A formula for the Whitehead group of a three-dimensional crystallographic group

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Abstract

We give an explicit formula for Whitehead group of a three-dimensional crystallographic group $\Gamma$ in terms of the Whitehead groups of the virtually infinite cyclic subgroups of $\Gamma$.

0 Introduction

The $K$-theory of integral group rings of crystallographic groups has been studied by Connolly and Koźniewski [8], Farrell and Hsiang [10], Farrell and Jones [11], Lück and Stamm [15], Quinn [20], Tsapogas [25]. Also, in [18], Pearson gives explicit computations of the lower algebraic $K$-theory of two-dimensional crystallographic groups.

In this article we give a simple formula for Whitehead group of a three-dimensional crystallographic group $\Gamma$ in terms of the Whitehead groups of the virtually infinite cyclic subgroups of $\Gamma$. Here is our main result:

Main Theorem: Let $\Gamma$ be a 3-crystallographic group. Then

$$\text{Wh}(\Gamma) = \bigoplus_{G \in VC_\infty(\Gamma)} \text{Wh}(G).$$

Moreover, the direct sum in the formula above is finite.
In the theorem above $VC_{\infty}(\Gamma)$ is the set of conjugacy classes of maximal virtually infinite cyclic subgroups of $\Gamma$.

**Corollary 1:** Let $\Gamma$ be a 3-crystallographic group. Then $Wh(\Gamma)$ is infinitely generated if and only if $\Gamma$ contains a maximal virtually infinite cyclic subgroup $G$ with $Wh(G)$ infinitely generated.

The following example answers a question proposed by F. T. Farrell: is there any 3-crystallographic group whose Whitehead group is infinitely generated?

**Example:** Consider the 2-crystallographic group $Pmm$. This group is generated by translations $\tau_p(e_i)(p) = p + e_i$, $p \in \mathbb{R}^2$, $e_1 = (1,0)$, $e_2 = (0,1)$ and reflections about the $x$ and $y$ axis. Then $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a subgroup of $Pmm$. Hence $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$ is a subgroup of the 3-crystallographic group $\Gamma = Pmm \times \mathbb{Z}$. In fact, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$ is maximal because $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z} = \Gamma^{(\mathbb{R}e_3)}$ (see Lemma 5.1) and the action of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$ on $\mathbb{R}e_3$ is cocompact. Also by the Bass-Heller-Swan formula $Wh(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}) = 2 \text{Nil}_1(\mathbb{Z}[\mathbb{Z}_2 \times \mathbb{Z}_2])$, which is infinitely generated (see Lemma 6.2). Therefore Corollary 1 implies that $Wh(Pmm \times \mathbb{Z})$ is infinitely generated.

**Corollary 2:** Let $\Gamma$ be a 2-crystallographic group. Then

$$2 \text{Nil}_1(\mathbb{Z}[\Gamma]) = 2 \left( \bigoplus_{F \in F(\Gamma)} \text{Nil}_1(\mathbb{Z}[F]) \right).$$

In the corollary above $F(\Gamma)$ is the set of conjugacy classes of maximal finite subgroups of $\Gamma$.

As in [5], [6], [15], [18], [26], the proof of our main theorem is accomplished using fundamental results of Farrell and Jones [11], “the Farrell-Jones Isomorphism Conjecture”, which holds for crystallographic groups. This conjecture states that the algebraic $K$-theory of $\mathbb{Z}[\Gamma]$ may be computed from the algebraic $K$-theory of the virtually cyclic subgroups of $\Gamma$ via an appropriate “assembly map”. Our approach is quite geometric and our main ingredient is
the construction of a somehow “concrete” model for the universal \((\Gamma, VC(\Gamma))\) space, for any \(n\)-crystallographic group. This construction is a variation of Farrell and Jones construction given in \([11]\) and \([13]\). It is interesting to note that some the results for 2-crystallographic groups in \([18]\) can be obtained in a simpler fashion using our geometric methods.

In this article we also classify, modulo isomorphism, the virtually infinite cyclic subgroups of a 3-crystallographic group.

This paper is organized as follows. In the Section 1, we introduce some definitions and propositions and state Farrell-Jones Isomorphism Conjecture. In the Section 2, we construct a model for the universal \((\Gamma, VC(\Gamma))\)-space, where \(\Gamma\) is a \(n\)-crystallographic group. In the Section 3, we classify, modulo isomorphism, the virtually infinite cyclic subgroups of a 3-crystallographic group. In the Section 4, we calculate the isotropy groups of open \(n\)-cells of the universal \((\Gamma, VC(\Gamma))\)-space. In the Section 5, we calculate the \(E^2_{i,j}\) terms of the spectral sequence, necessary to apply the Farrell-Jones Isomorphism Conjecture. Finally, in Section 6, we proof the results mentioned above.

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1 Preliminaries

First, let us recall some definitions and fix some notation.

A group \(\Gamma < O(n) \rtimes \mathbb{R}^n\) is \(n\)-crystallographic group if \(\Gamma\) acts properly discontinuously, cocompactly and by isometries on \(\mathbb{R}^n\).

Let \(\Gamma\) be a group and \(G < \Gamma\), i.e., \(G\) is a subgroup of \(\Gamma\). \(G\) is a maximal finite subgroup of \(\Gamma\), if \(G < G' < \Gamma\), with \(G'\) finite, implies \(G = G'\). Analogously, we define a maximal virtually infinite cyclic subgroup of \(\Gamma\).

Let \(X\) be a space equipped with an action of a group \(\Gamma\). Let \(B \subset X\). Then we write
\[ \Gamma^B = \{ \gamma \in \Gamma; \gamma b = b \text{ for all } b \in B \}. \]

\[ \Gamma^{(B)} = \{ \gamma \in \Gamma; \gamma B = B \}. \]

If \( G < \Gamma \), \( X^G = \{ x \in X; gx = x \text{ for all } g \in G \}. \)

If \( Y \) is another \( \Gamma \)-space, we write \( X \cong_\Gamma Y \) if \( X \) is \( \Gamma \)-homeomorphic to \( Y \).

A group \( G \) is virtually cyclic if it is finite or contains an infinite cyclic subgroup of finite index.

For a group \( \Gamma \), we write \( F(\Gamma) \) for the set of conjugacy classes of maximal finite subgroups of \( \Gamma \), and \( VC_\infty(\Gamma) \) the set of conjugacy classes of maximal virtually infinite cyclic subgroups of \( \Gamma \). \( VC(\Gamma) \) will denote the family of virtually cyclic subgroups of \( \Gamma \).

We will use the following result.

**Proposition 1.1** Let \( G \) be a subgroup of a \( n \)-crystallographic group \( \Gamma \). Then \( G \) virtually infinite cyclic if and only if there is at least one line \( l \) in \( \mathbb{R}^n \) left invariant by \( G \), and the action of \( G \) on \( l \) is cocompact.

**Proof.** This result is proven, with greater generality, in [11] p. 267. The proposition can also be shown directly, using some simple geometric arguments and the algebraic result of [23] p. 178-179.

Now, we shall state Farrell-Jones Isomorphism Conjecture (see [11]). Let \( X \) be a connected CW-complex and \( \Gamma = \pi_1(X) \). Let \( A \) denote a universal \((\Gamma, VC(\Gamma))\)-space and \( \tilde{X} \) be the universal covering space of \( X \). Let \( \Gamma \times \tilde{X} \times A \longrightarrow \tilde{X} \times A \) be the diagonal action. Then \( \rho : \mathcal{E}(X) \rightarrow B(X) \) and \( f : \mathcal{E}(X) \rightarrow X \) are defined to be the quotient of the standard projections \( \tilde{X} \times A \rightarrow A \) and \( \tilde{X} \times A \rightarrow \tilde{X} \) under the \( \Gamma \)-actions. Let \( \zeta_*() \) denote any of the \( \Omega \)-spectra-valued functors of [11] p.251-253. Then Farrell and Jones conjecture that

\[
\mathbb{H}_*(B(X), \zeta_*(\rho)) \xrightarrow{\zeta_*(f) \circ A_*} \zeta_*(X)
\]

is an equivalence of the \( \Omega \)-spectra, where \( A_* \) is an “assembly map” for the
simplicial stratified fibration $\rho : \mathcal{E}(X) \to \mathcal{B}(X)$ (see [11] p.257) and $\zeta_*(f)$ is the image of $f : \mathcal{E}(X) \to X$ under the functor $\zeta(*)$.

Let $\mathcal{P}(*,\_)$ denote the functor that maps $X$ to the $\Omega$-spectrum of stable topological pseudoisotopies on $X$. The following result was proven in [11]:

**Farrell-Jones Isomorphism Proposition for $\mathcal{P}(*,\_)$ 1.2:** The above conjecture is true for the functor $\mathcal{P}(*,\_)$ on the space $X$ provided that there exits a simply connected symmetric Riemannian manifold $M$ with non positive sectional curvature everywhere such that $M$ admits a properly discontinuous cocompact group action of $\Gamma = \pi_1(X)$ by isometries of $M$.

Follows that the Farrell-Jones Isomorphism Conjecture holds for crystallographic groups.

The relationship between $\mathcal{P}(*,\_)$ and lower algebraic $K$-theory is given by

**Proposition 1.3:** [11]

$$
\pi_j(\mathcal{P}(*,\_)(X)) = \begin{cases} 
K_{j+2}(\mathbb{Z}[\pi_1(X)]), & \text{if } j \leq -3 \\
K_0(\mathbb{Z}[\pi_1(X)]), & \text{if } j = -2 \\
Wh(\pi_1(X)), & \text{if } j = -1.
\end{cases}
$$

Finally, the following result aids in the calculation of the lower $K$-theory of $\Gamma$.

**Proposition 1.4:** (see [19]) Let $f : E \to X$ be a simplicially stratified fibration. Then there is a spectral sequence with $E^2_{i,j} = H_i(X, \pi_j \zeta_*(f))$ which abuts to $H_{i+j}(X, \zeta_*(f))$.

2 A model for the universal $(\Gamma, VC(\Gamma))$-space

In this section we construct a model for the universal $(\Gamma, VC(\Gamma))$-space, where $\Gamma$ is a crystallographic group and $VC(\Gamma)$ is the set of virtually cyclic subgroups of $\Gamma$. 

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Let $\Gamma$ be a n-crystallographic group. Then $\Gamma$ fits in the short exact sequence of groups

$$0 \to T \overset{i}{\to} \Gamma \overset{p}{\to} F \to 1$$

with $T$ isomorphic to $\mathbb{Z}^n$ and $F$ a finite subgroup of $O(n)$. Every $\gamma \in \Gamma$ acts on $\mathbb{R}^n$ as:

$$\gamma x = Jx + b, \text{ where } J = p\gamma \in O(n) \ b \in \mathbb{R}^n.$$ 

for all $x \in \mathbb{R}^n$. Note that

$$\gamma^{-1}x = J^{-1}x - J^{-1}b.$$ 

The elements in $T$ act on $\mathbb{R}^n$ by translations. For $h \in \mathbb{R}^n$ we write $\tau_h$ for the translation by $h$, i.e.,

$$\tau_h : \mathbb{R}^n \to \mathbb{R}^n, \tau_h x = x + h.$$ 

Let $H = \{h; \tau_h \in T\}$. Then $H < \mathbb{R}^n$ is an additive subgroup of $\mathbb{R}^n$.

Remark that, if $h \in H$ and $\gamma \in \Gamma$, then $(p\gamma)h \in H$. To see this, first notice that $\gamma \tau_h \gamma^{-1} \in \Gamma$. Then

$$(\gamma \tau_h \gamma^{-1})x = x + Jh = \tau_{Jh}x \Rightarrow Jh = (p\gamma)h \in H.$$ 

Define $C^n := \mathbb{R}^n / \sim$, where $x \sim y \Leftrightarrow x = \pm y$. Hence $C^n$ is homeomorphic to $C(\mathbb{R}P^{n-1})$, the open cone over the real projective space $\mathbb{R}P^{n-1}$. The vertex of $C^n$ is the point $[0] \in C^n$. A subset of the form $\{[\lambda x], \lambda \in \mathbb{R}\}$, $x \neq 0$ is called a ray of $C^n$.

Note that every element $[x] \in C^n$ has a well defined “norm” $||[x]|| = ||x|| = ||-x||$, that satisfies $||[\lambda x]|| = |\lambda||x||, \lambda \in \mathbb{R}$.

Note also that $\Gamma$ acts on $C^n$ via $p$:

$$\gamma[x] = [(p\gamma)x].$$

Consequently $\Gamma$ acts on $C^n \times \mathbb{R}^n$ diagonally:

$$\gamma([x], y) = (\gamma[x], \gamma y) = ([(p\gamma)x], \gamma y).$$
For each $h \in H < \mathbb{R}^n, h \neq 0$, write ( $\parallel$ means “parallel”)

$$l_h = \mathbb{R}h = \{th, t \in \mathbb{R}\}.$$  

$$\mathcal{L}_h = \{l, \text{ line in } \mathbb{R}^n; l \parallel l_h\}.$$  

$$\mathcal{L} = \bigcup_{h \in H} \mathcal{L}_h.$$  

$$\Lambda = \{\mathcal{L}_h, h \in H\}.$$  

Note that $\mathcal{L}_h$, with the quotient topology, is homeomorphic to $\mathbb{R}^{n-1}$. The action of $\Gamma$ on $C^n$ induce actions of $\Gamma$ on

(1) $\mathcal{L}$, because $l \parallel l_h \Rightarrow \gamma l \parallel l_{(\gamma)h}$, for $h \in H, \gamma \in \Gamma$.

(2) $\Lambda$, by $\gamma \mathcal{L}_h = \mathcal{L}_{(\gamma)h}$.

Note that the previous remark implies that the action is well-defined. Also, if $\gamma$ is a translation,

$$\gamma \mathcal{L}_h = \mathcal{L}_h, \forall h \in H.$$  

Since $p\gamma \in F$ and $F$ is finite, we have that the orbit $\Gamma \mathcal{L}_h$ of $\mathcal{L}_h$ is finite, $\forall h \in H$. Enumerate these orbits: $\Lambda_1, \Lambda_2, \Lambda_3, ...$ then:

(1) Each $\Lambda_k$ is finite.

(2) $\Gamma \Lambda_k = \Lambda_k$.

(3) $\bigcup_k \Lambda_k = \Lambda$.

For each $l \in \mathcal{L}$ define $c(l) \in C^n$, the height of $l$, in the following way: If $l \in \mathcal{L}$, then $l \parallel l_h$ for some $h \in H$. Hence there is $k$ such that $\mathcal{L}_h \in \Lambda_k$. Define $c(l) = \left[\frac{kh}{\|h\|}\right] \in C^n$. Note that $c(l)$ is well defined. Note also that $\gamma c(l) = c(\gamma l)$ and that $c$ is constant on each $\mathcal{L}_h$. 

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For $l \in \mathcal{L}$, write $\bar{l} = \{c(l)\} \times l \subset C^n \times \mathbb{R}^n$. We think of $\bar{l}$ as the line $l \subset \mathbb{R}^n$ lifted to a height $c(l) \in C^n$. Define $\bar{\mathcal{L}} = \{\bar{l}, l \in \mathcal{L}\}$, and $\bar{\mathcal{L}}_h = \{\bar{l}, l \in \mathcal{L}_h\}$. Note that $\bar{\mathcal{L}} = \bigcup_{h \in H} \bar{\mathcal{L}}_h$. Then $\bar{\mathcal{L}}$ is a set of disjoint “lifted lines” in $C^n \times \mathbb{R}^n$.

