SIMPLICIAL IDEALS, 2-LINEAR IDEALS AND ARITHMETICAL RANK

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Abstract. In the first part of this paper we study scrollers and linearly joined varieties. Scrollers were introduced in [BM4], linearly joined varieties are an extension of scroller and were defined in [EGHP], and they proved that scrollers are defined by homogeneous ideals having a 2-linear resolution. A particular class of varieties, of important interest in classical Geometry are Cohen–Macaulay varieties of minimal degree, they were classified by the successive contribution of Del Pezzo [DP], Bertini [B], and Xambo [X]. They appear naturally studying the fiber cone of of a codimension two toric ideals [GMS1], [GMS2], [BM1], [H], [HM]. Let $S$ be a polynomial ring and $I \subset S$ a homogeneous ideal defining a sequence of linearly-joined varieties.

• We compute the depth $S/I$, and the cohomological dimension $\text{cd}(I)$.
• We prove that under some hypothesis that $c(V) = \text{depth } S/I - 1$, where $c(V)$ is the connectedness dimension of the algebraic set defined by $I$.
• We characterize sets of generators of $I$, and give an effective algorithm to find equations, as an application we prove that $\text{ara} (I) = \text{projdim} (S/I)$ in the case where $V$ is a union of linear spaces, in particular this applies to any square free monomial ideal having a 2-linear resolution.
• In the case where $V$ is a union of linear spaces, the ideal $I$, can be characterized by a tableau, which is an extension of a Ferrer (or Young) tableau.
• We introduce a new class of ideals called simplicial ideals, ideals defining linearly-joined varieties are a particular case of simplicial ideals. All these results are new, and extend results in [BM4], [EGHP].

1 Introduction

Throughout this paper we will work with projective schemes $X \subset \mathbb{P}^r$, but we adopt the algebraic point of view, that is we consider a polynomial ring $S$, graded by its standard graduation and reduced homogeneous ideals. Our motivation comes from the study of the fiber cone $F(I)$ of a codimension two toric ideal, as it was shown in [GMS1], [GMS2].

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$F(I)$ appears to be a Cohen–Macaulay reduced ring having minimal multiplicity. Projective algebraic sets arithmetically Cohen–Macaulay with minimal degree were classified by the successive contribution of Del Pezzo [DP], Bertini [B], and Xambo [X], and were characterized homologically by Eisenbd-Goto [EG]. In the case of irreducible varieties of minimal degree, equations defining such varieties are given, but as was pointed by De Concini-Eisenbud-Procesi [CEP]: "the precise equations satisfied by reducible subvarieties of minimal degree remain mysterious", hence they "stop short of giving a normal form for the equations of each type. In [BM2], [BM4], we tried to answer to this question by describing a set of axioms satisfied by the ideal of any a linear union of scrolls. Some of our results were extended in [EGHP].

In [BM4], we have extended the notion of varieties of minimal degree to the notion of scrollers, where the algebraic set is not assumed to be equidimensional, Eisenbud-Green-Hulek-Popescu [EGHP] have defined, more generally, the notion of linearly joined varieties, without assuming that each irreducible component is a scroll, and they prove that scrollers (I use here our definition) are exactly the 2–regular (in the sense of Castelnuovo-Mumford) projective reduced algebraic sets. In this paper we continue to investigated about the structure of scrollers, in the first part of the paper we extend the characterization of the ideals of scrolls given in [BM4] to the case of linearly joined varieties, as a consequence we can compute some invariants of linearly joined varieties, as the depth, the connectedness dimension and the arithmetical rank, as a corollary we can give an effective algorithm to describe equations of linearly joined varieties, improving previous results in [BM4], [EGHP]. As an important corollary we get that for the ideal $J \subset S$ of any 2–regular algebraic set which is a union of linear spaces, the arithmetical rank of $J$ equals the projective dimension $\text{projdim}(S/J)$. In particular this is true for square free monomial ideals having a 2–linear resolution. Note that this results are independent on the characteristic of the field $K$.

In the second part of this paper, we extend the notion of linearly joined varieties to a linear-union of varieties, A reduced ideal $J \subset S$ defines a linear-union of varieties, if $J \subset S$ is the intersection of primes ideals $J_i = (M_i, (Q_i))$ for $i = 1, ..., l$, where $(Q_i)$ is the ideal of some sublinear space, satisfying the property:

$$J = (M_1, ..., M_s, \bigcap_{j=1}^{s}(Q_i)).$$

We define a class of linear-union of varieties, defined by "Simplicial ideals", a Simplicial ideal is a couple $(P_G, \tilde{\Delta})$ associated to a simplicial complex $\tilde{\Delta}$, on a set of vertices $G$, and with facets $G_1, ..., G_s$, with some properties. Recently, in her thesis work, my student Ha Minh Lam [H] has studied a class of Simplicial ideals, which variety is an intersection of scrolls, and she has proved that they are scrollers and have a 2-linear resolution. She also has studied the reduction number for some class of Simplicial ideals. We apply and extends some results of the first part of the paper to Simplicial ideals. In fact the methods developed here apply to a more general setting, this is part of my work in progress.

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2 Linearily joined varieties

Definition 1 (See [EGHP],[BM4]). An ordered sequence \( V_1, ..., V_l \subset \mathbb{P}^r \) of irreducible projective subvarieties is linearily joined if for any \( i = 1, ..., l-1 \) we have:

\[
V_{i+1} \cap (V_1 \cup ... \cup V_i) = \text{span}(V_{i+1}) \cap \text{span}(V_1 \cup ... \cup V_i)
\]

where \( \text{span}(V) \) is the smallest linear subspace of \( \mathbb{P}^r \) containing \( V \).

Linearily joined varieties were defined first in [BM4], assuming that each variety \( V_i \) is a scroll (there were called scrollers), then it was extended to the general case in [EGHP].

Here we follow the algebraic point of view developed in [BM4]. Let \( V \) a \( K \) vector space of dimension \( r+1 \), \( S = K[V] \), the polynomial ring corresponding to the projective space \( \mathbb{P}^r \). For any set \( Q \subset V \) we will denote by \( \langle Q \rangle \subset V \) the \( K \)-vector space generated by \( Q \) and by \( (Q) \subset S \) the ideal generated by \( Q \). For all \( m = 1, ..., l \), the irreducible variety \( V_m \) is defined by the reduced ideal \( J_m \subset S \). The linear variety \( L_m := \text{span}(V_m) \) is defined by an ideal generated by independent linear forms, so let \( Q_m \subset V \) be the linear space such that \( (Q_m) \) is the ideal defining \( L_m \). We can write \( J_m = (M_m, (Q_m)) \) where \( M_m \) is an ideal.

By [BM4] page 163, to show that the sequence of irreducible projective varieties \( V_1, ..., V_l \subset \mathbb{P}^r \) is linearily joined, we will say that the sequence of ideals \( J_1, ..., J_l \) is linearily joined, is equivalent to show that for all \( k = 2, ..., l \):

\[
J_k + \cap_{i=1}^{k-1} J_i = (Q_k) + (\cap_{i=1}^{k-1} Q_i).
\]

It follows from this relation that the sequence \( L_1, ..., L_l \) is also linearily joined. We denote \( L = L_1 \cup ... \cup L_l \) and \( Q := (Q_1) \cap ... \cap (Q_l) \), its defining ideal. The Theorem 2.1 of [BM4] page 163, can be extended to linearily joined varieties: more precisely

Definition 2 let \( D_1 = Q_1, D_i := \cap_{j=1}^{i} Q_j \). For all \( i = 2, ..., r \) let \( \langle \Delta_i \rangle \) be a linear space such that \( D_{i-1} = D_i \oplus \langle \Delta_i \rangle \), and let \( P_i \) be a linear space such that \( Q_i = P_i \oplus D_i \).

It follows from the definition that \( P_1 = 0, D_1 = 0, D_1 \supseteq D_2 \supseteq ... \supseteq D_l \), is a chain of subvector spaces, and \( D_i = \bigoplus_{j=i}^{l} \langle \Delta_j \rangle \). From [BM4] page 163, we have that for all \( i = 2, ..., r \), \( P_i \cap D_{i-1} = 0, Q_i + D_{i-1} = D_i \oplus \langle \Delta_i \rangle \oplus P_i \).

With the notations introduced before we have:

Theorem 1 The following conditions are equivalent:

1. the sequence of ideals \( J_1, ..., J_l \subset S := K[V] \) is linearily joined,

2. For all \( i = 1, ..., l \), there exist sublinear spaces \( D_i, P_i \subset V \), with \( D_1 = 0, P_1 = 0 \), and ideals \( M_i \subset K[V] \) such that

   a) for all \( i = 1, ..., l \), \( J_i = (M_i, Q_i) \)

   b) \( Q_i = D_i \oplus P_i \)

   c) \( D_1 \supseteq D_2 \supseteq ... \supseteq D_l \).

   d) \( M_i \subset (D_{i-1}) \) for all \( i = 2, ..., l \).
Proof Though the proof developed in [BM4, page 164] applies here, I will give a shorter proof of the implication “1. ⇒ 2.”.

The proof is by induction on \( l \). For \( l = 2 \), \( Q_1 = \langle \Delta_2 \rangle \oplus D_2, Q_2 = \mathcal{P}_2 \oplus D_2 \) with \( \langle \Delta_2 \rangle \cap \mathcal{P}_2 = 0 \), let \( S = K[V] \), where \( V \supset \langle \Delta_2 \rangle \oplus \mathcal{P}_2 \oplus D_2 \). The relation (*) implies that

\[
\mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots \oplus \mathcal{M}_l \subseteq (\langle \Delta_2 \rangle \oplus \mathcal{P}_2 \oplus D_2),
\]

and without changing the ideals \( J_1, J_2 \) we can consider the ideal \( \mathcal{M}_1 \) modulo the ideal \( (\langle \Delta_2 \rangle \oplus D_2) \), and the ideal \( \mathcal{M}_2 \) modulo the ideal \( (\langle \Delta_2 \rangle \oplus \mathcal{P}_2) \), so we get that \( \mathcal{M}_1 \subseteq (\mathcal{P}_1), \mathcal{M}_2 \subseteq (\mathcal{P}_2) \).

Now suppose that our assertion is true for \( l - 1 \), then

For \( i = 1, \ldots, l - 1, Q_i = Q'_i \oplus \mathcal{D}_{l-1} \)

\[
J_i + \cap_{i=1}^{l-1} J_i = (Q_i) + (\cap_{i=1}^{l-1} Q_i) = (\mathcal{P}_i \oplus \mathcal{D}_{l-1}),
\]

which implies that

\[
\mathcal{M}_1 + \mathcal{M}_2 + \cdots + \mathcal{M}_l \subseteq (\mathcal{P}_l \oplus \mathcal{D}_{l-1}),
\]

by induction hypothesis \( \mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_{l-1} \) are defined modulo \( \mathcal{D}_{l-1} \) and without changing the ideal \( J_i \) we can consider the ideal \( \mathcal{M}_i \) modulo the ideal \( (\mathcal{P}_i) \), and by the same arguments as above we get that for \( i = 1, \ldots, l - 1, \mathcal{M}_i \subseteq (\mathcal{P}_i) \), and \( \mathcal{M}_l \subseteq (\mathcal{P}_{l-1}) \).

