On quantum preimage attacks

Răzvan Roșie *

Abstract

We propose a preimage attack against cryptographic hash functions based on the speedup enabled by quantum computing. Preimage resistance is a fundamental property cryptographic hash functions must possess. The motivation behind this work relies in the lack of conventional attacks against newly introduced hash schemes such as the recently elected SHA-3 standard.

The proposed algorithm consists of two parts: a classical one running in \( O(\log_2 |S|) \), where \( S \) represents the searched space, and a quantum part that contains the bulk of the Deutsch-Jozsa circuit.

The mixed approach we follow makes use of the quantum parallelism concept to check the existence of an argument (preimage) for a given hash value (image) in the preestablished search space. For this purpose, we explain how a non-unitary measurement gate can be used to determine if \( S \) contains the target value. Our method is entirely theoretical and is based on the assumptions that a hash function can be implemented by a quantum computer and the key measurement gate we describe is physically realizable. Finally, we present how the algorithm finds a solution on \( S \). Keywords: preimage attacks, Keccak, SHA-3, quantum computing, Deutsch-Jozsa.

1 Introduction

In October 2012, a new hash algorithm was elected by NIST as the SHA-3 standard. Keccak, the winner of this selection process, is based on a “sponge” construction and is claimed to be able to provide a high level of security: the probability of success for any attack being asymptotic to that of a random oracle [7]. Since the growth of modern computer systems, the cryptographic hash functions became a constant need in activities involving authentication or pseudo-random number generation. The call for a single, omni-reliable and accepted algorithm that guarantees the modern security needs was therefore natural.

In this article, we investigate hash functions through the usage of quantum computers. Besides being elected as SHA-3, we use Keccak as a point of reference, for the fact that no reliable theoretical or practical pre-image or collision attacks have been proposed against the complete version of the algorithm. We give a brief overview of Keccak in Appendix A.

In section 2, we describe how to link the preimage attack with quantum computing, proposing an algorithm based on the Deutsch-Jozsa circuit. Finally, in section 3 we explain how a quantum preimage attack can be used against a hash function.

1.1 Motivation

Hash functions are deterministic cryptographic primitives, mapping arbitrarily long input bit-strings to fixed-sized bit-strings, called message digests or hash values. Essential properties that hash functions should exhibit are non-invertibility and close to uniform distribution of output values. There are three intrinsic properties that capture the nature of hash functions which we describe below:

1. **Preimage resistance**: Given a hash function \( h \) and a message digest \( y \), it should be computationally infeasible to find a message \( m \) such that \( h(m) = y \).

2. **Second preimage resistance**: Given a hash function \( h \) and an input message \( m_1 \), it should be computationally infeasible to find a different \( m_2 \) such that \( h(m_1) = h(m_2) \).

3. **Collision resistance**: It should be computationally infeasible to find two different messages \( m_1 \) and \( m_2 \) such that \( h(m_1) = h(m_2) \).

Asking if a hash scheme fulfills these requirements is, without doubts, a fascinating question. We are now able to state that our problem can be defined as a **preimage resistance** one. Given a message digest \( y \) and a hash function \( h \) fulfilling the above properties, finding an input \( x \) such that \( h(x) = y \), may be done either by searching the input space or by exploiting the construction details of \( h \). Related research [9] is directed to the collision finding problem, where the Birthday paradox gives an intuition of its complexity. If \( D \) represents the output set produced by \( h \), on average, we should inspect at least \( \sqrt{|D|} \) outputs before expecting the first collision. What we propose here, is to investigate a broad search space through the means offered by quantum computing, while tackling the problem of finding preimages.

2 Towards quantum preimage attacks

We start by formalizing the problem to solve: given a discrete search space \( S \) of size \( |S| \) consisting of the first
2.1 The Deutsch-Jozsa algorithm

The Deutsch-Jozsa algorithm is a fundamental construct in our proposal. In short, the Deutsch-Jozsa algorithm solves the following problem: given a boolean function \( f \), seen as an oracle, that is either balanced or constant, we can find the nature of \( f \) in constant time. The time improvement is exponential, compared with the classical deterministic solution. This is done by exploiting the property of superposition of quantum bits (qubits).

