Regularization Methods in Chiral Perturbation Theory

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Abstract

Chiral lagrangians describing the interactions of Goldstone bosons in a theory possessing spontaneous symmetry breaking are effective, non-renormalizable field theories in four dimensions. Yet, in a momentum expansion one is able to extract definite, testable predictions from perturbation theory. These techniques have yielded in recent years a wealth of information on many problems where the physics of Goldstone bosons plays a crucial role, but theoretical issues concerning chiral perturbation theory remain, to this date, poorly treated in the literature. We present here a rather comprehensive analysis of the regularization and renormalization ambiguities appearing in chiral perturbation theory at the one loop level. We discuss first on the relevance of dealing with tadpoles properly. We demonstrate that Ward identities severely constrain the choice of regulators to the point of enforcing unique, unambiguous results in chiral perturbation theory at the one-loop level for any observable which is renormalization-group invariant. We comment on the physical implications of these results and on several possible regulating methods that may be of use for some applications.

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1 Introduction

The phenomenon of the spontaneous breakdown of a continuous symmetry is one of the recurrent themes in modern physics. Randomly chosen examples exhibiting this property are the dynamical breakdown of the chiral symmetry in Quantum Chromodynamics[1, 2], the BCS theory of superconductivity[3], the breaking of the electroweak symmetry that gives masses to the $W^\pm$ and $Z$ intermediate vector bosons[4], or the appearance of the Néel state in strongly coupled electron systems[5].

A characteristic signal of the spontaneous breakdown of a continuous symmetry is the appearance of Goldstone bosons[6]. This is a very general phenomenon, taking place both for relativistic and non-relativistic systems. Stated in mathematical terms, it means that there are states whose energy vanishes as the three-momentum tends to zero. The dispersion relation connecting energy and momentum can be either linear ($E \sim |k|$, for a relativistic theory), or non-linear ($E \sim k^2$, for non-relativistic systems). From the point of view of field theory this amounts to stating the presence of zero-mass particles in the spectrum.

The implications of the spontaneous breakdown of a continuous symmetry go, however, well beyond the mere appearance of Goldstone bosons. The interactions of Goldstone bosons are largely dictated by very general considerations. On the one hand, interactions of Goldstone bosons necessarily contain at least one derivative. This can be understood as follows. The effective Hamiltonian describing the Goldstone bosons must be invariant under constant shifts of the Goldstone fields

$$\pi(x) \to \pi(x) + c$$

since such a field redefinition does not change the energy-momentum dispersion relation. The above invariance is ensured by derivative couplings. The fact that interactions are proportional to momenta implies that at zero momentum Goldstone bosons do not interact.

On the other hand, symmetries, even if they are partially broken, are powerful enough to dictate the form of the effective long distance lagrangian for these Goldstone modes. Since the work of Callan, Coleman, Wess and Zumino[7] we know that an economic and handy way of describing the interactions of Goldstone bosons is provided by a non-linear sigma model. The set of Goldstone bosons appearing after the breaking of a symmetry group $G$ to some subgroup $H$ can be grouped in a dimensionless matrix-valued field $U(x)$ belonging to the quotient space $G/H$. The effective lagrangian can be organized as a momentum expansion
with higher dimensional operators containing an increasing number of derivatives acting on the field $U(x)$.

The lagrangian governing the physics of Goldstone bosons is, except in very simple cases, non-linear. Thus we must deal with theories which are not very tractable from a quantum point of view in four dimensions. Can we, perhaps, ignore quantum corrections? The answer is except if we content ourselves with the crudest of approximations, no. The tree level approximation will be good only at very low energies. Loops give sizable contributions to many observables and, characteristically, they are amongst the most interesting ones. Perhaps the most striking examples are those where a process is actually forbidden at tree level, but allowed by loop corrections, such as the rare (but measured) decay $K_S \rightarrow \gamma\gamma$ [8] or, in the process $\gamma\gamma \rightarrow Z_L Z_L$, recently proposed [9] as an excellent way of testing possible departures from the minimal Standard Model[10]. (Using the so-called equivalence theorem[11], the scattering of longitudinal vector bosons can be expressed in terms of an equivalent process with the Goldstone bosons of the broken electroweak symmetry.)

We shall be concerned here with a particular pattern of symmetry breaking, in which the symmetry group $G = SU(N)_L \times SU(N)_R$ is broken to its diagonal subgroup $H = SU(N)_V$. Originally, we are dealing here with a lagrangian that depends on some matrix-valued field $M(x)$, which is left invariant under independent left and right multiplication of $M(x)$ by constant matrices of $SU(N)$. When this symmetry is broken to the diagonal group $SU(N)_V$, there appear $N^2 - 1$ Goldstone bosons, grouped in a matrix-valued field $U(x)$ belonging to the group $SU(N)_L \times SU(N)_R / SU(N)_V$. This breaking pattern, setting $N = 3$, is the one relevant in the realization of chiral symmetry in Quantum Chromodynamics at low energies. $N = 2$ corresponds to the breaking pattern of the global symmetry of the scalar sector of the Electroweak Theory, responsible for the generation of mass of the $W^\pm$ and $Z$ particles. The longitudinal components of these massive particles are precisely the Goldstone bosons associated to the broken generators. The effective lagrangians that correspond to the above breaking pattern are usually named chiral lagrangians.

The number of operators in the effective chiral lagrangian with the right invariance properties is relatively small if we do not consider the coupling to external fields and we assume that there are no terms that break explicitly the original chiral group $SU(N)_L \times SU(N)_R$. There is only one operator with two derivatives

$$L^{(2)} = \frac{f^2}{4} Tr \partial_{\mu} U^\dagger \partial^\mu U$$  \hspace{1cm} (1.2)
There are three operators with four derivatives:

$$\mathcal{L}^{(4)} = L_1 Tr(\partial^\mu U^\dagger \partial^\mu U)^2 + L_2 Tr(\partial^\mu U^\dagger \partial^\nu U) Tr(\partial^\mu U^\dagger \partial^\nu U)$$

$$+ L_3 Tr(\partial^\mu U^\dagger \partial^\nu U^\dagger \partial^\nu U)$$ (1.3)

and operators proliferate if we consider terms with six or more derivatives.

Notice that there is a dimensional constant $f_\pi$ in front of (1.2). In pion physics, $f_\pi$ is the pion decay constant. The very presence of this dimensional coupling constant plays a crucial role in chiral perturbation theory. Since we have a dimensional parameter at our disposal we can construct, with the help of the momenta, a dimensionless quantity $q^2/(4\pi f_\pi)^2$ which can be made arbitrarily small. We therefore have an adjustable parameter allowing us to determine a range of energies in which tree level is certainly more important than one-loop corrections, and these in turn are more important than two-loop corrections and so on. (In two dimensions, on the contrary, the equivalent to $f_\pi$ is dimensionless and we are left without any adjustable parameter. It is thus remarkable that two-dimensional chiral models are actually much more difficult to treat than four dimensional ones.)

$f_\pi$ sets an absolute scale with respect to which one measures all energy scales. Any effective theory possessing the same long-distance properties will necessarily lead to the same operator $\mathcal{L}^{(2)}$, with the same value for the coefficient in front. Because $\mathcal{L}^{(2)}$ is completely general, if we are somehow limited to performing low energy experiments we will learn very little, if anything at all, about the underlying theory from measuring $f_\pi$ very precisely. This is very clearly seen in attempts to derive a long distance effective lagrangian from QCD. This is, needless to say, a very difficult problem; a strongly coupled theory deep in the non-perturbative regime. Short of solving QCD exactly we can try to make some more or less justified simplifications, in the hope that we are somehow modelling QCD in an intermediate energy range. For instance, we can assume that, for some range of energies, interactions are dominated by the interchange of a scalar particle. Or, we can instead assume that vector and axial-vector particle interchange does dominate long distance interactions. The point is that in both cases we can always tune some parameters so as to reproduce the experimental value of $f_\pi$. How can we tell which of the two models is more correct? Certainly measuring $f_\pi$ will be of no help in this issue.

On the contrary, it has been found that integrating out the scalar, or the vector and axial-vector effective degrees of freedom lead to very different values for the $\mathcal{O}(p^4)$ coefficients. The values obtained from vector and axial-vector inter-
change are, on the whole and roughly speaking, much more compatible with the experimental results than those obtained from scalar interchange alone\cite{12}. So there is definitely some truth in assuming that at some intermediate scale massive vector particles dominate strong interactions. In the old days of hadronic physics this very fact used to be called vector-meson dominance\cite{13}.

We would certainly like to find the values of these constants $L_i$ directly from the QCD lagrangian or, generally speaking, for any theory that exhibits chiral symmetry breaking because, as we have just discussed, they contain the relevant information at low energies about whatever underlying physics is behind. For instance, it would be of utmost interest to be able to compare the theoretical and experimental values for these coefficients in the symmetry breaking sector of the Standard Model to verify or falsify different models for the scalar sector. A lot of work has been done recently in this field\cite{14, 15, 16, 17, 18}. But \textit{en rigueur} there are several theoretical issues that need be carefully clarified before one is entitled to make a detailed comparison with experiment and jump into conclusions.

One of the reasons is that the chiral lagrangian is a non-linear sigma model and it has a very bad ultraviolet behaviour in perturbation theory. Naive power counting allows even for quartic divergences and quadratic as well as logarithmic divergences are present\cite{19, 20}. As often happens in field theory, symmetries must play a crucial role in determining, how many of the possible counterterms that are allowed by naive power counting do actually appear. Even after taking this into account it should be obvious that quantum corrections in a non-linear theory of this type are badly divergent. In fact non-linear sigma models in four dimensions are not renormalizable in perturbation theory. This means that if we start with the simplest non-linear theory, the one described by $\mathcal{L}^{(2)}$, and we compute loops with it, new counterterms will be required at each order in perturbation theory. In particular, it will certainly be necessary to redefine the coefficients $L_i$ of $\mathcal{L}^{(4)}$ order by order in chiral perturbation theory to absorb some logarithmic ultraviolet divergences. In principle this is not very different from what one does in renormalizable theories, except that the number of counterterms here is, strictly speaking, infinite. However, at any given order in the adjustable parameter $q^2/(4\pi f_\pi)^2$ the number of possible counterterms is finite.

The renormalized parameters can be obtained from experiment. From comparison with the experimental data one will learn that, for instance, one of the $L_i$'s takes a given value when working in a given regularization and at a given subtraction scale $\mu$. Let's call the renormalized coefficients $L_i(\mu)$. The double dependence both on the regulator and on the scale means that none of the $L_i(\mu)$'s
have *per se* any physical meaning. Of course, for practical applications this is just fine. If we work consistently in the same regularization and renormalization scheme we can use the value of $L_i(\mu)$ just obtained in other processes and make definite, testable predictions. This is the standard procedure in pion physics.[1]

However, rather than just fitting the coefficients of the chiral lagrangian we would like to *compute* them. We would like to determine the values of, say, one of the $L_i$ in a given theory and compare this values with the data available. In order to do that we need a good theoretical control of the precise relation between those values of the renormalized coefficients $L_i(\mu)$ that can be extracted from the experiment and those that we can compute in a particular model. That the issue is far from obvious can be illustrated from recent estimations on the values of the $O(p^4)$ coefficients from QCD. After integrating out the quark and gluonic degrees of freedom, at leading order in $1/N_c$, $N_c$ being the number of colors, the authors of [21] found the following estimation for $L_1$, $L_2$ and $L_3$

$$
L_1 = \frac{N_c}{384\pi^2} \quad L_2 = \frac{N_c}{192\pi^2} \quad L_3 = -\frac{N_c}{96\pi^2}
$$ (1.4)

The leading gluonic corrections turn out to be zero for $L_1$ and $L_2$, but non-zero for $L_3$. Let’s ignore the gluonic corrections altogether for the present discussion. The point we need to retain is that the above values for the $L_i$ are finite, non-ambiguous and of $O(N_c)$ and remain so after including $O(N_c)$ gluonic corrections. Furthermore they appear to be independent of the regularization scheme used to derive them [21]. Now we want to compare these theoretical predictions with the experimental values for $L_1(\mu)$, $L_2(\mu)$ and $L_3(\mu)$. The latter are, however, renormalized coefficients that depend on some subtraction scale and on the regulator that has been used to render quantities finite in chiral perturbation theory. Is the comparison meaningful, or even possible?