Corollary 1

1. The sequence of ideals \( J_1, \ldots, J_l \subset S := K[V] \) is linearly joined.

2. For all \( i = 1, \ldots, l \), there exist sublinear spaces \( Q_i \subset V \), and ideals \( \mathcal{M}_i \subset K[V] \) such that

   a) for all \( i = 1, \ldots, l \), \( J_i = (\mathcal{M}_i, Q_i) \)

   b) the sequence of ideals \( Q_1, \ldots, Q_l \subset S := K[V] \) is linearly joined,

   c) \( \mathcal{M}_i \subseteq (Q_j) \) for all \( i \neq j, i, j \in \{1, \ldots, l\} \).

Proof

1. "1. ⇒ 2.,” is clear from the above theorem.

2. "2. ⇒ 1.” We know from [EGHP, Prop. 3.4], that \( J_1, \ldots, J_l \subset S := K[V] \) is linearly joined, if and only if \( Q_1, \ldots, Q_l \subset S := K[V] \) is linearly joined, and for all \( i \neq j, i, j \in \{1, \ldots, l\} \) the pair \( J_i, J_j \) is linearly joined. So it will be enough to prove that \( i \neq j \), the pair \( J_i, J_j \) is linearly joined, but \( J_i \cap J_j = (Q_i) + (Q_j) \), by hypothesis so our claim follows.

We also have from [BM4, page 164],

Corollary 2 For any sequence of ideals \( J_1, \ldots, J_l \) satisfying the axioms a) to e) we have:

\[
\bigcap_{j=1}^{k} J_j = (\mathcal{M}_1, \ldots, \mathcal{M}_k, \bigcap_{j=1}^{k} (Q_j))
\]

for all \( k = 1, \ldots, l \).
2.1 Equations of linearly-joined Hyperplane arrangements

As we have seen before if \( V_1, \ldots, V_l \) is a sequence of linearly joined irreducible varieties in \( \mathbb{P}^r \), then the sequence \( L_1, \ldots, L_l \) is also linearly joined. In this subsection we will study this situation. We denote \( L = L_1 \cup \ldots \cup L_l \) and \( Q := (Q_1) \cap \ldots \cap (Q_l) \) its defining ideal.

**Corollary 3** (See Theorem [7]) The following conditions are equivalent:

1. the sequence of ideals \((Q_1), \ldots, (Q_l)\) is linearly joined,
2. For all \( i = 1, \ldots, l \), there exist sublinear spaces \( D_i, P_i \), with \( D_1 = 0, P_1 = 0 \), such that
   - a) \( Q_i = D_i \oplus P_i \)
   - b) \( D_1 \supseteq D_2 \supseteq \ldots \supseteq D_l \),
   - c) \( \bigcap_{j=1}^{k-1} (Q_j) \subseteq (P_k, D_{k-1}) \) for all \( k = 2, \ldots, l \).

   For all \( k = 2, \ldots, l \) the sequence \( L_1, \ldots, L_k \) is linearly joined in the linear space spanned by \( L_1, \ldots, L_k \). Let \( D_{j,k} = \bigoplus_{i=j+1}^{k} (\Delta_i), Q_{j,k} := P_j \oplus D_{j,k} \). So we have that for \( j = 1, \ldots, k \), \( Q_j = Q_{j,k} \oplus D_{j,k} \), \( Q_{j,k} \) is the ideal defining \( L_j \) in \( (L_1, \ldots, L_k) \), and the sequence \( Q_{1,k}, \ldots, Q_{k,k} \) is linearly joined. As a consequence of the above corollary we have that \( \bigcap_{j=1}^{k-1} (Q_j) \subseteq (P_k) \) for all \( k = 2, \ldots, l \). We can now improve the Lemma 3.1 of [BM4]:

**Lemma 1** For any \( k = 2, \ldots, l \),

\[
\bigcap_{j=1}^{k} (Q_{j,k}) = \bigcup_{j=1}^{k} ((\Delta_j) \times P_j),
\]

where \( (\Delta_j) \times P_j \) is the ideal generated by all the products \( fg \), with \( f \in (\Delta_j), g \in P_j \).

**Proof** The proof is by induction on \( k \). For \( k = 2 \), \( Q_{1,2} = (\Delta_2), Q_{1,2} = P_2 \) and \( (\Delta_2) \cap P_2 = 0 \), and \( S = K[(\Delta_2) \oplus P_2] \). It is clear that \( (\Delta_2) \times P_2 \) is irreducible. The other inclusion follows working modulo \( (\Delta_2) \times P_2 \) and using the fact that \( (\Delta_2) \cap P_2 = 0 \).

Now suppose that our assertion is true for \( k - 1 \), then

\[
\bigcap_{j=1}^{k} (Q_{j,k}) = (P_k) \cap \bigcup_{j=1}^{k-1} ((\Delta_j) \times P_j, (\Delta_k)),
\]

but by the above corollary \( \bigcup_{j=1}^{k-1} ((\Delta_j) \times P_j) \subseteq (P_k) \), so \( \bigcup_{j=1}^{k} ((\Delta_j) \times P_j) \subseteq \bigcap_{j=1}^{k} (Q_{j,k}) \), again using the fact that \( (\Delta_k) \cap P_k = 0 \) we will have our statement. As a consequence of the Lemma we get the following characterization of equations of linearly joined hyperplane arrangements:

**Proposition 1** (See Theorem [7]) The following conditions are equivalent:

1. the sequence of ideals \((Q_1), \ldots, (Q_l)\) is linearly joined,
2. For all $i = 1, \ldots, l$, there exist sublinear spaces $\langle \Delta_i \rangle, P_i$, with $\langle \Delta_1 \rangle = 0, P_1 = 0$, such that

- a) $D_i := \bigoplus_{j=i+1}^{l} \langle \Delta_j \rangle$
- b) $Q_i = D_i \oplus P_i$
- c) For any $k = 2, \ldots, l$, and $j < k$ we have $\langle \Delta_j \rangle \times P_j \subset (P_k)$.

**Proof** The above Lemma implies that for all $k = 2, \ldots, l$

$$\bigcap_{j=1}^{k-1} (Q_j) = \bigcup_{j=2}^{k-1} (\langle \Delta_j \rangle \times P_j, D_{k-1}) \subset (P_k, D_{k-1}).$$

**Definition 3** (In view of the applications.) Given a sequence of linearly joined linear spaces $L_1 \cup \ldots \cup L_l \subset IP^r$, we will make an extension $\tilde{L}_1 \cup \ldots \cup \tilde{L}_l \subset IP((V \oplus V')^*)$. Let consider a sequence of linear spaces $H_1, \ldots, H_l$, such that $IP((V \oplus V')^*) = IP(V^*) \oplus \bigoplus_{i=1}^{l} H_i$ and set $\tilde{L}_i = L_i \oplus H_i$.

**Proposition 2** 1. The sequence $\tilde{L}_1 \cup \ldots \cup \tilde{L}_l \subset IP((V \oplus V')^*)$ is linearly joined. Moreover

$$\tilde{L}_i \cap (\tilde{L}_1 \cup \ldots \cup \tilde{L}_{i-1}) = L_i \cap (L_1 \cup \ldots \cup L_{i-1}).$$

2. From the algebraic point of view, let $V' = \bigoplus_{i=1}^{l} F_i$, such that $H_i$ is defined by the ideal $(\bigoplus_{j \neq i} F_j)$, so $\tilde{L}_i$ is defined by the ideal $(Q_i) := (Q_i \oplus (\bigoplus_{j \neq i} F_j))$. With this notation

$$\bigcap_{i=1}^{l} (\tilde{Q}_i) = \bigcap_{i=1}^{l} (Q_i) \cup \bigcup_{i=1}^{l} \tilde{Q}_i \times F_i.$$
2. We have the decomposition :
\[ \tilde{Q}_i = \tilde{P}_i \oplus \tilde{D}_i, \]
with
\[ \tilde{P}_i = P_i \oplus \bigoplus_{j < i} F_j, \quad \tilde{D}_i = D_i \oplus \bigoplus_{j > i} F_j. \]

Let \( \langle \Delta_{i+1} \rangle \) be a linear space such that \( D_i = D_{i+1} \oplus \langle \Delta_{i+1} \rangle \), so we have that \( \tilde{D}_i = D_{i+1} \oplus \langle \Delta_{i+1} \rangle \oplus F_{i+1} \), and applying the lemma the ideal \( \bigcap_{i=1}^l (\tilde{Q}_i) \) is generated by
\[ \bigcup_{i=2}^l (\langle \Delta_i \rangle \oplus F_i) \times (P_i \oplus \bigoplus_{j < i} F_j), \]
which is equal to:
\[ \bigcup_{i=2}^l (\langle \Delta_i \rangle) \times (P_i) \cup \bigcup_{i=2}^l (\langle \Delta_i \rangle) \times (\bigoplus_{j < i} F_j) \bigcup_{i=2}^l (F_i) \times (P_i) \bigcup_{i=2}^l (F_i) \times (\bigoplus_{j < i} F_j), \]
but
\[ \bigcup_{i=2}^l (F_i) \times (\bigoplus_{j < i} F_j) = \bigcup_{i=2}^l (F_i) \times (\bigoplus_{j \neq i} F_j), \]
and
\[ \bigcup_{i=2}^l ((\Delta_i)) \times (\bigoplus_{j < i} F_j) = \bigcup_{i=1}^{l-1} \bigcup_{j > i}^{l} (\bigoplus_{j < i} (\Delta_i)) \times (F_i) = \bigcup_{i=1}^{l-1} (D_i) \times (F_i), \]
so putting both computations together we get our claim.

2.2 Depth of linearly-joined varieties

We recall the following important facts :

**Remark 1**

1. Let \( S = K[G] \) be a ring of polynomials over a field \( K \), on a set of variables \( G \). For any subset \( \sigma \subseteq G \), with cardinal \( \text{card} \sigma \), the local cohomology group \( H_\text{m}^{\text{card} \sigma}(K[\sigma]) \), is an \( S \)-module isomorphic to
\[ K[\sigma^{-1}]_\sigma \simeq \bigoplus_{\alpha \in (1, \ldots, 1)^N + \text{card} \sigma} X^{-\alpha}. \]

It then follows that \( H_\text{m}^{\text{card} \sigma}(K[\sigma])_k = 0 \) for \( k < \text{card} \sigma \), and
\[ \text{dim} (H_\text{m}^{\text{card} \sigma}(K[\sigma])_{\text{card} \sigma}) = 1. \]

2. Let \( J \subseteq S \) be a reduced homogeneous ideal (for the standard grading in the polynomial ring) Suppose that \( J \) does not defines a linear space, let \( h = \text{depth} S/J \), then \( H_\text{m}^h(S/J)_{-(h-i)} \neq 0 \) for at least some \( i > 0 \).
3. For any two ideals $J_1, J_2 \subset S$ we have the following exact sequence:

$$0 \rightarrow S/J_1 \cap J_2 \rightarrow S/J_1 \oplus S/J_2 \rightarrow S/(J_1 + J_2) \rightarrow 0$$

which gives rise to the long exact sequence:

$$\rightarrow H^{h-1}_m(S/J_1 + J_2) \rightarrow H^h_m(S/J_1 \cap J_2) \rightarrow H^h_m(S/J_1) \oplus H^h_m(S/J_2) \rightarrow H^h_m(S/J_1 + J_2) \rightarrow$$

**Lemma 2** Let $X_1, X_2 \subset \mathbb{P}^w$ be a linearly joined sequence of projective subchemes (having a proper intersection). Let $J_1, J_2$ be the (reduced) ideals of definition of $X_1, X_2$, $L_1 = \text{span}(X_1), L_2 = \text{span}(X_2)$ and $(Q_1), (Q_2)$ its defining ideals. Then

$$\text{depth} \ S/J_1 \cap J_2 = \min \{ \text{depth} \ S/J_1, \ \text{depth} \ S/J_2, \ \dim S/(Q_1 + Q_2) + 1 \}.$$

Moreover since $\dim S/(Q_1 + Q_2) + 1 \leq \min \{ \dim S/J_1, \dim S/J_2 \}$ we have the particular cases

1. If $S/J_2$ is a Cohen–Macaulay ring then

$$\text{depth} \ S/J_1 \cap J_2 = \min \{ \text{depth} \ S/J_1, \ \dim S/(Q_1 + Q_2) + 1 \}.$$

2. If both $S/J_1, S/J_2$ are Cohen–Macaulay rings then

$$\text{depth} \ S/J_1 \cap J_2 = \dim S/(Q_1 + Q_2) + 1.$$

**Proof** We have that $J_1 + J_2 = (Q_1) + (Q_2)$, so $S/J_1 + J_2$ is isomorphic to a polynomial ring, let $h = \dim S/J_1 + J_2$, and $q = \min \{ \dim S/J_1, \dim S/J_2 \}$. It follows that $h + 1 \leq q$. So the last two assertions follow from the first one.

