Positive Solutions for a System of Coupled Semipositone Fractional Boundary Value Problems with Sequential Fractional Derivatives

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Abstract: We study the existence and multiplicity of positive solutions for a system of Riemann–Liouville fractional differential equations with sequential derivatives, positive parameters and sign-changing singular nonlinearities, subject to nonlocal coupled boundary conditions which contain Riemann–Stieltjes integrals and various fractional derivatives. In the proof of our main existence results we use the nonlinear alternative of Leray–Schauder type and the Guo–Krasnosel’skii fixed point theorem.

Keywords: Riemann–Liouville fractional differential equations; nonlocal boundary conditions; sign-changing functions; singular functions; existence; multiplicity

MSC: 34A08; 34B10; 34B16; 34B18; 45G15

1. Introduction

We consider the system of nonlinear ordinary fractional differential equations with sequential derivatives

\[
\begin{align*}
D_0^α_i \left(D_0^β_i u(t)\right) + λf(t, u(t), v(t)) &= 0, \quad t ∈ (0, 1), \\
D_0^β_i \left(D_0^γ_i v(t)\right) + µg(t, u(t), v(t)) &= 0, \quad t ∈ (0, 1),
\end{align*}
\]

supplemented with the nonlocal coupled boundary conditions

\[
\begin{align*}
u^{(j)}(0) &= 0, \quad j = 0, \ldots, n - 2; \quad D_0^β_i u(0) = 0, \quad D_0^γ_i u(1) = \sum_{i=1}^{p} \int_{0}^{1} D_0^γ_i v(t) dH_i(t), \\
v^{(j)}(0) &= 0, \quad j = 0, \ldots, m - 2; \quad D_0^β_i v(0) = 0, \quad D_0^γ_i v(1) = \sum_{i=1}^{q} \int_{0}^{1} D_0^γ_i u(t) dK_i(t),
\end{align*}
\]

where \(α_i, β_i, γ_i \in (0, 1], \quad β_1 ∈ (n - 1, n], \quad β_2 ∈ (m - 1, m], \quad n, m ∈ \mathbb{N}, \quad n, m ≥ 3, \quad p, q ∈ \mathbb{N}, \quad γ_i ∈ \mathbb{R} \) for all \(i = 0, 1, \ldots, p, \quad 0 ≤ γ_1 < γ_2 < \cdots < γ_p ≤ δ_0 < β_2 - 1, \quad δ_0 ≥ 1, \quad δ_i ∈ \mathbb{R} \) for all \(i = 0, 1, \ldots, q, \quad 0 ≤ δ_1 < δ_2 < \cdots < δ_q ≤ γ_0 < β_1 - 1, \quad γ_0 ≥ 1, \quad λ > 0, \quad µ > 0, \quad f \) and \(g \) are sign-changing continuous functions that may be singular at \(t = 0\) and/or \(t = 1\), the integrals from the boundary conditions \((2)\) are Riemann–Stieltjes integrals with \(H_i, i = 1, \ldots, p \) and \(K_j, j = 1, \ldots, q \) functions of bounded variation, and \(D_0^ζ_i \) denotes the Riemann–Liouville derivative of order \(ζ \) for \(ζ = α_1, β_1, α_2, β_2, γ_1 \) for \(i = 0, 1, \ldots, p, \delta_j \) for \(j = 0, 1, \ldots, q \).

Under some assumptions on the nonsingular/singular functions \(f \) and \(g \), we present intervals for parameters \(λ \) and \(µ \) such that problem \((1)\) and \((2)\) has at least one or two
positive solutions. So, our problem (1) and (2) is a semipositive problem. A positive solution of (1) and (2) is a pair of functions \((u, v) \in (C([0, 1], \mathbb{R}_+))^2\) which satisfy the system (1) and the boundary conditions (2), with \(u(t) > 0\) for all \(t \in [0, 1]\) or \(v(t) > 0\) for all \(t \in (0, 1]\). In the main existence results we apply the nonlinear alternative of Leray–Schauder type and the Guo–Krasnosel’skiĭ fixed point theorem, (see [1]). We present below some recent results related to our problem (1) and (2). The system

\[
\begin{cases}
D_{0+}^{\alpha_1}(\varphi_{q_1}(D_{0+}^{\beta_1} u(t)))+ \lambda f(t, u(t), v(t)) = 0, & t \in (0, 1), \\
D_{0+}^{\alpha_2}(\varphi_{q_2}(D_{0+}^{\beta_2} v(t)))+ \mu g(t, u(t), v(t)) = 0, & t \in (0, 1),
\end{cases}
\]

(3)

where \(\alpha_1, \alpha_2 > 1, \varphi_{q_i}(s) = |s|^{q_i-2}s, i = 1, 2, f \) and \(g \) are nonnegative nonsingular functions, supplemented with the boundary conditions (2) was studied in [2], by using the Guo–Krasnosel’skiĭ fixed point theorem. In [2], the authors present various intervals for parameters \(\lambda \) and \(\mu \), and conditions for the nonlinearities of the system such that positive solutions exist or not. The existence and multiplicity of positive solutions for the system (3) without parameters \((\lambda = \mu = 1)\) and with nonnegative nonlinearities \(f \) and \(g \) which can be singular at \(t = 0 \) and/or \(t = 1\), supplemented with the uncoupled boundary conditions

\[
\begin{cases}
u^{(j)}(0) = 0, & j = 0, \ldots, n-2; \quad D_{0+}^{\beta_1} v(0) = 0, \quad D_{0+}^{\gamma_i} v(1) = \sum_{i=1}^{p} \int_{0}^{1} D_{0+}^{\gamma_i} v(t) dH_i(t), \\
u^{(j)}(0) = 0, & j = 0, \ldots, m-2; \quad D_{0+}^{\beta_2}v(0) = 0, \quad D_{0+}^{\gamma_j}v(1) = \sum_{i=1}^{q} \int_{0}^{1} D_{0+}^{\gamma_i}v(t) dK_i(t),
\end{cases}
\]

(4)

where \(p, q \in \mathbb{N}, \gamma_i \in \mathbb{R}\) for all \(i = 0, \ldots, p, 0 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_p \leq \gamma_0 < \beta_1 - 1, \gamma_0 \geq 1, \delta_i \in \mathbb{R}\) for all \(i = 0, \ldots, q, 0 \leq \delta_1 < \delta_2 < \cdots < \delta_q \leq \delta_0 < \beta_2 - 1, \delta_0 \geq 1\), has been investigated in the paper [3], by using the Guo–Krasnosel’skiĭ fixed point theorem. The system (3) with the uncoupled multi-point boundary conditions

\[
\begin{cases}
\nu^{(j)}(0) = 0, & j = 0, \ldots, n-2; \quad D_{0+}^{\beta_1} u(0) = 0, \quad D_{0+}^{\gamma_i} u(1) = \sum_{i=1}^{N} \int_{0}^{1} D_{0+}^{\gamma_i} u(t) dH_i(t), \\
\nu^{(j)}(0) = 0, & j = 0, \ldots, m-2; \quad D_{0+}^{\beta_2}v(0) = 0, \quad D_{0+}^{\gamma_j}v(1) = \sum_{i=1}^{M} \int_{0}^{1} D_{0+}^{\gamma_i}v(t) dK_i(t),
\end{cases}
\]

(5)

where \(p_1, p_2, q_1, q_2 \in \mathbb{R}, p_1 \in [1, n-2], p_2 \in [1, m-2], q_1 \in [0, p_1], q_2 \in [0, p_2], \xi_i, a_i \in \mathbb{R}\) for all \(i = 1, \ldots, N (N \in \mathbb{N}), 0 \leq \xi_1 < \cdots < \xi_N \leq 1, \eta_i, b_i \in \mathbb{R}\) for all \(i = 1, \ldots, M (M \in \mathbb{N}), 0 < \eta_1 < \cdots < \eta_M \leq 1\), \(f \) and \(g \) are nonnegative and nonsingular functions was studied in [4]. In [4], the author presented conditions for \(f \) and \(g \) and intervals for positive parameters \(\lambda, \mu \) such that the problem (3) and (4) has at least one positive solution or it has no positive solutions. In [5], the author investigated the existence of solutions for the nonlinear system of fractional differential equations

\[
\begin{cases}
D_{0+}^{\alpha} x(t) + f(t, x(t), y(t), D_{0+}^{\beta_1} x(t), D_{0+}^{\beta_2} y(t)) = 0, & t \in (0, 1), \\
D_{0+}^{\beta} y(t) + g(t, x(t), y(t), D_{0+}^{\theta_1} x(t), D_{0+}^{\theta_2} y(t)) = 0, & t \in (0, 1),
\end{cases}
\]

(6)

with the coupled nonlocal boundary conditions

\[
\begin{cases}
x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \quad D_{0+}^{\gamma_i} x(1) = \sum_{i=1}^{p} \int_{0}^{1} D_{0+}^{\gamma_i} x(t) dH_i(t), \\
y(0) = y'(0) = \cdots = y^{(m-2)}(0) = 0, \quad D_{0+}^{\gamma_j} y(1) = \sum_{i=1}^{q} \int_{0}^{1} D_{0+}^{\gamma_i} y(t) dK_i(t),
\end{cases}
\]

where \(a, \beta \in \mathbb{R}, a \in (n-1, n), \beta \in (m-1, m], n, m \in \mathbb{N}, n \geq 2, m \geq 2, \theta_1, \theta_2, \sigma_1, \sigma_2 > 0, p, q \in \mathbb{N}, \gamma_i \in \mathbb{R}\) for all \(i = 0, \ldots, p, 0 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_p < \beta - 1, \gamma_0 \in [0, a - 1)\), \(\delta_i \in \mathbb{R}\) for all \(i = 0, \ldots, q, 0 \leq \delta_1 < \delta_2 < \cdots < \delta_q < a - 1, \delta_0 \in [0, \beta - 1)\), \(I_{\alpha-i}^\beta \) is the Riemann–Liouville integral of order \(\zeta \) (for \(\zeta = \theta_1, \sigma_1, \sigma_2, \sigma_2, \) \(f \) and \(g \) are nonlinear functions, and the integrals from the boundary conditions \((BC)\) are Riemann–Stieltjes integrals with \(H_i\) for \(i = 1, \ldots, p \) and \(K_i\) for \(i = 1, \ldots, q \) functions of bounded variation. She proved the existence of a unique solution of problem (5) and (6) by using the Banach contraction
mapping principle, and five existence results by applying the Leray–Schauder alternative theorem, the Krasnosel’skii theorem for the sum of two operators (for two results), the Schauder fixed point theorem, and the nonlinear alternative of Leray–Schauder type, respectively. In [6], the authors studied the existence of multiple positive solutions for the nonlinear fractional differential equation

\[ D_{0+}^α u(t) + f(t, u(t)) = 0, \quad t \in (0,1), \]

with the integral-differential boundary conditions

\[ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad D_{0+}^α u(1) = \sum_{i=1}^{m} \int_{0}^{1} D_{0+}^{β_i} u(t) \, dH_i(t), \]

where \( α \in \mathbb{R}, \, α \in (n - 1, n], \, m \in \mathbb{N}, \, n \geq 3, \, β_i \in \mathbb{R} \) for all \( i = 0, \ldots, m, \, 0 \leq β_1 < β_2 < \cdots < β_m \leq β_0 < α - 1, \, H_i, \, i = 1, \ldots, m \) are functions of bounded variation, and the nonlinearity \( f(t, u) \) may change sign and may be singular at the points \( t = 0, 1 \) and/or \( u = 0 \). In the proof of the main theorem, they used various height functions of \( f \) defined on special bounded sets, and two theorems from the fixed point index theory. In [7], the authors investigated the existence of positive solutions for the system of fractional differential equations

\[
\begin{align*}
D_{0+}^α u(t) + λ f(t, u(t), v(t)) &= 0, \quad t \in (0,1), \\
D_{0+}^β v(t) + μ g(t, u(t), v(t)) &= 0, \quad t \in (0,1),
\end{align*}
\]

subject to the coupled integral boundary conditions

\[
\begin{align*}
u(0) = ν'(0) &= \cdots = ν^{(m-2)}(0) = 0, \quad D_{0+}^p ν(1) = \int_{0}^{1} ν(s) \, dH(s), \\
v(0) = ν'(0) &= \cdots = ν^{(m-2)}(0) = 0, \quad D_{0+}^q ν(1) = \int_{0}^{1} ν(s) \, dK(s),
\end{align*}
\]

