DRINFELD TYPE PRESENTATIONS OF LOOP ALGEBRAS

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Abstract. Let \( g \) be the derived subalgebra of a Kac-Moody Lie algebra of finite type or affine type, \( \mu \) a diagram automorphism of \( g \) and \( \mathcal{L}(g, \mu) \) the loop algebra of \( g \) associated to \( \mu \). In this paper, by using the vertex algebra technique, we provide a general construction of current type presentations for the universal central extension \( \hat{g}[\mu] \) of \( \mathcal{L}(g, \mu) \). The construction contains the classical limit of Drinfeld’s new realization for (twisted and untwisted) quantum affine algebras [Dr] and the Moody-Rao-Yokonuma presentation for toroidal Lie algebras [MRY] as special examples. As an application, when \( g \) is of simply-laced type, we prove that the classical limit of the \( \mu \)-twisted quantum affinization of the quantum Kac-Moody algebra associated to \( g \) introduced in [CJKT1] is the universal enveloping algebra of \( \hat{g}[\mu] \).

1. Introduction and main results

1.1. The main result. Let \( A = (a_{ij})_{i,j \in I} \) be a generalized Cartan matrix of finite type or affine type, \( \mu \) a permutation of \( I \) with order \( N \) such that \( a_{\mu(i)\mu(j)} = a_{ij} \) for \( i, j \in I \), and \( g \) the derived subalgebra of the Kac-Moody Lie algebra associated to \( A \). It was known [GK] that \( g \) is generated by the Chevalley generators \( \alpha_i^\vee, e_i^\pm, i \in I \) and subject to the relations

\[
[\alpha_i^\vee, \alpha_j^\vee] = 0, \quad [\alpha_i^\vee, e_j^\pm] = \pm a_{ij} e_j^\mp, \quad [e_i^+, e_j^-] = \delta_{ij} \alpha_i^\vee, \quad i, j \in I, 
\]

\[
\text{ad}(e_i^\pm)^{1-a_{ij}}(e_j^\pm) = 0, \quad i, j \in I \text{ with } i \neq j.
\]

This presentation, known as the Serre-Gabber-Kac presentation of \( g \), is of fundamental importance in the study of the Kac-Moody Lie algebra. Let \( n_+ \) (resp. \( n_- \)) be the subalgebra of \( g \) generated by the elements \( e_i^+ \) (resp. \( e_i^- \)) for \( i \in I \). One of the advantages of this presentation is that the subalgebras \( n_+ \) and \( n_- \) are abstractly generated by these elements with the Serre relations (1.2). The Serre-Gabber-Kac presentation also implies that \( \mu \) induces an automorphism of \( g \), still denoted as \( \mu \).
(called diagram automorphism), such that
\begin{equation}
\mu(\alpha'_i) = \alpha_{\mu(i)}', \quad \mu(e_{\mu(i)}) = e^{\pm}_{\mu(i)}, \quad i \in I.
\end{equation}

Let \( \widehat{\mathfrak{g}} \) be the universal central extension of the loop algebra \( \mathcal{L}(\mathfrak{g}) = \mathbb{C}[t_1, t_1^{-1}] \otimes \mathfrak{g} \). When \( A \) is of finite type, \( \widehat{\mathfrak{g}} = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}k_1 \) is an untwisted affine Lie algebra; when \( A \) is of affine type, \( \widehat{\mathfrak{g}} \) is a toroidal Lie algebra with \( \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}k_1 \) as a (proper) subspace. Similar to \( [1, 3] \), \( \mu \) also induces an automorphism \( \widehat{\mu} \) of \( \widehat{\mathfrak{g}} \) such that (see \( \S 3.1 \))

\begin{equation}
\widehat{\mu}(t^m_i \otimes \alpha'_i) = \xi^{-m}t^m_i \otimes \alpha'_{\mu(i)}, \quad \widehat{\mu}(t^m_i \otimes e^+_i) = \xi^{-m}t^m_i \otimes e^+_{\mu(i)}, \quad \widehat{\mu}(k_1) = k_1,
\end{equation}

for \( m \in \mathbb{Z}, i \in I \) and \( \xi = e^{2\pi \sqrt{-1}/N} \). Let \( \mathfrak{g}[\mu] \) be the subalgebra of \( \widehat{\mathfrak{g}} \) fixed by \( \widehat{\mu} \), which is the Lie algebra concerned about in this paper. It was known \( [11, \text{ CJKT2}] \) that \( \mathfrak{g}[\mu] \) is the universal central extension of the (twisted) loop algebra

\begin{equation}
\mathcal{L}(\mathfrak{g}, \mu) = \text{Span}_\mathbb{C}\{t^m_i \otimes x_{(m)} \mid x \in \mathfrak{g}, m \in \mathbb{Z}\} \subset \mathcal{L}(\mathfrak{g})
\end{equation}

of \( \mathfrak{g} \) associated to \( \mu \), where \( x_{(m)} = \sum_{k \in \mathbb{Z}_N} \xi^{-km} \mu^k(x) \) and \( \mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z} \). The main goal of this paper is to provide a general construction of certain current type presentations for \( \mathfrak{g}[\mu] \), which are loop analogues of the Serre-Gabber-Kac presentation for \( \mathfrak{g} \).

To be more precise, let \( D = \text{diag}\{\epsilon_i, i \in I\} \) be a diagonal matrix of positive rational numbers such that \( DA \) is symmetric, and let \( \mathfrak{g}_+[\mu] \) (resp. \( \mathfrak{g}_-[\mu] \)) be the subalgebra of \( \mathfrak{g}[\mu] \) generated by the elements \( t^m_i \otimes e^+_{(m)} \) (resp. \( t^m_i \otimes e^-_{(m)} \)) for \( i \in I, m \in \mathbb{Z} \). We now recall a result of Drinfeld as motivation.

**Theorem 1.1.** Assume that \( A \) is of finite type. Then the affine Lie algebra \( \mathfrak{g}[\mu] \) is isomorphic to the Lie algebra generated by the elements

\begin{equation}
h_{i,m}, \quad x^\pm_{i,m}, \quad c, \quad i \in I, m \in \mathbb{Z}
\end{equation}

and subject to the relations \( (i, j \in I, m, n \in \mathbb{Z}) \)

\begin{align}
(\text{II}) \quad & h_{\mu(i), m} = \xi^m h_{i,m}, \quad [h_{i,m}, c] = 0, \quad [h_{i,m}, h_{j,n}] = \sum_{k \in \mathbb{Z}_N} \frac{mN}{\epsilon_j} a_{i\mu(k)} \xi^{km} \delta_{m+n, 0} c, \\
(\text{HX} \pm) \quad & [h_{i,m}, x^\pm_{j,n}] = \pm \sum_{k \in \mathbb{Z}_N} a_{i\mu(k)} \xi^{km} x^\pm_{j,m+n}, \quad [x^\pm_{j,n}, c] = 0, \\
(\text{XX}) \quad & [x^+_{i,m}, x^-_{j,n}] = \sum_{k \in \mathbb{Z}_N} \delta_{i\mu(k)} \xi^{km} (h_{j,m+n} + \frac{mN}{\epsilon_j} \delta_{m+n, 0}) c, \\
(\text{X} \pm) \quad & x^\pm_{\mu(i), m} = \xi^m x^\pm_{i,m}, \quad f_{ij}(z, w) \cdot [x^\pm_i(z), x^\pm_j(w)] = 0, \\
(\text{DS} \pm) \quad & [x^\pm_i(z_1), \ldots, x^\pm_i(z_{a_i}), x^\pm_j(w)] = 0, \quad \text{if } a_{ij} < 0 \text{ and } \mu(i) = i; \\
& [x^\pm_i(z_1), x^\pm_j(z_2), x^\pm_j(w)] = 0, \quad \text{if } a_{ij} = -1, a_{\mu(i)} = 0 \text{ and } \mu(j) \neq j; \\
& \frac{z^N-1}{z-1} [x^\pm_i(z_1), x^\pm_i(z_2), x^\pm_j(w)] = 0, \quad \text{if } a_{ij} = -1, a_{\mu(i)} = 0 \text{ and } \mu(j) = j; \\
& (z+1) [x^\pm_i(z_1), x^\pm_i(z_2), x^\pm_j(w)] = 0, \quad \text{if } a_{ij} = -1, a_{\mu(i)} = -1 \text{ and } j \neq \mu(i); \\
\end{align}
The isomorphism with $\hat{g}[\mu]$ is induced by the assignment
\[ h_{i,m} \mapsto t^m_1 \otimes \alpha^\vee_{i(m),None}, \quad x^+_{i,m} \mapsto t^m_1 \otimes e^+_{i(m)}, \quad c \mapsto k_1, \quad i \in I, \ m \in \mathbb{Z}. \]

Moreover, $\hat{n}_+[\mu]$ (resp. $\hat{n}_-[\mu]$) is isomorphic to the Lie algebra generated by $x^+_{i,m}$, (resp. $x^-_{i,m}$) for $i \in I$, $m \in \mathbb{Z}$ with relations $(X_+)$, $(DS_+)$ (resp. $(X_-)$, $(DS_-)$). The isomorphism with $\hat{n}_+[\mu]$ (resp. $\hat{n}_-[\mu]$) is induced by the assignment
\[ x^+_{i,m} \mapsto t^m_1 \otimes e^+_{i(m)}, \quad (\text{resp.} \ x^-_{i,m} \mapsto t^m_1 \otimes e^-_{i(m)}), \quad i \in I, \ m \in \mathbb{Z}. \]

The current algebra presentation of $\hat{g}[\mu]$ given in Theorem 1.1 is the classical limit of the Drinfeld’s new realization for (twisted and untwisted) quantum affine algebras ([B, Da1, Da2, ZJ]). When $\mu = 1$, a version of Theorem 1.1 was given in [G]. For the general case, a proof of Theorem 1.1 was given in [Da2].

Motivated by the Serre relations $(DS \pm)$ given in Theorem 1.1, we are interested in those Drinfeld type Serre relations in $\hat{g}[\mu]$ which have the form
\[ (P1) \sum_{\sigma \in S_{1-a_{ij}}} P_{ij,\sigma}(z_1, \cdots, z_{1-a_{ij}}, w) \cdot [e^+_i(z_{\sigma(1)}), \cdots, e^+_i(z_{\sigma(1-a_{ij})}), e^+_j(w)] = 0, \]

where $(i, j) \in I = \{(i, j) \in I \times I \mid a_{ij} < 0\}$, $e^+_i(z) = \sum_{m \in \mathbb{Z}} t^m_1 \otimes e^+_{i(m)} z^{-m-1}$ and $P_{ij,\sigma}(z_1, \cdots, z_{1-a_{ij}}, w)$ are some homogenous polynomials. Starting with any such Serre relations in $\hat{g}[\mu]$, we introduce the following definition.

**Definition 1.2.** Let
\[ P = \{ P_{ij,\sigma}(z_1, \cdots, z_{1-a_{ij}}, w) \mid (i, j) \in I, \ \sigma \in S_{1-a_{ij}} \} \]

be a family of homogenous polynomials which satisfies the condition $(P1)$. We define $\mathcal{D}_P(\mathfrak{g}, \mu)$ to be the Lie algebra generated by the elements as in (1.6) and subject to the relations $(H)$, $(HX \pm)$, $(XX)$, $(X \pm)$ as in Theorem 1.1 together with the following Serre relations $(i, j) \in I$

\[ (AS \pm) \quad (z_1^N - z_2^N) \cdot [x^+_i(z_1), [x^+_i(z_2), x^+_j(w)]] = 0, \quad \text{if } \mathfrak{g} \text{ is of type } A^{(1)}_1, \]
\[ (DS \pm)_P \sum_{\sigma \in S_{1-a_{ij}}} P_{ij,\sigma}(z_1, \cdots, z_{1-a_{ij}}, w) [x^+_i(z_{\sigma(1)}), \cdots, x^+_i(z_{\sigma(1-a_{ij})})] [x^+_i(z_{\sigma(1)}), x^+_j(w)] = 0, \]
where the polynomial \( f_{ij}(z, w) \) in \((X\pm)\) is now given by

\[
f_{ij}(z, w) = \begin{cases} 
\prod_{k \in \mathbb{Z}, \alpha_k \neq 0} (z - \xi^k w), & \text{if } A \text{ is not of type } A^{(1)}; \\
(z - w) \prod_{k \in \mathbb{Z}, \alpha_k < 0} (z - \xi^k w)^2, & \text{if } A \text{ is of type } A^{(1)}.
\end{cases}
\]

Similarly, we define \( D_P(n_+, \mu) \) (resp. \( D_P(n_-, \mu) \)) to be the Lie algebra generated by the elements \( x_{i,m}^+ \) (resp. \( x_{i,m}^- \)) for \( i \in I, m \in \mathbb{Z} \) and subject to the relations \((X^+), (AS^+), (DS^+) \mathcal{P} \) (resp. \((X^-), (AS^-), (DS^-) \mathcal{P} \)).

By definition, for \( m = \mathfrak{g} \) or \( n_+, n_- \), the assignment \((1.7)\) or \((1.8)\) determines a surjective Lie homomorphism, say \( \theta_{m, \mathcal{P}} : D_{\mathcal{P}}(m, \mu) \rightarrow \hat{\mathfrak{m}}[\mu] \) (see Lemma 3.6).

**Definition 1.3.** We say that the Lie algebra \( D_{\mathcal{P}}(\mathfrak{g}, \mu) \) is a Drinfeld type presentation of \( \hat{\mathfrak{g}}[\mu] \) if for every \( m = \mathfrak{g}, n_+ \) or \( n_- \), the Lie homomorphism \( \theta_{m, \mathcal{P}} \) is an isomorphism.

In view of Theorem 1.1 it is a natural question that for which suitable \( \mathcal{P} \) the Lie algebra \( D_{\mathcal{P}}(\mathfrak{g}, \mu) \) is a Drinfeld type presentation of \( \hat{\mathfrak{g}}[\mu] \) (especially when \( \mathfrak{g} \) is of affine type)? In this paper we give a surprising answer to this question by pointing out that the family \( \mathcal{P} \) only need to satisfy the following natural condition

\[
(P2) \quad \sum_{\sigma \in S_{1-nij}} P_{ij,\sigma}(w, \cdots, w, w) \neq 0, \quad \forall (i, j) \in I.
\]

Indeed, as the main result of our paper, we prove that

**Theorem 1.4.** Assume that the family \( \mathcal{P} \) satisfies the condition \((P2)\). Then the Lie algebra \( D_{\mathcal{P}}(\mathfrak{g}, \mu) \) is a Drinfeld type presentation of \( \hat{\mathfrak{g}}[\mu] \).

We say that a module \( W \) of a \( \mathbb{Z} \)-graded Lie algebra \( \mathfrak{p} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{p}_n \) is restricted if for every \( w \in W \), there exists a positive integer \( s \) such that \( \mathfrak{p}_n w = 0 \) for all \( n \geq s \). Note that there is a natural \( \mathbb{Z} \)-grading structure on the algebra \( D_{\mathcal{P}}(\mathfrak{g}, \mu) \) (resp. \( D_{\mathcal{P}}(n_+, \mu) \) or \( D_{\mathcal{P}}(n_-, \mu) \)) with \( (i \in I, m \in \mathbb{Z}) \)

\[
\deg(x_{i,m}^\pm) = m = \deg(h_{i,m}), \quad (\text{resp. } \deg(x_{i,m}^+) = m \text{ or } \deg(x_{i,m}^-) = m).
\]

By a standard result on restricted modules of \( \mathbb{Z} \)-graded Lie algebras (see Lemma 6.6), the proof of Theorem 1.4 can be reduced to the following theorem.

**Theorem 1.5.** Assume that the family \( \mathcal{P} \) satisfies the condition \((P2)\). Then for any restricted \( D_{\mathcal{P}}(m, \mu) \)-module \( W \), where \( m = \mathfrak{g}, n_+ \) or \( n_- \), there exists an \( \hat{\mathfrak{m}}[\mu] \)-module structure on \( W \) such that

\[
x.w = \theta_{m, \mathcal{P}}(x).w, \quad x \in D_{\mathcal{P}}(m, \mu), \quad w \in W.
\]

The main body of this paper is devoted to a proof of Theorem 1.5 which is based on the theory of \( \Gamma \)-vertex algebras and their quasi-modules developed by Li in [2], [3]. In §3.3, we construct a family \( \mathcal{P} \) of polynomials which satisfies the
conditions \((P1)\) and \((P2)\). When \(g\) is of finite type, the relations in \((DS\pm)_p\) are the same as that in \((DS\pm)\) except the last one, which is much simpler:

\[
[x_i^+(z_1), [x_i^+(z_2), x_j^+(w)]] = 0, \quad \text{if} \quad a_{ij} = -1, \; j = \mu(i).
\]

In particular, this shows that Theorem 1.1 is just a special case of Theorem 1.3. When \(g\) is of untwisted affine type and \(\mu = \text{Id}\), the presentation \(D_p(g, \mu)\) of the toroidal Lie algebra \(\hat{g}\) was first introduced by Moody-Rao-Yokonuma in \[MRY\] and was often called the MRY presentation of \(\hat{g}\). When \(g\) is not of type \(A_1^{(1)}\) and \(\mu = \text{Id}\), the presentation \(D_p(g, n_+)\) for \(\hat{n}_+\) was first proved in \[E\] for the purpose of understanding the classical limit of quantum current algebras associated to \(g\).

When \(A\) is of affine type and \(\mu\) is non-transitive, it was known \([FSS]\) that the \(\mu\)-folded matrix \(A = (\hat{a}_{ij})\) associated to \(A\) is also an affine generalized Cartan matrix. In this case, we constructed in \[CJKT2\] a current type presentation for \(\hat{g}[\mu]\) with a different Serre relation:

\[
[x_i^+(z_1), \cdots, x_i^+(z_1-\hat{a}_{ij}), x_j^+(w)] = 0, \quad \text{if} \quad \hat{a}_{ij} < 0.
\]

However, as pointed out in \[Da2\], when \(\mu\) is non-trivial, one cannot obtain a presentation for \(\hat{n}_+[\mu]\) by replacing the Serre relation \((DS\pm)_p\) with \((1.12)\).

1.2. The main motivation. The main motivation of this paper stems from the quantization theory of extended affine Lie algebras (EALAs for short). The notion of EALAs was first introduced by Høegh-Krohn and Torresani \([H-KT]\), and the theory of EALAs has been intensively studied for over twenty-five years (see \[AABGP, BGK, Ne\] and the references therein). An EALA \(E\) is by definition a complex Lie algebra, together with a Cartan subalgebra and a non-degenerate invariant symmetric bilinear form, that satisfies a list of natural axioms. The invariant form on \(E\) induces a semi-positive bilinear form on the \(\mathbb{R}\)-span of the root system \(\Phi\) of \(E\), and so \(\Phi\) divides into a disjoint union of the sets of isotropic and non-isotropic roots. Roughly speaking, the structure of an EALA is determined by its core, the subalgebra generated by non-isotropic root vectors.

One of the axioms for EALAs requires that the rank of the group generated by the isotropic roots is finite, and this rank is called the nullity of EALAs. The nullity 0 EALAs are nothing but the finite dimensional simple Lie algebras, the nullity 1 EALAs are precisely the affine Kac-Moody algebras \[ABGP\], and the nullity 2 EALAs are closely related to the Lie algebras studied by Saito and Slodowy in the work of singularity theory. Recently, a complete classification of the centerless cores (the core modulo its center) of nullity 2 EALAs was obtained in \[ABP\] (see also \[GP\]). Let \(\hat{E}_n\) denote the class of Lie algebras which are isomorphic to the core of some EALA with nullity \(n\) and are central closed. For convenience of the readers, we “describe” in the following Figure the relations between EALAs and the algebras concerned about in this paper:

\[
\begin{align*}
\hat{E}_0 & = \text{finite type } g, \\
\hat{E}_1 & = \text{affine type } g = \hat{g}[\mu], \; g : \text{finite type}.
\end{align*}
\]
$\hat{E}_2 = \hat{g}[\mu], \ g : \text{affine}, \ \mu : \text{non-transitive} + \hat{s}_{\ell}(\mathbb{C}_p), \ \ell \geq 2, \ p : \text{generic}$

Here, the notation $\hat{s}_{\ell}(\mathbb{C}_p)$ stands for the universal central extension of the special linear Lie algebra over the quantum torus associated to $p \in \mathbb{C}^\times$ ([BGK]).

Let $\mathcal{U}_h(\mathfrak{g})$ be the quantum Kac-Moody algebra over $\mathbb{C}[[\hbar]]$ associated to $\mathfrak{g}$, whose theory has been a tremendous success story. Similar to the classical case, quantum affine algebras also have two realizations: Drinfeld-Jimbo’s original realization and Drinfeld’s new realization as quantum affinizations of quantum finite algebras. By applying Drinfeld’s untwisted quantum affinization process to a quantum affine algebra $\mathcal{U}_h(\mathfrak{g})$, one obtains a quantum toroidal algebra $\mathcal{U}_h(\hat{\mathfrak{g}})$ ([GKV, J, Na, H1]).

