ON CUBIC FUNCTORS

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Abstract. We prove that the description of cubic functors is a wild problem in the sense of the representation theory. On the contrary, we describe several special classes of such functors (2-divisible, weakly alternative, vector spaces and torsion free ones). We also prove that cubic functors can be defined locally and obtain corollaries about their projective dimensions and torsion free parts.

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INTRODUCTION

Polynomial functors appeared in algebraic topology (cf. \[8\]) and proved to be useful for lots of questions in homotopy theory. That is why their study seems to be of interest. Mostly, one deals with “continuous” polynomial functors, which are defined by their values on free groups, and in what follows we consider only such ones. In \[7\] the author gave a description of all (finitely generated) quadratic functors. This description was done using the technique of the so called matrix problems, namely, representations of bunches of chains. The reduction to such a matrix problem depends on the presentation of quadratic functors as modules over a special ring, which happens to be an order in a semi-simple algebra. Then we used the procedure of \[6\]. Now we are going to pass the same way for cubic functors. We check that the corresponding ring is again an order in a semi-simple algebra. Unfortunately, in this case, the classification of modules is a wild problem, i.e., it includes, in some sense, the classification of all representations of all finitely generated algebras over the residue field \(\mathbb{Z}/2\) (cf. Section \[2\]). Thus, there can be no “good” description of all cubic functors.

On the contrary, such a description becomes possible (and rather analogous to that of quadratic functors) if we “make 2 invertible,” i.e., consider cubic modules over the ring \(\mathbb{Z}[\frac{1}{2}]\). Such a description is given in Section \[3\]. Just as for quadratic case, this classification problem is tame, i.e., indecomposable modules depend on some “discrete” combinatorial data and on at most one “continuous parametre,” which is an irreducible polynomial from \(\mathbb{Z}/2[\bar{t}]\) (or, the same, a closed point of \(\mathbb{A}^1_{\mathbb{Z}/2} = \text{Spec} \mathbb{Z}/2[\bar{t}]\)). We also consider weakly alternative cubic functors \(F\), i.e., those with \(F(\mathbb{Z}) = 0\). Their classification given in Section \[4\] is also tame, though this time the corresponding ring has both torsion and nilpotent ideals. At last, we give a classification of cubic functors with the image being vector spaces (Section \[5\]) as well as of torsion free ones (Section \[3\]). These problems also happens to be tame. We end up with one conjecture concerning polynomial functors of higher degrees that arises from the parallel between quadratic and 2-divisible cubic functors and some corollaries from this conjecture.

1. Generalities

We suppose all categories to be pre-additive, i.e., such that their morphism sets are abelian groups and the composition is bi-additive. On the contrary, we do not suppose the functors to be additive, though we always suppose that \(F(0) = 0\) for a zero morphism. Remind that a
fully additive category is an additive category such that every idempotent in it splits. If \( F : A \to B \) is a functor from an additive category \( A \) to a fully additive category \( B \) and \( A = \bigoplus_{k=1}^{n} A_k \) is an object from \( A \), consider the corresponding embeddings \( i_k : A_k \to A \) and projections \( p_k : A \to A_k \). Put \( e_k = i_k p_k \) and \( f(k) = F(e_k) \). Certainly, \( e_k \), hence, \( f(k) \), are pairwise orthogonal idempotents and \( \text{Im} f(k) \simeq F(A_k) \). Moreover, put \( f(kl) = F(e_k + e_l) - f(k) - f(l) \) (\( k < l \)). Then \( f(kl) \) are also idempotents, pairwise orthogonal and orthogonal to all \( f(k) \).

Hence, \( F(A) \) has a direct summand \( \bigoplus_{k} F(A_k) \oplus \bigoplus_{k<l} F_2(A_k | A_l) \), where one denotes \( F_2(A_k | A_l) = \text{Im} f(kl) \). Define recursively for any \( 1 \leq k_1 < k_2 < \cdots < k_r \leq n \)

\[
f(k_1k_2 \ldots k_m) = F(e_{k_1} + \ldots + e_{k_m}) - \sum_{r<m} \sum_{1 \leq l_1 < \cdots < l_r \leq m} f(k_{l_1}k_{l_2} \ldots k_{l_r})
\]

and \( F_m(A_1 | A_2 | \ldots | A_m) = \text{Im} f(k_1k_2 \ldots k_m) \). Then

\[
F(A) = \bigoplus_{m \leq n} \left( \bigoplus_{k_1k_2 \ldots k_m} F_m(A_{k_1} | A_{k_2} | \ldots | A_{k_m}) \right).
\]

One easily sees that \( F_m \) is indeed a functor \( A^m \to B \). It is called the \( m \)-th cross-effect of \( F \). Of course, \( F_1 = F \). If there is a positive integer \( n \) such that \( F_m = 0 \) for \( m > n \), one calls \( F \) a polynomial functor. The smallest possible \( n \) with this property is called the degree of \( F \).

Certainly, the functors of degree 1 are just additive ones. The functors of degree 1 are called linear, of degree 2 quadratic, of degree 3 cubic, etc. For an arbitrary pre-additive category \( A \), one can always consider its fully additive hull \( \text{add} A \) and we call polynomial functors \( A \to B \) those from \( \text{add} A \) to \( B \). Of course, for additive functors we get in this way nothing new as every additive functor \( A \to B \) can be uniquely (up to isomorphism) prolonged to an additive functor \( \text{add} A \to B \).

In what follows, we consider the case when \( A = \mathbb{Z} \), the ring of integers, and \( B = \text{R-Mod} \), the category of modules over a ring \( R \), mainly even \( B = \text{R-mod} \), the category of finitely generated \( \text{R-modules} \). Note that \( \text{add} \mathbb{Z} = \text{fab} \), the category of finitely generated free abelian groups. In this case polynomial functors are called polynomial \( \text{R-modules} \) and the category of finitely generated polynomial \( \text{R-modules} \) of degree at most \( n \) is denoted by \( \text{R-mod}^n \). Of course, \( \text{R-mod}^1 \) coincide with the category of finitely generated \( \text{R-modules} \). If \( R = \mathbb{Z} \), we are just speaking of “polynomial modules” not precising the target category.

To define a polynomial functor \( F : \text{fab} \to B \), one only has to define the following data:

- objects \( F_m = F_m(\mathbb{Z}|\mathbb{Z}| \ldots |\mathbb{Z}) \);
• morphisms \( h_k^m : F_m \to F_{m+1} \) that are compositions of the morphisms
\[
F_m(A_1 | A_2 | \ldots | A_m) \to F(A_1 \oplus A_2 \oplus \ldots \oplus A_m) \\
\to F(A_1 \oplus \ldots \oplus A_k \oplus A_k \oplus \ldots \oplus A_m) \\
\to F_{m+1}(A_1 | \ldots | A_k | A_k | \ldots | A_m),
\]
where all \( A_j = \mathbb{Z} \), the first morphism is the embedding of the direct summand, the last one is the projection onto the direct summand and the middle one is \( F(\gamma) \), \( \gamma|_{A_j} \) being identity if \( j \neq k \) and \( \gamma|_{A_k} \) being the diagonal embedding \( A_k \to A_k \oplus A_k \);
• morphisms \( p_k^m : F_{m+1} \to F_m \) that are compositions of the morphisms
\[
F_{m+1}(A_1 | \ldots | A_k | A_k | \ldots | A_m) \to F(A_1 \oplus \ldots \oplus A_k \oplus A_k \oplus \ldots \oplus A_m) \\
\to F_m(A_1 | A_2 | \ldots | A_m) \\
\to F_m(A_1 | A_2 | \ldots | A_m),
\]
where again all \( A_j = \mathbb{Z} \), the first morphism is the embedding of the direct summand, the last one is the projection onto the direct summand and the middle one is \( F(\beta) \), \( \beta|_{A_j} \) being identity for \( j \neq k \), while \( \beta|_{A_k} \) being the codiagonal (summation) mapping \( A_k \oplus A_k \to A_k \).

