On adelic model of boson Fock space

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We construct a canonical embedding of the Schwartz space on \( \mathbb{R}^n \) to the space of distributions on the adelic product of all the \( p \)-adic numbers. This map is equivariant with respect to the action of the symplectic group \( \text{Sp}(2n, \mathbb{Q}) \) over rational numbers and with respect to the action of rational Heisenberg group.

These notes contain two elements. First, we give a funny realization of a space of complex functions of a real variable as a space of functions of \( p \)-adic variable. Secondly, we try to clarify classical construction of modular forms through \( \theta \)-functions and Howe duality.

1. Introduction

1.1. Fields and rings. Below \( \mathbb{Q} \) denotes the rational numbers, \( \mathbb{Z} \) is the ring of integers, \( \mathbb{Q}_p \) is the field of \( p \)-adic numbers, \( \mathbb{Z}_p \subset \mathbb{Q}_p \) is the ring of \( p \)-adic integers. We denote the norm on \( \mathbb{Q}_p \) by \( | \cdot | \).

1.2. Adeles, (see [2], [11], [8]). An adele is a sequence

\[
(a_{\infty}, a_2, a_3, a_5, a_7, a_{11}, \ldots),
\]

where \( a_{\infty} \in \mathbb{R}, a_p \in \mathbb{Q}_p \) (\( p \) is a prime) and \( |a_p| = 1 \) for all \( p \) except a finite number of primes.

Our main object is the ring

\[
\mathbb{A} \subset \mathbb{Q}_2 \times \mathbb{Q}_3 \times \mathbb{Q}_5 \times \ldots
\]

consisting of the sequences

\[
a = (a_2, a_3, a_5, a_7, a_{11}, \ldots)
\]

satisfying the same conditions. The addition and multiplication in \( \mathbb{A} \) are defined coordinate-wise. Below the term "adeles" means the ring \( \mathbb{A} \). The space of sequences of the form \( (1.1) \) we denote by \( \mathbb{R} \times A \).

1.3. Convergence in \( \mathbb{A} \). A sequence \( a^{(j)} \) in \( \mathbb{A} \) converges to \( a \in \mathbb{A} \) iff

a) There is a finite set \( S \) of primes such that \( |a_p^{(j)}| = 1 \) for all \( p \notin S \) for all \( j \).

b) For each \( p \), the sequence \( a_p^{(j)} \) converges in \( \mathbb{Q}_p \).

The image of the diagonal embedding \( \mathbb{Q} \rightarrow \mathbb{A} \)

\[
\tau \mapsto (\tau, \tau, \tau, \ldots)
\]

is dense in \( \mathbb{A} \).

1.4. Integration. We define the Lesbegue measure \( da \) on the ring \( \mathbb{A} \) by two assumptions:

- the measure on \( \prod_p \mathbb{Z}_p \) is the product-measure
- the measure is translation-invariant.

This measure is \( \sigma \)-finite. We define the space \( L^2(\mathbb{A}^n) \) in the usual way. The Bruhat space \( B(\mathbb{A}^n) \) defined below is dense in \( L^2(\mathbb{A}^n) \).

1.5. Adelic exponents. For an adele \( a \in \mathbb{A} \), we define its exponent \( \exp(2\pi ia) \in \mathbb{C} \) by

\[
\exp(2\pi ia) = \prod_p \exp(2\pi ia_p).
\]
all the factors are roots of unity, only finite number of factors is \( \neq 1 \).

1.6. **Lattices.** A **lattice** \( L \) in a \( \mathbb{Q} \)-linear space \( \mathbb{Q}^n \) is an arbitrary additive subgroup isomorphic \( \mathbb{Z}^n \). Equivalently, a lattice is a group \( L \subset \mathbb{Q}^n \) having a form \( \bigoplus \mathbb{Z} f_j \), where \( f_j \) is a basis in \( \mathbb{Q}^n \). A **dual lattice** \( \hat{L} \) consists of \( y \in \mathbb{Q}^n \), such that \( \sum x_j y_j \in \mathbb{Z} \) for all the \( x \in L \).

A **lattice** in the \( p \)-adic linear space \( \mathbb{Q}_p^n \) is a set of the form \( \bigoplus \mathbb{Z}_p f_j \), where \( f_j \) is a basis in \( \mathbb{Q}_p^n \). The **standard lattice** is the set \( \mathbb{Z}_p^n \).

