Spin-foam fermions: PCT symmetry, Dirac determinant and correlation functions

Muxin Han and Carlo Rovelli
Centre de Physique Théorique, CNRS-Luminy Case 907, F-13288 Marseille, France
E-mail: Muxin.Han-At-cpt.univ-mrs.fr and rovelli-At-cpt.univ-mrs.fr

Received 2 August 2012, in final form 6 February 2013
Published 5 March 2013
Online at stacks.iop.org/CQG/30/075007

Abstract
We discuss fermion coupling in the framework of spin-foam quantum gravity. We analyze the gravity-fermion spin-foam model and its fermion correlation functions. We show that there is a spin-foam analogue of PCT symmetry for the fermion fields on a spin-foam model, which is proved for spin-foam fermion correlation functions. We compute the determinant of the Dirac operator for the fermions, where two presentations of the Dirac determinant are given in terms of diagram expansions. We compute the fermion correlation functions and show that they can be given by Feynman diagrams on the spin-foams, where the Feynman propagators can be represented by a discretized path integral of a world-line action along the edges of the underlying 2-complex.

PACS number: 04.60.Pp
(Some figures may appear in colour only in the online journal)

1. Introduction

Loop quantum gravity (LQG) is an attempt to make a background independent, non-perturbative quantization of four-dimensional general relativity (GR)—for reviews, see [1–3]. It is inspired by the classical formulation of GR as a dynamical theory of connections. Starting from this formulation, the kinematics of LQG is well studied and results in a successful kinematical framework (see the corresponding chapters in the books [1]). The framework of the dynamics in LQG is still largely open. There are two main approaches to the dynamics of LQG; they are (1) the operator formalism of LQG, which follows the spirit of Dirac quantization or reduced phase space quantization of a constrained dynamical system, and performs a canonical quantization of GR [4]; (2) the path integral formulation of LQG, which is currently understood in terms of the spin-foam formulation [3, 5–8]. The relation between these two approaches is well understood in the case of three-dimensional gravity [9], while for...
four-dimensional gravity, the situation is much more complicated and there are some recent attempts [10] for relating these two approaches.

A serious shortcoming of LQG and the spin-foam models has long been the difficulty of coupling matter quantum field theory (see the first two references in [3]), especially the coupling with fermions. It is still not clear so far about what is the behavior of the matter quantum fields on the quantum background described by LQG, and what are the quantum gravity corrections for matter quantum field theory. There were early pioneer works on coupling matter quantum field theory in canonical LQG [11] and in the context of spin-foam models in three and four dimensions, e.g. [12–14]. And there was recent progress in [15], where we define a very simple form of fermion and Yang–Mills couplings in the framework of a four-dimensional spin-foam formulation. Because of the simplicity of the fermion coupling, it is possible for us to further analyze the detailed properties of the quantum fermion fields coupling to spin-foam quantum gravity.

In this paper, we mainly discuss the fermion coupling in the framework of a four-dimensional Lorentzian EPRL model [6]. The EPRL model in LQG is mostly inspired by the four-dimensional Plebanski formulation of GR (Plebanski–Holst formulation by including the Barbero–Immirzi parameter \( \beta \)), which is a BF theory constrained by the condition that the \( B \) field should be ‘simple’, i.e. there is a tetrad field \( e \) such that \( B = \star (e \wedge e) \). In the EPRL model, the implementation of simplicity constraint is understood in the sense of [16]. More importantly, the semiclassical limit of the EPRL spin-foam model is shown to be well behaved in the sense of [18, 21].

Our analysis of fermion coupling in this work follows the definition in [15]. In section 2, we review the regularization procedure of the fermion action on a 2-complex \( \mathcal{K} \) and discuss its formal continuum limit. We also show that there is a way to express the Dirac fermion action (more precisely, the Dirac operator) in terms of spin-foam variables, so that the Dirac action is coupled into the spin-foam amplitudes. In this way, we describe the dynamics of the fermion quantum field theory on a quantum background geometry, which is described by the spin-foam model. Moreover, we define and discuss the fermion correlation function on spin-foams. In [12], it was mentioned that the (non-gauge-invariant) fermion correlation function vanishes in three-dimensional spin-foam quantum gravity (the same thing also happens in lattice gauge theory, see the first reference of [12] and the references therein), so one should make a certain gauge-fixing in order to define the correlation function properly. In four dimensions, we would find the similar vanishing result for the (non-gauge-invariant) fermion correlation function, if there was local \( SL(2, \mathbb{C}) \) gauge invariance. However, in the Lorentzian spin-foam model for pure gravity, a gauge-fixing has been implemented in order to make the vertex amplitude finite [22]; such a gauge-fixing breaks the local \( SL(2, \mathbb{C}) \) gauge invariance, and makes the fermion correlation functions well defined. After that we discuss the PCT symmetry of the spin-foam fermions. The invariance under the inversions of charge, parity and time simultaneously is believed to be a fundamental symmetry of nature. Here, we define a transformation \( \Theta \) for the gravity-fermion spin-foam partition function and its fermion correlation function, which can be viewed as a spin-foam analogue of the PCT transformation in standard quantum field theory. Then, a spin-foam PCT theorem is proved for the spin-foam fermion correlation functions, which states that the complex-conjugated fermion correlation function on a spin-foam background equals the correlation function of charge-conjugated fermions on a time-and-space reversed spin-foam background. This result is considered as a spin-foam analogue

\[ \text{footnote: The fermion coupling analyzed in this paper can also be translated into Euclidean signature and implemented in the Euclidean EPRL–FK model and the model defined in [8].} \]

\[ \text{footnote:} \]
of the celebrated PCT theorems proved for quantum field theory on Minkowski spacetime [23] and curved spacetime [24].

In section 3, we continue the computation for the gravity-fermion spin-foam model. If the integrations of the fermionic variables are carried out, it results in a determinant of the Dirac operator on the spin-foam model. This Dirac determinant contains the information about the interaction between the fermion field and the gravitational field. So in section 3 we provide two representations for computing the spin-foam Dirac determinant in terms of diagrams.

In section 4, we compute the n-point correlation functions of spin-foam fermions. It turns out that the resulting spin-foam fermion correlation functions can be understood as coupling free-fermion Feynman diagrams into the spin-foam amplitude, while each amplitude from a Feynman diagram depends on the spin-foam background geometry, which is summed over in the spin-foam amplitude. Because here we only consider the interaction between fermions and gravity, the Feynman diagrams coupled with the spin-foams are free fermion Feynman diagrams, which are completely factorized into Feynman propagators (matrix elements of inverse Dirac operator). And it turns out that the Feynman propagators can be expressed as a discretized path integration of a certain world-line action, where the world-lines are along the edges of the 2-complex underlying the spin-foam amplitudes. Our results confirm to some extend the early idea in [13] which proposes the inclusion of matter quantum fields by coupling their Feynman diagrams into the spin-foam model.

2. Gravity-fermion spin-foam model and PCT symmetry

2.1. Definition of spin-foam fermion

Given a 2-complex $K$ dual to a simplicial complex $\Delta$, we consider the discretization on the complex $K$ the classical Dirac action:

$$S_F := \int_M d^4x \, e \frac{i}{2} \left[ \overline{\psi} \gamma^\mu D_\mu \psi - D_\mu \overline{\psi} \gamma^\mu \psi \right] - m_0 \overline{\psi} \psi, \quad (2.1)$$

where $\gamma^\mu(x) = \gamma^I e^\mu_I(x)$ is the spinorial tetrad, $e(x) = \det(e^\mu_I)$, and $D_\mu$ is the covariant derivative for Dirac spinor, i.e.

$$D_\mu \psi = \partial_\mu \psi + \frac{1}{2} A_\mu^I S_I \psi. \quad (2.2)$$

Here, $S^I := \frac{1}{4} \{\gamma^I, \gamma^J\}$ is the Lie algebra generator of the Lorentz group.

We first consider the first term in the action; it can be written as

$$S_1 := \frac{i}{2} \int_M \overline{\psi} \gamma^\mu D_\mu \psi \wedge e^I \wedge e^K \wedge e^L S_I \quad (2.3)$$

which motivates us to make the following anzatz for the naive discretization of $S_1$. We assign a fermion $\psi_v$ to each vertex $v \in V(K)$ of the complex $K$, and make the following formally discretized $S_1$ [25, 15]:

$$S_1 \simeq 2i \sum_{e \in E(K)} V_e \overline{\psi_{b(e)}} \gamma^I n_I(e) [G_e \psi_{f(e)} - \psi_{b(e)}], \quad (2.4)$$

where $b(e)$ and $f(e)$ are respectively the begin and final points of $e$, $n_I(e) = \frac{e^I(v) e^I(w)}{\|e^I(v) e^I(w)\|}$ $(v = b(e))$ is a unit vector at the begin point of $e$, $V_e$ is a 3-volume associated with the edge $e$, which can be viewed as the volume of the tetrahedron (polyhedron) $\tau_e$ dual to $e$ \(^3\), and

$$G_e := \mathcal{P} e^{\frac{1}{2} \int \overline{\psi_{b(e)}} G \psi_{b(e)}}, \quad (2.5)$$

is an SL(2, $\mathbb{C}$) group element represented on the Dirac spinors.

\(^3\) Here, we use $\tau$ to denote both a tetrahedron and the center of a tetrahedron. We can make this notation because of their one-to-one correspondence.
When we check the formal continuum limit of equation (2.4), we consider a region $\Omega$, such that it is much larger than the scale of an elementary cell (e.g. a 4-simplex if $K$ is dual to a simplicial complex), but smaller than the scale over which the fermion field and gravitational field (and their derivatives) change significantly. In this region $\Omega$, along each edge $\langle e_i \rangle \equiv (\partial / \partial s_i )^\mu$,

$$
G_e \psi_{f(e)} \simeq \left[ 1 + \Delta s_i \hat{e}_i^\mu + \frac{1}{2} A_{ij}^\mu \mathcal{S}_{ij} \right] \left[ \psi_{b(e)} + \Delta s_i \dot{e}_i^\mu (D_\mu \psi)_{b(e)} \right] 
\simeq \psi_{b(e)} + \Delta s_i \dot{e}_i^\mu (D_\mu \psi)_{b(e)}.
$$

(2.6)

Then, at the region $\Omega$, the formal continuum limit of equation (2.4) is given by

$$
2i \sum_{e \subset \Omega} V_e \overline{\psi}_{b(e)} n_l(e) \gamma^\mu \Delta s_e \dot{e}_e^\mu (D_\mu \psi)_{b(e)} = 2i \sum_{e \subset \Omega} V_e \overline{\psi}_{b(e)} n_l(e) \gamma^\mu \Delta s_e \dot{e}_e^\mu \dot{e}_e^\mu (D_\mu \psi)_{b(e)}
$$

$$
= i \frac{1}{4} \text{Vol}(\Omega) \overline{\psi} \gamma^\mu D_\mu \psi,
$$

(2.7)

where we have used the averaging formula

$$
\sum_{e \subset \Omega} V_e n_l(e) \Delta s_e \dot{e}_e^\mu \dot{e}_e^\mu = \frac{1}{4} \text{Vol}(\Omega) \delta_i^j.
$$

(2.8)

To illustrate this averaging formula: firstly because there is a very large number of edges (with all the possible directions) in the region $\Omega^2$, the left-hand side of this formula is invariant under four-dimensional rotation, thus is proportional to $\delta_i^j$. Secondly, if we take the trace of the left-hand side, it gives the volume of the region $\Omega$. Here, we emphasize that the above argument of continuum limit is a formal (or naive) one, which helps us choose a certain discretization of the fermion action. The true continuum limit of an interacting discrete quantum theory is a delicate issue. In the context of lattice QCD, the correct continuum limit has only been proved in perturbation theory [32]. The continuum limit of spin-foam formulation is currently an active research direction in LQG. The analysis of the continuum limit of spin-foam fermion coupling at quantum level is beyond the scope of this paper.

Now, we consider the discretized $S_1$ in equation (2.4). The local SL(2, C) gauge transformations $U_\mu$ act as follows:

$$
G_e \mapsto U_{b(e)} G_e U^{-1}_{f(e)} \quad \psi_\mu \mapsto U_\mu \psi_\mu \quad \overline{\psi}_\mu \mapsto \overline{\psi}_\mu U^{-1}_\mu \quad n_l \mapsto \frac{i}{2} \text{tr}(\gamma^\mu U_\mu \gamma^\mu U^{-1}_\mu) n_l.
$$

(2.9)

The discretized $S_1$ is invariant under these gauge transformations. This can be seen by using the relation

$$
U_\mu \gamma^\mu U^{-1}_\mu = \gamma^\mu \frac{i}{2} \text{tr}(\gamma^\mu U_\mu \gamma^\mu U^{-1}_\mu).
$$

(2.10)

Similarly, the complex-conjugate term $S_2$ in equation (2.1) can be discretized as

$$
S_2 \simeq -\frac{i}{2} \sum_{e \in E(K)} V_e \left[ G_{SE} \psi_{f(e)} - \psi_{b(e)} \right] \gamma^\mu n_l(e) \psi_{b(e)},
$$

(2.11)

while the mass term $S_3$ is given by

$$
S_3 \simeq -m_0 \sum_{v \in V(K)} \tilde{V}_v \overline{\psi}_v \psi_v,
$$

(2.12)

where $\tilde{V}_v$ is the 4-volume associated with the 4-simplex dual to $v$. The expressions of 3-volume and 4-volume in terms of spin-foam variables are discussed in the paragraph close to equation (2.24).

One may also need to specify a distribution of links and their time-like/space-like nature in the case of Lorentzian signature, with some similar arguments as [26].
As a result, the formally discretized action reads
\[ S_{\epsilon}[\psi_{v}, g_{\epsilon}] \simeq 2i \sum_{e \in E(K)} V_{\epsilon} \left[ \overline{\psi}_{b(e)} \gamma^{I} n_{I}(e) G_{\epsilon} \psi_{f(e)} - \overline{\psi}_{f(e)} G_{\epsilon}^{-1} \gamma^{I} n_{I}(e) \psi_{b(e)} \right] - m_{0} \sum_{v \in V(K)} 4 V_{\epsilon} \overline{\psi}_{v} \psi_{v} \]
\[ = \sum_{e \in E(K)} S_{\epsilon}[\psi_{b(e)}, \psi_{f(e)}, g_{\epsilon}] + \sum_{v \in V(K)} S_{\epsilon}[\psi_{v}], \tag{2.13} \]

where \( G_{\epsilon} \) is the representation of \( g_{\epsilon} \in \text{SL}(2, \mathbb{C}) \) on Dirac spinors.

