EVER-EXPANDING, ISOTROPIZING, QUADRATIC COSMOLOGIES

S. Cotsakis, G.P. Flessas, P.G.L. Leach and L. Querella

† GEODYSYC, Department of Mathematics, University of the Aegean, Karlovassi 83 200, Greece
‡ Groupe de Cosmologie Théorique, Institut d’Astrophysique, Université de Liège, B–4000 Liège, Belgium

We consider some aspects of the global evolution problem of Hamiltonian homogeneous, anisotropic cosmologies derived from a purely quadratic action functional of the scalar curvature. We show that models can isotropize in the positive asymptotic direction and that quadratic diagonal Bianchi IX models do not recollapse and may be regular initially. Although the global existence and isotropization results we prove hold quite generally, they are applied to specific Bianchi models in an attempt to describe how certain dynamical properties uncommon to the general relativity case, become generic features of these quadratic universes. The question of integrability of the models is also considered. Our results point to the fact that the more general models are not integrable in the sense of Painlevé and for the Bianchi IX case this may be connected to the validity of a BKL oscillatory picture on approach to the singularity in sharp contrast with other higher order gravity theories that contain an Einstein term and show a monotonic evolution towards the initial singularity.

1. Introduction

The problem of describing the past and future asymptotic states of general cosmological models occupies a central position in mathematical cosmology. This question can be addressed either in the framework of general relativity and string theory or in that of higher-order and scalar-tensor theories. Among several different approaches that have been developed over the years to tackle the asymptotic problem, two are distinguished, namely, the Hamiltonian approach and the dynamical systems approach. Both have been and are being used widely and with great success when dealing with the asymptotic problem in the framework of general relativistic and string cosmology but less widely so in the case of higher-order cosmologies, that is, in the description of the asymptotic states of cosmological models derived from an action functional which depends nonlinearly on the curvature invariants.

In this paper we use the Hamiltonian formulation of homogeneous but anisotropic cosmologies derived from an action integral purely quadratic in the scalar curvature (quadratic cosmologies) developed by Demaret and Querella in [3] to prove several results pertaining to the global structure of the solution space of these systems. The main motivation for studying such systems is not so much due to their confrontation with the currently available empirical data (see, however, [1]) but to their different nature.

It is certainly true that recent high-precision tests such as the Hulse-Taylor binary pulsar (cf. [1]) have confirmed general relativity to an outstanding degree. This theory predicts that in the case of the binary pulsar PSR1913+16 the orbit slowly decays and the orbital period increases due to the loss of energy by emission of gravity waves and this has been observed with high precision. However, general relativity leads to singularities in the spacetimes of all known cosmological models – places where further predictions about the structure and evolution of the universe cannot be made. Quadratic curvature terms present in the gravitational action, which naturally arise in string-theoretic considerations as quantum gravitational corrections, may rectify this situation and lead to cosmological models free from such pathologies. One then envisions the general-relativistic (linear) action to be a low-energy, low spacetime curvature limit of some quantum gravity theory that would contain such higher-order curvature invariants and which has yet to be worked out. Of course the purely quadratic theories considered here do not reproduce the weak field limit of general relativity. However, such theories may gain importance in the high-energy regimes of the early universe and serve as prototypes from which more elaborate or natural, Lagrangian theories of gravity can be built and studied.

Due to their conformal equivalence [1], the dynamics of higher-order vacuum cosmologies can be regarded as closely related to that of general scalar field cosmologies in general relativity. However, it should be emphasized that this procedure, although practically useful, cannot imply a substitute for a full analysis of the original (higher-order) system since, for example, at those points on the manifold where the conformal transformation is singular, one cannot obtain a complete picture of the original dynamics when inverting the conformally related one. Therefore one has to rely solely on the original (higher order) field equations.
when analysing a higher-order system.

We consider vacuum Bianchi cosmologies of class A in the expanding direction and prove global existence results towards a future asymptotic state. We show how these quadratic cosmological systems isotropize and prove that Bianchi IX models in the expanding direction have a monotonically increasing volume and hence cannot recollapse. This result is in sharp contrast with the known behaviour of these systems in general relativity (cf. [10], [11]).

The plan of this paper is as follows. In the next section we cast the purely quadratic homogeneous cosmologies in a Hamiltonian form (cf. [1]) and write down the basic Hamiltonian cosmological equations. This Hamiltonian reduction process has an added advantage to that with the usual Lagrangian field equations, namely, the original fourth-order (Lagrangian) field equations are reduced to a first order system in the Hamiltonian variables. In Sec. 3, by defining new variables, we further reduce the Hamiltonian equations to a suitable autonomous dynamical system for the volume and anisotropy functions. Sec. 4 is the heart of the paper. We prove two theorems about the global existence of solutions to the basic equations in the expanding direction. These theorems are then used in Sec. 5 to prove an isotropization theorem for a class of quadratic, homogeneous, anisotropic cosmologies, and all previous results are applied in Sec. 6 to several specific Bianchi metrics in an attempt to highlight their dynamics. In particular, we show as a corollary of the basic theorems, that there exist non-recollapsing, ever-expanding Bianchi IX cosmologies in marked contrast to what is known about such systems in general relativity. We round up our analysis in Sec. 7 by listing the Lie point symmetries of quadratic cosmologies and performing for them the Painlevé test to decide on their algebraic integrability. We conclude in Sec. 8 wherein we discuss prospects for future work.

2. Purely quadratic Hamiltonian cosmologies

In this section we derive the general expressions for the super-Hamiltonian, supermomenta and canonical equations which are our starting point for the analysis of relevant cosmological models in subsequent sections.

Our beginning is the purely quadratic action

$$S = \frac{\beta_c}{8} \int R^2(-g)^{1/2} d^4x.$$ (1)

Following the ADM Hamiltonian prescription, we assume a foliation of spacetime into spacelike hypersurfaces and use the ADM coordinate basis to compute the components of various curvature tensors. The “0” index below refers to the normal component to the hypersurfaces, while the superscripts (4) and (3) indicate quantities defined over the 4-dimensional spacetime and the spacelike hypersurfaces, respectively. Most of the time, when there is no ambiguity, the last subscript will be omitted. This splitting of spacetime allows one to rewrite the action (1) in the form

$$S = \frac{\beta_c}{8} \int N^{(4)} R^2 (3) g^{1/2} d^4x,$$ (2)

with

$$R^{(4)} = R^{(3)} + K^2 - 3 \text{tr} K^2 - 2 \frac{\nabla^2 N}{N} - 2 g^{kl} \mathcal{L}_n K_{kl},$$

$$\mathcal{L}_n K_{kl} = \frac{1}{N} (K_{kl,0} - N^{i} K_{kli} - N^{i}_{[k} K_{li] - N^{i}_{[l} K_{ki]}),$$

$$K_{ij} = -\frac{1}{2} \mathcal{L}_n g_{ij} = -\frac{1}{2N} (g_{ij,0} - N_{ij} - N_{ji}),$$

where $K_{ij}$ and $R_{ij}$ are the extrinsic and intrinsic curvature tensors relative to the hypersurfaces mentioned above, the symbol $|$ denotes covariant differentiation on these 3-surfaces, $N$ and $N_i$ are the lapse and shift functions and $\mathcal{L}_n$ denotes the Lie derivative along the normal $\vec{n}$ to these hypersurfaces.