Finally, define $A = C^n \times \mathbb{R}^n / \cong$, where $x \cong y \iff x = y$ or $x, y \in \bar{l}$, for some $\bar{l} \in \bar{\mathcal{L}}$, i.e., $A$ is obtained from $C^n \times \mathbb{R}^n$ by identifying each line in $\bar{\mathcal{L}}$ to a point.

Let $Y = \bigcup_{l \in \mathcal{L}} \bar{l} \subset C^n \times \mathbb{R}^n$. Then $\Gamma Y = Y$ and $\Gamma((C^n \times \mathbb{R}^n) - Y) = (C^n \times \mathbb{R}^n) - Y$. Hence the action of $\Gamma$ on $C^n \times \mathbb{R}^n$ induces an action of $\Gamma$ on $A$. Let $\pi : C^n \times \mathbb{R}^n \to A$ be the collapsing map, and write $Z = \pi Y$. Then $\Gamma Z = Z$ and $\Gamma(A - Z) = A - Z$. Note that $\pi : (C^n \times \mathbb{R}^n) - Y \to A - Z$ is a $\Gamma$-equivariant homeomorphism. Since $\bar{\mathcal{L}} = \bigcup_{h \in H} \bar{\mathcal{L}}_h$, we have $Y = \bigcup_{h \in H} (\bigcup_{l \in \mathcal{L}_h} \bar{l})$. Consequently $Z = \pi Y = \bigcup_{h \in H} \pi Y_h = \bigcup_{h \in H} Z_h$, where $Y_h = \bigcup_{\bar{l} \in \bar{\mathcal{L}}_h} \bar{l}$ and $Z_h = \pi Y_h$.

The action of $\Gamma$ on $Z$

Since the connected components of $Z$ are the $Z_h$'s, we have that either $\gamma Z_h = Z_h$ or $\gamma Z_h \cap Z_h = \emptyset$, for $\gamma \in \Gamma$. Consider $\Gamma(Z_h) = \{\gamma \in \Gamma; \gamma Z_h = Z_h\}$. Note that $\Gamma(Z_h) = \Gamma(\mathcal{L}_h) = \Gamma(\mathcal{L}_h)$. We want to study the action of $\Gamma(Z_h)$ on $Z_h$, or equivalently, the action of $\Gamma(\mathcal{L}_h)$ on $\mathcal{L}_h$. For this we need an explicit homeomorphism between $\mathcal{L}_h$ and $\mathbb{R}^{n-1}$. Define $\alpha_h := (l_h)_{\perp} \subset \mathbb{R}^n$, i.e., $\alpha_h$ is the $(n-1)$-space orthogonal to the line spanned by $h$. Then for every $l \in \mathcal{L}_h$, we define $p_l$ to be the unique point such that $\{p_l\} := l \cap \alpha_h$. Write $\varphi(l) = p_l$. It is easy to verify that $\varphi$ is a homeomorphism between $\mathcal{L}_h$ and $\mathbb{R}^{n-1}$.

Let $\gamma \in \Gamma$. Then there are $J \in O(n), a \in \mathbb{R}^n$ such that $\gamma x = Jx + a$, for all $x \in \mathbb{R}^n$. We write $\gamma = (J, a)$. Then $\gamma \in \Gamma(\mathcal{L}_h)$ if and only if $Jh = \pm h$. This implies that $Jl_h = l_h$ and $J\alpha_h = \alpha_h$. Hence, $J|_{\alpha_h} : \alpha_h \to \alpha_h$ and $J|_{\alpha_h} \in O(\alpha_h)$ (\alpha_h with the scalar product of \mathbb{R}^n). Write $a = a_0 + \lambda h$, with $a_0 \in \alpha_h$, i.e., $a_0 = \text{Proj}_{\alpha_h} a$. Define $\bar{\gamma} : \alpha_h \to \alpha_h$, $\bar{\gamma} := \varphi \gamma \varphi^{-1}$. Hence we have the commutative diagram:

\[
\begin{array}{ccc}
\mathcal{L}_h & \xrightarrow{\gamma} & \mathcal{L}_h \\
\downarrow \varphi & & \downarrow \varphi \\
\alpha_h & \xrightarrow{\bar{\gamma}} & \alpha_h
\end{array}
\]
A simple calculation shows that $\tilde{\gamma} = (J|_{\alpha_h}, a_0)$, hence $\Gamma^{(L_h)}$ acts by isometries on $L_h$, where we consider $L_h$ as a vector space with scalar product obtained by identifying $L_h$ with $\alpha_h \subset \mathbb{R}^n$. Moreover, it is not difficult to show that this action is crystallographic, i.e., properly discontinuous and cocompact (this is proved, with more generality, in [11] p.267).

The triangulation of $A$

$C^n$ and $\mathbb{R}^n$ are $PL$ $\Gamma$-spaces, hence $C^n \times \mathbb{R}^n$ is a $PL$ $\Gamma$-space. Let $\mathcal{T}_C$ be a $\Gamma$-equivariant triangulation of $C^n$ and $\mathcal{T}_R$ a $\Gamma$-equivariant triangulation of $\mathbb{R}^n$. Let $\tilde{\mathcal{T}}$ be the cell structure on $C^n \times \mathbb{R}^n$ with product cells $\sigma_C \times \sigma_R, \sigma_C \in \mathcal{T}_C, \sigma_R \in \mathcal{T}_R$. Then $\tilde{\mathcal{T}}$ is $\Gamma$-equivariant. We can suppose that each $Y_h$ is a subcomplex of $C^n \times \mathbb{R}^n$ (because the projection of each $Y_h$ into $C^n$ is a point). Since the triangulations $\mathcal{T}_C$ and $\mathcal{T}_R$ are $\Gamma$-equivariant, we have that $\tilde{\mathcal{T}}$ induces a $\Gamma$-equivariant cell structure $\mathcal{T}$ on $A$ with the following properties:

(1) Each $Z_h$ is a subcomplex of $A$.

(2) If $\sigma$ is an open $k$-cell in $A - Z \cong (C^n \times \mathbb{R}^n) - Y$, then $\sigma = \sigma_C^i \times \sigma_R^j$, $k = i + j$, where $\sigma_C^i$ and $\sigma_R^j$ are open simplices in $C^n$ and $\mathbb{R}^n$, respectively.

We say that a 1-cell $\sigma_C^1$ in $C^n$ is a ray cell if $\sigma_C^1$ is contained in a ray of $C^n$.

**Theorem 2.1:** $\Gamma \times A \longrightarrow A$ is an universal $(\Gamma, VC(\Gamma))$-space.

**Proof:** We must verify the following properties:

(1) $A$ can be equipped with a cell structure $K$ such that $\Gamma \times A \longrightarrow A$ is a cellular action. Moreover, for $\gamma \in \Gamma$ and $\sigma \in K$ we have that $\gamma(\sigma) = \sigma$, implies $\gamma|_{\sigma}$=inclusion.

(2) For any $p \in A$ we have that $\Gamma^p \in VC(\Gamma)$.
For each subgroup $G \in VC(\Gamma)$ we have that $A^G$ is a nonempty contractible subcomplex of $K$.

(1) follows from the discussion above, taking $K = T$.

(To get that $\gamma(\sigma) = \sigma$, implies $\gamma|_\sigma = \text{inclusion}$, we may have to subdivide $T_C$ and $T_R$.) To verify (2) we first note that if $p \in A - Z$ then $\Gamma^p$ is a finite group. On the other hand, if $p \in Z$, then $p \in Z_h$, for some $h \in H$. Hence $\Gamma^p = \{ \gamma \in \Gamma : \gamma(q + IRh) = q + IRh \}$, where $\pi^{-1}(p) = q + IRh$. Thus $\Gamma^p$ acts properly discontinuously cocompactly by isometries on the line $q + IRh$.

Consequently, by prop. 1.1, $\Gamma^p \in VC(\Gamma)$.

Finally we verify (3). First consider the case where $G \in VC(\Gamma)$ contains an element $g \in G$ of infinite order, i.e., $G$ contains translations. Then $A^G \cap ((C^n \times IR^n) - Y) = \emptyset$. Hence $A^G \subset Z$. Moreover $A^G \subset Z_h$, where $h \in H$ is such that $\tau_h \in G$. Let $l = a + IRh$ be the line such that $Gl = l$, i.e., $G < \Gamma^{(l)}$ (see prop. 1.1). Also $[l] \in Z_h \subset Z \subset A$. Since $Gl = l$, we have that $G[l] = [l]$, that is $[l] \in A^G$. Follows that $A^G \neq \emptyset$.

Note that $G < \Gamma^{(Z_h)}$ and $A^G \subset Z_h$ is precisely the fixed point set of the $G$ action on $Z_h$. We know that the action of $\Gamma^{(Z_h)}$ on $Z_h$ is linear (in fact, it is crystallographic). Hence $A^G = \cap_{g \in G} A^{\{g\}}$, and each $A^{\{g\}}$ is a vector space. Therefore $A^G$ is contractible.

Consider now the case where $F \in VC(\Gamma)$ is finite.

Suppose first that $F$ is trivial, i.e. $A^F = A$. We prove that $A$ is contractible. (To prove this we cannot just use the fact that the collapsing map $\pi : C^n \times IR^n \to A$ is cell-like, because $\pi$ is not proper.)

For $0 \neq x \in IR^n$, let $\alpha_x$ be the plane orthogonal to IR$x$ in IR$n$, and for $v \in IR^n$ define $proj_x v$ to be the orthogonal projection of $v$ in $\alpha_x$. We define a homotopy $h_t$ on $C^n \times IR^n$ that moves a point $(\{x\}, v) \in C^n \times IR^n$, $x$ away from 0, toward $(\{x\}, proj_x v)$:

$h_t([x], v) = ([x], (\delta(||x||)t)proj_x v + (1 - \delta(||x||)t)v)$

where $\delta : [0, \infty) \to [0, 1]$ is a continuous function such that $\delta(t) = 0$ for $0 \leq t \leq 1/3$, $\delta(t) = 1$ for $2/3 \leq t$ and $0 < \delta(t) < 1$ for $1/3 < t < 2/3$. 

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Then \( h_0 \) is the identity and \( h_1(C^n \times \mathbb{R}^n) - (B_{2/3} \times \mathbb{R}^n) = \{(x, v) : v \in \alpha_x, \ 2/3 \leq ||x||\} \). Here \( B_{2/3} = \{x \in C^n, ||x|| < 2/3\} \).

It is straightforward to prove that \( h_t : C^n \times \mathbb{R}^n \to C^n \times \mathbb{R}^n \) defines a homotopy \( H_t : A \to A \), with \( H_t \pi = \pi h_t \), where \( \pi : C^n \times \mathbb{R}^n \to A \) is the collapsing map.

Since the collapsing occurs away from \( B_{2/3} \times \mathbb{R}^n \), we have that \( \pi|_{h_1(C^n \times \mathbb{R}^n)} \) is an embedding. Hence \( A = H_0 A \sim H_1 A = H_1 \pi(C^n \times \mathbb{R}^n) = \pi h_1(C^n \times \mathbb{R}^n) \cong_{\text{TOP}} h_1(C^n \times \mathbb{R}^n) \sim h_0(C^n \times \mathbb{R}^n) = C^n \times \mathbb{R}^n \sim \text{pt} \)

(\( \sim \) means “homotopy equivalent” and \( \cong_{\text{TOP}} \) means “homeomorphic”.) This proves that \( A \) is contractible.

Suppose now that \( F \) is any finite subgroup of \( \Gamma \). Hence there is \( x_0 \in \mathbb{R}^n \) such that \( F x_0 = x_0 \). Thus \( ([0], x_0) \in A^F \).

We prove that \( A^F \) is contractible. Let \( B = \pi^{-1}(A^F) \). Without loss of generality, we can suppose that \( x_0 = 0 \in \mathbb{R}^n \), that is, \( F 0 = 0 \). Then, every \( f \in F \) is linear. Define a subspace \( E_0 \) and a set \( E_1 \) in the following way:

(1) \( E_0 = \{y \in \mathbb{R}^n; fy = y \text{ for all } f \in F\} \).

(2) \( E_1 = \{y \in \mathbb{R}^n; fy = \pm y \text{ for all } f \in F \text{ and there is } f \in F \text{ such that } fy = -y\} \).

Note that \( E_1 \) is a cone (i.e. \( y \in E_1 \) implies \( \lambda y \in E_1, \lambda \in \mathbb{R} \)). Note also that \( E_0 \cap E_1 = \{0\} \). Moreover, if \( y \in E_1 \) then \( y \perp E_0 \).

Let \( PE_i \subset \mathbb{RP}^{n-1} \) be the projectivization of \( E_i \subset \mathbb{R}^n \) and \( C(PE_i) \subset C(\mathbb{RP}^{n-1}) = C^n \) the cone of \( PE_i \).

Claim 1: \( B = \{(C(PE_0) \cup C(PE_1)) \times E_0\} \cup \{\bigcup_{i \in I,F_i=I} I\} \).

Proof: Consider \( B_1 = B \cap [(C^n \times \mathbb{R}^n) - Y] \) and \( B_2 = B \cap Y \). Then \( B = B_1 \coprod B_2 \). Note that if \( ([x], y) \in (C(PE_0) \cup C(PE_1)) \times E_0 \), then \( F y = y \) and
$Fx = \pm x$. Hence $([x], y) \in B$. Note also that $B_2 = B \cap Y = \bigcup_{l \in \mathcal{L}, Fl = l} \bar{l} \subset B$. Therefore

$$\{ [C(PE_0) \cup C(PE_1)] \times E_0 \} \cup \bigcup_{l \in \mathcal{L}, Fl = l} \bar{l} \subset B.$$  

Conversely if $([x], y) \in B_1$, then $Fy = y$ and $Fx = \pm x$. Thus $y \in E_0$ and $x \in E_0$ or $E_1$. Hence $[x] \in C(PE_0) \cup C(PE_1)$. Therefore $([x], y) \in [C(PE_0) \cup C(PE_1)] \times E_0$. This proves the claim.

**Claim 2:** If $Fl = l$, then $l \cap E_0 \neq \emptyset$. Moreover either $l \subset E_0$ or $l \subset E_1 + y$, $\{y\} = l \cap E_0$.

**Proof:** Since $Fl = l$ and $F$ is finite, there is $y \in l$, such that $Fy = y$. Hence $y \in E_0$, and follows that $l \cap E_0 = \{y\} \neq \emptyset$.

Suppose $l \not\subset E_0$. We know that $l \parallel l_h$, for some $h \in H$. Then $y \pm h \in l$, where $\{y\} = l \cap E_0$. Since $fy = y$ for all $f \in F$, we have that $f(y + h) = y \pm h$, for all $f \in F$. Follows that $f(h) = \pm h$, for all $f \in F$. Since there is a $f \in F$ such that $f(y + h) = y - h$, follows that $fh = -h$ for some $f \in F$. Hence $h \in E_1$. Therefore $l \subset E_1 + y$. This proves the claim.