From large $N_c$ counting arguments it is not difficult to show that $f_\pi^2$ is of $O(N_c)$; therefore contributions from chiral loops (including the scale and regulator ambiguities) are down by powers of $1/N_c$. It is also known from chiral perturbation theory that there are combinations of renormalized coefficients, such as $L_1(\mu) - \frac{1}{2}L_2(\mu)$ which are renormalization group invariant at the one-loop level from the point of view of chiral lagrangians, i.e. where the logarithmic divergences of the effective theory cancel

$$
\mu \frac{\partial}{\partial \mu}(L_1(\mu) - \frac{1}{2}L_2(\mu)) = 0
$$ (1.5)

The combination $L_1(\mu) - \frac{1}{2}L_2(\mu)$ is not an observable by itself and in principle it needs not be regulator independent. In physical amplitudes is always accompan-
nied by a finite piece generated by one-loop perturbation theory; i.e. by interme-
diate states with at least two Goldstone bosons. Let us denote by $F$ this finite piece,
which is of $\mathcal{O}(1)$ in the $N_c$ expansion. The combination $A = F + L_1(\mu) - \frac{1}{2}L_2(\mu)$
is observable and, hence, should be independent of the regulator. In the best of
worlds we could determine this observable quantity $F$ from first principles, i.e.
from QCD. Then, after fixing the only free scale in the theory, $\Lambda_{QCD}$, from $f_\pi$, and
expanding the amplitude up to $\mathcal{O}(p^4)$ we would find a unique, non ambiguous,
value for the observable $A$. This value could then be expanded in powers of $1/N_c$.
From large $N_c$ arguments we know that the leading contribution to the effective
action is of order $\mathcal{O}(N_c)$, corresponding to contributions from intermediate states
with two quark lines. At order $p^4$ it would be given by the coefficients $L_i$ in
eq (1.4). In addition, although they have never been explicitly computed, there
are contributions from intermediate states with four or more quark lines. These
are of $\mathcal{O}(1)$ and must correspond, in the effective chiral theory, to $F$. Therefore,
while we cannot, strictly speaking, compare the values for $L_1(\mu)$ or $L_2(\mu)$ that we
get from experiment with the theoretical predictions of [21] because the former
have arbitrary subtractions that depend on the calculational scheme we use in
chiral perturbation theory and have nothing to do with the fundamental theory,
we expect to be able to compare the theoretical and experimental predictions
for the renormalization-group invariant combination $L_1(\mu) - \frac{1}{2}L_2(\mu)$. We should
mention that from [21] one concludes that, up to subleading gluonic corrections

$$L_1 - \frac{1}{2}L_2 \simeq 0$$

(1.6)

In fact, it is known that in $\pi\pi$ scattering the $J = 1, I = 1$ amplitude is very well
described if one ignores altogether the contributions from the $\mathcal{O}(p^4)$ operators,
which precisely turn out to be proportional to the combination $L_1 - \frac{1}{2}L_2$.

For the above identification to work a necessary condition must be met. The
value for the finite contribution $F$ that comes from the one loop chiral expan-
sion must be regulator independent. It must be a finite, unambiguous number.
Otherwise it would be very difficult to attach a precise physical meaning to combi-
nations like $L_1 - \frac{1}{2}L_2$, which we pretend to be able to compute in the fundamental
theory.

This brings us to the crux of the matter. When we compute in chiral pertur-
bation theory and obtain a quantity that is ultraviolet finite, and, in principle
observable, is this quantity regulator independent or not? If we were here dealing
with a renormalizable theory, the answer should be positively yes, but, of course,
we are dealing here with non-renormalizable quantum field theories, with a very
pathological ultraviolet behaviour as we have already mentioned. Another way of phrasing the question is whether renormalization-group invariants in chiral perturbation theory may depend on the way we regulate the theory. We will see in the next sections that chiral invariance plays a very crucial and subtle role in settling this issue in a very satisfactory manner. In principle the result, even for finite quantities, can be completely arbitrary. We will illustrate this in the particular example of $\pi\pi$ scattering, which we will discuss in detail in section 3. We will work in a rather general class of regulators in position space. In section 4 we will see how the potential ambiguities are resolved by demanding chiral invariance and that we can attach a definite physical meaning to a combination such as (1.5). We will propose a general rule that regulators preserving chiral invariance must fulfil. In section 5 we will extend the analysis to an arbitrary amplitude at one loop and we will see that the analysis of section 3 carries over to any one-loop process. Section 2 is devoted to general considerations about chiral perturbation theory. We will present our conclusions in section 6. In the Appendix we analyze, rather exhaustively, several regulators that may be of use in chiral perturbation theory.

Of particular interest is to implement the above ideas in the symmetry breaking sector of the Standard Model. Competing alternatives for the scalar sector of the Standard Model should lead to different values for the appropriate coupling constants $L_i$ [10]. We will not discuss this point in the present work, although we believe that our results will be of interest there. We note that a good theoretical control of the coefficients $L_i$ is of particular importance. The reason is twofold. First, the Electroweak Theory contains some interactions that explicitly break the global $SU(2) \times SU(2)$ symmetry, the number of possible counterterms is greatly augmented (up to eight new operators can be written up to $O(p^4)$, and this after restricting ourselves to the $CP$-even sector [14, 22]). Secondly, in contrast to pion physics, we have precious little experimental information concerning the scattering of longitudinal vector bosons, and we will, most likely, remain in the same position for some time.

Finally, an additional motivation to undertake this work has been the following. Technically, in chiral perturbation theory it is difficult to go beyond one loop. Yet, many interesting properties require such an analysis. Can we find regulators in which perturbation theory is more manageable? Obviously, for that we need to know, first of all, which is the most general regulator compatible with chiral symmetry. An interesting computational scheme we have analyzed is the so-called differential renormalization [23]. We will discuss in one of the appendices
its advantages and shortcomings.

To summarize, we have tried to give a comprehensive and complete account of the potential regularization ambiguities in non-linear sigma models at the one loop level. We will start with a brief summary of the basic building blocks of chiral perturbation theory. Since there are excellent references on this subject we will collect only the specific points that we will need in later developments.

2 Measure, Tadpoles and Counterterms

The non-linear sigma model can be described in euclidean space-time by the following partition function

\[ Z = \int d\mu(U) e^{-S(U)} \]  

(2.1)

Minkowskian amplitudes will be obtained, as usual, by performing a Wick rotation. As a rule, we will perform our calculations in euclidean space, but present our results for the amplitudes in Minkowski space-time. To make sense of the above expression we must, first of all, determine which is the correct measure to use. Then we must introduce a regulator in order to cut-off the high energy modes in the path-integral and define (2.1) properly. In fact the two points are not unrelated.

The obvious requirement that the measure of the path integral must fulfil is, of course, chiral invariance. Both the action and the measure must be invariant under \( U(x) \rightarrow LU(x)R^\dagger \), with \( L, R \) constant matrices of \( SU(N) \). \( d\mu(U) \) must therefore be an invariant group measure. The most convenient parametrization of \( U \), the field belonging to the group \( SU(N)_L \times SU(N)_R/SU(N) \) describing the Goldstone boson excitations, is given by \( U = \exp 2i\pi^a T^a/f_\pi \) with \( \pi^a \) being the pion fields and \( T^a \) a properly normalized set of hermitian generators of \( SU(N) \). The Goldstone boson fields — pions, for short — act as coordinates in the group.

In term of these coordinates we can construct the group metric\[24\]

\[ g_{ab} = \delta_{ab} + \frac{1}{3f_\pi^2}\pi^a\pi^d(\delta_{ad}\delta_{cb} - \delta_{ab}\delta_{cd}) + \frac{2}{45f_\pi^4}\pi^a\pi^d\pi^e\pi^f(\delta_{ed}\delta_{af} - \delta_{ef}\delta_{ae}\delta_{bf}) + \ldots \]  

(2.2)

The proper measure is then\[24\]

\[ d\mu(U) = \sqrt{\det g} \prod d\pi \]  

(2.3)

The prefactor \( \sqrt{\det g} \) can be exponentiated as

\[ \sqrt{\det g} = \exp \frac{1}{2}\delta^{(4)}(0)Tr \ln g \]  

(2.4)
Expanding the trace in the previous expression gives rise to a series of terms that, unlike \( S(U) \), do not contain any derivatives. By themselves they are not chiral invariant, but they are required to compensate the lack of chiral invariance of the “flat” measure \( \prod d\pi \). Of course \( \delta^{(4)}(0) \) does not make much sense; it must be regulated in the same way as the rest of the ultraviolet divergences that appear in the perturbative expansion of \( S(U) \). Even before doing that we can already see that the pieces proportional to \( \delta^{(4)}(0) \) originating from the measure cancel part of the tadpoles that are generated in chiral perturbation theory. For instance, the simplest tadpole, the one with two external legs, gives (we particularize to \( N = 2 \))

\[
\delta^{ab} \frac{1}{f^2_\pi} \left( \frac{2}{3} \delta^{(4)}(0) + \frac{2}{3} p^2 \Delta(0) \right)
\]

(2.5)

Where we have used that \( \Box \Delta(x) = -\delta^{(4)}(x) \). \( \Delta(x) \) is the pion propagator carrying a momentum \( p^\mu \) and \( a, b, ... \) are \( SU(2) \) indices. The contribution to this Green function originating from the measure is, at this order,

\[
\delta^{ab} \frac{1}{f^2_\pi} \left( -\frac{2}{3} \right) \delta^{(4)}(0)
\]

(2.6)

thus cancelling the piece proportional to \( \delta^{(4)}(0) \) in (2.5). The next term, of \( \mathcal{O}(1/f^4_\pi) \), in the expansion of the measure is

\[
(\delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{cb}) \frac{1}{f^4_\pi} \left( -\frac{4}{45} \right) \delta^{(4)}(0)
\]

(2.7)

and adds to the tadpole with four external legs, which on shell is equal to

\[
(\delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{cb}) \frac{1}{f^4_\pi} \left( -\frac{16}{45} \right) \delta^{(4)}(0) +
\]

\[
(\delta^{ab} \delta^{cd} s + \delta^{ac} \delta^{bd} t + \delta^{ad} \delta^{cb} u) \frac{1}{f^4_\pi} \left( \frac{10}{9} \right) \Delta(0)
\]

(2.8)

We have introduced the usual kinematical invariants

\[
s = (p_1 + p_2)^2 \quad t = (p_1 + p_3)^2 \quad u = (p_1 + p_4)^2
\]

(2.9)

\( a, b, c \) and \( d \) are the \( SU(2) \) indices carried by the pions with momentum \( p_1, p_2, p_3 \) and \( p_4 \), respectively. Both in (2.5) and (2.8) the \( \delta^{(4)}(0) \)'s originate from the laplacian acting on the propagator, \( \Box \Delta(0) = -\delta^{(4)}(0) \). It is natural to identify the \( \delta^{(4)}(0) \) from the measure with \( -\Box \Delta(0) \). When we regulate the short distance singularities on the theory with the help of some dimensional cut-off \( \epsilon \), \( \Box \Delta(0) \) will be finite and proportional to \( 1/\epsilon^4 \) and so will be \( \delta^{(4)}(0) \).
Apart from measure-induced terms and tadpoles there are many more contributions to the four point amplitude. In general, some additional pieces proportional to $1/\epsilon^4$ will be generated. (They will be discussed in detail in the next section after the evaluation of the diagrams in fig. 1.) Therefore we have contributions proportional to $1/\epsilon^4$ from very different sources: from the measure, from tadpoles and from “normal” Feynman diagrams. Chiral invariance, however dictates that the net result has to be zero, because a piece of the form $1/\epsilon^4$ cannot be absorbed by any chirally invariant counterterm. In dimensional regularization, or in any other similar method which automatically sets to zero all non-logarithmic divergences, the above requirement is fulfilled by construction. If a given regulator does lead to $1/\epsilon^4$ divergences it must be supplemented with appropriate counterterms to bring the results into agreement with chiral invariance. This will be our general philosophy. We shall write different conditions that a regulator must comply with. If a given prescription does not lead to results in agreement with the Ward identities of the theory, we will supplement it with suitable counterterms. We will see in detail how this procedure works for $\pi\pi$ scattering and then we will extend it to an arbitrary one-loop process.