The Hamiltonian form of (2) is then given, up to surface terms, by the integral

$$S = \int d^4x N [P_{ij} \mathcal{L}_n Q^{ij} + p^{ij} \mathcal{L}_n g_{ij} - \mathcal{H}(g, Q, p, P)](3)$$

with

$$\mathcal{H}(g, Q, p, P) = -2 p^{ij} P_{ij} + \frac{Q^2}{18 \beta_c g^{1/2}} + Q^{ij}_{|ij} + Q^{ij} R_{ij} - \frac{Q}{2} R + Q^{ij} P_{ij} P + \frac{Q}{2} (\text{tr} p^2 - p^2),$$ (4)

$$Q^{ij} = g^{1/2} \frac{\beta_c}{2} R^{(4)} g^{ij},$$

$$p^{ij} = -\frac{1}{2} \left( \mathcal{L}_n Q^{ij} - \frac{\delta \mathcal{H}}{\delta K_{ij}} \right),$$

$$P_{ij} = K_{ij},$$ (5)

where the trace and the traceless part of a tensor $A_{ij}$ are denoted respectively by $A$ and $A_{ij}^t$, $\text{tr} A^2 = A_{ij}^t A_{ij}$ and

$$\mathcal{H}(g, Q, K) = K Q^{ij} K_{ij} + Q^{ij} R_{ij} + Q^{ij}_{|ij} - \frac{Q^2}{18 \beta_c g^{1/2}} + \frac{Q}{2} (\text{tr} K^2 - K^2 - R),$$ (6)

wherein $Q^{ij}$ is a tensor introduced in order to reduce the order of the Lagrangian as in Ostrogradskii’s method. The canonical variables are $g_{ij}$ and $Q^{ij}$ and their conjugate momenta are $p^{ij}$ and $P_{ij}$, respectively.
We can also write the action (3) in the usual Hamiltonian form, where all the constraints are manifest, namely,

\[ S = \int d^4x \left[ P_{ij} \dot{Q}^{ij} + g^{ij} \dot{g}_{ij} - \mathcal{H}^* (g, Q, p, P) \right], \quad (7) \]

where we have introduced the total Hamiltonian constraint \( \mathcal{H}^* \) as

\[ \mathcal{H}^* = N \mathcal{H} + N^k \mathcal{H}_k = N^\mu \mathcal{H}_\mu, \quad (8) \]

\( \mathcal{H} \) and \( \mathcal{H}_k \) being the super-Hamiltonian and supermomenta (or spatial constraints) given by (3) and by

\[ \mathcal{H}_k = -Q^{ij} P_{ij|k} + 2(P_{ik} Q^{ij})_j - 2g_{ik} P^{ij}, \quad (9) \]

respectively.

The 3-metric of any diagonal Bianchi-type model is

\[ d\sigma^2 = e^{2\mu} \left[ e^{2(b_+ + \sqrt{3}b_-)}(\omega^1)^2 + e^{2(b_+ - \sqrt{3}b_-)}(\omega^2)^2 + e^{-4b_+}(\omega^3)^2 \right], \quad (10) \]

where \( \mu, b_+, b_- \) are functions of time only and the set \( \{ \omega^i \} \) is the basis of 1–forms with structure coefficients \( C^i_{jk} \),

\[ d\omega^i = \frac{1}{2} C^i_{jk} \omega^j \wedge \omega^k. \]

Following (3), we perform a canonical transformation from the original set of canonical variables \( \{ g, Q, p, P \} \) to the set \( \{ \mu, b_+, b_-; \Pi_\mu, \Pi_+, \Pi_-; Q_+, Q_-, Q_n, P_+, P_-, P_n \} \), where

\[ g_{11} = e^{2\mu} e^{2(b_+ + \sqrt{3}b_-)}, \]
\[ g_{22} = e^{2\mu} e^{2(b_+ - \sqrt{3}b_-)}, \]
\[ g_{33} = e^{2\mu} e^{-4b_+}, \]
\[ \Pi^1 = \frac{1}{12} (2\Pi_\mu + \Pi_+ + \sqrt{3}\Pi_-), \]
\[ \Pi^2 = \frac{1}{12} (2\Pi_\mu + \Pi_+ - \sqrt{3}\Pi_-), \]
\[ \Pi^3 = \frac{1}{6} (\Pi_\mu - \Pi_+), \]
\[ p^{ij} = \Pi^{ij} + P^{Tij} (Q^{Tjk} \frac{Q_n}{\sqrt{3}} + P_\mu \sqrt{3} Q^{ij} + P_- \sqrt{3} g_{ij}), \]
\[ Q^{ij} = Q^{Tij} \frac{Q_n}{\sqrt{3}} g^{ij}, \]
\[ P_{ij} = P_i \frac{P_n}{\sqrt{3}} g_{ij}, \]
\[ Q^{Tij} = \frac{1}{\sqrt{6}} \text{diag} \left( Q_+ + \sqrt{3}Q_-, Q_+ - \sqrt{3}Q_-, -2Q_+ \right), \]
\[ P_i \frac{P_n}{\sqrt{3}} g_{ij}, \]

In terms of these new variables the original Bianchi action can be written as

\[ S = \int d^4x \left[ \Pi_\mu \dot{\mu} + \Pi_+ \dot{\beta}_+ + \Pi_- \dot{\beta}_- + P_+ \dot{Q}_+ + P_- \dot{Q}_- + P_n \dot{Q}_n - \mathcal{H}^* \right] \]

with the 4-volume element \( d^4x = dt \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 \). The explicit form of the total Hamiltonian constraint \( \mathcal{H}^* \) depends on the particular Bianchi model considered. In the case \( L = R^2 \) we have to impose the constraint \( Q_\pm \approx 0 \). The super-Hamiltonian then reduces to

\[ \mathcal{H}_R = \frac{\Pi^2_+ + \Pi^2_-}{4\sqrt{3} Q_n} - \frac{1}{\sqrt{3}} P_n \Pi_\mu - \frac{2}{\sqrt{3}} P^2 Q_n + \frac{e^{-3\mu}}{6\beta_c} Q^2_n + \frac{e^{-2\mu}}{\sqrt{3}} Q_n V^*, \quad (11) \]

where \( V^* = V^*(\beta_+, \beta_-) \) is directly related to the usual Bianchi potentials (cf. [3], ch.10). Denoting in everything that follows by \( \lambda_y \) the derivative of the function \( X \) with respect to the variable \( y \), the canonical equations obtained from (3) are

\[ \dot{\mu} = -\frac{N}{\sqrt{3}} P_n, \]
\[ \dot{\beta}_+ = \frac{N}{2\sqrt{3}} \Pi_+, \]
\[ \dot{\Pi}_\mu = N \left[ \frac{1}{2\beta_c} e^{-3\mu} Q^2_n + \frac{2}{\sqrt{3}} e^{-2\mu} Q_n V^* \right], \]
\[ \dot{\Pi}_\pm = \frac{N}{\sqrt{3}} e^{-2\mu} Q_n V^*, \]
\[ \dot{Q}_n = -\frac{N}{\sqrt{3}} (\Pi_\mu + 4P_n Q_n), \]
\[ \dot{P}_n = \frac{N}{4\sqrt{3}} \left[ \left( \frac{\Pi_+}{Q_n} \right)^2 + \left( \frac{\Pi_-}{Q_n} \right)^2 + 8P^2_n \right] - \frac{N}{3\sqrt{3} \beta_c} e^{-3\mu} Q_n - \frac{N}{\sqrt{3}} e^{-2\mu} V^*. \]

### 3. Reduction to an autonomous form

By imposing the weak constraint \( \mathcal{H}_R \approx 0 \) and manipulating the basic Hamiltonian system (12) we obtain

\[ P_n Q_n = k - \Pi_\mu \quad (13) \]

where \( k \) is a constant of integration. When the scalar curvature is not constant, we can fix the temporal gauge by choosing \( (4^4 R) \approx -t \) to find

\[ Q_n = -\frac{\sqrt{3}}{2} \beta_c e^{3\mu}, \quad (14) \]
\[ N = \frac{3\beta_c}{2k} e^{3\mu}. \]