In the particular case $\mathfrak{g} = \hat{s}_{\ell+1}$, an additional parameter $p$ can be added in this quantum affinization process ([GKV] and then one gets a two parameter deformed algebra $\mathcal{U}_{h,p}(\hat{s}_{\ell+1})$, which is also called a quantum toroidal algebra. The theory of quantum toroidal algebras has been intensively studied since their discovery. In particular, the representation theory of quantum toroidal algebras is very rich and promising (see [H2] for a survey). Let $\mathcal{U}(\hat{E}_n)$ denote the class of universal enveloping algebras of the Lie algebras in $\hat{E}_n$. It is important for us to observe that all the quantum algebras mentioned above are related to EALAs by taking the classical limits:

One of the most fundamental problem in the theory of EALAs is that: just like the quantum finite, affine and toroidal algebras, what is the “right” $\hbar$-deformation
of the algebras in $\hat{E}_n$ for the general nullity $n$? The theory of quantum toroidal algebras suggests that the nullity 2 case is of particular interesting. In this case, due to the classification result obtained in [ABP], it suffices to consider the $\hbar$-deformation of $\hat{\mathfrak{g}}[\mu]$, where $\mathfrak{g}$ is of affine type and $\mu$ is non-transitive. Motivated by Drinfeld’s realization for twisted quantum affine algebras and the definition of quantum toroidal algebras, this can be achieved via the following two steps: (i). define the $\mu$-twisted quantum affinization $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$; (ii). prove that the classical limit of $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ is isomorphic to $\mathcal{U}(\hat{\mathfrak{g}}[\mu])$.

The second Figure given above suggests us that the classical limit of $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ should be certain Drinfeld type presentations of $\hat{\mathfrak{g}}[\mu]$ constructed in Theorem 1.4. Indeed, in [CJKT1], by the quantum vertex operators technique, we construct a new quantum algebra $\mathcal{U}_\hbar(\hat{\mathfrak{g}})$ when $\mathfrak{g}$ is of simply-laced affine type and $\mu$ is non-transitive. Due to Theorem 1.4, we prove in §6 that the classical limit of this quantum algebra is isomorphic to $\mathcal{U}(\hat{\mathfrak{g}}[\mu])$. The twisted quantum affinization of non-simply-laced quantum affine algebras will be defined in a forthcoming work, which together with Theorem 1.4 gives a complete answer to the previous fundamental problem on the quantization theory of nullity 2 EALAs.

1.3. The structure of our proof. Our original goal of this paper is to establish a suitable Drinfeld type presentation $\mathcal{D}_P(g, \mu)$ for $\hat{\mathfrak{g}}[\mu]$ when $\mathfrak{g}$ is of affine type. However, comparing to the finite type case, there are many new phenomenons appear in the affine type case: (i) there exist non-trivial diagram automorphisms on non-simply-laced affine Cartan matrices (even in the untwisted case); (ii) when $\mu$ is transitive, the $\mu$-folded matrix of $A$ is a zero matrix; (iii) the algebra $\hat{\mathfrak{g}}[\mu]$ is no longer a Kac-Moody algebra and so does not admit a standard triangular decomposition; (iv) $\hat{n}_+[\mu]$ is not the loop algebra of $n_+$ associated to $\mu$ but an infinite dimensional central extension of it. These difficulties make the method developed in [Da2] cannot be applied directly to the affine $\mathfrak{g}$ even in the simplest case that $P = p$. In this paper we employ the powerful vertex algebra approach to overcome these troubles. Moreover, this new method allows us to construct the Drinfeld type presentations of $\hat{\mathfrak{g}}[\mu]$ in a very general setting.

We now outline the structure of our proof for Theorem 1.4. Firstly, we construct a “universal” $\Gamma$-vertex algebra $V(\hat{\mathfrak{m}})$ associated to $\hat{\mathfrak{m}}$ and prove that the category of restricted modules for $\hat{\mathfrak{m}}[\mu]$ is equivalent to the category of quasi-modules for $V(\hat{\mathfrak{m}})$. The main tool of our proof is the theory of $\Gamma$-conformal Lie algebras developed in [G-KK, L2]. Next, due to Li’s local system theory, the locality relation $(X\pm)$ implies that for any restricted $\mathcal{D}_P(m, \mu)$-module $W$, there is an abstractly defined $\Gamma$-vertex algebra $\langle \mathcal{U}_{W,m} \rangle_\Gamma$ with $W$ as a quasi-module. By analyzing the structure of this $\Gamma$-vertex algebra, we prove that there is an $\hat{\mathfrak{m}}$-module structure on it, in where the condition $(P2)$ is used. Finally, using this fact we establish a $\Gamma$-vertex homomorphism $\varphi_{m,W}$ from $V(\hat{\mathfrak{m}})$ to $\langle \mathcal{U}_{W,m} \rangle_\Gamma$. Via this homomorphism, $W$ becomes a quasi-$V(\hat{\mathfrak{m}})$-module and hence a $\hat{\mathfrak{m}}[\mu]$-module. For convenience we “describe”
According to the motivation explained in §1.2, in this paper it is natural to simply assume that $g$ is of affine type. However, since Theorem 1.4 is also new when $g$ is of finite type and virtually no extra work results, we choose to work in both finite and affine types.

This paper is organized as follows. In §2, we review some basics on Kac-Moody algebras and toroidal Lie algebras. In §3, we introduce the definition of the algebra $\hat{g}[\mu]$ and establish a class of Drinfeld type Serre relations in $\hat{g}[\mu]$. As an application of Theorem 1.4, the classical limit of the twisted quantum affinization algebra introduced in [CJKT1] is determined in §4. §6 is devoted to the proofs of Theorem 1.4 and Theorem 1.5 and the basics on $\Gamma$-vertex algebras and their quasi-modules that are needed in the proof of Theorem 1.5 are collected in §5.

All the Lie algebras considered in this paper are over the field $\mathbb{C}$ of complex numbers. We denote the group of non-zero complex numbers, the set of non-zero integers, the set of positive integers and the set of non-negative integers by $\mathbb{C}^\times$, $\mathbb{Z}^\times$, $\mathbb{Z}_+$ and $\mathbb{N}$, respectively. For any $M \in \mathbb{Z}_+$, we set $\xi_M = e^{2\pi \sqrt{-1}/M}$. And, for a Lie algebra $\mathfrak{g}$, we will use the notation $U(\mathfrak{g})$ to stand for the universal enveloping algebra of $\mathfrak{g}$.

### 2. Preliminaries

In this section, we fix notations and review some basics on Kac-Moody algebras and toroidal Lie algebras.

#### 2.1. Basics on Kac-Moody algebras

Here we establish the notations we will use for finite dimensional simple Lie algebras and affine Kac-Moody algebras. We will use these materials throughout the rest of this paper.

Let $A = (a_{ij})_{i,j \in I}$, $\mu$ and $\hat{g}$ be as in §1.1. We denote by $\ell$ the rank of $A$ and set $I = \{1, \cdots, \ell\}$ (resp. $\{0,1,\cdots,\ell\}$) if $A$ is of finite (resp. affine) type. As a convention, we will label the generalized Cartan matrix $A$ using Tables Fin and Aff 1-3 of [K1, Chap 4]. If $A$ has label $X_\ell$ (the finite case) or $X_\ell(r)$ (the affine case) with $\ell, r \geq 1$ and $r = 1, 2, 3$, we say that $g$ (or $A$) has type $X_\ell$ or $X_\ell(r)$. As usual, we say that $g$ is an untwisted (resp. twisted) affine Kac-Moody algebra if $A$ has type $X_\ell^{(r)}$ with $r = 1$ (resp. $r > 1$).
Theorem 2.2, Exercise 2.5], there is a unique invariant symmetric bilinear form on the standard triangular decomposition of $g$, where $h = \oplus_{i\in I} \mathbb{C} \alpha_i^\vee$ and $n_+$, $n_-$ are the subalgebras of $g$ defined in §1.1. Let $\Delta$ be the root system (including 0) of $g$ and $Q$ the root lattice of $g$. Then there is a natural $Q$-grading $g = \oplus_{\alpha \in \Delta} g_\alpha$ on $g$ whose support is $\Delta$. Let $\Pi = \{\alpha_i, i \in I\}$ be the simple root system of $g$ such that $e_i^\pm \in g_{\pm \alpha_i}$ for $i \in I$, let $\Delta_+$ be the set of positive roots and let $\Delta_- = -\Delta_+$. Let $\Delta^x$ be the set of real roots in $\Delta$ and let $\Delta^0 = \Delta \setminus \Delta^x$. Note that $\mu$ induces an automorphism on $Q$ such that $\mu(\alpha_i) = \alpha_{\mu(i)}$ for $i \in I$.

Let $D = \text{diag}\{\epsilon_i, i \in I\}$ be a diagonal matrix as in §1.1. According to [K1, Theorem 2.2, Exercise 2.5], there is a unique invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $g$ such that

$$
\langle \alpha_i^\vee, \alpha_j^\vee \rangle = e_j^{-1} a_{ij}
$$

for $i, j \in I$. Recall the notion of normalized invariant form on $g$ introduced in [K1] (6.2.1), (6.4.2)]. From now on, we will fix the choice of the matrix $D$ such that if $g$ has type $X_{\ell}$ (resp. $X_{\nu(r)}$), then $\langle \cdot, \cdot \rangle$ is (resp. $r$ times of) the normalized invariant form on $g$.

2.2. More on affine Kac-Moody algebras. In this subsection, we assume that the algebra $g$ has affine type $X_{\nu(r)}$. Let $\hat{g}$ be a finite dimensional simple Lie algebra of type $X_{\nu(r)}$, $h$ a Cartan subalgebra of it and $\hat{\nu}$ a diagram automorphism of $\hat{g}$ with order $r$. For each $x \in \hat{g}$ and $m \in \mathbb{Z}$, we set

$$
x_{[m]} = r^{-1} \sum_{p \in \mathbb{Z}} \xi_r^{-mp} \nu^p(x) \quad \text{and} \quad \hat{g}_{[m]} = \text{Span}_\mathbb{C}\{x_{[m]} \mid x \in \hat{g}\}.
$$

Then it was shown in [K1, Chap. 8] that the affine Kac-Moody algebra $g$ can be realized as the Lie algebra

$$
\text{Aff}(\hat{g}, \nu) = (\sum_{m \in \mathbb{Z}} t_2^{-m} \otimes \hat{g}_{[m]}) \oplus \mathbb{C} k_2
$$

with the Lie bracket given by

$$
[t_2^{-m} \otimes x + a_1 k_2, t_2^{-m_2} \otimes y + a_2 k_2] = t_2^{-m_1 + m_2} \otimes [x, y] + \langle x, y \rangle \delta_{m_1 + m_2, 0} m_1 k_2,
$$

where $m_1, m_2 \in \mathbb{Z}, x \in \hat{g}_{[m_1]}, y \in \hat{g}_{[m_2]}, a_1, a_2 \in \mathbb{C}$ and $\langle \cdot, \cdot \rangle$ denotes the normalized invariant form on $\hat{g}$. We remark that the invariant form $\langle \cdot, \cdot \rangle$ on $g = \text{Aff}(\hat{g}, \nu)$ is given by (K1 (8.3.8))

$$
\langle t_2^{-m} \otimes x, t_2^{-n} \otimes y \rangle = \langle x, y \rangle \delta_{m+n, 0}, \quad \langle k_2, \text{Aff}(\hat{g}, \nu) \rangle = 0,
$$

for $m, n \in \mathbb{Z}$, $x \in \hat{g}_{[m]}$ and $y \in \hat{g}_{[n]}$.

Let $\hat{\Delta}$ be the root system (containing 0) of $\hat{g}$ with respect to $\hat{h}$ and let $\hat{g} = \oplus_{\alpha \in \hat{\Delta}} \hat{g}_\alpha$ be the corresponding root spaces decomposition of $\hat{g}$. Note that $\hat{\nu}$ induces...
an automorphism of the root lattice $\hat{Q}$ of $\mathfrak{g}$ such that $\hat{\nu}(\hat{\alpha}_i) = \hat{\alpha}_{\nu(i)}$ for $i \in \hat{I}$. Then the root lattice

$$Q = \hat{Q}[0] \oplus \mathbb{Z}\delta_2,$$

where $\hat{Q}[0] = \{\hat{\alpha}[0] = r^{-1} \sum_{\mu \in \mathbb{Z}^a} \nu^\mu(\hat{\alpha}) \mid \hat{\alpha} \in \hat{Q}\}$ and $\delta_2$ denotes the null root.

It was proved in [CJKT2, §2.2] that there exists an automorphism $\hat{\mu}$ of $\hat{\mathfrak{g}}$ and a homomorphism $\rho_\mu : Q \to \mathbb{Z}$ of abelian groups such that the action of $\mu$ on the real root spaces of $\mathfrak{g} = \text{Aff}(\hat{\mathfrak{g}}, \hat{\nu})$ is as follows

$$(2.3) \quad t_2^m \otimes x_{[m]} \mapsto t_2^{m+\rho_\mu(\hat{\alpha})} \otimes \hat{\mu}(x_{[m]}),$$

where $m \in \mathbb{Z}$, $\hat{\alpha} \in \hat{\Delta} \setminus \{0\}$ and $x \in \hat{\mathfrak{h}}_i$. We extend $\rho_\mu$ to a linear functional on $\hat{\mathfrak{h}}^*$ by $\mathbb{C}$-linearity and identify $\hat{\mathfrak{h}}$ with $\hat{\mathfrak{h}}^*$ by means of the normalized form on $\hat{\mathfrak{g}}$. Then $\rho_\mu$ can be viewed as a linear functional on $\hat{\mathfrak{h}}$, and the action of $\mu$ on the imaginary root spaces of $\mathfrak{g}$ is as follows

$$(2.4) \quad t_2^m \otimes h_{[m]} \mapsto t_2^m \otimes \hat{\mu}(h_{[m]}) + \delta_{m,0} \rho_\mu(h) k_2, \quad k_2 \mapsto k_2,$$

where $m \in \mathbb{Z}$ and $h \in \hat{\mathfrak{h}}$.

### 2.3. Twisted toroidal Lie algebras

We start with the definition of (twisted) multi-loop algebras. Let $\mathfrak{k}$ be a Lie algebra, and let $\sigma_1, \cdots, \sigma_s$ be pairwise commuting finite order automorphisms of it. Let $\mathbb{C}[t_1^{\pm 1}, \cdots, t_s^{\pm 1}]$ denote the algebra of Laurent polynomials in the commuting variables $t_1, \cdots, t_s$ over $\mathbb{C}$. Then the multi-loop Lie algebra of $\mathfrak{k}$ related to $\sigma_1, \cdots, \sigma_s$ is by definition the following subalgebra of $\mathbb{C}[t_1^{\pm 1}, \cdots, t_s^{\pm 1}] \otimes \mathfrak{k}$:

$$\mathcal{L}(\mathfrak{k}, \sigma_1, \cdots, \sigma_s) = \sum_{m_1, \cdots, m_s \in \mathbb{Z}} t_1^{m_1} \cdots t_s^{m_s} \otimes \mathfrak{k}(m_1, \cdots, m_s),$$

where $\mathfrak{k}(m_1, \cdots, m_s) = \{x \in \mathfrak{k} \mid \sigma_i(x) = \xi_{[M_i]}^{m_i} x, \ i = 1, \cdots, s\}$ and $M_i$ is the order of $\sigma_i$.

Suppose now that $\mathfrak{k}$ is a finite dimensional simple Lie algebra, which is equipped with a normalized invariant form $\langle \cdot, \cdot \rangle$. Let $\mathcal{K}_{M_1, \cdots, M_s}$ be the complex vector space spanned by the symbols

$$t_1^{m_1} \cdots t_s^{m_s} k_1, \cdots, t_1^{m_1} \cdots t_s^{m_s} k_s,$$

subject to the relation

$$\sum_{i=1}^s m_i t_1^{m_1} \cdots t_s^{m_s} k_i = 0,$$

where $m_i \in M_i \mathbb{Z}$ for all $i$. We define the twisted toroidal Lie algebra

$$\hat{\mathcal{L}}(\mathfrak{k}, \sigma_1, \cdots, \sigma_s) = \mathcal{L}(\mathfrak{k}, \sigma_1, \cdots, \sigma_s) \oplus \mathcal{K}_{M_1, \cdots, M_s},$$
with the Lie bracket given by
\[
[t_1^{m_1} \cdots t_s^{n_s} \otimes x, t_1^{n_1} \cdots t_s^{n_s} \otimes y]
\]
(2.5) 
\[= t_1^{m_1+n_1} \cdots t_s^{m_s+n_s} \otimes [x, y] + \langle x, y \rangle \left( \sum_{i=1}^{s} m_i t_1^{m_1+n_1} \cdots t_s^{m_s+n_s} k_i \right),\]
where \( x \in \mathfrak{t}_{(m_1, \ldots, m_s)} \), \( y \in \mathfrak{t}_{(n_1, \ldots, n_s)} \), \( m_1, \ldots, m_s, n_1, \ldots, n_s \in \mathbb{Z} \) and \( \mathcal{K}_{M_1, \ldots, M_s} \) is the center space. It was proved in [S] that \( \hat{\mathcal{L}}(\mathfrak{t}, \sigma_1, \ldots, \sigma_s) \) is central closed.

In this paper we will only use the algebra \( \hat{\mathcal{L}}(\mathfrak{t}, \sigma_1, \ldots, \sigma_s) \) for the special case that \( s = 1 \) or \( s = 2 \). Notice that, if \( s = 1 \), \( \mathcal{K}_{M_1} = \mathbb{C}k_1 \) is one dimensional. And, if \( s = 2 \), the elements
\[
t_1^{m_1} t_2^{m_2} k_1, \quad k_1, \quad t_1^{n_1} k_2, \quad m_1, n_1 \in M_1 \mathbb{Z}, \quad m_2 \in M_2 \mathbb{Z}^\times,
\]
form a basis of \( \mathcal{K}_{M_1, M_2} \).

3. The Lie algebra \( \hat{\mathfrak{g}}[\mu] \) and its Drinfeld type Serre relations

In this section, we introduce the definition of the Lie algebra \( \hat{\mathfrak{g}}[\mu] \) and establish a class of Drinfeld type Serre relations in \( \hat{\mathfrak{g}}[\mu] \).

3.1. The Lie algebra \( \hat{\mathfrak{g}}[\mu] \). From now on, when \( \mathfrak{g} \) is of affine type, we will often identify \( \mathfrak{g} \) with \( \text{Aff}(\hat{\mathfrak{g}}, \hat{\nu}) \) without further explanation. We set
\[
(3.1) \quad \hat{\mathfrak{g}} = \begin{cases} \hat{\mathcal{L}}(\mathfrak{g}, 1), & \text{if } \mathfrak{g} \text{ is of finite type;} \\ \hat{\mathcal{L}}(\mathfrak{g}, 1, \hat{\nu}), & \text{if } \mathfrak{g} \text{ is of affine type.} \end{cases}
\]

As a convention, when \( \mathfrak{g} \) is of affine type, we will also view \( \mathcal{L}(\mathfrak{g}) = \mathbb{C}[t_1, t_1^{-1}] \otimes \mathfrak{g} \) as a subspace of \( \hat{\mathfrak{g}} \) in the following way
\[
t_1^{m_1} \otimes x = t_1^{m_1} t_2^{m_2} \otimes \hat{x} + at_1^{m_1} k_2,
\]
for \( x = t_2^{m_2} \otimes \hat{x} + ak_2 \in \mathfrak{g} \), \( m_1 \in \mathbb{Z} \). For \( m_1, m_2 \in \mathbb{Z} \), we also set
\[
t_1^{m_1} t_2^{m_2} k_1 = \begin{cases} 0, & \text{if } \mathfrak{g} \text{ is of finite type;} \\ 0, & \text{if } \mathfrak{g} \text{ is of affine type and } m_2 \not\in r \mathbb{Z}^\times; \\ \frac{1}{m_2} t_1^{m_1} t_2^{m_2} k_1, & \text{if } \mathfrak{g} \text{ is of affine type and } m_2 \in r \mathbb{Z}^\times. \end{cases}
\]

Then it follows from (2.6) that the algebra \( \hat{\mathfrak{g}} \) is spanned by the elements
\[
t_1^{m_1} \otimes x, \quad k_1, \quad t_1^{n_1} t_2^{n_2} k_1', \quad x \in \mathfrak{g}, \quad m_1, n_1, n_2 \in \mathbb{Z}.
\]

Moreover, we have the following result.

Lemma 3.1. Let \( \alpha, \beta \in \Delta, \ x \in \mathfrak{g}_\alpha, \ y \in \mathfrak{g}_\beta \) and \( m_1, n_1 \in \mathbb{Z} \). If \( \alpha + \beta \in \Delta^\times \cup \{0\} \), then we have
\[
(3.2) \quad [t_1^{m_1} \otimes x, t_1^{n_1} \otimes y] = t_1^{m_1+n_1} \otimes [x, y] + m_1 \delta_{m_1, n_1} (x,y) k_1.
\]
If \( \mathfrak{g} \) is of affine type, \( x = t_2^{m_2} \otimes \hat{x}, \ y = t_2^{n_2} \otimes \hat{y} \) and \( \alpha + \beta \in \Delta^0 \setminus \{0\} \), then
\[
[t_1^{m_1} \otimes x, t_1^{n_1} \otimes y] = t_1^{m_1+n_1} \otimes [x, y] + \langle x, \hat{y} \rangle (m_1n_2 - m_2n_1) t_1^{m_1+n_1} t_2^{m_2+n_2} k_1.
\]

Proof. A direct verification by using (2.5). Notice that, when \( \mathfrak{g} \) is of affine type, one needs to use the fact (2.2). \( \square \)

Now it is obvious that the map
\[
\psi: \widehat{\mathfrak{g}} \rightarrow \mathcal{L}(\mathfrak{g}), \quad t_1^m \otimes x \mapsto t_1^m \otimes x, \ k_1 \mapsto 0, \ t_1^n t_2^{n_2} k_1' \mapsto 0
\]
is the universal central extension of \( \mathcal{L}(\mathfrak{g}) \). For \( x \in \mathfrak{g} \) and \( n \in \mathbb{Z} \), we introduce the formal series in \( \widehat{\mathfrak{g}}[[z, z^{-1}]] \) as follows:
\[
x(z) = \sum_{m \in \mathbb{Z}} (t_1^m \otimes x) z^{-m-1}, \quad t_2^n k_1'(z) = \sum_{m \in \mathbb{Z}} (t_1^n t_2^{n_2} k_1') z^{-m}, \quad k_1(z) = k_1.
\]
Using these formal series, one has the following reformulation of Lemma 3.1.

