Structured Nonconvex and Nonsmooth Optimization: Algorithms and Iteration Complexity Analysis

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Abstract

Nonconvex optimization problems are frequently encountered in much of statistics, business, science and engineering, but they are not yet widely recognized as a technology. A reason for this relatively low degree of popularity is the lack of a well developed system of theory and algorithms to support the applications, as is the case for its convex counterpart. This paper aims to take one step in the direction of disciplined nonconvex optimization. In particular, we consider in this paper some constrained nonconvex optimization models in block decision variables, with or without coupled affine constraints. In the case of no coupled constraints, we show a sublinear rate of convergence to an $\epsilon$-stationary solution in the form of variational inequality for a generalized conditional gradient method, where the convergence rate is shown to be dependent on the Hölderian continuity of the gradient of the smooth part of the objective. For the model with coupled affine constraints, we introduce corresponding $\epsilon$-stationarity conditions, and propose two proximal-type variants of the ADMM to solve such a model, assuming the proximal ADMM updates can be implemented for all the block variables except for the last block, for which either a gradient step or a majorization-minimization step is implemented. We show an iteration complexity bound of $O(1/\epsilon^2)$ to reach an $\epsilon$-stationary solution for both algorithms. Moreover, we show that the same iteration complexity of a proximal BCD method follows immediately. Numerical results are provided to illustrate the efficacy of the proposed algorithms for tensor robust PCA.

Keywords: Structured Nonconvex Optimization, $\epsilon$-Stationary Point, Iteration Complexity, Conditional Gradient Method, Alternating Direction Method of Multipliers, Block Coordinate Descent Method

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1 Introduction

In this paper, we consider the following nonconvex and nonsmooth optimization problem with multiple block variables:

$$\begin{align*}
\min \quad & f(x_1, x_2, \cdots, x_N) + \sum_{i=1}^{N-1} r_i(x_i) \\
\text{s.t.} \quad & \sum_{i=1}^{N} A_i x_i = b, \; x_i \in X_i, \; i = 1, \ldots, N - 1,
\end{align*}$$

(1.1)

where $f$ is differentiable and possibly nonconvex, and each $r_i$ is possibly nonsmooth and non-convex, $i = 1, \ldots, N - 1$; $A_i \in \mathbb{R}^{m \times n_i}$, $b \in \mathbb{R}^m$, $x_i \in \mathbb{R}^{n_i}$; and $X_i \subseteq \mathbb{R}^{n_i}$ are convex sets, $i = 1, 2, \ldots, N - 1$. A special case of (1.1) is when the affine constraints are absent, and there is no block structure of the variables, which leads to the following more compact form

$$\min \; \Phi(x) := f(x) + r(x), \; \text{s.t.} \; x \in S \subseteq \mathbb{R}^n,$$

(1.2)

where $S$ is a convex and compact set. In this paper, we propose several first-order algorithms for computing an $\epsilon$-stationary point (to be defined later) of (1.1) and (1.2), and analyze their iteration complexities. Throughout this paper, we assume that the sets of the stationary points to (1.1) and (1.2) are non-empty.

Problem (1.1) arises from a variety of interesting applications. For example, one of the nonconvex models for matrix robust PCA can be casted as follows (see, e.g., [45]), which seeks to decompose a given matrix $M \in \mathbb{R}^{m \times n}$ to a superposition of a low-rank matrix $Z$, a sparse matrix $E$ and a noise matrix $B$:

$$\min_{X,Y,Z,E,B} \quad \|Z - XY^\top\|_F^2 + \alpha R(E)$$

s.t. \quad $M = Z + E + B$

$$\|B\|_F \leq \eta,$$

(1.3)

where $X \in \mathbb{R}^{m \times r}$, $Y \in \mathbb{R}^{n \times r}$, with $r < \min(m, n)$ being the estimated rank of $Z$; $\eta > 0$ is the noise level, $\alpha > 0$ is a weighting parameter; $R(E)$ is a regularization function that can improve the sparsity of $E$. One of the widely used regularization functions is the $\ell_1$ norm, which is convex and nonsmooth. However, there are also many nonconvex regularization functions that are widely used in statistical learning and information theory, such as smoothly clipped absolute deviation (SCAD) [21], log-sum penalty (LSP) [15], minimax concave penalty (MCP) [50], and capped-$\ell_1$ penalty [51, 52], and they are nonsmooth at point 0 if composed with the absolute value function, which is usually the case in statistical learning. Clearly (1.3) is in the form of (1.1). Another example of the form (1.1) is the following nonconvex tensor robust PCA model (see, e.g., [48]), which seeks to decompose a given tensor $T \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$ into a superposition of a low-rank tensor $Z$, a sparse tensor $E$ and a noise tensor $B$:

$$\min_{X_i, \mathcal{C}, \mathcal{Z}, \mathcal{E}, B} \quad \|Z - \mathcal{C} \times_1 X_1 \times_2 X_2 \times_3 \cdots \times_d X_d\|_F^2 + \alpha R(\mathcal{E})$$

s.t. \quad $T = Z + \mathcal{E} + B$

$$\|B\|_F \leq \eta,$$

(1.4)

where $\mathcal{C}$ is the core tensor that has smaller size than $Z$, and $X_i$ are matrices with appropriate sizes, $i = 1, \ldots, d$. In fact, the “low-rank” tensor in the above model corresponds to the tensor with a small core; however a recent work [32] demonstrates that the CP-rank of the core regardless of its size could be as large as the original tensor. Therefore, if one wants to find the low CP-rank decomposition, then the following model is preferred:

$$\min_{X_i, \mathcal{Z}, \mathcal{E}, B} \quad \|Z - [X_1, X_2, \cdots, X_d]\|^2 + \alpha \mathcal{R}(\mathcal{E}) + \|B\|^2$$

s.t. \quad $T = Z + \mathcal{E} + B,$

(1.5)
for $X_i = [a_i^{1}, a_i^{2}, \cdots, a_i^{R}] \in \mathbb{R}^{m_i \times R}$, $1 \leq i \leq d$ and

$$\|X_1, X_2, \cdots, X_d\| := \sum_{r=1}^{R} a_1^{1,r} \otimes a_2^{2,r} \otimes \cdots \otimes a_d^{d,r}, \quad (1.6)$$

where “$\otimes$” denotes the outer product of vectors, and $R$ is an estimation of the CP-rank. In addition, the so-called sparse tensor PCA problem [1], which seeks the best sparse rank-one approximation for a given $d$-th order tensor $T$, can also be formulated in the form of (1.1):

$$\min_{x_1, x_2, \cdots, x_d} -T(x_1, x_2, \cdots, x_d) + \alpha \sum_{i=1}^{d} \mathcal{R}(x_i) \quad \text{s.t.} \quad x_i \in S_i = \{x \mid \|x\|_2^2 \leq 1\}, \quad i = 1, 2, \ldots, d, \quad (1.7)$$

where $T(x_1, x_2, \cdots, x_d) = \sum_{i_1, \ldots, i_d} T_{i_1, \ldots, i_d}(x_1)_{i_1} \cdots (x_d)_{i_d}$.

The convergence and iteration complexity for various nonconvex and nonsmooth optimization problems have recently attracted considerable research attention; see e.g. [3, 6–8, 10, 11, 19, 24, 25, 38]. In this paper, we propose several solution methods that use only the first-order information of the objective function, including generalized conditional gradient method, variants of alternating direction method of multipliers, and proximal block coordinate descent method, for solving (1.1) and (1.2). Specifically, we propose a generalized conditional gradient (GCG) method for solving (1.2). We prove that GCG can find an $\epsilon$-stationary point for (1.2) in $O(\epsilon^{-q})$ iterations under certain mild conditions, where $q$ is a parameter in the Hölder condition that characterizes the degree of smoothness of $f$. In other words, the speed of the algorithm’s convergence depends on the degree of “smoothness” of the objective function. It should be noted that a similar iteration bound that depends on the parameter $q$ was only recently reported in the context of convex optimization [13]. Furthermore, we show that if $f$ is concave, then GCG finds an $\epsilon$-stationary point for (1.2) in $O(1/\epsilon)$ iterations. For the affinely constrained problem (1.1), we propose two algorithms (called proximal ADMM-g and proximal ADMM-m in this paper) that can both be viewed as variants of the alternating direction method of multipliers (ADMM). Recently, there have been some emerging interests on the ADMM for nonconvex problems (see, e.g., [2, 29, 30, 35, 46, 47, 49]). The results in [35, 46, 47, 49] only proved the convergence of ADMM to a stationary point, and no iteration complexity analysis was provided. Moreover, the objective function is required to satisfy the so-called Kurdyka-Lojasiewicz (KL) property [9, 33, 39, 40] to ensure those convergence results. In [30], Hong, Luo and Razaviyayn analyzed the convergence of ADMM for solving nonconvex consensus and sharing problems. Note that they also analyzed the iteration complexity of ADMM for the consensus problem. However, they require the nonconvex part of the objective function to be smooth, and nonsmooth part to be convex. In contrast, $r_i$ in our model (1.1) can be nonconvex and nonsmooth at the same time. Moreover, we allow general set constraints $x_i \in \mathcal{X}_i$, $i = 1, \ldots, N - 1$, while the consensus problem in [30] only allows the set constraint for one block variable. The very recent work by Hong [29] discusses the iteration complexity of an augmented Lagrangian method for finding an $\epsilon$-stationary point for the following problem:

$$\min f(x), \text{ s.t. } Ax = b, x \in \mathbb{R}^n, \quad (1.8)$$

under the assumption that $f$ is differentiable. We will compare our results with [29] in more details in Section 3.

Throughout this paper, we make the following assumption.

**Assumption 1.1** All subproblems in our algorithms, though possibly nonconvex, can be solved to global optimality.

We shall show later that the solvability of our subproblems usually boils down to the computability of the proximal mapping with respect to the nonsmooth part of the objective function.
Besides, the proximal mappings of the aforementioned nonsmooth regularization functions, including the $\ell_1$ norm, SCAD, LSP, MCP and Capped-$\ell_1$ penalty, all admit closed-form solutions, and the explicit formulae can be found in [26].

Before proceeding, let us first summarize:

**Our contributions.**

(i) We provide a systematic study on how to define an $\epsilon$-stationary point of (1.1) and (1.2). For (1.1), our definition of $\epsilon$-stationary point covers the cases when each $r_i$ is convex, or $r_i$ is Lipschitz continuous (possibly nonconvex), or $r_i$ is lower semi-continuous (possibly nonconvex).

(ii) We propose a generalized conditional gradient method for solving (1.2) and analyze its iteration complexity for obtaining an $\epsilon$-stationary point of (1.2).

(iii) We propose two ADMM variants (proximal ADMM-g and proximal ADMM-m) for solving (1.1), under certain conditions on $A_N$. We also analyze their iteration complexities for obtaining an $\epsilon$-stationary point of (1.1).

(iv) As an extension, we also show how to use proximal ADMM-g and proximal ADMM-m to find an $\epsilon$-stationary point of (1.1) without assuming any assumption on $A_N$.

(v) As a by-product, we also propose a proximal block coordinate descent (BCD) method with cyclic order for solving (1.1) when the affine constraints are absent, and show that its iteration complexity can be obtained directly from that of proximal ADMM-g and proximal ADMM-m.

**Notation.** We use $\|x\|_2$ to denote the Euclidean norm of vector $x$, and $\|x\|^2_H$ to denote $x^\top H x$ for some positive definite matrix $H$. For set $S$ and scalar $p > 1$, we denote

$$\text{diam}_p(S) := \max_{x,y \in S} \|x - y\|_p, \tag{1.9}$$

where $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$. Without specification, we denote $\|x\| = \|x\|_2$ and $\text{diam}(S) = \text{diam}_2(S)$ for short. We use $\text{dist}(x, S)$ to denote the Euclidean distance of vector $x$ to set $S$. Given a matrix $A$, its spectral norm, largest singular value and smallest singular value are denoted by $\|A\|_2$, $\sigma_{\text{max}}(A)$ and $\sigma_{\text{min}}(A)$ respectively. We use $[a]$ to denote the largest integer that is less than or equal to scalar $a$.

**Organization.** The rest of this paper is organized as follows. In Section 2 we give the definition of $\epsilon$-stationary point of (1.2) and propose a generalized conditional gradient method that solves (1.2) and analyze its iteration complexity for obtaining such an $\epsilon$-stationary point of (1.2). In Section 3 we give three definitions of $\epsilon$-stationarity for (1.1) under different settings and propose two ADMM variants that solve (1.1) and analyze their iteration complexities to reach an $\epsilon$-stationary point of (1.1). In Section 4 we provide some extensions of the results in Section 3. In particular, we first show how to remove some of the conditions that we assume in Section 3, and then we propose a proximal BCD method to solve (1.1) without affine constraints and provide an iteration complexity analysis. In Section 5, we provide numerical results to illustrate the practical efficiency of the proposed algorithms.

## 2 A generalized conditional gradient method

In this section, we propose a GCG method for solving (1.2) and analyze its iteration complexity. The conditional gradient (CG) method, also known as the Frank-Wolfe method, was
originally proposed in [22], and regained a lot of popularity recently due to its capability of solving large-scale problems (see, [4,5,23,27,31,34,42]). However, these works focus on solving convex problems. Bredies et al. [14] considered a generalized conditional gradient method for solving nonconvex problems in Hilbert space, which is similar to our algorithm, but no iteration complexity was provided.

Throughout this section, we make the following assumption regarding to problem (1.2).

**Assumption 2.1** In (1.2), function \( r(x) \) is convex and nonsmooth, and the constraint set \( S \) is convex and compact. Moreover, \( f \) is differentiable and there exist some \( p > 1 \) and \( \rho > 0 \) such that

\[
 f(y) \leq f(x) + \nabla f(x) \top (y-x) + \frac{\rho}{2} \|y-x\|_p^p, \quad \forall x, y \in S. \tag{2.1}
\]

The inequality (2.1) is the so-called Hölder condition and was also used in other papers that discuss first-order algorithms (e.g., [20]). It can be shown that (2.1) holds for a variety of functions. In fact, we have the following results.

**Proposition 2.2**

(i) If \( f \) is concave, then (2.1) holds for any \( p > 0 \) and \( \rho > 0 \).

(ii) If the gradient of \( f \) satisfies

\[
 \|\nabla f(x) - \nabla f(y)\|_q^q \leq M \|x-y\|_p^p, \quad \forall x, y \in S, \tag{2.2}
\]

for some \( M > 0, 1 < p \leq 2 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \), then (2.1) holds.

