DISCONTINUOUS GALERKIN APPROXIMATIONS TO ELLIPTIC AND PARABOLIC PROBLEMS WITH A DIRAC LINE SOURCE

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Abstract. The analyses of interior penalty discontinuous Galerkin methods of any order $k$ for solving elliptic and parabolic problems with Dirac line sources are presented. For the steady state case, we prove convergence of the method by deriving \textit{a priori} error estimates in the $L^2$ norm and in weighted energy norms. In addition, we prove almost optimal local error estimates in the energy norm for any approximation order. Further, almost optimal local error estimates in the $L^2$ norm are obtained for the case of piecewise linear approximations whereas suboptimal error bounds in the $L^2$ norm are shown for any polynomial degree. For the time-dependent case, convergence of semi-discrete and of backward Euler fully discrete scheme is established by proving error estimates in $L^2$ in time and in space. Numerical results for the elliptic problem are added to support the theoretical results.

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1. Introduction

In this paper, we analyze interior penalty discontinuous Galerkin (dG) approximations to elliptic and parabolic problems with a Dirac measure concentrated on a line. Consider a convex domain $\Omega \subset \mathbb{R}^3$ containing a one-dimensional curve $\Lambda \subset \mathbb{R}$ which is strictly included in $\Omega$. The elliptic model problem reads

\begin{align}
-\Delta u &= f\delta_{\Lambda}, & \text{in } \Omega, \\
u &= 0, & \text{on } \partial\Omega,
\end{align}

where $f \in L^2(\Lambda)$ and $f\delta_{\Lambda}$ is a Dirac measure concentrated on $\Lambda$ defined as follows.

\[ \langle f\delta_{\Lambda}, v \rangle = \int_{\Lambda}fv\, ds, \quad \forall v \in C(\Omega). \]

For the parabolic problem, let $T$ be the final time, let $u^0$ be in $L^2(\Omega)$ and assume that $f$ belongs to $L^2(0, T; L^2(\Lambda))$. We consider the following problem.

\[ \partial_t u - \Delta u = f\delta_{\Lambda}, \quad \text{in } \Omega \times (0, T], \]
The main contributions of this work are as follows. For the elliptic problem, we show global convergence in the $L^2$ norm and in weighted energy norms. Further, in regions excluding the line $\Lambda$, we derive almost optimal $L^2$ error estimates for linear polynomials and suboptimal error bounds of order almost $k$ for dG approximations of degree $k \geq 2$. In addition, almost optimal error rates are established in local energy norms for approximations of any polynomial degree. For the parabolic problem, we show global convergence in the $L^2(0,T; L^2(\Omega))$ norm for both the semi-discrete approximation and for the backward Euler fully discrete scheme.

Partial differential equations with Dirac right-hand sides can model organ perfusion where blood vessels are considered as one dimensional fractures embedded in the tissue [13]. In this case, $f$ can be a function of the blood pressure in the vessel leading to a coupled 1D–3D problem for the pressures in the tissue and in the vessels [12, 13]. Medical applications of such formulations include modeling drug delivery to tissues with the help of implantable devices [11] and drug delivery to tumors where different treatment options are compared [6]. In addition, Dirac measures concentrated on lines arise in optimal control problems [24]. Thanks to favorable properties of dG methods, including local mass conservation and adaptability to complex domains [33], these methods are well suited to model physical phenomena such as organ perfusion. In this paper we study dG methods applied to (1.1), (1.2) and to (1.4)–(1.6).

The analysis of finite element approximations to model problems (1.1), (1.2) and (1.4)–(1.6) is non–standard since the true solution is not smooth enough in space, namely it does not belong to $H^1(\Omega)$ and it exhibits a logarithmic singularity near the line $\Lambda$ [2, 12, 27]. Nevertheless, continuous Galerkin (cG) approximations have been extensively studied; we refer to the work by Scott [34] and Casas [5] where global error bounds are established. More recently and in the context of optimal control problems, Gong et al. derived improved global $L^2$ error bounds [24]. Such bounds are polluted by the singularity of the true solution where the rate of convergence in the $L^2$ norm for any polynomial degree is at most $O(h)$ where $h$ is the mesh-size. For continuous Galerkin approximations to (1.4)–(1.6), global error estimates for semi-discrete and fully-discrete formulations are derived in [22, 23].

In addition, convergence of the cG approximations to the elliptic model problem (1.1), (1.2) has been investigated in different non-classical norms. For example, local $L^2$ optimal error estimates (up to a log factor for linear polynomials) are derived by Köppel et al. [26, 27], and local energy error estimates are obtained by Bertoluzza et al. [3]. Such improved estimates are possible since the solution is smooth in regions excluding the line $\Lambda$ [2]. In addition, D’Angelo obtained error estimates in weighted norms and showed that with graded meshes the finite element solution converges optimally in these norms [12]. We also mention the recent splitting technique to numerically approximate the model problem (1.1), (1.2) introduced by Gjerde et al. where the solution is split into an explicit singular part and an implicit smooth part [21]. A finite element discretization is then formulated for the smooth part and optimal error rates are recovered [21].

To the best of our knowledge, discontinuous Galerkin approximations to (1.1), (1.2) and to (1.4)–(1.6) are missing from the literature. However, there are papers which formulate and study dG methods for elliptic problems with Dirac sources concentrated at a point. To this end, we mention the work by Houston and Wihler where global a priori and a posteriori error bounds are derived [25], the work by Choi and Lee for local $L^2$ error estimates [8], and the recent paper by Leng and Chen where a priori and a posteriori error estimates for hybridizable dG are obtained [28]. The analysis of dG methods for elliptic problems with Dirac measures is particularly challenging since consistency of the numerical method cannot be assumed since the traces of the solution and its gradient are not well defined.

The rest of this paper is organized as follows. Weak formulations in usual and in weighted Sobolev spaces are presented and shown to be equivalent in Section 2. Then, Section 3 defines the cG and dG discrete solutions to model problem (1.1), (1.2). We show global convergence in the $L^2$ norm in Section 4 and in weighted dG norms in Section 5. The local convergence of the solution is analyzed in Section 6. We devote Section 7 to the analysis of dG formulations for (1.4)–(1.6). Numerical results for the elliptic problem are presented in Section 8.
2. Weak formulation

Fix $p_0 \in [1, 3/2)$ and $q_0$ be such that $1/q_0 + 1/p_0 = 1$. Let $W^{1,p_0}(\Omega)$ denote the usual Sobolev space and recall that

$$W^{1,p_0}(\Omega) = \{ v \in W^{1,p_0}(\Omega), \ v = 0 \text{ on } \partial\Omega \}.$$ 

The weak formulation for problem (1.1), (1.2) is [5]: Find $u \in W^{1,p_0}(\Omega)$ such that:

$$\int_{\Omega} \nabla u \cdot \nabla v = \int_{\Lambda} f v, \quad \forall v \in W^{1,p_0}(\Omega). \quad (2.1)$$

This weak formulation is well posed and a unique solution $u \in W^{1,p_0}(\Omega)$ for $p_0 \in [1, 3/2)$ exists [5]. Next, in a similar way to [12], we present another weak formulation of problem (1.1), (1.2) in weighted Sobolev spaces. Define the distance function to $\Lambda$:

$$d(x, \Lambda) = \text{dist}(x, \Lambda) = \min_{y \in \Lambda} ||x - y||, \quad \forall x \in \Omega. \quad (2.2)$$

We first remark that $d^\alpha$ is an $A_2$ weight for $|\alpha| < 2$ (see [17], Lem. 3.3) where $A_2$ is the Muckenhoupt class of weights satisfying:

$$A_2 = \left\{ w \in L^1_{loc}(\mathbb{R}^3), \sup_{B(x,r)} \left( \frac{1}{|B(x,r)|} \int_{B(x,r)} w \right) \left( \frac{1}{|B(x,r)|} \int_{B(x,r)} w^{-1} \right) < \infty \right\},$$

where the supremum is taken over all balls $B(x, r)$ centered at $x$ and of radius $r$. This implies that $d^\alpha$ belongs to $L^2(\Omega)$ if $|\alpha| < 1$. For $\alpha \in (-1, 1)$, define the weighted $L^2$ norm as follows.

$$\|u\|_{L^2_\alpha(\Omega)} = \left( \int_{\Omega} |u|^2 d^{2\alpha} \right)^{1/2}. \quad (2.3)$$

The $L^2_\alpha(\Omega)$ space and the weighted inner product are defined as:

$$L^2_\alpha(\Omega) = \{ v : \|v\|_{L^2_\alpha(\Omega)} < \infty \}, \quad (u, v)_\alpha = \int_{\Omega} uv d^{2\alpha}, \quad \forall u, v \in L^2_\alpha(\Omega).$$

Similarly, we introduce the weighted Sobolev spaces as:

$$H^m_\alpha(\Omega) = \{ u : D^\beta u \in L^2_\alpha(\Omega), |\beta| \leq m \}, \quad \dot{H}^m_\alpha(\Omega) = \{ u \in H^m_\alpha(\Omega), u|_{\partial\Omega} = 0 \}.$$

where $\beta$ is a multi-index and $D^\beta$ is the corresponding weak derivative. The weighted Sobolev semi-norms and norms are denoted by:

$$|u|^2_{H^m_\alpha(\Omega)} = \sum_{|\beta|=m} \|D^\beta u\|^2_{L^2_\alpha(\Omega)}, \quad \|u\|^2_{H^m_\alpha(\Omega)} = \sum_{k=0}^{m} |u|^2_{H^m_\alpha(\Omega)}.$$

**Lemma 1.** Let $\alpha$ be such that $-2/p_0 + 1 < \alpha < 2/p_0 - 1$. Then, the weak formulation (2.1) is equivalent to the following weak formulation: find $u_\alpha \in H^1_\alpha(\Omega)$ such that

$$\int_{\Omega} \nabla u_\alpha \cdot \nabla v = \int_{\Lambda} f v, \quad \forall v \in H^1_\alpha(\Omega). \quad (2.4)$$
Proof. Let \( u_\alpha \) be a solution of (2.4). The existence and uniqueness of \( u_\alpha \) is established in [12], see also [16]. Observe that the condition on \( \alpha \) implies that \( (\alpha p_0)/(2-p_0) = (\alpha q_0)/(q_0-2) \in (-1,1) \). Since \( d^\gamma \in L^1_{\text{loc}}(\mathbb{R}^3) \) for \( |\gamma| < 2 \), we use Hölder’s inequality and obtain
\[
\int_\Omega d^{-2\alpha} v^2 \leq \left( \int_\Omega d^{-2\alpha \frac{q_0}{q_0-2}} \right)^{(q_0-2)/q_0} \left\| v \right\|^2_{L^{q_0}(\Omega)} < \infty, \quad \forall v \in L^{q_0}(\Omega).
\]
This implies that \( W^{1,q_0}_0(\Omega) \subset \dot{H}^{1,\alpha}_0(\Omega) \). Hence \( u_\alpha \) satisfies (2.1) for all \( v \in W^{1,q_0}_0(\Omega) \). Similarly, for \( v \in L^2_\alpha(\Omega) \), we have
\[
\int_\Omega v^{p_0} = \int_\Omega v^{p_0} d^{p_0 \alpha} d^{-p_0 \alpha} \leq \left( \int_\Omega v^2 d^{2\alpha} \right)^{p_0/2} \left( \int_\Omega d^{-2\alpha \frac{p_0}{q_0}} \right)^{(2-p_0)/2} < \infty, \quad \forall v \in L^2_\alpha(\Omega).
\]
This implies that \( \dot{H}^{1,\alpha}_\alpha(\Omega) \subset W^{1,p_0}_0(\Omega) \). Thus, \( u_\alpha \) solves (2.1). Since the solution to (2.1) is unique (see [24], Thm. 2.1 case (ii)), we conclude that \( u_\alpha = u \).

3. Numerical approximations

Let \( \mathcal{E}_h \) denote a partition of \( \Omega \), made of simplices:
\[
\bigcup_{E \in \mathcal{E}_h} \bar{E} = \Omega.
\]
We assume that the line \( \Lambda \) crosses all element boundaries transversally. Namely, for all \( E \in \mathcal{E}_h \), the one-dimensional Lebesgue measure of \( \Lambda \cap \partial E \) is zero. The diameter of a given element \( E \) is denoted by \( h_E \) and the mesh size is denoted by \( h = \max_{E \in \mathcal{E}_h} h_E \). We assume that \( \mathcal{E}_h \) is regular in the sense that there exists a constant \( \rho > 0 \) such that
\[
\frac{h_E}{\rho_E} \leq \rho, \quad \forall E \in \mathcal{E}_h,
\]
where \( \rho_E \) is the maximum diameter of a ball inscribed in \( E \). In addition, we assume that \( \mathcal{E}_h \) is quasi-uniform: there is a constant \( \gamma > 0 \) independent of \( h \) such that
\[
h \leq \gamma h_E, \quad \forall E \in \mathcal{E}_h.
\]
The broken Sobolev space is denoted by \( H^m(\mathcal{E}_h) \) for \( m \geq 1 \), and the broken gradient is denoted by \( \nabla_h \). In the remaining of the paper, \( k \geq 1 \) is a fixed positive integer and \( C \) is a generic constant independent of \( h \).

3.1. Finite element approximation

Let \( W^k_\alpha(\mathcal{E}_h) \) be the finite element space defined as follows.
\[
W^k_\alpha(\mathcal{E}_h) = \left\{ w_h \in H^1_0(\Omega) : w_h|_E \in P^k(E), \quad \forall E \in \mathcal{E}_h \right\}.
\]
Here, \( P^k(E) \) denotes the space of polynomials of degree at most \( k \). Let \( u^\text{CG}_h \in W^k_\alpha(\mathcal{E}_h) \) be the finite element approximation to \( u \) satisfying
\[
\int_\Omega \nabla u^\text{CG}_h \cdot \nabla v_h = \int_\Lambda f v_h, \quad \forall v_h \in W^k_\alpha(\mathcal{E}_h).
\]
3.2. Discontinuous Galerkin approximation

We now introduce the interior penalty discontinuous Galerkin discrete solution [33]. We define the broken polynomial space as follows.

\[ V^k_h(\mathcal{E}_h) = \{ v_h \in L^2(\Omega) : v_h|_E \in \mathbb{P}^k(E), \forall E \in \mathcal{E}_h \}. \]  
(3.6)

We also denote by \( \Gamma_h \) the set of all interior faces in \( \mathcal{E}_h \). For each interior face \( e \), we associate a unit normal vector \( \mathbf{n}_e \) and we denote by \( E^1_e \) and \( E^2_e \) the two elements that share \( e \) such that the vector \( \mathbf{n}_e \) points from \( E^1_e \) to \( E^2_e \). We denote the average and the jump of a function \( v_h \in V^k_h(\mathcal{E}_h) \) by \( \{ v_h \} \) and \( [v_h] \) respectively.

\[ \{ v_h \} = \frac{1}{2} (v_h|_{E^1_e} + v_h|_{E^2_e}), \quad [v_h] = v_h|_{E^1_e} - v_h|_{E^2_e}, \quad \forall e \in \Gamma_h. \]  
(3.7)

If \( e \) belongs to the boundary of the domain, \( e = \partial \Omega \cap \partial E^1_e \), then we define the average and the jump as follows.

\[ [v] = \{ v \} = v|_{E^1_e}. \]  
(3.8)

Let \( u^\text{DG}_h \in V^k_h(\mathcal{E}_h) \) be the discontinuous Galerkin solution satisfying:

\[ a_e(u^\text{DG}_h, v_h) = \int_{\Omega} f v_h, \quad \forall v_h \in V^k_h(\mathcal{E}_h), \]  
(3.9)

where \( a_e(\cdot, \cdot) : V^k_h(\mathcal{E}_h) \times V^k_h(\mathcal{E}_h) \to \mathbb{R} \) is given by:

\[ a_e(u, v) = \sum_{E \in \mathcal{E}_h} \int_{E} \nabla u \cdot \nabla v - \sum_{e \in \Gamma_h \cup \partial \Omega} \int_{e} \nabla u \cdot \mathbf{n}_e [v] + \epsilon \sum_{e \in \Gamma_h \cup \partial \Omega} \int_{e} \nabla v \cdot \mathbf{n}_e [u] + \sum_{e \in \Gamma_h \cup \partial \Omega} \int_{e} \sigma_h [v][v]. \]  
(3.10)

In the above, \( \epsilon \in \{-1, 0, 1\} \), \( \sigma \) is a user specified parameter and \( \beta \geq 1 \) is a parameter to be specified in the subsequent sections. We define the following energy semi-norm. For \( B \subseteq \Omega \) or \( B = \overline{\Omega} \) and \( v_h \in V^k_h(\mathcal{E}_h) \),

\[ \| v_h \|^2_{\text{DG}(B)} = \sum_{E \in \mathcal{E}_h \cap B} \| \nabla v_h \|^2_{L^2(E \cap B)} + \sum_{e \in \Gamma_h \cap \partial B} \sigma h^{-1} \| [v_h] \|^2_{L^2(e \cap B)}. \]  
(3.11)

