Scaling Factor Inconsistencies in Neutrinoless Double Beta Decay

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The modern theory of neutrinoless double beta decay includes a scaling factor that has often been treated inconsistently in the literature. The nuclear contribution to the decay half life can be suppressed by 15-20% when scaling factors are mismatched. Correspondingly, \( \langle m_\nu \rangle \) is overestimated.

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I. INTRODUCTION

In recent years, experimental evidence for neutrino masses and mixing have lead to a concerted effort to refine the methods used in calculating reaction rates for double-beta decay. One of the largest uncertainties in these calculations is the determination of the nuclear matrix elements (NME). Together with kinematic factors and experimental bounds on the decay half life, an average neutrino mass can be extracted. An accurate determination of the NME is crucial and improvements in the QRPA and shell model techniques used to calculate them continue to be explored.

In the modern theory of neutrinoless double beta decay (\( \beta\beta(0\nu) \)) definitions of the NME include a scaling factor introduced such that the NME is dimensionless. When using these scaled NME to determine the \( \beta\beta(0\nu) \) decay rate, the scaling factor must be compensated for elsewhere. However, the scaling factor has not always been treated consistently in the literature. In this article, we detail how the scaling factor is introduced into the theory and illustrate the 15-20% suppression of the nuclear contribution when mismatched scaling factors are used.

II. NEUTRINOLESS DOUBLE BETA DECAY

In the simplest form of the weak Hamiltonian

\[
H_W = \frac{G}{\sqrt{2}} [j^+ L \bar{L}^+ + h.c.]
\]  

(1)

the half life of the \( 0^+ \to 0^+ \beta\beta(0\nu) \) is written as

\[
[T_{1/2}^{(0\nu)}]^{-1} = |M^{(0\nu)}|^2 G_{01} \left( \frac{\langle m_\nu \rangle}{m_e} \right)^2 .
\]  

(2)

where \( M^{(0\nu)} \) is the NME, \( G_{01} \) is the so-called phase space or kinematic factor and \( m_\nu \) and \( m_e \) are the neutrino and electron masses respectively. The NME is given by

\[
M^{(0\nu)} = M_{GT}^{(0\nu)} (1 - \chi_F) 
\]  

(3)

\[
M_{GT}^{(0\nu)} = \sum_a \langle 0_f^+ | \sum_n \tau_n^+ | N_a \rangle \langle N_a | \sum_m \tau_m^+ | 0_i^+ \rangle (\sigma_n \cdot \sigma_m) \frac{1}{2} (H_2 - H_1) 
\]  

(4)

\[
\chi_F = \sum_a \langle 0_f^+ | \sum_n \tau_n^+ | N_a \rangle \langle N_a | \sum_m \tau_m^+ | 0_i^+ \rangle \frac{1}{2} (H_2 - H_1) 
\]  

(5)

The sum is taken over intermediate states \( N_a \). A detailed explanation of the NME can be found in [1]. We will use notation adapted from [1] throughout this article.

The \( \nu \) potential induced by the virtual \( \nu \) exchange is given by

\[
H_k(r, E_a) = \frac{1}{2\pi^2} \int \frac{dq}{\omega} \frac{1}{\omega + A_k} e^{i\mathbf{q} \cdot \mathbf{r}} ,
\]  

(7)

\[
A_{1(2)} = E_a - \frac{1}{2} (E_i + E_f) \pm \frac{1}{2} (\epsilon_1 - \epsilon_2) .
\]  

(8)
The initial, intermediate and final state energies are denoted by \( E_i, E_a \) and \( E_f \) respectively and \( \epsilon_i \) denotes the energy of the \( i \)th electron.

