Geometric relative entropies and barycentric Rényi divergences

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Abstract

We give systematic ways of defining monotone quantum relative entropies and (multi-variate) quantum Rényi divergences starting from a set of monotone quantum relative entropies.

Interestingly, despite its central importance in information theory, only two additive and monotone quantum extensions of the classical relative entropy have been known so far, the Umegaki and the Belavkin-Staszewski relative entropies, which are the minimal and the maximal with these properties, respectively. Using the Kubo-Ando weighted geometric means, we give a general procedure to construct monotone and additive quantum relative entropies from a given one with the same properties; in particular, when starting from the Umegaki relative entropy, this gives a new one-parameter family of monotone (even under positive trace-preserving (PTP) maps) and additive quantum relative entropies interpolating between the Umegaki and the Belavkin-Staszewski ones on full-rank states.

In a different direction, we use a generalization of a classical variational formula to define multi-variate quantum Rényi quantities corresponding to any finite set of quantum relative entropies \((D^q)_x \in X\) and real weights \((P(x))_x \in X\) summing to 1, as

\[
Q_{P,q}(\rho_x) := \sup_{\tau \geq 0} \left\{ \text{Tr} \tau - \sum_x P(x) D^q_x (\tau \parallel \rho_x) \right\}.
\]

We analyze in detail the properties of the resulting quantity inherited from the generating set of quantum relative entropies; in particular, we show that monotone quantum relative entropies define monotone Rényi quantities whenever \(P\) is a probability measure. With the proper normalization, the negative logarithm of the above quantity gives a quantum extension of the classical Rényi divergences in the 2-variable case \((X = \{0, 1\}, P(0) = \alpha)\). We show that if both \(D^0\) and \(D^1\) are monotone and additive quantum relative entropies, and at least one of them is strictly larger than the Umegaki relative entropy then the resulting barycentric Rényi divergences are strictly between the log-Euclidean and the maximal Rényi divergences, and hence they are different from any previously studied quantum Rényi divergence.

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Dissimilarity measures of states of a system (classical or quantum) play a fundamental role in information theory, statistical physics, computer science, and various other disciplines. Probably the most relevant such measures for information theory are the Rényi divergences, defined for finitely supported probability distributions \( \rho, \sigma \) on a set \( X \) as
\[
D_\alpha(\rho \parallel \sigma) := \frac{1}{\alpha - 1} \log \sum_{x \in X} \rho(x)^\alpha \sigma(x)^{1-\alpha}
\] (I.1)
where \( \alpha \in [0, +\infty) \setminus \{1\} \) is a parameter. (For simplicity, here we assume all probability distributions and quantum states to have full support; we give the formulas for the general case in Section III.) See, for instance, \[11\] for the role of the Rényi divergences and derived information measures (entropy, divergence radius, channel capacity) in classical state discrimination, as well as source- and channel coding. The limit \( \alpha \to 1 \) yields the Kullback-Leibler divergence, or relative entropy
\[
\lim_{\alpha \to 1} D_\alpha(\rho \parallel \sigma) = D(\rho \parallel \sigma) := \sum_{x \in X} [\rho(x) \log \rho(x) - \rho(x) \log \sigma(x)]
\] (I.2)
Interestingly, the relative entropy in itself also determines the whole one-parameter family of Rényi divergences, as for every \( \alpha \in (0, 1) \cup (1, +\infty) \),
\[
D_\alpha(\rho \parallel \sigma) = \frac{1}{1 - \alpha} \min_{\omega \in \mathcal{P}(X)} \{\alpha D(\omega \parallel \rho) + (1 - \alpha) D(\omega \parallel \sigma)\}
\] (I.3)
where the optimization is over all finitely supported probability distributions \( \omega \) on \( X \) \[12\].

Due to the non-commutativity of quantum states, for any given \( \alpha \in (0, +\infty) \), there are infinitely many quantum extensions of the classical Rényi \( \alpha \)-divergence for pairs of quantum states, e.g., the measured, the maximal \[35\], or the Rényi \((\alpha, z)\)-divergences \[4\]. Of particular importance are the Petz-type \[48\] and the sandwiched \[45, 54\] Rényi divergences, which appear as exact quantifiers of the trade-off relations between the operationally relevant quantities in a number of information-theoretic problems, including state discrimination, classical-quantum channel coding, entanglement manipulation, and more \[3, 19, 20, 30, 33, 41, 42, 46\]. While so far only these two families of quantum Rényi divergences have found such explicit operational interpretations, it is nevertheless useful, for a number of different reasons, to consider other quantum extensions as well. Indeed, apart from their study being interesting from the purely mathematical point of view of matrix analysis, some of these quantities serve as useful tools in proofs to arrive at the operationally relevant Rényi information quantities in various problems; see, e.g., the
role played by the so-called log-Euclidean Rényi divergences $D_{α,∞}$ in determining the strong converse exponent in various problems [33, 41, 42], or the family of Rényi divergences $D^\#_α$ introduced in [17], where it was used to determine the strong converse exponent of binary channel discrimination.

Of course, only quantum extensions with a number of good mathematical properties may be interesting for quantum information theory, the most important being monotonicity, i.e., that for any two states $ρ, σ$, and completely positive trace-preserving (CPTP) map $Φ$, the data processing inequality (DPI) $D_α(Φ(ρ)||Φ(σ)) ≤ D_α(ρ||σ)$ holds. Another desirable property is additivity $D_α(ρ_1 ⊗ ρ_2||σ_1 ⊗ σ_2) = D_α(ρ_1||σ_1) + D_α(ρ_2||σ_2)$. However, quantum Rényi divergences without these properties might still be useful; indeed, $D_{α,∞}$ is additive, but not monotone for $α > 1$ (the range of $α$ values for which it was used in [41, 42]), and $D^\#_α$ is monotone, but not additive.

Quantum divergences with good mathematical properties also play an important role in the study of the problem of state convertibility, where the question is whether a set of states $(ρ_i)_{i ∈ I}$ can be mapped into another set of states $(q_i')_{i ∈ I}$ with a joint quantum operation. This problem can be studied in a large variety of settings; single-shot or asymptotic, exact or approximate, with or without catalysts, allowing arbitrary quantum operations or only those respecting some symmetry or being free operations of some resource theory, any combination of these, and more. Necessary conditions for convertibility can be obtained using multi-variate functions on quantum states with suitable mathematical properties; for instance, for exact single-shot convertibility, $F((ρ_i)_{i ∈ J}) ≥ F((q_i')_{i ∈ J})$ has to hold for any function $F$ whose variables are indexed by a subset $J ⊆ I$ and which is monotone non-increasing under the joint application of an allowed quantum operation on its arguments; the same has to hold also for multi-copy or catalytic single-shot convertibility, if $F$ is additionally additive on tensor products, and for asymptotic catalytic convertibility, if, moreover, $F$ is lower semi-continuous in its variables. (We refer to [16] for the precise definitions of the various versions of state convertibility.) Many of the quantum Rényi divergences provide such functions on pairs of states (i.e., $|J| = 2$), including the maximal Rényi $α$-divergences [35] with $α ∈ [0, 2]$, and the Rényi ($α, z$)-divergences [4] for certain values of $α$ and $z$ [55]. In the converse direction, sufficient conditions in terms of Rényi divergences have been given for the convertibility of pairs of commuting states in [9, 29, 31, 43, 52], and these have been extended very recently in [16] to a complete characterization of asymptotic as well as approximate catalytic convertibility between finite sets of commuting states in terms of the monotonicity of the multi-variate Rényi quantities

$$Q_α(ρ_1, ..., ρ_r) := \text{Tr}(ρ_1^{α_1} ... ρ_r^{α_r}),$$

(I.4)

where $α_1 + ... + α_r = 1$ and either all of them are non-negative or exactly one of them is positive. No sufficient conditions, however, are known in the general noncommutative case.

Motivated by the above, in this paper we set out to give systematic ways to define quantum Rényi divergences with good mathematical properties (in particular, monotonicity), for two and for more variables. We note that, to the best of our knowledge, this is the first time that multi-variate quantum Rényi divergences have been considered in the literature.

The structure of the paper is as follows. In Section II we give the necessary mathematical preliminaries. In Section III A we discuss in detail the definition and various properties of general multi-variate quantum divergences. Sections III B and III C contain brief reviews of the definitions and properties of the classical and the quantum Rényi divergences that we use in the paper. Section III D gives a high-level overview of the various ways we propose to define new quantum Rényi divergences from given ones, of which we work out in detail two approaches in this paper.

In Section IV we focus on quantum relative entropies. Interestingly, despite its central importance in information theory, only two additive and monotone quantum extensions of the classical relative entropy have been known so far, the Umegaki [53] and the Belavkin-Staszewski [6] relative entropies, which are the minimal and the maximal with these properties, respectively [35]. Here we give a general procedure to construct monotone and additive quantum relative entropies from a given one with the same properties; in particular, when starting from the Umegaki relative entropy, this gives a new one-parameter family of monotone (even under positive trace-preserving (PTP) maps) and additive quantum relative entropies interpolating between the Umegaki and the Belavkin-Staszewski ones on full-rank states.

In Section V we use a generalization of the classical variational formula in (I.3) to define quantum extensions of the multi-variate Rényi quantities (I.4) corresponding to any set of quantum relative entropies $(D^α_q)_{x ∈ X}$ and real weights $(P(x))_{x ∈ X}$ summing to 1, as

$$Q^b_α((q_x)_{x ∈ X}) := \sup_{τ ≥ 0} \left\{ \text{Tr} τ - \sum_x P(x)D^α_q(τ||q_x) \right\}.$$

(I.5)
We analyze in detail the properties of the resulting quantity inherited from the generating set of quantum relative entropies; in particular, we show that monotone quantum relative entropies define monotone Rényi quantities whenever $P$ is a probability measure. We also show that the negative logarithm of $Q^{b,q}_P(\{q_x\}_{x \in X})$ is equal to the $P$-weighted (left) relative entropy radius of the $\{q_x\}_{x \in X}$, i.e.,

$$-\log Q^{b,q}_P(\{q_x\}_{x \in X}) = \inf_{\omega} \sum_x P(x) D^{q_x}(\omega|q_x),$$

(1.6)

where the infimum is taken over all states on the given Hilbert space, and therefore we call the quantities in (1.5) barycentric Rényi quantities. With the proper normalization, the quantities in (1.6) give quantum extensions of the classical Rényi divergences (1.1) in the 2-variable case ($X = \{0, 1\}, P(0) = \alpha$), which we denote by $D^{b,q}_\alpha$. In Section VI we study the relation of the resulting (2-variable) quantum Rényi divergences to the known ones. It has been shown in [41] that if $D^{\alpha_0} = D^{\alpha_1} = D^{U_{\infty}}$ is the Umegaki relative entropy [53] then $D^{\alpha_0,\alpha_1}$ is equal to the log-Euclidean Rényi divergence $D_{\alpha,\infty}$. We show that if both $D^{\alpha_0}$ and $D^{\alpha_1}$ are monotone and additive quantum relative entropies, and at least one of them is strictly larger than $D^{U_{\infty}}$ then the resulting barycentric Rényi divergences are strictly between the log-Euclidean and the maximal Rényi divergences, and hence they are different from any previously studied quantum Rényi divergences.

II. PRELIMINARIES

For a finite-dimensional Hilbert space $\mathcal{H}$, let $\mathcal{B}(\mathcal{H})$ denote the set of all linear operators on $\mathcal{H}$, and let $\mathcal{B}(\mathcal{H})_{sa}, \mathcal{B}(\mathcal{H})_{\geq 0}, \mathcal{B}(\mathcal{H})_{> 0}$, and $\mathcal{B}(\mathcal{H})_{\geq 0}$ denote the set of self-adjoint, positive semi-definite (PSD), non-zero positive semi-definite, and positive definite operators, respectively. For an interval $I \subseteq \mathbb{R}$, let $\mathcal{B}(\mathcal{H})_{sa,J} := \{ A \in \mathcal{B}(\mathcal{H})_{sa} : \text{spec}(A) \subseteq J \}$, i.e., the set of self-adjoint operators on $\mathcal{H}$ with all their eigenvalues in $J$. Let $\mathcal{S}(\mathcal{H}) := \{ \varrho \in \mathcal{B}(\mathcal{H})_{\geq 0}, \text{Tr} \varrho = 1 \}$ denote the set of density operators, or states. For an operator $X \in \mathcal{B}(\mathcal{H})$, $\|X\|_\infty := \max \{ \|X \psi\| : \psi \in \mathcal{H}, \|\psi\| = 1 \}$ denotes the operator norm of $X$ (i.e., its largest singular value).

Similarly, for a finite set $I$, we will use the notation $\mathcal{F}(I) := C^I$ for the set of complex-valued functions on $I$, and $\mathcal{F}(I)_{\geq 0}, \mathcal{F}(I)_{\geq 0}, \mathcal{F}(I)_{> 0}$ for the set of non-negative, non-negative and not constant zero, and strictly positive functions on $I$. The set of probability density functions on $I$ will be denoted by $\mathcal{P}(I)$. When equipped with the maximum norm, $\mathcal{F}(I)$ becomes a commutative $C^*$-algebra, which we denote by $\ell^\infty(I)$. In the more general case when $I$ is an arbitrary non-empty set, we will also use the notations $\mathcal{P}_f(I)$ for the set of finitely supported probability measures, and $\mathcal{P}_f^+(I)$ for the set of finitely supported signed probability measures on $I$, i.e.,

$$\mathcal{P}_f^+(I) := \left\{ P \in \mathbb{R}^I : |\text{supp} P| < +\infty, \sum_{i \in I} P(i) = 1 \right\}, \quad \text{supp } P := \{ i \in I : P(i) \neq 0 \}.$$

For any non-empty set $X$, let

$$\mathcal{B}(X, \mathcal{H})_{\geq 0}, \quad \mathcal{B}(X, \mathcal{H})_{\geq 0}, \quad \mathcal{B}(X, \mathcal{H})_{> 0}, \quad \mathcal{S}(X, \mathcal{H}),$$

denote the set of functions mapping from $X$ into $\mathcal{B}(\mathcal{H})_{\geq 0}, \mathcal{B}(\mathcal{H})_{\geq 0}, \mathcal{B}(\mathcal{H})_{> 0}$, and $\mathcal{S}(\mathcal{H})$, respectively. Elements of $\mathcal{S}(X, \mathcal{H})$ are called classical-quantum channels, or cq channels, and we will use the terminology generalized classical-quantum channels, or gcq channels, for the elements of $\mathcal{B}(X, \mathcal{H})_{\geq 0}$. We will normally use the notation $W = (W_x)_{x \in X}$ to denote elements of $\mathcal{B}(X, \mathcal{H})_{\geq 0}$. We say that $W \in \mathcal{B}(X, \mathcal{H})_{\geq 0}$ is classical if $W_x W_y = W_y W_x$, $x, y \in X$, or equivalently, there exists an orthonormal basis $(e_i)_{i \in I}$ in $\mathcal{H}$ such that $W_x = \sum_{i \in I} \langle e_i, W_x e_i | e_i \rangle | e_i \rangle \langle e_i |$, $x \in X$. Equivalently, we may identify $W$ with the collection of functions $(\langle W_x(i) := \langle e_i, W_x e_i | e_i \⟩)_{i \in I} \in \mathcal{F}(X, I)_{\geq 0}$, where we use the notations

$$\mathcal{F}(X, I)_{\geq 0}, \quad \mathcal{F}(X, I)_{\geq 0}, \quad \mathcal{F}(X, I)_{> 0}, \quad \mathcal{P}(X, I),$$

for the sets of functions mapping elements of $X$ into functions $f_x \in \mathcal{F}(I), x \in X$, on the finite set $I$, such that all $f_x$ are non-negative/non-negative and not constant zero/strictly positive/probability density functions on $I$.
Operations on elements of $\mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0}$ are always meant pointwise; e.g., for any $W, W^{(1)}, W^{(2)} \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0}, V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, and $\sigma \in \mathcal{B}(\mathcal{K})_{\geq 0}$,

\begin{align}
VWV^* &=: (VW_x V^*)_x \in \mathcal{X}, \quad (\text{II.7}) \\
W^{(1)} \otimes W^{(2)} &=: \left( W^{(1)}_{x^1} \otimes W^{(2)}_{x^2} \right)_{x \in \mathcal{X}}, \quad (\text{II.8}) \\
W \hat{\otimes} \sigma &=: W \otimes \sigma := (W_x \otimes \sigma)_{x \in \mathcal{X}}. \quad (\text{II.9})
\end{align}

Note that here we only consider the (pointwise) tensor product of functions defined on the same set, and that this notion of tensor product is different from the one used to describe the parallel action of two cq channels, given by

\[ W^{(1)} \otimes W^{(2)} := \left( W^{(1)}_{x^1} \otimes W^{(2)}_{x^2} \right)_{(x^1, x^2) \in \mathcal{X}_1 \times \mathcal{X}_2}, \]

where $W^{(i)} \in \mathcal{B}(\mathcal{X}^{(i)}, \mathcal{H}^{(i)}), i = 1, 2$, and possibly $\mathcal{X}^{(1)} \neq \mathcal{X}^{(2)}, \mathcal{H}^{(1)} \neq \mathcal{H}^{(2)}$. The tensor product in (II.9) can be interpreted either in this setting, with $\mathcal{X}^{(1)} = \mathcal{X}, W^{(1)} = W$ and $\mathcal{X}^{(2)} = \{0\}, W^{(2)} = \sigma$, or as the pointwise tensor product between $W^{(1)} = W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0}$ and the constant function $W^{(2)} = \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0}$, $W^{(2)} = \sigma, x \in \mathcal{X}$.

The set of projections on $\mathcal{H}$ is denoted by $P(\mathcal{H}) := \{ P \in \mathcal{B}(\mathcal{H}) : P^2 = P = P^* \}$. For $P, Q \in P(\mathcal{H})$, the projection onto $(\text{ran } P) \cap (\text{ran } Q)$ is denoted by $P \wedge Q$. For a sequence of projections $P_1, \ldots, P_r \in P(\mathcal{H})$ summing to $I$, the corresponding pinching operation is

\[ B(\mathcal{H}) \ni X \mapsto \sum_{i=1}^r P_i XP_i. \]

For a self-adjoint operator $A$, let $P^A_a := 1_{\{a\}}(A)$ denote the spectral projection of $A$ corresponding to the singleton $\{a\} \subset \mathbb{R}$. (Here and henceforth $1_H$ stands for the characteristic (or indicator) function of a set $H$.) The projection onto the support of $A$ is $\sum_{a \geq 0} P^A_a$; in particular, if $A$ is positive semi-definite, it is equal to $\lim_{a \to 0^+} A^a := A^0$. In general, we follow the convention that real powers of a positive semi-definite operator $A$ are taken only on its support, i.e., for any $x \in \mathbb{R}$, $A^x := \sum_{a \geq 0} a^x P^A_a$. In particular, $A^{-1} := \sum_{a > 0} a^{-1} P^A_a$ stands for the generalized inverse of $A$, and $A^{-1} A = A A^{-1} = A^0$. For $A \in B(\mathcal{H})_{\geq 0}$ and a projection $P$ on $\mathcal{H}$ we write $A \in B(\mathcal{P}H)_{\geq 0}$ if $A^0 \leq P$.

For two PSD operators $\varrho, \sigma$, we write $\varrho \perp \sigma$ if $\text{ran } \varrho \perp \text{ran } \sigma$, which is equivalent to $\varrho \sigma = 0$, and further to $(\varrho, \sigma)_{HS} := \text{Tr } \varrho \sigma = 0$, and to $\varrho^0 \sigma^0 = 0$. In particular, it implies $\varrho^0 \wedge \sigma^0 = 0$, but not the other way around.

For two finite-dimensional Hilbert spaces $\mathcal{H}, \mathcal{K}$, we will use the notations PTP($\mathcal{H}, \mathcal{K}$) and CPTP($\mathcal{H}, \mathcal{K}$) for the set of positive trace-preserving linear maps and the set of completely positive trace-preserving linear maps, respectively, from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K})$. We will also use the notation $P^+(\mathcal{H}, \mathcal{K})$ for the set of (positive) linear maps from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{K})$ such that $\Phi(\varrho) \in \mathcal{B}(\mathcal{K})_{\geq 0}$ for all $\varrho \in \mathcal{B}(\mathcal{H})_{\geq 0}$. We will also consider (completely) positive maps of the form $\Phi : \mathcal{B}(\mathcal{H}) \to \ell^\infty(\mathbb{I})$ and $\Phi : \ell^\infty(\mathbb{I}) \to \mathcal{B}(\mathcal{H})$.

For a finite-dimensional Hilbert space $\mathcal{H}$ and a natural number $n$, we denote by

\[ \text{POVM}(\mathcal{H}, [n]) := \left\{ M = (M_i)_{i=1}^n \in \mathcal{B}(\mathcal{H})_{\geq 0}^n : \sum_{i=1}^n M_i = I \right\} \]

the set of $n$-outcome positive operator valued measures (POVMs) on $\mathcal{H}$. Any $M \in \text{POVM}(\mathcal{H}, [n])$ determines a CPTP map $\mathcal{M} : \mathcal{B}(\mathcal{H}) \to \ell^\infty([n])$ by

\[ \mathcal{M}(\cdot) := \sum_{i=1}^n (\text{Tr } M_i(\cdot)) 1_{\{i\}}. \]

For a differentiable function $f$ defined on an interval $J \subset \mathbb{R}$, let $f^{[1]} : J \times J \to \mathbb{R}$ be its first divided difference function, defined as

\[ f^{[1]}(a, b) := \begin{cases} \frac{f(a) - f(b)}{a - b}, & a \neq b, \\ f'(a), & a = b, \end{cases} \quad a, b \in J. \]
If \( f \) is a continuously differentiable function on an open interval \( J \subseteq \mathbb{R} \) then for any finite-dimensional Hilbert space \( \mathcal{H} \), \( A \mapsto f(A) \) is Fréchet differentiable on \( \mathcal{B}(\mathcal{H})_{sa,J} \), and its Fréchet derivative \((Df)[A] \) at a point \( A \in \mathcal{B}(\mathcal{H})_{sa,J} \) is given by

\[
(Df)[A](Y) = \sum_{i,j=1}^{r} f^{[i]}(a_i, a_j) P_i Y P_j, \quad Y \in \mathcal{B}(\mathcal{H})_{sa},
\]

for any \( P_1, \ldots, P_r \in \mathcal{P}(\mathcal{H}) \) and \( a_1, \ldots, a_r \in \mathbb{R} \) such that \( \sum_{i=1}^{r} P_i = I, \sum_{i=1}^{r} a_i P_i = A \). See, e.g., [8, Theorem V.3.3] or [22, Theorem 2.3.1].

By \( \log \) we denote the natural logarithm, and we use two different extensions of it to \([0, +\infty]\), defined as

\[
\log x := \begin{cases} 
-\infty, & x = 0, \\
\log x, & x \in (0, +\infty), \\
+\infty, & x = +\infty,
\end{cases}
\quad \tilde{\log} x := \begin{cases} 
0, & x = 0, \\
\log x, & x \in (0, +\infty), \\
+\infty, & x = +\infty.
\end{cases}
\]

Throughout the paper we use the convention

\[
0 \cdot (\pm \infty) := 0.
\]

For a function \( f : (0, +\infty) \to \mathbb{R} \), the corresponding \textit{operator perspective function} [14, 15, 23] \( P_f \) is defined on pairs of positive definite operators \( \varrho, \sigma \in \mathcal{B}(\mathcal{H})_{>0} \) as

\[
P_f(\varrho, \sigma) := \varrho^{1/2} f \left( \sigma^{-1/2} \varrho \sigma^{-1/2} \right) \sigma^{1/2},
\]

and it is extended to pairs of positive semi-definite operators \( \varrho, \sigma \) as \( P_f(\varrho, \sigma) := \lim_{\epsilon \to 0} P_f(\varrho + \epsilon I, \sigma + \epsilon I) \), whenever the limit exists. It is easy to see that for the \textit{transpose function} \( \tilde{f}(x) := xf(1/x), x > 0 \), we have

\[
P_f(\varrho, \sigma) = P_f(\sigma, \varrho),
\]

whenever both sides are well-defined. For any \( \gamma \in (0, 1) \), the choice \( \gamma_f := \text{id}_{[0, +\infty)} \) gives the \textit{Kubo-Ando} \( \gamma \)-\textit{weighted geometric mean}, denoted by \( P_{\gamma_f}(\varrho, \sigma) := \gamma \varrho\# \varrho, \varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0} \); see Section IV for more details.

The following is completely elementary; we state it explicitly because we use it multiple times in the paper.

**Lemma II.1** For any \( c \in \mathbb{R} \), the supremum of the function \([0, +\infty) \ni t \mapsto t - t \log t - tc =: f_c(t) \) is \( e^{-c} \), attained uniquely at \( t = e^{-c} \).

**Proof** We have \( f'(t) = -\log t - c = 0 \iff t = e^{-c} \), \( f''(t) = -1/t < 0 \), \( t \in (0, +\infty) \), from which the statement follows immediately. \( \square \)

The following minimax theorem is from [37, Corollary A.2].

**Lemma II.2** Let \( X \) be a compact topological space, \( Y \) be an ordered set, and let \( f : X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\} \) be a function. Assume that

(i) \( f(\cdot, y) \) is lower semi-continuous for every \( y \in Y \) and
(ii) \( f(x, \cdot) \) is monotonic increasing for every \( x \in X \), or \( f(x, \cdot) \) is monotonic decreasing for every \( x \in X \).

Then

\[
\inf_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \inf_{x \in X} f(x, y),
\]

and the infima in (II.13) can be replaced by minima.

The following might be known; however, we could not find a reference for it, so we provide a detailed proof. For simplicity, we include the Hausdorff property in the definition of a topological space, although the statement also holds without this.
Lemma II.3 Let $X$ be a topological space, $Y$ be an arbitrary set, and $f : X \times Y \to \mathbb{R} \cup \{\pm \infty\}$ be a function.

(i) If $f(\cdot, y)$ is lower semi-continuous for every $y \in Y$ then $\sup_{y \in Y} f(\cdot, y)$ is upper semi-continuous.

(ii) If $Y$ is a compact topological space, and $f$ is lower semi-continuous on $X \times Y$ w.r.t. the product topology, then $\inf_{y \in Y} f(\cdot, y)$ is lower semi-continuous.

(iii) If $Y$ is a compact topological space, and $f$ is continuous on $X \times Y$ w.r.t. the product topology, then $\inf_{y \in Y} f(\cdot, y)$ and $\sup_{y \in Y} f(\cdot, y)$ are both continuous.

Proof The assertion in (i) is trivial from the definition of lower semi-continuity.

To prove (ii), let $(x_i)_{i \in I} \subseteq X$ be a generalized sequence and $\overline{x} = \lim_{i \to \infty} x_i$. Since $f(x_i, \cdot)$ is lower semi-continuous on $Y$, and $Y$ is compact, there exists a $y_i \in Y$ such that $\inf_{y \in Y} f(x_i, y) = f(x_i, y_i)$. Let $(f(x_{\alpha(j)}, y_{\alpha(j)}))_{j \in J}$ be a subnet such that

$$\liminf_{i \to \infty} f(x_i, y_i) = \lim_{j \to \infty} f(x_{\alpha(j)}, y_{\alpha(j)}).$$

Since $Y$ is compact, there exists a subnet $(y_{\alpha(\beta(j))})_{j \in K} \subseteq Y$ converging to some $\overline{y} \in Y$. Then

$$\inf_{y \in Y} f(\overline{x}, y) \leq \liminf_{i \to \infty} f(x_i, y_i) = \lim_{j \to \infty} f(x_{\alpha(j)}, y_{\alpha(j)}) = \liminf_{i \to \infty} f(x_i, y_i) = \liminf_{i \to \infty} f(x_i, y),$$

where the second inequality follows from the lower semi-continuity of $f$ on $X \times Y$.

The assertion in (iii) follows immediately from (i) and (ii).

III. QUANTUM RÉNYI DIVERGENCES

A. Classical and quantum divergences

Let $\mathcal{X}$ be an arbitrary non-empty set. By an $\mathcal{X}$-variable quantum divergence $\Delta$ mean a function on collections of non-zero PSD matrices

$$\Delta : \cup_{d \in \mathbb{N}} \mathcal{B}(\mathcal{X}, \mathbb{C}^d)_{\geq 0} \to \mathbb{R} \cup \{\pm \infty\},$$

that is invariant under isometries, i.e., if $V : \mathbb{C}^{d_1} \to \mathbb{C}^{d_2}$ is an isometry then

$$\Delta(VWV^*) = \Delta(W), \quad W \in \mathcal{B}(\mathcal{X}, \mathbb{C}^{d_1})_{\geq 0}.$$

Due to the isometric invariance, $\Delta$ may be extended to collections of non-zero PSD operators on any finite-dimensional Hilbert space $\mathcal{H}$, by choosing any isometry $V : \mathcal{H} \to \mathbb{C}^d$ with large enough $d$, and defining

$$\Delta(W) := \Delta(VWV^*), \quad W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0}.$$

The isometric invariance property guarantees that this extension is well-defined, in the sense that the value of $\Delta(VWV^*)$ is independent of the choice of $d$ and $V$. Clearly, this extension is again invariant under isometries, i.e., for any $W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0}$ and $V : \mathcal{H} \to \mathcal{K}$ isometry, $\Delta(VWV^*) = \Delta(W)$. Note that this implies that $\Delta$ is invariant under extensions by pure states, i.e.,

$$\Delta(W \otimes |\psi\rangle\langle\psi|) = \Delta(W), \quad W \in \mathcal{B}(\mathcal{X}, \mathcal{H}),$$

where $\psi$ is an arbitrary unit vector in some finite-dimensional Hilbert space $\mathcal{K}$.

Analogously, an $\mathcal{X}$-variable classical divergence $\Delta$ is a function

$$\Delta : \cup_{d \in \mathbb{N}} \mathcal{F}(\mathcal{X}, [d])_{\geq 0} \to \mathbb{R} \cup \{\pm \infty\},$$
that is invariant under injective maps \( f : [d] \to [d'] \), in the sense that for any such map, and any \( w = \langle w_x \rangle_{x \in X} \in \mathcal{F}(X, [d]) \),
\[
\Delta(f_* w) = \Delta(w)
\]
where \( f_* w := \langle f_* w_x \rangle_{x \in X} \) with \( (f_* w_x)(i) := w_x(f^{-1}(i)), \ \forall i \in \text{ran} \ f, \) and \( (f_* w_x)(i) := 0 \) otherwise. Here, \( [d] := \{0, \ldots, d-1 \}, \ \forall d \in \mathbb{N}. \)

Due to the invariance property, a classical divergence can be uniquely extended to a collection of non-negative functions \( w = \langle w_x \rangle_{x \in X} \in \mathcal{F}(X, \mathcal{I}) \) with any finite set \( \mathcal{I} \) in the obvious way, and this extension will be invariant under permutations and under embeddings of the finite set \( \mathcal{I} \) into a larger one.

**Remark III.1** Divergences primarily serve as statistically motivated measures of how far away a collection of (classical or quantum) states are from each other. Allowing more general, non-normalized inputs is motivated partly by the fact that formulas e.g., for the Rényi divergences extend naturally to that setting, while from a more operational point of view, subnormalized states, for instance, might model the outputs of probabilistic protocols.

The notion of a classical or quantum divergence may also be extended by allowing some of arguments of the divergence to be zero, with the obvious modifications of the above definitions and those following below. This will be convenient for us in the definition of the \( \gamma \)-weighted geometric relative entropies in Section IV. Conversely, a divergence may be defined more restrictively, e.g., only on functions taking density operators as values (i.e., on elements of \( \mathcal{S}(X, \mathcal{H}) \)), as noted above, or only on functions taking positive definite operators as values (i.e., on elements of \( \mathcal{B}(X, \mathcal{H}) \)).

**Remark III.2** An \( X \)-variable divergence \( \Delta \) with \( X = [n] \) for some \( n \in \mathbb{N} \), is called an \( n \)-variable divergence (classical or quantum). In particular, in the case \( X = [2] = \{0, 1\} \), we call \( \Delta \) a binary divergence, and use the notations \( \varrho := W_0 \) and \( \sigma := W_1 \), and
\[
\Delta(\varrho||\sigma) := \Delta(W_0||W_1) := \Delta((W_0, W_1)).
\]

According to the following simple observation, any classical divergence \( \Delta \) has a unique extension to collections of commuting non-zero PSD operators, which we also denote by \( \Delta \).

**Lemma III.3** Let \( W \in \mathcal{B}(X, \mathcal{H}) \geq 0 \) be classical, and let \((e_i)_{i \in X} \) and \((f_i)_{i \in \mathcal{I}} \) be orthonormal bases jointly diagonalizing all \( W_x \). Then for any classical divergence \( \Delta \),
\[
\Delta \big((\langle e_i, W_x e_i \rangle)_{i \in X}, \langle f_i, W_x f_i \rangle)_{i \in \mathcal{I}} \big) = \Delta \big((\langle e_i, W_x e_i \rangle)_{i \in X} \big).
\]

**Proof** For every \( x \in X \), let \( (\lambda_{x,j})_{j \in J_x} \) be the different eigenvalues of \( W_x \). For every \( j \in \times_{x \in X} J_x \), let
\[
\mathcal{I}_j := \{ i \in \mathcal{I} : (e_i, W_x e_i) = \lambda_{x,j}, \ x \in X \}, \quad \mathcal{I}_j := \{ i \in \mathcal{I} : (f_i, W_x f_i) = \lambda_{x,j}, \ x \in X \}.
\]

Then
\[
|\mathcal{I}_j| = \dim (\cap_{x \in X} \ker (\lambda_{x,j}, J - W_x)) = |\mathcal{I}_j|,
\]
and therefore there exists a bijection \( h : \mathcal{I} \to \mathcal{I} \) such that for all \( j \in \times_{x \in X} J_x \), \( i \in \mathcal{I}_j \iff h(i) \in \mathcal{I}_j \). The equality in (III.14) then follows by the invariance of \( \Delta \) under bijections. \( \Box \)

We say that an \( X \)-variable quantum divergence \( \Delta^q \) is a quantum extension of an \( X \)-variable classical divergence \( \Delta \), if for any \( d \in \mathbb{N} \), and any \( w \in \mathcal{F}(X, [d]) \geq 0 \),
\[
\Delta^q \left( \sum_{i=0}^{d-1} w_x (i|/i)_{x \in X} \right) = \Delta(w), \quad (III.15)
\]
where \( (|i|)_{i \in [d]} \) is the canonical ONB of \( \mathbb{C}^d \). Clearly, this is equivalent to (III.15) holding for any \( w \in \mathcal{F}(X, \mathcal{I}) \geq 0 \) with an arbitrary finite set \( \mathcal{I} \), and \( (|i|)_{i \in X} \) an orthonormal system in an arbitrary finite-dimensional Hilbert space \( \mathcal{H} \). Obviously, for a commuting \( W \in \mathcal{B}(X, \mathcal{H}) \geq 0 \), the value of \( \Delta^q(W) \) coincides with (III.14).

