Gradient Estimates and Applications for SDEs in Hilbert Space with Multiplicative Noise and Dini Continuous Drift

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Abstract

Consider the stochastic evolution equation in a separable Hilbert space $\mathbb{H}$ with a nice multiplicative noise and a locally Dini continuous drift. We prove that for any initial data the equation has a unique (possibly explosive) mild solution. Under a reasonable condition ensuring the non-explosion of the solution, the strong Feller property of the associated Markov semigroup is proved. Gradient estimates and log-Harnack inequalities are derived for the associated semigroup under certain global conditions, which are new even in finite-dimensions.

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1 Introduction

Let $(\mathbb{H}, \langle \cdot, \cdot \rangle, | \cdot |)$ and $(\bar{\mathbb{H}}, \langle \cdot, \cdot \rangle_{\bar{\mathbb{H}}}, | \cdot |_{\bar{\mathbb{H}}})$ be two separable Hilbert spaces. Let $W = (W_t)_{t \geq 0}$ be a cylindrical Brownian motion on $\mathbb{H}$ with respect to a complete filtration probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, \mathbb{P})$. More precisely, $W_t = \sum_{n=1}^{\infty} B^n_t \bar{e}_n$ for a sequence of independent one-dimensional Brownian motions $\{B^n_t\}_{n \geq 1}$ with respect to $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, \mathbb{P})$, where $\{\bar{e}_n\}_{n \geq 1}$ is an orthonormal basis on $\mathbb{H}$. Consider the following semi-linear stochastic partial differential equation on $\mathbb{H}$:

\[ \text{d}X_t = \{AX_t + B_t(X_t) + b_t(X_t)\} \text{d}t + Q_t(X_t) \text{d}W_t, \tag{1.1} \]

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where \((A, \mathcal{D}(A))\) is a negative definite self-adjoint operator on \(\mathbb{H}\), \(B, b : [0, \infty) \times \mathbb{H} \rightarrow \mathbb{H}\) are measurable and locally bounded (i.e. bounded on bounded sets), and \(Q : [0, \infty) \times \mathbb{H} \rightarrow \mathcal{L}(\mathbb{H}; \mathbb{H})\) is measurable, where \(\mathcal{L}(\mathbb{H}; \mathbb{H})\) is the space of bounded linear operators from \(\mathbb{H}\) to \(\mathbb{H}\). Here, \(B\) and \(b\) stand for the regular part and the singular part of the drift respectively.

Let \(\|\cdot\|\) and \(\|\cdot\|_{HS}\) denote the operator norm and the Hilbert-Schmidt norm respectively, and let \(\mathcal{L}_{HS}(\mathbb{H}; \mathbb{H})\) be the space of all Hilbert-Schmidt operators from \(\mathbb{H}\) to \(\mathbb{H}\). Throughout the paper, we let \(A, B\) and \(Q\) satisfy the following two assumptions.

\((a1)\) \((A, \mathcal{D}(A))\) is a negative definite self-adjoint operator on \(\mathbb{H}\) such that \((-A)^{-1}\) is of trace class for some \(\varepsilon \in (0, 1)\); i.e. \(\sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty\) for \(0 < \lambda_1 \leq \lambda_2 \leq \cdots\) being all eigenvalues of \(-A\) counting multiplicities.

\((a2)\) \(B \in C([0, \infty) \times \mathbb{H}; \mathbb{H}), Q \in C([0, \infty) \times \mathbb{H}; \mathcal{L}(\mathbb{H}; \mathbb{H}))\) such that for every \((t, x) \in [0, \infty) \times \mathbb{H}\), \(B_t : \mathbb{H} \rightarrow \mathbb{H}\) is local Lipschitz continuous, \(Q_t \in C^2(\mathbb{H}; \mathcal{L}(\mathbb{H}; \mathbb{H}))\), \((Q_tQ_t^*)^t(x)\) is invertible and a.e. right-continuous in \(t \geq 0\), and

\[
\|\nabla B_t(x)\| + \|\nabla Q_t(x)\| + \|\nabla^2 Q_t(x)\| + \|Q_t(x)\| + \|(Q_tQ_t^*)^t(x)^{-1}\|
\]

is locally bounded in \((t, x) \in [0, \infty) \times \mathbb{H}\), where \(\|\nabla B_t(x)\|\) stands for the local Lipschitz constant of \(B_t\) at point \(x\).

Under \((a1)\) and \((a2)\), we first search for minimal conditions on \(b\) ensuring the existence and pathwise uniqueness of mild solutions to \((1.1)\), then study gradient estimates and Harnack inequalities of the associated semigroup.

Before moving on, we briefly recall some recent progresses made in this direction for \(\mathbb{H} = \mathbb{H}\), constant \(Q\) and \(B = 0\). By \((a1)\) and \((a2)\), the Ornstein-Uhlenbeck process

\[
Z^x_t := e^{At}x + \int_0^t e^{(t-s)A}QdW_s, \quad t \geq 0, \quad x \in \mathbb{H}
\]

is a continuous Markov process on \(\mathbb{H}\) having a unique invariant probability measure \(\mu\), see e.g. [9]. When \(Q = I\) (the identity operator), \(B = 0\) and \(b_t = b\) is independent of \(t\) satisfying a reasonable growth condition, the existence and uniqueness of mild solutions to \((1.1)\) are proved in [7] for \(\mu\)-a.e. starting points (see also [8] for the case with an additional gradient term). This improves the corresponding result derived in [6] where \(b\) is bounded. As for mild solutions to \((1.1)\) with arbitrary initial points, the existence and uniqueness have been proved in [4] when \(b\) is Hölder continuous.

We would also like to mention that for SDEs on \(\mathbb{R}^d\) with a nice non-degenerate multiplicative noise, the existence and uniqueness of solutions have been proved in [26] if the drift is in \(L^2_{loc}(d^{d+1})([0, \infty) \times \mathbb{R}^d)\). When the noise is non-degenerate and additive, this condition is weakened in [13] as that the drift belongs to \(L^q_{loc}([0, \infty) \rightarrow L^p_{loc}(\mathbb{R}^d))\) for some \(p, q \in [1, \infty]\) satisfying \(\frac{d}{p} + \frac{2}{q} < 1\). The main idea used in these two papers goes back to the arguments developed in [25], [27] using Sobolev regularities of the corresponding Kolmogorov equations. As already explained in e.g. [6] that such regularities are not available in infinite dimensions. Indeed, [6], [7] are attempts to extend these results to infinite-dimensions by using (local) boundedness conditions to replace the local integrability conditions.
By refining the argument developed from [5] for additive noise, and by carefully treating the operator-valued map $Q$, we find that the existence and uniqueness of mild solutions can be ensured by (a1) and (a2) provided $Q$ is asymptotically cylindrical and $b$ is locally Dini continuous. Precisely, for any $n \geq 1$, let $\pi_n : \mathbb{H} \rightarrow \mathbb{H}_n := \text{span}\{e_1, \ldots, e_n\}$ be the orthogonal projection, where $\{e_n\}_{n \geq 1}$ is the eigenbasis of $-A$ on $\mathbb{H}$ corresponding to the eigenvalues $\{\lambda_n\}_{n \geq 1}$. Moreover, let

$$D = \left\{ \phi : [0, \infty) \rightarrow [0, \infty) \text{ is increasing}, \phi^2 \text{ is concave}, \int_0^1 \frac{\phi(s)}{s} ds < \infty \right\}.$$ 

We shall need the following condition.

(a3) $b : [0, \infty) \times \mathbb{H} \rightarrow \mathbb{H}$ is measurable and locally bounded, and for any $n \geq 1$, there exists $\phi_n \in D$ such that

$$|b_t(x) - b_t(y)| \leq \phi_n(|x - y|), \quad t \in [0, n], \quad x, y \in \mathbb{H}, \quad |x| \vee |y| \leq n.$$ 

Moreover, for any $x \in \mathbb{H}$ and $s \geq 0$,

$$\lim_{n \rightarrow \infty} \|Q_s(x) - Q_s(\pi_n x)\|_{HS} := \lim_{n \rightarrow \infty} \sum_{k \geq 1} \left| \{Q_s(x) - Q_s(\pi_n x)\} \bar{e}_k \right|^2 = 0,$$

where $\{\bar{e}_k\}$ is an orthonormal basis on $\mathbb{H}$.

We remark that the condition $\int_0^1 \frac{\phi(s)}{s} ds < \infty$ is well known as Dini condition, due to the notion of Dini continuity. So, (1.2) implies that $b_t$ is Dini continuous on bounded sets in $\mathbb{H}$, locally uniformly in $t \geq 0$. Obviously, the class $D$ contains $\phi(s) := \frac{K}{\log^{1+\delta(c+s^{-1})}}$ for constants $K, \delta > 0$ and large enough $c \geq e$ such that $\phi^2$ is concave.

Next, a map $Q$ defined on $\mathbb{H}$ is called cylindrical if $Q(x) = Q(\pi_n x)$ holds for some $n \geq 1$ and all $x \in \mathbb{H}$. So, (1.3) means that $Q_s$ is asymptotically cylindrical under the Hilbert-Schmidt norm, uniformly in $s \geq 0$. We stress that assumptions (a2) and (a3) are satisfied by some infinite-dimensional models. For instance, when $\mathbb{H} = \mathbb{H}$ and $Q_s(x) = Q_0 + \varepsilon \tilde{Q}(x)$, where $Q_0 \in \mathcal{L}(\mathbb{H}; \mathbb{H})$ such that $Q_0 Q_0^* \varepsilon$ is invertible, $\tilde{Q} \in C^2(\mathbb{H}; \mathcal{L}(\mathbb{H}; \mathbb{H})) \cap C_b(\mathbb{H}; \mathcal{L}_{HS}(\mathbb{H}; \mathbb{H}))$ and $\varepsilon \in \mathbb{R}$, all conditions on $Q$ included in these two assumptions hold provided $|\varepsilon|$ is small enough.

In general, the mild solution (if exists) can be explosive. So, we consider mild solutions with life times.

**Definition 1.1.** A continuous adapted process $(X_t)_{t \in [0, \zeta)}$ is called a mild solution to (1.1) with life time $\zeta$, if $\zeta > 0$ is a stopping time such that $\mathbb{P}$-a.s. $\limsup_{t \uparrow \zeta} |X_t| = \infty$ holds on $\{\zeta < \infty\}$ and, $\mathbb{P}$-a.s.

$$X_t = e^{At} X_0 + \int_0^t e^{(t-s)A} (B_s + b_s)(X_s) ds + \int_0^t e^{(t-s)A} Q_s(X_s) dW_s, \quad t \in [0, \zeta).$$
If for any $x \in \mathbb{H}$, the equation (1.1) has a unique mild solution $X_t^x$ with $X_0 = x$ and infinite life time (i.e. the solution is non-explosive), then the associated Markov semigroup $P_t$ is defined as follows.

$$P_t f(x) := \mathbb{E} f(X_t^x), \quad f \in \mathcal{B}_b(\mathbb{H}), t \geq 0, x \in \mathbb{H},$$

where $\mathcal{B}_b(\mathbb{H})$ is the set of all bounded measurable real functions on $\mathbb{H}$. $P_t$ is called strong Feller if it sends $\mathcal{B}_b(\mathbb{H})$ into $C_b(\mathbb{H})$, the set of all bounded continuous real functions on $\mathbb{H}$.

Our first main result is the following.

**Theorem 1.1.** Assume (a1), (a2) and (a3).

1. For any $X_0 \in \mathcal{B}(\Omega \rightarrow \mathbb{H}; \mathcal{F}_0)$, the equation (1.1) has a unique mild solution $(X_t)_{t \in [0, \zeta)}$ with life time $\zeta$.

2. Let $\|Q_t\|_\infty := \sup_{x \in \mathbb{H}} \|Q_t(x)\|$ be locally bounded in $t \geq 0$. If there exist two positive increasing functions $\Phi, h : [0, \infty) \times [0, \infty) \rightarrow (0, \infty)$ such that $\int_1^\infty \frac{ds}{\Phi_t(s)} = \infty$ and

$$\langle (B_t + b_t)(x + y), x \rangle \leq \Phi_t(|x|^2) + h_t(|y|), \quad x, y \in \mathbb{H}, t \geq 0,$$

then the mild solution is non-explosive and $P_t$ is strong Feller for $t > 0$.

Without loss of generality, in Theorem 1.1 one may take $B = 0$ in Theorem 1.1. But the situation is different in the next result (Theorem 1.2) where the singular part $b$ is bounded in the space variable, so that the appearance of $B$ allows the whole drift $B_t + b_t$ unbounded and singular.

Comparing with the above mentioned results of [5, 6, 7], Theorem 1.1 contains the following several new points: (1) It works for multiplicative noise; (2) It works for arbitrary starting points and non-Hölder continuous drift; (3) The assertion on the strong Feller property is new, see also Remark 4.1 for a discussion on Harnack inequalities. Moreover, condition (1.4) is more general than

$$\langle b(x + y), x \rangle \leq C(|x|^2 + 1 + e^{p|y|}), \quad x, y \in \mathbb{H}$$

for some constants $C, p > 0$ which is used in [7, Theorem 16] to ensure the non-explosion of the solution. See [7, Remark 17] for an explanation on the reasonability of such a condition in infinite dimensions.

The main difficulty in the proof of Theorem 1.1 comes from the singular drift $b$. To overcome this difficulty, a regularization argument has been introduced in [5] and further developed in [6, 7], to reformulate the mild solution by using a regular functional instead of $b$. This functional is constructed by solving an equation involving in the resolvent associated to the corresponding regular equation, i.e. the equation (1.1) without $b$. Based on such a regularization formulation, the uniqueness can be proved as in [10] where the transport equation for Hölder continuous vector fields with a finite-dimensional multiplicative noise is concerned. See also [11, 12] and references therein for the study of singular SPDEs using regularization by the space-time white noise.
The key point in the proof of Theorem 1.1 is to realize the idea of [5] for the present situation where \(Q\) is non-constant and \(b\) is non-Hölder continuous. This is done by establishing necessary derivative estimates using minimal continuity conditions on \(b\).

Next, we consider gradient estimates and Harnack inequalities for the associated Markov semigroup \(P_t\). To this end, we need the following global versions of assumptions (a2) and (a3). For a real function \(f\) defined on \([0, T] \times \mathbb{H}\), let
\[
\|f\|_{T, \infty} = \sup_{t \in [0, T], x \in \mathbb{H}} |f(t, x)|.
\]
The same notation applies to \(\mathbb{H}\)-valued or operator-valued maps, for instance, \(\|Q\|_{T, \infty} = \sup_{[0, T] \times \mathbb{H}} \|Q\|\).

(a2') \(B\) and \(Q\) satisfy (a2), and moreover
\[
\|b\|_{T, \infty} + \|\nabla B\|_{T, \infty} + \|\nabla Q\|_{T, \infty} + \|\nabla^2 Q\|_{T, \infty} + \|Q\|_{T, \infty} + \|(QQ^*)^{-1}\|_{T, \infty} \leq \Psi(T)
\]
holds for some \(\Psi \in C([0, \infty))\) and all \(T \in [0, \infty)\).

(a3') \(Q\) satisfies (1.3). Moreover, for any \(T > 0\), there exists \(\phi \in \mathcal{D}\) such that
\[
|b_t(x) - b_t(y)| \leq \phi(|x - y|), \quad t \in [0, T], \ x, y \in \mathbb{H}.
\]

According to Theorem 1.1 under (a1), (a2') and (a3') the unique mild solution of (1.1) is non-explosive. Let \(P_t\) be the associated semigroup. Gradient estimates and log-Harnack inequalities presented in the next result are new even in finite-dimensions. Note that when \(b\) is Hölder continuous and \(\mathbb{H}\) is finite-dimensional, the (log) Harnack inequalities have been established recently in [14] using the regularization transform of [10] [11]. But Theorem 1.2 also applies to non-Hölder continuous drifts on infinite-dimensional \(\mathbb{H}\).

