The Josephson current through a long quantum wire

Domenico Giuliano$^{1,2}$ and Ian Affleck$^3$

$^1$ Dipartimento di Fisica, Università della Calabria Arcavacata di Rende I-87036, Cosenza, Italy
$^2$ INFN, Gruppo collegato di Cosenza, Arcavacata di Rende I-87036, Cosenza, Italy
$^3$ Department of Physics and Astronomy, University of British Columbia, Vancouver, BC, V6T 1Z1, Canada
E-mail: domenico.giuliano@fis.unical.it and iaffleck@phas.ubc.edu

Received 15 November 2012
Accepted 16 January 2013
Published 18 February 2013

Abstract. The dc Josephson current through a long SNS junction receives contributions from both Andreev bound states localized in the normal region and scattering states incoming from the superconducting leads. We show that in the limit of a long junction, this current, at low temperatures, can be expressed entirely in terms of properties of the Andreev bound states at the Fermi energy: the normal and Andreev reflection amplitudes at the left- and right-hand S–N interfaces. This has important implications for treating interactions in such systems.

Keywords: quantum wires (theory), quantum transport in one dimension
1. Introduction

As was shown by Josephson [1], a current can pass between two superconductors separated by a normal material, even with zero potential difference. At temperature $T$, this Josephson current is determined by the difference of the phase of the order parameter in the two superconductors, $\chi$,

$$I[\chi; T] = 2e \frac{dF}{d\chi},$$

(1)

where $F$ is the free energy [2]. Using the Bardeen–Cooper–Schrieffer (BCS) approximation in the superconducting leads and ignoring interactions in the normal region, $I(\chi)$ can be expressed as a sum over single quasi-particle energy levels, $E_n$,

$$I[\chi; T] = 2e \sum_n f(E_n) \frac{dE_n}{d\chi},$$

(2)

where $f(E) = 1/[e^{E/T} + 1]$ is the Fermi function and we measure energies from the chemical potential. In general, these states are of at least two distinct forms. There are Andreev bound states (ABSs) [3], with energies $|E| < \Delta$, where $\Delta$ is the superconducting gap, which are localized in the normal region and whose wavefunctions decay exponentially into the superconducting leads. There are also scattering states (SSs), with energies $|E| > \Delta$, corresponding to waves coming in from infinity in the superconducting leads and being reflected and transmitted. If the bottom of the band in the normal material is lower
than the bottom of the band in the superconducting leads, there are, in addition, normal bound states, localized in the normal region, which also decay exponentially in the leads. As remarked in [4] and later on discussed in detail in [5] (where, in the limit in which normal reflection processes at the S–N interfaces can be neglected, SSs are correctly taken into account, fixing the result of [4]), in general, all types of states contribute to the Josephson current. In addition, it is worth stressing that summing over all the states to get equation (2) may be quite a difficult task to achieve, since it turns out that the net current contains very small differences between very large terms [6].

A convenient way to compute equation (2) is to express the total current, summed over all types of states, as a contour integral in the complex energy plane, involving the $S$-matrix. In particular, using an adapted version of the formalism developed in [7, 8], we will show that, making a minimal set of reasonable assumptions about the properties of the $S$-matrix, on appropriately deforming the integration path, one may write the dc Josephson current as a sum over Matsubara frequencies which, when $T \to 0$, turns into an integral over the imaginary axis. This allows for getting rid of the wild oscillations in the integrand function arising at real values of the energy, thus paving the way to a systematic analysis of the long junction limit.

The limit of a long narrow normal region was considered in [9], using a nearest neighbor tight-binding model and initially ignoring interactions. In particular, it was assumed that the length of the normal region, $\ell$, was much greater than the coherence length, or equivalently than the finite size gap $\pi v_F/\ell \ll \Delta$, where $v_F$ is the Fermi velocity in the normal region. Furthermore, only $T = 0$ was considered. In this limit it appears natural to integrate out the gapped superconductors and derive an effective Hamiltonian for the normal region, with local pairing interactions induced by the proximity effect at its boundaries. Such an effective Hamiltonian was used to derive the Josephson current. In this approach, only ABSs are considered. Due to a remarkable cancellation between pairs of ABSs it was found that the current, to order $1/\ell$, could be expressed in terms of scattering amplitudes at the Fermi level only.

This approach was called into question by the results of [10]. There it was verified that the ABSs gave the entire Josephson current for long junctions in the unrealistic limit $\Delta > 2J$, where $4J$ is the bandwidth in the normal region. However, numerical results for intermediate length junctions seemed to suggest a significant contribution from SSs for $\Delta/(2J) < 1$.

In this paper we study general models of long non-interacting SNS junctions without integrating out the superconducting leads. We prove that, for $v_F/\ell$ and $T \ll \Delta$, the Josephson current can indeed be expressed in terms of data at the Fermi level only. We emphasize that states far from the Fermi energy make large contributions to the current; it is just that these nearly cancel for large $\ell$, leading to our main formula for the dc Josephson current,

$$I^{(0)}[\chi] = -\frac{4e v_F}{\pi \ell} \partial_\chi \vartheta^2(\chi), \quad (3)$$

with $\vartheta(\chi)$ being a real function of $\chi$, defined by

$$\vartheta(\chi) = \arccos\{\text{Re} \left[ \bar{N}_R^p N_L^p e^{2i\pi \varphi} + \bar{A}_R^p A_L^h \right] \}. \quad (4)$$

$\vartheta(\chi)$ is a real function of $\chi$, defined by

$$\vartheta(\chi) = \arccos\{\text{Re} \left[ \bar{N}_R^p N_L^p e^{2i\pi \varphi} + \bar{A}_R^p A_L^h \right] \}. \quad (4)$$
In equation (3), $\bar{N}_{R/L}^{p/h}, \bar{A}_{R/L}^{p/h}$ are respectively the normal and the Andreev single-particle/hole reflection amplitudes at the right/left-hand S–N interfaces evaluated at the Fermi level only, while $\alpha_F$ is the single-particle Fermi momentum in the central region. (In the following, we will denote by $\alpha_{p/h}$ the single-particle/hole momentum within the central region, respectively.) In particular, we apply our general result in equation (3) to the ‘Blonder–Tinkham–Klapwijk (BTK) model’ [11], obtaining an explicit formula for the current for $v_F/\ell \ll \Delta$. Then we show that our approach may readily be extended to tight-binding models, such as the one discussed in [9], whose results for the current we recover when $T = 0$ and $v_F/\ell \ll \Delta$. We also extend our contour methods to finite $T$, by expressing the resulting current in terms of a sum along the imaginary energy axis at the Matsubara frequencies, $E = i\omega_n \equiv i2\pi(n + 1/2)T$. As a result, we find that the current vanishes exponentially when $T \gg v_F/\ell$.

This finding is important because integrating out the superconductors provides a powerful method for including interaction effects in the normal region, based on boundary conformal field theory techniques [9]. (See also [12].) While [9, 12] only considered the dc Josephson current, the techniques introduced there can be extended [13] to the ac case by allowing for the phase of the boundary pairing interactions to evolve linearly in time, $\chi = eWt$, where $W$ is the voltage difference. A possible experimental realization of such a long SNS junction might be provided by a carbon nanotube between bulk superconductors. Using $v_F \approx 8.1 \times 10^5$ m s$^{-1}$, $\pi v_F/\ell \approx 0.5$ meV for $\ell = 3 \mu m$. Thus, obtaining sufficiently long clean nanotubes coupled to sufficiently high $T_c$ superconductors to satisfy $\pi v_F/\ell \ll \Delta$ may be near the limits of current nanotechnology.

The paper is organized as follows.

- In section 2, we employ a convenient version of the $S$-matrix approach to derive the general formula for the dc Josephson current across an SNS junction.
- In section 3, we apply the general formula to the specific case of a long SNS junction. We recast the final result in a systematic expansion in powers of $\ell^{-1}$ and, finally, derive equation (3) for the dc Josephson current.
- In section 4, we use equation (3) to compute the dc Josephson current in the continuum BTK model [11] and in the lattice tight-binding model for the SNS junction [9].
- In section 5, we discuss the generalization of our results to a finite temperature $T$.
- Section 6 contains conclusions.
- In the appendices, we provide mathematical details of our derivation.

2. The general formula for the dc Josephson current

To derive a general formula for the dc Josephson current across the SNS junction, we have to carefully sum over contributions from both ABSs and SSs [4, 5]. An effective way of performing the sum over both sets of states is provided by the $S$-matrix approach, which we extensively discuss in the following. The $S$-matrix approach has been shown to be quite useful in studying superconducting point contacts, as it allows for expressing the sum of the contributions from any set of states by means of just one formula [7, 8]. In general, getting a closed-form formula for the integral expressions one obtains in this way is quite hard, even in the simple case of a superconducting quantum point contact (‘short junction limit’) [7, 8]. On the other hand, in the following we show that the formulas
for the dc Josephson current greatly simplify in the complementary, long junction, limit. As remarked in [6], in this limit a huge complication arises from the fact that the net current is a ‘small quantity’ that arises from mutual cancellations of large, oscillating contributions. In fact, the large oscillations in the function giving the contributions to the dc Josephson current from states at a given energy make it extremely difficult to resort to a numerical calculation of the total current, even for very short junctions. In order to overcome such a problem, at the end of this section we will show how, using general mathematical properties of the $S$-matrix, it is possible to deform the integration path, so as to write the dc Josephson current as just one integral computed over the imaginary axis. This approach was originally introduced in the framework of a Green’s function approach by Ishii [5], who used it to show how, on carefully carrying out the sum over scattering states, a previous result obtained by Kulik [4] should be corrected, thus eventually getting a sawtooth-like dc Josephson current in the case in which there are no normal backscattering processes at the $S$–$N$ interfaces. Here, we employ an adapted version of the $S$-matrix approach, discussed in [7, 8] for a superconducting quantum point contact, which allows us to explicitly compute $I[x; T]$ for a generic long SNS matrix and to show that it depends on data at the Fermi level only. When $T \to 0$ and the single-particle backscattering at the Fermi level is purely Andreev-like at both $S$–$N$ interfaces, we recover Ishii’s sawtooth-like dc Josephson current. In particular, we consider a general SNS model in the non-interacting, BCS approximation with a gap function $\Delta(x)$ of magnitude $\Delta$ at $|x| \to \infty$ and 0 in the central region, $0 < x < \ell$. We also include a normal potential, $V(x)$, which vanishes at $|x| \to \infty$. A $4 \times 4$ transmission matrix, $M$, may be defined which relates the asymptotic wavefunction in the $S$ regions at $x \to \pm \infty$, $\tilde{A}^+ = M \tilde{A}^-$ with a Bogoliubov–DeGennes (BDG) wavefunction obeying, respectively,

$$
\begin{align*}
[u(x)] & \to \begin{bmatrix} \cos(\Psi/2) \\ -e^{-ix/2} \sin(\Psi/2) \end{bmatrix} [A_1^- e^{i\beta_p x} + A_2^- e^{-i\beta_h x}] \\
& \quad + \begin{bmatrix} -e^{ix/2} \sin(\Psi/2) \\ \cos(\Psi/2) \end{bmatrix} [A_3^- e^{-i\beta_h x} + A_4^- e^{i\beta_h x}],
\end{align*}
$$

