ON RANDOM WALKS IN LARGE COMPACT LIE GROUPS

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1. Introduction

In order to put the problem considered in this Note in perspective, we first recall some other relatively recent results around spectral gaps and generation in Lie groups.

It was shown in [B-G1] (resp. [B-G2]) that if Λ is a symmetric finite subset of SU(2) (resp. SU(d)) consisting of algebraic elements, such that the countable group Γ = ⟨Λ⟩ generated by Λ is dense, then the corresponding averaging operators

\[ T f = \frac{1}{|Λ|} \sum_{g \in Λ} f \circ g \]  

acting on \( L^2(G) \), has a uniform spectral gap (only depending on Λ). This result was generalized in [dS-B] to simple compact Lie groups.

It is not known if the assumption for Λ to be algebraic is needed, and one may conjecture that it is not. Short of providing uniform spectral gaps, Varju [V] established the following property which is the most relevant statement for what follows.

Proposition 1. Let \( G \) be a compact Lie group with semisimple connected component. Let \( μ \) be a probability measure on \( G \) such that \( \text{supp}(\tilde{μ} * μ) \), \( \tilde{μ} \) defined by \( \int f(x) d\tilde{μ}(x) = \int f(x^{-1}) dμ(x) \), generates a dense subgroup of \( G \). Then there is a constant \( c > 0 \) depending only on \( μ \) such that the following holds.

Let \( ϕ \in \text{Lip}(G), \|ϕ\|_2 = 1 \) and \( \int_G ϕ = 0 \). Then

\[ \left\| \int ϕ(h^{-1}g) dμ(h) \right\|_2 < 1 - c \log^{-A}(1 + \|ϕ\|_{\text{Lip}}) \]  

with \( A \) depending on \( G \).

Using (1.2) and decomposition of the regular representation of \( G \) in irreducible (though this may be avoided), one deduces easily from (1.2) that it takes time at most \( O(\log^A \frac{1}{ε}) \) as \( ε \to 0 \) for the random walk governed by \( μ \) to produce an \( ε \)-approximation of uniform measure on \( G \). Note that for \( G = U(d) \), this statement

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corresponds to the Solovay-Kitaev estimates on generation, cf. [D-N], which in fact turns out to be equivalent.

Let us focus on \( G = O(d) \) or \( U(d) \). While the exponent \( A \) in (1.2) is a constant, the prefactor \( c \) depends on \( \mu \), hence on \( G \), and seems to have received little attention. In this Note, we consider families of concrete random walks on \( O(d) \) or \( U(d) \) and are interested in the behaviour of \( c \) when \( d \to \infty \). The particular problem was brought to the author’s attention by T. Spencer (who was motivated by issues in random matrix theory that will not be pursued here). The general setting is as follows (we consider the \( U(d) \)-version). Identify \( d \) with the cyclic group \( \mathbb{Z}/d\mathbb{Z} \) and denote \( \nu_j \) the normalized Haar measure on \( U(2) \) acting on the space \( \{e_i, e_j\} \). Consider the random walk on \( U(d) \) given by

\[
Tf(x) = \frac{1}{d} \sum_{i=0}^{d-1} \int f(gx)\nu_{i,i+1}(dg)
\]

(variants may be treated similarly).

How long does it take for this random walk to become an \( \varepsilon \)-approximation of uniform measure on \( G \), with special emphasis on large \( d \)? Thus this is a particular instance of the more general issue formulated in the title. While we are unable to address the broader problem, specific cases may be analyzed in a satisfactory way (based partly on arguments that are also relevant to the general setting).

We prove

**Proposition 2.** In the above setting, \( \varepsilon \)-approximation of the uniform measure is achieved in time \( C(d \log \frac{1}{\varepsilon})^C \), with \( c \) a constant independent of \( d \).