We have the following exact sequences:

(A) $$0 \rightarrow H^i_m(S/J_1 \cap J_2) \rightarrow H^i_m(S/J_1) \oplus H^i_m(S/J_2) \rightarrow 0$$

for either $i < h$ or $i > h + 1$ and

(B) $$0 \rightarrow H^h_m(S/J_1 \cap J_2) \rightarrow H^h_m(S/J_1) \oplus H^h_m(S/J_2) \rightarrow H^h_m(S/J_1 + J_2) \rightarrow$$

$$\rightarrow H^{h+1}_m(S/J_1 \cap J_2) \rightarrow H^{h+1}_m(S/J_1) \oplus H^{h+1}_m(S/J_2) \rightarrow 0$$

If $\min \{ \text{depth} S/J_1, \ \text{depth} S/J_2 \} < h$ or $\min \{ \text{depth} S/J_1, \ \text{depth} S/J_2 \} > h$ we have that

$$\text{depth} \ S/J_1 \cap J_2 = \min \{ \text{depth} S/J_1, \text{depth} S/J_2, \ \dim S/(Q_1 + Q_2) + 1 \},$$

it remains to consider the case $\min \{ \text{depth} S/J_1, \ \text{depth} S/J_2 \} = h$, since the above sequence is graded we have for any integer $i > 0$:

$$0 \rightarrow H^h_m(S/J_1 \cap J_2)_{-h+i} \rightarrow H^h_m(S/J_1)_{-h+i} \oplus H^h_m(S/J_2)_{-h+i} \rightarrow H^h_m(S/J_1 + J_2)_{-h+i} \rightarrow$$

$$\rightarrow H^{h+1}_m(S/J_1 \cap J_2)_{-h+i} \rightarrow H^{h+1}_m(S/J_1)_{-h+i} \oplus H^{h+1}_m(S/J_2)_{-h+i} \rightarrow 0$$
but since $S/J_1 + J_2$ is isomorphic to a polynomial ring of dimension $h$,

$$H^h_m(S/J_1 + J_2) - h+1 = 0,$$

for any integer $i > 0$, so we get

$$H^h_m(S/J_1 \cap J_2) - h+1 \simeq H^h_m(S/J_1) - h+1 \oplus H^h_m(S/J_2) - h+1$$

Without restriction we can assume for example that $h = \text{depth } S/J_1$, but $h < \min\{ \text{dim } S/J_1, \text{dim } S/J_2 \}$, this implies that $J_1$ cannot define a linear space and so $H^h_m(S/J_1) - h+1 \neq 0$, for at least some $i > 0$, which implies $H^h_m(S/J_1 \cap J_2) - h+1 \neq 0$. So

$$\text{depth } S/J_1 \cap J_2 = h = \min\{ \text{depth } S/J_1, \text{depth } S/J_2, \text{dim } S/(Q_1 + Q_2) + 1 \}.$$

**Theorem 2** Let $V_1, ..., V_l \subset P^n$ be a linearly joined sequence of irreducible projective sub-varieties. Let $V = V_1 \cup ... \cup V_l$, $J$ the (reduced) ideal of definition of $V$, $L = L_1 \cup ... \cup L_l$ and $Q := (Q_1) \cap ... \cap (Q_l)$ its defining ideal. Then

1. $\text{depth } S/Q = \min_{i=1,...,t-1} \{ \text{dim } L_{i+1} \cap (L_1 \cup ... \cup L_i) \} + 2$.
2. $\text{depth } S/J = \min\{ \text{depth } S/J_1, ..., \text{depth } S/J_l, \text{depth } S/Q \}$.
3. Assume that for all $i = 1, ..., l$ the ring $S/J_i$ is Cohen-Macaulay, then we have $\text{depth } S/J = \text{depth } S/Q$.

**Proof** 1. The proof is by induction on $l$. If $l = 2$, both rings $S/(Q_1), S/(Q_2)$ are Cohen-Macaulay, so by the above lemma we have

$$\text{depth } S/Q_1 \cap Q_2 = \text{dim } S/(Q_1 + Q_2) + 1,$$

and $\text{dim } L_1 \cap L_2 = \text{dim } S/(Q_1 + Q_2) - 1$.

Now suppose that our claim is true for $l - 1$ and we will prove it for $l$. We can apply the above lemma to the ideals $\bigcap_{i=1}^{l-1} (Q_i), (Q_l)$ so $\text{depth } S/Q = \min\{ \text{depth } S/\bigcap_{i=1}^{l-1} (Q_i), \text{dim } S/(Q_l) + (\bigcap_{i=1}^{l-1} Q_i) + 1 \}$, but $\text{dim } L_l \cap (L_1 \cup ... \cup L_{l-1}) + 1 = \text{dim } S/(Q_l) + (\bigcap_{i=1}^{l-1} Q_i)$ and by induction hypothesis

$$\text{depth } S/\bigcap_{i=1}^{l-1} (Q_i) = \min_{i=1,...,l-2} \{ \text{dim } L_{i+1} \cap (L_1 \cup ... \cup L_i) \} + 2.$$ 

So the claim follows.

2. The proof is by induction on $l$. The case $l = 2$ follows from the above lemma. We suppose that our claim is true for $l - 1$ and we will prove it for $l$. We can apply the above lemma to the ideals $\bigcap_{i=1}^{l-1} (J_i), (J_l)$ so

$$\text{depth } S/J = \min\{ \text{depth } S/\bigcap_{i=1}^{l-1} (J_i), \text{depth } S/(J_l), \text{dim } S/(J_l) + (\bigcap_{i=1}^{l-1} J_i) + 1 \};$$

3. Assume that for all $i = 1, ..., l$ the ring $S/J_i$ is Cohen-Macaulay, then we have $\text{depth } S/J = \text{depth } S/Q$.
but \((\mathcal{J}_l) + (\bigcap_{i=1}^{l-1} \mathcal{J}_i) = (Q_l) + (\bigcap_{i=1}^{l-1} Q_i)\), so \(\dim S/(\mathcal{J}_l) + (\bigcap_{i=1}^{l-1} \mathcal{J}_i) = \dim \mathcal{L}_l \cap (\mathcal{L}_1 \cup \ldots \cup \mathcal{L}_{l-1}) + 1\), and by induction hypothesis

\[
\text{depth } S/\bigcap_{i=1}^{l} (\mathcal{J}_i) = \min \{ \text{depth } S/\mathcal{J}_1, \ldots, \text{depth } S/\mathcal{J}_l, \text{depth } S/\bigcap_{i=1}^{l-1} (Q_i) \}.
\]

So by using our claim 1, the claim 2. follows.

The proof of the claim 3. follows by the same arguments developed in the proof of the claim 2.

**Corollary 4** Let \(\mathcal{L}_1 \cup \ldots \cup \mathcal{L}_l \subset \mathbb{P}^r\) be a sequence of linearly joined linear spaces, and consider its extension \(\tilde{\mathcal{L}}_1 \cup \ldots \cup \tilde{\mathcal{L}}_l \subset \mathbb{P}((V \oplus V')^*)\), as defined in the Definition 3. Then

\[
\text{depth } K[V \oplus V']/\bigcap_{i=1}^{l} (\tilde{Q}_i) = \text{depth } K[V]/\bigcap_{i=1}^{l} (Q_i).
\]

From the proof of the Lemma 2, we also get:

**Corollary 5** Let \(\mathcal{J}_1, \ldots, \mathcal{J}_l \subset S\) be a sequence of ideals. Then

1. If \(S/\mathcal{J}_1, S/\mathcal{J}_2, S/(\mathcal{J}_1 + \mathcal{J}_2)\) are Cohen–Macaulay rings and \(\dim S/(\mathcal{J}_1 + \mathcal{J}_2) < \min \{ \dim S/\mathcal{J}_1, \dim S/\mathcal{J}_2 \}\), then

\[
\text{depth } S/\mathcal{J}_1 \cap \mathcal{J}_2 = \dim S/(\mathcal{J}_1 + \mathcal{J}_2) + 1.
\]

and \(S/\mathcal{J}_1 \cap \mathcal{J}_2\) is Cohen–Macaulay if and only if

\[
\dim S/\mathcal{J}_1 = \dim S/\mathcal{J}_2 = \dim S/(\mathcal{J}_1 + \mathcal{J}_2) + 1.
\]

2. If \(S/\mathcal{J}_1, \ldots, S/\mathcal{J}_l\) are Cohen–Macaulay rings of the same dimension \(d\) and for all \(i = 2, \ldots, l\), \(S/(\mathcal{J}_{i+1} + \bigcap_{j=1}^{i} \mathcal{J}_j)\) is a Cohen–Macaulay ring of dimension \(d - 1\) then

\(S/\bigcap_{i=1}^{l} \mathcal{J}_j\) is a Cohen–Macaulay ring of dimension \(d\).

**Remark 2** The proof of the Lemma 2 provides an effective way to compute local cohomology modules for linearly joined varieties, it should be interesting to study such kind of local cohomology modules.

The next result was proved in [EGHP], we give here a shorter proof.

**Theorem 3** Let \(V_1, \ldots, V_l \subset \mathbb{P}^r\) be a linearly joined sequence of irreducible projective subvarieties. Let \(V = V_1 \cup \ldots \cup V_l\), \(\mathcal{J}\) the (reduced) ideal of definition of \(V\), \(\mathcal{L} = \tilde{\mathcal{L}}_1 \cup \ldots \cup \mathcal{L}_l\) and \(Q := (Q_1) \cap \ldots \cap (Q_l)\) its defining ideal. then

\[
\text{reg } (\mathcal{J}) = \max \{ 2, \text{reg } (\mathcal{J}_1), \ldots, \text{reg } (\mathcal{J}_l) \}.
\]
Theorem 4 Let $V_1, ..., V_l \subset \mathbb{P}^r$ be a linearly joined sequence of irreducible projective subvarieties. Let $\mathcal{V} = V_1 \cup ... \cup V_l$, $\mathcal{J}_i$ (resp. $\mathcal{I}$) the (reduced) ideal of definition of $V_i$ (resp. $\mathcal{V}$), $\mathcal{L} = L_1 \cup ... \cup L_i$ and $\mathcal{Q} := (Q_1) \cap ... \cap (Q_l)$ its defining ideal. We assume that each $\mathcal{J}_i$ is a scci.

1. $\text{cd}(\mathcal{I}) = \max_{i=2, ..., l} \{ \dim_K (P_i + D_{i-1}) - 1 \}$.
2. $\text{cd}(\mathcal{I}) = \text{cd}(\mathcal{Q}) = \text{projdim} (S/\mathcal{Q})$. 