A **lattice** in the adelic module \( \mathbb{A}^n \) is a set of a form \( \bigoplus_p L_p \), where \( L_p \subset \mathbb{Q}_p^n \) are lattices, and \( L_p \) are the standard lattices for all \( p \) except a finite set.

For a lattice \( L \subset \mathbb{Q}^n \), consider its closure \( \overline{L} \subset \mathbb{A}^n \). It is a lattice, and moreover the map \( L \mapsto \overline{L} \) is a bijection of the set of all the lattices in \( \mathbb{Q}^n \) and the set of all the lattices in \( \mathbb{A}^n \).

1.7. **Bruhat test functions and distributions on \( \mathbb{A}^n \).** A **test function** \( f \) on \( \mathbb{Q}_p^n \) or on \( \mathbb{A}^n \) is a compactly supported locally constant complex-valued function.

The Bruhat space \( \mathcal{B}(\mathbb{Q}_p^n) \) (resp. \( \mathcal{B}(\mathbb{A}^n) \)) is the space of all the test functions.

A **distribution** is a linear functional on \( \mathcal{B}(\mathbb{Q}_p^n) \) (resp. \( \mathcal{B}(\mathbb{A}^n) \)). We denote the space of all the distributions by \( \mathcal{B}'(\mathbb{Q}_p^n) \) (resp. \( \mathcal{B}'(\mathbb{A}^n) \)).

1.8. **The second description of the spaces \( \mathcal{B} \).** Let \( S \) be a subset in \( \mathbb{Q}_p^n \) or \( \mathbb{A}^n \). Denote by \( \mathcal{I}_S \) the indicator function of \( S \), i.e.

\[
\mathcal{I}_S(x) = \begin{cases} 
1, & \text{if } x \in S \\
0, & \text{if } x \in S.
\end{cases}
\]

For a lattice \( L \) and a vector \( a \), the function \( \mathcal{I}_{L+a} \) is a test function. Each test function is a linear combination of functions of this type.

1.9. **Third description of the spaces \( \mathcal{B} \).** Consider the space \( \mathbb{Q}_p^n \) or \( \mathbb{A}^n \). Let \( K \subset L \) be lattices. Denote by \( \mathcal{B}(L|K) \) the space

a) \( f = 0 \) outside \( L \).

b) \( f \) is \( K \)-invariant.

The dimension of this space is the order of the quotient group \( L/K \), in particular the dimension is finite.

Then

\[
\mathcal{B}(\mathbb{Q}_p^n) = \bigcup_{K \subset L \subset \mathbb{Q}_p^n} \mathcal{B}(L|K; \mathbb{Q}_p), \quad \mathcal{B}(\mathbb{A}^n) = \bigcup_{K \subset L \subset \mathbb{A}^n} \mathcal{B}(L|K; \mathbb{A}).
\]

1.10. **The space \( \mathcal{M}(\mathbb{Q}^n) \).** We repeat literally the previous definition. For two lattices \( K \subset L \subset \mathbb{Q}^n \), denote by \( \mathcal{M}(L|K) \) the space of \( K \)-invariant functions on \( \mathbb{Q}^n \) supported by \( L \). We assume

\[
\mathcal{M}(\mathbb{Q}^n) = \bigcup_{K \subset L \subset \mathbb{Q}^n} \mathcal{M}(L|K).
\]

The space \( \mathcal{M}(\mathbb{Q}^n) \) is generated by the indicator functions \( \mathcal{I}_{L+a} \) of shifted lattices.

1.11. **The bijection \( \mathcal{M}(\mathbb{Q}^n) \leftrightarrow \mathcal{B}(\mathbb{A}^n) \).**

**Proposition 1.1.** a) Each function \( f \in \mathcal{M}(\mathbb{Q}^n) \) admits a unique continuous extension \( \overline{f} \) to a function on \( \mathbb{A}^n \).

b) The map \( f \mapsto \overline{f} \) is a bijection \( \mathcal{M}(\mathbb{Q}^n) \to \mathcal{B}(\mathbb{A}^n) \).