where \( α, \, β \in \mathbb{R}, \, α \in (n - 1, n], \, β \in (m - 1, m], \, n, \, m \in \mathbb{N}, \, n, \, m \geq 3, \, p, \, q \in \mathbb{R}, \, p \in [1, n - 2], \, q \in [1, m - 2], \) the integrals from (8) are Riemann–Stieltjes integrals with \( H \) and \( K \) functions of bounded variation, \( λ \) and \( μ \) are positive parameters, and \( f \) and \( g \) are sign-changing continuous functions which may be singular at \( t = 0 \) and/or \( t = 1 \). In [7], the authors present various assumptions on the nonlinearities \( f \) and \( g \) and intervals for \( λ \) and \( μ \) such that the problem (7) and (8) has at least one positive solution. In [8], the authors studied the existence and multiplicity of positive solutions for the system (7) with \( λ = μ = 1 \), subject to the coupled multi-point boundary conditions

\[
\begin{align*}
u(0) = ν'(0) &= \cdots = ν^{(m-2)}(0) = 0, \quad D_{0+}^{p_i} ν(1) = \sum_{i=1}^{N} a_i ν(t_i), \\
v(0) = ν'(0) &= \cdots = ν^{(m-2)}(0) = 0, \quad D_{0+}^{q_i} ν(1) = \sum_{i=1}^{M} b_i ν(t_i),
\end{align*}
\]

where \( p_1, \, p_2, \, q_1, \, q_2 \in \mathbb{R}, \, p_1 \in [1, n - 2], \, p_2 \in [1, m - 2], \, q_1 \in [0, p_2], \, q_2 \in [0, p_1] \), \( ξ_i, \, a_i \in \mathbb{R} \) for all \( i = 1, \ldots, N, \, (N \in \mathbb{N}) \), \( 0 < ξ_1 < \cdots < ξ_N < 1, \, η_i, \, b_i \in \mathbb{R} \) for all \( i = 1, \ldots, M, \, (M \in \mathbb{N}) \), \( 0 < η_1 < \cdots < η_M < 1 \), and the functions \( f \) and \( g \) are nonegative and they can be nonsingular or singular at the points \( t = 0 \) and/or \( t = 1 \). They used some theorems from the fixed point index theory and the Guo–Krasnosel’skii fixed point theorem. In [9], the author investigated the existence and nonexistence of positive solutions for a system with three Riemann–Liouville fractional differential equations with positive parameters, nonegative and nonsingular nonlinearities, supplemented with uncoupled multi-point boundary conditions, by using (for the existence) the Guo–Krasnosel’skii fixed point theorem. In [10], the authors studied the existence and nonexistence of positive solutions
for the system (7) with nonnegative and nonsingular functions $f$ and $g$, subject to the coupled boundary conditions

$$
\begin{align*}
&u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u'(1) = \int_0^1 v(s) \, dH(s), \\
&v(0) = v'(0) = \cdots = v^{(m-2)}(0) = 0, \quad v'(1) = \int_0^1 u(s) \, dK(s),
\end{align*}
$$

where $H$ and $K$ are nondecreasing functions. For other recent studies on fractional differential equations and systems see the papers [11–26], and the books [27,28]. We also mention the books [29–35], and their references, where the authors present applications of the fractional calculus and fractional differential equations in many scientific and engineering domains.

The semipositone boundary value problems are more difficult to solve than other problems with nonnegative and singular/nonsingular nonlinearities. Motivated by [6,7], in the present paper, we consider $\varrho_1 = \varrho_2 = 2$ in the system (3), and sign-changing and singular nonlinearities $f$ and $g$, with the general nonlocal boundary conditions (2). We were able to apply the change of functions (see Section 3 and problem (14) and (15)) only for these values of $\varrho_1$ and $\varrho_2$. So our paper was also motivated by the application of $p$-Laplacian operators in various fields such as fluid flow through porous media, nonlinear elasticity, glaciology, etc., (see [36] and its references).

The paper is organized as follows. In Section 2, we study a nonlocal boundary value problem for fractional differential equations with sequential fractional derivatives, and we give some properties of the associated Green functions. Section 3 is devoted to the main existence theorems for the positive solutions of problem (1) and (2). In Section 4, we present two examples which illustrate our results, and Section 5 contains the conclusions for the paper.

2. Preliminary Results

We consider the system of fractional differential equations

$$
\begin{align*}
&\frac{D^{\alpha_1}}{D^{\alpha_2}_0} \left( \frac{D^{\beta_1}}{D^{\beta_2}_0} u(t) \right) + h(t) = 0, \quad t \in (0, 1), \\
&\frac{D^{\alpha_1}}{D^{\alpha_2}_0} \left( \frac{D^{\beta_1}}{D^{\beta_2}_0} v(t) \right) + k(t) = 0, \quad t \in (0, 1),
\end{align*}
$$

subject to the coupled boundary conditions (2), where $h, k \in C(0, 1) \cap L^1(0, 1)$.

We denote by

$$
\begin{align*}
\Delta_1 = \sum_{i=1}^{\nu} \frac{1}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 \tau^{\beta_2 - \gamma_i - 1} \, dH_i(\tau), \\
\Delta_2 = \sum_{i=1}^{\nu} \frac{1}{\Gamma(\beta_1 - \delta_i)} \int_0^1 \tau^{\beta_1 - \delta_i - 1} \, dK_i(\tau), \\
\Delta = \frac{\Gamma(\beta_1) \Gamma(\beta_2)}{\Gamma(\beta_1 - \gamma_0) \Gamma(\beta_2 - \delta_0)} - \Delta_1 \Delta_2 \Gamma(\beta_1) \Gamma(\beta_2).
\end{align*}
$$

Using similar arguments as those used in the proof of [2] Lemma 2.3, we obtain the following result.

Lemma 1. If $\Delta \neq 0$, then the unique solution $(u, v) \in C[0, 1] \times C[0, 1]$ of problem (9) and (2) is given by

$$
\begin{align*}
u(t) &= \int_0^1 G_3(t, s) I_{\alpha_1} h(s) \, ds + \int_0^1 G_4(t, s) I_{\alpha_2} k(s) \, ds, \quad \forall t \in [0, 1],
\end{align*}
$$

subject to the coupled boundary conditions (2), where $h, k \in C(0, 1) \cap L^1(0, 1)$.

We denote by

$$
\begin{align*}
\Delta_1 = \sum_{i=1}^{\nu} \frac{1}{\Gamma(\beta_2 - \gamma_i)} \int_0^1 \tau^{\beta_2 - \gamma_i - 1} \, dH_i(\tau), \\
\Delta_2 = \sum_{i=1}^{\nu} \frac{1}{\Gamma(\beta_1 - \delta_i)} \int_0^1 \tau^{\beta_1 - \delta_i - 1} \, dK_i(\tau), \\
\Delta = \frac{\Gamma(\beta_1) \Gamma(\beta_2)}{\Gamma(\beta_1 - \gamma_0) \Gamma(\beta_2 - \delta_0)} - \Delta_1 \Delta_2 \Gamma(\beta_1) \Gamma(\beta_2).
\end{align*}
$$

Using similar arguments as those used in the proof of [2] Lemma 2.3, we obtain the following result.

Lemma 1. If $\Delta \neq 0$, then the unique solution $(u, v) \in C[0, 1] \times C[0, 1]$ of problem (9) and (2) is given by

$$
\begin{align*}
u(t) &= \int_0^1 G_3(t, s) I_{\alpha_1} h(s) \, ds + \int_0^1 G_4(t, s) I_{\alpha_2} k(s) \, ds, \quad \forall t \in [0, 1],
\end{align*}
$$

subject to the coupled boundary conditions (2), where $h, k \in C(0, 1) \cap L^1(0, 1)$.
where the Green functions $G_i$, $i = 1, \ldots, 4$ are

\begin{align}
G_1(t, s) &= g_1(t, s) + \frac{\Gamma(\beta_1 - \gamma_0)}{\Delta} \sum_{i=1}^{q} \int_0^1 g_{1i}(\tau, s) dK_i(\tau), \\
G_2(t, s) &= g_2(t, s) + \frac{\Gamma(\beta_1 - \gamma_0)}{\Delta} \sum_{i=1}^{q} \int_0^1 g_{2i}(\tau, s) dH_i(\tau), \\
G_3(t, s) &= g_3(t, s) + \frac{\Gamma(\beta_1 - \gamma_0)}{\Delta} \sum_{i=1}^{q} \int_0^1 g_{3i}(\tau, s) dK_i(\tau), \\
G_4(t, s) &= g_4(t, s) + \frac{\Gamma(\beta_1 - \gamma_0)}{\Delta} \sum_{i=1}^{q} \int_0^1 g_{4i}(\tau, s) dH_i(\tau),
\end{align}

for all $(t, s) \in [0, 1] \times [0, 1]$, and

\begin{align}
g_1(t, s) &= \frac{1}{\Gamma(\beta_1)} \int_0^1 \left( \int_0^1 \left( \int_0^1 \left( \int_0^1 \right) \right) \right) dK_i(\tau), \\
g_{1i}(t, s) &= \frac{1}{\Gamma(\beta_1)} \int_0^1 \left( \int_0^1 \left( \int_0^1 \left( \int_0^1 \right) \right) \right) dK_i(\tau), \\
g_2(t, s) &= \frac{1}{\Gamma(\beta_2)} \int_0^1 \left( \int_0^1 \left( \int_0^1 \left( \int_0^1 \right) \right) \right) dH_i(\tau), \\
g_{2i}(t, s) &= \frac{1}{\Gamma(\beta_2)} \int_0^1 \left( \int_0^1 \left( \int_0^1 \left( \int_0^1 \right) \right) \right) dH_i(\tau),
\end{align}

for all $i = 1, \ldots, q$ and $j = 1, \ldots, p$.

**Lemma 2.** Assume that $\Delta > 0$, $H_i : [0, 1] \to \mathbb{R}$, $i = 1, \ldots, p$ and $K_i : [0, 1] \to \mathbb{R}$, $j = 1, \ldots, q$ are nondecreasing functions, and there exists $\xi_0 \in \{1, \ldots, p\}$ such that $H_{i_0}(1) > H_{i_0}(0)$, and there exists $\eta_0 \in \{1, \ldots, q\}$ such that $K_{i_0}(1) > K_{i_0}(0)$. Then the functions $G_i$, $i = 1, \ldots, 4$ given by (12) have the properties:

(a1) $G_1(t, s) \leq J_1(s)$ for all $(t, s) \in [0, 1] \times [0, 1]$, where

\[ J_1(s) = h_1(s) + \frac{\Delta_1 \Gamma(\beta_2)}{\Delta} \sum_{i=1}^{q} \int_0^1 g_{1i}(\tau, s) dK_i(\tau), \]

with $h_1(s) = \frac{\Gamma(\beta_1)}{\Gamma(\beta_1)} \int_0^1 \left( \int_0^1 \left( \int_0^1 \left( \int_0^1 \right) \right) \right) dK_i(\tau)$, for all $s \in [0, 1]$;

(a2) $G_1(t, s) \leq \frac{\Delta_1 \Gamma(\beta_2)}{\Delta} \sum_{i=1}^{q} \int_0^1 g_{1i}(\tau, s) dK_i(\tau)$, for all $(t, s) \in [0, 1] \times [0, 1]$;

(b1) $G_2(t, s) \leq J_2(s)$ for all $(t, s) \in [0, 1] \times [0, 1]$, where

\[ J_2(s) = \frac{\Gamma(\beta_2)}{\Delta \Gamma(\beta_2 - \delta_0)} \sum_{i=1}^{q} \int_0^1 g_{2i}(\tau, s) dH_i(\tau), \]

with $\delta_0 = \frac{\Delta_1 \Gamma(\beta_2)}{\Delta} \sum_{i=1}^{q} \int_0^1 g_{1i}(\tau, s) dK_i(\tau)$, for all $s \in [0, 1]$;

(b2) $G_2(t, s) \leq \frac{\Delta_1 \Gamma(\beta_2)}{\Delta} \sum_{i=1}^{q} \int_0^1 g_{2i}(\tau, s) dH_i(\tau)$, for all $(t, s) \in [0, 1] \times [0, 1]$;

(c1) $G_3(t, s) \leq J_3(s)$ for all $(t, s) \in [0, 1] \times [0, 1]$, where

\[ J_3(s) = \frac{\Gamma(\beta_1)}{\Delta \Gamma(\beta_1 - \gamma_0)} \sum_{i=1}^{q} \int_0^1 g_{3i}(\tau, s) dK_i(\tau), \]

with $\gamma_0 = \frac{\Delta_1 \Gamma(\beta_1)}{\Delta} \sum_{i=1}^{q} \int_0^1 g_{3i}(\tau, s) dK_i(\tau)$, for all $s \in [0, 1]$;

(c2) $G_3(t, s) \leq \frac{\Delta_1 \Gamma(\beta_1)}{\Delta} \sum_{i=1}^{q} \int_0^1 g_{3i}(\tau, s) dK_i(\tau)$, for all $(t, s) \in [0, 1] \times [0, 1]$;

(d1) $G_4(t, s) \leq J_4(s)$ for all $(t, s) \in [0, 1] \times [0, 1]$, where

\[ J_4(s) = h_2(s) + \frac{\Delta_1 \Gamma(\beta_1)}{\Delta} \sum_{i=1}^{q} \int_0^1 g_{4i}(\tau, s) dH_i(\tau), \]

with $h_2(s) = \frac{\Gamma(\beta_2)}{\Gamma(\beta_2)} \int_0^1 \left( \int_0^1 \left( \int_0^1 \left( \int_0^1 \right) \right) \right) dH_i(\tau)$, for all $s \in [0, 1]$;

(d2) $G_4(t, s) \leq \frac{\Delta_1 \Gamma(\beta_1)}{\Delta} \sum_{i=1}^{q} \int_0^1 g_{4i}(\tau, s) dH_i(\tau)$, for all $(t, s) \in [0, 1] \times [0, 1]$,

where

\[ g_{1i}(t, s), g_{2i}(t, s), g_{3i}(t, s), g_{4i}(t, s) \text{ are defined as in (12).} \]
The properties from Lemma 2 follow easily from the properties of the functions $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3, \mathfrak{g}_4, i = 1, \ldots, q, j = 1, \ldots, p$ from (13), (see also [2,17]).