In particular, a cubic \( R \)-module \( F \), where \( R \) is a ring, is defined by a diagram of \( R \)-modules:

\[
\begin{array}{c}
F_1 \\
\uparrow h \quad \quad \downarrow h_1 \\
\leftarrow p \quad \quad \leftarrow p_1 \quad \quad \leftarrow p_2
\end{array}
\quad
\begin{array}{c}
F_2 \\
\uparrow h_2 \\
\leftarrow p_2
\end{array}
\quad
\begin{array}{c}
F_3
\end{array}
\]

(1)

One can show that such a diagram corresponds to a cubic module if and only if the following relations hold:
\[
\begin{align*}
&h_1p_2 = h_2p_1 = 0, \quad h_1h = h_2h, \quad pp_1 = pp_2, \\
&h_ip_ih_i = 2h_i, \quad p_ip_ih_i = 2p_i \quad (i = 1, 2), \\
&hph = 2(h + (p_1 + p_2)\overline{h}), \quad php = 2(p + \overline{p}(h_1 + h_2)), \\
&\overline{h}p + h_1 + h_2 = h_1p_1h_2p_2h_1 + h_2p_2h_1p_1h_2, \\
&h\overline{p} + p_1 + p_2 = p_1h_2p_2h_1p_1 + p_2h_1p_1h_2p_2,
\end{align*}
\]

where \( \overline{h} = h_1h = h_2h \) and \( \overline{p} = pp_1 = pp_2 \).
Consider the pre-additive category $A$ with three objects $1, 2, 3$ and generating morphisms
\[ h : 1 \rightarrow 2, \quad p : 2 \rightarrow 1, \quad h_i : 2 \rightarrow 3, \quad p_i : 3 \rightarrow 2 \quad (i = 1, 2) \]
subject to the relations (2). Then cubic $R$-modules are just linear functors $A \rightarrow R\text{-mod}$. Sometimes we identify $A$ with the endomorphism ring $\text{End}(1 \oplus 2 \oplus 3)$. Obviously, the categories of modules over $A$ and over this ring coincide.

2. Wildness

We are going to show that the description of all cubic functors is a wild problem in the sense of the representation theory (cf. [5]). Moreover, we show that it is even true for those cubic functors “freely generated by their 1-part,” or, the same, for $A(1,1)$-modules.

**Proposition 2.1.** The ring $A(1,1)$ is isomorphic to the subring of $\mathbb{Z}^3$ with the basis \{ $(1,1,1), (2,0,0), (0,6,0)$ \}.

**Proof.** Put $a = ph - \overline{ph}, b = ph$. One easily verifies that these two elements generate $A(1,1), ab = ba = 0, a^2 = 2a$ and $b^2 = 6b$. Hence, one gets an isomorphism by mapping $a \mapsto (2,0,0), b \mapsto (0,6,0)$. \[\square\]

The category of $A(1,1)$-modules can be embedded into that of $A$-modules. Namely, every $A(1,1)$-module $M$ gives rise to the cubic module $F$, where
\[ F_1 = M, \quad F_2 = A(1,2)A_{(1,1)}M \quad \text{and} \quad F_3 = A(1,3)A_{(1,1)}M \]
with the obvious action of morphisms. It is known (and easy to check) that this procedure really defines a functor, which is a full embedding.

Let $\Sigma_n$ be the free (non-commutative) algebra $\mathbb{Z}/4 \langle x_1, x_2, \ldots, x_n \rangle$ with $n$ generators over the residue ring $\mathbb{Z}/4$.

**Proposition 2.2.** Denote by $\varphi : A(1,1) \rightarrow \Sigma_2$ the homomorphism mapping $a$ to $2x_1$ and $b$ to $2x_2$. For any $\Sigma_2$-module $L$, let $\varphi L$ be the $A(1,1)$-module obtained from $L$ via the “change of rings” $\varphi$. Then, for any $\Sigma_2$-modules $L, L'$ which are free as $\mathbb{Z}/4$-modules,

- $\varphi L \simeq \varphi L'$ as $A(1,1)$-modules if and only if $L/2L \simeq L'/2L'$ as $\Sigma_2$-modules;
- $A(1,1)$-module $\varphi L$ is indecomposable if and only if so is $\Sigma_2$-module $L/2L$.

The proof is evident. \[\square\]

**Corollary 2.3.** For every $n$ there is a cubic module $M \in C(\Sigma_n)$ such that, for any $\Sigma_n$-modules $L, L'$ which are free over $\mathbb{Z}/4$,
\[ M_{\Sigma_n}L \simeq M_{\Sigma_n}L' \] if and only if \( L/2 \simeq L'/2 \);
\[ M_{\Sigma_n}L \text{ is indecomposable} \] if and only if \( L/2 \) is indecomposable.

**Proof.** One only has to consider the \( \Sigma_2-\Sigma_n \)-bimodule \( N \) which is free of rank \( n+1 \) over \( \Sigma_n \), while the action of \( \Sigma_2 \) is given by the following rules:
- \( x_1 \) is acting as the (upper) Jordan cell;
- \( x_2 \) is acting as the matrix
  \[
  \begin{pmatrix}
  0 & 0 & \ldots & 0 & 0 \\
  x_1 & 0 & \ldots & 0 & 0 \\
  0 & x_2 & \ldots & 0 & 0 \\
  \ldots & \ldots & \ldots & \ldots & \ldots \\
  0 & 0 & \ldots & x_n & 0
  \end{pmatrix}
  \]
Afterwards, one puts \( M = \varphi N \) with \( \varphi \) defined as in Proposition 2.2. All properties are verified immediately (cf., e.g., [5]). □

This Corollary shows that the classification of all cubic modules (even of \( A(1,1) \)-modules) is at least so complicated as the classification of \( n \)-tuples of matrices over the field \( \mathbb{Z}/2 \) up to conjugations, for all \( n \). It is just the situation when one calls such a problem “wild.”

### 3. 2-divisible case

In this section, we consider the case of cubic 2-divisible modules, i.e., cubic functors \( F : \text{fab} \to \mathbb{Z}'\text{-mod} \), where \( \mathbb{Z}' = \mathbb{Z}[1/2] \). It happens that here we are able to describe all (finitely generated) cubic modules. Moreover, this description is quite alike that of quadratic \( \mathbb{Z} \)-modules given in [6]. Certainly, 2-divisible cubic modules correspond to the modules over the category \( A' = A[1/2] \).

**Proposition 3.1.** The category \( A' \) is Morita equivalent to the direct product \( \mathbb{Z}' \times \mathbb{Z}' \times B \) where \( B \) is the subring in \( \mathbb{Z}' \times \text{Mat}(2,\mathbb{Z}') \times \text{Mat}(2,\mathbb{Z}') \times \mathbb{Z}' \) consisting of all quadruples of the form:

\[
\left( a, \begin{pmatrix} b_1 & 3b_2 \\ b_3 & b_4 \end{pmatrix}, \begin{pmatrix} c_1 & 3c_2 \\ c_3 & c_4 \end{pmatrix}, d \right)
\]

with

\[ a \equiv b_1, \ b_4 \equiv c_1, \ c_4 \equiv d \pmod{3} \]

**Remark.** One can see that a module over the ring \( B \) is given by a diagram of abelian (2-divisible) groups of the following shape:

\[
(3) \quad M_1 \xrightarrow{\alpha_1} M_2 \xleftarrow{\beta_1} M_2 \xrightarrow{\alpha_2} M_3.
\]
satisfying the relations:

\[ \alpha_1 \beta_1 \alpha_1 = 3 \alpha_1, \quad \beta_1 \alpha_1 \beta_1 = 3 \beta_1, \]
\[ \alpha_2 \beta_2 \alpha_2 = 3 \alpha_2, \quad \beta_2 \alpha_2 \beta_2 = 3 \beta_2, \]
\[ \beta_1 \beta_2 = 0, \quad \alpha_2 \alpha_1 = 0, \]
\[ \alpha_1 \beta_1 + \beta_2 \alpha_2 = 3 \text{id}_2. \]

We always identify B-modules with such diagrams.

**Proof.** Put \( A_i = A'(i,i) \) (\( i = 1, 2, 3 \)) and consider the following elements:

in \( A_1 \):

\[ f_1 = \frac{ph - \overline{ph}}{2}, \quad e_1 = \text{id}_1 - f_1, \quad a_1 = \frac{\overline{ph}}{2}; \]

in \( A_2 \):

\[ e_2 = \frac{p_1 h_1}{2}, \quad f_2 = \frac{p_2 h_2}{2}, \quad g = \text{id}_2 - e_2 - f_2, \]
\[ u = hp, \quad g_1 = \frac{gu}{2}, \quad g_2 = g - g_1, \]
\[ v_i = p_i h_j \ (i, j = 1, 2, \ i \neq j), \quad v_1' = \frac{5v_1 - v_1 v_2 v_1}{2}, \]
\[ a_2 = v_1 v_2, \quad b_2 = v_2 v_1; \]

in \( A_3 \):

\[ u_i = h_i p_i \ (i = 1, 2), \quad e_3 = \frac{u_1}{2}, \quad f_3 = \text{id}_3 - e_3, \]
\[ \overline{u} = \overline{hp} + e_3, \quad a_3 = 2u_2 e_3 - \overline{u}, \quad b_3 = 2e_3 u_2 - \overline{u}; \]

in \( A'(2,1) \):

\[ p' = p - \frac{\overline{p(h_1 + h_2)}}{2}. \]

On can verify that:

- elements \( e_i, f_i, g_i \) are orthogonal idempotents;
- elements \( e_1, f_1, a_1 \) form a \( Z' \)-basis of \( A_1 \);
- \( a_1 = e_1 a_1 e_1 \) and \( a_1^2 = 3a_1 \);
- elements \( e_2, f_2, g_1, g_2, a_2, b_2, v_1, v_2, v_1 b_2, a_2 v_1 \) form a \( Z' \)-basis of \( A_2 \);
- \( g A_2 = A_2 g = \langle g_1, g_2 \rangle \);
- \( a_2 = e_2 a_2 e_2, \quad b_2 = f_2 b_2 f_2, \quad a_2^2 = 3a_2, \quad b_2^2 = 3b_2 \);
- \( v_1'(v_2/2) = e_2, \quad (v_2/2)v_1' = f_2 \);
- \( p'(h/2) = f_1, \quad (h/2)p' = g_1 \).
• elements $e_3, f_3, a_3, b_3, a_3b_3, b_3a_3$ form a $\mathbb{Z}'$-basis of $A_3$;
• $a_3 = f_3a_3e_3$, $b_3 = e_3b_3f_3$, $a_3b_3a_3 = 3a_3$, $b_3a_3b_3 = 3b_3$;
• $p'(h/2) = f_1$ and $(h/2)p' = g_1$;
• $(p_1/2)h_1 = e_2$ and $h_1(p_1/2) = e_3$;
• $gp_i = h_ig = 0$ ($i = 1, 2$) and $g_2h = pg_2 = 0$;
• $f_2h_i = p_if_2 = 0$ ($i = 1, 2$).

Consider the projective $A'$-modules:

$$E_1 = A'(1,_)e_1, \quad F_1 = A'(1,_)f_1,$$

$$E_2 = A'(2,_)e_2, \quad F_2 = A'(2,_)f_2, \quad G_i = A'(2,_)g_i,$$

$$E_3 = A'(3,_)e_3, \quad F_3 = A'(3,_)f_3,$$

They are projective generators of the category of $A'$-modules. The above equalities imply that:

- $F_1 \simeq G_1$, $E_2 \simeq F_2$, $E_2 \simeq E_3$;
- $\text{Hom}(G_i, G_j) = 0$ if $i \neq j$;
- $\text{Hom}(G_i, E_j) = \text{Hom}(E_j, G_i) = \text{Hom}(G_i, F_3) = \text{Hom}(F_3, G_i) = 0$;
- $\text{Hom}(E_1, F_3) = \text{Hom}(F_3, E_1) = 0$;
- $\text{End} E_1 \simeq e_1A_1e_1 = \langle e_1, a_1 \rangle \simeq D$;
- $\text{End} E_2 \simeq e_2A_2e_2 = \langle e_2, a_2 \rangle \simeq D$;
- $\text{End} F_3 \simeq f_3A_3f_3 = \langle f_3, a_3b_3 \rangle \simeq D$;
- $\text{End} G_i = \mathbb{Z}'$;
- $\text{Hom}(E_3, F_3) = \langle b_3 \rangle \simeq \mathbb{Z}'$;
- $\text{Hom}(F_3, E_3) = \langle a_3 \rangle \simeq \mathbb{Z}'$;
- $\text{Hom}(E_1, E_3) = \langle \overline{p} \rangle \simeq \mathbb{Z}'$;
- $\text{Hom}(E_3, E_1) = \langle \overline{h} \rangle \simeq \mathbb{Z}'$.

Here $D$ denotes the ring $\mathbb{Z}'[t]/(t^2 - 3)$, which is isomorphic to the subring in $\mathbb{Z}' \times \mathbb{Z}'$ consisting of the pairs $\{(a, b) | a \equiv b \pmod{3}\}$. Therefore, the category of $A'$-modules is equivalent to that of $E$-modules, where $E = \text{End}(G_1 \oplus G_2 \oplus E_1 \oplus F_3 \oplus E_3) \simeq \mathbb{Z}' \times \mathbb{Z}' \times B$. \hfill \Box

One can verify that the $A'$-modules $G_i$ correspond to the functors $S^2$ and $\Lambda^2$, where $S^r$ and $\Lambda^r$ denote, as usually, the $r$-th symmetric and exterior powers. Actually, these functor are quadratic, and it is the main reason that they “stand apart” really cubic ones.

Now the theory of $2$-divisible cubic modules becomes quite parallel to that of quadratic modules [4] and one gets the main results just
following the same way. As the proofs are also almost the same, we replace them by the exact references to the corresponding items from [7]. We denote by \( e_i \ (i = 1, 2, 3) \) the natural idempotents in \( B \):

\[
\begin{align*}
e_1 &= \left( 1, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0 \right), \\
e_2 &= \left( 0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0 \right), \\
e_3 &= \left( 0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right).
\end{align*}
\]

Put \( B_i = Be_i \); they are just indecomposable projective \( B \)-modules. Denote by \( L_1 \) the projection of \( B_1 \) onto the first component, by \( L_2 \) its projection onto the second component; by \( L_3 \) the projection of \( B_2 \) onto the second component, by \( L_4 \) its projection onto the third component; by \( L_5 \) the projection of \( B_3 \) onto the third and by \( L_6 \) its projection onto the forth component. Then \( L_i \) are just irreducible torsion free \( B \)-modules. Put \( Z(p) = \{ a/b \mid a, b \in \mathbb{Z}, p \nmid b \} \), \( M(p) = Z(p)M \).

**Theorem 3.2.**

1. 2-divisible cubic modules \( M, N \) are isomorphic if and only if \( M(p) \simeq N(p) \) for every odd prime \( p \).

2. Given a set \( \{ N_p \} \ (p > 2) \) where \( N_p \) is a cubic \( \mathbb{Z}(p) \)-module, there is a 2-divisible cubic module \( M \) such that \( M(p) \simeq N_p \) for all \( p \) if and only if almost all (i.e., all but a finite set) of them are torsion free (maybe, zero) and \( \mathbb{Q}N_p \simeq \mathbb{Q}N_q \) for all \( p, q \). In this case, \( \mathbb{Q}M = \mathbb{Q}N_p \) and \( tM = \bigoplus_p tN_p \).

Cf. [7, Theorem 2.1].

Now we define our main personages: string and band modules. We rearrange a bit the description given in [7, Section 4] to make it more comprehensible. We start with some notations and definitions.

**Definition 3.3.**

1. Define an equivalence relation \(-\) on the set \( \{ 1, 2, 3, 4, 5, 6 \} \) with the only non-trivial equivalences \( 2 \sim 3 \) and \( 4 \sim 5 \) and a symmetric relation \( \sim \) (not equivalence!) putting \( 1 \sim 2, 3 \sim 4, 5 \sim 6 \).

2. Define the following mappings acting on every diagram of the form (3):

\[
\begin{align*}
\theta(11) &= 3id_1 - \beta_1 \alpha_1, \\
\theta(23) &= \alpha_1, \\
\theta(33) &= \beta_2 \alpha_2, \\
\theta(45) &= \alpha_2, \\
\theta(55) &= \alpha_2 \beta_2 \\
\theta(22) &= \beta_1 \alpha_1, \\
\theta(32) &= \beta_1, \\
\theta(44) &= \alpha_2 \beta_2, \\
\theta(54) &= \beta_2, \\
\theta(66) &= 3id_3 - \alpha_2 \beta_2.
\end{align*}
\]
3. Define the function $\nu : \{ \{1, 2\}, \{3, 4\}, \{5, 6\} \} \to \mathbb{N}$ putting $\nu \{1, 2\} = 1$, $\nu \{3, 4\} = 2$, $\nu \{5, 6\} = 3$.

4. A *primary polynomial* over a field $k$ is, by definition, a power of an irreducible polynomial with the leading coefficient 1.

Now the string and band modules can be defined as follows.

**Definition 3.4.** 1. A *string diagram* $D$ is a diagram of one of the following types:

(i) $\begin{array}{ccccccc} j_1 & j_2 & j_3 & \cdots & j_{2n-2} & j_{2n-1} \\ k_1 & k_2 & k_3 & \cdots & k_{2n-1} \\ i_1i_2 & i_3i_4 & i_2n-1 \end{array}$, or

(ii) $\begin{array}{ccccccc} j_2j_3 & \cdots & j_{2n-2}j_{2n-1} \\ k_2 & k_3 & \cdots & k_{2n-1} \\ i_2 & i_3i_4 & i_2n-1 \end{array}$, or

(iii) $\begin{array}{ccccccc} j_1 & j_2 & j_3 & \cdots & j_{2n} \\ k_1 & k_2 & \cdots & k_{2n} \\ i_1i_2 & i_2-1i_{2n} \end{array}$

where $k \in \mathbb{N}$, $i_m, j_m \in \{1, 2, 3, 4, 5, 6\}$, satisfying the following conditions:

- $i_{2m-1} \sim i_{2m}$ for every $m = 1, \ldots, n$.
  (This condition is empty for diagrams (i) and (ii) if $m = n$ and for diagrams (ii) if $m = 1$; nevertheless, in these cases we define $i_1$ or $i_2n$ so that this condition holds.)