2.3 Cohomological dimension

Let $I \subset S$ be an ideal (in our situation $S$ will be a graded polynomial ring). The cohomological dimension $\text{cd}(I)$ is the highest integer $q$ such that $H^q(S/I) \neq 0$. In this subsection we compute the cohomological dimension for some sequences of linearly joined ideals.

For any two ideals $J_1, J_2 \subset S$ we have the following exact sequence:

$$0 \rightarrow S/J_1 \cap J_2 \rightarrow S/J_1 \oplus S/J_2 \rightarrow S/(J_1 + J_2) \rightarrow 0$$

which gives rise to the long exact sequence:

$$\rightarrow H^{h-1}_{J_1 \cap J_2}(S) \rightarrow H^h_{J_1 + J_2}(S) \rightarrow H^h_{J_1}(S) \oplus H^h_{J_2}(S) \rightarrow H^h_{J_1 \cap J_2}(S) \rightarrow .$$

Let $\mathcal{J}_1, ..., \mathcal{J}_l \subset S$ be a linearly joined sequence of irreducible projective subvarieties. Let $\mathcal{V} = V_1 \cup ... \cup V_l$, $\mathcal{J}_i$ (resp. $\mathcal{I}$) the (reduced) ideal of definition of $V_i$ (resp. $\mathcal{V}$), $\mathcal{L} = L_1 \cup ... \cup L_i$ and $\mathcal{Q} := (Q_1) \cap ... \cap (Q_l)$ its defining ideal. We assume that each $\mathcal{J}_i$ is a scci.

1. $\text{cd}(\mathcal{I}) = \max_{i=2, ..., l} \{ \dim_K (P_i + D_{i-1}) - 1 \}$.
2. $\text{cd}(\mathcal{I}) = \text{cd}(\mathcal{Q}) = \text{projdim} (S/\mathcal{Q})$. 

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Since each \( \mathcal{J}_i \) is a stci, we have that \( \mathcal{J}_i \) can be generated up to radical by a regular sequence of \( \text{ht}(\mathcal{J}_i) \) elements. Let \( q_i := \text{ht}(\mathcal{J}_i) \) then \( \text{cd}(\mathcal{J}_i) = q_i \), also by definition of linearly joined ideals, (we use freely the notations of section 2.) for \( i = 2, \ldots, l \) we have \( (\mathcal{J}_i) + (\bigcap_{j=1}^{i-1} \mathcal{J}_j) \subset (\mathcal{P}_i + \mathcal{D}_{i-1}) \), wich implies that

\[
\text{cd}(\mathcal{J}_i) = \text{ht}(\mathcal{J}_i) < \text{ht}(\mathcal{P}_i + \mathcal{D}_{i-1}) = \dim_K (\mathcal{P}_i + \mathcal{D}_{i-1}) = \text{cd}(\mathcal{P}_i + \mathcal{D}_{i-1}).
\]

The same relation also implies \( \text{cd}(\mathcal{J}_1) = \text{ht}(\mathcal{J}_1) < \text{cd}(\mathcal{P}_2 + \mathcal{D}_1) \). The proof is by induction:

For \( i = 2 \), let \( h = \text{cd}(\mathcal{P}_2 + \mathcal{D}_1) \) we have

\[
\rightarrow H^{h-1}_{\mathcal{J}_1 \cap \mathcal{J}_2}(S) \rightarrow H^h_{\mathcal{J}_1 + \mathcal{J}_2}(S) \rightarrow H^h_{\mathcal{J}_1}(S) \oplus H^h_{\mathcal{J}_2}(S) \rightarrow H^h_{\mathcal{J}_1 \cap \mathcal{J}_2}(S) \rightarrow .
\]

which implies that \( \text{cd}(\mathcal{J}_1 \cap \mathcal{J}_2) = \text{cd}(\mathcal{P}_2 + \mathcal{D}_1) - 1 \). Now by induction suppose that \( i \geq 3 \) and

\[
\text{cd}(\bigcap_{j=1}^{i-1} \mathcal{J}_j) = \max_{j=2,\ldots,i-1} \{ \dim_K (\mathcal{P}_j + \mathcal{D}_{j-1}) - 1 \}.
\]

Let \( \mathcal{I} := \bigcap_{j=1}^{i-1} \mathcal{J}_j \), we consider the exact sequence:

\[
\rightarrow H^{h-1}_{\mathcal{J}_i \cap \mathcal{I}}(S) \rightarrow H^h_{\mathcal{J}_i + \mathcal{I}}(S) \rightarrow H^h_{\mathcal{J}_i}(S) \oplus H^h_{\mathcal{I}}(S) \rightarrow H^h_{\mathcal{J}_i \cap \mathcal{I}}(S) \rightarrow .
\]

where \( H^{j+1}_{\mathcal{J}_i \cap \mathcal{I}}(S) = 0, H^j_{\mathcal{J}_i}(S) = 0, \) for \( j \geq h \). This implies that

\[
\text{cd}(\bigcap_{j=1}^{i} \mathcal{J}_j) = \max_{j=2,\ldots,i} \{ \text{cd}(\bigcap_{j=1}^{j} \mathcal{J}_j), \ \dim_K (\mathcal{P}_j + \mathcal{D}_{j-1}) - 1 \} = \max_{j=2,\ldots,i} \{ \dim_K (\mathcal{P}_j + \mathcal{D}_{j-1}) - 1 \}.
\]

this completes the induction. The second assertion follows from our proof.

### 2.4 Connectedness dimension

**Definition 4** We recall the definition of connectedness dimension for a noetherian topological space \( T \):

\[
\text{c}(T) = \min \{ \dim Z : Z \subset T, Z \text{ is closed and } T \setminus Z \text{ is disconnected} \}.
\]

Let \( K \) be an algebraically closed field. For any ideal radical ideal \( \mathcal{I} \) in a polynomial ring \( S \) over \( K \), we set \( \text{c}(S/\mathcal{I}) \) for the connectedness dimension of the affine subvariety defined by \( \mathcal{I} \) in \( \text{Spec}(S) \).

Let remark that if \( V \subset \mathbb{P}^r \), is a projective variety defined by a homogeneous reduced ideal \( \mathcal{I} \subset S \), then \( \text{c}(S/\mathcal{I}) = \text{c}(V) + 1 \). In this section we will use the notations and the results of [7].

**Theorem 5** Let \( X \) be a topological space, \( A, B \subset X \) two closed subspaces having no relation of inclusion, then
1. \( c(A \cup B) \leq \dim (A \cap B) \);

2. if \( A \cap B \) is irreducible then \( c(A \cup B) \leq c(A) \);

3. If \( B \) is irreducible then \( c(A \cup B) \geq \min\{c(A), \dim (A \cap B)\} \);

4. If \( B, A \cap B \) are irreducible then \( c(A \cup B) = \min\{c(A), \dim (A \cap B)\} \).

Proof

1. It follows from the relation \( (A \cup B) \setminus (A \cap B) = (A \setminus (A \cap B)) \cup (B \setminus (A \cap B)) \).

2. Let \( Z \subset A \) be a closed set such that \( A \setminus Z \) is disconnected, it will be enough to prove that \( (A \cup B) \setminus Z \) is disconnected.

Assume that \( (A \cup B) \setminus Z \) is connected, by hypothesis we have that \( A \setminus Z = U_1 \cup U_2 \), with \( U_1, U_2 \) non empty closed sets in \( A \setminus Z \), such that \( U_1 \cap U_2 = \emptyset \). We have that \( (A \cup B) \setminus Z = U_1 \cup U_2 \) and \( U_1, U_2 \) are non empty closed sets in \( (A \cup B) \setminus Z \).

If \( U_1 \cap B \setminus Z = \emptyset \) then we can write \( (A \cup B) \setminus Z = U_1 \cup U_2 \setminus Z \), which proves that \( (A \cup B) \setminus Z \) is disconnected, we get the same conclusion if \( U_2 \cap B \setminus Z = \emptyset \). So we have that \( U_1 \cap B \setminus Z \neq \emptyset \) and \( U_2 \cap B \setminus Z \neq \emptyset \), but we have that

\[
(A \cap B) \setminus Z = (A \setminus Z) \cap (B \setminus Z) = (U_1 \cap B \setminus Z) \cap (U_2 \cap B \setminus Z)
\]

this is a contradiction since \( A \cap B \) is irreducible, showing our claim.

3. Let \( Z \subset A \cup B \) be a closed set such that \( A \cup B \setminus Z \) is disconnected, it will be enough to prove that either \( \dim Z \geq \dim (A \cap B) \) or \( \dim Z \geq c(A) \). By hypothesis we have that \( (A \cup B) \setminus Z = U_1 \cup U_2 \), with \( U_1, U_2 \) non empty closed sets in \( (A \cup B) \setminus Z \), such that \( U_1 \cap U_2 = \emptyset \), this implies that \( A \setminus (A \cap Z) = A \setminus Z = (U_1 \cap A) \cup (U_2 \cap A) \), if both \( U_1 \cap A, U_2 \cap A \) are non empty, this relation implies that \( \dim Z \geq c(A) \) and we get our claim. So we can assume that either \( U_1 \cap A = \emptyset \) or \( U_2 \cap A = \emptyset \). On the other hand we have that \( B \setminus Z = (U_1 \cap B) \cup (U_2 \cap B) \) and by assumptions \( B \) is irreducible so we have either \( U_1 \cap B = \emptyset \) or \( U_2 \cap B = \emptyset \). So we have \( A \setminus Z = U_i \) and \( B \setminus Z = U_j \) for \( \{i, j\} = \{1, 2\} \). These last two conditions imply that \( A \cap B \subset Z \), since \( A \setminus Z \) and \( B \setminus Z \) are disjoint, so we get that \( \dim Z \geq \dim (A \cap B) \) and our claim is proved.

4. Follows from the preceding assertions.

Theorem 6 Let \( X \) be a topological space, \( A_1, \ldots, A_l \subset X \) be a sequence of closed irreducible subspaces such that for any \( i = 1, \ldots, l - 1 \) the intersection \( A_{i+1} \cap (A_1, \ldots, A_i) \) is irreducible then:

\[
c(\bigcup_{i=1}^l A_i) = \min_{i=1}^{l-1} \{ \dim A_{i+1} \cap (A_1 \cup \ldots \cup A_i) \}.
\]

In particular let \( V_1, \ldots, V_l \subset \mathbb{P}^r \) be a linearly joined sequence of irreducible projective subvarieties, \( L_i = <V_i> \) for \( i = 1, \ldots, l \). Let \( V = V_1 \cup \ldots \cup V_l \), \( J \) the (reduced) ideal of definition of \( V \), \( L = L_1 \cup \ldots \cup L_l \) and \( Q := (Q_1) \cap \ldots \cap (Q_l) \) its defining ideal. Then
The proof is by induction on $c$ from the above theorem that $\mathcal{L} = \operatorname{L}$ the notations and results of section 4.1.

Let $\mathcal{R}$ be a ring graded by the standard graduation and $I \subseteq P$ a space defining $\mathcal{L}$. The generators of $\mathcal{L}$.

Theorem 7 We set $q_i = \sum_{p \in P_i} p$. Let $(P)$ be the ideal generated by $P$, then

$$\operatorname{rad}(P) = \operatorname{rad}(q_0, ..., q_r).$$

Remark 3 It should be interesting to study ideals $\mathcal{I}$ in a polynomial ring $S$, having the property $c(S/\mathcal{I}) = \operatorname{depth} S/\mathcal{I} - 1$. Let recall that Hartshorne have introduced and studied the varieties connected in codimension one.