**Lemma 3.** Assume that $\Delta > 0$, $H_i$, $i = 1, \ldots, p$ and $K_j$, $j = 1, \ldots, q$ are nondecreasing functions, there exists $i_0 \in \{1, \ldots, p\}$ such that $H_{i_0}(1) > H_{i_0}(0)$, and there exists $j_0 \in \{1, \ldots, q\}$ such that $K_{j_0}(1) > K_{j_0}(0)$, and $h, k \in C([0,1] \cap L^1(0,1)$ with $h(t) \geq 0, k(t) \geq 0$ for all $t \in (0,1)$. Then the solution $(u, v)$ of problem (9) and (2) given by (11) satisfies the inequalities $u(t) \geq 0$, $v(t) \geq 0$ for all $t \in [0, 1]$. Besides, we have the inequalities $u(t) \geq t^{b_1-1} u(\zeta)$ and $v(t) \geq t^{b_2-1} v(\zeta)$ for all $t, \zeta \in [0,1]$.

**Proof.** Under the assumptions of this lemma, by using relations (11) and Lemma 2, we deduce that $u(t) \geq 0$ and $v(t) \geq 0$ for all $t \in [0, 1]$. In addition, for all $t, \zeta \in [0,1]$, we find the following inequalities

\[
\begin{align*}
 u(t) & \geq t^{b_1-1} \left( \int_0^1 J_1(s) t_0^{b_1} h(s) \, ds + \int_0^1 J_2(s) t_0^{b_1} k(s) \, ds \right) \\
 & \geq t^{b_1-1} \left( \int_0^1 \mathfrak{g}_1(\zeta, s) t_0^{b_1} h(s) \, ds + \int_0^1 \mathfrak{g}_2(\zeta, s) t_0^{b_1} k(s) \, ds \right) \\
 & = t^{b_1-1} u(\zeta),
\end{align*}
\]

\[
\begin{align*}
 v(t) & \geq t^{b_2-1} \left( \int_0^1 J_3(s) t_0^{b_2} h(s) \, ds + \int_0^1 J_4(s) t_0^{b_2} k(s) \, ds \right) \\
 & \geq t^{b_2-1} \left( \int_0^1 \mathfrak{g}_3(\zeta, s) t_0^{b_2} h(s) \, ds + \int_0^1 \mathfrak{g}_4(\zeta, s) t_0^{b_2} k(s) \, ds \right) \\
 & = t^{b_2-1} v(\zeta).
\end{align*}
\]

\[\square\]

**3. Existence and Multiplicity of Positive Solutions**

In this section, we investigate the existence and multiplicity of positive solutions for problem (1) and (2) under various assumptions on the sign-changing nonlinearities $f$ and $g$ which may be singular at $t = 0$ and/or $t = 1$, and for some intervals for the parameters $\lambda$ and $\mu$. We present the assumptions that we will use in our results.

(11) $a_1, a_2 \in (0, 1], \beta_1 \in (n - 1, n], \beta_2 \in (m - 1, m], n, m \in \mathbb{N}, n, m \geq 3, p, q \in \mathbb{N},$ $\gamma_i \in \mathbb{R}$ for all $i = 0, 1, \ldots, p$, $0 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_p \leq \delta_0 < \beta_2 - 1, \delta_0 \geq 1$, $\delta_i \in \mathbb{R}$ for all $i = 0, 1, \ldots, q$, $0 \leq \delta_1 < \delta_2 < \cdots < \delta_q \leq \gamma_0 < \beta_1 - 1, \gamma_0 \geq 1$, $\lambda > 0, \mu > 0, H_i : [0, 1] \rightarrow \mathbb{R}$, $i = 1, \ldots, p$ and $K_j : [0, 1] \rightarrow \mathbb{R}$, $j = 1, \ldots, q$ are nondecreasing functions, there exists $i_0 \in \{1, \ldots, p\}$ such that $H_{i_0}(1) > H_{i_0}(0)$, there exists $j_0 \in \{1, \ldots, p\}$ such that $K_{j_0}(1) > K_{j_0}(0)$, and $\Delta > 0$ ($\Delta$ is given by (10)).

(12) The functions $f, g \in C([0,1] \times \mathbb{R}_+ \times \mathbb{R}_+)$ and there exist functions $\xi_1, \xi_2 \in C([0, 1], \mathbb{R}_+)$ such that $f(t, x, y) \geq -\xi_1(t)$ and $g(t, x, y) \geq -\xi_2(t)$ for any $t \in [0, 1]$, $x, y \in \mathbb{R}_+$.

(13) $f(t, 0, 0) > 0, g(t, 0, 0) > 0$ for all $t \in [0, 1]$.

(14) The functions $f, g \in C([0,1] \times \mathbb{R}_+ \times \mathbb{R}_+), f, g$ may be singular at $t = 0$ and/or $t = 1$, and there exist functions $\xi_1, \xi_2 \in C([0, 1], \mathbb{R}_+)$, $\chi_1, \chi_2 \in C([0, 1], \mathbb{R}_+)$, $\psi_1, \psi_2 \in C([0,1] \times \mathbb{R}_+ \times \mathbb{R}_+)$ such that

\[
-\xi_1(t) \leq f(t, x, y) \leq \chi_1(t) \psi_1(t, x, y), \quad -\xi_2(t) \leq g(t, x, y) \leq \chi_2(t) \psi_2(t, x, y),
\]

for all $t \in (0, 1), x, y \in \mathbb{R}_+$, with $0 < \int_0^1 \xi_i(t) \, dt < \infty, 0 < \int_0^1 \chi_i(t) \, dt < \infty, i = 1, 2$.

(15) There exist $0 < \theta_1 < \theta_2 < 1$ such that

\[
f_{\infty} = \lim_{t \rightarrow \infty} \min_{t \in [0, d_2]} f(t, x, y) = \infty \quad \text{or} \quad g_{\infty} = \lim_{t \rightarrow \infty} \min_{t \in [0, d_2]} g(t, x, y) = \infty.
\]
(16) There exists

$$R_1 > \max \left\{ 1, \frac{2\delta_1}{\Gamma(a_1 + 1)} \int_0^1 \zeta_1(\tau) \, d\tau + \frac{2\delta_2}{\Gamma(a_2 + 1)} \int_0^1 \zeta_2(\tau) \, d\tau, \frac{2\delta_3}{\Gamma(a_1 + 1)} \int_0^1 \zeta_1(\tau) \, d\tau + \frac{2\delta_4}{\Gamma(a_2 + 1)} \int_0^1 \zeta_2(\tau) \, d\tau \right\},$$

such that $f(t, x, y) \geq 0$, $g(t, x, y) \geq 0$ for all $t \in (0, 1)$ and $(x, y) \in ([0, \frac{R_0}{2}] \times [0, \infty)) \cup ([0, \infty) \times [0, \frac{R_1}{2}])$.

We consider the system of nonlinear fractional differential equations

$$\begin{cases}
D_{0+}^{\beta_1} \left( D_{0+}^{\alpha_1} x(t) \right) + \lambda f(t, [x(t) - \tilde{\zeta}_1(t)]^*, [y(t) - \tilde{\zeta}_2(t)]^*) + \tilde{\zeta}_1(t) = 0, \quad t \in (0, 1), \\
D_{0+}^{\beta_2} \left( D_{0+}^{\alpha_2} y(t) \right) + \mu g(t, [x(t) - \tilde{\zeta}_1(t)]^*, [y(t) - \tilde{\zeta}_2(t)]^*) + \tilde{\zeta}_2(t) = 0, \quad t \in (0, 1),
\end{cases}$$

with the boundary conditions

$$\begin{align}
x^{(j)}(0) &= 0, \quad j = 0, \ldots, n - 2; \quad D_{0+}^{\beta_1} x(0) = 0, \quad D_{0+}^{\alpha_1} x(1) = \sum_{i=1}^{q} \int_0^1 D_{0+}^{\gamma_i} y(t) \, dH_i(t), \\
y^{(j)}(0) &= 0, \quad j = 0, \ldots, m - 2; \quad D_{0+}^{\beta_2} y(0) = 0, \quad D_{0+}^{\alpha_2} y(1) = \sum_{i=1}^{q} \int_0^1 D_{0+}^{\gamma_i} x(t) \, dK_i(t),
\end{align}$$

where $w(t)^* = w(t)$ if $w(t) \geq 0$ and $w(t)^* = 0$ if $w(t) < 0$. Here $(\tilde{\zeta}_1(t), \tilde{\zeta}_2(t))$, $t \in [0, 1]$ given by

$$\begin{align}
\tilde{\zeta}_1(t) &= \lambda \int_0^t G_1(t, s) D_{0+}^{\alpha_1} \xi_1(s) \, ds + \mu \int_0^t G_2(t, s) D_{0+}^{\alpha_2} \xi_2(s) \, ds, \quad \forall t \in [0, 1], \\
\tilde{\zeta}_2(t) &= \lambda \int_0^t G_3(t, s) D_{0+}^{\alpha_1} \xi_1(s) \, ds + \mu \int_0^t G_4(t, s) D_{0+}^{\alpha_2} \xi_2(s) \, ds, \quad \forall t \in [0, 1],
\end{align}$$

is the solution of the system of fractional differential equations

$$\begin{cases}
D_{0+}^{\beta_1} \left( D_{0+}^{\alpha_1} \xi_1(t) \right) + \lambda \xi_1(t) = 0, \quad t \in (0, 1), \\
D_{0+}^{\beta_2} \left( D_{0+}^{\alpha_2} \xi_2(t) \right) + \mu \xi_2(t) = 0, \quad t \in (0, 1),
\end{cases}$$

with the boundary conditions

$$\begin{align}
\xi_1^{(j)}(0) &= 0, \quad j = 0, \ldots, n - 2; \quad D_{0+}^{\beta_1} \xi_1(0) = 0, \quad D_{0+}^{\alpha_1} \xi_1(1) = \sum_{i=1}^{q} \int_0^1 D_{0+}^{\gamma_i} \xi_2(t) \, dH_i(t), \\
\xi_2^{(j)}(0) &= 0, \quad j = 0, \ldots, m - 2; \quad D_{0+}^{\beta_2} \xi_2(0) = 0, \quad D_{0+}^{\alpha_2} \xi_2(1) = \sum_{i=1}^{q} \int_0^1 D_{0+}^{\gamma_i} \xi_1(t) \, dK_i(t).
\end{align}$$