For simplicity, we write \( \| \cdot \|^2_{\text{DG}} = \| \cdot \|^2_{\text{DG}(\Omega)} \). We also note that \( \| \cdot \|_{\text{DG}} \) defines a norm and the following Poincare inequality holds [15].

\[ \| v_h \|_{L^p(\Omega)} \leq C \| v_h \|_{\text{DG}}, \quad \forall 1 \leq p \leq 6, \quad \forall v_h \in V^k_h(\mathcal{E}_h). \]  
(3.12)

In the analysis, we will also use the following semi-norm. For \( v \in H^2(\mathcal{E}_h) \) and \( B \subseteq \Omega \) or \( B = \overline{\Omega} \),

\[ \| v \|^2_{\text{DG}(B)} = \| \nabla v \|^2_{L^2(B)} + \sum_{e \in \Gamma_h \cup \partial B} h \| [\nabla v] \|^2_{L^2(e \cap B)}. \]  
(3.13)

Similarly, denote \( \| \cdot \|^2_{\text{DG}} = \| \cdot \|^2_{\text{DG}(\Omega)} \). We then have the following continuity properties of the form \( a_e \) [7,33].

\[ a_e(v, w) \leq C \| v \|_{DG} \| w \|_{DG}, \quad a_e(v_h, w_h) \leq C \| v_h \|_{DG} \| w_h \|_{DG}, \quad \forall v, w \in H^2(\mathcal{E}_h), \quad \forall v_h, w_h \in V^k_h(\mathcal{E}_h). \]  
(3.14)

In addition, the following coercivity property

\[ a_e(w_h, w_h) \geq \frac{1}{2} \| w_h \|^2_{DG}, \quad \forall w_h \in V^k_h(\mathcal{E}_h), \]  
(3.15)

is valid for any value \( \sigma \geq 1 \) if \( \epsilon = +1 \) and for \( \sigma \) large enough if \( \epsilon = -1, 0 \). We recall the following important inverse inequalities, see Section 4.5 in [4].

\[ \| v_h \|_{L^p(\Omega)} \leq C h^{\frac{1}{p} - \frac{1}{2}} \| v_h \|_{L^2(\Omega)}, \quad \forall 1 \leq p \leq \infty, \quad \forall v_h \in V^k_h(\mathcal{E}_h). \]  
(3.16)
For the trace estimates, we will make use of the following
\[ \|v\|_{L^2(\partial)} \leq Ch^{-1/2}(\|v\|_{L^2(\Omega)} + h\|\nabla v\|_{L^2(\Omega)}), \quad \forall v \in \partial \Omega, \quad \forall v \in H^1(\Omega). \] (3.17)

For discrete functions, the above estimate reads
\[ \|v_h\|_{L^2(\partial)} \leq Ch^{-1/2}\|v_h\|_{L^2(\Omega)}, \quad \forall v \in \partial \Omega, \quad \forall v \in V_h^k(\Omega). \] (3.18)

Further, we recall that for any \( p \in [1, \infty] \),
\[ \|\nabla v_h\|_{L^p(\Omega)} \leq Ch^{-1}\|v_h\|_{L^p(\Omega)}, \quad \forall v \in V_h^k(\Omega). \] (3.19)

We end this section by recalling Lemma 4.1 proved by Chen and Chen [7]. Consider any two sets \( D, \tilde{D} \subset \Omega \) such that the distance between \( D \) and \( (\partial \tilde{D}\setminus\partial D) \) is strictly positive. Then, for \( h \) small enough, we have
\[ \|U - u_h^{DG}\|_{DG(D)} \leq C\left(h^{k}\|U\|_{H^{k+1}(\tilde{D})} + \|U - u_h^{DG}\|_{L^2(\tilde{D})}\right). \] (3.20)

4. Global error estimate in the \( L^2 \) norm

The goal of this section is to show a global \( L^2 \) estimate for the error \( u - u_h^{DG} \). We first recall important global \( L^2 \) estimates for the finite element discretization (3.5). For \( k = 1 \), Casas obtained the following estimate [5],
\[ \|u - u_h^{CG}\|_{L^2(\Omega)} \leq Ch^{1/2}\|f\|_{L^2(\Lambda)}. \] (4.1)

If the line \( \Lambda \) is a \( C^2 \) curve that does not intersect the boundary \( \partial \Omega \), the improved estimate
\[ \|u - u_h^{CG}\|_{L^2(\Omega)} \leq C(\theta)h^{1-\theta}\|f\|_{L^2(\Lambda)}, \quad 0 < \theta < \frac{1}{2}, \] (4.2)

was proved by Gong et al. for \( k = 1 \) in [24]. Similar arguments yield the same error bounds for \( k \geq 2 \). The parameter \( \theta \) arises from the fact that \( u \in W^{1, \frac{4}{3}}_0(\Omega) \) when \( 0 < \theta < 1/2 \). We follow the ideas of Scott [34] and Houston and Wihler [25] presented for a problem with a Dirac source concentrated at a point, and we construct an intermediate problem with an \( L^2 \) source term. Let \( T_\Lambda \subset \mathcal{E}_h \) be the set of elements that intersect the line \( \Lambda \),
\[ T_\Lambda = \{ E \in \mathcal{E}_h, \quad E \cap \Lambda \neq \emptyset \}. \]

Define \( f_h \in V_h^k(\mathcal{E}_h) \) as
\[ \forall E \in \mathcal{E}_h, \quad f_h|_E = \begin{cases} f_{h,E}, & \text{if } E \in T_\Lambda, \\ 0, & \text{otherwise}, \end{cases} \] (4.3)

where \( f_{h,E} \in \mathbb{P}^k(E) \) is defined as follows. For \( E \in T_\Lambda \),
\[ \int_E f_{h,E}v_h = \int_{E \cap \Lambda} f v_h, \quad \forall v_h \in \mathbb{P}^k(E). \] (4.4)

Clearly, the function \( f_{h,E} \) is well defined. Further, consider the following intermediate problem: find \( U \in H^1_0(\Omega) \) such that
\[ \begin{align*}
-\Delta U &= f_h, & \text{in } \Omega, \\
U &= 0, & \text{on } \partial \Omega.
\end{align*} \] (4.5, 4.6)

Since \( f_h \) belongs to \( L^2(\Omega) \), Lax-Milgram’s theorem yields existence and uniqueness of \( U \). In addition, since \( \Omega \) is convex, the function \( U \) belongs to \( H^2(\Omega) \). We proceed by obtaining a bound on \( f_h \) in the following lemma.
Lemma 2. The following estimate holds
\[ \| f_h \|_{L^2(\Omega)} \leq Ch^{-3/2} \| f \|_{L^2(\Lambda)}. \] (4.7)

In addition, if \( \Lambda \) is a \( C^2 \) curve and the mesh satisfies \( |\Lambda \cap E| \leq Ch \) for all \( E \in \mathcal{E}_h \), we have
\[ \| f_h \|_{L^2(\Omega)} \leq Ch^{-1} \| f \|_{L^2(\Lambda)}. \] (4.8)

Proof. With the definition of \( f_h \) given in (4.4), we have
\[ \| f_h \|_{L^2(\Omega)}^2 = \int_\Omega f_h^2 = \sum_{E \in \mathcal{E}_h} \int_E (f_h |_E)^2 = \sum_{E \in \mathcal{T}_h} \int_{E \cap \Lambda} f_h. \]

Using Hölder’s inequality, we obtain
\[ \int_{E \cap \Lambda} f_h \leq \| f_h \|_{L^\infty(E)} \| f \|_{L^1(E \cap \Lambda)}. \]

Hence, with (3.16) \( (q = \infty, p = 2) \), and (3.3), we obtain
\[ \| f_h \|_{L^2(\Omega)}^2 \leq \sum_{E \in \mathcal{T}_h} \| f_h \|_{L^\infty(E)} \| f \|_{L^1(E \cap \Lambda)} \leq Ch^{-3/2} \sum_{E \in \mathcal{T}_h} \| f_h \|_{L^2(E)} \| f \|_{L^1(E \cap \Lambda)} \leq Ch^{-3/2} \sum_{E \in \mathcal{T}_h} \| f_h \|_{L^2(E)} |\Lambda \cap E|^{1/2} \| f \|_{L^2(E \cap \Lambda)}. \]

If \( |\Lambda \cap E| \leq Ch \), we apply Hölder’s inequality for sums and obtain (4.8). Otherwise, we have (4.7). □

The following a priori error bounds hold.

Lemma 3. There exists a constant \( C \) independent of \( h \) such that
\[ \| U - u_h^{CG} \|_{L^2(\Omega)} + h \| \nabla (U - u_h^{CG}) \|_{L^2(\Omega)} \leq Ch^2 \| U \|_{H^2(\Omega)}, \] (4.9)
\[ \| U - u_h^{DG} \|_{DG} \leq Ch \| U \|_{H^2(\Omega)}. \] (4.10)

If in addition, \( \beta = 1 \) and \( \sigma \) is large enough if \( \epsilon = -1 \), or \( \beta > 3 \) and \( \sigma \) is large enough for \( \epsilon = 0 \) or \( \epsilon = 1 \), there exists a constant \( C \) independent of \( h \) such that
\[ \| U - u_h^{DG} \|_{L^2(\Omega)} \leq Ch^2 \| U \|_{H^2(\Omega)}. \] (4.11)

Proof. We have for any \( v_h \in V_h^k(\mathcal{E}_h) \),
\[ \int_\Omega f_h v_h = \sum_{E \in \mathcal{E}_h} \int_E f_h |E| v_h = \sum_{E \in \mathcal{T}_h} \int_{E \cap \Lambda} f v_h = \int_\Lambda f v_h. \]

Thus, since \( W_h^k(\mathcal{E}_h) \) is a subset of \( V_h^k(\mathcal{E}_h) \), the discrete functions \( u_h^{CG} \) and \( u_h^{DG} \) can be viewed as finite element and discontinuous Galerkin approximations to the intermediate problem (4.5). Since \( f_h \in L^2(\Omega) \), standard approximation and error bounds hold. In particular (4.9) and (4.10) hold. For a proof of (4.11), we refer to Theorem 2.13 in [33]. □

We are now ready to present and prove the main result of this section.
Theorem 1. Assume the penalty parameter \( \sigma \) is chosen so that (3.15) holds. In addition, if \( \epsilon = \{0, 1\} \), select \( \beta > 3 \), and if \( \epsilon = -1 \), choose \( \beta = 1 \). Then, there exists a constant \( C \) independent of \( h \) such that
\[
\|u - u_h^{DG}\|_{L^2(\Omega)} \leq Ch^{1/2}\|f\|_{L^2(\Lambda)}.
\] (4.12)

In addition, if \( \Lambda \) is a \( C^2 \) curve and \( |\Lambda \cap E| \leq Ch \) for all \( E \in \mathcal{E}_h \), we have the following improved estimate.
\[
\|u - u_h^{DG}\|_{L^2(\Omega)} \leq C(\theta) h^{1-\theta}\|f\|_{L^2(\Lambda)}, \quad 0 < \theta < 1/2.
\] (4.13)

Proof. We use triangle inequality to obtain:
\[
\|u - u_h^{DG}\|_{L^2(\Omega)} \leq \|u - u_h^{CG}\|_{L^2(\Omega)} + \|u_h^{CG} - U\|_{L^2(\Omega)} + \|U - u_h^{DG}\|_{L^2(\Omega)}.
\] (4.14)

We have for any \( v_h \in V_h^{k}(\mathcal{E}_h) \),
\[
\int_{\Omega} f_h v_h = \sum_{E \in \mathcal{E}_h} \int_{E} f_h v_h = \sum_{E \in \mathcal{T}_h} \int_{E \cap \Lambda} f v_h = \int_{\Lambda} f v_h.
\]

Since the domain \( \Omega \) is convex, we have the following elliptic regularity result:
\[
\|U\|_{H^2(\Omega)} \leq C\|f_h\|_{L^2(\Omega)}.
\] (4.15)

Using the bounds (4.9) and (4.11) in (4.14) yields:
\[
\|u - u_h^{DG}\|_{L^2(\Omega)} \leq \|u - u_h^{CG}\|_{L^2(\Omega)} + Ch^2\|f_h\|_{L^2(\Omega)}.
\] (4.16)

Bounds (4.1) and (4.7) give (4.12). Under the additional assumptions, bounds (4.2) and (4.8) yield (4.13). \( \square \)

Hereinafter, we will make the following assumption.

Assumption 1. We only consider the symmetric dG discretization \( (\epsilon = -1) \) and we set \( \beta = 1 \). We also assume that \( \Lambda \) is a \( C^2 \) curve, \( f \in L^2(\Lambda) \), and that \( |\Lambda \cap E| \leq Ch \), \( \forall E \in \mathcal{E}_h \).

For simplicity, we denote by \( a = a_{-1} \). Under Assumption 1, with (4.15) and (4.8), there is a constant \( C \) independent of \( h \) such that:
\[
h\|U\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Lambda)}.
\] (4.17)

5. Weighted energy estimate

We first show that the dG solution is stable in the weighted energy norm defined by:
\[
\|v\|_{DG,\alpha}^2 = \sum_{E \in \mathcal{E}_h} \|\nabla v\|_{L^2(E)}^2 + \sum_{e \in \Gamma_h \cup \partial \Omega} \frac{\sigma}{h} \|d_{\alpha}^e[v]\|_{L^2(e)}^2, \quad v \in H^1(\mathcal{E}_h), \quad \alpha \in (0, 1).
\] (5.1)

Lemma 4 (Stability). Let Assumption 1 hold. For \( \alpha \in (0, 1) \), there exists a constant \( C_\alpha \) independent of \( h \) but dependent on \( \max_{x \in \Omega} d_{2\alpha}(x) \) such that the dG solution, \( u_h^{DG} \), satisfies:
\[
\|u_h^{DG}\|_{DG,\alpha} \leq C_\alpha (\|f\|_{L^2(\Lambda)} + |u|_{H^1_0(\Omega)}).
\] (5.2)
Proof. Recall the intermediate problem (4.5). Since \( U \in H^2(\Omega) \cap H^1_0(\Omega) \), we immediately have with (4.10) and (4.17)

\[
\sum_{e \in \Gamma_h \cup \partial \Omega} \frac{\sigma}{h} \| d^{2\alpha} [u_h^{DG}] \|_{L^2(e)}^2 \leq \| d^{2\alpha} \|_{L^\infty(\Omega)}^2 \sum_{e \in \Gamma_h \cup \partial \Omega} \frac{\sigma}{h} \| [u_h^{DG} - U] \|_{L^2(e)}^2 \leq C \| d^{2\alpha} \|_{L^\infty(\Omega)}^2 \| f \|_{L^2(\Lambda)}^2.
\]

(5.3)

We use the triangle inequality, (4.9) and (4.17):

\[
\| \nabla U \|_{L^2(\Omega)} \leq \| d^{2\alpha} \|_{L^\infty(\Omega)} \| \nabla (U - u_h^{CG}) \|_{L^2(\Omega)} + \| \nabla u_h^{CG} \|_{L^2(\Omega)} \leq C_\alpha \| f \|_{L^2(\Lambda)} + \| \nabla u_h^{CG} \|_{L^2(\Omega)}.
\]

(5.4)

From Theorem 3.5 in [16] and Lemma 1, we have

\[
\| \nabla u_h^{CG} \|_{L^2(\Omega)} \leq C \| \nabla u \|_{L^2(\Omega)}, \quad \alpha \in (0, 1).
\]

(5.5)

This implies

\[
\| \nabla U \|_{L^2(\Omega)} \leq C_\alpha \| f \|_{L^2(\Lambda)} + C |u|_{H^1_0(\Omega)}.
\]

By the triangle inequality, equations (4.10), (4.17) and the above bound, we obtain

\[
\sum_{E \in \mathcal{E}_h} \| \nabla u_h^{DG} \|_{L^2(E)}^2 \leq 2 \sum_{E \in \mathcal{E}_h} \| \nabla (u_h^{DG} - U) \|_{L^2(E)}^2 + 2 \sum_{E \in \mathcal{E}_h} \| \nabla U \|_{L^2(E)}^2 \leq C_\alpha \| u_h^{DG} - U \|_{DG}^2 + 2 \| \nabla U \|_{L^2(\Omega)}^2 \leq C_\alpha (\| f \|_{L^2(\Lambda)} + |u|_{H^1_0(\Omega)})^2.
\]

(5.6)

We conclude the result by combining (5.3) and (5.6).

We have an a priori bound for \( U \) in the \( H^2_\alpha \) norm, which can be seen as a generalization of (4.17). We denote by \( d_E = \max_{x \in E} \| d(x, \Lambda) \| \) for \( E \in \mathcal{E}_h \).