The next type of divergences we have to worry about are the quadratic ones. Some of them are innocuous from the point of view of preserving the Ward identities—they can be absorbed by a redefinition of $f_\pi$. This is, for instance, the case of the two-legged tadpole (eq. (2.5)). Redefining

$$\frac{1}{f_\pi^2} \rightarrow \frac{1}{f_\pi^2} - \frac{1}{f_\pi^3} \left(\frac{2}{3}\right)\Delta(0)$$

(2.10)

eliminates this term. Once regulated $\Delta(0)$ will be proportional to $1/\epsilon^2$. Chiral invariance implies that this same redefinition must eliminate all the pieces proportional to $1/\epsilon^2$ in all amplitudes. For instance an amplitude with four external legs will contain terms in $1/\epsilon^2$ from tadpoles (eq. (2.8)) and from “normal” Feynman diagrams (fig. 1, see next section). Chiral invariance requires those to add-up in such a way that they can be absorbed by the redefinition (2.10). Quadratic divergences that cannot be eliminated via a redefinition of $f_\pi$ must be absent. As in the case of quartic divergences, we can take the pragmatic attitude of using an arbitrary regulator supplemented with suitable counterterms.

It turns out that from a computational point of view it is highly advantageous to enforce the condition $\Delta(0) = 0$ from the start. While our analysis will go through without this restriction, demanding $\Delta(0) = 0$ will simplify the calculations to a large extent. Therefore we shall assume from now on that our regulator is such that (perhaps with the help of a counterterm) $\Delta(0) = 0$. We
will see in the appendix how relaxing this condition does not change the results for the physically relevant part of the amplitude.

3 Non-linear $\sigma$ model to $\mathcal{O}(p^4)$: $\pi\pi$ scattering

We shall use the $\pi\pi$ scattering amplitude as a battleground to analyze in detail the regularization and renormalization ambiguities. We will consider the $SU(2)$ case for simplicity and use the notation and language of pion physics, although the conclusions are, obviously, more general. We will use the following notation to characterize $\pi\pi$ scattering

$$\pi^a(p_1) + \pi^b(p_2) \rightarrow \pi^c(p_3) + \pi^d(p_4) \quad (3.1)$$

where $p^i, i = 1, \ldots, 4$ are the four momenta of the pions, and $a, b, c = 1, 2, 3$ are isospin indices. The three Goldstone bosons form a $I = 1$ representation of the isospin group $SU(2)$. We will ignore throughout the additional contributions that one should include if we would take $SU(3)$ as the chiral symmetry group (as it is the case in the real world). We will also set all the masses of the Goldstone bosons to zero. Including the contributions from the kaons and mass corrections changes nothing at the conceptual level, but makes all calculations a lot more involved. There is an additional reason to stick to the massless $SU(2)$ case. As discussed in the introduction, the pions are here generic representatives of the Goldstone bosons generated in the spontaneous breaking of the $SU(2)_L \times SU(2)_R$ to its diagonal subgroup $SU(2)_V$. By just replacing $f_\pi \simeq 93$ MeV by $v \simeq 250$ GeV one is describing the scalar sector of the Standard Model of electroweak interactions\[11\]. The symmetry group there is indeed $SU(2)$ and, in addition, mass terms for the Goldstone bosons are explicitly ruled out by gauge invariance.

The amplitude for this process can be written in a form that shows explicitly the isospin structure\[23\]. In terms of the Mandelstam variables

$$\mathcal{A} = F(s, t, u) \delta^{ab} \delta^{cd} + F(u, t, s) \delta^{ac} \delta^{bd} + F(t, s, u) \delta^{ad} \delta^{bc} \quad (3.2)$$

In the effective chiral lagrangian, perturbation theory is organized as an expansion in inverse powers of $f_\pi$, the pion decay constant. Each pion field is accompanied by one of such powers. One loop corrections to $\pi\pi$ scattering are thus down by a factor $1/f_\pi^2$ with respect to tree level results. Since, on dimensional grounds, all operators contributing to the effective lagrangian must be of dimension four, it is clear that operators of $\mathcal{O}(p^4)$ like \[13\] have coefficients in

12
front of them which are down by a factor $1/f^2_\pi$ with respect to the only operator of $\mathcal{O}(p^2)$, \cite{12}. Thus to a given process, in particular $\pi\pi$ scattering, and up to one-loop order in chiral perturbation theory we must therefore consider several type of contributions. On the one hand, there are contributions from the $\mathcal{L}^{(2)}$ lagrangian, both at tree and one-loop level. In addition, the $\mathcal{L}^{(4)}$ lagrangian contributes only at tree level. Altogether, the Feynman diagrams that contribute to the process (3.1) up to one-loop order are depicted in fig. 1.

In the absence of any external fields and mass terms, $\mathcal{L}^{(2)}$ and $\mathcal{L}^{(4)}$ for $SU(2)$ reduce to the following operators

$$\mathcal{L}^{(2)} = \frac{f^2_\pi}{4} Tr(\partial_\mu U^\dagger \partial^\mu U)$$
$$\mathcal{L}^{(4)} = L_1 Tr(\partial_\mu U^\dagger \partial^\mu U)^2 + L_2 Tr(\partial_\mu U^\dagger \partial_\nu U) Tr(\partial^\mu U^\dagger \partial^\nu U)$$

(3.3)

If we now expand $U$ in powers of $1/f_\pi$ in (3.3) one ends up with

$$\mathcal{L}^{(2)} = + \frac{1}{2} \partial_\mu \pi^i \partial^\mu \pi^i + \frac{1}{6 f^2_\pi} \partial_\mu \pi^i \partial^\mu \pi^j \pi^k \pi^l (\delta^{il} \delta^{jk} - \delta^{ik} \delta^{jl}) +$$
$$+ \frac{1}{45 f^4_\pi} \pi^m \pi^n \pi^j \pi^k \partial_\mu \pi^r \partial^\mu \pi^s (\delta^{mn} \delta^{rs} \delta^{jk} - \delta^{mn} \delta^{jr} \delta^{ks}) + ...$$

$$\mathcal{L}^{(4)} = + \frac{4 L_1}{f^4_\pi} \partial_\mu \pi^i \partial^\mu \pi^i \partial_\nu \pi^j \partial^\nu \pi^j + \frac{4 L_2}{f^4_\pi} \partial_\mu \pi^i \partial_\nu \pi^j \partial^\mu \pi^j \partial^\nu \pi^j + ...$$

(3.4)

where it is apparent that $\mathcal{L}^{(2)}$ contains, in addition to a free kinetic piece, an interaction term. Upon iteration we will generate terms of $\mathcal{O}(1/f^4_\pi)$ and then we need to include $\mathcal{L}^{(4)}$ too, if only for that reason. Of course we need to include higher dimensional operators anyway because we are dealing with a non-linear theory, plagued with ultraviolet divergences. These divergences cannot be reabsorbed by a renormalization of the only coupling constant we have at our disposal in $\mathcal{L}^{(2)}$, namely $f_\pi$. We need additional counterterms. How will they look? Well, if we use a regulator that preserves the symmetry of the problem —which we certainly want to keep— counterterms at the one loop level, on locality, dimensionality and symmetry grounds will have to be necessarily of the same form as the operators contained in $\mathcal{L}^{(4)}$. Therefore constants like $L_1$ and $L_2$ will, in general, require an infinite logarithmic renormalization to make observables finite and to make contact with experiment.

All this is, of course, well known and discussed in textbooks on the subject\cite{19}. However, the discussion is usually left at this point, while our objective here is, for the reasons mentioned in the introduction, to answer the following two questions. Which is the most general regulator that preserves chiral invariance?
And then, what is universal in a general one-loop amplitude and what is regulator dependent?

We will thus be interested in calculating $F(s, t, u)$ in the most general way. The contributions from the effective lagrangian (3.4) to $F$ can be grouped in three parts

$$ F(s, t, u) = F_{\text{tree}}(s, t, u) + F^A(s, t, u) + F^B(s, t, u) \quad (3.5) $$

$F_{\text{tree}}$ contains the tree level contribution to the amplitude, both from the $L(2)$ and $L(4)$ pieces of the effective lagrangians. Explicitly,

$$ F_{\text{tree}}(s, t, u) = \frac{s}{f^2_\pi} + \frac{8}{f^4_\pi} (s^2 L_1 + (t^2 + u^2) L_2) \quad (3.6) $$

The one-loop contribution from $L(2)$ to the $\pi\pi$ scattering amplitude has been separated into two pieces, $F^A$ and $F^B$, for reasons that will be explained below. $F^A(s, t, u)$ is given by

$$ F^A(s, t, u) = \frac{1}{2f^4_\pi} \int d^4 z \left\{ s^2 \Delta^2 e^{iz(p_3 + p_4)} + 4(p_1^\mu p_2^\mu + p_3^\mu p_4^\mu) \Delta \partial_\mu \Delta (e^{iz(p_2 + p_4)} + e^{iz(p_2 + p_3)}) \right\} \quad (3.7) $$

It turns out that all logarithmic divergences are in $F^A$ and none is in $F^B(s, t, u)$, which includes the rest of the one-loop amplitude. Explicitly,

$$ F^B(s, t, u) = \frac{1}{2f^4_\pi} \left\{ \int d^4 z \left( \frac{2}{9} (\Box \Delta \Box \Delta) + \frac{2}{9} \Delta \Box^2 \Delta + \frac{4}{3} s \Delta \Box \Delta \right) e^{iz(p_3 + p_4)} ight. $$

$$ + \left( \frac{10}{9} (\Box \Delta \Box \Delta) - \frac{8}{9} \Delta \Box^2 \Delta + \frac{2}{3} \Delta \Box \Delta \right) e^{iz(p_2 + p_4)} $$

$$ + \left( \frac{10}{9} (\Box \Delta \Box \Delta) - \frac{8}{9} \Delta \Box^2 \Delta + \frac{2}{3} \Delta \Box \Delta \right) e^{iz(p_2 + p_3)} \right\} \quad (3.8) $$