2.5 Arithmetical rank of linearly-joined linear spaces

In this section for the computation of the arithmetical rank we use the following result of Schmitt and Vogel\cite{S-V}:

Lemma 3 Let $R$ be a commutative ring, with identity. Let $P$ be a finite subset of elements of $R$. Let $P_0, ..., P_r$ be subsets of $P$ such that:

1. $\bigcup_{i=0}^{r} P_i = P$;

2. $P_0$ has exactly one element;

3. If $p$ and $p''$ are different elements of $P_l$ ($0 \leq l \leq r$) there is an integer $l'$ with $0 \leq l' < l$ and an element $p' \in P_{l'}$ such that $(pp'')^m \in (p')$ for some positive integer $m$.

We set $q_l = \sum_{p \in P_l} p$. Let $(P)$ be the ideal generated by $P$, then

$$\operatorname{rad}(P) = \operatorname{rad}(q_0, ..., q_r).$$

Theorem 7 Let $V$ be a $K$-vector space of dimension $r + 1$, $S = K[V]$, the polynomial ring graded by the standard graduation and $\mathbb{P}^r$ the projective space associated to $S$. Let $\mathcal{L} = L_1 \cup ... \cup L_l \subset \mathbb{P}^r$ be a linearly joined sequence of sublinear spaces, $Q_i \subset V$ be a linear space defining $L_i$, for $i = 1, ..., l$, and $Q := (Q_1) \cap ... \cap (Q_l)$ the defining ideal of $\mathcal{L}$. We use the notations and results of section 4.1.

There exists an ordered subset $x_1, ..., x_n$ of $\bigcup_{2 \leq i \leq l} \Delta_i \cup P_i$ such that we can arrange the generators of $(Q_1) \cap (Q_2) \cap ... \cap (Q_l) = \bigcup_{2 \leq i \leq l} \Delta_i \times P_i$ into a triangle having $L(l) := \max_{2 \leq i \leq l} \{ \operatorname{card}(P_i) + \operatorname{card}(D_{i-1}) - 1 \}$ lines, as follows:

$$x_1 x_{1,0}$$

(1)
satisfying the properties:

1. For any positive integers \( j, m \) we have \( x_{j,0} = x_{m,0} \), in what follows we set \( x_{j,0} = x_n \).

2. All the products containing \( x_n \) appear in the left diagonal of the triangle.

3. All the products containing \( x_1 \) appear in the right diagonal of the triangle.

4. For any \( x_i \) appearing in the left diagonal, there is no holes in the right diagonal labelled \( i \) consisting of \( x_i x_{i,0}, \ldots, x_i x_{i,s_i} \) and the elements of the set \( \{x_{i,0}, x_{i,1}, \ldots, x_{i,s_i}\} \) are all linearly independent. In what follows we will set \( X_{i,j} \) be the linear space spanned by \( \{x_{i,0}, \ldots, x_{i,j}\} \).

5. For any \( m > i \) and \( k \) if there are two products \( x_i x_{i,k-i}, x_m x_{m,k-m} \) we have:
   - If \( x_m \in X_{i,s_i} \) then there exist some \( s < k \) such that \( x_m \in X_{i,s} \).
   - If \( x_m \not\in X_{i,s_i} \) then there exist some \( s < k \) such that \( x_{m,k+1-m} \in X_{i,s} \).

The proof is given by induction on \( l \), the number of irreducible components. For \( l = 2 \), take any basis \( P_2 \) of \( P_2 \), so we can range the elements in \( \Delta_2 \times P_2 \) in the following triangle of \( \text{card} (\Delta_2) + \text{card} (P_2) - 1 \) lines:
It is then clear that the theorem is true for $l = 2$.
Suppose that the theorem is true for $l - 1 \geq 2$, and we must prove it for $l$.
The proof is constructive and gives an algorithm to find a basis of $P_l$ and to compute
$\text{ara}((Q_1) \cap (Q_2) \cap \ldots \cap (Q_l))$.
By definition of $D_{l-1}$, we can write $(Q_i) = (Q_i', D_{i-1})$ for $i = 1, \ldots, l - 1$. By induction hypothesis the generators of $(Q_1') \cap (Q_2') \cap \ldots \cap (Q_{l-1}')$ are ranged in a triangle, satisfying
the theorem, then we will define an ordered basis $P_l$ of $P_l$, and we form a new triangle by
adding the quadratic elements in $\Delta_l \times P_l$ as a diagonal on the left or the right side of this triangle.

As a consequence we have that if $L(l-1)$ is the number of lines in the triangle corresponding to $(Q_1') \cap (Q_2') \cap \ldots \cap (Q_{l-1}')$, then:

$$
L(l) = \max\{ \text{card } (P_l) + \text{card } (\Delta_l) - 1, L(l-1) + \text{card } (\Delta_l) \}
$$

$$
= \max\{ \text{card } (P_l) + \text{card } (D_{l-1}) - 1 \}.
$$

Now let go to the proof. By induction hypothesis there exists an ordered subset $x_1, \ldots, x_n$ of $\bigcup_{2 \leq i \leq l-1} \Delta_i \cup P_l$ such that we can arrange the generators of $(Q_1') \cap (Q_2') \cap \ldots \cap (Q_{l-1}') = \bigcup_{2 \leq i \leq l-1} \Delta_i \times P_l$ into a triangle of $L(l-1)$ lines, as follows:

$$
x_1 x_{1,0}, \quad x_1 x_{1,1}, \quad x_2 x_{2,0}, \quad x_1 x_{1,1}, \quad x_3 x_{3,0}, \quad x_2 x_{2,1}, \quad x_1 x_{1,2}, \quad \ldots \quad \ldots
$$

$$
x_j x_{j,0}, \quad x_{j-1} x_{j-1,1}, \quad \ldots \quad x_1 x_{1,j-1}, \quad \ldots \quad \ldots
$$

where $x_{j,0} = x_n$ satisfying the properties in the theorem.
Let $x \in \Delta_l = D_{l-1}$. We consider two cases:
• $x_n \notin (P_l)$. We know that $\bigcup_{2 \leq i \leq l-1} \Delta_i \times P_i \subset (P_l)$, in particular for any product in the left diagonal $x_i x_n \in (P_l)$ we have that $x_i \in (P_l)$. Let the set $P_l$ a basis of $P_l$ containing the elements appearing in the left diagonal and multiplying $x_n$, note that by the point 4 of the theorem they are linearly independent.

We set $x_{n+1} = x$, and we add a left diagonal corresponding to all elements in $\Delta_l \times P_l$. Then we can range the elements in $\{x\} \times P_l \cup \bigcup_{2 \leq i \leq l-1} \Delta_i \times P_i$ into the triangle:

$$
\begin{align*}
&x_1 x_{n+1} \\
x_2 x_{n+1}, &x_1 x_n \\
x_3 x_{n+1}, &x_2 x_n, x_1 x_{1,1} \\
x_4 x_{n+1}, &x_3 x_n, x_2 x_{2,1}, x_1 x_{1,2} \\
&\vdots \\
&\vdots \\
x_j x_{n+1}, &x_{j-1} x_n, x_{j-2} x_{j-2,1}, \ldots x_1 x_{1,j-2} \\
&\vdots \\
&\vdots 
\end{align*}
$$

It is clear that we get the required properties in the theorem.

• Second case $x_n \in (P_l)$, Set $x_0 := x$, $x_{0,0} := x_n$. We define now by induction an ordered basis of $P_l$, and we add a right diagonal corresponding to all elements in $\{x\} \times P_l$. Then we can range the elements in $\{x\} \times P_l \cup \bigcup_{2 \leq i \leq l-1} \Delta_i \times P_i$ in the following triangle:

$$
\begin{align*}
&x_0 x_n \\
x_1 x_n, &x_0 x_{0,1} \\
x_2 x_n, &x_1 x_{1,1}, x_0 x_{0,2} \\
x_3 x_n, &x_2 x_{2,1}, x_1 x_{1,2}, x_0 x_{0,3} \\
&\vdots \\
&\vdots \\
x_j x_n, &x_{j-1} x_{j-1}, x_{j-2} x_{j-2,1}, \ldots x_1 x_{1,j-2}, x_0 x_{0,j-1} \\
&\vdots \\
&\vdots 
\end{align*}
$$

Let $H_s$: Suppose that we have defined $x_{0,0}, \ldots, x_{0,s}$ lineal independent elements in $P_l$, and there exists an integer $k > s$, such that for any product $x_l x_{l,j-l}$ with $l \leq j \leq k$, we have

a) If $x_l \in P_l$, then $x_l \in \langle P_l \rangle_s$, where $\langle P_l \rangle_s = \langle x_{0,0}, \ldots, x_{0,s} \rangle$;
b) If $x_l \notin P_l$ then $x_{l,j-l} \in \langle P_l \rangle_s$. 

We call $k(s)$ the biggest integer $k$ for which $\mathcal{H}_s$ is true.

The hypothesis $\mathcal{H}_0$ is clearly true.

We suppose that $\mathcal{H}_s$ is true. If $k(s) = L(l-1)+1$ then in order to finish the proof of the proposition, we complete to a basis of $\mathcal{P}_l$.

So suppose that $k(s) \neq L(l-1)+1$, let $1 \leq i \leq k(s)$ be the smallest integer such that for $x_ix_{i,k(s)}+1$ the statement in $\mathcal{H}_s$ is not true, so we have two cases:

- If $x_i \in \mathcal{P}_l$, first we show that necessarily $i = k(s)$. Suppose that $i < k(s)$ then $k(s) - i \geq 1$, the element $x_ix_{i,k(s)}$ appears in the line $k(s)$, so by induction hypothesis $x_i \in \langle \mathcal{P}_l \rangle_s$, in contradiction with the choice of $i$. In conclusion $i = k(s)$ and $x_ix_{i,k(s)} + 1 = x_{k(s)}x_{k(s)}$. We set $x_{0,s+1} := x_{k(s)}$, so $\mathcal{H}_{s+1}$ is verified in this case.

- If $x_i \notin \mathcal{P}_l$, then $x_{i,k(s)}+1 \in \mathcal{P}_l$, but $x_{i,k(s)}+1 \notin \langle \mathcal{P}_l \rangle_s$, we define $x_{0,s+1} := x_{i,k(s)}+1$. In order to verify $\mathcal{H}_{s+1}$ it will be enough to proof that for any $m > i$, and the element $x_mx_{m,k(s)}+1$ we have either $x_m \in \langle \mathcal{P}_l \rangle_s$, or $x_m \notin \mathcal{P}_l$ and $x_mx_{m,k(s)}+1 \in \langle \mathcal{P}_l \rangle_s$. We have to consider several cases:

  * If $k(s) + 1 - m = 0$, then $x_{k(s)}+1 = x_0$, so this case is clear;

  * if $k(s) + 1 - m \geq 1$, then by induction hypothesis, for the product $x_mx_{m,k(s)}+1$ we have either $x_m \in X_{i,k(s)}$ or $x_mx_{m,k(s)}+1 \in X_{i,k(s)}$. If $x_m \in X_{i,k(s)}$, then for each $j \leq k(s)$ we have that $x_{i,j} \in \langle \mathcal{P}_l \rangle_s$, it then follows that $x_m \in \langle \mathcal{P}_l \rangle_s$. If $x_mx_{m,k(s)}+1 \in X_{i,k(s)}$ we have again that $x_mx_{m,k(s)}+1 \in \langle \mathcal{P}_l \rangle_s$. The theorem is over.