(iii) (2.1) holds for the \( p \)-norm function

\[
 f(x) = \sum_{i=1}^n x_i^p, \quad \text{where } 1 < p \leq 2 \text{ and } x_i > 0, i = 1, \ldots, n. \tag{2.3}
\]

**Proof.** Part (i) is obvious. For (ii), let \( z = y - x \) and \( g(\alpha) = f(x + \alpha z) \), it follows that

\[
 f(y) - f(x) = \int_0^1 g'(\alpha)z \, d\alpha = \int_0^1 \nabla f(x + \alpha z) \top z \, d\alpha \\
 \leq \nabla f(x) \top z + \left\| \int_0^1 (\nabla f(x + \alpha z) - \nabla f(x)) \top z \, d\alpha \right\| \\
 \leq \nabla f(x) \top z + \int_0^1 \|\nabla f(x + \alpha z) - \nabla f(x)\|_q \|z\|_p \, d\alpha \\
 \leq \nabla f(x) \top z + M^{1/q} \|z\|_p^{1+\frac{1}{q}} \int_0^1 \alpha^{\frac{1}{q}} \, d\alpha \\
 = \nabla f(x) \top z + M^{1/q} \frac{1}{p} \|z\|_p^p,
\]

where the last equality is due to \( \frac{1}{q} + \frac{1}{p} = 1 \). Thus, the function with Lipschitz continuous gradient automatically satisfies inequality (2.1) for \( p = q = 2 \). In fact, condition (2.2) reflects the degree of the Hölderian continuity of \( \nabla f \), which was also considered in [20] to construct a so-called inexact first-order oracle. For Part (iii), we observe that the function is separable with respect to all \( x_i \), so it suffices to show that there exists some \( \rho \) such that:

\[
 v^p \leq u^p + (u^p)'(v-u) + \frac{\rho}{2} |v-u|^p = u^p + p u^{p-1} (v-u) + \frac{\rho}{2} |v-u|^p, \tag{2.4}
\]
when $1 < p \leq 2$. If $u = 0$, then the inequality trivially holds for any $\rho \geq 2$; otherwise we can divide both sides by $|u|^p$ and aim to prove an equivalent formulation:

$$k^p \leq 1 + p(k - 1) + \frac{\rho}{2}|k - 1|^p,$$

where $k = v/u$. To this end, define

$$g(k) := \begin{cases} 0 & \text{if } k = 1 \\ \frac{k^p - 1 - p(k - 1)}{|k - 1|^p} & \text{otherwise.} \end{cases}$$

Observe that $\lim_{k \to +\infty} g(k) = 1$ and by the L’Hospital rule

$$\lim_{k \to 1} g(k) = \begin{cases} 0 & \text{if } 1 < p < 2 \\ 1 & \text{if } p = 2, \end{cases}$$

so $g(k)$ is upper bounded on $\mathbb{R}$ and there exits some $\hat{\rho}$ such that (2.5) holds. Finally by letting $\rho = \max\{2, \hat{\rho}\}$, the inequality (2.4) follows. \hfill \Box

2.1 An $\epsilon$-stationary point for problem (1.2)

Our definition of an $\epsilon$-stationary point of (1.2) is given as follows.

**Definition 2.3** We call $x$ to be an $\epsilon$-stationary point ($\epsilon \geq 0$) of (1.2) if the following mixed variational inequality conditions is satisfied:

$$\psi_S(x) := \nabla f(x)^\top (y - x) + r(y) - r(x) \geq -\epsilon, \quad \forall y \in S. \quad (2.6)$$

If $\epsilon = 0$, we call $x$ to be a stationary point of (1.2).

When $\epsilon = 0$, the condition (2.6) is stronger than the commonly used KKT condition in the sense that it is a necessary condition for local minimum of (1.2). To see this, suppose that there exists some $y \in S$ such that $\nabla f(x)^\top (y - x) + r(y) - r(x) < 0$. Denote $d = y - x$. Then the directional derivative along direction $d$ at point $x$ satisfies

$$(f + r)'(x; d) = \lim_{\alpha \searrow 0} \frac{f(x + \alpha d) + r(x + \alpha d) - f(x) - r(x)}{\alpha}$$

$$\leq \lim_{\alpha \searrow 0} \frac{f(x + \alpha d) - f(x)}{\alpha} + \lim_{\alpha \searrow 0} \frac{(1 - \alpha)r(x) + \alpha r(x + d) - r(x)}{\alpha}$$

$$= \nabla f(x)^\top (y - x) + r(y) - r(x) < 0,$$

where the first inequality is due to the convexity of $r$. As a result, $x$ cannot be a local minimizer of problem (1.2).

We now compare our Definition 2.3 with some existing definitions of $\epsilon$-stationary point in the literature. For the smooth unconstrained problem $\{\min f(x)\}$, it is natural to define the $\epsilon$-stationary point using the criterion $\|\nabla f(x)\|_2 \leq \epsilon$. Nesterov [43] and Cartis et al. [17] showed that the gradient descent type methods with properly chosen step size need $O(1/\epsilon^2)$ iterations to find such a point. Moreover, Cartis et al. [16] constructed an example showing that the $O(1/\epsilon^2)$ iteration complexity is tight for the steepest descent type algorithm. However, the case for the constrained nonconvex optimization is more complicated. When $r \equiv 0$ in (1.2), i.e., the objective function is differentiable, Cartis et al. [18] proposed the following measure:

$$\chi_S(x) := \min_{x + d \in S, \|d\|_2 \leq 1} \|\nabla f(x)^\top d\| \leq \epsilon. \quad (2.7)$$
They showed that it requires no more than $O(1/\epsilon^2)$ iterations for the adaptive cubic regularization algorithm in [18] to find an $x$ satisfying (2.7). Ghadimi et al. [25] gave the following definition of an $\epsilon$-stationary point of (1.2). Define

$$P_S(x, \gamma) := \frac{1}{\gamma} (x - x^+), \quad \text{where } x^+ = \arg\min_{y \in S} \nabla f(x)^T y + \frac{1}{\gamma} V(y, x) + r(y), \quad (2.8)$$

where $\gamma > 0$ and $V$ is a prox-function. Ghadimi et al. [25] proposed a projected gradient algorithm to solve (1.2) and proved that it takes no more than $O(1/\epsilon^2)$ iterations to find an $x$ satisfying

$$\|P_S(x, \gamma)\|^2_2 \leq \epsilon. \quad (2.9)$$

We have the following proposition regarding the relationships among the three definitions of $\epsilon$-stationary point defined in (2.6), (2.7) and (2.9).

**Proposition 2.4** Consider (1.2).

(i) When $r(x) \equiv 0$, if $\psi_S(x) \geq -\epsilon$, then $\chi_S(x) \leq \epsilon$;

(ii) Suppose that the prox-function $V(y, x) = \|y-x\|_2^2/2$, then $\psi_S(x) \geq -\epsilon$ implies $\|P_S(x, \gamma)\|^2_2 \leq \epsilon/\gamma$. Conversely, if we further assume that the gradient function $\nabla f(x)$ is continuous, then $\|P_S(x, \gamma)\|^2_2 \leq \epsilon$ implies

$$\psi_S(x) \geq - (\gamma \tau + \gamma \varsigma + \text{diam}(S)) \sqrt{\epsilon}, \quad (2.10)$$

where $\tau = \max_{x \in S} \|\nabla f(x)\|_2$, $\varsigma = \max_{x \in S} \min_{z \in \partial r(x)} \|z\|_2$. Note that $\varsigma$ is finite, because $S$ is compact.

**Proof.** Part (i) follows from (2.6) and (2.7) by the following relationships:

$$\psi_S(x) \geq -\epsilon \implies \nabla f(x)^T (y - x) \geq -\epsilon, \forall \|y - x\|_2 \leq 1, y \in S \implies 0 \geq \min_{x + d \in S, \|d\|_2 \leq 1} \nabla f(x)^T d \geq -\epsilon \implies \chi_S(x) \leq \epsilon.$$

We now prove part (ii). Since $V(y, x) = \|y-x\|_2^2/2$, (2.8) implies that

$$\left( \nabla f(x) + \frac{1}{\gamma} (x^+ - x) + z \right)^T (y - x^+) \geq 0, \quad \forall y \in S, \quad (2.11)$$

where $z \in \partial r(x^+)$. By choosing $y = x$ in (2.11) one can get

$$\nabla f(x)^T (x - x^+) + r(x) - r(x^+) \geq \nabla f(x)^T (x - x^+) \geq \frac{1}{\gamma} \|x^+ - x\|_2^2. \quad (2.12)$$

Therefore, if $\psi_S(x) \geq -\epsilon$, then $\|P_S(x, \gamma)\|^2_2 \leq \frac{\epsilon}{\gamma}$ holds. To show the other direction, note that for $x, x^+ \in S$, one can choose $w \in \partial r(x)$ such that

$$\varsigma \|x - x^+\|_2 \geq w^T (x - x^+) \geq r(x) - r(x^+),$$

which together with (2.12) implies that

$$\nabla f(x)^T (y - x) + r(y) - r(x) + (\|\nabla f(x)\|_2^2 + \varsigma)\|x - x^+\|_2$$

$$\geq \nabla f(x)^T (y - x) + r(y) - r(x) + \nabla f(x)^T (x - x^+) + r(x) - r(x^+)$$

$$= \nabla f(x)^T (y - x^+) + r(y) - r(x^+)$$

$$\geq \nabla f(x)^T (y - x^+) \geq -\frac{1}{\gamma} (x^+ - x)^T (y - x^+) \geq -\frac{1}{\gamma} \|y - x^+\|_2 \|x^+ - x\|_2, \quad \forall y \in S,$$
where \( z \in \partial r(x^+) \), the second inequality follows from the convexity of \( r(x) \) and the third inequality is due to the optimality condition of (2.8). (2.10) follows by rearranging the terms in the above inequality. □

Under the conditions in Proposition 2.4, the relationship of these three definitions of \( \epsilon \)-stationary point of (1.2) is depicted in Figure 1, which shows that our definition (2.6) is to some extent more general than (2.7) and (2.9).

### 2.2 GCG method and its iteration complexity

For given point \( z \), we define a linearization of the objective function of (1.2) as:

\[
\ell(x; z) := f(z) + \nabla f(z)^\top (x - z) + r(x),
\]

which is obtained by linearizing the smooth part (function \( f \)) of \( \Phi \) in (1.2). Our GCG method for solving (1.2) is described in Algorithm 1.

**Algorithm 1** Generalized Conditional Gradient Algorithm (GCG) for solving (1.2)

**Require:** Given \( x^0 \in S \)

for \( k = 0, 1, \ldots \) do

[Step 1] \( y^k = \arg \min_{y \in S} \ell(y; x^k) \), and let \( d^k = y^k - x^k \);

[Step 2] \( \alpha_k = \arg \min_{\alpha \in [0, 1]} \alpha \nabla f(x^k)^\top d^k + \alpha \frac{p}{2} \|d^k\|_p^p + (1 - \alpha)r(x^k) + \alpha r(y^k) \);

[Step 3] Set \( x^{k+1} = (1 - \alpha_k)x^k + \alpha_k y^k \).

end for

Note that when the nonsmooth function \( r \) is absent, GCG differs from the classical CG method only by the choice of the step size \( \alpha_k \).

**Remark 2.5** According to Assumption 1.1, we assume that the problem in Step 1 of Algorithm 1 can be solved to global optimality. See [41] for problems arising from sparse PCA that satisfy this assumption.

Before we proceed to the main result on iteration complexity of GCG, we need the following lemma that gives a sufficient condition for \( \epsilon \)-stationary point of (1.2). This lemma is inspired by [24], and it indicates that if the progress gained by solving (2.14) is small, then \( z \) is close to a stationary point of (1.2).
Lemma 2.6 Define

\[ z_\ell := \text{argmin}_{x \in S} \ell(x; z). \]  

(2.14)

The improvement of the linearization at point \( z \) is defined as

\[ \Delta \ell_z := \ell(z; z) - \ell(z_\ell; z) = -\nabla f(z)^\top (z_\ell - z) + r(z) - r(z_\ell). \]

Given \( \epsilon \geq 0 \), for any \( z \in S \), if \( \Delta \ell_z \leq \epsilon \), then \( z \) is an \( \epsilon \)-stationary point of (1.2) as defined in Definition 2.3.

Proof. Since \( z_\ell \) is optimal to (2.14), we have

\[ \ell(y; z) - \ell(z_\ell; z) = \nabla f(z)^\top (y - z_\ell) + r(y) - r(z_\ell) \geq 0, \forall y \in S, \]

which implies that

\[ \nabla f(z)^\top (y - z) + r(y) - r(z) \]

\[ = \nabla f(z)^\top (y - z_\ell) + r(y) - r(z_\ell) + \nabla f(z)^\top (z_\ell - z) + r(z_\ell) - r(z) \]

\[ \geq \nabla f(z)^\top (z_\ell - z) + r(z_\ell) - r(z), \forall y \in S. \]

It then follows immediately that if \( \Delta \ell_z \leq \epsilon \), then \( \nabla f(z)^\top (y - z) + r(y) - r(z) \geq -\Delta \ell_z \geq -\epsilon. \)

We are now ready to give the main result of the iteration complexity of GCG (Algorithm 1) for obtaining an \( \epsilon \)-stationary point of (1.2).