Different quantum extensions of a classical divergence may be compared according to the following:
Definition III.4 For two \( \mathcal{X} \)-variable quantum divergences \( \Delta_1 \) and \( \Delta_2 \), we write

\[
\Delta_1 \leq \Delta_2, \quad \text{if} \quad \Delta_1(W) \leq \Delta_2(W)
\]

for any finite-dimensional Hilbert space \( \mathcal{H} \) and any \( W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0} \). Similarly, we write

\[
\Delta_1 < \Delta_2, \quad \text{if} \quad \Delta_1(W) < \Delta_2(W)
\]

for any finite-dimensional Hilbert space \( \mathcal{H} \) and any \( W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{> 0} \) such that \( W_x W_y \neq W_y W_x \) for some \( x, y \in \mathcal{X} \).

Remark III.5 Note that in the strict ordering of divergences above, we require the inputs to be non-commuting, so that we can compare different quantum extensions of the same classical divergence. Also, we require the inputs to be invertible, to avoid pathological cases.

Remark III.6 For the rest, by expressions like “for all \( W \in \mathcal{B}(\mathcal{H})_{\geq 0} \),” or “for all \( \varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0} \)” we mean that the given property holds for any finite-dimensional Hilbert space \( \mathcal{H} \), and we write it out explicitly when something is only supposed to be valid for a specific Hilbert space.

Before further discussing quantum extensions of classical divergences, we review a few important properties of general divergences. Let \( \Delta \) be an \( \mathcal{X} \)-variable quantum divergence. We say that \( \Delta \) is

- **non-negative** if \( \Delta(W) \geq 0 \) for all collections of density operators \( W \in \mathcal{S}(\mathcal{X}, \mathcal{H}) \), and it is **strictly positive** if it is non-negative and \( \Delta(W) = 0 \iff W_x = W_y, \ x, y \in \mathcal{X} \), again for density operators;
- **monotone under a given map** \( \Phi \in \mathcal{P}^+(\mathcal{H}, \mathcal{K}) \) for some finite-dimensional Hilbert spaces \( \mathcal{H}, \mathcal{K} \), if

\[
\Delta(\Phi(W)) \leq \Delta(W), \quad W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0},
\]

where \( \Phi(W) := (\Phi(W_x))_{x \in \mathcal{X}} \); in particular, it is monotone under CPTP maps/PTP maps/pinchings if monotonicity holds for any map in the given class for any two finite-dimensional Hilbert spaces \( \mathcal{H}, \mathcal{K} \), and it is **trace-monotone**, if

\[
\Delta(\text{Tr} W) \leq \Delta(W)
\]

for any \( W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0} \) on any finite-dimensional Hilbert space \( \mathcal{H} \);
- **jointly convex** if for all \( W^{(k)} \in \mathcal{B}(\mathcal{X}^{(k)}, \mathcal{H}^{(k)}) \), \( k \in [r] \), and probability distribution \( (p_k)_{k \in [r]} \),

\[
\Delta \left( \sum_{k \in [r]} p_k W^{(k)} \right) \leq \sum_{k \in [r]} p_k \Delta \left( W^{(k)} \right);
\]
- **additive**, if for all \( W^{(k)} \in \mathcal{B}(\mathcal{X}^{(k)}, \mathcal{H}^{(k)}) \), \( k = 1, 2 \),

\[
\Delta \left( W^{(1)} \otimes W^{(2)} \right) = \Delta \left( W^{(1)} \right) + \Delta \left( W^{(2)} \right),
\]

and **subadditive (superadditive)** if \( \text{LHS} \leq \text{RHS} \) (\( \text{LHS} \geq \text{RHS} \)) holds above;
- **weakly additive**, if for all \( W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0} \),

\[
\Delta \left( W^\otimes n \right) = n \Delta(W), \quad n \in \mathbb{N},
\]

and **weakly subadditive (superadditive)** if \( \text{LHS} \leq \text{RHS} \) (\( \text{LHS} \geq \text{RHS} \)) holds above;
- **block subadditive**, if for any \( W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0} \), and any sequence of projections \( P_1, \ldots, P_r \in \mathcal{P}(\mathcal{H}) \) summing to \( I \),

\[
\Delta \left( \sum_{i=1}^r P_i W P_i \right) \leq \sum_{i=1}^r \Delta(P_i W P_i).
\]

Conversely, if the inequality in the above always holds in the opposite direction then \( \Delta \) is called **block superadditive**, and if it is always an equality, then \( \Delta \) is **block additive**.
Remark III.7 We say that $\Delta$ is strictly trace-monotone, if equality in (III.18) implies that $W_x = W_y$, $x, y \in X$.

Remark III.8 Note that $z \otimes X \mapsto zX$ gives a canonical identification between $\mathbb{C} \otimes B(\mathcal{H})$ and $B(\mathcal{H})$. In particular, any additive quantum divergence $\Delta$ satisfies the scaling law

$$\Delta((tW_x)_{x \in X}) = \Delta(t) + \Delta(W), \quad W \in B(X, \mathcal{H})_{\geq 0}, t \in \mathcal{F}(\mathcal{X}, \{0\})_{> 0}. \quad (III.20)$$

A quantum divergence $\Delta$ is called homogeneous, if

$$\Delta((tW_x)_{x \in X}) = t \Delta((W_x)_{x \in X}), \quad W \in B(X, \mathcal{H})_{\geq 0}, \quad t \in (0, +\infty).$$

It is well known that a 2-variable quantum divergence that is monotone non-decreasing under partial traces is jointly concave whenever it has the additional properties of block superadditivity and homogeneity. The extension to multi-variable divergences is straightforward; we give a detailed proof for completeness.

Lemma III.9 Assume that an $X$-variable quantum divergence $\Delta$ is block superadditive, homogeneous, and monotone non-decreasing under partial traces. Then $\Delta$ is jointly concave and jointly superadditive.

Proof Let $W^{(i)} \in B(X, \mathcal{H})_{\geq 0}$, $i \in [r]$, let $(t_i)_{i \in [r]}$ be a probability distribution, and let $(|i\rangle)_{i \in [r]}$ be an orthonormal system in some Hilbert space $\mathcal{K}$. Then

$$\Delta\left(\sum_{i=0}^{r-1} t_i W^{(i)}_{x} \right)_{x \in X} = \Delta\left(\sum_{i=0}^{r-1} t_i W^{(i)} \otimes |i\rangle \langle i| \right)_{x \in X} \geq \Delta\left(\sum_{i=0}^{r-1} t_i W^{(i)} \otimes |i\rangle \langle i| \right)_{x \in X} \geq \sum_{i=0}^{r-1} \Delta\left(\sum_{i=0}^{r-1} t_i W^{(i)} \otimes |i\rangle \langle i| \right)_{x \in X} = \sum_{i=0}^{r-1} t_i \Delta\left(W^{(i)} \otimes |i\rangle \langle i| \right)_{x \in X} = \sum_{i=0}^{r-1} t_i \Delta\left(W^{(i)} \otimes |i\rangle \langle i| \right)_{x \in X},$$

where the first equality is obvious, the inequality is by the assumption that $\Delta$ is monotone non-decreasing under partial traces, the second inequality is due to the block superadditivity of $\Delta$, the second equality follows from homogeneity, and the last equality is due to the isometric invariance of $\Delta$. This proves joint concavity, and joint superadditivity follows from it immediately due to homogeneity. \[\square\]

Any classical divergence admits two canonical quantum extensions, the minimal and the maximal ones:

Example III.10 For a classical divergence $\Delta$,

$$\Delta^{\text{meas}}(W) := \sup \{ \Delta(M(W)) : M \in \text{POVM}(\mathcal{H}, |n\rangle, n \in \mathbb{N}) \}, \quad W \in B(X, \mathcal{H})_{\geq 0}, \quad (III.21)$$

gives a quantum extension of $\Delta$, called the measured, or minimal extension.

As introduced in [35] in the 2-variable case, a reverse test for $W \in B(X, \mathcal{H})_{\geq 0}$ is a pair $(w, \Gamma)$ with $w \in \mathcal{F}(\mathcal{X}, \mathbb{I})$ for some finite set $\mathbb{I}$, and $\Gamma : L^\infty(\mathbb{I}) \to B(\mathcal{H})$ a (completely) positive trace-preserving map such that $\Gamma(w) = W$. For a classical divergence $\Delta$,

$$\Delta^{\text{max}}(W) := \inf \{ \Delta(w) : (w, \Gamma) \text{ is a reverse test for } W \}, \quad W \in B(X, \mathcal{H})_{\geq 0}, \quad (III.22)$$

gives a quantum extension of $\Delta$, called the maximal extension.

It is straightforward to verify from their definitions that both $\Delta^{\text{meas}}$ and $\Delta^{\text{max}}$ are monotone under PTP maps, and for any quantum extension $\Delta^q$ of $\Delta$ that is monotone under CPTP maps,

$$\Delta^{\text{meas}} \leq \Delta^q \leq \Delta^{\text{max}}$$
holds.

It is also clear that if $\Delta$ is additive then $\Delta^{\text{meas}}$ is superadditive and $\Delta^{\text{max}}$ is subadditive, and the regularized measured and the regularized maximal $\Delta$-divergences

$$\Delta^{\text{meas}}(W) := \sup_{n \in \mathbb{N}} \frac{1}{n} \Delta^{\text{meas}}(W \otimes^n) = \lim_{n \to +\infty} \frac{1}{n} \Delta^{\text{meas}}(W \otimes^n)$$

(III.23)

$$\Delta^{\text{max}}(W) := \inf_{n \in \mathbb{N}} \frac{1}{n} \Delta^{\text{max}}(W \otimes^n) = \lim_{n \to +\infty} \frac{1}{n} \Delta^{\text{max}}(W \otimes^n), \quad W \in \mathcal{B}(\mathcal{X}, H)_{\geq 0},$$

(III.24)

are quantum extensions of $\Delta$ that are weakly additive. Obviously, $\Delta^{\text{meas}}$ and $\Delta^{\text{max}}$ are monotone under CPTP maps, and for any quantum extension $\Delta^q$ of $\Delta$ that is monotone under CPTP maps, and any $W \in \mathcal{B}(\mathcal{X}, H)_{\geq 0}$,

$$\exists \Delta^q(W) := \lim_{n \rightarrow +\infty} \frac{1}{n} \Delta^q(W \otimes^n) \implies \Delta^{\text{meas}}(W) \leq \Delta^q(W) \leq \Delta^{\text{max}}(W).$$

In particular, if $\Delta^q$ is additive then

$$\Delta^{\text{meas}} \leq \Delta^q \leq \Delta^{\text{max}}.$$ 

We will furthermore consider properties that only make sense for 2-variable divergences. In particular, we say that a 2-variable quantum divergence $\Delta$ is

- **anti-monotone in the second argument (AM)**, if for all $\varrho, \sigma_1, \sigma_2 \in \mathcal{B}(H)_{\geq 0}$,

  $$\sigma_1 \leq \sigma_2 \implies \Delta(\varrho \| \sigma_1) \geq \Delta(\varrho \| \sigma_2);$$

(III.25)

- **weakly anti-monotone in the second argument**, if for any $\varrho, \sigma \in \mathcal{B}(H)_{\geq 0}$,

  $$[0, +\infty) \ni \varepsilon \mapsto \Delta(\varrho \| \sigma + \varepsilon I)$$

  is decreasing;

(III.26)

- **regular**, if for any $\varrho, \sigma \in \mathcal{B}(H)_{\geq 0}$,

  $$\Delta(\varrho \| \sigma) = \lim_{\varepsilon \searrow 0} \Delta(\varrho \| \sigma + \varepsilon I);$$

(III.27)

- **strongly regular**, if for any $\varrho, \sigma \in \mathcal{B}(H)_{\geq 0}$, and any sequence of positive semi-definite operators $(\sigma_n)_{n \in \mathbb{N}}$ converging decreasingly to $\sigma$,

  $$\Delta(\varrho \| \sigma) = \lim_{n \to +\infty} \Delta(\varrho \| \sigma_n);$$

(III.28)

**Remark III.11** Note that

$$\text{AM + regularity} \implies \text{strong regularity.}$$

Indeed, assume that $\Delta$ is regular and anti-monotone its second argument. Let $(\sigma_n)_{n \in \mathbb{N}}$ be a sequence of positive semi-definite operators converging decreasingly to $\sigma$. Then

$$\sigma \leq \sigma_n = \sigma + \sigma_n - \sigma \leq \sigma + \|\sigma_n - \sigma\|_{\infty} I = \sigma + \varepsilon_n I,$$

whence

$$\Delta(\varrho \| \sigma) \geq \Delta(\varrho \| \sigma_n) \geq \Delta(\varrho \| \sigma + \varepsilon_n I), \quad n \in \mathbb{N}.$$ 

By the regularity assumption, the RHS above tends to $\Delta(\varrho \| \sigma)$ as $n \to +\infty$, whence also

$$\lim_{n \to +\infty} \Delta(\varrho \| \sigma_n) = \Delta(\varrho \| \sigma).$$

Thus, $\Delta$ is strongly regular.

**Remark III.12** All the above properties may be defined also for classical divergences, by requiring the PSD operators in the definitions to be commuting.
B. Classical Rényi divergences

The classical divergences of particular importance to us are the 2-variable divergences called relative entropy, or Kullback-Leibler divergence, and the classical Rényi divergences. For a finite set $I$, and $\varrho, \sigma \in F(I) \geq 0$ the relative entropy of $\varrho$ and $\sigma$ is defined as

$$D(\varrho \| \sigma) := \sum_{i \in I} \left[ \varrho(i) \log \varrho(i) - \varrho(i) \log \sigma(i) \right], \quad \text{supp } \varrho \subseteq \text{supp } \sigma,$$

(III.29)

The classical Rényi $\alpha$-divergences [49] are defined for $\varrho, \sigma \in F(I) \geq 0$ and $\alpha \in (0, 1) \cup (1, +\infty)$ as

$$D_\alpha(\varrho \| \sigma) := \frac{1}{\alpha - 1} \log Q_\alpha(\varrho \| \sigma) - \frac{1}{\alpha - 1} \log \sum_{i \in I} \varrho(i),$$

(III.30)

$$Q_\alpha(\varrho \| \sigma) := \lim_{\varepsilon \to 0} \sum_{i \in I} (\varrho(i) + \varepsilon)^\alpha(\sigma(i) + \varepsilon)^{1-\alpha}$$

$$= \left\{ \begin{array}{ll} \sum_{i \in I} \varrho(i)^\alpha \sigma(i)^{1-\alpha} & \alpha \in (0, 1) \text{ or } \supp \varrho \subseteq \supp \sigma, \\
\infty, & \text{otherwise}. \end{array} \right.$$  

(III.31)

For $\alpha \in \{0, 1, +\infty\}$, the Rényi $\alpha$-divergence is defined by the corresponding limit, and it is easy to see that

$$D_0(\varrho \| \sigma) := \lim_{\alpha \to 0} D_\alpha(\varrho \| \sigma) = -\log \sum_{i \in \supp \varrho} \sigma(i) + \log \sum_{i \in I} \varrho(i),$$

(III.32)

$$D_1(\varrho \| \sigma) := \lim_{\alpha \to 1} D_\alpha(\varrho \| \sigma) = \frac{1}{\log \sum_{i \in I} \sigma(i)} D(\varrho \| \sigma),$$

(III.33)

$$D_\infty(\varrho \| \sigma) := \lim_{\alpha \to +\infty} D_\alpha(\varrho \| \sigma) = \log \inf \{ \lambda > 0 : \varrho \leq \lambda \sigma \}.$$  

(III.34)

In particular, the Rényi 1-divergence is the same as the relative entropy up to normalization. We extend the definitions of the Rényi divergences to the case when the second argument is zero as

$$D_\alpha(\varrho \| 0) := +\infty, \quad \varrho \geq 0, \quad \alpha \in [0, +\infty),$$

(III.35)

and the definition of the relative entropy to the case when one or both arguments are zero as

$$D(0 \| \sigma) := 0 \quad \sigma \geq 0, \quad D(0 \| 0) := +\infty \quad \varrho \geq 0.$$  

(III.36)

For the study and applications of the (classical) Rényi divergences, the relevant quantity is actually $Q_\alpha$ (equivalently, $\psi_\alpha$); the normalizations in (III.30) are somewhat arbitrary, and are mainly relevant only for the limits in (III.32)–(III.34). For instance, one could alternatively use the symmetrically normalized Rényi $\alpha$-divergences defined for any $\varrho, \sigma \in F(I) \geq 0$ and $\alpha \in (0, 1) \cup (1, +\infty)$ as

$$\tilde{D}_\alpha(\varrho \| \sigma) := \frac{1}{\alpha(1-\alpha)} \left[ -\log Q_\alpha(\varrho \| \sigma) + \alpha \log \sum_{i \in I} \varrho(i) + (1-\alpha) \log \sum_{i \in I} \sigma(i) \right]$$

$$= \frac{1}{\alpha(1-\alpha)} \log Q_\alpha \left( \frac{\varrho}{\sum_{i \in I} \varrho(i)} \right) \left( \frac{\sigma}{\sum_{i \in I} \sigma(i)} \right).$$

For $\alpha \in \{0, 1\}$ these give

$$\tilde{D}_1(\varrho \| \sigma) := \lim_{\alpha \to 1} \tilde{D}_\alpha(\varrho \| \sigma) = D \left( \frac{\varrho}{\sum_{i \in I} \varrho(i)} \left\| \frac{\sigma}{\sum_{i \in I} \sigma(i)} \right. \right) = D(\varrho \| \sigma) - \log \sum_{i \in I} \varrho(i) + \log \sum_{i \in I} \sigma(i),$$

$$\tilde{D}_0(\varrho \| \sigma) := \lim_{\alpha \to 0} \tilde{D}_\alpha(\varrho \| \sigma) = D \left( \frac{\sigma}{\sum_{i \in I} \sigma(i)} \left\| \frac{\varrho}{\sum_{i \in I} \varrho(i)} \right. \right) = D(\sigma \| \varrho) + \log \sum_{i \in I} \varrho(i) - \log \sum_{i \in I} \sigma(i),$$

for any $\varrho, \sigma \in F(I)$.
while
\[ \hat{D}_{1,\infty}(\varrho||\sigma) := \lim_{\alpha \to \infty} \hat{D}_\alpha(\varrho||\sigma) = 0 \]
is not very interesting.

As mentioned already in the Introduction, the Rényi \( \alpha \)-divergences with \( \alpha \in (0,1) \cup (1, +\infty) \) can be recovered from the relative entropy as
\[ -\log Q_\alpha(\varrho||\sigma) = \inf_\omega \{ \alpha D(\omega||\varrho) + (1 - \alpha)D(\omega||\sigma) \}, \tag{III.37} \]
where the infimum is taken over all \( \omega \in \mathcal{P}(\mathcal{I}) \) with supp \( \omega \subseteq \text{supp} \varrho \), and it is uniquely attained at
\[ \omega_\alpha(\varrho||\sigma) := \sum_{i \in S} \frac{\varrho(i)^\alpha \sigma(i)^{1-\alpha}}{\sum_{j \in S} \varrho(j)^\alpha \sigma(j)^{1-\alpha}} 1_{\{i\}}, \tag{III.38} \]
where \( S := \text{supp} \varrho \cap \text{supp} \sigma \), provided that supp \( \varrho \subseteq \text{supp} \sigma \), or supp \( \varrho \cap \text{supp} \sigma \neq \emptyset \) and \( \alpha \in (0,1) \). The case \( \alpha \in (0,1) \) was discussed in [12] in the more general setting where \( \mathcal{I} \) is not finite, while the case \( \alpha > 1 \) was discussed in the finite-dimensional quantum case in [42]; see also Section V below.

It is natural to ask whether the concept of Rényi divergences can be generalized to more than two variables. Formulas (III.31) and (III.37) offer two different approaches to do that. In a very general setting, one may consider a set \( \mathcal{X} \) equipped with a \( \sigma \)-algebra \( \mathcal{A} \). Then for any \( w \in \mathcal{F}(\mathcal{X}, \mathcal{I}) \) and signed measure \( P \) on \( \mathcal{A} \) with \( P(\mathcal{A}) = 1 \), one may consider
\[ \hat{Q}_P(w) := \lim_{\varepsilon \searrow 0} \sum_{i \in \mathcal{I}} \exp \left( \int_{\mathcal{X}} \log(w_x(i) + \varepsilon) \, dP(x) \right), \tag{III.39} \]
or
\[ \hat{Q}_P(w) := \lim_{\varepsilon \searrow 0} \sum_{i \in \mathcal{I}} \exp \left( \int_{\mathcal{X}} \log((1 - \varepsilon)w_x(i) + \varepsilon) \, dP(x) \right), \tag{III.40} \]
where the latter is somewhat more natural when the \( w_x \) are probability density functions on \( \mathcal{I} \). In the most general case, various issues regarding the existence of the integrals and the limits arise, which are important from a mathematical, but not particularly relevant from a conceptual point, and hence for the rest we will restrict our attention to the case where \( P \) is finitely supported. In that case the integrals always exist, and the \( \varepsilon \searrow 0 \) limit can be easily determined as
\[ \hat{Q}_P(w) = \sum_{i \in \mathcal{I}} \left( \prod_{x: w_x(i) > 0} w_x(i)^{P(x)} \right) \cdot \begin{cases} 0, & \text{if } \sum_{x: w_x(i) = 0} P(x) > 0, \\ 1, & \text{if } \sum_{x: w_x(i) = 0} P(x) = 0, \\ +\infty, & \text{if } \sum_{x: w_x(i) = 0} P(x) < 0, \end{cases} \tag{III.41} \]

independently of whether (III.39) or (III.40) is used.

Alternatively, one may define
\[ \hat{Q}_P^{b,cl}(w) := \sup_{\tau \in \mathcal{P}(\mathcal{I})} \left\{ \sum_{i \in \mathcal{I}} \tau(i) - \int_{\mathcal{X}} D(\tau||w_x) \, dP(x) \right\}, \]
which is well-defined at least when \( P \) is a probability measure, all the \( w_x \) are probability density functions, and \( \mathcal{X} \ni x \mapsto D(\tau||w_x) \) is measurable. Again, we restrict to the case when \( P \) is finitely supported, but allow it to be a signed probability measure, in which case we use a slight modification of the above to define
\[ Q_P^{b,cl}(w) := \sup_{\tau \in \mathcal{P}(\mathcal{I}), \sup \sup \sup \sup \, w_x} \left\{ \sum_{i \in \mathcal{I}} \tau(i) - \sum_{x \in \mathcal{X}} P(x) D(\tau||w_x) \right\}. \tag{III.42} \]
We will show in Section VA that this is equivalent to (III.37) when \( \mathcal{X} = \{0,1\} \), \( P(0) = \alpha \in [0, +\infty) \). It is not too difficult to see that with the definition in (III.42), we have
\[ Q_P^{b,cl}(w) = +\infty \iff \bigcap_{x: P(x) > 0} \text{supp} w_x \nsubseteq \bigcap_{x: P(x) < 0} \text{supp} w_x. \]
Thus, while (III.41) and (III.42) coincide when \( P \) is a probability measure, they may differ when \( P \) can take negative values. We will discuss this in more detail in Section V H.
C. Quantum Rényi divergences

In this section we give a brief review of the (2-variable) quantum Rényi divergences most commonly used in the literature, which will also play an important role in the rest of the paper. We will discuss various ways to define multi-variate quantum Rényi divergences in Sections III D and V.

**Definition III.13** For any \( \alpha \in [0, +\infty] \), by a quantum Rényi \( \alpha \)-divergence we mean a quantum divergence that is a quantum extension of the classical Rényi \( \alpha \)-divergence. Similarly, by a quantum relative entropy we mean a quantum extension of the relative entropy.

**Remark III.14** Note that for any \( \alpha \in (0, 1) \cup (1, +\infty) \), there is an obvious bijection between quantum extensions of \( Q_\alpha \) and quantum extensions of \( D_\alpha \).

**Definition III.13** Since these values are fixed by definition, in the discussion of different quantum Rényi divergences and relative entropies below, it is sufficient to consider non-zero arguments most of the time.

**Remark III.15** Since 0 commutes with any other operator, any quantum Rényi \( \alpha \)-divergence \( D^\alpha_\alpha \) must satisfy

\[
D^\alpha_\alpha(\varrho||0) = +\infty, \quad \varrho \geq 0, \tag{III.43}
\]

according to (III.35), and any quantum relative entropy \( D^\varrho \) must satisfy

\[
D^\varrho(0||\sigma) = 0 \quad \sigma \geq 0, \quad D^\varrho(\varrho||0) = +\infty \quad \varrho \geq 0, \tag{III.44}
\]

according to (III.36).

**Remark III.16** Note that there is a bijection between quantum extensions of the Rényi 1-divergence and quantum extensions of the relative entropy, given in one direction by \( D^\varrho(\varrho||\sigma) := (\text{Tr } \varrho)D^\varrho_1(\varrho||\sigma) \), and in the other direction by \( D^\varrho_1(\varrho||\sigma) := D^\varrho(\varrho||\sigma)/\text{Tr } \varrho \), for any non-zero \( \varrho \).

The following examples of quantum Rényi \( \alpha \)-divergences are well studied in the literature. We review them in some detail for later use.

**Example III.17** For any \( \alpha \in [0, 1) \cup (1, +\infty) \) and \( z \in (0, +\infty) \), the Rényi \((\alpha, z)\)-divergence of \( \varrho, \sigma \in B(\mathcal{H})_{\geq 0} \) is defined as [4]

\[
D_{\alpha, z}(\varrho, \sigma) := \begin{cases} 
\frac{1}{\alpha-1} \log \text{Tr} \left( \frac{\varrho^{\frac{1}{z}} \sigma^{1-\frac{1}{z}} \varrho^{\frac{1}{z}}}{\varrho} \right) - \frac{1}{\alpha-1} \log \text{Tr } \varrho, & \alpha \in [0, 1) \text{ or } \varrho^0 \leq \sigma^0, \\
+\infty, & \text{otherwise.}
\end{cases} \tag{III.45}
\]

It is easy to see that it defines a quantum Rényi \( \alpha \)-divergence in the sense of Definition. \( D_{\alpha, 1}(\varrho||\sigma) \) is called the Petz-type (or standard) Rényi \( \alpha \)-divergence [48] of \( \varrho \) and \( \sigma \), and \( D_{\alpha, 0}(\varrho||\sigma) \) their sandwiched Rényi \( \alpha \)-divergence [45, 54]. The limit

\[
D_{\alpha, +\infty}(\varrho||\sigma) := \lim_{z \to +\infty} D_{\alpha, z}(\varrho||\sigma) \tag{III.46}
\]

\[
\begin{align*}
= & \left\{ \frac{1}{\alpha-1} \log \text{Tr } P e^{\alpha P(\log \varrho)P + P(\log \sigma)P} =: Q_{\alpha, +\infty}(\varrho||\sigma) \right\} - \frac{1}{\alpha-1} \log \text{Tr } \varrho, & \alpha \in (0, 1) \text{ or } \varrho^0 \leq \sigma^0, \\
+\infty, & \text{otherwise,}
\end{align*} \tag{III.47}
\]

where \( P := \varrho^0 \wedge \sigma^0 \), is also a quantum Rényi \( \alpha \)-divergence, often referred to as the log-Euclidean Rényi \( \alpha \)-divergence [4, 27, 40]. It is known [34, 38] that for any function \( z : (1 - \delta, 1 + \delta) \to (0, +\infty) \) such that \( \lim \inf_{\alpha \to 1} z(\alpha) > 0 \), and for any \( \varrho, \sigma \in B(\mathcal{H})_{\geq 0} \),

\[
\lim_{\alpha \to 1} D_{\alpha, z(\alpha)}(\varrho||\sigma) = \frac{1}{\text{Tr } \varrho} D^\text{Um}(\varrho||\sigma) =: D^\text{Um}_1(\varrho||\sigma),
\]

where the Umegaki relative entropy \( D^\text{Um}(\varrho||\sigma) \) is defined as

\[
D^\text{Um}(\varrho||\sigma) := \begin{cases} 
\text{Tr}(\varrho \log \varrho - \varrho \log \sigma), & \varrho^0 \leq \sigma^0, \\
+\infty, & \text{otherwise.}
\end{cases} \tag{III.48}
\]
In particular, for any \( z \in (0, +\infty) \), we define \( D_{1,z}(\varrho|\sigma) := D_{1}^{\text{Um}}(\varrho||\sigma) \).

For every \( \alpha \in (0, +\infty) \) and \( z \in (0, +\infty) \), the Rényi \((\alpha,z)\)-divergence is strictly positive \([36, \text{Corollary III.28}]\). The range of \((\alpha,z)\)-values for which \( D_{\alpha,z} \) is monotone under CPTP maps was studied in a series of works \([5, 18, 24, 48]\), and was finally characterized completely in \([55]\). It is clear from their definitions that for every \( \alpha \in (0, +\infty) \) and \( z \in (0, +\infty) \), the Rényi \((\alpha,z)\)-divergence is additive on tensor products.

**Example III.18** For any \( \alpha \in (0, +\infty) \) and \( \varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0} \), their measured Rényi \( \alpha \)-divergence \( D_{\alpha}^{\text{meas}}(\varrho||\sigma) \) is defined as a special case of (III.23) with \( \Delta = D_{\alpha} \), and their measured relative entropy \( D_{\alpha}^{\text{meas}}(\varrho||\sigma) = (\text{Tr} \varrho)D_{1}^{\text{meas}}(\varrho||\sigma) \) is also a special case of (III.21) with \( \Delta = D \). We have \( D_{0}^{\text{meas}} = D_{0,1}, D_{1/2}^{\text{meas}} = D_{1/2,1/2} \) (see \([47, \text{Chapter 9}]\) and for any \( \varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0} \),

\[
D_{\alpha}^{\text{meas}}(\varrho||\sigma) = D_{\alpha}^{\star}(\varrho||\sigma) := D_{\alpha}^{\text{meas}}(\varrho||\sigma) := \lim_{\alpha \to \infty} D_{\alpha}(\varrho||\sigma) = \log \inf \{ \lambda \geq 0 : \varrho \leq \lambda \sigma \}, \tag{III.49}
\]

where the last quantity was introduced in \([13]\) under the name max-relative entropy, and its equality to the limit above has been shown in \([45, \text{Theorem 5}]\). No explicit expression is known for \( D_{\alpha}^{\text{meas}} \) for other \( \alpha \) values.

Similarly, the regularized measured Rényi \( \alpha \)-divergence \( \overline{D}_{\alpha}^{\text{meas}}(\varrho||\sigma) \) is obtained as a special case of (III.23). Surprisingly, it has a closed formula for every \( \alpha \in (0, +\infty) \), given by

\[
\overline{D}_{\alpha}^{\text{meas}}(\varrho||\sigma) = \begin{cases} 
D_{\alpha}(\varrho||\sigma), & \alpha \in [1/2, +\infty], \\
\frac{1}{\alpha - 1}D_{1}(\varrho||\varrho) + \frac{1}{1 - \alpha} \log \frac{\text{Tr} \sigma}{\text{Tr} \varrho} = D_{1,\alpha}(\varrho||\sigma), & \alpha \in (0, 1/2), \\
D_{1,\alpha}(\varrho||\sigma), & \alpha = 0;
\end{cases} \tag{III.50}
\]

see \([26]\) for \( \alpha = 1, [40] \) for \( \alpha \in (1, +\infty) \), and \([21]\) for \( \alpha = (1/2, 1) \); the last expression for \( \alpha \in (0, 1/2) \) above was first observed by Péter Vrana in August 2022, to the best of our knowledge.

For every \( \alpha \in (0, 1) \), strict positivity of \( D_{\alpha}^{\text{meas}} \) is immediate from the strict positivity of the classical Rényi \( \alpha \)-divergence, which is a straightforward corollary of Hölder’s inequality, and strict positivity of \( D_{\alpha}^{\text{meas}} \) for \( \alpha \in [1, +\infty) \) follows from this and the easily verifiable fact that \( \alpha \mapsto D_{\alpha}^{\text{meas}} \) is monotone increasing. Strict positivity of \( \overline{D}_{\alpha}^{\text{meas}} \) follows from \(\overline{D}_{\alpha}^{\text{meas}} \leq D_{\alpha}^{\text{meas}} \).

For any \( \alpha \in (0, +\infty) \), the measured Rényi \( \alpha \)-divergence is superadditive on tensor products (see Example III.10), but not additive unless \( \alpha \in (0, 1/2, +\infty) \); see, e.g., \([23, \text{Remark 4.27}]\) and \([39, \text{Proposition III.13}]\) for the latter. On the other hand, for every \( \alpha \in [0, +\infty) \), the regularized measured Rényi \( \alpha \)-divergence is not only weakly additive (see Example III.10) but even additive on tensor products, according to Example III.17 and (III.50).

Monotonicity of \( D_{\alpha}^{\text{meas}} \) under PTP maps and of \( \overline{D}_{\alpha}^{\text{meas}} \) under CPTP maps is obvious by definition for every \( \alpha \in [0, +\infty) \) (see Example III.10). Moreover, \( \overline{D}_{\alpha}^{\text{meas}} \), \( \alpha \in [0, +\infty) \), are monotone even under PTP maps, according to \([5, 28, 44]\) and (III.50). In particular, the Umegaki relative entropy \( D_{\text{Um}} \) is monotone under PTP maps.

**Example III.19** For any \( \alpha \in (0, +\infty) \) and \( \varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0} \), their maximal Rényi \( \alpha \)-divergence \( D_{\alpha}^{\text{max}}(\varrho||\sigma) \) is defined as a special case of (III.22) with \( \Delta = D_{\alpha} \), and their maximal relative entropy \( D_{\alpha}^{\text{max}}(\varrho||\sigma) = (\text{Tr} \varrho)D_{1}^{\text{max}}(\varrho||\sigma) \) is also a special case of (III.22) with \( \Delta = D \).