So far the unit vector \( n^{I}(e) \) is located at the begin point \( v \) of each edge \( e \), and we assume \( n^{I}(e) \) to be time-like and future-directed, which means that it can be transformed into \((1, 0, 0, 0)\) by a proper orthochronous Lorentz transformation. We make a parallel transportation of \( n^{I}(e) \) from the begin point of \( e \) to a middle point \( \tau \), such that \( n^{I}(e) \) is transformed into \( n^{I}(\tau) = \delta^{I}_{0} = (1, 0, 0, 0) \), i.e. we consider a Lorentz transformation
\[ n^{I}(\tau) \frac{1}{2} \text{tr}(\gamma^{I} G_{\tau,v} \gamma^{I} G_{\tau,v}^{-1}) = \delta^{I}_{0} \Rightarrow n_{I}(e) \frac{1}{2} \text{tr}(\gamma^{I} G_{\tau,v} \gamma^{I} G_{\tau,v}^{-1}) = n_{I}(e) \frac{1}{2} \text{tr}(\gamma^{I} G_{\tau,v} \gamma^{I} G_{\tau,v}^{-1}) = \delta^{I}_{0}. \]

while
\[ \Lambda^{I}_{J} = \frac{1}{2} \text{tr}(\gamma^{I} G_{\tau,v} \gamma^{J} G_{\tau,v}^{-1}) \Rightarrow (\Lambda^{-1})^{I}_{J} = \frac{1}{2} \text{tr}(\gamma^{J} G_{\tau,v} \gamma^{I} G_{\tau,v}^{-1}). \tag{2.14} \]

As a result
\[ n_{I}(e) = \frac{1}{2} \text{tr}(\gamma^{I} G_{\tau,v} \gamma^{I} G_{\tau,v}^{-1}) \delta^{I}_{J}, \tag{2.16} \]

then
\[ \gamma^{I} n_{I}(e) = G_{\tau,v} \gamma^{I} G_{\tau,v}^{-1} \delta^{I}_{J} = G_{\tau,v} \gamma^{I} G_{\tau,v}^{-1}. \tag{2.17} \]

We define the Lorentz transformation \( G_{\tau,v} \) as the spin-foam \( \text{SL}(2, \mathbb{C}) \) holonomy \( g_{\tau,v} \) represented on the space of Dirac spinors. Recall that \( n^{I} = (1, 0, 0, 0) \) is the unit vector orthogonal to all the face bivectors of the tetrahedron (polyhedron) \( \tau \) by the simplicity constraint \[6, 16, 18\].

So this definition states that the internal vector \( n^{I}(e) \) coming from the tangent vector of the edge \( e \) is the normal of the tetrahedron (polyhedron) \( \tau \) viewed in the frame at the vertex \( v \). Therefore, under this definition the fermion action is expressed by
\[ S_{\epsilon} = 2i \sum_{e \in E(K)} V_{\epsilon} \left[ \overline{\psi}_{b(e)} G_{\epsilon} \gamma^{0} G_{\tau,f(e)} \psi_{f(e)} - \overline{\psi}_{f(e)} G_{\epsilon} \gamma^{0} G_{\tau,b(e)} \psi_{b(e)} \right] \]
\[ S_{\epsilon} = -m_{0} 4 V_{\epsilon} \overline{\psi}_{v} \psi_{v}. \tag{2.18} \]

We can also write \( S_{\epsilon} \) in terms of Weyl spinors. In Weyl basis,
\[ \psi = \begin{pmatrix} \xi^{A} \\ \psi^{A} \end{pmatrix}, \quad \gamma^{0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.19} \]

A representation of \( \text{SL}(2, \mathbb{C}) \) on Dirac spinors can thus be written as a tensor product between an \( \text{SL}(2, \mathbb{C}) \) representation on 2-spinors \( \xi^{A} \) and an \( \text{SL}(2, \mathbb{C}) \) representation on dual complex-conjugated 2-spinors \( \theta^{A} \). \(^{5}\)

Then, the action can be written as
\[ S_{\epsilon} = 2i \sum_{e \in E(K)} V_{\epsilon} \left[ \overline{\psi}_{b(e)} G_{\epsilon} \gamma^{0} G_{\tau,f(e)} \xi_{f(e)}^{A} + \theta_{b(e)}^{A} \overline{\xi}_{f(e)}^{A} G_{\epsilon} \psi_{f(e)}^{A} - \overline{\psi}_{f(e)}^{A} G_{\epsilon} \gamma^{0} G_{\tau,b(e)} \xi_{b(e)}^{A} \right] \]
\[ - \overline{\theta}_{f(e)}^{A} \overline{\xi}_{f(e)}^{A} G_{\epsilon} \psi_{b(e)}^{A} \theta_{b(e)}^{A} \right] \]
\[ S_{\epsilon} = -m_{0} 4 V_{\epsilon} \left[ \xi_{v}^{A} \theta_{v}^{A} + \theta_{v}^{A} \xi_{v}^{A} \right]. \tag{2.20} \]

\(^{5}\) Given \( g \in \text{SL}(2, \mathbb{C}) \) a \( 2 \times 2 \) complex matrix with a unit determinant, its representation on 2-spinors is given by \( \xi^{A} \mapsto g \xi^{A} g^{\dagger} \) and on dual complex-conjugated 2-spinors is given by \( \theta^{A} \mapsto \theta^{A} \overline{g} \) and on dual complex-conjugated 2-spinors is given by \( \theta^{A} \mapsto \theta^{A} g^{\dagger} \overline{\theta}^{A} \) (from \( \theta^{A} \mapsto \overline{\theta}^{A} \).
where $g_{ve}$ is an SL(2, $\mathbb{C}$) group element represented by a $2 \times 2$ complex matrix with unit determinant. Here $^t$ denote complex conjugate and transpose. In terms of spinor indices,

$$\xi^j_1, \xi^j_2, \xi^j_3 \equiv \delta_{BB} (\bar{\xi}_1)_A^B (g_2)_{A}^{B} (\bar{\xi}_1)^{Y} (\xi_2)^{A}$$

$$\theta^j_1, \theta^j_2, \theta^j_3 \equiv \delta_{BB} (g_1)_B^{A} (\bar{\xi}_2)_{A}^{B} (\bar{\theta}_1)^{Y} (\theta_2)^{A}.$$

Note that similar to equation (2.17), we also have

$$\sigma^{j}_{AA} n_1 (v) = \delta_{BB} (\bar{\xi}_1)_{A}^{B} (g_2)_{A}^{B} (\bar{\xi}_1)^{Y} (\xi_2)^{A}$$

In the discretized fermion action, the 3-volume $V_e$ is a function of the spins $j_f$ and the coherent intertwiner $i_e$ [28]. More explicitly, we can express the vertex amplitude $A_v(\xi_f, i_e, g_{ve})_{GR}$ in terms of Livine–Speziale coherent intertwiners $i_e = \{ j_f, n_{ef} \} [27]$, whose labels $(j_f, n_{ef})$ determine the geometry of a tetrahedron (or polyhedron) [28], where the spins $j_f$ and unit vectors $n_{ef}$ satisfy the closure condition $\sum_f j_f n_{ef} = 0$. So we write the 3-volume $V_e$ as a function of the labels $(j_f, n_{ef})$ in the case of a tetrahedron [28]:

$$V_e = \sqrt{\prod_{f} |A_{f} \cdot (\bar{A}_{f} \times \bar{A}_{f})|}, \quad \bar{A}_{f} \equiv \gamma j_f n_{ef}.$$  

For the 4-volume $^4V_e$ of a 4-simplex, we can write it as a function of the boundary data $(j_f, n_{ef})$ and the SL(2, $\mathbb{C}$) group element $g_{ve}$ by the following procedure. Given a unit 4-vector $(1, 0, 0, 0)$, we specify a so(3) sub-algebra inside the Lorentz Lie algebra (as a real Lie algebra), while we denote by the 4 $\times$ 4 matrices $L_{i} (i = 1, 2, 3)$ the so(3) generators in the Lorentz Lie algebra (representation). Given a tetrahedron dual to an edge $e$, its face bivectors are given by $B_{ef} = j_f n_{ef} L_{k}$. This bivector viewed from the frame of 4-simplex is defined by a parallel transportation $B_{ef} = g_{ve} B_{ef} g_{ve} = j_f n_{ef} \bar{G}_{ve} L_{k} g_{ve}$, where $g$ is the vector representation of $g \in \text{SL}(2, \mathbb{C})$. Two triangles $f, f'$ which do not belong to the same tetrahedron, we can write a 4-volume

$$^4V_e (f, f') = B_{ef} B_{ef'} \equiv j_f j_{f'} n_{ef} n_{ef'} g_{ve} L_{k} g_{ve} L_{k} g_{ve} g_{ve} L_{k} g_{ve}.$$  

where the inner product $\langle B_{ef}, B_{ef'} \rangle \equiv j_f j_{f'} n_{ef} n_{ef'} g_{ve} L_{k} g_{ve} L_{k} g_{ve} g_{ve} L_{k} g_{ve}$.

We have expressed all the quantities in the fermion action in terms of spin-foam variables. Let us now couple the fermions to spin-foam quantum gravity. The spin-foam model for pure gravity on a 2-complex $K$ with a boundary graph $\gamma$ [6, 7, 17] is

$$Z_{GR}(K)_{\gamma} = \sum_{j_f} \int d\mu_{j_f} (i_e) \int_{\text{SL}(2, \mathbb{C})} dg_{ve} \prod_{f} A_{f} (j_f, i_e, g_{ve}) \prod_{v} A_{v} (j_f, i_e, g_{ve}) \gamma \cdot f_{\gamma, j_f, i_e}.$$  

where $j_f$ is a boundary state on the boundary graph $\gamma$ in its spin-network representation, $d\mu_{j_f} (i_e) = d\mu_{j_f} (n_{ef})$ is an integral measure for the coherent intertwiner $i_e = \{ j_f, n_{ef} \}$ with

6 Using spinor language, $^t$ for $^ft$ is defined by $^t \xi^j_A = \bar{\gamma}^t \delta_{kA}$ and for $\theta_i$ is defined by $^t \theta_i^B = \delta_{kA} \bar{\gamma}^t \theta_{iA}$. Then the sum over all the intertwiners in the spin-foam model will be a integral over all the unit 3-vectors $n_{ef}$ satisfying the closure constraint [28].
fixed \( j_f \), defined in the first reference of [28], such that the integral \( \int d\mu_{j_f}(i_e) \) only integrates over the unit vectors \( \vec{n}_{i_f} \) and satisfy the closure condition \( \sum_f j_f \vec{n}_{i_f} = 0 \). Given a tetrahedron \( t_e \), the measure \( d\mu_{j_f}(i_e) \) can be written as

\[
d\mu_{j_f}(\vec{n}_{i_f}) = \prod_{f \subset t_e} d^3 \vec{n}_{i_f} \delta(3) \left( \sum_f j_f \vec{n}_{i_f} \right) \det(G(\vec{n}_{i_f})) \int_{H^3} \prod_f |\rho_h(\vec{n}_{i_f})|^{2(j_f+1)} dh, \tag{2.26}
\]

where the factor \( |\rho_h(\vec{n}_{i_f})|^2 \) is given by (the spin-1/2 coherent state \( |n_f\rangle \) is a normalized 2-component spinor)

\[
|\rho_h(\vec{n}_{i_f})|^2 = |n_f| h^3 h |n_f|^{-1}. \tag{2.27}
\]

The integral over \( H^3 = \text{SL}(2, \mathbb{C})/\text{SU}(2) \) describes an averaging of the coherent intertwiner along the action of \( H^3 \). The matrix \( G(\vec{n}_{i_f}) \) is the metric on the obits of the action

\[
G(\vec{n}_{i_f})_{ij} = \sum_{f \subset t_e} j_f (\delta^{ij} - n_{i_f} n^i_{i_f}). \tag{2.28}
\]

In order to couple the Dirac fermion to gravity, we add an edge amplitude to \( Z_{GR}(K) \)

\[
Z(K) := \sum_{j_f} \int d\mu_{j_f}(i_e) \int_{\text{SL}(2, \mathbb{C})} d\gamma_{ee} \int [D\psi_e D\overline{\psi}_e] \prod_f A_f[j_f] \prod_v A_v[j_v, i_v, g_{ee}, \psi_v]
\times \prod_v A_v[\psi_{be(e)}, \psi_{fe(e)}, g_{ee}, j_f, i_e, f_j, j_f, i_e, f_j, j_f]
\times \prod_v A_v[j_v, i_v, g_{ee}, \psi_v] = A_v[j_v, i_v, g_{ee}]_{\text{GR}} e^{iS_v}. \tag{2.29}
\]

The measure \( [D\psi_e D\overline{\psi}_e] \) is defined by the Grassmann integral

\[
[D\psi_e D\overline{\psi}_e] = \prod_{(e,a)} d\psi^a_e d\overline{\psi}^a_e. \tag{2.30}
\]

Note that here we follow the spin-foam formulation of the structure defined by the EPRL and FK models [6, 7], with the generalization by [17, 16] to arbitrary 2-complex\(^8\). Importantly, the factor \( \prod_v A_v[j_v, i_v, g_{ee}]_{\text{GR}} \) in these models (either in Euclidean or Lorenzian signature) can be written as either directly \( e^{iS} \) or \( e^{iS} d\mu_{t_e} \) with some additional integration variables [18–20] if the intertwiners \( i_v \) are the Livine–Speziale coherent intertwiners. Therefore, the pure gravity spin-foam amplitude can be written as a path integral form respecting a discrete “spin-foam action” \( S \). This spin-foam action is critical in the recent semiclassical analysis of spin-foam formulation [18–20, 29]. Therefore, here the coupling of fermion with spin-foam quantum gravity can be understood as a coupling in the level of action, i.e. the gravity-fermion spin-foam amplitude can be expressed in a path integral form with respect to an action \( S + S_e + S_{ve} \), where \( S \) is the gravity part and \( S_e + S_{ve} \) is the fermion part.

