On posterior distribution of Bayesian wavelet thresholding

Heng Lian

Division of Mathematical Sciences, SPMS,
Nanyang Technological University,
Singapore 637616

Abstract

We investigate the posterior rate of convergence for wavelet shrinkage using a Bayesian approach in general Besov spaces. Instead of studying the Bayesian estimator related to a particular loss function, we focus on the posterior distribution itself from a nonparametric Bayesian asymptotics point of view and study its rate of convergence. We obtain the same rate as in Abramovich et al. (2004) where the authors studied the convergence of several Bayesian estimators.

Key words: Besov spaces, Infinite-dimensional Bayesian procedure, Wavelet thresholding

1 Introduction

Infinite-dimensional Bayesian methods have become quite popular recently, due to both the computational and theoretical advances in this field. There are many results concerning posterior convergence using appropriate priors. These developments originate from the consideration of density estimation problems. In these problems, given the prior $\Pi_n$ on the set $\mathcal{P}$ of probability distributions, the posterior is a random measure:

$$
\Pi_n(B|X_1,\ldots,X_n) = \frac{\int_B \Pi^n p(X_i)d\Pi_n(P)}{\int \Pi^n p(X_i)d\Pi_n(P)}
$$

(1)

We say that the posterior is consistent if

Email address: henglian@ntu.edu.sg (Heng Lian).
\[ \Pi_n(P \in \mathcal{P} : d(P, P_0) > \epsilon |X) \to 0 \text{ in } P_0^n \text{ probability}. \]

where \( P_0 \) is the true distribution and \( d \) is some suitable distance function between probability measures.

To study rates of convergence, let \( \epsilon_n \) be a sequence decreasing to zero, we say the rate is at least \( \epsilon_n \) if for sufficiently large constant \( M \)

\[ \Pi_n(P : d(P, P_0) \geq M \epsilon_n |X) \to 0 \text{ in } P_0^n \text{ probability}. \]

It turns out the convergence rates are closely related to the existence of tests that separate the hypotheses in convex sets.

The most general result appeared recently in Ghosal and Van Der Vaart (2007), where the formulation includes both density estimation and regression problems, and the results also extend to non-iid cases such as stationary and non-stationary sequence of observations. In this general context, the definition of the posterior convergence rate is similar except the measure \( P \) and \( P_0 \) represent the distribution on the data and thus depends on sample size \( n \), and the observations are no longer i.i.d. so the likelihood used in (1) must be changed to a more general form accordingly.

Another relatively recent development in statistics is the investigation of wavelet method which has found numerous applications in engineering as well. There are many theoretical results explaining why wavelet transformation is effective, from both the frequentist and the Bayesian point of view. These well-known results include the now widely celebrated works of David Donoho and his collaborators (Donoho and Johnstone, 1994; Donoho et al., 1996). The property that distinguishes these works from previous results is that a single estimator can achieve the minimax rate over a range of function spaces including functions with inhomogeneous smoothness, whose minimax rate cannot be achieved by the simpler linear estimator. The sparsity of the coefficients for the function in an appropriate basis is the key to the success of the wavelet thresholding approach.

Bayesian approach to function estimation in Besov spaces has been investigated in Abramovich et al. (1998, 2004). In these approaches, after specifying an appropriate prior, the Bayesian estimator is obtained from the posterior and investigated from the frequentist point of view. In particular, they study the rate of convergence of different point estimators including the posterior mean and posterior median as well as other estimators derived from the posterior distribution. The theoretical results in Abramovich et al. (2004) show that some Bayesian estimators can achieve the better-than-linear rates if an appropriate prior is chosen that implicitly implements shrinkage or thresholding rule similar to the frequentist approach.
In a Bayesian framework, most researchers are more interested in the posterior as a distribution, instead of the point estimates derived from specific loss function. The convergence of the posterior distribution in this context has not been studied. This paper intends to fill this gap. Using the same prior as in Abramovich et al. (1998, 2004), we show that the posterior distribution has the same convergence rate as the point estimators proposed in those papers.

