A Strategy To Determine Operating Parameters in Tissue Engineering Hollow Fibre Bioreactors

Supplementary Material

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A Analytical Model Reduction

The full modelling equations throughout the module are given by

\[ \frac{\partial c_l}{\partial t} + \nabla \cdot (u c_l) = D_l \nabla^2 c_l \] in the lumen,
\[ \frac{\partial c_m}{\partial t} = D_m \nabla^2 c_m \] in the membrane,
\[ \frac{\partial c_e}{\partial t} = D_e \nabla^2 c_e - R(c_e) \] in the ECS \( \text{(1)} \)

where the reaction term is given by

\[ R(c_e) = \frac{V_{max} c_e}{c_e + K_m} \] \( \text{(2)} \).

On the lumen/membrane and membrane/ECS boundaries we prescribe continuity of concentration and flux, so that

\[ c_l = c_w \quad \text{and} \quad D_l \nabla c_l \cdot n = D_w \nabla c_w \cdot n \] on the lumen/membrane boundary, \( \text{(3)} \)
\[ c_w = c_e \quad \text{and} \quad D_w \nabla c_w \cdot n = D_e \nabla c_e \cdot n \] on the membrane/ECS boundary, \( \text{(4)} \)

where \( n \) is the unit outward pointing normal to the relevant surface. Finally, we impose a prescribed concentration \( c_{in} \) at the lumen inlet, and no diffusive flux of concentration out of the outer ECS boundary,

\[ c_l = c_{in} \quad \text{on} \quad z = 0, \quad \text{and} \quad D_e \nabla c_e \cdot n = 0 \quad \text{on the outer ECS boundary.} \] \( \text{(5)} \)

We mathematically reduce the full system of equations given by equations \( \text{(1)-(5)} \) to determine analytical expressions for the oxygen concentrations in each of the lumen, membrane and ECS.

First of all, we assume that the fibre is positioned symmetrically in the extra-lumen space and move to a radially symmetric setup described in cylindrical polar coordinates (defined by \( x = r \cos \theta, \ y = r \sin \theta \) and \( z = z \)), neglecting \( \theta \)-dependence so that \( c = c(r, z) \).

We non-dimensionalise the system above to reduce the number of parameters and to estimate the relative importance of the various terms. We set

\[ r = d \hat{r}, \quad z = L \hat{z}, \quad u_l = U \hat{u}, \quad t = \frac{L}{U} \hat{t}, \quad c = c_{in} \hat{c}, \] \( \text{(6)} \)

where \( d \) is the lumen radius, \( L \) is the length of the lumen, \( U \) is the mean velocity in the lumen and \( c_{in} \) is the lumen inlet oxygen concentration. From here onwards we drop the ‘hats’ on the dimensionless variables. The system now becomes

\[ \frac{\text{Pe}}{D_l} \left( \frac{\partial c_l}{\partial t} + 2 \left(1 - r^2 \right) \frac{\partial c_l}{\partial z} \right) = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial c_l}{\partial r} \right) + \epsilon^2 \frac{\partial^2 c_l}{\partial z^2} \quad \text{in} \quad 0 < r < 1, \] \( \text{(7)} \)
\[ \frac{\text{Pe} D_l}{D_w} \frac{\partial c_w}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial c_w}{\partial r} \right) + \epsilon^2 \frac{\partial^2 c_w}{\partial z^2} \quad \text{in} \quad 1 < r < R_w/d, \] \( \text{(8)} \)
\[ \frac{\text{Pe} D_l}{D_e} \frac{\partial c_e}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial c_e}{\partial r} \right) + \epsilon^2 \frac{\partial^2 c_e}{\partial z^2} - M \quad \text{in} \quad R_w/d < r < R_e/d, \] \( \text{(9)} \)
with boundary conditions