\textbf{T1.2} Theorem 1.2. Assume (a1), (a2') and (a3').

1. For any \(T > 0\) there exists a constant \(C(T) > 0\) such that
\[
|\nabla P_t f|^2 + \frac{P_t f^2 - (P_t f)^2}{t} \leq C(T) P_t |\nabla f|^2, \quad t \in (0, T], f \in C^1_b(\mathbb{H}).
\]

2. There exists a constant \(C > 0\) such that
\[
|\nabla P_t f|^2 \leq \frac{C}{t^{\wedge 1}} \{ P_t f^2 - (P_t f)^2 \}, \quad t > 0, f \in \mathcal{B}_b(\mathbb{H}),
\]
\[
P_t \log f(y) \leq \log P_t f(x) + \frac{C|x - y|^2}{t^{\wedge 1}}, \quad t > 0, x, y \in \mathbb{H}, 0 < f \in \mathcal{B}_b(\mathbb{H}),
\]
\[
P_t f(y) \leq P_t f(x) + |x - y| \sqrt[\wedge 1]{\frac{C}{t} P_t f^2(y)}, \quad x, y \in \mathbb{H}, t > 0, 0 \leq f \in \mathcal{B}_b(\mathbb{H}).
\]
Remark 1.1. (1) According to [17, 23], the key point in the proof of Theorem 1.2 is the gradient estimate
\[ |\nabla P_t f|^2 \leq C(T) P_t |\nabla f|^2. \]
As the regularization formula (2.30) still contains a non-Lipschitz term \( A u_s \), the standard argument in the literature is invalid. Our proof is new in this singular setting (see the proof of Lemma 6.1(2)).

(2) In the situation of Theorem 1.2 for any \( s \geq 0 \), let \( P_{s,t} f(x) = E f(X^x_{s,t}), x \in \mathbb{H}, t \geq s, f \in \mathcal{B}_b(\mathbb{H}) \), where \( (X^x_{s,t})_{t \geq s} \) is the unique mild solution to
\[
dX^x_{s,t} = \{ AX^x_{s,t} + (B_t + b_t)(X^x_{s,t}) \} dt + Q_t(X^x_{s,t}) dW_t, \quad t \geq s, X^x_{s,s} = x.
\]
Then the assertions in Theorem 1.2 hold for \( P_{s,s+t} \) in place of \( P_t \) with \( C(T) \) and \( C \) depending also on \( s \). If conditions in (a2') and (a3') are uniformly in \( T \) (i.e. they hold with \( T = \infty \) and \([0, \infty)\) in place of \([0, T]\)), then these constants are independent of \( s \) and, by the semigroup property, we may take \( C(T) = c_1 e^{c_2 T} \) for some constants \( c_1, c_2 > 0 \).

The inequality in Theorem 1.2(1) as well as (1.6) are well known due to Bakry-Emery under a curvature condition, and are easy to check in the regular case (i.e. \( b = 0 \)), see e.g. [3, 21] and references therein. The log-Harnack inequality (1.7) is introduced in [19] as a limit version of the dimension-free Harnack inequality founded in [18]. This inequality has a number of applications. For instance, it implies that the laws of \( X^x_t \) and \( X^y_t \) are equivalent and provides pointwise estimates on the Radon-Nikodym derivative; in the time-homogeneous case it implies that the invariant probability measure \( \mu \) (if exists) is unique and has full support on \( \mathbb{H} \), the semigroup has positive density (i.e. heat kernel) with respect to \( \mu \) (more generally, to an quasi-invariant measure), and it provides heat kernel estimates and entropy-cost inequalities of the semigroup; see, for instance, [20, §1.4] for details. Recently, a link of the log-Harnack inequality to the optimal transportation has been presented in [4]. Finally, according to [2, Proposition 2.3] and [22, Proposition 1.3] in a more general framework, the log-Harnack inequality (1.7) implies the gradient estimate (1.6), while (1.6) is equivalent to the Harnack type inequality (1.8).

The remainder of the paper is organized as follows. In Section 2, we present some gradient estimates on the semigroup for the corresponding O-U type equation, i.e. (1.1) with \( B = b = 0 \). These gradient estimates enable us to prove the desired regularization representation of the mild solution to (1.1) with non-Hölder drift \( b \). In Section 3, we prove the pathwise uniqueness using the regularization representation and, in Section 4, we investigate the strong Feller property and discuss Harnack inequalities for the semigroup. Results in Sections 3-4 are derived under some global conditions. Combining these results with a truncating argument, we prove Theorem 1.1 in Section 5. Finally, we prove Theorem 1.2 in Section 6 by using the regularization representation and finite-dimensional approximations.
2 Regularization representation of mild solutions

Since it is easy to construct a weak mild solution of (1.1), in the spirit of Yamada-Watanabe [24] the key point to prove the existence and uniqueness lies in the pathwise uniqueness. To prove the pathwise uniqueness of the mild solution to (1.1), we aim to construct a transform \( \theta : [0, T] \times \mathbb{H} \to \mathbb{H} \) such that

- For very \( t \in [0, T] \), \( \theta_t \) is a \( C^2 \)-diffeomorphism on \( \mathbb{H} \);
- If \( (X_t)_{t \in [0, T]} \) solves (1.1), then \( \{\theta_t(X_t)\}_{t \in [0, T]} \) solves a regular equation having pathwise uniqueness.

In this way we prove the pathwise uniqueness of (1.1). For readers’ convenience, we briefly explain the idea of the construction of \( \theta \) (see also [5]).

Write \( \theta_t(x) = x + u_t(x) \), \((t, x) \in [0, T] \times \mathbb{H} \). In order that \( \theta_t \) is a \( C^2 \)-diffeomorphism on \( \mathbb{H} \), we will take \( u_t \in C^1(\mathbb{H}; \mathbb{H}) \) such that \( \nabla u_t \) is Lipschitz continuous with \( \|\nabla u_t\|_\infty < 1 \).

By Itô’s formula we have, formally,

\[
\begin{align*}
\theta_t(X_t) = & \left\{ (\partial_t \theta_t)(X_t) + (L_t \theta_t)(X_t) \right\} dt + (\nabla \theta_t)(X_t) \left\{ Q_t(X_t) dW_t + (B_t(X_t)) dt \right\},
\end{align*}
\]

where \( X_t \) solves (1.1) and

\[
L_t := \frac{1}{2} \sum_{i,j} \langle Q_t Q_t^* e_i, e_j \rangle \nabla e_i \nabla e_j + \nabla A + \nabla b_t.
\]

To ensure that coefficients in (2.1) are regular as required by point (b), we set

\[
\partial_t \theta_t(x) = Ax - L_t \theta_t(x);
\]

i.e. \( \partial_t u_t = -L_t u_t - b_t \). In particular, with \( u_T = 0 \) we have

\[
\begin{align*}
\theta_t(X_t) & = \left\{ (\partial_t \theta_t)(X_t) + (L_t \theta_t)(X_t) \right\} dt + (\nabla \theta_t)(X_t) \left\{ Q_t(X_t) dW_t + (B_t(X_t)) dt \right\},
\end{align*}
\]

where \( \{P^0_{s,t}\}_{0 \leq s \leq t} \) is the semigroup associated to the O-U type equation

\[
\begin{align*}
\textbf{OU} \quad & dZ_{s,t}^x = AZ_{s,t}^x dt + Q_t(Z_{s,t}^x) dW_t, \quad t \geq s, Z_{s,s}^x = x.
\end{align*}
\]

It is well known that under assumptions (a1) and (a2'), the equation (2.3) has a unique mild solution which is non-explosive (see [9]). We have

\[
\begin{align*}
P^0_{s,t} f(x) = & E f(Z_{s,t}^x), \quad t \geq s \geq 0, f \in B_b(\mathbb{H}), x \in \mathbb{H}.
\end{align*}
\]

To ensure \( \|\nabla u_s\|_\infty < 1 \) as required by point (a), instead of (2.2) we consider

\[
\begin{align*}
\theta_t(X_t) & = \left\{ (\partial_t \theta_t)(X_t) + (L_t \theta_t)(X_t) \right\} dt + (\nabla \theta_t)(X_t) \left\{ Q_t(X_t) dW_t + (B_t(X_t)) dt \right\},
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\end{align*}
\]
for large enough $\lambda > 0$, which also ensures the desired regularity of the equation (2.1), see (6.3) and (6.4) below for details.

To verify the regularity properties of $u_s$ solving (2.4) for large $\lambda > 0$, we first consider derivative estimates on $P_{s,t}^0$. In the following result, (2.5) is more or less standard, but (2.6) is new.

**Lemma 2.1.** Assume (a1) and (a2') with $b = 0$. Let $T > 0$ be fixed.

1. There exists a constant $C > 0$ such that for any $f \in B_b(H)$,

   \[ |\nabla P_{s,t}^0 f(x) - \nabla P_{s,t}^0 f(y)| \leq C \left( \frac{|x - y|}{t - s} \wedge \frac{1}{\sqrt{t - s}} \right) \|f\|_{\infty}, \quad 0 \leq s < t \leq T, x, y \in \mathbb{H}. \]

2. There exists two constants $c_1, c_2 > 0$ such that for any increasing $\phi : [0, \infty) \to [0, \infty)$ with concave $\phi^2$,

   \[ \|\nabla^2 P_{s,t}^0 f\|_{\infty} := \sup_{x \in \mathbb{H}} \|\nabla^2 P_{s,t}^0 f(x)\| \leq \frac{c_1 \phi(c_2(t - s)^{\varepsilon/2})}{t - s}, \quad 0 \leq s < t \leq T \]

   holds for all $f \in B_b(H)$ satisfying

   \[ |f(x) - f(y)| \leq \phi(|x - y|), \quad x, y \in \mathbb{H}, \]

   where $\varepsilon \in (0, 1)$ is in (a1).

**Proof.** (1) We shall make use of the following Bismut formula

\[ \nabla \eta P_{s,t}^0 f(x) = \mathbb{E} \left[ \frac{f(Z_{s,t}^x)}{t - s} \int_s^t \langle \{Q_r^x(QQ^*)^{-1}_r(Z_{s,r}^x)\nabla \eta Z_{s,r}^x, dW_r \rangle \rangle \right] \]

for $x, \eta \in \mathbb{H}, t > s \geq 0$, and $f \in B_b(\mathbb{H})$. Here, by (a1) and (a2'), the derivative process $(\nabla \eta Z_{s,t}^x)_{t \geq s}$ is the unique mild solution to the linear equation

\[ d\nabla \eta Z_{s,t}^x = A\nabla \eta Z_{s,t}^x dt + (\nabla \nabla \eta Z_{s,t}^x Q_t)(Z_{s,t}^x) dW_t, \quad \nabla \eta Z_{s,s}^x = \eta, t \geq s, \]

so that

\[ \sup_{x \in \mathbb{H}, 0 \leq s \leq t \leq T} \mathbb{E} |\nabla \eta Z_{s,t}^x|^2 \leq c|\eta|^2, \quad \eta \in \mathbb{H} \]

holds for some constant $c > 0$; see, for instance, [9, Remark 9.5].

The formula (2.8) can be easily proved by using the Malliavin calculus. Here, we give a brief proof of this formula for $f \in C_b^1(\mathbb{H})$, which implies the same formula for $f \in B_b(\mathbb{H})$ by an approximation argument. Take

\[ h_v = \frac{1}{t - s} \int_s^t \{Q_r^x(QQ^*)^{-1}_r\}(Z_{s,r}^x) \nabla \eta Z_{s,r}^x dr, \quad v \in [s, t]. \]
In the same manner of [9, Remark 9.5], but using the Malliavin derivative \( D_h \) to replace the directional derivative \( \nabla \eta \), we see that the Malliavin derivative process \( (D_h Z_{s,r}^x)_{r \in [s,t]} \) is the unique mild solution to the equation

\[
dD_h Z_{s,r}^x = AD_h Z_{s,r}^x \, dr + (\nabla D_h Z_{s,r}^x) Q_r (Z_{s,r}^x) \, dW_r + Q_r (Z_{s,r}^x) h'_r \, dr, \quad D_h Z_{s,s}^x = 0, \ r \in [s,t].
\]

Combining this with (2.9) and the definition of \( h \), we see that both \( (\frac{t-s}{t-s} \nabla \eta Z_{s,t}^x)_{r \in [s,t]} \) and \( (D_h Z_{s,r}^x)_{r \in [s,t]} \) solve the equation

\[
dV_r = AV_r dr + (\nabla V_r) Q_r (Z_{s,r}^x) dW_r + \frac{1}{t-s} \nabla \eta Z_{s,r}^x dr, \quad r \geq s, V_s = 0.
\]

By the uniqueness of the mild solution to this equation, we obtain \( D_h Z_{s,t}^x = \frac{t-s}{t-s} \nabla \eta Z_{s,t}^x = \nabla \eta Z_{s,t}^x \). So, by the chain rule and the integration by parts formula in the Malliavin calculus, we arrive at

\[
\nabla \eta P_{s,t}^0 f(x) = \mathbb{E}(\nabla \eta Z_{s,t}^x f) (Z_{s,t}^x) = \mathbb{E}(\nabla D_h Z_{s,t}^x f) (Z_{s,t}^x)
\]

\[
= \mathbb{E} D_h (f (Z_{s,t}^x)) = \mathbb{E} \left[ f (Z_{s,t}^x) \int_s^t \langle h'_r, dW_r \rangle_\mathbb{H} \right].
\]

This implies (2.8).

