On the Conservativity of the Functor Assigning to a Motivic Spectrum its Motive

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Abstract
Given a 0-connective motivic spectrum $E \in \text{SH}(k)$ over a perfect field $k$, we determine $h_0$ of the associated motive $ME \in \text{DM}(k)$ in terms of $\pi_0(E)$. Using this we show that if $k$ has finite 2-étale cohomological dimension, then the functor $M : \text{SH}(k) \to \text{DM}(k)$ is conservative when restricted to the subcategory of compact spectra, and induces an injection on Picard groups. We extend the conservativity result to fields of finite virtual 2-étale cohomological dimension by considering what we call “real motives”.

1 Introduction
One of the starting points of classical stable homotopy theory is the Hurewicz theorem: we call a spectrum $E \in \text{SH}$ 0-connective if $\pi_i(E) = 0$ for all $i < 0$, where $\pi_i$ denote the stable homotopy groups. We may also associate with $E$ its homology groups $H_i(E) := H_i(C_*E)$, where $C_*E$ denotes the singular chain complex. The Hurewicz Theorem states that if $E$ is 0-connective then $H_i(E) = 0$ for all $i < 0$ and that there is a natural isomorphism $\pi_0(E) \cong H_0(E)$.

This has several nice consequences. For example, if $E \in \text{SH}$ is a connective spectrum (has only finitely many non-vanishing negative stable homotopy groups) and $C_*E \simeq 0$, then actually $E \simeq 0$. We call this the conservativity theorem.

For another example, recall that a spectrum $E$ is called invertible if there exists a spectrum $F$ such that $E \wedge F \simeq S$ ($S$ the sphere spectrum). The set of equivalence classes of invertible spectra forms an abelian group denoted $\text{Pic}(\text{SH})$; similarly we have the group $\text{Pic}(\text{D}(Ab))$. The functor $C_*$ induces a homomorphism $\text{Pic}(\text{SH}) \to \text{Pic}(\text{D}(Ab))$ which is injective. (It follows then quickly that it is an isomorphism, since $\text{Pic}(\text{D}(Ab))$ is easy to determine.) We call this the Pic-injectivity theorem.

We wish to investigate how to extend these results to the motivic world. Recall that in this world, for every field $k$ (at least) there is defined a category $\text{SH}(k)$ analogous to $\text{SH}$, but starting with smooth varieties as building blocks instead of topological spaces. In complete analogy with the classical situation, Morel, Ayoub and others have defined and studied a category $D^b(k)$ and a functor $C_*) : \text{SH}(k) \to D^b(k)$; see for example [27, Section 5.2]. Both of the categories $\text{SH}(k)$ and $D^b(k)$ carry a $t$-structure (known as the homotopy $t$-structure, not the conjectural motivic $t$-structure), and $C_*$ induces an isomorphisms of the hearts. (See Section 2 for our conventions regarding $t$-structures.) Let us write $\pi_i(E)$ for the homotopy objects of a spectrum $E \in \text{SH}(k)$ in this $t$-structure, and $L^i(E)$ for the homotopy objects of $C_*E$. Then Morel has proved that for 0-connective $E$ one has $\pi_0(E) \cong L_0(E)$ [29, Theorem 6.37]. This is a perfect analog of the classical Hurewicz theorem, and the conservativity and Pic-injectivity theorems follow in the same way as classically.

The problem with this formulation is that the category $D^b(k)$ is not yet very well understood. Moreover Voevodsky has constructed the category $\text{DM}(k)$ which some may consider to be a more accessible target category for a Hurewicz Theorem. But even though $\text{DM}(k)$ has a $t$-structure, and there is anice functor $M : \text{SH}(k) \to \text{DM}(k)$, the hearts of $\text{SH}(k)$ and $\text{DM}(k)$ are definitely not equivalent, so some new idea is needed. Write $L_i(E)$ for the homotopy objects of $ME$ (in the homotopy $t$-structure on $\text{DM}(k)$) for $E \in \text{SH}(k)$. Suppose $E \in \text{SH}(k)$ is 0-connective. Our first contribution is to identify $L_0(E)$.

Theorem (Motivic Hurewicz Theorem; see Theorem 12). Let $E \in \text{SH}(k)$ be 0-connective. Then in notation to be explained later, we have

\[ L_0(E) \cong \pi_0(E)/\eta. \]
Here we use implicitly that the heart of $DM(k)$ can be identified with a subcategory of the heart of $SH(k)$. This theorem shows that we lose some information in passing to the motive, and so cannot expect a perfect analogy with the classical situation. Nonetheless we can prove the following.

**Theorem** (Conservativity Theorem I; see Theorem 16). Let $k$ be a perfect field of finite 2-étale cohomological dimension and $E \in SH(k)$ be compact. If $ME \simeq 0$ then $E \simeq 0$.

(Recall that an object is called compact if $\text{Hom}_{SH(k)}(E, \bullet)$ commutes with arbitrary sums.) Combined with the Motivic Hurewicz Theorem, one easily obtains the following.

**Corollary** (Pic-injectivity Theorem; see Theorem 18). In the situation of the Theorem, the natural homomorphism $Pic(SH(k)) \to Pic(DM(k))$ is injective.

This result was one of the main motivations for our investigations. See later in this introduction for more details on applications. (We do not know if $Pic(SH(k)) \to Pic(DM(k))$ might also be surjective.)

The Conservativity Theorem (and by extension the Pic-injectivity Theorem) stated here is not optimal. Let us discuss to what extent the assumptions can be weakened.

In characteristic zero, the compactness assumption can be replaced by the technical assumptions of connectivity and “slice-connectivity” (see Section 4 for a definition of this term). In characteristic $p > 0$ we can also replace compactness by connectivity and slice-connectivity, but in this case we must additionally assume that the natural map $E \xrightarrow{\beta_p} E$ is an isomorphism.

If $k$ is a non-orderable field (i.e. one in which $-1$ is a sum of squares; we still assume $k$ perfect), then one may present $k$ as a colimit of perfect subfields with finite 2-étale cohomological dimension, and the theorem can then be deduced for such $k$, see [5]. Compactness instead of connectivity and slice-connectivity is then essential, however.

The author currently does not know how to weaken the perfectness assumption on $k$.

A further weakening is to consider orderable $k$, i.e. fields where $-1$ is not a sum of squares. (Such fields necessarily have characteristic zero.) Assume for simplicity that for any ordering of $k$, there is an order preserving embedding of $k$ into $\mathbb{R}$. Write $Sper(k)$ for the set of orderings of $k$. (Equivalently in this case, embeddings $k \to \mathbb{R}$.) If $\sigma \in Sper(k)$, then there is a so-called real realisation functor $R^2_{\sigma} : SH(k) \to SH$, related to considering the real points of a smooth variety, with their strong topology. We compose this with the singular chain complex functor to obtain the real motive $M^R_{\sigma} : SH(k) \to D(Ab)$. We can further consider homology with $\mathbb{Z}[1/2]$ coefficients and obtain $M^R_{\sigma}[1/2] : SH(k) \to D(\mathbb{Z}[1/2])$. We may then prove the following.

**Theorem** (Conservativity Theorem II; see Theorem 33). Let $k$ be a field of finite virtual 2-étale cohomological dimension with real embeddings for all orderings, as above. Let $E \in SH(k)$ be connective and slice-connective. If $0 \simeq ME \in DM(k)$ and additionally for all $\sigma \in Sper(k)$, $0 \simeq M^R_{\sigma}[1/2](E) \in D(\mathbb{Z}[1/2])$, then in fact $E \simeq 0$.

We point out right away that this theorem does not include a Pic-injectivity result; in the current form such a result seems unlikely to hold. The conditions on $k$ can again be relaxed. It turns out that for any orderable field $k$ and ordering $\sigma \in Sper(k)$, one may define a functor $M^R_{\sigma}[1/2] : SH(k) \to D(\mathbb{Z}[1/2])$ analogous to the real motive functors for $k \subset \mathbb{R}$; we still call them real motives. The theorem holds for all $k$ of characteristic zero and finite virtual 2-étale cohomological dimension, with the new definition of $M^R_{\sigma}[1/2]$. Details are explained before the proof of Theorem 33. One may also show that any orderable field is a colimit of orderable fields of finite virtual 2-étale cohomological dimension, so the theorem holds for arbitrary orderable fields, assuming that $E$ is compact. See again Bondarko’s work [5].

A word on applications. In the classical situation, Pic-injectivity immediately implies that $Pic(SH) = \mathbb{Z}$. In the motivic case, results are not nearly as strong, mainly because not much is known about $Pic(DM(k))$. However, combining conservativity and Pic-injectivity, many problems about (potentially) invertible objects in $SH(k)$ can be reduced to analogous questions in $DM(k)$. A prototypical example is as follows (the restriction to $k$ of characteristic zero can be overcome by inverting the exponential characteristic, we omit this here to simplify exposition):

**Proposition.** Let $E, F \in SH(k)$ be compact and suppose that $k$ is of finite 2-étale cohomological dimension and characteristic zero. Then $E, F$ are isomorphic invertible objects if and only if $ME, MF$ are.

**Proof.** It is clear that if $E \simeq F$ are invertible, then so are $ME, MF$. We show the converse.

Since $M$ is a monoidal functor between rigid monoidal categories, it preserves duals. Consider the natural map $DE \wedge E \xrightarrow{\alpha} \mathbb{I}_{SH(k)}$. If $ME$ is invertible, then $M\alpha$ is an isomorphism, and hence so is $\alpha$,
by conservativity. Thus $E$ is invertible. Similarly for $F$. But now $E, F \in \operatorname{Pic}(\text{SH}(k))$ and $E \cong F$ if and only if $ME \simeq MF$, by Pic-injectivity.

The conjectures of Po Hu on invertibility of Pfister quadrics, as stated in [16, Conjecture 1.4], can be cast in this form. We shall establish the analog of Hu’s conjecture in $\text{DM}(k)$ in a forthcoming article [3]. Together with the results of this work, this establishes Hu’s conjecture for $\text{SH}(k)$ whenever $k$ has finite 2-étale cohomological dimension.

If one only wishes to show that an object is invertible, we do not need Pic-injectivity. For example, one idea is that the reduced suspension spectrum of any smooth affine quadric $Q$ should yield an invertible object in $\text{SH}(k)$. Indeed this holds true under real and complex realisation, and also follows for Pfister Quadrics from Po Hu’s conjectures. Additionally it is true in the étale topology.

By arguments as in the above proposition, using the two conservativity theorems, it follows that if it suffices to show that the reduced motive $\tilde{M}Q$ is invertible and that for every ordering $\sigma$ of $k$, the reduced real motive $\tilde{M}^R_\sigma[1/2]Q$ is invertible. (For a variety $X/k$ we have the reduced suspension spectrum $\tilde{\Sigma}^\infty X := \text{hofib}(\Sigma^\infty X_+ \to \Sigma^\infty \text{Spec}(k)_+)$ and $\tilde{M}X := M(\tilde{\Sigma}^\infty X), \tilde{M}^R_\sigma[1/2]X := M^R_\sigma[1/2](\tilde{\Sigma}^\infty X).$) The former question is purely about motives, and dealt with in the same forthcoming work mentioned above. The latter question is purely topological and very easy. Thus we conclude: if $k$ is a field of characteristic 0 and $Q$ any smooth affine quadric over $k$, then $\Sigma^\infty Q$ is invertible in $\text{SH}(k)$. (More generally, this holds for any perfect field after inverting the characteristic.)

Here is a detailed description of the organisation of this paper. In Section 2 we prove a general result about $t$-categories and certain functors between them, which we call the abstract Hurewicz Theorem. It will be our replacement for the classical Hurewicz Theorem.

In Section 3, we recall the construction of $\text{SH}(k)$ and $\text{DM}(k)$ with the appropriate $t$-structures. We show how to combine the abstract Hurewicz Theorem with a fundamental result of Deglise to deduce our motivic Hurewicz Theorem.

In Section 4 we study the heart of the category $\text{SH}(k)$ in more detail. It is also known as the category of homotopy modules. We combine the careful study of Morel of strictly invariant sheaves with Levine’s work on Voevodsky’s slice filtration to show that every homotopy module $F_\bullet$ built using the unramified Milnor-Witt $K$-theory sheaves $K^\text{MW}_\sigma$. Combined with Voevodsky’s resolution of the Milnor conjectures, we can prove the conservativity theorem for $k$ of finite 2-étale cohomological dimension and with the exponential characteristic inverted.

In order to remove the need to invert the exponential characteristic, one uses a very similar (probably identical) filtration $F_\bullet H$. In this section we only state its properties; the actual construction is relegated to an appendix.

The Pic-injectivity theorem is obtained as an easy consequence of the conservativity result.

The rest of the paper deals with the case of orderable fields. Using methods very similar to the unordered case, one shows: if $E$ is connective and slice-connective and $ME \simeq 0$, then 2 is invertible on $E$. This is a crucial insight of Bondarko [5]. (This is where we need finite virtual 2-étale cohomological dimension.)

In Section 5 we tackle the problem of building a conservative functor for spectra on which 2 is invertible. We consider a category $\text{DM}_W(k, \mathbb{Z}[1/2])$ with a functor $F : \text{SH}(k) \to \text{DM}_W(k, \mathbb{Z}[1/2])$ factoring through $\text{SH}(k)_+$. It has already been studied in [1]. An easy argument using the abstract Hurewicz theorem shows that if $E \in \text{SH}(k)$ is connective and slice connective, $ME \simeq 0$ and also $FE \simeq 0$, then $E \simeq 0$. Consequently from this part on we concentrate on studying the category $\text{DM}_W(k, \mathbb{Z}[1/2])$. Using ideas from [17] we show that this category satisfies descent in the “real étale topology” in an appropriate sense. This allows us to assume that $k$ is real closed. Using ideas from semi-algebraic topology, we prove that the category $\text{DM}_W(k, \mathbb{Z}[1/2])$ is actually independent of the real closed field $k$, namely that $\text{DM}_W(k, \mathbb{Z}[1/2])$ is canonically equivalent to $D(\mathbb{Z}[1/2])$. Putting all of these facts together, we obtain the conservativity theorem for orderable fields.

The paper concludes with two appendices. In Appendix A, we recall some well-known results about compact objects in abelian and triangulated categories. In Appendix B we construct the missing filtration $\tilde{F}_\bullet$ mentioned earlier. (The results from appendix A are needed to establish that the filtration is finite.) This appendix is essentially a long and detailed computation with strictly homotopy invariant sheaves. As a bonus, we can reprove a version of rigidity for torsion objects in motivic homotopy theory [33], see Corollary 45.

The author wishes to thank Fabien Morel for suggesting this topic of investigation and providing many helpful insights. He also wishes to thank Mikhail Bondarko for useful discussions, and in particular for showing to him Lemma 10.
Notations and Conventions The signs $\simeq$ and $\cong$ shall denote isomorphisms. Whenever there is a category $C$ together with a homotopy category $Ho(C)$, we shall only apply $\simeq$ to mean an isomorphism in $Ho(C)$ and $\cong$ to mean an isomorphism in $C$.

An adjunction between categories $C, D$ with left adjoint $F$ and right adjoint $U$ shall be denoted $F : C \rightleftarrows D : U$.

2 The Abstract Hurewicz Theorem

In this short section we prove an abstract result about $t$-categories, by which we mean triangulated categories provided with a $t$-structure [4, Section 1.3]. It is (we feel) so natural that it has probably occurred elsewhere before. We also include some well-known results. Since all proofs are short and easy, we include them for the convenience of the reader.

We first review some notations. The most important thing to point out is that we use homological notation for $t$-categories, whereas most sources, such as the excellent [11, Section IV §4], use cohomological notation. The usual reindexing trick $(E_0 \leftrightarrow E^{-n})$ can be used to translate between the two.

In more detail, a $t$-category is a triple $(C, C_{\leq}, C_{\geq})$ consisting of a triangulated category $C$ and two full subcategories $C_{\leq} \subseteq C_{\geq} \subseteq C$ satisfying certain properties. One puts $C_{\leq} = C_{\leq}[n]$ and $C_{\geq} = C_{\geq}[n]$. The inclusion $C_{\leq} \hookrightarrow C$ has a left adjoint denoted $E \mapsto E_{\leq}$. Similarly the inclusion $C_{\geq} \hookrightarrow C$ has a right adjoint denoted $E \mapsto E_{\geq}$. Both are called truncation.

The full subcategory $C^{\geq} = C_{\geq} \cap C_{\leq}$ turns out to be abelian. It is called the heart of the $t$-category $C$. The homotopy objects are $\pi_0^t(E) = (E_{\leq})_{\geq} \simeq (E_{\geq})_{\leq} \in C^{\geq}$ and $\pi_1^t(E) = \pi_0^t(E[-i])$.

One has many more properties. For example $C_{\geq} \supset C_{\geq+1}$, $C_{\leq} \subset C_{\leq+1}$, there are functorial distinguished triangles $E_{\geq} \to E \to E_{\leq-1}$, the functor $\pi^t_\ast$ is homological (turns distinguished triangles into long exact sequences), etc.

Recall that a triangulated functor $F : C \to D$ between $t$-categories is called right-$t$-exact (respectively left-$t$-exact) if $F(C_{\geq}) \subseteq D_{\geq}$ (respectively if $F(C_{\leq}) \subseteq D_{\leq}$). It is called $t$-exact if it is both left and right-$t$-exact.

If $F : C \to D$ is any triangulated functor between $t$-categories, then it induces a functor $F^0 : C^0 \to D^0$. For computations, we often write $[A, B]$ instead of $\text{Hom}_C(A, B)$. We also note that the axioms for $t$-categories are self-dual. Passing from $C$ to $C^{op}$ (and $D$ to $D^{op}$) turns left-$t$-exact functors into right-$t$-exact functors, etc.

The notions of left and right $t$-exactness are partly justified by the following.

Proposition 1. Let $M : C \rightleftarrows D : U$ be adjoint functors between $t$-categories, with $M$ right-$t$-exact and $U$ left-$t$-exact. Then

$$M^0 : C^0 \rightleftarrows D^0 : U$$

is also an adjoint pair. In particular $M^0$ is right exact and $U^0$ is left exact.