The statement is trivial. More constructive variant of this is statements is \( \overline{\mathcal{I}_{L+a}} = \mathcal{I}_{\overline{L+a}} \).
1.12. Space $\mathcal{P}(\mathbb{R}^n)$ of Poisson distributions. Denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space on $\mathbb{R}^n$, i.e. the space of smooth functions $f$ on $\mathbb{R}^n$ satisfying the condition: for each $\alpha_1, \ldots, \alpha_n$, and each $N$
\[ \lim_{x \to \infty} \left( \sum x_j^2 \right)^N \frac{\partial^{\alpha_1}}{\partial^{\alpha_1}x_1} \cdots \frac{\partial^{\alpha_n}}{\partial^{\alpha_n}x_n} f(x) = 0. \]

By $\mathcal{S}'(\mathbb{R}^n)$ denote the space dual to $\mathcal{S}(\mathbb{R}^n)$, i.e., the space of all tempered distributions on $\mathbb{R}^n$.

Now we intend to define a certain subspace $\mathcal{P}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$. This space is spanned by functions
\[ \sum_{k_1, \ldots, k_n \in \mathbb{Z}} \delta(x - \sqrt{2\pi}(b + \sum_j k_j a_j)), \]
where $a_1, \ldots, a_n \in \mathbb{Q}^n$ are linear independent, $b \in \mathbb{Q}^n$.

**Lemma 1.2.** A countable sum $\psi$ of $\delta$-functions is an element of $\mathcal{P}(\mathbb{R}^n)$ iff there are two lattices $K \subset L \subset \mathbb{Q}^n$ such that $\psi$ is supported by $\sqrt{2\pi}L$ and $\psi$ is $\sqrt{2\pi}K$-invariant.

We denote by $\mathcal{P}(L|K) \subset \mathcal{P}(\mathbb{R}^n)$ the space of all the distributions satisfying this Lemma.

1.13. Canonical bijection $\mathcal{M}(\mathbb{Q}^n) \leftrightarrow \mathcal{P}(\mathbb{R}^n)$. Define a canonical bijective map $I_{\mathbb{R}}: \mathcal{M}(\mathbb{Q}^n) \to \mathcal{P}(\mathbb{R}^n)$. Let $f \in \mathcal{M}(\mathbb{Q}^n)$, let $M \subset L \subset \mathbb{Q}^n$ be corresponding lattices. We define the distribution $I_{\mathbb{R}}f \in \mathcal{P}(\mathbb{R}^n)$ as
\[ I_{\mathbb{R}}f(x) = \sum_{\xi \in L} f(\xi)\delta(x - \sqrt{2\pi}\xi). \]

We obtain the bijection $\mathcal{M}(\mathbb{Q}^n) \to \mathcal{P}(\mathbb{R}^n)$. Also, for each rational lattices $K \subset L$, we have a bijection
\[ \mathcal{M}(L|K) \leftrightarrow \mathcal{P}(L|K). \]

1.14. Observation. Thus we have the canonical bijection
\[ J_{\mathbb{R}^n} : \{ \text{space } \mathcal{P}(\mathbb{R}^n) \} \leftrightarrow \{ \text{adelic space } \mathcal{B}(\mathbb{A}^n) \}. \]

In particular, we have canonical embeddings
\[ \mathcal{S}(\mathbb{R}^n) \to \mathcal{B}'(\mathbb{A}^n), \quad \mathcal{B}(\mathbb{A}^n) \to \mathcal{S}'(\mathbb{R}^n). \]

1.15. The image of the Schwartz space in $\mathcal{B}'(\mathbb{A}^n)$.

**Proposition 1.3.** For $f \in \mathcal{S}(\mathbb{R}^n)$, the corresponding element $F \in \mathcal{B}'(\mathbb{A}^n)$ is
\[ F(a) = \sum_{\xi \in \mathbb{Q}^n} f(\xi)\delta_{\mathbb{A}}(a - \xi) \]
where $\delta_{\mathbb{A}}$ is the adelic delta-function.

**Proof.** Let $L \subset \mathbb{Q}$ be a lattice, $b \in \mathbb{Q}^n$. The value of the adelic distribution $F \in \mathcal{B}'(\mathbb{A}^n)$ on the adelic test function $I(L + b)$ is
\[ \sum_{\xi \in \mathbb{Q}^n \cap (L + b)} f(\xi) = \sum_{\xi \in (L + b)} f(\xi) \]
The last expression is the value of the Poisson distribution \( f_{RLa} \) on the Schwartz function \( f \).

\[ \square \]

1.16. Result of the paper. The space \( S(\mathbb{R}^n) \) is equipped with the canonical action of the real Heisenberg group \( \text{Heis}_n(\mathbb{R}) \) and the real symplectic group \( \text{Sp}(2n, \mathbb{R}) \) (in this sense, \( S(\mathbb{R}^n) \) is a bosonic Fock space mentioned in the title).

