ANYONIC BEHAVIOR OF QUANTUM GROUP FERMIONIC AND BOSONIC SYSTEMS

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Introduction

The role of quantum groups and quantum Lie algebras in physics has its origin in the theory of vertex models and the quantum inverse scattering method. From the mathematical point of view, two of the most important developments have been their understanding in terms of the theory of noncommutative Hopf algebras and their relation to non-commutative geometry.

In recent years the study of quantum groups and quantum algebras has greatly diversified into several areas of theoretical physics. Based on quantum group ideas, a considerable amount of work was devoted towards a formulation of the so called $q$-deformed physical systems. These approaches are attempts to develop more general formulations of quantum mechanics and field theory. The main motivation behind this type of projects resides

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1Talk given at International Conference on Orbis Scientiae 1997, January 23-26 Miami, Florida
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in searching for new roles that quantum groups could play in physics other than the theory of integrable models. A successful and consistent formulation of a theory involving quantum group symmetries will have the potential of having new features no present in the standard $q \to 1$ case. Besides, it will provide a more general, or alternative, framework to explain physical phenomena.

In this article we show the role that quantum group symmetries, in particular $SU_q(2)$, play in a thermodynamic system at high temperatures. We first display the quantum group covariant algebras, which will be used to build quantum group invariant hamiltonians, and then we will discuss the behavior of the corresponding quantum group gases at high temperatures, and show how the parameter $q$ interpolates between a wide range of attractive and repulsive systems.

**Quantum Group Covariant Algebras**

As it is well known, boson and fermions operators satisfy

\[
\phi_i \phi_j^\dagger - \phi_j^\dagger \phi_i = \delta_{ij} \\
\psi_i \psi_j^\dagger + \psi_j^\dagger \psi_i = \delta_{ij},
\]

which, for $i, j = 1, \ldots, N$, are covariant under $SU(N)$ transformations. For the case of unitary quantum group matrices $T$ the coefficients do not commute but satisfy for $N = 2$ the following algebraic relations

\[
T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
abla = q^{-1}ba, \quad a c = q^{-1}c a \\
b c = c b, \quad d c = q c d \\
abla = q b d, \quad d a - a d = (q - q^{-1}) b c \\
d e q T \equiv a d - q^{-1}b c = 1,
\]
with the unitary condition $[11] \pi = d, \overline{\pi} = q^{-1}c$ and $q \in \mathbb{R}$. Hereafter, we take $0 \leq q < \infty$.

A natural question to address is which are the quantum group analogues of Equation (1), which will tell us for example how to build quantum group invariant hamiltonians. The operator algebras covariant under the action of $SU_q(N)$ matrices were given in [12]

$$\Omega_j \Omega_s = \delta_{ij} \pm q^{\pm 1} R_{kijl} \Omega_k \Omega_l, \quad (4)$$

$$\Omega_i \Omega_k = \pm q^{\mp 1} R_{ijkl} \Omega_j \Omega_l, \quad (5)$$

where $\Omega = \Phi$, $\Psi$ and the upper (lower) sign applies to quantum group bosons $\Phi_i$ (quantum group fermions $\Psi_i$) operators. The $N^2 \times N^2$ matrix $R_{ijkl}$ is explicitly written as $[7]$

$$R_{ijkl} = \delta_{jk} \delta_{il} (1 + (q - 1) \delta_{ij}) + (q - q^{-1}) \delta_{ik} \delta_{jl} \theta(j - i), \quad (6)$$

where $\theta(j - i) = 1$ for $j > i$ and zero otherwise. Denoting the new fields as $\Omega'_i = \sum_{j=1}^N T_{ij} \Omega_j$, the $SU_q(N)$ transformation matrix $T$ and the $R$-matrix satisfy the well known algebraic relations $[13]$

$$RT_1 T_2 = T_2 T_1 R, \quad (7)$$

and

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}, \quad (8)$$

with the standard embedding $T_1 = T \otimes 1$, $T_2 = 1 \otimes T \in V \otimes V$ and $R_{23} = \delta_{ij} \delta_{j'k'} \in V \otimes V \otimes V$.