Proof. Let $A \in C^0$, $B \in D^0$. We compute

$$[M^0 A, B] \overset{(1)}{=} [\pi_0^D MA, B] \overset{(2)}{=} [(MA)_{\leq}, B] \overset{(3)}{=} [MA, B] \overset{(4)}{=} [A, UB] \overset{(5)}{=} [A, (UB)_{\geq}] \overset{(2)'}{=} [A, \pi_0^U B] \overset{(1)'}{=} [A, U^0 B].$$

Here (1) is by definition, (2) is because $A \in C_{\leq}$ and so $MA \in D_{\geq}$ by right-$t$-exactness of $M$, and (3) because $B \in D_{\leq}$. Equality (4) is by adjunction, and finally (5), (2)’), (1)’) reverse (3), (2), (1) with $M \leftrightarrow U$, left $\leftrightarrow$ right, etc.

The following lemma is a bit technical but naturally isolates a crucial step in the proof of the main theorem of this section.

Lemma 2. Let $M : C \rightleftarrows D : U$ be adjoint functors between $t$-categories. If $M$ is right-$t$-exact (respectively $U$ left-$t$-exact) then there is a natural isomorphism $(UE)_{\geq} \simeq (UE_{\geq})_{\geq} \simeq (ME_{\leq})_{\leq}$.

Proof. By duality, we need only prove one of the statements. Suppose that $U$ is left-$t$-exact. We compute for $T \in D_{\leq}$

$$[(ME)_{\leq}, T] \overset{(1)}{=} [ME, T] \overset{(2)}{=} [E, UT] \overset{(3)}{=} [E_{\leq}, UT] \overset{(2)'}{=} [ME_{\leq}, T] \overset{(1)'}{=} [(ME_{\leq})_{\leq}, T].$$
Since all equalities are natural, the result follows from the Yoneda lemma. Here (1) is because $T \in \mathcal{D}_{\leq n}$, (2) is by adjunction, and (3) is because $U$ is left-$t$-exact so $UT \in \mathcal{C}_{\leq n}$.

With this preparation, we can formulate our theorem.

**Theorem 3.** Let $M: \mathcal{C} \rightleftarrows \mathcal{D}: U$ be adjoint functors between $t$-categories, such that $M$ is right-$t$-exact and $U$ is left-$t$-exact. Then for $E \in \mathcal{C}_{>0}$ there is a natural isomorphism

$$\pi_0^\mathcal{D} ME \simeq M^\triangledown \pi_0^\mathcal{C} E.$$  

**Proof.** We have

$$M^\triangledown \pi_0^\mathcal{C} E = \pi_0^\mathcal{D} M \pi_0^\mathcal{C} E \overset{(1)}= \pi_0^\mathcal{D} ME \overset{(2)}= (ME)_{\leq 0} \overset{(3)}= (ME)_{\leq 0} \overset{(2)}= \pi_0^\mathcal{D} ME.$$  

This is the desired result. Here (1) is because $E \in \mathcal{C}_{>0}$, (2) is because $M$ is right-$t$-exact, and (3) is because of Lemma 2 applied to left-$t$-exactness of $U$.

This has the following useful corollary.

**Corollary 4.** In the above situation, if $M^\triangledown$ (or $U^\triangledown$) is an equivalence of categories, and the $t$-structure on $\mathcal{C}$ is non-degenerate, then the functor $M$ is conservative for connective objects.

**Proof.** If $X \in \mathcal{C}_{>0}$ and $FX = 0$ then $F^\triangledown(\pi_0^\mathcal{C} X) = \pi_0^\mathcal{D} FX = 0$, so $X \in \mathcal{C}_{>n+1}$. Iterating this argument we find that $\pi_i^\mathcal{C} X = 0$ for all $i$ and so $X = 0$ by non-degeneracy of the $t$-structure.

For convenience, we also include the following well-known observation.

**Lemma 5.** Let $M: \mathcal{C} \rightleftarrows \mathcal{D}: U$ be adjoint functors between $t$-categories. Then $M$ is right-$t$-exact if and only if $U$ is left-$t$-exact.

So in applying the theorem, only one of the two exactness properties has to be checked.

**Proof.** This is immediate by adjunction: $M$ is right-$t$-exact if and only if $M(\mathcal{C}_{>0}) \subset \mathcal{D}_{>0}$, which happens if and only if $[ME,F] = 0$ for all $E \in \mathcal{C}_{>0}, F \in \mathcal{D}_{<0}$. By adjunction $[ME,F] = [E,UF]$, and this vanishes if and only if $U(\mathcal{D}_{<0}) \subset \mathcal{C}_{<0}$, i.e. $F$ is left-$t$-exact.

To illustrate these rather abstract results, we show how to recover the classical Hurewicz theorem for spectra. For this, we consider understood the functor $C_*: \mathsf{SH} \rightarrow \mathsf{D}(\mathsf{Ab})$ and the respective $t$-structures. We write $\pi_i = \pi_i^{\mathsf{SH}}, H_i = \pi_i^{\mathsf{D}(\mathsf{Ab})}$. Since $C_*$ commutes with arbitrary sums (wedges), it has a right adjoint $U$, by Neeman’s version of Brown representability. Since $C_* S = \mathbb{Z}[0]$ is projective, an easy computation shows that $\pi_i UC = H_i C$ for any $C \in \mathsf{D}(\mathsf{Ab})$. Thus $U$ is $t$-exact (it is in fact the Eilenberg-MacLane spectrum functor). From Lemma 5 and Theorem 3 we conclude now that for $E \in \mathsf{SH}_{>0}$ we have $\pi_0^\mathcal{D} E = H_0 E$ (recall that $C_*^\triangledown$ is an isomorphism).

**Constructing $t$-structures.** In Section 5 we will have the need of transferring $t$-structures to localisations. The following Lemma applies in many situations where the homotopy objects are computed as homotopy sheaves, as will always be the case for us.

**Lemma 6.** Let $\mathcal{C}$ be a compactly generated triangulated $t$-category with coproducts, $G \subset \mathcal{C}$ a set of compact generators.

Let $i: U \subset \mathcal{C}$ be a colocalising subcategory with left adjoint $L: \mathcal{C} \rightarrow U$.

Assume the following:

(a) $G \subset \mathcal{C}_{>0}$,

(b) if $E \rightarrow F$ is a morphism in $\mathcal{C}$ such that for all $X \in G$ the induced map $[X,E] \rightarrow [X,F]$ is surjective, then $\pi_0^\mathcal{C} (E) \rightarrow \pi_0^\mathcal{C} (F)$ is also surjective,

(c) the $t$-structure is non-degenerate,

(d) homotopy objects commute with directed homotopy colimits,

(e) $L$ preserves compact objects,

(f) $iLG \subset \mathcal{C}_{>0}$.

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Then \( U \) has a unique \( t \)-structure such that \( i : U \to \mathcal{C} \) is \( t \)-exact.

**Proof.** Uniqueness is clear. We need to show existence.

I claim that for \( E \in U \) there exists a map \( E \to E' \) with \( E' \in U \), \( \pi_i^U(E') = 0 \) and \( \pi_i^U(E) = \pi_i^U(E') \) for \( i < 0 \). Indeed let \( A = \bigoplus_{X \in G} \bigoplus_{[X,E]} X \). Then there is a canonical map \( A \to E = iE \) and the cofibre \( LA \to E \to E' \) has the required property. Indeed \( \pi_i^U(E') = \pi_i^U(E) \) for \( i < 0 \) because \( \pi_i^U(A) = 0 \) by assumptions (a), (d). Also for \( X \in G \), given \( \alpha : X \to iE \) we get a factorisation \( X \to iLX \to iE \). Thus \([X,iLA] \to [X,iE]\) is surjective and by assumption (b), \( \pi_0^L(E') = 0 \).

Repeating the argument, given \( E \in U \) we get a diagram \( E = E_0 \to E_1 \to E_2 \to \ldots \) with \( E_i \in U \), and where \( E_i \to E_{i+1} \) has the property that \( \pi_i^U(E_i) = \pi_i^U(E_{i+1}) \) for \( k < i \) and \( \pi_i^U(E_{i+1}) = 0 \). Let \( E_{\infty} = \text{hocolim} E_i \). Since \( L \) preserves compact objects by (e), \( i \) preserves coproducts and so commutes with hocolim, whence \( E_{\infty} \) makes unambiguous sense (i.e. we may compute the homotopy colimit in \( U \) or \( \mathcal{C} \), with the same result).

But note that \( \pi_i^U(E) = \pi_i^U(E_k) \) for all \( i < 0 \) and all \( k \geq 0 \), whereas \( \pi_i^U(E_k) = 0 \) for \( 0 \leq i < k \). It follows from our assumption (d) that \( \pi_i^U(E_{\infty}) = 0 \) for all \( i \geq 0 \), and thus by non-degeneracy of the \( t \)-structure (c) we have \( E_{\infty} \in \mathcal{C}_{\geq 0} \). The map \( E \to E_{\infty} \) induces an isomorphism on \( \pi_i^U \) for all \( i < 0 \) and hence \( E_{\infty} \simeq E_{<0} \), again by (c).

Consequently we have shown that for \( E \in U \) we also have \( E_{<0} \in U \). It follows that \( E_{\geq 0} \in U \) (since \( U \) is triangulated) and one easily verifies that \( U \cap \mathcal{C}_{\geq 0}, U \cap \mathcal{C}_{<0} \) defines a \( t \)-structure on \( U \). The functor \( i : U \to \mathcal{C} \) is \( t \)-exact by design. \qed

## 3 The Case of Motivic Homotopy Theory: The Motivic Hurewicz Theorem

In this section we shall work with a fixed perfect ground field \( k \).

We now show how to apply the results of the previous section to motivic homotopy theory. This mainly consists in recalling definitions and providing finite references.

First we need to recall the construction of \( \text{SH}(k) \) and \( \text{DM}(k) \). We follow [34, Section 2]. Let \( \text{Sm}(k) \) be the category of smooth schemes over the perfect field \( k \) and \( \text{Cor}(k) \) the category whose objects are the smooth schemes and whose morphisms are the finite correspondences. We write \( \text{Shv}(k) \) (respectively \( \text{Shv}^{tr}(k) \)) for the categories of Nisnevich sheaves. Write \( R : \text{Sm}(k) \to \text{Shv}(k) \) and \( R_{tr} : \text{Cor}(k) \to \text{Shv}^{tr}(k) \) for the functors sending an object to the sheaf it represents.

There is a natural functor \( M : \text{Sm}(k) \to \text{Cor}(k) \) with \( M(X) = X \) and \( M(f) = \Gamma_f \), the graph of \( f \). This induces a functor \( U : \text{Shv}^{tr}(k) \to \text{Shv}(k) \) via \( (UF)(X) = F(MX) \). There is a left adjoint \( M : \text{Shv}(k) \to \text{Shv}^{tr}(k) \) to \( U \). It is the unique colimit-preserving functor such that \( M(RX) = R_{tr}(X) \). Write \( \text{Shv}_{\ast}(k) \) for the category of pointed sheaves. Then there is \( R_{+} : \text{Sm}(k) \to \text{Shv}_{\ast}(k) \) obtained by adding a disjoint base point. The objects in \( U(\text{Shv}_{\ast}^{tr}(k)) \) are canonically pointed (by zero) and one obtains a new adjunction \( \tilde{M} : \text{Shv}_{\ast}(k) \rightleftarrows \text{Shv}^{tr}(k) : U \).

We can pass to simplicial objects and extend \( M \) and \( U \) levelwise to obtain an adjunction \( \tilde{M} : \Delta^{op}\text{Shv}_{\ast}(k) \rightleftarrows \Delta^{op}\text{Shv}^{tr}(k) : U \). We denote by \( Spt(k) \) the category of \( S^{2 \cdot 1} := S^1 \wedge \mathbb{G}_m \)-spectra in \( \Delta^{op}\text{Shv}_{\ast}(k) \) and by \( Spt^{tr}(k) \) the category of \( M(S^{2 \cdot 1}) \)-spectra in \( \Delta^{op}\text{Shv}^{tr}(k) \). The adjunction still extends, so we obtain the following commutative diagram.

\[
\begin{array}{ccc}
\text{Sm}(k) & \xrightarrow{M} & \text{Cor}(k) \\
\downarrow R_+ & & \downarrow R_{tr} \\
\text{Shv}_{\ast}(k) & \xleftarrow{U} & \text{Shv}^{tr}(k) \\
\downarrow & & \downarrow \\
\Delta^{op}\text{Shv}_{\ast}(k) & \xrightarrow{M} & \Delta^{op}\text{Shv}^{tr}(k) \\
\downarrow \Sigma^\infty & & \downarrow \Sigma^\infty \\
\text{Spt}(k) & \leftarrow & \text{Spt}^{tr}(k)
\end{array}
\]

(See also [15, Diagram (4.1)].) One may put the projective local model structures on the four categories in the lower square and then the adjunctions become Quillen adjunctions, so pass through
localisation. Contracting the affine line yields the \( \mathbb{A}^1 \)-local model structures. The homotopy category of \( \text{Spt}(k) \) (in this model structure) is denoted \( \text{SH}(k) \) and is called the motivic stable homotopy category. Similarly the homotopy category of \( \text{Spt}^0(k) \) is denoted \( \text{DM}(k) \). It is essentially a bigger version of the category constructed by Voevodsky, as explained in [34, Section 2]. We have thus the following commutative diagram.

\[
\begin{array}{c}
\text{Sm}(k) \\
\downarrow^{\Sigma^\infty(\ast)} \quad \downarrow^M \\
\text{SH}(k) \\
\downarrow^U \\
\text{DM}(k)
\end{array}
\]

We recall that \( M \) (in all its incarnations) is a symmetric monoidal functor.

Next we need to define \( t \)-structures on \( \text{DM}(k) \) and \( \text{SH}(k) \). For \( E \in \text{SH}(k) \) (respectively \( E \in \text{DM}(k) \)) let \( \underline{\pi}_i(E)_j \) (respectively \( \underline{h}_i(E)_j \)) be the sheaf on \( \text{Sm}(k) \) (respectively on \( \text{Cor}(k) \)) associated with the presheaf \( V \mapsto [\Sigma^\infty(V_+) \land S^i, \mathbb{G}_m^{\ast j} \land E] \). Put

\[
\text{SH}(k)_{\geq 0} = \{ E \in \text{SH}(k) : \underline{\pi}_i(E)_j = 0 \text{ for } i < 0 \text{ and } j \in \mathbb{Z} \}
\]

\[
\text{SH}(k)_{\leq 0} = \{ E \in \text{SH}(k) : \underline{\pi}_i(E)_j = 0 \text{ for } i > 0 \text{ and } j \in \mathbb{Z} \},
\]

and similarly for \( \text{DM}(k) \). By [26, Section 5.2] this defines a \( t \)-structure on \( \text{SH}(k) \) called the homotopy \( t \)-structure. It is also true that \( \text{DM}(k)_{\geq 0}, \text{DM}(k)_{\geq 0} \) define a \( t \)-structure. This can be seen by repeating the arguments of [15, Section 2.1] for \( \text{DM}(k) \), noting that the connectivity theorem for \( \text{DM}(k) \) follows from Voevodsky’s cancellation theorem.

With all this setup out of the way, we can prove the result of this section.

**Lemma 7.** The functor \( U : \text{DM}(k) \rightarrow \text{SH}(k) \) is \( t \)-exact. In fact for \( E \in \text{DM}(k) \) we have \( \underline{\pi}_i(UE)_j = U(\underline{h}_i(E)_j) \) and \( U : \text{Shv}^t(k) \rightarrow \text{Shv}(k) \) detects zero objects.

**Proof.** It follows from the definitions of the \( t \)-structures that we need only prove the “in fact” part. Let \( \underline{\pi}_{pre}^i(E)_j \) be the presheaf \( V \mapsto [\Sigma^\infty(V_+) \land S^i, \mathbb{G}_m^{\ast j} \land E] \), and similarly for \( \underline{h}_{pre}^i(E)_j \). Then writing also \( U : \text{PreShv}(\text{Cor}(k)) \rightarrow \text{PreShv}(\text{Sm}(k)) \) we get immediately from the definitions that \( U(\underline{h}_{pre}^i(E)_j) = \underline{\pi}_{pre}^i(UE)_j \). So we need to show that \( U : \text{PreShv}(\text{Cor}(k)) \rightarrow \text{PreShv}(\text{Sm}(k)) \) commutes with taking the associated sheaf. This is well known, see e.g. [23, Theorem 13.1]. \( \square \)

**Corollary 8** (Preliminary form of the Motivic Hurewicz Theorem). Let \( E \in \text{SH}(k)_{\geq 0} \). Then \( ME \in \text{DM}(k)_{\geq 0} \) and \( h_0(ME)_* = M^\circ(\underline{\pi}_0(E)_*) \).

Here \( \underline{\pi}_0(E)_* \) denotes the homotopy object in \( \text{SH}(k)^{\circ} \), and similarly for \( h_0(ME)_* \).

**Proof.** We know that \( M \) is left adjoint to the \( t \)-exact functor \( U \). Hence \( M \) is right-\( t \)-exact by Lemma 5. Thus \( ME \in \text{DM}(k)_{\geq 0} \), and the result about homotopy objects is just a concrete incarnation of Theorem 3. \( \square \)

Next we explain how to identify \( M^\circ : \text{SH}(k)^{\circ} \rightarrow \text{DM}(k)^{\circ} \). To do this, recall the Hopf map \( \mathbb{A}^2 \setminus \{0\} \rightarrow \mathbb{P}^1 \). In \( \text{SH}(k) \) we have the isomorphism \( \Sigma^\infty(\mathbb{A}^2 \setminus \{0\}) \simeq \Sigma^\infty(\mathbb{P}^1 \land \mathbb{G}_m) \) and hence this defines a stable map \( \eta : \Sigma^\infty(\mathbb{G}_m)^\ast \rightarrow S \), where \( S \) is the sphere spectrum \( \Sigma^\infty(\text{Spec}(k)_+) \). If \( F \in \text{SH}(k)^{\circ} \) we put \( F(n) := \underline{\pi}_n(F \land \mathbb{G}_m^{\ast n}) \). Consequently there is a natural map \( \eta_F = \underline{\pi}_n(\eta \land \text{id}_F) : F(1) \rightarrow F \).

