JUMP-TYPE HUNT PROCESSES GENERATED BY LOWER BOUNDED SEMI-DIRICHLET FORMS

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Let $E$ be a locally compact separable metric space and $m$ be a positive Radon measure on it. Given a nonnegative function $k$ defined on $E \times E$ off the diagonal whose anti-symmetric part is assumed to be less singular than the symmetric part, we construct an associated regular lower bounded semi-Dirichlet form $\eta$ on $L^2(E; m)$ producing a Hunt process $X^0$ on $E$ whose jump behaviours are governed by $k$. For an arbitrary open subset $D \subset E$, we also construct a Hunt process $X^{D,0}$ on $D$ in an analogous manner. When $D$ is relatively compact, we show that $X^{D,0}$ is censored in the sense that it admits no killing inside $D$ and killed only when the path approaches to the boundary. When $E$ is a $d$-dimensional Euclidean space and $m$ is the Lebesgue measure, a typical example of $X^0$ is the stable-like process that will be also identified with the solution of a martingale problem up to an $\eta$-polar set of starting points. Approachability to the boundary $\partial D$ in finite time of its censored process $X^{D,0}$ on a bounded open subset $D$ will be examined in terms of the polarity of $\partial D$ for the symmetric stable processes with indices that bound the variable exponent $\alpha(x)$.

1. Introduction. Let $E$ be a locally compact separable metric space equipped with a metric $d$, $m$ be a positive Radon measure with full topological support and $k(x, y)$ be a nonnegative Borel measurable function on the space $E \times E \setminus$ diag, where diag denotes the diagonal set $\{(x, x) : x \in E\}$. A purpose of the present paper is to construct Hunt processes on $E$ and on its subsets with jump behaviors being governed by the kernel $k$ by using general results on a lower bounded semi-Dirichlet form on $L^2(E; m)$.

The inner product and the norm in $L^2(E; m)$ are denoted by $(\cdot, \cdot)$ and $\| \cdot \|$, respectively. Let $\mathcal{F}$ be a dense linear subspace of $L^2(E; m)$ such that $u \land 1 \in \mathcal{F}$ whenever $u \in \mathcal{F}$. A (not necessarily symmetric) bilinear form $\eta$ on $\mathcal{F}$ is called a lower bounded closed form if the following three conditions are satisfied: we set $\eta_\beta(u, v) = \eta(u, v) + \beta(u, v)$, $u, v \in \mathcal{F}$. There exists a $\beta_0 \geq 0$ such that:

(B.1) (lower boundedness); for any $u \in \mathcal{F}$, $\eta_{\beta_0}(u, u) \geq 0$.
(B.2) (sector condition); for any $u, v \in \mathcal{F}$,

$$|\eta(u, v)| \leq K \sqrt{\eta_{\beta_0}(u, u) \cdot \eta_{\beta_0}(v, v)}$$

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for some constant $K \geq 1$.

(B.3) (completeness); the space $\mathcal{F}$ is complete with respect to the norm $\eta^{1/2}(\cdot, \cdot)$ for some, or equivalently, for all $\alpha > \beta_0$.

For a lower bounded closed form $(\eta, \mathcal{F})$ on $L^2(E; m)$, there exist unique semi-groups $\{T_t; t > 0\}, \{\hat{T}_t; t > 0\}$ of linear operators on $L^2(E; m)$ satisfying

$$
(T_t f, g) = (f, \hat{T}_t g),
$$

(1.1)

$$
f, g \in L^2(E; m), \|T_t\| \leq e^{\beta_0 t}, \|\hat{T}_t\| \leq e^{\beta_0 t}, t > 0,
$$

such that their Laplace transforms $G_\alpha$ and $\hat{G}_\alpha$ are determined for $\alpha > \beta_0$ by

$$
G_\alpha f, \hat{G}_\alpha f \in \mathcal{F}, \quad \eta_\alpha(G_\alpha f, u) = \eta_\alpha(u, \hat{G}_\alpha f) = (f, u),
$$

$$
f \in L^2(E; m), u \in \mathcal{F}.
$$

See the first part of Section 3 for more details. $\{T_t; t > 0\}$ is said to be Markovian if $0 \leq T_t f \leq 1$, $t > 0$, whenever $f \in L^2(E; m), 0 \leq f \leq 1$. It was shown by Kunita [15] that the semigroup $\{T_t; t > 0\}$ is Markovian if and only if

$$
\eta(U u, u - U u) \geq 0 \quad \text{for any } u \in \mathcal{F},
$$

(1.2)

where $U u$ denotes the unit contraction of $u$: $U u = (0 \lor u) \land 1$. A lower bounded closed form $(\eta, \mathcal{F})$ on $L^2(E; m)$ satisfying (1.2) will be called a lower bounded semi-Dirichlet form on $L^2(E; m)$. The term "semi" is added to indicate that the dual semigroup $\{\hat{T}_t; t > 0\}$ may not be Markovian although it is positivity preserving. As we shall see in Section 3 for a lower bounded semi-Dirichlet form $\eta$ which is regular in the sense stated below, if the associated dual semigroup $\{\hat{T}_t; t > 0\}$ were Markovian, or equivalently, if $m$ were excessive, then $\eta$ is necessarily a nonnegative definite closed form, namely, $\beta_0$ in conditions (B.1), (B.3) [resp., (B.2)] can be retaken to be 0 (resp., 1).

A lower bounded semi-Dirichlet form $(\eta, \mathcal{F})$ is said to be regular if $\mathcal{F} \cap C_0(E)$ is uniformly dense in $C_0(E)$ and $\eta_\alpha$-dense in $\mathcal{F}$ for $\alpha > \beta_0$, where $C_0(E)$ denotes the space of continuous functions on $E$ with compact support. Carrillo-Menendez [8] constructed a Hunt process properly associated with any regular lower bounded semi-Dirichlet form on $L^2(E; m)$ by reducing the situation to the case where $\eta$ is nonnegative definite. We shall show in Section 4 that a direct construction is possible without such a reduction.

Later on, the nonnegative definite semi-Dirichlet form was investigated by Ma, Oberbeck and Röckner [16] and Fitzsimmons [10] specifically in a general context of the quasi-regular Dirichlet form and the special standard process. However, in producing the forms $\eta$ from nonsymmetric kernels $k$ corresponding to a considerably wide class of jump type Hunt processes in finite dimensions whose dual semigroups need not be Markovian, we will be forced to allow positive $\beta_0$.

To be more precise, we set for $x, y \in E, x \neq y$,

$$
k_\delta(x, y) := \frac{1}{2} \{k(x, y) + k(y, x)\} \quad \text{and} \quad k_\alpha(x, y) := \frac{1}{2} \{k(x, y) - k(y, x)\},
$$

(1.3)
that is, the kernel $k_s(x, y)$ denotes the symmetrized one of $k$, while $k_a(x, y)$ represents the anti-symmetric part of $k$. We impose four conditions $(2.1)$–$(2.4)$ on $k_s$ and $k_a$ stated below. Condition $(2.1)$ on $k_s$ is nearly optimal for us to work with the symmetric Dirichlet form $(1.4)$ defined below, while conditions $(2.2)$–$(2.4)$ require $k_a$ to be less singular than $k_s$.

Let conditions $(2.1)$–$(2.4)$ be in force on $k$. Denote by $C_{lip}^0(E)$ the space of uniformly Lipschitz continuous functions on $E$ with compact support. We also let

\[ E(u, v) := \int_{E \times E \setminus \text{diag}} (u(y) - u(x))(v(y) - v(x)) \times k_s(x, y) m(dx) m(dy), \]

\[ \mathcal{F}^r = \{ u \in L^2(E; m) : u \text{ is Borel measurable and } E(u, u) < \infty \}. \]

$(\mathcal{E}, \mathcal{F}^r)$ is a symmetric Dirichlet form on $L^2(E; m)$ and $\mathcal{F}^r$ contains the space $C_{lip}^0(E)$. We denote by $\mathcal{F}^0$ the $E_1$-closure of $C_{lip}^0(E)$ in $\mathcal{F}^r$. $(\mathcal{E}, \mathcal{F}^0)$ is then a regular Dirichlet form on $L^2(E; m)$ (cf. [13], Example 1.2.4, Theorem 3.1.1 and see also [23] and [24]).

For $u \in C_{lip}^0(E)$ and $n \in \mathbb{N}$, the integral

\[ \mathcal{L}^n u(x) := \int_{\{ y \in E : d(x, y) > 1/n \}} (u(y) - u(x))k(x, y) m(dy), \quad x \in E, \]

makes sense. We prove in Proposition 2.1 and Theorem 2.1 in Section 2 that the finite limit

\[ \eta(u, v) = - \lim_{n \to \infty} \int_E \mathcal{L}^n u(x) v(x) m(dx) \quad \text{for } u, v \in C_{lip}^0(E), \]

exists, $\eta$ extends to $\mathcal{F}^0 \times \mathcal{F}^0$ and $(\eta, \mathcal{F}^0)$ is a lower bounded semi-Dirichlet form on $L^2(E; m)$ with parameter $\beta_0 = 8(C_1 \lor C_2 C_3)(\geq 0)$ where $C_1$–$C_3$ are constants appearing in conditions $(2.2)$–$(2.4)$. Furthermore, the form $\mathcal{E}$ is shown to be a reference (symmetric Dirichlet) form of $\eta$ in the sense that, for each fixed $\alpha > \beta_0$,

\[ c_1 \mathcal{E}_1(u, u) \leq \eta(u, u) \leq c_2 \mathcal{E}_1(u, u), \quad u \in \mathcal{F}^0, \]

for some positive constants $c_1, c_2$ independent of $u \in \mathcal{F}^0$. Therefore, $(\eta, \mathcal{F}^0)$ becomes a lower bounded semi-Dirichlet form on $L^2(E; m)$ and gives rise to an associated Hunt process $X^0 = (X^0_t, P^0_x)$ on $E$. We call $X^0$ the minimal Hunt process associated with the form $\eta$. Equation $(1.6)$ indicates that the limit of $\mathcal{L}^n$ in $n$ plays a role of a pre-generator of $X^0$ informally.

If we define the kernel $k^*$ by

\[ k^*(x, y) := k(y, x), \quad x, y \in E, x \neq y, \]

and the form $\eta^*$ by $(1.5)$ and $(1.6)$ with $k^*$ in place of $k$, we have the same conclusions as above for $\eta^*$ (Corollary 2.1 of Section 2). In particular, there exists a minimal Hunt process $X^0_{\eta^*}$ associated with the form $\eta^*$. 


In the second half of Section 3, we are concerned with a killed dual semigroup \( \{e^{-\beta t}\tilde{T}_t; t > 0\} \), which can be verified to be Markovian for a large \( \beta > 0 \) but only for a restricted subfamily of the forms \( \eta \) considered in Section 2 (lower order cases). For a higher order \( \eta \), the killed dual semigroup may not be Markovian no matter how big \( \beta \) is. We shall also exhibit an example of a one-dimensional probability kernel \( k [\int_{\mathbb{R}^d} k(x, y) \, dy = 1] \) with \( m \) being the Lebesgue measure, for which the associated semi-Dirichlet form \( \eta \) is not nonnegative definite and accordingly the associated dual semigroup itself is non-Markovian.

When \( E = \mathbb{R}^d \) the \( d \)-dimensional Euclidean space and \( m(dx) = dx \) the Lebesgue measure on it, we shall verify in Section 5 that our requirements (2.1)–(2.4) on the kernel \( k(x, y) \) are fulfilled by

\[
k^{(1)}(x, y) = w(x)|x - y|^{-d-\alpha(x)},
\]
(1.9)
\[
k^{(1)*}(x, y) = w(y)|x - y|^{-d-\alpha(y)}, \quad x, y \in \mathbb{R}^d, x \neq y,
\]
for \( w(x) \) given by (5.1) and \( \alpha(x) \) satisfying the bounds (5.2). A Markov process corresponding to \( k^{(1)} \) is called a stable-like process and has been constructed by Bass [4] as a unique solution to a martingale problem. In this case, we shall prove that the minimal Hunt process associated with the corresponding form \( \eta \) is conservative and actually a solution to the same martingale problem, identifying it with the one constructed in [4] up to an \( \eta \)-polar set of starting points.

In Section 6, we consider an arbitrary open subset \( D \) of \( E \). Define \( m_D \) by \( m_D(B) = m(B \cap D) \) for any Borel set \( B \subseteq E \). By replacing \( E \) and \( m \) with \( D \) and \( m_D \), respectively, in (1.4), we obtain a symmetric Dirichlet form \( (\mathcal{E}_D, \mathcal{F}_D^0) \) on \( L^2(D; m_D) \). Denote by \( \overline{D} \) the closure of \( D \) and by \( C^\text{lip}_0(\overline{D}) \) the restriction to \( \overline{D} \) of the space \( C^\text{lip}_0(E) \). We also denote by \( C^\text{lip}_0(D) \) the space of uniformly Lipschitz continuous functions on \( D \) with compact support in \( D \). Let \( \mathcal{F}_D^0 \) and \( \mathcal{F}_D^1 \) be the \( \mathcal{E}_{D,1} \)-closures of \( C^\text{lip}_0(\overline{D}) \) and \( C^\text{lip}_0(D) \), respectively, in \( \mathcal{F}_D \). Then \( (\mathcal{E}_D, \mathcal{F}_D^0) \) is a regular symmetric Dirichlet form on \( L^2(D; m_D) \), while \((\mathcal{E}_D^0, \mathcal{F}_D^0)\) is a regular symmetric Dirichlet form on \( L^2(D; m_D) \) where \( \mathcal{E}_D^0 \) is the restriction of \( \mathcal{E}_D \) to \( \mathcal{F}_D^0 \times \mathcal{F}_D^0 \).

