LATTICES WHICH CAN BE REPRESENTED AS LATTICES OF INTERVALS.

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Abstract. We investigate lattices that can be represented as sub-
lattices of the lattice of all convex subsets of a linearly ordered set
\((X, \leq)\) and as lattices of convex subsets of \((X, \leq)\).

1. Introduction and the main result.

A lattice of sets is a family \(S\) of sets which is a lattice with respect
the inclusion relation \(\subseteq\). A ring of sets is a family of sets closed under
finite unions and intersections.

It is well known that a lattice is isomorphic to a ring of sets and
only if it is distributive \([2\) VII 8, or \(7\) Theorem 10.3]. On the other
hand any lattice is isomorphic to a lattice of sets closed under finite
intersections (see Section 3, Theorem 3.7).

A linear interval, or simply an interval, is a convex subset of linearly
ordered space \((X, \leq)\). The set of all intervals of \((X, \leq)\) is denoted by
\(\text{co}(X, \leq)\) or simply by \(\text{co}(X)\). The lattice \(\text{Co}(\mathcal{P}) = (\text{co}(X), \subseteq)\) is a
simple example of a non modular lattice of sets. If the set \(X\) is finite
then \(\text{Co}(\mathcal{P})\) is a planar lattice and its diagram was described by A.
R. Schweitzer \([14]\). The lattice \(\text{Co}(\mathcal{P})\), where \(\mathcal{P} = (X, \leq)\) is partial
ordered set, was studied by G. Birkhoff and M. K. Bennet \([3]\). The
class \(\text{SUB}\) of all lattices that can be embedded into some lattice of the
form \(\text{Co}(\mathcal{P})\) was studied by M. Semenova and F. Wehrung in \([15]\) which
proved that \(\text{SUB}\) forms a variety, defined by three indentites denoted
by \((S)\), \((U)\) and \((B)\). In \([10]\) the same authors extended this result for
sublattices of products of lattices of convex subsets \(\circ\)

We say that a lattice \(\mathcal{L} = (L, \leq)\) with 0 is representable as a lattice
of intervals if there exists a linearly ordered set \(\mathcal{P} = (X, \leq)\) and an
order-embedding \(f : (L, \leq) \rightarrow (\text{co}(X), \subseteq)\) such that \(f(0) = \emptyset\) and
\(f(a \land b) = f(a) \cap f(b)\). If moreover \(f(a \lor b) = \text{co}(f(a) \cup f(b))\), which

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means that $\mathcal{L}$ is isomorphic to a sublattice of the lattice of all intervals of $(X, \preceq)$, we shall say that $\mathcal{L}$ is \textit{faithfully representable as a lattice of intervals}. 

In this note we investigate lattices with 0 (finite or infinite) which can be represented or faithfully represented as lattices of intervals.

We give three lattice theoretical conditions which are described in the following definition:

\textbf{Definition 1.1.} A lattice $\mathcal{L} = (\mathcal{L}, \preceq)$ is said to be a \textit{loc-lattice} if it has the following properties:

(1) For every $a, b, c \in \mathcal{L}$ we have that $a \preceq b \lor c$ or $b \preceq a \lor c$ or $c \preceq a \lor b$.

(2) If $a, b, x$ are pairwise incomparable elements of $\mathcal{L}$ and $x \preceq a \lor b$ then $(a \lor x) \land (b \lor x) = x$.

(3) If $a, b, x$ are elements of $\mathcal{L}$ such that $x \preceq a \lor b$, $a \land x \neq 0$ and $b \land x \neq 0$ then $(a \land x) \lor (b \land x) = x$.

A semi-prime filter of a lattice $\mathcal{L} = (\mathcal{L}, \preceq)$ is a proper filter $F$ of $\mathcal{L}$ such that for every $a, b \in \mathcal{L}$ with $a \lor b \in \mathcal{L}$ either $a \in F$ or $b \in F$ or there exists an $c \in F$ such that $c \preceq a \lor b$ and $c \neq (a \land c) \lor (b \land c)$. The lattice $\mathcal{L} = (\mathcal{L}, \preceq)$ is said to be \textit{well separated} if for every semi-prime filter $F$ of $\mathcal{L}$ and every $x \in \mathcal{L} \setminus F$ there exists an $y \in F$ such that $x \not\preceq y$.

In figure \textit{fig1 (a)} we have an example of a loc-lattice with five semi-prime filters, $(a)^\uparrow, (b)^\uparrow, (c)^\uparrow, (f)^\uparrow, (g)^\uparrow$, where $(x)^\uparrow = \{y \in \mathcal{L} : x \preceq y\}$. This lattice is not well separated ($d \not\in (f)^\uparrow$ and for every $x \in (f)^\uparrow$, $d \preceq x$) and it is representable (but not faithfully representable) as lattice of intervals. The lattice of \textit{fig1 (b)} is a loc-lattice, it also has five semi-prime filters, $(a)^\uparrow, (b)^\uparrow, (c)^\uparrow, (f)^\uparrow, (g)^\uparrow$, it is not well separated and it is faithfully representable as a lattice of intervals.

The main result of this note is the following:
Theorem 1.2.

(a) Every loc-lattice \( \mathcal{L} = (L, \leq) \) is representable as a lattice of intervals.

(b) A well-separated lattice is faithfully representable as a lattice of intervals if and only if it is a loc-lattice.

A lattice of sets on a set \( X \) is a family \( M \) of subsets of \( X \) such that \((M, \subseteq)\) is a lattice. We say that the family \( M \) separates the set \( X \) if for every \( x, y \in X \) with \( x \neq y \) either there exists an \( M \in M \) with \( x \in M \) and \( y \notin M \) or there exists an \( M' \in M \) with \( x \notin M' \) and \( y \in M' \). We also say that the family \( M \) well-separates \( X \) if it separates \( X \) and for every \( M \in M \) and every \( x \in X \) with \( x \notin M \) there exists an \( N \in M \) with \( x \in N \) and \( N \not\subseteq M \).

In Section 2 we show (Theorem 2.2) that if \( M \) is family of subsets of a set \( X \) with some special properties - called a loc-lattice of sets (Definition 2.1)- which separates \( X \), then we may construct a linear ordering \( \leq \) of the set \( X \) such that every \( M \in M \) is a convex subset of \((X, \leq)\). Moreover if \( M \) well-separates \( X \) then \( M \) is a sublattice of the lattice of all convex sets of \((X, \leq)\).

In Section 3 we introduce the notion of the semi-prime filter of a lattice and we show that every lattice is isomorphic to a lattice \( M \) of subsets of a set \( X \) (the set of all semi-prime filters the lattice) which is closed under finite intersections and separates the points of \( X \). Moreover, if \( L \) is loc-lattice then \( L \) is isomorphic to a loc-lattice \((M, \subseteq)\) of subsets of a set \( X \) which separates \( X \) and Theorem 1.2 follows from the results of Section 2.

Finally in Section 4 we present some applications of Theorems 1.2 and 2.2 in general topology.

2. Loc-lattices of sets

A family of sets \( M \) is said to be a lattice of sets if \((M, \subseteq)\) is a lattice. The lattice sum of two elements \( A, B \) of \( M \) is denoted by \( A \lor B \) and their lattice product by \( A \land B \).

Definition 2.1. A family \( M \) of subsets of a set \( X \) is called a loc-lattice of sets if it is a lattice of subsets of \( X \) satisfying the following conditions:

(1) For every \( A, B \in M \), \( A \land B = A \cap B \).

(2) If \( A, B \in M \) and \( A \cap B \neq \emptyset \) then \( A \lor B = A \cup B \).

(3) For every \( A, B, C \in M \) we have that either \( A \subseteq B \lor C \) or \( B \subseteq A \lor C \) or \( C \subseteq A \lor B \).
(4) If \( A, B, C \) are pairwise incomparable elements of \((\mathcal{M}, \subseteq)\) and \( B \subseteq A \lor C \) then \( B = (A \lor B) \cap (C \lor B) \).

Given a family \( \mathcal{M} \) of subsets of a set \( X \) we define a binary relation \( L \subseteq X \times X \) as follows. We write \( xLy \) if for every \( M \in \mathcal{M} \) with \( x \in M \) we have that \( y \in M \). We say that a family \( \mathcal{M} \) separates the points of \( X \) (or that \( \mathcal{M} \) is a separating family for \( X \)) if for every \( x, y \in X \) with \( x \neq y \) we have that \( \neg(xLy) \) or \( \neg(yLy) \), where \( \neg(xLy) \) is the negation of \( xLy \). This notion of a separating family is introduced by A. Renyi \cite{renyi} which has showed that the minimal size of a separating family of a finite set \( X \) is exactly \( \lceil \log_2 |X| \rceil \). Let \( x, y \in X \). We say that \( x, y \) are completely separated if there exist \( A, B \in \mathcal{M} \) with \( x \in A, y \in B, x \notin B \) and \( y \notin A \) and that \( a, b \) are totally separated if there exist \( A, B \in \mathcal{M} \) with \( a \in A, b \in B \) and \( A \cap B = \emptyset \). We also say that the family \( \mathcal{M} \) completely separates (resp. totally separates) the set \( X \) if for every two distinct points of \( X \) are completely separated (resp. totally separated). Finally, we say that the family \( \mathcal{M} \) well separates the set \( X \) if for every \( M \in \mathcal{M} \) and \( x \notin M \) there exists an \( M' \in \mathcal{M} \) such that \( x \in M' \) and \( M' \not\subseteq M \). Note that if \( \mathcal{M} \) completely separates \( X \) then it well separates \( X \).

