ON EXPLICIT PARAMETRISATION OF
SPECTRAL CURVES FOR MOSER-CALOGERO
PARTICLES AND ITS APPLICATIONS

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Abstract. The system of $N$ classical particles on the line with the Weierstrass $\wp$ function as potential is known to be completely integrable. Recently D'Hoker and Phong found a beautiful parameterization by the polynomial of degree $N$ of the space of Riemann surfaces associated with this system. In the trigonometric limit of the elliptic potential these Riemann surfaces degenerate into rational curves. The D'Hoker-Phong polynomial in the limit describes the intersection points of the rational curves. We found an explicit determinant representation of the polynomial in the trigonometric case. We consider applications of this result to the theory of Toeplitz determinants and to geometry of the spectral curves. We also prove our earlier conjecture on the asymptotic behavior of the ratio of two symplectic volumes when the number of particles tends to infinity.

1. Introduction. Since its discovery in the middle of seventies by Moser and Calogero, the system of classical particles on the line with the Hamiltonian

$$H = \sum_{n=1}^{N} \frac{p_n^2}{2} - \frac{m^2 \sigma^2}{2} \sum_{n,r=1}^{N} \wp(q_n - q_r), \quad \text{where } \sigma^2 = \pm 1, \ m > 0,$$

has attracted a lot of attention from mathematicians and physicists. This system is completely integrable in the sense of Liouville, i.e., it has $N$ commuting integrals of motion. Algebro-geometrical methods in the study of this system were introduced by Krichever, [K1]. He showed that the Riemann surfaces —

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spectral curves, associated with this system are $N$–sheeted covers of the elliptic curve of the potential, the Weierstrass $\wp$ function. The geometry of such covers is fairly complicated and was not understood well until now.

In the papers of Seiberg–Witten and Donagi–Witten, [SW,DW], this system was considered in connection with Super-Symmetric Yang-Mills quantum field theory. Degeneration of the elliptic potential into a trigonometric one is interpreted by physicists as a classical limit of the quantum system. In such limit genus–$N$ spectral curves degenerate into genus–0 curves. Recently D’Hoker and Phong, [DP], found a beautiful parameterization by a polynomial $Q(p)$ of the space of spectral curves for particles with elliptic potential. In the trigonometric limit the zeros $k_n$ of the polynomial $Q(p) = m^N \prod_{n=1}^{N}(p - k_n)$ describe intersection points of the limiting rational curve. We have found a determinant representation of the polynomial $Q(p)$ in this limiting trigonometric case. Our description becomes completely explicit when the mechanical system is at the ground state: all particles at rest, placed equidistantly on the circle. In this case $Q(p)$ can be computed explicitly using the determinant representation. This provides us with complete information about the spectral curve. In all other cases, such as an elliptic or a system not at the ground state, the curve can be viewed as a deformation of this simple case.

In this paper we present a simple proof of the D’Hoker-Phong theorem, our determinant representation of $Q(p)$ and some of its’ various applications. The paper is organized as follows. In section 2 we present some well-known facts about particles with elliptic potential. In section 3 we give a simple proof of the D’Hoker–Phong theorem and prove our main result: an explicit formula for the polynomial $Q(p)$. In the three subsequent sections we consider various applications of our result: to the theory of Toeplitz determinants (section 4), to the geometry of spectral curves (section 5) and finally to the asymptotic behavior of the ratio of two symplectic volumes, when the number of particles tends to infinity (section 6).

2. Commutator formalism. Riemann surface. The $N$-particle Hamiltonian

$$H = \sum_{n=1}^{N} \frac{p_n^2}{2} - \frac{m^2 \sigma^2}{2} \sum_{n,r=1}^{N} \wp(q_n - q_r),$$

produces the equations of motion

\begin{align*}
\dot{q}_n &= \frac{\partial H}{\partial p_n} = p_n, \quad n = 1, \ldots, N, \\
\dot{p}_n &= -\frac{\partial H}{\partial q_n} = m^2 \sigma^2 \sum_{r \neq n} \wp'(q_n - q_r), \quad n = 1, \ldots, N.
\end{align*}

For attractive particles the parameter $\sigma = 1$ and for repulsive particles $\sigma = i$; the parameter $m > 0$ plays the role of mass.
For \( m = 2 \) the \( N \)-particle system can be embedded into elliptic solutions of the Kadomtzev-Petviashvilli (KP) equation. The key step is the following theorem

**Theorem 1.** [K1]. The equations

\[
\begin{align*}
\sigma \partial_y - \partial_x^2 + 2 \sum_{n=1}^{N} \wp(x - q_n(y)) &= 0, \\
\psi^+ \sigma \partial_y - \partial_x^2 + 2 \sum_{n=1}^{N} \wp(x - q_n(y)) &= 0
\end{align*}
\]

have solutions of the form

\[
\begin{align*}
\psi(x, y, k, z) &= \sum_{n=1}^{N} a_n(y, k, z) \Phi(x - q_n, z) e^{kx + \sigma^{-1/2}ky}, \\
\psi^+(x, y, k, z) &= \sum_{n=1}^{N} a_n^+(y, k, z) \Phi(-x + q_n, z) e^{-kx - \sigma^{-1/2}ky},
\end{align*}
\]

where \( \Phi(q, z) = \frac{\sigma(z-q)}{\sigma(z)\sigma(q)} \), if and only if the functions \( q_n(y) \) satisfy the system of equations (2.1).