\(^8\) The massive fermion coupling to spin-foam is defined on a 2-complex dual to simplicial complex, since the mass term involves the 4-volume, which is the 4-volume of 4-simplex when we deï¬¬fine it using spin-foam variable. If the fermion is massless, then the coupling can be made on an arbitrary 2-complex.
2.2. Fermion correlation functions

With the gravity-fermion spin-foam model defined above, we consider a correlation function with a number of \( \bar{\psi}_v \) and \( \psi_v \) insertions:

\[
(\bar{\psi}_{v_1} \cdots \bar{\psi}_{v_n} \psi_{v_{n+1}} \cdots \psi_{v_{n+m}})_{\text{Spin--foam}} := \sum_{j_f} \int d\mu_{j_f}(i_c) \int_{\text{SL}(2,\mathbb{C})} \text{dg}_{\text{ve}} \prod_f A_f[j_f]_{\text{GR}} \prod_v A_v[j_f, i_c, g_{\text{ve}}]_{\text{GR}} \cdot f_{j_f, j_f, i_c} \times \int [D\psi_v D\bar{\psi}_v] \bar{\psi}_{v_1} \cdots \bar{\psi}_{v_n} \psi_{v_{n+1}} \cdots \psi_{v_{n+m}} \exp(iS_F[K, j_f, i_c, g_{\text{ve}}, \psi_v]).
\]

The above definition of the correlation function needs some explanations. Given that (1) the vertex amplitude \( A_v[j_f, i_c, g_{\text{ve}}]_{\text{GR}} \) is invariant under the \( \text{SL}(2,\mathbb{C}) \) gauge transformation \( g_{\text{ve}} \mapsto \lambda_v g_{\text{ve}} \) \( (\lambda_v \in \text{SL}(2,\mathbb{C})) \) at the vertex \( v \); (2) the fermion action \( S_F[K, j_f, i_c, g_{\text{ve}}, \psi_v] \) is invariant under the \( \text{SL}(2,\mathbb{C}) \) gauge transformation \( g_{\text{ve}} \mapsto \Lambda_v g_{\text{ve}}, \psi_v \mapsto \Lambda_v \psi_v \) and \( \bar{\psi}_v \mapsto \bar{\psi}_v \Lambda_v^{-1} \) \( (\Lambda_v \text{ and } G_{\text{ve}} \text{ are the } \text{SL}(2,\mathbb{C}) \text{ elements } \lambda_v \text{ and } g_{\text{ve}} \text{ represented on the space of Dirac spinors}) \); (3) the measures \( d\mu_{j_f} \) and \( d\psi_v, d\bar{\psi}_v \) are invariant under the \( \text{SL}(2,\mathbb{C}) \) gauge transformation; we make a change of variables \( g_{\text{ve}} \mapsto g_{\text{ve}}^\lambda = \lambda_v g_{\text{ve}}, \psi_v \mapsto \psi_v^\lambda = \Lambda_v \psi_v \) and \( \bar{\psi}_v \mapsto \bar{\psi}_v^\lambda = \bar{\psi}_v \Lambda_v^{-1} \) of the integral (which does not change the integral at all), then use the gauge invariance of the vertex amplitude, fermion action, and the measures

\[
(\bar{\psi}_{v_1} \cdots \bar{\psi}_{v_n} \psi_{v_{n+1}} \cdots \psi_{v_{n+m}})_{\text{Spin--foam}} = \sum_{j_f} \int d\mu_{j_f}(i_c) \int_{\text{SL}(2,\mathbb{C})} d\mu_{j_f}^{\bar{\psi}} \prod_f A_f[j_f]_{\text{GR}} \prod_v A_v[j_f, i_c, g_{\text{ve}}^\lambda]_{\text{GR}} \cdot f_{j_f, j_f, i_c} \times \int [D\psi_v D\bar{\psi}_v] \bar{\psi}_{v_1} \cdots \bar{\psi}_{v_n} \psi_{v_{n+1}} \cdots \psi_{v_{n+m}} \exp(iS_F[K, j_f, i_c, g_{\text{ve}}^\lambda, \psi_v]).
\]

Therefore, naively one would have

\[
(\bar{\psi}_{v_1} \cdots \bar{\psi}_{v_n} \psi_{v_{n+1}} \cdots \psi_{v_{n+m}})_{\text{Spin--foam}} = (\bar{\psi}_v^\lambda \cdots \bar{\psi}_v^\lambda \psi_{v_{n+1}}^\lambda \cdots \psi_{v_{n+m}}^\lambda)_{\text{Spin--foam}} \times \Lambda_v^{n-1} \cdots \Lambda_v^{n-1},
\]

which results in that if \( v_1, \ldots, v_{n+m} \) are different vertices and the correlation function is non-gauge-invariant, then it vanishes because the gauge transformations \( \lambda_v \) are independent at different vertices. The same result was mentioned in [12] in the context of three-dimensional gravity, where it was suggested that one should either make a gauge-fixing to define the non-gauge-invariant correlation function, or instead consider the correlation functions of gauge invariant quantities, e.g. the fermion currents.

However, the above argument did not take into account the regularization of the Lorentzian vertex amplitude [22]. If we consider pure gravity amplitude only, and our vertex amplitude \( A_v[j_f, i_c, g_{\text{ve}}]_{\text{GR}} \) has the gauge invariant under \( g_{\text{ve}} \mapsto \lambda_v g_{\text{ve}} \), then the \( \text{SL}(2,\mathbb{C}) \) integrals

\[
\int_{\text{SL}(2,\mathbb{C})} d\mu \times A_v[j_f, i_c, g_{\text{ve}}]_{\text{GR}}
\]

(2.34)
are divergent, because we can always choose an edge $e_0$ at $v$ and a gauge $\lambda_v = g_v^{-1}$, such that $g_{ve} \mapsto g_{ve}^{-1} g_{ve} (e \neq e_0)$ and $g_{v0} \mapsto 1$; then, there is a redundant $SL(2, \mathbb{C})$ integral $\int dg_{v0}$ which gives the divergence. The same argument for divergence also applies to the gravity-fermion spin-foam model by the gauge invariance of the fermion action which gives the divergence. The same argument for divergence also applies to the gravity-fermion spin-foam model by the gauge invariance of the fermion action $S_f[K, j_f, i_e, \xi_{ve}, \psi_v]$. The way to remove the divergence is firstly to choose an edge $e_0$ at each vertex $v$, and fix the gauge $\lambda_v = g_v^{-1}$ at each vertex, such that $g_{ve} \mapsto g_{ve}^{-1} g_{ve} (e \neq e_0)$ and $g_{v0} \mapsto 1$, so the integrand does not depend on the variable $g_{v0}$ for each vertex. Then, the redundant integral $\int dg_{v0}$ should be removed at each vertex. It turns out that at least for the pure gravity spin-foam amplitude, the $SL(2, \mathbb{C})$ integrals lead to finite result after regularization for a large class of spin-foam vertices [22].

Therefore, the definition of fermion correlation function, equation (2.31), should be understood in terms of the gauge-fixed/regularized spin-foam amplitude, where there is an edge $e_0$ at each vertex $v$ such that the $SL(2, \mathbb{C})$ group element $g_{v0}$ is gauge fixed to be the identity $1 \in SL(2, \mathbb{C})$, and the integral of $g_{v0}$ is removed at each vertex $v$. The previous argument for the vanishing correlation function does not apply for the gauge-fixed/regularized spin-foam amplitude because the gauge-fixing breaks the $SL(2, \mathbb{C})$ gauge invariance. Thus, the correlation function (2.31) is not necessary vanishing. More explicitly, we could write the correlation function in the following way instead of equation (2.31):

\[
\langle \overline{\psi}_v \cdots \overline{\psi}_v \psi_{v+1} \cdots \psi_{v+n} \rangle_{\text{spin-foam}} = \sum_{j_f} \int d\mu_j (i_e) \int_{SL(2,\mathbb{C})} dg_{ve} \prod_f A_f(j_f)_{\text{GR}} \prod_v A_v(j_f, i_e, \xi_{ve})_{\text{GR}} \cdot f_{j_f, j_v, i_e} \\
\times \prod_{(i, e_v)} \delta_{SL(2,\mathbb{C})}(g_{v0}) \int [D\psi_{\bar{v}} D\overline{\psi}_{\bar{v}}] \overline{\psi}_{\bar{v}} \cdots \overline{\psi}_{\bar{v}+1} \psi_{v+1} \cdots \psi_{v+n} \times \exp(iS_f[K, j_f, i_e, \xi_{ve}, \psi_v]).
\]

(2.35)

More detailed computation on the fermion correlation function will be given in section 4.

2.3. PCT symmetry on spin-foam

We first briefly recall the notion of PCT symmetry for Dirac fermion on a flat and curved spacetime. Given the fermion field operator $\psi(t, \vec{x})$ on Minkowski spacetime, its parity-inversion, time-reversal and charge conjugation are defined by

\[
P \psi(t, \vec{x}) P^{-1} = \gamma^0 \psi(t, -\vec{x}) \quad T \psi(t, \vec{x}) T^{-1} = -\gamma^1 \gamma^3 \psi(-t, \vec{x}) \\
C \psi(t, \vec{x}) C^{-1} = -i \gamma^2 \psi^*(t, \vec{x}).
\]

(2.36)

Thus, the anti-unitary PCT transformation acts on the fermion field operator by (here we use $\psi^*$ to denote complex conjugation and $\psi^C$ to denote charge conjugation)

\[
\psi^C(t, \vec{x}) = (PCT) \psi(t, \vec{x}) (PCT)^{-1} = (-i)^3 \psi^*(t, -\vec{x}) = (-i)^3 \gamma^1 \gamma^3 \psi^*(t, \vec{x}).
\]

(2.37)

Then, the PCT theorem on Minkowski spacetime states that: given the Minkowski vacuum state $\Omega$, a class of local fields $\Phi(x)$ (can be composite fields) with $m$ primed spinor indices, and an anti-unitary PCT operator $(PCT)$

\[
(PCT) \Phi(x)(PCT)^{-1} = \Phi^C(x) = (-1)^m (-i)^F \Phi(-x)^*, \quad (PCT) \Omega = \Omega
\]

(2.38)

where $F = 0$ if $\Phi$ is bosonic field and $F = 1$ if $\Phi$ is fermionic field, the PCT theorem on Minkowski spacetime states that the complex-conjugated correlation function of a number of fields equals the correlation function of the corresponding charge-conjugated fields [23], i.e.

\[
\langle \Omega | \Phi^C_1(x_1) \cdots \Phi^C_m(x_m) | \Omega \rangle = \langle \Omega | \Phi_1(x_1) \cdots \Phi_m(x_m) | \Omega \rangle^*.
\]

(2.39)
On Minkowski spacetime, the map $\rho : (t, \vec{x}) \mapsto (-t, -\vec{x})$ defines an isometry preserving the spacetime orientation but reversing the time orientation. For a general (globally hyperbolic) spacetime $\mathcal{M}$ with metric $g_{\alpha\beta}$ and time orientation and spacetime orientation $o = (T, e_{a\beta yk})$, we let $\overline{\mathcal{M}}$ be the spacetime with the identical manifold structure and metric structure as $\mathcal{M}$, but its time orientation and spacetime orientation is given by $-o = (-T, e_{a\beta yk})$. Because there is no preferred vacuum state on a general curved spacetime, the PCT theorem on a general spacetime [24] is formulated in terms of operator-product-expansion (OPE) coefficients. Given a local field $\Phi(x)$ with $n$ unprimed spinor indices and $m$ primed spinor indices, we define its charge-conjugated field by

$$\Phi^{c}(x) := (-1)^{m}(-i)^{\frac{m}{2}} \Phi(x)^{*}. \quad (2.40)$$

Here, $x$ denotes the points on the spacetime manifold, and $\Phi^{c}(x)$ is a local field with $m$ unprimed spinor indices and $n$ primed spinor indices. We suppose that the $\alpha$ class of fields $\Phi_{\mathcal{M}}^{(j)}(x)$ ($j = 1, \ldots, n$) on the spacetime $\mathcal{M}$ has the following OPE in short geodesic distance:

$$\Phi_{\mathcal{M}}^{(1)}(y_{1}) \cdots \Phi_{\mathcal{M}}^{(n)}(y_{n}) \sim \sum_{(j)} c_{\mathcal{M},\alpha}^{(j)}(y_{1}, \ldots, y_{n}) \Phi_{\mathcal{M}}^{(j)}(x) \quad (2.41)$$

as $(y_{1}, \ldots, y_{n})$ approaching $x$, where the distributions $c_{\mathcal{M},\alpha}^{(j)}(y_{1}, \ldots, y_{n})$ are OPE structure coefficients. Then, the PCT theorem on a general curved spacetime implies that on the spacetime $\overline{\mathcal{M}}$ with the opposite time and space orientation, the OPE of the charge-conjugated fields $\Phi_{\overline{\mathcal{M}}}^{(j)c}(x)$ ($j = 1, \ldots, n$) is given by [24]

$$\Phi_{\overline{\mathcal{M}}}^{(1)c}(y_{1}) \cdots \Phi_{\overline{\mathcal{M}}}^{(n)c}(y_{n}) \sim \sum_{(j)} c_{\mathcal{M}}^{(j)c}(y_{1}, \ldots, y_{n})^{*} \Phi_{\overline{\mathcal{M}}}^{(j)c}(x), \quad (2.42)$$

whose OPE structure coefficients are the complex conjugation of $c_{\mathcal{M},\alpha}^{(j)}(y_{1}, \ldots, y_{n})$.

In the formalism of the spin-foam model, the quantization of spacetime $\mathcal{M}$ with metric $g_{\alpha\beta}$ is formulated by a spin-foam amplitude $Z(\mathcal{K})$ on a 2-complex $\mathcal{K}$. The reversal of time orientation while keeping spacetime orientation unchanged $o \mapsto -o$ can be formulated by simultaneously reversing all the internal edge orientations in the complex $\mathcal{K}$. The reason is the following. Given a spacetime $(\mathcal{M}, g_{\alpha\beta})$ with space and time orientation $o = (T, e_{a\beta yk})$, all the oriented orthonormal frames are given by $e_{I}^{\alpha}$, satisfying

$$g_{\alpha\beta} e_{I}^{\alpha} e_{J}^{\beta} = \eta_{IJ}, \quad e_{0}^{\alpha} \nabla_{2} T > 0, \quad e_{a\beta yk} e_{0}^{\alpha} e_{1}^{\alpha} e_{2}^{\beta} e_{3}^{\gamma} > 0. \quad (2.43)$$

The oriented orthonormal frames form the frame bundle $F(\mathcal{M})$ over $\mathcal{M}$ whose structure group is the proper orthochronous Lorentz group. However, for the spacetime $(\overline{\mathcal{M}}, g_{\alpha\beta})$ with opposite space and time orientation $-o = (-T, e_{a\beta yk})$, its frame bundle $F(\overline{\mathcal{M}})$ is naturally isomorphic to $F(\mathcal{M})$ by the map

$$I : F(\mathcal{M}) \to F(\overline{\mathcal{M}}), \quad e_{I}^{\alpha} \mapsto -e_{I}^{\alpha}. \quad (2.44)$$

We consider the spin-foam model defined on a 2-complex $\mathcal{K}$ dual to a simplicial complex imbedded in the spacetime manifold. Given a tetrahedron $\tau$ in the simplicial complex, all the area bivectors of the tetrahedron are orthogonal to a unit internal vector $n^{(e)} = (1, 0, 0, 0)$ by the simplicity constraint [6, 16, 18]. And this vector $n^{(e)}$ is given by the tangent vector of the edge $e$ dual to the tetrahedron $\tau$ up to a proper orthochronous Lorentz transformation, i.e. on the spacetime manifold $\mathcal{M}$ with orientation $o$

$$a \Lambda^{(e)} n^{(e)} = e^{a} e_{a}^{(e)} \quad (a > 0), \quad (2.45)$$

where $\Lambda^{(e)}$ is a proper orthochronous Lorentz transformation. However, if the spin-foam model is build on the spacetime manifold $\overline{\mathcal{M}}$ with orientation $-o$ (in another words, if the spin-foam
model is a quantization of the spacetime structure \( (M, g_{ab}, -\partial) \), the previous co-frame field \( e'_a \) changes into \( -e'_a \); thus, one has to change the previous tangent vector \( \dot{e}^a \) into \( -\dot{e}^a \), i.e. reverse the edge orientation to keep \( n' \) unchanged. Note that \( n' \) has to be fixed to be \((1, 0, 0, 0)\) for the definition of the spin-foam model.