The main result of this section is the following:

**Theorem 2.2.** Let \( X \) be a set and \( \mathcal{M} \) be a loc-lattice of subsets of a set \( X \) which separates the points of \( X \).

(a) There exists a linear ordering \( \leq \) of \( X \) such that every \( M \in \mathcal{M} \) is a convex subset of \( X \).

(b) If \( \mathcal{M} \) well separates \( X \) then for every \( A, B \in \mathcal{M}, A \lor B = \co(A \cup B) \) and so \((\mathcal{M}, \subseteq)\) is a sublattice of \((\Co(X), \subseteq)\).

The following lemma summarizes the basic properties of a loc-lattice of sets:

**Lemma 2.3.** Let \( \mathcal{M} \) be a loc-lattice of sets and \( A, B, C \in \mathcal{M} \setminus \{\emptyset\} \). Then

(1) If the sets \( A, B, C \) are pairwise incomparable, \( A \subseteq B \lor C \) and \( B \subseteq A \lor C \) then \( A \subseteq B \). Therefore, if the sets \( A, B, C \) are pairwise incomparable then only one of the relations \( A \subseteq B \lor C \), \( B \subseteq A \lor C \), \( C \subseteq A \lor C \) occurs.

(2) If \( A, B, C \) are pairwise disjoint and \( B \cap (A \lor C) \neq \emptyset \) then \( B \subseteq A \lor C \).

(3) If \( A \cap C \neq \emptyset \) and \( B \cap C \neq \emptyset \) then \( A \lor B \setminus A \cup B \subseteq C \).

(4) If \( C \cap (A \setminus B) \neq \emptyset \) and \( C \cap (B \setminus A) \neq \emptyset \) then \( A \cap B \subseteq C \).

**Proof.** (1) If \( A \subseteq B \lor C \) and \( B \subseteq A \lor C \) then \( A \subseteq (B \lor C) \cap (B \lor A) = B \), by Condition (4) of Definition 2.1.
(2) If \( A \subseteq B \cup C \) then by Condition (4) of Definition 2.1 we have that \((A \cup B) \cap (A \cup C) = A\). But then \( B \cap A = B \cap (A \cup C) \neq \emptyset \), a contradiction. If \( C \subseteq A \cup B \) then \( C = (A \cup C) \cap (B \cup C) \) and so \( B \cap (A \cup C) \subseteq (A \cup C) \cap (B \cup C) = C \), a contradiction. Therefore by Condition (3) of Definition 2.1 we must have \( B \subseteq A \cup C \).

(3) Since \( A \cap C \neq \emptyset \) and \( B \cap C \neq \emptyset \) we have that \( A \cup B \cap C = (A \cup B) \cap C \in \mathcal{M} \). So, \( A \cup B \subseteq (A \cap B) \cup C \).

(4) We may suppose that \( A \cap B \neq \emptyset \). The case \( C \cap A \subseteq (C \cap B) \cup (A \cap B) \) is impossible since then \( C \cap A \subseteq B \) which contradicts the assumption \( (C \setminus B) \cap A \neq \emptyset \). The case \( C \cap B \subseteq (C \cap A) \cup (A \cap B) \) is also impossible. Therefore, \( A \cap B \subseteq (C \cap A) \cup (C \cap B) \subseteq C \).

\[ \text{□} \]

A ternary relation in a set \( X \) is a subset \( T \) of \( X^3 \). We shall use the notation \((abc)_T\) instead of \((a, b, c) \in T\) and that \(\neg(abc)_T\) instead of \((a, b, c) \notin T\). We define a ternary relation in \( X \) setting \((abc)_M\) if for every \( M \in \mathcal{M} \) with \( a, c \in M \) we have also that \( b \in C \). Let

\[ T_M = \{(a, b, c) \in X^3 : (abc)_M\}. \]

Given a linear ordering \( \leq \) of \( X \) we set

\[ T_{\leq} = \{(a, b, c) \in X^3 : a \leq b \leq c \text{ or } c \leq b \leq a\}. \]

**Definition 2.4.** Let \( \mathcal{M} \) be a family of subsets of a set \( X \) and \( Y \subseteq X \). We say that a linear ordering \( \leq \) of \( Y \) of \( X \) is \( \mathcal{M} \)-consistent if \( T_{\leq} \subseteq T_M \).

Clearly in order to prove Theorem 2.2 we must find an \( \mathcal{M} \)-consistent linear ordering \( \leq \) of \( X \).

**Lemma 2.5.** Let \( \mathcal{M} \) be a loc-lattice of subsets of a set \( X \). Then for every \( a, b, c \in X \) either \((abc)_M\) or \((acb)_M\) or \((bac)_M\).

**Proof.** If \( \neg(abc)_M \) and \( \neg(acb)_M \) then there exist \( M, N \in \mathcal{M} \) such that \( \{a, c\} \subseteq M \), \( \{a, b\} \subseteq N \), \( b \notin M \) and \( c \notin N \). Let \( K \in \mathcal{M} \) such that \( b, c \in K \). Then \( K \cap (M \setminus N) \neq \emptyset \) and \( K \cap (N \setminus M) \neq \emptyset \). By Lemma 2.3 (4) we have that \( M \cap N \subseteq K \) and so \( a \in K \). \( \text{□} \)

If \( A, B \) are subsets of a linear ordered set \((X, \leq)\) we write \( A < B \) if for every \( a \in A \) and \( b \in B \) we have that \( a < b \). Similarly if \( a \in X \) and \( B \subseteq X \) we write \( a < M \) if \( a < b \) for every \( b \in B \).

**Lemma 2.6.** Let \( \mathcal{M} \) be a loc-lattice \( \mathcal{M} \) of subsets of \( X \) which well-separates \( X \) and \( \leq \) an \( \mathcal{M} \)-consistent linear ordering of \( X \). Then \( \mathcal{M} \) is a sublattice of the lattice \( \text{Co}(X) \) of intervals of \((X, \leq)\).
Proof. Clearly $\mathcal{M}$ is a subset of $\text{Co}(X)$. It remains to show that if $A, B \in \mathcal{M}$ then $A \lor B = \text{co}(A \cup B)$ where $\text{co}(A)$ denotes the convex hull of a subset $A$ of $X$. Since $A \lor B$ is convex we have that $\text{co}(A \cup B) \subseteq A \lor B$. Suppose that $\text{co}(A \cup B) \neq A \lor B$. Then $A \cap B = \emptyset$ and therefore either $A < B$ or $B < A$. Suppose that $A < B$. Let $c \in (A \lor B) \setminus \text{co}(A \cup B)$. Then we must have that either $c < A$ or $B < c$. Suppose that $B < c$ and let $C \in \mathcal{M}$ with $c \in C$ and $C \not\subseteq B$. The sets $A, B, C$ are incomparable, $B \subseteq A \lor C$ and $C \subseteq A \lor B$ which contradicts Lemma 2.3 (1).

The following Theorem, due to M. Altwegg [1] (see also [17]), is useful since it characterizes linear orderings of sets in terms of ternary relations. Note that by Lemma 2.3 $T_M$ always satisfies condition (3) of the Theorem of Altwegg.

**Theorem 2.7** (M. Altwegg). Let $T$ be a ternary relation in a set $X$ which satisfies the following postulates:

1. $(aba)_T$ if and only if $a = b$.
2. If $(abc)_T$ and $(bde)_T$ then either $(cbd)_T$ or $(eba)_T$.
3. For every $a, b, c \in X$ either $(abc)_T$ or $(bca)_T$ or $(cab)_T$.

Then there exists a linear ordering $\leq$ of $X$ such that for every $x, y, z \in X$ we have that $(xyz)_T$ if and only if $(xyz)_{\leq}$. Moreover, every other linear ordering $\leq'$ of $X$ satisfying the preceding condition is equal to $\leq$ or to the inverse order $\leq^*$ of $\leq$.

The ternary relation $T_M$ does not satisfy in general the conditions of Theorem 2.7 since for a loc-lattice $\mathcal{M}$ of subsets of $X$ there are probably many and even non-isomorphic $\mathcal{M}$-consistent linear orderings.

Indeed, the simplest example of a loc-lattice of subsets of a set $X$ is a chain of subsets of $X$. Suppose that $X = \mathbb{N}$ and that $\mathcal{M} = \{A_n : n \in \mathbb{N}\}$ where $A_n = \{1, \ldots, n\}$. Then the usual ordering $1 \leq 2 \leq 3 \leq \ldots$ of $\mathbb{N}$ and the ordering $\leq'$ given by

$$
\cdots \leq' 2n+2 \leq' 2n \leq' \cdots \leq' 2 \leq' 1 \leq' 3 \leq' \cdots \leq' 2n-1 \leq' 2n+1 \leq' \cdots
$$

are non-isomorphic $\mathcal{M}$-consistent linear orderings of $\mathbb{N}$.

