INDEFINITE NONLINEAR DIFFUSION PROBLEM IN POPULATION GENETICS

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Abstract. We study the following Neumann problem in one dimension,
\[
\begin{cases}
    u_t = du'' + g(x)u^2(1-u) \quad \text{in } (0,1) \times (0, \infty), \\
    0 \leq u \leq 1 \quad \text{in } (0,1) \times (0, \infty), \\
    u'(0,t) = u'(1,t) = 0 \quad \text{in } (0, \infty),
\end{cases}
\]
where \(g\) changes sign in \((0,1)\). This equation models the “complete dominance” case in population genetics of two alleles. It is known that this equation has a nontrivial stable steady state \(U_d\) for \(d\) sufficiently small. We show that \(U_d\) is a unique nontrivial steady state under a condition \(\int_0^1 g(x) \, dx \geq 0\) and some other additional condition.

1. Introduction. This article considers the “complete dominant” case of a migration selection model for the solution of gene frequency at a single locus with two alleles \(A_1, A_2\) initiated in [7] and [3]. This model is due to T. Nagylaki in 1975 [4]. We shall give a brief description of this model following the more recent presentation in [2] and [7].

Let \(u(x,t)\) be the frequency of allele \(A_1\) at time \(t\) and location \(x\) (thus \(0 \leq u \leq 1\)), and \(r_{ij}\) be the fitness (local selection coefficient) of the genotype \(A_i, A_j\) for \(i,j = 1,2\) and
\[
r_1 = r_{11}u + r_{12}(1-u)
\]
is the marginal selection coefficient of \(A_1\), and
\[
r = r_{11}u^2 + r_{12}u(1-u) + r_{21}(1-u)u + r_{22}(1-u)^2
\]
is the mean selection coefficient of the population. Now positing
\[
r_{11} = 1, \quad r_{12} = r_{21} = 1 - hg(x), \quad r_{22} = 1 - g(x),
\]
where \(g(x)\) reflects the “environmental variation” and \(0 \leq h \leq 1\) specifies the degree of dominance (assumed to be independent of the location), we have the selection term:
\[
S_1 = \lambda g(x)u(1-u)[hu + (1-h)(1-u)],
\]
where \(\lambda > 0\) is the ratio of selection intensity to the migration rate. Therefore, under some additional simplification assumptions, the migration-selection model describing the evolution of gene frequencies at a single locus with two alleles takes the following form
\[
\begin{cases}
    u_t = \Delta u + S_1(x,u) \quad \text{in } \Omega \times (0, \infty), \\
    \partial_\nu u = 0 \quad \text{on } \partial \Omega \times (0, \infty),
\end{cases}
\]

2010 Mathematics Subject Classification. 35J25, 35B25.

Key words and phrases. Reaction diffusion equation, singular perturbation, layers.
where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, the habitat $\Omega$ is a bounded domain with smooth boundary $\partial \Omega$ in $\mathbb{R}^n$, $\nu$ denotes the unit outward normal to $\partial \Omega$ and $\partial_{\nu}$ is the normal derivative on $\partial \Omega$. After scaling $t$, we have

$$
\begin{cases}
  u_t = d\Delta u + g(x)u(1-u)[hu + (1-h)(1-u)] & \text{in } \Omega \times (0, \infty), \\
  \partial_{\nu}u = 0 & \text{on } \partial \Omega \times (0, \infty). \\
\end{cases} \tag{1.4}
$$

It is clear that (1.4) has no nontrivial steady-states if $g(x)$ does not change sign in $\Omega$, i.e. in this case a steady state $u$ is either $u \equiv 0$ or $u \equiv 1$, implying that only one allele survives eventually. Thus, in order to sustain both alleles $A_1$ and $A_2$, the environmental variation has to be so significant that the selection reverses its direction at least once in $\Omega$, i.e. $g(x)$ changes sign at least once in $\Omega$ (Note that if $g(x) > 0$, then $r_{11} \geq r_{12} \geq r_{22}$ at location $x$; while $r_{11} \leq r_{12} \leq r_{22}$ (where $g(x) < 0$). Therefore we shall require in the rest of this paper that $g(x)$ changes sign in $\Omega$.

As was explained in [7] that all previous mathematical results in literature deal with the case $0 < h < 1$ and the “complete dominant” case $h = 1$ was left untouched until the publications of the papers [7] and [3] in 2010. In the case $h = 1$ we have $r_{12} = r_{22}$, i.e. the heterozygote $A_1A_2$ has the same fitness as the homozygote $A_2A_2$, we say that $A_2$ is completely dominant to $A_1$ (In the case $h = 0$, in the same way, $A_1$ is completely dominant to $A_2$). In the “completely” dominant case $h = 1$, (1.4) becomes

$$
\begin{cases}
  u_t = d\Delta u + g(x)u^2(1-u) & \text{in } \Omega \times (0, \infty), \\
  \partial_{\nu}u = 0 & \text{on } \partial \Omega \times (0, \infty). \\
\end{cases} \tag{1.5}
$$

This “completely dominant” case is not only mathematically challenging but also biologically important. In fact the following conjectures (a) to (c) have been proposed for a long time (See Lou and Nagylaki [1]):

(a) If $\int_{\Omega} g(x) \, dx = 0$, then for every $d > 0$, the problem (1.5) has unique nontrivial steady state which is globally asymptotically stable.

(b) If $\int_{\Omega} g(x) \, dx > 0$, then there exists $d_0 > 0$ such that for every $d \in (0, d_0)$, problem (1.5) has a unique nontrivial steady state which is globally asymptotically stable.

(c) If $\int_{\Omega} g(x) \, dx < 0$, then there exists $d_0 > 0$ such that for every $d \in (0, d_0)$, problem (1.5) has exactly two nontrivial steady states, one is asymptotically stable and the other is unstable.

There are two mathematically rigorous results towards the resolution of these important conjectures, [7] and [3]. In [7], the existence of a stable steady state as well as its limiting behavior (as $d$ tends to 0 or $\infty$) are obtained. Furthermore, in [3] the existence of at least two steady states, one stable and the other unstable, is established as well. However, the uniqueness part of the conjecture is still left open. Therefore, in this paper we survey the uniqueness part of (a) and (b) above, under the condition where the spatial dimension $n = 1$. A steady state solution of (1.5) satisfies

$$
\begin{cases}
  du'' + g(x)u^2(1-u) = 0 & \text{in } (0, 1), \\
  u'(0) = u'(1) = 0, \\
\end{cases} \tag{1.6}
$$

under the additional “nondegeneracy” condition on $g$:

All zeros on $[0, 1]$ are interior and nondegenerate; i.e.

(H) If $g(x_i) = 0$, then $x_i \in (0, 1)$ and $g'(x_i) \neq 0$, for $i = 1, 2, \cdots, m$. 

(1)
In [8], we have proved the uniqueness under the extra assumption that \( g \) has only one zero \( x_0 \) in \((0, 1)\), i.e. \( m = 1 \) in \((H)\) under the condition \( \int_0^1 g(x) \, dx \geq 0 \). In this paper we study the uniqueness in the case where \( g \) has multiple nondegenerate zeros in \((0, 1)\), again under the condition \( \int_0^1 g(x) \, dx \geq 0 \). However, in this case, we need some more conditions on \( g(x) \), in addition to \( \int_0^1 g(x) \, dx \geq 0 \) to prove the uniqueness of solutions.

The followings are our results on the uniqueness in [9]. To state our uniqueness result precisely, we first introduce the following auxiliary function \( p(x) \in C^1[0, 1] \) satisfying the following (P0) to (P3).

(P0) \( z_1 \) is the only zero of \( p(x) \) in \([0, \frac{1}{m}]\), and \( p'(z_1) \neq 0 \).

(P1) \( p(x) \) is symmetric in \( [\frac{k-1}{m}, \frac{k+1}{m}] \) with respect to \( x = \frac{k}{m} \) \( (k = 1, 2, \cdots, m-1) \).

(P2) \( p(x) \) is monotone in \( [\frac{k-1}{m}, \frac{k}{m}] \) \( (k = 1, 2, \cdots, m) \).

(P3) \( \int_0^1 p(x) \, dx \geq 0 \).

**Theorem 1.1.** Suppose that \( g(x) \geq p(x) \) with \( p(x) \) satisfying (P0) – (P3). Then, for every \( d \) small, \((1.6)\) has a unique nontrivial solution.

By the assumption \( g(x) \geq p(x) \) and the condition (P3), it holds that \( \int_0^1 g(x) \, dx \geq 0 \). (P0) is a condition corresponding to the nondegeneracy condition \((\mathcal{H})\).

In addition, the following result is important.

**Theorem 1.2.** Suppose that \( g \) changes its sign in \((0, 1)\) and that \((H)\) holds. Then, \((1.6)\) has a linearly stable nontrivial solution \( u_d \) for \( d \) sufficiently small. Furthermore, \( u_d \) has the following properties:

(i) On every compact subset \( S_1 \) of \([g < 0]\),

\[
C_1 d < U_d < C_2 d,
\]

where the constants \( C_1, C_2 \) depend on \( S_1 \).