Lemma 5. Let Assumption 1 hold. For \( \alpha \in (-1, 1) \), there exists a constant \( C \) depending on \( \alpha \) but independent of \( h \) such that

\[
\| U \|_{H^2_\alpha(\Omega)} \leq C h^{\alpha - 1} \| f \|_{L^2(\Lambda)}, \quad \alpha \in (-1, 1).
\]

(5.7)

Proof. Since \( d^{2\alpha} \in A_2 \), it follows from Theorem 3.1 in [32] that

\[
\| U \|_{H^2_\alpha(\Omega)} \leq C \| f \|_{L^2(\Lambda)}.
\]

(5.8)

Thus, to show (5.7), we find a bound on \( \| f_h \|_{L^2(\Omega)} \). Thanks to the shape-regularity of the mesh, for \( E \in \mathcal{T}_h \), \( c_E \leq d_E \leq C h_E \) (see [12], Lem. 3.1). Hence, using (5.10), (4.8) and (3.3), yield

\[
\| f_h \|_{L^2(\Omega)}^2 = \sum_{E \in \mathcal{T}_h} \| d^{\alpha} f_h \|_{L^2(E)}^2 \leq \sum_{E \in \mathcal{T}_h} \bar{d}_E^{2\alpha} \| f_h, E \|_{L^2(E)}^2 \leq C h^{2\alpha} \sum_{E \in \mathcal{T}_h} \| f_h, E \|_{L^2(E)}^2 \leq C h^{2\alpha - 2} \| f \|_{L^2(\Lambda)}^2.
\]

(5.9)

Substituting (5.9) in (5.8) yields (5.7).

The following equivalence of norms holds (see proof of Lem. 3.2 in [12]). There exist positive constants \( \gamma_1, \gamma_2 \) independent of \( h \) such that for \(-1 < \alpha < 1\), \( E \in \mathcal{E}_h \), and \( v_h \in \mathcal{P}_k(E) \),

\[
\gamma_1 \| d^{\alpha} v_h \|_{L^2(E)} \leq \bar{d}_E^{2\alpha} \| v_h \|_{L^2(E)} \leq \gamma_2 \| d^{\alpha} v_h \|_{L^2(E)}.
\]

(5.10)

In this section, we will make use of the following assumption (see [14], Thm. 3.4).

Assumption 2. The distance function satisfies the following bounds

\[
\| \nabla d \|_{L^\infty(\Omega)} \leq 1, \quad \| \nabla^2 d^2 \|_{L^\infty(\Omega)} \leq C.
\]

(5.11)
Lemma 6. Let Assumptions 1 and 2 hold. Let $k = 1$. For $\alpha \in (\frac{1}{2}, 1)$, there exists constants $C, C_*$ depending on $s$ and $\alpha$ but independent of $h$ such that if $\sigma > C_*$, 

\begin{align}
\| U - u_h^{DG} \|_{DG, \alpha} &\leq C \left( h^{\alpha} + h^{1 - \frac{3}{2} s(1 - \alpha)} \right), \quad \forall 1 < s < \frac{1}{1 - \alpha}.
\end{align}

Proof. Let $u_h^{CG} \in W_{1, \alpha}^1(\mathcal{E}_h)$ solve (3.5) for $k = 1$. We apply the triangle inequality.

\begin{align}
\| \nabla u_h^{DG} \|_{L^2_\delta(\Omega)} &+ \left( \sum_{e \in \Gamma_h \cup \partial \Omega} \frac{\sigma}{h} \| d^n [u_h^{DG}] \|_{L^2(e)}^2 \right)^{1/2} \\
&\leq \| \nabla (u - u_h^{CG}) \|_{L^2_\delta(\Omega)} + \| U - u_h^{DG} \|_{DG, \alpha} + \| \nabla (u_h^{CG} - U) \|_{L^2_\delta(\Omega)}.
\end{align}

Considering Lemma 1, the first term is bounded in Corollary 3.8 in [12] 

\begin{align}
\| \nabla (u - u_h^{CG}) \|_{L^2_\delta(\Omega)} &\leq C h^{\alpha - \delta} \| u \|_{V_{1, \alpha}^2(\Omega)}.
\end{align}

Bound (5.18) can also be derived from Theorem 3.5 in [16] and Theorem 3.6 in [12]. It remains to bound $\| U - u_h^{DG} \|_{DG, \alpha}$ and $\| \nabla (u_h^{CG} - U) \|_{L^2_\delta(\Omega)}$, which is the object of Lemmas 6 and 7 respectively. \hfill \Box

Lemma 6. Let Assumptions 1 and 2 hold. Let $k = 1$. For $\alpha \in (\frac{1}{2}, 1)$, there exists constants $C, C_*$ depending on $s$ and $\alpha$ but independent of $h$ such that if $\sigma > C_*$,

\begin{align}
\| U - u_h^{DG} \|_{DG, \alpha} &\leq C \left( h^{\alpha} + h^{1 - \frac{3}{2} s(1 - \alpha)} \right), \quad \forall 1 < s < \frac{1}{1 - \alpha}.
\end{align}
Proof. Let $\chi_h = \Pi_h U - u^\text{DG}_h$. With triangle inequality and the bounds (5.14), (4.10), (4.11), (4.17), we have

$$\|\chi_h\|_{L^2(\Omega)} + h\|\chi_h\|_{\text{DG}} \leq Ch^2\|U\|_{H^2(\Omega)} \leq Ch\|f\|_{L^2(\Lambda)}.$$  \hfill (5.20)

With several manipulations, as is done in [37], we have formally

$$\|\chi_h\|_{\text{DG},\alpha}^2 = a(\chi_h, d^{2\alpha}\chi_h) - 2\sum_{E \in \mathcal{E}_h} \int_E (d^\alpha \nabla \chi_h \cdot (\chi_h \nabla (d^\alpha)))$$

$$+ 2\sum_{e \in \Gamma_h \cup \partial\Omega} \int_e \{d^\alpha \nabla (d^\alpha \chi_h)\} \cdot \mathbf{n}_e[d^\alpha \chi_h] = \sum_{i=1}^3 T_i.$$  \hfill (5.21)

We now explain why each term $T_i$ above is well defined. From (5.12), (5.13), the term $T_1$ is well defined since $d^{2\alpha}\chi_h \in H^2(\mathcal{E}_h)$. Property (5.12) and Cauchy–Schwarz’s inequality guarantee that $T_2$ is well defined. For $T_3$, we write

$$\{d^\alpha \nabla (d^\alpha \chi_h)\} \cdot \mathbf{n}_e[d^\alpha \chi_h] = \{d^\alpha \nabla (d^\alpha \chi_h)\} \cdot \mathbf{n}_e[\chi_h].$$

Observe that since $\chi_h$ is a polynomial, the function $d^\alpha \nabla (d^\alpha \chi_h)$ belongs to $H^1(\mathcal{E}_h)^3$. Indeed we have

$$d^\alpha \nabla (d^\alpha \chi_h) = ad^{2\alpha-1}\chi_h \nabla d + d^{2\alpha} \nabla \chi_h,$$

and with (5.13), each term belongs to $H^1(E)$ for each mesh element $E$. This implies that $\|\{d^\alpha \nabla (d^\alpha \chi_h)\}\|_{L^2(e)}$ is bounded and the term $T_3$ is well defined. To handle the first term, we use the following Galerkin orthogonality

$$a(U - u^\text{DG}_h, v_h) = 0, \quad \forall v_h \in V^k_h(\mathcal{E}_h).$$  \hfill (5.22)

Let $\eta = \Pi_h U - U$ and $\xi = U - u^\text{DG}_h$ so that $\chi_h = \eta + \xi$. Since $[d^\alpha \eta] = 0$ a.e. on $e \in \Gamma_h \cup \partial\Omega$, we have

$$T_1 = a(\eta, d^{2\alpha}\chi_h) + a(\xi, d^{2\alpha}\chi_h - w_h)$$

$$= \sum_{E \in \mathcal{E}_h} \int_E \nabla \eta \cdot \nabla (d^{2\alpha}\chi_h) - \sum_{e \in \Gamma_h \cup \partial\Omega} \int_e \{d^\alpha \nabla \eta\} \cdot \mathbf{n}_e[d^\alpha \chi_h] + a(\xi, d^{2\alpha}\chi_h - w_h)$$

$$= \sum_{i=1}^3 T_{1,i},$$

where $w_h \in V^1_h(\mathcal{E}_h)$ is a piecewise Lagrange interpolant of $d^{2\alpha}\chi_h$ such that

$$\|d^{2\alpha}\chi_h - w_h\|_{\text{DG}} \leq Ch\|d^{2\alpha}\chi_h\|_{H^2(\mathcal{E}_h)}.$$  \hfill (5.23)

We begin by bounding $T_{1,3}$. With (3.14), (4.10), (5.23), we have

$$T_{1,3} = a(\xi, d^{2\alpha}\chi_h - w_h) \leq C\|\xi\|_{\text{DG}}\|d^{2\alpha}\chi_h - w_h\|_{\text{DG}} \leq Ch^2\|U\|_{H^2(\Omega)}\|d^{2\alpha}\chi_h\|_{H^2(\mathcal{E}_h)}.$$  \hfill (5.24)

Using (5.11) and (5.13), we obtain

$$\|d^{2\alpha}\chi_h\|_{H^2(\mathcal{E}_h)} \leq C\|d^{2\alpha-2}\chi_h\|_{L^2(\Omega)} + Ch\|d^{2\alpha-1}\nabla \chi_h\|_{L^2(\Omega)}.$$
\[
\leq C\|\chi_h\|_{L^{\frac{2}{2s(1-\alpha)}}(\Omega)} + \|d^{\alpha}\nabla_h \chi_h\|_{L^{\frac{2}{1-s(1-\alpha)}}(\Omega)}.
\]

For the second term, we first derive an inverse inequality for any \( \tau \in V_h^k(\mathcal{E}_h) \) and \( q \geq 2 \). With the local version of the inverse inequality (3.16), (5.10) and Jensen’s inequality, we have

\[
\|\chi_h\|_{L^{\frac{2}{2s(1-\alpha)}}(\Omega)} \leq C h^{-3s(1-\alpha)}\|\chi_h\|_{L^2(\Omega)} \leq C h^{-3s(1-\alpha)+1}\|f\|_{L^2(\Omega)}.
\]

For the second term, we first derive an inverse inequality for any \( \tau \in V_h^k(\mathcal{E}_h) \) and \( q \geq 2 \). With (5.14), (5.7), (4.11) and (4.17), we obtain

\[
\|d^{\alpha}\tau\|_{L^q(\Omega)} \leq \left( \sum_{E \in \mathcal{E}_h} d^p_{E,q} \|\tau\|_{L^q(E)}^q \right)^{1/q} \leq C h^{\frac{1}{q} - \frac{3}{2}} \left( \sum_{E \in \mathcal{E}_h} \|\tau\|_{L^q(E)}^q \right)^{1/q} \leq C h^{\frac{1}{q} - \frac{3}{2}} \|\tau\|_{L^q_h(\Omega)}.
\]

Hence, with (5.27), the second term in (5.25) is bounded as

\[
\|d^{\alpha}\nabla_h \chi_h\|_{L^{\frac{2}{1-s(1-\alpha)}}(\Omega)} \leq C h^{-\frac{3}{2}s(1-\alpha)}\|\nabla_h \chi_h\|_{L^2_h(\Omega)}.
\]

Thus, with (5.26) and (5.28), (5.25) reads

\[
|d^{\alpha}\chi_h|_{H^2(\mathcal{E}_h)} \leq C \left( h^{-3s(1-\alpha)+1} + h^{-\frac{3}{2}s(1-\alpha)}\|\nabla_h \chi_h\|_{L^2_h(\Omega)} \right).
\]

Thus, with (4.17) and (5.29), (5.24) reads

\[
T_{1,3} \leq C \left( h^{2-3s(1-\alpha)} + h^{1-\frac{3}{2}s(1-\alpha)}\|\nabla_h \chi_h\|_{L^2_h(\Omega)} \right).
\]

We now turn to \( T_{1,1} \) and \( T_{1,2} \). We write

\[
T_{1,1} = \sum_{E \in \mathcal{E}_h} \int_E \nabla \eta \cdot d^{\alpha}\nabla \chi_h + \int_E \nabla \eta \cdot 2\alpha d^{\alpha-1}\nabla \chi_h \leq \|\nabla \eta\|_{L^2_h(\Omega)} \|\nabla_h \chi_h\|_{L^2_h(\Omega)} + C \|\nabla \eta\|_{L^2_{2\alpha-1}(\Omega)} \|\chi_h\|_{L^2(\Omega)}.
\]

With (5.14), (5.7), (4.11) and (4.17), we obtain

\[
|T_{1,1}| \leq C h \|U\|_{H^2(\Omega)} \|\nabla_h \chi_h\|_{L^2(\Omega)} + C h^2 \|U\|_{H^2_{2\alpha-1}(\Omega)} \leq C h^\alpha \|\nabla_h \chi_h\|_{L^2(\Omega)} + C h^{2\alpha}.
\]

To handle \( T_{1,2} \), consider a mesh element \( E \) and let \( e \in \partial E \). Since \( d^{\alpha}\eta \) belongs to \( H^1_\alpha(\Omega) \), trace estimate (3.17) yields

\[
\|d^{\alpha}\nabla \eta\|_{L^2(e)} \leq C h^{-1/2}\|d^{\alpha}\nabla \eta\|_{L^2(E)} + C h^{1/2}\left( \|d^{\alpha}\nabla^2 \eta\|_{L^2(E)} + \|d^{\alpha-1}\nabla \eta\|_{L^2(E)} \right).
\]

Thus, with Cauchy–Schwarz’s inequality, (5.14) and (5.7), we obtain

\[
|T_{1,2}| \leq C \left( \|\nabla \eta\|_{L^2(\Omega)} + h \|U\|_{H^2(\Omega)} + h \|U\|_{H^2_{2\alpha-1}(\Omega)} \right) \|\chi_h\|_{DG,\alpha} \leq C h^\alpha \|\chi_h\|_{DG,\alpha}.
\]

For \( T_2 \), we apply Cauchy–Schwarz’s inequality and (5.11),

\[
|T_2| \leq \|\nabla_h \chi_h\|_{L^2(\Omega)} \|d^{\alpha-1}\chi_h\|_{L^2(\Omega)}
\]
With (5.14), Hölder’s inequality, the observation that \( d^\alpha + 1 \in L^{1/2} \Omega \), (5.20), and (3.16), we obtain
\[
|T_2| \leq \| \nabla_h \chi_h \|_{L_2(\Omega)} \| d^\alpha \|_{L^{1/2} \Omega} \| \chi_h \|_{L^{1/2} \Omega} \\
\leq C \| \nabla_h \chi_h \|_{L_2(\Omega)} h^{-\frac{2}{3}(1-\sigma)} \| \chi_h \|_{L_2(\Omega)} \leq C h^{-\frac{2}{3}(1-\sigma)+1} \| \nabla_h \chi_h \|_{L_2(\Omega)}.
\] (5.34)
Hence, with (5.30), (5.31), (5.32), (5.34), and Young’s inequality, we obtain
\[
|T_1| + |T_2| \leq \frac{1}{8} \| \chi_h \|^2_{DG,\alpha} + C \left( h^{2-3s(1-\sigma)} + h^{2\alpha} \right).
\] (5.35)
It remains to handle \( T_3 \). Fix a face \( e \in \Gamma_h \), shared by two elements, \( e = \partial E^1_e \cap \partial E^2_e \). We write
\[
\int_e (\nabla(d^\alpha \chi_h))|E^1_e \cdot n_e [d^\alpha \chi_h] = \int_e d^\alpha \nabla_h \chi_h |E^1_e \cdot n_e [d^\alpha \chi_h] + \int_e (\alpha d^\alpha \nabla \cdot n_e) \chi_h |E^1_e [d^\alpha \chi_h]
\]
\[
= A_{e,1} + A_{e,2}.
\]
For \( A_{e,1} \), recall the definition of \( \tilde{d}_E \). With (3.18) and (5.10), we have
\[
A_{e,1} \leq C d_{E^1_e} \| \nabla_h \chi_h \|_{L^2(E^1_e)} h^{-1/2} \| d^\alpha \chi_h \|_{L^2(e)} \leq C \gamma_2 \| \nabla_h \chi_h \|_{L^2(E^1_e)} h^{-1/2} \| d^\alpha \chi_h \|_{L^2(e)}.
\] (5.36)
Hence, with Young’s inequality, we obtain for a positive constant \( C_0 \)
\[
\sum_{e \in \Gamma_h \cup \partial \Omega} A_{e,1} \leq \frac{1}{16} \| \nabla_h \chi_h \|_{L^2(\Omega)}^2 + C_0 \sum_{e \in \Gamma_h \cup \partial \Omega} h^{-1} \| d^\alpha \chi_h \|_{L^2(e)}^2.
\] (5.37)
For the term \( A_{e,2} \), we have (5.11)
\[
A_{e,2} \leq \alpha d^\alpha \nabla \cdot n_e \chi_h |E^1_e L^2(\partial E^1_e) \| \chi_h \|_{L^2(\partial E^1_e)} \leq C \| d^\alpha \chi_h \|_{L^2(\partial E^1_e) \| \chi_h \|_{L^2(\partial E^1_e)}
\]
With the trace inequality (3.17), Hölder’s inequality and (5.11), we have
\[
\| d^\alpha \chi_h |_{E^1_e L^2(\partial E^1_e)} \leq C h^{-1/2} \| d^\alpha \chi_h \|_{L^2(\partial E^1_e)} + C \gamma_2 \| \nabla_h \chi_h \|_{L^2(\partial E^1_e)} \leq C h^{-1/2} \| d^\alpha \chi_h \|_{L^2(\partial E^1_e)} + C \gamma_2 \| \nabla_h \chi_h \|_{L^2(\partial E^1_e)}
\]
\[
+ C h^{-1/2} \| d^\alpha \chi_h \|_{L^2(\partial E^1_e)} \leq C \gamma_2 \| \nabla_h \chi_h \|_{L^2(\partial E^1_e)}.
\]
With similar arguments as the derivation of bound (5.29), with (5.20), (5.26), (5.28), and Hölder’s inequality, we obtain
\[
\sum_{e \in \Gamma_h \cup \partial \Omega} A_{e,2} \leq C \left( h^{-3s(1-\sigma)+2} + h^{-\frac{2}{3}s(1-\sigma)+1} \| \nabla_h \chi_h \|_{L_2(\Omega)}^2 \right) \left( \sum_{e \in \Gamma_h \cup \partial \Omega} h^{-1} \| \chi_h \|_{L_2(e)}^2 \right)^{1/2}.
\]
With Young’s inequality and the bound (5.20), this leads to
\[
\sum_{e \in \Gamma_h \cup \partial \Omega} A_{e,2} \leq \frac{1}{16} \| \nabla_h \chi_h \|_{L_2(\Omega)}^2 + C_0 \sum_{e \in \Gamma_h \cup \partial \Omega} h^{-1} \| d^\alpha \chi_h \|_{L_2(e)}^2 + C h^{-3s(1-\sigma)+2}.
\] (5.38)
Therefore we can bound \( T_3 \) with (5.37) and (5.38).
\[
|T_3| \leq \frac{1}{4} \| \nabla_h \chi_h \|_{L_2(\Omega)}^2 + C_0 \sum_{e \in \Gamma_h \cup \partial \Omega} h^{-1} \| d^\alpha \chi_h \|_{L_2(e)}^2 + C h^{-3s(1-\sigma)+2}.
\] (5.39)
We substitute (5.35), (5.39) in (5.21). With the assumption that \( \sigma > 4C_0 \), we obtain the result with an application of triangle’s inequality and the bound \( \| U - \Pi_h U \|_{DG,\alpha} \leq C h^\alpha. \)
Lemma 7. Let Assumptions 1 and 2 hold and let \( k = 1 \). For \( \alpha \in (1/2, 1) \), there exists a constant \( C \) depending on \( s \) and \( \alpha \) but independent of \( h \) such that