**Definition 2.8.** Let $X$ be a set and $\mathcal{M}$ a family of subsets of $X$. If $x_1, \ldots, x_n$ is a sequence of pairwise distinct points of $X$ we say that $(M_i)_{i=1}^n$ is a **representative family** for $(x_i)_{i=1}^n$ if for every $i, j \in \{1, \ldots, n\}$ with $j \neq i$ we have that $M_i \in \mathcal{M}$, $x_i \in M_i$ and $x_j \notin M_i$.

It is clear that if the family $\mathcal{M}$ is closed under finite intersections then for every finite sequence $(x_i)_{i=1}^n$ of pairwise completely separated points of $X$ there exists a representative family for $(x_i)_{i=1}^n$. 


Lemma 2.9. Let \( M \) be a loc lattice of subsets of a set \( X \), \( a, b, c \) three distinct points of \( X \) which are pairwise completely separated. Let \( (A, B, C) \in M^3 \) a representative family for the triple \( (a, b, c) \). Then

(i) \((abc)_M\) holds if and only \( B \subseteq A \lor C \).

(ii) If \( B \subseteq A \lor B \) then \( A \cap C = \emptyset, a \not\in B \lor C \) and \( c \not\in A \lor B \).

Proof. (i) Suppose that \( (abc)_M \) holds then for every representative family \( (A, B, C) \) of \( (a, b, c) \) we have that \( B \subseteq A \lor C \). If \( A \subseteq B \lor C \) then \( A = (A \lor B) \cap (A \lor C) \) and so \( A \cap B = B \cap (A \lor C) \). Since \( (abc)_M \) we have that \( b \in A \lor C \) and so \( b \in A \), a contradiction. In the same manner the case \( C \subseteq A \lor B \) is excluded. Therefore, we must have \( B \subseteq A \lor C \). Suppose now that there exists a representative family \( (A, B, C) \) for \( (a, b, c) \) such that \( B \subseteq A \lor C \) and \( (abc)_M \) does not hold. Then by Lemma 2.5 either \((bac)_M\) or \((cba)_M\) and by (a), either \( A \subseteq B \lor C \) or \( C \subseteq A \lor B \), but this contradicts Lemma 2.3 (1).

(ii) Suppose that \( A \cap C \neq \emptyset \) then \( A \lor C = A \lor C \) and since \( B \subseteq A \lor C \) we have that either \( b \in A \) or \( b \in C \), which contradicts the assumption that \( (A, B, C) \) is a representative family for \( (a, b, c) \). If \( a \in B \lor C \) since \( B = (A \lor B) \cap (B \lor C) \) we shall have \( a \in B \), a contradiction. Similarly we see that \( c \not\in A \).

Lemma 2.10. Let \( M \) be a loc-lattice of subsets of a set \( X \) which completely separates \( X \). Then \( M \) totally separates \( X \).

Proof. Let \( a, b \) be distinct points of \( X \) and \((A_1, B_1)\) a representative family for \( (a, b) \). Suppose that \( A_1 \cap B_1 \neq \emptyset \) and let \( c \in A_1 \cap B_1 \). Then we may choose a representative family \( (A, B, C) \) for \( (a, b, c) \) with \( A \subseteq A_1 \), \( B \subseteq B_1 \), \( C \subseteq A_1 \cap B_1 \). Since \( C \lor A \subseteq C \lor A_1 = A_1 \) we cannot have \( B \subseteq C \lor A \). Similarly, the case \( A \subseteq C \lor B \) is impossible. Therefore by Property (1) of Definition 1.1 we must have that \( C \subseteq A \lor B \). By Lemma 2.9 (2) we conclude that \( A \cap B = \emptyset \).

Lemma 2.11. Let \( M \) be a loc lattice of subsets of a set \( X \), \( a, b, c, d, e \) five distinct points of \( X \) such that every two of them are completely separated and let \((A, B, C, D, E)\) a representative family for \( (a, b, c, d, e) \). Then

1. \( B \subseteq A \lor C \) and \( C \subseteq B \lor D \) if and only if \( B \subseteq A \lor D \) and \( C \subseteq A \lor D \).
2. \( B \subseteq A \lor C \) and \( D \subseteq B \lor C \) if and only if \( B \subseteq A \lor D \) and \( D \subseteq A \lor C \).
3. If \( B \subseteq A \lor C \) and \( D \subseteq B \lor E \) then either \( B \subseteq C \lor D \) or \( B \subseteq A \lor E \).

Proof. (1) Suppose that \( B \subseteq A \lor C \) and \( C \subseteq B \lor D \). Then either \( A \subseteq B \lor D \) or \( B \subseteq A \lor D \) or \( D \subseteq A \lor B \). Since \( B \subseteq A \lor C \) and \( A, B, C \)
are pairwise incomparable we have that \( B = (A \lor B) \cup (B \lor C) \) and so \( c \notin A \lor B \). If \( D \subseteq A \lor B \) then \( A \lor B = (A \lor D) \cup (B \lor D) \) and since \( C \subseteq B \lor D \) we have that \( C \subseteq A \lor B \), a contradiction. In the same manner the case \( A \subseteq B \lor D \) is excluded. So, \( B \subseteq A \lor C \).

(2) Similar to (1).
(3) It follows readily from (1) and (2). □

**Proposition 2.12.** Let \( \mathcal{M} \) be a loc-lattice of subsets of a set \( X \) which completely separates the points of a subset \( Y \) of \( X \). Then there exists a consistent linear ordering \( \leq \) of \( Y \). Moreover, every other consistent linear ordering \( \leq' \) of \( Y \) is equal to \( \leq \) or equal to the inverse order of \( \leq \).

**Proof.** Let \( \mathcal{M}_Y = \{M \cap Y : M \in \mathcal{M}\} \). By Lemmas 2.5, 2.9 and 2.11 we see that the ternary relation \( T_{\mathcal{M}_Y} \) of \( Y \) satisfies the conditions of Altwegg’s Theorem 2.7. So we may find a \( \mathcal{M}_Y \)-consistent linear ordering of \( Y \) which will be \( \mathcal{M} \)-consistent too. □

By Lemma 2.6 we have as an immediate corollary the following.

**Theorem 2.13.** Let \( X \) be a set and \( \mathcal{M} \) be a loc-lattice of subsets of a set \( X \) which completely separates \( X \). Then there exists a linear ordering \( \leq \) of \( X \) such that \( \mathcal{M} \) is a sublattice of the lattice \( (\text{Co}(X), \subseteq) \) of all intervals of \( (X, \leq) \). Moreover, every other linear ordering \( \leq' \) of \( X \) satisfying the preceding condition is equal to \( \leq \) or to the inverse order \( \leq^* \) of \( \leq \).

The general case of Theorem 2.2 when the family \( \mathcal{M} \) does not completely separate the set \( X \) is much more complicated.

**Proof of Theorem 2.2**

Before we proceed to the proof of Theorem 2.2 we shall introduce some further notation. Let \( (X, \leq) \) be a linearly ordered set. If \( A \subseteq X \) we set

\[
A^- = \{x \in X : x < A\}, \quad A^+ = \{x \in X : A < x\}.
\]

A *section* of a linearly ordered set \( (X, \leq) \) a pair \( S = (A, B) \) of subsets of \( X \) such that \( A \cup B = X \) and \( A < B \) (for the theory of sections see also [18]). The the pairs \((\emptyset, X)\) and \((X, \emptyset)\) are considered as sections too. The set of all sections of a linearly ordered set \( (X, \leq) \) is denoted by \( \mathcal{S}(X, \leq) \). The section \( S = (A, B) \) is said to be

(1) a *gap* if \( A \) has no greatest element and \( B \) has no least element,
(2) a *jump* if \( A \) has a greatest element and \( B \) has a least element,
(3) a *left cut* if \( A \) has a greatest element and \( B \) has no least element,
(4) a right cut if $A$ has no greatest element and $B$ has a least element.

Given a section $S = (A, B)$ we also denote by $S^{(1)}$ the first member $A$ of the section and by $S^{(2)}$ the second member $B$ of the section.

We define a binary relation $L \subseteq X \times X$ on $X$ by the rule that $xLy$ if every element $M$ of $\mathcal{M}$ containing $x$ also contains $y$. It is trivial that $L$ is reflexive and transitive, and since $\mathcal{M}$ separates the set $X$ it is also antisymmetric; that is $L$ is a partial order on $X$. We write $\neg(xLy)$ if $xLy$ does not hold. Two points $x, y$ of $X$ is said to be independent if $\neg(xLy)$. A subset $Y$ of $X$ is said to be independent if every two distinct elements of $Y$ are independent.

