MONADS ON PROJECTIVE SPACE

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INTRODUCTION

A monad on projective $k$-space $\mathbb{P}^k$ over a field $K$ is a complex

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

of vector bundles on $\mathbb{P}^k$ where $\alpha$ is injective and $\beta$ is surjective. In this paper we classify completely when there exists monads on $\mathbb{P}^k$ whose maps are matrices of linear forms, i.e. monads

$$\mathcal{O}_{\mathbb{P}^k}(-1)^a \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^k}^b \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^k}(1)^c.$$

We prove the following.

**Main Theorem.** Let $k \geq 1$. There exists a monad as above if and only if at least one of the following holds.

1. $b \geq 2c + k - 1$ and $b \geq a + c$.
2. $b \geq a + c + k$.

If so, there actually exists a monad with the map $\alpha$ degenerating in expected codimension $b - a - c + 1$.

Thus in case 2 one sees that there exists a monad whose cohomology is a sheaf of constant rank $b - a - c$, i.e. a vector bundle.

Our way to this result came through work on curves in $\mathbb{P}^3$. Here monads of the form

$$\mathcal{O}_{\mathbb{P}^3}(-1)^n+r-2 \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{2n+r-1} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1)^n$$

naturally occur. The monad has cohomology $\mathcal{I}_C(r-2)$, the ideal sheaf of a space curve $C$ twisted with $r-2$. It was known that these monads existed for all $r \geq 3$ (with $C$ a smooth curve in fact). But it is fairly easy to see geometrically that the monad could not exist on $\mathbb{P}^3$ for $r \leq 2$.

Consider now the monad

$$\mathcal{O}_{\mathbb{P}^k}(-1)^{n+r-2} \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^k}^{2n+r-1} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^k}(1)^n.$$

When $k = 2$ it is easily seen that it exists for $r \geq 2$ but not for $r \leq 1$. When $k = 4$ considerations also indicated that it existed for $r \geq 4$ but not for $r \leq 3$. Whence we were lead to formulate the above result.

An interesting geometric consequence of the above result may be obtained by letting $k = r = 4$ in (1). By the theorem above, $\alpha$ may be assumed to degenerate in codimension 2 along a (locally Cohen-Macaulay) surface...
$S \subseteq P^1$. The cohomology of the monad is then $I_S(2)$. By standard exact sequences one easily sees that $H^1I_S(1) = n$. Thus $H^0O_S(1) = n + 5$ for $n \geq 0$. This means that $S$ embeds linearly normally into $P^{n+4}$ and that it projects isomorphically down to $P^4$. By a classical theorem of Severi it is known that the only smooth surface in $P^{n+4}$ where $n \geq 1$ that enjoys this property is the Veronese surface in $P^5$. In case $n = 1$ the surface $S$ above is a degeneration of the Veronese surface.

We would like to motivate the theorem in a more general context. Let $F$ be a sheaf on projective space $P^k$. Suppose $k - r = \text{lpd} F$, the local projective dimension of $F$. (If for instance $F = I_X$, the ideal sheaf of a smooth projective variety $X \subseteq P^k$ with $\text{dim} X = r$, this holds.) Then there are, [Wa] Proposition 1.3, canonical complexes

$$F_i^j : 0 \rightarrow F_i^{-(k-1-i)} \rightarrow F_i^{-(k-2-i)} \rightarrow \cdots \rightarrow F_i^0 \rightarrow \cdots \rightarrow F_i^1 \rightarrow 0$$

for $i = 0, \ldots, r$ with the following properties: Each $F_i^j$ is a finite sum of line bundles. The cohomology $H^j(F_i^j) = 0$ for $j \neq 0$ and $H^0(F_i^j) = F$. (When $i = 0$ this is the sheafification of a minimal resolution of the graded $K[x_0, \ldots, x_k]$-module $\oplus_{n \in Z} F(P^k, F(n))$.) Classification of when such complexes exist might be a way to understand what kind of algebraic or geometric objects which can exist on a projective space. The theorem above may be seen as a very small contribution to this.

