Free algebras, amalgamation and omitting types in BL algebras with operators

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Abstract. Let $K$ be a class of BL algebras with operators. BL algebras are algebraisations of many fuzzy logics, they are extensions of both Boolean algebras, and $MV$ algebras, the latter algebraize many-valued logic. We study atomicity of free algebras, the amalgamation property, and the algebraic counterpart of omitting types theorem for $K$. \[1\]

1 Introduction

A residuated lattice is an algebra

$$(L, \cup, \cap, *, \Rightarrow, 0, 1)$$

with four binary operations and two constants such that

(i) $(L, \cup, \cap, 0, 1)$ is a lattice with largest element 1 and the least element 0 (with respect to the lattice ordering defined the usual way: $a \leq b$ iff $a \cap b = a$).

(ii) $(L, *, 1)$ is a commutative semigroup with largest element 1, that is $*$ is commutative, associative, $1 * x = x$ for all $x$.

(iii) Letting $\leq$ denote the usual lattice ordering, we have $*$ and $\Rightarrow$ form an adjoint pair, i.e for all $x, y, z$

$$z \leq (x \Rightarrow y) \iff x * z \leq y.$$ 

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BL algebras, introduced and studied by Hajek [28], are what is called MTL algebras satisfying the identity \( x * (x \Rightarrow y) = x \cap y \). Both are residuated lattices with extra conditions. The propositional logic MTL was introduced by Esteva and Godo [12]. It has three basic connectives \( \rightarrow, \wedge \) and \&. We say that \( \mathcal{L} \) is a core fuzzy logic if \( \mathcal{L} \) expands MTL, \( \mathcal{L} \) has the Local Deduction Theorem \((LDT)\), and \( \mathcal{L} \) satisfies \((*)\) \( \phi \equiv \psi \vdash \chi(\phi) \equiv \chi(\psi) \) for all formulas \( \phi, \psi, \chi \). (Here \( \equiv \) is defined via \& and \( \Rightarrow \)). The \((LDT)\) says that for a theory \( T \) and a formula \( \phi \), whenever \( T \cup \{\phi\} \vdash \psi \), then there exists a natural number \( n \) such that \( T \vdash \phi^n \rightarrow \psi \). Here \( \phi^n \) is defined inductively by \( \phi^1 = \phi \) and \( \phi^n = \phi^{n-1} \& \phi \). Thus core fuzzy logics are axiomatic expansions of MTL having \( LDT \) and obeying the substitution rule \((*)\). The basic notions of evaluation, tautology and model for core fuzzy logics are defined the usual way. Let \( \mathcal{L} \) be a core fuzzy logic and \( I \) the set of additional connectives of \( \mathcal{L} \). An \( \mathcal{L} \) algebra is a structure \( \mathfrak{B} = (B, \cup, \cap, *, \Rightarrow, (c_B)_{c \in I}, 0, 1) \) such that \( (B, \cup, \cap, *, \Rightarrow, 0, 1) \) is an MTL algebra and each additional axiom of \( \mathcal{L} \) is a tautology of \( \mathfrak{B} \). Throughout the paper the operations of algebras are denoted by \( \cup, \cap, \Rightarrow \) * and the corresponding logical operations by \( \lor, \land, \rightarrow, \& \).

On the other hand \( MV \) algebras introduced by Chang in 1958 to provide an algebraic reflection of the completeness theorem of the Lukasiewicz infinite valued propositional logic, are BL algebras with the law of double negation. They can also be recovered from Boolean algebras by dropping idempotency. In recent years the range of applications of \( MV \) algebras has been enormously extended with profound interaction with other topics, ranging from lattice ordered abelian groups, \( C^* \) algebras, to fuzzy logic. In this paper we study \( MV \) algebras in connection to fuzzy (many valued) logic.

An \( MV \) algebra, has a dual behaviour; it can be viewed, in one of its facets, as a ‘non-idempotent’ generalization of a Boolean algebra possessing a strong lattice structure. The lack of idempotency enables \( MV \) algebras to be compared to monodial structures like monoids and abelian groups. Indeed, the category of \( MV \) algebras has been shown to be equivalent to the category of \( l \) groups. At the same time the lattice structure of Boolean algebras can be recovered inside \( MV \) algebras, by an appropriate term definability of primitive connectives. In this respect, they have a strong lattice structure (distributive and bounded), which make the techniques of lattice theory readily applicable to their study. As shown in this paper, in certain contexts when we replace the notion of a Boolean algebra with an \( MV \) algebra, the results survive such a replacement with some non-trivial modifications, and this can be accomplished in a somewhat unexpected manner.

Boolean algebras work as the equivalent semantics of classical propositional logic. To study classical first order logic, Tarski [22], [26] introduced cylindric algebras, while Halmos introduced polyadic algebras. Both of those can be viewed as Boolean algebras with extra operations that reflect algebraically
existential quantifiers.

Boolean algebras also have a neat and intuitive depiction, modulo isomorphisms; any Boolean algebra is an algebra of subsets of some set endowed with the concrete set theoretic operations of union, intersection and complements. Such a connection, a typical duality theorem, is today well understood. These nice properties mentioned above is formalized through the topology of Stone spaces that allows to select the right objects in the full power set of some set, the underlying set of the associated topological space. The representation theory of cylindric algebras, on the other hand, proves much more involved, and lacks such a strong well understood duality theorem like that of Boolean algebras. However, there is an extension of Stone duality to cylindric algebras, due to Comer, where he establishes a dual equivalence between cylindric algebras and certain categories of sheaves; but such a duality does not go deeper into the analysis of representability. There is a version of concrete (representable) algebras for cylindric algebras, with extra operations interpreted as projections, but this does not coincide with the abstract class of cylindric algebras. This is in sharp contrast to Boolean algebras. It is not the case that every cylindric algebra is representable in a concrete manner with the operations being set theoretic operations on relations. Not only that, but in fact the class of representable algebras need an infinite axiomatization in first order logic, and for any such axiomatization, there is an inevitable degree of complexity. On the other hand, polyadic algebras enjoy a strong representation theorem; every polyadic algebra is representable [11].

Cylindric and relation algebras were introduced by Tarski to algebraize first order logic. The structures of free cylindric and relation algebras are quite rich since they are able to capture the whole of first order logic, in a sense. One of the first things to investigate about these free algebras is whether they are atomic or not, i.e. whether their boolean reduct is atomic or not. By an atomic boolean algebra we mean an algebra for which below every non-zero element there is an atom, i.e. a minimal non-zero element. Throughout $n$ will denote a countable cardinal (i.e. $n \leq \omega$). More often than not, $n$ will be finite. $\mathbf{CA}_n$ stands for the class of cylindric algebras of dimension $n$. For a class $K$ of algebras, and a cardinal $\beta > 0$, $\mathfrak{Fr}_\beta K$ stands for the $\beta$-generated free $K$ algebra. In particular, $\mathfrak{Fr}_\beta \mathbf{CA}_n$ denotes the $\beta$-generated free cylindric algebra of dimension $n$. The following is known: If $\beta \geq \omega$, then $\mathfrak{Fr}_\beta \mathbf{CA}_n$ is atomless (has no atoms) [Pigozzi [22] 2.5.13]. Assume that $0 < \beta < \omega$. If $n < 2$ then $\mathfrak{Fr}_\beta \mathbf{CA}_n$ is finite, hence atomic, [22] 2.5.3(i). $\mathfrak{Fr}_\beta \mathbf{CA}_2$ is infinite but still atomic [Henkin, [22] 2.5.3(ii), 2.5.7(ii)]. If $3 \leq n < \omega$, then $\mathfrak{Fr}_\beta \mathbf{CA}_n$ has infinitely many atoms [Tarski, [22] 2.5.9], and it was posed as an open question, cf [22] problem 4.14, whether it is atomic or not. Here we prove, as a partial solution of problem 4.14 in [22], and among other things, that $\mathfrak{Fr}_\beta \mathbf{CA}_n$ is not atomic for $\omega > \beta > 0$ and $\omega > n \geq 4$. 

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In this paper we study atomicity of free $BL$ algebras with operators, we prove several amalgamation theorems for $MV$ algebras, obtaining several interpolation theorems for many valued logic. Intuitionistic logic, does not belong to fuzzy logic per se, though linear Heyting algebras do. However, here we give a deep representation theorem for Heyting algebras, culminating in an interpolation theorem for many predicate intuitionistic logics. We use Sheaf theoretic duality theory as worked out by Comer for cylindric algebras, but now applied to the Zarski topology defined on the prime spectrum of $BL$ algebras, to obtain some results on definability, mainly Beth definability for many valued logics. Finally, we give a new topological proof, using the celebrated Baire Category theorem, to the omitting types theorem for fuzzy logic, and we give several model theoretic consequence.

2 Basic Fuzzy Logic

The logics we start with arise typically from $t$ norms.

**Definition 2.1.** A $t$ norm is a binary operation $*$ on $[0, 1]$, i.e $(t : [0, 1]^2 \rightarrow [0, 1])$ such that

(i) $*$ is commutative and associative, that is for all $x, y, z \in [0, 1],$

$$x * y = y * x$$

$$(x * y) * z = x * (y * z).$$

(ii) $*$ is non decreasing in both arguments, that is

$$x_1 \leq x_2 \implies x_1 * y \leq x_2 * y$$

$$y_1 \leq y_2 \implies x * y_1 \leq x * y_2.$$

(iii) $1 * x = x$ and $0 * x = 0$ for all $x \in [0, 1]$.

The following are the most important (known) examples of continuous $t$ norms.

(i) Lukasiewicz $t$ norm: $x * y = max(0, x + y - 1)$

(ii) Godel $t$ norm $x * y = min(x, y)$

(iii) Product $t$ norm $x * y = x.y$

We have the following known result \cite{[23]} lemma 2.1.6
Theorem 2.2. Let $*$ be a continuous $t$ norm. Then there is a unique operation $x \implies y$ satisfying for all $x, y, z \in [0, 1]$, the condition $(x * z) \leq y$ iff $z \leq (x \implies y)$, namely $x \implies y = \max\{z : x * z \leq y\}$.

The operation $x \implies y$ is called the residuam of the $t$ norm. The residuam $\implies$ defines its corresponding unary operation of precomplement $(-)x = (x \implies 0)$. The Godel negation satisfies $(-)0 = 1$, $(-)x = 0$ for $x > 0$. Abstracting away from $t$ norms, we get BL algebras as defined in the introduction.

The following variant of the completeness theorem for core fuzzy logics is known:

Theorem 2.3. Let $\mathcal{L}$ be a core fuzzy logic, $\phi$ a formula and $T$ a theory. Then, the following conditions are equivalent

(i) $T \vdash \phi$
(ii) $e(\phi) = 1$ for each $\mathcal{L}$-algebra and each $\mathfrak{B}$ model $e$ of theory $T$
(iii) $e(\phi) = 1$ for each $\mathcal{L}$-chain $\mathfrak{B}$ and each $\mathfrak{B}$ model $e$ of theory $T$

Proof. [28], Thm 5 p.867. ■

Now we pass to predicate fuzzy logics, or predicate many valued logics. Let us assume from now on that $\mathcal{L}$ is some fixed core fuzzy logic. A predicate language consists of non-logical symbols and logical symbols. The non-logical symbols consist of a non-empty set of predicates, each together with a positive natural number - its arity, and a possibly empty set of constants. The logical symbols are a countable family of variables $x_1, \ldots, x_n \ldots$ connectives $\&, \to$, truth constants $0, 1$ and quantifiers $\forall, \exists$. Terms consist of variables and constants and nothing else. Atomic formulas have the form $P(t_1 \ldots t_n)$ where $P$ is a predicate of arity $n$ and $t_1 \ldots t_n$ are terms. If $\phi, \psi$ are formulas and $x$ is a variable, then $\phi \to \psi$, $\phi \& \psi$, $(\forall x)\psi$ $(\exists x)\phi$ are formulas. Other connectives are defined as follows:

- $\phi \land \psi$ is $\phi \& (\phi \to \psi)$,
- $\phi \lor \psi$ is $((\phi \to \psi) \to \psi) \land ((\psi \to \phi) \to \phi)$,
- $\neg \phi$ is $\phi \to 0$,
- $\psi \equiv \psi$ is $(\phi \to \psi) \& (\psi \to \phi)$.

For a linearly ordered $\mathcal{L}$ algebra, an $\mathbf{L}$ structure for a predicate language is $\mathfrak{M} = (M, P_M, c_m)$, where $M \neq \emptyset$, for each predicate $P$ of arity $n$, $P_M$ is an $n$-ary $\mathbf{L}$ fuzzy relation on $M$, that is $P_M : M^n \to \mathbf{L}$, and for each constant $c, c_m \in M$. One then defines for each formula $\phi$ the truth value $||\phi||_{\mathcal{M},v}$ of $\phi$ in $\mathfrak{M}$ determined by the $\mathbf{L}$ chain and evaluation $v$ of free variables the usual Tarskian way. In more detail, an $\mathfrak{M}$ evaluation is a map from $\omega$ to $M$. For
two evaluations \( v \) and \( v' \) and \( i \in \omega \) we write \( v \equiv_i v' \) iff \( v(j) = v'(j) \) for all \( j \neq i \). The value of a term given by \( M \), \( v \) is defined as follows \( \llbracket x_i \rrbracket_{M, v} = v(i) \) and \( \llbracket c \rrbracket_{M, v} = m_c \). Now we define the truth value \( \llbracket \phi \rrbracket_{M, v}^L \):
\[
\llbracket P(t_1, \ldots, t_n) \rrbracket_{M, v}^L = P_M(\llbracket t_1 \rrbracket_{M, v}^L, \ldots, \llbracket t_n \rrbracket_{M, v}^L),
\]
\[
\llbracket \phi \rightarrow \psi \rrbracket_{M, v}^L = \llbracket \phi \rrbracket_{M, v}^L \rightarrow \llbracket \psi \rrbracket_{M, v}^L,
\]
\[
\llbracket \phi \& \psi \rrbracket_{M, v}^L = \llbracket \phi \rrbracket_{M, v}^L \& \llbracket \psi \rrbracket_{M, v}^L,
\]
\[
\llbracket \forall x_i \phi \rrbracket_{M, v}^L = \bigwedge\{\llbracket \phi \rrbracket_{M, v'}^L : v' \equiv_i v\},
\]
\[
\llbracket \exists x_i \phi \rrbracket_{M, v}^L = \bigvee\{\llbracket \phi \rrbracket_{M, v'}^L : v' \equiv_i v\}.
\]

The structure \( M \) is \( L \) safe if the needed infima and suprema exist, i.e \( \llbracket \phi \rrbracket_{M, v}^L \) is defined for all \( \phi \) and \( v \). Let \( \phi \) be a formula and \( M \) be a safe \( L \) structure. The truth value of \( \phi \) in \( M \) is
\[
\llbracket \phi \rrbracket_{M}^L = \bigwedge\{\llbracket \phi \rrbracket_{M, v}^L : v \text{ is an } M \text{ evaluation } \}
\]

For each model \((M,L)\), let \( \mathcal{Alg}(M,L) \) be the subalgebra of \( L \) with domain \( \{\llbracket \phi \rrbracket_{M, v}^L : \phi, v\} \) of truth degrees of all formulas \( \phi \) under all \( M \) evaluations \( v \) of variables. \((M,L)\) is exhaustive if \( L = \mathcal{Alg}(M,L) \). Notions of free variables, substitution of a term for a variable, are defined like in classical first order logic. Given a safe structure \((M,L)\), a formula \( \phi(x_1 \ldots x_n) \) having free variables among the first \( n \) and \( s \in {}^n M \), \( s = (a_1 \ldots a_n) \), say, we write \( \llbracket \phi(a_1 \ldots a_n) \rrbracket_{M,L} \) or \( \llbracket \phi(s) \rrbracket_{M,L} \) for the value of the formula \( \phi \) in \( L \) when replacing the variables \( x_1 \ldots x_n \) by \( a_1 \ldots a_n \) respectively.

The following are logical axioms for quantifiers.
\begin{align*}
(\forall 1) & \quad (\forall x) \psi(x) \rightarrow \psi(t), t \text{ is substitutable for } x \text{ in } \psi(x),
(\exists 1) & \quad \psi(t) \rightarrow (\exists x) \psi(x), t \text{ is substitutable for } x \text{ in } \psi(x),
(\forall 2) & \quad (\forall x) (\psi \rightarrow \phi) \rightarrow (\psi \rightarrow (\forall x) \phi), x \text{ is not free in } \psi,
(\exists 2) & \quad (\forall x) (\psi \rightarrow \phi) \rightarrow ((\exists x) \psi \rightarrow \psi), x \text{ is not free in } \psi,
(\forall 3) & \quad (\forall x) (\psi \vee \phi) \rightarrow (\psi \vee (\forall x) \phi), x \text{ is not free in } \psi.
\end{align*}

Let \( \mathcal{L} \) be a core fuzzy logic that extends the basic propositional logic \( BL \). We associate with \( \mathcal{L} \) the corresponding predicate calculus \( \mathcal{L} \forall \) over a given signature \( S \) by taking as logical axioms
\begin{itemize}
  \item all formulas resulting from the axioms of \( \mathcal{L} \) by substituting arbitrary formulas of \( S \) for propositional variables, and the axioms (\( \forall 1 \) (\( \forall 2 \) (\( \forall 3 \),
(\exists 1) (\exists 2) for quantifiers and taking as deduction rules,
  \item modus ponens (from \( \phi, \phi \rightarrow \psi \) infer \( \psi \)) and,
  \item generalization (from \( \phi \) infer \( (\forall x) \phi \)).
\end{itemize}
Given this, the notions of proof, provability, theory, etc. are like classical logic.

**Definition 2.4.** Let $T$ be a theory. $T$ is linear if for every pair $\phi, \psi$ of sentences we have $T \vdash \phi \rightarrow \psi$ or $T \vdash \psi \rightarrow \phi$. We say that $T$ is Henkin if for each sentence $\phi = \forall x \psi$ such that $T \not\vdash \phi$, there is a constant $c$ such that $T \not\vdash \psi(c)$.

Set $[\phi]_T = \{ \psi : T \vdash \psi \equiv \phi \}$ and $L_T = \{ [\phi]_T : \phi$ a formula $\}$. The Lindenbaum algebra of the theory $T$ ($\mathfrak{F}m_T$) has domain $L_T$ and operations

$$f_{\mathfrak{F}m_T}([\phi_1]_T \ldots [\phi_n]_T) = [f(\phi_1 \ldots \phi_n)]_T.$$ 

Let $T$ be a Henkin Linear theory. The canonical model of theory $T$, denoted by $CM(T)$, is the pair $(CM(T), \mathfrak{F}m_T)$, where $\mathfrak{F}m_T$ is the Lindenbaum algebra of the theory $T$, the domain $CM(T)$ of $CM(T)$ consists of the constants, $c_{CM(T)} = c$ for each constant and $P_{CM(T)}(t_1 \ldots t_n) = [P(t_1 \ldots t_n)]_T$ for each predicate symbol $P$.

**Lemma 2.5.** Let $\mathcal{L}$ be a core fuzzy logic, $T$ a linear Henkin theory, and $\phi$ a formula with only one free variable $x$. Then

(i) $\mathfrak{F}m_T$ is an $\mathcal{L}$-chain,

(ii) $[\exists x \phi]_T = \bigvee [\phi(c)]$,

(iii) $[\forall x \phi]_T = \bigwedge [\phi(c)]$.

(iv) If $\phi$ is a sentence, then $||\phi||^{CM(T)} = [\phi]_T$. Thus $T \vdash \phi$ iff $CM(T) \models \phi$.

**Proof.** [28] lemma 6.

The previous lemma gives the following completeness theorem [14]:

**Theorem 2.6.** Let $\mathcal{L}^\forall$ be the predicate calculus given by a core fuzzy logic extending $BL$. Let $T$ be a theory over $\mathcal{L}^\forall$ and let $\phi$ be a formula of the language of $T$. $T$ proves $\phi$ if and only if for each linearly ordered $\mathcal{L}$-algebra $L$ and every safe $\mathcal{L}$-model $M$ of $L$, we have $||\phi||_M^L = 1$.

The following corollary is immediate [14].

**Theorem 2.7.** Let $\mathcal{L}$ be a core fuzzy logic, $T$ a theory and $\phi$ a formula. Then the following are equivalent:

(i) $T \vdash \phi$,

(ii) $(M, L) \models \phi$ for every model $(M, L)$ of $T$,

(iii) $(M, L) \models \phi$ for every exhaustive model $(M, L)$ of $T$. 

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In this paper we prove an omitting types theorem, but only for countable languages. The condition of countability of the language considered is sensible, because it is known that the omitting types theorem fails for uncountable languages for first order logic. Our proof is topological so we formulate and prove two known topological theorems. We assume familiarity with basic topological concepts such as basis, regular, compact .. etc.

**Definition 2.8.** An MV algebra is an algebra

\[ \mathfrak{A} = (A, \oplus, \odot, \neg, 0, 1) \]

where \( \oplus, \odot \) are binary operations, \( \neg \) is a unary operation and \( 0, 1 \in A \), such that the following identities hold:

1. \( a \oplus b = b \oplus a, \quad a \odot b = b \odot a \).
2. \( a \oplus (b \oplus c) = (a \oplus b) \oplus c, \quad a \odot (b \odot c) = (a \odot b) \odot c \).
3. \( a \oplus 0 = a, \quad a \odot 1 = a \).
4. \( a \oplus 1 = 1, \quad a \odot 0 = a \).
5. \( a \oplus \neg a = 1, \quad a \odot \neg a = 0 \).
6. \( \neg(a \oplus b) = \neg a \odot \neg b, \quad \neg(a \odot b) = \neg a \oplus \neg b \).
7. \( a = \neg \neg a \quad \neg 0 = 1 \).
8. \( \neg(a \oplus b) \oplus b = \neg(b \oplus a) \oplus a \).

MV algebras form a variety that is a subvariety of the variety of BL algebras introduced by Hajek, in fact MV algebras coincide with those BL algebras satisfying double negation law, namely that \( \neg \neg x = x \), and contains all Boolean algebras.

**Example 2.9.** A simple numerical example is \( A = [0, 1] \) with operations \( x \oplus y = \min(x+y, 1) \), \( x \odot y = \max(x+y-1, 0) \), and \( \neg x = 1-x \). In mathematical fuzzy logic, this MV-algebra is called the standard MV algebra, as it forms the standard real-valued semantics of Lukasiewicz logic. MV algebras can be obtained from Boolean algebras by dropping idempotency.

MV algebras also arise from the study of continuous t-norms.

**Definition 2.10.** A t norm is a binary operation \( * \) on \([0, 1]\), i.e \( t : [0, 1]^2 \rightarrow [0, 1] \) such that
(i) $*$ is commutative and associative, that is for all $x, y, z \in [0, 1],$
\begin{align*}
x \ast y &= y \ast x, \\
(x \ast y) \ast z &= x \ast (y \ast z).
\end{align*}

(ii) $*$ is non decreasing in both arguments, that is
\begin{align*}
x_1 \leq x_2 &\implies x_1 \ast y \leq x_2 \ast y, \\
y_1 \leq y_2 &\implies x \ast y_1 \leq x \ast y_2.
\end{align*}

(iii) $1 \ast x = x$ and $0 \ast x = 0$ for all $x \in [0, 1].$

The following are the most important (known) examples of continuous $t$ norms.

(i) Lukasiewicz $t$ norm: $x \ast y = \max(0, x + y - 1),$

(ii) Godel $t$ norm $x \ast y = \min(x, y),$

(iii) Product $t$ norm $x \ast y = x.y.$

We have the following known result [28] lemma 2.1.6

**Theorem 2.11.** Let $*$ be a continuous $t$ norm. Then there is a unique binary operation $x \rightarrow y$ satisfying for all $x, y, z \in [0, 1]$, the condition $(x \ast z) \leq y$ iff $z \leq (x \rightarrow y)$, namely $x \rightarrow y = \max\{z : x \ast z \leq y\}.$

The operation $x \rightarrow y$ is called the residuam of the $t$ norm. The residuam $\rightarrow$ defines its corresponding unary operation of precomplement $\neg x = (x \rightarrow 0)$.