Let \( g_{\sigma,ac} := \max \{ 0 \leq C \leq \varrho : C^{0} \leq \sigma^{0} \} = P_{\sigma}P - P_{\sigma}(P_{\sigma}^{\dagger}P_{\sigma})^{-1}P_{\sigma} \) be the absolutely continuous part of \( \varrho \) w.r.t. \( \sigma \) \([1]\), where \( P := \sigma^{0} \), and let \( \lambda_{i}, i \in [r] \), be the different eigenvalues of \( \sigma^{-1/2}g_{\sigma,ac}\sigma^{-1/2} \) with corresponding spectral projections \( P_{i} \). Let \( \mathcal{I} := [r] \cup \{ r + 1 \} \), let \( \tau_{\mathcal{I}} \in \mathcal{S}(\mathcal{H}) \) be arbitrary, and

\[
\tau_{1} := \begin{cases} 
g_{\varrho,ac} \frac{g_{\varrho,ac}}{D_{1}(g_{\varrho,ac})}, & \varrho^{0} \leq \sigma^{0}, \\
\tau_{0}, & \text{otherwise.}
\end{cases}
\]

According to \([35]\),

\[
\hat{\varrho}(i) := \begin{cases} 
\lambda_{i} \text{Tr} \sigma P_{i}, & i \in [r], \\
\text{Tr} \varrho - g_{\sigma,ac}, & i = r + 1,
\end{cases} \tag{III.51}
\]

\[
\tilde{\varrho}(1_{(i)}) := \begin{cases} 
\frac{\sigma^{1/2}P_{i}g^{1/2}}{\text{Tr} \sigma P_{i}}, & i \in [r], \text{Tr} \sigma P_{i} \neq 0, \\
\tau_{0}, & i \in [r], \text{Tr} \sigma P_{i} = 0, \\
\tau_{1}, & i = r + 1,
\end{cases} \tag{III.52}
\]
is a reverse test for \((\rho, \sigma)\) that is optimal for every \(D^{\max}_{\alpha}(\rho||\sigma), \alpha \in [0, 2] \cup \{+\infty\}\), and

\[
Q^{\max}_{\alpha}(\rho||\sigma) = Q_0(\tilde{\rho}||\tilde{\sigma}) = \frac{1}{\lambda} \sum_{i: \lambda_i > 0} \rho(i) \sigma(i) = \sum_{i: \lambda_i > 0} \lambda_i \rho P_i = \sum_{i: \lambda_i > 0} \lambda_i \rho P_i \geq 0, \quad \lambda \geq 0\]

where \(f_0 := \id_{[0, +\infty)}\), \(\eta(x) := x \log x, x \geq 0\). (For the expressions in terms of the perspective functions, see also [23, 25].) The expression in (III.56) is called the Belavkin-Staszewski relative entropy [6]. Note that the optimal reverse test above is independent of \(\alpha\). No explicit expression is known for \(D^{\max}_{\alpha}\) when \(\alpha \in (2, +\infty)\), in which case the above reverse test is known not to be optimal.

Strict positivity of \(D^{\max}_{\alpha}\) for all \(\alpha \in (0, +\infty)\) follows from that of \(D^{\max}_{\alpha}\) and the inequality \(D^{\max}_{\alpha} \leq D^{\max}_{\alpha}\), which is due to the monotonicity of the classical Rényi divergences under stochastic maps.

It is immediate from their definition that \(D^{\max}_{\alpha}\), \(\alpha \in [0, +\infty)\), are subadditive on tensor products (see Example III.10). For \(\alpha \in [0, 2] \cup \{+\infty\}\), \(D^{\max}_{\alpha}\) is even additive, as one can easily verify from the representation \(Q^{\max}_{\alpha}(\rho||\sigma) = \Tr \rho P_{\alpha}(\rho||\sigma), \rho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}\) in (III.53). However, additivity of \(D^{\max}_{\alpha}\) is not known for \(\alpha \in (2, +\infty)\). In particular, we have

\[
\overline{D}^{\max}_{\alpha} = \begin{cases} D^{\max}_{\alpha}, & \alpha \in [0, 2] \cup \{+\infty\}, \\ \leq D^{\max}_{\alpha}, & \alpha \in (2, +\infty). \end{cases}
\]

**Remark III.20** Note that, with the notations of Example III.19,

\[
\lim_{\alpha \searrow 0} Q^{\max}_{\alpha}(\rho||\sigma) = \Tr \sigma(\sigma^{-1/2} \rho,_{ac} \sigma^{-1/2})^0 = \Tr \sigma \sum_{i: \lambda_i > 0} P_i, \quad \text{as } \alpha \searrow 0\]

while

\[
Q^{\max}_{0}(\rho||\sigma) = Q_0(\tilde{\rho}||\tilde{\sigma}) = \sum_{i} \tilde{\rho}(i) \tilde{\sigma}(i) = \sum_{i: \lambda_i > 0} \lambda_i \Tr \rho P_i = \Tr \sum_{i: \lambda_i > 0} \lambda_i \Tr \rho P_i.
\]

Since

\[
P_i \sigma^{-1/2} \rho,_{ac} \sigma^{-1/2} P_i = \lambda_i P_i,
\]

we see that \(\lambda_i > 0 \implies P_i \not\perp \rho^0 \iff \Tr P_i > 0\), and hence (III.58) and (III.59) are equal to each other, i.e.,

\[
\lim_{\alpha \searrow 0} D^{\max}_{\alpha}(\rho||\sigma) = D^{\max}_{0}(\rho||\sigma), \quad \rho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}.
\]

**Example III.21** For any quantum Rényi \(\alpha\)-divergence \(D^{\alpha}_\rho\), its regularization on a pair \(\rho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}\) is defined as

\[
\overline{D}^{\alpha}_\rho(\rho||\sigma) := \lim_{n \to +\infty} \frac{1}{n} D^{\alpha}_\rho(\rho^\otimes n||\sigma^\otimes n),
\]

whenever the limit exists. If the limit exists for all \(\rho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}\), then \(\overline{D}^{\alpha}_\rho\) is a quantum Rényi \(\alpha\)-divergence that is weakly additive, and if \(D^{\alpha}_\rho\) is monotone under CPTP maps then so is \(\overline{D}^{\alpha}_\rho\).

**Remark III.22** Note that if \(D^{\alpha}_\rho\) is an additive quantum 1-divergence then the corresponding quantum relative entropy \(D^{\rho}\) is not additive; instead, it satisfies

\[
D^{\rho}(\rho \otimes \sigma^2||\sigma_1 \otimes \sigma_2) = (\Tr \rho^2) D^{\rho}(\rho_1||\sigma_1) + (\Tr \rho_1) D^{\rho}(\rho_2||\sigma_2)
\]
for any $\varrho_k, \sigma_k \in \mathcal{B}(\mathcal{H}_k)_{\geq 0}$, $k = 1, 2$. Thus, the natural notion of regularization for a quantum relative entropy $D^q$ on a pair $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ is

$$D(\varrho \| \sigma) := \langle \text{Tr} \, \varrho \rangle D^q_1(\varrho \| \sigma),$$

which is well-defined whenever $D^q_1(\varrho \| \sigma)$ is. Clearly, if $D(\varrho \| \sigma)$ is well-defined for all $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ then it gives a quantum relative entropy that is weakly additive, and if $D^q$ is monotone under CPTP maps then so is $D(\varrho \| \sigma)$.

**Remark III.23** According to Remark III.10, for any given $\alpha \in [0, +\infty]$, and any quantum Rényi $\alpha$-divergence $D^q_\alpha$ that is monotone under CPTP maps,

$$D^\text{meas}_\alpha \leq D^q_\alpha \leq D^\text{max}_\alpha. \quad (\text{III.60})$$

If the regularization of $D^q_\alpha$ is well-defined then we further have

$$D^\text{meas}_\alpha \leq D^q_\alpha \leq D^\text{max}_\alpha; \quad (\text{III.61})$$

in particular, this is the case if $D^q_\alpha$ is additive, when we also have $D^q_\alpha = D^\alpha_\alpha$.

Likewise, for any quantum relative entropy $D^q$ that is monotone under CPTP maps,

$$D^\text{meas}_q \leq D^q \leq D^\text{max}_q. \quad (\text{III.62})$$

and if the regularization of $D^q$ is well-defined then we further have

$$D^\text{meas}_q \leq D^q \leq D^\text{max}_q = D^\text{max}_q. \quad (\text{III.63})$$

Note that $D^\text{meas}_q = D^\text{meas}_q = D^\text{max}_q = D^\text{max}_q$ is the unique quantum extension of $D^q$ that is monotone under (completely) positive trace-preserving maps, as it was observed in [50], and this unique extension also happens to be additive. On the other hand, for any other $\alpha \in [0, +\infty)$, there are infinitely many different monotone and additive quantum Rényi $\alpha$-divergences; see, e.g., Example III.17.

**Remark III.24** According to Remark III.8, any additive quantum Rényi $\alpha$-divergence $D^q_\alpha$ satisfies the scaling law

$$D^q_\alpha(t\varrho \| s\varrho) = D^q_\alpha(\varrho \| \sigma) + D_\alpha(t \| s) = D^q_\alpha(\varrho \| \sigma) + \log t - \log s. \quad (\text{III.64})$$

In particular, this holds for $D_{\alpha,z}$, $\alpha \in [0, +\infty)$, $z \in (0, +\infty)$, and $D^\text{max}_\alpha$, $\alpha \in [0, 2) \cup \{ +\infty \}$. It is easy to verify that $D^\text{meas}_\alpha$ also satisfies (III.64) for every $\alpha \in (2, +\infty)$, where additivity is not known, and $D^\text{meas}_\alpha$ also satisfies (III.64) for every $\alpha \in (0, +\infty)$, even though they are not additive unless $\alpha \in \{ 0, 1/2, +\infty \}$.

Note that a quantum Rényi $1$-divergence $D^q_1$ satisfies the scaling law (III.64) if and only if the corresponding quantum relative entropy $D^q$ satisfies the scaling law

$$D^q(t\varrho \| s\varrho) = tD^q(\varrho \| \sigma) + (\text{Tr} \, \varrho)D(t \| s), \quad (\text{III.65})$$

which in turn equivalent to

$$D^q(t\varrho \| \sigma) = (t\log t) \text{Tr} \, \varrho + t D^q(\varrho \| \sigma), \quad (\text{III.66})$$

$$D^q(\varrho \| s\varrho) = D^q(\varrho \| \sigma) - (\log s) \text{Tr} \, \varrho. \quad (\text{III.67})$$

**Remark III.25** By definition, a quantum Rényi $\alpha$-divergence $D^q_\alpha$ is trace-monotone, if

$$D^q_\alpha(\varrho \| \sigma) \geq D_\alpha(\text{Tr} \, \varrho \| \text{Tr} \, \sigma) \quad (= \log \text{Tr} \, \varrho - \log \text{Tr} \, \sigma) \quad (\text{III.68})$$

for any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$, and it is strictly trace-monotone if equality holds in (III.68) if and only if $\varrho = \sigma$. Likewise, a quantum relative entropy $D^q$ is trace-monotone, if

$$D^q(\varrho \| \sigma) \geq D(\text{Tr} \, \varrho \| \text{Tr} \, \sigma) \quad (= (\text{Tr} \, \varrho) \log \text{Tr} \, \varrho - (\text{Tr} \, \varrho) \log \text{Tr} \, \sigma). \quad (\text{III.69})$$

for any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$, and it is strictly trace-monotone if equality holds in (III.69) if and only if $\varrho = \sigma$. Obviously, any trace-monotone Rényi $\alpha$-divergence or relative entropy is non-negative. Moreover, is is easy to see that if a quantum Rényi $\alpha$-divergence $D^q_\alpha$ satisfies the scaling law (III.64) then it is non-negative (strictly positive) if and only if it is (strictly) trace-monotone, and similarly, if a quantum relative entropy $D^q$ satisfies the scaling law (III.65) then it is non-negative (strictly positive) if and only if it is (strictly) trace-monotone.
Remark III.26 If a quantum relative entropy $D^q$ satisfies the trace monotonicity (III.69) then for any $\tau, \sigma \in B(H)_{\geq 0}$,

$$D^q(\tau\|\sigma) \geq -(\text{Tr}\,\tau)\log \frac{\text{Tr}\,\sigma}{\text{Tr}\,\tau} \geq \text{Tr}\,\tau \left( 1 - \frac{\text{Tr}\,\sigma}{\text{Tr}\,\tau} \right) = \text{Tr}\,\tau - \text{Tr}\,\sigma,$$

(III.70)

and equality holds everywhere when $\tau = \sigma$. As an immediate consequence of this, for any $\sigma \in B(H)_{\geq 0}$,

$$\text{Tr}\,\sigma = \max_{\tau \in B(H)_{\geq 0}} \{ \text{Tr}\,\tau - D^q(\tau\|\sigma) \} = \max_{\tau \in B(H)_{\geq 0}} \{ \text{Tr}\,\tau - D^q(\tau\|\sigma) \},$$

(III.71)

$$\log \text{Tr}\,\sigma = \max_{\tau \in B(H)_{\geq 0}} \left\{ \log \text{Tr}\,\tau - \frac{1}{\text{Tr}\,\tau} D^q(\tau\|\sigma) \right\} = \max_{\tau \in B(H)_{\geq 0}} \left\{ \log \text{Tr}\,\tau - \frac{1}{\text{Tr}\,\tau} D^q(\tau\|\sigma) \right\}.$$  

(III.72)

Note that $\tau$ is a maximizer for (III.71) if and only if $\text{Tr}\,\tau = \text{Tr}\,\sigma$ and $D^q(\tau\|\sigma) = 0$ (since the second inequality in (III.70) holds as an equality if and only if $\text{Tr}\,\tau = \text{Tr}\,\sigma$, and if $D^q$ also satisfies the scaling property (III.65) then $\tau$ is a maximizer for (III.72) if and only if $D \left( \frac{\text{Tr}\,\tau}{\text{Tr}\,\sigma} \| \frac{\text{Tr}\,\tau}{\text{Tr}\,\sigma} \right) = 0$. If $D^q$ is strictly trace monotone then $\tau = \sigma$ is the unique maximizer for all the expressions in (III.71)-(III.72).

The variational formula (III.71) has already been noted in [51, Lemma 6] in the case $D^q = D^\text{Um}$.

Remark III.27 It is easy to see from their definitions that $D^\text{meas}$, $D^\text{Um}$, and $D^\text{max}$ are all regular and anti-monotone in their second argument (due to the operator anti-monotonicity of log and operator anti-monotonicity of the inverse [8]), i.e.,

$$D^q(\rho\|\sigma + \varepsilon I) \succ D^q(\rho\|\sigma) \quad \varepsilon \searrow 0.$$  

(III.73)

By Remark III.11, they are also strongly regular. It is clear from (III.48) and (III.56) that for any fixed $\varepsilon > 0$, $B(H)_{\geq 0} \ni (\rho, \sigma) \rightarrow D^q(\rho\|\sigma + \varepsilon I)$ is continuous when $q = \text{Um}$ or $q = \text{max}$. Hence, by (III.73), $D^\text{Um}$ and $D^\text{max}$ are both jointly lower semi-continuous in their arguments. In particular, the classical relative entropy is jointly lower semi-continuous, whence $D^\text{meas}$, as the supremum of lower semi-continuous functions, is also jointly lower semi-continuous.

Remark III.28 It is clear from (III.48) and (III.56) that $D^\text{Um}$ and $D^\text{max}$ are block additive. For $D^\text{meas}$, we have block sub-additivity. Indeed, let $\varrho_k, \sigma_k \in B(H_k)_{\geq 0}, k = 1, 2$. For any $(M_i)_{i=1}^n \in \text{POVM}(H_1, [n])$, $(N_i)_{i=1}^n \in \text{POVM}(H_2, [n])$,

$$D \left( \sum_{i=1}^n (\text{Tr}\,\varrho_1 M_i) (\text{Tr}\,\sigma_1 N_i) \right) + D \left( \sum_{i=1}^n (\text{Tr}\,\varrho_2 N_i) (\text{Tr}\,\sigma_2 N_i) \right),$$

$$\geq D \left( \sum_{i=1}^n (\text{Tr}\,\varrho_1 M_i + \text{Tr}\,\varrho_2 N_i) (\text{Tr}\,\sigma_1 M_i + \text{Tr}\,\sigma_2 N_i) \right),$$

$$= D \left( \sum_{i=1}^n (\text{Tr}(\varrho_1 \oplus \varrho_2) (M_i \oplus N_i)) (\text{Tr}(\sigma_1 \oplus \sigma_2) (M_i \oplus N_i)) \right),$$

where the inequality follows from the joint subadditivity of the relative entropy (a consequence of joint convexity and homogeneity). Taking the supremum over $(M_i)_{i=1}^n \in \text{POVM}(H_1, [n])$, $(N_i)_{i=1}^n \in \text{POVM}(H_2, [n])$, and then over $n$, we get

$$D^\text{meas}(\varrho_1\|\sigma_1) + D^\text{meas}(\varrho_2\|\sigma_2)$$

$$\geq \sup_{n \in \mathbb{N}} \sup_{M,N} D \left( \sum_{i=1}^n (\text{Tr}(\varrho_1 \oplus \varrho_2) (M_i \oplus N_i)) (\text{Tr}(\sigma_1 \oplus \sigma_2) (M_i \oplus N_i)) \right),$$

$$= \max_{n \in \mathbb{N}} \sup_{T \in \text{POVM}(H_1 \oplus H_2, [n])} D \left( \sum_{i=1}^n (\text{Tr}(\varrho_1 \oplus \varrho_2) T_i) (\text{Tr}(\sigma_1 \oplus \sigma_2) T_i) \right),$$

$$= D^\text{meas}(\varrho_1 \oplus \varrho_2\|\sigma_1 \oplus \sigma_2).$$

The first equality above follows from the simple fact that for any $T \in B(H_1 \oplus H_2)$, $\text{Tr}(\varrho_1 \oplus \varrho_2) T = \text{Tr}(\varrho_1 \oplus \varrho_2) (T_1 \oplus T_2)$, where $T_k = P_k T P_k$, with $P_k$ the projection onto $H_k$ in $H_1 \oplus H_2$.

D. Weighted geometric means and induced divergences

Recall that for non-negative numbers $a, b \in [0, +\infty)$ and some $\gamma \in (0, 1)$, their $\gamma$-weighted geometric mean is

$$G_\gamma(a\|b) := a^{\gamma} b^{1-\gamma}.$$
This can be extended to any \( a, b \in (0, +\infty) \) and \( \gamma \in \mathbb{R} \) by the same formula. Note that for \( \varrho, \sigma \in (\mathbb{R}^X)_{>0} \) and \( \alpha \in (0, 1) \cup (1, +\infty) \), \( Q_{\alpha} \) in (III.31) can be expressed as
\[
Q_{\alpha}(\varrho||\sigma) = \sum_{x \in X} G_{\alpha}(\varrho(x)||\sigma(x)).
\]

**Definition III.29** For any \( \gamma \in \mathbb{R} \), a non-commutative \( \gamma \)-weighted geometric mean is a function on pairs of positive definite matrices,
\[
G_{\gamma}^d : \cup_{d \in \mathbb{N}} (B(C^d)>0 \times B(C^d)>0) \to \cup_{d \in \mathbb{N}} B(C^d)>0
\]
such that

(i) \( G_{\gamma}^d (B(C^d)>0 \times B(C^d)>0) \subseteq B(C^d)>0, \ d \in \mathbb{N}; \)

(ii) \( G_{\gamma}^d \) is covariant under isometries, i.e., if \( V : C^{d_1} \to C^{d_2} \) is an isometry then
\[
G_{\gamma}^d(V\varrho V^*\sigma V^*) =VG_{\gamma}^d(\varrho||\sigma)V^*, \quad \varrho, \sigma \in B(C^{d_1})>0;
\]

(iii) if \( \varrho = \sum_{i=1}^{d} \varrho(i)|i\rangle\langle i| \) and \( \sigma = \sum_{i=1}^{d} \sigma(i)|i\rangle\langle i| \) are diagonal in the same ONB then
\[
G_{\gamma}^d \left( \sum_{i=1}^{d} \varrho(i)|i\rangle\langle i| , \sum_{i=1}^{d} \sigma(i)|i\rangle\langle i| \right) = \sum_{i=1}^{d} \varrho(i)^{\gamma} \sigma(i)^{1-\gamma} |i\rangle\langle i|.
\]

Similarly to quantum divergences, a non-commutative \( \gamma \)-weighted geometric mean can be uniquely extended to pairs of positive definite operators on an arbitrary finite-dimensional Hilbert space; we denote this extension also by \( G_{\gamma}^d \).

**Example III.30** For any \( \gamma \in \mathbb{R} \), \( z \in (0, +\infty) \), and \( \varrho, \sigma \in B(H)>0 \), let
\[
G_{\gamma,z}(\varrho||\sigma) := \left( \varrho^{\frac{1}{2}} \sigma^{\frac{1-\gamma}{2}} \varrho^{\frac{1}{2}} \right)^z,
\]
\[
\bar{G}_{\gamma,z}(\varrho||\sigma) := \left( \sigma^{\frac{1}{2}} \varrho^{\frac{1}{2}} \sigma^{\frac{1-\gamma}{2}} \right)^z,
\]
\[
\hat{G}_{\gamma,1}(\varrho||\sigma) := \varrho^{\frac{1}{2}} \sigma^{\frac{1}{2}} \left( \sigma^{1/2} \varrho^{1/2} \sigma^{-1/2} \right)^{\gamma} \sigma^{1/2},
\]
\[
\hat{G}_{\gamma,z}(\varrho||\sigma) := \left( \sigma^{\frac{1}{2}} \#_\gamma \varrho^{\frac{1}{2}} \right)^z,
\]
\[
\hat{G}_{\gamma,+\infty}(\varrho||\sigma) := \lim_{\gamma \to +\infty} \hat{G}_{\gamma,z}(\varrho||\sigma) = e^{\gamma \log \varrho + (1-\gamma) \log \sigma},
\]
where \( \#_\gamma \) denotes the Kubo-Ando \( \gamma \)-weighted geometric mean [32] (see Section IV for a more detailed exposition), and the last equality is due to [27, Lemma 3.3]. It is easy to see that these are all non-commutative \( \gamma \)-weighted geometric means.

For any non-commutative \( \alpha \)-weighted geometric mean \( G_{\alpha}^d \),
\[
Q_{G_{\alpha}^d}(\varrho||\sigma) := \text{Tr} G_{\alpha}^d(\varrho||\sigma) \quad (\text{III.74})
\]
defines an extension of the classical \( Q_{\alpha} \), and hence also of the classical Rényi divergence \( D_{\alpha} \), to pairs of positive definite operators. This can then be further extended to pairs of non-zero positive semi-definite operators by
\[
Q_{G_{\alpha}^d}(\varrho||\sigma) := \lim_{\varepsilon \to 0} \text{Tr} G_{\alpha}^d(\varrho + \varepsilon I||\sigma + \varepsilon I), \quad (\text{III.75})
\]
provided that the limit exists. This is the case, e.g., for \( G_{\alpha,1}, \hat{G}_{\alpha,1}, \alpha \in (0, 1) \cup (1, +\infty) \), \( z \in (0, +\infty) \), for \( \hat{G}_{\alpha,1}, \alpha \in (0, 1) \cup (1, +\infty) \), and \( \hat{G}_{\alpha,\infty}, \alpha \in (0, 1) \cup (1, +\infty) \), and we have
\[
Q_{\alpha,1}(\varrho||\sigma) = Q_{\alpha,1}(\varrho||\sigma) = Q_{\alpha,1}(\varrho||\sigma), \quad \alpha \in (0, 1) \cup (1, +\infty), \ z \in (0, +\infty), \quad (\text{III.76})
\]
\[
Q_{\alpha,\infty}(\varrho||\sigma) = Q_{\alpha,\infty}(\varrho||\sigma), \quad \alpha \in (0, 1) \cup (1, +\infty), \quad (\text{III.77})
\]
\[
Q_{\alpha}^{\text{max}}(\varrho||\sigma) = Q_{\alpha}^{\text{max}}(\varrho||\sigma), \quad \alpha \in (0, 1) \cup (1, 2), \quad (\text{III.78})
\]
for any \( \varrho, \sigma \in B(H)>0 \); see Examples III.17 and III.19, and [41] for the validity of (III.75) for \( G_{\alpha}^d = \hat{G}_{\alpha,\infty} \). The equality in (III.78) follows from (III.54) and the fact that \( \sigma \#_\alpha \varrho = \sigma \#_\alpha \varrho_{\sigma,\alpha} \), \( \alpha \in (0, 1) \).
Remark III.31 For every $\alpha \in (0, 1)$, $[1, +\infty] \ni z \mapsto Q_{\alpha,z}^\max(\|\|) = Q_{\alpha,z}^\max(\|\|)$ interpolates monotonically between $Q_{\alpha}^\max(\|\|)$ and $Q_{\alpha,\infty}(\|\|)$, according to [2, Corollary 2.4], and hence

$$[1, +\infty] \ni z \mapsto D_{\alpha,z}^\max(\|\|) := \frac{1}{\alpha - 1} \log Q_{\alpha,z}^\max(\|\|) - \frac{1}{\alpha - 1} \log \text{Tr} \varrho$$

interpolates monotonically decreasingly between $D_{\alpha}^\max(\|\|)$ and $D_{\alpha,\infty}(\|\|)$.

Remark III.32 Note that the regularization in (III.75) might work even without the trace, whence it defines an extension of the given geometric mean $G_\gamma$ to pairs of positive semi-definite operators as

$$G_\gamma^\ell(\|\|) := \lim_{\varepsilon \to 0} G_\gamma^\ell(\varrho + \varepsilon I\|\| + \varepsilon I).$$

This is the case, for instance, for any $\gamma \in (0, 1)$ and $G_{\gamma,z}, G_{\gamma,z}, z \in (0, +\infty)$, as well as $G_{\gamma,z}, z \in (0, +\infty]$.

Next, we explore some ways to define (potentially) new quantum Rényi divergences and relative entropies from combining some given quantum Rényi divergences/relative entropies with some non-commutative weighted geometric means.

First, given $\alpha \in (0, +\infty), \gamma \in (0, \min\{1, \alpha\})$, a non-commutative $\gamma$-weighted geometric mean $G_\gamma^\ell$, and a quantum Rényi divergence $D_{\alpha}^\ell$,

$$D_{\alpha}^{\ell,s}(\|\|) := \frac{1}{1 - \gamma} D_{\alpha}^{\ell} (\| \| G_\gamma^\ell(\|\|))$$

defines an extension of the classical Rényi $\alpha$-divergence to pairs of positive definite operators $\varrho, \sigma$, and further extension to pairs of positive semi-definite operators may be considered analogously to (III.75). This concept was first introduced in [10] in the special case $D_{\alpha}^\ell = D_{\alpha,\alpha}$ for some $\alpha > 1$ and $G_\gamma^\ell = G_{\gamma,1}$. The special case $\alpha = 1$ gives that any quantum relative entropy $D_{\alpha}$ and any non-commutative $\gamma$-weighted geometric mean $G_\gamma^\ell$ with some $\gamma \in (0, 1)$ defines a quantum relative entropy on pairs of positive definite operators via

$$D_{\alpha}^{\gamma,s}(\|\|) := \frac{1}{1 - \gamma} D_{\alpha}^{\gamma} (\| \| G_\gamma^\ell(\|\|)).$$

Remark III.33 If $D_{\alpha}^\ell$ satisfies the scaling law (III.64) then for any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_\geq 0$ such that $Q_{\gamma}^\ell(\|\|) = \text{Tr} G_\gamma^\ell(\|\|) \neq 0$, (III.79) can be rewritten as

$$D_{\alpha}^{\gamma,s}(\|\|) = \frac{1}{1 - \gamma} \left[ D_{\alpha}^{\gamma} (\| \| \text{Tr} G_\gamma^\ell(\|\|)) + \log \text{Tr} \varrho - \log \text{Tr} G_\gamma^\ell(\|\|) \right]$$

$$= \frac{1}{1 - \gamma} D_{\alpha}^{\gamma} (\| \| \text{Tr} G_\gamma^\ell(\|\|)) + D_{\alpha}^{\gamma} (\|\|).$$

Likewise, if $D_{\alpha}^{\ell}$ satisfies the scaling law (III.65), and $\text{Tr} G_\gamma^\ell(\|\|) \neq 0$, then (III.80) can be rewritten as

$$D_{\alpha}^{\gamma,s}(\|\|) = \frac{1}{1 - \gamma} \left[ D_{\alpha}^{\gamma} (\| \| \text{Tr} G_\gamma^\ell(\|\|)) + \log \text{Tr} \varrho - \log \text{Tr} G_\gamma^\ell(\|\|) \right]$$

$$= \left( \text{Tr} \varrho \right) \frac{1}{1 - \gamma} D_{\alpha}^{\gamma} (\| \| \text{Tr} G_\gamma^\ell(\|\|)) + D_{\alpha}^{\gamma} (\|\|).$$

In particular, if $D_{\gamma}^{\ell}$ is strictly positive and $D_{\alpha}^{\ell}$ (resp. $D_{\alpha}^{\gamma}$) is non-negative, then $D_{\alpha}^{\ell,s}(\|\|)$ (resp. $D_{\alpha}^{\gamma,s}(\|\|)$) is strictly positive.

Another way of obtaining new quantum Rényi divergences using non-commutative geometric means is by starting with two quantum relative entropies $D_{\alpha}^{\ell}$ and $D_{\alpha}^{\gamma}$, and two non-commutative $\alpha$-weighted geometric means $G_{\alpha}^\ell$ and $G_{\alpha}^\gamma$ with some $\alpha \in (0, 1) \cup (1, +\infty)$, and defining

$$D_{\alpha}^{\ell,\gamma}(\|\|) := \frac{\alpha}{1 - \alpha} \left( \frac{G_{\alpha}^\ell(\|\|)}{\text{Tr} G_{\alpha}^\ell(\|\|)} \right) + \frac{\alpha}{1 - \alpha} \left( \frac{G_{\alpha}^\gamma(\|\|)}{\text{Tr} G_{\alpha}^\gamma(\|\|)} \right) - \frac{1}{\alpha - 1} \log \text{Tr} \varrho$$
for any $\varrho, \sigma \in \mathcal{B}(\mathcal{H}) > 0$, extending again to $\varrho, \sigma \in \mathcal{B}(\mathcal{H}) > 0$ as in (III.75), if possible.

Instead of writing normalized geometric means in the first arguments of the quantum relative entropies in (III.83), one may optimize over arbitrary states, which leads to

$$
D^{b(q_0, q_1)}_{\alpha}(\varrho \| \sigma) := \frac{1}{1 - \alpha} \inf_{\omega \in \mathcal{S}(\varrho, \sigma)} \left\{ \alpha D^{q_0}(\omega \| \varrho) + (1 - \alpha) D^{q_1}(\omega \| \sigma) \right\} - \frac{1}{\alpha - 1} \log \text{Tr} \varrho. \tag{III.84}
$$

This again gives a quantum Rényi $\alpha$-divergence, which we call the barycentric Rényi $\alpha$-divergence corresponding to $D^{q_0}$ and $D^{q_1}$. (The restriction that the support of the state $\omega$ is contained in the support of $\varrho$ is only necessary when $\alpha > 1$; see Section V for more details.)

It may happen that the quantities in (III.83) and (III.84) coincide; this is the case, for instance, when $D^{q_0} = D^{q_1} = D^{Um}$ and $G^{q_0}_\alpha = G^{q_1}_\alpha = \bar{G}_{\alpha, +\infty}$, as was shown in [40]. We explain this in a bit more detail in Example III.34 below. Of course, this is not to be expected in general, and we show in Section VI C that in the case $D^{q_0} = D^{q_1} = D^{\text{max}}$ and $G^{q_0}_\alpha = G^{q_1}_\alpha = \bar{G}_{\alpha, 1}$, the two quantities are different for every $\alpha \in (0, 1)$.

**Example III.34** It was shown in [40, Theorem 3.6] that for $D^{q_0} = D^{q_1} = D^{Um}$ and $G^{q_0}_\alpha = G^{q_1}_\alpha = \bar{G}_{\alpha, +\infty}$, the quantum Rényi $\alpha$-divergences in (III.83) and (III.84) are both equal to $D_{\alpha, +\infty}$ in (III.47). That is, for any $\varrho, \sigma \in \mathcal{B}(\mathcal{H}) > 0$, and any $\alpha \in (0, 1)$,

$$
D^{Um}_{\alpha}(\bar{G}_{\alpha, +\infty})(\varrho \| \sigma) := \frac{\alpha}{1 - \alpha} D^{Um}(\omega^U_{\alpha, +\infty}(q || \varrho)) + D^{Um}(\omega^U_{\alpha, +\infty}(q || \sigma)) - \frac{1}{\alpha - 1} \log \text{Tr} \varrho
$$

$$
= D^{b(Um, Um)}_{\alpha}(\varrho \| \sigma) := \frac{1}{1 - \alpha} \inf_{\omega \in \mathcal{S}(\mathcal{H})} \left\{ \alpha D^{Um}(\omega || \varrho) + (1 - \alpha) D^{Um}(\omega || \sigma) \right\} - \frac{1}{\alpha - 1} \log \text{Tr} \varrho
$$

$$
= D_{\alpha, +\infty}(\varrho \| \sigma) = \frac{1}{\alpha - 1} \log \text{Tr} S e^{\alpha S(\log \varrho) S + (1 - \alpha) S(\log \sigma) S} - \frac{1}{\alpha - 1} \log \text{Tr} \varrho,
$$

where $S := \varrho^0 \& \sigma^0$, and

$$
\omega^U_{\alpha, +\infty}(q || \sigma) := S e^{\alpha S(\log \varrho) S + (1 - \alpha) S(\log \sigma) S} / Q_{\alpha, +\infty}(q || \sigma). \tag{III.85}
$$

Moreover, the above equalities hold also for $\alpha > 1$, provided that $P \neq 0$.

Whether the quantities introduced above have good mathematical properties depends of course on the initial quantum Rényi divergences/relative entropies and the given geometric means. In this paper we analyze two of the above constructions in detail: the relative entropies obtained by the procedure in (III.80) using the Kubo-Ando weighted geometric means (Section IV), and the barycentric Rényi divergences (III.84) (Sections V and VI).