For a more specific motivation consider the complex

$$F_2 : 0 \rightarrow O_{P^4}(-1)^{n+r-1} \rightarrow O_{P^4}^{2n+m+r+1} \rightarrow O_{P^4}(1)^{n+2m} \rightarrow O_{P^4}(2)^m \rightarrow 0.$$ 

We let the grading of the complex correspond to the twist of the line bundles. Suppose the complex has cohomology only in degree 0. The cohomology will then be a sheaf $E$ of rank 2. If this is to be a vector bundle one must have $c_2(E) = 0$ and $c_4(E) = 0$. This gives two equations relating $m, n$ and $r$. The following integer values of $r$ (and maybe others) will give integer solutions for $m$ and $n$.

a. For $r = 1$ one gets $(m, n) = (0, 0)$ or $(m, n) = (0, -1)$. When $(m, n) = (0, 0)$ one gets the vector bundle $O_{P^4}\oplus 2$. When $(m, n) = (0, -1)$ one gets the bundle $O_{P^4}(1) \oplus O_{P^4}(-1)$. Here one has to shift a term across an arrow if the summand is of negative order.

b. For $r = 0$ one gets $(m, n) = (0, 0)$ or $(m, n) = (2, 6)$. When $(m, n) = (0, 0)$ one gets the bundle $O_{P^4}\oplus O_{P^4}(-1)$. When $(m, n) = (2, 6)$ one actually gets the $F_2$ complex of the Horrocks-Mumford bundle.

c. For $r = -1$ one gets $(m, n) = (0, 0)$ or $(m, n) = (11, 22)$. When $(m, n) = (0, 0)$ one gets the bundle $O_{P^4}(-1)^2$. When $(m, n) = (11, 22)$ one might get a potentially new rank 2 bundle $E$ on $P^4$ with $F_2$ complex $0 \rightarrow O_{P^4}(-1)^{20} \rightarrow O_{P^4}^{55} \rightarrow O_{P^4}(1)^{44} \rightarrow O_{P^4}(2)^{11} \rightarrow 0$.

The bundle $E$ would have $c_1(E) = -2$ and $c_2(E) = 12$. 
The next step in classifying complexes on projective space might be to classify complexes

\[ 0 \to \mathcal{O}_{\mathbb{P}^k}(-2)^c \overset{\beta}{\to} \mathcal{O}_{\mathbb{P}^k}(-1)^b \overset{\alpha}{\to} \mathcal{O}_{\mathbb{P}^k}a \]

where the cokernel in degree 0 is \( I_X(2c - b) \) for a locally Cohen-Macaulay subscheme \( X \subseteq \mathbb{P}^k \) of codimension 2. This means that \( \beta \) is surjective and \( \alpha \) degenerates in codimension 2. Note that we must have the relation \( b + 1 = a + c \). (The Main Theorem deals with the case \( a + c \leq b \).) We may then write the complex as

\[ \mathcal{O}_{\mathbb{P}^k}(-2)^n \overset{\beta}{\to} \mathcal{O}_{\mathbb{P}^k}(-1)^{2n+r} \overset{\alpha}{\to} \mathcal{O}_{\mathbb{P}^k}^{n+r+1}. \]

**Conjecture.** Let \( k \geq 2 \). The complex above with \( \beta \) surjective and \( \alpha \) degenerating in codimension 2 exists if and only if \( r \geq 0 \) and \( n \leq \binom{r+3-k}{2} \).

For instance if \( k = 2 \) this is readily seen to hold. When \( n = \binom{r+1}{2} \) and \( S = K[x_0, x_1, x_2] \) the complex above corresponds to the resolution of the power \( m^r \subseteq S \) where \( m = (x_0, x_1, x_2) \).

When \( k = 3 \) the necessity of the above condition on \( n \) is also readily seen geometrically to hold.

Let us just say some words about the organization of the paper. In the first section we give the existence of the monads. This is based on an immediate explicit construction. The main work of the section is to verify the statement about the degeneracy locus.

In the second section we show the necessity of the conditions given. This is an argument based on the study of degeneracy loci.

We work over an arbitrary field \( K \).