Basically, one could expect this to be a more general phenomenon (though some additional assumptions are clearly needed). In some sense, it would give a continuous version of the conjecture of Babai and Seress [B-S] predicting poly-logarithmic diameter for the family of non-Abelian finite simple groups (independently of the choice of generators). Important progress in this direction for the symmetric group appears in [H-S]. Independently of Spencer’s question, related spectral gap and mixing time issues for specific random walks in large (not necessarily compact) linear groups appear inos in the theory of Anderson localization for ‘quasi-one-dimensional’ methods in Math Phys.

Consider the strip \( \mathbb{Z} \times \mathbb{Z}/d\mathbb{Z} \) and a random Schrödinger operator \( \Delta + \lambda V \) with \( \Delta \) the usual lattice Laplacian on \( \mathbb{Z} \times \mathbb{Z}/d\mathbb{Z} \), \( V \) a random potential and \( \lambda > 0 \) the disorder. This model is wellknown to exhibit pure point spectrum with so-called Anderson localization for the eigenfunctions. The issue here is how the localization length (or equivalently, the Lyapounov exponents in the transfer matrix approach) depend on \( d \) when \( d \to \infty \).
The classical approach based on Furstenberg’s random matrix product theory (acting on extension powers of \( \mathbb{R}^d \)), cf. [B-L], is not quantitative and sheds no light on the role of \( d \). In fact, the first explicit lower bound on Lyapounov exponents seems to appear in [B] (using different techniques based on Green’s function analysis), with, roughly speaking exponential dependence on \( d \) (while the ‘true’ behaviour is believed to be rather of the form \( d^{-C} \)). Clearly understanding the mixing time for the random walk in the symplectic group \( Sp(2d) \) associated to the transfer matrix is crucial. Note that this group is non-compact, which is an added difficulty (for very small \( \lambda \), depending on \( d \), [B-S] provides the precise asymptotic of the exponents, based on a multi-dimensional extension of the Figoli-Pastur approach).

2. SOME PRELIMINARY COMMENTS

The proof of Proposition 1 in [V] exploits the close relation between ‘generation’ and ‘restricted spectral gaps’. This point of view is also the key idea here in establishing

**Proposition 1’** Let \( T \) be defined by (1.3). Then there is the following estimate

\[
\|Tf\|_2 < 1 - (cd)^{-C}(\log(1 + \|f\|_{\text{Lip}}))^{-A}
\]

for \( f \in \text{Lip}(G) \), \( \|f\|_2 = 1 \), \( \int_G f = 0 \).

Here \( C \) and \( A \) are constants (denoted differently, because of their different appearance in the argument).

Unlike in [V], we tried to avoid the use of representation theory. The reason for this is the following. If one relies on decomposition of the regular representation of \( G \) in irreducibles and the Peter-Weyl theorem, one is faced with convergence issues of the generalized Fourier expansion of functions on \( G \) of given regularity. Conversely, we also need to understand the regularity of matrix coefficients of the representations of increasing dimension. While these are classical issues, understanding the role of the dimension \( d \) does not seem to have been addressed explicitly.

3. PROOF OF PROPOSITION 1’

According to (1.3), denote

\[
\nu = \frac{1}{d} \sum_{i=0}^{d-1} \nu_{i,i+1}
\]

Thus \( \nu = \tilde{\nu} \) and \( T \) is the corresponding averaging operator.

Let \( f \in \text{Lip}(G) \), \( \|f\|_2 = 1 \) and \( \int_S f = 0 \). Assuming

\[
\left\| \int \tau_g f \nu(dg) \right\|_2^2 = \|Tf\|_2^2 > 1 - \varepsilon
\]
denoting $(\tau_g f(x) = f(gx))$ our aim is to obtain a lower bound on $\epsilon$.