If $l \subset E_0$, then $l \parallel l_h = IRh$, $h \in E_0$. Thus $c(l) \in C(PE_0)$. Consequently, if $a \in l$ we have $(c(l), a) \in C(PE_0) \times E_0$. Hence $\bar{l} \subset C(PE_0) \times E_0$. On the other hand, if $l \subset E_1 + y$, $l \parallel l_h$, $h \in E_1$. Thus $c(l) \in C(PE_1)$. If $a \in l$, then $(c(l), a) \in [C(PE_0) \cup C(PE_1)] \times E_0$ if and only if $a \in E_0$, i.e., $a = y$. Therefore $\bar{l} \cap [C(PE_0) \cup C(PE_1)] \times E_0 = \{\text{point}\}$.

Follows from the discussion above that collapsing every line $\bar{l}$, with $l$ not in $E_0$, to its (unique) point of intersection with $[C(PE_0) \cup C(PE_1)] \times E_0$, we obtain a homeomorphism of $A$ onto the space $\pi([C(PE_0) \cup C(PE_1)] \times E_0)$. Moreover, since there is no collapsing among points in $C(PE_1) \times E_0$, deforming $\pi(C(PE_1) \times E_0)$ to $\{[0]\} \times E_0$ we see that $A$ is homotopy equivalent to $\pi(C(PE_0) \times E_0)$, which is obtained from $C(PE_0) \times E_0$ by collapsing to a point every line $\bar{l} \subset C(PE_0) \times E_0$, with $l \subset E_0$, $l \in \mathcal{L}$. But an argument similar to the one used before (to prove that $A = \pi(C^n \times IR^n)$ is contractible) shows that $\pi(C(PE_0) \times E_0)$ is contractible. This proves property (3). Therefore $\Gamma \times A \longrightarrow A$ is an universal $(\Gamma, V C(\Gamma))$-space. This proves the theorem.
3 Virtually infinite cyclic subgroups in dimension three

In this section we calculate, modulo isomorphism, the virtually infinite cyclic subgroups of a 3-crystallographic group.

In what follows, $\mathbb{Z}_i$ will denote the cyclic group of order $i$, $D_i$ the dihedral group of order $2i$, $S_n$ the permutation group of order $n$! and $A_n$ will denote the alternating group of order $n!/2$.

We will use the following facts about a 3-crystallographic group $\Gamma$ and a finite subgroup $F$ of $\Gamma$.

(a) If $\gamma \in \Gamma$ has finite order, then $1 \leq |\gamma| \leq 6$ and $|\gamma| \neq 5$. This is due to the crystallographic restriction (see [24], p.32).

(b) If $F$ is finite, then $F$ is trivial or isomorphic to one of the following groups: $\mathbb{Z}_i$, $D_i$, $\mathbb{Z}_i \times \mathbb{Z}_2$, $D_i \times \mathbb{Z}_2$, $A_4$, $A_4 \times \mathbb{Z}_2$, $S_4$, $S_4 \times \mathbb{Z}_2$, $i = 2, 3, 4, 6$ (see [24], p.49).

(c) If $F$ is isomorphic to one of the groups $A_4$, $A_4 \times \mathbb{Z}_2$, $S_4 \times \mathbb{Z}_2$, then $Fl \neq l$ for all lines in $\mathbb{R}^3$, i.e., $F$ does not leave any line in $\mathbb{R}^3$ invariant. In particular $(\mathbb{R}^3)^F = \{x \in \mathbb{R}^3; \gamma x = x, \text{ for all } \gamma \in F\} = \{\text{point}\}$ (see [24], p.48).

(d) If $F$ fixes a point $x_0$ and leaves invariant a line $l$ that contains $x_0$ (or equivalently, a plane $\alpha$ that contains $x_0$) then, by (c) and (b) above, follows that $F$ is trivial or isomorphic to one of the groups $\mathbb{Z}_i$, $D_i$, $\mathbb{Z}_i \times \mathbb{Z}_2$, $D_i \times \mathbb{Z}_2$, $i = 2, 3, 4, 6$.

Now, let $G$ be a virtually infinite cyclic subgroup of a 3-crystallographic group $\Gamma$. Recall that $G$ leaves invariant a line $l$ in $\mathbb{R}^3$, i.e., $G < \Gamma^{(l)}$, and $G$ acts cocompactly on $l$ (see prop. 1.1). Hence we have two possibilities:

(1) $G$ acts by translations and by reflections on $l$, i.e., $G$ has a dihedral action on $l$.
(2) $G$ acts just by translations on $l$.

In the first case, we have a surjection $G \to D_\infty$, and in the second case we have a surjection $G \to \mathbb{Z}$. In any case, we have

$$0 \to F \hookrightarrow G \xrightarrow{\rho} H \to 0,$$

where $H$ is isomorphic to $D_\infty$ or $\mathbb{Z}$, $\rho(g) = g|_l : l \to l$, and $F$ is the subgroup of all elements $g \in G$, that fix $l$ pointwise, i.e., $g|_l = 1_l$. Hence $F$ acts on a plane $\alpha$ orthogonal to $l$ (choose any plane orthogonal to $l$). By (a) above we have that $F$ is trivial or isomorphic to $\mathbb{Z}$ or $D_i$, $i = 2, 3, 4, 6$. We analyze the two cases separately.

$$H \cong D_\infty$$

Then $G$ has a dihedral action on $l$. Write $D_\infty = \mathbb{Z}_2^a \ast \mathbb{Z}_2^b$, where $a$ and $b$ are reflections about some points $p_a, p_b \in l$, $p_a \neq p_b$. Thus $G = G^a \ast_F G^b$, where $G^a = \rho^{-1}(\mathbb{Z}_2^a)$ and $G^b = \rho^{-1}(\mathbb{Z}_2^b)$ and $F \hookrightarrow G^a, F \hookrightarrow G^b$ are the inclusions (see [23], p.178).

**Lemma 3.1:** Let $F$ and $G$ as above. Then

(1) $F$ has index two in $G^a$ and $G^b$, i.e. $|G^a/F| = |G^b/F| = 2$.

(2) $G^a$ and $G^b$ are the isotropies of $p_a$ and $p_b$ respectively, i.e. $G^a = G^{\{p_a\}}$ and $G^b = G^{\{p_b\}}$.

**Proof of (1):** Follows from the definitions that $G^a/F \cong G^b/F \cong \mathbb{Z}_2$.

**Proof of (2):** Let $\tilde{a} \in G$, such that $\rho(\tilde{a}) = a$. If $g \in G^a$, then $g = f$ or $g = \tilde{a}f$, for some $f \in F$. Since $\tilde{a}, f \in G^{\{p_a\}}$, follows that $g \in G^{\{p_a\}}$. Conversely, if $g \in G^{\{p_a\}}$, $gp_a = p_a$, hence $\rho(g)(p_a) = p_a$. Thus $\rho(g)$ is the identity or $a \in D_\infty$. This proves $G^a = G^{\{p_a\}}$. The proof of $G^b = G^{\{p_b\}}$ is identical.

**Proposition 3.2:** Let $G$ be a subgroup of a 3-crystallographic group. Suppose
that $G$ has a dihedral action on a line $l$. Then $G$ is isomorphic to $G^a \ast_F G^b$ where the possibilities for $(F, G^a, G^b)$ are shown in table 1.

$$
\begin{array}{|c|c|c|}
\hline
F & G^a & G^b \\
\hline
\text{trivial} & \mathbb{Z}_2 & \mathbb{Z}_2 \\
\mathbb{Z}_2 & \mathbb{Z}_4 & \mathbb{Z}_4 \\
 & D_2 & \mathbb{Z}_4 \\
 & D_3 & D_2 \\
\mathbb{Z}_3 & \mathbb{Z}_6 & \mathbb{Z}_6 \\
 & D_3 & \mathbb{Z}_6 \\
 & D_3 & D_3 \\
\mathbb{Z}_4 & \mathbb{Z}_4 \times \mathbb{Z}_2 & \mathbb{Z}_4 \times \mathbb{Z}_2 \\
 & D_1 & \mathbb{Z}_4 \times \mathbb{Z}_2 \\
 & D_3 & D_4 \\
\mathbb{Z}_6 & \mathbb{Z}_6 \times \mathbb{Z}_2 & \mathbb{Z}_6 \times \mathbb{Z}_2 \\
 & D_6 & \mathbb{Z}_6 \times \mathbb{Z}_2 \\
 & D_6 & D_6 \\
D_2 & D_2 \times \mathbb{Z}_2 & D_2 \times \mathbb{Z}_2 \\
 & D_1 & D_2 \times \mathbb{Z}_2 \\
 & D_1 & D_4 \\
D_3 & D_3 \times \mathbb{Z}_2 & D_3 \times \mathbb{Z}_2 \\
D_4 & D_4 \times \mathbb{Z}_2 & D_4 \times \mathbb{Z}_2 \\
D_6 & D_6 \times \mathbb{Z}_2 & D_6 \times \mathbb{Z}_2 \\
\hline
\end{array}
$$

Table 1: Dihedral action

Remark: In table 1 we do not give the inclusions $F \hookrightarrow G^a$ or $F \hookrightarrow G^b$ because it is straightforward to show that if $i_1 : F \rightarrow G^a$, $i_2 : F \rightarrow G^a$ are one-to-one, then there is $\phi \in Aut(G^a)$ such that $i_2 = \phi \circ i_1$. Analogously for $G^b$. Hence the isomorphism class of $G^a \ast_F G^b$ does not depend on the particular inclusions.

Proof: Suppose that $G$ acts cocompactly on the line $l$. By the discussion above, $0 \rightarrow F \hookrightarrow G \xrightarrow{\rho} \mathbb{Z}_2 \ast \mathbb{Z}_2 \rightarrow 0$ and $G = G^a \ast_F G^b$, where

(1) $F$ is trivial or isomorphic to $\mathbb{Z}_i$, $D_i$, $i = 2, 3, 4, 6$. 

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(2) $F$ has index two in $G^a$ and $G^b$.

(3) $F = \{ g \in G ; g|_l = 1 \}$, $G^a = G^{\{p_a\}}$ and $G^b = G^{\{p_b\}}$ for some $p_a, p_b \in I$, $p_a \neq p_b$.

Note that, since $G^a$, $G^b$ fix a point in $\mathbb{R}^3$, $G^a$, $G^b$ are isomorphic to one of the following groups $\mathbb{Z}_i, \mathbb{Z}_i \times \mathbb{Z}_2, D_i, D_i \times \mathbb{Z}_2, A_4, A_4 \times \mathbb{Z}_2, S_4, S_4 \times \mathbb{Z}_2, i = 2, 3, 4, 6$. By (1) and (2) above and property (d) (mentioned at the beginning of this section) we have the following possibilities, shown in table 2:

| $F$     | $G^a$     | $G^b$     |
|---------|-----------|-----------|
| trivial | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |
| $\mathbb{Z}_2$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |
|         | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
|         | $D_2 \times \mathbb{Z}_2$ | $D_2 \times \mathbb{Z}_2$ |
| $\mathbb{Z}_3$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |
|         | $D_2 \times \mathbb{Z}_2$ | $D_2 \times \mathbb{Z}_2$ |
| $\mathbb{Z}_4$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |
|         | $D_2 \times \mathbb{Z}_2$ | $D_2 \times \mathbb{Z}_2$ |
| $\mathbb{Z}_6$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |
|         | $D_2 \times \mathbb{Z}_2$ | $D_2 \times \mathbb{Z}_2$ |
| $D_2$   | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |
|         | $D_2 \times \mathbb{Z}_2$ | $D_2 \times \mathbb{Z}_2$ |
| $D_3$   | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |
|         | $D_2 \times \mathbb{Z}_2$ | $D_2 \times \mathbb{Z}_2$ |
| $D_4$   | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |
|         | $D_2 \times \mathbb{Z}_2$ | $D_2 \times \mathbb{Z}_2$ |
| $D_6$   | $\mathbb{Z}_2 \times \mathbb{Z}_2$ | $\mathbb{Z}_2 \times \mathbb{Z}_2$ |
|         | $D_2 \times \mathbb{Z}_2$ | $D_2 \times \mathbb{Z}_2$ |

Table 2:
The group \( \mathbb{Z}_4 \times \mathbb{Z}_2 \) acts by isometries on \( \mathbb{R}^3 \) in a unique way (modulo conjugation). That is, the \( \mathbb{Z}_4 \) factor fixes a line \( \tilde{l} \) pointwise and acts on a plane \( \tilde{\alpha} \) orthogonal to \( \tilde{l} \), by rotations. The factor \( \mathbb{Z}_2 \) acts trivially on \( \tilde{\alpha} \) and by reflection on \( \tilde{l} \). Note that the only line in \( \mathbb{R}^3 \) fixed (as a set) by \( \mathbb{Z}_4 \times \mathbb{Z}_2 \) is \( \tilde{l} \). If \( F = D_2 \) and \( G = \mathbb{Z}_4 \times \mathbb{Z}_2 \), then \( \mathbb{Z}_2 \) acts trivially on \( \tilde{l} \) and by reflections on \( \tilde{\alpha} \). Also \( \mathbb{Z}_4 \) acts trivially on \( l \) and by rotations on \( \alpha \). But then, by (3) above, we should have \( D_2 \cong \mathbb{Z}_4 \), which is a contradiction. Therefore the groups \( (\mathbb{Z}_4 \times \mathbb{Z}_2)^* D_2 \), \( (\mathbb{Z}_4 \times \mathbb{Z}_2)^* D_4 \) and \( (\mathbb{Z}_4 \times \mathbb{Z}_2)^* D_2 (D_2 \times \mathbb{Z}_2) \) do not occur geometrically as virtually infinite cyclic subgroups of a 3-crystallographic group \( \Gamma \). This proves the proposition.

\[
H \cong \mathbb{Z}
\]

Then \( G \) acts only by translations on \( l \) and fits in the exact sequence

\[
0 \to F \to G \to H \to 0,
\]

with \( F \) trivial or isomorphic to \( \mathbb{Z}_i \) or \( D_i \), \( i = 2, 3, 4, 6 \). Thus \( G = F * F \cong F \rtimes \varphi \mathbb{Z} \) for some automorphism \( \varphi : F \to F \) (see [23], p.178).

To give an action of \( \mathbb{Z} \) on \( F \), means to give an automorphism \( \varphi \) of \( F \). Since we want to classify the groups \( F * F \cong F \rtimes \varphi \mathbb{Z} \) modulo isomorphism, \( \varphi \) must be an outer automorphism, i.e., \( \varphi \in Out(F) \). Moreover if \( \varphi_1, \varphi_2 \) are conjugate in \( Out(F) \), \( F \rtimes \varphi_1 \mathbb{Z} \) and \( F \rtimes \varphi_2 \mathbb{Z} \) are isomorphic. A direct calculation shows that:

(1) \( Out(F) \) is trivial, for \( F \) isomorphic to \( \mathbb{Z}_2 \) or \( D_3 \).

(2) \( Out(F) \cong \mathbb{Z}_2 \), for \( F \) isomorphic to \( \mathbb{Z}_4, \mathbb{Z}_6, D_4 \) or \( D_6 \).

(3) \( Out(D_2) \cong D_3 \).

Note that \( D_3 \) has only three conjugacy classes: \([1], [\varphi], [\phi]\), where \( \varphi : D_2 \to D_2 \) has order 2 and \( \phi : D_2 \to D_2 \) has order 3.

When does a group of the form \( F \rtimes \varphi \mathbb{Z} \), with \( F \) trivial or isomorphic to \( \mathbb{Z}_i \) or \( D_i \), \( i = 2, 3, 4, 6 \), occur as a virtually infinite cyclic subgroup of a
3-crystallographic group?