By making the same replacement in (1.5) and (1.6), we get a form \( \eta_D \) on \( C^\text{lip}_0(\overline{D}) \times C^\text{lip}_0(\overline{D}) \), which extends to \( \mathcal{F}_D^0 \times \mathcal{F}_D^0 \) to be a regular lower bounded semi-Dirichlet form on \( L^2(\overline{D}; m_D) \) possessing \( \mathcal{E}_D \) as its reference form, yielding an associated Hunt process \( X_{\overline{D}} \) on \( \overline{D} \). We also consider the restriction \( \eta_D^0 \) of \( \eta_D \) to \( \mathcal{F}_D^0 \times \mathcal{F}_D^0 \) so that \( (\eta_D^0, \mathcal{F}_D^0) \) is a regular lower bounded semi-Dirichlet form on \( L^2(D; m_D) \) possessing \( \mathcal{E}_D^0 \) as its reference form. We shall show in Section 6 that the part process \( X^{D,0} \) of \( X_{\overline{D}} \) on \( D \), namely, the Hunt process obtained from \( X_{\overline{D}} \) by killing upon hitting the boundary \( \partial D \), is properly associated with \( (\eta_D^0, \mathcal{F}_D^0) \).

We shall also prove in Section 6 that \( X_{\overline{D}} \) admits no jump from \( D \) to \( \partial D \), and furthermore when \( D \) is relatively compact, \( X_{\overline{D}} \) is conservative so that \( X^{D,0} \) admits
no killing inside $D$ and its sample path is killed only when it approaches to the boundary $\partial D$. $X^{D,0}$ is accordingly different from the part process of $X^0$ on the set $D$ in general because the sample path of $X^0$ may jump from $D$ to $E \setminus D$ resulting in a killing inside $D$ of its part process. By adopting $k^*$ instead of $k$, we get in an analogous manner Hunt processes $X^{D,*}$ on $\overline{D}$ and $X^{D,0,*}$ on $D$ satisfying the same properties as above.

When $(\mathcal{E}, \mathcal{F}^r)$ is the Dirichlet form on $L^2(\mathbb{R}^d)$ of a symmetric stable process on $\mathbb{R}^d$, the space $\mathcal{F}^0$ is identical with $\mathcal{F}^r$. In this case, for an arbitrary open set $D \subset \mathbb{R}^d$, the symmetric Hunt process on $D$ associated with $(\mathcal{E}_D^0, \mathcal{F}_D^0)$ is a censored stable process on $D$ in the sense of Bogdan, Burdzy and Chen [7]. It was further shown in [7] that, if $D$ is a $d$-set, then the space $\mathcal{F}_D^r$ coincides with $\mathcal{F}_D^r$ so that the symmetric Hunt process on $\overline{D}$ associated with $(\mathcal{E}_D, \mathcal{F}_D^r)$ was called a reflecting stable process over $\overline{D}$.

For the nonsymmetric kernel $k^{(1)}$ on $\mathbb{R}^d$ as (1.9), associated Hunt processes $X^{D,0}, X^{D,0,*}$ on an arbitrary open set $D \subset \mathbb{R}^d$ may well be called censored stable-like processes in view of the stated properties of them. However, it is harder in this case to identify the space $\mathcal{F}_D^r$ with $\mathcal{F}_D^r$, and accordingly we call the associated Hunt processes $X^{\overline{D}}, X^{\overline{D}}$ over $\overline{D}$ modified reflecting stable-like processes analogously to the Brownian motion case (cf. [11]). At the end of Section 6, we give sufficient conditions in terms of the upper and lower bounds of the variable exponent $\alpha(x)$ for the approachability in finite time of the censored stable-like processes to the boundary.

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2. Construction of a lower bounded semi-Dirichlet form from $k$. Throughout this section, we make the following assumptions on a nonnegative Borel measurable function $k(x, y)$ on $E \times E \setminus \text{diag}$:

$$M_s \in L^2_{\text{loc}}(E; m) \quad \text{for } M_s(x) = \int_{y \neq x} (1 \wedge d(x, y)^2) k_s(x, y) m(dy),$$

(2.1) \[ x \in E, \]

$$C_1 := \sup_{x \in E} \int_{d(x, y) \geq 1} |k_a(x, y)| m(dy) < \infty,$$

(2.2) \[ \text{and there exists a constant } \gamma \in (0, 1) \text{ such that} \]

$$C_2 := \sup_{x \in E} \int_{d(x, y) < 1} |k_a(x, y)|^\gamma m(dy) < \infty,$$

(2.3) \[ \text{and furthermore, for some constant } C_3 \geq 0, \]

$$|k_a(x, y)|^{2-\gamma} \leq C_3 k_s(x, y) \quad \text{for any } x, y \in E$$

(2.4) \[ \text{with } 0 < d(x, y) \leq 1. \]
For each $n \in \mathbb{N}$, define $L^n u$ for $u \in C^{\text{lip}}_0(E)$ by (1.5) and $\eta^n(u, v)$ for $u, v \in C^{\text{lip}}_0(E)$ by
\[
\eta^n(u, v) := - \int_E L^n u(x)v(x)m(dx),
\]
the integral on the right-hand side being absolutely convergent by (2.1). We note that any $u \in C^{\text{lip}}_0(E)$ belongs to the domain $F^\gamma$ of the form $\mathcal{E}$ defined by (1.4). In fact, if we denote by $K$ the support of $u$, then $\mathcal{E}(u, u)$ is dominated by twice the integral of $(u(x) - u(y))^2 k_s(x, y)m(dx)m(dy)$ on $K \times E$, which is finite by (2.1).

$\mathcal{E}(u, v)$ admits also an alternative expression for $u, v \in C^{\text{lip}}_0(E)$,
\[
\mathcal{E}(u, v) := \iint_{E \times E \setminus \text{diag}} (u(y) - u(x))(v(y) - v(x))k(x, y)m(dx)m(dy),
\]
because the right-hand side of the above can be seen to be equal to the same integral with $k(y, x)$ in place of $k(x, y)$ by interchanging the variables $x, y$, and we arrive at the expression in (1.4) by averaging. In particular, $\mathcal{E}(u, v) = \lim_{n \to \infty} \mathcal{E}^n(u, v)$ for $u, v \in C^{\text{lip}}_0(E)$ where
\[
\mathcal{E}^n(u, v) := \iint_{d(x, y) > 1/n} (u(y) - u(x))(v(y) - v(x))k(x, y)m(dx)m(dy).
\]

**Proposition 2.1.** Assume (2.1)–(2.4). Then for all $u, v \in C^{\text{lip}}_0(E)$, the limit
\[
\eta(u, v) = \lim_{n \to \infty} \eta^n(u, v)
\]
exists. Moreover, the limit has the following expression:
\[
\eta(u, v) = \frac{1}{2} \mathcal{E}(u, v) + \iint_{y \neq x} (u(x) - u(y))v(y)k_a(x, y)m(dx)m(dy),
\]
where $\mathcal{E}$ is defined by (1.4) and the integral on the right-hand side is absolutely convergent.

**Proof.** For $u, v \in C^{\text{lip}}_0(E)$, we have
\[
\eta^n(u, v) - \eta^n(v, u) = - \int_{d(x, y) > 1/n} (u(y) - u(x))v(x)k(x, y)m(dx)m(dy)
\]
\[+ \int_{d(x, y) > 1/n} (v(y) - v(x))u(x)k(x, y)m(dx)m(dy)
\]
\[= - \int_{d(x, y) > 1/n} u(y)v(x)k(x, y)m(dx)m(dy)
\]
\[+ \int_{d(x, y) > 1/n} v(y)u(x)k(x, y)m(dx)m(dy)
\]
\[= 2 \int_{d(x, y) > 1/n} u(x)v(y)k_a(x, y)m(dx)m(dy),
\]
and further
\[ \eta^n(u, v) + \eta^n(v, u) \]
\[ = - \iint_{d(x, y) \geq 1/n} (u(y) - u(x)) v(x) k(x, y) m(dx) m(dy) \]
\[ - \iint_{d(x, y) \geq 1/n} (v(y) - v(x)) u(x) k(x, y) m(dx) m(dy) \]
\[ = \iint_{d(x, y) \geq 1/n} (u(y) - u(x)) (v(y) - v(x)) k(x, y) m(dx) m(dy) \]
\[ - \iint_{d(x, y) \geq 1/n} (u(y) - u(x)) v(y) k(x, y) m(dx) m(dy) \]
\[ - \iint_{d(x, y) \geq 1/n} (v(y) - v(x)) u(x) k(x, y) m(dx) m(dy) \]
\[ = \mathcal{E}^n(u, v) - 2 \iint_{d(x, y) \geq 1/n} u(y) v(y) k_a(x, y) m(dx) m(dy). \]

By adding up the obtained identities, we get for \( u, v \in C_0^{\text{lip}}(E) \),
\[ 2\eta^n(u, v) = \mathcal{E}^n(u, v) + 2 \iint_{d(x, y) > 1/n} (u(x) - u(y)) v(y) \times k_a(x, y) m(dx) m(dy). \]
(2.8)

Since \( \mathcal{E}^n(u, v) \) converges to \( \mathcal{E}(u, v) \) as \( n \to \infty \), it remains to see that the second term of the right-hand side also converges absolutely as \( n \to \infty \) for each \( u, v \in C_0^{\text{lip}}(E) \).

From the Schwarz inequality and (2.2), we see that
\[
\iint_{d(x, y) > 1/n} |(u(x) - u(y)) v(y) k_a(x, y)| m(dx) m(dy) 
\leq \iint_{1/n < d(x, y) < 1} |u(x) - u(y)| \cdot |v(y)| |k_a(x, y)|^{\gamma/2} 
\times |k_a(x, y)|^{1-\gamma/2} m(dx) m(dy) 
+ \iint_{d(x, y) \geq 1} |u(x) - u(y)| \cdot |v(y)| k_s(x, y)^{1/2} |k_a(x, y)|^{1/2} m(dx) m(dy) 
\leq \sqrt{\iint_{1/n < d(x, y) < 1} (u(x) - u(y))^2 |k_a(x, y)|^{2-\gamma} m(dx) m(dy)} 
\times \sqrt{\iint_{1/n < d(x, y) < 1} v(y)^2 |k_a(x, y)|^{\gamma} m(dx) m(dy)} 
+ \sqrt{C_1 \|v\| \iint_{d(x, y) \geq 1} (u(x) - u(y))^2 k_s(x, y) m(dx) m(dy)}.
\]
So, by making use of assumptions (2.3) and (2.4) and an elementary inequality
\[ \sqrt{A} + \sqrt{B} \leq \sqrt{2} \sqrt{A + B} \]
holding for \( A \geq 0 \) and \( B \geq 0 \), we have
\[
\int \int_{d(x,y)>1/n} |(u(x) - u(y))v(y)k_d(x,y)|m(dx)m(dy) \\
\leq \sqrt{2} \sqrt{C_1 \vee C_2 C_3} ||v|| \cdot \sqrt{\mathcal{E}^n(u, u)}.
\]
Then taking \( n \to \infty \),
\[
\int \int_{y\neq x} |(u(x) - u(y))v(y)k_d(x,y)|m(dx)m(dy) \\
\leq \sqrt{2} \sqrt{C_1 \vee C_2 C_3} ||v|| \cdot \sqrt{\mathcal{E}(u, u)} < \infty
\]
as was to be proved. \( \square \)

For \( u, v \in C_0^{\text{lip}}(E) \), set
\[ \eta_\beta(u, v) = \eta(u, v) + \beta(u, v), \quad \beta > 0, \]
and
\[
B(u, v) := \int \int_{x \neq y} (u(x) - u(y))v(y)k_d(x,y)m(dx)m(dy).
\]
Then equation (2.7) reads
\[
\eta(u, v) = \frac{1}{2} \mathcal{E}(u, v) + B(u, v), \quad u, v \in C_0^{\text{lip}}(E),
\]
while we get from the proof of the preceding proposition
\[
|B(u, v)| \leq C_4 ||v|| \sqrt{\mathcal{E}(u, u)},
\]
where \( C_4 = \sqrt{2} \cdot \sqrt{C_1 \vee C_2 C_3} \). Now we put \( \beta_0 := 4(C_4)^2 = 8(C_1 \vee C_2 C_3) \).

From equation (2.10) and the bound (2.11), we have for \( u \in C_0^{\text{lip}}(E) \),
\[
\eta_{\beta_0}(u, u) = \frac{1}{4}\mathcal{E}_{\beta_0}(u, u) + \frac{1}{4}\mathcal{E}(u, u) + \frac{3}{4}\beta_0 ||u||^2 + B(u, u) \\
\geq \frac{1}{4}\mathcal{E}_{\beta_0}(u, u) + \sqrt{3}C_4 \sqrt{\mathcal{E}(u, u)} ||u|| + B(u, u) \geq \frac{1}{4}\mathcal{E}_{\beta_0}(u, u).
\]

Further, for \( u, v \in C_0^{\text{lip}}(E) \),
\[
|\eta(u, v)| \leq \frac{1}{2}|\mathcal{E}(u, v)| + |B(u, v)| \\
\leq \frac{1}{2}\sqrt{\mathcal{E}(u, u)} \sqrt{\mathcal{E}(v, v)} + C_4 ||v|| \sqrt{\mathcal{E}(u, u)} \\
\leq \frac{1}{2} (\sqrt{\mathcal{E}(v, v)} + 2C_4 ||v||) \sqrt{\mathcal{E}(u, u)} \\
\leq \frac{\sqrt{2}}{2} \sqrt{\mathcal{E}_{\beta_0}(v, v)} \sqrt{\mathcal{E}_{\beta_0}(u, u)}.
\]
So it also follows that

\[
|\eta(u, v)| \leq 2\sqrt{2}\sqrt{\frac{\eta_{\beta_0}(u, u)}{\eta_{\beta_0}(v, v)}}
\]

and

\[
\frac{1}{2}\mathcal{E}_{\beta_0}(u, u) \leq \eta_{\beta_0}(u, u) \leq 2 + \sqrt{2}\mathcal{E}_{\beta_0}(u, u), \quad u, v \in C_{0}^{\text{lip}}(E).
\]

Let \(\mathcal{F}^0\) be the \(\mathcal{E}_1\)-closure of \(C_{0}^{\text{lip}}(E)\) in \(\mathcal{F}\). Since \(\mathcal{F}^0\) is complete with respect to \(\mathcal{E}_\alpha\) for any \(\alpha > 0\), the estimates obtained above readily lead us to the first conclusion of the following theorem.

