Some peculiarities in response to filling up the Fermi sphere with quarks

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received 8 May 2009; accepted in final form 4 September 2009
published online 1 October 2009

PACS 11.10.-z – Field theory
PACS 11.15.Tk – Other nonperturbative techniques

Abstract – Considering quarks as the quasi-particles of the model Hamiltonian with four-fermion interaction we study response to the process of filling up the Fermi sphere with quarks, calculate the vacuum pressure and demonstrate the existence of filled-in state degenerate with the vacuum one.

Despite the highest theoretical status of Quantum Chromodynamics (QCD) nowadays there are still a lot of problems in hadron physics in which its predicting power and possibilities of the quantitative calculations are rather poor. Such a situation gives rise to the dedicated approaches (effective models) complementary to QCD which successfully clarify its particular aspects but it is very often unclear to which extent their specific features could be shared by the fundamental first-principle QCD. Moreover, sometimes these effective models lead to the unexpected results which evoke the intrinsic theoretical interest and could be indicative for examing QCD. It concerns, first of all, the conceptual idea of a complicated structure of the QCD vacuum [1] having populated by intensive stochastic gluon fields with nontrivial topological properties which form the basis for treating the quark behaviour. Studying the corresponding cooled lattice configurations gives evidence of this component contributing to the physical characteristics of hadrons and, for example, the instantons in the singular gauge to fit the data turns out so helpful and practical that allows one to evaluate the respective ensemble density and the distinctive size of a saturating configuration. Both estimates are in fairly good agreement with the corresponding results of instanton liquid model. Nevertheless, the keen search of such configurations as confining ones is still going on and collects more and more convincing confirmations that the construction of self-consistent ensemble of such configurations is quite serious problem, indeed (see, for example, the estimate for the (anti-)instanton ensemble obtained in ref. [2]).

In the present paper we develop such a model to study an ensemble of quarks as the quasi-particles of the effective Hamiltonian (density) with four-fermion interaction

$$\mathcal{H} = -\bar{q}(i\gamma \nabla + im)q - q\bar{\epsilon}^\alpha \gamma_\alpha q \int dy \bar{q}' \bar{\epsilon}^b \gamma_\alpha q' g^2(A^a_{\alpha}A^b_{\alpha}),$$  \hspace{1cm} (1)

taken in the form of a product of two coloured currents placed in the spatial points \(x\) and \(y\) which are connected by a form factor being translation invariant function of spatial variables. In eq. (1) \(q = q(x), \bar{q} = \bar{q}(x), q' = q(y), \bar{q}' = \bar{q}(y)\) are the (anti-)quark operators,

$$q_{\alpha i}(x) = \int \frac{dp}{(2\pi)^3} \frac{1}{(2|p_4|)^{1/2}} \left[ a(p, s, c)u_{\alpha i}(p, s, c) e^{i\vec{p}\vec{x}} + b^+(p, s, c) v_{\alpha i}(p, s, c) e^{-i\vec{p}\vec{x}} \right],$$  \hspace{1cm} (2)

where \(p_4^2 = -\vec{p}^2 - m^2\), \(i\) is the colour index, \(\alpha\) is the spinor index in the coordinate space, \(a^+, a\) and \(b^+, b\) are the creation and annihilation operators of quarks and anti-quarks, \(a|0\rangle = 0, b|0\rangle = 0\) and \(|0\rangle\) is the vacuum state of the free Hamiltonian. Whenever a summation over indices \(s\) and \(c\) is meant, the index \(s\) describes two spin polarization of the quark and index \(c\) plays the similar role for the colour. \(t^a = \lambda^a/2\) are the generators of colour gauge group \(SU(N_c)\) and \(m\) is the current quark mass. The Hamiltonian density is considered in