An important observation is that \( M(\eta) \) is the zero map. One may show that this implies that for \( F \in \text{DM}(k)^{\circ} \), we have \( 0 = \eta_{MF} : UF(1) \rightarrow UF \). We denote by \( \text{SH}(k)^{\circ, \eta=0} \) the full subcategory of \( \text{SH}(k)^{\circ} \) consisting of objects \( F \in \text{SH}(k)^{\circ} \) with \( \eta_F = 0 \).

**Theorem 9** (Deglise [8]). Let \( k \) be a perfect field. The functor \( U : \text{DM}(k)^{\circ} \rightarrow \text{SH}(k)^{\circ, \eta=0} \) is an equivalence of categories.

**Proof.** Modulo identifying \( \text{SH}(k)^{\circ} = \Pi_\ast(k) \) and \( \text{DM}(k)^{\circ} = \Pi_\ast^0(k) \) this is Theorem 1.3.4 of Deglise. The first identification is explained in [26, Theorem 5.2.6]. The second one is obtained by adapting loc. cit. \( \square \)

**Corollary 10.** For \( F \in \text{SH}(k)^{\circ, \eta=0} \) we have \( UM^\circ F = F \).
Proof. By the theorem we may write $F = UF'$. Using the fact that $M^{\triangleright}$ is left adjoint to $U (= U^{\triangleright})$ by Proposition 1, we compute $[M^{\triangleright}F, T] = [M^{\triangleright}UF', T] = [UF', UT] = [F', T]$, where the last equality is because $U$ is fully faithful (by the theorem). Thus $M^{\triangleright}F = F'$ by the Yoneda lemma, and finally $U M^{\triangleright}F = UF' = F$.

Corollary 11. For $F \in \text{SH}(k)^{\triangleright}$ we have $UM^{\triangleright}(F) = F/\eta$, where $F/\eta$ denotes the cokernel of $\eta_F : F(1) \to F$ in the abelian category $\text{SH}(k)^{\triangleright}$.

Proof. We have the right exact sequence

$$F(1) \to F \to F/\eta \to 0.$$  

Since $M^{\triangleright}$ is left adjoint it is right exact. Also $U$ is exact, so we get the right exact sequence

$$UM^{\triangleright}F(1) \to UM^{\triangleright}F \to UM^{\triangleright}(F/\eta) \to 0.$$  

The first arrow is zero and $UM^{\triangleright}(F/\eta) = F/\eta$ by the previous corollary (note that $F/\eta \in \text{SH}(k)^{\triangleright, n=0}$). The result follows.

We thus obtain the Hurewicz theorem for $\text{SH}(k) \to \text{DM}(k)$.

Theorem 12 (Final Version of the Motivic Hurewicz Theorem). Let $k$ be a perfect field and $E \in \text{SH}(k)^{\geq 0}$. Then $ME \in \text{DM}(k)^{\geq 0}$ and modulo the identification of $\text{DM}(k)^{\triangleright}$ as a full subcategory of $\text{SH}(k)^{\triangleright}$ (via Theorem 9) we have

$$\mathbb{L}_0(ME)_* = \mathbb{L}_0(E)_*/\eta.$$  

Proof. Combine Corollary 8 with Corollary 11.

4 Homotopy Modules and the Slice Filtration

In essentially all of this section, the base field $k$ will be assumed perfect. We restate this assumption with each theorem, but not necessarily otherwise.

We now study in more detail the heart $\text{SH}(k)^{\triangleright}$. First recall some notation. By $\text{Sm}(k)$ we denote the symmetric monoidal category of smooth varieties over a field $k$, monoidal operation being cartesian product. We write $\text{Shv}(k)$ for the abelian closed symmetric monoidal category of sheaves of abelian groups on $\text{Sm}(k)$ in the Nisnevich topology. The monoidal product comes from the ordinary tensor product. We study it because of the following result. (In fact this presheaf is already a Nisnevich sheaf, but we do not need this observation.) One has $Z\times X \tensor ZY \equiv Z(X \times Y)$, i.e. $Z\bullet$ is a monoidal functor.

If $X \in \text{Sm}(k)$ and $x \in X$ is a rational point, there is a natural splitting $Z\times X \cong Z(\text{Spec}(k)) \tensor Z(X,x)$ (defining the last term). We put $ZG_m = Z(A^1 \setminus \{0\}, 1)$. An easy computation shows that for any $F \in \text{Shv}(k)$, $U \in \text{Sm}(k)$ one has a natural splitting $F(U \times G_m) = F(U) \oplus \Omega F(U)$ (defining the last term). In fact $\Omega F$ is easily seen to be a sheaf. One verifies without difficulty that $\Omega F \cong \text{Hom}(ZG_m, F)$. (Here $\text{Hom}$ denotes the right adjoint of $\otimes$.) In other words the notation $\Omega F = F_{-1}$ is often used; we avoid this for sake of clarity.

A sheaf $F \in \text{Shv}(k)$ is called strictly $A^1$-invariant (or strictly invariant if the context is clear) if for all $U \in \text{Sm}(k), n \in \mathbb{N}$ the natural map $H^n_{\text{Nis}}(U, F) \to H^n_{\text{Nis}}(U \times A^1,F)$ is an isomorphism. The full subcategory of $\text{Shv}(k)$ consisting of strictly invariant sheaves is denoted $\text{HI}(k)$.

By a homotopy module we mean a collection $F_* \in \text{HI}(k), * \in \mathbb{Z}$ together with isomorphisms $F_n \cong \Omega F_{n+1}$. A morphism of homotopy modules $f_* : F_* \to G_*$ is a collection of morphisms $f_n : F_n \to G_n$ such that $F_{n+1} = f_n$ under the natural identifications. We denote the category of homotopy modules by $\text{HI}_*(k)$. We study it because of the following result.

Theorem 13 (Morel [26], Theorem 5.2.6). Let $k$ be perfect and $E \in \text{SH}(k)$. Then for each $n \in \mathbb{Z}$, the collection $\mathbb{L}_n(E)_*$ is a homotopy module. The functor $\text{SH}(k)^{\triangleright} \to \text{HI}_*(k), E \mapsto \mathbb{L}_0(E)_*$ is an equivalence of categories.
In particular the category $\text{HI}_i(k)$ is abelian. We can clarify the abelian structure as follows. Recall that $\text{HI}(k)$ is the heart of the $S^1$-stable $\mathbb{A}^1$-homotopy category [26, Lemma 4.3.7(2)]. By construction, the inclusion $\text{HI}(k) \to \text{Shv}(k)$ is exact. This means that for any morphism of strictly invariant sheaves, the kernel and image (computed in $\text{Shv}(k)$) are strictly invariant. Finally the functor $u^n : \text{HI}(k) \to \text{HI}(F, N \to F_n)$ is also exact.

Homotopy modules have a lot of structure. The zeroth homotopy module of the sphere spectrum is denoted $K_0^\text{MW} := \pi_0(S)_*$ and called unramified Milnor-Witt $K$-theory. It has been explicitly described by Morel [29, Chapter 3]. For $n \geq 1$ there exist natural surjections $\mathbb{Z}[m] \to K_n^\text{MW}$. Let $F_n \in \text{HI}(k)$, $m \in \mathbb{Z}$. The isomorphism $\Omega F_n \cong F_{n-1}$ induces by adjunction a pairing $\mathbb{Z}[m] \otimes F_{n-1} \to F_n$. It turns out that this pairing factorises through the surjection $\mathbb{Z}[m] \to K_n^\text{MW}.$

Now quite generally given any three sheaves $F, G, H$ there exists a natural morphism $\text{Hom}(F, G) \otimes H \to \text{Hom}(F \otimes G, H)$. Thus the pairing $K_n^\text{MW} \otimes F_n \to F_{n+1}$ induces $\Omega^2 K_n^\text{MW} \otimes F_n \to \Omega^2 F_{n+1}$, i.e. $K_{n-1}^\text{MW} \otimes F_n \to F_{n-1}$. The graded ring sheaf $K_n^\text{MW}$ is generated by $K_1^\text{MW}$ and $\eta \in K_1^\text{MW}$, so there is at most one possible extension of these two pairings to a total pairing $K_n^\text{MW} \otimes F_* \to F_{*+1}$. It turns out that this pairing always exists. In fact $\text{HI}(k) \cong \text{SH}(k)^\text{eff}$ inherits a monoidal structure we denote $\wedge$ to avoid confusion. For any $F, G \in \text{HI}(k)$ there exists a natural morphism $F_n \otimes G_m \to (F \wedge G)_{n+m}$. Since $K_n^\text{MW} = \pi_0(S)_*$ is the unit object, we obtain a pairing $K_n^\text{MW} \otimes F_* \to (K_n^\text{MW} \wedge F)_{n+m} \cong F_{n+m}$. One may prove that these two pairings just constructed coincide.

Homotopy modules also have transfers, though in a weaker sense than Voevodsky’s sheaves with transfers. For this, denote by $F_k$ the category of fields of finite transcendence degree over $k$. If $L \in F_k$ (or more generally $L$ essentially smooth over $k$) and $F \in \text{Shv}(k)$, then $F(L)$ can be defined by an appropriate colimit. If $L'/L$ is any finite extension (still $L' \in F_k$), $n \in \mathbb{Z}$ and $F \in \text{HI}(k)$, then Morel has constructed the cohomological transfer [29, Chapter 4] $\tau_{L'/L} : F_n(L') \to F_n(L)$. (Actually, this only works in characteristic not two. See Subsection B.5 for the correct definition in full generality.)

If $F$ is a strictly invariant sheaf then $F$ is unramified [28, Lemma 6.4.4]. (In particular for connected $X \in \text{Sm}(k)$, the natural map $F(X) \to F(k(X))$ is injective.) The notion of unramified presheaves is reviewed in more detail at the beginning of Subsection B.1.) Thus $F$ is $0$ if and only if $F(L) = 0$ for all $L \in F_k$. Since $\text{HI}(k)$ is abelian it follows that if $F$ is a strictly invariant sheaf and $G, H \subset F$ are strictly invariant subsheaves, then $G = H$ if and only if $G(L) = H(L)$ for all $L \in F_k$ (see the beginning of Subsection B.1 for the definition of $F(L)$ for $L$ not of finite type over $k$).

Now let $F \otimes G \to H$ be a pairing, where $H$ is a strictly invariant sheaf with transfers for finite field extensions. We shall write

$$(FG)^{tr}(L) := \langle \tau_{L'/L}(F(L')G(L')) \rangle_{L'/L \text{ finite} \subset H(L)}.$$

It follows from the above discussion that there exists at most one strictly invariant sheaf $(FG)^{tr} \subset H$ with the above sections over fields.

Now recall the slice filtration [39, Section 2]. Write $\text{SH}(k)^{eff}(i)$ for the localising subcategory of $\text{SH}(k)$ generated by $(X^\infty \times X)_+ \wedge \mathbb{G}_m^\wedge$ for all $X \in \text{Sm}(k)$. The inclusion $\text{SH}(k)^{eff}(i) \to \text{SH}(k)$ commutes with arbitrary sums by construction and so has a right adjoint $f_i$ by Neeman’s version of Brown representability. The object $f_iE$ is called the $i$-th slice cover of $E$. It is easy to see that there is a commutative diagram of natural transformations

\[
\begin{array}{ccc}
f_i & \longrightarrow & f_{i-1} \\
\downarrow & & \downarrow \\
id & \longrightarrow & id. \\
\end{array}
\]

We call $E$ such that $E \in \text{SH}(k)^{eff}(n)$ for some $n$ (equivalently $E = f_n E$) slice-connective.

Suppose that $E \in \text{SH}(k)_{\geq 0}$. We want to define a filtration on $\pi_0(E)_*$. We shall put

$F_N f_0(E)_* := \text{im}(\pi_0(f_{-N} E)_* \to \pi_0(E)_*) \subset \pi_0(E)_*.$

(Imag computed in the abelian category $\text{HI}_i(k)$.) There is now the following interesting result. (We warn the reader that Levine uses somewhat different indexing conventions than we do.)

**Theorem 14** (Levine [20], slightly adapted Theorem 2). Let $k$ be a perfect field of characteristic different from 2 and $E \in \text{SH}(k)_{\geq 0}$. Then for $m \geq i$ any perfect field extension $F/k$ we have

$$\langle F f_0(E)_* \rangle_{m}(F) = (k_1^\text{MW} \pi_0(E)_*)^{tr}(F).$$

(1)
Note that if \( k \) has characteristic zero, then the above theorem implies that there is a (unique) strictly invariant sheaf \( (K_i^{MW} H_m)^{tr} \) with sections over fields given by formula (1). In characteristic \( p > 0 \) the result is not quite strong enough, but it is good enough if we invert \( p \). In Appendix B, we prove the following variant.

**Theorem 15.** Let \( H_\ast \in HI_\ast(k) \) be a homotopy module over the perfect field \( k \).

1. For all \( m, n \geq 0 \) there exists a (unique) strictly invariant sheaf \( K_i^{MW} H_m \) which has sections over fields given by formula (1).

2. We have that \( (K_i^{MW} H_m)^{tr} = \Omega(K_i^{MW} H_m)^{tr} \subset H_{n+m} \) for all \( m \in \mathbb{Z}, n \geq 0 \).

Put
\[
(F_n H)_m = \begin{cases} (K_i^{MW} H_n)^{tr} : m > n \\ H_m : m \leq n. \end{cases}
\]

3. \( (F_n H)_\ast \) is a homotopy submodule of \( H_\ast \).

4. If \( H = \overline{\pi}_\ast (E)_\ast \), for \( E \in SH(k)^{gp} \), then the filtration \( F_\ast H \) is finite: there exists \( N >> 0 \) with \( F_N H = H \).

(Here \( SH(k)^{gp} \) denotes the set of compact objects in \( SH(k) \).) What this theorem says is that even though we may not be able to understand the filtration \( F_\ast H \), we can construct a different filtration \( \tilde{F}_\ast H \) which agrees on perfect fields. There is one set-back compared to Levine’s theorem: the filtration \( \tilde{F}_\ast H \) is completely algebraic, not geometric in origin. In particular if \( E \in SH(k)^{gp} \) for some \( N \) (i.e. \( E \) is slice-connective), then it is immediate that \( F_N \overline{\pi}_\ast E_\ast = \overline{\pi}_\ast E_\ast \), but we do not have such a nice characterisation for \( \tilde{F}_\ast \). However see (4) of the Theorem.

We write \( SH(k)_e \) for the e-localised version of \( SH(k) \). This is the full (colocalising) monoidal triangulated subcategory of all objects \( E \in SH(k) \) such \( E \xrightarrow{\sim} E \) is an isomorphism; equivalently all the homotopy sheaves \( \overline{\pi}_\ast (E) \) are uniquely e-divisible.

We can finally prove our first conservativity theorem.

**Theorem 16 (Conservativity I).** Let \( k \) be a perfect field of finite 2-étale cohomological dimension and exponential characteristic \( e \), and \( E \in SH(k) \). Assume that either (a) \( E \in SH(k)_e \) is connective and slice-connective, or (b) \( E \) is compact.

Then if \( ME = 0 \), one also has \( E = 0 \).

More specifically, if either (a) or (b) holds, and \( ME \in DM(k)_n \) then also \( E \in SH(k)_n \).

**Proof.** As a preparatory remark, let us note that compact objects are connective and slice-connective. Indeed \( SH(k) \) is generated (as a localising subcategory) by the compact objects \( \Sigma^\infty X_\ast (i) \) for \( i \in \mathbb{Z} \) and \( X \in Sm(k) \) (see e.g. [34], Lemma 2.27 and paragraph thereafter); it follows from general results [31, Lemma 2.2] that \( SH(k)^{gp} \) is the thick triangulated subcategory generated by the same objects. It follows that \( E \in SH(k)^{gp} \) is obtained using finitely many operations from (finitely many) objects of the form \( \Sigma^\infty X_\ast (i) \), and all of these are connective [28] and slice-connective. It remains to observe that the subcategories of connective and slice-connective spectra are also thick.

We shall prove: if \( E \in SH(k)_n \), (a) or (b) holds, and \( \overline{\pi}_\ast (ME)_\ast = 0 \) then \( E \in SH(k)_{n+1} \). This is the “more specifically” part. It follows that if actually \( ME = 0 \) then \( \overline{\pi}_\ast (E)_j = 0 \) for all \( i \) and \( j \) and so \( E \simeq 0 \), since weak equivalences in \( SH(k) \) are detected by \( \overline{\pi}_\ast (\bullet)_\ast \), essentially by construction [26, Proposition 5.1.14].

The first claim of the theorem then follows, since under (a) we assume \( E \) connective, and under (b) we assume \( E \) compact and all compact objects are connective.

To explain the general argument, let us assume for now that \( k \) has characteristic zero. We will indicate at the end what needs to be changed in positive characteristic.

By unramifiedness of strictly invariant sheaves, it is enough to show that \( \overline{\pi}_\ast (E)_\ast (K) = 0 \) for all finitely generated field extensions \( K/k \). We consider the filtration \( F_\ast \overline{\pi}_\ast (E)_\ast \). We know that \( F_N \overline{\pi}_\ast (E)_\ast = \overline{\pi}_\ast (E)_\ast \) for some \( N \) sufficiently large, by slice-connectivity. I claim that for \( r > 0 \) we have \( \overline{\pi}_\ast (E)_{N+r} (K) = 0 \). This claim proves the result. Indeed we have \( \overline{\pi}_\ast (E)_\ast / \eta = 0 \) by the Hurewicz Theorem 12, so \( \eta \) is surjective, whence the claim implies that \( \overline{\pi}_\ast (E)_\ast (K) = 0 \) for all \( i \).
We compute
\[
\pi_0(E)_{N+r}(K) = (\lambda K_{MW}^p \pi_0(E)_{N})^{tr}(K) \\
= (K_{MW}^p \eta^* \pi_0(E)_{N+r})^{tr}(K) \\
= (L^r \pi_0(E)_{N+r})^{tr}(K).
\]

Here the first equality is by Theorem 14, the second is by surjectivity of \( \eta \), and the third is essentially by definition. Hence we make use of the sheaf of unramified fundamental ideals \( I \) and its powers \( I^r \). We can define it quickly as \( I = im(\eta : K_{MW}^p \to K_{MW}^p) \). It is easy to see that \( I^r = im(\eta^r : K_{MW}^p \to K_{MW}^p) \), and also that if \( K \) is any field, then \( I^r(K) \subset K_{MW}^p(K) \cong GW(K) \) is given by \( I(K)^r \), the \( r \)-th power of the fundamental ideal. It is thus enough to show that for some \( r >> 0 \) and for all \( L/K \) finite, we have \( I(L)^r = 0 \). Now \( K \) is finitely generated over \( k \), so \( K \) has finite 2-éthale cohomological dimension [37, Theorem 28 of Chapter 4], say \( R \). Then \( L \) also has 2-éthale cohomological dimension bounded by \( R \) by loc. cit. Finally the resolution of the Milnor conjectures (see [25] for an overview) implies that \( I(L)^l = 0 \) for \( l > R \). This concludes the proof in characteristic zero.