Theorem 2.7 For any \( \epsilon \in (0, \text{diam}_p^p(S)\rho) \), GCG finds an \( \epsilon \)-stationary point of (1.2) within

\[ \left\lceil \frac{2(\Phi(x^0) - \Phi^*)}{q\epsilon^{\rho/p} + 1} \rho \right\rceil \]

iterations, where \( \frac{1}{p} + \frac{1}{q} = 1. \)

Proof. For ease of presentation, we denote \( D := \text{diam}_p(S) \) and \( \Delta \ell^k := \Delta \ell_{x^k} \). By Assumption 2.1, using the fact that \( \frac{Dp}{\rho} < 1 \), and by the definition of \( \alpha_k \) in Algorithm 1, we have

\[ \left( \frac{\epsilon}{Dp} \right)^{\frac{1}{p-1}} \Delta \ell^k \leq \frac{1}{2^{\rho/(p-1)}} \left( \frac{\epsilon}{D} \right)^{\frac{1}{p-1}} \]

\[ \leq -\left( \frac{\epsilon}{Dp} \right)^{\frac{1}{p-1}} \left( \nabla f(x^k)^\top (y^k - x^k) + r(y^k) - r(x^k) \right) - \frac{\rho}{2} \left( \frac{\epsilon}{Dp} \right)^{\frac{1}{p-1}} \| y^k - x^k \|_p \]

\[ \leq -\alpha_k \left( \nabla f(x^k)^\top (y^k - x^k) + r(y^k) - r(x^k) \right) - \frac{\rho\alpha^p}{2} \| y^k - x^k \|_p \]

\[ \leq -\nabla f(x^k)^\top (x^{k+1} - x^k) + r(x^k) - r(x^{k+1}) - \frac{\rho}{2} \| x^{k+1} - x^k \|_p \]

\[ \leq f(x^k) - f(x^{k+1}) + r(x^k) - r(x^{k+1}) = \Phi(x^k) - \Phi(x^{k+1}), \]

(2.15)

where the third inequality is due to the convexity of function \( r \) and the fact that \( x^{k+1} - x^k = \alpha_k (y^k - x^k) \), and the last inequality is due to (2.1). Furthermore, (2.15) immediately yields

\[ \Delta \ell^k \leq \left( \frac{\epsilon}{Dp} \right)^{\frac{1}{p-1}} \left( \Phi(x^k) - \Phi(x^{k+1}) \right) + \frac{\epsilon}{2}. \]

(2.16)

For any integer \( K > 0 \), summing (2.16) over \( k = 0, 1, \ldots, K \), yields

\[ \min_{k \in \{0, 1, \ldots, K\}} \Delta \ell^k \leq \sum_{k=1}^{K} \Delta \ell^k \leq \left( \frac{\epsilon}{Dp} \right)^{\frac{1}{p-1}} \left( \Phi(x^0) - \Phi(x^{K+1}) \right) + \frac{\epsilon}{2} K \]

\[ \leq \left( \frac{\epsilon}{Dp} \right)^{\frac{1}{p-1}} \left( \Phi(x^0) - \Phi^* \right) + \frac{\epsilon}{2} K, \]

9
where $\Phi^*$ is the optimal value of (1.2). It is easy to see that by setting $K = \left[ \frac{2(\Phi(x^0) - \Phi^*)(D^p)^p}{\epsilon^q} \right]$, the above inequality implies $\Delta \ell_{x^k} \leq \epsilon$, where $k^* := \arg\min_{k \in \{1, \ldots, K\}} \Delta \ell^k$. According to Lemma 2.6, $x^k$ is an $\epsilon$-stationary point of (1.2) as defined in Definition 2.3.

We have the following immediate corollary when $f$ is a concave function.

**Corollary 2.8** When $f$ is a concave function, if we set $\alpha_k = 1$ for all $k$ in GCG (Algorithm 1), then it returns an $\epsilon$-stationary point of (1.2) within $\left\lceil \frac{\Phi(x^0) - \Phi^*}{\epsilon} \right\rceil$ iterations.

**Proof.** By setting $\alpha_k = 1$ in Algorithm 1 we know that $x^{k+1} = y^k$ for all $k$. Since $f$ is concave, it holds that

$$\Delta \ell^k = -\nabla f(x^k)^\top (x^{k+1} - x^k) + r(x^k) - r(x^{k+1}) \leq \Phi(x^k) - \Phi(x^{k+1}).$$

Summing this inequality over $k = 0, 1, \ldots, K$ yields

$$K \min_{k \in \{0, 1, \ldots, K\}} \Delta \ell^k \leq \Phi(x^0) - \Phi^*,$$

which leads to the desired result immediately. \qed

### 3 Variants of ADMM for solving nonconvex problems with affine constraints

In this section, we propose two variants of the ADMM (Alternating Direction Method of Multipliers) for solving the general problem (1.1), and prove their iteration complexities for obtaining an $\epsilon$-stationary point (to be defined later) under certain conditions. Throughout this section, we assume the following two assumptions regarding problem (1.1).

**Assumption 3.1** The partial gradient of the function $f$ with respect to $x_N$ is Lipschitz continuous with Lipschitz constant $L > 0$, i.e., for any $(x_1^1, \ldots, x_N^1)$ and $(x_1^2, \ldots, x_N^2) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_{N-1} \times \mathbb{R}^{n_N}$, it holds that

$$\|\nabla_N f(x_1^1, x_2^1, \ldots, x_N^1) - \nabla_N f(x_1^2, x_2^2, \ldots, x_N^2)\| \leq L \| (x_1^1 - x_1^2, x_2^1 - x_2^2, \ldots, x_N^1 - x_N^2)\|,$$

which implies that for any $(x_1, \ldots, x_{N-1}) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_{N-1}$ and $x_N, \hat{x}_N \in \mathbb{R}^{n_N}$, it holds that

$$f(x_1, \ldots, x_{N-1}, x_N) \leq f(x_1, \ldots, x_{N-1}, \hat{x}_N) + (x_N - \hat{x}_N)^\top \nabla_N f(x_1, \ldots, x_{N-1}, \hat{x}_N) + \frac{L}{2} \|x_N - \hat{x}_N\|^2. \quad (3.2)$$

**Assumption 3.2** $f$ and $r_i, i = 1, \ldots, N - 1$ are all lower bounded, and we denote

$$f^* = \min_{x_1 \in \mathcal{X}_1, i = 1, \ldots, N-1; x_N \in \mathbb{R}^{n_N}} \{f(x_1, x_2, \ldots, x_N)\}$$

and $r^*_i = \min_{x_i \in \mathcal{X}_i} \{r_i(x_i)\}$ for $i = 1, 2, \ldots, N - 1$.

### 3.1 Preliminaries

To characterize the optimality conditions of (1.1) when $r_i$ is nonsmooth and nonconvex, we need to recall the definition of generalized gradient (see, e.g., [44]).

**Definition 3.3** Let $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous function. Suppose $h(\bar{x})$ is finite for a given $\bar{x}$. For $v \in \mathbb{R}^n$, we say that
and the following inequality that holds for any vectors \(a\) and \(v\):

\[
\lim_{x \neq \bar{x} \to \bar{x}} \inf_{x \to \bar{x}} \frac{h(x) - h(\bar{x}) - \langle v, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0;
\]

(ii) \(v\) is a general subgradient of \(h\) at \(\bar{x}\), written \(v \in \partial h(\bar{x})\), if there exist sequences \(\{x^k\}\) and \(\{v^k\}\) such that \(x^k \to \bar{x}\) with \(h(x^k) \to h(\bar{x})\), and \(v^k \in \partial h(x^k)\) with \(v^k \to v\) when \(k \to \infty\).

The following proposition lists some well known facts on semi-continuous functions that will be used in our analysis later.

**Proposition 3.4** Let \(h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}\) and \(g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}\) be proper lower semi-continuous functions. Then it holds that:

(i) (Theorem 10.1 in [44]) The Fermat’s rule remains true: if \(\bar{x}\) is a local minimum of \(h\), then \(0 \in \partial h(\bar{x})\).

(ii) If \(h\) is continuously differentiable function, then \(\partial(h + g)(x) = \nabla h(x) + \partial g(x)\).

(iii) (Exercise 10.10 in [44]) If \(h\) is locally Lipschitz continuous at \(\bar{x}\), then \(\partial(h + g)(x) \subset \partial h(x) + \partial g(x)\).

As a direct consequence of Proposition 3.4, we have the following optimality conditions based on variational inequality (VI) for a general constrained optimization problem.

**Lemma 3.5** Suppose \(h(x)\) is locally Lipschitz continuous, \(X\) is a closed and convex set, and \(\bar{x}\) is a local minimum of \(\min_{x \in X} h(x)\). Then there exists \(v \in \partial h(\bar{x})\) such that \((x - \bar{x})^\top v \geq 0, \forall x \in X\).

**Proof.** Note that \(\min_{x \in X} h(x)\) is equivalent to \(\min_x h(x) + \delta_X(x)\), where \(\delta_X(x) = 0\) if \(x \in X\) and \(\delta_X(x) = 0\) otherwise. According to Proposition 3.4, it holds that

\[0 \in \partial(h + \delta_X)(\bar{x}) \subset \partial h(\bar{x}) + \partial \delta_X(\bar{x}) = \partial h(\bar{x}) + N_X(\bar{x}),\]

where \(N_X(\bar{x})\) is the normal cone of \(X\) at point \(\bar{x}\). That is, there exits \(v \in \partial h(\bar{x})\) such that \(-v \in N_X(\bar{x})\). By invoking the definition of normal cone of a convex set, we obtain the desired result. \(\qed\)

In our analysis, we frequently use the following identity that holds for any vectors \(a, b, c, d\),

\[
(a - b)^\top (c - d) = \frac{1}{2} (\|a - d\|^2 - \|a - c\|^2 + \|b - c\|^2 - \|b - d\|^2),
\]

and the following inequality that holds for any vectors \(a, b\), and scalar \(\xi > 0\)

\[
a^\top b \leq \frac{1}{4\xi} \|a\|^2 + \xi \|b\|^2/2.
\]

### 3.2 An \(\epsilon\)-stationary point for problem (1.1)

We now introduce notions of \(\epsilon\)-stationarity for (1.1) in the following three settings:

(i) **Setting 1:** \(r_i\) is a convex function, and \(\mathcal{X}_i\) is a compact set, for \(i = 1, \ldots, N - 1\);

(ii) **Setting 2:** \(r_i\) is Lipschitz continuous, and \(\mathcal{X}_i\) is a compact set, for \(i = 1, \ldots, N - 1\);

(iii) **Setting 3:** \(r_i\) is lower semi-continuous, and \(\mathcal{X}_i = \mathbb{R}^{n_i}\), for \(i = 1, \ldots, N - 1\).
Definition 3.6 (\(\epsilon\)-stationary point of (1.1) in Setting 1) Under the conditions in Setting 1, for \(\epsilon \geq 0\), we call \((x_1^*, \ldots, x_N^*, \lambda^*)\) to be an \(\epsilon\)-stationary point of (1.1) if for any \((x_1, \ldots, x_N, \lambda) \in X_1 \times \cdots \times X_{N-1} \times \mathbb{R}^{n_N} \times \mathbb{R}^m\), it holds that

\[
 r_i(x_i) - r_i(x_i^*) + (x_i - x_i^*)^T \left[ \nabla_i f(x_1^*, \ldots, x_N^*) - A_i^T \lambda^* \right] \geq -\epsilon, \quad i = 1, \ldots, N - 1, \tag{3.5}
\]

\[
 \left\| \nabla_N f(x_1^*, \ldots, x_{N-1}^*, x_N) - A_N^T \lambda^* \right\| \leq \epsilon, \tag{3.6}
\]

\[
 \sum_{i=1}^N A_i x_i^* - b \leq \epsilon. \tag{3.7}
\]

If \(\epsilon = 0\), we call \((x_1^*, \ldots, x_N^*, \lambda^*)\) to be a stationary point of (1.1).

Definition 3.7 (\(\epsilon\)-stationary point of (1.1) in Setting 2) Under the conditions in Setting 2, for \(\epsilon \geq 0\), we call \((x_1^*, \ldots, x_N^*, \lambda^*)\) to be an \(\epsilon\)-stationary point of (1.1) if for any \((x_1, \ldots, x_N, \lambda) \in X_1 \times \cdots \times X_{N-1} \times \mathbb{R}^{n_N} \times \mathbb{R}^m\), (3.6), (3.7) and the following hold:

\[
 (x_i - x_i^*)^T \left[ g_i^* + \nabla_i f(x_1^*, \ldots, x_N^*) - A_i^T \lambda^* \right] \geq -\epsilon, \quad i = 1, \ldots, N - 1, \tag{3.8}
\]

where \(g_i^* \in \partial r_i(\hat{x}_i)\) is a general subgradient of \(r_i\) at \(\hat{x}_i\) as defined in Definition 3.3, \(i = 1, \ldots, N - 1\). If \(\epsilon = 0\), we call \((x_1^*, \ldots, x_N^*, \lambda^*)\) to be a stationary point of (1.1).

If \(X_i = \mathbb{R}^{n_i}\) for \(i = 1, \ldots, N - 1\), then the VI kind conditions in Definition 3.7 reduce to the following one.

Definition 3.8 (\(\epsilon\)-stationary point of (1.1) in Setting 3) Under the conditions in Setting 3, for \(\epsilon \geq 0\), we call \((x_1^*, \ldots, x_N^*, \lambda^*)\) to be an \(\epsilon\)-stationary point of (1.1) if (3.6), (3.7) and the following hold:

\[
 \text{dist} \left( -\nabla_i f(x_1^*, \ldots, x_N^*) + A_i^T \lambda^*, \partial r_i(x_i^*) \right) \leq \epsilon, \quad i = 1, \ldots, N - 1, \tag{3.9}
\]

where \(\partial r_i(x_i^*)\) is the general subgradient of \(r_i\) at \(x_i^*, \ i = 1, 2, \ldots, N - 1\). If \(\epsilon = 0\), we call \((x_1^*, \ldots, x_N^*, \lambda^*)\) to be a stationary point of (1.1).

The three settings of problem (1.1) considered in this section and their corresponding definitions of \(\epsilon\)-stationary point, are summarized in Table 1.

Table 1: \(\epsilon\)-stationary point of (1.1) in three settings

| Setting | \(r_i, i = 1, \ldots, N - 1\) | \(X_i, i = 1, \ldots, N - 1\) | \(\epsilon\)-stationary point |
|---------|-----------------|-----------------|-----------------|
| Setting 1 | convex | \(X_i \subset \mathbb{R}^{n_i}\) compact | Definition 3.6 |
| Setting 2 | Lipschitz continuous | \(X_i \subset \mathbb{R}^{n_i}\) compact | Definition 3.7 |
| Setting 3 | lower semi-continuous | \(X_i = \mathbb{R}^{n_i}\) | Definition 3.8 |

A very recent work of Hong [29] proposes a definition of \(\epsilon\)-stationary point of problem (1.8), and analyzes the iteration complexity of a proximal augmented Lagrangian method for obtaining such a solution. Specifically, \((x^*, \lambda^*)\) is called an \(\epsilon\)-stationary point of (1.8) in [29] if \(Q(x^*, \lambda^*) \leq \epsilon\), where

\[
 Q(x, \lambda) := \| \nabla_x L_\beta(x, \lambda) \|^2 + \| Ax - b \|^2,
\]

and \(L_\beta(x, \lambda) := f(x) - \lambda^T (Ax - b) + \frac{\beta}{2} \| Ax - b \|^2\) is the augmented Lagrangian function of (1.8). Note that [29] assumes that \(f\) is differentiable and has bounded gradient in (1.8). The following result reveals that an \(\epsilon\)-stationary point in [29] is equivalent to an \(O(\sqrt{\epsilon})\)-stationary point of (1.1) as defined in Definition 3.8 with \(r_i = 0\) and \(f\) being differentiable. Note that there is no set constraint in (1.8), so the definition of \(\epsilon\)-stationary point in [29] does not apply to Definitions 3.6 and 3.7.
The desired result then follows immediately.