We describe the model and present the main theorem in Section 2. Some possible extensions for our result are discussed in the final section.

2 Main result

Consider the white noise model

\[ dX(t) = f(t)dt + \sigma_n dW(t) \]  

(2)

where \( \sigma_n^2 = 1/n \), \( f \in B_{p,q}^s[0, 1] \) and \( W \) is the standard Brownian motion. Using wavelet basis on \([0,1]\) with sufficient regularity, the function \( f \) can be expanded as

\[ f = \sum_{j=0}^{j_0-1} \sum_{k=0}^{2^j-1} \alpha_{j_0,k} \phi_{j_0,k} + \sum_{j\geq j_0} \sum_{k=0}^{2^j-1} \beta_{j,k} \psi_{j,k} \]

where \( \phi_{j_0,k} \) are the scaling functions and \( \psi_{j,k} \) are the mother wavelets at resolution \( j \), and \( j_0 \) is the lowest resolution in the expansion. We assume \( j_0 = 0 \) for simplicity of notation below.

The Besov spaces include the well-known Sobolev and Hölder classes of function and also nearly contains the space of functions of bounded variation. The norm for the Besov space with parameter \( s > \max(0, 1/p - 1/2) \), \( 1 \leq p \leq \infty \), and \( 1 \leq q \leq \infty \) is defined as

\[ \|f\|_{B_{p,q}^s} = \|P_0(f)\|_{L^p} + \left( \sum_{j\geq 0} (2^{js}) \|Q_j(f)\|_{L^p} \right)^{1/q} \]

where \( P_0(f) = \alpha_{00} \phi_{00} \) is the projection of \( f \) on the “approximation space”, and \( Q_j(f) = \sum_{k=0}^{2^j-1} \beta_{j,k} \psi_{j,k} \) is the projection of \( f \) onto the “detail space”.

In terms of the coefficients in the wavelet expansion, the Besov norm can be equivalently defined by
\[ \|f\|_{B^s_{p,q}} \simeq \|\beta\|_{B^s_{p,q}} = |\alpha_{00}| + \left\{ \sum_{j=0}^{\infty} 2^j(s+1/2-1/p)q \|\beta_j\|_p^q \right\}^{1/q} \]

Note that for cases where \( q = \infty \) the usual change to the sup norm is required.

By abuse of notation, we also define \( \beta' \) to be the sequence \( \beta' \) such that \( \beta'_{jk} = \beta_{jk} \) when \( j \leq J \) and \( \beta'_{jk} = 0 \) when \( j > J \).

The white noise model (2) is closely related to the nonparametric regression model (Brown and Low, 1996; Donoho et al, 1995):

\[ Y_i = f\left( \frac{i}{n} \right) + z_i \]

with standard normal noise. We choose to work with (2) for its simplicity of formulation.

After wavelet transformation for (2), we get the Gaussian sequence model:

\[
\begin{align*}
X_{00} &= \alpha_{00}^0 + z_{00}/\sqrt{n} \\
X_{jk} &= \beta_{jk}^0 + z_{jk}/\sqrt{n}, j \geq 0, k = 0, 1, \ldots, 2^j - 1
\end{align*}
\]

where the superscript 0 indicates the true parameter.

Using Bayesian approach for Gaussian sequence estimation, we put a prior on \( \beta_{jk}^0 \):

\[ \beta_{jk}^0 \sim \pi_j N(0, \alpha_j^2) + (1 - \pi_j)\delta_0 \] (3)

with hyperparameters \( \alpha_j^2 \simeq 2^{-\alpha j}, \pi_j \simeq 2^{-\gamma j} \), for some \( \alpha > 1 \), and \( \gamma > 0 \). This prior is proposed in Abramovich et al (1998), and Abramovich et al. (2004) investigated the optimality of some Bayesian estimators with this prior. The choice of the hyperparameters must satisfy some conditions for the prior to put positive mass on \( B^s_{p,q} \) (Abramovich et al, 1998, Theorem 1), although this is not our focus here. In the following we assume \( \alpha \) and \( \gamma \) satisfy these conditions. We also assume the value \( \alpha_{00}^0 \) is known for simplicity, which does not affect our asymptotic result.