Clearly (3.2) implies that
\[
\left\langle f, \int \tau_g f(\nu * \nu)(dg) \right\rangle > 1 - \epsilon
\]
and
\[
\int \|f - \tau_g f\|^2_2(\nu * \nu)(dg) < 2\epsilon. \tag{3.3}
\]

Fix $\epsilon_1 > 0$ to be specified later and denote $B_{\epsilon_1}$ an $\epsilon_1$-neighborhood (for the operator norm) of $Id$ in $U(d)$. It is clear from (3.1) that $\nu(B_{\epsilon_1}) \gtrsim \epsilon_1^4$ and hence (3.3) implies
\[
\int \|f - \tau_{g'} f\|^2_2 \nu(dg) \lesssim \epsilon_1^{-4} \epsilon \tag{3.4}
\]
for some $g' \in B_{\epsilon_1}$. Next, partitioning $U(2)$ in $\epsilon_1$-cells $\Omega_{\alpha}$ and denoting $\Omega_{\alpha,i} = \{g \in U(d) ; g(e_j) = e_j \text{ for } j \notin \{i, i + 1\} \text{ and } g|_{[e_i, e_{i+1}]} \in \Omega_{\alpha}\}$ observe that $\nu(\Omega_{\alpha,i}) \geq \frac{1}{4} \epsilon_1^4$ so that by (3.4)
\[
\int_{\Omega_{\alpha,i}} \|f - \tau_{g'} f\|^2_2 \nu(dg) \lesssim d \epsilon_1^{-8} \epsilon \ll 1. \tag{3.5}
\]
Exploiting (3.5), it is clear that we may introduce a collection $\mathcal{G} \subset U(d)$ with the following properties
\[
\|f - \tau_g f\|_2 \lesssim \sqrt{d} \epsilon_1^{-4} \sqrt{\epsilon} \text{ for } g \in \mathcal{G}. \tag{3.6}
\]
and
\[
\|f - \tau_g f\|_2 \lesssim \sqrt{d} \epsilon_1^{-4} \sqrt{\epsilon} \text{ for } g \in \mathcal{G}.
\]

Given an element $\gamma \in U(2)$ and $1 \leq i < j \leq d$, denote $\gamma_{ij} \in U(d)$ the element defined by
\[
\begin{cases}
\gamma_{ij}(e_k) = e_k \text{ for } k \notin \{i, j\} \\
\gamma_{ij}|_{[e_i, e_{i+1}]} = \gamma.
\end{cases}
\tag{3.7}
\]
Then, for each $\gamma \in U(2)$ and $1 \leq i \leq d$, there is $g \in \mathcal{G}$ s.t.
\[
\|g - \gamma_{i,i+1}\|_2 < \epsilon_1. \tag{3.8}
\]

At this point, we will invoke generation. Since $\int_{U(d)} f = 0$,
\[
\int_{U(d)} \|f - \tau_g f\|^2_2 dg = 2
\]
and we take some $h_0 \in U(d)$ s.t.
\[
\|f - \tau_{h_0} f\|_2 \geq \sqrt{2}.
\]
If $\|h_0 - h_1\| < \delta \sim \frac{1}{\|f\|_{\text{Lip}}}$, then
\[
\|\tau_{h_0} f - \tau_{h_1} f\|_2 \leq (\|f\|_{\text{Lip}} \delta)^{\frac{1}{2}} < \frac{1}{2}
\]
and consequently
\[ \| f - \tau h_1 f \|_2 > 1 \text{ of } \| h_0 - h_1 \| < \delta. \] (3.9)

In order to get a contradiction, we need to produce a word \( h_1 = g_1 \cdots g_\ell \); \( g_1, \ldots, g_\ell \in G \) such that
\[ \| h_0 - g_1 \cdots g_\ell \| < \delta \] (3.10)
and
\[ \ell < \frac{\varepsilon_4^4}{\sqrt{d} \sqrt{\varepsilon}} \] (3.11)

Indeed, (3.6) implies then that
\[ \| f - \tau h_1 f \|_2 \leq \| f - \tau g_1 f \|_2 + \cdots + \| f - \tau g_\ell f \|_2 < 1. \]

For \( 1 \leq i < d \), let \( \sigma_{i,i+1} \in \text{Sym}(d) \) be the permutation of \( i \) and \( i+1 \).