\[
\| \nabla (U - u_h^{CG}) \|_{L^2(\Omega)} \leq C \left( h^\alpha + h^{1 - \frac{s}{2} (1 - \alpha)} \right), \quad \forall 1 < s < \frac{1}{1 - \alpha}.
\]  

(5.40)

Proof. Let \( \zeta_h = \Pi_h U - u_h^{CG} \). We have

\[
\sum_{E \in \mathcal{E}_h} \int_E d^{2\alpha} \nabla \zeta_h \cdot \nabla \zeta_h = \sum_{E \in \mathcal{E}_h} \int_E \nabla \zeta_h \cdot \nabla (d^{2\alpha} \zeta_h) - 2 \sum_{E \in \mathcal{E}_h} \int_E d^{2\alpha} \zeta_h \nabla \zeta_h \cdot \nabla (d^{2\alpha}) = X_1 + X_2.
\]

Let \( w_h \) be the continuous Lagrange interpolant of \( d^{2\alpha} \zeta_h \). We have

\[
\| \nabla (d^{2\alpha} \zeta_h - w_h) \|_{L^2(\Omega)} \leq C h |d^{2\alpha} \zeta_h|_{H^2(\mathcal{E}_h)}.
\]  

(5.41)

Using the Galerkin orthogonality of the finite element method, we write

\[
X_1 = \sum_{E \in \mathcal{E}_h} \int_E \nabla (U - u_h^{CG}) \cdot \nabla (d^{2\alpha} \zeta_h - w_h) - \sum_{E \in \mathcal{E}_h} \int_E \nabla (U - \Pi_h U) \cdot \nabla (d^{2\alpha} \zeta_h).
\]

The terms in the right-hand side are bounded using similar arguments as in (5.25)–(5.31). We obtain

\[
X_1 \leq \frac{1}{4} \| \nabla \zeta_h \|_{L^2(\Omega)}^2 + C \left( h^{2-3s(1-\alpha)} + h^{2\alpha} \right).
\]

For \( X_2 \), similar arguments to the bound (5.34) for the term \( T_2 \) hold:

\[
X_2 \leq C h^{1-\frac{1}{2}s(1-\alpha)} \| \nabla \zeta_h \|_{L^2(\Omega)}.
\]

We skip some details for brevity. The result is concluded by using triangle inequality. \( \square \)

6. LOCAL \( L^2 \) AND ENERGY ERROR ESTIMATES

We show that the dG solution converges with an almost optimal rate in regions excluding the line \( \Lambda \) for \( k = 1 \) in Section 6.1. For \( k \geq 2 \), we show that the dG solution converges with a rate of \( k \) in Section 6.2. In this section, we make the following assumption on the weak solution to (2.1).

Assumption 3. For any neighborhood \( N \) of \( \Lambda \), namely \( \Lambda \subset N \subset \overline{N} \subset \Omega \), the weak solution \( u \) belongs to \( H^2(\Omega \setminus N) \).

This assumption is justified in the following two cases. If \( f \in H^2(\Lambda) \), then \( u \in H^2(\Omega \setminus N) \). This result was established using a splitting technique by Gjerde et al. [20]. Further, Ariche et al. show that if \( f \in L^2(\Lambda) \) and \( \Lambda \) is of class \( C^4 \), then \( u \) belongs to a Kondratiev’s type space [2]. This implies that \( u \in H^2(\Omega \setminus N) \), see also [12].

We first establish a local a priori bound on the solution of the intermediate problem (4.5).

Lemma 8. Assume 1 and 3 hold. Let \( N_0 \) and \( N_1 \) be nested neighborhoods of \( \Lambda \) satisfying

\[
\Lambda \subset N_0 \subset \overline{N_0} \subset N_1 \subset \Omega.
\]

There exist \( h_0 > 0 \) and a constant \( C \) independent of \( h \) such that for all \( h \leq h_0 \)

\[
\| U \|_{H^2(\Omega \setminus N_1)} \leq C \left( \| f \|_{L^2(\Lambda)} + \| u \|_{H^2(\Omega \setminus N_0)} \right).
\]  

(6.1)
Proof. There exists a neighborhood $N_{1/2}$ of $\Lambda$ such that

$$\mathcal{N}_0 \subset N_{1/2} \subset \mathcal{N}_{1/2} \subset N_1 \subset \mathcal{N}_1 \subset \Omega.$$ 

Define a mollifier function $\phi \in C^\infty(\Omega)$ which is equal to 1 in $\Omega \setminus N_1$ and to 0 in $N_{1/2}$. Recall that by definition of $U$ (4.5) and $f_h$ (4.4), there exists $h_0 > 0$ such that for $h \leq h_0$, we have

$$-\Delta U = 0, \quad \text{in} \quad \Omega \setminus N_0.$$ 

In addition, set $g$ as follows.

$$g = \Delta(U\phi), \quad \text{in} \quad \Omega. \quad (6.2)$$

Clearly, $g \in L^2(\Omega)$ and

$$g = \phi \Delta U + 2\nabla U \cdot \nabla \phi + U \Delta \phi = \begin{cases} 0, & \text{in} \ N_{1/2}, \\ 2\nabla U \cdot \nabla \phi + U \Delta \phi, & \text{in} \ N_1 \setminus N_{1/2}, \\ 0, & \text{in} \ \Omega \setminus N_1. \end{cases} \quad (6.3)$$

Hence, with Cauchy–Schwarz’s inequality, we obtain

$$\|g\|_{L^2(\Omega)} \leq C\|U\|_{H^1(N_1 \setminus N_{1/2})} \left(\|\nabla \phi\|_{L^2(N_1 \setminus N_{1/2})} + \|\Delta \phi\|_{L^2(N_1 \setminus N_{1/2})}\right) \leq C\|U\|_{H^1(N_1 \setminus N_{1/2})}^2. \quad (6.4)$$

In the above, the constant $C$ depends on the choice of the cut-off function $\phi$ but it is independent of $h$ for all $h \leq h_0$. We remark that $U\phi$ vanishes on the boundary $\partial\Omega$. By convexity of the domain and the above bound, we have

$$\|U\phi\|_{H^2(\Omega)} \leq C\|g\|_{L^2(\Omega)} \leq C\|U\|_{H^1(N_1 \setminus N_{1/2})}^2. \quad (6.5)$$

By the definition of $\phi$, the above bound, and the triangle inequality (with $u_h^{CG} \in W_1^k(\mathcal{E}_h)$ satisfying (3.5) for $k = 1$), we obtain

$$\|U\|_{H^2(\Omega \setminus N_1)} = \|U\phi\|_{H^2(\Omega \setminus N_1)} \leq \|U\phi\|_{H^2(\Omega)} \leq C\|U\|_{H^1(N_1 \setminus N_{1/2})}^2 \leq C\left(\|U - u_h^{CG}\|_{H^1(N_1 \setminus N_{1/2})} + \|u - u_h^{CG}\|_{H^1(N_1 \setminus N_{1/2})} + \|u\|_{H^1(N_1 \setminus N_{1/2})}\right). \quad (6.6)$$

A standard finite element bound (4.9), the convexity of the domain and (4.8) yield

$$\|U - u_h^{CG}\|_{H^1(\Omega)} \leq Ch\|U\|_{H^2(\Omega)} \leq Ch\|f_h\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Lambda)}. \quad (6.7)$$

To bound the second term in (6.6), we use Theorem 9.1 in [36].

$$\|u - u_h^{CG}\|_{H^1(N_1 \setminus N_{1/2})} \leq \|u - u_h^{CG}\|_{H^1(\Omega \setminus N_1)} \leq C\left(\|u\|_{H^2(\Omega \setminus N_0)} + \|u - u_h^{CG}\|_{L^2(\Omega)}\right). \quad (6.8)$$

Using the global bound (4.2) for $\theta = \frac{1}{2}$, we obtain for

$$\|u - u_h^{CG}\|_{H^1(N_1 \setminus N_{1/2})} \leq C\left(h\|u\|_{H^2(\Omega \setminus N_0)} + h^{1/2}\|f\|_{L^2(\Lambda)}\right). \quad (6.9)$$

Substituting (6.7) and (6.9) in (6.6) yields the result. \hfill \Box
6.1. Local $L^2$ bound for $k = 1$

Let $N$ be a neighborhood of $\Lambda$ such that $\overline{N} \subset \Omega$. There exist sets $N_0, N_1, N_2, N_3$ such that

$$\Lambda \subset N_0 \subset N_1 \subset N_2 \subset N_3 \subset N \subset \Omega.$$ 

It is important to note that the choice of the above sets is fixed and does not depend on the mesh. The main result of this section is the following local $L^2$ estimate.

**Theorem 3.** Let $k = 1$ and let Assumptions 1–3 hold. There exist $h_0 \geq 0$ and a constant $C(\theta)$ independent of $h$ such that for $0 < \theta < \frac{1}{2}$ and all $h \leq h_0$

$$\|u - u_h^{DG}\|_{L^2(\Omega \setminus N)} \leq C(\theta) h^{2-\theta} + Ch^2 |\ln(h)|. \quad (6.10)$$

The proof of this estimate also relies on establishing local bounds for the continuous and discontinuous discretizations of the intermediate problem (4.5). As before, this will be established in several Lemmas.

**Lemma 9.** Let Assumptions 1–3 hold. There exist $h_0 > 0$ and a constant $C(\theta)$ independent of $h$ such that for all $h \leq h_0$

$$\|U - u_h^{DG}\|_{L^2(\Omega \setminus N)} \leq C(\theta) h^{2-\theta}, \quad \forall 0 < \theta < \frac{1}{2}. \quad (6.11)$$

**Proof.** Define the characteristic function associated to $\Omega \setminus N$:

$$\chi_{\Omega \setminus N}(x) = \begin{cases} 1, & x \in \Omega \setminus N, \\ 0, & x \in N. \end{cases}$$

For readability, in this proof, we drop the dependence on $\theta$ in the constant “$C(\theta)$” and use $C$ instead. Set $\xi = U - u_h^{DG}$ and consider the auxiliary problem:

$$-\Delta w = \xi \chi_{\Omega \setminus N}, \quad \text{in } \Omega, \quad (6.12)$$
$$w = 0, \quad \text{on } \partial \Omega. \quad (6.13)$$

Clearly, since $\xi \chi_{\Omega \setminus N}$ belongs to $L^2(\Omega)$, the function $w$ belongs to $H^2(\Omega) \cap H^1_0(\Omega)$. Multiplying (6.12) by $\xi$ and integrating over $\Omega$, we obtain

$$\|\xi\|_{L^2(\Omega \setminus N)}^2 = \sum_{E \in \mathcal{E}_h} \int_E \nabla \xi \cdot \nabla w - \sum_{e \in \Gamma_h \cup \partial \Omega} \int_e \{\nabla w\} \cdot n_e [\xi] = a(\xi, w). \quad (6.14)$$

Let $S_h w \in W^1_h(\mathcal{E}_h)$ be the Scott–Zhang interpolant of $w$. With the consistency property (5.22), we have

$$\|\xi\|_{L^2(\Omega \setminus N)}^2 = a(\xi, w - S_h w)$$

$$= \sum_{E \in \mathcal{E}_h} \int_E \nabla \xi \cdot \nabla (w - S_h w) - \sum_{e \in \Gamma_h \cup \partial \Omega} \int_e \{\nabla (w - S_h w)\} \cdot n_e [\xi]$$

$$= \Theta_1 + \Theta_2. \quad (6.15)$$

We proceed by providing bounds for $\Theta_1$ and $\Theta_2$. We follow [8, 26], split $\Theta_1$ into two terms, and use Holder’s inequality,

$$\Theta_1 = \sum_{E \in \mathcal{E}_h} \int_{E \cap N_2} \nabla \xi \cdot \nabla (w - S_h w) + \sum_{E \in \mathcal{E}_h} \int_{E \cap (\Omega \setminus N_2)} \nabla \xi \cdot \nabla (w - S_h w)$$
Using (6.20), we have

\[ \leq \|\nabla (w - S_h w)\|_{L^\infty(\Omega_N)} \sum_{E \in \mathcal{E}_h^0} \|\nabla \xi\|_{L^1(E \cap N_2)} + \|\nabla_h \xi\|_{L^2(\Omega \setminus N_2)} \|\nabla (w - S_h w)\|_{L^2(\Omega \setminus N_2)} \]

\[ = \Theta_1 + \Theta_1^1. \tag{6.16} \]

Fix \( \theta \in (0, 1/2) \), define \( \alpha = 1 - \theta^2 \), which implies that \( 3/4 < \alpha < 1 \). Take \( s = 2/(3\theta) \) in Lemma 6. We have

\[ \|\xi\|_{DG, \alpha} \leq C h^{1-\theta}. \tag{6.17} \]

Hence, with Cauchy–Schwarz’s inequality and the fact that \( d^{-\alpha} \in L^2(\Omega) \), (recall \( d \) is the distance function defined in (2.2)), we obtain

\[ \sum_{E \in \mathcal{E}_h} \|\nabla \xi\|_{L^1(E \cap N_2)} \leq \sum_{E \in \mathcal{E}_h} \|d^{-\alpha}\|_{L^2(E \cap N_2)} \|\nabla \xi\|_{L^2(E \cap N_2)} \leq C \|\nabla_h \xi\|_{L^2(\Omega)} \leq C h^{1-\theta}. \tag{6.18} \]

In addition, observe that since \(-\Delta w = 0\) in \( N_3 \), Theorem 8.10 in [19] and elliptic regularity due to the convexity of the domain yield

\[ \|w\|_{W^{4,2}(N_3)} \leq C \|w\|_{H^2(\Omega)} \leq C \|\xi\|_{L^2(\Omega \setminus N)}. \tag{6.19} \]

Hence, by a Sobolev embedding result and approximation properties there is \( h_1 > 0 \) such that for all \( h \leq h_1 \)

\[ \|\nabla (w - S_h w)\|_{L^\infty(N_2)} \leq C h \|w\|_{W^{4,2}(N_3)} \leq C h \|w\|_{W^{4,2}(N_3)} \leq C h \|\xi\|_{L^2(\Omega \setminus N)}. \tag{6.20} \]

With (6.18) and (6.20), we obtain

\[ |\Theta_1^1| \leq C h^{2-\theta} \|\xi\|_{L^2(\Omega \setminus N)}. \tag{6.21} \]