**Lemma 3.2.** Let \( \alpha, \beta \in \Delta, \ x \in \mathfrak{g}_\alpha, \ y \in \mathfrak{g}_\beta \). If \( \alpha + \beta \in \Delta^\times \cup \{0\} \), then
\[
[x(z), y(w)] = [x, y] w^{-1} \delta \left( \frac{w}{z} \right) + \langle x, y \rangle k_1(w) \frac{\partial}{\partial w} z^{-1} \delta \left( \frac{w}{z} \right)
\]

If \( \mathfrak{g} \) is of affine type, \( x = t_2^{m_2} \otimes \hat{x}, \ y = t_2^{n_2} \otimes \hat{y} \) and \( \alpha + \beta \in \Delta^0 \setminus \{0\} \), then
\[
x(z), y(w) = \left( [x, y] w + \langle x, \hat{y} \rangle m_2 \frac{\partial}{\partial w} t_2^{m_2+n_2} k_1'(w) \right) z^{-1} \delta \left( \frac{w}{z} \right)
\]
\[
+ \langle x, \hat{y} \rangle (m_2 + n_2) t_2^{m_2+n_2} k_1'(w) \frac{\partial}{\partial w} z^{-1} \delta \left( \frac{w}{z} \right),
\]
where \( \delta(z) = \sum_{m \in \mathbb{Z}} z^m \) is the usual \( \delta \)-function.

We define a \( \mathbb{Q} \times \mathbb{Z} \)-grading \( \widehat{\mathfrak{g}} = \bigoplus_{(\alpha, n) \in \mathbb{Q} \times \mathbb{Z}} \widehat{\mathfrak{g}}_{\alpha, n} \) on \( \widehat{\mathfrak{g}} \) by letting
\[
t_1^m \otimes x \in \widehat{\mathfrak{g}}_{\alpha, m}, \quad k_1 \in \widehat{\mathfrak{g}}_{0,0}, \quad t_1^n t_2^{n_2} k_1' \in \widehat{\mathfrak{g}}_{n_2, n_1},
\]
for \( x \in \mathfrak{g}_\alpha, \ \alpha \in \Delta \) and \( m_1, n_1, n_2 \in \mathbb{Z} \). This grading induces a natural triangular decomposition
\[
\widehat{\mathfrak{g}} = \widehat{n}_+ \oplus \widehat{\mathfrak{h}} \oplus \widehat{n}_-
\]
of \( \widehat{\mathfrak{g}} \), where \( \widehat{\mathfrak{h}} = \bigoplus_{m \in \mathbb{Z}} \widehat{\mathfrak{g}}_{0,m} = \text{Span}_\mathbb{C} \{ t_1^m \otimes h, \ k_1 | h \in \mathfrak{h}, m \in \mathbb{Z} \} \) is a Heisenberg algebra and \( \widehat{n}_\pm = \bigoplus_{n \in \Delta_\pm, m \in \mathbb{Z}} \widehat{\mathfrak{g}}_{\alpha,m} \). When \( \mathfrak{g} \) is of finite type, \( \widehat{n}_\pm = \mathbb{C}[t_1, t_1^{-1}] \otimes n_\pm \) is the loop algebra of \( n_\pm \). However, when \( \mathfrak{g} \) is of affine type,
\[
\widehat{n}_+ = \text{Span}_\mathbb{C} \{ t_1^m \otimes x, \ t_1^{m_1} t_2^{m_2} k_1' | x \in n_\pm, m, m_1 \in \mathbb{Z}, \pm m_2 \in \mathbb{Z}_+ \}
\]
is an (infinite dimensional) central extension of \( \mathbb{C}[t_1, t_1^{-1}] \otimes n_+ \).

Observe that the algebra \( \widehat{\mathfrak{g}} \) is generated by the following elements
\[
t_1^m \otimes \hat{e}_i, \quad t_1^m \otimes \hat{a}_i^\vee, \quad k_1, \quad i \in I, \ m \in \mathbb{Z}.
\]
As was indicated in (1.3), we have the following result.
Lemma 3.3. The action

\begin{align}
\begin{aligned}
t^m_1 \otimes e^\pm_i &\mapsto \xi^{-m_1} t^m_1 \otimes e^\pm_{\mu(i)}, \\
t^m_1 \alpha^\vee_i &\mapsto \xi^{-m_1} t^m_1 \alpha^\vee_{\mu(i)}, \\
k_1 &\mapsto k_1
\end{aligned}
\end{align}

for \( i \in I, \ m \in \mathbb{Z}, \) defines (uniquely) an automorphism \( \hat{\mu} \) of \( \hat{g}. \)

Proof. The lemma is obviously when \( g \) is of finite type. For the affine case, this lemma was proved in \( [\text{CJKT}2, \text{Lemma } 3.2] \). For convenience of the readers, we describe the explicit action of \( \hat{\mu} \) on \( \hat{g} \) as follows:

\begin{align}
\begin{aligned}
t^m_{1} \otimes x &\mapsto \xi^{-m_1} t^m_{1} \otimes \mu(x), \\
k_1 &\mapsto k_1,
\end{aligned}
\end{align}

where \( m_1 \in \mathbb{Z}, \ x \in g_\alpha \) and \( \alpha \in \Delta^\times \cup \{0\}, \) and (the following actions exist only when \( g \) is of affine type)

\begin{align}
\begin{aligned}
t^m_1 \otimes h &\mapsto \xi^{-m_1} (t^m_1 \otimes \mu(h) - m_1 \rho_\mu(\hat{h}) t^{m_2}_1 k'_1), \\
t^m_1 t^m_{2} k_1 &\mapsto \xi^{-m_1} t^m_1 t^m_{2} k_1,
\end{aligned}
\end{align}

where \( m_1, n_1, n_2 \in \mathbb{Z}, \ h = t^m_{2} \otimes \hat{h}, \) \( m_2 \in \mathbb{Z}^\times \) and \( \hat{h} \in \hat{h}_{[m_2]} \).

We define \( \hat{g}[\mu] \) to be the subalgebra of \( \hat{g} \) fixed by \( \hat{\mu}. \) It was known \( (K1, \text{CJKT}2) \) that the restriction map (see \( \text{[3.6]} \))

\[ \psi_{[\hat{g}[\mu]]} : \hat{g}[\mu] \to \mathcal{L}(g, \mu) \]

is the universal central extension of \( \mathcal{L}(g, \mu). \) We remark that the automorphism \( \hat{\mu} \) preserves the decomposition \( \text{[3.7]} \) of \( \hat{g}. \) So we have

\[ \hat{g}[\mu] = \hat{n}_+[\mu] \oplus \hat{h}[\mu] \oplus \hat{n}_-[-\mu], \]

where \( \hat{n}_+[\mu] = \hat{g}[\mu] \cap \hat{n}_+ \) and \( \hat{h}[\mu] = \hat{g}[\mu] \cap \hat{h}. \) Using \( \text{[3.8]} \) and \( \text{[3.9]} \), one knows that the Lie algebra \( \hat{g}[\mu] \) is generated by the following elements

\begin{align}
\begin{aligned}
t^m_{1} \otimes e^\pm_{i(m)}, \\
t^m_{1} \otimes \alpha^\vee_{i(m)}, \\
k_1, \quad i \in I, \ m \in \mathbb{Z},
\end{aligned}
\end{align}

where the notation \( x_{(m)} \) was given in \( \text{[1.5]} \). Moreover, the subalgebras \( \hat{n}_\pm[\mu] \) (resp. \( \hat{h}[\mu] \)) of \( \hat{g} \) are generated by the elements \( t^m_{1} \otimes e^\pm_{i(m)} \) (resp. \( t^m_{1} \otimes \alpha^\vee_{i(m)}, k_1 \)) for \( i \in I \) and \( m \in \mathbb{Z}. \) The following result can be checked directly by using \( \text{[3.2]} \).

Lemma 3.4. Let \( i, j \in I \) and \( m, n \in \mathbb{Z}. \) Then

\begin{align}
\begin{aligned}
t^m_1 \otimes \alpha^\vee_{\mu(i)(m)} &\equiv \xi^m t^m_1 \otimes \alpha^\vee_{\mu(i)(m)}, \\
t^m_1 \otimes e^\pm_{\mu(i)(m)} &\equiv \xi^m t^m_1 \otimes e^\pm_{\mu(i)(m)}, \\
[t^m_1 \otimes \alpha^\vee_{i(m)}, k_1] &\equiv 0 = [t^m_1 \otimes e^\pm_{i(m)}, k_1], \\
[t^m_1 \otimes \alpha^\vee_{i(m)}, t^m_1 \otimes \alpha^\vee_{j(n)}] &\equiv \sum_{k \in \mathbb{Z}_N} \frac{mN}{\epsilon_j} a_{\mu(k)} \xi^{km} \delta_{m+n,0} k_1, \\
[t^m_1 \otimes \alpha^\vee_{i(m)}, t^m_1 \otimes e^\pm_{j(n)}] &\equiv \pm \sum_{k \in \mathbb{Z}_N} a_{\mu(k)} \xi^{km} t^m_1 t^n_1 \otimes e^\pm_{j(m+n)},
\end{aligned}
\end{align}
\[ [\mathit{l}_1^m \otimes e_i^+(m), \mathit{t}_1^n \otimes e_j^-(n)] = \sum_{k \in \mathbb{Z}} \delta_{i,j} \delta^{(j)}(\mathit{i}^m) \mathit{e}_j^{km}(\mathit{i}^m + n \otimes \alpha_j^{(m+n)} + \frac{mN}{n} \delta_{m+n,0} k_1). \]

As in \((3.5)\), for \(x \in \mathfrak{g}\) and \(n \in \mathbb{Z}\), we introduce the formal series in \(\hat{\mathfrak{g}}[\mu][[z, z^{-1}]]\) as follows:

\[
\mathit{x}(z) = \sum_{m \in \mathbb{Z}} \left( \sum_{p \in \mathbb{Z}} \hat{\mathfrak{g}}^p(t_1^m \otimes x) \right) z^{-m-1}, \quad t_1^m k_1'(z) = \sum_{m \in \mathbb{N}} t_1^m t_2^n k_1' z^{-m}, \quad k_1(z) = k_1.
\]

Note that, if \(x \in \mathfrak{g}_\alpha\) with \(\alpha \in \Delta^\times \cup \{0\}\), then it follows from \((3.10)\) that

\[
\mathit{x}(z) = \sum_{m \in \mathbb{Z}} t_1^m \otimes x(m) z^{-m-1}.
\]

This is not true when \(\mathfrak{g}\) is of affine type and \(x \in \mathfrak{g}_\alpha\) with \(\alpha \in \Delta^0 \setminus \{0\}\). However, the following still holds true (see \((3.11)\))

\[
[x(z), y(w)] = \left[ \sum_{m \in \mathbb{Z}} t_1^m \otimes x(m) z^{-m-1}, \sum_{n \in \mathbb{Z}} t_1^n \otimes y(n) w^{-n-1} \right]
\]

for all \(x, y \in \mathfrak{g}\). Using this observation and Lemma 3.2, one can easily verify the following result.

**Lemma 3.5.** For \(\alpha, \beta \in \Delta, \mathit{x} \in \mathfrak{g}_\alpha\), and \(\mathit{y} \in \mathfrak{g}_\beta\), one has that

\[
[x(z), y(w)] = \sum_{k \in \mathbb{Z}} \left[ \mu^k(x), y \right](w) z^{-1} \left( \xi^{-k} \right)
\]

\[
+ \sum_{k \in J_0} \left( \mu^k(x), y \right)(w) \frac{\partial}{\partial w} z^{-1} \left( \xi^{-k} \right)
\]

\[
+ \sum_{k \in J_1} \left( \hat{\mathfrak{g}}^k(x), \hat{\mathfrak{g}}^k(y) \right) \left( m_k + n \otimes \nu_2 \right) z^{-1} \left( \xi^{-k} \right)
\]

\[
+ (m_k + n) t_2^{m+n} k_1' z^{-1} \left( \xi^{-k} \right).
\]

Here, \(J_0 = \{k \in \mathbb{Z}_N \mid \mu^k(\alpha) + \beta = 0\}\) and \(J_1 = \{k \in \mathbb{Z}_N \mid \mu^k(\alpha) + \beta \in \Delta^0 \setminus \{0\}\}\). The set \(J_1\) exists only when \(\mathfrak{g}\) is of affine type, and in this case the notations \(\hat{\mathfrak{g}}^k, \hat{\mathfrak{g}}^k, m_k, n \in \mathbb{Z}, k \in J_1\) are defined by the rule

\[
\mu^k(x) = t_2^{m_k} \otimes \hat{x}_k \quad \text{and} \quad y = t_2^n \otimes \hat{y}.
\]

Finally, recall the Lie algebras \(\mathcal{D}_P(\mathfrak{g}, \mu), \mathcal{D}_P(n_+, \mu)\) and \(\mathcal{D}_P(n_-, \mu)\) defined in Definition \((1.12)\). Then we have the following result.

**Lemma 3.6.** The assignment \((1.7)\) (resp. \((1.8)\)) determines a surjective Lie homomorphism, say \(\theta_{\mathfrak{g}}^P\) (resp. \(\theta_{n_+}^P\) or \(\theta_{n_-}^P\)), from \(\mathcal{D}_P(\mathfrak{g}, \mu)\) (resp. \(\mathcal{D}_P(n_+, \mu)\) or \(\mathcal{D}_P(n_-, \mu)\)) to \(\hat{\mathfrak{g}}[\mu]\) (resp. \(\hat{n}_+[\mu]\) or \(\hat{n}_-[\mu]\)).
Proof. It suffices to show that the elements in (3.13) satisfy the defining relations of \( D_P(g, \mu) \). The relations \((HH), (XX), (HX \pm)\) are implied by Lemma 3.4, the relation \((X \pm)\) is implied by Lemma 3.5 and the relation \((DS \pm)\) is implied by the condition \((P1)\). Now we check the relation \((AS \pm)\), and so \( g \) is of type \( A_1(1) \).

Note that in this case \( \alpha \mu^k(i) + \alpha \mu^l(i) + \alpha_j \) is not a null root for any \( i_1, i_2 \in \mathbb{Z}_N \) and \( (i, j) \in I \). This together with Lemma 3.5 implies that

\[
[e^\pm_i(z_1), [e^\pm_i(z_2), e^\pm_j(w)]] = \sum_{k_1, k_2 \in \mathbb{Z}_N} [e^\pm_{\mu^k_1(i)}, [e^\pm_{\mu^k_2(i)}, e^\pm_j]](w) z_1^{-1} \delta \left( \frac{\xi - k_1 w}{z_1} \right) z_2^{-1} \delta \left( \frac{\xi - k_2 w}{z_2} \right).
\]

Therefore, we have that

\[
(z_1^N - z_2^N) \cdot [e^\pm_i(z_1), [e^\pm_i(z_2), e^\pm_j(w)]] = 0,
\]

as required. \(\square\)

Remark 3.7. Let \( i, j \in I \). Then \( \alpha_i + \alpha_j \in \Delta^0 \) if and only if \( g \) is of type \( A_1(1) \) and \( i \neq j \). Therefore, when \( g \) is of type \( A_1(1) \), the defining relation \((X \pm)\) of \( D_P(g, \mu) \) is different from that of other types.

3.2. More on diagram automorphisms. Before deducing the Drinfeld type Serre relations in \( \widehat{g}[\mu] \), in this subsection we collect some elementary properties on the automorphism \( \mu \) for later use.

Let \( \langle \mu \rangle \) be the subgroup of the permutation group on \( I \) generated by \( \mu \). For \( i \in I \), let \( O(i) \) be the orbit of \( I \) containing \( i \) under the action of \( \langle \mu \rangle \). The following result is about the edges joining the vertices in the same orbit of \( I \).

Lemma 3.8. For \( i \in I \), exactly one of the following holds

(a) the elements \( \alpha_p, p \in O(i) \) are pairwise orthogonal;
(b) \( O(i) = \{i, \mu(i)\} \) and \( a_{\mu(i)} = -1 = a_{\mu(i)} \);
(c) \( A \) has type \( A_1(1) \) and \( O(i) = I \).

Proof. The lemma is proved by checking the claim for each possible \( A \) and each diagram automorphism \( \mu \) on \( A \). For a list of automorphisms on affine generalized Cartan matrices, see [ABP, Tables 2, 3] for example. \(\square\)

For \( i \in I \), set

\[
(3.14) \quad s_i = \begin{cases} 
1, & \text{if (a) holds in Lemma 3.8} \\
2, & \text{if (b) holds in Lemma 3.8} \\
3, & \text{if (c) holds in Lemma 3.8}
\end{cases}
\]

The automorphism \( \mu \) is said to be transitive if \( \mu \) acts transitively on the set \( I \). Observe that, if \( A \) is of affine type, then \( \mu \) is non-transitive if and only if \( s_i \leq 2 \) for all \( i \in I \).
For $i,j \in I$, we set
\[(3.15) \quad \Gamma_{ij}^- = \{ k \in \mathbb{Z}_N | a_{ik} < 0 \},\]
and introduce the numbers
\[(3.16) \quad \text{Card } O_i, \quad d_i = N/N_i, \quad N_{ij} = \gcd(N_i, N_j), \quad d_{ij} = \text{Card } \Gamma_{ij}^-.
\]
The following result is about the edges joining the vertices in two different orbits.

**Lemma 3.9.** Let $i,j \in I$ and $k \in \mathbb{Z}_N$. Assume that $i \notin O(j)$ and $a_{ij} < 0$. Then exactly one of the following holds
\[
\text{(a)} \quad a_{ik}(j) = 0; \\
\text{(b)} \quad a_{ik}(j) = a_{ij} \text{ and } N_{ij} \text{ divides } k.
\]
In particular, in this case $\Gamma_{ij}^-$ is a subgroup of $\mathbb{Z}_N$ with order $d_{ij} = N/N_{ij}$.

**Proof.** Let $i,j$ be as in lemma. Note that in this case $\{s_i, s_j\} = \{1\}$ or $\{1, 2\}$, and so we may (and do) assume that $s_j = 1$. In what follows we list all the possible Dynkin diagrams $S_{ij}$ of $O(i) \oplus O(j)$ and the numbers $N_{ij}$:

- (i) if $s_i = 1$, then $S_{ij}$ is one of the types $A_2, A_3, D_4(1), A_2 \times A_2, B_2 \times B_2$, and $A_2 \times A_2 \times A_2$ and in each case $N_{ij} = 1, 1, 1, 1, 2, 2$ and 3, respectively;
- (ii) if $s_i = 2$, then $S_{ij}$ is one of the types $A_4, C_3(1), D_4(2), D_5(1)$, and $A_2(1)$ and in each case $N_{ij} = 2, 2, 2, 2$ and 1, respectively.

Using this and a case by case argument, one can get the desired result. \(\square\)

Recall from § 1.1 that $I = \{(i, j) \in I \times I | a_{ij} < 0\}$. For $(i, j) \in I$ and $k = (k_1, \ldots, k_{1-a_{ij}}) \in (\mathbb{Z}_N)^{1-a_{ij}}$, we introduce the notation
\[
\alpha_{ij}(k) = \alpha_{k_1(i)} + \cdots + \alpha_{k_{1-a_{ij}}(i)} + \alpha_j \in Q.
\]
The rest of this subsection is devoted to give the sufficient and necessary condition for $\alpha_{ij}(k)$ belongs to $\Delta$ (resp. $\Delta^+$; resp. $\Delta^0$). Firstly, one has that

**Lemma 3.10.** Let $(i, j) \in I$ and $k = (k_1, \ldots, k_{1-a_{ij}}) \in (\mathbb{Z}_N)^{1-a_{ij}}$. Then

- (a) if $s_i = 1$, then $\alpha_{ij}(k) \in \Delta$ if and only if $-k_1, \ldots, -k_{1-a_{ij}} \in \Gamma_{ij}^-$ and at least two of $\mu^{k_1}(i), \ldots, \mu^{k_{1-a_{ij}}}(i)$ are distinct;
- (b) if $s_i = 2$ and $i \notin O(j)$, then $\alpha_{ij}(k) \notin \Delta$ for all $k \in (\mathbb{Z}_N)^{1-a_{ij}}$;
- (c) if $s_i = 2$, $i \notin O(j)$ and $N_{ij} = 2$, then $\alpha_{ij}(k) \in \Delta$ if and only if all except one of $k_1, \ldots, k_{1-a_{ij}}$ contained in $\Gamma_{ij}^-$;
- (d) if $s_i = 2$, $i \notin O(j)$ and $N_{ij} = 1$, then $\alpha_{ij}(k) \in \Delta$ if and only if $k_1 \neq k_2$;
- (e) if $s_i = 3$ and $\varphi = 1$, then $\alpha_{ij}(k) \in \Delta$ if and only if there is exactly one, say $k_i$, of $k_1, k_2, k_3$ such that $\mu^k(i) = j$;
- (f) if $s_i = 3$ and $\varphi = 2$, then $\alpha_{ij}(k) \in \Delta$ if and only if $\mu^{k_1}(i), \mu^{k_2}(i), j$ are pairwise distinct;
- (g) if $s_i = 3$ and $\varphi \geq 3$, then $\alpha_{ij}(k) \in \Delta$ if and only if the Dynkin diagram of $\{\mu^{k_1}(i), \mu^{k_2}(i), j\}$ is of type $A_3$.  