**Theorem 8** In this theorem $K$ is considered to be algebraically closed. Let $\mathcal{L}_1, ..., \mathcal{L}_l \subset \mathbb{P}^n$ be a linearly joined sequence of linear spaces, let $Q_i$ be the ideal of definition of $\mathcal{L}_i$, and $Q := (Q_1) \cap ... \cap (Q_l)$ then:

$$c(S/Q) = \dim S - \text{ara} \ Q - 1.$$  

$$\text{ara} \ Q = \dim S - \text{depth} \ (S/Q).$$  

$$\text{ara} \ Q = \text{projdim} \ (S/Q) = \text{cd} \ (Q).$$

Proof. It follows from the Theorem 6 that $c(S/Q) = \text{depth} \ (S/Q) - 1$, and from the above theorem we have that

$$\text{ara} \ Q \leq \dim S - \text{depth} \ (S/Q) = \dim S - c(S/Q) - 1,$$

on the other hand, by [LS], 19.5.3

$$c(S/Q) \geq \dim S - \text{ara} \ Q - 1,$$

so we have both equalities in the claim. Recall that by the Auslander-Buchsbaum’s theorem $\dim S - \text{depth} \ (S/Q) = \text{projdim} \ (S/Q)$. The last assertion follows from the Theorem 4.

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Example 2  Consider again the example 1.  \(S = K[a, b, c, x, y, z, u]\), and 
\[J_1 = (a, b, c); J_2 = (y, a, b); J_3 = (x, z - u, b, c); J_4 = (x - u, y - u, a, c).\]

\(\bigcap_{i=1}^{4} J_i\) is generated by the following terms ordered in a triangle):

\[cb\]
\[ca \quad ab\]
\[cy \quad ax \quad b(x - u)\]
\[cz \quad a(z - u) \quad b(y - u)\]

So \(\bigcap_{i=1}^{4} J_i = \text{rad}(cb, ca + ab, cy + ax + b(x - u), cz + a(z - u) + b(y - u))\).

As a corollary we get the following important theorem.

**Theorem 9**  For any square free monomial ideal \(Q \subset S\) having a 2–linear resolution, we have
\[\text{ara}(Q) = \dim S - \text{depth}(S/Q).\]
Moreover computing \(\text{depth}(S/Q)\), \(\text{ara}(Q)\) and a set of generators up to radical for \(Q\) is effective.

Let remark that Herzog-Hibi-Zheng have proved in [HHZ] that any square free monomial ideal \(Q \subset S\) having a 2–linear resolution, has the property that any power \(Q^k\) has a linear resolution. In a work in progress we are trying to extend this result to any linearly joined hyperplane arrangements.

### 2.6 Linearly joined tableau, Ferrer’s tableaux.

**Definition 5**  Suppose that for all \(i = 1, ..., l\), there exist subsets \(\Delta_i, P_i,\) with \(\Delta_1 = \emptyset, P_1 = \emptyset,\) such that \(P_i \cap \bigcup_{j=i+1}^{l} \Delta_j = \emptyset.\) We will say that the elements in \(\bigcup_{2 \leq i \leq l} \Delta_i \times P_i\) are ranged in a "linearly joined tableau" if there exists an ordered subset \(x_1, ..., x_n\) of \(\bigcup_{2 \leq i \leq l} \Delta_i \cup P_i\) such that we can arrange the elements of \(\bigcup_{2 \leq i \leq l} \Delta_i \times P_i\) into a triangle satisfying the properties of the Theorem 7.

We have the followig consequence:

**Corollary 6**  Any linearly joined sequence of hyperplane arrangements, with ideal \(Q \subset S,\) determines a linearly joined tableau and reciprocally. Moreover \(\text{projdim} (S/Q)\) is the number of lines in a linearly joined tableau.

Note that a linearly joined tableau should be unique up to some operations. This is part of a work in progress.

The monomial ideals associated to Ferrer’s tableaux or diagram are a particular case of the above construction. Using the notations of the Corollary 3 we can describe a Ferrer’ideal.

A Ferrers diagram is a way to represent partitions of a natural number \(N\). Let \(N, m\) be a natural number. A partition of \(N\) is a sum of natural numbers: \(N = \lambda_1 + \lambda_2 + ... + \lambda_m,\)
where \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \). A partition is described by a Young diagram which consists of \( m \) rows, with the first row containing \( \lambda_1 \) boxes, the second row containing \( \lambda_2 \) boxes, etc. Each row is left-justified. Let \( \lambda_{m+1} = 0, \delta_0 = 0, \) and \( \delta_1 \) be the highest integer such that \( \lambda_1 = \ldots = \lambda_{\delta_1} \), and by induction we define \( \delta_{i+1} \) as the highest integer such that \( \lambda_{\delta_i+1} = \ldots = \lambda_{\delta_{i+1}} \), and set \( l \) such that \( \delta_{l-1} = m \). Let \( n = \lambda_1 \), we consider two disjoint sets of variables: \( \{x_1, x_2, \ldots, x_m\}, \{y_1, y_2, \ldots, y_n\} \). For \( i = 0, \ldots, l - 2 \) let

\[
\Delta_{l-i} = \{x_{\delta_i+1}, \ldots, x_{\delta_i+1}\}, \Pi_{i+2} = \{y_{\lambda_{m-i+1}+1}, \ldots, y_{\lambda_{m-i}}\}.
\]

and \( P_{i+2} = \bigcup_{j=2}^{i+2} \Pi_j \). The Ferrer’s ideal corresponding to the Ferrer’s tableau is generated by

\[
I_F = (\bigcup_{i=2}^{l} \Delta_i \times P_i).
\]

So we have the following:

**Proposition 3** Let \( l > 1 \) be a natural number and for \( i = 1, \ldots, l \) consider two families of subsets \( \Delta_i, \Pi_i \) such that \( V = \bigoplus_{i=2}^{l} (\Delta_i \oplus (\Pi_i)) \) is a decomposition into linear spaces, and let

\[
P_k = \bigoplus_{i=2}^{k} (\Pi_i), D_{k-1} = \bigoplus_{i=k}^{l} (\Delta_i),
\]

\( Q_k = P_k \oplus D_k \), and \( (Q_k) \subset K[V] \) be the ideal generated by \( Q_k \). Then the linearly joined ideal \( Q = \bigoplus_{k=1}^{l} (Q_k) \) is a Ferrer’s ideal. Reciprocally it is immediate to see that any Ferrer’s ideal is obtained in this way. Moreover Ferrer’s ideal are characterized as those linearly joined ideals (see Corollary 3) \( Q = \bigoplus_{k=1}^{l} (Q_k) \), with \( Q_k = P_k \oplus D_k \), arrangements of linear spaces for which we have the inclusions \( P_2 \subset P_3 \subset \ldots P_l \) and such that \( V = D_1 \oplus P_l \).

**Proof** Applying the algorithm described in the proof of the Theorem 7 gives the following tableau, we recognize a Ferrer’s tableau, reciprocally any Ferrer’s tableau gives rise to such decomposition.
As a consequence we have

**Corollary 7** For any Ferrer’s ideal \( I_\lambda \subset S \), with label \( \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m \)

1. The minimal primary decomposition of \( I_\lambda \) is given in the above proposition.

2. \( \text{projdim} \left( \frac{S}{I_\lambda} \right) = \max_{i=1}^m \{ \lambda_i + i - 1 \} \).

3. \( \text{ara} \left( I_\lambda \right) = \text{cd} \left( I_\lambda \right) = \text{projdim} \left( \frac{S}{I_\lambda} \right) \). In fact \( \text{projdim} \left( \frac{S}{I_\lambda} \right) \) is the number of diagonals in a Ferrer’s tableau.

4. \( c(S/I_\lambda) = \sum_{i=1}^m (\lambda_m - \lambda_i) + (m - i) \).

Items 1. and 2. were proved by Corso-Nagel in [CN].

### 2.7 Generalized trees, square free monomial ideals

Let \( K \) be a field, and let \( R = K[x_1, ..., x_n] \) be the ring of polynomials. Let \( \Delta \) be a simplicial complex of dimension \( d \), on the vertex set \( V = \{x_1, ..., x_n\} \). Let \( I_\Delta \) be the ideal of \( R = K[x_1, ..., x_n] \) generated by the products of those sets of variables which are not faces of \( \Delta \). The ring \( S = K[x_1, ..., x_n]/I_\Delta \) is called the Stanley-Reisner ring of \( \Delta \) over \( K \). It holds that \( \dim S = d + 1 \). The graph associated to \( \Delta \) will be the (1-dimensional) graph \( G(\Delta) \) on the vertex set \( V \) whose edges are the 1-dimensional faces of \( \Delta \), (often named the 1-skeleton of \( \Delta \)).

Vice versa, if \( G \) is a graph on the vertex set \( V \), we shall consider the simplicial complex associated to \( G \), denoted by \( \Delta(G) \), whose maximal faces are all subsets \( F \) of \( V \) such that the complete graph on \( F \) is a subgraph of \( G \).

**Definition 6** A generalized \( d \)-tree on \( V \) is a graph defined recursively as follows:

- (a) The complete graph on a set of \( d + 1 \) elements of \( V \), is a \( d \)-tree.
• (b) Let $G$ be a graph on the vertex set $V$. Suppose that there exists some vertex $v \in V$ such that:
  
  - the restriction $G'$ of $G$ to $V' = V \setminus \{v\}$ is a generalized $d$–tree, and
  
  - there exists a subset $V'' \subset V'$ of exactly $1 \leq j \leq d$ vertices, such that the restriction of $G$ to $V''$ is a complete graph, and
  
  - $G$ is the graph generated by $G'$ and the complete graph on $V'' \cup \{v\}$.

The vertex $v$ in the above definition will be called an extremal vertex. If always $j = d$ in the above definition then we say that $G$ is a $d$–tree. In all this paper we will use the terminology generalized tree, instead of generalized $d$–tree.

We can quote the following theorems of Fröberg [Fr]:

**Theorem 10** The Stanley Reisner ring of $\Delta$ is a Cohen-Macaulay ring of minimal degree if and only if

- the graph $G(\Delta)$ is a $d$–tree and
- $\Delta = \Delta(G(\Delta))$.

**Theorem 11** The Stanley Reisner ring of $\Delta$ has a 2-linear resolution if and only if

- the graph $G(\Delta)$ is a generalized tree and
- $\Delta = \Delta(G(\Delta))$.

Let $\Delta$ a simplicial complex as in the above theorem, in the rest of this paper we will say that $\Delta$ is a generalized tree.