The kinematic factor \( G_{01} \) in Eq. (2) is defined as

\[
G_{01} = \frac{a_{0\nu}}{m_e^2 \ln 2} \int d\Omega_{0\nu} \ F_0(Z, \epsilon_1) \ F_0(Z, \epsilon_2),
\]

where

\[
a_{0\nu} = \frac{(G g_A)^4 m_e^3}{64 \pi^5} \quad (10)
\]

\[
d\Omega_{0\nu} = m_e^{-5} \ p_1 \ p_2 \ \epsilon_1 \ \epsilon_2 \ \delta(\epsilon_1 + \epsilon_2 + E_f - E_i) \ d\epsilon_1 \ d\epsilon_2 \ d(\hat{p}_1 \cdot \hat{p}_2). \quad (11)
\]

We have assumed \( S = 0 \) electron wave functions with no \( r \) dependence. The Fermi functions, \( F_0(Z, \epsilon) \), depend upon the charge of the daughter nucleus, \( Z \), and the energy of the \( i \)th electron, \( \epsilon_i \).

In the early \( \beta\beta(0\nu) \) calculations, the NME and kinematic factor were defined as above. The NME were given in units of \( \text{fm}^{-1} \) and the kinematic factors in units of \( \text{yr}^{-1} \text{fm}^2 \) (for example, [2]). In the mid-eighties, a scaling factor was introduced into the \( \beta\beta(0\nu) \) theory. The \( \nu \) potential, Eq. (7), is scaled by a factor of \( \chi_F \). The kinematic potential, Eq. (11), is scaled by a factor of \( R = r_0 A^{1/3} \) such that the NME are dimensionless:

\[
h_+ = \frac{R}{2} (H_2 + H_1) \quad (12)
\]

\[
\tilde{M}_{0\nu} = \tilde{M}^{(0\nu)} (1 - \chi_F) \quad (13)
\]

\[
\tilde{M}^{(0\nu)} = \sum_a \langle 0^+_f | \sum_n \tau^+_n || N_a \rangle \sum_m \tau^+_m || 0^+_i \rangle \sigma_n \cdot \sigma_m h_+ \quad (14)
\]

\[
\chi_F = \sum_a \langle 0^+_f | \sum_n \tau^+_n || N_a \rangle \sum_m \tau^+_m || 0^+_i \rangle h_+ \quad (15)
\]

\[
. \quad (16)
\]

The scaling factor, \( R \), in \( h_+ \) is compensated for by introducing \( 1/R^2 \) into the definition of the kinematic factor, Eq. (2):

\[
G_{01}^S(R) = \frac{a_{0\nu}}{(m_e R)^2 \ln 2} \int d\Omega_{0\nu} \ F_0(Z, \epsilon_1) \ F_0(Z, \epsilon_2) \quad (17)
\]

The \( G_{01}^S(R) \) have been calculated by many authors; for example, [1] using \( R = 1.2 A^{1/3} \) and [2] using \( R = 1.1 A^{1/3} \). Though the underlying physics of the kinematic factor is unchanged, the published values of \( G_{01}^S(R) \) are significantly different due to different choices of \( R \). Provided that the NME and kinematic factor are calculated using the same scaling factor, these differences are irrelevant. However, the \( R \) used in calculating the NME have not always been carried consistently to the kinematic factor.

In recent years, numerous calculations of NME have been performed using \( h_+ \). As techniques develop, it is customary to compare with previously published values. These comparisons are often made by defining \( C_{00} \):

\[
[T_{1/2}^{(0\nu)}]^{-1} = C_{00} \left( \frac{m_e}{m_e} \right)^2 \quad (18)
\]

\[
C_{00} = |\tilde{M}^{(0\nu)}|^2 G_{01}^S(R) \quad (19)
\]

If scaling is treated consistently, \( C_{00} \) is independent of the scaling factor. However, in citing previous calculations of the NME, the scaling factor has not always been accounted for properly. Often, \( G_{01}^S(R = 1.2 A^{1/3}) \) is used to calculate \( C_{00} \) regardless of the scaling factor used in determining the NME. If one combines NME calculated with \( R = 1.1 A^{1/3} \) fm with \( G_{01}^S(R = 1.2 A^{1/3}) \), \( C_{00} \) is suppressed by \((1.1/1.2)^2 \sim 20\%\). Correspondingly, this leads to an overestimation of \( \langle m_{\nu} \rangle \).

