TOWARDS A CLUSTER STRUCTURE ON TRIGONOMETRIC ZASTAVA

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(WITH AN APPENDIX BY GALYNA DOBROVOLSKA)

Abstract. We study a moduli problem on a nodal curve of arithmetic genus 1, whose solution is an open subscheme in the zastava space for projective line. This moduli space is equipped with a natural Poisson structure, and we compute it in a natural coordinate system. We compare this Poisson structure with the trigonometric Poisson structure on the transversal slices in an affine flag variety. We conjecture that certain generalized minors give rise to a cluster structure on the trigonometric zastava.

1. INTRODUCTION

1.1. Zastava and euclidean monopoles. Let $G$ be an almost simple simply connected algebraic group over $\mathbb{C}$. We denote by $\mathcal{B}$ the flag variety of $G$. Let us also fix a pair of opposite Borel subgroups $B, B_-$ whose intersection is a maximal torus $T$. Let $\Lambda$ denote the cocharacter lattice of $T$; since $G$ is assumed to be simply connected, this is also the coroot lattice of $G$. We denote by $\Lambda^+ \subset \Lambda$ the sub-semigroup spanned by positive coroots.

It is well-known that $H_2(\mathcal{B}, \mathbb{Z}) = \Lambda$ and that an element $\alpha \in H_2(\mathcal{B}, \mathbb{Z})$ is representable by an effective algebraic curve if and only if $\alpha \in \Lambda^+$. Let $\overset{\circ}{\mathbb{Z}}^\alpha$ denote the space of maps $C = \mathbb{P}^1 \to \mathcal{B}$ of degree $\alpha$ sending $\infty \in \mathbb{P}^1$ to $B_- \in \mathcal{B}$. It is known [7] that this is a smooth symplectic affine algebraic variety, which can be identified with the hyperkähler moduli space of framed $G$-monopoles on $\mathbb{R}^3$ with maximal symmetry breaking at infinity of charge $\alpha$ [11], [12].

The monopole space $\overset{\circ}{\mathbb{Z}}^\alpha$ has a natural partial compactification $\overset{\circ}{\mathbb{Z}}^\alpha$ (zastava scheme). It can be realized as the moduli space of based quasi-maps of degree $\alpha$; set-theoretically it can be described in the following way:

$$ Z^\alpha = \bigsqcup_{0 \leq \beta \leq \alpha} \overset{\circ}{\mathbb{Z}}^\beta \times \mathbb{A}^{\alpha-\beta}, $$

where for $\gamma \in \Lambda_+$ we denote by $\mathbb{A}^\gamma$ the space of all colored divisors $\sum \gamma_i x_i$ with $x_i \in \mathbb{A}^1$, $\gamma_i \in \Lambda_+$ such that $\sum \gamma_i = \gamma$.

The zastava space is equipped with a factorization morphism $\pi_\alpha : Z^\alpha \to \mathbb{A}^\alpha$ whose restriction to $\overset{\circ}{\mathbb{Z}}^\alpha \subset Z^\alpha$ has a simple geometric meaning: for a based map $\varphi \in \overset{\circ}{\mathbb{Z}}^\alpha$ the colored divisor $\pi_\alpha(\varphi)$ is just the pullback of the colored Schubert divisor $D \subset \mathcal{B}$ equal to the complement of the open $B$-orbit in $\mathcal{B}$. The morphism $\pi_\alpha : \overset{\circ}{\mathbb{Z}}^\alpha \to \mathbb{A}^\alpha$ is the Atiyah-Hitchin integrable system (with respect to the above symplectic structure): all the fibers of $\pi_\alpha$ are Lagrangian.
A system of étale birational coordinates on $\tilde{Z}^\alpha$ was introduced in [7]. Let us recall the definition for $G = SL(2)$. In this case $\tilde{Z}^\alpha$ consists of all maps $\mathbb{P}^1 \to \mathbb{P}^1$ of degree $\alpha$ which send $\infty$ to 0. We can represent such a map by a rational function $\frac{P}{Q}$ where $Q$ is a monic polynomial of degree $\alpha$ and $R$ is a polynomial of degree $< \alpha$. Let $w_0, \ldots, w_\alpha$ be the zeros of $Q$. Set $y_r = R(w_r)$. Then the functions $(y_1, \ldots, y_\alpha, w_1, \ldots, w_\alpha)$ form a system of étale birational coordinates on $\tilde{Z}^\alpha$, and the above mentioned symplectic form in these coordinates reads $\Omega_{\text{rat}} = \sum_{r=1}^\alpha \frac{dy_r \wedge dw_r}{y_r}$.

For general $G$ the definition of the above coordinates is quite similar. In this case given a point in $\tilde{Z}^\alpha$ we can define polynomials $R_i, Q_i$ where $i$ runs through the set $I$ of vertices of the Dynkin diagram of $G$, $\alpha = \sum a_i \alpha_i$, and

1. $Q_i$ is a monic polynomial of degree $a_i$,
2. $R_i$ is a polynomial of degree $< a_i$.

Hence, we can define (étale, birational) coordinates $(y_{i,r}, w_{i,r})$ where $i \in I$ and $r = 1, \ldots, a_i$. Namely, $w_{i,1}$ are the roots of $Q_i$, and $y_{i,r} = R_i(w_{i,r})$. The Poisson brackets of these coordinates with respect to the above symplectic form are as follows: $\{w_{i,r}, w_{j,s}\}_\text{rat} = 0$, $\{w_{i,r}, y_{j,s}\}_\text{rat} = d_i \delta_{i,j} \delta_{r,s} y_{j,s}$, and finally $\{y_{i,r}, y_{j,s}\}_\text{rat} = (\tilde{\alpha}_i, \tilde{\alpha}_j) \frac{dy_{i,r} \wedge dy_{j,s}}{y_{i,r} - y_{j,s}}$ for $i \neq j$, and finally $\{y_{i,r}, y_{i,s}\}_\text{rat} = 0$. Here $\tilde{\alpha}_i$ is a simple root, $\langle \ , \ \rangle$ is the invariant scalar product on $(\text{Lie } T)^*$ such that the square length of a short root is 2, and $d_i = (\tilde{\alpha}_i, \tilde{\alpha}_i)/2$.

Now recall that the standard rational $r$-matrix for $\mathfrak{g} = \text{Lie } G$ gives rise to a Lie bialgebra structure on $\mathfrak{g}[z^\pm 1]$ corresponding to the Manin triple $\mathfrak{g}[z], z^{-1} \mathfrak{g}[z^{-1}], \mathfrak{g}[z^\pm 1]$. This in turn gives rise to a Poisson structure on the affine Grassmannian $Gr_G = G[z^\pm 1]/G[z]$. The transversal slices $W^\lambda_\mu$ from a $G[z]$-orbit $Gr^\mu_G$ to another orbit $Gr^\lambda_G$ (here $\mu \leq \lambda$ are dominant coweights of $G$) are examples of symplectic leaves of the above Poisson structure. According to [11], the zastava spaces are “stable limits” of the above slices. More precisely, for $\alpha = \lambda - \mu$ there is a birational Poisson map $s^\lambda_\mu : W^\lambda_\mu \to Z^\alpha$.

### 1.2. Trigonometric zastava and monopoles.

We have an open subset $\mathbb{G}^\alpha_m \subset \mathbb{A}^\alpha$ (colored divisors not meeting 0 in $\mathbb{A}^1$), and we introduce the open subscheme of trigonometric zastava $\mathbb{Z}^\alpha := \pi_\alpha^{-1} \mathbb{G}^\alpha_m \subset Z^\alpha$, and its smooth open affine subvariety of trigonometric monopoles $\mathcal{Z}^\alpha := \mathbb{Z}^\alpha \cap Z^\alpha$. These schemes are solutions of the following modular problems.

Let $C^\dagger$ be an irreducible nodal curve of arithmetic genus 1 obtained by gluing the points $0, \infty \in C = \mathbb{P}^1$, so that $\pi : C \to C^\dagger$ is the normalization. Let $c \in C^\dagger$ be the singular point. The moduli space $\text{Bun}_T^0(C^\dagger)$ of $T$-bundles on $C^\dagger$ of degree 0 is canonically identified with the Cartan torus $T$ itself. We fix a $T$-bundle $\mathcal{F}_T$ which corresponds to a regular point $t \in T_{\text{reg}}$. Then $\mathcal{Z}^\alpha$ is the moduli space of the following data:

1. a trivialization $\tau_t$ of the fiber of $\mathcal{F}_T$ at the singular point $c \in C^\dagger$;
2. a $B$-structure $\phi$ in the induced $G$-bundle $\mathcal{F}_G = \mathcal{F}_T \times G$ of degree $\alpha$ which is transversal to $\mathcal{F}_B = \mathcal{F}_T \times B$ at $c$.

The scheme $\mathbb{Z}^\alpha$ is the moduli space of the similar data with the only difference: we allow a $B$-structure in (b) to be generalized (i.e. to acquire defects at certain points of $C^\dagger$), but require it to have no defect at $c$. 

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As a regular Cartan element $t$ varies, the above moduli spaces become fibers of a single family. More precisely, we consider the following moduli problem:

(t) a $T$-bundle $\mathcal{F}_T$ of degree 0 on $C^1$ corresponding to a regular element of $T$;
(a,b) as above;
(c) a trivialization $f_\ell$ at $c$ of the $T$-bundle $\phi_T$ induced from the $B$-bundle $\phi$ in (b).

This moduli problem is represented by a scheme $\hat{Y}^\alpha \subset Y^\alpha$ (depending on whether the $B$-structure in (b) is genuine or generalized). Note that $Y^\alpha$ is equipped with an action of $T \times T$ changing the trivializations in (a,c). We prove that $\hat{Y}^\alpha$ is a smooth affine variety equipped with a natural projection $\varpi : \hat{Y}^\alpha \to \hat{\mathcal{Z}}^\alpha$, and we construct a nondegenerate bivector field on $\hat{Y}^\alpha$ arising from a differential in a spectral sequence involving the tangent and cotangent bundles of $\hat{Y}^\alpha$ (this construction is a trigonometric degeneration of the construction in § for elliptic curves; its rational analogue was worked out in [17]). This bivector field descends to $\hat{\mathcal{Z}}^\alpha$ under the projection $\varpi : \hat{Y}^\alpha \to \hat{\mathcal{Z}}^\alpha$. The Poisson brackets of the coordinates in the corresponding symplectic structure of $\hat{\mathcal{Z}}^\alpha$ are as follows:

\[
\{w_{i,r}, w_{j,s}\}_{\text{trig}} = 0, \quad \{w_{i,r}, y_{j,s}\}_{\text{trig}} = \delta_i \delta_j \delta_{rs} w_{ij,rs}, \quad \{y_{i,r}, y_{j,s}\}_{\text{trig}} = \left(\delta_i \delta_j \frac{(w_{i,r}+w_{j,s})y_{i,r}+y_{j,s}}{2(w_{i,r}-w_{j,s})}\right)
\]

for $i \neq j$, and finally $\{y_{i,r}, y_{i,s}\}_{\text{trig}} = 0$. In particular, the projection $\pi_\alpha : \hat{\mathcal{Z}}^\alpha \to \mathcal{G}^\alpha_m$ is the trigonometric Atiyah-Hitchin integrable system (for $G = SL(2)$ this system goes back at least to [5]).

Now recall that the standard trigonometric $r$-matrix for $\mathfrak{g}$ gives rise to a Lie bialgebra structure on $\mathfrak{g}((z^{-1}))$ which in turn gives rise to a Poisson structure on the affine flag variety $\mathcal{F}_G$ (the quotient of $G[z^{\pm 1}]$ with respect to an Iwahori subgroup). The symplectic leaves of this Poisson structure are the intersections $\mathcal{F}_\ell^w$ of the opposite Iwahori orbits (aka open Richardson varieties). Here $w, y$ are elements of the affine Weyl group $W_a = W \ltimes \Lambda$. For dominant coweights $\mu \leq \lambda \in \Lambda$, and the longest element $w_0$ of the finite Weyl group $W$, the projection $pr : \mathcal{F}_w^{w_0 \times \lambda} \to \mathcal{W}_\mu^\lambda$ is an open embedding, and the composition $s_\mu^\lambda \circ pr : \mathcal{F}_w^{w_0 \times \lambda} \to Z^\alpha$ is a symplectomorphism onto its image $\hat{\mathcal{Z}}^\alpha \subset Z^\alpha$ equipped with the trigonometric symplectic structure.

1.3. Cluster aspirations. It seems likely that the construction due to B. Leclerc [16] extends from the open Richardson varieties in the finite flag varieties to the case of the affine flag varieties, and provides $\mathcal{F}_\ell^w$ with a cluster structure. This structure can be transferred from $\mathcal{F}_w^{w_0 \times \lambda}$ to $\hat{\mathcal{Z}}^\alpha$ via the above symplectomorphism. In case of $G = SL(2)$ the resulting cluster structure on the moduli space of trigonometric monopoles was discovered in [9], which served as the starting point of the present note. It seems likely that for general $G$ the Gaiotto-Witten superpotential on $Z^\alpha$ (see e.g. [3]) restricted to $\hat{\mathcal{Z}}^\alpha$ is totally positive in the above cluster structure (see Section 5 for details).

Remark 1.4. Implicit in the above discussion when $\alpha$ is dominant (as a coweight of $G$) is an affine open embedding $\hat{\mathcal{Z}}^\alpha \subset Z^\alpha \hookrightarrow \mathbb{A}^{2|\alpha|}$ into an affine space. Here is a modular interpretation: $\mathbb{A}^{2|\alpha|}$ is the moduli space of $B$-bundles $\phi_B$ on $\mathbb{P}^1$ equipped with a trivialization.
(φ_B)_{∞} \sim B of the fiber at ∞ ∈ P^1, such that the induced T-bundle (under projection B → T) has degree α.

1.5. Contents. In Section 2 the moduli spaces τZα ⊂ Zα and τYα ⊂ Yα are introduced. In Section 3 a bivector field on Yα is introduced and the resulting Poisson bracket { , }_{trig} on τZα is computed in coordinates (w_{i,r}, y_{i,r}). The technicalities of this computation occupy the bulk of the present note. In Section 4 we compare the rational Poisson bracket on the slices \( W^p \subset Gr_G \) with the Poisson bracket { , }_{rat} on the monopole moduli space \( \hat{Z}^\alpha \). We also compare the trigonometric Poisson bracket on the slices \( \mathcal{F}_{w_0}^{\mu} \subset \mathcal{F}_G \) with the Poisson bracket { , }_{trig} on the trigonometric monopole moduli space \( \hat{Z}^\alpha \). Section 5 contains a few comments on the cluster structures. Finally, the Appendix Section 6 by Galyna Dobrovolska identifies our cluster structure on \( \hat{Z}^\alpha \) for \( G = SL_2 \) with the one of [9].

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2. Trigonometric zastava

2.1. Line bundles on a nodal curve C^\dagger. Let C^\dagger be an irreducible nodal curve of arithmetic genus 1, and let \( \pi : C \to C^\dagger \) be its normalization. Then C is a projective line. We fix a coordinate function z on C such that the preimage of the node \( \zeta \in C^\dagger \) consists of the points 0, ∞ ∈ C. We have Pic^\dagger(C^\dagger) = G_m: any line bundle on C^\dagger is obtained by descent from the one on C gluing its fibers at 0 and ∞; a degree 0 line bundle on C is trivial, so its fibers at 0 and ∞ are canonically identified. Moreover, the above choice of a coordinate z on C gives rise to an identification Pic^n(C^\dagger) = \( G_m \) for any \( n \in \mathbb{Z} \). If \( n > 0 \), and s is a section of \( \mathcal{L} \in \text{Pic}^n(C^\dagger) \) not vanishing at \( \zeta \), then \( \text{div}(s) \in \text{Sym}^n(C^\dagger \setminus \zeta) = \text{Sym}^n(G_m) \). (This identification makes use of the coordinate z); we have a multiplication morphism \( m : \text{Sym}^n(G_m) \to G_m \), and finally \( \mathcal{L} \mapsto m(\text{div}(s)) \in G_m \) (the result is independent of the choice of a section s). If \( \mathcal{L} \in \text{Pic}^n(C^\dagger) \) for \( n < 0 \), then \( \mathcal{L} \) goes to the inverse of the class of \( \mathcal{L}^{-1} \).

For the canonical line bundle \( \omega_C^{\dagger} \), we have the following exact sequence:

\[
0 \to \omega_C^{\dagger} \to \pi_* \omega_C(\{0\} + \{∞\}) \xrightarrow{\rho_\pi} \mathbb{C}_\zeta \to 0
\]

(2.1)

where \( \rho_\pi(\xi) = \text{Res}_0(\xi) + \text{Res}_\infty(\xi) \). The line bundle \( \omega_C^{\dagger} \) is trivial, with trivializing section \( z^{-1}dz \). In what follows we will freely use the above identification \( \omega_C^{\dagger} \cong \mathcal{O}_{C^\dagger} \).

We define the theta-characteristic \( \theta \in \text{Pic}^0(C^\dagger) \) as a unique nontrivial line bundle such that \( \theta^2 = \omega_C^{\dagger} \). It enters the following exact sequence:

\[
0 \to \theta \to \pi_* \omega_C(\{0\} + \{∞\}) \xrightarrow{\rho_\pi} \mathbb{C}_\zeta \to 0
\]

(2.2)
where $\rho_-(\xi) = \text{Res}_0(\xi) - \text{Res}_\infty(\xi)$.