(5)

for $x \to -\infty$, and

$$
\begin{align*}
[u(x)] & \to \begin{bmatrix} \cos(\Psi/2) \\ -e^{ix/2} \sin(\Psi/2) \end{bmatrix} [A_1^+ e^{i\beta_p (x-\ell)} + A_2^+ e^{-i\beta_h (x-\ell)}] \\
& \quad + \begin{bmatrix} -e^{-ix/2} \sin(\Psi/2) \\ \cos(\Psi/2) \end{bmatrix} [A_3^+ e^{-i\beta_h (x-\ell)} + A_4^+ e^{i\beta_h (x-\ell)}],
\end{align*}
$$

(6)

for $x \to +\infty$. The particle and hole momenta, for energy $E$, are $\beta^2_{p/h} = 2m_S \{\mu \pm (E^2 - \Delta^2)^{1/2}\}$, with $m_S$ the electron effective mass in the $S$ regions, $\mu$ the chemical potential, and $\Psi \equiv -\arcsin(\Delta/E)$. (For simplicity, we take an energy-independent gap, $\Delta$, but our results can be extended to more realistic models.) Equations (5) and (6) apply also to the ABS regime, in which $E^2 - \Delta^2 < 0$. In this case, the phases of the arguments of the complex square root functions are always chosen so that $\text{Im}(\beta_p) \geq 0$ and $\text{Im}(\beta_h) \leq 0$ [7]. The $S$-matrix, which expresses outgoing waves ($A_1^+, A_3^+, A_2^-, A_4^-$) in terms of incoming waves, can be expressed in terms of $M$. (An alternative way of writing $\text{det}[S]$ has been introduced in [7], where it was shown that, in the so-called ‘Andreev approximation’, discussed below...
in detail, one gets \( \det[S] = \det[I - \alpha_E^2 r_A s_0(E) r_A s_0(-E)] \), with \( \alpha_E = \exp[-i \arccos(E/\Delta)] \),
\( r_A = \begin{bmatrix} e^{i2\Delta} & 0 \\ 0 & e^{-i2\Delta} \end{bmatrix} \), and \( s_0(E) \) the \((2 \times 2)\) scattering matrix for the whole system, in the limit of normal leads—\( \Delta \to 0 \). In particular we find it convenient to express \( \det[S] \) as a ratio,

\[
\det[S] = \frac{M_{1,1} M_{3,3} - M_{1,3} M_{3,1}}{M_{2,2} M_{4,4} - M_{2,4} M_{4,2}} = \frac{\mathcal{F}(E; \chi)}{\mathcal{G}(E; \chi)}, \tag{7}
\]

where \( \mathcal{F}(E; \chi) \) and \( \mathcal{G}(E; \chi) \) may be regarded as functions of \( E \) in the complex \( E \)-plane (see appendix C for the explicit derivation of equation (7)). We chose them to obey several convenient properties, which are crucial for our derivation (and appear to be generally met in physically relevant models).

(i) They are always finite for finite \( E \). This can be easily achieved by shifting poles of \( \mathcal{G} \) into zeros of \( \mathcal{F} \) and vice versa.

(ii) They have no common zeros. (Possible common zeros (e.g. \( E_0 \)) could always be cancelled by a redefinition, \( \mathcal{F}(E; \chi) \to \mathcal{F}(E; \chi)/(E - E_0) \), \( \mathcal{G}(E; \chi) \to \mathcal{G}(E; \chi)/(E - E_0) \), without changing equation (7)).

(iii) \( \mathcal{F}(E; \chi) = \mathcal{G}^*(E; \chi) \). Here this equation refers to complex conjugating the function without complex conjugating its argument, \( E \). This condition is consistent with the requirement that \( |\det[S]| = 1 \) for scattering states.

(iv) \( \mathcal{G}(E; \chi) \) can be defined to have branch cuts along the real \( E \)-axis, corresponding to nonzero density of scattering states in the leads. This is due to the fact that \( \mathcal{G}(E; \chi) \) depends on \( E \) via \( \beta_p \) and \( \beta_h \) and that they become double-valued functions of \( E \) for \( |E| > \Delta \).

(v) \( \partial_E \ln \mathcal{G}(E; \chi) \) vanishes rapidly at \( |E| \to \infty \) along any ray not parallel to the real axis. This condition is crucial to allow for conveniently deforming the integration path in the energy plane when computing \( \mathcal{I}^0(\chi) \).

(vi) \( \mathcal{G}(E; \chi) \) is real in the bound state region: the real axis with \( -\Delta \leq E \leq \Delta \).

These conditions appear to determine \( \mathcal{F}(E; \chi) \) and \( \mathcal{G}(E; \chi) \) uniquely except for an over all multiplicative constant factor. Moreover, they imply that zeros of \( \mathcal{G}(E; \chi) \) correspond to poles of \( \det[S] \). These conditions imply that there are no poles of \( \det[S] \) off the real axis. We are actually dealing with a two-sheeted Riemann surface, due to the branch cuts. We may regard \( \mathcal{F}(E; \chi) \) as being \( \mathcal{G}(E; \chi) \) on the second sheet of the Riemann surface. With the definition we gave of \( \beta_p(E), \beta_h(E) \), the zeros of \( \mathcal{G}(E; \chi) \), corresponding to poles of \( \det[S] \), occur either on the real axis, or else off-axis on the second sheet of the Riemann surface. This property of \( \det[S] \) follows from general principles. Since the \( S \)-matrix can be derived from the retarded Green’s function it should have no singularities in the upper half-plane. Since

\[
\mathcal{G}(E; \chi) = \mathcal{G}^*(E; \chi) \tag{8}
\]

in the BS region, we can use the Schwarz reflection principle to define its unique analytic continuation to the entire first sheet of the Riemann surface, where it obeys equation (8). Thus, if \( \mathcal{G}(E; \chi) \) had a zero in the lower half-plane, at \( E_0 \), it would have to have a twin at energy \( E_0^* \) in the upper half-plane. This would violate this basic property of \( S \), telling us that no such zeros exist. (Notice that the functions \( \mathcal{F}(E; \chi) \) and \( \mathcal{G}(E; \chi) \) can be equally

\[\text{doi:10.1088/1742-5468/2013/02/P02034}\]
The Josephson current through a long quantum wire

well defined in different models, such as the one describing the Josephson current in ballistic superconductor–graphene systems [14].

The ABSs correspond to poles of the $S$-matrix and therefore to the zeros of $G(E)$. This allows us to write the contribution to the ground state energy from ABSs as

$$E_{\text{ABS}}^{(0)} = -\frac{1}{2\pi i} \oint_{\Gamma_{\text{ABS}}} dE \ln G(E; \chi),$$

where the contour $\Gamma_{\text{ABS}}$ in the complex energy plane surrounds the negative energy ABSs.

To compute the contribution to the ground state energy arising from the SSs, $E_{\text{SS}}^{(0)}$, we put the system in a large box $-L/2 < x < \ell + L/2$, requiring $u(x)$ and $v(x)$ to obey vanishing boundary conditions. This gives an equation of the form

$$\sum_{n=\pm1, m=\pm1} \mu_{n,m} e^{iL(n\beta_p + m\beta_h)} + \mu_{0,0} = 0,$$

where the coefficients $\mu_{n,m}$ depend on the transmission matrix. In appendix C, we outline the derivation of equation (10) and, in particular, of the coefficients $\mu_{n,m}$ in terms of the transmission matrix $M$.

Here we notice that defining

$$\zeta \equiv e^{iL(\beta_h - \beta_p)}, \quad \eta \equiv e^{iL(\beta_p + \beta_h)}$$

and multiplying equation (10) by $\zeta$ gives a quadratic equation in $\zeta$ where only the term linear in $\zeta$ depends on $\eta$. It then follows that the product of the two roots is independent of $\eta$ and is given by

$$\prod_{a=1}^{2} \exp iL(\beta_h^a - \beta_p^a) = \frac{\mu_{1,-1}}{\mu_{1,1}} = \det[S].$$

Equation (12) then implies

$$G(E; \chi) = \prod_{a=1}^{2} \exp \left[ -\frac{i}{2} L(\beta_h^a - \beta_p^a) \right].$$