The space \( \mathcal{B}(\mathbb{A}^n) \) is equipped with the canonical action of the adelic Heisenberg group \( \text{Heis}_n(\mathbb{A}) \) and the adelic symplectic group \( \text{Sp}(2n, \mathbb{A}) \).

There are canonical embeddings

\[
\text{Heis}_n(\mathbb{Q}) \rightarrow \text{Heis}_n(\mathbb{R}), \quad \text{Heis}_n(\mathbb{Q}) \rightarrow \text{Heis}_n(\mathbb{A}),
\]

\[
\text{Sp}(2n, \mathbb{Q}) \rightarrow \text{Sp}(2n, \mathbb{R}), \quad \text{Sp}(2n, \mathbb{Q}) \rightarrow \text{Sp}(2n, \mathbb{A});
\]

in all the cases the images are dense.

Theorem 1.4. a) The map \( J_{\mathbb{R}A} \) commutes with the action of \( \text{Heis}_n(\mathbb{Q}) \).

b) The map \( J_{\mathbb{R}A} \) commutes with the action of \( \text{Sp}(2n, \mathbb{Q}) \).

Corollary 1.5. For \( f \in S(\mathbb{R}) \) denote by \( \hat{f} \) its Fourier transform. Then the adelic Fourier transform of the distribution (1.3) is

\[
\text{const} \cdot \sum_{\xi \in \mathbb{Q}^n} \hat{f}(\xi) \delta_{\mathbb{R}^n}(a - \xi)
\]

Theorem 1.6. For each \( f \in \mathcal{P}(\mathbb{R}^n) \), there is a congruence subgroup in \( \text{Sp}(2n, \mathbb{Z}) \) that fixes \( f \).

1.17. Another description of the operator \( J_{\mathbb{R}A} \). Consider the space \( \mathbb{R}^n \times \mathbb{A}^n \) (in fact, it is the adelic space in the usual sense). Consider the tensor product \( S(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{A}^n) \), and consider the linear functional (Poisson–Weil distribution) on this space given by

\[
K(x, \xi) = \sum_{\xi \in \mathbb{Q}^n} \delta_{\mathbb{R}^n}(x + \xi) \delta_{\mathbb{A}^n}(a - \xi)
\]

Our operator \( S(\mathbb{R}^n) \rightarrow \mathcal{B}'(\mathbb{A}^n) \) is the pairing

\[
f(x) \mapsto F(a) = \{ K(x, a), f(x) \}
\]

2. Rational Heisenberg group

2.1. Heisenberg group. By \( \text{Heis}_n \) we denote the group of \((1+n+1) \times (1+n+1)\)-matrices

\[
R(v_+, v_-, \alpha) = \begin{pmatrix}
1 & v_+ & \alpha + \frac{1}{2}v_+ v_-^t \\
0 & 1 & v_-^t \\
0 & 0 & 1
\end{pmatrix}.
\]

Here \( v_+, v_- \) are matrices-rows, \( v_-^t \) is a matrix-column, the sign \( ^t \) is the transposition. We have

\[
R(v_+, v_-, \alpha)R(w_+, w_-, \beta) = R(v_+ + w_+, v_- + w_-, \alpha + \beta + \frac{1}{2}(v_+ w_-^t - w_+ v_-^t))
\]

We consider 4 Heisenberg groups, \( \text{Heis}_n(\mathbb{Q}) \), \( \text{Heis}_n(\mathbb{R}) \), \( \text{Heis}(\mathbb{Q}_p) \), \( \text{Heis}_n(\mathbb{A}) \), this means that matrix elements of 2.1 are elements of \( \mathbb{Q}, \mathbb{R}, \mathbb{Q}_p, \mathbb{A} \).

\[2\] All the definitions are given below.
The group $\text{Heis}_n(\mathbb{Q})$ is a dense subgroup in $\text{Heis}_n(\mathbb{R})$, $\text{Heis}_n(\mathbb{Q}_p)$ $\text{Heis}_n(\hat{\mathbb{A}})$.