From the above two sequences we have the natural embeddings $\pi_*\omega_C \hookrightarrow \omega_{C^\dagger} \hookrightarrow \pi_*\omega_C(\{0\} + \{\infty\})$ and $\pi_*\omega_C \hookrightarrow \theta \hookrightarrow \pi_*\omega_C(\{0\} + \{\infty\})$. Noting that $\omega_C(\{0\} + \{\infty\}) = \mathcal{O}_C$, we combine the above embeddings into the following exact sequence:

$$0 \to \pi_*\omega_C \to \theta \oplus \omega_{C^\dagger} \to \pi_*\mathcal{O}_C \to 0 \quad (2.3)$$

We also have natural embeddings $\pi_*\omega_C = \pi_*\mathcal{O}_C(-\{0\} - \{\infty\}) \hookrightarrow \pi_*\pi_*\mathcal{O}_C(-\{0\}) \hookrightarrow \pi_*\mathcal{O}_C$ and $\pi_*\omega_C = \pi_*\mathcal{O}_C(-\{0\} - \{\infty\}) \hookrightarrow \pi_*\pi_*\mathcal{O}_C(-\{\infty\}) \hookrightarrow \pi_*\mathcal{O}_C$. They combine into the following exact sequence:

$$0 \to \pi_*\omega_C \to \pi_*\Xi \to \pi_*\mathcal{O}_C \to 0 \quad (2.4)$$

where $\Xi$ stands for $\mathcal{O}_C(-\{0\}) \oplus \mathcal{O}_C(-\{\infty\})$.

2.2. A group $G$. Let $G$ be an almost simple simply connected algebraic group over $\mathbb{C}$. We denote by $\mathcal{B}$ the flag variety of $G$. Let us also fix a pair of opposite Borel subgroups $B$, $B_-$ whose intersection is a maximal torus $T$ (thus we have $\mathcal{B} = G/B = G/B_-$). We denote by $T^\text{reg} \subset T$ the open subset formed by the regular elements.

Let $\Lambda$ (resp. $\Lambda^\vee$) denote the cocharacter (resp. character) lattice of $T$; since $G$ is assumed to be simply connected, this is also the coroot lattice of $G$. We denote by $\Lambda_+ \subset \Lambda$ the sub-semigroup spanned by positive coroots. We say that $\alpha \geq \beta$ (for $\alpha, \beta \in \Lambda$) if $\alpha - \beta \in \Lambda_+$. The simple coroots are $\{\alpha_i\}_{i \in I}$; the simple roots are $\{\check{\alpha}_i\}_{i \in I}$: the fundamental weights are $\{\check{\omega}_i\}_{i \in I}$. We consider the invariant bilinear form $(,)$ on the weight lattice $\Lambda^\vee$ such that the square length of a short simple root $(\check{\alpha}_i, \check{\alpha}_i) = 2$. We set $\check{d}_i := \frac{(\check{\alpha}_i, \check{\alpha}_i)}{2}$. We fix the Chevalley generators $(E_i, F_i, H_i)_{i \in I}$ of $\mathfrak{g}$. An irreducible $G$-module with a dominant highest weight $\lambda \in \Lambda^+_\vee$ is denoted $V_\lambda$; we fix its highest vector $v_\lambda$. For a weight $\check{\mu} \in \Lambda^\vee$ the $\check{\mu}$-weight subspace of a $G$-module $V$ is denoted $V(\check{\mu})$. Finally, $W$ is the Weyl group of $G, T$; the simple reflections are denoted $s_i, \ i \in I$, and $w_0 \in W$ is the longest element.

The identification $\text{Pic}^0(C^\dagger) = \mathbb{G}_m$ (resp. $\text{Pic}^{\alpha}(C^\dagger) \cong \mathbb{G}_m$, depending on the choice of coordinate $z$) of Section 2.1 gives rise to the identification $\text{Bun}^0_T(C^\dagger) = T$ (resp. $\text{Bun}^{\alpha}_T(C^\dagger) \cong T$). We denote by $\text{Bun}^{\text{reg}}_T(C^\dagger) \subset \text{Bun}^0_T(C^\dagger)$ the open subset corresponding to $T^\text{reg} \subset T$ under the above identification.

2.3. The moduli space $Y^\alpha$. Given a $T$-bundle $\mathcal{F}_T \in \text{Bun}^\text{reg}_T(C^\dagger)$ we denote by $\mathcal{F}_B$ (resp. $\mathcal{F}_G$) the corresponding induced $B$-bundle (resp. $G$-bundle).

**Definition 2.4.** Given $\alpha \in \Lambda_+$, we define $\mathcal{Y}^\alpha$ (resp. $Y^\alpha$) as the moduli space of the following data:

(a) a regular $T$-bundle $\mathcal{F}_T \in \text{Bun}^\text{reg}_T(C^\dagger)$;

(b) a trivialization $\tau_\epsilon$ of the fiber of $\mathcal{F}_T$ at $\epsilon \in C^\dagger$;

(c) a $B$-structure (resp. a generalized $B$-structure, see e.g. [1]) $\phi$ in $\mathcal{F}_G$ of degree $\alpha$ (that is, the induced $T$-bundle $\phi_T$ has degree $\alpha$), such that $\phi$ has no defect at $\epsilon \in C^\dagger$ (is not generalized) and is transversal to $\mathcal{F}_B$ at $\epsilon$;

(d) a trivialization $f_\epsilon$ of the induced $T$-bundle $\phi_T$ at $\epsilon$.

We have a natural action of $T \times T$ on $\mathcal{Y}^\alpha \subset Y^\alpha$: the first (resp. second) copy of $T$ acts via the change of trivialization $\tau_\epsilon$ (resp. $f_\epsilon$).
We also have a morphism $(p, q) : Y^\alpha \to \text{Bun}_T^\reg(C^\dagger) \times \text{Bun}_T^\reg(C^\dagger) \cong T \times T^\reg$ sending $(T, \tau, \phi, f_c)$ to $(\phi T, T)$. 

Finally, we have a morphism $\varpi : Y^\alpha \to Z^\alpha$ to the zastava space (see e.g. [3]) defined as follows. Recall that $Z^\alpha$ is the moduli space of triples $(\mathcal{F}_G^\alpha, \phi^\alpha)$ where $\mathcal{F}_G^\alpha$ is a $G$-bundle on $C$, while $\phi^\alpha$ (resp. $\phi^t\alpha$) is a $B$-structure in $\mathcal{F}_G^\alpha$ of degree 0 (resp. a generalized $B$-structure in $\mathcal{F}_G^\alpha$ of degree $\alpha$) such that $\phi^\alpha$ has no defect at $\infty \in C$, and is transversal to $\phi^{t\alpha}$ at $\infty \in C$. Now $\varpi$ sends $(T, \tau, \phi, f_c)$ to $(\mathcal{F}_G^\alpha := \pi^*\mathcal{F}_G, \phi^\alpha := \pi^*\phi, \phi^{t\alpha} := \pi^*\phi)$. Note that $\hat{Y}^\alpha = \varpi^{-1}(\hat{Z}^\alpha)$ (recall that the open subset $\hat{Z}^\alpha \subset Z^\alpha$ is formed by the triples $(\mathcal{F}_G^\alpha, \phi^\alpha)$ such that $\phi^{t\alpha}$ has no defect, i.e. is a usual as opposed to generalized $B$-structure. The moduli space $\hat{Z}^\alpha$ is isomorphic to the moduli space of degree $\alpha$ based maps from $(C, \infty)$ to $(B, B_-)$.

**Proposition 2.5.** $Y^\alpha$ is represented by a scheme.

**Proof.** Recall the scheme $\hat{\mathcal{M}}^\alpha_B$ introduced in [2.3]. It is the moduli space of degree $\alpha$ generalized $B$-structures $\phi^\alpha$ in the trivial $G$-bundle on $C$, equipped with a trivialization $f_\infty$ at $\infty \in C$ of the induced $T$-bundle $\phi^\alpha_C$. We claim that $Y^\alpha$ is a locally closed subscheme in $T^\reg \times \hat{\mathcal{M}}^\alpha_B$. In effect, given a regular $T$-bundle $T \in \text{Bun}_T^\reg(C^\dagger) = T^\reg$, its trivialization $\pi_\tau$ at $\tau \in C^\dagger$ gives rise to a trivialization $\pi_\infty$ of $\pi^*\mathcal{F}_T$ at $\infty \in C$ which extends uniquely to a trivialization of $\pi^*\mathcal{F}_C$ on $C$, and hence to a trivialization of $\pi^*\mathcal{F}_G$ on $C$. Now $\phi^\alpha_C := \pi^*\phi$ is a generalized $B$-structure in $\pi^*\mathcal{F}_G$, and the trivialization $f_c$ gives rise to a trivialization $f_\infty$ of $\phi^\alpha_C$ at $\infty \in C$. Note that $\phi^\alpha_C$ has no defect neither at $0 \in C$ nor at $\infty \in C$, and its values $\phi^\alpha_C(0), \phi^\alpha_C(\infty) \in B$ are both transversal to $B \in B$. Conversely, given $(t, \phi^\alpha, f_\infty) \in T^\reg \times \hat{\mathcal{M}}^\alpha_B$ such that $\phi^\alpha$ has no defect neither at $0 \in C$ nor at $\infty \in C$, and $\phi^\alpha(0) = t\phi^\alpha(\infty) \in B$ is transversal to $B$, we construct $(T, \tau, \phi, f_c) \in Y^\alpha$ by descent from $C$ to $C^\dagger$.

**2.6. A reduction of $Y^\alpha$.** Recall the factorization morphism $\pi_\alpha : Z^\alpha \to \mathbb{A}^\alpha = (C \setminus \{\infty\})^\alpha$ (see e.g. [3]). We have an open embedding $G^\alpha_0 = (C \setminus \{0, \infty\})^\alpha \subset \mathbb{A}^\alpha.

**Definition 2.7.** We define the trigonometric zastava space as $\hat{1}Z^\alpha := \pi_\alpha^{-1}(G^\alpha_0) \subset Z^\alpha$. We define the trigonometric monopole moduli space as $\hat{\hat{Z}}^\alpha := \hat{1}Z^\alpha \cap \hat{Z}^\alpha$: a dense open smooth subscheme of the trigonometric zastava $\hat{Z}^\alpha$.

Recall the action of $T \times T$ on $Y^\alpha$, and the morphism $(p, q) : Y^\alpha \to T \times T^\reg$ introduced in Section 2.2. The action of $1 \times T$ on $Y^\alpha$ is clearly free. The morphism $\varpi : Y^\alpha \to Z^\alpha$ of Section 2.3 is clearly $(1 \times T)$-equivariant and gives rise to the same named morphism $\varpi : Y^\alpha/(1 \times T) \to Z^\alpha$. We fix $t_0 \in T^\reg$.

**Proposition 2.8.** For any $t_0 \in T^\reg$ we have an isomorphism $\varpi : q^{-1}(t_0)/(1 \times T) \cong \hat{1}Z^\alpha$, and $\varpi : (q^{-1}(t_0) \cap \hat{1}Y^\alpha)/(1 \times T) \cong \hat{\hat{Z}}^\alpha$.

**Proof.** The locally closed embedding $Y^\alpha \hookrightarrow T^\reg \times \hat{\mathcal{M}}^\alpha_B$ constructed in the proof of Proposition 2.5 gives rise to the locally closed embedding $q^{-1}(t_0)/(1 \times T) \hookrightarrow t_0 \times \hat{\mathcal{M}}^\alpha_B$ where $\mathcal{M}^\alpha_B$ is the moduli space of degree $\alpha$ quasimaps from $C$ to $B$. More precisely, the image of $q^{-1}(t_0)/(1 \times T)$ is the locally closed subscheme $\mathcal{M}^\alpha_B \subset \hat{\mathcal{M}}^\alpha_B$ formed by the quasimaps that
have no defects at 0, ∞ ∈ C, their values φC(0) and φC(∞) are both transversal to B, and φC(0) = t₀φC(∞).

An open subset of B formed by the Borels transversal to B (the big Schubert cell) is a free orbit U ⋊ {B⁻}, and we will identify it with U (the unipotent radical of B). So for φC ∈ QM₀B we have φC(∞) = n ⋊ {B⁻} for n ∈ U, and we will simply write φC(∞) = n ∈ U.

Note that G acts on QM₀B, and U acts on QM₀B, and n⁻¹φC(∞) = B⁻, i.e. n⁻¹φC ∈ Zα is a based quasimap.

A moment of reflection shows that ω(t₀, τ, φ, f) = (φC(∞))⁻¹φC, and the condition of transversality of B and φC(0) guarantees that the value at 0 ∈ C of (φC(∞))⁻¹φC is also transversal to B, i.e. (φC(∞))⁻¹φC ∈ 1Zα. Thus we have a well defined morphism q⁻¹(t₀)/(1 × T) ⊆ QM₀B → 1Zα, and we have to prove that it is an isomorphism, i.e. that for a based quasimap φ : (C, ∞) → (B, B⁻) without defect at 0 ∈ C with φ(0) ∈ U ⋊ {B⁻} ⊆ B there exists a unique φC ∈ QM₀B such that φ = (φC(∞))⁻¹φC.

Let φ(0) = n′ ⋊ {B⁻} for some n′ ∈ U. We are looking for the desired φC in the form n⁻¹φ, n ∈ U. So we must have Adtn0(n) = n⁻¹n′, that is t₀n⁻¹t⁻¹₀ = n⁻¹n′ ⇔ [n, t₀] = n′. It remains to recall the following well known

**Lemma 2.9.** Let t₀ ∈ Treg. Then the commutator map U → U, n → [n, t₀] is an isomorphism (of algebraic varieties).

**Proof.** Filtering U by its lower central series, one can introduce a system of coordinates (x_{i,j})_{1≤i≤h, 1≤j≤h−1} on the affine space U such that for the inversion morphism U → U, n → n⁻¹ we have (x_{i,j})⁻¹ = (y_{i,j}(x)), and y_{i,j}(x) = −x_{i,j} + P_{i,j}(x_{i', j'}), \ 1≤i≤h, 1≤j≤i for a certain polynomial P_{i,j}. Moreover, for the multiplication morphism m : U × U → U we have m((x_{i,j}(x'), (x''_{i,j})) = (x_{i,j}(P_{i,j})), and x_{i,j}(P_{i,j} x''_{i,j}) = x_{i,j} + x''_{i,j} + Q_{i,j}(x_{i', j'}, x_{i'', j''}), \ 1≤i≤h, 1≤j≤i for a certain polynomial Q_{i,j}. Finally, for the adjoint action Adtn : U → U we have Adtn(x_{i,j}) = (w_{i,j}(x)), and w_{i,j}(x) = a_{i,j}x_{i,j} for a certain number a_{i,j} ≠ 1 (due to the regularity assumption on t₀).

Now given n' = (x_{i,j}) ∈ U we can construct a unique n = (x_{i,j}) ∈ U such that [n, t₀] = n' recursively starting from i = 1, and going to i = h − 1.

The proposition is proved.

**Corollary 2.10.** Yα is an irreducible scheme with an open dense smooth subscheme ̂Yα.

**Proof.** 1Zα is open in the irreducible zastava scheme Zα possessing an open dense smooth subscheme ̂Zα.

3. A trigonometric symplectic structure

3.1. Coordinates on Yα. Recall the locally closed embedding Yα → QM₀B introduced in the proof of Proposition 2.5. Via the Plücker embedding, QM₀B is a locally closed subscheme in \[ \prod_{i∈I} V_α ∧ Γ(C, O_C(a_i)) \] (notations of Section 2.2) where α = \[ \sum_{i∈I} a_i α_i \]. In particular, we have the coefficients Q_i, R_i, S_{ij} ∈ Γ(C, O_C(a_i)) of the highest, prehighest and next highest
vectors $v_{\omega_i}, F_j v_{\omega_i}, F_j F_i v_{\omega_i}$. Thus $Q_i, R_i, S_{ij}$ are the regular functions on $Y^\alpha$ with coefficients in the space of degree $\leq a_i$ polynomials in $z$. The conditions in Section 2.3(c) ensure that $\deg Q_i = a_i$, and $Q_i(0) \neq 0$.

It follows from [7, Remark 2] and Proposition 2.8 that (the coefficients of) $(Q_i, R_i)_{i \in I}$ form a rational coordinate system on $Y^\alpha$. Let us denote by $B_i$ (resp. $b_i$) the leading coefficient (resp. constant term) of $Q_i$, so that $Q_i = B_i z^{a_i} + \ldots + b_i$. Similarly, we have $R_i = C_i z^{a_i} + \ldots + c_i$. Note that $B_i \neq 0 \neq b_i$. Following [7, 3.3], we introduce a rational étale coordinate system on $Y^\alpha$. Namely, $(w_{i,r})_{i \in I}^{1 \leq r \leq a_i}$ are the ordered roots of $Q_i$, and $y_{i,r} := B_i^{-1} R_i(w_{i,r})$. The desired coordinate system is formed by $(B_i, C_i, w_{i,r}, y_{i,r})_{i \in I}^{1 \leq r \leq a_i}$. It follows from [7, Remark 2] and Proposition 2.8 that these functions do form a coordinate system on an unramified covering of the open subset of $\check{Y}^\alpha$ where all the roots of all the polynomials $Q_i, i \in I$, are distinct.

We describe the $T \times T$-action on $Y^\alpha$, and the morphisms $(p, q): Y^\alpha \to T \times T$ (see Section 2.3) in the above coordinates. Note that the collection of fundamental weights $\check{\omega}_i: T \to \mathbb{G}_m$ identifies $T$ with $\mathbb{G}_m^I$. We have $(t_1, t_2) \cdot (Q_i, R_i) = (\check{\omega}_i(t_1 t_2) Q_i, \check{\omega}_i(t_1 t_2) \check{\alpha}_i(t_1)^{-1} R_i)$, and $\check{\omega}_j(p(Q_i, R_i)_{i \in I}) = B_j^{-1} b_j$, and $\check{\alpha}_j(q(Q_i, R_i)_{i \in I}) = B_j^{-1} c_j b_j C_j$.