At energies for which both $\beta_p$ and $\beta_h$ are real, in particular for $-\sqrt{\mu^2 + \Delta^2} \leq E < -\Delta$, we may write the solutions of equation (12) in the form

$$\beta_p^a = \frac{\pi m_p}{L} + \frac{\sigma_p^a}{L}, \quad \beta_h^a = \frac{\pi m_h}{L} + \frac{\sigma_h^a}{L},$$

with

$$(\beta_p^a)^2 / 2m_S - \mu = \mu - (\beta_h^a)^2 / 2m_S.$$  

Here $m_p, m_h$ are integers and the phase shifts obey $0 \leq \sigma_p^a, \sigma_h^a \leq \pi$. Thus, the phase shifts obey

$$G(E; \chi) = \prod_a e^{i(1/2)[\sigma_p^a - \sigma_h^a]}.$$

By adapting the derivation of [9] to the continuum model, we now derive the contribution to the total ground state energy arising from states with energy between $-\sqrt{\mu^2 + \Delta^2}$ and $-\Delta$. From the definition of $\beta_p^a, \beta_h^a$ in equation (14), as $L \to \infty$, one obtains $(\beta_p^a)^2 / 2m_S - \mu = \mu - (\beta_h^a)^2 / 2m_S$ and $\beta_p \sigma_p^a = -\beta_h \sigma_h^a$, with $\beta_p/h = \pi m_p/h / L$. Defining $\beta_{p,l(u)}, \beta_{h,l(u)}$ to be
the values of $\beta_p, \beta_h$ corresponding to $-\sqrt{\mu^2 + \Delta^2}$ and $-\Delta$, respectively, one then finds that the total ground state energy arising from states with energy between $-\sqrt{\mu^2 + \Delta^2}$ and $-\Delta$, $E_{SS}$, may either be written as $E_{SS}^1 = \mathcal{E}_{SS \uparrow} + (1/\pi m_S) \int_{\beta_p \uparrow}^{\beta_p \downarrow} d\beta_p \beta_p \sigma_p^a$ or as $E_{SS}^1 = \mathcal{E}_{SS \downarrow} - (1/\pi m_S) \int_{\beta_h \uparrow}^{\beta_h \downarrow} d\beta_h \beta_h \sigma_h^a$, with $\mathcal{E}_{SS \uparrow}, \mathcal{E}_{SS \downarrow}$ being independent of $\chi$. Taking the mean of the two equivalent expressions for $E_{SS}^1$, we may then write

$$E_{SS}^1 = \mathcal{E}_{SS}^1 + \frac{1}{2\pi} \int_{-\sqrt{\mu^2 + \Delta^2}}^{-\Delta} dE \sum_a [\sigma_p^a - \sigma_h^a]$$

$$= \mathcal{E}_{SS} - \frac{1}{2\pi i} \int_{-\sqrt{\mu^2 + \Delta^2}}^{-\Delta} dE \ln \left[ \mathcal{G}^*(E; \chi) / \mathcal{G}(E; \chi) \right], \quad (17)$$

with $\mathcal{E}_{SS}$ being independent of $\chi$. Remarkably, equation (17) can be readily extended to $E < -\sqrt{\mu^2 + \Delta^2}$, in which case one obtains $\mathcal{G}(E; \chi) = \prod_i e^{-i\phi_i}$. A procedure similar to the one leading to equation (17) yields the contribution to the total ground state energy arising from states with energy $E < -\sqrt{\mu^2 + \Delta^2}$, $E_{SS}^2$, which is given by

$$E_{SS}^2 = \mathcal{E}_{SS}^2 - \frac{1}{\pi} \int_{-\infty}^{-\sqrt{\mu^2 + \Delta^2}} dE \sum_a \sigma_h^a$$

$$= \mathcal{E}_{SS}^2 - \frac{1}{2\pi i} \int_{-\infty}^{-\sqrt{\mu^2 + \Delta^2}} dE \ln \left[ \mathcal{G}^*(E; \chi) / \mathcal{G}(E; \chi) \right], \quad (18)$$

with $\mathcal{E}_{SS}$ being independent of $\chi$. Adding equations (17) and (18), we obtain the total contribution to the ground state energy from scattering states in the form

$$E_{SS}^{(0)} = \mathcal{E}_{SS}^0 - \frac{1}{2\pi i} \int_{-\infty}^{-\Delta} dE \ln \left[ \mathcal{G}^*(E; \chi) / \mathcal{G}(E; \chi) \right], \quad (19)$$

where $\mathcal{E}_{SS}^0$ is independent of $\chi$. Using equation (8), the second term in equation (19) can also be written as a contour integral in the complex energy plane, like equation (9), with the contour now running on both sides of the branch cut in $\mathcal{G}$ along the real $E$-axis from $-\infty$ to $-\Delta$. These two terms can be combined, allowing us to write a simple unified formula for the zero-temperature Josephson current,

$$I^{(0)}[\chi] = -\frac{2e}{2\pi i} \int_{\Gamma} dE \partial_\chi \{ \ln \mathcal{G}(E; \chi) \}, \quad (20)$$

where the contour $\Gamma$ runs infinitesimally above and below the negative $E$ axis.

Equation (20) is exact and, in principle, as long as $\mathcal{G}(E; \chi)$ is known, it may be used to compute $I^{(0)}[\chi]$ for any values of the system parameters. However, in general it is of no great usefulness for practical purposes as, typically, when $E$ lies on the real axis, $\mathcal{G}(E; \chi)$ turns out to be a rapidly oscillating function of $E$, which makes it quite hard to figure out reliable approximations in possibly relevant regimes (such as, for instance, the ‘long junction’ limit). In addition, oscillations also make any attempt to numerically estimate $I^{(0)}[\chi]$ fail, except possibly in some very specific cases, such as the short junction limit. A way to greatly improve the convergence properties of the integral in equation (20) is to deform the integration path $\Gamma$ by using the facts that, on the physical Riemann sheet, $\mathcal{G}(E; \chi)$ has no zeros off the real axis and $\partial_\chi \ln \mathcal{G}(E; \chi)$ vanishes rapidly at $|E| \rightarrow \infty$. Thus, to trade equation (20) for a more tractable formula, we use the fact that, due to the

doi:10.1088/1742-5468/2013/02/P02034
properties of $\mathcal{G}(E; \chi)$ discussed above, $\Gamma$ can be deformed into a single line running along the imaginary $E$-axis from $-\infty$ to $\infty$. (See figure 1.) As a result, one eventually obtains the general formula

$$I^{(0)}[\chi] = \frac{2e}{2\pi} \int_{-\infty}^{\infty} d\omega \partial_{\chi} \{ \ln \mathcal{G}(i\omega : \chi) \}. \quad (21)$$

Equation (21) is particularly amenable for explicitly computing $I^{(0)}[\chi]$ for at least two reasons. First of all, integrating over the imaginary axis greatly improves the convergence properties of the integral, as it allows for getting rid of the oscillations in the integrand functions. Moreover, as we will show in the explicit examples discussed in the following, it enables us to compute $I^{(0)}[\chi]$ in a systematic expansion in inverse powers of the length of the junction (that is, of the size $\ell$ of the normal region $C$), eventually letting us show that, to leading order in $\ell^{-1}$, terms in $I_{SS}^{(0)}[\chi]$ and $I_{ABS}^{(0)}[\chi]$ cancel with each other, so that $I^{(0)}[\chi]$ can be expressed in terms of ABSs at the Fermi level only.
3. The dc Josephson current in a long SNS junction

We assume that our system is made of two superconductors at phase difference $\chi$ separated by a long central normal region $C$ of length $\ell$, defined by $0 < x < \ell$, as sketched in figure 2. We assume that the gap function $\Delta(x)$ makes an abrupt transition from $\Delta_{\epsilon} \pm \frac{\epsilon}{2}$ to 0 in the central region and include a normal potential energy function $V(x)$, which we assume makes an abrupt transition from 0 in the leads to $V_C$ in the central region. Here ‘abrupt’ means rapid on the scale of $\ell$. We assume $V_C < \mu$ so that the central region is metallic. The mass in $C$ is written as $m$. The wavefunctions in the central region, far from the interfaces, may be written as

$$
\begin{bmatrix}
u(x) \\ v(x) \end{bmatrix} = \begin{bmatrix} C_1 \exp(i\alpha_p x) + C_2 \exp(-i\alpha_p x) \\ C_3 \exp(-i\alpha_h x) + C_4 \exp(i\alpha_h x) \end{bmatrix},
$$

(22)

with $\alpha_{p/h} = \left\{2m(\mu - V_C \pm \Delta)\right\}^{1/2}$. Thus, we define the transmission matrices for the left and right interfaces, $L$ and $R$, by $\tilde{C} = L \tilde{A}^-, \tilde{A}^+ = R \cdot M^C \tilde{C}$, in terms of which $M = R \cdot M^C \cdot L$, where $M^C$ is the transmission matrix of $C$, given by

$$
M^C = \begin{bmatrix}
\exp(i\alpha_p \ell) & 0 & 0 & 0 \\
0 & \exp(-i\alpha_p \ell) & 0 & 0 \\
0 & 0 & \exp(-i\alpha_h \ell) & 0 \\
0 & 0 & 0 & \exp(i\alpha_h \ell) 
\end{bmatrix}.
$$

(23)

Figure 2. A sketch of $V(x), \Delta(x)$ in the SNS system. The central region $C$ has length $\ell$ and is defined by $0 < x < \ell$. The $L$- and $R$-interfaces are assumed to be ‘sharp’, that is, $V(x), \Delta(x)$ and the single-particle mass are assumed to vary over typical length scales $\ll \ell$. In this case scattering is localized at the interfaces and is fully encoded within the particle and hole normal- and Andreev-scattering amplitudes at the interfaces, $N_p/\hbar L/R, A_p/\hbar L/R$. 

The Josephson current through a long quantum wire

doi:10.1088/1742-5468/2013/02/P02034
By the definition of $F$ and $G$ in equation (7) and of the $R$- and $L$-transmission matrices, one finds that the following explicit formulas hold:

$$
F(E; \chi) = F_{0,0}(\chi; E) + F_{1,1}(E)e^{i[\alpha_p-\alpha_h]\ell} + F_{-1,-1}(E)e^{-i[\alpha_p-\alpha_h]\ell} + F_{1,-1}(E)e^{i[\alpha_p+\alpha_h]\ell} + F_{-1,1}(E)e^{-i[\alpha_p+\alpha_h]\ell},
$$

$$
G(E; \chi) = G_{0,0}(\chi; E) + G_{1,1}(E)e^{i[\alpha_p-\alpha_h]\ell} + G_{-1,-1}(E)e^{-i[\alpha_p-\alpha_h]\ell} + G_{1,-1}(E)e^{i[\alpha_p+\alpha_h]\ell} + G_{-1,1}(E)e^{-i[\alpha_p+\alpha_h]\ell},
$$

with

$$
F_{0,0}(\chi) = -(R_{1,1}R_{3,2} - R_{1,2}R_{3,1})(L_{1,3}L_{2,1} - L_{1,1}L_{2,3})
- (R_{1,3}R_{3,4} - R_{1,4}R_{3,3})(L_{3,3}L_{4,1} - L_{3,1}L_{4,3}),
$$