Abstracting away from $t$ norms, we get

**Definition 2.12.** A residuated lattice is an algebra
\[(L, \cup, \cap, \ast, \rightarrow, 0, 1)\]

with four binary operations and two constants such that

(i) $(L, \cup, \cap, 0, 1)$ is a lattice with largest element 1 and the least element 0 (with respect to the lattice ordering defined the usual way: $a \leq b$ iff $a \cap b = a$).

(ii) $(L, \ast, 1)$ is a commutative semigroup with largest element 1, that is $\ast$ is commutative, associative, $1 \ast x = x$ for all $x.$
(iii) Letting $\leq$ denote the usual lattice ordering, we have $\ast$ and $\rightarrow$ form an adjoint pair, i.e. for all $x, y, z$

$$z \leq (x \rightarrow y) \iff x \ast z \leq y.$$ 

A result of Hajek, is that an $MV$ algebra is a prelinear commutative bounded integral residuated lattice satisfying the additional identity $x \cup y = (x \rightarrow y) \rightarrow y$. In case of an $MV$ algebra, $\ast$ is the so-called strong conjunction which we denote here following standard notation in the literature by $\odot$. $\cap$ is called weak conjunction. The other operations are defined by $\neg a = a \rightarrow 0$ and $a \oplus b = \neg(\neg a \odot \neg b)$. The operation $\cup$ is called weak disjunction, while $\oplus$ is called strong disjunction. The presence of weak and strong conjunction is a common feature of substructural logics without the rule of contraction, to which Lukasiewicz logic belongs.

We now turn to describing some metalogical notions, culminating in formulating our main results in logical form. However, throughout the paper, our investigations will be purely algebraic, using the well developed machinery of algebraic logic. There are two kinds of semantics for systems of many-valued logic. Standard logical matrices and algebraic semantics. We shall only encounter algebraic semantics. From a philosophical, especially epistemological point of view the semantic aspect of logic is more basic than the syntactic one, because it is mainly the semantic core which determines the choice of suitable syntactic versions of the corresponding system of logic.

3 Free algebras in $BL$ algebras with operators

In this section we study atomicity of $BL$ algebras with extra operations. This notion is important in cylindric algebras, and lack of atomicity has been linked to Godel's incompleteness theorem.

**Definition 3.1.** Let $K$ be variety of $BAO$'s. Let $\mathfrak{L}$ be the corresponding multimodal logic. We say that $\mathfrak{L}$ has the *Gödel's incompleteness property* if there exists a formula $\phi$ that cannot be extended to a recursive complete theory. Such formula is called incompletable.

Let $\mathfrak{L}$ be a general modal logic, and let $\mathfrak{Fm}_\equiv$ be the Tarski-Lindenbaum formula algebra on finitely many generators.

**Theorem 3.2.** (Essentially Nemeti's) If $\mathfrak{L}$ has G.I., then the algebra $\mathfrak{Fm}_\equiv$ is not atomic.

**Proof.** Assume that $\mathfrak{L}$ has G.I. Let $\phi$ be an incompletable formula. We show that there is no atom in the Boolean algebra $\mathfrak{Fm}$ below $\phi/\equiv$. Note that
because \( \phi \) is consistent, it follows that \( \phi/ \equiv \) is non-zero. Now, assume to the contrary that there is such an atom \( \tau/ \equiv \) for some formula \( \tau \). This means that \( (\tau \land \bar{\phi})/ \equiv \tau/ \equiv \). Then it follows that \( \vdash (\tau \land \phi) \rightarrow \phi \), i.e. \( \vdash \tau \rightarrow \phi \). Let \( T = \{\tau, \phi\} \) and let \( \text{Conseq}(T) = \{\psi \in Fm : T \vdash \psi\} \).

\( \text{Conseq}(T) \) is short for the consequences of \( T \). We show that \( T \) is complete and that \( \text{Conseq}(T) \) is decidable. Let \( \psi \) be an arbitrary formula in \( Fm \).

Then either \( \tau/ \equiv \psi/ \equiv \) or \( \tau/ \equiv \neg \psi/ \equiv \) because \( \tau/ \equiv \) is an atom. Thus \( T \vdash \psi \) or \( T \vdash \neg \psi \). Here it is the exclusive or, i.e. the two cases cannot occur together. Clearly \( \text{Conseq}(T) \) is recursively enumerable. By completeness of \( T \) we have \( Fm \equiv \setminus \text{Conseq}(T) = \{\neg \psi : \psi \in \text{Conseq}(T)\} \), hence the complement of \( \text{Conseq}(T) \) is recursively enumerable as well, hence \( T \) is decidable. Here we are using the trivial fact that \( Fm \) is decidable. This contradiction proves that \( Fm \equiv \) is not atomic.

In the following theorem, we give a unified perspective on several classes of algebras, studied in algebraic logic. Such algebras are cousins of cylindric algebras; though the differences, in many cases, can be subtle and big.

(1) holds for diagonal free cylindric algebras, cylindric algebras, Pinter’s substitution algebras (which are replacement algebras endowed with cylindrifiers) and quasipolyadic algebras with and without equality when the dimension is \( \leq 2 \). (2) holds for Boolean algebras; we do not know whether it extends any further. (3) holds for such algebras for all finite dimensions.

**Theorem 3.3.** Let \( K \) be a variety of \( BL \) algebras with finitely many operators.

(1) Assume that \( K = V(\text{Fin}(K)) \), and for any \( \mathcal{B} \in K \) and \( b' \in \mathcal{B} \), there exists a regular \( b \in \mathcal{B} \) such that \( \mathfrak{g}^\mathcal{B}\{b'\} = \mathfrak{g}^\mathcal{B}\{b\} \). If \( \mathfrak{A} \) is finitely generated, then \( \mathfrak{A} \) is atomic, hence the finitely generated free algebras are atomic. In particular, if \( K \) is a discriminator variety, with discriminator term \( d \), then finitely generated algebras are atomic. (One takes \( b' = d(b) \)).

(2) Assume that \( V \) is a BAO and that the condition above on principal ideals, together with the condition that that if \( b'_1 \) and \( b'_2 \) are the generators of two given ideals happen to be a partition (of the unit), then \( b_0, b_1 \) can be chosen to be also a partition. Then \( \mathfrak{S}_\beta K_\alpha \times \mathfrak{S}_\beta K_\alpha \cong \mathfrak{S}_\beta \{\beta+1\}K \). In particular if \( \beta \) is infinite, and \( \mathfrak{A} = \mathfrak{S}_\beta K \), then \( \mathfrak{A} \times \mathfrak{A} \cong \mathfrak{A} \).

(3) Assume that \( \beta < \omega \), and assume the above condition on principal ideals. Suppose further that for every \( k \in \omega \), there exists an algebra \( \mathfrak{A} \in K \), with at least \( k \) atoms, that is generated by a single element. Then \( \mathfrak{S}_\beta K \) has infinitely many atoms.

(4) Assume that \( K = V(\text{Fin}(K)) \). Suppose \( \mathfrak{A} \) is \( K \) freely generated by a finite set \( X \) and \( \mathfrak{A} = \mathfrak{S}gY \) with \( |Y| = |X| \). Then \( \mathfrak{A} \) is \( K \) freely generated by \( Y \).
Proof. (1) Assume that \( a \in A \) is non-zero. Let \( h : A \to B \) be a homomorphism of \( A \) into a finite algebra \( B \) such that \( h(a) \neq 0 \). Let \( I = \ker h \).

We claim that \( I \) is a finitely generated ideal. Let \( R_I \) be the congruence relation corresponding to \( I \), that is \( R_I = \{(a, b) \in A \times A : h(a) = h(b)\} \).

Let \( X \) be a finite set such that \( X \) generates \( A \) and \( h(X) = B \). Such a set obviously exists. Let \( X' = X \cup \{x + y : x, y \in X \} \cup \{-x : x \in X\} \) \( \cup \bigcup_{f \in \Gamma} \{f(x) : x \in X\} \). Let \( R = \mathcal{G}_B(R_I \cap X \times X') \). Clearly \( R \) is a finitely generated congruence and \( R_I \subseteq R \). We show that the converse inclusion also holds.

For this purpose we first show that \( R(X) = \{a \in A : \exists x \in X(x, a) \in R\} = A \). Assume that \( xRa \) and \( yRb \), \( x, y \in X \) then \( x + yRa + b \), but there exists \( z \in X \) such that \( h(z) = h(x + y) \) and \( zR(x + y) \), hence \( zR(a + b) \), so that \( a + b \in R(X) \). Similarly for all other operations. Thus \( R(X) = A \). Now assume that \( a, b \in A \) such that \( h(a) = h(b) \).

Then there exist \( x, y \) such that \( xRa \) and \( xRb \). Since \( R \subseteq \ker h \), we have \( h(x) = h(a) = h(b) = h(y) \) and so \( xRy \), hence \( aRb \) and \( R_I \subseteq R \).

So \( I = \mathcal{I}_B \{b\} \) for some element \( b \). Then there exists \( b \in A \) such that \( \mathcal{I}_B \{b\} = \mathcal{I}_B \{b'\} \). Since \( h(b) = 0 \) and \( h(a) \neq 0 \), we have \( a - b = 0 \). If \( a - b = 0 \), then \( h(a) - h(b) = 0 \).

Now \( h(A) \cong A/\mathcal{I}_B \{b\} \) as \( K \)-algebras. Let \( \mathcal{R}_B A = \{x : x \leq -b\} \). Let \( f : A/\mathcal{I}_B \{b\} \to \mathcal{R}_B A \) be defined by \( \bar{x} \mapsto x. -b \). Then \( f \) is an isomorphism of Boolean algebras (recall that the operations of \( \mathcal{R}_B A \) are defined by relativizing the Boolean operations to \(-b\)). Indeed, the map is well defined, by noting that if \( x\delta y \in \mathcal{I}_B \{b\} \), where \( \delta \) denotes symmetric difference, then \( x. -b = y. -b \) because \( x, y \leq b \).

Since \( \mathcal{R}_B A \) is finite, and \( a - b = \mathcal{R}_B A \) is non-zero, then there exists an atom \( x \in \mathcal{R}_B A \) below \( a \), but clearly \( \mathcal{A}(\mathcal{R}_B A) \subseteq \mathcal{A} A \) and we are done.

(2) Let \( (g_i : i \in \beta + 1) \) be the free generators of \( A = \mathfrak{F}_{\beta+1} K \). We first show that \( \mathcal{R}_B A \) is freely generated by \( \{g_i, g_\beta : i < \beta\} \). Let \( B \) be in \( K \) and \( y \in \mathcal{B} \). Then there exists a homomorphism \( f : A \to B \) such that \( f(g_i) = y_i \) for all \( i < \beta \) and \( f(g_\beta) = 1 \). Then \( f \upharpoonright \mathcal{R}_B A \) is a homomorphism such that \( f(g_i, g_\beta) = y_i \). Similarly \( \mathcal{R}_B A \) is freely generated by \( \{g_i, -g_\beta : i < \beta\} \). Let \( B_0 = \mathcal{R}_B A \) and \( B_1 = \mathcal{R}_B A \). Let \( t_0 = g_\beta \) and \( t_1 = -g_\beta \). Let \( x_i \) be such that \( J_i = \mathcal{I}_B \{t_i\} = \mathcal{I}_B \{x_i\} \), and \( x_0.x_1 = 0 \). Exist by assumption. Assume that \( z \in J_0 \cap J_1 \). Then \( z \leq x_i \), for \( i = 0, 1 \), and so \( z \leq 0 \). Thus \( J_0 \cap J_1 = \{0\} \).

Let \( y \in A \times A \), and let \( z = (y_0.x_0 + y_1.x_1) \), then \( y_i.x_i = z.x_i \) for each \( i = \{0, 1\} \) and so \( z \in \bigcap y_0/J_0 \cap y_1/J_1 \). Thus \( A/J_i \cong B_i \), and so \( A \cong B_0 \times B_1 \).

(3) Let \( A = \mathfrak{F}_\beta K \). Let \( B \) have \( k \) atoms and generated by a single element.
Then there exists a surjective homomorphism \( h : \mathfrak{A} \to \mathfrak{B} \). Then, as in the first item, \( \mathfrak{A}/\mathfrak{g}_{\mathfrak{B}}\{b\} \cong \mathfrak{B} \), and so \( \mathfrak{R}_0\mathfrak{B} \) has \( k \) atoms. Hence \( \mathfrak{A} \) has \( k \) atoms for any \( k \) and we are done.

(4) Let \( \mathfrak{A} = \mathfrak{f}_X K \), let \( \mathfrak{B} \in \text{Fin}(K) \) and let \( f : X \to \mathfrak{B} \). Then \( f \) can extended to a homomorphism \( f' : \mathfrak{A} \to \mathfrak{B} \). Let \( \bar{f} = f' \mid Y \). If \( f,g \in X B \) and \( \bar{f} = \bar{g} \), then \( f' \) and \( g' \) agree on a generating set \( Y \), so \( f' = g' \), hence \( f = g \). Therefore we obtain a one to one mapping from \( X B \) to \( Y B \), but \( |X| = |Y| \), hence this map is surjective. In other words for each \( h \in Y B \), there exists a unique \( f \in X B \) such that \( \bar{f} = h \), then \( f' \) with domain \( \mathfrak{A} \) extends \( h \). Since \( \mathfrak{f}_X K = \mathfrak{f}_X (\text{Fin}(K)) \) we are done.

\[ \square \]

For cylindric algebras, diagonal free cylindric algebras Pinter’s algebras and quasipolyadic equality, though free algebras of > 2 dimensions contain infinitely many atoms, they are not atomic. (The diagonal free case of cylindric algebras is a very recent result, due to Andréka and Némethi, that has profound repercussions on the foundation of mathematics.) We, next, state two theorems that hold for such algebras, in the general context of \( \text{BAO} \)’s. But first a definition.

**Definition 3.4.** Let \( K \) be a class of \( \text{BAO} \) with operators \((f_i : i \in I)\). Let \( \mathfrak{A} \in K \). An element \( b \in A \) is called **hereditary closed** if for all \( x \leq b \), \( f_i(x) = x \).

In the presence of diagonal elements \( d_{ij} \) and cylindrifications \( c_i \) for indices \(< 2, -c_0 - d_{01}\), is hereditary closed.

**Theorem 3.5.**

(1) Let \( \mathfrak{A} = \mathfrak{g}_X \) and \( |X| < \omega \). Let \( b \in \mathfrak{A} \) be hereditary closed. Then \( \text{At} \mathfrak{A} \cap \mathfrak{R}_b \mathfrak{A} \leq 2^n \). If \( \mathfrak{A} \) is freely generated by \( X \), then \( \text{At} \mathfrak{A} \cap \mathfrak{R}_b \mathfrak{A} = 2^n \).

(2) If every atom of \( \mathfrak{A} \) is below \( b \), then \( \mathfrak{A} \cong \mathfrak{R}_b \mathfrak{A} \times \mathfrak{R}_b \mathfrak{A} \), and \( |\mathfrak{R}_b \mathfrak{A}| = 2^{2^n} \). If in addition \( \mathfrak{A} \) is infinite, then \( \mathfrak{R}_b \mathfrak{A} \) is atomless.

**Proof.** Assume that \( |X| = m \). We have \( |\text{At} \mathfrak{A} \cap \mathfrak{R}_b \mathfrak{A}| = |\{ Y \sim \sum (X \sim Y).b \} | \leq m^2 \). Let \( \mathfrak{B} = \mathfrak{R}_b \mathfrak{A} \). Then \( \mathfrak{B} = \mathfrak{g}_B \{ x_i.b : i < m \} = \mathfrak{g}_B \{ x_i.b : i < \beta \} \) since \( b \) is hereditary fixed. For \( \Gamma \subseteq m \), let

\[
x_\Gamma = \prod_{i \in \Gamma} (x_i.b). \prod_{i \in m \setminus \Gamma} (x_i - b).
\]

Let \( \mathfrak{C} \) be the two element algebra. Then for each \( \Gamma \subseteq m \), there is a homomorphism \( f : \mathfrak{A} \to \mathfrak{C} \) such that \( fx_i = 1 \) iff \( i \in \Gamma \). This shows that \( x_\Gamma \neq 0 \) for every \( \Gamma \subseteq m \), while it is easily seen that \( x_\Gamma \) and \( x_\Delta \) are distinct for distinct \( \Gamma, \Delta \subseteq m \). We show that \( \mathfrak{A} \cong \mathfrak{R}_b \mathfrak{A} \times \mathfrak{R}_b \mathfrak{A} \).
Let $B_0 = Rl_\mathfrak{A}$ and $B_1 = Rl_{-\mathfrak{A}}$. Let $t_0 = b$ and $t_1 = -b$. Let $J_i = Ig \{ t_i \}$. Assume that $z \in J_0 \cap J_1$. Then $z \leq t_i$ for $i = 0, 1$, and so $z = 0$. Thus $J_0 \cap J_1 = \{ 0 \}$. Let $y \in A \times A$, and let $z = (y_0.t_0 + y_1.t_1)$, then $y_i.x_i = z.x_i$ for each $i = \{ 0, 1 \}$ and so $z \in \bigcap y_0/J_0 \cap y_1/J_1$. Thus $\mathfrak{A}/J_i \cong \mathfrak{B}_i$, and so $\mathfrak{A} \cong \mathfrak{B}_0 \times \mathfrak{B}_1$.

The above theorem holds for free cylindric and quasi-polyadic equality algebras. The second part (all atoms are zero-dimensional) is proved by Madársz and Németi.

The following theorem holds for any class of $BAO$’s.

**Theorem 3.6.** The free algebra on an infinite generating set is atomless.

**Proof.** Let $X$ be the infinite freely generating set. Let $a \in A$ be non-zero. Then there is a finite set $Y \subseteq X$ such that $a \in Ig^\mathfrak{A} Y$. Let $y \in X \sim Y$. Then by freeness, there exist homomorphisms $f : \mathfrak{A} \to \mathfrak{B}$ and $h : \mathfrak{A} \to \mathfrak{B}$ such that $f(\mu) = h(\mu)$ for all $\mu \in Y$ while $f(y) = 1$ and $h(y) = 0$. Then $f(a) = h(a) = a$. Hence $f(a.y) = h(a. - y) = a \neq 0$ and so $a.y \neq 0$ and $a. - y \neq 0$. Thus $a$ cannot be an atom.

**4 Amalgamation in $MV$ algebras**

Theorems 4.2, 4.4, 4.5 to come, give a flavour of the interconnections between the local properties of $CP$ and $SIP$ (on free algebras) and the global property of superamalgamation (of the entire class). Maksimova and Madársz proved that if interpolation holds in free algebras of a variety, then the variety has the superamalgamation property. Using a similar argument, we prove this implication in a slightly more general setting. But first an easy lemma:

**Lemma 4.1.** Let $K$ be a class of $BAO$’s. Let $\mathfrak{A}, \mathfrak{B} \in K$ with $\mathfrak{B} \subseteq \mathfrak{A}$. Let $M$ be an ideal of $\mathfrak{B}$. We then have:

1. $Ig^\mathfrak{A} M = \{ x \in A : x \leq b \text{ for some } b \in M \}$

2. $M = Ig^\mathfrak{A} M \cap \mathfrak{B}$

3. if $\mathfrak{C} \subseteq \mathfrak{A}$ and $N$ is an ideal of $\mathfrak{C}$, then $Ig^\mathfrak{A} (M \cup N) = \{ x \in A : x \leq b \oplus c \text{ for some } b \in M \text{ and } c \in N \}$

4. For every ideal $N$ of $\mathfrak{A}$ such that $N \cap B \subseteq M$, there is an ideal $N'$ in $\mathfrak{A}$ such that $N \subseteq N'$ and $N' \cap B = M$. Furthermore, if $M$ is a maximal ideal of $\mathfrak{B}$, then $N'$ can be taken to be a maximal ideal of $\mathfrak{A}$. 

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Proof. Only (iv) deserves attention. The special case when \( n = \{ 0 \} \) is straightforward. The general case follows from this one, by considering \( \mathfrak{A}/N, \mathfrak{B}/(N \cap \mathfrak{B}) \) and \( M/(N \cap \mathfrak{B}) \), in place of \( \mathfrak{A}, \mathfrak{B} \) and \( M \) respectively.

The previous lemma will be frequently used without being explicitly mentioned.

Theorem 4.2. Let \( K \) be a class of BAO’s such that \( HK = SK = K \). Assume that for all \( \mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in K \), inclusions \( m : \mathfrak{C} \to \mathfrak{A}, n : \mathfrak{C} \to \mathfrak{B}, \) there exist \( \mathfrak{D} \) with SIP and \( h : \mathfrak{D} \to \mathfrak{C}, h_1 : \mathfrak{D} \to \mathfrak{A}, h_2 : \mathfrak{D} \to \mathfrak{B} \) such that for \( x \in h^{-1}(\mathfrak{C}) \),

\[
    h_1(x) = m \circ h(x) = n \circ h(x) = h_2(x).
\]

Then \( K \) has SUPAP.

\[
\begin{array}{ccc}
\mathfrak{A} & \xrightarrow{h_1} & \mathfrak{D} \\
\downarrow h & & \downarrow h \\
\mathfrak{C} & \xrightarrow{m} & \mathfrak{D} \\
\downarrow h & & \downarrow h \\
\mathfrak{B} & \xrightarrow{n} & \mathfrak{D}
\end{array}
\]

Proof. Let \( \mathfrak{D}_1 = h_1^{-1}(\mathfrak{A}) \) and \( \mathfrak{D}_2 = h_2^{-1}(\mathfrak{B}) \). Then \( h_1 : \mathfrak{D}_1 \to \mathfrak{A} \), and \( h_2 : \mathfrak{D}_2 \to \mathfrak{B} \).

Let \( M = \ker h_1 \) and \( N = \ker h_2 \), and let \( h_1 : \mathfrak{D}_1/M \to \mathfrak{A}, h_2 : \mathfrak{D}_2/N \to \mathfrak{B} \) be the induced isomorphisms.

Let \( l_1 : h^{-1}(\mathfrak{C})/h_1^{-1}(\mathfrak{C}) \cap M \to \mathfrak{C} \) be defined via \( \bar{x} \to h(x) \), and \( l_2 : h^{-1}(\mathfrak{C})/h_1^{-1}(\mathfrak{C}) \cap N \to \mathfrak{C} \) be defined via \( \bar{x} \to h(x) \). Then those are well defined, and hence \( k^{-1}(\mathfrak{C}) \cap M = h^{-1}(\mathfrak{C}) \cap N \). Then we show that \( \mathfrak{P} = \mathfrak{Ig}(M \cup N) \) is a proper ideal and \( \mathfrak{D}/\mathfrak{P} \) is the desired algebra. Now let \( x \in \mathfrak{Ig}(M \cup N) \cap \mathfrak{D}_1 \). Then there exist \( b \in M \) and \( c \in N \) such that \( x \leq b \oplus c \). Thus \( x - b \leq c \). But \( x - b \in \mathfrak{D}_1 \) and \( c \in \mathfrak{D}_2 \), it follows that there exists an interpolant \( d \in \mathfrak{D}_1 \cap \mathfrak{D}_2 \) such that \( x - b \leq d \leq c \). We have \( d \in N \) therefore \( d \in M \), and since \( x \leq d \oplus b \), therefore \( x \in M \). It follows that \( \mathfrak{Ig}(M \cup N) \cap \mathfrak{D}_1 = M \) and similarly \( \mathfrak{Ig}(M \cup N) \cap \mathfrak{D}_2 = N \). In particular \( P = \mathfrak{Ig}(M \cup N) \) is a proper ideal.

Let \( k : \mathfrak{D}_1/M \to \mathfrak{D}/P \) be defined by \( k(a/M) = a/P \) and \( h : \mathfrak{D}_2/N \to \mathfrak{D}/P \) by \( h(a/N) = a/P \). Then \( k \circ m \) and \( h \circ n \) are one to one and \( k \circ m \circ f = h \circ n \circ g \).