The algorithm we refer to as Deutsch-Jozsa, is the result of several successive improvements of the initial version proposed by David Deutsch. We assume a minimal familiarity with the basic concepts of quantum computer science (interested readers may want to consult [3] for a brief introduction) and make use of the standard notations, where \( \otimes \) stands for the tensor Kronecker product over Hilbert spaces and \( \oplus \) for addition over \( \mathbb{Z}_n \). Several building concepts that serve in our description of next sections are defined below.

Definition 2. Balanced functions

A function \( f : \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2 \) is called balanced if \( 2^{n-1} \) inputs map to 0 and \( 2^{n-1} \) inputs map to 1.

Deutsch’s algorithm restricts its view to the trivial case of a balanced function where \( n = 1 \), and is a special case of the more general Deutsch-Jozsa algorithm. We focus on the more general problem, with the intention to use a derived version of it while inspecting hash functions.

Definition 3. A “binary” inner product

The binary inner product \( \langle \mathbf{x}, \mathbf{y} \rangle \) is defined as a function from \( \mathbb{Z}_2^n \times \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2 \) in the following way: \( \langle \mathbf{x}, \mathbf{y} \rangle = (x_0 \wedge y_0) \oplus (x_0 \wedge y_1) \oplus \ldots \oplus (x_{n-1} \wedge y_{n-1}) \).

\[ \frac{i\hbar}{dt} |\psi(t)\rangle = H |\psi(t)\rangle \]

that governs the dynamics in an isolated quantum system, enforces that all quantum operations used are unitary, hence reversible and norm-preserving. On the other hand, few classical boolean gates are reversible and can be represented as unitary operators. One example is the logical \textit{NOT}. To represent the NAND (which together with the \textit{NOT} are universal gates for classical circuits) as a unitary operator, we should observe that in classical boolean circuits, some physical information is not used (i.e. the output has half the dimension of the input). This suggest a correlation between logical and physical reversibility. If we take into consideration the unused, lost data, we can represent the NAND as an unitary operator. In practice, the realization of quantum hash circuits based on boolean functions may be a difficult task, given the obfuscated structure of most hash functions. However, Keccak presents a clear design having only 5 inner block permutations applied on the state of the algorithm, in 24 successive rounds. A lightweight boolean implementation of this algorithm [8] requires a relatively small number of gates, and this is a promising result in the quantum world (a small number of quantum gates can make such a circuit realizable).
The usefulness of the binary inner product is seen when introducing an $n$-dimensional Hadamard operator. The basic 1-qubit Hadamard gate is represented by $H = \frac{1}{\sqrt{2}} (|1\rangle - |0\rangle)$. Apart from its simplicity, this operator is required to put the qubit in superposition: $\frac{1}{\sqrt{2}} (|1\rangle + |0\rangle) = \frac{1}{\sqrt{2}} (|a\rangle + |\beta\rangle)$.

The general Hadamard gate can be obtained from the tensor product of several unidimensional Hadamard gates. It can be shown that it has the following form:

**Definition 4.** The $n$-dimensional Hadamard operator: $H_{ij} = \frac{1}{\sqrt{2}} (-1)^{i-j}$, where $[i,j]$ stands for the binary inner product.

### 2.2 Overview of Deutsch-Jozsa circuit

Below, we present the content of the state after each phase of the Deutsch-Jozsa algorithm.

![Figure 2: The Deutsch-Jozsa circuit](image)

The first state of the algorithm, $|\phi_0\rangle$ consists of the initial $n+1$ input qubits that are required by the algorithm. As seen in Figure 2, the leading $n$ qubits are represented by the tensor product of $n$ $|0\rangle$ qubits. The tensor product is the “binding” operator. We write $|0^n\rangle = |0\rangle \otimes \ldots \otimes |0\rangle$, using Dirac’s notation.

$|\phi_0\rangle = |0^n\rangle \otimes |1\rangle$

The result of applying the Hadamard operators on the two groups of qubits, is revealed in $|\phi_1\rangle$ - the qubits are in superposition.