Under the assumptions (11) and (12), or (11) and (14) we have $\tilde{\zeta}_1(t) \geq 0$, $\tilde{\zeta}_2(t) \geq 0$ for all $t \in [0, 1]$. We shall prove that there exists a solution $(x, y)$ for the boundary value problem (14) and (15) with $x(t) \geq \tilde{\zeta}_1(t)$ and $y(t) \geq \tilde{\zeta}_2(t)$ on $[0, 1]$, $x(t) > \tilde{\zeta}_1(t)$ or $y(t) > \tilde{\zeta}_2(t)$ on $(0, 1]$. In this case $(u, v)$ with $u(t) = x(t) - \tilde{\zeta}_1(t)$ and $v(t) = y(t) - \tilde{\zeta}_2(t)$ for all $t \in [0, 1]$ represent a positive solution of the boundary value problem (1) and (2). Indeed, by (14)–(17) we have for $t \in (0, 1)$

$$\begin{align}
D_{0+}^{\beta_1} \left( D_{0+}^{\alpha_1} u(t) \right) &= D_{0+}^{\beta_1} \left( D_{0+}^{\alpha_1} x(t) \right) - D_{0+}^{\beta_1} \left( D_{0+}^{\alpha_1} \tilde{\zeta}_1(t) \right) \\
&= -\lambda f(t, [x(t) - \tilde{\zeta}_1(t)]^*, [y(t) - \tilde{\zeta}_2(t)]^*) - \lambda \tilde{\zeta}_1(t) + \lambda \tilde{\zeta}_1(t) = -\lambda f(t, u(t), v(t)), \\
D_{0+}^{\beta_2} \left( D_{0+}^{\alpha_2} v(t) \right) &= D_{0+}^{\beta_2} \left( D_{0+}^{\alpha_2} y(t) \right) - D_{0+}^{\beta_2} \left( D_{0+}^{\alpha_2} \tilde{\zeta}_2(t) \right) \\
&= -\mu g(t, [x(t) - \tilde{\zeta}_1(t)]^*, [y(t) - \tilde{\zeta}_2(t)]^*) - \mu \tilde{\zeta}_2(t) + \mu \tilde{\zeta}_2(t) = -\mu g(t, u(t), v(t)),
\end{align}$$

respectively.
and
\[
u^{(j)}(0) = x^{(j)}(0) - \zeta^{(j)}(0) = 0, \quad j = 0, \ldots, n - 2,\]
\[D_{0+}^{\beta_1} u(0) = D_{0+}^{\alpha_1} x(0) - D_{0+}^{\alpha_1} \xi_1(0) = 0,
\]
\[D_{0+}^{\gamma_1} u(1) = D_{0+}^{\gamma_1} x(1) - D_{0+}^{\gamma_1} \zeta_1(1) = \sum_{i=1}^{p} \int_{0}^{1} D_{0+}^{\gamma_i} y(t) dH_i(t),\]
\[\quad = \sum_{i=1}^{p} \int_{0}^{1} D_{0+}^{\gamma_i} v(t) \ dH_i(t),\]
\[v^{(j)}(0) = y^{(j)}(0) - \zeta_2^{(j)}(0) = 0, \quad j = 0, \ldots, m - 2,\]
\[D_{0+}^{\beta_2} v(0) = D_{0+}^{\beta_2} y(0) - D_{0+}^{\beta_2} \zeta_2(0) = 0,
\]
\[D_{0+}^{\delta_2} v(1) = D_{0+}^{\delta_2} y(1) - D_{0+}^{\delta_2} \zeta_2(1) = \sum_{i=1}^{q} \int_{0}^{1} D_{0+}^{\delta_i} x(t) dK_i(t),
\]
\[\quad = \sum_{i=1}^{q} \int_{0}^{1} D_{0+}^{\delta_i} u(t) dK_i(t).\]

So, in what follows we shall investigate the boundary value problem (14) and (15). By using Lemma 1 (relations (11)), the problem ((14) and (15)) is equivalent to the system

\[
\left\{
\begin{array}{l}
x(t) = \lambda \int_{0}^{1} G_1(t,s) f(s, [x(s) - \zeta_1(s)]^*, [y(s) - \zeta_2(s)]^*) + \zeta_1(s) \ ds \\
\quad + \mu \int_{0}^{1} G_2(t,s) f(s, [x(s) - \zeta_1(s)]^* + \zeta_2(s)) ds, \quad \forall t \in [0, 1],
\end{array}
\right.
\]
\[
\left\{
\begin{array}{l}
y(t) = \lambda \int_{0}^{1} G_3(t,s) f(s, [x(s) - \zeta_1(s)]^*, [y(s) - \zeta_2(s)]^*) + \zeta_1(s) \ ds \\
\quad + \mu \int_{0}^{1} G_4(t,s) f(s, [x(s) - \zeta_1(s)]^* + \zeta_2(s)) ds, \quad \forall t \in [0, 1].
\end{array}
\right.
\]

We consider the Banach space \( X = C[0, 1] \) with the supremum norm \( \| \cdot \| \), and the Banach space \( Y = X \times X \) with the norm \( \|(u, v)\|_Y = \|u\| + \|v\| \).

We define the cone
\[\mathcal{P} = \{ (u, v) \in Y, \ u(t) \geq t^{\beta_1-1} \|u\|, \ v(t) \geq t^{\beta_2-1} \|v\|, \ \forall t \in [0, 1] \}.\]

For \( \lambda, \mu > 0 \), we introduce the operators \( A_1, A_2 : Y \to X \) and \( A : Y \to Y \) defined by
\[A(x, y) = (A_1(x, y), A_2(x, y)), \ (x, y) \in Y \]
with
\[A_1(x, y)(t) = \lambda \int_{0}^{1} G_1(t,s) f(s, [x(s) - \zeta_1(s)]^*, [y(s) - \zeta_2(s)]^*) + \zeta_1(s) \ ds \\
\quad + \mu \int_{0}^{1} G_2(t,s) f(s, [x(s) - \zeta_1(s)]^* + \zeta_2(s)) ds, \quad \forall t \in [0, 1],
\]
\[A_2(x, y)(t) = \lambda \int_{0}^{1} G_3(t,s) f(s, [x(s) - \zeta_1(s)]^*, [y(s) - \zeta_2(s)]^*) + \zeta_1(s) \ ds \\
\quad + \mu \int_{0}^{1} G_4(t,s) f(s, [x(s) - \zeta_1(s)]^* + \zeta_2(s)) ds, \quad \forall t \in [0, 1].
\]

It is easy to see that \((x, y) \in \mathcal{P} \) is a solution of problem (14) and (15) if and only if \((x, y)\) is a fixed point of operator \(A\).

**Lemma 4.** If (11) and (12), or (11) and (14) hold, then operator \( A : \mathcal{P} \to \mathcal{P} \) is a completely continuous operator.

**Proof.** The operators \( A_1 \) and \( A_2 \) are well-defined. To prove this, let \((x, y) \in \mathcal{P}\) be fixed with \( \|(x, y)\|_Y = L \), that is \( \|x\| + \|y\| = L \). Then, we have
\[ [x(s) - \zeta_1(s)]^* \leq x(s) \leq \|x\| \leq \|(x, y)\|_Y = L, \forall s \in [0, 1],
\]
\[ [y(s) - \zeta_2(s)]^* \leq y(s) \leq \|y\| \leq \|(x, y)\|_Y = L, \forall s \in [0, 1].\]
If (I1) and (I2) hold, we deduce
\[
\mathcal{A}_1(x, y)(t) \leq \lambda \int_0^1 \mathcal{J}_1(s) t_{\beta_1}^\alpha \left( f(s, [x(s) - \xi_1(s)]^*, [y(s) - \xi_2(s)]^*) + \xi_1(s) \right) ds \\
+ \mu \int_0^1 \mathcal{J}_2(s) t_{\beta_2}^\alpha \left( g(s, [x(s) - \xi_1(s)]^*, [y(s) - \xi_2(s)]^*) + \xi_2(s) \right) ds \\
\leq \frac{2\lambda \tilde{M}_1}{\Gamma(a_1 + 1)} \int_0^1 \mathcal{J}_1(s) ds + \frac{2\mu \tilde{M}_1}{\Gamma(a_2 + 1)} \int_0^1 \mathcal{J}_2(s) ds < \infty, \quad \forall t \in [0, 1],
\]
\[
\mathcal{A}_2(x, y)(t) \leq \lambda \int_0^1 \mathcal{J}_3(s) t_{\beta_1}^\alpha \left( f(s, [x(s) - \xi_1(s)]^*, [y(s) - \xi_2(s)]^*) + \xi_1(s) \right) ds \\
+ \mu \int_0^1 \mathcal{J}_4(s) t_{\beta_2}^\alpha \left( g(s, [x(s) - \xi_1(s)]^*, [y(s) - \xi_2(s)]^*) + \xi_2(s) \right) ds \\
\leq \frac{2\lambda \tilde{M}_1}{\Gamma(a_1 + 1)} \int_0^1 \mathcal{J}_3(s) ds + \frac{2\mu \tilde{M}_1}{\Gamma(a_2 + 1)} \int_0^1 \mathcal{J}_4(s) ds < \infty, \quad \forall t \in [0, 1],
\]
where \( \tilde{M}_1 = \max\{\max_{t \in [0,1]} |f(t, u, v)|, \max_{t \in [0,1]} |g(t, u, v)|, \max_{t \in [0,1]} \xi_1(t), \max_{t \in [0,1]} \xi_2(t)\} \).

If (I1) and (I4) hold, we obtain for all \( t \in [0, 1] \)
\[
\mathcal{A}_1(x, y)(t) \leq \lambda \int_0^1 \mathcal{J}_1(s) t_{\beta_1}^\alpha \left( f(s, [x(s) - \xi_1(s)]^*, [y(s) - \xi_2(s)]^*) + \xi_1(s) \right) ds \\
+ \mu \int_0^1 \mathcal{J}_2(s) t_{\beta_2}^\alpha \left( g(s, [x(s) - \xi_1(s)]^*, [y(s) - \xi_2(s)]^*) + \xi_2(s) \right) ds \\
\leq \lambda \int_0^1 \mathcal{J}_1(s) \left( \frac{1}{\Gamma(a_1)} \int_0^s (s - \tau)^{a_1 - 1} [\chi_1(\tau) \psi_1(\tau, [x(\tau) - \xi_1(\tau)]^*, [y(\tau) - \xi_2(\tau)]^*) + \xi_1(\tau)] d\tau \right) ds \\
+ \mu \int_0^1 \mathcal{J}_2(s) \left( \frac{1}{\Gamma(a_2)} \int_0^s (s - \tau)^{a_2 - 1} [\chi_2(\tau) \psi_2(\tau, [x(\tau) - \xi_1(\tau)]^*, [y(\tau) - \xi_2(\tau)]^*) + \xi_2(\tau)] d\tau \right) ds \\
\leq \frac{\lambda \tilde{M}_2}{\Gamma(a_1 + 1)} \int_0^1 \mathcal{J}_1(s) ds + \frac{\mu \tilde{M}_2}{\Gamma(a_2 + 1)} \int_0^1 \mathcal{J}_2(s) ds < \infty,
\]
where \( \tilde{M}_2 = \max\{\max_{t \in [0,1], x,y \in [0,1]} \psi_1(t, x, y), \max_{t \in [0,1], x,y \in [0,1]} \psi_2(t, x, y), 1\} \), \( J_{10} = \max_{s \in [0,1]} \mathcal{J}_1(s), J_{20} = \max_{s \in [0,1]} \mathcal{J}_2(s) \).

In a similar manner we find
\[
\mathcal{A}_2(x, y)(t) \leq \frac{\lambda \tilde{M}_2 J_{30}}{\Gamma(a_1 + 1)} \int_0^1 (1 - \tau)^{a_1} (\chi_1(\tau) + \xi_1(\tau)) d\tau \\
+ \frac{\mu \tilde{M}_2 J_{40}}{\Gamma(a_2 + 1)} \int_0^1 (1 - \tau)^{a_2} (\chi_2(\tau) + \xi_2(\tau)) d\tau < \infty,
\]
where \( J_{30} = \max_{s \in [0,1]} \mathcal{J}_3(s), J_{40} = \max_{s \in [0,1]} \mathcal{J}_4(s) \). Therefore \( \mathcal{A}_1(x, y) \) and \( \mathcal{A}_2(x, y) \) are well-defined.

Besides, by Lemma 3, we deduce that
\[
\mathcal{A}_1(x, y)(t) \geq t^{\beta_1 - 1} \mathcal{A}_1(x, y)(t'), \quad \mathcal{A}_2(x, y)(t) \geq t^{\beta_2 - 1} \mathcal{A}_2(x, y)(t'), \quad \forall t, t' \in [0, 1],
\]
and so $A_1(x,y)(t) \geq t^{b_1-1}\|A_1(x,y)\|$, $A_2(x,y)(t) \geq t^{b_2-1}\|A_2(x,y)\|$, $\forall t \in [0,1]$. We obtain $(A_1(x,y), A_2(x,y)) \in \mathcal{P}$, and hence $\mathcal{A}(\mathcal{P}) \subset \mathcal{P}$.

By using standard arguments, we conclude that operator $\mathcal{A} : \mathcal{P} \to \mathcal{P}$ is a completely continuous operator. □

**Theorem 1.** We suppose that (11) – (13) hold. Then there exist the constants $\lambda_0 > 0$ and $\mu_0 > 0$ such that for any $\lambda \in (0,\lambda_0]$ and $\mu \in (0,\mu_0]$, the boundary value problem (1) and (2) has at least one positive solution.