**Theorem 2.1.** Assume (2.1)–(2.4). Then the form \(\eta\) defined by Proposition 2.1 extends from \(C_{0}^{\text{lip}}(E) \times C_{0}^{\text{lip}}(E)\) to \(\mathcal{F}^0 \times \mathcal{F}^0\) to be a lower bounded closed form on \(L^2(E; m)\) satisfying (B.1)–(B.3) with \(\beta_0 = 8(C_1 \vee C_2 C_3)\), \(K = 2\sqrt{2}\) and possessing \((\mathcal{E}, \mathcal{F}^0)\) as a reference form in the sense of (1.7).

Furthermore, the pair \((\eta, \mathcal{F}^0)\) is a regular lower bounded semi-Dirichlet form on \(L^2(E; m)\).

We note that the above constant \(\beta_0\) is equal to 0 if \(k\) is symmetric: \(k(x, y) = k(y, x), (x, y) \in E \times E \setminus \text{diag}\).

**Proof of Theorem 2.1.** It suffices to prove the contraction property (1.2) for the present pair \((\eta, \mathcal{F}^0)\). We first show this for \(u \in C_{0}^{\text{lip}}(E)\). Note that \(Uu \in C_{0}^{\text{lip}}(E)\) and, for \(n \in \mathbb{N}\),

\[
\eta^n(Uu, u - Uu) = -\int\int_{d(x, y) > 1/n} (Uu(y) - Uu(x))(u(x) - Uu(x))k(x, y)m(dx)m(dy)
\]

\[
= \int\int_{\{d(x, y) > 1/n\} \cap \{x : u(x) \geq 1\}} (1 - Uu(y))(u(x) - 1)k(x, y)m(dx)m(dy)
\]

\[
- \int\int_{\{d(x, y) > 1/n\} \cap \{x : u(x) \leq 0\}} Uu(y)u(x)k(x, y)m(dx)m(dy)
\]

\[
\geq 0.
\]

Then, we have by Proposition 2.1

\[
\eta(Uu, u - Uu) = \lim_{n \to \infty} \eta^n(Uu, u - Uu) \geq 0.
\]

Following a method in [17], Lemma 4.9, we next prove (1.2) for any \(u \in \mathcal{F}^0\). Choose a sequence \(\{u_\ell\} \subset C_{0}^{\text{lip}}(E)\) which is \(\mathcal{E}_1\)-convergent to \(u\). Then

\[
\|Uu_\ell - Uu\| \to 0, \quad \ell \to \infty,
\]
because \( U \) is easily seen to be a continuous operator from \( L^2(E; m) \) to \( L^2(E; m) \). Fix \( \alpha > \beta_0 \). We then get from (1.7) the boundedness

\[
\sup_{\ell} \eta_\alpha(Uu_\ell, Uu_\ell) \leq C_2 \sup_{\ell} \mathcal{E}_1(u_\ell, u_\ell) < \infty.
\]

On the other hand, using the dual resolvent \( \hat{G}_\alpha \) associated with the lower bounded closed form \((\eta, \mathcal{F}^0)\), we see from equation (3.1) below that, for any \( g \in L^2(E; m) \),

\[
\eta_\alpha(Uu_\ell, \hat{G}_\alpha g) = (Uu_\ell, g) \to (Uu, g) = \eta_\alpha(Uu, \hat{G}_\alpha g), \quad \ell \to \infty.
\]

Since \( \{\hat{G}_\alpha g : g \in L^2(E, m)\} \) is \( \eta_\alpha \)-dense in \( \mathcal{F}^0 \), we can conclude by making use of the above \( \eta_\alpha \)-bound of \( \{Uu_\ell\} \) and the sector condition (B.2) that \( \{Uu_\ell\} \) is \( \eta_\alpha \)-weakly convergent to \( Uu \) as \( \ell \to \infty \). In particular, by the above \( \eta_\alpha \)-bound and (B.2) again, we have

\[
(2.15) \quad \eta_\alpha(Uu_\ell, u_\ell) \to \eta_\alpha(Uu, u), \quad \ell \to \infty.
\]

We consider the dual form \( \widehat{\eta} \) and the symmetrizing form \( \tilde{\eta} \) of \( \eta \) defined by

\[
\widehat{\eta}(u, v) = \eta(v, u), \quad \tilde{\eta}(u, v) = \frac{1}{2}(\eta(u, v) + \eta(v, u)), \quad u, v \in \mathcal{F}^0.
\]

In the same way as above, we can see that \( \{Uu_\ell\} \) converges as \( \ell \to \infty \) to \( Uu \) \( \tilde{\eta}_\alpha \)-weakly and consequently \( \tilde{\eta}_\alpha \)-weakly. Since \((\tilde{\eta}_\alpha, \mathcal{F}^0)\) is a nonnegative definite symmetric bilinear form, it follows that

\[
(2.16) \quad \eta_\alpha(Uu, Uu) = \tilde{\eta}_\alpha(Uu, Uu) \leq \liminf_{\ell \to \infty} \tilde{\eta}_\alpha(Uu_\ell, Uu_\ell) = \liminf_{\ell \to \infty} \eta_\alpha(Uu_\ell, Uu_\ell).
\]

We can then obtain (1.2) for \( u \in \mathcal{F}^0 \) from (2.14), (2.15) and (2.16) as

\[
\eta(Uu, u - Uu) \geq \lim_{\ell \to \infty} \eta(Uu_\ell, u_\ell) - \liminf_{\ell \to \infty} \eta(Uu_\ell, Uu_\ell) \\
= \limsup_{\ell \to \infty} \eta(Uu_\ell, u_\ell - Uu_\ell) \geq 0.
\]

For the kernel \( k^* \) defined by (1.8), we have obviously

\[
(2.17) \quad k^*_s(x, y) = k_s(x, y) \quad \text{and} \quad k^*_a(x, y) = -k_a(x, y), \quad x, y \in E, x \neq y.
\]

Hence, if the kernel \( k(x, y) \) satisfies (2.1)–(2.4), so does the kernel \( k^*(x, y) \). Define \( \eta^* \) as in Proposition 2.1 with \( k^*(x, y) \) in place of \( k(x, y) \). The same calculations made above for \( k(x, y) \) remain valid for \( k^*(x, y) \). Note also that the domain \( \mathcal{F}^{0*} \) is the same as \( \mathcal{F}^0 \) since the symmetric form \( \mathcal{E}^* \) defined by \( k^* \) is also the same as \( \mathcal{E} \). Thus, we can have the following corollary.

**Corollary 2.1.** Assume conditions (2.1)–(2.4) hold. Then the pair \((\eta^*, \mathcal{F}^0)\) is also a regular lower bounded semi-Dirichlet form on \( L^2(E; m) \).
3. Markov property of dual semigroups. First, we fix a general lower bounded closed form \((\eta, \mathcal{F})\) on \(L^2(E; m)\) satisfying (B.1)–(B.3) and make several remarks on it. The last condition (B.3) is equivalent to

\[(B.3)' \quad (\tilde{\eta}_\beta, \mathcal{F}) \text{ is a closed symmetric form on } L^2(E; m),\]

where \(\tilde{\eta}\) denotes the symmetrization of the form \(\eta : \tilde{\eta}(u, v) = \frac{1}{2}(\eta(u, v) + \eta(v, u)).\)

\(\eta_\beta\) is therefore a coercive closed form in the sense of [17], Definition 2.4, so that, by [17], Theorem 2.8, there exist uniquely two families of linear bounded operators \(\{G_\alpha\}_{\alpha > \beta_0}, \{\hat{G}_\alpha\}_{\alpha > \beta_0}\) on \(L^2(E; m)\) such that, for \(\alpha > \beta_0, G_\alpha(L^2(E; m)), \hat{G}_\alpha(L^2(E; m)) \subset \mathcal{F}\) and

\[\eta_\alpha(G_\alpha f, u) = (f, u) = \eta_\alpha(u, \hat{G}_\alpha f), \quad f \in L^2(E; m), u \in \mathcal{F}.\] (3.1)

In particular, \(G_\alpha\) and \(\hat{G}_\alpha\) are mutually adjoint:

\[\langle G_\alpha g, f \rangle = \langle g, \hat{G}_\alpha f \rangle, \quad f, g \in L^2(E; m), \alpha > \beta_0.\] (3.2)

We call \(\{G_\alpha; \alpha > \beta_0\}\) (resp., \(\{\hat{G}_\alpha; \alpha > \beta_0\}\)) the resolvent (resp., dual resolvent) associated with \((\eta, \mathcal{F})\).

Accordingly we see in exactly the same way as the proof of Theorem 2.8 of [17] that there exist strongly continuous contraction semigroups \(\{S_t; t > 0\}, \{\hat{S}_t; t > 0\}\) of linear operators on \(L^2(E; m)\) such that, for \(\alpha > 0, f \in L^2(E; m),\)

\[G_{\beta_0 + \alpha} f = \int_0^\infty e^{-\alpha t} S_t f \, dt, \quad \hat{G}_{\beta_0 + \alpha} f = \int_0^\infty e^{-\alpha t} \hat{S}_t f \, dt.\]

We then set \(T_t = e^{\beta_0 t} S_t, \quad \hat{T}_t = e^{\beta_0 t} \hat{S}_t\) to get strongly continuous semigroups \(\{T_t; t > 0\}, \{\hat{T}_t; t > 0\}\) satisfying

\[G_\alpha f = \int_0^\infty e^{-\alpha t} T_t f \, dt, \quad \hat{G}_\alpha f = \int_0^\infty e^{-\alpha t} \hat{T}_t f \, dt, \quad \alpha > \beta_0.\] (3.3)

as well as (1.1).

We call \(\{T_t; t > 0\}\) (resp., \(\{\hat{T}_t; t > 0\}\)) the semigroup (resp., dual semigroup) on \(L^2(E; m)\) associated with the lower bounded closed form \((\eta, \mathcal{F})\). We introduce the dual form \(\tilde{\eta}\) of \(\eta\) by

\[\tilde{\eta}(u, v) = \eta(v, u), \quad u, v \in \mathcal{F}.\]

Then \((\tilde{\eta}, \mathcal{F})\) is a lower bounded closed form on \(L^2(E; m)\) with which \(\{\hat{T}_t; t > 0\}\) and \(\{\hat{G}_\alpha; \alpha > \beta_0\}\) are the associated semigroup and resolvent, respectively.

Suppose \((\eta, \mathcal{F})\) is a lower bounded semi-Dirichlet form, namely, it satisfies the contraction property (1.2) additionally. As in the proof of the corollary to Theorem 4.1 of [15] or the proof of Theorem 4.4 of [17], we can then readily verify that the family \(\{\alpha G_\alpha; \alpha > \beta_0\}\) is Markovian, which is in turn equivalent to the Markovian property of \(\{T_t; t > 0\}\). Together with \(\{T_t; t > 0\}\), its Laplace transform then determines a bounded linear operator \(G_\alpha\) on \(L^\infty(E; m)\) for every \(\alpha > 0\)
and \( \{\alpha G_{\alpha}; \alpha > 0\} \) becomes Markovian. Further, \( \{\tilde{T}_t; t > 0\} \) is positivity preserving in view of (1.1).

Suppose additionally that \( (\eta, \mathcal{F}) \) is regular. Then the associated Markovian semigroup and resolvent can be represented by the transition function \( \{P_t; t > 0\} \) and the resolvent \( \{R_{\alpha}; \alpha > 0\} \) of the associated Hunt process \( X \) specified in Theorem 2 of the next section: \( P_t f = T_t f, t > 0, \) and \( R_{\alpha} f = G_{\alpha} f, \alpha > 0, \) for any \( f \in \mathcal{B}_b(E) \cap L^2(E; m) \). We call a \( \sigma \)-finite measure \( \mu \) on \( E \) excessive relative to \( X \) if \( \mu P_t \leq \mu \) for any \( t > 0 \). The next lemma was already observed in Silverstein [20].

**LEMMA 3.1.** Let \( \eta \) be a regular lower bounded semi-Dirichlet form on \( L^2(E; m) \).

(i) The following three conditions are mutually equivalent:

1. \( m \) is excessive relative to \( X \).
2. The dual semigroup \( \{\tilde{T}_t; t > 0\} \) is Markovian.
3. \( \eta(u - U u, U u) \geq 0 \) for any \( u \in \mathcal{F} \).

(ii) If one of the three conditions in (i) is satisfied, then \( \eta \) is nonnegative definite and the constant \( \beta_0 \) in conditions (B.1), (B.3) [resp., (B.2)] can be retaken to be 0 (resp., 1).

**Proof.** (i) 3 is the Markovian criterion (1.2) for the dual semigroup. If 2 is satisfied, then for any \( f \in L^2(E; m) \) with \( 0 \leq f \leq 1 \), \( 0 \leq \tilde{T}_t f \leq 1 \) so that \( (f, P_th) = (\tilde{T}_t f, h) \leq (1, h) \) for any \( h \in \mathcal{B}_+ \cap L^2(E; m) \), from which 1 follows. The converse can be shown similarly.