If \( e = 2 \) then the functor \( SH(k)_2 \to DM(k, \mathbb{Z}[1/2]) \) is conservative on connective objects by Corollary 4 (see the discussion of \( SH(k)_2 \) in the last paragraph of this section), and the theorem follows.

If \( e > 1 \) and assumption (b) holds (i.e. \( E \) is compact), then the same argument as in characteristic zero applies, but with \( F_s \) replaced by \( \tilde{F}_s \) and the reference to Theorem 14 replaced by Theorem 15.

If \( e > 2 \) and assumption (a) holds, we still need to show that for \( K/k \) finitely generated we have \( \pi_0(E)_*(K) = 0 \). Let \( K^p/K \) be the perfect closure. By Lemma 17 below, the natural map \( \pi_0(E)_*(K) \to \pi_0(E)_*(K^p) \) is injective. It is thus enough to show that the latter group is zero. For this we use the same argument as in characteristic zero; the only additional thing we need is to show that \( K^p \) has finite 2-éthale cohomological dimension, but this follows from finite dimension of \( K \) since they have the same éthale site [38, Tags 04DZ and 01S4].

**Lemma 17.** Let \( k \) be a perfect field of characteristic \( p > 0 \) and \( H_* \) a homotopy module on which \( p \) is invertible. Let \( K/k \) be a finitely generated field and \( K^p/k \) a purely inseparable extension. Then \( H_* (K) \to H_* (K^p) \) is injective.

**Proof.** Let \( L/k \) be a finitely generated field extension and \( x \in L \) not a \( p \)-th root. Write \( L' = L(x^{1/p}) \). I claim that \( H_*(L) \to H_*(L') \) is injective. Once this is done we conclude that \( H_*(K) \to H_*(K^p) \) is injective for any purely inseparable finitely generated extension (being a composite of finitely many extensions of the form \( L'/L \)) and hence for any purely inseparable extension by continuity. (See the beginning of Subsection B.1 for the continuous extension of a homotopy module from a presheaf on \( Sm(k) \) to essentially smooth schemes.)

In order to prove the claim, we shall use the transfer \( tr_{L'/L} : H_*(L') \to H_*(L) \). This satisfies the projection formula: if \( a \in H_*(L) \) then \( tr_{L'/L} \cdot \alpha_{L'} = tr_{L'/L} \cdot (1) \alpha \), where \( 1 \in GW(L') \) is the unit. (This is because transfer comes from an actual map of pro-spectra \( \Sigma^\infty Spec(L)_+ \to \Sigma^\infty Spec(L')_+ \).) Hence it is enough to show that \( t := tr_{L'/L} (1) \) is a unit in \( GW(L)[1/p] \). I claim that

\[
t = \sum_{i=1}^p ((-1)^{i+1}).
\]

Indeed this may be checked by direct computation, using the fact that (geometric) transfers on \( K_*^{MW} \) coincide with Scharlau transfers, as follows from their definition [29, Section 4.2] and Scharlau’s reciprocity law [35, Theorem 4.1]. We will explain this at the end of the proof.

Next we need to show that \( t \in GW(L)[1/p] \) is invertible. For this, recall the fibre product decomposition
\[
GW(L) \xrightarrow{dim} \mathbb{Z} \\
\xrightarrow{cl} W(L) \xrightarrow{\cdot 2} \mathbb{Z}/2.
\]

Here \( W(L) = GW(L)/h \) is the Witt ring, \( h = (1,-1) \) is the hyperbolic plane, \( dim \) is the dimension homomorphism (determined by the property that \( dim(a) = 1 \) for all \( a \in L^\times \)) and \( cl \) is the canonical surjection (as is \( \mathbb{Z} \to \mathbb{Z}/2 \)). Both injectivity and surjectivity of the map \( GW(L) \to W(L) \times_{\mathbb{Z}/2} \mathbb{Z} \) follow from the fact that \( GW(L) h = \mathbb{Z}h \) [18, Lemma 1.16].
Thus to show that \( t \in GW(L)[1/p] \) is invertible, it is enough to consider the canonical images \( \dim(t) \in \mathbb{Z}[1/p] \) and \( cl(t) \in W(L)[1/p] \). We know that \( \dim(t) = p \) is invertible by design. If \( p = 2 \) then \( W(L)[1/p] = 0 \) [24, Theorem III.3.6], so \( cl(t) \) is a unit. Otherwise we have that \( t = \frac{e}{p}((1) + (-1)) + 1 \) and so \( cl(t) = 1 \) is also invertible. This concludes the proof, modulo the claim (2).

**Computation of \( t \)**. Recall that the transfer of a simple extension \( GW(L') \to GW(L) \) is defined by considering the surjection

\[
K^M_1(L(U)) \cong \bigoplus_{P \in (A_1)^{(1)}} GW(L[U]/P) =: R \to 0.
\]

Now write \( L' = L[U]/P_0 \) for some irreducible polynomial \( P_0 \), pick \( \alpha \in GW(L') \), consider the element \( X \in R \) with \( XP_0 = \alpha \) and \( XP = 0 \) for \( P \neq P_0 \) and pick a lift \( X' \in K^M_1(L(U)) \). There exists a canonical boundary \( \partial^\infty : K^M_1(L(U)) \to GW(L) \) and \( tr(x) := \partial^\infty(X') \). The fact that this is well-defined is not at all obvious and is discussed in [29, Section 4.2].

Let \( f_P : L[U]/P \) be the \( L \)-linear map with \( f_P(u^i) = 0 \) for \( i = 0, 1, \ldots, \deg(P) - 2 \) and \( f_P(u^{\deg(P)-1}) = 1 \). Here \( u \) is the image of \( U \). Then given a bilinear space \( B \) over \( L[U]/P \), we can view \( B \) as a space over \( L \) and transport the bilinear form to have values in \( L \) by applying \( f_P \). The result is denoted \( f_P \ast B \). This extends by linearity to yield \( f_P \ast : GW(L[U]/P) \to GW(L) \).

Let \( Y = (Y_P)_P \in R \). Then the Scharlau reciprocity theorem tells us that \( \partial^\infty(Y) = \sum_P f_P \ast \partial^P(Y) \). In particular, for \( \alpha \in GW(L') \) we apply this to the lift \( X' \) to obtain \( tr(x) = f_P \ast(x) \).

In our case, \( x \) corresponds to the one-dimensional \( L' \)-vector space \( V \) with basis \( e \) and bilinear form \( B(ae, be) = ab \) for \( a, b \in L' \). We have \( P(U) = U - x \). Thus as an \( L \)-vector space \( V \) has basis \( e, u, \ldots, w^{r-1}e \), where \( u \in L' \) is the image of \( U \). It follows that

\[
f_P \ast V \left( \sum_i a_i u^i e, \sum_j b_j w^j e \right) = \sum_i a_i b_{p-1-i}.
\]

Thus \( f_P \ast V \) can be decomposed in the evident way into a sum of two-dimensional bilinear spaces all of which are hyperbolic, and possibly a single one-dimensional space with the standard form (this happens if and only if \( p \) is odd). Hence we arrive at the claimed formula \( t = \sum_{i=1}^p ((-1)^{i-1}) \).

We note that Theorem 16 can definitely fail if \( k \) has infinite 2-étale cohomological dimension. See [5, Example 2.1.2(4)] for an example.

We also obtain the Pic-injectivity result:

**Theorem 18** (Pic-injectivity). Let \( k \) be a perfect field of finite 2-étale cohomological dimension and \( E \in SH(k) \) be invertible. If \( ME \cong Z \) then \( E \cong S \).

(Here \( Z, S \) denote the respective monoidal units of \( DM(k), SH(k) \)).

**Proof.** Invertible objects are compact because the unit is, so we can use the conservativity theorem. Since \( Z \in DM(k)_{\geq 0} \) it follows from the “more specifically” part that \( E \in SH(k)_{\geq 0} \). Let \( a \in \text{Hom}(Z, ME) \) be an isomorphism. We have \( \text{Hom}(Z, ME) = \mathbb{H}_0(ME)_0(k) = \mathbb{Z}(E)_0(k)/\eta_0(k) = \mathbb{Z}(E)_0(k)/\eta_0(E)_1(k) \). It follows that there exists \( \tilde{a} \in \mathbb{Z}(E)_0(k) = \text{Hom}(S, E) \) such that \( M\tilde{a} = a \). Conservativity implies that \( \tilde{a} \) is an isomorphism. This concludes the proof.

The remainder of this paper deals with the situation in which the 2-étale cohomological dimension cannot be finite, namely the orderable fields. We will not worry about imperfectness problems, since we are really only interested in characteristic zero. We begin with the following.

**Lemma 19** (Bondarko [5]). Let \( k \) be of characteristic zero and finite virtual 2-étale cohomological dimension.

Let \( E \in SH(k) \) be connective and slice-connective, and suppose that \( 2^m \text{id}_E = 0 \), for some \( m \in \mathbb{N} \). Then if \( ME \in DM(k)_{\geq n} \) also \( E \in SH(k)_{\geq n} \). In particular if \( ME = 0 \) then \( E = 0 \).

**Proof.** We may assume that \( E \in SH(k)_{\geq 0} \) and prove that \( \mathbb{Z}(E)_* = 0 \). By induction it will follow that \( \mathbb{Z}(E)_* = 0 \) for all \( i \), and so \( E = 0 \) because the homotopy \( t \)-structure is non-degenerate.

Suppose that \( E \in SH(k)^{\leq 1}/(-N) \) (this is true for some \( N \) because \( E \) is assumed slice-connective.) By the same arguments as in the proof of Theorem 16 it suffices to show that for any finitely generated extension \( K/k \) there exists \( r \gg 0 \) such that for any finite extension \( L/K \) we have that \( I(L)^r \mathbb{Z}(E)_{N+r}(L) = 0 \).
Write $vcd_2$ for the virtual 2-étale cohomological dimension. Then $vcd_2(K) < \infty$ and $vcd_2(L) \leq vcd_2(K)$ [37, Theorem 28 of Chapter 4]. If $r > vcd_2(K) + m$ then $I(L)^r = I(L)^r - m 2^m$ [10]. But then $I(L)^r \mathcal{E}_{N+r}(L) = 0$ because $\mathcal{E}_{N+r}(L)$ is a 2$m$-torsion group. This concludes the proof. \hfill \square

We will have to deal with several different monoidal categories at once. In order to keep notations straight, we will write $\mathbb{I} = \mathbb{I}_C$ for the unit in any monoidal category $C$, omitting the reference to $C$ if it is clear from context. Thus for example $\mathbb{I} \mathcal{SH}(k) = S$.

The unit $\mathbb{I} \mathcal{SH}(k)$ is the suspension $S_2 = \text{hocolim}(S \to S \to S \to \ldots)$ and the localisation $\mathcal{SH}(k) \to \mathcal{SH}(k)^+ \subset \mathcal{SH}(k)$ is given by smash product with $S_2$.

Recall that the category $\mathcal{SH}(k)^+ \subset \mathcal{SH}(k)$ naturally splits into two parts. Indeed $\text{End}(\mathbb{I}) \mathcal{SH}(k)^+ = GW(k)[1/2] = W(k)[1/2] \oplus Z[1/2]$ [27, Theorem 6.2.1, Remark 6.1.6 (b)] [29, Chapter 3 and Theorem 6.40]. More specifically one puts $\epsilon = -(-1) \in \text{End}(\mathbb{I})$. Then $\epsilon^2 = 1$ and hence $\epsilon_{\pm} = (\pm 1)/2$ are idempotents and induce the decomposition. We write $\mathcal{SH}(k)^+_{\pm}$ for those $E \in \mathcal{SH}(k)$ where $\epsilon$ acts as -1, and $\mathcal{SH}(k)^-_{\pm}$ for those $E$ where $\epsilon$ acts as +1; then every object $E \in \mathcal{SH}(k)^+$ can be written uniquely as $E = E^+ \oplus E^-$, with $E^\pm \in \mathcal{SH}(k)^{\pm}$. Note that $\eta[-1] = -\epsilon - 1 = -2$ on $\mathcal{SH}(k)^2$, so $\eta$ is invertible on $\mathcal{SH}(k)^2$.

**Corollary 20.** Let $k$ be of characteristic zero and finite virtual 2-étale cohomological dimension. If $E \in \mathcal{SH}(k)$ is connective and slice-connective, and $ME = 0$, then $E \in \mathcal{SH}(k)^2$.

**Proof.** Let $F$ fit in the distinguished triangle $E \to E \to F$. We have 4 id$_F = 0$ and so the above Lemma applies: $F = 0$. We conclude that $E \to E \to E$ is an isomorphism, whence $E \in \mathcal{SH}(k)^2$. Then $ME = M(E^+) \oplus M(E^-)$. We have $M(E^-) = 0$ (on $UM$) the map $\eta$ is zero, but it is also an isomorphism since it comes from $\eta$ on $E^-$. Thus $M(E^+) = 0$. But $M : \mathcal{SH}(k)^+ \to \mathcal{DM}(k, Z[1/2])$ is conservative on connective objects by Corollary 4 and Theorem 9 (note that $\eta = 0$ on $K^M_W[1/2] = K^M[1/2]$). Thus $E^+ = 0$. This concludes the proof. \hfill \square

Note also that if $k$ is of arbitrary characteristic but has finite 2-étale cohomological dimension then $k$ is non-orderable, so $W(k)[1/2] = 0$ [24, Theorem III.3.6] and $\mathcal{SH}(k)^2 = 0$. Thus $\mathcal{SH}(k)^+ \to \mathcal{DM}(k, Z[1/2])$ is conservative on connective objects in this case. This finishes an argument in the proof of Theorem 33.

## 5 Witt Motives, Real Motives and the Real Étale Topology

We will now use the category of Witt motives [1, Section 4]. Recall that for any scheme $X$, one may define the Witt ring $W(X)$ [19]. This is the ring of isometry classes of symmetric bilinear forms on vector bundles on $X$, modulo hyperbolic bundles. The associated sheaf $W$ restricted to smooth varieties over a field is unramified [32, Theorem A]. This is the sheaf of unramified Witt groups. One may check that it coincides with the sheaf defined by Morel [29, Example 3.34, Lemma 3.10].

To construct the category of Witt motives, start with the category of complexes of sheaves of $W[1/2]$-modules. This category is abelian, so we may form its derived category $\mathcal{D}$, which is even closed symmetric monoidal. Consider the subcategory $\mathcal{E}$ of those objects $E$ such that the natural maps $E \to \text{Hom}(A^1, E)$ (coming from pull-back) and $E \to \text{Hom}(ZG_m, E)$ (coming from the structure as a $W$-module and the identification $W = \text{Hom}(ZG_m, W)$) are quasi-isomorphisms. This is the category $\mathcal{DM}_W(k, Z[1/2])$ of (2-inverted) Witt motives. We summarise its basic properties.

**Lemma 21.** Let $k$ be a field of characteristic zero. The category $\mathcal{DM}_W(k, Z[1/2])$ is tensor triangulated and compact-rigidly generated. There is an adjunction $F : \mathcal{SH}(k)^2 \rightleftarrows \mathcal{DM}_W(k, Z[1/2]) : U$ with $F$ monoidal.

**Proof.** We can describe $\mathcal{DM}_W(k, Z[1/2])$ as the homotopy category a certain Bousfield localization of the model category of chain complexes of presheaves of $W[1/2]$-modules. Let us write $C(W[1/2], k)$ for this model category of presheaves of $W$-modules and $LC(W[1/2], k)$ for the Bousfield localization such that $\text{Ho}(LC(W[1/2], k)) = \mathcal{DM}_W(k, Z[1/2])$.

When using appropriate model structures on $C(W[1/2], k)$ and $\mathcal{SH}(k)$, there is a Quillen adjunction $\text{SH}(k) \rightleftarrows LC(W[1/2], k)$. Passing to $G_m$-spectra, we obtain a Quillen adjunction $\text{SH}(k) \rightleftarrows \text{Sp}(LC(W[1/2], k), G_m)$. But in $\mathcal{DM}_W(k, Z[1/2])$ the object corresponding to $G_m$ is $\otimes$-invertible by design, so $\text{Ho}(\text{Sp}(LC(W[1/2], k), G_m)) \simeq \mathcal{DM}_W(k, Z[1/2])$ [13, Theorem 5.1]. We thus have an adjunction $\text{SH}(k) \rightleftarrows \mathcal{DM}_W(k, Z[1/2]) : U$. It is clear that $U(\mathcal{DM}_W(k, Z[1/2])) \subset \mathcal{SH}(k)^2$, so we have an adjunction as displayed. It follows (since $k$ has characteristic zero) that $\mathcal{DM}_W(k, Z[1/2])$ is rigidly generated.
To prove that $\text{DM}_W(k, \mathbb{Z}[1/2])$ is compactly generated note first that $\text{Ho}(C(\mathbb{W}[1/2], k))$ is compactly generated (since we deal here with presheaves [6, Example 1.5 and Theorem 5.4]). I claim that $\text{LC}(\mathbb{W}[1/2], k)$ can be obtained by inverting three kinds of maps: $X \times \mathbb{A}^1 \to X$, $X \to X \times \mathbb{G}_m$ (for $X \in \text{Sm}(k)$) and the elementary Nisnevich squares. Here $X \to X \times \mathbb{G}_m$ comes from the point $-1$ of $\mathbb{G}_m$. Once we know that it will follow that $\text{DM}_W(k, \mathbb{Z}[1/2])$ is still compactly generated. This is because we still have a “bounded descent structure” in the sense of [6, Theorem 5.4].