We also consider the sieve prior as in Shen and Wasserman (2001), and define the prior \( \Pi_n \) by \( \Pi_n(A) = \sum_m \lambda_m \Pi_n^m(A) \) where \( \lambda_m \propto 2^{-\mu m} \) for some \( \mu > 0 \), and \( \Pi_n^m \) is a prior on \( \beta_{jk} \) such that \( \beta_{jk} \sim N(0, 2^{-\alpha j}) \) when \( j \leq m \) and \( \beta_{jk} = 0 \) when \( j > m \).

The main result we obtain in this paper is the following:
Theorem 1 Consider a bounded subset of the Besov space: \( B_{p,q}^s(B) = \{ \beta \in B_{p,q}^s[0,1], ||\beta||_{B_{p,q}^s} < B \} \) and \( \beta^0 \in B_{p,q}^s(B) \). Let \( \alpha = 2s + 1 \) for \( p \geq 2 \), and \( \alpha = (2s + 2 - 2/p) \) for \( 1 \leq p < 2 \). Then for sufficiently large constant \( M \), under the prior (3), we have

\[
\Pi_n \left( \sum_{j,k} (\beta_{jk} - \beta_{jk}^0)^2 > M \epsilon_n^2 | X_{jk} \right) \rightarrow 0 \text{ in probability},
\]

where \( \epsilon_n^2 = (\log n)^2 n^{-2s/(2s+1)} \) when \( p \geq 2 \), and \( \epsilon_n^2 = (\log n)^2 n^{-(2s+1-2/p)/(2s+2-2/p)} \) when \( 1 \leq p < 2 \).

Remark: The above rate of convergence is the same as in Abramo vich et al. (2004) for posterior mean and posterior median, except an extra log factor in our case, which we think might be an artifact of our proofs.

Proof of Theorem 1. In the proof, we use \( C \) to denote generic constant whose value can change in difference locations. We make use of the general result for Bayesian posterior rate of convergence (Theorem 6 in Ghosal and Van Der Vaart (2007)), although we only use a simpler version which corresponds to Theorem 2.1 in Ghosal et al. (2000) in the iid case. Two conditions for the theorems must be verified:

(I) \( \log D(\epsilon_n, B_{p,q}^s(B), ||-||_2) \leq n \epsilon_n^2 \), where \( D(\epsilon, F, ||-||) \) is the \( \epsilon \)-covering number of the space \( F \) with norm \( ||-|| \).

(II) \( \Pi_n^B(\beta \in B_{p,q}^s(B) : ||\beta - \beta^0||_2 \leq \epsilon_n^2) \geq \exp\{-CN \epsilon_n^2\} \), where \( \Pi_n^B \) denote the prior distribution as in (3) constrained on \( B_{p,q}^s(B) \) by renormalization.

Corollary 2 in Nickl and Potscher (2007) gives the bracketing entropy number for Besov spaces as \( H_B(\epsilon, B_{p,q}^s(B), ||-||_2) \approx \epsilon^{-1/s} \). Since bracketing entropy number is an upper bound for usual entropy, \( \epsilon_n \) defined in the statement of the theorem obviously satisfies condition (I).

Condition (II) is verified as follows:

Since \( \beta^0 \in B_{p,q}^s(B) \), there exists \( \delta \) such that \( ||\beta^0||_{B_{p,q}^s} \leq B - \delta \). Let \( J = (\log n)/\alpha \). We have

\[
\Pi_n^B(||\beta - \beta^0||_2^2 \leq \epsilon_n^2) \\
\geq \Pi_n(||\beta - \beta^0||_2^2 \leq \epsilon_n^2, ||\beta||_{B_{p,q}^s} < B) \\
\geq \Pi_n\left( \sum_{j=0}^J \sum_k (\beta_{jk} - \beta_{jk}^0)^2 \leq \epsilon_n^2/2, |P_{\beta^0}||_{B_{p,q}^s} < B - \delta/2 \right) .
\]
\[ \Pi_n \left( \sum_{j=J+1}^{\infty} \sum_k (\beta_{jk} - \beta_{jk}^0)^2 \leq \epsilon_n^2 / 2, ||\beta - P_J \beta||_{B_{p,q}} < \delta / 2 \right) \]