By some further straightforward algebraic manipulations it is possible to reduce the set of canonical equations to a non-autonomous dynamical system.
of three coupled equations for the physical variables $\mu(t)$, $\beta_\pm(t)$:
\[
\begin{align*}
\mu'' + \mu' + A_1 t^2 e^{6\mu} - A_2 t e^{4\mu} V &= 0, \\
\beta'' + \beta' + A_3 t e^{4\mu} V_\beta &= 0,
\end{align*}
\] (16)
where $A_i$ are strictly positive constants given by
\[
A_1 = \frac{9\rho}{16}, \quad A_2 = \frac{3\rho}{2}, \quad A_3 = \frac{3\rho}{8}, \quad \rho = \frac{\beta^2}{k^2}
\] (17)
with $\beta^2$ and $k^2$ arbitrary constants. (The canonical variables $\mu$ and $\beta_\pm$ become uncoupled when the potential term vanishes, i.e., for Bianchi-type I model.) We can rewrite the dynamical system (16) in the so-called Taub gauge, $d\lambda = e^{3\mu(t)} dt$, where $\lambda$ is the physical time with $0 \leq \lambda = \lambda(t) \leq \infty$ and, assuming invertibility, $t = t(\lambda)$, we can write $\mu(t) = \mu(\lambda)$, $\beta_+(t) = \beta_+(\lambda)$ or $\beta_-(t) = \beta_-(\lambda)$ and $V = V(\beta_+(t), \beta_-(t)) = V(\beta_+(\lambda), \beta_-(\lambda))$.

The treatment of the system of ODEs (16) is greatly facilitated by transforming it into an autonomous system. For this purpose we introduce new variables $(t^*, w, y, z)$ as follows:
\[
\begin{align*}
\mu(t) &= w(t^*), \quad \beta_+(t) = y(t^*), \\
\beta_-(t) &= z(t^*), \quad t = \varphi(t^*),
\end{align*}
\] (18)
where $\varphi$ is a smooth function of $t^*$. On inserting (18) into (16) we obtain:
\[
\begin{align*}
\frac{\ddot{w}\varphi}{\varphi^2} - \frac{\dot{w}\dot{\varphi}^2}{\varphi^3} + \frac{\ddot{w}}{\varphi} + A_1 e^{6w} \varphi^2 &= - A_2 e^{4w} \varphi V(y, z) = 0, \\
\frac{\ddot{\beta}}{\varphi^2} - \frac{\dot{\beta}\dot{\varphi}^2}{\varphi^3} + \frac{\dot{\beta}}{\varphi} + A_3 e^{4w} V_\beta(y, z) \varphi &= 0,
\end{align*}
\] (19,20)
where the dot denotes differentiation with respect to $t^*$ and
\[
\beta = y(t^*) \quad \text{or} \quad z(t^*).
\] (21)
In (19) and (20) we eliminate $\dot{w}$ and $\dot{\beta}$, respectively, by requiring that
\[
\frac{\ddot{\varphi}}{\varphi^3} = \frac{1}{\varphi}
\] (22)
and hence
\[
\varphi(t^*) = C e^{k_1 t^*}.
\] (23)
Eq. (23) is the general solution of (22) with $C$ and $k_1$ arbitrary constants. On the substitution of (23) into (19) and (20) we deduce that
\[
\begin{align*}
\frac{\ddot{w}}{Ck_1^2} + A_1 C^2 e^{6w + 3k_1 t^*} - A_2 V(y, z) C e^{4w + 2k_1 t^*} &= 0, \\
\frac{\ddot{\beta}}{Ck_1^2} + A_3 V_\beta(y, z) C e^{4w + 2k_1 t^*} &= 0
\end{align*}
\] (24)
From (24) it is obvious that by introducing a new dependent variable $x(t^*)$ through
\[
\dot{w}(t^*) = \frac{1}{2}[x(t^*) - k_1 t^*]
\] (25)
and, since it suffices without loss of generality and for our purposes, by taking in (23)-(25)
\[
C = k_1 = -1,
\] (26)
the basic system (24) finally assumes the autonomous form
\[
\begin{align*}
\ddot{x} + \alpha_1 e^{3x} + \alpha_2 e^{2x} V(y, z) &= 0, \\
\dot{y} + \alpha_3 e^{2x} V_\beta(y, z) &= 0, \\
\ddot{z} + \alpha_4 e^{2x} V_\beta(y, z) &= 0, \\
\mu(t) &= \frac{1}{2}[x(t^*) + t^*], \quad \beta_+(t) = y(t^*), \\
\beta_-(t) = z(t^*), \quad t = - e^{-t^*}
\end{align*}
\] (27-30)
with $\alpha_1 = -2A_1$, $\alpha_2 = -2A_2$, $\alpha_3 = A_3$. All functions in this system can also be regarded as functions of $\lambda$.

**Notation:** In the following we use the two basic time variables $t^*$ and $\lambda$. A prime will denote differentiation with respect to $\lambda$ and an overdot differentiation with respect to $t^*$.

### 4. Global existence

In this section we show that under very plausible hypotheses on the potential $V(y, z)$, the solutions of the autonomous dynamical system (27)-(30) can be defined and smoothly extended in the expanding direction for the whole positive $\lambda$-halfline. The main result of this section is given in the following two theorems. Theorem 1 basically says that $t^*$ is a compactified time parameter meaning that its interval of definition is a compact interval on the real line and Theorem 2 proves that the interval of definition of the time variable $\lambda$ is the interval $[0, \infty)$.

**Theorem 1.** If the potential $V = V(\beta_+(\lambda), \beta_-(\lambda))$ is a smooth function of $\beta_+(\lambda)$ and $\beta_-(\lambda)$ and satisfies for all real $\beta_+(\lambda)$ and $\beta_-(\lambda)$ the condition
\[
V = V(\beta_+(\lambda), \beta_-(\lambda)) \geq M, \quad M \in \mathbb{R},
\] (31)
and in the case $M < 0$ the arbitrary constant $\rho > 0$ given in (47) fulfills
\[
\rho < \frac{36(x_0')^2}{27e^{2x_0} - 256M^3},
\] (32)
then there exist solutions $\mu(\lambda)$ and $\beta_+(\lambda)$, $\beta_-(\lambda)$ to (14), which are monotonically increasing and decreasing, respectively, defined on the interval $0 \leq \lambda \leq \lambda_{\text{max}}$, where
\[
\begin{align*}
\lambda_{\text{max}} &= \int_{-e^{-16}}^{-e^{-T_*}} \exp \left\{ \frac{3}{2} x [\ln(-w)] \right\} (-w)^{-3/2} dw,
\end{align*}
\] (33)
Theorem 2. If Theorem 1 is valid and, in addition, in the case \( \lim_{\lambda \to \lambda_{\text{max}}} \beta_+(\lambda) = -\infty \) and (or) \( \lim_{\lambda \to \lambda_{\text{max}}} \beta_-(\lambda) = -\infty \), the derivatives \( V_{\beta} \) are absolutely bounded, that is, for \( n, k_1 \in \mathbb{R}^+, k_2 \in \mathbb{R}^+, \beta_+(m) \in \mathbb{R}^-, \beta_-(m) \in \mathbb{R}^-, \) and \( \beta_+(\lambda) \in (-\infty, \beta_+(m)], \beta_-(\lambda) \in (-\infty, \beta_-(m)] \), the conditions
\[
|V_{\beta_+\beta_+^n}| < k_1, \quad |V_{\beta_-\beta_-^n}| < k_2
\]
hold, then necessarily
\[
\lambda_{\text{max}} = \infty, \quad \lim_{\lambda \to \infty} \mu(\lambda) = \infty.
\]

