MULTIDIMENSIONAL BREEDEN-LITZENBERGER REPRESENTATION FOR STATE PRICE DENSITIES

JARNO TALPONEN AND LAURI VIITASAARI

Abstract. In this article, we consider European options of type \( h(X_1^T, X_2^T, \ldots, X_n^T) \) depending on many underlying assets. We study how such options can be valued in terms of simple vanilla options in different market models. We consider different approaches and derive several pricing formulas for a wide class of functions \( h: \mathbb{R}_+^n \to \mathbb{R} \). We also give multidimensional version of the result of Breeden and Litzenberger [2] on the relation between derivatives of the call price and the risk-neutral density of the underlying asset.

1. Introduction

Option valuation is one of the most central problems in financial mathematics. However, in many models of interest the option valuation cannot be solved in closed form and thus different approaches have been developed. For instance one can use partial differential equations (PDE) or partial integro-differential (PIDE) methods, Monte Carlo methods, or tree methods. One approach to value more complicated options is to determine its value in terms of the values of simple options of the underlying such as call options and digital options. In the work of Breeden and Litzenberger [2] the authors showed that if the second derivative of the call option price \( V^C(K) \) with respect to the strike exists and is continuous, then the price of European option \( f(X^T) \) is given by

\[
V^f = \int_0^\infty f(a) \frac{d^2}{da^2} V^C(a) da
\]

where we treated the short interest rate as 0 for the sake of simplicity. Thus the second derivative of the strike price of the call is the state price density of the underlying asset \( X^T \). This result has significant applications especially to static hedging which is a field of active research. For more details, see for instance Carr [4] and references therein.

Bick [1] extended the result of Breeden and Litzenberger to a case where either the payoff function or the price of a call has continuous second derivative with respect to its strike price except in a finite set of points \( (s_k)_{k=0}^N \) in which the left- and right derivatives exist and are finite. In particular,
Bick showed that
\[ V^f = B_T^{-1} f(0) + \int_0^\infty f''(a) V^C(a) da \]
\[ + B_T^{-1} \sum_{k=0}^{N} \Delta^- f(s_k) Q(X_T \geq s_k) \]
\[ + B_T^{-1} \sum_{k=0}^{N} \Delta^+ f(s_k) Q(X_T > s_k) \]
\[ + \sum_{k=0}^{N} (f'(s_k^+) - f'(s_k^-)) V^C(s_k), \]
(1.2)

where \( B_T \) denotes the bond function, \( Q \) is the given pricing measure and \( \Delta^- \) and \( \Delta^+ \) denotes the jump of the payoff function \( f \). For later studies on the relation between call options and general options, see also Jarrow \[9\], who derived a characterisation theorem for the distribution function of the underlying asset, and Brown and Ross \[3\], who consider a model with finite state space and showed that a wide class of options are a portfolio of call options with different strike prices. In similar spirit, Cox and Rubinstein \[5\] introduced a method for approximating continuous functions with piecewise linear functions, which are a portfolio of call options with different strikes. They also considered the pricing error of this approximation, and suggested that one should find approximation which is the best in the sense of maximum absolute difference. However, this may cause problems when considering infinite state space.

Recently the results of in [2] and [1] have been extended by the second named author [11] to cover the case where \( f \) is only once piecewise differentiable. In particular, in [11] it is shown that
\[ V^f = B_T^{-1} \mathbb{E}_Q[f(X_T)] \]
\[ = B_T^{-1} f(0) + \int_0^\infty f'(a) V^C(da) \]
\[ + B_T^{-1} \sum_{k=0}^{N} \Delta^- f(s_k) Q(X_T \geq s_k) \]
\[ + B_T^{-1} \sum_{k=0}^{N} \Delta^+ f(s_k) Q(X_T > s_k), \]
(1.3)

where the measure \( V^C(da) \) always exists since \( V^C(a) \) is decreasing function in strike. Barrier-type options were also considered in this context.

To summarize, there exists vast array of studies in the relation between call and digital option values and values for more general options. However, all the mentioned studies consider market models with bond and one stock and similar results in multidimensional case are not so well-known. In this article we give multidimensional versions of the Breeden-Litzenberger type representation of the state-price density, widely understood. In particular, under some natural assumptions we give a pricing formula for European options \( h(X_T) = f(X^1_T, X^2_T, \ldots, X^n_T) \) for a wide class of payoff functions \( f \),
including the rainbow and basket options. In particular, our results cover all continuous functions \( h \) for which the partial derivative \( \partial_{\sigma(1)} \ldots \partial_{\sigma(n)} h \) exists in the sense of distributions for every \( m = 1, \ldots, n \) and every permutation \( \sigma \) of integers \( \{1, \ldots, n\} \). For options which are not of this form we consider standard mollifying techniques with respect to Lebesgue measure. The benefit of this is that in this case the resulting smooth function does not depend on the underlying asset \( X_T \) or the particular choice of the measure \( Q \). We also derive multidimensional version of the Breeden-Litzenberger type representation of the state price density. In other words, if the distribution of the underlying asset \( X_T \) is absolutely continuous with respect to Lebesgue measure, we show that then the density can be derived from the partial derivatives of the price of rainbow options

\[
  h_p(\mathbf{x}, K) = \left( \left( \sum_{i=1}^{n} (x_i - K_i)^+ \right)^{1/p} - K \right)^+,
\]

where \( 0 < p < \infty \).

The benefit of our results is that they cover a wide class of models. In particular, we only assume that at least one pricing measure for \( X_T \) exists. We do not assume that it is unique. Moreover, we consider general underlying assets \( X_T \). Hence our results are valid in models which may be complete or incomplete, or discrete or continuous in time or the state space.

The problem of inferring the state-price density from observed prices of the derivatives can also be regarded as an inverse problem. One plausible approach would be interpreting to pricing functional a rather general integral operator

\[
  \Phi(f, Q) = \int f \, dQ
\]

which may be invertible on the latter coordinate if a sufficiently wide class of payoff functions \( f \) is included. Instead of inverting the operator forcibly, e.g. by discretizing the operator and then inverting the resulting matrix numerically, we will apply some subtle properties of the payoff function class in question.

The rest of the paper is organised as follows. In section 2 we introduce our notation, assumptions and we present our main results. In subsection 2.2 we consider approximation of more general payoff functions with mollifiers and subsection 2.3 is devoted to multidimensional version of Breeden and Litzenberger result. In section 3 we give a result related to uniqueness of the pricing measure \( Q \).