The above two terms are dealt with in the following two lemmas, which provide a lower bound of \( e^{-Cn\epsilon_n^2} \) and the theorem is proved. \( \square \)

**Lemma 1** \( \Pi_n(\sum_{j=J+1}^{\infty} \sum_k (\beta_{jk} - \beta_{jk}^0)^2 \leq \epsilon_n^2 / 2, ||\beta - P_J \beta||_{B_{p,q}} < \delta / 2) \) is bounded away from 0.

**Proof.** Since \( \sum_{j>J} \sum_k (\beta_{jk}^0)^2 \leq \sum_{j>J} C^2 s^2 \leq \epsilon_n^2 / 8 \), where \( s' = s \) for \( p \geq 2 \) and \( s' = s + 1 / 2 - 1 / p \) otherwise, we have

\[
\Pi_n \left( \sum_{j=J+1}^{\infty} \sum_k (\beta_{jk} - \beta_{jk}^0)^2 \leq \epsilon_n^2 / 2 \right) \\
\geq \Pi_n \left( \sum_{j>J,k} \beta_{jk}^2 \leq \epsilon_n^2 / 8 \right) \\
\geq 1 - 8E[ \sum_{j>J,k} \beta_{jk}^2 ] / \epsilon_n^2 \\
\geq 1 - C \cdot 2^{-(\alpha - 1)j} / \epsilon_n^2 \\
\to 1
\]

On the other hand, \( \Pi_n(||\beta - P_J \beta||_{B_{p,q}} < \delta / 2) \geq \Pi_n(||\beta||_{B_{p,q}} < \delta / 2) =: t > 0 \) when \( \alpha \) and \( \beta \) are chose appropriately such that \( \Pi_n(B_{p,q}) > 0 \) (this is possible by Abramovich et al. (1998)).

Thus \( \Pi_n(\sum_{j=J+1}^{\infty} \sum_k (\beta_{jk} - \beta_{jk}^0)^2 \leq \epsilon_n^2 / 2, ||\beta - P_J \beta||_{B_{p,q}} < \delta / 2) \to t > 0 \) as \( n \to \infty \). \( \square \)

**Lemma 2** \( \Pi_n(\sum_{j=0}^{J} \sum_k (\beta_{jk} - \beta_{jk}^0)^2 \leq \epsilon_n^2 / 2, ||P_J \beta||_{B_{p,q}} < B - \delta / 2) \geq e^{-Cn\epsilon_n^2} \)

**Proof.** This probability can be bounded from below using the techniques in Section 5 of Shen and Wasserman (2001).

First we show

\[
\Pi_n(\sum_{j=0}^{J} \sum_k (\beta_{jk} - \beta_{jk}^0)^2 \leq \epsilon_n^2 / 2, ||P_J \beta||_{B_{p,q}} < B - \delta / 2) \\
\geq \Pi_n(\sum_{j=0}^{J} \sum_k (\beta_{jk} - \beta_{jk}^0)^2 \leq c^2 \tau_n^2 / \log n) \\
\geq e^{-c^2 \tau_n^2 / \log n})
\]

for a small enough constant \( c \), where \( \tau_n = n^{-(s+1/2-1/p)/(2s+1)} \) when \( p \geq 2 \) and \( \tau_n = n^{-s/(2s+2-2/p)} \) when \( 1 \leq p < 2 \). Notice we obviously have \( \tau_n = O(\epsilon_n) \).
Case 1: \( p \geq 2 \).