### 1. Existence

We shall prove the existence of the monads by providing an explicit construction. Denote by \( X_{r,r+n} \) the \( r \) by \( r + n \) matrix

\[
\begin{pmatrix}
x_0 & x_1 & \cdots & x_n \\
x_0 & x_1 & \cdots & x_n \\
\vdots & \ddots & \ddots & \ddots \\
x_0 & x_1 & \cdots & x_n 
\end{pmatrix}
\]

A basic fact we may note is that \( X_{r,r+n} \) degenerates in rank if and only if all \( x_i = 0 \). Similarly denote by \( Y_{r,r+m} \) the \( r \) by \( r + m \) matrix

\[
\begin{pmatrix}
y_0 & y_1 & \cdots & y_m \\
y_0 & y_1 & \cdots & y_m \\
\vdots & \ddots & \ddots & \ddots \\
y_0 & y_1 & \cdots & y_m 
\end{pmatrix}
\]

Let \( \sigma_k = \sum_{k=i+j} x_i y_j \). Form the \( r \) by \( r + m + n \) matrix
The following lemma is easily verified.

**Lemma 1.**

\[
X_{r,r+n} \cdot Y_{r+n,r+n+m} = Y_{r,r+m} \cdot X_{r+m,r+m+n} = \Sigma_{r,r+n+m}.
\]

**Note.** We have been notified that V. Ancona and G. Ottaviani used the same matrices and lemma (with \(n = m\)) in [An-Ot].

Let

\[
\begin{align*}
S &= K[x_0, \ldots, x_n, y_0, \ldots, y_m], \\
P^N &= \text{Proj } S
\end{align*}
\]

and \(P^N = \text{Proj } S\) where \(N = n + m + 1\). We may form the complex

\[
(2) \quad \mathcal{O}_{P^N}(-1)^{r+n+m} \xrightarrow{\alpha} \mathcal{O}_{P^N}^{2r+n+m} \xrightarrow{\beta} \mathcal{O}_{P^N}(1)^r,
\]

where the maps \(\beta\) and \(\alpha\) are given by the matrices

\[
B = \begin{bmatrix} X_{r,r+n} & Y_{r,r+m} \end{bmatrix}, \quad A = \begin{bmatrix} Y_{r+n,r+n+m} \\ -X_{r+m,r+m+n} \end{bmatrix}.
\]

We easily see that the map \(\beta\) is surjective. The whole existence part of the Main Theorem may be derived from this complex as we shall shortly see, but first we investigate the degeneracy loci of the map \(\alpha\).

Let \(Z_d \subseteq P^N\) be the locus where \(A\) degenerates to rank \(r + n + m - d\).

**Lemma 2.**

1. If \(d > \max(n, m)\) then \(Z_d = \emptyset\).
2. If \(\min(n, m) < d \leq \max(n, m)\) then \(d = \max(n, m)\) and \(\text{cod } Z_d = \min(n, m) + 1\).
3. If \(d \leq \min(n, m) + 1\) then \(\text{cod } Z_d \geq d\).

In particular we see that if \(|n - m| \leq 1\) then \(\text{cod } Z_d \geq d\) for all \(d \geq 0\).

**Proof.**

1. Suppose \(d > \max(n, m)\). If \(\text{rk } A \leq r + n + m - d\) then \(\text{rk } A < r + n\) and \(\text{rk } A < r + m\). Then it is easily seen that all \(x_i = 0\) and all \(y_i = 0\).

2. In this case assume \(m < n\). Since \(d > m\) we get \(\text{rk } A \leq r + n + m - d < r + n\). This gives that all \(y_i = 0\). Since \(d \leq n\) we also see that \(\text{rk } A = r + n + m - d \geq r + m\). We see that we must have \(\text{rk } A = r + m\) and \(d = n\).

3. We will show that there is a linear subspace of \(P^N\) of dimension \(d - 1\) where \(A\) does not degenerate to rank \(r + n + m - d\). This will prove the third part of the lemma.

Let \(x_i = t_i\) for \(i = 0, \ldots, d - 1\) and \(x_i = 0\) for \(i \geq d\). Let \(y_{m-i} = t_{d-1-i}\) for \(i = 0, \ldots, d - 1\) and \(y_{m-i} = 0\) for \(i \geq d\). This gives a linear subspace
$L_{d-1} \subseteq \mathbb{P}^N$ of dimension $d-1$. With these substitutions the matrix $A$ now takes a form (letting $t = (t_0, \ldots, t_{d-1})$)

$$A_t = \begin{bmatrix} T_1 \\ -T_2 \end{bmatrix},$$

where

$$T_1 = \begin{pmatrix} 0 & \ldots & 0 & t_0 & \ldots & t_{d-1} \\ 0 & \ldots & 0 & t_0 & \ldots & t_{d-1} \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & t_0 & \ldots & t_{d-1} \end{pmatrix}$$

and

$$T_2 = \begin{pmatrix} t_0 & \ldots & t_{d-1} & 0 & \ldots & 0 \\ t_0 & \ldots & t_{d-1} & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ t_0 & \ldots & t_{d-1} & 0 & \ldots & 0 \end{pmatrix}.$$ 