In particular, for $N = 2$, Equations (4) and (5) are simply written

a) $SU_q(2) -$ fermions

$$\{\Psi_2, \overline{\Psi}_2\} = 1 \quad (9)$$

$$\{\Psi_1, \overline{\Psi}_1\} = 1 - (1 - q^{-2}) \overline{\Psi}_2 \Psi_2 \quad (10)$$
\[ \Psi_1 \Psi_2 = -q \Psi_2 \Psi_1 \]  
\[ \overline{\Psi}_1 \overline{\Psi}_2 = -q \overline{\Psi}_2 \overline{\Psi}_1 \]  
\[ \{\Psi_1, \Psi_1\} = 0 = \{\Psi_2, \Psi_2\}, \]  
\[ (11) \]  
\[ (12) \]  
\[ (13) \]

b) \( SU_q(2) - \text{bosons} \)

\[ \Phi_2 \overline{\Phi}_2 - q^2 \Phi_2 \Phi_2 = 1 \]  
\[ (14) \]
\[ \Phi_1 \overline{\Phi}_1 - q^2 \Phi_1 \Phi_1 = 1 + (q^2 - 1) \overline{\Phi}_2 \Phi_2 \]  
\[ (15) \]
\[ \Phi_2 \Phi_1 = q \Phi_1 \Phi_2 \]  
\[ (16) \]
\[ \Phi_2 \overline{\Phi}_1 = q \overline{\Phi}_1 \Phi_2, \]  
\[ (17) \]

which for \( q = 1 \) become the fermion and boson algebras respectively. These operator relations are very different than those satisfied by the so called \( q \)-fermions [14] and \( q \)-bosons [15, 16], which are written respectively as

c) \( q \)-fermions

\[ bb^\dagger + qb^\dagger b = q^N \]  
\[ (18) \]
\[ b^\dagger b = [N] \]  
\[ (19) \]
\[ bb^\dagger = [1 - N] \]  
\[ (20) \]
\[ b^2 = 0 = b^{\dagger 2}, \]  
\[ (21) \]

where the bracket \([x] = \frac{q^x - q^{-x}}{q - q^{-1}}\).

d) \( q \)-bosons

\[ a_i a_i^\dagger - q^{-1} a_i^\dagger a_i = q^N, \quad [a_i, a_j^\dagger] = 0 = [a_i, a_j], \]  
\[ (22) \]

It is simple to check that Equations (18)-(22) are not quantum group co-

variant, and therefore a quantum group action on the operators \( b_i \) and \( a_j \)
cannot be defined. Hereafter, we discuss the thermodynamic properties of

the systems described by the simplest quantum group invariant hamiltonians.
Quantum Group Fermion and Boson Models

Quantum Group Fermion Gas

From Equation (13) we see that for quantum group fermions the occupation numbers are restricted to \( m = 0 \) or \( 1 \), and therefore \( SU_q(N) \)-fermions satisfy the Pauli exclusion principle. For a given \( \kappa \), a normalized state is simply written as
\[
\Psi_2^m \Psi_1^n |0\rangle \quad n, m = 0, 1,
\]
and the operator \( M_i \equiv \Psi_i \Psi_i \) satisfy
\[
[M_2, \Psi_1] = 0 = M_1 \Psi_2 - q^2 \Psi_2 M_1.
\]
A representation of the \( \Psi \) operators in terms of ordinary fermions \( \psi_j \) is simply given by the following relations
\[
\Psi_m = \psi_m \prod_{l=m+1}^{N} \left( 1 + (q^{-1} - 1) M_l \right),
\]
\[
\Psi_m = \psi_m^\dagger \prod_{l=m+1}^{N} \left( 1 + (q^{-1} - 1) M_l \right),
\]
where \( M_l = \psi_l^\dagger \psi_l \).

The simplest Hamiltonian one can write in terms of the operators \( \Psi_i \) is simply the one that becomes the free fermion Hamiltonian for \( q = 1 \). It is given by
\[
H_F = \sum_\kappa \varepsilon_\kappa (M_{1,\kappa} + M_{2,\kappa}),
\]
where \( M_{i,\kappa} = \Psi_i \Psi_i \) and \( \{ \Psi_{i,\kappa}, \Psi_{j,\kappa'} \} = 0 \) for \( \kappa \neq \kappa' \). With use of the fermion representation in Equations (25) and (26), the original Hamiltonian becomes the interacting fermion Hamiltonian
\[
H_F = \sum_\kappa \varepsilon_\kappa \left( M_{1,\kappa} + M_{2,\kappa} + (q^{-2} - 1) M_{1,\kappa} M_{2,\kappa} \right).
\]
We see that the parameter \( q \neq 1 \) mixes the two degrees of freedom in a non-trivial way through a quartic interaction term. The grand partition function for this model is simply written as