Let $Y$ be a subset, $\leq$ a linear ordering of $Y$, $\mathcal{G}$ the set of all sections and $\phi : X \setminus Y \to \mathcal{G}$ any mapping from the remainder set $X \setminus Y$ to the set $\mathcal{G}$. Suppose further that for any $S \in \mathcal{G}$ we have defined a linear ordered set $(X_S, \leq_S)$ with $X_S \subseteq \phi^{-1}(S)$. Then we may extend the linear ordering $\leq$ of $Y$ to a linear ordering $\leq'$ of $Y \cup (\bigcup_{S \in \mathcal{G}} X_S)$ by the following rules:

1. If $x, y$ in $Y$ we set $x \leq' y$ if and only if $x \leq y$.

2. If $x \in X_S$ with $S = (A, B)$ and $y \in Y$ we set $x \leq' y$ if $y \in B$ and $y \leq' x$ if $y \in A$.

3. If $x \in X_S$ with $S = (A, B)$, $x' \in X_{S'}$ with $S' = (A', B')$ and $S \neq S'$ we set $x \leq x'$ if and only if $A \subset A'$.

4. If $x \in X_S$ with $S = (A, B)$, $x' \in X_{S'}$ with $S' = (A', B')$ and $S = S'$ we set $x \leq x'$ if and only if $x \leq_S x'$.

**Definition 2.14.** We shall call the linear ordering $\leq'$ the extension of $\leq$ to the set $Y \cup (\bigcup_{S \in \mathcal{G}} X_S)$.

**Definition 2.15.** Let $\mathcal{M}$ be a family of subsets of a set $X$. A triple $(Y, \leq, \phi)$ is said to be an $\mathcal{M}$-consistent triple if the following conditions are fulfilled:

1. $Y$ is a subset of $X$ and $\leq$ is an $\mathcal{M}$-consistent linear ordering of $Y$.

2. $\phi$ is a map from the set $X \setminus Y$ to the set $\mathcal{G} = \mathcal{G}(Y, \leq)$ of all sections of $(Y, \leq)$ so that for every family $(X_S, \leq_S)_{S \in \mathcal{G}}$ with $X_S \subseteq \phi^{-1}(S)$ and $\leq_S$ to be an $\mathcal{M}$-consistent linear ordering...
of $X_S$, the extension of $\leq$ to the set $\bigcup_{S \in \mathcal{S}} X_S \cup Y$ is also $\mathcal{M}$-consistent.

Let $\text{Ord}$ the class of all ordinal numbers. Given a loc-family $\mathcal{M}$ of subsets of $X$ which separates the points of $X$ we shall construct a family $(Y_\xi, \leq_\xi, \phi_\xi)_{\xi \in \text{Ord}}$ with the following properties:

1. For every $\xi$, $(Y_\xi, \leq_\xi, \phi_\xi)$ is an $\mathcal{M}$-consistent triple.
2. If $\zeta \leq \xi$ then $(Y_\zeta, \leq_\zeta)$ is an extension of $(Y_\xi, \leq_\xi)$.
3. If $Y_\xi \neq X$ then $Y_{\xi+1} \neq Y_\xi$.

Clearly if a such family $(Y_\xi, \leq_\xi, \phi_\xi)_{\xi \in \text{Ord}}$ exists, then for some $\xi \in \text{Ord}$ we shall have that $Y_\xi = X$ and Theorem 2.2 has been proved.

**Step 0.**

In this step of the proof we shall construct an $\mathcal{M}$-consistent triple $(Y_0, \leq_0, \phi_0)$.

By Zorn’s Lemma the set $X$ contains a maximal independent subset $Y$. If $Y$ is an independent subset of $X$ then by Proposition 2.12 there exists an $\mathcal{M}$-consistent linear ordering of the set $Y$. Let $Y$ be a maximal independent subset of $X$ and $\leq$ an $\mathcal{M}$-consistent linear ordering of $Y$. If $|Y| = 1$ then it is easy to see that the relation $L$ is an $\mathcal{M}$-consistent linear ordering of $X$. So we may assume that $|Y| \geq 2$.

For every $x \in X \setminus Y$ we set

$$L(x) = L_Y(x) = \{ y \in Y : xLy \}, \quad M(x) = M_Y(x) = \{ y \in Y : yLx \}.$$

Note that for every $x \in X \setminus Y$ we have that $L(x) = \emptyset$ if and only if $M(x) = \emptyset$ and that $L(x)$ is a convex subset of $(Y, \leq)$. By Lemma 2.9 (ii) we have that $|M(x)| \leq 2$ and if $M(x) = \{ y, z \}$ with $y < z$ then $z$ is the immediate successor of $y$.

We classify the points of $X \setminus Y$ in eight classes or types by the following rules: A point $x \in X \setminus Y$ is said to be a point of

- **Type Ia** if $L(x) \neq \emptyset$, $L^+(x) \neq \emptyset$ and $L^-(x) \neq \emptyset$,
- **Type Ib** if $L(x) \neq \emptyset$, $L^+(x) \neq \emptyset$ and $L^-(x) = \emptyset$,
- **Type Ic** if $L(x) \neq \emptyset$, $L^+(x) = \emptyset$ and $L^-(x) \neq \emptyset$,
- **Type Id** if $L(x) = Y$,
- **Type IIa** if $|M(x)| = 2$,
- **Type IIb** if $M(x) = \{ y \}$ and $y$ is neither the first nor the last element of $Y$,
- **Type IIc** if $M(x) = \{ y \}$ and $y$ is the first element of $Y$,
- **Type IId** if $M(x) = \{ y \}$ and $y$ is the last element of $Y$. 


For every $x \in X \setminus Y$ with $L(x) \neq \emptyset$. We set

\[ L_1(x) = \{ y \in L(x) : \text{there exists a } y' \in L^{-}(x) \text{ with } - (y, y'), M \} , \]

\[ L_2(x) = \{ y \in L(x) : \text{there exists a } y' \in L^{+}(x) \text{ with } - (y, y'), M \} . \]

**Lemma 2.16.** Let $x \in X \setminus Y$ such that $L(x) \neq \emptyset$ and $L(x) \neq Y$. Then the set $L_1(x)$ is an initial segment of $L(x)$ (if it is not empty) and $L_2(x)$ is a final segment of $L(x)$ (if it is not empty). Moreover,

\[ L_1(x) \cap L_2(x) = \emptyset \text{ and } L_1(x) \cup L_2(x) = L(x). \]

**Proof.** Let $y \in L_1(x)$ and $y' \in L(x)$ with $y' < y$. Then there exists a $z < L(x)$ and $M \in \mathcal{M}$ with $z, y \in M$ and $x \notin M$. Since $z < y' < y$ and $\leq$ is consistent we have that $y' \in M$ and so $y' \in L_1(x)$. This shows that $L_1(x)$ is an initial segment of $L(x)$. By the same reasoning we show that $L_2(x)$ is a final segment of $L(x)$.

Suppose that there exists $y \in L_1(x) \cap L_2(x)$ and let $y_1, y_2 \in Y$ with $y_1 < L(x) < y_2$. We may find $M_1, M_2, M \in \mathcal{M}$ with $y_1, y \in M_1$, $y_2, y \in M_2$, such that $(M_1, M_2, M)$ is a representative triple for $(y_1, y_2, x)$. Clearly $M_1 \not\subseteq M \cup M_2 = M \cup M_2$ and $M_2 \not\subseteq M \cup M_1 = M \cup M_1$. But then $M \subseteq M_1 \cup M_2$ and $M_1 \cap M_2 \neq \emptyset$ which contradicts Lemma 2.6.

Finally suppose that there exists a $y \in L(x) \setminus (L_1(x) \cup L_2(x))$. Let $y_1, y_2 \in Y$ with $y_1 < L(x) < y_2$ and let $(M_1, M_2, M)$ be a representative triple for $(y_1, y_2, x)$ with $x \notin M$. Since $y \notin L_1(x)$ we have that $x \in M \cup M_1$ and since $x \notin L_2(x)$ we have that $x \in M \cup M_2$. But then $x \in (M \cup M_1) \cap (M \cup M_2) = M$, a contradiction. \qed

For every $x \in X \setminus Y$ we associate a section $S_x = (A_x, B_x)$ of $Y$ as follows:

(i) If $x$ is a point of type $I_a$ we set

\[ A_x = L_1^{-}(x) \cup L_1(x) = L_2^{-}(x), \]

\[ B_x = L_1^{+}(x) = L_2(x) \cup L_2^{+}(x). \]

(ii) If $x$ is a point of type $I_b$ we set

\[ A_x = L_2^{-}(x), \]

\[ B_x = L_2(x) \cup L_2^{+}(x). \]
(iii) If \( x \) is a point of type \( \text{I}_c \) we set
\[
A_x = L_1^-(x) \cup L_1(x), \\
B_x = L_1^+(x).
\]
Let \( W \) be the set of all points of \( X \) of type \( \text{I}_d \), \( U_1 \) the set of all points of \( X \setminus Y \) of type \( \text{I}_b \) with \( A_x = \emptyset \) and \( U_2 \) the set of all points of \( X \setminus Y \) of type \( \text{I}_c \) with \( B_x = \emptyset \). For every \( U \subseteq X \) we set
\[
d(U) = \sup \{|V| : V \text{ is an independent subset of } U\}
\]

Lemma 2.17. Let \( U \) is a subset of \( X \setminus Y \) with \( \bigcap_{x \in U} L(x) \neq \emptyset \) then \( d(U) \leq 2 \). Moreover \( d(U_1) \leq 1 \) and \( d(U_2) \leq 1 \).