$$
F_{1,1} = (R_{1,1}R_{3,3} - R_{1,3}R_{3,1})(L_{1,1}L_{3,3} - L_{1,3}L_{3,1}),
$$

$$
F_{-1,-1} = (R_{1,2}R_{3,4} - R_{1,4}R_{3,2})(L_{2,1}L_{4,3} - L_{4,1}L_{2,3}),
$$

$$
F_{1,-1} = (R_{1,1}R_{3,4} - R_{1,3}R_{1,4})(L_{1,1}L_{4,3} - L_{1,3}L_{4,1}),
$$

$$
F_{-1,1} = (R_{1,2}R_{3,3} - R_{3,2}R_{1,3})(L_{2,1}L_{3,3} - L_{3,1}L_{3,2}),
$$

and

$$
G_{0,0}(\chi) = -(R_{2,1}R_{4,2} - R_{2,2}R_{4,1})(L_{1,4}L_{2,2} - L_{1,2}L_{2,4})
- (R_{2,3}R_{4,4} - R_{2,4}R_{4,3})(L_{3,4}L_{4,2} - L_{3,2}L_{4,4}),
$$

$$
G_{1,1} = (R_{2,1}R_{4,3} - R_{2,3}R_{4,1})(L_{1,2}L_{3,4} - L_{3,2}L_{1,4}),
$$

$$
G_{-1,-1} = (R_{2,2}R_{4,4} - R_{2,4}R_{4,2})(L_{2,2}L_{4,4} - L_{2,4}L_{4,2}),
$$

$$
G_{1,-1} = (R_{2,1}R_{4,4} - R_{4,1}R_{2,4})(L_{1,2}L_{4,4} - L_{1,4}L_{4,2}),
$$

$$
G_{-1,1} = (R_{2,2}R_{4,3} - R_{4,2}R_{2,3})(L_{2,3}L_{4,4} - L_{3,2}L_{2,4}),
$$

and the explicit dependence upon $\chi$ set according to the definition of the $R$- and $L$-matrices. Using general properties of the transmission matrices, arising from the continuity equation for the probability current, it is not difficult to show that equations (25) and (26) are consistent with the identity $F(E; \chi) = G^*(E; \chi)$, that $G(E; \chi)$ is real for real $E$ and $-\Delta \leq E \leq \Delta$, and that the branch cuts of $G(E; \chi)$ lie on the real axis, from $E \to -\infty$ to $E = -\Delta$ and from $E = \Delta$ to $E \to \infty$. From equation (24), one then sees that

$$
\partial_{\chi} \ln G(E; \chi) = \frac{\partial_{\chi} G_{0,0}(E; \chi)}{G(E; \chi)}.
$$

Because, as $E$ goes to infinity along any ray not parallel to the real axis, either the imaginary part of $\alpha_p$ or the imaginary part of $\alpha_h$ goes to $-\infty$, from equation (27) we find that $\partial_{\chi} \ln G(E; \chi)$ exponentially vanishes as $|E| \to \infty$ off the real axis. Using equation (21) for the current and taking into account that, for large $\ell$, contributions to the integral with $|\omega| \geq V$ are strongly suppressed, we may approximate $\alpha_p$ and $\alpha_h$ as

$$
\alpha_p \approx \alpha_F + i\omega \sqrt{\frac{m}{2(\mu - V_C)}}, \quad \alpha_h \approx \alpha_F - i\omega \sqrt{\frac{m}{2(\mu - V_C)}},
$$

with $\alpha_F = \sqrt{2m(\mu - V_C)}$. Using equation (28), the zero-temperature Josephson current to leading order in $\ell^{-1}$ is then given by
The Josephson current through a long quantum wire

\[ I^{(0)}[\chi] = -\frac{e}{\pi \ell} \sqrt{\frac{\mu - V_C}{2m}} \int_0^\infty dz \partial\chi \tilde{G}_{0,0}(\chi) \{ \tilde{G}_{1,1} e^{-z} + \tilde{G}_{-1,-1} e^{z} \]  
+ \tilde{G}_{1,-1} e^{2i\alpha_F \ell} + \tilde{G}_{-1,1} e^{-2i\alpha_F \ell} + \tilde{G}_{0,0}(\chi) \}^{-1}, \]  
(29)

with the coefficients \( \tilde{G}_{a,b} \) being defined as the coefficients \( G_{a,b} \) evaluated at \( \omega = 0 \), that is, setting \( \alpha_p = \alpha_L = \alpha_F, \beta_p = \beta_L = \{2m_S[\mu+i\Delta]\}^{1/2} \). Computing the integral in equation (29), one eventually finds

\[ I^{(0)}[\chi] = \frac{e v_F}{\pi \ell} \partial\chi \text{ln}^2 \left[ \frac{u_+(\chi)}{u_-(\chi)} \right], \]  
(30)

with \( v_F = \alpha_F/m \) and \( u_\pm(\chi) \) being the roots of the second-degree equation

\[ \tilde{G}_{1,-1} u^2 + [\tilde{G}_{1,1} e^{2i\alpha_F \ell} + \tilde{G}_{-1,1} e^{-2i\alpha_F \ell} + \tilde{G}_{0,0}(\chi)] u + \tilde{G}_{1,1} = 0. \]  
(31)

Because of the particle–hole symmetry at the Fermi level, one finds \( \tilde{G}_{1,-1} = \tilde{G}_{1,1} \), which implies the identity \( u_+ (\chi) u_- (\chi) = 1 \) that we used to derive equation (30). In order to prove that equation (30) yields equation (3), we have to rewrite equation (31) in terms of the normal- and Andreev-scattering amplitudes at the Fermi level. To do so, we relate the scattering amplitudes at both interfaces to the \( R \)- and \( L \)-matrix elements. This may be readily done starting from the definition of the normal- and Andreev-scattering amplitudes. The result is

\[ N_R^p(E) = \frac{R_{2,4} R_{4,1} - R_{2,1} R_{4,4}}{R_{2,2} R_{4,4} - R_{2,4} R_{4,2}}, \quad A_R^p(E) = \frac{R_{2,1} R_{4,2} - R_{2,2} R_{4,1}}{R_{2,2} R_{4,4} - R_{2,4} R_{4,2}}, \]  
\[ N_R^h(E) = \frac{R_{2,2} R_{4,4} - R_{2,4} R_{4,2}}{R_{2,2} R_{4,4} - R_{2,4} R_{4,2}}, \quad A_R^h(E) = \frac{R_{2,1} R_{4,2} - R_{2,2} R_{4,1}}{R_{2,2} R_{4,4} - R_{2,4} R_{4,2}}, \]  
(32)

and

\[ N_L^p(E) = \frac{L_{1,2} L_{4,4} - L_{1,1} L_{4,2}}{L_{2,2} L_{4,4} - L_{2,1} L_{4,2}}, \quad A_L^p(E) = \frac{L_{3,2} L_{4,4} - L_{3,1} L_{4,2}}{L_{2,2} L_{4,4} - L_{2,1} L_{4,2}}, \]  
\[ N_L^h(E) = \frac{L_{2,2} L_{4,4} - L_{2,1} L_{4,2}}{L_{2,2} L_{4,4} - L_{2,1} L_{4,2}}, \quad A_L^h(E) = \frac{L_{3,2} L_{4,4} - L_{3,1} L_{4,2}}{L_{2,2} L_{4,4} - L_{2,1} L_{4,2}}. \]  
(33)

Using equations (32) and (33) specified at \( E = 0 \), we may then rewrite equation (31) as

\[ u^2 + 1 - \{ \tilde{N}_R^p N_L^p e^{2i\alpha_F \ell} + \tilde{A}_R^p \tilde{A}_L^h \text{c.c.} \} u = 0, \]  
(34)

with, as specified below, the overbar meaning that all the scattering amplitudes in equation (34) are evaluated at \( E = 0 \). By definition, one has that \( \tilde{A}_R^p, \tilde{A}_L^h \propto e^{i(\frac{\pi}{2})\chi} \), where \( \chi \) stands for factors that are independent of \( \chi \). As a result, \( \tilde{A}_R^p \tilde{A}_L^h \propto e^{i\chi} \) and, on setting \( u_\pm(\chi) = e^{\pm i\vartheta(\chi)} \), we obtain

\[ \vartheta(\chi) = \arccos[\text{Re} \{ \tilde{N}_R^p N_L^p e^{2i\alpha_F \ell} + \tilde{A}_R^p \tilde{A}_L^h \}], \]  
(35)

which finally gives our main result in equation (3). Our result makes it feasible to compute the dc Josephson current in several models of physical interest. We are now going to analyze two of them, as examples, in the next section.
4. Explicit calculation of the dc Josephson current in models of physical interest

To illustrate the effectiveness of our result, we now compute $I^{(0)}[\chi]$ in two models of physical interest, also showing how results previously obtained in the literature for specific values of the system parameters may be straightforwardly recovered within our approach.

As a first example, we consider a model system whose S–N interfaces may be thought of as a generalization of the one studied in [11], that is, we assume that $m$ is uniform, $\Delta(x)$ abruptly changes at the S–N interfaces and the normal potential energy function $V(x)$ is given by

$$V(x) = V_0[\delta(x) + \delta(x - \ell)] + V_C \theta(x) \theta(\ell - x),$$

where $\delta$ and $\theta$ are the Dirac delta-function and Heavyside step function respectively. At the Fermi level, the result for the normal reflection amplitudes is

$$\bar{N}_R^p = \bar{N}_L^p = -\frac{(\beta_F - \alpha_F - iZ)(\beta_F^* + \alpha_F + iZ)}{\alpha_F^2 + |\beta_F - iZ|^2},$$

with $\beta_F = \{2m(\mu + i\Delta)\}^{1/2}$ (that is, the momentum $\beta_p$ for $E = 0$) and $Z = 2mV_0$. The Andreev reflection amplitudes at the Fermi level are given by

$$\bar{A}_R^p = \bar{A}_L^h = \frac{i e^{i(\chi/2)} \alpha_F (\beta_F + \beta_F^*)}{\alpha_F^2 + |\beta_F - iZ|^2}. \tag{38}$$