\[
Z_F = \prod_\kappa \sum_{n=0}^1 \sum_{m=0}^1 e^{-\beta \varepsilon_\kappa (n+m-(1-q^{-2})mn) \epsilon \mu (n+m)}
\]

\[
= \prod_\kappa \left(1 + 2e^{-\beta (\varepsilon_\kappa - \mu)} + e^{-\beta (\varepsilon_\kappa (q^{-2}+1)-2\mu)}\right),
\]

which for \( q = 1 \) becomes the square of a single-fermion-type grand partition function. For a high temperature (or low density) gas, we expand the grand partition function \( Z_F \) in terms of the fugacity \( z \ll 1 \)

\[
\ln Z_F = 4V (2m\pi/h^2\beta)^{3/2} \left[\frac{z^2}{2} - \alpha(q)\frac{z^2}{2} + \gamma(q)\frac{z^3}{3!} + ...\right],
\]

where the functions \( \alpha(q) \) and \( \gamma(q) \) are

\[
\alpha(q) = \frac{1}{2^{3/2}} - \frac{1}{2(q^{-2}+1)^{3/2}}
\]

\[
\gamma(q) = \frac{4}{3^{3/2}} - \frac{3}{(q^{-2}+2)^{3/2}}.
\]

Calculating the average number of particles \( \langle M \rangle = \frac{1}{\beta} \left(\frac{\partial \ln Z_F}{\partial \mu}\right)_{T,V} \) and reverting the equation to write the fugacity in terms of \( \langle M \rangle \) gives for Equation (31)

\[
\ln Z_F = \langle M \rangle \left[1 + \frac{\alpha(q)\langle M \rangle}{2V}\lambda_T^3 - \frac{\langle M \rangle^2}{16V^2}\lambda_T^6 \Lambda + ...\right],
\]

where \( \Lambda = \frac{8\gamma(q)}{3} + 16\alpha^2(q) \) and \( \lambda_T = (h^2\beta/2\pi m)^{1/2} \).

From this equation we can obtain the internal energy \( U = -\frac{\partial \ln Z_F}{\partial \beta} + \mu \langle M \rangle \), the heat capacity \( C_v = \left(\frac{\partial U}{\partial T}\right)_V \) and the entropy \( S = \frac{U - \mu \langle M \rangle}{T} + k \ln Z_F \) as functions of \( \langle M \rangle \). The corresponding equations are

\[
U = \frac{3\langle N \rangle}{2\beta} \left[1 + \frac{\langle M \rangle}{2V}\lambda_T^3 \alpha(q) - \frac{\langle M \rangle^2}{16V^2}\lambda_T^6 \Lambda + ...ight],
\]

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\[ C_v = \frac{3\langle M \rangle k}{2} \left[ 1 - \frac{\langle M \rangle}{4V} \lambda_7^2 \alpha(q) + \frac{\langle M \rangle^2}{8V^2} \lambda_7^6 \Lambda + \ldots \right], \quad (34) \]

\[ S = \langle M \rangle k \left[ \frac{5}{2} - \ln \left( \frac{\langle M \rangle}{2V} \lambda_7^3 \right) + \frac{\langle M \rangle}{4V} \lambda_7^3 \alpha(q) + \ldots \right]. \quad (35) \]

The equation of state is given by the equation

\[ pV = kT\langle M \rangle \left[ 1 + \frac{\langle M \rangle}{2V} \lambda_7^2 \alpha(q) + \ldots \right]. \quad (36) \]

Clearly, all these functions become, for \( q = 1 \), the thermodynamic functions for an ideal fermion gas with two species. The sign of the second virial coefficient depends on the value of \( q \), implying then that the parameter \( q \) interpolates between repulsive and attractive systems. Figure 1 shows a graph of the coefficient \( \alpha(q) \). The function \( \alpha(q) \) takes values in the interval \( 2^{-5/2} \leq \alpha \leq 2^{-3/2} \) for \( 0 \leq q \leq 1 \), vanishes at \( q = 1.96 \) and it gets its lowest value \( \alpha(q) = -2^{-5/2}(\sqrt{2} - 1) \) in the limit \( q \to \infty \). It is important to remark that the second virial coefficient for the ideal boson gas case \( B_{bosons} = -2^{-7/2} \beta^{3/2} < B(q \to \infty, T) = -2^{-5/2}(\sqrt{2} - 1) \beta^{3/2} \), and therefore free bosons are not described in this model.