We now prove that \( \mathfrak{D}/P \) is actually a superamalgam. i.e we prove that \( K \) has the superamalgamation property. Assume that \( k \circ m(a) \leq h \circ n(b) \). There exists \( x \in \mathfrak{D}_1 \) such that \( x/P = k(m(a)) \) and \( m(a) = x/M \). Also there exists \( z \in \mathfrak{D}_2 \) such that \( z/P = h(n(b)) \) and \( n(b) = z/N \). Now \( x/P \leq z/P \) hence \( x - z \in P \).
Therefore there is an \( r \in M \) and an \( s \in N \) such that \( x - r \leq z \oplus a \). Now \( x - r \in \mathfrak{D}_1 \) and \( z \oplus a \in \mathfrak{D}_2 \), it follows that there is an interpolant \( u \in \mathfrak{D}_1 \cap \mathfrak{D}_2 \) such that \( x - r \leq u \leq z \oplus a \). Let \( t \in \mathfrak{C} \) such that \( m \circ f(t) = u/M \) and \( n \circ g(t) = u/N \). We have \( x/P \leq u/P \leq z/P \). Now \( m(f(t)) = u/M \geq x/M = m(a) \).

Thus \( f(t) \geq a \). Similarly \( n(g(t)) = u/N \leq z/N = n(b) \), hence \( g(t) \leq b \). By total symmetry, we are done. \( \square \)

The intimate relationship between \( CP \) on free algebras generating a certain variety and the \( AP \) for such varieties, has been worked out extensively by Pigozzi for various classes of cylindric algebras. Here we prove an implication in one direction for \( BAO \)'s. Notice that we do not assume that our class is a variety.

**Lemma 4.3.** Let \( L \supseteq L_{BA} \) be a functional signature, and \( V \) a variety of \( L - BAO \)'s. Let \( d(x) \) be a unary \( L \) term. Then the following are equivalent:

1. \( d \) is a discriminator term of \( \text{Sir}V \), so that \( V \) is a discriminator variety.
2. all equations of the following for are valid in \( V \):
   1. \( x \leq d(x) \)
   2. \( d(d(x)) \leq d(x) \)
   3. \( f(x) \leq d(x) \) for all \( f \in L \sim L_{BA} \)

**Theorem 4.4.** Let \( K \) be such that \( HK = SK = K \). If \( K \) has the amalgamation property, then the \( V(K) \) free algebras, on any set of generators, have \( CP \).

*Proof.* For \( R \in \text{Co}\mathfrak{A} \) and \( X \subseteq A \), by \( (\mathfrak{A}/R)^{(X)} \) we understand the subalgebra of \( \mathfrak{A}/R \) generated by \( \{ x/R : x \in X \} \). Let \( \mathfrak{A}, X_1, X_2, R \) and \( S \) be as specified in in the definition of \( CP \). Define

\[
\theta : \mathfrak{G}^3(X_1 \cap X_2) \rightarrow \mathfrak{G}^3(X_1)/R
\]

by

\[
a \mapsto a/R.
\]

Then \( \ker \theta = R \cap ^2 \mathfrak{G}^3(X_1 \cap X_2) \) and \( \text{Im} \theta = (\mathfrak{G}^3(X_1)/R)^{(X_1 \cap X_2)} \). It follows that

\[
\tilde{\theta} : \mathfrak{G}^3(X_1 \cap X_2)/R \cap ^2 \mathfrak{G}^3(X_1 \cap X_2) \rightarrow (\mathfrak{G}^3(X_1)/R)^{(X_1 \cap X_2)}
\]

defined by

\[
a/R \cap ^2 \mathfrak{G}^3(X_1 \cap X_2) \mapsto a/R
\]
is a well defined isomorphism. Similarly
\[ \tilde{\psi} : \mathcal{Sg}^A(X_1 \cap X_2) / S \cap 2\mathcal{Sg}^A(X_1 \cap X_2) \to (\mathcal{Sg}^A(X_2) / S)^{(X_1 \cap X_2)} \]
defined by
\[ a/S \cap 2\mathcal{Sg}^A(X_1 \cap X_2) \mapsto a/S \]
is also a well defined isomorphism. But
\[ R \cap 2\mathcal{Sg}^A(X_1 \cap X_2) = S \cap 2\mathcal{Sg}^A(X_1 \cap X_2), \]
Hence
\[ \phi : (\mathcal{Sg}^A(X_1) / R)^{(X_1 \cap X_2)} \to (\mathcal{Sg}^A(X_2) / S)^{(X_1 \cap X_2)} \]
defined by
\[ a/R \mapsto a/S \]
is a well defined isomorphism. Now \((\mathcal{Sg}^A(X_1) / R)^{(X_1 \cap X_2)}\) embeds into \(\mathcal{Sg}^A(X_1) / R\) via the inclusion map; it also embeds in \(\mathcal{Sg}^A(X_1) / R\) via \(i \circ \phi\) where \(i\) is also the inclusion map. For brevity let \(\mathfrak{A}_0 = (\mathcal{Sg}^A(X_1) / R)^{(X_1 \cap X_2)}, \mathfrak{A}_1 = \mathcal{Sg}^A(X_1) / R\) and \(\mathfrak{A}_2 = \mathcal{Sg}^A(X_2) / S\) and \(j = i \circ \phi\). Then \(\mathfrak{A}_0\) embeds in \(\mathfrak{A}_1\) and \(\mathfrak{A}_2\) via \(i\) and \(j\) respectively. Then there exists \(\mathfrak{B} \in V\) and monomorphisms \(f\) and \(g\) from \(\mathfrak{A}_1\) and \(\mathfrak{A}_2\) respectively to \(\mathfrak{B}\) such that \(f \circ i = g \circ j\). Let
\[ \tilde{f} : \mathcal{Sg}^A(X_1) \to \mathfrak{B} \]
be defined by
\[ a \mapsto f(a/R) \]
and
\[ \tilde{g} : \mathcal{Sg}^A(X_2) \to \mathfrak{B} \]
be defined by
\[ a \mapsto g(a/R). \]
Let \(\mathfrak{B}'\) be the algebra generated by \(Imf \cup Img\). Then \(\tilde{f} \cup \tilde{g} \downharpoonright X_1 \cup X_2 \to \mathfrak{B}'\)
is a function since \(\tilde{f}\) and \(\tilde{g}\) coincide on \(X_1 \cap X_2\). By freeness of \(\mathfrak{A}\), there exists \(h : \mathfrak{A} \to \mathfrak{B}'\) such that \(h \upharpoonright_{X_1 \cup X_2} = \tilde{f} \cup \tilde{g}\). Let \(T = kerh\). Then it is not hard to check that
\[ T \cap 2\mathcal{Sg}^A(X_1) = R \] and \(T \cap 2\mathcal{Sg}^A(X_2) = S. \)
\\
Finally we show that \(CP\) implies a weak form of interpolation.

**Theorem 4.5.** If an algebra \(\mathfrak{A}\) has \(CP\), then for \(X_1, X_2 \subseteq \mathfrak{A}\), if \(x \in \mathcal{Sg}^A X_1\) and \(z \in \mathcal{Sg}^A X_2\) are such that \(x \leq z\), then there exists \(y \in \mathcal{Sg}^A(X_1 \cap X_2), n \in \omega\) and a term \(\tau\) such that \(x \leq y \leq \tau(z^n)\). If \(Ig^{2\mathfrak{A}}\{z\} = Ig^\mathfrak{A}\{z\}\), then \(\tau\) can be chosen to be the identity term. In particular, if \(z\) is closed, or \(\mathfrak{A}\) comes from a discriminator variety, then the latter case occurs.
Proof. Now let \( x \in \mathcal{Sg}(X_1), z \in \mathcal{Sg}(X_2) \) and assume that \( x \leq z \). Then
\[
x \in (\mathcal{Ig}(z)) \cap \mathcal{Sg}(X_1).
\]
Let
\[
M = \mathcal{Ig}^{(X_1)} \{z\} \text{ and } N = \mathcal{Ig}^{\mathcal{Sg}(X_2)}(M \cap \mathcal{Sg}(X_1 \cap X_2)).
\]
Then
\[
M \cap \mathcal{Sg}(X_1 \cap X_2) = N \cap \mathcal{Sg}(X_1 \cap X_2).
\]
By identifying ideals with congruences, and using the congruence extension property, there is an ideal \( P \) of \( \mathfrak{A} \) such that
\[
P \cap \mathcal{Sg}(X_1) = N \text{ and } P \cap \mathcal{Sg}(X_2) = M.
\]
It follows that
\[
\mathcal{Ig}(N \cup M) \cap \mathcal{Sg}(X_1) \subseteq P \cap \mathcal{Sg}(X_1) = N.
\]
Hence
\[
(\mathcal{Ig}(z)) \cap A^{(X_1)} \subseteq N.
\]
and we have
\[
x \in \mathcal{Ig}^{\mathcal{Sg}(X_1)} \mathcal{Ig}^{\mathcal{Sg}(X_2)} \{z\} \cap \mathcal{Sg}(X_1 \cap X_2).
\]
This implies that there is an element \( y \) such that
\[
x \leq y \in \mathcal{Sg}(X_1 \cap X_2)
\]
and \( y \in \mathcal{Ig}^{\mathcal{Sg}(X)} \{z\} \), hence the first required. The second required follows follows, also immediately, since \( y \leq z \), because \( \mathcal{Ig}(z) = \mathfrak{A}_1 \mathfrak{A} \).

5 An application to Heyting algebras.

For an algebra \( \mathfrak{A} \), \( \text{End}(\mathfrak{A}) \) denotes the set of endomorphisms of \( \mathfrak{A} \) (homomorphisms of \( \mathfrak{A} \) into itself), which is a semigroup under the operation \( \circ \) of composition of maps.

Definition 5.1. A transformation system is a quadruple \((\mathfrak{A}, I, G, S)\) where \( \mathfrak{A} \) is an algebra, \( I \) is a set, \( G \) is a subsemigroup of \((I, \circ)\) and \( S \) is a homomorphism from \( G \) into \( \text{End}(\mathfrak{A}) \).

Throughout the paper, \( \mathfrak{A} \) will always be a Heyting algebra. If we want to study predicate intuitionistic logic, then we are naturally led to expansions of Heyting algebras allowing quantification. But we do not have negation in the classical sense, so we have to deal with existential and universal quantifiers each separately.
Definition 5.2. Let $\mathfrak{A} = (A, \lor, \land, \to, 0)$ be a Heyting algebra. An existential quantifier $\exists$ on $A$ is a mapping $\exists : \mathfrak{A} \to \mathfrak{A}$ such that the following hold for all $p, q \in A$:

1. $\exists(0) = 0$,
2. $p \leq \exists p$,
3. $\exists(p \land \exists q) = \exists p \land \exists q$,
4. $\exists(\exists p \to \exists q) = \exists p \to \exists q$,
5. $\exists(\exists p \lor \exists q) = \exists p \lor \exists q$,
6. $\exists \exists p = \exists p$.

Definition 5.3. Let $\mathfrak{A} = (A, \lor, \land, \to, 0)$ be a Heyting algebra. A universal quantifier $\forall$ on $A$ is a mapping $\forall : \mathfrak{A} \to \mathfrak{A}$ such that the following hold for all $p, q \in A$:

1. $\forall 1 = 1$,
2. $\forall p \leq p$,
3. $\forall(p \to q) \leq \forall p \to \forall q$,
4. $\forall \forall p = \forall p$.

Now we define our algebras. Their similarity type depends on a fixed in advance semigroup. We write $X \subseteq_\omega Y$ to denote that $X$ is a finite subset of $Y$.

Definition 5.4. Let $\alpha$ be an infinite set. Let $G \subseteq \alpha\alpha$ be a semigroup under the operation of composition of maps. An $\alpha$ dimensional polyadic Heyting $G$ algebra, a $GPHA_\alpha$ for short, is an algebra of the following form

$$(A, \lor, \land, \to, 0, s_\tau, c_{(J)}, q_{(J)})_{\tau \in G, J \subseteq_\omega \alpha}$$

where $(A, \lor, \land, \to, 0)$ is a Heyting algebra, $s_\tau : \mathfrak{A} \to \mathfrak{A}$ is an endomorphism of Heyting algebras, $c_{(J)}$ is an existential quantifier, $q_{(J)}$ is a universal quantifier, such that the following hold for all $p \in A$, $\sigma, \tau \in [G]$ and $J, J' \subseteq_\omega \alpha$:

1. $s_{Id}p = p$.

2. $s_{\sigma \tau}p = s_\sigma s_\tau p$ (so that $s : \tau \mapsto s_\tau$ defines a homomorphism from $G$ to $End(\mathfrak{A})$; that is $(A, \lor, \land, \to, 0, G, s)$ is a transformation system).

3. $c_{(J \cup J')}p = c_{(J)}c_{(J')}p$, $q_{(J \cup J')}p = q_{(J)}c_{(J')}p$. 
(4) \( c_J q_J p = q_J c_J p \), \( q_J c_J p = c_J p \).

(5) If \( \sigma \upharpoonright \tau \sim J \upharpoonright J \), then \( s_\sigma c_J p = s_\tau c_J p \) and \( s_\sigma q_J p = s_\tau q_J p \).

(6) If \( \sigma \upharpoonright \sigma^{-1}(J) \) is injective, then \( c_{(J)} s_\sigma p = s_\tau c_{\sigma^{-1}(J)} p \) and \( q_{(J)} s_\sigma p = s_\tau q_{\sigma^{-1}(J)} p \).

**Definition 5.5.** Let \( \alpha \) and \( G \) be as in the previous definition. By a \( G \) polyadic equality algebra, a \( GPHAE_\alpha \) for short, we understand an algebra of the form 

\[
(A, \lor, \land, \to, 0, s_\tau, c_{(J)}, q_{(J)}, d_{ij})_{\tau \in G, J \subseteq \omega, i, j \in \alpha}
\]

where \((A, \lor, \land, \to, 0, s_\tau, c_{(J)}, q_{(J)})_{\tau \in G, J \subseteq \omega, i, j \in \alpha}\) is a \( GPHA_\alpha \) and \( d_{ij} \in A \) for each \( i, j \in \alpha \), such that the following identities hold for all \( k, l \in \alpha \) and all \( \tau \in G \):

1. \( d_{kk} = 1 \)
2. \( s_\tau d_{kl} = d_{r(k), \tau(l)} \).
3. \( x \cdot d_{kl} \leq s_{[k|l]} x \)

Here \([k|l]\) is the replacement that sends \( k \) to \( l \) and otherwise is the identity.

In our definition of algebras, we depart from [26] by defining polyadic algebras on sets rather than on ordinals. In this manner, we follow the tradition of Halmos. We refer to \( \alpha \) as the dimension of \( \mathfrak{A} \) and we write \( \alpha = \dim \mathfrak{A} \). Borrowing terminology from cylindric algebras, we refer to \( c_{\{i\}} \) by \( c_i \) and \( q_{\{i\}} \) by \( q_i \). However, we will have occasion to impose a well order on dimensions thereby dealing with ordinals.

**Remark 5.6.** When \( G \) consists of all finite transformations, then any algebra with a Boolean reduct satisfying the above identities relating cylindrifications, diagonal elements and substitutions, will be a quasipolyadic equality algebra of infinite dimension.

Next, we collect some properties of \( G \) algebras that are more handy to use in our subsequent work. In what follows, we will be writing \( GPHA \) (\( GPHAE \)) for all algebras considered.

**Theorem 5.7.** Let \( \alpha \) be an infinite set and \( \mathfrak{A} \in GPHA_\alpha \). Then \( \mathfrak{A} \) satisfies the following identities for \( \tau, \sigma \in G \) and all \( i, j, k \in \alpha \).

1. \( x \leq c_i x = c_i c_i x, \; c_i (x \lor y) = c_i x \lor c_i y, \; c_i c_j x = c_j c_i x. \)
   That is \( c_i \) is an additive operator (a modality) and \( c_i, c_j \) commute.

2. \( s_\tau \) is a Heyting algebra endomorphism.

3. \( s_\tau s_\sigma x = s_\tau a x \) and \( s_\tau d x = x \).
4. \( s_\tau c_i x = s_\tau[i|j]c_i x \).

Recall that \( \tau[i|j] \) is the transformation that agrees with \( \tau \) on \( \alpha \setminus \{i\} \) and \( \tau[i|j](i) = j \).

5. \( s_\tau c_i x = c_j s_\tau x \) if \( \tau^{-1}(j) = \{i\} \), \( s_\tau q_i x = q_j s_\tau x \) if \( \tau^{-1}(j) = \{i\} \).

6. \( c_i s_{[i|j]} x = s_{[i|j]} x \), \( q_i s_{[i|j]} x = s_{[i|j]} x \)

7. \( s_{[i|j]} c_i x = c_i x \), \( s_{[i|j]} q_i x = q_i x \).

8. \( s_{[i|j]} c_k x = c_k s_{[i|j]} x \), \( s_{[i|j]} q_k x = q_k s_{[i|j]} x \) whenever \( k \notin \{i, j\} \).

9. \( c_i s_{[j|i]} x = c_j s_{[i|j]} x \), \( q_i s_{[j|i]} x = q_j s_{[i|j]} x \).

**Proof.** The proof is tedious but fairly straightforward. □

Obviously the previous equations hold in \( GPHAE_\alpha \). Following cylindric algebra tradition and terminology, we will be often writing \( s_j^\alpha \) for \( s_{[i|j]} \).

**Remark 5.8.** For \( GPHA_\alpha \) when \( G \) is rich or \( G \) consists only of finite transformation it is enough to restrict our attention to replacements. Other substitutions are definable from those.

### 5.1 Neat reducts and dilations

Now we recall the important notion of neat reducts, a central concept in cylindric algebra theory, strongly related to representation theorems. This concept also occurs in polyadic algebras, but unfortunately under a different name, that of compressions.

Forming dilations of an algebra, is basically an algebraic reflection of a Henkin construction; in fact, the dilation of an algebra is another algebra that has an infinite number of new dimensions (constants) that potentially eliminate cylindrifications (quantifiers). Forming neat reducts has to do with restricting or compressing dimensions (number of variables) rather than increasing them. (Here the duality has a precise categorical sense which will be formulated in the part 3 of this paper as an adjoint situation).

**Definition 5.9.**

(1) Let \( \alpha \subseteq \beta \) be infinite sets. Let \( G_\beta \) be a semigroup of transformations on \( \beta \), and let \( G_\alpha \) be a semigroup of transformations on \( \alpha \) such that for all \( \tau \in G_\alpha \), one has \( \bar{\tau} = \tau \cup Id \in G_\beta \). Let \( \mathfrak{B} = (B, \lor, \land, \to, 0, c_i, s_\tau)_{\iota \in \beta, \tau \in G_\beta} \) be a \( G_\beta \) algebra.

(i) We denote by \( Rd_\alpha \mathfrak{B} \) the \( G_\alpha \) algebra obtained by dicarding operations in \( \beta \sim \alpha \). That is \( Rd_\alpha \mathfrak{B} = (B, \lor, \land, \to, 0, c_i, s_\tau)_{\iota \in \alpha, \tau \in G_\alpha} \). Here \( s_\tau \) is evaluated in \( \mathfrak{B} \).
(ii) For $x \in B$, then $\Delta x$, the dimension set of $x$, is defined by $\Delta x = \{i \in \beta : c_i x \neq x\}$. Let $A = \{x \in B : \Delta x \subseteq \alpha\}$. If $A$ is a subuniverse of $R^a_B$, then $A$ (the algebra with universe $A$) is a subreduct of $B$, it is called the neat $\alpha$ reduct of $B$ and is denoted by $\mathfrak{Nr}_\alpha B$.

(2) If $A \subseteq \mathfrak{Nr}_\alpha B$, then $B$ is called a dilation of $A$, and we say that $A$ neatly embeds in $B$. if $A$ generates $B$ (using all operations of $B$), then $B$ is called a minimal dilation of $A$.

The above definition applies equally well to $GPHA\alpha$.

Remark 5.10. In certain contexts minimal dilations may not be unique (up to isomorphism), but what we show next is that in all the cases we study, they are unique, so for a given algebra $A$, we may safely say the minimal dilation of $A$.

For an algebra $A$, and $X \subseteq A$, $\mathcal{G}^a A X$ or simply $\mathcal{G} A X$, when $A$ is clear from context, denotes the subalgebra of $A$ generated by $X$. The next theorems apply equally well to $GPHA\alpha$ with easy modifications which we state as we go along.

Lemma 5.11. Let $G_I$ be the semigroup of finite transformations on $I$. Let $A \in G_\alpha PHA_\alpha$ be such that $\alpha \sim \Delta x$ is infinite for every $x \in A$. Then for any set $\beta$, such that $\alpha \subseteq \beta$, there exists $B \in G_\beta PHA_\beta$, such that $A \subseteq \mathfrak{Nr}_\alpha B$.

Proof. Let $\alpha \subseteq \beta$. We assume, loss of generality, that $\alpha$ and $\beta$ are ordinals with $\alpha < \beta$. The proof is a direct adaptation of the proof of Theorem 2.6.49(i) in [22]. First we show that there exists $B \in G_{\alpha+1} PHA_{\alpha+1}$ such that $A$ embeds into $\mathfrak{Nr}_\alpha B$, then we proceed inductively. Let

$$R = Id \upharpoonright (\alpha \times A) \cup \{(k, x, (\lambda, y)) : k, \lambda < \alpha, x, y \in A, \lambda \notin \Delta x, y = s_{[k\mid \lambda]} x\}.$$ 

It is easy to see that $R$ is an equivalence relation on $\alpha \times A$. Define the following operations on $((\alpha \times A)/R)$ with $\mu, i, k \in \alpha$ and $x, y \in A$:

$$(\mu, x)/R \lor (\mu, y)/R = (\mu, x \lor y)/R,$$

$$(\mu, x)/R \land (\mu, y)/R = (\mu, x \land y)/R,$$

$$(\mu, x)/R \rightarrow (\mu, y)/R = (\mu, x \rightarrow y)/R,$$

$$(c_i((\mu, x))/R) = (\mu, c_i x)/R, \quad \mu \in \alpha \setminus \{i\},$$

$$(s_{[j\mid i]}((\mu, x))/R) = (\mu, s_{[j\mid i]} x)/R, \quad \mu \in \alpha \setminus \{i, j\}.$$ 

It can be checked that these operations are well defined. Let

$$C = ((\alpha \times A)/R, \lor, \land, \rightarrow, 0, c_i, s_{[i\mid j]}), i, j \in \alpha,$$
and let
\[ h = \{(x, (\mu, x)/R) : x \in A, \mu \in \alpha \sim \Delta x \}. \]

Then \( h \) is an isomorphism from \( \mathfrak{A} \) into \( \mathfrak{C} \). Now to show that \( \mathfrak{A} \) neatly embeds into \( \alpha + 1 \) extra dimensions, we define the operations \( c_\alpha, s_{[i]|\alpha} \) and \( s_{[\alpha]|i} \) on \( \mathfrak{C} \) as follows:

\[
c_\alpha = \{((\mu, x)/R, (\mu, c_\mu x)/R) : \mu \in \alpha, x \in B\},
\]

\[
s_{[i]|\alpha} = \{((\mu, x)/R, (\mu, s_{[i]|\alpha} x)/R) : \mu \in \alpha \setminus \{i\}, x \in B\},
\]

\[
s_{[\alpha]|i} = \{((\mu, x)/R, (\mu, s_{[\alpha]|i} x)/R) : \mu \in \alpha \setminus \{i\}, x \in B\}.
\]

Let
\[ \mathfrak{B} = ((\alpha \times A)/R, \lor, \land, \to, c_\alpha, s_{[i]|\alpha}, s_{[\alpha]|i})_{i,j \leq \alpha}. \]

Then
\[ \mathfrak{B} \in G_{\alpha+1}PA_{\alpha+1} \text{ and } h(\mathfrak{A}) \subseteq \mathfrak{N}_\alpha \mathfrak{B}. \]

It is not hard to check that the defined operations are as desired. We have our result when \( G \) consists only of replacements. But since \( \alpha \sim \Delta x \) is infinite one can show that substitutions corresponding to all finite transformations are term definable. For a finite transformation \( \tau \in ^1_0 \alpha \) we write \([u_0|v_0, u_1|v_1, \ldots, u_{k-1}|v_{k-1}]\) if \( \sup \tau = \{u_0, \ldots, u_{k-1}\}, u_0 < u_1 \ldots < u_{k-1} \) and \( \tau(u_i) = v_i \) for \( i < k \). Let \( \mathfrak{A} \in GPHA_\alpha \) be such that \( \alpha \sim \Delta x \) is infinite for every \( x \in A \). If \( \tau = [u_0|v_0, u_1|v_1, \ldots, u_{k-1}|v_{k-1}] \) is a finite transformation, if \( x \in A \) and if \( \pi_0, \ldots, \pi_{k-1} \) are in this order the first \( k \) ordinals in \( \alpha \sim (\Delta x \cup Rg(u) \cup Rg(v)) \), then

\[
s_\tau x = s_{\pi_0}^{u_0} \ldots s_{\pi_{k-1}}^{u_{k-1}} s_{v_0}^{\pi_0} \ldots s_{v_{k-1}}^{\pi_{k-1}} x.
\]

The \( s_\tau \)'s so defined satisfy the polyadic axioms, cf \text{[22]} Theorem 1.11.11. Then one proceeds by a simple induction to show that for all \( n \in \omega \) there exists \( \mathfrak{B} \in G_{\alpha+n}PHA_{\alpha+n} \) such that \( \mathfrak{A} \subseteq \mathfrak{N}_\alpha \mathfrak{B} \). For the transfinite, one uses ultraproducts \text{[22]} theorem 2.6.34.

