VERTICAL SQUARE FUNCTIONS AND OTHER OPERATORS ASSOCIATED WITH AN ELLIPTIC OPERATOR

CRUZ PRISUELOS ARRIÑAS

Abstract. We study the vertical and conical square functions defined via elliptic operators in divergence form. In general, vertical and conical square functions are equivalent operators just in $L^2$. But when this square functions are defined through the heat or Poisson semigroup that arise from an elliptic operator, we are able to find open intervals containing 2 where the equivalence holds. The intervals in question depend ultimately on the range where the semigroup is uniformly bounded or has off-diagonal estimates. As a consequence we obtain new boundedness results for some square functions. Besides, we consider a non-tangential maximal function associated with the Poisson semigroup and extend the known range where that operator is bounded. Our methods are based on the use of extrapolation for Muckenhoupt weights and change of angle estimates. All our results are obtained in the general setting of a degenerate elliptic operator, where the degeneracy is given by an $A_2$ weight, in weighted Lebesgue spaces. Of course they are valid in the unweighted and/or non-degenerate situations, which can be seen as special cases, and provide new results even in those particular settings.

We also consider the square root of a degenerate elliptic operator in divergence form $L$, and improve the lower bound of the interval where this operator is known to be bounded on $L^p(vdw)$. Finally, we give unweighted boundedness results for the degenerate operators under consideration.

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1. Introduction

The study of the operators that arise from an elliptic operators $L$ in divergence form has been of great interest specially after the solution of the Kato problem in [3]. In [1] P. Auscher developed a complete study of the boundedness and off-diagonal estimates of the heat semigroup generated by $L$, as well as its gradient. He also proved boundedness and other norm inequalities for the square root of $L$, the Riesz transform, and two representative vertical functions. These results were extended to weighted Lebesgue spaces for Muckenhoupt weights by P. Auscher and J.M. Martell in [5, 6, 7]. Recently, in [12], D. Cruz-Uribe, J.M. Martell, and C. Rios carried on a similar study for degenerate elliptic operators which are defined by introducing a degeneracy in the elliptic operator in terms of a Muckenhoupt weight $w \in A_2$. In [13], D. Cruz-Uribe and C. Rios considered these degenerate elliptic operators and solved the Kato square root problem under the assumption that the associated heat kernel satisfies classic Gaussian upper bounds.

Coming back to the square function associated with the operator $L$, in [4], P. Auscher, S. Hofmann, and J.M. Martell studied boundedness in weighted Lebesgue spaces for vertical and conical square functions associated with the gradient of the heat and Poisson semigroups generated by $L$. Besides, they showed how the vertical and conical operators are related in general. Specifically, they showed that the norm on $L^p(w)$ of the conical operator controls the norm on $L^p(w)$ of the vertical one for all $0 < p < 2$ and $w \in RH_{(2/p)'}$, and the reverse inequality holds for all $2 < p < \infty$ and $w \in A_{p/2}$. In particular, we have that these operators are equivalent on $L^2(\mathbb{R}^n)$.

Furthermore, in [19] J.M. Martell and the author of this paper studied weighted norm estimates and boundedness of the conical square functions associated with the operator $L$ defined via the heat or Poisson semigroup, or their gradients. In order to obtain the boundedness of the conical square functions associated with the Poisson semigroup it was essential to compare their norms on $L^p(w)$ with the corresponding norms of the conical square functions associated with the heat semigroup. This work was extended for degenerate elliptic operators in [9]. The weighted boundedness of the non-tangential maximal functions associated with the heat and Poisson semigroup was proved in [20] in the context of Hardy spaces.

In this work our aim is to complete this theory in the most general way that has been considered so far. More precisely, given a Muckenhoupt weight $w \in A_2$ we consider a second order divergence form degenerate elliptic operator defined by

$$L_w f := -w^{-1} \text{div}(wA\nabla f).$$

If the reader is interested only in the non-degenerate case, we note that all our results and proofs are valid replacing the weight $w$ with a constant equal to one. For example, when $w \equiv 1$ in the previous definition we obtain the uniformly elliptic operator $L f = -\text{div}(A\nabla f)$, (see the complete definitions in Section 2).

In the papers mentioned above the boundedness of the vertical and conical square functions were considered independently. This is natural for, instance, when we apply the general norm comparison results between vertical and conical operators proved in [4, Proposition 2.1, Proposition 2.3] to $s_{2,1H}$ and $S_{2,1H}$ (see the definitions below), the boundedness of the conical square function implies boundedness for the vertical one just for values of $p$ less than 2, while we know that this vertical square function could be bounded for values of $p$ up to infinity (see [4]), as in the case of
$L = -\Delta$. In Theorems 3.1 and 3.2 of this paper we prove sharper norm comparison results, in the particular case of considering vertical and conical square functions defined via the heat or Poisson semigroup generated by $L_w$. As a consequence of those results, we obtain boundedness of the vertical square functions defined in (2.26) and (2.27) directly from the boundedness of the conical square functions defined in (2.28)-(2.31). To illustrate how our results improve the known comparison result proved in [4, Proposition 2.3] for this class of operators, let us formulate Theorems 3.1 and 3.2 in the particular case of $w \equiv 1$ and $v \in A_{\infty}$, and for the square functions:

$$s_{2,H}f(x) := \left( \int_0^\infty |r^2 L e^{-r^2 t} f(y)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}, \quad g_{0,H}f(x) := \left( \int_0^\infty |\nabla e^{-r^2 t} f(y)|^2 \frac{dt}{t} \right)^{\frac{1}{2}},$$

$$S_{2,H}f(x) := \left( \int_{\Gamma(x)} |r^2 L e^{-r^2 t} f(y)|^2 \frac{dy}{r^{p+1}} \right)^{\frac{1}{2}}, \quad G_{0,H}f(x) := \left( \int_{\Gamma(x)} |\nabla e^{-r^2 t} f(y)|^2 \frac{dy}{r^{p+1}} \right)^{\frac{1}{2}},$$

where $\Gamma(x) := \{(y,t) \in \mathbb{R}^{n+1} : |x-y| < t\}$, is the cone of aperture one with vertex at $x$. For other definitions see Sections 2.1 and 2.2.

**Theorem 1.1.** Given $f \in L^2(\mathbb{R}^n)$ and $v \in A_{\infty}$, we have

(a) $\|s_{2,H}f\|_{L^p(v)} \leq \|S_{2,H}f\|_{L^p(v)}$, for $p \in (0, p_+(L))$ and $v \in RH_{\left(\frac{a_{\infty}}{p}\right)}$,

(b) $\|S_{2,H}f\|_{L^p(v)} \leq \|s_{2,H}f\|_{L^p(v)}$, for $p \in (p_-(L), \infty)$ and $v \in A_{\frac{a_{\infty}}{p}}$,

(c) $\|g_{0,H}f\|_{L^p(v)} \leq \|G_{0,H}f\|_{L^p(v)}$, for $p \in (0, q_+(L))$ and $v \in RH_{\left(\frac{a_{\infty}}{p}\right)}$,

(d) $\|G_{0,H}f\|_{L^p(v)} \leq \|g_{0,H}f\|_{L^p(v)}$, for $p \in (q_-(L), \infty)$ and $v \in A_{\frac{a_{\infty}}{p}}$.

In particular, for all $p \in (p_-(L), p_+(L))$ and $v \in A_{\frac{a_{\infty}}{q-(L)}} \cap RH_{\left(\frac{a_{\infty}}{p}\right)}$, we have

$$\|s_{2,H}f\|_{L^p(v)} \approx \|S_{2,H}f\|_{L^p(v)},$$

and, for all $p \in (q_-(L), q_+(L))$ and $v \in A_{\frac{a_{\infty}}{q+(L)}} \cap RH_{\left(\frac{a_{\infty}}{p}\right)}$, we have

$$\|g_{0,H}f\|_{L^p(v)} \approx \|G_{0,H}f\|_{L^p(v)}.$$

If we had applied the results in [4, Proposition 2.1], we would have obtained (b) and (d) for $p \in (2, \infty)$ and $v \in A_{\frac{p}{2}}$; and (a) and (c) for $p \in (0, 2)$ and $v \in RH_{\left(\frac{a_{\infty}}{p}\right)}$, which are clearly smaller intervals (see Section 2.1). We recall that the above norm comparison results are between vertical and conical operators. Norm comparison results between vertical square function are easy consequences of the boundedness or off-diagonal estimates of the heat or Poisson semigroup, the Riesz transform $\nabla L^{-\frac{1}{2}}$, and the subordination formula. They have been observed and used, as needed, in some papers such as [4, 17]. As for norm comparison result between conical square functions see [19, 9].

Coming back to Theorems 3.1 and 3.2, their proofs are obtained from Proposition 4.2. This is a general norm comparison result between a particular class of vertical and conical operators. This proposition follows from off-diagonal and change of angle estimates, and extrapolation for Muckenhoupt weights. Theorem 3.1 implies that the vertical and conical square function associated with the heat or Poisson semigroup are equivalent operators in $L^p(vdhw)$ for the values of $p$ for which the heat semigroup is uniformly bounded. Analogously, when $w \equiv 1$, by Theorem 3.2 we obtain these equivalences for the vertical and conical square functions associated with the gradient of the heat or Poisson semigroup, for the values of $p$ for which the gradient of the heat semigroup is uniformly bounded. In the case that $w \in A_{2}$ the use of Poincaré inequality narrows the interval and the class of weights where the equivalences hold. Anyhow, in Remark 4.31 we see that the vertical square functions considered in Theorem 3.2 can be controlled in norm by conical square functions associated with the heat semigroup (without gradient), in the range where the gradient of the heat
semigroup is uniformly bounded. Similarly, we could prove norm comparison results, between the conical square functions considered in Theorem 3.2 and vertical square functions associated with the heat semigroup, in bigger intervals than those considered in Theorem 3.2, parts (b) and (d). Although the proofs are much more long and intricate, and without an application in sight their small contribution to this work does not seem to be worth the effort, specially for the reader.

Moreover, in Theorems 3.5 and 3.6 we infer the boundedness of the vertical square functions from the boundedness of the conical ones. We recall that the boundedness of representative vertical and conical square functions has been studied in several papers already mentioned: [1, 5, 4, 19, 9, 12]. We also observe that (in addition to even powers) we allow odd powers of the square root of the operator \( L_w \) in the definitions of the square functions that we deal with (see (2.26)-(2.31)), a possibility that has not always been considered in the aforementioned papers.

We now turn towards the non-tangential maximal functions associated with the operator \( L_w \). In Theorem 3.7 we study boundedness of \( N_p^w \) (see the definition below in (2.32)). In [10] this has been recently considered; the proof there follows the lines of [20] in the weighted non-degenerate case. Here we modify that proof to improve the range of boundedness, even in the unweighted non-degenerate case (i.e., \( w \equiv 1 \equiv v \)), see [17, (6.49)] and [21]. More specifically, this improvement follows from the comparison result in Theorem 4.20. To explain this better let us consider, for instance, that \( w \equiv 1 \) and \( v \in A_{\infty} \). So far we knew that \( N_p \) can be extended to a bounded operator on \( L^p(v) \) for all \( p \in (p_-(L), p_+(L)) \) and \( v \in A_{\frac{n}{p-1}} \cap RH (\frac{\|\mu_0\|}{p}) \). However, in Theorem 4.20 we obtain the inequality (see the definition of \( N_H \) in (2.32)): 

\[
\| N_p f \|_{L^p(v)} \leq \| N_H f \|_{L^p(v)} + \| S_{2,H} f \|_{L^p(v)}, \quad \forall \ p \in (p_-(L), p_+(L)^*) \text{ and } v \in A_{\frac{n}{p-1}} \cap RH (\frac{\|\mu_0\|}{p}) \text{'}. 
\]

From the boundedness of \( N_H \) and \( S_{2,H} \) (see [19, 20]), this implies that \( N_p \) can be extended to a bounded operator on \( L^p(v) \) for all \( p \in (p_-(L), p_+(L)^*) \) and \( v \in A_{\frac{n}{p-1}} \cap RH (\frac{\|\mu_0\|}{p}) \) (see definition (2.21)). In turn, if \( v \equiv 1 \) this implies that \( N_p \) can be extended to a bounded operator on \( L^p(\mathbb{R}^n) \) for all \( p \in (p_-(L), p_+(L)^*) \). The gist of the proof is in obtaining in the second term of the sum in (1.2) the \( L^p(v) \) norm of a conical square function, while so far we only knew the above inequality replacing that norm with the \( L^p(v) \) norm of a vertical square function which ultimately has worse boundedness properties than the conical one. We are able to achieve this improvement by the use of extrapolation for Muckenhoupt weights.

Finally we consider the square root of the operator \( L_w \). The properties of this operator as been widely studied, specially after the resolution of the Kato conjecture in any dimension in [3], when \( w \equiv 1 \). That is, 

\[
\| \sqrt{L} f \|_{L^2(\mathbb{R}^n)} \equiv \| \nabla f \|_{L^2(\mathbb{R}^n)}.
\]

In the case that \( w \in A_2 \), this was solved in [13]. The extension to \( L^p \), for a general \( p \), was done in [1] in the unweighted non-degenerate situation, and in [5] in the weighted non-degenerate case. Besides, in [12] the authors proved the unweighted degenerate version of (1.3) for a general \( p \). That is, for \( w \in A_2 \), 

\[
\| \sqrt{L_w} f \|_{L^p(w)} \equiv \| \nabla f \|_{L^p(w)}.
\]

In that paper, the weighted degenerate case (that is, \( w \in A_2 \) and \( L^p(vdw) \) with \( v \in A_{\infty}(w) \)) was also considered, but in view of the previous results in the unweighted or weighted non-degenerate case, we expect that the range of boundedness obtained in [12, Proposition 6.1] regarding the inequality 

\[
\| \sqrt{L_w} f \|_{L^p(vdw)} \leq \| \nabla f \|_{L^p(vdw)},
\]

...
can be improved. We do so in Theorem 3.8, by seeing the product weight \( v \cdot w \) as a weight in \( A_{\infty} \) (see Remark 2.15), and then applying the Calderón-Zygmund decomposition in Lemma 5.6 and interpolation in weighted Lebesgue spaces and in Sobolev spaces (see [8]).

The organization of this paper is as follows. In Section 2 we list some properties and useful results for the present work about Muckenhoupt weights and elliptic operators. In Section 3 we formulate our comparison and boundedness results. In Section 4 we obtain some auxiliary results. In Section 5 we prove the theorems established in Section 3. Finally in Section 6, we provide unweighted estimates for degenerate operators. That is, we consider \( w \in A_2 \) and \( v \equiv 1 \).

2. Preliminaries

First of all we note that when we say unweighted degenerate case we mean that we consider \( w \in A_2 \) and \( v \equiv 1 \), weighted degenerate case stands for \( w \in A_2 \) and \( v \in A_{\infty}(w) \), unweighted non-degenerate case indicates \( w \equiv 1 \equiv v \), and weighted non-degenerate case refers to \( w \equiv 1 \) and \( v \in A_{\infty} \).

Next, we specify our notation. We denote by \( n \) the dimension of the underlying space \( \mathbb{R}^n \) and we always assume that \( n \geq 2 \). Let \( \mu \) be a measure in \( \mathbb{R}^n \), given a set \( E \subset \mathbb{R}^n \), we write

\[
\mu(E) := \int_E d\mu.
\]

Moreover, for \( 1 \leq p < \infty \), we denote the Lebesgue space \( L^p(\mathbb{R}^n, d\mu) \) by \( L^p(\mu) \). Throughout the paper the measure \( \mu \) will be given by a weight \( w \in A_\infty \) or a product of weights \( vw \), where \( w \in A_\infty \) and \( v \in A_{\infty}(w) \), see (2.5) and Remark 2.15, and the definitions below. In the latter case, we shall use independently the notation \( vd\nu \) or \( d(vw) \) depending on whether we want to emphasize the fact that \( w \in A_\infty \) and \( v \in A_{\infty}(w) \) or to see \( vw \) as a weight in \( A_\infty \). In the same line we can write \( L^p(vd\nu) \) or \( L^p(vw) \).

Besides, for every ball \( B \subset \mathbb{R}^n \), we define the annuli of \( B \) as

\[
C_j(B) := 4B, \quad C_j(B) := 2^{j+1}B \setminus 2^jB, \quad \text{for} \quad j \geq 2,
\]

where for any \( \lambda > 0 \) we denote \( \lambda B \) as the ball with the same center as \( B \) and radius \( \lambda \) times the radius of \( B \). Furthermore, abusing notation

\[
\int_{\lambda B} f \, d\mu := \frac{1}{\mu(\lambda B)} \int_{\lambda B} f \, d\mu, \quad \int_{C_j(B)} f \, d\mu := \frac{1}{\mu(2^{j+1}B)} \int_{C_j(B)} f \, d\mu \quad \text{for} \quad j \geq 2.
\]

Additionally, we denote by \( C, c, \) or \( \theta \) any positive constant that may depend on several parameters, but without altering the result of the main computation. In some cases, we indicate such a dependence by adding a subindex.

Finally, we define \( \mathbb{R}^{n+1}_+ := \{(x,t) : x \in \mathbb{R}^n, t > 0\} \) the upper-half space, and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \).

2.1. Muckenhoupt weights. In this section we present some properties of Muckenhoupt weights, for further details see [14, 15, 16].

A Muckenhoupt weight \( w \) is a non-negative, locally integrable function. We say that \( w \) belongs to an \( A_p \) class, denoted by \( w \in A_p \) if, for \( 1 < p < \infty \),

\[
[w]_{A_p} := \sup_B \left( \int_B w(x) \, dx \right) \left( \int_B w(x)^{1-p'} \, dx \right)^{p-1} < \infty,
\]

and, \( w \in A_1 \) if

\[
[w]_{A_1} := \sup_B \left( \int_B w(x) \, dx \right) \left( \text{ess sup}_{x \in B} w(x)^{-1} \right) < \infty.
\]

Here and below the suprema run over the collection of balls \( B \subset \mathbb{R}^n \).
The weight \( w \) can also belong to a Reverse Hölder class denoted by \( w \in RH_s \). We say that \( w \in RH_s \) if, for \( 1 < s < \infty \),
\[
[w]_{RH_s} := \sup_B \left( \frac{\int_B w(x) \, dx}{\int_B w(x)^s \, dx} \right)^{1/s} < \infty,
\]
and \( w \in RH_\infty \) if
\[
[w]_{RH_\infty} := \sup_B \left( \frac{\int_B w(x) \, dx}{\int_B w(x)^s \, dx} \right)^{1/s} \left( \text{ess sup}_{x \in B} w(x) \right) < \infty.
\]
We denote by \( A_\infty \) the collection of all the weights
\[
A_\infty := \bigcup_{1 \leq p < \infty} A_p = \bigcup_{1 < s \leq \infty} RH_s.
\]

An important property is that if \( w \in RH_s, 1 < s \leq \infty \), then
\[
\frac{w(E)}{w(B)} \leq [w]_{RH_s} \left( \frac{|E|}{|B|} \right)^{1/s}, \quad \forall E \subset B,
\]
where \( B \) is any ball in \( \mathbb{R}^n \). Analogously, if \( w \in A_p, 1 \leq p < \infty \), then
\[
\left( \frac{|E|}{|B|} \right)^p \leq [w]_{A_p} \frac{w(E)}{w(B)}, \quad \forall E \subset B.
\]
This implies in particular that \( w \) is a doubling measure:
\[
w(\lambda B) \leq [w]_{A_p} \lambda^n w(B), \quad \forall B, \forall \lambda > 1.
\]

As a consequence of this doubling property, we have that with the ordinary Euclidean distance \( | \cdot |, (\mathbb{R}, \text{d}w, | \cdot |) \) is a space of homogeneous type. In this setting we can define new classes of weights \( A_p(w) \) and \( RH_s(w) \) by replacing the Lebesgue measure in the definitions above with \( d\lambda \): e.g., \( v \in A_p(w) \) for \( 1 < p < \infty \) if
\[
[v]_{A_p(w)} = \sup_B \left( \frac{\int_B v(x) \, dw}{\int_B v(x)^{1-p'} \, dw} \right)^{p-1} < \infty,
\]
and \( v \in RH_s(w) \) for \( 1 < s \leq \infty \) if
\[
[v]_{RH_s(w)} = \sup_B \left( \frac{\int_B v(x) \, dw}{\int_B v(x)^s \, dw} \right) < \infty.
\]
From these definitions, it follows at once that there is a “duality” relationship between the weighted and unweighted \( A_p \) and \( RH_s \) conditions:
\[
w^{-1} \in A_p(w) \text{ if and only if } w \in RH_{p'} \quad \text{and} \quad w^{-1} \in RH_{s'}(w) \text{ if and only if } w \in A_s.
\]

It is also important to observe that the weights in the \( A_p \) and \( RH_{p'} \) classes have a self-improving property: if \( w \in A_p \), there exists \( \varepsilon > 0 \) such that \( w \in A_{p-\varepsilon} \), and similarly if \( w \in RH_{p'} \), then \( w \in RH_{(s-\delta)_s} \) for some \( \delta > 0 \). Thus, given \( w \in A_\infty \), we define the infimum of those values by
\[
r_w := \inf \left\{ p : w \in A_p \right\}, \quad s_w := \inf \left\{ q : w \in RH_q \right\}.
\]
It should be noted that some authors prefer to define \( s_w \) as the conjugate exponent of ours. That is, by \( \sup \left\{ q : w \in RH_q \right\} \), see for instance [6, Lemma 4.1] or [12].

Related to \( r_w \) and \( s_w \) we define the following intervals. Given \( 0 \leq p_0 < q_0 \leq \infty \) and \( w \in A_\infty \), [6, Lemma 4.1] implies that
\[
W_{w}(p_0, q_0) := \left\{ p \in (p_0, q_0) : w \in A_p \cap RH_{p_0} \right\} = \left( \frac{p_0 r_w}{s_w}, \frac{q_0}{s_w} \right).
\]
If \( p_0 = 0 \) and \( q_0 < \infty \) it is understood that the only condition that stays is \( w \in RH_1(\frac{p_0}{q_0}) \). Analogously, if \( 0 < p_0 \) and \( q_0 = \infty \) the only assumption is \( w \in A_{\frac{p_0}{q_0}} \). Finally \( W_w(0, \infty) = (0, \infty) \).

In the same way, for a weight \( v \in A_{\infty}(w) \), with \( w \in A_{\infty} \) we set

\[
(2.11) \quad r_v(w) := \inf \{ r : v \in A_r(w) \} \quad \text{and} \quad s_v(w) := \inf \{ s : v \in RH_s(w) \}.
\]

For \( 0 \leq p_0 < q_0 \leq \infty \) and \( v \in A_{\infty}(w) \), by a similar argument to that of [6, Lemma 4.1], we have

\[
(2.12) \quad W_w^n(p_0, q_0) := \left\{ p \in (p_0, q_0) : v \in A_{\frac{p_0}{q_0}}(w) \cap RH_{\frac{p_0}{q_0}}(w)^c \right\} = \left( p_0 r_v(w), \frac{q_0}{s_v(w)} \right).
\]

If \( p_0 = 0 \) and \( q_0 < \infty \), as before, it is understood that the only condition that stays is \( v \in RH_{\frac{q_0}{p_0}}(w) \).

Analogously, if \( 0 < p_0 \) and \( q_0 = \infty \) the only assumption is \( v \in A_{\frac{p_0}{q_0}}(w) \). Finally \( W_w^n(0, \infty) = (0, \infty) \).

Additionally, note that for every \( w \in A_p \), \( v \in A_q(w) \), \( 1 \leq p, q < \infty \), it follows that

\[
(2.13) \quad \left( \frac{[E]}{|B|} \right)^{pq} \leq [w]_{A_p}^q \left( \frac{w(E)}{w(B)} \right) \leq [w]_{A_q(w)}^q \left( \frac{vw(E)}{vw(B)} \right) \quad \forall E \subset B.
\]

Analogously, if \( w \in RH_p \) and \( v \in RH_q(w) \) \( 1 < p, q \leq \infty \), one has

\[
(2.14) \quad \frac{vw(E)}{vw(B)} \leq [v]_{RH_q(w)} \left( \frac{w(E)}{w(B)} \right) \leq [v]_{RH_q(w)} [w]_{RH_p} \left( \frac{[E]}{|B|} \right)^{\frac{1}{p} \frac{1}{q} - 1}, \quad \forall E \subset B.
\]

From these inequalities we can guess that given a weight \( w \in A_{\infty} \) and \( v \in A_{\infty}(w) \) the product of \( v \) and \( w \) may belong to \( A_{\infty} \). In fact, we obtain the following:

Remark 2.15. Given \( w \in A_{\infty} \) and \( v \in A_{\infty}(w) \), we have that \( r_{vw} \leq r_w r_v(w) \). The equality holds when \( r_w = 1 = r_v(w) \), and the converse inequality is false in general. For instance, consider \( w(x) := |x|^p \) and \( v := w^{-1} \). We have that \( r_{vw}(w) = 2 \) but \( r_{vw} = 1 \), since \( vw \equiv 1 \).