3.2. The tangent bundle. Our goal in this Section is to describe the tangent space $T_y Y^\alpha$ at $y = (\mathcal{F}_T, \tau, \phi, f_t) \in \check{Y}^\alpha$. We denote by $\mathfrak{g}, \mathfrak{b}, \mathfrak{u}, \mathfrak{t}$ the Lie algebras of $G, B, U, T$. Given a $T$-bundle $\mathcal{F}_T$ we denote the vector bundle associated to the adjoint action of $T$ on $\mathfrak{g}$ by $\mathfrak{g}^T$. It is a direct sum of two subbundles corresponding to the trivial (resp. nontrivial) eigenvalues of $T$ on $\mathfrak{g}: \mathfrak{g}^T = \mathfrak{t}^T \oplus \mathfrak{r}^T$. Note that $\mathfrak{t}^T = \mathfrak{t} \otimes \mathcal{O}_{C^T}$. A $B$-structure $\phi$ on $\mathcal{F}_G$ gives rise to a vector subbundle $\mathfrak{b}^\phi \subset \mathfrak{g}^T$. The adjoint action of $B$ on $\mathfrak{u}$ gives rise to a subbundle $\mathfrak{u}^\phi \subset \mathfrak{b}^\phi$. We denote the quotient bundle by $\mathfrak{b}^\phi = \mathfrak{b} \otimes \mathcal{O}_{C^T}$: a trivial bundle where $\mathfrak{h} = \mathfrak{b} / \mathfrak{u}$ is the abstract Cartan. The Killing form identifies the dual vector bundle $\mathfrak{b}^{\*\phi}$ with the quotient bundle $\mathfrak{g}^T / \mathfrak{u}^\phi =: (\mathfrak{g} / \mathfrak{u})^\phi$. For a vector bundle $\mathcal{V}$ on $C^\dagger$ we denote by $\mathcal{V}_c$ the skyscraper quotient of $\mathcal{V}$ by the ideal sheaf of the point $c \in C^\dagger$.

We consider the following complex $K_y^\bullet$ of coherent sheaves on $C^\dagger$: it lives in degrees $-1, 0$, and $K_y^{-1} = (\mathfrak{t} \otimes \mathfrak{h}) \otimes \pi_* \mathcal{O}_C$, while $K_y^0 = (\mathfrak{g} / \mathfrak{u})^\phi$. The differential $d: K_y^{-1} \to K_y^0$ is a direct sum of $d': \mathfrak{t} \otimes \pi_* \mathcal{O}_C \to (\mathfrak{g} / \mathfrak{u})^\phi$ and $d'': \mathfrak{h} \otimes \pi_* \mathcal{O}_C \to (\mathfrak{g} / \mathfrak{u})^\phi$. Here $d'$ is the composition of $\mathfrak{t} \otimes \pi_* \mathcal{O}_C \hookrightarrow \mathfrak{t} \otimes \mathcal{O}_{C^T} = \mathfrak{t}^T$ (see Section 2.1) and $\mathfrak{t}^T \to \mathfrak{g}^T \to (\mathfrak{g} / \mathfrak{u})^\phi$, while $d''$ is the composition $\mathfrak{h} \otimes \pi_* \mathcal{O}_C \hookrightarrow \mathfrak{h} \otimes \mathcal{O}_{C^T} = \mathfrak{h}^\phi \hookrightarrow (\mathfrak{g} / \mathfrak{u})^\phi$.

**Proposition 3.3.** There is a canonical isomorphism $T_y Y^\alpha \cong H^0(C^\dagger, K_y^\bullet)$.

**Proof.** We consider the following complex $K_y^\bullet$ of coherent sheaves on $C^\dagger$: it lives in degrees $-1, 0$, and $K_y^{-1} = \mathfrak{b}^\phi$, while $K_y^0 = \mathfrak{t}^T \oplus \mathfrak{h}^\phi$. The differential from $K_y^{-1}$ to $K_y^0$ is a direct sum of $d': \mathfrak{b}^\phi \to \mathfrak{t}^T$ and $d'': \mathfrak{b}^\phi \to \mathfrak{h}^\phi$ where $d'$ is the composition $\mathfrak{b}^\phi \hookrightarrow \mathfrak{g}^T \to \mathfrak{t}^T \to \mathfrak{t}^T$, and $d''$ is the composition $\mathfrak{b}^\phi \hookrightarrow \mathfrak{h}^\phi \to \mathfrak{h}^\phi$.

Then $T_y Y^\alpha \cong H^0(C^\dagger, K_y^\bullet)$. Now consider yet another complex $\check{K}_y^\bullet$ living in degrees $-1, 0$ such that $\check{K}_y^{-1} = K_y^{-1}$ and $\check{K}_y^0 = K_y^0 \oplus \mathfrak{r}^T$, and the differential equals $d' + d'' + d'''$ where $d''' : \mathfrak{b}^\phi \to \mathfrak{r}^T$ is the composition $\mathfrak{b}^\phi \hookrightarrow \mathfrak{g}^T \to \mathfrak{r}^T$. We have a canonical morphism
"K∗ → K∗ inducing an isomorphism on cohomology H0(C†, "K∗) → H0(C†, K∗) since H1(C†, tF) = 0 due to the regularity assumption on FT.

Also we have a canonical quasiisomorphism "K∗ → K∗ where "K−1 = K−1 ⊕ (t ⊕ h) ⊗ πω+C, and "K∗ = (t ⊕ h) ⊗ O_C ⊕ tF. The new components of the differential are as follows: (t ⊕ h) ⊗ πω+C → (t ⊕ h) ⊗ O_C (see Section 2.1), and bφ → hφ, and the composition bφ → gF → tF.

Finally, note that "K∗ = hφ ⊕ gF, and (hφ ⊕ gF)/d(bφ) ∼= (g/u)φ, so we have a canonical quasiisomorphism "K∗ → K∗.

The composition of the morphisms induced on H0(C†, ?) by the above quasiisomorphisms is the desired isomorphism T∗Yα = H0(C†, K∗) ∼= H0(C†, K∗).

3.4. The cotangent bundle. Let us describe the Serre dual complex L∗ := D∗K∗. It lives in degrees 0, 1, and L0 = bφ, while L1 = (t ⊕ h) ⊗ πO. The differential d : L0 → L1 is a direct sum of d′ : bφ → t ⊗ πO and d′′ : bφ → h ⊗ πO. Here d′′ is the composition bφ → hφ = h ⊗ O_C → h ⊗ πO (see Section 2.1), while d′ is the composition bφ → gF → tF = t ⊗ O_C → t ⊗ πO. Now Proposition 3.3 has the following immediate

Corollary 3.5. There is a canonical isomorphism T∗Yα ∼= H0(C†, L∗).

3.6. Some differentials. Here we describe the differentials of the morphism (p, q) : Yα → T × T reg and of the action of T × T on Yα introduced in Section 2.3.

Note that for a regular T-bundle FT on C† the tangent space FT,Bun reg(C†) is canonically isomorphic to H1(C†, tF) = H1(C†, t ⊗ O_C) ∼= t (here the second isomorphism is Id ⊗ Tr for the trace isomorphism Tr : H1(C†, O_C) → H1(C†, ω_C) → H1(C†, C)). The scalar product (, ) on t∗ (Section 2.2) identifies t with t∗. Together with the Serre duality H1(C†, O_C)∗ = H0(C†, O_C) this gives rise to a canonical isomorphism FT,Bun reg(C†) ∼= H0(C†, O_C) ≃ t. Similarly, for a degree α T-bundle FT we have canonical isomorphisms T∗Yα ∼= H0(C†, hφ) = h and T∗μ μ′,Bun reg(C†) ∼= H0(C†, h ⊗ O_C) = h.

We have distinguished triadings (g/u)φ → K∗ → (t ⊕ h) ⊗ πω+C[1] and (t ⊕ h) ⊗ πO[-1] → L∗ → bφ. They give rise to a morphism dp : T∗Yα = H0(C†, K∗) → H1(C†, (t ⊕ h) ⊗ πω+C) → H1(C†, (t ⊕ h) ⊗ πω+C) → H1(C†, h ⊗ O_C) = T∗μ μ′,Bun reg(C†) where the middle arrow arises from the natural morphism πω+C → ω_C (see Section 2.1), and the next arrow arises from the projection t ⊕ h → h. Similarly, we have dφ : T∗Yα = H0(C†, K∗) → H1(C†, (t ⊕ h) ⊗ πω+C) → H1(C†, (t ⊕ h) ⊗ ω_C) → H1(C†, t ⊕ h) ⊗ O_C) = T∗μ μ′,Bun reg(C†). Dually, we have d′ : T∗μ μ′,Bun reg(C†) = H0(C†, (t ⊕ h) ⊗ O_C) → H0(C†, (t ⊕ h) ⊗ ω_C) → H1(C†, t ⊕ h) ⊗ O_C) = T∗μ μ′,Bun reg(C†) → H0(C†, (t ⊕ h) ⊗ πO) → H0(C†, (t ⊕ h) ⊗ πO) → H0(C†, (t ⊕ h) ⊗ 0) → H0(C†, L∗) = T∗Yα and d′′ : T∗μ μ′,Bun reg(C†) = H0(C†, t ⊗ O_C) → H0(C†, t ⊕ h) ⊗ O_C) → H0(C†, (t ⊕ h) ⊗ 0) → H0(C†, L∗) = T∗Yα.

We also have a morphism a1 : T∗μ μ′ = t = H0(C†, t ⊕ O_C) → H0(C†, tF) → H0(C†, (g/u)φ) → H0(C†, K∗) = T∗Yα and a2 : T∗μ μ′ = h = H0(C†, h ⊗ O_C) = H0(C†, (g/u)φ) → H0(C†, K∗) = T∗Yα.
Lemma 3.7. a) \((a_1, a_2) : t \oplus h \rightarrow T_y \hat{Y}^\alpha\) is the differential of the action \(T \times T \times \hat{Y}^\alpha \rightarrow \hat{Y}^\alpha\) (see Section 2.3);

b) \((d_p, d_q) : T_y \hat{Y}^\alpha \rightarrow T_{\phi T} \text{Bun}_T(G) \oplus T_{\phi T} \text{Bun}_T(\text{reg})\) is the differential of \((p, q) : \hat{Y}^\alpha \rightarrow \text{Bun}_T(\text{reg}) \times \text{Bun}_T(\text{reg}).\)

c) \((d_p^*, d_q^*) : T_{\phi T} \text{Bun}_T(G) \oplus T_{\phi T} \text{Bun}_T(\text{reg}) \rightarrow T_{\phi T} \hat{Y}^\alpha\) is the codifferential of \((p, q) : \hat{Y}^\alpha \rightarrow \text{Bun}_T(\text{reg}) \times \text{Bun}_T(\text{reg}).\)

Proof. Clear from the construction. \(\square\)

3.8. A bivector field. We consider the following bicomplex \(M_y^{\bullet \bullet}\):

\[
\begin{array}{ccc}
\mathfrak{b}^\phi & \longrightarrow & (t \oplus h) \otimes \pi_* \mathcal{O}_C \\
\downarrow & & \downarrow \\
(t \oplus h) \otimes \pi_* \omega_C & \longrightarrow & \mathfrak{v}^\tau \oplus (t \oplus h) \otimes (\theta \oplus \mathcal{O}_{C^!}) \\
\downarrow & & \downarrow \\
(t \oplus h) \otimes \pi_* \omega_C & \longrightarrow & (\mathfrak{g}/u)^\phi
\end{array}
\]

(3.1)

Here the middle term lives in bidegree \((0, 0)\), the first line is nothing but \(L^\bullet\) of Section 3.4 while the last line is nothing but \(K^\bullet\) of Section 3.2. The left vertical arrow is the identity morphism, as well as the right vertical arrow. The middle line is a direct sum of the complex consisting of \(\mathfrak{v}^\tau\) in degree 0, and of the exact complex (2.3) tensored with \(t \oplus h\). Note that the middle term can be rearranged as \(b^\phi \oplus g^\tau \oplus (t \oplus h) \otimes \theta\). Now the middle column is a direct sum of the complex consisting of \((t \oplus h) \otimes \theta\) in degree 0, and of the exact complex \(b^\phi \rightarrow b^\phi \oplus g^\tau \rightarrow (\mathfrak{g}/u)^\phi\).

It follows (looking at the columns of (3.1)) that the total complex \(\text{Tot} M_y^{\bullet \bullet}\) has only one cohomology in degree 0, and \(H^0(\text{Tot} M_y^{\bullet \bullet}) = (t \oplus h) \otimes \theta\). Since \(H^*(C^!, (t \oplus h) \otimes \theta) = 0\), we deduce that \(H^*(C^!, \text{Tot} M_y^{\bullet \bullet}) = 0\). Now let us look at the rows of \(M_y^{\bullet \bullet}\). The hypercohomology of \(C^!\) with coefficients in the middle row vanishes since \(H^*(C^!, \mathfrak{v}^\tau) = 0\) (due to the regularity assumption on \(T_T\)). Hence the second differential in the spectral sequence converging to \(H^*(C^!, \text{Tot} M_y^{\bullet \bullet}) = 0\) from the hypercohomology of \(C^!\) with coefficients in the rows is \(d_2 : H^0(C^!, L^\bullet) \rightarrow H^0(C^!, K^\bullet)\).

Finally, due to Proposition 3.3 and Corollary 3.5 we can view the above differential as \(d_2 : T_y \hat{Y}^\alpha \rightarrow T_y \hat{Y}^\alpha\).

3.9. Calculation of the bivector field: preparation. We follow the strategy of [7], and eventually reduce our calculation to the one of loc. cit. The result of the somewhat lengthy calculation of \(d_2\) is contained in Lemma 3.10 Remark 3.11 and Proposition 3.14.

Given a character \(\lambda \in \Lambda^\vee\), we consider the composed homomorphism \(B \rightarrow T \rightarrow \mathbb{G}_m\), and denote the associated (to the \(B\)-torsor \(\phi\)) line bundle on \(C^!\) by \(L^\phi\). For an irreducible \(G\)-module \(V_\lambda\), the associated (to the \(G\)-torsor \(T_G\)) vector bundle on \(C^!\) is denoted \(V^\tau_\lambda\). If \(\hat{\lambda}\) is a fundamental weight \(\hat{\omega}_i\), then we have an isomorphism \(V^{*\lambda}_{\hat{\omega}_i} \cong V_{\hat{\omega}_i}\) for an involution
\[ I \xrightarrow{\sim} I, \ i \mapsto i^* \]. If we extend the involution \( \tilde{\omega}_i \mapsto \tilde{\omega}_i^* \) by linearity to the weight lattice \( \Lambda^\vee, \ \lambda \mapsto \lambda^*, \) then this involution preserves the scalar product \( \langle \cdot, \cdot \rangle \) of Section 2.2.

We have a natural embedding of vector bundles on \( C^\dagger \) : \( (g/u)^\phi \hookrightarrow \bigoplus_{i \in I} V_{i^*}^{\tilde{\omega}_i} \otimes \mathcal{L}_{-\tilde{\omega}_i}^\phi \), and the dual surjection \( \bigoplus_{i \in I} V_{i^*}^{\tilde{\omega}_i} \otimes \mathcal{L}_{-\tilde{\omega}_i}^\phi \to b^\phi \). They give rise to the following morphisms of two-term complexes of coherent sheaves on \( C^\dagger \):

\[
\begin{array}{ccc}
(t \oplus h) \otimes \pi_* \omega_C & \longrightarrow & (g/u)^\phi \\
\downarrow & & \downarrow \\
(t \oplus h) \otimes \pi_* \Xi & \longrightarrow & (g/u)^\phi \otimes \pi_* \mathcal{O}_C
\end{array}
\]

(3.2)

\[
\begin{array}{ccc}
(t \oplus h) \otimes \pi_* \Xi & \longrightarrow & (g/u)^\phi \otimes \pi_* \mathcal{O}_C \\
\downarrow & & \downarrow \\
(t \oplus h) \otimes \pi_* \Xi & \longrightarrow & (\bigoplus_{i \in I} V_{i^*}^{\tilde{\omega}_i} \otimes \mathcal{L}_{-\tilde{\omega}_i}^\phi) \otimes \pi_* \mathcal{O}_C
\end{array}
\]

(3.3)

Note that the top row of (3.2) coincides with the bottom row \( K_y^* \) of the bicomplex (3.1), while the bottom row of (3.3) coincides with the top row \( L_y^* \) of the bicomplex (3.1). So composing the vertical arrows of (3.3), (3.2) with the vertical arrows of (3.1) we obtain a bicomplex \( M_y^{**} \):

\[
\begin{array}{ccc}
(t \oplus h) \otimes \pi_* \omega_C & \longrightarrow & (t \oplus h) \otimes \pi_* \Xi \\
\downarrow & & \downarrow \\
(t \oplus h) \otimes \pi_* \omega_C & \longrightarrow & (t \oplus h) \otimes \pi_* \Xi
\end{array}
\]

(3.4)

Note that \( H^*(C^\dagger, (t \oplus h) \otimes \pi_* \Xi) = 0 \), so just as in Section 3.8 the second differential in the spectral sequence converging to \( H^*(C^\dagger, \text{Tot } M_y^{**}) \) from the hypercohomology of \( C^\dagger \) with coefficients in the rows is

\[
d^2_2 : H^1(C^\dagger, \bigoplus_{i \in I} V_{i^*}^{\tilde{\omega}_i} \otimes \mathcal{L}_{-\tilde{\omega}_i}^\phi) \otimes \pi_* \omega_C \to H^0(C^\dagger, \bigoplus_{i \in I} V_{i^*}^{\tilde{\omega}_i} \otimes \mathcal{L}_{-\tilde{\omega}_i}^\phi) \otimes \pi_* \mathcal{O}_C).
\]

(3.5)

**Lemma 3.10.** The following diagram commutes:
\[ H^0(C^\dagger, L_y^\bullet) \leftarrow H^1(C^\dagger, (\bigoplus_{i \in I} V_{\omega_i}^{\mathcal{T}} \otimes \mathcal{L}^\phi_{-\omega_i}) \otimes \pi_\ast \omega_C) \]

\[ \mathcal{d}_2 \downarrow \quad \mathcal{d}_2' \downarrow \]

\[ H^0(C^\dagger, H_y) \longrightarrow H^0(C^\dagger, (\bigoplus_{i \in I} V_{w_i}^{\mathcal{T}} \otimes \mathcal{L}^\phi_{\omega_i}) \otimes \pi_\ast \omega_C) \]

**Proof.** Clear. \[ \square \]

**Remark 3.11.** In what follows we will be occupied with the calculation of \( d_2' : H^1(C^\dagger, (\bigoplus_{i \in I} V_{\omega_i}^{\mathcal{T}} \otimes \mathcal{L}^\phi_{-\omega_i}) \otimes \pi_\ast \omega_C) \rightarrow H^0(C^\dagger, (\bigoplus_{i \in I} V_{\omega_i}^{\mathcal{T}} \otimes \mathcal{L}^\phi_{\omega_i}) \otimes \pi_\ast \omega_C) \). Let us presently comment in which sense does it calculate the desired \( d_2 : H^0(C^\dagger, L_y^\bullet) \rightarrow H^0(C^\dagger, K_y^\bullet) \).

