Radial geodesics as a microscopic origin of black hole entropy.

III: Exercise with the Kerr-Newman black hole

V.V.Kiselev*†

* Russian State Research Center “Institute for High Energy Physics”, Pobeda 1, Protvino, Moscow Region, 142281, Russia
† Moscow Institute of Physics and Technology, Institutskii per. 9, Dolgoprudnyi Moscow Region, 141700, Russia
E-mail: kiselev@th1.ihep.su

Abstract. We specify an angular motion on geodesics to reduce the problem to the case of radial motion elaborated in previous chapters. An appropriate value of entropy for a charged and rotating black hole is obtained by calculating the partition function on thermal geodesics confined under horizons. The quantum aggregation is classified in a similar way to the Reissner–Nordstrøm black hole.

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1. Continuation of Preface

A rotation of Kerr–Newman black hole involves a new feature in the description of geodesics responsible for the entropy of black hole: a nonzero projection of angular momentum on the axis of rotation is permitted by the symmetry of the problem. Therefore, we start with a specification of angular motion on geodesics of conserved orbital momentum in Section 2. Then, the procedure has a little to differ from the Reissner–Nordstrøm black hole. The difference is reduced to a particular dependence of mass sum on the polar angle, that allows us to evaluate the partition function and entropy for a cool state of aggregation in agreement with the Bekenstein–Hawking formula [1–3] in Section 3. A short summary is situated in Section 4.

2. Angular motion and reduced geodesics

The Kerr–Newman metric of charged and rotating black hole can be written in the form

\[
\mathrm{ds}^2_{KN} = \frac{\Delta}{\Sigma} \, \mathrm{d}\omega_t^2 - \frac{\Sigma}{\Delta} \, \mathrm{d}r^2 - \Sigma \, \mathrm{d}\theta^2 - \frac{\sin^2 \theta}{\Sigma} \, \mathrm{d}\omega_\phi^2,
\]

with

\[
\begin{align*}
\mathrm{d}\omega_t &= \mathrm{d}t - a \sin^2 \theta \, \mathrm{d}\phi, \\
\mathrm{d}\omega_\phi &= (r^2 + a^2) \, \mathrm{d}\phi - a \, \mathrm{d}t, \\
\Sigma &= r^2 + a^2 \cos^2 \theta, \\
\Delta &= (r - r_+)(r - r_-),
\end{align*}
\]

(1)
where the black hole parameters: mass \( M \), charge \( Q \), and angular momentum \( J \), are given by

\[
M = \frac{1}{2}(r_+ + r_-), \quad Q^2 + a^2 = r_+ r_-, \quad J = a M. \tag{3}
\]

In order to proceed with the Hamilton–Jacobi equation for a test particle with a mass \( m \)

\[
g^{\mu \nu} \partial_\nu S_{HJ} \partial_\mu S_{HJ} - m^2 = 0, \tag{4}
\]
one has to invert the metric in terms of \( \{ t, \phi \} \) given by the following elements:

\[
g_{tt} = \frac{1}{\Sigma} (\Delta - a^2 \sin^2 \theta), \quad g_{t\phi} = -a \frac{\sin^2 \theta}{\Sigma} (\Delta - r^2 - a^2), \quad g_{\phi\phi} = \frac{\sin^2 \theta}{\Sigma} (a^2 \sin^2 \theta - r^2 - a^2). \tag{5}
\]

For the corresponding \( 2 \times 2 \)-matrix \( \hat{g} \) we get the determinant

\[
\det \hat{g} = \Delta \frac{\sin^2 \theta}{\Sigma^2} [-a^4 \sin^4 \theta - (r^2 + a^2)^2 + 2(r^2 + a^2)a^2 \sin^2 \theta], \tag{6}
\]
which enters the inverse matrix elements

\[
g^{tt} = \frac{1}{\det \hat{g}} g_{\phi\phi}, \quad g^{t\phi} = -\frac{1}{\det \hat{g}} g_{t\phi}, \quad g^{\phi\phi} = \frac{1}{\det \hat{g}} g_{tt}. \tag{7}
\]