The only part of the claim that needs proof is that inverting $X \to X \times \mathbb{G}_m$ has the same effect as passing to complexes such that $\epsilon_E : E^* \to \text{Hom}(\mathbb{G}_m, E^*)$. In order to understand this we may pass to $\text{Spt}(C(\mathbb{W}[1/2], k))$ first, i.e. make $\mathbb{G}_m \otimes$-invertible (indeed both kinds of localizations make $\mathbb{G}_m \otimes$-invertible). Then one sees that inverting $X \to X \times \mathbb{G}_m$ corresponds to inverting the element $[-1] \in [S^0, \mathbb{G}_m]$ whereas passing to complexes $E$ on which $\epsilon_E$ is an equivalence means inverting $\eta \in [S^0, \mathbb{G}_m]$. However in $\text{SH}(\mathbb{k})_2$ we have the equality $\eta[-1] = -2$, whereas in $\text{SH}(\mathbb{k})_2$ we have $2[-1]^2 = 0$ and so $[-1]$ is nilpotent (see [29, Section 3.1] for these kinds of calculations). Thus inverting either element has the same effect. \end{proof}

We shall make good use of the next result, even though the proof is very easy. Recall that an object $E$ in a symmetric monoidal category $\mathcal{C}$ is called rigid if there exists an object $DE$ such that the functors $\otimes E, \otimes DE$ are both left and right adjoint to one another. Rigid objects are preserved by monoidal functors since they are detected by the zig-zag equations [22, Theorem 2.6]. If $\mathcal{C}$ is a tensor triangulated category then the subcategory $\mathcal{C}^{rig}$ of rigid objects is a thick tensor triangulated subcategory [14, Theorem A.2.5].

Lemma 22. Let $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ be an adjunction, where $\mathcal{C}, \mathcal{D}$ are symmetric monoidal triangulated categories. Assume that $F$ is a symmetric monoidal, triangulated functor and that the composite $GF$ preserves coproducts, that $\otimes_\mathcal{C}, \otimes_\mathcal{D}$ commute with coproducts (in both variables separately) and that $\mathcal{C}$ is rigid-compactly generated.

Then for any $E \in \mathcal{C}$ there is a natural isomorphism $GFE \to E \otimes G1D$. In particular if $1\mathcal{C} \to G1D$ is an isomorphism, then $F$ is fully faithful.

We note that for example if $\mathcal{D}$ is also compact-rigidly generated, then $F$ preserves compact objects and $G$ preserves coproducts, so $GF$ preserves coproducts. Moreover the tensor product commutes with coproducts as seen since the monoidal structure is closed.

Proof. Let $E \in \mathcal{C}$. There is a natural map $GFE \to E \otimes G1$ defined as follows. By adjunction and monoidality of $F$, we only need $F \to F \otimes FG1$. Now $F \cong F \otimes 1$, so it suffices to find $1 \to FG1$. For this we use that $F1 \cong 1$, so we are trying to find $1 \to FG1$, for which it suffices by functoriality to find $1 \to GF$. This is clear by adjunction.

Let $E \in \mathcal{C}$ be rigid. I claim that $GFE \to E \otimes G1$ is an isomorphism. To do this, let $T \in \mathcal{C}$. We compute

$$[T, E \otimes G1] \cong [T \otimes DE, G1] \cong [FT \otimes DFE, 1] \cong [FT, FE] \cong [T, GFE].$$

Here we have used that $F$ is monoidal and so preserves duals. The claim follows by the Yoneda lemma.

Now let $A$ be the class of objects in $E \in \mathcal{C}$ such that $GFE \to E \otimes G1$ is an isomorphism. It is closed under isomorphisms and cones (since $G, F$ are triangulated functors) and also under arbitrary sums (since $GF$ preserves coproducts). Finally it contains all rigid objects, by the claim. Consequently $A$ contains all objects of $\mathcal{C}$, by compact-rigid generation.

For the last part, if $1 \to G1$ is an isomorphism then it follows that $GF \cong 1d$, and $F$ is fully faithful. \end{proof}

This allows us to prove a somewhat deeper property of $\text{DM}_W(k, \mathbb{Z}[1/2])$.

Proposition 23. The category $\text{DM}_W(k, \mathbb{Z}[1/2])$ admits a $t$-structure such that in the adjunction $F : \text{SH}(\mathbb{k})_2 \leftrightarrows \text{DM}_W(k, \mathbb{Z}[1/2]) : U$ the functor $U$ is $t$-exact and induces an equivalence on the hearts.

In particular $F$ is conservative on connective objects.

Proof. We will use again the notation of the proof of Lemma 21.

We wish to use the obvious analogue of the homotopy $t$-structure. We shall use Lemma 6. In order to do this we need to know that if $X \in \text{Spt}(k)$, then the fibrant replacement of $\mathbb{W}[1/2] \otimes X \in C(\mathbb{W}[1/2], k)$ is non-negative in the standard $t$-structure. The proof of Lemma 21 shows that it is enough to know that $FUX \in \text{SH}(\mathbb{k})_{10}$. But by Lemma 22 for this it is enough to know that $U \in \text{SH}(\mathbb{k})_{10}$.

The unit in $C(\mathbb{W}[1/2], k)$ is just the sheaf $\mathbb{W}[1/2]$. This is already strictly homotopy invariant and $\epsilon$-local, so globally equivalent to its own fibrant replacement! In particular it is non-negative.
This finishes the proof that $DM_W(k, Z[1/2])$ admits a homotopy $t$-structure. By design $U$ is $t$-exact. If $F_\ast \in SH(k)_2^\boxtimes$ then $F$ consists of strictly invariant sheaves which are $\epsilon$-local, and consequently $F$ defines an object $e(F) \in DM_W(k, Z[1/2])^\boxtimes$. (All of the sheaves $F_i$ are canonically isomorphic, so this is well defined.) The functors $e$ and $U : DM_W(k, Z[1/2])^\boxtimes \to SH(k)_2^\boxtimes$ are manifestly inverse to each other.

The last sentence just follows from Corollary 4 and Lemma 5.

We now have to use the real étale topology [36, (1.2)]. To any scheme $X$ one may functorially associate a topological space (in fact locally ringed space) $R(X)$, called its “realification”. If $X = Spec(A)$ then one writes $Spec(A) := R(Spec(A))$. The points of $R(X)$ are pairs $(P, \alpha)$ where $P \in X$ and $\alpha$ is an ordering of the residue field $k(P)$. We call a family of étale maps $(\alpha_i : X_i \to X)$ a real étale covering (or rét-covering) if the induced maps $R(\alpha_i) : R(X_i) \to R(X)$ are jointly surjective. For example, if $X = Spec(k)$ where $k$ is a field, then $R(X)$ is the set of orderings of $k$, and a family of separable field extension $l_i/k$ defines a rét-covering $(Spec(l_i) \to Spec(k))$ if and only if every ordering of $k$ extends to one of the $l_i$.

The rét-coverings define a topology on schemes [36, (1.2)]. We write $a_{rét}$ for the associated sheaf functor, in various situations. Write $W$ for the presheaf of Witt groups.

**Lemma 24** ([17], Lemma 6.4 (2) and its proof). The sheaf $a_{Nis}W[1/2]$ is a sheaf in the rét-topology on $Sch/\mathbb{Z}[1/2]$: for a scheme $X$ such that $2$ is invertible on $X$, the natural map

$$(a_{Nis}W[1/2])(X) \to (a_{rét}W[1/2])(X)$$

is an isomorphism. Moreover so is

$$(a_{rét}Z[1/2])(X) \to (a_{rét}W[1/2])(X).$$

**Corollary 25.** For any field $k$ of characteristic not two, the sheaf $W[1/2]$ on $Sm(k)$ is a sheaf in the rét-topology. Moreover the natural map $Z[1/2] \to W[1/2]$ induces an isomorphism $a_{rét}Z[1/2] = W[1/2]$.

**Proof.** The (Nisnevich) sheaf of unramified Witt groups $W$ is just the sheaf associated (in the Zariski topology) with the presheaf $W$ of Witt groups [32, Theorem A].

We can consider a rét-version of the category $DM_W(k, Z[1/2])$. This is constructed in precisely the same way, just that we start with rét-sheaves instead of Nisnevich sheaves. There is an evident adjunction $L : DM_W(k, Z[1/2]) \dashv DM_W(k, Z[1/2])^{rét} : R$ induced by $a_{rét}$ and the forgetful functor. Note that since $W[1/2] = a_{rét}Z[1/2]$, the category $DM_W(k, Z[1/2])^{rét}$ is really just built from the derived category of chain complexes on $Shv(Sm(k)^{rét})$, the Witt notation is mostly a coincidence.

**Lemma 26.** The canonical functor $L : DM_W(k, Z[1/2]) \to DM_W(k, Z[1/2])^{rét}$ is an equivalence of categories.

**Proof.** I claim that $DM_W(k, Z[1/2])^{rét}$ is compactly generated. The category $D(Shv(Sm(k)^{rét}))$ is compactly generated by [7, Proposition 1.1.9], since the rét-cohomological dimension of a scheme $X$ is bounded by the Zariski dimension of $X$ [36, Theorem 7.6]. The category $DM_W(k, Z[1/2])^{rét}$ is a reflective localization of $D(Shv(Sm(k)^{rét}))$ which has a left adjoint $L'$. The essential image of $\iota$ consists of the local objects, i.e. those $E \in D(Shv(Sm(k)^{rét}))$ such that $\alpha^* : [T_2, E[i]] \to [T_1, E[i]]$ is an isomorphism for all $i \in \mathbb{Z}$ and appropriate maps $\alpha : T_1 \to T_2 \in D(Shv(Sm(k)^{rét}))$. Namely, as in the proof of Lemma 21, the maps $\alpha$ we consider here are induced by $X \times \mathbb{A}^1 \to X$ and $X \to X \times \mathbb{G}_m$, for $X \in Sm(k)$. In particular, these are maps between compact objects. Since $\iota$ is fully faithful, it is easy to see that the image under $L'$ of a generating set for $D(Shv(Sm(k)^{rét}))$ is a generating set for $DM_W(k, Z[1/2])^{rét}$. Since local objects are characterised in terms of maps out of certain compact objects, they are closed under arbitrary sums. This implies that $\iota$ preserves arbitrary sums, and hence $L'$ preserves compact objects. Thus the claim is proved.

It follows that $DM_W(k, Z[1/2])^{rét}$ is compact-rigidly generated (since the monoidal functor $L : DM_W(k, Z[1/2]) \to DM_W(k, Z[1/2])^{rét}$ preserves rigid objects) and so we may apply Lemma 22. It is thus enough to show that $RL \simeq \mathbb{I}$. But as we have said before the sheaf $W[1/2] \in LC(W[1/2], k)$ is already fibrant, i.e. $H_{Nis}^n(X \times \mathbb{A}^1, W[1/2]) = H_{Nis}^n(X, W[1/2])$ and $H_{Nis}^n(X \times \mathbb{G}_m, W[1/2]) = H_{Nis}^n(X, W[1/2]-1) = H_{Nis}^n(X, W[1/2])$. Since Nisnevich and rét-cohomology of rét-sheaves agree [36, Proposition 19.2.1] and $W[1/2]$ already is a rét-sheaf by Corollary 25 we conclude that $W[1/2]$ is also fibrant as an object of $L^{rét}C(W[1/2], k)$, i.e. in the model category underlying $DM_W(k, Z[1/2])^{rét}$. Consequently $RL \simeq \mathbb{I}$ and we are done.

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As we have noted above, the category $\text{DM}_W(k, \mathbb{Z}[1/2])^{\text{rét}}$ is obtained as a localization of $D(\text{Shv}(\text{Sm}(k)^{\text{rét}}), \mathbb{Z}[1/2])$. We might thus attempt to use Lemma 6 to build a $t$-structure on $\text{DM}_W(k, \mathbb{Z}[1/2])^{\text{rét}}$ such that $\text{DM}_W(k, \mathbb{Z}[1/2])^{\text{rét}} \hookrightarrow D(\text{Shv}(\text{Sm}(k)^{\text{rét}}), \mathbb{Z}[1/2])$ is $t$-exact. The heart of this $t$-structure will, by design, be a full subcategory of $\text{Shv}(\text{Sm}(k)^{\text{rét}})$. If this is possible, we shall say that $\text{DM}_W(k, \mathbb{Z}[1/2])^{\text{rét}}$ admits a homotopy $t$-structure. Note that we have already constructed a $t$-structure on $\text{DM}_W(k, \mathbb{Z}[1/2])$ which we now know is equivalent to $\text{DM}_W(k, \mathbb{Z}[1/2])^{\text{rét}}$. However, it is not clear a priori that these two $t$-structures are the same.

**Lemma 27.** The category $\text{DM}_W(k, \mathbb{Z}[1/2])^{\text{rét}}$ admits a homotopy $t$-structure, and the equivalence $\text{DM}_W(k, \mathbb{Z}[1/2])^{\text{rét}} \simeq \text{DM}_W(k, \mathbb{Z}[1/2])$ is $t$-exact.

**Proof.** We have “forgetful” functors

$$D(\text{Shv}(\text{Sm}(k)^{\text{rét}}), \mathbb{Z}[1/2]) \xrightarrow{U_1} D(W[k]-\text{Mod}) \xrightarrow{U_2} D(\text{Shv}(\text{Sm}(k)^{\text{Nis}})).$$

All three categories have canonical $t$-structures, and the left adjoints are precisely objects in the heart of $\text{Shv}(\text{Sm}(k)^{\text{rét}})$, which we now identify as in the corollary are precisely objects in the heart of $\text{SH}(k)^{\text{rét}}$, which is the same as the $\mathbb{A}^1$-invariant and $G_m$-invariant rét-sheaves by the two results.

This result has the following interesting consequence. Recall the functor $\Omega$ from the beginning of Section 4.

**Corollary 28.** Let $k$ be a field of characteristic zero.

Let $F$ be a Nisnevich sheaf of $W[k]$-modules on $\text{Sm}(k)$ which is strictly homotopy invariant and such that the natural map $F \to \Omega F$ is an isomorphism. Then for any $p \geq 0$ and $X \in \text{Sm}(k)$ we have $H^p_{\text{Nis}}(X, F) = H^p_{\text{rét}}(X, F)$.

**Proof.** This follows from Proposition 23 and Lemma 27. Indeed the sheaves $F$ as in the corollary are precisely objects in the heart of $\text{SH}(k)^{\text{rét}}$, which is the same as the $\mathbb{A}^1$-invariant and $G_m$-invariant rét-sheaves by the two results.

From now on, we will identify $\text{DM}_W(k, \mathbb{Z}[1/2])$ and $\text{DM}_W(k, \mathbb{Z}[1/2])^{\text{rét}}$.

**Corollary 29.** Let $k$ have characteristic zero and $l_j/k$ be field extensions, $j \in J$ some indexing set. Assume that every ordering of $k$ extends to an ordering of one of the $l_j$, i.e. that $\{\text{Spec}(l_j) \to \text{Spec}(k)\}_j$ is a rét-covering. Then the functor

$$\text{DM}_W(k, \mathbb{Z}[1/2]) \to \prod_{j \in J} \text{DM}_W(l_j, \mathbb{Z}[1/2])$$

is conservative.

**Proof.** The base change functor is $t$-exact and so commutes with taking homotopy sheaves. By the previous corollary all of the homotopy sheaves are sheaves in the rét-topology, so it suffices to show that

$$\text{Shv}_{\text{rét}}(\text{Sm}(k)) \to \prod_j \text{Shv}_{\text{rét}}(\text{Sm}(l_j))$$

is conservative. This is essentially clear, since $\{\text{Spec}(l_j) \to \text{Spec}(k)\}_j$ is a rét-covering. □

The above corollary shows that for our purposes, in studying $\text{DM}_W(k, \mathbb{Z}[1/2])$ we may assume that $k$ is real closed. It turns out that in this case the field is essentially irrelevant. Before we can prove this, we need to recall a result from semi-algebraic topology. This combines the work of several people.

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Theorem 30 (Coste-Roy, Delfs, Scheiderer). Let $K/k$ be an extension of real closed fields and $X/k$ a separated scheme of finite type. Then for any presheaf of abelian groups $F$ (on $X$) and any $p \geq 0$, the natural map

$$H^p_{\acute{e}t}(X, F) \to H^p_{\acute{e}t}(X_K, F_K)$$

is an isomorphism, where $F_K$ denotes the pulled back sheaf.

Proof. Associated with $X$ we have the real étale site $X_{\acute{e}t}$, the topological space $R(X)$, and a further site which we shall denote $G(X)$ and which is called the geometric space associated with $X$ in [9] (this reference requires the finite type and separatedness assumptions). The toposes associated with $X_{\acute{e}t}$ and $R(X)$ are equivalent [36, Theorem 1.3], and the toposes associated with $G(X)$ and $R(X)$ are also equivalent [9, Proposition 1.4]. Finally, the comparison result is proved for cohomology in $G(X)$ in [9, Theorem 6.1] (take $\Phi = X$).

Proposition 31. Let $f : \text{Spec}(l) \to \text{Spec}(k)$ be an extension of real closed fields. The functor $f^* : \text{DM}^W(k, \mathbb{Z}[1/2]) \to \text{DM}^W(l, \mathbb{Z}[1/2])$ is an equivalence of categories. Moreover, both categories are generated by the monoidal unit.

Proof. We first show that the functor is fully faithful. We wish to apply Lemma 22. Certainly the functor $f^*$ is monoidal with right adjoint $f_*$, and both $\text{DM}^W(k, \mathbb{Z}[1/2])$ and $\text{DM}^W(l, \mathbb{Z}[1/2])$ are compact-rigidly generated (by Lemma 21). It remains to show that $f_*$ is an equivalence of categories. Moreover, both categories are compact-rigidly generated by the monoidal unit.