It is easy to see that for \( y \in \check{Y}^\alpha \) lying in the open subset \( U^\alpha \subset \check{Y}^\alpha \) formed by all the quadruples \((\mathcal{F}_T, \tau, \phi, f_l)\) such that \( t^\alpha \cap b^\alpha = 0 \subset g_T^\alpha \) the morphism \( H^1(C^\dagger, (\bigoplus_{i \in I} V_{\omega_i}^{\mathcal{T}} \otimes \mathcal{L}^\phi_{-\omega_i}) \otimes \pi_\ast \omega_C) \rightarrow H^0(C^\dagger, L_y^\bullet) \) is surjective, and the morphism \( H^0(C^\dagger, K_y^\bullet) \rightarrow H^0(C^\dagger, (\bigoplus_{i \in I} V_{\omega_i}^{\mathcal{T}} \otimes \mathcal{L}^\phi_{\omega_i}) \otimes \pi_\ast \omega_C) \) is injective. Since we are only going to calculate our \( d_2 \) generically, the only trouble is that for some \( \alpha \) the open subset \( U^\alpha \) may happen to be empty. Indeed, for \( \alpha = \sum_{i \in I} a_i \alpha_i \), we have \( U^\alpha = \emptyset \) if \( a_i = 0 \) for some \( i \in I \).

So in what follows we assume \( a_i > 0 \, \forall i \in I \) (otherwise the moduli space \( \check{Y}^\alpha \) essentially reduces to the one for a semisimple Lie algebra \( g^\prime \) of smaller rank).

### 3.12. Reduction to a calculation on \( C \)

The goal of this Section is a description of \( d_2 ' (3.5) \) in terms of \( C \), see Corollary 3.15.

Note that \( \pi_\ast (h^\phi \oplus g^{\mathcal{T}}) \cong (h \oplus g) \otimes \omega_C \), and \( \pi_\ast ((t \oplus h) \otimes \theta) \cong (t \oplus h) \otimes \pi_\ast \omega_C \). Hence \( \pi_\ast \pi_\ast (h^\phi \oplus g^{\mathcal{T}} \oplus (t \oplus h) \otimes \theta) \cong (h \oplus g \oplus t \oplus h) \otimes \pi_\ast \omega_C \). The morphisms \( \mathcal{r}_T \oplus (t \oplus h) \otimes (\theta \otimes \omega_C) = h^\phi \oplus g^{\mathcal{T}} \oplus (t \oplus h) \otimes \theta \rightarrow (t \oplus h) \otimes \pi_\ast \omega_C \) (resp. \( \mathcal{r}_T \oplus (t \oplus h) \otimes (\theta \otimes \omega_C) = h^\phi \oplus g^{\mathcal{T}} \oplus (t \oplus h) \otimes \theta \rightarrow (t \oplus h) \otimes \pi_\ast \omega_C \) of \( (3.3) \) factor through the canonical morphism \( h^\phi \oplus g^{\mathcal{T}} \oplus (t \oplus h) \otimes \theta \rightarrow (\bigoplus_{i \in I} V_{\omega_i}^{\mathcal{T}} \otimes \mathcal{L}^\phi_{\omega_i}) \otimes \pi_\ast \omega_C \) of \( (3.4) \) to the following bicomplex \( M_y^{\mathcal{M}^\bullet} \) (3.4) to the following bicomplex \( ^\prime M_y^{\mathcal{M}^\bullet} \): \( \bigoplus_{i \in I} V_{\omega_i}^{\mathcal{T}} \otimes \mathcal{L}^\phi_{-\omega_i} \otimes \pi_\ast \omega_C \rightarrow (t \oplus h) \otimes \pi_\ast \Xi \)

\[ \downarrow \]

\[ (t \oplus h) \otimes \pi_\ast \omega_C \rightarrow (h \oplus g \oplus t \oplus h) \otimes \pi_\ast \omega_C \rightarrow (t \oplus h) \otimes \pi_\ast \omega_C \quad (3.6) \]

\[ \downarrow \]

\[ (t \oplus h) \otimes \pi_\ast \Xi \rightarrow (\bigoplus_{i \in I} V_{\omega_i}^{\mathcal{T}} \otimes \mathcal{L}^\phi_{\omega_i}) \otimes \pi_\ast \omega_C \]

(The morphisms from all the terms of \( ^\prime M_y^{\mathcal{M}^\bullet} \) to the corresponding terms of \( ^{\prime\prime} M_y^{\mathcal{M}^\bullet} \) except for the middle ones are identities). Just as in Section 3.9 we obtain the second differential

\[ d_2'' : H^1(C^\dagger, (\bigoplus_{i \in I} V_{\omega_i}^{\mathcal{T}} \otimes \mathcal{L}^\phi_{-\omega_i}) \otimes \pi_\ast \omega_C) \rightarrow H^0(C^\dagger, (\bigoplus_{i \in I} V_{\omega_i}^{\mathcal{T}} \otimes \mathcal{L}^\phi_{\omega_i}) \otimes \pi_\ast \omega_C) \quad (3.7) \]

in the spectral sequence converging to \( H^\bullet(C^\dagger, \operatorname{Tot} ^{\prime\prime} M_y^{\mathcal{M}^\bullet}) \) from the hypercohomology of \( C^\dagger \) with coefficients in the rows. It follows that \( d_2'' = d_2 ' \) of (3.5).
Similarly to Lemma 3.10 we deduce the following mutative diagrams:

\[
\begin{array}{c}
\bigoplus_{i \in I} V_{\dot{\omega}_i^*} \otimes L_{-\dot{\omega}_i}^\phi \otimes \pi_* \omega_C \rightarrow (h \oplus g \oplus t \oplus h) \otimes \pi_* O_C \\
\end{array}
\]

(respectively \( (t \oplus h) \otimes \pi_* \Xi \rightarrow (h \oplus g \oplus t \oplus h) \otimes \pi_* O_C \)) of (3.11) factor through the natural morphism \( (h \oplus g \oplus t \oplus h) \otimes \pi_* \Xi \rightarrow (h \oplus g \oplus t \oplus h) \otimes \pi_* \Xi \) (resp. \( (t \oplus h) \otimes \pi_* \omega_C \rightarrow (h \oplus g \oplus t \oplus h) \otimes \pi_* \Xi \)). Hence we obtain a morphism to the bicomplex \( "M_{y}^{\bullet \bullet} \) to the following bicomplex \( "M_{y}^{\bullet \bullet} \) of (3.12):

\[
\begin{array}{c}
\bigoplus_{i \in I} V_{\dot{\omega}_i^*} \otimes L_{-\dot{\omega}_i}^\phi \otimes \pi_* \omega_C \\
\end{array}
\]

Now the morphisms \( (\bigoplus_{i \in I} V_{\dot{\omega}_i^*} \otimes L_{-\dot{\omega}_i}^\phi) \otimes \pi_* \omega_C \rightarrow (h \oplus g \oplus t \oplus h) \otimes \pi_* O_C \) (resp. \( (t \oplus h) \otimes \pi_* \Xi \rightarrow (h \oplus g \oplus t \oplus h) \otimes \pi_* O_C \)) factor through the natural morphism \( (h \oplus g \oplus t \oplus h) \otimes \pi_* \Xi \rightarrow (h \oplus g \oplus t \oplus h) \otimes \pi_* \Xi \) (see Section 2.1) and \( (\bigoplus_{i \in I} V_{\dot{\omega}_i^*} \otimes L_{-\dot{\omega}_i}^\phi) \otimes \pi_* \omega_C \rightarrow \) (\( (h \oplus g \oplus t \oplus h) \otimes \pi_* \Xi \)) (resp. \( (t \oplus h) \otimes \pi_* \omega_C \rightarrow (h \oplus g \oplus t \oplus h) \otimes \pi_* \Xi \)). Hence we obtain a morphism to the bicomplex \( "M_{y}^{\bullet \bullet} \) to the following bicomplex \( "M_{y}^{\bullet \bullet} \) of (3.12):

\[
\begin{array}{c}
\bigoplus_{i \in I} V_{\dot{\omega}_i^*} \otimes L_{-\dot{\omega}_i}^\phi \\
\end{array}
\]

By composing the vertical arrows of (3.10) with the vertical arrows of the following commutative diagrams:

\[
\begin{array}{c}
\bigoplus_{i \in I} V_{\dot{\omega}_i^*} \otimes L_{-\dot{\omega}_i}^\phi \otimes \pi_* \omega_C \rightarrow (t \oplus h) \otimes \pi_* \Xi \\
\end{array}
\]

\[
\begin{array}{c}
\bigoplus_{i \in I} V_{\dot{\omega}_i^*} \otimes L_{-\dot{\omega}_i}^\phi \otimes \pi_* \Xi \\
\end{array}
\]

\[
\begin{array}{c}
(t \oplus h) \otimes \pi_* \Xi \\
\end{array}
\]

\[
\begin{array}{c}
(t \oplus h) \otimes \pi_* \Xi \\
\end{array}
\]

Similarly to Lemma 3.10 we deduce the following

\[
\begin{array}{c}
(t \oplus h) \otimes \pi_* \Xi \\
\end{array}
\]
Lemma 3.13. The following diagram commutes:

\[
\begin{array}{c}
H^1(C^\dagger, (\bigoplus_{i \in I} V_{\omega_i}^{\ast} \otimes L_{-\omega_i}^\phi) \otimes \pi_s \Xi) \quad \xrightarrow{d_2} \quad H^1(C^\dagger, (\bigoplus_{i \in I} V_{\omega_i}^{\ast} \otimes L_{-\omega_i}^\phi) \otimes \pi_s \omega_C) \\
\downarrow \quad \downarrow d''_2 \\
H^0(C^\dagger, (\bigoplus_{i \in I} V_{\omega_i}^{\ast} \otimes L_{-\omega_i}^\phi) \otimes \pi_s \Xi) \quad \xrightarrow{d_2'} \quad H^0(C^\dagger, (\bigoplus_{i \in I} V_{\omega_i}^{\ast} \otimes L_{-\omega_i}^\phi) \otimes \pi_s \omega_C)
\end{array}
\]

where \(d_2' : H^1(C^\dagger, (\bigoplus_{i \in I} V_{\omega_i}^{\ast} \otimes L_{-\omega_i}^\phi) \otimes \pi_s \Xi) \to H^0(C^\dagger, (\bigoplus_{i \in I} V_{\omega_i}^{\ast} \otimes L_{-\omega_i}^\phi) \otimes \pi_s \omega_C)\) is the second differential in the spectral sequence converging to \(H^\bullet(C^\dagger, \text{Tot } \mathcal{M}_y^{\bullet \bullet})\) from the hypercohomology of \(C^\dagger\) with coefficients in the rows.

By the projection formula \((\bigoplus_{i \in I} V_{\omega_i}^{\ast} \otimes L_{-\omega_i}^\phi) \otimes \pi_s \omega_C \cong \pi_s \pi^\ast(\bigoplus_{i \in I} V_{\omega_i}^{\ast} \otimes L_{-\omega_i}^\phi) = \pi_s \bigoplus_{i \in I} V_{\omega_i} \otimes \pi^\ast L_{-\omega_i}^\phi\), and similarly \((\bigoplus_{i \in I} V_{\omega_i}^{\ast} \otimes L_{-\omega_i}^\phi) \otimes \pi_s \Xi \cong \pi_s (\bigoplus_{i \in I} V_{\omega_i} \otimes \pi^\ast L_{-\omega_i}^\phi) \otimes \Xi\), and \((\bigoplus_{i \in I} V_{\omega_i}^{\ast} \otimes L_{-\omega_i}^\phi) \otimes \pi_s \omega_C \cong \pi_s ((\bigoplus_{i \in I} V_{\omega_i} \otimes \pi^\ast L_{-\omega_i}^\phi) \otimes \omega_C)\), and \((\bigoplus_{i \in I} V_{\omega_i}^{\ast} \otimes L_{-\omega_i}^\phi) \otimes \pi_s \Xi \cong \pi_s ((\bigoplus_{i \in I} V_{\omega_i} \otimes \pi^\ast L_{-\omega_i}^\phi) \otimes \Xi)\).

Now since the upper-right and lower-left terms \((t \oplus h) \otimes \pi_s \Xi\) of the bicomplex \((3.10)\) are acyclic sheaves on \(C^\dagger\), the differential \(d_2'\) coincides with the second differential from the spectral sequence arising from the following complex on \(C^\dagger\):

\[
\left(\bigoplus_{i \in I} V_{\omega_i}^{\ast} \otimes L_{-\omega_i}^\phi\right) \otimes \pi_s \Xi \rightarrow (h \oplus g \oplus t \oplus h) \otimes \pi_s \Xi \rightarrow \left(\bigoplus_{i \in I} V_{\omega_i}^{\ast} \otimes L_{-\omega_i}^\phi\right) \otimes \pi_s \Xi
\]

which is by construction a direct sum of the one term complex \((t \oplus h) \otimes \pi_s \Xi\) in degree zero and the following complex:

\[
\left(\bigoplus_{i \in I} V_{\omega_i}^{\ast} \otimes L_{-\omega_i}^\phi\right) \otimes \pi_s \Xi \rightarrow (h \oplus g) \otimes \pi_s \Xi \rightarrow \left(\bigoplus_{i \in I} V_{\omega_i}^{\ast} \otimes L_{-\omega_i}^\phi\right) \otimes \pi_s \Xi
\]

which in turn is nothing but the direct image \(\pi_s N_Y^\bullet\) of the following complex \(N_Y^\bullet\) of vector bundles on \(C\):

\[
\left(\bigoplus_{i \in I} V_{\omega_i} \otimes \pi^\ast L_{-\omega_i}^\phi\right) \otimes \Xi \rightarrow (h \oplus g) \otimes \Xi \rightarrow \left(\bigoplus_{i \in I} V_{\omega_i} \otimes \pi^\ast L_{-\omega_i}^\phi\right) \otimes \Xi
\]

Lemma 3.14. The following diagram commutes:

\[
\begin{array}{c}
H^1(C^\dagger, (\bigoplus_{i \in I} V_{\omega_i}^{\ast} \otimes L_{-\omega_i}^\phi) \otimes \pi_s \Xi) \quad \xrightarrow{d_2} \quad H^1(C, (\bigoplus_{i \in I} V_{\omega_i}^{\ast} \otimes \pi^\ast L_{-\omega_i}^\phi) \otimes \Xi) \\
\downarrow \quad \downarrow d''_2 \\
H^0(C^\dagger, (\bigoplus_{i \in I} V_{\omega_i}^{\ast} \otimes L_{-\omega_i}^\phi) \otimes \pi_s \Xi) \quad \xrightarrow{d_2'} \quad H^0(C, (\bigoplus_{i \in I} V_{\omega_i}^{\ast} \otimes \pi^\ast L_{-\omega_i}^\phi) \otimes \Xi)
\end{array}
\]

where \(d_2' : H^1(C^\dagger, (\bigoplus_{i \in I} V_{\omega_i} \otimes \pi^\ast L_{-\omega_i}^\phi) \otimes \Xi) \to H^0(C, (\bigoplus_{i \in I} V_{\omega_i} \otimes \pi^\ast L_{-\omega_i}^\phi) \otimes \Xi)\) is the second differential in the spectral sequence converging to \(H^\bullet(C^\dagger, \text{Tot } \mathcal{M}_{Y^\bullet})\) from the cohomology of \(C\) with coefficients in its terms.

Recall that our goal is to compute the differential \((3.5)\): \(H^1(C, (\bigoplus_{i \in I} V_{\omega_i}^{\ast} \otimes \pi^\ast L_{-\omega_i}^\phi) \otimes \omega_C) = H^1(C^\dagger, (\bigoplus_{i \in I} V_{\omega_i}^{\ast} \otimes L_{-\omega_i}^\phi) \otimes \pi_s \omega_C) \xrightarrow{d'_3} H^0(C^\dagger, (\bigoplus_{i \in I} V_{\omega_i}^{\ast} \otimes L_{-\omega_i}^\phi) \otimes \pi_s \omega_C) = H^0(C, (\bigoplus_{i \in I} V_{\omega_i} \otimes \pi^\ast L_{-\omega_i}^\phi) \otimes \omega_C).\) The bottom line of the present Section is the following.