The operator \( L_2 \) with the elliptic potential

\[
L_2 = \partial_x^2 - u(x, y) = \partial_x^2 - 2 \sum_{n=1}^{N} \wp(x - q_n(y))
\]

enters into the commutator formula

\[
[\sigma \partial_y - L_2, \partial_t - L_3] = 0,
\]

with \( L_3 = \partial_x^3 - \frac{3}{2} u \partial_x - w \). These imply that \( u \) satisfies the KP equation

\[
\frac{3}{4} \sigma^2 u_{yy} = (u_t + \frac{3}{2} u u_x - \frac{1}{4} u_{xxx})_x.
\]

For arbitrary mass \( m > 0 \), the system (2.1) is equivalent to the matrix equation \( \sigma L = [L, M] \), with \( N \times N \) matrices \( M \) and \( L \) having entries

\[
\begin{align*}
L_{nr} &= \sigma p_n \delta_{nr} + m \Phi(q_n - q_r, z)(1 - \delta_{nr}), \\
M_{nr} &= \left( -\wp(z) + m \sum_{s \neq n} \wp(q_n - q_s) \right) \delta_{nr} + m \Phi'(q_n - q_r, z)(1 - \delta_{nr}).
\end{align*}
\]

\( \sigma(z) \) denotes the Weierstrass function. See section 7 for definitions.
The matrix $L$ can be simplified using a gauge transformation

$$L = G\tilde{L}G^{-1},$$

where $G_{nr} = e^{\zeta(z)q_n}\delta_{nr}$ and $	ilde{L}_{nr} = \sigma p_{nr}\delta_{nr} + m\Phi_0(q_n - q_m, z)(1-\delta_{nm})$, $\Phi_0(q, z) = \frac{\sigma(z-q)}{\sigma(z)\sigma(q)}$. The spectrum of $L$ is preserved and this determines the curve

$$\Gamma \equiv \{(k, z) : R(k, z) = \det(L + mk) = \det(\tilde{L} + mk) = 0\}.$$

We denote by lower and upper Greek letters the points $(k, z)$ on the curve $\Gamma$, for example, $\gamma, \Pi, \text{etc.}$.

**Remark.** An exchange of positions and velocities of particles labeled, say, $n$ and $r$ does not affect the curve. It simply permutes the $n$-th and $r$-th columns and rows of the matrix $L$.

The following facts about the curve can be found in [K1].

(i.) The determinant $R(k, z)$ can be written in the form

$$R(k, z) = \sum_{n=0}^{N} r_n(z)k^n,$$

where $r_n(z)$ are elliptic functions of $z$. This means that $\Gamma$ is an $N$–sheeted covering of the elliptic curve $\Gamma_0$.

(ii.) In the vicinity of zero

$$R(k, z) = m^N \left[ k - \left( \frac{N-1}{z} + k_1^{(0)} + \ldots \right) \right] \prod_{n=2}^{N} \left[ k - \left( -\frac{1}{z} + k_n^{(0)} + \ldots \right) \right].$$

The points $\Pi_n, \ n = 1, \ldots, N$ above $z = 0$ are called infinities. The infinity $\Pi_1$ corresponds to the “upper sheet”, where $k(z) = \frac{N-1}{z} + O(1)$.

(iii.) In the elliptic case, for the generic configuration of $N$ particles the genus of the curve $\Gamma$ is $N$. In the trigonometric case the curve is rational.

The following lemma describes symmetries of the matrix $L$. We assume that $2\omega$ is real and $2\omega'$ is pure imaginary.

**Lemma 2.** (i.) $\sigma = 1$. Let $\tau_1$ is defined as: $\tau_1(k, z) = (\bar{k}, \bar{z})$. Then

$$(L + mk)(\tau_1\gamma) = (L + mk)(\gamma).$$

(ii.) $\sigma = i$. Let $\tau_i$ is defined as: $\tau_i(k, z) = (-\bar{k}, -\bar{z})$. Then

$$(L + mk)(\tau_i\gamma) = -(L + mk)^T(\gamma).$$
Proof. (i). The statement follows from the identities
\[(L + mk)_{nn}(\tau_1 \gamma) = p_n + mk(\gamma) = (L + mk)_{nn}(\gamma),\]
\[(L + mk)_{nr}(\tau_1 \gamma) = m\Phi(q_n - qr, \bar{z}) = m\Phi(q_n - qr, z) = (L + mk)_{nr}(\gamma).\]

(ii). The proof is similar to that in the attractive case. □

Lemma 2 implies that in the attractive/repulsive case there exists an antiholomorphic involution \(\tau_1 / \tau_i\) on the curve \(\Gamma\).

For some special configurations of particles there is also an additional symmetry.

Lemma 3. Let \(\tau_-\) is defined as: \(\tau_-(k, z) = (-k, -z)\). If \(p_n = 0\) for all \(n = 1, \ldots, N\); then
\[(L + mk)(\tau_- \gamma) = -(L + mk)^T(\gamma).\]

Proof. It is enough to note that
\[(L + mk)_{nn}(\tau_- \gamma) = -mk(\gamma) = -(L + mk)_{nn}(\gamma),\]
\[(L + mk)_{nr}(\tau_- \gamma) = m\Phi(q_n - qr, -z) = -m\Phi(q_r - q_n, z) = -(L + mk)_{rn}(\gamma).\]

□

Lemma 3 implies that if \(p \equiv 0\), then there is an involution \(\tau_-\) on the curve.