Proof. Suppose that there exist three independent points \( x_1, x_2, x_3 \in U \) then we can find a representative triple \( (M_1, M_2, M_3) \) for \( (x_1, x_2, x_3) \). But then \( M_1 \cap M_2 \cap M_3 \neq \emptyset \) which contradicts Lemma 2.9. Suppose that there exist two independent points \( x_1, x_2 \in U_1 \). We may suppose that \( L(x_1) \subseteq L(x_2) \). Let \( y \in L(x_2) \), \( y' > L(x_2) \). Then we may find a representative triple \( (M, M_1, M_2) \) for \( (y', x_1, x_2) \) with \( y \in M \). But then \( y \in M \cap M_1 \cap M_2 \) which again contradicts Lemma 2.9. \( \square \)

Lemma 2.18. Every point of \( U_1 \) is independent from any point of \( U_2 \).

Proof. Suppose that there exist \( x_1 \in U_1 \) and \( x_2 \in U_2 \) such that \( x_1 \mathcal{L} x_2 \). Then we see that \( L(x_1) = Y \), a contradiction. \( \square \)

(iv_1) There exist \( w_0 \in W \) and \( x_1 \in U_1 \) such that \( \neg(w \mathcal{L} x_1) \).
In that case clearly \( \neg(x_1 \mathcal{L} w_0) \) (otherwise \( L(x_1) = Y \)) and so by Lemmas 2.17 and 2.18 we have that \( w \mathcal{L} x_2 \) for every \( x_2 \in U_2 \). We set
\[
S_{w_0} = (Y, \emptyset).
\]
Let any \( w \in W \). If \( w \) is independent from \( w_0 \) then \( w \mathcal{L} x_1 \) from every \( x_1 \in U_1 \) and we set
\[
S_w = (\emptyset, Y).
\]
If \( w \) is not independent from \( w_0 \) we set
\[
S_w = S_{w_0} = (Y, \emptyset).
\]

(iv_2) There exist \( w_0 \in W \) and \( x_2 \in U_2 \) such that \( \neg(w \mathcal{L} x_2) \).
In that case clearly \( \neg(x_2 \mathcal{L} w_0) \) (otherwise \( L(x_2) = Y \)) and so by Lemmas 2.17 and 2.18 we have that \( w \mathcal{L} x_1 \) for every \( x_1 \in U_2 \). We set
\[
S_{w_0} = (\emptyset, Y).
\]
Let any \( w \in W \). If \( w \) is independent from \( w_0 \) then \( wLx_2 \) from every \( x_2 \in U_2 \) and we set

\[
S_w = (Y, \emptyset).
\]

If \( w \) is not independent from \( w_0 \) we set

\[
S_w = S_{w_0} = (Y, \emptyset).
\]

(iv) \( d(W) = 2 \) and for every \( w \in W, x_1 \in U_2 \) and \( x_2 \in U_2 \) we have that \( (wLx_1) \) and \( (wLx_2) \).

We select two independent points \( w_1, w_2 \) of \( W \) and we set

\[
S_{w_1} = (\emptyset, Y), \quad S_{w_2} = (Y, \emptyset).
\]

For any other \( w \in W \) we set \( S_w = S_{w_1} \) if the points \( w, w_1 \) are not completely separated and \( S_w = S_{w_2} \) if the points \( w, w_2 \) are not completely separated.

(v) \( x \) is point of type \( \Pi_a \) and \( M(x) = \{y_1, y_2\} \). In that case we set

\[
A_x = \{ y \in Y : y \leq y_1 \}, \quad B_x = \{ y \in Y : y_2 \leq y \}.
\]

Lemma 2.19. If \( x \) is a point of type \( \Pi_b \) with \( M(x) = \{y_x\} \) then one and only one of the following cases occurs:

1. For every \( y > y_x \) we have \( \neg(xy_x y)M \).
2. For every \( y < y_x \) we have \( \neg(xy_x y)M \).

Proof. Suppose that for some \( y > y_x \) we have that \( (xy_x y)M \). Let \( y' < y_x \) and let \( (M', M, A) \) a representative triple for \( (y', y, x) \) with \( y_x \notin A \).

Then \( A = (M' \cap A) \cap (M \cap A) \) since \( y_x \notin A \) and \( y_x \in M \cap A \) we have that \( y_x \notin M' \cap A \) and so \( \neg(xy_x y')M \). The same argument shows that there exist no \( y_1, y_2 \) with \( y_1 < y_x < y_2 \) such that \( (xy_x y_1)M \) and \( (xy_x y_2)M \).

(vi) \( x \) is a point of type \( \Pi_b \) and for every \( y > y_x \) we have \( \neg(xy_x y)M \).

In that case we set

\[
A_x = \{ y \in Y : y \leq y_x \}, \quad B_x = \{ y \in Y : y_x < y \}.
\]

(vii) \( x \) is a point of type \( \Pi_b \) and for every \( y < y_x \) we have \( \neg(xy_x y)M \).

In that case we set

\[
A_x = \{ y \in Y : y < y_x \}, \quad B_x = \{ y \in Y : y_x \leq y \}.
\]

(vii) \( x \) is a point of type \( \Pi_c \), \( M(x) = \{y_0\} \) where \( y_0 \) is the first element of \( Y \) and for some \( y > y_0 \) we have that \( \neg(xy_0 y)M \) then we set

\[
A_x = \{ y_0 \}, \quad B_x = \{ y \in Y : y_0 < y \}.
\]
Lemma 2.20. Let $y \in M$ and for every $y > y_0$ we have that $(xy_0y)_M$ then we set $A_x = \emptyset, B_x = Y$.

Lemma 2.21. If $x, x' \in X \setminus Y$ with $S_x = (A_x, B_x)$. Then for every $y_1 \in A_x, y_2 \in B_x$ we have that $(y_1y_2)_M$.

Proof. Suppose that $x$ is a point of type I, then we may find $y_1 \leq y_1' \leq y_2 \leq y_2' \leq y_1 \leq L(x) < t'_2$. Let $M_1, M_2, M \in \mathcal{M}$ with $y_2' \in M_2, y_1, y_1' \in M_1, x \in M x \notin M_1 \cup M_2, y_1' \notin M$ and $y_2' \notin M$. Since $L(x) \subseteq M$ there exists a $y \in M$ with $y_1' < y < y_2'$. The triple $(M_1, M_2)$ is a representative family for $(y_1, y, y_2)$ and so $M \subseteq M_1 \cup M_2$. If $N \in \mathcal{M}$ with $y_1', y_2 \in N$ then $N \subseteq M_1 \subseteq N$ and $N \subseteq M_2 \subseteq N$. By Lemma 2.20 (3) we have that $M_1 \cup M_2 \subseteq N$ and so $x \in N$. This shows that $(y_1y_2)_M$. The other cases can also be checked easily.

Lemma 2.21. If $x, x' \in X \setminus Y$ with $L_1(x) \cap L_1(x') \neq \emptyset$ and $L_2(x) \cap L_2(x') \neq \emptyset$ then $x = x'$. In particular, for every section $S = (A, B)$ of $(Y, \leq)$ there exists at most one point $x$ such that $S_x = S, L(x) \cap A \neq \emptyset$ and $L(x) \cap B \neq \emptyset$.

Proof. Clearly $S_x = S_{x'}$. If $M \in \mathcal{M}$ with $x \in M$ then $M \subseteq A \neq \emptyset$ and $M \cap B \neq \emptyset$ and by Lemma 2.20 we shall have that $x' \in M$. So $xLx'$. Similarly, $x'Lx$ and so $x = x'$, since the family $\mathcal{M}$ separates the set $X$.

Let us call the points of type II, the points $x$ of type II such that $S_x = (\emptyset, Y)$ (i.e. the points of Case (viii) ) and the points $x$ of type II with $S_x = (Y, \emptyset)$ (i.e. the points of Case (x) ) exceptional points and the other points of $X \setminus Y$ ordinary points. We set $Y_0 = Y \cup \{ x : x \in X \setminus Y$ and $x$ is an ordinary point }.

Our next step is to extend the linear ordering $\leq$ to an $\mathcal{M}$-consistent linear ordering $\leq_0$ of $Y_0$. 
Before doing this for every section \( S = (A, B) \) of \((Y, \leq)\) we shall define a consistent linear ordering \( \leq_S \) to the set \( \tilde{S}_{ord} \) of all ordinary points \( x \) of \( X \setminus Y \) with \( S_x = S \).

1. \( S = (\emptyset, Y) \). In that case an ordinary point \( x \in \tilde{S}_{ord} \) is a point of type \( I_b \) or of the type \( I_d \). By Lemma 2.27 and the previous construction we have that for any \( x, x' \in \tilde{S}_{ord} \) either \( x \x L x' \) or \( x' \x L x \). We set \( x \leq_S y \) if \( x \x L y \).

2. \( S = (Y, \emptyset) \). In that case for \( x, y \in \tilde{S} \) we set \( x \leq y \) if and only if \( y \x L x \).