A

imal ADMM-m, that solve (1.1) with some further assumptions on

Our proximal ADMM-g solves (1.1) under the condition that

3.3 Proximal gradient-based ADMM (proximal ADMM-g)

According to Assumption 1.1, we assume that the subproblems in Step 1 of

Remark 3.10 According to Assumption 1.1, we assume that the subproblems in Step 1 of Algorithm 2 can be solved to global optimality. In fact, this can be achieved by choosing an appropriate $H_i$ such that the associated objective function is strongly convex. In addition, when the coupled objective is absent or can be linearized, after choosing some proper matrix $H_i$, the solution of the corresponding subproblem is given by the proximal mappings of $r_i$. As we mentioned earlier, many nonconvex regularization functions such as SCAD, LSP, MCP and Capped-$\ell_1$ adopt closed-form proximal mappings.
Algorithm 2 Proximal Gradient-based ADMM (proximal ADMM-g) for solving (1.1) with $A_N = I$

**Require:** Given $(x_N^0, x_{N-1}^0, \ldots, x_1^0) \in X_1 \times \cdots \times X_{N-1} \times \mathbb{R}^{nN}$, $\lambda^0 \in \mathbb{R}^m$

for $k = 0, 1, \ldots$

[Step 1] $x_N^{k+1} := \arg\min_{x_N \in X_N} \mathcal{L}_\beta(x_1^{k+1}, \ldots, x_N^{k+1}, x_{i+1}^k, \ldots, x_N^k, \lambda^k) + \frac{1}{2} \|x_i - x_i^k\|^2 H_i$ for some positive definite matrix $H_i$, $i = 1, \ldots, N - 1$

[Step 2] $x_N^{k+1} := x_N^k - \gamma \nabla_N \mathcal{L}_\beta(x_1^{k+1}, x_2^{k+1}, \ldots, x_N^{k+1}, \lambda^k)$

[Step 3] $\lambda^{k+1} := \lambda^k - \beta \left( \sum_{i=1}^N A_i x_i^{k+1} - b \right)$

end for

Before we present the main result on the iteration complexity of proximal ADMM-g, we need some lemmas.

**Lemma 3.11** Suppose the sequence $\{(x_1^k, \ldots, x_N^k, \lambda^k)\}$ is generated by Algorithm 2. The following inequality holds

$$\|\lambda^{k+1} - \lambda^k\|^2 \leq 3 \left[ \beta - \frac{1}{\gamma} \right]^2 \|x_N^k - x_N^{k+1}\|^2 + 3 \left[ \left( \beta - \frac{1}{\gamma} \right)^2 + L^2 \right] \|x_N^{k-1} - x_N^k\|^2 + 3L^2 \sum_{i=1}^{N-1} \|x_i^{k+1} - x_i^k\|^2.$$  \hspace{1cm} (3.10)

**Proof.** Note that Steps 2 and 3 of Algorithm 2 yield that

$$\lambda^{k+1} = \left( \beta - \frac{1}{\gamma} \right) (x_N^k - x_N^{k+1}) + \nabla_N f(x_1^{k+1}, \ldots, x_N^{k+1}, x_N^k).$$ \hspace{1cm} (3.11)

Combining (3.11) and (3.1) yields that

$$\|\lambda^{k+1} - \lambda^k\|^2 \leq \|\left( \nabla_N f(x_1^{k+1}, \ldots, x_N^{k+1}, x_N^k) - \nabla_N f(x_1^k, \ldots, x_N^k, x_N^{k-1}) \right) + \left[ \beta - \frac{1}{\gamma} \right] (x_N^k - x_N^{k+1})$$

$$- \left[ \beta - \frac{1}{\gamma} \right] (x_N^{k-1} - x_N^k) \|^2$$

$$\leq 3 \|\nabla_N f(x_1^{k+1}, \ldots, x_N^{k+1}, x_N^k) - \nabla_N f(x_1^k, \ldots, x_N^k, x_N^{k-1})\|^2 + 3 \left[ \left( \beta - \frac{1}{\gamma} \right)^2 + L^2 \right] \|x_N^{k-1} - x_N^k\|^2$$

$$+ 3 \left[ \beta - \frac{1}{\gamma} \right]^2 \|x_N^{k-1} - x_N^k\|^2$$

$$\leq 3 \left[ \beta - \frac{1}{\gamma} \right]^2 \|x_N^k - x_N^{k+1}\|^2 + 3 \left( \left( \beta - \frac{1}{\gamma} \right)^2 + L^2 \right) \|x_N^{k-1} - x_N^k\|^2 + 3L^2 \sum_{i=1}^{N-1} \|x_i^{k+1} - x_i^k\|^2.$$ \hspace{1cm} \square

We now define the following function, which will play a crucial role in our analysis:

$$\Psi_G (x_1, x_2, \ldots, x_N, \lambda, \bar{x}) = \mathcal{L}_\beta(x_1, x_2, \ldots, x_N, \lambda) + \frac{3}{\beta} \left[ \left( \beta - \frac{1}{\gamma} \right)^2 + L^2 \right] \|x_N - \bar{x}\|^2.$$ \hspace{1cm} (3.12)

**Lemma 3.12** Suppose the sequence $\{(x_1^k, \ldots, x_N^k, \lambda^k)\}$ is generated by Algorithm 2. In Algorithm 2, assume

$$\beta > \max \left( \frac{18\sqrt{3} + 6}{13} L, \max_{i=1,2,\ldots,N-1} \frac{6L^2}{\sigma_{\min}(H_i)} \right).$$ \hspace{1cm} (3.13)
Then \( \Psi_G(x_1^{k+1}, \ldots, x_N^{k+1}, \lambda^{k+1}, x_N^k) \) monotonically decreases over \( k \geq 0 \), where \( \gamma > 0 \) is set as

\[
\gamma \in \left\{ \begin{array}{ll}
\left( \frac{\sqrt{13\beta^2-123L^2-72L^2}-2\beta}{93\beta^2-123L^2-72L^2}, +\infty \right), & \text{if } \beta \in \left( \frac{2+2\sqrt{13}}{3} L, +\infty \right) \\
\left( 2\beta-\frac{\sqrt{13\beta^2-123L^2-72L^2}}{72L^2+123L^2-93\beta^2}, \frac{\beta^2-\sqrt{13\beta^2-123L^2-72L^2}}{72L^2+123L^2-93\beta^2} \right), & \text{if } \beta \in \left( \frac{18\sqrt{3}+6}{13} L, \frac{2+2\sqrt{13}}{3} L \right]. 
\end{array} \right.
\]  

(3.14)

**Proof.** From Step 1 of Algorithm 2 it is easy to obtain that

\[
\mathcal{L}_\beta(x_1^{k+1}, \ldots, x_{N-1}^{k+1}, x_N^k) \leq \mathcal{L}_\beta(x_1^k, \ldots, x_N^k, \lambda^k) - \sum_{i=1}^{N-1} \frac{1}{2} \|x_i^k - x_i^{k+1}\|^2_{H_i}.
\]  

(3.15)

From Step 2 of Algorithm 2 we get that

\[
0 = (x_N^k - x_{N-1}^{k+1})^\top \left[ \nabla f(x_1^{k+1}, \ldots, x_N^k) - \lambda^k + \beta \left( \sum_{i=1}^{N-1} A_i x_i^{k+1} + x_N^k - b \right) - \frac{1}{\gamma} (x_N^k - x_{N-1}^{k+1}) \right]
\]

\[
\leq f(x_1^{k+1}, \ldots, x_N^{k+1}, x_N^k) - f(x_1^{k+1}, \ldots, x_{N-1}^{k+1}, x_N^k) + \frac{L}{2} \|x_N^k - x_{N-1}^{k+1}\|^2 - \left( x_N^k - x_{N-1}^{k+1} \right)^\top \lambda^k
\]

\[
+ \frac{\beta}{2} \|x_N^k - x_{N-1}^{k+1}\|^2 + \beta \left( \sum_{i=1}^{N-1} A_i x_i^{k+1} + x_N^k - b \right) - \frac{\beta}{2} \left( \sum_{i=1}^{N-1} A_i x_i^{k+1} + x_N^k - b \right)^2
\]

\[
= \mathcal{L}_\beta(x_1^{k+1}, \ldots, x_{N-1}^{k+1}, x_N^k, \lambda^k) - \mathcal{L}_\beta(x_1^{k+1}, \ldots, x_{N-1}^{k+1}, x_N^k, \lambda^k) + \left( \frac{L + \beta}{2} - \frac{1}{\gamma} \right) \|x_N^k - x_{N-1}^{k+1}\|^2,
\]  

(3.16)

where the inequality follows from (3.2) and (3.3). Moreover, the following equality holds trivially

\[
\mathcal{L}_\beta(x_1^{k+1}, \ldots, x_{N-1}^{k+1}, x_N^k, \lambda^k) = \mathcal{L}_\beta(x_1^{k+1}, \ldots, x_{N-1}^{k+1}, x_N^k, \lambda^k) + \frac{1}{\beta} \|\lambda^k - \lambda^{k+1}\|^2.
\]  

(3.17)

Combining (3.15), (3.16), (3.17) and (3.10) yields that

\[
\mathcal{L}_\beta(x_1^{k+1}, \ldots, x_{N-1}^{k+1}, x_N^k, \lambda^k) - \mathcal{L}_\beta(x_1^{k}, \ldots, x_N^k, \lambda^k)
\]

\[
\leq \left( \frac{L + \beta}{2} - \frac{1}{\gamma} \right) \|x_N^k - x_{N-1}^{k+1}\|^2 - \sum_{i=1}^{N-1} \frac{1}{2} \|x_i^k - x_i^{k+1}\|^2_{H_i} + \frac{1}{\beta} \|\lambda^k - \lambda^{k+1}\|^2
\]

\[
\leq \left( \frac{L + \beta}{2} - \frac{1}{\gamma} + \frac{3}{\beta} \left[ \beta - \frac{1}{\gamma} \right]^2 \right) \|x_N^k - x_{N-1}^{k+1}\|^2 + \frac{3}{\beta} \left[ \beta - \frac{1}{\gamma} \right]^2 + \frac{L^2}{\beta} \|x_N^k - x_N^{k+1}\|^2
\]

\[+ \sum_{i=1}^{N-1} (x_i^k - x_i^{k+1})^\top \left( \frac{3L^2}{\beta} - \frac{1}{2} H_i \right) (x_i^k - x_i^{k+1}), \]

which further implies that

\[
\Psi_G(x_1^{k+1}, \ldots, x_N^{k+1}, \lambda^{k+1}, x_N^k) - \Psi_G(x_1^{k}, \ldots, x_N^{k}, \lambda^k, x_N^{k-1})
\]

\[
\leq \left( \frac{L + \beta}{2} - \frac{1}{\gamma} + \frac{6}{\beta} \left[ \beta - \frac{1}{\gamma} \right]^2 + \frac{3L^2}{\beta} \right) \|x_N^k - x_{N-1}^{k+1}\|^2 - \sum_{i=1}^{N-1} \|x_i^k - x_i^{k+1}\|^2_{H_i - \frac{3L^2}{\beta} I}.
\]  

(3.18)

It is easy to verify that when \( \beta > \frac{18\sqrt{3}+6}{13} L \), then \( \gamma \) defined as in (3.14) ensures that \( \gamma > 0 \) and

\[
\frac{L + \beta}{2} - \frac{1}{\gamma} + \frac{6}{\beta} \left[ \beta - \frac{1}{\gamma} \right]^2 + \frac{3L^2}{\beta} < 0.
\]  

(3.19)
Therefore, choosing $\beta > \max \left(\frac{18\sqrt{3}+6}{13} L, \max_{i=1,2,\ldots,N-1} \frac{6L^2}{\sigma_{\min}(H_i)} \right)$ and $\gamma$ as in (3.14) guarantees that $\Psi_G(x_1^{k+1}, \ldots, x_N^{k+1}, \lambda^{k+1}, x_N^k)$ monotonically decreases over $k \geq 0$. In fact, (3.19) can be verified as follows. By denoting $z = \beta - \frac{1}{\gamma}$, (3.19) is equivalent to

$$12z^2 + 2\beta z + (6L^2 + \beta L - \beta^2) < 0,$$

which holds when $\beta > \frac{18\sqrt{3}+6}{13} L$ and

$$-\beta - \sqrt{13\beta^2 - 12\beta L - 72L^2} < z < -\beta + \sqrt{13\beta^2 - 12\beta L - 72L^2},$$

i.e.,

$$-2\beta - \sqrt{13\beta^2 - 12\beta L - 72L^2} < -\frac{1}{\gamma} < -2\beta + \sqrt{13\beta^2 - 12\beta L - 72L^2},$$

which holds when $\gamma$ is chosen as in (3.14).

\[ \square \]

**Lemma 3.13** Suppose the sequence $\{(x_1^k, \ldots, x_N^k, \lambda^k)\}$ is generated by Algorithm 2. Under the same conditions as in Lemma 3.12, for any $k \geq 0$, we have

$$\Psi_G(x_1^{k+1}, \ldots, x_N^{k+1}, \lambda^{k+1}, x_N^k) \geq \sum_{i=1}^{N-1} r_i^* + f^*,$$

where $r_i^*$ and $f^*$ are defined in Assumption 3.2.

**Proof.** Note that from (3.11), we have

$$L_\beta(x_1^{k+1}, \ldots, x_N^{k+1}, \lambda^{k+1})$$

$$= \sum_{i=1}^{N-1} r_i(x_i^{k+1}) + f(x_1^{k+1}, \ldots, x_N^{k+1}) - \left( \sum_{i=1}^{N-1} A_i x_i^{k+1} + x_N^{k+1} - b \right) \top \nabla_N f(x_1^{k+1}, \ldots, x_N^{k+1})$$

$$+ \frac{\beta}{2} \left\| \sum_{i=1}^{N-1} A_i x_i^{k+1} + x_N^{k+1} - b \right\|^2 - \left( \sum_{i=1}^{N-1} A_i x_i^{k+1} + x_N^{k+1} - b \right) \top \left[ (\beta - \frac{1}{\gamma}) (x_N^k - x_N^{k+1}) \right.\left. + (\nabla_N f(x_1^{k+1}, \ldots, x_N^{k+1}, x_N^k) - \nabla_N f(x_1^{k+1}, \ldots, x_N^{k+1})) \right]$$

$$\geq \sum_{i=1}^{N-1} r_i(x_i^{k+1}) + f(x_1^{k+1}, \ldots, x_N^{k+1}, b - \sum_{i=1}^{N-1} A_i x_i^{k+1}) + \left( \frac{\beta}{2} - \frac{6}{L} \right) \left\| \sum_{i=1}^{N-1} A_i x_i^{k+1} + x_N^{k+1} - b \right\|^2$$

$$- \frac{3}{\beta} \left[ (\beta - \frac{1}{\gamma})^2 + L^2 \right] \left\| x_N^k - x_N^{k+1} \right\|^2$$

$$\geq \sum_{i=1}^{N-1} r_i^* + f^* - \frac{3}{\beta} \left[ (\beta - \frac{1}{\gamma})^2 + L^2 \right] \left\| x_N^k - x_N^{k+1} \right\|^2,$$

where the first inequality follows from (3.2) and (3.4) with $\xi = 3/(2\beta)$, and the second inequality is due to $\beta \geq 3L/2$. The desired result follows from the definition of $\Psi_G$ in (3.12).

\[ \square \]

Now we are ready to give the iteration complexity of Algorithm 2 for finding an $\epsilon$-stationary point of (1.1).