**Proof.** Let $\omega \in (0,1)$ be fixed. By (12) and (13), there exists $R_0 \in (0,1]$ such that

$$f(t,x,y) \geq \omega f(t,0,0), \; g(t,x,y) \geq \omega g(t,0,0), \; \forall t \in [0,1], \; x,y \in [0,R_0].$$

We define

$$f_0 = \max_{(t,x,y) \in [0,1] \times [0,R_0]} \{f(t,x,y) + \zeta_1(t)\}, \; g_0 = \max_{(t,x,y) \in [0,1] \times [0,R_0]} \{g(t,x,y) + \zeta_2(t)\},$$

$$A_1 = \int_0^1 J_1(s) ds, \; A_2 = \int_0^1 J_2(s) ds, \; A_3 = \int_0^1 J_3(s) ds, \; A_4 = \int_0^1 J_4(s) ds, \; \lambda_0 = \min \left\{ \frac{\mu_1^0}{\lambda_0}, \frac{\mu_2^0}{\lambda_0}, \frac{\mu_3^0}{\lambda_0}, \frac{\mu_4^0}{\lambda_0} \right\}, \; \mu_0 = \min \left\{ \frac{\mu_1^0}{\lambda_0}, \frac{\mu_2^0}{\lambda_0}, \frac{\mu_3^0}{\lambda_0}, \frac{\mu_4^0}{\lambda_0} \right\}.$$

We see that $f_0 \geq \max_{(t,x,y) \in [0,1]} \{\omega f(t,0,0) + \zeta_1(t)\} > 0$ and $g_0 \geq \max_{(t,x,y) \in [0,1]} \{\omega g(t,0,0) + \zeta_2(t)\} > 0$. We will show that for any $\lambda \in (0,\lambda_0]$ and $\mu \in (0,\mu_0]$ problem (1) and (2) has at least one positive solution. For this, let $\lambda \in (0,\lambda_0]$ and $\mu \in (0,\mu_0]$ be arbitrary, but fixed for the moment. We define the set $E = \{(x,y) \in \mathcal{P}, \; ||(x,y)|| = R_0 \}$. We suppose that there exist $(x,y) \in \partial E$ such that $x = vA_1(x,y)$ and $y = vA_2(x,y)$.

We deduce that

$$[x(t) - \zeta_1(t)]^* = x(t) - \zeta_1(t) \leq x(t) \leq R_0, \; \text{if } x(t) - \zeta_1(t) \geq 0,$$

$$|x(t) - \zeta_1(t)]^* = 0, \; \text{for } x(t) - \zeta_1(t) < 0, \; \forall t \in [0,1],$$

$$[y(t) - \zeta_2(t)]^* = y(t) - \zeta_2(t) \leq y(t) \leq R_0, \; \text{if } y(t) - \zeta_2(t) \geq 0,$$

$$[y(t) - \zeta_2(t)]^* = 0, \; \text{for } y(t) - \zeta_2(t) < 0, \; \forall t \in [0,1].$$

Then by Lemma 2, for all $t \in [0,1]$, we obtain

$$x(t) = vA_1(x,y)(t) \leq A_1(x,y)(t)$$

$$= \lambda \int_0^1 G_1(t,s) f_0^0 ds + \mu \int_0^1 G_2(t,s) g_0^0 ds + \frac{\lambda}{\Gamma(a_1 + 1)} \int_0^1 J_1(s) f_0 ds + \frac{\lambda}{\Gamma(a_2 + 1)} \int_0^1 J_2(s) g_0 ds$$

$$\leq \frac{\lambda}{\Gamma(a_1 + 1)} \int_0^1 J_1(s) f_0 ds + \frac{\mu}{\Gamma(a_2 + 1)} \int_0^1 J_2(s) g_0 ds,$$

$$y(t) = vA_2(x,y)(t) \leq A_2(x,y)(t)$$

$$= \lambda \int_0^1 G_3(t,s) f_0^0 ds + \mu \int_0^1 G_4(t,s) g_0^0 ds + \frac{\lambda}{\Gamma(a_1 + 1)} \int_0^1 J_3(s) f_0 ds + \frac{\mu}{\Gamma(a_2 + 1)} \int_0^1 J_4(s) g_0 ds$$

$$\leq \frac{\lambda}{\Gamma(a_1 + 1)} \int_0^1 J_3(s) f_0 ds + \frac{\mu}{\Gamma(a_2 + 1)} \int_0^1 J_4(s) g_0 ds.$$

Then $x \leq \frac{R_0}{\lambda}$ and $y \leq \frac{R_0}{\mu}$. So $R_0 = ||(x,y)||_Y = ||x|| + ||y|| \leq \frac{R_0}{\lambda}$$, which is a contradiction.

Therefore, by the nonlinear alternative of Leray–Schauder type, we conclude that $\mathcal{A}$ has a fixed point $(x_0,y_0) \in \mathcal{F}$. That is, $(x_0,y_0) = \mathcal{A}(x_0,y_0)$ or equivalently $x_0 = A_1(x_0,y_0)$,
Theorem 2. We suppose that \( \lambda \) for all one positive solution. Let
\[
y_0 = A_2(x_0, y_0), \quad \| (x_0, y_0) \|_Y = \| x_0 \| + \| y_0 \| \leq R_0 \text{ with } x_0(t) \geq t^{\beta_1-1} \| x_0 \|, \quad y_0(t) \geq t^{\beta_2-1} \| y_0 \| \text{ for all } t \in [0, 1].
\]
In addition, by (18), we deduce
\[
x_0(t) = A_1(x_0, y_0)(t) \geq \lambda \int_0^1 G_1(t, s) I_{0+}(t) \omega f(s, 0, 0) + \zeta_1(s) ds + \mu \int_0^1 G_2(t, s) I_{0+}(t) \omega g(s, 0, 0) + \zeta_2(s) ds 
\]
\[
\geq \lambda \int_0^1 G_1(t, s) I_{0+}(t) \omega f(s, 0, 0) + \zeta_1(s) ds = \xi_1(t), \quad \forall \ t \in [0, 1],
\]
\[
y_0(t) = A_2(x_0, y_0)(t) \geq \lambda \int_0^1 G_3(t, s) I_{0+}(t) \omega f(s, 0, 0) + \zeta_1(s) ds + \mu \int_0^1 G_4(t, s) I_{0+}(t) \omega g(s, 0, 0) + \zeta_2(s) ds 
\]
\[
\geq \lambda \int_0^1 G_3(t, s) I_{0+}(t) \omega f(s, 0, 0) + \zeta_1(s) ds = \xi_2(t), \quad \forall \ t \in [0, 1],
\]
Therefore \( x_0(t) \geq \xi_1(t), y_0(t) \geq \xi_2(t) \) for all \( t \in [0, 1] \), \( x_0(t) \geq \xi_1(t), y_0(t) \geq \xi_2(t) \) for all \( t \in (0, 1] \). Let \( u_0(t) = x_0(t) - \xi_1(t) \) and \( v_0(t) = y_0(t) - \xi_2(t) \) for all \( t \in [0, 1] \). Then
\[
u_0(t) \geq 0, v_0(t) \geq 0 \text{ for all } t \in [0, 1], \quad u_0(t) > 0, v_0(t) > 0 \text{ for all } t \in (0, 1].
\]
So \((u_0, v_0)\) is a positive solution of problem (1) and (2).

Theorem 2. We suppose that (11), (14), (15) and (16) hold. Then there exist \( \lambda^* > 0 \) and \( \mu^* > 0 \) such that for any \( \lambda \in (0, \lambda^*] \) and \( \mu \in (0, \mu^*] \), the boundary value problem (1) and (2) has at least one positive solution.

Proof. We consider the positive number \( R_1 \) given by (16), and we define the set \( \Omega_1 = \{ (x, y) \in Y, \| (x, y) \|_Y < R_1 \} \). We introduce the positive constants
\[
\lambda^* = \min \left\{ 1, \frac{R_1}{4M_1} \left( \int_0^1 \mathcal{J}_1(s) I_{0+}^{\alpha_1} (\chi_1(s) + \zeta_1(s)) ds \right)^{-1}, \frac{R_1}{4M_1} \left( \int_0^1 \mathcal{J}_3(s) I_{0+}^{\alpha_1} (\chi_1(s) + \zeta_1(s)) ds \right)^{-1} \right\},
\]
\[
\mu^* = \min \left\{ 1, \frac{R_1}{4M_2} \left( \int_0^1 \mathcal{J}_2(s) I_{0+}^{\alpha_2} (\chi_2(s) + \zeta_2(s)) ds \right)^{-1}, \frac{R_1}{4M_2} \left( \int_0^1 \mathcal{J}_4(s) I_{0+}^{\alpha_2} (\chi_2(s) + \zeta_2(s)) ds \right)^{-1} \right\},
\]
with \( M_1 = \max \{ \max_{t \in [0, 1]} |u, v, 0| = 0, u = v \leq R_1 \psi_3(t, u, v), 1 \} \), \( M_2 = \max \{ \max_{t \in [0, 1]} |u, v, 0| = 0, u = v \leq R_1 \psi_3(t, u, v), 1 \} \).

Let \( \lambda \in (0, \lambda^*] \) and \( \mu \in (0, \mu^*] \). Then for any \((x, y) \in \mathcal{P} \cap \partial \Omega_1\) and \( s \in [0, 1] \) we have
\[
[x(s) - \xi_1(s)]^* \leq x(s) \leq \| x \| \leq R_1, \quad [y(s) - \xi_2(s)]^* \leq y(s) \leq \| y \| \leq R_1.
\]
Then for any \((x, y) \in \mathcal{P} \cap \partial \Omega_1\), we obtain

\[
\|A_1(x, y)\| \leq \lambda \int_0^1 J_1(s) I_{0+}^{\alpha_1} (\chi_1(s) \psi_1(x, [x(s) - \xi_1(s)]^*, [y(s) - \xi_2(s)]^*) + \xi_1(s)) \, ds
\]
\[
+ \mu \int_0^1 J_2(s) I_{0+}^{\alpha_2} (\chi_2(s) \psi_2(s, [x(s) - \xi_1(s)]^*, [y(s) - \xi_2(s)]^*) + \xi_2(s)) \, ds
\]
\[
\leq \lambda^* M_1 \int_0^1 J_1(s) I_{0+}^{\alpha_1} (\chi_1(s) + \xi_1(s)) \, ds + \mu^* M_2 \int_0^1 J_2(s) I_{0+}^{\alpha_2} (\chi_2(s) + \xi_2(s)) \, ds
\]
\[
\leq \frac{R_1}{4} + \frac{R_2}{4} = \frac{R_3}{2},
\]

\[
\|A_2(x, y)\| \leq \lambda \int_0^1 J_3(s) I_{0+}^{\alpha_1} (\chi_1(s) \psi_1(x, [x(s) - \xi_1(s)]^*, [y(s) - \xi_2(s)]^*) + \xi_1(s)) \, ds
\]
\[
+ \mu \int_0^1 J_4(s) I_{0+}^{\alpha_2} (\chi_2(s) \psi_2(s, [x(s) - \xi_1(s)]^*, [y(s) - \xi_2(s)]^*) + \xi_2(s)) \, ds
\]
\[
\leq \lambda^* M_1 \int_0^1 J_3(s) I_{0+}^{\alpha_1} (\chi_1(s) + \xi_1(s)) \, ds + \mu^* M_2 \int_0^1 J_4(s) I_{0+}^{\alpha_2} (\chi_2(s) + \xi_2(s)) \, ds
\]
\[
\leq \frac{R_1}{4} + \frac{R_2}{4} = \frac{R_3}{2}.
\]

Hence

\[
\|A(x, y)\| = \|A_1(x, y)\| + \|A_2(x, y)\| \leq R_1 = \|(x, y)\|_Y, \quad \forall (x, y) \in \mathcal{P} \cap \partial \Omega_1.
\]  

(19)

Next for \(\theta_1, \theta_2\) given in (15), we choose a constant \(\Lambda > 0\) such that

\[
\Lambda \geq \max \left\{ \frac{1}{\theta_1^{\delta_2 - 2}}, \frac{1}{\theta_1^{\delta_1 + \alpha_2 - 2}}, \frac{4 \Gamma(\alpha_1 + 1)}{\lambda} \left( \int_{\theta_1}^{\theta_2} (s - \theta_1)\delta_1 J_1(s) \, ds \right)^{-1}, \frac{4 \Gamma(\alpha_2 + 1)}{\mu} \left( \int_{\theta_1}^{\theta_2} (s - \theta_1)\delta_2 J_2(s) \, ds \right)^{-1} \right\}.
\]

Then by the assumption (15) we deduce that there exists a constant \(M_0 > 0\) such that

\[
f(t, u, v) \geq \Lambda (u + v) \quad \text{or} \quad g(t, u, v) \geq \Lambda (u + v), \quad \forall t \in [\theta_1, \theta_2], \: u, v \geq 0, \: u + v \geq M_0.
\]

Now we define

\[
R_2 = \max \left\{ \frac{4 M_0}{\theta_1^{\delta_1 - 1}}, \frac{4 \delta_1}{\Gamma(\alpha_1 + 1)} \int_0^1 \xi_1(s) \, ds + \frac{4 \delta_2}{\Gamma(\alpha_2 + 1)} \int_0^1 \xi_2(s) \, ds, \frac{4 \delta_3}{\Gamma(\alpha_1 + 1)} \int_0^1 \xi_1(s) \, ds + \frac{4 \delta_4}{\Gamma(\alpha_2 + 1)} \int_0^1 \xi_2(s) \, ds \right\},
\]

and let \(\Omega_2 = \{(x, y) \in Y, \: \|(x, y)\|_Y < R_2\}\).