To simplify the notation, assume that \( l \) is the \( z \)-axis. Follows that we can take \( \alpha = l^\perp \cong \mathbb{R}^2 \cong \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3 \). We can also assume that every \( g \in G \) is of the form \( g = (L, ne_3) \), \( L \in O(2) \subset O(3) \), \( n \in \mathbb{Z} \), \( e_3 = (0,0,1) \) (this means \( gx = Lx + ne_3 \), for all \( x \in \mathbb{R}^3 \)) and the action of \( G \) on \( l = \mathbb{R}e_3 \) is generated by \( \tau_{e_3} \), where \( \tau_{e_3} \) denotes translation by \( e_3 \). Since \( \rho \) is onto, there is \( J \in O(2) \subset O(3) \) such that \( (J, e_3) \in G \). Then the action of \( H \cong \mathbb{Z} \) on \( F \) is generated by conjugation by \( g_0 = (J, e_3) \), that is, \( g_0^{-1}fg_0 = J^{-1}fJ = \varphi f \). Hence \( F \rtimes \varphi \mathbb{Z} \) occurs geometrically if there is a \( J \in O(2) \) such that \( J^{-1}fJ = \varphi f \) for all \( f \in F \).

**Proposition 3.3:** Let \( G \) be a subgroup of a 3-crystallographic group. Suppose that \( G \) acts cocompactly by translations on a line \( l \). Then \( G \) is isomorphic to \( F \rtimes F_\varphi \), where the possibilities for \( (F, F \rtimes F_\varphi) \) are shown in table 3:

| \( F \) | \( F \rtimes F_\varphi \) |
|--------|------------------|
| trivial | \( \mathbb{Z} \) |
| \( \mathbb{Z}_2 \) | \( \mathbb{Z}_2 \times \mathbb{Z} \) |
| \( \mathbb{Z}_3 \) | \( \mathbb{Z}_3 \times \mathbb{Z} \) |
| & \( \mathbb{Z}_3 \rtimes \varphi \mathbb{Z} \) |
| \( \mathbb{Z}_4 \) | \( \mathbb{Z}_4 \times \mathbb{Z} \) |
| & \( \mathbb{Z}_4 \rtimes \varphi \mathbb{Z} \) |
| \( \mathbb{Z}_6 \) | \( \mathbb{Z}_6 \times \mathbb{Z} \) |
| & \( \mathbb{Z}_6 \rtimes \varphi \mathbb{Z} \) |
| \( D_2 \) | \( D_2 \times \mathbb{Z} \) |
| & \( D_2 \rtimes \varphi \mathbb{Z} \) |
| \( D_3 \) | \( D_3 \times \mathbb{Z} \) |
| \( D_4 \) | \( D_4 \times \mathbb{Z} \) |
| & \( D_4 \rtimes \varphi \mathbb{Z} \) |
| \( D_6 \) | \( D_6 \times \mathbb{Z} \) |
| & \( D_6 \rtimes \varphi \mathbb{Z} \) |

Table 3: Action by translations

where \( \varphi \neq 1 \in \text{Out}(F) \), and for \( F = D_2 \), \( \varphi \) has order 2.
Proof: By the discussion above, we must determine an automorphism \( \varphi : F \to F \) and \( J \in O(2) \) such that \( J^{-1} fJ = \varphi f \), for all \( f \in F \).

If \( F \) is trivial, then \( G = \mathbb{Z} \).

If \( F = \mathbb{Z}_i \), \( i = 3, 4, 6 \), then \( \text{Out}(\mathbb{Z}_i) = \text{Aut}(\mathbb{Z}_i) = \{ 1_{\mathbb{Z}}, \varphi \} \), where \( \mathbb{Z}_i = \langle t; t^i = 1 \rangle \) and \( \varphi : \mathbb{Z}_i \to \mathbb{Z}_i \), defined by \( \varphi(t) = t^{i-1} \) and \( t \in \mathbb{Z}_i \) acts on \( \mathbb{R}^2 \) rotating an angle \( 2\pi/i \). Hence, geometrically, we obtain the groups \( \mathbb{Z}_i \rtimes 1_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z}_i \times \mathbb{Z} \), \( i = 2, 3, 4, 6 \) by taking \( J = 1_{\mathbb{R}^2} \). Also, we obtain \( \mathbb{Z}_i \rtimes \varphi \mathbb{Z} \) by taking \( J(x, y) = (x, -y) \), for \( i = 3, 4, 6 \).

Let \( F = D_2 = \{ 1, r_x, r_y, -1 \} \), where \( r_x, r_y \) denote reflections about the \( x \) and \( y \) axes, respectively, and \( (-1) \) denotes rotation by \( \pi \). Then

\[
\text{Out}(D_2) = \text{Aut}(D_2) \cong D_3 = \langle \varphi, \phi; \phi^3 = 1, \varphi^2 = 1, (\varphi\phi)^2 = 1 \rangle.
\]

Since there are three conjugacy classes in \( D_3 \), \([1],[\varphi],[\phi] \), there are at most (modulo isomorphism) three possibilities: \( D_2 \rtimes 1_{D_2} \mathbb{Z} \), \( D_2 \rtimes \varphi \mathbb{Z} \), and \( D_2 \rtimes \varphi^i \mathbb{Z} \). But there is no \( J \in O(2) \) such that \( J^{-1}(-1)J \neq (-1) \). Hence the group \( D_2 \rtimes \varphi \mathbb{Z} \), does not occur geometrically. Therefore there are only two possibilities: \( D_2 \rtimes 1_{D_2} \mathbb{Z} = D_2 \times \mathbb{Z} \), \( D_2 \times \varphi \mathbb{Z} \) which we obtain geometrically taking \( J = 1_{\mathbb{R}^2} \) and \( J(x, y) = (x, -y) \), respectively.

If \( F \) is isomorphic to \( \mathbb{Z}_2 \) or \( D_3 \), then \( \text{Out}(F) = 1 \) and we obtain the groups \( \mathbb{Z}_2 \times \mathbb{Z} \) and \( D_3 \times \mathbb{Z} \).

If \( F \) is isomorphic to \( D_4 \) or \( D_6 \), then \( F = \langle s, t; t^i = 1, s^2 = 1, (st)^2 = 1 \rangle \), \( i = 4, 6 \), and we have \( \text{Out}(D_4) \cong \text{Out}(D_6) \cong \{ 1, \varphi \} \), where \( \varphi(s) = st \) and \( \varphi(t) = t \), \( i = 4, 6 \). Hence there are two possibilities: \( D_i \rtimes 1_{D_i} \mathbb{Z} = \mathbb{Z}_i \times \mathbb{Z} \) and \( D_i \rtimes \varphi \mathbb{Z} \), \( i = 4, 6 \), and we can show that they occur geometrically by taking \( J = 1_{\mathbb{R}^2} \) or equal to a rotation by an angle \( \pi/i \), for \( i = 4, 6 \). This proves the proposition.
4 Isotropy in dimension three

In this section we calculate the possible isotropy groups $\Gamma^{\sigma} = \{\gamma \in \Gamma; \gamma\sigma = \sigma\}$, for an open cell $\sigma$ in the universal $(\Gamma, V C(\Gamma))$-space $A$.

We will need the following lemma.

**Lemma 4.1** Let $G$ be a virtually infinite cyclic subgroup of a 3-crystallographic group $\Gamma$. If $G$ leaves invariant two different lines, then $G$ is trivial or isomorphic to $\mathbb{Z}, D_\infty, \mathbb{Z} \times \mathbb{Z}_2, D_\infty \times \mathbb{Z}_2$.

**Proof:** Suppose that $Gl = l$ and $G\tilde{l} = \tilde{l}$, with $l \neq \tilde{l}$, and $G$ acts cocompactly on $l$ (see prop. 1.1). Hence $l \parallel \tilde{l}$ (if $l, \tilde{l}$ are not contained in a plane, $G$ is trivial). Thus $G$ leaves invariant the plane $\alpha$ that contains $l, \tilde{l}$. Moreover $G$ leaves invariant all lines in $\alpha$ which are parallel to $l$. Note that if $g \in G$ acts trivially on $l$, then $g$ acts trivially on $\alpha$. Let $\beta$ be the plane that contains $l$ and is orthogonal to the plane $\alpha$. Then $G$ leaves invariant $\beta$. Note that if $g \in G$ acts trivially on $\beta$, then $g$ acts trivially on $\mathbb{R}^3$, i.e., $g$ is the identity. Hence $G$ is a virtually infinite cyclic group acting faithfully on $\mathbb{R}^2$. Therefore, $G$ is isomorphic to $\mathbb{Z}, D_\infty, \mathbb{Z} \times \mathbb{Z}_2, D_\infty \times \mathbb{Z}_2$ (see [18]). This proves the lemma.

Recall that every (open) cell in $A - Z = (C^3 \times \mathbb{R}^3) - Y$ is a product of (open) simplices $\sigma_C \times \sigma_R$ (see sect. 2).

**Isotropy of 0-cells**

Let $\sigma^0$ be a 0-cell in $A$ and $\gamma \in \Gamma$. We have two cases:

**First case: $\sigma^0 \subset A - Z$.**

Recall that $A - Z$ is $\Gamma$-homeomorphic to $(C^3 \times \mathbb{R}^3) - Y$. Then $\sigma^0 = \sigma^0_C \times \sigma^0_R$. If $\gamma\sigma^0 = \sigma^0$, we have that $\gamma\sigma^0_C = \sigma^0_C$ and $\gamma\sigma^0_R = \sigma^0_R$. Since $\gamma$ fixes the point $\sigma^0_R$ in $\mathbb{R}^3$, $\gamma$ is a rotation. Thus $\Gamma^{\sigma} < O(3)$. Hence we have two possibilities:

1. If $\sigma^0_C$ is not the vertex $[0]$ of $C^3$ (the vertex does not determine any
direction), \( \gamma \) fixes a direction in \( \mathbb{R}^3 \), the direction determined by \( \sigma_C^0 \neq [0] \). Therefore \( \gamma \) leaves invariant a line \( l \) in \( \mathbb{R}^3 \) (the line that passes through \( \sigma_R^0 \) and has direction determined by \( \sigma_C^0 \)). By (d) at the beginning of section 3, follows that \( \Gamma^{\sigma^0} \) is trivial or isomorphic to one of groups \( D_i, \mathbb{Z}_i, D_i \times \mathbb{Z}_2, \mathbb{Z}_i \times \mathbb{Z}_2, i = 2, 3, 4, 6 \).

\( (2) \) If \( \sigma_C^0 \), is the vertex \([0]\) of \( C^3 \), \( \sigma_C^0 \) does not determine any direction in \( \mathbb{R}^3 \). In this case the only thing we can say is that \( \gamma \) fixes a point in \( \mathbb{R}^3 \). Therefore, by (b) at the beginning of section 3, \( \Gamma^{\sigma^0} \) is trivial or isomorphic to one of groups \( D_i, \mathbb{Z}_i, A_4, S_4, D_i \times \mathbb{Z}_2, \mathbb{Z}_i \times \mathbb{Z}_2, A_4 \times \mathbb{Z}_2, S_4 \times \mathbb{Z}_2, i = 2, 3, 4, 6 \).

Second case: \( \sigma^0 \subset \mathbb{Z} \).

Remark that \( \mathbb{Z} = \bigcup_{h \in H} \mathbb{Z}_h \). Then \( \sigma^0 \subset \mathbb{Z}_h \) for some \( h \in H \). If \( \gamma \sigma^0 = \sigma^0 \), \( \gamma \) fixes a point in \( \mathbb{Z}_h \). Recall that a point in \( \mathbb{Z}_h \) is obtained by collapsing a line in \( C^3 \times \mathbb{R}^3 \). Let \( l \) be the line the collapses to \( \sigma^0 \). Then \( \Gamma^{\sigma^0} = \Gamma^{(l)} \).

**Isotropy of 1-cells**

Let \( \sigma^1 \) be an open 1-cell in \( A \), and \( \gamma \in \Gamma \). We have two cases:

**First case: \( \sigma^1 \subset A - \mathbb{Z} \).**

We have two possibilities: \( \sigma^1 = \sigma_C^0 \times \sigma_R^1 \) or \( \sigma^1 = \sigma_C^1 \times \sigma_R^0 \).

\( (1) \) Suppose \( \sigma^1 = \sigma_C^0 \times \sigma_R^1 \). If \( \gamma \sigma^1 = \sigma^1 \) then \( \gamma \sigma_R^1 = \sigma_R^1 \). Hence \( \gamma \) fixes a line pointwise. Therefore \( \gamma \) leaves invariant any plane orthogonal to this line and acts by rotation or reflection on these planes. By the crystallographic restriction, we have that \( \Gamma^{\sigma^1} \) is trivial or isomorphic to one of the groups \( D_i, \mathbb{Z}_i, i = 2, 3, 4, 6 \).

\( (2) \) Suppose \( \sigma^1 = \sigma_C^1 \times \sigma_R^0 \). Thus if \( \gamma \sigma^1 = \sigma^1 \) then \( \gamma \sigma_R^0 = \sigma_R^0 \) and \( \gamma \sigma_C^1 = \sigma_C^1 \). Hence \( \gamma \) fixes one point in \( \mathbb{R}^3 \) and at least, one direction in \( \mathbb{R}^3 \) (a direction determined by a point in \( \sigma_C^1 \), different from the vertex \([0]\)). By (d) of section 3, \( \Gamma^{\sigma^1} \) is trivial or isomorphic to one of the groups.
\(D_i, \mathbb{Z}_i, D_i \times \mathbb{Z}_2, \mathbb{Z}_i \times \mathbb{Z}_2, i = 2, 3, 4, 6.\)

**Second case** \(\sigma^1 \subset Z.\)

Since \(Z = \bigcup_{h \in H} Z_h, \sigma^1 \subset Z_h\) for some \(h \in H.\) Then if \(\gamma \sigma^1 = \sigma^1,\) \(\gamma\) fixes pointwise a line in \(Z_h.\) Hence, \(\Gamma \sigma^1\) fixes, at least, two points in \(Z_h.\) Therefore \(\Gamma \sigma^1\) leaves invariant, at least, two lines in \(\mathbb{R}^3.\) Then Lemma 4.1 implies that \(\Gamma \sigma^1\) is trivial or isomorphic to \(\mathbb{Z}, D_\infty, \mathbb{Z} \times \mathbb{Z}_2, D_\infty \times \mathbb{Z}_2.\)

**Isotropy of 2-cells**

Let \(\sigma^2\) be an open 2-cell in \(A\) and \(\gamma \in \Gamma \sigma^2.\) We have two cases:

**First case:** \(\sigma^2 \subset A - Z.\)

We have three possibilities: \(\sigma^2 = \sigma_0^0 \times \sigma_R^2, \sigma^2 = \sigma_C^2 \times \sigma_R^0\) or \(\sigma^2 = \sigma_C^1 \times \sigma_R^1.\)

(1) Suppose \(\sigma^2 = \sigma_0^0 \times \sigma_R^2.\) If \(\gamma \sigma^2 = \sigma^2\) then \(\gamma \sigma_R^2 = \sigma_R^2,\) and follows that \(\gamma\) fixes a plane in \(\mathbb{R}^3\) pointwise. Hence, \(\Gamma \sigma^2\) is trivial or isomorphic to \(\mathbb{Z}_2.\)

(2) Suppose \(\sigma^2 = \sigma_C^2 \times \sigma_R^0.\) Thus if \(\gamma \sigma^2 = \sigma^2\) then \(\gamma \sigma_C^2 = \sigma_C^2,\) and \(\gamma \sigma_R^0 = \sigma_R^0.\) Hence \(\gamma\) fixes infinitely many directions in \(\mathbb{R}^3,\) and fixes a point in \(\mathbb{R}^3.\) Then \(\gamma\) leaves invariant at least a plane in \(\mathbb{R}^3\) and acts on it (at most) by rotation by \(\pi.\) Also, \(\gamma\) acts (at most) by reflection on the line orthogonal to this plane. Therefore, \(\Gamma \sigma^2\) is trivial or isomorphic to \(\mathbb{Z}_2\) or \(\mathbb{Z}_2 \times \mathbb{Z}_2.\)

(3) Suppose \(\sigma^2 = \sigma_C^1 \times \sigma_R^1.\) Thus if \(\gamma \sigma^2 = \sigma^2\) then \(\gamma \sigma_R^1 = \sigma_R^1.\) Hence \(\gamma\) fixes a line in \(\mathbb{R}^3\) pointwise. Then \(\gamma\) acts on the plane orthogonal to this line (if \(\sigma_C^1\) determines the same direction as \(\sigma_R^1).\) Therefore, \(\Gamma \sigma^2\) is trivial or isomorphic to one of \(D_i\) or \(\mathbb{Z}_i, i = 2, 3, 4, 6.\)

Note that if \(\Gamma \sigma^2\) is isomorphic to \(D_i\) or \(\mathbb{Z}_i, i = 2, 3, 4, 6,\) then \(\sigma_C^1\) is a ray cell, and the direction that \(\sigma_C^1\) determines is exactly the direction of \(\sigma_R^1.\) In this case, we say that \(\sigma^2\) is a **special 2-cell**.
Second case $\sigma^2 \subset Z$.