(ii) By the Schwarz inequality,

\[
(R_{\alpha} f(x))^2 \leq R_{\alpha} 1(x) R_{\alpha} f^2(x) \leq \frac{1}{\alpha} R_{\alpha} f^2(x), \quad x \in E, \ f \in \mathcal{B}_b(E) \cap L^2(E; m).
\]

Assuming 1 of (i), an integration with respect to \( m \) yields \( \alpha^2 \|G_{\alpha} f\|^2 \leq \|f\|^2 \), the \( L^2 \)-contraction property of \( \alpha G_{\alpha} \). In view of [17], Theorem 2.13, \( \eta(u, u) = \lim_{\alpha \to \infty} \alpha(u - \alpha G_{\alpha} u, u) \in \mathcal{F} \), which particularly implies that \( \eta(u, u) \geq 0, u \in \mathcal{F} \), and \( \{\eta_{\alpha}; \alpha > 0\} \) become equivalent on \( \mathcal{F} \).

We now return to the setting of the preceding section that \( (\eta, \mathcal{F}^0) \) is defined in terms of the kernel \( k \) satisfying conditions (2.1)–(2.4). By Proposition 2.1, \( \tilde{\eta}(u, v) = \frac{1}{2} \mathcal{E}(v, u) + B(v, u) \) where \( B \) is defined by (2.9) on \( \mathcal{F}^0 \times \mathcal{F}^0 \). On the other hand, we have from (2.17) that \( \eta^*(u, v) = \frac{1}{2} \mathcal{E}(u, v) - B(u, v) \) and consequently

\[
(3.4) \quad \tilde{\eta}(u, v) = \eta^*(u, v) + (B(u, v) + B(v, u)), \quad u, v \in \mathcal{F}^0.
\]

We know from Theorem 2.1 and Corollary 2.1 that both \( (\eta, \mathcal{F}^0) \) and \( (\eta^*, \mathcal{F}^0) \) are regular lower bounded semi-Dirichlet forms. In order to get a similar property
for the dual form \( \hat{\eta} \), we need to impose on the kernel \( k \) stronger conditions than (2.1)–(2.4) making the additional term on the right-hand side of (3.4) controllable.

In the rest of this section, we assume that the kernel \( k \) satisfies the condition

\[
M_s \in L^2_{\text{loc}}(E; m) \quad \text{for } M_s(x) = \int_{y \neq x} (1 \wedge d(x, y))k_s(x, y)m(dy),
\]

(3.5)

\( x \in E, \)

in place of (2.1), and further satisfies condition (2.2) as well as (2.3) for \( \gamma = 1 \) so that

\[
\frac{\beta_1}{2} := \sup_{x \in E} \int_{x \neq y} |k_d(x, y)|m(dy)
\]

(3.6)

\[
= \sup_{x \in E} \frac{1}{2} \int_{x \neq y} |k(x, y) - k(y, x)|m(dy) < \infty.
\]

Notice that condition (2.4) for \( \gamma = 1 \) is always satisfied with \( C_3 = 1 \).

Then the integrals

\[
\mathcal{L}u(x) = \int_{y \neq x} (u(y) - u(x))k(x, y)m(dy) \quad \text{and}
\]

(3.7)

\[
\mathcal{L}^*u(x) = \int_{y \neq x} (u(y) - u(x))k^*(x, y)m(dy),
\]

converge for \( u \in C^\text{lip}_0(E), x \in E \), and we get from Proposition 2.1 the identities

(3.8) \( \eta(u, v) = -(\mathcal{L}u, v), \quad \eta^*(u, v) = -(\mathcal{L}^*u, v), \quad u, v \in C^\text{lip}_0(E). \)

Furthermore,

\[
K(x) := 2 \int_{y \neq y} k_d(x, y)m(dy)
\]

(3.9)

\[
= \int_{y \neq x} (k(x, y) - k(y, x))m(dy), \quad x \in E,
\]

defines a bounded function on \( E \) and (3.4) readily leads us to

\[
\hat{\eta}(u, v) = \eta^*(u, v) + (u, K v), \quad u, v \in \mathcal{F}^0,
\]

which combined with (3.7) means that \( \widehat{\mathcal{L}} = \mathcal{L}^* - K \) is the formal adjoint of \( \mathcal{L} \). \( \hat{\eta} \) does not necessarily satisfy the contraction property (1.2), but the form

\[
\hat{\eta}_\beta(u, v) = \eta^*(u, v) + (u, (K + \beta) v), \quad \beta \geq \beta_1,
\]

does because so does the form \( \eta^* \) by Corollary 2.1 and \( K + \beta \geq 0 \) if \( \beta \geq \beta_1 \). So we have the following proposition.
PROPOSITION 3.1. Assume that (3.5) and (3.6) hold. Then $(\overline{\eta}_\beta, \mathcal{F}^0)$, which is the dual of $(\eta_\beta, \mathcal{F}^0)$, is a regular lower bounded semi-Dirichlet form on $L^2(E; m)$ provided that $\beta \geq \beta_1$.

This proposition means that, under conditions (3.5) and (3.6), $\{e^{-\beta t} \hat{T}_t; t > 0\}$ is Markovian for the dual semigroup $\{\hat{T}_t; t > 0\}$ associated with $\eta$ when $\beta \geq \beta_1$. If (3.6) fails, the dual semigroup of $\{e^{-\beta t} T_t; t > 0\}$ may not be Markovian no matter how large $\beta$ is.

A nonnegative Borel function $k$ on $E \times E$ is said to be a probability kernel if $\int_E k(x, y) m(dy) = 1, x \in E$. A probability kernel $k$ with the additional property $\sup_{x \in E} \int_D k(y, x) m(dy) < \infty$ (3.10) satisfies conditions (3.5) and (3.6) and $\eta$ defined by (3.8) yields a regular lower bounded semi-Dirichlet form on $L^2(E; m)$. We now give an example of such a kernel on $\mathbb{R}^1$ with $m$ being the Lebesgue measure for which the associated semi-Dirichlet form $\eta$ is not nonnegative definite so that, according to Lemma 3.1, the associated dual semigroup $\{\hat{T}_t, t > 0\}$ is not Markovian although $\{e^{-\beta t} \hat{T}_t; t > 0\}$ is Markovian for a large $\beta > 0$ in view of Proposition 3.1. A transition probability density function with respect to the Lebesgue measure of the one-dimensional Brownian motion with a mildly localized drift serves to be an example of such a kernel $k$.

Consider a diffusion $Y$ on $\mathbb{R}^1$ with generator $\mathcal{G}u = \frac{1}{2} u'' + \lambda b(x) u'$ where $\lambda$ is a positive constant and $b$ is a function in $C^1_C(\mathbb{R}^1)$ not identically 0. Then $\mathcal{G} = \frac{d}{dm} \cdot \frac{d}{ds}$ for

$$dm(x) = m(x) dx, \quad ds(x) = 2m(x)^{-1} dx,$$

where

$$m(x) = 2 \exp \left\{ 2 \lambda \int_0^x b(y) dy \right\},$$

namely, $Y$ is a diffusion with canonical scale $s$ and canonical (speed) measure $dm$.

The following facts about $Y$ are taken from [12]. Since $m(x)$ is bounded from above and from below by positive constants, both $\pm \infty$ are nonapproachable in the sense that $s(\pm \infty) = \pm \infty$. Therefore, $Y$ is recurrent and consequently conservative: $q_t(x, E) = 1, x \in E$, where $\{q_t; t > 0\}$ denotes the transition function of $Y$. $Y$ is $m$-symmetric and its Dirichlet form $(\mathcal{E}^Y, \mathcal{F}^Y)$ on $L^2(\mathbb{R}^1, m)$ is given by

$$\mathcal{E}^Y(u, v) = \frac{1}{2} \int_{\mathbb{R}^1} u'(x)v'(x)m(x) dx,$$

$$\mathcal{F}^Y = \{u \in L^2(\mathbb{R}^1; m): u \text{ is absolutely continuous and } \mathcal{E}^Y(u, u) < \infty \} (= H^1(\mathbb{R}^1)).$$
For \( u \in C^1_0(\mathbb{R}^1) \), \( \mathcal{E}^Y(u, \frac{u}{m}) \) is seen to be equal to \( \frac{1}{2} \int_{\mathbb{R}^1} ((u')^2 - 2\lambda bu'u) \, dx \) and so

\[
\mathcal{E}^Y(u, \frac{u}{m}) = \frac{1}{2} \left( \int_{\mathbb{R}^1} (u')^2 \, dx + \lambda \int_{\mathbb{R}^1} b'u^2 \, dx \right).
\]

There is a finite interval \( I \subset \mathbb{R}^1 \) where \( b' \) is strictly negative. Choose \( u_0 \in C^1_0(\mathbb{R}^1) \) not identically zero and with support being contained in \( I \). We can then make a choice of \( \lambda > 0 \) such that the right-hand side of the above equation is negative for \( u = u_0 \).

Since \( q_t \) maps \( L^2(\mathbb{R}^1; m) \) into \( \mathcal{F}^Y \subset C(\mathbb{R}^1) \), \( q_t(x, \cdot) \) is absolutely continuous with respect to \( m \) and hence with respect to the Lebesgue measure for each \( x \in \mathbb{R}^1 \). Denote by \( q_t(x, y) \) its density with respect to the Lebesgue measure so that

\[
\int_{\mathbb{R}^1} q_t(x, y) \, dy = 1, \quad x \in \mathbb{R}^1, \quad \text{with} \quad q_t(y, x) = m(x)q_t(x, y) \frac{1}{m(y)}.
\]

Equality (3.10) follows from (3.11).

4. Associated Hunt process and martingale problem. Let \((\eta, \mathcal{F})\) be a regular lower bounded semi-Dirichlet form on \( L^2(\mathbb{E}; m) \) as is defined in Section 1. For the symmetrization \( \tilde{\eta} \), \((\tilde{\eta}, \mathcal{F}^0)\) is then a closed symmetric form on \( L^2(\mathbb{E}; m) \) but not necessarily a symmetric Dirichlet form. A symmetric Dirichlet form \( \mathcal{E} \) on \( L^2(\mathbb{E}; m) \) with domain \( \mathcal{F} \) will be called a reference (symmetric Dirichlet) form of \( \eta \) if, for each fixed \( \alpha > \beta > 0 \),

\[
c_1 \mathcal{E}_1(u, u) \leq \eta(u, u) \leq c_2 \mathcal{E}_1(u, u), \quad u \in \mathcal{F},
\]

for some positive \( c_1, c_2 \) independent of \( u \in \mathcal{F} \). \( \mathcal{E} \) is then a regular Dirichlet form. In what follows, we assume that \( \eta \) admits a reference form \( \mathcal{E} \). This assumption is really unnecessary (cf. [16, 19]) but convenient to simplify some arguments. The regular lower bounded semi-Dirichlet form \((\eta, \mathcal{F}^0)\) constructed in Section 2 from a kernel \( k \) satisfying (2.1)–(2.4) has a reference form \((\mathcal{E}, \mathcal{F}^0)\) defined right after (1.4).

In formulating an association of a Hunt process with \( \eta \), Carrillo Menendez adopted a functional capacity theorem due to Ancona [2]. More specifically, denote by \( \mathcal{O} \) the family of all open sets \( A \subset \mathbb{E} \) with \( \mathcal{L}_A = \{u \in \mathcal{F}: \exists 1 \text{ m-a.e. on } A\} \neq \)

(4.2) \( e_A \in \mathcal{L}_A, \quad \eta_\alpha(e_A, w) \geq \eta_\alpha(e_A, e_A) \) for any \( w \in \mathcal{L}_A \).

A set \( N \subset E \) is called \( \eta \)-polar if there exist decreasing \( A_n \in \mathcal{O} \) containing \( N \) such that \( e_{A_n} \) is \( \eta_\alpha \)-convergent to 0 as \( n \to \infty \). A numerical function \( u \) on \( E \) is called \( \eta \)-quasi-continuous if there exist decreasing \( A_n \in \mathcal{O} \) such that \( e_{A_n} \) is \( \eta_\alpha \)-convergent to 0 as \( n \to \infty \) and \( u|_{E \setminus A_n} \) is continuous for each \( n \).

The capacity \( \text{Cap} \) for the reference form \( \mathcal{E} \) is defined by

\[
\text{Cap}(A) = \inf \{ \mathcal{E}_1(u, u) : u \in \mathcal{L}_A, A \in \mathcal{O} \}.
\]

It then follows from (4.1) that

\[
c_1 \text{Cap}(A) \leq \eta_\alpha(e_A, e_A) \leq c_2 K_\alpha^2 \text{Cap}(A), \quad A \in \mathcal{O},
\]

because (4.2) and (B.2) imply \( \eta_\alpha(e_A, e_A) \leq K_\alpha^2 \eta_\alpha(w, w), w \in \mathcal{L}_A \). Equation (4.3) means that a set \( N \) is \( \eta \)-polar iff it is \( \mathcal{E} \)-polar in the sense that \( \text{Cap}(N) = 0 \), and a function \( u \) is \( \eta \)-quasi-continuous iff it is \( \mathcal{E} \)-quasi-continuous in the sense that there exist decreasing \( A_n \in \mathcal{O} \) with \( \text{Cap}(A_n) \downarrow 0 \) as \( n \to \infty \) and \( u|_{E \setminus A_n} \) is continuous for each \( n \). Every element of \( \mathcal{F} \) admits its \( \eta \)-quasi-continuous \( m \)-version. If \( \{u_n\} \subset \mathcal{F} \) is \( \eta_\alpha \)-convergent to \( u \in \mathcal{F} \) and if each \( u_n \) is \( \eta \)-quasi-continuous, then (4.1) implies that a subsequence of \( \{u_n\} \) converges \( \eta \)-q.e., namely, outside some \( \eta \)-polar set, to an \( \eta \)-quasi-continuous version of \( u \). We shall occasionally drop \( \eta \) from the terms \( \eta \)-polar, \( \eta \)-q.e. and \( \eta \)-quasi-continuity for simplicity.