**Theorem 3.14** Suppose the sequence $\{(x_1^k, \ldots, x_N^k, \lambda^k)\}$ is generated by Algorithm 2. Assume $\beta$ satisfies (3.13) and $\gamma$ satisfies (3.14). Denote $\kappa_1 := \frac{3}{\beta^2} \left[ (\beta - \frac{1}{\gamma})^2 + L^2 \right], \kappa_2 :=
By summing (3.18) over \( k \), 

\[
(\|\beta - \frac{1}{\gamma}\| + L)^2, \quad \kappa_3 := \left( L + \beta \sqrt{N} \max_{1 \leq i \leq N} \|A_i\|_2^2 + \max_{1 \leq i \leq N} \|H_i\|_2 \right)^2, \quad \kappa_4 := \max_{1 \leq i \leq N-1} (\text{diam}(X_i))^2
\]

and 

\[
\tau := \min \left\{ \left(-\frac{L + \beta - 1}{2} - \frac{6}{\beta} \left( \beta - \frac{1}{\gamma} \right)^2 + \frac{3L^2}{\beta} \right), \min_{i=1, \ldots, N-1} \left\{ -\frac{3L^2}{\beta} - \sigma_{i\min}(H_i) \right\} \right\} > 0.
\]

Letting 

\[
K := \left\{ \left[ 2 \max_{1 \leq k \leq k+1} \frac{N}{\tau^2} \left( \Psi_G(x_1^k, \ldots, x_N^k, \lambda^k, x_0^k) - \sum_{i=1}^{N-1} r_i^* - f^* \right) \right] \right\}, \quad \text{for Settings 1, 2}
\]

and denoting 

\[
k \hat{} = \arg \min_{2 \leq k \leq K+1} \sum_{i=1}^{N} \left( \|x_i^k - x_i^{k+1}\|^2 + \|x_i^{k-1} - x_i^k\|^2 \right),
\]

then \((x_1^k, \ldots, x_N^k, \lambda^k)\) is an \( \epsilon \)-stationary point of optimization problem (1.1) with \( A_N = I \), according to Definitions 3.6, 3.7 and 3.8 under the conditions of Settings 1,2 and 3 in Table 1, respectively.

Proof. For ease of presentation, denote 

\[
\theta_k := \sum_{i=1}^{N} (\|x_i^k - x_i^{k+1}\|^2 + \|x_i^{k-1} - x_i^k\|^2).
\]

By summing (3.18) over \( k = 1, \ldots, K \), we obtain that 

\[
\Psi_G(x_1^{K+1}, \ldots, x_N^{K+1}, \lambda^{K+1}, x_N^{K}) - \Psi_G(x_1^1, \ldots, x_N^1, \lambda^1, x_N^0) \leq -\tau K \sum_{k=1}^{N} \|x_i^k - x_i^{k+1}\|^2,
\]

where \( \tau \) is defined in (3.20). By invoking Lemmas 3.12 and 3.13, we get 

\[
\min_{2 \leq k \leq K+1} \theta_k \leq \frac{1}{\tau K} \left[ \Psi_G(x_1^1, \ldots, x_N^1, \lambda^1, x_N^0) + \Psi_G(x_1^2, \ldots, x_N^2, \lambda^2, x_N^1) - 2 \sum_{i=1}^{N} r_i^* - 2f^* \right]
\]

\[
\leq \frac{2}{\tau K} \left[ \Psi_G(x_1^1, \ldots, x_N^1, \lambda^1, x_N^0) - \sum_{i=1}^{N} r_i^* - f^* \right].
\]

We now give upper bounds to the terms in (3.6) and (3.7) through \( \theta_k \). Note that (3.11) implies that 

\[
\|\lambda^{k+1} - \nabla_N f(x_1^{k+1}, \ldots, x_N^{k+1})\| 
\leq \|\beta - \frac{1}{\gamma}\| \|x_N^k - x_N^{k+1}\| + \|\nabla_N f(x_1^{k+1}, \ldots, x_N^{k+1}) - \nabla f(x_1^{k+1}, \ldots, x_N^{k+1})\|
\leq \left( \|\beta - \frac{1}{\gamma}\| + L \right) \|x_N^k - x_N^{k+1}\|,
\]

which yields 

\[
\|\lambda^{k+1} - \nabla_N f(x_1^{k+1}, \ldots, x_N^{k+1})\|^2 \leq \left( \|\beta - \frac{1}{\gamma}\| + L \right)^2 \theta_k.
\]

(3.24)
From Step 3 of Algorithm 2 and (3.10) it is easy to see that
\[
\left\| \sum_{i=1}^{N-1} A_i x_i^{k+1} + x_N^{k+1} - b \right\|^2 = \frac{1}{\beta^2} \| \lambda^{k+1} - \lambda^k \|^2
\]
\[
\leq \frac{3}{\beta^2} \left[ \beta - \frac{1}{\gamma} \right]^2 \| x_N^k - x_N^{k+1} \|^2 + \frac{3}{\beta^2} \left[ \left( \beta - \frac{1}{\gamma} \right)^2 + L^2 \right] \| x_N^{k-1} - x_N^k \|^2 + \frac{3L^2}{\beta^2} \sum_{i=1}^{N-1} \| x_i^k - x_i^{k+1} \|^2
\]
\[
\leq \frac{3}{\beta^2} \left[ \left( \beta - \frac{1}{\gamma} \right)^2 + L^2 \right] \theta_k.
\]
(3.25)

We now give upper bounds to the terms in (3.5), (3.8) and (3.9) under the three settings in Table 1, respectively.

**Setting 3.** Because \( r_i \) is lower semi-continuous and \( X_i = \mathbb{R}^n, i = 1, \ldots, N - 1 \), it follows from Step 1 of Algorithm 2 that there exists a general subgradient \( g_i \in \partial r_i(x_i^{k+1}) \) such that
\[
\text{dist} \left( -\nabla_i f(x_1^{k+1}, \ldots, x_N^{k+1}) + A_i^\top \lambda^{k+1}, \partial r_i(x_i^{k+1}) \right) \leq \| g_i + \nabla_i f(x_1^{k+1}, \ldots, x_N^{k+1}) - A_i^\top \lambda^{k+1} \|
\]
\[
\leq \| \nabla_i f(x_1^{k+1}, \ldots, x_N^{k+1}) - \nabla_i f(x_1^{k+1}, \ldots, x_i^{k+1}, x_{i+1}^k, \ldots, x_N^k) \|
\]
\[
+ \beta A_i^\top \left( \sum_{j=i+1}^{N} A_j (x_j^{k+1} - x_j^k) - H_i(x_i^{k+1} - x_i^k) \right)
\]
\[
\leq L \sqrt{\sum_{j=i+1}^{N} \| x_j^k - x_j^{k+1} \|^2 + \beta \| A_i \|_2 \sum_{j=i+1}^{N} \| A_j \|_2 \| x_j^{k+1} - x_j^k \| + \| H_i \|_2 \| x_i^{k+1} - x_i^k \|_2}
\]
\[
\leq \left( L + \beta \sqrt{N} \max_{1 \leq i \leq N} [\| A_i \|_2^2 + \max_{1 \leq i \leq N} \| H_i \|_2^2] \right) \sqrt{\theta_k}
\]

By combining (3.26), (3.24) and (3.25) we conclude that Algorithm 2 returns an \( \epsilon \)-stationary point of (1.1) according to Definition 3.8 under conditions in Setting 3 in Table 1.

**Setting 2.** Under this setting, we know \( r_i \) is Lipschitz continuous and \( X_i \subset \mathbb{R}^n \) is convex and compact. Because \( f(x_1, \ldots, x_N) \) is differentiable, we know \( r_i(x_i) + f(x_1, \ldots, x_N) \) is locally Lipschitz continuous with respect to \( x_i \) for \( i = 1, 2, \ldots, N - 1 \). Similar to (3.26), for any \( x_i \in X_i \), Step 1 of Algorithm 2 yields that
\[
\left( x_i - x_i^{k+1} \right)^\top \left[ g_i + \nabla_i f(x_1^{k+1}, \ldots, x_N^{k+1}) - A_i^\top \lambda^{k+1} \right]
\]
\[
\geq \left( x_i - x_i^{k+1} \right)^\top \left[ \nabla_i f(x_1^{k+1}, \ldots, x_N^{k+1}) - \nabla_i f(x_1^{k+1}, \ldots, x_i^{k+1}, x_{i+1}^k, \ldots, x_N^k) \right]
\]
\[
+ \beta A_i^\top \left( \sum_{j=i+1}^{N} A_j (x_j^{k+1} - x_j^k) - H_i(x_i^{k+1} - x_i^k) \right)
\]
\[
\geq -L \text{diam}(X_i) \sqrt{\sum_{j=i+1}^{N} \| x_j^k - x_j^{k+1} \|^2 - \beta \| A_i \|_2 \text{diam}(X_i) \sum_{j=i+1}^{N} \| A_j \|_2 \| x_j^{k+1} - x_j^k \|}
\]
\[
- \text{diam}(X_i) \| H_i \|_2 \| x_i^{k+1} - x_i^k \|_2
\]
\[
\geq - \left( \beta \sqrt{N} \max_{1 \leq i \leq N} \| A_i \|_2^2 + L + \max_{1 \leq i \leq N} \| H_i \|_2 \right) \max_{1 \leq i \leq N-1} \{ \text{diam}(X_i) \} \sqrt{\theta_k},
\]
where \( g_i \in \partial r_i(x_i^{k+1}) \) is a general subgradient of \( r_i \) at \( x_i^{k+1} \). By combining (3.27), (3.24) and (3.25) we conclude that Algorithm 2 returns an \( \epsilon \)-stationary point of (1.1) according to Definition 3.6 under conditions in Setting 2 in Table 1.

**Setting 1.** Under this setting, \( r_i \) is convex, so \( g_i \) in (3.27) becomes a subgradient of \( r_i \) at \( x_i^{k+1} \). Therefore, for \( i = 1, 2, \cdots, N-1 \) and any \( x_i \in \mathcal{X}_i \) we have that

\[
    r_i(x_i) - r_i(x_i^{k+1}) + (x_i - x_i^{k+1})^T \left[ \nabla_i f(x_i^{k+1}, \cdots, x_N^{k+1}) - A_i^T \lambda^{k+1} \right]
\]

(3.28)

\[
    \geq (x_i - x_i^{k+1})^T \left[ g_i + \nabla_i f(x_i^{k+1}, \cdots, x_N^{k+1}) - A_i^T \lambda^{k+1} \right]
\]

(3.29)

By combining (3.28), (3.24) and (3.25) we conclude that Algorithm 2 returns an \( \epsilon \)-stationary point of (1.1) according to Definition 3.6 under the conditions of Setting 1 in Table 1. \( \square \)

**Remark 3.15** Note that the potential function \( \Psi_G \) defined in (3.12) is related to the augmented Lagrangian function. The augmented Lagrangian function has been used as a potential function in analyzing the convergence of nonconvex splitting and ADMM methods in [2,28–30,35]. See [29] for a more detailed discussion on this.

### 3.4 Proximal majorization ADMM (proximal ADMM-m)

Our proximal ADMM-m solves (1.1) under the condition that \( A_N \) is full row rank. In this section, we use \( \sigma_N \) to denote the smallest eigenvalue of \( A_N A_N^\top \). Note that \( \sigma_N > 0 \) because \( A_N \) is full row rank. Our proximal ADMM-m can be described as in Algorithm 3, where \( U(x_1, \cdots, x_{N-1}, x_N, \lambda, \bar{x}) \) is defined as

\[
    U(x_1, \cdots, x_{N-1}, x_N, \lambda, \bar{x}) = f(x_1, \cdots, x_{N-1}, \bar{x}) + (x_N - \bar{x})^\top \nabla_N f(x_1, \cdots, x_{N-1}, \bar{x}) + \frac{L}{2} \| x_N - \bar{x} \|^2 - \left\langle \lambda, \sum_{i=1}^N A_i x_i - b \right\rangle + \frac{\beta}{2} \left\| \sum_{i=1}^N A_i x_i - b \right\|^2 .
\]

**Algorithm 3** Proximal majorization ADMM (proximal ADMM-m) for solving (1.1) with \( A_N \) being full row rank

**Require:** Given \((x_1^0, x_2^0, \cdots, x_N^0) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_{N-1} \times \mathbb{R}^{nN}, \lambda^0 \in \mathbb{R}^m\)

for \( k = 0, 1, \ldots \) do

[Step 1] \( x_i^{k+1} := \operatorname{argmin}_{x_i \in \mathcal{X}_i} \mathcal{L}_g(x_1^{k+1}, \cdots, x_{i-1}^{k+1}, x_i, x_{i+1}^{k}, \cdots, x_N^{k}, \lambda^k) + \frac{L}{2} \| x_i - x_i^k \|^2 \) for some positive definite matrix \( H_i, i = 1, \ldots, N-1 \)

[Step 2] \( x_N^{k+1} := \operatorname{argmin}_{x_N} U(x_1^{k+1}, \cdots, x_{N-1}^{k+1}, x_N, \lambda^k, x_N^k) \)

[Step 3] \( \lambda^{k+1} := \lambda^k - \beta \left( \sum_{i=1}^N A_i x_i^{k+1} - b \right) \)

end for

It is worth noting the proximal ADMM-m and proximal ADMM-g differ only in Step 2: Step 2 of proximal ADMM-g takes a gradient step of the augmented Lagrangian function with respect to \( x_N \), while Step 2 of proximal ADMM-m requires to minimize a quadratic function of \( x_N \).