3. D’Hoker-Phong parametrisation. Determinant Formula. Following D’Hoker-Phong, [DP], let us introduce
\[h_k(z) = \frac{\partial_k \theta_1(\frac{z}{2\omega})}{\theta_1(\frac{z}{2\omega})}, \quad k = 0, 1, \ldots\]

The function \(h_1(z) = \zeta(z) - \frac{2}{\omega} z\) has periodicity properties
\[h_1(z + 2\omega) = h_1(z),\]
\[h_1(z + 2\omega') = h_1(z) - \frac{i\pi}{\omega}.\]

Let us also introduce a multivalued function \(p(\gamma)\) on the curve \(\Gamma\) as
\[p(\gamma) \equiv k(\gamma) + h_1(z(\gamma)).\]

The function \(p(\gamma)\) has a simple pole at \(\Pi_1\) only and does not change under the shift \(z \rightarrow z + 2\omega\). It is defined up to an integer multiple of \(\frac{i\pi}{\omega}\) which corresponds to the increment of \(h_1\) under the vertical shift \(z \rightarrow z + 2\omega'\). It is called quasimomentum, because \(\psi(x, y, \gamma)\) defined in Theorem 1 has the Bloch property in the \(x\)-variable:
\[\psi(x + 2\omega, y, \gamma) = e^{p(\gamma)2\omega} \psi(x, y, \gamma).\]

So far we represented \(\Gamma\) as a curve in the direct product of the \(k\) and \(z\) variables. Now we want to write an equation for \(\Gamma\) in the direct product of \(p\) and \(z\).
Theorem 4. [DP]. There exists a polynomial $Q(p)$ of degree $N$,

$$Q(p) = m^N \sum_{n=0}^{N} p^{N-n} Q_n = m^N \prod_{n=1}^{N} (p - k_n),$$

such that

$$\frac{\theta_1 \left( \frac{z-\partial}{\omega} \right)}{\theta_1 \left( \frac{z}{2\omega} \right)} Q(p) = \sum_{n=0}^{\infty} \frac{h_n(z)}{n!} (-\partial)^n Q(p) = R(p - h_1(z), z).$$

Proof. * Performing the change of variables $k(\gamma) = p(\gamma) - h_1(z(\gamma))$ we have $R(p - h_1(z), z) = m^N \prod_{n=1}^{N} (p - p_n(z))$. The roots $p_n(z)$ do not change under the shift $z \rightarrow z + 2 \omega$. They transform as

$$p_{s_r}(z + 2\omega') = p_r(z) - \frac{i\pi}{\omega}$$

under the vertical shift $z \rightarrow z + 2\omega'$; $(s_1, \ldots, s_N)$ is some permutation of sheets. Similarly the operator

$$\frac{\theta_1 \left( \frac{z-\partial}{\omega} \right)}{\theta_1 \left( \frac{z}{\omega} \right)}$$

does not change under the horizontal shift and for the vertical shifts we have

$$\frac{\theta_1 \left( \frac{z+2\omega'-\partial}{2\omega} \right)}{\theta_1 \left( \frac{z+2\omega'}{\omega} \right)} = \frac{\theta_1 \left( \frac{z-\partial}{\omega} \right)}{\theta_1 \left( \frac{z}{\omega} \right)} e^{\frac{i\pi}{2}\partial}.$$

Because of these, the polynomial

$$Q(p, z) \equiv m^N \sum_{n=0}^{N} p^{N-n} Q_n(z) = \frac{\theta_1 \left( \frac{z}{\omega} \right)}{\theta_1 \left( \frac{z}{2\omega} \right)} R(p - h_1(z), z)$$

has elliptic coefficients $Q_n(z)$. The coefficients of the polynomial on the right hand side

$$R(p - h_1(z), z) = m^N \left[ p^N + \left( \sum_{n=1}^{N} -p_n(z) \right) p^{N-1} + \cdots + \left( \prod_{n=1}^{N} -p_n(z) \right) \right]$$

have singularities at $z = 0$ not higher than a simple pole. The operator

$$\theta \left( \frac{z-\partial}{\omega} \right) / \theta \left( \frac{z}{2\omega} \right)$$

does not decrease the order of a singularity. Therefore, the

*This short proof was communicated to me by I. Krichever*
elliptic coefficients \( Q_n(z) \) have a singularity at \( z = 0 \) not higher than a simple pole, and these singularities do not depend on \( z \) at all. □

When we want to emphasize the dependence of all functions on modular parameter \( \tau \) we write \( h(z) = h(z|\tau) \), etc. In the trigonometric limit \( \tau \to +i\infty \) an elliptic curve \( \Gamma_0 \) degenerates into a sphere with two punctures, parametrized by \( w = e^{-\frac{i\pi}{w}} \). North (\( \beta \)) and south (\( \nu \)) punctures correspond to \( w = \infty \) and \( w = 0 \). The function \( p(\gamma) \) can be used as a global parameter on the limiting rational curve. The equation of the curve

\[
\theta_1 \left( \frac{z - \partial}{2\omega_1} \bigg| \tau \right) Q(p|\tau) = 0,
\]

using the definition of \( \theta_1 \) can be written as

\[
\sum_n (-1)^n h^{n(n-1)} w^{-n} Q \left( p + \frac{i\pi}{2\omega} - \frac{i\pi n}{\omega} \bigg| \tau \right) = 0.
\]

In the limit \( \tau = +i\infty \) it becomes

\[
Q \left( p + \frac{i\pi}{2\omega} +i\infty \right) - w^{-1} Q \left( p - \frac{i\pi}{2\omega} +i\infty \right) = 0.
\]

For each point \( \nu_n \in \Gamma, n = 1, \ldots, N \) above \( w = 0 \) there exists a point \( \beta_{s_n} \in \Gamma, \) above \( w = \infty \) such that

\[
(3.1) \quad p(\nu_n) - \frac{i\pi}{2\omega} = k_n = p(\beta_{s_n}) + \frac{i\pi}{2\omega}.
\]

It is easy to check that \( \nu_n \) and \( \beta_{s_n} \) correspond to a simple crossing.