$|\phi_1\rangle = \frac{1}{\sqrt{2^n}} \left( \sum_{x \in \{0,1\}^n} |x\rangle \right) \otimes \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$

The central part of the algorithm makes use of quantum parallelism enabled by qubits’ superposition, to evaluate the function $f$, represented as the oracle $U_f$. $|\phi_2\rangle$ stands for the output of $U_f$ (Figure 3). Converting $\sum_{x \in \{0,1\}^n} |x\rangle$ into $\sum_{x \in \{0,1\}^n} (-1)^{f(x)}$ through the means of $U_f$, can be justified by the output produced by $U_f$ given that $|x\rangle(|f(x)\rangle - |1\rangle \oplus f(x)\rangle) = (-1)^{f(x)} |x\rangle (|0\rangle - |1\rangle)$.

$|\phi_2\rangle = \frac{1}{\sqrt{2^n}} \left( \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle \right) \otimes \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$

![Figure 3: A representation of a boolean function $f$ as a quantum oracle.](image)

The last part applies an $n$-qubit Hadamard gate to the leading $n$ qubits, and is used to make possible bringing the qubits in one of the classical states, $|0\rangle$ or $|1\rangle$, therefore allowing quantum devices to measure them as ordinary bits. The last qubit is no longer used and can be ignored when writing the last state.

$|\phi_3\rangle = \frac{1}{2^n} \left( \sum_{x \in \{0,1\}^n} (-1)^{f(x)} \sum_{y \in \{0,1\}^n} (-1)^{|x\rangle y\rangle} \right)$

$|\phi_4\rangle = \frac{1}{2^n} \left( \sum_{x \in \{0,1\}^n} \sum_{y \in \{0,1\}^n} (-1)^{f(x)\otimes|x\rangle y\rangle} \right)$

Finally, when the last (and measurable) state of the circuit is inspected, it can be seen that a $|0^n\rangle$ is measured with the following probability:

$Pr(|0^n\rangle) = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |2$

If the function $f$ is constant, the measurement of the last state will produce $+1|0^n\rangle$, or $-1|0^n\rangle$, depending on $f$ being constant to 0 or 1. Similarly, if $f$ is balanced, when measuring $|\phi_3\rangle$, the result will collapse to $0|0^n\rangle$. To conclude, the Deutsch-Jozsa circuit allows for a clear and deterministic distinction between balanced and constant functions.

### 2.3 Deutsch-Jozsa applied in preimage existence checking

As stated previously, we intend to apply a modified version of the Deutsch-Jozsa algorithm on a search space $S$ using an oracle obtained from a cryptographic hash function $h$ and a message digest $t$ generated by $h$, with the declared purpose to identify if the searched space $S$ contains a preimage $x$ of the target $t$.

#### 2.3.1 From hash to target functions

The first technical problem concerns the construction of the Deutsch-Jozsa circuit, that handles multiple boolean inputs to single boolean output functions. To enable preimage existence checks, we make use of the target function $T$ (definition 1), rather than $h$ itself. Notice that $T$ was constructed to serve our purpose, vanishing for a preimage value $x$. Computing the function $T_i$ given $(h, t)$ can be easily done by applying OR gates in a Wimbledon-knockout tournament style to the bits resulting from the XOR between $t$ and the output of $h$, as depicted in Figure 4.

![Figure 4: A representation of a boolean function $f$ as a quantum oracle.](image)
2.3 Deutsch-Jozsa applied in preimage existence checking

2.3.2 Adapting Deutsch-Jozsa circuit to check for existence of preimages

The second problem that needs to be solved regards the applicability of the Deutsch-Jozsa algorithm given the design of modern hash functions. In short, given a decent hash function, we expect a close to zero ratio between the number of roots $x$ of the target function $T_I$ and the cardinality of input space $\mathbb{S}$. Based on this observation, we re-examine the states of the Deutsch-Jozsa, particularly $|\phi_2\rangle$ and $|\phi_3\rangle$ (we are only interested in the states after the application of $U_{T_I}$). Consider $|\phi_2\rangle$:

$$|\phi_2\rangle = \frac{1}{\sqrt{2^n}} \left( \sum_{x \in \{0,1\}^n} (-1)^{T_I(x)} |x\rangle \right) \otimes \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$$