We provide some lemmas that are useful in analyzing the iteration complexity of proximal ADMM-m for solving (1.1).

**Lemma 3.16** Suppose the sequence \( \{(x_1^k, \cdots, x_N^k, \lambda^k)\} \) is generated by Algorithm 3. The following inequality holds

\[
\left\| \lambda^{k+1} - \lambda^k \right\|^2 \leq \frac{3L^2}{\sigma_N} \left\| x_N^k - x_N^{k+1} \right\|^2 + \frac{6L^2}{\sigma_N} \left\| x_N^{k-1} - x_N^k \right\|^2 + \frac{3L^2}{\sigma_N} \sum_{i=1}^{N-1} \left\| x_i^k - x_i^{k+1} \right\|^2 .
\]

(3.29)
Proof. From the optimality conditions of Step 2 of Algorithm 3, we have

\[
0 = \nabla_N f(x_1^{k+1}, \ldots, x_N^{k+1}) - A_N^\top \lambda^k + \beta A_N^\top \left( \sum_{i=1}^N A_i x_i^{k+1} - b \right) - L(x_N^k - x_N^{k+1})
\]

\[
= \nabla_N f(x_1^{k+1}, \ldots, x_N^{k+1}) - A_N^\top \lambda^k - L(x_N^k - x_N^{k+1}),
\]

where the second equality is due to Step 3 of Algorithm 3. Therefore, we have

\[
\|\lambda^{k+1} - \lambda^k\|^2 \leq \sigma_N^{-1} \|A_N^\top \lambda^k + A_N^\top \lambda^k\|^2
\]

\[
\leq \sigma_N^{-1} \|\nabla_N f(x_1^{k+1}, \ldots, x_N^{k+1}) - \nabla_N f(x_1^{k}, \ldots, x_N^{k+1})\| - L(x_N^k - x_N^{k+1}) + L(x_N^{k+1} - x_N^k)\|^2
\]

\[
\leq \frac{3}{\sigma_N} \|\nabla_N f(x_1^{k+1}, \ldots, x_N^{k+1}) - \nabla_N f(x_1^{k}, \ldots, x_N^{k+1})\|^2 + \frac{3L^2}{\sigma_N} (\|x_N^k - x_N^{k+1}\|^2 + \|x_N^{k+1} - x_N^k\|^2)
\]

\[
\leq \frac{3L^2}{\sigma_N} \|x_N^k - x_N^{k+1}\|^2 + \frac{6L^2}{\beta \sigma_N} \|x_N^k - x_N^{k+1}\|^2 + \frac{3L^2}{\sigma_N} \sum_{i=1}^{N-1} \|x_i^k - x_i^{k+1}\|^2.
\]

We define the following function that will be used in the analysis of proximal ADMM-m:

\[
\Psi_L(x_1, \ldots, x_N, \lambda, \bar{x}) = \mathcal{L}_\beta(x_1, \ldots, x_N, \lambda) + \frac{6L^2}{\beta \sigma_N} \|x_N - \bar{x}\|^2.
\]

Similar as the function used in proximal ADMM-g, we can prove the monotonicity and boundedness of function \(\Psi_L\).

**Lemma 3.17** Suppose the sequence \(\{(x_1^k, \ldots, x_N^k, \lambda^k)\}\) is generated by Algorithm 3. In Algorithm 3, assume

\[
\beta > \max \left\{ \frac{18L}{\sigma_N}, \max_{1 \leq i \leq N-1} \left\{ \frac{6L^2}{\sigma_N \sigma_{\min}(H_i)} \right\} \right\}, \tag{3.30}
\]

Then \(\Psi_L(x_1^{k+1}, \ldots, x_N^{k+1}, \lambda^{k+1}, x_N^k)\) monotonically decreases over \(k > 0\).

**Proof.** By Step 1 of Algorithm 3 one observes that

\[
\mathcal{L}_\beta(x_1^{k+1}, \ldots, x_N^{k+1}, x_N^k, \lambda^k) \leq \mathcal{L}_\beta(x_1^k, \ldots, x_N^k, \lambda^k) - \sum_{i=1}^{N-1} \frac{1}{2} \|x_i^k - x_i^{k+1}\|_{H_i}^2, \tag{3.31}
\]

while by Step 2 of Algorithm 3 we have

\[
0 = (x_N^k - x_N^{k+1})^\top \left[ \nabla_N f(x_1^{k+1}, \ldots, x_N^{k+1}, x_N^k) - A_N^\top \lambda^k + \beta A_N^\top \left( \sum_{i=1}^N A_i x_i^{k+1} - b \right) - L(x_N^k - x_N^{k+1}) \right]
\]

\[
\leq f(x_1^{k+1}, \ldots, x_N^{k+1}, x_N^k) - f(x_1^k, \ldots, x_N^k) - \frac{L}{2} \|x_N^k - x_N^{k+1}\|^2 - \left( \sum_{i=1}^N A_i x_i^{k+1} + A_N x_N^k - b \right)^\top \lambda^k
\]

\[
+ \left( \sum_{i=1}^N A_i x_i^{k+1} - b \right)^\top \lambda^k + \frac{\beta}{2} \left( \sum_{i=1}^N A_i x_i^{k+1} + A_N x_N^k - b \right) - \frac{\beta}{2} \left( \sum_{i=1}^N A_i x_i^{k+1} - b \right)^2
\]

\[
- \frac{\beta}{2} \|A_N x_N^k - A_N x_N^{k+1}\|^2
\]

\[
\leq \mathcal{L}_\beta(x_1^{k+1}, \ldots, x_N^{k+1}, x_N^k, \lambda^k) - \mathcal{L}_\beta(x_1^k, \ldots, x_N^k, \lambda^k) - \frac{L}{2} \|x_N^k - x_N^{k+1}\|^2, \tag{3.32}
\]
where the first inequality is due to (3.2) and (3.3). Moreover, from (3.29) we have
\[
L_\beta(x_1^{k+1}, \ldots, x_N^{k+1}, \lambda^{k+1}) - L_\beta(x_1^k, \ldots, x_N^k, \lambda^k) = \frac{1}{\beta} \| \lambda^k - \lambda^{k+1} \|^2
\]
\[
\leq \frac{3L^2}{\beta\sigma_N} \| x_N^k - x_N^{k+1} \|^2 + \frac{6L^2}{\beta\sigma_N} \| x_N^{k-1} - x_N^k \|^2 + \frac{3L^2}{\beta\sigma_N} \sum_{i=1}^{N-1} \| x_i^k - x_i^{k+1} \|^2. \tag{3.33}
\]
Combining (3.31), (3.32) and (3.33) yields that
\[
L_\beta(x_1^{k+1}, \ldots, x_N^{k+1}, \lambda^{k+1}) - L_\beta(x_1^k, \ldots, x_N^k, \lambda^k)
\leq \left( \frac{3L^2}{\beta\sigma_N} - \frac{L}{2} \right) \| x_N^k - x_N^{k+1} \|^2 + \sum_{i=1}^{N-1} \| x_i^k - x_i^{k+1} \|^2 \sum_{i=1}^{\sigma_{\min}(H_i)} + \frac{6L^2}{\beta\sigma_N} \| x_N^{k-1} - x_N^k \|^2,
\]
which further implies that
\[
\Psi_L(x_1^{k+1}, \ldots, x_N^{k+1}, \lambda^{k+1}, x_N^k) - \Psi_L(x_1^k, \ldots, x_N^k, \lambda^k, x_N^{k-1})
\leq \left( \frac{9L^2}{\beta\sigma_N} - \frac{L}{2} \right) \| x_N^k - x_N^{k+1} \|^2 + \sum_{i=1}^{N-1} \left( \frac{3L^2}{\beta\sigma_N} - \frac{\sigma_{\min}(H_i)}{2} \right) \| x_i^k - x_i^{k+1} \|^2 < 0,
\tag{3.34}
\]
where the second inequality is due to (3.30). This completes the proof. \(\square\)

The following lemma shows that the function \(\Psi_L\) is lower bounded.

**Lemma 3.18** Suppose the sequence \(\{ (x_1^k, \ldots, x_N^k, \lambda^k) \}\) is generated by Algorithm 3. Under the same conditions as in Lemma 3.17, the sequence \(\{ \Psi_L(x_1^{k+1}, \ldots, x_N^{k+1}, \lambda^{k+1}, x_N^k) \}\) is bounded from below.

**Proof.** From Step 3 of Algorithm 3 we have
\[
\Psi_L(x_1^{k+1}, \ldots, x_N^{k+1}, \lambda^{k+1}, x_N^k) \geq L_\beta(x_1^{k+1}, \ldots, x_N^{k+1}, \lambda^{k+1})
\]
\[
= \sum_{i=1}^{N-1} r_i(x_i^{k+1}) + f(x_1^{k+1}, \ldots, x_N^{k+1}) - \sum_{i=1}^{N} A_i x_i^{k+1} - b = \sum_{i=1}^{N} \left( \sum_{i=1}^{N} A_i x_i^{k+1} - b \right) \lambda^{k+1} + \frac{\beta}{2} \left( \sum_{i=1}^{N} A_i x_i^{k+1} - b \right)^2
\]
\[
= \sum_{i=1}^{N-1} r_i(x_i^{k+1}) + f(x_1^{k+1}, \ldots, x_N^{k+1}) - \frac{1}{\beta} (\lambda^k - \lambda^{k+1})^\top \lambda^{k+1} + \frac{1}{2\beta} \| \lambda^k - \lambda^{k+1} \|^2
\]
\[
= \sum_{i=1}^{N-1} r_i(x_i^{k+1}) + f(x_1^{k+1}, \ldots, x_N^{k+1}) - \frac{1}{2\beta} \| \lambda^k \|^2 + \frac{1}{2\beta} \| \lambda^{k+1} \|^2 + \frac{1}{\beta} \| \lambda^k - \lambda^{k+1} \|^2 \tag{3.35}
\]
\[
\geq \sum_{i=1}^{N-1} r_i^* + f^* - \frac{1}{2\beta} \| \lambda^k \|^2 + \frac{1}{2\beta} \| \lambda^{k+1} \|^2,
\]
where the third equality follows from (3.3). Summing this inequality over \(k = 0, 1, \ldots, K - 1\) for any integer \(K \geq 1\) yields that
\[
\frac{1}{K} \sum_{k=0}^{K-1} \Psi_L(x_1^{k+1}, \ldots, x_N^{k+1}, \lambda^{k+1}, x_N^k) \geq \sum_{i=1}^{N-1} r_i^* + f^* - \frac{1}{2\beta} \| \lambda^0 \|^2.
\]

Lemma 3.17 stipulates that \(\{ \Psi_L(x_1^{k+1}, \ldots, x_N^{k+1}, \lambda^{k+1}, x_N^k) \}\) is a monotonically decreasing sequence; the above inequality thus further implies that the entire sequence is bounded from below. \(\square\)

We are now ready to give the iteration complexity of proximal ADMM-m, whose proof is similar to that of Theorem 3.14.

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Theorem 3.19 Suppose the sequence \( \{(x^k, \cdots, x^k_N, \lambda^k)\} \) is generated by proximal ADMM-m (Algorithm 3), and \( \beta \) satisfies (3.30). Denote

\[
\kappa_1 := \frac{6L^2}{\beta^2\sigma_N}, \kappa_2 := 4L^2, \kappa_3 := \left( L + \beta\sqrt{N} \max_{1 \leq i \leq N} \|[A_i]\|_2 \right) + \max_{1 \leq i \leq N} \|H_i\|_2 \right) ^2, \kappa_4 := \max_{1 \leq i \leq N-1} (\text{diam}(\mathcal{X}_i))^2
\]

and

\[
\tau := \min \left\{ -\left( \frac{9L^2}{\beta\sigma_N} - \frac{L}{2} \right), \min_{i=1, \ldots, N-1} \left\{ -\left( \frac{3L^2}{\beta\sigma_N} - \frac{\sigma_{\text{min}}(H_i)}{2} \right) \right\} \right\} > 0. \quad (3.36)
\]

Letting

\[
K := \begin{cases} 
\left\lfloor \frac{2\max\{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}}{\tau e^2} \right\rfloor \left( \Psi_L(x^1_1, \cdots, x^1_N, \lambda^1, x^0_N) - \sum_{i=1}^{N-1} r_i^* - f^* \right) \right), & \text{for Settings 1,2} \\
\left\lfloor \frac{2\max\{\kappa_1, \kappa_2, \kappa_3\}}{\tau e^2} \right\rfloor \left( \Psi_L(x^1_1, \cdots, x^1_N, \lambda^1, x^0_N) - \sum_{i=1}^{N-1} r_i^* - f^* \right), & \text{for Setting 3}
\end{cases}
\]

and denoting \( \hat{k} := \min_{2 \leq k \leq K+1} \sum_{i=1}^{N} \left( \|x^k_i - x^{k+1}_i\|^2 + \|x_i^{k+1} - x^{k}_i\|^2 \right) \), then \( (x^1_1, \cdots, x^K_N, \lambda^k) \) is an \( \epsilon \)-stationary point of problem (1.1) with \( A_N \) being full row rank, according to Definitions 3.6, 3.7 and 3.8 under the conditions of Settings 1,2 and 3 in Table 1, respectively.

Proof. By summing (3.34) over \( k = 1, \ldots, K \), we obtain that

\[
\Psi_L(x_1^{K+1}, \cdots, x_N^{K+1}, \lambda^{K+1}, x_N^{K}) - \Psi_L(x_1^1, \cdots, x_N^1, \lambda^1, x_N^0) \leq -\tau \sum_{k=1}^{K} \sum_{i=1}^{N} \|x^k_i - x^{k+1}_i\|^2, \quad (3.38)
\]

where \( \tau \) is defined in (3.36). From Lemma 3.18 we know that there exists a constant \( \Psi_L^* \) such that \( \Psi(x_1^{k+1}, \cdots, x_N^{k+1}, \lambda^{k+1}, x_N^{k}) \geq \Psi_L^* \) holds for any \( k \geq 1 \). Therefore,

\[
\min_{2 \leq k \leq K+1} \theta_k \leq \frac{2}{\tau K} \left[ \Psi_L(x_1^1, \cdots, x_N^1, \lambda^1, x_N^0) - \Psi_L^* \right], \quad (3.39)
\]

where \( \theta_k \) is defined in (3.22), i.e., for \( K \) defined as in (3.37), \( \theta_k = O(e^2) \).