Next we need to show that the functor is essentially surjective. To do this it suffices to show that $\text{DM}^W(k, \mathbb{Z}[1/2])$ is generated by the monoidal unit $\mathbb{1}$. For this it is enough to show that if $F \in \text{DM}^W(k, \mathbb{Z}[1/2])$ is a sheaf such that $F(k) = 0$ then $F = 0$. Now $F$ is unramified, so it is enough to show that $F(K) = 0$ for every finitely generated field extension $K/k$. But Corollary 28 applies and thus $F$ is a rét-sheaf. Hence it suffices to show that $F(K^r) = 0$ for every real closure $f : \text{Spec}(K^r) \to \text{Spec}(K)$. But this just says that $[f^* \mathbb{1}, f^* F] = 0$, which we know is true because $[f^* \mathbb{1}, f^* F] = [\mathbb{1}, F] = F(k)$ by the fully faithfulness we already proved.

Proposition 32. The category $\text{DM}^W(k, \mathbb{Z}[1/2])$ is canonically equivalent to $D(\mathbb{Z}[1/2])$, where $k$ is any real closed field.

Proof. There exists an adjunction of sites $e : \mathbb{Z}[1/2]-\text{Mod} \rightleftarrows \text{Shv}(\text{Sm}(k)_{r\acute{e}t}, \mathbb{Z}[1/2]) : r$, where $(rF) = F(\text{Spec}(k))$ and $e$ is the constant sheaf functor. Composing with the localisation adjunction, we get $\tilde{e} : D(\mathbb{Z}[1/2]) \rightleftarrows \text{DM}^W(k, \mathbb{Z}[1/2])_{r\acute{e}t} \rightleftarrows \text{DM}^W(k, \mathbb{Z}[1/2]) : \tilde{r}$. Now $\mathbb{Z}[1/2] = \alpha_{r\acute{e}t} \mathbb{Z}[1/2]$ is already A1-Gm-local, so $\tilde{r} e = e$. By a further application of Lemma 22 we conclude that $\tilde{e}$ is fully faithful. Proposition 31 implies that $\tilde{e}$ has dense image and so is essentially surjective. This concludes the proof.

Now let $k$ be an arbitrary field (of characteristic zero) and $\sigma$ an ordering of $k$. Write $k_{\sigma}$ for a real closure of $(k, \sigma)$. We denote by $M_{\sigma}[1/2] : \text{SH}(k) \to D(\mathbb{Z}[1/2])$ the composite $\text{SH}(k) \to \text{SH}(k_{\sigma}) \to \text{DM}^W(k_{\sigma}, \mathbb{Z}[1/2]) \simeq D(\mathbb{Z}[1/2])$ and call it the real motive associated with $\sigma$. We have thus proved the following:

Theorem 33 (Conservativity II). Let $k$ be a field of characteristic zero. Assume that $k$ has finite virtual 2-étale cohomological dimension. Write $\text{Sper}(k)$ for the set of orderings of $k$.

If $E \in \text{SH}(k)$ is connective and slice-connective, and we have that $0 \simeq ME \in \text{DM}(k)$ and that for each $\sigma \in \text{Sper}(k)$, $0 \simeq M_{\sigma}[1/2](E) \in D(\mathbb{Z}[1/2])$, then $E \simeq 0$.

Proof. Combine Corollary 20, Proposition 23, Corollary 29 and Proposition 32.

Remark 1. If $\sigma : k \subset \mathbb{R}$ is an embedding of ordered fields then one might define $M_{\sigma}[1/2](E) \in D(\mathbb{Z}[1/2])$ as $\text{C}_*(\text{R}_{k}(E), \mathbb{Z}[1/2])$, where $\text{R}_{k}$ denotes “real realisation” [30, Section 3.3.3] and $\text{C}_*$ denotes the singular complex. It is not hard to find a canonical equivalence $M_{\sigma}[1/2](E) \simeq M_{\sigma}[1/2](E)$, but we do not need this here.
Remark 2. The author contends that the results in this section can be proved more directly by using transfer considerations. For example to prove Corollary 28, it seems like a good first step to prove: if $k$ is a field of characteristic zero and $H_*$ is a homotopy module which is also a module over $\mathbb{W}[1/2]$ (i.e. 2 and $\eta$ act invertibly on $H_*$), then for any finitely generated field extension $K/k$ and any étale cover $\{\text{Spec}(L_i) \to \text{Spec}(K)\}_{i \in I}$, the map

$$H_*(K) \to \prod_{i \in I} H_*(L_i)$$

is injective. Since $H_*$ has transfers satisfying the projection formula, for this it is enough to prove that the ideal inside $W(K)[1/2]$ generated by $tr_{L_i/K}(W(L_i))[1/2])$ is all of $W(K)[1/2]$. This can be checked directly, using that $W(K)[1/2] \approx \mathbb{Z}[1/2]^{\text{Spec}(K)}$ (and similarly for the $L_i$).

Added during revision: this hope has now been fulfilled [2].

A Compact Objects in Abelian and Triangulated Categories

Here we collect some well-known but hard to reference facts about compact objects. Recall that if $\mathcal{C}$ is a category with (countable) colimits then an object $X \in \mathcal{C}$ is called \textit{(countably) compact} if for every (countable) family $\{T_i\}_{i \in A}$ in $\mathcal{C}$ the natural map $\text{colim}_i \text{Hom}(X, T_i) \to \text{Hom}(X, \text{colim}_i T_i)$ is an isomorphism.

Since triangulated categories do not usually have filtered colimits, this definition would not make much sense. Instead an object $X$ in a triangulated category $\mathcal{C}$ with (countable) coproducts is called \textit{(countably) compact} if for every (countable) filtering diagram $\{T_{\lambda}\}_{\lambda \in \Lambda}$ in $\mathcal{C}$ the natural map

$$\bigoplus_{\lambda} \text{Hom}(X, T_{\lambda}) \to \text{Hom}(X, \bigoplus_{\lambda} T_{\lambda})$$

is an isomorphism [31, Definition 0.1].

Finally recall sequential countable homotopy colimits in triangulated categories. Suppose $\{T_n, f_n : T_n \to T_{n+1}\}_{n \in \mathbb{N}}$ is a sequence in the triangulated category $\mathcal{C}$. There is a natural map

$$\text{id} - s : \bigoplus_n T_n \to \bigoplus_n T_n, \quad (t_1, t_2, \ldots) \mapsto (t_1, t_2 - f_1 t_1, t_3 - f_2 t_2, \ldots).$$

A cone on this map is denoted $\text{hocolim}_n T_n$ [31, Definition 1.4].

Suppose $X$ is a countably compact object in the triangulated category $\mathcal{C}$ and $\{T_n\}$ is a sequence. After choosing a presentation as a cone, there is a natural map $\text{colim}_n \text{Hom}(X, T_n) \to \text{Hom}(X, \text{hocolim}_n T_n)$. This map is an isomorphism [31, Lemma 1.5].

Lemma 34. Let $\mathcal{C}$ be a $t$-category with countable coproducts and such that $\mathcal{C}^\ominus$ satisfies Ab5 and the inclusion $\mathcal{C}^\ominus \hookrightarrow \mathcal{C}$ preserves countable coproducts. If $\{T_n\} \in \mathcal{C}^\ominus$ is a sequence of objects then there is an isomorphism

$$\text{hocolim}_n T_n \simeq \text{colim}_n T_n.$$

Here $\text{hocolim}_n T_n$ is computed in $\mathcal{C}$ by the above formula, and $\text{colim}_n T_n$ is computed in the abelian category $\mathcal{C}^\ominus$.

We recall that Ab5 means that filtered colimits are exact. This is true in many abelian categories, including $\text{HL}_*(k) = \text{SH}_*(k)^\ominus$.

Proof. The point is that because of Ab5 the sequence

$$0 \to \bigoplus_n T_n \xrightarrow{id - s} \bigoplus_n T_n \to \text{colim}_n T_n \to 0$$

is exact on the left (exactness at all other places follows from the definition of colimit). In any $t$-category short exact sequences in the heart yield distinguished triangles in the triangulated category, so we are done by the definition of hocolim.

Corollary 35. In the above situation, if $E \in \mathcal{C}_{\geq 0}$ is countably compact then so is $\pi_0^C E \in \mathcal{C}^\ominus$.

Proof. Let $\{T_n\}$ be a sequence in $\mathcal{C}^\ominus$. We compute

$$\text{Hom}(\pi_0^C E, \text{colim}_n T_n) \xrightarrow{(1)} \text{Hom}(E, \text{colim}_n T_n) \xrightarrow{(2)} \text{Hom}(E, \text{hocolim}_n T_n) \xrightarrow{(3)} \text{colim}_n \text{Hom}(E, T_n) \xrightarrow{(4)} \text{colim}_n \text{Hom}(\pi_0^C E, T_n).$$
This is the required result. (Here (1) is because \( \text{colim}_n T_n \in C_{\leq 0} \), (2) is by the lemma, (3) is by compactness of \( E \) and (4) is because \( T_n \in C_{\leq 0} \).) □

B Construction of \( F \cdot H \); General Case

In this section \( k \) is a perfect field; we shall prove Theorem 15 by constructing the relevant sheaves by hand and establishing the necessary properties. We are trying to make this appendix reasonably self-contained, in particular the first and introductory subsection. There is no way around the fact however that it relies very heavily on the first half of [29], so when proving technical results we will freely reference that book.

B.1 Overview of the Proof

Suppose for a moment that we can show (1) to (3). In Appendix A we proved that if \( E \in \text{SH}(k)^{\text{opt}}_0 \) then also \( \prod_0(E)_* \in \text{HI}_*(k) \) is (countably) compact. (We also clarified the meaning of compactness there.) Statement (4) then follows from the observation that

\[
H_* = \bigcup_n F_n H_*. 
\]

(Indeed \( (F_n H)_n = H_n \) by definition.)

Thus we need only establish the following.

**Theorem 36.** Let \( k \) be a perfect field, \( M_* \in \text{HI}_*(k) \), and \( n > 0, m \in \mathbb{Z} \). Then there exists a strictly invariant sheaf \( (\mathcal{K}_n^{\text{MW}} M_m)^{tr} \) with sections over fields given by formula (1), and it satisfies \( \Omega(\mathcal{K}_n^{\text{MW}} M_m)^{tr} = (\mathcal{K}_{n-1}^{\text{MW}} M_m)^{tr} \). (We put \( (\mathcal{K}_n^{\text{MW}} M_m)^{tr} = M_m \).)

In this section we shall outline the strategy, postponing the proofs of certain technical lemmas to separate subsections at the end. As a very highlevel overview, we will construct a presheaf \( (\mathcal{K}_n^{\text{MW}} M_m)^{tr} \) which has the correct sections on fields, show that it is strictly invariant, and finally deduce that \( \Omega((\mathcal{K}_n^{\text{MW}} M_m)^{tr}) = (\mathcal{K}_{n-1}^{\text{MW}} M_m)^{tr} \).

Recall that if \( F \) is a presheaf of sets on \( Sm(k) \) and \( X \) is an essentially smooth scheme, i.e. the inverse limit in the category of schemes of a filtering diagram of smooth schemes with affine transition morphisms \( X \cong \lim_i X_i \), then the colimit \( F(X) := \text{colim}_i F(X_i) \) is well-defined independent of the presentation \( X \cong \lim_i X_i \) [12, Proposition 8.13.5]. One thus obtains an extension of \( F \) to the category of essentially smooth schemes which is continuous in the sense that it turns filtering inverse limits with affine transition morphisms into colimits.

Recall also that a presheaf \( F \) on \( Sm(k) \) is called unramified if (a) for every \( X \in Sm(k) \) with connected components \( X_\alpha \) (i.e. \( \alpha \in X^{(0)} \)) we have \( F(X) = \prod_\alpha F(X_\alpha) \), (b) for every \( U \subset X \) everywhere dense open \( (X \in Sm(k)) \) the pullback \( F(X) \to F(U) \) is injective, and finally (c) for every \( X \in Sm(k) \) connected, the map \( F(X) \to \cap_{x \in X^{(1)}} F(O_{X,x}) \) is bijective [29, Definition 2.1]. Here the intersection is taken inside \( F(k(X)) \) (into which each of the groups embeds by (b)). Note that we are routinely being somewhat cavalier about the arguments of \( F \); if \( A \) is a ring then we write \( F(A) := F(\text{Spec}(A)) \). Also if \( X \) is a scheme and \( x \in X \) is a point, we write \( F(x) := F(k(x)) = F(\text{Spec}(k(x))) \), where \( k(x) \) denotes the residue field of \( x \). This should always be unambiguous.

Write \( \mathcal{F}_k \) for the category of field extensions of \( k \) of finite transcendence degree. Note that \( \mathcal{F}_k^{\text{op}} \) is equivalent to a full subcategory of the category of essentially smooth \( k \)-schemes. Hence any presheaf of sets on \( Sm(k) \) defines a continuous (covariant) functor \( F|_{\mathcal{F}_k} : \mathcal{F}_k \to \text{Set} \).

Suppose that \( K \in \mathcal{F}_k \) and \( O_\nu \subset K \) is a discrete valuation ring (dvr) with residue field \( k(\nu) \). (We will always assume that \( \nu(x) \) is trivial.) Then we get a pullback \( F(O_\nu) \to F(K) \), as well as the specialisation \( s : F(O_\nu) \to F(k(\nu)) \). If \( F \) is invariant, then it is believable that this data (for varying \( K, O_\nu \)) almost determines \( F \). One might then try to define an unramified subsheaf \( I \subset F \) by specifying \( I(K) \subset F(K) \), putting \( I(O_\nu) = F(O_\nu) \cap I(K) \), and trying to restrict the specialisation maps. (If \( F \) is part of a homotopy module and \( I \) is to be a homotopy sub-module, then one may show that this definition of \( I(O_\nu) \) is the only one possible.) The result below says that up to a compatibility condition, this strategy does indeed work.

Recall the notion of an excellent ring [21, Definition 2.35]. Most rings which occur in practice are excellent. Instead of considering all dvr’s, it turns out that it suffices to consider excellent dvr’s.
Proposition 37 (proof in Subsection B.3). Let $M$ be a strictly invariant sheaf and $I : \mathcal{F}_\varepsilon \to \text{Ab}$ a continuous subfunctor of $\mathcal{M}_f$. Suppose that for each $K \in \mathcal{F}_\varepsilon$ and for each excellent dvr $\mathcal{O}_v \subset K$ we have that $s(I(K) \cap M(\mathcal{O}_v)) \subset I(\kappa(v))$.

Then there exists a unique sub-presheaf $I$ of $M$ extending $I : \mathcal{F}_\varepsilon \to \text{Ab}$ and with $I(\mathcal{O}_v) = I(K) \cap M(\mathcal{O}_v)$ for each $K \in \mathcal{F}_\varepsilon$ and dvr $\mathcal{O}_v \subset K$. It is a Nisnevich sheaf and we have $I(X) = M(X) \cap I(X^{(0)})$, for all $X \in \text{Sm}(k)$.

The difficulty with applying this result is in showing that $s(I(K) \cap M(\mathcal{O}_v)) \subset I(\kappa(v))$, since $M(\mathcal{O}_v)$ is not usually very accessible. If $M$ is part of a homotopy module $M_\ast$, then the situation is somewhat better. Namely if $K \in \mathcal{F}_\varepsilon$, $\mathcal{O}_v \subset K$ is a dvr and $\pi$ is a uniformizer of $\mathcal{O}_v$, then there exists a boundary morphism $\partial^\pi : M_\ast(K) \to M_{\ast-1}(\kappa(v))$. We may use the boundary to extend the specialisation maps to all of $M_\ast(K)$. Recall that $M_\ast(K)$ is a module over $K^{\text{MW}}(K)$ and that every element $a \in K^\ast$ defines elements $(a) \in K^{\text{MW}}_0(K), [a] \in K^{\text{MW}}_0(K)$.

Proposition 38 (proof in Subsection B.4). Let $\mathcal{O}_v \subset K$ be a dvr with uniformizer $\pi$, $M_\ast$ a homotopy module. Write $s : M_\ast(\mathcal{O}_v) \to M_\ast(\kappa(v))$ for the specialisation and $\partial^\pi : M_\ast(K) \to M_{\ast-1}(\kappa(v))$ for the boundary. Put

$$s^\pi : M_\ast(K) \to M_\ast(\kappa(v)), s^\pi(m) = (-1)^{\partial^\pi([\pi][m]).$$

Then for $m \in M_\ast(\mathcal{O}_v) \subset M_\ast(K)$ we have $s^\pi(m) = s(m)$.

It is thus important to analyse the interaction of the boundary maps with transfers.

Lemma 39 (proof in Subsection B.5). Let $\mathcal{O}_v \subset K$ be an excellent dvr with uniformizer $\pi$. Then $\partial^\pi([K^{\text{MW}}_n M_\ast]^{\text{tr}}(K) \subset [K^{\text{MW}}_{n-1} M_\ast]^{\text{tr}}(\kappa(v))) \subset [K^{\text{MW}}_{n+\ast} M_\ast]^{\text{tr}}(\kappa(v)))$, for each $n > 0$.

Corollary 40. The (so far hypothetical) sheaf $(K_\ast M_\ast)^{\text{tr}} \subset M_{\ast+n}$ exists and is unramified.

Proof. We want to apply Proposition 37. For this we need to show that if $\mathcal{O}_v \subset K$ is an excellent dvr then $s([K^{\text{MW}}_n M_\ast]^{\text{tr}}(K) \cap M_{\ast+n} (\mathcal{O}_v)) \subset [K^{\text{MW}}_n M_\ast]^{\text{tr}}(\kappa(v)))$. Using Proposition 38 we know that $s(x) = s^\pi(x) = (-1)^{\partial^\pi([\pi][x])}$ and so $s([K^{\text{MW}}_n M_\ast]^{\text{tr}}(K) \cap M_{\ast+n} (\mathcal{O}_v)) \subset \partial^\pi([K^{\text{MW}}_n M_\ast]^{\text{tr}}(K) \subset [K^{\text{MW}}_{n+\ast} M_\ast]^{\text{tr}}(\kappa(v)))$ by the lemma. (Here we are using the projection formula in the form $[\pi][tr(x)] = tr([\pi][x])$, see Subsection B.5.)