We consider firstly that \(f_{\infty} = \infty\), so we have

\[
f(t, u, v) \geq \Lambda (u + v), \quad \forall t \in [\theta_1, \theta_2], \: u, v \geq 0, \: u + v \geq M_0.
\]  

(20)

Let \((x, y) \in \mathcal{P} \cap \partial \Omega_2\). So \(\|(x, y)\|_Y = R_2\) or equivalently \(\|x\| + \|y\| = R_2\). This last relation gives us \(\|x\| \geq R_2/2\) or \(\|y\| \geq R_2/2\). We suppose firstly that \(\|x\| \geq R_2/2\). Then we obtain
\[ x(t) - x_1(t) = x(t) - \lambda \int_0^1 \mathcal{G}_1(t,s) I^{\rho_1}_{0+} \xi_1(s) \, ds - \mu \int_0^1 \mathcal{G}_2(t,s) I^{\rho_2}_{0+} \xi_2(s) \, ds \]
\[ \geq x(t) - \delta_1 t^{\beta_1-1} \int_0^1 I^{\rho_1}_{0+} \xi_1(s) \, ds - \delta_2 t^{\beta_1-1} \int_0^1 I^{\rho_2}_{0+} \xi_2(s) \, ds \]

\[ = x(t) - t^{\beta_1-1} \left( \frac{\delta_1}{\Gamma(\alpha_1 + 1)} \int_0^1 (1 - \tau)^{\alpha_1 - 1} \xi_1(\tau) \, d\tau + \frac{\delta_2}{\Gamma(\alpha_2 + 1)} \int_0^1 (1 - \tau)^{\alpha_2 - 1} \xi_2(\tau) \, d\tau \right) \]
\[ \geq x(t) - \frac{2 R_2}{\delta_1} x(t) \left( \frac{\delta_1}{\Gamma(\alpha_1 + 1)} \int_0^1 \xi_1(\tau) \, d\tau + \frac{\delta_2}{\Gamma(\alpha_2 + 1)} \int_0^1 \xi_2(\tau) \, d\tau \right) \]
\[ \geq \frac{1}{2} x(t) \geq 0, \quad \forall t \in [0, 1]. \]

Therefore we deduce

\[ [x(t) - x_1(t)]^* = x(t) - x_1(t) \geq \frac{1}{2} \lambda x(t) \geq \frac{1}{2} t^{\beta_1-1} ||x|| \geq \frac{R_2}{4} t^{\beta_1-1} \]
\[ \geq \frac{R_2}{4} \rho^{\beta_1-1} - M_0, \quad \forall t \in [\theta_1, \theta_2]. \]

Hence

\[ [x(t) - x_1(t)]^* + [y(t) - x_2(t)]^* \geq [x(t) - x_1(t)]^* = x(t) - x_1(t) \geq M_0, \quad \forall t \in [\theta_1, \theta_2]. \] (22)

Then by (20) and (22), we find

\[ f(t, [x(t) - x_1(t)]^*, [y(t) - x_2(t)]^*) \geq \Lambda([x(t) - x_1(t)]^* + [y(t) - x_2(t)]^*) \]
\[ \geq \Lambda x(t) - x_1(t)]^* \geq \frac{1}{2} x(t), \quad \forall t \in [\theta_1, \theta_2]. \]

Hence for any \( t \in [\theta_1, \theta_2] \) we obtain

\[ \mathcal{A}_1(x, y)(t) \geq \lambda \int_{\theta_1}^{\theta_2} \mathcal{G}_1(t,s) \, ds \left( f(s, [x(s) - x_1(s)]^*, [y(s) - x_2(s)]^*) + \xi_1(s) \right) ds \]
\[ \geq \lambda \int_{\theta_1}^{\theta_2} \mathcal{G}_1(t,s) \left( \frac{1}{\Gamma(\alpha_1)} \int_{\theta_1}^{s} (s - \tau)^{\alpha_1 - 1} f(\tau, [x(\tau) - x_1(\tau)]^*, [y(\tau) - x_2(\tau)]^*) + \xi_1(\tau) \right) d\tau ds \]
\[ \geq \lambda \int_{\theta_1}^{\theta_2} \mathcal{G}_1(t,s) \left( \frac{1}{\Gamma(\alpha_1)} \int_{\theta_1}^{s} (s - \tau)^{\alpha_1 - 1} \Lambda([x(\tau) - x_1(\tau)]^*) d\tau \right) ds \]
\[ \geq \lambda \int_{\theta_1}^{\theta_2} \mathcal{G}_1(t,s) \left( \frac{1}{\Gamma(\alpha_1)} \int_{\theta_1}^{s} (s - \tau)^{\alpha_1 - 1} \Lambda \frac{R_2}{4} t^{\beta_1-1} d\tau \right) ds \]
\[ \geq \frac{\Lambda R_2}{4 \Gamma(\alpha_1 + 1)} \int_{\theta_1}^{\theta_2} (s - \theta_1)^{\alpha_1} \mathcal{J}_1(s) \, ds \geq R_2. \]

Then \( ||\mathcal{A}_1(x, y)|| \geq ||(x, y)||_y \) and therefore we conclude

\[ ||\mathcal{A}(x, y)||_y \geq ||(x, y)||_y, \quad \forall (x, y) \in \mathcal{P} \cap \partial \Omega_2. \] (23)

If for \((x, y) \in \mathcal{P} \cap \partial \Omega_2\) one has the case \( ||y|| \geq R_2/2 \), then
\[ y(t) - \xi_2(t) = y(t) - \lambda \int_0^1 G_3(t, s) \rho_1^0 \xi_1(s) \, ds - \mu \int_0^1 G_4(t, s) \rho_1^2 \xi_2(s) \, ds \]
\[ \geq y(t) - \delta_3 \theta_2^{\rho_1 - 1} \int_0^1 \rho_1^0 \xi_1(s) \, ds - \delta_4 \theta_2^{\rho_1 - 1} \int_0^1 \rho_1^2 \xi_2(s) \, ds \]
\[ = y(t) - \theta_2^{\rho_1 - 1} \left( \frac{\delta_3}{\Gamma(\alpha_1 + 1)} \int_0^1 (1 - \tau)^{\alpha_1} \xi_1(\tau) \, d\tau + \frac{\delta_4}{\Gamma(\alpha_2 + 1)} \int_0^1 (1 - \tau)^{\alpha_2} \xi_2(\tau) \, d\tau \right) \]
\[ \geq y(t) - \frac{2}{R_2} y(t) \left( \frac{\delta_3}{\Gamma(\alpha_1 + 1)} \int_0^1 \xi_1(\tau) \, d\tau + \frac{\delta_4}{\Gamma(\alpha_2 + 1)} \int_0^1 \xi_2(\tau) \, d\tau \right) \]
\[ \geq y(t) \left[ 1 - \frac{2}{R_2} \left( \frac{\delta_3}{\Gamma(\alpha_1 + 1)} \int_0^1 \xi_1(\tau) \, d\tau + \frac{\delta_4}{\Gamma(\alpha_2 + 1)} \int_0^1 \xi_2(\tau) \, d\tau \right) \right] \]
\[ \geq \frac{1}{2} y(t) \geq 0, \quad \forall \, t \in [0, 1]. \] (24)

Therefore for any \( t \in [\theta_1, \theta_2] \) we deduce
\[ \|y(t) - \xi_2(t)\|^* = \|y(t) - \xi_2(t)\| \geq \frac{1}{2} y(t) \geq \frac{1}{2} \theta_2^{\rho_1 - 1} ||y|| \geq \frac{R_2}{4} \theta_2^{\rho_1 - 1} \geq M_0. \]

So
\[ |x(t) - \xi_1(t)|^* + |y(t) - \xi_2(t)|^* \geq |y(t) - \xi_2(t)|^* = y(t) - \xi_2(t) \geq M_0, \quad \forall \, t \in [\theta_1, \theta_2]. \] (25)

Then by (20) and (25), we find
\[ f(t, x(t) - \xi_1(t))^*, [y(t) - \xi_2(t)]^* \geq \Lambda([x(t) - \xi_1(t)]^* + [y(t) - \xi_2(t)]^*) \]
\[ \geq \Lambda y(t) - \xi_2(t))^* \geq \frac{1}{2} y(t), \quad \forall \, t \in [\theta_1, \theta_2]. \]

It follows that for any \( t \in [\theta_1, \theta_2] \) we have
\[ A_1(x, y)(t) \geq \lambda \int_0^1 G_1(t, s) \rho_1^0 \left( f(s, [x(s) - \xi_1(s)]^*, [y(s) - \xi_2(s)]^*) + \xi_1(s) \right) \, ds \]
\[ \geq \lambda \int_{\theta_1}^{\theta_2} G_1(t, s) \left( \frac{1}{\Gamma(\alpha_1)} \int_{\theta_1}^{\theta_2} (s - \tau)^{\alpha_1 - 1} f(\tau, [x(\tau) - \xi_1(\tau)]^*, [y(\tau) - \xi_2(\tau)]^*) \, d\tau \right) \, ds \]
\[ \geq \lambda \int_{\theta_1}^{\theta_2} G_1(t, s) \left( \frac{1}{\Gamma(\alpha_1)} \int_{\theta_1}^{\theta_2} (s - \tau)^{\alpha_1 - 1} \Lambda([y(\tau) - \xi_2(\tau)]^*) \, d\tau \right) \, ds \]
\[ \geq \lambda \Lambda R_2 \theta_2^{\rho_1 + \rho_2 - 2} \int_{\theta_1}^{\theta_2} \left( s - \theta_1 \right)^{\alpha_1 - 1} J_1(s) \, ds \geq R_2. \]

Then \( \|A_1(x, y)\| \geq \|(x, y)\|\_y \) and we obtain again the relation (23).

We assume now that \( g_0 = \infty \), and so we have
\[ g(t, u, v) \geq \Lambda(u + v), \quad \forall \, t \in [\theta_1, \theta_2], \; u, v \geq 0, \; u + v \geq M_0. \] (26)

Let \( (x, y) \in \mathcal{P} \cap \partial \Omega_2 \). So \( \|(x, y)\|\_y \geq R_2 \) or equivalently \( \|x\| + \|y\| \geq R_2 \). This last relation gives us \( \|x\| \geq R_2 / 2 \) or \( \|y\| \geq R_2 / 2 \). We suppose firstly that \( \|x\| \geq R_2 / 2 \). Then we obtain as in the first case that \( x(t) - \xi_1(t) \geq x(t) / 2 \geq 0 \) for all \( t \in [0, 1] \), \( |x(t) - \xi_1(t)|^* \geq R_2 \theta_2^{\rho_1 - 1} / 4 \geq M_0 \) for all \( t \in [\theta_1, \theta_2] \), and \( |x(t) - \xi_1(t)|^* + |y(t) - \xi_2(t)|^* \geq M_0 \).
for all \( t \in [\theta_1, \theta_2] \). Hence we deduce \( g(t, [x(t) - \xi_1(t)]^*, [y(t) - \xi_2(t)]^*) \geq \Lambda x(t)/2 \) for all \( t \in [\theta_1, \theta_2] \). Using (26), it follows for any \( t \in [\theta_1, \theta_2] \) that

\[
A_1(x, y)(t) \geq \mu \int_{\theta_1}^{\theta_2} G_2(t, s) \left( \frac{1}{\Gamma(\alpha_2)} \int_{0}^{1} (s - \tau)^{\alpha_2 - 1} \Lambda \left( |y(t) - \xi_2(t)|^* \right) d\tau \right) ds \\
\geq \mu \frac{\Delta R_2 r_{\theta_1}^{2\beta_1 - 2}}{4 \Gamma(\alpha_2 + 1)} \int_{\theta_1}^{\theta_2} (s - \theta_1)^{\alpha_2} J_2(s) ds \geq R_2.
\]

Then \( ||A_1(x, y)|| \geq ||(x, y)||_Y \) and therefore we obtain relation (23).