\[ \begin{align*}
  c_l &= c_w \quad \text{and} \quad \frac{\partial c_l}{\partial r} = \frac{D_w}{D_l} \frac{\partial c_w}{\partial r} \quad \text{on} \quad r = 1, \\
  c_w &= c_e \quad \text{and} \quad \frac{\partial c_w}{\partial r} = \frac{D_e}{D_w} \frac{\partial c_e}{\partial r} \quad \text{on} \quad r = R_w/d, \\
  \frac{\partial c_e}{\partial r} &= 0 \quad \text{on} \quad r = R_e/d, \\
  c_l (r,0) &= 1,
\end{align*} \]

where all parameters are defined in the main paper.

### A.1 Solution in the Membrane and ECS

Next we solve the system described by equations (7)–(13) in steady state to predict the oxygen distribution throughout the module. We integrate the steady-state versions of equations (8) and (9) for the concentration in the wall and ECS to give

\[ \begin{align*}
  c_w &= A(z) \ln r + B(z), \\
  c_e &= \frac{M}{4} r^2 + C(z) \ln r + D(z),
\end{align*} \]

where \( A(z), B(z), C(z) \) and \( D(z) \) are arbitrary functions of \( z \). Inputting the no-flux boundary condition out of the ECS outlet (from equation (12)), together with the continuity of concentration and flux boundary conditions on \( r = R_w/d \) (equation (11)) yields

\[ \begin{align*}
  c_m &= \frac{MD_e}{2D_m d^2} \left( R_m^2 - R_e^2 \right) \ln r + B(z), \\
  c_e &= \frac{M}{4d^2} \left[ (dr)^2 - R_m^2 + \frac{2D_e}{D_m} (R_m^2 - R_e^2) \ln \left( \frac{R_m}{d} \right) + 2R_e^2 \ln \left( \frac{R_m}{dr} \right) \right] + B(z),
\end{align*} \]

where \( B(z) \) remains unknown. To determine \( B \), and the solution in the lumen, we must now solve equation (7) (in steady-state), subject to the boundary conditions

\[ \frac{\partial c_l}{\partial r} (1,z) = \gamma, \quad c_l (r,0) = 1, \]

where \( \gamma \) is defined by

\[ \gamma = \frac{MD_e}{2D_l d^2} \left[ R_m^2 - R_e^2 \right], \]

and we also require that \( \frac{\partial c_l}{\partial r} \) is bounded at \( r = 0 \).
A.2 Solution in the Lumen

We perform a change of variables to give a homogeneous derivative boundary condition on \( r = 1 \). We let

\[ c_t = c + \gamma r + 1, \quad (19) \]

upon which the system in the lumen becomes

\[ 2 \mathrm{Pe} \frac{\partial c}{\partial z} = \frac{1}{r(1-r^2)} \frac{\partial}{\partial r} \left( r \frac{\partial c}{\partial r} \right) + \frac{\gamma}{r(1-r^2)} \quad \text{in} \quad 0 < r < 1, \quad (20) \]

\[ \frac{\partial c}{\partial r} (1, z) = 0, \quad (21) \]

\[ \frac{\partial c}{\partial r} \text{ finite as } r \to 0, \quad (22) \]

\[ c (r, 0) = -\gamma r. \quad (23) \]

A.2.1 Solution of the Homogeneous Counterpart to (20)–(22)

First of all, we consider the homogeneous counterpart to (20) given by

\[ 2 \mathrm{Pe} \frac{\partial c}{\partial z} = \frac{1}{r(1-r^2)} \frac{\partial}{\partial r} \left( r \frac{\partial c}{\partial r} \right) \quad \text{in} \quad 0 < r < 1, \quad (24) \]

and solve via separation of variables by seeking a solution of the form \( c(r, z) = f(r) g(z) \). This yields

\[ \frac{1}{r(1-r^2)} \frac{d}{dr} \left( r \frac{df}{dr} \right) = \frac{2 \mathrm{Pe}}{g} \frac{dg}{dz}. \quad (25) \]