Let us introduce the \( N \times N \) matrix \( \mathcal{L} \) such that

\[
\mathcal{L}_{nr} = \sigma p_n \delta_{nr} + m R(q_n - q_r)(1 - \delta_{nr}), \quad \text{where} \quad R(q) \equiv \frac{\pi}{2\omega \sin \frac{\pi q}{2\omega}}.
\]

**Theorem 5.** The polynomial \( Q(p|\tau) \) for \( \tau = +i\infty \) is given by the formula

\[
Q(p| +i\infty) = \det (\mathcal{L} + mp).
\]

**Proof.** It is enough to prove that

\[
(3.1) \quad \lim_{\tau \to +i\infty} \det \left( \tilde{L}(z \mid \tau) + mp \right) \bigg| _{z = \frac{\tau}{2\omega}} = \det (\mathcal{L} + mp),
\]
(3.2) \[ \lim_{\tau \to +i\infty} h_1(z|\tau) \bigg|_{z = \frac{\tau}{2}} = -\frac{i\pi}{2\omega}, \]

(3.3) \[ \lim_{\tau \to +i\infty} \frac{\theta_1 \left( \frac{z - \partial}{2\omega} \right)}{\theta_1 \left( \frac{z}{2\omega} \right)} \bigg|_{z = \frac{\tau}{2}} = e^{\frac{i\pi}{2\omega}}. \]

After that, the D’Hoker-Phong formula taken at \( z = \frac{\tau}{2} \):

\[
\theta_1 \left( \frac{z - \partial}{2\omega} \right) \bigg|_{z = \frac{\tau}{2}} Q(p|\tau) = \det \left( \bar{L}(z|\tau) + m(p - h_1(z)) \right) \bigg|_{z = \frac{\tau}{2}}
\]
implies the statement, when \( \tau \to +i\infty \).

In order to prove (3.1), note that

\[
\Phi_0(q,z|\tau) \bigg|_{z = \frac{\tau}{2}} = \frac{1}{\text{sn}(q|\tau)} e^{-\eta' q}
\]
and

\[
\lim_{\tau \to +i\infty} \frac{1}{\text{sn}(q|\tau)} = R(q).
\]

To prove (3.2), note that

\[
\partial^k \theta_1 \left( \frac{z}{2\omega} \right) = i \sum_n (-1)^n h(\frac{2n-1}{2\omega})^2 \left[ \frac{i\pi}{2\omega} (2n-1) \right]^k e^{\frac{i\pi}{2\omega} (2n-1)}.
\]

Therefore,

\[
\left. \partial^k \theta_1 \left( \frac{z}{2\omega} \right) \right|_{z = \frac{\tau}{2}} = i \left[ \frac{i\pi}{2\omega} \right]^k e^{-i\pi \tau / 4} + O \left( e^{\frac{i\pi}{2\omega}} \right),
\]
when \( \tau \to +i\infty \). Finally,

\[
\left. h_k(z|\tau) \right|_{z = \frac{\tau}{2}} = -\left[ \frac{i\pi}{2\omega} \right]^k.
\]

This completes the proof of (3.2).

In order to prove the last identity (3.3) we use the previous result

\[
\lim_{\tau \to +i\infty} \frac{\theta_1 \left( \frac{z - \partial}{2\omega} \right)}{\theta_1 \left( \frac{z}{2\omega} \right)} \bigg|_{z = \frac{\tau}{2}} = \lim_{\tau \to +i\infty} \sum_{k=0}^{\infty} \frac{h_k(z|\tau)}{k!} (-\partial)^k = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{i\pi}{2\omega} \right)^k = e^{\frac{i\pi}{2\omega}}.
\]

In fact, only the first \( N+1 \) terms in all sums matter, since the degree of each polynomial in \( p \) does not exceed \( N \). This justifies interchange of the limit and sum. The last identity (3.3) is proved. □

There are two essentially distinct cases. For \( \sigma = i \) the matrix \( L \) is skew-adjoint and all \( k_n \), roots of the polynomial \( Q(p) \), are pure imaginary. For \( \sigma = 1 \) the skew-adjointness is lost, but similar to Lemma 3, if \( p \equiv 0 \) the skew-adjointness of \( L \) is restored and all \( k_n \) are pure imaginary.
4. Spectrum of the Toeplitz operator. For a finite number of repulsive particles ($\tau \leq +i\infty$) there exists a point or set of points in the phase space where the Hamiltonian $H(q,p)$, bounded from below, achieves its minimum. Such a point $\mathfrak{p}_n, \mathfrak{q}_n$, $n = 1, \ldots, N$ is called a “ground state”. In our case the Hamiltonian is rotationally invariant and ground state is a one-parameter family: $\mathfrak{p}_n = 0$, $\mathfrak{q}_n - \mathfrak{q}_{n-1} = \frac{2\omega}{N}$, for all $n = 1, \ldots, N$.

Obviously, the matrix $\mathcal{L}$ at the ground state is Toeplitz and skew-symmetric. Let us introduce $e^T = (e^1, \ldots, e^N)^T$, an eigenvector of $\mathcal{L}$ : $(\mathcal{L} + km)e = 0$ corresponding to the eigenvalue $k$ and normalized such that $e^1 = 1$.

**Lemma 6.** Let $\sigma = i$. If the system is at the ground state, then the eigenvalues and eigenvectors of the matrix $\mathcal{L}$ are given by the formulas

$$k_n = \frac{\pi}{2\omega i}(N + 1 - 2n), \quad n = 1, \ldots, N;$$

and

$$e^j_n = e^{i\frac{\pi}{N}(j-1)(1-2n)}, \quad j = 1, \ldots, N; \quad n = 1, \ldots, N.$$

To prove the Lemma we introduce some formulas from Fourier analysis on the additive group $G_N = \mathbb{Z}/N\mathbb{Z}$ of the ring of residues modulo $N$.

i. Characters $\mu \in \hat{G}_N$ are all defined as $\mu_n = e^{i\beta_n}$, $\beta_n = \frac{2\pi}{N}(n-1)$, $n = 1, \ldots, N$.