In view of (2.9) and (2.11), it is immediate to see that the inequality \( r_{vw} \leq r_w r_v(w) \) follows from the fact that, for \( 1 \leq p, q < \infty \), \( w \in A_p \), and \( v \in A_q(w) \), the product of the weights \( v \) and \( w \) belongs to the Muckenhoupt class \( p \) times \( q \). That is \( vw \in A_{pq} \). This is an easy consequence of Hölder’s inequality. Indeed, assuming that \( p \neq 1 \) and \( q \neq 1 \) (the cases \( p = 1 \) and/or \( q = 1 \) follow similarly), since \( pq - 1 > q - 1 \) and \( \left( \frac{p-1}{q-1} \right)^{p-1} = \frac{p}{q} \),

\[
\left( \frac{1}{|B|} \int_B (vw)^{1-(pq)} \right)^{pq-1} \leq \left( \frac{1}{|B|} \int_B w^{-1} \right)^{1-\frac{1}{p}} \left( \frac{1}{w(B)} \int_B \left( \frac{1}{w(B)} \right)^{1-\frac{1}{q}} w^{\frac{1}{q}} \right)^{q-1} \left( \frac{1}{|B|} \int_B w^{-1} \right)^{q(p-1)} \leq \left( \frac{1}{|B|} \int_B w^{-1} \right)^{1-\frac{1}{p}} \left( \frac{1}{w(B)} \int_B vw \right)^{1-\frac{1}{q}} = \left( \frac{1}{|B|} \int_B vw \right)^{1-\frac{1}{q}}.
\]

Next, we observe that, under particular assumptions, we can compare the average of a function with respect to the measure given by a weight \( w \in A_{\infty} \), with that with respect to a product of weights \( w \in A_{\infty} \) and \( v \in A_{\infty}(w) \). More specifically:

Remark 2.16. Given \( 0 < \bar{p} \leq \bar{q} < \infty \), note that, if \( v \in A_{\bar{p}}(w) \) then,

\[
(2.17) \quad \left( \frac{1}{C_{\bar{p},(B)}} \int_{C_{\bar{p},(B)}} |f(x)|^{\bar{p}} dw(x) \right)^{\frac{1}{\bar{p}}} \leq \left( \frac{1}{C_{\bar{p},(B)}} \int_{C_{\bar{p},(B)}} |f(x)|^{\bar{p}} (vw)(x) \right)^{\frac{1}{\bar{p}}} \quad \forall B \subset \mathbb{R}^n, \quad j \geq 1.
\]
On the other hand, if \( v \in RH\left( \frac{2}{\tilde{q}} \right) \)'(\( w \)) then

\[
(2.18) \quad \left( \frac{1}{|C_j(B)|} \int_{C_j(B)} |f(x)|^{\tilde{p}} d\tilde{v}(vw(x)) \right)^{\frac{1}{\tilde{p}}} \leq \left( \frac{1}{|C_j(B)|} \int_{C_j(B)} |f(x)|^{\tilde{p}} d\tilde{w}(x) \right)^{\frac{1}{\tilde{p}}}, \quad \forall B \subset \mathbb{R}^n, \quad j \geq 1.
\]

We detail the case \( \tilde{p} < \tilde{q} \). The case \( \tilde{p} = \tilde{q} \) follows similarly.

We obtain (2.17) applying Hölder’s inequality and (2.6)

\[
\left( \frac{1}{|C_j(B)|} \int_{C_j(B)} |f(x)|^{\tilde{p}} d\tilde{v}(vw(x)) \right)^{\frac{1}{\tilde{p}}} = \left( \frac{1}{|C_j(B)|} \int_{C_j(B)} |f(x)|^{\tilde{p}} v(x)^{-\frac{\tilde{p}}{\tilde{q}}} dw(x) \right)^{\frac{1}{\tilde{p}}}
\leq \left( \frac{1}{|C_j(B)|} \int_{C_j(B)} |f(x)|^{\tilde{p}} v(x) dw(x) \right)^{\frac{1}{\tilde{p}}} \left( \frac{1}{|C_j(B)|} \int_{C_j(B)} v(x)^{1-\frac{\tilde{p}}{\tilde{q}}} dw(x) \right)^{-\frac{1}{\tilde{q}}} = \left( \frac{1}{|C_j(B)|} \int_{C_j(B)} |f(x)|^{\tilde{p}} d(vw(x)) \right)^{\frac{1}{\tilde{p}}}.
\]

Similarly, we obtain (2.18) applying Hölder’s inequality and (2.7)

\[
\left( \frac{1}{|C_j(B)|} \int_{C_j(B)} |f(x)|^{\tilde{p}} d\tilde{v}(vw(x)) \right)^{\frac{1}{\tilde{p}}} = \left( \frac{w(2j+1)}{vw(2j+1)} \right)^{\frac{1}{\tilde{p}}} \left( \frac{1}{|C_j(B)|} \int_{C_j(B)} |f(x)|^{\tilde{p}} v(x) dw(x) \right)^{\frac{1}{\tilde{p}}}
\leq \left( \frac{1}{|C_j(B)|} \int_{C_j(B)} v(x) dw(x) \right)^{-\frac{1}{\tilde{p}}} \left( \frac{1}{|C_j(B)|} \int_{C_j(B)} |f(x)|^{\tilde{p}} dw(x) \right)^{\frac{1}{\tilde{p}}} \left( \frac{1}{|C_j(B)|} \int_{C_j(B)} v(x)(\frac{\tilde{p}}{\tilde{q}})^{-1} dw(x) \right)^{\frac{\tilde{q}}{\tilde{p}}}
\leq \left( \frac{1}{|C_j(B)|} \int_{C_j(B)} |f(x)|^{\tilde{p}} dw(x) \right)^{\frac{1}{\tilde{p}}}.
\]

We also introduce the Hardy-Littlewood maximal function

\[
Mf(x) := \sup_{B(x,r)} \frac{1}{|B|} \int f(y) \, dy.
\]

By the classical theory of weights, \( w \in A_p, \ 1 < p < \infty \), if and only if \( M \) is bounded on \( L^p(w) \).

Likewise, given \( w \in A_\infty \), we can introduce the weighted maximal operator \( M^w \):

\[
(2.19) \quad M^w f(x) := \sup_{B(x,r)} \frac{1}{|B|} \int |f(y)| \, dw(y).
\]

Since \( w \) is a doubling measure, one can also show that \( v \in A_p(w), \ 1 < p < \infty \), if and only if \( M^w \) is bounded on \( L^p(vdw) \).

Furthermore, for any \( p \in (0, \infty) \) and a weight \( w \in A_\infty \), we define

\[
(2.20) \quad (p)_{w,*} := \frac{nrw}{nrw + p},
\]

and, for \( k \in \mathbb{N} \),

\[
(2.21) \quad p^{k,*}_w := \begin{cases} \frac{nrw}{nrw - kp} & \text{if } nrw > kp, \\ \infty & \text{otherwise.} \end{cases}
\]

If we consider \( w \equiv 1 \), since \( r_w = 1 \), we have that

\[
(p)_* := \frac{np}{n + p}, \quad \text{and} \quad p^{k,*}_n := \begin{cases} \frac{np}{n - kp} & \text{if } n > kp, \\ \infty & \text{otherwise.} \end{cases}
\]
We write $p_w^* := p_w^{1,*}$, or $p^* := p^{1,*}$, when $w \equiv 1$.

2.2. **Elliptic operators.** Consider $A$ an $n \times n$ matrix of complex and $L^\infty$-valued coefficients defined on $\mathbb{R}^n$, satisfying the following uniform ellipticity (or “accretivity”) condition: there exist $0 < \lambda \leq \Lambda < \infty$ such that

\begin{equation}
\lambda |\xi|^2 \leq \text{Re} A(x) \xi \cdot \bar{\xi} \quad \text{and} \quad |A(x) \xi \cdot \bar{\xi}| \leq \Lambda |\xi||\bar{\xi}|,
\end{equation}

for all $\xi, \zeta \in \mathbb{C}^n$ and almost every $x \in \mathbb{R}^n$. We have used the notation $\xi \cdot \zeta = \xi_1 \zeta_1 + \cdots + \xi_n \zeta_n$ and therefore $\xi \cdot \bar{\zeta}$ is the usual inner product in $\mathbb{C}^n$. Given a Muckenhoupt weight $w \in A_2$ we define a second order divergence form degenerate elliptic operator by

\begin{equation}
L_w f := -w^{-1} \text{div}(w A \nabla f).
\end{equation}

In the non-degenerate case we define $Lf := -\text{div}(A \nabla f)$, and replace $L_w$ with $L$ everywhere below.

The operator $-L_w$ generates a $C_0$–semigroup of contractions on $L^2(w)$ which is called the heat semigroup $\{e^{-tL_w}\}_{t \geq 0}$. We also consider the Poisson semigroup $\{e^{-t\sqrt{L_w}}\}_{t \geq 0}$ defined via the subordination formula:

\begin{equation}
e^{-t\sqrt{L_w}} f(y) = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} u^\frac{1}{2} e^{-u} e^{-\frac{t^2}{4u}} L_w f(y) \frac{du}{u}.\end{equation}

We denote by $(p_-(L_w), p_+(L_w))$ the maximal open interval on which the heat semigroup $\{e^{-tL_w}\}_{t \geq 0}$ is uniformly bounded on $L^p(w)$:

\begin{align*}
p_-(L_w) &:= \inf \left\{ p \in (0, 1) : \sup_{t > 0} \|e^{-tL_w}\|_{L^p(w) \to L^p(w)} < \infty \right\}, \\
p_+(L_w) &:= \sup \left\{ p \in (0, 1) : \sup_{t > 0} \|e^{-tL_w}\|_{L^p(w) \to L^p(w)} < \infty \right\}.
\end{align*}

Note that in place of the semigroup $\{e^{-tL_w}\}_{t \geq 0}$ we are using its rescaling $\{e^{-tL_w}\}_{t \geq 0}$. We do so because all the “heat” square functions, defined below, are written using the latter and because in the context of off-diagonal estimates, discussed in the next section, this will simplify some computations.

According to [12] (see also [1]),

\begin{equation}
p_-(L_w) \leq (2^*_w)' < 2 < 2_w^* \leq p_+(L_w).
\end{equation}

For all $m \in \mathbb{N}$ and $K \in \mathbb{N}_0$, we define the vertical square functions associated with the heat semigroup by

\begin{equation}
s_{m,\text{H}}^w f(y) = \left( \int_{0}^{\infty} |(t \sqrt{L_w})^m e^{-tL_w} f(y)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \quad \text{and} \quad g_{K,\text{H}}^w f(y) = \left( \int_{0}^{\infty} \|\nabla(t \sqrt{L_w})^K e^{-tL_w} f(y)\|^2 \frac{dt}{t} \right)^{\frac{1}{2}};
\end{equation}

and with the Poisson semigroup

\begin{equation}
s_{m,\text{P}}^w f(y) = \left( \int_{0}^{\infty} |(t \sqrt{L_w})^m e^{-t\sqrt{L_w}} f(y)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \quad \text{and} \quad g_{K,\text{P}}^w f(y) = \left( \int_{0}^{\infty} \|\nabla(t \sqrt{L_w})^K e^{-t\sqrt{L_w}} f(y)\|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.
\end{equation}

The conical square functions are defined by

\begin{equation}
S_{m,\text{H}}^w f(y) = \left( \int_{\Gamma(x)} |(t \sqrt{L_w})^m e^{-tL_w} f(y)|^2 \frac{dw(y)dt}{tw(B(y,t))} \right)^{\frac{1}{2}},
\end{equation}

\begin{equation}
G_{K,\text{H}}^w f(y) = \left( \int_{\Gamma(x)} |\nabla(t \sqrt{L_w})^K e^{-tL_w} f(y)|^2 \frac{dw(y)dt}{tw(B(y,t))} \right)^{\frac{1}{2}},
\end{equation}

\begin{equation}
E_{m,\text{H}}^w f(y) = \left( \int_{\Gamma(x)} |(t \sqrt{L_w})^m e^{-t\sqrt{L_w}} f(y)|^2 \frac{dw(y)dt}{tw(B(y,t))} \right)^{\frac{1}{2}},
\end{equation}

\begin{equation}
G_{K,\text{P}}^w f(y) = \left( \int_{\Gamma(x)} |\nabla(t \sqrt{L_w})^K e^{-t\sqrt{L_w}} f(y)|^2 \frac{dw(y)dt}{tw(B(y,t))} \right)^{\frac{1}{2}}.
\end{equation}
(2.30) \[ S_{m,p}^w f(y) = \left( \int \int_{\Gamma(x)} |(t \sqrt{L_w})^m e^{-t \sqrt{L_w}} f(y)|^2 \frac{dw(y)dt}{tw(B(y,t))} \right)^{1/2}, \]

(2.31) \[ G_{K,p}^w f(y) = \left( \int \int_{\Gamma(x)} |(t \sqrt{L_w})^K e^{-t \sqrt{L_w}} f(y)|^2 \frac{dw(y)dt}{tw(B(y,t))} \right)^{1/2}. \]

Note that we allow that the square root of the operator \( L_w \) is raised to odd powers, in contrast to \([19, 9]\), where only even powers were considered.

Besides, the non-tangential maximal functions are defined by

(2.32) \[ N_{1}^w f(x) = \left( \sup_{t > 0} \int_{|x-y| < t} |e^{-t \sqrt{L_w}} f(y)|^2 dw(y) \right)^{1/2}, \]

(2.33) \[ \sqrt{L_w} f(y) = \frac{1}{\sqrt{t}} \int_0^\infty s L_w e^{-s t \sqrt{L_w}} f(y) \frac{ds}{s}. \]

Finally, we note the following result obtained in \([9]\) (see also \([4, 19]\)) where the authors proved boundedness for the conical square functions defined in (2.28)-(2.31) considering even powers of the square root of the operator \( L_w \).

**Theorem 2.34.** Given \( w \in A_2 \) and \( v \in A_{\infty}(w) \), for every \( m \in \mathbb{N} \) and \( K \in \mathbb{N}_0 \), there hold:

(a) The conical square functions \( S_{2m,H}^w \) and \( G_{2K,H}^w \) can be extended to bounded operators on \( L^p(vdw) \), for all \( p \in W_p(m, (L_w, \infty)) \).

(b) The conical square functions \( S_{2m,P}^w \) and \( G_{2K,P}^w \) can be extended to bounded operators on \( L^p(vdw) \), for all \( p \in W_p(m, (L_w, 1)) \) and for all \( p \in W_p(m, (L_w, 2K+1)) \) respectively.

### 2.3. Off-diagonal estimates.

**Definition 2.35.** Let \( \{T_t\}_{t>0} \) be a family of sublinear operators and let \( 1 < p < \infty \). Given a doubling measure \( \mu \) we say that \( \{T_t\}_{t>0} \) satisfies \( L^p(\mu) - L^q(\mu) \) full off-diagonal estimates, denoted by \( T_t \in \mathcal{O}(L^p(\mu) - L^q(\mu)) \), if there exist constants \( C, c > 0 \) such that for all closed sets \( E \) and \( F \), all \( f \in L^p(\mathbb{R}^n) \), and all \( t > 0 \) we have

(2.36) \[ \left( \int_E |T_t (1_E f)(x)|^p d\mu \right)^{1/p} \leq C e^{-\frac{c(\text{d}(E,F))^2}{t}} \left( \int_E |f(x)|^p d\mu \right)^{1/p}, \]

where \( \text{d}(E,F) = \inf \{|x-y| : x \in E, y \in F\} \).

Besides, set \( \Upsilon(s) = \max\{|s, s^{-1}\} \) for \( s > 0 \) and recall the notation in (2.1) and (2.2).

**Definition 2.37.** Given \( 1 \leq p \leq q \leq \infty \) and any doubling measure \( \mu \), we say that a family of sublinear operators \( \{T_t\}_{t>0} \) satisfies \( L^p(\mu) - L^q(\mu) \) off-diagonal estimates on balls, denoted by \( T_t \in \mathcal{O}(L^p(\mu) - L^q(\mu)) \), if there exist \( \theta_1, \theta_2 > 0 \) and \( c > 0 \) such that for all \( t > 0 \) and for all balls \( B \) with radius \( r_B \),

(2.38) \[ \left( \int_B |T_t (1_B f)|^q d\mu \right)^{1/q} \leq \Upsilon \left( \frac{r_B}{\sqrt{t}} \right)^{\theta_2} \left( \int_B |f|^p d\mu \right)^{1/p}, \]
and for \( j \geq 2, \)

\[
(2.39) \quad \left( \iint_B |T(f\mathbf{1}_{C_j(B)})|^q d\mu \right)^{\frac{1}{q}} \leq 2^{\theta_1 q} \left( \frac{2^j r_B}{\sqrt{t}} \right)^{\theta_2} e^{-\frac{\epsilon t r_B^2}{2}} \left( \iint_B |f|^p d\mu \right)^{\frac{1}{p}},
\]

and

\[
(2.40) \quad \left( \iint_{C_j(B)} |T(f\mathbf{1}_B)|^q d\mu \right)^{\frac{1}{q}} \leq 2^{\theta_1 q} tL \left( \frac{2^j r_B}{\sqrt{t}} \right)^{\theta_2} e^{-\frac{\epsilon t r_B^2}{2}} \left( \iint_B |f|^p d\mu \right)^{\frac{1}{p}}.
\]

Throughout the paper we shall use the following results about off-diagonal estimates on balls:

**Lemma 2.41** ([7, Section 2], [12, Sections 3 and 7]). Let \( L_w \) be a degenerate elliptic operator as in (2.23) with \( w \in A_2 \). There hold:

(a) If \( p_-(L_w) < p \leq q < p_+(L_w) \), then \( e^{-tL_w} \) and \((tL_w)^m e^{-tL_w}\) belong to \( O(L^p(w) - L^q(w)) \), for every \( m \in \mathbb{N} \).

(b) Let \( p_-(L_w) < p \leq q < p_+(L_w) \). If \( f \in A_{p/(p_-(L_w))} \cap RH_{p/(p_+(L_w))/q’}(w) \), then \( e^{-tL_w} \) and \((tL_w)^m e^{-tL_w}\) belong to \( O(L^p(wv) - L^q(wv)) \), for every \( m \in \mathbb{N} \).

(c) There exists an interval \( K(L_w) \) such that if \( p, q \in K(L_w) \), \( p \leq q \), we have that \( \sqrt{n}\nabla e^{-tL_w} \in O(L^p(w) - L^q(w)) \). Moreover, denoting respectively by \( q_-(L_w) \) and \( q_+(L_w) \) the left and right endpoints of \( K(L_w) \), we have that \( q_-(L_w) = p_-(L_w) \), and \( 2 < q_+(L_w) \leq q_+(L_w)’ \leq p_+(L_w) \).

(d) Let \( q_-(L_w) < p \leq q < q_+(L_w) \). If \( f \in A_{p/q_-(L_w)} \cap RH_{q_+(L_w)/q’}(w) \), then \( \sqrt{n}\nabla e^{-tL_w} \in O(L^p(wv) - L^q(wv)) \).

(e) If \( p = q \) and \( \mu \) a doubling measure then \( \mathcal{T}(L^p(\mu) - L^p(\mu)) \) and \( O(L^p(\mu) - L^p(\mu)) \) are equivalent.

Furthermore, in the following result, which is a weighted version of [20, (5.12)] (see also [17]), we show off-diagonal estimates for the family \( \{T_{t,s,f}\}_{t,s > 0} : \{e^{-tL_w} - e^{-t(x^2 + s^2)L_w}\}_{t,s > 0} \).

**Proposition 2.42.** For \( 0 < t, s < \infty, M \in \mathbb{N} \), and for \( E_1, E_2 \subset \mathbb{R}^n \) closed, given \( p \in (p_-(L_w), p_+(L_w)) \), and \( f \in L^p(w) \) such that \( \text{supp}(f) \subset E_1 \), we have that \( \|T_{t,s,f}\|_{L^p(w)} \) satisfies the following \( L^p(w) - L^p(w) \) off-diagonal estimates:

\[
\tag{2.43} \left\| T_{t,s,f} \right\|_{L^p(w)} \leq \left( \frac{s^2}{t^2} \right)^{M} e^{-\frac{\alpha E_1 E_2^2}{t + s^2}} \|f\|_{L^p(w)}. \]

**Proof.** Note that we have

\[
\left\| T_{t,s,f} \right\|_{L^p(w)} = \left\| T_E \left( e^{-tL_w} - e^{-t(x^2 + s^2)L_w} \right)^M f \right\|_{L^p(w)} = \left\| T_E \left( \int_0^t \cdots \left( \int_0^x \cdots \left( \int_0^{r_1} \cdots \left( \int_0^{r_M} e^{-r_1\cdots-r_M+M^2 r^2} L_w f \right) dr_1 \cdots dr_M \right) dr_1 \cdots dr_M \right) \right) \right\|_{L^p(w)} \leq \frac{dr_1 \cdots dr_M}{(r_1 + \cdots + r_M + M^2)^M} \left\| T_E \right\|_{L^p(w)}
\]

where we have used that \((tL_w)^m e^{-tL_w} \in \mathcal{T}(L^p(w) - L^p(w))\), for all \( p \in (p_-(L_w), p_+(L_w)) \), (see Lemma 2.41, part (e)).
Remark 2.44. From Proposition 2.42, we immediately obtain that, for all \( p \in (p_\ast(L_w), p_\ast(L_w)) \) and \( f \in L^p(w) \),
\[
\|T_{t,f}f\|_{L^p(w)} \leq \left( \frac{x^2}{t^2} \right)^M \|f\|_{L^p(w)} \quad t, s > 0.
\]

We conclude this section by introducing the following off-diagonal estimates on Sobolev spaces. The proof follows as in [10], see also [1] for the unweighted non-degenerate case.

Lemma 2.45. Let \( q \in (q_\ast(L_w), q_\ast(L_w)) \), \( p \) such that \( \max \{ r_w, (q_\ast(L_w))_{w,v} \} < p \leq q \), and \( \alpha > 0 \). Then, for every \( (x, t) \in \mathbb{R}^{n+1}_+ \), there exists \( \theta > 0 \) such that
\[
(2.46) \quad \left( \int_{B(x, \alpha t)} |\nabla e^{-tL_w} f(z)|^q \, dw(z) \right)^{\frac{1}{q}} \leq \Upsilon(\alpha)^{\theta} \sum_{j \geq 1} e^{-c_4j} \left( \int_{B(x, 2^{j+1}\alpha t)} |\nabla S_1 f(z)|^p \, dw(z) \right)^{\frac{1}{p}},
\]
where \( S_1 \) can be equal to \( e^{-tL_w} \) or the identity, for all \( t > 0 \).

2.4. Extrapolation and change of angle. In our proofs the use of extrapolation and change of angle formulas will be essential. In this section we formulate these results.

The following extrapolation result for \( w \equiv 1 \) can be found in [11, Chapter 2], [6], and see also [19, Lemma 3.3]. The proof can be easily obtained by adapting the arguments there replacing everywhere the Lebesgue measure with the measure given by \( w \) and the Hardy-Littlewood maximal function with its weighted version \( M^w \) introduced in (2.19). Further details are left to the interested reader.

Theorem 2.47. [9, Theorem A.1] Let \( \mathcal{T} \) be a given family of pairs \((f, g)\) of non-negative and not identically zero measurable functions.