Then, we introduce two integrals of motion: an energy \( \mathcal{E} \) and an angular momentum \( \mu \), which determine the action in the form

\[
S_{HJ} = -\mathcal{E} t + \mu \phi + S_{HJ}(r), \tag{8}
\]
at \( \dot{\theta} \equiv 0 \). From (4) we deduce

\[
\left( \frac{\partial S_{HJ}}{\partial r} \right)^2 = \frac{\Delta}{\Sigma} \left[ \mathcal{E}^2 g^{tt} + 2 \mathcal{E} \mu g^{t\phi} + \mu^2 g^{\phi\phi} - m^2 \right] \equiv \frac{\Delta}{\Sigma} \mathcal{H}. \tag{9}
\]
which results in

\[
S_{HJ}(r) = \int_{r_0}^{r(t)} dr \sqrt{\frac{\mathcal{H} \Delta}{\Sigma}}. \tag{10}
\]
The trajectory is implicitly determined by equations

\[
\frac{\partial S_{HJ}}{\partial \mathcal{E}} = t_0 = -t + \int_{r_0}^{r(t)} dr \sqrt{\frac{\Delta}{h \Sigma} [\mathcal{E} g^{tt} + \mu g^{t\phi}]}, \tag{11}
\]
\[
\frac{\partial S_{HJ}}{\partial \mu} = \phi_0 = \phi + \int_{r_0}^{r(t)} dr \sqrt{\frac{\Delta}{h \Sigma} [\mathcal{E} g^{t\phi} + \mu g^{\phi\phi}]}. \tag{12}
\]
Taking the derivative of (11), (12) with respect to the time $\dot{t}$, we get
\begin{align}
1 &= \dot{r} \sqrt{\frac{\Delta}{\Sigma}} [\mathcal{E} g^{tt} + \mu g^{t\phi}], \\
\dot{\phi} &= -\dot{r} \sqrt{\frac{\Delta}{\Sigma}} [\mathcal{E} g^{\phi t} + \mu g^{\phi\phi}], 
\end{align}
\tag{13}
\tag{14}
determining the angular motion by
\begin{equation}
\dot{\phi} = -\frac{\mathcal{E} g^{t\phi} + \mu g^{\phi\phi}}{\mathcal{E} g^{tt} + \mu g^{t\phi}}.
\tag{15}
\end{equation}

Further, we use a relation specifying the angular motion by
\begin{equation}
\mu = \mathcal{E} a \sin^2 \theta,
\tag{16}
\end{equation}
that gives
\begin{equation}
\dot{\phi} = \frac{a}{r^2 + a^2}.
\tag{17}
\end{equation}

Note, for such the angular velocity $d\omega_\phi = 0$, and the interval takes the form
\begin{equation}
ds^2 = \Sigma \frac{\Delta}{(r^2 + a^2)^2} (dt^2 - dr_*^2),
\tag{18}
\end{equation}
with
\begin{equation}
dr_* = \frac{r^2 + a^2}{\Delta} \, dr,
\tag{19}
\end{equation}
yielding
\begin{equation}
r_* = r + \frac{r^2 + a^2}{r_+ - r_-} \ln \left[ \frac{r}{r_+} - 1 \right] - \frac{r^2 + a^2}{r_+ - r_-} \ln \left[ \frac{r}{r_-} - 1 \right].
\tag{20}
\end{equation}

Then, we can repeat the Hamilton–Jacobi formalism for the interval of reduced motion in (18) and find
\begin{equation}
\frac{1}{m^2} \left( \frac{\partial S_{\text{HJ}}}{\partial r_*} \right)^2 = \mathcal{E}_A - U(r),
\tag{21}
\end{equation}
with $\mathcal{E}_A = 1/A$, 
\begin{equation}
U(r) = \frac{\Sigma \cdot \Delta}{(r^2 + a^2)^2},
\tag{22}
\end{equation}
and
\begin{equation}
\frac{dt}{dr_*} = \frac{\mathcal{E}_A}{\sqrt{\mathcal{E}_A - U(r)}}.
\tag{23}
\end{equation}

Therefore, on the geodesics we get the causal interval
\begin{equation}
ds_A^2 = \frac{U^2(r)}{\mathcal{E}_A - U(r)} \frac{(r^2 + a^2)^2}{\Delta^2} \, dr^2.
\tag{24}
\end{equation}

For the ground state we will consider further, $\mathcal{E}_A \rightarrow 0$ and
\begin{equation}
ds^2 = \frac{r^2 + a^2 \cos^2 \theta}{(r_+ - r)(r - r_-)} \, dr^2, \quad r_- < r < r_+, \tag{25}
\end{equation}
\[\dagger\] As usual $\partial_t f(t) = \dot{f}$.  