Given the fact that $T_I$ may or may not have roots on $\mathbb{S}$, after the unitary transform $U_{T_I}$ is applied, the state $|\phi_2\rangle$ can end up in one of the following situations (we use the explicit ket vector representation of the first $n$ qubits):

- $|\phi_2\rangle = \frac{1}{\sqrt{2^n}} [-1, -1, \ldots, -1]^T$, if $\forall x \in \mathbb{S}$, $T_I(x) \neq 0$
- $|\phi_2\rangle = \frac{1}{\sqrt{2^n}} [-1, -1, \ldots, 1, \ldots, -1]^T$, if $\exists x \in \mathbb{S}$, s.t. $T_I(x) = 0$

An expected small number of $x$ for which $T_I(x) = 0$ implies that $|\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)}|^2$ is close to $1$ for $n$ large. When the last Hadamard transform is applied ($|\phi_3\rangle$ is obtained) and the measurement is performed, the probability to detect if $T_I$ has roots on the input space is close to $0$. To overcome this obstacle, a feasible way to check for existence of preimages in $\mathbb{S}$ is needed. In quantum world, this translates in finding an algorithm able to distinguish between the two possible forms of $|\phi_2\rangle$ presented above. One approach that might be immediately spotted: this problem may be transformed into a searching one and solved by Grover’s circuit in $2^{n/2}$ iterations. An alternative solution is proposed below.

2.3.3 Decision by failure

We continue the theoretical journey with the most important part, that introduces a simple way to distinguish between two state $|\psi\rangle = \frac{1}{\sqrt{1/2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$ and $|\phi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle$ where one (more) entries in the second state are positive (only for this problem we assume the entries in the kets representing $|\psi\rangle$, $|\phi\rangle$ belong to $\mathbb{R}$). We refer to these two states as being two states equivalent up to a relative phase. Consider the following non-unitary but still Hermitian operator:

$$O = pA_{2^n} - (p - 1)I_{2^n}$$

where $I_{2^n}$ stands for the identity matrix, $A_{2^n}$ is the average matrix with $a_{i,j} = \frac{1}{2^n}$ and $p \in \mathbb{R}$.

**Proposition 1.** The non-unitary, self-adjoint operator $O$ preserves the entries of a normalized and constant state represented by $|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$.

**Proof:** Let $|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$. Hence:

$$|O\psi\rangle = (pA_{2^n} - (p - 1)I_{2^n})|\psi\rangle \Leftrightarrow$$

$$|O\psi\rangle = pA_{2^n}|\psi\rangle - (p - 1)I_{2^n}|\psi\rangle \Leftrightarrow$$

$$|O\psi\rangle = p(|\psi\rangle - (p - 1)|\psi\rangle) = |\psi\rangle$$

Due to the fact that $O$ is non-unitary, it cannot be used as part of the circuit, but only as a measurement operator. The need for unitary operators in quantum circuits is a consequence of the postulates of quantum mechanics, that enforce the norm-preservation of quantum states throughout the system. We show how to use the newly introduced operator as a measurement one.

Consider the average result of measuring that a quantum system is in state $|\psi\rangle = \frac{1}{\sqrt{2^n}} [-1, -1, \ldots, -1]^T$ (constant and real entries), using the measurement observable $O$: $\langle O \rangle = \langle \psi | O | \psi \rangle = |\psi\rangle = 1$. The expectation (average of measured values) is $1$ since $O$ is constructed to preserve the entries in a ket with constant elements such as $|\psi\rangle$. On the other hand, $O$ does not preserve the norm and entries in $|\phi\rangle$, (distinct and real entries).

**Proposition 2.** The expectation value $\langle O \rangle_\phi$ of a non-unitary, self-adjoint operator $O$ applied on a state $|\phi\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} \pm |x\rangle$ is in general different by $1$.

The proof is given in Appendix B. A stronger effect may be obtained if we use $O' = O^q$, for some positive integer $q$. As expected, $O'$ preserves the entries of a normalized constant ket vector $|\psi\rangle$. The proof is immediate: $\langle O' \psi \rangle = O_{n-1} \cdots O_2 O_1 |\psi\rangle = O_{n-1} \cdots O_2 O_1 |\psi\rangle = \cdots = O|\psi\rangle = |\psi\rangle$. In the general case, the action of a dispersion operator such as $O$ will be in amplifying the absolute difference between positive and negative entries, while preserving kets with constant entries unchanged. Hence, for a non-constant ket, the mean of observed values will be different by $1$ and dependent on parameter $p$. Several measurements need to be performed to ensure that enough data is available for calculating the mean.