If $d = 1$ then clearly $\text{rk} A = r + n + m$, so suppose $d > 1$. If $t_{d-1} = 0$ then a submatrix of $A_t$ is of the form

$$A'_t = \begin{bmatrix} T'_1 \\ -T'_2 \end{bmatrix},$$

which is the matrix corresponding to the case $n' = n$, $m' = m - 1$ and $t' = (t_0, \ldots, t_{d-2})$. By induction $\text{rk} A'_t > r + n' + m' - (d-1) = r + n + m - d$.

In the following for a vector $v = (v_1, \ldots, v_{r+n+m})$ in $\mathbb{K}^{r+n+m}$, let

$$\min(v) = \min\{i \mid v_i \neq 0\}.$$ 

Assume now that there exists a $t^0$ with $t^0_{d-1} \neq 0$ such that $\text{rk} A_{t^0} \leq r + n + m - d$. Then there is a $d$-dimensional subspace of $v$ in $\mathbb{K}^{r+n+m}$ such that $A_{t^0} \cdot v = 0$. Among these there must be a non-zero $v$ such that $\min(v) \geq d$. But from the equation $T_2 \cdot v = 0$ and the fact that $t^0_{d-1} \neq 0$ we see that we must in fact have $\min(v) \geq r + m + d$. In particular $\min(v) \geq m + 1$. But this is impossible since we have $T_1 \cdot v = 0$ with $t^0_{d-1} \neq 0$. \[\square\]

Let

$$\mathcal{O}_{\mathbb{P}^N}(-1)^{r+n+m-s} \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}^N}(-1)^{r+n+m}$$

be a general injection. Recall the map $\alpha$ from [2].

**Lemma 3.** Suppose $\text{cod} Z_d \geq d$ for $d = 1, \ldots, s + 1$ and $\text{cod} Z_d \geq s + 1$ for $d \geq s + 1$. Then $\alpha \circ \phi$ degenerates in codimension $s + 1$.

In particular we see that if $|n - m| \leq 1$ then $\alpha \circ \phi$ will degenerate in codimension $s + 1$. 
Proof. Let $E = \ker \beta$. Dualizing

$$O_{\mathbb{P}^N}(-1)^{r+n+m} \xrightarrow{\alpha} E$$

we get

$$E^\vee(-1) \xrightarrow{\alpha^\vee(-1)} O_{\mathbb{P}^N}^{r+n+m}.$$ Let

$$T_0 = \operatorname{cok} \alpha^\vee(-1).$$

Then $T_0$ is generated by the sections coming from $O_{\mathbb{P}^N}^{r+n+m}$. We get for $T_0$ a flattening stratification of locally closed subschemes (see [Mu] section 8 or [E] section 20.2)

$$\mathbb{P}^N = \bigcup_{d \geq 0} Z_{d,0}$$
such that $T_0|_{Z_{d,0}}$ is locally free of rank $d$. Note that $Z_{d,0} = Z_d$.

Take a general section of $O_{\mathbb{P}^N}^{r+n+m}$. This gives a section of $T_0$ and a sequence

$$O_{\mathbb{P}^N} \rightarrow T_0 \rightarrow T_1 \rightarrow 0.$$ Let

$$\mathbb{P}^N = \bigcup_{d \geq 0} Z_{d,1}$$
be a flattening stratification for $T_1$. We must have

$$Z_{d,1} \subseteq Z_{d,0} \cup Z_{d+1,0}.$$ Since the sections of $O_{\mathbb{P}^N}^{r+n+m}$ generate $T_0|_{Z_{d,0}}$, we see that

$$\operatorname{cod} (Z_{d,1} \cap Z_{d,0}) > \operatorname{cod} Z_{d,0}$$
for $d \geq 1$. Thus in this case