Note \( ||\beta_j||_p \leq ||\beta_j||_2 \) when \( p \geq 2 \). Conditioned on the event \( \sum_{j=0}^{J} \sum_{k} (\beta_{jk} - \beta_{jk}^0)^2 \leq c^2 \tau^2_n / \log n \), the \( B_{p,q}^s \) norm for \( P_j \beta - P_j \beta^0 \) can be bounded as follows:

\[
||P_j \beta - P_j \beta^0||_{B_{p,q}^s} \\
= \left( \sum_{j=0}^{J} 2^{j(1/2-1/p)q} ||\beta_j - \beta_j^0||_{p}^{1/q} \right) \\
\leq \left( \sum_{j=0}^{J} 2^{j(1/2-1/p)q} ||\beta_j - \beta_j^0||_{2}^{1/q} \right) \\
\leq 2^{J(s+1/2-1/p)} \left( \sum_{j=0}^{J} ||\beta_j - \beta_j^0||_{2}^{1/q} \right) \\
\leq 2^{J(s+1/2-1/p)} J^{\max(1/q-1/2,0)} ||P_j \beta - P_j \beta^0||_2 \\
\leq n^{(s+1/2-1/p)/(2s+1)} \cdot c \tau_n
\]

Case 2: \( 1 \leq p < 2 \).

Since \( ||\beta_j||_p \leq 2^{1/(p-1/2)} ||\beta_j||_2 \) when \( 1 \leq p < 2 \), the \( B_{p,q}^s \) norm for \( P_j \beta - P_j \beta^0 \) can be bounded as follows:

\[
||P_j \beta - P_j \beta^0||_{B_{p,q}^s} \\
\leq \left( \sum_{j=0}^{J} 2^{j(1/2-1/p)q} ||\beta_j - \beta_j^0||_{p}^{1/q} \right) \\
\leq \left( \sum_{j=0}^{J} 2^{jsq} ||\beta_j - \beta_j^0||_{2}^{1/q} \right) \\
\leq 2^{Js} \left( \sum_{j=0}^{J} ||\beta_j - \beta_j^0||_{2}^{1/q} \right) \\
\leq 2^{Js} J^{\max(1/q-1/2,0)} ||P_j \beta - P_j \beta^0||_2 \\
\leq n^{s/(2s+2-2/p)} \cdot c \tau_n
\]

Summarizing the above two cases, \( ||P_j \beta - P_j \beta^0||_{B_{p,q}^s} \) will be less than \( \delta/2 \) when \( c \) is sufficiently small, and \([\Pi]\) is proved by noticing \( ||P_j \beta||_{B_{p,q}^s} \leq ||P_j \beta - P_j \beta^0||_{B_{p,q}^s} + ||\beta^0||_{B_{p,q}^s} < B - \delta/2 \)

What is left is to lower bound \( \Pi_n (\sum_{j=0}^{J} \sum_{k} (\beta_{jk} - \beta_{jk}^0)^2 \leq c^2 \tau^2_n / \log n) \)

Obviously the above prior probability is smallest when \( \pi_j = 1 \) and thus the prior is a normal distribution. Let \( \delta_n^2 = c^2 \tau^2_n / \log n \) for simplicity of notation.
If we denote by $K = \sum_{j=0}^{J} j 2^j \approx (\log_2 n)^{n/\alpha}$ the total number of variables $\beta_{jk}$, with $0 < j \leq J$, and let $\Delta = \exp\{-\sum_{j=0}^{J} \sum_{k} 2^{\alpha j} (\beta_{jk}^0)^2\}$, $A = \{w_{jk} : 0 \leq j \leq J, 0 \leq k \leq 2^j - 1, ||w||_2^2 \leq \delta_n^2\}$, we have