Recall that the \( L^2 \)-resolvent \( \{G_\alpha; \alpha > \beta_0\} \) associated with \( \eta \) determines the resolvent \( \{G_\alpha; \alpha > 0\} \) on \( L^\infty(E; m) \) with \( \|G_\alpha f\|_\infty \leq \frac{1}{\alpha} \|f\|_\infty, \quad \alpha > 0, \quad f \in L^\infty(E; m) \).

**Lemma 4.1.** Suppose \( G_\beta f \) admits a quasi-continuous \( m \)-version \( R_\beta f \) for a fixed \( \beta > \beta_0 \) and for every bounded Borel \( f \in L^2(E; m) \). Then, for any \( \alpha \) with \( 0 < \alpha \leq \beta_0 \) and for any bounded Borel \( f \in L^2(E; m) \),

\[
R_\alpha f(x) = \sum_{k=1}^{\infty} (\beta - \alpha)^{k-1} R_\beta^k f(x)
\]

converges q.e. and defines a quasi-continuous \( m \)-version of \( G_\alpha f \). Further the resolvent equation

\[
R_\alpha f - R_\beta f + (\alpha - \beta)R_\alpha R_\beta f = 0
\]

holds q.e. for any bounded Borel \( f \in L^2(E; m) \).
PROOF. Choose a regular nest \( \{ F_\ell \} \) so that \( R_k^{\beta} f \in C(\{ F_\ell \}) \) for \( k \geq 1 \). Define \( v_n(x) = \sum_{k=1}^{n} (\beta - \alpha)^{k-1} R_k^{\beta} f(x) \). By the resolvent equation for \( \{ G_\alpha ; \alpha > 0 \} \), we have

\[
G_\alpha f = v_n + (\beta - \alpha)^n G_\beta^n G_\alpha f.
\]

The \( L^\infty \)-norm of the second term of the right-hand side is dominated by \( \frac{1}{\alpha} (\frac{\beta - \alpha}{\beta})^n \| f \|_\infty \), which tends to 0 as \( n \to \infty \). Therefore, \( \{ v_n \} \) is convergent uniformly on each set \( F_\ell \) to a quasi-continuous version of \( G_\alpha f \). The resolvent equation is clear. \( \square \)

THEOREM 4.1. There exist a Borel \( \eta \)-polar set \( N_0 \subset E \) and a Hunt process \( X = (X_t, P_x) \) on \( E \setminus N_0 \) which is properly associated with \( (\eta, F) \) in the sense that \( R_\alpha f \) is a quasi continuous version of \( G_\alpha f \) for any \( \alpha > 0 \) and any bounded Borel \( f \in L^2(E; m) \). Here \( R_\alpha \) is the resolvent of \( X \) and \( G_\alpha \) is the resolvent associated with \( \eta \).

This theorem was proved in [8] first by assuming that \( \beta_0 = 0 \) and then reducing the situation to this case. Actually the proof can be carried out without such a reduction. Indeed, after constructing the kernel \( \hat{V}_\lambda \) of [8], Proposition II.2.1, for every rational \( \lambda > \beta_0 \) ([8], Proposition II.2.2) can be shown first for every rational \( \lambda > \beta_0 \), and then for every \( 0 < \lambda \leq \beta_0 \) by using Lemma 4.1. The rest of the arguments in [8] then works in getting to Theorem 4.1.

Our next concern will be exceptional sets and fine continuity for the Hunt process \( X = (X_t, P_x) \) appearing in Theorem 4.1. Denote by \( B(E) \) the family of all Borel sets of \( E \). For \( B \in B(E) \), we let

\[
\sigma_B = \inf\{ t > 0 : X_t \in B \}, \quad \sigma_{\bar{B}} = \inf\{ t > 0 : X_t \notin B \}, \quad \inf \emptyset = \infty.
\]

\( A \in B(E) \) is called \( X \)-invariant if

\[
P_x(\sigma_{E \setminus A} \land \sigma_{E \setminus A} < \infty) = 0 \quad \forall x \in A.
\]

\( N \in B(E) \) is called properly exceptional (with respect to \( X \)) if \( m(N) = 0 \) and \( E \setminus N \) is \( X \)-invariant.

A set \( N \subset E \) is called \( m \)-polar if there exists \( N_1 \supset N, N_1 \in B(E) \) such that \( P_m(\sigma_{N_1} < \infty) = 0 \). Any properly exceptional set is \( m \)-polar.

THEOREM 4.2.

(i) For \( A \in \mathcal{O} \), the function \( p_\alpha^A \) defined by \( p_\alpha^A(x) = E_x[e^{-\alpha \sigma_A}], x \in E \setminus N_0, \) is a quasi-continuous version of \( e_A, \alpha > \beta_0 \).

(ii) For any \( \eta \)-polar set \( B \), there exists a Borel properly exceptional set \( N \) containing \( N_0 \cup B \).
(iii) If $u$ is $\eta$-quasi-continuous, then there exists a Borel properly exceptional set $N \supset N_0$ such that, for any $x \in E \setminus N$,

\[ P_x \left( \lim_{i \uparrow t} u(X_{t_i}) = u(X_t) \forall t \geq 0 \text{ and } \lim_{i \uparrow t} u(X_{t_i}) = u(X_{t_-}) \forall t \in (0, \zeta) \right) = 1, \]

where $\zeta$ is the lifetime of $X$. In particular, $u$ is finely continuous with respect to the restricted Hunt process $X|_{E \setminus N}$.

(iv) Any $X$-semi-polar set is $\eta$-polar.

(v) A set $N \subset E$ is $\eta$-polar if and only if $N$ is $m$-polar.

**Proof.** (i) A function $u \in L^2(E; m)$ is said to be $\alpha$-excessive if $u \geq 0$, $\beta G_{\alpha \beta} u \leq u, \beta > 0$. A function $u \in \mathcal{F}$ is $\alpha$-excessive iff $\eta_\alpha(u, v) \geq 0$ for all non-negative $v \in \mathcal{F}$ (cf. [16], Theorem 2.4). In particular, $e_A$ is $\alpha$-excessive and further $v = e_A \wedge p^\alpha_0$ is an $\alpha$-excessive function in $\mathcal{F}$ (cf. [16], Theorem 2.6). Hence, $\eta_\alpha(v, e_A - v) \geq 0$. Since $v \in \mathcal{L}_A$, $\eta_\alpha(e_A, e_A - v) \leq 0$ so that $v = e_A$ and $e_A \leq p^\alpha_0$.

The converse inequality can be obtained as in the proof of Theorem 6.1 below by using the optional sampling theorem for a supermartingale but with time parameter set being a finite set.

Since the quasi-continuous function $\beta R_{\alpha + \beta} p^\alpha_A$ converges to $p^\alpha_A$ as $\beta \to \infty$ pointwise and in $\eta_\alpha$, we get the quasi-continuity of $p^\alpha_A$.

(ii) Choose a decreasing sets $A_n \in \mathcal{O}$ with $A_n \supset B$, $\text{Cap}(A_n) \to 0, n \to \infty$ and put $B_1 = \bigcap_n A_n$. By (4.1) and (i), $\lim_{n \to \infty} p^\alpha_{A_n} = 0$ q.e. so that

\[ P_x (\sigma_{B_1} \wedge \tilde{\sigma}_{B_1} < \infty) = 0, \quad x \in E \setminus N_1, \]

for some polar set $N_1$. Choose next a decreasing sets $A'_n \in \mathcal{O}$ containing $B_1 \cup N_1 \cup N_0$ with $\text{Cap}(A'_n) \to 0, n \to \infty$ and put $B_2 = \bigcap_n A'_n$. Then the above identity holds for $x \in E \setminus B_2$. Moreover, the above identity holds true for $B_2$ in place of $B_1$ and for some polar set $N_2$ in place of $N_1$. Repeating this procedure, we get an increasing sequence $\{B_k\}$ of $G_\delta$-sets which are polar sets such that

\[ P_x (\sigma_{B_k} \wedge \tilde{\sigma}_{B_k} < \infty) = 0, \quad x \in E \setminus B_{k+1}. \]

It then suffices to put $N = \bigcup_k B_k$.

(iii) Choose decreasing $A_n \in \mathcal{O}$ such that $\text{Cap}(A_n) \to 0, n \to 0$, and $u|_{E \setminus A_n}$ is continuous for each $n$. Let $N$ be a properly exceptional set constructed in (ii) starting with this sequence $\{A_n\}$. Then, for any $x \in E \setminus N$, $\lim_{n \to \infty} p^\alpha_{A_n}(x) = 0$ and consequently $P_x (\lim_{n \to \infty} \sigma_{A_n} = \infty) = 1$, which readily implies (4.4).

(iv) We reproduce a proof by Silverstein [20]. For $B \in \mathcal{B}(E)$, consider the entry time $\tilde{\sigma}_B = \inf \{t \geq 0 : X_t \in B\}$ and the function $\hat{p}^\alpha_B(x) = E_x[e^{-\alpha \tilde{\sigma}_B}], x \in E, \alpha > \beta_0$. Let $K$ be a compact thin set: $K$ admits no regular point relative to $X$. It suffices to show that $K$ is $\eta$-polar.

Choose relatively compact open sets $\{G_n\}$ such that $G_n \supset \overline{G}_{n+1}$ and $\bigcap_n G_n = K$. Due to the quasi-left continuity of $X$, $p^\alpha_{G_n}(x) = \hat{p}^\alpha_{G_n}(x)$ then decreases to
\( \hat{p}_K^\alpha(x) \) as \( n \to \infty \) for each \( x \in E \). By (i) and (4.1) and (4.2), the sequence \( \{ \hat{p}_G^\alpha_n \} \) is \( \mathcal{E}_1 \)-bounded so that the Cesàro mean sequence \( f_n \) of its suitable subsequence is \( \mathcal{E}_1 \)-convergent. Since \( f_n \) are quasi-continuous and converges to \( \hat{p}_K^\alpha \) pointwise as \( n \to \infty \), we conclude that \( \hat{p}_K^\alpha \) is a quasi-continuous element of \( \mathcal{F} \). On the other hand, the quasi-continuous function \( \beta R_{r^*} + \hat{p}_K^\alpha \) converges to \( p_K^\alpha \) as \( \beta \to \infty \) pointwise and in \( \eta_\alpha \) so that \( p_K^\alpha \) is also a quasi-continuous version of \( \hat{p}_K^\alpha \). Therefore, \( p_K^\alpha = \hat{p}_K^\alpha \) q.e. and in particular \( K \) is \( \eta \)-polar.

(v) “only if” part follows from (ii). To show “if” part, assume that \( K \) is a compact \( m \)-polar set. Then \( p_K^\alpha = 0 \) \( m \)-a.e. Choose for \( K \) relatively compact open sets \( \{ G_n \} \) as in the proof of (iv) so that the Cesàro mean \( f_\ell \) of a certain subsequence \( \{ p_{G_n \ell}^\alpha \} \) is \( \mathcal{E}_1 \)-convergent to \( p_K^\alpha \) as \( \ell \to \infty \) which is now a zero element of \( \mathcal{F}^0 \). Since \( f_\ell \geq 1 \) \( m \)-a.e. on \( G_n \ell \), we have \( \text{Cap}(K) \leq \text{Cap}(G_n \ell) \leq \mathcal{E}_1(f_\ell, f_\ell) \) and we get \( \text{Cap}(K) = 0 \) by letting \( \ell \to \infty \). For any Borel \( m \)-polar set \( N \), we have \( \text{Cap}(N) = \sup \{ \text{Cap}(K) : K \subset N, K \text{ is compact} \} = 0. \) \( \square \)

Clearly, the restriction of \( X \) outside its properly exceptional set is again a Hunt process properly associated with \( \eta \).

Our final task in this section is to relate the Hunt process of Theorem 4.1 to a martingale problem.

We consider the case where \( \eta \) admits the expression

\[
\eta(f, g) = -(\mathcal{L} f, g), \quad f \in \mathcal{D}(\mathcal{L}), g \in \mathcal{F},
\]

for a operator \( \mathcal{L} \) with domain \( \mathcal{D}(\mathcal{L}) \) satisfying the following:

(L.1) \( \mathcal{D}(\mathcal{L}) \) is a linear subspace of \( \mathcal{F} \cap C_0(E) \),

(L.2) \( \mathcal{L} \) is a linear operator sending \( \mathcal{D}(\mathcal{L}) \) into \( L^2(E; m) \cap C_b(E) \),

(L.3) there exists a countable subfamily \( \mathcal{D}_0 \) of \( \mathcal{D}(\mathcal{L}) \) such that each \( f \in \mathcal{D}(\mathcal{L}) \) admits \( f_n \in \mathcal{D}_0 \) such that \( f_n, \mathcal{L} f_n \) are uniformly bounded and converge pointwise to \( f, \mathcal{L} f \), respectively, as \( n \to \infty \).

We also consider an additional condition that

(L.4) there exists \( f_n \in \mathcal{D}(\mathcal{L}) \) such that \( f_n, \mathcal{L} f_n \) are uniformly bounded and converge to 1, 0, respectively, as \( n \to \infty \).

**Theorem 4.3.** Assume that \( \eta \) admits the expression (4.5) with \( \mathcal{L} \) satisfying conditions (L.1), (L.2), (L.3).