**IV. $\gamma$-WEIGHTED GEOMETRIC RELATIVE ENTROPIES**

Recall that for positive definite operators $\varrho, \sigma \in \mathcal{B}(\mathcal{H}) > 0$ and $\gamma \in (0, 1)$, the Kubo-Ando $\gamma$-weighted geometric mean [32] is defined as

$$
\sigma^{#_{\gamma}} := \sqrt[\gamma]{\varrho^{1-\gamma} \sigma^{1-\gamma}} \sigma^{1-\gamma}, \tag{IV.86}
$$

$$
P_{\alpha}^{#_{\gamma}}(\varrho, \sigma) = P_{\alpha^{#_{\gamma}}}^{#_{\gamma}}(\sigma, \varrho) \tag{IV.87}
$$

$$
\varrho^{#_{\gamma}} := \sqrt[\gamma]{\varrho^{-1} \sigma^{1-\gamma} \varrho^{1-\gamma}} = \varrho^{#_{1-\gamma}} \sigma^{#_{\gamma}}, \tag{IV.88}
$$

(see (II.12) for the equality in (IV.87)), and extended to general PSD operators as

$$
\sigma^{#_{\gamma}} := \lim_{\varepsilon \searrow 0} (\sigma + \varepsilon I)^{#_{\gamma}}(\sigma + \varepsilon I).
$$
The following are well known [32] or easy to see:

\[(\sigma \#_{\gamma} \varrho)^0 = \sigma^0 \land \varrho^0;\]  
\[\sigma \#_{\gamma} \varrho = \varrho \#_{1-\gamma} \sigma;\]  
\[\varrho_1 \leq \varrho_2, \sigma_1 \leq \sigma_2 \implies \sigma_1 \#_{\gamma} \varrho_1 \leq \sigma_2 \#_{\gamma} \varrho_2;\]  
\[\varrho^0 \leq \sigma^0 \implies \sigma \#_{\gamma} \varrho = \sigma^{1/2} \left(\sigma^{-1/2} \varrho \sigma^{-1/2}\right)^{1/\gamma} \sigma^{1/2};\]  
\[\varrho^0 \geq \sigma^0 \implies \sigma \#_{\gamma} \varrho = \varrho^{1/2} \left(\varrho^{-1/2} \sigma \varrho^{-1/2}\right)^{1-\gamma} \varrho^{1/2}.\]

The $\gamma$-weighted geometric means are monotone continuous in the sense that for any functions $(0, +\infty) \ni \varepsilon \mapsto \varrho_\varepsilon \in \mathcal{B}(\mathcal{H})_{\geq 0}$ and $(0, +\infty) \ni \varepsilon \mapsto \sigma_\varepsilon \in \mathcal{B}(\mathcal{H})_{\geq 0}$ that are monotone decreasing in the PSD order,

\[\left(\sigma_\varepsilon \#_{\gamma} \varrho_\varepsilon\right) \searrow (\lim_{\varepsilon \searrow 0} \sigma_\varepsilon) \#_{\gamma} (\lim_{\varepsilon \searrow 0} \varrho_\varepsilon), \quad \gamma \in (0, 1),\]

as $\varepsilon \searrow 0$.

Note that for any $\varrho_k, \sigma_k \in \mathcal{B}(\mathcal{H}_k)_{\geq 0}, k = 1, 2$, and any sequence $\varepsilon_1 > \varepsilon_2 > \ldots \to 0$,

\[\left[(\sigma_1 + \varepsilon_n I) \otimes (\sigma_2 + \varepsilon_n I)\right] \#_{\gamma} \left[(\varrho_1 + \varepsilon_n I) \otimes (\varrho_2 + \varepsilon_n I)\right] = \left[(\sigma_1 + \varepsilon_n I)\#_{\gamma} (\varrho_1 + \varepsilon_n I)\right] \otimes \left[(\sigma_2 + \varepsilon_n I)\#_{\gamma} (\varrho_2 + \varepsilon_n I)\right],\]

and taking the limit $n \to +\infty$ yields, by (IV.94), that

\[(\sigma_1 \otimes \sigma_2)\#_{\gamma} (\varrho_1 \otimes \varrho_2) = (\sigma_1 \#_{\gamma} \varrho_1) \otimes (\sigma_2 \#_{\gamma} \varrho_2).\]

The following is well known; we state it for later use:

**Lemma IV.1** For any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ and any $\gamma \in (0, 1)$,

\[\text{Tr}(\sigma \#_{\gamma} \varrho) \leq \text{Tr} \varrho^\gamma \sigma^{1-\gamma} \leq (\text{Tr} \varrho)^\gamma (\text{Tr} \sigma)^{1-\gamma},\]

and equality holds in the second inequality if and only if $\varrho$ and $\sigma$ are linearly dependent. In particular, for states $\varrho, \sigma \in \mathcal{S}(\mathcal{H})$,

\[\text{Tr}(\sigma \#_{\gamma} \varrho) \leq 1, \quad \text{and} \quad \text{Tr}(\sigma \#_{\gamma} \varrho) = 1 \iff \varrho = \sigma.\]

**Proof** The second inequality in (IV.96) and its equality condition is a simple consequence of Hölder’s inequality; for the first inequality, see, e.g., [23, Example 4.5]. The assertion in (IV.97) follows immediately from (IV.96).

The inequality between the first and the last terms in (IV.96) is a special case of the following lemma. For a proof of the lemma, see, e.g., [23, Proposition 3.30].

**Lemma IV.2** For any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ and any positive linear map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$,

\[\Phi(\varrho \#_{\gamma} \sigma) \leq \Phi(\varrho)\#_{\gamma} \Phi(\sigma).\]

The following is a special case of (III.80):

**Definition IV.3** Let $D^\varrho$ be a quantum relative entropy. For every $\gamma \in [0, 1)$, and every $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$, let

\[D^\varrho_{\#_{\gamma}}(\varrho \| \sigma) := \begin{cases} \frac{1}{1-\gamma} D^\varrho(\varrho \| \sigma \#_{\gamma} \varrho), & \gamma \in (0, 1), \\ D^\varrho(\varrho \| \sigma), & \gamma = 0. \end{cases}\]

We call $D^\varrho_{\#_{\gamma}}$ the $\gamma$-weighted geometric $D^\varrho$.

**Remark IV.4** Note that by (IV.89), $\sigma \#_{\gamma} \varrho = 0$ might happen even if both $\varrho$ and $\sigma$ are quantum states (thus, in particular, are non-zero); in this case the value of $D^\varrho_{\#_{\gamma}}(\varrho \| \sigma)$ is $+\infty$, according to Remark III.15.
Remark IV.5 Assume that $D^q$ satisfies the scaling property (III.65). Then for any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ such that $\varrho^0 \& \sigma^0 \neq 0$,

$$D^{q,\#\gamma}(\varrho\|\sigma) = (\text{Tr} \varrho) \left[ \frac{1}{1 - \gamma} D^q \left( \frac{\varrho}{\text{Tr} \varrho} \left| \frac{\sigma^\# \gamma \varrho}{\text{Tr} \sigma^\# \gamma \varrho} \right| \right) + D_{\gamma}^{\text{max}}(\varrho\|\sigma) \right], \quad \gamma \in (0,1).$$

This is a special case of (III.82).

It is clear that for any $\gamma \in (0,1)$, $D^{q,\#\gamma}$ is a quantum relative entropy. In the following we show how certain properties of $D^q$ are inherited by $D^{q,\#\gamma}$.

(i) If $D^q$ is additive on tensor products then so is $D^{q,\#\gamma}$, due to (IV.95).

(ii) If $D^q$ is block subadditive/superadditive/additive, then so are $D^{q,\#\gamma}$, $\gamma \in (0,1)$. This follows immediately from the fact that for any sequence of projections $P_1, \ldots, P_r$ summing to $I$, we have $(\sum_{i=1}^r P_i \varrho P_i)^\# \gamma (\sum_{i=1}^r P_i \sigma P_i) = \sum_{i=1}^r (P_i \varrho P_i)^\# \gamma (P_i \sigma P_i)$.

(iii) If $D^q$ satisfies the support condition

$$D^q(\varrho\|\sigma) < +\infty \iff \varrho^0 \leq \sigma^0$$

then so do $D^{q,\#\gamma}$, $\gamma \in (0,1)$. Indeed, by (IV.89), $(\sigma^\# \gamma \varrho)^0 = \varrho^0 \& \sigma^0$, and clearly, $\varrho^0 \leq \varrho^0 \& \sigma^0 \iff \varrho^0 \leq \sigma^0$, whence $D^{q,\#\gamma}(\varrho\|\sigma) < +\infty \iff \varrho^0 \leq \sigma^0$.

(iv) If $D^q$ satisfies either of the scaling laws (III.66) or (III.67) then so do $D^{q,\#\gamma}$, $\gamma \in (0,1)$. As a consequence, if $D^q$ satisfies the scaling law (III.65) then so do $D^{q,\#\gamma}$, $\gamma \in (0,1)$. These can be verified by straightforward computations, which we omit.

(v) If $D^q$ satisfies trace monotonicity (III.69) then $D^{q,\#\gamma}$, $\gamma \in (0,1)$, also satisfy trace monotonicity (III.69); moreover, they are strictly positive. Indeed, if $\sigma^\# \gamma \varrho \neq 0$ then

$$D^{q,\#\gamma}(\varrho\|\sigma) = \frac{1}{1 - \gamma} D^q(\varrho\|\sigma^\# \gamma \varrho) \quad \text{(IV.98)}$$

$$\geq \frac{1}{1 - \gamma} \left[ (\text{Tr} \varrho) \log \text{Tr} \varrho - (\text{Tr} \varrho) \log \frac{\text{Tr}(\sigma^\# \gamma \varrho)}{(\text{Tr} \varrho)^{(\text{Tr} \sigma)^{-\gamma}}} \right] \quad \text{(IV.99)}$$

$$\geq \frac{1}{1 - \gamma} \left[ (\text{Tr} \varrho) \log \text{Tr} \varrho - (\text{Tr} \varrho) \log \left( (\text{Tr} \varrho)^\gamma (\text{Tr} \sigma)^{1-\gamma} \right) \right] \quad \text{(IV.100)}$$

$$= (\text{Tr} \varrho) \log \text{Tr} \varrho - (\text{Tr} \varrho) \log \text{Tr} \sigma, \quad \text{(IV.101)}$$

where the first inequality follows from the assumed trace monotonicity of $D^q$, and the second inequality follows from Lemma IV.1. If $\sigma^\# \gamma \varrho = 0$ then either $\varrho = 0$ and thus $D^{q,\#\gamma}(\varrho\|\sigma) = \frac{1}{1 - \gamma} D^q(\varrho\|\sigma) = 0 = D(0\|0) = D(\text{Tr} \varrho \| \text{Tr} \sigma)$, or $\varrho \neq 0$, whence $D^{q,\#\gamma}(\varrho\|\sigma) = \frac{1}{1 - \gamma} D^q(\varrho\|0) = +\infty \geq D(\text{Tr} \varrho \| \text{Tr} \sigma)$ holds trivially. This shows that $D^{q,\#\gamma}$, $\gamma \in (0,1)$, satisfy trace monotonicity, and hence they are also non-negative, according to Remark III.25.

Assume now that $\text{Tr} \varrho = \text{Tr} \sigma = 1$ and $\varrho \neq \sigma$. If $\varrho^0 \& \sigma^0 = 0$ then $D^{q,\#\gamma}(\varrho\|\sigma) = \frac{1}{1 - \gamma} D^q(\varrho\|\sigma) = +\infty > 0$ holds trivially. Otherwise (IV.98)-(IV.101) hold, and the inequality in (IV.100) is strict, according to Lemma IV.1, whence $D^{q,\#\gamma}(\varrho\|\sigma) > 0$ for every $\gamma \in (0,1)$. This proves that $D^{q,\#\gamma}$, $\gamma \in (0,1)$, are strictly positive.

(vi) If $D^q$ satisfies the scaling law (III.67) and it is non-negative then $D^{q,\#\gamma}$, $\gamma \in (0,1)$, are strictly positive. Indeed, let $\varrho, \sigma \in \mathcal{S}(\mathcal{H})$ be unequal states. If $\sigma^\# \gamma \varrho = 0$ then $D^{q,\#\gamma}(\varrho\|\sigma) = \frac{1}{1 - \gamma} D^q(\varrho\|0) = +\infty > 0$ is obvious. Assume for the rest that $\sigma^\# \gamma \varrho \neq 0$. Then

$$(1 - \gamma) D^{q,\#\gamma}(\varrho\|\sigma) = D^q \left( \varrho \left| \frac{\sigma^\# \gamma \varrho}{\text{Tr} \sigma^\# \gamma \varrho} \right| \text{Tr} \sigma^\# \gamma \varrho \right) = D^q \left( \varrho \left| \frac{\sigma^\# \gamma \varrho}{\text{Tr} \sigma^\# \gamma \varrho} \right| \right) - (\text{Tr} \varrho) \log \frac{\text{Tr} \sigma^\# \gamma \varrho}{<1} > 0,$$

where the strict inequality is due to Lemma IV.1.
(vii) If $D^q$ is strongly regular then so are $D^{q,\#_\gamma}$, $\gamma \in (0,1)$. This follows immediately from the fact that if $\sigma_n$, $n \in \mathbb{N}$, is a sequence of PSD operators converging monotone decreasingly to $\sigma$ then $\sigma_n\#_\gamma q$ converges monotone decreasingly to $\sigma\#_\gamma q$; see (IV.94).

(viii) If $D^q$ is anti-monotone in its second argument, then so are $D^{q,\#_\gamma}$, $\gamma \in (0,1)$. Indeed, this follows immediately from (IV.91).

(ix) Assume that $D^q$ is anti-monotone in its second argument. If $q, \sigma \in B(\mathcal{H})_{\geq 0}$ and $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ is a positive linear map such that

$$D^q(\Phi(\varrho)\|\Phi(\sigma)) \leq D^q(\varrho\|\sigma) \quad \text{then} \quad D^{q,\#_\gamma}(\Phi(\varrho)\|\Phi(\sigma)) \leq D^{q,\#_\gamma}(\varrho\|\sigma), \quad \gamma \in (0,1).$$

Indeed,

$$D^{q,\#_\gamma}(\Phi(\varrho)\|\Phi(\sigma)) = \frac{1}{1-\gamma} D^q(\Phi(\varrho)\|\Phi(\sigma)\#_\gamma \Phi(\varrho)) \leq \frac{1}{1-\gamma} D^q(\Phi(\varrho)\|\Phi(\sigma\#_\gamma q)) \leq \frac{1}{1-\gamma} D^q(\varrho\|\sigma\#_\gamma q),$$

where the first inequality follows from Lemma IV.2, the second inequality from anti-monotonicity of $D^q$ in its second argument, and the last inequality from the monotonicity of $D^q$ under the given class of maps. In particular, $D^{q,\#_\gamma}$, $\gamma \in (0,1)$, are monotone under any class of positive maps under which $D^q$ is monotone.

(xi) Since the $\gamma$-weighted geometric means are jointly concave in their arguments [32], it is easy to see that if $D^q$ is anti-monotone in its second argument and it is jointly convex, then so are $D^{q,\#_\gamma}$, $\gamma \in (0,1)$.

Indeed, (IV.102) follows immediately from the definition (III.16) of the ordering of divergences, as does (IV.103) from (III.17), where in the latter case we also need that $\varrho \sigma \neq \sigma q \implies \varrho(\sigma\#_\gamma q) \neq (\sigma\#_\gamma q)\varrho$ for invertible $\varrho, \sigma$. This is straightforward to verify; indeed, by (IV.90),

$$\varrho(\sigma\#_\gamma q) = (\sigma\#_\gamma q)\varrho \iff \varrho^{1/2} (q^{-1/2} \varrho q^{-1/2})^{1-\gamma} q^{1/2} = \varrho^{1/2} (q^{-1/2} \varrho q^{-1/2})^{1-\gamma} q^{1/2} q \iff \varrho (q^{-1/2} \varrho q^{-1/2})^{1-\gamma} q = (q^{-1/2} \varrho q^{-1/2})^{1-\gamma} q \iff \varrho (q^{-1/2} \varrho q^{-1/2}) = (q^{-1/2} \varrho q^{-1/2}) \varrho \iff \varrho(\sigma) = \sigma q.$$

Remark IV.6 By Remark IV.5 and the strict positivity of $D^q_{\max}$, if $D^q$ satisfies the scaling law (III.65) and it is non-negative then $D^{q,\#_\gamma}$ is strictly positive. Note that (v) and (vi) above establish strict positivity of $D^{q,\#_\gamma}$ under slightly weaker conditions.

Remark IV.7 According to Remark III.23, if $D^{q,\#_\gamma}$ is monotone under CPTP maps then

$$D^{\text{meas}} \leq D^{q,\#_\gamma} \leq D_{\max},$$

and if $D^{q,\#_\gamma}$ is also additive on tensor products then

$$D^{U\text{m}} \leq D^{q,\#_\gamma} \leq D_{\max},$$

for any $\gamma \in (0,1)$. In particular,

$$D^{U\text{m}} \leq D^{U\text{m},\#_\gamma} \leq D_{\max,\#_\gamma} \leq D_{\max}, \quad \gamma \in (0,1),$$

where we also used (xi) above.
Lemma IV.8 Assume that $D^q$ is anti-monotone in its second argument and regular. Let $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ be the limits of monotone decreasing sequences $(\varrho_\varepsilon)_{\varepsilon > 0}$ and $(\sigma_\varepsilon)_{\varepsilon > 0}$, respectively. Then, for every $\gamma \in (0,1)$,

$$\frac{1}{1 - \gamma} D^q(\varrho \| \sigma_\varepsilon \#_\gamma \varrho_\varepsilon) \nearrow D^q(\varrho \| \sigma) \quad \text{as} \quad \varepsilon \searrow 0, \quad \text{(IV.106)}$$

$$D^q(\varrho \#_\gamma (\varrho \| \sigma)) \nearrow D^q(\varrho \| \sigma) \quad \text{as} \quad \varepsilon \searrow 0. \quad \text{(IV.107)}$$

**Proof** By the assumptions and Remark III.11, $D^q$ is strongly regular. By (IV.94), both $\sigma_\varepsilon \#_\gamma \varrho_\varepsilon$ and $\sigma_\varepsilon \#_\gamma \varrho$ converge monotone decreasingly to $\sigma \#_\gamma \varrho$ as $\varepsilon \searrow 0$. From these, the assertions follow immediately. \[\square\]

Remark IV.9 (IV.107) is a special case of the anti-monotonicity and strong regularity stated in (viii) and (vii) above.

Corollary IV.10 If $D^q$ is anti-monotone in its second argument, regular, and jointly lower semi-continuous in its arguments, then $D^q(\#_\gamma)$ has the same properties for every $\gamma \in (0,1)$.

**Proof** Anti-monotonicity and (strong) regularity have been covered in (viii) and (vii) above (see also Remark III.11). For every $\varepsilon > 0$, $(\varrho, \sigma) \mapsto (\sigma + \varepsilon I) \#_\gamma (\varrho + \varepsilon I)$ is continuous (due to the continuity of functional calculus), and thus $(\varrho, \sigma) \mapsto \frac{1}{1 - \gamma} D^q(\varrho \| (\sigma + \varepsilon I) \#_\gamma (\varrho + \varepsilon I))$ is lower semi-continuous. Thus, by (IV.106), $(\varrho, \sigma) \mapsto D^q(\varrho \| \sigma)$ is the supremum of lower semi-continuous functions, and therefore is itself lower semi-continuous. \[\square\]

Example IV.11 Combining Remark III.27 and Lemma IV.8 yields that (IV.106)–(IV.107) hold for $D^q = D^{\text{meas}}, D^{\text{Um}}, D^{\text{max}}$, and by Corollary IV.10, $D^{\text{meas},\#_\gamma}, D^{\text{Um},\#_\gamma},$ and $D^{\text{max},\#_\gamma}$, are jointly lower semi-continuous in their arguments for every $\gamma \in (0,1)$.

For a given relative entropy $D^q$, the $\gamma$-weighted relative entropies $D^q(\#_\gamma), \gamma \in (0,1)$, give (potentially) new quantum relative entropies. On the other hand, as Proposition IV.13 below shows, iterating this procedure does not give further quantum relative entropies. We will need the following simple lemma for the proof.

Lemma IV.12 For any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ and any $\gamma_1, \gamma_2 \in (0,1)$,

$$(\sigma \#_{\gamma_2} \varrho) \#_{\gamma_1} \varrho = \sigma \#_{1 - (1 - \gamma_1)(1 - \gamma_2)} \varrho.$$ \quad \text{(IV.108)}$$

**Proof** First, assume that $\varrho, \sigma$ are positive definite. Then the statement follows as

$$(\sigma \#_{\gamma_2} \varrho) \#_{\gamma_1} \varrho = \sigma^{1/2} \left( \varrho^{-1/2} \left( \sigma \#_{\gamma_2} \varrho \right) \varrho^{-1/2} \right)^{1 - \gamma_1} \varrho^{1/2}$$

$$= \sigma^{1/2} \left( \varrho^{-1/2} \left( \varrho^{1/2} \left( \varrho^{-1/2} \sigma \varrho^{-1/2} \right)^{1 - \gamma_2} \varrho^{1/2} \right)^{1 - \gamma_1} \varrho^{1/2} \right)$$

$$= \sigma^{1/2} \left( \varrho^{-1/2} \sigma \varrho^{-1/2} \right)^{(1 - \gamma_1)(1 - \gamma_2)} \varrho^{1/2}$$

$$= \sigma \#_{1 - (1 - \gamma_1)(1 - \gamma_2)} \varrho.$$ \quad \text{(IV.109)}$$

Next, we consider the general case. By the joint concavity and homogeneity of the weighted geometric Kubo-Ando mean (see, e.g., [22, Corollary 3.2.3]),

$$(\sigma + \varepsilon I) \#_{\gamma} (\varrho + \varepsilon I) \geq \sigma \#_{\gamma} \varrho + (\varepsilon I) \#_{\gamma} (\varepsilon I) = \sigma \#_{\gamma} \varrho + \varepsilon I, \quad \varepsilon > 0,$$ \quad \text{(IV.110)}$$

holds for any $\gamma \in (0,1)$. On the other hand,

$$(\sigma + \varepsilon I) \#_{\gamma} (\varrho + \varepsilon I) = \sigma \#_{\gamma} \varrho + (\varrho + \varepsilon I) \#_{\gamma} (\sigma + \varepsilon I) - \sigma \#_{\gamma} \varrho$$

$$\leq \sigma \#_{\gamma} \varrho + I \|(\sigma + \varepsilon I) \#_{\gamma} (\varrho + \varepsilon I) - \sigma \#_{\gamma} \varrho\|_{\infty} = \sigma \#_{\gamma} \varrho + f(\varepsilon)I,$$ \quad \text{(IV.111)}$$

where (IV.109) is satisfied. The required inequality follows from the above. \[\square\]
where \( f : [0, +\infty) \rightarrow [0, +\infty) \) is monotone increasing and \( 0 = f(0) = \lim_{\varepsilon \downarrow 0} f(\varepsilon) \), according to (IV.94).

Thus,
\[
(\sigma \#_{\gamma_1, q} \#_{\gamma_2, q} = \lim_{\varepsilon \downarrow 0} (\sigma \#_{\gamma_2, q} + \varepsilon I) \#_{\gamma_1, q} (q + \varepsilon I)
\leq \lim_{\varepsilon \downarrow 0} ((\sigma + \varepsilon I) \#_{\gamma_2, q} (q + \varepsilon I)) \#_{\gamma_1, q} (q + \varepsilon I)
= \lim_{\varepsilon \downarrow 0} (\sigma + \varepsilon I) \#_{1-(1-\gamma_1)(1-\gamma_2)} (q + \varepsilon I)
= \sigma \#_{1-(1-\gamma_1)(1-\gamma_2)} q,
\]
where the first and the last equalities follow from the monotone continuity (IV.94) of \#, for any \( \gamma \in (0, 1) \), the inequality is due to (IV.110), and the second equality is due to (IV.109). Similarly,
\[
(\sigma \#_{\gamma_1, q} \#_{\gamma_2, q} = \lim_{\varepsilon \downarrow 0} (\sigma \#_{\gamma_2, q} + f(\varepsilon) I) \#_{\gamma_1, q} (q + f(\varepsilon) I)
\geq \lim_{\varepsilon \downarrow 0} ((\sigma + f(\varepsilon) I) \#_{\gamma_2, q} (q + f(\varepsilon) I)) \#_{\gamma_1, q} (q + f(\varepsilon) I)
= \lim_{\varepsilon \downarrow 0} (\sigma + f(\varepsilon) I) \#_{1-(1-\gamma_1)(1-\gamma_2)} (q + f(\varepsilon) I)
= \sigma \#_{1-(1-\gamma_1)(1-\gamma_2)} q,
\]
where the inequality follows from (IV.111), and the rest follow as in the previous argument. This gives (IV.108).

**Proposition IV.13** For any quantum relative entropy \( D^q \), and any \( \gamma_1, \gamma_2 \in (0, 1) \),
\[
D(q \#_{\gamma_1}) \#_{\gamma_2} = D(q \#_{1-(1-\gamma_1)(1-\gamma_2)}).
\]

**Proof** For any \( q, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0} \),
\[
D(q \#_{\gamma_1}) \#_{\gamma_2} (\sigma \| \sigma) = \frac{1}{1 - \gamma_2} D(q \#_{\gamma_1}) (\sigma \| \sigma \#_{\gamma_2} q)
= \frac{1}{1 - \gamma_2} \cdot \frac{1}{1 - \gamma_1} D(q \| (\sigma \#_{\gamma_2} q) \#_{\gamma_1} \sigma)
= \frac{1}{1 - \gamma_2} \cdot \frac{1}{1 - \gamma_1} D(q \| (\sigma \#_{1-(1-\gamma_1)(1-\gamma_2)} \sigma) \| \sigma)
= D(q \#_{\gamma_1}) (\sigma \#_{1-(1-\gamma_1)(1-\gamma_2)} \sigma),
\]
where all equalities apart from the third one are by definition, and the third equality is due to Lemma IV.12.

Next, we study the \( \gamma \)-weighted geometric relative entropies corresponding to the largest and the smallest additive and CPTP-monotone quantum relative entropies, i.e., to \( D^\text{max} \) and \( D^\text{Um} \). The first case turns out to be trivial, while in the second case we get that the \( \gamma \)-weighted geometric Umegaki relative entropies essentially interpolate between \( D^\text{Um} \) and \( D^\text{max} \) in a monotone increasing way.

**Proposition IV.14** For any \( \gamma \in (0, 1) \),
\[
D^\text{max, \#}_{\gamma} = D^\text{max}.
\]

**Proof** We need to show that for any \( q, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0} \),
\[
\frac{1}{1 - \gamma} D^\text{max}(q \| \sigma \#_{\gamma} q) = D^\text{max}(q \| \sigma).
\]
If \( q \not\leq \sigma^0 \) then both sides are \( +\infty \) (see (iii) above), and hence for the rest we assume that\( q \leq \sigma^0 \).
Assume first that \( \varrho \) and \( \sigma \) are invertible. Then

\[
\frac{1}{1 - \gamma} D_{\max}(\varrho|\sigma \# \gamma) = \frac{1}{1 - \gamma} \text{Tr} \varrho \log \left( \varrho^{1/2} (\sigma \# \gamma^{-1}) \varrho^{1/2} \right) = \frac{1}{1 - \gamma} \text{Tr} \varrho \log \left( \varrho^{1/2} \left( \varrho^{-1/2} (\varrho^{1/2} \sigma^{-1} \varrho^{1/2})^{1 - \gamma} \varrho^{-1/2} \right)^{1/2} \right) = \frac{1}{1 - \gamma} \text{Tr} \varrho \log (\varrho^{1/2} \sigma^{-1} \varrho^{1/2}) = D_{\max}(\varrho|\sigma),
\]

where the first equality is due to (III.56), the second equality follows from (IV.90), and the rest are obvious. If we only assume that \( \varrho^0 \leq \sigma^0 \) then we have

\[
D_{\max}(\varrho|\sigma)^\gamma = \lim_{\varepsilon \to 0} \frac{1}{1 - \gamma} D_{\max}(\varrho|\sigma + \varepsilon I \# \gamma (\varrho + \varepsilon I)) = \lim_{\varepsilon \to 0} \text{Tr} \varrho \log \left( (\varrho + \varepsilon I)^{1/2} (\sigma + \varepsilon I)^{-1} (\varrho + \varepsilon I)^{1/2} \right) = \text{Tr} \varrho \log (\varrho^{1/2} \sigma^{-1} \varrho^{1/2}) = D_{\max}(\varrho|\sigma),
\]

where the first equality was stated in Example IV.11, the second equality follows as in (IV.112), the third equality is easy to verify, and the last equality follows by (III.56).

\[\square\]

**Lemma IV.15** Assume that \( D^q \) is a quantum relative entropy such that \( D^q \leq D^q_{\# \gamma} \) for every \( \gamma \in (0, 1) \). Then \( (0, 1) \ni \gamma \mapsto D^q_{\# \gamma} \) is monotone increasing.

**Proof** Let \( 0 < \gamma_1 \leq \gamma_2 < 1 \), and let \( \gamma := \frac{\gamma_2 - \gamma_1}{\gamma_1 - \gamma_2} \), so that \( 1 - \gamma = \frac{1 - \gamma_1}{1 - \gamma_2} \). For any \( \varrho, \sigma \in \mathcal{B}(\mathcal{H}) \geq 0 \),

\[
D^q_{\# \gamma_1}(\varrho|\sigma) = \frac{1}{1 - \gamma_1} D^q(\varrho|\sigma \# \gamma_1 \varrho) \leq \frac{1}{1 - \gamma_1} D^q_{\# \gamma}(\varrho|\sigma \# \gamma_1 \varrho) = \frac{1}{1 - \gamma_1} \frac{1}{1 - \gamma} D^q(\varrho|\sigma \# \gamma_1 \varrho \# \gamma \varrho) = \frac{1}{1 - \gamma_1} \frac{1}{1 - \gamma} D^q(\varrho|\sigma \# \gamma_2 \varrho) = \frac{1}{1 - \gamma_2} \frac{1}{1 - \gamma} D^q(\varrho|\sigma \# \gamma_2 \varrho) = D^q_{\# \gamma_2}(\varrho|\sigma),
\]

where the first, the second, and the last equalities are by definition, the inequality is by assumption, while the third and the fourth equalities follow from Lemma IV.12 and the definition of \( \gamma \). \(\square\)

**Proposition IV.16** Let \( \varrho, \sigma \in \mathcal{B}(\mathcal{H}) \geq 0 \). Then

\[
(0, 1) \ni \gamma \mapsto D^{\text{Um}, \# \gamma}(\varrho|\sigma) \text{ is monotone increasing}, \tag{IV.113}
\]

and if \( \varrho, \sigma \) are positive definite then

\[
D^{\text{Um}, \# \gamma}(\varrho|\sigma) \prec D^{\text{Um}}(\varrho|\sigma), \quad \text{as } \gamma \searrow 0, \tag{IV.114}
\]

\[
D^{\text{Um}, \# \gamma}(\varrho|\sigma) \succ D^{\max}(\varrho|\sigma), \quad \text{as } \gamma \nearrow 1. \tag{IV.115}
\]

**Proof** Since \( D^{\text{Um}} \) is monotone under CPTP maps, so are \( D^{\text{Um}, \# \gamma}, \gamma \in (0, 1) \), according to (ix) above, and hence, by (IV.105), \( D^{\text{Um}} \leq D^{\text{Um}, \# \gamma}, \gamma \in (0, 1) \). By Lemma IV.15, \( (0, 1) \ni \gamma \mapsto D^{\text{Um}, \# \gamma} \) is monotone increasing.

Let \( \varrho, \sigma \in \mathcal{B}(\mathcal{H}) \geq 0 \). Then (IV.114) follows simply by the continuity of functional calculus, so we only have to prove (IV.115). Note that the definition of \( \sigma \# \gamma \varrho \) in (IV.86), and hence also the definition of
$D^\text{Um}(\varrho \| \sigma \#_\gamma \varrho)$ make sense for any $\gamma \in \mathbb{R}$, and both are (infinitely many times) differentiable functions of $\gamma$. Moreover, $D^\text{Um}(\varrho \| \sigma \#_1 \varrho) = D^\text{Um}(\varrho \| \varrho) = 0$. Hence,

\[
\lim_{\gamma \searrow 1} D^\text{Um,\#}_\gamma (\varrho \| \sigma) = - \lim_{\gamma \searrow 1} \frac{D^\text{Um}(\varrho \| \sigma \#_\gamma \varrho) - D^\text{Um}(\varrho \| \sigma \#_1 \varrho)}{\gamma - 1} \\
= \left. - \frac{d}{d\gamma} D^\text{Um}(\varrho \| \sigma \#_\gamma \varrho) \right|_{\gamma = 1} = - \left. \frac{d}{d\gamma} (\text{Tr} \varrho \log \varrho - \text{Tr} \varrho \log (\sigma \#_\gamma \varrho)) \right|_{\gamma = 1} \\
= \left. - \text{Tr} \varrho (D \log) \| \sigma \#_1 \varrho \right|_{\gamma = 1} \left( \frac{d}{d\gamma} (\sigma \#_\gamma \varrho) \right)_{\gamma = 1} \\
= \sum_{i,j} \log^1(\lambda_i, \lambda_j) \left. \text{Tr} \varrho P_i \sigma^{1/2}(\sigma^{-1/2} \varrho \sigma^{-1/2}) \log(\sigma^{-1/2} \varrho \sigma^{-1/2}) \right|_{\sigma = 1/2} P_j \\
= \sum_{i,j} \log^1(\lambda_i, \lambda_j) \left. \text{Tr} P_i \sigma^{-1/2} \log(\sigma^{-1/2} \varrho \sigma^{-1/2}) \right|_{\sigma = 1/2} P_j \\
= \sum_{i} \log^1(\lambda_i, \lambda_i) \left. \text{Tr} P_i \sigma^{-1/2} \log(\sigma^{-1/2} \varrho \sigma^{-1/2}) \right|_{\sigma = 1/2} P_j \\
= \left. \text{Tr} \varrho \sigma^{-1/2} \varrho \sigma^{-1/2} \log(\varrho \sigma^{-1/2}) \right|_{\sigma = 1/2} P_j \\
= D^{\max}(\varrho \| \sigma),
\]

where $P_1, \ldots, P_r \in \mathbb{P}(\mathcal{H})$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ are such that $\sum_{i=1}^r P_i = I$, $\sum_{i=1}^r \lambda_i P_i = \varrho$, and in the fifth equality we used (II.10), while the rest of the steps are straightforward. \hfill \Box

**Corollary IV.17** Let $D^\varrho$ be a quantum relative entropy that is monotone under CPTP maps and additive on tensor products. For any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{>0}$,

\[
\lim_{\gamma \searrow 1} D^{\varrho \#_\gamma}(\varrho \| \sigma) = D^{\max}(\varrho \| \sigma). \tag{IV.116}
\]

If, moreover, $D^\varrho$ is continuous on $\mathcal{B}(\mathcal{H})_{>0} \times \mathcal{B}(\mathcal{H})_{>0}$ then for any $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{>0}$,

\[
\lim_{\gamma \searrow 0} D^{\varrho \#_\gamma}(\varrho \| \sigma) = D^\varrho(\varrho \| \sigma). \tag{IV.117}
\]

In particular, $(D^{\varrho \#_\gamma})_{\gamma \in (0,1)}$ continuously interpolates between $D^\varrho$ and $D^{\max}$ when the arguments are restricted to be invertible.