The LHS of equation (25) is dependent on \( r \) only, whilst the RHS is dependent on \( z \) only. Therefore both sides must constant, and indeed this constant must be negative (the concentration must decay as \( z \) increases from the inlet at \( z = 0 \)). Defining this constant to be \( -\lambda^2 \), the system for \( f(r) \) becomes

\[ \frac{d}{dr} \left( r \frac{df}{dr} \right) + \lambda^2 r (1-r^2) f = 0 \quad \text{for} \quad 0 < r < 1, \quad (26) \]

with boundary conditions

\[ \frac{df}{dr} (1) = 0 \quad \text{and} \quad \frac{df}{dr} \text{ finite as } r \to 0. \quad (27) \]

This is a Sturm-Liouville problem – for further details please refer to Abramowitz & Stegun (1965). To solve for \( f(r) \), we perform two transformations

\[ X = \lambda r^2, \quad \Theta = \exp \left( \lambda r^2 / 2 \right) f(r), \quad (28) \]
which transform (26) to
\[ X \frac{d^2 \Theta}{dX^2} + (1 - X) \frac{d \Theta}{dX} - \left( \frac{1}{2} - \frac{\lambda}{4} \right) \Theta = 0, \]  
(29)
which is known as the Kummer equation. The Kummer equation is normally written in the form
\[ x \frac{d^2 y}{dx^2} + (\nu - x) \frac{dy}{dx} - \mu y = 0; \]  
(30)
here we have \( \nu = 1 \) and \( \mu = \frac{1}{2} - \frac{\lambda}{4} \). The Kummer equation has two linearly independent solutions, denoted KummerM \((\mu, \nu, x)\) and KummerU \((\mu, \nu, x)\) (both hypergeometric functions), and so our solution here can be written as a linear sum of the two,
\[ \Theta = E \text{KummerM} \left( \frac{1}{2} - \frac{\lambda}{4}, 1, X \right) + F \text{KummerU} \left( \frac{1}{2} - \frac{\lambda}{4}, 1, X \right), \]  
(31)
where \( E \) and \( F \) are integration constants to be determined. A brief summary of the properties of the KummerM and KummerU functions required here is given below; for further details please refer to Abramowitz & Stegun (1965).

**Important Properties of the KummerM and KummerU Functions**

1. Both KummerM\((\mu, \nu, x)\) and KummerU\((\mu, \nu, x)\) are finite in the limit as \( x \to 0 \).
2. \( \frac{df}{dr} \text{KummerM} (\mu, \nu, x) = \frac{\mu}{\nu} \text{KummerM} (\mu + 1, \nu + 1, x) \).
3. KummerM \((\mu, \nu, x) = \exp (x) \text{KummerM} (\nu - \mu, \nu, -x) \).

The requirement that \( df/\text{dr} \) is finite as \( r \to 0 \) yields \( F = 0 \) (to avoid a \( 1/\sqrt{X} \) singularity as \( X \to 0 \)) – this uses property 1. The boundary condition at \( r = 1 \) from (27) provides the following *eigenvalue equation* for \( \lambda \),
\[ \left( 1 - \frac{\lambda}{2} \right) \text{KummerM} \left( \frac{1}{2} + \frac{\lambda}{4}, 2, -\lambda \right) - \text{KummerM} \left( \frac{1}{2} + \frac{\lambda}{4}, 1, -\lambda \right) = 0. \]  
(32)
The determination of this equation uses property 2. above to evaluate the derivative, and 3. to reduce oscillations in the eigenvalue equation. Equation (32) has infinitely many solutions for \( \lambda \) which may be determined by solving Equation (32) numerically; we denote them by \( \lambda_n \), where \( n = 0 \ldots \infty \). Each eigenvalue \( \lambda_n \) corresponds to a different solution for \( f \) (an *eigenfunction*), which we now denote by \( f_n \). These functions \( f_n \) are given by
\[ f_n = E_n \exp \left( \frac{\lambda_n r^2}{2} \right) \text{KummerM} \left( \frac{1}{2} + \frac{\lambda_n}{4}, 1, -\lambda_n r^2 \right), \]  
(33)
where Sturm-Liouville theory dictates that the coefficients \( E_n \) may be fixed used the normalization condition
\[ \int_{r=0}^{1} s (r) f_n^2 (r) \, dr = 1, \]  
(34)
for each \( n = 0 \ldots \infty \), where \( s (r) = r \left( 1 - r^2 \right) \) are the \textit{weight functions} of the Sturm-Liouville equation (26). Therefore,