(i) There exists then a Borel properly exceptional set \( N \) containing \( N_0 \) such that, for every \( f \in \mathcal{D}(\mathcal{L}) \),

\[
M_t^{[f]} = f(X_t) - f(X_0) - \int_0^t (\mathcal{L} f)(X_s) \, ds, \quad t \geq 0,
\]

is a \( P_x \)-martingale for each \( x \in E \setminus N \).

(ii) If the additional condition (L.4) is satisfied, then the Hunt process \( X \big|_{E \setminus N} \) is conservative.
Proof. (i) Take \( f \in D(\mathcal{L}) \) and \( g \in L^2(E; m) \). By (4.5) and (3.2), we have, for \( \alpha > \beta_0 \),

\[
(G_\alpha \mathcal{L} f, g) = (\mathcal{L} f, \widehat{G}_\alpha g) = -\eta(f, \widehat{G}_\alpha g)
\]

\[
= -\eta_\alpha(f, \widehat{G}_\alpha g) + \alpha(f, \widehat{G}_\alpha g)
\]

\[
= -(f, g) + \alpha(G_\alpha f, g).
\]

Thus, \((G_\alpha \mathcal{L} f, g) = (\alpha G_\alpha f - f, g)\) holds for any \( g \in \mathcal{F} \) and

\[
\frac{1}{\alpha} G_\alpha(\mathcal{L} f)(x) = G_\alpha f(x) - \frac{f(x)}{\alpha}, \quad m\text{-a.e.}
\]

We denote by \( \{P_t; t \geq 0\} \) and \( \{R_\alpha; \alpha > 0\} \) the transition function and the resolvent of \( X \), respectively:

\[
P_t h(x) = \mathbb{E}_x[h(X_t)], \quad R_\alpha h(x) = \int_0^\infty e^{-\alpha t} P_t h(x) \, dt.
\]

Since \( X \) is properly associated with \( \eta \) by Theorem 4.1, we get

\[
\frac{1}{\alpha} R_\alpha(\mathcal{L} f)(x) = R_\alpha f(x) - \frac{f(x)}{\alpha}, \quad \text{q.e.}
\]

Hence, by virtue of Theorem 4.2(ii), there exists a Borel properly exceptional set \( N \) such that

\[
\int_0^\infty e^{-\alpha t} \left( \int_0^t P_s(\mathcal{L} f)(x) \, ds \right) \, dt = \int_0^\infty e^{-\alpha t} (P_t f(x) - f(x)) \, dt, \quad x \in E \setminus N,
\]

holds for any \( \alpha \in \mathbb{Q}_+ \) with \( \alpha > \beta_0 \) and for any \( f \in D_0 \).

Since \( P_t h(x) \) is a right continuous in \( t \geq 0 \) for any \( h \in C_b(E) \), we get

\[
(4.7) \quad P_t f(x) - f(x) = \int_0^t P_s(\mathcal{L} f)(x) \, ds, \quad t \geq 0, \quad x \in E \setminus N,
\]

holding for any \( f \in D_0 \). By virtue of condition (L.3), we conclude that the equation (4.7) holds true for any \( f \in D(\mathcal{L}) \). Equation (4.7) implies that, for any \( f \in D(\mathcal{L}) \), the functional \( M_{t[f]}^1, t \geq 0 \), defined by (4.6) is a mean zero, square integrable additive functional of the Hunt process \( X|_{E \setminus N} \) so that it is a \( P_x \)-martingale for each \( x \in E \setminus N \).

(ii) Under the additional condition (L.4), we let \( n \to \infty \) in equation (4.7) with \( f_n \) in place of \( f \) arriving at \( P_t 1 = 1, t \geq 0 \). \( \Box \)

Theorem 4.3 will enable us in the next section to relate our Hunt process to the solution of a martingale problem in a specific case.
5. Stable-like process. In this section, we consider the case that $E = \mathbb{R}^d$ and $m(dx) = dx$ is the Lebesgue measure on $\mathbb{R}^d$. For a positive measurable function $\alpha(x)$ defined on $\mathbb{R}^d$, Bass introduced the following integro-differential operator in [5] (see also [4, 6]): for $u \in C^2_b(\mathbb{R}^d)$,

$$L u(x) = w(x) \int_{h \neq 0} (u(x + h) - u(x) - \nabla u(x) \cdot h 1_B((h)) |h|^{-d - \alpha(x)}dh,$$

where $w(x)$ is a function chosen so that $L e^{iux} = -|u|\alpha(x) e^{iux}$ and $C^2_b(\mathbb{R}^d)$ denotes the set of twice-differentiable bounded functions. If $\alpha$ is Lipschitz continuous, bounded below by a constant which is greater than 0, and bounded above by a constant which is less than 2, then he constructed a unique strong Markov process associated with $L$ by solving the $L$-martingale problem for every starting point $x \in \mathbb{R}^d$. Using the theory of stochastic differential equation with jumps, Tsuchiya [22] also succeeded in constructing the Markov process associated with $L$ (see also [18]). Note that the weight function $w(x)$ is given by

$$w(x) = \frac{\Gamma((1 + \alpha(x))/2)\Gamma((\alpha(x) + d)/2)\sin(\pi \alpha(x)/2)}{2^{1-\alpha(x)}\pi^{d/2+1}}$$

(see, e.g., [3]).

Put $k(x, y) = w(x)|x - y|^{-d - \alpha(x)}$, $x, y \in \mathbb{R}^d$ with $x \neq y$. Then this falls into our case when we consider the following conditions: there exist positive constants $\underline{\alpha}, \overline{\alpha}, M$ and $\delta$ so that for $x, y \in \mathbb{R}^d$,

$$0 < \underline{\alpha} \leq \alpha(x) \leq \overline{\alpha} < 2, \overline{\alpha} < 1 + \frac{\alpha}{2} \quad \text{and} \quad |\alpha(x) - \alpha(y)| \leq M|x - y|^\delta$$

(5.2) for $\delta$ with $0 < \frac{1}{2} (2\overline{\alpha} - \underline{\alpha}) < \delta \leq 1$.

**Proposition 5.1.** Assume (5.2) holds. Then conditions (2.1)–(2.4) are satisfied by the function

$$k(x, y) = w(x)|x - y|^{-d - \alpha(x)}, \quad x, y \in \mathbb{R}^d, x \neq y.$$  

(5.3)

**Proof.** Note first that, from equation (5.1) defining the weight $w(x)$, we easily see that there exist constants $c_i$ ($i = 1, 2, 3$) so that for $x, y \in \mathbb{R}^d$,

$$c_1 \leq w(x) \leq c_2, \quad |w(x) - w(y)| \leq c_3|\alpha(x) - \alpha(y)|.$$

Then

$$k(x, y) = \frac{1}{2}(w(x)|x - y|^{-d - \alpha(x)} + w(y)|x - y|^{-d - \alpha(y)})$$

$$\leq \begin{cases} M|x - y|^{-d - \overline{\alpha}}, & |x - y| \leq 1, \\ M|x - y|^{-d - \underline{\alpha}}, & |x - y| > 1. \end{cases}$$
This and the condition $0 < \alpha \leq \alpha < 2$ imply that condition (2.1) is fulfilled because the function $M_s$ in it is bounded. Condition (2.2) is also valid as $|k_a(x, y)| \leq k_s(x, y)$.

On the other hand, since

$$k_a(x, y) = w(x)|x - y|^{-d - \alpha(x)} - w(y)|x - y|^{-d - \alpha(y)}$$
$$= (w(x) - w(y))|x - y|^{-d - \alpha(x)}$$
$$+ w(y)|x - y|^{-d}(|x - y|^{-\alpha(x)} - |x - y|^{-\alpha(y)})$$

and

$$|x - y|^{-\alpha(x)} - |x - y|^{-\alpha(y)} = \int_{\alpha(y)}^{\alpha(x)} |x - y|^{-u} \frac{1}{\ln|x - y| - 1} du,$$

we see that for $|x - y| < 1$,

$$|k_a(x, y)| \leq |w(x) - w(y)| \cdot |x - y|^{-d - \alpha(x)}$$
$$+ w(y)|x - y|^{-d}|\alpha(x) - \alpha(y)| \cdot |x - y|^{-((\alpha(x) + \alpha(y)) - 1)}$$
$$\leq M \left(|x - y|^{-d - \alpha + \delta} + |x - y|^{-d - \alpha + \delta} \frac{1}{\ln|x - y| - 1}\right)$$
$$\leq M' |x - y|^{-d - \alpha + \delta} \frac{1}{\ln|x - y| - 1}.$$ 

So if $\gamma$ satisfies

$$\gamma (d + \alpha - \delta) - (d - 1) < 1,$$

then condition (2.3) holds. As for condition (2.4), note that

$$k_s(x, y) \geq M' |x - y|^{-d - \alpha}, \quad |x - y| < 1.$$ 

So, (2.4) is valid when

$$(d + \alpha - \delta)(2 - \gamma) < d + \alpha.$$ 

Therefore, conditions (2.3) and (2.4) hold provided that $\gamma$ satisfies

$$\frac{d + 2\alpha - 2\delta - \alpha}{d + \alpha - \delta} < \gamma < \frac{d}{d + \alpha - \delta}.$$ 

Let $(\eta, F^0)$ be the regular lower bounded semi-Dirichlet form on $L^2(\mathbb{R}^d)$ associated with the kernel (5.3) satisfying (5.2) according to Theorem 2.1. Let $X = (X_t, P_x)$ be the Hunt process on $\mathbb{R}^d$ properly associated with $(\eta, F)$ by Theorem 4.1.
Define a linear operator \( L \) by
\[
\begin{cases}
\mathcal{D}(L) = C^2_0(\mathbb{R}^d), \\
L u(x) = \int_{h \neq 0} (u(x + h) - u(x) - \nabla u(x) \cdot h \mathbf{1}_{B_1(0)}(h)) \frac{w(x)}{|h|^{d + \alpha(x)}} dh, \\
x \in \mathbb{R}^d.
\end{cases}
\]

\( C^2_0(\mathbb{R}^d) \) is a linear subspace of \( \mathcal{F}^0 \cap C_0(\mathbb{R}^d) \) and, by condition (5.2), we can see that \( L \) maps \( C^2_0(\mathbb{R}^d) \) into \( L^2(\mathbb{R}^d) \cap C_b(\mathbb{R}^d) \). As any continuously differentiable function and its derivatives can be simultaneously approximated by polynomials and their derivatives uniformly on each rectangles (cf. [9], Chapter II), conditions (L.1), (L.2), (L.3) in the preceding section on \( L \) are fulfilled. We can easily verify that the present \( L \) satisfies condition (L.4) as well.

Since the vector valued function \( hw(x)1_{B_1(0)}(h)|h|^{-d - \alpha(x)} \) is odd with respect to the variable \( h \) for each \( x \in \mathbb{R}^d \), we get for \( u \in C^2_0(\mathbb{R}^d) \),
\[
\eta^n(u, v) = -\int \int_{|x - y| > 1/n} (u(y) - u(x)) v(x) \frac{w(x)}{|x - y|^{d + \alpha(x)}} dx dy
\]
\[
= -\int \int_{|h| > 1/n} (u(x + h) - u(x)) v(x) \frac{w(x)}{|h|^{d + \alpha(x)}} dx dh
\]
\[
= -\int \int_{|h| > 1/n} (u(x + h) - u(x) - \nabla u(x) \cdot h 1_{B_1(0)}(h)) v(x) \times \frac{w(x)}{|h|^{d + \alpha(x)}} dx dh.
\]

By letting \( n \to \infty \), we have
\[
\eta(u, v) = -(Lu, v),
\]
that is, \( \eta \) is related to \( L \) by (4.5).

By virtue of Theorem 4.3, there exists a Borel properly exceptional set \( N \subset \mathbb{R}^d \) so that \( X|\mathbb{R}^d \setminus N \) is conservative and, for each \( x \in \mathbb{R}^d \setminus N \),
\[
M^{|f|}_t = f(X_t) - f(X_0) - \int_0^t (Lf)(X_s) ds, \quad t \geq 0,
\]
is a martingale under \( P_x \) for every \( f \in C^2_0(\mathbb{R}^d) \). Approximating \( f \in C^2_b(\mathbb{R}^d) \) by a uniformly bounded sequence \( \{f_n\} \subset C^2_0(\mathbb{R}^d) \) such that \( \{Lf_n\} \) is uniformly bounded and convergent to \( Lf \), we see that (4.6) remains valid for \( f \in C^2_b(\mathbb{R}^d) \) and \( M^{|f|}_t \) is still a martingale under \( P_x \) for \( x \in \mathbb{R}^d \setminus N \). For each \( x \in \mathbb{R}^d \setminus N \), the measure \( P_x \) is thus a solution to the martingale problem for the operator \( L \) of (5.4) starting at \( x \) so that \( P_x \) coincides with the law constructed by Bass [5] because of the uniqueness also due to [5].
Remark 5.1. Let

\[ k^*(x, y) = \frac{w(y)}{|x - y|^{d+\alpha(y)}}, \quad x, y \in \mathbb{R}^d, x \neq y. \]

Under condition (5.2), the form \( \eta^* \) corresponding to the kernel \( k^* \) is a regular lower bounded semi-Dirichlet form on \( L^2(\mathbb{R}^d) \) by virtue of Proposition 5.1 and Corollary 2.1. By Theorem 4.1, \( \eta^* \) admits a properly associated Hunt process \( X^* \) on \( \mathbb{R}^d \). Furthermore, we can have an explicit expression

\[ \eta^*(u, v) = -(\mathcal{L}^* u, v) \]

for \( u \in C_0^2(\mathbb{R}^d) \) and \( v \in \mathcal{F}^0 \) with

\[
\mathcal{L}^* u(x) = \int_{h \neq 0} (u(x + h) - u(x) - \nabla u(x) \cdot h 1_{B_1(0)}(h)) \frac{w(x + h)}{|h|^{d+\alpha(x+h)}} dh \\
+ \frac{1}{2} \int_{0<|h|<1} \nabla u(x) \cdot h \left( \frac{w(x + h)}{|h|^{d+\alpha(x+h)}} - \frac{w(x - h)}{|h|^{d+\alpha(x-h)}} \right) dh, \quad x \in \mathbb{R}^d.
\]

In a lower order case as is considered in Section 3, both \( \mathcal{L} \) and \( \mathcal{L}^* \) admit simpler expressions (3.7) and \( \mathcal{L}^* - K \) is a formal adjoint of \( \mathcal{L} \) for a function \( K \) defined by (3.9).