Since $Z = \bigcup_{h \in H} Z_h$, then $\sigma^2 \subset Z_h$ for some $h \in H$. If $\gamma \sigma^2 = \sigma^2$, $\gamma$ fixes $Z_h$ pointwise. Hence $\gamma$ leaves invariant all lines parallel to $h$. Since $\Gamma \sigma^2$ acts cocompactly on $l_h$, follows that $\Gamma \sigma^2$ acts faithfully on $l_h$. Therefore, $\Gamma \sigma^2$ is trivial or isomorphic to $D_\infty$ or $\mathbb{Z}$.

Isotropy of 3-cells

Let $\sigma^3$ be an open 3-cell in $A$ and $\gamma \in \Gamma^{\sigma^3}$. Since $Z = \bigcup_{h \in H} Z_h$ and $Z_h \cong \mathbb{R}^2$, we have that $\sigma^3 \subset A - Z$. Hence, there are four possibilities: $\sigma^3 = \sigma^3_C \times \sigma^3_R$, $\sigma^3 = \sigma^1_C \times \sigma^2_R$, $\sigma^3 = \sigma^2_C \times \sigma^1_R$ or $\sigma^3 = \sigma^3_C \times \sigma^0_R$.

(1) Suppose $\sigma^3 = \sigma^0_C \times \sigma^3_R$. If $\gamma \sigma^3 = \sigma^3$, then $\gamma \sigma^3_R = \sigma^3_R$. Hence $\gamma$ fixes the whole $\mathbb{R}^3$ pointwise. Therefore, $\Gamma^{\sigma^3}$ is trivial.

(2) Suppose $\sigma^3 = \sigma^1_C \times \sigma^2_R$. If $\gamma \sigma^3 = \sigma^3$, then $\gamma \sigma^2_R = \sigma^2_R$ and $\gamma \sigma^1_C = \sigma^1_C$. Hence $\gamma$ fixes pointwise a plane $\alpha$ and fixes a direction in $\mathbb{R}^3$. Then $\gamma$ acts trivially or by reflection on the line orthogonal to $\alpha$. Therefore, $\Gamma^{\sigma^3}$ is trivial or isomorphic to $\mathbb{Z}_2$.

(3) Suppose $\sigma^3 = \sigma^2_C \times \sigma^1_R$. If $\gamma \sigma^3 = \sigma^3$, then $\gamma \sigma^1_R = \sigma^1_R$ and $\gamma \sigma^2_C = \sigma^2_C$. Hence $\gamma$ fixes a line pointwise and fixes infinitely many directions in $\mathbb{R}^3$. Then $\gamma$ leaves invariant the plane orthogonal to the line fixed by $\gamma$ and acts on it (at most) by rotation by $\pi$. Therefore, $\Gamma^{\sigma^3}$ is trivial or isomorphic to $\mathbb{Z}_2$.

(4) Suppose $\sigma^3 = \sigma^3_C \times \sigma^0_R$. If $\gamma \sigma^3 = \sigma^3$, then $\gamma \sigma^0_R = \sigma^0_R$ and $\gamma \sigma^3_C = \sigma^3_C$. Hence $\gamma$ fixes a point and fixes all directions in $\mathbb{R}^3$. Therefore, $\Gamma^{\sigma^3}$ is trivial or isomorphic to $\mathbb{Z}_2$.

Isotropy of 4-cells

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Let \( \sigma^4 \) be an open 4-cell in \( A \) and \( \gamma \in \Gamma \). Since \( Z = \bigcup_{h \in H} Z_h \) and \( Z_h \cong \mathbb{R}^2 \), we have \( \sigma^4 \subset A - Z \). Hence, there are three possibilities: \( \sigma^4 = \sigma^1_C \times \sigma^3_R \), \( \sigma^4 = \sigma^2_C \times \sigma^2_R \) or \( \sigma^4 = \sigma^3_C \times \sigma^1_R \).

(1) Suppose \( \sigma^4 = \sigma^1_C \times \sigma^3_R \). If \( \gamma \sigma^4 = \sigma^4 \), then \( \gamma \sigma^3_R = \sigma^3_R \). Hence \( \gamma \) fixes whole \( \mathbb{R}^3 \) pointwise. Therefore, \( \Gamma^{\sigma^4} \) is trivial.

(2) Suppose \( \sigma^4 = \sigma^2_C \times \sigma^2_R \). If \( \gamma \sigma^4 = \sigma^4 \), then \( \gamma \sigma^2_R = \sigma^2_R \) and \( \gamma \sigma^2_C = \sigma^2_C \). Hence \( \gamma \) fixes a plane pointwise. Therefore, \( \Gamma^{\sigma^4} \) is trivial or isomorphic to \( \mathbb{Z}_2 \).

(3) Suppose \( \sigma^4 = \sigma^3_C \times \sigma^1_R \). If \( \gamma \sigma^4 = \sigma^4 \), then \( \gamma \sigma^1_R = \sigma^1_R \) and \( \gamma \sigma^3_C = \sigma^3_C \). Hence \( \gamma \) fixes a line pointwise and all directions in \( \mathbb{R}^3 \). Therefore, \( \Gamma^{\sigma^4} \) is trivial.

**Isotropy of 5-cells**

Let \( \sigma^5 \) be an open 5-cell in \( A \) and \( \gamma \in \Gamma \). Since \( Z = \bigcup_{h \in H} Z_h \) and \( Z_h \cong \mathbb{R}^2 \), we have \( \sigma^5 \subset A - Z \). Hence, there are two possibilities: \( \sigma^5 = \sigma^2_C \times \sigma^3_R \) or \( \sigma^5 = \sigma^3_C \times \sigma^2_R \).

(1) Suppose \( \sigma^5 = \sigma^2_C \times \sigma^3_R \). If \( \gamma \sigma^5 = \sigma^5 \), then \( \gamma \sigma^3_R = \sigma^3_R \). Hence \( \gamma \) fixes whole \( \mathbb{R}^3 \) pointwise. Therefore, \( \Gamma^{\sigma^5} \) is trivial.

(2) Suppose \( \sigma^5 = \sigma^3_C \times \sigma^2_R \). If \( \gamma \sigma^5 = \sigma^5 \), then \( \gamma \sigma^2_R = \sigma^2_R \) and \( \gamma \sigma^3_C = \sigma^3_C \). Hence \( \gamma \) fixes a plane pointwise and all directions in \( \mathbb{R}^3 \). Therefore, \( \Gamma^{\sigma^5} \) is trivial.

**Isotropy of 6-cells**

Let \( \sigma^6 \) be an open 6-cell in \( A \) and \( \gamma \in \Gamma \). Since \( Z = \bigcup_{h \in H} Z_h \) and \( Z_h \cong \mathbb{R}^2 \), we have \( \sigma^6 \subset A - Z \). Hence \( \sigma^6 = \sigma^3_C \times \sigma^3_R \). If \( \gamma \sigma^6 = \sigma^6 \), then
\( \gamma \sigma_R^3 = \sigma_R^3 \). Hence, \( \Gamma^{\sigma_6} \) is trivial.

5 Calculation of \( H_i(A/\Gamma, \mathcal{P}_*(\rho)) \)

Recall that a cell \( \tilde{\sigma} \) in \( A/\Gamma \) corresponds to an orbit \( \Gamma \sigma, \sigma \in A \). In what follows we will use the same notation for a cell \( \sigma \) and its orbit \( \Gamma \sigma \). We know that there is a spectral sequence with \( E_{p,q}^2 = H_p(A/\Gamma, \pi_q(\mathcal{P}_*(\rho))) \) (see Prop. 1.2 and 1.4) which abuts to \( H_{p+q}(A/\Gamma, \mathcal{P}_*(\rho)) \). We are interested in the case \( p + q = -1 \). By Prop. 1.3, [7] and [12], \( \pi_q(\mathcal{P}_*(\rho)) = 0 \), if \( q + 2 \leq -2 \). Hence \( H_{p+q}(A/\Gamma, \mathcal{P}_*(\rho)) = 0 \), if \( q + 2 \leq -2 \). Then, the possible non zero terms of the spectral sequence with \( p + q = -1 \) are \( E_{0,-1}^2 = H_0(A/\Gamma, \pi_{-1}(\mathcal{P}_*(\rho))) \), \( E_{1,-2}^2 = H_1(A/\Gamma, \pi_{-2}(\mathcal{P}_*(\rho))) \) and \( E_{2,-3}^2 = H_2(A/\Gamma, \pi_{-3}(\mathcal{P}_*(\rho))) \).

Calculation of the term \( E_{0,-1}^2 \).

Let \( \sigma_i^0 \) and \( \sigma_j^1 \) denote the 0-cells and 1-cells of \( A \). Consider the associated cellular chain complex \( \ldots \leftarrow C_0 \leftarrow \partial C_1 \leftarrow \ldots \), for \( E_{0,-1}^2 \), where \( C_0 = \bigoplus_i Wh(\Gamma^{\sigma_i^0}) \), \( C_1 = \bigoplus_j Wh(\Gamma^{\sigma_j^1}) \). By the calculations of Section 4 (isotropy of 1-cells), \( \Gamma^{\sigma_1} \) is trivial or isomorphic to one of groups \( \mathbb{Z} \times \mathbb{Z}_2, D_\infty \times \mathbb{Z}_2, D_i, \mathbb{Z}, D_\infty, \mathbb{Z}_4 \times \mathbb{Z}_2, D_i \times \mathbb{Z}_2, i = 2, 3, 4, 6 \). In [29], Whitehead proves that \( Wh(F) = 0 \) if \( F \) is trivial or isomorphic to \( \mathbb{Z}_i, i = 2, 3, 4 \). In [7], Carter proves that \( Wh(\mathbb{Z}_6) = 0 \). In [3], Bass proves that \( Wh(\mathbb{Z}) = Wh(D_\infty) = 0 \). In [18], Pearson proves that \( Wh(\mathbb{Z} \times \mathbb{Z}_2) = Wh(D_\infty \times \mathbb{Z}_2) = 0 \). In [2], Bass proves that \( Wh(F) = \mathbb{Z}^{r-q} + SK_1(\mathbb{Z}[F]) \), when \( F \) is a finite group, where \( r \) is the number of real representations irreducible of \( F \) and \( q \) is the number of rational representations irreducible. By Theorems 14.1 and 14.2 of [17], we have \( SK_1(\mathbb{Z}[F]) = 0 \), if \( F \) is isomorphic to \( D_1, D_i \times \mathbb{Z}_2, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_6 \times \mathbb{Z}_2, i = 2, 3, 4, 6 \). A direct calculation shows that \( r = q \) for these groups. Then
\(W h(\Gamma^\sigma) = 0\). Hence \(C_1 = \bigoplus_j W h(\Gamma^\sigma_j) = 0\). Therefore,

\[\mathcal{E}^2_{0,-1} = H_0(A/\Gamma, \pi_{-1}(P_+(\rho))) = \bigoplus_i W h(\Gamma^\sigma_i).\]

The following lemmata show that the term \(\mathcal{E}^2_{0,-1}\) can be written in terms of the maximal virtually cyclic subgroups of \(\Gamma\).

**Lemma 5.1:** Let \(l \in \mathcal{L}\) and suppose that \(\Gamma^{(l)}\) is a virtually cyclic infinite subgroup of \(\Gamma\). If \(\mathcal{L}^{\Gamma^{(l)}} = \{l\}\), then \(\Gamma^{(l)}\) is a maximal virtually finite cyclic subgroup of \(\Gamma\).

**Proof:** Suppose that \(G\) is virtually infinite cyclic and that \(\Gamma^{(l)} < G\). Since \(G\) leaves invariant at least one line \(l\), we have \(\Gamma^{(l)} < G < \Gamma^{(\tilde{l})}\). Hence \(l = \tilde{l}\).

This proves the lemma.

**Lemma 5.2:** Let \(p \in \mathbb{R}^3\). If \(\Gamma^p\) fixes only \(p\), then \(\Gamma^p\) is a maximal finite subgroup of \(\Gamma\).

**Proof:** Suppose that \(F\) is a finite subgroup of \(\Gamma\) and that \(\Gamma^p < F\). Since \(F\) is finite, it fixes at least a point \(\tilde{p}\) in \(\mathbb{R}^3\), then \(\Gamma^p < F < \Gamma^{\tilde{p}}\). Hence \(p = \tilde{p}\).

This proves the lemma.

**Lemma 5.3:** Let \(G\) be a virtually infinite cyclic subgroup of \(\Gamma\). If \(W h(G) \neq 0\), then \(\mathcal{L}^G = \{l\}\) or, equivalently, \(G = \Gamma^l\), for some unique line \(l\).

**Proof:** If \(\mathcal{L}^G \supset \{l, \tilde{l}\}, l \neq \tilde{l}\), by Lemma 4.1, \(G\) is isomorphic to \(\mathbb{Z}, D_\infty, \mathbb{Z} \times \mathbb{Z}_2, D_\infty \times \mathbb{Z}_2\). Follows that \(W h(G) = 0\) (see [18]). This proves the lemma.

Recall that \(\Gamma^z, z \in Z\) is a virtually infinite cyclic group (because a point in \(Z\) is obtained by collapsing a line in \(C^3 \times \mathbb{R}^3\), see prop. 1.1). Hence, lemmata 5.1 and 5.3 imply that if \(W h(\Gamma^z) \neq 0\), then \(\Gamma^z\) is maximal virtually infinite cyclic. Therefore, we can write

\[\mathcal{E}^2_{0,-1} = \left[ \bigoplus_{H \in F(\Gamma)} W h(H) \right] \oplus \left[ \bigoplus_{H \in V C_\infty(\Gamma)} W h(H) \right],\]
where \( F(\Gamma) \) and \( VC_\infty(\Gamma) \) denote the sets of conjugacy classes of maximal finite subgroups of \( \Gamma \) and maximal virtually infinite cyclic subgroups of \( \Gamma \), respectively.