3. Let \( S = (A, B) \) to be a section of \((Y, \leq)\) such that \( A \neq \emptyset \) and \( B \neq \emptyset \). We set

\[
\tilde{S} = \{ x \in X \setminus Y : S_x = S \},
\]

\[
\tilde{S}_1 = \{ x \in \tilde{S} : L(x) \cap A \neq \emptyset \}, \quad \tilde{S}_2 = \{ x \in \tilde{S} : L(x) \cap B \neq \emptyset \}
\]

\[
\tilde{S}_3 = \{ x \in \tilde{S} : M(x) \cap A \neq \emptyset \}, \quad \tilde{S}_4 = \{ x \in \tilde{S} : M(x) \cap B \neq \emptyset \}.
\]

**Lemma 2.22.** For every \( y \in A, y' \in B, x \in \tilde{S}_1 \cup \tilde{S}_2, z \in \tilde{S}_3, z' \in \tilde{S}_4 \) we have that \((yxy')_M, (yxz')_M, (zxy')_M\) and \((zxz')_M\).

**Proof.** The fact that \((yxy')_M\) follows from Lemma 2.20. Let \( z \in \tilde{S}_3, y \in B \) and \( y_0 \) to be the last element of \( A \), which exists since \( \tilde{S}_3 \neq \emptyset \). Suppose that there exists a \( M \in \mathcal{M} \) with \( z, y \in M \) and \( x \notin M \). Let \( N \in \mathcal{M} \) with \( y_0 \in N \) and \( x \notin N \). Then \( z \in N \) and so \( M \cup N = M \cup N \). But then \( y, y_0 \in M \cup N \) and \( x \notin M \cup N \) which contradicts Lemma 2.20. The other cases are treated similarly. \( \square \)

**Lemma 2.23.** \( d(\tilde{S}_1) = d(\tilde{S}_2) = 1 \).

**Proof.** Suppose that there exist two independent points \( x, x' \) of \( \tilde{S}_1 \). Since \( S \neq (\emptyset, Y) \) then the (non-empty) sets \( L(x), L_1(x') \) are not initial segments of \( Y \). Also we may suppose that \( L_1(x) \subset L_1(x') \). So there exist \( y \in L_1(x), y' \in L_1(x') \) and \( M \in \mathcal{M} \) such that \( y, y' \in M \) and \( x, x' \notin M \). The set \( \{ y', x \} \) is independent. Therefore there exists a representative \((N, N', M)\) for \((x, x', y')\) with \( y \subseteq M \). But then \( N \cap N' \cap M \neq \emptyset \) which contradicts Lemma 2.29. So \( d(\tilde{S}_1) = 1 \). In the same manner we show that \( d(\tilde{S}_2) = 1 \). \( \square \)

The ordinary points of \( S \) are those of \( \tilde{S}_1 \cup \tilde{S}_2 \), \( \tilde{S}_3 \setminus \tilde{S}_4 \) and \( \tilde{S}_4 \setminus \tilde{S}_3 \).

Given \( K, L \subseteq \tilde{S} \) whenever we say that we set \( K \leq_S L \), we shall mean that we define that \( a \leq_S b \) for any \( a \in K \) and any \( b \in L \).
We set 
\[ \tilde{S}_3 \setminus \tilde{S}_4 \leq S \tilde{S}_1 \leq S \tilde{S}_2 \leq S \tilde{S}_4 \setminus \tilde{S}_3. \]

If \( x, x' \in \tilde{S}_3 \setminus \tilde{S}_4 \) we set \( x \leq_S x' \) if for every \( y \in B \) we have that \( (xx'y)_M \) (see Lemma 2.19).

If \( x, x' \in \tilde{S}_4 \setminus \tilde{S}_3 \) we set \( x \leq_S x' \) if for every \( y \in A \) we have that \( (xx'y)_M \) (see Lemma 2.19).

If \( x, x' \in \tilde{S}_1 \) we set \( x \leq_S x' \) if \( x' \not\subseteq Lx \). If \( x, x' \in \tilde{S}_2 \) we set \( x \leq_S x' \) if \( xLx' \).

By Lemmas 2.20 and 2.23 we see that \( \leq_S \) is an \( M \)-consistent linear ordering of all the ordinary points of \( \tilde{S} \).

Let \( X_{ord} \) to be the set of all ordinary points of \( X \setminus Y \) and let \( Y_0 = Y \cup X_{ord} \).

We extend the linear ordering \( \leq \) of \( Y \) to an \( M \)-consistent linear ordering \( \leq \) of \( Y_0 \) as follows:

1. If \( x, y \in Y \) then we set \( x \leq_0 y \) if and only if \( x \leq y \).

2. If \( x \in X_{ord} \) and \( y \in Y \) we set \( x \leq_0 y \) if \( y \in B_x \).

3. If \( x \in X_{ord} \) and \( y \in Y \) we set \( y \leq_0 x \) if \( y \in A_x \).

4. If \( x, x' \in X_{ord} \) and \( S_x \neq S_{x'} \) then we set \( x \leq_0 x' \) if \( A_x \subseteq A_{x'} \).

5. If \( x, x' \in X_{ord} \) and \( S_x = S_{x'} = S \) then we set \( x \leq_0 x' \) if \( x \leq_S x' \).

Using Lemma 2.22 we may check that \( \leq \) is an \( M \)-consistent linear ordering of \( Y \).

Now we define the mapping \( \phi_0 : X \setminus Y_0 \rightarrow \tilde{S} \) from the set of all exceptional points of \( X \) to the set \( \tilde{S} \) of all sections of \( (Y_0, \leq_0) \) as follows:

Let \( x \in X \setminus Y_0 \).

1. If \( S_x = (\emptyset, Y) \) then we set \( \phi_0(x) = (\tilde{A}, \tilde{B}) \) where 
   \[ \tilde{A} = \{ z \in S_x : L(x) \neq \emptyset \} \quad \text{and} \quad \tilde{B} = \tilde{Y} \setminus \tilde{A}. \]

2. If \( S_x = (Y, \emptyset) \) then \( \phi(x) = (A, B) \) where 
   \[ \tilde{B} = \{ z \in S_x : L(x) \neq \emptyset \} \quad \text{and} \quad \tilde{A} = \tilde{Y} \setminus \tilde{B}. \]
Step \( \alpha \) Let \( \alpha \in \text{Ord} \) with \( \alpha > 0 \) and suppose that for any ordinal \( \xi < \alpha \) an \( \mathcal{M} \)-consistent triple \( (Y_\xi, \leq_\xi, \phi_\xi)_{\xi<\alpha} \) is constructed such that if \( \zeta \leq \xi \) then \( (Y_\zeta, \leq_\zeta, \phi_\zeta) \) is an extension of \( (Y_\xi, \leq_\xi, \phi_\xi) \). We consider two cases:

**Case 1.** The ordinal \( \alpha \) is a limit ordinal.

We set

\[
Y_\alpha = \bigcup_{\xi<\alpha} Y_\xi.
\]

The ordering \( \leq_\alpha \) is defined by

\[
y \leq_\alpha z \text{ if there exists } \xi < \alpha \text{ with } y, z \in Y_\xi \text{ and } y \leq_\xi z.
\]

Finally for every \( x \in X \setminus Y_\alpha \) we set

\[
\phi_\alpha(x) = \left( \bigcup_{\xi<\alpha} \phi_\xi^{(1)}(x), \bigcup_{\xi<\alpha} \phi_\xi^{(2)}(x) \right)
\]

It is straightforward to check that the triple \( (Y_\alpha, \leq_\alpha, \phi_\alpha) \) is an \( \mathcal{M} \)-consistent triple.

**Case 2.** There exists an ordinal \( \beta \) such that \( \alpha = \beta + 1 \).

For every \( S \in \mathfrak{G}(Y_\beta, \leq_\beta) \) such that \( \phi_\beta^{-1}(S) \neq \emptyset \) we consider the family

\[
\mathcal{M}_S = \{ M \cap \phi_\beta^{-1}(S) : M \in \mathcal{M} \}
\]

which is the restriction of \( \mathcal{M} \) in the set \( \phi_\beta^{-1}(S) \). Clearly, \( \mathcal{M}_S \) is a local-lattice of subsets of \( \phi_\beta^{-1}(S) \).

We repeat the construction of **Step 0** for every pair \( (\phi_\beta^{-1}(S), \mathcal{M}_S), S \in \mathfrak{G}(Y_\beta, \leq_\beta) \), and we obtain for every section \( S \) of \( (Y_\beta, \leq_\beta) \) an \( \mathcal{M}_S \)-consistent triple \( (X_S, \leq_S, \phi_S) \). Let

\[
U = \bigcup_{S \in \mathfrak{G}(Y_\beta, \leq_\beta)} X_S, \quad Y_\alpha = Y_\beta \cup U.
\]

Let \( \leq_\alpha \) to be the extension of \( \leq_\beta \) to the set \( Y_\alpha \). The mapping \( \phi_\alpha : X \setminus Y_\alpha \to \mathfrak{G}(Y_\alpha, \leq_\alpha) \) is defined as follows:

1. If \( Y_\alpha = X \) then we set \( \phi_\alpha = \emptyset \).
2. If \( Y_\alpha \neq X \) and \( x \in X \setminus Y_\alpha \) then \( x \in X \setminus Y_\beta \). Let \( S = \phi_\beta(x) \) and \((A_S, \leq_S, \phi_S)\) be the corresponding \( \mathcal{M}_S \)-consistent triple. If \( \phi_S(x) = (A_S, B_S) \) we set \( \phi_\alpha(x) = (A, B) \) with
   \[
   A = \{ y \in Y_\alpha : \text{There exists } z \in A_S \text{ with } y \leq_\alpha z \}
   \]
   \[
   B = \{ y \in Y_\alpha : \text{There exists } z \in B_S \text{ with } z \leq_\alpha y \}
   \]
It is straightforward to check that \((Y_\alpha, \leq, \phi_\alpha)\) is an \(M\)-consistent triple. Clearly if \(Y_\beta \neq X\) we have that \(Y_{\beta+1} \neq Y_\beta\) and the proof of part (a) of Theorem 2.2 is complete. The part (b) of Theorem 2.2 follows from part (a) and the Lemma 2.6.