\[
\Pi_n(\sum_{j=0}^{J} \sum_{k} (\beta_{jk} - \beta_{jk}^0)^2 \leq \delta_n^2) = \frac{1}{2\pi} \left(\frac{\pi}{\sqrt{\Delta}}\right)^{K/2} \prod_{j=0}^{J} (2^{\alpha j/2})^{2j} \int_A \exp\{-\frac{1}{2} \sum_{j,k} 2^{\alpha j}(w_{jk} + \beta_{jk}^0)^2\} \cdot \exp\{-\sum_{j,k} 2^{\alpha j}(w_{jk})^2\} \cdot \frac{\delta_n^K \cdot \pi^{K/2}}{\Gamma(K/2)} \int_0^1 u^{(K/2)-1} \exp\{-2^{\alpha J}\delta_n^2 \cdot u\} du \geq \Delta \frac{1}{2\pi} \left(\frac{\pi}{\sqrt{\Delta}}\right)^{K/2} \prod_{j=0}^{J} (2^{\alpha j/2})^{2j} \int_0^1 u^{(K/2)-1} e^{-u} du \geq 2^{-c\alpha J K/F(2^{\alpha J} \delta_n^2; K/2)} \geq e^{-c\alpha J \delta_n^{2}} \geq e^{-c\alpha J \delta_n^{2}}.
\]

In the above we used Lemma 3 in Shen and Wasserman (2001) as well as the inequality $F(b; \alpha) = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \int_0^b x^{\alpha-1} e^{-x} dx \approx e^b e^{-b^\alpha} \alpha^{-\alpha - \alpha/2}$ which also appeared in that paper. □

If we use the sieve prior presented right before Theorem 1, the same conclusion still holds.

**Theorem 2** The result of Theorem 1 is still true with the sieve prior.

**Proof.** The entropy bound for condition (I) is unchanged. With the same $J = \log_2 n/\alpha$, we have

\[
\Pi_n^B(||\beta - \beta^0||_2^2 \leq \epsilon_n^2) \geq \Pi_n(||\beta - \beta^0||_2^2 \leq \epsilon_n^2, ||\beta||_{B_{p,q}} < B) \geq \lambda_J \Pi_n^{J}(\sum_{j=0}^{J} \sum_{k} (\beta_{jk} - \beta_{jk}^0)^2 \leq \epsilon_n^2/2, ||P_J \beta||_{B_{p,q}} < B) \cdot 
\Pi_n^{J}(\sum_{j=J+1}^{\infty} \sum_{k} (\beta_{jk} - \beta_{jk}^0)^2 \leq \epsilon_n^2/2)
\]

In the second probability above the event is actually deterministic since $\beta_{jk} = \cdots$
0 when $j > J$ under the prior $\Pi_n^J$, and $\sum_{j > J, k}(\beta_{jk}^0)^2 \leq \epsilon_n^2/2$. So the probability of this term is 1 and Lemma 1 is not needed.

For the first probability, the lower bound is exactly the same as above. So the lower bound for the prior probability $\Pi_n^B(\|\beta - \beta^0\|_2^2 \leq \epsilon_n^2)$ is bounded below by $\lambda_J e^{-Cn^2 \epsilon_n}$, and $\lambda_J = n^{-\mu/\alpha}$ is obviously ignorable (can be incorporated into the constant $C$ in $e^{-Cn^2 \epsilon_n}$) in this case. □

If we focus on bounded functions only, then we can get the rate of convergence for the posterior mean and posterior median.

**Corollary 1** Consider the case where $\|\beta^0\|_2 < 1$ and the prior on $\beta$ is also renormalized to put mass 1 on the set $\{\beta : \|\beta\|_2 < 1\}$. Denote by $\hat{\beta}$ and $\tilde{\beta}$ the posterior mean and posterior median respectively. We have $\|\hat{\beta} - \beta^0\|_2 = O(\epsilon_n)$ and $\|\tilde{\beta} - \beta^0\|_2 = O(\epsilon_n)$ in probability.

**Proof.** First note that with slight modifications, Theorem 1 and Theorem 2 are still true when the prior is constrained to unit $l^2$ balls.

The result for posterior mean is well-known (Barron et al., 1999; Ghosal et al., 2000) since the $l^2$ loss is bounded under the current assumptions.