Table III includes several \( C_{00} \) predictions for \( ^{76}\text{Ge} \) \( \beta\beta(0\nu) \). This table was originally published in a recent review article [1] and adapted from [2]. The mismatch of scaling factors has occurred several times in the literature. We include revisions to Table 2 of [2] because it is one of the more thorough listings of NME calculations.

In Table III we include both the previously published \( C_{00} \) and the revised values obtained using the correctly scaled \( G_{01}^S(R) \). For clarity, the value of \( r_0 \) used in the original publication is included. In some instances \( r_0 \) was not clearly stated; the \( r_0 \) value was extracted from \( T_{1/2} \), \( \langle m_{\nu} \rangle \) or stated \( G_{01}^S(R) \) values given in the original paper assuming that the scaling was originally treated consistently.
Using the simple $S = 0$ electron wave functions with no $r$ dependence, we obtain $G_0^S (R = 1.24^{1/3}) = 6.46 \cdot 10^{-15}$ yr$^{-1}$ and $G_0^S (R = 1.14^{1/3}) = 7.78 \cdot 10^{-15}$ yr$^{-1}$. These values are within a few percent of those published previously by [1] and [3]. Using the appropriate kinematic factor, most $C_{00}$ determined using NME with $r_0 = 1.1$ are changed by $\approx 20\%$. Assuming a half life of $4 \cdot 10^{27}$ yr the spread of predicted $\langle m_\nu \rangle$ values is unchanged, 0.022-0.068. However, several predicted neutrino masses are reduced.

| $C_{00}$ x 10$^{-14}$ (yr$^{-1}$) | $r_0$ (fm) | Method | $\langle m_\nu \rangle$ (eV) | Reference |
|----------------------------------|------------|--------|-----------------------------|-----------|
| 11.2 x 10$^{-14}$               | 1.2        | QRPA   | 0.024                       | 1.1$^2$   |
| 6.97 x 10$^{-14}$               | NS         | QRPA   | 0.031                       | 8         |
| 7.51 x 10$^{-14}$               | NS         | Number-projected QRPA | 0.029     |
| 7.19 x 10$^{-14}$               | 1.1        | QRPA   | 0.030                       | 3         |
| 12.1 x 10$^{-14}$               | NS         | QRPA   | 0.023                       | 9         |
| 13.3 x 10$^{-14}$               | NS         | QRPA   | 0.022                       | 10        |
| 8.34 x 10$^{-14}$               | 1.2        | QRPA   | 0.028                       | 11        |
| 1.89 - 12.8 x 10$^{-14}$        | 1.2        | QRPA   | 0.059 - 0.023               | 12        |
| 5.02 - 5.93 x 10$^{-14}$        | 1.1        | QRPA   | 0.036 - 0.033               | 13        |
| 8.61 x 10$^{-14}$               | 1.2        | QRPA   | 0.028                       | 5         |
| 1.40 x 10$^{-14}$               | 1.1        | QRPA with np pairing       | 0.068     |
| 5.59 x 10$^{-14}$               | 1.1        | QRPA with forbidden        | 0.034     |
| 10.1 x 10$^{-14}$               | 1.1        | RQRPA | 0.025                       | 15        |
| 6.10 x 10$^{-14}$               | 1.1        | RQRPA with forbidden       | 0.033     |
| 6.77 - 7.72 x 10$^{-14}$        | 1.1        | RQRPA | 0.031 - 0.029               | 13        |
| 2.26 - 9.04 x 10$^{-14}$        | 1.2        | RQRPA | 0.054 - 0.027               | 12        |
| 4.48 x 10$^{-14}$               | 1.1        | RQRPA with forbidden       | 0.038     |
| 2.71 x 10$^{-14}$               | 1.1        | Full RQRPA                  | 0.049     |
| 3.72 - 8.75 x 10$^{-14}$        | 1.2        | Full RQRPA                  | 0.042 - 0.027 |
| 6.66 - 9.43 x 10$^{-14}$        | 1.2        | Second QRPA                 | 0.031 - 0.026 |
| 0.34 - 0.40 x 10$^{-14}$        | 1.2        | Self-consistent QRPA*       | 0.139 - 0.128 |
| 28.8 x 10$^{-14}$               | 1.2        | VAMPIR*                     | 0.015     |
| 15.6 x 10$^{-14}$               | NS         | Shell model truncation*     | 0.020     |
| 7.03 - 16.2 x 10$^{-14}$        | 1.2        | Shell model truncation*     | 0.030 - 0.020 |
| 1.94 x 10$^{-14}$               | 1.2        | Large-scale shell model     | 0.058     |