Corollary 3.15. The following diagram commutes:
\[
H^1(C, (\bigoplus_{i \in I} V_{\omega_i}^*) \otimes \pi^* \mathcal{L}_{\omega_i}^\phi) \otimes \omega_C) \longrightarrow H^1(C, (\bigoplus_{i \in I} V_{\omega_i}^*) \otimes \pi^* \mathcal{L}_{\omega_i}^\phi) \otimes \Xi)
\]
\[
d_2^C
\]
\[
H^0(C, (\bigoplus_{i \in I} V_{\omega_i}, \otimes \pi^* \mathcal{L}_{\omega_i}^\phi) \otimes \omega_C)
\]
\[
\longleftarrow H^0(C, (\bigoplus_{i \in I} V_{\omega_i}, \otimes \pi^* \mathcal{L}_{\omega_i}^\phi) \otimes \Xi)
\]

3.16. **Calculation on C.** The differential \(d_2^C\) of Corollary 3.15 was computed in [7]. To formulate the result, we introduce homogeneous coordinates \(z_1, z_2\) on \(C\) such that \(z = z_1/z_2\), so that \(z_1 = 0\) (resp. \(z_2 = 0\)) is an equation of \(0 \in C\) (resp. \(\infty \in C\)). We also introduce another copy of the curve \(C\) with homogeneous coordinates \(u_1, u_2\), and \(u := u_1/u_2\). The differential \(d_2^C\) has “matrix elements” \(\widetilde{D}_{\omega_i, \omega_j} : H^1(C, V_{\omega_i}^* \otimes \pi^* \mathcal{L}_{\omega_i}^\phi \otimes \Xi) \rightarrow H^0(C, V_{\omega_i}^* \otimes \pi^* \mathcal{L}_{\omega_i}^\phi \otimes \Xi)\). Note that \(\pi^* \mathcal{L}_{-\omega_i} \cong \mathcal{O}_C(-a_i)\), and \(\pi^* \mathcal{L}_{\omega_i} \cong \mathcal{O}_C(a_j)\), while \(\Xi \cong \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)\) is Serre selfdual, so that \(\widetilde{D}_{\omega_i, \omega_j} \in V_{\omega_i}^* \otimes V_{\omega_j} \otimes H^0(C, \mathcal{O}_C(a_i - 1) \oplus \mathcal{O}_C(a_j - 1)) \otimes H^0(C, \mathcal{O}_C(a_i - 1) \oplus \mathcal{O}_C(a_j - 1))\). Decomposing \(V_{\omega_i}^* \otimes V_{\omega_j}\) according to the weights of \(T\), for \(\lambda \in \Lambda^V\) we obtain a matrix element \(\widetilde{D}_{\omega_i, \omega_j}^\lambda\) which is defined as the weight \(\lambda^*\)-component of \(\widetilde{D}_{\omega_i, \omega_j}\). Then according to [7, 3.8], \(\widetilde{D}_{\omega_i, \omega_j}^\lambda = 0 = \widetilde{D}_{\omega_i, \omega_j}^\lambda - 2\alpha_i\), and if \(i \neq j\), then \(\widetilde{D}_{\omega_i, \omega_j} = 0 = \widetilde{D}_{\omega_i, \omega_j} - \alpha_i\), while

\[
\widetilde{D}_{\omega_i, \omega_j} = \frac{Q_i(z_1, z_2)R_i(u_1, u_2) - R_i(z_1, z_2)Q_i(u_1, u_2)}{z_1 u_2 - z_2 u_1}
\]

\[
\text{(3.16)}
\]

Here the homogeneous polynomials \(Q_i, R_i, S_{ij}\) are but the homogenizations of the same named polynomials of one variable introduced in Section 3.1, and the above matrix coefficients are “scalar” \(2 \times 2\) matrices with respect to the decomposition \(\Xi \cong \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)\).

Now to compute the desired \(d_2^C\) it remains to describe the horizontal arrows of the commutative diagram of Corollary 3.15. The lower one \(H^0(C, (\bigoplus_{i \in I} V_{\omega_i}^* \otimes \pi^* \mathcal{L}_{\omega_i}^\phi) \otimes \Xi) \rightarrow H^0(C, \mathcal{O}_C(a_i - 1) \oplus \mathcal{O}_C(a_j - 1)) \cong H^0(C, \pi^* \mathcal{L}_{\omega_i}^\phi \otimes \Xi) \rightarrow H^0(C, \pi^* \mathcal{L}_{\omega_i}^\phi) \cong H^0(C, \mathcal{O}_C(a_i))\) which takes a pair \((P_1(z_1, z_2), P_2(z_1, z_2))\) of homogeneous degree \(a_i - 1\) polynomials to \(z_1 P_1 + z_2 P_2\). The upper arrow \(H^1(C, (\bigoplus_{i \in I} V_{\omega_i}^* \otimes \pi^* \mathcal{L}_{\omega_i}^\phi) \otimes \omega_C) \rightarrow H^1(C, \mathcal{O}_C(a_i))\) arises from the dual embedding \(H^0(C, \mathcal{O}_C(a_i))^* \cong H^1(C, \pi^* \mathcal{L}_{\omega_i}^\phi \otimes \omega_C) \rightarrow H^1(C, \pi^* \mathcal{L}_{\omega_i}^\phi \otimes \Xi) \cong H^0(C, \mathcal{O}_C(a_i - 1) \oplus \mathcal{O}_C(a_i - 1))^*\). Namely, if we think of \(H^0(C, \mathcal{O}_C(a_i))^*\) as of the homogeneous degree \(a_i\) differential operators in \((u_1, u_2)\), then an operator \(O \) goes to \((O_1, O_2)\) such that \(O_1(P) := O(u_2 \cdot P)/2\), while \(O_2(P) := O(u_1 \cdot P)/2\).
Proposition 3.17. The matrix elements of the differential $d_2^*$ (3.5) are

$$d_2^\alpha_{i,i} = \tilde{d}_i(F_i \ast v_{i,\ast} \otimes v_{i,\ast} - v_{i,\ast} \otimes F_i \ast v_{i,\ast}) \otimes \frac{(z + u)(Q_i(z)R_i(u) - R_i(z)Q_i(u))}{2(z - u)}$$ (3.20)

$$d_2^\alpha_{i,j} = (\alpha_i, \alpha_j)(\alpha_i, \alpha_j)^{-1} v_{i,\ast} \otimes F_i \ast F_j \ast v_{j,\ast} + F_i \ast v_{i,\ast} \otimes F_j \ast v_{j,\ast} + (\alpha_j, \alpha_i)^{-1} F_j \ast F_i \ast v_{i,\ast} \otimes v_{j,\ast}) \otimes \frac{(z + u)(R_i(z)R_j(u) - Q_i(z)S_{ij}(u) - S_{ij}(z)Q_j(u))}{2(z - u)}$$ (3.21)

while $D_{\alpha_i,\alpha_j} = 0 = D_{\alpha_i,\alpha_i}^\alpha$, and if $i \neq j$, then $D_{\alpha_i,\alpha_j} = 0 = D_{\alpha_j,\alpha_j}^\alpha$.

3.18. Calculation of the Poisson bracket. The differential $d_2$ (Section 3.8) defines a bivector field on $\hat{Y}^\alpha$ (i.e. a bidifferential operation on the coordinate ring of $\hat{Y}^\alpha$). We denote the bivector by $\mathfrak{B}$ and the corresponding bidifferential operation on the coordinate ring of $\hat{Y}^\alpha$ simply by $\{,\}$ (though it is not a Poisson bracket on $\hat{Y}^\alpha$).

Proposition 3.19. We have

$$\{w_{i,r}, w_{j,s}\} = 0,$$

$$\{w_{i,r}, y_{j,s}\} = \tilde{d}_i \delta_{ij} \delta_{rs} w_{i,r} y_{j,s},$$

$$\{y_{i,r}, y_{j,s}\} = (1 - \delta_{ij})(\alpha_i, \alpha_j) \frac{w_{i,r} + w_{j,s}}{2(w_{i,r} - w_{j,s})} y_{i,r} y_{j,s}.$$ 

Proof. By Proposition 3.17 on $\hat{Y}^\alpha$ we have

$$\{Q_i(z), Q_j(u)\} = 0,$$ (3.22)

$$\{Q_i(z), R_j(u)\} = -\tilde{d}_i \delta_{ij} \frac{z + u}{2(z - u)} (Q_i(z)R_j(u) - R_i(z)Q_j(u)),$$ (3.23)
Corollary 3.21. In the coordinates  
\[ \{ R_i(z), R_j(u) \} = (1 - \delta_{ij})(\alpha_i, \alpha_j) \frac{z + u}{2(z - u)} (R_i(z)R_j(u) - Q_i(z)S_{ji}(u) - S_{ij}(z)Q_j(u)). \]  
(3.24)  

The relation \( \{ w_{i,r}, w_{j,s} \} = 0 \) is obvious from (3.22). We have \( \{ B_i, B_j \} = 0 \) and \( \{ w_{i,r}, B_j \} = 0 \)  
from (3.22) as well. Substituting \( u = w_{j,s} \) to (3.23), we get \( \{ B_i, y_{j,s} \} = -\frac{\delta_{ij}}{2} B_i y_{j,s} \) and  
\( \{ w_{i,r}, y_{j,s} \} = \delta_{ij} \delta_{rs} w_{i,r} y_{j,s} \). Finally, substituting \( z = w_{i,r}, u = w_{j,s} \) to (3.24), we get  
\( \{ y_{i,r}, y_{j,s} \} = (1 - \delta_{ij})(\alpha_i, \alpha_j) \frac{w_{i,r} + w_{j,s}}{w_{i,r} - w_{j,s}} y_{i,r} y_{j,s}. \) \( \square \)  

The \( 1 \times T \) action on \( \hat{Y} \) preserves this bivector field, hence it gives a well-defined bivector field on \( \hat{Y}/(1 \times T) \). Moreover, the following is true:  

**Corollary 3.20.** The map \( \varpi : \hat{Y} \rightarrow T\hat{Z} \) agrees with the bivector field \( \mathfrak{B} \) on \( \hat{Y} \) (in the sense that for \( f_1, f_2 \in \mathbb{C}[T\hat{Z}] \) we have \( \{ \varpi^*(f_1), \varpi^*(f_2) \} = \varpi^*(f) \) for some \( f \in \mathbb{C}[\hat{Z}] \)). So we get a bivector field on \( T\hat{Z} = \varpi(\hat{Y}) \).  

**Proof.** Note that the functions \( w_{i,r}, y_{j,s} \) form a (rational étale) coordinate system on \( T\hat{Z} \). So the only thing to be checked is that the bracket of any pair of pullbacks of these functions  
is a pullback of some function on \( T\hat{Z} \). But this immediately follows from Proposition 3.19. \( \square \)  

Slightly abusing notation we denote the image of \( \mathfrak{B} \) on \( T\hat{Z} \) also by \( \mathfrak{B} \).  

**Corollary 3.21.** In the coordinates \( w_{i,r}, y_{j,s} \) on \( T\hat{Z} \) the bivector field reads  
\[ \mathfrak{B} = \sum_{i,r} \frac{\partial}{\partial w_{i,r}} \wedge \frac{\partial}{\partial y_{i,r}} + \sum_{i \neq j} \sum_{r,s} \frac{1}{2} (\alpha_i, \alpha_j) \frac{w_{i,r} + w_{j,s}}{w_{i,r} - w_{j,s}} \frac{\partial}{\partial y_{i,r}} \wedge \frac{\partial}{\partial y_{j,s}}. \]  

**Corollary 3.22.** The bivector field \( \mathfrak{B} \) on \( T\hat{Z} \) is Poisson, i.e. \( [\mathfrak{B}, \mathfrak{B}] = 0 \). This Poisson structure extends uniquely to \( \hat{Z} \).  

**Proof.** The first claim is immediate from the explicit formula of Corollary 3.21. We have a smooth open subvariety \( T\hat{Z} \subset \hat{Z} \subset \hat{Z} \) formed by the based quasimaps with defect at most a simple coroot, see e.g. [1, proof of Proposition 5.1]. Its complement has codimension 2 in \( \hat{Z} \). Now \( \hat{Z} \) is normal by [1, Corollary 2.10], so it suffices to check that the bivector field on \( \hat{Z} \) extends as a Poisson structure to \( \hat{Z} \). Moreover, it suffices to check this at the generic points of the boundary components \( \hat{Z} \setminus \hat{Z} \) (given by equations \( y_{i,r} = 0 \)) where the claim is evident from the explicit formula of Corollary 3.21. \( \square \)  

**Corollary 3.23.** The Poisson structure \( \mathfrak{B} \) on \( T\hat{Z} \) is nondegenerate. The corresponding symplectic form reads  
\[ \Omega_{\text{trig}} := \mathfrak{B}^{-1} = \sum_{i,r} \frac{dy_{i,s} \wedge dw_{i,r}}{d_i w_{i,r} y_{i,r}} + \sum_{i \neq j} \sum_{r,s} \frac{1}{2d_i d_j} \frac{w_{i,r} + w_{j,s} \frac{dw_{i,r} \wedge dw_{j,s}}{w_{i,r} w_{j,s}}}{w_{i,r} w_{j,s}}. \]
4. Transversal slices in the affine flag variety

4.1. Schubert cells in the affine flag varieties. We have an embedding of the affine Grassmannian of $G$ (thin one: an ind-scheme) into the Kashiwara affine Grassmannian (thick one: an infinite type scheme): $\text{Gr} = G((z))/G[[z]] \cong G[[z]]^{\pm 1}/G[z] \hookrightarrow G((z^{-1}))/G[z] = \text{Gr}$. The subgroup of currents $G[[z]]$ (resp. $G[[z^{-1}]]$) taking value in $B$ (resp. $B_-$) at $z = 0$ (resp. $z = \infty$) is the Iwahori group $\text{Iw}$ (resp. $\text{Iw}_-$). The unipotent radical of $\text{Iw}$ (resp. $\text{Iw}_-$) is denoted $\text{N}$ (resp. $\text{N}_-$). We have an embedding of the affine flag variety of $G$ (thin one: an ind-scheme) into the Kashiwara affine flag variety (thick one: an infinite type scheme): $\mathcal{F} = G((z))/\text{Iw} \cong G[[z]]^{\pm 1}/(G[z]\cap \text{Iw}) \hookrightarrow G((z^{-1}))/((G[z]\cap \text{Iw}) = \mathcal{F}_\ell$. The natural projection $\text{pr} : \mathcal{F}_\ell \rightarrow \text{Gr}$ (as well as its restriction $\text{pr} : \mathcal{F}_\ell \rightarrow \text{Gr}$) is a fibration with fibers $\mathcal{B}$.

The set of $T$-fixed points in $\mathcal{F}_\ell$ (resp. $\text{Gr}$) is in a natural bijection with the affine Weyl group $W_a = W \ltimes \Lambda$. For $w \in W_a$ we will denote the corresponding $T$-fixed point by the same symbol $w$; its $\text{N}$-orbit (resp. $\text{N}_-$-orbit) will be denoted by $\mathcal{F}_\ell^w \subset \mathcal{F}_\ell$ (resp. $\mathcal{F}_\ell^w \subset \mathcal{F}_\ell$): a thin (resp. thick) Schubert cell. The intersection $\mathcal{F}_\ell^w := \mathcal{F}_\ell^w \cap \mathcal{F}_\ell^y$ (an open Richardson variety, aka transversal slice) is nonempty iff $w \leq y$ in the Bruhat order. Similarly, for $\lambda \in \Lambda \subset \text{Gr}$ the $\text{N}$-orbit $\text{N} \cdot \lambda$ (resp. $\text{N}_- \cdot \lambda$) will be denoted by $X^\lambda \subset \text{Gr}$ (resp. $X_\lambda \subset \text{Gr}$) as an element in the affine Weyl group; it is the minimal length representative of its $\text{N}$-orbit (resp. $\text{N}_-$-orbit) will be denoted by $\text{Gr}^\lambda = \bigcup_{\nu \in W \cdot \lambda} X^\nu$. The set of coweight $\lambda \in \Lambda^+ \subset \Lambda$ the $G((z))$-orbit $\text{Gr}^\lambda := G((z)) \cdot \lambda$ is a union $\text{Gr}^\lambda = \bigcup_{w \in W \cdot \lambda} X^w$.

Recall the notations of [I] 2.4: $G_1 \subset G[[z]]$ is the kernel of evaluation at $z = \infty$, and $W_\mu := G_1 \cdot \mu \subset \text{Gr}$ for $\mu \in \Lambda$. If $\mu$ is dominant, then $W_\mu = X_\mu$. If $\lambda \geq \mu$ is also dominant, then the transversal slice $W_\mu^\lambda := \text{Gr}^\lambda \cap W_\mu$ of loc. cit. is a union $W_\mu^\lambda = \bigcup_{\nu \in W \cdot \lambda} X^\nu \cap X_\mu$.

Given a dominant $\eta \in \Lambda^+ \subset \Lambda \subset W_a$, we consider $-\eta$ as an element in the affine Weyl group; it is the minimal length representative of its left $W$-coset, and the maximal length representative of its right $W$-coset. Furthermore, $\eta$ is the minimal length representative of its left $W$-coset, and the minimal length representative of its right $W$-coset. The projection $\text{pr} : \mathcal{F}_\ell^w \rightarrow \text{Gr}^T = \Lambda$ realizes $\Lambda$ as the set of left $W$-cosets in $W_a$. Hence for a dominant $\lambda \in W_a/W$, the affine Weyl group element $w_0(\lambda) \in \Lambda \subset W_a$ (resp. $w_0(\lambda) \times w_0 = w_0 \times \lambda \in W \ltimes \Lambda = W_a$) is the minimal length (resp. maximal length) representative of the left $W$-coset $\lambda$. In particular, $\text{pr} : \mathcal{F}_\ell^w \sim \rightarrow X^\lambda$, and $\text{pr} : \mathcal{F}_\ell^{w_0(\lambda)} \sim \rightarrow X_\lambda$.

Finally, for $\lambda \geq \mu$, the intersection $X^\lambda \cap X_\mu$ is open in the slice $W_\mu^\lambda \subset \text{Gr}$, and $\text{pr}$ is an open embedding of the open Richardson variety $\mathcal{F}_\ell^{w_0(\lambda) \times \mu} = \mathcal{F}_\ell^{w_0 \times \lambda} \cap \mathcal{F}_\ell^{w_0 \times \mu}$ into $X^\lambda \cap X_\mu$. All in all,

$$\text{pr} : \mathcal{F}_\ell^{w_0 \times \lambda} \hookrightarrow W_\mu^\lambda$$

is an open embedding.

4.2. A modular interpretation. Recall the morphism $s_\mu^\lambda : W_\mu^\lambda \rightarrow Z^\alpha$ of [I] Theorem 2.8 (here $\alpha = \lambda - \mu$). Recall the open subscheme of trigonometric monopoles $\mathcal{D}^\alpha \subset Z^\alpha$ introduced in Definition 2.7.

**Proposition 4.3.** The composition $s_\mu^\lambda \circ \text{pr} : \mathcal{F}_\ell^{w_0 \times \lambda} \rightarrow Z^\alpha$ is an open embedding with the image $\mathcal{D}^\alpha \subset Z^\alpha$.  

