The Eremenko–Lyubich constant

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Abstract
Eremenko and Lyubich proved that an entire function $f$ whose set of singular values is bounded is expanding at points where $|f(z)|$ is large. These expansion properties have been at the centre of the subsequent study of this class of functions, now called the Eremenko–Lyubich class. We improve the estimate of Eremenko and Lyubich, and show that the new estimate is asymptotically optimal. As a corollary, we obtain an elementary proof that functions in the Eremenko–Lyubich class have lower order at least $1/2$.

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1 INTRODUCTION

Eremenko and Lyubich [5] introduced the class $B$ of transcendental entire functions whose set of critical and asymptotic values is bounded. The dynamics of functions in this class, now called the Eremenko–Lyubich class, has been studied intensively in recent years; see, for example, [2, 3, 9, 10, 12, 13] and the survey [14]. The key property of functions in this class is that they satisfy strong expansion properties near infinity. More precisely, let $R > 0$ be so large that $|f(0)| \leq R$ and such that all critical and asymptotic values of $f$ have modulus at most $R$. Then there is a holomorphic function

$$F \colon \mathcal{T} \to \mathbb{H}_{\ln R} := \{x + iy : x > \ln R\}$$

with $\exp \circ F = f \circ \exp$, where

$$\mathcal{T} = \{\zeta \in \mathbb{C} : |f(\exp(\zeta))| > R\} = \{\zeta \in \mathbb{C} : \Re F(\zeta) > \ln R\}.$$
The map $F$ is a conformal isomorphism when restricted to any connected component of $\mathcal{T}$; these components are also called the (logarithmic) tracts of $F$.

Eremenko and Lyubich [5, Lemma 1] use the Koebe quarter theorem to show that

$$|F'(\zeta)| \geq \frac{\text{Re} F(\zeta) - \ln R}{4\pi}$$

(1.1)

for all $\zeta \in \mathcal{T}$; in other words,

$$|f'(z)| \geq \frac{(\ln |f(z)| - \ln R) \cdot |f(z)|}{4\pi |z|}$$

(1.2)

whenever $|f(z)| > R$.

The inequality (1.1) has played a fundamental role in the study of the class $\mathcal{B}$. It therefore seems natural to ask whether the above estimates can be improved. The purpose of this note is to establish the optimal constant in (1.1) and (1.2).

**Theorem 1.1** (Optimal expansion estimates). Let $f$ and $F$ be as above. Then

$$|F'(\zeta)| \geq \frac{\text{Re} F(\zeta) - \ln R}{2}$$

(1.3)

for all $\zeta \in \mathcal{T}$, and hence

$$|f'(z)| \geq \frac{(\ln |f(z)| - \ln R) \cdot |f(z)|}{2|z|}$$

(1.4)

whenever $|f(z)| > R$.

Moreover, the constant 2 is optimal. Indeed, for $f(z) = \cos \sqrt{z}$,

$$\lim_{r \to +\infty} \frac{|f'(-r)| \cdot r}{(\ln |f(-r)|) \cdot |f(-r)|} = \frac{1}{2}.$$  

(1.5)

Every component of $\mathcal{V} := \exp(\mathcal{T})$ is simply connected, and $f$ is bounded on $\partial \mathcal{V}$. By an old result of Wiman [15], it follows that $f$ must have order $\rho(f) \geq 1/2$, where

$$\rho(f) = \limsup_{r \to \infty} \frac{\ln \ln M(r, f)}{\ln r}.$$  

Here $M(r, f) = \max_{|z|=r} |f(z)|$. That $\rho(f) \geq 1/2$ also follows from the famous $\cos \pi \rho$ theorem, proved later by Wiman [16]. See also [6, p. 119], and compare [8, p. 1788] and the proof of [1, Corollary 2]. In fact, $f$ must even have lower order at least $1/2$, that is,

$$\liminf_{r \to \infty} \frac{\ln \ln M(r, f)}{\ln r} \geq \frac{1}{2}.$$  

(1.6)

According to Heins [7], this result is also due to Wiman; see [11, Lemma 3.5], where (1.6) is proved using a version of the Ahlfors distortion theorem. As an application of Theorem 1.1, we obtain a new elementary proof of (1.6).
Corollary 1.2 (Lower order of functions in $B$). Let $f \in B$. Then $f$ has lower order at least $1/2$. More precisely, there are constants $c > 0$ and $r_0 > 1$ such that

$$\ln M(r, f) \geq c \cdot \sqrt{r}$$

for $r \geq r_0$.

2 PROOF OF THE ESTIMATE

Lemma 2.1 (Expansion in logarithmic coordinates). Suppose that $T \subset \mathbb{C}$ is a simply-connected domain such that $\exp |_T$ is injective, let $\rho \in \mathbb{R}$ and suppose that $\varphi : T \to \mathbb{H}_\rho$ is a conformal isomorphism. Then

$$|\varphi'(\zeta)| \geq \frac{\text{Re} \varphi(\zeta) - \rho}{2}$$

for all $\zeta \in T$.