\[
E_n^2 = 1 / \int_{r=0}^{1} r \left( 1 - r^2 \right) \exp \left( \lambda_n r^2 \right) \left( \text{KummerM} \left( \frac{1}{2} + \frac{\lambda_n}{4}, 1, -\lambda_n r^2 \right) \right)^2 \mathrm{d}r.
\]

### A.2.2 Solution of the Non-Homogeneous Problem (20)–(22)

Now we return to the non-homogeneous problem given by (20)–(22), and pick up the non-homogeneous term \( \gamma / r \left( 1 - r^2 \right) \) using the radial solution, \( f \). Sturm-Liouville theory states that this non-homogeneous term can be expressed as a linear sum of the eigenfunctions \( f_n \), which act as basis functions (analogous to a Fourier series expansion). Therefore,

\[
\frac{\gamma}{r \left( 1 - r^2 \right)} = \sum_{n=0}^{\infty} F_n f_n (r),
\]

where the \( F_n \) are constants given by

\[
F_n = \int_{r=0}^{1} \frac{\gamma s (r)}{r \left( 1 - r^2 \right)} f_n (r) \mathrm{d}r = \int_{r=0}^{1} \gamma f_n (r) \mathrm{d}r.
\]

Next we express the solution for \( c \) as a linear sum over all the eigenvalues and eigenfunctions (admissible because the equations (20)–(23) are linear),

\[
c = \sum_{n=0}^{\infty} f_n (r) \, g_n (z),
\]

where the \( g_n (z) \) are undetermined. Substituting the two sum expressions (36) and (38) into the Equation (20) for \( c \) and using Equation (26) to simplify the derivative terms in \( f_n \) yields the following ordinary differential equation for each \( g_n \) \((n = 0 \ldots \infty)\)

\[
2 \text{Pe} \frac{dg_n}{dz} + \lambda_n^2 g_n (z) = F_n.
\]

The solution to (39) can be written in the form

\[
g_n = G_n \exp \left( -\frac{\lambda_n^2}{2 \text{Pe}} z \right) + \frac{F_n}{\lambda_n^2},
\]

where the integration constants \( G_n \) are undetermined. Indeed, substituting for \( c \) from Equation (38) into the remaining boundary condition (23) yields

\[
\sum_{n=0}^{\infty} g_n (0) \, f_n (r) = -\gamma r.
\]

Expressing \( g_n (0) \) as a Sturm-Liouville sum yields

\[
g_n (0) = \int_{0}^{1} -\gamma r^2 \left( 1 - r^2 \right) f_n (r) \mathrm{d}r,
\]
where the expression for $g_n$ given by Equation (40) means that

$$g_n(0) = G_n + \frac{F_n}{\lambda_n^2}.$$  \hfill (43)

Therefore the constants $G_n$ are given by

$$G_n = -\frac{F_n}{\lambda_n^2} + \int_0^1 -\gamma r^2 (1 - r^2) f_n(r) \, dr.$$  \hfill (44)

The final solution for $c_1$ is summarized in the main paper.