A natural question to address is whether a similar interpolation occurs at \( D = 2 \). Repeating the previous procedure leads to the equation of state

\[ pA = kT\langle M \rangle \left( 1 + \frac{1}{4(1 + q^2)} \frac{\langle M \rangle}{A} \lambda_7^2 + \ldots \right), \quad (37) \]

wherein the second virial coefficient is positive for all values of \( q \), showing that this model, at \( D = 2 \), describes only interacting fermionic systems.
Quantum Group Boson Gas

A representation of the quantum group bosons in terms of boson operators $\phi_i$ and $\phi_j^\dagger$, according to Equations (14)-(17), is simply given by

\begin{align*}
\Phi_2 &= (\phi_2^\dagger)^{-1}\{N_2\} \quad (38) \\
\Phi_1 &= (\phi_1^\dagger)^{-1}\{N_1\}q^{N_2} \quad (40) \\
\overline{\Phi}_1 &= \phi_1^\dagger q^{N_2}, \quad (41)
\end{align*}

**FIG. 1** The coefficient $\alpha(q)$ for the interval $0 \leq q \leq 5$. The line at $q=1.96$ divides the two regions: $\alpha(q)>0$ and $\alpha(q)<0$ which correspond to fermionic and boson-like behaviors respectively.
where the bracket $\{ x \} = \frac{1}{1-q^2}$ and the boson number operator $N_i = \phi_i^\dagger \phi_i$. Therefore, the simplest quantum group invariant Hamiltonian $\mathcal{H}_B$ is

$$\mathcal{H}_B = \sum_\kappa \varepsilon_\kappa (N_{1,\kappa} + N_{2,\kappa}), \quad (42)$$

with $[\Phi_{i,\kappa}, \Phi_{j,\kappa'}] = 0$ for $\kappa \neq \kappa'$, becomes the interacting bosonic Hamiltonian

$$\mathcal{H}_B = \sum_\kappa \varepsilon_\kappa \{ \phi_{1,\kappa}^\dagger \phi_{1,\kappa} + \phi_{2,\kappa}^\dagger \phi_{2,\kappa} \}, \quad (43)$$

with the bracket $\{ x \}$ as defined below Equation (41). Now, it is simple to write the grand partition function $Z_B$ for this model. Introducing the chemical potential $\mu$ in the usual way gives

$$Z_B = \prod_\kappa \sum_{n=0}^\infty \sum_{m=0}^\infty e^{-\beta \varepsilon_\kappa (n+m)} e^{\beta \mu (n+m)}, \quad (44)$$

such that after rearrangement of equal power terms it simplifies to the expression

$$Z_B = \prod_\kappa \sum_{m=0}^\infty (m+1) e^{-\beta \varepsilon_\kappa m} z^m. \quad (45)$$

In $D = 3$ the first few terms in powers of $z$ read

$$\ln Z_B = \frac{4\pi V}{h^3} \int_0^\infty dp [2 e^{-\beta \varepsilon_\kappa} + (6 e^{-\beta \varepsilon_\kappa} - 4 e^{-\beta \varepsilon_\kappa^2}) \frac{z^2}{2} + (24 e^{-\beta \varepsilon_\kappa^3} - 36 e^{-\beta \varepsilon_\kappa^2} + 16 e^{-\beta \varepsilon_\kappa^3}) \frac{z^3}{3!} + ...], \quad (46)$$

such that performing the elementary integrations gives

$$\ln Z_B = \frac{4\pi V}{h^3} \left( \sqrt{\frac{m}{2}} \frac{3}{\beta} \right)^{3/2} \frac{1}{\sqrt{\frac{m}{2}}} \sqrt{\frac{2m}{\beta}} \delta(q) z^2 + ... \right), \quad (47)$$

where $\delta(q) = \frac{1}{4} \left( \frac{3}{1+q^2} \right)^{3/2} - \frac{1}{\sqrt{2}}$. Calculating the average number of particles $\langle N \rangle = \frac{1}{\beta} \left( \frac{\beta \ln z}{\partial \mu} \right)_{T,V}$ and reverting the equation we find for the fugacity