Notice that $F^B$ contains only propagators and laplacian operators acting on propagators, any of the contributions containing at least a laplacian. We have arrived at this unique decomposition separating the integrals which can produce a laplacian acting on a propagator inside the integral from the ones which cannot. In order to do that one has to use identities like

$$ \Delta \Box \Delta = \frac{1}{2} \Box \Delta^2 - \partial_\mu \Delta \partial_\mu \Delta $$

$$ \Delta \Box^2 \Delta = \frac{1}{2} \Box^2 \Delta^2 - \Box \Delta \Box \Delta - 4 \partial_\mu \Delta \partial_\mu \Box \Delta - 2 \partial_{\mu \nu} \Delta \partial_{\mu \nu} \Delta $$

$$ \Delta \partial_{\mu \nu} \Delta = \frac{1}{2} \partial_{\mu \nu} \Delta^2 - (\partial_\mu \Delta)(\partial_\nu \Delta) \quad (3.9) $$
In all these manipulations it is assumed that the propagator \( \Delta(z) \) is regulated in some unspecified way to make it sufficiently regular at short distances so that boundary terms can be safely neglected. Being more precise, let us now calculate \( F^A(s, t, u) \) using a regulated propagator defined in the following way

\[
\Delta(z) = \frac{1}{4\pi^2} \int_0^\infty e^{-tz^2} f^\epsilon(t) dt.
\]

(3.10)

There are some obvious requirements that \( f^\epsilon(t) \) must fulfil. One has just been mentioned: the propagator must be regular (for a finite cut-off) at short distances, giving rise to well defined integrals (albeit dependent on the cut-off and on the form of the regulating function \( f^\epsilon \)). Furthermore, by removing the cut off one must recover the usual free propagator for a scalar particle

\[
\lim_{\epsilon \to 0} f^\epsilon(t) = 1 \quad \Rightarrow \quad \Delta(z) \xrightarrow{\epsilon \to 0} \frac{1}{4\pi^2 z^2}
\]

(3.11)

On dimensional grounds, \( f^\epsilon(t) = f(e^2 t) \). If we now introduce the expression for the propagator (3.10) into \( F^A(s, t, u) \) and \( F^B(s, t, u) \) everything can be written, obviously, in terms of integrals of the regulating function \( f^\epsilon \). For instance,

\[
F^A(s, t, u) = \frac{1}{32\pi^2} \{ s^2 I_1^\epsilon(s) + t^2 I_1^\epsilon(t) + u^2 I_1^\epsilon(u) + (8t^2 \frac{d}{dt} - 8s)(\frac{1}{2} I_2^\epsilon(t) - I_3^\epsilon(t)) + (8u^2 \frac{d}{du} - 8s)(\frac{1}{2} I_2^\epsilon(u) - I_3^\epsilon(u)) \}
\]

(3.12)

where

\[
I_1^\epsilon(s) = \int_0^\infty \int_0^\infty dt dt' f^\epsilon(t) f^\epsilon(t') \frac{1}{(t + t')^2} e^{-s/(4(t + t'))}
\]

\[
I_2^\epsilon(s) = \int_0^\infty \int_0^\infty dt dt' f^\epsilon(t) f^\epsilon(t') \frac{1}{(t + t')^2} e^{-s/(4(t + t'))}
\]

\[
I_3^\epsilon(s) = \int_0^\infty \int_0^\infty dt dt' f^\epsilon(t) f^\epsilon(t') \frac{t^2}{(t + t')^3} e^{-s/(4(t + t'))}
\]

(3.13)

These integrals are all divergent when the cut-off is removed. \( I_1^\epsilon \) is logarithmically divergent and \( I_2^\epsilon \) and \( I_3^\epsilon \) diverge quadratically as well as logarithmically. All three are dominated by an end-point singularity, which fixes the coefficient of the log in an unique manner independently of the detailed shape of the function \( f^\epsilon(t) \) (see Appendix A). Here we, of course, recover a familiar result in field theory, namely the coefficient of the logarithmic singularity is universal, independent of the regulating function \( f^\epsilon \). This is a well known result in renormalizable theories, like QED, where the logarithmic singularity is the dominant one. It is much
less clear, but it is nevertheless true, in a theory like this one where the leading divergences are quadratic (there may even exist quartic divergences, depending on the type of regulator, as we will see in a moment).

We can write

\[ I_1^\varepsilon(s) = -\log s \varepsilon^2 + f + O(\varepsilon^2) \]

\[ I_2^\varepsilon(s) = \frac{s}{4} \log s \varepsilon^2 + \frac{k_1}{\varepsilon^2} + cs + O(\varepsilon^2) \]

\[ I_3^\varepsilon(s) = \frac{s}{12} \log s \varepsilon^2 + \frac{k_2}{\varepsilon^2} + ds + O(\varepsilon^2) \] (3.14)

Except for the logarithmic coefficients all the other quantities \((f, k_1, k_2, c, d)\) depend, of course, on the explicit form of \(f^\varepsilon(t)\). In fact not all these coefficients are independent. There exist a relation between \(f\) and \(c\) that can be found easily from the fact that \(dI_2/ds = -\frac{1}{4}I_1\),

\[ f = -4c - 1 \] (3.15)

By inserting \(I_1, I_2\) and \(I_3\) into \(F^A\) one gets (already in Minkowski space)

\[ F^A(s, t, u) = -\frac{1}{96\pi^2 f^4} \left\{ 3s^2 (\log -s \varepsilon^2 - \beta_1) \right. \]

\[ + t(t - u)(\log -t \varepsilon^2 - \beta_2) + u(u - t)(\log -u \varepsilon^2 - \beta_2) \]

\[ + 24 \frac{s}{\varepsilon^2} (2k_2 - k_1) \} \] (3.16)

where

\[ \beta_1 = -\frac{4}{3} - 12d \quad \beta_2 = -1 - 12d \] (3.17)

The last quadratic divergent term in (3.16) can be combined with the four-legged tadpole and absorbed in a redefinition of \(f_\pi\), as discussed in section 2. This would uniquely fix the combination \(2k_2 - k_1\) in a given scheme. (The general form of \(F^A\) was originally given by Lehmann[29].)

In order to complete the expression for the amplitude we still need to compute \(F^B(s, t, u)\). First of all, it is easy to prove that \(F^B\) does not contain any log divergence. From dimensional arguments we see that \(F^B\) contains, potentially at least, quartic divergences which are totally forbidden by chiral symmetry since it is impossible to write a chirally invariant counterterm without derivatives (as it would be required to absorb a \(1/\varepsilon^4\) divergence, on dimensional grounds). So chiral symmetry has something to say on \(F^B\). In fact it is not difficult to see that in dimensional regularization, which complies with the chiral Ward identities, \(F^B \equiv \)
Indeed, in dimensional regularization $\Box \Delta(x) = -\delta^{(n)}(x)$ and substituting this result into (3.8) we get

$$F^B(s, t, u) = \frac{2}{9f_\pi^4}(2\delta^{(n)}(0) + s\Delta(0))$$

which is zero since in dimensional regularization

$$\Delta(0) = \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2} = 0 \quad \delta^{(n)}(0) = \int \frac{d^n k}{(2\pi)^n} 1 = 0$$

Note that the coefficient of $\delta(0)$ is such that, combined with the tadpole and the contribution from the measure, cancels out. This is as it should be. The fact that $F^B = 0$ in dimensional regularization is not an accident, rather we will see in the next section that the requirement that our regulator satisfies the Ward identities is enough to guarantee that the contribution from $F^B$ to the scattering amplitude vanishes when the cut-off is removed. Let us then set $F^B = 0$. The complete amplitude reduces then to $F^{\text{tree}} + F^A$.

One can decompose the amplitude (3.2) into three isospin channels $I = 0, 1$ and 2. These amplitudes with well defined isospin are given by the following combinations

$$T(0) = 3F(s, t, u) + F(u, t, s) + F(t, s, u)$$
$$T(1) = F(t, s, u) - F(u, t, s)$$
$$T(2) = F(t, s, u) + F(u, t, s)$$

Let’s see what is the contribution from $F^{\text{tree}}$ (including the $O(p^4)$ contribution) and $F^A$ to $T(0)$ and $T(2)$

$$T(0) = \frac{2s}{f_\pi^2} + \frac{1}{96\pi^2 f_\pi^4} \{48 \frac{s}{\epsilon^2}(k_1 - 2k_2) - 12s^2 \log -se^2 + (2tu - 8t^2) \log -te^2$$
$$+ (2tu - 8u^2) \log -ue^2 - (t^2 + u^2)(24 + 240d) - tu(26 + 240d)\}$$
$$+ \frac{16}{f_\pi^2} \{L_1(2(t^2 + u^2) + 3tu) + \frac{L_2}{2}(3(t^2 + u^2) + 2tu)\}$$

$$T(2) = -\frac{s}{f_\pi^2} + \frac{1}{96\pi^2 f_\pi^4} \{-24 \frac{s}{\epsilon^2}(k_1 - 2k_2) - 3s^2 \log -se^2 + (ts - 4t^2) \log -te^2$$
$$+ (us - 4u^2) \log -ue^2 - (t^2 + u^2)(9 + 96d) - tu(8 + 96d)\}$$
$$+ \frac{8}{f_\pi^2} \{L_1(t^2 + u^2) + \frac{L_2}{2}(3(t^2 + u^2) + 4tu)\}$$

It is well known (and it can easily be checked from the above expression) that the contribution from $F^A$ to both $T(0)$ and $T(2)$ is ultraviolet divergent, while
the contribution from the one-loop chiral diagrams to $T(1)$ is, in fact, finite. The divergences that appear in $T(0)$ and $T(2)$ can be removed by the following subtractions

$$L_1 \to L_1 = L_1(\mu) + \frac{1}{32\pi^2} \left(\frac{1}{12}\right) \log \epsilon^2 \mu^2 \quad L_2 \to L_2 = L_2(\mu) + \frac{1}{32\pi^2} (-\frac{1}{6}) \log \epsilon^2 \mu^2$$

(3.23)

With this prescription we can then find the value of $L_1(\mu)$ and $L_2(\mu)$ at the fixed (but arbitrary) scale $\mu$ from the experimental value of, for instance, the phase shifts in the $I = 0$ and $I = 2$ channels. Of course the values of $L_1(\mu)$ and $L_2(\mu)$ will be totally dependent on the regulator one has used by an arbitrary constant because the loop parts of divergent amplitudes do depend on the way the cut-off is introduced. In addition we may, in principle, add any constant we want to the subtractions implied by (3.23). This would amount to a change in the renormalization prescription, implying a modification in the value of the renormalized coefficients $L_i(\mu)$. Let’s now see what are the consequences of this renormalization procedure on the $I = 1$ channel.

If we add all the contributions to the $I = 1$ isospin one channel from $F(s, t, u)$ (tree level and one-loop) that remain after imposing chiral symmetry invariance (3.6) and (3.16) one arrives to the following finite expression

$$T(1) = \frac{t - u}{f_\pi^2} - \frac{(t - u)}{96\pi^2 f_\pi^4} \{s \log (-s + t) \log -t + u \log -u - 3s(\beta - 256\pi^2(L_1 - \frac{L_2}{2}))\}$$

(3.24)

$\beta$ is defined as $\beta = \beta_2 - \beta_1$ and its value from (3.17) is

$$\beta = \frac{1}{3}$$

(3.25)

independently of the regulator one has chosen. This remarkable result shows that $\beta$ is a universal quantity provided the regulator preserves chiral invariance. On the other hand, both $T(0)$ and $T(2)$ depend on various combinations of the two finite numbers $\beta_1$ and $\beta_2$, which do depend on the function $f'(t)$ one has chosen to regularize the amplitude. $\beta$, and consequently, the one-loop contribution to $T(1)$ is not only cut-off independent, but also regulator independent. In order to obtain this result it has been crucial to impose chiral symmetry invariance.