We break the proof of these theorems into the two Lemmata below. We define
\[
p_1 = \dot{x}, \quad p_2 = \dot{y}, \quad p_3 = \dot{z}.
\]
Then (27)–(28) become
\[
p_1p_1x + \alpha_1e^{3x} + \alpha_2e^{2x}V(y, z) = 0,
p_2p_2y + \alpha_3e^{2x}V_y = 0,
p_3p_3z + \alpha_3e^{2x}V_z = 0,
\]
and after integration the system (38) yields by virtue of (37):
\[
\dot{x} = \left( -\frac{4}{3}A_1e^{3x} + 4A_3 \int e^{2x}V(y, z)dx \right)^{1/2},
\]
\[
\dot{y} = -\left( -2A_3 \int e^{2x}V_y dy \right)^{1/2},
\]
\[
\dot{z} = -\left( -2A_3 \int e^{2x}V_z dz \right)^{1/2}.
\]

Each choice of positive and negative signs in front of the square roots in (39) and (41) is taken in order to generate increasing \( x(t^*) \) and decreasing \( y(t^*) \) and \( z(t^*) \) as functions of \( t^* \), respectively, and will be justified at the end of this section. Since the system (27)–(29) is autonomous, the initial value of \( t^*, t_i^* \), can be arbitrary, and we consider here \( t_i^* = t_0^* \). Therefore, by incorporating into our system (27)–(29) the physically plausible assumption that \( V(y, z), V_y(y, z) \) and \( V_z(y, z) \) are continuous for all \( y \) and \( z \) in the vicinity of \( t^* = t_0^* \), we can invoke the existence theorem for the dynamical system (27)–(29) and establish the validity in a neighbourhood \( N \) of \( t^* = t_0^* \) of a unique, \( C^2 \) solution \( (x(t^*), y(t^*), z(t^*)) \), satisfying the initial conditions
\[
x(t_0^*) = x_0, \quad \dot{x}(t_0^*) = \dot{x}_0 > 0,
y(t_0^*) = y_0, \quad \dot{y}(t_0^*) = \dot{y}_0 < 0,
z(t_0^*) = z_0, \quad \dot{z}(t_0^*) = \dot{z}_0 < 0,
\]
where the numbers \( x_0, y_0, z_0 \), are arbitrary and \( \dot{x}_0, \dot{y}_0, \dot{z}_0 \), are signed as shown, \( x(t^*) \) and \( y(t^*), z(t^*) \) being monotonically increasing and decreasing functions of \( t^* \), respectively.

Equivalently, due to the monotonicity of \( x(t^*) \), we may choose \( x \) as the independent variable and express the solution (22) as
\[
y = y(x), \quad z = z(x),
\]
with the roles of \( t^* \) and \( x \) interchanged in the first of Eqs. (12). Analogous to (14) relations can be written down by considering \( y \) or \( z \) as independent variables.

Before we consider the existence proof, we proceed to compactify the time interval \([0, \lambda_{\text{max}}]\). From (33)–(1) we deduce by using (34) and (14)
\[
T^* - t_0^* = \int_{t_0^*}^{x_1} G^{-1/2},
\]
\[
x_0 \leq x(t^*) \leq x_1 = x(T^*)
\]
where
\[
G(x) = \dot{x}_0^2 + \left( 3 \beta_2 \right)(e^{3x} - e^{3x_0}) + 6 \beta_1 \int e^{2x}V(y, w, z(w))dw,
\]
\[
T^* - t_0^* = -\int_{t_0^*}^{y_1^*} H^{-1/2}
\]
\[
y_1 = y(T^*) \leq y(t^*) \leq y_0;
\]
here \( H(y) = \dot{y}_0^2 + (3/4) \rho \int_0^{y_0} e^{2x(w)}V_y dw \) and
\[
T^* - t_0^* = -\int_{t_0^*}^{z_1} K^{-1/2},
\]
\[
z_1 = z(T^*) \leq z(t^*) \leq z_0,
\]
where
\[
K(z) = \dot{z}_0^2 + (3/4) \rho \int_0^{z_0} e^{2x(z)}V_z dw.
\]
In (45) the upper limit \( x_1 \) in the integral denotes the maximum \( x = x(t^*) \) — the value for which the solution (14) is defined, i.e., \( x_0 \leq x \leq x_1, x_0 \leq w \leq x_1 \), which implies that the solution (12) holds for \( t_0^* \leq t^* \leq T^* \) with \( x_1 = x(T^*), y_1 = y(T^*) = y(x_1) \) and \( z_1 = z(T^*) = z(x_1) \). We observe now that by virtue of dt = e^{-\lambda(t)}dt and (39)
\[
\lambda(t) = \int_{t_0^*}^{t^*} e^{3\beta_2(w)}dw
\]
a bijective mapping of the physical interval $0 \leq \lambda \leq \lambda_{\text{max}}$ onto $t_0^* \leq t^* \leq T^*$. For this mapping to be physically legitimate and useful for the treatment of potentials $V(y, z)$ of interest entering (14) or, equivalently, (27)–(29) that

$$
\lambda_{\text{max}} = \infty.
$$

(49)

In conjunction with (19) we observe that the integrand in (18) can diverge only at $-e^{-T^*}$. This occurs either for $\lim_{t' \to T^*} x(t') = \infty$, irrespective of whether $T^*$ is finite or infinite, or for $T^* = \infty$ ($w = 0$). In both cases (19) holds.

We first consider the two possibilities regarding the values of $T^*$, i.e., finite or infinite, and in this respect state the following lemma for (13–17), which is in fact valid for a class of physically interesting potentials $V(y, z)$.

**Lemma 1.** If the potential $V = V(\beta_+(\lambda), \beta_-(\lambda))$ satisfies for all real $\beta_+(\lambda)$ and $\beta_-(\lambda)$ the condition

$$
V = V(\beta_+(\lambda), \beta_-(\lambda)) \geq M, \quad M \in \mathbb{R}
$$

(50)

and in the case $M < 0$ the arbitrary constant $\rho > 0$ given in (17) fulfills

$$
\rho < \frac{36 (x'_0)^2}{27e^{x_0} - 256M^3},
$$

(51)

$x_0$ and $x'_0$ having been introduced in (16), then $T^*$ and thereby the maximal interval of existence, $[t_0^*, T^*]$, of solution (14) are finite.

**Proof.** Since by assumption $V \geq M$, $M \in \mathbb{R}$, for all real $\beta_+(\lambda), \beta_-(\lambda)$, then due to (30) also $V(y, z) \geq M$, $M \in \mathbb{R}$, for all real $y$ and $z$, and we obtain the estimate:

$$
G(x) \geq (\dot{x}_0)^2 + \frac{3}{4}\rho(e^{3x} - e^{3x_0}) + 6\rho M \int_{x_0}^{x} e^{2w}dw
$$

$$
= (\dot{x}_0)^2 - \frac{3}{4}\rho e^{3x_0} - 3\rho M e^{2x_0} + \frac{3}{4}\rho e^{2x}(e^x + 4M),
$$

(52)

and we call the expression appearing in the last line $F(x)$. Now to further simplify the notation, let us call $F_x(x)$ the ‘value’ of $F(x)$ when the exponential in the fourth term multiplying the round bracket is $e^{2x_0}$. We distinguish the following cases:

**A)** $M + e^{x_0}/4 \geq 0$.

Then, since $x = x(t^*)$ is an increasing function of $t^*$, the inequality $M + e^x/4 > 0$ is valid and, owing to (24), for $x \geq x_0$

$$
F \geq F_{x_0} = (\dot{x}_0)^2 - \frac{3}{4}\rho e^{3x_0} + \frac{3}{4}\rho e^{2x_0}e^x > 0.
$$

(53)

Eqs. (52) and (53) yield for (16) the estimate

$$
0 < T^* - t_0^* = \int_{x_0}^{x_1} G^{-1/2} \leq \int_{x_0}^{x_1} F^{-1/2}
$$

$$
\leq \int_{x_0}^{x_1} F_{x_0}^{-1/2} = J_{x_0}(x_1).
$$

(54)

**B)** $M + e^{x_0}/4 < 0$ ($M < 0$),

Consider $f(x) = \frac{3}{4}\rho e^{2x}(4M + e^x)$.