\section*{B \quad Eigenvalues and Normalization Constants for the Sturm-Liouville Problem}

The eigenvalues $\lambda_n$, and normalization constants $E_n$ (for $n = 0, \ldots, \infty$) as given by Equations (32) and (35) are independent of the geometry of the bioreactor and the cell population properties. The values of $\lambda_n$ and $E_n$ for $n = 0, \ldots, 49$ are summarized in Table 2.

| $n$ | $\lambda_n$ | $E_n$  |
|-----|-------------|--------|
| 0   | 5.06751     | 4.58666|
| 1   | 9.15761     | 6.0934 |
| 2   | 13.1972     | 7.29187|
| 3   | 17.2202     | 8.31844|
| 4   | 21.2355     | 9.23112|
| 5   | 25.2465     | 10.0611|
| 6   | 29.2549     | 10.8275|
| 7   | 33.2615     | 11.5431|
| 8   | 37.2669     | 12.2168|
| 9   | 41.2714     | 12.8552|
| 10  | 45.2752     | 13.4633|
| 11  | 49.2785     | 14.0451|
| 12  | 53.2813     | 14.6037|
| 13  | 57.2838     | 15.1417|
| 14  | 61.286      | 15.6612|
| 15  | 65.288      | 16.164 |
| 16  | 69.2898     | 16.6517|
| 17  | 73.2914     | 17.1254|
| 18  | 77.2929     | 17.5864|
| 19  | 81.2943     | 18.0356|
Table 1: The values of $\lambda_n$ and $E_n$ for $n = 0, \ldots, 49$

|   |   |   |
|---|---|---|
| 20 | 85.2955 | 18.4739 |
| 21 | 89.2967 | 18.902 |
| 22 | 93.2978 | 19.3206 |
| 23 | 97.2988 | 19.7304 |
| 24 | 101.3 | 20.1318 |
| 25 | 105.301 | 20.5254 |
| 26 | 109.301 | 20.9115 |
| 27 | 113.302 | 21.2907 |
| 28 | 117.303 | 21.6632 |
| 29 | 121.304 | 22.0294 |
| 30 | 125.304 | 22.3896 |
| 31 | 129.305 | 22.7441 |
| 32 | 133.305 | 23.0932 |
| 33 | 137.306 | 23.4371 |
| 34 | 141.306 | 23.776 |
| 35 | 145.307 | 24.1101 |
| 36 | 149.307 | 24.4397 |
| 37 | 153.308 | 24.7649 |
| 38 | 157.308 | 25.0859 |
| 39 | 161.309 | 25.4028 |
| 40 | 165.309 | 25.7158 |
| 41 | 169.31 | 26.025 |
| 42 | 173.31 | 26.3306 |
| 43 | 177.31 | 26.6327 |
| 44 | 181.311 | 26.9315 |
| 45 | 185.311 | 27.2269 |
| 46 | 189.311 | 27.5192 |
| 47 | 193.312 | 27.8084 |
| 48 | 197.312 | 28.0946 |
| 49 | 201.312 | 28.377 |