Denote \( \tilde{\sigma}_{i,i+1} \) the corresponding unitary operator. Since
\[ \{ \sigma_{i,i+1}; i = 1, \ldots, d-1 \} \]
is a generating set for \( \text{Sym}(d) \) consisting of cycles of bounded length, it follows from a result in [D-F] that the corresponding Cayley graph on \( \text{Sym}(d) \) has diameter at most \( Cd^2 \). In particular, given \( i, j \not\in \mathbb{Z}/d\mathbb{Z}, i \neq j \), \( \tilde{\sigma}_{i,j} \) may be realized as a composition of a string of elements \( \tilde{\sigma}_{i,i+1} \) of length at most \( Cd^2 \). In view of (3.7), this implies that if \( \gamma \in U(2) \) and \( 1 \leq i < j \leq d \), then
\[ \| \gamma_{ij} - g \| < cd^2 \varepsilon_1 \] (3.12)
for some \( g \in G_{\ell_1}, \ell_1 < cd^2 \).

Let \( \kappa > 0, \)
\[ \kappa^2 > cd^2 \varepsilon_1. \] (3.13)

Adopting the Lie-algebra point of view, the preceding implies that given \( s, t \in \mathcal{R}, |s|, |t| < 1 \) and \( z \in \mathbb{C}, |z| < 1 \), then
\[ \text{dist} \left( I + \kappa (isz(e_i \otimes e_i) + it(e_j \otimes e_j) + z(e_i \otimes e_j) + \bar{z}(e_j \otimes e_i)), \mathcal{G}_{\ell_1} \right) < \kappa^2 \] (3.14)
and therefore
\[ \text{dist} \left( I + \kappa A, \mathcal{G}_{d^2 \ell_1} \right) < d^2 \kappa^2 \] (3.15)
for skew symmetric \( A, \| A \| \leq 2\pi \).

Let \( h \in U(d), h = e^A \) with \( A \) as above. Taking \( \kappa = \frac{1}{r} \), we have
\[ e^A = (e^{rac{1}{r} A})^r = \left( 1 + \frac{1}{r} A \right)^r + O \left( \frac{1}{r} \right) \]
and therefore, by (3.15)
\[ \text{dist} \left( h_0, \mathcal{G}_{rd^2 \ell_1} \right) \leq rd^2 \kappa^2 = \frac{d^2}{r}. \] (3.16)
Taking \( \kappa = \frac{1}{r} = d^{-C} \) and \( \varepsilon_1 = d^{-2C-2} \), (3.16) ensure that
\[ \text{dist} \left( h, \mathcal{G}_{d^2 \varepsilon_1} \right) < d^{-C} \text{ for all } h \in U(d). \] (3.17)
Next, we rely on the Solovay-Kitaev commutator technique to produce approximations at smaller scale, see [?]. This procedure is in fact dimensional free (see the comment in [D-N] following Lemma 2 in order to eliminate a polynomial prefactor in $d$ - which actually would be harmless if we start from scales $\varepsilon_0 = d^{-C}$). The conclusion is that

$$\text{dist} (h, G_\ell) < \tau \quad \text{for all } h \in U(d)$$

may be achieved with

$$\ell < d^{C_1} \left( \log \frac{1}{\tau} \right)^A.$$

Returning to (3.10), (3.11), we obtain the condition

$$d^{C_0} \log^A (1 + \|f\|_{\text{Lip}}) < \frac{\varepsilon_1^A}{\sqrt{\varepsilon} \sqrt{d}} = d^{-C_2} \varepsilon^{-\frac{1}{2}}$$

(3.19)

and Proposition 1′ follows.

4. PROOF OF PROPOSITION 2

The disadvantage of our approach is that $T$ is not restricted to finite dimensional invariant subspaces of $L^2(G)$ so that strictly speaking, one can not rely on a spectral gap argument to control the norm of iterates of $T$.