Proof. We first prove the assertion (a) and so assume now that $s_i = 1$. Let
\[ \alpha = (\sum_{p \in \mathcal{O}(i)}^m m_\alpha \alpha_p) + \alpha_j \] for some $m_\alpha \in \mathbb{N}$. By using Lemma 3.8 (a) and an
induction argument on the number $N_i$, one can easily check that $\alpha \in \Delta$ if and
only if $m_\alpha \leq a_{pj}$ for all $p \in \mathcal{O}(i)$. It is easy to see that this implies the assertion
(a) and we omit the details. The assertion (b) is implied by Lemma 3.8 (b), and
the assertions (c),(d) can be proved by checking all the possible Dynkin diagrams
$S_{ij}$ of $\mathcal{O}(i) \cup \mathcal{O}(j)$ given in the proof of Lemma 3.9.

For the assertion (e), one only needs to notice that if $g$ is of type $A_1^{(1)}$, then
$m_0a_0 + m_1a_1 \in \Delta$ for some $m_0, m_1 \in \mathbb{N}$ with $m_0 + m_1 = 4$ if and only if $m_0 = m_1 = 2$. The assertion (f) follows from the fact that: if $g$ is of type $A_2^{(1)}$, then
$m_0a_0 + m_1a_1 + m_2a_2 \in \Delta$ for some $m_0, m_1, m_2 \in \mathbb{N}$ with $m_0 + m_1 + m_2 = 3$ if and
only if $m_0 = m_1 = m_2 = 1$. Finally, the assertion (g) is obvious. \qed

As an immediate by-product of Lemma 3.10 we have that

**Corollary 3.11.** Let $(i, j) \in \mathbb{I}$ and $k = (k_1, \ldots, k_{1-a_{ij}}) \in (\mathbb{Z}_N)^{1-a_{ij}}$. If $\alpha_{ij}(k) \in \Delta$, then there exist $s, t = 1, \ldots, 1 - a_{ij}$ such that $\mu^{k_s(i)} \neq \mu^{k_t(i)}$.

Now, for $(i, j) \in \mathbb{I}$, we define three subsets of $(\mathbb{Z}_N)^{1-a_{ij}}$ as follows
\[ \Upsilon_{ij} = \{ k \in (\mathbb{Z}_N)^{1-a_{ij}} | \alpha_{ij}(k) \in \Delta \}; \]
\[ \Upsilon_{ij}^\times = \{ k \in (\mathbb{Z}_N)^{1-a_{ij}} | \alpha_{ij}(k) \in \Delta^\times \}; \]
\[ \Upsilon_{ij}^0 = \{ k \in (\mathbb{Z}_N)^{1-a_{ij}} | \alpha_{ij}(k) \in \Delta^0 \}. \]

The following result, together with Lemma 3.10, determines the three sets given
above.

**Lemma 3.12.** Let $(i, j) \in \mathbb{I}$. Then the following results hold true

(a) if $A$ is not one of types $A_1^{(1)}$ and $A_2^{(1)}$, then $\Upsilon_{ij} = \Upsilon_{ij}^\times$;
(b) if $A$ is of type $A_1^{(1)}$ or $A_2^{(1)}$, then $\Upsilon_{ij} = \Upsilon_{ij}^0$.

**Proof.** Assume that $\alpha_{ij}(k) = (\sum_{p \in \mathcal{O}(i)}^m m_\alpha \alpha_p) + \alpha_j \in \Delta^0$ and so $g$ is of affine type.
Let $a_s, s \in I$ be the labels in the diagrams of [K] Chap.4, Table Aff 1-3]. Recall
that $a_s, s \in I$ are the (unique) positive integers such that $\delta_2 = \sum_{s \in I} a_s \alpha_s$. In
view of this, if $j \notin \mathcal{O}(i)$, then $\alpha_{ij}(k) \in \Delta^0$ if and only if
\[ I = \mathcal{O}(i) \cup \{ j \}, \mathcal{O}(j) = \{ j \}, a_j = 1 \text{ and } m_p = a_p \text{ for all } p \in \mathcal{O}(i). \]

Using the Dynkin diagrams $S_{ij}$ given in the proof of Lemma 3.9, one can easily
check that $\alpha_{ij}(k) \in \Delta^0$ if and only if $g$ is of type $A_2^{(1)}$ with $N = 2$, $N_i = 2$ and
$N_j = 1$. For the case $j \in \mathcal{O}(i)$, one has that $I = \mathcal{O}(i)$, $g$ has type $A_{N-1}^{(1)}$ and $a_s = 1$
for all $s \in I$. Then the assertion immediately follows from Lemma 3.10 (e),(f) and
(g). \qed
3.3. Drinfeld type Serre relations in \( \widehat{\mathfrak{g}}[\mu] \). In this subsection we construct a class of Drinfeld type Serre relations in \( \widehat{\mathfrak{g}}[\mu] \).

For \((i, j) \in \mathbb{I}\), we define the Drinfeld polynomial

\[
(3.17) \quad p_{ij}(z, w) = \prod_{k \in \Omega_{ij}^X} (z - \zeta^k w) \cdot \prod_{k \in \Omega_{ij}^0} (z - \zeta^k w)^2,
\]

where \( \Omega_{ij}^X \) and \( \Omega_{ij}^0 \) are the subsets of \( \mathbb{Z}_N \) defined as follows: for any \( k \in \mathbb{Z}_N \), \( k \in \Omega_{ij}^X \) (resp. \( \Omega_{ij}^0 \)) if and only if there is a tuple \( k \in \mathbb{Y}_{ij}^X \) (resp. \( \mathbb{Y}_{ij}^0 \)) such that \( k = k_s - k_t \) for some \( 1 \leq s, t \leq m_{ij} \) with \( \mu^{k_s}(i) \neq \mu^{k_t}(i) \).

Using Lemma 3.8, Lemma 3.9 and Lemma 3.10, we have the following explicit description of the polynomials \( p_{ij}(z, w) \).

**Lemma 3.13.** Let \((i, j) \in \mathbb{I}\). If \( j \in \mathcal{O}(i) \), then

\[
p_{ij}(z, w) = \begin{cases} 
1, & \text{if } s_i \leq 2; \\
\left(\frac{z - w^N}{z - w}\right)^2, & \text{if } N = 2, 3 \text{ and } s_i = 3; \\
\prod_{k \in \mathbb{P}_{ij}^\pm} (z - \zeta^k w)(z - \zeta^{2k} w), & \text{if } N \geq 6 \text{ and } s_i = 3.
\end{cases}
\]

And, if \( j \notin \mathcal{O}(i) \), then

\[
p_{ij}(z, w) = \frac{z^{s_i d_i} - w^{s_i d_i}}{z^{d_i} - w^{d_i}} \cdot \frac{z^{d_{ij}} - w^{d_{ij}}}{z^{d_i} - w^{d_i}} = \begin{cases} 
\frac{z^{N/N_{ij}} - w^{N/N_{ij}}}{z^{N/N_{ij}} - w^{N/N_{ij}}}, & \text{if } s_i = 1; \\
z^{N/N_{ij}} + w^{N/N_{ij}}, & \text{if } s_i = 2 \text{ and } N_{ij} = 2; \\
(z + w)^2, & \text{if } s_i = 2 \text{ and } N_{ij} = 1.
\end{cases}
\]

When \( \mathfrak{g} \) is of finite type and \((i, j) \in \mathbb{I}\) with \( i \notin \mathcal{O}(j) \), it follows from Lemma 3.13 that the polynomial \( p_{ij}(z, w) \) defined in (3.17) coincides with that appeared in the relation \( (DS\pm) \) (see Theorem 1.1). For this reason, \( p_{ij}(z, w) \) is called the the Drinfeld polynomial.

Now we establish a class of Drinfeld type Serre relations in \( \widehat{\mathfrak{g}}[\mu] \) as follows.

**Proposition 3.14.** For \((i, j) \in \mathbb{I}\), one has that

\[
(3.18) \quad \prod_{1 \leq s < t \leq 1-a_{ij}} p_{ij}(z_s, z_t) \cdot [e_{i}^{\pm}(z_1), \ldots, [e_{i}^{\pm}(z_{1-a_{ij}}), e_{j}^{\pm}(w)]] = 0.
\]

**Proof.** The proposition is implied by the definition of \( p_{ij}(z, w) \), Corollary 3.11 and the lemma below. \( \square \)

**Lemma 3.15.** Let \((i, j) \in \mathbb{I}\). If \( A \) is not of type \( A_{1}^{(1)} \) or \( A_{2}^{(1)} \), then

\[
[e_{i}^{\pm}(z_{1}), \ldots, [e_{i}^{\pm}(z_{1-a_{ij}}), e_{j}^{\pm}(w)]]
\]
If \( A \) is of type \( A_1^{(1)} \) or \( A_2^{(1)} \), then
\[
[e^+_{\mu, k_1}(i), \ldots, e^+_{\mu, k_m(i)}] (w) \prod_{1 \leq i \leq s} z_i^{-1} \delta \left( \frac{\xi - k_1 w}{z_i} \right).
\]

If \( A \) is of type \( A_1^{(1)} \) or \( A_2^{(1)} \), then

\[
[e^+_{\mu, k_1}(i), \ldots, e^+_{\mu, k_m(i)}] (w) = \sum_{k = (k_1, \ldots, k_n)} \left[ e^+_{\mu, k_1}(i), \ldots, e^+_{\mu, k_m(i)} \right] (w) \prod_{1 \leq i \leq s} z_i^{-1} \delta \left( \frac{\xi - k_1 w}{z_i} \right).
\]

\[
= \sum_{k = (k_1, \ldots, k_n)} \left[ e^+_{\mu, k_1}(i), \ldots, e^+_{\mu, k_m(i)} \right] (w) \prod_{1 \leq i \leq s} z_i^{-1} \delta \left( \frac{\xi - k_1 w}{z_i} \right) + \frac{\partial}{\partial w} k_1 (w)
\]

where the elements \( m_{k_1}, n_{k'} \in \mathbb{Z} \), \( x^\pm_{k_1}, y^\pm_{k'} \in \hat{g} \) are determined by the following rule
\[
e^\pm_{\mu, k_1} (i) = t^\pm_{2^m k_1} \otimes x^\pm_{k_1}, \quad [e^+_{\mu, k_2(i)} \otimes \cdots, [e^+_{\mu, k_{-1}(i)}, e^\pm_j] = t^\pm_{2^m k'} \otimes y^\pm_{k'}.
\]

\[\text{Proof.}\] It is directly verified by using Lemma 3.5.

For \((i, j) \in \Pi \) and \( \sigma \in S_{1-a_{ij}} \), we set
\[
p_{ij, \sigma}(z_1, \ldots, z_{1-a_{ij}}) = \begin{cases} \prod_{1 \leq s < t \leq 1-a_{ij}} p_{ij}(z_s, z_t), & \text{if } \sigma = 1; \\ 0, & \text{if } \sigma \neq 1. \end{cases}
\]

Then it follows from Proposition 3.14 that the family
\[
\mathcal{P} = \{ P_{ij, \sigma}(z_1, \ldots, z_{1-a_{ij}}, w) = p_{ij, \sigma}(z_1, \ldots, z_{1-a_{ij}}) \mid (i, j) \in \Pi, \sigma \in S_{1-a_{ij}} \}
\]
of homogenous polynomials satisfies the conditions \((P1)\) and \((P2)\). Notice that, when \( \mu = 1 \), the Serre relations \((DS \pm)_p \) are the usual Serre relations
\[
[\hat{x}^\pm_1(z_1), \ldots, [\hat{x}^\pm_{1-a_{ij}}, \hat{x}^\pm_j(w)]] = 0, \quad (i, j) \in \Pi.
\]

More generally, for each \((i, j) \in \Pi \), let \( f_{ij}(z_1, \ldots, z_{1-a_{ij}}, w) \) be an arbitrarily given homogenous polynomial such that \( f(w, \ldots, w, w) \neq 0 \). Then, according to Theorem 1.4 and Proposition 3.14 one knows that \( D_f(g, \mu) \) is a Drinfeld type presentation of \( \hat{g}[\mu] \), where

\[
f = \{ f_{ij, \sigma}(z_1, \ldots, z_{1-a_{ij}}, w) \mid (i, j) \in \Pi, \sigma \in S_{1-a_{ij}} \},
\]

with
\[
f_{ij, \sigma}(z_1, \ldots, z_{1-a_{ij}}, w) = f_{ij}(z_1, \ldots, z_{1-a_{ij}}, w) \cdot p_{ij, 1}(z_1, \ldots, z_{1-a_{ij}}),
\]

if \( \sigma = 1 \) and \( f_{ij, \sigma}(z_1, \ldots, z_{1-a_{ij}}, w) = 0 \) if \( \sigma \neq 1 \). Thus one can obtain in this way a large class of Drinfeld type presentations for \( \hat{g}[\mu] \).
4. Classical limit of twisted quantum affinization algebras

In [CJKT1] we introduced a class of twisted quantum affinization algebras by constructing their vertex representations. As an application of Theorem 1.4, in this section we determine the classical limit of these quantum algebras. Throughout this section, we assume that $A$ is simply-laced and that $\mu$ is not a transitive automorphism of $A^{(1)}_\varphi$, $\varphi \geq 2$.

We first recall the twisted quantum affinization algebra introduced in [CJKT1]. Let $\hbar$ be an indeterminate over $\mathbb{C}$, and $\mathbb{C}[[\hbar]]$ the power series ring of $\hbar$. Set $q = e^{\hbar}$, and let

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = \left(\sum_{m \geq 0} \frac{2n^{2m+1}}{(2m+1)!} \hbar^{2m}\right) \left(\sum_{m \geq 0} \frac{2}{(2m+1)!} \hbar^{2m}\right)^{-1} \in \mathbb{C}[[\hbar]]$$

for $n \in \mathbb{Z}$.

For $i, j \in I$, we introduce the polynomials

$$F_{ij}^\pm(z, w) = \prod_{k \in \mathbb{Z}^N; a_{\mu(k)}(j) \neq 0} (z - \xi^k q^{\pm a_{\mu(k)}(j)} w) ,$$

$$G_{ij}^\pm(z, w) = \prod_{k \in \mathbb{Z}^N; a_{\mu(k)}(j) \neq 0} (q^{\pm a_{\mu(k)}(j)} z - \xi^k w) .$$

In the case that $a_{ij} < 0$ and $i \notin \mathcal{O}(j)$, we also introduce the polynomials

$$p_{ij}^\pm(z, w) = (z^{d_i} + q^{\mp d_i} w^{d_i})^{s_i-1} q^{\mp 2d_i} z^{d_i} - w^{d_i} .$$

Furthermore, for $i \in I$ with $s_i = 2$, we set

$$p_i^\pm(z_1, z_2, z_3) = q^{\mp d_i} z_1^{d_i} - (q^{\pm d_i} + 1) z_2^{d_i} + q^{\pm 2d_i} z_3^{d_i} .$$

The twisted quantum affinization algebra $U_{\hbar}(\widehat{\mathfrak{g}}_{\mu})$ defined in [CJKT1] is a $\mathbb{C}[[\hbar]]$-algebra topologically generated by the elements

$$h_{i,m}, x_{i,m}^\pm, c, \quad i \in I, \ m \in \mathbb{Z},$$

and subject to the relations

(Q0) $x_{\mu(i),m}^\pm = \xi^m x_{i,m}^\pm, \quad h_{\mu(i),m} = \xi^m h_{i,m},$

(Q1) $c$ is central,

(Q2) $[h_{i,0}, h_{j,m}] = 0,$

(Q3) $[h_{i,0}, x_{j,m}^\pm] = \pm \sum_{k \in \mathbb{Z}^N} a_{\mu(k)}(j) x_{j,m}^\pm,$

(Q4) $[h_{i,m}, h_{j,m'}] = \delta_{m+m',0} \frac{1}{m} \sum_{k \in \mathbb{Z}^N} \xi^{mk}[ma_{\mu(k)}]_q q^{mc} - q^{-mc} \frac{q^{mc} - q^{-mc}}{q - q^{-1}}, \text{ if } m \neq 0,$
\[ [h_{i,m}, x_{j,n}^\pm] = \pm \frac{1}{m} \sum_{k \in \mathbb{Z}_N} \xi^{mk}[ma_{m\nu(j)}]q^{-\frac{1}{2}|m|c_{x_{j,m+n}}} \text{ if } m \neq 0, \]

\[ [x_i^+(z), x_j^-(w)] = \frac{1}{q - q^{-1}} \sum_{k \in \mathbb{Z}_N} \delta_{i,m} \]
\[
\times \left( \phi_i^+(q^{-\frac{1}{2}c}z) \delta \left( \frac{q^c \xi^k w}{z} \right) - \phi_i^-(q^{\frac{1}{2}c}z) \delta \left( \frac{q^{-c} \xi^k w}{z} \right) \right),
\]

\[ F_{ij}^+(z, w) x_j^+(z) x_i^+(w) = G_{ij}^+(z, w) x_i^+(z) x_j^+(w), \]

\[ \sum_{\sigma \in S_2} \left\{ p_{ij}^+(z_{\sigma(1)}, z_{\sigma(2)}) (x_i^+(z_{\sigma(1)}) x_i^+(z_{\sigma(2)}) x_j^+(w) - [2]_q a_{ij} x_i^+ (z_{\sigma(1)}) x_j^+(z_{\sigma(2)}) x_i^+(z_{\sigma(2)})
\]
\[
+ x_j^+(w)x_i^+(z_{\sigma(1)}) x_i^+(z_{\sigma(2)}) \right\} = 0, \text{ if } a_{ij} < 0 \text{ and } i \notin \mathcal{O}(j), \]

\[ \sum_{\sigma \in S_3} \left\{ p_{ij}^+(z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}) x_i^+(z_{\sigma(1)}) x_i^+(z_{\sigma(2)}) x_i^+(z_{\sigma(3)}) \right\} = 0, \text{ if } s_i = 2, \]

where \( i, j \in I, m, m' \in \mathbb{Z}^\times, n \in \mathbb{Z}, x_i^\pm(z) = \sum_{m \in \mathbb{Z}} x_{i,m} z^{-m} \text{ and } \]
\[ \phi_i^\pm(z) = q^{\pm h_{i,0}} \exp \left( \pm(q - q^{-1}) \sum_{m > 0} h_{i,m} z^{-m} \right). \]

Here and as usual, for any \( a \in \text{Span}_\mathbb{C}\{h_{i,0}, c \mid i \in I\}, \) we used the notation \( q^a = e^{ab}. \)

When \( A \) is of finite type, the quantum algebra \( \mathcal{U}_0(\mathfrak{g}_\mu) \) was first introduced by Drinfeld ([Dr]) for the purpose of providing a current algebra realization for the twisted quantum affine algebra.

Notice that the relation (Q6) is equivalent to the following relations:

\[ [x_{i,n}, x_{j,n'}^+] = \sum_{k \in \mathbb{Z}_N} \delta_{i,m} \xi^{kn}[a_{m\nu(j)}]q^{-\frac{n-m}{2}c} h_{j,n+n'}, \text{ if } n + n' > 0, \]

\[ [x_{i,n}, x_{j,n'}^-] = \sum_{k \in \mathbb{Z}_N} \delta_{i,m} \xi^{kn}[a_{m\nu(j)}]q^{-\frac{n+m}{2}c} h_{j,n+n'}, \text{ if } n + n' < 0, \]

\[ [x_{i,n}, x_{j,-n}] = \sum_{k \in \mathbb{Z}_N} \delta_{i,m} \xi^{kn}[a_{m\nu(j)}] \left( q^{nc} h_{j,0} + cq^{h_{i,0}} q^{2nc - q^{-nc}} \right), \]

where \( i, j \in I \) and \( n, n' \in \mathbb{Z}. \) Then it is immediate to see that the defining relations of the classical limit \( \mathcal{U}_0(\mathfrak{g}_\mu)/h\mathcal{U}_0(\mathfrak{g}_\mu) \) of \( \mathcal{U}_0(\mathfrak{g}_\mu) \) are the same as the defining relations
of $D_{\mathcal{P}}(g, \mu)$ with the family $\mathcal{P}$ defined as follows $((i, j) \in \mathbb{I} \text{ and } \sigma \in S_2)$

$$P_{ij,\sigma}(z_1, z_2, w) = \begin{cases} p_{ij}(z_1, z_2), & \text{if } i \notin \mathcal{O}(j), \ \sigma = 1; \\ 1, & \text{if } i \notin \mathcal{O}(j), \ \sigma = (12); \\ p_i(z_{\sigma(1)}, z_{\sigma(2)}, -w), & \text{if } i \in \mathcal{O}(j). \end{cases}$$

We remark that the polynomial $p_{ij}(z, w)$ defined above coincides with that defined in (3.17). Thus, by using Proposition 3.14, one can easily check that the family $\mathcal{P}$ defined above satisfies the conditions ($P_1$) and ($P_2$). Now, as a consequence of Theorem 1.4, one can immediately get that

Theorem 4.1. The classical limit $U_0(\hat{\mathfrak{g}}_\mu)/\hbar U_0(\hat{\mathfrak{g}}_\mu)$ of $U_0(\hat{\mathfrak{g}}_\mu)$ is isomorphic to the universal enveloping algebra of $\hat{\mathfrak{g}}[\mu]$.