If the operator $O$ can be physically implemented for $p$ close to $2^n$ (as suggested in Appendix B), then there is a way of deciding if the input space $\mathbb{S}$ contains or not at least a root of the target function $T_I$ (i.e preimage of $h$):

- $\langle O \rangle = 1$, if $\exists x \in \mathbb{S}$ s.t. $T_I(x) = 0$
- $\langle O \rangle \neq 1$, if $\forall x \in \mathbb{S}$ s.t. $T_I(x) = 0$

![Figure 5: The adapted version of the Deutsch-Jozsa circuit, using $O$ as a measurement observable.](image)

![Figure 4: Computing $T_I$ from $h$ using a tree of OR gates. Notice the roots of $T_I$ are preimages for $h(t)$.](image)
3 Quantum preimage attacks

In this section, we formalize a quantum preimage attack algorithm. We start by assuming the existence of a hash function $h$, producing fixed sized outputs of binary length $N$. We can handle hash algorithms that produce classes of outputs by defining a family of hash functions for each configuration of the algorithm. In the case of Keccak, we can define $h_{224}, h_{256}, h_{384}$ and $h_{512}$, for the available configurations of the algorithm described in Appendix A. For an accurate description of the method, we assume the existence of a subset $S$ of the infinite input space, such that $S = \{0, 1, \ldots, 2^n - 1\}$ (in general $n \neq N$), and a target function $T$ as the one defined in definition 1. The outcome of this section is an algorithm able to identify a preimage value $x$ (given that such a value exists) of the input space of $h$, fulfilling the constraint: $h(x) = t$ or equivalently $T_i(x) = 0$.

An algorithm for computing preimages of hash functions (high level overview)

- Given:
  - $h : \mathbb{Z}_2^N \to \mathbb{Z}_2^N$ - a hash function producing message digests of binary size $N$
  - an input set $S = \{0, 1, 2, \ldots, 2^n - 1\}$, $n > 1$
  - a positive target integer $t$
  - a quantum computer with $n + 1$ available qubits allowed to query an oracle implementing $T_i$
  - a result array $r$.

- Step 1: Select one $x \in S$. If $T_i(x)$ is 0, output $x$ and halt. Otherwise, run the adapted Deutsch-Jozsa algorithm with $O = pA_2^n - (p - 1)I_2^n$ and $p = 2^n$. If the expectation is (close to) 1, report that no preimage $x$ exists in $S$. Halt.

- Step 2: If $|S| > 1$, go to Step 3. Otherwise, go to Step 6.

- Step 3: Split $S$ in two equally sized sets $S_0$ and $S_1$. Select $x_0 \in S_0$ and $x_1 \in S_1$. If $T_i(x_0)$ or $T_i(x_1)$ are 0, print $(x_0, x_1)$ and halt. Otherwise, go to Step 4.

- Step 4: Run the quantum adapted Deutsch-Jozsa algorithm on $S_0$. If $S_0$ contains a preimage, the expectation will be different by 1, add 0 to the result array $r$, update $S = S_0$ and go to Step 2.

- Step 5: Run the quantum adapted Deutsch-Jozsa algorithm on $S_1$. If $S_1$ contains a preimage, the expectation will be different by 1, add 1 to the result array $r$, update $S = S_1$ and go to Step 2.

- Step 6: Construct the preimage $x$ by concatenating the bits in the result array $r$. Halt.

As regards the complexity of the algorithm, it is logarithmic in the size of the input space $S$. The correctness of the classical algorithm relies on the correctness of the binary search problem. We do not need to check for middle values, ($|S|$ is power of 2). The sanity checks - values from each subset are tested for being preimages - are used to handle the extreme cases when all elements in $S$ are preimages of $t$ (quantum expectation will be 1, same as if $S$ would have no preimage). Good cryptographic hash functions exhibit a close to uniform distribution of bits [4]. Given the Law of large numbers, we expect that the iteration of the quantum part of the algorithm to be repeated for a constant number of times. This is not a great computational obstruction, and its purpose is to approximate the expectation with a good accuracy.