Next we need to prove that the unramified Nisnevich sheaf $(K_\ast M_\ast)^{\text{tr}} \subset M_{\ast+n}$ is strictly homotopy invariant, i.e. that $H^p(X \times \mathbb{A}_k, (K_\ast M_\ast)^{\text{tr}}) \to H^p(X, (K_\ast M_\ast)^{\text{tr}})$ is an isomorphism for all $n$ and all $X \in \text{Sm}(k)$. A deep theorem of Morel says that it is only necessary to prove this for $p = 0, 1$. This together with some diagram chasing yields the following general criterion.

Lemma 41 (proof in Subsection B.3). Let $M$ be a strictly invariant sheaf and $I$ a subsheaf such that $I(X) = M(X) \cap I(X^{(0)})$. Then $I$ is strictly invariant if and only if for all $L \in \mathcal{F}_\varepsilon$ we have $H^1(L, I) = 0$.

In order to use this, we need to get down to the nitty gritty details of constructing an actual resolution of $(K_\ast M_\ast)^{\text{tr}}$ on $\mathbb{A}_k^1$. In order to motivate this, recall that any homotopy module $M_\ast$ has a Rost-Schmid resolution [29, Chapter 5]. This is essentially built out of the boundary maps $\partial^\pi$. Unfortunately these depend on the choice of uniformizer, which is very inconvenient. It is possible to remove this problem by twisting. Let us discuss this technique first.

Recall that if $M$ is a sheaf with a $\mathbb{Z}[G_m]$-module structure and $\mathcal{L}$ is a line bundle on $X$, then one writes $M(\mathcal{L})$ for the sheaf tensor product $M|X \otimes_{\mathcal{O}_X} \mathcal{L}^\times$, where $\mathcal{L}^\times(U)$ is the free abelian group on the trivialisations of $\mathcal{L}|U$. The sections are denoted $M(U, \mathcal{L}) := M(\mathcal{L})(U)$. If $\mathcal{L}$ is trivial on the affine $U$, then $M(U, \mathcal{L}) = M(U) \otimes_{\mathcal{O}_X(U)} \mathcal{L}(U)^\times$, where $\mathcal{L}(U)^\times$ is the set of generators of the free rank one $\mathcal{O}(U)$-module $\mathcal{L}(U)$. Thus $M(U, \mathcal{L})$ is isomorphic to $M(U)$ for such $U$, but not canonically so.

If $M_\ast$ is a homotopy module then each $M_n$ is a $\mathbb{Z}[G_m]$-module, via the natural map $\mathbb{Z}[G_m] \to K^{\text{MW}}_0 = \langle a \rangle$. The only $\mathbb{Z}[G_m]$-module structure we shall use for twisting. It follows that $M_n(\mathcal{L}^\times \otimes \mathcal{L}^\times) \simeq M_n(\mathcal{L})$, canonically.

If $x \in X$, we put $\Lambda_x^\times = \Lambda^{\text{max}}(m_x/m_x^2)$, where $m_x \subset \mathcal{O}_{X,x}$ is the defining ideal and $\Lambda^{\text{max}}$ denotes the maximal non-zero exterior power. This is a one-dimensional $k(x)$-vector space.

Suppose that $X \in \text{Sm}(k)$ is connected and $x \in X^{(1)}$. Choose a uniformizer $\pi$ of the dvr $\mathcal{O}_x$ (which we view as an element of $m_x$) and define a homomorphism

$$\partial_x : M_n(\kappa(x)) \to M_{n-1}(\kappa(x), \Lambda_x^\times), m \mapsto m \otimes \pi.$$
It turns out that this is independent of the choice of \( \pi \) [29, Lemma 5.10]. The Rost-Schmid complex then starts as

\[
M_*(X) \rightarrow M_*(k(X)) \oplus_{x \in X^{(1)}} \bigoplus_{m = 1} M_{-1}(\kappa(x), \Lambda^X_x) \rightarrow \ldots
\]

The next term involves \( M_{-2}(x) \) with \( x \in X^{(2)} \) and so if \( X \) has dimension one, then the complex has to stop. It is thus plausible to try to resolve \((K^{MW}_{n-1} M_m)^{\text{tr}}(X)\) as follows:

\[
0 \rightarrow (K^{MW}_{n} M_m)^{\text{tr}}(X) \rightarrow (K^{MW}_{n} M_m)^{\text{tr}}(X^{(0)}) \oplus_{x \in X^{(1)}} (K^{MW}_{n-1} M_m)^{\text{tr}}(k(x), \Lambda^X_x) \rightarrow 0.
\] (3)

Certainly the first map is injective because \((K^{MW}_{n} M_m)^{\text{tr}}(X)\) is unramified. The boundary map \( \partial \) is the same as for \( M_{m+n} \). This makes sense (i.e. the map lands in the group on the right) by Lemma 39. Since \( M_k \) is a homotopy module its Rost-Schmid complex is a resolution. Consequently in the complex (3) the kernel of \( \partial \) is \( M_{m+n}(X) \cap (K^{MW}_{n} M_m)^{\text{tr}}(X^{(0)}) \). But by construction, i.e. the last part of Proposition 37, this is the same as \((K^{MW}_{n} M_m)^{\text{tr}}(X)\).

Note that the various complexes (3) assemble to a complex of presheaves on \( X_{\text{Nis}} \). Moreover, the two resolving presheaves on the right are acyclic Nisnevich sheaves [29, Lemma 5.42]. Thus the following lemma will show that the Rost-Schmid complex is an acyclic resolution:

**Lemma 42** (proof in Subsection B.5). Let \( X \) be the Henselization of a smooth variety over \( k \) in a point of codimension one. Then the differential

\[
(K^{MW}_{n} M_m)^{\text{tr}}(X^{(0)}) \oplus_{x \in X^{(1)}} (K^{MW}_{n-1} M_m)^{\text{tr}}(k(x), \Lambda^X_x)
\]

is surjective.

The next step is then to use this to show that \( H^1(k^{1}_{K}, (K^{MW}_{n} M_m)^{\text{tr}}) = 0 \). This is the content of the following lemma:

**Lemma 43** (proof in Subsection B.5). Let \( K \in F_k \). The differential

\[
(K^{MW}_{n} M_m)^{\text{tr}}(K(T)) \rightarrow \bigoplus_{x \in (A^1_k)^{(1)}} (K^{MW}_{n-1} M_m)^{\text{tr}}(k(x), \Lambda^1_x)
\]

is surjective.

We have thus established that \((K^{MW}_{n} M_m)^{\text{tr}}(X) \subseteq M_{m+n}\) is a strictly homotopy invariant subsheaf. It remains to show that we have constructed a homotopy module, i.e. that \( \Omega((K^{MW}_{n} M_m)^{\text{tr}}) = (K^{MW}_{n-1} M_m)^{\text{tr}}. \)

Note that both sides are canonically subsheaves of \( M_{m+n-1} \) so this makes sense. This is essentially formal:

**Corollary 44.** We have \( \Omega((K^{MW}_{n} M_m)^{\text{tr}}) = (K^{MW}_{n-1} M_m)^{\text{tr}}. \)

**Proof.** If \( F \) is any strictly homotopy invariant Nisnevich sheaf, then in [29, Chapter 5] there is constructed for any \( X \) essentially k-smooth Nisnevich sheaf \( \Omega F \) such that \( \Omega((K^{MW}_{n} M_m)^{\text{tr}}(K)) \rightarrow \bigoplus_{x \in X^{(1)}} (\Omega F)(x) \rightarrow \ldots \)

Applying this to \( X = A^1_k \) and using strict homotopy invariance shows that \( \Omega F(K) = \partial^T_0 (F(K(T))) \), where \( \partial^T_0 \) is the boundary map corresponding to the point \( 0 \in A^1_k \).

This construction is functorial in \( F \). Given a subsheaf \( F \rightarrow G \), we obtain an injection \( \Omega F \rightarrow \Omega G \), and we find that \( \Omega F(K) = \partial^T_0 (F(K(T))) \subset \Omega(G)(K) \), where \( \partial^T_0 \) can mean either the boundary in the Rost-Schmid complex of \( F \) or \( G \), because they are compatible.

Thus if \( K \in F_k \) then \( \Omega((K^{MW}_{n} M_m)^{\text{tr}}(K)) = \partial^T_0 ((K^{MW}_{n} M_m)^{\text{tr}}(K(T))) = (K^{MW}_{n-1} M_m)^{\text{tr}}(K) \). Indeed the first equality holds for every strictly invariant sheaf, as we just explained, and the second holds by Lemma 37.

Thus \( (K^{MW}_{n} M_m)^{\text{tr}}(K) \) and \( (K^{MW}_{n-1} M_m)^{\text{tr}}(K) \) are two strictly invariant subsheaves of \( M_{m+n-1} \) with the same sections on fields and hence equal. (If \( F \) is strictly invariant so is \( \Omega F \) [29, Lemma 2.32 and Theorem 5.46].)

All parts of Theorem 36 are now proved (modulo the postponed technical points).
B.2 An Application

We cannot resist giving the following application. It is not used in the rest of the text.

**Corollary 45** (compare [33]). Let $M_*$ be a homotopy module which is torsion of exponent prime to the characteristic of $k$, and $L/k$ an extension of algebraically closed fields. Then $M_*(L) = M_*(K)$.

**Proof.** We shall call any sheaf $F$ with $F(L) = F(K)$ rigid. Rigid sheaves are stable under extensions, limits and colimits.

Let $K_0 = \ker(\eta^*: M_{*+1} \to M_*)$. Then there are exact sequences $0 \to K_n \to K_{*+1} \to K_{*+1}/K_n \to 0$. Since $\eta$ acts trivially on $K_{*+1}/K_n$, this homotopy module has transfers by Theorem 9 and so is rigid [23, Theorem 7.20]. Also $\eta$ acts trivially on $K_1$, so this module is rigid by the same argument. By induction we find that $K_n$ is rigid for all $n$ (recall that rigid modules are stable under extensions), and hence $K_\infty := \text{colim}_n K_n$ is rigid. It is thus sufficient to show that $M/K_\infty$ is rigid. We may thus assume that $\eta: M_{*+1} \to M_*$ is injective.

Next consider the exact sequences $0 \to K(n) := \eta^*(M_{*+1})/\eta^*(M_*) \to M_*/\eta^*(M_{*+1}) \to M_*/\eta^*(M_{*+1}) 
\to 0$. By induction on $n$ (starting at $n = 0$) using the fact that $\eta$ acts trivially on $K(n)$ and so $K(n)$ is rigid as before, one finds that $M_*/\eta^*(M_{*+1})$ is rigid for all $n$. Let $\eta^\infty(M)_* = \cap_n \eta^*(M_{*+1})$. It follows that $M_*/\eta^\infty(M)_*$ is rigid, being an inverse limit of rigid sheaves. It remains to show that $\eta^\infty(M)$ is rigid.

Now $\eta$ was assumed injective on $M$. From this it follows that $\eta$ is an isomorphism on $\eta^\infty(M)$. So in particular from now on we may assume that $\eta$ is surjective on $M$.

Consider $F_1 M/F_0 M =: M'$. It is straightforward to check that $\eta$ acts trivially on this module, so it is rigid. On the other hand $(F_1 M)_1(K) = M_1(K)$ whereas $(F_0 M)_1(K) = K_1^{MW}(K) M_0(K) = K_1^{MW}(K) \eta M_1(K) = I(K) M_1(K) = 0$, since $K$ is algebraically closed. Thus $M'((K)_1 = M((K)_1)$ and similarly for $L$, so we conclude that $M_1$ is rigid. The general result follows by replacing $M_*$ by $M_{*+n}$.

B.3 Constructing Strictly Invariant Subsheaves: Proofs of Proposition 37 and Lemma 41

**Proposition 37.** Let $M$ be a strictly invariant sheaf and $I: \mathcal{F}_k \to \text{Ab}$ a continuous subfunctor of $M|\mathcal{F}_k$. Suppose that for each $K \in \mathcal{F}_k$ and for each excellent dvr $\mathcal{O}_v \subset K$ we have that $s(I(K) \cap M(\mathcal{O}_v)) \subset I(\kappa(v))$.

There exists a unique unramified subpresheaf $I$ of $M$ extending $I: \mathcal{F}_k \to \text{Ab}$ and with $I(\mathcal{O}_v) = I(K) \cap M(\mathcal{O}_v)$ for each $K \in \mathcal{F}_k$ and excellent dvr $\mathcal{O}_v \subset K$. It is a Nisnevich sheaf and we have $I(X) = M(X) \cap I(X^{(0)})$.

**Proof of Proposition 37.** We use [29, Theorem 2.11]. This says that the functor $I: \mathcal{F}_k \to \text{Ab}$ together with the data $I(\mathcal{O}_v)$ and the specialisations inherited from $M$ extends to a (unique) unramified presheaf, provided the data satisfies certain axioms called (A1) to (A4). In fact the theorem establishes an equivalence of categories, so $I$ will be a subsheaf of $M$. Also while it is not stated in the source, the proof of the theorem only ever uses the value of $I(\mathcal{O}_v)$ on geometrically constructed dvrs, i.e. localisations of finitely generated rings over a field, and such rings are excellent [21, Theorem 2.39(a, c)]. Hence we only need the data $I(\mathcal{O}_v)$ for excellent dvrs.

It is easy to check that the axioms (A1) to (A4) for $I$ are all direct consequences of the axioms for $M$ and our definitions, so we know that $I$ exists as a presheaf, and that it has the correct sections over fields and over dvrs.

The formula $I(X) = M(X^{(0)}) \cap M(X)$ for all $X$ follows from the formula being true for $X = \text{Spec}(\mathcal{O}_v)$, because $I$ is unramified. Indeed we compute for $X$ connected

$$I(X) = \bigcap_{x \in X^{(1)}} I(O_{X,x}) = \bigcap_{x \in X^{(1)}} (I(k(X)) \cap M(O_{X,x}))$$

$$= I(k(X)) \cap \bigcap_{x \in X^{(1)}} M(O_{X,x}) = I(k(X)) \cap M(X).$$

(The first equality is because $I$ is unramified, the second is by definition, and the fourth is because $M$ is unramified.) The case of non-connected $X$ follows directly from this because for an unramified presheaf $F$ we have $F(X \coprod Y) = F(X) \oplus F(Y)$, by definition.

To prove that $I$ is a sheaf, let $\{U_\alpha\}_\alpha$ be a Nisnevich cover of $X$ and $\{i_\alpha\}_\alpha$ a compatible family, $i_\alpha \in I(U_\alpha)$. Then there exists a unique “glued” section $i \in M(X)$ with $i|_{U_\alpha} = i_\alpha$. We need to show that
Let $i \in I(X)$, i.e. that $i \in I(X(0))$. Let $\xi \in X(0)$. Since $\{U_{\alpha}\}_\alpha$ is a Nisnevich cover there exists $\alpha_\xi$ such that $\xi \in U_{\alpha_\xi}$. It then follows that $i|_{U_{\alpha_\xi}} = (i|_{U_{\alpha_\xi}})|_i = i_{\alpha_\xi}|_i \in I(\xi)$. This concludes the proof.

**Lemma (41).** Let $M$ be a strictly invariant sheaf and $I$ a subsheaf such that $I(X) = M(X) \cap I(X(0))$ for all $X$. Then $I$ is strictly invariant if and only if for all $L \in F_k$ we have $H^1(\mathcal{L}_L, I) = 0$.

**Proof of Lemma 41.** We note that an arbitrary presheaf $F$ is $\mathbb{A}^1$-invariant if and only if for each $X$, the map $F(X \times \mathbb{A}^1) \to F(X)$ induced by the zero section $X \to X \times \mathbb{A}^1$ is injective. (Pullback along the zero section shows that $F(X)$ is a retract of $F(X \times \mathbb{A}^1)$.) In particular $I$ is invariant, being a subsheaf of an invariant sheaf.

By [29, Theorem 5.46] we then know that $I$ is strictly invariant if and only if $H^1(\mathbb{A}^1, I)$ is invariant. Let $C = M/I$ be the Nisnevich quotient sheaf and consider the following commutative diagram with exact rows induced by the zero section $X \to \mathbb{A}^1 \times X$. A chase reveals that $\alpha$ is injective if and only if $\beta$

\[
\begin{array}{c}
H^0(\mathbb{A}^1 \times X, I) \to H^0(\mathbb{A}^1 \times X, M) \to H^0(\mathbb{A}^1 \times X, C) \to H^1(\mathbb{A}^1 \times X, I) \to H^1(\mathbb{A}^1 \times X, M) \\
\downarrow \cong \downarrow \cong \downarrow \alpha \downarrow \beta \cong
\end{array}
\]

is injective. Hence $I$ is strictly invariant if and only if $C$ is invariant.

Now let $X$ be essentially smooth over $k$ and Nisnevich local (i.e. the spectrum of a Henselian ring) with fraction field $K$, and consider the following commutative diagram with exact rows.

\[
\begin{array}{c}
I(X) \to M(X) \to C(X) \to 0 \\
\downarrow \downarrow \downarrow \\
I(K) \to M(K) \to C(K) \to 0
\end{array}
\]

Since the left square is Cartesian by the assumption that $I(X) = M(X) \cap I(K)$, the morphism $\gamma$ is injective. Now let $X$ be arbitrary. Since $C$ is a Nisnevich sheaf, we conclude that the composite $C(X) \to \bigoplus_{x \in X} C(O_{X,x}) \to \bigoplus_{x \in X} C(Frac(O_{X,x}))$ is injective, and hence so is $C(X) \to \bigoplus_{F \to X} C(F)$. Here the sum is over all fields $F \in F_k$ and all $F$-points of $X$. Now consider the following commutative diagram.

\[
\begin{array}{c}
C(X \times \mathbb{A}^1) \xrightarrow{\alpha} \bigoplus_{F \to X} C(F \times \mathbb{A}^1) \to \bigoplus_{F \to X \times \mathbb{A}^1} C(F) \\
\downarrow \downarrow \downarrow \\
C(X) \to \bigoplus_{F \to X} C(F)
\end{array}
\]

The top right map comes from the fact that $\text{Hom}(\text{Spec}(F), X \times \mathbb{A}^1) = \text{Hom}(\text{Spec}(F), X) \times \text{Hom}(\text{Spec}(F), \mathbb{A}^1)$. By what we just said the top composite is injective and hence so is $\alpha$. Thus the left vertical map is injective if the right one is. That is, $C$ is invariant if (and only if) $C(F \times \mathbb{A}^1) \to C(F)$ is injective for all $F \in F_k$, which (going back to the first diagram) is the same as $H^1(F \times \mathbb{A}^1, I) = 0$. This concludes the proof.