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Proof. Recall from the end of the proof of [1] Theorem 2.8 that the restriction of $s^\lambda_\mu$ to $\mathcal{S}^\lambda \cap \overline{\mathcal{W}}^{\lambda}_\mu$ is an isomorphism onto $\mathcal{O}^\alpha \subset \mathcal{Z}^\alpha$. Here $\overline{\mathcal{W}}^{\lambda}_\mu = \bigcup_{\mu \leq \nu \leq \lambda} \mathcal{W}^\nu_\mu$, and $\mathcal{S}^\lambda$ is a semiinfinite orbit (whose intersection with $\mathcal{W}^\nu_\mu$ is dense in $\mathcal{W}^{\lambda}_\mu$). According to [18] Theorem 3.2, (3.6)], for $\nu < \lambda$ we have $\mathcal{S}^\lambda \cap \text{Gr}^\nu = \emptyset$, and $\mathcal{S}^\lambda \cap \text{Gr}^\lambda = \mathcal{X}^\lambda$. It follows that $\mathcal{S}^\lambda \cap \overline{\mathcal{W}}^{\lambda}_\mu = \mathcal{X}^\lambda \cap \mathcal{X}_\mu$. It remains to check that the open subset $\text{pr}(\mathcal{F}_w^{\lambda \times \mu}) \subset \mathcal{X}^\lambda \cap \mathcal{X}_\mu$ goes to the open subset $\mathcal{O}^\alpha \subset \mathcal{Z}^\alpha$ under the isomorphism $s^\lambda_\mu: \mathcal{X}^\lambda \cap \mathcal{X}_\mu = \mathcal{S}^\lambda \cap \overline{\mathcal{W}}^{\lambda}_\mu \sim \mathcal{O}^\alpha$.

To this end recall from [1] the proof of Theorem 2.8 the modular interpretation of our slices. For the sake of exposition we assume that both $\lambda$ and $\mu$ are regular dominant coweights. The argument in the general case is similar but requires introducing more notations. First of all, $\text{Gr}$ is the moduli space of $G$-bundles $\mathcal{F}_G$ on $C$ equipped with a trivialization $\sigma$ in the formal neighbourhood of $\infty \in C$. Second, $\text{Fl}$ is the moduli space of triples $(\mathcal{F}_G, \sigma, F)$ where $(\mathcal{F}_G, \sigma) \in \text{Gr}$, and $F$ is a $B$-structure in the fiber of $\mathcal{F}_G$ at $0 \in C$. Third, $\mathcal{X}_\mu = \mathcal{W}_\mu \subset \text{Gr}$ is formed by the pairs $(\mathcal{F}_G, \sigma)$ such that the isomorphism type of $\mathcal{F}_G$ is $\mu$, and the value of the Harder-Narasimhan flag of $\mathcal{F}_G$ at $\infty \in C$ is $B_\mu \subset B$ (with respect to the trivialization $\sigma$ at $\infty \in C$). Now $\text{Fl}_{w_\mu} \subset \text{Fl}$ is formed by the triples $(\mathcal{F}_G, \sigma, F)$ such that $(\mathcal{F}_G, \sigma) \in \mathcal{X}_\mu$, and $F$ is the value of the Harder-Narasimhan flag of $\mathcal{F}_G$ at $0 \in C$. Furthermore, $\text{Gr}$ is the moduli space of $G$-bundles $\mathcal{F}_G$ on $C$ equipped with a trivialization $\zeta$ over $C \setminus 0$, while $\mathcal{F}_\ell$ is the moduli space of triples $(\mathcal{F}_G, \zeta, F)$ where $(\mathcal{F}_G, \zeta) \in \text{Gr}$, and $F$ is a $B$-structure in the fiber of $\mathcal{F}_G$ at $0 \in C$. The projection $\text{pr}: \mathcal{F}_\ell \to \text{Gr}$ admits a section $s$ over $\text{Gr}^\lambda$; we define $F$ as the fiber of the extension through $0 \in C$ (due to properness of $B$) of the trivial $B$-structure $B \subset B$ in the trivial $G$-bundle over $C \setminus 0$ transferred to $\mathcal{F}_G$ via $\zeta$. Finally, $\mathcal{F}_w^{\lambda \times \mu} \subset \mathcal{F}_\ell$ is formed by the triples $(\mathcal{F}_G, \zeta, F)$ such that $(\mathcal{F}_G, \zeta) \in X^\lambda$, and $F$ is transversal to $s(\mathcal{F}_G, \zeta)$.

Finally, comparing with the definition of $s^\lambda_\mu$ in the proof of [1] Theorem 2.8, we see that $s^\lambda_\mu(pr(\mathcal{F}_w^{\mu \times \lambda})) \subset \mathcal{O}^\alpha$ is formed by all the based maps $(C, \infty) \to (B, B_\mu)$ whose value at $0 \in C$ is transversal to $B$.

The proposition is proved.

4.4. Stabilization. Let $\mu, \nu \in \Lambda^+$ be dominant coweights. According to [14] 2E], we have the inclusion of stabilizers $\text{St}_\mu \subset \text{St}_{\mu + \nu} \subset G_1 \subset G[[z^{-1}]]$, so the identity morphism $G_1 \to G_1$ induces a morphism $s^\mu_\mu + \nu: X_\mu = G_1/\text{St}_\mu \to G_1/\text{St}_{\mu + \nu} = X_{\mu + \nu}$. According to loc. cit., $s^\mu_{\mu + \nu}$ restricts to the same named morphism $\mathcal{W}^\lambda_\mu \to \mathcal{W}^{\lambda + \nu}_{\mu + \nu}$ for any $\Lambda^+ \ni \lambda \geq \mu$. Similarly, we have $\text{St}_{w_\mu} \subset \text{St}_{w_\mu \times (\mu + \nu)} \subset \text{N}_-$, and the identity morphism $\text{N}_- \to \text{N}_-$ induces a morphism $\sigma^\mu_\mu + \nu: \text{Fl}_{w_\mu} = \text{N}_-/\text{St}_{w_\mu} \to \text{N}_-/\text{St}_{w_\mu \times (\mu + \nu)} = \text{Fl}_{w_\mu \times (\mu + \nu)}$ which restricts to the same named morphism $\mathcal{F}_w^{\lambda \times \mu} \to \mathcal{F}_w^{\lambda \times (\mu + \nu)}$ for any $\Lambda^+ \ni \lambda \geq \mu$. The following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{F}_w^{\lambda \times \mu} & \xrightarrow{\sigma^\mu_\mu + \nu} & \mathcal{F}_w^{\mu \times (\mu + \nu)} \\
\text{pr} & & \text{pr} \\
\mathcal{W}^\lambda_\mu & \xrightarrow{s^\mu_\mu + \nu} & \mathcal{W}^{\lambda + \nu}_{\mu + \nu}
\end{array}
\]
Moreover, from the construction of \( s_\mu^\lambda : W_\mu^\lambda \to Z^\alpha \) in [1] Lemma 2.7, Theorem 2.8 (where \( \alpha = \lambda - \mu \)) it follows immediately that the following diagrams commute as well:

\[
\begin{array}{c}
W_\mu^\lambda \xrightarrow{s_\mu^\lambda \eta^{\mu+\nu}} W_{\mu+\nu}^\lambda \\
\downarrow s_\nu^\mu \downarrow \\
Z^\alpha \cong Z^\alpha
\end{array}
\]  

(4.3)

\[
\begin{array}{c}
\mathcal{F}_{w_0 \times \lambda}^{\mu+\nu} \xrightarrow{\sigma_{\mu+\nu}^\nu \circ \text{pr}} \mathcal{F}_{w_0 \times (\lambda+\nu)}^{\mu} \\
\downarrow s_\nu^\mu \circ \text{pr} \downarrow \\
\overset{\varnothing}{Z^\alpha} \cong \overset{\varnothing}{Z^\alpha}
\end{array}
\]  

(4.4)

It follows in particular that \( \sigma_{\mu+\nu}^\nu : \mathcal{F}_{w_0 \times \lambda}^{\mu+\nu} \xrightarrow{\sim} \mathcal{F}_{w_0 \times (\lambda+\nu)}^{\mu} \) is an isomorphism.

4.5. \( s_\mu^\lambda \) in coordinates. We will use the generalized minors of [1] 2A to construct regular functions on the open Richardson varieties; however, our notation differs slightly from the one of loc. cit. Namely, given an irreducible \( G \)-module \( V_\lambda \) with highest weight \( \lambda \in \Lambda_+^\mu \) and highest vector \( v_\lambda \), its dual \( V^*_\lambda \) is isomorphic to \( V_{\lambda^\ast} \) with the lowest weight \( -\lambda \), and the lowest vector \( v_{-\lambda} \) such that \( \langle v_{-\lambda}, v_\lambda \rangle = 1 \). Given \( w, y \in W \), we define the following regular function on \( G \)

\[
\Delta_{w,\lambda,\lambda}(g) := \langle \overline{w} v_{-\lambda}, g \overline{v}_\lambda \rangle
\]

(4.5)

where \( \overline{w}, \overline{v} \in G \) are the lifts of \( w, v \) defined in loc. cit.

Following loc. cit. we consider the regular functions \( \Delta_{w,\lambda,\lambda}^{(s)} \), \( s \in \mathbb{Z} \), on \( G((z^{-1})) \) defined as follows:

\[
\Delta_{w,\lambda,\lambda}^{(s)}(g(z)) = \sum_{s=-\infty}^{\infty} \Delta_{w,\lambda,\lambda}(g(z))z^{-s}
\]

(4.6)

More generally, to any \( v \in V_\lambda \) and \( \beta \in V_\lambda^* \) we can assign the generalized minor \( \Delta_{\beta,\nu}(z) := \langle \beta, g(z)v \rangle \). We also denote by \( \Delta_{\beta,\nu}^{(s)} \) the coefficient at \( z^{-s} \) of the power series \( \Delta_{\beta,\nu}(z) \).

Recall from [1] 2.6 that \( St_\mu \subset G_1 \subset G[[z^{-1}]] \) is the stabilizer of \( \mu \in \Lambda_+ \subset \Lambda = Gr^T \). Similarly, for \( w \in W_0 \) we denote by \( St_w \subset N_- \) the stabilizer of \( w \in W_0 = \mathcal{F}^T \). We have \( X^\mu_\mu = G_1/\text{St}_\mu \) and \( \text{Fl}_w = N_-/\text{St}_w \). In case \( w = w_0 \times \mu \) (see Section 4.1), we have \( \text{St}_\mu = G_1 \cap \text{St}_{w_0 \times \mu} \), and the natural morphism \( \text{Fl}_{w_0 \times \mu} = N_-/\text{St}_{w_0 \times \mu} \to G_1/\text{St}_\mu = X_\mu^\mu \) is an isomorphism. According to [1] Lemma 2.19, the functions \( \Delta_{w_0,\lambda,\lambda}^{(s)}(z), \Delta_{w_0,\lambda,\lambda}^{(s)}(z), s > 0, i \in I \) restricted to \( G_1 \) (resp. \( N_- \)) are \( St_\mu \)-invariant (resp. \( St_{w_0 \times \mu} \)-invariant); hence they may be viewed as the functions on \( \text{Fl}_{w_0 \times \mu} \cong X_\mu^\mu \).

Now let \( \Lambda^+ \ni \lambda \geq \mu \), and \( \Lambda_+ \ni \alpha = \sum_{i \in I} a_i \alpha_i := \lambda - \mu \). Recall the isomorphism \( s_\mu^\lambda \circ \text{pr} : \mathcal{F}_{w_0 \times \lambda}^{\mu} \xrightarrow{\sim} \overset{\varnothing}{Z^\alpha} \) of Proposition 4.3. Recall also the regular polynomial-valued functions \( Q_i, R_i \) on \( Z^\alpha \) (see e.g. [7] 3.3): \( Q_i = z^{a_i} + q_i z^{a_i-1} + \ldots \) (resp. \( R_i = r_i z^{a_i-1} + \ldots \)) is the highest (resp. prehighest) Plücker coordinate on the space of based quasimaps (in notations of loc. cit. \( Q_i = \phi_{\hat{w}_i}^{\omega_{i}}, R_i = \phi_{\hat{w}_i}^{\omega_{i}} \). Following loc. cit. and Section 3.3 we also
consider a rational étale coordinate system on \( Z^\alpha \). Namely, \((w_{i,r})_{1 \leq i \leq e_i}\) are the ordered roots of \( Q_i \), and \( y_{i,r} := R_i(w_{i,r}) \).

**Proposition 4.6.** Under the isomorphism \( s^\lambda_{\mu} \circ \text{pr} : \mathcal{Z}_{\mu}^{\mu} \times \mathcal{Z}_{\mu}^{\mu} \to \mathcal{Z}_{\mu}^{\mu} \) we have \( Q_i = \sum_{s=0}^{a_i} \Delta^{(s)}_{\omega_i, \omega_i} z^{a_i-s} \), \( R_i = \sum_{s=0}^{a_i} \Delta^{(s)}_{\omega_i, \omega_i} z^{a_i-s} \).

**Proof.** Follows at once from the commutative diagram [1, (2.3)] (and the definition of \( \pi_{\mu, n} \) in [1 Lemma 2.7]).

**4.7. Rational Poisson bracket revisited.** We fix a basis \( e_\alpha, e_{-\alpha}, h_i \) in \( \mathfrak{g} \) where \( i \in I \), and \( \alpha \in \mathbb{R}^+ \) is a positive coroot (and the weight of \( e_\alpha \) is the dual root \( \check{\alpha} \); in particular, \( e_{\alpha_i} = E_i \) of Section 2.2 and \( e_{-\alpha_i} = \check{E}_i F_i \)). We assume \((e_\alpha, e_{-\alpha}) = 1, (h_i, h_j) = \delta_{ij} \). Then the Lie bialgebra structure on \( \mathfrak{g}((z^{-1})) \) is determined by the classical rational \( r \)-matrix

\[
r_{\text{rat}}(z,u) := \frac{1}{z-u} \left( \sum_{\alpha > 0} e_\alpha \otimes e_{-\alpha} + e_{-\alpha} \otimes e_\alpha + \sum_{i \in I} h_i \otimes h_i \right),
\]

see e.g. [4, Section 6.4]. This determines a Poisson group structure on \( G((z^{-1})) \) such that \( G_1 \) is a Poisson subgroup.

**Proposition 4.8.** [14, Proposition 2.13] The rational Poisson bracket \( \{, \}_{\text{rat}} \) on the subgroup \( G_1 \) is

\[
\{ \Delta_{\beta_1, v_1}(z), \Delta_{\beta_2, v_2}(u) \}_{\text{rat}} = \frac{1}{z-u} \left( \sum_{\alpha > 0} \Delta_{\beta_1, e_\alpha v_1}(z) \Delta_{\beta_2, e_{-\alpha} v_2}(u) + \sum_{\alpha > 0} \Delta_{\beta_1, e_{-\alpha} v_1}(z) \Delta_{\beta_2, e_\alpha v_2}(u) + \sum_{i \in I} \Delta_{\beta_1, h_i v_1}(z) \Delta_{\beta_2, h_i v_2}(u) \right) - \frac{1}{z-u} \left( \sum_{\alpha > 0} \Delta_{e_\alpha, \beta_1, v_1}(z) \Delta_{e_{-\alpha}, \beta_2, v_2}(u) + \sum_{\alpha > 0} \Delta_{e_\alpha, \beta_2, v_2}(u) + \sum_{i \in I} \Delta_{h_i, \beta_1, v_1}(z) \Delta_{h_i, \beta_2, v_2}(u) \right).
\]

According to [14], this Poisson structure on \( G_1 \) induces a Poisson structure on transversal slices \( \mathcal{W}^\lambda_{\mu} \) in the affine Grassmannian \( \mathbf{Gr} = G((z^{-1}))/G[z] \). On the other hand, recall a symplectic structure on \( \mathcal{Z}^\alpha \) defined in [7]. It extends uniquely to a Poisson bracket \( \{, \}^{\mathcal{Z}} \) on \( \mathcal{Z}^\alpha \) by the same argument as in the proof of Corollary [2.22].

The following theorem confirms expectations of [14, Remark 2.11].

**Theorem 4.9.** The map \( s_{\mu} : \mathcal{W}^\lambda_{\mu} \to \mathcal{Z}^\alpha \) is Poisson.

**Proof.** The field of rational functions on \( \mathcal{Z}^\alpha \) coincides with the field of rational functions in the Fourier coefficients of the functions \( Q_i(z), R_i(z) \). Hence it is sufficient to show that the Poisson bracket of the coefficients \( Q_i(z), R_i(z) \) is the same on \( \mathcal{W}^\lambda_{\mu} \) and \( \mathcal{Z}^\alpha \). Let us introduce the following generalized minors: \( S_{ij}(z) := \Delta_{E_{ij} E_{iv_1}, v_1} = (E_j \check{E}_i v_1, g(z) v_1) \). According to [21 (7) and (8)], the Poisson bracket of the (polynomial-valued) functions \( Q_i(z), R_i(z) \) is given by

\[
\{ Q_i(z), Q_j(u) \}_{\text{rat}}^{\mathcal{Z}} = 0,
\]

\[
\{ Q_i(z), R_j(u) \}_{\text{rat}}^{\mathcal{Z}} = -\check{d}_i \delta_{ij} \left( \frac{1}{z-u} Q_i(z) R_j(u) - \frac{1}{z-u} R_i(z) Q_j(u) \right),
\]
\[ \{ R_i(z), R_j(u) \}_{\text{rat}}^Z = \]
\[ = (1 - \delta_{ij})((\hat{a}_i, \hat{a}_j) \frac{1}{z-u} R_i(z)R_j(u) + \frac{1}{z-u} Q_i(z)S_{ji}(u)) + \frac{1}{z-u} S_{ij}(z)Q_j(u). \]
(4.10)

On the other hand, the Fourier coefficients of the pullbacks $s^\mu_* Q_i = z^{a_i} + \sum_{s=1}^{a_i} \Delta^{(s)}_{i\hat{a}_i, \hat{a}_i} z^{a_i-s}$ and $s^\mu_* R_i = \sum_{s=1}^{a_i} \Delta^{(s)}_{s_i \hat{a}_i, \hat{a}_i} z^{a_i-s}$ obey the same relations by Proposition 4.8.

4.10. Trigonometric Poisson bracket. The standard Lie bialgebra structure on $\mathfrak{g}((z^{-1}))$ (see e.g. [4, 6.2.1, 6.5]) gives rise to a Poisson structure on $\mathfrak{g}[[z]]$, whose symplectic leaves are the open Richardson varieties $\mathcal{F}_{y}^{u}$ (see [17, Corollary 2.9]).