(ii) On every compact subset \( S_2 \) of \([g > 0]\),

\[
C_3 \exp \left( -\frac{C_4}{\sqrt{d}} \right) < 1 - U_d < C_5 \exp \left( -\frac{C_6}{\sqrt{d}} \right),
\]

where the constants \( C_3, C_4, C_5, C_6 \) depend on \( S_2 \).

The stability and the facts that \( U_d \to 0 \) in \([g < 0]\) and \( U_d \to 1 \) in \([g > 0]\) were already established earlier in [7] for any dimension \( N \). However, the rates of convergence and linearized stability have not been obtained. These rates, \((1.7)\) and \((1.8)\) and linearized stability are essential to prove the uniqueness of \( U_d \).

This article is organized as follows. In Sections 2 - 3, we will show a proof of uniqueness result Theorems 1.1. In Section 4, we will introduce recent results Theorems 4.1 - 4.3, which give counterexamples of Conjectures (b) and (c).

2. **Construction of upper and lower solutions.** In this section, we will construct, for every \( d > 0 \) small, a pair of upper solution \( U^* \) and lower solution \( U_* \), both exhibit transition layer near every zero \( x_i \) of \( g \ (i \in \{1, 2, \cdots, m\}) \) and \( U^* > U_* \) on \([0, 1]\). This will guarantee the existence of a solution \( U_d \) such that \( U^* \geq U_d \geq U_* \).
Thus $U_d$ also exhibits desired transition layer at $x_i$. Upper solution $U^*$ and lower solution $U_*$ will be useful in later sections in serving as a barrier for our proof of the uniqueness of solutions.

Setting $d = e^3$, (1.6) takes the following form:

$$
\begin{cases}
    e^3 u'' + g(x)u^2(1 - u) = 0 \quad \text{in } (0, 1), \\
    w'(0) = w'(1) = 0,
\end{cases}
$$

(2.1)

where $g(x) \in C^1[0, 1]$ has zeros denoted by $\{x_i \in (0, 1) | i = 1, 2, \ldots, m\}$.

We will first construct a lower solution of (2.1) with a transition layer of width $\epsilon$ near each $x_i$.

By the nondegeneracy condition (H) in the Introduction, there are two cases $g'(x_i) > 0$ and $g'(x_i) < 0$. We first assume $g'(x_i) > 0$, since the case $g'(x_i) < 0$ is treated in the same way clearly. Letting $\phi$ be the unique solution of the following ODE problem (For the uniqueness of $\phi$, see Appendix in [7])

$$
\begin{cases}
    \phi'' + z\phi^2(1 - \phi) = 0 \quad \text{in } (-\infty, \infty), \\
    \phi(-\infty) = 0, \quad \phi(\infty) = 1,
\end{cases}
$$

(2.2)

we have the following properties of $\phi$, whose proofs are given in Appendix A in [8].

**Lemma 2.1.** $\phi$ is monotone increasing in $(-\infty, \infty)$, and there exist positive constants $C_i, i = 1, \ldots, 6$, $\lambda_j, j = 1, 2, 3$, and $R$ such that the following hold:

$$
1 - C_1 \exp(-\lambda_1 z^{\frac{2}{3}}) < \phi(z) < 1 - C_2 \exp(-\lambda_2 z^{\frac{2}{3}}) \quad \text{for } z > R,
$$

(2.3)

$$
\phi'(z) < C_3 \exp(-\lambda_3 z^{\frac{2}{3}}) \quad \text{for } z > R.
$$

(2.4)

$$
-\frac{C_4}{z^3} < \phi(z) < -\frac{C_5}{z^3} \quad \text{for } z < -R,
$$

(2.5)

$$
\phi'(z) < \frac{C_6}{z^4} \quad \text{for } z < -R.
$$

(2.6)

Let $L$ be a large constant to be chosen later, and we define two $C^1$ functions as follows:

$$
\eta(z) = \frac{|z|^2}{1 + |z|},
$$

$$
\theta(z) = \begin{cases}
    \exp(-\lambda_2(L + 1)^{\frac{2}{3}})\eta(z - L), & z \geq L, \\
    0, & -L \leq z \leq L, \\
    \frac{\kappa}{L\epsilon^4}g(z + L), & z \leq -L,
\end{cases}
$$

where $\lambda_2$ is the constant in (2.3), $\kappa$ is a constant satisfying $\kappa > C_5$ and $C_5$ is the constant in (2.5).

For the case $g'(x_i) > 0$, we can construct a lower solution near $x_i$ in the same way as [8]. Define

$$
u(x) = \phi \left( \frac{B(x_i)(x - x_i)}{\epsilon} - L^4 \right) - \theta \left( \frac{B(x_i)(x - x_i)}{\epsilon} - L^4 \right)
$$

where $B(x_i) = |g'(x_i)|^{\frac{1}{3}}$, and let $\xi_1, \xi_2$ satisfy

$$
u(\xi_1) = 0, \quad \xi_1 \leq x_i + \frac{\epsilon}{B(x_i)}(-L + L^4),
$$

(2.7)

$$
u'(\xi_2) = 0, \quad \xi_2 \geq x_i + \frac{\epsilon}{B(x_i)}(L + L^4).
$$

(2.8)

Completely by the same way as we have obtained (2.11) and (2.12) in [8], we can show the following lemma:
Lemma 2.2. There exists $L^*_0 > 0$ such that for all $L > L^*_0$, $\xi^*_1$, $\xi^*_2$ are uniquely determined and satisfy

$$\xi^*_1 \in \left( x_i + \frac{\epsilon}{B(x_i)} (-2L + L^4), x_i + \frac{\epsilon}{B(x_i)} (-L + L^4) \right),$$

$$\xi^*_2 \in \left( x_i + \frac{\epsilon}{B(x_i)} (L + L^4), x_i + \frac{\epsilon}{B(x_i)} (PL + L^4) \right).$$

for $\epsilon$ sufficiently small. Moreover, we have the estimate

$$u(\xi^*_1) < 1 - C_2 \exp(-\lambda_2 (PL)^{\frac{3}{2}}),$$

$$u(\xi^*_2) > 1 - C_1 \exp(-\lambda_1 L^{\frac{3}{2}}),$$

where

$$P = 2 \left( \frac{\lambda_2}{\lambda_3} \right)^{\frac{3}{2}} > 2 \quad (\lambda_2 > \lambda_3)$$

and $\lambda_1$, $\lambda_2$, $\lambda_3$, $C_1$, $C_2$ are as in Lemma 2.1.

For the general case ($g'(x_i) > 0$ and $g'(x_i) < 0$), a lower solution is constructed in the following way:

Set $B(x_i) = |g'(x_i)|^{\frac{1}{2}}$ and choose $\mu_0 > 0$ small such that

$$\mu_0 < \frac{1}{4} \min_{i \in \{1, 2, \ldots, m-1\}} \frac{|x_{k+1} - x_k|}{4}, \quad \mu_0 < \frac{1}{2} \min \{ x_1, 1 - x_m \}. $$

Define a signed distance function defined in $\{ x \in [0, 1] \mid -\mu_0 \leq x - x_i \leq \mu_0 \}$:

$$d_i(x) = \begin{cases} 
  x - x_i & \text{if } g'(x_i) > 0, \\
  -x + x_i & \text{if } g'(x_i) < 0.
\end{cases}$$

(2.14)

We also define

$$u(x) = \phi \left( B(x_i) \frac{d_i(x)}{\epsilon} - L^4 \right) - \theta \left( B(x_i) \frac{d_i(x)}{\epsilon} - L^4 \right).$$

Let $\xi^*_1$, $\xi^*_2$ satisfy

$$u(\xi^*_1) = 0, \quad d_i(\xi^*_1) \leq \frac{\epsilon}{B(x_i)} (-L + L^4),$$

$$u(\xi^*_2) = 0, \quad d_i(\xi^*_2) \geq \frac{\epsilon}{B(x_i)} (L + L^4).$$

(2.15)

(2.16)

Clearly Lemma 2.2 is generalized as follows:

**Lemma 2.3.** There exists $L^*_0 > 0$ such that for all $L > L^*_0$, $\xi^*_1$, $\xi^*_2$ are uniquely determined and satisfy

$$\frac{\epsilon}{B(x_i)} (-2L + L^4) \leq d_i(\xi^*_1) \leq \frac{\epsilon}{B(x_i)} (-L + L^4),$$

$$\frac{\epsilon}{B(x_i)} (L + L^4) \leq d_i(\xi^*_2) \leq \frac{\epsilon}{B(x_i)} (PL + L^4)$$

for $\epsilon$ sufficiently small, where $P$ is in (2.13). Moreover, we have the estimate

$$u(\xi^*_1) < 1 - C_2 \exp(-\lambda_2 (PL)^{\frac{3}{2}}),$$

$$u(\xi^*_2) > 1 - C_1 \exp(-\lambda_1 L^{\frac{3}{2}}).$$

(2.19)

(2.20)

Set

$$\Phi(u) \equiv \epsilon^3 u'' + g(x)f(u) \quad \text{and} \quad f(u) = u^2(1-u).$$

(2.21)

The following lemma is given in Lemma 2.2 in [8].
Lemma 2.4. There exists $L^i > 0$ such that for all $L > L^i$, $\Phi(u(x)) > 0$ holds on $\{ x \in (0, 1) \mid d_i(\xi^i_1) < d_i(x) < d_i(\xi^i_2) \}$ for $z$ sufficiently small.