2. Main results

Let \( S^k_t \) denote the stock price processes and \( X^k_t \) the underlying assets of an option. As examples, \( X^k_t \) can be a functional of \( S^k_t \)'s like the average

\[
  X^k_t = \frac{1}{t} \int_0^t S^k_u \, du \quad \text{representing Asian option},
\]

\[
  X^k_t = \sup_{u \leq t} S^k_u \quad \text{representing Lookback option},
\]

\[
  X^k_t = \max_{1 \leq k \leq d} S^k_t \quad \text{representing Rainbow option or}
\]

\[
  X^k_t = \sum_{k=1}^{d} \alpha_k S^k_t \quad \text{representing Basket option. Throughout the article,} \quad B_t \quad \text{denotes the bond given by an non-decreasing deterministic function with} \quad B_0 = 1 \quad \text{(all the results can be extended to stochastic interest rate models, with obvious}
changes in theorems). A vector \( (x_1, \ldots, x_n) \) is denoted by \( \mathbf{x} \). Similarly, \( \mathbf{X}_t \) denotes the vector \( (X^1_t, \ldots, X^n_t) \).

We assume that our model is, to some extent, free of arbitrage which means that there exists at least one pricing measure \( Q \) such that for each claim \( C \), the discounted value at time \( t \) is given by \( \mathbb{E}^Q [B_t^{-1}C | \mathcal{F}_t] \). For more details on mathematics of arbitrage, see [6] and [10] and references therein. In the notation, we usually omit the dependence on \( Q \) and \( \mathbb{E} \) stands for expectation with respect to \( Q \). We also assume that for given maturity \( T \) we have \( X_k^T \in L^1(Q) \) and \( X_k^T \geq 0 \) almost surely. Moreover, the price of a European option with payoff profile \( h(X^1_T, \ldots, X^n_T) \) is denoted by \( V^h \). We present our result for prices only i.e. values at time \( t = 0 \). However, our results could be extended to cover values at arbitrary time \( t \) with obvious changes. Note also that we assume the maturity \( T \), but omit it on the notation.

**Definition 2.1.** For a function \( f : \mathbb{R}_+ \rightarrow \mathbb{R} \), we denote \( f \in \Pi^Q(X_T) \) if the following conditions are satisfied:

1. \( f \) is continuously differentiable except on at most countable set of points \( 0 \leq s_0 < s_1 < \ldots < s_\alpha < \ldots (\alpha < \mu \text{ countable ordinals}) \) in which \( f \) and \( f' \) have jump-discontinuities,
2. \( f(X_T) \in L^1(Q) \),
3. \( f \) satisfies
   \[
   \lim_{x \to \infty} |f(x-)Q(X_T \geq x)| = 0
   \]
   and,
4. the integral
   \[
   \int_0^\infty f'(a)Q(X_T > a)da
   \]
   is finite.

**Definition 2.2.** We denote by \( \mu_{c,-} \) and \( \mu_{c,+} \) the counting measures, so that for a given function \( f : \mathbb{R}_+ \rightarrow \mathbb{R} \), we have

1. \( \mu_{c,-} \)
   \[
   \int_0^x f(y)d\mu_{c,-}(y) = \sum_{y \leq x} \Delta_- f(y),
   \]
   where \( \Delta_- f(y) = f(y) - f(y-) \),
2. \( \mu_{c,+} \)
   \[
   \int_0^x f(y)d\mu_{c,+}(y) = \sum_{y < x} \Delta_+ f(y),
   \]
   where \( \Delta_+ f(y) = f(y+) - f(y) \).

The jump from the left at 0 is defined as \( \Delta_- f(0) = 0 \).

We will also need the following counting measures.

**Definition 2.3.** We denote by \( |\mu|_{c,-} \) and \( |\mu|_{c,+} \) the counting measures such that for a given function \( f : \mathbb{R}_+ \rightarrow \mathbb{R} \), we have
\begin{align}
(1) & \quad \int_0^x f(y) d\mu_{c_-(y)} = \sum_{y \leq x} |\Delta_- f(y)|, \\
(2) & \quad \int_0^x f(y) d\mu_{c_+(y)} = \sum_{y < x} |\Delta_+ f(y)|. 
\end{align}

For a given measure $Q$ and underlying process $X_t$ we consider the following class of payoff functions.

**Definition 2.4.** For a function $h : \mathbb{R}_+^n \to \mathbb{R}$, we write $h \in \Pi^Q(X_T)$ if the following conditions are satisfied:

1. \( h(x) = \sum_{k=1}^m \prod_{j=1}^n f_{k,j}(x_j), \)
   where for every $k = 1, \ldots, m$ and $j = 1, \ldots, n$ we have $f_{k,j} \in \Pi^Q(X_T^j)$.
2. for every $k = 1, \ldots, m$, every $i = 1, \ldots, n$, and every permutation $\sigma = (\sigma(1), \ldots, \sigma(n))$ of integers $1, \ldots, n$ we have
   \( \prod_{j=1}^i f_{k,\sigma(j)}(X_T^{\sigma(j)}) \in L^1(Q). \)
   In particular, $h(X_T) \in L^1(Q)$.
3. for every $k = 1, \ldots, m$ and every $i = 1, \ldots, n$
   \( \lim_{b \to \infty} |f_{k,i}(b-)| \mathbb{E} \left[ \mathbf{1}_{X_T \geq b} \prod_{j=1}^{i-1} |f_{k,j}(X_T^j)| \right] = 0. \)

We note that all polynomials are of form (2.6).

We also need some operators for further use. For the rest of the paper $\partial_k$ denotes the usual partial derivative with respect to variable $x_k$. We find it convenient to use multi-indices to formulate the main results. Recall that a multi-index is a vector $a \in \{0, 1, 2, \ldots\}^n$ which encodes the order of each pure multiple partial derivative in a mixed higher order partial derivative. In what follows all multi-indices $a$ satisfy $|a| := \max_i a_i \leq 1$, being binary sequences, which means that they can also be regarded as subsets of $\{1, \ldots, n\}$. Recall the following standard notation: $|a| := a_1 + \ldots + a_n$.

**Definition 2.5.** For a function $h : \mathbb{R}_+^n \to \mathbb{R}$ we define operator $0_k$ by

\( 0_k h(x) = h(x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n). \)

We will apply the multi-index notation for the operators $0_k$ as well. So, for example $0^k \partial^b h(x_1, x_2, x_3, x_4) = \partial_{x_1} \partial_{x_2} h(0, 0, x_3, x_4)$ for $n = 4$, $a = (1, 1, 0, 0)$ and $b = (0, 0, 1, 1)$. One may also consider $e_1, e_2 \in a; e_3, e_4 \in b; e_1, e_2, e_3, e_4 \in a + b$. 