**Lemma 5.12.** With \( \mathfrak{A} \) and \( \mathfrak{B} \) as in the previous lemmad for all \( X \subseteq A \), one has \( \mathfrak{S}^a \mathfrak{X} = \mathfrak{N}_\alpha \mathfrak{S}^a \mathfrak{X} \).

**Proof.** let \( \mathfrak{A} \subseteq \mathfrak{N}_\alpha \mathfrak{B} \) and \( A \) generates \( \mathfrak{B} \) then \( \mathfrak{B} \) consists of all elements \( s_\sigma x \) such that \( x \in A \) and \( \sigma \) is a finite transformation on \( \beta \) such that \( \sigma \upharpoonright \alpha \) is one to one \text{[22]} lemma 2.6.66. Now suppose \( x \in \mathfrak{N}_\alpha \mathfrak{S}^a \mathfrak{X} \) and \( \Delta x \subseteq \alpha \), then there exist \( y \in \mathfrak{S}^a \mathfrak{X} \) and a finite transformation \( \sigma \) of \( \beta \) such that \( \sigma \upharpoonright \alpha \) is one to one and \( x = s_\sigma y \). Let \( \tau \) be a finite transformation of \( \beta \) such that \( \tau \upharpoonright \alpha = Id \) and \( (\tau \circ \sigma) \alpha \subseteq \alpha \). Then \( x = s_\tau^a x = s_\tau^a s_\sigma y = s_\tau^a s_\sigma y = s_\tau^a s_\sigma y \). In the presence of diagonal elements, one defines them in the bigger algebra (the dilation) precisely as in \text{[22]}, theorem 2.6.49(i). The next lemma formulated only for \( GPHA_\alpha \) will be used in proving our main (algebraic) result. The
proof works without any modifications when we add diagonal elements. The
lemma says, roughly, that if we have an \( \alpha \) dimensional algebra \( \mathfrak{A} \), and a set \( \beta \)
containing \( \alpha \), then we can find an extension \( \mathfrak{B} \) of \( \mathfrak{A} \) in \( \beta \) dimensions, specified
by a carefully chosen subsemigroup of \( \beta \beta \), such that \( \mathfrak{A} = \mathfrak{A} \mathfrak{B} \) and for all
\( b \in B \), \( |\Delta b \sim \alpha| < \omega \). \( \mathfrak{B} \) is not necessarily the minimal dilation of \( \mathfrak{A} \), because
the large subsemigroup chosen maybe smaller than the semigroup used to form
the unique dilation. It can happen that this extension is the minimal dilation,
but in the case we consider all transformations, the constructed algebra is
only a proper subreduct of the dilation obtained basically by discarding those
elements \( b \) in the original dilation for which \( \Delta b \sim \alpha \) is infinite.

5.2 Algebraic Proofs of main theorems

Our work in this section is closely related to that in [8]. Our main theorem is
a typical representability result, where we start with an abstract (free) algebra,
and we find a non-trivial homomorphism from this algebra to a concrete algebra
based on Kripke systems (an algebraic version of Kripke frames).

The idea (at least for the equality-free case) is that we start with a theory
(which is defined as a pair of sets of formulas, as is the case with classical
intuitionistic logic), extend it to a saturated one in enough spare dimensions, or
an appropriate dilation (lemma 5.15), and then iterate this process countably
many times forming consecutive (countably many) dilations in enough spare
dimensions, using pairs of pairs (theories), cf. lemma 5.16; finally forming an
extension that will be used to construct desired Kripke models (theorem ??).
The extensions constructed are essentially conservative extensions, and they
will actually constitute the set of worlds of our desired Kripke model.

The iteration is done by a subtle zig-zag process, a technique due to Gabbay
[16]. When we have diagonal elements (equality), constructing desired Kripke
model, is substantially different, and much more intricate.

All definitions and results up to lemma 5.18, though formulated only for
the diagonal-free case, applies equally well to the case when there are diagonal
elements, with absolutely no modifications. (The case when diagonal elements
are present will be dealt with in part 2).

Definition 5.13. Let \( \mathfrak{A} \in GPH \Lambda_\alpha \).

(1) A theory in \( \mathfrak{A} \) is a pair \( (\Gamma, \Delta) \) such that \( \Gamma, \Delta \subseteq \mathfrak{A} \).

(2) A theory \( (\Gamma, \Delta) \) is consistent if there are no \( a_1, \ldots, a_n \in \Gamma \) and \( b_1, \ldots, b_m \in \Delta \) \( (m, n \in \omega) \) such that

\[
a_1 \land \ldots a_n \leq b_1 \lor \ldots b_m.
\]
Not that in this case, we have $\Gamma \cap \Delta = \emptyset$. Also if $F$ is a filter (has the finite intersection property), then it is always the case that $(F, \{0\})$ is consistent.

(3) A theory $(\Gamma, \Delta)$ is complete if for all $a \in A$, either $a \in \Gamma$ or $a \in \Delta$.

(4) A theory $(\Gamma, \Delta)$ is saturated if for all $a \in A$ and $j \in \alpha$, if $c_j a \in \Gamma$, then there exists $k \in \alpha \sim \Delta a$, such that $s_k^j a \in \Gamma$. Note that a saturated theory depends only on $\Gamma$.

**Lemma 5.14.** Let $\mathfrak{A} \in GPHA_\alpha$ and $(\Gamma, \Delta)$ be a consistent theory.

(i) For any $a \in A$, either $(\Gamma \cup \{a\}, \Delta)$ or $(\Gamma, \Delta \cup \{a\})$ is consistent.

(ii) $(\Gamma, \Delta)$ can be extended to a complete theory in $\mathfrak{A}$.

**Proof.**

(i) Cf. [8]. Suppose for contradiction that both theories are inconsistent. Then we have $\mu_1 \wedge a \leq \delta_1$ and $\mu_2 \leq a \wedge \delta_2$ where $\mu_1$ and $\mu_2$ are some conjunction of elements of $\Gamma$ and $\delta_1, \delta_2$ are some disjunction of elements of $\Delta$. But from $(\mu_1 \wedge a \to \delta_1) \wedge (\mu_2 \to a \vee \delta_2) \leq (\mu_1 \wedge \mu_2 \to \delta_1 \vee \delta_2)$, we get $\mu_1 \wedge \mu_2 \leq \delta_1 \vee \delta_2$, which contradicts the consistency of $(\Gamma, \Delta)$.

(ii) Cf. [8]. Assume that $|A| = \kappa$. Enumerate the elements of $\mathfrak{A}$ as $(a_i : i < \kappa)$. Then we can extend $(\Gamma, \Delta)$ consecutively by adding $a_i$ either to $\Gamma$ or $\Delta$ while preserving consistency. In more detail, we define by transfinite induction a sequence of theories $(\Gamma_i, \Delta_i)$ for $i \in \kappa$ as follows. Set $\Gamma_0 = \Gamma$ and $\Delta_0 = \Delta$. If $\Gamma_i, \Delta_i$ are defined for all $i < \mu$ where $\mu$ is a limit ordinal, let $\Gamma_\mu = (\bigcup_{i \in \mu} \Gamma_i, \bigcup_{i \in \mu} \Delta_i)$. Now for successor ordinals. Assume that $(\Gamma_i, \Delta_i)$ are defined. Set $\Gamma_{i+1} = \Gamma_i \cup \{a_i\}, \Delta_{i+1} = \Delta_i$ in case this is consistent, else set $\Gamma_{i+1} = \Gamma_i$ and $\Delta_{i+1} = \Delta_i \cup \{a_i\}$. Let $T = \bigcup_{i \in \kappa} T_i$ and $F = \bigcup_{i \in \kappa} F_i$, then $(T, F)$ is as desired.

**Lemma 5.15.** Let $\mathfrak{A} \in GPHA_\alpha$ and $(\Gamma, \Delta)$ be a consistent theory of $\mathfrak{A}$. Let $I$ be a set such that $\alpha \subseteq I$ and let $\beta = |I \sim \alpha| = \max(|A|, |\alpha|)$. Then there exists a minimal dilation $\mathfrak{B}$ of $\mathfrak{A}$ of dimension $I$, and a theory $(T, F)$ in $\mathfrak{B}$, extending $(\Gamma, \Delta)$ such that $(T, F)$ is saturated and complete.

**Proof.** Let $I$ be provided as in the statement of the lemma. By lemma 5.11, there exists $\mathfrak{B} \in GPHA_I$ such that $\mathfrak{A} \subseteq \text{Nr}_\alpha \mathfrak{B}$ and $\mathfrak{A}$ generates $\mathfrak{B}$. We also have for all $X \subseteq \mathfrak{A}$, $\mathfrak{G}^\alpha X = \text{Nr}_\alpha \mathfrak{G}^\alpha X$. Let $\{b_i : i < \kappa\}$ be an enumeration of the elements of $\mathfrak{B}$; here $\kappa = |B|$. Define by transfinite recursion a sequence
\((T_i, F_i)\) for \(i < \kappa\) of theories as follows. Set \(T_0 = \Gamma\) and \(F_0 = \Delta\). We assume inductively that

\[
|\beta| \sim \bigcup_{x \in T_i} \Delta x \cup \bigcup_{x \in F_i} \Delta x \geq \omega.
\]

This is clearly satisfied for \(F_0\) and \(T_0\). Now we need to worry only about successor ordinals. Assume that \(T_i\) and \(F_i\) are defined. We distinguish between two cases:

1. \((T_i, F_i \cup \{b_i\})\) is consistent. Then set \(T_{i+1} = T_i\) and \(F_{i+1} = F_i \cup \{b_i\}\).

2. If not, that is if \((T_i, F_i \cup \{b_i\})\) is inconsistent. In this case, we distinguish between two subcases:

   (a) \(b_i\) is not of the form \(c_j p\). Then set \(T_{i+1} = T_i \cup \{b_i\}\) and \(F_{i+1} = F_i\).

   (b) \(b_i = c_j p\) for some \(j \in I\). Then set \(T_{i+1} = T_i \cup \{c_j p, s_u p\}\) where \(u \notin \Delta p \cup \bigcup_{x \in T_i} \bigcup_{x \in F_i} \Delta x\) and \(F_{i+1} = F_i\).

Such a \(u\) exists by the inductive assumption. Now we check by induction that each \((T_i, F_i)\) is consistent. The only part that needs checking, in view of the previous lemma, is subcase (b). So assume that \((T_i, F_i)\) is consistent and \(b_i = c_j p\). If \((T_{i+1}, F_{i+1})\) is inconsistent, then we would have for some \(a \in T_i\) and some \(\delta \in F_i\) that \(a \land c_j p \land s_u p \leq \delta\). From this we get \(a \land c_j p \leq \delta\), because \(s_u p \leq c_j p\). But this contradicts the consistency of \((T_i \cup \{c_j p\}, F_i)\).

Let \(T = \bigcup_{i \in \kappa} T_i\) and \(F = \bigcup_{i \in \kappa} F_i\), then \((T, F)\) is consistent. We show that it is saturated. If \(c_j p \in T\), then \(c_j p \in T_{i+1}\) for some \(i\), hence \(s_u p \in T_{i+1} \subseteq T\) and \(u \notin \Delta p\). Now by lemma 5.14 we can extend \((T, F)\) is \(\mathcal{B}\) to a complete theory, and this will not affect saturation, since the process of completion does not take us out of \(\mathcal{B}\).

The next lemma constitutes the core of our construction; involving a zig-zag Gabbay construction, it will be used repeatedly, to construct our desired representation via a set algebra based on a Kripke system defined in 5.2.

**Lemma 5.16.** Let \(\mathfrak{A} \in GPHA_\alpha\) be generated by \(X\) and let \(X = X_1 \cup X_2\). Let \((\Delta_0, \Gamma_0)\), \((\Theta_0, \Gamma_0^*)\) be two consistent theories in \(\mathfrak{Sg}^{\mathfrak{A}} X_1\) and \(\mathfrak{Sg}^{\mathfrak{A}} X_2\), respectively such that \(\Gamma_0 \subseteq \mathfrak{Sg}^{\mathfrak{A}} (X_1 \cap X_2)\), \(\Gamma_0^* \subseteq \Gamma_0^*\). Assume further that \((\Delta_0 \cap \Theta_0 \cap \mathfrak{Sg}^{\mathfrak{A}} X_1 \cap \mathfrak{Sg}^{\mathfrak{A}} X_2, \Gamma_0)\) is complete in \(\mathfrak{Sg}^{\mathfrak{A}} X_1 \cap \mathfrak{Sg}^{\mathfrak{A}} X_2\). Suppose that \(I\) is a set such that \(\alpha \subseteq I\) and \(|I| \sim \alpha| = max(|A_1|, |A_2|)\). Then there exist a dilation \(\mathcal{B} \in GPHA_I\) of \(\mathfrak{A}\), and theories \(T_1 = (\Delta_\omega, \Gamma_\omega)\), \(T_2 = (\Theta_\omega, \Gamma_\omega^*)\) extending \((\Delta_0, \Gamma_0), (\Theta_0, \Gamma_0^*)\), such that \(T_1\) and \(T_2\) are consistent and saturated in \(\mathfrak{Sg}^{\mathfrak{A}} X_1\) and \(\mathfrak{Sg}^{\mathfrak{A}} X_2\), respectively, \((\Delta_\omega \cap \Theta_\omega, \Gamma_\omega)\) is complete in \(\mathfrak{Sg}^{\mathfrak{A}} X_1 \cap \mathfrak{Sg}^{\mathfrak{A}} X_2\), and \(\Gamma_\omega \subseteq \Gamma_\omega^*\).

**Proof.** Like the corresponding proof in [3], we will build the desired theories in a step-by-step zig-zag manner in a large enough dilation whose dimension
is specified by $I$. The spare dimensions play a role of added witnesses, that will allow us to eliminate quantifiers, in a sense. Let $\mathfrak{A} = \mathfrak{A}_0 \in GPHA_\alpha$. The proof consists of an iteration of lemmata 5.14 and 5.15. Let $\beta = \max(|A|, |\alpha|)$, and let $I$ be such that $|I \sim \alpha| = \beta$.

We distinguish between two cases:

Assume that $G$ is strongly rich or $G$ contains consists of all finite transformations. In this case we only deal with minimal dilations. We can write $\beta = I \sim \alpha$ as $\bigcup_{n=1}^\infty C_n$ where $C_i \cap C_j = \emptyset$ for distinct $i$ and $j$ and $|C_i| = \beta$ for all $i$. Then iterate first two items in lemma 5.11. Let $A_1 = A(C_1) \in G_{a \cup C_1} PHA_{a \cup C_1}$ be a minimal dilation of $A$. Let $\mathfrak{A}_1 = \mathfrak{A}(C_1)$ be a minimal dilation of $\mathfrak{A}$, so that $\mathfrak{A} = \mathfrak{M}_\alpha \mathfrak{A}_1$. Let $\mathfrak{A}_2 = \mathfrak{A}(C_1)(C_2)$ be a minimal dilation of $\mathfrak{A}_1$, so that $\mathfrak{A}_1 = \mathfrak{M}_\alpha \mathfrak{A}_2$. Generally, we define inductively $\mathfrak{A}_n = \mathfrak{A}(C_1)(C_2)\ldots(C_n)$ to be a minimal dilation of $\mathfrak{A}_{n-1}$, so that $\mathfrak{A}_{n-1} = \mathfrak{M}_\alpha \mathfrak{A}_1 \ldots \mathfrak{A}_n$. Notice that for $k < n$, $\mathfrak{A}_k$ is a minimal dilation of $\mathfrak{A}_k$. So we have a sequence of algebras $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2 \ldots$. Each element in the sequence is the minimal dilation of its preceding one.

Now that we have a sequence of extensions $\mathfrak{A}_0 \subseteq \mathfrak{A}_1 \ldots$ in different increasing dimensions, we now form a limit of this sequence in $I$ dimensions. We can use ultraproducts, but instead we use products, and quotient algebras. First form the Heyting algebra, that is the product of the Heyting reduc ts of the constructed algebras, that is take $\mathfrak{C} = \prod_{n=0}^\infty RdA_n$, where $Rd\mathfrak{A}_n$ denotes the Heyting reduct of $\mathfrak{A}_n$ obtained by discarding substitutions and cylindrifiers. Let $M = \{f \in C : (\exists n \in \omega)(\forall k \geq n)f_k = 0\}$.

Then $M$ is a Heyting ideal of $\mathfrak{C}$. Now form the quotient Heyting algebra $\mathfrak{D} = \mathfrak{C}/M$. We want to expand this Heyting algebra algebra by cylindrifiers and substitutions, i.e to an algebra in $GPHA_I$. Towards this aim, for $\tau \in G$, define $\phi(\tau) \in \mathfrak{C}C$ as follows:

$$\phi(\tau)f) = s_{\tau|dim\mathfrak{A}_n}f_n$$

if $\tau(dim(\mathfrak{A}_n)) \subseteq dim(\mathfrak{A}_n)$. Otherwise

$$\phi(\tau)f) = f_n.$$

For $j \in I$, define

$$c_jf_n = c_{\mathfrak{A}_n|dim\mathfrak{A}_n \cap \{j\}}f_n,$$

and

$$q_jf_n = q_{\mathfrak{A}_n|dim\mathfrak{A}_n \cap \{j\}}f_n.$$

Then for $\tau \in G$ and $j \in I$, set

$$s_{\tau}(f/M) = \phi(\tau)f/M,$$

$$c_j(f/M) = (c_jf)/M.$$
and

\[ q_j(f/M) = (q_j f)/M. \]

Then, it can be easily checked that, \( \mathfrak{A}_n = (\mathfrak{A}, s_r, c_j, q_j) \) is a \( GPH A I \), in which every \( \mathfrak{A}_n \) neatly embeds. We can and will assume that \( \mathfrak{A}_n = \mathfrak{N}_{\alpha \cup C_1 \ldots \cup C_n} \mathfrak{A}_\infty \). Also \( \mathfrak{A}_\infty \) is a minimal dilation of \( \mathfrak{A}_n \) for all \( n \). During our ‘zig-zagging’ we shall be extensively using lemma 5.12.

From now on, fix \( \mathfrak{A} \) to be as in the statement of lemma 5.16 for some time to come. So \( \mathfrak{A} \in GPHA_\infty \) is generated by \( X \) and \( X = X_1 \cup X_2 \). (\( \Delta_0, \Gamma_0 \)), (\( \Theta_0, \Gamma_0^* \)) are two consistent theories in \( \mathfrak{S} \mathfrak{g}^3 X_1 \) and \( \mathfrak{S} \mathfrak{g}^3 X_2 \), respectively such that \( \Gamma_0 \subseteq \mathfrak{S} \mathfrak{g}^3 (X_1 \cup X_2) \), \( \Gamma_0 \subseteq \Gamma_0^* \). Finally \( (\Delta_0 \cap \Theta_0 \cap \mathfrak{S} \mathfrak{g}^3 X_1 \cap \mathfrak{S} \mathfrak{g}^3 X_2, \Gamma_0) \) is complete in \( \mathfrak{S} \mathfrak{g}^3 X_1 \cap \mathfrak{S} \mathfrak{g}^3 X_2 \). Now we have:

\[
\Delta_0 \subseteq \mathfrak{S} \mathfrak{g}^3 X_1 \subseteq \mathfrak{S} \mathfrak{g}^{\mathbf{C}(C_1)} X_1 \subseteq \mathfrak{S} \mathfrak{g}^{\mathbf{C}(C_1)(C_2)} X_1 \subseteq \mathfrak{S} \mathfrak{g}^{\mathbf{C}(C_1)(C_2)(C_3)} X_1 \ldots \subseteq \mathfrak{S} \mathfrak{g}^3 X_1.
\]

\[
\Theta_0 \subseteq \mathfrak{S} \mathfrak{g}^3 X_2 \subseteq \mathfrak{S} \mathfrak{g}^{\mathbf{C}(C_1)} X_2 \subseteq \mathfrak{S} \mathfrak{g}^{\mathbf{C}(C_1)(C_2)} X_2 \subseteq \mathfrak{S} \mathfrak{g}^{\mathbf{C}(C_1)(C_2)(C_3)} X_2 \ldots \subseteq \mathfrak{S} \mathfrak{g}^3 X_2.
\]

In view of lemmata 5.14, 5.15 extend \( (\Delta_0, \Gamma_0) \) to a complete and saturated theory \( (\Delta_1, \Gamma_1') \) in \( \mathfrak{S} \mathfrak{g}^{\mathbf{C}(C_1)} X_1 \). Consider \( (\Delta_1, \Gamma_0) \). Zig-zagging away, we extend our theories in a step by step manner. The proofs of the coming Claims, 1, 2 and 3, are very similar to the proofs of the corresponding claims in [8], which are in turn an algebraic version of lemmata 4.18-19-20 in [16], with one major difference from the former. In our present situation, we can cylindrify on only finitely many indices, so we have to be careful, when talking about dimension sets, and in forming neat reducts (or compressions). Our proof then becomes substantially more involved. In the course of our proof we use extensively lemmata 5.11 and 5.12 which are not formulated in [8] because we simply did not need them when we had cylindrifications on possibly infinite sets.

**Claim 1.** The theory \( T_1 = (\Theta_0 \cup (\Delta_1 \cap \mathfrak{S} \mathfrak{g}^{\mathbf{C}(C_1)} X_2), \Gamma_0^*) \) is consistent in \( \mathfrak{S} \mathfrak{g}^{\mathbf{C}(C_1)} X_2 \).