Lemma 2.4 tells us that $u$ is a lower solution for (2.1) in the interval $\{ x \in [0, 1] \mid d_i(\xi^i_1) < d_i(x) < d_i(\xi^i_2) \}$. We will extend $u$ to the entire interval $(0, 1)$ in the following way.

Set $\alpha_1(L) = C_2 \exp(-\lambda_2(PL)^{3/2})$. Let $\lambda(x)$ be a smooth function in $[0, 1]$ such that

(i) $\lambda(0) = 0$, $\lambda(1) = 1$,
(ii) $\lambda'(x) \geq 0$ in $(0, 1)$, $\lambda(0) = \lambda'(1) = 0$,
(iii) There exists $M > 0$ such that $|\lambda''(z)| < M$ in $(0, 1]$.

Let $\beta > 0$ be an arbitrarily small constant such that $5\beta < \mu_0$ and set $\chi(x) = \lambda \left( \frac{d_i(x) - \beta}{\beta} \right)$. Then $\chi(x)$ is a smooth function in $\{ x \in [0, 1] \mid \beta < d_i(x) \leq 2\beta \}$.

Define

$$
U^i(x) = \begin{cases} 
0 & \text{if } -3\beta \leq d_i(x) \leq d_i(\xi^i_1), \\
\frac{u(x)}{\lambda(\xi^i_1)} & \text{if } d_i(\xi^i_1) < d_i(x) \leq d_i(\xi^i_2), \\
\frac{(1 - \chi(x))u(\xi^i_2) + (1 - \alpha_1(L))\chi(x)}{1 - \alpha_1(L)} & \text{if } \beta \leq d_i(x) \leq 2\beta, \\
1 - \alpha_1(L) & \text{if } 2\beta \leq d_i(x) \leq 3\beta.
\end{cases}
$$

Note that $U^i(x)$ is not $C^1$ function, however, it is a lower solution in the weak sense.

We obtain the following lemma:

Lemma 2.5. There exists $L^i > 0$ such that for all $L > L^i$, $\Phi(U^i(x)) > 0$ holds on the interval $-3\beta \leq d_i(x) \leq 3\beta$ for $z$ sufficiently small.

Proof. Lemma 2.4 shows that $\Phi(U^i(x)) = \Phi(u(x)) \geq 0$ for $d_i(\xi^i_1) \leq d_i(x) \leq d_i(\xi^i_2)$.

It is obvious that $\Phi(1 - \alpha_1(L)) \geq 0$ holds for $g(x) > 0$ and that $\Phi(0) = 0$ holds.

We have only to prove that $\Phi(U^i(x)) > 0$ in $(x_i + \beta, x_i + 2\beta)$. On this interval, it holds that $g(x) \geq C'\beta$, hence, $C'_i$ (i = 1, 2, · · · ) is a constant independent of $\epsilon, L, \beta$. Note that both $u(\xi^i_2)$ and $1 - \alpha_1(L)$ are close to 1 and $u(\xi^i_2) < 1 - \alpha_1(L)$, therefore, it holds that $\int((1 - \chi(x))u(\xi^i_2) + \chi(x)(1 - \alpha_1(L)) \geq f(1 - \alpha_1(L))$.

Consequently,

$$
\begin{align*}
\Phi(U^i(x)) & = \Phi((1 - \chi_i)u(\xi^i_2) + \chi_i(1 - \alpha_1(L)) \\
& = e^3\chi''_i(x)(1 - \alpha_1(L) - u(\xi^i_2)) + g(x)f((1 - \chi_i)u(\xi^i_2) + \chi_i(1 - \alpha_1(L)) \\
& \geq -e^3\lambda''_i\left(\frac{d_i(x) - \beta}{\beta}\right) + C'_i\beta f(1 - \alpha_1(L)) \\
& \geq -e^3\frac{\lambda''_i}{\beta} + C'_i\beta \exp(-\lambda_2(PL)^{3/2}).
\end{align*}
$$

This shows that $\Phi(U^i(x)) > 0$ holds for $z$ sufficiently small. \hfill \Box

Let

$$
dist(x) = \min_{i=1, 2, \ldots, m} |x - x_i|.
$$

(2.23)

Note that $\text{dist}(x)$ is a nonnegative function defined in the entire interval $[0, 1]$ and $d_i(x)$ is a locally defined function near $x_i$. Finally we set

$$
U_*(x) = \begin{cases} 
0 & \text{in } \{ x \in [0, 1] \mid \text{dist}(x) \geq 3\beta, \ g(x) < 0 \}, \\
U^i(x) & \text{in } \{ x \in [0, 1] \mid \text{dist}(x) < 3\beta, \ g(x) < 0 \}, \\
1 - \alpha_1(L) & \text{in } \{ x \in [0, 1] \mid \text{dist}(x) \geq 3\beta, \ g(x) > 0 \}.
\end{cases}
$$

(2.24)
Now we have the following proposition.

**Proposition 2.6.** There exists $\hat{L} > 0$ such that for all $L > \hat{L}$, $U_\epsilon$ is a lower solution for (2.1) for $\epsilon > 0$ sufficiently small.

An upper solution is constructed in a similar fashion as the lower solution, thus we shall give a brief explanation.

Let

$$
\hat{\theta}(z) = \begin{cases} 
\frac{C_2}{2} \exp(-\lambda(L + 1)^2)\eta(z - L), & z \geq L, \\
0, & -L \leq z \leq L, \\
\frac{\hat{\kappa}}{L^4} \eta(z + L), & z \leq -L,
\end{cases} \tag{2.25}
$$

where $\hat{\kappa} > C_6$. Here $C_2$, $C_6$, $\lambda_2$ are in Lemma 2.1 and

$$
\bar{u}(x) = \phi \left( \frac{B(x_i)d_i(x)}{\epsilon} + L^4 \right) + \hat{\theta} \left( \frac{B(x_i)d_i(x)}{\epsilon} + L^4 \right). \tag{2.26}
$$

Similarly, let $\xi_3^i, \xi_4^i$ satisfy

$$
d_i(\xi_3^i) \leq \frac{\epsilon}{B(x_i)}(-L - L^4), \quad \xi' = 0, \tag{2.27}
$$

$$
d_i(\xi_4^i) \geq \frac{\epsilon}{B(x_i)}(L - L^4), \quad \bar{u}(\xi_4^i) = 1, \tag{2.28}
$$

In the same way as we have obtained $\xi_3$ and $\xi_4$ in (2.26) – (2.28) in [8], we obtain the following lemma:

**Lemma 2.7.** There exists $L_0^1 > 0$ such that for all $L > L_0^1$, $\xi_3^i$ and $\xi_4^i$ are uniquely determined and satisfy

$$
\frac{\epsilon}{B(x_i)}(-3L - L^4) \leq d_i(\xi_3^i) \leq \frac{\epsilon}{B(x_i)}(-L - L^4), \tag{2.29}
$$

$$
\frac{\epsilon}{B(x_i)}(L - L^4) \leq d_i(\xi_4^i) \leq \frac{\epsilon}{B(x_i)}(PL - L^4) \tag{2.30}
$$

for $\epsilon$ sufficiently small. Moreover, we have the estimate

$$
\frac{C_4}{(3L)^3} \leq \bar{u}(\xi_3) \leq \frac{C_5}{L^3}. \tag{2.31}
$$

In the similar way as lower solution, we set $\alpha_2(L) = \frac{C_4}{(3L)^3}$ and $\bar{\chi}_i(x) = \lambda$

$$(\frac{-d_i(x) + \beta}{\beta}).$$

Define

$$
\bar{U}^i(x) = \begin{cases} 
\alpha_2(L) & \text{if } -3\beta \leq d_i(x) \leq -2\beta, \\
\bar{\chi}_i(x)\alpha_2(L) + (1 - \bar{\chi}_i(x))\bar{u}(\xi_3^i) & \text{if } -2\beta \leq d_i(x) \leq -\beta, \\
\bar{u}(\xi_3^i) & \text{if } -\beta \leq d_i(x) \leq d_i(\xi_3^i), \\
\bar{u}(x) & \text{if } d_i(\xi_3^i) \leq d_i(x) \leq d_i(\xi_4^i), \\
1 & \text{if } d_i(\xi_4^i) \leq d_i(x) \leq 3\beta.
\end{cases} \tag{2.32}
$$

In the same way as we have Lemma 2.5, we obtain

**Lemma 2.8.** There exists $L_1^1 > 0$ such that for all $L > L_1^1$, $\Phi(\bar{U}^i(x)) > 0$ holds on the interval $-3\beta \leq d_i(x) \leq 3\beta$ for $\epsilon$ sufficiently small.
Finally we set
\[
U^*(x) = \begin{cases} 
\alpha_2(L) & \text{in } \{x \in [0,1] \mid \text{dist}(x) \geq 3\beta, \ g(x) < 0\}, \\
\bar{U}^*(x) & \text{in } \{x \in [0,1] \mid \text{dist}(x) < 3\beta\}, \\
1 & \text{in } \{x \in [0,1] \mid \text{dist}(x) \geq 3\beta, \ g(x) > 0\}. 
\end{cases}
\] (2.33)
and obtain the following proposition.