But Proposition 1′ nevertheless permit to derive easily the following

**Proposition 3.** Assume $f \in \text{Lip}(G)$, $\|f\|_2 = 1$, $\int_G f = 0$. Let $0 < \rho < \frac{1}{2}$. Then

$$\|T^\ell f\|_2 < \rho$$

(4.1)

provided

$$\ell > cd^{C_0} \log^A (1 + \|f\|_{\text{Lip}}) \left( \log \frac{1}{\rho} \right)^{A+1}.$$  

(4.2)

**Proof.** Let $B = \|f\|_{\text{Lip}}$. Clearly $\|T^\ell f\|_{\text{Lip}} \leq B$ also.

Fix some $\ell$ and let $f_1 = \frac{T^\ell f}{\|T^\ell f\|_2}$. Hence $\|f_1\|_{\text{Lip}} \leq \frac{B}{\|T^\ell f\|_2}$.

Applying Proposition 1′, it follows that

$$\|T^{\ell+1} f\|_2 \leq \|T^\ell f\|_2 (1 - \varepsilon_\ell)$$

with

$$\varepsilon_\ell = cd^{-C} \left( \log \left( 1 + \frac{B}{\|T^\ell f\|_2} \right) \right)^{-A} > cd^{-C} \left( \log (1 + B) \right)^{-A} \left( \log \left( 1 + \frac{1}{\|T^\ell f\|_2} \right) \right)^{-A}.$$

Hence, assuming $\|T^\ell f\|_2 > \rho$, we obtain

$$\rho < (1 - cd^{-C} \left( \log (1 + B) \right)^{-A} \left( \log \frac{1}{\rho} \right)^{-A})^\ell$$

implying (4.2).  \qed
Proof of Proposition 2.

Apply Proposition 3 with \( \log B \sim \log \frac{1}{\varepsilon} \) and \( \log \frac{1}{\rho} \sim d^2 \log \frac{1}{\varepsilon} \).

5. Variants

The previous argument is clearly very flexible and may be applied in other situations. For instance in (1.3), one may replace the uniform measure \( \nu_{i,i+1} \) by any finitely supported probability measure on \( SU(2) \) which support is sufficiently dense (meaning that there is \( \varepsilon \)-approximation of arbitrary elements using words of length at most \( C(\log \frac{1}{\varepsilon})^C \), using Solovay-Kitaev for instance). Note that here \( \nu_{i,i+1} \) may not be known to display a uniform spectral gap as convolution operator on \( L^2(U(2)) \).

There is another model that may be treated using these methods and may be closer to T. Spencer’s original question. We now consider time dependent random walks, one example would be the following. At time \( k \in \mathbb{Z}_+ \), introduce \( T_k \) as
\[
\frac{1}{2}(\tau_g + \tau_{g^{-1}})
\]
where we first pick some \( i \in \mathbb{Z}/d\mathbb{Z} \) and then choose a random element \( g \in U_{i,i+1}(2) \) according to uniform measure. In this situation, one obtains random walks on \( U(d) \) indexed by an additional probability space \( \otimes (\mathbb{Z}/d\mathbb{Z} \otimes U(2)) \)
\[
T^\omega = \cdots T_k T_{k-1} \cdots T_1
\] (5.1)
and may ask for the typical mixing time of a realization.

Rather straightforward adjustments of the arguments appearing in the proof of Proposition 1’ combined with some Markovian considerations permit us to establish the analogue of Proposition 2 for \( T^\omega \). Thus

**Proposition 4.** Let \( T^\omega \) be defined by (5.1). Then, with large probability in \( \omega \), \( \varepsilon \)-approximation of uniform measure on \( U(d) \) may be achieved in time \( C(d \log \frac{1}{\varepsilon})^C \).

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P. Varju also reported the following somewhat related question of A. Lubotzky: Does \( SU(d) \) admit a finite set of generators with a spectral gap that is uniform in \( d \)?

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