(a) Suppose that for some fixed exponent \( p_0 \), \( 1 \leq p_0 < \infty \), and every weight \( v \in A_{p_0}(w) \),
\[
\int_{\mathbb{R}^n} f(x)^{p_0} v(x) \, dw(x) \leq C_{w,v,p_0} \int_{\mathbb{R}^n} g(x)^{p_0} v(x) \, dw(x), \quad \forall (f, g) \in \mathcal{T}.
\]
Then, for all \( 1 < p < \infty \), and for all \( v \in A_p(w) \),
\[
\int_{\mathbb{R}^n} f(x)^p v(x) \, dw(x) \leq C_{w,v,p} \int_{\mathbb{R}^n} g(x)^p v(x) \, dw(x), \quad \forall (f, g) \in \mathcal{T}.
\]

(b) Suppose that for some fixed exponent \( q_0 \), \( 1 \leq q_0 < \infty \), and every weight \( v \in RH_{q_0}(w) \),
\[
\int_{\mathbb{R}^n} f(x)^{\frac{1}{q_0}} v(x) \, dw(x) \leq C_{w,v,q_0} \int_{\mathbb{R}^n} g(x)^{\frac{1}{q_0}} v(x) \, dw(x), \quad \forall (f, g) \in \mathcal{T}.
\]
Then, for all \( 1 < q < \infty \) and for all \( v \in RH_q(w) \),
\[
\int_{\mathbb{R}^n} f(x)^{\frac{1}{q}} v(x) \, dw(x) \leq C_{w,v,q} \int_{\mathbb{R}^n} g(x)^{\frac{1}{q}} v(x) \, dw(x), \quad \forall (f, g) \in \mathcal{T}.
\]

(c) Suppose that for some fixed exponent \( r_0 \), \( 0 < r_0 < \infty \), and every weight \( v \in A_{r_0}(w) \),
\[
\int_{\mathbb{R}^n} f(x)^{r_0} v(x) \, dw(x) \leq C_{w,v,r_0} \int_{\mathbb{R}^n} g(x)^{r_0} v(x) \, dw(x), \quad \forall (f, g) \in \mathcal{T}.
\]
Then, for all \( 0 < r < \infty \) and for all \( v \in A_{r_0}(w) \),
\[
\int_{\mathbb{R}^n} f(x)^r v(x) \, dw(x) \leq C_{w,v,r} \int_{\mathbb{R}^n} g(x)^r v(x) \, dw(x), \quad \forall (f, g) \in \mathcal{T}.
\]
(d) Suppose that for some fixed exponents \( p_0 \) and \( q_0 \), given \( 0 < p_0 < p < q_0 < \infty \), and every weight \( v \in A_{\frac{m}{r}}(w) \cap RH\left(\frac{w}{r}\right)^{\ast}(w) \),

\[
\int_{\mathbb{R}^n} f(x)^p \nu(x)dw(x) \leq C_{w,v,p} \int_{\mathbb{R}^n} g(x)^p \nu(x)dw(x), \quad \forall (f,g) \in \mathcal{F}.
\]

Then, for all \( p_0 < q < q_0 \), and for all \( v \in A_{\frac{m}{r}}(w) \cap RH\left(\frac{m}{r}\right)^{\ast}(w) \),

\[
\int_{\mathbb{R}^n} f(x)^q \nu(x)dw(x) \leq C_{w,v,q} \int_{\mathbb{R}^n} g(x)^q \nu(x)dw(x), \quad \forall (f,g) \in \mathcal{F}.
\]

In our proofs we will use these extrapolation results to prove inequalities of the type

\[
\|Q_0 f\|_{L^\infty(\nu dw)} \leq \|Q_1 f\|_{L^\infty(\nu dw)}, \quad \forall p \in W^m_{\nu}(\hat{p},q_w^{k^*}),
\]

where \( w \in A_2 \), \( v \in A_\infty(w) \), \( 0 \leq \hat{p} < q_1 < \max\{2,q_1\} < \hat{q} \leq \infty \), \( k \in \mathbb{N} \), \( Q_0 \) and \( Q_1 \) are operators acting over a function \( f \), and \( q_w^{k^*} \) is defined in (2.21). Hence, it will be enough to show

\[
\|Q_0 f\|_{L^{q_1}(\nu dw)} \leq \|Q_1 f\|_{L^{q_1}(\nu dw)}, \quad \forall \nu v_0 \in A_{q_1}(w) \cap RH\left(\frac{k^*}{q_1}\right)^{\ast}(w).
\]

The fact that \( v_0 \in RH\left(\frac{k^*}{q_1}\right)^{\ast}(w) \) yields an important consequence that we see in the following remark.

**Remark 2.48.** Let \( w \in A_2 \), \( \hat{k} \in \mathbb{N} \), \( 1 < q_1 < \hat{q} \), \( \max\{2,q_1\} < \hat{q} \), and \( v_0 \in RH\left(\frac{k^*}{q_1}\right)^{\ast}(w) \) then we can find \( \hat{\tau}, q_0, \) and \( r \) such that \( r_w < \hat{\tau} < 2, 2 < q_0 < \hat{q} \), \( \max\{q_0/2,q_1/2\} \leq r < \infty \), \( v_0 \in RH(2r/q_1)^{\ast}(w) \) so that

\[
(2.49) \quad \hat{k} + \frac{n \hat{\tau}}{2r} - \frac{n \hat{\tau}}{q_0} > 0.
\]

Indeed, for \( n r_w > \hat{k} \hat{q} \), note that \( s_{v_0}(w) < \frac{n r_w q_1 (n w - k \hat{q})}{n q_0} \). Hence, we can find \( 2 > \hat{\tau} > r_w \) close enough to \( r_w \), \( \varepsilon_0 > 0 \) small enough and \( 2 < q_0 < \hat{q} \), close enough to \( \hat{q} \) so that \( \max\{q_1,q_0/s_{v_0}(w)\} < \frac{n r_w q_0}{2(1 + \varepsilon_0 / (n w - k \hat{q} q_0))} \). Besides, define \( r := \frac{q_0}{2(1 + \varepsilon_0 / (n w - k \hat{q} q_0))} \). Then, \( q_0/2 < r < \infty \), \( v_0 \in RH(2r/q_1)^{\ast}(w) \), and

\[
\hat{k} + \frac{n \hat{\tau}}{2r} - \frac{n \hat{\tau}}{q_0} > 0.
\]

If now \( n r_w \leq \hat{k} \hat{q} \), our condition on the weight \( v_0 \) becomes \( v_0 \in A_\infty(w) \). Then, we take \( r > s_{v_0}(w) \max\{q_1/2,1\} \) and \( q_0 \) satisfying \( \max\left\{ \frac{2}{q_0}, \frac{q_0}{q_0^2} \right\} < \max(\hat{q},2r) \) if \( \hat{q} < \infty \), or \( q_0 = 2r \) if \( \hat{q} = \infty \). Therefore, we have that \( 2 < q_0 < \hat{q}, q_0/2 \leq r < \infty \), and \( v_0 \in RH(2r/q_1)^{\ast}(w) \). Besides,

\[
\hat{k} + \frac{n r_w}{2r} - \frac{n r_w}{q_0} > \frac{n r_w}{q_0} \geq 0.
\]

Then, taking \( r_w < \hat{\tau} < 2 \) close enough to \( r_w \) we get (2.49).

Moreover, note that if we know that there exists \( 0 < \hat{p} < q_1 \) so that \( v_0 \in A_{q_1}(w) \), then we can find \( p_0, \hat{p} < p_0 < q_1 \), close enough to \( \hat{p} \) so that \( v_0 \in A_{q_1}(w) \).

Finally we present the change of angle results that we shall use. The following proposition is a version of [19, Proposition 3.30] in the weighted degenerate case.
Proposition 2.50. [9, Proposition A.2] Let \( w \in A_\tau \) and \( v \in RH^\tau(w) \) with \( 1 \leq \tau, r < \infty \). For every \( 1 \leq q \leq r, \ 0 < \beta \leq 1, \) and \( t > 0 \), there holds

\[
\int_{\mathbb{R}^n} \left( \int_{B(x,\beta t)} |h(y, t)| \frac{dv(y)}{w(B(y, \beta t))} \right)^{\frac{1}{q}} v(x) dv(x) \leq \beta^\tau \left( \frac{1}{\tau} \right)^{\frac{1}{q}} \int_{\mathbb{R}^n} \left( \int_{B(x, t)} |h(y, t)| \frac{dv(y)}{w(B(y, t))} \right)^{\frac{1}{q}} v(x) dv(x).
\]

Consider now the following operator:

\[
\mathcal{A}^w_F(x) := \left( \int_{\Gamma(x)} |F(y, t)|^2 \frac{dv(y) dt}{tw(B(y, t))} \right)^{\frac{1}{2}}, \quad \forall x \in \mathbb{R}^n,
\]

where \( F \) is a measurable function defined in \( \mathbb{R}^{n+1}_+ \), \( \alpha > 0 \), and \( \Gamma^\alpha(x) \) is the cone of aperture \( \alpha \) with vertex at \( x \), \( \Gamma^\alpha(x) := \{(y, t) \in \mathbb{R}^{n+1}_+ : |x - y| < \alpha t\} \).

Proposition 2.51. [9, Proposition 4.9] Let \( 0 < \alpha \leq \beta < \infty \).

(i) For every \( w \in A_\tau \) and \( v \in A_r(w) \), \( 1 \leq \tau, \tilde{\tau} < \infty \), there holds

\[
\|\mathcal{A}^w_F\|_{L^p(v^\alpha dw)} \leq C \left( \frac{\beta}{\alpha} \right)^{\frac{\alpha}{\beta}} \|\mathcal{A}^w_F\|_{L^p(v^\beta dw)} \quad \text{for all} \quad 0 < p \leq 2r,
\]

where \( C \geq 1 \) depends on \( n, p, \tau, \tilde{\tau}, [w]_{A_r}, \) and \([v]_{A_\tau(w)}\), but it is independent of \( \alpha \) and \( \beta \).

(ii) For every \( w \in RH^\tau \) and \( v \in RH^\tau(w) \), \( 1 \leq s, \tilde{s} < \infty \), there holds

\[
\|\mathcal{A}^w_F\|_{L^p(v^\alpha dw)} \leq C \left( \frac{\alpha}{\beta} \right)^{\frac{s}{sp}} \|\mathcal{A}^w_F\|_{L^p(v^\beta dw)} \quad \text{for all} \quad \frac{2}{s} \leq p < \infty,
\]

where \( C \geq 1 \) depends on \( n, p, s, \tilde{s}, [w]_{RH^\tau}, \) and \([v]_{RH^\tau(w)}\), but it is independent of \( \alpha \) and \( \beta \).

The previous proposition was proved in the unweighted non-degenerate case in [2] and in the weighted non-degenerate case in [19, Proposition 3.2].

3. Main results

In this section we present our main results. In particular in Section 3.1 we establish comparison in weighted degenerate Lebesgue spaces for the vertical and conical square functions defined in (2.26)-(2.31). In Section 3.2 we formulate boundedness results for the operators defined in (2.26)-(2.33).

3.1. Norm comparison for vertical and conical square functions. In this section we study the values of \( p \) where the vertical and conical square functions defined in (2.26)-(2.31) are comparable on \( L^p(v^\alpha dw) \).

We first consider the vertical and conical square functions defined via the heat or the Poisson semigroup.

Theorem 3.1. Given \( w \in A_2, \ v \in A_\infty(w), \) and \( f \in L^2(w), \) for every \( m \in \mathbb{N}, \) we have

(a) \( \|s^w_{m,H,f}\|_{L^p(v^\alpha dw)} \leq \|s^w_{m,H,f}\|_{L^p(v^\beta dw)}, \) for \( p \in \mathcal{W}_v^\alpha(0, p_m(L_w)) \);

(b) \( \|s^w_{m,H,f}\|_{L^p(v^\alpha dw)} \leq \|s^w_{m,H,f}\|_{L^p(v^\beta dw)}, \) for \( p \in \mathcal{W}_v^\alpha(p_m(L_w), \infty) \);

(c) \( \|s^w_{m,P,f}\|_{L^p(v^\alpha dw)} \leq \|s^w_{m,P,f}\|_{L^p(v^\beta dw)}, \) for \( p \in \mathcal{W}_v^\alpha(p_m(L_w), p_m(L_w)) \);

(d) \( \|s^w_{m,P,f}\|_{L^p(v^\alpha dw)} \leq \|s^w_{m,P,f}\|_{L^p(v^\beta dw)}, \) for \( p \in \mathcal{W}_v^\alpha(p_m(L_w), p_m(L_w)) \).

In particular, for all \( p \in \mathcal{W}_v^\alpha(p_m(L_w), p_m(L_w)) \), we have

\[
\|s^w_{m,H,f}\|_{L^p(v^\alpha dw)} = \|s^w_{m,H,f}\|_{L^p(v^\beta dw)}, \quad \text{and} \quad \|s^w_{m,P,f}\|_{L^p(v^\alpha dw)} \approx \|s^w_{m,P,f}\|_{L^p(v^\beta dw)}.
\]
As for the vertical and conical square functions defined via the gradient of the heat or the Poisson semigroup, we have the following result.

**Theorem 3.2.** Given \( w \in A_2, v \in A_\infty(w) \), and \( f \in L^2(w) \), for every \( K \in \mathbb{N}_0 \), we have

1. \[ \|G_{K,H}^w f\|_{L^p(vdw)} \leq \|G_{K,H}^w f\|_{L^p(vdw)} \text{ for } p \in \mathcal{W}_w^p(0, q_+(L_w)); \]
2. \[ \|G_{K,H}^w f\|_{L^p(vdw)} \leq \|G_{K,H}^w f\|_{L^p(vdw)} \text{ for } p \in \mathcal{W}_w^p(\max(r_w, q_-(L_w)), \infty); \]
3. \[ \|G_{K,P}^w f\|_{L^p(vdw)} \leq \|G_{K,P}^w f\|_{L^p(vdw)} \text{ for } p \in \mathcal{W}_w^p(\max(r_w, q_-(L_w)), q_+(L_w)); \]
4. \[ \|g_{K,P}^w f\|_{L^p(vdw)} \leq \|g_{K,P}^w f\|_{L^p(vdw)} \text{ for } p \in \mathcal{W}_w^p(\max(r_w, q_-(L_w)), q_+(L_w)). \]

In particular, for all \( p \in \mathcal{W}_w^p(\max(r_w, q_-(L_w)), q_+(L_w)) \), we have

\[ \|g_{K,H}^w f\|_{L^p(vdw)} \approx \|G_{K,H}^w f\|_{L^p(vdw)} \quad \text{and} \quad \|g_{K,P}^w f\|_{L^p(vdw)} \approx \|G_{K,P}^w f\|_{L^p(vdw)}. \]

**Remark 3.3.** The additional restriction \( \max(r_w, q_-(L_w)) < p < \infty \) and \( v \in A_{\infty(\Delta_{N_0},\Delta_{\infty})}(w) \) in Theorem 3.2 (see (2.12)), comes from the use of the Poincaré inequality (see Lemma 2.45). Note that in the non-degenerate case (i.e., \( w \equiv 1 \)) we have that \( r_w = 1 \). Then, we would obtain that for every \( K \in \mathbb{N}_0 \), and \( p \in \mathcal{W}_w^p(q_-(L), q_+(L)), \)

\[ \|g_{K,H}^w f\|_{L^p(vdw)} \approx \|G_{K,H}^w f\|_{L^p(vdw)} \quad \text{and} \quad \|g_{K,P}^w f\|_{L^p(vdw)} \approx \|G_{K,P}^w f\|_{L^p(vdw)}. \]

### 3.2. Boundedness results.

In our first result we study boundedness for the conical square functions defined in (2.28)-(2.31) allowing odd powers of the square root of the operator \( L_m \). Recall that the case of even powers was considered in [9, 19] (see Theorem 2.34).

**Theorem 3.4.** Given \( w \in A_2 \) and \( v \in A_\infty(w) \), for every \( m \in \mathbb{N} \), the conical square functions \( S_{2m-1,H}^w \), \( S_{2m-1,P}^w \), \( G_{2m-1,H}^w \), and \( G_{2m-1,P}^w \) can be extended to bounded operators on \( L^p(vdw) \), for all \( p \in \mathcal{W}_w^p(p_-(L_m), p_+(L_m)^{2m-1,*}). \)

As a consequence of the above results and Theorem 2.34, in the following results, we obtain boundedness of the vertical square functions defined in (2.26)-(2.27).

**Theorem 3.5.** Given \( w \in A_2 \) and \( v \in A_\infty(w) \), for every \( m \in \mathbb{N} \), \( K \in \mathbb{N}_0 \), and \( p \in \mathcal{W}_w^p(p_-(L_m), p_+(L_m)) \) the operators \( s_{m,H}^w \) and \( s_{K,P}^w \) can be extended to bounded operators on \( L^p(vdw) \).

**Theorem 3.6.** Given \( w \in A_2 \) and \( v \in A_\infty(w) \), for every \( m \in \mathbb{N} \), \( K \in \mathbb{N}_0 \), and \( p \in \mathcal{W}_w^p(q_-(L_m), q_+(L_m)) \) the operators \( g_{m,H}^w \) and \( g_{K,P}^w \) can be extended to bounded operators on \( L^p(vdw) \).

Finally, in the next results, we improve the range of values of \( p \) where \( \mathcal{N}_p^w \) and \( \sqrt{\mathcal{L}_m} \) are respectively known to be bounded on \( L^p(vdw) \). Besides, note that Theorem 3.7 improves the range of values of \( p \) where \( \mathcal{N}_p^w \) is bounded even in the unweighted or weighted non-degenerate cases (see [17, 20, 21]). The boundedness of \( \sqrt{\mathcal{L}_m} \) on \( L^p(vdw) \) was also studied in [12]. Additionally, see [5] for the weighted non-degenerate case, and [1] for the unweighted non-degenerate case.

**Theorem 3.7.** Given \( w \in A_2 \) and \( v \in A_\infty(w) \), the non-tangential maximal function \( \mathcal{N}_p^w \) can be extended to a bounded operator on \( L^p(vdw) \), for all \( p \in \mathcal{W}_w^p(p_-(L_m), p_+(L_m)^w). \)

**Theorem 3.8.** Given \( w \in A_2 \) and \( v \in A_\infty(w) \), assume that \( \mathcal{W}_w^p(\max(r_w, p_-(L_m)), p_+(L_m)) \neq \emptyset \). Then, the operator \( \sqrt{\mathcal{L}_m} \) can be extended to a bounded operator from \( W^{1,2}(vdw) \) to \( L^p(vdw) \), for all \( p \in \mathcal{W}_w^p(\max\{r_w, (p_-(L_m))(w)^*\}, p_+(L_m)). \)

The space \( W^{1,2}(vdw) \) is defined as the completion of \( \{ f \in C_0^\infty(\mathbb{R}^n) : \nabla f \in L^1(vdw) \} \) under the semi-norm \( \|f\|_{W^{1,2}(vdw)} := \|\nabla f\|_{L^1(vdw)} \).
4. Auxiliary results

In this section we obtain some results that will simplify the proofs of the theorems formulated in the previous section.

First of all, consider the following conical and vertical square functions:

\[ \mathcal{V}F(x) := \left( \int_0^\infty |T_\mathcal{B}F(y,t)|^{2\frac{dt}{t}} \right)^{\frac{1}{2}} \quad \text{and} \quad \mathcal{S}F(x) := \left( \int_0^\infty \int_\mathbb{R} |T_\mathcal{B}F(y,t)|^2 \frac{dw(y)\,dt}{tw(B(y,t))} \right)^{\frac{1}{2}}, \]

where \( \{T_t\}_{t>0} \) is a family of sublinear operators and \( F \) is a measurable function defined in \( \mathbb{R}^{n+1}_+ \).

Note that given \( w \in A_\infty \) and \( 2 < q_0 < \infty \), for all \( v_0 \in RH_{\frac{q_0}{p_0}} (w) \), there holds

\[ \|\mathcal{V}F\|_{L^2(v_0dw)} \leq \left( \int_\mathbb{R}^n \int_0^\infty \left( \int_\mathcal{B}_t \int_\mathbb{R} |T_\mathcal{B}F(y,t)|^2 \frac{dt}{t} \right)^{\frac{2}{q_0}} \frac{dw(y)\,dt}{tw(B(y,t))} \right)^{\frac{1}{2}} \cdot \]

Indeed, by Fubini’s theorem, (2.5), and (2.18), we get

\[ \|\mathcal{V}F\|_{L^2(v_0dw)} = \left( \int_0^\infty \int_\mathbb{R} |T_\mathcal{B}F(y,t)|^2 \int_\mathcal{B}_t \frac{d(v_0w)(y)}{v_0w(B(y,t))} \frac{v_0(y)\,dy\,dt}{t} \right)^{\frac{1}{2}} \]

\[ \leq \left( \int_0^\infty \int_\mathbb{R} \int_\mathcal{B}_t |T_\mathcal{B}F(y,t)|^2 d(v_0w)(y) v_0(y)\,dy\,dt \right)^{\frac{1}{2}} \]

From this and under some assumptions on the family \( \{T_t\}_{t>0} \), we obtain comparison results between \( \mathcal{V} \) and \( \mathcal{S} \), as we see in the following proposition.

**Proposition 4.2.** Given \( w \in A_\infty \) and \( v \in A_\infty (w) \). Let \( \{T_t\}_{t>0} \) be a family of sublinear operators and \( 0 < p_0 < 2 < q_0 < \infty \). Consider \( B := B(x,t) \), for \( (x,t) \in \mathbb{R}^{n+1}_+ \), a measurable function \( F \) defined in \( \mathbb{R}^{n+1}_+ \), and the following conditions:

(i) For any constant \( c > 0 \) there exists a constant \( C > 0 \) such that \( F(y,ct) = CF(y,t) \);

(ii) \( w(B)^{\frac{q_0}{p_0}} \|1_B T_\mathcal{B}F\|_{L^2(w)} \leq \sum_{j \geq 1} e^{-c4^j} w(2^{j+1}B)^{\frac{q_0}{p_0}} \|1_{2^{j+1}B} T_\mathcal{B}F\|_{L^2(w)} \);

(iii) \( w(B)^{\frac{q_0}{p_0}} \|1_B T_\mathcal{B}F\|_{L^2(w)} \leq \sum_{j \geq 1} e^{-c4^j} w(2^{j+1}B)^{2} \|1_{2^{j+1}B} T_\mathcal{B}F\|_{L^2(w)} \).

Then, assuming that \( F \) satisfies condition (i), there hold:

(a) If \( \{T_t\}_{t>0} \) satisfies condition (ii),

\[ \|\mathcal{S}F\|_{L^p(v_0dw)} \leq \|\mathcal{V}F\|_{L^p(v_0dw)}, \quad \forall p \in W^{p,\infty}(p_0, \infty). \]

(b) If \( \{T_t\}_{t>0} \) satisfies condition (iii),

\[ \|\mathcal{V}F\|_{L^p(v_0dw)} \leq \|\mathcal{S}F\|_{L^p(v_0dw)}, \quad \forall p \in W^{p,\infty}(0, q_0). \]

In particular, if \( F \) satisfies condition (i) and \( \{T_t\}_{t>0} \) satisfies conditions (ii), and (iii), we have

\[ \|\mathcal{V}F\|_{L^p(v_0dw)} = \|\mathcal{S}F\|_{L^p(v_0dw)}, \quad \forall p \in W^{p,\infty}(p_0, q_0). \]

**Proof.** We shall proceed by extrapolation to prove both part (a) and part (b). Indeed, to obtain part (a), in view of (2.12), and from Theorem 2.47, part (a), it is enough to prove

\[ \|\mathcal{S}F\|_{L^2(v_0dw)} \leq \|\mathcal{V}F\|_{L^2(v_0dw)}, \quad \forall v_0 \in A_\infty^{\frac{q_0}{p_0}} (w). \]

(4.3)
As for proving part (b), in view of (2.12), and from Theorem 2.47, part (b), we just need to show
\begin{equation}
\|VF\|_{L^2(v_0 dw)} \lesssim \|SF\|_{L^2(v_0 dw)}, \quad \forall v_0 \in RH_{\frac{m}{n}}.
\end{equation}

We first prove (4.3). By (2.5), condition (ii), and (2.17) (recall that $v_0 \in A_{\frac{m}{n}}(w)$), we get
\begin{align*}
SF(x) & \leq \left( \int_0^\infty \int_{B(x,t)} |T_{\frac{\partial}{\partial x}} F(y,t)|^2 \frac{dw(y)dt}{t} \right)^{\frac{1}{2}} \\
& \leq \sum_{j \geq 1} e^{-c_1j} \left( \int_0^\infty \left( \int_{B(x,2^{j+1}t)} |T_{\frac{\partial}{\partial x}} F(y,t)|^2 dw(y) \right)^{\frac{2}{\nu_0}} \frac{dt}{t} \right)^{\frac{1}{2}} \\
& \leq \sum_{j \geq 1} e^{-c_1j} \left( \int_0^\infty \left( \int_{B(x,2^{j+1}t)} |T_{\frac{\partial}{\partial x}} F(y,t)|^2 v_0(y)dw(y) \right)^{\frac{2}{\nu_0}} \frac{dt}{t} \right)^{\frac{1}{2}} \\
& \leq \left( \int_{\mathbb{R}^n} \left( \int_0^\infty |T_{\frac{\partial}{\partial x}} F(y,t)|^2 v_0(y)dw(y) \right)^{\frac{2}{\nu_0}} \frac{dt}{t} \right)^{\frac{1}{2}} = \|VF\|_{L^2(v_0 dw)},
\end{align*}
where we have used again (2.5) in the last inequality.