**Proposition 2.9.** There exists \( \hat{L} > 0 \) such that for all \( L > \hat{L} \), \( U^* \) is a upper solution for (2.1) for \( \epsilon > 0 \) sufficiently small.

**Proof of Theorem 1.2.** By Propositions 2.6 and 2.9, there exists a solution \( U_d \) of (2.1) such that \( U_* < U_d < U^* \). The linearized stability of \( U_d \) follows from Lemma 4.1 in [9]. The estimates in the theorem follow from Lemmas 3.1, 3.2 and 3.5 in [9].

3. Uniqueness. In this section we will show uniqueness of the solution which we have constructed in Section 2. Then we prove Theorem 1.1 in the last part of this section. We will use many lemmas in [9] for the proof.

Let \( \{x_1, x_2, \cdots, x_m\} \) be zeros of \( g(x) \). We set \( x_0 = 0, \ x_{m+1} = 1 \) for convenience.

Let
\[
G^+ = \{x \in [0,1] \mid g(x) > 0\}, 
G^- = \{x \in [0,1] \mid g(x) < 0\}.
\]
By Lemma 3.5 in [9], any solution \( u_\epsilon \) is close to 0 in \( G^- \). By Lemmas 3.1 and 3.2 in [9], either of the following holds for each \( (x_k, x_{k+1}) \), \( u_\epsilon(x) \) is close to 0 in \( (x_k, x_{k+1}) \) or \( u_\epsilon(x) \) is close to 1 in \( (x_k, x_{k+1}) \). Therefore the equation (2.1) can have only 3 types of solutions in the following:

(T1) \( u_\epsilon \) is close to 0 in \( G^+ \)
(T2) \( u_\epsilon \) is close to 1 in \( G^+ \)
(T3) There exists \( S \), a subset of \( G^+ \) such that
\[
S = \bigcup_{k \in K} (x_k, x_{k+1}) \subset G^+ \quad (K \subset \{0,1,2,\cdots,m\}), 
S \neq G^+,
\]
\( u_\epsilon \) is close to 1 in \( S \) and \( u_\epsilon \) is close to 0 in \( G^+ \setminus S \).

The uniqueness will be proved in the following way. In Lemma 3.1 below, we will show that the case (T1) is impossible under the condition \( \int_0^1 g(x) \, dx \geq 0 \). In Lemma 3.2 below, we will show that a solution satisfying (T2) is unique. In Lemma 3.4 below, we will show that the case (T3) is impossible under the condition (P0) – (P3) in Section 1, Introduction.

We will provide the following lemma. Let \( u_\epsilon \) be a nontrivial solution of (2.1).

**Lemma 3.1.** Assume \( \int_0^1 g(x) \, dx \geq 0 \) holds. Then there exists \( \bar{x} \) such that \( u_\epsilon(\bar{x}) \geq \frac{2}{3} \).

**Proof.** Assume \( u_\epsilon(x) < \frac{2}{3} \) for \([0,1]\). By the uniqueness of solutions of ODE, we have \( 0 < u_\epsilon(x) < 1 \). Using equation (2.1) and the assumption of the lemma, we obtain
\[
-\int_0^1 \frac{u''_\epsilon}{u^2_\epsilon(1-u_\epsilon)} \, dx = \frac{1}{\epsilon^3} \int_0^1 g(x) \, dx \geq 0.
\]
The left-hand side of the above equality is
\[
-\int_0^1 \frac{u''_\epsilon}{u^2_\epsilon(1-u_\epsilon)} \, dx = -\left[ \frac{u'_\epsilon}{u^2_\epsilon(1-u_\epsilon)} \right]_0^1 - \int_0^1 \frac{(2-3u_\epsilon)(u'_\epsilon)^2}{u^2_\epsilon(1-u_\epsilon)^2} \, dx
= -\int_0^1 \frac{(2-3u_\epsilon)(u'_\epsilon)^2}{u^2_\epsilon(1-u_\epsilon)^2} \, dx < 0,
\]
since \( u_\epsilon(x) < \frac{2}{3} \) holds for \([0,1]\). This is a contradiction. The lemma is proved. \[\square\]

The following lemma shows that a nontrivial solution satisfying (T2) above is unique. By Lemmas 3.8, 3.9, and 3.10 [9], any solution \( u_\epsilon(x) \) characterized by (T2) should satisfy \( U_\epsilon(x) < u_\epsilon(x) < U^*(x) \) where \( U^*(x) \) and \( U_\epsilon(x) \) are a pair of an upper and a lower solutions constructed in Section 2. In the following Lemma 3.2 we will show that a nontrivial solution which stays in between \( U^*(x) \) and \( U_\epsilon(x) \) is unique.

**Lemma 3.2.** (2.1) has only one solution in \( P = \{ u \in C[0,1] | U_\epsilon(x) < u_\epsilon(x) < U^*(x) \} \), where \( U^*(x) \) and \( U_\epsilon(x) \) are a pair of an upper and a lower solutions in Section 2.

**Proof.** Set \( p > 1 \) max \( |g(x)| \). For given \( u \in C[0,1] \) we define a map \( w = Au, A: C[0,1] \rightarrow C[0,1] \) by

\[
\begin{cases}
-\epsilon^2 w'' + pw = pu + g(x)u^2(1-u) & \text{in } (0,1), \\
w'(0) = w'(1) = 0.
\end{cases}
\]

There is one to one correspondence between a solution of (2.1) and a fixed point of \( A \). By maximum principle, or the variation of constants for the second order ordinally differential equations, \( A \) is order preserving in \( P \) (i.e. if \( u_1 \geq u_2 \) \((u_1, u_2) \in P\), then \( Au_1 \geq Au_2 \)). Since \( U^* \) and \( U_\epsilon \) are a pair of an upper and a lower solutions in strict sense, it holds that \( AU_\epsilon > U_\epsilon \) \( AU^* < U^* \).

Choose any \( v_0 \in P \) and \( \lambda \in [0,1] \), we define

\[ \begin{align*}
H(u, \lambda) &= (1 - \lambda)Au + \lambda v_0. \\
H(u, \lambda) &= (1 - \lambda)Au + \lambda v_0 < (1 - \lambda)AU^* + \lambda U_\epsilon \leq (1 - \lambda)U^* + \lambda U^* = U^*. \\
\end{align*} \]

In the same way we have \( H(u, \lambda) > U_\epsilon \) for \( u \in P \). Therefore, \( H(\cdot, \lambda) \) maps into \( P \) and \( H(u, \lambda) \) does not have any fixed point on \( \partial P \) for any \( \lambda \in [0,1] \). \( \deg(I - H(\cdot, \lambda), P) \) is well defined. By the homotopy invariance property of the degree theory, we have

\[ \deg(I - A, P) = \deg(I - H(\cdot, \lambda), P) = \deg(I - v_0, P) = 1. \quad (3.1) \]

Let \( T(\cdot, u_\epsilon) \) be a derivation of \( A \) with respect to \( u \) at \( u = u_\epsilon \). i.e. \( \omega = T(\phi, u_\epsilon) \)

\[ T(\cdot, u_\epsilon) : C[0,1] \rightarrow C[0,1] \] such that

\[
\begin{cases}
-\epsilon^3 \omega'' + pw = p\phi + g(x)(2u_\epsilon - 3u_\epsilon^2)\phi, \\
\omega'(0) = \omega'(1) = 0.
\end{cases}
\]

The following Lemma 3.3 directly follows from Lemma 4.1 in [9].

**Lemma 3.3.** Let \( u_\epsilon \) be a solution of (2.1) and \( u_\epsilon \in P \). Then \( u_\epsilon \) is linearly stable.

This lemma is equivalent to the fact that all the eigenvalues of

\[
\begin{cases}
\epsilon^3 \psi'' + g(x)(2u_\epsilon - 3u_\epsilon^2)\psi = \lambda \psi, \\
\psi'(0) = \psi'(1) = 0.
\end{cases}
\]

is negative, i.e., all the eigenvalues of \( T(\cdot, u_\epsilon) \) is less than 1. Therefore, \( T(\cdot, u_\epsilon) \) is invertible and \( u_\epsilon \) is an isolated fixed point. Thus the fixed point index of \( A \) at \( u = u_\epsilon \) is well defined (Here, fixed point index means degree of \( A \) in the neighborhood of \( u_\epsilon \)) and we have

\[ \text{Index}(I - A, u_\epsilon) = 1. \quad (3.2) \]
Suppose that (2.1) has more than one solution \( u_i \in P \) \((i \in I)\). Then \( u_i \) \((i \in I)\) are fixed points of \( A \) in \( P \). By (3.2), for any fixed point \( u_i \) \((i \in I)\), it holds that

\[
\text{Index}(I - A, u_i) = 1 \quad \text{for} \quad i \in I.
\] (3.3)

By (3.3) and the assumption that \( A \) has more than one fixed point in \( P \), we have

\[
\sum_{i \in I} \text{Index}(I - A, u_i) \geq 2.
\] (3.4)

On the other hand, it holds from (3.1) that

\[
1 = \deg(I - A, P) = \sum_{i \in I} \text{Index}(I - A, u_i).
\]

This contradicts (3.4). Now we have shown that the solution of (2.1) in \( P \) is unique.

Lemma 3.2 is proved.

In the next lemma, we will show that any solution which stays close to zero is unstable.