**Example 3** Let $I_1$ be the Stanley-Reisner ideal defined by the simplicial complex:

```
  a  b  d
  
  a  c  f

  a  e  c
```

then the generators of $I_1$ can be ranged in the following linearly joined tableau:

- $fd$ (1)
- $ad, \ f e$ (2)
- $de, \ ae, \ fb$ (3)
- $fa$ (4)
Example 4 Let $I_2$ be the Stanley-Reisner ideal defined by the simplicial complex:

then the generators of $I_2$ can be ranged in the following linearly joined tableau: (1st step adding $g$)

- $gd$  
- $fd, ge$  
- $ad, fe, gb$  
- $de, ae, fb, ga$

(2nd step adding $g, h$)

- $hd$  
- $gd, he$  
- $fd, ge, hb$  
- $ad, fe, gb, ha$  
- $de, ae, fb, ga, hc$  
- $fa, gc$

(3rd step adding $g, h, i$)

- $id$  
- $hd, ie$  
- $gd, he, ib$  
- $fd, ge, hb, ia$  
- $ad, fe, gb, ha, ic$  
- $de, ae, fb, ga, hc$  
- $fa, gc$
Example 5 Let $I_3$ be the Stanley-Reisner ideal defined by the simplicial complex:

![Diagram of a simplicial complex]

then the generators of $I_3$ can be ranged in the following linearly joined tableau:

(4th step adding $g, h, i, j$)

\[
\begin{align*}
jd & \quad (-2) \\
id, ji & \quad (-2) \\
hd, ie, jh & \quad (-1) \\
gd, he, ib, jg & \quad (0) \\
f_d, ge, h_b, ia, je & \quad (1) \\
ad, fe, gb, ha, ic, jb & \quad (2) \\
d_e, ae, fb, ga, hc, ja & \quad (3) \\
fa, gc, & \quad j_c & \quad (4)
\end{align*}
\]

3 Linear-union of varieties, simplicial ideals

Definition 7 A reduced ideal $J \subset S$ defines a linear-union of affine varieties, if $J$ is the intersection of primes ideals $J_i = (M_i, (Q_i))$ for $i = 1, \ldots, l$, where $(Q_i)$ is the ideal of some sublinear space, satisfying the property:

\[
J = (M_1, \ldots, M_s \bigcap_{i=1}^{s} (Q_i)).
\]

Definition 8 Let consider a set of variables $G$ and a decomposition $G = \bigcup_{i=1}^{s} G_i$ into distinct sets $G_i$. For any $i = 1, \ldots, s$, let $L_i \subset K[G]$ be a set of polynomials, such that $L_i \subset (G_i)^2$. For $i \neq j$, we set $G_{i,j} = G_i \cap G_j$, $L_{i,j} = L_i \cap (G_{i,j})$, $I_{i,j} \subset K[G]$ be the ideal generated by
If all our ideals are homogeneous $L_{i,j}, I_{i,i} \subset K[G]$ be the ideal generated by $L_i \setminus \cup_{j \neq i} L_j$, We set $I_i = \sum_{j=1}^{s} I_{i,j}, I_G = \sum_{j=1}^{s} I_j$ and

$$\mathcal{P}_G = (I_G, \bigcap_{j=1}^{s} (G \setminus G_j)).$$

We assume

- $I_i$ is a prime ideal for any $i$
- For any $k, l$ and $j \neq k, l$ we have $I_{k,l} \subset (G \setminus G_j)$.

We call $\mathcal{P}_G$ a simplicial ideal.

We can prove our first theorem:

**Theorem 12** Simplicial ideals define linear-union of varieties, more precisely : if $\mathcal{P}_G$ is a simplicial ideal then

$$\mathcal{P}_G = \bigcap_{j=1}^{s} (I_j, (G \setminus G_j)).$$

We prove the two inclusions:

- " $\subset "$: Since for $k \neq j, l$ we have that $I_{j,l} \subset (G \setminus G_k)$, it follows that $I_G = I_k + \sum_{j,l \neq k} I_{j,l} \subset (I_k + (G \setminus G_k))$, on the other hand for any $k, \cap_{j=1}^{s} (G \setminus G_j) \subset (G \setminus G_k)$, and so $\mathcal{P}_G = (I_G, \cap_{j=1}^{s} (G \setminus G_j)) \subset \cap_{k=1}^{s} (I_k, (G \setminus G_k))$.

- " $\supset "$: Let $\mathcal{P} \supset \mathcal{P}_G$ be a minimal associated prime of $\mathcal{P}_G$, then $\mathcal{P} \supset \cap_{j=1}^{s} (G \setminus G_j)$ and since $(G \setminus G_j)$ is a prime ideal, there exist some $l$ such that $\mathcal{P} \supset (G \setminus G_l)$, on the other hand since $\mathcal{P} \supset \mathcal{P}_G \supset I_l$, it follows that $\mathcal{P} \supset (I_l, G \setminus G_l)$, and $(I_l, G \setminus G_l)$ is a prime ideal containing $\mathcal{P}$ by the first item. In conclusion the minimal associated primes of $\mathcal{P}$ are the prime ideals $(I_l, G \setminus G_l)$ for $l = 1, \ldots, s$.

Secondly we compute for $l = 1, \ldots, s$, the $(I_l, G \setminus G_l)$-primary component of $\mathcal{P}$. In fact we will prove that

$$\mathcal{P}_{(I_l, G \setminus G_l)} = (I_l, G \setminus G_l)(I_l, G \setminus G_l),$$

which will imply that $\mathcal{P}$ is reduced and we will get our claim. Let $j \neq l$, since there exist at least one element $x \in G_l \setminus G_j$, and $I_l \subset (G_l)^2$, we get that $x \notin (I_l, G \setminus G_l)$, and $(G \setminus G_j)(I_l, G \setminus G_l) = (1)$, this implies that

$$\mathcal{P}_{(I_l, G \setminus G_l)} = (I_G, \cap_{j=1}^{s} (G \setminus G_j))(I_l, G \setminus G_l) = (I_l, (G \setminus G_l))(I_l, G \setminus G_l),$$

because $I_{k,l} \subset (G \setminus G_l)$ for any $l, k, l \neq k, l$, and we are done.

**Corollary 8**

$$\dim (\mathcal{R}/\mathcal{P}_G) = \max_{l=1,\ldots,s} \{ \dim (\mathcal{R}/(I_l, G \setminus G_l)) \} = \max_{l=1,\ldots,s} \{ \dim (K[G_l]/(I_l)) \}.$$ 

If all our ideals are homogeneous

$$\deg(\mathcal{R}/\mathcal{P}_G) = \sum_{\dim (\mathcal{R}/(I_l, G \setminus G_l)) = \dim (\mathcal{R}/\mathcal{P}_G)} \deg(\mathcal{R}/(I_l, G \setminus G_l)).$$
which proves the equality proposed.

We illustrate the definition of simplicial ideals by the following examples. In these examples we can apply the methods developed above for a linearly joined sequence of ideals, in order to compute \( \text{projdim}(K[G]/\mathcal{P}_G) \), \( \text{depth}(K[G]/\mathcal{P}_G) \), \( c(K[G]/\mathcal{P}_G) \) and \( cd(\mathcal{P}_G) \). For all of them \( K[G]/\mathcal{P}_G \) will be a a Cohen–Macaulay ring, by the Corollary\(^5\). We introduce some methods in order to compute the arithmetical rank. We will use it in the next section.

**Example 6** Let \( G_1 = \{d, b, c, y_1, y_2\} \), \( G_2 = \{a, b, c, y_1, y_2, z_1, z_2\} \), \( G_3 = \{e, a, c, z_1, z_2\} \) and \( I_{1,2} \) be the ideal generated by the \( 2 \times 2 \) minors of the matrix \( M_1 \), \( I_{2,3} \) be the ideal generated by the \( 2 \times 2 \) minors of the matrix \( M_2 \), where

\[
M_1 = \begin{pmatrix} b & y_1 & y_2 \\ y_1 & y_2 & c \end{pmatrix},
M_2 = \begin{pmatrix} a & z_1 & z_2 \\ z_1 & z_2 & c \end{pmatrix},
\]

we can check easily the hypothesis in the definition of a simplicial ideal. Note that \( I_2 := I_{1,2} + I_{2,3} \) is a prime ideal because it is the toric ideal of the variety parametrized by

\[
b = s^3, c = t^3, a = u^3, y_1 = s^2t, y_2 = st^2, z_1 = u^2t, z_2 = ut^2,
\]

then:

\[
\mathcal{P}_G := (I_{1,2}, I_{2,3}, da, de, be, dz_1dz_2, ey_1, ey_2) = (I_{1,2}, a, e, z_1, z_2) \cap (I_{1,2}, I_{2,3}, d, e) \cap (I_{2,3}, b, d, y_1, y_2),
\]

and

\[
\mathcal{P}_G = \sqrt{(I_{1,2}, I_{2,3}, da, de, be)},
\]

remark that \( z_1^2 = az_2 \mod I_{2,3} \), so \( (dz_1)^2 = (da)(dz_2) \mod I_{2,3} \), this shows that \( dz_1 \in \sqrt{(I_{1,2}, I_{2,3}, da, de, be)} \), in the same way we can prove that \( dz_2, ey_1, ey_2 \in \sqrt{(I_{1,2}, I_{2,3}, da, de, be)} \), which proves the equality proposed.

Let remark that:

\[
\text{ht}(\mathcal{P}_G) = 6, \text{ara}(I_{1,2}) = \text{ht}I_{1,2} = 2, \text{ara}(I_{2,3}) = \text{ht}I_{2,3} = 2,
\]

and \( (da, de, be) = \sqrt{(de, da + be)} \), it then follows that \( \mathcal{P}_G \) is a stci.

**Example 7** Let \( G = \{a, b, c, d, e, f, g, h, i, l, m\} \), \( G_1 = \{d, b, c, f, g, l\} \), \( G_2 = \{a, b, c, f, g, h, i\} \), \( G_3 = \{e, a, c, h, i, m\} \) and \( L_1 \) be the set of \( 2 \times 2 \) minors of the matrix \( M_1 \), \( L_3 \) be the set of \( 2 \times 2 \) minors of the matrix \( M_2 \), where

\[
M_1 = \begin{pmatrix} b & f & g & l \\ f & g & c & d \end{pmatrix},
M_2 = \begin{pmatrix} a & h & i & m \\ h & i & c & e \end{pmatrix}.
\]

We have that \( I_1, I_3 \subset (G_2) \), and we have that \( I_2 := I_{1,2} + I_{2,3} \). \( I_1, I_2, I_3 \) are prime because they are toric ideals, and

\[
\mathcal{P}_G := (I_1, I_3, ad, al, be, bm, de, dh, di, dm, ef, eg, el, fm, gm, hl, il, lm)
\]

is equal to

\[
(I_1, a, e, m, h, i) \cap (I_1, I_3, d, e, l, m) \cap (I_3, b, d, f, g, l),
\]
and \( \text{projdim}(K[G]/\mathcal{P}_G) = 8 \). On the other hand
\[
\mathcal{P}_G = \sqrt{(I_1, I_3, ad, al, be, bm, de, dm, el, lm)},
\]
remark that \( h^2 = ai \pmod{I_3}, \) so \((dh)^2 = (da)(di) \pmod{I_3}, \) this shows that \( dh \in \sqrt{(I_1, I_3, ad, al, be, bm, de, dm, el, lm)} \), in the same way we can prove our assertion.

Let remark that:
\[
\text{ht}(\mathcal{P}_G) = 8, \; \text{ara}(I_1) = \text{ht}(I_1) = 3, \; \text{ara}(I_3) = \text{ht}(I_3) = 3,
\]
and \( \text{ara}(\mathcal{P}_G) = 2 \).