For \( \Theta_1^2 \), we apply Lemma 4.1 by Chen and Chen [7] (see (3.20) with \( D = \Omega \setminus N_1 \) and \( \tilde{D} = \Omega \setminus N_2 \)). There exists \( h_2 \geq 0 \) such that for all \( h \leq h_2 \)

\[ \|\nabla_h \xi\|_{L^2(\Omega \setminus N_2)} \leq C h \|U\|_{H^2(\Omega \setminus N_1)} + C \|\xi\|_{L^2(\Omega \setminus N_1)}. \]

With Lemma 8, (4.11), and (4.17), we have

\[ \|\nabla_h \xi\|_{L^2(\Omega \setminus N_2)} \leq C h \left( \|f\|_{L^2(\Lambda)} + \|u\|_{H^2(\Omega \setminus N_2)} \right) + C h^2 \|U\|_{H^2(\Omega)} \leq C h \left( \|f\|_{L^2(\Lambda)} + \|u\|_{H^2(\Omega \setminus N_2)} \right). \]

With approximation properties and an elliptic bound, we have

\[ \|\nabla (w - S_h w)\|_{L^2(\Omega \setminus N)} \leq C h \|w\|_{H^2(\Omega)} \leq C h \|\xi\|_{L^2(\Omega \setminus N)}. \]

So we combine the bounds above:

\[ |\Theta_1^2| \leq C h^2 \|\xi\|_{L^2(\Omega \setminus N)}. \tag{6.22} \]

Similarly, we split and bound \( \Theta_2 \). For any domain \( \mathcal{O} \), let \( \Gamma_h(\mathcal{O}) \) denote the set of all faces \( e \) such that \( e \cap \mathcal{O} \neq \emptyset \) and let \( \Gamma_h^\circ(\mathcal{O}) \) be the complementary set of faces, namely \( \Gamma_h^\circ(\mathcal{O}) = (\Gamma_h \cup \{e : e \subset \partial\Omega\}) \setminus \Gamma_h(\mathcal{O}) \). There exists \( h_3 > 0 \) such that for all \( h \leq h_3 \):

\[ |\Theta_2| \leq \|\nabla (w - S_h w)\|_{L^\infty(\Omega_N)} \sum_{e \in \Gamma_h(\mathcal{N}_1)} \|\xi\|_{L^1(e)} + \sum_{e \in \Gamma_h^\circ(\mathcal{N}_1)} \|\nabla (w - S_h w)\cdot n_e\|_{L^2(e)} \|\xi\|_{L^2(e)} = \Theta_1^2 + \Theta_1^2. \]

Using (6.20), we have

\[ \Theta_1^2 \leq C h \|\xi\|_{L^2(\Omega \setminus N)} \sum_{e \in \Gamma_h(\mathcal{N}_1)} \|\xi\|_{L^1(e)}. \]
To handle the second factor in the left-hand side of the inequality above, we introduce a tubular domain $B_h$ containing $\Lambda$. That is, $B_h$ is the set of elements $E$ such that for any $x \in E$, the distance $d(x, \Lambda) \leq 2h$. This implies that the number of elements in $B_h$ is bounded above by $Ch^{-1}$ for some constant $C$ independent of $h$.

$$
\sum_{e \in \Gamma_h(N_1 \setminus B_h)} \|\xi\|_{L^1(e)} \leq C \left( \sum_{e \in \Gamma_h(B_h)} h \|d^{-\alpha}\|^2_{L^2(E)} \right)^{1/2} \|\xi\|_{DG} \leq Ch\|\xi\|_{DG}.
$$

Any face $e \in \Gamma_h(N_1 \setminus B_h)$ belongs to two elements, say $E^1_\epsilon$ and $E^2_\epsilon$. Since $d_{E}^{\alpha-1} \leq h^{-\alpha-1}$, the function $d^{-\alpha}$ belongs to $H^1(E^i_\epsilon)$, for $i = 1, 2$. With the trace inequality (3.17) and with (5.11)

$$
\sum_{e \in \Gamma_h(N_1 \setminus B_h)} \|\xi\|_{L^1(e)} \leq C \left( \sum_{e \in \Gamma_h(N_1 \setminus B_h)} h \|d^{-\alpha}\|^2_{L^2(E)} \right)^{1/2} \|\xi\|_{DG, \alpha}
$$

$$
\leq C \left( \sum_{e \in \Gamma_h(N_1 \setminus B_h)} \left( \|d^{-\alpha}\|^2_{L^2(E_1^{\epsilon} \cup E_2^{\epsilon})} + h^2 \|d_{E}^{-\alpha-1}\|^2_{L^2(E_1^{\epsilon} \cup E_2^{\epsilon})} \right) \right)^{1/2} \|\xi\|_{DG, \alpha}
$$

$$
\leq C \sum_{e \in \Gamma_h(N_1 \setminus B_h)} \|d^{-\alpha}\|^2_{L^2(E_1^{\epsilon} \cup E_2^{\epsilon})} \|\xi\|_{DG, \alpha}
$$

$$
\leq C \|d^{-\alpha}\|^2_{L^2(\Omega)} \|\xi\|_{DG, \alpha}.
$$

Hence, we use (6.17), (6.20) and the fact $\|\xi\|_{DG} \leq Ch\|U\|_{H^2(\Omega)} \leq C$. We have

$$
|\theta^1_2| \leq Ch^{2-\theta} \|\xi\|_{L^2(\Omega \setminus N)}.
$$

To handle $\theta^2_2$, we use (3.17), approximation properties, Lemma 4.1 in [7] (see (3.20) with $D = \Omega \setminus N_2$ and $\tilde{D} = \Omega \setminus N$), and (6.1). We have

$$
|\theta^2_2| \leq C \left( \sum_{e \in \Gamma^N_h(N_2)} \|\nabla(w - S_h w)^2_{L^2(E_1^{\epsilon} \cup E_2^{\epsilon})} + h^2 \|\nabla^2 w\|^2_{L^2(E_1^{\epsilon} \cup E_2^{\epsilon})} \right)^{1/2} \|\xi\|_{DG(\Omega \setminus N_2)}
$$

$$
\leq Ch|w|_{H^2(\Omega)} \left( h\|U\|_{H^2(\Omega \setminus N)} + \|\xi\|_{L^2(\Omega \setminus N)} \right).
$$

With (4.11) and (4.17), we have

$$
\|\xi\|_{L^2(\Omega \setminus N)} \leq Ch\|f\|_{L^2(\Lambda)}.
$$

Thus, with (6.19), we obtain

$$
|\theta^2_2| \leq Ch^2 \|\xi\|_{L^2(\Omega \setminus N)}.
$$

Combining bounds (6.21)–(6.23), (6.25) with (6.15) yields the result.

The next step is to bound the local $L^2$ norm of the error $U - u_h^{CG}$.

**Lemma 10.** Let Assumptions 1–3 hold. There exist $h_0 > 0$ and a constant $C(\theta)$ independent of $h$ such that for all $h \leq h_0$

$$
\|U - u_h^{CG}\|_{L^2(\Omega \setminus N)} \leq C(\theta) h^{2-\theta}, \quad 0 < \theta < \frac{1}{2}.
$$

(6.26)
Therefore, with approximation properties and convexity of the domain, we have

\[ -\Delta z = (U - u_h^{CG}) \chi_{\Omega \setminus N}, \quad \text{in } \Omega, \]
\[ z = 0, \quad \text{on } \partial \Omega, \]

where \( \chi_{\Omega \setminus N} \) is the characteristic function associated to \( \Omega \setminus N \). Let \( S_h z \) denote the Scott–Zhang interpolant of \( z \). We multiply (6.27) by \((U - u_h^{CG})\) and integrate by parts.

\[
\|U - u_h^{CG}\|^2_{L^2(\Omega \setminus N)} = \int \nabla z \cdot \nabla (U - u_h^{CG}) = \int \nabla (z - S_h z) \cdot \nabla (U - u_h^{CG})
\]
\[
\leq C \|\nabla (z - S_h z)\|_{L^\infty(N_1)} \|\nabla (U - u_h^{CG})\|_{L^1(N_1)}
\]
\[
+ \|\nabla (z - S_h z)\|_{L^2(\Omega \setminus N_1)} \|\nabla (U - u_h^{CG})\|_{L^2(\Omega \setminus N_1)}. \tag{6.29}
\]

The first term is handled like \( \Theta_1 \). Let \( \alpha = 1 - \theta^2 \) and use Lemma 7 with \( s = 2/(3\theta) \) to obtain for \( h \) small enough:

\[
\|\nabla (z - S_h z)\|_{L^\infty(N_1)} \|\nabla (U - u_h^{CG})\|_{L^1(N_1)} \leq Ch \|\nabla w\|_{W^{2,\infty}(N_2)} \|\nabla (U - u_h^{CG})\|_{L^2(\Omega)}
\]
\[
\leq C h^{2-\theta} \|U - u_h^{CG}\|_{L^2(\Omega \setminus N)}. \tag{6.30}
\]

For the second term, we use Theorem 9.1 in [36], (6.1), (4.9), (4.15) and (4.8).

\[
\|\nabla (U - u_h^{CG})\|_{L^2(\Omega \setminus N_1)} \leq C \left( h \|U\|_{H^2(\Omega \setminus N_0)} + \|U - u_h^{CG}\|_{L^2(\Omega)} \right) \leq C h. 
\]

Therefore, with approximation properties and convexity of the domain, we have

\[
\|z - S_h z\|_{L^2(\Omega \setminus N_1)} \|\nabla (U - u_h^{CG})\|_{L^2(\Omega \setminus N_1)} \leq C h^2 \|z\|_{H^2(\Omega)} \leq C h^2 \|U - u_h^{CG}\|_{L^2(\Omega \setminus N)}. \tag{6.31}
\]

Bound (6.26) immediately follows from (6.29) to (6.31). \( \square \)

**Proof of Theorem 3.** The result follows by the triangle inequality:

\[
\|u - u_h^{DG}\|_{L^2(\Omega \setminus N)} \leq \|u - u_h^{CG}\|_{L^2(\Omega \setminus N)} + \|u_h^{CG} - U\|_{L^2(\Omega \setminus N)} + \|U - u_h^{DG}\|_{L^2(\Omega \setminus N)}. \tag{6.32}
\]

The first term is bounded in [27]:

\[
\|u - u_h^{CG}\|_{L^2(\Omega \setminus N)} \leq C h^2 |\ln h|. 
\]

The result then follows by using Lemmas 9 and 10. \( \square \)

### 6.2. Local \( L^2 \) bounds for \( k \geq 2 \)

In this section, we use duality arguments to obtain a local \( L^2 \) estimate for \( k \geq 2 \). The main difference from the previous section is that these results hold on a convex sub-domain \( B \) with \( \overline{B} \subset \Omega \) that does not contain the line \( \Lambda \). This is in contrast to the set \( N \) in Theorem 3 which is a neighborhood of the line \( \Lambda \). We use negative norms, recalled here. The domain \( B \) is assumed to have sufficiently smooth boundary, for instance \( B \) can be chosen as a sphere. For any integer \( m \geq 0 \) and for \( v \in L^2(\Omega) \),

\[
\|v\|_{H^{-m}(B)} = \sup_{\phi \in H^m_0(B)} \frac{|\int_B v \phi|}{\|\phi\|_{H^m(B)}}, \quad B \subseteq \Omega. \tag{6.33}
\]

The main result of this section is given in Theorem 4. To begin this analysis, we first establish general local results for the dG approximation. Such results are shown with techniques adapted from Nitsche and Schatz.
In addition, for any convex domain $B \subseteq \Omega$ with sufficiently smooth boundary, we introduce the operator $Q_B : L^2(B) \to H^2(\Omega) \cap H_0^3(\Omega)$ with $Q_B(\phi) = v$ such that $v$ solves
\begin{align*}
-\Delta v &= \phi \quad \text{in } B \\
v &= 0, \quad \text{on } \partial B.
\end{align*}
(6.34)
(6.35)

The following elliptic regularity result holds [18]. For any integer $m \geq 0$,
\begin{equation}
\|Q_B(\phi)\|_{H^{m+2}(B)} \leq C\|\phi\|_{H^m(\Omega)}.
\end{equation}
(6.36)

**Lemma 11.** Let Assumption 1 hold. Let $B, B_1$ be open convex sets satisfying $B \subset \overline{B} \subset B_1 \subset \overline{B_1} \subset \Omega$. Assume $\partial B$, $\partial B_1$, and $\partial \Omega$ are sufficiently smooth so that (6.36) holds in the respective domains. There exists $h_0 > 0$ such that for any integer $m \geq 0$ and all $0 < h \leq h_0$
\begin{equation}
\|U - u_h^{DG}\|_{H^{-m}(\Omega)} \leq C\left(h^{\min(k,m+1)}\|U - u_h^{DG}\|_{DG(B)} + \|U - u_h^{DG}\|_{H^{-m-1}(B)}\right).
\end{equation}
(6.37)

In addition, we have
\begin{equation}
\|U - u_h^{DG}\|_{L^2(\Omega)} \leq C\left(h\|U - u_h^{DG}\|_{DG(B)} + \|U - u_h^{DG}\|_{H^{-m}(B)}\right).
\end{equation}
(6.38)

The constant $C$ is independent of $h$.

**Proof.** Fix an integer $m \geq 0$ and denote $\xi = U - u_h^{DG}$. Let $\omega \in C_0^\infty(\Omega)$ with $\omega = 1$ in $B$ and $\omega = 0$ in $\Omega \setminus B_0$ where $B \subset B_0 \subset \overline{B_0} \subset B_1$. Note that supp$(\omega) \subset B_0$. We have
\begin{equation}
\|\xi\|_{H^{-m}(\Omega)} = \|\omega\xi\|_{H^{-m}(\Omega)} \leq \sup_{\phi \in H^m_0(\Omega)} \frac{|\int_\Omega \omega \xi \phi|}{\|\phi\|_{H^m(\Omega)}}.
\end{equation}
(6.39)

Fix $\phi \in H^m_0(\Omega)$ and define $v = Q_\Omega(\phi)$. We multiply (6.34) with $\omega \xi$ and integrate by parts. Since $v \in H^2(\Omega)$, we have
\begin{equation}
\int_\Omega \omega \xi \phi = \sum_{E \in \mathcal{E}_h} \int_E \nabla v \cdot \nabla (\omega \xi) - \sum_{e \in \Gamma_h \cup \partial \Omega} \int_e \{\nabla v\} \cdot n_e \omega \xi = a(\omega \xi, v).
\end{equation}
(6.40)

In view of (6.40) and (6.36), (6.39) yields
\begin{equation}
\|\xi\|_{H^{-m}(\Omega)} \leq C \sup_{v \in H^{m+2}(\Omega)} \frac{|a(\omega \xi, v)|}{\|v\|_{H^{m+2}(\Omega)}}.
\end{equation}
(6.41)

Observe that
\begin{equation*}
a(\omega \xi, v) = \sum_{E \in \mathcal{E}_h} \int_E \xi \nabla \omega \cdot \nabla v + \sum_{E \in \mathcal{E}_h} \int_E \nabla \xi \cdot (\nabla (\omega v) - v \nabla \omega) - \sum_{e \in \Gamma_h \cup \partial \Omega} \int_e \{\nabla (\omega v) - v \nabla \omega\} \cdot n_e \xi.
\end{equation*}

In addition, with integration by parts and the fact that $v \nabla \omega$ is continuous, we have
\begin{equation}
- \sum_{E \in \mathcal{E}_h} \int_E \nabla \xi \cdot (v \nabla \omega) = \sum_{E \in \mathcal{E}_h} \int_E \xi \nabla \cdot (v \nabla \omega) - \sum_{e \in \Gamma_h \cup \partial \Omega} \int_e \{v \nabla \omega\} \cdot n_e \xi.
\end{equation}
(6.42)
Hence, we obtain
\[ a(\omega \xi, v) = a(\xi, \omega v) + I(\xi, \omega v), \] (6.43)
with
\[ I(\xi, \omega v) = \sum_{E \in \mathcal{E}_h} \int_E \xi (\nabla \omega \cdot \nabla v + \nabla \cdot (v \nabla \omega)). \]

For \( E \in \mathcal{E}_h \) with \( E \cap B_1 \neq \emptyset \), let \( y_{h, E} \in P^k(E) \) be the Lagrange interpolant of \( \omega v \) satisfying
\[ \|\omega v - y_{h, E}\|_{H^d(E)} \leq C h^{\min(k+1,m+2-d)} \|\omega v\|_{H^{m+2}(E)}; \quad 0 \leq d \leq 2. \] (6.44)

Then, define \( \chi_h \in V_h^k(\mathcal{E}_h) \) as \( \chi_h|_E = y_{h, E} \) if \( \omega v|_E \neq 0 \) a.e in \( E \). Otherwise, \( \chi_h|_E = 0 \). By construction, for \( h \) small enough, all the terms involving elements and edges that do not intersect \( B_1 \) vanish. Using (5.22) and continuity properties, we have
\[ a(\xi, \omega v) = a(\xi, \omega v - \chi_h) \leq C \|\xi\|_{DG(B_1)} \|\omega v - \chi_h\|_{DG(B_1)}. \] (6.45)

From trace estimates and (6.44), we have
\[ \|\omega v - \chi_h\|_{DG(B_1)} \leq C h^{\min(k,m+1)} \|\omega v\|_{H^{m+2}(B_1)} \leq C h^{\min(k,m+1)} \|v\|_{H^{m+2}(B_1)}. \] (6.46)

Therefore, (6.45) becomes
\[ a(\xi, \omega v) \leq C h^{\min(k,m+1)} \|\xi\|_{DG(B_1)} \|v\|_{H^{m+2}(B_1)}. \] (6.47)

For the second term in (6.43), since \( \omega \in C^\infty(\Omega) \) with \( \text{supp}(\omega) \subset B_0 \),
\[ I(\xi, \omega v) \leq C \|\xi\|_{H^{-m-1}(B_1)} \|v\|_{H^{m+2}(B_1)}. \] (6.48)

With (6.47) and (6.48), (6.41) yields (6.37). To show (6.38), define a finite sequence of nested convex sets \( D_0 = B \subset D_1 \subset \cdots \subset D_{m-1} = B_1 \) such that \( \bar{D}_i \subset D_{i+1} \). Applying (6.37) with \( s = 0 \) for the sets \( D_0 \subset D_1 \) yields:
\[ \|\xi\|_{L^2(B)} \leq C h \|\xi\|_{DG(D_1)} + \|\xi\|_{H^{-1}(D_1)}. \] (6.49)

Iteratively applying bound (6.37) to the last term in the above inequality yields (6.38).