**Lemma 3.4.** Let \( u_\epsilon \) be a solution of (2.1) and \( 0 < u_\epsilon(x) < 1 - \frac{1}{\sqrt{2}} \). Then \( u_\epsilon \) is an unstable solution.

**Proof.**

\[
L(\hat{w}) = \int_0^1 e^3|\hat{w}'(x)|^2 - g(x)(2u_\epsilon(x) - 3u_\epsilon^2(x))\hat{w}(x)^2 \, dx.
\]

We will show \( L(u_\epsilon) < 0 \) to show \( u_\epsilon \) is unstable. For convenience, we denote \( u_\epsilon \) by \( u \) in this proof. It holds that

\[
L(u) = \int_0^1 e^3|u'(x)|^2 - g(x)(2u(x)^3 - 3u(x)^4) \, dx.
\] (3.5)

By the equation (2.1), we have

\[
e^3 \int_0^1 \frac{u''}{u - 1} \, dx = \int_0^1 g(x)u^2 \, dx.
\]

Note that \( \left( \frac{u'}{u - 1} \right)' = \frac{u''}{u - 1} - \frac{u'^2}{(u - 1)^2} \), we have

\[
\int_0^1 g(x)u^2 \, dx = e^3 \left[ \frac{u'}{u - 1} \right]_0^1 + e^3 \int_0^1 \frac{|u'|^2}{(u - 1)^2} \, dx = e^3 \int_0^1 \frac{|u'|^2}{(u - 1)^2} \, dx.
\] (3.6)

We also obtain by integrating the equation (2.1)

\[
0 = e^3 |u'|_0^1 = \int_0^1 g(x)u^2 \, dx - \int_0^1 g(x)u^3 \, dx.
\] (3.7)

Multiplying the equation (2.1) by \( u_\epsilon \), integrating that over \([0, 1]\), we have

\[-e^3 \int_0^1 u''u \, dx = \int_0^1 g(x)(u^3 - u^4) \, dx.
\]

Therefore, we have

\[
e^3 \int_0^1 |u'|^2 \, dx = -e^3 |u'|_0^1 - e^3 \int_0^1 u''u \, dx = \int_0^1 g(x)u^3 \, dx - \int_0^1 g(x)u^4 \, dx.
\] (3.8)

By (3.7), we obtain

\[
\int_0^1 g(x)u^2 \, dx = \int_0^1 g(x)u^3 \, dx = S.
\]
This and (3.8) show that
\[ \int_0^1 g(x)u'^4 \, dx = S - \epsilon_1^3 \int_0^1 |u'|^2 \, dx. \]  
(3.9)

By (3.5) and (3.9), we have
\[ \mathcal{L}(u) = \epsilon_1^3 \int_0^1 |u'|^2 \, dx - 2S + 3 \int_0^1 g(x)u'^4 \, dx = S - 2\epsilon_1^3 \int_0^1 |u'|^2 \, dx. \]

It holds from (3.6) that
\[ \mathcal{L}(u) = \epsilon_1^3 \int_0^1 \left( \frac{1}{(u - 1)^2} - 2 \right) |u'|^2 \, dx. \]

Since \( u < 1 - \frac{1}{\sqrt{2}} \), we obtain \( \mathcal{L}(u) < 0 \). The lemma is proved. \( \Box \)

The following Lemma 3.5 shows that (T3) is impossible for some classes of \( g(x) \).

Let \( p(x) \in C^4[0, 1] \) be a function satisfying (P0) to (P3) in Introduction.

**Lemma 3.5.** Assume (H). Suppose \( g(x) \geq p(x) \) and let \( p(x) \) satisfy (P0) – (P3). The following 2 cases cannot hold simultaneously.

(i) There exists \( k \in \{0, 1, 2, \cdots, m\} \) such that \( g(x) > 0 \) in \((x_k, x_{k+1})\) and \( u_\epsilon(x) \) is close to 0 in \((x_k, x_{k+1})\).

(ii) There exists \( \ell \in \{0, 1, 2, \cdots, m\} \) such that \( g(x) > 0 \) in \((x_\ell, x_{\ell+1})\) and \( u_\epsilon(x) \) is close to 1 in \((x_\ell, x_{\ell+1})\).

The condition \( g(x) \geq p(x) \) and (P3) imply \( \int_0^1 g(x) \, dx \geq 0 \). (P0) comes from the nondegeneracy condition (H) in the introduction (\( z_1 \) in (P0) is the same as \( x_1 \) in (H)).

We also make a remark on the conditions (T1) - (T3). The solution satisfying (T1) stays close to 0 in the first case. On the other hand, the solution satisfying (T2) has a steep gap near each point of \( \partial G^+ \setminus \{0, 1\} \). Moreover, the solution satisfying (T3) has a steep gap near each point of \( S \setminus \{0, 1\} \). We call these steep gaps “layers”.

**Proof of Lemma 3.5.** We will first show the case \( g(x) = p(x) \) in Step 1 and then \( g(x) > p(x) \) in Step 2.

**Step 1.** The case \( g(x) = p(x) \)

We will first show the case \( g(x) = p(x) \). In the proof, we will use Theorem 1.1 in [3] on stability property of \( u = 0 \). We can also obtain a proof without using this stability property (we use degree theory on a positive cone instead). However, we use their theorem to make the proof simpler and clearer. Their theorem is concerned with the problem on higher dimension, therefore, we will rewrite the theorem to one dimensional case to suite our problem.

**Theorem LNS.** For problem (2.1), the following statement holds. If \( \int_0^1 g(x) \, dx \geq 0 \), then solution \( u = 0 \) is unstable. If \( \int_0^1 g(x) \, dx < 0 \), then solution \( u = 0 \) is stable.

(1) In the case \( m = 2 \), \( g(x) = p(x) \).

There are two cases \( g(0) = g(1) < 0 \) and \( g(0) = g(1) > 0 \). In the case \( g(0) = g(1) < 0 \), \( g(x) \) has only one maximum. It is trivial that (ci) and (cii) cannot hold simultaneously.

In the case \( g(0) = g(1) > 0 \), suppose that there exists \( u_\epsilon(x) \) is close to 0 in \((0, x_1)\), and \( u_\epsilon(x) \) is close to 1 in \((x_2, 1)\). Since \( g(x) \) is symmetric with respect to \( x = \frac{1}{2} \),
$u_e(1-x)$ is also a solution of (2.1). Define $u_{min}(x) = \min\{u_e(x), u_e(1-x)\}$. Since both $u_e(x)$ and $u_e(1-x)$ are upper solutions, $u_{min}(x)$ is also an upper solution. Note that $u_{min}(x)$ is close to 0. On the other hand, 0 is a lower solution. Therefore, there is at least one stable solution in between 0 and $u_{min}(x)$. However, under the condition (P3), there is no solution nearby 0 except for 0 itself by Lemma 3.1. Therefore, 0 is a stable solution. This contradicts Theorem LNS under the condition (P3).

(2) In the case $m \geq 3$, $g(x) = p(x)$.

Suppose that $u_e$ is close to 0 and that $g(x) > 0$ are satisfied in $(x_i, x_{i+1})$. We also suppose that $u_e$ has at least one layer. There are 3 possibilities as follows:

(i) $u_e$ has at least one layer in $(0, x_i)$, and $u_e$ has at least one layer in $(x_{i+1}, 1)$.

(ii) $u_e$ does not have any layer in $(0, x_i)$.

(iii) $u_e$ does not have any layer in $(x_{i+1}, 1)$.

We will obtain a contradiction for each case.

(i) Denote

$$x_j = \min\{z \in (x_{i+1}, 1) \mid g(z) = 0 \text{ and } u_e \text{ has a layer at } x = z\},$$

$$x_k = \max\{z \in (0, x_i) \mid g(z) = 0 \text{ and } u_e \text{ has a layer at } x = z\}.$$ 

Note that both $j - (i + 1)$ and $i - k$ are odd numbers. There are only two possibilities $u'_e \left(\frac{j}{m}\right) \geq 0$ or $u'_e \left(\frac{k}{m}\right) < 0$ (Note that $\frac{j+i+1}{2m} = \frac{i}{m}$). We will first consider the case $u'_e \left(\frac{j}{m}\right) \geq 0$. Define $u_{min}(x) = \min\{u_e(x), u_e\left(\frac{i+k-1}{m} - x\right)\}$ in $[\frac{k-1}{m}, \frac{i}{m}]$.

We remark that $g(x)$ is symmetric with respect to $x = \frac{i+k-1}{2m}$ in $[\frac{k-1}{m}, \frac{i}{m}]$. (Since $i - k$ is an odd number, $i + k - 1$ is an even number). Therefore, $u_e\left(\frac{i+k-1}{m} - x\right)$ is also a solution in $\left[\frac{k-1}{m}, \frac{i}{m}\right]$. We will check boundary conditions of $u_{min}(x)$.

$$u'_{min}\left(\frac{i}{m}\right) = u'_e\left(\frac{i}{m}\right) \geq 0,$$

$$u'_{min}\left(\frac{k-1}{m}\right) = u'_e\left(\frac{i+k-1}{m} - x\right) \bigg|_{x=\frac{i}{m}} = -u'_e\left(\frac{i}{m}\right) \leq 0.$$ 

We have shown that $u_{min}(x)$ is an upper solution in $[\frac{k-1}{m}, \frac{i}{m}]$. It is clear that 0 is a lower solution. Therefore, there exists at least one stable solution $w(x)$ near 0 on this interval. However, we have the condition

$$\int_{\frac{k-1}{m}}^{\frac{i}{m}} g(x) \, dx = \int_{\frac{k-1}{m}}^{\frac{i}{m}} p(x) \, dx = \frac{i-k+1}{m} \int_0^1 p(x) \, dx \geq 0. \quad (3.10)$$

Under the condition (3.10), there is no nontrivial solution nearby 0 on the interval $[\frac{k-1}{m}, \frac{i}{m}]$ by Lemma 3.1. Therefore, $w(x) = 0$ and 0 has to be a stable solution. This contradicts Theorem LNS and the condition (3.10).