Equation (35) now gives

$$\vartheta(\chi) = \arccos \left\{ \text{Re} \left[ -\cos(\chi) \left[ \frac{\alpha_F (\beta_F + \beta_F^*)}{\alpha_F^2 + |\beta_F - iZ|^2} \right]^2 \right. \right.$$

$$\left. \left. - \left( \frac{(\beta_F - \alpha_F - iZ)(\beta_F^* + \alpha_F + iZ)}{\alpha_F^2 + |\beta_F - iZ|^2} \right)^2 e^{2i\omega_F \ell} \right] \right\}, \tag{39}$$

where $\alpha_F = \sqrt{2m(\mu - V_C)}$. An interesting point is how Ishii’s sawtooth current [5] may be recovered from our result in equation (39), taken in an appropriate limit. First of all, let us remark that, as shown in figure 3, in order to obtain Ishii’s result from the formula for $\vartheta(\chi)$ in equation (35), one has to take the system in the limit of perfect Andreev scattering at zero energy, $\bar{N}_L/R = 0$, $|\bar{A}_R^p/h| = 1$. (On the other hand, notice that one obtains a complete suppression of the Josephson current when the Andreev reflection amplitude vanishes at either interface.) Thus, we see that two conditions must be satisfied for perfect Andreev reflection and hence a sawtooth current: $Z = \text{Im} \beta_F = \sqrt{m(\mu^2 + \Delta^2 - \mu)}$, $V_C = -Z^2/(2m)$. These two conditions are readily met if one assumes $V_C = Z = 0$ (corresponding to S–N interfaces without barrier normal scattering potentials [11]) and $\Delta/\mu \approx 0$ (that is, the so-called ‘Andreev approximation’, consisting in assuming no normal scattering at the interfaces at zero energy; in the absence of a barrier potential this is quite a harmless approximation, given the typical values for $\mu$ and $\Delta$ in an ordinary superconductor). As both approximations are made in Ishii’s derivation, we see how the result of [5] may be regarded as just a special case of our equation (39).

---

4 See Likharev [15] for a discussion of the physical applicability of the model with stepwise changes in the physical parameters as a function of the position.
Figure 3. $I^{(0)}[\chi]$ versus $\chi$ for the two-interface BTK model for different values of the system parameters, with $\vartheta(\chi)$ computed with equation (39). The parameters have been chosen so that $\alpha_F = \text{Re} \beta_F$ (arbitrary units), $\text{Im} \beta_F / \text{Re} \beta_F = 0.77$, $\alpha_F \ell = 13\pi$, while $Z$ is varied, so as to change the ratio between the normal and Andreev reflection coefficients. In particular, from bottom to top we set $Z / \text{Im} \beta_F = 0, 0.5, 0.8, 1$. Notice that Ishii’s sawtooth behavior for $I^{(0)}[\chi]$ is recovered only when $Z$ is fine-tuned to be equal to $\text{Im} \beta_F$. On the other hand, for $Z = 0$, the mismatch between the Fermi momenta in the central region and in the superconducting leads (due to $\text{Im} \beta_F$ being different from zero) yields a nonzero normal reflection coefficient, as evidenced by the sinusoidal dependence of $I^{(0)}[\chi]$ on $\chi$.

The second example we consider is related to the fact that while in carrying out our derivation we mainly referred to a continuum one-dimensional model of an SNS system just because of the wide applicability of such a model, the requirements on $\mathcal{F}(E; \chi)$ and $\mathcal{G}(E; \chi)$ we made in section 2 are quite general, so we expect our approach to successfully apply to a wide class of models such as, for instance, the paradigmatic tight-binding Hamiltonian studied in [9]. In particular, we now derive the dc Josephson current for a particular lattice model Hamiltonian for a central region consisting of $\ell - 1$ sites connected to two infinite bulk superconductors at a phase difference of $\chi$ [9]. For such a system, the amplitudes $u_j, v_j$ become functions of the lattice site $j$, and the BdG equations are given by

$$
\begin{align*}
E u_j &= -\tau_{j,j+1} u_{j+1} - \tau_{j,j-1} u_{j-1} - \mu u_j + V_j u_j + \Delta_j v_j, \\
E v_j &= \tau_{j,j+1} v_{j+1} + \tau_{j,j-1} v_{j-1} + \mu v_j - V_j v_j + \Delta_j^* u_j,
\end{align*}
$$

with the lattice hopping amplitudes being given by

$$
\tau_{j,j+1} = \begin{cases} 
  t_S & \text{for } j \leq -1 \text{ and } j \geq \ell, \\
  J & \text{for } j \in \{1, \ldots, \ell - 2\}, \\
  t'' & \text{for } j = 0, \ell - 1,
\end{cases}
$$

$$
doi:10.1088/1742-5468/2013/02/P02034
$$
The Josephson current through a long quantum wire

the superconducting gap being given by

$$
\Delta_j = \begin{cases} 
\Delta e^{i\chi/2} & \text{for } j \leq 0, \\
\Delta e^{-i\chi/2} & \text{for } j \geq \ell, \\
0 & \text{for } j \in \{1, \ldots, \ell - 1\},
\end{cases}
$$

and the potential being given by

$$
V_j = \begin{cases} 
V_C, & \text{for } 1 \leq j \leq \ell - 1, \\
0 & \text{otherwise}.
\end{cases}
$$

The construction of the function $G(E; \chi)$ is readily achieved by following the same procedure as we used in the continuum case. However, the (lattice) particle and hole momenta within C and within the leads are now related to the (negative) energy $E$, by means of the lattice dispersion relations, that is

$$
-2tS \cos(\beta_h) - \mu = \sqrt{E^2 - \Delta^2},
$$

$$
-2tS \cos(\beta_p) - \mu = \sqrt{E^2 - \Delta^2},
$$

and

$$
-2J \cos(\alpha_p) + V_C - \mu = E,
$$

$$
-2J \cos(\alpha_h) - V_C + \mu = E.
$$

(In [9], the definition of $\beta_h$ was shifted by $\pi$. ) $F$ and $G$ can be defined as in equations (24) and (26) and can be seen to possess the properties listed in section 2 with the following modifications. The branch cuts now run from $\Delta$ to $\tilde{E}_S = \sqrt{(2tS + \mu)^2 + \Delta^2}$ and from $-\Delta$ to $-\tilde{E}_S$. Furthermore, if $\tilde{E}_S < 2J + V_C - \mu$ there are normal bound states (NBSs) in the energy range $\tilde{E}_C < |E| < 2J + V_C - \mu$, as indicated in figure 4. Looking at the most general case in which both ABSs and NBSs contribute to $I(0)[\chi]$, we obtain

$$
I(0)[\chi] = I_{\text{ABS}}(0)[\chi] + I_{\text{SS}}(0)[\chi] + I_{\text{NBS}}(0)[\chi],
$$

with

$$
I_{\text{ABS}}(0)[\chi] = -\frac{2e}{2\pi i} \int_{\Gamma_{\text{ABS}}} dE \partial_\chi \{ \ln G(E; \chi) \},
$$

$$
I_{\text{SS}}(0)[\chi] = -\frac{2e}{2\pi i} \int_{\Gamma_{\text{SS}}} dE \partial_\chi \{ \ln G(E; \chi) \},
$$

$$
I_{\text{NBS}}(0)[\chi] = -\frac{2e}{2\pi i} \int_{\Gamma_{\text{NBS}}} dE \partial_\chi \{ \ln G(E; \chi) \},
$$

and the path $\Gamma_{\text{ABS}}$ defined as in section 2, $\Gamma_{\text{SS}}$ being a path surrounding the branch cut lying over the real axis from $E = -\tilde{E}_S$ to $E = -\Delta$ and $\Gamma_{\text{NBS}}$ being a path surrounding the NBSs. (See figure 4(a) for a sketch of the integration paths.) Clearly, $I_{\text{NBS}}(0)[\chi] = 0$ if there are no NBSs. Considering the path $\Gamma$ made by $\Gamma_{\text{ABS}} \cup \Gamma_{\text{SS}} \cup \Gamma_{\text{NBS}}$, all run through clockwise, and by the outer closed path $\tilde{\Gamma}$ (figure 4(b)), made by the arc $\Sigma$ closed along the imaginary axis, as $G(E; \chi)$ has no poles in the region of the complex plane bounded by $\Gamma$, sending the radius of $\Sigma$ to infinity, we readily obtain equation (21). When computing the integral, we consider again that, for large $\ell$, we may solve equation (45) for $\alpha_p, \alpha_h$ with $E = i\omega$, by setting

$$
\alpha_p \approx \alpha_F + \frac{i\omega}{v_F}, \quad \alpha_h \approx \alpha_F - \frac{i\omega}{v_F},
$$

$\text{doi:10.1088/1742-5468/2013/02/P02034}$
with $-2J \cos(\alpha_F) + V_C - \mu = 0$ and $v_F = 2J \sin(\alpha_F)$. By direct calculation, one finds that the normal reflection amplitudes at the Fermi level for particle-like states are given by

$$N_R^p = N_L^p = \frac{(e^{-i\beta} - e^{-i\alpha_F - \lambda})(e^{i\beta} - e^{-i\alpha_F - \lambda})}{(e^{2\beta_I} + e^{-2\lambda} - 2 \cos(\alpha_F) \cos(\beta_R)e^{\beta_I - \lambda})^2},$$

(48)

with $\beta \equiv \beta_R + i\beta_I = \beta_p(E = 0) = \beta_S^*(E = 0)$, $\cos(\beta) = -\mu/(2t_S + i(\Delta/2t_S))$, and $(t'')^2 = Jt's^{-\lambda}$. At variance, the Andreev reflection amplitudes at the Fermi level are given by

$$A_R^p = A_L^h = \frac{2ie^{i(\lambda/2)} \sin(\alpha_F) \sin(\beta_R)e^{\beta_I - \lambda}}{(e^{2\beta_I} + e^{-2\lambda} - 2 \cos(\alpha_F) \cos(\beta_R)e^{\beta_I - \lambda})^2},$$

(49)

As a result, we obtain that again $I^{(0)}[\chi]$ is given by equation (30), with $u_\pm(\chi)$ being the roots of the equation

$$u^2 + 1 - 2u \left( \Re \left[ \frac{(e^{-i\beta} - e^{-i\alpha_F - \lambda})(e^{i\beta} - e^{-i\alpha_F - \lambda})^2e^{2i\alpha_F t}}{(e^{2\beta_I} + e^{-2\lambda} - 2 \cos(\alpha_F) \cos(\beta_R)e^{\beta_I - \lambda})^2} \right] \right. - 4 \cos(\chi) \sin^2(\alpha_F) \sin^2(\beta_R)e^{2\beta_I - 2\lambda} \bigg) = 0,$$

(50)
The Josephson current through a long quantum wire

which implies that \( \vartheta(\chi) \) is now

\[
\vartheta(\chi) = \arccos \left\{ -4 \cos(\chi) \sin^2(\beta_F) \sin(2\beta_I) \cos^2(2\lambda) \left[ e^{2\beta_I} + e^{-2\lambda} - 2 \cos(\beta_F) \cos(\beta_R) \right] - \lambda \right\}.
\]

From equation (51) we see that, again, as happens in the continuum model, in the lattice model one may also tune the system to the perfect Andreev point, at which \( I^{(0)}[\chi] \) takes a sawtooth dependence on \( \chi \), and that, in order to do so, one needs two tuning parameters. Indeed, as is seen from equation (51), a sawtooth formula for \( \vartheta(\chi) \) is achieved once one sets \( \lambda + \beta_I = 0 \) and \( \alpha_F = \pm \beta_R \). These conditions are the analogs of the conditions for the continuum model, with now \( t'' \) and \( V_C \) playing the role of tuning parameters (a different choice for the tuning parameters was made in [9], where an additional normal scattering potential \( V \{ \delta_{j,1} + \delta_{j,\ell-1} \} \) was added at the interfaces, but \( V_C \) was set to zero).