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the Euclidean space and $\gamma_\mu$ denotes the hermitian Dirac matrices, $\mu, \nu = 1, 2, 3, 4$. $(A^\mu_r A^\nu_r)(x)$ stands for the form factor $(A^\mu_r A^\nu_r)(x) = \delta^{ab} \frac{2}{i\pi^2} \left[ \delta_{\mu\nu} I(x) - J_{\mu\nu}(x) \right]$, and the second term here is spanned onto the vector of relative distance. Actually the effective Hamiltonian (1) results from the averaged description of quarks influenced by intensive stochastic gluon field $A^\mu_r$ (for example, by an (anti)-instanton ensemble) and has been used to analyse the ground state of such a system $|\sigma\rangle$ with the Bogolyubov trial function composed by the quark–anti-quark pairs with opposite momenta and with vacuum quantum numbers [3], i.e.

$$|\sigma\rangle = T|0\rangle,$$

$$T = \Pi_p \exp \{ \varphi^+ (p, s) b^+ (-p, s) + a(p, s) b (-p, s) \}.$$  

Disentangling the exponentials eq. (3) can be presented by the standard form as a mixed state with the vacuum quantum numbers

$$|\sigma\rangle = \Pi_p \langle \cos \varphi + \sin \varphi a^+ (p, s) b^+ (-p, s) |0\rangle.$$

In this formula and below, in order to simplify the notations, we refer to only one complex index $s$ which means both polarization indices (spin and colour). The parameter $\varphi(p)$ describing the pairing strength is determined by the minimum of the mean energy $E = \langle \sigma | H | \sigma \rangle$, $H = \int \text{d}x \mathcal{H}$. Introducing the dressing transformation $T$ we define the creation and annihilation operators of quasi-particles as $A = TaT^\dagger$, $B^+ = Tb^+ T^\dagger$, and the quark field operators are presented in the following form:

$$q(x) = \int \frac{dp}{(2\pi)^3 \sqrt{2|p|}} \left[ A(p, s) U(p, s)e^{ipx} + B^+(p, s)V(p, s)e^{-ipx} \right],$$

$$\bar{q}(x) = \int \frac{dp}{(2\pi)^3 \sqrt{2|p|}} \left[ A^+(p, s) \bar{U}(p, s)e^{-ipx} + B(p, s)\bar{V}(p, s)e^{ipx} \right].$$

Then the spinors $U$ and $V$ are given as

$$U(p, s) = \cos(\varphi) u(p, s) - \sin(\varphi) v(-p, s),$$

$$V(p, s) = \sin(\varphi) u(-p, s) + \cos(\varphi) v(p, s),$$

where $\bar{U}(p, s) = U^+(p, s)\gamma_4$, $\bar{V}(p, s) = V^+(p, s)\gamma_4$ are the Dirac conjugated spinors.

Now our central issue could be formulated in the following way— to construct the state filled in by quasi-particles (the letter determinant)

$$|N\rangle = \prod_{|p| < p_F, s} A^+(P; s)|\sigma\rangle,$$

which possesses the minimal mean energy $\langle N | H | N \rangle$ (surely, we assume the quasi-particles are stable). From here on we use capital letters to designate the momenta and polarizations of quasi-particles which fill in the Fermi sphere and draw the small letters in the other cases. In eq. (4) $P_F$ stands for the Fermi momentum and the polarization runs over all possible values. It allows us to optimize the dressing transformation and, as a consequence, to follow up the modifications of quasi-particles being influenced by the process of filling in the Fermi sphere. Eventually it fixes the form of the charge operator (particle number operator) $|N^q | q |N\rangle$.

Omitting the details of calculating the corresponding matrix elements, let us define the partial energy density per one quark degree of freedom, as

$$w_{vac} = \langle N^q | q | N \rangle.$$

We perform a natural regularization subtracting the mean energy of ensemble $E_{vac}$, $E = E/V$, where $E$ is the total energy of ensemble. Using the diagonal matrix element of $H$ we have for the state with the single quasi-particle following ref. [3]

$$w_1 = w_{vac} + |P| \cos \theta$$

$$+ 2G \int \frac{dp}{(2\pi)^3} \frac{Pq}{|Pq|} \left( \sin \theta - \frac{m}{p} \cos \theta \right) \left( \sin \theta - \frac{m}{q} \cos \theta \right) (I - J/4),$$

where $I = \bar{I}(p + q)$, $J_{ij} = J_{ij}(p + q)$, $J = \Sigma_{i=1}^3 J_{ii}$, $\theta = 2\varphi(\theta^\prime = \theta(q))$ and the term $w_{vac}$ describes the vacuum contribution, and the colour index for convenience is taken $G = \frac{2G^2}{N_c}$.