The $I = 1$ amplitude is, of course, observable. $T(1)$ can be expanded into partial waves and by using the effective range approximation one can find for the phase shift in the $I = 1$ channel and $l = 1$ wave (this is the only one one has to include due to Bose statistics)

$$\cot(\delta_{l=1}^{I=1}) = \frac{96\pi f_\pi^2}{s} - \frac{3}{\pi} \left(\beta - 256\pi^2(L_1 - \frac{L_2}{2}) - \frac{1}{9}\right)$$

(3.26)
The quantity on the r.h.s has to be a scheme independent quantity. The first term is the tree contribution and is non ambiguous. The second term $\beta - 256\pi^2(L_1 - \frac{1}{2}L_2)$ should be, as a consequence, a scheme independent quantity but as we have just shown $\beta$ is regulator independent so $L_1(\mu) - \frac{1}{2}L_2(\mu)$ is regulator independent too.

Eq. (3.26) has exactly the same form written either in terms of the renormalized $L_i(\mu)$ or the unrenormalized $L_i$ coefficients if we choose the subtraction prescription (3.23). If we take a renormalization prescription different from the one implied by (3.23), i.e. if we modify the finite part of the subtractions, $\beta_1$ and $\beta_2$ are modified in such a way that the amplitudes $T(0)$ and $T(2)$ remain numerically the same. The value of $\beta = \beta_2 - \beta_1$ also gets modified accordingly although, obviously, numerically (3.26) remains the same. This is a zero-sum game, we may well choose to transfer finite parts back and forth between $\beta$ and the renormalized value of $L_1(\mu) - \frac{1}{2}L_2(\mu)$ if we so wish, but the fact is that in any chiral invariant regularization scheme $\beta = 1/3$ is an unambiguous prediction of chiral perturbation theory and a comparison between the theoretical and experimental value for $L_1 - \frac{1}{2}L_2$ is a priori meaningful.

4 Ward identities

Let’s now see how the chiral Ward identities impose severe constraints on our regulator. The use of a regulating function $f^i(t)$ for the propagator eq. (3.10) implies a modification of the momentum space inverse propagator

$$p^2 \rightarrow p^2 + p^2 F(\epsilon^2 p^2)$$  (4.1)

The free part of the effective lagrangian $L^{(2)}$ then changes in the following way

$$L^{(2)} = \frac{1}{2} \pi^i(\Box + F(\epsilon^2 \Box) \Box) \pi_i + ...$$  (4.2)

The dots stand for the (unmodified) interaction terms. The point to analyze now is whether under a chiral transformation, the action is still chiral invariant or not. The answer is that, in general, is not. Under a left chiral transformation ($L \neq 1, R = 1$) the pion field transforms as

$$\delta \pi^a(x) = \frac{f_\pi}{2} \omega^a - \frac{1}{2} \epsilon^{abc} \omega^b \pi^c + \frac{1}{6f_\pi}(\delta^{ab} \delta^{cd} - \delta^{ad} \delta^{bc})\omega^d \pi^b \pi^c + O(1/f_\pi^2)$$  (4.3)

For a right chiral transformation ($L = 1, R \neq 1$) there is a change in the sign of the second term on the r.h.s.
For an arbitrary $F$ the action is not chiral invariant under this transformation, but rather $\delta S \neq 0$. If we now require chiral symmetry to hold, it would imply either to include counterterms to compensate the noninvariance, i.e. the non-zero value of $\delta S$ or conditions on $F$ (or, equivalently, on $f^i(t)$). Let’s demand invariance at the level of $S$-matrix elements

$$\langle \delta S a^{\alpha_1}(p_1)...a^{\alpha_n}(p_n) \rangle = 0$$  \hspace{1cm} (4.4)$$

Since we are interested in $\pi\pi$ scattering at the one loop level, a simple counting of powers of $f_\pi$ shows that the relevant diagrams to analyze to the order we are calculating are the ones shown in fig. 2. They lead to the following matrix element

$$\int d^4x \int d^4y \langle 0 | \delta \pi^a(x) F(\epsilon^2 \square) \pi^a(x) L_{\text{int}}(y) a^{\alpha_1}(p_1) a^{\beta_1}(p_2) a^{\gamma_1}(p_3) | 0 \rangle$$  \hspace{1cm} (4.5)$$

Adding and subtracting 1 to $F(\epsilon^2 \square)$ and using that

$$\square (1 + F(\epsilon^2 \square)) \Delta(x) = -\delta^{(4)}(x)$$  \hspace{1cm} (4.6)$$

where $\Delta$ is the regulated propagator, and imposing $\Delta(0) = 0$ (see the discussion in section 2), one immediately sees that all terms will necessarily contain laplacians acting on the propagators. In fact one arrives exactly to the same type of integrals that appear in (3.8). If one expands them in order to separate the quartic and quadratic divergences and the finite parts one gets

$$T_1(s) = \int d^4z \Delta \Delta e^{ipz} = \frac{A}{\epsilon^4} + \frac{Bs}{\epsilon^2} + G_1 s^2 + O(\epsilon^2)$$

$$T_2(s) = \int d^4z \Delta \Delta e^{ipz} = \frac{C}{\epsilon^4} + \frac{Ds}{\epsilon^2} + G_2 s^2 + O(\epsilon^2)$$

$$T_3(s) = \int d^4z \Delta \Delta e^{ipz} = \frac{E}{\epsilon^2} + G_3 s + O(\epsilon^2)$$  \hspace{1cm} (4.7)$$

The contributions proportional to $\delta^{(4)}(0)$ from (4.5) cancel exactly the terms from the measure, so we do not need to worry about them.

Notice that the previous decomposition does not include any logarithmic divergence. That this is so, can be proved easily by introducing the general propagator (3.10) into (4.7),

$$T_1 = \int_0^\infty \int_0^\infty dt dt' f^i(t) f^i(t') \frac{e^{-s/(4(t+t'))}}{(t+t')^6} \left( st^4 t' + \frac{s^2}{16} t^2 t'^2 + 12(t^4 t'^2 + t^3 t'^3) \right)$$

$$T_2 = \int_0^\infty \int_0^\infty dt dt' f^i(t) f^i(t') \frac{e^{-s/(4(t+t'))}}{(t+t')^6} \left( \frac{3s}{2} (t^4 t' + t^3 t'^2) + \frac{s^2}{16} t^4 + 12(t^4 t'^2 + t^3 t'^3) \right)$$

20
\[ T_3 = \frac{1}{4} \int_0^\infty \int_0^\infty dt dt' f^i(t) f^i(t') \frac{e^{-s/(4(t+t'))}}{(t + t')^4} \left( -4t't^2 - \frac{st^2}{4} \right) \] (4.8)

and following the analysis of appendix A one can see that the logs cancel.

Not all the coefficients in (4.7) are free. There are some relations amongst them

\[ A = C \quad 2D = 2B - E \] (4.9)

By introducing (4.7) and (4.9) in the matrix element one ends up with

\[
\langle \delta S a^\dagger(\mathbf{p}_1) a^\dagger(\mathbf{p}_2) a^\dagger(\mathbf{p}_3) \rangle = \mathcal{O}(\frac{1}{f^2_{\pi^0}}) - \frac{1}{f^2_{\pi^0}} \frac{\omega_b}{36} \delta(k_0) \delta^{\beta\alpha} \times \left\{ \frac{8A}{\epsilon^4} + \frac{1}{\epsilon^2} [2p_1^2(2D + 7E) + (2D - 4E)(p_2^2 + p_3^2)] + (G_1 + G_2 + 3G_3)(2p_1^4 + p_2^4 + p_3^4) + (9G_1 - 9G_2 - 3G_3)(p_1^2 p_2^2 + p_1^2 p_3^2) + p_2^2 p_3^2(-18G_1 + 18G_2 - 2G_3) + R(p_1, \epsilon) + \mathcal{O}(\epsilon^2) \right\} + (\alpha \leftrightarrow \beta \quad \mathbf{p}_1 \leftrightarrow \mathbf{p}_2) + (\alpha \leftrightarrow \gamma \quad \mathbf{p}_1 \leftrightarrow \mathbf{p}_3) + \mathcal{O}(1/f^2_{\pi^0}) = 0 \] (4.10)

where \( R(p_1, \epsilon) \) collects the contribution that comes from the action of \( F(\epsilon^2 \Box) \) over the external legs, which (like the simplifying assumption \( \Delta(0) = 0 \)) do not modify the conclusion for the pieces of \( \mathcal{O}(p^4) \). If we set to zero all the different tensorial structures, we obtain a set of equations for the coefficients \( A, B, \ldots \). It turns out that the only solution of this system of equations turns out to be

\[
A = C = 0 \\
B = D = E = 0 \\
G_1 = G_2 = G_3 = 0, \] (4.11)

implying that, up to terms that vanish when the cut-off is removed, \( T_1 = T_2 = T_3 = 0 \) if our regulator is to comply with chiral invariance. But these integrals are exactly the same appearing in \( F^B \), so from the requirement of chiral invariance of the regulator we conclude that \( F^B(s, t, u) = 0 \).

From the preceding analysis we can extract a set of rules that are bound to yield results respecting chiral invariance in chiral perturbation theory.

• Take the amplitude in position space, either calculating the matrix element directly in position space or Fourier transforming from momentum space.

• Separate all the integrals that can lead to a laplacian acting on a propagator from the ones that cannot before regularizing.

• Use \( \Box \Delta(x) = -\delta(x) \) inside the integrals
Integrate the delta functions and end up with integrals that do not contain any $\Box$ inside, and terms like $\Delta(0)$ and $\Box\Delta(0)$.

Choose any regulator good enough for the remaining integrals and such that verifies $\Delta(0) = 0$ and $\Box\Delta(0) = 0$. For instance,

$$\Delta(x) = \frac{1}{x^2} \left( 1 - \left( \frac{x}{a\epsilon} \right)^4 K_1 \left( \frac{x}{a\epsilon} \right)^4 \right)$$  \hspace{0.5cm} (4.12)

If the regulator does not automatically cancel all tadpoles, we can always remove them by suitable counterterms. This has no effect on the relevant finite parts.

## 5 A General One-loop Process

In this section we will show that the previous procedure can be extended to an arbitrary one-loop process with $m$ external legs. That is, we will see that in chiral perturbation theory, provided one is respectful with the chiral invariance of the theory, the loop part of finite amplitudes is automatically independent of the regulator.

The most general diagram with $n$ vertices and $m$ external propagators that one can construct from the lagrangian

$$\mathcal{L}^{(2)} = \frac{f_\pi^2}{4} \text{Tr}(\partial_\mu U^\dagger \partial_\mu U)$$  \hspace{0.5cm} (5.1)

has $n$ internal propagators, $2n$ derivatives and $p^i = q^i + r^i + s^i + \ldots$, $i = 1, \ldots n$ external momenta. $q^i$, $r^i$, etc. label the different external momenta flowing into vertex $i$. The conservation rule $\sum_{i=1}^n p^i = 0$ holds. A generic diagram is shown in fig. 3.