We first formulate the PCT invariance of the spin-foam fermions in the following formal way. We define an anti-unitary PCT operator \( \theta \), such that it acts on fermion field operators by

\[
\theta \psi_v \theta^{-1} = (-i) \gamma^5 \gamma^0 \overline{\psi}_v, \quad \theta \overline{\psi}_v \theta^{-1} = (-i) \gamma^0 \gamma^5 \psi_v,
\]

which are charge-conjugate field operators. If we consider the fermion action \( S_F(\mathcal{K}) \) (defined on a complex \( \mathcal{K} \)) is a composite operator from the fermion field operators \( \psi_v \) and \( \overline{\psi}_v \), then \( S_F \) is PCT invariant in the sense that

\[
\theta S_F(\mathcal{K}) \theta^{-1} = S_F(\mathcal{K}^{-1}).
\]

It is indeed the case. We consider the bilinear form \( 2iV_v \overline{\psi}_v G_{uv} \gamma^0 G_{uv} \psi_v \) appearing in the discretized fermion action by using \( \gamma^5, \gamma^0 \) and the fact that \( \gamma^5 \gamma^0 \gamma^5 = \gamma^0 \gamma^0 \gamma^5 \), as well as the fact that the assumption \( \theta \) is anti-unitary

\[
\theta [2iV_v \overline{\psi}_v G_{uv} \gamma^0 G_{uv} \psi_v] \theta^{-1} = -2iV_v (-i) \gamma^0 \gamma^5 G_e^* \gamma^0 G_e (-i) \gamma^5 \gamma^0 \overline{\psi}_v = -2iV_v \gamma^0 G_e^* \gamma^0 G_e^* \overline{\psi}_v = 2iV_v \overline{\psi}_v G_e \gamma^0 G_e \psi_v,
\]

where we have treated \( V_v \) and \( G_{uv} \) as \( c \)-numbers (gravity as external field), and we also used the fact that \( \gamma^5 \) commutes with \( G \) and the relation \( \gamma^0 G^* \gamma^0 = (G^{\alpha^{-1}})^* = (G^{-1})^* \). Here, we have shown that the action of \( \theta \) on the bilinear form interchanges the vertices \( v \) and \( v' \). Hence, under this transformation of variables, \( S_\epsilon [\psi_{f(e)}, \psi_{f(e')}] \) transforms to

\[
S_\epsilon [\psi_{f(e)}, \psi_{f(e')}] \rightarrow 2V_v \overline{\psi}_{f(e')} G_{f(e') \epsilon} \gamma^0 G_{\epsilon \epsilon} \psi_{f(e)} - \overline{\psi}_{f(e')} G_{f(e') \epsilon} \gamma^0 G_{\epsilon \epsilon} \psi_{f(e)} = S_\epsilon [\psi_{f(e^-1)}, \psi_{f(e^{-1})}],
\]

where we use that \( b(e) = f(e^{-1}) \) and \( f(e) = b(e-) \). On the other hand, it is easy to see that the mass term \( S_e = -m_0 \epsilon V_v \overline{\psi}_v \psi_v \) is invariant under \( \theta \). Hence, we obtain the PCT invariance of the spin-foam fermion in the sense of

\[
\theta S_F(\mathcal{K}) \theta^{-1} = S_F(\mathcal{K}^{-1}).
\]

Now consider the gravity-fermion spin-foam model. We define the following transformation \( \Theta \), which is an analogue of PCT transformation.

**Definition 2.1.** Given a 2-complex \( \mathcal{K} \) with a boundary graph \( \gamma \), and the gravity-fermion spin-foam

\[
Z(\mathcal{K})_f = \sum_{j_f} \int \mu_{j_f} (\alpha_e) \int_{SL(2, \mathbb{C})} d \gamma_{\epsilon \epsilon} \prod_f A_f |j_f G_R \prod_v A_v |j_f, \epsilon, \gamma_{\epsilon \epsilon} G_R \cdot f_{j_f, \epsilon, \epsilon} \times \int [D \psi_v, D \overline{\psi}_v] \exp(iS_F[\mathcal{K}, j_f, \epsilon, \gamma_{\epsilon \epsilon}, \psi_v]),
\]

where \( f_{j_f, \epsilon, \epsilon} \) is a boundary state in its spin-network representation, we define a spin-foam analogue of PCT transformation \( \Theta \) by the following.

- \( \Theta \) reverses the orientations of all the internal edges in the complex \( \mathcal{K} \), i.e. \( \Theta : \mathcal{K} \mapsto \mathcal{K}^{-1} \);
- \( \Theta \) changes all the gravity vertex amplitudes \( A_v |j_f, \epsilon, \gamma_{\epsilon \epsilon} G_R \) into their complex conjugates \( A_v |j_f, \epsilon, \gamma_{\epsilon \epsilon} G_R \);
- The boundary state \( f_{j_f, \epsilon, \epsilon} \) transforms into its complex conjugate \( f^*_{j_f, \epsilon, \epsilon} \).
\[ \Theta changes the fermion action on the exponential \text{i}S_F(K) \text{ into} \]
\[
\theta(\text{i}S_F(K))^{-1} = -\text{i}S_F(K^{-1});
\]
\[ \text{for each fermion } \psi_v \text{ at a vertex, } \Theta : \psi_v \mapsto \psi_v^C := (-\text{i})\gamma^5 \psi_v^* = (-\text{i})\gamma^5 \gamma^\alpha \psi_v^* \text{ (charge conjugation), or in terms of Weyl spinors:} \]
\[
\Theta : \begin{pmatrix} \xi_v \\ \theta_v \end{pmatrix} \mapsto (-\text{i}) \begin{pmatrix} \xi_v^* \\ -\theta_v^* \end{pmatrix}. \]

The second and third transformations are motivated by the anti-unitarity of the PCT operator. We consider heuristically the pure gravity spin-foam amplitude as a physical inner product between some certain in-state and out-state \( f_{\text{in}} \) and \( f_{\text{out}} \), which describe the boundary data of quantum gravity

\[ Z_{\text{GR}}(K_f) = \langle f_{\text{out}} \cdot f_{\text{in}} \rangle_{\text{phys}}. \]

Then, heuristically the PCT transformation reverses the in-state and out-state by

\[ \Theta : \langle f_{\text{out}} \cdot f_{\text{in}} \rangle_{\text{phys}} \mapsto \langle \theta f_{\text{out}} \cdot \theta f_{\text{in}} \rangle_{\text{phys}} = \langle f_{\text{out}} \cdot f_{\text{in}} \rangle_{\text{phys}}^* = \langle f_{\text{in}} \cdot f_{\text{out}} \rangle_{\text{phys}}, \]

where we implicitly use the fact that the path integral measure \( d\zeta_{\text{sc}} \) is real, since the SL(2, \( C \)) Haar measure \( dg \) can be written explicitly by

\[ dg = \frac{d\beta d\gamma d\eta d\delta}{|\delta|^2} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \]

which is manifestly invariant under \( g \mapsto g^* \). The other integration measure \( d\mu_{jj}(r_\nu) \) is a real measure [28].

Here, we argue that the PCT transformation of pure gravity spin-foam amplitude

\[ \Theta Z_{\text{GR}}(K_f) = Z_{\text{GR}}(K_f^{-1})^* \]

is a spin-foam analogue of the spacetime \( \mathcal{M} \) with reversed time and space orientation \( -\sigma = (-T, e_{\text{phys}}) \). We consider the semiclassical behavior of the pure gravity spin-foam amplitude. For a given 4-simplex, the corresponding vertex amplitude \( A_v[j_f, i_v]_{\text{GR}} \) can be represented in terms of Livine–Speziale (LS) coherent intertwiners [27], i.e. use LS coherent intertwiner \( |j_f, n_f \rangle \) for the SU(2) intertwiner \( i_v \). Thus, we can denote the vertex amplitude by

\[ A_v[j_f, n_f]_{\text{GR}} = \int \prod_n \text{d}g_{\text{sc}} A_v[j_f, n_f, g_{\text{sc}}]_{\text{GR}}, \]

where \( n_f \in S^2 \) is a unit 3-vector associated with each triangle/face \( f \). The large-\( j \) asymptotics of the vertex amplitude \( A_v[j_f, n_f]_{\text{GR}} \) was studied in [18] when \( j_f, n_f \) satisfy the closure condition. If \( j_f \) goes to be large,

\[ A_v[j_f, n_f]_{\text{GR}} \sim N_+ e^{\frac{4}{3} \sum_i \beta_j i \cdot \beta_f} + N_- e^{-\frac{4}{3} \sum_i \beta_j i \cdot \beta_f} \]

where \( N_\pm \) are independent of \( j_f \), and \( \beta_f \) is the extrinsic angle between the two tetrahedra sharing the triangle \( f \), i.e. for two tetrahedra \( \tau, \tau' \) sharing a triangle \( f \), \( n'_f(\tau) \) and \( n'_f(\tau') \) are two unit 4-vectors respectively orthogonal to all the face bivectors of \( \tau \) and \( \tau' \). The extrinsic angle \( \beta_f \) is defined by \( \cos \beta_f = -n_f(\tau) \cdot n'_f(\tau') \). Then,

\[ \sum_f \beta_j i \cdot \beta_f[j_f] = S_{\text{Regge}}[j_f] \]

is the Regge action in a single 4-simplex with triangle areas \( A_f = 8\pi G\hbar \beta_j f \) (\( \beta \) is the Barbero–Immirzi parameter).
The vertex amplitude can also be written in the holomorphic representation [21] with boundary states to be the complexifier coherent states. Given the boundary graph \( \gamma \) of the vertex amplitude, and for a given link \( l \), we associate a complexifier coherent state \( \psi_{\gamma}^{(h)}(h) \) [30] where
\[
\psi_{\gamma}^{(h)}(h) = \sum_{j} (2j + 1) e^{-j(j+1)/2} \chi^j(h),
\]
where \( t \) is a dimensionless classicality proportional to \( \ell_p^2 \) and \( g_l \in \text{SL}(2, \mathbb{C}) \) is the complexified phase space coordinate \( g_l = h_l e^{\ell p / \ell_p^2} \). These coherent states (with all possible \( g_l \)) form an over-complete basis of the Hilbert space \( L^2(\text{SU}(2)) \). The representation of the vertex amplitude on the complexifier coherent states is called the holomorphic representation of the vertex amplitude, denoted by \( A_v[g_f, g_{ve}]_{\text{GR}} \) and is regarded as an amplitude with boundary data \( g_f \) (a link \( l \) of the boundary graph uniquely corresponds to a face \( f \)). In the holomorphic representation and as the limit \( \ell_p \to 0 \) [21, 30],
\[
A_v[g_f]_{\text{GR}} \sim \sum_{j_f} A_v[j_f, u_{j_f}]_{\text{GR}} \exp \left[ -i \sum_f j_f \xi_f \right] \exp \left[ - \sum_f \left( j_f - \frac{A_f}{\ell_p^2} \right)^2 \frac{t}{2} \right] + \sum_f \left( \frac{A_f}{\ell_p^2} \right)^2 \frac{t}{2},
\]
where \( A_f \) is the area of the triangle \( f \) evaluated at the phase space point \( g_f \) and \( \xi_f = \beta \vartheta \) is the extrinsic angle evaluated at the phase space point \( g_f \). From this, we see that as the limit \( \ell_p \to 0 \), the large spins \( j_f \sim \frac{A_f}{\ell_p^2} \) dominate the contributions of \( A_v[g_f, g_{ve}]_{\text{GR}} \). We then consider the large spin contributions and insert in the large-\( j \) asymptotics of the vertex amplitude equation (2.59)
\[
A_v[g_f]_{\text{GR}} \sim \sum_{p=\pm 1} \sum_{j_f} N_p \exp \left[ i \sum_f j_f (p \beta \vartheta_f - \xi_f) \right] \exp \left[ - \sum_f \left( j_f - \frac{A_f}{\ell_p^2} \right)^2 \frac{t}{2} \right] + \sum_f \left( \frac{A_f}{\ell_p^2} \right)^2 \frac{t}{2},
\]
where the factor \( \exp \left[ i \sum_f j_f (p \beta \vartheta_f - \xi_f) \right] \) is a rapid oscillating phase as \( j_f \) is large, so that the sum over \( j_f \) suppresses unless the boundary data \( \xi_f \) coincide with \( \beta \vartheta_f \) or \( -\beta \vartheta_f \). Without losing generality, we assume the boundary data \( \xi = \beta \vartheta_f \); thus, in \( A_v[g_f, g_{ve}]_{\text{GR}} \) the terms with \( p = +1 \) are preserved in the sum, corresponding to \( e^{i \vartheta_{\text{Regge}}} \), while the terms with \( p = -1 \) suppress in the sum over \( j_f \). Therefore,
\[
A_v[g_f]_{\text{GR}} \sim \sum_{j_f} N_p \exp \left[ i \sum_f j_f (\beta \vartheta_f - \xi_f) \right] \exp \left[ - \sum_f \left( j_f - \frac{A_f}{\ell_p^2} \right)^2 \frac{t}{2} \right] + \sum_f \left( \frac{A_f}{\ell_p^2} \right)^2 \frac{t}{2}.
\]
If we make the PCT transformation \( \Theta \) on the spin-foam amplitude \( A_v[g_f, g_{ve}]_{\text{GR}} \), then the reversal of the internal edge orientations does not affect the amplitude while the complex

---

9 The fact that the two different exponentials \( e^{i \vartheta_{\text{Regge}}} \) and \( e^{-i \vartheta_{\text{Regge}}} \) from the spin-foam model correspond to two different spacetime orientations has been argued in [31, 20]. The fermion coupling considered here only couples to a specified spacetime orientation, which is consistent with the formulation of usual quantum field theory (in curved spacetime) [24].
conjugation gives

\[ \Theta A_v^1g_f|\Theta A_v^1g_f|_{GR} = \Theta A_v^1g_f|_{GR} \]

\[ \sim \sum_{j_f} N^*_f \exp \left[ -i \sum_j j_f (\beta \theta_f - \xi_f) \right] \exp \left[ -\sum_j \left( \frac{j_f - A_f}{\ell^2_{P\beta}} \right)^2 \right] \]

\[ + \sum_j \left( \frac{A_f}{\ell^2_{P\beta}} \right)^2 \frac{2}{f} \]

(2.65)



where effectively the terms corresponding to \( e^{-iS_{Regge}} \) are preserved, while the terms corresponding to \( e^{iS_{Regge}} \) are suppressed. The flip from \( e^{iS_{Regge}} \) to \( e^{-iS_{Regge}} \) corresponds to the flip of the extrinsic angles from \( \theta_f \) to \(-\theta_f\), which is a discrete analogue of flipping the sign of the extrinsic curvature by flipping the time orientation.