We now give upper bounds to the terms in (3.6) and (3.7) through \( \theta_k \). Note that (3.30) implies that

\[
\|A_N^\top \lambda^{k+1} - \nabla_N f(x_1^{k+1}, \cdots, x_N^{k+1})\| \leq L \|x_N^{k+1} - x_N^k\| + \|\nabla_N f(x_1^{k+1}, \cdots, x_N^{k+1}) - \nabla_N f(x_1^1, \cdots, x_N^1)\| \leq 2L \|x_N^{k+1} - x_N^k\|,
\]

which implies that

\[
\|A_N^\top \lambda^{k+1} - \nabla_N f(x_1^{k+1}, \cdots, x_N^{k+1})\|^2 \leq 4L^2 \theta_k. \quad (3.40)
\]

By Step 3 of Algorithm 3 and (3.29) we have

\[
\left\| \sum_{i=1}^{N} A_i x_i^{k+1} - b \right\|^2 \leq \frac{1}{\beta^2} \|\lambda^{k+1} - \lambda^k\|^2 \quad (3.41)
\]

\[
\leq \frac{3L^2}{\beta^2\sigma_N} \|x_N^{k+1} - x_N^k\|^2 + \frac{6L^2}{\beta^2\sigma_N} \|x_N^{k-1} - x_N^k\|^2 + \frac{3L^2}{\beta^2\sigma_N} \sum_{i=1}^{N-1} \|x_i^k - x_i^{k+1}\|^2 \leq \frac{6L^2}{\beta^2\sigma_N} \theta_k.
\]

The remaining proof is to give upper bounds to the terms in (3.5), (3.8) and (3.9). Since the proof steps are very similar to the that of Theorem 3.14, we shall only provide the key inequalities below.
Setting 3. Under conditions in Setting 3 in Table 1, the inequality (3.26) becomes
\[
\text{dist}\left(-\nabla_i f(x_i^{k+1}, \ldots, x_N^{k+1}) + A_i^\top \lambda^{k+1}, \partial r_i(x_i^{k+1})\right) \\
\leq \left(L + \beta \sqrt{N} \max_{1 \leq i \leq N} \|A_i\|_2 + \max_{1 \leq i \leq N} \|H_i\|_2\right) \sqrt{\theta_k}.
\] (3.42)

By combining (3.42), (3.40) and (3.41) we conclude that Algorithm 3 returns an \(\epsilon\)-stationary point of (1.1) according to Definition 3.8 under conditions in Setting 3 in Table 1.

Setting 2. Under conditions in Setting 2 in Table 1, the inequality (3.27) becomes
\[
\left(x_i - x_i^{k+1}\right)^\top \left[g_i + \nabla_i f(x_i^{k+1}, \ldots, x_N^{k+1}) - A_i^\top \lambda^{k+1}\right] \\
\geq - \left(\beta \sqrt{N} \max_{1 \leq i \leq N} \|A_i\|_2^2 + L + \max_{1 \leq i \leq N} \|H_i\|_2\right) \max_{1 \leq i \leq N-1} [\text{diam}(\mathcal{X}_i)] \sqrt{\theta_k}.
\] (3.43)

By combining (3.43), (3.40) and (3.41) we conclude that Algorithm 3 returns an \(\epsilon\)-stationary point of (1.1) according to Definition 3.7 under conditions in Setting 2 in Table 1.

Setting 1. Under conditions in Setting 1 in Table 1, inequality (3.28) becomes
\[
r_i(x_i) - r_i(x_i^{k+1}) + \left(x_i - x_i^{k+1}\right)^\top \left[\nabla_i f(x_i^{k+1}, \ldots, x_N^{k+1}) - A_i^\top \lambda^{k+1}\right] \\
\geq - \left(\beta \sqrt{N} \max_{1 \leq i \leq N} \|A_i\|_2^2 + L + \max_{1 \leq i \leq N} \|H_i\|_2\right) \max_{1 \leq i \leq N-1} [\text{diam}(\mathcal{X}_i)] \sqrt{\theta_k}.
\] (3.44)

By combining (3.44), (3.40) and (3.41) we conclude that Algorithm 3 returns an \(\epsilon\)-stationary point of (1.1) according to Definition 3.6 under conditions in Setting 1 in Table 1. \(\square\)

4 Extensions

4.1 Relax the assumption on the last block variable \(x_N\)

It is noted that in (1.1), we have some restrictions on the last block variable \(x_N\), i.e., \(r_N \equiv 0\), \(\mathcal{X}_N = \mathbb{R}^{n_N}\) and \(A_N = I\) or is full row rank. A more general problem to consider is

\[
\begin{align*}
\min_{x_1, x_2, \ldots, x_N} & \quad f(x_1, x_2, \ldots, x_N) + \sum_{i=1}^{N} r_i(x_i) \\
\text{s.t.} & \quad \sum_{i=1}^{N} A_i x_i = b, \quad x_i \in \mathcal{X}_i, \ i = 1, \ldots, N,
\end{align*}
\] (4.1)

with the same assumptions as in (1.1), plus \(\mathcal{X}_N \subset \mathbb{R}^{n_N}\) being a compact set. Note that we do not assume \(A_N = I\) nor that \(A_N\) is full row rank. Moreover, note that if \(r_N = 0\) and \(\mathcal{X}_N = \mathbb{R}^{n_N}\), then (4.1) reduces to (1.1). In the following, we shall briefly illustrate how to use proximal ADMM-m to find an \(\epsilon\)-stationary point of (4.1), and proximal ADMM-g can be applied in the same manner. We introduce the following problem that is closely related to (4.1):

\[
\begin{align*}
\min_{x_1, x_2, \ldots, x_N} & \quad f(x_1, x_2, \ldots, x_N) + \sum_{i=1}^{N} r_i(x_i) + \frac{\mu}{2} \|x_{N+1}\|^2 \\
\text{s.t.} & \quad \sum_{i=1}^{N} A_i x_i + x_{N+1} = b, \quad x_i \in \mathcal{X}_i, \ i = 1, \ldots, N,
\end{align*}
\] (4.2)

where \(\mu > 0\). Now proximal ADMM-m is ready to be used for solving (4.2) because \(A_{N+1} = I\) and \(x_{N+1}\) is unconstrained. We have the following iteration complexity result for proximal ADMM-m to obtain an \(\epsilon\)-stationary point of (4.1).
Theorem 4.1 Suppose the sequence \( \{(x^k_1, \cdots, x^k_N, \lambda^k)\} \) is generated by proximal ADMM-m for solving (4.2) with \( \mu = 1/\epsilon^2 \), where \( \epsilon > 0 \) is the given tolerance. Under the same conditions and using the same notation as in Theorem 3.19, \( (x^k_1, \cdots, x^k_N, \lambda^k) \) is an \( \epsilon \)-stationary point of (4.2), and \( (x^k, \cdots, x^k_N, \lambda^k) \) is an \( \epsilon \)-stationary point of (4.1), according to Definitions 3.6, 3.7 and 3.8 under conditions in Settings 1,2 and 3 in Table 1, respectively.

Proof. We only show the case for Setting 3 in Table 1, i.e., Definition 3.8, and the other two settings can be proved similarly. Note that if \( (x^k_1, \cdots, x^k_N, \lambda^k) \) satisfies (3.9), (3.6) and (3.7) with \( N \) being replaced by \( N + 1 \), then it is an \( \epsilon \)-stationary point of (4.2). Therefore, according to Theorem 3.19, we have

\[
\text{dist} \left( -\nabla_i f(x^k_1, \cdots, x^k_N) + A_i^T \lambda^k, \partial r_i(x^k_1) \right) \leq \epsilon, \quad \text{for } i = 1, \ldots, N, \tag{4.3}
\]

\[
\left\| \sum_{i=1}^N A_i x^k_i + x^k_{N+1} - b \right\| \leq \epsilon. \tag{4.4}
\]

By combining (3.29) and (3.38) (note we need to replace \( N \) by \( N + 1 \) in these two inequalities), we obtain

\[
\sum_{k=1}^K \left\| \lambda^{k+1} - \lambda^k \right\|^2 \leq \frac{6L^2}{\sigma_N} \sum_{k=1}^K \sum_{i=1}^{N+1} \left( \left\| x^k_{N+1} - x^k_i \right\|^2 + \left\| x^{k+1}_i - x^k_i \right\|^2 \right)
\]

\[
\leq \frac{12L^2}{\sigma_N} \Psi_L(x^1_1, \cdots, x^k_{N+1}, \lambda^1, x^{0}_{N+1}) - \sum_{i=1}^{N+1} r^*_i - f^*,
\]

which implies \( \lambda^{k+1} - \lambda^k \rightarrow 0 \) as \( k \rightarrow \infty \), and thus \( \left\| \lambda^{k+1} \right\|^2 - \left\| \lambda^k \right\|^2 \) is bounded. This fact combined with (3.35) yields that

\[
\Psi_L(x^k_1, \cdots, x^k_{N+1}, \lambda^k, x^k_{N+1}) = \sum_{i=1}^{N+1} r_i(x^k_i) + f(x^k_1, \cdots, x^k_N) + \frac{\left\| x^{k+1}_{N+1} \right\|^2}{2\epsilon^2} - \frac{1}{2\beta} \left\| \lambda^{k+1} \right\|^2 + \frac{1}{2\beta} \left\| \lambda^k \right\|^2 + \frac{1}{2\beta} \left\| \lambda^{k+1} \right\|^2
\]

\[\geq \sum_{i=1}^{N+1} r^*_i + f^* + \frac{\left\| x^{k+1}_{N+1} \right\|^2}{2\epsilon^2} - \frac{1}{2\beta} \left\| \lambda^{k+1} \right\|^2 + \frac{1}{2\beta} \left\| \lambda^k \right\|^2.
\]

Moreover, recall Lemma 3.17 and Lemma 3.18 state that \( \left\{ \Psi_L(x^{k+1}_1, \cdots, x^{k+1}_{N+1}, \lambda^{k+1}, x^{k}_{N+1}) \right\} \) is monotonically decreasing and lower bounded, thus \( \left\{ \frac{\left\| x^{k+1}_{N+1} \right\|^2}{2\epsilon^2} \right\} \) is bounded as well. Therefore, from (4.4) we get

\[
\left\| \sum_{i=1}^{N+1} A_i x^k_i - b \right\| \leq \left\| \sum_{i=1}^{N+1} A_i x^k_i + x^k_{N+1} - b \right\| + \left\| x^k_{N+1} \right\| \leq C\epsilon, \tag{4.5}
\]

for some constant \( C > 0 \). Finally, (4.3) and (4.5) imply that \( (x^k_1, \cdots, x^k_N, \lambda^k) \) is an \( \epsilon \)-stationary point of (4.1), according to Definition 3.8 under the conditions of Setting 3 in Table 1. \( \square \)

4.2 Proximal BCD (Block Coordinate Descent)

In this section, we propose a proximal block coordinate descent method for solving the following variant of (1.1) and prove its iteration complexity:

\[
\min \quad F(x_1, x_2, \cdots, x_N) := f(x_1, x_2, \cdots, x_N) + \sum_{i=1}^N r_i(x_i) \tag{4.6}
\]

s.t. \( x_i \in \mathcal{X}_i, \quad i = 1, \ldots, N, \)
where $f$ is differentiable, $r_i$ is nonsmooth, and $\mathcal{X}_i \subset \mathbb{R}^{n_i}$ is a closed convex set for $i = 1, 2, \ldots, N$. Note that $f$ and $r_i$ can be nonconvex functions. Our proximal BCD method for solving (4.6) is described in Algorithm 4.

**Algorithm 4 A proximal BCD method for solving (4.6)**

**Require:** Given $(x_1^0, x_2^0, \ldots, x_N^0) \in \mathcal{X}_1 \times \cdots \times \mathcal{X}_N$

for $k = 0, 1, \ldots$ do

Update block $x_i$ in a cyclic order, i.e., for $i = 1, \ldots, N$ ($H_i$ positive definite):

$$
x_i^{k+1} := \arg\min_{x_i \in \mathcal{X}_i} F(x_1^{k+1}, \ldots, x_{i-1}^{k+1}, x_i, x_{i+1}^{k}, \ldots, x_N^{k}) + \frac{1}{2} \|x_i - x_i^k\|^2_{H_i}. \tag{4.7}
$$

end for

Similar as settings in Table 1, depending on the properties of $r_i$ and $\mathcal{X}_i$, the $\epsilon$-stationary point of (4.6) can be defined as follows.

**Definition 4.2** $(x_1^*, \ldots, x_N^*, \lambda^*)$ is called an $\epsilon$-stationary point of (4.6), if

(i) $r_i$ is convex, $\mathcal{X}_i$ is convex and compact, and for any $x_i \in \mathcal{X}_i$, $i = 1, \ldots, N$, it holds that

$$
r_i(x_i) - r_i(x_i^*) + (x_i - x_i^*)^\top \left[ \nabla_i f(x_1^*, \ldots, x_N^*) - A_i^\top \lambda^* \right] \geq -\epsilon;
$$

(ii) or, if $r_i$ is Lipschitz continuous, $\mathcal{X}_i$ is convex and compact, and for any $x_i \in \mathcal{X}_i$, $i = 1, \ldots, N$, it holds that ($g_i = \partial r_i(x_i^*)$ denotes a generalized subgradient of $r_i$)

$$
(x_i - x_i^*)^\top \left[ \nabla_i f(x_1^*, \ldots, x_N^*) + g_i - A_i^\top \lambda^* \right] \geq -\epsilon;
$$

(iii) or, if $r_i$ is lower semi-continuous, $\mathcal{X}_i = \mathbb{R}^{n_i}$ for $i = 1, \ldots, N$, it holds that

$$
\text{dist} \left( -\nabla_i f(x_1^*, \ldots, x_N^*) + A_i^\top \lambda^*, \partial r_i(x_i^*) \right) \leq \epsilon.
$$

We now show that the iteration complexity of Algorithm 4 can be obtained from that of proximal ADMM-g. By introducing an auxiliary variable $x_{N+1}$ and an arbitrary vector $b \in \mathbb{R}^m$, problem (4.6) can be equivalently rewritten as

$$\min f(x_1, x_2, \ldots, x_N) + \sum_{i=1}^N r_i(x_i) \quad \text{subject to} \quad x_{N+1} = b, x_i \in \mathcal{X}_i, i = 1, \ldots, N. \tag{4.8}$$

It is easy to see that applying proximal ADMM-g to solve (4.8) (with $x_{N+1}$ being the last block variable) reduces exactly to Algorithm 4. Hence, we have the following iteration complexity result of Algorithm 4 for obtaining an $\epsilon$-stationary point of (4.6).