B.) $-8M/3 \leq e^{x_0} < -4M$. Then $f(x)$ increases for $x \geq x_0$. Thus

$$
\min F = F(x_0) = (x'_0)^2 > 0.
$$

Therefore

$$
F(x) > 0, \quad x \geq x_0
$$

and due to (52) we obtain for (15)

$$
0 < T^* - t_0^* = \int_{x_0}^{x_1} G < \int_{x_0}^{x_1} F dx + J_1(x_1),
$$

(55)

where

$$
J_1(x_1) = J_{\ln(-4M)}(x_1).
$$

(56)

We note that by construction (55) as well as the estimates (54) and (55) hold for all $x_0, \dot{x}_0 > 0$, $y_0, \dot{y}_0 < 0$ and $z_0, \dot{z}_0 < 0$. Also the condition (23) is not particularly restrictive since it involves a relation between the arbitrary constants $\rho > 0$, $x_0$ and $\dot{x}_0 > 0$.

Now, by virtue of Lemma 1 and the discussion following (19), we observe that the remaining possibility for the validity of (19) is that $\lim_{t' \to T^*} x(t') = \infty$. Consequently the ensuing developments focus on the proof of this fact for all $x_0, \dot{x}_0 > 0$, $y_0, \dot{y}_0 < 0$ and $z_0, \dot{z}_0 < 0$.

To this end we use an immediate and important consequence of Lemma 1, namely that (19) at least one of $\lim_{t' \to T^*} |x(t')|$, $\lim_{t' \to T^*} |y(t')|$, $\lim_{t' \to T^*} |z(t')|$ is $\infty$. If for a given $V(y, z)$ the analytic structure of either one of both of the quantities $\partial V(y, z)/\partial y$ and $\partial V(y, z)/\partial z$ is such as to exclude in the context of (16) and (24) for all initial conditions $x_0, \dot{x}_0 > 0$, $y_0, \dot{y}_0 < 0$ and $z_0, \dot{z}_0 < 0$ the cases $y = -\infty$ or $z = -\infty$ or $y = -\infty$ and $z = -\infty$, then clearly in (16) and
$y_1 \in \mathbb{R}$ and $z_1 \in \mathbb{R}$, whence necessarily $\lim_{t^* \to T^*} x(t^*) = \infty$.

If, however, the previous cases cannot be as above a priori eliminated, upon taking into account the analytic form of some physically interesting potentials $V(y, z)$, we have the following

**Lemma 2.** Suppose that for the initial conditions $x_0, \dot{x}_0 > 0$, $y_0, \dot{y}_0 < 0$ and $z_0, \dot{z}_0 < 0$, to either

$\lim_{y \to -\infty} \frac{w}{y}$ or $y = -\infty$ and $z = -\infty$ appear for either $w \in [-\infty, y_m]$ or $w \in [-\infty, z_m]$, where $y_m \in \mathbb{R}^-$ and $z_m \in \mathbb{R}^-$, the derivatives $V_y, V_z$ in (17) and (18) are absolutely bounded, that is, for

$|V_y w'| \leq k_1$ and $|V_z w'| \leq k_2$. (58)

Then, for all initial conditions $x_0, \dot{x}_0 > 0$, $y_0, \dot{y}_0 < 0$ and $z_0, \dot{z}_0 < 0$ we have

$\lim_{t^* \to T^*} x(t^*) = \infty$. (60)

**Proof.** The proof is carried out by reductio ad absurdum. Assume firstly the existence of initial conditions $x_0, \dot{x}_0 > 0$, $y_0, \dot{y}_0 < 0$ and $z_0, \dot{z}_0 < 0$ so that concurrently

$w \to y_1 =$ \lim_{t^* \to T^*} x(t^*) \in \mathbb{R},$

$\lim_{t^* \to T^*} z(t^*) = z_1 \in \mathbb{R}$ and

$\lim_{t^* \to T^*} y(t^*) = y_1 = -\infty.$

We use (18) and (19) to obtain

$\dot{y} = -H^{1/2}$ and $\dot{t} = -H^{1/2}(-\infty)$. (61) (62)

Obviously now for $y_m < 0$, owing to (58) and since $x_0 \leq x(w) \leq x_1$ for $-\infty \leq w \leq y_0$, we obtain for the integral in (2)

$\int_{-\infty}^{y_0} e^{2x(w)V_y} \leq \int_{-\infty}^{y_0} e^{2x(w)V_y} dw$

$= \int_{-\infty}^{y_m} e^{2x(w)V_y} dw + \int_{y_m}^{y_0} e^{2x(w)V_y} dw$

$\leq \int_{y_m}^{y_0} e^{2x(w)V_y} dw + k_1 e^{2x(z_1)} \int_{-\infty}^{y_m} e^{-n} dw$. (63)

Due to the continuity of $V_y$ as a function of $y = y(t^*)$ and $z = z(t^*) = z(y)$ (owing to the monotonicity of $y(t^*)$), with $-\infty < y(t^*) \leq y_0$ and $z_1 \leq z(t^*) \leq y_0$, the integral

$\int_{y_m}^{y_0} e^{2x(w)V_y} dw$

exists. Consequently, the integral on the left-hand side of (33), by virtue of the estimate on the right-hand side of (33), also converges. Therefore (22) yields $\lim_{t^* \to T^*} y = -a_1, a_1 \in \mathbb{R}^+$, $a_1$ depending of course on $x_0, \dot{x}_0 > 0, y_0, \dot{y}_0 < 0, z_0, \dot{z}_0 < 0$. This relation implies that in some left neighbourhood of $T^*$ we have

$y = -a(t^*, x_0, \dot{x}_0 > 0, y_0, \dot{y}_0 < 0, z_0, \dot{z}_0 < 0) < 0,$

with $|a(t^*, x_0, \dot{x}_0 > 0, y_0, \dot{y}_0 < 0, z_0, \dot{z}_0 < 0)| < \infty$ and

$\lim_{t^* \to T^*} a(t^*) = a_1$, whereupon by integration we deduce that in fact $\lim_{t^* \to T^*} y(t^*)$ is finite.

We thus arrive at a contradiction with our hypothesis for the existence of appropriate $x_0, \dot{x}_0 > 0, y_0, \dot{y}_0 < 0$ and $z_0, \dot{z}_0 < 0$ such that $\lim_{w \to y_1} x(w) = \lim_{t^* \to T^*} x(t^*) = x_1 \in \mathbb{R}$, $\lim_{t^* \to T^*} y(t^*) = y_1$ and $\lim_{t^* \to T^*} z(t^*) = z_1 = -\infty$ or, owing to (58) and (59), such that $\lim_{w \to y_1} x(w) = \lim_{t^* \to T^*} x(t^*) = x_1 \in \mathbb{R}$, $\lim_{t^* \to T^*} y(t^*) = y_1 = -\infty$ and $\lim_{t^* \to T^*} z(t^*) = -\infty$.