Proof. Post-composing with a translation, we may assume that $\rho = 0$. Let $\mathbb{D}$ denote the unit disc, and consider the Möbius transformation

$$M : \mathbb{D} \to \mathbb{H}_0; \quad z \mapsto \frac{(1 - z)}{(1 + z)}.$$

Let $\zeta \in T$. The map

$$g : \mathbb{D} \to \exp(T - \zeta); \quad z \mapsto \exp\left(\varphi^{-1}(\text{Re}(\varphi(\zeta)) \cdot M(z) + i \cdot \text{Im}(\varphi(\zeta) - \zeta)\right)$$

is a conformal isomorphism with $g(0) = 1$. Since $0 \notin \exp(T - \zeta)$, the Koebe quarter theorem implies

$$4 \geq |g'(0)| = \frac{\text{Re}(\varphi(\zeta)) \cdot |M'(0)|}{|\varphi'(\zeta)|} = \frac{2 \text{Re}(\varphi(\zeta))}{|\varphi'(\zeta)|}.$$

So $|\varphi'(\zeta)| \geq \text{Re} \varphi(\zeta)/2$, as claimed. \qed

Proof of Theorem 1.1. If $T$ is a connected component of $\mathcal{T}$, then $T$ and $\varphi := F|_T$ satisfy the hypotheses of Lemma 2.1. This proves (1.3). To deduce (1.4) from (1.3), differentiate the functional equation $\exp \circ F = f \circ \exp$.

For $f(z) = \cos \sqrt{z}$ and $r > 0$, we have

$$|f'(-r)| = \frac{|\sin i \sqrt{r}|}{2 \sqrt{r}}.$$

As $r \to \infty$,

$$|\sin i \sqrt{r}| = \frac{e^{\sqrt{r}}}{2} + O(e^{-\sqrt{r}}) = |f(-r)| + O(e^{-\sqrt{r}}).$$
Moreover,

\[ \ln |f(-r)| = \sqrt{r} - \ln 2 + O(1/|f(-r)|^2). \]

Hence

\[ |f'(-r)| - \frac{(\ln |f(-r)| + \ln 2) \cdot |f(-r)|}{2r} = O(e^{-\sqrt{r}/\sqrt{r}}) \]

as \( r \to \infty \). We may rewrite this as

\[ \frac{|f'(-r)| \cdot r}{(\ln |f(-r)|) \cdot |f(-r)|} - \frac{1}{2} = \frac{\ln 2}{2 \ln |f(-r)|} + O\left( \frac{e^{-\sqrt{r}} \cdot r}{(\ln |f(-r)|) \cdot |f(-r)| \cdot \sqrt{r}} \right) = O(r^{-\frac{1}{2}}), \]

which proves (1.5).

For future reference, we also record the following reformulation of Lemma 2.1 in terms of the hyperbolic density of \( T \); see, for example, [4] for background on hyperbolic geometry.

**Corollary 2.2** (Hyperbolic density of logarithmic tracts). Suppose that \( T \subset \mathbb{C} \) is a simply connected domain such that \( \exp|_T \) is injective, and let \( \rho_T \) denote the density of the hyperbolic metric of \( T \). Then

\[ \rho_T(\zeta) \geqslant \frac{1}{2} \quad \text{for all } \zeta \in T. \quad (2.1) \]

**Proof.** Let \( \varphi: T \to \mathbb{H}_0 \) be a conformal isomorphism. The hyperbolic density of \( \mathbb{H}_0 \) is given by \( \rho_{\mathbb{H}_0}(\omega) = 1 / \text{Re} \omega \). The hyperbolic density \( \rho_T \) satisfies

\[ \rho_T(\zeta) = \rho_{\mathbb{H}_0}(\omega) \cdot |\varphi'(\zeta)| = \frac{|\varphi'(\zeta)|}{\text{Re} \varphi(\zeta)} \geqslant \frac{1}{2} \]

by Lemma 2.1.

The following implies Corollary 1.2.

**Corollary 2.3** (Lower order in logarithmic coordinates). Suppose that \( T \subset \mathbb{C} \) is a simply-connected domain such that \( \exp|_T \) is injective, let \( \rho \in \mathbb{R} \) and suppose that \( \varphi: T \to \mathbb{H}_\rho \) is a conformal isomorphism with \( \varphi^{-1}(t) \to \infty \) as \( t \to +\infty, \ t > 0 \). Then there are constants \( \gamma > 0 \) and \( \sigma_0 > 0 \) such that

\[ \sup_{\text{Re} \zeta = \sigma} \ln \text{Re} \varphi(\zeta) \geqslant \frac{\sigma}{2} - \gamma \]

for \( \sigma \geqslant \sigma_0 \).
\textbf{Proof.} Without loss of generality, we may suppose that \( \rho = 0 \). Set \( \zeta_0 := \varphi^{-1}(1) \) and \( \gamma := \text{Re} \zeta_0/2 \) and \( \sigma_0 := \text{Re} \zeta_0 \). Let \( \sigma > \sigma_0 \); by the assumption on \( \varphi \) and the intermediate value theorem, there is \( t > 1 \) such that \( \text{Re} \varphi^{-1}(t) = \sigma \). Set \( \zeta := \varphi^{-1}(t) \). By Lemma 2.1,

\[
|\zeta - \zeta_0| \leq \int_{1}^{t} |(\varphi^{-1})'(x)| \, dx \leq \int_{1}^{t} \frac{2 \, dx}{x} = 2 \ln t = 2 \ln \text{Re} \varphi(\zeta).
\]

So

\[
\ln \text{Re} \varphi(\zeta) \geq \frac{|\zeta - \zeta_0|}{2} = \frac{\sigma}{2} - \gamma.
\]

\[ \square \]

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