2.2. The standard representations of Heisenberg groups. These representations are given by almost the same formulae for the rings $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Q}_p$, $\hat{\mathbb{A}}$, but these formulae differs by position of factors $2\pi$.

Real case. The group $\text{Heis}_n(\mathbb{R})$ acts in the Schwartz space $S(\mathbb{R}^n)$ on $\mathbb{R}^n$ by the transformations

$$T_R(v_+, v_-, \alpha) f(x) = f(\sqrt{2\pi}(x + v_+)) \exp\{2\pi ixv_0^- + 2\pi i(n + 1/2)w_+v_0^-\}. $$

This formula also defines unitary operators in $L^2(\mathbb{R}^n)$ and continuous transformations of the space $S'(\mathbb{R}^n)$ of the space of tempered distributions on $\mathbb{R}^n$.

Adelic case. The group $\text{Heis}_n(\hat{\mathbb{A}})$ acts on the space $\mathcal{B}(\hat{\mathbb{A}}^n)$ by the formula

$$T(v_+, v_-, \alpha) f(x) = f(x + v_+) \exp\{2\pi i(xv_0^- + \alpha + 1/2w_+v_0^-)\}. $$

This formula also defines unitary operators in $L^2(\hat{\mathbb{A}}^n)$ and continuous operators in the space $\mathcal{B}'(\hat{\mathbb{A}}^n)$ of adelic distributions.

$p$-adic case. The action of $\text{Heis}_n(\mathbb{Q}_p)$ on $\mathcal{B}(\mathbb{Q}_p)$ and $\mathcal{B}'(\mathbb{Q}_p)$ is defined by the same formula.

Rational case. The group $\text{Heis}_n(\mathbb{Q})$ acts in the space $\mathcal{M}(\mathbb{Q}^n)$ via the same formula (2.3).

2.3. Relations between the standard representations of $\text{Heis}_n(\cdot)$.

Proposition 2.1. a) The subgroup $\text{Heis}_n(\mathbb{Q}) \subset \text{Heis}_n(\mathbb{R})$ preserves the space $\mathcal{P}(\mathbb{R}^n)$.

b) The canonical map $I_\mathbb{R} : \mathcal{M}(\mathbb{Q}^n) \to \mathcal{P}(\mathbb{R}^n)$ commutes with the action of $\text{Heis}_n(\mathbb{Q})$.

c) The canonical bijection $\mathcal{M}(\mathbb{Q}^n) \to \mathcal{B}(\hat{\mathbb{A}}^n)$ commutes with the action of $\text{Heis}_n(\mathbb{Q})$.

This statement is more-or-less obvious. It also implies Theorem 1.4.a.

2.4. Irreducibility.

Lemma 2.2. The representation of $\text{Heis}_n(\mathbb{Q})$ in $\mathcal{M}(\mathbb{Q}^n)$ is irreducible. Any operator $A : \mathcal{M}(\mathbb{Q}^n) \to \mathcal{M}(\mathbb{Q}^n)$ commuting with the action of $\text{Heis}_n(\mathbb{Q})$ is a multiplication by a constant.

Proof. First, we present an alternative description of the space $\mathcal{M}(L|K)$, it consists of functions fixed with respect to operators

$$T_v f(x) = f(x + v), \quad v \in K, \quad (2.4)$$

$$S_w f(x) = f(x) \exp(2\pi ixw^t), \quad w \in L^0. \quad (2.5)$$

The space $\mathcal{M}(L|K)$ is point-wise fixed by the group $G(L|M)$ generated by these operators.

The space $\mathcal{M}(L|K)$ is invariant with respect to the group $D(L|K)$ generated by the operators $T_v$, where $v \in L$, and $S_w$, where $w \in K^0$.

Hence the quotient-group $A(L|M) = D(L|M)/G(L|M)$ acts in $\mathcal{M}(L|M)$. In fact, this group is generated by the same operators $T_v$, $S_w$, see (2.4)–(2.5), but now we consider $v$ as an element of $L/M$ and $w$ as an element of $M^0/L^0$ (in fact, $A(L|M)$ is a finite Heisenberg group).