The pairing is: for any $s \in G_N$, $\mu \in \hat{G}_N$ : $(s, \mu) = \mu^s$.

ii. For any complex function $a = \{a_s, s \in G_N\}$ its Fourier transform is $\hat{a}(\mu) = \sum_{s \in G_N} a_s(s, \mu)$, and the inverse transform is $a_s = \frac{1}{N} \sum_{\mu \in \hat{G}_N} \hat{a}(\mu)(-s, \mu)$.

iii. For any complex functions $a, b$ the Plancherel identity holds: $\sum_{s \in G_N} a_s \overline{b_s} = \frac{1}{N} \sum_{\mu \in \hat{G}_N} \hat{a}(\mu)\hat{b}(\mu)$.

Proof. To simplify calculations let $m = \frac{2\omega i}{\pi}$ and also re-scale the spectral parameter $k \rightarrow \frac{\pi}{2\omega i} k$. General case easily follows from this.

**Step 1.** Let $e^T_+ = (e^1, \ldots, e^N)^T$ is an eigenvector corresponding to the eigenvalue $k$. We will prove that $e^T_- = (e^N, \ldots, e^1)^T$ is an eigenvector corresponding to the eigenvalue $-k$. Indeed, for any $j$

$$\sum_{r \neq j} \mathcal{L}_{jr} e^r_+ + \mathcal{L}_{j,N-j+1} e^{N-j+1}_+ + ke^j_+ = 0.$$

Then,

$$\sum_{r \neq j} \mathcal{L}_{j,N-r+1} e^{N-r+1}_+ + \mathcal{L}_{j,N-j+1} e^{N-j+1}_+ + ke^j_- = 0.$$
At the ground state \( L_{N-j+1,N-r+1} = -L_{jr} \). Introducing \( e^n_\pi = \pi^{N-n+1} \), we obtain
\[
- \sum_{r \neq j, r \neq N-j+1} L_{N-j+1,r} e^r_\pi - L_{N-j+1,j} e^j_\pi - (-k)e^{N-j+1}_\pi = 0.
\]
The statement is proved.

**Step 2.** For \( j \neq r \) we have
\[
L_{jr} = mR(\overline{q}_j - \overline{q}_r) = \frac{(\sqrt{2i})^2 \mu_j^{1/2} \mu_r^{1/2}}{\mu_j - \mu_r}.
\]
Let \( D, \tilde{L} \) be the \( N \times N \) matrices with entries
\[
D_{jr} = \sqrt{2i} \mu_j^{1/2} \delta_{jr},
\]
\[
\tilde{L}_{jr} = -\frac{k}{2} \mu_j^{-1} \delta_{jr} + \frac{1}{\mu_j - \mu_r}(1 - \delta_{jr}).
\]
Then, \( L + mk = D\tilde{L}D \) and the eigenvalue problem can be written as
\[
(L + mk)e = D\tilde{L}De = \tilde{L}e = 0,
\]
where \( \tilde{e} = De \). For any \( j \)
\[
-\frac{k}{2} \mu_j^{-1} \tilde{e}^j + \sum_{r \neq j} \frac{\tilde{e}^r}{\mu_j - \mu_r} = 0.
\]
Introducing \( u_j = \mu_j^{-1} \tilde{e}^j \) and using \( \mu_j/\mu_r = \mu_{j-r+1} \), we obtain
\[
-\frac{k}{2} u_j + \sum_{r \neq j} \frac{u_r}{\mu_{j-r+1} - 1} = 0.
\]
This is a convolution equation which can be solved using Fourier analysis. Introducing
\[
\hat{U}(\mu) = \sum_{s \in G_N} u_s(s, \mu), \quad \hat{M}(\mu) = \sum_{s \in G_N - \{N\}} \frac{1}{\mu_{s+1} - 1}(s, \mu),
\]
we arrive at the main equation
\[
\hat{U}(\mu)(2\hat{M}(\mu) - k) = 0.
\]
Step 3. Note $\mu_n = \mu_2^{n-1}$. For $n \geq 2$ we have

$$
\hat{M}(\mu_n) = \sum_{s \in G_N \setminus \{N\}} \mu_{n+1}^s - 1 \frac{\mu_n^s}{\mu_{s+1}}
$$

$$
= \sum_{s \in G_N \setminus \{N\}} \mu_2^{(n-1)s} - 1 + 1 \frac{\mu_s^s}{\mu_2^s} - 1
$$

$$
= \hat{M}(\mu_1) + \sum_{s \in G_N \setminus \{N\}} \sum_{k=0}^{n-2} (\mu_2^s)^k
$$

$$
= \hat{M}(\mu_1) + \sum_{k=1}^{n-2} \sum_{s \in G_N \setminus \{N\}} (\mu_2^k)^s + \sum_{s \in G_N \setminus \{N\}} 1^s.
$$

If $r^N = 1$ and $r \neq 1$, then $r + r^2 + \ldots + r^{N-1} = -1$. Therefore

$$
\hat{M}(\mu_n) = \hat{M}(\mu_1) + N + 1 - n.
$$

From this formula we see that the function $\hat{M}(\mu)$ takes different values at the different points of the character group $\hat{G}_N$. Let $k_n = 2\hat{M}(\mu_n)$, $n = 1, \ldots, N$ in the main equation of Step 2. Then, $k_n = k_1 + 2(N + 1 - n)$, for $n \geq 2$. The eigenvalue $k_1$ is the smallest and $k_2$ is the largest. Using Step 1: $-k_1 = k_2$, we finally obtain

$$
k_1 = -N + 1,
$$

$$
k_n = N + 3 - 2n, \quad n = 2, \ldots, N.
$$

Step 4. Let, for any $n$

$$
\hat{U}_n(\mu_r) = N \sqrt{2} i e^{\frac{2\pi i}{N} (n-1)} \delta_{nr}, \quad \mu_r \in \hat{G}_N.
$$

Inverting the Fourier transform

$$
w_n^j = \frac{1}{N} \sum_{\mu \in \hat{G}_N} \hat{U}_n(\mu)(-j, \mu) = \sqrt{2} i e^{\frac{2\pi i}{N} (n-1)} \mu_n^{-j}.
$$

