ON DIVISIBILITY OF SUMS OF APÉRY POLYNOMIALS

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Abstract. For any positive integers $m$ and $\alpha$, we prove that
\[ \sum_{k=0}^{n-1} \epsilon^k (2k+1) A_k^{(\alpha)}(x)^m \equiv 0 \pmod{n}, \]
where $\epsilon \in \{1, -1\}$ and
\[ A_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k}^{\alpha} \left( \binom{n+k}{k} \right)^{\alpha} x^k. \]

1. Introduction

The Apéry number $A_n$ is defined by
\[ A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \left( \binom{n+k}{k} \right)^2. \]
Those numbers play an important role in Apéry’s ingredient proof \cite{4} of the irrationality of $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$. In 2000, Ahlgren and Ono \cite{1} solved a conjecture of Beukers \cite{2} and showed that for odd prime $p$,
\[ A_{(p-1)/2} \equiv a(p) \pmod{p^2}, \]
where $a(n)$ is the Fourier coefficient of $q^n$ in the modular form $\eta(2z)^4 \eta(4z)^4$.

Recently, Sun \cite{7} defined the Apéry polynomial as
\[ A_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \left( \binom{n+k}{k} \right)^2 x^k, \]
and proved several new congruences for the sums of $A_n(x)$. For example,
\[ \sum_{k=0}^{n-1} (2k+1) A_k(x) \equiv 0 \pmod{n}. \]

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for every positive integer $n$. In fact, he showed that
\[
\frac{1}{n} \sum_{k=0}^{n-1} (2k + 1)A_k(x) = \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n+k}{k} \binom{2k}{k} x^k.
\]
Furthermore, Sun proposed the following conjecture.

**Conjecture 1.1.** For $m \in \{1, 2, 3, \ldots\}$,
\[
\sum_{k=0}^{n-1} \epsilon^k (2k + 1)A_k(x)^m \equiv 0 \pmod{n},
\]
where $\epsilon \in \{1, -1\}$.

In [3], Guo and Zeng proved that
\[
\frac{1}{n} \sum_{k=0}^{n-1} (-1)^k (2k+1)A_k(x) = (-1)^{n-1} \sum_{k=0}^{n-1} \binom{2k}{k} x^k \sum_{j=0}^{k} \binom{k}{j} \binom{k+j}{j} \binom{n-1}{k+j} \binom{n+k+j}{k+j}.
\]
On the other hand, in [6], Sun also define the central Delannoy polynomial
\[
D_n(x) = \sum_{k=0}^{n-1} \binom{n}{k} \binom{n+k}{k} x^k.
\]
He showed that
\[
\frac{1}{n} \sum_{k=0}^{n-1} (2k + 1)D_k(x) = \sum_{k=0}^{n-1} \binom{n}{k+1} \binom{n+k}{k} x^k.
\]
Sun also conjectured that
\[
\frac{1}{n} \sum_{k=0}^{n-1} (2k + 1)D_k(x)^m
\]
is always an integer.

In fact, motivated by [5] and [11 eq. (1.7)], we may define the generalized Apéry polynomial
\[
A_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} ^{\alpha} \binom{n+k}{k} ^{\alpha} x^k,
\]
where $\alpha$ is a positive integer. (In [3], Guo and Zeng called such polynomial as the Schmidt polynomial.) In the same paper, Guo and Zeng also proved that fact all $\alpha \geq 1$, there exist explicit formulas for
\[
\frac{1}{n} \sum_{k=0}^{n-1} (2k + 1)A_k^{(\alpha)}(x) \quad \text{and} \quad \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k (2k + 1)A_k^{(\alpha)}(x).
\]
However, no explicit formula is known for
\[
\frac{1}{n} \sum_{k=0}^{n-1} (2k + 1)A^\alpha_k(x)^m \quad \text{and} \quad \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k(2k + 1)A^\alpha_k(x)^m
\]
when \(m \geq 2\).