Let us show that the representation of $A(L|M)$ in the space $\mathcal{M}(L|M)$ is irreducible. The subgroup of $A(L|M)$ generated by the operators $S_w$ has a simple specter, its eigenvectors are $\delta$-functions on $L/M$. Hence any invariant subspace is
spanned by some collection of $\delta$-functions. But $T_v$-invariance implies the triviality of an invariant subspace.

Now the both statements of the Lemma become obvious. □

3. Weil representation

On the Weil representation, see [11], [7], [3], [6].

3.1. Symplectic groups. Consider a ring $K = \mathbb{R}, \mathbb{Q}, \mathbb{Q}_p, \mathbb{A}, \mathbb{Z}, \mathbb{Z}_p$. Consider the space $K^n \oplus K^n$ equipped with a skew-symmetric bilinear form with the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. By $\text{Sp}(2n, K)$ we denote the group of all the operators in $K^n \oplus K^n$ preserving this form, we write its elements as block matrices $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

An element of the adelic symplectic group $\text{Sp}(2n, \mathbb{A})$ also can be considered as a sequence $(g_2, g_3, g_5, \ldots)$, where $g_p \in \text{Sp}(2n, \mathbb{Q}_p)$, and $g_p \in \text{Sp}(2n, \mathbb{Z}_p)$ for all the $p$ except finite number.

3.2. Automorphisms of the Heisenberg groups. Let $K = \mathbb{R}, \mathbb{Q}, \mathbb{Q}_p, \mathbb{A}$. The symplectic group $\text{Sp}(2n, K)$ acts on the Heisenberg group $\text{Heis}_n(K)$ by automorphisms

$$\sigma(g) : \{v_+ \oplus v_-\} \oplus \alpha \mapsto \left\{ (v_+ \oplus v_-) \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\} \oplus \alpha,$$

see (2.2).

3.3. Real case.

Theorem 3.1. a) For each $g \in \text{Sp}(2n, \mathbb{R})$, there is a unique up to a factor unitary operator $\text{We}(g) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ such that for each $h \in \text{Heis}_n$

$$T(\sigma(h)) = \text{We}(g)^{-1}T(h)\text{We}(g).$$

b) For each $g_1, g_2 \in \text{Sp}(2n, \mathbb{R})$,

$$\text{We}(g_1)\text{We}(g_2) = c(g_1, g_2)\text{We}(g_1g_2),$$

where $c(g_1, g_2) \in \mathbb{C}$. Moreover, there is a choice of $\text{We}(g)$, such that $c(g_1, g_2) = \pm 1$ for all $g_1, g_2$.

Thus $\text{We}()$ is a projective representation of $\text{Sp}(2n, \mathbb{R})$. It is named the Weil representation.

It is easy to write the operators $\text{We}(g)$ for some special matrices $g$,

$$\text{We} \left( \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \right) f(x) = |\det(A)|^{-1/2} f(xA^t),$$

$$\text{We} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(y) \exp\{i xy\} \, dy,$$

$$\text{We} \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} f(x) = \exp\left( \frac{i}{2} xBx^t \right) \{ f(x) \},$$

where the matrix $B$ is symmetric, $B = B^t$.

Since these elements generate the whole group $\text{Sp}(2n, \mathbb{R})$, our formulae allow to obtain $\text{We}(g)$ for an arbitrary $g \in \text{Sp}(2n, \mathbb{R})$.

Theorem 3.2. The space $\mathcal{P}(\mathbb{R}^{2n})$ is invariant with respect to the action of the group $\text{Sp}(2n, \mathbb{Q})$. 
Proof. Obviously, $\mathcal{P}(\mathbb{R}^n)$ is invariant with respect to operators (3.2), (3.4) with rational matrices $A, B$.

By the Poisson summation formula, $\mathcal{P}(\mathbb{R}^n)$ is invariant with respect to the Fourier transform (3.3).

It can be readily checked that the group $\text{Sp}(2n, \mathbb{R})$ is generated by elements of these 3 types, and this finishes the proof. $\square$

3.4. $p$-adic Weil representation. For the group $\text{Sp}(2n, \mathbb{Q}_p)$, the literal analog of Theorem 3.2 is valid. In this case the operators $\text{We}(g)$ are unitary in $L^2(\mathbb{Q}_p^n)$ and preserve the Bruhat space $\mathcal{B}(\mathbb{Q}_p^n)$.