**Example 8** Let \( G_1 = \{d, b, c, f, g, l\}, G_2 = \{a, b, c, f, g, h, i\}, G_3 = \{e, a, c, h, i, m\} \) and \( L \) be the set \( \{h^2 - ai, f^2 - bg, ch - i^2, cf - g^2, bc - f g, ac - hi, b^2 d - i^3, ae^2 - m^3\} \), this set is a generator of the toric ideal \( I_T \) parametrized by \( u^3 - a, s^3 - b, t^3 - c, v^3 - d, w^3 - e, s^2 t - f, st^2 - g, tu^2 - h, t^2 u - i, s^2 v - l, u w^2 - m \). We have that
\[
L_1 = \{f^2 - bg, cf - g^2, bc - f g, b^2 d - i^3\},
\]
\[
L_2 = \{h^2 - ai, f^2 - bg, ch - i^2, cf - g^2, bc - f g, ac - hi\},
\]
\[
L_3 = \{h^2 - ai, ch - i^2, ac - hi, ae^2 - m^3\}.
\]
The ideals \( I_1, I_2, I_3, \) generated respectively by \( L_1, L_2, L_3 \) are prime, because they are toric, Then we have that:
\[
\mathcal{P}_G := (I_T, da, de, dm, dh, di, be, bm, al, ef, el, f m, gm, hl, il, lm)
\]
is equal to
\[
(I_1, a, e, h, i, m) \cap (I_2, d, e, l, m) \cap (I_3, b, d, f, g, l),
\]
On the other hand
\[
\mathcal{P}_G = \sqrt{I_T, da, de, be},
\]
remark that \( h^2 = ah \pmod{I_3}, \) so \((dh)^2 = (da)(di) \pmod{I_3}, \) this shows that \( dh \in \sqrt{(I_T, da, de, be)} \), in the same way we can prove that \( dm, di, bm, al, ef, el, f m, gm, hl, il, lm \in \sqrt{(I_T, da, de, be)}, \) which proves the equality proposed. Also we have that \( \text{ara}(I_1) \leq 3, \text{ara}(I_3) \leq 3, \) so \( \text{ara}(I_T) \leq 6, \) which implies that: \( 8 = \text{ht}(\mathcal{P}_G) \leq \text{ara}(\mathcal{P}_G) \leq \text{ara}(I_T) + 2 \leq 8. \) So \( \mathcal{P}_G \) is a set theoretically complete intersection and we can give explicitly the generators up to the radical. Note that \( I_T \) is also a stci.

Now we study the Cohen–Macaulay property. In this example we have that
\[
(I_1, a, h, i, e, m) + (I_2, d, l, e, m) = (f^2 - bg, cf - g^2, bc - f g, a, h, i, d, l, e, m)
\]
so the quotient ring \( K[G]/((I_1, a, h, i, e, m) + (I_2, d, le, m)) \simeq K[b, c, f, g]/(f^2 - bg, cf - g^2, bc - f g) \) is Cohen–Macaulay of dimension two, this will imply that \( K[G]/((I_1, a, h, i, e, m) \cap (I_2, d, le, m)) \) is Cohen–Macaulay of dimension three.
\[(I_1, a, h, i, e, m) ∩ (I_2, d, l, e, m) + (I_3, b, d, f, g, l) = (h^2 - ai, ch - i^2, ac - hi, b, d, e, f, g, l, m)\]

so the quotient ring \(K[G]/(((I_1, a, h, i, e, m) ∩ (I_2, d, l, e, m)) + (I_3, b, d, f, g, l)) \simeq K[a, c, h, i]/(h^2 - ai, ch - i^2, ac - hi)\) is Cohen–Macaulay of dimension two, this will imply from the Corollary \(\text{that } K[G]/(P_G)\) is Cohen–Macaulay of dimension three.

4 Ara of some simplicial ideals

In the Definition 9 we have extended any sequence of linearly joined linear spaces. In the case of Stanley-Reisner ideal associated to a simplicial complex we can give a more general definition.

**Definition 9** Let \(\triangle(F)\) be a simplicial complex, with set of vertices \(F\) and facets \(F_1, ..., F_s\). Let denote by \(I_{\triangle(F)}\), the Stanley-Reisner ideal associated to \(\triangle(F)\). Consider a family of disjoints sets \(F'_l, 1 \leq l \leq s\) and disjoints also from \(F\), we define new sets \(G_l = F_l \cup F'_l\), and let \(\triangle(G)\) be the simplicial complex with vertices \(G = \bigcup_{1 \leq l \leq s} G_l\) and facets \(G_1, ..., G_s\). We call \(\triangle(G)\) a extension of \(\triangle(F)\).

Let remark that in the above situation \(I_{\triangle(G)}\), is generated by \(I_{\triangle(F)}\), and all products \(yz\) such that \(y \in F'_l, y \in G_j \setminus F_i\) for all \(i \neq j\). This is clear since by hypothesis, for all \(i \neq j\) and \(y \in F'_l, z \in G_j \setminus F_i\) the edge \([y, z]\) is not in any facet of \(\triangle(G)\) so the product \(yz\) belongs to \(I_{\triangle(G)}\), and the other generators of \(I_{\triangle(G)}\) have its support in \(F\).

It follows from the Proposition 2 that if \(\triangle(F)\) is a generalized tree then \(\triangle(G)\) is a generalized tree, and \(\text{depth}(K[F]/I_{\triangle(F)}) = \text{depth}(K[G]/I_{\triangle(G)})\).

In the following theorem, \(I_i\) will be a toric ideal on the variables \(G_i\), and we assume that \(I_i\) is fully parametrized on the set \(F_i\), that is for any \(y \in F'_i\), there exists some natural number \(m\) such that \(y^m - x_{F_i} \in I_i\), where \(x_{F_i}\) is a monomial with support the set \(F_i\). In particular this implies that \(\dim K[G_i]/I_i = \text{card } F_i\), for all \(i\). Let remark that the ideal \(P_G = (I_1, ..., I_s, I_{\triangle(G)})\) fulfills the conditions to be a simplicial ideal, and it follows that \(P_G = \bigcap_{i=1}^s (I_i, G \setminus G_i)\).

**Theorem 13** Let consider as above \(\triangle(H), \triangle(G)\), toric ideals \(I_i\) on the variables \(G_i\), that are fully parametrized on the set \(F_i\) and \(P_G = (I_1, ..., I_s, I_{\triangle(G)})\). Then

1. \(P_G = \text{rad} (\sum_{i=1}^s I_i + I_{\triangle(F)})\) and \(\text{ara}(P_G) \leq \sum_{i=1}^s \text{ara}(I_i) + \text{ara}(I_{\triangle(F)})\).

2. If \(\triangle(F)\) is a generalized tree and each ideal \(I_i\) is a stci then
   \[
   \text{cd}(P_G) = \text{ara}(P_G) = \text{card } G - \text{depth}(K[F]/I_{\triangle(F)}).
   \]
   \[
   c(K[G]/P_G) = \text{card } G - \text{ara}(P_G) - 1.
   \]

3. In particular if \(\triangle(F)\) is a \(d\)-tree, and each ideal \(I_i\) is a stci then \(P_G\) is a stci.
4. If $\triangle(F)$ is a generalized tree, each ideal $I_i$ is a stci, and $K[G_i]/I_i$ is Cohen–Macaulay, then:

\[
c(K[G]/\mathcal{P}_G) = \text{card } G - \text{ara } (\mathcal{P}_G) - 1.
\]

\[
\text{cd}(\mathcal{P}_G) = \text{ara } (\mathcal{P}_G) = \text{projdim } (K[G]/\mathcal{P}_G).
\]

Proof.

1. It will be enough to prove that for all $i \neq j$ and $y \in F'_i, z \in G_j \setminus F_i$, $yz \in \text{rad } (I_1, \ldots, I_s, I_{\triangle(F)})$. Since for every $i$, $I_{i,i} = I_i$ is a simplicial toric ideal, and $F_i$ is a set of parameters of $I_i$, for every element $y \in F'_i$, there exists some natural number $m$ such that $y^m - x_{F_i} \in I_i$, where $x_{F_i}$ is a monomial with support $F_i$, let $z \in F_j \setminus F_i$, it follows that $zx_{F_i} \in I_{\triangle(F)}$, which implies that $z^m y^m \in (I_i, I_{\triangle(F)})$, now let $z \in F'_j \setminus F_i$, then there exists an integer $\mu$ such that $z^\mu - x_{F_j} \in I_j$, where $x_{F_j}$ is a monomial with support $F_j \neq F_i$. Let remark that we can choose $m = \mu$. We have that

\[
(y^m - x_{F_i})(z^m - x_{F_j}) = y^m z^m - y^m x_{F_j} - z^m x_{F_i} + x_{F_i} x_{F_j},
\]

which implies that $y^m z^m \in (I_i, I_j, I_{\triangle(F)})$, as a consequence $\mathcal{P}_G = \text{rad } (\sum_i I_i + I_{\triangle(F)})$.

2. If each $I_i$ is a stci then $\text{ara } (I_i) = \text{card } (F'_i)$, and if $\triangle(F)$ is a generalized tree, then also $\triangle(G)$, is a generalized tree, \text{depth } (K[G]/I_{\triangle(G)}) = \text{depth } (K[F]/I_{\triangle(F)})$, and

\[
\text{ara } (I_{\triangle(F)}) = \text{card } F - \text{depth } (K[F]/I_{\triangle(F)}),
\]

so

\[
\text{ara } (\mathcal{P}_G) \leq \sum_i \text{card } (F'_i) + \text{card } F - \text{depth } (K[F]/I_{\triangle(F)}) = \text{card } G - \text{depth } (K[F]/I_{\triangle(F)}) =
\]

\[
= \text{card } G - \text{depth } (K[G]/I_{\triangle(G)}) = \text{card } G - c(K[G]/\mathcal{P}_G) - 1,
\]

since by the Theorem 6, $\text{depth } (K[G]/I_{\triangle(G)}) = c(K[G]/I_{\triangle(G)}) = c(K[G]/\mathcal{P}_G)$, it follows then:

\[
\text{ara } (\mathcal{P}_G) \leq \text{card } G - c(K[G]/\mathcal{P}_G) - 1,
\]

or

\[
\text{c}(K[G]/\mathcal{P}_G) \leq \text{card } G - \text{ara } (\mathcal{P}_G) - 1,
\]

on the other hand, by [B-S.], 19.5.3

\[
\text{c}(K[G]/\mathcal{P}_G) \leq \text{card } G - \text{ara } (\mathcal{P}_G) - 1,
\]

so we have the equality.

3. in particular if $\triangle(F)$ is a $d$–tree then $K[F]/I_{\triangle(F)}$ is a Cohen–Macaulay ring, so

\[
\text{depth } (K[F]/I_{\triangle(F)}) = \text{dim } (K[F]/I_{\triangle(F)}) = \text{dim } (K[G]/\mathcal{P}_G)
\]

\[
\text{ara } (\mathcal{P}_G) \leq \text{card } G - \text{dim } (K[G]/\mathcal{P}_G) = \text{ht}(\mathcal{P}_G),
\]

which implies that $\text{ara } (\mathcal{P}_G) = \text{ht}(\mathcal{P}_G)$, and so $\mathcal{P}_G$ is a stci.
4. If for every $i$, $K[G_i]/I_i$ is Cohen–Macaulay, then:

\[
\text{depth}(K[F]/I_{\Delta(F)}) = \text{depth}(K[G]/I_{\Delta(G)}) = \text{depth}(K[G]/P_G),
\]

so our statement follows from 2.

The statements about cohomological dimension follow from the Theorem 4.

Remark 4 1. We recall that in [BMT] it was proved that any (simplicial) toric ideal $I$ fully parametrized is a stci if $\text{char}(K) = p > 0$ and almost stci if $\text{char}(K) = p > 0$. So we can apply the above theorem to find a large class of examples for which $\text{ara}(P_G) < \text{projdim}(K[G]/P_G)$. This will be published later.

2. It follows from my work [M], and the work of Robbiano-Valla [RV1], [RV1] that any simplicial toric ideal in codimension two, arithmetically Cohen–Macaulay is a stci. So we can apply the above theorem to this family.

3. The examples given in this paper sustain that there is a more general version of the above Theorem, this is part of a work in progress.

4. It should be interesting to study homogeneous ideals in a polynomial ring $S$ for which

\[
c(S/I) = \dim S - \text{ara}(I) - 1.
\]

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