In this paper, we shall prove

**Theorem 1.1.** For any positive integers \(m\) and \(\alpha\),
\[
\sum_{k=0}^{n-1} (2k + 1)A^\alpha_k(x)^m \equiv 0 \pmod{n}, \quad (1.2)
\]
and
\[
\sum_{k=0}^{n-1} (-1)^k(2k + 1)A^\alpha_k(x)^m \equiv 0 \pmod{n}. \quad (1.3)
\]

In the next sections, we shall use \(q\)-congruences to prove (1.2) and (1.3) respectively.

## 2. Proof of (1.2)

For an integer \(n\), define the \(q\)-integer
\[
[n]_q = \frac{1 - q^n}{1 - q}.
\]
Clearly \(\lim_{q \to 1} [n]_q = n\). For a non-negative integer \(k\), the \(q\)-binomial coefficient \([n\atop k]_q\) is given by
\[
[n\atop k]_q = \frac{\prod_{1 \leq j \leq k} [n - j + 1]_q}{\prod_{1 \leq j \leq k} [j]_q}.
\]
In particular, \([n\atop 0]_q = 1\). Also, we set \([n\atop k]_q = 0\) if \(k < 0\). It is easy to see that \([n\atop k]_q\) is a polynomial in \(q\) with integral coefficients, since
\[
[n + 1\atop k]_q = q^k [n\atop k]_q + [n\atop k-1]_q.
\]

Below we introduce the notion of \(q\)-congruences. Suppose that \(a, b, n\) are integers and \(a \equiv b \pmod{n}\). Then over the polynomial ring \(\mathbb{Q}(q)\), we have
\[
\frac{1 - q^a}{1 - q} = \frac{1 - q^b}{1 - q} = q^a \cdot \frac{1 - q^{b-a}}{1 - q} \equiv 0 \pmod{\frac{1 - q^n}{1 - q}},
\]
i.e., \([a]_q \equiv [b]_q \pmod{[n]_q}\). Furthermore, for the \(q\)-binomial coefficients, we have the following \(q\)-Lucas congruence.
Lemma 2.1. Suppose that \( d > 1 \) is a positive integer. Suppose that \( a, b, h, l \) are integers with \( 0 \leq b, l \leq d - 1 \). Then
\[
\left[ \frac{ad + b}{hd + l} \right]_q \equiv \left( \frac{a}{h} \right) \left[ \frac{b}{l} \right]_q \pmod{\Phi_d(q)},
\]
where \( \Phi_d(q) \) is the \( d \)-th cyclotomic polynomial.

Define the generalized \( q \)-Apéry polynomial
\[
A_k^{(\alpha)}(x; q) = \sum_{j=0}^{k} q^{(\frac{j}{2})-jk} \left[ \frac{k}{j} \right]_q \left[ \frac{k+j}{j} \right]_q x^j.
\]

In order to prove (1.2), it suffices to show that

\[ n - 1 \sum_{k=0}^{n-1} q^{n-1-k}[2k + 1]_q A_k^{(\alpha)}(x; q) \equiv 0 \pmod{[n]_q}. \tag{2.1} \]

Let us explain why (2.1) implies (1.2). Since \([n]_q\) is a primitive polynomial (i.e., the greatest divisor of all coefficients of \([n]_q\) is 1), by the Gauss lemma, there exists a polynomial \( H(x, q) \) with integral coefficients such that
\[
\sum_{k=0}^{n-1} q^{n-1-k}[2k + 1]_q A_k^{(\alpha)}(x; q) = [n]_q H(x, q). \tag{2.2}
\]

Substituting \( q = 1 \) in (2.2), we get (1.2).