When we consider the fermions in the spin-foam model, in terms of the semiclassical limit, the fermion coupling \( e^{i\theta} \) should couple with the \( e^{iS_{Regge}} \)-terms (with positive coupling constant) on the spacetime manifold \( \mathcal{M} \). A more precise argument relies on the large-j asymptotic analysis for the gravity-fermion spin-foam model on a large number of 4-simplices, which will be studied in the future publication. A more precise argument relies on the large-j asymptotic analysis for the gravity-fermion spin-foam model on a large number of 4-simplices, which will be studied in the future publication.

First of all we note that the fermion measure is invariant under charge conjugate

\[ [D\psi_v^c, D\psi_v^c] = [D\psi_v^c, D\psi_v^c]^c. \]

(2.67)

We then make a change of variables for the fermionic integration

\[ \psi_v \rightarrow \psi_v = -i\beta \psi_v, \quad \bar\psi_v \rightarrow \bar\psi_v = -i\beta \bar\psi_v \]

(2.68)

which does not change the integration. Then

\[ \{ \bar\psi_v^c, \cdots, \bar\psi_v^c \} \]

\[ = \sum_{j_f, i_f} \int_{SL(2, \mathbb{C})} \prod_f A_f j_f |_{GR} \prod_v A_v j_f, \kappa_v, g_{ve} |_{GR} \cdot f_{j_f, i_f} \]

\[ \times \left( D\psi_v^c, D\psi_v^c \right) \psi_v^c, \cdots, \psi_v^c \exp(-iS_F[K^{-1}, j_f, i_f, g_{ve}, \psi_v]) \]

(2.69)

\[ \psi_v \rightarrow \psi_v = -i\beta \psi_v, \quad \bar\psi_v \rightarrow \bar\psi_v = -i\beta \bar\psi_v \]

(2.68)

which does not change the integration. Then

\[ \{ \bar\psi_v^c, \cdots, \bar\psi_v^c \} \]

\[ = \sum_{j_f, i_f} \int_{SL(2, \mathbb{C})} \prod_f A_f j_f |_{GR} \prod_v A_v j_f, \kappa_v, g_{ve} |_{GR} \cdot f_{j_f, i_f} \]

\[ \times \left( D\psi_v^c, D\psi_v^c \right) \psi_v^c, \cdots, \psi_v^c \exp(-iS_F[K^{-1}, j_f, i_f, g_{ve}, \psi_v]) \]

(2.69)

\[ \psi_v \rightarrow \psi_v = -i\beta \psi_v, \quad \bar\psi_v \rightarrow \bar\psi_v = -i\beta \bar\psi_v \]

(2.68)

which does not change the integration. Then

\[ \{ \bar\psi_v^c, \cdots, \bar\psi_v^c \} \]

\[ = \sum_{j_f, i_f} \int_{SL(2, \mathbb{C})} \prod_f A_f j_f |_{GR} \prod_v A_v j_f, \kappa_v, g_{ve} |_{GR} \cdot f_{j_f, i_f} \]

\[ \times \left( D\psi_v^c, D\psi_v^c \right) \psi_v^c, \cdots, \psi_v^c \exp(-iS_F[K^{-1}, j_f, i_f, g_{ve}, \psi_v]) \]

(2.69)

\[ \psi_v \rightarrow \psi_v = -i\beta \psi_v, \quad \bar\psi_v \rightarrow \bar\psi_v = -i\beta \bar\psi_v \]

(2.68)

which does not change the integration. Then

\[ \{ \bar\psi_v^c, \cdots, \bar\psi_v^c \} \]

\[ = \sum_{j_f, i_f} \int_{SL(2, \mathbb{C})} \prod_f A_f j_f |_{GR} \prod_v A_v j_f, \kappa_v, g_{ve} |_{GR} \cdot f_{j_f, i_f} \]

\[ \times \left( D\psi_v^c, D\psi_v^c \right) \psi_v^c, \cdots, \psi_v^c \exp(-iS_F[K^{-1}, j_f, i_f, g_{ve}, \psi_v]) \]

(2.69)

\[ \psi_v \rightarrow \psi_v = -i\beta \psi_v, \quad \bar\psi_v \rightarrow \bar\psi_v = -i\beta \bar\psi_v \]

(2.68)

which does not change the integration. Then

\[ \{ \bar\psi_v^c, \cdots, \bar\psi_v^c \} \]

\[ = \sum_{j_f, i_f} \int_{SL(2, \mathbb{C})} \prod_f A_f j_f |_{GR} \prod_v A_v j_f, \kappa_v, g_{ve} |_{GR} \cdot f_{j_f, i_f} \]

\[ \times \left( D\psi_v^c, D\psi_v^c \right) \psi_v^c, \cdots, \psi_v^c \exp(-iS_F[K^{-1}, j_f, i_f, g_{ve}, \psi_v]) \]

(2.69)

\[ \psi_v \rightarrow \psi_v = -i\beta \psi_v, \quad \bar\psi_v \rightarrow \bar\psi_v = -i\beta \bar\psi_v \]

(2.68)

which does not change the integration. Then

\[ \{ \bar\psi_v^c, \cdots, \bar\psi_v^c \} \]

\[ = \sum_{j_f, i_f} \int_{SL(2, \mathbb{C})} \prod_f A_f j_f |_{GR} \prod_v A_v j_f, \kappa_v, g_{ve} |_{GR} \cdot f_{j_f, i_f} \]

\[ \times \left( D\psi_v^c, D\psi_v^c \right) \psi_v^c, \cdots, \psi_v^c \exp(-iS_F[K^{-1}, j_f, i_f, g_{ve}, \psi_v]) \]

(2.69)

\[ \psi_v \rightarrow \psi_v = -i\beta \psi_v, \quad \bar\psi_v \rightarrow \bar\psi_v = -i\beta \bar\psi_v \]

(2.68)

which does not change the integration. Then

\[ \{ \bar\psi_v^c, \cdots, \bar\psi_v^c \} \]

\[ = \sum_{j_f, i_f} \int_{SL(2, \mathbb{C})} \prod_f A_f j_f |_{GR} \prod_v A_v j_f, \kappa_v, g_{ve} |_{GR} \cdot f_{j_f, i_f} \]

\[ \times \left( D\psi_v^c, D\psi_v^c \right) \psi_v^c, \cdots, \psi_v^c \exp(-iS_F[K^{-1}, j_f, i_f, g_{ve}, \psi_v]) \]

(2.69)
Again, we consider the bilinear form \(2V_c \bar{\psi}_c G_{ce} \gamma^0 G_{ec} \psi'_c\), appearing in the discretized fermion action,

\[
2V_c \bar{\psi}_c G_{ce} \gamma^0 G_{ec} \psi'_c = 2V_c (\psi_c^\dagger \gamma^0 \gamma^5 G_{ce} \gamma^0 G_{ec} (\psi_c^\dagger \gamma^0 \gamma^5 G_{ec} \psi'_c)
\]

\[
= 2V_c \psi_c^\dagger \gamma^0 G_{ce} \gamma^0 \psi'_c
\]

\[
= 2V_c \psi_c^\dagger \gamma^0 G_{ce} \gamma^0 G_{ec} \psi'_c
\]

\[
= -2iV_c \bar{\psi}_c G_{ce} \gamma^0 G_{ec} \psi'_c.
\]

(2.70)

where we used the fact that \(\gamma^5\) commutes with \(G\) and the relation \(\gamma^0 G \gamma^0 = (G^{-1})^\dagger\), as well as the anticommutativity of \(\psi\). \(S_c[\psi_{b(e)}, \psi_{f(e)}, \delta_{ve}]\) reads

\[
S_c[\psi_{b(e)}, \psi_{f(e)}, \delta_{ve}] = -2iV_c [\bar{\psi}_{f(e)} G_{fe}^0 \gamma^0 G_{eb(e)}^0 \psi_{*}^* - \bar{\psi}_{b(e)} G_{be}^0 \gamma^0 G_{ef(e)}^0 \psi_{*}^*] \]

\[
= S_{e^{-1}}[\psi_{b(e)}, \psi_{f(e)}, \delta_{ve}]^*.
\]

(2.71)

which can be viewed as a nontrivial transformation property of the spin-foam fermion action, and such a property results in the PCT symmetry of the spin-foam fermion. For the mass term,

\[
S_c[\psi_{*}^*] = -m_0^2 V_c \bar{\psi}_c \psi'_c = mV_c \psi'_c = S_c[\psi_{*}].
\]

(2.72)

Therefore, we obtain that

\[
S_F[\mathcal{K}^{-1}, j_f, i_e, g_{ve}, \psi_{*}^*] = S_F[\mathcal{K}, j_f, i_e, g_{ve}, \psi_{*}]^*.
\]

(2.73)

Then, as a result

\[
\left(\bar{\psi}_{v_1}^{C} \cdots \bar{\psi}_{v_1}^{C} \psi_{v_{1+1}}^{C} \cdots \psi_{v_{vean}}^{C}\right)_{\text{Spin–foam}(\mathcal{M})} \]

\[
= \sum_{j_f, i_e} \int d\delta_{ve} \prod_f A_f[J_f]_{GR} \prod_v A_v[J_v, i_e, g_{ve}]_{GR} \cdot f_{*,j_f,i_e}^* \]

\[
\times \left[D\psi_{v_1}^* D\bar{\psi}_{v_1} \psi_{v_{1+1}}^* \cdots \psi_{v_{vean}}^* \right] \exp(iS_F[\mathcal{K}, j_f, i_e, g_{ve}, \psi_{*}])^*.
\]

(2.74)

We summarize the result as a spin-foam analogue of PCT theorem.

**Proposition 2.1** (PCT invariance for spinfoam fermions). The correlation functions of fermions on the spin-foam analogue of a spacetime \(\mathcal{M}\) with a certain time and space orientation \(o = (T, e_{a,b,y,k})\) equals the complex-conjugated correlation functions of charge-conjugated fermions on the spin-foam analogue of the spacetime \(\overline{\mathcal{M}}\) with an opposite time and space orientation \(-o = (-T, e_{a,b,y,k})\), i.e.

\[
\left(\bar{\psi}_{v_1}^{C} \cdots \bar{\psi}_{v_1}^{C} \psi_{v_{1+1}}^{C} \cdots \psi_{v_{vean}}^{C}\right)_{\text{Spin–foam}(\mathcal{M})} = \left(\bar{\psi}_{v_1}^{C} \cdots \bar{\psi}_{v_1}^{C} \psi_{v_{1+1}}^{C} \cdots \psi_{v_{vean}}^{C}\right)_{\text{Spin–foam}(\overline{\mathcal{M}})}.
\]

(2.75)

### 3. Determinant of Dirac operator on spin-foam

To simplify the formula, now we consider a massless chiral Weyl fermion on the complex \(\mathcal{K}\)

\[
S_c[\xi_{b(e)}, \xi_{f(e)}, g_{z,b(e)}, g_{z,f(e)}] = 2iV_c [\xi_{b(e)}^* g_{z,b(e)}^* \xi_{f(e)}^* - \xi_{f(e)} g_{z,f(e)}^* \xi_{b(e)}^*] \]

\[
S_F[\xi_{*}^*, \delta_{ve}] = \sum_{v,e} S_c[\psi_{b(e)}, \psi_{f(e)}, g_{z,b(e)}, g_{z,f(e)}] = \sum_{v,e} i\xi_{v}^* \bar{D}_{v,e} \xi_{v},
\]

(3.1)

where the Dirac operator has the following matrix element:

\[
\bar{D}_{b(e),f(e)}^{A} = 2V_c \delta_{b B} \bar{G}_{t(b,e)} A \delta_{z(t,f,e)} \bar{G}_{z(b,e)} A
\]

\[
\bar{D}_{f(e),b(e)}^{A} = -2V_c \delta_{B b} \bar{G}_{t(f,e)} A \delta_{z(t,f,e)} \bar{G}_{z(b,e)} A
\]

(3.2)
Figure 1. The building blocks of polymers: a monomer at a single index point and a polymer line connecting two different index points.

and all the other matrix elements vanish, where we see that $\mathcal{D}$ is an anti-Hermitian matrix

$$
\mathcal{D}_{\nu,\nu'} = -\bar{\mathcal{D}}_{\nu,\nu'}.
$$

(3.3)

We define the Grassmann path integral measure by

$$
D\mu[\xi, \xi^\dagger] := \prod_v d(\xi_v) d(\xi_v^\dagger) = \prod_v d(\xi_v) d(\xi_v^\dagger) z.
$$

(3.4)

Obviously, the path integral of the fermion action gives the determinant of the Dirac operator (Dirac determinant)

$$
\det \mathcal{D}[\xi, \xi^\dagger] = \int D\mu[\xi, \xi^\dagger] e^{i\mathcal{S}_F[\xi, \xi^\dagger]} = \int D\mu[\xi, \xi^\dagger] e^{-\sum_v \xi_v \bar{\mathcal{D}}_{\nu,\nu'} \xi_{\nu'}}.
$$

(3.5)

3.1. Polymer representation

We briefly recall the polymer representation of the Grassmann Gauss integral [32]. Given a matrix $Q$ such that

$$
Q_{ij} = M_i \delta_{ij} - K_{ij},
$$

(3.6)

where $M_i$ are diagonal contributions and $K_{ij}$ are off-diagonal contributions, so the diagonal elements of $K_{ij}$ are assumed to vanish $K_{ii} = 0$. In terms of the Grassmann Gauss integral:

$$
\det Q = \int d\eta_1 \cdots d\eta_N e^{-\sum_i M_i \eta_i + \sum_{i<j} \pi K_{ij} \eta_i \eta_j}.
$$

(3.7)

This determinant can be represented in the following way: we first draw $N$ points as a lattice representing the indices $i = 1, \ldots, N$. Then, we draw all possible polymer diagrams using the building blocks in figure 1, such that each index point has precisely one incoming and outgoing line (a monomer is counted both as a incoming and outgoing line). We denote the set of all possible polymer diagrams by $\mathcal{P}$ and denote a polymer diagram by $z$.