For posterior median, since the $l^2$ loss is now bounded and the posterior probability $\Pi_n^B(\|\beta - \beta^0\|_2^2 \geq M\epsilon_n^2 \|X\|)$ converges to zero at least at the order $\epsilon_n^2$ (implicit in the proof of Ghosal et al. (2000), Theorem 2.1), we have $E\|\beta - \beta^0\|_2^2 = O(\epsilon_n^2)$ in probability, where the expectation is over the posterior distribution of $\beta$. Then we use the simple fact that for any random variable $X$, $E[X^2] \leq a^2$ implies $|\text{median}(X)| \leq 2a$. This can be seen by $P(|X| > 2a) \leq E(X^2)/(4a^2) < 1/2$. Now replacing $X$ by $\beta_{jk} - \beta_{jk}^0$, and summing over $j$ and $k$, we get the convergence rate for $\tilde{\beta}$. □

### 3 Discussion

Using the approach of Ghosal et al. (2000), Ghosal and Van Der Vaart (2007), we have investigated the convergence rate of the posterior distribution for Gaussian white noise model in Besov spaces. Investigation of posterior distribution rather than the Bayes estimators seems to be more desirable from a philosophical and practical point of view, since the posterior distribution can be directly utilized to assess the uncertainty of the Bayesian inference. As shown in Abramovich et al. (2004), their Bayes factor estimator can achieve a better rate of convergence (although it is still not optimal within the whole range $1 \leq p < 2$). Using the prior (3) we cannot hope to achieve this rate since it was shown in Abramovich et al. (2004) that the posterior mean can-
not achieve this faster rate and the rate for the posterior distribution is no faster than that of the posterior mean.

The loss function used in this investigation is the simplest $l_2$ loss. The extension to more general $l_p$ norm is left for further research. The derived rate is the same as in Abramovich et al. (2004) up to an extra log term and is suboptimal in the inhomogeneous cases $1 \leq p < 2$. Heavy-tailed distributions like double exponential are successfully used in Johnstone and Silverman (2003) to achieve better rates and it was argued that the implicit thresholding in normal mixture are too heavy on high-resolution levels. We believe optimal rates for posterior distribution are achievable with similar heavy-tailed distributions.

References

Abramovich, F., Amato, U., Angelini, C., 2004. On optimality of bayesian wavelet estimators. Scandinavian Journal of Statistics 31 (2), 217–234.

Abramovich, F., Sapatinas, T., Silverman, B. W., 1998. Wavelet thresholding via a bayesian approach. Journal of the Royal Statistical Society Series B-Statistical Methodology 60, 725–749, part 4.

Barron, A., Schervish, M. J., Wasserman, L., 1999. The consistency of posterior distributions in nonparametric problems. Annals of Statistics 27 (2), 536–561.

Brown, L. D., Low, M. G., 1996. Asymptotic equivalence of nonparametric regression and white noise. Annals of Statistics 24 (6), 2384–2398.

Donoho, D. L., Johnstone, I. M., 1994. Ideal spatial adaptation by wavelet shrinkage. Biometrika 81 (3), 425–455.

Donoho, D. L., Johnstone, I. M., Kerkyacharian, G., Picard, D., 1995. Wavelet shrinkage - asymptopia. Journal of the Royal Statistical Society Series B-Methodological 57 (2), 301–337.

Donoho, D. L., Johnstone, I. M., Kerkyacharian, G., Picard, D., 1996. Density estimation by wavelet thresholding. Annals of Statistics 24 (2), 508–539.

Ghosal, S., Ghosh, J. K., Van der Vaart, A. W., 2000. Convergence rates of posterior distributions. Annals of Statistics 28 (2), 500–531.

Ghosal, S., Van Der Vaart, A., 2007. Convergence rates of posterior distributions for noniid observations. Annals of Statistics 35 (1), 192–223.

Johnstone, I. M., Silverman, B. W., 2005. Empirical bayes selection of wavelet thresholds. Annals of Statistics 33 (4), 1700–1752.

Nickl, R., Potscher, B. M., 2007. Bracketing metric entropy rates and empirical central limit theorems for function classes of besov- and sobolev-type. Journal of Theoretical Probability 20 (2), 177–199.

Shen, X. T., Wasserman, L., 2001. Rates of convergence of posterior distributions. Annals of Statistics 29 (3), 687–714.