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while the increment of time per cycle is given by

$$\Delta_c t_E = 2\pi (r_+ + r_-).$$  \hspace{1cm} (26)

Further, due to (18) and (20) we can introduce two consistent maps under the horizons in terms of Kruskal isotropic variables in a manner of our treatment in Chapter II, with inverse temperatures

$$\beta_+ = 4\pi \frac{r_+^2 + a^2}{r_+ - r_-}, \quad \beta_- = 4\pi \frac{r_-^2 + a^2}{r_+ - r_-},$$

and a quantum ratio of horizon areas

$$\frac{A_+}{A_-} = \frac{4\pi (r_+^2 + a^2)}{4\pi (r_-^2 + a^2)} = l \in \mathbb{N}. \hspace{1cm} (27)$$

The winding numbers on geodesics are equal to the same values as in Chapter II,

$$n_+ = \frac{2l}{l-1}, \quad n_- = \frac{2}{l-1}. \hspace{1cm} (29)$$

The increment of interval per cycle $\Delta_c s(\cos \theta)$ follows from (25), while at $\theta = \pi/2$ we easily get

$$\Delta_c s(0) = \pi (r_+ + r_-). \hspace{1cm} (30)$$

Thus, we can follow the analogy with the Reissner–Nordstrøm black hole.

3. Entropy

Introducing a sum of particles moving at a fixed value of angle $\theta$ in the maps $\pm$,

$$\sigma_{\pm}(\cos \theta) = \sum_{\pm} mc \bigg|_{\theta}, \hspace{1cm} (31)$$

for pure “ice” state of aggregation we get the partition function

$$\ln Z_+ = -n_+ \Delta_c s(\cos \theta) \sigma_+(\cos \theta). \hspace{1cm} (32)$$

In order to get the most probable configuration the product $\Delta_c s(\cos \theta) \sigma_+(\cos \theta)$ should be invariant under the variation of $\theta$ and take its minimal value, which is reached at $\cos \theta = 0$, so that

$$\ln Z_+ = -n_+ \Delta_c s(0) \sigma_+(0) = -\frac{\beta_+}{2} \sigma_+(0). \hspace{1cm} (33)$$

Thus, in the thermal equilibrium, the sum of masses $\sigma(\cos \theta)$ adjusts its value in order to compensate the dependence of $\Delta_c s$ on $\cos \theta$. An example of such the adjustment is shown in Fig. 1.

From (33) and Chapter II we deduce the expressions

$$\sigma_+(0) = 2M - \frac{1}{2\beta_+} A_+, \hspace{1cm} (34)$$

and the entropy

$$S_+ = \frac{1}{4} A_+, \hspace{1cm} (35)$$
valid due to the standard relation between the temperature and ‘surface gravity’ (see discussion in review [4]),

\[ T_+ = 4 \frac{\partial M}{\partial A_+}, \quad \text{at } dQ^2 \equiv 0, \quad dJ \equiv 0. \quad (36) \]

![Graph of \( \sigma_+(\cos \theta) \)]

**Figure 1.** The variation of \( \sigma_+ \) versus \( \cos \theta \) at \( a = 1, r_- = 2, r_+ = 3 \).

A discrimination of two phases of aggregation is the same as for the charged black hole.

4. Conclusion

In present chapter we have shown how the consideration of rotating black hole is reduced to the charged one of Reissner and Nordstrøm. The only actual difference is the adjustment of mass sum versus the polar angle.

Finally, let us make a short note on the extremal black hole with \( r_+ = r_- \), that corresponds to solitonic BPS states in superstrings [5]. We have found that the extremum takes place, when the sphere of euclidian time and radius degenerates to a torus with an infinitely small radius (two poles of sphere are glued, but the radius tends to zero). The corresponding winding number tends to infinity for the ground state. However, coming from such the singular ground state to an excited state, one could adjust the level of winding number (the value of \( x \)) and tune the sum of particle masses in order to preserve the final value of entropy, which is independent of particular value of temperature at the given area of external horizon. Due to the thermal quantization of ratio of horizon areas, the limit of extremal black hole cannot be reached continuously: one has got a quantum jump of \( r_- \).

Thus, we finalize considering the basics of method for the evaluation of black hole entropy by calculating the microscopic partition function on classical geodesics confined under the horizons by their thermal quantization. We hope that the tool provides a new
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insight to the entropy of black hole, not excluding some new particular problems and questions.

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