Analogs of formulae (3.2), (3.3) also can be easily written,

\begin{align*}
(3.5) \quad \text{We} \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} f(x) &= |\det(A)|^{-1/2} f(x A^{-1}), \\
(3.6) \quad \text{We} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f(x) &= \int_{\mathbb{Q}_p^n} f(y) \exp\{2\pi i x y'\} dy, \\
(3.7) \quad \text{We} \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} f(x) &= \exp\{\pi i x B x'\} f(x).
\end{align*}

Remark. After an appropriate normalization of operators $\text{We}(g)$, we can obtain

\begin{equation}
W(g)\mathcal{I}_{\mathbb{Z}_p^n} = \mathcal{I}_{\mathbb{Z}_p^n}, \quad \text{where } g \in \text{Sp}(2n, \mathbb{Z}_p),
\end{equation}

see also Proposition 4.3.

3.5. Adelic Weil representation. We have

\[ L^2(\mathbb{A}^n) = \bigotimes_p \left( L^2(\mathbb{Q}_p^n), \mathcal{I}_{\mathbb{Z}_p^n} \right), \quad \mathcal{B}(\mathbb{A}^n) = \bigotimes_p \left( L^2(\mathbb{Q}_p^n), \mathcal{I}_{\mathbb{Z}_p^n} \right), \]

in the first case we have a tensor product in the category of Hilbert spaces, in the second case we have a tensor product in the category of abstract linear spaces.

Remark. To define a tensor products of an infinite family of spaces $V_j$, we need in a distinguished unit vector $e_j$ in each space, the tensor product space $\bigotimes V_j$ is spanned by products $v_1 \otimes v_2 \otimes \ldots$, where $v_j = e_j$ for all $j$ except a finite set. $\square$

The Weil representation of $\text{Sp}(2n, \mathbb{A})$ is defined as $W(g) = \bigotimes W(g^{(p)})$. These operators are unitary in $L^2(\mathbb{A}^n)$ and preserve the dense subspace $\mathcal{B}(\mathbb{A}^n)$. For almost all $\mathcal{I}_{\mathbb{Z}_p^n}$, we have $\text{We}(g^{(p)})\mathcal{I}_{\mathbb{Z}_p^n} = \mathcal{I}_{\mathbb{Z}_p^n}$ and this allows to define tensor products of operators.

3.6. Proof of Theorem 1.4.b. Transfer the representations of $\text{Sp}(2n, \mathbb{Q})$ from the spaces $\mathcal{B}(\mathbb{A}^n), \mathcal{M}(\mathbb{R}^n)$ to the space $\mathcal{M}(\mathbb{Q}^n)$. We obtain two representations of $\text{Sp}(2n, \mathbb{Q})$ in $\mathcal{M}(\mathbb{Q}^n)$, say $\text{We}_1(g)$, $\text{We}_2(g)$. These operators satisfy the commutation relations

\[ T(\sigma(h)) = \text{We}_1(g)^{-1} T(h) \text{We}_1(g), \quad T(\sigma(h)) = \text{We}_2(g)^{-1} T(h) \text{We}_2(g). \]

Hence $\text{We}_1(g)^{-1} \text{We}_2(g)$ commutes with $T(h)$. By Lemma 2.2 $\text{We}_2(g) = \lambda(g) \text{We}_1(g)$, where $\lambda \in \mathbb{C}$. $\square$

4. Addendum. Constructions of modular forms

Here we explain the standard construction of modular forms from theta-functions and Howe duality, see [9], [1], [5], [4].
4.1. Congruence subgroups. Consider the group $\text{Sp}(2n, \mathbb{Z})$ of symplectic matrices $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with integer elements. For a positive integer $N$, denote by $\Gamma_N$ the principal congruence-subgroup consisting of matrices $g \in \text{Sp}(2n, \mathbb{Z})$ such that $N$ divides all the matrix elements of $g - 1$. A congruence subgroup in $\text{Sp}(2n, \mathbb{Z})$ is any subgroup including a principal congruence-subgroup.

For the following statement, see, for instance, [10]. Theorem 4.1. The subgroup in $U_1 \subset \text{Sp}(2n, \mathbb{Z})$ generated by matrices

\begin{align*}
\begin{pmatrix} 1 + l\alpha & 0 \\ 0 & (1 + l\alpha)^{t-1} \end{pmatrix}, \begin{pmatrix} 1 & l\beta \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ l\gamma & 1 \end{pmatrix},
\end{align*}

where $\alpha, \beta, \gamma, (1 + l\delta)^{-1}$ are integer matrices, is a congruence subgroup.