6. Associated Hunt processes on open subsets and on their closures. We make the same assumptions on \( E, m, k \) as in Section 2. Let \( D \) be an arbitrary open subset of \( E \) and \( \overline{D} \) be the closure of \( D \), \( m_D \) is defined to be \( m_D(B) = m(B \cap D), B \in \mathcal{B}(E) \) and \( (u, v)_D \) denotes the inner product of \( L^2(D, m_D) \) (=\( L^2(\overline{D}, m_D) \)). Consider the related function spaces \( C_{lip}^0(\overline{D}) \) and \( C_{lip}^0(D) \) introduced in Section 1. Define

\[
\begin{align*}
\mathcal{E}_D(u, v) & := \int_{D \times D \setminus \text{diag}} (u(y) - u(x))(v(y) - v(x)) \times k_s(x, y)m_D(dx)m_D(dy), \\
\mathcal{F}^0_D & = \{ u \in L^2(D; m_D) : u \text{ is Borel measurable and } \mathcal{E}_D(u, u) < \infty \},
\end{align*}
\]

and let \( \mathcal{F}_{\overline{D}}^0 \) and \( \mathcal{F}_D^0 \) be the \( \mathcal{E}_{D,1} \)-closures of \( C_{lip}^0(\overline{D}) \) and \( C_{lip}^0(D) \) in \( \mathcal{F}^0_D \), respectively. \( (\mathcal{E}_D, \mathcal{F}_D^0) [\text{resp., } (\mathcal{E}_D, \mathcal{F}_{\overline{D}}^0)] \) is a regular symmetric Dirichlet form on \( L^2(\overline{D}, m_D) \) [resp., \( L^2(D; m_D) \)] where \( \mathcal{E}^0_D \) denotes the restriction of \( \mathcal{E}_D \) to \( \mathcal{F}^0_D \times \mathcal{F}^0_D \). Furthermore, in view of [13], Theorem 4.4.3, we have the identity

\[ \mathcal{F}^0_D = \{ u \in \mathcal{F}_{\overline{D}}^0 : \tilde{u} = 0, \mathcal{E}_D\text{-q.e. on } \partial D \}, \]

where \( \tilde{u} \) denotes an \( \mathcal{E}_D \)-quasi continuous version of \( u \in \mathcal{F}_{\overline{D}}^0 \). We keep in mind that a subset of \( D \) is polar for \( (\mathcal{E}_D, \mathcal{F}^0_D) \) if it is for \( (\mathcal{E}_D, \mathcal{F}_{\overline{D}}^0) \), and the restriction to \( D \) of a quasi continuous function with respect to the latter is quasi-continuous with respect to the former.

Now define for \( u \in C_{lip}^0(\overline{D}) \) and \( n \in \mathbb{N} \)

\[
\mathcal{L}^0_D u(x) := \int_{\{ y \in D : d(x, y) > 1/n \}} (u(y) - u(x))k(x, y)m_D(dy), \quad x \in D.
\]
Then, just as in Proposition 2.1 and Theorem 2.1 of Section 2, we conclude that the finite limit
\begin{equation}
\eta_D(u, v) = -\lim_{n \to \infty} \int_D L_n^D u(x)v(x)m_D(dx)
\end{equation}
exists, \( \eta_D \) extends to \( \overline{\mathcal{F}}_D \times \overline{\mathcal{F}}_D \) and \((\eta_D, \overline{\mathcal{F}}_D)\) becomes a regular lower bounded semi-Dirichlet form on \( L^2(\overline{D}; m_D) \) possessing \((\mathcal{E}_D, \mathcal{F}_D)\) as its reference symmetric Dirichlet form. In parallel with \((\eta_D, \overline{\mathcal{F}}_D)\), the space \((\eta_0^D, \overline{\mathcal{F}}_D)\) becomes a regular lower bounded semi-Dirichlet form on \( L^2(D; m_D) \) possessing \((\mathcal{E}_0^D, \mathcal{F}_0^D)\) as its reference symmetric Dirichlet form. Here \( \eta_0^D \) is the restriction of \( \eta_D \) to \( \overline{\mathcal{F}}_D \times \overline{\mathcal{F}}_D \).

Let \( X^\overline{D} = (X_t, P_x) \) be a Hunt process on \( \overline{D} \) properly associated with the form \((\eta_D, \overline{\mathcal{F}}_D)\) on \( L^2(\overline{D}; m_D) \). Denote by \( X^{D,0} = (X^{D,0}_t, P_x) \) the part process of \( X^\overline{D} \) on \( D \), namely, \( X^{D,0}_t \) is obtained from \( X_t \) by killing upon hitting the boundary \( \partial D \):
\begin{align*}
X^{D,0}_t &= X_t, \quad t < \sigma_{\partial D}; \\
X^{D,0}_t &= \Delta, \quad t \geq \sigma_{\partial D}.
\end{align*}

\( X^{D,0} \) is a Hunt process with state space \( D \).

**Theorem 6.1.** The part process \( X^{D,0} \) of \( X^\overline{D} \) on \( D \) is properly associated with the regular lower bounded semi-Dirichlet form \((\eta_0^D, \overline{\mathcal{F}}_D)\) on \( L^2(D; m_D) \).

**Proof.** Let \( \{R_\alpha; \alpha > 0\} \) be the resolvent of \( X^\overline{D} \). \( \sigma \) will denote the hitting time of \( \partial D \) by \( X^\overline{D} \) : \( \sigma = \sigma_{\partial D} \). Put, for \( \alpha > 0 \) and \( x \in \overline{D} \),
\begin{align*}
R^{D,0}_\alpha f(x) &= E_x \left[ \int_0^\sigma e^{-\alpha t} f(X_t) dt \right], \\
H^{D,0}_\alpha u(x) &= E_x [e^{-\alpha \sigma} u(X_\sigma)], \quad x \in \overline{D},
\end{align*}

\( \{R^{D,0}_\alpha |_{D}; \alpha > 0\} \) is the resolvent of the part process \( X^{D,0} \) of \( X^\overline{D} \) on \( D \).

We need to prove that, for any \( \alpha > \beta_0 \) and any \( f \in \mathcal{B}(\overline{D}) \cap L^2(\overline{D}, m_D) \),
\begin{equation}
R^{D,0}_\alpha f \text{ is } \eta_0^D \text{-quasi-continuous,}
\end{equation}
\begin{equation}
R^{D,0}_\alpha f \in \overline{\mathcal{F}}_D, \quad \eta_0^D(\alpha R^{D,0}_\alpha f, v) = (f, v)_D \quad \text{for any } v \in \mathcal{F}_D^0.
\end{equation}

We denote by \( \mathcal{G} \) the space appearing in the right-hand side of (6.2). Notice that \( \mathcal{E}_D \)-q.e. (resp., \( \mathcal{E}_D \)-quasi-continuity) is now a synonym of \( \eta_D \)-q.e. (resp., \( \eta_D \)-quasi-continuity). As the set of points of \( \partial D \) that are irregular for \( \partial D \) is known to be semi-polar, we have \( P_x(\sigma = 0) = 1 \) and so \( R^{D,0}_\alpha f(x) = 0 \) for \( \eta_D \)-q.e. \( x \in \partial D \) owing to Theorem 4.2(iv). Since
\begin{align*}
R_\alpha f \text{ is } \eta_D \text{-quasi-continuous,}
R_\alpha f \in \mathcal{F}_D, \quad \eta_{D,\alpha}(R_\alpha f, v) = (f, v)_D \quad \text{for any } v \in \mathcal{F}_D.
\end{align*}
and
\( R_\alpha f(x) = R_{\alpha}^{D,0} f(x) + H_\alpha^{\partial D} R_\alpha f(x) \), \( x \in \overline{D} \),

we see that, for the proof of (6.5), it is enough to show that
\[
H_\alpha^{\partial D} R_\alpha f \text{ is } \eta_D\text{-quasi-continuous},
\]
(6.7)
\[
H_\alpha^{\partial D} R_\alpha f \in F_{\overline{D}} \qquad \eta_{D,\alpha}(H_\alpha^{\partial D} R_\alpha f, v) = 0 \quad \text{for any } v \in G.
\]

To this end, we fix \( \alpha > \beta_0 \), \( f \in B_+^{D} \cap L^2(D; m_D) \) and put \( u = R_\alpha f \). Consider a closed convex subset of \( F_{\overline{D}} \) defined by
\[
L_{u,\partial D} = \{ v \in F_{\overline{D}}, \tilde{v} \geq \tilde{u} \text{ q.e. on } \partial D \}.
\]

Let \( u_\alpha \) be the \( \eta_{D,\alpha}\)-projection of 0 on \( L_{u,\partial D} \):
\[
u_{u,\alpha}(u_\alpha, v - u_\alpha) \geq 0, \quad \text{for any } v \in L_{u,\partial D}.
\]

Both \( u \) and \( u_\alpha \) are \( \alpha \)-excessive elements of \( F_{\overline{D}} \). By making use of the function \( v = u_\alpha \wedge u \) as in the proof of Proposition 3.1(i), we readily get
\[
\tilde{u}_\alpha = u \text{ q.e. on } \partial D, \quad \eta_{D,\alpha}(u_\alpha, v) = 0 \quad \text{for any } v \in G.
\]
(6.8)

Finally, we prove that
\[
H_\alpha^{\partial D} u \text{ is } \eta_D\text{-quasi continuous,} \quad H_\alpha^{\partial D} u = u_\alpha,
\]
which leads us to the desired property (6.7). By (6.6), \( H_\alpha^{\partial D} u \) is an \( \alpha \)-excessive function dominated by \( u \in F_{\overline{D}} \) so that \( H_\alpha^{\partial D} u \) is a quasi-continuous element of \( F_{\overline{D}} \). Further \( H_\alpha^{\partial D} u = u \text{ q.e. on } \partial D \) by (6.6) and an observation made preceding it.

Let \( v = H_\alpha^{\partial D} u \wedge u_\alpha \). Then \( \tilde{v} = H_\alpha^{\partial D} u \wedge \tilde{u}_\alpha = u \text{ q.e. on } \partial D \) so that \( \eta_{D,\alpha}(u_\alpha, u_\alpha - v) = 0 \) by (6.8). On the other hand, \( v \) is \( \alpha \)-excessive and so \( \eta_{D,\alpha}(v, u_\alpha - v) \geq 0 \). Consequently, \( \eta_{\alpha}(u_\alpha - v, u_\alpha - v) \leq 0 \) and we get the inequality \( u_\alpha \leq H_\alpha^{\partial D} u \).

To get the converse inequality, consider a bounded nonnegative Borel function \( h \) on \( D \) with \( \int_D h dm = 1 \). Denote by \( \{ p_t; t \geq 0 \} \) the transition function of \( X_{\overline{D}} \). We choose a Borel measurable quasi-continuous version \( \tilde{u}_\alpha \) of \( u_\alpha \in F_{\overline{D}} \). We set \( \tilde{u}_\alpha(\Delta) = 0 \) for the cemetery \( \Delta \) of \( X_{\overline{D}} \). Since \( u_\alpha \) is \( \alpha \)-excessive, \( e^{-\alpha t} p_t \tilde{u}_\alpha \leq \tilde{u}_\alpha \text{ m-a.e.} \), and we can see that the process \( \{ Y_t = e^{-\alpha t} \tilde{u}_\alpha(X_t); t \geq 0 \} \) is a right continuous positive supermartingale under \( P_{h,m} \) in view of Theorem 4.2(iii). For any compact set \( K \subset \partial D \), we get from the optional sampling theorem and (6.8),
\[
E_{h,m}[Y_{\sigma_K}] = E_{h,m}[e^{-\alpha \sigma_K} \tilde{u}_\alpha(X_{\sigma_K})] \\
= E_{h,m}[e^{-\alpha \sigma_K} u(X_{\sigma_K})] \leq E_{h,m}[Y_0] \\
= (h, u_\alpha)_D.
\]

By choosing \( K \) such that \( \sigma_K \downarrow \sigma \text{ in } P_{h,m}\text{-a.e.} \), we obtain \( (h, H_\alpha^{\partial D} u)_D \leq (h, u_\alpha)_D \) and \( H_\alpha^{\partial D} u \leq u_\alpha \). □
As a preparation for the next lemma, we take any open set \( G \subset D \) and denote by \( m_G \) the restriction of \( m \) to \( G \). Let \( \mathcal{F}_G^0 \) be the \( \mathcal{E}_{D,1} \)-closure of \( \mathcal{C}_1^0(G) \) in \( \mathcal{F}_D \) and \( \eta_G^0 \) be the restriction of \( \eta_D \) to \( \mathcal{F}_G^0 \times \mathcal{F}_G^0 \). Then, just as above,

\[
\mathcal{F}_G^0 = \{ u \in \mathcal{F}_D : \tilde{u} = 0 \ \text{q.e. on} \ \overline{D} \setminus G \}
\]

and \( (\eta_G^0, \mathcal{F}_G^0) \) becomes a regular lower bounded semi-Dirichlet form on \( L^2(G; m_G) \) with which the part process \( X^G,0 \) on \( G \) is properly associated. The resolvent of \( X^G,0 \) will be denoted by \( R_{\alpha}^{G,0} \).