3. Semi prime filters and Proof of Theorem 1.2

In this final section we show that every lattice is isomorphic to a lattice \(M\) of subsets of a set \(X\) which is closed under finite intersections and separates the points of \(X\) (Theorem 3.7). We this representation in order to prove Theorem 1.2.

Let \(L = (L, \preceq)\) be a lattice. A filter of \(L\) is a proper subset \(F\) of \(L\) such that: If \(a \in F\) and \(a \preceq b\) then \(b \in F\), and if \(a, b \in F\) then \(a \land b \in F\). An ideal of \(L\) is a proper subset \(I\) of \(L\) such that: If \(a \in F\) and \(b \preceq a\) then \(b \in F\), and if \(a, b \in F\) then \(a \lor b \in F\). A prime filter is a filter \(F\) such that if \(a, b \in L\) and \(a \lor b \in F\) then \(a \in F\) or \(b \in F\).

Definition 3.1. Let \(L = (L, \preceq)\) be a lattice and \(a, b \in L\). An element \(x \in L\) is said to be an internal element of \(a, b\) if \(x \preceq a \lor b\) and \((a \land x) \lor (b \land x) \neq x\). By \(I(a, b)\) we shall denote the set of all internal elements of \(a, b\) and by \(\overline{I}(a, b)\) the set \(I(a, b) \cup \{a, b\}\).

Definition 3.2. A filter \(F\) of a lattice \(L = (L, \preceq)\) is called a semi prime filter if for every \(a, b \in L\) with \(a \lor b \in F\) we have that \(\overline{I}(a, b) \cap F \neq \emptyset\).

Definition 3.3. A lattice \(L = (L, \preceq)\) is said to be

(a) well separated if for every semi prime filter \(F\) of \(L\) and every \(a \in L \setminus F\) there exists \(b \in F\) such that \(a \not\preceq b\);
(b) completely separated if for any \(F, G\) semi prime filters of \(L\) with \(F \neq G\) we have that \(F \not\subseteq G\) and \(G \not\subseteq F\).

Remark 3.4. A completely separated lattice is well separated. Indeed, if \(F\) is a semi prime filter and \(a \in L \setminus F\) we select a semi prime filter \(G\) with \(a \in G\). Since \(F \not\subseteq G\) there exists an element \(b \in F \setminus G\). It is plain that \(a \not\preceq b\) and \(b \not\preceq a\). The lattice \((M, \subseteq)\) with \(M = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}\) is an example of a well separated loc-lattice which is not completely separated. Note also that if \(L\) is completely separated loc-lattice then \(L\) is isomorphic to a completely separated loc-lattice of sets, which by Lemma 2.10 is totally separated. We conclude that a loc-lattice \(L\) is completely separated if and only if for any \(F, G\) distinct semi prime filters of \(L\) there exist \(a \in F\) and \(b \in G\) with \(a \land b = 0\).
Lemma 3.5. Given a lattice $\mathcal{L} = (L, \preceq)$, an ideal $I$ of $\mathcal{L}$ and a filter $F$ of $\mathcal{L}$ such that $F \cap I = \emptyset$ there exists a semi prime filter $\tilde{F}$ such that $F \subseteq \tilde{F}$ and $\tilde{F} \cap I = \emptyset$.

Proof. Let $F(\mathcal{L})$ the set of all filters of $\mathcal{L}$ and

$$\mathcal{F} = \{ G \in F(\mathcal{L}) : F \subseteq G \text{ and } G \cap I = \emptyset \}.$$ 

Let $\tilde{F}$ be a maximal element of $(\mathcal{F}, \subseteq)$. We shall show that $\tilde{F}$ is a semi prime filter. For every filter $F$ and $x \in L$ we set

$$F(x) = \{ z \in L : \text{there exists } y \in F \text{ such that } y \wedge x \preceq z \}.$$ 

Clearly, $F(x)$ is a filter, $F \subseteq F(x)$ and $x \in F(x)$. Let $a, b \in L$ such that $a \vee b \in \tilde{F}$ and $\{ a, b \} \cap \tilde{F} = \emptyset$. Since $a, b \notin \tilde{F}$ and by the maximality of $\tilde{F}$ we have that $I \cap \tilde{F}(a) \neq \emptyset$ and $I \cap \tilde{F}(b) \neq \emptyset$. Let $a', b' \in F$ such that $a \wedge a', b \wedge b' \in I$. Let $c = (a \vee b) \wedge (a' \vee b')$. Clearly $c \in F$, $c \preceq a \vee b$ and $(c \wedge a) \vee (c \wedge b) \in I$. So, $c \neq (c \wedge a) \vee (c \wedge b)$ and therefore $c \in \mathcal{I}(a, b)$. \hfill \Box

Definition 3.6. Let $\mathcal{L} = (L, \preceq)$ be a lattice and let $X_\mathcal{L}$ be the set of all semi prime filters of $\mathcal{L}$. We call the mapping $f : L \to \mathcal{P}(X_\mathcal{L})$ defined by $f(x) = \{ F : x \in F \}$ the Stone map of $\mathcal{L}$. The topology $\tau$ of $X_\mathcal{L}$ which has as a basis the set $f(L)$ is called the Stone topology of $X_\mathcal{L}$. The space $(X_\mathcal{L}, \tau)$ is a $T_0$ topological space called the Stone space of $\mathcal{L}$.

Theorem 3.7. Let $\mathcal{L} = (L, \preceq)$ be a lattice and $f : L \to \mathcal{P}(X_\mathcal{L})$ the Stone map of $\mathcal{L}$. Then $f$ is an isomorphism onto $(f(L), \subseteq)$. Moreover $f(L), \subseteq$ is a set lattice closed under finite intersections, $f(L)$ separates $X_\mathcal{L}$ and for every $a, b \in L$ we have that

$$f(a) \wedge f(b) = f(a \wedge b) = f(a) \cap f(b)$$

and

$$f(a) \vee f(b) = f(a \vee b) = \bigcup \{ f(x) : x \in \mathcal{I}(a, b) \}.$$ 

Proof. Clearly, if $a, b \in L$ and $a \preceq b$ then $f(a) \subseteq f(b)$. Suppose that $f(a) \subseteq f(b)$ and $a \not\preceq b$. By Lemma 3.5 there exists an $F \in X_\mathcal{L}$ such that $\{ x \in L : a \preceq x \} \subseteq F$ and $F \cap \{ x \in L : x \preceq b \} = \emptyset$, in particular $a \in F$ and $b \notin F$. But then $f \in f(a) \setminus f(b)$, a contradiction. So, $a \preceq b$ if and only if $f(a) \subseteq f(b)$ and $f$ is an isomorphism onto $f(L)$. Let $F, G \in X_\mathcal{L}$ with $F \neq G$. Then either $F \not\subseteq G$ or $G \not\subseteq F$. Suppose that $F \not\subseteq G$. Clearly, the family $f(L)$ separates $X_\mathcal{L}$. Finally the equations (1), (2) are obvious from that facts that for every semi-prime filter $F$ and every points $a, b$ of $L$ we have that $a \wedge b \in F$ if and only if $a \in F$ and $b \in F$ and that $a \vee b \in F$ if and only if there exists $x \in \mathcal{I}(a, b)$ such that $x \in F$. \hfill \Box
Remark 3.8. By duality, it is clear that every lattice \( \mathcal{L} = (L, \preceq) \) can also be represented as lattice of of subsets of a set \( Y_\mathcal{L} \), closed under finite unions. The set \( Y_\mathcal{L} \) will be the set of all semi prime filters of the dual lattice \( L^* = (L, \preceq^*) \), where \( \preceq^* \) is the inverse order of \( \preceq \). The elements of \( Y_\mathcal{L} \) are called the **semi prime ideals** of \( \mathcal{L} \).

Lemma 3.9. Let \( \mathcal{L} = (L, \preceq) \) be a lattice and let \( f : L \to \wp(X_\mathcal{L}) \) be the Stone map of \( \mathcal{L} \). Then \( (f(L), \subseteq) \) is a loc-lattice of sets. Moreover, if \( \mathcal{L} \) is well separated then the family \( f(L) \) well separates the set \( X_\mathcal{L} \).