Another important observation is that, by setting \( \mu = V_C = 0 \) (and, accordingly, \( \beta_R = \alpha_F = \pi/2 \)), equation (30) yields

\[
I^{(0)} = -\frac{e v_F}{2\pi \ell} \partial_\chi \left\{ \arccos^2 \left[ \frac{4J^2\Delta_B^2 \cos(\chi) + (J^2 - \Delta_B^2)^2}{(J^2 + \Delta_B^2)^2} \right] \right\},
\]

with

\[
\Delta_B = (t'')^2 \left\{ \frac{E_S - \Delta}{2L^2} \right\}.
\]

and, clearly, \( E_S = \sqrt{4t''^2 + \Delta^2} \). Equation (52) is equal to equation (3.13) of [9], despite the fact that the latter was derived within an effective two-boundary model Hamiltonian, obtained by trading the superconducting leads for effective boundary interactions which, well below the superconducting gap, become nearly energy independent. Indeed, the main result of our formalism is that, in principle, when \( \ell \gg 1 \), it allows for performing calculations of the dc Josephson current by just focusing on states near the Fermi level. This is crucial in motivating the step of trading the actual interface model for an effective boundary Hamiltonian, which is much simpler to deal with, especially in the case of an interacting central region, to which our approach is likely to apply as well [13]. Within the effective boundary Hamiltonian formulation one readily sees, for instance, that allowing the central region to host an effectively attractive interaction between electrons should allow for the system to dynamically self-tune to the perfect Andreev reflection point as \( \ell \) becomes large [9].

5. Finite-temperature generalization

It is easy to generalize our contour result to compute the dc Josephson current at finite temperature \( T \), \( I[\chi; T] \). Now [5, 7, 8] the contour \( \Gamma \) gets deformed into a sum of circles around the poles of the Fermi function at \( \omega_n = 2\pi T(n + 1/2) \), yielding

\[
I[\chi; T] \approx 2 e T \sum_{n=-\infty}^{\infty} \frac{\partial_n \text{Re} [\bar{A}_R^p A_L^p]}{\cosh (2\omega_n \ell/v_F) - \text{Re} [N_R^p N_L^p e^{2i\alpha_F \ell} + A_R^p A_L^p]}.
\]
The Josephson current through a long quantum wire

Of course, this gives our $T = 0$ result at $T \ll v_F/\ell$, where we may approximate the sum by an integral. At $T \gg v_F/\ell$ we may approximate the sum by the two terms with $\omega_n = \pm \pi T$,

$$I[\chi;T] \approx 8eTe^{-2\pi T\ell/v_F} \partial_\chi \text{Re} [\bar{A}_R A_L^h] + O(e^{-6\pi T\ell/v_F}).$$

This becomes exponentially small when $T \gg v_F/\ell$.

For general values of the ratio $v_F/T\ell$, the sum in equation (54) can easily be performed numerically. Representative results are shown in figure 5, where we report $I[\chi;T]$ versus $\chi$ for the BTK model system we studied in section 4 for different values of $T$, at fixed system parameters (see the caption). In particular, we see that there is a rapid reduction of the current as soon as the ratio $r = 4\pi T/\ell \sim 1.5$, which is consistent with the exponential decay evidenced in equation (55).

6. Conclusions

By using the analytic properties of the scattering matrix for an SNS system we have expressed the Josephson current, which has contributions from both bound and scattering states, as a single contour integral in the complex energy plane. In the limit of a long central region we then proved that the current can be expressed in terms of properties of the Andreev bound states at the Fermi energy only: namely the normal and Andreev-scattering amplitudes. The result holds at finite temperature provided that $T, v_F/\ell \ll \Delta$. This result shows that the Josephson current is a universal quantity in this long length low-temperature limit and justifies the low energy Hamiltonian approach used in [9], which was crucial for treating Luttinger liquid interaction effects. It also paves the way towards an extension to the non-equilibrium (AC) Josephson effects.

Figure 5. $I[\chi;T]$ versus $\chi$ for the two-interface BTK model for various values of $T$, with the system parameters chosen so that Re $\beta_F = \text{Im} \beta_F$, $\ell = 20$, $v_F = 1$, $Z = 0.02$, $\alpha_F \ell = 13\pi$. The results are displayed for different values of $r = 4\pi T/v_F$. From top to bottom, we have set $r = 0$ (corresponding to $I^{(0)}[\chi]$), 0.35, 0.75, 1.5. For $r > 1.5$ $I[\chi;T]$ becomes negligible compared to $I^{(0)}[\chi]$. 

Of course, this gives our $T = 0$ result at $T \ll v_F/\ell$, where we may approximate the sum by an integral. At $T \gg v_F/\ell$ we may approximate the sum by the two terms with $\omega_n = \pm \pi T$,

$$I[\chi;T] \approx 8eTe^{-2\pi T\ell/v_F} \partial_\chi \text{Re} [\bar{A}_R A_L^h] + O(e^{-6\pi T\ell/v_F}).$$

This becomes exponentially small when $T \gg v_F/\ell$.

For general values of the ratio $v_F/T\ell$, the sum in equation (54) can easily be performed numerically. Representative results are shown in figure 5, where we report $I[\chi;T]$ versus $\chi$ for the BTK model system we studied in section 4 for different values of $T$, at fixed system parameters (see the caption). In particular, we see that there is a rapid reduction of the current as soon as the ratio $r = 4\pi T/\ell \sim 1.5$, which is consistent with the exponential decay evidenced in equation (55).
The Josephson current through a long quantum wire

Acknowledgment

We would like to thank C W J Beenakker, J-S Caux, A Tagliacozzo and A Zagoskin for helpful discussions and correspondence. DG would like to thank the Department of Physics and Astronomy of the University of British Columbia for the kind hospitality at various stages of this work. This research was supported in part by NSERC and CIFAR.

Appendix A. Derivation of equations (32) and (33)

Throughout our derivation, equations (32) and (33) of section 3 are quite crucial, as they allow us to relate the normal and Andreev reflection amplitudes at the S–N interfaces to the transmission matrices L- and R. In this appendix we derive them in detail, starting from the right-hand S–N interface. By definition, the normal and Andreev reflection amplitudes for a particle-like solution of the BdG equations at the right-hand interface are defined by considering a solution that, within C, is given by

\[
\begin{bmatrix}
  u(x) \\
  v(x)
\end{bmatrix}
_{\text{p,R}} = \begin{bmatrix}
  e^{i\alpha_p(x-\ell)} + N_R^p(E)e^{-i\alpha_p(x-\ell)} \\
  A_R^p(E)e^{i\alpha_n(x-\ell)}
\end{bmatrix}.
\]  

(A.1)

Similarly, the normal and Andreev reflection amplitudes for a hole-like solution of the BdG equations at the right-hand interface are defined by considering a solution that, within C, is given by

\[
\begin{bmatrix}
  u(x) \\
  v(x)
\end{bmatrix}
_{\text{h,R}} = \begin{bmatrix}
  A_R^h(E)e^{-i\alpha_p(x-\ell)} \\
  e^{-i\alpha_n(x-\ell)} + N_R^h(E)e^{i\alpha_n(x-\ell)}
\end{bmatrix}.
\]  

(A.2)

(Note that equations (A.1) and (A.2) correspond to a particle or hole respectively, incident on the right interface, reflected as a particle or hole.) By definition of the R-matrix, we find that, in the lead R, the amplitudes corresponding to the solution in equation (A.1) are given by

\[
\begin{align*}
A_1^+ &= \begin{bmatrix}
  R_{1,1} + R_{1,2}N_R^p(E) + R_{1,4}A_R^p(E) \\
  R_{2,1} + R_{2,2}N_R^p(E) + R_{2,4}A_R^p(E)
\end{bmatrix}, \\
A_2^+ &= \begin{bmatrix}
  R_{3,1} + R_{3,2}N_R^p(E) + R_{3,4}A_R^p(E) \\
  R_{4,1} + R_{4,2}N_R^p(E) + R_{4,4}A_R^p(E)
\end{bmatrix}, \\
A_3^+ &= \begin{bmatrix}
  R_{1,1} + R_{1,2}N_R^h(E) + R_{1,4}A_R^h(E) \\
  R_{2,1} + R_{2,2}N_R^h(E) + R_{2,4}A_R^h(E)
\end{bmatrix}, \\
A_4^+ &= \begin{bmatrix}
  R_{3,1} + R_{3,2}N_R^h(E) + R_{3,4}A_R^h(E) \\
  R_{4,1} + R_{4,2}N_R^h(E) + R_{4,4}A_R^h(E)
\end{bmatrix},
\end{align*}
\]

(A.3)

while the ones corresponding to the solution in equation (A.2) are given by

\[
\begin{align*}
A_1^- &= \begin{bmatrix}
  R_{1,2}A_R^h(E) + R_{1,3} + R_{1,4}N_R^h(E) \\
  R_{2,2}A_R^h(E) + R_{2,3} + R_{2,4}N_R^h(E)
\end{bmatrix}, \\
A_2^- &= \begin{bmatrix}
  R_{3,2}A_R^h(E) + R_{3,3} + R_{3,4}N_R^h(E) \\
  R_{4,2}A_R^h(E) + R_{4,3} + R_{4,4}N_R^h(E)
\end{bmatrix}, \\
A_3^- &= \begin{bmatrix}
  R_{1,2}A_R^h(E) + R_{1,3} + R_{1,4}N_R^h(E) \\
  R_{2,2}A_R^h(E) + R_{2,3} + R_{2,4}N_R^h(E)
\end{bmatrix}, \\
A_4^- &= \begin{bmatrix}
  R_{3,2}A_R^h(E) + R_{3,3} + R_{3,4}N_R^h(E) \\
  R_{4,2}A_R^h(E) + R_{4,3} + R_{4,4}N_R^h(E)
\end{bmatrix},
\end{align*}
\]

(A.4)

As we are eventually interested in computing \(I^{(0)}[\chi]\) with the formula in equation (3), we focus on solutions with energy \(|E| < \Delta\). Clearly, acceptable solutions in \(R\) must not contain terms exponentially growing as \(x \to \infty\). According to equation (6), this implies \(A_2^+ = A_4^+ = 0\). The corresponding relations, derived from equations (A.3) and (A.4), allow...
The Josephson current through a long quantum wire

for fully determining \( N^p_R(E), A^p_R(E) \) and \( N^h_R(E), A^h_R(E) \), respectively. As a result, one obtains equation (32).