Define

\[
H_{\alpha}^{\tilde{D} \setminus G} u(x) = E_x[e^{-\alpha \tau_G} v(X_{\tau_G})], \quad x \in \overline{D}.
\]

As (6.7), we have, for \( u = R_{\alpha} f, f \in \mathcal{B}(\overline{D}) \cap L^2(\overline{D}; m_D), \alpha > \beta_0, \)

\[
H_{\alpha}^{\tilde{D} \setminus G} u \in \mathcal{F}_D, \quad \eta_D,\alpha(H_{\alpha}^{\tilde{D} \setminus G} u) = 0 \quad \text{for any} \ v \in \mathcal{F}_G^0,
\]

and the bound \( \eta_D,\alpha(H_{\alpha}^{\tilde{D} \setminus G} u) \leq \eta_D,\alpha(u, u) \). We can easily see that (6.10) holds true for any \( u \in \mathcal{F}_D \cap C_0(\overline{D}) \) where \( C_0(\overline{D}) \) denotes the restrictions to \( \overline{D} \) of functions in \( C_0(E) \). In fact, by the resolvent equation, (6.10) is true for \( R_{\beta} u, \beta > \beta_0 \), in place of \( u \). Since \( \{ \beta_n R_{\beta_n} u \} \) converges to \( u \) pointwise as well as in \( \eta_D,\alpha \)-metric as \( \beta_n \to \infty \), so does the sequence \( \{ \beta_n H_{\alpha}^{\tilde{D} \setminus G} R_{\beta_n} u \} \), arriving at the validity of (6.10) for such \( u \).

**Lemma 6.1.** Let \( G \) be a relatively compact open set with \( \overline{G} \subset D \). Then for any \( v \in \mathcal{F}_D \cap C_0(\overline{D}) \) with \( \text{supp}[v] \subset \overline{D} \setminus \overline{G} \), it follows for \( \alpha > \beta_0 \) that

\[
E_x[e^{-\alpha \tau_G} v(X_{\tau_G})] = R_{\alpha}^{G,0} g_v(x) \quad \text{for q.e.} \ x \in G,
\]

where \( \tau_G = \sigma_{\tilde{D} \setminus G} \wedge \xi \) is the first leaving time from \( G \) and \( g_v \) is a function given by

\[
g_v(x) = 1_G(x) \int_{\overline{D} \setminus \overline{G}} k(x, y)v(y)m_D(dy), \quad x \in \overline{D}.
\]

**Proof.** Take any \( u \in \mathcal{F}_D \cap C_0(\overline{D}) \) such that \( \text{supp}[u] \subset G \). From (6.3) and (6.4), we then have

\[
\eta_D(u, v) = -\int_{G \times (\overline{D} \setminus \overline{G})} u(y)v(x)k(x, y)m_D(dx)m_D(dy).
\]

We can now proceed as in [13], page 163. The function \( g_v \) defined by (6.12) belongs to \( L^2(G; m_G) \) on account of condition (2.1) on the kernel \( k \). Therefore, we
obtain from (6.13)
\[
\eta_{G,\alpha}^0(R^{G,0}_\alpha g_v, u) = \int_G g_v(x)u(x)m_G(dx)
\]
\[
= \int_{G \times (\overline{D}\setminus G)} u(x)v(y)k(x, y)m_D(dx)m_D(dy)
\]
\[
= -\eta_D(v, u) = -\eta_{D,\alpha}(v, u)
\]
\[
= -\eta_{G,\alpha}^0(v - H^{\overline{D}\setminus G}_\alpha v, u), \quad \alpha > \beta_0.
\]
the last identity being a consequence of (6.10). Since \(\mathcal{F}^{\overline{D}} \cap C_0(G)\) is \(\eta_{G,\alpha}^0\)-dense in \(\mathcal{F}_G^0\), we get
\[
H^{\overline{D}\setminus G}_\alpha v(x) = H^{\overline{D}\setminus G}_\alpha v(x) - v(x) = R^{G,0}_\alpha g_v(x) \quad \text{for } m_G\text{-a.e. on } G.
\]
We then obtain (6.11) because \(H^{\overline{D}\setminus G}_\alpha v\) and \(R^{G,0}_\alpha g_v\) are \(\eta_{G}^0\)-quasi-continuous by (6.10). □

**THEOREM 6.2.**

(i) \(X^{\overline{D}} = (X_t, P_x)\) admits no jump from \(D\) to \(\partial D\):
\[
P_x(X_{t-} \in D, X_t \in \partial D \text{ for some } t > 0) = 0 \quad \text{for q.e. } x \in D.
\]

(ii) If \(D\) is relatively compact, then \(X^{\overline{D}}\) is conservative: denoting by \(\zeta\) the lifetime of \(X^{\overline{D}}\),
\[
P_x(\zeta = \infty) = 1 \quad \text{for q.e. } x \in \overline{D}.
\]

(iii) If \(D\) is relatively compact, then \(X^{D,0} = (X^{D,0}_t, P_x)\) admits no killing inside \(D\): denoting by \(\zeta^0\) the lifetime of \(X^{D,0}\),
\[
P_x(X^{D,0}_\zeta^0 \in D, \zeta^0 < \infty) = 0 \quad \text{for q.e. } x \in D.
\]

**PROOF.** (i) For any open set \(G\) as Lemma 6.1 and any compact subset \(F\) of \(\partial D\), we can find a uniformly bounded sequence \(\{v_n\} \subset \mathcal{F}^{\overline{D}} \cap C_0(\overline{D})\) with support being contained in a common compact subset of \(\overline{D} \setminus G\) and \(\lim_{n \to \infty} v_n = 1_F\). Then \(g_{v_n}(x)\) are uniformly bounded and converge to \(g_1f(x) = 0\) as \(n \to \infty\). Therefore, by letting \(n \to \infty\) in (6.11) with \(v_n\) in place of \(v\), we get \(P_x(X_{tG} \in F) = 0\) for q.e. \(x \in G\). Since \(G\) and \(F\) are arbitrary with the stated properties, we have (6.14).

(ii) When \(D\) is relatively compact, \(1 \in C^{\text{lip}}_0(\overline{D})\) so that we see from (6.3) and (6.4) that \(1 \in \mathcal{F}^{\overline{D}}\) and \(\eta_D(1, v) = 0\) for any \(v \in \mathcal{F}^{\overline{D}}\). We have therefore, for any \(\alpha > \beta_0\) and \(f \in L^2(\overline{D}, m_D)\),
\[
0 = \eta_D(1, \hat{G}_\alpha f) = (1, f)_D - \alpha(1, \hat{G}_\alpha f)_D = (1 - \alpha R_\alpha 1, f)_D,
\]
where $\hat{\mathcal{G}}$ is the dual resolvent. This implies that $\alpha R_\alpha 1 = 1_{MD}$-a.e. for $\alpha > \beta_0$ and consequently q.e. on $\overline{D}$ because $R_\alpha 1$ is quasi-continuous. Equation (6.15) is proven.

(iii) This is an immediate consequence of (i), (ii) as $X^{D,0}$ is the part process of $X^\overline{D}$ on $D$. □

We conjecture that the property (6.16) for $X^{D,0}$ holds true without the assumption of the relative compactness of $D$ and especially for the minimal process $X^0$ on $E$.

Finally, we consider the case where $E$ is $\mathbb{R}^d$ and $m$ is the Lebesgue measure on it. For $\alpha \in (0, 2)$ and an arbitrary open set $D \subset \mathbb{R}^d$, we make use of the Lévy kernel

$$k^{[\alpha]}(x, y) = \frac{\alpha 2^{\alpha-1} \Gamma((\alpha + d)/2)}{\pi^{d/2} \Gamma(1 - \alpha/2)} \frac{1}{|x - y|^{d+\alpha}}, \quad x, y \in \mathbb{R}^d,$$

of the symmetric $\alpha$-stable process to introduce the Dirichlet form

$$\begin{align*}
\mathcal{E}^{[\alpha]}_D(u, v) := & \int_{D \setminus \text{diag}} (u(y) - u(x))(v(y) - v(x))k^{[\alpha]}(x, y) \, dx \, dy, \\
\mathcal{F}^{[\alpha],r}_D = & \{ u \in L^2(D) : u \text{ is Borel measurable and } \mathcal{E}^{[\alpha]}_D(u, u) < \infty \},
\end{align*}$$

on $L^2(D)$ based on the Lebesgue measure on $D$. Denote by $\mathcal{F}^{[\alpha]}_D$ the $\mathcal{E}^{[\alpha]}_{D,1}$-closure of $\mathcal{E}^{[\alpha]}_{D,0}$ in $\mathcal{F}^{[\alpha],r}_D$. For $s \in (0, d]$, a Borel subset $\Gamma$ of $\mathbb{R}^d$ is said to be an $s$-set if there exist positive constants $c_1, c_2$ such that for all $x \in \Gamma$ and $r \in (0, 1]$, $c_1 r^s \leq \mathcal{H}^s(\Gamma \cap B(x, r)) \leq c_2 r^s$, where $\mathcal{H}^s$ denotes the $s$-dimensional Hausdorff measure on $\mathbb{R}^d$ and $B(x, r)$ is the ball of radius $r$ centered at $x \in \mathbb{R}^d$.

If the open set $D$ is a $d$-set, then, by making use of Jonsson–Wallin’s trace theorem [14] as in [7], one can show that $\mathcal{F}^{[\alpha]}_D = \mathcal{F}^{[\alpha],r}_D$ and moreover that a subset of $\overline{D}$ is $\mathcal{E}^{[\alpha]}_D$-polar iff it is polar with respect to the symmetric $\alpha$-stable process on $\mathbb{R}^d$.

Let us consider the kernel $k^{(1)}$ of (1.9) for $w(x)$ given by (5.1) and $\alpha(x)$ satisfying condition (5.2). In particular, it is assumed that

$$0 < \underline{\alpha} \leq \alpha(x) \leq \overline{\alpha} < 2$$

for some constant $\alpha, \overline{\alpha}$. $k^{(1)}$ satisfies conditions (2.1)–(2.4) by Proposition 5.1 and one can associate with it the regular lower bounded semi-Dirichlet form $\eta_D$ (resp., $\eta^0_D$) on $L^2(D; 1_D \, dx)$ [resp., $L^2(D)$] possessing as its reference form $\mathcal{E}^{[\alpha]}_D$ (resp., $\mathcal{E}^{[\alpha]}_D$) defined right after (6.1) for $k^{(1)}$ and the Lebesgue measure in place of $k$ and $m$. 
Suppose \( D \) is bounded, then there exist positive constants \( c_3, c_4 \) with 
\[
c_3 k^{[2]}(x, y) \leq k^{(1)}_2(x, y) \leq c_4 k^{[2]}(x, y), \quad x, y \in \overline{D},
\]
so that 
\[
(6.18) \quad c_3 \mathcal{E}^\alpha_D(u, u) \leq \mathcal{E}_D(u, u) \leq c_4 \mathcal{E}^\alpha_D(u, u), \quad u \in C^\alpha_0(\overline{D}).
\]

For the kernel \( k^{(1)} \), the Hunt process \( \bar{X}^D \) on \( D \) associated with \((\eta_D, \mathcal{F}_D)\) is called a modified reflecting stable-like process, while its part process \( X^{D,0} \) on \( D \), which is associated with \((\eta^0_D, \mathcal{F}^0_D)\), is called a censored stable-like process.

**PROPOSITION 6.1.** Assume that \( D \) is a bounded open \( d \)-set.

(i) If \( \partial D \) is polar with respect to the symmetric \( \alpha \)-stable process on \( \mathbb{R}^d \), then the censored stable-like process \( X^{D,0} = (X^{D,0}_t, P_x, \xi^0) \) is conservative and it does not approach to \( \partial D \) in finite time:
\[
(6.19) \quad P_x(\xi^0 = \infty) = 1, \quad P_x(X^{D,0}_t \in \partial D \text{ for some } t > 0) = 0.
\]

(ii) If \( \partial D \) is nonpolar with respect to the symmetric \( \alpha \)-stable process on \( \mathbb{R}^d \), then the censored stable-like process \( X^{D,0} \) satisfies
\[
(6.20) \quad \int_D P_x(X^{D,0}_{\xi^0_0} \in \partial D, \xi^0 < \infty)h(x)\,dx = \int_D P_x(\xi^0 < \infty)h(x)\,dx > 0
\]
for any strictly positive Borel function \( h \) on \( D \) with \( \int_D h(x)\,dx = 1 \).

**PROOF.** (i) Since \( \mathcal{E}_D \) is a reference form of \((\eta_D, \mathcal{F}_D)\), we see that \( \partial D \) is \( \eta_D \)-polar by (6.18) and the stated observation in [7]. The assertions of (i) then follows from Theorem 4.2(ii) and Theorem 6(ii).

(ii) \( \partial D \) is not \( \eta_D \)-polar by (6.18) and accordingly not \( m \)-polar with respect to the process \( \bar{X}^D \) by Theorem 4.2(v), where \( m \) is the Lebesgue measure on \( D \). Taking Theorem 6.2(i), (iii) into account, we then get (6.20). \( \square \)

The polarity of a set \( N \subset \mathbb{R}^d \) with respect to the symmetric \( \alpha \)-stable process is equivalent to \( C^{\alpha/2,2}(N) = 0 \) for the Bessel capacity \( C^{\alpha/2,2} \) (cf. Section 2.4 of the second edition of [13]). The latter has been well studied in [1] in relation to the Hausdorff measure and the Hausdorff content. For instance, when \( \alpha \leq d \) and \( \partial D \) is a \( s \)-set, \( \partial D \) is polar in this sense if and only if \( \alpha + s \leq d \). Of course, we get the same results as above for the second kernel \( k^{(1)*} \) in (1.9).

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