Definition 2.6. Let $h \in \Pi^n_Q(\overline{X}_T)$ and let $z$, $d$, $r$, and $l$ (short for 'zero', 'derivative', 'right jump' and 'left jump', respectively) be multi-indices such that $|z + d + r + l| = n$ and $|z + d + r + l| = 1$. We consider a functional $A_{z,d,r,l} : \Pi^n_Q(\overline{X}_T) \to \mathbb{R}$ (implicitly depending on $Q$ and $\overline{X}_T$) given by

$$A_{z,d,r,l}(h) = \int_{\mathbb{R}^{n-|z|}} Q^z \partial^d h(\overline{y}) \]

$$\prod_{\sigma \in d+r} dy_{\sigma} \prod_{\sigma \in r} d\mu_{c,+}(y_{\sigma}) \prod_{\sigma \in l} d\mu_{c,-}(y_{\sigma}).$$

We will also require a similar positive functional:

$$|A|_{z,d,r,l} = \int_{\mathbb{R}^{n-|z|}} |Q^z \partial^d h(\overline{y})| \]

$$\prod_{\sigma \in d+r} dy_{\sigma} \prod_{\sigma \in r} d|\mu|_{c,+}(y_{\sigma}) \prod_{\sigma \in l} d|\mu|_{c,-}(y_{\sigma}).$$

For further use we also consider restrictions of operators $A_{z,d,r,l}$ and $|A|_{z,d,r,l}$ to a subset $K^{n-|z|} \subset \mathbb{R}^{n-|z|}$ i.e.

$$A_{z,d,r,l}^K(h) = \int_{K^{n-|z|}} Q^z \partial^d h(\overline{y}) \]

$$\prod_{\sigma \in d+r} dy_{\sigma} \prod_{\sigma \in r} d\mu_{c,+}(y_{\sigma}) \prod_{\sigma \in l} d\mu_{c,-}(y_{\sigma}),$$

and $|A|_{z,d,r,l}^K$ is defined similarly.

The definition of the operators admittedly appears complicated at first sight. However, it is rather natural, we simply start with a function $h$ and choose $|z|$ variables which we set to zero. Next we choose $|d|$ variables and compute partial derivatives with respect to these variables. Next we choose $|r|$ variables and consider right jumps with respect to these variables and for the remaining $|l|$ variables we consider left jumps. Finally, we weight the resulting function with probability where for partial derivatives and right jumps we consider strict tails and for left jumps we consider tail probabilities of the form $Q(X_T^n \geq y_m)$, and integrate over these variables. The functional $A_{z,d,r,l}$ computes this and later on we may sum over every possible permutation. Moreover, if $|A|_{z,d,r,l}$ is finite, then also $A_{z,d,r,l}$ is finite and well-defined. Now we are ready to formulate some of our main theorems.

Theorem 2.1. Let $h \in \Pi^n_Q(\overline{X}_T)$. If

$$|A|_{z,d,r,l}(h) < \infty$$

(2.9)
for each combination of multi-indices $z, d, r, \text{ and } l$ such as above, then the price of a European option with payoff $h(X_T)$ is given by

\begin{equation}
V^h = B_T^{-1} \sum_{|z+d+r+l|=n} A_{z,d,r,l}(h).
\end{equation}

We note that according to the above result we may price option of given type as follows: for every variable $y_k$ we either set it to zero, take partial derivative or consider jump from right or left and then integrate with respect to corresponding measure. The price is obtained by summing over all possible combinations. As a result we obtain $4^n$ terms. However, usually payoff functions are continuous at least with respect to some of the variables. Hence many of the terms vanish.

**Example 2.1.** As an example set $n = 2$ and consider up-and-in Barrier call option with strike $K$ and barrier $H$ given by

$f(S_T, X_T) = (S_T - K)^+1_{X_T \geq H}$,

where $X_T = \sup_{0 \leq u \leq T} S_u$. The price of this option is given by

$V^f = B_T^{-1} \int_K^\infty \mathbb{Q}(S_T > y \land X_T \geq H)dy$.

This result is already established in [11].

The proof of Theorem 2.1 is based on the following lemma which is an extension of results in [11]. The proof is presented in the appendix.

**Lemma 2.1.** Let $f \in \Pi_{\mathbb{Q}}(X_T)$, and $Y \in L^1(\mathbb{Q})$ such that

\begin{equation}
\int_0^\infty |f'(a)|\mathbb{E}[1_{X_T > a}|Y]|da < \infty,
\end{equation}

\begin{equation}
\int_0^\infty |f(a)|\mathbb{E}[1_{X_T > a}|Y]|d\mu_{c,+}(a) < \infty,
\end{equation}

\begin{equation}
\int_0^\infty |f(a)|\mathbb{E}[1_{X_T \geq a}|Y]|d\mu_{c,-}(a) < \infty,
\end{equation}

and

\begin{equation}
\lim_{b \to \infty} |f(b-)|\mathbb{E}[1_{X_T \geq b}|Y]| = 0.
\end{equation}

Then

\begin{equation}
\mathbb{E}[f(X_T)Y] = f(0)\mathbb{E}[Y] + \int_0^\infty f'(a)\mathbb{E}[1_{X_T > a}|Y]da
\end{equation}

\begin{equation}
+ \int_0^\infty f(a)\mathbb{E}[1_{X_T > a}|Y]d\mu_{c,+}(a)
\end{equation}

\begin{equation}
+ \int_0^\infty f(a)\mathbb{E}[1_{X_T \geq a}|Y]d\mu_{c,-}(a).
\end{equation}
Proof of Theorem 2.1. By linearity it is sufficient to consider function
\[ h(\mathbf{x}) = \prod_{k=1}^{n} f_k(x_k). \]

We put \( Y_1 = \prod_{k=1}^{n-1} f_k(X^k_T) \) and apply Lemma 2.1 for payoff \( f_n(X^n_T)Y_1 \) to obtain
\[
V^f_t = B^{-1} \int_0^\infty f_n(a) \mathbb{E}[1_{X^n_T > a} Y_1] da
+ B^{-1} \int_0^\infty f_n(a) \mathbb{E}[1_{X^n_T > a} Y_1] d\mu_{c,+}(a)
+ B^{-1} \int_0^\infty f_n(a) \mathbb{E}[1_{X^n_T > a} Y_1] d\mu_{c,-}(a).
\]

Now we can compute \( \mathbb{E}[1_{X^n_T > a} Y_1] \) (the term \( \mathbb{E}[1_{X^n_T > a} Y_1] \) is treated similarly) by setting \( Y_2 = 1_{X^n_T > a} \prod_{k=1}^{n-2} f_k(X^k_T) \) and applying Lemma 2.1 for \( f_{n-1}(X^{n-1}_T)Y_2 \). Indeed, assumption (2.7) implies that (2.14) is satisfied for every \( Y_i \). Moreover, (2.9) implies that assumptions (2.11)-(2.13) are satisfied for every \( Y_i \). Hence, by proceeding similarly and applying Lemma 2.1 repeatedly, we obtain the result.

In many practical cases the payoff function is continuous but not of form (2.6). By taking a sequence of functions \( h_k \in \Pi^n(Q(X_T)) \) we obtain similar results for limiting functions having enough smoothness. For discontinuous functions the jump parts may cause problems. However, we can approximate discontinuous functions with continuous ones. This is the topic of the next subsection.

