ON THE CACHAZO-DOUGLAS-SEIBERG-WITTEN
CONJECTURE FOR SIMPLE LIE ALGEBRAS

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ABSTRACT. Recently, motivated by supersymmetric gauge theory, Cachazo, Douglas, Seiberg, and Witten proposed a conjecture about finite dimensional simple Lie algebras, and checked it in the classical cases. We prove the conjecture for type $G_2$, and also verify a consequence of the conjecture in the general case.

1. THE CDSW CONJECTURE

Let $g$ be a simple finite dimensional Lie algebra over $\mathbb{C}$. We fix an invariant form on $g$ and do not distinguish $g$ and $g^*$. Let $g$ be the dual Coxeter number of $g$. Consider the associative algebra $R = \bigwedge (g \oplus g)$. This algebra is naturally $\mathbb{Z}_+$-bigraded (with the two copies of $g$ sitting in degrees $(1,0)$ and $(0,1)$, respectively). The degree $(2,0), (1,1), \text{and } (0,2)$ components of $R$ are $\bigwedge^2 g, g \otimes g, \bigwedge^2 g$, respectively, hence each of them canonically contains a copy of $g$. Let $I$ be the ideal in $R$ generated by these three copies of $g$, and $A = R/I$.

The associative algebra $A$ may be interpreted as follows. Let $\Pi g$ denote $g$ regarded as an odd vector space. Then $R$ may be thought of as the algebra of regular functions on $\Pi g \times \Pi g$. We have the supercommutator map $\{\cdot, \cdot\} : \Pi g \times \Pi g \to g$ given by the formula $\{X, Y\} = XY + YX$, where the products are taken in the universal enveloping algebra (this is a morphism of supermanifolds). The ideal $I$ in $R$ is then defined by the relations $\{X, X\} = 0, \{X, Y\} = 0, \{Y, Y\} = 0$, so $A$ is the algebra of functions on the superscheme defined by these equations.

In [CDSW,W], the following conjecture is proposed, and proved for classical $g$:

**Conjecture 1.1.** (i) The algebra $A^g$ of $g$-invariants in the algebra $A$ is generated by the unique invariant element $S$ of degree $(1,1)$ (namely, $S = Tr|_{V}(XY)$, where $V$ is a non-trivial irreducible finite dimensional representation of $g$).

(ii) $S^g = 0$.

(iii) $S^{g-1} \neq 0$.

Here we prove the conjecture for $g$ of type $G_2$. We believe that the method of the proof should be relevant in the general case.

We also prove in general the following result, which is a consequence of the conjecture.

**Proposition 1.2.** Any homogeneous element of $A^g$ is of degree $(d,d)$ for some $d$; therefore, the algebra $A^g$ is purely even, and the natural action of $sl(2)$ on it (by linear transformations of $X,Y$) is trivial.

**Remark.** Conjecture 1.1 has the following cohomological interpretation. Let $g[x,y]$ denote the Lie algebra of polynomials of $x,y$ with values in $g$. Consider the algebra of relative cohomology $H^*(g[x,y], g, \mathbb{C})$. It is graded by the cohomological
degree $D$ and the $x,y$-degree $d$. It is clear that for any nonzero cochain, one has $D \leq d$. Thus the homogeneous elements of the algebra $H^\bullet(g[x,y], g, \mathbb{C})$ for which $D = d$ form a subalgebra $E$. Conjecture 1.1 then states that $E$ is generated by an element $S$ of degree 2 with the defining relation $S^g = 0$.

Note that in this formulation, one can obviously replace $g[x,y]$ with its quotient by any ideal spanned by homogeneous elements of $x,y$-degree $d \geq 3$.

\section{Proof}

The main idea of the proof is to consider first the algebra $B$ of functions on the superscheme of $X \in \Pi g$ such that $\{X, X\} = 0$. This algebra was intensively studied by Kostant, Peterson, and others (see [K]) and is rather well understood. In particular, it is known that as a $g$-module, $B$ is a direct sum of $2^{2k}(g)$ non-isomorphic simple $g$-modules $V_a$ parametrized by abelian ideals $a$ in a Borel subalgebra $b$ of $g$. Namely, if $d$ is such an ideal of dimension $d$ then it defines (by taking its top exterior power) a nonzero vector $v_d$ in $\wedge^d g$ (defined up to scaling). This vector generates an irreducible submodule in $\wedge^d g$ (with highest weight vector $v_d$), and the sum of these submodules is a complement to the kernel of the projection $\wedge g \to B$. In fact, this is the unique invariant complement, because in each degree $d$ it is the eigenspace of the quadratic Casimir $C$ with eigenvalue $d$ (the largest possible eigenvalue on $\wedge^d g$; here the Casimir is normalized so that $C|\mathfrak{g} = 1$). Thus the (direct) sum of $V_a$ is canonically identified with $B$ as a $g$-module.