This Lie bialgebra structure on $\mathfrak{g}((z^{-1}))$ is determined by the classical $r$-matrix
\[ r_{\text{trig}}(z, u) := \frac{1}{z-u} \left( z \left( \sum_{\alpha>0} e_\alpha \otimes e_{-\alpha} + \frac{1}{2} \sum_{i \in I} h_i \otimes h_i \right) + u \left( \sum_{\alpha>0} e_{-\alpha} \otimes e_\alpha + \frac{1}{2} \sum_{i \in I} h_i \otimes h_i \right) \right). \]
(4.11)

see e.g. [4, (6.6)].

**Proposition 4.11.** The Poisson bracket of the functions $\Delta^{(s)}_{\beta, \nu}$ on the Iwahori subgroup $\text{Iw}_{-}$ is
\[ \{ \Delta_{\beta_1, \nu_1}(z), \Delta_{\beta_2, \nu_2}(u) \}_{\text{trig}} = \]
\[ = \frac{1}{z-u} \left( z \left( \sum_{\alpha>0} \Delta_{\beta_1, e_\alpha \nu_1}(z) \Delta_{\beta_2, e_{-\alpha} \nu_2}(u) + u \sum_{\alpha>0} \Delta_{\beta_1, e_{-\alpha} \nu_1}(z) \Delta_{\beta_2, e_\alpha \nu_2}(u) + \right) \right) \]
\[ + \frac{z + u}{2} \sum_{i \in I} \Delta_{\beta_1, h_i \nu_1}(z) \Delta_{\beta_2, h_i \nu_2}(u) - \]
\[ - \frac{1}{z-u} \left( z \left( \sum_{\alpha>0} \Delta_{e_\alpha \beta_1, \nu_1}(z) \Delta_{e_{-\alpha} \beta_2, \nu_2}(u) + u \sum_{\alpha>0} \Delta_{e_{-\alpha} \beta_1, \nu_1}(z) \Delta_{e_\alpha \beta_2, \nu_2}(u) + \right) \right) \]
\[ + \frac{z + u}{2} \sum_{i \in I} \Delta_{h_i \beta_1, \nu_1}(z) \Delta_{h_i \beta_2, \nu_2}(u) \]

**Proof.** This follows from the Belavin-Drinfeld formula for trigonometric $r$-matrix (see e.g. [4, (6.6)]). Indeed, following [14, Proposition 2.13] we note that the cobracket on $\mathfrak{g}((z^{-1}))$ is coboundary, namely it is given by the map
\[ a(t) \mapsto [a(z) \otimes 1 + 1 \otimes a(u), r_{\text{trig}}(z, u)], \]
where the $r$-matrix is given by (4.11). By the standard procedure this gives a structure of Poisson group on $G((z^{-1}))$. We note that the Iwahori subgroup $\text{Iw}_{-} \subset G((z^{-1}))$ is a Poisson subgroup, hence the bracket of any two functions on it is the restriction of the bracket of any extensions of these functions to $G((z^{-1}))$. The rest of the proof is a word-to-word repetition of that of [13, Proposition 2.13].

By an abuse of notation, we will denote the rational étale functions $w_{i,r} \circ s_{\mu}^\lambda \circ p_r$, $y_{i,r} \circ s_{\mu}^\lambda \circ p_r$ (notations of Section 4.5) on $\mathcal{F}_{w_0 \times \lambda}$ simply by $w_{i,r}, y_{i,r}$. 

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Proposition 4.12. We have
\[
\{w_{i,r}, w_{j,s}\}_{\text{trig}} = 0,
\]
\[
\{w_{i,r}, y_{j,s}\}_{\text{trig}} = \delta_{ij} \delta_{rs} w_{i,r} y_{j,s},
\]
\[
\{y_{i,r}, y_{j,s}\}_{\text{trig}} = (1 - \delta_{ij}) (\bar{\alpha}_i, \bar{\alpha}_j) \frac{w_{i,r} + w_{j,s}}{2(w_{i,r} - w_{j,s})} y_{i,r} y_{j,s}.
\]

Proof. Consider the functions \( Q_i(z) = \Delta^{(0)}_{\omega_i, \omega_i} \prod_{r=1}^n (z - w_{i,r}), \) \( R_i(z) = \Delta^{(0)}_{\omega_i, \omega_i} \sum_{r=1}^n y_{i,r} (z - w_{i,r}) Q_i(z) \).

According to Proposition 3.3, we have \( Q_i = \sum_{s=0}^{a_i} \Delta^{(s)}_{\omega_i, \omega_i} z^{a_i-s}, \) \( R_i = \sum_{s=0}^{a_i} \Delta^{(s)}_{\omega_i, \omega_i} z^{a_i-s}. \)

Set \( B_i := \Delta^{(0)}_{\omega_i, \omega_i} \) and recall the generalized minors \( S_{ij}(z) \) introduced in the proof of Theorem 4.9. \( S_{ij}(z) := \Delta_{E_i, E_j, \omega_i, \omega_i} (E_j E_i - \omega_i, g(z) - \omega_j). \) Then by Proposition 4.11 we have
\[
\{Q_i(z), Q_j(u)\}_{\text{trig}} = 0,
\]
\[
\{Q_i(z), R_j(u)\}_{\text{trig}} = -\bar{d}_i \bar{d}_j \frac{z + u}{2(z - u)} Q_i(z) R_j(u) - \frac{u}{z - u} R_i(z) Q_j(u),
\]
\[
\{R_i(z), R_j(u)\}_{\text{trig}} =
= (1 - \delta_{ij}) (\bar{\alpha}_i, \bar{\alpha}_j) \frac{z + u}{2(z - u)} R_i(z) R_j(u) + \bar{d}_i \bar{d}_j \frac{z}{z - u} Q_i(z) S_{ij}(u) + \bar{d}_i \bar{d}_j \frac{u}{z - u} S_{ij}(z) Q_j(u).
\]

The relation \( \{w_{i,r}, w_{j,s}\}_{\text{trig}} = 0 \) is obvious from (4.12). Substituting \( u = w_{j,s} \) to (4.13), we get \( \{B_i, y_{j,s}\}_{\text{trig}} = -\bar{d}_i \bar{d}_j B_i y_{j,s} \) and \( \{w_{i,r}, y_{j,s}\}_{\text{trig}} = \bar{d}_i \delta_{ij} \delta_{rs} w_{i,r} y_{j,s}. \) Finally, substituting \( z = w_{i,r}, u = w_{j,s} \) to (4.14), we get \( \{y_{i,r}, y_{j,s}\}_{\text{trig}} = (1 - \delta_{ij}) (\bar{\alpha}_i, \bar{\alpha}_j) \frac{w_{i,r} + w_{j,s}}{2(w_{i,r} - w_{j,s})} y_{i,r} y_{j,s}. \)

Theorem 4.13. The isomorphism \( s^\lambda \circ pr : T^\mu_{w_0 x_0} \overset{\sim}{\to} \hat{\mathcal{Z}}^\alpha \) of Proposition 4.3 is a symplectomorphism.

Proof. Indeed, by Proposition 4.12 and Proposition 3.19 the Poisson brackets \( \{., .\} \) on \( \hat{\mathcal{Z}}^\alpha \) and \( \{., \}_{\text{trig}} \) on \( T^\mu_{w_0 x_0} \) are given by the same formulas on coordinate functions \( w_{i,r}, y_{i,r}. \)

Remark 4.14. Note that the formulas (4.13) and (4.14) are different from (3.23) and (3.24), so the morphism \( s^\lambda \circ pr : T^\mu_{w_0 x_0} \overset{\sim}{\to} \hat{\mathcal{Z}}^\alpha \) does not extend to a Poisson morphism \( \mathfrak{I}w_\_ \to \hat{Y}^\alpha. \)

5. A speculation on cluster structure

5.1. An affine Lie algebra. Let \( \hat{\mathfrak{g}} \) be the universal central extension of the polynomial loop algebra \( \mathfrak{g}[z^{\pm 1}]: \)
\[
0 \to \mathbb{C} \to \hat{\mathfrak{g}} \to \mathfrak{g}[z^{\pm 1}] \to 0
\]
Let \( \mathfrak{g}_{\text{aff}} = \hat{\mathfrak{g}} \otimes \mathbb{C}d \) be the semidirect product of \( \hat{\mathfrak{g}} \) with the degree operator. Then \( \mathfrak{g}_{\text{aff}} \) is an untwisted affine Kac-Moody Lie algebra. It has a triangular decomposition \( \mathfrak{g}_{\text{aff}} = \mathfrak{n}_- \oplus \mathfrak{t}_{\text{aff}} \oplus \mathfrak{n} \) where \( \mathfrak{n}_- = \text{Lie } \mathcal{N}_- \cap \mathfrak{g}[z^{-1}], \) \( \mathfrak{n} = \text{Lie } \mathcal{N} \cap \mathfrak{g}[z], \) and \( \mathfrak{t}_{\text{aff}} \) is the affine Cartan subalgebra. The
fundamental weights will be denoted \( \varpi_i \), \( i \in I_a := I \cup \{ i_0 \} \). The corresponding fundamental integral representations (where \( n \) acts locally nilpotently) will be denoted \( V_{\varpi_i} \), and their restricted duals (where \( n_+ \) acts locally nilpotently) will be denoted \( V_{\varpi_i}^* \). We choose the highest weight vectors \( v_{\varpi_i} \in V_{\varpi_i} \) and the lowest weight vectors \( v_{-\varpi_i} \in V_{\varpi_i}^* \) such that \( \langle v_{-\varpi_i}, v_{\varpi_i} \rangle = 1 \). Note that the action of \( n_- \) (resp. \( n_+ \)) on \( V_{\varpi_i}^* \) (resp. \( V_{\varpi_i} \)) integrates to the action of \( N_- \) (resp. \( N_+ \)). Given \( w, y \in W_0 \) and \( i \in I_a \) we define the following regular function on \( N_- \) (a generalized minor):

\[
\Delta_{w,\varpi_i,\varpi_j}(g) := \langle \overline{w} v_{-\varpi_i}, g \overline{y} v_{\varpi_j} \rangle
\]

where \( \overline{w}, \overline{y} \in G_{aff} \) are the lifts of \( w, y \) defined similarly to \([14, 2A]\).

5.2. An initial seed. B. Leclerc defines in \([16]\) a cluster structure on the open Richardson varieties in the flag varieties of simple Lie algebras of types \( ADE \), but presumably the construction can be extended to the affine Lie algebras of arbitrary types. Here we describe the initial seed for \( \mathcal{F}_{w_0,\lambda}^{\mu} \) following \([16\text{, Section 5, 4.8.3, Corollary 4.4}]\). We choose a reduced expression in the affine Weyl group: \( \lambda = s_{i_1} \ldots s_{i_l} \) where \( i_1, \ldots, i_l \in I_a \), and \( l = 2|\lambda| \) is the length of \( \lambda \in \Lambda^+ \subset \Lambda_+ \subset \Lambda \subset W_0 \) (for \( \lambda = \sum_{i \in I} a_i \alpha_i \) we have \( 2|\lambda| = 2 \sum_{i \in I} a_i \)).

Note that \( i_1 = i_0 \) (the affine simple reflection). Then the initial seed consists of all the (irreducible factors of the) generalized minors \( \Delta_{w_0 s_{i_1} \ldots s_{i_l}, \varpi_{r}, \varpi_{s}, w_0, w_0} \), \( 1 \leq r \leq l \) (they are well defined as functions on \( \mathcal{F}_{w_0,\lambda}^{\mu} \) according to loc. cit.). Among them, which divide \( \prod_{i \in I_a} \Delta_{w_0, \varpi_i, \varpi_i, w_0, w_0} \) are the frozen variables.

5.3. An exchange matrix. The rows of the exchange matrix \( B \) are numbered by \( 1 \leq r \leq l \), and the columns are numbered by those \( 1 \leq s \leq l \) for which there exists \( r > s \) such that \( i_r = i_s \) (the minimal among such \( r \) is denoted \( s^+ \)). The matrix entries are as follows:

- \( b_{r,s}^+ = -b_{s^+,s} = -1 \); and for \( s < r < s^+ \) such that for any \( r < r' < s^+ \) we have \( i_r \neq i_{r'} \), the matrix entry \( b_{r,r'} = C_{i_{r},i_{r'}} \), and \( b_{r,s} = C_{i_{r},i_{s}} \) (here \( (C_{i,j})_{i,j \in I_a} \) is the Cartan matrix of \( \mathfrak{g} \)).

All the other matrix entries are zero.

According to \([16\text{, Section 6}]\), this cluster structure on \( \mathcal{F}_{w_0,\lambda}^{\mu} \) is compatible with the symplectic structure of Section 4.10 on \( \mathcal{F}_{w_0,\lambda}^{\mu} \) in the sense of \([8\text{, Section 4.1}]\). Taking \( \mu = 0 \) and \( \alpha = \lambda \), and transferring the cluster structure via the isomorphism \( s_0^\lambda \circ \text{pr} : \mathcal{F}_{w_0,\lambda}^{\mu} \xrightarrow{\sim} \hat{\mathcal{Z}}^\alpha \) we obtain a cluster structure on \( \hat{\mathcal{Z}}^\alpha \) compatible with its symplectic structure (see Theorem 4.13).

5.4. Destabilization. Let \( \nu \in \Lambda^+ \) be a dominant coweight. Then the open Richardson variety \( \mathcal{F}_{w_0,\lambda}^{\mu}(\lambda+\nu) \) also has a cluster structure with the initial cluster given by certain generalized minors, and with the same exchange matrix as in Section 5.3. However, the stabilization map \([4.13] \sigma: \mathcal{F}_{w_0,\lambda}^{\mu} \xrightarrow{\sim} \mathcal{F}_{w_0,\nu}^{w_0,\lambda+\nu} \) does not take the initial seed of \( \mathcal{F}_{w_0,\lambda}^{\mu} \) to the initial seed of \( \mathcal{F}_{w_0,\nu}^{w_0,\lambda+\nu} \) (already in the simplest example of 2-dimensional slices for \( \mathfrak{g} = \mathfrak{sl}_2 \) where both variables are frozen, cf. \([14\text{, Example 2.12}]\)).

We consider the following action of \( \mathbb{Z}^l \) on \( \hat{\mathcal{Z}}^\alpha \): the generator \((0, \ldots, 0, 1, 0, \ldots, 0) \) (1 at the \( i \)-th place) acts in the Plücker coordinates \((Q_j, R_j)_{j \in I}\) of Section 4.5 by an automorphism \( \eta_i(Q_j, R_j) = (Q_j, z_i^{\delta_j} R_j - \delta_j \eta_i a_0 - 1 Q_j) \) (it is easy to check that this is indeed a biregular automorphism of \( \hat{\mathcal{Z}}^\alpha \)). The frozen variables of the cluster structure on \( \hat{\mathcal{Z}}^\alpha \) are \((F_{\alpha, q_j, 0})_{j \in I}\)
where $F_a$ is the equation of the boundary of zastava, see e.g. [3, Section 5]. Clearly, $\eta_i$ takes $(F_a,q_{j,0})_{j \in I}$ to $(F_{a,q_{j,0}},q_{j,0})_{j \in I}$, i.e. does not preserve the frozen variables. However, it seems likely that $\eta_i$ is an almost cluster transformation: a composition of a few mutations, and the above change of frozen variables. Furthermore, if we set $\eta_\nu := \prod_{i \in I} \eta_i^{n_i}$ for $\nu = \sum_{i \in I} n_i \alpha_i$, then the cluster structure transferred to $\check{Z}^\lambda$ from the isomorphism $s_0^{\nu+\nu} \circ \text{pr} : \mathcal{F}_{w_0}^{\nu+\nu} \to \check{Z}^\lambda$ differs from the reference one (transferred from the isomorphism $s_0^{\nu} \circ \text{pr} : \mathcal{F}_{w_0}^{\nu} \to \check{Z}^\lambda$) by the automorphism $\eta_\nu$.

5.5. $g = \mathfrak{sl}_2$. For $g = \mathfrak{sl}_2$, a positive coroot $\alpha$ is but a positive integer $a$, and a cluster structure on $\check{Z}^a$ was defined in [9, Section 5] (where $\check{Z}^a$ is denoted $\check{R}_a$). According to Theorem 5.1 this cluster structure is a particular case of the one of Section 5.4. In particular, the exchange matrix $B(z)$, $(z) = (2,0,\ldots,0)$ of [9, (5.16)] coincides with the exchange matrix of Section 5.3. Note that the cluster variables of [9] are certain minors of a Hankel matrix composed of the coefficients of the formal series $R(z)/Q(z) \in \mathbb{C}[[z^{-1}]]$ (where $R,Q$ are the Plücker coordinates of Section 4.5). It would be nice to have such an explicit formula for the cluster variables for general $g$. Also, the automorphism $\eta$ of Section 5.4 is nothing but the shift of [9, Lemma 5.4.(i)] (a transformation from the type $A_{a-1}$ Q-system, cf. the paragraph before [9, Remark 6.2]).

5.6. Gaiotto-Witten superpotential. Let $K_i(z), i \in I$, be a collection of monic polynomials, $K_i(z) = z^i + \kappa_{i, i-1} z^{i-1} + \ldots + \kappa_{i,0}$. The data of $\{K_i(z)\}_{i \in I}$ is equivalent to the data of
(a) an ordered collection $\Lambda$ of dominant coweights $\lambda_1, \ldots, \lambda_N$;
(b) an ordered configuration $(z_1, \ldots, z_N)$ of points in $\mathbb{A}^1$. Namely, given the above data we set $K_i(z) := \prod_{1 \leq n \leq N} (z - z_n)^{\lambda_i, \alpha_i}$. We denote by $\mathbb{A}_\Lambda$ the moduli space of the above configurations of distinct points $z_n$.