2.1. Pricing with distributions. Recall that for each continuous functions \( h : \mathbb{R}^n_+ \to \mathbb{R} \) all the mixed partial derivatives \( \partial^\beta h \) exist in the sense of distributions, see [8] for discussion. Therefore for every continuous function \( g \) with compact support there exists a sequence of smooth (test) functions \( h_n \), obtained by applying the Stone-Weierstrass Theorem on compact sets, such that
\[
\int_{\mathbb{R}^n_+} g(\mathbf{y}) \partial^\beta h(d\mathbf{y}) = \lim_n \int_{\mathbb{R}^n_+} g(\mathbf{y}) \partial^\beta h_n(\mathbf{y}) d\mathbf{y}.
\]

The order of taking the partials does not matter in the above formula because of the possibility of approximating with polynomials. Thus each of the functionals \( A_{z,d,r,l} : \Pi^n(Q(X_T)) \to \mathbb{R} \) can be extended naturally to \( \tilde{A}_{z,d,r,l} \) with the range of all continuous functions \( h \). Indeed, if \( h \) is continuous, we set \( A_{z,d,r,l} = 0 \) whenever \( r \neq 0 \) or \( l \neq 0 \). The functionals \( A_{z,d,0,0} \) are defined naturally since partial derivatives of \( h \) exist. Evidently the same condition holds for \( |A_{z,d,r,l}| \).

Theorem 2.2. Let \( h : \mathbb{R}^n_+ \to \mathbb{R} \) be a continuous payoff function such that
(2.16) \[ |A_{z,d,0,0}(h)| < \infty \]
for all multi-indices $|z + d| = n$, $\|z + d\| = 1$. Then the price of a European option $h(x_T)$ is given by

\begin{equation}
V^h = B_T^{-1} \sum_{|z+d|=n \atop \|z+d\|=1} A_{z,d,0,0}(h).
\end{equation}

**Proof.** Assume first that $h \in C^\infty_0(\mathbb{R}^n_+)$ and let $N$ be a number such that $\text{supp}(h) \subset [0, N]^n$. From real analysis we know that in a compact set $[0, N + \delta]^n$ we can approximate $h$ with a sequence of polynomials $T_n(x)$ uniformly such that the partial derivatives of $T_n$ converge to partial derivatives of $h$ also. Now, setting $T_n(x) = 0$ outside $[0, N + \delta]^n$, by using suitable coordinate-wise convolution, we have $T_n(x) \in \Pi^n_0(X_T)$ for every $n$. Hence the claim for $h \in C^\infty_0(\mathbb{R}^n_+)$ follows.

Assume next that $h$ is merely continuous. By assumption (2.16) there exists a compact set $K \subset \mathbb{R}^{n-1}$, a finite union of suitable smaller compact sets $K_{z,d}$, such that

$$
\sum_{|z+d|=n \atop \|z+d\|=1} |\tilde{A}_{z,d,0,0}(h)| < \epsilon,
$$

where

$$
\tilde{A}_{z,d,0,0}(h) = \int_{\mathbb{R}^n_+ \backslash |z|} 0^x \partial^d h(y)Q \left( \bigwedge_{\sigma, d} (X^\sigma_T > y) \right) \prod_{\sigma, d} dy_\sigma.
$$

Since $h$ is continuous, we can take a sequence $h_k \in C^\infty_0(\mathbb{R}^n_+)$ such that $h_k$ converges to $h$ uniformly on compact sets and all the partial derivatives converge in the space of distributions $\mathcal{D}'(\mathbb{R}^n_+)$ accordingly. Thus

$$
\tilde{A}_{z,d,0,0}(h_k) \rightarrow \tilde{A}_{z,d,0,0}(h)
$$

as $k \rightarrow \infty$ for all multi-indices $|z + d| = 1$, $\|z + d\| = 1$. Hence the claim follows as $\epsilon > 0$ was arbitrary. \qed

**Example 2.2.** Consider a spread option $f(x_T^1, x_T^2) = (X_T^1 - X_T^2)^+$. Now $f(x, y)$ is continuous, $f(0, 0) = f_y(0, y) = 0$, and $f_x(x, 0) = 1$. Moreover, $f_{xy}(x, y)$ exists in the sense of distributions and equals $-\delta_x(y)$, where $\delta_x(y)$ is the Dirac delta function at $x$. Hence we obtain

$$
V^f = B_T^{-1} \int_0^\infty Q(X^1_T > y)dy - B_T^{-1} \int_{\mathbb{R}^2_+} Q(X^1_T > x \wedge X^2_T > y)dz\delta_x(dy)
$$

$$
= B_T^{-1}E[X^1_T] - B_T^{-1} \int_0^\infty Q(X^1_T > y \wedge X^2_T > y)dy.
$$

**Example 2.3.** Consider a payoff function $f(x, y) = 1_{x \leq y}$. Now $f$ is not continuous nor of form (2.2). However, we have

$$
f(x, y) = \lim_{\epsilon \rightarrow 0^+} \frac{(x - y + \epsilon)^+ - (x - y)^+}{\epsilon}.
$$
Hence, by the previous example and the dominated convergence theorem, we may calculate formally as follows

\[
\mathbb{E}[f(X_T^1, X_T^2)] = -\lim_{\epsilon \to 0^+} \int_0^\infty \frac{Q(X_T^1 > y \wedge X_T^2 > y + \epsilon) - Q(X_T^1 > y \wedge X_T^2 > y)}{\epsilon} dy
\]

\[
= \int_0^\infty Q(X_T^1 > y, X_T^2 \in dy).
\]

Another way to price such options is to use mollifiers as in next subsection.

2.2. Approximation with smooth functions. Our main theorem explains how options with sufficient smoothness can be priced. In this section we consider general integrable functions \( h \) and consider how to apply our main theorems for pricing such options. We use mollifiers with respect to the Lebesgue measure. The benefit is that this mollifier does not depend on \( \mathbf{X} \), and hence not on the particular choice of \( Q \).

We use the standard mollifier given by

\[
\rho(x) = c1_{|x| < 1} e^{-\frac{1}{1-|x|^2}},
\]

where \( |\cdot| \) denotes the standard Euclidean norm and \( c \) is a constant such that

\[
\int_{\mathbb{R}^n} \rho(x) dx = 1.
\]

Now we have \( \rho \in C_0^\infty \). Let now \( h \) be an arbitrary function. Formally we set

\[
h_\epsilon(x) = \int_{\mathbb{R}^n} \rho(y) h(x - \epsilon y) dy.
\]

A standard result of real analysis states that for sufficiently small \( \epsilon \), \( h_\epsilon \) is infinitely differentiable on compact subsets. Moreover, if \( h \) is continuous, then \( h_\epsilon \to h \) uniformly on compact subsets. We also recall the following fact from real analysis (cf. [13]).

Lemma 2.2. Let \( \mu \) be a positive Radon measure on \( \mathbb{R}^n \). For any \( h \in L^1(\mathbb{R}^n, \mu) \) and any \( \epsilon > 0 \), there exists a function \( \varphi \in C_0(\mathbb{R}^n) \) such that

\[
||h - \varphi||_\mu < \epsilon.
\]

We now proceed to consider our case.