- $j_{2m+1} \sim j_{2m}$ for every $m = 1, \ldots, n - 1$.

- $i_k - j_k$ for every $m = 1, \ldots, 2n$.
  (This condition is empty for diagrams (i) and (ii) if $m = 2n$ and for diagrams (ii) if $m = 1$.)

2. The *string $B$-module* corresponding to a string diagram $D$ is the $B$-module $M = M^D$ with the generators

$g_1, g_2, \ldots, g_n, \quad g_m \in M_\nu \{i_{2m-1}, i_{2m} \}$,

and the defining relations

$3^{k_{2m}} \theta(i_{2m}j_{2m})g_m = 3^{k_{2m+1}} \theta(i_{2m+1}j_{2m+1})g_{m+1}$

for $i = 0, \ldots, n$; we put here $g_0 = g_{n+1} = 0$ and omit the relation with $m = n$ for diagrams (i), (ii) and with $m = 1$ for diagrams (ii).
3. A **string cubic module** is one corresponding to a string \( B \)-module via the Morita equivalence of Proposition 3.1.

**Definition 3.5.** 1. A diagram \( D \) of type (iii) is said to be **non-periodic** if it cannot be obtained by a repetition of a smaller diagram \( D_1 \) of the same type: \( D \neq D_1 D_1 \ldots D_1 \).

2. A **band data** is a pair \( B = (D, f(t)) \) consisting of a non-periodic diagram of type (iii) with the additional condition:
   - \( j_{2n} \sim j_1 \),
   and of a primary polynomial \( f(t) = \lambda_1 + \lambda_2 t + \cdots + \lambda_{d^d-1} t^d \) over the residue field \( \mathbb{Z}/3 \) with \( \lambda_1 \neq 0 \).

3. The **band \( B \)-module** corresponding to a band data \( B \) is the \( B \)-module \( M = M^B \) with the generators
   \[ g_{ml} \ (m = 1, \ldots, n, l = 1, \ldots, d), \quad g_{ml} \in M_{\nu(i_{2m-1}, i_{2m})}, \]
   and the defining relations:
   \[ 3^{k_{2m}} \theta(i_{2m}j_{2m}) g_{ml} = 3^{k_{2m+1}} \theta(i_{2m+1}j_{2m+1}) g_{m+l, l} \quad \text{if} \quad 1 \leq m < n; \]
   \[ 3^{k_{2n}} \theta(i_{2n}j_{2n}) g_{nl} = 3^{k_{1}} \theta(i_{1}j_{1}) g_{1,l+1} \quad \text{if} \quad 1 \leq l < d; \]
   \[ 3^{k_{2n}} \theta(i_{2n}j_{2n}) g_{nd} = -3^{k_{1}} \theta(i_{1}j_{1}) \sum_{\nu=1}^{d} \lambda_{\nu} g_{1\nu}. \]

4. A **band cubic module** is one corresponding to a band \( B \)-module via the Morita equivalence of Proposition 3.1.

Put also \( L(i, p, k) = L_i/p^{k} L_i \), where \( k \) is a positive integer, \( p > 3 \) is a prime number, \( i \in \{1, 2, 4, 6\} \); \( \Lambda^2(p, k) = \Lambda^2/p^k \Lambda^2 \) and \( S^2(p, k) = S^2/p^k S^2 \), where \( p \geq 3 \) is again a prime number.

**Theorem 3.6.** 1. The string and band cubic modules defined above, the modules \( L(i, p, k) \) and the modules \( \Lambda^2(p, k), S^2(p, k) \) are just all finitely generated indecomposable 2-divisible cubic modules.

2. The only isomorphisms between these modules are the following:
   - \( M^D \simeq M^{D^*} \), where \( D^* \) denotes the diagram symmetric to \( D \), the latter being of type (ii) or (iii);
   - \( M^B \simeq M^{B^{(s)}} \), where \( B^{(s)} = (D^{(s)}, f(t)) \) and \( D^{(s)} \) is the \( s \)-th shift of the diagram \( D \), i.e., the diagram
     \[ j_{2s+1} \quad k_{2s+1} \quad j_{2s+2} \quad k_{2s+2} \quad \ldots \quad j_{2s} \quad k_{2s} \]
     \[ i_{2s+1} j_{2s+2} \quad k_{2s+2} \quad i_{2s+1} j_{2s} \quad k_{2s} \]
     \[ \lambda_1 i_{2s} d f(1/t); \]
   - \( M^B \simeq M^{B^*^{(s)}} \), where \( (D, f(t))^* = (D^*, f^*(t)) \) and \( f^*(t) = \lambda_1^{-1} t^d f(1/t); \)
Corollary 3.7. 1. A 2-divisible cubic module $M$ is of finite projective dimension if and only if it contains no string summands of types (i) and (iii). In this case either $M$ is projective (hence, a direct sum of modules isomorphic to $B_i$) or $	ext{pr. dim } M = 1$. Otherwise, $	ext{pr. dim } M = \infty$.

2. Every 2-divisible cubic module $M$ has a periodic projective resolution

$$\cdots \to P_{n+1} \xleftarrow{\gamma_n} P_n \xrightarrow{\gamma_{n-1}} \cdots \xleftarrow{\gamma_1} P_1 \xrightarrow{\gamma_0} P_0 \to M \to 0$$

of period 6, namely, such that $\gamma_{n+6} = \gamma_n$ for every $n \geq 2$.

3. If $M$ is an indecomposable non-projective 2-divisible cubic module, $tM$ is its torsion part, then

$$M/tM \simeq \begin{cases} L_{i_{2n}} & \text{if } M \text{ is a string module of type (i)} \\ L_{i_1} \oplus L_{i_{2n}} & \text{if } M \text{ is a string module of type (ii)} \\ 0 & \text{otherwise} \end{cases}$$

(Remind that here $i_{2n}$ denotes the unique index such that $i_{2n-1} \sim i_{2n}$ and $i_1$ denotes the unique index such that $i_2 \sim i_1$.)

Cf. [7, Corollaries 5.3, 5.5].

4. Weakly alternative functors

Consider now the case of cubic $\mathbb{Z}$-modules $F$ such that $F(\mathbb{Z}) = 0$. We call them weakly alternative. They are actually modules over the category $A^a = A/(id_1)$. The functors $\Lambda^3$ and $\Lambda^2 Id$ are of this sort. Here again, we are able to obtain a complete description.

Denote by $C_0$ the subring in $\mathbb{Z} \times \text{Mat}(2, \mathbb{Z})$ consisting of the pairs $(a, B)$ with $b_{12} \equiv 0$, $a \equiv b_{11} \pmod{3}$.

Proposition 4.1. The category $A^a$ is Morita equivalent to the semi-direct product $C = (\mathbb{Z} \times C_0) \rtimes T$ where $T$ is the elementary 2-group with three generators $\xi, \eta, \theta$ such that the following relations hold:

$$\theta = \xi \eta, \quad \eta \xi = 0,$$

$$\eta e = \eta, \quad e \xi = \xi,$$

where $e = (1, 0, 0)$,

$$\xi (0, a, B) = a \xi \quad \text{and} \quad (0, a, B) \eta = a \eta$$

for any pair $(a, B) \in C_0$. 

Proof. The category $A^a$ has two objects: $2, 3$, and is generated by the morphisms $h_i : 2 \to 3$, $p_i : 3 \to 2$ (i=1,2) subject to the following relations, cf. diagram (I):

$$h_i p_i h_i = 2 h_i, \quad p_i h_i p_i = 2 p_i, \quad h_i p_j = 0 \text{ if } i \neq j,$$
$$h_1 + h_2 = h_1 p_1 h_2 p_1 + h_2 p_2 h_1 p_1 h_2,$$
$$p_1 + p_2 = p_1 h_2 p_1 h_1 + p_2 h_1 p_1 h_2,$$

that imply:

$$h_i p_i = h_i p_j p_j h_i p_i,$$
$$2 p_i = 2 p_i h_j p_i h_i,$$
$$2 h_i = 2 h_i p_j p_i h_i$$

(always $i, j \in \{1, 2\}, i \neq j$). Then $e_i = p_i h_j p_j h_i$ are orthogonal idempotents in $A^a(2, 2)$ which are conjugate as $e_1 p_1 h_2 = p_1 h_2 e_2$ and $e_2 p_2 h_1 = p_2 h_1 e_1$. Moreover, one can check that $e_3 p_1 h_2 = p_1 h_2 e_3 = e_3 p_2 h_1 = p_2 h_1 e_3$ where $e_3 = 1 - e_1 - e_2$. Denote this product by $\theta$. Then $2 \theta = 0$ and $p_i h_i = \theta + 2 e_i$, so $A^a(2, 2)$ is Morita equivalent to $\mathbb{Z} \times \mathbb{Z}[\theta]$.