If for \((x, y) \in P \cap \Omega_2\) one has the case \( ||y|| \geq R_2/2 \), then \( y(t) - \xi_2(t) \geq y(t)/2 \geq 0 \) for all \( t \in [0, 1] \), \( |y(t) - \xi_2(t)|^* \geq R_2 \theta_2^{\beta_2 - 1}/4 \geq M_0 \) for all \( t \in [\theta_1, \theta_2] \), and \( (x(t) - \xi_1(t))^* + |y(t) - \xi_2(t)|^* \geq M_0 \) for all \( t \in [\theta_1, \theta_2] \). So we find \( g(t, [x(t) - \xi_1(t)]^*, [y(t) - \xi_2(t)]^*) \geq \Lambda y(t)/2 \) for all \( t \in [\theta_1, \theta_2] \). By using again the relation (26), we deduce for any \( t \in [\theta_1, \theta_2] \) that

\[
A_1(x, y)(t) \geq \mu \int_{\theta_1}^{\theta_2} G_2(t, s) \left( \frac{1}{\Gamma(\alpha_2)} \int_{0}^{1} (s - \tau)^{\alpha_2 - 1} \Lambda \left( |y(t) - \xi_2(t)|^* \right) d\tau \right) ds \\
\geq \mu \frac{\Delta R_2 r_{\theta_1}^{2\beta_1 + \beta_2 - 2}}{4 \Gamma(\alpha_2 + 1)} \int_{\theta_1}^{\theta_2} (s - \theta_1)^{\alpha_2} J_2(s) ds \geq R_2.
\]

Then \( ||A_1(x, y)|| \geq ||(x, y)||_Y \), and hence we obtain again relation (23).

Therefore by the relations (19) and (23) and the Guo–Krasnosel’skii fixed point theorem, we conclude that \( A \) has a fixed point \((x_1, y_1) \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)\), that is \( R_1 \leq ||(x_1, y_1)||_Y \leq R_2 \) with \( x_1(t) \geq t^{\beta_1 - 1} ||x_1|| \) and \( y_1(t) \geq t^{\beta_2 - 1} ||y_1|| \) for all \( t \in [0, 1] \). If \( ||x_1|| \geq R_1/2 \) and \( ||y_1|| \geq R_1/2 \), then we deduce in a similar manner as (21) and (24) that

\[
x_1(t) - \xi_1(t) \geq x_1(t) \left[ 1 - \frac{2}{R_1} \left( \frac{\delta_1}{\Gamma(\alpha_1 + 1)} \right) \int_{0}^{1} \xi_1(\tau) d\tau + \frac{\delta_2}{\Gamma(\alpha_2 + 1)} \int_{0}^{1} \xi_2(\tau) d\tau \right] \\
\geq t^{\beta_1 - 1} ||x_1|| \left[ 1 - \frac{2}{R_1} \left( \frac{\delta_1}{\Gamma(\alpha_1 + 1)} \right) \int_{0}^{1} \xi_1(\tau) d\tau + \frac{\delta_2}{\Gamma(\alpha_2 + 1)} \int_{0}^{1} \xi_2(\tau) d\tau \right] \\
= \Lambda_1 t^{\beta_1 - 1}, \quad \forall t \in [0, 1],
\]

and

\[
y_1(t) - \xi_2(t) \geq y_1(t) \left[ 1 - \frac{2}{R_1} \left( \frac{\delta_3}{\Gamma(\alpha_1 + 1)} \right) \int_{0}^{1} \xi_1(\tau) d\tau + \frac{\delta_4}{\Gamma(\alpha_2 + 1)} \int_{0}^{1} \xi_2(\tau) d\tau \right] \\
\geq t^{\beta_2 - 1} ||y_1|| \left[ 1 - \frac{2}{R_1} \left( \frac{\delta_3}{\Gamma(\alpha_1 + 1)} \right) \int_{0}^{1} \xi_1(\tau) d\tau + \frac{\delta_4}{\Gamma(\alpha_2 + 1)} \int_{0}^{1} \xi_2(\tau) d\tau \right] \\
= \Lambda_2 t^{\beta_2 - 1}, \quad \forall t \in [0, 1],
\]
where

\[
\begin{align*}
\tilde{\lambda}_1 &= \frac{R_1}{2} - \left( \frac{\delta_1}{\Gamma(a_1 + 1)} \int_0^1 \xi_1(\tau) \, d\tau + \frac{\delta_2}{\Gamma(a_2 + 1)} \int_0^1 \xi_2(\tau) \, d\tau \right) > 0, \\
\tilde{\lambda}_2 &= \frac{R_1}{2} - \left( \frac{\delta_3}{\Gamma(a_1 + 1)} \int_0^1 \xi_1(\tau) \, d\tau + \frac{\delta_4}{\Gamma(a_2 + 1)} \int_0^1 \xi_2(\tau) \, d\tau \right) > 0.
\end{align*}
\] 

(29)

If \( \|x_1\| \geq R_1/2 \) and \( \|y_1\| \leq R_1/2 \), then by (27) we have \( x_1(t) - \tilde{\xi}_1(t) \geq \tilde{\lambda}_1 t^{\beta_1-1} \) with \( \tilde{\lambda}_1 \) given by (29). In addition, because \( [x_1(s) - \tilde{\xi}_1(s)]^+ \leq x_1(s) - \tilde{\xi}_1(s) \leq x_1(s) \leq R_2 \) and \([y_1(s) - \tilde{\xi}_2(s)]^+ \leq y_1(s) - \tilde{\xi}_2(s) \leq y_1(s) \leq R_1/2 \) for all \( s \in [0, 1] \), we obtain by using (16) that

\[
y_1(t) = A_2(x_1, y_1)(t) = \lambda \int_0^1 G_3(t, s) l_{a_1}^1 f(s, [x_1(s) - \tilde{\xi}_1(s)]^+, [y_1(s) - \tilde{\xi}_2(s)]^+) + \xi_1(s) \, ds \\
+ \mu \int_0^1 G_4(t, s) l_{a_2}^2 g(s, [x_1(s) - \tilde{\xi}_1(s)]^+) + \xi_2(s) \, ds \\
\geq \lambda \int_0^1 G_3(t, s) l_{a_1}^1 \xi_1(s) \, ds + \mu \int_0^1 G_4(t, s) l_{a_2}^2 \xi_2(s) \, ds = \tilde{\xi}_2(t), \forall t \in [0, 1].
\]

If \( \|y_1\| \geq R_1/2 \) and \( \|x_1\| \leq R_1/2 \), then by (28) we have \( y_1(t) - \tilde{\xi}_2(t) \geq \tilde{\lambda}_2 t^{\beta_2-1} \) with \( \tilde{\lambda}_2 \) given by (29). Besides, because \( [x_1(s) - \tilde{\xi}_1(s)]^+ \leq x_1(s) - \tilde{\xi}_1(s) \leq x_1(s) \leq R_1/2 \) and \([y_1(s) - \tilde{\xi}_2(s)]^+ \leq y_1(s) - \tilde{\xi}_2(s) \leq y_1(s) \leq R_2 \) for all \( s \in [0, 1] \), we obtain by using again (16) that

\[
x_1(t) = A_1(x_1, y_1)(t) = \lambda \int_0^1 G_1(t, s) l_{a_1}^1 f(s, [x_1(s) - \tilde{\xi}_1(s)]^+) + \xi_1(s) \, ds \\
+ \mu \int_0^1 G_2(t, s) l_{a_2}^2 g(s, [x_1(s) - \tilde{\xi}_1(s)]^+) + \xi_2(s) \, ds \\
\geq \lambda \int_0^1 G_1(t, s) l_{a_1}^1 \xi_1(s) \, ds + \mu \int_0^1 G_2(t, s) l_{a_2}^2 \xi_2(s) \, ds = \tilde{\xi}_1(t), \forall t \in [0, 1].
\]

Let \( u_1(t) = x_1(t) - \tilde{\xi}_1(t) \) and \( v_1(t) = y_1(t) - \tilde{\xi}_2(t) \) for all \( t \in [0, 1] \). Then by the above remarks, the pair \((u_1, v_1)\) is a positive solution of problems (1) and (2) with \( u_1(t) \geq \tilde{\lambda}_1 t^{\beta_1-1} \) and \( v_1(t) \geq 0 \) for all \( t \in [0, 1] \), or \( u_1(t) \geq 0 \) and \( v_1(t) \geq \tilde{\lambda}_2 t^{\beta_2-1} \) for all \( t \in [0, 1] \). □

**Theorem 3.** We suppose that (11), (13), (15), (16) and (14'). The functions \( f, g \in C([0, 1] \times \mathbb{R}_+ \times \mathbb{R}_+) \) and there exist functions \( \xi_1, \xi_2, \chi_1, \chi_2 \in C([0, 1], \mathbb{R}_+) \), \( \psi_1, \psi_2 \in C([0, 1] \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+) \) such that

\[-\xi_1(t) \leq f(t, x, y) \leq \chi_1(t) \psi_1(t, x, y), \quad -\xi_2(t) \leq g(t, x, y) \leq \chi_2(t) \psi_2(t, x, y),\]

for all \( t \in [0, 1] \), \( x, y \in \mathbb{R}_+ \), with \( \int_0^1 \xi_i(t) \, dt > 0 \), \( \int_0^1 \chi_i(t) \, dt > 0 \), \( i = 1, 2 \), hold. Then the boundary value problems (1) and (2) has at least two positive solutions for \( \lambda > 0 \) and \( \mu > 0 \) sufficiently small.

**Proof.** Because assumption (14') implies assumptions (12) and (14), we can apply Theorem 1 and Theorem 2. Then we deduce that for \( 0 < \lambda \leq \min\{\lambda_0, \lambda^*\} \) and \( 0 < \mu \leq \min\{\mu_0, \mu^*\} \), problems (1) and (2) has at least two positive solutions \((u_0, v_0)\) and \((u_1, v_1)\) with \( \|(u_0 + \xi_1, v_0 + \xi_2)\|_Y \leq 1 \) and \( \|(u_1 + \xi_1, v_1 + \xi_2)\|_Y > 1 \), where \((\xi_1, \xi_2)\) is the solution of problem (16) and (17). □

**4. Examples.**

Let \( a_1 = 1/2 \), \( a_2 = 1/3 \), \( \beta_1 = 9/4 \) \( (n = 3) \), \( \beta_2 = 16/5 \) \( (m = 4) \), \( p = 2 \), \( q = 1 \), \( \gamma_0 = 6/5 \), \( \gamma_1 = 1/3 \), \( \gamma_2 = 7/6 \), \( \delta_0 = 3/2 \), \( \delta_1 = 2/3 \), \( H_1(t) = 3t \) for all \( t \in [0, 1] \), \( H_2(t) = \{1, t \in [0, 1/2)\}; 3/2, t \in [1/2, 1]\} \), \( K_1(t) = \{2, t \in [0, 1/4)\}; 18/7, t \in [1/4, 1]\}. \)
We consider the system of fractional differential equations

\[
\begin{align*}
D_{0+}^{1/2} \left( D_{0+}^{9/4} u(t) \right) + \lambda f(t, u(t), v(t)) &= 0, \quad t \in (0, 1), \\
D_{0+}^{1/3} \left( D_{0+}^{16/5} v(t) \right) + \mu g(t, u(t), v(t)) &= 0, \quad t \in (0, 1),
\end{align*}
\]

subject to the coupled boundary conditions

\[
\begin{align*}
&\begin{cases}
u(0) = 0, & D_{0+}^{9/4} u(0) = 0, & D_{0+}^{5/5} u(1) = 3 \int_0^1 D_{0+}^{1/3} v(t) \, dt + \frac{1}{2} D_{0+}^{7/6} (\frac{1}{2}), \\
v(0) = v'(0) = 0, & D_{0+}^{16/5} v(0) = 0, & D_{0+}^{3/2} v(1) = \frac{1}{2} D_{0+}^{2/3} u(\frac{1}{4}).
\end{cases}
\end{align*}
\]