It is not difficult to check that
\[
[n]_q = \prod_{d \mid n, d > 1} \Phi_d(q).
\]

The advantage of \( q \)-congruences is that we only need to prove that
\[
\sum_{k=0}^{n-1} q^{n-1-k}[2k + 1]_q A_k^{(\alpha)}(x; q) \equiv 0 \pmod{\Phi_d(q)}
\]
for every divisor \( d > 1 \) of \( n \). Note that
\[
\left[ \frac{k+j}{j} \right]_q = \frac{(1 - q^{k+1})(1 - q^{k+2}) \cdots (1 - q^{k+j})}{(1 - q)(1 - q^2) \cdots (1 - q^j)}
\]
\[
= (-1)^j q^{j^2 + \binom{j+1}{2}} (1 - q^{-1})(1 - q^{-2}) \cdots (1 - q^{-j})
\]
\[
= (-1)^j q^{j^2 + \binom{j+1}{2}} \left[ \frac{-k - 1}{j} \right]_q.
\]
Suppose that \(d > 1\) is a divisor of \(n\). Let \(h = n/d\). Write \(k = ad + b\) where \(0 \leq b \leq d - 1\). Then by Lemma \(2.1\)

\[
A_{ad+b}^{(\alpha)}(x; q) = \sum_{-\infty < s < +\infty} (1)_{q}^{(s)} \left( \frac{a}{s} \right)_{q}^{(t)} \left[ \frac{(a-1)(d+b-1)}{dy+t} \right]_{q}^{(\alpha)} x^{sd+t}.
\]

Hence,

\[
\sum_{k=0}^{n-1} [2k+1]_{q}^{(a)} A_{k}^{(\alpha)}(x; q)^m = \sum_{0 \leq a \leq h-1} \sum_{0 \leq b \leq d-1} q^{hd-1-ad-b}[2ad + 2b + 1]_{q} A_{ad+b}^{(\alpha)}(x; q)^m
\]

\[
\equiv \sum_{0 \leq a \leq h-1} \sum_{0 \leq b \leq d-1} q^{-1-b}[2b + 1]_{q} B_{a,b,d}^{(\alpha)}(x; q)^m \pmod{\Phi_d(q)},
\]

where

\[
B_{a,b,d}^{(\alpha)}(x; q) = \sum_{-\infty < s < +\infty} (1)_{q}^{(s)} \left( \frac{a}{s} \right)_{q}^{(t)} \left[ \frac{(a-1)}{t} \right]_{q}^{(\alpha)} \left[ \frac{(d+b-1)}{t} \right]_{q}^{(\alpha)} x^{sd+t}.
\]

Similarly, since \(k = ad + b \iff n - k - 1 = (h - a - 1)d + (d - b - 1)\) and \(B_{a,b,d}^{(\alpha)}(x; q) = B_{a,d-b-1,d}^{(\alpha)}(x; q)\), we have

\[
\sum_{k=0}^{n-1} q^{k[2n-2k-1]} A_{n-k-1}^{(\alpha)}(x; q)^m = \sum_{0 \leq a \leq h-1} \sum_{0 \leq b \leq d-1} q^{b[-2b-1]} \frac{B_{h-a-1,d-b-1,d}^{(\alpha)}(x; q)^m}{a' = h - a - 1}
\]

\[
\sum_{0 \leq a' \leq h-1} \sum_{0 \leq b \leq d-1} q^{b[-2b-1]} B_{a',b,d}^{(\alpha)}(x; q)^m \pmod{\Phi_d(q)).
\]

Note that

\[
q^{-1-b}[2b + 1]_{q} + q^{b][-2b - 1]_{q} = q^{-1-b} - q^{b} + q^{b} - q^{-b-1} = 0.
\]
Therefore,
\[
2 \sum_{k=0}^{n-1} q^{n-1-k}[2k + 1]qA_k^{(\alpha)}(x; q)m
= \sum_{k=0}^{n-1} q^{n-1-k}[2k + 1]qA_k^{(\alpha)}(x; q)m + \sum_{k=0}^{n-1} q^k[2n - 2k - 1]qA_{n-1-k}^{(\alpha)}(x; q)m
\equiv 0 \pmod{\Phi_d(q)}.
\]
This concludes the proof of Theorem 2.1.

3. PROOF OF (1.3)

The proof of (1.3) is a little complicated.