As mentioned above, the Whitehead groups of \( \mathbb{Z}_i, D_i, \mathbb{Z}_i \times \mathbb{Z}_2, D_i \times \mathbb{Z}_2, \) \( i = 2, 3, 4, 6 \) vanish. We show now that the Whitehead groups of the other finite subgroups of 3-crystallographic groups also vanish. The groups \( A_4, S_4, A_4 \times \mathbb{Z}_2, S_4 \times \mathbb{Z}_2 \), act on \( \mathbb{R}^3 \) fixing only a point (see (d) at the beginning of Section 3). Then, by Lemma 5.2, these groups are maximal. By [17], Theorem 14.1, follows that \( Wh(S_4) = Wh(S_4 \times \mathbb{Z}_2) = 0 \). By [17], Theorem 14.6, follows that \( SK_1(\mathbb{Z}[A_4]) = 0 \). By [2], \( Wh(A_4) = \mathbb{Z}^{r-q} \oplus SK_1(\mathbb{Z}[A_4]) \). A direct calculation shows that \( r = q \). Hence \( Wh(A_4) = 0 \).

The proof of the following lemma was suggested to us by F.T. Farrell.

**Lemma 5.4:** If \( F = A_4 \times \mathbb{Z}_2 \), then \( Wh(F) = 0 \).

**Proof:** Recall that \( Wh(F) = \mathbb{Z}^y \oplus SK_1(\mathbb{Z}[F]) \). Since the elementary subgroups of \( A_4 \times \mathbb{Z}_2 \) are either cyclic or the sum of cyclic subgroups of order 2, by [17], examples 1 and 2 on page 14, Theorem 5.3 and page 7, we have that \( SK_1(\mathbb{Z}[F]) = 0 \). Now the rank of the torsion-free part of \( Wh(F) \) is \( y = r - q \), where \( r \) is the number of conjugacy classes of sets \( \{g, g^{-1}\} \) and \( q \) is the number of conjugacy classes of cyclic subgroups of \( F \) (see [10]). We have six cyclic subgroups (modulo conjugation) of \( A_4 \times \mathbb{Z}_2 \): the trivial subgroup, three subgroups isomorphic to \( \mathbb{Z}_2 \), one subgroup isomorphic to \( \mathbb{Z}_3 \) and one subgroup isomorphic to \( \mathbb{Z}_6 \). On the other hand, in \( A_4 \) there are exactly three elements of order 2 (all of them are conjugate) and eight elements of order 3. Let \( a, b \in A_4 \) with \( a^3 = b^3 = 1 \) and \( a, b \) non trivial. A direct calculation shows that either \( a = b^2 \) or \( a \) and \( b \) are conjugate. Hence there are six of conjugacy classes of sets \( \{g, g^{-1}\} \) in \( A_4 \times \mathbb{Z}_2 \). Therefore, \( r = q \) and \( Wh(A_4 \times \mathbb{Z}_2) = 0 \). This proves the lemma.

Follows that \( Wh(F) = 0 \), if \( F \) is a finite subgroup of a 3-crystallographic group. Hence we can write:

\[
Wh(\Gamma) = \bigoplus_{G \in VC_\infty(\Gamma)} Wh(G).
\]
Calculation of the term $\mathcal{E}_{2,-3}^2$.

Consider the associated cellular chain complex $\ldots \leftarrow C_1 \leftarrow \partial C_2 \leftarrow \ldots$, for $\mathcal{E}_{2,-3}^2$, where $C_1 = \bigoplus_j K_{-1}(\mathbb{Z}[\Gamma^j])$ and $C_2 = \bigoplus_i K_{-1}(\mathbb{Z}[\Gamma^i])$. By Section 4 (isotropy of 2-cells) $\Gamma^j$ is trivial or isomorphic to $\mathbb{Z}, D_i, \mathbb{Z}$ or $D_{\infty}$. By [3], we have that $K_{-1}(\mathbb{Z}[\mathbb{G}]) = 0$, if $\mathbb{G}$ is isomorphic to $D_2, \mathbb{Z}, D_{\infty}$ or $\mathbb{D}_i$, $i = 2, 3, 4$, and that $K_{-1}(\mathbb{Z}[\mathbb{Z}_6]) = \mathbb{Z}$. Hence, it is enough to study the case in which $\sigma^2$ is a special 2-cell with isotropy $D_6$ (in the other cases $K_{-1}$ vanishes), i.e., $\sigma^2 = \sigma^1_C \times \sigma^1_R$, where $\sigma^1_C$ is a ray cell that determines the same direction as $\sigma^1_R$ and the isotropy of $\sigma^1_R$ is $D_6$.

In what follows the bar denotes “closure”.

**Lemma 5.5:** Let $\sigma^2 = \sigma^1_C \times \sigma^1_R$, $\tilde{\sigma}^2 = \tilde{\sigma}^1_C \times \tilde{\sigma}^1_R$ be special 2-cells in $A - Z$. If $\overline{\sigma^2} \cap \overline{\sigma^2}$ is an 1-cell of the form $\sigma^0_C \times \sigma^1_R$, then $\sigma^1_R = \tilde{\sigma}^1_R$, and $\sigma^1_C, \tilde{\sigma}^1_C$ lie in the same ray in $C^3$.

**Proof:** Since $\overline{\sigma^2} \cap \overline{\sigma^2} = \overline{\sigma^0_C \times \sigma^1_R}$ we have $\sigma^1_R = \tilde{\sigma}^1_R$. Hence $\sigma^1_C$ and $\tilde{\sigma}^1_C$ determine the same direction as $\sigma^1_R = \tilde{\sigma}^1_R$. Therefore $\sigma^1_C$ and $\tilde{\sigma}^1_C$ lie in the same ray. This proves the lemma.

It follows from the lemma above that the only special 2-cell that contains a one cell of the form $[0] \times \sigma^1_R$ is the special 2-cell $\overline{\sigma^1_C \times \sigma^1_R}$, where $\sigma^1_C$ is the ray cell that contains the cone point $[0]$ and determines the same direction as $\sigma^1_R$.

We will prove that $\partial : C_2 \to C_1$ is injective. It will then follow that $\mathcal{E}_{2,-3}^2 = 0$.

Fix $\sigma^1_R$ and enumerate all ray cells $\sigma^1_C$ which determine the same direction as $\sigma^1_R$. Then we have the family of 2-cells $\{\sigma^2_i = (\sigma^1_C, \times \sigma^1_R)\}$. Note that $\cup_i(\overline{\sigma^1_C})_i$ is the ray $r \cong [0, \infty)$, determined by the direction of $\sigma^1_R$. Denote by $l$ the line that contains $\sigma^1_R$. After reindexing the cells, we can identify each
$(\sigma_C^1)_i$ with the interval $(i, i + 1)$. Therefore $\sigma^2_i = (i, i + 1) \times \sigma^1_R$.

From the definition of the relation $\cong$ over $C^3 \times \mathbb{R}^3$ (Section 2) we get that we have to collapse at most one cell of the form $\{n\} \times \sigma_R^1$ to one point, where $n > 0$ depends on $\sigma_R^1$. In fact, if $l$ is parallel to $l_h \in \mathcal{L}_h \in \Lambda_n$, for some $h \in H$, then $c(l) = [\frac{nh}{|l_h|}]$. Hence $\{n\} \times \sigma_R^1$ collapses to one point. Therefore $n$ depends on $\sigma_R^1$. We write $n = n(\sigma_R^1)$. Then, either $\cup \sigma_i^2 = [0, \infty) \times \sigma_R^1$, or $\cup \sigma_i^2$ is obtained from $[0, \infty) \times \sigma_R^1$ collapsing $\{n\} \times \sigma_R^1$ to a point $\{n\}$ (if $l \parallel l_h$, for some $h$), where $n = n(\sigma_R^1) > 0$.

Lemma 5.6: $\partial : C_2 \rightarrow C_1$ is injective.

Proof: Let $\sum_k m_k \sigma_k^2 \in C_2$ with $\partial(\sum_k m_k \sigma_k^2) = 0$. Since $m_k \in K_{-1}(\mathbb{Z}[\Gamma^{\sigma_i^2}])$, we can suppose that $\sigma_k^2$ is a special 2-cell, that is, it is of the form $\sigma_C^1 \times \sigma_R^1$, with $\sigma_C^1$ a ray cell that determines the same direction as $\sigma_R^1$ and the isotropy of $\sigma_R^1$ is $D_6$ (in the other cases $K_{-1}(\mathbb{Z}[\Gamma^{\sigma_i^2}]) = 0$). Thus we can write

$$\sum_k m_k \sigma_k^2 = \sum_{\sigma_R^1} \left( \sum_i n_i((\sigma_C^1)_i \times \sigma_R^1) \right).$$

We will prove that $m_k = 0$, for all $k$, by proving that for each $\sigma_R^1$, all $n_i$ are zero. Fix one $\sigma_R^1$. Like before, identify $(\sigma_C^1)_i$ with $(i, i + 1)$, and note that $\Gamma^{(i,i+1) \times \sigma_R^1} = \Gamma^{(i+1) \times \sigma_R^1} = \Gamma^{\sigma_R^1} \cong D_6$, for all $i + 1 \neq n(\sigma_R^1)$, because $(\sigma_C^1)_i = (i, i + 1)$ is a ray cell having the same direction as $\sigma_R^1$. Hence $n_i \in K_{-1}(\mathbb{Z}[\Gamma^{\sigma_R^1}])$ and the boundary maps for the coefficients are just the identity:

$$K_{-1}(\mathbb{Z}[\Gamma^{[(i,i+1) \times \sigma_R^1]}]) \xrightarrow{\partial} K_{-1}(\mathbb{Z}[\Gamma^{(i+1) \times \sigma_R^1}]) \xrightarrow{id} K_{-1}(\mathbb{Z}[\Gamma^{\sigma_R^1}]).$$

Since all $\sigma_k^2$ are special 2-cells, Lemma 5.5 implies that, for $i \geq 0$ and $i + 1 \neq n(\sigma_R^1)$, $[i, i + 1] \times \sigma_R^1$ and $[i + 1, i + 2] \times \sigma_R^1$ are the only closed 2-cells in the family $\{\sigma_j^2\}$ whose boundaries contain $\{i + 1\} \times \sigma_R^1$. Hence, since we are assuming

$$\partial(\sum_i n_i((\sigma_C^1)_i \times \sigma_R^1)) = 0,$$

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we have that \( n_i = n_{i+1} \). But \([0, 1] \times \sigma_R^{-1}\) is the only closed 2-cell of the family \( \{ [i, i+1] \times \sigma_R^{-1} \} \) whose boundary contains \([0] \times \sigma_R^{1}\). Hence \( n_0 = 0 \). Therefore \( n_i = 0, i < n(\sigma_R^1) \). On the other hand, if \( n_{i_0} \neq 0, i_0 > n(\sigma_R^1) \) we would have \( n_i \neq 0 \) for all \( i > i_0 \), which shows that the sum \( \sum_i n_i((i, i+1) \times \sigma_R^{1}) \) is infinite. Hence \( n_i = 0 \) for all \( i \). This proves the lemma.

**Calculation of the term \( \mathcal{E}_{1,-2}^2 \).**

Consider the associated cellular chain complex, \( \ldots \leftarrow C_0 \leftarrow C_1 \leftarrow \ldots \), for \( \mathcal{E}_{1,-2}^2 \), where \( C_0 = \bigoplus_j \tilde{K}_0(\mathbb{Z}[\Gamma^0]) \) and \( C_1 = \bigoplus_j \tilde{K}_0(\mathbb{Z}[\Gamma^1]) \). From Section 4 (isotropy of 1-cells), \( \Gamma^0 \) is trivial or isomorphic to \( \mathbb{Z}_i, D_i, \mathbb{Z}, D_\infty, \mathbb{Z}_i \times \mathbb{Z}_2, D_i \times \mathbb{Z}_2, \mathbb{Z} \times \mathbb{Z}_2, D_\infty \times \mathbb{Z}_2, i = 2, 3, 4, 6 \). By [21] (see also [22]), \( \tilde{K}_0(\mathbb{Z}[F]) = 0 \) for \( F \) isomorphic to \( D_i \) or \( \mathbb{Z}_i \), \( i = 2, 3, 4 \). By [21], \( \tilde{K}_0(\mathbb{Z}_i[\mathbb{Z}_0]) = 0 \). By [3], \( \tilde{K}_0(\mathbb{Z}[F]) = 0 \) if \( F \) is isomorphic to \( D_\infty \) or \( \mathbb{Z} \). By [13], \( \tilde{K}_0(\mathbb{Z}[F]) = 0 \) if \( F \) is isomorphic to \( D_\infty \times \mathbb{Z}_2, \mathbb{Z} \times \mathbb{Z}_2, \) or \( D_6 \). \( \tilde{K}_0(\mathbb{Z}[F]) \) does not vanish for \( F \) isomorphic to \( \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_6 \times \mathbb{Z}_2, D_2 \times \mathbb{Z}_2, D_4 \times \mathbb{Z}_2 \) or \( D_6 \times \mathbb{Z}_2 \) (see [3] and [4]). Hence it is enough to study the case \( \sigma^1 \subset A - Z \cong \Gamma (\mathbb{Z}^3 \times \mathbb{R}^3) - Y \), in which \( \sigma^1 = \sigma^1_C \times \sigma^0_R \), where \( \sigma^1_C \) is a ray cell and \( \sigma^0_R \) is a 0-cell (in the other cases \( \tilde{K}_0 \) vanishes).

In the following lemma we consider \( D_\infty = \mathbb{Z}_2 \ast \mathbb{Z}_2 \), and write \( D_\infty = \mathbb{Z}_2^a \ast \mathbb{Z}_2^b \) to distinguish the two factors. Let \( i^a : \mathbb{Z}_2^a \hookrightarrow \mathbb{Z}_2^a \ast \mathbb{Z}_2^b \), \( i^b : \mathbb{Z}_2^b \hookrightarrow \mathbb{Z}_2^a \ast \mathbb{Z}_2^b \) be the canonical inclusions, let \( \beta : \mathbb{Z}_2^a \ast \mathbb{Z}_2^b \rightarrow \mathbb{Z}_2^a \ast \mathbb{Z}_2^b \) be homomorphism such that \( \beta(a) = a \) and \( \beta(b) = 0 \). Then \( \beta \circ i^a = 1_{\mathbb{Z}_2^a} \) and \( \beta \circ i^b = 0 \). Let \( F \) be a group. Define the group homomorphisms

\[\alpha^a : \mathbb{Z}_2^a \times F \rightarrow (\mathbb{Z}_2^a \ast \mathbb{Z}_2^b) \times F, \quad \alpha^b : \mathbb{Z}_2^b \times F \rightarrow (\mathbb{Z}_2^a \ast \mathbb{Z}_2^b) \times F\]

by

\[\alpha^a = i^a \times 1_F, \quad \alpha^b = i^b \times 1_F.\]

Let

\[\alpha^a : \tilde{K}_0(\mathbb{Z}[\mathbb{Z}_2^a \times F]) \rightarrow \tilde{K}_0(\mathbb{Z}[(\mathbb{Z}_2^a \ast \mathbb{Z}_2^b) \times F]),\]

\[\alpha^b : \tilde{K}_0(\mathbb{Z}[\mathbb{Z}_2^b \times F]) \rightarrow \tilde{K}_0(\mathbb{Z}[(\mathbb{Z}_2^a \ast \mathbb{Z}_2^b) \times F])\]

denote the induced homomorphisms, at the \( \tilde{K}_0 \) level.
Lemma 5.7: Let $F$ be a group with $\tilde{K}_0(\mathbb{Z}[F]) = 0$. Then $\alpha^a_*$ and $\alpha^b_*$ are injective. Moreover $\text{Im}(\alpha^a_*) \cap \text{Im}(\alpha^b_*) = \{0\}$.