Now, according to (2.8), (2.10) and (a2'), there exists a constant \( c > 0 \) such that

\[
|\nabla P_{s,t}^0 f|^2(x) \leq \frac{c}{t-s} P_{s,t}^0 f^2(x), \quad 0 \leq s < t \leq T, x \in \mathbb{H}, f \in \mathcal{B}_b(\mathbb{H}).
\]

Next, writing \( P_{s,t}^0 = \frac{P_{s,\frac{t+s}{2}}^0}{P_{\frac{t+s}{2},t}} P_{\frac{t+s}{2},t}^0 \) by the Markov property, and applying (2.8) to \( \frac{t+s}{2} \) and \( P_{\frac{t+s}{2},t}^0 f \) instead of \( t \) and \( f \), we obtain

\[
\nabla \eta P_{s,t}^0 f(x) = \mathbb{E} \left[ \left( \frac{P_{\frac{t+s}{2},t}^0 f}{(t-s)/2} \right) (Z_{s,t}^x) \int_s^{\frac{t+s}{2}} \langle \{ Q_r^\ast (QQ^\ast)^{-1} \} (Z_{s,r}) \nabla \eta Z_{s,r}^x, dW_r \rangle_\mathbb{H} \right].
\]

So, for any \( \eta' \in \mathbb{H} \), we can prove

\[
\frac{1}{2} (\nabla \eta \eta' \nabla \eta P_{s,t}^0 f)(x)
\]

\[
= \mathbb{E} \left[ \left( \frac{\nabla \eta \eta' Z_{s,t}^{\frac{t+s}{2}} P_{\frac{t+s}{2},t}^0 f}{(t-s)/2} \right) (Z_{s,t}^x) \int_s^{\frac{t+s}{2}} \langle \{ Q_r^\ast (QQ^\ast)^{-1} \} (Z_{s,r}) \nabla \eta Z_{s,r}^x, dW_r \rangle_\mathbb{H} \right]
\]

\[
+ \mathbb{E} \left[ \left( \frac{P_{\frac{t+s}{2},t}^0 f}{(t-s)/2} \right) (Z_{s,t}^x) \int_s^{\frac{t+s}{2}} \langle (\nabla \eta \eta' Z_{s,t}^{\frac{t+s}{2}} \{ Q_r^\ast (QQ^\ast)^{-1} \}) (Z_{s,r}) \nabla \eta Z_{s,r}^x, dW_r \rangle_\mathbb{H} \right]
\]

\[
+ \mathbb{E} \left[ \left( \frac{P_{\frac{t+s}{2},t}^0 f}{(t-s)/2} \right) (Z_{s,t}^x) \int_s^{\frac{t+s}{2}} \langle Q_r^\ast (QQ^\ast)^{-1} \} (Z_{s,r}) \nabla \eta' \nabla \eta Z_{s,r}^x, dW_r \rangle_\mathbb{H} \right].
\]
To verify this formula, we need to apply the dominated convergence theorem. In the spirit of [Remark 9.5], (a1), (a2') and (2.9) imply that \( γ_r := \nabla_η \nabla_η Z^r_{s,r} \) is the unique mild solution to the equation

\[
dγ_r = Aγ_r dr + \left\{ (\nabla_γ, Q_r)(Z^r_{s,r}) + (\nabla_γ Z^r_{s,r}, \nabla_γ Z^r_{s,r} Q_r)(Z^r_{s,r}) \right\} dW_r, \quad r \geq s, γ_s = 0,
\]

so that

\[
(2.15) \sup_{x \in H, 0 \leq r \leq T} \mathbb{E}|\nabla_η \nabla_η Z^x_{s,r}|^2 \leq c|η|^2|η'|^2, \quad η, η' \in H
\]

holds for some constant \( c > 0 \). Combining this with (2.12), (2.10) and (a2'), we derive (2.14) from (2.13) by using the dominated convergence theorem. Moreover, (2.14) implies

(3) Applying (2.16) to \( \tilde{f} := f - f(e^{(t-s)A}x) \) in place of \( f \), we obtain

\[
|\nabla_η \nabla_η P^0_{s,t} f(x) - |\nabla_η \nabla_η P^0_{s,t} f(y) | \leq \frac{c|\eta|^2|\eta'|^2}{(t-s)^2} P^0_{s,t} f^2(x), \quad x, η, η' \in H, 0 \leq s < t \leq T, f \in B(H)
\]

for some constant \( c > 0 \). In particular,

\[
|\nabla P^0_{s,t} f(x) - \nabla P^0_{s,t} f(y) | \leq \frac{c|\eta|^2|\eta'|^2}{t-s} \| f \|_\infty, \quad x, y \in H, 0 \leq s < t \leq T, f \in B(H)
\]

holds for some constant \( c > 0 \). Combining this with (2.12) we prove (2.5).

As a straightforward consequence of Lemma 2.1, we have the following result on the resolvent

\[
(R^λ_{s,t} f)(x) := \int_s^t e^{-(r-s)λ} P^0_{s,r} f_r(x) dr, \quad x \in H, λ \geq 0, t \geq s \geq 0, f \in B([0, \infty) \times H).
\]
Lemma 2.2. Assume (a1) and (a2') with $b = 0$. Let $T > 0$ be fixed.

1. There exists a constant $C > 0$, such that for any $f \in \mathcal{B}_b([0,T] \times \mathbb{H})$,
\[ |\nabla(R^\lambda_{s,t}f)(x) - \nabla(R^\lambda_{s,t}f)(y)| \leq C\|f\|_\infty |x - y| \log \left(e + \frac{1}{|x - y|}\right), \quad \lambda \geq 0, 0 \leq s \leq t \leq T. \]

2. For any $\phi \in \mathcal{D}$, there exists a decreasing function $\delta_\phi : [0, \infty) \to (0, \infty)$ such that
\[ \lim_{\lambda \to \infty} \delta_\phi(\lambda) = 0, \quad \|\nabla^2 R^\lambda_{s,t}f\|_\infty \leq \delta_\phi(\lambda), \quad \lambda \geq 0, 0 \leq s \leq t \leq T \]
for all $f \in \mathcal{B}_b([0,T] \times \mathbb{H})$ satisfying
\[ |f_t(x) - f_t(y)| \leq \phi(|x - y|), \quad x, y \in \mathbb{H}, t \in [0,T]. \]

Proof. By Lemma 2.1(1) and the definition of $R^\lambda_{s,t}f$, there exist constants $C_1, C_2 > 0$ such that for any $f \in \mathcal{B}_b([0,T] \times \mathbb{H})$,
\[
\begin{align*}
|\nabla(R^\lambda_{s,t}f)(x) - \nabla(R^\lambda_{s,t}f)(y)| &\leq C_1\|f\|_\infty \int_s^t \frac{e^{-(r-s)\lambda}}{\sqrt{r-s}} \wedge \frac{|x - y|}{r-s} dr \\
&\leq C_1\|f\|_\infty \left( \int_0^{|x - y|^2 \wedge e^{-1}} \frac{dr}{\sqrt{r}} + |x - y| \int_1^{T\vee 1} \frac{dr}{r} + |x - y| \int_1^{T\vee 1} e^{-\lambda r} dr \right) \\
&\leq C_2\|f\|_\infty |x - y| \log \left(e + \frac{1}{|x - y|}\right), \quad x, y \in \mathbb{H}, 0 \leq s \leq t \leq T.
\end{align*}
\]

Then (1) is proved.

Next, since for $\phi \in \mathcal{D}$ we have $\int_0^T \frac{\phi(c_2 s^{\zeta/2})}{s} ds < \infty$, Lemma 2.1(2) implies the second assertion for
\[ \delta_\phi(\lambda) := c_1 \int_0^T \frac{e^{-\lambda s} \phi(c_2 s^{\zeta/2})}{s} ds \downarrow 0 \text{ as } \lambda \uparrow \infty. \]

In the next result, we characterize the solution $u_s$ to (2.4) which will be used to formulate the mild solution to (1.1) (see Proposition 2.5 below). To prove the formulation in infinite-dimensions, we shall adopt an approximation argument based on the second assertion of the following result.

Lemma 2.3. Assume (a1) and (a2'), and let $T > 0$ be fixed. Then there exists a constant $\lambda(T) > 0$ such that the following assertions hold.

1. For any $\lambda \geq \lambda(T)$, the equation (2.4) has a unique solution $u \in C([0,T]; C^1_b(\mathbb{H}; \mathbb{H}))$. 

(2) Assume that
\[
\lim_{r \to 0} \sup_{|x-y| \leq r, t \in [0, T]} |b_t(x) - b_t(y)| = 0.
\]

Let \(P_{s,t}^{(n)}\) be defined as \(P_{s,t}^0\) for \(Q \circ \pi_n\) in place of \(Q\), and let \(b^{(n)} = b \circ \pi_n\). Then for any \(\lambda \geq \lambda(T)\) and \(n \geq 1\), the equation
\[
\sup_{n \geq 1} \|\nabla u^{(n)}\|_{T, \infty} + \|u^{(n)}\|_{T, \infty} \leq \delta(\lambda),
\]
holds for some \(\phi \in \mathcal{D}\), such that \(\lim_{\lambda \to \infty} \delta(\lambda) = 0\).

Proof. We first observe that although Lemma 2.2(3) works also for \(\mathbb{H}\)-valued functionals. For instance, if \(f, e\) satisfies (1.5) for any unit \(e \in \mathbb{H}\) the real function \(\langle f, e \rangle\) satisfies (2.18) as well, so that Lemma 2.2(3) implies
\[
\sup_{|\eta|/|\eta| \leq 1} \|\nabla_\eta \nabla_\eta R^{\lambda}_{s,t} f, e\|_{\infty} = \sup_{|\eta|/|\eta| \leq 1} \|\nabla_\eta \nabla_\eta R^{\lambda}_{s,t} f, e\|_{\infty} \leq \delta(\lambda) |e|, \quad 0 \leq s \leq t \leq T.
\]
That is, Lemma 2.2(3) works also for \(\mathbb{H}\)-valued functions. Below we prove assertions (1) and (2) respectively.

(1) Let \(\mathcal{H} = C([0, T]; C_b^1(\mathbb{H}; \mathbb{H}))\), which is a Banach space under the norm
\[
\|u\|_{\mathcal{H}} := \|u\|_{T, \infty} + \|\nabla u\|_{T, \infty} = \sup_{t \in [0, T], x \in \mathbb{H}} |u_t(x)| + \sup_{t \in [0, T], x \in \mathbb{H}} \|\nabla u_t(x)\|, \quad u \in \mathcal{H}.
\]

For any \(u \in \mathcal{H}\), define
\[
(\Gamma u)_s(x) = \int_s^T e^{-\lambda(t-s)} P_{s,t}^0(\nabla b_t u_t + b_t) (x) dt, \quad s \in [0, T].
\]
By the fixed-point theorem, it suffices to show that for large enough \(\lambda > 0\), the map \(\Gamma\) is contractive on \(\mathcal{H}\). For any \(u, \bar{u} \in \mathbb{H}\), by the definition of \(\Gamma\) we have
\[
\|\Gamma u - \Gamma \bar{u}\|_{T, \infty} \leq \int_0^T e^{-\lambda t} \|b\|_{T, \infty} \|\nabla u - \nabla \bar{u}\|_{T, \infty} dt = \frac{\|b\|_{T, \infty}}{\lambda} \|\nabla u - \nabla \bar{u}\|_{T, \infty}.
\]
Next, by (2.12) and the definition of $\Gamma$, we have

\[ \|\nabla (\Gamma u - \Gamma \tilde{u})\|_{T,\infty} \leq C_1 \|b\|_{T,\infty} \|\nabla u - \nabla \tilde{u}\|_{T,\infty} \int_0^T \frac{e^{-\lambda t}}{\sqrt{t}} \, dt \leq \frac{C_2}{\sqrt{\lambda}} \|\nabla u - \nabla \tilde{u}\|_{T,\infty} \]

for some constants $C_1, C_2 > 0$. Combining this with (2.23) we may find $\lambda_0(T) > 0$ such that the operator $\Gamma$ is a contraction operator on $H$ when $\lambda \geq \lambda_0(T)$.

(2) Obviously, if $(B, b, Q)$ satisfies (a2'), so does $(B \circ \pi_n, b \circ \pi_n, Q \circ \pi_n)$ uniformly in $n \geq 1$. By (1), $(u^{(n)})_{n \geq 1}$ are well defined for $\lambda \geq \lambda_0(T)$. Due to (2.12), there exists a constant $C > 0$ such that

\[ |\nabla u^{(n)}| \leq C \int_s^T \frac{e^{-\lambda(t-s)}}{\sqrt{t-s}} (|\nabla u^{(n)}|_{T,\infty} + 1) \, dt, \quad n \geq 1, s \in [0, T]. \]

Taking $\lambda_1(T) \geq \lambda_0(T)$ such that $C \int_0^T \frac{e^{-\lambda_1(T)s}}{\sqrt{s}} \, ds \leq \frac{1}{2}$, we obtain

\[ \|\nabla u^{(n)}\|_{T,\infty} \leq 2C \int_0^T \frac{e^{-\lambda s}}{\sqrt{s}} \, ds, \quad \lambda \geq \lambda_1(T), n \geq 1. \]

Combining this with the definition of $u^{(n)}$ we prove

\[ \sup_{n \geq 1} (\|\nabla u^{(n)}\|_{T,\infty} + \|u^{(n)}\|_{T,\infty}) \leq \delta(\lambda), \quad \lambda \geq \lambda_1(T) \]

for some function $\delta$ with $\delta(\lambda) \to 0$ as $\lambda \to \infty$. Moreover, we have

\[ u_s - u^{(n)} = \int_s^T e^{-\lambda(t-s)} P_{s,t} \{ \nabla b^{(n)}(u_t - u^{(n)}_t) \} \, dt \]

\[ + \int_s^T e^{-\lambda(t-s)} P_{s,t} \{ \nabla b^{(n)} - b^{(n)} \} u_t + b_t^{(n)} \, dt \]

\[ + \int_s^T e^{-\lambda(t-s)} \left( P_{s,t} - P_{s,t}^{(n)} \right) \{ \nabla b^{(n)} u^{(n)}_t + b^{(n)}_t \} \, dt. \]

To prove the limits in (2.21), let

\[ g_s(x) = \limsup_{n \to \infty} \|\nabla u_s - \nabla u^{(n)}_s\|(x), \quad s \in [0, T], x \in \mathbb{H}. \]

Then $g \in \mathbb{B}_b([0, T] \times \mathbb{H})$. Combining (2.24) with (2.12) which also holds for $P_{s,t}^{(n)}$ uniformly in $n \geq 1$, we obtain

\[ g_s \leq C_1 \limsup_{n \to \infty} \int_s^T \frac{e^{-\lambda(t-s)}}{\sqrt{t-s}} \left( \left( P_{s,t}^{(n)} \| u_t - u^{(n)}_t \|^{2/2} + \left( P_{s,t}^{(n)} \| b_t - b^{(n)}_t \|^{2/2} \right) \right) \, dt \]

\[ + \limsup_{n \to \infty} \int_s^T e^{-\lambda(t-s)} \left\{ \nabla \left( P_{s,t}^{(n)} - P_{s,t} \right) \{ \nabla b^{(n)} u^{(n)}_t + b^{(n)}_t \} \right\} \, dt \]

for some constant $C_1 > 0$. Obviously, by the dominated convergence theorem we have

\[ \lim_{n \to \infty} \int_s^T \frac{e^{-\lambda(t-s)}}{\sqrt{t-s}} \left( P_{s,t}^{(n)} \| b_t - b^{(n)}_t \|^{2/2} \right) \, dt = 0. \]
Moreover, by (2.19) and Lemma 2.2(1), for every $t \in [0, T]$, the following Lemma 2.4 applies to
\[ f_{n} := \nabla b_{t}^{(n)} u_{t}^{(n)} + b_{t}^{(n)} , \quad n \geq 1 , \]
so that
\[ \lim_{n \to \infty} \left\| \nabla \left( P_{s,t}^{0} - P_{s,t}^{(n)} \right) \{ \nabla b_{t}^{(n)} u_{t}^{(n)} + b_{t}^{(n)} \} \right\| = 0 . \]
Combining this with
\[ \left\| \nabla \left( P_{s,t}^{0} - P_{s,t}^{(n)} \right) \{ \nabla b_{t}^{(n)} u_{t}^{(n)} + b_{t}^{(n)} \} \right\| \leq \frac{C}{\sqrt{t-s}} \]
for some constant $C > 0$ according to (2.12) which also holds for $P_{s,t}^{(n)}$ in place of $P_{s,t}^{0}$, we can apply the dominated convergence theorem to obtain
\[ \limsup_{n \to \infty} \int_{s}^{T} e^{-\lambda(t-s)} \left\| \nabla \left( P_{s,t}^{0} - P_{s,t}^{(n)} \right) \{ \nabla b_{t}^{(n)} u_{t}^{(n)} + b_{t}^{(n)} \} \right\| dt = 0 . \]
Thus, it follows from (2.25) that
\[ g_{s} \leq C_{1} \int_{s}^{T} e^{-\lambda(t-s)} \left( P_{s,t}^{0} g_{t} \right)^{2} \left( \frac{1}{\sqrt{t-s}} \right) dt \leq \frac{C_{2}}{\sqrt{\lambda}} \| g \|_{T,\infty}, \quad s \in [0, T], \lambda \geq \lambda_{1}(T) \]
holds for some constant $C_{2} > 0$. Taking $\lambda_{2}(T) = \lambda_{1}(T) \vee (4C_{2}^{2})$, we obtain $\| g \|_{T,\infty} \leq \frac{1}{2} \| g \|_{T,\infty}$ for $\lambda \geq \lambda_{2}(T)$. Since $g$ is bounded, this implies
\[ \lim_{n \to \infty} \| \nabla u_{s} - \nabla u_{s}^{(n)} \| = 0, \quad s \in [0, T] \]
provided $\lambda \geq \lambda_{2}(T)$. Combining this with (2.21) and Lemma 2.4 below, and using again the dominated convergence theorem, we obtain $\limsup_{n \to \infty} | u_{s} - u_{s}^{(n)} | = 0$. Therefore, (2.21) holds for $\lambda \geq \lambda_{2}(T)$.