Theorem 2.3. Assume that \( h(\mathbf{X}_T) \in L^1(\mathbb{Q}) \). Then the following are equivalent:

1. \( V^{h_\epsilon} \to V^h \),

2. \( \mathbb{E}|h_\epsilon(\mathbf{X}_T) - h(\mathbf{X}_T)| \to 0 \),

3. for every \( \bar{\epsilon} \) there exists a compact set \( K \) and a constant \( \eta > 0 \) such that

\[
\sup_{0 < \epsilon < \eta} |\mathbb{E}[h_\epsilon(\mathbf{X}_T)]1_{\mathbf{X}_T \in \mathbb{R}^n \setminus K}| < \bar{\epsilon}.
\]
Proof. Without loss of generality we can omit the bond.

(3) ⇒ (2): Assume we have (2.22) and fix \( \tilde{\epsilon} \). By (2.22) we can take compact subset \( K \subset \mathbb{R}^n_+ \) such that

\[
\mathbb{E}|h_{\epsilon}(X_T) - h(X_T)|1_{X_T \in \mathbb{R}^n_+ \setminus K} < \frac{\tilde{\epsilon}}{4}.
\]

Moreover, by Lemma 2.2 we can take continuous \( \varphi \) such that

\[
\mathbb{E}|h(X_T) - \varphi(X_T)| < \frac{\tilde{\epsilon}}{4}.
\]

We obtain

\[
\mathbb{E}|h_{\epsilon}(X_T) - \varphi(X_T)|1_{X_T \in \mathbb{R}^n_+ \setminus K} < \frac{\tilde{\epsilon}}{4}.
\]

Since \( \varphi \) is continuous, the second term is bounded by \( \frac{\tilde{\epsilon}}{4} \) for sufficiently small \( \epsilon \). To finish the proof we obtain by continuity of \( \varphi \), compactness of \( K \) and assumption (2.23) that

\[
\mathbb{E}|h_{\epsilon}(X_T) - \varphi(X_T)|1_{X_T \in K} < \frac{\tilde{\epsilon}}{4}.
\]

(2) ⇒ (1): this implication is obvious.

(1) ⇒ (3): Assume that we have (1) and (2.22) does not hold. Let \( \tilde{\epsilon} > 0 \) be fixed. Since \( h \) is integrable, we can find compact set \( K \) such that

\[
\mathbb{E}|h(X_T)|1_{X_T \in \mathbb{R}^n_+ \setminus K} < \frac{\tilde{\epsilon}}{2}.
\]

Moreover, by (1) we have

\[
\mathbb{E}|h_{\epsilon}(X_T) - h(X_T)|1_{X_T \in \mathbb{R}^n_+ \setminus K} < \frac{\tilde{\epsilon}}{2}
\]

for \( \epsilon \) sufficiently small. Hence we also have

\[
\mathbb{E}|h_{\epsilon}(X_T)|1_{X_T \in \mathbb{R}^n_+ \setminus K} < \tilde{\epsilon}
\]

and we have a contradiction. This completes the proof. \( \square \)

Note that condition (3) is closely related to the notion of uniform integrability with respect to pair \((B, h)\), a notion which was introduced in [12] in one-dimensional case.

In many practical case the Condition (2.22) is satisfied. The next result gives an easily verifiable sufficient condition, under which we have (2.22).

**Corollary 2.1.** If for every \( \tilde{\epsilon} > 0 \) there exist a compact set \( K \in \mathbb{R}^n_+ \) and a constant \( \eta \) such that

\[
(2.23) \quad \sup_{0 < \epsilon < \eta} \sup_{|y| \leq 1} \mathbb{E}|h(X_T - \epsilon y)|1_{X_T \in \mathbb{R}^n_+ \setminus K} < \tilde{\epsilon},
\]

Then we also have (2.22).
Many options of interest have payoff functions which are polynomially bounded, i.e. there exists a polynomial $p(x)$ such that

$$|h(x)| \leq p(x), \forall x.$$ 

If we now have

$$\mathbb{E}[p(X_T)] < \infty,$$

then we also have (2.23).

### 2.3 Absolute continuity with respect to Lebesgue measure.

In a general model the measure $Q$ is not necessarily absolutely continuous with respect to Lebesgue measure (more precisely, the law of $X_T$ under $Q$). Yet, if the payoff function has enough smoothness as it does in many practical cases, one may apply our Theorem 2.1.

However, typically the state-price density is absolutely continuous with respect to the Lebesgue and then we have nice representations for it. It was shown by Breeden and Litzenberger [2] that in one dimensional case the risk-neutral density can be obtained by taking the second derivative of the strike price in the call’s price functional. In this section we derive similar result for multidimensional case. We omit the interest rate for simplicity.

Let $0 < p < \infty$. We define a function $h_p : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}$ by

$$h_p(x, \overline{K}, K) = \left( \sum_{i=1}^{n} ((x_i - K_i)^+)^{1/p} - K \right)^+.$$

We will denote by $V^p(\overline{K}, K)$ the corresponding price of European rainbow option with payoff $h_p(X_T, \overline{K}, K)$. We will also consider function $h_{\infty}$ given as a limit

$$h_{\infty} = \lim_{p \to \infty} h_p,$$

and the corresponding price given by $V^\infty$. We begin with some results on relation between prices $V^p$ with different values of $p$.

**Lemma 2.3.** Let $(\overline{K}, K) \in \mathbb{R}_+^{n+1}$ be arbitrary. For every $x \in \mathbb{R}^n$ we have

$$h_{\infty}(x, \overline{K}, K) = \left( \max_i (x_i - K_i)^+ - K \right)^+,$$

and

$$V^\infty(\overline{K}, K) = \lim_{p \to \infty} V^p(\overline{K}, K).$$

**Proposition 2.1.** Let $0 < p \leq \infty$. Then we have

$$\sum_{i=1}^{n} \lim_{K_j \to \infty} V^p(K_1, \ldots, K_n, 0) = V^1(K_1, \ldots, K_n, 0)$$

for all $K_1, \ldots, K_n$. The similar conclusion holds for the corresponding payoff functions with limits taken pointwise.