These results, according to the remarks prior to Lemma 2, immediately imply (60). The proof of Lemma 2 is now complete and with this, the proofs of Theorems 1 and 2 are also complete.
are defined for $0 \leq \lambda < \infty$ and that they are monotonically increasing and decreasing functions of $\lambda$, respectively, fulfilling the initial conditions

$$
\mu_0 = \frac{1}{2}(x_0 + t_0^*), \quad \mu'_0 = \frac{1}{2}(x_0 + 1) \exp[t_0^* - \frac{3}{2}(x_0 + t_0^*)] > 0,
$$

$$
\beta_{+,0} = y_0, \quad \beta'_{+,0} = y_0 \exp[t_0^* - \frac{3}{2}(x_0 + t_0^*)] < 0, \quad \beta_0(0) = z_0, \quad \beta'_{-,0} = z_0 \exp[t_0^* - \frac{3}{2}(x_0 + t_0^*)] < 0,
$$

whereby, also by virtue of (66), (67) and (68),

$$
\lim_{\lambda \to \infty} \mu(\lambda) = \infty, \quad \mu_0 \leq \mu(\lambda) < \infty,
$$

$$
T^* - t_0^* = \int_{x_0}^{x} G^{-1/2}, \quad \lim_{t^* \to t^*} x(t^*) = x_1 = \infty;
$$

$$
\lim_{\lambda \to \infty} \beta_+(\lambda) = y_1, \quad y_1 < \beta_+ \leq \beta_0,
$$

$$
\lim_{\lambda \to \infty} \lim_{t^* \to t^*} y(T^*) = y_1;
$$

$$
\lim_{\lambda \to \infty} \lim_{t^* \to t^*} z(T^*) = z_1.
$$

We stress that (61) – (62) remain valid for all $x_0, \dot{x}_0 > 0, \dot{y}_0, \dot{y}_0 < 0$ and $z_0, \dot{z}_0 < 0$.

5. Isotropization theorem for quadratic cosmologies

In view of the results of the previous section, it is now possible to prove an isotropization theorem for those cosmologies that have potentials complying with the assumptions of Theorems 1 and 2. We consider the scalar shear parameter $\Sigma$ and show that it becomes vanishingly small in the expanding direction. We have

**Theorem 3.** The validity of Theorems 1 and 2 implies that the shear fulfills

$$
\lim_{\lambda \to \infty} \Sigma^2(\lambda) = \lim_{\lambda \to \infty} \left[ (\beta'_{+}/\mu')^2 + (\beta'_{-}/\mu')^2 \right] = 0. \quad (73)
$$

**Proof.** We first observe that due to (64), (59) and (60)

$$
\beta'_{+}/\mu' = 2\dot{y}/(\dot{x} + 1), \quad (74)
$$

$$
\beta'_{-}/\mu' = 2\dot{z}/(\dot{x} + 1), \quad (75)
$$

with (cf. (38), (43))

$$
\dot{x} = G^{1/2}. \quad (76)
$$

On recalling that $\lim_{w \to y_1} e^{2x(w)} = \infty$, we consider (61) and observe that the following possibilities exist:

1) The integral in (61) for $w \to y_1$ diverges, in which case (cf. (58) for the case $y_1 = -\infty$ or $z_1 = -\infty$ or $y_1 = -\infty$ and $z_1 = -\infty$) at the most

$$
\lim_{t^* \to t^*} \dot{y} \sim -\lim_{t^* \to t^*} e^{2x(t^*)}. \quad (77)
$$

2) The integral in (61) for $w \to y_1$ converges and thus

$$
\lim_{t^* \to T} \dot{y} = -a, \quad a \in \mathbb{R}^+ . \quad (78)
$$

Let it be pointed out that by using standard criteria for the convergence of integrals one can easily construct examples for possibilities 1) and 2) above. Cases (77) and (78), which are of course compatible with (58) (i.e., (44)), are the ones possible for the behaviour of $\lim_{t^* \to T} \dot{y}$, in view of $y(t^*)$ being monotonically decreasing for $t^* \in (t_0^*, T^*)$, with (73) corresponding to $\beta'_{+}/\mu' = 0$, $\beta'_{-}/\mu' < 0$.

Equivalently from (76) we deduce that, owing to $V(y, z) \geq M$, $M \in \mathbb{R}$, at the least

$$
\lim_{t^* \to T} \dot{x} \sim \lim_{t^* \to T} e^{2x(t^*)/2}. \quad (79)
$$

Therefore (77) – (79) yield for (74)

$$
\lim_{\lambda \to \infty} \beta'_{+}/\mu' = 0. \quad (80)
$$

Along the lines that led to (80) it is verified that (73) satisfies

$$
\lim_{\lambda \to \infty} \beta'_{-}/\mu' = 0. \quad (81)
$$

Eqs. (80) and (81) establish the correctness of (73). \(\square\)

6. Ever-expanding quadratic Bianchi IX cosmologies

In this section, we apply Theorems 1, 2 and 3 above to some anisotropic quadratic cosmologies and show as a corollary of the general theory developed in the previous sections that there exist ever-expanding Bianchi IX universes in the purely $R^2$ theory, which may also be taken to be regular initially in sharp contrast with the situation in general relativity.

As a warm-up exercise, let us consider first the easier cases of models of types I, II and V.

**Case A:** $V = 0$ (Bianchi I, $M = 0$).

Theorems 1 and 3 trivially hold but Theorem 2 is devoid of any meaning in the present simple case since, by setting $t_0^* = 0$ in all the relevant formulae, without loss of generality we obtain

$$
\lim_{t^* \to T} x(t^*) = \infty,
$$

$$
T^* = -\frac{\ln S}{3\sqrt{C_1}}, \quad S = \frac{\sqrt{A} - \sqrt{C_1}}{\sqrt{A} + \sqrt{C_1}}.
$$
where \( A = C_1 + \frac{3}{4} \rho e^{3 \lambda_0} \) and \( S \in (0, 1) \). The case when \( C_1 \) in (82) is not positive leads identically to the results of the case \( C_1 > 0 \). It is probably interesting to point out that the case \( V(y, z) = C = \text{const} \), which is of no physical importance but fulfills the conditions for the validity of Theorem 1, yields after a lengthy albeit exact calculation by means of elliptic integrals the exact finite value of \( T^* \) with \( \lim_{t^* \to T^*} x(t^*) = \infty \) and, due to (29) and (31), \( \lim_{t^* \to T^*} y(t^*) = y_1 \in \mathbb{R} \) and \( \lim_{t^* \to T^*} z(t^*) = z_1 \in \mathbb{R} \).

**Case B:** \( V = \exp[a (\beta_+ (\lambda) + \sqrt{3} \beta_- (\lambda))] \) (Bianchi II).

Eq. (21) holds with \( M = 0 \), while (24) and (27) are satisfied due to the analytic form of \( V \) and to the fact that \( \beta_+ (\lambda) \) and \( \beta_- (\lambda) \) are monotonically decreasing functions of \( \lambda \). Consequently Theorems 1 to 3 are valid. Note here that the possibility \( \lim_{\lambda \to \infty} \beta_+ (\lambda) = -\infty \) or \( \lim_{\lambda \to \infty} \beta_- (\lambda) = -\infty \) or both is not excluded.

**Case C:** \( V = e^{4 \beta_+ (\lambda)} \) (Bianchi V).

It is evident that here Theorems 1 to 3 also hold and the possibility \( \lim_{\lambda \to \infty} \beta_+ (\lambda) = -\infty \) may appear.