**Proof** By (IV.104) and the preservation of ordering stated in (xi) above, we have

\[
D^\text{Um,\#}_\gamma (\varrho \| \sigma) \leq D^{\varrho \#_\gamma}(\varrho \| \sigma) \leq D^{\max,\#}_\gamma(\varrho \| \sigma) = D^{\max}(\varrho \| \sigma),
\]

where the last equality follows from Proposition IV.14. Taking the limit $\gamma \searrow 1$ and using (IV.115) yields (IV.116). The limit in (IV.117) is obvious from the assumed continuity and that $\lim_{\gamma \searrow 0} \varrho \#_\gamma \sigma = \sigma$ when $\varrho$ and $\sigma$ are invertible. \hfill \Box

**Remark IV.18** Let $\sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ and $\psi \in \text{ran} \sigma$ be a unit vector. Then

\[
\sigma \#_\gamma |\psi\rangle \langle \psi| = \sigma^{1/2} \left( \sigma^{-1/2} |\psi\rangle \langle \psi| \sigma^{-1/2} \right)^{\gamma} \sigma^{1/2} = |\psi\rangle \langle \psi| \langle \psi, \sigma^{-1}\psi \rangle^\gamma = 1,
\]

whence

\[
D^\text{Um,\#}_\gamma (|\psi\rangle \langle \psi| \| \sigma) = - \frac{1}{1-\gamma} \text{Tr} |\psi\rangle \langle \psi| \log \left( |\psi\rangle \langle \psi| \langle \psi, \sigma^{-1}\psi \rangle^\gamma \langle \psi \| \langle \psi\rangle \langle \psi| \sigma^{-1} \langle \psi\rangle \langle \psi| \right) \\
= \log \langle \psi, \sigma^{-1}\psi \rangle = \text{Tr} |\psi\rangle \langle \psi| \log (|\psi\rangle \langle \psi| \sigma^{-1} \langle \psi\rangle \langle \psi|) \\
= D^{\max}(|\psi\rangle \langle \psi| \| \sigma)
\]

for every $\gamma \in (0,1)$, while

\[
D^\text{Um}(|\psi\rangle \langle \psi| \| \sigma) = - \text{Tr} |\psi\rangle \langle \psi| \log \sigma = \langle \psi, (\log \sigma^{-1}) \psi \rangle.
\]
Thus, we get that
\[ D^{Um}(\psi)(\psi \parallel \sigma) \leq D^{Um,\#_\gamma}(\psi)(\psi \parallel \sigma) = D^{\text{max}}(\psi)(\psi \parallel \sigma), \quad \gamma \in (0, 1), \]
and the inequality is strict if \( \psi \) is not an eigenvector of \( \sigma \). In particular, this shows that the condition that \( \rho \) and \( \sigma \) are invertible cannot be completely omitted in (IV.114).

**Remark IV.19** Obviously, \([0, 1] \ni t \mapsto D^{(t)} := (1 - t) D^{Um} + t D^{\text{max}}\) interpolates continuously and monotone increasingly between \( D^{Um} \) and \( D^{\text{max}} \), and the same is true for \([0, 1] \ni \gamma \mapsto D^{Um,\#_\gamma} \), according to Proposition IV.16. The two families, however, are different. Indeed, the example in Remark IV.18 shows that if a unit vector \( \psi \in \text{ran} \sigma \) is not an eigenvector of \( \sigma \) then
\[ D^{(t)}(\psi)(\psi \parallel \sigma) < D^{\text{max}}(\psi)(\psi \parallel \sigma) = D^{Um,\#_\gamma}(\psi)(\psi \parallel \sigma), \quad t, \gamma \in (0, 1). \]
Since \( D^q(\rho \parallel \sigma) = \lim_{\varepsilon \downarrow 0} D^q(\rho + \varepsilon I \parallel \sigma + \varepsilon I) \) holds for both \( D^q = D^{Um} \) and \( D^q = D^{\text{max}} \) and any \( \rho, \sigma \in B(\mathcal{H})_{\geq 0} \), the above argument also shows that for any \( t, \gamma \in (0, 1) \) there exist invertible \( \rho, \sigma \) such that \( D^{(t)}(\rho \parallel \sigma) < D^{Um,\#_\gamma}(\rho \parallel \sigma) \).

**V. DIVERGENCE RADIi AND BARYCENTRIC RÉNYI DIVERGENCES**

In the rest of the paper (i.e., in the present section and in Section VI) we use the term “quantum relative entropy” in a more restrictive (though still very general) sense than in the previous sections. Namely, a quantum divergence \( D^q \) will be called a quantum relative entropy if, on top of being a quantum extension of the classical relative entropy, it is also non-negative, it satisfies the scaling law (III.65), and the following "support condition":
\[ D^q(\rho \parallel \sigma) < +\infty \iff \rho^0 \leq \sigma^0. \] (V.118)

Note that by Remark III.25, any quantum relative entropy in the above sense is also trace-monotone. In particular, no quantum relative entropy can take the value \( -\infty \).

**Example V.1** It is easy to verify that that \( D^{Um}, D^{\text{meas}} \) and \( D^{\text{max}} \) are all quantum relative entropies in the above more restrictive sense.

**A. Definitions**

**Definition V.2** Let \( W \in B(\mathcal{X}, \mathcal{H})_{\geq 0} \) be a gsc channel, let \( P \in \mathcal{P}_f^+(\mathcal{X}) \) and
\[ S_+ := \bigwedge_{x : P(x) > 0} W^0_x, \quad S_- := \bigwedge_{x : P(x) < 0} W^0_x, \]
and for every \( x \in \mathcal{X} \), let \( D^{q_x} \) be a quantum relative entropy. We define
\[ Q^b_{P,q}(W) := \sup_{\tau \in B(S_+, \mathcal{H})_{\geq 0}} \left\{ \text{Tr} \tau - \sum_{x \in \mathcal{X}} P(x) D^{q_x}(\tau \parallel W_x) \right\}, \] (V.119)
\[ q_{P,q}(W) := \log Q^b_{P,q}(W), \] (V.120)
\[ R_{D^q,\text{left}}(W, P) := \inf_{\omega \in S(S_+, \mathcal{H})} \sum_{x \in \mathcal{X}} P(x) D^{q_x}(\omega \parallel W_x). \] (V.121)

Here, \( q := (q_x)_{x \in \mathcal{X}} \), \( D^q := (D^{q_x})_{x \in \mathcal{X}} \), and \( R_{D^q,\text{left}}(W, P) \) is the \( P \)-weighted left \( D^q \)-radius of the image of \( W \).

**Remark V.3** Note that by definition,
\[ P(x) \geq 0, \ x \in \mathcal{X} \implies S_- = I, \quad P(x) \leq 0, \ x \in \mathcal{X} \implies S_+ = I. \]
Definition V.4 Let $D^q = (D^{q_0}, D^{q_1})$ be quantum relative entropies. For any two non-zero PSD operators $\varrho, \sigma \in B(H)_{\geq 0}$, and any $\alpha \in [0, +\infty)$, let

$$Q^b_{\alpha, q}(\varrho\|\sigma) := \sup_{\tau \in B((\varrho^q H)_{\geq 0})} \{ \text{Tr} \tau - \alpha D^{q_0}(\tau\|\varrho) - (1 - \alpha) D^{q_1}(\tau\|\sigma) \},$$

$$\psi^b_{\alpha, q}(\varrho\|\sigma) := \log Q^b_{\alpha, q}(\varrho\|\sigma),$$

$$D^b_{\alpha, q}(\varrho\|\sigma) := \frac{\psi^b_{\alpha, q}(\varrho\|\sigma) - \psi^b_{\alpha, q}(\varrho\|\sigma)}{\alpha - 1},$$

where we define the last quantity only for $\alpha \in [0, 1) \cup (1, +\infty)$. $D^b_{\alpha, q}(\varrho\|\sigma)$ is called the barycentric Rényi $\alpha$-divergence of $\varrho$ and $\sigma$ corresponding to $D^q$.

Remark V.5 When $D^{q_0} = D^{q_1} = D^q$, we will use the simpler notation $D^b_{\alpha, q}$ instead of $D^b_{\alpha, (q_0, q_1)}$.

Remark V.6 Note that with the choice $X = \{0, 1\}$, $W_0 = \varrho$, $W_1 = \sigma$, and $P(0) = \alpha$, (V.122) and (V.123) are special cases of (V.119) and (V.120), respectively, when $\alpha \in (0, +\infty)$, and we will show in Lemma V.9 that also (V.124) is a special case of (V.121) in this case. When $\alpha = 0$, the restriction $\tau^0 \leq S_+$ in (V.119) would give $\tau^0 \leq \sigma^0$, while we use $\tau^0 \leq \varrho^0$ in (V.122). The reason for this is to guarantee the continuity of $D^b_{\alpha, q}$ at 0; see Proposition V.29.

Remark V.7 It is easy to see that when $P$ is a probability measure, the supremum in (V.119) and the infimum in (V.121) can be equivalently taken over $B(H)_{\geq 0}$ and $S(H)$, respectively, i.e.,

$$Q^b_{P, q}(W) = \sup_{\tau \in B(H)_{\geq 0}} \left\{ \text{Tr} \tau - \sum_{x \in X} P(x) D^{q_0}(\tau\|W_x) \right\},$$

$$R^b_{P, q}(W, P) = \inf_{\varrho \in S(H)_{\geq 0}} \sum_{x \in X} P(x) D^{q_0}(\tau\|W_x),$$

and in the 2-variable case we have

$$Q^b_{\alpha, q}(\varrho\|\sigma) = \sup_{\tau \in B(H)_{\geq 0}} \{ \text{Tr} \tau - \alpha D^{q_0}(\tau\|\varrho) - (1 - \alpha) D^{q_1}(\tau\|\sigma) \},$$

and

$$\sum_{\tau \in B((\varrho^q H)_{\geq 0})} \text{Tr} \tau - \alpha D^{q_0}(\tau\|\varrho) - (1 - \alpha) D^{q_1}(\tau\|\sigma), \quad \alpha \in (0, 1).$$

In the general case, the restriction $\tau^0 \leq S_+$ is introduced to avoid the appearance of infinities of opposite signs in $\sum_{x \in X} P(x) D^{q_0}(\tau\|W_x)$. In the 2-variable case (V.122), the restriction $\tau^0 \leq \varrho^0$ also serves to guarantee that $Q^b_{\alpha, q}$ is a quantum extension of $Q^b_{1, q}$ for $\alpha > 1$, which would not be true, for instance, if it was replaced by $\tau^0 \leq \varrho^0 \wedge \sigma^0$; see Sections VB and VH.

Remark V.8 Note that (V.122) can be seen as a 2-variable extension of the variational formula (III.71). In particular, we have

$$Q^b_{1, q}(\varrho\|\sigma) = \max_{\tau \in B((\varrho^q H)_{\geq 0})} \{ \text{Tr} \tau - D^{q_0}(\tau\|\varrho) \} = \text{Tr} \varrho,$$

where the first equality is by definition (V.122), and the second equality is due to (III.71). Thus,

$$\psi^b_{1, q}(\varrho\|\sigma) = \log \text{Tr} \varrho,$$

and for every $\alpha \in [0, 1) \cup (1, +\infty)$,

$$D^b_{\alpha, q}(\varrho\|\sigma) = \frac{1}{\alpha - 1} \log Q^b_{\alpha, q}(\varrho\|\sigma) - \frac{1}{\alpha - 1} \log \text{Tr} \varrho.$$

By Remark III.26, the maximum in (V.129) is attained at $\tau$ if and only if $\text{Tr} \tau = \text{Tr} \varrho$ and $D^{q_0}(\tau\|\varrho) = 0$; in particular, $\tau = \varrho$ is the unique maximizer in (V.129) when $D^{q_0}$ is strictly positive.

At $\alpha = 0$, (V.122) and (III.71) give

$$\sigma^0 \leq \varrho^0 \implies Q^b_{0, q}(\varrho\|\sigma) = \text{Tr} \sigma \implies D^b_{0, q}(\varrho\|\sigma) = \log \text{Tr} \varrho - \log \text{Tr} \sigma.$$

In general the above equalities do not hold; see Proposition V.35.
Lemma V.9 (i) In the setting of Definition V.2,

\[-\psi^{b,q}_P(W) = R_{D^q,\text{left}}(W, P)\]  

(V.133)

Moreover, if \( S_+ \leq S_- \) then a \( \tau \in D(S_+ \mathcal{H}) \geq 0 \) is optimal in (V.119) if and only if

\[Q^{b,q}_P(W) = \text{Tr} \tau \quad \text{and} \quad \sum_{x \in \mathcal{X}} P(x) \, D^{q_x}(\tau \| W_x) = 0,\]  

(V.134)

and if \( \tau \neq 0 \) is optimal in (V.119) then \( \omega := \tau / \text{Tr} \tau \) is optimal in (V.133). Conversely, for any \( \omega \) that is optimal in (V.133) \( \tau := e^{-R_{D^q,\text{left}}(W, P)} \omega \) is optimal in (V.119).

(ii) In the setting of Definition V.4,

\[\psi^{b,q}_\alpha(\rho \| \sigma) = - \inf_{\omega \in \mathcal{S}(\rho^\alpha \mathcal{H})} \{ \alpha \, D^{q_\alpha}(\omega \| \rho) + (1 - \alpha) \, D^{q_\alpha}(\omega \| \sigma) \}, \quad \alpha \in [0, +\infty).\]  

(V.135)

Assume for the rest that \( \alpha \in [0, 1] \) or \( \rho^0 \leq \sigma^0 \). Then \( \tau \) is optimal in (V.122) if and only if

\[Q^{b,q}_\alpha(\rho \| \sigma) = \text{Tr} \tau \quad \text{and} \quad \alpha \, D^{q_\alpha}(\tau \| \rho) + (1 - \alpha) \, D^{q_\alpha}(\tau \| \sigma) = 0,\]  

(V.136)

and if \( \tau \neq 0 \) is optimal in (V.122) then \( \omega := \tau / \text{Tr} \tau \) is optimal in (V.135). Conversely, for any \( \omega \) that is optimal in (V.135), \( \tau := e^{\psi^{b,q}_\alpha(\rho \| \sigma) \omega} \) optimal in (V.122).

Proof (i) Assume first that \( S_+ = 0 \). Then the only admissible \( \tau \in D(S_+ \mathcal{H}) \geq 0 \) in (V.119) is \( \tau = 0 \), whence \( Q^{b,q}_P(W) = 0 \), according to (III.44), and thus \( \psi^{b,q}_P(W) = -\infty \). On the other hand, the infimum in (V.121) is taken over the empty set, and hence it is equal to +\( \infty \). Thus, (V.133) and (V.134) hold.

Assume next that \( S_+ \neq 0 \). If there exists an \( x \in \mathcal{X} \) such that \( P(x) < 0 \) and \( S_+ \not\in W_x \) then taking \( \tau := \omega := S_+/\text{Tr} S_+ \) yields

\[Q^{b,q}_P(W) = \text{Tr} \tau - \sum_{x: P(x) > 0} P(x) \, D^{q_x}(\tau \| W_x) - \sum_{x: P(x) < 0} P(x) \, D^{q_x}(\tau \| W_x) = +\infty,\]  

(\( \in \mathbb{R} \))

and

\[R_{D^q,\text{left}}(W, P) \leq \sum_{x: P(x) > 0} P(x) \, D^{q_x}(\omega \| W_x) + \sum_{x: P(x) < 0} P(x) \, D^{q_x}(\omega \| W_x) = -\infty,\]  

(\( \in \mathbb{R} \))

(where we used that \( D^{q_x} \) does not take the value \(-\infty\)), whence (V.133) holds.

Finally, if \( 0 \neq S_+ \leq S_- \) then the proof follows easily from representing a positive semi-definite operator \( \tau \in D(S_+ \mathcal{H}) \geq 0 \) as a pair \( (\omega, t) \in \mathcal{S}(S_+ \mathcal{H}) \times [0, +\infty) \). Indeed, we have

\[Q^{b,q}_P(W) = \sup_{\omega \in \mathcal{S}(S_+ \mathcal{H})} \sup_{t \in [0, +\infty)} \left\{ \text{Tr} \, t \omega - \sum_{x \in \mathcal{X}} P(x) \, D^{q_x}(t \omega \| W_x) \right\} = c(\omega),\]  

(V.137)

where the first equality is by definition, and the second equality follows from the scaling property (III.66). Note that \( c(\omega) \neq \pm \infty \) by assumption, and the inner supremum in (V.137) is equal to \( e^{-c(\omega)} \), attained at \( t = e^{-c(\omega)} \), according to Lemma II.1. From these, all the remaining assertions in (i) follow immediately.

The assertions in (ii) are special cases of the corresponding ones in (i) when \( \alpha \in (0, +\infty) \) (also taking into account (V.128) when \( \alpha \in (0, 1) \)). The case \( \alpha = 0 \) can be verified analogously to the above; we omit the easy details. \( \square \)

Remark V.10 Clearly, when \( \alpha > 1 \) and \( \rho^0 \not\leq \sigma^0 \) then the set of optimal \( \tau \) operators in (V.122) is exactly \( D(\rho^0 \mathcal{H}) \geq 0 \), and the set of optimal \( \omega \) states in (V.135) is exactly \( \mathcal{S}(\rho^0 \mathcal{H}) \).
Corollary V.11 Assume that the supremum in (V.119) is a maximum. Then

\[ Q^b,q_{\alpha}(W) = \max \left\{ \text{Tr} \tau : \tau \in B(\mathcal{H})_{\geq 0}, \sum_{x \in X} P(x) D^{q_\tau}(\tau||W_x) = 0 \right\}. \]

Likewise, if the supremum in (V.122) is a maximum, and \( \alpha \in [0,1] \) or \( \varrho^0 \leq \sigma^0 \), then

\[ Q^b,q_{\alpha}(\varrho||\sigma) = \max \left\{ \text{Tr} \tau : \tau \in B(\varrho^0 \mathcal{H})_{\geq 0}, \alpha D^{q_\tau}(\varrho||\sigma) + (1 - \alpha) D^{q_\tau}(\tau||\sigma) = 0 \right\}. \]

Proof Immediate from the characterizations of the optimal \( \tau \) in (V.134) and (V.136). \( \square \)

Remark V.12 Under the conditions given in Lemma V.9, for the supremum in (V.119) to be a maximum, it is sufficient if the infimum in (V.133) is a minimum. For the latter, a natural sufficient condition is that each \( D^{q_\tau} \) with \( x \in \text{supp} P \) is lower semi-continuous in its first argument (when \( P \) is a probability measure), or continuous in its first argument with its support dominated by the support of a fixed second argument (when \( P \) can take negative values), since the domain of optimization, namely, \( B(\varrho^0 \mathcal{H}) \), is a compact set.

Examples of quantum relative entropies that are lower semi-continuous in their first argument (in fact, in both of their arguments), include \( D^{\text{max}} \), \( D^{\text{min}} \), and their \( \gamma \)-weighted versions, as well \( D^{\text{max}} \), and obviously, all possible convex combinations of these. \( D^{\text{min}} \) and \( D^{\text{max}} \) are also clearly continuous in their first argument when its support is dominated by the support of a fixed second argument.

Remark V.13 Using (V.130) and the scaling law (III.67), (V.124) can be rewritten as

\[ D^b,q_{\alpha}(\varrho||\sigma) = \frac{1}{\alpha - 1} \log Q^b,q_{\alpha}(\varrho||\sigma) - \frac{1}{\alpha - 1} \log \text{Tr} \varrho \]

On the other hand, using Lemma V.9 we get that for every \( \alpha \in [0,1) \cup (1, +\infty) \),

\[ D^b,q_{\alpha}(\varrho||\sigma) = \frac{1}{1 - \alpha} \log Q^b,q_{\alpha}(\varrho||\sigma) - \frac{1}{1 - \alpha} \log \text{Tr} \varrho \]

where the first equality is by (V.131), the second equality follows from (V.135), and the third and the fourth equalities from the scaling laws (III.66)–(III.67). Moreover, for \( \alpha \in (0,1) \), the infimum can be taken over \( \mathcal{S}(\mathcal{H}) \), i.e.,

\[ D^b,q_{\alpha}(\varrho||\sigma) = \frac{1}{1 - \alpha} \log Q^b,q_{\alpha}(\varrho||\sigma) - \frac{1}{1 - \alpha} \log \text{Tr} \varrho \]

because if \( \omega^0 \notin \varrho^0 \), then \( D^{q_\tau}(\omega||\varrho) = D^{q_\tau}(\omega||\tau) = +\infty \). The situation is different for \( \alpha = 0 \); see, e.g., (V.160).

The formulas in (V.139)–(V.141) explain the term “barycentric Rényi divergence”.

B. Barycentric Rényi divergences are quantum Rényi divergences

In this section we show that the barycentric Rényi \( \alpha \)-divergences are quantum Rényi divergences for every \( \alpha \in (0,1) \), provided that the defining quantum relative entropies are monotone under pinchings. This latter condition does not pose a serious restriction: indeed, all the concrete quantum relative entropies that we consider in this paper (e.g., measured, Umegaki, maximal, and the \( \gamma \)-weighted versions of these) are monotone under PTP maps, and hence also under pinchings.

Isometric invariance holds even without this mild restriction, and also for \( \alpha > 1 \):
Lemma V.14 All the quantities in (V.119)–(V.124) are invariant under isometries, and hence are all quantum divergences.

Proof We prove the statement only for $Q_P^{b,a}$, as for the other quantities it either follows from that, or the proof goes the same way. Let $W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0}$ be a gcq channel, $P \in \mathcal{P}_f(\mathcal{X})$, and $V : \mathcal{H} \to \mathcal{K}$ be an isometry. Obviously, $\hat{S}_+ := \wedge_x P(x) > 0 (V W_x V^*)^0 = V(\wedge_x P(x) > 0 W_x^0) V^* = V S_+ V^*$, and for any $\tau \in \mathcal{B}(\hat{S}_+ \mathcal{H})_{\geq 0}$ there exists a unique $\hat{\tau} \in \mathcal{B}(S_+ \mathcal{H})_{\geq 0}$ such that $\tau = V \hat{\tau} V^*$. Thus,

$$Q_P^{b,a}(V W V^*) = \sup_{\tau \in \mathcal{B}(\hat{S}_+ \mathcal{K})_{\geq 0}} \left\{ \text{Tr} \tau - \sum_{x \in \mathcal{X}} P(x) D^{q_2}(\tau \| V W_x V^*) \right\}$$

(V.144)

$$= \sup_{\hat{\tau} \in \mathcal{B}(S_+ \mathcal{H})_{\geq 0}} \left\{ \text{Tr} V \hat{\tau} V^* - \sum_{x \in \mathcal{X}} P(x) D^{q_2}(V \hat{\tau} V^* \| V W_x V^*) \right\}$$

(V.145)

$$= \sup_{\hat{\tau} \in \mathcal{B}(S_+ \mathcal{H})_{\geq 0}} \left\{ \text{Tr} \hat{\tau} - \sum_{x \in \mathcal{X}} P(x) D^{q_2}(\tau \| W_x) \right\}$$

(V.146)

$$= Q_P^{b,a}(W),$$

(V.147)

where the third equality follows by the isometric invariance of the relative entropies. $\square$

Recall that $D^q$ is said to be monotone under pinchings if

$$D^q \left( \sum_{i=1}^r P_i \bar{q}_i \| \sum_{i=1}^r P_i \bar{q}_i P_i \right) \leq D^q(\bar{q} \| \bar{q})$$

for any $\bar{q}, \bar{q} \in \mathcal{B}(\mathcal{H})_{\geq 0}$ and $P_1, \ldots, P_r \in \mathcal{P}(\mathcal{H})$ such that $\sum_{i=1}^r P_i = I$.

Lemma V.15 Let $W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0}$ be a gcq channel that is classical on the support of some $P \in \mathcal{P}_f(\mathcal{X})$, i.e., there exists a ONB $(e_i)_{i=0}^{d-1}$ in $\mathcal{H}$ such that $W_x = \sum_{i=0}^{d-1} \tilde{W}_x(i) |e_i\rangle \langle e_i|$, where $\tilde{W}_x(i) := \langle e_i, W_x e_i\rangle$, $i \in [d]$, $x \in \text{supp} P$. If all $D^{q_2}, x \in \text{supp} P$, are monotone under pinchings then

$$Q_P^{b,a}(W) = \sum_{i \in \tilde{S}} \prod_{x \in \text{supp} P} \tilde{W}_x(i)^{P(x)} ,$$

(V.148)

where $\tilde{S} := \bigcap_{x \in \text{supp} P} \text{supp} \tilde{W}_x$ and sup $\tilde{W}_x = \{ i \in [d] : \tilde{W}_x(i) > 0 \}$; moreover, there exists a unique optimal $\tau$ in (V.119), given by

$$\tau^b_P(W) := \tau_P(\tilde{W}) := \sum_{i \in \tilde{S}} |e_i\rangle \langle e_i| \prod_{x \in \text{supp} P} \tilde{W}_x(i)^{P(x)} .$$

(V.149)

Proof If $S_+ = 0$ then $Q_P^{b,a}(W) = 0$, and the RHS of (V.148) is an empty sum, whence the equality in (V.148) holds trivially.

Thus, for the rest we assume that $S_+ \neq 0$. Let $\mathcal{E}(\cdot) := \sum_{i=1}^d |e_i\rangle \langle e_i| \cdot |e_i\rangle \langle e_i|$ be the pinching corresponding to the joint eigenbasis of the $W_x$, $x \in \text{supp} P$, guaranteed by the classicality assumption. For any $\tau \in \mathcal{B}(S_+ \mathcal{H})_{\geq 0}$,

$$\text{Tr} \tau - \sum_{x \in \mathcal{X}} P(x) \frac{D^{q_2}(\tau \| W_x)}{\geq D^{q_2}(\mathcal{E}(\tau) \| \mathcal{E}(W_x))} \leq \text{Tr} \tau - \sum_{x \in \mathcal{X}} P(x) D^{q_2}(\mathcal{E}(\tau) \| \mathcal{E}(W_x)) = \text{Tr} \mathcal{E}(\tau) - \sum_{x \in \mathcal{X}} P(x) D^{q_2}(\mathcal{E}(\tau) \| \mathcal{E}(W_x)),$$

where the inequality follows from the monotonicity of the $D^{q_2}$ under pinchings. Thus, the supremum in (V.119) can be restricted to $\tau$ operators that can be written as $\tau = \sum_{i=1}^d \tilde{\tau}(i) |e_i\rangle \langle e_i|$ with some
\[
\tilde{\tau}(i) \in [0, +\infty), \ i \in [d].\] Clearly, \(\tau^0 \leq S^0_+\) is equivalent to \(\text{supp} \tilde{\tau} \subseteq \tilde{S}\). For any such \(\tau\),
\[
\text{Tr} \tau - \sum_{x \in \mathcal{X}} P(x) D^{q^*}(\tau\|W_x) = \text{Tr} \tau - \sum_{x \in \mathcal{X}} P(x) \sum_{i \in S} [\hat{\tau}(i) \log \hat{\tau}(i) - \tilde{\tau}(i) \log W_x(i)]
= \sum_{i \in S} [\hat{\tau}(i) - \tilde{\tau}(i) \log \tilde{\tau}(i) + \tau(i) \sum_{x \in \text{supp} P} P(x) \log \tilde{W}_x(i)].
\]
The supremum of this over all such \(\tau\) is
\[
\sum_{i \in S} e^{\sum_{x \in \text{supp} P} P(x) \log \tilde{W}_x(i)} = \sum_{i \in S} \prod_{x \in \text{supp} P} \tilde{W}_x(i)^{P(x)}
\]
which is uniquely attained at the \(\tau = \tau^q_P(W)\) given in (V.149), according to Lemma II.1. This proves (V.148).

**Corollary V.16** In the setting of Lemma V.15, the \(P\)-weighted left \(D^q\)-radius of the image of \(W\) can be given explicitly as
\[
R_{D^q,\text{left}}(W, P) = -\log \sum_{i \in S} \prod_{x \in \text{supp} P} \tilde{W}_x(i)^{P(x)},
\]
and if \(\tilde{S} \neq \emptyset\) then there is a unique \(\omega\) attaining the infimum in (V.121) (called the \(P\)-weighted left \(D^q\)-center), given by
\[
\omega^q_P(W) := \frac{\tau^q(W, P)}{\text{Tr} \tau^q(W, P)} = \sum_{i \in S} |e_i\rangle \langle e_i| \frac{\prod_{x \in \text{supp} P} \tilde{W}_x(i)^{P(x)}}{\sum_{j \in S} \prod_{x \in \text{supp} P} W_x(j)^{P(x)}} := \omega_P(\tilde{W}). \quad (V.150)
\]

**Proof** Immediate from Lemmas V.9 and V.15. \(\Box\)

Lemma V.15 yields immediately the following:

**Corollary V.17** Assume that \(\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}\) commute, and hence can be written as \(\varrho = \sum_{i = 1}^d \tilde{\sigma}(i) |e_i\rangle \langle e_i|\), \(\sigma = \sum_{i = 1}^d \tilde{\sigma}(i) |e_i\rangle \langle e_i|\), in some ONB \((e_i)_{i = 1}^d\). If \(D^{q^0}\) and \(D^{q^1}\) are monotone under pinchings then
\[
Q^a_\varrho,\sigma(\varrho\|\sigma) = Q_\alpha(\tilde{\sigma}\|\tilde{\sigma}) = \sum_{i = 1}^d \tilde{\sigma}(i)^\alpha \tilde{\sigma}(i)^{1-\alpha}, \quad \alpha \in (0, 1),
\]
and there exists a unique optimal \(\tau\) in (V.122), given by
\[
\tau^a_\varrho(\varrho\|\sigma) := \tau_\alpha(\tilde{\sigma}\|\tilde{\sigma}) := \sum_{i = 1}^d |e_i\rangle \langle e_i| \tilde{\sigma}(i)^\alpha \tilde{\sigma}(i)^{1-\alpha}. \quad (V.151)
\]

As a special case of Corollary V.16, we get the following:

**Corollary V.18** In the setting of Corollary V.17, if \(\varrho^0 \land \sigma^0 \neq 0\) then for every \(\alpha \in (0, 1)\) there exists a unique \(\omega\) attaining the infima in (V.138)–(V.141), given by
\[
\omega^q_\varrho(\varrho\|\sigma) := \frac{\tau^q_\varrho(\varrho\|\sigma)}{\text{Tr} \tau^q_\varrho(\varrho\|\sigma)} = \sum_{i \in S} |e_i\rangle \langle e_i| \frac{\tilde{\sigma}(i)^\alpha \tilde{\sigma}(i)^{1-\alpha}}{\sum_{j \in S} \tilde{\sigma}(j)^\alpha \tilde{\sigma}(j)^{1-\alpha}} := \omega_\alpha(\tilde{\sigma}\|\tilde{\sigma}). \quad (V.152)
\]

**Proof** Immediate from Corollary V.17 and Lemma V.9. \(\Box\)

**Remark V.19** Note that \(\tau^q_\varrho(\varrho\|\sigma)\) in (V.151) and \(\omega^q_\varrho(\varrho\|\sigma)\) in (V.152) are independent of \(D^{q^0}\) and \(D^{q^1}\), as long as both of them are monotone under pinchings.

Lemma V.14 and Corollary V.17 together give the following:
Proposition V.20 If $D^{q_0}$ and $D^{q_1}$ are two quantum relative entropies that are monotone under pinchings then for every $\alpha \in (0, 1)$ the corresponding barycentric Rényi $\alpha$-divergence $D^{b, q}_\alpha$ is a quantum Rényi $\alpha$-divergence in the sense of Definition III.13.

Remark V.21 Note that in the classical case the barycentric Rényi $\alpha$-divergence is equal to the unique classical Rényi $\alpha$-divergence also for $\alpha > 1$; see (III.37). On the other hand, if $D^{q_0} \neq D^{q_1}$ then it may happen that $D^{b, q}_\alpha$ is not a quantum Rényi $\alpha$-divergence for some $\alpha > 1$; see Remark V.30.

Note that for a fixed $i \in \cap_{x \in \text{supp} \ P} \text{supp} \ W_x$, the expression $\prod_{x \in \text{supp} \ P} W_x(i)^{P(x)}$ in (V.148) is the weighted geometric mean of $(W_x(i))_{x \in \text{supp} \ P}$ with weights $(P(x))_{x \in \text{supp} \ P}$. This motivates the following:

Definition V.22 If $D^q$, $W$, and $P$ are such that there exists a unique optimizer $\tau =: \tau^q_P(W)$ in (V.119) then this $\tau$ is called the $P$-weighted $D^q$-geometric mean of $(W_x)_{x \in \text{supp} \ P}$, and is also denoted by $G^D(q \mid P) := \tau^q_P(W)$.

Similarly, if there exists a unique optimizer $\tau =: \tau^q(P \mid \sigma)$ in (V.122) then it is called the $\alpha$-weighted $D^q$-geometric mean of $\varrho$ and $\sigma$, and is also denoted by $G^D_{\alpha}(\varrho \mid \sigma) := \tau^q_{\alpha}(\varrho \mid \sigma)$.

Remark V.23 Note that if $G^D_{\alpha}(\varrho \mid \sigma)$ exists then $Q^{b, q}_{\alpha}(\varrho \mid \sigma) = \text{Tr} G^D_{\alpha}(\varrho \mid \sigma)$, which can be seen as a special case of (III.74).

In classical statistics, the family of states $(\omega_{\alpha}(\varrho \mid \sigma))_{\alpha \in (0, 1)}$ given in (V.152) is called the Hellinger arc. (Note that if $\tilde{\varrho}$ and $\tilde{\sigma}$ are probability distributions with equal supports then the Hellinger arc connects them in the sense that $\lim_{\alpha \downarrow 0} \omega_{\alpha}(\tilde{\varrho} \mid \tilde{\sigma}) = \tilde{\varrho}$, $\lim_{\alpha \uparrow 1} \omega_{\alpha}(\tilde{\varrho} \mid \tilde{\sigma}) = \tilde{\sigma}$.) This motivates the following:

Definition V.24 Assume that $\varrho, \sigma \in B(\mathcal{H}) \geq 0$ and $D^{q_0}$, $D^{q_1}$ are such that for every $\alpha \in (0, 1)$ there exists a unique state $\omega =: \omega_{\alpha}^q(\varrho \mid \sigma)$ attaining the infimum in (V.135) (equivalently, in (V.138)–(V.141)). Then $(\omega_{\alpha}^q(\varrho \mid \sigma))_{\alpha \in (0, 1)}$ is called the $D^q$-Hellinger arc for $\varrho$ and $\sigma$.