5. Basics on $\Gamma$-vertex algebras and their quasi-modules

In this section we collect some basic materials on $\Gamma$-vertex algebras and their quasi-modules needed in the proof of Theorem 1.5.

Throughout this section, let $\Gamma$ be a fixed group equipped with a character $\phi : \Gamma \rightarrow \mathbb{C}^\times$. We write $\mathbb{C}_\Gamma[z,w]$ for the subalgebra of the polynomial algebra $\mathbb{C}[z,w]$ generated by the monomials $z - \alpha w, \alpha \in \phi(\Gamma)$. For a vector space $W$, we denote by $W((z_1, \cdots, z_s))$ the space of lower truncated (infinite) integral power series in the commuting variables $z_1, \cdots, z_s$ with coefficients in $W$, and set $E(W) = \text{Hom}(W, W((z)))$. For each pair $(\alpha, n) = ((\alpha_1, \cdots, \alpha_s), (n_1, \cdots, n_s)) \in (\mathbb{C}^\times)^s \times \mathbb{N}^s, \ s \geq 1$,

we denote that

$$\delta^{(\alpha, n)}(z_1, \cdots, z_s, w) = \prod_{1 \leq i \leq s} \frac{1}{n_i!} \left( \alpha_k^{-1} \frac{\partial}{\partial w} \right)^{n_i} z_i^{-1} \delta \left( \alpha_k \frac{w}{z_i} \right).$$

When $\alpha = 1 = (1, \cdots, 1)$, we also write $\delta^{(n)}(z_1, \cdots, z_s, w) = \delta^{(1,n)}(z_1, \cdots, z_s, w)$.

5.1. Definitions. In this subsection we recall the definitions of $\Gamma$-vertex algebras and their quasi-modules introduced in \cite{L2, L3}.

A $\Gamma$-vertex algebra is a vector space $V$ equipped with a distinguished vector $\mathbb{1}$ (called the vacuum vector), a linear map

$$Y : V \rightarrow E(V), \ v \mapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1},$$

and a group homomorphism

$$R : \Gamma \rightarrow \text{GL}(V), \ g \mapsto R_g,$$

such that the following axioms hold: for $g \in \Gamma, v \in V$,

$$Y(\mathbb{1}, x) = 1_V, \ Y(v, x) \mathbb{1} \in V[[x]] \quad \text{and} \quad \lim_{z \to 0} Y(v, x) \mathbb{1} = v,$$

$$(5.2) \quad Y(1, x) = 1_V.$$
\[ R_g(1) = 1 \quad \text{and} \quad R_gY(v, z)R_g^{-1} = Y(R_g(v), \phi(g)^{-1}z), \]

and for \( u, v \in V, \)

\[
\begin{align*}
  &z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y(u, z_1)Y(v, z_2) - z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y(v, z_2)Y(u, z_1) \\
  &= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y(Y(u, z_0)v, z_2).
\end{align*}
\]

In other words, a \( \Gamma \)-vertex algebra is a vertex algebra together with a \( \Gamma \)-action as in (5.1) such that the axiom (5.3) holds.

A linear map \( \varphi \) from a \( \Gamma \)-vertex algebra \((V, Y, 1, R)\) to another \( \Gamma \)-vertex algebra \((V', Y', 1', R')\) is called a \( \Gamma \)-vertex algebra homomorphism if for \( g \in \Gamma, v, w \in V, \)

\[
\varphi(1) = 1', \quad \varphi(Y(v, z)w) = Y'(\varphi(v), z)\varphi(w) \quad \text{and} \quad \varphi(R_g(v)) = R'_g(\varphi(v)).
\]

The following is an analogue of [LL, Proposition 5.7.9].

**Proposition 5.1.** Let \( \varphi \) be a linear map from a \( \Gamma \)-vertex algebra \((V, Y, 1, R)\) to a \( \Gamma \)-vertex algebra \((V', Y', 1', R')\) such that \( \varphi(1) = 1' \) and let \( T \) be a generating subset of \( V \) as a vertex algebra. Assume that for \( a \in T, g \in \Gamma \) and \( v \in V, \)

\[
\varphi(Y(a, z)v) = Y'(\varphi(a), z)\varphi(v),
\]

\[
\varphi(R_g(v)) = R'_g(\varphi(v)) \implies \varphi(R_g(Y(a, z)v)) = R'_g(Y'(\varphi(a), z)\varphi(v)).
\]

Then \( \varphi \) is a \( \Gamma \)-vertex algebra homomorphism.

**Proof.** In view of [LL, Proposition 5.7.9], the condition (5.5) implies that \( \varphi \) is a vertex algebra homomorphism. Thus, it remains to show that \( \varphi(R_g(v)) = R'_g(\varphi(v)) \) for \( g \in \Gamma, v \in V. \) This assertion can be easily checked by the assertion (5.5), the condition (5.6) and the fact that \( V \) is generated by \( T. \)

Let \((V, Y, 1, R)\) be a \( \Gamma \)-vertex algebra. A quasi-module for \( V \) is a vector space \( W \) equipped with a linear map

\[ Y_W : V \to E(W), \quad v \mapsto Y_W(v, z) \]

such that the following conditions hold: for \( g \in \Gamma, v \in V, \)

\[
Y_W(1, z) = 1_W, \quad Y_W(R_g v, z) = Y_W(v, \phi(g)z),
\]

and for \( u, v \in V, \) there exists a polynomial \( 0 \neq f(z_1, z_2) \in \mathbb{C}[z_1, z_2] \) such that

\[
\begin{align*}
  &z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) f(z_1, z_2)Y_W(u, z_1)Y_W(v, z_2) - z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) f(z_1, z_2) \\
  &\cdot Y_W(v, z_2)Y_W(u, z_1) = z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) f(z_1, z_2)Y_W(Y(u, z_0)v, z_2).
\end{align*}
\]

When \( \Gamma = \{1\} \), a quasi-module for a \( \Gamma \)-vertex algebra is nothing but a module for a vertex algebra.
Finally, we remark that if \( \varphi : V \rightarrow V' \) is a homomorphism of \( \Gamma \)-vertex algebras and \((W, Y'_W)\) is a quasi-\(V'\)-module, then \( W \) is naturally a quasi-\(V\)-module with the module action \( Y'_W = Y_W \circ \varphi \).

5.2. \( \Gamma \)-vertex algebras arising from \( \Gamma \)-local subspaces. In this subsection we first recall a general construction of \( \Gamma \)-vertex algebras and their quasi-modules developed in [L2], and then prove a result about the vanishing of certain (multi-)commutators among vertex operators (see Lemma 5.4).

Let \( W \) be a given vector space. Formal series \( a(z), b(z) \in \mathcal{E}(W) \) are said to be mutually \( \Gamma \)-local if there exists a non-zero polynomial \( f(z, w) \in \mathbb{C}_\Gamma[z, w] \) such that

\[
(5.10) \quad f(z, w) [a(z), b(w)] = 0.
\]

In [L2], for any mutually \( \Gamma \)-local pair \( a(z), b(z) \in \mathcal{E}(W) \) and \( \alpha \in \mathbb{C}^\times \), the author defined a corresponding generating function

\[
\mathcal{Y}_\alpha(a(z), w)b(z) = \sum_{n \in \mathbb{Z}} (a(z)_{(\alpha, n)} b(z)) w^{-n-1} \in \mathcal{E}(W)[[w, w^{-1}]].
\]

See [L2, Definition 3.4] for details. When \( \alpha = 1 \), we often write

\[
\mathcal{Y}(a(z), w)b(z) = \mathcal{Y}_1(a(z), w)b(z) \quad \text{and} \quad a(z)_n b(z) = a(z)_{(1,n)} b(z).
\]

A subspace \( U \) of \( \mathcal{E}(W) \) is said to be \( \Gamma \)-local if any pair \( a(z), b(z) \in U \) are mutually \( \Gamma \)-local. For any \( \alpha \in \mathbb{C}^\times \), a \( \Gamma \)-local subspace \( U \) of \( \mathcal{E}(W) \) is said to be closed under \( \mathcal{Y}_\alpha \) if \( a(z)_{(\alpha, n)} b(z) \in U \) for any pair \( a(z), b(z) \in U \) and \( n \in \mathbb{Z} \). We define a linear map \( R : \Gamma \rightarrow \text{GL} \mathcal{E}(W) \) by the rule

\[
(5.9) \quad R_g(a(z)) = a(\phi(g)z), \quad g \in \Gamma, a(z) \in \mathcal{E}(W).
\]

The following result was proved in [L3, Theorem 2.9]

**Proposition 5.2.** Let \( U \) be a \( \Gamma \)-local subspace of \( \mathcal{E}(W) \). Then there exists a smallest \( \Gamma \)-local subspace \( \langle U \rangle_\Gamma \), that contains \( 1_W \) and \( U \) and that is closed under \( \mathcal{Y}_\alpha \) for any \( \alpha \in \phi(\Gamma) \). Moreover, \((\langle U \rangle_\Gamma, 1_W, \mathcal{Y}, R)\) is a \( \Gamma \)-vertex algebra and \((W, Y_{\mathbb{M}})\) is a natural faithful quasi-module for it, with

\[
(5.10) \quad Y_{\mathbb{M}}(a(z), w) = a(w), \quad a(z) \in \langle U \rangle_\Gamma, w \in W.
\]

Let \( U \) be as in Proposition 5.2. We remark that, if \( U \) is also invariant under the \( \Gamma \)-action \( (5.9) \), then it follows from [L2, Proposition 4.12] that

\[
(5.11) \quad \langle U \rangle_\Gamma = \langle U \rangle_{(1)} = \text{Span}_\mathbb{C} \{ a^{(1)}_{r_1} \cdots a^{(s)}_{r_s} 1_W \mid a^{(i)} \in U, r_i \in \mathbb{Z}, 1 \leq i \leq s, s \geq 0 \}.
\]

Namely, \( \langle U \rangle_\Gamma \) is generated by \( U \) as a vertex algebra. We now mention a result of Li which plays an essential role in determining the structure of various vertex algebras arising from \( \Gamma \)-local sets.
Lemma 5.3. Let \( a(z), b(z) \in \mathcal{E}(W) \) and let \( 0 \neq f(z, w) \in \mathbb{C}_1[z, w] \). If the \( \Gamma \)-locality (5.8) holds, then
\[
(w_1 - w_2)^s [\mathcal{Y}(a(z), w_1), \mathcal{Y}(b(z), w_2)] = 0,
\]
where \( s \) is the order of the zero of \( f(z, w) \) at \( z = w \). Moreover, if
\[
f(z, w) = \prod_{k=1}^{l}(z - \alpha_k w)^{r_k}, \quad \alpha_k \in \phi(\Gamma), \; r_k \in \mathbb{Z}_+,
\]
then one has that
\[
[a(z), b(w)] = \sum_{k=1}^{l} \sum_{n=0}^{r_k-1} a(w)(\alpha_k, n)b(w) \delta(\alpha_k, n)(z, w),
\]
(5.13)
\[
[\mathcal{Y}(a(z), w_1), \mathcal{Y}(b(z), w_2)] = \sum_{k=1}^{l} \delta_{\alpha_k, 1} \sum_{n=0}^{r_k-1} (\mathcal{Y}(a(z), n) b(z, w_2)) \delta(n)(w_1, w_2).
\]
(5.14)

Proof. The locality (5.12) was proved in [L2, Proposition 4.8], the commutator formula (5.13) was proved in [L2, Proposition 3.13], and the commutator formula (5.14) is implied by (5.12) and (5.13). \( \square \)

The rest part of this subsection is devoted to a proof of the following result.

Lemma 5.4. Let \( U \) be a \( \Gamma \)-local subset of \( \mathcal{E}(W) \) and \( a_1(z), \cdots, a_s(z), b(z) \in U \) for some \( s \in \mathbb{Z}_+ \). Assume that there exist polynomials \( g(z_1, \cdots, z_s, w), g'(z_1, \cdots, z_s, w) \) in \( \mathbb{C}[z_1, \cdots, z_s, w] \) such that
\[
g(w, \cdots, w, w) \neq 0,
\]
and such that
\[
g(z_1, \cdots, z_s, w)g'(z_1, \cdots, z_s, w)[a_1(z_1), \cdots, [a_s(z_s), b(w)]] = 0.
\]
Then the following relation holds true
\[
g'(w_1, \cdots, w_s) [\mathcal{Y}(a_1(z), w_1), \cdots, [\mathcal{Y}(a_s(z), w_s), \mathcal{Y}(b(z), w)]] = 0.
\]
(5.17)

Before proving Lemma 5.4, we present two of its by-products, which will play a key role in our proof of Theorem 1.5.

Corollary 5.5. Let \( U \) be a \( \Gamma \)-local subset of \( \mathcal{E}(W) \) and \( a_1(z), a_2(z), b(z) \in U \). Assume that there exists a positive integer \( M \) such that
\[
(z_1^M - z_2^M) [a(z_1), [a(z_2), b(w)]] = 0.
\]
Then the following relation holds true
\[
(w_1 - w_2) [\mathcal{Y}(a_1(z), w_1), [\mathcal{Y}(a_2(z), w_2), \mathcal{Y}(b(z), w)]] = 0.
\]
(5.18)

Proof. By applying Lemma 5.4 with \( g(z_1, z_2, w) = z_1 - z_2 \) and \( g'(z_1, z_2, w) = \frac{z_1^M - z_2^M}{z_1 - z_2} \). \( \square \)
Corollary 5.6. Let $U$ be a $\Gamma$-local subset of $\mathcal{E}(W)$ and $a_1(z), \ldots, a_s(z), b(z) \in U$ for some $s \in \mathbb{Z}_+$. Assume that there exist polynomials $h_{ij}(z, w) \in \mathbb{C}[z, w], 1 \leq i < j \leq s$ and $g_{\sigma}(z_1, \ldots, z_s, w) \in \mathbb{C}[z_1, \ldots, z_s, w], \sigma \in S_s$ such that
\begin{equation}
(5.18) \quad h_{ij}(w, w) \neq 0, \quad h_{ij}(z, w) [a_i(z), a_j(w)] = 0, \quad i, j = 1, \ldots, s,
\end{equation}
and moreover
\begin{equation}
(5.19) \quad \sum_{\sigma \in S_s} g_{\sigma}(w, \ldots, w, w) \neq 0,
\end{equation}
\begin{equation}
(5.20) \quad \sum_{\sigma \in S_s} g_{\sigma}(z_1, \ldots, z_s, w) [a_1(z_{\sigma(1)}), \ldots, [a_s(z_{\sigma(s)}), b(w)]] = 0.
\end{equation}
Then the following relation holds true
\begin{equation}
(5.21) \quad [\mathcal{Y}(a_1(z), w_1), \ldots, [\mathcal{Y}(a_{s-1}(z), w_{s-1}), [\mathcal{Y}(a_s(z), w_s), \mathcal{Y}(b(z), w)]]) = 0.
\end{equation}
\begin{proof}
We need to introduce the polynomials
\begin{align*}
h(z_1, \ldots, z_s) &= \prod_{1 \leq i < j \leq s} \prod_{1 \leq a < b \leq s} h_{ij}(z_a, z_b), \\
g(z_1, \ldots, z_s, w) &= h(z_1, \ldots, z_s) \cdot \left( \sum_{\sigma \in S_s} g_{\sigma}(z_1, \ldots, z_s, w) \right).
\end{align*}
From (5.18) and (5.19), it follows that
\begin{equation}
(5.22) \quad g(w, \ldots, w, w) \neq 0.
\end{equation}
Moreover, by using (5.18) and the Jacobi identity, one has that
\begin{equation}
(5.23) \quad h(z_1, \ldots, z_s) \left( [a_1(z_1), \ldots, [a_s(z_s), b(z)]] - [a_1(z_{\sigma(1)}), \ldots, [a_s(z_{\sigma(s)}), b(z)]] \right) = 0,
\end{equation}
for all $\sigma \in S_s$. This together with (5.20) gives that
\begin{equation}
(5.24) \quad g(z_1, \ldots, z_s, w) [a_1(z_1), \ldots, [a_s(z_s), b(z)]] = 0.
\end{equation}
So the assertion is implied by Lemma 5.4 (with $g'(z_1, \ldots, z_s, w) = 1), (5.22)$, and (5.23). \qed

Now we turn to prove Lemma 5.4. We start with a slight generalization of Lemma 5.3.

Lemma 5.7. Let $U$ be a $\Gamma$-local subset of $\mathcal{E}(W)$ and $a_1(z), \ldots, a_s(z), b(z) \in U$ for some $s \in \mathbb{Z}_+$. Then the formal series $[a_1(z_1), \ldots, [a_s(z_s), b(z)]]$ is a finite summation of the form
\begin{equation}
(5.25) \quad \sum_{(\alpha, n) \in (\phi(\Gamma))^* \times \mathbb{N}^s} c_{\alpha, n}(w) \delta^{(\alpha, n)}(z_1, \ldots, z_s, \bar{z})
\end{equation}
for some uniquely determined formal series $c_{\alpha, n}(z) \in \langle U \rangle_{\Gamma}$. Moreover,
\begin{equation}
[\mathcal{Y}(a_1(z), w_1), \ldots, [\mathcal{Y}(a_s(z), w_s), \mathcal{Y}(b(z), w)]]
\end{equation}
is a finite summation of the form
\[(5.25) \sum_{(\alpha, n) \in (\phi(\Gamma))^s \times \mathbb{N}^e} \delta_{\alpha, 1} Y(c_{1, n}(z), w) \delta^{(n)}(w_1, \cdots, w_s, w).\]

**Proof.** When \(s = 1\), the assertion is implied by Lemma 5.8. For the general case, one can prove the assertion by using Lemma 5.3 and an induction argument. We omit the details. \(\square\)

We now record two well-known properties of \(\delta\)-functions for later use, whose proofs can be found respectively in [L1, § 2] and [L2, (2.3.51)].

**Lemma 5.8.** Let \(f_1(w), \cdots, f_l(w) \in W[[w, w^{-1}]]\) and let \((\alpha_1, n_1), \cdots, (\alpha_l, n_l)\) be some distinct pairs in \((\phi(\Gamma))^s \times \mathbb{N}^e\), where \(s \in \mathbb{Z}_+\). Then
\[
\sum_{k=1}^l f_k(w) \delta^{(\alpha_k, n_k)}(z_1, \cdots, z_s, w) = 0
\]
if and only if \(f_k(w) = 0\) for all \(k = 1, \cdots, l\).

**Lemma 5.9.** For any Laurent polynomial \(f(w_1, w_2)\) and \(\alpha \in \mathbb{C}^e\), one has that
\[
f(w_1, w_2) \left( \frac{\partial}{\partial w_1} \right)^n \delta\left( \frac{w_1}{w_2} \right) = \sum_{k=0}^n (-1)^k \binom{n}{k} \left( \frac{\partial}{\partial w_1} \right)^k f(w_2, \alpha w_2) \left( \frac{\partial}{\partial w_1} \right)^{n-k} \delta\left( \frac{w_1}{w_2} \right).\]

Here we are ready to finish the proof of Lemma 5.4. Let \(a_1(z), \cdots, a_s(z), b(z)\) and \(g(z_1, \cdots, z_s, w), g'(z_1, \cdots, z_s, w)\) be as in Lemma 5.4. Due to Lemma 5.4, we may write the the commutators
\[
[a_1(z_1), \cdots, [a_s(z_s), b(w)]] \quad \text{and} \quad [Y(a_1(z_1), w_1), \cdots, [Y(a_s(z_s), w_s), Y(b(z), w)]]
\]
as in \(5.23\) and \(5.25\), respectively. Given an \((\alpha, n) \in (\phi(\Gamma))^s \times \mathbb{N}^e\). Then it follows from Lemma 5.9 that the formal series
\[
A(z_1, \cdots, z_s, w) \delta^{(\alpha, n)}(z_1, \cdots, z_s, w) \quad (A = g \text{ or } g')
\]
is a finite summation of the form
\[(5.26) \sum_{m \leq n \in \mathbb{N}^e} A_{\alpha, n, m}(w) \delta^{(\alpha, m)}(z_1, \cdots, z_s, w)
\]
for some \(A_{\alpha, n, m}(w) \in \mathbb{C}[w]\), where \(m \preceq n\) means that \(m_k \leq n_k\) for all \(1 \leq k \leq s\).