**Proof.** By Theorem 3.7 the Stone mapping \( f \) is an isomorphism so \((f(L), \subseteq)\) is lattice of subsets of \( X_\mathcal{L} \) and satisfies the properties (1), (3) and (4) of Definition 2.1. We show that \( f(L) \) satisfies property (2) of Definition 2.1. We note that \( f(a) = \emptyset \) if and only if \( a = 0 \). Suppose that for \( A = f(a), B = f(b) \) we have that \( A \cap B \neq \emptyset \). Then \( a \wedge b \neq 0 \) which implies that \( I(a, b) = \emptyset \). Indeed, if \( I(a, b) \neq \emptyset \) there exists an internal element \( c \) of \( a, b \). Then \( a, b, c \) must be incomparable and so by the Property (2) of Definition 1.1 we have that \( c = (a \lor c) \land (b \lor c) \) which implies that \( a \land b \preceq c \). But then \( a \land c \neq 0 \) and \( b \land c \neq 0 \) and by the property (3) of Definition 1.1 we shall have that \( c = (a \land c) \lor (b \land c) \), a contradiction. So,

\[
A \lor B = f(a) \lor f(b) = \bigcup_{c \in I(a, b)} f(c) = f(a) \cup f(b) = A \cup B.
\]

Suppose that the lattice \( \mathcal{L} \) is well separated and let \( A = f(a) \in f(L) \) and \( F \notin A \). Let \( b \in L \) such that \( b \npreceq a \) and \( b \in F \). Clearly \( F \in B = f(b) \) and \( B \subseteq A \). Therefore the family \( f(L) \) well separates \( X_\mathcal{L} \). \( \square \)

It is plain that Theorem 1.2 follows from Theorem 2.2 and Lemma 3.9.

4. SOME APPLICATIONS IN GENERAL TOPOLOGY.

An **orderable topological space** is a topological space \((X, \tau)\) such that there exists a linear ordering \( \leq \) of \( X \) with the property that the open intervals of \((X, \leq)\) is a base for \( \tau \). A **weakly orderable topological space** is a \( T_0 \) topological space \((X, \tau)\) such that there exists a linear ordering \( \leq \) of \( X \) with the property \( \tau \) has a base consisting of convex sets. A **generalized orderable space** or a **suborderable space** ([1]) is a Hausdorff topological space \((X, \tau)\) such that there exists a linear ordering \( \leq \) of \( X \) with the property that \( \tau \) has a basis of convex subsets of \( X \). It is known ([1]) that the class of generalized ordered spaces coincides with the class of subspaces of linearly ordered topological spaces. A linear ordering of \( \leq \) of \( X \) is said a **(Dedekind) complete linear ordering** if
every nonempty subset of $X$ with an upper bound has a least upper bound (supremum). A complete orderable topological space is a topological space $(X, \tau)$ such that there exists a complete linear ordering $\leq$ of $X$ with the property that the open intervals of $(X, \leq)$ is a base for $\tau$. An immediate consequence of Theorems 1.2 and 2.2 is the following:

**Theorem 4.1.**

1. Every loc-lattice is isomorphic to a basis of a weakly orderable space.
2. A $T_0$ topological space $(X, \tau)$ is weakly orderable if and only if has a basis which is a loc-lattice of subsets of $X$.

The problem of characterization of orderable spaces is considered by many authors. R. L. Moore [12] and A. Wallace [19] characterized the orderable continua, S. Eilenberg [8] characterized the connected metric orderable spaces, H. Herrlich [10] the countable and the totally disconnected metric orderable spaces. A characterization of general orderable and suborderable spaces is given by J. van Dalen and E. Wattel [5] and E. Deák [6] (see also [9], for another approach). We can obtain further characterizations of orderable and suborderable spaces using Theorems 1.2 and 2.2. Before this we shall investigate which elements of a lattice can be represented as open intervals.

An element $a$ of a lattice $(L, \preceq)$ is said to be accessible from below if there exists a subset $A$ of $L$ such that $\bigvee A = a$ and $a \notin A$. We say that $a$ is inaccessible from below if it is not accessible from below. Respectively we say that $a$ is accessible from above if there exists a subset $A$ of $L$ such that $\bigwedge A = a$ and $a \notin A$ and that $a$ is inaccessible from above if it is not accessible from above.

An open interval of linear ordered set $(X, \mathcal{L})$ is a subset of $X$ of the form $(a, b) = \{x \in X : a < x < b\}$ or $(a, \rightarrow) = \{x \in X : a < x\}$ or $(\leftarrow, a) = \{x \in X : x < a\}$ or $X$.

**Lemma 4.2.** Let $\mathcal{L} = (L, \preceq)$ be a totally separated loc-lattice and let $a \in L$ be an inaccessible from above element of $L$. If $f : L \to \wp(X_{\mathcal{L}})$ is the Stone map of $\mathcal{L}$ and $\leq$ the $f(L)$-consistent linear ordering of $X_{\mathcal{L}}$ then $f(a)$ is an open interval

**Proof.** Suppose that $X_{\mathcal{L}} \neq f(a)$ and that the set $f(a)^+ = \{x \in X_{\mathcal{L}} : f(a) < x\}$ is not empty. We shall show that $f(a)$ has a first element. Suppose that $f(a)$ has no a first element. Let $\mathcal{M} = \{f(a) \lor f(x) : f(x) \subseteq f(a)^+\}$. Then $\bigcap \mathcal{M} = f(a)$. Indeed, if there exists $x \in \bigcap \mathcal{M} \setminus f(a)$ then we select an $x' \in f(a)^+$ such that $x' < x$ and $x, x' \in L$ such that $x \in f(x), x' \in f(x')$ and $f(x) \cap f(x') = \emptyset$. Then $x \notin f(a) \lor f(x')$ which contradicts the assumption that $x \in \bigcap \mathcal{M}$. This implies that $f(a) = \bigcap \{f(x) : f(a) \subseteq f(x), f(a) \neq f(x)\}$. Since $f$ is an embedding
from \((L, \preceq)\) onto \((f(L), \subseteq)\) we conclude that 
\[ a = \bigwedge \{ x \in L : a \prec x \}, \]
which contradicts the assumption that \(a\) is inaccessible from above.

Similarly, if the set \(f(a)^- = \{ x \in X_L : x < f(a) \}\) is not empty we show that the set \(f(a)^-\) has a last element \(a\). \(\square\)

**Lemma 4.3.** Let \(\mathcal{L} = (L, \preceq)\) be a totally separated loc-lattice such that every \(a \in L\) is inaccessible from above. If \(f : L \to \wp(X_L)\) is the Stone map of \(\mathcal{L}\) and \(\preceq\) the \(f(L)\)-consistent linear ordering of \(X_L\), then \((X_L, \tau)\) is a complete linear ordered topological space.

**Proof.** By Lemma 4.2 for every \(a \in L\) we have that \(f(a)\) is an open interval. Since \(\mathcal{L} = (L, \preceq)\) is a totally separated loc-lattice, then the family \(f(L)\) totally separates the set \(X_L\) and therefore \(f(L)\) is a basis for the topology \(\tau_{\preceq}\) generated by the open intervals of \((X_L, \preceq)\). So \(\tau = \tau_{\preceq}\) and so \((X_L, \tau)\) is an ordered space. It remains to show that \((X, \preceq)\) has no gaps. Suppose that \(S = (S_1, S_2)\) is a gap of \((X_L, \preceq)\). It is easy to see that the set 
\[ F_S = \{ a \in L : f(a) \cap S_1 \neq \emptyset \text{ and } f(a) \cap S_2 \neq \emptyset \} \]
is a semi prime filter of \(\mathcal{L}\) and so an element of \(X_L\). We set \(x = F_S\). Then either \(x \in S_1\) and so \(x\) is the last element of \(S_1\) or \(x \in S_2\) and so \(x\) is the first element of \(S_2\). Every case contradicts the assumption that \(S\) is a gap. \(\square\)

By Lemma 4.3 and Theorems 1.2 and 2.2 we easily obtain the following:

**Theorem 4.4.**

1. Every totally separated loc-lattice such that every \(a \in L\) is inaccessible from above is isomorphic to a basis of a complete orderable space.

2. A \(T_1\) topological space \((X, \tau)\) is orderable if and only if has a basis \(\mathcal{M}\) such that \((\mathcal{M}, \subseteq)\) is a loc-lattice of subsets of \(X\) such that every \(M \in \mathcal{M}\) is inaccessible from above.

**Remark 4.5.** A family \(\mathcal{M}\) of sets is called interlocking (see [5]) provided that every set \(M \in \mathcal{M}\) which is an intersection of strictly larger members of \(\mathcal{M}\) has a representation as a union of strictly smaller members of \(\mathcal{M}\). So, we may call a lattice \(\mathcal{L} = (L, \preceq)\) to be an interlocking lattice if every element of \(L\) either is inaccessible from above or it is accessible from above and below. In such lattices the sets \(f(a), a \in L\) are either open intervals or unions of open intervals. Therefore an interlocking and totally separated lattice has a representation as a basis of an ordered space.
LATTICES REPRESENTED AS LATTICES OF INTERVALS

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