**Proof of Claim 1.** Assume that \( T_1 \) is inconsistent. Then for some conjunction \( \theta_0 \) of elements in \( \Theta_0 \), some \( E_1 \in \Delta_1 \cap \mathfrak{S} \mathfrak{g}^{\mathbf{C}(C_1)} X_2 \), and some disjunction \( \mu_0^* \) in \( \Gamma_0^* \), we have \( \theta_0 \land E_1 \leq \mu_0^* \), and so \( E_1 \leq \theta_0 \to \mu_0^* \). Since \( \theta_0 \in \Theta_0 \subseteq \mathfrak{S} \mathfrak{g}^3 X_2 \) and \( \mu_0^* \subseteq \Gamma_0^* \subseteq \mathfrak{S} \mathfrak{g}^3 X_2 \subseteq \mathfrak{M}_{\alpha \cup C_1} \mathfrak{A} \), therefore, for any finite set \( D \subseteq C_1 \sim \alpha \), we have \( c_{(D)} \theta_0 = \theta_0 \) and \( c_{(D)} \mu_0^* = \mu_0^* \). Also for any finite set \( D \subseteq C_1 \sim \alpha \), we have \( c_{(D)} E_1 \leq c_{(D)} (\theta_0 \to \mu_0^*) = \theta_0 \to \mu_0^* \). Now \( E_1 \in \Delta_1 \), hence \( E_1 \in \mathfrak{S} \mathfrak{g}^{\mathbf{C}(C_1)} X_1 \). By definition, we also have \( E_1 \in \mathfrak{S} \mathfrak{g}^{\mathbf{C}(C_1)} X_2 \). By lemma 5.12 there exist finite sets \( D_1 \) and \( D_2 \) contained in \( C_1 \sim \alpha \), such that

\[
c_{(D_1)} E_1 \in \mathfrak{M}_\alpha \mathfrak{S} \mathfrak{g}^{\mathbf{C}(C_1)} X_1
\]

and

\[
c_{(D_2)} E_1 \in \mathfrak{M}_\alpha \mathfrak{S} \mathfrak{g}^{\mathbf{C}(C_1)} X_2.
\]

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\[ D = D_1 \cup D_2. \] Then \( D \subseteq C_1 \sim \alpha \) and we have:

\[ c_{(D)}E_1 \in \mathcal{N}_\alpha \mathsf{Sg}^{\alpha(C_1)}X_1 = \mathsf{Sg}^{\mathcal{N}_\alpha \alpha(C_1)}X_1 = \mathsf{Sg}^\alpha X_1 \]

and

\[ c_{(D)}E_1 \in \mathcal{N}_\alpha \mathsf{Sg}^{\alpha(C_1)}X_2 = \mathsf{Sg}^{\mathcal{N}_\alpha \alpha(C_1)}X_2 = \mathsf{Sg}^\alpha X_2, \]

that is to say

\[ c_{(D)}E_1 \in \mathsf{Sg}^\alpha X_1 \cap \mathsf{Sg}^\alpha X_2. \]

Since \((\Delta_0 \cap \Theta_0 \cap \mathsf{Sg}^\alpha X_1 \cap \mathsf{Sg}^\alpha X_2, \Gamma_0)\) is complete in \(\mathsf{Sg}^\alpha X_1 \cap \mathsf{Sg}^\alpha X_2\), we get that \(c_{(D)}E_1\) is either in \(\Delta_0 \cap \Theta_0\) or \(\Gamma_0\). We show that either way leads to a contradiction, by which we will be done. Suppose it is in \(\Gamma_0\). Recall that we extended \((\Delta_0, \Gamma_0)\) to a complete saturated extension \((\Delta, \Gamma')\) in \(\mathsf{Sg}^{\alpha(C_1)}X_1\).

Since \(\Gamma_0 \subseteq \Gamma'_1\), we get that \(c_{(D)}E_1 \in \Gamma'_1\) hence \(c_{(D)}E_1 \notin \Delta_1\) because \((\Delta_1, \Gamma'_1)\) is saturated and consistent. But this contradicts that \(E_1 \in \Delta_1\) because \(E_1 \leq c_{(D)}E_1\). Thus, we can infer that \(c_{(D)}E_1 \in \Delta_0 \cap \Theta_0\). In particular, it is in \(\Theta_0\), and so \(\theta_0 \rightarrow \mu_0^* \in \Theta_0\). But again this contradicts the consistency of \((\Theta_0, \Gamma'_0)\).

Now we extend \(T_1\) to a complete and saturated theory \((\Theta_2, \Gamma'_2)\) in \(\mathsf{Sg}^{\alpha(C_1)}(C_2)X_2\). Let \(\Gamma_2 = \Gamma'_2 \cap \mathsf{Sg}^{\alpha(C_1)}(C_2)X_1\).

**Claim 2.** The theory \(T_2 = (\Delta_1 \cup (\Theta_2 \cap \mathsf{Sg}^{\alpha(C_1)}(C_2))X_1), \Gamma_2)\) is consistent in \(\mathsf{Sg}^{\alpha(C_1)}(C_2)X_1\).

**Proof of Claim 2.** If the Claim fails to hold, then we would have some \(\delta_1 \in \Delta_1, E_2 \in \Theta_2 \cap \mathsf{Sg}^{\alpha(C_1)}(C_2)X_1\), and a disjunction \(\mu_2 \in \Gamma_2\) such that \(\delta_1 \wedge E_2 \rightarrow \mu_2\), and so \(\delta_1 \leq (E_2 \rightarrow \mu_2)\) since \(\delta_1 \in \Delta_1 \subseteq \mathsf{Sg}^{\alpha(C_1)}X_1\). But \(\mathsf{Sg}^{\alpha(C_1)}X_1 \subseteq \mathcal{N}_\alpha \mathsf{Sg}^{\alpha(C_1)}(C_2)X_1\), therefore for any finite set \(D \subseteq C_2 \sim C_1\), we have \(q(D)\delta_1 = \delta_1\). The following holds for any finite set \(D \subseteq C_2 \sim C_1\),

\[ \delta_1 \leq q(D)(E_2 \rightarrow \mu_2). \]

Now, by lemma 5.12 there is a finite set \(D \subseteq C_2 \sim C_1\), satisfying

\[ \delta_1 \rightarrow q(D)(E_2 \rightarrow \mu_2) \in \mathcal{N}_\alpha \mathsf{Sg}^{\alpha(C_1)}(C_2)X_2, \]

\[ = \mathsf{Sg}^{\mathcal{N}_\alpha \alpha(C_1)\alpha(C_1)(C_2)}X_2, \]

\[ = \mathsf{Sg}^{\alpha(C_1)}X_2. \]

Since \(\delta_1 \in \Delta_1\), and \(\delta_1 \leq q(D)(E_2 \rightarrow \mu_2)\), we get that \(q(D)(E_2 \rightarrow \mu_2)\) is in \(\Delta_1 \cap \mathsf{Sg}^{\alpha(C_1)}X_2\). We proceed as in the previous claim replacing \(\Theta_0\) by \(\Theta_2\) and the existential quantifier by the universal one. Let \(E_1 = q(D)(E_2 \rightarrow \mu_2)\). Then \(E_1 \in \mathsf{Sg}^{\alpha(C_1)}X_1 \cap \mathsf{Sg}^{\alpha(C_2)}X_2\). By lemma 5.12 there exist finite sets \(D_1\) and \(D_2\) contained in \(C_1 \sim \alpha\) such that

\[ q(D_1)E_1 \in \mathcal{N}_\alpha \mathsf{Sg}^{\alpha(C_1)}X_1, \]

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and
\[ q_{(D_2)}E_1 \in \mathfrak{N}_\alpha \mathfrak{S}_g^{\mathfrak{a}(C_1)} X_2. \]

Le \( J = D_1 \cup D_2 \). Then \( J \subseteq C_1 \sim \alpha \), and we have:
\[ q_{(J)}E_1 \in \mathfrak{N}_\alpha \mathfrak{S}_g^{\mathfrak{a}(C_1)} X_1 = \mathfrak{S}_g^{\mathfrak{a}(C_1)} X_1 \]
and
\[ q_{(J)}E_1 \in \mathfrak{N}_\alpha \mathfrak{S}_g^{\mathfrak{a}(C_1)} X_2 = \mathfrak{S}_g^{\mathfrak{a}(C_1)} X_2. \]

That is to say,
\[ q_{(J)}E_1 \in \mathfrak{S}_g^{\mathfrak{a}} X_1 \cap \mathfrak{S}_g^{\mathfrak{a}} X_2. \]

Now \( (\Delta_0 \cap \Theta_2 \cap \mathfrak{S}_g^{\mathfrak{a}} X_1 \cap \mathfrak{S}_g^{\mathfrak{a}} X_2, \Gamma_0) \) is complete in \( \mathfrak{S}_g^{\mathfrak{a}} X_1 \cap \mathfrak{S}_g^{\mathfrak{a}} X_2 \), we get that \( q_{(J)}E_1 \) is either in \( \Delta_0 \cap \Theta_2 \) or \( \Gamma_0 \). Suppose it is in \( \Gamma_0 \). Since \( \Gamma_0 \subseteq \Gamma_1 \), we get that \( q_{(J)}E_1 \notin \Delta_1 \), because \( (\Delta_1, \Gamma_1) \) is saturated and consistent. Here, recall that, \( (\Delta, \Gamma') \) is a saturated complete extension of \( (\Gamma, \Delta) \). But this contradicts that \( E_1 \in \Delta_1 \). Thus, we can infer that \( q_{(J)}E_1 \in \Delta_0 \cap \Theta_2 \). In particular, it is in \( \Theta_2 \). Hence \( q_{(D \cup J)}(E_2 \rightarrow \mu_2) \in \Theta_2 \), and so \( E_2 \rightarrow \mu_2 \in \Theta_2 \) since \( q_{(D \cup J)}(E_2 \rightarrow \mu_2) \leq E_1 \rightarrow \mu_2 \). But this is a contradiction, since \( E_2 \in \Theta_2 \), \( \mu_2 \in \Gamma_2^* \) and \( (\Theta_2, \Gamma_2^*) \) is consistent.

Extend \( T_2 \) to a complete and saturated theory \( (\Delta_3, \Gamma'_3) \) in \( \mathfrak{S}_g^{\mathfrak{a}(C_1)(C_2)(C_3)} X_1 \) such that \( \Gamma_2 \subseteq \Gamma'_3 \). Again we are interested only in \( (\Delta_3, \Gamma_2) \).

**Claim 3**. The theory \( T_3 = (\Theta_2 \cup \Delta_3 \cap \mathfrak{S}_g^{\mathfrak{a}(C_1)(C_2)(C_3)} X_2, \Gamma_2^* \) is consistent in \( \mathfrak{S}_g^{\mathfrak{a}(C_1)(C_2)(C_3)} X_2 \).

**Proof of Claim 3.** Seeking a contradiction, assume that the Claim does not hold. Then we would get for some \( \theta_2 \in \Theta_2 \), \( E_3 \in \Delta_3 \cap \mathfrak{S}_g^{\mathfrak{a}(C_1)(C_2)(C_3)} X_2 \) and some disjunction \( \mu_2^* \in \Gamma_2^* \), that \( \theta_2 \land E_3 \leq \mu_2^* \). Hence \( E_3 \leq \theta_2 \rightarrow \mu_2^* \). For any finite set \( D \subseteq C_3 \sim (C_1 \cup C_2) \), we have \( c_{(D)}E_3 \leq \theta_2 \rightarrow \mu_2^* \). By lemma 5.12, there is a finite set \( D_3 \subseteq C_3 \sim (C_1 \cup C_2) \), satisfying
\[ c_{(D_3)}E_3 \in \mathfrak{N}_{\alpha, \cup, C_1 \cup C_2} \mathfrak{S}_g^{\mathfrak{a}(C_1)(C_2)(C_3)} X_1 = \mathfrak{S}_g^{\mathfrak{a}(C_1)(C_2)(C_3)} X_1 = \mathfrak{S}_g^{\mathfrak{a}(C_1)(C_2)} X_1. \]
If \( c_{(D_3)}E_3 \in \Gamma_2^* \), then it in \( \Gamma_2 \), and since \( \Gamma_2 \subseteq \Gamma'_3 \), it cannot be in \( \Delta_3 \). But this contradicts that \( E_3 \in \Delta_3 \). So \( c_{(D_3)}E_3 \in \Theta_2 \), because \( E_3 \leq c_{(D_3)}E_3 \), and so \( (\theta_2 \rightarrow \mu_2^*) \in \Theta_2 \), which contradicts the consistency of \( (\Theta_2, \Gamma_2^*) \).

Likewise, now extend \( T_3 \) to a complete and saturated theory \( (\Delta_4, \Gamma'_4) \) in \( \mathfrak{S}_g^{\mathfrak{a}(C_1)(C_2)(C_3)(C_4)} X_2 \) such that \( \Gamma_3 \subseteq \Gamma'_4 \). As before the theory \( (\Delta_3, \Theta_4 \cap \mathfrak{S}_g^{\mathfrak{a}(C_1)(C_2)(C_3)(C_4)} X_1, \Gamma_4) \) is consistent in \( \mathfrak{S}_g^{\mathfrak{a}(C_1)(C_2)(C_3)(C_4)} X_1 \). Continue, inductively, to construct \( (\Delta_5, \Gamma'_5) \), \( (\Delta_5, \Gamma_4) \) and so on. We obtain, zigzagging along, the following sequences:
\[ (\Delta_0, \Gamma_0), (\Delta_1, \Gamma_0), (\Delta_3, \Gamma_2) \ldots \]
such that

1. \((\theta_{2n}, \Gamma_{2n}^*)\) is complete and saturated in \(\mathbb{Sg}^{n(C1)\ldots(C2n)}X_2\),
2. \((\Delta_{2n+1}, \Gamma_{2n})\) is a saturated theory in \(\mathbb{Sg}^{n(C1)\ldots(C2n+1)}X_1\),
3. \(\Theta_{2n} \subseteq \Theta_{2n+2}, \Gamma_{2n}^* \subseteq \Gamma_{2n+2}^*\) and \(\Gamma_{2n} = \Gamma_{2n}^* \cap \mathbb{Sg}^{n(C1)\ldots(C2n)}X_1\),
4. \(\Delta_0 \subseteq \Delta_1 \subseteq \Delta_3 \subseteq \ldots\)

Now let \(\Delta = \bigcup_n \Delta_n, \Gamma = \bigcup_n \Gamma_n, \Gamma^* = \bigcup_n \Gamma_n^*\) and \(\Theta = \bigcup_n \Theta_n\). Then we have \(T_1 = (\Delta, \Gamma), T_2 = (\Theta, \Gamma^*)\) extend \((\Delta, \Gamma), (\Theta, \Gamma^*)\), such that \(T_1\) and \(T_2\) are consistent and saturated in \(\mathbb{Sg}^{n}X_1\) and \(\mathbb{Sg}^{n}X_2\), respectively, \(\Delta \cap \Theta\) is complete in \(\mathbb{Sg}^{n}X_1 \cap \mathbb{Sg}^{n}X_2\), and \(\Gamma \subseteq \Gamma^*\). We check that \((\Delta \cap \Theta, \Gamma)\) is complete in \(\mathbb{Sg}^{n}X_1 \cap \mathbb{Sg}^{n}X_2\). Let \(a \in \mathbb{Sg}^{n}X_1 \cap \mathbb{Sg}^{n}X_2\). Then there exists \(n\) such that \(a \in \mathbb{Sg}^{n}X_1 \cap \mathbb{Sg}^{n}X_2\). Now \((\Theta_{2n}, \Gamma_{2n}^*)\) is complete and so either \(a \in \Theta_{2n}\) or \(a \in \Gamma_{2n}^*\). If \(a \in \Theta_{2n}\) it will be in \(\Delta_{2n+1}\) and if \(a \in \Gamma_{2n}^*\) it will be in \(\Gamma_{2n}\). In either case, \(a \in \Delta \cap \Theta\) or \(a \in \Gamma\).

**Definition 5.17.**  (1) Let \(\mathfrak{A}\) be an algebra generated by \(X\) and assume that \(X = X_1 \cup X_2\). A pair \((\Delta, \Gamma) (T, F)\) of theories in \(\mathbb{Sg}^{n}X_1\) and \(\mathbb{Sg}^{n}X_2\) is a matched pair of theories if \((\Delta \cap T \cap \mathbb{Sg}^{n}X_1 \cap \mathbb{Sg}^{n}X_2, \Gamma \cap F \cap \mathbb{Sg}^{n}X_1 \cap \mathbb{Sg}^{n}X_2)\) is complete in \(\mathbb{Sg}^{n}X_1 \cap \mathbb{Sg}^{n}X_2\).

2. A theory \((T, F)\) extends a theory \((\Delta, \Gamma)\) if \(\Delta \subseteq T\) and \(\Gamma \subseteq F\).

3. A pair \((T_1, T_2)\) of theories extend another pair \((\Delta_1, \Delta_2)\) if \(T_1\) extends \(\Delta_1\) and \(T_2\) extends \(\Delta_2\).

The following Corollary follows directly from the proof of lemma \(5.16\).

**Corollary 5.18.** Let \(\mathfrak{A} \in GPHA_{\alpha}\) be generated by \(X\) and let \(X = X_1 \cup X_2\). Let \(((\Delta_0, \Gamma_0), (\Theta_0, \Gamma_0^*))\) be a matched pair in \(\mathbb{Sg}^{n}X_1\) and \(\mathbb{Sg}^{n}X_2\), respectively. Let \(I\) be a set such that \(\alpha \subseteq I\), and \(|I| = \max(|A|, |\alpha|)\). Then there exists a dilation \(\mathfrak{B} \in GPHA_{\alpha}\) of \(\mathfrak{A}\), and a matched pair, \((T_1, T_2)\) extending \(((\Delta_0, \Gamma_0), (\Theta_0, \Gamma_0^*))\), such that \(T_1\) and \(T_2\) are saturated in \(\mathbb{Sg}^{n}X_1\) and \(\mathbb{Sg}^{n}X_2\), respectively.

We next define set algebras based on Kripke systems. We stipulate that undirect products (in the universal algebraic sense) are the representable algebras, which the abstract axioms aspire to capture. Here Kripke systems (a direct generalization of Kripke frames) are defined differentially than those defined in \(\mathfrak{S}\), because we allow relativized semantics. In the clasical case, such algebras reduce to products of set algebras.\(^2\)

\(^2\)The idea of relativization, similar to Henkin’s semantics for second order logic, has proved a very fruitful idea in the theory of cylindric algebras.
Definition 5.19. Let \( \alpha \) be an infinite set. A Kripke system of dimension \( \alpha \) is a quadruple \( \mathfrak{K} = (K, \leq \{X_k\}_{k \in K}, \{V_k\}_{k \in K}) \), such that \( V_k \subseteq X_k \), and

1. \((K, \leq)\) is preordered set,

2. For any \( k \in K \), \( X_k \) is a non-empty set such that 
   \[ k \leq k' \implies X_k \subseteq X_{k'} \text{ and } V_k \subseteq V_{k'} \].

Let \( \mathfrak{O} \) be the Boolean algebra \( \{0, 1\} \). Now Kripke systems define concrete polyadic Heyting algebras as follows. Let \( \alpha \) be an infinite set and \( G \) be a semigroup of transformations on \( \alpha \). Let \( \mathfrak{K} = (K, \leq \{X_k\}_{k \in K}, \{V_k\}_{k \in K}) \) be a Kripke system. Consider the set 
\[ \mathfrak{F}_\mathfrak{K} = \{(f_k : k \in K); f_k : V_k \rightarrow \mathfrak{O}, k \leq k' \implies f_k \leq f_{k'}\} \].

If \( x, y \in X_k \) and \( j \in \alpha \) we write \( x \equiv_j y \) if \( x(i) = y(i) \) for all \( i \neq j \). We write \( (f_k) \) instead of \( (f_k : k \in K) \). In \( \mathfrak{F}_\mathfrak{K} \) we introduce the following operations:

\[ (f_k) \lor (g_k) = (f_k \lor g_k) \]
\[ (f_k) \land (g_k) = (f_k \land g_k) \]

For any \( (f_k) \) and \( (g_k) \in \mathfrak{F} \), define 
\[ (f_k) \rightarrow (g_k) = (h_k), \]
where \( (h_k) \) is given for \( x \in V_k \) by \( h_k(x) = 1 \) if and only if for any \( k' \geq k \) if \( f_{k'}(x) = 1 \) then \( g_{k'}(x) = 1 \). For any \( \tau \in G \), define
\[ s_\tau : \mathfrak{F} \rightarrow \mathfrak{F} \]
by
\[ s_\tau(f_k) = (g_k) \]

where
\[ g_k(x) = f_k(x \circ \tau) \]
for any \( k \in K \) and \( x \in V_k \).

For any \( j \in \alpha \) and \( (f_k) \in \mathfrak{F} \), define
\[ c_j(f_k) = (g_k), \]
where for \( x \in V_k \)
\[ g_k(x) = \bigvee \{f_k(y) : y \in V_k, y \equiv_j x\}. \]

Finally, set
\[ q_j(f_k) = (g_k) \]
where for \( x \in V_k \),
\[ g_k(x) = \bigwedge \{f_l(y) : k \leq l, y \in V_k, y \equiv_j x\}. \]

The diagonal element \( d_{ij} \) is defined to be the tuple \( (f_k : k \in K) \) where for \( x \in V_k \), \( f_k(x) = 1 \) iff \( x_i = x_j \).

The algebra \( \mathbf{F}_\mathbf{K} \) is called the set algebra based on the Kripke system \( \mathbf{K} \).
5.3 Diagonal Free case

We now deal with the case when $G$ is the semigroup of all finite transformations on $\alpha$. In this case, we stipulate that $\alpha \sim \Delta x$ is infinite for all $x$ in algebras considered. To deal with such a case, we need to define certain free algebras, called dimension restricted. Those algebras were introduced by Henkin, Monk and Tarski. The free algebras defined the usual way, will have the dimensions sets of their elements equal to their dimension, but we do not want that. For a class $K$, $S$ stands for the operation of forming subalgebras of $K$, $PK$ that of forming direct products, and $HK$ stands for the operation of taking homomorphic images. In particular, for a class $K$, $HSPK$ stands for the variety generated by $K$.

Our dimension restricted free algebras, are an instance of certain independently generated algebras, obtained by an appropriate relativization of the universal algebraic concept of free algebras. For an algebra $\mathfrak{A}$, we write $R \in Con\mathfrak{A}$ if $R$ is a congruence relation on $\mathfrak{A}$.

**Definition 5.20.** Assume that $K$ is a class of algebras of similarity $t$ and $S$ is any set of ordered pairs of words of $\mathfrak{F}^t_\alpha$, the absolutely free algebra of type $t$. Let

$$Cr^{(S)}_\alpha K = \cap \{ R \in Con\mathfrak{F}^t_\alpha, \mathfrak{F}^t_\alpha / R \in SK, S \subseteq R \}$$

and let

$$\mathfrak{F}^{(S)}_\alpha K = \mathfrak{F}^t_\alpha / Cr^{(S)}_\alpha K.$$ 

$\mathfrak{F}^{(S)}_\alpha K$ is called the free algebra over $K$ with $\alpha$ generators subject to the defining relations $S$.

As a special case, we obtain dimension restricted free algebra, defined next.

**Definition 5.21.** (1) Let $\delta$ be a cardinal. Let $\alpha$ be an ordinal, and let $G$ be the semigroup of finite transformations on $\alpha$. Let $G\alpha^t_\delta$ be the absolutely free algebra of $\delta$ generators and of type $GPHA_\alpha$. Let $\rho \in ^\delta \wp(\alpha)$. Let $L$ be a class having the same similarity type as $GPHA_\alpha$. Let

$$Cr^{(\rho)}_\delta L = \bigcap \{ R : R \in Con_{\alpha} G\delta^t, G\delta^t / R \in SP L, c^r_k G\delta^t / R \in \eta / R \mbox{ for each } \eta < \delta \}$$

and

$$G\delta^t_\delta L = G\delta^t / Cr^{(\rho)}_\delta L.$$ 

The ordinal $\alpha$ does not figure out in $Cr^{(\rho)}_\delta L$ and $G\delta^t_\delta L$ though it is involved in their definition. However, $\alpha$ will be clear from context so that no confusion is likely to ensue.
(2) Assume that $\delta$ is a cardinal, $L \subseteq GPHA_\alpha$, $\mathfrak{A} \in L$, $x = \langle x_\eta : \eta < \beta \rangle \in \delta A$ and $\rho \in \delta \varphi(\alpha)$. We say that the sequence $x$ $L$-freely generates $\mathfrak{A}$ under the dimension restricting function $\rho$, or simply $x$ freely generates $\mathfrak{A}$ under $\rho$, if the following two conditions hold:

(i) $\mathfrak{A} = \mathfrak{Sg}_\beta^A Rg(x)$ and $\Delta^A x_\eta \subseteq \rho(\eta)$ for all $\eta < \delta$.

(ii) Whenever $\mathfrak{B} \in L$, $y = \langle y_\eta : \eta < \delta \rangle \in \delta \mathfrak{B}$ and $\Delta^\mathfrak{B} y_\eta \subseteq \rho(\eta)$ for every $\eta < \delta$, then there is a unique homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, such that $h \circ x = y$.

The second item says that dimension restricted free algebras has the universal property of free algebras with respect to algebras whose dimensions are also restricted. The following theorem can be easily distilled from the literature of cylindric algebra.

**Theorem 5.22.** Assume that $\delta$ is a cardinal, $L \subseteq GPHA_\alpha$, $\mathfrak{A} \in L$, $x = \langle x_\eta : \eta < \delta \rangle \in \delta A$ and $\rho \in \delta \varphi(\alpha)$. Then the following hold:

(i) $\mathfrak{S}v_\rho^A L \in GPHA_\alpha$ and $x = \langle \eta/Cr_\rho^A L : \eta < \delta \rangle$ $\text{SPL}$-freely generates $\mathfrak{A}$ under $\rho$.