**Theorem 4.** Assume 1. Fix \( \theta \in (0, \frac{1}{2}) \), \( k \geq 1 \), and two convex sets \( B \) and \( \bar{B} \) with \( \bar{B} \subset \bar{B} \subset \bar{B} \subset \Omega \) with \( \Lambda \subset \Omega \). Assume that \( u \in H^k(\bar{B}) \) and that \( \partial B, \partial \bar{B}, \) and \( \partial \Omega \) are sufficiently smooth. There exist \( h_0 > 0 \) and a constant \( C(\theta) \) independent of \( h \) such that
\[ \|u - u_h^{DG}\|_{L^2(B)} \leq C(\theta) h^{k-\theta}. \] (6.50)

**Remark 1.** We remark that this result is not optimal. However, for \( k \geq 2 \), it is an improvement to the order of convergence provided in Theorem 1. In addition, it allows us to show almost optimal estimates for the local energy norm, see Section 6.3.

**Proof.** First, we apply the triangle inequality to obtain
\[ \|u - u_h^{DG}\|_{L^2(B)} \leq \|u - u_h^{DG}\|_{L^2(B)} + \|u_h^{DG} - U\|_{L^2(B)} + \|U - u_h^{DG}\|_{L^2(B)}. \] (6.51)

The remainder of the proof will consist of bounding each of the above terms. We divide this task into several steps. We select convex sets \( B_0, B_1, \ldots, B_k \) with smooth boundaries such that \( B \subset B_0, \bar{B}_i \subset B_{i+1} \) for \( i = 0, \ldots, k-1 \), and \( \bar{B}_k \subset \bar{B} \). For simplicity, in this proof, we drop the dependence on \( \theta \) and use \( C \) instead.
**Step 1.** Bounding $\|u - u_h^{CG}\|_{L^2(B)}$: Since $W_h^k(\mathcal{E}_h) \subset W_0^{1,q}(\Omega)$, we have the following Galerkin orthogonality property.

$$\int_{\Omega} \nabla (u - u_h^{CG}) \cdot \nabla v_h = 0, \quad \forall v_h \in W_h^k(\mathcal{E}_h). \tag{6.52}$$

Thus, we apply Theorem 5.1 in [30]. There exists $h_1 \geq 0$ such that for all $h \leq h_1$, we have

$$\|u - u_h^{CG}\|_{L^2(B)} \leq C \left( h^k \|u\|_{H^k(B_0)} + \|u - u_h^{CG}\|_{H^{-k}(\Omega)} \right). \tag{6.53}$$

To estimate the second term, fix $\phi \in H_0^k(\Omega)$. Observe that with a Sobolev embedding result and (6.36), we have

$$\|Q_\Omega(\phi)\|_{W^{k+1,4}(\Omega)} \leq C \|Q_\Omega(\phi)\|_{H^{k+2}(\Omega)} \leq C \|\phi\|_{H^{k}(\Omega)}.$$  

We denote by $v_h$ the Scott–Zhang interpolant of $Q_\Omega(\phi)$; we have

$$\|\nabla (Q_\Omega(\phi) - v_h)\|_{L^2(\Omega)} \leq C h^k \|Q_\Omega(\phi)\|_{W^{k+1,4}(\Omega)} \leq C h^k \|\phi\|_{H^{k}(\Omega)}.$$  

We multiply (6.34) by $u - u_h^{CG}$ and integrate by parts. By (6.52), we have

$$\int_{\Omega} (u - u_h^{CG}) \phi = \int_{\Omega} \nabla (Q_\Omega(\phi) - v_h) \cdot \nabla (u - u_h^{CG}) \leq \|\nabla (Q_\Omega(\phi) - v_h)\|_{L^2(\Omega)} \|\nabla (u - u_h^{CG})\|_{L^{4/3}(\Omega)} \leq C h^k \|\phi\|_{H^{k}(\Omega)} \|\nabla (u - u_h^{CG})\|_{L^{4/3}(\Omega)}. \tag{6.54}$$

Let $S_h u$ be the Scott–Zhang interpolant of $u$. With the stability of the interpolant (3.19), and (4.2), we have

$$\|\nabla (u - u_h^{CG})\|_{L^{4/3}(\Omega)} \leq \|\nabla (u - S_h u)\|_{L^{4/3}(\Omega)} + \|\nabla (S_h u - u_h^{CG})\|_{L^{4/3}(\Omega)} \leq C |u|_{W^{1,4/3}(\Omega)} + h^{-\theta} \|f\|_{L^2(\Lambda)}. \tag{6.55}$$

With (6.55) and (6.54), we have

$$\|u - u_h^{CG}\|_{H^{-k}(\Omega)} \leq C h^{-\theta}. \tag{6.56}$$

From (6.56) and (6.53), we have

$$\|u - u_h^{CG}\|_{L^2(B)} \leq C h^{-\theta}. \tag{6.57}$$

**Step 2.** Bounding $\|U - u_h^{CG}\|_{L^2(B)}$: Let $N$ be a neighborhood of $\Lambda$ such that $\mathcal{B}_k \subset \Omega \setminus N$. There exists $h_2 > 0$ such that for all $h \leq h_2$, $-\Delta U = 0$ in $\Omega \setminus N$. Theorem 8.10 in [19] and Lemma 8 yield:

$$\|U\|_{H^{k+1}(B_k)} \leq C \|U\|_{H^{1}(\Omega \setminus N)} \leq C. \tag{6.58}$$

An application of Theorem 5.1 in [30] yields, for $h$ small enough, say $h \leq h_2$, for some $h_2 \geq 0$:

$$\|U - u_h^{CG}\|_{L^2(B)} \leq C h^k \|U\|_{H^{k}(B_0)} + C \|U - u_h^{CG}\|_{H^{-k}(\Omega)}. \tag{6.59}$$

We perform a similar duality argument as above. For any $\phi \in H_0^k(\Omega)$, we denote $z = Q_\Omega \phi$ and $S_h z$ the Scott–Zhang interpolant of $z$,

$$\int_{\Omega} (U - u_h^{CG}) \phi = \int_{\Omega} \nabla (z - S_h z) \cdot \nabla (U - u_h^{CG}) \leq C h^k \|z\|_{H^{k+1}(\Omega)} \|\nabla (U - u_h^{CG})\| \leq C h^k \|\phi\|_{H^{k}(\Omega)} \|\nabla (U - u_h^{CG})\|. \tag{6.60}$$
Theorem 5. Let Assumptions 1–3 hold and fix a set that does not intersect with the boundary. Theorem 5. Let Assumptions 1–3 hold and fix a set that does not intersect with the boundary. is a stronger result in the sense that it is valid up to the boundary of $\Omega$ whereas (6.66) is valid for a domain $\Omega$.

6.3. Local energy estimate

**Step 3.** Bounding $\|U - u_h^{DG}\|_{L^2(B)}$. We denote $\xi = U - u_h^{DG}$ and we iteratively use (3.20) and (6.38) for the nested sets $B \subset B_0 \subset \ldots \subset B_k$. We obtain

$$
\|\xi\|_{L^2(B)} \leq C\left(h^{k+1}\|U\|_{H^{k+1}(B_k)} + h^k\|\xi\|_{L^2(\Omega)}\right) + C\|\xi\|_{H^{-k}(\Omega)}.
$$

(6.62)

To estimate $\|\xi\|_{H^{-k}(\Omega)}$, we also use a duality argument. Let $\phi \in H_0^k(\Omega)$ be given and let $v = Q_\Omega \phi$. We multiply (6.34) by $v$, integrate by parts, use (5.22), the symmetry of $a(\cdot, \cdot)$, and (4.10).

$$
\int_{\Omega} \phi \xi = a(v, \xi) = a(v - S_h v, \xi) \leq C\|v - S_h v\|_{DG}\|\xi\|_{DG} \leq C h^k \|v\|_{H^{k+1}(\Omega)} \leq C h^k \|\phi\|_{H^k(\Omega)}.
$$

(6.63)

This implies that

$$
\|\xi\|_{H^{-k}(\Omega)} \leq C h^k.
$$

(6.64)

With the global estimate (4.11), the bound (6.58), and the above bound, we finally have that

$$
\|\xi\|_{L^2(B)} \leq C h^k.
$$

(6.65)

This concludes the proof.

**6.3. Local energy estimate**

With the local $L^2$ results of the previous sections, we show a local energy estimate. The second bound (6.65) is a stronger result in the sense that it is valid up to the boundary of $\Omega$ whereas (6.66) is valid for a domain that does not intersect with the boundary.

**Theorem 5.** Let Assumptions 1–3 hold and fix $\theta \in (0, \frac{1}{2})$. There exist $h_0 > 0$ and a constant $C(\theta)$ independent of $h$ such that for all $h \leq h_0$ and for any neighborhood $N \subset \Omega$ with $\Lambda \subset N$:

$$
\|u - u_h^{DG}\|_{DG(\Omega \setminus N)} \leq C(\theta) h^{1-\theta}, \quad k = 1.
$$

(6.65)

In addition, let $B$ and $\hat{B}$ be given as in Theorem 4. Under the same assumptions as in Theorem 4, the following estimate holds.

$$
\|u - u_h^{DG}\|_{DG(B)} \leq C(\theta) h^{k-\theta}, \quad k \geq 1.
$$

(6.66)

**Proof.** We show (6.66). By the triangle inequality, we have

$$
\|u - u_h^{DG}\|_{DG(B)} \leq \|u - u_h^{CG}\|_{DG(B)} + \|u_h^{CG} - U\|_{DG(B)} + \|U - u_h^{DG}\|_{DG(B)}.
$$

(6.67)

We proceed by providing bounds on each of the terms above. Let $B_0$ be a convex set with smooth boundary such that $\bar{B} \subset B_0 \subset \overline{B_0} \subset \hat{B}$. Theorem 9.1 in [36] applied to problems (1.1) and (4.5) results in the following two bounds. There exists $h_0 > 0$ such that for all $h \leq h_0$,

$$
\|u - u_h^{CG}\|_{DG(B)} = \|\nabla(u - u_h^{CG})\|_{L^2(B)} \leq C\left(h^{k}\|u\|_{H^{k+1}(B_0)} + \|u - u_h^{CG}\|_{L^2(B_0)}\right),
$$

(6.68)
\[ \|U - u_h^{DG}\|_{DG(B)} = \|\nabla(U - u_h^{DG})\|_{L^2(B)} \leq C \left( h^k \|U\|_{H^{k+1}(B_0)} + \|U - u_h^{DG}\|_{L^2(B_0)} \right). \]  

(6.69)

We apply Lemma 4.1 by Chen and Chen [7]: (3.20) with \( D = B \) and \( \tilde{D} = B_0 \). We obtain:

\[ \|U - u_h^{DG}\|_{DG(B)} \leq C \left( h^k \|U\|_{H^{k+1}(B_0)} + \|U - u_h^{DG}\|_{L^2(B_0)} \right). \]

(6.70)

Employing bounds (6.68)–(6.70) in (6.67), we obtain

\[ \|u - u_h^{DG}\|_{DG(B)} \leq C h^k \left( \|u\|_{H^{k+1}(B_0)} + \|U\|_{H^{k+1}(B_0)} \right) \]

\[ + C \left( \|u - u_h^{DG}\|_{L^2(B_0)} + \|U - u_h^{DG}\|_{L^2(B_0)} \right). \]

(6.71)

Using (6.57), (6.61) and (6.64) in (6.71) yields,

\[ \|u - u_h^{DG}\|_{DG(B)} \leq C h^k \left( \|u\|_{H^{k+1}(B_0)} + \|U\|_{H^{k+1}(B_0)} \right) + C(\theta) h^{k-\theta}. \]

(6.72)

We conclude that (6.66) holds by using bound (6.58) in the above estimate. The proof of bound (6.65) follows the same lines: we apply (6.70) with \( B = \Omega \setminus N \) and \( B_0 = \Omega \setminus \tilde{N} \) where \( N \subset \tilde{N} \).

\[ \square \]

7. THE PARABOLIC PROBLEM

In this section, we consider the time dependent problem (1.4)–(1.6) with a Dirac line source. The domain \( \Omega \) is assumed to be convex, the curve \( \Lambda \) is a \( C^4 \) curve such that \( |E \cap \Lambda| \leq Ch \) for any \( E \in \mathcal{E}_h \). A very weak solution \( u \) to (1.4)–(1.6) can be defined \textit{via} the method of transposition, see [22, 23]. To this end, for a given function \( g \in L^2(0, T; L^2(\Omega)) \), define the backward in time parabolic problem:

\[ -\partial_t \psi - \Delta \psi = g, \quad \text{in } \Omega \times (0, T], \]

(7.1)

\[ \psi = 0, \quad \text{on } \partial \Omega \times (0, T], \]

(7.2)

\[ \psi(T) = 0, \quad \text{in } \{T\} \times \Omega. \]

(7.3)

The solution \( \psi \) belongs to \( L^2(0, T; H^2(\Omega)) \) and the following bounds hold (see Thm. 5, in Sect. 7.1.3 and Thm. 4 in Sect. 5.9.2 in [18])

\[ \|\psi\|_{L^\infty(0, T; H^2(\Omega))} \leq C \left( \|\psi\|_{L^2(0, T; H^2(\Omega))} + \|\partial_t \psi\|_{L^2(0, T; L^2(\Omega))} \right) \leq C \|g\|_{L^2(0, T; L^2(\Lambda))}. \]

(7.4)

If for all \( g \in L^2(0, T; L^2(\Omega)) \), \( u \) satisfies

\[ \int_0^T \int_\Omega u g = \int_0^T \int_\Lambda f \psi + \int_\Omega u^0 \psi(0), \]

(7.5)

where \( \psi \in L^2(0, T; H^2(\Omega)) \) solves (7.1)–(7.3), then \( u \) is referred to as a very weak solution to (1.4)–(1.6). From a Sobolev inequality and (7.4), we have

\[ \int_0^T \int_\Omega u g \leq \|f\|_{L^2(0, T; L^2(\Lambda))} \|\psi\|_{L^2(0, T; L^\infty(\Omega))} + \|u^0\|_{L^2(\Omega)} \|\psi\|_{L^\infty(0, T; L^2(\Omega))} \]

\[ \leq C \left( \|f\|_{L^2(0, T; L^2(\Lambda))} \|\psi\|_{L^2(0, T; H^2(\Omega))} + \|u^0\|_{L^2(\Omega)} \|\psi\|_{L^\infty(0, T; L^2(\Omega))} \right) \]

\[ \leq C \left( \|f\|_{L^2(0, T; L^2(\Lambda))} + \|u^0\|_{L^2(\Omega)} \right) \|g\|_{L^2(0, T; L^2(\Omega))}. \]
Hence, the right hand side of (7.5) defines a bounded linear functional on \(L^2(0,T;L^2(\Omega))\). Thus, with the Lax-Milgram Theorem, a unique solution \(u\) exists in the sense of (7.5). In addition, if \(u^0 \in H^1(\Omega)\), then the very weak solution \(u\) belongs to \(L^2(0,T;W^{1,q}(\Omega)) \cap H^1(0,T;W^{-1,q}(\Omega))\) for \(q \in (1,2)\) and satisfies [23]

\[
\int_0^T (\partial_t u,v) + \int_0^T (\nabla u, \nabla v)\Omega = \int_0^T f v, \quad \forall v \in L^2(0,T;W^{1,q}_0(\Omega)).
\]

(7.6)

We denote by \((\cdot,\cdot)_\Omega\) the \(L^2\) inner product over \(\Omega\). In the above, \(q'\) is the conjugate pair of \(q\), \(W^{-1,q}(\Omega)\) is the dual space of \(W^{1,q}_0(\Omega)\), and \((\cdot,\cdot)\) denotes the duality pairing between \(L^2(0,T;W^{1,q}_0(\Omega))\) and \(L^2(0,T;W^{-1,q}(\Omega))\). We also define the following norm:

\[
\|v\|^2_{L^2(0,T;V^k(\mathcal{E}_h))} = \int_0^T \|v\|^2_{DG}, \quad \forall v \in L^2(0,T;H^2(\mathcal{E}_h)).
\]

(7.7)

7.1. Semi-discrete formulation

We introduce the continuous in time dG approximation \(u^k(t)\) which belongs to \(V^k_h(\mathcal{E}_h)\) for all \(t > 0\) and satisfies:

\[
\int_\Omega \frac{\partial}{\partial t} u^k(t)v + a(u^k(t),v) = \int_\Omega f(t)v, \quad \forall t > 0, \quad \forall v \in V^k_h(\mathcal{E}_h),
\]

(7.8)

\[
\int_\Omega u^k(0)v = \int_\Omega u^0v, \quad \forall v \in V^k_h(\mathcal{E}_h).
\]

(7.9)

We recall that \(a\) is the symmetric bilinear form \((\cdot,\cdot)\) in (3.10) and \(\beta = 1\). We also introduce the dG approximation \(\psi_h(t) \in V^k_h(\mathcal{E}_h)\) to \(\psi(t)\) the solution of (7.1)–(7.3).

\[
-\int_\Omega \frac{\partial}{\partial t} \psi_h(t)v + a(\psi_h(t),v) = \int_\Omega g(t)v, \quad \forall 0 \leq t < T, \quad \forall v \in V^k_h(\mathcal{E}_h),
\]

(7.10)

\[
\psi_h(T) = 0.
\]

(7.11)

The main goal of this section is to establish a global estimate in \(L^2(0,T;L^2(\Omega))\) for the error \(u^k(t) - u\), see Theorem 6. We first establish estimates for the error \(\psi_h(t) - \psi(t)\). Such estimates that depend on the time derivative of \(\psi\) are standard [33]. Here, we follow the arguments in [9] and derive error bounds with constants that depend only on \(\alpha\) and not on \(\partial_t\psi\).