**Case D:** (Bianchi IX).

We now move on to the Bianchi IX case which from the point of view of dynamics is the most interesting. The usual Bianchi IX potential is

\[
V = \frac{1}{2} e^{4 \beta_+ (\lambda)} \left[ \cosh \left( 4 \sqrt{3} \beta_- (\lambda) \right) - 1 \right] + \frac{1}{4} e^{-8 \beta_+ (\lambda)} - e^{-2 \beta_+ (\lambda)} \cosh \left( 2 \sqrt{3} \beta_- (\lambda) \right). 
\]

Now, on introducing the variables \( y(t^*) \) and \( z(t^*) \), by virtue of (21) we obtain for the Bianchi IX potential

\[
\frac{4 V'(y, z)}{3} + 1 = V'(y, z),
\]

where \( V'(y, z) \) is the positive-definite potential used in (10). Therefore \( M = -3/4 \), and Theorem 1 retains its validity for the Bianchi IX potential. To decide whether Theorems 2 and 3 are valid, we need

\[
V_y = 2 e^{4 y} \left[ \cosh \left( 4 \sqrt{3} z \right) - 1 \right] - 2 e^{-8 y} \\
+ 2 e^{-2 y} \cosh(2 \sqrt{3} z),
\]

\[
V_z = 2 \sqrt{3} \left[ e^{4 y} \sinh(4 \sqrt{3} z) - e^{-2 y} \sinh(2 \sqrt{3} z) \right].
\]

\[
(85)
\]

In the ensuing developments we consider all the possibilities. Unless otherwise stated, all limits are taken in the direction \( y \to -\infty, z \to -\infty \).

\( a) \) We assume that there exist initial conditions \( x_0, y_0 > 0, y_0, y_0 < 0, z_0, \tilde{z}_0 < 0 \) such that

\[
\lim_{t^* \to T^*} x(t^*) = x_1 \in \mathbb{R},
\]

\[
\lim_{t^* \to T^*} y(t^*) = -\infty, \quad \text{and}
\]

\[
\lim_{t^* \to T^*} z(t^*) = z_1 \in \mathbb{R}.
\]

Then from (84) we obtain

\[
\lim_{y \to y_1} V_y = \lim_{z \to z_1} V_z = -\infty,
\]

\[
\lim_{y \to y_1} V_z = -\infty, \quad \text{and}
\]

\[
\lim_{z \to z_1} z(t^*) = z_1 \in \mathbb{R}.
\]

\( b) \) Suppose now that for appropriate \( x_0, \dot{x}_0 > 0, y_0, y_0 < 0, z_0, \tilde{z}_0 < 0 \), we have \( \lim_{t^* \to T^*} x(t^*) = x_1 \in \mathbb{R}, \lim_{t^* \to T^*} y(t^*) = y_1 \in \mathbb{R} \). Then (85) yields

\[
\lim_{y \to y_1} V_y = \lim_{z \to z_1} V_z = \infty,
\]

\[
\lim_{y \to y_1} V_z = -\infty. \quad \text{(87)}
\]

As in possibility \( a) \) above, due to the condition (88), (89) ceases to hold for the Bianchi IX potential. Moreover, since from (11) and (33),

\[
\hat{z} = -K^{1/2},
\]

\[
\lim_{t^* \to T^*} \hat{z} = -K^{1/2} (\infty), \quad \text{(89)}
\]

we are again led to (89), so that the square root in (10) becomes complex. Thus this case is also excluded.

\( c) \) Finally, let for some specific initial conditions, \( x_0, \dot{x}_0 > 0, y_0, y_0 < 0, z_0, \tilde{z}_0 < 0 \), the result

\[
\lim_{t^* \to T^*} (x, y, z) = (x_1, -\infty, -\infty),
\]

be valid. From (85) we deduce

\[
\lim V_y = \lim \left( e^{-4 |y| + 4 \sqrt{3} |z|} + e^{2 |y| + 2 \sqrt{3} |z|} - 2 e^{8 |y|} \right),
\]

\[
\lim V_z = \lim \sqrt{3} \left( -e^{-4 |y| + 4 \sqrt{3} |z|} + e^{2 |y| + 2 \sqrt{3} |z|} \right). \quad \text{(91)}
\]

In the following we investigate how (21) and (24) impinge on the validity of (88) and (89), and, consequently, on the validity of Theorems 2 and 3.
There are various possibilities to be considered. Firstly, if for certain initial conditions $x_0, \dot{x}_0 > 0, y_0, \dot{y}_0 < 0$ and $z_0, \dot{z}_0 < 0$, we have

$$\lim(|z|/|y|) < \sqrt{3},$$

then (91) and (92) yield

$$\lim V_y = \lim(-\exp[8|y|]) = -\infty$$

and

$$\lim V_z = \lim \exp[2|y| + 2\sqrt{3}|z|] = \infty,$$

respectively. Eqs. (93) and (94) demonstrate that (58) and (59) are not fulfilled. In addition, by virtue of (93) in (29), the integral tends to $-\infty$, which renders the square root complex. Therefore this case cannot occur.

Secondly, suppose that for a set of initial conditions, $x_0, \dot{x}_0 > 0, y_0, \dot{y}_0 < 0$ and $z_0, \dot{z}_0 < 0$, the relation

$$\lim(|z|/|y|) > \sqrt{3}$$

holds. From (91) and (92) we obtain

$$\lim V_y = \lim \exp \left[8|y| + 12|y| \left(\frac{|z|}{\sqrt{3}|y|} - 1\right)\right] = \infty$$

and

$$\lim V_z = \lim \left[-\exp \left(2|y| + 2\sqrt{3}|z|\right)\right] = -\infty,$$

respectively. Proceeding as in i) above, we conclude that this possibility too is excluded. Lastly, if there exist initial conditions, $x_0, \dot{x}_0 > 0, y_0, \dot{y}_0 < 0$ and $z_0, \dot{z}_0 < 0$, generating

$$\lim(|z|/|y|) = \sqrt{3},$$

(91) and (92) show that

$$\lim V_y = 0, \quad \lim V_z = 0.$$

From (14), (28) and (29) we infer by integration in the vicinity of $T^*$, as in the proof of Lemma 2, that $\lim_{t^* \to T^*} y(t^*)$ and $\lim_{t^* \to T^*} z(t^*)$ are finite, thus arriving at a contradiction with the supposition made at the outset of the present case c).

Cases a) – c) above immediately show that for the Bianchi IX potential and for all initial conditions, $x_0, \dot{x}_0 > 0, y_0, \dot{y}_0 < 0$ and $z_0, \dot{z}_0 < 0$, we have

$$\lim_{t^* \to T^*} x(t^*) = \infty, \lim_{t^* \to T^*} y(t^*) = y_1 \in \mathbb{R} \quad \text{and} \quad \lim_{t^* \to T^*} z(t^*) = z_1 \in \mathbb{R}.$$ We thus arrive at (60) without having to resort to Lemma 2, and precisely such a possibility was announced prior to Lemma 2.

It is straightforward by following the proof of Theorem 3, since now $V_y$ and $V_z$ are bounded for $y \in [y_0, \dot{y}_0], z \in [z_0, \dot{z}_0]$, to conclude that it holds as (34) is based essentially on $\lim_{w \to y_0} y(w) = \lim_{w \to z_1} x(w) = \lim_{w \to T^*} x(t^*) = \infty$. We thus have

**Corollary 1.** For the potentials considered in Theorems 1 to 3 we obtain for the volume $V_L(\lambda) = e^{2\mu(\lambda)}$

by using (14) that $V_L(\lambda)$ is a monotonically increasing function of $\lambda$, $0 \leq \lambda < \infty$ and, owing to (24) and (58), fulfills

$$V_L(\lambda = 0) = \exp[2\mu(\lambda = 0)] = \exp(x_0 + t^*_0),$$

$$\lim_{\lambda \to \infty} V_L(\lambda) = \lim_{\lambda \to \infty} e^{2\mu(\lambda)} = \infty.$$  

**Remark.** Had we chosen in (39) the minus sign, we would not have been able to prove the basic Lemma 1. A respective calculation has been carried out and it does not lead to any sensible results. Further, the plus sign in either (14) or (11) or in both would generate increasing functions $y(t^*)$ or (and) $z(t^*)$, thus preventing us from formulating (58) or (59) or both in view of the analytic form of the Bianchi potentials. Finally, it is plausible to use $x(t^*)$ as the independent variable in (14) since it is actually a time variable according to (13).