$$\operatorname{cod} Z_{d,1} \geq \operatorname{cod} Z_{d,0} + 1$$ or

$$\operatorname{cod} Z_{d,1} \geq \operatorname{cod} Z_{d+1,0}.$$ In this way we may proceed to $T_s$. This sheaf has again a flattening stratification

$$\mathbb{P}^N = \bigcup_{d \geq 0} Z_{d,s}$$
such that when $d \geq 1$ we have

$$\operatorname{cod} Z_{d,s} \geq \operatorname{cod} Z_{d+a,0} + s - a$$
for some $a$ with $0 \leq a \leq s$. By hypothesis we get $\operatorname{cod} Z_{d,s} \geq s + 1$ for $d \geq 1$. 

The above process gives a diagram

\[
\begin{array}{c}
\mathcal{O}^{s}_{\mathbb{P}^{N}} \quad \mathcal{O}^{s+1}_{\mathbb{P}^{N}} \\
\mathcal{E}^{-1} \quad \mathcal{E}^{-1} \\
\mathcal{O}^{r+n+m}_{\mathbb{P}^{N}} \quad \mathcal{O}^{r+n+m-s}_{\mathbb{P}^{N}} \quad T_{0} \quad T_{s}
\end{array}
\]

We may take \( \varphi^{-1} \) to be the lower middle vertical map. Then \( \alpha \circ \varphi \) degenerates in \( \text{Supp} T_{s} \subseteq \bigcup_{d \geq 1} Z_{d,s} \) which has codimension \( \geq s+1 \). But since \( s \) is the difference of ranks between \( \mathcal{E}^{-1} \) and \( \mathcal{O}^{r+n+m-s}_{\mathbb{P}^{N}} \), we know that the codimension is at most \( s+1 \). Thus it is exactly \( s+1 \).

Proof of the existence part of the Main Theorem. Given \( a, b, c \) and \( k \) satisfying part 1 of the Main Theorem. Let \( r = c \). Choose \( n \) and \( m \) with \( |n-m| \leq 1 \) such that \( b - 2c = n + m \). By (3) there exists a monad

\[
\mathcal{O}^{N}_{\mathbb{P}^{N}}(1) \quad \mathcal{O}^{a}_{\mathbb{P}^{k}} \quad \mathcal{O}^{b}_{\mathbb{P}^{k}} \quad \mathcal{O}^{c}_{\mathbb{P}^{k}}
\]

with \( N = n + m + 1 \). By Lemmata 2 and 3 we may assume that \( \alpha \) degenerates in codimension \( b - c + a + 1 \). By restricting to a general subspace \( \mathbb{P}^{k} \subseteq \mathbb{P}^{N} \) we get part 1 of the Main Theorem.

To prove the existence in part 2 we may assume that \( b \leq 2c + k - 1 \) since else we may refer to part 1. Since \( b \geq a + c + k \) we get \( a \leq c - 1 \). Thus

\[
b \geq a + c + k \geq 2a + 1 + k \geq 2a + k - 1.
\]

But then by part 1 there exists a monad

\[
\mathcal{O}^{k}_{\mathbb{P}^{k}}(1) \quad \mathcal{O}^{b}_{\mathbb{P}^{k}} \quad \mathcal{O}^{c}_{\mathbb{P}^{k}}
\]

with \( \beta \) degenerating in codimension \( b - a - c + 1 \geq k + 1 \). But then \( \beta \) does not degenerate. Dualizing we get a monad

\[
\mathcal{O}^{k}_{\mathbb{P}^{k}}(1) \quad \mathcal{O}^{b}_{\mathbb{P}^{k}} \quad \mathcal{O}^{c}_{\mathbb{P}^{k}}
\]

with \( \alpha \) not degenerating. \( \square \)

2. Necessity of Conditions

Suppose now we have given a monad

\[
\mathcal{O}^{k}_{\mathbb{P}^{k}}(1) \quad \mathcal{O}^{b}_{\mathbb{P}^{k}} \quad \mathcal{O}^{c}_{\mathbb{P}^{k}}.
\]