4 Conclusion

The main contribution of this work resides in proposing an algorithm for finding preimages given a value produced by a known hash function. The need for such a proposal relies in the lack of practical results through the means of classical cryptanalysis techniques against the recently introduced hash functions, such as Keccak. In particular, we make use of the binary search algorithm and extend it in a novel fashion, by linking it with the highly theoretical area of quantum computing.

A slight variation of the Deutsch-Jozsa algorithm is used as the core quantum part in our process of finding preimages in the input set.

The approaches described by this work are entirely theoretical and obviously limited by the practical feasibility of the suggested measurement observable. From this point of view, the applicability of our proposal is highly constrained. However, the problem discussed here remains open, and eventualty, a reliable, universally accepted quantum computer, will be used in testing it.

References

[1] D. Deutsch, Quantum theory, the Church-Turing principle and the universal quantum computer, Proceedings of the Royal Society of London, A 400, 1985

[2] D. Deutsch, R. Jozsa, Rapid solution of problems by quantum computation, Proceedings of the Royal Society of London, A 439: 553, 1992

[3] N. David Mermin, Quantum Computer Science - An Introduction, Cambridge University Press, 2007

[4] B. Preneel, Analysis and Design of Cryptographic Hash Functions, 2003

[5] H. Wiklicky, Quantum Computing Lecture Notes, Imperial College, London, 2014

[6] G. Bertoni, J. Daemen, M. Peeters and G. Van Assche, The Keccak reference, 2011

[7] G. Bertoni, J. Daemen, M.1 Peeters, and G. Van Assche The Keccak SHA-3 submission, 2011

[8] E. B. Kavun , T. Yalcin, A Lightweight Implementation of Keccak Hash Function for Radio-Frequency Identification Applications, 2010
A.3 Keccak-f

Keccak-f encapsulates the real hashing algorithm. The number of rounds increases by the size of \( w \) by the following rule: \( 12 + 2 \cdot l, \) with \( l = \log_{2}(w) \). The submitted version [6] has 24. A round is a composition of five permutations (described below) over the state and can be mathematically written as: \( R = \theta \circ \rho \circ \pi \circ \chi \circ \iota. \)

A.3.1 Theta

\( \theta \) is the first permutation that is applied on the sponge. It is aimed to diffusion, and without \( \theta \), Keccak will not provide any data diffusion at all. In short, the value of each bit in the state matrix is obtained as an XOR between the sums (over \( GF(2) \)) of two other values of each bit in the state matrix. \( \theta \) is mathematically written as:

\[
\sum_{y'=0}^{4} a[x][y'][z] + \sum_{y'=0}^{4} a[x-1][y'][z] + \sum_{y'=0}^{4} a[x][y'][z-1].
\]

A.3.2 Rho

Rho performs a rotation over the lanes in the state using a predefined constant. For each bit in the state, at a given round \( n \), we have: \( a[x][y][z] = a[x][y][z + T(n)] \).
A.3.3 Pi

II is a transform applied over the lanes. Practically, the lanes are reordered in the following way: \( a[x][y][z] = a[x'][y'][z] \) where \( x = y' \) and \( y = 2x' + 3y' \).

A.3.4 Chi

\( \chi \) is the only non-linear function. Without it, Keccak-\( f \) will be a linear map over the GF(2). \( \chi \) operates on each row in the state, so it can be viewed as an S-box applied over each row in the state. Mathematically, \( \chi \) can be described as: \( a[x][y][z] = a[x][y][z] \) and \( a[x][y][z] = 2x' + 3y' \).

A.3.5 Iota

The main purpose of \( \iota \) is to disrupt the symmetry of Keccak It consists of the addition of round constants. Without it, the round function would be translation-invariant in the \( z \) axis and all rounds would be equal making Keccak-\( f \) subject to attacks exploiting symmetry such as slide attacks.

A.4 Squeezing phase

The last phase of Keccak, is the squeezing phase; at this stage, the output is generated: the entries in several lanes are taken from the state and concatenated as the output string. If the length of the message digest is not sufficient, a new round of iterations are applied over the state, and the new message digest is appended to the previous.