For any \(m \preceq n\), set
\[
\begin{align*}
c_{\alpha, n, m}(w) &= g_{\alpha, n, m}(w) c_{\alpha, n}(w) \in W[[w, w^{-1}]], \\
b_{n, m}(w) &= g_{1, n, m}(w) Y(c_{1, n}(z), w) \in E(W)[[w, w^{-1}]].
\end{align*}
\]
Then one can conclude from (5.24) and (5.26) (with $A = g'$) that the formal series
\[ g'(z_1, \ldots, z_s, w) [a_1(z_1), \ldots, [a_s(z_s), b(w)]] \]
is a finite summation of the form
\[ \sum_{(\alpha, n) \in (\phi(\Gamma))^s \times N^s} c_{\alpha, n}'(w) \delta^{(\alpha, n)}(z_1, \ldots, z_s, w), \]
where $c_{\alpha, n}'(w) = \sum_{n' > n} c_{\alpha, n', n}(w)$. Similarly, by applying (5.25) and (5.26) (with $A = g$), we know that the formal series
\[ g'(w_1, \ldots, w_s, w) [\mathcal{V}(a_1(z), w_1), \ldots, [\mathcal{V}(a_s(z), w_s), \mathcal{V}(b(z), w)]] \]
has the following expression
\[ \sum_{n \in N^s} b_n'(w) \delta^{(n)}(w_1, \ldots, w_s, w), \]
where $b_n'(w) = \sum_{n' > n} b_{n', n}(w)$.

Thus, it suffices to prove that $b_n'(w) = 0$ for all $n \in N^s$. Indeed, assume conversely that the finite set $\{ n \in N^s \mid b_n'(w) \neq 0 \}$ is non-empty. Let us take a maximal element $n_0$ in this finite set with respect to the partial order $\preceq$. We remark that for $\alpha \in \phi(\Gamma)$ and $m, n \in N^s$ with $m \not\preceq n$, the formal series $c_{\alpha, n, m}(z)$ may be not contained in $(U)_\Gamma$. But these formal series belong to any maximal $\Gamma$-local subspace, say $U'$, of $\mathcal{E}(W)$ that contains $1_w$ and $U$. Moreover, as operators on $\text{End}(U')[[w, w^{-1}]]$, one has by definition (II.2 Definition 3.4) that
\[ \mathcal{V}(c_{1, n, m}(z), w) = b_{n, m}(w). \]
This in particular shows that $c_{1, n_0}'(w) \neq 0$ and that $n_0$ is also maximal in the set $\{ n \in N^s \mid c_n'(w) \neq 0 \}$. Now, from (5.27) and (5.26) (with $A = g$), we know that the formal series
\[ g(z_1, \ldots, z_s, w)g'(z_1, \ldots, z_s, w) [a_1(z_1), \ldots, [a_s(z_s), b(w)]] \]
is a finite summation of the form
\[ \sum_{(\alpha, n) \in (\phi(\Gamma))^s \times N^s} c_{\alpha, n}''(w) \delta^{(\alpha, n)}(z_1, \ldots, z_s, w), \]
where $c_{\alpha, n}''(w) = \sum_{n' > n} g_{\alpha, n', n}(w) c_{\alpha, n'}'(w)$. This together with Lemma 5.8 and the assumption (5.16) gives that
\[ c_{\alpha, n}''(w) = 0, \quad \forall (\alpha, n) \in (\phi(\Gamma))^s \times N^s. \]
But, by the maximality of $n_0$ and the assumption (5.15), one has that
\[ c_{1, n_0}'(w) = g_{1, n_0, n_0}(w) c_{1, n_0}'(w) = g(w, w, \ldots, w) c_{1, n_0}'(w) \neq 0, \]
a contradiction to (5.29). This finishes the proof of Lemma 5.4.
5.3. $\Gamma$-conformal Lie algebras and their universal $\Gamma$-vertex algebras. In this subsection we recall another general construction of $\Gamma$-vertex algebras and their quasi-modules given in [L3].

Recall that a conformal Lie algebra $(C, Y_-, T)$ ([K2]), also known as a vertex Lie algebra ([DLM] [P]), is a vector space $C$ equipped with a linear operator $T$ and a linear map $Y_- : C \to \text{Hom}(C, z^{-1}C[z^{-1}])$, $u \mapsto Y_-(u, z) = \sum_{n \geq 0} u(n) z^{-n-1}$ (5.30) such that for any $u, v \in C$,

\begin{align*}
[T, Y_-(u, z)] &= Y_-(Tu, z) = \frac{d}{dz}Y_-(u, z), \\
Y_-(u, z)v &= \text{Sing} \left( e^{zT}Y_-(v, -z)u \right), \\
[Y_-(u, z), Y_-(v, w)] &= \text{Sing}(Y_-(Y_-(u, z - w)v, w)),
\end{align*}

(5.31)

where Sing stands for the singular part.

A conformal Lie algebra structure on a vector space $C$ exactly amounts to a Lie algebra structure on the quotient space $\hat{C} = C[t, t^{-1}] \otimes C / (1 \otimes T + \frac{d}{dt} \otimes 1)(C[t, t^{-1}] \otimes C)$ of $C[t, t^{-1}] \otimes C$. Let us denote by

$$
\rho : \mathbb{C}[t, t^{-1}] \otimes C \to \hat{C}, \quad t^m \otimes u \mapsto u(m), \quad u \in C, \ m \in \mathbb{Z}
$$

the natural quotient map. The following result was proved in [P, Remark 4.2].

**Lemma 5.10.** Let $C$ be a vector space equipped with a linear operator $T$ and a linear map $Y_-$ as in (5.30) such that (5.31) holds. Then $C$ is a conformal Lie algebra if and only if there is a Lie algebra structure on $\hat{C}$ such that

\begin{align*}
[u(m), v(n)] &= \sum_{i \geq 0} \binom{m}{i} (u(i)v)(m + n - i),
\end{align*}

(5.32)

for $u, v \in C$, $m, n \in \mathbb{Z}$.

Let $C$ be a conformal Lie algebra. Set

$$
\hat{C}^{(-)} = \rho(t^{-1}C[t^{-1}] \otimes C) \quad \text{and} \quad \hat{C}^{(+)} = \rho(C[t] \otimes C).
$$

Then both $\hat{C}^{(+)}$ and $\hat{C}^{(-)}$ are subalgebras of $\hat{C}$ and

\begin{equation}
\hat{C} = \hat{C}^{(+)} \oplus \hat{C}^{(-)}
\end{equation}

(5.33)

is a polar decomposition of $\hat{C}$. Moreover, the map

\begin{equation}
C \to \hat{C}^{(-)}, \quad u \mapsto u(-1)
\end{equation}

(5.34)

is an isomorphism of vector spaces ([P, Theorem 4.6]).
Consider the induced \( \hat{C} \)-module
\[
V_C = \mathcal{U}(\hat{C}) \otimes_{\mathcal{U}(\hat{C}^+)} \mathbb{C},
\]
where \( \mathbb{C} \) is the one dimensional trivial \( \hat{C}^+ \)-module. Set \( \mathbb{1} = 1 \otimes 1 \). Identify \( C \) as a subspace of \( V_C \) through the linear map \( u \mapsto u(-1) \mathbb{1} \). Then it was proved in [P] that there exists a unique vertex algebra structure on \( V_C \) with \( \mathbb{1} \) as the vacuum vector and with
\[
Y(u, z) = u(z) = \sum_{n \in \mathbb{Z}} u(n) z^{-n-1}
\]
for \( u \in C \). In the literature, \( V_C \) is often called the universal vertex algebra associated to \( C \).

**Lemma 5.11.** Let \( C_0 \) be a subset of \( C \) such that \( \rho(C[t, t^{-1}] \otimes C_0) \) generates the associated Lie algebra \( \hat{C} \). Then \( C_0 \) generates \( V_C \) as a vertex algebra.

**Proof.** Since \( C \) generates \( V_C \) as a vertex algebra, we only need to show that \( C_0 \) generates \( C \) as a conformal Lie algebra. Using (5.32), one knows that any element in the subalgebra of \( \hat{C} \) generated by \( \rho(C[t, t^{-1}] \otimes C_0) \) is a finite summation \( \sum u^{(i)}(n_i) \) for some \( u^{(i)} \in \langle C_0 \rangle \), \( n_i \in \mathbb{Z} \), where \( \langle C_0 \rangle \) indicates the conformal Lie subalgebra of \( C \) generated by \( C_0 \). As \( \rho(C[t, t^{-1}] \otimes C_0) \) generates the Lie algebra \( \hat{C} \), for any \( u \in C \), there exist finitely many \( u^{(i)} \in \langle C_0 \rangle \) and \( n_i \in \mathbb{Z} \) such that
\[
(5.35) \quad u(-1) = \sum u^{(i)}(n_i).
\]
Note that \( \hat{C}^+ \cap \hat{C}^- = 0 \) and \( k!u(-k+1) = T^k(u)(-1) \) for \( u \in C \), \( k \in \mathbb{Z}_+ \). Thus, we may (and do) assume that all the integers \( n_i \) appeared in (5.35) are \(-1\). Then, due to the isomorphism given in (5.34), one gets that \( \langle C_0 \rangle = C \), as desired. \( \square \)

Recall from [L3] (see also [G-KK]) that a \( \Gamma \)-conformal Lie algebra \((C, Y_-, T, R)\) is a conformal Lie algebra \((C, Y_-, T)\) equipped with a group homomorphism
\[
R : \Gamma \to \text{GL}(C), \quad g \mapsto R_g
\]
such that for any \( u, v \in C \),
\[
(5.36) \quad TR_g = \phi(g)R_g T,
\]
\[
(5.37) \quad R_g Y_-(u, z) R_g^{-1} = Y_-(R_g u, \phi(g)^{-1} z)),
\]
\[
Y_-(R_g u, z) v = 0 \quad \text{for all but finitely many} \ g \in \Gamma.
\]

As was pointed out in [G-KK], the following result shows that a \( \Gamma \)-conformal Lie algebra exactly amounts to a conformal Lie algebra equipped with a \( \Gamma \)-action on the associated Lie algebra by automorphisms.
Lemma 5.12. Let \((C, Y, T)\) be a conformal Lie algebra and let \(R : \Gamma \to \text{GL}(C)\) be a group homomorphism such that (5.36) holds. Then \((C, Y, T, R)\) is a \(\Gamma\)-conformal Lie algebra if and only if for each \(g \in \Gamma\), the map

\[(5.38) \quad \hat{R}_g : \hat{C} \to \hat{C}; \quad u(m) \mapsto \phi(g)^{m+1}(R_g(u)(m)), \quad u \in C, \ m \in \mathbb{Z} \]

is an automorphism of \(\hat{C}\) and for each \(u, v \in C\), \([\hat{R}_g(u)(z), v(w)] = 0\) for all but finitely many \(g \in \Gamma\).

Proof. Using (5.36), one can easily check that the map \(\hat{R}_g\) is well-defined. Moreover, for any \(u, v \in C\) and \(g \in \Gamma\),

\[
[\hat{R}_g(u)(z), \hat{R}_g(v)(w)] = [R_g(u)(\phi(g)^{-1}z), R_g(v)(\phi(g)^{-1}w)] = \sum_{i \geq 0} \phi(g)^{i+1} (R_g(u)(i)R_g(v)) (\phi(g)^{-1}w)\delta^{(i)}(z, w).
\]

On the other hand,

\[
\hat{R}_g([u(z), v(w)]) = \sum_{i \geq 0} \hat{R}_g(u(i)v)(w)\delta^{(i)}(z, w) = \sum_{i \geq 0} R_g(u(i)v)(\phi(g)^{-1}w)\delta^{(i)}(z, w).
\]

Thus, \(\hat{R}_g\) is an automorphism of \(\hat{C}\) if and only if for any \(i \in \mathbb{Z}\),

\[
R_g(u(i)v) = \phi(g)^{i+1}(R_gu)(i)(R_gv),
\]

Notice that this identity is equivalent to that in (5.37) and so we complete the proof of lemma.

Suppose now that \(C\) is a \(\Gamma\)-conformal Lie algebra. Then for each \(g \in \Gamma\), the automorphism \(\hat{R}_g\) on \(\hat{C}\) preserves the polar decomposition (5.33) and hence induces a linear automorphism, still denoted by \(R_g\), on \(V_C \cong U(\hat{C}^{(-)})\). The following result was proved in [L3, Lemma 4.16].

Lemma 5.13. If \(C\) is a \(\Gamma\)-conformal Lie algebra, then the universal vertex algebra \(V_C\) associated to \(C\) is a \(\Gamma\)-vertex algebra with the \(\Gamma\)-action given by

\[
R : \Gamma \to \text{GL}(V_C), \quad g \mapsto R_g, \ g \in \Gamma.
\]

Following [L3], we define a new operation \([\cdot, \cdot]_{\Gamma}\) on \(\hat{C}\) by letting

\[
[a, b]_{\Gamma} = \sum_{g \in \Gamma} [\hat{R}_g(a), b]
\]

for \(a, b \in \hat{C}\). Under this operation, the quotient space

\[
\hat{C}_{\Gamma} = \hat{C}/\text{Span}_C\{\hat{R}_g(a) - a \mid a \in \hat{C}, g \in \Gamma\}
\]

becomes a Lie algebra ([L3, Lemma 4.1]).

Remark 5.14. If \(\Gamma = \langle g \rangle\) is a finite cyclic group, then it is easy to see that \(\hat{C}_\Gamma\) is isomorphic to the subalgebra of \(\hat{C}\) fixed by \(\hat{g}\).
For any \( u \in C, m \in \mathbb{Z} \), we will also denote the image of \( u(m) \) in \( \hat{C}_\Gamma \) by itself. Note that there is a natural \( \mathbb{Z} \)-grading structure on \( \hat{C}_\Gamma \) such that \( \deg(u(m)) = m \) for \( u \in C, m \in \mathbb{Z} \). For any restricted \( \hat{C}_\Gamma \)-module \( W \) and \( u \in C \), we set
\[
    u(z) = \sum_{m \in \mathbb{Z}} u(m) z^{-m-1} \in \mathcal{E}(W).
\]

The following result was proved in [L3, Theorem 4.17].

**Proposition 5.15.** Let \( C \) be a \( \Gamma \)-conformal Lie algebra. Assume that the character \( \phi : \Gamma \to \mathbb{C}^\times \) is injective. Then any restricted module \( W \) for \( \hat{C}_\Gamma \) is naturally a quasimodule for the \( \Gamma \)-vertex algebra \( V_C \) with \( Y_W(u, z) = u(z) \) for \( u \in C \). On the other hand, any quasimodule \( W \) for the \( \Gamma \)-vertex algebra \( V_C \) is naturally a restricted module for the Lie algebra \( \hat{C}_\Gamma \) with \( u(z) = Y_W(u, z) \) for \( u \in C \).

6. **Proof of Theorem 1.4 and Theorem 1.5**

In this section we prove the main results of this paper (Theorems 1.4 and 1.5) stated in the Introduction. Throughout this section, let \( \Gamma = \langle \mu \rangle \) and let \( \phi : \Gamma \to \mathbb{C}^\times \) be the character determined by the rule \( \phi(\mu) = \xi^{-1} \).

6.1. **Proof of Theorem 1.5.** As in Theorem 1.5 let \( m \) be one of the algebras \( g, n_+ \) and \( n_- \), and \( W \) an arbitrary restricted \( \mathcal{D}_P(m, \mu) \)-module. We define a subspace \( U_{m, W} \) of \( \mathcal{E}(W) \) as follows
\[
    U_{n_+, W} = \sum_{i \in I} \mathbb{C} x_i^+(z), \quad U_{n_-, W} = \sum_{i \in I} \mathbb{C} x_i^-(z) \quad \text{and} \quad U_{g, W} = U_{n_+, W} \oplus U_{n_-, W}.
\]

Then one can conclude from the defining relation \( (X \pm) \) of \( \mathcal{D}_P(m, \mu) \) that \( U_{m, W} \) is a \( \Gamma \)-local subspace of \( \mathcal{E}(W) \). Therefore, it follows from Proposition 5.2 that \( U_{m, W} \) generates a \( \Gamma \)-vertex algebra \( \langle U_{m, W} \rangle_\Gamma, 1_W, \mathcal{Y}, \mathfrak{R} \rangle \) and that \( (W, Y_{2m}) \) is a quasimodule of \( \langle U_{m, W} \rangle_\Gamma \) with
\[
    Y_{2m}(x_i^+(z), w) = x_i^+(w), \quad i \in I.
\]

Moreover, it follows from (5.9) and the first relation in \( (X \pm) \) that
\[
    \mathfrak{R}_{\mu^k}(x_i^+(z)) = x_i^+(\xi^k z) = \xi^k x_i^{\pm}(\mu^k z), \quad i \in I, k \in \mathbb{Z}_N.
\]

When \( \mu = 1 \), recall the usual Serre relation \( (DS)_P \) given in (3.21). The following result will be proved in §6.3.

**Proposition 6.1.** \( \mathcal{D}_p(g, 1) \) is a Drinfeld type presentation of \( \hat{g} \).

Assume now that Proposition 6.1 holds. Then we can prove the following result.

**Lemma 6.2.** Let \( m \) be one of the algebras \( g, n_+ \) and \( n_- \). Then there is an \( \hat{m} \)-module structure on the \( \Gamma \)-vertex algebra \( \langle U_{m, W} \rangle_\Gamma \) with the action determined by
\[
    e_i^+(w) = \mathcal{Y}(x_i^+(z), w), \quad i \in I.
\]
Furthermore, as an \( \hat{m} \)-module, \( \langle U_{m,W} \rangle_\Gamma \) is generated by \( 1_W \) and the elements in

\[ \{ t_i^m \otimes e_i^\pm \mid i \in I, m \in \mathbb{N} \} \cap \hat{m} \]

act trivially on \( 1_W \).

**Proof.** We first prove that the action \( (6.3) \) determines an \( \hat{m} \)-module structure on \( \langle U_{m,W} \rangle_\Gamma \) with \( m = n_+ \). In view of Proposition \( 6.1 \) it suffices to show that the operators \( \mathcal{Y}(x_i^\pm(z), w) \), \( i \in I \) satisfy the relations

\[ f_{ij}(w_1 - w_2) \cdot [\mathcal{Y}(x_i^\pm(z), w_1), \mathcal{Y}(x_j^\mp(z), w_2)] = 0, \quad \text{for } i, j \in I, \]

and if \( i, j \in I \) satisfy the following addition relation

\[ (w_1 - w_2) [\mathcal{Y}(x_i^\pm(z), w_1), \mathcal{Y}(x_j^\mp(z), w_2)] = 0, \quad \text{for } i \neq j. \]

Indeed, the relation \( (6.5) \) is implied by the defining relation \((X+)\) of \( \mathcal{D}_P(n_+, \mu) \) and \( (5.12) \), the relation \( (6.6) \) follows from the defining relation \((DS+)\) of \( \mathcal{D}_P(n_+, \mu) \), the condition \((P2)\) and Corollary \( 5.6 \) and the relation \( (6.7) \) is implied by the defining relation \((AS\pm)\) of \( \mathcal{D}_P(n_+, \mu) \) and Corollary \( 5.5 \). Similarly, one can prove that \( \langle U_{n_-,W} \rangle_\Gamma \) is an \( \hat{n}_- \)-module with the action determined by \( (6.3) \).

Now we turn to consider the case that \( m = g \). In this case the formal series

\[ h_i(z) = \sum_{m \in \mathbb{Z}} h_{i,m} z^{-m-1}, \quad i \in I, \]

are also contained in \( \langle U_{g,W} \rangle_\Gamma \). We are going to prove that the action

\[ e_i^\pm(w) = \mathcal{Y}(x_i^\pm(z), w), \quad \alpha_i^\pm(w) = \mathcal{Y}(h_i(z), w), \quad i \in I, \quad k_1 = N \mathcal{Y}(c(z), w) \]

determines a \( \hat{g} \)-module structure on \( \langle U_{g,W} \rangle_\Gamma \). This action is well-defined as

\[ \frac{\partial}{\partial z} \mathcal{Y}(c(z), w) = \mathcal{Y}(\frac{\partial}{\partial z} c(z), w) = 0. \]

Again by using Proposition \( 6.1 \) in addition the relations in \( (6.5) \) and \( (6.6) \), it remains to prove that the following commutation relations:

\[ [\mathcal{Y}(h_i(z), w_1), \mathcal{Y}(h_j(z), w_2)] = e_j^{-1} a_{ij} \mathcal{Y}(c(z), w_2) \delta^{(1)}(w_1, w_2); \]

\[ [\mathcal{Y}(h_i(z), w_1), \mathcal{Y}(x_j^\pm(z), w_2)] = \pm a_{ij} \mathcal{Y}(x_j^\pm(z), w_2) \delta^{(0)}(w_1, w_2); \]

\[ [\mathcal{Y}(h_i(z), w_1), \mathcal{Y}(c(z), w_2)] = 0 = [\mathcal{Y}(x_i^\pm(z), w_1), \mathcal{Y}(c(z), w_2)]; \]

\[ [\mathcal{Y}(x_i^\pm(z), w_1), \mathcal{Y}(x_j^\pm(z), w_2)] = \delta_{i,j} \mathcal{Y}(h_j(z), w_2) \delta^{(0)}(w_1, w_2) \]

\[ + e_j^{-1} \mathcal{Y}(c(z), w_2) \delta^{(1)}(w_1, w_2); \]

for \( i, j \in I \). Now we rewritten the defining relation \((H)\) of \( \mathcal{D}_P(g, \mu) \) as follows

\[ [h_i(z), c(w)] = 0, \quad [h_i(z), h_j(w)] = \sum_{k \in \mathbb{Z}_N} \frac{N}{e_j} a_{i,j}^{k} c(w_2) \delta^{(k,1)}(w_1, w_2). \]
This together with Lemma 5.3 gives (6.8). The relations (6.9)-(6.11) can be proved in a similar way by using the remaining defining relations of \( \mathcal{D}_P(g, \mu) \) and Lemma 5.3, and we omit the details.