In case $u'_e\left(\frac{j}{m}\right) < 0$, define $u_{min}(x) = \min\{u_e(x), u_e\left(\frac{i+j}{m} - x\right)\}$ in $[\frac{j}{m}, \frac{i}{m}]$. We remark that $g(x)$ is symmetric with respect to $x = \frac{i+j}{2m}$ in $[\frac{j}{m}, \frac{i}{m}]$. (Since $j + 1 - i$ is an odd number, $i + j$ is an even number). Therefore, $u_e\left(\frac{i+j}{m} - x\right)$ is also a solution in $[\frac{j}{m}, \frac{i}{m}]$. We will check boundary conditions of $u_{min}(x)$.

$$u'_{min}\left(\frac{i}{m}\right) = u'_e\left(\frac{i}{m}\right) < 0,$$

$$u'_{min}\left(\frac{j}{m}\right) = u'_e\left(\frac{i+j}{m} - x\right) \bigg|_{x=\frac{i}{m}} = -u'_e\left(\frac{i}{m}\right) > 0.$$
We have shown that \( u_{\text{min}}'(x) \) is an upper solution in \([\frac{i}{m}, \frac{j}{m}]\). It is clear that 0 is a lower solution. Therefore, there exists at least one stable solution near 0. However, we have the condition
\[
\int_{\frac{i}{m}}^{\frac{j}{m}} g(x) \, dx = \int_{\frac{i}{m}}^{\frac{j}{m}} p(x) \, dx = \frac{j-i}{m} \int_{0}^{1} p(x) \, dx \geq 0. \tag{3.11}
\]
Under the condition (3.11), there is no nontrivial solution nearby 0 on the interval \([\frac{i}{m}, \frac{j}{m}]\) by Lemma 3.1. Therefore, 0 is a stable solution. This contradicts Theorem LNS under the condition (3.11).

(ii) Since \( u_\varepsilon \) does not have any layers in \((0, x_i)\), there is at least one layer in \((x_{i+1}, 1)\). Denote
\[
x_j = \min\{z \in (x_{i+1}, 1) \mid g(z) = 0 \text{ and } u_\varepsilon \text{ has a layer at } x = z\},
\]
Note that \( u_\varepsilon \) has exactly one layer in \([\frac{i}{m}, \frac{j}{m}]\) and \( u_\varepsilon \) is close to 0 in \([0, \frac{i}{m}]\). There are only two possibilities \( u_\varepsilon' \left( \frac{i}{m} \right) \geq 0 \) or \( u_\varepsilon' \left( \frac{j}{m} \right) < 0 \). In the case \( u_\varepsilon' \left( \frac{i}{m} \right) \geq 0 \), \( u_\varepsilon \) is an upper solution in \([0, \frac{i}{m}]\), and 0 is a lower solution on this interval. Therefore, there exists at least one stable solution near 0. On the other hand, we have the condition
\[
\int_{0}^{\frac{i}{m}} g(x) \, dx = \int_{0}^{\frac{i}{m}} p(x) \, dx = \frac{i}{m} \int_{0}^{1} p(x) \, dx \geq 0. \tag{3.12}
\]
Under the condition (3.12), there is no nontrivial solution nearby 0 on the interval \([0, \frac{i}{m}]\) by Lemma 3.1. Therefore, 0 is a stable solution. This contradicts Theorem LNS under the condition (3.12).

In case \( u_\varepsilon' \left( \frac{i}{m} \right) < 0 \), define \( u_{\text{min}}(x) = \min\{u_\varepsilon(x), u_\varepsilon(\frac{i+j}{m} - x)\} \). We remark that \( g(x) \) is symmetric with respect to \( x = \frac{i+j}{2m} \) in \([\frac{i}{m}, \frac{j}{m}]\). (Since \( j + 1 - i \) is an odd number, \( i + j \) is an even number). Therefore, \( u_\varepsilon(\frac{i+j}{m} - x) \) is also a solution in \([\frac{i}{m}, \frac{j}{m}]\).

We will check boundary conditions of \( u_{\text{min}}(x) \).
\[
\begin{align*}
\left. u_{\text{min}}'(x) \right|_{x=\frac{i}{m}} &= u_\varepsilon' \left( \frac{i}{m} \right) \geq 0, \\
\left. u_{\text{min}}'(x) \right|_{x=\frac{j}{m}} &= u_\varepsilon' \left( \frac{j}{m} \right) = u_\varepsilon' \left( \frac{i+j}{m} - x \right) \bigg|_{x=\frac{i}{m}} = -u_\varepsilon' \left( \frac{i}{m} \right) > 0.
\end{align*}
\]
We have shown that \( u_{\text{min}}'(x) \) is an upper solution in \([\frac{i}{m}, \frac{j}{m}]\). It is clear that 0 is a lower solution. Therefore, there exists at least one stable solution near 0. However, we have the condition
\[
\int_{\frac{i}{m}}^{\frac{j}{m}} g(x) \, dx = \int_{\frac{i}{m}}^{\frac{j}{m}} p(x) \, dx = \frac{j-i}{m} \int_{0}^{1} p(x) \, dx \geq 0. \tag{3.13}
\]
Under the condition (3.13), there is no nontrivial solution nearby 0 on the interval \([\frac{i}{m}, \frac{j}{m}]\) by Lemma 3.1. Therefore, 0 is a stable solution. This contradicts Theorem LNS under the condition (3.13).

(iii) We obtain a contradiction in the same way as (ii) (we consider \( u_\varepsilon(1-x) \) instead of \( u_\varepsilon(x) \), then this case is reduced to the case (ii)).

**Step 2.** The case \( g(x) \geq p(x) \) in \((0, 1)\) and \( g(x) > p(x) \) for some \( x \in (0, 1) \).

We will prove the case \( g(x) > p(x) \), where \( p(x) \) satisfies (P0) – (P3). Suppose that there exists \( u_\varepsilon \), a solution of (2.1), satisfying (ci) and (cii). We will show that
\(u_\epsilon\) is an upper solution of

\[
\begin{align*}
\epsilon^3 u'' + p(x)u^2(1-u) &= 0, \\
u'(0) &= u'(1) = 0.
\end{align*}
\tag{3.14}
\]

To show that we set \(\Psi(u) = \epsilon^3 u'' + p(x)u^2(1-u)\), we have

\[
\Psi(u_\epsilon) = (-g(x) + p(x))u_\epsilon^2 (1-u_\epsilon) < 0 \quad \text{in} \quad (0,1).
\tag{3.15}
\]

We will find a lower solution in the following way. Denote the zeros of \(p(x)\) by \(\{x_i \mid i = 1, 2, \ldots, m\}\) in the same way as in Step 1. We now define a lower solution of (3.14). We recall Section 2, Proposition 2.5 (See also Lemmas 2.2 - 2.4), we can construct a lower solution of (3.14) which has a layer near each zero of \(p(x)\), \(\{x_i \mid i = 1, 2, \ldots, m\}\). Denote this layered lower solution by \(U_\ast(x)\). Set \(x_0 = 0\) and \(x_{m+1} = 1\) for convenience. We define

\[
\mathcal{Z} = \{\ell \in \{0, 1, 2, \ldots, m\} \mid \text{There exists } s \in (x_\ell, x_{\ell+1}) \text{ such that } u_\epsilon(s) > \frac{1}{2},\]

where \(u_\epsilon\) is a solution of (2.1). We also define

\[
w(x) = \begin{cases} U_\ast(x) & \text{in } \bigcup_{\ell \in \mathcal{Z}} (x_\ell, x_{\ell+1}), \\
0 & \text{in } [0,1] \setminus \bigcup_{\ell \in \mathcal{Z}} (x_\ell, x_{\ell+1}).\end{cases}
\tag{3.16}
\]

**Lemma 3.6.** Let \(u_\epsilon\) be any nontrivial solution of (2.1), \(g(x) \geq p(x)\) in \((0,1)\) and \(g(x) > p(x)\) for some \(x \in (0,1)\), and \(w\) be defined in (3.16). Then \(w(x) < u_\epsilon(x)\) \((0 \leq x \leq 1)\) for \(\epsilon > 0\) sufficiently small.