Finally, let $b$ satisfy (1.5) for some $\phi \in D$. Then, by Lemma 2.2(1), there exists a constant $c > 0$ such that the functions
\[ f_{t}^{(n)} := \nabla b_{t}^{(n)} u_{t}^{(n)} + b_{t}^{(n)} , \quad n \geq 1, t \in [0, T] \]
satisfy (2.7) with $\bar{\phi}(s) := c \sqrt{\phi^{2}(s)} + s$ in place of $\phi$. Obviously, $\phi \in D$ implies $\bar{\phi} \in D$. So, (2.22) follows from Lemma 2.2(3). In conclusion, Lemma 2.3 holds for $\lambda(T) = \lambda_{2}(T)$.

**Lemma 2.4.** Let $P_{s,t}^{(n)}$ be in Lemma 2.3. For any sequence $\{ f_{n} \}_{n \geq 1} \subset C_{b}(\mathbb{H};\mathbb{H})$ such that
\[ \sup_{n \geq 1} \| f_{n} \|_{\infty} < \infty, \quad \delta_{r} := \sup_{|x-y| \leq r, n \geq 1} | f_{n}(x) - f_{n}(y) | \downarrow 0 \text{ as } r \downarrow 0 , \]
there holds
\[ \lim_{n \to \infty} \left( | P_{s,t}^{0} f_{n} - P_{s,t}^{(n)} f_{n} | + \| \nabla P_{s,t}^{0} f_{n} - \nabla P_{s,t}^{(n)} f_{n} \| \right) = 0, \quad 0 \leq s < t . \]
Proposition 2.5. Let \((Z_{s,t}^{n,x})_{t \geq s}\) solve the equation

\[
dZ_{s,t}^{n,x} = AZ_{s,t}^{n,x} \, dt + Q_t(\pi_n Z_{s,t}^{n,x}) \, dW_t, \quad t \geq s, \quad Z_{s,s}^{n,x} = x.
\]

We have \(P_{s,t} f(x) = E f(Z_{s,t}^{n,x}), \ f \in \mathcal{B}_b(\mathbb{H})\). By (a2′) it is easy to see that

\[
\lim_{n \to \infty} E \left( |Z_{s,t}^{n,x} - Z_{s,t}^{x}|^2 + \|\nabla Z_{s,t}^{n,x} - \nabla Z_{s,t}^{x}\|^2 \right) = 0.
\]

Then for any \(r > 0\),

\[
\limsup_{n \to \infty} \left| P_{s,t} f_n(x) - P_{s,t}^0 f_n(x) \right| \leq \delta_r + \limsup_{n \to \infty} E\left( |f_n(Z_{s,t}^{n,x}) - f_n(Z_{s,t}^{x})|1_{\{|Z_{s,t}^{n,x} - Z_{s,t}^{x}| \geq r\}} \right) = \delta_r.
\]

By letting \(r \to 0\) we prove \(\lim_{n \to \infty} |P_{s,t}^0 f_n - P_{s,t} f_n| = 0\).

Next, by (2.8) and the corresponding formula for \(P_{s,t} f_n\), for any \(\eta \in \mathbb{H}\) we have

\[
\begin{align*}
|\nabla_{\eta} P_{s,t} f_n(x) - \nabla_{\eta} P_{s,t}^0 f_n(x)| &\leq E \left[ \frac{f_n(Z_{s,t}^{x}) - f_n(Z_{s,t}^{n,x})}{t-s} \int_s^T \langle Q_r(QQ^*)^{-1}_r(Z_{s,r}) \nabla_{\eta} Z_{s,r}, \, dW_r \rangle_{\mathbb{H}} \right] \\
&+ E \left[ \frac{f_n(Z_{s,t}^{n,x})}{t-s} \int_s^T \langle Q_r(QQ^*)^{-1}_r(Z_{s,r}) \nabla_{\eta} Z_{s,r} - Q_r^*(QQ^*)^{-1}_r(\pi_n Z_{s,r}^{n,x}) \nabla_{\eta} Z_{s,r}^{n,x}, \, dW_r \rangle_{\mathbb{H}} \right] \\
&=: J_n + J'_n.
\end{align*}
\]

Similarly to (2.28), we can prove \(\lim_{n \to \infty} J_n = 0\) uniformly in \(|\eta| \leq 1\). Moreover, since

\[
\sup_{n \geq 1} \|f_n\|_\infty + \|Q^*(QQ^*)^{-1}\|_{T,\infty} + \|\nabla Q^*(QQ^*)^{-1}\|_{T,\infty} < \infty,
\]

from (2.27) we see that \(\lim_{n \to \infty} J'_n = 0\) uniformly in \(|\eta| \leq 1\). Therefore,

\[
\lim_{n \to \infty} \|\nabla P_{s,t}^0 f_n(x) - \nabla P_{s,t} f_n(x)\| = 0.
\]

Finally, we present the regularization representation of the mild solution as explained in the beginning of this section. When \(Q\) is constant and \(B = 0\), this result is essentially due to [5] [6] [7]. Recall that \(Q\) is called cylindrical if there exists \(n \geq 1\) such that \(Q(x) = Q(\pi_n x)\) for all \(x \in \mathbb{H}\).

**Proposition 2.5.** Assume (a1), (a2′) and either (a3′) or

(a3′′) \(Q\) is cylindrical and \(b_s \in C_b(\mathbb{H}; \mathbb{H})\) for \(s \geq 0\).
For any $T > 0$, there exists a constant $\lambda(T) > 0$ such that for any stopping time $\tau$, any adapted continuous process $(X_t)_{t \in [0, \tau \wedge T]}$ on $\mathbb{H}$ with $\mathbb{P}$-a.s.

\[ X_t = e^{tA}X_0 + \int_0^t e^{(t-s)A}\{B_s + b_s\}(X_s)ds + \int_0^t e^{(t-s)A}Q_s(X_s)dW_s, \quad t \in [0, \tau \wedge T], \]

and any $\lambda \geq \lambda(T)$, there holds $\mathbb{P}$-a.s.

\[ X_t = e^{tA}(X_0 + u_0(X_0)) - u_t(X_t) + \int_0^t e^{(t-s)A}\{Q_s + (\nabla u_s)Q_s\}(X_s)ds \]

\[ + \int_0^t \left\{ (\lambda - A)e^{(t-s)A}u_s + e^{(t-s)A}(B_s + \nabla B_s u_s) \right\}(X_s)ds, \quad t \in [0, \tau \wedge T], \]

where $u$ solves (2.31), and $(\nabla u_z)z := \nabla_z u_z$ for $z \in \mathbb{H}$.

**Proof.** As in the proof of [6] Theorem 7 (see also the proof of [7] Theorem 2), we first make finite-dimensional approximations such that Itô’s formula applies. For every $\lambda > 0$, let

\[ B_s^{(n)} = \pi_n B_s \circ \pi_n, \quad b_s^{(n)} = \pi_n b_s \circ \pi_n, \quad Q_s^{(n)} = \pi_n Q_s \circ \pi_n, \quad n \geq 1, s \geq 0. \]

Let $(P_{s,t}^{(n)})_{t \geq s, n \geq 0}$ be the semigroup of the following SDE on $\mathbb{H}_n$ (note that $A = \pi_n A$ holds on $\mathbb{H}_n$):

\[ dZ_{s,t}^{(n,z)} = \{ A Z_{s,t}^{(n,z)} \} dt + Q_t^{(n)}(Z_{s,t}^{(n,z)})dW_t, \quad Z_{s,s}^{(n,z)} = z \in \mathbb{H}_n, t \geq s. \]

We have

\[ P_{s,t}^{(n)} f(z) = \mathbb{E} f(Z_{s,t}^{(n,z)}), \quad f \in \mathcal{B}_b(\mathbb{H}_n), t \geq s \geq 0, z \in \mathbb{H}_n. \]

It is easy to see that $(B^{(n)}, Q^{(n)})$ satisfies (a2') for $\mathbb{H}_n$ in place of $\mathbb{H}$. Let $\lambda(T) > 0$ be such that assertions in Lemma 2.3 hold. Then for any $\lambda \geq \lambda(T)$, there exists unique $u^{(n)} \in C([0, T]; C_b^1(\mathbb{H}_n; \mathbb{H}_n))$ satisfying

\[ u_s^{(n)} = \int_s^T e^{-\lambda(t-s)} P_{s,t}^{(n)}(\nabla b_t^{(n)}u_t^{(n)} + b_t^{(n)}) dt, \quad s \in [0, T]. \]

Let $G_r^{(n)} = \nabla b_r^{(n)}u_r^{(n)} + b_r^{(n)}, r \geq 0$. To regularize this functional, we fix $\delta > 0$ and let

\[ F_{s,r}(z) = P_{s,\delta+r}^{(n)} G_r^{(n)}(\pi_n z), \quad 0 \leq s \leq r \leq T, z \in \mathbb{H}. \]

Then $F_{s,r} = F_{s,r} \circ \pi_n$ and, by (2.16) which also holds for $(P_{s,t}^{(n)}, \mathbb{H}_n)$ in place of $(P_{s,t}^{0}, \mathbb{H})$,

\[ \sup_{0 \leq s \leq r \leq T} \left\{ \| F_{s,r} \|_\infty + \| \nabla F_{s,r} \|_\infty + \| \nabla^2 F_{s,r} \|_\infty \right\} < \infty. \]

So, by (2.31) and Itô’s formula, for any $0 \leq s \leq r \leq T$, we have

\[ dF_{s,r}(Z_{r,t}^{(n,z)}) = L_t^{(n)} F_{s,r}(Z_{r,t}^{(n,z)}) dt + \langle \nabla F_{s,r}(Z_{r,t}^{(n,z)}), Q_t^{(n)}(Z_{r,t}^{(n,z)}) dW_t \rangle, \quad t \geq r, \]
Let, for any second-order differentiable function $F$ on $\mathbb{H}$,

$$L^{(n)}_t F(z) := \frac{1}{2} \sum_{i,j=1}^{n} \langle Q_i Q^*_i e_i, e_j \rangle (z) \nabla_{e_i} \nabla_{e_j} F(z) + \langle Az, \nabla F(z) \rangle$$

$$= \frac{1}{2} \sum_{k \geq 1} (\nabla^2_{Q_k(z) e_k} F)(z) + \langle Az, \nabla F(z) \rangle, \quad z \in \mathbb{H}.$$  

Here, $\nabla^2_e := \nabla_e \nabla_e$ for $e \in \mathbb{H}$, and $\{e_k\}_{k \geq 1}$ is an orthonormal basis on $\mathbb{H}$. By (2.34), $Q \in C([0, \infty) \times \mathbb{H}; \mathcal{L}(\mathbb{H}; \mathbb{H}))$ and noting that $F_{s,r} = F_{s,r} \circ \pi_n$, we have $L^{(n)}_s F_{s,r} \in C_b([0, T] \times \mathbb{H})$ for any $T > 0$. So, it follows from (2.35) and the a.e. right-continuity of $Q_t Q^*_t$ that

$$\frac{d}{ds} F_{s,r}(z) := - \lim_{v \downarrow 0} \frac{F_{s-v,r}(z) - F_{s,r}(z)}{v} = - \lim_{v \downarrow 0} \frac{\mathbb{E} F_{s,r}(Z_{s-v,s}^{(n,z)}) - F_{s,r}(z)}{v}$$

$$= - \lim_{v \downarrow 0} \frac{1}{v} \mathbb{E} \int_{s-v}^{s} (L^{(n)}_t F_{s,r})(Z_{s-v,t}^{(n,z)}) dt = -L^{(n)}_s F_{s,r}(z), \quad r > 0, \text{ a.e. } s \in (0, r].$$

Let

$$u^{(n,\delta)}_s = \int_0^T e^{-\lambda(t-s)} (P^{(n)}_{s,t+\delta} G_t^{(n)}) \circ \pi_n dt = \int_s^T e^{-\lambda(t-s)} F_{s,t} dt, \quad s \in [0, T].$$

It follows from (2.34) and (2.36) that

$$\partial_s u^{(n,\delta)}_s = (\lambda - L^{(n)}_s) u^{(n,\delta)}_s - (P^{(n)}_{s,s+\delta} G^{(n)}_s) \circ \pi_n$$

$$= (\lambda - L^{(n)}_s) u^{(n,\delta)}_s - (P^{(n)}_{s,s+\delta} \{\nabla_{b^{(n)}_s u^{(n)}_s} + b^{(n)}_s\}) \circ \pi_n.$$

On the other hand, by (2.29), $X^{(n)}_s := \pi_n X_s$ solves the following equation on $\mathbb{H}_n$:

$$dX^{(n)}_s = AX^{(n)}_s ds + \pi_n \{B_s + b_s(X_s)\} ds + \pi_n Q_s(X_s)dW_s, \quad s \in [0, \tau \wedge T].$$

Then, by $u^{(n,\delta)} = u^{(n,\delta)} \circ \pi_n$ and Itô’s formula,

$$du^{(n,\delta)}_s(X^{(n)}_s) = (\nabla Q_s(X_s) dW_s u^{(n,\delta)}_s)(X^{(n)}_s) + \{\partial_s u^{(n,\delta)}_s(X^{(n)}_s) + (L^{(n)}_s u^{(n,\delta)}_s + \nabla_{\pi_n B_s u^{(n,\delta)}_s})\}(X^{(n)}_s) ds,$$

Combining this with (2.38) and noting that $(\nabla_{b^{(n)}_s u^{(n)}_s}) \circ \pi_n = (\nabla b^{(n)}_s u^{(n)}_s) \circ \pi_n$, we obtain

$$(P^{(n)}_{s,s+\delta} b^{(n)}_s(X^{(n)}_s)) ds$$

$$= \lambda u^{(n,\delta)}_s(X^{(n)}_s) - du^{(n,\delta)}_s(X^{(n)}_s) + \{\nabla_{b^{(n)}_s(X^{(n)}_s)} u^{(n,\delta)}_s - P^{(n)}_{s,s+\delta} \nabla_{b^{(n)}_s u^{(n,\delta)}_s}\}(X^{(n)}_s) ds$$

$$+ (\nabla Q_s(X_s) dW_s u^{(n,\delta)}_s)(X^{(n)}_s) + \frac{1}{2} \sum_{k \geq 1} (\nabla^2_{Q_s(X_s) - Q_s(X^{(n)}_s)}) \nabla_{\pi_n B_s u^{(n,\delta)}_s}(X^{(n)}_s) ds$$

$$+ (\nabla_{\pi_n B_s(X_s)} u^{(n,\delta)}_s(X^{(n)}_s)) ds, \quad s \in [0, \tau \wedge T].
Finally, we complete proof by using (a3') and (a3'') respectively.

(i) Assume (a3’). Then \((QQ') (X_t^{(n)}) - (QQ') (X_t) = 0\) for large \(n\), so that (2.39) reduces to \(\mathbb{P}\)-a.s.

\[
\int_{t_1}^{t_2} \left( P_{s,s+\delta}^{(n)} b_s^{(n)} \right) (X_s^{(n)}) ds
\]

for \(0 \leq t_1 \leq t_2 \leq T \wedge \tau\). We claim that when \(\delta \downarrow 0\) this yields \(\mathbb{P}\)-a.s.

\[
\int_{t_1}^{t_2} b_s^{(n)} (X_s^{(n)}) ds = \int_{t_1}^{t_2} \left\{ \lambda u_s^{(n)} + \nabla \pi_n B_s (X_s) u_s^{(n)} \right\} (X_s^{(n)}) ds
\]

for \(0 \leq t_1 \leq t_2 \leq T \wedge \tau\).