Just in the same way, one can verify that $A^a(3, 3) = \langle f_1, f_2, \alpha_1, \alpha_2, \beta \rangle$ where $f_1 = h_1 p_1 h_2 p_2, f_2 = 1 - f_1$, $\alpha_i = h_i p_i$, $\beta = \alpha_2 \alpha_1$. Moreover, $f_i \alpha_i = \alpha_i f_j = \alpha_i$, $\alpha_1 \alpha_2 = 3 f_1$ and $\beta^2 = 3 \beta$. Therefore, $A^a(3, 3)$ is isomorphic to the subring $C_0$.

Let $u = p_1 h_2 p_2, v = h_1 p_1 h_2 p_2 h_1$. Then $e_1 u = u f_1, f_1 v = v e_1, u v = e_1, v u = f_1$, so $e_1$ and $f_1$ are conjugate in $A^a$. Put $\xi = p_1 - p_1 h_2 p_2 h_1 p_1, \eta = h_1 - h_1 p_1 h_2 p_1 h_2$. One can also check that

$$2 \xi = 0, 2 \eta = 0, \eta \xi = 0, \xi \eta = \theta, \xi f_3 = \xi, f_3 \eta = \eta,$$

$$e_3 A^a(3, 2) = \langle \xi \rangle, \quad A^a(2, 3) e_3 = \langle \eta \rangle.$$

Hence, putting $E^a_3 = A^a(2, \_ \_ e_3$ we get that $A^a$ is Morita equivalent to the endomorphism ring $\text{End}(E^a_3 \oplus A^a(3, \_ \_))$ which is isomorphic to $C$.

Describe now (finitely generated) $C$-modules or, the same, weakly alternative cubic functors. If $M$ is such a module, put $\overline{M} = M/TM$. It is a module over $\mathbb{Z} \times C$, so $\overline{M} = M_1 \oplus M_2$, where $M_1$ is an abelian group with $C_0 M_1 = 0$ and $M_2$ is a $C_0$-module. Again, the same observations as in [3, Theorem 2.1] yield the following result.

**Proposition 4.2.**

1. Weakly alternative cubic modules $M, N$ are isomorphic if and only if $M_{(p)} \simeq N_{(p)}$ for every prime $p$.

2. Given a set $\{N_p\}$ where $N_p$ is a weakly alternative cubic $\mathbb{Z}_{(p)}$-module, there is a weakly alternative cubic module $M$ such that
$M(p) \simeq N_p$ for all $p$ if and only if almost all of them are torsion free (maybe, zero) and $\mathbb{Q}N_p \simeq \mathbb{Q}N_q$ for all $p,q$. In this case $\mathbb{Q}M = \mathbb{Q}N_p$ and $\mathfrak{t}M = \bigoplus_p \mathfrak{t}N_p$.

Let $\overline{C} = C/T = \mathbb{Z} \times C_0$, $C_1 = L_1 = \overline{C}e$, $C_2 = \overline{C}(0,1,e_{11})$ and $C_3 = L_4 = \overline{C}(0,0,e_{22})$. They are all indecomposable projective $\overline{C}$-modules. Put also $L_2 = \overline{C}(0,1,0)$ and $L_3 = \overline{C}(0,0,e_{11})$. They are all non-projective indecomposable torsion free $\overline{C}$-modules. If $p > 2$, the localization $C(p)$ is torsion free. If, moreover, $p > 3$, this localization is hereditary, hence, any $C(p)$-module splits into a direct sum of a torsion free and a torsion one, the latter being a direct sum of modules isomorphic to $L(i,k,p) = L_i/p^k$ for $i \in \{1,2,3\}$, $k \in \mathbb{N}$.

The description of $C(3)$-modules is similar to that of $B$-modules in the preceding section. One only has to consider the set $\{1,2,3,4\}$ instead of $\{1,2,3,4,5,6\}$ with the relations — and $\sim$ defined as follows: $3 - 4$ and $2 \sim 3$.

At last, $C(2) = C' \times C''$ where $C' = (\mathbb{Z}(2) \times \mathbb{Z}(2)) \rtimes T$ and $C'' \simeq \text{Mat}(2,\mathbb{Z}(2))$. A $C''$-module is a direct sum of several copies of $(L_3)_2$ and of modules isomorphic to $L(3,k,2) = L_3/2^k$. A $C'$-module $W$ is given by a diagram of $\mathbb{Z}(2)$-modules of the form:

$$
\begin{array}{c}
W_1 \\
\uparrow \xi \\
W_2
\end{array}
\quad \quad
\begin{array}{c}
W_2 \\
\downarrow \eta
\end{array}
\quad \quad
\begin{array}{c}
W_1
\end{array}
$$

such that $2\xi = 0$, $2\eta = 0$, $\eta \xi = 0$. Split both $W_1$ and $W_2$ into a direct sum of cyclic modules $C_i = \mathbb{Z}/2^i$ and $C_\infty = \mathbb{Z}(2)$. Then one can check that such a diagram is a direct sum of diagrams $W(\omega)$ and $W(\omega,\pi)$. Here $\omega$ is a (finite) word of the form

$$
\ldots \xi^{i_m} \eta_j \xi^{i_m+1} \eta_j \xi \ldots
$$

containing no subwords of the form $\infty \xi$, $\infty \eta$, $\eta_1 \xi$. In $W(\omega,\pi)$, $\omega$ must be of the form

$$
\begin{array}{c}
\eta_1 \xi^{i_1} \eta_j \xi^{i_2} \eta_j \xi \ldots \eta_j
\end{array}
$$

with the same restrictions as above and $\pi(t) \neq t^n$ is a primary polynomial over $\mathbb{Z}/2$. Namely, if $W = W(\omega)$, then $W_1 = \bigoplus_m C_{i_m}$, $W_2 = \bigoplus_m C_{j_m}$, while $\xi(C_{i_m}) \subseteq C_{j_{m-1}}$, $\eta(C_{j_m}) \subseteq C_{i_m}$ and the induced mappings are the unique non-zero ones of period 2 (we denote them by $\gamma$). If $W = W(\omega,\pi)$ and $\deg \pi = n$, then $W_1 = \bigoplus_m nC_{i_m}$, $W_2 = nC_j \oplus (\bigoplus_m nC_{j_m})$; $\xi(nC_{i_m}) \subseteq nC_{j_{m-1}}$, $\eta(nC_{j_m}) \subseteq nC_{i_m}$, $\xi(nC_{i_1}) \subseteq nC_j$, $\eta(nC_j) \subseteq nC_{i_m}$ and the induced mappings are given by the matrices $\gamma I$, except the last one, given by $\gamma \Phi$ where $\Phi$ is the Frobenius cell corresponding to the polynomial $\pi$. 

Note that the torsion free part of $W(\omega, \pi)$ is zero, that of $W(\omega)$ consists of at most one cyclic summand, and that of an indecomposable $C_{(3)}$-module either is trivial, or consists of one or of two cyclic summands (for string modules $M^D$ of type, respectively, (i) or (ii), cf. page 10). So, in accordance with Proposition 4.2, the indecomposable $C$-modules $M$ that are not torsion have the following local components $M(p)$ (we only describe $M(2)$ and $M(3)$ as all other ones are torsion free, hence, uniquely defined):

1. $M(2) = W(\omega)$ where $\omega$ contains $\xi^\infty$, $M(3) = M^D$ where $D$ is a string of type (i) or (ii) with $i_{2n-1} = 2$ or $i_2 = 2$ (if both $i_2 = i_{2n-1} = 2$ in a string of type (ii), there are two possibilities for such $M$).
2. $M(2) = W(\omega) \oplus W(\omega')$ where both $\omega$ and $\omega'$ contain $\xi^\infty$, $M(3) = M^D$ where $D$ is a string of type (ii) with $i_2 = i_{2n-1} = 2$.
3. $M(3)$ is a string module of type (i) or (ii), $M(2)$ is torsion free (thus, uniquely defined).

(Note that the case when $M(3)$ is torsion free is a part of case (1) above.)

5. Cubic vector spaces

Now we consider the cubic vector spaces, i.e., cubic functors $F : \text{fab} \to \text{k-mod}$ where $k$ is a field. If $\text{char } k \neq 2$ they are a partial case of the functors considered in Section 3; hence, we always suppose that $\text{char } k = 2$. In this case a cubic functor $F$ is given by a diagram of $k$-vector spaces of the same shape (1) with the relations:

\begin{align*}
  h_1p_2 + h_2p_1 &= 0, \quad h_1h = h_2, \quad pp_1 = pp_2, \\
  h_i p_i h_i &= 0, \quad p_i h_i p_i = 0 \quad (i = 1, 2), \\
  hph &= 0, \quad php &= 0, \\
  \overline{h}p + h_1 + h_2 &= h_1p_1h_2p_2h_1 + h_2p_2h_1p_1h_2, \\
  h\overline{p} + p_1 + p_2 &= p_1h_2p_2h_1p_1 + p_2h_1p_1h_2p_2
\end{align*}

(5)

(just as before, $\overline{h} = h_1h = h_2$ and $\overline{p} = pp_1 = pp_2$). Denote by $C$ the $k$-linear category with objects 1, 2, 3 and generating morphisms $h : 1 \to 2$, $p : 2 \to 1$, $h_i : 2 \to 3$, $p_i : 3 \to 1$ ($i = 1, 2$) satisfying the relations (5). So, a cubic vector space is the same as a $C$-module (i.e., a linear functor $C \to \text{k-mod}$).