**Lemma 12.** Let Assumption 1 hold. There exists a constant \(C\) independent of \(h\) such that

\[
\|\psi(0) - \psi_h(0)\|_{L^2(\Omega)} + \|\psi - \psi_h\|_{L^2(0,T;DG)} \leq C h \left(\|\psi\|_{L^\infty(0,T;H^1(\Omega))} + \|\psi\|_{L^2(0,T;H^2(\Omega))}\right).
\]

(7.12)

**Proof.** The proof applies the arguments in [9] to a dG discretization of the backward problem. Define \(R_h\psi(t) \in V^k_h(\mathcal{E}_h)\) as the elliptic projection of \(\psi(t)\)

\[
a(R_h\psi(t) - \psi(t),v) = 0, \quad \forall v \in V^k_h(\mathcal{E}_h), \quad \forall t \in (0,T].
\]

(7.13)

From the consistency property of the dG discretization (7.13) and (7.10), we have the following relation.

\[
-(\partial_t\psi(t) - \partial_t\psi_h(t),v)\Omega + a(R_h\psi(t) - \psi_h(t),v) = 0, \quad \forall v \in V^k_h(\mathcal{E}_h).
\]

(7.14)

Let \(P_h\psi(t)\) be the \(L^2\) projection of \(\psi(t)\) onto \(V^k_h(\mathcal{E}_h)\). Thus, with the above, we can write

\[
-\frac{1}{2} \frac{d}{dt} \|\psi - \psi_h\|^2_{L^2(\Omega)} + a(R_h\psi(t) - \psi_h(t),R_h\psi(t) - \psi_h(t))
\]
It is clear that Assumption 1 holds. Assume that the last inequality follows from the approximation properties and triangle inequality. The final result follows.

Proof. The proof extends the arguments of Theorem 2.5 in [35] given for the continuous Galerkin discretization and adapts it to the backward parabolic problem. We define two linear operators

\[ R \] and \[ \Delta \] symmetric. Indeed, for any \( z, w \in L^2(\Omega) \),

\[ (Rz, w)_\Omega = a(Rz, w) = a(z, Rw) = (z, Rw)_\Omega. \] (7.18)

We also define the discrete Laplacian operator \( \Delta_h : V_h^k(\mathcal{E}_h) \to V_h^k(\mathcal{E}_h) \) satisfying

\[ (\Delta_h w_h, v)_\Omega = -a(w_h, v), \quad \forall v \in V_h^k(\mathcal{E}_h). \]

Since \( a \) is coercive, we also have that \( Q_h(\Delta_h w_h) = -w_h \). With the discrete Laplacian, we can write (7.10) as

\[ -\partial_t \psi_h(t) - \Delta_h \psi_h(t) = P_h g(t). \]
Applying the operator $Q_h$ to the above equality, we obtain
\[-Q_h \partial_t \psi_h(t) + \psi_h(t) = Q_h P_h g(t) = Q_h g(t).\]

On the continuous level, we also have
\[-Q \frac{\partial}{\partial t} \psi(t) + \psi(t) = Q g(t).\]

Define $e_h = \psi_h - \psi$ and $\rho_h = -\psi - Q_h (\Delta \psi)$, then
\[-Q_h \partial_t e_h + e_h = Q_h g + (Q_h - Q) \partial_t \psi - Q g = (Q - Q_h) (-\partial_t \psi - g) = (Q - Q_h) (\Delta \psi) = \rho_h. \tag{7.19}\]

The last equality is obtained with (7.17). This implies
\[-(-Q_h \partial_t e_h, e_h)_\Omega + \frac{1}{2} \|e_h\|_{L^2(\Omega)}^2 \leq \frac{1}{2} \|\rho_h\|_{L^2(\Omega)}^2.\]

Since $Q_h$ is self-adjoint and $Q_h$ commutes with the derivative in time operator, we obtain
\[-\frac{\partial}{\partial t}(e_h, Q_h e_h)_\Omega + \|e_h\|_{L^2(\Omega)}^2 \leq \|\rho_h\|_{L^2(\Omega)}^2. \tag{7.20}\]

We integrate from $t = 0$ to $t = T$ and observe that by coercivity we have
\[(e_h, Q_h e_h)_\Omega = a(Q_h e_h, Q_h e_h) \geq \frac{1}{2} \|Q_h e_h\|_{DG}^2.\]

Hence, since $e_h(T) = 0$,
\[\frac{1}{2} \|Q_h e_h(0)\|_{DG}^2 + \int_0^T \|e_h\|_{L^2(\Omega)}^2 \leq \int_0^T \|\rho_h\|_{L^2(\Omega)}^2. \tag{7.21}\]

In addition, note that by consistency of the dG discretization
\[a(Q_h (-\Delta \psi), v) = (-\Delta \psi, v) = a(\psi, v), \quad \forall v \in V_h^k(\mathcal{E}_h).\]

Thus, we have, if $\psi$ belongs to $L^2(0, T; H^s(\Omega))$
\[\|\rho_h\|_{L^2(\Omega)} = \|\psi + Q_h (\Delta \psi)\|_{L^2(\Omega)} \leq C h^{\min(k+1, s)} \|\psi\|_{L^2(0,T;H^s(\Omega))}.\]

We can then conclude with (7.21). \qed

With Lemmas 12 and 13, we show the main result of this section.

**Theorem 6.** Let $u$ be the very weak solution to (1.4)–(1.6) and let $u_h^{DG}$ satisfies (7.8) and (7.9). Assume 1 holds. There exists a constant $C$ independent of $h$ such that for any $\theta \in (0, \frac{1}{2}),$
\[\|u_h^{DG} - u\|_{L^2(0,T;L^2(\Omega))} \leq C(\theta) h^{1-\theta} \left( \|f\|_{L^2(0,T;L^2(\Omega))} + \|u^0\|_{L^2(\Omega)} \right). \tag{7.22}\]

**Proof.** The proof is based on a duality argument and follows similar techniques as the proof of Theorem 3.4 in [22]. Define $\chi(t) = u_h^{DG}(t) - u(t)$. Fix $g \in L^2(0, T; L^2(\Omega))$ and let $\psi$ solve (7.1)–(7.3). With (7.5), consistency of the dG discretization for (7.1)–(7.3), and the definition of $u_h^{DG}(0)$ (see (7.9)), we have
\[\int_0^T (\chi, g)_{\Omega} = \int_0^T (u_h^{DG}, -\partial_t \psi - \Delta \psi)_{\Omega} - \int_0^T \int_{\Lambda} f \psi - (u^0, \psi(0))_{\Omega}\]
\[
\begin{align*}
R &= \int_0^T (\partial_t \psi, u_h^{DG}) + \int_0^T a(\psi, u_h^{DG}) - \int_0^T \int_\Lambda f \psi - (u^0, \psi(0))_\Omega \\
&= \int_0^T (-\partial_t \psi, u_h^{DG}) + \int_0^T a(\psi, u_h^{DG}) - \int_0^T \int_\Lambda f \psi - (u^0, \psi(0))_\Omega \\
&= (\psi_h(0), u_h^{DG}(0)) + \int_0^T (\partial_t u_h^{DG}, \psi_h) + \int_0^T a(\psi_h, u_h^{DG}) - \int_0^T \int_\Lambda f \psi - (u^0, \psi(0))_\Omega \\
&= (u^0, \psi_h(0) - \psi(0)) + \int_0^T \int_\Lambda (f \psi - \psi(0)) = R_1 + R_2.
\end{align*}
\]

For \( R_1 \), we use Cauchy–Schwarz’s inequality, Lemma 12 and (7.4):
\[
|R_1| \leq \|u^0\|_{L^2(\Omega)} |\psi_h(0) - \psi(0)|_{L^2(\Omega)} \leq Ch\|u^0\|_{L^2(\Omega)} \|g\|_{L^2(0,T;L^2(\Omega))}.
\] (7.23)

For the term \( R_2 \), we use the following trace inequality valid for any \( 2 < q < 3 \) and \( q \leq r < q/(3-q) \) (see [1], Thm. 4.12 and [29], Prop. 2.3).
\[
\|v\|_{L^r(\Lambda)} \leq C(q)\|v\|_{W^{1,q}(\Omega)}, \quad \forall v \in W^{1,q}(\Omega).
\] (7.24)

We denote by \( L_h \psi \) the Lagrange interpolant of \( \psi \) in \( W^k_h(\mathcal{E}_h) \). From Theorem 3.1.6 in [10], we have
\[
\|\psi - L_h \psi\|_{W^{1,q}(E)} \leq C(q)\psi_{H^2(E)}, \quad \forall E \in \mathcal{E}_h.
\] (7.25)

From the above bound and Jensen’s inequality, we obtain
\[
\|\psi - L_h \psi\|_{W^{1,q}(\Omega)} = \left( \sum_{E \in \mathcal{E}_h} \|\psi - L_h \psi\|_{W^{1,q}(E)} \right)^{1/q} \leq \frac{\mathcal{H}^{3/2}}{\mathcal{H}} \left( \sum_{E \in \mathcal{E}_h} \|\psi\|_{H^2(E)}^q \right)^{1/q} \leq \mathcal{H}^{3/2} \|\psi\|_{H^2(\Omega)}.
\] (7.26)

Let \( r \) and \( q \) satisfy the conditions in (7.24) and let \( r' \) be the conjugate exponent of \( r \) \((1/r + 1/r' = 1)\). Note that \( L_h \psi \in W^{1,q}(\Omega) \). Hence, with (7.24) and (7.26), we obtain
\[
\|\psi - L_h \psi\|_{L^r(\Lambda)} \leq C(q)\|\psi - L_h \psi\|_{W^{1,q}(\Omega)} \leq C(q)\mathcal{H}^{3/2} \|\psi\|_{H^2(\Omega)}.
\] (7.27)

With Cauchy–Schwarz’s inequality (3.16), and (7.27), we have
\[
\int_\Lambda f(\psi_h - \psi) = \sum_{E \in \mathcal{T}_h} \int_{E \cap \Lambda} f(\psi_h - L_h \psi_h) + \int_\Lambda f(L_h \psi_h - \psi)
\leq \sum_{E \in \mathcal{T}_h} \|f\|_{L^1(E) \cap \Lambda} \|\psi_h - L_h \psi_h\|_{L^\infty(E)} + \|f\|_{L^{r'}(\Lambda)} \|L_h \psi_h - \psi\|_{L^{r}(\Lambda)}
\leq C \sum_{E \in \mathcal{T}_h} |E \cap \Lambda|^{1/2} \|f\|_{L^2(E) \cap \Lambda} \mathcal{H}^{-3/2} \|\psi_h - L_h \psi_h\|_{L^2(E)} + C(q)\mathcal{H}^{-3/2} \|f\|_{L^{r'}(\Lambda)} \|\psi\|_{H^2(\Omega)}
\leq C\mathcal{H}^{-1} \|f\|_{L^2(\Lambda)} \|\psi_h - L_h \psi_h\|_{L^2(\Lambda)} + C(q)\mathcal{H}^{-3/2} \|f\|_{L^2(\Lambda)} \|\psi\|_{H^2(\Omega)}.
\] (7.28)

The last inequality holds since \( r' < 2 \). From Lemma 13, approximation properties, and (7.4), it then follows that
\[
|R_2| \leq C\mathcal{H}^{-1} \|f\|_{L^2(0,T;L^2(\Lambda))} \|\psi_h - L_h \psi_h\|_{L^2(0,T;L^2(\Omega))} + C(q)\mathcal{H}^{-3/2} \|f\|_{L^2(0,T;L^2(\Lambda))} \|\psi\|_{L^2(0,T;H^2(\Omega))}
\leq C\mathcal{H} \|f\|_{L^2(0,T;L^2(\Lambda))} \|\psi\|_{L^2(0,T;H^2(\Omega))} + C(q)\mathcal{H}^{-3/2} \|f\|_{L^2(0,T;L^2(\Lambda))} \|\psi\|_{L^2(0,T;H^2(\Omega))}.
\]
\[ \leq C h \| f \|_{L^2(0,T;L^2(\Omega))} \| g \|_{L^2(0,T;L^2(\Omega))} + C(q) h^{\frac{3}{2} - \frac{
}{2}} \| f \|_{L^2(0,T;L^2(\Omega))} \| g \|_{L^2(0,T;L^2(\Omega))}. \]

For any \( \theta \in (0, 1/2) \), choose \( q = 6/(3 - 2\theta) \). The bound for \( R_2 \) becomes
\[
| R_2 | \leq C(\theta) h^{1 - \theta} \| f \|_{L^2(0,T;L^2(\Omega))} \| g \|_{L^2(0,T;L^2(\Omega))}. \tag{7.29}
\]

We remark that
\[
\| \chi \|_{L^2(0,T;L^2(\Omega))} = \sup_{g \in L^2(0,T;L^2(\Omega)), g \neq 0} \frac{| \int_0^T (\chi, g) \Omega |}{\| g \|_{L^2(0,T;L^2(\Omega))}}.
\]

Therefore, with (7.23) and (7.29), we can conclude. \( \square \)

### 7.2. Fully discrete formulation

In this section, we consider a backward Euler discretization of problem (1.4)–(1.6). To simplify notation, we drop the subscript DG on the discrete solution, namely \( u_h \).

Assume that 1 holds. There exists a constant \( C \).