**Proof.** Since \(w(x) = 0 < u_\epsilon(x)\) in \(x \in [0,1] \setminus \bigcup_{\ell \in \mathcal{Z}} (x_\ell, x_{\ell+1})\), we have only to prove \(u_\epsilon(x) > w(x)\) in \((x_\ell, x_{\ell+1})\) for \(\ell \in \mathcal{Z}\). There are 3 possibilities, \(\ell = m\) or \(\ell = 0\) or \(0 < \ell < m\). Since the case \(\ell = 0\) is treated in the same way as \(\ell = m\), we only have to consider two cases \(\ell = m\) and \(0 < \ell < m\) in the following. Let \(\beta\) be a small constant satisfying

\[
0 < \beta < \min\left\{\frac{1}{8}(\frac{\ell}{m} - x_\ell), \frac{1}{8}(x_\ell - \frac{\ell-1}{m})\right\} \tag{3.17}
\]

\[
p'(x) > 0 \quad \text{in} \quad (x_\ell - 5\beta, x_\ell + 5\beta). \tag{3.18}
\]

In the following we will show

(i) In the case \(\ell = m\),

\[
u_\epsilon(x) > w(x) \quad \text{in} \quad (x_m + 3\beta, 1), \tag{3.19}
\]

(ii) In the case \(0 < \ell < m\),

\[
u_\epsilon(x) > w(x) \quad \text{in} \quad (x_\ell + 3\beta, x_{\ell+1} - 3\beta). \tag{3.20}
\]

Recall Section 2, by (2.19), (2.22) and (2.24), we have

\[
U_\ast(x) \leq 1 - \alpha_1(L) \quad \text{in} \quad [0,1], \tag{3.21}
\]

where \(\alpha_1(L) = C_2 \exp(-\lambda_2(PL)^\frac{1}{2})\) is a small constant independent of \(\epsilon > 0\).

(i) In the case \(\ell = m\), it holds that \(g(x) > 0\) in \((x_m, 1)\). Lemma 3.1 of [9] shows

\[
u_\epsilon(x) > 1 - C_2 \exp(-C_1 \epsilon^{-\frac{3}{2}}) \quad \text{in} \quad (x_m + 3\beta, 1). \tag{3.22}
\]
(ii) In the case $\ell < m$, we remark that $u_\epsilon$ is a solution of (2.1). Since $g(x) > 0$ in $(x_{\ell}, x_{\ell+1})$, Lemma 3.2 shows
\[ u_\epsilon(x) > 1 - C_2 \exp(-C_1 \epsilon^{-\frac{3}{2}}) \quad (x_{\ell} + 3\beta, x_{\ell+1} - 3\beta). \] (3.23)

(3.22), (3.23) and (3.21) show (3.19) and (3.20).

We consider the interval $x_{\ell} \leq x \leq x_{\ell} + 5\beta$ for $\ell = 1, 2, \ldots, m$. Set
\[ V(x, q) = U_\epsilon(x - q) \quad (x_{\ell} \leq x \leq x_{\ell} + 5\beta, 0 \leq q \leq 4\beta) \quad (\ell = 1, 2, \ldots, m). \]
We also set $\Psi(u) = \epsilon^3 u'' + p(x)u^2(1 - u)$. Since $p(x)$ is monotone increasing in $[x_{\ell} - 5\beta, x_{\ell} + 5\beta]$,
\[ \Psi(V(x, q)) = (p(x) - p(x - q))(U_\epsilon(x - q))^2(1 - U_\epsilon(x - q)) \geq 0. \] (3.24)

We will check boundary conditions. We have
\[ V(x_{\ell}, q) = U_\epsilon(x_{\ell} - q) = 0 < u_\epsilon(x_{\ell}) \quad (0 \leq q \leq 4\beta). \] (3.25)

Again, (3.22), (3.23) and (3.21) show
\[ V(x_{\ell} + 5\beta, q) = U_\epsilon(x_{\ell} + 5\beta - q) \leq 1 - \alpha_1(L) - C_2 \exp(-C_1 \epsilon^{-\frac{3}{2}}) < u_\epsilon(x_{\ell} + 5\beta). \] (3.26)

Therefore, $V(x, q)$ is an upper solution. We will next show
\[ V(x, 4\beta) = U_\epsilon(x - 4\beta) < u_\epsilon(x) \quad (x_{\ell} \leq x \leq x_{\ell} + 5\beta). \] (3.27)

By (3.22), (3.23) and (3.21), we have
\[ V(x, 4\beta) = U_\epsilon(x - 4\beta) \leq 1 - \alpha_1(L) - C_2 \exp(-C_1 \epsilon^{-\frac{3}{2}}) < u_\epsilon(x) \quad (x_{\ell} + 3\beta \leq x \leq x_{\ell} + 5\beta). \] (3.28)

On the other hand, for $x_{\ell} \leq x \leq x_{\ell} + 3\beta$ it holds that
\[ V(x, 4\beta) = U_\epsilon(x - 4\beta) = 0 < u_\epsilon(x) \quad (x_{\ell} \leq x \leq x_{\ell} + 3\beta). \] (3.29)

Then (3.28) and (3.29) imply (3.27). From (3.24) – (3.27) and the comparison principle on ordinary differential equation, or the strong maximum principle, it follows that
\[ V(x, q) < u_\epsilon(x) \quad (x_{\ell} \leq x \leq x_{\ell} + 5\beta, \ 0 \leq q \leq 4\beta). \]

Setting $q = 0$, we have
\[ w(x) = U_\epsilon(x) < u_\epsilon(x) \quad (x_{\ell} \leq x \leq x_{\ell} + 5\beta). \] (3.30)

Since $w(x) = 0 < u_\epsilon(x)$ holds in $0 \leq x \leq x_{\ell}$, (3.19), (3.20) and (3.30) completes the proof of Lemma 3.6. 

Now by Lemma 3.6 and the fact that $u_\epsilon$ is an upper solution and $w$ is a lower solution, there exists a solution $w(x)$ of (3.14) with $p(x)$ satisfying (P0) – (P3), where $w(x)$ satisfies
\[ w(x) \text{ is close to 0 in } [0, 1] \setminus \bigcup_{\ell \in \mathbb{Z}} (x_{\ell}, x_{\ell+1}), \]
\[ w(x) \text{ is close to 1 in } \bigcup_{\ell \in \mathbb{Z}} (x_{\ell}, x_{\ell+1}). \] (3.31)

It holds from the assumption that $u_\epsilon$ has at least one layer that $\bigcup_{\ell \in \mathbb{Z}} (x_{\ell}, x_{\ell+1})$ is neither empty set nor $[0, 1]$. By Step 1, such a solution $w(x)$ satisfying (3.31) cannot exist. This is a contradiction. The proof of Lemma 3.5 is complete. 

Proof of Theorem 1.1. The uniqueness obtained by Lemmas 3.1, 3.2 and 3.5 proves Theorem 1.1 under the condition $g(x) \geq p(x)$, where $p(x)$ satisfies (P0) – (P3).
For general $g(x)$, where $g(x)$ does not satisfy $g(x) \geq p(x)$, we can obtain some information using the similar idea as the proof of Lemma 3.5. For example, the following lemma shows that the condition $\int_0^1 g(x) \, dx \geq 0$ allows only one kind of one bump solutions. We denote zeros of $g(x)$ by $\{x_i \mid g(x_i) = 0, i = 1, 2, \cdots, m\}$.

**Lemma 3.7.** Assume $\int_0^1 g(x) \, dx \geq 0$. Let both $u_\epsilon(x)$ and $v_\epsilon(x)$ be solutions of (2.1). Assume that

\[
\begin{aligned}
    &\quad \text{if } \epsilon \rightarrow 0, \\
    &u_\epsilon \text{ is close to } 1 \text{ in } (x_i, x_{i+1}), \\
    &v_\epsilon \text{ is close to } 1 \text{ in } (x_j, x_{j+1}),
\end{aligned}
\]

Then it holds that $i = j$.

**Proof of Lemma 3.7.** Assume that $i \neq j$ to obtain a contradiction. Set $u_{\min}(x) = \min(u_\epsilon(x), v_\epsilon(x))$. If $i \neq j$, $u_{\min}(x)$ is close to 0. Since both $u_\epsilon$ and $v_\epsilon$ are solutions of (2.1), and $u_\epsilon \neq v_\epsilon$, $u_{\min}$ is an upper solution in strict sense. Therefore, there is a stable solution $u(x)$ such that $0 < u(x) < u_{\min}(x)$. However, by Lemma 3.1, there is no solution except for $u = 0$ under the condition $\int_0^1 g(x) \, dx \geq 0$. Therefore, $u = 0$ is a stable solution. This contradicts Theorem LNS given in Proof of Lemma 3.5.

**4. Concluding remarks.** Recently we have found that Part (c) of the conjectures in Introduction is false. We will present the counterexample for Conjecture (c) in this section. In fact, (1.6) has more than three nontrivial solutions for some $g(x)$. We will find such $g(x)$ in the following way.

First, let $b(x) \in C^2[0, 1]$ be a function satisfying

(B1) There exists $x_0 \in (0, 1)$ such that $b(x) < 0$ in $[0, x_0)$, $b(x) > 0$ in $(x_0, 1]$, and $b'(x_0) > 0$,

(B2) $b'(0) = b'(1) = 0$,

(B3) $\int_0^1 b(x) \, dx < 0$.