7. Lie point symmetries and Painlevé analysis of some Bianchi models

To round off our discussion it is instructive to consider the question of algebraic integrability of the models considered previously. This is certainly far from being of purely academic interest since one needs to know if the known solutions occupy a large part of the phase space or whether there exist regions where non-integrability manifests itself in a non-trivial way.

In such a case usually one has the phenomenon of the formation of singular envelopes (cf. [1]) whereas any non-integrable regions are enveloped by the known solutions.

One method to determine the integrability of a system is by a performance of the so-called *singularity, or Painlevé, analysis* in an effort to examine whether or not there exists a Laurent expansion of the solution about a movable pole which contains the number of arbitrary constants necessary for a general solution. Any other singularities are not permitted except in the case of branch point singularities which give rise to what is called the weak Painlevé property. A system which is integrable in the sense of Painlevé has its general solution analytic except at the pole-like singularity (for more details and a nice introduction to the singularity analysis of dynamical systems we refer the reader to [13], ch. 8).

It is interesting that even for these quadratic models where, as we saw earlier, the global dynamics can be very different from the corresponding one in general relativity, integrability is not a common feature in these examples except for trivial cases.
We first examine the system (101) for Lie point symmetries. For the computation we use Program LIE due to Head [8, 12].

**Case A:** $V(y, z) = 0$ (Bianchi I), $M = 0$.

The equations are particularly simple, being

\[
\ddot{x} + \alpha_1 e^{3x} = 0, \\
\ddot{y} = 0, \\
\ddot{z} = 0.
\]

(99)

There are nine Lie point symmetries: $\partial_t, \partial_y, \partial_z, t\partial_y, t\partial_z, y\partial_y, z\partial_y, z\partial_z$. We observe that most of the symmetry is due to the second and third members of (99), both of which are trivially integrable. The first has the solution

\[
x(t) = -\frac{2}{3} \log \left\{ (\alpha_1/1)^{1/2} \sinh \left[ \frac{3}{2} (21)^{1/2} (t - t_0) \right] \right\},
\]

where $I$ and $t_0$ are constants of integration.

**Case B:** $V = \exp(4(y + \sqrt{3}z))$, (Bianchi II)

The system is now

\[
\ddot{x} - 9Ke^{3x} - 24K e^{2x+4(y+\sqrt{3}z)} = 0, \\
\ddot{y} + 12K e^{2x+4(y+\sqrt{3}z)} = 0, \\
\ddot{z} + 12\sqrt{3}Ke^{2x+4(y+\sqrt{3}z)} = 0.
\]

(100)

Upon introduction of the new variables

\[
v = y + \sqrt{3}z, \\
w = -\sqrt{3}y + z
\]

(101)

the system (100) takes the simpler form

\[
\ddot{x} - 9Ke^{3x} - 24K e^{2x+4v} = 0, \\
\ddot{v} + 48Ke^{2x+4v} = 0, \\
\ddot{w} = 0.
\]

(102)

The parameter $K = 8\rho$, where $\rho$ was introduced in (17). The Lie point symmetries of (102) are $\partial_t, \partial_w, t\partial_w, w\partial_v$.

We observe a considerable reduction in the number of Lie point symmetries compared with Case A. The solution of the third of (102) is trivial, but of the first two not at all obvious. There is a first integral

\[
J = \frac{1}{2} (x^2 - v^2) - 3Ke^{3x} - 12K e^{2x+4v}.
\]

(103)

Eqs. (102a) and (102b) are not in a suitable form for applying the Painlevé test. Under the transformation

\[
r(T) = 27Ke^{4(2x+v)}, \\
s(T) = 27Ke^{3x}, \\
T = \frac{2}{3} it
\]

(104)

(the constants are chosen for later numerical convenience and do not affect the essence of the analysis) we obtain the system

\[
2r - 4r^2 + 6r^3s = 0, \\
s\ddot{s} - s^2 - s^3 + 6r = 0.
\]

(105)
theories that contain an Einstein term. This is due to the conformal equivalence of such theories to general relativity with a scalar field matter source and this case is known to have a monotonic approach to the initial Bianchi IX singularity. Here, however, we see the interesting result that in the purely quadratic theory (that is, without the Einstein term) some sort of non-integrability returns. It remains to be seen if this lack of integrability of the Bianchi IX spacetime found here is related to a possible chaotic behaviour in the spacetime geometry similar to that which occurs in general relativity. If true, such a result would imply, for instance, that the pure $R^2$ theory is in a sense closer to general relativity than other quadratic extensions of it containing the linear Einstein term.

8. Conclusion

The behaviour of vacuum quadratic cosmologies in the expanding direction through the Hamiltonian approach advanced here can be generalized to include matter fields, and this could be the next problem to tackle in this respect. How does the inclusion of a scalar field or a perfect fluid affect the isotropization theorem proved in Sec. 5? Is the non-recollapse of the Bianchi IX model in quadratic gravity stable to matter inclusion or to perturbations of the purely quadratic scalar curvature Lagrangian? In general gelativity all homogeneous, vacuum or matter-filled, closed cosmologies satisfying the usual energy conditions recollapse (cf. [10, 11]). We have shown that the purely quadratic diagonal Bianchi IX models are ever-expanding (i.e., volume-increasing) and so they provide a counterexample to a possible closed universe recollapse conjecture in higher-order gravity theories.

Another question which can be of interest is to analyse these systems in terms of the expansion-normalized variables of Wainwright (cf. [14]) and compare the results with those arrived at here. This would nicely complement the isotropization theorem proved in this paper about the scalar shear variable.

Acknowledgement

We thank Dr. A. Lyberopoulos for drawing our attention to Ref. [9] and many useful discussions. PGLL thanks the National Research Foundation of South Africa and the University of Natal for their continuing support.

References

[1] J.D. Barrow and S. Cotsakis, *Phys Lett B* 214, 515–518 (1988).
[2] J.D. Barrow and S. Cotsakis, *Phys Lett B* 232, 172 (1989).
[3] G.V. Bicknell, *J Phys A* 7, 1061 (1974).
[4] S. Cotsakis and P.G.L. Leach, *J Phys A* 27, 1625 (1994).
[5] J. Demaret and L. Querella, *Class Quant Grav* 12, 3085–3101 (1995).
[6] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series and Products* Alan Jeffrey ed (Academic Press, New York, 1980).
[7] S.W. Hawking and R. Penrose, *The Nature of Space and Time*, (Princeton University Press, Princeton 1996).
[8] A.K. Head, *Comp Phys Commun* 77, 241-248 (1993).
[9] M.W. Hirsch and S. Smale, *Differential Equations, Dynamical Systems, and Linear Algebra* (Academic Press, San Diego, 1974).
[10] X. Lin and R.M. Wald, *Phys Rev D* 40, 3280–3286 (1989).
[11] X. Lin and R.M. Wald, *Phys Rev D* 4, 2444–2448 (1990).
[12] J. Sherring, A.K. Head and G.E. Prince, *Math Comp Model* 25, 153-164 (1997).
[13] M. Tabor, *Chaos and Integrability in Nonlinear Dynamics: An Introduction* (Academic Press, 1989).
[14] J. Wainwright and G.F.R. Ellis, *Dynamical Systems in Cosmology* (Cambridge University Press, Cambridge, 1997).