**Theorem 4.3** Suppose the sequence $\{(x_1^k, \ldots, x_N^k)\}$ is generated by proximal BCD (Algorithm 4). Denote

$$\kappa_3 := (L + \max_{1 \leq i \leq N} \|H_i\|)^2, \quad \kappa_4 := \max_{1 \leq i \leq N} (\text{diam}({\mathcal{X}}_i))^2.$$

Letting

$$K := \begin{cases}
\left[ \frac{\kappa_3}{N} (\Psi_G(x_1^k, \ldots, x_N^k, \lambda, x_0^k) - \sum_{i=1}^N r_i^k - f^*) \right] & \text{for Settings 1, 2} \\
\left[ \frac{\kappa_3}{\epsilon^2} (\Psi_G(x_1^k, \ldots, x_N^k, \lambda, x_0^k) - \sum_{i=1}^N r_i^k - f^*) \right] & \text{for Setting 3}
\end{cases}
$$

with $\tau$ being defined in (3.20), and $\hat{k} := \min_{1 \leq k \leq K} \sum_{i=1}^N \|x_i^k - x_i^{k+1}\|$, then $(x_1^\hat{k}, \ldots, x_N^\hat{k})$ is an $\epsilon$-stationary point of problem (4.6).
Proof. Note that \( A_1 = \cdots = A_N = 0 \) and \( A_{N+1} = I \) in problem (4.8). By applying proximal ADMM-g with \( \beta > \max \left\{ 18L, \max_{1 \leq i \leq N} \left\{ \frac{\delta_i^2}{\min(H_i)} \right\} \right\} \), Theorem 3.14 holds. In particular, (3.26), (3.27) and (3.28) are valid in different settings with \( \beta \max_{i+1 \leq j \leq N+1} \|A_j\|_2 \|A_i\|_2 = 0 \) for \( i = 1, \ldots, N \), which leads to the choices of \( \kappa_3 \) and \( \kappa_4 \) in the above. Moreover, we do not need to consider the optimality with respect to \( x_{N+1} \) and the violation of the affine constraints, thus \( \kappa_1 \) and \( \kappa_2 \) are excluded in the expression of \( K \), and the conclusion follows. \( \square \)

5 Numerical Results

5.1 Robust Tensor PCA Problem

We consider the following nonconvex and nonsmooth model of robust tensor PCA with \( \ell_1 \) norm regularization for third-order tensor of dimension \( I_1 \times I_2 \times I_3 \). Given an initial estimate \( R \) of the CP-rank, we aim to solve the following problem:

\[
\begin{align*}
\text{min}_{A,B,C,Z,E,B} & \quad \|Z - [A, B, C]\|^2 + \alpha \|E\|_1 + \|B\|^2 \\
\text{s.t.} & \quad Z + E + B = T,
\end{align*}
\]

(5.1)

where \( A \in \mathbb{R}^{I_1 \times R} \), \( B \in \mathbb{R}^{I_2 \times R} \), \( C \in \mathbb{R}^{I_3 \times R} \), \([A, B, C]\) is defined in (1.6). The augmented Lagrangian function of (5.1) is given by

\[
L_\beta(A, B, C, Z, E, B, \Lambda) = \|Z - [A, B, C]\|^2 + \alpha \|E\|_1 + \|B\|^2 - (\Lambda, Z + E + B - T) + \frac{\beta}{2} \|Z + E + B - T\|^2.
\]

The following identities are useful for our presentation later.

\[
\|Z - [A, B, C]\|^2 = \|Z_{(i)} - A(C \odot B)^\top\|^2 = \|Z_{(2)} - B(C \odot A)^\top\|^2 = \|Z_{(3)} - C(B \odot A)^\top\|^2,
\]

where \( Z_{(i)} \) stands for the mode-\( i \) unfolding of tensor \( Z \) and \( \odot \) stands for the Khatri-Rao product of matrices.

Note that there are six block variables in (5.1), and we choose \( B \) as the last block variable. A typical iteration of proximal ADMM-g for solving (5.1) can be described as follows (we chose \( H_i = \delta_i I_i \), with \( \delta_i > 0, i = 1, \ldots, 5 \)):

\[
\begin{align*}
A^{k+1} &= \left( (Z)_{(1)}^k (C^k \odot B^k) + \frac{\alpha}{R} A^k \right) \left( (C^k)\top C^k \circ ((B^k)\top B^k) + \frac{\alpha}{R} I \right)^{-1} \\
B^{k+1} &= \left( (Z)_{(2)}^k (C^k \odot A^{k+1}) + \frac{\alpha}{R} B^k \right) \left( (C^k)\top C^k \circ ((A^{k+1})\top A^{k+1}) + \frac{\alpha}{R} I \right)^{-1} \\
C^{k+1} &= \left( (Z)_{(3)}^k (B^{k+1} \odot A^{k+1}) + \frac{\alpha}{R} C^k \right) \left( (B^{k+1})\top B^{k+1} \circ ((A^{k+1})\top A^{k+1}) + \frac{\alpha}{R} I \right)^{-1} \\
E^{k+1} &= S \left( \frac{\beta}{\kappa_1} (T_{(1)} + \frac{1}{\kappa_4} \Lambda_{(1)}^k - B_{(1)}^k) - Z_{(1)}^k \right) + \frac{\alpha}{\kappa_3} E_{(1)}^k + \frac{\alpha}{\kappa_5} \delta_5 (Z_{(1)}^k) \circ \Lambda_{(1)}^k - \beta (E_{(1)}^k + B_{(1)}^k - T_{(1)}) \\
Z^{k+1} &= 1 + 2\delta_5 \beta \left( 2A^{k+1} (C^{k+1} \odot B^{k+1}) + 2\delta_5 (Z_{(1)}^k) \circ \Lambda_{(1)}^k - \beta (E_{(1)}^k + B_{(1)}^k - T_{(1)}) \right) \\
B^{k+1} &= B_{(1)}^k - \gamma \left( 2B_{(1)}^k - \Lambda_{(1)}^k + \beta (E_{(1)}^k + Z_{(1)}^k + B_{(1)}^k - T_{(1)}) \right) \\
\Lambda_{(1)}^{k+1} &= \Lambda_{(1)}^k - \beta \left( Z_{(1)}^{k+1} + E_{(1)}^{k+1} + B_{(1)}^{k+1} - T_{(1)} \right)
\end{align*}
\]

where \( \circ \) is the matrix Hadamard product and \( S \) stands for the soft shrinkage operator. The updates in proximal ADMM-m are almost the same as proximal ADMM-g except \( B_{(1)} \) is updated as

\[
B_{(1)}^{k+1} = \frac{1}{L + \beta} \left( (L - 2)B_{(1)}^k + \Lambda_{(1)}^k - \beta (E_{(1)}^k + Z_{(1)}^k + T_{(1)}) \right).
\]
On the other hand, note that (5.1) can be equivalently written as

$$\min_{A,B,C,Z,\mathcal{E}} \|Z - [A, B, C]\|^2 + \alpha \|\mathcal{E}\|_1 + \|Z + \mathcal{E} - T\|^2,$$

(5.2)

which can be solved by the classical BCD method as well as our proximal BCD (Algorithm 4).

In the following we shall compare the numerical performance of BCD, proximal BCD, proximal ADMM-g and proximal ADMM-m for solving (5.1). We let $\alpha = 2 / \max\{\sqrt{t_1}, \sqrt{t_2}, \sqrt{t_3}\}$ in model (5.1). We apply proximal ADMM-g and proximal ADMM-m to solve (5.1), and apply BCD and proximal BCD to solve (5.2). In all the four algorithms we set the maximum iteration number to be 2000, and the algorithms are terminated either when the maximum iteration number is reached or when $\theta_k$ as defined in (3.22) is less than $10^{-6}$. The parameters used in the two ADMM variants are specified in Table 2 and we note that $L = 2$ in (5.1).

| $H_i, \ i = 1, \ldots, 5$ | $\beta$   | $\gamma$ |
|--------------------------|----------|----------|
| proximal ADMM-g          | $L \cdot I$ | $2L$    |
| proximal ADMM-m          | $L \cdot I$ | $2.5L$ |

Table 2: Choices of parameters in the two ADMM variants.

In the experiment, we randomly generate 20 instances for fixed tensor dimension and CP-rank. Suppose the low-rank part $Z^0$ is of rank $R_{CP}$. It is generated by

$$Z^0 = \sum_{r=1}^{R_{CP}} a^{1,r} \otimes a^{2,r} \otimes a^{3,r},$$

where vectors $a^{i,r}$ are generated from standard Gaussian distribution for $i = 1, 2, 3$, $r = 1, \ldots, R_{CP}$. Moreover, a sparse tensor $\mathcal{E}^0$ is generated with cardinality of 0.001 $I_1 I_2 I_3$ such that each nonzero component follows from standard Gaussian distribution. Finally, we generate noise $B^0 = 0.001 \ast \mathcal{B}$, where $\mathcal{B}$ is a Gaussian tensor. Then we set $T = Z^0 + \mathcal{E}^0 + B^0$ as the observed data in (5.1). We report the average performance of 20 instances of the four algorithms with initial guess $R = R_{CP}$ and $R = R_{CP} + 1$ in Tables 3 and 4, respectively.

| $R_{CP}$ | Tensor Size 10 $\times$ 10 $\times$ 30 | proximal ADMM-g | proximal ADMM-m | BCD | proximal BCD |
|----------|----------------------------------------|------------------|------------------|-----|--------------|
| Iter.    | Err.        | Num | Iter. | Err. | Num | Iter. | Err. | Num | Iter. | Err. | Num |
| 3        | 371.80      | 0.0362 | 19    | 395.25 | 0.0362 | 19 | 678.15 | 0.7095 | 1 | 292.80 | 0.0362 | 19 |
| 10       | 632.10      | 0.0320 | 17    | 566.15 | 0.0320 | 17 | 1292.10 | 0.9133 | 0 | 356.00 | 0.0154 | 19 |
| 15       | 529.25      | 0.0165 | 18    | 545.05 | 0.0165 | 18 | 1458.65 | 0.9224 | 0 | 753.75 | 0.0404 | 15 |

| Tensor Size 15 $\times$ 25 $\times$ 30 |
|----------------------------------------|------------------|------------------|-----|--------------|
| Iter. | Err. | Num | Iter. | Err. | Num | Iter. | Err. | Num | Iter. | Err. | Num |
| 5     | 516.30 | 0.0163 | 19    | 636.85 | 0.0341 | 17 | 611.25 | 0.8597 | 0 | 434.25 | 0.0358 | 18 |
| 10    | 671.80 | 0.0345 | 17    | 723.20 | 0.0385 | 17 | 1223.60 | 0.9072 | 0 | 592.60 | 0.0335 | 17 |
| 20    | 776.70 | 0.0341 | 16    | 922.25 | 0.0412 | 15 | 1716.05 | 0.9544 | 0 | 916.90 | 0.0416 | 14 |

| Tensor Size 30 $\times$ 50 $\times$ 70 |
|----------------------------------------|------------------|------------------|-----|--------------|
| Iter. | Err. | Num | Iter. | Err. | Num | Iter. | Err. | Num | Iter. | Err. | Num |
| 8     | 909.05 | 0.1021 | 13    | 1004.30 | 0.1066 | 13 | 1094.05 | 0.9271 | 0 | 798.05 | 0.1059 | 13 |
| 20    | 1304.65 | 0.1233 | 7     | 1386.75 | 0.1387 | 6 | 1635.80 | 0.9668 | 0 | 1102.85 | 0.1444 | 5 |
| 40    | 1261.25 | 0.0623 | 10    | 1387.40 | 0.0779 | 7 | 2000.00 | 0.9798 | 0 | 1096.80 | 0.0610 | 9 |

Table 3: Numerical results for tensor robust PCA with initial guess $R = R_{CP}$

In Tables 3 and 4, “Err.” denotes the averaged relative error of the low-rank tensor over 20 instances $\frac{\|Z^* - Z^0\|_F}{\|Z^0\|_F}$ where $Z^*$ is the solution returned by the corresponding algorithm; “Iter.” denotes the averaged number of iterations over 20 instances; “Num” records the number of solutions (out of 20 instances) that have relative error less than 0.01.

Tables 3 and 4 suggest that BCD mostly converges to a local solution rather than the global optimal solution, while the other three methods are much better in finding the global optimum. It is interesting to note that the results presented in Table 4 are better than that of Table 3 when a slightly larger basis is allowed in tensor factorization. Moreover, in this case, proximal BCD usually consumes less number of iterations than the two ADMM variants.
| RC_F | proximal ADMM-g | proximal ADMM-m | BCD | proximal BCD |
|------|----------------|----------------|-----|--------------|
| Iter. | Err. | Num | Iter. | Err. | Num | Iter. | Err. | Num | Iter. | Err. | Num | Iter. | Err. | Num | Iter. | Err. | Num |
| 3    | 1830.65 | 0.0032 | 20 | 1758.90 | 0.0032 | 20 | 462.90 | 0.7763 | 0 | 1734.85 | 0.0032 | 20 |
| 10   | 1493.20 | 0.0029 | 20 | 1586.00 | 0.0029 | 20 | 1277.15 | 0.9135 | 0 | 1137.15 | 0.0029 | 20 |
| 15   | 1336.65 | 0.0078 | 19 | 1486.40 | 0.0031 | 20 | 1453.30 | 0.9224 | 0 | 945.05 | 0.0106 | 19 |
| 5    | 1267.10 | 0.0019 | 20 | 1291.95 | 0.0019 | 20 | 609.45 | 0.8597 | 0 | 1471.10 | 0.0019 | 20 |
| 10   | 1015.25 | 0.0019 | 20 | 1121.00 | 0.0164 | 19 | 1220.50 | 0.9072 | 0 | 1121.40 | 0.0019 | 20 |
| 20   | 814.95  | 0.0019 | 20 | 888.40 | 0.0019 | 20 | 1716.30 | 0.9544 | 0 | 736.70 | 0.0020 | 20 |
| 8    | 719.45  | 0.0009 | 20 | 608.25 | 0.0009 | 20 | 1094.10 | 0.9271 | 0 | 508.05 | 0.0327 | 18 |
| 20   | 726.95  | 0.0088 | 19 | 817.20 | 0.0220 | 17 | 1635.10 | 0.9668 | 0 | 539.25 | 0.0254 | 17 |
| 16   | 1063.55 | 0.0270 | 16 | 1122.75 | 0.0322 | 15 | 1998.05 | 0.9798 | 0 | 649.10 | 0.0246 | 16 |

Table 4: Numerical results for tensor robust PCA with initial guess $R = RC_F + 1$

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