We start our calculation in position space and label by $z_1, \ldots, z_n$ the vertex coordinates. Taking into account the way the derivatives act on the propagators, using $x_i = z_i - z_{i+1}$ $i = 1 \ldots n - 1$ and integrating by parts according to the set of rules of section 3 one arrives to the following general decomposition for any diagram, at one loop,

$$\delta^{a_1 a_2} \delta^{a_3 a_4} \ldots \delta^{a_{m-1} a_m} F^{(nm)}(q^i, r^j, \ldots) + \text{permutations}$$  \hspace{0.5cm} (5.2)

where we have to include all possible permutations of isospin indices and momenta. The $F^{(nm)}$ have the form

$$F^{(nm)} = \frac{1}{f_\pi} \left\{ \sum_{i=0}^n g^{(nm)}_{(ni)}(q_1, r_1, \ldots) I^{(n)}_{\mu_1 \ldots \mu_i}(p_1, p_2, \ldots p_{n-1}) \right\}$$

$$+ \sum_{k=2}^{n-1} \left\{ \sum_{i=0}^{k-1} g^{(nm)}_{(ki)}(q_1, r_1, \ldots) I^{(k)}_{\mu_1 \ldots \mu_i}(p_1 - p_k, p_2 - p_k, \ldots p_{k-1} - p_k) \right\}$$
where \( I_{\mu_1 \ldots \mu_6}^{(s)} (p_1, \ldots, p_{s-1}) \) is the integral

\[
\int d^4x_1 \ldots d^4x_{s-1} e^{i(\sum_{j=1}^{s-1} p_j \cdot x_j)} \Delta(x_1) \ldots \Delta(x_{s-1}) \partial_{\mu_1 \ldots \mu_s} \Delta(x_1 + \ldots + x_{s-1})
\]  

(5.4)

Let us see how this decomposition works in a simple example. Let’s suppose that we want to describe a process that involves 6 external particles. This process will receive contribution from diagrams with 3 and 2 vertices. Let us concentrate only in the first diagram for which \( n = 3 \) and \( m = 6 \) depicted in Fig. 4. Using the interaction lagrangian (5.1), integrating by parts and using that \( \Box \Delta(x) = -\delta^{(4)}(x) \) the contribution from the diagram is, apart from isospin indices,

\[
F^{(3,6)} = \frac{1}{\beta_0} \left\{ g_{(3,6)}^{(3,6)} \int d^4x_1 d^4x_2 e^{i(p_1 \cdot x_1 + p_2 \cdot x_2)} \Delta(x_1) \Delta(x_2) \Delta(x_1 + x_2) 
\right.
\]

+ \( g_{(3,1)}^{(3,6)} \int d^4x_1 d^4x_2 e^{i(p_1 \cdot x_1 + p_2 \cdot x_2)} \Delta(x_1) \Delta(x_2) \frac{\partial}{\partial x_1^{\mu_1}} \Delta(x_1 + x_2) \)

+ \( g_{(3,2)}^{(3,6)} \int d^4x_1 d^4x_2 e^{i(p_1 \cdot x_1 + p_2 \cdot x_2)} \Delta(x_1) \Delta(x_2) \frac{\partial}{\partial x_1^{\mu_1}} \frac{\partial}{\partial x_1^{\mu_2}} \Delta(x_1 + x_2) \)

+ \( g_{(3,3)}^{(3,6)} \int d^4x_1 d^4x_2 e^{i(p_1 \cdot x_1 + p_2 \cdot x_2)} \Delta(x_1) \Delta(x_2) \frac{\partial}{\partial x_1^{\mu_1}} \frac{\partial}{\partial x_1^{\mu_2}} \frac{\partial}{\partial x_1^{\mu_3}} \Delta(x_1 + x_2) \)

+ \( g_{(2,0)}^{(3,6)} \int d^4x_1 e^{i(p_1 \cdot x_1)} \Delta(x_1) \Delta(x_1) \)

+ \( g_{(2,1)}^{(3,6)} \int d^4x_1 e^{i(p_1 \cdot x_1)} \Delta(x_1) \frac{\partial}{\partial x_1^{\mu_1}} \Delta(x_1) \)

+ \( g_{(2,2)}^{(3,6)} \int d^4x_1 e^{i(p_1 \cdot x_1)} \Delta(x_1) \frac{\partial}{\partial x_1^{\mu_1}} \frac{\partial}{\partial x_1^{\mu_2}} \Delta(x_1) \}

(5.5)

In the last three terms the equation \( \Box \Delta(x) = -\delta^{(4)}(x) \) has been used once. Of course it is also easy to see that, after neglecting terms like \( \Delta(0) \) or \( \Box \Delta(0) \), the \( \pi \pi \rightarrow \pi \pi \) amplitude, which then reduces to the \( F^A \) term (eq. 3.7), is also of the generic form (5.3).

The decomposition given in (5.3) shows what type of integrals appear in the calculation of a given diagram after the application of the set of rules given in section 3. Of all these integrals we will be only interested in those that by power counting are potentially divergent, since the integrals that are convergent by naive power counting are certainly independent of the regulator. It is clear from the non-linear nature of (5.1) that a physical process with \( m \) external particles gets contributions, at the one loop level, from diagrams with \( n = m/2 \) internal lines all the way down to diagrams with \( n = 2 \) (we do not take into account
tadpole diagrams). A given physical process with \( m \) external particles receives contributions from the same type of integrals that appear for a process with \( m-2 \), \( m-4 \), etc. external particles (all the way down to 4, again ignoring tadpoles). The question is: are there new divergent integrals allowed by power counting for increasing values of \( m \) or does the appearance of new types of divergent integrals stops at some point? Fortunately, the number of divergent integrals that appear is rather limited. For instance, the process with \( m = 4 \) has (excluding tadpoles) only one class of divergent diagram that leads to two independent integrals. A process with \( m = 6 \) gets contribution from diagrams with \( n = 3 \) and \( n = 2 \) with four type of divergent integrals, the two that contributed for \( m = 4 \) plus two new ones. For a process with \( m = 8 \) legs diagrams with \( n = 4 \), \( 3 \) and \( 2 \) vertices contribute. Only one new divergent integral appears, making a total of 5 different divergent integrals. For \( m = 10 \) and beyond no new divergent integrals appear.

We are thus confronted with only 5 possible divergent integrals for any process in chiral perturbation theory at the one loop level.

\[
I^{(2)}(p_1) = \int d^4 x e^{iP_1 x} \Delta(x)^2 \tag{5.6}
\]

\[
I^{(2)}_{\mu \nu}(p_1) = \int d^4 x e^{iP_1 x} \Delta(x) \partial_{\mu \nu} \Delta(x) \tag{5.7}
\]

\[
I^{(3)}_{\mu_1 \mu_2}(p_1, p_2) = \int d^4 x_1 d^4 x_2 e^{i(P_1 x_1 + P_2 x_2)} \Delta(x_1) \Delta(x_2) \partial_{\mu_1 \mu_2} \Delta(x_1 + x_2) \tag{5.8}
\]

\[
I^{(3)}_{\mu_1 \mu_2 \mu_3}(p_1, p_2) = \int d^4 x_1 d^4 x_2 d^4 x_3 e^{i(P_1 x_1 + P_2 x_2 + P_3 x_3)} \Delta(x_1) \Delta(x_2) \Delta(x_3) \partial_{\mu_1 \mu_2 \mu_3} \Delta(x_1 + x_2 + x_3) \tag{5.9}
\]

\[
I^{(4)}_{\mu_1 \mu_2 \mu_3 \mu_4}(p_1, p_2, p_3) = \int d^4 x_1 d^4 x_2 d^4 x_3 e^{i(P_1 x_1 + P_2 x_2 + P_3 x_3)} \Delta(x_1) \Delta(x_2) \Delta(x_3) \partial_{\mu_1 \mu_2 \mu_3 \mu_4} \Delta(x_1 + x_2 + x_3) \tag{5.10}
\]

All of them contain only logarithmic divergences except for the second one which has an additional quadratic divergence. If one expands (5.8) (5.9) and (5.10) in terms of all the possible tensorial structures, one sees that all integrals can be split into a finite (by power counting) part, which is non-ambiguous, plus some terms that contain the divergent contributions. In fact, all the divergences are concentrated in only two pieces \( I^{(2)} \) and \( I^{(2)}_{\mu \nu} \), which turn out to be proportional to the integrals that appear in \( \pi \pi \) scattering, namely \( I^{(2)} \) and \( I^{(2)}_{\mu \nu} \)

\[
I^{(3)}_{\mu_1 \mu_2}(p_1, p_2) = \frac{1}{4} g_{\mu_1 \mu_2} I^{(2)}(p_1 - p_2) + \text{finite}
\]

\[
I^{(3)}_{\mu_1 \mu_2 \mu_3}(p_1, p_2) = -\frac{i}{12} ((p_1 + p_2)^{\mu_3} g_{\mu_1 \mu_2} + (p_1 + p_2)^{\mu_2} g_{\mu_1 \mu_3} + \ldots)
\]
\[(p_1 + p_2)^{\mu_1} g_{\mu_2 \mu_3} \] \[I^{(2)}(p_1 - p_2) + \text{finite} \]

\[I^{(4)}_{\mu_1 \mu_2 \mu_3 \mu_4}(p_1, p_2, p_3) = \frac{1}{24} (g_{\mu_1 \mu_2} g_{\mu_3 \mu_4} + g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} + g_{\mu_1 \mu_4} g_{\mu_2 \mu_3}) I^{(2)}(p_1 - p_2) + \text{finite} \]

This implies that having regularized the one-loop process with two vertices (section 3), where the \(I^{(2)}\) and \(I^{(2)}_{\mu_1 \mu_2}\) integrals appear, one has by the same token regularized all the one-loop processes. In a way this should not be too surprising; recall that all logarithmic divergences at the one loop level can be eliminated by a redefinition of just two coupling constants of the \(\mathcal{L}^{(4)}\) lagrangian.

For any amplitude \(\mathcal{A}\) with \(m\) external particles one can construct a number of linear combinations of the different \(F^{(nm)}(s_1, s_2, s_3, \ldots)\) (\(s_i\) are the invariant quantities that one can construct with \(2m\) independent momenta) such that the chiral logs cancel. Because it is a finite quantity, if we call this combination \(\mathcal{B}\), it obviously verifies

\[\mu \frac{\partial}{\partial \mu} \mathcal{B} = 0\]

\(\mu\) being the renormalization scale. For the \(\pi \pi\) scattering amplitude, that we have analyzed in previous sections, we would have \(m = 4\), all the amplitudes can be expressed in terms of \(F^{(2,4)}(s, t, u)\) plus permutations of \(s, t\) and \(u\). The combination that leads to a finite amplitude is, obviously, \(T(1)\), the \(I = 1\) channel amplitude (3.24)

\[\mathcal{B} = F^{(2,4)}(s_2, s_1, s_3) - F^{(2,4)}(s_1, s_2, s_1)\]

The functions \(F^{(nm)}\) contain divergent (and, hence, regulator dependent) integrals as well as finite, unambiguous terms. From the discussion leading to eq. (5.11) we learn that all divergent integrals can be reduced unambiguously to two integrals \(I^{(2)}\) and \(I^{(2)}_{\mu_1 \mu_2}\). We can still go further and isolate the logarithmic divergences of these integrals into one structure. Using a tensorial decomposition of \(I^{(2)}\) and \(I^{(2)}_{\mu_1 \mu_2}\) we can write,

\[I^{(2)} = \int d^4 x \Delta(x)^2 e^{ipx} = \frac{1}{16 \pi^2} B\]

\[I^{(2)}_{\mu \nu} = \int d^4 x \Delta(x) \partial_{\mu \nu} \Delta(x) e^{ipx} = \frac{1}{16 \pi^2} [A(g_{\mu \nu} p^2 - 4p_\nu p_\mu) + D g_{\mu \nu} \frac{1}{\epsilon^2} + C p_\nu p_\mu]\]

The expression for \(A, B, C\) and \(D\) can be deduced by using the propagator (3.10) and the integrals (3.13) and their solution (3.14)

\[A = -\frac{1}{12} \log \frac{s^2}{\epsilon} + 2d - c\]

\[B = -\log \frac{s^2}{\epsilon^2} - 4c - 1\]
\[ C = 12d - 4c + \frac{1}{3} \]
\[ D = 2k_2 - k_1 \]  \hspace{1cm} (5.16)

All the arbitrariness in choosing one regulating function \( f^*(t) \) or another is encoded, except for a redefinition of \( f_\pi \), in the coefficients \( c \) and \( d \). The key point is that all the logarithmic divergences are concentrated in only one structure for each integral, \( A \) for \( I^{(2)} \) and \( B \) for \( I^{(2)}_{\mu_1\mu_2} \).