TABLE I: A comparison of the nuclear contributions to $T_{1/2}$, $C_{00}$, and the resulting $\langle m_\nu \rangle$ values assuming $T_{1/2} = 4.0 \cdot 10^{27}$ yr. Included are the values of $r_0$ used in each NME calculation. The $r_0$ has been explicitly stated in the original publication unless otherwise indicated as: NS (no scaling/C$_{00}$ published in original paper); $^1$ ($r_0$ inferred from published values of $\langle m_\nu \rangle$, $T_{1/2}$ or $G_{01}$) and $^2$ (Published $C_{00}$; $r_0$ inferred from published values of $\langle m_\nu \rangle$, $T_{1/2}$ or $G_{01}$).

It is important to point out $G_{01}$, defined without the scaling factor, does depend on $R$ through the Fermi functions of the electron wave functions. The appropriate choice of $R$ depends upon the nucleus being considered and should be chosen such that experimental values of the mean square radius $\langle r^2 \rangle$ are reproduced. For a uniform charge distribution, $R^2 = 5/3\langle r^2 \rangle$. In Table (II) we give the unscaled $G_{01}$ calculated using $R$ values fit to experimental root-mean-square radii when possible $^2$. Comparing $G_{01}$ for $^{150}$Nd, it is clear that the unscaled $G_{01}$ are not very sensitive to the choice of $R$. The significant differences obtained by $^1$ and $^2$ for $G_{01}(R)$ are predominately due to the scaling factor.

To avoid confusion in the future, we strongly encourage that further calculations of NME be published with either no scaling factor included, or a clear indication of what choice the authors have made for $r_0$. 
| Nucleus | $\langle r^2 \rangle^{1/2}$ | $r_0$ | $G_{01}^{0\nu}$ |
|---------|-----------------|-----|----------------|
| $^{48}$Ca | 3.470 | 1.23 | 1.236 $\times 10^{-12}$ |
| $^{76}$Ge | 3.451 | 1.23 | 1.236 $\times 10^{-12}$ |
| $^{82}$Se | 4.081 | 1.24 | 1.663 $\times 10^{-13}$ |
| $^{96}$Zr | no data | 1.20 | 7.748 $\times 10^{-13}$ |
| $^{100}$Mo | 4.360 | 1.24 | 1.792 $\times 10^{-12}$ |
| $^{114}$Cd | 4.430 | 1.23 | 1.436 $\times 10^{-12}$ |
| $^{126}$Te | no data | 1.20 | 6.609 $\times 10^{-14}$ |
| $^{130}$Te | no data | 1.20 | 1.661 $\times 10^{-12}$ |
| $^{136}$Xe | no data | 1.20 | 1.825 $\times 10^{-12}$ |
| $^{150}$Nd | 5.048 | 1.23 | 8.719 $\times 10^{-12}$ |
|           | 5.015 | 1.22 | 8.750 $\times 10^{-12}$ |
|           | 4.948 | 1.20 | 8.813 $\times 10^{-12}$ |

TABLE II: $0^+ \rightarrow 0^+ \beta\beta(0\nu)$ kinematic factor $G_{01}$ calculated at the specified values of $r_0$ obtained by fitting the experimental $\langle r^2 \rangle^{1/2}$ when available [20].

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