Hence, by Fubini’s theorem, changing the variable $t$ into $\sqrt{2}t$, and by condition (i), we conclude that
\begin{align*}
\|SF\|_{L^2(v_0 dw)} & \leq \sum_{j \geq 1} e^{-c_1j} \left( \int_0^\infty \int_{\mathbb{R}^n} |T_{\frac{\partial}{\partial x}} F(y,t)|^2 \frac{d(v_0w)(y)}{v_0w(B(y,2^{j+1}t))} v_0(y)dw(y) \frac{dt}{t} \right)^{\frac{1}{2}} \\
& \leq \sum_{j \geq 1} e^{-c_1j} \left( \int_0^\infty \left( \int_{\mathbb{R}^n} |T_{\frac{\partial}{\partial x}} F(y,\sqrt{2}t)|^2 v_0(y)dw(y) \right)^{\frac{2}{\nu_0}} \frac{dt}{t} \right)^{\frac{1}{2}} \\
& \leq \left( \int_{\mathbb{R}^n} \left( \int_0^\infty |T_{\frac{\partial}{\partial x}} F(y,t)|^2 v_0(y)dw(y) \right)^{\frac{2}{\nu_0}} \frac{dt}{t} \right)^{\frac{1}{2}} = \|VF\|_{L^2(v_0 dw)}.
\end{align*}

We next prove (4.4). To this end, first apply (4.1) and condition (iii). Then, changing the variable $t$ into $\sqrt{2}t$, by condition (i), (2.5), and Proposition 2.51, we obtain that
\begin{align*}
\|VF\|_{L^2(v_0 dw)} & \leq \left( \int_{\mathbb{R}^n} \left( \int_0^\infty \left( \int_{B(x,t)} |T_{\frac{\partial}{\partial x}} F(y,t)|^2 v_0(y)dw(y) \right)^{\frac{2}{\nu_0}} \frac{dt}{t} v_0(x)dw(x) \right)^{\frac{1}{2}} \\
& \leq \sum_{j \geq 1} e^{-c_1j} \left( \int_{\mathbb{R}^n} \left( \int_0^\infty \left( \int_{B(x,2^{j+1}t)} |T_{\frac{\partial}{\partial x}} F(y,t)|^2 v_0(y)dw(y) \right)^{\frac{2}{\nu_0}} \frac{dt}{t} v_0(x)dw(x) \right)^{\frac{1}{2}} \\
& \leq \sum_{j \geq 1} e^{-c_1j} \left( \int_{\mathbb{R}^n} \left( \int_0^\infty \left( \int_{B(x,2^{j+1}t)} |T_{\frac{\partial}{\partial x}} F(y,\sqrt{2}t)|^2 v_0(y)dw(y) \right)^{\frac{2}{\nu_0}} \frac{dt}{t} v_0(x)dw(x) \right)^{\frac{1}{2}} \\
& \leq \sum_{j \geq 1} e^{-c_1j} \left( \int_{\mathbb{R}^n} \left( \int_0^\infty \left( \int_{B(x,2^{j+1}t)} |T_{\frac{\partial}{\partial x}} F(y,t)|^2 tw(B(y,t)) v_0(y)dw(y) \right)^{\frac{1}{2}} \\
& \leq \sum_{j \geq 1} e^{-c_1j} 2^{j\beta_0} \|SF\|_{L^2(v_0 dw)} \\
& \leq \|SF\|_{L^2(v_0 dw)},
\end{align*}
which concludes the proof.
In the following proposition, for every \( m \in \mathbb{N} \), we compare the norms on \( L^p(vd\nu) \) of \( S_{m+1,f}^w \) and \( S_{m+1,f}^w \). This will allow us to obtain Theorem 4.4 for \( S_{m+1,f}^w \), from Theorem 2.34. This result is proved in [10] for \( m = 1 \), (see also [22, Proposition 9.4] [18, Corollary 4.17 and (5.21)], where the weighted and unweighted non-degenerate cases were considered respectively).

**Proposition 4.5.** Given \( w \in A_2 \), \( v \in A_w(w) \), \( m \in \mathbb{N} \), and \( f \in L^2(w) \), there hold

(a) \( \|S_{m+1,f}^w\|_{L^p(vd\nu)} \leq \|S_{m,f}^w\|_{L^p(vd\nu)} \) for all \( p \in W_m^w(0, p_+(L_w)^{m+1,*}) \);

(b) \( \|S_{m,f}^w\|_{L^p(vd\nu)} \leq \|S_{m+1,f}^w\|_{L^p(vd\nu)} \) for all \( p \in W_m^w(0, p_+(L_w)^{m,*}) \).

In particular, we have

\( \|S_{m,f}^w\|_{L^p(vd\nu)} \approx \|S_{m+1,f}^w\|_{L^p(vd\nu)} \) for all \( p \in W_m^w(0, p_+(L_w)^{m,*}) \).

**Proof.** We shall use extrapolation to prove both inequalities. Indeed, Theorem 2.47, part (b) (or Theorem 2.47, part (c), if \( p_+(L_w)^{m+1,*} = \infty \) or \( p_+(L_w)^{m,*} = \infty \)) allows us to obtain (a) from

\[
\|S_{m+1,f}^w\|_{L^2(vd\nu)} \leq \|S_{m,f}^w\|_{L^2(vd\nu)} \quad \forall v_0 \in RH \left( \frac{p_+(L_w)^{m+1,*}}{2} \right)(w)
\]

and (b) from

\[
\|S_{m,f}^w\|_{L^2(vd\nu)} \leq \|S_{m+1,f}^w\|_{L^2(vd\nu)} \quad \forall v_0 \in RH \left( \frac{p_+(L_w)^{m,*}}{2} \right)(w).
\]

Note that Remark 2.48 with \( q_1 = 2, \tilde{q} = p_+(L_w), \) and \( \tilde{k} \) equal to \( m + 1 \) or \( m \) implies that we can find \( \tilde{r}, q_0, \) and \( r \) such that \( r_w < \tilde{r} < 2, 2 < q_0 < p_+(L_w), \) and \( q_0/2 \leq r < \infty \) so that \( v_0 \in RH_r(w) \) and

\[
\tilde{k} + \frac{n \tilde{r}}{2r} - \frac{n \tilde{r}}{q_0} > 0.
\]

We also observe that for \( M \in \mathbb{N} \) large enough the above inequality gives

\[
\tilde{k} + \frac{n \tilde{r}}{2r} - \frac{n \tilde{r}}{q_0} - \frac{1}{M} > 0.
\]

After these observations, we prove (4.6) and (4.7).

We first prove (4.6). By (2.33), Minkowski’s integral inequality, and (2.5) (note that \( B(x, t) \subset B(y, 2t) \), for all \( y \in B(x, t) \)), we obtain

\[
S_{m+1,f}^w(x) \leq \left( \int_0^\infty \left( \int_{B(x,t)} |sL_w e^{-\tau^2 L_w(t \sqrt{L_w})^m e^{-\tau^2 L_w} f(y)|^2 d\nu(y) \right)^{\frac{1}{2}} \frac{ds}{s} \right)^{\frac{1}{2}} \frac{dt}{tw(B(x,t))} \right)^{\frac{1}{2}}
\]

\[
+ \left( \int_0^\infty \left( \int_{B(x,t)} |sL_w e^{-\tau^2 L_w(t \sqrt{L_w})^m e^{-\tau^2 L_w} f(y)|^2 d\nu(y) \right)^{\frac{1}{2}} \frac{ds}{s} \right)^{\frac{1}{2}} \frac{dt}{tw(B(x,t))} \right)^{\frac{1}{2}} =: I + II.
\]

In the case that \( s < t \), for \( F(y, t) := (t \sqrt{L_w})^m e^{-\tau^2 L_w} f(y) \), we use the fact that \( \tau L_w e^{-\tau L_w} \in \mathcal{F}(L^2(w) - L^2(w)) \)

\[
I = \left( \int_0^\infty \left( \int_0^t \frac{st}{t^2/2 + s^2} \left( \int_{B(x,t)} (s^2 + t^2/2) L_w e^{-(s^2 + t^2/2) L_w} F(y, t)^2 d\nu(y) \right)^{\frac{1}{2}} \frac{ds}{s} \right)^{\frac{1}{2}} \frac{dt}{tw(B(x,t))} \right)^{\frac{1}{2}}
\]

\[
\leq \sum_{j=1}^\infty e^{-c_kj} \left( \int_0^\infty \left( \int_0^t \frac{s ds}{t} \right)^{\frac{1}{2}} \frac{ds}{tw(B(x,t))} \right)^{\frac{1}{2}} \frac{dt}{tw(B(x,t))} \right)^{\frac{1}{2}}
\]
where in the last inequality we have changed the variable $t$ into $\sqrt{2}t$ and used (2.5). Then, applying change of angle (Proposition 2.51), we conclude that

$$
\|I\|_{L^2(v_0 \omega)} \leq \sum_{j \geq 1} \epsilon^{-c4j} \left( \int_0^\infty \int_0^{t_j} \frac{d \eta_B(y,t)}{tw(B(y,t))} \right)^{\frac{1}{2}},
$$

As for the estimate of $II$, consider $\tilde{F}(y,s) := (s \sqrt{\Lambda_m})^{m+2} e^{-\frac{1}{2}L_m} f(y)$, apply Cauchy-Schwartz’s inequality in the integral in $s$, the fact that $e^{-\frac{1}{2}L_m} \in \mathcal{F}(L^2(w) - L^2(w))(2.5)$, Jensen’s inequality in the integral in $y (q_0 > 2)$, Fubini’s theorem, and (2.5) to obtain

$$
II = \left( \int_0^\infty \left( \int_0^s \left( \int_0^{t_j} \left( \int_{B(x,t)} |e^{-\frac{1}{2}L_m} \tilde{F}(y,s)|^2 \frac{d \eta_B(y,t)}{w(B(x,t))} \right)^{\frac{1}{2}} ds \right) \frac{dt}{t} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}
$$

Note now that applying Propositions 2.50 with $\beta = t/s < 1$ and $q = q_0/2$, and (2.5),

$$
\int_{\mathbb{R}^n} \left( \int_{B(x,2^{j+1}st/s)} |\tilde{F}(y,s)|^{q_0} \frac{d \eta_B(y,t)}{w(B(y,2^{j+1}st/s))} \right)^{\frac{1}{q_0}} v_0(x) d\omega(x)
$$

recall the choices of $r, \tilde{r}$, and $q_0$ at the beginning of the proof. Besides, since $\tau L_m e^{-\frac{1}{2}L_m} \in \mathcal{F}(L^2(w) - L^2(w))$, for $\tilde{F}(y,s) := (s \sqrt{\Lambda_m})^{m} e^{-\frac{1}{2}L_m}$

$$
\left( \int_{B(x,2^{j+1}st/s)} |\tilde{F}(y,s)|^{q_0} d\omega(y) \right)^{\frac{1}{q_0}} \leq \left( \int_{B(x,2^{j+1}st/s)} \frac{2 L_m e^{-\frac{1}{2}L_m}}{2} |\tilde{F}(y,s)|^{q_0} d\omega(y) \right)^{\frac{1}{q_0}} \leq \sum_{j \geq 1} \epsilon^{-c4j} 2^{\beta j} \left( \int_{B(x,2^{j+1}st/s)} |\tilde{F}(y,s)|^{q_0} d\omega(y) \right)^{\frac{1}{2}}.
$$

Consequently, applying Fubini’s theorem, (4.8) with $m = 1$, changing the variable $s$ into $\sqrt{2}s$, and by (2.5) and change of angle (Proposition 2.51), we get

$$
\|II\|_{L^2(v_0 \omega)}
$$
\[
\leq \sum_{j \geq 1} e^{-c'4} \sum_{l \geq 1} e^{-c'4} \left( \int_{\mathbb{R}^n} \int_0^\infty \int_0^s \left( \frac{t}{s} \right)^{2(m+1)+n} |F(y, s)|^2 \frac{ds}{s} \left\| \mathcal{L}_m \right\|_{L^2(y \omega dy)} \right) \]

\[
\leq \sum_{j \geq 1} e^{-c'4} \sum_{l \geq 1} e^{-c'4} \left( \int_{\mathbb{R}^n} \int_0^\infty \int_{B(x, 2^{j-1} \eta x)} \left| F(y, \sqrt{2}s) \right|^2 \frac{dw(y) ds}{sw(B(y, s))} \right)^{\frac{1}{2}} \]

\[
\leq \sum_{j \geq 1} e^{-c'4} \sum_{l \geq 1} e^{-c'4} 2^{(l-j)\theta \eta n} \| S_{m,H}^w \|_{L^2(y \omega dy)} \]

\[
\leq \| S_{m,H}^w \|_{L^2(y \omega dy)}. \]

As for proving (4.7), by (2.5), (2.33), and Minkowski’s integral inequality, we obtain

\[
\mathcal{S}_{m,H}^w f(x) \leq \left( \int_0^\infty \left( \int_{B(x,t)} \left| tsL_n e^{-s^2L_w} (t \sqrt{L_w})^{m-1} e^{-r^2L_w} f(y) \right|^2 \frac{dw(y)}{s} \right)^{\frac{1}{2}} \frac{ds}{s} \right)^{\frac{1}{2}} \frac{dt}{tw(B(x,t))} \]

\[
+ \left( \int_0^\infty \left( \int_{B(x,t)} \left| tsL_n e^{-s^2L_w} (t \sqrt{L_w})^{m-1} e^{-r^2L_w} f(y) \right|^2 \frac{dw(y)}{s} \right)^{\frac{1}{2}} \frac{ds}{s} \right)^{\frac{1}{2}} \frac{dt}{tw(B(x,t))} \]\n
\[=: \tilde{I} + \tilde{II}. \]

We first estimate \( \tilde{I} \). Using that \( s < t \) and applying the fact that \( e^{-tL_w} \in \mathcal{F}(L^2(w) - L^2(w)) \), and (2.5), we have

\[
\tilde{I} = \left( \int_0^\infty \left( \int_t^s \left( \int_{B(x,t)} \left| e^{-s^2L_w} (t \sqrt{L_w})^{m-1} e^{-r^2L_w} f(y) \right|^2 \frac{dw(y)}{s} \right)^{\frac{1}{2}} \frac{ds}{s} \right)^{\frac{1}{2}} \frac{dt}{tw(B(x,t))} \right)^{\frac{1}{2}} \]

\[
\leq \sum_{j \geq 1} e^{-c'4} \left( \int_0^\infty \left( \int_t^s \left( \int_{B(x,t)} \left| e^{-s^2L_w} (t \sqrt{L_w})^{m-1} e^{-r^2L_w} f(y) \right|^2 \frac{dw(y)}{s} \right)^{\frac{1}{2}} \frac{ds}{s} \right)^{\frac{1}{2}} \frac{dt}{tw(B(x,t))} \right)^{\frac{1}{2}} \]

\[
\leq \sum_{j \geq 1} e^{-c'4} \left( \int_0^\infty \left( \int_t^s \left( \int_{B(x,t)} \left| e^{-s^2L_w} (t \sqrt{L_w})^{m-1} e^{-r^2L_w} f(y) \right|^2 \frac{dw(y)}{s} \right)^{\frac{1}{2}} \frac{ds}{s} \right)^{\frac{1}{2}} \frac{dt}{tw(B(x,t))} \right)^{\frac{1}{2}}. \]

Therefore, applying change of angle (Proposition 2.51), we get

\[
\| \tilde{I} \|_{L^2(y \omega dy)} \leq \sum_{j \geq 1} e^{-c'4} 2^{j \theta \eta n} \left\| S_{m+1,H}^w \right\|_{L^2(y \omega dy)} \leq \left\| S_{m+1,H}^w \right\|_{L^2(y \omega dy)}. \]

The estimate of \( \tilde{II} \) is very similar to that of \( II \) (in the proof of (4.6)), so we skip some details. We apply again the fact that \( e^{-tL_w} \in \mathcal{F}(L^2(w) - L^2(w)) \), Cauchy-Schwartz’s inequality in the integral in \( s \), Jensen’s inequality in the integral in \( y \), Fubini’s theorem, and (2.5) in order to obtain

\[
\tilde{II} \leq \sum_{j \geq 1} e^{-c'4} \left( \int_0^\infty \left( \int_t^s \left( \int_{B(x,t)} \left| (s \sqrt{L_w})^{m+1} e^{-s^2L_w} f(y) \right|^2 \frac{dw(y)}{s} \right)^{\frac{1}{2}} \frac{ds}{s} \right)^{\frac{1}{2}} \frac{dt}{t} \right)^{\frac{1}{2}} \]

\[
\leq \sum_{j \geq 1} e^{-c'4} \left( \int_0^\infty \int_t^s \left( \int_{B(x,t)} \left| (s \sqrt{L_w})^{m+1} e^{-s^2L_w} f(y) \right|^2 \frac{dw(y)}{s} \right)^{\frac{1}{2}} \frac{ds}{s} \right)^{\frac{1}{2}} \frac{dt}{t} \]

\[
\leq \sum_{j \geq 1} e^{-c'4} \left( \int_0^\infty \int_t^s \left( \int_{B(x,t)} \left| (s \sqrt{L_w})^{m+1} e^{-s^2L_w} f(y) \right|^2 \frac{dw(y)}{s} \right)^{\frac{1}{2}} \frac{ds}{s} \right)^{\frac{1}{2}} \frac{dt}{s} \]
Note that, Proposition 2.50 with $\beta = t/s < 1$ and $q = q_0/2$, and (2.5) imply

$$
\int_{\mathbb{R}^n} \left( \int_{B(x,2^{i+1}s/t)} \| (s \sqrt{L_m} e^{-s^2L_w} f(y))^{q_0} \frac{dw(y)}{B(y, 2^{i+1}s/t)} \right)^{\frac{2}{q_0}} v_0(x) dw(x) \\
\leq \left( \frac{2^{i+1}}{s} \right)^{n^2} \left( \sum_{j \geq i} \int_{\mathbb{R}^n} \left( \int_{B(x,2^{i+1}s/t)} \| (s \sqrt{L_m} e^{-s^2L_w} f(y))^{q_0} \frac{dw(y)}{B(y, 2^{i+1}s/t)} \right)^{\frac{2}{q_0}} v_0(x) dw(x) \right)^{\frac{2}{q_0}}
$$

Besides, since $e^{-rL_w} \in O(L^2(w) \rightarrow L^{q_0}(w))$

$$
\left( \int_{B(x,2^{i+1}s/t)} \| (s \sqrt{L_m} e^{-s^2L_w} f(y))^{q_0} \frac{dw(y)}{B(y, 2^{i+1}s/t)} \right)^{\frac{2}{q_0}} \leq \sum_{j \geq i} e^{-c_12j} \left( \int_{B(x,2^{i+1}s/t)} \| (s \sqrt{L_m} e^{-s^2L_w} f(y))^{q_0} \frac{dw(y)}{B(y, 2^{i+1}s/t)} \right)^{\frac{2}{q_0}}.
$$

Hence, applying Fubini’s theorem, (4.8) with $k = m$, changing the variable $s$ into $\sqrt{2}s$, and by (2.5) and Proposition 2.51 we have

$$
\| \hat{H} \|_{L^2(v_0 dw)} \leq \sum_{j \geq i} e^{-c_1d} \int_{\mathbb{R}^n} \left( \int_{B(x,2^{i+1}s/t)} \| (s \sqrt{L_m} e^{-s^2L_w} f(y))^{q_0} \frac{dw(y)}{B(y, 2^{i+1}s/t)} \right)^{\frac{2}{q_0}} v_0(x) dw(x)
$$

This and the estimate obtained for $\| \hat{F} \|_{L^2(v_0 dw)}$ give us (4.7).

Our next result will be useful in the proof of Theorem 3.1 parts (c) and (d). Given a measurable function $F$ defined in $\mathbb{R}^{n+1}$, consider the following vertical and conical operators:

$$
\tilde{V}F(x) := \left( \int_{\mathbb{R}^n} \int_0^\infty |F(y,t)|^2 \frac{dt}{t} v_0(x) dw(x) \right)^{\frac{1}{2}}
$$

and

$$
\tilde{S}F(x) := \left( \int_{\mathbb{R}^n} \int_0^\infty |F(y,t)|^2 \frac{dw(y)dt}{tw(B(y,t))} v_0(x) dw(x) \right)^{\frac{1}{2}}.
$$

**Lemma 4.9.** Given $w \in A_2$, $0 < p_0 < 2 < q_0 < \infty$, $r \geq q_0/2$, $v_0 \in A_w \mathcal{M}(w) \cap RH_{r_0}(w)$, $\alpha \geq 1$, $0 < u < 1/4$, and $F$ a measurable function defined in $\mathbb{R}^{n+1}$, let $\tilde{r} > r_w$ and let $\{\mathcal{T}_r\}_{r>0}$ be a family of sublinear operators such that $\mathcal{T}_r \in O(L^{p_0}(w) - L^{q_0}(w))$. Then,

$$
\int_{\mathbb{R}^n} \int_0^\infty \int_{B(x,\alpha r)} |\mathcal{T}\tilde{r}_{\mathcal{T}} F(y,t)|^2 \frac{dw(y)dt}{tw(B(y,t))} v_0(x) dw(x) \right)^{\frac{1}{2}}
$$

$$
\leq \left( \int_{\mathbb{R}^n} \int_0^\infty \left( \int_{B(x,\alpha r)} |\mathcal{T}\tilde{r}_{\mathcal{T}} F(y,t)|^2 \frac{dw(y)dt}{tw(B(y,t))} \right)^{\frac{1}{2}} v_0(x) dw(x) \right)^{\frac{1}{2}}
$$

$$
\leq u^{n^2} \left( \frac{2^{i+1}}{s} \right)^{n^2} \min \left\{ \| \tilde{S}F \|_{L^2(v_0 dw)} , \| \tilde{V}F \|_{L^2(v_0 dw)} \right\}.
$$

**Proof.** We fix $w, v_0, q_0, r, \alpha, \tilde{r}$, and $u$ as in the statement of the lemma and denote:

$$
I := \left( \int_{\mathbb{R}^n} \int_0^\infty \int_{B(x,\alpha r)} |\mathcal{T}\tilde{r}_{\mathcal{T}} F(y,t)|^2 \frac{dw(y)dt}{tw(B(y,t))} v_0(x) dw(x) \right)^{\frac{1}{2}}.
$$

Then, by (2.5), Jenssen’s inequality \((q_0 > 2)\), and Fubini’s theorem

\[
I \leq \left( \int_0^\infty \int_{\mathbb{R}^n} \left( \int_{B(x,at)} |T_{\frac{2}{4\alpha}} F(y, t)|^{q_0} \frac{dw(y)}{w(B(y, at))} \right)^{\frac{1}{q_0}} v_0(x)dw(x) \frac{dt}{t} \right)^{\frac{1}{2}} =: \tilde{I},
\]

which proves the first inequality in (4.10).

Thus, by Proposition 2.50 with \(\beta = 2 \sqrt{u} < 1\) and \(q = q_0/2\), and (2.5), we have (4.12)

\[
\tilde{I} = \left( \int_0^\infty \int_{\mathbb{R}^n} \left( \int_{B(x,2 \sqrt{at}/2 \sqrt{u})} |T_{\frac{2}{4\alpha}} F(y, t)|^{q_0} \frac{dw(y)}{w(B(y, 2 \sqrt{at}/2 \sqrt{u}))} \right)^{\frac{1}{q_0}} v_0(x)dw(x) \frac{dt}{t} \right)^{\frac{1}{2}}
\]

\[
\leq u^{q_0} \left( \frac{1}{\frac{q_0}{2} - \frac{1}{2}} \right) \sum_{i \geq 1} e^{-C_i t} \left( \int_0^\infty \int_{\mathbb{R}^n} \left( \int_{B(x,2^{i+1}at/2 \sqrt{u})} |F(y, t)|^{q_0} dw(y) \right)^{\frac{1}{q_0}} v_0(x)dw(x) \frac{dt}{t} \right)^{\frac{1}{2}},
\]

where in the last inequality we have also used that \(T_{\tau} \in O(L^{p_0}(w) - L^{\infty}(w))\). Now note that, on the one hand, we have

\[
\left( \int_{B(x,2^{i+1}at/2 \sqrt{u})} |F(y, t)|^{q_0} dw(y) \right)^{\frac{1}{q_0}} \leq M_{p_0}^w \left( F(\cdot, t) \right)(x), \quad (x, t) \in \mathbb{R}^{n+1}.
\]

On the other hand, by Fubini’s theorem and since \(\frac{2^{i+1}a}{2 \sqrt{u}} > 1\) we also have

\[
\left( \int_{B(x,2^{i+1}at/2 \sqrt{u})} |F(y, t)|^{p_0} dw(y) \right)^{\frac{1}{p_0}} = \left( \int_{B(x,2^{i+1}at/2 \sqrt{u})} \int_{B(y,t)} dw(\xi) |F(y, t)|^{p_0} dw(y) \right)^{\frac{1}{p_0}}
\]

\[
\leq \left( \int_{B(x,2^{i+1}at/2 \sqrt{u})} \int_{B(y,t)} |F(y, t)|^{p_0} dw(y) dw(\xi) \right)^{\frac{1}{p_0}}
\]

\[
\leq \left( \int_{B(x,2^{i+1}at/2 \sqrt{u})} |F(y, t)|^{2} dw(y) \right)^{\frac{1}{p_0}} \left( \int_{B(y,t)} dw(\xi) \right)^{\frac{1}{p_0}}
\]

\[
\leq M_{p_0}^w \left( \left( \int_{B(y,t)} |F(y, t)|^{2} dw(y) \right)^{\frac{1}{2}} \right)(x),
\]

where in the first inequality we have used that for \(y \in B(\xi, t)\) we have that \(B(\xi, t) \subset B(y, 2t)\) and (2.5), and in the second one Jensen’s inequality, since \(2 > p_0\).