Indeed, since \(\lim_{\delta \downarrow 0} P_{t,t+\delta}^{(n)} f = f\) holds for \(t \geq 0\) and \(f \in C_b (\mathbb{H}; \mathbb{H})\), by the boundedness and continuity of \(b_s^{(n)}\) and \(G_s^{(n)}\), and noting that (2.33) and (2.37) imply

\[
\lim_{\delta \downarrow 0} \int_{t_1}^{t_2} \left( P_{s,s+\delta}^{(n)} b_s^{(n)} \right) (X_s^{(n)}) ds = \int_{t_1}^{t_2} b_s^{(n)} (X_s^{(n)}) ds, \quad 0 \leq t_1 \leq t_2 \leq T \wedge \tau.
\]

Moreover, combining (2.42) with (2.12) which also holds for \(P_{s,t}^{(n)}\) in place of \(P_{s,t}^{0}\), we obtain

\[
\lim_{\delta \downarrow 0} \| \nabla (u_s^{(n,\delta)} - u_s^{(n)}) \| \leq \lim_{\delta \downarrow 0} \int_{t_1}^{t_2} \frac{c}{\sqrt{t-s}} \sqrt{P_{s,t}^{(n)} | P_{t,t+\delta}^{(n)} G_t^{(n)} - G_t^{(n)} |^2} dt = 0
\]

due to the dominated convergence theorem. Thus, \(\mathbb{P}\)-a.s.

\[
\lim_{\delta \downarrow 0} \int_{t_1}^{t_2} \left\{ \nabla b_s (X_s) u_s^{(n,\delta)} \right\} (X_s^{(n)}) ds = \int_{t_1}^{t_2} \left\{ \nabla b_s (X_s) u_s^{(n)} \right\} (X_s^{(n)}) ds,
\]

\[
\lim_{\delta \downarrow 0} \int_{t_1}^{t_2} \left( \nabla Q_s (X_s) dw_s u_s^{(n,\delta)} \right) (X_s^{(n)}) = \int_{t_1}^{t_2} \left( \nabla Q_s (X_s) dw_s u_s^{(n)} \right) (X_s^{(n)}), \quad 0 \leq t_1 \leq t_2 \leq T \wedge \tau.
\]

So, to deduce (2.40) from (2.42) with \(\delta \downarrow 0\), it remains to prove

\[
\lim_{\delta \downarrow 0} \int_{t_1}^{t_2} \left\{ \nabla b_s u_s^{(n,\delta)} - P_{s,s+\delta}^{(n)} \nabla b_s u_s^{(n)} \right\} ds = 0, \quad 0 \leq t_1 \leq t_2 \leq T.
\]
This follows since by the boundedness of \( b \), the uniform boundedness and continuous of \( \nabla_{b_s^n} u_s^{(n)} \), \( G_s^{(n)} \), and (2.43), we have

\[
\limsup_{\delta \downarrow 0} \int_0^T \left| \nabla_{b_s} u_s^{(n,\delta)} - P_{s,s+\delta}^{(n)} \nabla_{b_s} u_s^{(n)} \right| \, ds \\
\leq \limsup_{\delta \downarrow 0} \int_0^T \left( \left| \nabla_{b_s} (u_s^{(n,\delta)} - u_s^{(n)}) \right| + \left| P_{s,s+\delta}^{(n)} \nabla_{b_s} u_s^{(n)} - \nabla_{b_s} u_s^{(n)} \right| \right) \, ds = 0.
\]

Now, writing (2.41) as

\[
b_t^{(n)}(X_t^{(n)}) dt = \{ \lambda u_t^{(n)} + \nabla_{\pi_n B_s(X_t)} u_t^{(n)} \} (X_t^{(n)}) dt \\
- du_t^{(n)}(X_t^{(n)}) + (\nabla u_t^{(n)})(X_t^{(n)}) Q_t^{(n)}(X_t) \, dW_t, \quad 0 \leq t \leq \tau \wedge T,
\]

we conclude that \( \mathbb{P} \)-a.s. for all \( t \in [0, T \wedge \tau] \) and large enough \( n \),

\[
X_t - e^{tA} X_0 - \int_0^t e^{(t-s)A} B_s(X_s) \, ds = \int_0^t e^{(t-s)A} b_s(X_s) \, ds + \int_0^t e^{(t-s)A} Q_s(X_s) \, dW_s \\
= \int_0^t e^{(t-s)A} b_s^{(n)}(X_s^{(n)}) \, ds + \int_0^t e^{(t-s)A} Q_s(X_s) \, dW_s + \int_0^t e^{(t-s)A} \{ b_s(X_s) - b_s^{(n)}(X_s^{(n)}) \} \, ds \\
= e^{tA} u_0^{(n)}(X_0^{(n)}) - u_t^{(n)}(X_t^{(n)}) + \int_0^t (\lambda - A) e^{(t-s)A} u_s^{(n)}(X_s^{(n)}) \, ds \\
+ \int_0^t e^{(t-s)A} \{ b_s(X_s) - b_s^{(n)}(X_s^{(n)}) + (\nabla_{\pi_n B_s(X_s)} u_s^{(n)})(X_s^{(n)}) \} \, ds \\
+ \int_0^t e^{(t-s)A} \{ Q_s(X_s) + (\nabla u_t^{(n)})(X_t^{(n)}) Q_s^{(n)}(X_s) \} \, dW_s.
\]

Since \( b_s \) and \( Q_s \) are bounded and continuous, and \( \| u^{(n)} \|_\infty + \| \nabla u^{(n)} \|_\infty \) is bounded in \( n \) by Lemma 2.3(2), with \( n \to \infty \) this implies (2.30) provided

\[\textbf{D*0} \quad (2.44) \quad \lim_{n \to \infty} u^{(n)} \circ \pi_n = u, \quad \lim_{n \to \infty} \int_0^T \| \nabla u_s - \nabla u_s^{(n)} \circ \pi_n \| \, ds = 0,\]

where the first limit implies \( \int_0^t (\lambda - A) e^{(t-s)A} u_s^{(n)}(X_s^{(n)}) \, ds \to \int_0^t (\lambda - A) e^{(t-s)A} u_s(X_s) \, ds \) weakly in \( \mathbb{H} \) as \( n \to \infty \). To prove (2.44) using Lemma 2.3(2), let \( (Z_{s,t}^{(n,z)})_{t \geq s} \) solve the equation (2.26) for \( z \) in place of \( x \). Since \( \pi_n A = A \) holds on \( \mathbb{H}_n \), we see that \( \pi_n Z_{s,t}^{(n,z)} \) solves (2.31) for \( \pi_n z \) in place of \( z \). Thus, \( \pi_n Z_{s,t}^{(n,z)} = Z_{s,t}^{(n,\pi_n z)} \), so that

\[
P_{s,t}^{(n)}(\pi_n z) = P_{s,t}^{(n)}(f \circ \pi_n)(z), \quad z \in \mathbb{H}, \quad f \in \mathcal{B}_b(\mathbb{H}).
\]

Combining this with (2.33) and \( b^{(n)} \circ \pi_n = b^{(n)} \), we conclude that \( u^{(n)} := u^{(n)} \circ \pi_n \) solves (2.20). Therefore, (2.44) follows from Lemma 2.3(2).
(ii) Assume (a3'). Then (1.3) and (1.5) hold for some \( \phi \in \mathcal{D} \). By Lemma 2.3(2), \( \|\nabla u^{(n)}\|_\infty + \|\nabla^2 u^{(n)}\|_\infty \) is bounded in \( n \geq 1 \). Since

\[
u_s^{(n,\delta)} = \int_s^T e^{-\lambda(t-s)} (P_{s,t+\delta}^{(n)} G_t^{(n)}) \circ \pi_n dt,
\]

and as explained in the proof of (2.22) that \( f_t := 1_{[s,T]}(t) (P_{s,t+\delta}^{(n)} G_t^{(n)}) \circ \pi_n \) satisfies (2.7) for some \( \tilde{\phi} \in \mathcal{D} \), we have \( \sup_{n \geq 1, \delta \in (0,1)} \|\nabla^2 u^{(n,\delta)}\|_\infty < \infty \) according to Lemma 2.3(2). Combining this with (1.3), we obtain

\[
\limsup_{n \to \infty} \limsup_{\delta \downarrow 0} \int_0^{\tau^T \wedge T} \sum_{k \geq 1} \left| \nabla^2 \{Q_s(X_s^{(n)}) - Q_s(X_s)\} \tilde{e}_k \nu_s^{(n,\delta)}(X_s^{(n)}) \right| ds = 0.
\]

Therefore, repeating the argument in case (i) we prove (2.30).

\[\square\]

### 3 Pathwise uniqueness

In this section, we prove the pathwise uniqueness of mild solutions under (a1), (a2'), and either (a3') or the following stronger version of (a3).

(a3') \( Q \) is cylindrical, i.e. \( Q = Q \circ \pi_n \); \( b \in \mathcal{B}_b([0,\infty) \times \mathbb{R}; \mathbb{R}) \) such that (1.5) holds for some \( \phi \in \mathcal{D}_0 \).

#### P3.1 Proposition 3.1

Assume (a1), (a2') and (a3'). Let \((X_t)_{t \geq 0}, (Y_t)_{t \geq 0}\) be two adapted continuous process on \( \mathcal{H} \) with \( X_0 = Y_0 \). For any \( n \geq 1 \), let

\[
\tau_n^X = n \wedge \inf\{t \geq 0 : |X_t| \geq n\}, \quad \tau_n^Y = n \wedge \inf\{t \geq 0 : |Y_t| \geq n\}.
\]

If \( \mathbb{P}\)-a.s. for all \( t \in [0, \tau_n^X \wedge \tau_n^Y] \) there holds

\[
X_t = e^{tA} x + \int_0^t e^{(t-s)A} (B_s + b_s)(X_s)ds + \int_0^t e^{(t-s)A} Q_s(X_s)dW_s,
\]

\[
Y_t = e^{tA} x + \int_0^t e^{(t-s)A} (B_s + b_s)(Y_s)ds + \int_0^t e^{(t-s)A} Q_s(Y_s)dW_s,
\]

then \( \mathbb{P}\)-a.s. \( X_t = Y_t \) for all \( t \in [0, \tau_n^X \wedge \tau_n^Y] \). In particular, \( \mathbb{P}\)-a.s. \( \tau_n^X = \tau_n^Y \).

**Proof.** For any \( m \geq 1 \), let

\[
\tau_m = \tau_n^X \wedge \tau_n^Y \wedge \inf\{t \geq 0 : |X_t - Y_t| \geq m\}.
\]

It suffices to prove that for any \( T > 0 \) and \( m \geq 1 \),

\[
\int_0^T \mathbb{E}[1_{\{s < \tau_m\}} |X_s - Y_s|^2] ds = 0.
\]

A1 (3.1)
Let \( \lambda > 0 \) be large enough such that assertions in Lemma 2.3 and Proposition 2.5 hold. By (2.3) for \( \tau = \tau_m \), we have \( \mathbb{P}\)-a.s.

\[
X_t - Y_t = u_t(Y_t) - u_t(X_t) + \int_0^t (\lambda - A)e^{(t-s)A}(u_s(X_s) - u_s(Y_s))ds \\
+ \int_0^t e^{(t-s)A}\{(B_s + \nabla B_s)u_s(X_s) - (B_s + \nabla B_s)u_s(Y_s)\}ds \\
+ \int_0^t e^{(t-s)A}(\nabla u_s(X_s) - \nabla u_s(Y_s))Q_s(X_s)dW_s \\
+ \int_0^t e^{(t-s)A}(\nabla u_s(Y_s) + I)(Q_s(X_s) - Q_s(Y_s))dW_s, \quad t \in [0, \tau_m \wedge T].
\]

Since \( b \) and \( u \) are bounded on \([0, T] \times \mathbb{H}, \) by (2.4) and (2.12) we may find a constant \( C > 0 \) such that

\[
\|\nabla u_t\|_{\infty} \leq C \int_0^T \frac{e^{-\lambda s}}{\sqrt{s}} ds \leq \frac{1}{5}, \quad t \in [0, T]
\]

for large \( \lambda > 0 \). Combining this with (3.2) we obtain \( \mathbb{P}\)-a.s.

\[
|X_t - Y_t| \leq \frac{5}{4} \int_0^t (\lambda - A)e^{(t-s)A}(u_s(X_s) - u_s(Y_s))ds \\
+ \frac{5}{4} \int_0^t e^{(t-s)A}(B_s(X_s) - B_s(Y_s) + \nabla B_s(X_s) - \nabla B_s(Y_s)u_s(X_s))ds \\
+ \frac{5}{4} \int_0^t e^{(t-s)A}(\nabla u_s(Y_s) + I)(Q_s(X_s) - Q_s(Y_s))dW_s \\
+ \frac{5}{4} \int_0^t e^{(t-s)A}(\nabla B_s(Y_s)u_s(X_s) - \nabla B_s(Y_s)u_s(Y_s))ds \\
+ \frac{5}{4} \int_0^t e^{(t-s)A}(\nabla u_s(X_s) - \nabla u_s(Y_s))Q_s(X_s)dW_s, \quad t \in [0, \tau_m \wedge T].
\]

Moreover, by (a1) there exists some function \( \varepsilon(\lambda) \downarrow 0 \) as \( \lambda \uparrow \infty \) such that

\[
\int_0^r e^{-2\lambda t}E\left|1_{\{t < \tau_m\}} \int_0^t e^{(t-s)A}(\nabla u_s(X_s) - \nabla u_s(Y_s))Q_s(X_s)dW_s\right|^2 dt \\
\leq \|Q\|_{T, \infty}^2 \int_0^r e^{-2\lambda t} dt \int_0^r \|e^{A(t-s)}\|_{H^2}^2 E\left|1_{\{s < \tau_m\}} |\nabla u_s(X_s) - \nabla u_s(Y_s)|^2 \right| ds \\
= \|Q\|_{T, \infty}^2 \int_0^r e^{-2\lambda s} E\left|1_{\{s < \tau_m\}} |\nabla u_s(X_s) - \nabla u_s(Y_s)|^2 \right| ds \int_0^r \|e^{A(t-s)}\|_{H^2}^2 e^{-2\lambda(t-s)} dt \\
\leq \varepsilon(\lambda) \int_0^r e^{-2\lambda s} E\left|1_{\{s < \tau_m\}} |\nabla u_s(X_s) - \nabla u_s(Y_s)|^2 \right| ds, \quad r \in [0, T].
\]
Similarly, since \(3.3\) and (a2') imply \(\|\nabla u\|_{T,\infty} + \|\nabla Q\|_{T,\infty} < \infty\),

\[
\int_0^r e^{-2\lambda t} \mathbb{E} \left| 1_{\{t<\tau_m\}} \int_0^t e^{(t-s)A} \left( \nabla u_s(Y_s) + I \right) \left( Q_s(X_s) - Q_s(Y_s) \right) ds \right| dt
\]

\[
\leq \varepsilon(\lambda) \int_0^r e^{-2\lambda s} \mathbb{E} \left| 1_{\{s<\tau_m\}} X_s - Y_s \right|^2 ds, \quad r \in [0, T]
\]

holds for the same type \(\varepsilon(\lambda)\). Combining this with \(3.4\) and \(3.5\), and using (a2') and (3.3), we may find a constant \(C_0 > 0\) such that for large enough \(\lambda > 0\)

\[
\eta_r := \int_0^r e^{-2\lambda t} \mathbb{E} \left| 1_{\{t<\tau_m\}} X_t - Y_t \right|^2 dt
\]

\[
A3 \tag{3.6}
\]

\[
\leq \frac{25}{2} \int_0^r e^{-2\lambda t} \mathbb{E} \left| 1_{\{t<\tau_m\}} \int_0^t (\lambda - A) e^{(t-s)A} (u_s(X_s) - u_s(Y_s)) ds \right|^2 dt + C_0 \int_0^r \eta_t dt
\]

\[
+ C_0 \int_0^r e^{-2\lambda s} \mathbb{E} \left| 1_{\{s<\tau_m\}} \|\nabla u_s(X_s) - \nabla u_s(Y_s)\| \right|^2 ds, \quad r \in [0, T].
\]