Proof: Consider the following diagram of groups and homomorphisms

$$
\begin{array}{ccc}
\mathbb{Z}_2^a \times F & \xrightarrow{\alpha^a} & (\mathbb{Z}_2^a \ast \mathbb{Z}_2^b) \times F \\
& \uparrow \alpha^b & \downarrow \gamma \\
& \mathbb{Z}_2^b \times F & \\
\end{array}
$$

Here $\gamma = \beta \times 1_F$. Note that $\gamma \circ \alpha^a = 1_{\mathbb{Z}_2^a \times F}$, and $\gamma \circ \alpha^b = 0$. Applying the $\tilde{K}_0$ functor, we get

$$
\tilde{K}_0(\mathbb{Z}[\mathbb{Z}_2^a \times F]) \xrightarrow{\alpha^a} \tilde{K}_0(\mathbb{Z}[\mathbb{Z}_2^a \ast \mathbb{Z}_2^b \times F]) \xrightarrow{\gamma_*} \tilde{K}_0(\mathbb{Z}[\mathbb{Z}_2^a \times F]).
$$

Then $\gamma_* \circ \alpha^a_*$ is the identity. Hence $\alpha^a_*$ is injective. Analogously $\alpha^b_*$ is injective. If $x \in \text{Im}(\alpha^a_*) \cap \text{Im}(\alpha^b_*)$, then $x = \alpha^a_* y$ and $x = \alpha^b_* z$. Hence $\gamma_* x = \gamma_* \circ \alpha^a_* y = y$ and $\gamma_* x = \gamma_* \circ \alpha^b_* z = 0$. Therefore $x = 0$. This proves the lemma.

Lemma 5.8: Consider the 1-cells $\sigma^1_C \times \sigma^0_R$, $\tilde{\sigma}^1_C \times \tilde{\sigma}^0_R$ of $A - Z$, where $\sigma^1_C, \tilde{\sigma}^1_C$ are ray cells. Suppose that $(\sigma^1_C \times \sigma^0_R) \cap (\tilde{\sigma}^1_C \times \tilde{\sigma}^0_R) \neq \emptyset$.

(a) If $\sigma^0_R = \tilde{\sigma}^0_R$ then either $\sigma^1_C$ and $\tilde{\sigma}^1_C$ lie in the same ray (i.e., determine same direction in $\mathbb{R}^3$) or $\sigma^1_C \cap \tilde{\sigma}^1_C = [0]$.

(b) If $\sigma^0_R \neq \tilde{\sigma}^0_R$ then $\sigma^1_C \cap \tilde{\sigma}^1_C \neq \emptyset$, $\sigma^1_C, \tilde{\sigma}^1_C$ lie in the same ray and the line $l$ that contains $\sigma^0_R$ and $\tilde{\sigma}^0_R$ has the direction determined by $\sigma^1_C$. Moreover $l \in \mathcal{L}$ and $(\sigma^1_C \times \sigma^0_R) \cap (\tilde{\sigma}^1_C \times \tilde{\sigma}^0_R) = [\bar{l}]$.

Proof (a): If $\sigma^0_R = \tilde{\sigma}^0_R$, then $\sigma^1_C$ and $\tilde{\sigma}^1_C$ lie in the same cone based on $\sigma^0_R = \tilde{\sigma}^0_R$. Since the collapsing map $\pi : C^a \times \mathbb{R}^a \to A$ is injective on all cones
In $C^m \times \{v\}$, $v \in \mathbb{R}^3$, we have that $(\sigma_C^1 \times \sigma_R^0) \cap (\overline{\sigma_C^1} \times \overline{\sigma_R^0}) \neq \emptyset$, implies either $\sigma_C^1 \times \sigma_R^0 = \overline{\sigma_C^1} \times \overline{\sigma_R^0}$ or $(\sigma_C^1 \times \overline{\sigma_R^0}) \cap (\overline{\sigma_C^1} \times \sigma_R^0) = \sigma_C^0 \times \sigma_R^0$. If $\sigma_R^0 = \emptyset$, then $\overline{\sigma_C^1} \cap \overline{\sigma_C^0} = \emptyset$. Otherwise $\sigma_C^1$ and $\overline{\sigma_C^1}$ lie in the same ray, because $\sigma_C^1 \cap \overline{\sigma_C^0} = \sigma_C^0$ and $\sigma_C^1$, $\overline{\sigma_C^1}$ are ray cells.

(b): If $\sigma_R^0 \neq \overline{\sigma_R^0}$, consider the line $l$ that contains $\sigma_R^0$ and $\overline{\sigma_R^0}$. Recall that we collapse only the lines $\overline{l}$ with $l \in \mathcal{L}$. Note that the height at which we collapse depends only on the direction of the line, and that if a point in a ray is identified with a point in another ray, then these rays determine the same direction. Hence if $(\sigma_C^1 \times \overline{\sigma_R^0}) \cap (\overline{\sigma_C^1} \times \sigma_R^0) \neq \emptyset$ with $\sigma_R^0 \neq \overline{\sigma_R^0}$, we have that $l \in \mathcal{L}$. Moreover $(\sigma_C^1 \times \sigma_R^0) \cap (\overline{\sigma_C^1} \times \sigma_R^0) = [\overline{l}]$ and $\overline{\sigma_C^1} \cap \overline{\sigma_C^0} \neq \emptyset$. This proves the lemma.

The next lemma implies that $\mathcal{E}_{1,-2}^2 = 0$.

**Lemma 5.9:** $\partial_1 : C_1 \rightarrow C_2$ is injective.

**Proof:** Let $\sum_k m_k \sigma_C^1_k \in C_1$ with $\partial(\sum_k m_k \sigma_C^1_k) = 0$. Since $m_k \in \tilde{K}_0(\mathbb{Z}[\Gamma^1])$, we can suppose that $\sigma_C^1_k$ is of the form $\sigma_C^1 \times \sigma_R^0$, with $\sigma_C^1$ a ray cell (in the other cases $\tilde{K}_0(\mathbb{Z}[\Gamma^1]) = 0$). Thus we can write

$$\sum_k m_k \sigma_C^1_k = \sum_{\sigma_R^0, r} \left( \sum_i n_i, r (\sigma_C^1, i, r \times \sigma_R^0) \right).$$

where we assume that $U_i(\overline{\sigma_C^1})_i, r$ is the ray $r$ and that the $(\sigma_C^1)_i, r$ are enumerated in such a way that $(\sigma_C^1)_i, r := (\sigma_C^1)_i, r \cap (\sigma_C^1)_i, r \neq \emptyset$. Note that all $\Gamma^{(\sigma_C^1), i, r \times \sigma_R^0}$ are equal, for all $i$ and $\sigma_R^0$, $r$ fixed. We will suppress the subindex $r$ to alleviate the heavy notation, e.g we will write $n_i$ instead of $n_i, r$.

We will prove that $m_k = 0$, for all $k$, by proving that for each $\sigma_R^0$ and $r$ fixed, all $n_i$ are zero.

Suppose there is $\sigma_R^0$ and $r$ for which there is a $n_i$ with $n_i \neq 0$. Since $n_i \in \tilde{K}_0(\mathbb{Z}[\Gamma^{(\sigma_C^1), i, r \times \sigma_R^0}])$ then $\tilde{K}_0(\mathbb{Z}[\Gamma^{(\sigma_C^1), i, r \times \sigma_R^0}]) \neq 0$. Hence $\Gamma^{(\sigma_C^1), i, r \times \sigma_R^0}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_6, \mathbb{Z}_2 \times D_2, \mathbb{Z}_2 \times D_4, \mathbb{Z}_2 \times D_6$, i.e., $\Gamma^{(\sigma_C^1), i, r \times \sigma_R^0}$ is
isomorphic to $\mathbb{Z}_2 \times F$, where $F$ is isomorphic to $\mathbb{Z}_4, \mathbb{Z}_6, D_2, D_4$ or $D_6$.

Let $l$ be the line with direction determined by the direction of $r = \bigcup_i (\sigma^1_C)_i$ and that contains $\sigma^0_R$. Let $\alpha$ be the plane orthogonal to $l$ that contains $\sigma^0_R$. Let $G = \Gamma(t)$. Note that $\Gamma(\sigma_C^1) \times \sigma^0_R = G^\sigma_R$, for all $i$. Moreover $G^\sigma_R$ acts by reflection on $l$ (if the action was trivial, $G^\sigma_R = \Gamma(\sigma_C^1) \times \sigma^0_R$ would be isomorphic to $\mathbb{Z}_4, D_i, i = 2, 3, 4, 6$). Also for $\Gamma(\sigma_C^1) \times \sigma^0_R \cong \mathbb{Z}_2 \times F$, the $\mathbb{Z}_2$ factor acts by reflection on $l$, and $F$ acts on the plane $\alpha$. Note that $F = G^l$. Hence we have two cases: $l/G$ is not compact or $l/G$ is compact.

**First case:** $l/G$ is not compact.

Then $G$ contains no translations. Hence $G = G^\sigma_R = \Gamma(\sigma_C^1) \times \sigma^0_R \cong \mathbb{Z}_2 \times F$, which is finite. Therefore, $l \notin L$. Since $r = \bigcup_i (\sigma^1_C)_i$ is the ray determined by $l$, we have that $(r \times \sigma^0_R) \cap Z = \emptyset$ (we only collapse lines $\tilde{l}$ with $l \in L$). Hence $\Gamma(\sigma_C^1) \times \sigma^0_R = \Gamma(\sigma_C^1 \times \sigma^0_R) = G$ and the boundary maps $\tilde{K}_0(\mathbb{Z}[\Gamma(\sigma_C^1) \times \sigma^0_R]) \to K_0(\mathbb{Z}[\Gamma(\sigma_C^1 \times \sigma^0_R)])$ at the coefficient level are just the identity.

Now recall that we are assuming some $n_i \neq 0 \in \tilde{K}_0(\mathbb{Z}[G])$. Since $(r \times \sigma^0_R) \cap Z = \emptyset$, (b) of lemma 5.8 cannot happen. Hence, by (a) of lemma 5.8, we have that $n_{i+1} \neq 0 \in \tilde{K}_0(\mathbb{Z}[G])$. In the same way we prove that $n_j \neq 0 \in \tilde{K}_0(\mathbb{Z}[G])$ for all $j \geq i$ which is a contradiction, because the sum above is finite.

**Second case:** $l/G$ is compact.

Then $G$ contains translations. Moreover, since $G$ acts by reflection on $l$, we have that the action of $G$ on $l$ is a dihedral action. Then $G = G^\sigma_R * F G^\sigma_R$ where $F = G^l$ is isomorphic to $\mathbb{Z}_4, \mathbb{Z}_6, D_2, D_4$, and $\tilde{\sigma}_R \in l$, $\tilde{\sigma}_R \neq \sigma^0_R$. Also, since $G$ contains translations, $l \in L$. Thus $l \in L_m$ for some $m$. Since $r = \bigcup_i (\sigma^1_C)_i$ is the ray determined by $l$, by lemma 5.8 (b) we have that $(r \times \sigma^0_R) \cap Z = (r \times \tilde{\sigma}_R) \cap Z = (r \times \sigma^0_R) \cap (r \times \tilde{\sigma}_R) = \{[\tilde{l}]\} = \{[m] \times \sigma^0_R\} = \{[m] \times \sigma^0_R\}$ ($l$ is collapsed to a point). Write $\sigma^0 = \{[\tilde{l}]\}$, for this 0-cell. Note that $G = \Gamma(t) = \Gamma([m] \times \sigma^0_R)$ and that $\Gamma[\sigma_C^1 \times \sigma^0_R = \Gamma(\sigma_C^1 \times \sigma^0_R = G^\sigma_R$ for all $i$ and $i+1 \neq m$. Then the boundary maps (at the coefficient level)
The $\tilde{K}_0(\mathbb{Z}[\Gamma^{(\sigma^1_{\mathcal{C}}, \times \sigma^0_{\mathcal{R}})}]) \rightarrow \tilde{K}_0(\mathbb{Z}[\Gamma^{(\sigma^0_{\mathcal{C}}),_{i+1} \times \sigma^0_{\mathcal{R}}})])$ are just the identity for $i+1 \neq m$.

That is, we have, for $i+1 \neq m$, the following diagram:

\[
\begin{array}{ccc}
\tilde{K}_0(\mathbb{Z}[\Gamma^{(\sigma^1_{\mathcal{C}}, \times \sigma^0_{\mathcal{R}})}]) & \xrightarrow{\partial} & \tilde{K}_0(\mathbb{Z}[\Gamma^{(\sigma^0_{\mathcal{C}}),_{i+1} \times \sigma^0_{\mathcal{R}}})]) \\
\tilde{K}_0(\mathbb{Z}[G^{\sigma^0_{\mathcal{R}}})]) & \xrightarrow{id} & \tilde{K}_0(\mathbb{Z}[G^{\sigma^0_{\mathcal{R}}})]) \\
\end{array}
\]

For $i+1 = m$, we have

\[
\begin{array}{ccc}
\tilde{K}_0(\mathbb{Z}[\Gamma^{(\sigma^1_{\mathcal{C}}, \times \sigma^0_{\mathcal{R}})}]) & \xrightarrow{\partial} & \tilde{K}_0(\mathbb{Z}[\Gamma^{\sigma^0_{\mathcal{R}}})]) \\
\tilde{K}_0(\mathbb{Z}[G^{\sigma^0_{\mathcal{R}}})]) & \xrightarrow{\alpha_*} & \tilde{K}_0(\mathbb{Z}[G]) \\
\end{array}
\]

where $\alpha_*$ is induced by the inclusion $\alpha : G^{\sigma^0_{\mathcal{R}}} \to G = G^{\sigma^0_{\mathcal{R}}}_r \ast_F G^{\sigma^0_{\mathcal{R}}}_s$. By Lemma 5.7 $\alpha_*$ is injective. Analogously, $\tilde{\alpha}_*$ is injective, where $\tilde{\alpha} : G^{\sigma^0_{\mathcal{R}}}_r \to G = G^{\sigma^0_{\mathcal{R}}}_r \ast_F G^{\sigma^0_{\mathcal{R}}}_s$ is the inclusion.

**Claim:** $\text{Im}(\alpha_*) \cap \text{Im}(\tilde{\alpha}_*) = \{0\}$.

**Proof:** Note that $|G^{\sigma^0_{\mathcal{R}}}_r/F| = |G^{\sigma^0_{\mathcal{R}}}_s/F| = 2$. If $G^{\sigma^0_{\mathcal{R}}}_r \cong G^{\sigma^0_{\mathcal{R}}}_s \cong \mathbb{Z}_2 \times F$, the claim follows from Lemma 5.7, because $\tilde{K}_0(\mathbb{Z}[F]) = 0$ for $F$ isomorphic to $\mathbb{Z}_4, \mathbb{Z}_6, D_2, D_4$ or $D_6$. In the other cases we have $\tilde{K}_0(\mathbb{Z}[G^{\sigma^0_{\mathcal{R}}}_s]) = 0$. This proves the claim.