Consequently, letting \(\mathcal{C} F(x, t)\) be equal to \(M_{p_0}^w \left( \left( \int_{B(y,t)} |F(y, t)|^{2} dw(y) \right)^{\frac{1}{2}} \right)(x)\) or \(M_{p_0}^w \left( F(\cdot, t) \right)(x)\), (4.12), the boundedness of \(M_{p_0}^w\) on \(L^2(v_0 dw)\) (recall that \(v_0 \in A^{\infty}(w)\)), Fubini’s theorem, and (2.5) imply

\[
\tilde{I} \leq u^{q_0} \left( \frac{1}{\frac{q_0}{2} - \frac{1}{2}} \right) \left( \int_0^\infty \int_{\mathbb{R}^n} |\mathcal{C} F(x, t)|^2 v_0(x)dw(x) \frac{dt}{t} \right)^{\frac{1}{2}}
\]

\[
\leq u^{q_0} \left( \frac{1}{\frac{q_0}{2} - \frac{1}{2}} \right) \min \left\{ \|S F \|_{L^2(v_0 dw)}, \|\nabla F \|_{L^2(v_0 dw)} \right\},
\]

which, in view of (4.11), completes the proof of (4.10). \(\square\)
Remark 4.13. Given \( w \in A_2, \max\{r_w, (q_- (L_w))_{w^{-1}}\} < p_0 < 2 < q_0 < q_+ (L_w) \), and \( v_0, \alpha \geq 1, u \), and \( F \) as in the statement of Lemma 4.9, we have that

\[
\left( \int_{B(\tau^2)} \left| \nabla e^{-\frac{\tau^2}{4}} F(y, t) \right|^2 \frac{dw(y)}{tw(B(\tau^2))} v_0(x)dw(x) \right)^{\frac{1}{2}} \leq \left( \int_{0}^{\infty} \left( \int_{B(\tau^2)} \left| \nabla e^{-\frac{\tau^2}{4}} F(y, t) \right|^{\frac{q_0}{w(B(\tau^2))}} \right)^{\frac{1}{q_0}} v_0(x)dw(x) \frac{dt}{t} \right)^{\frac{1}{2}} \leq u^{n\beta} \left( \frac{1}{2} - \frac{1}{m_0} \right) \min \left\{ \| \nabla F \|_{L^2(v_0 dw)}, \| \nabla \nabla F \|_{L^2(v_0 dw)} \right\}.
\]

The first inequality in (4.14) follows as (4.11).

As for the second inequality, note that, for every \( t > 0 \), by Lemma 2.45 with \( q = q_0, p = p_0 \), and \( S_t \) equal to the identity, we have that

\[
\left( \int_{B(x, \tau^2/2 \sqrt{n})} |\nabla e^{-\frac{\tau^2}{4}} F(y, t)|^{q_0} dw(y) \right)^{\frac{1}{q_0}} \leq \sum_{i \geq 1} e^{-\psi_i} \left( \int_{B(x, 2^{i+1} \tau^2/2 \sqrt{n})} |\nabla F(y, t)|^{p_0} dw(y) \right)^{\frac{1}{p_0}}.
\]

Therefore,

\[
\left( \int_{0}^{\infty} \left( \int_{B(x, \tau^2/2 \sqrt{n})} |\nabla e^{-\frac{\tau^2}{4}} F(y, t)|^{q_0} dw(y) \right)^{\frac{1}{q_0}} v_0(x)dw(x) \frac{dt}{t} \right)^{\frac{1}{2}} \leq \sum_{i \geq 1} e^{-\psi_i} \left( \int_{0}^{\infty} \left( \int_{B(x, 2^{i+1} \tau^2/2 \sqrt{n})} |\nabla F(y, t)|^{p_0} dw(y) \right)^{\frac{1}{p_0}} v_0(x)dw(x) \frac{dt}{t} \right)^{\frac{1}{2}}.
\]

Proposition 2.50 and the above inequality imply

\[
\left( \int_{0}^{\infty} \left( \int_{B(x, \tau^2/2 \sqrt{n})} |\nabla e^{-\frac{\tau^2}{4}} F(y, t)| \frac{dw(y)}{w(B(\tau^2))} \right)^{\frac{1}{2}} v_0(x)dw(x) \frac{dt}{t} \right)^{\frac{1}{2}} \leq u^{n\beta} \left( \frac{1}{2} - \frac{1}{m_0} \right) \sum_{i \geq 1} e^{-\psi_i} \left( \int_{0}^{\infty} \left( \int_{B(x, 2^{i+1} \tau^2/2 \sqrt{n})} |\nabla F(y, t)|^{p_0} dw(y) \right)^{\frac{1}{p_0}} v_0(x)dw(x) \frac{dt}{t} \right)^{\frac{1}{2}}.
\]

This substitutes (4.12) in the proof of Lemma 4.9. Hereafter, the proof follows as the proof of that lemma, but writing \( \nabla F \) in place of \( F \).

Note now that proceeding as in the proof of [9, Theorem 3.5, part (b)], we obtain the following comparison result between the conical square functions defined in (2.28) and (2.30), even for odd values of \( m \).

Proposition 4.15. Given \( w \in A_2, v \in A_\infty(w), m \in \mathbb{N}, \) and \( f \in L^2(w) \), there holds

\[
\| S_{m,p}^w f \|_{L^p(vdw)} \leq \| S_{m,H}^w f \|_{L^p(vdw)}, \text{ for all } p \in W_\nu^w(0, p^*(L_w)^{m+1,\nu}).
\]

Besides, with the aim of obtaining boundedness for the conical square functions defined via the gradient of the heat or the Poisson semigroup, we show the following comparison result.

Proposition 4.16. Given \( w \in A_2, v \in A_\infty(w), m \in \mathbb{N}, K \in \mathbb{N}_0, \) and \( f \in L^2(w) \), there hold

(a) \( \| G_{m,H}^w f \|_{L^p(vdw)} \leq \| S_{m,H}^w f \|_{L^p(vdw)}, \) for all \( 0 < p < \infty; \)

(b) \( \| G_{K,F}^w f \|_{L^p(vdw)} \leq \| S_{K+2,H}^w f \|_{L^p(vdw)} + \| G_{K,H}^w f \|_{L^p(vdw)}, \) for all \( p \in W_\nu^w(0, p^*(L_w)^{K+1,\nu}). \)
Proof. The proof of part (a) follows as the proof of [9, Theorem 3.3]. Indeed, use (2.5) and apply the fact that \( \sqrt{\nabla e^{-\tau L_w}} \in \mathcal{O}(L^2(w) - L^2(w)) \); then, change the variable \( t \) into \( \sqrt{t} \), and apply again (2.5) and Proposition 2.51 to obtain

\[
\|G_{m,H}^w f\|_{L^p(vdw)} \leq \left( \int_{\mathbb{R}^n} \left( \int_0^\infty \frac{\|\nabla e^{-\frac{t}{2}} (t \sqrt{L_w})^m e^{-\frac{t}{2} L_w} f(y)\|}{t} \frac{dw(y)dt}{t} \right)^{\frac{q}{2}} \right)^{\frac{1}{q}} v(x)dw(x)
\]

\[
\leq \sum_{j \geq 1} e^{-c4^j} \left( \int_{\mathbb{R}^n} \left( \int_0^\infty \int_{B(x,2^{j+2}t)} |(t \sqrt{L_w})^m e^{-\frac{t}{2} L_w} f(y)|^2 \frac{dw(y)dt}{tw(B(y,t))} \right)^{\frac{q}{2}} \right)^{\frac{1}{q}} v(x)dw(x)
\]

\[
\leq \sum_{j \geq 1} e^{-c4^j} 2^{j\theta_0 w} \|S_{m,H}^w f\|_{L^p(vdw)} \leq \|S_{m,H}^w f\|_{L^p(vdw)}.
\]

In order to prove part (b), first of all note that

\[
G_{K,H} f(x) = \left( \int_0^\infty \int_{B(x,t)} |\nabla (t \sqrt{L_w})^K (e^{-t \sqrt{L_w}} - e^{-t^2 L_w}) f(y)| \frac{dw(y)dt}{tw(B(y,t))} \right)^{\frac{1}{2}} + G_{K,H} f(y).
\]

So we just need to prove that

\[
\|Q_K f\|_{L^p(vdw)} \leq \|S_{K+2,H}^w f\|_{L^p(vdw)}, \quad \forall p \in \mathbb{W}_w^\lambda (0, p_+(L_w)^{K+1,\ast}).
\]

We show this by extrapolation. In particular, by Theorem 2.47, part (b) (or Theorem 2.47, part (c) if \( p_+(L_w)^{K+1,\ast} = \infty \), (4.17) follows from the inequality

\[
\|Q_K f\|_{L^2(v_0 dw)} \leq \|S_{K+2,H}^w f\|_{L^2(v_0 dw)}, \quad \forall v_0 \in RH_p \left( \frac{p_+(L_w)^{K+1,\ast}}{2} \right)^{\gamma}(w).
\]

To this end, fix \( w \in A_2 \), \( f \in L^2(w) \), and \( v_0 \in RH \left( \frac{p_+(L_w)^{K+1,\ast}}{2} \right)^{\gamma}(w) \), and note that by Remark 2.48, with \( q_1 = 2, \tilde{q} = p_+(L_w) \) and \( \tilde{K} = K + 1 \), we can find \( q_0, \tilde{t}, \) and \( r \) such that \( q_0 < \tilde{t} \), \( 2 < q_0 < p_+(L_w) \), \( q_0/2 \leq r < \infty \), \( v_0 \in RH_p, (w) \), and

\[
(4.19) \quad K + 1 + \frac{n\tilde{r}}{2r} - \frac{n\tilde{t}}{q_0} > 0.
\]

Keeping this in mind, by the subordination formula (2.24) and Minkowski’s integral inequality, we have

\[
Q_K f(x) \leq \int_0^{\frac{1}{4}} u^\frac{1}{2} e^{-u} I(u) \frac{du}{u} + \int_{\frac{1}{4}}^{\infty} u^\frac{1}{2} e^{-u} I(u) \frac{du}{u} =: I + II,
\]

where

\[
I(u) := \left( \int_0^\infty \int_{B(x,t)} |\nabla (t \sqrt{L_w})^K (e^{-t \sqrt{L_w}} - e^{-t^2 L_w}) f(y)| \frac{dw(y)dt}{tw(B(y,t))} \right)^{\frac{1}{2}}.
\]

We first estimate II, for \( 1/4 < u < \infty \), Cauchy-Schwartz’s inequality implies

\[
\left| \int (t \sqrt{L_w})^K e^{-t^2 L_w} - t \nabla (t \sqrt{L_w})^K e^{-t^2 L_w} f(y) \right| \leq \int_{\mathbb{R}^n} |\partial_t \nabla (t \sqrt{L_w})^K e^{-t^2 L_w} f(z)| ds
\]

\[
\leq \int_{\mathbb{R}^n} \left| t \nabla (t \sqrt{L_w})^K \left( s \sqrt{L_w} \right)^2 e^{-t^2 L_w} f(y) \right| \frac{ds}{s}.
\]
\[
\lesssim u^\frac{1}{2} \left( \int_{\mathbb{R}^n} \frac{dt}{t} \frac{dw(y)}{sw(B(y,t))} \right)^{\frac{1}{2}}.
\]

Then, for \( u > 1/4 \), we estimate \( I(u) \) using (2.5), Fubini’s theorem, the fact that \( s < t < 2\sqrt{u}s \), and that \( \sqrt[n]{\nabla e^{-tL_w}} \in \mathcal{F}(L^2(w) - L^2(w)) \). Besides, we change the variable \( s \) into \( \sqrt{s} \) to obtain

\[
I(u) \leq u^{\theta_K} \left( \int_0^\infty \int_{B(x,t)} |s\nabla(s \sqrt{L_w})^{K+2} e^{-\frac{1}{2}L_w} f(y)|^{2} \frac{dw(y)}{sw(B(y,t))} \frac{dt}{t} \right)^{\frac{1}{2}}.
\]

\[
\leq \sum_{j \geq 1} e^{-c_4j} u^{\theta_K} \left( \int_0^\infty \int_{B(x,t)} |s\nabla(s \sqrt{L_w})^{K+2} e^{-\frac{1}{2}L_w} f(y)|^{2} \frac{dw(y)}{sw(B(y,t))} \frac{dt}{t} \right)^{\frac{1}{2}}.
\]

Hence, by Minkowski’s integral inequality and change of angle (Proposition 2.51)

\[
\|I\|_{L^2(\gamma,dw)} \leq \sum_{j \geq 1} e^{-c_4j} 2^{j\theta_{K_0,n}} \left( \int_0^\infty \int_{B(x,t)} |s\nabla(s \sqrt{L_w})^{K+2} e^{-\frac{1}{2}L_w} f(y)|^{2} \frac{dw(y)}{sw(B(y,t))} \frac{dt}{t} \right)^{\frac{1}{2}}.
\]

As for the estimate of \( I \), note that for \( 0 < u < 1/4 \), again by Cauchy- Schwartz’s inequality

\[
\left| \frac{d}{ds} \right|^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} \frac{dw(y)}{sw(y)} \right)^{\frac{1}{2}} \leq \left( \int_{\mathbb{R}^n} \frac{d}{ds} \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} \frac{dw(y)}{sw(y)} \right)^{\frac{1}{2}}.
\]

The above estimate, Fubini’s theorem, (2.5), and Jensen’s inequality \((q_0 > 2)\) imply

\[
I \lesssim \left( \int_0^t \int_{\mathbb{R}^n} \left| s\nabla(s \sqrt{L_w})^{K+2} e^{-\frac{1}{2}L_w} f(y) \right|^{q_0} \frac{dw(y)}{sw(B(y,t/s))} \frac{dt}{t} \right)^{\frac{1}{2}}.
\]

Besides, note that by Proposition 2.50, with \( \beta = t/s < 1 \) and \( q = q_0/2 \),

\[
\int_{\mathbb{R}^n} \left( \int_{B(x,t/s)} \left| s\nabla(s \sqrt{L_w})^{K+2} e^{-\frac{1}{2}L_w} f(y) \right|^{q_0} \frac{dw(y)}{sw(B(y,t/s))} \right)^{\frac{2}{q_0}} v_0(x) dw(x)
\]

\[
\lesssim \left( \frac{1}{s} \right)^{n} \left( \frac{1}{s} \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} \left| s\nabla(s \sqrt{L_w})^{K+2} e^{-\frac{1}{2}L_w} f(y) \right|^{q_0} \frac{dw(y)}{sw(B(y,t/s))} \right)^{\frac{2}{q_0}} v_0(x) dw(x).
\]

Thus, for \( C_1 := 2K + 2 + n\left( \frac{1}{4} - \frac{2}{q_0} \right) \), we apply first Fubini’s theorem, and the above inequality. Then, by (2.5), changing the variable \( s \) into \( 2s \), (4.19), the fact that \( \sqrt[n]{\nabla e^{-tL_w}} \in O(L^2(w) - L^{q_0}(w)) \), and by Proposition 2.51, we get

\[
\|I\|_{L^2(\gamma,dw)}
\]
which finishes the proof. □

In the next theorem we obtain a norm comparison result for $\mathcal{N}_f^w$. This will be used in the proof of Theorem 3.7.

**Theorem 4.20.** Given $w \in A_2$, $v \in A_{\infty}(w)$, and $f \in L^2(w)$, there holds

$$
||\mathcal{N}_f^w||_{L^p(vdw)} \leq ||\mathcal{N}_f^w||_{L^p(vdw)} + ||S_{K+2,H}^w||_{L^2(vdw)},
$$

for all $p \in W_r^p (p- (L_w), p, (L_w)^r)$. $\forall p \in W_r^p (p- (L_w), p, (L_w)^r)$.

**Proof.** First of all, fix $w, v, f$, and $p$ as in the statement of the theorem. Then, recalling the definition of $\mathcal{N}_f^w$ in (3.22), note that

$$
(4.21) \quad \mathcal{N}_f^w f(x) \leq \mathcal{N}_f^w f(x) + \sup_{t>0} \biggl( \int_{B(x,t)} \left| e^{-\frac{t}{4} L_w} (e^{t^2 L_w} f(z)) \right|^2 dw(z) \biggr)^{\frac{1}{2}} =: \mathcal{N}_f^w f(x) + \sup_{t>0} I.
$$

Then, by the subordination formula in (2.24) and Minkowski’s integral inequality,

$$
(4.22) \quad I \leq \int_0^1 u^{\frac{1}{2}} \left( \int_{B(x,t)} \left| e^{-\frac{t}{4} L_w} (e^{t^2 L_w} f(z)) \right|^2 dw(z) \right) \frac{1}{u} \frac{du}{u} + \int_1^\infty e^{-u} u^{\frac{1}{2}} \left( \int_{B(x,t)} \left| e^{-\frac{t}{4} L_w} (e^{t^2 L_w} f(z)) \right|^2 dw(z) \right) \frac{1}{u} \frac{du}{u} := I_1 + I_2.
$$

Similarly as in the proof of [20, Proposition 7.1, part (b)] (see also [10]), we have that

$$
I_2 \leq \int_1^\infty e^{-cu} \left| S_{2,H}^w f(x) \right| \frac{du}{u}.
$$

Hence, by Minkowski’s integral inequality and Proposition 2.51,

$$
(4.23) \quad ||I_2||_{L^p(vdw)} \leq \int_1^\infty e^{-cu} ||S_{2,H}^w f||_{L^p(vdw)} \frac{du}{u} \leq ||S_{2,H}^w||_{L^p(vdw)}.
$$

In order to estimate $I_1$, take $p_0$ such that $p- (L_w) < p_0 < \min[2, p]$ close enough to $p- (L_w)$ so that $v \in A_{\infty}(w)$, and note that the fact that $e^{-t L_w} \in O(L^{p_0}(w) - L^2(w))$ implies

$$
(4.24) \quad I_1 \leq \sum_{j \geq 1} e^{-ct^j} \left( \int_{B(x,j^j+1)} \left| e^{-\frac{j^j}{4} L_w} (e^{j^j L_w} f(z)) \right|^2 dw(z) \right)^{\frac{1}{2}} \frac{du}{u}.
$$

Besides, for $0 < u < \frac{1}{2}$, it holds from Hölder’s inequality that

$$
\left| e^{-\frac{j^j}{4} L_w} (e^{j^j L_w} f(z)) \right| \leq \int_{\frac{j^j}{4}}^{\frac{1}{2}} \left| \partial_s e^{-s^2 L_w} f(z) \right| ds.
$$
Therefore,

\[
I_1 \leq \sum_{j \geq 1} e^{-c^j \int_0^1 u^2 (\log u^{-\frac{1}{2}})^{\frac{1}{\alpha}} (\int_{2^j}^{2^{j+1}} |s^2 L_w e^{-s^2 L_w f(z)}| \frac{ds}{s})^\frac{p_0}{r} dw(z) dy}\cdot \frac{\int_0^1 u^\frac{1}{2} (\log u^{-\frac{1}{2}})^{\frac{1}{\alpha}} \, du}{u}.
\]

Note now that, for \( \frac{1}{\sqrt{2}} < s < \frac{1}{\sqrt{a}} \),

\[
\{(z, y) \in \mathbb{R}^n \times \mathbb{R}^n : z \in B(x, 2^{j+1} r), y \in B(z, 2 \sqrt{u} s) \} \subset \{(z, y) \in \mathbb{R}^n \times \mathbb{R}^n : y \in B(x, 2^{j+2} r), z \in B(y, 2 \sqrt{u} s) \},
\]

and for \( z \in B(y, 2 \sqrt{u} s) \) it holds that \( B(y, 2 \sqrt{u} s) \subset B(z, 4 \sqrt{u} s) \). Then, Fubini’s theorem and (2.5) imply

\[
\int_{B(x, 2^{j+1} r)} \int_{\mathbb{R}^n} |s^2 L_w e^{-s^2 L_w f(z)}|^{p_0} dw(z) \frac{ds}{s} \, dy = \int_{\mathbb{R}^n} \int_{B(x, 2^{j+1} r)} |s^2 L_w e^{-s^2 L_w f(z)}|^{p_0} dw(z) \frac{ds}{s} \, dy
\]

\[
\leq \int_{\mathbb{R}^n} \int_{B(2r, 2 \sqrt{u} s)} |s^2 L_w e^{-s^2 L_w f(z)}|^{p_0} dw(z) \frac{ds}{s} \, dy \cdot II.
\]

Next, apply Fubini’s theorem, (2.5), Jensen’s inequality in the integral in \( z \) \((2 > p_0)\), and Hölder’s inequality in the integral in \( s \) with \( 2/p_0 \) and \((2/p_0)’\), and again (2.5). Then,

\[
II \leq \int_{B(x, 2^{j+1} r)} \int_{\mathbb{R}^n} |s^2 L_w e^{-s^2 L_w f(z)}|^{p_0} dw(z) \frac{ds}{s} \, dy
\]

\[
\leq \int_{B(x, 2^{j+1} r)} \int_{\mathbb{R}^n} \left( \int_{B(2r, 2 \sqrt{u} s)} |s^2 L_w e^{-s^2 L_w f(z)}|^2 dw(z) \frac{ds}{s} \right) \frac{dy}{dy}
\]

\[
\leq \int_{B(x, 2^{j+1} r)} \left( \int_{\mathbb{R}^n} \left( \int_{B(2r, 2 \sqrt{u} s)} |s^2 L_w e^{-s^2 L_w f(z)}|^2 dw(z) \frac{ds}{s} \right) \frac{dy}{dy} \right)^{\frac{p_0}{2}} \cdot (\log u^{-\frac{1}{2}})^{\frac{2-p_0}{2}}
\]

Consequently,

\[
I_1 \leq \int_0^1 u^\frac{1}{2} (\log u^{-\frac{1}{2}})^{\frac{1}{\alpha}} \mathcal{M}_{p_0}^{w} \left( \mathbb{Z}_H^{\sqrt{u}w} f \right) (x) \frac{du}{u},
\]

where

\[
\mathbb{Z}_H^{\sqrt{u}w} f(y) := \left( \int_0^\infty \int_{B(y, 2 \sqrt{u} s)} |s^2 L_w e^{-s^2 L_w f(z)}|^2 dw(z) \frac{ds}{sw(B(z, 2 \sqrt{u} s))} \right)^{\frac{1}{2}}.
\]

Note now that since \( v \in A_{p_0} \) \((w)\) the maximal operator \( \mathcal{M}_{p_0}^{w} \) is bounded on \( L^p(ydw) \). Hence, by Minkowski’s integral inequality

\[
(4.24) \quad \|I_1\|_{L^p(ydw)} \leq \int_0^1 u^\frac{1}{2} (\log u^{-\frac{1}{2}})^{\frac{1}{\alpha}} \| \mathbb{Z}_H^{\sqrt{u}w} f \|_{L^p(ydw)} \frac{du}{u}.
\]
In order to estimate the norm on $L^p(\nu dw)$ of $\mathcal{Z}_H^{2,\sqrt{n}}f$ we shall proceed by extrapolation. To this end, note that for $2 < q_0 < p_+(L_w)$, $r_w < \tilde{r} < 2$ and $r \geq q_0/2$, for all $v_0 \in RH_\tilde{r}(w)$, by Jensen’s inequality, Fubini’s theorem, Proposition 2.50 with $\beta = 2 \sqrt{n} < 1$ and $q = q_0/2$, and (2.5), we obtain
\[
\left( \int_{\mathbb{R}^n} \left| \mathcal{Z}_H^{2,\sqrt{n}}f(y) \right|^2 v_0(y) dw(y) \right)^{\frac{1}{2}} \leq \left( \int_0^{\infty} \left( \int_{B(y,2\sqrt{s})} \left| s^2 L_w e^{-s^2 L_w f(z)} \right|^q \frac{dw(z)}{w(B(z,2\sqrt{s}))} \right)^{\frac{q}{q_0}} \left( \frac{d}{s} \right)^{\frac{q_0}{q}} v_0(y) dw(y) \right)^{\frac{1}{2}} \leq u^{\tilde{r}} v^{\tilde{r}} \left( \int_0^{\infty} \left( \int_{B(y,s)} \left| e^{-\frac{s^2}{4}L_w s^2 L_w e^{-\frac{s^2}{4}L_w f(z)} \right|^q \frac{dw(z)}{w(B(z,s))} \right)^{\frac{q_0}{q}} \left( \frac{d}{s} \right)^{\frac{q_0}{q}} v_0(y) dw(y) \right)^{\frac{1}{2}} =: III.
\]
Now, apply the fact that $e^{-\tilde{r}L_w} \in O(L^2(w) - L^0(w))$, change the variable $s$ into $\sqrt{s}$, and apply Proposition 2.51 to get
\[
III \leq \sum_{j \geq 1} e^{-c_1 j} u^{\tilde{r}} v^{\tilde{r}} \left( \int_{\mathbb{R}^n} \left( \int_{B(y,\sqrt{s})} \left| e^{-\frac{s^2}{4}L_w s^2 L_w e^{-\frac{s^2}{4}L_w f(z)} \right|^q \frac{dw(z)}{w(B(z,\sqrt{s}))} \right)^{\frac{q_0}{q}} \left( \frac{d}{s} \right)^{\frac{q_0}{q}} v_0(y) dw(y) \right)^{\frac{1}{2}} \leq \sum_{j \geq 1} e^{-c_1 j} 2^{\tilde{r} j} u^{\tilde{r}} v^{\tilde{r}} \left( \int_{\mathbb{R}^n} \left| S_{2,\tilde{r}}^w f(y) \right|^2 v_0(y) dw(y) \right)^{\frac{1}{2}} \leq \left( \int_{\mathbb{R}^n} u^{c(r,q_0)} S_{2,\tilde{r}}^w \left| f(y) \right|^2 v_0(y) dw(y) \right)^{\frac{1}{2}},
\]
where $c(r,q_0) := n\tilde{r} \left( \frac{4}{r} - \frac{1}{q_0} \right)$.