We wish to prove the numerical conditions on \( a, b, c \) and \( k \) given in the Main Theorem. The image of

\[
\Gamma(\mathcal{O}^{k}_{\mathbb{P}^{k}}(1)) \quad \Gamma(\mathcal{O}^{b}_{\mathbb{P}^{k}}(1)) \quad \Gamma(\mathcal{O}^{c}_{\mathbb{P}^{k}}(1))
\]

determines a subspace \( V \subseteq \Gamma(\mathcal{O}^{k}_{\mathbb{P}^{k}}(1)) \) which generates the bundle \( \mathcal{O}^{k}_{\mathbb{P}^{k}}(1) \) since \( \beta \) is surjective. Also \( \dim V \geq c + k \) since otherwise \( \beta \) would degenerate.
in a non-empty subscheme of codimension \( \dim V - c + 1 \) by [Fu] 14.4.13. Let \( U \subseteq V \) be a general subspace of dimension \( c + k - 1 \). Then the map

\[
U \otimes \mathcal{O}_{\mathbb{P}^k} \to \mathcal{O}_{\mathbb{P}^k}(1)^c
\]

degenerates in dimension 0, again by [Fu] 14.4.13. Fix a splitting

\[
\Gamma(\mathcal{O}_{\mathbb{P}^k}^b) \to V.
\]

Let \( W = \Gamma(\mathcal{O}_{\mathbb{P}^k}^b)/U \) and \( S = K[x_0, \ldots, x_k] \). We get a diagram of free \( S \)-modules.

\[
\begin{array}{ccc}
U \otimes S & \longrightarrow & U \otimes S \\
\downarrow & & \downarrow^p \\
S(-1)^a & \longrightarrow & S^b \quad \longrightarrow \quad S(1)^c \\
| & | & | \\
S(-1)^a & \longrightarrow & W \otimes S
\end{array}
\]

Let \( \tilde{p} \) and \( \tilde{q} \) denote the corresponding maps of sheaves. We note that there is a surjection

\[
\text{cok} \tilde{q} \to \text{cok} \tilde{p} \to 0.
\]

Since \( \tilde{p} \) degenerates in expected codimension, by [Bu-Ei] Theorem 2.3 we have an equality

\[
\text{Fitt}_1(\text{cok} \tilde{p}) = \text{Ann}(\text{cok} \tilde{p})
\]

where \( \text{Fitt}_1(\text{cok} \tilde{p}) \) is the first Fitting ideal generated by the \( c \times c \) minors of the matrices (locally) representing \( \tilde{p} \). We now get

\[
\text{Fitt}_1(\text{cok} \tilde{q}) \subseteq \text{Ann}(\text{cok} \tilde{q}) \subseteq \text{Ann}(\text{cok} \tilde{p}) = \text{Fitt}_1(\text{cok} \tilde{p})
\]

where the first inclusion is valid if we replace \( \text{cok} \tilde{q} \) by any coherent sheaf, [Ei] Proposition 20.7.a. This gives

\[
\text{Fitt}_1(\text{cok} q) \subseteq \Gamma_* \text{Fitt}_1(\text{cok} \tilde{q}) \subseteq \Gamma_* \text{Fitt}_1(\text{cok} \tilde{p}).
\]

Since \( p \) degenerates in expected codimension \( k \), and \( S \) is a Cohen-Macaulay ring, \( S/\text{Fitt}_1(\text{cok} p) \) will be a Cohen-Macaulay ring of dimension 1 by [Ei] Theorem 18.18 or [Fu] Theorem 14.4.c. Thus the irrelevant maximal ideal \( \mathfrak{m} \subseteq S \) is not an associated prime of \( \text{Fitt}_1(\text{cok} p) \) and this is thus a saturated ideal. This gives

\[
\Gamma_* \text{Fitt}_1(\text{cok} p) = \text{Fitt}_1(\text{cok} p).
\]

Since now \( \text{Fitt}_1(\text{cok} p) \) is generated by polynomials of degree \( \geq c \), no polynomial in \( \text{Fitt}_1(\text{cok} q) \) will have degree \( < c \). Note that since \( \alpha \) is injective and \( S^b \to W \otimes S \) is a general quotient, the map \( q \) may be assumed to generically have maximal rank. If therefore \( q \) is generically surjective, we must have

\[
\dim W \geq c.
\]
Otherwise we have
\[ \dim W > a. \]
Since \( \dim W = b - c - k + 1 \) this gives
\[ b \geq 2c + k - 1 \]
or
\[ b \geq a + c + k. \]
This proves the necessity of the conditions in the Main Theorem.

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