A similar analysis applies to the left-hand S–N interface. By definition, the normal and Andreev reflection amplitudes for a particle-like solution of the BdG equations are defined by considering a solution that, within \( C \), is given by

\[
\begin{bmatrix}
    u(x) \\
    v(x)
\end{bmatrix}_{p,L} = \begin{bmatrix}
    N^p_L(E)e^{i\alpha_p x} + e^{-i\alpha_p x} \\
    A^p_L(E)e^{-i\alpha_n x}
\end{bmatrix}.
\] (A.5)

Similarly, the normal and Andreev reflection amplitudes for a hole-like solution of the BdG equations are defined by considering a solution that, within \( C \), is given by

\[
\begin{bmatrix}
    u(x) \\
    v(x)
\end{bmatrix}_{h,L} = \begin{bmatrix}
    A^h_L(E)e^{i\alpha_h x} \\
    N^h_L(E)e^{-i\alpha_n x} + e^{i\alpha_n x}
\end{bmatrix}.
\] (A.6)

To compute \( I^{(0)}[\chi] \) one needs the reflection amplitudes at the Fermi level. Thus, one has to consider solutions of the BdG equations like the ones in equations (A.5) and (A.6) at energy \( |E| < \Delta \). Within \( L \), these solutions must contain no exponentially growing terms. Thus, we must set \( A_2^-, A_4^- = 0 \) in equation (5), getting

\[
\begin{bmatrix}
    N^p_L(E) \\
    1 \\
    A^p_L(E) \\
    0
\end{bmatrix} = \begin{bmatrix}
    L_{1,2} A_2^- + L_{1,4} A_4^- \\
    L_{2,2} A_2^- + L_{2,4} A_4^- \\
    L_{3,2} A_2^- + L_{3,4} A_4^- \\
    L_{4,2} A_2^- + L_{4,4} A_4^-
\end{bmatrix},
\] (A.7)

and

\[
\begin{bmatrix}
    A^h_L(E) \\
    0 \\
    N^h_L(E) \\
    1
\end{bmatrix} = \begin{bmatrix}
    L_{1,2} A_2^- + L_{1,4} A_4^- \\
    L_{2,2} A_2^- + L_{2,4} A_4^- \\
    L_{3,2} A_2^- + L_{3,4} A_4^- \\
    L_{4,2} A_2^- + L_{4,4} A_4^-
\end{bmatrix}.
\] (A.8)

Getting rid of \( A_2^-, A_4^- \), from equations (A.7) and (A.8) one eventually obtains equation (33) for \( N^p_L(E) \) and \( A^p_L(E) \).

Appendix B. Derivation of the normal and Andreev reflection amplitudes in the examples of section 4

We now outline the calculation of the reflection amplitudes used to compute \( I^{(0)}[\chi] \) for the two-interface continuum model and for the lattice model studied in section 4.

The calculation of the reflection amplitudes in the continuum model can be performed within the framework of the BdG equations, as discussed in [11]. To be specific, let us focus on the right-hand S–N interface. For the model we consider in section 4, this is described by assuming a coordinate-dependent gap function \( \tilde{\Delta}(x) \) and a potential \( \tilde{V}(x) \) given by

\[
\tilde{\Delta}(x) = \Delta e^{-(i/2)\theta(x-\ell)}, \quad \tilde{V}(x) = V_0 \delta(x-\ell) + V_C \theta(\ell-x).
\] (B.1)
The wavefunctions $u(x), v(x)$ for a state with energy $E$ must solve the time-independent BdG equations

\[
Eu(x) = \left[ -\frac{1}{2m} \frac{d^2}{dx^2} - \mu + \tilde{V}(x) \right] u(x) + \tilde{\Delta}(x)v(x), \\
Ev(x) = \tilde{\Delta}^*(x)u(x) - \left[ -\frac{1}{2m} \frac{d^2}{dx^2} - \mu + \tilde{V}(x) \right] v(x).
\]  

To compute $N^p_R(E), A^p_R(E)$ we require $u(x), v(x)$ to obey boundary conditions such that, within $C$, they are given by equation (A.1), with $\alpha_p = \sqrt{2m(E + \mu - V_C)}$ and $\alpha_h = \sqrt{2m(-E + \mu - V_C)}$, while, within $R$, they are given by equation (6), with $B_2 = B_4 = 0$. In order to obey equation (B.2), at the interface $u(x), v(x)$ must satisfy the continuity condition

\[
\begin{bmatrix}
  u(\ell^-) \\
  v(\ell^-)
\end{bmatrix} = \begin{bmatrix}
  u(\ell^+) \\
  v(\ell^+)
\end{bmatrix},
\]

and, in addition, the discontinuity in the derivatives must cancel the term due to the localized normal scattering potential at the interface, that is

\[
\begin{bmatrix}
  u'(\ell^-) \\
  v'(\ell^-)
\end{bmatrix} - \begin{bmatrix}
  u'(\ell^+) \\
  v'(\ell^+)
\end{bmatrix} = -Z \begin{bmatrix}
  u(\ell) \\
  v(\ell)
\end{bmatrix},
\]

with $Z = 2mV_0$. Equations (B.3) and (B.4) provide a set of four algebraic equations in the four unknowns $B_1, B_3, N^p_R(E), A^p_R(E)$. Getting rid of $B_1, B_3$, one obtains a set of two algebraic equations in the unknowns $N^p_R(E), A^p_R(E)$. Solving the resulting equations for $E = 0$, one obtains the formulas we give in equations (37) and (38). The amplitudes $N^h_R(E), A^h_R(E)$ are obtained following the same procedure, but using the formula in equation (A.2) for the solution within $C$. In the same way, looking for solutions of equation (B.2), but now with $\tilde{\Delta}(x) = \Delta e^{i(\chi/2)}\theta(-x)$ and $\tilde{V}(x) = V_0\delta(x) + V_C\theta(x)$, one obtains the particle-like and hole-like reflection amplitudes at the left-hand interface.

To derive the normal and Andreev reflection amplitudes in the lattice model, one may follows exactly the same procedure as discussed above for the continuum model, except that, now, the lattice BdG equations in equation (40) must be used. For instance, in order to compute $N^p_R(E), A^p_R(E)$ in the lattice model, one considers equation (40) with the hopping given by

\[
\tau_{j,j+1} = \begin{cases}
  J & \text{for } j \leq \ell - 2, \\
  t_S & \text{for } j \geq \ell, \\
  t'' & \text{for } j = \ell - 1,
\end{cases}
\]

the superconducting gap given by

\[
\Delta_j = \begin{cases}
  0 & \text{for } j \leq \ell - 1, \\
  \Delta e^{i(\chi/2)} & \text{for } j \geq \ell,
\end{cases}
\]

\[\text{doi:10.1088/1742-5468/2013/02/P02034} \]
and the potential by

$$V_j = \begin{cases} V_C & \text{(for } j \leq \ell - 1), \\ 0 & \text{(otherwise)}. \end{cases}$$  \hfill (B.7)

For $j \leq \ell - 1$, the solution with energy $E$ obeying the appropriate boundary condition is given by

$$\begin{bmatrix} u_j \\ v_j \end{bmatrix}_p,R = \begin{bmatrix} e^{\alpha_p(j-\ell)} + N^pREE^{-\alpha_p(j-\ell)} \\ A^pREE^{-\alpha_p(j-\ell)} \end{bmatrix},$$  \hfill (B.8)

with $\alpha_p, \alpha_h$ given in equation (45). Within $R$, one instead obtains

$$\begin{bmatrix} u_j \\ v_j \end{bmatrix}_S = \begin{bmatrix} \cos \left( \frac{\Psi}{2} \right) & -e^{(i/2)x} \sin \left( \frac{\Psi}{2} \right) \\ -e^{(i/2)x} \sin \left( \frac{\Psi}{2} \right) & \cos \left( \frac{\Psi}{2} \right) \end{bmatrix} \begin{bmatrix} -e^{-(i/2)x} \sin \left( \frac{\Psi}{2} \right) \\ \cos \left( \frac{\Psi}{2} \right) \end{bmatrix} e^{-i\beta_h(j-\ell)},$$  \hfill (B.9)

with $\beta_p, \beta_h$ given in equation (44) and

$$\cos(\Psi) = \frac{2t_S \cos(\beta_p) + \mu}{E} = -\frac{2t_S \cos(\beta_h) - \mu}{E}, \quad \sin(\Psi) = \frac{\Delta}{E}. \hfill (B.10)$$

As discussed in detail in [9], in the lattice-interface model described by equations (40), (B.5), (B.6) and (B.7), the matching conditions at the interface, given in equations (B.3) and (B.4) for the continuum model, are just substituted by the lattice equations for $u_{\ell-1}, v_{\ell-1}$ and $u_\ell, v_\ell$. Requiring the solution in equations (B.8) and (B.9) to satisfy equation (40) for $u_{\ell-1}, v_{\ell-1}$ and $u_\ell, v_\ell$ and getting rid of $B_1, B_3$, one obtains a set of equations for $N^p_R, A^p_R$ whose solutions, at the Fermi level, provide us with the result in equations (48) and (49). Following the same procedure, after substituting $\begin{bmatrix} u_j \\ v_j \end{bmatrix}_p,R$ in equation (B.8) with

$$\begin{bmatrix} u_j \\ v_j \end{bmatrix}_h,R = \begin{bmatrix} A^h_REE^{-\alpha_p(j-\ell)} \\ e^{-\alpha_h(j-\ell)} + N^h_REE^{\alpha_h(j-\ell)} \end{bmatrix},$$  \hfill (B.11)

one obtains $N^h_R, A^h_R$. Similarly, one readily obtains $N^p_L, A^p_L$ and $N^h_L, A^h_L$ as well.