For the ensemble of quasi-particles (for simplicity we consider only the situation when the correlator $J_{\mu\nu} = 0$) we obtain the following expression for the partial energy:

$$\langle N | w_1 | N \rangle = \int_{P_F} \frac{dp}{(2\pi)^3} |P| + \int_{P_F} \frac{dp}{(2\pi)^3} \left( \sin \theta - \frac{m}{p} \cos \theta \right)$$

$$- G \int_{P_F} \frac{dp}{(2\pi)^3} \left( \sin \theta - \frac{m}{q} \cos \theta \right) \left( \sin \theta - \frac{m}{q} \cos \theta \right) I.$$

We perform a natural regularization subtracting the contribution of free Hamiltonian $H_0$. As a result there appears a unit in the bracket containing cos $\theta$ in the first line. It could have rather interesting interpretation if compared to the vacuum mean energy

$$w_{vac} = \langle N | w \rangle = \int \frac{dp}{(2\pi)^3} \left( \sin \theta - \frac{m}{p} \cos \theta \right) \times \frac{dq}{(2\pi)^3} \left( \sin \theta' - \frac{m}{q} \cos \theta' \right) I.$$
It is easy to see that for the state with the filled-in Fermi sphere the angles of pairing could be defined by the condition of functional minimum (5) only for momenta larger than the Fermi momentum $P_F$ (a similar infrared cutoff was used in [4]). Then the quarks composing the Fermi sphere look like the free (non-interacting) ones, as seen from the first term of eq. (5). Now let us calculate the quark chemical potential which, by definition, is an energy necessary for adding (removing) one quasi-particle to (from) a system. It allows us to obtain a well-known self-consistent gap equation for the dynamical quark mass. For the parameter set given by ref. [6], and limit the integration interval over momentum with the quantity $|p| < \Lambda$ ($\Lambda = 631$ MeV). Then the functional of eq. (5) is written in the following form (inessential terms contributing to the constant values are omitted)

$$w = w_0 + \int_{\text{Fermi}} \frac{dp}{(2\pi)^3} |p_4| (1 - \cos \theta)$$

$$- \int_{\text{Fermi}} \frac{dp}{(2\pi)^3} G \frac{p}{|p_4|} \left( \sin \theta - \frac{m}{p} \cos \theta \right)$$

$$\times \int_{\text{Fermi}} \frac{dq}{(2\pi)^3} \frac{q}{|q_4|} \left( \sin \theta' - \frac{m}{q} \cos \theta' \right),$$

(6)

where $w_0 = \int_{\text{Fermi}} \frac{dp}{(2\pi)^3} |p_4|$ is the contribution coming from the free quarks and $m = 5.5$ MeV. The equation to calculate the equilibrium angle $\theta$ reads as

$$\left( p^2 + m^2 \right) \sin \theta - M (p \cos \theta + m \sin \theta) = 0,$$

(7)

where the constituent quark mass is

$$M = 2G \int_{\text{Fermi}} \frac{dp}{(2\pi)^3} \frac{p}{|p_4|} \left( \sin \theta - \frac{m}{p} \cos \theta \right).$$

(8)