Lemma 14. Assume that 1 holds. There exists a constant \( C \) independent of \( \tau \) and \( h \) such that the following estimate holds. For \( 1 \leq m \leq N_T \),
\[
\sum_{n=1}^{m} \| u^n_h - u^{n-1}_h \|^2_{L^2(\Omega)} + \tau \sum_{n=1}^{m} \| u^n_h - u^{n-1}_h \|^2_{DG} + \tau \| u^m_h \|^2_{DG} \leq C \tau h^{-2} \left( \| u^0 \|^2_{L^2(\Omega)} + \| f \|^2_{L^2(0,T;L^2(\Omega))} \right). \tag{7.33}
\]
Proof. Let \( v = u_h^n - u_h^{n-1} \) in (7.30). Using the symmetry of \( a \), we obtain
\[
\|u_h^n - u_h^{n-1}\|_{L^2(\Omega)}^2 + \frac{\tau}{2} (a(u_h^n, u_h^n) - a(u_h^{n-1}, u_h^{n-1}) + a(u_h^n - u_h^{n-1}, u_h^n - u_h^{n-1})) = \tau \int_{\Lambda} f(t^n)(u_h^n - u_h^{n-1}).
\]

We observe that by Hölder’s inequality and (3.16),
\[
\int_{\Lambda} f(t^n)(u_h^n - u_h^{n-1}) \leq \sum_{E \in \mathcal{T}_h} |E \cap \Lambda|^{1/2} \|f(t^n)\|_{L^2(E)} \|u_h^n - u_h^{n-1}\|_{L^\infty(E)} \leq C \sum_{E \in \mathcal{T}_h} h^{-1} \|f(t^n)\|_{L^2(E \cap \Lambda)} \|u_h^n - u_h^{n-1}\|_{L^2(E)}.
\]

With the coercivity (3.15) and the above bound, we obtain
\[
\|u_h^n - u_h^{n-1}\|_{L^2(\Omega)}^2 + \frac{\tau}{4} a(u_h^n, u_h^n) - a(u_h^{n-1}, u_h^{n-1}) + \frac{\tau}{4} \|u_h^n - u_h^{n-1}\|_{DG}^2 \leq C \tau^2 h^{-2} \|f(t^n)\|_{L^2(\Lambda)}^2 + \frac{1}{2} \|u_h^n - u_h^{n-1}\|_{L^2(\Omega)}^2.
\]

With the continuity of \( a \) (3.14), an inverse inequality and the stability of the \( L^2 \) projection, we have
\[
a(u_h^n, u_h^0) \leq C \|u_h^0\|_{DG}^2 \leq Ch^{-2} \|u_h^0\|_{L^2(\Omega)}^2 \leq Ch^{-2} \|u^0\|_{L^2(\Omega)}^2. \tag{7.34}
\]

With the above bound, we conclude the proof. \( \square \)

Proof of Theorem 7. The proof uses some techniques from the proof of Theorem 3.4 in [23]. We first fix \( g \in L^2(0, T; L^2(\Omega)) \) and consider \( \psi \) the solution of (7.1)–(7.3). From (7.5), we have
\[
\int_0^T (u_h, \tau - u, g)_\Omega = \sum_{n=1}^{N_T} \int_{t^{n-1}}^{t^n} (u_h^n, g) - (u_0^0, \psi(0))_\Omega - \int_0^T \int_{\Lambda} f \psi.
\]

We rewrite the first term in the right-hand side as
\[
\int_{t^{n-1}}^{t^n} (u_h^n, g)_\Omega = \int_{t^{n-1}}^{t^n} (u_h^n - \partial_t \psi - \Delta \psi)_\Omega = -(u_h^n, \psi(t^n) - \psi(t^{n-1}))_\Omega + \int_{t^{n-1}}^{t^n} a(u_h^n, \psi)
\]
\[
= (u_h^n - u_h^{n-1}, \psi(t^{n-1}))_\Omega - ((u_h^n, \psi(t^n))_\Omega - (u_h^{n-1}, \psi(t^{n-1}))_\Omega) + \int_{t^{n-1}}^{t^n} a(u_h^n, \psi).
\]

Since \( \psi(T) = 0 \), (7.35) reads
\[
\int_0^T (u_h, \tau - u, g)_\Omega = \sum_{n=1}^{N_T} (u_h^n - u_h^{n-1}, \psi(t^{n-1}))_\Omega + \sum_{n=1}^{N_T} \int_{t^{n-1}}^{t^n} a(u_h^n, \psi)
\]
\[
- (u_0^0 - u_0^0, \psi(0))_\Omega - \int_0^T \int_{\Lambda} f \psi. \tag{7.36}
\]
For each \( t \in (t^{n-1}, t^n] \), choose \( v = R_h \psi(t) \) in \((7.30)\) (recall that \( R_h \psi \) is defined by \((7.13)\)). Integrate the resulting equation from \( t^{n-1} \) to \( t^n \), sum from \( n = 1 \) to \( n = N_T \), and divide by \( \tau \). We obtain

\[
\sum_{n=1}^{N_T} \int_{t^{n-1}}^{t^n} a(u_h^n, R_h \psi(t)) = -\frac{1}{\tau} \sum_{n=1}^{N_T} \int_{t^{n-1}}^{t^n} (u_h^n - u_h^{n-1}, R_h \psi(t))_\Omega + \sum_{n=1}^{N_T} \int_{t^{n-1}}^{t^n} f(t^n) R_h \psi(t). \tag{7.37}
\]

With the definition of \((7.13)\), \((7.36)\) becomes

\[
\int_0^T (u_{h,\tau} - u, g)_\Omega = \frac{1}{\tau} \sum_{n=1}^{N_T} \int_{t^{n-1}}^{t^n} (u_h^n - u_h^{n-1}, \psi(t^{n-1}) - R_h \psi(t))_\Omega - (u^0 - u_h^0, \psi(0))_\Omega + \sum_{n=1}^{N_T} \int_{t^{n-1}}^{t^n} (f(t^n) R_h \psi(t) - f(t) \psi(t)) = E_1 + E_2 + E_3. \tag{7.38}
\]

For \( E_1 \), we introduce \( \psi(t) \) and write

\[
(u_h^n - u_h^{n-1}, \psi(t^{n-1}) - R_h \psi(t))_\Omega = -\left( u_h^n - u_h^{n-1}, \psi(t) - R_h \psi(t) + \int_{t^{n-1}}^{t} \partial_t \psi \right)_\Omega. \]

Therefore, using error bounds of the elliptic projection, we obtain

\[
|E_1| \leq C\tau^{-1} h^2 \sum_{n=1}^{N_T} \int_{t^{n-1}}^{t^n} \|u_h^n - u_h^{n-1}\|_{L^2(\Omega)} \|\psi(t)\|_{H^2(\Omega)}
+ \tau^{-1} \sum_{n=1}^{N_T} \int_{t^{n-1}}^{t^n} \|u_h^n - u_h^{n-1}\|_{L^2(\Omega)} \left( t - t^{n-1} \right)^{1/2} \|\partial_t \psi\|_{L^2(t^{n-1}, t^n; L^2(\Omega))}
\leq C\tau^{-1} h^2 \sum_{n=1}^{N_T} \|u_h^n - u_h^{n-1}\|_{L^2(\Omega)} \|\psi\|_{L^2(t^{n-1}, t^n; H^2(\Omega))} + C\tau^{-1} \sum_{n=1}^{N_T} \|u_h^n - u_h^{n-1}\|_{L^2(\Omega)} \|\partial_t \psi\|_{L^2(t^{n-1}, t^n; L^2(\Omega))}
\leq C \left( \sum_{n=1}^{N_T} \|u_h^n - u_h^{n-1}\|_{L^2(\Omega)}^2 \right)^{1/2} \left( \tau^{-1/2} h^2 \|\psi\|_{L^2(0,T;H^2(\Omega))} + \tau^{1/2} \|\partial_t \psi\|_{L^2(0,T;L^2(\Omega))} \right). \tag{7.39}
\]

With Lemma 14 and \((7.4)\), \((7.39)\) reads

\[
|E_1| \leq C(\tau h^{-1} + h) \|g\|_{L^2(0,T;L^2(\Omega))} \left( \|f\|_{L^2(0,T;L^2(\Omega))} + \|u^0\|_{L^2(\Omega)} \right). \tag{7.40}
\]

The term \( E_2 \) is easily handled since \( u_h^0 \) is the \( L^2 \) projection of \( u^0 \). We use approximation properties of the Lagrange operator \( L_h \) and \((7.4)\)

\[
E_2 = (u^0 - u_h^0, \psi(0) - L_h \psi(0))_\Omega \leq C h \|u^0\|_{L^2(\Omega)} \|\psi(0)\|_{H^1(\Omega)} \leq C h \|u^0\|_{L^2(\Omega)} \|g\|_{L^2(0,T;L^2(\Omega))}. \tag{7.41}
\]

For the term \( E_3 \), we write

\[
\int_{\Lambda} (f(t^n) R_h \psi(t) - f(t) \psi(t)) = \sum_{E \in T_h} \int_{E \cap \Lambda} (f(t^n) - f(t)) R_h \psi(t) + \sum_{E \in T_h} \int_{E \cap \Lambda} f(t) (R_h \psi(t) - \psi(t)) = \mathcal{W}_1 + \mathcal{W}_2.
\]

For \( \mathcal{W}_1 \), we Hölder’s inequality, \((3.16)\) \((g = \infty, p = 6)\) and \((3.12)\). We obtain

\[
|\mathcal{W}_1| \leq \|f(t^n) - f(t)\|_{L^1(\Lambda)} \|R_h \psi(t)\|_{L^\infty(\Omega)} \leq C h^{-\frac{3}{2}} \|f(t^n) - f(t)\|_{L^1(\Lambda)} \|R_h \psi(t)\|_{L^6(\Omega)}.
\]
Therefore, with (7.38) and the bounds (7.40), (7.41) and (7.44), we conclude that for any non-zero $g$ we have

$$W_1 \leq C(t^n - t)^{1/2}h^{-\frac{1}{2}}\|\partial_t f\|_{L^2(t,t';L^1(\Lambda))}\|\psi(t)\|_{H^2(\Omega)}.$$  

(7.42)

For $W_2$, we apply a similar argument as for the derivation of (7.28) (by introducing the Lagrange interpolant $L_h\psi$) and obtain for any $2 < q < 3$

$$W_2 \leq Ch^{-1}\|f(t)\|_{L^2(\Lambda)}\|R_h\psi(t) - L_h\psi(t)\|_{L^2(\Omega)} + C(q)h^{\frac{3}{2}-\frac{1}{2}}\|f(t)\|_{L^2(\Lambda)}\|\psi(t)\|_{H^2(\Omega)}.$$  

(7.43)

Hence, with approximation properties, choosing $q = 6/(3-2\theta)$ for $0 < \theta < 1/2$, and (7.4), the bound on $E_3$ reads

$$|E_3| \leq C_\tau h^{-\frac{1}{2}}\|\partial_t f\|_{L^2(0,T;L^1(\Lambda))}\|\psi\|_{L^2(0,T;H^2(\Omega))} + Ch^{-1}\|f\|_{L^2(0,T;L^2(\Lambda))}\|R_h\psi - L_h\psi\|_{L^2(0,T;L^2(\Omega))}$$

$$+ C(\theta)h^{1-\theta}\|f\|_{L^2(0,T;L^2(\Lambda))}\|\psi\|_{L^2(0,T;H^2(\Omega))}$$

$$\leq (C_\tau h^{-\frac{1}{2}}\|\partial_t f\|_{L^2(0,T;L^1(\Lambda))} + C(\theta)h^{1-\theta}\|f\|_{L^2(0,T;L^2(\Lambda))})\|g\|_{L^2(0,T;L^2(\Omega))}. \quad (7.44)$$

Therefore, with (7.38) and the bounds (7.40), (7.41) and (7.44), we conclude that for any non-zero $g \in L^2(0,T;L^2(\Omega))$

$$\frac{\int_0^T(u_h,\tau - u,g)_{\Omega} \|g\|_{L^2(0,T;L^2(\Omega))}}{C(\tau h^{-1} + h)\|f\|_{L^2(0,T;L^2(\Omega))} + \|u^0\|_{L^2(\Omega)}}$$

$$\leq (C_\tau h^{-1}\|\partial_t f\|_{L^2(0,T;L^1(\Lambda))} + C(\theta)h^{1-\theta}\|f\|_{L^2(0,T;L^2(\Lambda))}). \quad (7.45)$$

We conclude by taking supremum over all $g$. \hfill $\square$

8. Numerical results for elliptic problem

We employ the method of manufactured solutions to test the convergence rates of the scheme (3.9). The domain is $(0,1) \times (0,1) \times (0,0.25)$ and the line $\Lambda$ is the vertical line passing through the point $(2/3,1/3,0)$. The function $f$ is chosen to be the constant function equal to 1. The exact solution is defined by

$$u(x, y, z) = -\frac{1}{2\pi} \ln \left( \left( \frac{x - 2}{3} \right)^2 + \left( \frac{y - 1}{3} \right)^2 \right)^{1/2}. \quad (8.1)$$

We compute the numerical errors on a series of uniformly refined meshes made of tetrahedra. We vary the mesh size and the polynomial degree. The parameters in the definition of the bilinear form are chosen: $\epsilon = -1$, $\beta = 1$. For $k = 1$, we choose $\sigma = 5$ and for $k = 2$, the penalty value is $\sigma = 12$. Figure 1 shows the dG solution for $k = 1$; the size of the mesh is $h = 1/16$ and the domain has been sliced for visualization. Table 1 displays the $L^2$ errors and convergence rates for the numerical solution with $k = 1$ and $k = 2$. When errors are computed over the whole domain $\Omega$, they converge with a rate equal to one, which is consistent with our bound (4.13). Next, we verify the accuracy of the solution away from the line singularity by computing the $L^2$ error in two subdomains $B = (0.25, 0.5) \times (0.5, 0.75) \times (0, 0.25)$ and $B_0 = (0.0, 0.25) \times (0.75, 0.1) \times (0, 0.25)$. Table 1 shows the errors in the $L^2$ norm over $B$ and over $B_0$ as the mesh is uniformly refined. Errors converge with a rate equal to 2, which is optimal for piecewise linear approximations and suboptimal for piecewise quadratic approximation. The numerical rates are consistent with (6.10) for $k = 1$ and (6.50) for $k = 2$. We also remark that the errors in $B$ and in $B_0$ are several order of magnitude smaller than the errors in $\Omega$. 

\[\]
Figure 1. View on sliced domain of the dG approximation obtained on mesh of size $h = 1/16$.

Table 1. Numerical errors and convergence rates for the numerical solution over the whole domain and the two subdomains.

| $k$ | $h$ | $||u - u_h^\text{DG}||_{L^2(\Omega)}$ | $||u - u_h^\text{DG}||_{L^2(B)}$ | $||u - u_h^\text{DG}||_{L^2(B_0)}$ |
|-----|-----|------------------------------------|---------------------------------|----------------------------------|
| 1   | 1/4 | 6.99e-03                           | 1.28e-04                        | 2.54e-05                        |
|     | 1/8 | 2.28e-03                           | 3.00e-05                        | 6.70e-06                        | 1.92    |
|     | 1/16| 1.33e-03                           | 6.60e-06                        | 1.84e-06                        | 1.86    |
|     | 1/32| 7.12e-04                           | 1.63e-06                        | 5.05e-07                        | 1.87    |
| 2   | 1/4 | 1.14e-02                           | 1.09e-04                        | 4.37e-06                        |
|     | 1/8 | 4.27e-03                           | 1.98e-05                        | 7.48e-07                        | 2.55    |
|     | 1/16| 1.56e-03                           | 6.22e-06                        | 1.11e-07                        | 2.75    |
|     | 1/32| 6.14e-04                           | 1.50e-06                        | 1.77e-08                        | 2.65    |

Tables 2 and 3 display the $L^2$ norm of the weighted broken gradient of the error for the values $\alpha = 0.51$ and $\alpha = 0.99$ respectively and for $k = 1$ and $k = 2$. We observe that the convergence rates increase as we increase the polynomial degree. We note that our results in Section 5 are valid only for $k = 1$. For the case $\alpha = 0.51$, the numerical rates are better than the rates predicted by Theorem 2 whereas for the case $\alpha = 0.99$, the numerical rates match the predicted theoretical rates.

To show the robustness of the scheme (3.9), we now consider a sinusoidal-like curve $\Lambda$ made of segments. The numerical parameters are the same as for the manufactured solution but here, we do not know the exact solution. Figure 2 displays the DG solution on a mesh of size $h = 1/10$.

9. Conclusions

Convergence of the class of interior penalty discontinuous Galerkin methods applied to elliptic and parabolic equations with Dirac line-source is proved by deriving error estimates in different norms. Almost optimal error bounds are shown in regions away from the line singularity. The proofs of the error estimates are technical and utilize dual problems and weighted Sobolev spaces. Stronger results are obtained for the case of piecewise linear approximation since local error bounds are valid in regions that may reach the boundary of the domain. In the general case of approximation of degree $k \geq 2$, local error bounds are suboptimal and valid in regions strictly included in the domain. Most of the paper is dedicated to the analysis of the elliptic problem and convexity of the domain is assumed. For the parabolic problem, global error bounds in $L^2$ in time and in space are shown. Future work would address relaxing the convexity assumption and obtaining local error bounds for the time-dependent problem.
Table 2. Numerical error and rates for the weighted energy norm with $\alpha = 0.51$. 

| $k$  | $h$ | Error     | Rate |
|------|-----|-----------|------|
| 1/4  | 1/4 | 6.79e−02  |      |
| 1/8  | 4.90e−02 | 0.47    |      |
| 1/16 | 3.48e−02 | 0.49    |      |
| 1/32 | 2.46e−02 | 0.50    |      |

Table 3. Numerical error and rates for the weighted energy norm with $\alpha = 0.99$. 

| $k$  | $h$ | Error     | Rate |
|------|-----|-----------|------|
| 1/4  | 1/4 | 2.78e−02  |      |
| 1/8  | 1.56e−02 | 0.83    |      |
| 1/16 | 8.53e−03 | 0.87    |      |
| 1/32 | 4.64e−03 | 0.88    |      |

Figure 2. Sliced view of the numerical solution for a piecewise linear curve $\Lambda$. 

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