Example V.25 As it was already mentioned in Example III.34, [41, Theorem 3.6] shows that if $D^{q_0} = D^{q_1} = D^{q_m}$ and $\varrho, \sigma \in B(\mathcal{H}) \geq 0$ are such that $\varrho^0 \wedge \sigma^0 \neq 0$ then the corresponding Hellinger arc exists, and is given by (III.85). The exact same proof as in [41, Theorem 3.6] yields that if $D^{q_m} = D^{q_m^c}$, $x \in \text{supp} \ P$, and $S_+ \leq S_-$, then (V.119) has the unique maximizer

$$
\tau^{q_m}_{\alpha} P(W) = S_+ e^{\sum_{x \in \text{supp} \ P} S_+(\log W_x) S_+},
$$

and consequently, (V.121) has the unique maximizer

$$
\omega^{q_m}_{\alpha} P(W) = \frac{S_+ e^{\sum_{x \in \text{supp} \ P} S_+(\log W_x) S_+}}{\text{Tr} S_+ e^{\sum_{x \in \text{supp} \ P} S_+(\log W_x) S_+}}.
$$

Regarding Definition V.22, there are at least two natural questions. First, whether for given $D^q$, $W$ and $P$, $\tau^q P(W)$ exists and whether it can be expressed by an explicit formula. Second, whether for a given non-commutative $P$-weighted geometric mean $G_P$ (this can be defined for more than 2 variables analogously to Definition III.29), there exist quantum relative entropies $D^q$ such that $G_P(W) = \tau^q P(W)$ for any gcq channel $W$. We leave these questions for future work.

C. Homogeneity and scaling

Note that the normalized relative entropies $D^{q_0}$ and $D^{q_1}$ satisfy the scaling property (III.64) by assumption. This property is inherited by all the corresponding barycentric Rényi divergences $D^{b, q}_\alpha$. More generally, we have the following:
Lemma V.26 For any \( P \in \mathcal{P}^+_f(\mathcal{X}) \), any gcq channel \( W \in \mathcal{B}(\mathcal{X}, \mathcal{H}) \geq 0 \) and any \( t \in (0, +\infty)^X \),

\[
Q^b_{P}((t_x W_x)_{x \in X}) = \left( \prod_{x \in \text{supp} P} t^P_x \right) Q^b_{P}(W),
\]

(\text{V.153})

\[
-\psi^b_{P}((t_x W_x)_{x \in X}) = -\psi^b_{P}(W) - \sum_x P(x) \log t_x.
\]

(\text{V.154})

In particular, \( Q^b_{\alpha} \) is homogeneous.

\textbf{Proof} (V.154) is straightforward to verify from the definition (V.120), and the scaling law (III.65), and (V.153) follows immediately from it.

Corollary V.27 The barycentric Rényi divergences satisfy the scaling law (III.64), i.e.,

\[
D^b_{\alpha}(t \| s) = D^b_{\alpha}(\| s) + \log t - \log s,
\]

(\text{V.155})

for every \( \varrho, \sigma \in \mathcal{B}(\mathcal{H}) \geq 0 \), \( t, s \in (0, +\infty) \), \( \alpha \in [0, +\infty] \).

\textbf{Proof} Immediate from Lemma V.26, or alternatively, from (V.141).

D. Monotonicity in \( \alpha \) and limiting values

Monotonicity in the parameter \( \alpha \) is a characteristic property of the classical Rényi divergences, which is inherited by the measured, the regularized measured, and the maximal Rényi divergences, and it also holds for the Petz-type Rényi divergences. The representations in Lemma V.9 and Remark V.13 show that barycentric Rényi divergences have the same monotonicity property.

Proposition V.28 (i) For any \( W \in \mathcal{B}(\mathcal{X}, \mathcal{H}) \geq 0 \), the maps

\[ P \mapsto Q^b_{P}(W) \quad \text{and} \quad P \mapsto \log Q^b_{P}(W) \]

are convex, and

\[ P \mapsto R_{D^b_{\alpha}, \text{left}}(W, P) \]

is concave, on \( \mathcal{P}^+_f(\mathcal{X}) \).

(ii) For any fixed \( \varrho, \sigma \in \mathcal{B}(\mathcal{H}) \geq 0 \),

\[ \alpha \mapsto \psi^b_{\alpha}(\| \sigma) \quad \text{is convex on } [0, +\infty), \quad \text{and} \]

\[ \alpha \mapsto D^b_{\alpha}(\| \sigma) \quad \text{is monotone increasing on } [0, 1) \cup (1, +\infty) \).

\textbf{Proof} (i) By Definition V.121, \( P \mapsto R_{D^b_{\alpha}, \text{left}}(W, P) \) is the infimum of affine functions in \( P \), and hence it is concave. By (V.133), this implies the convexity of \( P \mapsto \log Q^b_{P}(W) \), from which the convexity of \( P \mapsto Q^b_{P}(W) \) follows immediately.

(ii) By (V.135), \( \alpha \mapsto \psi^b_{\alpha}(\| \sigma) \) is the supremum of affine functions, and hence convex. (The convexity on \( (0, +\infty) \) also follows as a special case of the above.) The monotonicity of \( \alpha \mapsto D^b_{\alpha}(\| \sigma) \) follows from this convexity by definition (V.124).

\[ D^b_{1} = \sup_{\alpha \in (0, 1)} D^b_{\alpha}(\| \sigma) = \lim_{\alpha \searrow 1} D^b_{\alpha}(\| \sigma), \]

(\text{V.156})

\[ D^b_{1+} = \inf_{\alpha > 1} D^b_{\alpha}(\| \sigma) = \lim_{\alpha \searrow 1} D^b_{\alpha}(\| \sigma), \]

(\text{V.157})

\[ D^b_{\infty} = \sup_{\alpha > 1} D^b_{\alpha}(\| \sigma) = \lim_{\alpha \searrow +\infty} D^b_{\alpha}(\| \sigma), \]

(\text{V.158})

where the equalities follow from the monotonicity established in Proposition V.28. Using the representations in Remark V.13, it is easy to show the following:
Proposition V.29 Let $\varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$.

(i) We have

\[ D_{0}^{h, q}(\varrho, \sigma) = \inf_{\alpha \in (0, 1)} D_{\alpha}^{h, q}(\varrho, \sigma) = \lim_{\alpha \downarrow 0} D_{\alpha}^{h, q}(\varrho, \sigma) \quad \text{(V.159)} \]

\[ = \inf_{\omega \in \mathcal{S}(\mathcal{H})} \left\{ D^{q_{1}} \left( \omega \left\| \sigma \right\|_{\text{Tr} \varrho} \right) \right\} + \log \text{Tr} \varrho - \log \text{Tr} \sigma, \quad \text{(V.160)} \]

\[ D_{\infty}^{h, q}(\varrho, \sigma) = \sup_{\omega \in \mathcal{S}(\mathcal{H})} \left\{ D^{q_{1}} \left( \omega \left\| \sigma \right\|_{\text{Tr} \varrho} \right) - D^{q_{0}} \left( \omega \left\| \varrho \right\|_{\text{Tr} \varrho} \right) \right\} + \log \text{Tr} \varrho - \log \text{Tr} \sigma \quad \text{(V.161)} \]

\[ = \sup_{\omega \in \mathcal{S}(\mathcal{H})} \left\{ D^{q_{1}} (\omega \left\| \sigma \right\|) - D^{q_{0}} (\omega \left\| \varrho \right\|) \right\}. \quad \text{(V.162)} \]

(ii) If $D^{q_{0}}$ is strictly positive and $D^{q_{0}} (\cdot \left\| \varrho \right\|)$ and $D^{q_{1}} (\cdot \left\| \sigma \right\|$ are both lower semi-continuous on $\mathcal{S}(\mathcal{H})$ then

\[ D_{1}^{h, q}(\varrho, \sigma) = \frac{1}{\text{Tr} \varrho} D^{q_{1}} (\varrho \left\| \sigma \right\|). \quad \text{(V.163)} \]

If, moreover, $D^{q_{1}} (\cdot \left\| \sigma \right\|$ is upper semi-continuous on $\mathcal{S}(\mathcal{H})$ then

\[ D_{1}^{h, q} = D_{1}^{h, q}(\varrho, \sigma). \quad \text{(V.164)} \]

Proof (i) The second equality in (V.159) is obvious from the monotonicity established in Proposition V.28, and the rest of the equalities in (V.159)–(V.160) follow as

\[ \inf_{\alpha \in (0, 1)} D_{\alpha}^{h, q}(\varrho, \sigma) = \inf_{\alpha \in (0, 1)} \inf_{\omega \in \mathcal{S}(\mathcal{H})} \left\{ \frac{\alpha}{1 - \alpha} D^{q_{0}} \left( \omega \left\| \frac{\varrho}{\text{Tr} \varrho} \right\| \right) + D^{q_{1}} \left( \omega \left\| \frac{\sigma}{\text{Tr} \sigma} \right\| \right) \right\} + \log \text{Tr} \varrho - \log \text{Tr} \sigma \]

where the first equality is due to (V.141), the second equality is trivial, the third equality follows from the non-negativity of $D^{q_{0}}$, the fourth equality follows from the scaling law (III.64), and the last equality by definition.

The equalities in (V.161)–(V.162) follow as

\[ \sup_{\alpha > 1} D_{\alpha}^{h, q}(\varrho, \sigma) = \sup_{\alpha > 1} \sup_{\omega \in \mathcal{S}(\mathcal{H})} \left\{ \frac{\alpha}{1 - \alpha} D^{q_{0}} \left( \omega \left\| \frac{\varrho}{\text{Tr} \varrho} \right\| \right) + D^{q_{1}} \left( \omega \left\| \frac{\sigma}{\text{Tr} \sigma} \right\| \right) \right\} + \log \text{Tr} \varrho - \log \text{Tr} \sigma \]

where the first equality is due to (V.141), the second one is trivial, the third one follows from the non-negativity of $D^{q_{0}} \left( \omega \left\| \frac{\varrho}{\text{Tr} \varrho} \right\| \right)$, and the last equality is due to the scaling property (III.67).
The equality in (V.163) follows as

\[
\sup_{\alpha \in (0,1)} D^{b,\alpha}_\alpha(\varrho||\sigma) = \sup_{\alpha \in (0,1)} \inf_{\sigma \in S(\varrho^*\varrho)} \left\{ \frac{\alpha}{1-\alpha} D^{\varrho_0}(\omega || \frac{\varrho}{\Tr \varrho}) + D^{\varrho_1}(\omega || \sigma) \right\} + \log \Tr \varrho
\]

where the first equality is due to (V.140), and the second one follows from the minimax theorem in Lemma II.2, using the fact that \(\alpha \mapsto \frac{\alpha}{1-\alpha}\) is monotone increasing on \((0,1)\). The third equality follows from the fact that \(\sup_{\alpha \in (0,1)} \frac{\alpha}{1-\alpha} = +\infty\), and that \(D^{\varrho_0}\) is strictly positive, and the last equality is immediate from the scaling law (III.66).

Finally, to prove (V.164), first note that if \(\varrho^0 \nleq \sigma^0\) then \(D^{b,\alpha}_\alpha(\varrho||\sigma) = +\infty\) for every \(\alpha > 1\); indeed, by (V.139),

\[
D^{b,\alpha}_\alpha(\varrho||\sigma) \geq \frac{\alpha}{1-\alpha} D^{\varrho_0}(\varrho||\varrho) + D^{\varrho_1}(\varrho||\varrho) - \frac{1}{\alpha-1} \log \Tr \varrho = +\infty.
\]  

(V.165)

(See also Corollary V.34.) Hence,

\[
D^{1,\alpha}_\alpha(\varrho||\sigma) = \inf_{\alpha > 1} D^{b,\alpha}_\alpha(\varrho||\sigma) = \inf_{\alpha > 1} +\infty = +\infty = \frac{1}{\Tr \varrho} D^{\varrho_1}(\varrho||\sigma) = D^{b,\alpha}_1(\varrho||\sigma),
\]

where the fourth equality is due to the support condition (V.118), and the last equality is (V.163). Hence, for the rest we assume that \(\varrho^0 \leq \sigma^0\), so that \(D^{\varrho_1}(\omega || \varrho)\) and \(D^{\varrho_1}(\omega || \sigma)\) are both finite for \(\omega \in S(\varrho^0\varrho)\). Moreover, by assumption, \(\omega \mapsto \frac{\alpha}{1-\alpha} D^{\varrho_0}(\omega || \varrho) + D^{\varrho_1}(\omega || \sigma)\) is upper semi-continuous on \(S(\varrho^0\varrho)\) for every \(\alpha > 1\). Then

\[
D^{b,\alpha}_\alpha(\varrho||\sigma) = \inf_{\alpha > 1} D^{b,\alpha}_\alpha(\varrho||\sigma)
\]

\[
= \inf_{\alpha > 1} \sup_{\omega \in S(\varrho^0\varrho)} \left\{ \frac{\alpha}{1-\alpha} D^{\varrho_0}(\omega || \frac{\varrho}{\Tr \varrho}) + D^{\varrho_1}(\omega || \sigma) \right\} + \log \Tr \varrho
\]

\[
= \sup_{\omega \in S(\varrho^0\varrho)} \inf_{\alpha > 1} \left\{ \frac{\alpha}{1-\alpha} D^{\varrho_0}(\omega || \varrho) + D^{\varrho_1}(\omega || \sigma) \right\} + \log \Tr \varrho
\]

\[
= D^{\varrho_1}(\varrho || \sigma) + \log \Tr \varrho = \frac{1}{\Tr \varrho} D^{\varrho_1}(\varrho || \sigma) = D^{b,\alpha}_1(\varrho||\sigma),
\]

where the second equality is due to (V.140), the third one follows from the minimax theorem in Lemma II.2, the fourth equality follows from the strict positivity of \(D^{\varrho_0}\) and the fact that \(\inf_{\alpha > 1} \alpha/(1-\alpha) = -\infty\), the fifth one is due to the scaling law (III.66), and the last equality is (V.163).

\[\square\]

**Remark V.30** Clearly, for any \(\varrho \in \mathcal{B}(\mathcal{H})_{>0}\) and any quantum Rényi \(\alpha\)-divergence \(D^{b}_\alpha(\varrho||\varrho) = 0\). On the other hand, if \(D^{\varrho_0}\) and \(D^{\varrho_1}\) are such that for any two invertible non-commuting \(\omega_1, \omega_2 \in \mathcal{B}(\mathcal{H})_{>0}\), \(D^{\varrho_0}(\omega_1||\omega_2) > D^{\varrho_0}(\omega_1||\omega_2)\), then by (V.162), \(D^{b,\alpha}_\alpha(\varrho||\varrho) > 0\), \(\varrho \in \mathcal{B}(\mathcal{H})_{>0}\), and hence \(D^{b,\alpha}_\alpha\) is not a quantum Rényi \(\alpha\)-divergence for any large enough \(\alpha\). For instance, this is the case if \(D^{\varrho_0} = D^{\varrho_1}\) and \(D^{\varrho_0} = D^{\varrho_1}\); see, e.g., [23, Proposition 4.7].

**Remark V.31** It is well known and easy to verify that for commuting states \(\varrho\) and \(\sigma\), the unique Rényi \(\alpha\)-divergences satisfy

\[Q_\alpha(\varrho||\sigma) = Q_{1-\alpha}(\sigma||\varrho), \quad \alpha \in (0,1).\]

As a consequence, if \(D^{q}_{1-\alpha}\) is a quantum Rényi \((1-\alpha)\)-divergence for some \(\alpha \in (0,1)\) then

\[
\tilde{D}^{q}_{\alpha}(\varrho||\sigma) := \frac{1}{1-\alpha} \left[ \alpha D^{q}_{1-\alpha}(\varrho||\sigma) + \log \Tr \varrho - \log \Tr \sigma \right]
\]
defines a quantum Rényi α-divergence. Given a collection \((D_\alpha^q)_{\alpha \in (0,1)}\) of quantum Rényi α-divergences, we call \((\hat{D}_\alpha^q)_{\alpha \in (0,1)}\) the dual collection. The measured, the regularized measured and the maximal Rényi divergences are easily seen to be self-dual, as are the Petz-type (or standard) Rényi divergences (see Section III.C for the definitions).

For the barycentric Rényi divergences, it is straightforward to verify from (V.142) that
\[
D_\alpha^{b,(q_0,q_1)}(\varrho \| \sigma) = D_\alpha^{b,(q_1,q_0)}(\varrho \| \sigma).
\]

In particular, \(D_\alpha^{b,(q_0,q_1)}\) is self-dual when \(D_\alpha^{q_0} = D_\alpha^{q_1}\).

Combining this duality with (V.163) we obtain that if \(D_\alpha^{q_1}\) is strictly positive and \(D_\alpha^{q_1}(\cdot \| \|_\mathfrak{T})\) and \(D_\alpha^{q_0}(\cdot \| \sigma)\) are both lower semi-continuous on \(\mathcal{S}(\varrho \| \mathcal{H})\) then
\[
\frac{1}{\text{Tr} \varrho} D_\alpha^{q_0}(\varrho \| \sigma) = \lim_{\alpha \nearrow 1} D_\alpha^{b,(q_1,q_0)}(\varrho \| \sigma) = \lim_{\alpha \nearrow 1} \hat{D}_\alpha^{b,(q_0,q_1)}(\varrho \| \sigma) = \lim_{\alpha \rightarrow 0, \alpha > 0} \left[ \left(1 - \alpha\right) D_\alpha^{b,(q_0,q_1)}(\varrho \| \sigma) + \log \text{Tr} \varrho - \log \text{Tr} \sigma \right]. \tag{V.166}
\]

Due to this and Proposition V.29, if both \(D_\alpha^{q_0}\) and \(D_\alpha^{q_1}\) are strictly positive and lower semi-continuous in their first variable, then they can both be recovered from \((D_\alpha^{b,q})_{\alpha \in (0,1)}\) by taking limits at \(\alpha \searrow 0\) and at \(\alpha \nearrow 1\), respectively. In particular, if \((D_\alpha^{q_0}, D_\alpha^{q_1}) \neq (D_\alpha^{0,q_1}, D_\alpha^{0,q_1})\) then \(D_\alpha^{b,(q_0,q_1)} \neq D_\alpha^{b,(q_0,q_1)}\) for \(\alpha\) close enough to 0 from above or to 1 from below.

E. Non-negativity and finiteness

Here we show that the barycentric Rényi divergences are pseudo-distances in the sense that \(D_\alpha^{b,q}\) is non-negative for every \(\alpha \in [0, +\infty]\), and it is strictly positive for every \(\alpha \in (0, +\infty]\) under some mild conditions on \(D_\alpha^{q_0}\) and \(D_\alpha^{q_1}\).

We start more generally with giving bounds on the multi-variate Rényi quantities in Definition V.2, for which the following easy observation will be useful.

**Lemma V.32** Let \(D^q\) be a quantum relative entropy. For any \(\sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}\) and any state \(\omega \in \mathcal{S}(\mathcal{H})\),
\[
- \log \text{Tr} \sigma \leq D^q(\varrho \| \sigma). \tag{V.167}
\]

If, moreover, \(D^q\) is anti-monotone in its second argument, then
\[
D^q(\varrho \| \sigma) \begin{cases} 
\leq - \log \lambda_{\min}(\sigma) + \text{Tr} \omega \log \omega & , \quad \omega^0 \leq \sigma^0, \\
= +\infty, & \text{otherwise.} \tag{V.168}
\end{cases}
\]

**Proof** The inequality in (V.167) is immediate from the trace monotonicity of \(D^q\), and \(\omega^0 \nleq \sigma^0 \implies D^q(\omega \| \sigma) = +\infty\) in (V.168) follows from the support condition (V.118). Hence, we are left to prove the upper bound in (V.168).

Let \(\lambda_{\min}(\sigma)\) denote the smallest non-zero eigenvalue of \(\sigma\). By assumption, \(D^q\) is anti-monotone in its second argument, whence \(\sigma \geq \lambda_{\min}(\sigma)\sigma^0\) implies that
\[
\begin{align*}
D^q(\omega \| \sigma) & \leq D^q(\omega \| \lambda_{\min}(\sigma)\sigma^0) \\
& = - \log \lambda_{\min}(\sigma) + D^q(\omega \| \sigma^0) \\
& = - \log \lambda_{\min}(\sigma) + D(\omega \| \sigma^0) \\
& = \text{Tr} \omega \log \omega, \\
& = - \log \lambda_{\min}(\sigma) + \text{Tr} \omega \log \omega,
\end{align*}
\]
where \(D\) is the unique extension of the classical relative entropy to commuting pairs of operators (see Lemma III.3), the first equality follows from the scaling property (III.67), the second equality from the fact that \(\omega\) and \(\sigma^0\) commute due to the assumption that \(\omega^0 \leq \sigma^0\), and the last equality is by the definition of \(D\). \(\Box\)
Proposition V.33  (i) Let \( P \in \mathcal{P}_f(\mathcal{X}) \). Then, for any \( W \in \mathcal{B}(\mathcal{X}, \mathcal{H}) \geq 0 \),

\[
- \sum_x P(x) \log \text{Tr} W_x \leq -\psi_{P,q}^b(W),
\]

and if each \( D^{q_x} \), \( x \in \text{supp} P \), is anti-monotone in its second argument then

\[
-\psi_{P,q}^b(W) \begin{cases} 
\leq \sum_x P(x)(-\log \lambda_{\text{min}}(W_x)) < +\infty, & S_+ \neq 0, \\
= +\infty, & \text{otherwise}.
\end{cases}
\]

In particular,

\[
-\psi_{P,q}^b(W) = +\infty \iff S_+ = 0 \iff Q_{P,q}^b(W) = 0.
\]

(ii) Assume that \( P \in \mathcal{P}_f^+(\mathcal{X}) \) is such that \( P(x) < 0 \) for some \( x \in \mathcal{X} \). If for each \( x \) such that \( P(x) > 0 \), \( D^{q_x} \) is anti-monotone in its second argument then

\[
\sum_{x: P(x)>0} P(x) \log \lambda_{\text{min}}(W_x) + \sum_{x: P(x)<0} P(x) \log \text{Tr} W_x \leq \psi_{P,q}^b(W).
\]

If for each \( x \) such that \( P(x) < 0 \), \( D^{q_x} \) is anti-monotone in its second argument then

\[
\psi_{P,q}^b(W) \begin{cases} 
\leq \sum_{x: P(x)>0} P(x) \log \text{Tr} W_x + \sum_{x: P(x)<0} P(x) \log \lambda_{\text{min}}(W_x), & S_+ \leq S_-, \\
= +\infty, & \text{otherwise};
\end{cases}
\]

in particular,

\[
\psi_{P,q}^b(W) = +\infty \iff S_+ \leq S_- \iff Q_{P,q}^b(W) = +\infty.
\]

**Proof** The inequalities in (V.169), (V.170), (V.172), and (V.173) are obvious from (V.133), (V.167), and (V.168), and (V.171) and (V.174) follow immediately. \( \square \)

As a special case, we get the following characterization of the finiteness of the 2-variable barycentric Rényi divergences. This gives an easy way to show that certain quantum Rényi divergences cannot be represented as barycentric Rényi divergences, as we show in Section V H.

**Corollary V.34** We have

\[
D_{\alpha}^{b,q}(\varrho\|\sigma) = +\infty \begin{cases} 
\iff \varrho^0 \land \sigma^0 = 0, & \text{when } \alpha \in (0,1), \\
\iff \varrho^0 \not\leq \sigma^0, & \text{when } \alpha > 1.
\end{cases}
\]

If \( D^{\sigma^0} \) is anti-monotone in its second argument then the one-sided implication above is also an equivalence.

**Proof** The case \( \alpha \in (0, +\infty) \) is immediate from Proposition V.33. The case \( \alpha = 0 \) follows similarly; we leave the details to the reader. \( \square \)

Finally, we turn to the strict positivity of the 2-variable barycentric Rényi divergences.

**Proposition V.35** Let \( \varrho, \sigma \in \mathcal{B}(\mathcal{H}) \geq 0 \).

(i) For every \( \alpha \in [0, +\infty] \),

\[
D_{\alpha}^{b,q}(\varrho\|\sigma) \geq \log \text{Tr} \varrho - \log \text{Tr} \sigma.
\]

(ii) If \( \sigma^0 \leq \varrho^0 \) then (V.176) holds with equality for \( \alpha = 0 \). If \( D^{\sigma^0} \) is strictly positive and \( D^{\sigma^0}(\cdot\|\cdot) \) is lower semi-continuous on \( S(\sigma^0) \) then \( \sigma^0 \leq \varrho^0 \) is also necessary for equality in (V.176) for \( \alpha = 0 \).

(iii) We have (a) \( \implies \) (b) \( \iff \) (c) \( \iff \) (d) in the following:

(a) Equality holds in (V.176) for every \( \alpha \in [0, +\infty] \).

(b) Equality holds in (V.176) for some \( \alpha \in (0, +\infty] \).
(c) Equality holds in (V.176) for every $\alpha \in [0, 1]$.

(d) $\|\rho/\sigma\|_{\text{Tr}} = \|\sigma/\tau\|_{\text{Tr}}$.

If $D^{\rho_0}$ and $D^{\rho_1}$ are strictly positive, and $D^{\rho_0}(\|\rho/\sigma\|_{\text{Tr}} \rho)$ and $D^{\rho_1}(\|\sigma/\tau\|_{\text{Tr}} \tau)$ are lower semi-continuous on $S(\rho^0 \mathcal{H})$, then we have (a) $\iff$ (b) $\iff$ (c) $\iff$ (d).

On the other hand, if $D^{\rho_0} = D^{\rho_1}$ then the implication (d) $\implies$ (a) holds.

**Proof**

(i) The inequality in (V.176) for $\alpha \in [0, 1)$ is immediate from (V.141) and the non-negativity of $D^{\rho_0}$ and $D^{\rho_1}$. From this (V.176) follows also for $\alpha \in [1, +\infty)$, using the monotonicity in Proposition V.28.

(ii) Let $\alpha = 0$. If $\rho^0 \leq \rho^1$ then (V.176) holds with equality, according to (V.132). Assume now that $D^{\rho_0}(\|\sigma\|)$ is lower semi-continuous on $S(\rho^0 \mathcal{H})$, so that there exists an $\omega_0 \in S(\rho^0 \mathcal{H})$ such that $\inf_{\omega \in S(\rho^0 \mathcal{H})} D^{\rho_0}(\omega ) = D^{\rho_0}(\omega_0)$. Now, if $D^{\rho_1}$ is strictly positive then, by (V.160), $D^{\rho_1}(\rho_0) > \text{Tr} \rho - \text{Tr} \sigma$ unless $\omega_0 = \rho/\sigma$ or $\nu = \rho/\sigma$, which in turn implies that $\rho^0 = \rho^1$. If $\rho^0 = \rho^1$ then choosing $\omega := \rho/\sigma$ yields that the infimum in (V.141) is zero, and hence equality holds in (V.176), for every $\alpha \in [0, 1)$, and also for $\alpha = 1$, by taking the limit. This proves (d) $\implies$ (c).

Next, we prove (b) $\implies$ (d) under the assumption that $D^{\rho_0}$ and $D^{\rho_1}$ are strictly positive, and $D^{\rho_0}(\|\rho/\sigma\|_{\text{Tr}} \rho)$ and $D^{\rho_1}(\|\sigma/\tau\|_{\text{Tr}} \tau)$ are lower semi-continuous on $S(\rho^0 \mathcal{H})$. Note that equality in (V.176) for some $\alpha \in (0, +\infty)$ implies equality in (V.176) for some $\alpha \in (0, 1)$, due to the inequality in (V.176) and the monotonicity in Proposition V.28. In particular, the infimum in (V.141) is zero. By the semi-continuity assumptions, there exists an $\omega_0$ where that infimum is attained. Strict positivity of $D^{\rho_0}$ and $D^{\rho_1}$ then implies $\omega_0 = \rho/\sigma$, and $\inf_{\omega \in S(\rho^0 \mathcal{H})} D^{\rho_0}(\omega ) = D^{\rho_0}(\omega_0)$, proving (d).

Finally, if $D^{\rho_0} = D^{\rho_1}$ and $\rho/\sigma = \sigma/\tau$ then (V.176), (V.161), and the monotonicity established in Proposition V.28, yield that for every $\alpha \in [0, +\infty)$,

$$\text{log Tr} \rho - \text{log Tr} \tau \leq D^{\rho_0}(\|\rho/\sigma\|_{\text{Tr}} \rho) \leq D^{\rho_1}(\|\rho/\sigma\|_{\text{Tr}} \rho) = \text{log Tr} \rho - \text{log Tr} \sigma,$$

proving the implication (d) $\implies$ (a).

Proposition V.35 yields immediately the following:

**Corollary V.36**

For every $\alpha \in [0, +\infty)$, $D^{\rho_0}_{\alpha}$ is non-negative. If $D^{\rho_0}$ and $D^{\rho_1}$ are both strictly positive and lower semi-continuous in their first arguments then $D^{\rho_0}_{\alpha}$ is strictly positive for every $\alpha \in (0, +\infty]$.

**Remark V.37**

Using the scaling property (V.155), (V.176) can be rewritten equivalently as

$$D^{\rho_0}_{\alpha} \left( \frac{\rho}{\text{Tr} \rho} \left\| \frac{\sigma}{\text{Tr} \sigma} \right\| \right) \geq 0, \quad \alpha \in [0, +\infty].$$

**Remark V.38**

Note that (V.176) tells exactly that the trace-monotonicity of $D^{\rho_0}$ and $D^{\rho_1}$ is inherited by all the generated barycentric Rényi divergences $D^{\rho_0}_{\alpha}$, $\alpha \in [0, +\infty]$.

Since the barycentric Rényi divergences satisfy the scaling property (III.64) according to Corollary V.27, their non-negativity is in fact equivalent to trace-monotonicity, as noted in Remark III.25.

**F. Monotonicity under CPTP maps and joint convexity**

One of the most important properties of quantum divergences is monotonicity under quantum operations (i.e., CPTP maps). Many of the important quantum divergences are monotone under more general trace-preserving maps, e.g., dual Schwarz maps in the case of Petz-type Rényi divergences for $\alpha \in [0, 2]$ [48], or PTP maps in the case of the sandwiched Rényi divergences for $\alpha \geq 1/2$ [5, 28, 44], and the measured as well as the maximal Rényi divergences for $\alpha \in [0, +\infty]$, by definition. It is easy to see that for $\alpha \in [0, 1]$, the barycentric Rényi $\alpha$-divergences are monotone under the same class of PTP maps as their generating quantum relative entropies. More generally, we have the following:

**Proposition V.39**

If all $D^{\rho_x}$, $x \in \mathcal{X}$, are monotone non-increasing under a trace non-decreasing positive map $\Phi \in P^+(\mathcal{H}, \mathcal{K})$ then $Q^{\rho_{\Phi}}$ is monotone non-decreasing, and $R_{D^{\rho_x},\text{left}}$ is monotone non-increasing under $\Phi$, i.e., for every gscq channel $W \in \mathcal{B}(\mathcal{X}, \mathcal{H})_{\geq 0}$ and every $P \in \mathcal{P} \mathcal{Y}(\mathcal{X})$,

$$Q^{\rho_{\Phi}}(\Phi(W)) \geq Q^{\rho_{\Phi}}(W), \quad R_{D^{\rho_x},\text{left}}(\Phi(W), P) \leq R_{D^{\rho_x},\text{left}}(W, P).$$
Proof We have
\[ Q^{b,q}_\alpha(\Phi(W)) = \sup_{\tau \in \mathcal{B}(\mathcal{H}) \geq 0} \left\{ \text{Tr} \tau - \sum_{x \in \mathcal{X}} P(x) D^{q_x}(\tau \| \Phi(W_x)) \right\} \]
\[ \geq \sup_{\tau \in \mathcal{B}(\mathcal{H}) \geq 0} \left\{ \text{Tr} \Phi(\tau) - \sum_{x \in \mathcal{X}} P(x) D^{q_x}(\Phi(\tau) \| \Phi(W_x)) \right\} \]
\[ \geq \sup_{\tau \in \mathcal{B}(\mathcal{H}) \geq 0} \left\{ \text{Tr} \tau - \sum_{x \in \mathcal{X}} P(x) D^{q_x}(\tau \| W_x) \right\} \]
\[ = Q^{b,q}_\alpha(W), \]
where the equalities are by definition (V.119) and by (V.125), the first inequality is obvious, and the second one follows from the assumptions. This proves (V.177), and (V.178) follows immediately by Lemma V.9. □

Proposition V.40 If \( D^{q_0} \) and \( D^{q_1} \) are monotone under a trace non-decreasing map \( \Phi \in \mathcal{P}^+(\mathcal{H}, \mathcal{K}) \) then for every \( \alpha \in [0,1] \), \( Q^{b,q}_\alpha \) is monotone non-decreasing under \( \Phi \), i.e., for every \( \varrho, \sigma \in \mathcal{B}(\mathcal{H}) \geq 0 \),
\[ Q^{b,q}_\alpha(\Phi(\varrho) \| \Phi(\sigma)) \geq Q^{b,q}_\alpha(\varrho \| \sigma). \] (V.179)

If, moreover, \( \Phi \) is trace-preserving, then for every \( \alpha \in [0,1] \), \( D^{b,q}_\alpha \) is monotone non-increasing under \( \Phi \), i.e., for every \( \varrho, \sigma \in \mathcal{B}(\mathcal{H}) \geq 0 \),
\[ D^{b,q}_\alpha(\Phi(\varrho) \| \Phi(\sigma)) \leq D^{b,q}_\alpha(\varrho \| \sigma). \] (V.180)

Vice versa, if \( D^{q_0} \) and \( D^{q_1} \) are strictly positive, and lower semi-continuous in their first variables, and \( D^{b,q}_\alpha, \alpha \in (0,1), \) are monotone non-increasing under a trace-preserving positive map \( \Phi \in \mathcal{P}^+(\mathcal{H}, \mathcal{K}) \), then \( D^{q_0} \) and \( D^{q_1} \) are monotone non-increasing under the same map.

Proof Proposition V.39 yields (V.179) as a special case when \( \alpha \in (0,1] \), and the case \( \alpha = 0 \) follows by a trivial modification of the proof. From this, (V.180) follows immediately. The last assertion follows due to (V.163) and (V.166). □

Remark V.41 By assumption, both \( D^{q_0} \) and \( D^{q_1} \) are trace-monotone, and hence so are \( D^{b,q}_\alpha, \alpha \in [0,1], \) according to Proposition V.40 above. This gives an alternative proof of (V.176).

Remark V.42 The above proof for Proposition V.39 only works when \( P \) is a probability measure (i.e., there is no \( x \) such that \( P(x) < 0 \)), which translates to \( \alpha \in [0,1] \) in Proposition V.40. These conditions cannot be removed in general; indeed, it was shown in [41, Lemma 3.17] that \( D^{b,U_{\alpha}} \) is not monotone under CPTP maps (in fact, not even under pinching) for any \( \alpha > 1 \), even though \( D^{U_{\alpha}} \) is monotone.

Proposition V.43 Let \( P \in \mathcal{P}_f(\mathcal{X}) \), and assume that at least one of the following holds:

(i) \( D^{q_x}, x \in \text{supp} P, \) are monotone under partial traces and are block subadditive.

(ii) \( D^{q_x}, x \in \text{supp} P, \) are jointly convex in their variables.

Then \( W \mapsto Q^{b,q}_P(W) \) and \( W \mapsto \psi^{b,q}_P(W) \) are concave, and \( W \mapsto R_{D^{b,q}_{\text{left}}}(W;P) \) is convex on \( \mathcal{B}(\mathcal{X}, \mathcal{H}) \geq 0 \).