Since

\[
I_t := e^{-2\lambda t} \mathbb{E} \left| 1_{\{t<\tau_m\}} \int_0^t (\lambda - A) e^{(t-s)A} (u_s(X_s) - u_s(Y_s)) ds \right|^2
\]

\[
= \sum_{i=1}^\infty e^{-2\lambda t} \left| (\lambda + \lambda_i) \int_0^{t \wedge \tau_m} e^{-(t \wedge \tau_m - s)\lambda_i} (u_s(X_s) - u_s(Y_s), e_i) ds \right|^2
\]

\[
\leq \sum_{i=1}^\infty \left( \int_0^t (\lambda + \lambda_i) e^{-(t-s)(\lambda + \lambda_i)} ds \right) \times \int_0^{t \wedge \tau_m} (\lambda + \lambda_i) e^{-(t \wedge \tau_m - s)(\lambda + \lambda_i) - 2\lambda s} (u_s(X_s) - u_s(Y_s), e_i)^2 ds
\]

\[
\leq \sum_{i=1}^\infty \int_0^{t \wedge \tau_m} (\lambda + \lambda_i) e^{-(t \wedge \tau_m - s)(\lambda + \lambda_i) - 2\lambda s} (u_s(X_s) - u_s(Y_s), e_i)^2 ds,
\]

it follows from (3.3) that

\[
\mathbb{E} \int_0^r I_t dt
\]

\[
\leq \sum_{i=1}^\infty (\lambda + \lambda_i) \mathbb{E} \int_0^t dt \int_0^{t \wedge \tau_m} e^{-(t-s)(\lambda + \lambda_i) - 2\lambda s} (u_s(X_s) - u_s(Y_s), e_i)^2 ds
\]

\[
A4 \tag{3.7}
\]

\[
= \sum_{i=1}^\infty \mathbb{E} \int_0^{t \wedge \tau_m} e^{-2\lambda s} (u_s(X_s) - u_s(Y_s), e_i)^2 ds \int_s^t (\lambda + \lambda_i) e^{-(t-s)(\lambda + \lambda_i)} dt
\]

\[
\leq \mathbb{E} \int_0^r e^{-2\lambda s} 1_{\{s<\tau_m\}} |u_s(X_s) - u_s(Y_s)|^2 ds
\]

\[
\leq \frac{1}{25} \int_0^r e^{-2\lambda s} \mathbb{E} \left| 1_{\{s<\tau_m\}} X_s - Y_s \right|^2 ds = \frac{1}{25} \eta_r.
\]

22
Next, by the boundedness of $b$ and Lemma 2.3(1), we have $\|\nabla b u + b\|_{T,\infty} < \infty$. So, according to Lemma 2.2(1) and (2.4), there exists a constant $C_1 > 0$ such that

$$\|\nabla u_s(x) - \nabla u_s(y)\| \leq C_1|x - y| \log(e + |x - y|^{-1}), \quad s \in [0, T], x, y \in \mathbb{H}.$$  

If (1.5) holds for some $\phi \in \mathcal{D}$, then by Lemma 2.2(1) and $\|b\|_{T,\infty} + \|\nabla u\|_{T,\infty} < \infty$, we conclude that

$$f_t := \nabla b_t u_t + b_t, \quad t \in [0, T]$$

satisfies (2.18) with $\phi$ replaced by $\tilde{\phi}(s) := c\sqrt{\phi(s)^2 + s}$, which is in $\mathcal{D}$ as well. Therefore, by (a1) and Lemma 2.3(2), when $\lambda$ is large enough we have

$$C_0 \int_0^T \mathbb{E}\left[1_{\{s \leq r_m\}} \|\nabla u_s(X_s) - \nabla u_s(Y_s)\|^2\right] ds \leq \frac{1}{4} \eta_r, \quad r \in [0, T].$$

Substituting this and (3.7) into (3.6), we arrive at

$$\eta_r \leq \frac{3}{4} \eta_r + C_0 \int_0^r \eta_t dt, \quad r \in [0, T].$$

Hence,

$$\eta_r \leq 4C_0 \int_0^r \eta_t dt, \quad r \in [0, T].$$

By the Gronwall inequality we obtain $\eta_T = 0$, which is equivalent to the desired (3.1).

4 Strong Feller property and Harnack inequality

In this section, we investigate the strong Feller property and discuss Harnack inequalities of the semigroup associated to the equation (1.1).

**Proposition 4.1.** Let $B = 0$, $b_t \in C_b(\mathbb{H}; \mathbb{H})$, and $Q_t \in C^1_b(\mathbb{H}; \mathcal{L}(\mathbb{H}; \mathbb{H}))$ for every $t \geq 0$. Assume

$$\|b\|_{T,\infty} + \|Q\|_{T,\infty} + \|\nabla Q\|_{T,\infty} + \|(QQ^*)^{-1}\|_{T,\infty} < \infty, \quad T > 0.$$  

If, for any $x \in \mathbb{H}$ and any cylindrical Brownian motion $(W_t)_{t \geq 0}$, the equation (1.1) has a unique mild solution, then the associated Markov semigroup $P_t$ is strong Feller for $t > 0$.

**Proof.** For fixed $z \in \mathbb{H}, T > 0$ and $f \in \mathcal{B}_b(\mathbb{H})$, we intend to prove

$$\lim_{x \to z} P_T f(x) = P_T f(z).$$

To this end, we formulate $P_T$ using the mild solution to the regular equation

$$dZ_t^x = AZ_t^x dt + Q_t(Z_t^x) dW_t, \quad Z_0^x = x.$$
More precisely, we have
\[ Z^x_t := e^{tA}x + \int_0^t e^{(t-s)A}Q_s(Z^x_s)\,dW_s \]
\[ = e^{tA}x + \int_0^t e^{(t-s)A}b_s(Z^x_s)\,ds + \int_0^t e^{(t-s)A}Q_s(Z^x_s)\,dW^x_s, \quad t \in [0, T], \]
where
\[ W^x_t := W_t - \int_0^t \{ Q_s'(QQ^*)^{-1}b_s \}(Z^x_s)\,ds, \quad t \in [0, T]. \]

By the Girsanov theorem, \((W^x_t)_{t \in [0, T]}\) is a cylindrical Brownian motion on \(\overline{\mathbb{H}}\) under probability \(dQ^x := R_T^x\,dP\), where
\[ R_T^x := \exp \left[ \int_0^T \left\langle \{ Q_s'(QQ^*)^{-1}b_s \}(Z^x_s), dW_s \right\rangle_{\overline{\mathbb{H}}} - \frac{1}{2} \int_0^T \left\{ Q_s'(QQ^*)^{-1}b_s \}(Z^x_s)^2 \right\}_D ds \right]. \]

Then \((Z^x_t, W^x_t)_{t \in [0, T]}\) is a weak mild solution to (1.1), so that
\[ P_T f(x) = \mathbb{E}[f(Z^x_T)R_T^x], \quad x \in \mathbb{H}. \]

By the boundedness and continuity of \(Q_s'(QQ^*)^{-1}b_s\), and noting that \(Z^x_t\) is continuous in \(x\), we conclude that
\[ \lim_{x \to z} |P_T f(x) - \mathbb{E}[f(Z^x_T)R_T^x]| \leq \|f\|_\infty \lim_{x \to z} \mathbb{E}|R_T^x - R_T^z| = 0. \]

Next, to prove the continuity of \(\mathbb{E}[f(Z^x_T)R_T^x]\) in \(x\), we approximate \(b\) by \(C^1_b\) maps such that Malliavin calculus can be applied. Since \(b\) is bounded and continuous in the space variable, we may find a sequence \(\{b^{(n)}\}_{n \geq 1} \subset \mathcal{B}_b([0, T] \times \overline{\mathbb{H}})\) such that \(b^{(n)} \in C^1_b(\mathbb{H}; \mathbb{H})\) for \(s \in [0, T]\) with \(\|\nabla b^{(n)}\|_{T, \infty} < \infty\) for every \(n \geq 1\), \(\sup_{n \geq 1} \|b^{(n)}\|_{T, \infty} < \infty\), and \(\lim_{n \to \infty} b^{(n)} = b\) holds on \([0, T] \times \overline{\mathbb{H}}\). Let
\[ R_{T,n}^x := \exp \left[ \int_0^T \left\langle \{ Q_s'(QQ^*)^{-1}b^{(n)} \}(Z^x_s), dW_s \right\rangle_{\overline{\mathbb{H}}} - \frac{1}{2} \int_0^T \left\{ Q_s'(QQ^*)^{-1}b^{(n)} \}(Z^x_s)^2 \right\}_D ds \right], \quad n \geq 1. \]

It is easy to see that \(R_{T,n}^x\) is Malliavin differentiable and
\[ \lim_{n \to \infty} \sup_{x \in \mathbb{H}} \left| \mathbb{E}[f(Z^x_T)R_{T,n}^x] - \mathbb{E}[f(Z^x_T)R_{T,n}^x] \right| \leq \|f\|_\infty \lim_{n \to \infty} \mathbb{E}|R_{T,n}^x - R_{T,n}^z| = 0. \]

Now, for \(\eta \in \mathbb{H}\), let \(h\) be in (2.11) with \(s = 0\) and \(t = T\) such that \(\nabla_{\eta} Z^x_T = D_h Z^x_T\) according to the proof of Lemma 2.1. Then, for \(\nabla_{\eta}\) being taken with respect to the variable \(x\), it follows from the integration by parts formula that for any \(f \in C^1_b(\overline{\mathbb{H}}),\)
\[ \nabla_{\eta} \mathbb{E}[f(Z^x_T)R_{T,n}^x] = \mathbb{E}\left[ (\nabla_{\eta} Z^x_T f)(Z^x_T)R_{T,n}^x \right] = \mathbb{E}\left[ D_h\{ f(Z^x_T) \} R_{T,n}^x \right] \]
\[ = \mathbb{E}\left[ D_h\{ f(Z^x_T) R_{T,n}^x \} - \mathbb{E}\left[ f(Z^x_T) D_h R_{T,n}^x \right] \right] \]
\[ = \mathbb{E}\left\{ f(Z^x_T) \left( R_{T,n}^x \int_0^T \langle h'_x, dW_t \rangle_{\overline{\mathbb{H}}} - D_h R_{T,n}^x \right) \right\}, \quad f \in C^1_b(\mathbb{H}; \mathbb{H}). \]

Up to an approximation argument this implies that \(|\nabla_{\eta} \mathbb{E}[f(Z^x_T)R_{T,n}^x]| < \infty\) for any \(f \in \mathcal{B}_b(\mathbb{H})\), where the derivative \(\nabla_{\eta}\) is taken with respect to \(x\). In particular, \(\mathbb{E}[f(Z^x_T)R_{T,n}^x]\) is continuous in \(x\). Combining this with (4.4) and (4.5), we prove (4.2). \(\square\)
Remark 4.1. Using coupling by change of measures as in [20, Chapter 4], in the situation of Proposition 4.1, we may derive the dimension-free Harnack inequality in the sense of [18]. Here, instead of repeating the coupling arguments therein, we intend to show that (4.3) together with known Harnack inequalities of $P_T$ implies the corresponding inequalities for $P_T$. For instance, when $Q_t(x) = Q_t$ does not depend on $x$, by [20, Theorem 3.2.1] for $K = 0$ and $\lambda_T := \sup_{t \in [0, T]} \|Q_t^*(Q_t)^{-1}\|^2$, the Harnack inequality

$$(P_T f(y))^p \leq P_T f^p(x) \exp\left[\frac{p|x - y|^2}{2\lambda_T(p - 1)}\right], \quad p > 1, x, y \in \mathbb{H}, f \in \mathcal{B}_+(\mathbb{H})$$

holds, where $\mathcal{B}_+(\mathbb{H})$ is the set of all positive measurable functions on $\mathbb{H}$. On the other hand, by (4.3) and Hölder’s inequality, we have

$$(P_T f(x))^p \leq P_T f^p(x)\left(\mathbb{E}(R_T^p:\frac{1}{1 - p})\right)^{p - 1},
(P_T f(x))^p \leq \left(\mathbb{E}f^p(Z_T^p)R^x\left(\mathbb{E}(R_T^p:\frac{1}{1 - p})\right)^{p - 1} = P_T f(x)\left(\mathbb{E}(R^x:\frac{1}{1 - p})\right)^{p - 1}, \quad p > 1.$$

Combining these with (4.6) we obtain

$$(P_T f(x))^p \leq (P_T f^p(y))^p\left(\mathbb{E}(R_T^p:\frac{1}{1 - p})\right)^{p - 1}\exp\left[\frac{p^2|x - y|^2}{2\lambda_T(p - 1)}\right]$$

for any $p > 1$ and $f \in \mathcal{B}_+(\mathbb{H})$. When the noise is multiplicative, we may derive the Harnack inequality from [20, Theorem 3.4.1(2)] for large enough $p > 1$.

Comparing with (4.6), the Harnack inequality included in (4.7) is worse for short distance since

$$\lim_{y \to x} \left(\mathbb{E}(R_T^p:\frac{1}{1 - p})\right)^{p - 1}\left(\mathbb{E}(R_T^p:\frac{1}{1 - p})\right)^{p - 1}\exp\left[\frac{p^2|x - y|^2}{2\lambda_T(p - 1)}\right] > 1.$$

In particular, it does not imply the strong Feller property as (4.6) does. See Section 6 for the study of the log-Harnack inequality which is sharp for short distance as in the regular case.

5 Proof of Theorem 1.1

Throughout this section, we assume (a1), (a2) and either (a3). Using $b$ to replace $b + B$, we may and do assume that $B = 0$.

(a) We first assume further that (a2') and (a3') hold. In this case, for any $X_0 \in \mathcal{B}(\Omega \to \mathbb{H}; \mathcal{F}_0)$, the equation (1.1) has a weak mild solution as shown in the proof of Proposition 4.1 for $X_0$ in place of $x$. On the other hand, by Proposition 3.1 we have the pathwise uniqueness of the mild solution. So, by the Yamada-Watanabe principle [24] (see [15, Theorem 2] or
for the result in infinite dimensions), the equation (1.1) with \( B = 0 \) has a unique mild solution. Moreover, in this case the solution is non-explosive.