Using the identities $w_n^j = \mu_n^{-j} \tilde{e}^j$, $\tilde{e}^j = \sqrt{2} i \mu_j^{1/2} e^j$, we arrive at the formula $e_n^j = e^{\frac{2\pi i}{N}(j-1)(3-2n)}$. Shifting $n \rightarrow n + 1$, completes the proof. □
When the number of particles with $m = 1$ tends to infinity for the ground state with $v = \frac{2\omega}{N} = 1$ the matrix $\mathcal{L}$ has the limit

$$
\mathcal{L}_\infty = \begin{vmatrix}
0 & -\frac{1}{2} & -\frac{1}{3} & -\frac{1}{4} & \cdots \\
1 & 0 & -\frac{1}{2} & -\frac{1}{3} & \cdots \\
\frac{1}{3} & \frac{1}{3} & 0 & -\frac{1}{2} & \cdots \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{vmatrix}
$$

The corresponding Toeplitz operator $T_\varphi$ has the symbol $\varphi(t) = i\pi - it, \quad 0 \leq t < 2\pi$. To emphasize the dependence of $k_n$ on $N$ we write $k_{n,N}$. For any continuous function $f$ on $iR^1$ we have

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(k_{n,N}) = \frac{1}{2\pi} \int_{0}^{2\pi} f(\varphi(t))dt.
$$

Such distribution of eigenvalues is called canonical, see [W]. It is usually proved with the aid of Szegö theorem. Our case is not covered by the conditions of the classical theorem, but the conclusion for this symbol is true also.

5. Real ovals of anti-involution. Images of infinities. In the trigonometric case the curve $\Gamma$ is rational and $p(Q), \ Q \in \Gamma$ can be used as a uniformization parameter. On the curve $\Gamma$ anti-involution $\tau_1/\tau_i$ can be expressed in terms of the parameter $p$.

**Lemma 7.** (i). $\sigma = 1$. Then $p(\tau_1 Q) = \overline{p(Q)}$.

(ii). $\sigma = i$. Then $p(\tau_i Q) = -\overline{p(Q)}$.

**Proof.** (i). The statement follows from identity

$$
p(\tau_1 Q) = \overline{k(Q)} + h_1(\overline{z(Q)}) = \overline{p(Q)}.
$$

(ii). Similarly

$$
p(\tau_i Q) = -\overline{k(Q)} + h_1(-\overline{z(Q)}) = -\overline{p(Q)}.
$$

□

In the attractive case, as a corollary of the Lemma, we obtain, that $\Im p = 0$ is a fixed oval of anti-involution $\tau_1$, and in the repulsive case $\Re p = 0$ is a fixed oval of anti-involution $\tau_i$. 
Consider the repulsive case and let us look at the images of infinities $\Pi$. For $n \geq 2$ we have

$$p(\Pi_n) = k_n(\gamma) + h_1(\gamma)|_{\gamma=\Pi_n} = -\frac{1}{z} + k_n^{(0)} + O(z) + \left. \frac{1}{z} \right|_{z=0} = k_n^{(0)}.$$ 

As is proved in Lemma 5 of [V2], all $k_n^{(0)}$ are pure imaginary and therefore all images of infinities $\Pi_2, \ldots, \Pi_N$ lie on the fixed oval $\Re p = 0$ of anti-involution $\tau_i$. For $n = 1$ and $\gamma$ in the vicinity of $\Pi_1$

$$p(\gamma) = k(\gamma) + h_1(\gamma) = \frac{N - 1}{z} + O(1) + \frac{1}{z} = \frac{N}{z} + O(1).$$

Therefore $\Pi_1$ corresponds infinity of the uniformization parameter $p$.

For example, for $N = 2$ at the ground state $k_2^{(0)} = 0$ and due to Lemma 6 we have the fig 1.

When one pumps energy into the system the loops are perturbed, but $k_n^{(0)}$ are still pure imaginary (fig 2). When the modular parameter $\tau$ changes from $+i\infty$ (trigonometric case) to some finite imaginary number (elliptic case), the loops “move inside” and become fixed ovals around holes, as is shown in fig 3, 4.

For $N > 2$ the picture is similar, but $N$ loops are present.

The case of attractive particles is more complicated, because skew adjointness of the matrix $L$ is lost. It is restored for $p \equiv 0$. 

![Fig. 1](image1.png)  
![Fig. 2](image2.png)
In the trigonometric case, for both repulsive and attractive particles the images of infinities $\Pi_2, \ldots, \Pi_N$ are moduli of the spectral curves. We will prove this statement in the elliptic case. First, we derive the trace formula relating total momentum $P$ and $k_n^{(0)}$, $n = 2, \ldots, N$.

Computing $\sum_{\Pi} \text{res} k^2 \, dz = 0$, we have $\sum_{n=1}^{N} k_n^{(-1)} k_n^{(0)} = 0$. Using $k_n^{(-1)} = -1$, $n \geq 2$ and $k_1^{(-1)} = N - 1$, we obtain $N k_1^{(0)} = \sum_{n=1}^{N} k_n^{(0)}$. On the other hand $\text{tr} L = \sigma P = -m \sum_{n=1}^{N} k_n^{(0)}$. By comparing the last two expression we arrive at the trace formula

$$P = \sum_{n=2}^{N} J_n, \quad \text{where} \quad J_n = -k_n^{(0)} \frac{mN}{\sigma (N-1)}.$$ 

For another trace formula relating $H$ and $k_n^{(-1)}$ see [V2]. Now, we are ready to prove

**Lemma 8.** In the elliptic case $J_n$, $n = 2, \ldots, N$, supplemented by $Q_N$, the constant term of the polynomial $Q(p)$, are moduli of the spectral curves.