The last two equations imply that

\begin{align*}
  \overline{h}\overline{p} + h_ip_i &= h_ip_ih_jp_jh_ip_j \quad (i, j = 1, 2; \ i \neq j),
\end{align*}
whence \( p_i h p = h p h_i = 0 \) for \( i = 1, 2 \). Hence, the elements \( e_i = h_i p_i h_j p_j \) as well as the elements \( f_i = p_i h_j p_j h_i \) \( (i, j = 1, 2; \ i \neq j) \) are orthogonal idempotents, respectively, in \( C(3, 3) \) and in \( C(2, 2) \). Thus, in \( \text{add} \ C, \ 3 \cong 3_0 \oplus 3_1 \oplus 3_2 \) and \( 2 \cong 2_0 \oplus 2_1 \oplus 2_2 \), so that the identity morphisms of \( 3_i \) are identified with \( e_i \) (with \( e_0 = 1 - e_1 - e_2 \)) and those of \( 2_i \) are identified with \( f_i \) (with \( f_0 = 1 - f_1 - f_2 \)). In what follows, we write \( C(x, y) \) for the set of morphisms \( x \to y \) in \( \text{add} \ C \).

An easy calculation shows that the four objects \( 2_i, 3_i \ (i = 1, 2) \) are isomorphic in \( \text{add} \ C \). For instance, as \( p_1 h_2 p_2 = p_1 h_2 p_2 h_1 p_1 h_2 p_2 \), this element lies in \( C(3_1, 2_1) \); the element \( h_1 p_1 h_2 p_2 h_1 \) lies in \( C(2_1, 3_1) \) and their products are just \( f_1 \) and \( e_1 \), whence \( 2_1 \cong 3_1 \), etc. So, we only have to take into account one of these objects, say \( 3_1 \). One can also easily check that \( C(3_1, 3_1) = k e_1 \), while \( C(x, 3_1) = C(3_1, x) = 0 \) if \( x \in \{ 1, 2_0, 3_0 \} \). Thus, the category \( C \) is Morita equivalent to the direct product of the trivial category with one object \( 3_1 \) and the full subcategory \( C^* \) of \( \text{add} \ C \) with the objects \( 1, 2_0, 3_0 \). One easily check that the cubic module \( T \) corresponding to the (unique) indecomposable representation of the trivial part is the following one:

\[
T_1 = 0, \quad T_2 = \langle u_1, u_2 \rangle, \quad T_3 = \langle v_1, v_2 \rangle;
\]

\[
(6) \quad h_1(u_1) = v_1, \quad h_1(u_2) = 0; \quad h_2(u_1) = 0, \quad h_2(u_2) = v_2;
\]

\[
p_1(v_1) = 0, \quad p_1(v_2) = u_1; \quad p_2(v_1) = u_2, \quad p_2(v_2) = 0.
\]

(This cubic module corresponds to the functor \( k \Lambda^2 \text{Id} \).) So, from now on, we only consider the representations of \( C^* \) and for every morphism \( \alpha \) from \( C \) we denote by the same letter \( \alpha \) its restriction onto \( C^* \). As such restrictions of \( e_i \) and \( f_i \) are zero for \( i = 1, 2 \), one obtains the relations:

\[
h p + p_1 + p_2 = 0, \quad h p + h_1 + h_2 = 0,
\]

so we may exclude \( p_2, h_2 \) from the generating set. Therefore, \( C^* \)-modules are just diagrams of vector spaces

\[
(7) \quad F_1 \xleftarrow{h} F_2 \xrightarrow{h_1} F_3 \xleftarrow{p}
\]

with the relations

\[
(8) \quad h ph = p h p = h_1 p_1 h_1 = p_1 h_1 p_1 = 0, \quad h_1 p_1 = h_1 h p p_1.
\]

(One easily checks that they imply all relations \((5)\) if we put \( h_2 = h_1 + h_1 h p \) and \( p_2 = p_1 + h p p_1 \).)

Consider the subdiagram

\[
F_1 \xleftarrow{h} F_2 \xrightarrow{p} F_2.
\]
As \( hph = php = 0 \), it decomposes into a direct sum of the following shape:

\[
\begin{array}{ccccccc}
U_1 & U_2 & U_3 & U_4 & U_5 & U_6 \\
\Rightarrow & \downarrow & \downarrow & \Uparrow & \checkmark & \Uparrow \\
V_6 & V_1 & V_2 & V_3 & V_4 & V_5 \\
\end{array}
\]

\((U_i \text{ are the direct summands of } F_1, V_i \text{ those of } F_2, \text{ the arrows show the action of } h \text{ and } p \text{ when it is non-zero, the corresponding maps being isomorphisms.})\) It means that \( h \) and \( p \) are given by the following matrices:

\[
h = \begin{pmatrix}
I & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad p = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & I \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\((I \text{ denotes the identity matrices.})\)

With respect to the decomposition of \( F_2 \), \( h_1 \) and \( p_1 \) are given by the matrices \( H = (H_1, H_2, H_3, H_4, H_5, H_6) \) and \( P = (P_1, P_2, P_3, P_4, P_5, P_6)^\top \) where \( H_i : V_i \to F_3, P_i : F_3 \to V_i \). For these matrices the following conditions hold:

- the number of rows of \( H \) equals the number of columns of \( P \);
- the number of columns of \( H_i \) is the same as the number of rows of \( P_i \) for every \( i \);
- the number of columns of \( H_1 \) is the same as the number of rows of \( H_6 \).

When one applies the isomorphisms of the spaces \( F_i \) which do not destroy the form \((9)\) of \( h \) and \( p \), they result in elementary transformations of the columns of the matrices \( H \) and \( P \) such that:

- the transformations of \( P \) are contragredient to those of \( H \) (e.g., when we add the \( k \)-th column of \( H \) to the \( l \)-th one, we have to subtract the \( l \)-th column of \( P \) from the \( k \)-th one, etc.);
- the transformations inside \( H_1 \) are the same as those inside \( H_6 \);
- one can only add columns of \( H_i \) to those of \( H_j \) if \( i \leq j \) and \((i, j) \neq (3, 4)\).
Using such transformations, one can reduce the matrix $H$ to the following form:

\[
\begin{pmatrix}
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

(the vertical lines denote the borders of the matrices $H_i$). Certainly, we have to subdivide in the same manner the rows and columns of the matrix $P$. Denote the corresponding blocks by $P_{ij}$ ($1 \leq i \leq 17$, $1 \leq j \leq 10$); the horizontal (vertical) stripes of $P$ will be denoted by $R_i$ (respectively, $S_j$). Then the relations (8) for $h_1$ and $p_1$ are equivalent to the following conditions for these blocks:

- $P_{ij} = 0$ if $i \in \{4, 6, 9, 12, 14, 16, 18\}$,
- $P_{ij} = 0$ if $i \in \{3, 17\}$ and $j \neq 10$,
- $P_{8j} = P_{11,j}$ for all $j$,
- $P_{3,10} = P_{17,10}$ and $P_{i}P_{3,10} = 0$ for all $i$.

Thus, we only have to consider the matrix $\overline{P}$ obtained from $P$ by crossing out all zero horizontal stripes as well as the stripes $R_3$ and $R_8$. A straightforward calculation shows that the automorphisms of the spaces $F_3$ and $V_i$ which do not destroy the shape of $H$ give rise to the elementary transformations of the matrix $\overline{P}$ satisfying the following conditions:

- the transformations inside $R_1$ are the same as those inside $R_{15}$;
- the transformations inside $S_2$ are the same as those inside $S_9$;
- the transformations inside $R_2$ are contragredient to those inside $S_8$;
- the transformations inside $R_{11}$ are contragredient to those inside $S_4$;
- the transformations inside $R_{17}$ are contragredient to those inside $S_1$;
- one can add the columns of $S_i$ to those of $S_j$ if and only if $i \leq j$ and $(i, j) \neq (5, 6)$;
• one can add the rows of $R_i$ to those of $R_j$ if and only if $i \geq j$
and $(i, j) \neq (10, 7)$.