For each polymer diagram, we write down its contribution for $\det Q$:

$$
R_z = (-1)^{\text{Number of Polymer Loops}} \prod_l M_l \prod_{j<k} K_{jk}.
$$

(3.8)

where the index point $i$ is attached by a monomer $M_i$ and the index points $j$ and $k$ are connected by a polymer line $K_{jk}$, while the polymer lines forming closed polymer loops. Then the determinant $\det Q$ equals the sum over all possible polymer contributions:

$$
\det Q = \sum_{z \in \mathcal{P}} R_z.
$$

(3.9)
Now we consider the determinant of our Dirac operator \( \mathcal{D} \). The diagonal elements of \( \mathcal{D} \) are zeros; thus, the monomer contribution is not allowed. For the off-diagonal elements:

\[
\mathcal{D}_{(b(e),A);(f(e),A)} = 2V_{e} \delta_{BB} (\xi_{b(e)})^{B}_{A} (\xi_{f(e)})^{B}_{A} \\
\mathcal{D}_{(f(e),A);(b(e),A)} = -2V_{e} \delta_{BB} (\xi_{f(e)})^{B}_{A} (\xi_{b(e)})^{B}_{A}.
\] (3.10)

We draw the lattice of all the index points \( j \equiv (v,A) \), which is \( V(\mathcal{K}) \times \mathbb{Z}_{2} \). Then, we draw all possible polymer loop diagrams using polymer lines \( \mathcal{D}_{ij} \), such that each index point has precisely one incoming and outgoing line. For each polymer diagram \( z \in \mathcal{P} \),

\[
R_{z} = (-1)^{\text{Number of Polymer Loops}} \prod_{j<k} (-\mathcal{D}_{jk})
\]

\[
= (-1)^{\text{Number of Polymer Loops}} \prod_{(v',A') \neq (v,A)} (1)\delta^{v'}_{v} (2V_{e}) \delta_{BB} (\xi_{v'})^{B}_{A} (\xi_{v})^{B}_{A},
\] (3.11)

where \( \delta^{v'}_{v} = 1 \) if \( v' = b(e) \) and \( v = f(e) \) otherwise \( \delta^{v'}_{v} = 0 \). Finally, the Dirac determinant is represented as a sum over all polymer contributions:

\[
\det \mathcal{D} = \sum_{z \in \mathcal{P}} R_{z}.
\] (3.12)

For the massive Dirac fermion, the lattice of all the index points is \( V(\mathcal{K}) \times \{1, 2, 3, 4\} \). And we not only need to consider the polymer lines, but also need to consider the monomers \( iM_{j\equiv(v,A)} = \imath m_{j} \gamma_{5} \). For each polymer diagram \( z \in \mathcal{P} \), we have

\[
R_{z} = (-1)^{\text{Number of Polymer Loops}} \prod_{l} iM_{l} \prod_{j<k} (-\mathcal{D}_{jk})
\]

\[
= (-1)^{\text{Number of Polymer Loops}} \prod_{(v,a)} (i\imath m_{v} \gamma_{5}) \prod_{(v',a') \neq (v,a)} (-1)^{\delta^{v'}_{v}} (2V_{e}) [G_{v'v} G_{v}]_{a' a},
\] (3.13)

and

\[
\det(\mathcal{D} + iM) = \sum_{z \in \mathcal{P}} R_{z}.
\] (3.14)

### 3.2. \( \epsilon \)-loop representation

Here, we compute the Grassmann path integral in a more explicit manner. Because of the Grassmann variables, on each edge the exponentiated fermion action has a nine-term expansion

\[
e^{iS} = \left[ 1 - \xi_{b(e)}^{\dagger} \mathcal{D}_{b(e),f(e)} \xi_{f(e)} + \frac{1}{2} (\xi_{b(e)})^{2} \mathcal{D}_{b(e),f(e)} (\xi_{f(e)})^{2} \right] \times \left[ 1 - \xi_{f(e)}^{\dagger} \mathcal{D}_{f(e),b(e)} \xi_{b(e)} + \frac{1}{2} (\xi_{f(e)})^{2} \mathcal{D}_{f(e),b(e)} (\xi_{b(e)})^{2} \right]
\]

\[
= \left[ 1 - \xi_{b(e)}^{\dagger} \mathcal{D}_{b(e),f(e)} \xi_{f(e)} - \xi_{f(e)}^{\dagger} \mathcal{D}_{f(e),b(e)} \xi_{b(e)} + \frac{1}{2} (\xi_{b(e)})^{2} \mathcal{D}_{b(e),f(e)} (\xi_{f(e)})^{2} + \frac{1}{2} (\xi_{f(e)})^{2} \mathcal{D}_{f(e),b(e)} (\xi_{b(e)})^{2} \right]
\]

\[
+ \frac{1}{2} (\xi_{b(e)})^{2} \mathcal{D}_{b(e),f(e)} (\xi_{f(e)})^{2} + \frac{1}{2} (\xi_{f(e)})^{2} \mathcal{D}_{f(e),b(e)} (\xi_{b(e)})^{2} - \frac{1}{2} (\xi_{f(e)})^{2} \mathcal{D}_{f(e),b(e)} (\xi_{b(e)})^{2}.
\] (3.15)

Given a Grassmann integration at a vertex \( v \), it only affects the \( e^{iS} \)'s with the \( e \)s connecting to \( v \). We have

\[
\int \prod_{e \in \partial(v) = v} d(\xi_{e})^{4} \prod_{e' \in \partial(v) = v} d(\xi_{e'})^{4} = \prod_{e \in \partial(v) = v} e^{iS_{e}} \prod_{e' \in \partial(v) = v} e^{iS_{e'}}.
\] (3.16)
We know that the only nonvanishing Grassmann integral is
\[ \int d\xi d\xi^+ e^{A\xi B} e^{A^\dagger B^\dagger} = \int d\xi d(\xi^+) d\xi^+ d\xi e^{\xi B} e^{A^\dagger B^\dagger} = -e^{AB} e^{AB}. \] (3.17)
Therefore, there are a few possibilities for the contributions for the integral at each vertex \( v \).

**Example 1.** Consider two outgoing edges \( e_1, e_2 \) and two incoming edges \( e_3, e_4, v = b(e_1) = b(e_2) = f(e_3) = f(e_4) \), we have a nonvanishing integral

\[
\int d\xi_i d\xi_i^+ \frac{1}{2} \left( e^A e^B \xi^A \xi^B \right) [\xi_i B_{A}(f(e_1)) f(2) e_{f(i)}, v] [\xi_i B_{A}(f(e_2)) f(3) e_{f(i)}, v] = -e^{AB} e^{AB}.
\]

Insert in the matrix element of the Dirac operator
\[
\frac{1}{2} e^{AD} D_{b(e_1),A \Lambda} e_{f(i), A} \delta_{B_{A}(f(e_1)),D} e^{BC} \]

the above integral equals
\[
= -16 V_\Lambda b \left( e_{f(i)} \right)^{B_A} e_{f(i),\Lambda} \delta^{B_A} D_{b(e_1),A \Lambda} e_{f(i),\Lambda} \delta_{B_{A}(f(e_1)),D} e^{BC} \]

\[
= \frac{1}{2} e^{AD} D_{b(e_1),A \Lambda} e_{f(i), A} \delta_{B_{A}(f(e_1)),D} e^{BC} \]

**Example 2.** Consider two outgoing edges \( e_1, e_2, b(e_1) = b(e_2) = v \), there is a nonvanishing integral

\[
\int d\xi_i d\xi_i^+ \frac{1}{2} \left( e^A e^B \xi^A \xi^B \right) [\xi_i B_{A}(f(e_1)) f(2) e_{f(i)}, v] [\xi_i B_{A}(f(e_2)) f(3) e_{f(i), v}]
\]

\[
= -\frac{1}{2} e^{AD} D_{b(e_1),A \Lambda} e_{f(i), A} \delta_{B_{A}(f(e_1)),D} e^{BC} \]

**Example 3.** Consider an outgoing edge \( e_2 \) and an incoming edge \( e_1, b(e_1) = f(e_2) = v \), there is a nonvanishing integral

\[
\int d\xi_i d\xi_i^+ \frac{1}{2} \left( e^A e^B \xi^A \xi^B \right) [\xi_i B_{A}(f(e_1)) f(2) e_{f(i)}, v] [\xi_i B_{A}(f(e_2)) f(3) e_{f(i), v}]
\]

\[
= -\frac{1}{2} e^{AD} D_{b(e_1),A \Lambda} e_{f(i), A} \delta_{B_{A}(f(e_1)),D} e^{BC} \]

\[
= -\frac{1}{2} e^{AD} D_{b(e_1),A \Lambda} e_{f(i), A} \delta_{B_{A}(f(e_1)),D} e^{BC} \]

\[
= -\frac{1}{2} e^{AD} D_{b(e_1),A \Lambda} e_{f(i), A} \delta_{B_{A}(f(e_1)),D} e^{BC} \]

\[
= -\frac{1}{2} e^{AD} D_{b(e_1),A \Lambda} e_{f(i), A} \delta_{B_{A}(f(e_1)),D} e^{BC} \]
Example 4. For each single outgoing edge \( e, v = b(e) \), we have the integral
\[
\int d\xi_e d\xi^e \left[ \frac{1}{4} [\xi^e D_{v,e}(f(e)) A \xi^e B D_{v,e}(f(e)) B \xi^e A D_{v,e}(f(e)) A \xi^e B] \right] = -4V^e [\xi^e B D_{v,e}(f(e)) A \xi^e B] \]

Example 5. For each single incoming edge \( e, v = f(e) \), in the same way
\[
\int d\xi_e d\xi^e \left[ \frac{1}{4} [\xi^e D_{v,e}(b(e)) A \xi^e B D_{v,e}(b(e)) B \xi^e A D_{v,e}(b(e)) A \xi^e B] \right] = -4V^e [\xi^e B D_{v,e}(b(e)) A \xi^e B] \]

All the contributions of the determinant \( \det D \) can be obtained by the integrals for each vertex, similar to the previous examples. And they can be represented graphically.

- We draw an arrow-line for each \( \xi^e D_{v,e} \xi^e \) in \( S_v \) and associate \( \xi^e \) with its source and \( \xi \) to its target. The arrow-line (figure 2) represents \( D_{v,e} \) with \( A' \) at its source and \( A \) at its target. (However, in the following graphic representation we often ignore the \( A', A \) label for the arrow-line, in order to simplify the graph.)
- We represent each edge by \( S_v \) in figure 3, where we have a minus sign \( (-1) \) when the orientation of a fermion arrow line in \( S_v \) coincides with the orientation of the corresponding edge \( e \). The parallel double-arrows correspond to the terms \( (\xi^e D \xi)^2 \) in the expression of \( S_v \) (equation (3.15)).
- We assign the weights to the arrows and double-arrows (figure 4).
Figure 4. The weights for arrow and double-arrow.

Figure 5. A $\varepsilon$-contractor connects two sources or two targets.

- We represent $\varepsilon^{AB}$ (resp. $\varepsilon^{A'B'}$) by contractors connecting the two targets (resp. two sources) of two arrow-lines (figure 5). Note that the $\varepsilon$-contractors not only can contract the arrow-lines from different edges, but can also contract the double-arrows from a single edge.
- Each vertex $v \in V(K)$ must choose precisely two incoming and two outgoing fermion arrow-lines from the edges connecting $v$. These fermion arrow-lines are contracted by two $\varepsilon$-contractors. There are seven types of contractions shown in figure 6. Note that it requires precisely the same number of incoming and outgoing arrows because one need precisely two $\xi$ and two $\bar{\xi}$ to make the integral nonvanish.

Type 1. Four fermion arrow-lines are from four different edges.
Type 2. Two arrow-lines are from two different edges, but a double arrow is from another single edge.
Type 3. A pair of opposite oriented arrows from a single edge, and two arrows from two different edges.
Type 4. Two pairs of opposite oriented arrows from two different edges.
Type 5. A double-arrow and a single arrow from a single edge, and another arrow from another edge.
Type 6. Two double-arrows are from two different edges.
Type 7. Two double-arrows are from the same edge.

(1) The contraction at each vertex makes the arrow-lines form close loops. There are two type of loops, i.e. nontrivial loops and trivial loops, see figure 8. We call these loops the ‘$\varepsilon$-loops’ because the neighboring edges have opposite directions. Note that a double-arrow from a single edge only can form a trivial $\varepsilon$-loop by the above contraction rule. And each $\varepsilon$-loop must contain even number of arrow-lines by construction.

(2) We denote an $n$-gon nontrivial $\varepsilon$-loop by $L_n$ ($n \geq 2$ is even) and a trivial $\varepsilon$-loop by $T$. Each trivial $\varepsilon$-loop contributes

$$T = -2V_\varepsilon^2 \varepsilon^{AB} \bar{\varepsilon}^{AB} = -4V_\varepsilon^2.$$  (3.25)
Figure 6. The typical contractions for each vertex.

Figure 7. A 4-gon nontrivial $\varepsilon$-loop and a trivial $\varepsilon$-loop.

Figure 8. A 4-gon nontrivial $\varepsilon$-loop.

Let us consider a non-trivial $\varepsilon$-loop, for example a 4-gon $\varepsilon$-loop with vertices $v_1, v_2, v_3, v_4$ (cyclic ordered) and $v_1$ is a source node for two arrows

$$L_4 = \int d\xi^A_1 d\xi^B_1 d\xi^C_1 d\xi^D_1 \left( \varepsilon_{12A'B'} \xi^A_1 \xi^B_2 \xi^C_3 \xi^D_4 \right) \left( -\varepsilon_{AB} \right) \xi^A_1 \xi^B_2 \xi^C_3 \xi^D_4$$

$$= \varepsilon_{D'A'} \xi^A_1 \xi^B_2 \xi^C_3 \xi^D_4$$
Figure 9. A simple diagram with two nontrivial loops.

\[
= (-1)^{16} V_\epsilon V_\psi V_\nu (1 - \epsilon_1) \sum \text{tr} \left[ (g_\epsilon g_\psi g_\nu) \right]
\]

where \(\epsilon \) is the number of arrows whose orientations coincide with the orientations of the associated edges.

Generalize it to an \(n\)-gon \(\epsilon\)-loop diagram \(L_n\) (choose \(v_1\) to be a source node, \(v_i, e_i\) are cyclic ordered)

\[
L_n = (-1)^n \prod_{i=1}^n 2V_\epsilon (1 - \epsilon_1) \text{tr} \left[ (g_{e_{1i}} e_{e_{1i}}) \right] \cdots
\]

where \(g_{e_i}\) is the SL(2, \(\mathbb{C}\)) holonomy from the middle point \(\tau_i\) to the middle point \(\tau_{i'}\).