(b) In general, take \( \psi \in C_b^\infty([0, \infty)) \) such that \( 0 \leq \psi \leq 1, \psi(r) = 1 \) for \( r \in [0, 1] \) and \( \psi(r) = 0 \) for \( r \geq 2 \). For any \( m \geq 1, t \geq 0 \) and \( z \in \mathbb{H} \), let

\[
b_t^{[m]}(z) = b_t \wedge m(z) \psi(|z|/m),
\]

and

\[
Q_t^{[m]}(z) = \begin{cases} Q_t(\psi(|\pi_n z|/m)\pi_n z), & \text{if } Q = Q \circ \pi_n \text{ for some } n \geq 1, \\ Q_t(\psi(|z|/m)z), & \text{otherwise.} \end{cases}
\]

By (a2) we see that (a2') holds for \( B = 0 \) and \( Q^{[m]} \) in place of \( Q \). Moreover, (a3) implies that \( (Q^{[m]}, b^{[m]}) \) satisfies (a3'). Then by (a), (1.1) for \( B = 0 \) and \( (b^{[m]}, Q^{[m]}) \) in place of \( (b, Q) \) has a unique mild solution \( X_t^{(m)} \) starting at \( X_0 \) which is non-explosive. Let

\[
\tau_n = n \wedge \inf\{t \geq 0 : |X_t^{(n)}| \geq n\}, \quad n \geq 1.
\]

Since \( b_s^{[m]}(z) = b_s(z) \) and \( Q_s^{[m]}(z) = Q_s(z) \) hold for \( s \leq m \) and \( |z| \leq m \), by Proposition 3.1 for any \( n, m \geq 1 \) we have \( X_t^{(m)} = X_t^{(m)} \) for \( t \in [0, \tau_n \wedge \tau_m] \). In particular, \( \tau_m \) is increasing in \( m \). Let \( \zeta = \lim_{m \to \infty} \tau_m \) and

\[
X_t = \sum_{m=1}^{\infty} 1_{[\tau_{m-1}, \tau_m)}(t) X_t^{(m)}, \quad \tau_0 := 0, t \in [0, \zeta).
\]

Then it is easy to see that \( (X_t^\zeta)_{t \in [0, \zeta]} \) is a mild solution to (1.1) for \( B = 0 \) with life time \( \zeta \) and, due to Proposition 3.1, the mild solution is unique. We prove Theorem 1.1 (1) for \( B = 0 \).

(c) Let \( \|Q\|_{T, \infty} < \infty \) for \( T > 0 \), and let (1.4) hold for some positive increasing \( \Phi, h \) such that \( \int_1^\infty \frac{ds}{\Phi(s)} = \infty, t \geq 0 \). Let \( (X_t)_{t \in [0, \zeta]} \) be a mild solution to (1.1) for \( B = 0 \) with life time \( \zeta \). Let \( \xi_t = \int_0^t e^{(t-s)A}Q_s(X_s)dw_s \), which is an adapted continuous process on \( \mathbb{H} \) up to the life time \( \zeta \). Then \( Y_t := X_t - \xi_t \) is the mild solution to the equation

\[
dY_t = (AY_t + b_t(Y_t + \xi_t))dt, \quad Y_0 = X_0, \quad t < \zeta.
\]

Due to (1.4) for \( B = 0 \), the increasing property of \( h, \Phi, \) and \( A \leq 0 \), this implies that for any \( T > 0 \),

\[
d|Y_t|^2 \leq 2 \langle Y_t, b_t(Y_t + \xi_t) \rangle dt \leq 2(\Phi_{T \wedge \zeta}(|Y_t|^2) + h_{T \wedge \zeta}(|\xi_t|))dt, \quad |Y_0|^2 = |X_0|^2, t < \zeta \wedge T.
\]

Letting

\[
(5.1) \quad \Psi_T(s) = \int_1^s \frac{dr}{2\Phi_{T \wedge \zeta}(r)}, \quad \alpha_T = |X_0|^2 + 2 \int_0^{T \wedge \zeta} h_{T \wedge \zeta}(|\xi_s|)ds, \quad T > 0,
\]
we obtain
\[ |Y_t|^2 \leq \alpha_T + 2 \int_0^t \Phi_T(|Y_r|^2) dr, \quad T > 0, t \in [0, T \wedge \zeta). \]

By Bihari’s inequality, this implies
\[ |Y_t|^2 \leq \Psi_T^{-1}(\Psi_T(\alpha_t) + t), \quad T > 0, t \in [0, \zeta \wedge T). \]

Moreover, (a1) and \( \|Q\|_{T, \infty} < \infty \) yield
\[ \mathbb{P} \sup_{t \in [0, T \wedge \zeta)} |\xi_t|^2 < \infty, \quad T > 0, \]
so that on the set \( \{ \zeta < \infty \} \) we have \( \mathbb{P}\)-a.s.
\[ \limsup_{t \uparrow \zeta} |Y_t| = \limsup_{t \uparrow \zeta} |X_t| = \infty. \]

We conclude that \( \mathbb{P}(\zeta < \infty) = 0 \), i.e. \( X_t \) is non-explosive. Indeed, on the set \( \{ \zeta \leq T \}, \) (5.3) implies \( \mathbb{P}\)-a.s.
\[ \alpha_T = |X_0|^2 + 2 \int_0^\zeta h(\xi_s) ds < \infty, \]
so that (5.2) and (5.4) imply
\[ \infty = \limsup_{t \uparrow \zeta} |Y_t|^2 \leq \Psi_T^{-1}(\Psi_T(\alpha_T) + T) < \infty, \]
where the last step is due to the fact that \( \Psi_T(r) \uparrow \infty \) as \( r \uparrow \infty \), which implies \( \Psi_T^{-1}(r) \leq \infty \)
for any \( r \in (0, \infty) \). This contradiction means that \( \mathbb{P}(\zeta \leq T) = 0 \) holds for all \( T \in (0, \infty) \).

Hence, \( \mathbb{P}(\zeta < \infty) = 0 \).

Finally, let \( \alpha_T(x) \) be defined in (5.1) as \( \alpha_T \) for \( X_0 = x \). By the local boundedness of
\( \|Q_t\|_{\infty} \), \( \alpha_T(x) \) is \( \mathbb{P}\)-a.s. locally bounded on \([0, \infty) \times \mathbb{H}\). Then, applying (5.2) to \( X_0 = y \) we conclude that for any \( x \in \mathbb{H} \) and \( T > 0 \), \( \mathbb{P}\)-a.s.
\[ \Xi^x := \sup_{t \in [0, T], |y-x| \leq 1} |X_t^y| < \infty, \]
where \( X_t^y \) is the mild solution for \( X_0 = y \). Let \( X_t^{(n,z)} \) solve (1.1) with \( X_0 = z \) for \( B = 0 \) and
\( (b^{[n]}, Q^{[n]}) \) in place of \( (b, Q) \), and let
\[ P_t^{(n)} f(z) = \mathbb{E} f(X_t^{(n,z)}), \quad t \geq 0, \ f \in \mathcal{B}(\mathbb{H}), \ z \in \mathbb{H}, \ n \geq 1. \]

By Proposition 4.1, \( P_T^{(n)} \) is strong Feller. Since \( X_t^{(n,z)} = X_t^z \) for \( t \leq \tau_n^z \), where \( \tau_n^z := n \wedge \inf\{ t \geq 1 : |X_t^{(n,z)}| \geq n \} \), it follows that
\[ |P_T f(y) - P_T f(x)| \leq |P_T^{(n)} f(y) - P_T^{(n)} f(x)| + 2 \|f\|_{\infty}(\mathbb{P}(\tau_n^x \leq T) + \mathbb{P}(\tau_n^y \leq T)) \]
\leq |P_T^{(n)} f(y) - P_T^{(n)} f(x)| + 4 \|f\|_{\infty}(\Xi^x \geq n), \ n > T, |y-x| \leq 1.

Since \( P_T^{(n)} \) is strong Feller, this implies
\[ \limsup_{y \to x} |P_T f(y) - P_T f(x)| \leq 4 \|f\|_{\infty}(\Xi^x \geq n), \ n > T, \ f \in \mathcal{B}(\mathbb{H}). \]

Letting \( n \to \infty \) and using (5.5), we obtain \( \limsup_{y \to x} |P_T f(y) - P_T f(x)| = 0 \) for \( f \in \mathcal{B}(\mathbb{H}) \).

Thus, \( P_T \) is strong Feller.
6 Proof of Theorem 1.2

Throughout this section, we assume (a1), (a2’) and (a3’). The idea of the proof is to transform (1.1) into an equation with regular coefficients, so that gradient estimates for the solution of the new equation can be derived. To this end, we use the regularization representation (2.30). Let us fix \( T > 0 \). By Lemma 2.3, we take large enough \( \lambda(T) > 0 \) such that for any \( \lambda \geq \lambda(T) \) the unique solution \( u \) to (2.4) satisfies

\[
\|\nabla^2 u\|_{T,\infty} + \|\nabla u\|_{T,\infty} \leq \frac{1}{8}.
\]

By (6.1), for any \( t \in [0,T] \),

\[
H \ni x \mapsto \theta_t(x) := x + u_t(x) \in H
\]
is a diffeomorphism with

\[
\frac{7}{8} \leq \|\nabla \theta\|_{T,\infty} \leq \frac{9}{8}, \quad \frac{8}{9} \leq \|\nabla \theta^{-1}\|_{T,\infty} \leq \frac{8}{7}.
\]

Now, let \( X_t^x \) solve (1.1) for \( X_0 = x \). By (2.30), \( Y_t^x := \theta_t(X_t^x) \) satisfies

\[
Y_t^x = e^{tA}Y_0^x + \int_0^t e^{(t-s)A}\{ (\lambda - A)u_s + B_s + \nabla B_s u_s \} \circ \theta_s^{-1}(Y_s^x)ds
\]

\[
+ \int_0^t e^{(t-s)A}\{ Q_s + (\nabla u_s)Q_s \} \circ \theta_s^{-1}(Y_s^x) dW_s, \quad t \in [0,T].
\]

Thus, letting

\[
\tilde{b}_t = \{ B_t + \nabla B_t u_t + (\lambda - A)u_t \} \circ \theta_t^{-1}, \quad \tilde{Q}_t = \{ Q_t + (\nabla u_t)Q_t \} \circ \theta_t^{-1},
\]

\( X_t^x := Y_t^{\theta_0^{-1}(x)} \) is a mild solution to the equation

\[
d\tilde{X}_t^x = A\tilde{X}_t^x dt + \tilde{b}_t(\tilde{X}_t^x) dt + \tilde{Q}_t(\tilde{X}_t^x) dW_t, \quad t \in [0,T], \tilde{X}_0^x = x.
\]

Let \( \tilde{P}_t f(x) = \mathbb{E} f(\tilde{X}_t^x) \). We have

\[
\tilde{P}_t f(x) := \mathbb{E} f(\tilde{X}_t^x) = \mathbb{E} (f \circ \theta_t^{-1})(Y_t^x)
\]

\[
= \mathbb{E} (f \circ \theta_t^{-1})(\tilde{X}_t^{\theta_0(x)}) = (\tilde{P}_t f \circ \theta_t)(\theta_0(x)), \quad f \in \mathcal{B}_b(H), t \in [0,T], x \in H.
\]

We first study gradient estimates and the log-Harnack inequality for \( \tilde{P}_t \). To this end, one may wish to apply the corresponding results derived recently in (23). However, in the present case the assumption (A1) in (23) is not available, i.e. our conditions do not imply the existence of \( K \in L^2([0,T];dt) \) such that

\[
|e^{tA}(\tilde{b}_s(x) - \tilde{b}_s(y))| \leq K(t)|x - y|, \quad t, s \in [0,T], x, y \in H.
\]
Lemma 6.1. Assume (a1), (a2') and (a3'). For any $T > 0$ and large enough $\lambda > \lambda(T)$, there exists a constant $C > 0$ such that the following assertions hold.

1. If in addition $\|B\|_{T, \infty} < \infty$, then (6.7) holds.
2. $|\nabla \tilde{P}_t^{(n)} f|^2 \leq C \tilde{P}_t^{(n)} |\nabla f|^2$, $n \geq 1, t \in [0, T], f \in C^1_b(\mathbb{H}_n)$.
3. $\frac{1}{C} |\nabla \tilde{P}_t^{(n)} f|^2 \leq \tilde{P}_t^{(n)} f^2 - (\tilde{P}_t^{(n)} f)^2 \leq C t \tilde{P}_t^{(n)} |\nabla f|^2$, $n \geq 1, t \in [0, T], f \in C^1_b(\mathbb{H}_n)$.
4. $\tilde{P}_t^{(n)} \log f(y) \leq \log \tilde{P}_t^{(n)} f(x) + \frac{C|x-y|^2}{t}$, $n \geq 1, t \in [0, T], 0 < f \in \mathcal{B}_b(\mathbb{H}_n)$.

Proof. For simplicity, we omit $x$ and $\pi_n x$ from the subscripts, i.e. we write $(\tilde{X}_t, \tilde{X}_t^{(n)})$ instead of $(\tilde{X}_t^x, \tilde{X}_t^{(n, \pi_n x)})$. The essential part of the proof is for (1) and (2), since (3) and (4) can be deduced from (2) by using standard arguments.

By (6.4) and (6.6), we have

\[ I_n := \mathbb{E} \int_0^T e^{-2pt} |\tilde{X}_t - \tilde{X}_t^{(n)}|^2 dt \]
\[ \leq 5|x - \pi_n x|^2 + 5 \mathbb{E} \int_0^T e^{-2pt} \left| \int_0^t e^{(t-s)A} \left\{ \tilde{b}_s(\tilde{X}_s) - \tilde{b}_s^{(n)}(\tilde{X}_s) \right\} ds \right|^2 dt \]
\[ + 5 \mathbb{E} \int_0^T e^{-2pt} \left( \int_0^t e^{(t-s)A} \left\{ \tilde{b}_s^{(n)}(\tilde{X}_s) - \tilde{b}_s^{(n)}(\tilde{X}_s^{(n)}) \right\} ds \right)^2 dt \]
\[ + 5 \mathbb{E} \int_0^T e^{-2pt} \int_0^t \left\| e^{(t-s)A} \left\{ \tilde{Q}_s(\tilde{X}_s) - \tilde{Q}_s^{(n)}(\tilde{X}_s) \right\} \right\|_{HS}^2 ds \]
\[ + 5 \mathbb{E} \int_0^T e^{-2pt} \int_0^t \left\| e^{(t-s)A} \left\{ \tilde{Q}_s^{(n)}(\tilde{X}_s) - \tilde{Q}_s^{(n)}(\tilde{X}_s^{(n)}) \right\} \right\|_{HS}^2 ds, \quad p \geq 1. \]
Obviously, (a1), (a2') and (a3') imply sup_{t \in [0,T], n \geq 1} \mathbb{E}(|X_t|^2 + |\dot{X}_t^{(n)}|^2) < \infty, so that

\[ I := \limsup_{n \to \infty} I_n < \infty. \]

Moreover, (a2') and (6.1) imply that \( \bar{Q} \) is bounded. So, it follows from (6.4), (6.2), sup_{t \in [0,T]} \mathbb{E}|X_t|^2 < \infty, and (a1) that

\[
\begin{aligned}
&\sup_{t \in [0,T]} \mathbb{E} \left| \int_0^t e^{(t-s)A} \dot{b}_s(\bar{X}_s) ds \right|^2 \\
&\leq 3 \sup_{t \in [0,T]} \left\{ \mathbb{E}|\dot{X}_t|^2 + |\bar{X}_0|^2 + \mathbb{E} \left| \int_0^t e^{(t-s)A} \bar{Q}_s(\bar{X}_s) dW_s \right|^2 \right\} \\
&\leq 3 \sup_{t \in [0,T]} \left\{ \|\nabla \theta\|_{T,\infty}^2 \mathbb{E}|X_t^{\theta_0^{-1}(x)}|^2 + |\theta_0^{-1}(x)|^2 + \|\bar{Q}\|_{T,\infty}^2 \int_0^t \|e^{(t-s)A}\|_{HS}^2 ds \right\} < \infty.
\end{aligned}
\]