Obviously, one may suppose that the matrix $P_{17,10}$ is of the form
$\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$. Then the whole stripe $S_1$ has the form $(0 \ 0')$, the
number of columns in $S_1'$ being the same as that of zero rows in $P_{17,10}$.
Moreover, using the elementary transformations described above, one
may suppose that the remaining part of the stripe $S_{10}$ is of the form
$(S_{10}' \ 0)$, the number of columns in $S_{10}'$ being the same as in the zero
part of $P_{17,10}$. In what follows, we omit the dash and write $S_1$, $S_{10}$, $P_1$ and $P_{10}$
for the remaining (non-zero) parts of the corresponding matrices. Of course, we should no more consider the stripe $R_{17}$.

Now one immediately sees that we are again in the situation consid-
ered in $\mathbb{F}$. Namely, we have got the following semi-chains:

$$
\mathcal{E} = \{ R_i \mid R_1 > R_2 > R_3 > R_4 > R_5 > R_7 > R_{11} > R_{13} > R_{15}, R_5 > R_{10} > R_{11} \},
$$
$$
\mathfrak{F} = \{ S_j \mid S_1 < S_2 < S_3 < S_4 < S_5 < S_7 < S_8 < S_9 < S_{10}, S_4 < S_6 < S_7 \}
$$
with the involution $\sigma$ such that

$$
\sigma(R_1) = R_{15}, \quad \sigma(R_2) = S_8, \quad \sigma(R_{11}) = S_4, \quad \sigma(S_2) = S_9
$$

and $\sigma(x) = x$ if $x \notin \{ R_1, R_2, R_{11}, R_{15}, S_2, S_4, S_8, S_9 \}$. Hence, one
can deduce from $\mathbb{F}$ a list of canonical forms of the matrix $P$ and,
therefore, of cubic functors $\text{fab} \to \text{k-mod}$. It can again be arranged
in the form of “strings and bands,” though a trifle more sophisticated
than in the preceding sections.

**Definition 5.1.** Put $\mathfrak{X} = (\mathcal{E} \cup \mathfrak{F}) \setminus \{ R_{10}, S_0 \}$. Write $x - y$ if $x \in \mathcal{E}$, $y \in \mathfrak{F}$ or vice versa; $x \sim y$ if $\sigma(x) = y \neq x$ or $x = y \in \{ R_7, S_5 \}$. Call $R_7, S_5$ special elements. An $\mathfrak{X}$-word (or simply word) is a sequence $w = x_1 r_2 x_2 r_3 \ldots r_n x_n$, where $x_k \in \mathfrak{X}$, $r_k \in \{ \sim, - \}$, satisfying the following conditions:

• $x_{k-1} r_k x_k$ (in the above defined sense) for all $k = 2, \ldots, n$;
• $r_k \neq r_{k-1}$ for all $k = 3, \ldots, n$;

Such a word is called full if the following conditions hold:

• either $r_2 = \sim$ or $x_1 \not\sim y$ for every $y \neq x_1$;
• either $r_n = \sim$ or $x_n \not\sim y$ for every $y \neq x_n$.

A full word is called special if $x_1$ or $x_n$, but not both of them, is a
special element, bispecial if both $x_1$ and $x_n$ are special, and ordinary
if neither $x_1$ nor $x_n$ is special. A word $w$ is called non-symmetric if
$w \neq w^*$, where $w^*$ is the inverse word: $w^* = x_n r_n x_{n-1} \ldots x_2 r_2 x_1$. 
An $X$-word is called cyclic if $r_2 = r_n = -$ and $x_n \sim x_1$. A cyclic word is called aperiodic if it is not of the form $v \sim v \sim \cdots \sim v$ for a shorter word $v$. The $s$-th shift of a cyclic word $w$ is the word $w^{(s)} = x_{2s+1}r_{2s+2}x_{2s+2} \cdots x_n \sim x_1r_2 \cdots r_2x_{2s}$. A cyclic word $w$ is said to be shift-symmetric if $w^{(s)}$ is symmetric for some $s$. Note that the length $n$ of a cyclic word is always even.

**Definition 5.2.** A string datum $D$ is:

- either an ordinary non-symmetric $X$-word $w$ (ordinary string datum);
- or a pair $(w, \delta)$ consisting of a special word $w$ and $\delta \in \{0, 1\}$ (special string datum);
- or a quadruple $(w, \delta_1, \delta_2, m)$ consisting of a bispecial non-symmetric word $w$, $\delta_1, \delta_2 \in \{0, 1\}$ and $m \in \mathbb{N}$ (bispecial string datum).

Put $D^* = w^*$ in the first case, $D^* = (w^*, \delta)$ in the second and $D^* = (w^*, \delta_2, \delta_1, m)$ in the third one.

A band datum $B$ is a pair $(w, \pi(t))$ consisting of an aperiodic cyclic word $w$ and of a primary polynomial $\pi(t) \in k[t]$ (i.e., a power of an irreducible one) such that $\pi(t) \neq t^d$ if $w$ is not shift-symmetric and $\pi(t) \neq (t - 1)^d$ if $w$ is shift-symmetric. Put $B^{(s)} = (w^{(s)}, f(t))$ and $B^* = (w^*, \lambda^{-1}t^d\pi(1/t))$, where $w$ is not shift-symmetric, $d = \deg \pi$ and $\lambda = \pi(0)$.

Now the results of [1] immediately imply the following

**Theorem 5.3.**

1. Every string data $D$ defines an indecomposable cubic vector space $V^D$ called string cubic space.
2. Every band data $B$ defines an indecomposable cubic vector space $V^B$ called band cubic space.
3. Every indecomposable cubic vector space, except $k\Lambda^2\text{Id}$, is isomorphic either to a string or to a band one.
4. The only isomorphisms between string and band spaces are the following:
   - $V^D \simeq V^{D^*}$ where $D$ is a string data;
   - $V^B \simeq V^{B^{(s)}}$ and $V^B \simeq V^{B^*^{(s)}}$ where $B$ is a band data and $s \in \mathbb{N}$.

Moreover, one can deduce from [1] and the preceding considerations an explicit construction of string and band spaces, though it is rather cumbersome and we will not include it here.
6. Torsion free cubic modules

In this section, we consider torsion free cubic modules, i.e., cubic functors \( f_{ab} \rightarrow f_{ab} \). Again, we first study them locally, i.e., describe cubic functors \( f_{ab} \rightarrow f_{ab(p)} \), the latter being the category of torsion free finitely generated \( \mathbb{Z}_{(p)} \)-modules. Denote also by \( \mathbb{Z}_p \) the ring of \( p \)-adic integers. The latter has the advantage of being complete, that guarantees lifting idempotent endomorphisms modulo \( p \). Note that the calculations of Section 3 easily imply that \( A \mathbb{Q} \cong \mathbb{Q}^3 \times \text{Mat}(2, \mathbb{Q}) \times \text{Mat}(4, \mathbb{Q})^2 \). In particular, it is a semi-simple split \( \mathbb{Q} \)-algebra, i.e., \( \text{End} V \cong \mathbb{Q} \) for every simple module \( V \). Thus, it follows from the standard results of the theory of lattices over orders (cf. [9, Chapter 4, §1]) that every torsion free \( A_{(p)} \)-module is actually a completion of a torsion free \( A_{(p)} \)-module. Hence, lifting idempotents is possible for \( A_{(p)} \)-modules too. Together with the calculations of Section 3 it implies the following result.

**Proposition 6.1.** The ring \( A_{(2)} \) is isomorphic to the subring of \( \mathbb{Z}_3^3 \times \text{Mat}(2, \mathbb{Z}_2) \times \text{Mat}(4, \mathbb{Z}_2)^2 \) consisting of all sextuples \((a_1, a_2, a_3, B, C, D)\) satisfying the following congruences modulo 2:

\[
\begin{align*}
a_1 &\equiv b_{11} \equiv c_{11}, \\
a_2 &\equiv b_{22} \equiv c_{22} \equiv c_{33}, \\
a_3 &\equiv c_{44}, \\
b_{12} &\equiv 0 \text{ and } c_{ij} \equiv 0 \text{ if } i < j.
\end{align*}
\]

(*)

One can easily check that this ring is an example of the so called Backström order [10], i.e., its Jacobson radical coincides with that of an hereditary order \( H \). Thus, torsion free \( A_{(2)} \)-modules are in a natural one-to-one correspondence with the representations of a quiver \( Q \) over the field \( \mathbb{Z}/2 \). Namely, the vertices of \( Q \) are just simple \( A_{(2)} \)-modules \( A_1, \ldots, A_r \) and simple \( H \)-modules \( H_1, \ldots, H_s \), all arrows are from some \( A_i \) to some \( H_j \) and the number of such arrows equals the multiplicity of \( A_i \) in \( H_j \). In our example \( H \) consists of the sextuples satisfying the congruences (*) only, \( r = 4, s = 10 \) and the quiver \( Q \) consists of 4 connected components:

\[
\begin{array}{ccccccccc}
H_1 & \leftarrow & A_1 & \longrightarrow & H_4 & \quad & H_5 \\
 \downarrow & & & & \uparrow & & \\
H_6 & & & & & & \\
H_2 & \leftarrow & A_2 & \longrightarrow & H_7 & \quad & H_8 \\
 \downarrow & & & & \uparrow & & \\
H_3 & \leftarrow & A_3 & \longrightarrow & H_9 & & \\
A_4 & \longrightarrow & H_{10} & & & & \\
\end{array}
\]
The description is the natural one with respect to the description of \( \mathbf{A}_2 \) and \( \mathbf{H} \) above.) This quiver is tame and the list of its representations is well known (cf., e.g., [2]). Hence, we can derive the description of torsion free \( \mathbf{A}_2 \)-modules.