**Appendix C. Derivation of equation (7) and of equation (10)**

In this appendix, we outline the derivation of equation (7), which is crucial for our proof. To do so, we have to go through the derivation of equation (10), which gives the necessary condition to be satisfied by solutions of BdG equations in a one-dimensional box defined by $-L/2 < x < \ell + L/2$. Thus, we will also outline the procedure for deriving the formulas for the coefficients $\mu_{n,m}$. The starting point is noticing that a solution of the BdG equations in the one-dimensional box, $\begin{bmatrix} u(x) \\ v(x) \end{bmatrix}$, must obey vanishing boundary conditions at both boundaries of the box, that is

$$\begin{bmatrix} u(-L/2) \\ v(-L/2) \end{bmatrix} = \begin{bmatrix} u(L/2) \\ v(L/2) \end{bmatrix} = 0.$$  Clearly, $\begin{bmatrix} u(x) \\ v(x) \end{bmatrix}$ takes the form

$$\begin{bmatrix} u(x) \\ v(x) \end{bmatrix} = \begin{bmatrix} u(x) \\ v(x) \end{bmatrix}.$$  

doi:10.1088/1742-5468/2013/02/P02034
given in equation (5) for $x \rightarrow -L/2$ and the form given in equation (6) for $x \rightarrow \ell + L/2$. Thus, vanishing boundary conditions imply the system of algebraic equations
\begin{equation}
\cos(\Psi/2)[A_1^+ e^{(i/2)\beta_p L} + A_1^- e^{(i/2)\beta_p L}] - e^{i\chi/2} \sin(\Psi/2)[A_3^- e^{(i/2)\beta_h L} + A_4^- e^{-(i/2)\beta_p L}] = 0,
\end{equation}
\begin{equation}
- e^{i\chi/2} \sin(\Psi/2)[A_1^+ e^{(i/2)\beta_p L} + A_2^- e^{(i/2)\beta_p L}] \\
+ \cos(\Psi/2)[A_3^- e^{(i/2)\beta_h L} + A_4^- e^{-(i/2)\beta_p L}] = 0,
\end{equation}
and
\begin{equation}
\cos(\Psi/2)[A_1^+ e^{(i/2)\beta_p L} + A_3^- e^{-(i/2)\beta_h L}] - e^{-i\chi/2} \sin(\Psi/2) \\
\times [A_3^+ e^{-(i/2)\beta_h L} + A_4^+ e^{(i/2)\beta_p L}] = 0,
\end{equation}
\begin{equation}
- e^{i\chi/2} \sin(\Psi/2)[A_1^+ e^{(i/2)\beta_p L} + A_2^+ e^{-(i/2)\beta_h L}] + \cos(\Psi/2) \\
\times [A_3^+ e^{-(i/2)\beta_h L} + A_4^+ e^{(i/2)\beta_p L}] = 0.
\end{equation}

By definition of the transmission matrix $M$, one finds that $A_1^+, A_2^+, A_3^+, A_4^+$ are related to $A_1^-, A_2^-, A_3^-, A_4^-$ by $\bar{A}^+ = M \bar{A}^-$. Using this last equation to express $A_1^+, A_2^+, A_3^+, A_4^+$ in terms of $A_1^-, A_2^-, A_3^-, A_4^-$, the system given by equations (C.1) and (C.2) can be traded for a homogeneous $4 \times 4$ system in the unknowns $A_1^-, A_2^-, A_3^-, A_4^-$, given by the equations
\begin{equation}
e^{-(i/2)\beta_p L} A_1^- + e^{(i/2)\beta_p L} A_2^- = 0, \quad e^{(i/2)\beta_h L} A_3^- + e^{-(i/2)\beta_h L} A_4^- = 0,
\end{equation}
and
\begin{equation}
\{ e^{(i/2)\beta_p L} M_{1,1} + e^{-(i/2)\beta_p L} M_{2,1} \} A_1^- + \{ e^{(i/2)\beta_p L} M_{1,2} + e^{-(i/2)\beta_p L} M_{2,2} \} A_2^- \\
+ \{ e^{(i/2)\beta_p L} M_{1,3} + e^{-(i/2)\beta_p L} M_{2,3} \} A_3^- \\
+ \{ e^{(i/2)\beta_p L} M_{1,4} + e^{-(i/2)\beta_p L} M_{2,4} \} A_4^- = 0,
\end{equation}
\begin{equation}
\{ e^{-(i/2)\beta_h L} M_{3,1} + e^{(i/2)\beta_h L} M_{4,1} \} A_1^- + \{ e^{-(i/2)\beta_h L} M_{3,2} + e^{(i/2)\beta_h L} M_{4,2} \} A_2^- \\
+ \{ e^{-(i/2)\beta_h L} M_{3,3} + e^{(i/2)\beta_h L} M_{4,3} \} A_3^- \\
+ \{ e^{-(i/2)\beta_h L} M_{3,4} + e^{(i/2)\beta_h L} M_{4,4} \} A_4^- = 0.
\end{equation}

Equations (C.3) and (C.4) constitute a system of the form $\mathcal{M} \bar{A}^- = 0$, with the matrix $\mathcal{M}$ being a known function of the energy $E$ and the transmission matrix elements. Nonzero solutions for $\bar{A}^-$ are then only obtained if $\det[\mathcal{M}] = 0$. This latter condition yields equation (10) and explicitly defines the coefficients $\mu_{n,m}$. In particular, by direct calculation one readily obtains
\begin{equation}
\frac{\mu_{1,-1}}{\mu_{-1,1}} = \frac{M_{1,1} M_{3,3} - M_{1,3} M_{3,1}}{M_{2,2} M_{4,4} - M_{2,4} M_{4,2}}.
\end{equation}

In order to explicitly show that $\det[S]$ is equal to the right-hand side of equation (C.5), we now perform a different manipulation on the system of equations (C.1) and (C.2). In particular, we use the definition of the $S$-matrix,
\begin{equation}
\begin{bmatrix}
\sqrt{\nu_p} A_1^+ \\
\sqrt{\nu_p} A_2^- \\
\sqrt{\nu_h} A_3^+ \\
\sqrt{\nu_h} A_4^-
\end{bmatrix}
= S \begin{bmatrix}
\sqrt{\nu_p} A_1^- \\
\sqrt{\nu_p} A_2^+ \\
\sqrt{\nu_h} A_3^- \\
\sqrt{\nu_h} A_4^+
\end{bmatrix},
\end{equation}
\text{do}\textit{i}:10.1088/1742-5468/2013/02/P02034
The Josephson current through a long quantum wire

with $v_p/h = |dE/d\beta_p/h|$, to express $A_1^+, A_2^-, A_3^+, A_4^- \in$ terms of $A_1^-, A_2^+, A_3^-, A_4^+$. As a result, one obtains an algebraic homogeneous system of the form

$$
\mathcal{M}' \begin{bmatrix}
A_1^- \\
A_2^+ \\
A_3^- \\
A_4^+
\end{bmatrix} = 0.
$$

(C.7)

Equation (C.7) takes nonzero solutions only if

$$
\det[\mathcal{M}'] = 0 = \sum_{n,m=\pm 1} \nu_{n,m} e^{i[n\beta_p+m\beta_h]L} + \nu_{0,0}.
$$

(C.8)

Comparing equation (C.8) with equation (10), one sees that the $\nu_{n,m}$s must be equal to the $\mu_{n,m}$s, apart from an over all multiplicative constant. In particular, this implies

$$
\frac{\mu_{1,-1}}{\mu_{-1,1}} = \frac{\nu_{1,-1}}{\nu_{-1,1}} \Rightarrow \det[S] = \frac{M_{1,1}M_{3,3} - M_{1,3}M_{3,1}}{M_{2,2}M_{4,4} - M_{2,4}M_{4,2}},
$$

(C.9)

that is, equation (7).

References

[1] Josephson D B, 1962 Phys. Lett. 1 251
[2] de Gennes P-G, 1966 Superconductivity of Metals and Alloys (New York: Benjamin)
Anderson P W, 1963 Ravello Lectures on the Many-Body Problem ed E R Gianello (New York: Academic)
[3] Andreev A F, 1964 Sov. Phys.—JETP 19 1228
[4] Kulik I O, 1970 Sov. Phys.—JETP 30 944
[5] Ishii C, 1970 Prog. Theor. Phys. 44 1525
[6] Bardeen J and Johnson J L, 1972 Phys. Rev. B 5 72
[7] Beenakker C W J, 1991 Phys. Rev. Lett. 67 3836
Beenakker C W J, 1992 Transport Phenomena in Mesoscopic Systems ed H Fukuyama and T Ando (Berlin: Springer) arXiv:cond-mat/0406127
[8] Furusaki A and Tsukada M, 1991 Solid State Commun. 78 299
[9] Affleck I, Caux J-S and Zagoskin A M, 2000 Phys. Rev. B 62 1433
[10] Perfetto E, Stefanucci G and Cini M, 2009 Phys. Rev. B 80 205408
[11] Blonder G E, Tinkham M and Klapwijk T M, 1982 Phys. Rev. 25 4515
[12] Titov M, Muller M and Belzig W, 2006 Phys. Rev. Lett. 97 257006
[13] Giuliano D and Affleck I, in progress
[14] Hagymási I, Kormányos S and Cserti J, 2010 Phys. Rev. B 82 134516
[15] Likharev K K, 1979 Rev. Mod. Phys. 51 101

doi:10.1088/1742-5468/2013/02/P02034