It allows us to obtain a well-known self-consistent gap equation for the dynamical quark mass. For the parameters used the dynamical quark mass $M_q = M - m$, at zero Fermi momentum is $M_q = -335$ MeV and for the quark condensate $\langle N \bar{q} q | N \rangle = \frac{N_c}{2} \int_{\text{Fermi}} \frac{dp}{(2\pi)^3} (p \sin \theta - m \cos \theta)$, we have $\langle \sigma \bar{q} q | \sigma \rangle = -i(247$ MeV)$^3$. The constant characterizing a strength of four-fermion interaction was taken as $G\Lambda^2/(2\pi^2) = 1.34$. In fig. 1 the dynamical quark mass as a function of the Fermi momentum is depicted. For comparison the data are presented for current quark mass $m = 5.5$ MeV (the solid line) and the dashed line corresponds to the calculation in the chiral limit. For the NJL model, in particular, the quark chemical potential equals

$$\mu = |P^F_4| \cos \theta_F + M \frac{P_F \sin \theta_F - m \cos \theta_F}{|P^F_4|}.$$

(9)

When the Fermi momentum reaches the zero value the chemical potential quantity coincides with dynamical quark mass $\mu(0) = |M| - Mm/|M| = |M| + m$. In fig. 2 the quark chemical potential is depicted as a function of the Fermi momentum for the configurations analogous to the ones shown in fig. 1. The chemical potential dependence on the Fermi momentum showing up could be interpreted as the effect of rapid decrease of the dynamical quark mass with the Fermi momentum increasing. Then using eq. (7) the chemical potential may be presented as
The dynamical quark mass ($|M_q|$) is plotted as a function of the chemical potential. The solid line corresponds to the current quark mass $m = 5.5 \text{ MeV}$. The dashed one shows the behaviour in the chiral limit.

\[ \mu = \frac{MP_F}{|P_F| \sin \theta_F}, \]
and taking into account the identity
\[ (|P_1|^2 - Mm)^2 + M^2 P^2 = [P^2 + (M - m)^2]|P_4|^2, \]
we come to the noteworthy definition of the chemical potential
\[ \mu = |P_2^2 + M_q^2|^{1/2}. \]

Let us remind that for the free fermion gas the chemical potential increases monotonically with the Fermi momentum growing. The curious feature of the NJL model is the appearance of state almost degenerate with the vacuum state while the process of filling up the Fermi sphere reaches to the momenta close to the dynamical quark mass value (the similar value is peculiar to the momentum of quark inside a baryon, see, for example, [7]). This state density with the factor 3 (which expresses the relation between baryonic and quark degrees of freedom) absorbed corresponds to a normal nuclear density ($n \sim 0.12/\text{fm}^3$), and chiral condensate could be estimated as $|\langle \bar{q}q \rangle|^{1/3} \sim 100 \text{ MeV}$. In the chiral limit the chemical potential is close to the discussed value and is even smaller than the vacuum one. The coincidence of the chemical potentials occurs at the values of current quark mass around $2 \text{ MeV}$. In fact, fig. 2 shows that the u-quark bond looks stronger than one of the d-quark. For clarity in fig. 3 the dynamical quark mass, is plotted as a function of chemical potential. The pressure of the quark ensemble
\[ P = -\frac{dE}{dV} = -\frac{\partial E}{\partial V} + P_F \frac{\partial E}{\partial P_F} = -E + \mu n, \]
is depicted in fig. 4 as a function of the Fermi momentum where $n = N/V$ is the quark density. The quark pressure at the values of the Fermi momentum close to the quantity of dynamical quark mass is approximately degenerate with the vacuum pressure (slightly lower than the vacuum one).

|\begin{figure}[ht]
|\begin{minipage}{0.5\textwidth}
Fig. 3: The dynamical quark mass ($|M_q|$) as a function of the chemical potential. The solid line corresponds to the current quark mass $m = 5.5 \text{ MeV}$. The dashed one shows the behaviour in the chiral limit.
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|\begin{minipage}{0.5\textwidth}
Fig. 4: The pressure of the quark ensemble as a function of the Fermi momentum. The solid line corresponds to the current quark mass $m = 5.5 \text{ MeV}$. The dashed line shows the behaviour in the chiral limit.
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\end{figure}|