Proof Since all \( D^{q_x} \) satisfy the scaling law (III.65), they are in particular homogeneous, and thus, by Lemma III.9, (i) implies (ii). Assume therefore (ii). Then
\[ \mathcal{B}(\mathcal{H}) \geq 0 \ni \tau \mapsto \text{Tr} \tau - \sum_x P(x) D^{q_x}(\tau \| W_x) \]
is jointly concave in \( \tau \) and \( W \), and hence its supremum over \( \tau \) is concave in \( W \). This proves the concavity of \( W \mapsto Q^{b,q}_P(W) \). The concavity of \( W \mapsto \psi^{b,q}_P(W) \) then follows immediately, which in turn implies the convexity of \( W \mapsto R_{D^{b,q}_{\text{left}}}(W;P) \) due to (V.133). □
As a special case, we get the following:

**Corollary V.44** Let $\alpha \in (0,1)$, and assume that at least one of the following holds:

(i) $D^{\alpha}, x \in \{0,1\}$, are monotone under partial traces and are block subadditive.

(ii) $D^{\alpha}, x \in \{0,1\}$, are jointly convex in their variables.

Then $(\varrho, \sigma) \mapsto Q^b_{\alpha,q}(\varrho, \sigma)$ and $(\varrho, \sigma) \mapsto \psi^b_{\alpha,q}(\varrho, \sigma)$ are concave, and $(\varrho, \sigma) \mapsto D^b_{\alpha,q}(\varrho, \sigma)$ is convex on $B(\mathcal{X}, \mathcal{H})_{\geq 0}$. The same conclusions hold if $\alpha = 0$ and the properties in (i) or (ii) are assumed for $D^{\alpha}$, or if $\alpha = 1$ and the properties in (i) or (ii) are assumed for $D^{\alpha}$.

**G. Lower semi-continuity and regularity**

Note that when $\mathcal{X}$ is finite then $B(\mathcal{X}, \mathcal{H}) = B(\mathcal{H})^\mathcal{X}$ is a finite-dimensional vector space, and hence there exists a unique norm topology on it, which is what we implicitly refer to in statements about (semi-)continuity on this space.

**Proposition V.45** If $P \in \mathcal{P}_f(\mathcal{X})$ and $D^{\alpha}, x \in \text{supp } P$, are all jointly lower semi-continuous in their variables then $W \mapsto Q^b_P(W)$ is upper semi-continuous, and $W \mapsto R_{D^\alpha_{left}}(W, P)$ is lower semi-continuous on $B(\text{supp } P, \mathcal{H})_{\geq 0}$.

**Proof** By assumption,

$$B(\text{supp } P, \mathcal{H})_{\geq 0} \times \mathcal{S}(\mathcal{H}) \ni (W, \omega) \mapsto \sum_{x \in \mathcal{X}} P(x)D^{\alpha}(\omega\|W_x)$$

is lower semi-continuous, and hence, by Lemma II.3, its infimum over $\omega \in \mathcal{S}(\mathcal{H})$ is lower semi-continuous, too, proving the assertion about $R_{D^\alpha_{left}}$. The assertion about $Q^b_P$ then follows immediately due to (V.133).

**Corollary V.46** If $D^{\alpha}$ and $D^{\alpha}$ are jointly lower semi-continuous in their variables then for any $\alpha \in (0,1)$, $Q^b_{\alpha,q}$ is jointly upper semi-continuous and $D^b_{\alpha,q}$ is jointly lower semi-continuous in their arguments.

**Proof** The case $\alpha \in (0,1)$ follows as a special case of Proposition V.45. For $\alpha = 1$, $Q^b_{1,q}(\varrho\|\sigma) = \text{Tr } \varrho$ by (V.129), and continuity holds trivially, while $D^b_{1,q}$ is the supremum of lower semi-continuous functions according to the above and (V.156), and hence is itself lower semi-continuous.

**Proposition V.47** Let $P \in \mathcal{P}_f(\mathcal{X})$ and assume that $D^{\alpha}, x \in \text{supp } P$, are weakly anti-monotone in their second arguments. Then

$$\begin{align*}
(0, +\infty) \ni \varepsilon & \mapsto Q^b_P((W_x + \varepsilon I)_{x \in \mathcal{X}}) \quad \text{is monotone increasing,} & (V.181) \\
(0, +\infty) \ni \varepsilon & \mapsto R^b_{D^\alpha_{left}}((W_x + \varepsilon I)_{x \in \mathcal{X}}) \quad \text{is monotone decreasing} & (V.182)
\end{align*}$$

for any $W \in B(\mathcal{X}, \mathcal{H})_{\geq 0}$. If, moreover, $D^{\alpha}, x \in \text{supp } P$, are regular in the sense of (III.27), and lower semi-continuous in their first arguments, then for any $W \in B(\mathcal{X}, \mathcal{H})_{\geq 0}$,

$$\begin{align*}
Q^b_P(W) & = \lim_{\varepsilon \searrow 0} Q^b_P((W_x + \varepsilon I)_{x \in \mathcal{X}}) = \inf_{\varepsilon > 0} Q^b_P((W_x + \varepsilon I)_{x \in \mathcal{X}}), & (V.183) \\
R^b_{D^\alpha_{left}}(W) & = \lim_{\varepsilon \searrow 0} R^b_{D^\alpha_{left}}((W_x + \varepsilon I)_{x \in \mathcal{X}}) = \sup_{\varepsilon > 0} R^b_{D^\alpha_{left}}((W_x + \varepsilon I)_{x \in \mathcal{X}}). & (V.184)
\end{align*}$$
Proof The monotonicity assertions in (V.181)–(V.182) are obvious, as are the second equalities in (V.183)–(V.184). The first equality in (V.184) follows as
\[
\sup_{\varepsilon > 0} R_{\epsilon}^{b}(W_{x} + \varepsilon I) = \sup_{\varepsilon > 0} \int_{x = x}^{n} P(x)D_{\varepsilon}^{b}(\omega \| W_{x} + \varepsilon I) = \sup_{\omega \in S(\mathcal{S}_x \mathcal{H})} \sum_{x} P(x)D_{\varepsilon}^{b}(\omega \| W_{x} + \varepsilon I) = \sup_{\omega \in S(\mathcal{S}_x \mathcal{H})} \sum_{x} P(x)D_{\varepsilon}^{b}(\omega \| W_{x}) = R_{\epsilon}^{b}(W),
\]
where the first and the last equalities are by definition, the second equality follows from the minimax theorem in Lemma II.2, and the third equality from the regularity of the $D_{\varepsilon}^{b}$. From this, the first equality in (V.183) follows by (V.133).

In the 2-variable case we have the following:

**Corollary V.48** Assume that $D_{\varepsilon}^{0}$ and $D_{\varepsilon}^{1}$ are weakly anti-monotone in their second arguments. Then
\begin{equation}
(0, +\infty) \ni \varepsilon \mapsto Q_{\alpha}^{b}(\omega \| \sigma + \varepsilon I) \text{ is monotone increasing} \tag{V.185}
\end{equation}
for any $\omega, \sigma \in B(\mathcal{H})_{\geq 0}$ and $\alpha \in [0, 1]$. If, moreover, $D_{\varepsilon}^{0}$ and $D_{\varepsilon}^{1}$ are regular in the sense of (III.27), and lower semi-continuous in their first arguments, then for any $\omega, \sigma \in B(\mathcal{H})_{\geq 0}$,
\begin{align*}
Q_{\alpha}^{b}(\omega \| \sigma) &= \lim_{\varepsilon \to 0} Q_{\alpha}^{b}(\omega \| \sigma + \varepsilon I) = \inf_{\varepsilon > 0} Q_{\alpha}^{b}(\omega \| \sigma + \varepsilon I), \quad \alpha \in (0, 1], \tag{V.186} \\
D_{\alpha}^{b}(\omega \| \sigma) &= \lim_{\varepsilon \to 0} D_{\alpha}^{b}(\omega \| \sigma + \varepsilon I), \quad \alpha \in (0, 1). \tag{V.187}
\end{align*}

**Proof** The monotonicity assertion in (V.185) is again obvious by definition, and the equalities in (V.186)–(V.187) follow as special cases of (V.183)–(V.184).

**Remark V.49** Note that monotonicity of $(0, +\infty) \ni \varepsilon \mapsto D_{\varepsilon}^{b}(\omega \| \sigma + \varepsilon I)$ is not true in general, as $\varepsilon \mapsto \frac{1}{1-\alpha} \log Q_{\alpha}^{b}(\omega \| \sigma + \varepsilon I)$ is monotone decreasing, while $\frac{1}{1-\alpha} \log Tr(\omega \| \sigma + \varepsilon I)$ is monotone increasing.

**Proposition V.50** Assume that $D_{\varepsilon}^{1}$ is weakly anti-monotone in its second argument. Then
\begin{align*}
(0, +\infty) \ni \varepsilon \mapsto Q_{\alpha}^{b}(\omega \| \sigma + \varepsilon I) \text{ is monotone increasing}, \tag{V.188} \\
(0, +\infty) \ni \varepsilon \mapsto D_{\alpha}^{b}(\omega \| \sigma + \varepsilon I) \text{ is monotone decreasing} \tag{V.189}
\end{align*}
for any $\omega, \sigma \in B(\mathcal{H})_{\geq 0}$ and $\alpha \in [0, 1]$. If, moreover, $D_{\varepsilon}^{1}$ is regular in the sense of (III.27), and lower semi-continuous in its first argument, then for any $\omega, \sigma \in B(\mathcal{H})_{\geq 0}$,
\begin{align*}
Q_{\alpha}^{b}(\omega \| \sigma) &= \lim_{\varepsilon \to 0} Q_{\alpha}^{b}(\omega \| \sigma + \varepsilon I) = \inf_{\varepsilon > 0} Q_{\alpha}^{b}(\omega \| \sigma + \varepsilon I), \quad \alpha \in [0, 1], \tag{V.190} \\
D_{\alpha}^{b}(\omega \| \sigma) &= \lim_{\varepsilon \to 0} D_{\alpha}^{b}(\omega \| \sigma + \varepsilon I) = \sup_{\varepsilon > 0} D_{\alpha}^{b}(\omega \| \sigma + \varepsilon I), \quad \alpha \in [0, 1]. \tag{V.191}
\end{align*}

**Proof** The proof is essentially the same as for Proposition V.47 and Corollary V.48, and hence we omit most of it, and only mention that the $\alpha = 1$ case in (V.189) and (V.191) follow from the respective statements for $\alpha \in [0, 1]$ using (V.156).

**H. Finiteness and non-examples**

Corollary V.34 gives an easily verifiable condition for a quantum Rényi $\alpha$-divergence not being a barycentric Rényi $\alpha$-divergence, as follows:
Proposition V.51 Let $D_\alpha^n$ be a quantum Rényi $\alpha$-divergence for some $\alpha \in (0, 1)$ with the property that $D_\alpha^n(\rho || \sigma) = +\infty \iff \rho \perp \sigma$. Then there exist no quantum relative entropies $D_\alpha^{\rho_0}$ and $D_\alpha^{\rho_1}$ with which $D_\alpha^n = D_\alpha^{b,q}$.

Proof Let $\rho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ be such that $\rho^0 \perp \sigma^0 = 0$ and $\rho \not\perp \sigma$. Then
\[
D_\alpha^{b,q}(\rho || \sigma) = +\infty > D_\alpha^n(\rho || \sigma)
\] (V.192)
for any quantum relative entropies $D_\alpha^{\rho_0}$ and $D_\alpha^{\rho_1}$, according to Corollary V.34. Since such pairs exist in any dimension larger than 1, we get that $D_\alpha^{b,q} \neq D_\alpha^n$.

Corollary V.52 $D_{\alpha,z}$ is not a barycentric Rényi $\alpha$-divergence for any $\alpha \in (0, 1)$ and $z \in (0, +\infty)$.

Proof It is obvious by definition that for any $\alpha \in (0, 1)$ and $z \in (0, +\infty)$, and any $\rho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$, $D_{\alpha,z}(\rho || \sigma) = +\infty \iff \rho \perp \sigma$, and hence the assertion follows immediately from Proposition V.51.

Corollary V.53 The measured Rényi $\alpha$-divergence $D_\alpha^{\text{meas}}$ is not a barycentric Rényi $\alpha$-divergence for any $\alpha \in (0, 1)$.

Proof According to Proposition V.51 we only need to prove that for any $\alpha \in (0, 1)$ and any $\rho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$, $D_\alpha^{\text{meas}}(\rho || \sigma) = +\infty \iff \rho \perp \sigma$. This is well known and easy to verify, but we give the details for the readers’ convenience. If $\rho \perp \sigma$ then the measurement $M_0 := \rho^0$, $M_1 := I - \rho^0$ gives $D_\alpha^{\text{meas}}(\rho || \sigma) \geq D_\alpha(M(\rho) || M(\sigma)) = +\infty$. If $\rho \not\perp \sigma$ then we have $D_\alpha^{\text{meas}}(\rho || \sigma) \leq D_{\alpha,1}(\rho || \sigma) < +\infty$, where the first inequality is due to the monotonicity of the Petz-type Rényi $\alpha$-divergence under measurements [48].

One might have the impression that the strict inequality in (V.192) is the result of some pathology, and would not happen if the operators had full support, and both Rényi divergences took finite values on them. This, however, is not the case, at least if we assume some mild and very natural continuity and regularity properties of $D_\alpha^n$ and the quantum relative entropies $D_\alpha^{\rho_0}$ and $D_\alpha^{\rho_1}$.

Indeed, Proposition V.51 and Corollary V.46 yield the following:

Corollary V.54 Let $D_\alpha^n$ be a quantum Rényi $\alpha$-divergence for some $\alpha \in (0, 1)$, such that $D_\alpha^n(\rho || \sigma) = +\infty \iff \rho \perp \sigma$, and assume that $D_\alpha^n$ is jointly continuous in its arguments. Let $D_\alpha^{\rho_0}$ and $D_\alpha^{\rho_1}$ be quantum relative entropies that are jointly lower semi-continuous in their arguments. Then for any two $\rho_0, \rho_1 \in \mathcal{B}(\mathcal{H})_{\geq 0}$ such that $\rho_0^0 \perp \sigma_0^0 = 0$ and $\rho_0 \not\perp \sigma_0$, and for any $\rho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$ in a neighbourhood of $(\rho_0, \sigma_0)$,
\[
D_\alpha^{b,q}(\rho || \sigma) > D_\alpha^{\rho_0}(\rho || \sigma),
\] (V.193)
and $D_\alpha^{b,q}(\rho || \sigma) < +\infty$ for any pair of invertible elements in the neighbourhood. In particular, $D_\alpha^n \neq D_\alpha^{b,q}$.

Proof Let $M > D_\alpha^n(\rho_0 \parallel \sigma_0)$ be a finite number. By the (semi-)continuity assumptions, $(\rho, \sigma) \in \mathcal{B}(\mathcal{H})_{\geq 0}^2$ : $D_\alpha^n(\rho || \sigma) < M < D_\alpha^{b,q}(\rho || \sigma)$ is an open subset of $\mathcal{B}(\mathcal{H})_{\geq 0}^2$ containing $(\rho_0, \sigma_0)$, and for any of its elements $(\rho, \sigma)$, the inequality (V.193) holds. The assertion about the invertible pairs is obvious from Corollary V.34.

Likewise, Proposition V.51 and Corollary V.48 yield the following:

Corollary V.55 Let $D_\alpha^n$ be a quantum Rényi $\alpha$-divergence for some $\alpha \in (0, 1)$, such that $D_\alpha^n(\rho || \sigma) = +\infty \iff \rho \perp \sigma$, and such that $D_\alpha^n$ is regular in the sense that $D_\alpha^n(\rho || \sigma) = \lim_{\varepsilon \to 0} D_\alpha^n(\rho + \varepsilon I || \sigma + \varepsilon I)$ for any $\rho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0}$. Let $D_\alpha^{\rho_0}$ and $D_\alpha^{\rho_1}$ be quantum relative entropies that are lower semi-continuous in their first arguments, weakly anti-monotone in their second arguments, and regular. Then for any $\rho_0, \sigma_0 \in \mathcal{B}(\mathcal{H})_{\geq 0}$ such that $\rho_0^0 \perp \sigma_0^0 = 0$ and $\rho_0 \not\perp \sigma_0$, and for any $\varepsilon > 0$ small enough,
\[
+\infty > D_\alpha^{b,q}(\rho_0 + \varepsilon I || \sigma_0 + \varepsilon I) > D_\alpha^n(\rho_0 + \varepsilon I || \sigma_0 + \varepsilon I).
\] (V.194)
In particular, $D_\alpha^n \neq D_\alpha^{b,q}$.

Proof Let $M > D_\alpha^n(\rho_0 \parallel \sigma_0)$ be a finite number. By Corollary V.34 and Corollary V.48, there exists some $\varepsilon_1 > 0$ such that $+\infty > D_\alpha^{b,q}(\rho_0 + \varepsilon I || \sigma_0 + \varepsilon I) > M$ for every $0 < \varepsilon \leq \varepsilon_1$. By the assumed regularity of $D_\alpha^n$, there exists some $\varepsilon_2 > 0$ such that $D_\alpha^n(\rho_0 + \varepsilon I || \sigma_0 + \varepsilon I) < M$ for every $0 < \varepsilon < \varepsilon_2$. Hence (V.194) holds for every $0 < \varepsilon < \min\{\varepsilon_1, \varepsilon_2\}$. □
Example V.56 For every $\alpha \in (0, 1)$ and $z \in (0, +\infty)$, $D_{\alpha, z}$ satisfies the conditions in Corollaries V.54 and V.55, and hence both corollaries apply to it.

Example V.57 As discussed above, for every $\alpha \in (0, 1)$, $D^\text{meas}_\alpha(\rho\|\sigma) = +\infty \iff \rho \perp \sigma$. Hence, Corollaries V.54 and V.55 can be applied to $D^\text{meas}_\alpha$ if it is continuous in its arguments. To show this, let us fix a finite-dimensional Hilbert space $\mathcal{H}$ and an orthonormal system $(e_i)_{i=1}^d$ in it. Let $\mathcal{U}(\mathcal{H})$ denote the set of unitaries on $\mathcal{H}$. Continuity of the classical Rényi $\alpha$-divergence $D_\alpha$ yields that the function

$$B(\mathcal{H})_\geq \times B(\mathcal{H})_\geq \times \mathcal{U}(\mathcal{H}) \ni (\rho, \sigma, U) \mapsto D_\alpha \left( \left\langle (e_i, U^* \rho U e_i) \right\rangle_{i=1}^d \left\| (e_i, U^* \rho U e_i) \right\|_{i=1}^d \right)$$

is continuous. Hence,

$$D^\text{meas}_\alpha(\rho\|\sigma) = \sup_{U \in \mathcal{U}(\mathcal{H})} D_\alpha \left( \left\langle (e_i, U^* \rho U e_i) \right\rangle_{i=1}^d \left\| (e_i, U^* \rho U e_i) \right\|_{i=1}^d \right)$$

is continuous in $\rho$ and $\sigma$ according to Lemma II.3. (For the equality in (V.195), see [7, Theorem 4].)

### VI. EXAMPLES OF BARYCENTRIC RÉNYI DIVERGENCES

In this section we consider the relations among various known quantum Rényi $\alpha$-divergences and barycentric Rényi $\alpha$-divergences obtained from specific quantum relative entropies.

Example VI.1 It is clear from Definition V.4 and (V.156) that

$$D^{\alpha_0} \leq D^{\alpha_1}, \quad D^{\beta_0} \leq D^{\beta_1} \implies D_{\alpha}^{b, q} \leq D_{\alpha}^{b, q}, \quad \alpha \in [0, 1]. \quad (VI.196)$$

One might expect that the strict ordering of relative entropies yields a strict ordering of the corresponding variational Rényi divergences. This, however, is not true in complete generality, as Example VI.6 below shows. On the other hand, $D_{\alpha}^{b, q}(\rho\|\sigma) < D_{\alpha}^{b, d}(\rho\|\sigma)$ might nevertheless hold if some extra conditions are imposed on the inputs $\rho$ and $\sigma$, which is still satisfied by generic pairs of inputs, as we show in Sections VIA–VIC below.

The following is also easy to verify:

Proposition VI.2 Let $D^{\alpha_0}$ and $D^{\alpha_1}$ be quantum relative entropies that are monotone under CPTP maps. Then

$$D^\text{meas}_\alpha \leq D^{b, \text{meas}}_\alpha \leq D^{b, q}_\alpha \leq D^{b, \text{max}}_\alpha \leq D^{\text{max}}_\alpha, \quad \alpha \in [0, 1]. \quad (VI.197)$$

Proof The second and the third inequalities are immediate from (III.62) and (VI.196). Since $D^\text{meas}_\alpha$ and $D^{\text{max}}_\alpha$ are monotone under CPTP maps, so are $D^{b, \text{meas}}_\alpha$ and $D^{b, \text{max}}_\alpha$ as well, according to Proposition V.40. Hence, the first and the last inequalities in (VI.197) follow immediately from (III.60).

We have seen in Corollary V.53 that the first inequality in (VI.197) is not an equality, i.e., for any $\alpha \in (0, 1)$, the smallest barycentric Rényi $\alpha$-divergence generated by CPTP-monotone quantum relative entropies is above and not equal to the smallest CPTP-monotone quantum Rényi $\alpha$-divergence. We will show in Section VIA that the same happens “at the top of the spectrum”, i.e., the last inequality in (VI.197) is not an equality, either.

A. Umegaki relative entropy and smaller relative entropies

Proposition VI.3 Let $D^{\alpha_0}, D^{\alpha_1} \leq D^{\text{Um}},$ and assume that at least one of them is strictly smaller than $D^{\text{Um}}$. Then for any two non-commuting invertible positive operators $\rho, \sigma \in B(\mathcal{H})_\geq$,

$$D_\alpha^{b, q}(\rho\|\sigma) < D_\alpha^{b, \text{Um}}(\rho\|\sigma) \quad (= D_{\alpha, +\infty}(\rho\|\sigma)), \quad \alpha \in (0, 1). \quad (VI.198)$$

In particular,

$$D^\text{meas}_\alpha \leq \left\{ \begin{array}{l} D_\alpha^{b, (\text{meas,Um})} \\ D_\alpha^{b, (\text{Um,meas})} \end{array} \right\} < D_\alpha^{b, \text{Um}}. \quad (VI.199)$$
Proof} Recall the definition of $\omega_\alpha := \omega_{\alpha}^{\text{Um}}(\rho\|\sigma)$ in (III.85). It is easy to see that if it commutes with $\rho$ or $\sigma$ then $\rho$ and $\sigma$ have to commute with each other. Indeed, assume that $\omega_\alpha$ commutes with $\rho$; then $\rho$ also commutes with any function of $\omega_\alpha$, in particular, with $(1-\alpha)\rho + \alpha\sigma$, and hence it also commutes with $\sigma$. The same argument works if $\omega_\alpha$ commutes with $\sigma$.

Hence, in our setting $\omega_\alpha$ does not commute with either of $\rho$ or $\sigma$, and therefore

$$D_{\alpha}^{\text{b,Um}}(\rho\|\sigma) = \frac{\alpha}{1-\alpha} D_{\alpha}^{\text{Um}}(\omega_\alpha\|\sigma) + D_{\alpha}^{\text{Um}}(\omega_\alpha\|\sigma) - \frac{1}{\alpha - 1} \log \text{Tr} \rho,$$

where the equality follows from the definition of $\omega_\alpha$, the strict inequality follows as $D_{\alpha}^{\text{Um}}(\omega_\alpha\|\sigma) > D_{\alpha}^{\text{q}}(\omega_\alpha\|\sigma)$ or $D_{\alpha}^{\text{Um}}(\omega_\alpha\|\sigma) > D_{\alpha}^{\text{q}}(\omega_\alpha\|\sigma)$ by assumption, and the last inequality is due to (V.139). This proves (VI.198) and the strict inequality in (VI.199), while the first inequality in (VI.199) is obvious from Example VI.1.

\section{Maximal relative entropy and a smaller relative entropy}

\begin{theorem}
Let $D_{\alpha}^{\text{q}}$ and $D_{\alpha}^{\text{q}}$ be quantum relative entropies that are block sub-additive and for any non-commuting PSD operators $\omega_1, \omega_2$ with $\omega_1 \leq \omega_2$, $D_{\alpha}^{\text{q}}(\omega_1\|\omega_2) < D_{\alpha}^{\text{max}}(\omega_1\|\omega_2)$ or $D_{\alpha}^{\text{q}}(\omega_1\|\omega_2) < D_{\alpha}^{\text{max}}(\omega_1\|\omega_2)$. Let $\rho, \sigma \in B(H)_{\geq 0}$ be such that $\rho^0 \land \sigma^0 \neq \{0\}$, and $\rho$ and $\sigma$ do not have a common eigenvector. Then

$$D_{\alpha}^{\text{b,q}}(\rho\|\sigma) < D_{\alpha}^{\text{b,max}}(\rho\|\sigma), \quad \alpha \in (0, 1).$$

\end{theorem}

\begin{proof}
Let $\rho, \sigma$ be as given in the statement, and let $\alpha \in (0, 1)$ be fixed.

Due to the lower semi-continuity of $D_{\alpha}^{\text{max}}$ (see, e.g., [38, Appendix A]), there exists an $\omega_\alpha \in S((\rho^0 \land \sigma^0)H)$ such that

$$D_{\alpha}^{\text{b,max}}(\rho\|\sigma) = \frac{\alpha}{1-\alpha} D_{\alpha}^{\text{max}}(\omega_\alpha\|\rho) + D_{\alpha}^{\text{max}}(\omega_\alpha\|\sigma) + \kappa_{\rho,\alpha},$$

where $\kappa_{\rho,\alpha} := (\log \text{Tr} \rho)/(1-\alpha)$; see (V.139). Hence, in order to prove the strict inequality in (VI.200), it is sufficient to show that $\omega_\alpha$ does not commute with $\rho$ or $\sigma$.

Let $A := \{\rho\} \cap \{\sigma\}$ be the *-subalgebra of operators commuting with both $\rho$ and $\sigma$, and let $P_1, \ldots, P_r$ be a sequence of minimal projections in $A$ summing to $I$ (in particular, $P_i \perp i=1 \neq j \neq P_j$). Let $\Phi(\cdot) = \sum_{i=1}^r P_i(\cdot)P_i$ be the corresponding pinching operation. Then

$$D_{\alpha}^{\text{b,max}}(\rho\|\sigma) = \frac{\alpha}{1-\alpha} D_{\alpha}^{\text{max}}(\omega_\alpha\|\rho) + D_{\alpha}^{\text{max}}(\omega_\alpha\|\sigma) + \kappa_{\rho,\alpha},$$

where the first inequality follows by the monotonicity of $D_{\alpha}^{\text{max}}$ under CPTP maps, the second equality is due to $\Phi(\rho) = \rho$ and $\Phi(\sigma) = \sigma$, and the second inequality is due to (V.139). Hence, we may assume that $\omega_\alpha = \Phi(\omega_\alpha) = \sum_i P_i\omega_\alpha P_i$. Using then the block additivity of $D_{\alpha}^{\text{max}}$, we get

$$D_{\alpha}^{\text{b,max}}(\rho\|\sigma) = \sum_{i=1}^r \frac{\alpha}{1-\alpha} D_{\alpha}^{\text{max}}(P_i\omega_\alpha P_i\|P_i\rho P_i) + D_{\alpha}^{\text{max}}(P_i\omega_\alpha P_i\|P_i\rho P_i) + \kappa_{\rho,\alpha}.$$
\[ D_{\alpha}^{b, \text{max}}(\varrho \| \sigma) = \sum_{i=1}^{r} \frac{\alpha}{1-\alpha} D^{\text{max}}(P_i \omega_i P_i \| P_i \varrho P_i) + D^{\text{max}}(P_i \omega_i P_i \| P_i \sigma P_i) + \kappa_{\varrho, \alpha} \]

\[
> \sum_{i=1}^{r} \frac{\alpha}{1-\alpha} D^{\text{bq}}(P_i \omega_i P_i \| P_i \varrho P_i) + D^{\text{bq}}(P_i \omega_i P_i \| P_i \sigma P_i) + \kappa_{\varrho, \alpha} \\
\geq \frac{\alpha}{1-\alpha} D^{\text{bq}} \left( \sum_{i=1}^{r} P_i \omega_i P_i \right) \left\| \sum_{i=1}^{r} P_i \varrho P_i \right\|_{\sigma} + D^{\text{bq}} \left( \sum_{i=1}^{r} P_i \omega_i P_i \right) \left\| \sum_{i=1}^{r} P_i \sigma P_i \right\|_{\sigma} + \kappa_{\varrho, \alpha} \\
\geq D_{\alpha}^{b, \text{q}}(\varrho \| \sigma), \quad (VI.201) 
\]

where the strict inequality follows by assumption, the first inequality is due to the block sub-additivity of \( D^{\text{bq}} \) and \( D^{\text{bq}} \), and the last inequality is due to (V.139).

Assume therefore that \( P_i \omega_i P_i \) is a constant multiple of \( P_i \) for all \( i \). Then there exists at least one \( i \) such that \( P_i \omega_i P_i = cP_i \) with some \( c \in (0, +\infty) \). For the rest we may restrict the Hilbert space to \( \text{ran} \ P_i \), and use the notations \( I \) for \( P_i \), \( \varrho \) for \( P_i \varrho P_i \), and \( \bar{\sigma} \) for \( P_i \sigma P_i \). Note that the finiteness of \( D_{\alpha}^{b, \text{max}}(\varrho \| \sigma) \) in (VI.201) implies that \( D^{\text{max}}(cI \| \bar{\varrho}) < +\infty \), \( D^{\text{max}}(cI \| \bar{\sigma}) < +\infty \), whence \( \bar{\varrho}^0 = I = \bar{\sigma}^0 \), according to (III.56). By the definition of \( \omega_i \), for any self-adjoint traceless operator \( X \in \mathcal{B}(\text{ran} \ P_i) \), and any \( t \in (-c/ \| X \|, c/ \| X \|) \),

\[ f_X(t) := \frac{\alpha}{1-\alpha} D^{\text{max}}(cI + tX \| \bar{\varrho}) + D^{\text{max}}(cI + tX \| \bar{\sigma}) \geq \frac{\alpha}{1-\alpha} D^{\text{max}}(cI \| \bar{\varrho}) + D^{\text{max}}(cI \| \bar{\sigma}). \quad (VI.202) \]

By the joint convexity of \( D^{\text{max}} \) [35], \( f_X(\cdot) \) is a convex function, and hence, by the above, it has a global minimum at \( t = 0 \). Since it is also differentiable at \( t = 0 \), as we show below, we get that

\[ f_X'(0) = 0, \quad X \in \mathcal{B}(\text{ran} \ P_i)_{sa}, \quad \text{Tr} \ X = 0. \]

We have

\[
\frac{d}{dt} D^{\text{max}}(cI + tX \| \bar{\varrho}) \bigg|_{t=0} = \frac{d}{dt} \text{Tr} \ \bar{\varrho}^{1/2}(cI + tX) \bar{\varrho}^{-1/2} \log \left( \bar{\varrho}^{-1/2}(cI + tX) \bar{\varrho}^{-1/2} \right) \bigg|_{t=0} = \text{Tr} \ \bar{\varrho}^{1/2}X \bar{\varrho}^{-1/2} \log \left( \bar{\varrho}^{-1/2}(cI \bar{\varrho}^{-1/2}) \right) + \text{Tr} \ \bar{\varrho}^{1/2}cI \bar{\varrho}^{-1/2} \frac{d}{dt} \log \left( \bar{\varrho}^{-1/2}(cI + tX) \bar{\varrho}^{-1/2} \right) \bigg|_{t=0}. 
\]

Let \( \bar{\varrho} = \sum_{i=1}^{r} \lambda_i Q_i \) be the spectral decomposition of \( \bar{\varrho} \). Then the Fréchet derivative of \( \log \) at \( c\bar{\varrho}^{-1} \) is the linear operator

\[ D \log(c\bar{\varrho}^{-1}) : A \mapsto \sum_{i,j=1}^{r} \log^{[1]}(c/\lambda_i,c/\lambda_j) Q_i A Q_j, \quad A \in \mathcal{B}(\text{ran} \ P_i), \]

where

\[ \log^{[1]}(x,y) = \begin{cases} \log x - \log y, & x \neq y, \\ 1/x, & x = y, \end{cases} \]
is the first divided difference function of log (see (II.10)). Thus,

\[
\begin{align*}
\text{Tr} \, cI \frac{d}{dt} \log \left( \tilde{q}^{-1/2} (cI + tX) \tilde{q}^{-1/2} \right) & \\
& = \text{Tr} \, cI \sum_{i,j=1}^{r} \log [1] \left( c/\lambda_i, c/\lambda_j \right) Q_i \tilde{q}^{-1/2} X \tilde{q}^{-1/2} Q_j \\
& = c \sum_{i,j=1}^{r} \log [1] \left( c/\lambda_i, c/\lambda_j \right) \lambda_i^{-1/2} \lambda_j^{-1/2} \text{Tr} \, Q_i X Q_j \\
& = c \sum_{i=1}^{r} \lambda_i \frac{1}{c} \text{Tr} \, Q_i X = \text{Tr} \, X.
\end{align*}
\]

By an exactly analogous computation for \( \frac{d}{dt} D^{\max}(cI + tX\|\tilde{q}) \bigg|_{t=0} \), we finally get

\[
0 = f'_X(0) = \text{Tr} \left[ \frac{\alpha}{1 - \alpha} \left( I + \log(c\tilde{q}^{-1}) \right) + I + \log(c\tilde{q}^{-1}) \right] = - \text{Tr} \left[ \frac{\alpha}{1 - \alpha} \log \tilde{q} + \log \tilde{\sigma} \right],
\]

for any \( X \in B(\text{ran} P_i)_{sa} \) with \( \text{Tr} \, X = 0 \). This is equivalent to the existence of some \( \kappa \in \mathbb{R} \) such that

\[
\frac{\alpha}{1 - \alpha} \log \tilde{q} + \log \tilde{\sigma} = \kappa I,
\]

i.e.,

\[
\tilde{\sigma} = e^{\kappa \tilde{q}^{\frac{\alpha}{1 - \alpha}}}. \tag{50}
\]

In particular, \( \tilde{q} \) and \( \tilde{\sigma} \) have a common eigenvector \( \psi \in \text{ran} P_i \), which is also a common eigenvector of \( q \) and \( \sigma \), contradicting our initial assumptions. \( \square \)

**Example VI.5** Let \( D^{q_0} \) and \( D^{q_1} \) be both block subadditive, and assume that \( D^{q_0} = D^{U_m} \) or \( D^{q_1} = D^{U_m} \). If \( q, \sigma \in B(\mathcal{H})_{\geq 0} \) are such that \( q^0 \wedge \sigma^0 \neq \{0\} \), and \( q \) and \( \sigma \) do not have a common eigenvector, then

\[
D^{b,q}_\alpha (q\|\sigma) < D^{b,\max}_\alpha (q\|\sigma), \quad \alpha \in (0, 1).
\]

Indeed, by [23, Theorem 4.3], for any non-commuting PSD operators \( \omega_1, \omega_2 \) with \( q_0^0 \leq \omega_1^0, D^{U_m}(\omega_1\|\omega_2) < D^{\max}(\omega_1\|\omega_2) \), and hence the claim follows by Theorem VI.4.