(3) We draw all possible close \(\epsilon\)-loop diagrams \(\{\Gamma_F\}\) by using the possible edge contributions (figure 3) and the possible vertex contributions (figure 6). For each \(\epsilon\)-loop diagram, it consists of a certain number of trivial \(\epsilon\)-loops \(T(i)\), \(i = 1, \ldots, |T_{\epsilon'}|\), and a certain number of nontrivial \(\epsilon\)-loops \(L_n(i)\), \(i = 1, \ldots, |L_{\epsilon'}|\), where \(T_{\epsilon'}\) and \(L_{\epsilon'}\) are respectively the sets of all trivial and nontrivial loops in \(\Gamma_F\). Therefore, we can write that the determinant of the Dirac operator is a sum over all possible loop diagrams

\[
\det \mathcal{D} = \sum_{\{T_{\epsilon'}\}} \prod_{i=1}^{T_{\epsilon'}} T(i) \prod_{i=1}^{L_n(i)} L_n(i)
\]

where \(\{\rightarrow\}\) denote the set of all the arrows in the \(\epsilon\)-loop diagram \(\Gamma_F\), so \(\prod_{\{\rightarrow\}} V_\epsilon\) means that the product of all the \(V_\epsilon\) associated with the edges corresponds to all the arrows, and also note that for an edge there can be two arrows.

22
One could follow the following two steps to construct each term in the sum equation (3.28).

Step 1. We first ignore all the trivial $\epsilon$-loops. All nontrivial $\epsilon$-loops can be constructed by using the first four terms of $S_c$ in figure 3 and the types 1–5 vertex contributions in figure 6 (ignoring trivial loops). But one should make sure that each vertex has four fermion arrows (two incoming and two outgoing arrows) or has two fermion arrows (both incoming or both outgoing).

Step 2. We add trivial $\epsilon$-loops such that all the vertex has precisely four fermion arrows, two of which are incoming and two of which are outgoing.

4. Fermion correlation functions on spin-foam

4.1. Computing fermion n-point functions

Now we consider the correlation functions of a massless Weyl fermion on spin-foam

$$\langle \xi_{v_1}^+ \cdots \xi_{v_n}^+ \xi_{v_{n+1}} \cdots \xi_{v_m} \rangle_{\text{Spin–foam}}$$

$$: = \sum_{f_f,j_f, i_f} \int d\xi_{v_f} \prod_f A_f \prod_{v_f} A_{v_f} \prod \delta_{\text{SL}(2,C)}(g_{v_f})$$

$$\times \int D\mu[\xi, \eta] \xi_{v_1}^+ \cdots \xi_{v_n}^+ \xi_{v_{n+1}} \cdots \xi_{v_m} \exp(iS_F[K, j_f, i_f, g_{v_f}, \xi_v]).$$

(4.1)

The Weyl fermion action reads

$$S_c[\xi_{bf(e)}, \xi_{f(e)}, g_{\tau_f(e)}, g_{\tau_{f'}(e)}] = 2i\lambda[\xi_{bf(e)} g_{\tau_f(e)} \xi_{f(e)} - \xi_{f(e)} g_{\tau_{f'}(e)} g_{\tau_f(e)}]$$

$$\exp(iS_f[\xi_v, g_{v_f}]) = - \sum_{v_f}^{\text{fermion}} D_{v_f, v} \xi_{v_f} - \eta_{v_f} g_{v_f}.$$ (4.2)

where the Dirac operator has the following matrix element:

$$D_{f_b, f_c} = 2\lambda[\xi_{bf(b)} g_{\tau_f(e)} \xi_{f(e)} - \xi_{f(e)} g_{\tau_{f'}(e)} g_{\tau_f(e)}]$$

We employ the standard textbook technique to evaluate the correlation functions. We define a generating functional

$$Z(K, \eta^+, \eta_f) := \sum_{j_f, i_f} \int d\xi_{v_f} \prod_{(v_f, v_{f'})} \delta_{\text{SL}(2,C)}(g_{v_f}) \prod_f A_f \prod_{v_f} A_{v_f} \prod \delta_{\text{SL}(2,C)}(g_{v_f})$$

$$\times \int D\mu[\xi, \eta] \exp \left( - \sum_{v_f, v_{f'}} \eta_{v_f} D_{v_f, v}[j_f, i_f, g_{v_f}] \xi_{v_f} + \sum_{v_f} \eta_{v_f} \xi_{v_f} + \sum_{v_f} \eta_{v_f} \eta_{v_f} \right)$$

$$= \sum_{j_f, i_f} \int d\xi_{v_f} \prod_{(v_f, v_{f'})} \delta_{\text{SL}(2,C)}(g_{v_f}) \prod_f A_f \prod_{v_f} A_{v_f} \prod \delta_{\text{SL}(2,C)}(g_{v_f})$$

$$\times \exp \left( \sum_{v_f, v_{f'}} \eta_{v_f} D_{v_f, v}[j_f, i_f, g_{v_f}] \eta_{v_f} \right).$$ (4.4)

Then, the correlation function is given by the functional derivative, e.g. the two-point function (fermion propagator) is given by

$$\langle \xi_{v_1} \xi_{v_2} \rangle_{\text{Spin–foam}} = \left. \frac{\delta^2}{\delta \eta_{v_2} \delta \eta_{v_1}} Z(K, \eta^+, \eta_f) \right|_{\eta = \eta'} = \sum_{j_f, i_f} \int d\xi_{v_f} \prod_{(v_f, v_{f'})} \delta_{\text{SL}(2,C)}(g_{v_f}) \prod_f A_f \prod_{v_f} A_{v_f} \prod \delta_{\text{SL}(2,C)}(g_{v_f})$$

$$\times \prod_f A_{v_f} \xi_{v_f} \exp \left( \sum_{v_f, v_{f'}} \eta_{v_f} D_{v_f, v}^{-1}[j_f, i_f, g_{v_f}] \eta_{v_f} \right).$$ (4.5)
An $n$-point correlation function is given by
\[
\langle \xi_{v_1} \cdots \xi_{v_n} \xi_{v_{n+1}}^\dagger \cdots \xi_{v_{2n}}^\dagger \rangle_{\text{spin–foam}} = 0 \quad \text{if} \quad m \neq n
\]
\[
\langle \xi_{v_1} \cdots \xi_{v_n} \xi_{v_{n+1}}^\dagger \cdots \xi_{v_{2n}}^\dagger \rangle_{\text{spin–foam}} = \sum_{\sigma} (-1)^{\sigma} \frac{\delta^2}{\delta \eta_{v_{n+1}}} \cdots \frac{\delta^2}{\delta \eta_{v_{2n}}} Z(K, \eta^1, \eta_f) \bigg|_{\eta^\prime = \eta_f = 0}
\]
\[
= \sum_{f, j, \Gamma} d_{\text{GSF}}^{\Gamma} \prod_{(\nu,v)} \delta_{\text{SL}(2,C)}(g_{\nu,v}) \prod_f A_f[j_f] \prod_v A_v[j, i, g, \eta] \cdot f_{\gamma, j, i, v} \prod_{\sigma} (-1)^{\sigma} D_{\xi_{v_{n+1}}}^{-1} [j_f, i, g, \eta] \cdots \prod_{\sigma} D_{\xi_{v_{2n}}}^{-1} [j_f, i, g, \eta] \quad (4.6)
\]
where $\sigma$ denotes the permutation \((1, 2, \ldots, n) \rightarrow (\sigma(1), \sigma(2), \ldots, \sigma(n))\).

The Dirac determinant in equation (4.6) has been studied in the previous section. So we are
going to find an expression for the inverse of the spin-foam Dirac operator $D_{\xi_{v_{n+1}}}^{-1} [j_f, i, g, \eta]$. Here, we put in a small regulator $\varepsilon > 0$ and consider the inverse matrix $\varepsilon^{-1} (D + \varepsilon)^{-1}$. Since the spin-foam Dirac operator $D$ is anti-Hermitian, the spectrum of operator $D + \varepsilon$ lies in the right-half complex plane; thus $e^{-(D + \varepsilon)t} \in [0, \infty)$ gives a contraction semigroup on a
finite-dimensional vector space (with the usual norm). Then, we have the following strong
operator-equation as a consequence of the Hille–Yosida theorem [33]:
\[
(D + \varepsilon)^{-1} = \int_0^\infty dL e^{-(D + \varepsilon)L}, \quad (4.9)
\]
which is also known as Schwinger’s proper time representation in physics literatures. Given
two vertices $v, v'$
\[
(D + \varepsilon)^{-1}_{v, v'} = \int_0^\infty dL e^{-\varepsilon L} \left[ e^{-(D)L}_{v, v'} \right] \quad (4.10)
\]
\[
= \int_0^\infty dL e^{-\varepsilon L} \sum_{\text{Path}_{v \rightarrow v'}} \frac{(-1)^k}{k!} L^k \sum_{\text{Path}_{v \rightarrow v'}} \cdots \sum_{\text{Path}_{v \rightarrow v'}} (s_{v_{2k-1}}^+ g_{v_{2k-1}} g_{v_{2k}} \cdots)
\]
\[
i \frac{1}{\hat{p} - m + i \varepsilon} \quad (4.8)
\]
We see that the regulator $\varepsilon$ corresponds to a Feynman regulator for the free quantum field.
where $D$ is nonvanishing only when $v, v'$ are neighboring vertices, $\sum_{\text{Path}_{v \to v'}}$ denotes the sum over all the oriented paths from $v$ to $v'$ along the edges of the 2-complex $K$, and the number $O_p$ denotes the number of edges on the path, whose orientations are opposite to the path.

For a Dirac fermion, the regulator $\varepsilon$ should be replaced by $iM_{v,v'} + \varepsilon = i\delta_{v,v}^4V_v m_0 + \varepsilon$, where $^4V_v$ is the volume of a 4-simplex at $v$ and $m_0$ is the fermion (bare) mass; in this case, the inverse Dirac operator

\[
(D + iM + \varepsilon)^{-1} = \int_0^\infty dL \ e^{-(iM + \varepsilon)L} \sum_{\text{Path}_{v \to v'}} \frac{(-1)^k}{k!} L^k
\]

\[
\times \prod_{j=1}^k (2V_{v_j})(G_{v_{j+1}v_j}G_{v_{j+1}v_{j+2}}G_{v_{j+2}v_{j+3}}) \cdots (G_{v_{n-1}v_n}G_{v_{n}v_{n+1}})(-1)^{O_p}.
\]

(4.11)

### 4.2. World-line representation

There is another representation of $(D + \varepsilon)^{-1} v, v'$ in terms of an discretized world-line action, which is physically interesting. This representation is obtained by discretizing the exponential $e^{-D L}$

\[
(D + \varepsilon)^{-1} v, v' = \int_0^\infty dL e^{-L L} \sum_{\text{Path}_{v \to v'}} [\epsilon^{-\frac{1}{2}}] v, v' \cdots [\epsilon^{-\frac{1}{2}}] v, v'.
\]

(4.12)

In case the number of vertices of $K$ goes to be large,

\[
\int_0^\infty dL \ e^{-L} \lim_{n \to \infty} \sum_{\text{Path}_{v \to v'}} \left[ 1 - \frac{L}{n} \rho \right] \left[ 1 - \frac{L}{n} \rho \right] \cdots \left[ 1 - \frac{L}{n} \rho \right] v, v'.
\]

\[
= \int_0^\infty dL \ e^{-L} \lim_{n \to \infty} \sum_{\text{Path}_{v \to v'}} \left[ \delta_{v, v_1} - \frac{2\varepsilon_{v_1}}{\gamma_{v_1}} V_{v_1} \frac{L}{n} g_{v_1, v_1} \right] \left[ \delta_{v_1, v_2} - \frac{2\varepsilon_{v_2}}{\gamma_{v_2}} V_{v_2} \frac{L}{n} g_{v_2, v_2} \right] \cdots
\]

\[
\left[ \delta_{v_{n-1}, v} - \frac{2\varepsilon_{v_n}}{\gamma_{v_n}} V_{v_n} \frac{L}{n} g_{v_n, v_n} \right].
\]

(4.13)

where $\text{Path}_{v \to v'}^n$ denotes the set of paths passing through $n - 1$ vertices except $v$ and $v'$, $\varepsilon_{v,v'} = 1$ if the orientation of $e$ coincides with $(v, v')$ and $\varepsilon_{v,v'} = -1$ otherwise. We make a change of variable $L \mapsto \frac{l}{\ell_p}$ to make $L$ having a dimension of length/time

\[
\frac{1}{\ell_p^4} \int_0^\infty dL \ e^{-\frac{L}{\ell_p}} \lim_{n \to \infty} \sum_{\text{Path}_{v \to v'}} \left[ \delta_{v, v_1} - \varepsilon_{v, v_1} \frac{2V_{v_1} \frac{l_1}{\ell_p}}{\ell_p} g_{v_1, v_1} \right] \cdots \left[ \delta_{v_{n-1}, v} - \varepsilon_{v_{n-1}, v} \frac{2V_{v_n} \frac{l_n}{\ell_p}}{\ell_p} g_{v_n, v_n} \right].
\]

(4.14)

where $l_1 + l_2 + \cdots + l_n = L$. Recall that

\[
g_{v,v}^+ g_{v,v}^- = n_\sigma(\epsilon) \sigma^\sigma = n_\sigma(\epsilon) \sigma^{\sigma}(\epsilon).
\]

(4.15)

---

12 On Minkowski spacetime, the discussion of the world-line representation of bosonic propagator often can be found in string theory textbooks, e.g. [34]. For the fermionic propagator on Minkowski spacetime, the discussion of world-line representation can be found in e.g. [35] and the reference therein.
where \( n^e(e) \) is the normalized tangent vector along the edge \( e \) at the begin point \( b(e) \); then \( \xi_{\nu_\nu}(e) n^\nu(e) \) is the normalized tangent vector along the edge \((v, v')\) at \( b(e) \). Therefore, in the limit \( n \to \infty \)

\[
(\mathcal{D} + \varepsilon)^{-1} = \frac{1}{\ell_p^4} \int_0^\infty dL e^{-\frac{L}{n}} \sum_{\gamma \in \text{Path}_{n,v,v'}} \mathcal{P} \exp \left[ -\frac{2}{\ell_p^4} \int_0^L dL n_\gamma(\gamma(l)) \sigma^a(\gamma(l)) V(\gamma(l)) \right]
\]

(4.16)

where \( n_\gamma(\gamma(l)) \) is the normalized tangent vector along the path \( \gamma(l) \), \( V(\gamma(l)) \) is the 3-volume of the tetrahedron \( \tau(l) \) at \( \gamma(l) \). We then make a change of variable and define \( t = 1/L \)

\[
(\mathcal{D} + \varepsilon)^{-1} = \frac{1}{\ell_p^4} \int_0^\infty dL e^{-\frac{L}{n}} \sum_{\gamma \in \text{Path}_{n,v,v'}} \mathcal{P} \exp \left[ -\frac{2}{\ell_p^4} \int_0^1 dt L n_\gamma(\gamma(t)) \sigma^a(\gamma(t)) V(\gamma(t)) \right].
\]

(4.17)

This expression can be written as a gauge-fixed path integral \( (n^a(t) = n^a(\gamma(t)), \sigma^a(t) = \sigma^a(\gamma(t))) \)

\[
= \frac{1}{\ell_p^4} \int_0^\infty dL \int [De(t)] \int_{\gamma(0)=v}^{\gamma(1)=v'} [D\gamma(t)] \prod_{t \in [0,1]} \delta(e(t) - L) e^{-\frac{L}{n} \ell_p^4 dt} e(t)
\]

\[
\times \mathcal{P} e^{-\frac{L}{n} \ell_p^4 dt} e(t)n_\gamma(\gamma(t))\sigma^a(\gamma(t)) V(\gamma(t)),
\]

(4.18)

where \( f_{\gamma(0)=v}^{\gamma(1)=v'}[D\gamma(t)] = \sum_{\text{Path}_{n,v,v'}} \) is nothing but a counting measure. If we define a world-line action (matrix)

\[
S_W[e, \gamma, j, i, e, v] = \frac{2}{\ell_p^4} \int_0^1 dt \left[ e(t)n_\gamma(t)\sigma^a(t)V(t) + \frac{1}{2} \varepsilon e(t) \right]
\]

(4.20)

and consider \( e(t) \) as a world-line metric, then the inverse spin-foam Dirac operator is a discretized path integral of this world-line action on the spin-foam background

\[
(\mathcal{D} + \varepsilon)^{-1} = \frac{1}{\ell_p^4} \int_0^\infty dL \int [De(t)] \int_{\gamma(0)=v}^{\gamma(1)=v'} [D\gamma(t)] \prod_{t \in [0,1]} \delta(e(t) - L) \mathcal{P} e^{-S_W[e, \gamma, j, i, e, v]}.
\]

(4.21)

Obviously, \( \prod_{t \in [0,1]} \delta(e(t) - L) \) is a gauge-fixing for the world-line reparametrization invariance, and the parameter \( L \) is a world-line Teichmüller parameter, which is a gauge fixing left-over.