For the furthermore statement, it follows from (6.2) that \( U_{m,W} \) is a \( \Gamma \)-submodule of \( \mathcal{E}(W) \) and hence \( \langle U_{m,W} \rangle_\Gamma = \langle U_{m,W} \rangle_{\{1\}} \) (see (5.11)). This implies that the \( \hat{m} \)-module \( \langle U_{m,W} \rangle_\Gamma \) is generated by \( 1_W \). So it remains to prove that the elements in (6.4) act trivially on \( 1_W \), or equivalently, to prove that

\[
\mathcal{Y}(x_i^+(z), w).1_W \in \langle U_{m,W} \rangle_\Gamma[[w]].
\]

But this is just the creation property of the vacuum vector \( 1_W \) (see (5.2)), and so we complete the proof.

Let \( \hat{m} \) be one of the algebras \( \hat{g}, \hat{n}_+ \) and \( \hat{n}_- \). We define \( \hat{m}^{(+)} \) to be the subalgebra of \( \hat{m} \) generated by the elements in (6.4). Form the induced \( \hat{m} \)-module

\[
V(\hat{m}) = U(\hat{m}) \otimes_{U(\hat{m}^{(+)})} \mathbb{C},
\]

where \( \mathbb{C} \) stands for the one dimensional trivial \( \hat{m}^{(+)} \)-module. Then it follows from Lemma 6.2 and the universality of the \( \hat{m} \)-module \( V(\hat{m}) \) that there is a (unique) surjective \( \hat{m} \)-module homomorphism

\[
\varphi_{m,W} : V(\hat{m}) \to \langle U_{m,W} \rangle_\Gamma,
\]

such that \( \varphi_{m,W}(1) = 1_W \). In what follows, we will often view

\[
T_m = \{ e_i^+ | i \in I \} \cap m
\]

as a subset of \( V(\hat{m}) \) in sense of that

\[
e_i^+ = (t_i^{-1} \otimes e_i^+) \otimes 1, \quad i \in I.
\]

Since \( \varphi_{m,W} \) is an \( \hat{m} \)-module homomorphism, one gets that

\[
\varphi_{m,W}(e_i^+(w).1) = e_i^+(w).\varphi_{m,W}(1) = \mathcal{Y}(x_i^+(z), w).1_W, \quad i \in I.
\]

By the creation property on the vacuum vector (see (5.2)), this implies that

\[
\varphi_{m,W}(e_i^+) = x_i^+(z), \quad i \in I.
\]

The following result will be proved in §6.4.

**Proposition 6.3.** Let \( \hat{m} \) be one of the algebras \( \hat{g}, \hat{n}_+ \) and \( \hat{n}_- \). Then there exists a \( \Gamma \)-vertex algebra structure on the induced \( \hat{m} \)-module \( V(\hat{m}) \) such that \( 1 = 1 \otimes 1 \) is the vacuum vector and such that \( T_m \) generates \( V(\hat{m}) \) as a vertex algebra. The vertex operators on \( T_m \) are given by

\[
Y(e_i^+, z) = e_i^+(z) = \sum_{m \in \mathbb{Z}} (t_i^m \otimes e_i^+) z^{-m-1}, \quad i \in I,
\]

and the \( \Gamma \)-action on \( T_m \) is determined by

\[
R_\mu(e_i^+) = \xi^{-1} e_{\mu(i)}, \quad i \in I.
\]
Moreover, any quasi-module $W$ for the $\Gamma$-vertex algebra $V(\hat{m})$ is naturally a restricted module for the Lie algebra $\hat{\mathfrak{m}}[\mu]$ with

$$e_i^+(z) = \sum_{m \in \mathbb{Z}} (e_1^m \otimes e_{i(m)}^+) z^{-m-1} = Y_W(e_i^+, z), \quad i \in I.$$  

On the other hand, any restricted module $W$ for the Lie algebra $\hat{\mathfrak{m}}[\mu]$ is naturally a quasi-module for the $\Gamma$-vertex algebra $V(\hat{m})$ with $Y_W(e_i^+, z) = e_i^+(z)$ for $i \in I$.

We continue the proof of Theorem\textsuperscript{[1.5]} by assuming that Proposition\textsuperscript{[6.3]} holds.

**Lemma 6.4.** For any $m = g, n_+$ or $n_-$, the $\hat{\mathfrak{m}}$-module homomorphism $\varphi_{m,W} : V(\hat{m}) \to \langle U_{m,W} \rangle_\Gamma$ is a $\Gamma$-vertex algebra homomorphism

**Proof.** By the general principle given in Proposition\textsuperscript{[5.1]} one only need to prove that for $a \in T_m$ and $v \in V(\hat{m})$,

$$\varphi_{m,W}(Y(a, w)v) = \mathcal{Y}(\varphi_{m,W}(a), w) \varphi_{m,W}(v),$$

and that for $a \in T_m$, $k \in \mathbb{Z}_N$ and $v \in V(\hat{m})$,

$$\varphi_{m,W}(R_{\mu^k}(Y(a, w)v)) = \mathfrak{R}_{\mu^k}(\mathcal{Y}(\varphi_{m,W}(a), w) \varphi_{m,W}(v)),$$

provided that

$$\varphi_{m,W}(R_{\mu^k}(v)) = \mathfrak{R}_{\mu^k}(\varphi_{m,W}(v)).$$

We first prove the identity\textsuperscript{[6.17]}. Indeed, for $i \in I$ and $v \in V(\hat{m})$, one has that

$$\varphi_{m,W}(Y(e_i^+, w)v) = \varphi_{m,W}(e_i^+(w)v) \quad \text{by (6.14)}$$

$$= e_i^+(w) \varphi_{m,W}(v) \quad \text{(as $\varphi_{m,W}$ is a module homomorphism)}$$

$$= \mathcal{Y}(x_i^+(z), w) \varphi_{m,W}(v) = \mathcal{Y}(\varphi_{m,W}(e_i^+), w) \varphi_{m,W}(v). \quad \text{by (6.2), (6.13)}.$$  

Next we prove that the condition (6.19) implies the identity (6.18). For $i \in I$, $k \in \mathbb{Z}_N$ and $v \in V(\hat{m})$, one has that

$$\varphi_{m,W} \circ R_{\mu^k}(Y(e_i^+, w)v) = \varphi_{m,W}(Y(R_{\mu^k}(e_i^+), w) R_{\mu^k}(v)) \quad \text{by (5.3)}$$

$$= \xi^k \varphi_{m,W}(Y(e_{\mu^k(i)}^+, w) R_{\mu^k}(v)) = \xi^k \varphi_{m,W}(e_{\mu^k(i)}^+(w) R_{\mu^k}(v)) \quad \text{by (6.15), (6.14)}$$

$$= \xi^k \varphi_{m,W}(e_{\mu^k(i)}^+(w), R_{\mu^k}(v)) = \xi^k \varphi_{m,W}(x_{\mu^k(i)}^+(z), w) \varphi_{m,W}(R_{\mu^k}(v)) \quad \text{by (6.3)}$$

$$= \mathcal{Y}(\mathfrak{R}_{\mu^k}(x_{\mu^k(i)}^+(z)), w) \varphi_{m,W}(R_{\mu^k}(v)) = \mathcal{Y}(\mathfrak{R}_{\mu^k}(x_{\mu^k(i)}^+(z)), w) \mathfrak{R}_{\mu^k}(\varphi_{m,W}(v)) \quad \text{by (6.2), (6.19)}$$

$$= \mathfrak{R}_{\mu^k}(\mathcal{Y}(x_{\mu^k(i)}^+(z), w) \varphi_{m,W}(v)) = \mathfrak{R}_{\mu^k}(\mathcal{Y}(e_{\mu^k(i)}^+(z), \varphi_{m,W}(v))) \quad \text{by (6.3), (6.3)}$$

$$= \mathfrak{R}_{\mu^k}(\varphi_{m,W}(e_{\mu^k(i)}^+(z)v)) = \mathfrak{R}_{\mu^k} \circ \varphi_{m,W}(Y(e_{\mu^k(i)}^+, w)v), \quad \text{by (6.14)}$$

as desired. \hfill \Box

Now we are ready to complete the proof of Theorem\textsuperscript{[1.5]}. Firstly, recall that the restricted $\mathcal{D}_P(m, \mu)$-module $W$ is naturally a quasi-$\langle U_{m,W} \rangle_\Gamma$-module under the action [6.1]. Next, by Lemma\textsuperscript{6.4} via the $\Gamma$-vertex algebra homomorphism $\varphi_{m,W}$, we have
the quasi-\((U_{m,W})_1\)-module \((W,Y_W)\) becomes a quasi-\(V(\hat{m})\)-module with \(Y_W(v,z) = Y_{\hat{m}}(\varphi_{m,W}(v),z)\), \(v \in V(\hat{m})\). In particular, by (6.3), one gets that

\[(6.20)\]
\[Y_W(e_i^+,w) = Y_{\hat{m}}(x_i^+(z),w), \quad i \in I.\]

Finally, in view of Proposition 6.3, the quasi-\(V(\hat{m})\)-module \((W,Y_W)\) admits an \(\hat{m}[\mu]\)-module structure with action (6.16). In summary, by the actions (6.1, 6.20) and (6.16), we have proved that there is an \(\hat{m}[\mu]\)-module structure on \(W\) with

\[e_i^+(w) = Y_W(e_i^+,w) = Y_{\hat{m}}(x_i^+(z),w) = x_i^+(w), \quad i \in I.\]

This finishes the proof of Theorem 1.5 as the set

\[\{t_i^m \otimes e_i^{+}\mid i \in I, m \in \mathbb{Z}\} \cap \hat{m}[\mu]\]

generates the Lie algebra \(\hat{m}[\mu]\).

6.2. **Proof of Theorem 1.4**. In this subsection we give the proof of Theorem 1.4. For this purpose, we need to prove two simple results on restricted modules of \(\mathbb{Z}\)-graded Lie algebras.

**Lemma 6.5.** Let \(p = \bigoplus_{n \in \mathbb{Z}}p_n\) be a \(\mathbb{Z}\)-graded Lie algebra and let \(a \in p\). Assume that \(a.w = 0\) for any restricted \(p\)-module \(W\) and any \(w \in W\). Then \(a = 0\).

**Proof.** Let \(a \in p\) be as in lemma and assume conversely that \(a \neq 0\). Then one may take a sufficiently large integer \(s\) such that \(a \notin p_{\geq s}\), where \(p_{\geq s} = \bigoplus_{n \geq s}p_n\). Consider the left \(U(p)\)-module \(U(p)/U(p)p_{\geq s}\), via the left multiplication action. Note that this \(p\)-module is restricted and so we have

\[a \in \{x \in p \mid x.(U(p)/U(p)p_{\geq s}) = 0\} = p \cap U(p)p_{\geq s}.\]

However, one can conclude from the PBW basis theorem that

\[p \cap U(p)p_{\geq s} = p_{\geq s},\]

which implies \(a \in p_{\geq s}\), a contradiction. \(\square\)

**Lemma 6.6.** Let \(p\) be a \(\mathbb{Z}\)-graded Lie algebra, and let \(f : p \to q\) be a homomorphism of Lie algebras. Assume that for any restricted \(p\)-module \(W\), there is a \(q\)-module structure on it with

\[(6.21)\]
\[a.w = f(a).w, \quad a \in p, \quad w \in W.\]

Then \(f\) is an injective map.

**Proof.** If \(a \in \ker f\), then for any restricted \(p\)-module \(W\), the relation (6.21) gives that \(a.w = 0\) for all \(w \in W\). Thus the assertion follows from Lemma 6.5. \(\square\)

Now, recall from Lemma 3.6 that we already have the following surjective Lie homomorphisms

\[\theta m, p : D_p(m, \mu) \to \hat{m}[\mu], \quad m = g, n_+, \text{ and } n_.\]

By combining Theorem 1.3 with Lemma 6.6, one finds that these homomorphisms are also injective. This completes the proof of Theorem 1.4.
6.3. Proof of Proposition 6.1. When \( g \) is of not type \( A_1^{(1)} \), Proposition 6.1 was proved in [E]. For convenience of the readers, in this subsection we give a proof of Proposition 6.1 for any \( g \). So throughout this subsection we assume that \( \mu = \text{Id} \).

Set \( D(m) = D_p(m, \text{Id}) \) and \( \theta_m = \theta_{m,p} \) for \( m = g, n_+ \) or \( n_- \). Then we need to prove that all \( \theta_m \) are isomorphisms. Firstly, we have that

**Proposition 6.7.** The Lie homomorphism \( \theta \) : \( D(g) \rightarrow \hat{g} \) is an isomorphism.

**Proof.** Notice that, for any restricted \( D(g) \)-module \( W \), the relation

\[
(z - w)[x_i^+(z), x_i^+(w)] = 0, \quad i \in I,
\]

implies that (see Lemma 5.3)

\[
x_i^+(z), x_i^+(w) = c_i^+(w)\delta^{(0)}(z, w),
\]

for some \( c_i^+(z) \in \mathcal{E}(W) \). But, on the other hand, one has that

\[
x_i^+(z), x_i^+(w) = -[x_i^+(w), x_i^+(z)] = -c_i^+(z)\delta^{(0)}(w, z) = -c_i^+(w)\delta^{(0)}(z, w).
\]

This gives that \( c_i^+(w) = 0 \) for \( i \in I \) and so we have that (see Lemma 5.3)

\[
x_i^+(z), x_i^+(w) = 0, \quad i \in I.
\]

When \( g \) is of untwisted affine type, by replacing the relation (6.22) with (6.23), the presentation \( D(g) \) of \( \hat{g} \) was proved in [MRY]. When \( g \) is of other types, the proof given in [MRY] is also valid. We left the details to the interesting readers. \( \square \)

Let \( \hat{D}(g) \) be the algebra generated by the same generators of \( D(g) \) with the defining relations \((H), (HX\pm)\) and \((XX)\), and let \( \hat{D}(g)_{+}, \hat{D}(g)_{-} \) and \( \hat{D}(g)_{0} \) be the subalgebras of \( \hat{D}(g) \) generated by \( x_{i,m}^{+}, x_{i,m}^{-} \) and \( h_{i,m}, c (i \in I, m \in \mathbb{Z}) \), respectively. The following result is standard.

**Lemma 6.8.** \( \hat{D}(g) = \hat{D}(g)_{+} \oplus \hat{D}(g)_{0} \oplus \hat{D}(g)_{-} \) and the algebra \( \hat{D}(g)_{+} \) (resp. \( \hat{D}(g)_{-} \)) are free generated by the elements \( x_{i,m}^{+} \) (resp. \( x_{i,m}^{-} \)), \( i \in I, m \in \mathbb{Z} \).

We denote by \( \hat{D}(g) \) the quotient algebra of \( \hat{D}(g) \) modulo the relation \((X\pm)\), and denote by \( \overline{D}(g) \) the quotient algebra of \( \hat{D}(g) \) modulo the relation \((AS\pm)\), \( \hat{D}(g) \neq \overline{D}(g) \) only if \( g \) is of type \( A_1^{(1)} \). Then \( D(g) \) is the quotient algebra of \( \overline{D}(g) \) obtained by modulo the relation \((DS\pm)_{\mu} \). Similar to \( D(g) \), all the algebras \( \hat{D}(g) \), \( \hat{D}(g) \) and \( \overline{D}(g) \) are natural \( \mathbb{Z} \)-graded.

**Lemma 6.9.** (i) In \( \overline{D}(g) \), one has that

\[
f_{ij}(z, w) \cdot [x_i^+(z), x_j^+(w)], x_k^+(w')] = 0, \quad i, j, k \in I.
\]

(ii) If \( g \) is of type \( A_1^{(1)} \), then in \( \overline{D}(g) \) one has that

\[
(z_1 - z_2)[x_i^+(z_1), x_j^+(z_2), x_k^+(w)], x_k^+(w')] = 0, \quad (i, j) \in I, k \in I.
\]
(iii) In $\mathcal{D}(\mathfrak{g})$, one has that

$$\left[ [x^+_i(z_1), \cdots, [x^+_i(z_{-a_{ij}}), x^+_j(w)]]_{x^+_k(w')} \right] = 0, \quad (i, j) \in \mathbb{I}, \quad k \in I.$$  

Proof. By applying the relations $(XX)$ and $(HX \pm)$ in $\widetilde{\mathcal{D}}(\mathfrak{g})$, one gets that

$$[[x^+_i(z), x^+_j(w)], x^+_k(w')] = \pm \delta_{i,k} [h_i(z), x^+_j(w)] \delta^{(0)}(z, w') \pm \delta_{j,k} [x^+_i(z), h_j(w)] \delta^{(0)}(w, w') = \pm \delta_{i,k} a_{ij} x^+_j(w) - \delta_{j,k} a_{ji} x^+_i(z) \delta^{(0)}(z, w) \delta^{(0)}(w, w').$$

Then the identity (6.24) is implied by the fact that $\delta_{i,k} a_{ij} x^+_j(w) - \delta_{j,k} a_{ji} x^+_i(z) = 0$ if $a_{ij} \geq 0$, and that $(z - w) \delta^{(0)}(z, w) = 0$ if $a_{ij} < 0$.

Note that a similar argument of (6.23) shows that for all $i \in I$, the relations $[x^+_i(z), x^+_i(w)] = 0$ hold in $\widetilde{\mathcal{D}}(\mathfrak{g})$ and $\mathcal{D}(\mathfrak{g})$. Therefore, in the rest of the proof, we can and do assume that

$$f_{ii}(z, w) = 1, \quad \text{for } i \in I$$

in the relation $(X \pm)$.

For the identities (6.25) and (6.26), we only need to check the case that $k = i$ or $k = j$. If $k = j$, then the identities (6.23) and (6.26) follow from the relation $(XX)$ with $i = j$. So assume now that $k = i$. Let $W$ be an arbitrary restricted $\widetilde{\mathcal{D}}(\mathfrak{g})$-module. In view of the relation $(X \pm)$ in $\widetilde{\mathcal{D}}(\mathfrak{g})$ and the equation (6.27), one finds that as operators on $W$, the formal series

$$x^+_i(z_1) \cdots x^+_a(z_a), i_1, \cdots, i_a \in I,$$

lie in the space $\text{Hom}(W, W((z_1, \cdots, z_a)))$.

We first consider the case that $\mathfrak{g}$ is not of type $A^{(1)}_1$ and so $\widetilde{\mathcal{D}}(\mathfrak{g}) = \mathcal{D}(\mathfrak{g})$. By definition, we have that

$$[x^+_i(z), x^+_j(w)] =: x^+_i(z)x^+_j(w) : (z - w)^{-1} - (w - z)^{-1}$$

$$= : x^+_i(z)x^+_j(w) : \delta^{(0)}(z, w) =: x^+_i(w)x^+_j(w) : \delta^{(0)}(z, w).$$

More generally, one can easily check that for any $a \geq 1$,

$$x^+_i(z), : x^+_i(w) \cdots x^+_a(w)x^+_j(w) : = : x^+_i(w) \cdots x^+_a(w)x^+_j(w) : \delta^{(0)}(z, w).$$

This implies that for any $a \geq 1$,

$$[x^+_i(z_1), \cdots, [x^+_i(z_a), x^+_j(w)]]$$

$$= : x^+_i(w) \cdots x^+_a(w)x^+_j(w) : \delta^{(0)}(z_1, \cdots, z_a, w).$$

(6.28)
For convenience, set $b = 1 - a_{ij}$. Then, by applying the relations $(XX)$ and $(HX\pm)$, we have that

\[
[[x_i^+(z_1), \ldots, [x_i^+(z_b), x_j^+(w)], x_i^+(w')]]
= \pm \sum_{1 \leq a \leq b} [x_i^+(z_1), \ldots, [x_i^+(z_{a-1}), [h(x), [x_i^+(z_{a+1}), \ldots,
\cdot [x_i^+(z_b), x_j^+(w)]]]]\delta(0)(z_a, w')
= 2 \sum_{1 \leq a < a' \leq b} X(z_1, \ldots, z_a, \ldots, z_{a'}, w)\delta(0)(z_a, z_{a'})\delta(0)(z_a, w')
+ a_{ij} \sum_{1 \leq a < b} X(z_1, \ldots, z_a, \ldots, z_b, w)\delta(0)(z_a, w)\delta(0)(z_a, w').
\]

where for any $a = 1, \ldots, b$,

\[
X(z_1, \ldots, z_a, \ldots, z_b, w) = [x_i^+(z_1), \ldots, [x_i^+(z_{a-1}), [x_i^+(z_{a+1}), \ldots, [x_i^+(z_b), x_j^+(w)]]]].
\]

Moreover, it follows from \((6.28)\) that for $a, a' = 1, \ldots, b$ with $a < a'$,

\[
X(z_1, \ldots, z_a, \ldots, z_{a'}, w)\delta(0)(z_a, z_{a'})\delta(0)(z_a, w')
= X(z_1, \ldots, z_a, \ldots, z_b, w)\delta(0)(z_a, w)\delta(0)(z_a, w')
= : x_i^+(w) \cdots x_i^+(w)x_j^+(w) : \delta(0)(z_1, \ldots, z_a, w)\delta(0)(w, w').
\]

Thus, by combining these facts, we obtain that the formal series

\[
[[x_i^+(z_1), \ldots, [x_i^+(z_b), x_j^+(w)], x_i^+(w')]]
= (b(b - 1) + a_{ij}b) : x_i^+(w) \cdots x_i^+(w)x_j^+(w) : \delta(0)(z_1, \ldots, z_a, w)\delta(0)(w, w')
= 0
\]

on any restricted $\hat{D}(g)$-module. This together with Lemma \((6.5)\) proves the identity \((6.26)\) with $k = i$, as required.