**Proof.** It is easy to see that the statement about the payoff functions holds in the sense that the limits are taken pointwise. Let now $K_1 \subset \ldots K_n \subset \ldots$ be a sequence of compact sets such that $K_n \uparrow \mathbb{R}^n$. Now (2.26) follows by
considering expectations \( \mathbb{E}[h_p] \mathbf{1}_{X_T \in K_n} \) and applying monotone convergence theorem. □

**Lemma 2.4.** Suppose that \( Q <\ll m \) on \([0, \infty)^n\). Then

\[
\frac{dQ}{dm}(K_1, \ldots, K_n) = \frac{\partial^n}{\partial K_1 \cdots \partial K_n} \mathbb{Q}\left( \bigwedge_i (X^i_T \leq K_i) \right) \quad m\text{-}a.e.
\]

**Proof.** Let \( \frac{dQ}{dm} \) be the Radon-Nikodym derivative of \( Q \). Observe that

\[
\mathbb{Q}\left( \bigwedge_i (X^i_T \leq K_i) \right) = \int 1_{X^1_T \leq K_1} \cdots 1_{X^n_T \leq K_n} \frac{dQ}{dm} \, dm
\]

\[
= \int_0^{K_1} \cdots \int_0^{K_n} \frac{dQ}{dm} (x_1, \ldots, x_n) \, dx_n \ldots dx_1.
\]

According to Lebesgue’s differentiation theorem the right hand side is differentiable with respect to \( K_1 \) for \( m_1\text{-}a.e. \) \( K_1 \geq 0 \). The set of tuples \( (K_1, x_2, \ldots, x_n) \), where this differentiability fails is \( m \)-null, so that we may disregard it. By proceeding in this manner and differentiating \( n \) times we obtain the statement. □

**Theorem 2.4.** Suppose that \( Q <\ll m \) on \([0, \infty)^n\). Then

\[
\frac{dQ}{dm}(K_1, \ldots, K_n) = \frac{\partial^n}{\partial K_1 \cdots \partial K_n} \sum_{i=1}^n \frac{\partial}{\partial K_i} V^\infty(K_1, \ldots, K_n, 0) \quad m\text{-}a.e.
\]

**Proof.** Denote by \( 1 = (1, 1, \ldots, 1, 0) \in \mathbb{R}^{n+1} \). Here we apply the elementary fact that in the case with continuous partials the directional derivative can be calculated by taking the inner product of a gradient and a direction vector.

It is easy to see that the above partials are continuous in an open subset of the state space \( \mathbb{R}^n_+ \) with \( m \)-null complement. The directional derivative of the payoff function satisfies

\[
D_1 h^\infty(X^1_T, \ldots, X^n_T, K_1, \ldots, K_n, 0) = -\max_i 1_{X^i_T > K_i}
\]

when defined. This limit on the left hand side is both defined and uniform on compact subsets of

\[
\bigvee_i (X^i_T < K_i) \lor \neg \bigwedge_i (X^i_T = K_i)
\]

which has clearly \( Q \)-null complement. By the uniform convergence and the fact that \( Q \) is a Radon measure we get

\[
\mathbb{E}\left( -\max_i 1_{X^i_T > K_i} \right) = \mathbb{E}(D_1 h^\infty(\overline{X}_T, K_1, \ldots, K_n, 0))
\]

\[
= D_1 \mathbb{E}(h^\infty(\overline{X}_T, K_1, \ldots, K_n, 0)) = D_1 V^\infty(K_1, \ldots, K_n, 0).
\]

On the other hand, \( 1 - \mathbb{E}\left( -\max_i 1_{X^i_T > K_i} \right) = \mathbb{Q}\left( \bigwedge_i (X^i_T \leq K_i) \right) \). The argument is finished by Lemma 2.4 Θ

**Proposition 2.2.** The following equality holds

\[
\frac{\partial^+}{\partial K_j} V^1(K_1, \ldots, K_n, 0) = -\mathbb{Q}(X^j_T \geq K_j).
\]
Similarly,
\[ \frac{\partial^+}{\partial K_j} h_1(X_1^T, \ldots, X_n^T, K_1, \ldots, K_n, 0) = -1_{X_j^T \geq K_j}. \]

Proof. The proof is similar to that of Theorem 2.4. It is easy to see that the right derivative in the latter statement coincides with \(-1_{X_j^T \geq K_j}\). In fact,
\[ h_1(x_1, \ldots, x_n, K_1, \ldots, K_j + \epsilon, \ldots, K_n, 0) - h_1(x_1, \ldots, x_n, K_1, \ldots, K_n, 0) \]
\[ \epsilon \]
tends to \(-1_{X_j^T \geq K_j}\) as \(\epsilon \to 0^+\) and even uniformly so in compact subsets of \((X_T^T = K_j)\). Therefore
\[ \frac{\partial^+}{\partial K_j} \mathbb{E}(h_1(X_1^T, \ldots, X_n^T, K_1, \ldots, K_n, 0)) = \frac{\partial^+}{\partial K_j} V^1(K_1, \ldots, K_n, 0). \]
\[ \square \]

Theorem 2.5. Suppose that \(\mathbb{Q} \ll m\) on \([0, \infty)^n\). Then
\[ \frac{d\mathbb{Q}}{dm}(K_1, \ldots, K_n) = \lim_{K \to 0^+} \frac{\partial^{n+1}}{\partial K_1 \cdots \partial K_n \partial \mathbb{K}} V^p(K_1, \ldots, K_n, \mathbb{K}) \quad m\text{-a.e.} \]
for \(0 < p \leq \infty\).

Proof. We observe that for a given \(K\) the limit
\[ \frac{\partial}{\partial \mathbb{K}} h_p(X_1^T, \ldots, X_n^T, K_1, \ldots, K_n, K) = -1_{h_p(X_T^T, \mathbb{K}, 0) > 0} \]
exists and is uniform on compact subsets of \((h_p(X_T^T, \mathbb{K}, 0) = K)\) whose complement is \(m\)-null. Hence,
\[ \frac{\partial}{\partial \mathbb{K}} \mathbb{E}(h_p(X_1^T, \ldots, X_n^T, K_1, \ldots, K_n, K)) = -\mathbb{Q}(h_p > 0), \]
since \(\mathbb{Q}\) is a Radon measure. By using the \(\sigma\)-additivity of \(\mathbb{Q}\) we obtain that \(\lim_{K \to 0^+} \mathbb{Q}(h_p(X_T^T, \mathbb{K}, K) > 0) = \mathbb{Q}(\lim_{K \to 0^+} h_p(X_T^T, \mathbb{K}, K) > 0)\) exists for \((K_1, \ldots, K_n)\). Thus, by keeping the definition of \(h_p\) in mind, it is easy to see that
\[ \lim_{K \to 0^+} \frac{\partial}{\partial \mathbb{K}} V^p(\mathbb{K}, K) = -\mathbb{Q} \left( \bigvee_i (X_i^T > K_i) \right) = \mathbb{Q} \left( \bigwedge_i (X_i^T \leq K_i) \right) - 1. \]
The argument is finished similarly as in the proof of Theorem 2.4 and the order of taking the limit can be changed according to the monotone convergence theorem. \[ \square \]

We note that for \(n = 2\) and \(p = 1\) the above state price density can be expressed in an alternative form due to the fact that
\[ D_{(-1,-1,1)} V^1(K_1, K_2, K) = -\mathbb{Q}(X_1^T \geq K_1 \land X_2^T \geq K_2). \]
That is,
\[
\frac{dQ}{dm}(K_1, K_2) = \frac{\partial^2}{\partial K_1 \partial K_2} \left( \frac{\partial V^1}{\partial K_1} + \frac{\partial V^1}{\partial K_2} - \frac{\partial V^1}{\partial K} \right) m\text{-a.e.}
\]

3. ON THE UNIQUENESS OF ARBITRAGE-FREE PRICES

The main theme in this paper has been deducing the pricing kernel from observed prices of European style options with several underlying assets.