$$z \approx \frac{1}{2} \left( \frac{h^2}{2m\pi kT} \right)^{3/2} \frac{\langle N \rangle}{V} - \delta(q) \left( \frac{h^2}{2m\pi kT} \right)^3 \left( \frac{\langle N \rangle}{V} \right)^2. \quad (48)$$
The internal energy, heat capacity and entropy functions in terms of the average number of particles $\langle N \rangle$ and $q$ read

\[
U = \frac{3\langle N \rangle}{2\beta} \left[ 1 - \frac{\langle N \rangle}{V} \lambda T^3 \delta(q) + \frac{\langle N \rangle^2}{V^2} \lambda T^6 \left( 4\delta^2(q) - \frac{\Gamma(q)}{12} \right) + \ldots \right], \quad (49)
\]

\[
C_v = \frac{3k\langle N \rangle}{2} \left[ 1 + \frac{\langle N \rangle}{2V} \lambda T^2 \delta(q) - 2\frac{\langle N \rangle^2}{V^2} \lambda T^6 \left( 4\delta^2(q) - \frac{\Gamma(q)}{12} \right) + \ldots \right], \quad (50)
\]

\[
S = k\langle N \rangle \left[ \frac{5}{2} - \ln \left( \frac{\langle N \rangle}{2V} \lambda T^3 \right) - \frac{\langle N \rangle}{2V} \lambda T^3 \delta(q) + \ldots \right], \quad (51)
\]

where the function $\Gamma(q) = \frac{12}{3q^2} - \frac{18}{(2+q^2)3q^2} + \frac{8}{3q^2}$. The equation of state for this model is more interesting than for the $SU_q(2)$ fermion gas. For $D = 3$, the equation of state is given by

\[
pV = kT\langle N \rangle \left( 1 - \frac{\langle N \rangle}{V} \lambda T^3 \delta(q) + \ldots \right). \quad (52)
\]

As expected, at $q = 1$ the coefficient $\delta(1) = 2^{-7/2}$, which is the numerical factor in the second virial coefficient for a free boson gas with two species. The free fermion $\delta(q) = -2^{7/2}$ and ideal gas $\delta(q) = 0$ cases are reached at $q \approx 1.78$ and $q \approx 1.27$ respectively.

A similar calculation for $D = 2$ leads to the equation of state

\[
pA = kT\langle N \rangle \left( 1 - \frac{\langle N \rangle}{A} \lambda T^2 \eta(q) + \ldots \right), \quad (53)
\]

with $\eta(q) = \frac{(2-q^2)}{4(1+q^2)}$. Figure 2 shows a graph of the coefficient $\eta(q)$ as a function of the parameter $q$ for $D = 2$. The coefficient $\eta(q)$ in Equation (53) takes values in the interval $[-\frac{1}{4}, \frac{1}{2}]$. At $D = 2$ this model behaves as a fermion gas at $q = \sqrt{5}$. 

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Since the $SU_q(2)$ boson gas at $D = 2$ also interpolates completely between bosons and fermions, we can find a relation between the parameter $q$ and the statistical parameter $\alpha$ for an anyon gas \cite{19} of two species. This relation is given by

$$\alpha = 1 - \sqrt{\frac{5 - q^2}{2(1 + q^2)}},$$

(54)

where $0 \leq \alpha \leq 1$. The parameter $q$ interpolates within a larger range of attractive and repulsive systems than the $\alpha$ parameter does.

**FIG. 2.** The coefficient $\eta(q)$ for the interval $0 \leq q \leq 5$. At the values $q=1$ and $q=\sqrt{5}$ the system behaves as a free boson and fermion gas respectively.
Discussion

In this article we have discussed the high temperature behavior of quantum group gases. We considered the two simplest quantum group invariant Hamiltonians, which are those that become for \( q = 1 \) the free fermion or boson gases with two species. A representation of the quantum group fermions in terms of ordinary fermions leads to a fermion system with a quartic interaction whose coupling constant vanishes as \( q \to 1 \). At high temperatures we analyzed the equation of state at \( D = 2 \) and \( D = 3 \) spatial dimensions. At \( D = 2 \) the second virial coefficient is always positive for all values of \( q \), therefore in two dimensional space this model describes only interacting fermion systems. At \( D = 3 \) the sign of the second virial coefficient depends of the value of \( q \), showing then that the parameter \( q \) interpolates between repulsive and attractive behavior. The ideal gas case corresponds to \( q = 1.96 \) and the system becomes repulsive for \( q < 1.96 \). For \( q > 1.96 \) the system becomes attractive, but as \( q \to \infty \) the free boson limit is not reached, and therefore this model does not interpolate completely between the free fermion and free boson cases.