Hence, by Theorem 2.47, part (b), we have
\[
(4.25) \quad \int_{\mathbb{R}^n} \left| \mathcal{Z}_H^{2,\sqrt{n}}f(y) \right|^p v(y) dw(y) \leq \left( \int_{\mathbb{R}^n} u^{c(r,q_0)} S_{2,\tilde{r}}^w \left| f(y) \right|^p v(y) dw(y) \right)^{\frac{1}{p}}
\]
for all $r_w < \tilde{r} < 2$, $2 < q_0 < p_+(L_w)$, $r \geq q_0/2$, $1 < \tilde{r} < \infty$, and $v \in RH_\tilde{r}(w)$. Moreover, for $v \in RH_\tilde{r}(w)$, Remark (2.48) with $p_1 = p$, $\tilde{k} = 1$, and $\tilde{q} = p_+(L_w)$, implies that there exist $r_w < \tilde{r} < 2$, $2 < q_0 < p_+(L_w)$, $r \geq q_0/2$, and $1 < \tilde{r} = 2r/p < \infty$ so that $v \in RH_\tilde{r}(w)$, and
\[
(4.26) \quad 1 + \frac{n\tilde{r}}{2r} - \frac{n\tilde{r}}{q_0} > 0.
\]
Consequently, by (4.25) we conclude that given $v \in RH_\tilde{r}(w)$, there exist $r_w < \tilde{r} < 2$, $2 < q_0 < p_+(L_w)$, $r \geq q_0/2$ such that (4.26) is satisfied and
\[
\int_{\mathbb{R}^n} \left| \mathcal{Z}_H^{2,\sqrt{n}}f(y) \right|^p v(y) dw(y) \leq \left( \int_{\mathbb{R}^n} u^{c(r,q_0)} S_{2,\tilde{r}}^w \left| f(y) \right|^p v(y) dw(y) \right)^{\frac{1}{p}}
\]
Plugging this into (4.24) allows us to obtain
\[
\|I_1\|_{L^p(\nu dw)} \leq \int_0^1 \frac{1}{u} u^{\tilde{r} + n\tilde{r} \left( \frac{1}{4r} - \frac{1}{2q_0} \right)} \frac{du}{u} \left\| S_{2,\tilde{r}}^w \right\|_{L^p(\nu dw)}.
\]
Note now that in view of (4.26), we can take $M \in \mathbb{N}$ large enough so that
\[
\frac{1}{2} + n\tilde{r} \left( \frac{1}{4r} - \frac{1}{2q_0} \right) - \frac{1}{4M} > 0.
\]
Therefore,
\[ \|I\|_{L^p(vdw)} \leq \int_0^\frac{1}{2} u^{\frac{1}{2} + \sigma_0 + \frac{1}{2}} \left( M \log u^{-1} + \frac{1}{2} \right) \frac{du}{u} \left\| \mathcal{S}^w_{2,H} f \right\|_{L^p(vdw)} \leq \int_0^\frac{1}{2} u^{\frac{1}{2} + \sigma_0 + \frac{1}{2}} \frac{du}{u} \left\| \mathcal{S}^w_{2,H} f \right\|_{L^p(vdw)} \leq \left\| \mathcal{S}^w_{2,H} f \right\|_{L^p(vdw)}. \]

This (4.21), (4.22), and (4.23) imply
\[ \left\| N^w_{\mathcal{T}} f \right\|_{L^p(vdw)} \leq \left\| N^w_{\mathcal{H}} f \right\|_{L^p(vdw)} + \left\| \mathcal{S}^w_{2,H} f \right\|_{L^p(vdw)}. \]

The boundedness of the non-tangential maximal function associated with the heat semigroup, \( N^w_{\mathcal{H}} \), is proved in [10] in the weighted degenerate case. The proof is easy and consist in controlling the norm on \( L^p(vdw) \) of \( N^w_{\mathcal{H}} \) by that of an adequate Hardy-Littlewood maximal operator. This is the argument used in [20, Proposition 7.1] in the weighted non-degenerate case, see also [17, Section 6].

**Theorem 4.27.** [10] Given \( w \in A_2 \) and \( v \in A_{\infty}(w) \), the non-tangential maximal function \( N^w_{\mathcal{H}} \) can be extended to a bounded operator on \( L^p(vdw) \), for all \( p \in \mathcal{W}^w(p_-(L_w), \infty) \).

To conclude this section we observe some trivial norm comparison results between vertical square functions that will make the proof of Theorem 3.6 easier.

**Remark 4.28.** Given \( w \in A_2 \) and \( v \in A_{\infty}(w) \), as a consequence of the boundedness of the Riesz transform \( \nabla L_w^{-\frac{1}{2}} \) (see [12] and also [1, 5]), for every \( m \in \mathbb{N} \) and \( f \in L^2(w) \), we have that, for all \( \mathcal{W}^w(m, \infty), q_-(L_w), q_+(L_w) \),
\[ (4.29) \quad \|g^w_{m-1,1,H} f\|_{L^p(vdw)} \leq \|s^w_{m,H} f\|_{L^p(vdw)} \quad \text{and} \quad \|g^w_{m-1,1,P} f\|_{L^p(vdw)} \leq \|s^w_{m,P} f\|_{L^p(vdw)}. \]

We prove the above inequalities by extrapolation. In particular, by Theorem 2.47, part (e), it is enough to prove that
\[ \|g^w_{m-1,1,H} f\|_{L^2(v_0 dw)} \leq \|s^w_{m,H} f\|_{L^2(v_0 dw)} \quad \text{and} \quad \|g^w_{m-1,1,P} f\|_{L^2(v_0 dw)} \leq \|s^w_{m,P} f\|_{L^2(v_0 dw)}, \]
for all \( v_0 \in A_{\frac{1}{q(L_w)^{1/m}}} \cap \mathcal{RH}\left(\frac{w}{q(L_w)^{1/m}}\right) \). This follows applying Fubini’s theorem and the boundedness of the Riesz transform on \( L^2(v_0 dw) \). Indeed, letting \( F(x,t) \) be equal to \( e^{-t^2 L_w} f(x) \) or \( e^{-t \sqrt{L_w}} f(x) \), we get
\[ \int_{\mathbb{R}^n} \int_0^\infty |\nabla (t \sqrt{L_w})^{m-1} F(x,t)|^2 \frac{dt}{t} v_0(x) dw(x) = \int_{\mathbb{R}^n} \int_0^\infty |\nabla L_w^{-\frac{1}{2}} (t \sqrt{L_w})^{m} F(x,t)|^2 v_0(x) dw(x) \frac{dt}{t} \leq \int_{\mathbb{R}^n} \int_0^\infty |(t \sqrt{L_w})^{m} F(x,t)|^2 \frac{dt}{t} v_0(x) dw(x). \]

Besides, as it has been observed in several papers before (see, for instance, [4, 17]), by applying (2.24), Minkowski’s integral inequality, and the change of variable \( t \) into \( 2 \sqrt{u} \), we also have that
\[ (4.30) \quad \mathcal{S}^w_{m,P} f(x) \leq \int_0^\infty u^m e^{-u} \frac{du}{u} \mathcal{S}^w_{m,H} f(x) \leq \mathcal{S}^w_{m,H} f(x), \quad x \in \mathbb{R}^n. \]

**Remark 4.31.** Note that (4.29) and Theorem 3.1 parts (a) and (c) imply that, for all \( K \in \mathbb{N}_0, w \in A_2 \), and \( p \in \mathcal{W}^w(m, \infty), q_-(L_w), q_+(L_w) \),
\[ \|g^w_{K+1,1,H} f\|_{L^p(vdw)} \leq \|S^w_{K+1,1,H} f\|_{L^p(vdw)} \quad \text{and} \quad \|g^w_{K+1,1,P} f\|_{L^p(vdw)} \leq \|S^w_{K+1,1,P} f\|_{L^p(vdw)}. \]

5. **Proof of the main results**

### 5.1. Norm comparison for vertical and conical square functions.
5.1. Proof of Theorem 3.1, parts (a) and (b). Fix $w \in A_2$, $f \in L^2(w)$, and $m \in \mathbb{N}$. In order to prove parts (a) and (b) note that for $T_t = e^{-tL_w}$, $F(y,t) = (t \sqrt{L_w})^m f(y)$, and any $p_-(L_w) < p_0 < 2 < q_0 < p_+(L_w)$, the conditions (i) – (iii) in Proposition 4.2 are satisfied. Then, given $0 < p < p_+(L_w)$ and $v \in RH_p(w)$, since we can always find $\max[2, p] < q_0 < p_+(L_w)$ close enough to $p_+(L_w)$ so that $v \in RH_{\frac{p}{p-1}}(w)$, in view of (2.12), Proposition 4.2, part (b), implies
\[
\|s_{m,H}^w f\|_{L^p(vdw)} \leq \|s_{m,H}^w f\|_{L^p(vdw)}.
\]
On the other hand, given $p_-(L_w) < p < \infty$ and $v \in A_{\frac{p}{p-1}}(w)$, since we can always find $p_-(L_w) < p_0 < \min[2, p]$ close enough to $p_-(L_w)$ so that $v \in A_{\frac{p}{p-1}}(w)$, in view of (2.12), Proposition 4.2, part (a), implies
\[
\|s_{m,H}^w f\|_{L^p(vdw)} \leq \|s_{m,H}^w f\|_{L^p(vdw)}.
\]

5.1.2. Proof of Theorem 3.1, part (c). Fix $w \in A_2$, $f \in L^2(w)$, and $m \in \mathbb{N}$. We shall proceed by extrapolation. In particular, note that by Theorem 2.47, part (d) (or part (a) if $p_+(L_w) = \infty$) it is enough to show that
\[
\|s_{m,p}^w f\|_{L^2(vdw)} \leq \|s_{m,p}^w f\|_{L^2(vdw)}, \quad \forall v_0 \in A_{\frac{p}{p-1}}(w) \cap RH_{\frac{p}{p-1}}(w).
\]
In order to prove this inequality, first of all note that since $v_0 \in A_{\frac{p}{p-1}}(w) \cap RH_{\frac{p}{p-1}}(w)$ we can take $p_-(L_w) < p_0 < 2 < q_0 < p_+(L_w)$ so close to $p_-(L_w)$ and $p_+(L_w)$ so that $v_0 \in A_{\frac{p}{p-1}}(w) \cap RH_{\frac{q_0}{q_0-1}}(w)$. Keeping this choice of $p_0$ and $q_0$, change the variable $t$ into $2t$ and apply the subordination formula (2.24) to get
\[
\|s_{m,p}^w f\|_{L^2(vdw)} \leq \left( \int_{\mathbb{R}} \int_0^\infty (t \sqrt{L_w})^m e^{-t \sqrt{L_w}} f(y)^2 \frac{dt}{t} v_0(y) dw(y) \right)^{\frac{1}{2}} \\
\leq \int_0^{\frac{1}{2}} u^{\frac{1}{2}} I(u) \frac{du}{u} + \int_{\frac{1}{2}}^{\infty} u^{\frac{1}{2}} e^{-u} I(u) \frac{du}{u} =: I + II,
\]
where in the second inequality we have used Minkowski’s integral inequality and
\[
I(u) := \left( \int_{\mathbb{R}} \int_0^\infty e^{-\frac{t}{2} L_w} (t \sqrt{L_w})^m e^{-t \sqrt{L_w}} f(y)^2 \frac{dt}{t} v_0(y) dw(y) \right)^{\frac{1}{2}}.
\]
Moreover, for each $u > 0$, recalling our choice of $q_0$, the inequality (4.1), with $T_{T_2} = e^{-\frac{t}{2} L_w}$ and $F(y,t) = (t \sqrt{L_w})^m e^{-t \sqrt{L_w}} f(y)$, and (2.5) yield
\[
I(u) \leq \left( \int_{\mathbb{R}} \int_0^\infty (\int_{B(x,t)}) e^{-\frac{t}{2} L_w} (t \sqrt{L_w})^m e^{-t \sqrt{L_w}} f(y)^2 \frac{dt}{t} v_0(x) dw(x) \right)^{\frac{1}{2}}.
\]
Hence, by Lemma 4.9 with $T_r = e^{-r L_w}$, $F(y,t) = (t \sqrt{L_w})^m e^{-t \sqrt{L_w}} f(y)$, $\alpha = 1$, and $r = q_0/2$
\[
I \leq \int_0^{\frac{1}{2}} u^{\frac{1}{2}} I(u) \frac{du}{u} \leq \int_0^{\frac{1}{2}} u^{\frac{1}{2}} I(u) \|s_{m,p}^w f\|_{L^2(vdw)} \leq \|s_{m,p}^w f\|_{L^2(vdw)}.
\]
Finally, (5.2), Fubini’s theorem, the fact that $e^{-\frac{t}{2} L_w} \in O(L^2(w) - L^m(w))$, (2.5), and Proposition 2.51 imply
\[
II \leq \sum_{j=1}^{\infty} e^{-c j} \left( \int_{\mathbb{R}} \int_0^\infty u^{\frac{1}{2}} e^{-u} \frac{du}{u} \left( \int_{\mathbb{R}} \int_0^\infty (\int_{B(x,t)}) (t \sqrt{L_w})^m e^{-t \sqrt{L_w}} f(y)^2 \frac{dt}{t} v_0(x) dw(x) \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}.
\]
\[ \leq \sum_{j \geq 1} e^{-c4j} 2^{j\theta_{p,w}} \| S_{m,p}^w f \|_{L^2(\psi_0 dw)} \leq \| S_{m,p}^w f \|_{L^2(\psi_0 dw)}. \]

This and (5.3) imply (5.1) and the proof is complete. \(\square\)

5.1.3. **Proof of Theorem 3.1, part (d).** Fix \( w \in A_2, f \in L^2(w), \) and \( m \in \mathbb{N}. \) We shall use extrapolation to prove this result. In particular, by Theorem 2.47, part (d) (or part (a) if \( p_+(L_w)_w^* = \infty \)) it is enough to show that

\[ \| S_{m,p}^w f \|_{L^2(\psi_0 dw)} \leq \| S_{m,p}^w f \|_{L^2(\psi_0 dw)}, \quad \forall v_0 \in A_2 \begin{array}{c} \frac{2}{p_+(L_w)}(w) \cap RH_j \left( \frac{p_+(L_w)}{2} \right)(w). \end{array} \]

To prove this inequality, first of all note that since \( v_0 \in A_2 \begin{array}{c} \frac{2}{p_+(L_w)}(w) \cap RH_j \left( \frac{p_+(L_w)}{2} \right)(w) \end{array} \) by Remark 2.48 (with \( \tilde{p} = p_-(L_w), \tilde{q} = p_+(L_w), q_1 = 2, \) and \( \tilde{k} = 1)\), we can find \( p_0, \tilde{r}, r, q_0 \) such that \( r_0 < \tilde{r} < 2, \)

\[ 1 < \frac{q_0}{2} \leq r < \infty, v_0 \in A_2 \begin{array}{c} \frac{2}{p_+(L_w)}(w) \cap RH_{r'}(w), \end{array} \]

and

\[ 1 + \tilde{r}n \left( \frac{1}{2r} - \frac{1}{q_0} \right) > 0. \]

Changing the variable \( t \) into \( 2t \) and applying the subordination formula (2.24), and Minkowski’s integral inequality, we get

\[ \| S_{m,p}^w f \|_{L^2(\psi_0 dw)} \leq \int_0^\infty \frac{1}{2} u^2 I(u) \frac{du}{u} + \int_0^\infty \frac{1}{2} u^2 e^{-u} I(u) \frac{du}{u} =: I + II, \]

where

\[ I(u) := \left( \int_{B(2r,2)} \int_0^\infty \int_{B(x,2)} \left| e^{-\frac{r}{4} L_0} \left( t \sqrt{L_0} \right)^m e^{-t} \sqrt{\psi_0} f(y)^2 \frac{dv(y)}{t w(B(y,2t))} v_0(x) dw(x) \right|^2 \frac{dt}{t} dv(y) \right)^{\frac{1}{2}}. \]

For \( 0 < u < 1/4, \) since \( v_0 \in A_2 \begin{array}{c} \frac{2}{p_+(L_w)}(w) \cap RH_{r'}(w) \end{array} \) with \( r \geq q_0/2, \) Lemma 4.9 with \( T_{T} = e^{-tL_0}, \)

\[ F(y, t) = (t \sqrt{L_0})^m e^{-t} \sqrt{\psi_0} f(y), \] \( 2 \) implies

\[ I(u) \leq u^n \left( \int_{B(2r,2)} \int_0^\infty \left| (t \sqrt{L_0})^m e^{-t} \sqrt{\psi_0} f(y)^2 \frac{dt}{t} v_0(y) dw(y) \right|^2 \frac{du}{u} \right) = u^n \left( \int_{B(2r,2)} \int_0^\infty \left( \frac{dv(y)}{t w(B(y,2t))} \right) \right) \leq u^n \left\| S_{m,p}^w f \right\|_{L^2(\psi_0 dw)}. \]

Therefore, (5.5) yields

\[ I \leq \int_0^\infty \frac{1}{2} u^2 + u^n \left( \frac{1}{2} - \frac{q_0}{2} \right) \frac{du}{u} \left\| S_{m,p}^w f \right\|_{L^2(\psi_0 dw)} \leq \left\| S_{m,p}^w f \right\|_{L^2(\psi_0 dw)}. \]

To estimate \( II \) apply (2.5), the fact that \( e^{-tL_0} \in O(L^{p_0}(w) - L^2(w)), \) (2.17) (recall that \( v_0 \in A_2 \begin{array}{c} \frac{2}{p_+(L_w)} \end{array} \)), and Fubini’s theorem to conclude

\[ II \leq \sum_{j \geq 1} e^{-c4j} \int_{\mathbb{R}^n} \int_0^\infty \left( \int_{B(x,2^{j+1}t)} \left| (t \sqrt{L_0})^m e^{-t} \sqrt{\psi_0} f(y)^2 \right| dv_0(y) \right) \frac{dt}{t} v_0(x) dw(x) \left\| S_{m,p}^w f \right\|_{L^2(\psi_0 dw)}. \]

\[ \leq \sum_{j \geq 1} e^{-c4j} \left( \int_{\mathbb{R}^n} \int_0^\infty \int_{B(x,2^{j+1}t)} \left| (t \sqrt{L_0})^m e^{-t} \sqrt{\psi_0} f(y)^2 \right| dv_0(y) \frac{dt}{t} v_0(x) dw(x) \right)^{\frac{1}{2}} \leq \left\| S_{m,p}^w f \right\|_{L^2(\psi_0 dw)}. \]

\[ \square \]
5.1.4. Proof of Theorem 3.2, parts (a) and (b). Fix \( w \in A_2, f \in L^2(w) \), and \( K \in \mathbb{N}_0 \). Following the notation in Proposition 4.2 consider \( T_1 = \nabla e^{i t^2 L_w} \) and \( F(y, t) = t(t \sqrt{t} \omega)^m f(y) \). Note that \( F \) satisfies condition (i) in that proposition. Besides, for \( \max \{ r_w, (q_-(L_w))_{w, s} \} < p_0 < 2 < q_0 < q_+(L_w) \), by Lemma 2.45 with \( \alpha = 1 \) and \( S_i = e^{-\frac{2}{\sqrt{t}} L_w} \), we have that \( T_i \) satisfies conditions (ii) and (iii) in Proposition 4.2.

Moreover, note that for all \( 0 < p < q_+(L_w) \) and \( v \in RH_{\left( \frac{q_0}{\sqrt{t}} \right)}(w) \), we can find \( q_0 \) satisfying \( \max \{ 2, p \} < q_0 < q_+(L_w) \) so that \( v \in RH_{\left( \frac{q_0}{\sqrt{t}} \right)}(w) \). Hence, Proposition 4.2, part (b), implies

\[
\|g_{K, H}^w f\|_{L^p(w)} < \|G_{K, H}^w f\|_{L^p(w)},
\]

which proves part (a).

If we now take \( \max \{ r_w, (q_-(L_w))_{w, s} \} < p < \infty \) and \( v \in A_{\max \{ r_w, (q_-(L_w))_{w, s} \}}(w) \), we can find \( p_0 \) satisfying \( \max \{ r_w, (q_-(L_w))_{w, s} \} < p_0 < \min \{ 2, p \} \) (recall that \( r_w < 2 \) and \( (q_-(L_w))_{w, s} < q_-(L_w) < 2 \)) so that \( v \in A_{\frac{1}{p_0}}(w) \). Hence, Proposition 4.2, part (a), implies

\[
\|G_{K, H}^w f\|_{L^p(w)} < \|g_{K, H}^w f\|_{L^p(w)},
\]

which proves part (b).

5.1.5. Proof of Theorem 3.2, part (c). We proceed as in the proof of Theorem 3.1, part (c), but in this case we need to prove, for every \( K \in \mathbb{N}_0 \),

\[
\|g_{K, P, f}^w\|_{L^2(w)} < \|G_{K, P, f}^w\|_{L^2(w)}\quad \forall v_0 \in A_{\max \{ r_w, (q_-(L_w))_{w, s} \}}(w) \cap RH_{\left( \frac{q_0}{\sqrt{t}} \right)}(w),
\]

instead of (5.1). To this end, fix \( p_0 \) and \( q_0 \) satisfying \( \max \{ r_w, (q_-(L_w))_{w, s} \} < p_0 < 2 < q_0 < q_+(L_w) \) so that \( v_0 \in A_{\frac{1}{p_0}}(w) \cap RH_{\left( \frac{q_0}{\sqrt{t}} \right)}(w) \). From now on, the proof follows the lines of that of Theorem 3.1, part (c), replacing \( p_+(L_w) \) with \( q_+(L_w) \), using (4.1) with \( T_2 = \nabla e^{-\frac{2}{\sqrt{t}} L_w} \) and \( F(y, t) = t(t \sqrt{t} \omega)^K e^{-t \sqrt{t} \omega} f(y) \), and Remark 4.13 with the previous \( F(y, t) \), instead of Lemma 4.9, and Lemma 2.45 with \( q = q_0, p = 2, \alpha = 2 \sqrt{\omega} \), and \( S_i \) equal to the identity, instead of the fact that \( e^{-\frac{2}{\sqrt{t}} L_w} \in O(L^2(w) - L^2(w)) \).