Now, for a constant $k \geq 0$, we define

\[
b_{ex}(x : k) = \begin{cases} 
    b(-(2 + k)x + 1) & 0 \leq x \leq \frac{1}{2 + k}, \\
    b(0) & \frac{1}{2 + k} \leq x \leq \frac{1 + k}{2 + k}, \\
    b((2 + k)x - 1 - k) & \frac{1 + k}{2 + k} \leq x \leq 1.
\end{cases}
\]

(4.1)

Note that $b_{ex}(x; \ell)$ is symmetric with respect to $x = \frac{1}{2}$ and $b_{ex}(x; k)$ has two zeros

\[
z_1(k) = \frac{1 - x_0}{2 + k}, \quad z_2(k) = \frac{x_0 + 1 + k}{2 + k}.
\]

Finally, set $g(x) = b_{ex}(x; k)$ in (1.6). By Theorem 1.2, $U_d(x)$ satisfies the following:

(i) $U_d \rightarrow 1$ uniformly in any compact set in $(0, z_1(k)) \cup (z_2(k), 1)$, as $d \rightarrow 0$

(ii) $U_d \rightarrow 0$ uniformly in any compact set in $(z_1(k), z_2(k))$, as $d \rightarrow 0$.

On the other hand, by [3] and [8], there exists at least one solution $\omega_d(x)$ of (1.6) with $g(x) = b_{ex}(x; k)$ satisfying $\omega_d \rightarrow 0$ uniformly in any compact set in $(0, 1)$, as $d \rightarrow 0$.

In addition to $U_d(x)$ and $\omega_d(x)$, the following theorems show that many other nontrivial solutions of (1.6) exist for some $k$. 
Theorem 4.1. Let \( b(x) \in C^1[0, 1] \) be any function satisfying (B1) - (B3), and set \( g(x) = b_{xx}(x; k) \), where \( b_{xx}(x; k) \) is given by (4.1). Then there exists \( k^* (> 0) \) such that for \( k = k^* \) and \( d \) sufficiently small, (1.6) has a solution \( u_d(x) \) satisfying (i) and (ii) below:

(i) On every compact subset \( S \) of \( [0, z_2(k)] \),
\[
C_1 d < u_d(x) < C_2 d.
\]

(ii) On every compact subset \( S \) of \( (z_2(k), 1] \),
\[
C_3 \exp \left( -\frac{C_4}{\sqrt{d}} \right) < 1 - u_d(x) < C_5 \exp \left( -\frac{C_6}{\sqrt{d}} \right),
\]
where the constants \( C_1, \ldots, C_6 \) depend on \( S \).

Note that \( C_1, \ldots, C_6 \) are not the same as those of Theorem 1.2.

Once we find a solution \( u_q \) satisfying (i) and (ii) in Theorem 4.1, we are able to find many solutions as we will show in the following Theorems 4.2 and 4.3.

Let \( k^* \) be the constant in Theorem 4.1. The next theorem shows the existence of two different solutions, both of which have only one layer.

Theorem 4.2. Suppose that \( g(x) \in C^1[0, 1] \) satisfies (H) and \( g(x) < b_{xx}(x; k^*) \) in \([0, 1]\). We also suppose \( g(x) \) has exactly two zeros \( \zeta_1, \zeta_2 \) such that \( g(\zeta_1) = g(\zeta_2) = 0 \) and \( 0 < \zeta_1 < z_1(k) < z_2(k) < \zeta_2 < 1 \). Then for sufficiently small \( d \), (1.6) has two solutions \( u_i(x) \) (\( i = 1, 2 \)) satisfying (i) and (ii).

(i) On every compact subset \( S \) of \( [0, \zeta_2) \) (resp. \( (\zeta_1, 1] \)),
\[
C_1 d < u_i(x) < C_2 d.
\]

(ii) On every compact subset \( S \) of \( (\zeta_2, 1] \) (resp. \( [0, \zeta_1) \)),
\[
C_3 \exp \left( -\frac{C_4}{\sqrt{d}} \right) < 1 - u_i(x) < C_5 \exp \left( -\frac{C_6}{\sqrt{d}} \right),
\]
where the constants \( C_1, \ldots, C_6 \) depend on \( S \). Moreover, \( u_1(x) < u_2(x) \) holds and \( u_1(x) \) is stable.

Note that \( C_1, \ldots, C_6 \) are not the same as those of Theorem 1.2.

The next theorem is for the existence of solutions close to 0 when \( g(x) \) is symmetric with respect to \( \frac{1}{2} \).

Theorem 4.3. Suppose that \( g(x) \in C^1[0, 1] \) satisfies (H), \( g(x) < b_{xx}(x; k^*) \), and that \( g(x) \) has exactly two zeros and satisfies \( g(1 - x) = g(x) \) in \([0, 1]\). Then there exists \( C_1, C_2 \) such that for sufficiently small \( d \), (1.6) has at least three solutions \( w_i(x) \) (\( i = 1, 2, 3 \)) satisfying \( C_1 d < w_i(x) < C_2 d \) in \([0, 1]\).

Suppose that \( g(x) \) satisfies the same assumption as that of Theorem 4.3. By Theorem 4.1 and 4.2, we obtain 4 solutions. Since \( h(x) \) is symmetric, 4 solutions are described as \( u_1(x), u_1(1 - x), u_2(x), u_2(1 - x) \). All of them have only one layer. On the other hand, Theorem 4.3 shows the existence of 3 solutions which stay close to 0. With \( U_d(x) \), the equation (1.6) has at least 8 nontrivial solutions under the condition.

In this section we will show the outline of the proof of Theorem 4.1, which shows the existence of the third solution. We will first introduce Lemma 4.4 below, which is equivalent to Theorem 4.1. By setting
\[
\xi = (2 + k)x - 1 - k, \quad \epsilon^3 = (2 + k)^2 d \quad (k \geq 0), \quad U(\xi) = u(x), \quad (4.2)
\]
the equation (1.6) with a condition \( g(x) = b_{ex}(x; \ell) \) is rewritten as

\[
\begin{aligned}
\epsilon^3 U_{\xi\xi} + h_{ex}(\xi; k) U^2(1 - U) &= 0 \quad \text{in} \quad (-k - 1, 1), \\
U'(-k - 1) = U'(1) &= 0,
\end{aligned}
\]

with \( k \geq 0 \), where

\[
h_{ex}(\xi; k) = \begin{cases}
b(-\xi - k) & -k - 1 \leq \xi \leq -k, \\
b(0) & -k \leq \xi \leq 0, \\
b(\xi) & 0 \leq \xi \leq 1.
\end{cases}
\]

Then Theorem 4.1 is rewritten as the following lemma.

**Lemma 4.4.** For \( \epsilon \) sufficiently small, (4.3) has a nontrivial solution \( U(\xi) \) satisfying

(i) On every compact subset \( S \) of \([-k - 1, x_0]\),

\[
C_1 \epsilon^3 < U(\xi) < C_2 \epsilon^3.
\]

(ii) On every compact subset \( S \) of \((x_0, 1]\),

\[
C_3 \exp(-C_4 \epsilon^{-\frac{3}{2}}) < 1 - U(\xi) < C_5 \exp(-C_6 \epsilon^{-\frac{3}{2}}),
\]

where the constants \( C_1 - C_6 \) depend on \( S \).

The idea to prove this lemma is as follows. We first consider auxiliary problem of (4.3).

\[
\begin{aligned}
\epsilon^3 u'' + h(x; k) u^2(1 - u) &= 0 \quad \text{in} \quad (-k, 1), \\
\ell(u(-k)) = \ell(u(1)) &= 0.
\end{aligned}
\]

with \( k \geq 0 \), Here \( h(x; k) \) is defined as

\[
h(x; k) = \begin{cases}
b(0) & -k \leq x \leq 0, \\
b(x) & 0 \leq x \leq 1.
\end{cases}
\]

Especially, \( h(x; 0) = b(x) \). Note that

\[
h_{ex}(x; k) = h(x; k) \quad \text{in} \quad [-k, 1], \quad h_{ex}(x; k) = b(-x - k) \quad \text{in} \quad [-k - 1, -k].
\]

In [10] we find a solution of \((E)_k\) with one layer, which we denote by \( u_\epsilon(x; k) \). We will also find a solution of \((E)_0\), which is close to 0 in \([0, 1]\), denoted by \( w_\epsilon(x) \). Suppose there exists \( k^* > 0 \) such that

\[
w_\epsilon(0) = u_\epsilon(-k^*; k^*).
\]

Then we can define

\[
U_{ex}(\xi; k^*) = \begin{cases}
w_\epsilon(-\xi - k^*) & -k^* - 1 \leq \xi \leq -k^*, \\
u_\epsilon(\xi; k^*) & -k^* \leq \xi \leq 1,
\end{cases}
\]

and show that \( U_{ex}(\xi; k^*) \) is a solution satisfying (i) and (ii) in Lemma 4.4. Since both \( u_\epsilon(x) \) and \( u_\epsilon(x; k) \) satisfy Neumann zero boundary condition, the standard regularity argument shows that \( U_{ex} \) is smooth and is a solution of (4.3) in \([-k - 1, 1]\).

For the rigorous proof, we refer [10]. Sections 3 - 5 in [10] are devoted to prove (4.6). The proof of Theorems 4.2 and 4.3 are given in Section 6 in [10].
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Received April 2019; 1st revision August 2019; final revision January 2020.

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