}{1-zp} \left[ \Phi_2(z, p) + \Phi_1(z, p) \right] \]

\[
3 \sum_{n=1}^{\infty} \left( \frac{2n}{n} \right)^{-1} \frac{1}{n^2} S_1 W_2 z^n = \int_0^1 ds \frac{z}{1-zp} \left[ \Phi_2(z, p) - \Phi_1(z, p) - \Phi_3(z, p) \right] \]

\[
\sum_{n=1}^{\infty} \left( \frac{2n}{n} \right)^{-1} \frac{1}{n^2} \left[ W_3 - 3W_{1,2} \right] z^n = \int_0^1 ds \frac{z}{1-zp} \left[ \Phi_4 - \Phi_5 - \Phi_2 + \Phi_1 + \Phi_3 \right] \]

\[
\sum_{n=1}^{\infty} \left( \frac{2n}{n} \right)^{-1} \frac{1}{n^2} \left[ 3W_{1,(1+1)} + 2W_{2,1} \right] z^n = \int_0^1 ds \frac{z}{1-zp} \left[ \Phi_5(z, p) + \Phi_1(z, p) \right],
\]

where

\[
\Phi_1(z, p) = \log\left( \frac{s}{1-s} \right) \left[ \text{Li}_2\left( \frac{zs^2}{1-zp} \right) + \frac{1}{2} \log^2\left( \frac{1-zs}{1-zp} \right) + \log(1-zp) \log\left( \frac{1-zs}{1-zp} \right) \right] - 2 \left[ \text{Li}_3(pz) - S_{1,2}(pz) - \log(1-zp) \text{Li}_2(pz) \right]
\]

\[
\Phi_2(z, p) = \text{Li}_3\left( \frac{zs^2}{1-zp} \right) + S_{1,2}\left( \frac{zs^2}{1-zp} \right) - S_{1,2}(pz) - \log(1-zp) \left[ \text{Li}_2\left( \frac{zs^2}{1-zp} \right) \right] - \frac{1}{2} \log^3(1-zp)
\]

\[
\Phi_3(z, p) = 2S_{1,2}\left( \frac{zs^2}{1-zp} \right) - 2\text{Li}_3(zp) + 2 \log\left( \frac{1-zs}{1-zp} \right) \text{Li}_2\left( \frac{zs^2}{1-zp} \right) + \frac{1}{3} \log^3(1-zs) - \log^2(1-zs) \log(1-zp) + \log(1-zs) \log^2(1-zp)
\]

\[
\Phi_4(z, p) = 2\text{Li}_3\left( \frac{zs^2}{1-zp} \right) - \text{Li}_2\left( \frac{zs^2}{1-zp} \right) \log\left( \frac{s}{(1-s)(1-zp)} \right) + 2\text{Li}_3(pz) + S_{1,2}(pz) + \frac{5}{6} \log^3(1-zp)
\]

\[
9\Phi_5(z, p) = 2 \left[ \text{Li}_3(pz) - 6S_{1,2}(pz) - 3 \log(1-zp) \text{Li}_2(pz) - \frac{1}{6} \log^3(1-zp) \right] - 4 \left[ \text{Li}_3\left( \frac{zs^2}{1-zp} \right) + 2S_{1,2}\left( \frac{zs^2}{1-zp} \right) \right] - \text{Li}_2\left( \frac{zs^2}{1-zp} \right) \log\left( \frac{s}{(1-s)(1-zp)^2} \right).
\]
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**Figure captions**

**Fig. 1:** 2-loop selfenergy- and vertex diagrams considered in the paper. Line numbering and kinematics.

**Fig. 2:** 2-loop selfenergy diagrams evaluated in this work. Solid lines denote propagators with the mass $M$; dashed lines denote massless propagators.

**Fig. 3:** 2-loop vertex diagrams evaluated in this work. Kinematics as in Fig. 1 ($p_1^2 = p_2^2 = 0$). Solid lines denote propagators with mass $M$; dashed lines denote massless propagators.
\[ p_1^2 = 0, \ p_2^2 = 0 \]

Fig. 1
Fig. 3
Fig. 2