**Proof.** Using the representation of $R(k, z)$ in the vicinity of zero and $h_1(z) = \frac{1}{z} + O(1)$ we have

$$R(p - h_1(z), z) = m^N \left( p - \frac{N}{z} - k_1^{(0)} + O(z) \right) \prod_{n=2}^{N} \left( p - k_n^{(0)} + O(z) \right).$$
Therefore
\[
\text{res}_{z=0} R(p - h_1(z), z) = -Nm^N \prod_{n=2}^{N} (p - k_n^{(0)}).
\]

From the D'Hoker-Phong formula
\[
\sum_{k=0}^{\infty} \text{res}_{z=0} \frac{h_k(z)}{k!} (-\partial)^k Q(p) = -Nm^N \prod_{n=2}^{N} (p - k_n^{(0)}).
\]

Since \(\text{res} h_k(0) = 0\) for \(k\) even and \(\text{res} h_1(0) = 1\) we can recover from \(J\)’s the derivative of \(Q(p)\). The constant term \(Q_N\) allows us to reconstruct the whole polynomial \(Q(p)\). \(\square\)

6. Symplectic volumes. Let us introduce in a general elliptic case the new Hamiltonian
\[
\bar{H} = \sum_{n=1}^{N} \frac{p_n^2}{2} - \frac{m^2 \sigma^2}{2} \sum_{n,r=1}^{N} \bar{\phi}(q_n - q_r),
\]

where \(\bar{\phi} = \varphi + \frac{\eta}{\omega}\). In the trigonometric limit \(\bar{\phi}(q) = R^2(q)\).

Consider now the trigonometric case. Singularities of the curve \(\Gamma\) are simple crossings, as it is shown in fig 5.

Form the identity (3.1) is follows that \(k(\nu_n) = k(\beta_{sn}) = k_n\). We will derive a set of new trace formulas.

Lemma 9. In the trigonometric case the following formulas hold
\[
P = \sum_{n=1}^{N} I_n, \quad \text{where} \quad I_n = -\frac{m}{\sigma} k_n,
\]
\[
\bar{H} = \sum_{n=1}^{N} I_n', \quad \text{where} \quad I_n' = \frac{m^2 \sigma^2}{2} k_n^2.
\]
\textbf{Proof.} For \( Q(p) \) we have

\[
Q(p) = m^N \prod_{n=1}^{N} (p - k_n)
\]

\[
= m^N \left[ p^N + p^{N-1} \left( \sum_{n=1}^{N} -k_n \right) + p^{N-2} \left( \sum_{n,n'=1}^{N} k_n k_n' \right) + \ldots \right].
\]

On the other hand we can write the polynomial \( Q(p) \) as a Fredholm expansion for the determinant given by Theorem 5

\[
Q(p) = \det(L + mp) = (mp)^N + (mp)^{N-1} \left( \sum_{n=1}^{N} \sigma p_n \right)
\]

\[
+ (mp)^{N-2} \left( \sum_{n,n'=1}^{N} \sigma^2 p_n p_{n'} - m^2 R(q_n - q_{n'})R(q_{n'} - q_n) \right) + \ldots.
\]

Comparing coefficients after simple algebra, we obtain the result. \( \square \)

In the elliptic case a simple crossing becomes a handle, as shown in fig 6. In this case one can relate \( P, \bar{H} \) to the periods of \( kdz, k^2dz \) over \( a \)-circles of \( \Gamma \).

\textbf{Lemma 10.} In the elliptic case these formulas hold:

\[
P = \sum_{n=1}^{N} I_n, \quad \text{where} \quad I_n = -\frac{m}{\sigma} \int_{a_n} k(z) \frac{dz}{2\omega},
\]

\[
\bar{H} = \sum_{n=1}^{N} I'_n, \quad \text{where} \quad I'_n = \frac{m^2 \sigma^2}{2} \int_{a_n} k^2(z) \frac{dz}{2\omega}.
\]
Proof. * Let $c_n$ be small contours around $\Pi_n$, $n = 1, \ldots, N$. Using Cauchy’s theorem

$$\sum_{n=1}^{N} \int_{c_n} k(z) h_1(z) dz + \sum_{n=1}^{N} \int_{a_n} k(z) (h_1(z + 2\omega') - h_1(z)) dz = 0$$

Note that $h_1(z + 2\omega') - h_1(z) = -\frac{i\pi}{\omega}$ and the second sum reduces to

$$\ldots = \sum_{n=1}^{N} -\frac{i\pi}{\omega} \int_{a_n} k(z) dz.$$

To evaluate the first sum we use the asymptotics of $k$ near infinities $\Pi$:

$$\int_{c_n} k(z) h_1(z) dz = \int_{c_n} \left( \frac{k_n^{(-1)}}{z} + k_n^{(0)} + O(z) \right) \times \left( \frac{1}{z} + O(z) \right) dz = 2\pi i k_n^{(0)}.$$

As a result we derive

$$\sum_{n=1}^{N} k_n^{(0)} = \sum_{n=1}^{N} \int_{a_n} k(z) \frac{dz}{2\omega}.$$

On another hand we have

$$\text{tr } L = \sigma P = -m \sum_{n=1}^{N} k_n^{(0)}.$$

The first formula is proved.