**Theorem 3.1.**
\[
\sum_{k=0}^{n-1} (-1)^k q^{n-1-k}[2k + 1]qA_k^{(\alpha)}(x; q^2)
\]

is divisible by
\[
\prod_{\substack{d|n \text{ is odd}}} \Phi_d(q) \cdot \prod_{\substack{d|n \text{ is even}}} \Phi_d(q^2).
\]

Clearly we only need to prove that
\[
\sum_{k=0}^{n-1} (-1)^k q^{n-1-k}[2k + 1]qA_k^{(\alpha)}(x; q^2)m + \sum_{k=0}^{n-1} (-1)^{n-1-k}q^k[2n - 2k - 1]qA_{n-1-k}^{(\alpha)}(x; q^2)m
\]
is divisible by \(\Phi_d(q)\) for odd \(d > 1\) and by \(\Phi_d(q^2)\) for even \(d\) respectively.

**Lemma 3.1.** If \(d \geq 1\) is odd, then \(\Phi_d(q)\) divides \(\Phi_d(q^2)\). If \(d\) is even, then \(\Phi_d(q^2) = \Phi_{2d}(q)\).

**Proof.** We know that for \(d > 1\),
\[
\Phi_d(q) = \prod_{\xi \text{ is } d\text{-th primitive root of unity}} (q - \xi).
\]
Suppose that \(d\) is odd and \(\xi\) is an arbitrary \(d\)-th primitive root of unity. Then \(\xi^2\) also is a \(d\)-th primitive root of unity, i.e., \(\Phi_d(\xi^2) = 0\). Hence \(\Phi_d(q)\) divides \(\Phi_d(q^2)\). Similarly, if \(d\) is even and \(\xi\) is a \(2d\)-th primitive root of unity, then \(\xi^2\) is a \(d\)-th primitive root of unity. So \(\Phi_{2d}(q)\) divides \(\Phi_d(q^2)\). Note that now \(\deg \Phi_{2d} = \phi(2d) = 2\phi(d) = 2 \deg \Phi_d\), where \(\phi\) is the Euler totient function. We must have \(\Phi_d(q^2) = \Phi_{2d}(q)\). \(\square\)
Suppose that $d > 1$ is an odd divisor of $n$. Let $h = n/d$. Then
\[
\sum_{k=0}^{n-1} (-1)^k q^{n-1-k}[2k + 1]_q A_k^{(\alpha)}(x; q^2)^m
\]
\[
\equiv \sum_{0 \leq a \leq h-1 \atop 0 \leq b \leq d-1} (-1)^{ad+b} q^{hd-1-ad-b}[2(ad + b) + 1]_q B_{a,b,d}^{(\alpha)}(x; q^2)^m \pmod{\Phi_d(q^2)}
\]
\[
\equiv \sum_{0 \leq a \leq h-1 \atop 0 \leq b \leq d-1} (-1)^{ad+b} q^{-1-b}[2b + 1]_q B_{a,b,d}^{(\alpha)}(x; q^2)^m \pmod{\Phi_d(q)}.
\]

and
\[
\sum_{k=0}^{n-1} (-1)^{n-1-k}[2n - 2k - 1]_q q^k A_{n-1-k}^{(\alpha)}(x; q^2)^m
\]
\[
\equiv \sum_{0 \leq a \leq h-1 \atop 0 \leq b \leq d-1} (-1)^{hd-1-ad-b} q^{ad+b}[2sd - 2(ad + b) - 1]_q B_{h-a-1,d-b-1,d}^{(\alpha)}(x; q^2)^m
\]
\[
\equiv \sum_{0 \leq a' \leq h-1 \atop 0 \leq b \leq d-1} (-1)^{a'd+d-1-b}[-2b - 1]_q B_{a',b,d}^{(\alpha)}(x; q^2)^m \pmod{\Phi_d(q)}.
\]

Since $d$ is odd,
\[
(-1)^{ad+b} q^{-1-b}[2b + 1]_q + (-1)^{ad+d-1-b} q^{b}[-2b - 1]_q = 0.
\]