(ii) In order that $\mathfrak{A} \cong \mathfrak{S}v_\rho^A L$ it is necessary and sufficient that there exists a sequence $x \in \delta A$ which $L$ freely generates $\mathfrak{A}$ under $\rho$.

**Proof.** [22] theorems 2.5.35, 2.5.36, 2.5.37.

Note that when $\rho(i) = \alpha$ for all $i$ then $\rho$ is not restricting the dimension, and we recover the notion of ordinary free algebras. That is for such a $\rho$, we have $\mathfrak{S}v_\rho^A GPHA_\alpha \cong \mathfrak{S}v_\rho^A GPHA_\alpha$.

Now we formulate the analogue of theorem ?? for dimension restricted agebras, which adresses infinitely many cases, because we have infinitely many dimension restricted free algebras having the same number of generators.

Now we formulate the analogue of theorem ?? for dimension restricted agebras, which adresses infinitely many cases, because we have infinitely many dimension restricted free algebras having the same number of generators.

**Theorem 5.23.** Let $G$ be the semigroup of finite transformations on an infinite set $\alpha$ and let $\delta$ be a cardinal $> 0$. Let $\rho \in \delta \varphi(\alpha)$ be such that $\alpha \sim \rho(i)$ is infinite for every $i \in \delta$. Let $\mathfrak{A}$ be the free $G$ algebra generated by $X$ restricted by $\rho$; that is $\mathfrak{A} = \mathfrak{S}v_\rho^A GPHA_\alpha$, and suppose that $X = X_1 \cup X_2$. Let $(\Delta_0, \Gamma_0)$, $(\Theta_0, \Gamma_0)$ be two consistent theories in $\mathfrak{Sg}_\rho^3 X_1$ and $\mathfrak{Sg}_\rho^3 X_2$, respectively. Assume that $\Gamma_0 \subseteq \mathfrak{Sg}_\rho^3 (X_1 \cap X_2)$ and $\Gamma_0 \subseteq \Gamma_0^*$. Assume, further, that $(\Delta_0 \cap \Theta_0 \cap \mathfrak{Sg}_\rho^3 X_1 \cap \mathfrak{Sg}_\rho^3 X_2, \Gamma_0)$ is complete in $\mathfrak{Sg}_\rho^3 X_1 \cap \mathfrak{Sg}_\rho^3 X_2$. Then there exist a Kripke system $\mathfrak{K} = (K, \leq \{ X_k \}_{k \in K} \{ V_k \}_{k \in K}),$ a homomorphism $\psi : \mathfrak{A} \rightarrow \mathfrak{S}K$, $k_0 \in K$, and $x \in V_{k_0}$, such that for all $p \in \Delta_0 \cup \Theta_0$ if $\psi(p) = (f_k)$, then $f_{k_0}(x) = 1$ and for all $p \in \Gamma_0^*$ if $\psi(p) = (f_k)$, then $f_{k_0}(x) = 0$.  

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Proof. We use lemma 4.16 extensively. Assume that $\alpha$, $G$, $A$ and $X_1$, $X_2$ and everything else in the hypothesis are given. Let $I$ be a set containing $\alpha$ such that $\beta = |I \sim \alpha| = max(|A|, |\alpha|)$. If $G$ is strongly rich, let $(K_n : n \in \omega)$ be a family of pairwise disjoint sets such that $|K_n| = \beta$. Define a sequence of algebras $A = A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \ldots$, such that $A_{n+1}$ is a minimal dilation of $A_n$ and $dim(A_{n+1}) = dim(A_n \cup K_n)$. If $G = \alpha\alpha$, then let $(K_n : n \in \omega)$ be a family of pairwise disjoint sets, such that $|K_1| = \beta$ and $|K_n| = \omega$ for $n \geq 1$, and define a sequence of algebras $A = A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \ldots$, such that $A_1$ is a minimal extension of $A$, and $A_{n+1}$ is a minimal dilation of $A_n$ for $n \geq 2$, with $dim(A_{n+1}) = dim(A_n \cup K_n)$.

We denote $dim(A_n)$ by $I_n$ for $n \geq 1$. Recall that $dim(A_0) = dim(A = \alpha)$.

We interrupt the main stream of the proof by two consecutive claims. Not to digress, it might be useful that the reader at first reading, only memorize their statements, skip their proofs, go on with the main proof, and then get back to them. The proofs of Claims 1 and 2 to follow are completely analogous to their statements, skip their proofs, go on with the main proof, and then get back to them.

Claim 1. Let $n \in \omega$. If $((\Delta, \Gamma), (T, F))$ is a matched pair of saturated theories in $\mathcal{G}X_1^n$ and $\mathcal{G}X_2^n$, then the following hold. For any $a, b \in \mathcal{G}X_1^n$ if $a \rightarrow b \notin \Delta$, then there is a matched pair $((\Delta', \Gamma'), (T', F'))$ of saturated theories in $\mathcal{G}X_1^{n+1}$ and $\mathcal{G}X_2^{n+1}$, respectively, such that $\Delta \subseteq \Delta'$, $T \subseteq T'$, $a \in \Delta'$ and $b \notin \Delta'$.

Proof of Claim 1. Since $a \rightarrow b \notin \Delta$, we have $(\Delta \cup \{a\}, b)$ is consistent in $\mathcal{G}X_1^n$. Then by lemma 5.14, it can be extended to a complete theory $(\Delta', T')$ in $\mathcal{G}X_1^n$. Take

$$\Phi = \Delta' \cap \mathcal{G}X_1^n \cap \mathcal{G}X_2^n,$$

and

$$\Psi = T' \cap \mathcal{G}X_1^n \cap \mathcal{G}X_2^n.$$
$(\Phi, \Psi)$ is inconsistent which is impossible. Thus $(T \cup \Phi, \Psi)$ is consistent. Now the pair $((\Delta', T')(T \cup \Phi, \Psi))$ satisfy the conditions of lemma 5.16. Hence this pair can be extended to a matched pair of saturated theories in $\mathcal{Sg}^{3n+1}_X1 \cap \mathcal{Sg}^{3n+1}_X 2$. This pair is as required by the conclusion of lemma 5.16.

Claim 2. Let $n \in \omega$. If $((\Delta, \Gamma), (T, F))$ is a matched pair of saturated theories in $\mathcal{Sg}^{3n}_X1 \cap \mathcal{Sg}^{3n}_X 2$, then the following hold. For $x \in \mathcal{Sg}^{3n}_X1$ and $j \in I_n = dim \mathfrak{A}_n$, if $q_j x \not\in \Delta$, then there is a matched pair $((\Delta', \Gamma'), (T', F'))$ of saturated theories in $\mathcal{Sg}^{3n+2}_X1 \cap \mathcal{Sg}^{3n+2}_X 2$ respectively, $u \in I_{n+2}$ such that $\Delta \subseteq \Delta'$, $T \subseteq T'$ and $s'_u x \not\in \Delta'$.

Proof. Assume that $x \in \mathcal{Sg}^{3n}_X1$ and $j \in I_n$ such that $q_j x \not\in \mathcal{Sg}^{3n}_X1$. Then there exists $u \in I_{n+1} \sim I_n$ such that $(\Delta, s'_u x)$ is consistent in $\mathcal{Sg}^{3n+1}_X1$. So $(\Delta, s'_u x)$ can be extended to a complete theory $(\Delta', T')$ in $\mathcal{Sg}^{3n+1}_X1$. Take

$$\Phi = \Delta' \cap \mathcal{Sg}^{3n+1}_X1 \cap \mathcal{Sg}^{3n+1}_X 2,$$

and

$$\Psi = T' \cap \mathcal{Sg}^{3n+1}_X1 \cap \mathcal{Sg}^{3n+1}_X 2.$$

Then $(\Phi, \Psi)$ is complete in $\mathcal{Sg}^{3n+1}_X1 \cap \mathcal{Sg}^{3n+1}_X 2$. We shall show that $(T \cup \Phi, \Psi)$ is consistent in $\mathcal{Sg}^{3n+1}_X2$. If not, then there exist $\theta \in T$, $\phi \in \Phi$ and $\psi \in \Psi$, such that $\theta \wedge \phi \leq \psi$. Hence, $\theta \leq \phi \rightarrow \psi$. Now

$$\theta = q_j(\theta) \leq q_j(\phi \rightarrow \psi).$$

Since $(T, F)$ is saturated in $\mathcal{Sg}^{3n}_X2$, it thus follows that

$$q_j(\phi \rightarrow \psi) \in T \cap \mathcal{Sg}^{3n}_X1 \cap \mathcal{Sg}^{3n}_X 2 = \Delta \cap \mathcal{Sg}^{3n}_X1 \cap \mathcal{Sg}^{3n}_X 2.$$ 

So $q_j(\phi \rightarrow \psi) \in \Delta'$ and consequently we get $q_j(\phi \rightarrow \psi) \in \Phi$. Also, we have, $\phi \in \Phi$. But $(\Phi, \Psi)$ is complete, we get $\psi \in \Phi$ and this contradicts that $\psi \in \Psi$. Now the pair $((\Delta', \Gamma'), (T \cup \Phi, \Psi))$ satisfies the hypothesis of lemma 5.16 applied to $\mathcal{Sg}^{3n+1}_X1 \cap \mathcal{Sg}^{3n+1}_X 2$. The required now follows from the conclusion of lemma 5.16.

Now that we have proved our claims, we go on with the proof. We prove the theorem when $G$ is a strongly rich semigroup, because in this case we deal with relativized semantics, and during the proof we state the necessary modifications for the case when $G$ is the semigroup of all transformations. Let

$$K = \{((\Delta, \Gamma), (T, F)) : \exists n \in \omega \text{ such that } (\Delta, \Gamma), (T, F)\}$$

be a matched pair of saturated theories in $\mathcal{Sg}^{3n}_X1 \cap \mathcal{Sg}^{3n}_X 2$. We have $((\Delta_0, \Gamma_0), (\Theta_0, \Gamma_0^*))$ is a matched pair but the theories are not saturated. But by lemma 5.16 there are $T_1 = (\Delta_\omega, \Gamma_\omega)$, $T_2 = (\Theta_\omega, \Gamma_\omega^\omega)$ extending
When theories are matched pairs, \( \psi \) concisely, we write \( x \) to avoid tiresome notation, we shall denote the map those will be pasted using the freeness of \( \bar{\tau} \) detail, in this case, we take for \( k \) the least such number, so \( n \) is unique to \( i \). Before going on we introduce a piece of notation. For a set \( M \) and a sequence \( p \in \alpha M \), \( \alpha M(p) \) is the following set

\[
\{ s \in \alpha M : |\{ i \in \alpha : s_i \neq p_i \}| < \omega \}.
\]

Let

\[
\mathfrak{R} = (K, \leq, \{ M_i \}, \{ V_i \})_{i \in \mathcal{R}}
\]

where \( V_i = \bigcup_{p \in G_n} \alpha M_i(p) \), and \( G_n \) is the strongly rich semigroup determining the similarity type of \( \mathfrak{A}_n \), with \( n \) the least number such \( i \) is a saturated matched pair in \( \mathfrak{A}_n \). The order \( \leq \) is defined as follows: If \( i_1 = ((\Delta_1, \Gamma_1)), (T_1, F_1)) \) and \( i_2 = ((\Delta_2, \Gamma_2), (T_2, F_2)) \) are in \( \mathfrak{R} \), then define

\[
i_1 \leq i_2 \iff M_{i_1} \subseteq M_{i_2}, \Delta_1 \subseteq \Delta_2, T_1 \subseteq T_2.
\]

This is, indeed as easily checked, a preorder on \( K \).

We define two maps on \( \mathfrak{A}_1 = \mathcal{S}g^{31}X_1 \) and \( \mathfrak{A}_2 = \mathcal{S}g^{31}X_2 \) respectively, then those will be pasted using the freeness of \( \mathfrak{A} \) to give the required single homomorphism, by noticing that they agree on the common part, that is on \( \mathcal{S}g^{3}(X_1 \cap X_2) \).

Set \( \psi_1 : \mathcal{S}g^{31}X_1 \to \mathfrak{R} \) by \( \psi_1(p) = (f_k) \) such that if \( k = ((\Delta, \Gamma), (T, F)) \) \( \in K \) is a matched pair of saturated theories in \( \mathcal{S}g^{3n}X_1 \) and \( \mathcal{S}g^{3n}X_2 \), and \( M_k = \dim \mathfrak{A}_n \), then for \( x \in V_k = \bigcup_{p \in G_n} \alpha M_k(p) \),

\[
f_k(x) = 1 \iff x_{\cup(\text{Id}_{M_k \sim \alpha})} p \in \Delta \cup T.
\]

To avoid tiresome notation, we shall denote the map \( x \cup \text{Id}_{M_k \sim \alpha} \) simply by \( \bar{x} \) when \( M_k \) is clear from context. It is easily verifiable that \( \bar{x} \) is in the semigroup determining the similarity type of \( \mathfrak{A}_n \) hence the map is well defined. More concisely, we write

\[
f_k(x) = 1 \iff x_{\bar{x}} p \in \Delta \cup T.
\]

The map \( \psi_2 : \mathcal{S}g^{31}X_2 \to \mathfrak{R} \) is defined in exactly the same way. Since the theories are matched pairs, \( \psi_1 \) and \( \psi_2 \) agree on the common part, i.e. on \( \mathcal{S}g^{3}(X_1 \cap X_2) \). Here we also make the tacit assumption that if \( k \leq k' \) then \( V_k \subseteq V_{k'} \) via the embedding \( \tau \mapsto \tau \cup \text{Id} \).

When \( G \) is the semigroup of all transformations, with no restrictions on cardinalities, we need not relativize since \( \bar{\tau} \) is in the big semigroup. In more detail, in this case, we take for \( k = ((\Delta, \Gamma), (T, F)) \) a matched pair of saturated
theories in $\mathcal{G}_n^1 X_1, \mathcal{G}_n^2 X_2$, $M_k = \dim \mathfrak{A}_n$ and $V_k = \mathcal{A}_n$ and for $x \in \mathfrak{A}_n$, we set

$$f_k(x) = 1 \iff \mathfrak{s}_x^{\mathfrak{A}_n}(p \in \Delta \cup T).$$

Before proving that $\psi$ is a homomorphism, we show that

$$k_0 = ((\Delta, \Gamma), (\Theta, \Gamma^*))$$

is as desired. Let $x \in V_{k_0}$ be the identity map. Let $p \in \Delta_0 \cup \Theta_0$, then $s_x p = p \in \Delta \cup \Theta$, and so if $\psi(p) = (f_k)$ then $f_{k_0}(x) = 1$. On the other hand if $p \in \Gamma^*$, then $p \notin \Delta \cup \Theta$, and so $f_{k_0}(x) = 0$. Then the union $\psi$ of $\psi_1$ and $\psi_2$, $k_0$ and $\text{Id}$ are as required, modulo proving that $\psi$ is a homomorphism from $\mathfrak{A}$, to the set algebra based on the above defined Kripke system, which we proceed to show. We start by $\psi_1$. Abusing notation, we denote $\psi_1$ by $\psi$, and we write a matched pair in $\mathfrak{A}_n$ instead of a matched pair of saturated theories in $\mathcal{G}_n^1 X_1, \mathcal{G}_n^2 X_2$, since $X_1$ and $X_2$ are fixed. The proof that the postulated map is a homomorphism is similar to the proof in [8] baring in mind that it is far from being identical because cylindrifiers and their duals are only finite.

(i) We prove that $\psi$ preserves $\wedge$. Let $p, q \in A$. Assume that $\psi(p) = (f_k)$ and $\psi(q) = (g_k)$. Then $\psi(p) \wedge \psi(q) = (f_k \wedge g_k)$. We now compute $\psi(p \wedge q) = (h_k)$. Assume that $x \in V_k$, where $k = ((\Delta, \Gamma), (T, F))$ is a matched pair in $\mathfrak{A}_n$ and $M_k = \dim \mathfrak{A}_n$. Then

$$h_k(x) = 1 \iff \mathfrak{s}_x^{\mathfrak{A}_n}(p \wedge q) \in \Delta \cup T$$

$$\iff \mathfrak{s}_x^{\mathfrak{A}_n} p \wedge \mathfrak{s}_x^{\mathfrak{A}_n} q \in \Delta \cup T$$

$$\iff \mathfrak{s}_x^{\mathfrak{A}_n} p \in T \cup \Delta \text{ and } \mathfrak{s}_x^{\mathfrak{A}_n} q \in \Delta \cup T$$

$$\iff f_k(x) = 1 \text{ and } g_k(x) = 1$$

$$\iff (f_k \wedge g_k)(x) = 1$$

$$\iff (\psi(p) \wedge \psi(q))(x) = 1.$$  

(ii) $\psi$ preserves $\to$. (Here we use Claim 1). Let $p, q \in A$. Let $\psi(p) = (f_k)$ and $\psi(q) = (g_k)$. Let $\psi(p \to q) = (h_k)$ and $\psi(p) \to \psi(q) = (h'_k)$. We shall prove that for any $k \in \mathfrak{A}$ and any $x \in V_k$, we have

$$h_k(x) = 1 \iff h'_k(x) = 1.$$  

Let $x \in V_k$. Then $k = ((\Delta, \Gamma), (T, F))$ is a matched pair in $\mathfrak{A}_n$ and $M_k = \dim \mathfrak{A}_n$. Assume that $h_k(x) = 1$. Then we have

$$\mathfrak{s}_x^{\mathfrak{A}_n} (p \to q) \in \Delta \cup T,$$
from which we get that

\[(*) \quad \mathbf{s}_x^{\mathfrak{A}_n} p \rightarrow \mathbf{s}_x^{\mathfrak{A}_n} q \in \Delta \cup T.\]

Let \(k' \in K\) such that \(k \leq k'\). Then \(k' = ((\Delta', \Gamma'), (T', F'))\) is a matched pair in \(\mathfrak{A}_m\) with \(m \geq n\). Assume that \(f_{k'}(x) = 1\). Then, by definition we have (**)

\[\mathbf{s}_x^{\mathfrak{A}_m} p \in \Delta' \cup T'.\]

But \(\mathfrak{A}_m\) is a dilation of \(\mathfrak{A}_n\) and so

\[\mathbf{s}_x^{\mathfrak{A}_m} p = \mathbf{s}_x^{\mathfrak{A}_n} p \text{ and } \mathbf{s}_x^{\mathfrak{A}_m} q = \mathbf{s}_x^{\mathfrak{A}_n} q.\]

From (*) we get that,

\[\mathbf{s}_x^{\mathfrak{A}_m} p \rightarrow \mathbf{s}_x^{\mathfrak{A}_m} q \in \Delta' \cup T'.\]

But, on the other hand, from (**), we have \(\mathbf{s}_x^{\mathfrak{A}_m} q \in \Delta' \cup T'\), so

\[f_{k'}(x) = 1 \implies g_{k'}(x) = 1.\]

That is to say, we have \(h_{k'}(x) = 1\). Conversely, assume that \(h_k(x) \neq 1\), then

\[\mathbf{s}_x^{\mathfrak{A}_n} p \rightarrow \mathbf{s}_x^{\mathfrak{A}_n} q \notin \Delta \cup T,\]

and consequently

\[\mathbf{s}_x^{\mathfrak{A}_n} p \rightarrow \mathbf{s}_x^{\mathfrak{A}_n} q \notin \Delta.\]

From Claim 1, we get that there exists a matched pair \(k' = ((\Delta', \Gamma')(\langle T', F' \rangle))\) in \(\mathfrak{A}_{n+2}\), such that

\[\mathbf{s}_x^{\mathfrak{A}_{n+2}} p \in \Delta' \text{ and } \mathbf{s}_x^{\mathfrak{A}_{n+2}} q \notin \Delta'.\]

We claim that \(\mathbf{s}_x^{\mathfrak{A}_{n+2}} q \notin T'\), for otherwise, if it is in \(T'\), then we would get that

\[\mathbf{s}_x^{\mathfrak{A}_{n+2}} q \in \mathcal{Sg}^{\mathfrak{A}_{n+2}} X_1 \cap \mathcal{Sg}^{\mathfrak{A}_{n+2}} X_2.\]

But

\[(\Delta' \cap T' \cap \mathcal{Sg}^{\mathfrak{A}_{n+2}} X_1 \cap \mathcal{Sg}^{\mathfrak{A}_{n+2}} X_2, \Gamma' \cap F' \cap \mathcal{Sg}^{\mathfrak{A}_{n+2}} X_1 \cap \mathcal{Sg}^{\mathfrak{A}_{n+2}} X_2)\]

is complete in \(\mathcal{Sg}^{\mathfrak{A}_{n+2}} X_1 \cap \mathcal{Sg}^{\mathfrak{A}_{n+2}} X_2\), and \(\mathbf{s}_x^{\mathfrak{A}_{n+2}} q \notin \Delta' \cap T'\), hence it must be the case that

\[\mathbf{s}_x^{\mathfrak{A}_{n+2}} q \in \Gamma' \cap F'.\]

In particular, we have

\[\mathbf{s}_x^{\mathfrak{A}_{n+2}} q \in F'.\]

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which contradicts the consistency of $(T', F')$, since by assumption $s^{\mathfrak{A}_n+2}_x q \in T'$. Now we have
\[
s^{\mathfrak{A}_n+2}_x q \notin \Delta' \cup T',
\]
and
\[
s^{\mathfrak{A}_n+2}_x p \in \Delta' \cup T'.
\]
Since $\Delta' \cup T'$ extends $\Delta \cup T$, we get that $h'_k(x) \neq 1$.

(iii) $\psi$ preserves substitutions. Let $p \in \mathfrak{A}$. Let $\sigma \in G$. Assume that $\psi(p) = (f_k)\text{ and }\psi(s_{\sigma}p) = (g_k)$. Assume that $M_k = \dim \mathfrak{A}_n$ where $k = ((\Delta, \Gamma), (T, F))$ is a matched pair in $\mathfrak{A}_n$. Then, for $x \in V_k$, we have
\[
g_k(x) = 1 \iff s^{\mathfrak{A}_n}_x s^{\mathfrak{A}}_{\sigma}p \in \Delta \cup T
\]
\[
\iff s^{\mathfrak{A}_n}_x s^{\mathfrak{A}}_{\sigma}p \in \Delta \cup T
\]
\[
\iff s^{\mathfrak{A}_n}_x s^{\mathfrak{A}}_{\sigma}p \in \Delta \cup T
\]
\[
\iff f_k(x \circ \sigma) = 1.
\]

(iv) $\psi$ preserves cylindrifications. Let $p \in \mathfrak{A}$. Assume that $m \in I$ and assume that $\psi(c_m p) = (f_k)\text{ and }c_m \psi(p) = (g_k)$. Assume that $k = ((\Delta, \Gamma), (T, F))$ is a matched pair in $\mathfrak{A}_n$ and that $M_k = \dim \mathfrak{A}_n$. Let $x \in V_k$. Then
\[
f_k(x) = 1 \iff s^{\mathfrak{A}_n}_x c_m p \in \Delta \cup T.
\]
We can assume that
\[
s^{\mathfrak{A}_n}_x c_m p \in \Delta.
\]
For if not, that is if
\[
s^{\mathfrak{A}_n}_x c_m p \notin \Delta \text{ and } s^{\mathfrak{A}_n}_x c_{(m)} p \in T,
\]
then
\[
s^{\mathfrak{A}_n}_x c_m p \in \mathfrak{S}^{\mathfrak{A}_n}g_1 X_1 \cap \mathfrak{S}^{\mathfrak{A}_n}g_2 X_2,
\]
but
\[
(\Delta \cap T \cap \mathfrak{S}^{\mathfrak{A}_n}g_1 X_1 \cap \mathfrak{S}^{\mathfrak{A}_n}g_2 X_2, \Gamma \cap F \cap \mathfrak{S}^{\mathfrak{A}_n}g_1 X_1 \cap \mathfrak{S}^{\mathfrak{A}_n}g_2 X_2)
\]
is complete in $\mathfrak{S}^{\mathfrak{A}_n}g_1 X_1 \cap \mathfrak{S}^{\mathfrak{A}_n}g_2 X_2$, and
\[
s^{\mathfrak{A}_n}_x c_m p \notin \Delta \cap T,
\]
it must be the case that
\[
s^{\mathfrak{A}_n}_x c_m p \in \Gamma \cap F.
\]
In particular,

\[ s_{x}^{\Delta}c_{m}p \in F. \]

But this contradicts the consistency of \((T, F)\).