We obtain \( \Delta_1 \approx 0.83043938, \Delta_2 \approx 0.28544249, \) and \( \Delta \approx 2.45375814 > 0. \) So assumption (11) is satisfied. In addition we deduce

\[
\begin{align*}
&g_1(t,s) = \frac{1}{\Gamma(\frac{9}{4})} \begin{cases}
t^{5/4} (1-s)^{1/20} - (t-s)^{5/4}, & 0 \leq s \leq t \leq 1, \\
t^{5/4} (1-s)^{1/20}, & 0 \leq t \leq s \leq 1,
\end{cases} \\
g_{11}(t,s) = \frac{1}{\Gamma(\frac{19}{12})} \begin{cases}
t^{7/12} (1-s)^{1/20} - (t-s)^{7/12}, & 0 \leq s \leq t \leq 1, \\
t^{7/12} (1-s)^{1/20}, & 0 \leq t \leq s \leq 1,
\end{cases} \\
g_2(t,s) = \frac{1}{\Gamma(\frac{19}{15})} \begin{cases}
t^{11/5} (1-s)^{7/10} - (t-s)^{11/5}, & 0 \leq s \leq t \leq 1, \\
t^{11/5} (1-s)^{7/10}, & 0 \leq t \leq s \leq 1,
\end{cases} \\
g_{21}(t,s) = \frac{1}{\Gamma(\frac{28}{15})} \begin{cases}
t^{28/15} (1-s)^{7/10} - (t-s)^{28/15}, & 0 \leq s \leq t \leq 1, \\
t^{28/15} (1-s)^{7/10}, & 0 \leq t \leq s \leq 1,
\end{cases} \\
g_{22}(t,s) = \frac{1}{\Gamma(\frac{31}{30})} \begin{cases}
t^{31/30} (1-s)^{7/10} - (t-s)^{31/30}, & 0 \leq s \leq t \leq 1, \\
t^{31/30} (1-s)^{7/10}, & 0 \leq t \leq s \leq 1,
\end{cases} \\
G_1(t,s) = g_1(t,s) + \frac{45}{2} \left[ \frac{45}{4} \Gamma(\frac{16}{5}) \Gamma(\frac{43}{43}) \Gamma(\frac{1}{2}) \right] \frac{1}{61/30} g_{11}(\frac{1}{4}, s), \\
G_2(t,s) = \frac{\Gamma(\frac{19}{15})}{\Gamma(\frac{11}{10})} \left[ \frac{45}{2} \Gamma(\frac{16}{5}) \Gamma(\frac{43}{43}) \Gamma(\frac{1}{2}) \right] \frac{1}{61/30} g_{11}(\frac{1}{4}, s), \\
G_3(t,s) = \frac{\Gamma(\frac{11}{5})}{\Gamma(\frac{9}{4})} \frac{1}{61/30} g_{11}(\frac{1}{4}, s), \\
G_4(t,s) = g_2(t,s) + \frac{1}{\Gamma(\frac{9}{4})} \left[ \frac{45}{2} \Gamma(\frac{16}{5}) \Gamma(\frac{43}{43}) \Gamma(\frac{1}{2}) \right] \frac{1}{61/30} g_{11}(\frac{1}{4}, s), \\
h_1(t) = \frac{1}{\Gamma(\frac{9}{4})} \left[ (1-s)^{1/20} - (1-s)^{5/4} \right], \\
h_2(t) = \frac{1}{\Gamma(\frac{19}{15})} \left[ (1-s)^{7/10} - (1-s)^{11/5} \right],
\end{align*}
\]

for all \( t, s \in [0, 1]. \) We also find \( \delta_1 \approx 1.11677471, \delta_2 \approx 0.90284119, \delta_3 \approx 0.13538798, \delta_4 \approx 0.52199968. \) Besides, we obtain

\[
\mathcal{J}_1(s) = \begin{cases}
h_1(s) + \frac{1}{\Gamma(\frac{19}{12})} \left[ \frac{45}{2} \Gamma(\frac{16}{5}) \Gamma(\frac{43}{43}) \Gamma(\frac{1}{2}) \right] \frac{1}{61/30} g_{11}(\frac{1}{4}, s), & 0 \leq s < \frac{1}{4}, \\
h_1(s) + \frac{1}{\Gamma(\frac{19}{12})} \left[ \frac{45}{2} \Gamma(\frac{16}{5}) \Gamma(\frac{43}{43}) \Gamma(\frac{1}{2}) \right] \frac{1}{61/30} g_{11}(\frac{1}{4}, s), & \frac{1}{4} \leq s \leq 1,
\end{cases}
\]
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Example 1. We consider the functions

\[ f(t, x, y) = (x + y)^4 + \cos x, \quad g(t, x, y) = (x + y)^{1/5} + \cos y, \quad t \in [0, 1], \quad x, y \geq 0. \] (32)

Here we have \( \zeta_1(t) = \zeta_2(t) = 1 \) for all \( t \in [0, 1] \), \( f(t, x, y) \geq \cos x \geq -1 \), \( g(t, x, y) \geq \cos y \geq -1 \) for all \( t \in [0, 1], \ x, y \geq 0 \). So assumption (12) is satisfied. The assumption (13) is also satisfied because \( f(t, 0, 0) = 1 > 0 \) and \( g(t, 0, 0) = 1 > 0 \) for all \( t \in [0, 1] \). We consider \( \omega = 1/2 < 1 \) and \( R_0 = 1 \). Then

\[ f(t, x, y) \geq \omega f(t, 0, 0) = \frac{1}{2}, \quad g(t, x, y) \geq \omega g(t, 0, 0) = \frac{1}{2}, \]

for all \( t \in [0, 1], \ x, y \geq 0 \). In addition we obtain

\[ f_0 = \max_{t, x, y \in [0, 1]} \{ f(t, x, y) + \zeta_1(t) \} = \max_{t, x, y \in [0, 1]} \{ (x + y)^4 + \cos x + 1 \} \approx 17.54030231, \]

\[ g_0 = \max_{t, x, y \in [0, 1]} \{ g(t, x, y) + \zeta_2(t) \} = \max_{t, x, y \in [0, 1]} \{ (x + y)^{1/5} + \cos y + 1 \} \approx 3.01750724. \]

Besides, after some computations, we find \( \Lambda_1 = \int_0^1 J_1(s)ds \approx 0.763435044, \quad \Lambda_2 = \int_0^1 J_2(s)ds \approx 0.30091922, \quad \Lambda_3 = \int_0^1 J_3(s)ds \approx 0.10756388, \quad \Lambda_4 = \int_0^1 J_4(s)ds \approx 0.15023461, \) and then \( \lambda_0 = \min \{ \frac{\Gamma(3/2)}{8\Lambda_1^2 t^2}, \frac{\Gamma(4/3)}{8\Lambda_2^2 t^2} \} \approx 0.009 \) and \( \mu_0 = \min \{ \frac{\Gamma(3/2)}{8\Lambda_3^2 t^2}, \frac{\Gamma(4/3)}{8\Lambda_4^2 t^2} \} \approx 0.121 \). By Theorem 1, for any \( \lambda \in (0, \lambda_0] \) and \( \mu \in (0, \mu_0] \), we conclude that problem (30) and (31) with the nonlinearities (32) has at least one positive solution \( (u_1(t), v_1(t)), \ t \in [0, 1] \). Because assumption \( (4') \) is satisfied \( \chi_1(t) = \chi_2(t) = 1, \quad \psi_1(t, x, y) = (x + y)^4 + 1, \quad \psi_2(t, x, y) = (x + y)^{1/5} + 1 \) for all \( t \in [0, 1], \ x, y \geq 0 \), and assumption (15) is also satisfied \( (f_0 = \infty) \), by Theorem 3 we deduce that problem (30) and (31) with the functions \( f \) and \( g \) given by (32) has at least two positive solutions \( (u_1(t), v_1(t)), (u_2(t), v_2(t)), \ t \in [0, 1] \) for \( \lambda \) and \( \mu \) sufficiently small.

Example 2. We consider the functions

\[ f(t, x, y) = \frac{1}{\sqrt[4]{5} \sqrt[4]{1-t}}, \quad \frac{(x-20)^2 + (y-10)^2}{25}, \quad \frac{1}{\sqrt[4]{5} \sqrt[4]{(1-t)^2}}, \quad \frac{(x-7)^4 + (y-9)^2}{16}, \quad \frac{1}{\sqrt[4]{5} \sqrt[4]{(1-t)^2}}. \] (33)
for all \( t \in (0, 1) \), \( x, y \geq 0 \). The functions \( f \) and \( g \) are continuous, and singular at \( t = 0 \) and \( t = 1 \).

In addition, we obtain the inequalities

\[
\begin{align*}
f(t, x, y) & \geq -\frac{1}{\sqrt{t} \sqrt{1-t}}; \quad g(t, x, y) \geq -\frac{1}{\sqrt{t} \sqrt{1-t}}, \\
f(t, x, y) & \leq \frac{1}{\sqrt{t^3} \sqrt{1-t}} \left( (x-20)^2 + (y-10)^2 \right), \\
g(t, x, y) & \leq \frac{1}{\sqrt{t} \sqrt{(1-t)^2}} \left( (x-7)^4 + (y-9)^2 \right).
\end{align*}
\]

for all \( t \in (0, 1) \) and \( x, y \geq 0 \). Here we have \( \zeta_1(t) = \frac{1}{\sqrt{1-t}}, \zeta_2(t) = \frac{1}{\sqrt{1-t}}, \chi_1(t) = \frac{1}{\sqrt{1-t}}, \chi_2(t) = \frac{1}{\sqrt{(1-t)^2}} \). for all \( t \in (0, 1) \), \( \psi_1(t, x, y) = \frac{(x-20)^2 + (y-10)^2}{25}, \psi_2(t, x, y) = \frac{(x-7)^4 + (y-9)^2}{16} \) for \( t \in [0, 1], x, y \geq 0 \). Besides we find \( \int_0^1 \psi_1(t) \, dt = B\left(\frac{1}{2}, \frac{3}{2}\right) \in (0, \infty), \int_0^1 \psi_2(t) \, dt = B\left(\frac{3}{2}, \frac{3}{2}\right) \in (0, \infty) \). where \( B(p, q) = \int_0^1 s^{p-1}(1-s)^{q-1} \, ds, p, q > 0 \) is the first Euler function (the beta function).

Therefore assumption (14) is satisfied.

Next, because

\[
\frac{(x-20)^2 + (y-10)^2}{3} - \frac{1}{3}(x+y)^2 \geq 900,
\]

we obtain for the function \( f \) the inequality

\[
\frac{f(t, x, y)}{x+y} \geq \frac{1}{25\sqrt{t} \sqrt{1-t}} \left[ \frac{1}{3}(x+y) - \frac{900}{x+y} \right] - \frac{1}{\sqrt{t} \sqrt{1-t}} (x+y), \quad t \in (0, 1), \ x+y > 0.
\]

Hence for \( \theta_1, \theta_2 \in (0, 1), \ \theta_1 < \theta_2, \) we find \( f_{\infty} = \lim_{x+y \to \infty} \min_{t \in [0, \theta_2]} \frac{f(t, x, y)}{x+y} = \infty. \) Then assumption (15) is satisfied.

We also have \( 2\left( \frac{\delta_1}{\Gamma(\alpha+1)} \right) \int_0^\theta_1 \xi_1(s) \, ds + \frac{\delta_2}{\Gamma(\alpha+1)} \int_0^\theta_2 \xi_2(s) \, ds \right) \approx 9.7974951 \) and \( 2\left( \frac{\delta_1}{\Gamma(\alpha+1)} \right) \int_0^\theta_1 \xi_1(s) \, ds + \frac{\delta_4}{\Gamma(\alpha+1)} \int_0^\theta_2 \xi_2(s) \, ds \right) \approx 2.685269. \) Then we choose \( R_1 = 10 \) which satisfies the inequality from assumption (16). For this constant, we deduce the inequalities

\[
\begin{align*}
f(t, x, y) & \geq \frac{1}{\sqrt{t} \sqrt{1-t}} \left[ (x-20)^2 + (y-10)^2 - 1 \right] \geq 0, \\
g(t, x, y) & \geq \frac{1}{\sqrt{t} \sqrt{(1-t)^2}} \left[ (x-7)^4 + (y-9)^2 - 1 \right] \geq 0,
\end{align*}
\]

for all \( t \in (0, 1) \) and \( (x, y) \in ([0, 5] \times [0, \infty)) \cup ([0, \infty) \times [0, 5]) \), so the assumption (16) is satisfied.

We also find \( M_1 = \max \{ \max_{t \in [0, 1], u,v \geq 0, u+v \leq 10} \psi_1(t, u, v), 1 \} = 20 \) and \( M_2 = \max \{ \max_{t \in [0, 1], u,v \geq 0, u+v \leq 10} \psi_2(t, u, v), 1 \} = 1241/8. \) In addition, after some computations, we obtain
Conclusions

In this paper, we discussed the existence and multiplicity of positive solutions for a system of Riemann–Liouville fractional differential equations with sequential derivatives, positive parameters and sign-changing singular nonlinearities, supplemented with non-local coupled boundary conditions involving fractional derivatives and Riemann–Stieltjes...
integrals. We also present some properties of the associated Green functions, and two examples are finally given to illustrate the obtained results.

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