The algebra $A$ is obtained from $B \otimes B$ by taking a quotient by the additional relation $\{X, Y\} = 0$ and then taking invariants. In other words, $A = (B \otimes B)^g/L$, where $L$ is the space of invariants in the ideal $L$ in $B \otimes B$ given by the relation $\{X, Y\} = 0$.

Therefore, let us look more carefully at the algebra $(B \otimes B)^g$. From the above we see that a basis of $(B \otimes B)^g$ is given by the elements $z_a$, the canonical elements in $V_a \otimes V_a^*$. The element $z_a$ sits in bidegree $(d, d)$, where $d$ is the dimension of $a$. This implies Proposition 1.2 (the $sl(2)$ action on $A^g$ is trivial because the $sl(2)$-weight of a vector in $A$ of degree $(d_1, d_2)$ is $d_1 - d_2$).

Now we proceed to prove the conjecture for $G_2$. Let $\omega_1$ and $\omega_2 = \theta$ be the fundamental weights of $G_2$ ($\theta$ is the highest root), and $\alpha_1, \alpha_2$ the corresponding simple roots (so that $\theta = 3\alpha_1 + 2\alpha_2$). The abelian ideals in the corresponding Borel subalgebra are $a_0 = 0$, $a_1 = \mathbb{C}e_{\alpha_1}$, $a_2 = \mathbb{C}e_\theta \oplus \mathbb{C}e_{3\alpha_1 + \alpha_2}$, $a_3 = \mathbb{C}e_\theta \oplus \mathbb{C}e_{2\alpha_1 + \alpha_2} \oplus \mathbb{C}e_{2\alpha_1 + \alpha_2}$. Thus the $g$-module $B$ has 4 irreducible components: $V_0 = \mathbb{C}$ in degree 0, $V_1 = g$ in degree 1, $V_2 = V(3\omega_1)$ in degree 2, and $V_3 = V(\omega_1 + 2\omega_2)$ in degree 3.

Let $S$ be the invariant element of degree $(1,1)$ in $\wedge g \otimes \wedge g$. Clearly the projection of powers of $S$ onto $B \otimes B$ is nonzero in degree $0,1,2,3$ (this follows from the fact that for each degree $d$ one has a canonical decomposition $\wedge^d g = V_{a_d} \oplus \text{Ker}(\wedge^d g \to B[d])$, where $B[d]$ is the degree $d$ component of $B$).

The dual Coxeter number of $G_2$ is 4. Thus, our job is just to show that the ideal generated by the relation $\{X, Y\} = 0$ in $B \otimes B$ contains no nonzero $g$-invariants.

A $g$-invariant of degree $(d+1, d+1)$ in this ideal is a linear combination of elements of the form $C_{w} := \sum f_{ijk}(x_i, x_j)w(x_k)$, where $x_i$ is an orthonormal basis of $g$, $f_{ijk}$ are the structure constants of $g$ in this basis, and $w : g \to \text{End}(V_a) = V_a \otimes V_a^*$ is a homomorphism of representations (here for brevity $a := a_d$). It is easy to check
that the only homomorphisms $w$ relevant to our situation (i.e., for $d = 1, 2$) are given by the action of $g$ on $V_a$.

So the result follows from the following lemma:

**Lemma 2.1.** If $w$ is the action map then $C_w$ is zero in $B \otimes B$.

**Proof.** Let us regard $C_w$ as an operator on $\wedge^{d+1}(g)$. Then it has the form

$$C_w = \sum_{i,j,k} f_{ijk} W_x, L_{x_i} I_{x_j},$$

where $I, L, W$ are the operators of contraction, Lie derivative, and wedging, respectively. This can be more shortly written as

$$C_w = \sum W_x, L_{[x_i, x_j]} I_{x_j}.$$

We claim that if $v \in V_a$ then $C_w v$ is zero (this would imply that $C_w$ is zero in $B \otimes B$). Set $y = p_1 \wedge ... \wedge p_{d+1}$. Then we get

$$C_w y = \sum_{i,j,r,s} (-1)^{r+s} x_i \wedge ([x_i, p_r], p_s) - ([x_i, p_s], p_r) \wedge p_1 \wedge ... \hat{p}_r ... \hat{p}_s ... \wedge p_{d+1} =$$

$$\sum_{i,j,r,s} (-1)^{r+s} x_i \wedge [p_s, p_r] x_i \wedge p_1 \wedge ... \hat{p}_r ... \hat{p}_s ... \wedge p_{d+1}.$$  

This implies the result, since $V_a$ coincides with the kernel of the map $\wedge^{d+1} g \to g \otimes \wedge^{d-1} g$, given by

$$y \mapsto \sum_{r,s} (-1)^{r+s} [p_s, p_r] \otimes p_1 \wedge ... \hat{p}_r ... \hat{p}_s ... \wedge p_{d+1}.$$

□

References

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1This has to do with the fact that the highest weight of $V_2$ does not involve $\omega_2$. The situation is exactly the same for type $B_2$, except that it is not true there that all $w$ come from the action of $g$; this results in $S^3 = 0$ for $B_2$, while $S^3 \neq 0$ for $G_2$. 