4.2. The subgroup $\Gamma_{1,2}$. Denote by $\Gamma_{1,2}$ the subgroup of $\text{Sp}(2n, \mathbb{Z})$ consisting of $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that the matrices $A^tC$ and $B^tD$ have even elements on the diagonals. For the following theorem, see [5]. Theorem 4.2. The group $\Gamma_{1,2}$ is generated by matrices

$\begin{pmatrix} A & 0 \\ 0 & A^{t-1} \end{pmatrix}, \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix},$

where the matrices $B, C$ have even diagonals.

Denote

$\Delta(x) = \sum_{k_1, \ldots, k_n} \prod_j \delta(x_j - \sqrt{2\pi}k_j)$

Proposition 4.3. The restriction of the Weil representation of $\text{Sp}(2n, \mathbb{R})$ to $\Gamma_{1,2}$ is a linear representation. Moreover, we can normalize the operators $\text{We}(g), g \in \Gamma_{1,2}$, such that

$\text{We}(g)\Delta = \Delta$ (4.2)

Proof. First, $\Delta$ is an eigenvector for operators $\text{We}(g), g \in \Gamma_{1,2}$. It is easy to verify this for generators of $\Gamma_{1,2}$, and hence this is valid for all $g$. Now we can choose the normalization (4.2). Now $\text{We}(g)$ became a linear representation of $\Gamma_{1,2}$. $\square$

4.3. Congruence subgroups and the space $\mathcal{P}(\mathbb{R}^n)$.

Theorem 4.4. The stabilizer of each element of $\mathcal{P}(\mathbb{R}^n)$ in the group $\Gamma_{1,2}$ is a congruence subgroup.

Proof. It is easy to verify (see Theorem 4.1) that the subgroup $U_{2N^2}$ fix all the vectors of $\mathcal{P}(N^{-1}\mathbb{Z}^n|NZ^n)$. $\square$

4.4. Modular forms of the weight $1/2$. Denote by $W_n$ the Siegel upper half-plane, i.e., the set of $n \times n$ complex matrices satisfying the condition $\frac{1}{2}(z - z^*) > 0$. The group $\text{Sp}(2n, \mathbb{R})$ acts in the space of holomorphic functions on $W_n$ by the following operators

$T_{1/2} \begin{pmatrix} A & B \\ C & D \end{pmatrix} f(x) = f((A + zC)^{-1}(B + zD)) \det(A + zC)^{-1/2}$ (4.3)
Consider the operator

\[ J_\chi(z) = \{ \exp(\frac{1}{2}xzx^t), \chi \} \]

from \( S'(\mathbb{R}^n) \) to our space of holomorphic functions. It is easy to verify, that this operator intertwines the Weil representation and the representation \( T_{1/2} \).

By Proposition 4.3 for \( g \in \Gamma_{1,2} \), we can normalize the operators \( T'_{1/2}(g) = \lambda(g)T_{1/2}(g) \), \( \lambda(g) \in \mathbb{C} \) and obtain a linear representation of \( \Gamma_{1,2} \) (in fact, \( \lambda(g) \) ranges in 8-th roots of 1).

**Proposition 4.5.** Let \( \chi \in \mathcal{P}(\mathbb{R}^n) \) be a Poisson distribution, \( \Phi = J_\chi \). There is a congruence subgroup \( \Gamma \subset \Gamma_{1,2} \) such that

\[ T'_{1/2}(g)\Phi = \Phi \]

where \( g \in \Gamma \).

In fact, Theorem 4.4 provides lot of possibilities to produce modular forms. For instance, consider some embedding \( I : SL(2, \mathbb{R}) \to Sp(2n, \mathbb{R}) \) such that \( i(SL(2, \mathbb{Q})) \subset Sp(2n, \mathbb{Q}) \). Assume that the restriction of the Weil representation to \( SL(2, \mathbb{R}) \) contains a subrepresentation \( V \) of a discrete series\(^3\). Then we can consider projection of the space \( \mathcal{P}(\mathbb{R}^n) \) to \( V \).

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\(^3\)On representations of \( SL(2, \mathbb{R}) \), see, for instance [2].