To prove the second trace formula one should integrate $k^2 dz$ over the same contour. \(\square\)

The meromorphic 2–forms

$$\Omega = -\frac{m}{\sigma} \sum_{n=1}^{N} dk(\gamma_n) \wedge dz(\gamma_n) = \sum_{n=1}^{N} dI_n \wedge d\phi_n$$

$$\Omega' = \frac{m^2 \sigma}{2} \sum_{n=1}^{N} dk^2(\gamma_n) \wedge dz(\gamma_n) = \sum_{n=1}^{N} dI'_n \wedge d\phi_n$$

on the space of spectral curves can be written following [KP,K2] in a different way, namely

*For another proof see [DP].
**Theorem 11.** In the elliptic case the meromorphic 2-forms can be written as

\[
\Omega = -\frac{m}{\sigma} \sum_{n=1}^{N} \text{res}_{\Pi_n} \langle a^*(\delta L - mk) \wedge \delta a \rangle \, dz,
\]

\[
\Omega' = \frac{m^2 \sigma^2}{2} \sum_{n=1}^{N} \text{res}_{\Pi_n} k \langle a^*(\delta L - mk) \wedge \delta a \rangle \, dz.
\]

The form \(\Omega\) can be written as \(\sum dq_n \wedge dp_n\), but for the higher form \(\Omega'\) an explicit expression is not known.

In the paper [V1] we considered a Jacobian between symplectic volumes

\[
\text{Jac} \equiv \frac{N}{N'} \wedge \Omega \wedge \Omega' = \prod_{n=1}^{N} \frac{\partial I_n}{\partial I'_n}.
\]

The quantity \(N^{-1} \log \text{Jac}\), when \(N \to \infty\), has a meaning of entropy or, in probabilistic terms, rate function, see [L,Va].

**Lemma 12.** For repulsive particles in the trigonometric case at the ground state the following asymptotics hold *

\[
\frac{1}{N} \log \text{Jac} = \log N + \log v + \log \frac{2}{\pi m} + o(1), \quad \text{when } N \to \infty;
\]

where \(v = \frac{2\omega}{N}\).

**Proof.** Consider the case when \(N\) is divisible by 4. At the ground state Lemma 6 implies

\[
I_n = \frac{m\pi}{Nv} (N + 1 - 2n), \quad I'_n = \frac{m^2 \pi^2}{2N^2 v^2} (N + 1 - 2n)^2.
\]

Therefore,

\[
\text{Jac} = \prod_{n=1}^{N} \frac{\partial I_n}{\partial I'_n} = \left( \frac{2v}{m\pi} \right)^N \prod_{n=1}^{N} \frac{N}{N + 1 - 2n}
\]

and

\[
\frac{1}{N} \log \text{Jac} = \log v + \log \frac{2}{m\pi} + \log N - \frac{2}{N} \log(N - 1)!!
\]

Using the formula

\[
\frac{2}{N} \log(N - 1)!! = \log N - 1 + o(1)
\]

*This formula was first conjectured in [V1].
we have
\[ \frac{1}{N} \log \text{Jac} = \log v + \log \frac{2}{m \pi} + 1 + o(1). \]

The remark of section 2 implies that in fact we consider action-angle variables on the fundamental domain \( q_1 < q_2 < \ldots < q_N \) of the configuration space \([0, 2\omega)^N\). Therefore the Jacobian has to be multiplied by \( N! \). This and \( N^{-1} \log N! = \log N - 1 + o(1) \) imply the statement.

The case of general \( N \) can be considered similarly. \( \square \)

7. **Appendix.** The Weierstrass function \( \sigma(z) \) has periods \( 2\omega \) and \( 2\omega' \) and is defined as
\[ \sigma(z) = z \prod \{(1 - \frac{z}{\omega}) \exp(\frac{z}{\omega} + \frac{z^2}{2\omega^2})\}, \]
where \( \omega = 2\omega n + 2\omega'n' \) and \( \prod \) is taken with \( n, n' \in \mathbb{Z}^1; \ n^2 + n'^2 > 0; \) \( \sigma(z) \) has zeros at the points of the lattice \( 2\omega n + 2\omega'n' \). In the vicinity of zero it has the expansion
\[ \sigma(z) = z + O(z^5). \]
Under the shift \( z \to z + 2\omega \) the function \( \sigma(z) \) transforms as
\[ \sigma(z + 2\omega) = -\sigma(z) \exp(2\eta(z + \omega)). \]

For the shift \( z \to z + 2\omega' \) one should replace \( 2\eta \) by \( 2\eta' \) in the formula above. The functions \( \zeta(z) \) and \( \wp(z) \) are defined as:
\[ \zeta(z) = \frac{d}{dz} \log \sigma(z), \quad \wp(z) = -\frac{d}{dz} \zeta(z). \]
They have respective expansions at zero of the form
\[ \zeta(z) = \frac{1}{z} + O(z^3), \quad \wp(z) = \frac{1}{z^2} + O(z^2); \]
and periodicity properties
\[ \wp(z + 2\omega) = \wp(z), \quad \zeta(z + 2\omega) = \zeta(z) + 2\eta. \]
For the shift \( z \to z + 2\omega' \) one should replace \( 2\eta \) by \( 2\eta' \) in the formula above. The Legendre’s identity is \( \eta \omega' - \eta' \omega = \frac{i\pi}{2} \).

The Jacobi functions \( \theta_0, \theta_1 \) are defined as
\[ \theta_0 \left( \frac{z}{2\omega} \right) = \sum_{n=-\infty}^{+\infty} (-1)^n h^{n^2} e^\frac{i\pi 2n z}{2\omega}, \]
\[ \theta_1 \left( \frac{z}{2\omega} \right) = \sum_{n=-\infty}^{+\infty} (-1)^n h^{(2n-1)^2} e^\frac{i\pi (2n-1) z}{2\omega}, \]
where $h = e^{i\pi \tau}$. The relation between $\theta_1(\frac{z}{2\omega})$ and $\sigma(z)$ is:

$$\sigma(z) = \frac{2\omega}{\theta_1'(0)} \theta_1 \left( \frac{z}{2\omega} \right) e^{\frac{\eta z^2}{2\omega}}.$$ 

The Jacobi $sn(z)$ is defined as

$$sn(z) = \frac{\theta_0(0) \theta_1 \left( \frac{z}{2\omega} \right)}{\theta_1'(0) \theta_0 \left( \frac{z}{2\omega} \right)}.$$ 

For more information see [HC].

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