Recall the Gaiotto-Witten superpotential $W_{-}^{\Lambda, \alpha}$: a multivalued holomorphic function on $\mathfrak{h}^\vee \times \check{Z}^\alpha \times \mathbb{A}_\Lambda$ (see e.g. [3, 1.8]). We will denote by $\overline{W}_{-}^{\Lambda, \alpha}$ the restriction of $W_{-}^{\Lambda, \alpha}$ to $0 \times \check{Z}^\alpha \times \mathbb{A}_\Lambda$. In the coordinates $w_{i,r}, y_{i,r}$ of Section 4.5 we have

$$\overline{W}_{-}^{\Lambda, \alpha}(w, y, K) = \sum_{i,r} y_{i,r} K_i(w_{i,r}) - \log F_a + \sum_{1 \leq m < n \leq N} \lambda_m \cdot \lambda_n \log(z_m - z_n) \tag{5.3}$$

where $F_a$ is the equation of the zastava boundary $\partial Z^\alpha = Z^\alpha \setminus \check{Z}^\alpha$ (see e.g. [4, Section 5]). Let $\mathbb{C}[[z^{-1}]] \ni \frac{R(z)}{Q(z)} := \sum_{p=0}^{\infty} h_i p z^{-p-1}$. Then

$$\overline{W}_{-}^{\Lambda, \alpha}(w, y, K) = \sum_{i,p} h_i p \kappa_{i,p} - \log F_a + \sum_{1 \leq m < n \leq N} \lambda_m \cdot \lambda_n \log(z_m - z_n) \tag{5.4}$$

In case $g = \mathfrak{sl}_2$, the boundary equation $F_a$ is a frozen cluster variable of the cluster structure on $\check{Z}^\alpha$ of [9, Section 5], and all the coefficients $h_p$ are cluster variables according to [9, Lemma 5.3, Proposition 5.4]. Hence $\overline{W}_{-}^{\Lambda, \alpha}|_{K=K_0}$ is a constant $\ell(K_0) := \ldots$
Moreover, according to Proposition 4.6, we have $s(\Delta_{V,0})$ plus a totally positive function on $\Gamma z^a$ for a monic polynomial $K_0$ with nonnegative coefficients $\kappa_p$. We expect a similar positivity for general $g$ (in particular, $F_a$ is a frozen cluster variable).

6. Appendix. $G = SL_2$: identification with the cluster structure of \textbf{[3]}

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Recall that $\widehat{sl}_2$ has two fundamental representations, which we denote by $V_{w_1}$ and $V_{w_0}$ in accordance with the notation of Section 5.1. Recall the generalized minors $\Delta_{w_0,s_1,...,s_{i-1},w_{i-1},w_0}$ from Section 5.2. Since for $g = sl_2$ we have $w_0 = s_1$, these generalized minors are of the form $\Delta_{s_1(s_0,s_1)^r,w_1,w_1}$ and $\Delta_{s_1(s_0,s_1)^r,w_0,w_0}$.

Given a pair of polynomials $Q(z) = z^a + q_0 z^{a-1} + \ldots + q_0$, $R(z) = r_{a-1} z^{a-1} + r_{a-2} z^{a-2} + \ldots + r_0$ representing a point of $\Gamma z^a$, we can find a unique pair of polynomials $F(z) = z^a + f_{a-1} z^{a-1} + \ldots + f_0$, $D(z) = d_{a-1} z^{a-1} + \ldots + d_0$ such that $QF - RD = z^{2a}$. Then both the matrix $g(Q,R) := \begin{pmatrix} z^{-a} F(z) & z^{-a} D(z) \\ z^{-a} R(z) & z^{-a} Q(z) \end{pmatrix} \in \widehat{SL}_2$ and its inverse $g(Q,R)^{-1} = \begin{pmatrix} z^{-a} Q(z) & z^{-a} D(z) \\ -z^{-a} R(z) & z^{-a} F(z) \end{pmatrix} \in \widehat{SL}_2$ actually lie in $N_- \subset \widehat{SL}_2$ (notations of Section 5.1).

Moreover, according to Proposition 4.6, we have $s_0^\lambda \circ \text{pr}(g(Q,R)) = (Q,R)$. Here $\lambda = a\alpha$ is a multiple of the simple coroot of $sl_2$, and $g(Q,R) \in \mathcal{F}_{s_1 \times a \alpha} \subset Fl_{s_1} = N_- \cdot s_1$ is the point $g(Q,R) \cdot s_1$.

On the other hand, following \textbf{[3]}, we consider the Taylor expansion at $\infty \in \mathbb{P}^1$ of $R(z) = \frac{c_0}{z^2} + \frac{c_2}{z^4} + \ldots + \frac{c_{2a-2}}{z^{2a}} + \ldots$. We form the corresponding Hankel matrix using the elements $c_0, c_2, \ldots, c_{2a-2}$, namely the $a \times a$ matrix $[c_{j+k}]_{j,k=0}^{a-1}$. We consider two kinds of minors of this matrix, the principal minors $C_1, \ldots, C_a$ of sizes $1, \ldots, a$, and the minors $D_1, \ldots, D_{a-1}$ of sizes $1, \ldots, a - 1$ which are obtained from the principal minors of the same size by a shift of all entries by one unit to the right (or, equivalently, by a shift of all entries by one unit down). We will also denote these minors by $C_{\alpha}(Q,R)$, $D_{\alpha}(Q,R)$ when we want to stress the dependence on $Q, R$. These Hankel minors (also called Hankel determinants) arise as cluster variables in the cluster corresponding to $(\varepsilon) = (0,0,\ldots,0)$ in \textbf{[3]}. See also the survey \textbf{[10]} for general properties of Hankel matrices.

In this appendix we prove the following theorem:

**Theorem 6.1.** (a) The generalized minor $\Delta_{s_1(s_0,s_1)^r,w_1,w_1}(g(Q,R))$ is equal (up to a change of sign) to the Hankel minor $C_{\varepsilon}(Q,R)$.

(b) The generalized minor $\Delta_{s_1(s_0,s_1)^r,w_0,w_0}(g(Q,R))$ is equal (up to a change of sign) to the Hankel minor $D_{\varepsilon}(Q,R)$.

Before starting the proof of Theorem 6.1, we will recall a theorem of Kronecker which we will use.

First, for two polynomials $Q(z) = z^a + q_{a-1} z^{a-1} + \ldots + q_0$, $R(z) = r_{a-1} z^{a-1} + r_{a-2} z^{a-2} + \ldots + r_0$ we will write the $(2a - 1) \times (2a - 1)$ Sylvester matrix (the determinant of which
computes the resultant of \(Q\) and \(R\) in the following form:

\[
\begin{bmatrix}
1 & q_{a-1} & \ldots & \ldots & q_0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & q_{a-1} & \ldots & q_1 & q_0 & 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 1 & q_{a-1} & \ldots & q_1 & q_0 & 0 & 0 \\
0 & 0 & \ldots & 0 & r_{a-1} & r_{a-2} & \ldots & r_1 & r_0 & 0 \\
0 & 0 & \ldots & r_{a-1} & r_{a-2} & \ldots & r_0 & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & r_{a-1} & \ldots & r_0 & 0 & \ldots & \ldots & 0 \\
r_{a-1} & r_{a-2} & \ldots & r_0 & 0 & 0 & \ldots & 0 \\
\end{bmatrix}
\]

Next we define for each \(i\) an odd sub-resultant \(R_i\) (which coincides with the notion of sub-resultant in [19 (2.6)]) to be the minor of the Sylvester matrix which is obtained by removing the same number \(i\) of columns and rows at the top, the bottom, the right, and the left. We also define an even sub-resultant \(S_i\) to be the minor of the Sylvester matrix obtained by removing the middle row, the same number \(i\) of rows at the top and the bottom, and removing \(i\) columns at the left, and \(i+1\) columns at the right.

Now we can state the following formula of Kronecker [15] (cf. [19 Corollary 3.2] for a modern reference) expressing sub-resultants in terms of Hankel determinants of the Taylor expansion of the ratio of two polynomials:

**Proposition 6.2.** (L. Kronecker) \(R_i = C_{a-i}\).

We will also recall some facts from the theory of infinite-dimensional Lie algebras which we will need in the course of the proof (we will follow the exposition in [13] and use the notation of loc. cit.).

We start with an infinite-dimensional vector space vector space \(V = \oplus \mathbb{C} v_j\). For each \(m \in \mathbb{Z}\) we have the infinite-dimensional vector space \(F^{(m)}\) with a vacuum vector \(\psi_m = v_m \wedge v_{m-1} \wedge \ldots\) and a basis given by \(v_{i_0} \wedge v_{i_1} \wedge \ldots\) (such that \(i_0 > i_1 > \ldots\)). We define a group \(GL_\infty\) as the group of invertible infinite matrices with entries \(a_{i,j}\) (\(i,j \in \mathbb{Z}\)) such that all but finitely many of \(a_{i,j} - \delta_{i,j}\) are zero. The group \(GL_\infty\) acts in \(F^{(m)}\) as follows: \(A(v_{i_0} \wedge v_{i_1} \wedge \ldots) = Av_{i_0} \wedge Av_{i_1} \wedge \ldots = \sum_{j_0 > j_1 > \ldots} \det A_{j_0,j_1,\ldots} v_{j_0} \wedge v_{j_1} \wedge \ldots\), where \(A_{j_0,j_1,\ldots}\) denotes the matrix located on the intersection of the rows \(j_0, j_1, \ldots\) and columns \(i_0, i_1, \ldots\) of the matrix \(A\).

Consider the standard \(n\)-dimensional representation \(U\) of \(\mathfrak{sl}_n\) with basis \(u_1, u_2, \ldots, u_n\) (in this appendix we will only use \(n = 2\)). Note that according to [13 (9.8)], one can define an action of \(\mathfrak{sl}_n\) on \(V\) in the following way, where an element \(A = \sum_i A_i z^i \in \mathfrak{sl}_n\) acts as the infinite matrix below (note that this action is obtained from the representation \(U[z^{\pm 1}]\) of \(\mathfrak{sl}_n\) by identifying a basis element \(v_j\) of \(V\) with \(z^j \cdot u_r\) where \(j = kn+r\), and \(r \in \{1, 2, \ldots, n\}\)):

\[
\begin{bmatrix}
\vdots & \ldots & \ldots & \ldots & \ldots \\
\ldots & A_{-1} & A_0 & A_1 & \ldots \\
\ldots & \ldots & A_{-1} & A_0 & A_1 \\
\ldots & \ldots & \ldots & A_{-1} & A_0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\end{bmatrix}
\]
Note that the images of the matrices in $\hat{s}_n$ obtained in this way have finitely many non-zero diagonals. Therefore by [1,4 Section 4.4] the action of $\hat{s}_n$ in $F^{(m)}$ is given by the same formula as for $GL_\infty$, namely $A(v_{i_0} \wedge v_{i_1} \wedge ... = \sum_{j_0>j_1>...} \det A_{j_0,j_1,...}^{i_0,i_1,...} v_{j_0} \wedge v_{j_1} \wedge ...$ for $A \in \mathfrak{sl}_n$. This way for $m = 0, 1, 2, ..., n - 1$ we obtain all the fundamental representations $V_{\omega_m}$, where $V_{\omega_m}$ is the irreducible sub-representation of $\hat{s}_n$ in $F^{(m)}$ which is generated by the vacuum vector $\psi_m = v_m \wedge v_{m-1} \wedge ...$

Finally, note that the action of $\mathfrak{n}_-$ in $F^{(m)}$ is not integrable (in general, $g v_{i_0} \wedge v_{i_1} \wedge ...$ is an infinite sum for an element $g \in \mathfrak{n}_-$). However, for any basis element $v_{j_0} \wedge v_{j_1} \wedge ...$ of $F^{(m)}$ its coefficient in the (infinite) expansion of $g v_{i_0} \wedge v_{i_1} \wedge ...$ in the elements of the basis of $F^{(m)}$ is well-defined and can be computed by the same formula as for $g \in GL_\infty$, namely it is equal to $\det A_{j_0,j_1,...}^{i_0,i_1,...}$ Note that here we calculate $\det A_{j_0,j_1,...}^{i_0,i_1,...}$ in the following way. By definition there exists $N$ such that for $k < N$ we have $j_k = i_k = k + m$; then $\det A_{j_0,j_1,...}^{i_0,i_1,...} = \det A_{j_0,j_1,...}^{i_0,i_1,...} |_{N}$, which is a finite determinant. The justification for this formula is that the infinite matrix with rows $j_0, j_1, ...$ and columns $i_0, i_1, ...$ can be divided into four blocks, where the two diagonal blocks are the finite block with rows $j_0, j_1, ..., j_N$ and columns $i_0, i_1, ..., i_N$ and the infinite lower-triangular block with 1’s on the diagonal with rows $j_N - 1, j_N - 2, ...$ and columns $i_N - 1, i_N - 2, ...$

Now we are ready to prove our theorem.

**Proof of Theorem 6.1.** Using exterior powers, the computation of generalized minors reduces to the computation of finite minors because the infinite matrices we use are upper triangular up to a finite portion. As a result we obtain finite minors which stand in certain rows and columns of the infinite matrix, depending on the element of the Weyl group which appears in the definition of a particular generalized minor.

For example, for the element of the Weyl group given by $s_0 s_1 s_0 s_1$ and the fundamental representation $V_{\omega_1}$, we obtain the following minors:

\[
\begin{bmatrix}
-5 & 1 \\
-4 & 0 \\
-3 & 1 \\
-2 & 0 \\
-1 & 1 \\
0 & 0 \\
1 & 1 \\
2 & 0 \\
3 & 1 \\
4 & 0 \\
5 & 1 \\
6 & 0 \\
7 & 1 \\
8 & 0
\end{bmatrix}
\]

After we collect the entries at the intersections of the marked rows and columns, we obtain the following submatrix (which after transposing it and exchanging the order and the signs of some the rows will be exactly the $(a - 3)$-th sub-resultant $R_{a-3}$ for the polynomials $Q$ and $R$):

\[
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1
\end{bmatrix}
\]
We see that the finite minors we obtain up to permutation of rows and transposition are exactly the sub-resultants in the case of odd number of rows. Indeed, for the element of the Weyl group given by \((s_0 s_1)^r\) and the fundamental representation \(V_{\varpi_1}\), we obtain that the finite minor consists of \(2r + 1\) rows numbered \(2r, 2r - 2, \ldots, -2r + 2\) and \(2r + 1\) columns numbered 0, -1, \ldots, -2r + 2. If we permute the columns so that the odd rows stand on the left in the same order as they were before and even rows stand on the right in the same order as they were before, we obtain exactly an odd sub-resultant as defined above.

Now we use the above theorem of Kronecker to conclude that the generalized minors for the fundamental representation \(V_{\varpi_1}\) of \(\tilde{s}_2\) are equal to the corresponding principal Hankel minors.

Now we turn to the generalized minors of the basic representation \(V_{\varpi_0}\). For example, for the element \(s_0 s_1 s_0 s_1\) of the Weyl group and the fundamental representation \(V_{\varpi_0}\) we have the following submatrix:

| -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|----|----|----|----|----|---|---|---|---|---|---|---|---|---|
|    | 1  |    |    |    | 1 | 1 |   |   |   |   |   |   |   |
| -3 | 0  |    |    |    |   |   | 1 |   |   |   |   |   |   |
| -2 | 1  |    |    |    |   |   |   | 1 |   |   |   |   |   |
| -1 | 0  |    |    |    |   |   |   |   | 1 |   |   |   |   |
|  0 | 1  |    |    |    |   |   |   |   |   | 1 |   |   |   |
|  1 | 0  |    |    |    |   |   |   |   |   |   | 1 |   |   |
|  2 | 1  |    |    |    |   |   |   |   |   |   |   | 1 |   |
|  3 | 0  |    |    |    |   |   |   |   |   |   |   |   | 1 |
|  4 | 1  |    |    |    |   |   |   |   |   |   |   |   |   |
|  5 | 0  |    |    |    |   |   |   |   |   |   |   |   |   |
|  6 | 1  |    |    |    |   |   |   |   |   |   |   |   |   |
|  7 | 0  |    |    |    |   |   |   |   |   |   |   |   |   |
|  8 | 1  |    |    |    |   |   |   |   |   |   |   |   |   |

After we collect the entries at the intersections of the marked rows and columns we obtain the following submatrix (which after transposing and exchanging the order of the rows is the \((a - 3)\)-th sub-resultant \(S_{a-3}\) for the polynomials \(Q\) and \(R\)):

| 0  | 1  | 0  | 1 |
|----|----|----|---|
| \(r_{a-1}\) | \(q_{a-1}\) | \(r_{a-2}\) | \(q_{a-2}\) |
| \(r_{a-3}\) | \(q_{a-3}\) | \(r_{a-4}\) | \(q_{a-4}\) |
| \(r_{a-5}\) | \(q_{a-5}\) | \(r_{a-6}\) | \(q_{a-6}\) |

More generally, for the element of the Weyl group given by \((s_0 s_1)^r\) and the fundamental representation \(V_{\varpi_0}\) we obtain that the finite minor consists of \(2r + 1\) rows numbered \(2r, 2r - 2, \ldots, -2r + 2\) and \(2r\) rows numbered 0, -1, \ldots, -2r + 1. If we permute the columns so that the odd rows stand on the left in the same order as they were before and even rows stand on the right in the same order as they were before, we obtain exactly an even sub-resultant as defined above.

Finally, we can reduce the case of even number of rows to the case of odd number of rows in the following way. Note that the even sub-resultant \(S_i\) of the polynomials \(R(z)\) and \(Q(z)\) is equal to a usual sub-resultant (cf. [19, (2.6)]) of the polynomials \(R(z)\) and \(zQ(z)\). We claim that this (usual) sub-resultant of the polynomials \(R(z)\) and \(zQ(z)\) is equal to the determinant of the original Hankel minor \(D_{a-i-1}\). Indeed, by the theorem of Kronecker mentioned above, the usual sub-resultants of \(R(z)\) and \(zQ(z)\) equal the corresponding (principal) Hankel minors for \(R(z)\) and \(zQ(z)\). To conclude, we notice that the equality \(R(z)/Q(z) = z(R(z)/(zQ(z)))\) implies that the principal Hankel minors for \(R(z), zQ(z)\) are
equal to the Hankel minors for $R(z), Q(z)$ which are obtained by shifting the corresponding principal Hankel minors by one unit to the right.

\[\square\]

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