B Proof of Proposition 2

Let \( |\phi\rangle \) denote a quantum state with \( u \) entries mapping to \( \frac{1}{\sqrt{2^n}} \) and \( v \) entries mapping to \( \frac{1}{\sqrt{2^n}} \) such that \( u + v = 2^n \). Let \( O = pA_2^n - (p-1)I_2^n \), with \( A_2^n \) and \( I_2^n \) the average and identity operators. Let \( |r\rangle = |O\phi\rangle \). We write \( r_i \) for the \( i \)-th entry in \( |r\rangle \) and compute it as follows.

\[
\begin{align*}
    r_i &= p \cdot \sum_j a_{i,j} \cdot \phi_j - (p-1) \cdot \phi_i \\
    r_i &= p \cdot \frac{1}{2^n} \sum_j \phi_j - (p-1) \cdot \phi_i \\
    r_i &= p \cdot \frac{1}{2^n} \left( \frac{1}{\sqrt{2^n}} - \frac{1}{\sqrt{2^n}} \right) - (p-1) \cdot \phi_i \\
    r_i &= p \cdot \frac{1}{2^n} \left( \frac{2^n - 2u}{2^n} \right) - (p-1) \cdot \phi_i \\
    r_i &= p \cdot \frac{1}{2^n} - p \cdot \frac{2u}{2^n \cdot \sqrt{2^n}} - (p-1) \cdot \phi_i \\
    \text{Case 1: } \phi_i &= \frac{1}{\sqrt{2^n}} \\
    r_i &= p \cdot \frac{1}{\sqrt{2^n}} - p \cdot \frac{2u}{2^n \cdot \sqrt{2^n}} - (p-1) \cdot \frac{1}{\sqrt{2^n}} \\
    r_i &= \frac{1}{\sqrt{2^n}} - p \cdot \frac{2u}{2^n \cdot \sqrt{2^n}} \\
    \text{Case 2: } \phi_i &= -\frac{1}{\sqrt{2^n}} \\
    r_i &= p \cdot \frac{1}{\sqrt{2^n}} - p \cdot \frac{2u}{2^n \cdot \sqrt{2^n}} + (p-1) \cdot \frac{1}{\sqrt{2^n}} \\
    r_i &= (2p-1) \cdot \frac{1}{\sqrt{2^n}} - p \cdot \frac{2u}{2^n \cdot \sqrt{2^n}} \\
\end{align*}
\]

We now compute the expectation (the mean of measured results) \( \langle O \rangle_\phi = \langle O | \phi \rangle = \langle \phi | r \rangle \) as follows:

\[
\begin{align*}
    \langle \phi | r \rangle &= u - \frac{1}{\sqrt{2^n}} \left[ \frac{2p - 1}{2^n} - \frac{2pu}{2^{n+1}} \right] + v - \frac{1}{\sqrt{2^n}} \left[ \frac{1}{2^n} - \frac{2pu}{2^n} \right] \\
    \langle \phi | r \rangle &= \frac{u \cdot (1 - 2p)}{2^n} + \frac{2pu^2}{2^{n+1}} + \frac{u \cdot (1 - 2p)}{2^n} - \frac{2pu}{2^n} \\
    \langle \phi | r \rangle &= u - 2pu + v + \frac{2pu(u - v)}{2^n} \\
    \langle \phi | r \rangle &= 1 + \frac{2pu(2u - 2^n)}{2^n} \\
    \langle \phi | r \rangle &= 1 + 4p \cdot \frac{u(u - 2^n)}{2^n} \\
\end{align*}
\]

For a hash function, we assume \( u \neq 0 \) (otherwise the problem is trivial). If we select \( p = 2^n \) (notice that \( p = 2^n \) will make \( pA \) the unit matrix), we get the expectation modeled by:

\[
\langle \phi | r \rangle = \begin{cases} 
    1 + 4 \cdot \frac{u(u - 2^n)}{2^n} & \text{if } u \not\in \{0, 2^n\} \\
    1, & \text{otherwise}
\end{cases}
\]

In the first case, the expectation is different by 1, and this result allows to distinguish between two states that are equivalent up to a relative phase.