The vacuum pressure is of order $40-50 \text{ MeV/fm}^3$ and corresponds well to the value extracted from the bag models, see, for example, ref. [7]. Actually, all the NJL results could be received in the mean-field approximation because some terms in the mean energy definition eq. (6) can be rewritten (using eq. (9) again) as the functions of dynamical and current quark masses in the following form:

\[ \frac{p \sin \theta - m \cos \theta}{|p_4|} = \frac{M_q}{|p^2 + M_q^2|^{1/2}}, \]
\[ |p_4| \cos \theta = \frac{p^2 - mM_q}{|p^2 + M_q^2|^{1/2}}. \]

In particular, drawing these definitions we are able to demonstrate that at the extremal curves the following representation of functional is valid:

\[ w = w_{\text{vac}} + \int_{2}^\Lambda \frac{dp}{(2\pi)^3} E_q n - \tilde{\mu} \tilde{N} - \tilde{T} \tilde{S}, \]
\[ w_{\text{vac}} = \int_{2}^\Lambda \frac{dp}{(2\pi)^3} (|p_4| - E_q) + \frac{M_q + m}{4G}, \]
\[ \tilde{S} = -\int_{2}^\Lambda \frac{dp}{(2\pi)^3} [n \ln n - (1 - n) \ln(1 - n)]. \]

$n$ is the density of ensemble of the quasi-particles with the mass $M_q$ which is defined at fixed chemical potential $\tilde{\mu}$ (as a matter of fact it precisely corresponds to the chemical potential $\mu$ which we calculated here since we knew the Fermi momentum, see also [8]) and temperature $\tilde{T}$. In the considered situation $\tilde{T} = 0$. Equation (7) (together
with eq. (8)) which defines the dynamical quark mass is identical with the well-known gap equation (in our case the stable branch corresponds to a negative value of quark mass)

$$M_q + m = 2G \int \frac{dp}{(2\pi)^3} \frac{M_g}{E_q} (1 - n).$$

It is clear just the approach exploiting the direct construction of the Fermi sphere, which we present here, made it possible to discover the solution branches unnoticed in previous analyses of this equation. They are shown in fig. 3 and correspond to the multiphiered solutions as a function of chemical potential.

Obviously in the NJL model the Fermi momentum is allowed to increase until the magnitude of cutoff parameter $\Lambda_{NJL}$, although, in principle, such an extrapolation is possible into the region of higher Fermi momenta in the models with less naive form factor behaviour. In order to trace back the dependence of all results on the form factor we consider the model (in a sense, opposite to tracebackthedependenceofallresultsontheform

In summary we would like to emphasize that the considered approach allowed us to find new branch of mean-field equation solution overlooked before. Its results coincide with the ones of standard technique which based on the concept of bare quark chemical potential $\mu_{u,d}$ at large magnitude of the Fermi momenta only (large chemical potentials of bare quarks) and much more suitable for studying an interrelation between the chiral symmetry breaking and deconfinement in confining models. Apparently our estimate of the effects responding to the process of filling up the Fermi sphere entails a hope to understand a routine feature of hadron world, namely, the fact of quark equilibrium in the vacuum. The chemical potential degeneracy and specific behaviour of the quark pressure (with one new essential element which is just the presence of instability region $dP/dP_F < 0$) justify, in principle, the conventional bag model. Very crude estimate of the quasi-particle density which can be done shows it is small but entirely finite (what is a key moment). At first sight, this result makes a hint at a possibility to consider the states (4) as a natural “building” material for baryon octet and to develop an approach à la the Thomas-Fermi model [10] (including the obvious corrections similar to those in the atomic physics) or to adapt properly the phenomenological density functional theory [11]. However, the answer which we reached on this way up to now is not satisfactory and we believe the idea of constructing soliton (Skyrme)-type energy functional [12] is more relevant.

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The unyielding character of our referee and his highly professional report are greatly appreciated. This work was supported by the INTAS Grant 04-84-398 and the Grant of National Academy of Sciences of Ukraine.

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