If there is an explicit formula for the pricing measure, then, of course, the measure must be unique. One may ask if the price information about some class of derivatives sufficiently determines the measure, although no explicit formula may not be available (cf. [3]).

The following fact is probably evident to the specialists in the field and it can be obtained rather immediately from the considerations in the previous section.

**Proposition 3.1.** The values \( C_1(K_1, \ldots, K_n, 0) \) determine \( Q \) uniquely if \( S_i \) are independent.

**Proof.** This fact is based on disintegrating the measure similarly as in the proof of Lemma 2.4.

Next we will attempt to see how some partial information obtained from the prices of a class of derivatives translates to a kind of partial uniqueness of the pricing measure. The information accumulation can be conveniently encoded in terms of sub-\( \sigma \)-algebras, as is customary in the probability theory.

The following result roughly states that derivatives obtained by multiplying European call options determine the pricing measure on the sub-\( \sigma \)-algebra they generate.

**Theorem 3.1.** Let \( \mathcal{F} \) be a set of non-negative Borel functions on the state space \( \mathbb{R}_+^n \) considered as European style payoff. Let \( Q \) be a set of Borel probability measures on the same state space. Assume that \( \mathbb{E}_Q \prod_{i=1}^n (f_i - K_i)^+ \) exists and does not depend on the particular choice of \( Q \in \mathcal{Q} \) for \( K_1, \ldots, K_n \in \mathbb{R}_+ \) and \( f_1, \ldots, f_n \in \mathcal{F} \). Suppose that \( g \) is a \( \sigma(\mathcal{F}) \)-measurable payoff function. Then \( g \) has \( Q \)-expectation if and only it has expectation with respect to all the measures in the family. Moreover, the value \( \mathbb{E}_Q g \), when defined, does not depend on the particular choice of \( Q \in \mathcal{Q} \).

Recall that \( \sigma(\mathcal{F}) \) is the smallest \( \sigma \)-algebra containing the sets \( f^{-1}(U) \) for \( f \in \mathcal{F} \) and \( U \subset \mathbb{R} \) open and the statement that \( g \) is \( \Sigma \)-measurable means that \( g^{-1}(U) \in \Sigma \) for each open \( U \subset \mathbb{R} \). The payoff profiles \( f_i \) and \( g \) can be seen as random variables with respect to the different measures \( Q \in \mathcal{Q} \). For example putting \( g = \mathbb{E}_{Q_0}(h|\sigma(\mathcal{F})) \) gives a typical \( \sigma(\mathcal{F}) \)-measurable random variable. We will apply Dynkin’s lemma which is well suited for analyzing the uniqueness of measures, see [7].

**Proof.** We denote by \( \mathcal{D} \) the collection of all Borel sets \( A \) such that \( Q(A) \) does not depend on the particular choice of \( Q \). Note that \( \mathcal{D} \) is closed with respect to taking complements, since \( Q \) are probability measures. By the \( \sigma \)-additivity of the measures we observe that \( \mathcal{D} \) is closed with respect to taking countable unions of disjoint subsets. Thus \( \mathcal{D} \) is a Dynkin system.
We note that $\sigma(F)$ is generated by the sets $f \geq K$ with $f \in F$, $K \geq 0$. In what follows we restrict our attention to $\sigma(F)$, i.e. the measures and measurable functions are considered with this $\sigma$-algebra. We aim to show that $\sigma(F) \subset D$. This suffices in order to obtain the statement for $\sigma(F)$-simple functions $g$. Indeed, then it follows that the measures in $Q$ restricted to $\sigma(F)$ coincide. If $g$ should be integrable with respect to a measure $Q_0 \in Q$, then $E_{Q_0}(g)$ can be approximated by expectations of simple functions $E_{Q_0}(g_n)$ with $g_n \not\sim g$ $Q_0$-a.s. as $n \to \infty$. It follows that the values $E_Q(g_n)$ coincide for different choices of $Q$. Note that $g_n \not\sim g$ $Q$-a.s. as $n \to \infty$ by the equivalence of the measures. Thus, $E_Q(g) = E_{Q_0}(g)$ by the Monotone Convergence Theorem. Next we invoke Dynkin’s Theorem, which yields that if the sets of the form

$$
\bigwedge_{i=1}^{n} M_i \leq f_i < K_i
$$

are included in $D$, then also $\sigma(F) \subset D$. Indeed, the collection of sets in (3.1) are closed with respect to taking finite intersections and they also $\sigma$-generate sets $M < f_i < K$ and consequently the $\sigma$-algebra $\sigma(F)$ as well. Recall that an indicator function\(1_{M_i \leq f_i < K_i}\) can be written as\(1_{M_i \leq f_i < K_i} = \lim_{\epsilon \to 0^+} g_\epsilon\) where

$$
g_\epsilon = \prod_{i=1}^{n} \frac{(f_i - M_i + \epsilon)^+ - (f_i - K_i + \epsilon)^+ + (f_i - K_i)^+}{\epsilon}.
$$

This means that $\prod_{i=1}^{n} 1_{M_i \leq f_i < K_i}$ can be written as $\lim_{\epsilon \to 0^+} g_\epsilon$, where

$$
Q(\bigwedge_{i=1}^{n} M_i \leq f_i < K_i) = \lim_{\epsilon \to 0^+} E_Q(g_\epsilon)
$$

does not depend on $Q$. Indeed, the functions $g_\epsilon$ have expectations according to the assumptions. Thus the sets (3.1) are included in $D$. \hfill \Box

**Acknowledgements.**

Lauri Viitasaari thanks the Finnish Doctoral Programme in Stochastics and Statistics for financial support.

**Appendix A. Proof of Lemma 2.1**

Proof is based on following lemmas.

**Lemma A.1.** Assume that

$$
\prod_{k=1}^{m} X^{\sigma(k)}_T \in L^1(Q)
$$


for every \( m = 1, \ldots, n \) and every permutation \( \sigma \). Then the expected value of
\[
h(X_T; \mathbf{y}) = \prod_{k=1}^{n} (X_T^k - y_k)^+
\]
is absolutely continuous with respect to Lebesgue measure.