Thus, by the dominated convergence theorem, (6.8) implies

\[
I \leq \limsup_{n \to \infty} \mathbb{E} \int_0^T e^{-2pt} \left| \int_0^t e^{(t-s)A} \{ \dot{b}_s^{(n)}(\bar{X}_s) - \dot{b}_s^{(n)}(\bar{X}_s^{(n)}) \} ds \right|^2 dt \\
+ 5 \limsup_{n \to \infty} \mathbb{E} \int_0^T e^{-2pt} \int_0^t \|e^{(t-s)A} \{ \dot{Q}_s^{(n)}(\bar{X}_s) - \dot{Q}_s^{(n)}(\bar{X}_s^{(n)}) \} \|_{HS}^2 ds =: I' + I''.
\]

By (2.22) and (a2') we have

\[ \sup_{n \geq 1} \|\nabla \bar{Q}^{(n)}\|_{T,\infty}^2 \leq C_1 \]

for some constant \( C_1 > 0 \), so that

\[
I'' \leq 5C_1 \limsup_{n \to \infty} \int_0^T e^{-2pt} \int_0^t \|e^{(t-s)A}\|_{HS}^2 \mathbb{E}|\bar{X}_s - \bar{X}_s^{(n)}|^2 ds \\
\leq 5C_1 \limsup_{n \to \infty} \int_0^T e^{-2pt} \mathbb{E}|\bar{X}_s - \bar{X}_s^{(n)}|^2 ds \int_0^T \|e^{(t-s)A}\|_{HS}^2 e^{-2p(t-s)} dt \leq c(p) I,
\]

where according to (a1),

\[ c(p) := 5C_1 \int_0^T \|e^{tA}\|_{HS}^2 e^{-2pt} dt \to 0 \text{ as } p \to \infty. \]
Taking large enough $p > 1$ such that $c(p) \leq \frac{1}{2}$, and substituting this into (6.10), we arrive at

$$I \leq 10 \limsup_{n \to \infty} \sum_{i=1}^{\infty} \mathbb{E} \int_{0}^{T} e^{-2pt} \left| \int_{0}^{t} e^{-(t-s)(p+\lambda_i)} (\bar{b}_s(\bar{X}_s) - \bar{b}_s(\bar{X}_s^{(n)}), e_i) ds \right|^2 \leq 10 \limsup_{n \to \infty} \sum_{i=1}^{\infty} \mathbb{E} \int_{0}^{T} \left( \int_{0}^{t} e^{-(t-s)(p+\lambda_i)} (p + \lambda_i) ds \right) \times \left( \int_{0}^{t} e^{-(t-s)(p+\lambda_i)} - 2ps \right) \left( \bar{b}_s(\bar{X}_s) - \bar{b}_s(\bar{X}_s^{(n)}), e_i \right)^2 ds \leq 10 \limsup_{n \to \infty} \sum_{i=1}^{\infty} \mathbb{E} \int_{0}^{T} \left( \frac{e^{-2ps}}{(p + \lambda_i)^2 - 2ps} (\bar{b}_s(\bar{X}_s) - \bar{b}_s(\bar{X}_s^{(n)}), e_i) \right)^2 ds.$$

(6.12)

Noting that $\langle \bar{b}_s, e_i \rangle = (\lambda + \lambda_i) \langle u_s \circ \theta_s^{-1}, e_i \rangle + \langle (B_s + \nabla B_s u_s) \circ \theta_s^{-1}, e_i \rangle$, for $p \geq \lambda$ this implies

$$I \leq 20 \limsup_{n \to \infty} \mathbb{E} \int_{0}^{T} e^{-2ps} \left| u_s \circ \theta_s^{-1}(\bar{X}_s) - u_s \circ \theta_s^{-1}(\bar{X}_s^{(n)}) \right|^2 ds + 20 \limsup_{n \to \infty} \mathbb{E} \int_{0}^{T} e^{-2ps} \left| (B_s + \nabla B_s u_s) \circ \theta_s^{-1}(\bar{X}_s) - (B_s + \nabla B_s u_s) \circ \theta_s^{-1}(\bar{X}_s^{(n)}) \right|^2 ds \leq 20 \left( \frac{\|\nabla(u \circ \theta^{-1})\|_{T,\infty}^2 + \|\nabla(B + \nabla B u) \circ \theta^{-1}\|_{T,\infty}^2}{p^2} \right) I \leq \frac{1}{2} I.$$

(6.13)

for large enough $p \geq \lambda$, since due to (6.1) and (6.2) we have $\|\nabla(u \circ \theta^{-1})\|_{T,\infty} \leq \frac{l}{7}$. Combining this with (6.9), we prove $I = 0$ which is equivalent to (6.7).

(2) By (6.6), (6.11) and (a1), we have

$$\mathbb{E} [\bar{X}_t^{(n,x)} - \bar{X}_t^{(n,y)}]^2 \leq 3|x - y|^2 + 3 \int_{0}^{t} e^{(t-s)A} \left\{ \bar{b}_s^{(n)}(\bar{X}_s^{(n,x)}) - \bar{b}_s^{(n)}(\bar{X}_s^{(n,y)}) \right\} ds \leq C_2 \int_{0}^{t} \mathbb{E} [\bar{X}_s^{(n,x)} - \bar{X}_s^{(n,y)}]^2 ds, \quad t \in [0, T], n \geq 1, x, y \in \mathbb{H}_n$$

(6.14)

for some constant $C_2 > 0$. On the other hand, similarly to (6.12) and (6.13), for large enough $p > 0$, there holds

$$3 \mathbb{E} \int_{0}^{T} e^{-2pt} \left| \int_{0}^{t} e^{(t-s)A} \left\{ \bar{b}_s^{(n)}(\bar{X}_s^{(n,x)}) - \bar{b}_s^{(n)}(\bar{X}_s^{(n,y)}) \right\} ds \right|^2 dt \leq \frac{1}{2} \int_{0}^{T} e^{-2pt} \mathbb{E} [\bar{X}_t^{(n,x)} - \bar{X}_t^{(n,y)}]^2 dt, \quad n \geq 1, x, y \in \mathbb{H}_n.$$
Combining this with (6.14), we obtain
\[
\int_0^t e^{-2ptE} |X_t^{(n,x)} - X_t^{(n,y)}|^2 dt \\
\leq 6 \int_0^T e^{-2pt} |x-y|^2 dt + 2C_2 \int_0^T e^{-2pt} dt \int_0^T \mathbb{E}|X_s^{(n,x)} - X_s^{(n,y)}|^2 ds \\
\leq 6 \int_0^T e^{-2pt} |x-y|^2 dt + 2C_2 \int_0^T e^{-2pt} \mathbb{E}|X_s^{(n,x)} - X_s^{(n,y)}|^2 ds \int_s^T e^{-2p(t-s)} dt \\
\leq 6 \int_0^T e^{-2pt} |x-y|^2 dt + \frac{C_2}{p} \int_0^T e^{-2pt} \mathbb{E}|X_t^{(n,x)} - X_t^{(n,y)}|^2 dt.
\]
Taking large enough \( p_0 = 2C_2 \), such that \( \frac{C_2}{p} \leq \frac{1}{2} \) for \( p \geq p_0 \), we obtain
\[
\int_0^T e^{-2ptE} |X_t^{(n,x)} - X_t^{(n,y)}|^2 dt \leq C|x-y|^2 \int_0^T e^{-2pt} dt, \quad n \geq 1, x, y \in \mathbb{H}_n, p \geq p_0
\]
for some constant \( C > 0 \). Since a finite measure on \([0,T]\) is determined by its Laplace transform, this implies that for any \( n \geq 1 \) and \( x, y \in \mathbb{H}_n \), \( \mathbb{E}|X_t^{(n,x)} - X_t^{(n,y)}|^2 \leq C|x-y|^2 \) holds for a.e. \( t \in [0,T] \). By the continuity of \( \mathbb{E}|X_t^{(n,x)} - X_t^{(n,y)}|^2 \) in \( t \), we prove
\[
\mathbb{E}|X_t^{(n,x)} - X_t^{(n,y)}|^2 \leq C|x-y|^2, \quad t \in [0,T], x, y \in \mathbb{H}_n, n \geq 1.
\]
Then for any \( f \in C_b^1(\mathbb{H}_n) \) and \( (t,x) \in [0,T] \times \mathbb{H}_n \),
\[
|\nabla \bar{P}_t^{(n)} f(x)|^2 := \lim_{y \rightarrow x} \sup_{y \rightarrow y} \frac{|\bar{P}_t^{(n)} f(x) - \bar{P}_t^{(n)} f(y)|^2}{|x-y|^2} \leq \lim_{y \rightarrow x} \sup_{y \rightarrow y} \frac{\mathbb{E}|f(X_t^{(n,x)} - f(X_t^{(n,y)}))^2}{|x-y|^2} \leq \{ \bar{P}_t^{(n)} |\nabla f|^2(x) \} \lim_{y \rightarrow x} \sup_{y \rightarrow y} \frac{\mathbb{E}|X_t^{(n,x)} - X_t^{(n,y)}|^2}{|x-y|^2} \leq C \bar{P}_t^{(n)} |\nabla f|^2(x).
\]
(3) By an approximation argument, we may and do assume that \( \bar{b}^{(n)} \in C([0,T];C_b^2(\mathbb{H}_n;\mathbb{H}_n)) \) and \( f \in C^2(\mathbb{H}_n) \). By (a2') and (2.22), there exist constants \( c_1, c_2 > 0 \) such that
\[
(6.15) \quad c_2 I^{(n)} \geq \bar{Q}_t^{(n)}(\bar{Q}_t^{(n)})^* \geq c_1 I^{(n)},
\]
where \( I^{(n)} \) is the identity on \( \mathbb{H}_n \). Let \( \bar{P}_{s,t}^{(n)} \) be the in-homogenous Markov semigroup associated to (6.8). We have
\[
P_t^{(n)} f^2 - (P_t^{(n)} f)^2 = \int_0^t \frac{d}{ds} P_s^{(n)} (P_s^{(n)} f)^2 ds = \int_0^t P_s^{(n)} (Q_s^{(n)} (Q_s^{(n)})^* \nabla P_s^{(n)} f, \nabla P_s^{(n)} f) ds.
\]
Combining this with (6.15) and (2), we prove (3). For instance, regarding \( s \) as the starting time, we see that (2) also holds for \( \bar{P}_{s,t}^{(n)} \) in place of \( \bar{P}_t^{(n)} \), so that
\[
P_t^{(n)} f^2 - (P_t^{(n)} f)^2 \leq c_2 \int_0^t \bar{P}_s^{(n)} |\nabla \bar{P}_s^{(n)} f|^2 ds \\
\leq c_2 C \int_0^t \bar{P}_s^{(n)} \bar{P}_s^{(n)} |\nabla f|^2 ds = c_2 C \bar{P}_t^{(n)} |\nabla f|^2.
\]
(4) As in (3), we assume that \( \bar{b}^{(n)} \in C([0,T]; C^2_b(\mathbb{R}_n; \mathbb{R}_n)) \). It suffices to prove for \( f \in C^2_b(\mathbb{R}_n) \) which is strictly positive such that \( \nabla f = 0 \) outside a bounded set. Take \( \gamma_s = x + \frac{s}{t}(y - x), s \in [0,t] \). By (6.15) and (2), we have

\[
\frac{d}{ds} \bar{P}_s^{(n)} \log \bar{P}_s^{(n)} f(\gamma_s) = -P_s(\bar{Q}_s^{(n)}(\bar{Q}_s^{(n)})^*) \nabla \log \bar{P}_s^{(n)} f(\gamma_s) + \frac{1}{t}(y - x, \nabla P_s^{(n)} \log \bar{P}_s^{(n)} f(\gamma_s)) \leq \frac{|x - y|}{t} \left| \nabla \bar{P}_s^{(n)} f(\gamma_s) - C_3 \bar{P}_s^{(n)} \right| \left| \nabla \bar{P}_s^{(n)} f(\gamma_s) \right|^2 \leq \frac{|x - y|^2}{(2C_3t)^2}, s \in [0,t]
\]

for some constant \( C_3 > 0 \). Integrating over \([0,t]\) we prove (4) for some constant \( C > 0 \). □

**Proof of Theorem 1.2** (a) We first assume that \( \|B\|_{L_\infty} < \infty \) for \( T > 0 \). In this case we observe that due to Lemma 6.1(1), assertions in Lemma 6.1 (2)-(4) hold for \( \bar{P}_t \) instead of \( P_t \). To save space we only prove the first inequality in (3), the proofs for others are similar and even simpler. Let \( f \in C^1_b(\mathbb{R}) \). Since \( \bar{P}_t 1 = 1 \), this inequality is equivalent to

\[
|\nabla \bar{P}_t f|^2 \leq \frac{C}{t} P_t f^2.
\]

Since \( P_t f \in C_b(\mathbb{R}) \) which is true even for \( f \in \mathcal{B}_b(\mathbb{R}) \) according to the strong Feller property of \( P_t \) and the relation (6.5), this inequality follows from

\[
\frac{|\bar{P}_t f(x) - \bar{P}_t f(y)|^2}{|x - y|^2} \leq \frac{C}{t} \int_0^1 \bar{P}_t f^2(x + s(y - x))ds.
\]

By Lemma 6.1(3), we have

\[
\frac{|(\bar{P}_t^{(n)} f)(\pi_n x) - (\bar{P}_t^{(n)} f)(\pi_n y)|^2}{|x - y|^2} \leq \left( \int_0^1 |\nabla \bar{P}_t^{(n)} f| \circ \pi_n(x + s(y - x))ds \right)^2 \leq \frac{C}{t} \int_0^1 (\bar{P}_t^{(n)} f^2) \circ \pi_n(x + s(y - x))ds.
\]

Moreover, Lemma 6.1(1) implies

\[
\lim_{n \to \infty} \int_0^T |\bar{P}_t f - (\bar{P}_t^{(n)} f) \circ \pi_n|^2 dt = 0, \quad f \in C^1_b(\mathbb{R}).
\]

So, by letting \( n \to \infty \) (up to a subsequence) in (6.17) we prove (6.16) for a.e. \( t \in [0,T] \) with respect to the Lebesgue measure. By the continuity of \( \bar{X}_t, \bar{P}_t f \) is continuous in \( t \). Therefore, (6.16) holds for all \( t \in [0,T] \).

Now, according to (6.5) and (6.2), we only need to prove Theorem 1.2 for \( \bar{P}_t f \) in place of \( P_t \). By the above observation, Theorem 1.2(1) as well as (1.6) and (1.7) with \( t \in (0,1] \) hold for \( \bar{P}_t \) in place of \( P_t \). Then the proof is complete by the following two facts: (a) Due to the semigroup property and Jensen’s inequality, if (1.6) and (1.7) hold for \( t \in (0,1] \), then they also hold for all \( t > 0 \); (b) According to [22 Proposition 1.3], (1.6) is equivalent to (1.8).
(b) In general, let \( \tilde{P}^{(n)}_t \) be the semigroup associated to (1.1) for \( \tilde{B}^{(n)} := B \circ \psi_n \) in place of \( B \), where
\[
\psi_n(x) := \left(1 \wedge \frac{n}{|x|}\right)x, \quad x \in \mathbb{H}.
\]
Then \( \|\tilde{B}^{(n)}\|_{T,\infty} < \infty \) for \( T > 0 \) and \((a2')\) holds for \( \tilde{B}^{(n)} \) in place of \( B \) with the same function \( \Psi \). According to (a), assertions in Lemma 6.1 (2)-(4) hold for \( \tilde{P}^{(n)}_t \) in place of \( \bar{P}^{(n)}_t \). Moreover, by the uniqueness and non-explosion of solutions to the equation \((
abla \nabla \nabla \nabla)\), we have
\[
\lim_{n \to \infty} \tilde{P}^{(n)}_t f = P_t f, \quad t \geq 0, \quad f \in C_b(\mathbb{H}).
\]
Therefore, as explained in (a) that these assertions also hold for \( P_t \).

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