5.1.6. Proof of Theorem 3.2, part (d). We proceed as in the proof of Theorem 3.1, part (d), but now we prove, for every \( K \in \mathbb{N}_0 \),

\[
\|G_{K, P, f}^w\|_{L^2(w)} < \|g_{K, P, f}^w\|_{L^2(w)}\quad \forall v_0 \in A_{\max \{ r_w, (q_-(L_w))_{w, s} \}}(w) \cap RH_{\left( \frac{q_0}{\sqrt{t}} \right)}(w),
\]

instead of (5.4). To this end, note that, by Remark 2.48 (with \( \tilde{p} = \max \{ r_w, (q_-(L_w))_{w, s} \}, \tilde{q} = q_+(L_w), q_1 = 2, \) and \( k = 1 \)), we can find \( p_0, \tilde{r}, q_0 \) such that \( r_w < \tilde{r} < 2, 1 < \frac{q_0}{\tilde{r}} \leq r < \infty \), and \( v_0 \in A_{\frac{1}{p_0}}(w) \cap RH_{\tilde{r}}(w) \), and

\[
1 + \tilde{r} \left( \frac{1}{2r} - \frac{1}{q_0} \right) > 0.
\]

Hereafter, we can follow the proof of Theorem 3.1, part (d), but using Remark 4.13 with \( F(y, t) = t(t \sqrt{t} \omega)^K e^{-t \sqrt{t} \omega} f(y) \), instead of Lemma 4.9, and Lemma 2.45 with \( p = p_0, q = 2, \alpha = 2 \sqrt{\omega} \), and \( S_i \) equal to the identity, instead of the fact that \( e^{-t L_w} \in O(L^p(w) - L^2(w)) \).

5.2. Boundedness results.

Proof of Theorem 3.4. The proof follows at once by Theorem 2.34 and Propositions 4.5, part (b), 4.15, and 4.16.
Proof of Theorem 3.5. The boundedness of \( g_{m,H}^{w} \) follows by Theorem 3.1, part (a), and Theorems 3.4 and 2.34, since \( p_{+}(L_{w}) < p_{+}(L_{w})^{m,s} \), for all \( m \in \mathbb{N} \).

The boundedness of \( g_{m,P}^{w} \) follows by (4.30) and the boundedness of \( g_{m,H}^{w} \).

Proof of Theorem 3.6. The boundedness of \( g_{m,H}^{w} \) and \( g_{m,P}^{w} \) follow by (4.29) and Theorem 3.5, since \( W_{p}^{w}(q_{-}(L_{w}), q_{+}(L_{w})) \subset W_{p}^{w}(p_{-}(L_{w}), p_{+}(L_{w})) \).

Proof of Theorem 3.7. The proof follows by Theorem 4.20 and the fact that \( N^{w}_{q,H} \) and \( S^{w}_{2,H} \) are bounded operators on \( L^{p}(v_{wdw}) \) for all \( p \in W_{p}^{w}(p_{-}(L_{w}), \infty) \) (see Theorem 2.34, and [10] or [20, Proposition 7.1]).

Finally, we prove Theorem 3.8. To this end, we need the following Calderón-Zygmund decomposition for functions on weighted Sobolev spaces

**Lemma 5.6.** [5, Lemma 6.6]. Let \( n \geq 1, \alpha > 0, \sigma \in A_{\infty}, \) and \( 1 \leq p < \infty \) such that \( \sigma \in A_{p} \). Assume that \( f \in S \) such that \( \| \nabla f \|_{L^{p}(|\sigma|)} < \infty \). Then, there exist a collection of balls \( \{B_{i}\}_{i} \) with radii \( r_{B_{i}} \), smooth functions \( \{b_{i}\}_{i} \), and a function \( g \in L^{1}_{\text{loc}}(\sigma) \) such that

\[
\tag{5.7}
f = g + \sum_{i=1}^{\infty} b_{i}
\]

and the following properties hold:

\[
\tag{5.8}
|\nabla g(x)| \leq C \sigma, \text{ for } \mu - \text{a.e. } x,
\]

\[
\tag{5.9}
supp b_{i} \subset B_{i} \quad \text{and} \quad \int_{B_{i}} |\nabla b_{i}(x)|^{p} |\sigma(x)| \leq C \alpha^{p} \sigma(B_{i}),
\]

\[
\tag{5.10}
\sum_{i=1}^{\infty} \sigma(B_{i}) \leq \frac{C}{\alpha^{p}} \int_{\mathbb{R}^{n}} |\nabla f(x)|^{p} |\sigma(x)|,
\]

\[
\tag{5.11}
\sum_{i=1}^{\infty} 1_{4B_{i}} \leq N,
\]

where \( C \) and \( N \) depend only on \( n, p, \) and \( \sigma \). In addition, for \( 1 \leq q < p_{\sigma}^{*} \), where \( p_{\sigma}^{*} \) is defined in (2.21),

\[
\tag{5.12}
\left( \int_{B_{i}} |b_{i}|^{q} |\sigma| \right)^{\frac{1}{q}} \leq \alpha r_{B_{i}}.
\]

**Proof of Theorem 3.8.** In [12, Proposition 6.1] the authors proved that, for all \( f \in S \) and \( p \in W_{p}^{w}(\max\{r_{w}, p_{-}(L_{w})\}, p_{+}(L_{w})) = \left( r_{w}(w) \max\{r_{w}, p_{-}(L_{w})\}, \frac{p_{+}(L_{w})}{s_{w}(w)} \right) \), it holds

\[
\tag{5.13}
\| \sqrt{\nu_{w} f} \|_{L^{p}(v_{wdw})} \leq \| \nabla f \|_{L^{p}(v_{wdw})}.
\]

Here we extend this boundedness for all

\[
p \in W_{p}^{w}(\max\{r_{w}, (p_{-}(L_{w}))_{w,s}\}, p_{+}(L_{w})) = \left( r_{w}(w) \max\{r_{w}, (p_{-}(L_{w}))_{w,s}\}, \frac{p_{+}(L_{w})}{s_{w}(w)} \right),
\]

(we recall that \( (p_{-}(L_{w}))_{w,s} < p_{-}(L_{w}) \), see (2.20)).

First of all, note that we may assume that \( r_{w} < p_{-}(L_{w}) \). Otherwise,

\[
\max\{r_{w}, p_{-}(L_{w})\} = \max\{r_{w}, (p_{-}(L_{w}))_{w,s}\}
\]
and by [12, Proposition 6.1] (see (5.13)) the proof would be complete. Hence, assuming that $r_w < p_-(L_w)$, let us extend (5.13) for all $p \in \mathcal{W}_p^w(\max\{r_w, (p_-(L_w))_{w,s}\}, p_+(L_w))$.

To lighten the proof we denote $p_- := p_-(L_w)$ and $p_+ := p_+(L_w)$. Besides, we fix $p$ satisfying:

\begin{equation}
\tau_v(w) \max\{r_w, (p_-)_{w,s}\} < p < \tau_v(w)p_- = \tau_v(w) \max\{r_w, p_-\}.
\end{equation}

We shall show that

\begin{equation}
vw \left( \left\{ x \in \mathbb{R}^n : \sqrt{L_w f(x)} > \alpha \right\} \right) \leq \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f(x)|^p v(x)dw(x), \quad \forall \alpha > 0.
\end{equation}

This, together with (5.13), will allow us to conclude the proof Theorem 3.8 by interpolation (see [8] and recall that by Remark 2.15 $vw \in A_{\infty}$). Thus, let us prove (5.15).

To this end, note that the interval

\begin{equation}
\left( \tau_v(w)p_- \min \left\{ \frac{p_+}{s_0(w)}, \frac{p_v^*}{s_0(w)} \right\} \right) \neq \emptyset.
\end{equation}

Indeed, by hypothesis we have that $\mathcal{W}_p^w(\max\{r_w, p_-\}, p_+) \neq \emptyset$. Then, in view of (2.12) and recalling that we are assuming that $r_w < p_-$, our hypothesis implies that

\begin{equation}
\tau_v(w)p_- < \frac{p_+}{s_0(w)}.
\end{equation}

Therefore, we just need to show that

\begin{equation}
\tau_v(w)p_- < p_v^*.
\end{equation}

In order to prove this, notice that we can assume that $nr_{vw} > p$ (otherwise $p_v^* = \infty$ and the inequality is trivial). Hence, by (5.14), (2.21), and Remark 2.15,

\begin{equation}
\frac{1}{p_v^*} = \frac{1}{p} - \frac{1}{nr_{vw}} < \frac{1}{\tau_v(w)(p_-)_{w,s}} - \frac{1}{nr_{vw}} = \frac{nr_w + p_-}{\tau_v(w)p_-nr_w} - \frac{1}{nr_{vw}} = \frac{1}{\tau_v(w)p_-} - \frac{1}{nr_{vw}} \left( 1 - \frac{r_w}{r_w \tau_v(w)} \right) \leq \frac{1}{\tau_v(w)p_-}.
\end{equation}

Therefore, in view of (5.16), we can take

\begin{equation}
\tau_v(w)p_- < p_1 < \min \left\{ \frac{p_+}{s_0(w)}, \frac{p_v^*}{s_0(w)} \right\}.
\end{equation}

In particular,

\begin{equation}
v \in A_{\frac{p_1}{s_0}(w)} \cap RH \left( \frac{\mu^w}{s_0} \right)(w).
\end{equation}

Next, fix $\alpha > 0$ and take a Calderón-Zygmund decomposition of $f$ at height $\alpha$ as in Lemma 5.6, for $\sigma = vw$, and $p$ as in (5.14). Note that, by Remark 2.15 $vw \in A_{\infty}$ and $r_{vw} \leq r_w \tau_v(w) < p$. Moreover, let $b_1, g$, and $\{B_i\}$ be the functions and the collection of balls given by Lemma 5.6, and let $M \in \mathbb{N}$. We define $B_{r_{b_i}} := (I - e^{-r_{b_i}/L_w})M$ and $A_{r_{b_i}} := I - B_{r_{b_i}} = \sum_{k=1}^M C_k M e^{-v_{b_i}/r_{b_i}}$. Hence, we can write $f = g + \sum_{i \in \mathbb{N}} A_{r_{b_i}} b_i + \sum_{i \in \mathbb{N}} B_{r_{b_i}} b_i =: g + \hat{b} + \hat{b}$, and then

\begin{equation}
vw \left( \left\{ x \in \mathbb{R}^n : \sqrt{L_w f(x)} > \alpha \right\} \right) \leq vw \left( \left\{ x \in \mathbb{R}^n : \sqrt{L_w g(x)} > \frac{\alpha}{3} \right\} \right) + vw \left( \left\{ x \in \mathbb{R}^n : \sqrt{L_w \hat{b}(x)} > \frac{\alpha}{3} \right\} \right) + vw \left( \left\{ x \in \mathbb{R}^n : \sqrt{L_w \hat{b}}(x) > \frac{\alpha}{3} \right\} \right) =: I + II + III.
\end{equation}

In order to estimate $I$, first recall that $p < p_1$ (see (5.14) and (5.17)). Then, apply Chebyshev’s inequality, (5.13), and properties (5.8)-(5.11) to obtain

\begin{equation}
I \leq \frac{1}{\alpha^{p_1}} \int_{\mathbb{R}^n} |\nabla g(x)|^{p_1} v(x)dw(x) \leq \frac{1}{\alpha^{p_1}} \int_{\mathbb{R}^n} |\nabla g(x)|^p v(x)dw(x) \leq \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla g(x)|^p v(x)dw(x)
\end{equation}
In order to estimate $II$ and $III$ we shall use the following inequality:

\[
\left( \sum_{i \in \mathbb{N}} \int_{B_i} \left( \mathcal{M}^w(|u|^{p_i}(x)) \right)^{\frac{1}{p_i}} v(x)dw(x) \right)^{p_1} \leq \left( \int_{\bigcup_{i \in \mathbb{N}} B_i} \left( \mathcal{M}^w(|u|^{p_i}(x)) \right)^{\frac{1}{p_i}} v(x)dw(x) \right)^{p_1}
\]

\[
\leq v\omega(\bigcup_{i \in \mathbb{N}} B_i)||u||^{p_1}_{L^{p_i}(vw)} \leq \frac{1}{\alpha^{p_1}} \int_{\mathbb{R}^n} |v(x)|^{p_1} dw(x)
\]

where $u \in L^{p_i}(v\omega)$ such that $||u||_{L^{p_i}(v\omega)} = 1$. The inequality follows by Kolmogorov’s inequality (see [16, Exercise 2.1.5]), and follow the proof suggested there replacing the Lebesgue measure with the measure given by the weight $v\omega$. Besides, in the last inequality we have applied (5.10).

After this observation let us estimate $II$. By Chebyshev’s inequality, (5.13) and the definition of $\tilde{b}$, we have

\[
II \leq \frac{1}{\alpha^{p_1}} \int_{\mathbb{R}^n} |\sqrt{L_{\tilde{b}}}(x)|^{p_1} v(x)dw(x) \leq \frac{1}{\alpha^{p_1}} \int_{\mathbb{R}^n} |\tilde{b}(x)|^{p_1} v(x)dw(x)
\]

By Hölder’s inequality and the fact that $\sqrt{\tau \mathcal{M}^{L_{\tilde{b}}}} \in \mathcal{F}(L^{p_1}(v\omega) - L^{p_1}(v\omega))$ (see (5.17) and Lemma 2.41), we estimate the integral in $x$ as follows

\[
\int_{C_j(B_i)} |\nabla e^{-kr_{\tilde{b}_j} L_{\tilde{b}}}(x)||u(x)||v(x)dw(x)
\]

\[
\leq \frac{1}{r_{B_i}} \left( \int_{C_j(B_i)} |\sqrt{L_{\tilde{b}_j}}|^{p_1} d(v\omega)(x) \right)^{\frac{1}{p_1}} \left( \int_{C_j(B_i)} |u(x)|^{p_1} d(v\omega)(x) \right)^{\frac{1}{p_1}}
\]

\[
\leq \frac{e^{-c_{p_1}/2}}{r_{B_i}} v\omega(2^{j+1}B_i) \left( \int_{B_i} |b_i(x)|^{p_1} d(v\omega)(x) \right)^{\frac{1}{p_1}} \left( \int_{2^{j+1}B_i} |u(x)|^{p_1} d(v\omega)(x) \right)^{\frac{1}{p_1}}
\]

\[
\leq \alpha e^{-c_{p_1}/2} v\omega(B_i) \inf_{x \in B_i} \left( \mathcal{M}^w(|u|^{p_1}(x)) \right)^{\frac{1}{p_1}},
\]

where in the last inequality we have used (5.12) (see (5.17)), and (2.5). Plugging this into (5.22) and applying (5.21), we have

\[
II \leq \left( \sup_{||u||_{L^{p_i}(v\omega)}} \sum_{i \in \mathbb{N}} \sum_{k \geq 1} C_{k,M} \sum_{j \geq 1} e^{-c_{p_1}/2} \int_{B_i} \left( \mathcal{M}^w(|u|^{p_1}(x)) \right)^{\frac{1}{p_1}} v(x)dw(x) \right)^{p_1}
\]

\[
\leq \left( \sup_{||u||_{L^{p_i}(v\omega)}} \sum_{i \in \mathbb{N}} \int_{B_i} \left( \mathcal{M}^w(|u|^{p_1}(x)) \right)^{\frac{1}{p_1}} v(x)dw(x) \right)^{p_1} \leq \frac{1}{\alpha^{p_1}} \int_{\mathbb{R}^n} |v(x)|^{p_1} dw(x).
\]

Finally, we estimate $III$. By (5.10) and (2.5), we have

\[
III \leq v\omega(\bigcup_{i \in \mathbb{N}} 4B_i) + v\omega \left( \left\{ x \in \mathbb{R}^n : \sqrt{L_{\tilde{b}}}(x) > \frac{\alpha}{3} \right\} \right)
\]

\[
\leq \frac{1}{\alpha^{p_1}} \int_{\mathbb{R}^n} |v(x)|^{p_1} dw(x) + III,
\]
where

\[ III := \mathcal{W} \left( \left\{ x \in \mathbb{R}^n \setminus \bigcup_{i \in \mathbb{N}} 4B_i : \sqrt{L_{\nu} b(x)} \geq \frac{\alpha}{3} \right\} \right). \]

Again by Chebyshev’s inequality and duality, proceeding as in the estimate of term II, we have

\[ (5.25) \]

\[ III \leq \frac{1}{\alpha^{p_1}} \left( \sup_{\|u\|_{L^{p_1}(\mathcal{W})} = 1} \sum_{m \in \mathbb{N}} \left( \int_{\mathcal{C}(B_i)} |\sqrt{L_{\nu} B_{r_B} b_j(x)}|^{p_1} v(x) dw(x) \right)^{\frac{1}{p_1}} \|u1_{\mathcal{C}(B_i)}\|_{L^{p_1}(\mathcal{W})} \right)^{p_1}. \]

We estimate \( III_{ij} \) by using (2.33) and Minkowski’s integral inequality:

\[ (5.26) \]

\[ III_{ij} \leq \left( \int_{\mathcal{C}(B_i)} \left( \int_0^\infty \left| tL_{\nu} e^{-tL_{\nu}} B_{r_B} b_j(x) \right| \frac{dt}{t} \right)^{p_1} v(x) dw(x) \right)^{\frac{1}{p_1}} \]

\[ \leq \int_0^\infty \left( \int_{\mathcal{C}(B_i)} \left| tL_{\nu} e^{-tL_{\nu}} B_{r_B} b_j(x) \right|^{p_1} v(x) dw(x) \right)^{\frac{1}{p_1}} \frac{dt}{t}. \]

We compute the above integral in \( x \) by using functional calculus. The notation is taken from [1], [5. Section 7], and [12]. We write \( \theta \in [0, \pi/2) \) for the supremum of \( |\arg(zL_{\nu} f_{L^{p_1}(\mathcal{W})} f)| \) over all \( f \) in the domain of \( L_{\nu} \). Let \( 0 < \theta < \theta < \vartheta < \mu < \pi/2 \) and note that, for a fixed \( t > 0 \), \( \phi(z, t) := e^{-t^2 z (1 - e^{-r_B^2 t})^M} \) is holomorphic in the open sector \( \Sigma_{\mu} = \{ z \in \mathbb{C} \setminus \{ 0 \} : |\arg(z)| < \mu \} \) and satisfies \( |\phi(z, t)| \lesssim |z|^M (1 + |z|)^{-2M} \) (with implicit constant depending on \( \mu, t > 0, r_B, \) and \( M \)) for every \( z \in \Sigma_{\mu} \). Hence, we can write

\[ \phi(L_{\nu}, t) = \int_{\Gamma} e^{-\lambda z} \eta(z, t) \, dz, \quad \text{where} \quad \eta(z, t) = \int_\gamma e^{\xi z} \phi(z, t) \, d\xi. \]

Here \( \Gamma = \partial \Sigma_{\mu=0} \) with positive orientation (although orientation is irrelevant for our computations) and \( \gamma = \mathbb{R}_+ e^{i \text{sign} \text{Im}(z)} \nu \). It is not difficult to see that for every \( z \in \Gamma \),

\[ |\eta(z, t)| \lesssim \frac{r_{B,i}^M}{(1 + i^2 t)^{M+1}}. \]

By these observations, the fact that \( zL_{\nu} e^{-zL_{\nu}} \in O(L^{p_1}(\mathcal{W}) - L^{p_1}(\mathcal{W})) \) (see (5.18)), (5.12) (recall (5.17)) and since \( j \geq 2 \), we have

\[ (5.27) \]

\[ \left( \int_{\mathcal{C}(B_i)} \left| tL_{\nu} e^{-tL_{\nu}} B_{r_B} b_j(x) \right|^{p_1} v(x) dw(x) \right)^{\frac{1}{p_1}} \]

\[ \leq \mathcal{W}(2R_{B,i}^j) \int_0^\infty \left( \int_{\mathcal{C}(B_i)} |zL_{\nu} e^{-zL_{\nu}} B_{r_B} b_j(x)|^{p_1} d(vw)(x) \right)^{\frac{1}{p_1}} \frac{r_{B,i}^M t}{(1 + i^2 t)^{M+1}} \frac{|dz|}{|z|} \]

\[ \leq \mathcal{W}(2R_{B,i}^j) \int_0^\infty \left( \int_{\mathcal{C}(B_i)} \left( \frac{2j^{1/R_{B,i}^j}}{|z|^{1/2}} \right)^{p_1} e^{-z/2} \frac{2j^{1/R_{B,i}^j} r_{B,i}^M t}{(1 + i^2 t)^{M+1}} \frac{|dz|}{|z|} \left( \int_{B_i} |b_j(x)|^{p_1} d(vw)(x) \right)^{\frac{1}{p_1}}\right)^{\frac{1}{p_1}} \]

\[ \leq \alpha r_{B,i} \mathcal{W}(2R_{B,i}^j)^{\frac{1}{p_1}} 2^{\theta j} \int_0^\infty \left( \int_{\mathcal{C}(B_i)} \left( \frac{2j^{1/R_{B,i}^j}}{|z|^{1/2}} \right)^{p_1} e^{-z/2} \frac{r_{B,i}^M t}{(s + i^2 t)^{M+1}} \frac{ds}{s} \right)^{\frac{1}{p_1}}. \]
\[
\leq ar_B, vw(2^{j+1}B) \frac{1}{2} 2^{\theta_2} \int_0^\infty \frac{Y(s)^{\theta_2} e^{-cs^2}}{(1/s^2 + t^2)^{M+1}} \frac{ds}{s},
\]
where in the last inequality we have changed the variable \(s\) into \(4j r_B^2 / s^2\). Plugging this and (5.27) into (5.26), and changing the variable \(t\) into \(2/r_B t\), we obtain, for \(M \in \mathbb{N}\) such that \(2M > \theta_2\),

\[
III_{j} \leq \alpha vw(2^{j+1}B) \frac{1}{2} 2^{-2(2M+1-\theta_1)} \int_0^\infty \int_0^\infty \frac{t}{2} (1/s^2 + t^2)^{M+1} \frac{ds}{s}.
\]

Consequently, in view of (5.25), by (2.13) and taking \(M \in \mathbb{N}\) large enough satisfying \(2M > \max\{\theta_2, \theta_1 + r_w r_v(w)n - 1\}\), we get

\[
III \leq \left( \sup_{\|\eta\| \in L^p(|vdw|), 1} \sum_{\ell \in \mathbb{N}} \inf_{x \in B_i} \left( M^{vw}(|\eta|)^{\rho_i}(x) \right) \right)^{p_1}
\]

\[
\leq \left( \sup_{\|\eta\| \in L^p(|vdw|), 1} \int_{\cup_{i \in \mathbb{N}} B_i} \left( M^{vw}(|\eta|)^{\rho_i}(x) \right) \frac{1}{2} v(x) dw(x) \right)^{p_1} \leq \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f(x)|^p v(x) dw(x),
\]

where in the last inequality we have used (5.21). This and (5.24) imply that

\[
III \leq \frac{1}{\alpha^p} \int_{\mathbb{R}^n} |\nabla f(x)|^p v(x) dw(x),
\]

which, in turn, together with (5.19), (5.20), and (5.23), implies (5.15), and the proof is complete. \(\square\)

### 6. Unweighted boundedness for degenerate operators

In this section we prove unweighted results for degenerate operators. That is, we show boundedness on \(L^p(\mathbb{R}^n)\) for the degenerate operators defined in (2.26)-(2.33). We obtain this from our results on \(L^p(|vdw|)\), by taking \(v = w^{-1}\). The statements of our results are written so that the ranges where we obtain those boundedness results depend on the weight \(w\), but do not depend on the operator \(L_w\).

Before being more precise, we observe that since we assume that \(n \geq 2\), we have that \(2^*_w = \infty\) if and only if \(n = 2\) and \(r_w = 1\). Otherwise,

\[
2^*_w = \frac{nr_w}{nr_w - 2}.
\]

In particular,

\[
2^*_w = \begin{cases} 
\frac{2nr_w}{nr_w - 2} & \text{if } nr_w > 2, \\
nr_w & \text{if } nr_w = 2
\end{cases} \quad \text{and} \quad (2^*_w)^* = \begin{cases} 
\frac{2nr_w}{nr_w - 2} & \text{if } nr_w > 2, \\
1 & \text{if } nr_w = 2.
\end{cases}
\]

Besides, for all \(M \in \mathbb{N}\)

\[
(2^*_w)^M = 2^{M+1,*},
\]

Finally, from [12] we know that

\[
p_-(L_w) = q_-(L_w) < (2^*_w)^* < 2 < \min\{2^*_w, q_+(L_w)\} \leq \max\{2^*_w, q_+(L_w)\} < q_+(L_w)^* < p_+(L_w).
\]
Corollary 6.4. Given \( w \in A_2 \), for \( m \in \mathbb{N} \) and \((2_w')s_w < p < \frac{2n}{r_w} \), the vertical square functions \( s_{w,H}^m \) and \( s_{m,p}^w \) can be extended to bounded operators on \( L^p(\mathbb{R}^n) \). In particular, we have this in the following cases:

(a) If \( r_w = 1 \), for \( n = 2 \), \( w \in A_1 \cap RH_{p'} \), \( 1 < p < \infty \); and for \( n > 2 \), \( w \in A_1 \cap RH_{(\frac{m+2}{2m})'} \), \( \frac{2n}{n+2} < p < \frac{2n}{n-2} \).