For Dirac fermion, the fermion world-line action reads

\[
S_\psi[e, \gamma, j, i, e, v] = \frac{2}{\ell_p^4} \int_0^1 dt \left[ e(t)n_\gamma(t)\psi^a(t)V(t) + \frac{1}{2} (im_0 \psi V(t) + \varepsilon)e(t) \right],
\]

(4.22)

where the discretized version of \( n_\gamma(t)\psi^a(t) \) is \( \xi_{\nu e}G_{\nu e}\psi^0G_{\nu e} \). Then, the inverse spin-foam Dirac operator is a discretized version of world-line path integral

\[
(\mathcal{D} + iM + \varepsilon)^{-1} = \frac{1}{\ell_p^4} \int_0^\infty dL \int [De(t)] \int_{\gamma(0)=v}^{\gamma(1)=v'} [D\gamma(t)] \prod_{t \in [0,1]} \delta(e(t) - L)
\]

\[
\times \mathcal{P} e^{-S_\psi[e, \gamma, j, i, e, v]}.
\]

(4.23)

\[\text{Actually, it can also be considered as a ‘world-tube’ action because of the 3-volume } V(t) \text{ i.e.}\]

\[
S_W = \frac{2}{\ell_p^4} \int_0^1 dt e(t) \int_{(\gamma(t))} d^3s \sqrt{det q} \left[ n_\gamma(t)\sigma^a(t) + \frac{1}{2} \varepsilon \right],
\]

(4.19)

where we redefine the regulator \( \varepsilon \) such that \( \varepsilon = eV(t) \).
5. Conclusion and discussion

We have defined and discussed the fermion quantum field coupled with spin-foam quantum gravity, and defined and explored the properties of the fermion correlation functions on spin-foams, where we have shown that there is a spin-foam analogue of PCT symmetry for spin-foam fermions. The concrete evaluation of the fermion correlations function has also been performed, and the main building blocks, the Dirac determinant and the inverse Dirac operator have been computed. We have shown that the spin-foam fermion correlation functions can be represented as the Feynman diagrams of fermion world-lines imbedded in the spin-foam amplitudes. In this paper, we have considered only the interaction between fermions and gravity, so the Feynman diagram imbedded in the spin-foams is factorized into disconnected propagators. We expect that even for interacting matter quantum fields, a similar structure holds, i.e. the matter field correlation functions could be represented (at least perturbatively) by Feynman diagrams of the interacting fields imbedded in the spin-foam amplitudes.

In closing, we present a remark about the species doubling problem for lattice fermions. The spin-foam fermions are defined with a discrete setting, similar to the fermions in lattice field theory. It is well known that the formal discretization of the fermion action on a lattice suffers the problem of species doubling (see any textbook on lattice field theory e.g. [32], see also [36]), while the problem is resolved when the discretized Dirac operator satisfies the Ginsparg–Wilson relation. Such a Dirac operator can be constructed from the formal discretization in an overlap formulation (overlap fermions) by Neuberger [37], which gives an exact chiral symmetry and anomaly calculation (see [36] for a summary). The overlap fermions can also be defined on a curved lattice in the presence of external gravitational field [38], where the chiral symmetry and anomaly calculation are reproduced correctly. Because of these results, one might consider if the overlap formulation should be employed to define the spin-foam fermion, instead of the formal discretization used in this work. First of all such an idea could be realized straightforwardly in the formulation of overlap fermion, following the technique for example in [38]. However, the overlap formulation (and Ginsparg–Wilson relation) makes correction for the formal discretized Dirac operator by additional terms proportional (and higher order of) to the lattice spacing \( a \), which is a semiclassical concept in the context of the spin-foam model. Thus, it seems to us that it is unnatural to implement those corrections fundamentally. But it would be interesting to see if those corrections can emerge from some certain semiclassical approximations of the spin-foam model. It is not hopeless in our opinion for the following reasons: the summing over all the geometries in the spin-foam model make it hopeful that the fermion doublers are canceled in a similar way to those on a flat random lattice. Some evidences for this have been shown in the context of fermion on Regge gravity [25], where the fermion propagator is computed numerically and display excellent agreement with the continuum field theory.

Acknowledgments

The authors would like to thank E Bianchi, E Magliaro, E Livine, C Perini and W Wieland for fruitful discussions. MH would also like to thank Song He for discussions and his comments from a lattice-field-theorist’s perspective.

References

[1] Thiemann T 2007 Modern Canonical Quantum General Relativity (Cambridge: Cambridge University Press)
Rovelli C 2004 Quantum Gravity (Cambridge: Cambridge University Press)
[2] Rovelli C 2010 Loop quantum gravity: the first twenty five years arXiv:1012.4707

[3] Rovelli C 2010 A new look at loop quantum gravity arXiv:1004.1780

[4] Thiemann T 1998 Quantum spin dynamics (QSD) Class. Quantum Grav. 15 839–73 (arXiv: gr-qc/9606089)

[5] Han M and Thiemann T 2010 Commuting simplicity and closure constraints for 4D spin foam models arXiv:1010.5444

[6] Engle J, Pereira R and Rovelli C 2007 The loop-quantum-gravity vertex-amplitude Phys. Rev. Lett. 99 161301

[7] Freidel L and Krasnov K 2008 New spin foam model for 4d gravity Class. Quantum Grav. 25 125018

[8] Han M and Thiemann T 2009 On the relation between operator constraint–master constraint–reduced phase space–and path integral quantisation arXiv:0911.3428

[9] Dowdall R and Fairbairn W 2010 Observables in 3d spinfoam quantum gravity with fermions arXiv:1003.1847

Fairbairn W 2007 Fermions in three-dimensional spinfoam quantum gravity Gen. Rel. Grav. 39 427–76

Livine E and Ryan J 2008 N=2 supersymmetric spin foams in three dimensions Class. Quantum Grav. 25 175014

Oriti D and Ryan J 2006 Group field theory formulation of 3d quantum gravity coupled to matter fields Class. Quantum Grav. 23 653–76
[13] Freidel L and Louapre D 2004 Ponzano–Regge model revisited: I. Gauge fixing, observables and interacting spinning particles. *Class. Quantum Grav.* **21** 5685–726
Freidel L and Livine E 2006 Ponzano–Regge model revisited: III. Feynman diagrams and effective field theory. *Class. Quantum Grav.* **23** 2021–62
Baratin A and Freidel L 2007 Hidden quantum gravity in 3d Feynman diagrams. *Class. Quantum Grav.* **24** 2027–60
Baratin A and Freidel L 2007 Hidden quantum gravity in 4d Feynman diagrams: emergence of spin foams. *Class. Quantum Grav.* **24** 2027–60
[14] Mikovic A 2003 Spin foam models of Yang–Mills theory coupled to gravity. *Class. Quantum Grav.* **20** 239–46
Oriti D and Pfeiffer H 2002 A spin foam model for pure gauge theory coupled to quantum gravity. *Phys. Rev. D* **66** 124010
Mikovic A 2002 Spin foam models of matter coupled to gravity. *Class. Quantum Grav.* **19** 2335–54
[15] Bianchi E, Han M, Magliaro E, Perini C, Rovelli C and Wieland W 2010 Spinfoam fermion arXiv: 1012.4719
[16] Ding Y, Han M and Rovelli C 2010 Generalized spinfoams arXiv:1011.2149 [gr-qc]
Ding Y and Rovelli C 2010 Physical boundary Hilbert space and volume operator in the Lorentzian new spin-foam theory arXiv:1006.1294 [gr-qc]
[17] Kamiński W, Kisielowski M and Lewandowski J 2010 Spinfoams for all loop quantum gravity. *Class. Quantum Grav.* **27** 095006
Kamiński W, Kisielowski M and Lewandowski J 2010 The EPRL intertwiners and corrected partition function. *Class. Quantum Grav.* **27** 165020
[18] Barrett J W, Dowdall R J, Fairbairn W J, Hellmann F and Pereira R 2009 Lorentzian spin foam amplitudes: graphical calculus and asymptotics. *J. Math. Phys.* **50** 112504
[19] Conrady F and Freidel L 2008 Path integral representation of spin foam models of 4d gravity. *Class. Quantum Grav.* **25** 245010
Bonzom V and Livine E R 2009 A Lagrangian approach to the Barrett–Crane spin foam model. *Phys. Rev. D* **79** 064034
Bonzom V 2009 Spin foam models for quantum gravity from lattice path integrals *Phys. Rev. D* **80** 064028
[20] Han M and Zhang M 2012 Asymptotics of spinfoam amplitude on simplicial manifold: Euclidean theory. *Class. Quantum Grav.* **29** 165004 (arXiv:1109.0506)
Han M and Zhang M 2011 Asymptotics of spinfoam amplitude on simplicial manifold: Lorentzian theory arXiv:1109.0499
Bianchi E, Magliaro E and Perini C 2010 Spinfoams in the holomorphic representation. *Phys. Rev. D* **82** 124031
Engle J and Pereira R 2009 Regularization and finiteness of the Lorentzian LQG vertices. *Phys. Rev. D* **79** 084034
Kaminski W 2010 All 3-edge-connected relativistic BC and EPRL spin-networks are integrable arXiv:1010.5384
Streater R F and Wightman A S 1964 *PCT, Spin and Statistics, and All That* (Reading, MA: Benjamin)
Haag R 1992 *Local Quantum Physics* (Berlin: Springer)
[21] Bianchi E, Dona P and Speziale S 2010 Polyhedra in loop quantum gravity arXiv:1009.3402v1[gr-qc]
Bianchi E and Haggard H M 2011 Discreteness of the volume of space from Bohr-sommerfeld quantization. *Phys. Rev. Lett.* **107** 011301
Freidel L, Krasnov K and Livine E R 2010 Holomorphic factorization for a quantum tetrahedron. *Commun. Math. Phys.* **293** 85–125
Hollands S and Wald R M 2010 Axiomatic quantum field theory in curved spacetime. *Commun. Math. Phys.* **293** 85–125
Hollands S 2004 A general PCT theorem for the operator product expansion in curved spacetime. *Commun. Math. Phys.* **244** 209–44
Ren H 1988 Matter fields in lattice gravity. *Nucl. Phys. B* **301** 661
Bombelli L, Henson J and Sorkin R D 2009 Discreteness without symmetry breaking: a theorem. *Mod. Phys. Lett. A* **24** 2579–87
Livre E R and Speziale S 2007 A new spinfoam vertex for quantum gravity. *Phys. Rev. D* **76** 084028
Conrady F and Freidel L 2009 Quantum geometry from phase space reduction. *J. Math. Phys.* **50** 123510
Bianchi E, Donà P and Speziale S 2010 Polyhedra in loop quantum gravity arXiv:1009.3402v1[gr-qc]
Bianchi E and Hagbard H M 2011 Discreteness of the volume of space from Bohr-sommerfeld quantization. *Phys. Rev. Lett.* **107** 011301
Freidel L, Krasnov K and Livine E R 2010 Holomorphic factorization for a quantum tetrahedron. *Commun. Math. Phys.** **297** 45–93
Conrady F and Freidel L 2008 On the semiclassical limit of 4d spin foam models. *Phys. Rev. D* **78** 104023
Dittrich B and Ryan J P 2008 Phase space descriptions for simplicial 4d geometries arXiv:0807.2806[gr-qc]
Thiemann T 2001 Gauge field theory coherent states (GCS): I. General properties. *Class. Quantum Grav.* **18** 2025
Thiemann T and Winkler O 2001 Gauge field theory coherent states (GCS): II. Peakedness properties. *Class. Quantum Grav.* **18** 2561
[31] Livine E R and Oriti D 2003 Implementing causality in the spin foam quantum geometry *Nucl. Phys.* B 663 231–79
Rovelli C et al 2005 Background independence in a nutshell *Class. Quantum Grav.* 22 2971–90
Rovelli C and Wilson-Ewing E 2012 Discrete symmetries in covariant LQG *Phys. Rev.* D 86 064002
Rovelli C et al 2012 Divergences and orientation in spinfoams (arXiv:1207.5156 [gr-qc])

[32] Montvay I and Münster G 1997 *Quantum Fields on a Lattice* (Cambridge Monographs on Mathematical Physics) (Cambridge: Cambridge University Press)

[33] Reed M and Simon B 1970 *Modern Methods in Mathematical Physics* vol 2 (New York: Academic)

[34] Kritsis E 2007 *String Theory in a Nutshell* (Princeton, NJ: Princeton University Press)

[35] Migdal A A 1986 Momentum loop dynamics and random surface in QCD *Nucl. Phys.* B 265 594
Alexandrou C, Rosenfelder R and Schreiber A W 1999 Worldline path integral for the massive Dirac propagator:
a four-dimensional approach *Phys. Rev.* A 59 3

[36] Fujikawa K and Suzuki H 2004 *Path Integrals and Quantum Anomalies* (Oxford: Oxford University Press)

[37] Neuberger H 1998 *Phys. Lett.* B 417 141
Neuberger H 1998 *Phys. Lett.* B 427 353

[38] Hayakawa M, So H and Suzuki H 2006 Overlap lattice fermion in a gravitational field *Prog. Theor. Phys.* 116 197 (arXiv:hep-lat/0604003)