**Example VI.6** Let \( D^{q_0} \) and \( D^{q_1} \) be quantum relative entropies as in Theorem VI.4. Let

\[
q := p \oplus (1 - p) \langle \psi_1 | \psi_1 \rangle \quad \text{and} \quad \sigma := q \oplus (1 - q) \langle \psi_2 | \psi_2 \rangle
\]

be PSD operators on \( \mathcal{H} = \mathbb{C} \oplus \mathbb{C}^d \) for some \( d > 1 \), where \( p, q \in (0, 1) \), and \( | \psi_1, \psi_2 \rangle \in \mathbb{C}^d \) are unit vectors that are neither parallel nor orthogonal. Then \( q^0 \wedge \sigma^0 = 1 \oplus 0 \), and hence the unique optimal \( \omega_\alpha \) for any \( \alpha \in (0, 1) \) and any barycentric Rényi \( \alpha \)-divergence is \( \omega_0 = 1 \oplus 0 \). Thus,

\[
D^{b,\max}_\alpha (q\|\sigma) = \frac{\alpha}{1 - \alpha} D^{\max}(q_0\|q) + D^{\max}(\omega_\alpha\|\sigma) + \kappa_{\theta,\alpha}
\]

\[
= \frac{\alpha}{1 - \alpha} (- \log p) - \log q + \kappa_{\theta,\alpha}
\]

\[
= \frac{\alpha}{1 - \alpha} D^{q_0}(q_0\|q) + D^{q_1}(\omega_\alpha\|\sigma) + \kappa_{\theta,\alpha}
\]

\[
= D^{b,q}_\alpha (q\|\sigma).
\]

This shows that the assumption that \( q \) and \( \sigma \) do not have a common eigenvector cannot be completely omitted in Theorem VI.4.
C. Maximal Rényi divergences vs. the barycentric maximal Rényi divergences

By Proposition VI.2, for every \( \alpha \in (0,1) \), \( D_{\alpha}^{b,\max} \leq D_{\alpha}^{\max} \). Our aim in this section is to show that equality does not hold. In fact, we conjecture the stronger relation \( D_{\alpha}^{b,\max} < D_{\alpha}^{\max}, \alpha \in (0,1) \), supported by numerical examples. We will prove this below in the special case where the inputs are 2-dimensional. Of course, this already gives at least that

\[
D_{\alpha}^{b,\max} \leq D_{\alpha}^{\max}, \quad \alpha \in (0,1).
\]

Let \( \varrho, \sigma \in \mathcal{B}(\mathcal{H})_{\geq 0} \) be such that \( \varrho^0 \leq \sigma^0 \), and recall the definition of the maximal Rényi \( \alpha \)-divergence and the optimal reverse test \((\hat{\varrho}, \hat{\sigma}, \hat{\Gamma})\) from Example III.19. Let \( \omega_\alpha := \omega_\alpha(\hat{\varrho}||\hat{\sigma}) = \sum_{x \in X} \hat{p}(x)\alpha \hat{q}(x)^{1-\alpha}1_{\{x\}} \) be as in (V.152). Then

\[
\frac{1}{\alpha - 1} \log \text{Tr} \varrho + D_{\alpha}^{\max}(\varrho||\sigma) = \frac{1}{\alpha - 1} \log \text{Tr} \varrho + D_{\alpha}^{\max}(\hat{\varrho}||\hat{\sigma}) = \frac{\alpha}{1 - \alpha} D(\omega_\alpha || \hat{\varrho}) + D(\omega_\alpha || \hat{\sigma}) \geq \frac{\alpha}{1 - \alpha} D_{\alpha}^{\max}(\hat{\Gamma}(\omega_\alpha) || \hat{\Gamma}(\hat{\varrho})) + D_{\alpha}^{\max}(\hat{\Gamma}(\omega_\alpha) || \hat{\Gamma}(\hat{\sigma})) = \frac{\alpha}{1 - \alpha} D_{\alpha}^{\max}(\hat{\Gamma}(\omega_\alpha) || \varrho) + D_{\alpha}^{\max}(\hat{\Gamma}(\omega_\alpha) || \sigma),
\]

where the first two equalities follow from Example III.19 and (III.37)–(III.38), the inequality is due to the monotonicity of \( D_{\alpha}^{\max} \) under positive trace-preserving maps, and the third equality is by the definition of \( \hat{\Gamma} \). By (III.38) and (III.51)–(III.54),

\[
\omega_\alpha = \sum_{i=1}^r \lambda_i^\alpha \text{Tr} \sigma P_i 1_{\{i\}}, \quad (VI.207)
\]

\[
Q_{\alpha}^{\max}(\varrho||\sigma) = Q_{\alpha}(\hat{\varrho}||\hat{\sigma}) = \sum_{i=1}^r \lambda_i^\alpha \text{Tr} \sigma P_i \sigma - \text{Tr} \sigma(\sigma^{-1/2} \varrho \sigma^{-1/2})^\alpha = \text{Tr} \sigma \#_\alpha \varrho, \quad (VI.208)
\]

\[
\hat{\Gamma}(\omega_\alpha) = \frac{1}{Q_{\alpha}^{\max}(\varrho||\sigma)} \sum_i \lambda_i^\alpha \sigma^{1/2} P_i \sigma^{-1/2} = \frac{1}{Q_{\alpha}^{\max}(\varrho||\sigma)} \omega^{1/2}(\sigma^{-1/2} \varrho \sigma^{-1/2})^\alpha \sigma^{-1/2} =: \sigma \#_\alpha \varrho, \quad (VI.209)
\]

where \( \sigma \#_\alpha \varrho \) is the \( \alpha \)-weighted Kubo-Ando geometric mean of \( \varrho \) and \( \sigma \) (see Section IV).

Lemma VI.7 Assume that \( \varrho \) and \( \sigma \) are invertible. Then the inequality in (VI.205) holds as an equality, and thus

\[
D_{\alpha}^{\max}(\varrho||\sigma) = \frac{\alpha}{1 - \alpha} D_{\alpha}^{\max} \left( \frac{\sigma \#_\alpha \varrho}{Q_{\alpha}^{\max}(\varrho||\sigma)} || \varrho \right) + D_{\alpha}^{\max} \left( \frac{\sigma \#_\alpha \varrho}{Q_{\alpha}^{\max}(\varrho||\sigma)} || \sigma \right) - \frac{1}{\alpha - 1} \log \text{Tr} \varrho. \quad (VI.210)
\]

Proof Let \( Q_{\alpha}^{\max} := Q_{\alpha}^{\max}(\varrho||\sigma) \). Recall that for invertible \( \varrho \) and \( \sigma \),

\[
\sigma \#_\alpha \varrho = \varrho \#_{1-\alpha} \sigma = \varrho^{1/2}(\varrho^{-1/2} \varrho \varrho^{-1/2})^\alpha \varrho^{-1/2}; \quad (VI.211)
\]

see (IV.86)–(IV.90). Thus, by (III.55),

\[
D_{\alpha}^{\max} \left( \frac{\sigma \#_\alpha \varrho}{Q_{\alpha}^{\max}(\varrho||\sigma)} || \varrho \right) = \frac{1}{Q_{\alpha}^{\max}} \text{Tr} \varrho \varrho^{-1/2} \varrho^{-1/2} \varrho^{-1/2} \log \left( \frac{\varrho^{-1/2} \varrho \varrho^{-1/2} \varrho^{-1/2} \varrho^{-1/2}}{Q_{\alpha}^{\max}} \right) = \frac{\log Q_{\alpha}^{\max}}{Q_{\alpha}^{\max}} \text{Tr} \varrho \varrho^{-1/2} \varrho^{-1/2} \varrho^{-1/2} \log \left( \frac{\varrho^{-1/2} \varrho \varrho^{-1/2} \varrho^{-1/2}}{Q_{\alpha}^{\max}} \right) + \frac{1-\alpha}{Q_{\alpha}^{\max}} \text{Tr} \varrho \varrho^{-1/2} \varrho^{-1/2} \varrho^{-1/2} \log(\varrho^{-1/2} \varrho \varrho^{-1/2}) \quad (VI.212)
\]
where in the last equality we used that the transpose function of \( f(\cdot) := (\cdot)^{1−α} \log(\cdot) \) is \( \tilde{f}(\cdot) := −(\cdot)α \log(\cdot) \), whence

\[
\text{Tr} \varrho(\varrho^{-1/2} \sigma \varrho^{-1/2})^{1−α} \log(\varrho^{-1/2} \sigma \varrho^{-1/2}) = −\text{Tr} \sigma(\varrho^{-1/2} \sigma \varrho^{-1/2})^α \log(\varrho^{-1/2} \sigma \varrho^{-1/2}), \tag{VI.213}
\]

according to (II.12). Similarly,

\[
D^\text{max}\left( \frac{\sigma^#\alpha \varrho}{Q_\alpha^\text{max}(\varrho\|\sigma)} \right) = \frac{1}{Q_\alpha^\text{max}} \text{Tr} \sigma(\varrho^{-1/2} \sigma \varrho^{-1/2})^α \log \left( \frac{\sigma^{-1/2} \varrho \sigma^{-1/2}}{Q_\alpha^\text{max}} \right) = \frac{\log Q_\alpha^\text{max}}{Q_\alpha^\text{max}} \text{Tr} \sigma(\varrho^{-1/2} \sigma \varrho^{-1/2})^α + \frac{α}{Q_\alpha^\text{max}} \text{Tr} \sigma(\varrho^{-1/2} \sigma \varrho^{-1/2})^α \log(\varrho^{-1/2} \sigma \varrho^{-1/2}). \tag{VI.214}
\]

From (VI.212) and (VI.214) we obtain that the RHS of (VI.210) is

\[
\frac{1}{α−1} \log Q_\alpha^\text{max} − \frac{1}{α−1} \log \text{Tr} \varrho = D_α^\text{max}(\varrho\|\sigma),
\]

where the equality follows from (III.54). \( \square \)

**Remark VI.8** Using (VI.212)–(VI.214), a straightforward computation gives

\[
D^\text{max}(\tilde{Γ}(ω_α)\|\varrho) = D(ω_α\|\tilde{p}) = −\log Q_α^\text{max}(\varrho\|\sigma) + (1−α) \sum_i ω_α(i) \log \frac{\hat{q}(i)}{\hat{p}(i)}
\]

\[
D^\text{max}(\tilde{Γ}(ω_α)\|\sigma) = D(ω_α\|\tilde{p}) = −\log Q_α^\text{max}(\varrho\|\sigma) − α \sum_i ω_α(i) \log \frac{\hat{q}(i)}{\hat{p}(i)}.
\]

From these, the equality in (VI.215) can also be verified directly.

**Remark VI.9** A different way of proving equality in (VI.215) is by noting that

\[
\tilde{Γ}_2(\omega_α^2/\tilde{p}) = Q_α^\text{max}(\varrho\|\sigma)^{-2} \sum_i \lambda_i^{2α−1} \sigma^{1/2} P_i^{α1/2} Q_α^\text{max}(\varrho\|\sigma)^{-2} \sigma^{1/2} (\varrho^{-1/2} \sigma \varrho^{-1/2})^{2α−1} \sigma^{1/2},
\]

\[
\tilde{Γ}_2(\omega_α) = Q_α^\text{max}(\varrho\|\sigma)^{-2} \sum_{i,j} \lambda_i^α \lambda_j^α \sigma^{1/2} P_i^{α1/2} \sigma^{1/2} \equiv \sum_k \lambda_k^{−1} P_k = \delta_{i,j} \lambda_k^{−1} P_k
\]

\[
Q_α^\text{max}(\varrho\|\sigma)^{-2} \sum_i \lambda_i^{2α−1} \sigma^{1/2} P_i^{α1/2} \sigma^{1/2} = \tilde{Γ}_2(\omega_α^2/\tilde{p}).
\]

Hence, by [23, Theorem 3.34], \( D(ω_α\|\tilde{p}) = D^\text{max}(\tilde{Γ}(ω_α)\|\varrho) \). A completely analogous computation yields \( D(ω_α\|\tilde{q}) = D^\text{max}(\tilde{Γ}(ω_α)\|\sigma) \).

Our aim now is to prove that \( \tilde{σ}_#^\alpha \varrho \) is not an optimal \( ω \) in the variational formula (V.139) for \( D^\text{h, max}_α \). We prove this (at least in the 2-dimensional case) by showing that any state \( ω \) on the line segment connecting \( \tilde{σ}_#^\alpha \varrho \) and the maximally mixed state \( π_\mathcal{H} := I/d, d := \dim \mathcal{H} \), that is close enough to \( \tilde{σ}_#^\alpha \varrho \) but is not equal to it, gives a strictly lower value than the RHS of (VI.210) when substituted into \( \frac{α}{1−α} D^\text{max}(\cdot\|\varrho) + D^\text{max}(\cdot\|\sigma) − \frac{1}{α−1} \log \text{Tr} \varrho \).

**Lemma VI.10** Let \( \varrho, \sigma \in B(\mathcal{H})_{>0} \), and let \( P_1, \ldots, P_r \in P(\mathcal{H}) \) and \( λ_1, \ldots, λ_r \in \mathbb{R} \), be such that \( \sum_{i=1}^r P_i = I \), and

\[
σ^{-1/2} \varrho σ^{-1/2} = \sum_{i=1}^r λ_i P_i.
\]
Then
\[
\partial_{\pi_H} := \frac{d}{dt} \left[ \alpha D^{\max} \left( (1-t)\sigma \# \tilde{\alpha} \sigma + t \pi_H \| \sigma \| \right) + (1-\alpha) D^{\max} \left( (1-t)\sigma \# \tilde{\alpha} \sigma + t \pi_H \| \sigma \| \right) \right]_{t=0}
\]
\[
= -1 + \frac{1}{d} \sum_{i,j} \text{Tr} P_i \sigma P_j \sigma^{-1} \left[ \frac{\alpha \log [1]}{\lambda_1^\alpha - \lambda_j^\alpha} \lambda_1^\alpha - 1 + (1-\alpha) \log [1] \right] \lambda_1^\alpha),
\]
where
\[
\Lambda_{\alpha,i,j} = \begin{cases} 
\alpha (1-\alpha) (\log \lambda_i - \log \lambda_j) \frac{(\lambda_i - \lambda_j)}{(\lambda_i^\alpha - \lambda_j^\alpha)(\lambda_i^{\alpha-1} - \lambda_j^{\alpha-1})}, & \lambda_i \neq \lambda_j, \\
1, & \lambda_i = \lambda_j.
\end{cases}
\]

**Proof** We defer the slightly lengthy proof to Appendix A. □

Our aim is therefore to prove that \( \partial_{\pi_H} < 0 \). For this, we will need the following:

**Lemma VI.11** The following equivalent inequalities are true: for every \( \alpha \in (0, 1) \),
\[
\frac{\log \lambda - \log \eta}{\lambda - \eta} > \frac{1}{\alpha} \frac{\lambda^\alpha - \eta^\alpha}{\lambda - \eta}, \quad \lambda, \eta \in (0, +\infty), \lambda \neq \eta,
\]
\[
\frac{\log x}{x-1} > \frac{1}{\alpha} \frac{x^{\alpha-1} - 1}{x - 1}, \quad x \in (0, +\infty) \setminus \{1\},
\]
\[
\int_0^1 \frac{1}{tx + 1 - t} \, dt > \int_0^1 \frac{1}{(tx + 1 - t)^{1-\alpha}} \, dt, \quad x \in (0, +\infty) \setminus \{1\}.
\]

**Proof** It is straightforward to verify that the above inequalities are equivalent to each other. The inequality in (VI.219) follows from the strict concavity of the power functions, as
\[
\int_0^1 \frac{1}{(tx + 1 - t)^\gamma} \, dt < \left( \int_0^1 \frac{1}{tx + 1 - t} \, dt \right)^\gamma, \quad \gamma \in (0, 1).
\] □

**Corollary VI.12** In the setting of Lemma VI.10,
\[
\Lambda_{\alpha,i,j} > 1, \quad i \neq j.
\]

**Proof** Immediate from (VI.217). □

Note that we may take the \( P_i \) in Lemma VI.10 to be rank 1, i.e., \( P_i = |e_i\rangle\langle e_i|, i = 1, \ldots, d \), for some orthonormal eigenbasis of \( \sigma^{-1/2} \tilde{\sigma} \sigma^{-1/2} \). Then (VI.215) can be rewritten as
\[
\partial_{\pi_H} = -1 + \frac{1}{d} \sum_{i,j} \langle e_i, \sigma e_j \rangle \langle e_j, \sigma^{-1} e_i \rangle \cdot \Lambda_{\alpha,i,j}
\]
\[
= -1 + \langle u, (S \ast (S^{-1})^T \ast \Lambda_\alpha) u \rangle,
\]
where \( u = \frac{1}{\sqrt{d}}(1, 1, \ldots, 1) \) and \( A \ast B \) denotes the component-wise (also called Hadamard, or Schur) product of two matrices \( A \) and \( B \).

Next, note that
\[
(S^{-1})_{j,i} = (-1)^{i+j} \frac{\det([S]_{i,j})}{\det S},
\]
where \([S]_{i,j}\) is the matrix that we get by omitting the \( i \)-th row and \( j \)-th column of \( S \). Thus, (VI.221) can be rewritten as
\[
\partial_{\pi_H} = -1 + \frac{1}{d} \sum_{i=1}^d \frac{1}{\det S} \sum_{j=1}^d (-1)^{i+j} S_{i,j} \det([S]_{i,j}) \Lambda_{\alpha,i,j}.
\]
Note that for every $i$,

$$\frac{1}{\det S} \sum_{j=1}^{d} (-1)^{i+j} S_{i,j} \det([S]_{i,j}) = (SS^{-1})_{i,i} = 1.$$ 

**Theorem VI.13** Let $\varrho, \sigma \in B(H)_{>0}$, where $\dim H = 2$, and assume that $\varrho \sigma \neq \sigma \varrho$. Then

$$D_{\alpha}^{b,\max}(\varrho||\sigma) < D_{\alpha}^{b}(\varrho||\sigma), \quad \alpha \in (0, 1).$$

**Proof** By Corollary V.27, we may assume that $\text{Tr} \varrho = \text{Tr} \sigma = 1$. By the above, it is sufficient to prove that $\partial_{\varpi_N} < 0$. Let $(e_1, e_2)$ be an orthonormal eigenbasis of $\sigma^{-1/2} \varrho \sigma^{-1/2}$ with corresponding eigenvalues $\lambda_1, \lambda_2$. By assumption, $\varrho \sigma \neq \sigma \varrho$, which implies that $\lambda_1 \neq \lambda_2$. (In fact, $\lambda_1 = \lambda_2 \iff \varrho = c \sigma$ for some $c > 0$, in which case $c = \lambda_1 = \lambda_2$.) Writing out everything in the ONB $(e_1, e_2)$, we have

$$S = \frac{1}{2} \begin{bmatrix} 1 + z & x - iy \\ x + iy & 1 - z \end{bmatrix}$$

with some $r := (x, y, z) \in \mathbb{R}^3$ such that $\|r\|^2 = x^2 + y^2 + z^2 < 1$, and

$$(S^{-1})^T = \frac{4}{1 - \|r\|^2} \begin{bmatrix} 1 - z & -x + iy \\ -x - iy & 1 + z \end{bmatrix},$$

whence

$$S \ast (S^{-1})^T = \frac{1}{1 - \|r\|^2} \begin{bmatrix} 1 - z^2 & -(x^2 + y^2) \\ -(x^2 + y^2) & 1 - z^2 \end{bmatrix}.$$  

Hence, by (VI.221) and the symmetry $\Lambda_{\alpha,1,2} = \Lambda_{\alpha,2,1}$,

$$\partial_{\varpi_N} = -1 + \frac{1}{1 - \|r\|^2} \left[ 1 - z^2 - (x^2 + y^2) \Lambda_{\alpha,1,2} \right].$$

(VI.223)

Since $\sigma$ is not diagonal in the given ONB (otherwise it would commute with $\varrho$), we have $(x^2 + y^2) > 0$. Combining this with Corollary VI.12, we get $\partial_{\varpi_N} < 0$, as required. $\square$

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**Appendix A: Proof of Lemma VI.10.**

**Proof** Let us introduce the notation

$$(\varrho/\sigma) := \sigma^{-1/2} \varrho \sigma^{-1/2} = \sum_i \lambda_i P_i.$$ 

Let

$$X := \varrho^{1/2} \sigma^{-1/2}, \quad \text{and} \quad X = U |X| = U \sum_i \lambda_i^{1/2} P_i \quad (A.1)$$
be its polar decomposition. Then

$$q^{-1/2} q^{-1/2} = (X^{-1})^*(X^{-1}) = U \left( \sum_i \lambda_i^{-1} P_i \right) U^* = \sum_i \lambda_i^{-1} U P_i U^* \tag{A.2}$$

is a spectral decomposition of $q^{-1/2} q^{-1/2}$. Recall from (VI.209) that

$$\sigma^\# \alpha \otimes = \frac{1}{Q_{\alpha}^\text{max}} \sigma^\#_\alpha \otimes = \frac{1}{Q_{\alpha}^\text{max}} q^{\#_1 - \alpha} \sigma,$$

where $Q_{\alpha}^\text{max} := Q_{\alpha}^\text{max}(\sigma | \sigma)$, and note the following identities:

$$\sigma^{-1/2} \sigma^\#_\alpha \otimes \sigma^{-1/2} = \frac{1}{Q_{\alpha}^\text{max}} (\sigma^{-1/2} \sigma^{-1/2}) = \sum_i \frac{\lambda_i^{\alpha}}{Q_{\alpha}^\text{max}} P_i,$$

$$q^{-1/2} q^{-1/2} = \frac{1}{Q_{\alpha}^\text{max}} (q^{-1/2} q^{-1/2}) 1 - \alpha = \sum_i \frac{\lambda_i^{\alpha-1}}{Q_{\alpha}^\text{max}} R_i,$$

where in the last line we used (VI.211).

Recall that $\pi_H = I/d$ denotes the maximally mixed state on $\mathcal{H}$. We have

$$\frac{d}{dt} D_{\alpha}^{\text{max}} \left| t = 0 \right. = \text{Tr} \sigma^{1/2} \left( \pi_H - \sigma^\#_\alpha \otimes \right) \sigma^{-1/2} \log \frac{\sigma^{-1/2} \sigma^{-1/2}}{\sigma^{-1/2} \sigma^{-1/2}} = \frac{1}{Q_{\alpha}^\text{max}} \sigma^{-1/2} \sigma^{-1/2} \sum_{i,j} \log \left( \frac{\lambda_i^{\alpha}}{Q_{\alpha}^\text{max}}, \frac{\lambda_j^{\alpha}}{Q_{\alpha}^\text{max}} \right) P_i \sigma^{-1/2} \left( \pi_H - \sigma^\#_\alpha \otimes \right) \sigma^{-1/2} P_j$$

$$
= - \log Q_{\alpha}^\text{max} + \frac{\alpha}{d} \text{Tr} \log (\sigma^{-1/2} \sigma^{-1/2})
+ \frac{1}{Q_{\alpha}^\text{max}} \text{Tr} \sigma (\sigma^{-1/2} \sigma^{-1/2}) = \frac{\alpha}{Q_{\alpha}^\text{max}} \text{Tr} \sigma (\sigma^{-1/2} \sigma^{-1/2}) \log (\sigma^{-1/2} \sigma^{-1/2})
+ \frac{1}{d Q_{\alpha}^\text{max}} \sum_{i,j} \log \left( \frac{\lambda_i^{\alpha}}{Q_{\alpha}^\text{max}}, \frac{\lambda_j^{\alpha}}{Q_{\alpha}^\text{max}} \right) \text{Tr} \sigma (\sigma^{-1/2} \sigma^{-1/2}) = \frac{\alpha}{Q_{\alpha}^\text{max}} \sum_{i,j} \log \left( \frac{\lambda_i^{\alpha}}{Q_{\alpha}^\text{max}}, \frac{\lambda_j^{\alpha}}{Q_{\alpha}^\text{max}} \right) \text{Tr} \sigma (\sigma^{-1/2} \sigma^{-1/2})
- \frac{1}{Q_{\alpha}^\text{max}} \text{Tr} \sigma (\sigma^{-1/2} \sigma^{-1/2}) = \frac{\alpha}{Q_{\alpha}^\text{max}} \sum_{i,j} \log \left( \frac{\lambda_i^{\alpha}}{Q_{\alpha}^\text{max}}, \frac{\lambda_j^{\alpha}}{Q_{\alpha}^\text{max}} \right) \lambda_i^{\alpha} \text{Tr} \sigma P_i \sigma^{-1/2} P_j$$

An exactly analogous calculation yields

$$\frac{d}{dt} D_{\alpha}^{\text{max}} \left| t = 0 \right. = \frac{1}{d} \text{Tr} \log (\sigma^{-1/2} \sigma^{-1/2}) - \frac{1}{Q_{\alpha}^\text{max}} \text{Tr} \sigma (\sigma^{-1/2} \sigma^{-1/2}) \log (\sigma^{-1/2} \sigma^{-1/2})
+ \frac{1}{d Q_{\alpha}^\text{max}} \sum_{i,j} \log \left( \frac{\lambda_i^{\alpha-1}}{Q_{\alpha}^\text{max}}, \frac{\lambda_j^{\alpha-1}}{Q_{\alpha}^\text{max}} \right) \lambda_i^{\alpha-1} \text{Tr} \sigma R_i \sigma^{-1/2} R_j - 1.$$
Thus,

\[
\partial_{\pi_H} = \frac{d}{dt} \left[ \alpha D_{\max} \left( (1-t)\sigma_{\#\alpha} + t\pi_H \|\sigma\| \right) + (1-\alpha) D_{\max} \left( (1-t)\sigma_{\#\alpha} + t\pi_H \|\sigma\| \right) \right]_{t=0}
= \frac{\alpha(1-\alpha)}{d} \left[ \text{Tr} \log(\varrho^{1/2}\sigma \varrho^{1/2}) + \text{Tr} \log(\sigma^{1/2} \varrho^{1/2}) \right] \sum_i \log \lambda_i^{-1} \shorteq 0
- \frac{\alpha(1-\alpha)}{Q_{\alpha} \max} \left[ \text{Tr} \varrho (\varrho^{1/2}\varrho^{1/2})^{1-\alpha} \log(\varrho^{1/2}\varrho^{1/2}) + \text{Tr} \varrho (\sigma^{1/2} \varrho^{1/2})^{\alpha} \log(\sigma^{1/2} \varrho^{1/2}) \right] \sum_i \log \lambda_i^{-1} \shorteq 0
+ \frac{\alpha}{d} \sum_{i,j} \log \left( \lambda_i^{1-\alpha} \lambda_j^{\alpha-1} \right) \lambda_i \lambda_j \text{Tr} \varrho R_i \varrho^{-1} R_j + \frac{1-\alpha}{d} \sum_{i,j} \log \left( \lambda_i^{\alpha} \lambda_j \right) \lambda_i \text{Tr} \varrho R_i \varrho^{-1} R_j - 1
= \frac{\alpha}{d} \sum_{i,j} \log \left( \lambda_i^{1-\alpha} \lambda_j^{\alpha-1} \right) \lambda_i \lambda_j \text{Tr} \varrho R_i \varrho^{-1} R_j + \frac{1-\alpha}{d} \sum_{i,j} \log \left( \lambda_i^{\alpha} \lambda_j \right) \lambda_i \text{Tr} \varrho R_i \varrho^{-1} R_j - 1,
\]

(A.3)

where the first expression above is equal to 0 due to (A.2), and the second expression is equal to 0 according to (VI.213).

Note that by (A.1),

\[
U = X |X|^{-1} = \varrho^{1/2} \sigma^{-1/2} (\varrho/\sigma)^{-1/2},
\]

whence

\[
U^* = (\varrho/\sigma)^{-1/2} \sigma^{-1/2} \varrho^{1/2} \quad \| \quad U^{-1} = (\varrho/\sigma)^{1/2} \sigma^{1/2} \varrho^{-1/2},
\]

which in turn yields

\[
U = (U^{-1})^* = \varrho^{-1/2} \sigma^{1/2} (\varrho/\sigma)^{1/2}.
\]

Thus,

\[
\text{Tr} \varrho R_i \varrho^{-1} R_j = \text{Tr} \varrho (U P_i U^*) \varrho^{-1} (U P_j U^*)
= \text{Tr} \varrho \underbrace{\varrho^{1/2} \varrho^{1/2}}_{=U} \underbrace{P_i \varrho^{1/2} \varrho^{1/2}}_{=U^*} \underbrace{\varrho^{-1} \varrho^{1/2} \varrho^{1/2}}_{=U} \underbrace{\varrho^{-1} \varrho^{1/2} \varrho^{1/2}}_{=U^*} \underbrace{P_j \varrho^{1/2} \varrho^{1/2}}_{=P_j} \underbrace{\varrho^{1/2}}_{=U}
= \text{Tr} \varrho \underbrace{P_i \varrho^{-1} P_j} \underbrace{\varrho^{-1} P_j \varrho^{1/2} \varrho^{1/2}}_{=P_j} \underbrace{\varrho^{1/2}}_{=U}
= \text{Tr} \varrho P_i \varrho^{-1} P_j.
\]

Writing this back into (A.3), we get

\[
\partial_{\pi_H} = -1 + \frac{1}{d} \sum_{i,j} \text{Tr} \varrho P_i \varrho^{-1} P_j \left[ \alpha \log \left( \lambda_i^{1-\alpha} \lambda_j^{\alpha-1} \right) \lambda_i \lambda_j + (1-\alpha) \log \left( \lambda_i^{\alpha} \lambda_j \right) \lambda_i \right] \shorteq \Lambda_{\alpha,i,j}.
\]

(A.4)

It follows by a straightforward computation that \(\Lambda_{\alpha,i,j}\) can be written as in (VI.216). Note that \(\Lambda_{\alpha}\) is symmetric, i.e., \(\Lambda_{\alpha,i,j} = \Lambda_{\alpha,j,i}\). Exchanging the indices \(i\) and \(j\) in (A.4) yields (VI.215).

[1] W. N. Anderson, Jr. and G. E. Trapp. Shor ted operators. II. SIAM Journal on Applied Mathematics, 28(1):60–71, 1975.
[35] K. Matsumoto. A new quantum version of f-divergence. In Nagoya Winter Workshop 2015: Reality and Measurement in Algebraic Quantum Theory, pages 229–273, 2018.

[36] Milán Mosonyi. The strong converse exponent of discriminating infinite-dimensional quantum states. Communications in Mathematical Physics, 2023. arXiv:2107.08036.

[37] Milán Mosonyi and Fumio Hiai. On the quantum Rényi relative entropies and related capacity formulas. IEEE Transactions on Information Theory, 57(4):2474–2487, April 2011.

[38] Milán Mosonyi and Fumio Hiai. Some continuity properties of quantum Rényi divergences. arXiv:2209.00646, 2023.

[39] Milán Mosonyi and Fumio Hiai. Test-measured Rényi divergences. IEEE Transactions on Information Theory, 69(2):1074–1092, 2023. arXiv:2201.05477.

[40] Milán Mosonyi and Tomohiro Ogawa. Quantum hypothesis testing and the operational interpretation of the quantum Rényi relative entropies. Communications in Mathematical Physics, 334(3):1617–1648, 2015. arXiv:1309.3228.

[41] Milán Mosonyi and Tomohiro Ogawa. Strong converse exponent for classical-quantum channel coding. Communications in Mathematical Physics, 355(1):373–426, June 2017. arXiv:1409.3562.

[42] Milán Mosonyi and Tomohiro Ogawa. Divergence radii and the strong converse exponent of classical-quantum channel coding with constant compositions. IEEE Transactions on Information Theory, 67(3):1668–1698, 2021. arXiv:1811.10599.

[43] X. Mu, L. Pomatto, P. Strack, and O. Tamuz. From Blackwell dominance in large samples to Rényi divergences and back again. Econometrica, 89(1):475–506, 2020.

[44] Alexander Müller-Hermes and David Reeb. Monotonicity of the quantum relative entropy under positive maps. Ann. Henri Poincaré, 18:1777–1788, 2017.

[45] Martin Müller-Lennert, Frédéric Dupuis, Oleg Szehr, Serge Fehr, and Marco Tomamichel. On quantum Rényi entropies: A new generalization and some properties. Journal of Mathematical Physics, 54(12):122203, December 2013. arXiv:1306.3142.

[46] Hiroshi Nagaoka. The converse part of the theorem for quantum Hoeffding bound. arXiv:quant-ph/0611289, November 2006.

[47] Michael A. Nielsen and Isaac L. Chuang. Quantum Computation and Quantum Information. Cambridge University Press, 2000.

[48] Dénes Petz. Quasi-entropies for finite quantum systems. Reports in Mathematical Physics, 23:57–65, 1986.

[49] Alfréd Rényi. On measures of entropy and information. In Proc. 4th Berkeley Sympos. Math. Statist. and Prob., volume I, pages 547–561. Univ. California Press, Berkeley, California, 1961.

[50] M. Tomamichel. Quantum Information Processing with Finite Resources, volume 5 of Mathematical Foundations, SpringerBriefs in Math. Phys. Springer, 2016.

[51] Joel A. Tropp. From joint convexity of quantum relative entropy to a concavity theorem of Lieb. Proceedings of the American Mathematical Society, 140(5):1757–1760, 2011.

[52] Sadi Turgut. Catalytic transformations for bipartite pure states. Journal of Physics A: Mathematical and Theoretical, 40(40):12185, 2007. arXiv:0707.0444.

[53] H. Umegaki. Conditional expectation in an operator algebra, IV: Entropy and information. Kôdai Math. Sem. Rep., 14:59–85, 1962.

[54] Mark M. Wilde, Andreas Winter, and Dong Yang. Strong converse for the classical capacity of entanglement-breaking and Hadamard channels via a sandwiched Rényi relative entropy. Communications in Mathematical Physics, 331(2):593–622, October 2014. arXiv:1306.1586.

[55] Haonan Zhang. From Wigner-Yanase-Dyson conjecture to Carlen-Frank-Lieb conjecture. Advances in Mathematics, 365:107053, 2020. arXiv:1811.01205.