Assuming that \(s_{x}c_{m}p \in \Delta\), we proceed as follows. Let

\[ \lambda \in \{ \eta \in I_{n} : x^{-1}\{ \eta \} = \eta \} \sim \Delta p. \]

Let

\[ \tau = x \upharpoonright I_{n} \sim \{ m, \lambda \} \cup \{ (m, \lambda)(\lambda, m) \}. \]

Then, by item (5) in theorem 5.7, we have

\[ c_{\lambda}s_{x}^{\Delta}c_{m}p = s_{x}^{\Delta}c_{m}p = s_{x}^{\Delta}c_{m}p \in \Delta. \]

We introduce a piece of helpful notation. For a function \(f\), let \(f(m \to u)\) is the function that agrees with \(f\) except at \(m\) where its value is \(u\). Since \(\Delta\) is saturated, there exists \(u \not\in \Delta\) such that

\[ s_{\lambda}(x)(m \to u) \in \Delta, \text{ and so } s_{\lambda}(x)(m \to u)p \in \Delta. \]

This implies that \(x \in c_{m}f(p)\) and so \(g_{k}(x) = 1\).

Conversely, assume that \(g_{k}(x) = 1\) with \(k = ((\Gamma, \Delta)), (T, F)\) a matched pair in \(\mathfrak{A}_{n}\). Let \(y \in V_{k}\) such that \(y \equiv_{m} x\) and \(\psi(p)y = 1\). Then \(s_{y}p \in \Delta \cup T\). Hence \(s_{y}c_{m}p \in \Delta \cup T\) and so \(s_{x}c_{m}p \in \Delta \cup T\), thus \(f_{k}(x) = 1\) and we are done.

(v) \(\psi\) preserves universal quantifiers. (Here we use Claim 2). Let \(p \in A\) and \(m \in I\). Let \(\psi(p) = (f_{k}), q_{m}\psi(p) = (g_{k})\) and \(\psi(q_{m}p) = (h_{k})\). Assume that \(h_{k}(x) = 1\). We have \(k = ((\Delta, \Gamma)), (T, F)\) is a matched pair in \(\mathfrak{A}_{n}\) and \(x \in V_{k}\). Then

\[ s_{x}^{\Delta}q_{m}p \in \Delta \cup T, \]

and so

\[ s_{y}^{\Delta}q_{m}p \in \Delta \cup T \text{ for all } y \in I M_{k}, y \equiv_{m} x. \]

Let \(k' \geq k\). Then \(k' = ((\Delta', \Gamma'), (T', F'))\) is a matched pair in \(\mathfrak{A}_{l}\ l \geq n, \Delta \subseteq \Delta'\) and \(T \subseteq T'\). Since \(p \geq q_{m}p\) it follows that

\[ s_{y}^{\Delta}p \in \Delta' \cup T' \text{ for all } y \in I M_{k}, y \equiv_{m} x. \]

Thus \(g_{k}(x) = 1\). Now conversely, assume that \(h_{k}(x) = 0\), \(k = ((\Delta, \Gamma)), (T, F)\) is a matched pair in \(\mathfrak{A}_{n}\), then, we have

\[ s_{x}^{\Delta}q_{m}p \notin \Delta \cup T, \]

and so

\[ s_{x}^{\Delta}q_{m}p \notin \Delta. \]

Let

\[ \lambda \in \{ \eta \in I_{n} : x^{-1}\{ \eta \} = \eta \} \sim \Delta p. \]
Let
\[ \tau = x \mid I_n \sim \{m, \lambda\} \cup \{(m, \lambda)(\lambda, m)\}. \]

Then, like in the existential case, using polyadic axioms, we get
\[ q_\lambda s_\tau p = s_\tau q_m p = s_x q_m p \notin \Delta \]

Then there exists \( u \) such that \( s_\tau s_\tau p \notin \Delta \). So \( s_\tau s_\tau p \notin T \), for if it is, then by the previous reasoning since it is an element of \( \mathcal{S}\mathcal{g}^{3n+2}X_1 \cap \mathcal{S}\mathcal{g}^{3n+2}X_2 \) and by completeness of \( (\Delta \cap T, \Gamma \cap F) \) we would reach a contradiction. The we get that \( s_{(x(m \rightarrow n))} p \notin \Delta \cup T \) which means that \( g_k(x) = 0 \), and we are done.

\[ \Box \]

**Theorem 5.24.** Let \( G \) be the semigroup of finite transformations on an infinite set \( \alpha \) and let \( \delta \) be a cardinal > 0. Let \( \rho \in \delta \phi(\alpha) \) be such that \( \alpha \sim \rho(i) \) is infinite for every \( i \in \delta \). Let \( \mathfrak{A} \) be the free \( G \) algebra generated by \( X \) restricted by \( \rho \); that is \( \mathfrak{A} = \mathfrak{F}_0 G \mathcal{P} H \mathcal{A}_n \), and suppose that \( X = X_1 \cup X_2 \). Let \( (\Delta_0, \Gamma_0), (\Theta_0, \Gamma_0) \) be two consistent theories in \( \mathcal{G}\mathcal{g}^3X_1 \) and \( \mathcal{G}\mathcal{g}^3X_2 \), respectively. Assume that \( \Gamma_0 \subseteq \mathcal{G}\mathcal{g}^3(X_1 \cap X_2) \) and \( \Gamma_0 \subseteq \Gamma_0^* \). Assume, further, that \( (\Delta_0 \cap \Theta_0 \cap \mathcal{G}\mathcal{g}^3X_1 \cap \mathcal{G}\mathcal{g}^3X_2, \Gamma_0) \) is complete in \( \mathcal{G}\mathcal{g}^3X_1 \cap \mathcal{G}\mathcal{g}^3X_2 \). Then there exist a Kripke system \( \mathfrak{K} = (K, \leq \{X_k\}_{k \in K}\{V_k\}_{k \in K}) \), a homomorphism \( \psi : \mathfrak{A} \rightarrow \mathfrak{F}_K \), \( k_0 \in K \), and \( x \in V_{k_0} \), such that for all \( p \in \Delta_0 \cup \Theta_0 \) if \( \psi(p) = (f_k) \), then \( f_{k_0}(x) = 1 \) and for all \( p \in \Gamma_0^* \) if \( \psi(p) = (f_k) \), then \( f_{k_0}(x) = 0 \).

**Proof.** We state the modifications in the above proof of theorem 5.22. Form the sequence of minimal dilations \( (\mathfrak{A}_n : n \in \omega) \) built on the sequence \( (K_n : n \in \omega) \), with \( |K_n| = \beta, \beta = |I \sim \alpha| = max(|A|, \alpha) \) with \( I \) is a superset of \( \alpha \). If \( i = ((\Delta, \Gamma), (T, F)) \) is a matched pair of saturated theories in \( \mathcal{G}\mathcal{g}^{3n}X_1 \) and \( \mathcal{G}\mathcal{g}^{3n}X_2 \), let \( M_i = dim \mathfrak{A}_n \), where \( n \) is the least such number, so \( n \) is unique to \( i \). Define \( K \) as in in the proof of theorem 5.22, that is, let
\[ K = \{(\Delta, \Gamma), (T, F) : \exists n \in \omega \text{ such that } (\Delta, \Gamma), (T, F) \}

is a a matched pair of saturated theories in \( \mathcal{G}\mathcal{g}^{3n}X_1, \mathcal{G}\mathcal{g}^{3n}X_2 \).

Let
\[ \mathfrak{K} = (K, \leq, \{M_i\}, \{V_i\})_{i \in \mathfrak{K}}, \]
where now \( V_i = ^\alpha M_i^{(\alpha)} = \{s \in ^\alpha M : |i \in \alpha : s_i \neq i| < \omega\} \), and the order \( \leq \) is defined by: If \( i_1 = ((\Delta_1, \Gamma_1), (T_1, F_1)) \) and \( i_2 = ((\Delta_2, \Gamma_2), (T_2, F_2)) \) are in \( \mathfrak{K} \), then
\[ i_1 \leq i_2 \iff M_{i_1} \subseteq M_{i_2}, \Delta_1 \subseteq \Delta_2, T_1 \subseteq T_2. \]
This is a preorder on $K$. Set $\psi_1 : \mathfrak{G}^n_1 \rightarrow \mathfrak{S}_K$ by $\psi_1(p) = (f_k)$ such that $f_k = (\Delta, \Gamma, \langle T, F \rangle) \in K$ is a matched pair of saturated theories in $\mathfrak{G}^n_1$ and $\mathfrak{G}^n_2$, and $M_k = \dim \mathfrak{A}_n$, then for $x \in V_k = aM^{(i)}_k$, $f_k(x) = 1 \iff s_{x}^{\Gamma} \in \Delta \cup T$.

Define $\psi_2$ analogously. The rest of the proof is identical to the previous one. □

It is known that the condition $\Gamma \subseteq \Gamma'$ cannot be omitted. On the other hand, to prove our completeness theorem, we need the following weaker version of theorem ??, with a slight modification in the proof, which is still a step-by-step technique, though, we do not ‘zig-zag’.

**Lemma 5.25.** Let $A \in GPHA$. Let $(\Delta_0, \Gamma_0)$ be consistent. Suppose that $I$ is a set such that $\alpha \subseteq I$ and $|I | = \max(|A|, |\alpha|)$.

1. Then there exists a dilation $B \in GPHA_I$ of $A$, and theory $T = (\Delta_\omega, \Gamma_\omega)$, extending $(\Delta_0, \Gamma_0)$, such that $T$ is consistent and saturated in $B$.

2. There exists $K = (K_n : n \in \omega)$, a homomorphism $\psi : A \rightarrow K$, $k_0 \in K$, and $x \in V_0$, such that for all $p \in \Delta_0$ if $\psi(p) = (f_k)$, then $f_k(x) = 1$ and for all $p \in \Gamma_0$ if $\psi(p) = (g_k)$, then $g_k(x) = 0$.

**Proof.** We deal only with the case when $G$ is strongly rich. The other cases can be dealt with in a similar manner by undergoing the obvious modifications, as indicated above. As opposed to theorem ??, we use theories rather than pairs of theories, since we are not dealing with two subalgebras simultaneously. (i) follows from [5.15]. Now we prove (ii). The proof is a simpler version of the proof of ??, Let $I$ be a set such that $\beta = |I | = \max(|A|, |\alpha|)$. Let $(K_n : n \in \omega)$ be a family of pairwise disjoint sets such that $|K_n| = \beta$. Define a sequence of algebras $\mathfrak{A} = \mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \mathfrak{A}_3 \subseteq \mathfrak{A}_4 \subseteq \mathfrak{A}_5 \subseteq \mathfrak{A}_6$ such that $\mathfrak{A}_n$ is a minimal dilation of $A_n$ and $\dim (\mathfrak{A}_n) = \dim \mathfrak{A}_n$. We denote $\dim (\mathfrak{A}_n)$ by $I_n$ for $n \geq 1$. If $(\Delta, \Gamma)$ is saturated in $\mathfrak{A}_n$, then the following analogues of Claims 1 and 2 in theorem ?? hold: For any $a, b \in \mathfrak{A}_n$ if $a \rightarrow b \notin \Delta$, then there is a saturated theory $(\Delta', \Gamma')$ in $\mathfrak{A}_{n+1}$ such that $\Delta \subseteq \Delta'$, $a \in \Delta'$ and $b \notin \Delta'$. If $(\Delta, \Gamma)$ is saturated in $\mathfrak{A}_n$, then for all $x \in \mathfrak{A}_n$ and $j \in I_n$, if $a \notin \Delta$, then there $(\Delta', \Gamma')$ of saturated theories in $\mathfrak{A}_{n+2}$, $u \in I_{n+2}$ such that $\Delta \subseteq \Delta'$, and $s_j^u x \notin \Delta'$. Now let

$$K = \{ (\Delta, \Gamma) : \exists n \in \omega \text{ such that } (\Delta, \Gamma) \text{ is saturated in } \mathfrak{A}_n \}.$$

If $i = (\Delta, \Gamma)$ is a saturated theory in $\mathfrak{A}_n$, let $M_i = \dim (\mathfrak{A}_n)$, where $n$ is the least such number, so $n$ is unique to $i$. If $i_1 = (\Delta_1, \Gamma_1)$ and $i_2 = (\Delta_2, \Gamma_2)$ are in $K$, then set

$$i_1 \leq i_2 \iff M_{i_1} \subseteq M_{i_2}, \Delta_1 \subseteq \Delta_2.$$
This is a preorder on $K$; define the kripke system $\mathcal{R}$ based on the set of worlds $K$ as before. Set $\psi: A \rightarrow \mathcal{L}_n$ by $\psi_k(p) = (f_k)$ such that if $k = (\Delta, \Gamma) \in \mathcal{R}$ is saturated in $A_n$, and $M_k = dim A_n$, then for $x \in V_k = \bigcup_{\rho \in G_n} \alpha M_k^{(\rho)}$,

$$f_k(x) = 1 \iff s_{x \cup (id_{M_k \sim \alpha})}^{\alpha} p \in \Delta.$$ Let $k_0 = (\Delta_\omega, \Gamma_\omega)$ be defined as a complete saturated extension of $(\Delta_0, \Gamma_0)$ in $A_1$, then $\psi$, $k_0$ and $id$ are as desired. The analogues of Claims 1 and 2 in theorem ?? are used to show that $\psi$ so defined preserves implication and universal quantifiers.

6 Presence of diagonal elements

All results, in Part 1, up to the previous theorem, are proved in the absence of diagonal elements. Now let’s see how far we can go if we have diagonal elements. Considering diagonal elements, as we shall see, turn out to be problematic but not hopeless.

Our representation theorem has to respect diagonal elements, and this seems to be an impossible task with the presence of infinitary substitutions, unless we make a compromise that is, from our point of view, acceptable. The interaction of substitutions based on infinitary transformations, together with the existence of diagonal elements tends to make matters ‘blow up’; indeed this even happens in the classical case, when the class of (ordinary) set algebras ceases to be closed under ultraproducts. The natural thing to do is to avoid those infinitary substitutions at the start, while finding the interpolant possibly using such substitutions. We shall also show that in some cases the interpolant has to use infinitary substitutions, even if the original implication uses only finite transformations.

So for an algebra $A$, we let $RdA$ denote its reduct when we discard infinitary substitutions. $RdA$ satisfies cylindric algebra axioms.

**Theorem 6.1.** Let $G$ be the semigroup of finite transformations on an infinite set $\alpha$ and let $\delta$ be a cardinal $> 0$. Let $\rho \in \delta_\omega(\alpha)$ be such that $\alpha \sim \rho(i)$ is infinite for every $i \in \delta$. Let $A$ be the free $G$ algebra with equality generated by $X$ restricted by $\rho$; that is $A = \mathfrak{F}E^{G PHA \alpha}_o$, and suppose that $X = X_1 \cup X_2$. Let $(\Gamma_0, \Theta_0, \Gamma_0^*)$ be two consistent theories in $\mathfrak{S}^{G^2}X_1$ and $\mathfrak{S}^{G^2}X_2$, respectively. Assume that $\Gamma_0 \subseteq \mathfrak{S}^{G^2}(X_1 \cap X_2)$ and $\Gamma_0 \subseteq \Gamma_0^*$. Assume, further, that $(\Delta_0 \cap \Theta_0 \cap \mathfrak{S}^{G^2}X_1 \cap \mathfrak{S}^{G^2}X_2, \Gamma_0)$ is complete in $\mathfrak{S}^{G^2}(X_1 \cap X_2)$. Then there exist a Kripke system $\mathcal{R} = (K, \leq \{X_k\}_{k \in K}, \{V_k\}_{k \in K})$, a homomorphism $\psi: A \rightarrow \mathcal{L}_n$, $k_0 \in K$, and $x \in V_{k_0}$, such that for all $p \in \Delta_0 \cup \Theta_0$ if $\psi(p) = (f_k)$, then $f_{k_0}(x) = 1$ and for all $p \in \Gamma_0$ if $\psi(p) = (f_k)$, then $f_{k_0}(x) = 0$. 

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Proof. The first half of the proof is almost identical to that of lemma ??.

We highlight the main steps, for the convenience of the reader, except that we only deal with the case when $G$ is strongly rich. Assume, as usual, that $\alpha$, $G$, $\mathfrak{A}$ and $X_1$, $X_2$, and everything else in the hypothesis are given. Let $I$ be a set such that $\beta = |I| = \max(|A|, |\alpha|)$. Let $(K_n : n \in \omega)$ be a family of pairwise disjoint sets such that $|K_n| = \beta$. Define a sequence of algebras $\mathfrak{A} = \mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2 \ldots \subseteq \mathfrak{A}_n \ldots$ such that $\mathfrak{A}_{n+1}$ is a minimal dilation of $\mathfrak{A}_n$ and $\dim(\mathfrak{A}_{n+1}) = \dim \mathfrak{A}_n \cup K_n$. We denote $\dim(\mathfrak{A}_n)$ by $I_n$ for $n \geq 1$. The proofs of Claims 1 and 2 in the proof of ?? are the same.

Now we prove the theorem when $G$ is a strongly rich semigroup. Let

$$K = \{((\Delta, \Gamma), (T, F)) : \exists n \in \omega \text{ such that } (\Delta, \Gamma), (T, F) \text{ is a matched pair of saturated theories in } \mathfrak{S}_g^{Rd_{\mathfrak{A}}^n}X_1, \mathfrak{S}_g^{Rd_{\mathfrak{A}}^n}X_2 \}.$$

We have $((\Delta_0, \Gamma_0), (\Theta_0, \Gamma^*_0))$ is a matched pair but the theories are not saturated. But by lemma $5.16$ there are $T_1 = (\Delta_\omega, \Gamma_\omega), T_2 = (\Theta_\omega, \Gamma^*_\omega)$ extending $(\Delta_0, \Gamma_0), (\Theta_0, \Gamma^*_0)$, such that $T_1$ and $T_2$ are saturated in $\mathfrak{S}_g^{Rd_{\mathfrak{A}_1}X_1}$ and $\mathfrak{S}_g^{Rd_{\mathfrak{A}_2}X_2}$, respectively. Let $k_0 = ((\Delta_\omega, \Gamma_\omega), (\Theta_\omega, \Gamma^*_\omega))$. Then $k_0 \in K$, and $k_0$ will be the desired world and $x$ will be specified later; in fact $x$ will be the identity map on some specified domain.

If $i = ((\Delta, \Gamma), (T, F))$ is a matched pair of saturated theories in $\mathfrak{S}_g^{Rd_{\mathfrak{A}_n}X_1}$ and $\mathfrak{S}_g^{Rd_{\mathfrak{A}_n}X_2}$, let $M_i = \dim \mathfrak{A}_n$, where $n$ is the least such number, so $n$ is unique to $i$. Let

$$K = (K, \leq, \{M_i\}, \{V_i\})_{i \in \mathfrak{A}},$$

where $V_i = \bigcup_{p \in G_n,p \text{ a finitary transformation}} \alpha M_p^{(b)}$ (here we are considering only substitutions that move only finitely many points), and $G_n$ is the strongly rich semigroup determining the similarity type of $\mathfrak{A}_n$, with $n$ the least number such $i$ is a saturated matched pair in $\mathfrak{A}_n$, and $\leq$ is defined as follows: If $i_1 = ((\Delta_1, \Gamma_1)), (T_1, F_1))$ and $i_2 = ((\Delta_2, \Gamma_2), (T_2, F_2))$ are in $K$, then set

$$i_1 \leq i_2 \iff M_{i_1} \subseteq M_{i_2}, \Delta_1 \subseteq \Delta_2, T_1 \subseteq T_2.$$

We are not yet there, to preserve diagonal elements we have to factor out $K$ by an infinite family equivalence relations, each defined on the dimension of $\mathfrak{A}_n$, for some $n$, which will actually turn out to be a congruence in an exact sense. As usual, using freeness of $\mathfrak{A}$, we will define two maps on $\mathfrak{A}_1 = \mathfrak{S}_g^{Rd_{\mathfrak{A}_1}X_1}$ and $\mathfrak{A}_2 = \mathfrak{S}_g^{Rd_{\mathfrak{A}_2}X_2}$, respectively; then those will be pasted to give the required single homomorphism.

Let $i = ((\Delta, \Gamma), (T, F))$ be a matched pair of saturated theories in $\mathfrak{S}_g^{Rd_{\mathfrak{A}_n}X_1}$ and $\mathfrak{S}_g^{Rd_{\mathfrak{A}_n}X_2}$, let $M_i = \dim \mathfrak{A}_n$, where $n$ is the least such number, so $n$ is unique to $i$. For $k,l \in \dim \mathfrak{A}_n = I_n$, set $k \sim_i l$ iff $d^{\mathfrak{A}_n}_{kl} \in \Delta \cup T$. This is well defined since $\Delta \cup T \subseteq \mathfrak{A}_n$. We omit the superscript $\mathfrak{A}_n$. These are infinitely
many relations, one for each $i$, defined on $I_n$, with $n$ depending uniquely on $i$, we denote them uniformly by $\sim$ to avoid complicated unnecessary notation. We hope that no confusion is likely to ensue. We claim that $\sim$ is an equivalence relation on $I_n$. Indeed, $\sim$ is reflexive because $d_{ii} = 1$ and symmetric because $d_{ij} = d_{ji}$; finally $E$ is transitive because for $k, l, u < \alpha$, with $l \notin \{k, u\}$, we have

$$d_{kl} \cdot d_{lu} \leq c_l(d_{kl} \cdot d_{lu}) = d_{ku},$$

and we can assume that $T \cup \Delta$ is closed upwards. For $\sigma, \tau \in V_k$, define $\sigma \sim \tau$ iff $\sigma(i) \sim \tau(i)$ for all $i \in \alpha$. Then clearly $\sigma$ is an equivalence relation on $V_k$.

Let $W_k = V_k/ \sim$, and $\mathfrak{K} = (K, \leq, M_k, W_k)_{k \in K}$, with $\leq$ defined on $K$ as above. We write $h = [x]$ for $x \in V_k$ if $x(i)/ \sim = h(i)$ for all $i \in \alpha$; of course $X$ may not be unique, but this will not matter. Let $F_{\mathfrak{K}}$ be the set algebra based on the new Kripke system $\mathfrak{K}$ obtained by factoring out $\mathfrak{K}$.

Set $\psi_1 : \mathfrak{G}_n^{Rel} X_1 \to F_{\mathfrak{K}}$ by $\psi_1(p) = (f_k)$ such that if $k = ((\Delta, \Gamma), (T, F)) \in K$ is a matched pair of saturated theories in $\mathfrak{G}_n^{Rel} X_1$ and $\mathfrak{G}_n^{Rel} X_2$, and $M_k = \text{dim} A_n$, with $n$ unique to $k$, then for $x \in W_k$

$$f_k([x]) = 1 \leftrightarrow s_{x \cup (id_{M_k})^{-1}}^n p \in \Delta \cup T,$$

with $x \in V_k$ and $[x] \in W_k$ is define as above.

To avoid cumbersome notation, we write $s_x^n p$, or even simply $s_x p$, for $s_{x \cup (id_{M_k})^{-1}}^n p$. No ambiguity should arise because the dimension $n$ will be clear from context.