For \( SU_q(2) \) bosons the results are more interesting. A representation of the quantum group boson operators in terms of ordinary bosons leads to a hamiltonian in terms of ordinary boson interactions involving powers of the number operators and \( \ln q \). For \( D = 2 \) and \( D = 3 \) the \( q \) parameter interpolates completely between the free boson and free fermion cases. For \( D = 2 \), a comparison with the anyon statistical parameter shows that the parameter \( q \) interpolates within a larger range of systems.

Thus, at high temperatures the interactions that result by imposing \( SU_q(2) \) symmetry in the simplest hamiltonian are such that these models, and in particular the quantum group boson model, offer an alternative approach in describing systems obeying fractional statistics in two and three spatial dimensions.
References

[1] See for example M. Jimbo, ed., Advanced Series in Mathematical Physics, Vol. 10 Yang-Baxter equation in integrable systems, (World Scientific, Singapore, 1990).

[2] H. Saleur and J.-B. Zuber, Integrable lattice models and quantum groups, in: Proceedings of the Trieste Spring School String Theory and Quantum Gravity, eds. M. Green, R. Iengo, S. Randjbar-Daemi, E. Sezgin and H. Verlinde, (World Scientific, Singapore, 1992) and references therein.

[3] L. Fadeev, Integrable models in (1+1)- dimensional quantum field theory in Recent Advances in Field Theory and Statistical Mechanics, Les Houches 1982, eds. J.-B. Zuber and R. Stora, (North Holland 1984).

[4] V. Drinfeld, Hopf algebras and the quantum Yang-Baxter equation, Sov. Math. Dokl. 32:254 (1985).

[5] S. L. Woronowicz, Twisted SU(2) group. An example of a non-commutative differential calculus, Publ. RIMS 23:117 (1987).

[6] Yu. I. Manin, Multiparametric quantum deformation of the general linear supergroup, Comm. Math. Phys. 123:163 (1989).

[7] J. Wess and B. Zumino, Covariant differential calculus on the quantum hyperplane, Nucl. Phys. B18(Proc. Suppl.):302 (1990).

[8] M. R. Ubriaco, Quantum deformations of quantum mechanics, Mod. Phys. Lett. A8:89 (1993); and references therein.

[9] M. R. Ubriaco, Complex q analysis and scalar field theory on a q lattice, Mod. Phys. Lett. A9:1121 (1994); and references therein.
[10] A. Sudbery, $SU_q(n)$ gauge theory, *Phys. Lett.* B375:75 (1996).

[11] S. Vokos, B. Zumino and J. Wess, Properties of quantum $2 \times 2$ matrices, in: *Symmetry in Nature*, Scuola Normale Superiore Publ., Pisa (1989).

[12] M. R. Ubriaco, Quantum group Schrödinger field theory, *Mod. Phys. Lett.* A8:2213 (1993); A10:2223(E) (1995).

[13] L. A. Takhtajan, Quantum groups and integrable models, *Adv. Stud. Pure Math.* 19:1 (1989).

[14] Y.J. Ng, Comment on the $q$-analogues of the harmonic oscillator, *J. Phys.* A:23, 1203 (1990).

[15] A. J. Macfarlane, On $q$-analogues of the quantum harmonic oscillator and the quantum group $SU_q(2)$, *J. Phys.* A22:4581 (1989).

[16] L. C. Biedenharn, The quantum group $SU_q(2)$ and a $q$-analog of the boson operators, *J. Phys.* A22:873 (1989).

[17] M. R. Ubriaco, High and low temperature behavior of a quantum group fermion gas, *Mod. Phys. Lett.* A11:2325 (1996).

[18] M. R. Ubriaco, Anyonic behavior of quantum group gases, *Phys. Rev E* 55:291 (1997).

[19] D. Arovas, Topics in fractional statistics, in: *Geometric Phases in Physics*, A. Shapere and F. Wilczek, ed., (World Scientific, Singapore, 1989).