C Worked Example of the Operating Equations: Cardiomyocytes

C.1 ECS Depth Fixed
| $n$ | $B_n$          | $C_n$          |
|-----|---------------|---------------|
| 0   | 0.0113993     | −0.00237277   |
| 1   | 0.0051295     | −0.000420139  |
| 2   | 0.00297741    | −0.00014473   |
| 3   | 0.00195891    | −6.78109 × 10^{-5} |
| 4   | 0.00139676    | −3.76759 × 10^{-5} |
| 5   | 0.00105284    | −2.33015 × 10^{-5} |
| 6   | 0.000826219   | −1.55143 × 10^{-5} |
| 7   | 0.00066838    | −1.09022 × 10^{-5} |
| 8   | 0.000553649   | −7.98395 × 10^{-6} |
| 9   | 0.000467386   | −6.04048 × 10^{-6} |
| 10  | 0.000400727   | −4.69235 × 10^{-6} |
| 11  | 0.000348041   | −3.72553 × 10^{-6} |
| 12  | 0.0003056     | −3.01269 × 10^{-6} |
| 13  | 0.000270857   | −2.47466 × 10^{-6} |
| 14  | 0.000242016   | −2.06033 × 10^{-6} |
| 15  | 0.000217784   | −1.73566 × 10^{-6} |
| 16  | 0.000197207   | −1.47736 × 10^{-6} |
| 17  | 0.000179567   | −1.26907 × 10^{-6} |
| 18  | 0.000164319   | −1.09911 × 10^{-6} |
| 19  | 0.000151039   | −9.58919 × 10^{-7} |
| 20  | 0.000139394   | −8.42176 × 10^{-7} |
| 21  | 0.000129119   | −7.44109 × 10^{-7} |
| 22  | 0.000120003   | −6.61076 × 10^{-7} |
| 23  | 0.000111874   | −5.90262 × 10^{-7} |
| 24  | 0.00010459    | −5.29469 × 10^{-7} |
| 25  | 9.80348 × 10^{-5} | −4.7696 × 10^{-7} |
| 26  | 9.21125 × 10^{-5} | −4.31349 × 10^{-7} |
| 27  | 8.67419 × 10^{-5} | −3.91525 × 10^{-7} |
| 28  | 8.18546 × 10^{-5} | −3.56584 × 10^{-7} |
| 29  | 0.000077393   | −3.25789 × 10^{-7} |
| 30  | 7.33076 × 10^{-5} | −2.98534 × 10^{-7} |
| 31  | 6.95563 × 10^{-5} | −2.74317 × 10^{-7} |
| 32  | 6.61026 × 10^{-5} | −2.52721 × 10^{-7} |
| 33  | 0.000062915   | −2.33393 × 10^{-7} |
| 34  | 5.99662 × 10^{-5} | −2.16041 × 10^{-7} |
| 35  | 5.72322 × 10^{-5} | −2.00413 × 10^{-7} |
| 36  | 5.46921 × 10^{-5} | −1.86298 × 10^{-7} |
| 37  | 5.23275 × 10^{-5} | −1.73514 × 10^{-7} |
| 38  | 5.01221 × 10^{-5} | −1.61905 × 10^{-7} |
| 39  | 4.80617 × 10^{-5} | −1.51338 × 10^{-7} |
| 40  | 4.61334 × 10^{-5} | −1.41696 × 10^{-7} |
Table 2: Worked example of the operating equations: cardiomyocytes. The values of $B_n$ and $C_n$ for $n = 0, \ldots, 49$ for neonatal rat cardiomyocytes with $R_e = 215 \mu m$