The finite combination of the \( F^{(a,m)} \) that defines the corresponding \( B \) can be splitted into two pieces. One contains all the finite contributions that come from the manifestly convergent integrals. Of course, as these integrals are well defined \emph{per se}; they don’t need to be regularized, so their value is fixed and is nonambiguous (scheme independent). The second piece would contain the finite parts that accompany the divergences of \( A \) and \( B \). There will exist only one combination of \( A \) and \( B \) that is finite; from (5.16) one can see that it is \( A - \frac{1}{12}B \). This unique combination fixes the way the scheme dependent quantities \( c \) and \( d \) appear in \( B \). Symbolically

\[ B = \sum_\alpha C_\alpha F^\alpha[I^{(2)}, I^{(2)}_{\mu\nu}, \text{ finite}] = \sum_\alpha C_\alpha F^\alpha[A, B, C, \text{ finite}] \]  \hspace{1cm} (5.17)

where the index \( \alpha \) represents all diagrams, including permutations of indices, that contribute to the finite quantity \( B \). From the previous discussion,

\[ B = \text{ finite } + (A - \frac{1}{12}B)(\text{ finite}) \]
\[ = \text{ finite } + \frac{1}{6}(12d - 4c + \frac{1}{2})(\text{ finite}) \]  \hspace{1cm} (5.18)

where “finite” means some quantity that can be written in terms of convergent integrals alone—hence, unambiguous. All the non-universal dependence on the type of regulator is potentially encoded in the combination \( 12d - 4c \). But this particular structure \( 12d - 4c \) is exactly the same one finds from \( I_{\mu\nu} \) by contracting with the metric \( g_{\mu\nu} \)

\[ \int d^4x \Delta \Box \Delta e^{iP_x} = \frac{1}{16\pi^2} \left[ \frac{4}{\epsilon^2} (2k_2 - k_1) + p^2(12d - 4c + \frac{1}{3}) \right] \]  \hspace{1cm} (5.19)

In section 4 we have shown that this integral has to be zero, on chiral invariance grounds, so then

\[ 12d - 4c = -\frac{1}{3} \]  \hspace{1cm} (5.20)
and putting this result into (5.18) one finally ends up with an expression for $B$ that does not contain any remnants of the arbitrariness of the regulating function $f'(t)$ one has chosen

$$B = \text{finite} + \frac{1}{36}(\text{finite})$$

(5.21)

Renormalization group invariants are automatically independent of the regulator in chiral perturbation theory, at least at the one loop level, provided the regulator respects the chiral symmetry.

6 Conclusions

Chiral lagrangians provide a consistent framework to describe the interactions amongst Goldstone bosons. These are non-linear, non-renormalizable theories in four dimensions. To make sense of these theories and to compare calculations and experiment we must, first of all, absorb the rather severe ultraviolet divergences into effective couplings. In spite of the non-renormalizable character of the theory this can be accomplished order by order in a momentum expansion and the renormalized effective coupling obtained from comparison with the experimental data. This program has been successfully applied to very many different physical applications with very satisfactory results.

If one wants to be more ambitious and compare the effective couplings that can be deduced from experiment with theoretical predictions, or, even if for more formal reasons one wants to somehow attach more field theoretical respectability to chiral perturbation theory, one needs to know to what extent the results obtained in such non-renormalizable theories may depend on the way the cut-off is introduced. We have presented an in-depth study of these issues here. We have found that, generally speaking, observables which are renormalization-group invariants are completely independent of the regulator, provided the latter respects the Ward identities of the theory. We have proven that only at the one loop level, but we trust it must hold at higher orders. Likewise, we have not considered the addition of gauge fields to the chiral lagrangian, but we believe, too, that this should not change matters.

As a conclusion, combinations of the $O(p^4)$ coefficients that are renormalization group invariant, have unambiguous, finite values that can be extracted from experiment and compared with predictions from some more fundamental theory to which the chiral lagrangian is an approximation at long distances. This is of particular importance in applications of chiral lagrangian techniques to the sym-
metry breaking sector of the Standard Model to discern, for instance, between a strongly interacting Higgs or more exotic possibilities.

Our approach has been to use a general regulator in position space and demand the fulfilment of the chiral Ward identities. To simplify matters we have demanded an extra condition for this propagator, namely $\Delta(0) = 0$, but this is only a technical point. Then we isolate from the different amplitudes a part $A$ that is non-zero and contains the relevant information and a part $B$ which must vanish upon restricting ourselves to chirally invariant regulators. It matters little where the regulator complies directly with the chiral Ward identities or these have to be enforced by counterterms. This is only a semantic distinction. We have noted, that counterterms to restore chiral invariance have to be considered in any case since they are generated by the measure anyway. When we consider the non-zero $A$ part, a universal result is obtained, even if the regulator we use is non-chirally invariant. We can work out this $A$ part in the regulator we please.

Finally, we have collected a number of regulators, including the recently proposed differential renormalization, and several technical details in the appendices in the hope that this material can be of use to the reader.

We hope to have clarified some of the issues raised in the introduction.

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A End Point Singularities

In this appendix we will comment on the evaluation of the integrals in (3.13). These integrals contain all the information needed in order to determine the coefficients of the chiral logarithms and the scheme independent quantity $\beta$.

The key point is that all three integrals have a universal dependence either on $\log s$ or $s \log s$ that is easy to determine. Let us recall the minimum requirements on the regulating function $f^\epsilon(t)$. First, the function $f^\epsilon(t)$ must define a well-behaved propagator everywhere (this implies, in particular, that $f(t) \sim 1/t^{(2+\alpha)}$, $\alpha \geq 0$), and second, when removing the cut-off one must recover the usual propagator, forcing $f(0) = 1$. It is useful to perform a change of variables on these integrals.

$$v = t + t' \quad u = t$$

(A.1)

The $I^\epsilon_1$ integral, in particular, having rescaled the cut off $\epsilon$, reads

$$I^\epsilon_1 = \int_0^\infty dv e^{-\frac{s\epsilon^2}{4v}} \frac{1}{v^2} g(v)$$

(A.2)

where $g(v) = \int_0^v du f(u) f(v - u)$. From the requirements on $f^\epsilon(t)$

$$g(v) \sim 0 \quad g(v) \sim v$$

(A.3)

If we set $\epsilon = 0$ we find a logarithmically divergent integral dominated by the singularity at $v = 0$. We can split the range of integration in two, from 0 to a certain value $c$ and from $c$ to $\infty$. The last one will be convergent, while in the first one we can approximate $g(v) \sim v$ if we choose $c$ to be small enough. Then

$$I^\epsilon_1(s) = -\log \frac{se^2}{4} + \text{finite} + O(s\epsilon^2)$$

(A.4)

The coefficient of the logarithm is uniquely determined from the obvious requirements described above. All other pieces in the integral, in particular, the finite part depend completely on the chosen function $f^\epsilon(t)$.

This also holds for quadratically divergent integrals, for instance, $I^\epsilon_3(s)$. On dimensional grounds,

$$I^\epsilon_3(s) = \frac{a}{\epsilon^2} + bs \log se^2 + cs + O(se^2)$$

(A.5)

If we now derive $I^\epsilon_3$ with respect to $s$ we obtain

$$\frac{d}{ds} I^\epsilon_3(s) = b \log se^2 + \text{finite} + O(\epsilon^2)$$

(A.6)
Which is again logarithmically divergent and, hence, we just follow the same steps as in the previous case, finding

\[ I_3^\varepsilon(s) = \frac{a}{\varepsilon^2} + \frac{s}{12} \log s \varepsilon^2 + cs + O(\varepsilon^2) \]  

(A.7)

B Some Regulators

We propose here some regulated propagators fulfilling the properties (3.11) and check that indeed all of them lead to the same value for the \( F^A(s, t, u) \) amplitude.

**Regulator 1.**

\[ \Delta(x) = \frac{1}{x^2 + \varepsilon^2} \]  

(B.1)

This corresponds to a \( f^\varepsilon(t) = e^{-t \varepsilon^2} \). Substituting into (3.13) gives the following values for the integrals,

\[ I_1^\varepsilon(s) = -(\log \frac{s \varepsilon^2}{4} + 2\gamma_E) \]
\[ I_2^\varepsilon(s) = \frac{1}{\varepsilon^2} + \frac{s}{4} (\log \frac{s \varepsilon^2}{4} + 2\gamma_E - 1) \]
\[ I_3^\varepsilon(s) = \frac{1}{3\varepsilon^2} + \frac{s}{12} (\log \frac{s \varepsilon^2}{4} + 2\gamma_E - 1) \]  

(B.2)

Putting this into (3.12) one gets for \( F^A(s, t, u) \) in Minkowski space,

\[ F^A(s, t, u) = -\frac{1}{96\pi^2 f^4} \{3s^2(\log -\frac{s \varepsilon^2 e^{2\gamma_E}}{4} + \frac{1}{3}) + t(t-u)(\log -\frac{t \varepsilon^2 e^{2\gamma_E}}{4}) + u(u-t)(\log -\frac{u \varepsilon^2 e^{2\gamma_E}}{4}) \} \]  

(B.3)

implying that \( \beta_1 = -2\gamma_E - \frac{1}{3} + \log 4 \) and \( \beta_2 = -2\gamma_E + \log 4 \). So, we recover the correct value of \( \beta \) which is \( \frac{1}{3} \). On the other hand, if using this regulator, we try to calculate the \( F^B(s, t, u) \) part of the amplitude, which, as we have shown, chiral symmetry requires to be zero, we find quartic and quadratic divergences as well as finite parts. This is not a good regulator for the whole amplitude, but it is good enough for the part that contains the scheme independent information, which we have identified.

**Regulator 2.**

\[ \Delta(x) = \frac{1}{x^2}(1 - \frac{x}{a \varepsilon} K_1 \left( \frac{x}{a \varepsilon} \right)) \]  

(B.4)
has a nice expression in momentum space:

\[ \Delta(p) = \frac{1}{p^2 + a\epsilon^2 p^4} \quad (B.5) \]

Power counting in momentum space shows that this propagator regulates all but the integrals that are quartically divergent. This does not influence \( F^A \) in any case and it also reproduces the value \( \beta = 1/3 \). This regulator can be generalized easily to an arbitrary polynomial in momenta, that corresponds in position space to a series of modified Bessel functions. In particular for polynomials of order six or greater it will be able to regulate the quartic divergences too.