Now, we turn to consider the case that $g$ is of type $A_1^{(1)}$. Then by definition we have that

\[
[x_i^+(z), x_j^+(w)] = x_i^+(z)x_j^+(w) : \delta(1)(z, w) + \left( \frac{\partial}{\partial z} : x_i^+(z)x_j^+(w) : \right) |_{z=w}\delta(0)(z, w).
\]

From the relation $(X\pm)$ and the equation \((6.27)\), we also have

\[
(z - w)^2 x_i^+(z) \left( \frac{\partial^{n_1 + \cdots + n_k}}{\partial z_1^{n_1} \cdots \partial z_k^{n_k}} : x_i^+(z_1) \cdots x_i^+(z_k)x_j^+(w) : \right) |_{z_1=\cdots=z_k=w}
\]
lies in the space $\text{Hom}(W, W((z, w)))$ for any $k \geq 0$ and $n_1, \ldots, n_k = 0, 1$. This implies that

$$
[x_t^+(z_1), [x_t^+(z_2), x_j^+(w)]] = [x_t^+(w)x_j^+(w) : \delta^{(1,1)}(z_1, z_2, w) + \left( \frac{\partial}{\partial z_1} : x_t^+(z_1)x_j^+(w) : \right) |_{z_1 = w} \delta^{(0,1)}(z_1, z_2, w) + \left( \frac{\partial}{\partial z_2} : x_t^+(w)x_j^+(z_2) : \right) |_{z_2 = w} \delta^{(1,0)}(z_1, z_2, w) + \left( \frac{\partial^2}{\partial z_1 \partial z_2} : x_t^+(z_1)x_j^+(z_2) : \right) |_{z_1 = z_2 = w} \delta^{(0,0)}(z_1, z_2, w).
$$

(6.29)

By using the relations $(XX)$ and $(HX \pm)$, we get that

$$
[x_t^+(w'), [x_t^+(z_1), [x_t^+(z_2), x_j^+(w)]]] = -2 : x_t^+(w)x_j^+(w) : \delta^{(1,0,0)}(w', z_1, z_2, w) + 2 \left( \frac{\partial}{\partial z} : x_t^+(z)x_j^+(w) : \right) |_{z = w} \delta^{(0,0,0)}(w', z_1, z_2, w),
$$

and

$$
[x_t^+(w'), [x_t^+(z_1), [x_t^+(z_2), [x_t^+(z_3), x_j^+(w)]]]] = 6 \left( \frac{\partial}{\partial z} : x_t^+(w)x_t^+(z)x_j^+(w) : \right) |_{z = w} \delta^{(1,1)}(w', z_1, z_2, z_3, w) + 2 \sum_{s=2}^4 \left( \frac{\partial}{\partial z} : x_t^+(w)x_t^+(z)x_j^+(w) : \right) |_{z = w} \delta^{(1,s)}(w', z_1, z_2, z_3, w) + 2 \sum_{1 \leq s < t \leq 4} : x_t^+(w)x_t^+(w)x_j^+(w) : \delta^{(1,t)}(w', z_1, z_2, z_3, w),
$$

(6.30)

(6.31)

where $1_s = (\delta_{1s}, \ldots, \delta_{4s})$ and $1_{s,t} = (\delta_{1s} + \delta_{1t}, \ldots, \delta_{4s} + \delta_{4t})$. Then the identity is implied by Lemma 6.5, the relation (6.30) and the fact that $(z-w)\delta^{(0)}(z-w) = 0$.

For the identity (6.29), one can conclude from (6.29) that

$$
: x_t^+(w)x_t^+(w)x_j^+(w) := 0 = \left( \frac{\partial}{\partial z_1} : x_t^+(z_1)x_t^+(w)x_j^+(w) : \right) |_{z_1 = w}
$$

(6.32)

on any restricted $\mathcal{D}(g)$-module. Then it is easy to see that the identity (6.26) is implied by Lemma 6.5, the relation (6.31) and the relation 6.32 with $k = i$, as required.

Let $\mathcal{D}(g)^+$ and $\mathcal{D}(g)^-$ denote respectively the subalgebras of $\mathcal{D}(g)$ generated by the elements $x_{i,m}^+$ and $x_{i,m}^-$ for $i \in I, m \in \mathbb{Z}$. By combining Lemma 6.8 with Lemma 6.9, one immediately gets that
Lemma 6.10. The algebra $\mathcal{D}(\mathfrak{g})_+$ (resp. $\mathcal{D}(\mathfrak{g})_-$) is isomorphic to the Lie algebra abstractly generated by the elements $x_{i,m}^+$ (resp. $x_{i,m}^-$), $i \in I, m \in \mathbb{Z}$ and subject to the relations $(X^+), (AS^+), (DS^+)_p$ (resp. $(X^-), (AS^-), (DS^-)_p$).

Finally, from Proposition 5.7 and Lemma 6.10 it follows that both $\theta_{n+}$ and $\theta_{n-}$ are isomorphisms. This completes the proof of Proposition 6.1.

6.4. Proof of Proposition 6.3. In this subsection, based on the theory of $\Gamma$-conformal Lie algebras developed in §5.2, we give a proof of Proposition 6.3. We first consider the conformal Lie algebra $C_\mathfrak{a}$ associated to $\mathfrak{a}$. As a vector space

$$C_\mathfrak{a} = (\mathbb{C}[T] \otimes \mathfrak{g}) \oplus \mathbb{C}k_1,$$

where $\mathfrak{g} = \mathfrak{g} \oplus \sum_{m \in \mathbb{Z}} \mathbb{C}t^m k_1' \subset \mathfrak{a}$. We also introduce two subspaces of $C_\mathfrak{a}$ as follows

$$C_{n+} = \mathbb{C}[T] \otimes \mathfrak{n}_+ \quad \text{and} \quad C_{n-} = \mathbb{C}[T] \otimes \mathfrak{n}_-,$$

where $\mathfrak{n}_+ = \mathfrak{n}_+ \oplus \sum_{m \in \mathbb{Z}_+} \mathbb{C}t^m k_1'$ and $\mathfrak{n}_- = \mathfrak{n}_- \oplus \sum_{m \in \mathbb{Z}_+} \mathbb{C}t^m k_1'$. Let $T$ be the operator on $C_\mathfrak{a}$ defined by

$$(6.33) \quad T(T^m \otimes x) = T^{m+1} \otimes x, \quad T(k_1) = 0,$$

where $m \in \mathbb{N}$ and $x \in \mathfrak{g}$. We define

$$Y_- : C_\mathfrak{a} \to \text{Hom}(C_\mathfrak{a}, z^{-1}C_\mathfrak{a}[z^{-1}]), \quad x \mapsto \sum_{n \in \mathbb{N}} x(n) z^{-n-1}$$

to be the unique linear map such that the property (5.31) holds and such that the non-trivial $n$-products on $\mathfrak{g} \oplus k_1$ are as follows: for any $\alpha, \beta \in \Delta$, $x \in \mathfrak{g}_{\alpha}$ and $y \in \mathfrak{g}_{\beta}$, if $\alpha + \beta \in \Delta^\times \cup \{0\}$, then

$$(6.34) \quad x^{(0)}y = [x,y], \quad x^{(1)}y = \langle x,y \rangle k_1;$$

if $\mathfrak{g}$ is of affine type, $x = t_2^{m_2} \otimes \dot{x}$, $y = t_2^{n_2} \otimes \dot{y}$ and $\alpha + \beta \in \Delta^0 \setminus \{0\}$, then

$$(6.35) \quad x^{(0)}y = [x,y] + \langle \dot{x}, \dot{y} \rangle m_2(T \otimes t_2^{m_2+n_2}k_1'), \quad x^{(1)}y = (m_2 + n_2) \langle \dot{x}, \dot{y} \rangle t_2^{m_2+n_2}k_1'.$$

Lemma 6.11. The triple $(C_\mathfrak{a}, T, Y_-)$ defined above is a conformal Lie algebra, and $C_{n+}$, $C_{n-}$ are two conformal Lie subalgebras of it. Moreover, the linear map $i_\mathfrak{a} : \widehat{C}_\mathfrak{g} \to \widehat{\mathfrak{a}}$ defined by

$$(6.36) \quad x(m) \mapsto t_1^m \otimes x, \quad t_2^{n_2}k_1'(m) \mapsto t_1^{m+1}t_2^{n_2}k_1', \quad k_1(m) \mapsto \delta_{m,-1}k_1$$

for $x \in \mathfrak{g}$ and $m, n \in \mathbb{Z}_+$, is an isomorphism of Lie algebras. Similarly, the linear maps $i_{n+} = i_\mathfrak{a}|_{\widehat{C}_{n+}} : \widehat{C}_{n+} \to \widehat{\mathfrak{n}_+}$ and $i_{n-} = i_\mathfrak{a}|_{\widehat{C}_{n-}} : \widehat{C}_{n-} \to \widehat{\mathfrak{n}_-}$ are isomorphisms of Lie algebras.

Proof. Using (6.33), it is easy to see that all the maps $i_\mathfrak{a}$, $i_{n+}$ and $i_{n-}$ are isomorphisms of vector spaces. Then $\widehat{C}_\mathfrak{g}$ admits a Lie algebra structure transferring from $\widehat{\mathfrak{a}}$. In view of Lemma 5.2, the Lie bracket on $\widehat{C}_\mathfrak{g}$ is given by (5.32) with the $n$-products defined by (6.34) and (6.35). Thus, by Lemma 5.10, $(C_\mathfrak{a}, T, Y_-)$ is a conformal Lie algebra. Moreover, it is obvious that $C_{n+}$ and $C_{n-}$ are invariant.
under the \(n\)-products \((6.34)\) and \((6.35)\). This implies that they are conformal Lie subalgebras of \(C_g\) and the associated Lie algebras are isomorphic to \(\hat{n}_\pm\).

Recall the subset \(T_m \subset C_m\) defined in \((6.12)\). Using Lemma \ref{lemma:5.11} one gets that

**Lemma 6.12.** For any \(m = g, n_+\) or \(n_-\), \(T_m\) is a generating subset of the universal vertex algebra \(V_{C_m}\) associated to \(C_m\).

We now define a linear transformation \(R_\mu\) on \(C_g\) such that

\[
R_\mu(T^n \otimes x) = \xi^{n+1} T^n \otimes \mu(x), \quad R_\mu(k_1) = k_1
\]

for \(n \in \mathbb{N}, x \in g, \alpha \in \Delta^\times \cup \{0\}\), and such that (the following actions exist only when \(g\) is of affine type)

\[
R_\mu(T^n \otimes h) = \xi^{n+1} (T^n \otimes \mu(h) + \rho_\mu(\hat{h}) T^{n+1} \otimes t_2^{m_2} k_1'),
\]

\[
R_\mu(T^n \otimes t_2^{m_2} k_1') = \xi^n T^n \otimes t_2^{m_2} k_1,
\]

for \(n \in \mathbb{N}, h = t_2^{m_2} \otimes \hat{h}\) with \(m_2 \neq 0, \hat{h} \in \hat{h}_{[m_2]}\) and \(n_2 \in \mathbb{Z}\).

**Lemma 6.13.** The map \(R_\mu\) has order \(N\) and satisfies the axiom \(TR_\mu = \xi^{-1} R_\mu T\).

*Proof.* The assertion is obvious when \(g\) is of finite type. For the case that \(g\) is of affine type, recall the automorphism \(\hat{\mu}\) on \(\hat{g}\) and the linear functional \(\rho_\mu\) on \(\hat{h}\) introduced in §2.2. Then one can conclude from the fact \(\mu^N = 1\) and \([\text{CJKT2}]\) Lemma 2.1 (b),(c)] that

\[
\sum_{k \in \mathbb{Z}_N} \rho_\mu(\hat{h}_k) = 0, \quad \forall \hat{h} \in \hat{h}.
\]

Using this and the action \((2.4)\), one can easily check that \(R_\mu^N(T^n \otimes h) = T^n \otimes h\) for \(n \in \mathbb{N}, h = t_2^{m_2} \otimes \hat{h}\) with \(m_2 \neq 0\) and \(\hat{h} \in \hat{h}_{[m_2]}\). This implies the first assertion in lemma and the second one is obvious.

In view of Lemma \ref{lemma:6.13} we now have the following group homomorphism

\[
R : \Gamma \to \text{GL}(C_g), \quad \mu^n \mapsto R_\mu^n = (R_\mu)^n, \quad n \in \mathbb{Z}_N,
\]

which satisfies the condition that

\[
(6.37) \quad TR_\mu^n = \xi^{-n} R_\mu T, \quad n \in \mathbb{Z}_N.
\]

**Lemma 6.14.** \((C_g, T, Y_-, R)\) is a \(\Gamma\)-conformal Lie algebra and \(C_{n_+}, C_{n_-}\) are \(\Gamma\)-conformal Lie subalgebras of \(C_g\). Moreover, the associated Lie algebra \((\hat{C}_m)_{\Gamma}\) is isomorphic to the Lie algebra \([\hat{m}]_{[\mu]}\) for \(m = g, n_+\) or \(n_-\).

*Proof.* Recall the linear map \(\hat{R}_\mu : \hat{C}_g \to \hat{C}_g\) induced by \(R_\mu\) defined in \((5.38)\). By using the explicit action of \(\hat{\mu}\) on \(\hat{g}\) given in Lemma \ref{lemma:3.12} one can check that \(\hat{R}_\mu = \hat{\mu} \circ \hat{t}_g^{-1}\). So \(\hat{R}_\mu\) is an automorphism of \(\hat{C}_g\). This together with Lemma \ref{lemma:5.12} and \((6.37)\) proves that \((C_g, T, Y_-, R)\) is a \(\Gamma\)-conformal Lie algebra. Note that both \(C_{n_+}\) and \(C_{n_-}\) are stable under the map \(R_\mu\). So they are \(\Gamma\)-conformal Lie subalgebras.
of $C_g$, which completes the proof of the first assertion. The second assertion follows from Remark 5.14 and the moreover statement in Lemma 6.11.

Combining Lemma 6.14 with Lemma 5.13, one knows that $V_{C_m}$ admits a natural $\Gamma$-vertex algebra structure. Recall the subalgebra $\hat{m}(+)$ of $m$ defined in §6.1 and the subalgebra $\hat{C}_m(+)\hat{C}_m$ defined in §5.3.

Lemma 6.15. For any $m = g, n_+$ or $n_-$, one has that $\hat{m}(+) = i_m(\hat{C}_m(+)\hat{C}_m)$.

Proof. It is easy to see that
$$i_m(\hat{C}_m(+)\hat{C}_m) = \text{Span}_C\{t_1^m \otimes x, t_1^{n+1}t_2^n k_1 | x \in g, n \in Z, m \in N\} \cap \hat{m}$$

is the subalgebra of $\hat{m}$ generated by the elements in (6.4), and hence coincides with $\hat{m}(+)$, as required. □

Now we are ready to finish the proof of Proposition 6.3. In view of Lemma 6.15, there is a natural $\Gamma$-vertex algebra structure on $V(\hat{m})$ transferring from $V_{C_m}$, with $T_m$ as a generating subset (Lemma 6.12). This proves the first assertion of Proposition 6.3. The second assertion of Proposition 6.3 is implied by Lemma 6.14 and Proposition 5.15, as desired.

References

[AABGP] B. Allison, S. Azam, S. Berman, Y. Gao, and A. Pianzola. Extended affine Lie algebras and their root systems, 126. Mem. Amer. Math. Soc., 1997.

[ABGP] B. Allison, S. Berman, Y. Gao, and A. Pianzola. A characterization of affine Kac-Moody Lie algebras. Comm. Math. Phys., 185, 671–688, 1997.

[ABP] B. Allison, S. Berman, and A. Pianzola. Multiloop algebras, iterated loop algebras and extended affine Lie algebras of nullity 2. J. Eur. Math. Soc., 16, 327–385, 2014.

[B] J. Beck. Braid group action and quantum affine algebras. Comm. Math. Phys., 165, 555–568, 1994.

[BGK] S. Berman, Y. Gao, and Y. Krylyuk. Quantum tori and the structure of elliptic quasi-simple Lie algebras. J. Funct. Anal., 135, 339–389, 1996.

[CJKT1] F. Chen, N. Jing, F. Kong, and S. Tan. Twisted quantum affinizations and their vertex representations. J. Math. Phys., 59, 2018.

[CJKT2] F. Chen, N. Jing, F. Kong, and S. Tan. Twisted toroidal Lie algebras and Moody-Rao-Yokonuma presentation. arXiv:1901.09627

[Da1] I. Damiani. Drinfeld realization of affine quantum algebras: the relations. Publ. Res. Inst. Math. Sci., 48, 661–733, 2012.

[Da2] I. Damiani. From the Drinfeld realization to the Drinfeld-Jimbo presentation of affine quantum algebras: injectivity. Publ. Res. Inst. Math. Sci., 51, 131–171, 2015.

[DLM] C. Dong, H. Li, and G. Mason. Regularity of rational vertex operator algebras. Adv. Math., 132, 148–166, 1997.

[Dr] V. Drinfeld. A new realization of Yangians and quantized affine algebras. In Soviet Math. Dokl, 36, 212–216, 1988.

[E] B. Enriquez. PBW and duality theorems for quantum groups and quantum current algebras. J. Lie Theory, 13, 21–64, 2003.

[FSS] J. Fuchs, B. Schellekens, and C. Schweigert. From Dynkin diagram symmetries to fixed point structures. Comm. Math. Phys., 180, 39–97, 1996.
[GK] O. Gabber and V. Kac. On defining relations of certain infinite-dimensional Lie algebras. *Bulletin of the American Mathematical Society*, 5, 185–189, 1981.

[G] H. Garland. The arithmetic theory of loop algebras. *J. Algebra*, 53, 480–551, 1978.

[GP] P. Gille and A. Pianzola. Torsors, reductive group schemes and extended affine Lie algebras. *Mem. Amer. Math. Soc.*, 226, vi+112, 2013.

[GKV] V. Ginzburg, M. Kapranov, and E. Vasserot. Langlands reciprocity for algebraic surfaces. *Math. Res. Lett.*, 2, 147–160, 1995.

[G-KK] M. Golenishcheva-Kutuzova and V. Kac. $\Gamma$-conformal algebras. *J. Math. Phys.*, 39, 2290–2305, 1998.

[H1] D. Hernandez. Representations of quantum affinizations and fusion product. *Transform. Groups*, 10, 163–200, 2005.

[H2] D. Hernandez. Quantum toroidal algebras and their representations. *Selecta Math.*, 14, 701–725, 2009.

[H-KT] R. Høegh-Krohn and B. Torresani. Classification and construction of quasisimple Lie algebras. *J. Funct. Anal.*, 89, 106–136, 1990.

[J] N. Jing. Quantum Kac-Moody algebras and vertex representations. *Lett. Math. Phys.*, 44, 261–271, 1998.

[K1] V. Kac. *Infinite dimensional Lie algebras*. Cambridge University Press, 1994.

[K2] V. Kac. *Vertex algebras for beginners*. 10, Amer. Math. Soc., 1998.

[LL] J. Lepowsky and H. Li. *Introduction to vertex operator algebras and their representations*. 227. Birkhäuser Boston Incoporation, 2004.

[L1] H. Li. Local systems of twisted vertex operators, vertex superalgebras and twisted modules. In C. Dong and G. Mason, editors, *Moonshine, the Monster and Related Topics*, 193 of *Contemp. Math.*, 203–236, Mount Holyoke, 1996. Summer Research Conference, Amer. Math. Soc.

[L2] H. Li. A new construction of vertex algebras and quasi-modules for vertex algebras. *Adv. Math.*, 202, 232–286, 2006.

[L3] H. Li. On certain generalizations of twisted affine Lie algebras and quasimodules for $\Gamma$-vertex algebras. *J. Pure Appl. Algebra*, 209, 853–871, 2007.

[MRY] R. Moody, S. E. Rao, and T. Yokonuma. Toroidal Lie algebras and vertex representations. *Geom. Dedicata*, 35, 283–307, 1990.

[Na] H. Nakajima. Quiver varieties and finite dimensional representations of quantum affine algebras. *J. Amer. Math. Soc.*, 14, 145–238, 2001.

[Ne] E. Neher. Extended affine Lie algebras. *C.R. Math. Acad. Sci. Soc. R. Can.*, 26, 90–96, 2004.

[P] M. Primc. Vertex algebras generated by Lie algebras. *J. Pure Appl. Algebra*, 135, 253–293, 1999.

[S] J. Sun. Universal central extensions of twisted forms of split simple Lie algebras over rings. *J. Algebra*, 322, 1819–1829, 2009.

[VV] M. Varagnolo and E. Vasserot. Double-loop algebras and the Fock space. *Invent. Math.*, 133, 133–159, 1998.

[ZJ] H. Zhang and N. Jing. Drinfeld realization of twisted quantum affine algebras. *Commun. Algebra*, 35, 3683–3698, 2007.
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