The description of indecomposable torsion free \( \mathbf{A}_3 \)-modules is given in Section 3. For any other prime \( p \), the localization \( \mathbf{A}_{(p)} \) is just \( \mathbb{Z}_p^3 \times \text{Mat}(2, \mathbb{Z}_p) \times \text{Mat}(4, \mathbb{Z}_p)^2 \), hence, there are exactly 6 indecomposable (and irreducible) torsion free modules. Therefore, the standard “local–global” procedure [3, 4] implies the following result on torsion free cubic modules.

**Proposition 6.2.** Torsion free cubic modules are in one-to-one correspondence with the pairs \((M_2, M_3)\), where \( M_p \) is a torsion free \( \mathbf{A}_{(p)} \)-module \((p = 2, 3)\) and \( \mathbb{Q}M_2 \simeq \mathbb{Q}M_3 \).

**Proof.** It follows from [3] Chapter 4 that such a pair \((M_2, M_3)\) always defines a cubic module \( M \) up to genus. (Remind that two modules \( M, N \) belong to the same genus if \( M_p \simeq N_p \) for all prime \( p \).) Note that \( \Gamma = \mathbb{Z}_p^3 \times \text{Mat}(2, \mathbb{Z}) \times \text{Mat}(4, \mathbb{Z})^2 \) is a maximal order containing \( \mathbf{A} \) and two torsion free \( \Gamma \)-modules belonging to the same genus are isomorphic. Applying the results of [3], we see that the isomorphism classes of modules belonging to the same genus as \( M \) are in one-to-one correspondence with the double cosets

\[
\text{Aut}(\mathbb{Q}M) \setminus \text{Aut}(\mathbb{Q}M_2) \times \text{Aut}(\mathbb{Q}M_3) / \text{Aut}(M_2) \times \text{Aut}(M_3),
\]

where \( \mathbb{Q}M \) denotes the \( \Gamma \)-submodule in \( \mathbb{Q}M \) generated by \( M \). As \( \mathbf{A}^6 \Gamma \), we can replace these cosets by

\[
\overline{\text{Aut}}(\mathbb{Q}M) \setminus \text{Aut}(\mathbb{Q}M/6\mathbb{Q}M) / \text{Aut}(M/6\mathbb{Q}M),
\]

where \( \overline{\text{Aut}}(\mathbb{Q}M) \) denotes the image of \( \text{Aut}(\mathbb{Q}M) \) in \( \text{Aut}(\mathbb{Q}M/6\mathbb{Q}M) \). But \( \text{End}(\mathbb{Q}M) \) is just a direct product of matrix algebras \( \text{Mat}(n, \mathbb{Z}) \), and any matrix invertible modulo 6 is the image of an invertible integer matrix. Hence, \( \overline{\text{Aut}}(\mathbb{Q}M) \) is isomorphic to \( \text{Aut}(\mathbb{Q}M/6\mathbb{Q}M) \), so every genus only contains one module up to isomorphism.

Using the arguments analogous to those of [3, Theorem 2.1], one gets the following corollary extending Theorem 3.2 to all cubic functors.

**Corollary 6.3.** 1. Cubic modules \( M, N \) are isomorphic if and only if \( M_{(p)} \simeq N_{(p)} \) for every prime \( p \).

2. Given a set \( \{N_p\} \) where \( N_p \) is a cubic \( \mathbb{Z}_{(p)} \)-module, there is a cubic module \( M \) such that \( M_{(p)} \simeq N_p \) for all \( p \) if and only if almost all of them are torsion free (maybe, zero) and \( \mathbb{Q}N_p \simeq \mathbb{Q}N_q \) for all \( p, q \). In this case \( \mathbb{Q}M = \mathbb{Q}N_p \) and \( tM = \bigoplus_p tN_p \).
It seems plausible that the analogous result is no more true for the polynomial functors of degree 4, but at the moment we do not have a corresponding counterexample.

7. One Conjecture

The descriptions of quadratic modules and of cubic 2-divisible modules as well as some other evidence give rise to the following conjecture concerning polynomial modules of higher degrees.

Put $\mathbb{Z}_{\leq p} = \mathbb{Z}[1/(p - 1)!]$ and denote by $A^{(p)}$ the subring of the direct product $\mathbb{Z}_{\leq p} \times \mathbb{Z}_{\leq p} \times \text{Mat}(2, \mathbb{Z}_{\leq p})^{p-1}$ consisting of all $(p + 1)$-tuples $(a, b, C^1, \ldots, C^{p-1})$

such that $a \equiv c^1_{11} \pmod{p}$, $b \equiv c^p_{22} \pmod{p}$ and $c^i_{22} \equiv c^{i+1}_{11} \pmod{p}$ for every $i = 1, \ldots, p - 2$.

Conjecture 7.1. The category $\mathcal{M}(p) = \mathbb{Z}_{\leq p} \text{-mod}^p$ is equivalent to the category of modules over $A^{(p)} \times \mathbb{Z}_{\leq p}^r$ for appropriate $r$ (depending on $p$).

As the ring $A^{(p)}$ fits the conditions of §3, it would give a complete description of such polynomial functors in strings and bands terms quite similar to that of Section §3. Moreover, easy calculations show that this conjecture would imply the following properties of such polynomial functors analogous to those of quadratic and 2-divisible cubic functors:

- Two modules from $\mathcal{M}(p)$ are isomorphic if and only if all their localizations are isomorphic.
- Given a set $\{N_q\}$ of polynomial functors ($N_q \in \mathbb{Z}_{(q)} \text{-mod}^p$, $q$ runs through all primes $\geq p$), there is a functor $M \in \mathcal{M}(p)$ such that $M_{(q)} \simeq N_q$ for all $q$ if and only if almost all $N_q$ are torsion free and $\mathbb{Q}N_q \simeq \mathbb{Q}N_{q'}$ for all $q, q'$. In this case $tM = \bigoplus_q tN_q$ and $\mathbb{Q}M \simeq \mathbb{Q}N_q$.
- Any functor from $\mathcal{M}(p)$ is either projective, or of projective dimension 1, or of infinite projective dimension.
- Any functor from $\mathcal{M}(p)$ has a periodic projective resolution of period $2p$.
- If a functor $F \in \mathcal{M}(p)$ is indecomposable and non-projective, its torsion free part is a direct sum of at most two irreducible torsion free modules.

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References

[1] V. Bondarenko. Representations of bundles of semichained sets and their applications. Algebra i Analiz 3, no. 5 (1991), 38–61. (English translation: St. Petersburg Math. J. 3 (1992), 973–996.)

[2] V. Dlab and C. M. Ringel. Indecomposable representations of graphs and algebras. Mem. Am. Math. Soc. 173 (1976).

[3] W. Dreckmann. Notes on periodicity, to appear.

[4] Y. Drozd. Adèles and integral representations. Izvestia Acad. Sci. USSR 33 (1969) 1080–1088. (English translation: Math. USSR Izvestija 3 (1969) 1019–1026.)

[5] Y. Drozd. Tame and wild matrix problems. In: Representations and quadratic forms, Kiev, 1979, 39–74. (English translation in: AMS Translations, v. 128, 1986, 31–55.)

[6] Y. Drozd. Finite modules over pure Noetherian algebras. Trudy Mat. Inst. Steklov Acad. Sci. USSR 183 (1990) 56–68. (English translation: Proc. Steklov Institute of Mathematics 183 (1991) 97–108.)

[7] Y. Drozd. Finitely generated quadratic modules. Manuscripta mathematica 104 (2000).

[8] S. Eilenberg and S. MacLane. On the groups $H(\pi, n)$, II. Ann. Math. 60 (1954), 49–139.

[9] K. W. Roggenkamp and V. Huber-Dyson. Lattices over Orders, I. Lecture Notes in Math. 115, Springer–Verlag, Berlin, 1970.

[10] C. M. Ringel and K. W. Roggenkamp. Indecomposable representations of orders and Dynkin diagrams. C. R. Math. Rep. Acad. Sci. Canada 1 (1978), 91–94.