Now, recall that we are assuming that some $n_i \neq 0 \in \tilde{K}_0(\mathbb{Z}[\Gamma^{(\sigma^1_{\mathcal{C}}, \times \sigma^0_{\mathcal{R}})}]) = \tilde{K}_0(\mathbb{Z}[G^{\sigma^0_{\mathcal{R}}}_s])$. If $i+1 \neq m$, Lemma 5.8 (a) implies that $n_i = n_{i+1}$, for $i+1 \neq m$. In the other case, $i+1 = m$, Lemma 5.8 (b) and the claim above imply that $n_{m-1} = n_m$. Then $0 \neq n_i = n_{i+1}$, for all $i$, which is a contradiction because the sum above is finite. This proves the lemma.

**6 Proof of the Results**

Here we proof the results mentioned in the introduction.
Proof of the Main Theorem. First a claim.

Claim The terms $\mathcal{E}^\infty_{p,q}$ vanish if $p + q = -1$ with the exception of

$$
\mathcal{E}^\infty_{0,-1} = \bigoplus_{G \in VC_\infty(\Gamma)} Wh(G).
$$

Proof. To calculate the terms $\mathcal{E}^\infty_{0,-1}$ we need of the terms $\mathcal{E}^2_{2,-2}$ and $\mathcal{E}^2_{3,-3}$. Recall that if $\sigma^2$ is a 2-cell in $A$, $\Gamma^{\sigma^2}$ is trivial or isomorphic to $\mathbb{Z}, D_\infty, D_4, \mathbb{Z}_4$, $i = 2, 3, 4, 6$. Then $\tilde{K}_0(\mathbb{Z}[\Gamma^{\sigma^2}]) = 0$ for any these groups. Hence $\mathcal{E}^2_{2,-2} = 0$. Recall also that if $\sigma^3$ is a 3-cell in $A$, $\Gamma^{\sigma^3}$ is trivial or isomorphic to $\mathbb{Z}_2$. Then $K_{-1}(\mathbb{Z}[\Gamma^{\sigma^3}]) = 0$. Hence $\mathcal{E}^2_{3,-3} = 0$. Therefore

$$
\mathcal{E}^2_{0,-1} = \mathcal{E}^3_{0,-1} = \ldots = \mathcal{E}^\infty_{0,-1} = \bigoplus_{G \in VC_\infty(\Gamma)} Wh(G).
$$

In Section 5 we proved that $\mathcal{E}^2_{1,-2}$ and $\mathcal{E}^2_{2,-3}$ vanish. Hence $\mathcal{E}^\infty_{1,-2}$ and $\mathcal{E}^\infty_{2,-3}$ vanish. This proves the claim.

Recall that our spectral sequence, for $p + q = -1$, converges to $H_{-1}(A/\Gamma, P_*(\rho)) := \pi_{-1}(\mathbb{H}(A/\Gamma, P_*(\rho)))$, where $P_*(\rho)$ is the stable pseudoisotopy functor (see [13]). By the claim, propositions 1.2, 1.3, 1.4 we have that

$$
Wh(\Gamma) = Wh(\pi_1(X)) = \pi_{-1}(P_*(X)) = \bigoplus_{G \in VC_\infty(\Gamma)} Wh(G),
$$

where $X$ is a space such that $\pi_1(X) = \Gamma$.

It remains to prove that the sum in the formula is finite. This is implied by the following lemma:

Lema 6.1: Let $\Gamma$ be a 3-crystallographic group. Then, the subset $\{G < \Gamma; G \in VC_\infty, Wh(G) \neq 0\}$ is finite.

Proof: If $G \in VC_\infty$, with $Wh(G) \neq 0$, then $G$ leaves invariant only one line in $\mathbb{R}^3$ (see Lemma 5.3). Let $l_G$ be the line invariant by $G$. 

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Claim 1: \( l_G \) is contained in the 1-skeleton of the triangulation \( \Gamma_R \) of \( \mathbb{R}^3 \).

Proof of Claim 1: Since \( G \in VC_\infty \), we have the sequence \( 0 \to F \to G \to H \to 0 \), with \( H \) isomorphic to \( \mathbb{Z} \) or \( D_\infty \) and \( F \) a finite subgroup of \( \Gamma \). Recall that \( F \) fixes \( l_G \) pointwise. Recall also that the \( \Gamma \)-action on \( \mathbb{R}^3 \) is simplicial. Moreover, we can assume (subdividing if necessary) that \( \gamma \sigma = \sigma \) implies \( \gamma|_\sigma = 1|_\sigma \), where \( \sigma \) is a simplex and \( \gamma \in \Gamma \). Hence:

If \( \sigma^3 \) is a open 3-cell in \( \Gamma_R \) and \( l_G \cap \sigma^3 \neq \emptyset \), then \( \gamma \sigma^3 = \sigma^3 \) for all \( \gamma \in F \). Hence \( F \) fixes pointwise \( \mathbb{R}^3 \) and follows that \( F \) is the trivial group. Therefore, \( G \) is isomorphic to \( \mathbb{Z} \) or \( D_\infty \) and \( Wh(G) = 0 \).

If \( \sigma^2 \) is a open 2-cell in \( \Gamma_R \) and \( l_G \cap \sigma^2 \neq \emptyset \), we have that \( \gamma \sigma^2 = \sigma^2 \) for all \( \gamma \in F \). Then \( F \) fixes pointwise a plane \( \alpha \) in \( \mathbb{R}^3 \) and acts (at most) by reflection in \( \alpha^\perp \). Therefore \( F \) is the trivial group or isomorphic to \( \mathbb{Z}_2 \). By propositions 3.2 and 3.3, we have that \( G \) is isomorphic to one of the following groups: \( \mathbb{Z} \times \mathbb{Z}_2 \), \( D_2 *_{\mathbb{Z}_2} D_2 \), \( D_2 *_{\mathbb{Z}_2} \mathbb{Z}_4 \), \( \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4 \). Consider the group \( G = G_i *_{\mathbb{Z}_2} G_2 \), where \( G_i, i = 1, 2 \) are finite. By [27], we obtain the exact sequence

\[
Wh(\mathbb{Z}_2) \to Wh(G_1) \oplus Wh(G_2) \to Wh(G) \to \tilde{K}_0(\mathbb{Z}[\mathbb{Z}_2]).
\]

Since \( Wh(\mathbb{Z}_2), Wh(\mathbb{Z}_4), Wh(D_2) \) and \( \tilde{K}_0(\mathbb{Z}[\mathbb{Z}_2]) \) are trivial, we have that \( Wh(D_2 *_{\mathbb{Z}_2} D_2), Wh(D_2 *_{\mathbb{Z}_2} \mathbb{Z}_4), Wh(\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_4) \) are trivial. Also, \( Wh(\mathbb{Z} \times \mathbb{Z}_2) \) is trivial (see [13]). This proves the claim.

Claim 2: Let \( \Gamma^{(l_1)} \) and \( \Gamma^{(l_2)} \) be the isotropy groups of the lines \( l_1 \) and \( l_2 \) respectively. Assume that \( l_i \) is the only line left invariant by \( \Gamma^{(l_i)} \) for \( i = 1, 2 \). Then there is \( \gamma \in \Gamma \) such that \( \gamma l_1 = l_2 \) if and only if \( \Gamma^{(l_1)} \) and \( \Gamma^{(l_2)} \) are conjugate.

Proof of Claim 2: If \( \gamma \in \Gamma \) is such that \( \gamma \Gamma^{(l_1)} \gamma^{-1} = \Gamma^{(l_2)} \) and \( g_1 \in \Gamma^{(l_1)} \), \( g_2 \in \Gamma^{(l_2)} \) with \( g_2 = \gamma g_1 \gamma^{-1} \), we have that: \( g_2(\gamma l_1) = (\gamma g_1 \gamma^{-1})(\gamma l_1) = \gamma g_1 l_1 = \gamma l_1 \). Therefore \( \Gamma^{(l_2)} \) fixes the line \( \gamma l_1 \). Since \( \Gamma^{(l_2)} \) fixes only one line, follows that \( \gamma l_1 = l_2 \). Conversely, if \( \gamma l_1 = l_2 \) and \( g_1 \in \Gamma^{(l_1)} \), then \( \gamma g_1 \gamma^{-1} l_2 = \gamma g_1 l_1 = \gamma l_1 = l_2 \). Hence \( \Gamma^{(l_1)} \) is conjugate to \( \Gamma^{(l_2)} \). This proves the claim.

Recall that the \( \Gamma \)-action on \( \mathbb{R}^3 \) is cocompact. Then, there is a finite subcomplex \( D \subset \mathbb{R}^3 \), such that for all \( x \in \mathbb{R}^3 \) there are \( x_D \in D \) and \( \gamma \in \Gamma \)
with $\gamma x_D = x$. Hence, if $l$ is a line in $\mathbb{R}^3$, there are a line $l_D$ and $\gamma \in \Gamma$ with $l_D \cap D \neq \emptyset$ and $\gamma l_D = l$. Since $D$ is finite, $D$ intercepts only a finite number of 1-cells of the triangulation $T_R$. By Claims 1 and 2, follows that, modulo conjugation, there is a finite number of groups $G \in VC_\infty$ with $Wh(G) \neq 0$. This proves the lemma and the Main Theorem.

**Proof of Corollary 1:** Follows directly from the Main Theorem.

The following result was used in the example mentioned in the introduction.

**Lemma 6.2:** $\text{Nil}_1(\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}_2])$ is infinitely generated.

**Proof:** Consider the Cartesian square with all maps surjective:

\[
\begin{array}{ccc}
\mathbb{Z}[G] & \longrightarrow & \mathbb{Z}[\mathbb{Z}_2] \\
\downarrow & & \downarrow \\
\mathbb{Z}[[\mathbb{Z}_2]] & \longrightarrow & F_2[\mathbb{Z}_2].
\end{array}
\]

where $F_2$ denotes the field with two elements. By [14], $\text{Nil}_1(\mathbb{Z}[[\mathbb{Z}_2]]) = 0$. Hence the Mayer-Vietoris sequence of that square produce an epimorphism

\[
\text{Nil}_2(F_2[\mathbb{Z}_2]) \longrightarrow \text{Nil}_1(\mathbb{Z}[G]) \longrightarrow 0.
\]

Since $\text{Nil}_2(F_2[\mathbb{Z}_2])$ is non trivial (this follows from van der Kallen technique [28]), we have that $\text{Nil}_1(\mathbb{Z}[G]) \neq 0$. Since $\text{Nil}_1(\mathbb{Z}[G]) \neq 0$, we have that $\text{Nil}_1(\mathbb{Z}[G])$ is infinitely generated (see [9]). This proves the lemma.

To prove corollary 2 we need two lemmata.

**Lemma 6.3:** Let $\Gamma$ be a n-crystallographic group and $G$ a maximal virtually infinite cyclic group of $\Gamma \times \mathbb{Z}$. Then (at least) one of the following holds:

1. $G$ is isomorphic to a virtually infinite cyclic subgroup of $\Gamma$.
2. $G = F \times \mathbb{Z}$, $F < \Gamma$, with $F$ maximal finite in $\Gamma$.

**Proof:** Let $\pi : \Gamma \times \mathbb{Z} \rightarrow \Gamma$ be the projection. Then $\text{Ker}(\pi) = \mathbb{Z}$. We have two possibilities: $\pi(G)$ finite or $\pi(G)$ infinite.
Claim: If $\pi(G)$ is infinite then $\pi(G) \cong G$.

Proof of Claim: In fact, if $\pi|_G$ is not one-to-one then we get a exact sequence $0 \to \text{Ker}(\pi|_G) \to G \to \pi(G) \to 0$ with $\text{Ker}(\pi|_G)$ and $\pi(G)$ infinite. This is impossible since $G$ is virtually cyclic. This proves the claim.

Suppose $\pi(G)$ finite. Write $F = \pi(G)$. Then $G < F \times \mathbb{Z}$. But $G$ is maximal, hence $G = F \times \mathbb{Z}$. Certainly, $F$ has to be maximal. This proves the lemma.

Lemma 6.4: $F$ is maximal finite in $\Gamma$ if and only if $F \times \mathbb{Z}$ is maximal virtually infinite cyclic in $\Gamma \times \mathbb{Z}$.

Proof: If $F \times \mathbb{Z}$ is maximal virtually infinite cyclic in $\Gamma \times \mathbb{Z}$ then, certainly, $F$ is maximal finite in $\Gamma$.

Suppose now that $F$ is maximal finite in $\Gamma$ and let $H$ be a virtually infinite cyclic subgroup of $\Gamma$, with $F \times \mathbb{Z} \subset H$. Let $\pi : \Gamma \times \mathbb{Z} \to \Gamma$ be the projection. If $\pi(H)$ is infinite then we have the exact sequence

$$0 \to (\text{Ker} \pi) \cap H \to H \to \pi(H) \to 0$$

This is a contradiction because $\{0\} \times \mathbb{Z} = (\text{Ker} \pi) \cap H$ and $H$ is virtually infinite cyclic. Follows that $\pi(H)$ is finite. Since $F$ is maximal finite in $\Gamma$, we have that $F = \pi(H)$. Therefore $H = F \times \mathbb{Z}$. This proves the lemma.

We will use the following fact (see [18]):

(*) $Wh(G) = \tilde{K}_0(G) = 0$, for $G$ a finite or virtually infinite cyclic subgroup of 2-crystallographic group.

Recall that $F(\Gamma)$ denotes the set of conjugacy classes of maximal finite subgroups of $\Gamma$.

Proof of corollary 2: Let $\Gamma$ be a 2-crystallographic group. Bass-Heller-Swan-formula (see [28] p.152) implies that $Wh(\Gamma \times \mathbb{Z}) = Wh(\Gamma) \oplus \tilde{K}_0(\mathbb{Z}[\Gamma]) \oplus 2\text{Nil}_1(\mathbb{Z}[\Gamma])$. But $Wh(\Gamma), \tilde{K}_0(\mathbb{Z}[\Gamma])$ vanish for any 2-crystallographic group (see (*) above). Hence $Wh(\Gamma \times \mathbb{Z}) = 2\text{Nil}_1(\mathbb{Z}[\Gamma])$. Note that $F$ is conjugate to $F'$ in $\Gamma$ if and only if $F \times \mathbb{Z}$ is conjugate to $F' \times \mathbb{Z}$ in $\Gamma \times \mathbb{Z}$. Hence, by
the Main Theorem, Lemmata 6.3, 6.4, and the Bass-Heller-Swan formula we have

\[ Wh(\Gamma \times \mathbb{Z}) = \bigoplus_{G \in \mathcal{V}C_{\infty}(\Gamma \times \mathbb{Z})} Wh(G) = \]

\[ \bigoplus_{F \in \mathcal{F}(\Gamma)} Wh(F \times \mathbb{Z}) = \bigoplus_{F \in \mathcal{F}(\Gamma)} 2\text{Nil}_1(\mathbb{Z}[F]) = 2 \left( \bigoplus_{F \in \mathcal{F}(\Gamma)} \text{Nil}_1(\mathbb{Z}[F]) \right). \]
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