(b) If \( r_w > 1 \), for \( 1 < r \leq 2 \), \( w \in A_r \cap RH_{(\frac{m+2}{2m})'} \), \( \frac{2nr}{nr+2} < p < \frac{2n}{n-2} \).

Proof. Fix \( w \in A_2 \), \( m \in \mathbb{N} \), and \((2_w')s_w < p < \frac{2n}{r_w} \). Then, by (6.3), we have that \( p-(L_w)s_w < p < \frac{p-(L_w)}{r_w} \). Consequently, by (2.9),

\[
 w \in RH_{(\frac{m+2}{2m})'} \cap A_{\frac{p-(L_w)}{p}},
\]

which in view of (2.8) yields

\[
 w^{-1} \in A_{\frac{p-(L_w)}{p}}(w) \cap RH_{(\frac{m+2}{2m})'}(w).
\]

Hence taking \( v = w^{-1} \) in Theorem 3.5, we conclude that \( s_{w,H}^m \) and \( s_{m,p}^w \) can be extended to bounded operators on \( L^p(\mathbb{R}^n) \).

Assume now that \( r_w = 1 \), in particular \( w \in A_2 \), and note that, since we are assuming that \( n \geq 2 \) by (6.1),

\[
 2_w^* = \begin{cases} \frac{2m}{n+2} & \text{if } n > 2, \\ \infty & \text{if } n = 2; \end{cases} \quad \text{and} \quad (2_w')^* = \begin{cases} \frac{2n}{n+2} & \text{if } n > 2, \\ 1 & \text{if } n = 2. \end{cases}
\]

Thus, for \( n = 2 \) since \( 2_w^* = \infty \), the conditions \( 1 < p < \infty \) and \( w \in A_1 \cap RH_{p'} \), can be written as \((2_w')s_w < p < \frac{2n}{r_w} \). If now \( n > 2 \), note that again the conditions \( \frac{2n}{n+2} < p < \frac{2n}{n-2} \) and \( w \in A_1 \cap RH_{(\frac{m+2}{2m})'} \), can be written as \((2_w')s_w < p < \frac{2n}{r_w} \).

Assume next that \( r_w > 1 \) and \( w \in A_r \cap RH_{(\frac{m+2}{2m})'} \), for \( 1 < r \leq 2 \) and \( \frac{2nr}{nr+2} < p < \frac{2n}{n-2} \). This can be written as \( w \in A_r \subseteq A_2 \) and, since \( r_w < r \),

\[
 (2_w')s_w = \frac{2nrw}{nr+2}s_w < \frac{2n}{nr+2}s_w < p < \frac{2nr}{rw(nr-2)} < \frac{2nrw}{rw(nrw-2)} = \frac{2w}{r_w}.
\]

\[\square\]

Corollary 6.5. Given \( w \in A_2 \), for \( K \in \mathbb{N}_0 \) and \((2_w')s_w < p \leq \frac{2}{r_w} \), the vertical square functions \( g_{K,H}^w \) and \( g_{K,p}^w \) can be extended to bounded operators on \( L^p(\mathbb{R}^n) \). In particular, we have this in the following cases:

(a) If \( r_w = 1 \), for \( n = 2 \), \( w \in A_1 \cap RH_{p'} \), and \( 1 < p \leq 2 \); and, for \( n > 2 \), \( w \in A_1 \cap RH_{(\frac{m+2}{2m})'} \), and \( \frac{2n}{n+2} < p \leq 2 \).

(b) If \( r_w > 1 \), for \( 1 < r \leq 2 \), \( w \in A_r \cap RH_{(\frac{m+2}{2m})'} \), \( \frac{2nr}{nr+2} < p \leq \frac{2}{r} \).

Proof. Fix \( w \in A_2 \), \( K \in \mathbb{N}_0 \), and \((2_w')s_w < \frac{2}{r_w} \). Then, by (6.3), we have that \( q-(L_w)s_w < p < \frac{q-(L_w)}{r_w} \). Consequently, by (2.9),

\[
 w \in RH_{(\frac{m+2}{2m})'} \cap A_{\frac{q-(L_w)}{p}},
\]

which in view of (2.8) yields

\[
 w^{-1} \in A_{\frac{q-(L_w)}{p}}(w) \cap RH_{(\frac{m+2}{2m})'}(w).
\]
Hence taking \( v = w^{-1} \) in Theorem 3.6, we conclude that \( g_{K,H}^w \) and \( g_{M,P}^w \) can be extended to bounded operators on \( L^p(\mathbb{R}^n) \).

In particular if \( r_w = 1 \), for \( n = 2 \) since \( 2^*_w = \infty \) (see (6.1)), the conditions \( 1 < p \leq 2 \) and \( w \in A_1 \cap RH_{p'} \), can be written as \((2^*_w)'s_w < p \leq \frac{2}{r_w} \). If now \( n > 2 \), we have that \((2^*_w)' = \frac{2n}{nr+2} \). Then, the conditions \( \frac{2n}{nr+2} < p \leq 2 \) and \( w \in A_1 \cap RH(\frac{\log w}{\log n})' \), can be written as \((2^*_w)'s_w < p \leq \frac{2}{r_w} \) (see (6.1)).

If now \( r_w > 1 \), \( 1 < r \leq 2 \), and \( w \in A_r \cap RH(\frac{\log w}{\log n})' \), with \( \frac{2nr}{nr+2} < p \leq \frac{2}{r} \). This can be written as \( w \in A_r \cap A_2 \) and, since \( r_w < r \),

\[
(2^*_w)'s_w = \frac{2nr_w}{nrw+2} s_w < \frac{2nr}{nr+2} s_w < p \leq \frac{2}{r} < \frac{2}{r_w}.
\]

Corollary 6.6. Given \( w \in A_2 \), for \( m \in \mathbb{N} \) and \((2^*_w)'s_w < p < \frac{2^{2m,s}}{r_w} \), the conical square functions \( S_{2m-1,1}^w \), \( S_{2m-1,p}^w \), \( G_{2m-1,1}^w \), and \( G_{2m-1,p}^w \) can be extended to bounded operators on \( L^p(\mathbb{R}^n) \). In particular, this holds in the following cases:

(a) If \( r_w = 1 \), for \( n = 2 \), \( w \in A_1 \cap RH_{p'} \), and \( 1 < p < \infty \); and, for \( n > 2 \), \( w \in A_1 \cap RH(\frac{\log w}{\log n})' \), and \( \frac{2n}{nr+2} < p < 2^{2m,s} \).

(b) If \( r_w > 1 \), for \( 1 < r \leq 2 \), \( w \in A_r \cap RH(\frac{\log w}{\log n})' \), and \( \frac{2nr}{nr+2} < p < \infty \), if \( nr \leq 4m \), and \( \frac{2nr}{nr+2} < p < \frac{2n}{nr-4m} \), if \( nr > 4m \).

Note that \( \left( \frac{2nr}{nr+2}, \frac{2n}{nr-4m} \right) \) is a not empty interval for \( n < 4m + 1 \) and \( 1 < r \leq 2 \).

Proof. Fix \( w \in A_2 \), \( m \in \mathbb{N} \), and \((2^*_w)'s_w < p < \frac{2^{2m,s}}{r_w} \). Then, by (6.3) and (6.2), we have that \( p_{-}(L_w)s_w < p < \frac{\mu(L_w)^{2m-1,s}}{r_w} \). Consequently, by (2.9),

\[
w \in RH(\frac{p}{p_{-}(L_w)^{2m-1,s}}) \cap A_{\frac{\mu(L_w)^{2m-1,s}}{p}}.
\]

which in view of (2.8) yields

\[
w^{-1} \in A_{\frac{p}{p_{-}(L_w)^{2m-1,s}}}(w) \cap RH(\frac{\mu(L_w)^{2m-1,s}}{p})(w).
\]

Hence taking \( v = w^{-1} \) in Theorem 3.4, we conclude that \( S_{2m-1,1}^w \), \( S_{2m-1,p}^w \), \( G_{2m-1,1}^w \), and \( G_{2m-1,p}^w \) can be extended to bounded operators on \( L^p(\mathbb{R}^n) \).

If now we assume that \( r_w = 1 \) (in particular \( w \in A_2 \)). Then, for \( n = 2 \) since \( 2^*_w = \infty \) (see (6.1)) we have that \( 2^{2m,s} = \infty \) (see (6.2)). Thus, the conditions \( 1 < p < \infty \) and \( w \in A_1 \cap RH_{p'} \), can be written as \((2^*_w)'s_w < p < \frac{2^{2m,s}}{r_w} \). Similarly, in view of (6.1), if \( n > 2 \), the conditions \( \frac{2n}{nr+2} < p < 2^{2m,s} \) and \( w \in A_1 \cap RH(\frac{\log w}{\log n})' \), can be written as

\[
(2^*_w)'s_w = \frac{2n}{n+2} s_w < p < 2^{2m,s} = \frac{2^{2m,s}}{r_w}.
\]

Assume next that \( r_w > 1 \), \( 1 < r \leq 2 \), and \( w \in A_r \cap RH(\frac{\log w}{\log n})' \), in particular, notice that \( w \in A_2 \).

In the case that \( nr \leq 4m \) we take \( \frac{2nr}{nr+2} < p < \infty \). Then, since \( 2^{2m,s} = \infty \), we in fact have

\[
(2^*_w)'s_w = \frac{2nr_w}{nrw+2} s_w < \frac{2nr}{nr+2} s_w < p < \infty = \frac{2^{2m,s}}{r_w}.
\]
As for the case that \( nr > 4m \), we take \( \frac{2n}{n+2} < p < \frac{2n}{nr-4m} \). Thus, since \( r_w < r \), we get
\[
(2^n_w)'s_w = \frac{2nr_w}{nr_w + 2s_w} < \frac{2nr}{nr + 2}s_w < \frac{2nr}{r_w(nr - 4m)} < \frac{2nr_w}{r_w(nr - 4m)} = \frac{2m^*}{r_w}.
\]

\( \square \)

**Corollary 6.7.** Given \( w \in A_2 \), for \( (2^n_w)'s_w < p < \infty \) we have that \( N_H^w \) can be extended to a bounded operator on \( L^p(\mathbb{R}^n) \). In particular, this holds in the following cases:

(a) If \( r_w = 1 \), for \( n = 2 \), \( w \in A_1 \cap RH_\rho' \), and \( 1 < p < \infty \); and for \( n > 2 \), \( w \in A_1 \cap RH_\rho' \left( \frac{(\alpha+2n)_w}{2m} \right) ', \)

and \( \frac{2n}{n+2} < p < \infty \).

(b) If \( r_w > 1 \), for \( 1 < r < 2 \), \( w \in A_r \cap RH_\rho' \left( \frac{(\alpha+2n)_w}{2m} \right) ', \) and \( \frac{2nr}{nr+2} < p < \infty \).

**Proof.** Fix \( w \in A_2 \) and \( (2^n_w)'s_w < p < \infty \). Then, by (6.3) and (6.2), we have that \( p(L_w)s_w < \infty \). Consequently, by (2.9),
\[
w \in RH_\rho' \left( \frac{p}{p-\frac{m}{4m}} \right),'
\]

which in view of (2.8) yields
\[
w^{-1} \in A_1 \frac{p}{p-\frac{m}{4m}}(w).
\]

Hence taking \( \nu = w^{-1} \) in Theorem 4.27, we conclude that \( N_H^w \) can be extended to a bounded operator on \( L^p(\mathbb{R}^n) \).

Assume now that \( r_w = 1 \) (in particular, \( w \in A_2 \)). For \( n = 2 \), since \( 2^n_w = \infty \) (see (6.1)), the conditions \( 1 < p < \infty \) and \( w \in A_1 \cap RH_\rho' \), can be written as \( (2^n_w)'s_w < p < \infty \). If now \( n > 2 \), in view of (6.1), the conditions \( \frac{2n}{n+2} < p < \infty \) and \( w \in A_1 \cap RH_\rho' \left( \frac{(\alpha+2n)_w}{2m} \right) ', \) can be written as \( (2^n_w)'s_w < p < \infty \).

Assume next that \( r_w > 1 \) and \( w \in A_r \cap RH_\rho' \left( \frac{(\alpha+2n)_w}{2m} \right) ', \) for \( 1 < r < 2 \) and \( \frac{2nr}{nr+2} < p < \infty \). In particular, \( w \in A_2 \). Besides, since \( r_w < r \) in view on (6.1), we can write the previous conditions as
\[
(2^n_w)'s_w = \frac{2nr_w}{nr_w + 2s_w} < \frac{2nr}{nr + 2}s_w < p < \infty.
\]

\( \square \)

**Corollary 6.8.** Given \( w \in A_2 \), for \( (2^n_w)'s_w < p < \frac{2^m_w}{w} \) we have that \( N_H^w \) can be extended to a bounded operator on \( L^p(\mathbb{R}^n) \). In particular, this holds in the following cases:

(a) If \( r_w = 1 \), for \( n = 2 \), \( w \in A_1 \cap RH_\rho' \), and \( 1 < p < \infty \); and for \( n > 2 \), \( w \in A_1 \cap RH_\rho' \left( \frac{(\alpha+2n)_w}{2m} \right) ', \)

and \( \frac{2n}{n+2} < p < \frac{2^m_w}{w} \).

(b) If \( r_w > 1 \), for \( 1 < r < 2 \), \( w \in A_r \cap RH_\rho' \left( \frac{(\alpha+2n)_w}{2m} \right) ', \) and \( \frac{2nr}{nr+2} < p < \infty \), if \( nr \leq 4 \), or
\[
\frac{2nr}{nr+2} < p < \frac{2n}{nr-4}, \text{ if } nr > 4.
\]

Note that the interval \( \left( \frac{2n}{nr+2}, \frac{2n}{nr-4} \right) \) is not empty for \( n < 5 \) and \( 1 < r \leq 2 \).

**Proof.** Fix \( w \in A_2 \) and \( (2^n_w)'s_w < p < \frac{2^m_w}{w} \). Then, by (6.3) and (6.2), we have that \( p(L_w)s_w < \infty \). Consequently, by (2.9),
\[
w \in RH_\rho' \left( \frac{p}{p-\frac{m}{4m}} \right) \cap A_1 \frac{p}{p-\frac{m}{4m}} \left( \frac{(\alpha+2n)_w}{2m} \right) ',
\]

which in view of (2.8) yields
\[
w^{-1} \in A_1 \frac{p}{p-\frac{m}{4m}}(w) \cap RH_\rho' \left( \frac{p}{p-\frac{m}{4m}} \right) \left( \frac{(\alpha+2n)_w}{2m} \right) '(w).
\]
Hence taking $v = w^{-1}$ in Theorem 3.7, we conclude that $\mathcal{N}_w^\alpha$ can be extended to a bounded operator on $L^p(\mathbb{R}^n)$.

To obtain parts (a) and (b) we proceed as in the proof Corollary 6.6 taking $m = 1$. □

**Corollary 6.9.** Given $w \in A_2$, for $\max \{ r_w, ((2_w^*)')_{w,s} \} s_w < p < \frac{2^w}{r_w}$, we have that $\sqrt{L_w}$ can be extended to a bounded operator on $L^p(\mathbb{R}^n)$. In particular, this is the case in the following situations:

(a) For $p = 2$, $1 \leq r_w < \frac{1}{n} + 1$ and $s_w < \min \left\{ \frac{2}{r_w}, \frac{nr_w + 4}{nr_w} \right\}$, we define \((s_w')' = \max \left\{ \left( \frac{2}{r_w} \right)', \frac{nr_w}{4} + 1 \right\} \).}

(b) If $r_w = 1$ and $n = 2$ for $w \in A_1 \cap RH_{p'}$ and $1 < p < \infty$, if $r_w = 1$ and $2 < n \leq 4$, for $w \in A_1 \cap RH_{p'}$ and $1 < p < \frac{2n}{n-2}$.

(c) If $r_w = 1$ and $n > 4$, for $w \in A_1 \cap RH \left( \frac{\alpha}{n+4} \right)$ and $2n < p < \frac{2n}{n-2}$.

(d) If $r_w > 1$, for $1 < r \leq 2$, $w \in A_r \cap RH \left( \frac{\alpha}{rn+4} \right)$ and $\max \left\{ r, \frac{2n}{rn+4} \right\} < p < \frac{2n}{n-2}$.

**Proof.** Fix $w \in A_2$ and $\max \{ r_w, ((2_w^*)')_{w,s} \} s_w < p < \frac{2^w}{r_w}$. Then, by (6.3) and (6.2), we have that $\max \{ r_w, (p_r)_{w,s} \} s_w < p < \frac{2^w}{r_w}$. Consequently, by (2.9),

$$w \in RH \left( \frac{\alpha}{max\{r,w\}} \right) \cap A_{\frac{\alpha}{max\{r,w\}}},$$

which in view of (2.8) yields

$$w^{-1} \in A_{\frac{\alpha}{max\{r,w\}}} (w) \cap RH \left( \frac{\alpha}{\max\{r,w\}} \right)'(w).$$

Hence taking $v = w^{-1}$ in Theorem 3.8, we conclude that $\sqrt{L_w}$ can be extended to a bounded operator on $L^p(\mathbb{R}^n)$.

Now, note that, for $p = 2$, $1 \leq r_w < \frac{1}{n} + 1$ and $s_w < \min \left\{ \frac{2}{r_w}, \frac{nr_w + 4}{nr_w} \right\}$ imply that

$$\max \left\{ r_w, ((2_w^*)')_{w,s} \right\} s_w = \max \left\{ r_w, \frac{2nr_w}{nr_w + 4} \right\} s_w < 2 < \frac{2n}{nr_w - 2} = \frac{2nr_w}{r_w(nr_w - 2)} \leq \frac{2^w}{r_w}.$$

To see that (b) and (c) imply that $w \in A_2$ and $\max \{ r_w, ((2_w^*)')_{w,s} \} s_w < p < \frac{2^w}{r_w}$, it is enough to notice that for $r_w = 1$, if $2 \leq n \leq 4$ we have that $\max \{ r_w, ((2_w^*)')_{w,s} \} = r_w = 1$. Moreover, by (6.1), $2^w = \infty$, if $n = 2$, and $2^w = \frac{2n}{n-2}$, if $n > 2$.

On the other hand, if $r_w = 1$ and $n > 4$, we have that $\max \{ r_w, ((2_w^*)')_{w,s} \} = ((2_w^*)')_{w,s} = \frac{2n}{n+4}$. Besides, as we have observed above, $2n/(n-2) = 2n + 2n/2w$.

Finally, part (d) follows from the following observation: for $1 < r_w < r \leq 2$,

$$\max \left\{ r, \frac{2n}{nr + 4} \right\} > \max \left\{ r_w, \frac{2rn}{nr + 4} \right\} = \max \left\{ r_w, ((2_w^*)')_{w,s} \right\},$$

and

$$\frac{2n}{nr - 2} < \frac{2nr_w}{r_w(nr_w - 2)} = \frac{2^w}{r_w}.$$

□

**Remark 6.10.** Note that in Corollary 6.9, part (a), we improve the range obtained in [12, Theorem 11.4]. To see this, note that we define $s_w$ as the conjugate exponent of the one defined in [12] using the same notation.

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REFERENCES

[1] AUSCHER, P. On necessary and sufficient conditions for $L^p$-estimates of Riesz transforms associated to elliptic operators on $\mathbb{R}^n$ and related estimates. Mem. Amer. Math. Soc. 186, 871 (2007), xviii+75.

[2] P. Auscher. Change of Angle in tent spaces. C. R. Math. Acad. Sci. Paris. 349 (2011), no. 5-6, 297-301.

[3] AUSCHER, P., HOFMANN, S., LACEY, M., McIntosh, A., and TCHAMITCHIAN, P. The solution of the Kato square root problem for second order elliptic operators on $\mathbb{R}^n$. Ann. of Math. (2) 156, 2 (2002), 633–654.

[4] AUSCHER, P., HOFMANN, S., and MARTELL, J.-M. Vertical versus conical square functions, Trans. Amer. Math. Soc. 364, 10 (2012), 5469–5489.

[5] AUSCHER, P., and MARTELL, J. M. Weighted norm inequalities, off-diagonal estimates and elliptic operators. III. Harmonic analysis of elliptic operators. J. Funct. Anal. 241, 2 (2006), 703–746.

[6] AUSCHER, P., and MARTELL, J. M. Weighted norm inequalities, off-diagonal estimates and elliptic operators. I. General operator theory and weights. Adv. Math. 212, 1 (2007), 225–276.

[7] AUSCHER, P., and MARTELL, J. M. Weighted norm inequalities, off-diagonal estimates and elliptic operators. II. Off-diagonal estimates on spaces of homogeneous type. J. Evol. Equ. 7, 2 (2007), 265–316.

[8] BADR, N. Real interpolation of Sobolev spaces. Math. Scand. 105, 2 (2009), 235–264.

[9] CHEN, L., MARTELL, J. M., and PRISUELOS-ARRIBAS, C. Conical square functions for degenerate elliptic operators. arXiv:1610.05952 (2016). To appear in Advances in Calculus of Variations.

[10] CHEN, L., MARTELL, J. M., and PRISUELOS-ARRIBAS, C. The weighted regularity problem for degenerate elliptic operators. Preprint (2018).

[11] D.V. Cruz-Uribe, J.M. Martell, C. Perez Weights Extrapolation and the Theory of Rubio de Francia, Operator Theory: Advances and Applications, 215. Birkhäuser/Springer Basel AG, Basel, 2011.

[12] CRUZ-URIBE, D., MARTELL, J. M., and RIOS, C. On the Kato problem and extensions for degenerate elliptic operators. arXiv:1510.06790. Analysis and PDE, 11 (2018), 3, 609–660.

[13] CRUZ-URIBE, D., and RIOS, C. The Kato problem for operators with weighted ellipticity. Trans. Amer. Math. Soc. 367, 7 (2015), 4727–4756.

[14] DUANDIKOETXEA, J. Fourier analysis, vol. 29 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2001. Translated and revised from the 1995 Spanish original by David Cruz-Uribe.

[15] GARCÍA-CUERVA, J., and RUBIO DE FRANCIA, J. L. Weighted norm inequalities and related topics, vol. 116 of North-Holland Mathematics Studies. North-Holland Publishing Co., Amsterdam, 1985.

[16] Grafakos, L. Classical Fourier analysis, second edition. Graduate Texts in Mathematics, 249. Springer, New York, (2008).

[17] HOFMANN, S., and MAYBORODA, S. Hardy and BMO spaces associated to divergence form elliptic operators. Math. Ann. 344, 1 (2009), 37–116.

[18] HOFMANN, S., MAYBORODA, S., and McIntosh, A. Second order elliptic operators with complex bounded measurable coefficients in $L^p$, Sobolev and Hardy spaces. Ann. Sci. Éc. Norm. Supér. (4) 44, 5 (2011), 723–800.

[19] MARTELL, J. M., and PRISUELOS-ARRIBAS, C. Weighted Hardy spaces associated with elliptic operators. Part I: weighted norm inequalities for conical square functions. (2011). Trans. Amer. Math. Soc. 369(6), (2017), 4193-4233.

[20] MARTELL, J. M., and PRISUELOS-ARRIBAS, C. Weighted Hardy spaces associated with elliptic operators. Part II: Characterizations of $H^1_0(w)$. arXiv:1701.00920 (2017).

[21] MAYBORODA, S. The connections between Dirichlet, regularity and Neumann problems for second order elliptic operators with complex bounded measurable coefficients. Adv. Math. 225, 4 (2010), 1786–1819.

[22] PRISUELOS-ARRIBAS, C. Weighted Hardy spaces associated with elliptic operators. Part III: Characterizations of $H^1_0(w)$ and the weighted Hardy space associated with the Riesz transform. arXiv:1702.04648 To appear in Journal of Geometric Analysis (2018).

Cruz Prisuelos Arribas, Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM, Consejo Superior de Investigaciones Científicas, C/ Nicolás Cabrera, 13-15, E-28049 Madrid, Spain

E-mail address: cruz.prisuelos@icmat.es