C.2 Lumen Flow Velocity and Length Fixed

| $n$ | $H_n$         | $J_n$         |
|-----|---------------|---------------|
| 0   | $-4.22 \times 10^{-17}$ | $-8.80 \times 10^{-2}$ |
| 1   | $5.64 \times 10^{-50}$ | $2.87 \times 10^{-2}$ |
| 2   | $-8.76 \times 10^{-14}$ | $-1.45 \times 10^{-2}$ |
| 3   | $1.74 \times 10^{-193}$ | $8.81 \times 10^{-3}$ |
| 4   | $-4.61 \times 10^{-294}$ | $-5.96 \times 10^{-3}$ |
| 5   | 0.00          | $4.32 \times 10^{-3}$ |
| 6   | 0.00          | $-3.29 \times 10^{-3}$ |
| 7   | 0.00          | $2.59 \times 10^{-3}$ |
| 8   | 0.00          | $-2.10 \times 10^{-3}$ |
| 9   | 0.00          | $1.74 \times 10^{-3}$ |
| 10  | 0.00          | $-1.46 \times 10^{-3}$ |
| 11  | 0.00          | $1.25 \times 10^{-3}$ |
| 12  | 0.00          | $-1.08 \times 10^{-3}$ |
| 13  | 0.00          | $9.47 \times 10^{-4}$ |
| 14  | 0.00          | $-8.36 \times 10^{-4}$ |
| 15  | 0.00          | $7.44 \times 10^{-4}$ |
| 16  | 0.00          | $-6.67 \times 10^{-4}$ |
| 17  | 0.00          | $6.01 \times 10^{-4}$ |
| 18  | 0.00          | $-5.45 \times 10^{-4}$ |
| 19  | 0.00          | $4.96 \times 10^{-4}$ |
| 20  | 0.00          | $-4.54 \times 10^{-4}$ |
| 21  | 0.00          | $4.18 \times 10^{-4}$ |
| 22  | 0.00          | $-3.85 \times 10^{-4}$ |
| 23  | 0.00          | $3.56 \times 10^{-4}$ |
| n  | $H_n$      | $J_n$       |
|----|------------|------------|
| 24 | $-3.31 \times 10^{-4}$ | $-3.31 \times 10^{-4}$ |
| 25 | $3.08 \times 10^{-4}$   | $3.08 \times 10^{-4}$   |
| 26 | $-2.88 \times 10^{-4}$ | $-2.88 \times 10^{-4}$ |
| 27 | $2.69 \times 10^{-4}$   | $2.69 \times 10^{-4}$   |
| 28 | $-2.53 \times 10^{-4}$ | $-2.53 \times 10^{-4}$ |
| 29 | $2.37 \times 10^{-4}$   | $2.37 \times 10^{-4}$   |
| 30 | $-2.24 \times 10^{-4}$ | $-2.24 \times 10^{-4}$ |
| 31 | $2.11 \times 10^{-4}$   | $2.11 \times 10^{-4}$   |
| 32 | $-2.00 \times 10^{-4}$ | $-2.00 \times 10^{-4}$ |
| 33 | $1.89 \times 10^{-4}$   | $1.89 \times 10^{-4}$   |
| 34 | $-1.79 \times 10^{-4}$ | $-1.79 \times 10^{-4}$ |
| 35 | $1.70 \times 10^{-4}$   | $1.70 \times 10^{-4}$   |
| 36 | $-1.62 \times 10^{-4}$ | $-1.62 \times 10^{-4}$ |
| 37 | $1.54 \times 10^{-4}$   | $1.54 \times 10^{-4}$   |
| 38 | $-1.47 \times 10^{-4}$ | $-1.47 \times 10^{-4}$ |
| 39 | $1.40 \times 10^{-4}$   | $1.40 \times 10^{-4}$   |
| 40 | $-1.34 \times 10^{-4}$ | $-1.34 \times 10^{-4}$ |
| 41 | $1.28 \times 10^{-4}$   | $1.28 \times 10^{-4}$   |
| 42 | $-1.23 \times 10^{-4}$ | $-1.23 \times 10^{-4}$ |
| 43 | $1.18 \times 10^{-4}$   | $1.18 \times 10^{-4}$   |
| 44 | $-1.13 \times 10^{-4}$ | $-1.13 \times 10^{-4}$ |
| 45 | $1.09 \times 10^{-4}$   | $1.09 \times 10^{-4}$   |
| 46 | $-1.05 \times 10^{-4}$ | $-1.05 \times 10^{-4}$ |
| 47 | $1.01 \times 10^{-4}$   | $1.01 \times 10^{-4}$   |
| 48 | $-9.70 \times 10^{-5}$ | $-9.70 \times 10^{-5}$ |
| 49 | $9.34 \times 10^{-5}$   | $9.34 \times 10^{-5}$   |

Table 3: Worked example of the operating equations: cardiomyocytes. The values of $H_n$ and $J_n$ for $n = 0, \ldots, 49$ for neonatal rat cardiomyocytes with $\text{Pe} = 1/3$

References

Abramowitz, M. & Stegun, I. (Eds.) (1965). *Handbook of Mathematical Functions*. Nat. Bur. Standards.