**Proof.** The claim follows directly from observation that
\[
|(X_T^k - y_k)^+ - (X_T - y_k)^+| \leq h.
\]
\[
\square
\]

**Lemma A.2.** Let \( \alpha \geq 0, \alpha < \beta < \infty \) and consider a function \( g \) of the form \( g(x) = f(x)1_{\alpha \leq x \leq \beta} \), where \( f \) is continuous on \([\alpha, \beta]\) and continuously differentiable on \((\alpha, \beta)\). If \( Y \in L^1 \), then
\[
\mathbb{E}[g(X_T)Y] = f(\alpha)\mathbb{E}[1_{X_T \geq \alpha}Y] - f(\beta)\mathbb{E}[1_{X_T < \beta}Y] + \int_{\alpha}^{\beta} f'(a)\mathbb{E}[1_{X_T > a}Y]da.
\]

**Proof.** The proof is essentially the same as presented in [11] for case \( Y = 1 \).

Indeed, since \( g \) is continuous on \([\alpha, \beta]\), we can approximate it with
\[
g_n(x) = f(\alpha)1_{x=\alpha} + \sum_{k=1}^{n} (c_k x + b_k)1_{ak < x \leq ak+1}
\]
(A.2)

where \( \alpha = a_1 < a_2 < \ldots < a_{n+1} = \beta \) is a partition of the interval \([\alpha, \beta]\) and the coefficients are given by
\[
c_k = \frac{f(ak+1) - f(ak)}{ak+1 - ak},
\]
and
\[
b_k = f(ak+1) - c_ka_{k+1} = f(ak) - c_kak.
\]

Simple computations and taking expectation yields
\[
\mathbb{E}[g_n(X_T)Y] = f(\alpha)\mathbb{E}[1_{X_T \geq \alpha}Y] - f(\beta)\mathbb{E}[1_{X_T < \beta}Y] + \sum_{k=1}^{n} c_k \left[\mathbb{E}[(X_T - ak)^+Y] - \mathbb{E}[(X_T - ak+1)^+Y]\right].
\]

Now applying Mean value theorem, continuity of \( f' \), and the fact that \( \mathbb{E}[(X_T - a)^+Y] \) is a monotonic function with respect to \( a \) we obtain
\[
\sum_{k=1}^{n} c_k \left[\mathbb{E}[(X_T - ak)^+Y] - \mathbb{E}[(X_T - ak+1)^+Y]\right] \rightarrow \int_{\alpha}^{\beta} f'(a)d\mathbb{E}[(X_T-a)^+Y].
\]

Moreover, by Lemma [A.1] we have
\[
\int_{\alpha}^{\beta} f'(a)d\mathbb{E}[(X_T-a)^+Y] = \int_{\alpha}^{\beta} f'(a)\mathbb{E}[1_{X_T > a}Y]da.
\]

It remains to note that \( g_n \) converges to \( g \) pointwise. Hence the result follows by Dominated convergence theorem. \( \square \)
Lemma A.3. Let \( \alpha \geq 0, \alpha < \beta < \infty \) and consider a function \( g^0(x) = f(x)I_{\alpha < x < \beta} \), where \( f \) is continuous on \([\alpha, \beta]\) and continuously differentiable on \((\alpha, \beta)\). If \( Y \in L^1 \), then
\[
\mathbb{E}[g^0(X_T)Y] = f(\alpha)\mathbb{E}[1_{X_T \geq \alpha}Y] - f(\beta)\mathbb{E}[1_{X_T > \beta}Y]
\]
\begin{equation}
(A.3)
\end{equation}

Proof. Define a new function \( g \) by
\[
g(x) = \begin{cases} 
  f(\alpha+), & x = \alpha \\
  f(x), & x \in (\alpha, \beta) \\
  f(\beta-), & x = \beta.
\end{cases}
\]

By Lemma A.2, we obtain
\[
\mathbb{E}[g(X_T)Y] = f(\alpha)\mathbb{E}[1_{X_T \geq \alpha}Y] - f(\beta)\mathbb{E}[1_{X_T > \beta}Y]
\]
\[
+ \int_\alpha^\beta f'(a)\mathbb{E}[1_{X_T > a}Y]\,da.
\]

Noting that \( g^0(x) = g(x) - g(\alpha)1_{x=\alpha} - g(\beta)1_{x=\beta} \) we obtain the result. \( \square \)

Proof of Lemma 2.1. Put \( g_b(x) = f(x)1_{0 \leq x < b} \), and set \( s_{n+1} = b \). By assumptions, we may write
\[
g_b(x) = \sum_{k=0}^n f(x)1_{s_k < x < s_{k+1}} + \sum_{k=0}^n f(x)1_{x = s_k}.
\]

Applying Lemma A.3 for terms on the first sum and direct computations yields
\[
\mathbb{E}[f(X_T)Y] = f(0)\mathbb{E}[Y] + \int_0^b f'(a)\mathbb{E}[1_{X_T > a}Y]\,da
\]
\[
+ \int_0^b f(a)\mathbb{E}[1_{X_T > a}Y]\,d\mu_{\mathbb{C}_{+}}(a)
\]
\[
+ \int_0^b f(a)\mathbb{E}[1_{X_T > a}Y]\,d\mu_{\mathbb{C}_{-}}(a)
\]
\[
- f(b-)\mathbb{E}[1_{X_T > b}Y].
\]

Letting \( b \) tend to infinity and applying Dominated convergence result together with assumptions gives the result. \( \square \)

References

[1] Bick, A. (1982). Comments on the valuation of derivative assets, Journal of Financial Economics, 10: 331-345.
[2] Breeden, D.T., Litzenberger, R.H. (1978). Prices of state-contingent claims implicit in option prices, Journal of Business, 51: 621-651.
[3] Brown, D.J., Ross, S.A. (1991). Spanning, valuation and options, Economic Theory, 1: 3-12.
[4] Carr, P., Picron, J. (1999). Static Hedging of Timing Risk, Journal of Derivatives, 3: 57-70.
[5] Cox, J., Rubinstein, M. (1985). Options Markets, Prentice-Hall.
[6] F. Delbaen and W. Schachermayer, A general version of the fundamental theorem of asset pricing, Math. Ann. 300 (1994) 463-520.
[7] P. Halmos, 1950. Measure theory. Van Nostrand and Co.
[8] L. Hörmander, 1983, The Analysis of Partial Differential Operators I, Distribution Theory and Fourier Analysis, A Series of Comprehensive Studies in Mathematics, 256, Springer-Verlag.
[9] Jarrow, R. (1986). A Characterization theorem for unique risk neutral probability measures, Economic Letters, 22: 61-65.
[10] F. Delbaen and W. Schachermayer, The fundamental theorem of asset pricing for unbounded stochastic processes, Math. Ann. 312 (1998) 215-250.
[11] Viitasaari, L. (2012). Option prices with call prices, ArXiv: 1207.6205.
[12] Viitasaari, L. (2012). Rate of convergence for discrete approximation of option prices, ArXiv: 1207.6756.
[13] Rudin, W. (1987). Real and complex analysis, 3ed., McGraw-Hill.

DEPARTMENT OF PHYSICS AND MATHEMATICS, UNIVERSITY OF EASTERN FINLAND, P.O. BOX 111, 80101 JOENSUU, TALPONEN@IKI.FI

DEPARTMENT OF MATHEMATICS AND SYSTEM ANALYSIS, HELSINKI UNIVERSITY OF TECHNOLOGY, P.O. BOX 11100, FIN-00076 AALTO, FINLAND