Regulator 3.
We can rederive the results of dimensional regularization simply by considering the integrals appearing in (3.7 and 3.8) in \( n \) dimensions and introducing \( f^\epsilon(t) = (\frac{t}{\pi})^{n/2-2} \), with \( n = 4 - \epsilon \), whose Laplace transform (3.10) reproduces the \( n \)-dimensional \( x \)-space propagator

\[ \Delta(x) = \frac{1}{4\pi^{n/2}x^{n-2}} \Gamma\left(\frac{n}{2} - 1\right) \quad (B.6) \]

The integrals (3.13) in their \( n \)-dimensional version reduce then to \( \Gamma \)-function type integrals, whose evaluation gives the familiar result \( \beta_1 = 11/6, \beta_2 = 13/6, \) and \( \beta = 1/3 \).

Regulator 4.
Dimensional regularization is based in analytically continuing the amplitudes to a complex number of dimensions. We can try other type of regulators based in analytic continuation too. We have, for instance, checked that

\[ \Delta(x) = \frac{1}{x^2} J_\epsilon(\frac{ex}{\nu}) \quad (B.7) \]

with \( \nu \) being some arbitrary scale and \( \epsilon \) some dimensionless number yields \( \beta = 1/3 \) upon analytic continuation to \( \epsilon = 0 \).

C Differential Renormalization
We have also investigated the evaluation of \( F^A(s, t, u) \) by using the differential renormalization method. This method was introduced in [23] and applied with success to rather involved calculations in both massless and massive \( \lambda \phi^4 \) [28] and
QED\textsuperscript{[29]}. It works directly in position space and it offers some computational advantages, so it is worth investigating its application to non-renormalizable non-linear theories like the chiral model.

The method consists in writing the bare amplitude in position space and proceed to regulate the short distance divergences that arise when two points approach each other by expressing the products of propagators as derivatives of less singular functions with a well-defined Fourier transform. Then one performs the Fourier transform by integration by parts and disposes of the surface terms. The basic identities we will need are

\[
\begin{align*}
\frac{1}{x^4} &= -\frac{1}{4} \frac{\Box}{x^2} \log x^2 \mathcal{M}^2 \\
\frac{1}{x^6} &= -\frac{1}{32} \frac{\Box}{x^2} \log x^2 \mathcal{M}^2 \\
\text{(C.1)}
\end{align*}
\]

These identities are strictly valid except for \( x = 0 \), so in differential renormalization one, in a way, makes a minimal surgery on the original theory—just one point.

We will also need the Fourier transform

\[
\int d^4 x e^{i P x} \frac{1}{x^2} \log x^2 \mathcal{M}^2 = -\frac{4 \pi^2}{p^2} \log \frac{p^2}{M^2}
\]

where \( M = 2 \mathcal{M}/\gamma_E \). \( M \) is an integration constant of the differential equations (C.1). Note that the method yields renormalized amplitudes directly; there’s really no cut-off anywhere. (In a sense, differential renormalization provides an implementation of the BPHZ procedure.) In fact there is really no reason why the integration constants for the two differential equations implied by (C.1) should be the same. In fact, generically, they are not. It is known that in QED there are well-determined relations between different scales, which are dictated by the Ward identities of the theory. This is crucial to recover the correct value for the axial anomaly \textsuperscript{[29]}.

By using these techniques in the evaluation of the \( F^A(s, t, u) \) part of the amplitude one ends up, in euclidean space, with

\[
F^A(s, t, u) = -\frac{1}{96 \pi^2 f_2^2} \left\{ 3 s^2 \log \frac{s}{M_1^2} + 2 t^2 \log \frac{t}{M_1^2} + s t \log \frac{t}{M_2^2} + 2 u^2 \log \frac{u}{M_1^2} + s u \log \frac{u}{M_2^2} \right\}
\]

where one can indeed recognize the correct coefficients of the logarithms. In deriving the previous expression we have used a different scale \( M \) for each identity.
If one defines an adimensional quantities $\lambda_1$ as the ratio of the square of these $M$'s

$$\lambda_1 = \frac{M_1^2}{M_2^2}$$  \hspace{1cm} (C.4)

The $\beta$ parameter obtained from the $F^A(s,t,u)$ amplitude (which we know should be equal to $1/3$) is

$$\beta = -\frac{1}{3} \log \lambda_1$$  \hspace{1cm} (C.5)

Unlike QED, however, there is no way of fixing the value of $\lambda_1$ from symmetry principles\[29\]. The standard Ward identities for the effective action gives no information at all to fix this $\log \lambda_1$ since there is only one diagram at this order in chiral perturbation theory, apart from tadpole diagrams that are set to zero automatically by chiral arguments.

One could try to see whether we can extract some information on $\lambda_1$ from requiring the conservation of the chiral current, namely

$$\frac{\partial}{\partial x_\mu} \langle 0| T j^b_\mu(x) \pi^a(z_1) \pi^b(z_2) \pi^c(z_3)|0 \rangle = \langle 0| T \delta^b \mathcal{L} \pi^a(z_1) \pi^b(z_2) \pi^c(z_3)|0 \rangle + \delta(x-z_1) \langle 0| T \delta^b \pi^a(z_1) \pi^b(z_2) \pi^c(z_3)|0 \rangle + \delta(x-z_2) \langle 0| T \delta^b \pi^a(z_1) \pi^b(z_2) \pi^c(z_3)|0 \rangle + \delta(x-z_3) \langle 0| T \delta^b \pi^c(z_3) \pi^a(z_1) \pi^b(z_2)|0 \rangle$$  \hspace{1cm} (C.6)

where $\delta \pi^a$ is (1.3) and the lagrangian is invariant under this transformation, i.e. $\delta^b \mathcal{L} = 0$. The associated Noether current is (expanding in powers of $1/f_\pi$)

$$j_\mu = j_\mu^{(2)} + j_\mu^{(4)} + ...$$

with

$$j_\mu^{(2)} = \frac{f_\pi \omega^i}{2} \partial_\mu \pi^i$$

$$j_\mu^{(4)} = \frac{1}{3 f_\pi} \omega^i \partial_\mu \pi^k \pi^j \pi^l (\delta^{i_k} \delta^{j_l} - \delta^{i_l} \delta^{j_k})$$  \hspace{1cm} (C.7)

etc. The first term on the r.h.s of the equality (C.6) is zero because of chiral symmetry invariance. To the order we are working the matrix element of $j_\mu$ in the three pion state is the relevant one. There are two contributions of the same order in $1/f_\pi$ to this matrix element, namely $j_\mu^{(2)}$ with two insertions of the interaction lagrangian, and $j_\mu^{(4)}$ with just one insertion. The corresponding diagrams are depicted in fig. 5.

The evaluation of these expressions is tedious but quite straightforward. There appear integrals containing all the different scales. At the end however the coefficient multiplying $\log \lambda_1$ vanishes, making it impossible to determine $\beta$ from
chiral invariance arguments. It seems that if one uses the differential renormalization procedure to calculate processes in chiral perturbation theory there is a loss of information in throwing away the surface terms that appear when pulling out the laplacians. These surface terms cannot be recovered, it seems, by using Ward identities. It is perhaps worth pointing out that differential renormalization automatically delivers renormalized amplitudes and here we are dealing with a non-renormalizable theory.

D Constructing Finite Observables

There is only a combination of $O(p^4)$ operators where the logarithmic divergences cancel, namely $L_1 - \frac{1}{2}L_2$. Therefore, by expanding the operators of the $L^{(4)}$ lagrangian (3.3) to higher orders in $1/f_\pi$ one can find the observables that are finite at the one loop level in chiral perturbation theory and, hence, their finite parts are unambiguously predicted. It is not very complicated to construct a computer code to perform this analysis and we have done this. We thus have a systematic way of finding finite amplitudes at the one loop level. As an example we will show the first two cases.

(a) For $m = 4$ external legs there are three kinematical invariants that we denote here by $s_1, s_2,$ and $s_3$ (the $s, t, u$ Mandelstam variables). The only finite amplitude is proportional to the following combination of momenta

$$A \sim (\delta^{a_1 a_2} \delta^{a_3 a_4} tu + \delta^{a_1 a_3} \delta^{a_2 a_4} su + \delta^{a_1 a_4} \delta^{a_2 a_3} st)$$  \hspace{1cm} (D.1)

where one recognizes the $T(1)$ amplitude discussed at length in section 3.

(b) The case $m = 6$ is a bit longer. There are 10 kinematical invariants invariant $s_i$, $i = 1..10$ given by

$$s_1 = (p_1 + p_2)^2 \quad s_2 = (p_1 + p_3)^2 \quad s_3 = (p_1 + p_4)^2$$ $$s_4 = (p_1 + p_5)^2 \quad s_5 = (p_2 + p_3)^2 \quad s_6 = (p_2 + p_4)^2$$ $$s_7 = (p_2 + p_5)^2 \quad s_8 = (p_3 + p_4)^2 \quad s_9 = (p_3 + p_5)^2$$ $$s_{10} = (p_4 + p_5)^2$$  \hspace{1cm} (D.2)

and the (only) finite amplitude is proportional to

$$A \sim \delta^{a_1 a_2} \delta^{a_3 a_4} \delta^{a_5 a_6} (s_2 s_3 + s_2 s_5 + s_3 s_6 + s_4 s_9 + s_4 s_{10} + s_5 s_6 + s_7 s_9 + s_7 s_{10}) - 2(s_2 s_7 + s_2 s_{10} + s_3 s_7 + s_3 s_9 + s_4 s_5 + s_4 s_6 + s_5 s_{10} + s_6 s_9) + 3(s_1 s_9 + s_1 s_{10} + s_4 s_8 + s_7 s_8) + 4(s_3 s_5 + s_2 s_6 - s_1 s_8 - s_4 s_7 - s_9 s_{10})$$ $$+ 14 \text{ permutations}$$  \hspace{1cm} (D.3)
The permutations consist in exchanging the indices $a_i$ in all possible ways and change the $s_j$ invariants accordingly under an interchange of the corresponding $p_i$. For instance, if one permuts $a_4 \leftrightarrow a_5$ it implies to change $p_4 \leftrightarrow p_5$ and as a consequence $s_3 \leftrightarrow s_4, s_6 \leftrightarrow s_7, s_8 \leftrightarrow s_9$. The $O(p^4)$ lagrangian contributes to this finite amplitude with a proportionality factor equal to $(8/3)(L_1 - \frac{1}{2}L_2)$.

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Figure Captions

Fig. 1. Diagrams contributing to $\pi\pi$ scattering amplitude at tree and one-loop order. (a): tree level from $\mathcal{L}^2$. (b), (c) and (d): one-loop level (s,t,u channels) from $\mathcal{L}^2$. (e): tree level from $\mathcal{L}^4$.

Fig. 2. Ward Identity. The black box represents the insertion of $\delta S$ to the appropriate order. (a): tree level diagram. (b): tadpole. (c): one-loop with one insertion of $\delta S$ and one vertex from $\mathcal{L}^2$.

Fig. 3. General one-loop diagram with $n$ vertices and $m$ external legs.

Fig. 4. One of the diagrams contributing to a 6 pion process. It has $n=3$ vertices and $m=6$ external legs.

Fig. 5. Diagrams that correspond to the Ward Identity (C.6). (a) and (b) contribute to the l.h.s of the W.I. and (c) to the r.h.s. Here the black box is the insertion of the current $j_\mu$ defined in eq. (C.7) to the appropriate order.