So $\Phi_d(q)$ divides
\[
\sum_{k=0}^{n-1} (-1)^k q^{n-1-k}[2k + 1]_q A_k^{(\alpha)}(x; q^2)^m + \sum_{k=0}^{n-1} (-1)^{n-1-k} q^k[2n - 2k - 1]_q A_{n-1-k}^{(\alpha)}(x; q^2)^m.
\]

Suppose that $d$ is an even divisor of $n$. Then
\[
\sum_{k=0}^{n-1} (-1)^k q^{n-1-k}[2k + 1]_q A_k^{(\alpha)}(x; q^2)^m
\]
\[
\equiv \sum_{0 \leq a \leq h-1 \atop 0 \leq b \leq d-1} (-1)^{ad+b} q^{hd-1-(ad+b)}[2(ad + b) + 1]_q B_{a,b,d}^{(\alpha)}(x; q^2)^m
\]
\[
\equiv \sum_{0 \leq a \leq h-1 \atop 0 \leq b \leq d-1} (-1)^{ad+b} q^{hd-ad-1-b}[2b + 1]_q B_{a,b,d}^{(\alpha)}(x; q^2)^m \pmod{\Phi_d(q^2)}.
\]
And
\[
\sum_{k=0}^{n-1} (-1)^{n-1-k} q^k [2n - 2k - 1] q A_{n-1-k}^{(\alpha)}(x; q^2)^m
\]
\[
\equiv \sum_{0 \leq a \leq h-1, 0 \leq b \leq c-1} (-1)^{ah-d-1} q^{ad+b} [2hd - 2(ad + b) - 1] q B_{h-a-1,d-b-1,d}^{(\alpha)}(x; q^2)^m
\]
\[
\equiv \sum_{0 \leq a' \leq h-1, 0 \leq b \leq d-1} (-1)^{a'd+d-1-b} q^{hd-a'd-d+b} [-2b - 1] q B_{a',b,d}^{(\alpha)}(x; q^2)^m \pmod{\Phi_d(q^2)}.
\]

Note that \(\Phi_d(q^2) = \Phi_{2d}(q)\) divides \(1 + q^d = (1 - q^{2d})/(1 - q^d)\), i.e.,
\[
q^d \equiv -1 \pmod{\Phi_d(q^2)}.
\]

We have
\[
(-1)^{ad+b} q^{hd-ad-1-b} [2b + 1] q + (-1)^{ad+d-1-b} q^{hd-ad-d+b} [-2b - 1] q
\]
\[
\equiv (-1)^{ad+b} q^{hd-ad} (q^{-1-b}[2b + 1] q + q^b[-2b - 1] q)
\]
\[
= 0 \pmod{\Phi_d(q^2)}.
\]

That is, \(\Phi_d(q^2)\) divides
\[
\sum_{k=0}^{n-1} (-1)^k q^{n-1-k}[2k + 1] q A_k^{(\alpha)}(x; q^2)^m + \sum_{k=0}^{n-1} (-1)^{n-1-k} q^k [2n - 2k - 1] q A_{n-1-k}^{(\alpha)}(x; q^2)^m.
\]

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References

[1] S. Ahlgren and K. Ono, A Gaussian hypergeometric series evaluation and Apéry number congruences, J. Reine Angew. Math., 518 (2000), 187-212.

[2] F. Beukers, Another congruence for the Apéry numbers, J. Number Theory, 25 (1987), 201-210.

[3] V. J. W. Guo and J. Zeng, Proof of some conjectures of Z.-W. Sun on congruences for Apéry polynomials, preprint, arXiv:1101.0983.

[4] A. van der Poorten, A proof that Euler missed ... Apéry’s proof of the irrationality of \(\zeta(3)\), Math. Intelligencer, 1 (1978/79), 195-203.

[5] A.L. Schmidt, Generalized q-Legendre polynomials, J. Comput. Appl. Math., 49(1993), 243-249.

[6] Z.-W. Sun, Congruences involving generalized central trinomial coefficients, preprint, arXiv:1008.3887.

[7] Z.-W. Sun, On the sum of Apéry polynomials and related congruences, preprint, arXiv:1101.1946.

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