**B.4 Interaction of Boundary and Specialisation: Proof of Proposition 38**

We will actually prove the following more general statement, which we need later anyway. Recall that for $a \in K^\times$ there are elements $\langle a \rangle \in K^{MW}_0(K), [a] \in K^{MW}_1(K)$ [29, Section 3.1].

**Proposition 46.** Let $\mathcal{O}_K \subset K$ be a dvr with uniformiser $\pi$, $M_\ast$ a homotopy module. Write $s : M_\ast(\mathcal{O}_v) \to M_\ast(\kappa(v))$ for the specialisation and $\partial^\pi : M_\ast(K) \to M_{\ast-1}(\kappa(v))$ for the boundary. Put

\[
s^\pi : M_\ast(K) \to M_\ast(\kappa(v)), s^\pi(m) = (-1)^{\partial^\pi([-\pi]m)}.
\]

Then the following hold.

(i) For $m \in M_\ast(\mathcal{O}_v)$ we have $s^\pi(m) = s(m)$.

(ii) For $a \in \mathbb{A}^{MW}_\ast(K), m \in M_\ast(K)$ we have $s^\pi(\alpha m) = s^\pi(\alpha)s^\pi(m)$.
(iii) For \( \alpha \in K^{MW}_n(K) \), \( m \in M_*(K) \) we have
\[
\partial^\sigma(\alpha m) = \partial^\sigma(\alpha)s^\sigma(m) + s^\sigma(\alpha)\partial^\sigma(m) + (\partial^\sigma(\alpha)[-1])\partial^\sigma(m).
\]

Proof. We begin with two preliminaries. Firstly, if \( M_* = K^{MW}_* \), then all of the relations hold by [29, Lemma 3.16].

Secondly, suppose that \( \alpha \in K^{MW}_n(O_v) \) and \( m \in M_*(K) \) (respectively \( \alpha \in K^{MW}_n(K) \) and \( m \in M_*(O_v) \)). Then we have \( \partial^\sigma(\alpha m) = s(\alpha)\partial^\sigma(m) \) (respectively \( \partial^\sigma(\alpha m) = \partial^\sigma(\alpha)s(m) \)). To see this, one goes back to the geometric construction of \( \partial^\sigma \) as in [29, Corollary 3.25]. That is we observe that after canonical identifications, \( \partial^\sigma \) corresponds to the boundary map \( \partial : H^0(K, M_*) \to H^1_v(O_v, M_*) \) in the long exact sequence for cohomology with support. Our claim then follows from the observation that for any sheaf of rings \( K \) and \( K \)-module \( M \), the boundary map in cohomology with support satisfies our claim. This is easy to prove using e.g. GodeMENT resolutions.

Statement (i) follows now from \( \partial^\sigma([\pi]) = (-1) \).

We can prove statements (ii) and (iii) most quickly together. To do this, let \( K^{MW}_*(K) ) \) be the graded-\( \epsilon \)-commutative ring with \( \xi \) in degree one, \( \xi^2 = \xi[[-1]] \) and let \( M_*(K) ) = M_*(K) \oplus M_{*-1}(K) \) be the evident \( K^{MW}_*(K) ) \) module. Put \( \Theta : M_*(K) \to M_*(K(\epsilon)) ) \), \( \Theta(\alpha) = s(\alpha) + \partial(\alpha)\xi \). (See [29, Lemma 3.16].) Then (ii) and (iii) are equivalent to \( \Theta(\alpha m) = \Theta(\alpha)\Theta(m) \). It is sufficient to prove this on generators \( \alpha \) of \( K^{MW}_n(K) \). Moreover (ii) and (iii) for \( \alpha \in K^{MW}_n(O_v) \) hold by the second preliminary point. It follows from [29, Theorem 3.22, Lemma 3.14] and standard formulas in Milnor-Witt K-theory that \( K^{MW}_n(K) \) is generated by \( K^{MW}_n(O_v) \) together with \( [\pi] \). Hence to establish (ii) and (iii) we need only consider \( \alpha = \pi \).

These are easy computations:
\[
s^\sigma([\pi]m) = \partial^\sigma([-\pi][\pi]m) = 0 \text{ because } -\pi[\pi] = 0 \quad \text{[29, Lemma 3.7 1]} \quad \text{and also } s^\sigma([\pi]) = 0, \quad \text{so } s^\sigma([\pi]m)s^\sigma(m) = 0 = s^\sigma([\pi]m) \text{ as desired.}
\]
Similarly \( \partial^\sigma([\pi]m) = \partial^\sigma([-1] + \langle -\pi \rangle)m = -\partial^\sigma(m) + (-1)\partial^\sigma([-\pi]m) \) and this is the same as \( \partial^\sigma([\pi])s^\sigma(m) + s^\sigma([\pi])\partial^\sigma(m) + \partial^\sigma([-\pi])[\pi]m \) because \( \partial^\sigma([-\pi]) = 1 \) and \( s^\sigma([-\pi]) = 0 \). \( \square \)

B.5 Boundary and Transfer: Proof of Lemmas 39, 42 and 43

Finally we come to transfers. Write \( \omega \) for the canonical sheaf. We will use it for twisting. By [29, Chapter 5], there are absolute transfers \( \text{Tr}_K^L : M_*(L, \omega) \to M_*((L, \omega)) \) for any finite extension \( f : \text{Spec}(L) \to \text{Spec}(K) \) in \( \mathcal{F}_k \). These satisfy many properties, for example the projection formula \( \text{Tr}_K^L (f^*(\alpha)m) = \alpha \text{Tr}_K^L (m) \) whenever \( \alpha \in K^{MW}_n(K), m \in M_*(L) \). It follows that for a line bundle \( L \) on \( K \) there exist natural transfers \( M_*(L, f^!(L) \otimes \omega) \to M_*((K, L \otimes \omega)) \). Let \( \nu(f) = f \otimes f^!(\omega)^{-1} \). Then we find the twisted transfer \( M_*(L, \nu(f)) \to M_*((K)) \). We shall need the following results, which says that the transfers on fields extend to arbitrary finite morphisms.

**Theorem 47** ([29], Corollary 5.30). Let \( f : X' \to X \) be a finite morphism of essentially k-smooth schemes. Then there is a commutative diagram
\[
\begin{array}{cccc}
M_*(X', \nu(f)) & \longrightarrow & M_*(X'(0), \nu(f)) & \longrightarrow & \bigoplus_{x \in X'^{(1)}} M_{*-1}(k(x), \nu(f) \otimes \Lambda^X_x) \\
\downarrow & & \downarrow \text{Tr} & & \downarrow \text{Tr} \\
M_*(X) & \longrightarrow & M_*(X(0)) & \longrightarrow & \bigoplus_{x \in X^{(1)}} M_{*-1}(k(x), \Lambda^X_x).
\end{array}
\]

Here the middle vertical arrow is the transfer just defined. If \( x \in X'^{(1)} \) then we have \( \nu(f) \otimes \Lambda^X_x \simeq \omega \otimes f^!(\omega_X)^{-1} \) and hence there is a natural transfer \( M_{*-1}(k(x), \nu(f) \otimes \Lambda^X_x) \to M_{*-1}(k(f(x)), \Lambda^X_{f(x)}) \).

The right vertical map is just the evident sum of such transfers. The left horizontal maps are injective, so the theorem asserts that the right square is commutative and that the unique vertical map on the left making the diagram commutative exists.

We now put
\[
(\mathcal{K}^MW_m)^{gr}(K) = \langle \text{Tr}_K^L (K^{MW}_n(L, \nu(L/K))M_m(L)) \rangle_{L/K}.
\]

This is the “correct” definition of \( (\mathcal{K}^MW_m)^{gr}(K) \), using twisted absolute transfers. But note that if we just choose some isomorphisms \( \omega_K \simeq K \) and \( \omega_L \simeq L \), we get the composite \( K_n(L)m(L) \subset M_{n+m}(L) \simeq M_{n+m}(L, \omega_L) \to M_{m+n}(K, \omega_K) \simeq M_{m+n}(K) \) and this has the same image as the “correct” transfer, so the twists do not actually matter in some sense. (Note also that in the definition of \( (\mathcal{K}^MW_m)^{gr}(K) \), it makes no difference if we view the right hand side as the subgroup generated or the sub-GW(K)-module generated. This follows from the projection formula.)
Lemma (39). Let \( \mathcal{O}_v \subset K \) be an excellent dvr with uniformizer \( \pi \). Then \( \partial^\pi(K_n^{MW}M_m)^{tr}(K) \subset (K_{n-1}^{MW}M_m)^{tr}(\kappa(v)) \subset M_{n+m-1}(\kappa(v)) \), for each \( n > 0 \).

**Proof of Lemma 39.** Let \( f : \text{Spec}(L) \to \text{Spec}(K) \) be a finite extension and \( A \) the integral closure of \( \mathcal{O}_v \) in \( L \). Then \( A \) is semilocal, smooth over \( k \) (indeed \( A \) is normal and one-dimensional) and finite over \( \mathcal{O}_v \) [21, Theorem 2.39(d)]. Let \( x_1, \ldots, x_n \) be the closed points. From Theorem 47 we get the commutative diagram

\[
\begin{array}{ccc}
M_{n+m}(L, \nu(f)) & \longrightarrow & \bigoplus_{i=1}^{n} M_{n+m-1}(k(x_i), \Lambda^A_{x_i} \otimes \nu(f)) \\
\downarrow \Sigma \text{Tr} & & \downarrow \Sigma \text{Tr} \\
M_{n+m}(K) & \longrightarrow & M_{n+m-1}(\kappa(v), \Lambda^{\mathcal{O}_v})
\end{array}
\]

The choice of \( \pi \) trivialises \( \Lambda^A_{\mathcal{O}_v} \) and the bottom horizontal arrow becomes \( \partial^\pi \). Let \( \alpha \in K_n^{MW}(L, \nu(f)) \) and \( m \in M_{n}(L) \). Then \( \partial^\pi(\text{Tr}(\alpha m)) = \Sigma \text{Tr}(\partial(\alpha)m) \) and we see that it is enough to prove that \( \partial(\alpha)(K_m^{MW}(L, \nu(f)))M_{m}(L) \subset K_{n-1}^{MW}(k(x_i), \Lambda^A_{x_i} \otimes \nu(f))M_{m}(k(x_i)) \). We note that \( \nu(f) \) is (non-canonically) trivial so we can ignore that twist.

We may thus forget about the extension \( L/K \) and just prove that \( \partial^\pi(K_n^{MW}(K)M_m(K)) \subset K_{n-1}(K)M_m(K) \) (renaming \( L \) to \( K \)). But this follows from Proposition 46 part (iii) and the fact that \( K_{>0}^{MW}(K) \) is generated in degree 1.

**Lemma (42).** Let \( X \) be the Henselization of a smooth variety over \( k \) in a point of codimension one. Then the differential

\[
(K_n^{MW}M_m)^{tr}(X(0)) \xrightarrow{0} \bigoplus_{x \in X^{(1)}} (K_{n-1}^{MW}M_m)^{tr}(k(x), \Lambda^X_{x})
\]

is surjective.

**Proof of Lemma 42.** Let \( X \) have generic point \( x \) and closed point \( x \). We need to prove that \( \partial : (K_n^{MW}M_m)^{tr}(x) \to (K_{n-1}^{MW}M_m)^{tr}(x, \Lambda^X_x) \) is surjective. Let \( l/\kappa(x) \) be a finite extension. Writing this as a sequence of simple extensions and lifting minimal polynomials of generators (which remain irreducible by Gauss’ lemma), we can find a finite extension \( L/K \) with \( [L:K] = [l/\kappa(x)] \) such that if \( f : X \to X \) is the integral closure of \( X \) in \( L \) then \( l \) is one of the residue fields of \( X \). Note that \( X \) is connected (\( \text{Spec} X \subset L \) is a domain), essentially \( k \)-smooth (since normal) and finite over \( X \) [21, Theorem 2.39(d)] (\( X \) is excellent [12, Corollaire 18.7.6]), so is in fact Nisnevich local (i.e. the spectrum of a Henselian local ring) [38, Tag 04GH]. Let \( \tilde{x} \) be the unique point above \( x \), so that \( l = \kappa(\tilde{x}) \). From Theorem 47 and Lemma 39 we get the commutative diagram

\[
\begin{array}{ccc}
(K_n^{MW}M_m)(k(\tilde{x}), \nu(f)) & \longrightarrow & (K_{n-1}^{MW}M_m)(\kappa(\tilde{x}), \nu(f) \otimes \Lambda^X_{\tilde{x}}) \\
\downarrow \text{Tr} & & \downarrow \text{Tr} \\
(K_n^{MW}M_m)^{tr}(k(X)) & \longrightarrow & (K_{n-1}^{MW}M_m)^{tr}(k(x), \Lambda^X_x).
\end{array}
\]

Observing that \( \nu(f) \) is (non-canonically) trivial (and that \( l \) was arbitrary) it is thus enough to show that \( (K_n^{MW}M_m)(\xi) \xrightarrow{0} (K_{n-1}^{MW}M_m)(\kappa(x), \Lambda^X_x) \) is surjective (renaming \( \tilde{x} \) to \( X \)).

Since \( k \) is perfect \( \kappa(x)/k \) is smooth. It is then the hard to see (using the fact that \( X \) is Henselian) that \( x \to X \) admits a section, and so \( s : M_m(X) \to M_m(\kappa(x)) \) is surjective. Also \( \partial : K_n^{MW}(\xi) \to K_{n-1}(\kappa(x), \Lambda^X_x) \) is surjective because \( K_{n}^{MW} \) is a homotopy module. So for \( \alpha \in K_{n-1}(\kappa(x), \Lambda^X_x) \) and \( m \in M_m(\kappa(x)) \) we can find \( \alpha' \in K_{n}^{MW}(\xi) \) and \( m' \in M_m(X) \) with \( \partial(\alpha') = \alpha, s(m') = m \) and thus \( \partial(\alpha'm') = \alpha m \) by a (very) special case of Proposition 46 part (iii).

This concludes the proof.

**Lemma (43).** Let \( K \in \mathcal{F}_k \). The differential

\[
(K_n^{MW}M_m)^{tr}(K(T)) \to \bigoplus_{x \in (\Lambda^X_k)^{(1)}} (K_{n-1}^{MW}M_m)^{tr}(k(x), \Lambda^X_x)
\]

is surjective.
Proof of Lemma 4.3. Fix $x_0 \in (\mathcal{A}_K^1)^{(1)}$. It suffices to prove that given $\alpha \in (K_{n-1}^{MW} M_m)^{tr}(\kappa(x_0), \Lambda_{x_0}^1)$ there exists $\beta \in (K_n^{MW} M_m)(K(T))$ such that $\partial_{\alpha}(\beta) = \alpha$ and $\partial_{\beta}(\beta) = 0$ for $y \neq x_0$. We may suppose that $\alpha$ is obtained by transfer from a finite extension $L/\kappa(x)$. Then $L/K$ is also finite.

Letting $f : \mathcal{A}_L^1 \to \mathcal{A}_K^1$ be the canonical map, as before we get a commutative diagram

\[
\begin{array}{ccc}
(K_n^{MW} M_m)(L(T), \nu(f)) & \longrightarrow & \bigoplus_{y \in (\mathcal{A}_L^1)^{(1)}} (K_{n-1}^{MW} M_m)(k(y), \nu(f) \otimes \Lambda_y^1) \\
\downarrow_{T_r} & & \downarrow_{T_r} \\
(K_n^{MW} M_m)^{tr}(K(T)) & \longrightarrow & \bigoplus_{x \in (\mathcal{A}_K^1)^{(1)}} (K_{n-1}^{MW} M_m)^{tr}(k(x), \Lambda_x^1).
\end{array}
\]

We note that there is at least one point $y_0 \in \mathcal{A}_L^1$ above $x_0 \in \mathcal{A}_K^1$ with $\kappa(y_0) = \kappa(x_0)$ and that $\nu(f)$ is (non-canonically) trivial. Hence it suffices to prove: given $y_0 \in (\mathcal{A}_L^1)^{(1)}$ with $\kappa(y) = L$ and $\alpha \in (K_{n-1}^{MW} M_m)(\kappa(y_0), \Lambda_{y_0}^1)$ there exists $\beta \in (K_n^{MW} M_m)(L(T))$ with $\partial_{\kappa(y)}(\beta) = \alpha$ and $\partial_{\beta}(\beta) = 0$ for $y \neq y_0$.

We may assume that $\alpha = cm$ with $c \in K_{n-1}^{MW} (\kappa(y_0), \Lambda_{y_0}^1)$ and $m \in M_m(\kappa(y_0))$. Since $K_n^{MW}$ is a homotopy module there exists $c' \in K_{n}^{MW} (L(T))$ with $\partial_{\kappa(y)}(c') = c$ and $\partial_{\beta}(c') = 0$ for $y \neq y_0$. Moreover since $\kappa(y_0) = \kappa(y)$ the natural map $M_m(L) \to M_m(\kappa(y))$ is an isomorphism and we may choose $m' \in M_m(L)$ mapping to $m$. Then $m' \in M_m(\mathcal{A}_L^1,y)$ for all $y$ and hence $\partial_{\kappa(y)}(c'm') = \partial_{\beta}(c'm') = 0$ for all $y$ (including $y_0$). Thus $\partial_{\kappa(y)}(c'm') = cm$ and $\partial_{\beta}(c'm') = 0$ for $y \neq y_0$. This concludes the proof. \qed

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