We need to check that $\psi_1$ is well defined. It suffices to show that if $\sigma, \tau \in V_k$ if $\sigma \sim \tau$ and $p \in A_n$, with $n$ unique to $k$, then

$$s_{\tau} p \in \Delta \cup T \text{ iff } s_{\sigma} p \in \Delta \cup T.$$

This can be proved by induction on the cardinality of $J = \{i \in I_n : \sigma i \neq \tau i\}$, which is finite since we are only taking finite substitutions. If $J$ is empty, the result is obvious. Otherwise assume that $k \in J$. We recall the following piece of notation. For $\eta \in V_k$ and $k, l < \alpha$, write $\eta(k \mapsto l)$ for the $\eta' \in V$ that is the same as $\eta$ except that $\eta'(k) = l$. Now take any

$$\lambda \in \{\eta \in I_n : \sigma^{-1}\{\eta\} = \tau^{-1}\{\eta\} = \{\eta\}\} \setminus \Delta x.$$

This $\lambda$ exists, because $\sigma$ and $\tau$ are finite transformations and $A_n$ is a dilation with enough spare dimensions. We have by cylindric axioms (a)

$$s_{\sigma} x = s_{\sigma k}^\lambda s_{\sigma(k \mapsto \lambda)} p.$$

We also have (b)

$$s_{\tau k}^\lambda (d_{\lambda, \sigma k} \land s_{\sigma} p) = d_{\tau k, \sigma k} s_{\sigma} p,$$

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and (c) 

\[ s_{\tau k}^\lambda (d_{\lambda,\sigma k} \land s_{\sigma(k\to \lambda)p}) \]

\[ = d_{\tau k,\sigma k} \land s_{\sigma(k\to \tau k)p}. \]

and (d) 

\[ d_{\lambda,\sigma k} \land s_{\sigma k}^\lambda s_{\sigma(k\to \lambda)p} = d_{\lambda,\sigma k} \land s_{\sigma(k\to \lambda)p} \]

Then by (b), (a), (d) and (c), we get,

\[ d_{\tau k,\sigma k} \land s_{\sigma}p = s_{\tau k}^\lambda (d_{\lambda,\sigma k} \cdot s_{\sigma}p) \]

\[ = s_{\tau k}^\lambda (d_{\lambda,\sigma k} \land s_{\sigma k}^\lambda s_{\sigma(k\to \lambda)p}) \]

\[ = s_{\tau k}^\lambda (d_{\lambda,\sigma k} \land s_{\sigma(k\to \lambda)p}) \]

\[ = d_{\tau k,\sigma k} \land s_{\sigma(k\to \tau k)p}. \]

The conclusion follows from the induction hypothesis. Now \( \psi_1 \) respects all quasipolyadic equality operations, that is finite substitutions (with the proof as before; recall that we only have finite substitutions since we are considering \( Sg^{Rd_n(X_1)} \)) except possibly for diagonal elements. We check those:

Recall that for a concrete Kripke frame \( F_W \) based on \( W = (W, \leq, V_k, W_k) \), we have the concrete diagonal element \( d_{ij} \) is given by the tuple \((g_k : k \in K)\) such that for \( y \in V_k \), \( g_k(y) = 1 \) iff \( y(i) = y(j) \).

Now for the abstract diagonal element in \( A \), we have \( \psi_1(d_{ij}) = (f_k : k \in K) \), such that if \( k = ((\Delta, \Gamma), (T, F)) \) is a matched pair of saturated theories in \( Sg^{Rd_n(X_1), Sg^{Rd_n(X_2)}} \) with \( n \) unique to \( i \), we have \( f_k([x]) = 1 \) iff \( s_{\sigma d_{ij} \in \Delta \cup T} \) (this is well defined \( \Delta \cup T \subseteq A_n \)).

But the latter is equivalent to \( d_{x(i),x(j)} \in \Delta \cup T \), which in turn is equivalent to \( x(i) \sim x(j) \), that is \([x](i) = [x](j)\), and so \( (f_k) \in d_{ij}^F \). The reverse implication is the same.

We can safely assume that \( X_1 \cup X_2 = X \) generates \( A \). Let \( \psi = \psi_1 \cup \psi_2 \upharpoonright X \). Then \( \psi \) is a function since, by definition, \( \psi_1 \) and \( \psi_2 \) agree on \( X_1 \cap X_2 \). Now by freeness \( \psi \) extends to a homomorphism, which we denote also by \( \psi \) from \( A \) into \( F_{\hat{R}} \). And we are done, as usual, by \( \psi \), \( k_0 \) and \( Id \in V_{k_0} \).

Theorem 5.24 generalizes as is, to the expanded structures by diagonal elements. That is to say, we have:

**Theorem 6.2.** The free dimension restricted free algebras have the interpolation property

Assume that \( \theta_1 \in \mathfrak{S}_g^{Rd_n(X_1)} \) and \( \theta_2 \in \mathfrak{S}_g^{Rd_n(X_2)} \) such that \( \theta_1 \leq \theta_2 \). Let \( \Delta_0 = \{ \theta \in \mathfrak{S}_g^{Rd_n(X_1 \cap X_2)} : \theta_1 \leq \theta \} \). If for some \( \theta \in \Delta_0 \) we have \( \theta \leq \theta_2 \), then we are done. Else \( (\Delta_0, \{ \theta_2 \}) \) is consistent. Extend this to a complete theory \( (\Delta_1, \Gamma_2) \) in \( \mathfrak{S}_g^{Rd_n(X_2)} \). Consider \( (\Delta, \Gamma) = (\Delta_2 \cap \mathfrak{S}_g^{Rd_n(X_1 \cap X_2)}, \Gamma_2 \cap \mathfrak{S}_g^{Rd_n(X_1 \cap X_2)}) \). Then \( (\Delta \cup \{ \theta_1 \}, \Gamma) \) is consistent. For otherwise, for some \( F \in \Delta, \mu \in \Gamma, \) we would
have \((F \land \theta_1) \to \mu\) and \(\theta_1 \to (F \to \mu)\), so \((F \to \mu) \in \Delta_0 \subseteq \Delta_2\), which is impossible. Now \((\Delta \cup \{\theta_1\}, \Gamma)\) \((\Delta_2, \Gamma_2)\) are consistent with \(\Gamma \subseteq \Gamma_2\) and \((\Delta, \Gamma)\) complete in \(\mathbb{S}^{\alpha}X_1 \cap \mathbb{S}^{\alpha}X_2\). So by theorem 5.24 \((\Delta_2 \cup \{\theta_1\}, \Gamma_2)\) is satisfiable at some world in some set algebra based on a Kripke system, hence consistent. But this contradicts that \(\theta_2 \in \Gamma_2\), and we are done.

7 Sheaf Duality, epimorphisms and omitting types

**Definition 7.1.** Let \(\mathfrak{B}\) be an algebra. A filter of \(\mathfrak{B}\) is a nonempty subset \(F \subseteq A\) such that for all \(a, b \in B\),

(i) \(a, b \in F\) implies \(a \ast b \in F\).

(ii) \(a \in F\) and \(a \leq b\) imply \(b \in F\).

It easy to check that if \(F\) is a filter on \(A\) then \(1 \in F\) and whenever \(a, a \implies b \in F\) then \(b \in F\). Also \(a \ast b \in F\) if and only if \(a \cap b \in F\) iff \(a \in F\) and \(b \in F\). A filter \(F\) is proper if \(F \neq A\) and it is easy to see that a filter \(F\) is proper iff \(0 \notin F\).

**Definition 7.2.** A filter \(P\) of \(A\) is prime provided that it is a prime filter of the underlying lattice \(L(\mathfrak{B})\) of \(\mathfrak{B}\), that is \(a \cup b \in P\) implies \(a \in P\) or \(b \in P\). This is equivalent to the statement that for all \(a, b \in \mathfrak{B}\), \(a \implies b \in P\) or \(b \implies a \in P\). A proper filter \(F\) is maximal if it is not properly contained in any other proper filter.

We let \(\text{Max}(\mathfrak{B})\) denote the set of maximal filters and \(\text{Spec}(\mathfrak{B})\) the family of prime filters. Then it is not hard to actually show that \(\text{Max}(\mathfrak{B}) \subseteq \text{Spec}(\mathfrak{B})\) [24]. For a set \(X \subseteq \mathfrak{B}\), \(\mathfrak{F}^\mathfrak{B}X\) denotes the filter generated by \(X\). A filter \(F\) is called principal, if \(F = \mathfrak{F}\{a\} = \{x \in B : x \geq a\}\). The following notions are taken from [24]. Proofs are also found in [24]. Let \(\mathfrak{B}\) be a non-trivial algebra. For each \(X \subseteq \mathfrak{B}\), we set

\[
V(X) = \{P \in \text{Spec}(X) : X \subseteq P\}.
\]

Then the family \(\{V(X)\}_{X \subseteq \mathfrak{B}}\) of subsets of \(\text{spec}(\mathfrak{B})\) satisfies the axioms for closed sets in a topological space. The resulting topology is called the Zariski topology, and the resulting topological space is called the prime spectrum of \(\mathfrak{B}\). We write \(V(a)\) for the more cumbersome \(V(\{a\})\). For any \(X \subseteq \mathfrak{B}\), let

\[
D(X) = \{P \in \text{Spec}(X) : X \not\subseteq P\}
\]
Then \{D(X)\}_{X \subseteq A} is the family of open sets of the Zariski topology. We write \(D(a)\) for \(D(\{a\})\). The minimal spectrum of \(\mathcal{B}\) is the topology induced by the Zariski topology on \(\text{Max}(\mathcal{B})\). For \(X \subseteq \mathcal{B}\) and \(a \in \mathcal{B}\), let

\[V_M(X) = V(X) \cap \text{Max}(\mathcal{B})\]

\[D_M(a) = V(a) \cap \text{Max}(\mathcal{B}), \quad \text{and} \quad D_M(a) = D(a) \cap \text{Max}(\mathcal{B}).\]

In other words,

\[V_M(a) = \{F \in \text{Max}(\mathcal{B}) : a \in F\}\]

and

\[D_M(a) = \{F \in \text{Max}(\mathcal{B}) : a \notin F\}.\]

**Lemma 7.3.** Let \(\mathcal{B}\) be an algebra. Let \(a, b \in \mathcal{B}\). Then the following hold:

(i) \(D_M(a) \cap D_M(b) = D_M(a \cup b)\).

(ii) \(D_M(a) \cup D_M(b) = D_M(a \cap b) = D_M(a \ast b)\).

(iii) \(D_M(X) = \text{Max}(\mathcal{B}) \iff \mathcal{B}^X = \mathcal{B}\).

(iv) \(D_M(\bigcup_{i \in I} X_i) = \bigcup_{i \in I} D_M(X_i)\).

(v) \(V_M(a) \cap V_M(b) = V_M(a \cap b)\).

(vi) \(a \leq b\) if and only if \(V_M(a) \subseteq V_M(b)\).

**Proof.** [24] proposition 2.8. We only prove one side of the last item, since it is not mentioned in [24]. Assume that \(V_a \subseteq V_b\). If it is not the case that \(a \leq b\), then we may assume that \(a \cap (b \implies 0)\) is not 0. Hence there is a proper maximal filter \(F\), such that \(a \cap (b \implies 0) \in F\). Hence \(a \in F\) and \(b \not\to 0\) is in \(F\). But this implies that \(b \notin F\) lest \(0 \in F\). Hence \(F \in V_a\) and \(F \notin V_b\). This is a contradiction, and the required is proved.

**Theorem 7.4.** Let \(\mathcal{B}\) be an algebra.

(i) \(\{D_M(a)\}_{a \in \mathcal{B}}\) is a basis for a compact Hausdorff topology on \(\text{Max}(\mathcal{B})\).

(ii) Furthermore if \(a = \bigvee a_i\), then \(V_M(a) \sim \bigcup V_M(a_i)\) is a nowhere dense subset of \(\text{Max}(\mathcal{B})\). Similarly if \(a = \bigwedge a_i\), then \(\bigcap V_M(a_i) \sim V_M(a)\) is nowhere dense.

(iii) If \(\mathcal{B}\) is countable, then \(\text{Max}(\mathcal{B})\) is a Polish space.

**Proof.**
(i) We include the proof for self completeness and also because the ‘nowhere density’ part is completely new, and as we shall see in a while it will play a pivotal role in the proof of the omitting types theorem. That $\text{Max}(\mathfrak{B})$ is compact and Hausdorff is proved in [21], theorem 2.9, the proof goes as follows: Assume that

$$\text{Max}(\mathfrak{B}) = \bigcup_{i \in I} D_M(a_i) = D_M(\bigcup_{i \in I} a_i).$$

Then $\mathfrak{B} = \mathfrak{F}\{\bigcup_{i \in I} a_i\}$, hence $0 \in \mathfrak{F}\{\bigcup_{i \in I} a_i\}$. There is an $n \geq 1$ and $i_1, \ldots, i_n \in I$ such that $a_{i_1} \ast \ldots a_{i_n} = 0$. But

$$\text{Max}(\mathfrak{B}) = D_M(0) = D_M(a_{i_1} \ast \ldots a_{i_n}) = D_M(a_{i_1}) \cup \ldots D_M(a_{i_n}).$$

Hence every cover is reducible to a finite subcover. Hence the space is compact. Now we show that it is Hausdorff. Let $M, N$ be distinct maximal filters. Let $x \in M \sim N$ and $y \in N \sim M$. Let $a = x \implies y$ and $b = y \implies x$. Then $a \notin M$ and $b \notin N$. Hence $M \in D_M(a)$ and $N \in D_M(b)$. Also $D_M(a) \cap D_M(b) = D_M(a \lor b) = D_M(1) = \emptyset$. We have proved that the space is Hausdorff.

(ii) Now assume that $a = \bigvee a_i$ and $V_M(a) \sim \bigcup V_M(a_i)$ is not nowhere dense. Then there exists $d$ such that $D_M(d) \subseteq V_M(a) \sim V_M(a_i)$ Hence

$$V_M(a_i) \subseteq V_M(a) \sim D_M(d) = V_M(a) \cap V_M(d) = V_M(a \cap d).$$

It follows that $a \cap d = a$ so $a \leq d$. Then $D_M(d) \subseteq D_M(a)$. So we have, $D_M(d) \subseteq D_M(a) \cap V_M(a) = \emptyset$ contradiction. Conversely assume that $a = \bigwedge a_i$ and assume that

$$D_M(d) \subseteq \bigcap V_M(a_i) \sim V_M(a).$$

Let $e = d \rightarrow 0$. Then $V_M(e) = D_M(d)$. Now we have

$$V_M(e) \subseteq \bigcap V_M(a_i) \sim V_M(a).$$

Taking complements twice, we get

$$V_M(e) \subseteq D_M(a) \sim \bigcup D_M(a_i)$$

Then $V_M(e) \subseteq D_M(a) \sim D_M(a_i)$. So

$$D_M(a_i) \subseteq D_M(a) \sim V_M(e) = D_M(a) \cap D_M(e) = D_M(a \cup e).$$

Hence $V_M(a \cup e) \subseteq V_M(a_i)$. So $a \cup e \leq a_i$ for each $i$. Thus $a \cup e = a$ from which we get that $e \leq a$. Hence $V_M(e) \subseteq V_M(a)$. But $V_M(e) \subseteq D_M(a)$ it follows that $V_M(e) = \emptyset$. But $V_M(e) = D_M(d)$ and we are done.
(iii) If $\mathcal{B}$ is countable, then $\text{Max}\mathcal{B}$ is second countable, so the required follows.

We start by a concrete example addressing variants and extension first order logics. The following discussion applies to $L_n$ (first order logic with $n$ variables), $L_{\omega,\omega}$ (usual first order logic), rich logics, Keislers logics with and without equality, finitray logics of infinitary relations; the latter three logics are infinitary extensions of first order logic, though the former and the latter have a finitary flavour, because quantification is taken only on finitely many variables. These logics have an extensive literature in algebraic logic. Let us start with the concrete example of usual first order logic. $\mathcal{L}_n$ denotes a relational first order language (we have no function symbols) with $n$ constants, $n \leq \omega$, and as usual a sequence of variables of order type $\omega$.

Example 7.5. Let $\mathfrak{sn}_{\mathcal{L}_n}$ denote the set of all $\mathcal{L}_n$ sentences, and fix an enumeration $(c_i : i < n)$ of the constant symbols. We assume that $T \subseteq \mathfrak{sn}_{\mathcal{L}_0}$. Let $X_T = \{ \Delta \subseteq \mathfrak{sn}_{\mathcal{L}_0} : \Delta \text{ is complete} \}$. This is simply the underlying set of the Priestly space, equivalently the Stone space, of the Boolean algebra $\mathfrak{sn}_{L_0}/T$. For each $\Delta \in X_T$, let $\mathfrak{sn}_{\mathcal{L}_n}/\Delta$ be the corresponding Tarski-Lindenbaum quotient algebra, which is a (representable) cylindric algebra of dimension $n$. The $i$th cylindrifier $c_i$ is defined by $c_i \phi/\Delta = \exists x(c_i|x)$, where the latter is the formula obtained by replacing the $i$th constant if present by the first variable $x$ not occurring in $\phi$, and then applying the existential quantifier $\exists x$. Let $\delta T$ be the following disjoint union $\bigcup_{\Delta \in X_T} \{ \Delta \} \times \mathfrak{sn}_{\mathcal{L}_n}/\Delta$. Define the following topologies, on $X_T$ and $\delta T$, respectively. On $X_T$ the Priestly (Stone) topology, and on $\delta T$ the topology with base $B_{\phi,\psi} = \{ \Delta, [\phi]_{\Delta}, \psi \in \Delta, \Delta \in X_T \}$. Then $(X_T, \delta T)$ is a sheaf, and its dual consisting of the continuous sections, $\Gamma(T, \Delta)$, with operations defined pointwise, is actually isomorphic to $\mathfrak{sn}_{\mathcal{L}_n}/T$.

Example 7.6. By the same token, let $\mathcal{L}$ be the predicate language for $BL$ algebras, $\mathfrak{fm}$ denotes the set of $L$ formulas, and $\mathfrak{sn}$ denotes the set of all sentences (formulas with no free variables). This for example includes $MV$ algebras; that are, in turn, algebraisations of many valued logics. Let $X_T$ be the Zarski (equivalently the Priestly) topology on $\mathfrak{sn}/T$ based on $\{ \Delta \in \text{Spec}(\mathfrak{sn}) : a \notin \Delta \}$. Let $\delta T = \bigcup_{\Delta \in X_T} \{ \Delta \} \times \mathfrak{fm}_\Delta$. Then again, we have $(X_T, \delta T)$ is a sheaf, and its dual consisting of the continuous sections with operations defined pointwise, $\Gamma(T, \Delta)$ is actually isomorphic to $\mathfrak{fm}_T$.

This situation is very similar to the one in algebraic geometry of describing the ring associated with the affine variety in terms of the local rings given at at point of the variety.

This needs further clarification. Let us formalize the above concrete examples in an abstract more general setting, that allows further applications. Let
Let $\mathfrak{A}$ be a bounded distributive lattice with extra operations $(f_i : i \in I)$. $\mathfrak{A}$ denotes the distributive bounded lattice $\mathfrak{A} = \{ x \in \mathfrak{A} : f_i x = x, \forall i \in I \}$, where the operations are the natural restrictions. (Idempotency of the $f_i$s guarantees that this is well defined). If $\mathfrak{A}$ is a locally finite algebra of formulas of first order logic or predicate modal logic or intuitionistic logic, or any predicate logic where the $f_i$s are interpreted as the existential quantifiers, then $\mathfrak{A}$ is the Boolean algebra of sentences.

Let $K$ be class of bounded distributive lattices with extra operations $(f_i : i \in I)$. We describe a functor that associates to each $\mathfrak{A} \in K$, and $J \subseteq I$, a pair of topological spaces $(X(\mathfrak{A}, J), \delta(\mathfrak{A})) = \mathfrak{A}_d$, where $\delta(\mathfrak{A})$ has an algebraic structure, as well; in fact it is a direct product of distributive lattices, that turn out to be simple (have no proper congruences) under favourable circumstances, in which case $\delta(\mathfrak{A})$ is a semi-simple lattice carrying a product topology. This pair is called the dual space of $\mathfrak{A}$. For $J \subseteq I$, let $\mathfrak{N}\mathfrak{r}_J \mathfrak{A} = \{ x \in A : f_i x = x \forall i \notin J \}$, with operations $f_i : i \in J$. $X(\mathfrak{A}, J)$ is the usual dual space of $\mathfrak{N}\mathfrak{r}_J \mathfrak{A}$, that is, the set of all prime ideals of the lattice $\mathfrak{N}\mathfrak{r}_J \mathfrak{A}$, this becomes a Priestly space (compact, Hausdorff and totally disconnected), when we take the collection of all sets $N_a = \{ x \in X(\mathfrak{A}, J) : a \notin x \}$, and their complements, as a base for the topology.

For a set $X$ of an algebra $\mathfrak{A}$ we let $\text{Co}^\mathfrak{A} X$ denote the congruence relation generated by $X$ (in the universal algebraic sense). This is defined as the intersection of all congruence relations that have $X$ as an equivalence class. Now we turn to defining the second component; this is more involved. For $x \in X(\mathfrak{A}, J)$, let $G_x = \mathfrak{A}/\text{Co}^\mathfrak{A} x$ and $\delta(\mathfrak{A}) = \bigcup\{ G_x : x \in X(\mathfrak{A}) \}$. This is clearly a disjoint union, and hence it can also be looked upon as the following product $\prod_{x \in \mathfrak{A}} G_x$ of algebras. This is not semi-simple, because $x$ is only prime, least maximal in $\mathfrak{N}\mathfrak{r}_J \mathfrak{A}$. But the semi-simple case will deserve special attention.

The projection $\pi : \delta(\mathfrak{A}) \rightarrow X(\mathfrak{A})$ is defined for $s \in G_x$ by $\pi(s) = x$. Here $G_x = \pi^{-1} x$ is the stalk over $x$. For $a \in A$, we define a function $\sigma_a : X(\mathfrak{A}) \rightarrow \delta(\mathfrak{A})$ by $\sigma_a(x) = a/\text{Co}^\mathfrak{A} x \in G_x$.

Now we define the topology on $\delta(\mathfrak{A})$. It is the smallest topology for which all these functions are open, so $\delta(\mathfrak{A})$ has both an algebraic structure and a topological one, and they are compatible.

We can turn the glass around. Having such a space we associate a bounded distributive lattice in $K$. Let $\pi : G \rightarrow X$ denote the projection associated with the space $(X, G)$, built on $\mathfrak{A}$. A function $\sigma : X \rightarrow G$ is a section of $(X, G)$ if $\pi \circ \sigma$ is the identity on $X$.

Dually, the inverse construction uses the sectional functor. The set $\Gamma(X, G)$ of all continuous sections of $(X, G)$ becomes a $BLO$ by defining the operations pointwise, recall that $G = \prod G_x$ is a product of bounded distributive lattices.

The mapping $\eta : \mathfrak{A} \rightarrow \Gamma(X(\mathfrak{A}, J), \delta(\mathfrak{A}))$ defined by $\eta(a) = \sigma_a$ is as easily checked an isomorphism. Note that under this map an element in $\mathfrak{N}\mathfrak{r}_J \mathfrak{A}$
of lattices $\Gamma(\lambda, \mu)$ is a pair $(\lambda, \mu)$ where $\lambda : Y \rightarrow X$ is a continous map and $\mu$ is a continous map $Y + \lambda \mathcal{L} \rightarrow \mathcal{G}$ such that $\mu_y = \mu(y, -)$ is a homomorphism of $\mathcal{L}_{\lambda(y)}$ into $\mathcal{G}_y$. We consider $Y + \lambda \mathcal{L} = \{(y, t) \in Y \times \mathcal{L} : \lambda(y) = \pi(t)\}$ as a subspace of $Y \times \mathcal{L}$. That is, it inherits its topology from the product topology on $Y \times \mathcal{L}$.

A sheaf morphism $(\lambda, \mu) = H : (Y, \mathcal{G}) \rightarrow (X, \mathcal{L})$ produces a homomorphism of lattices $\Gamma(H) = \Gamma(X, \mathcal{L}) \rightarrow \Gamma(Y, \mathcal{G})$ the natural way: for $\sigma \in \Gamma(X, \mathcal{L})$ define $\Gamma(H)\sigma$ by $(\Gamma(H)\sigma)(y) = \mu(y, \sigma(\lambda y))$ for all $y \in Y$. A sheaf morphism $h^d : \mathbb{A}^d \rightarrow \mathbb{A}^d$ can also be associated with a homomorphism $h : \mathbb{A} \rightarrow \mathbb{B}$. Define $h^d = (h^*, h^o)$ where for $y \in X(\mathbb{B})$, $h^*(y) = h^{-1} \cap Zd\mathbb{A}$ and for $y \in X(\mathbb{B})$ and $a \in A$

$$h^0(h, a/\mathcal{I}g^\mathbb{A}h^*(y)) = h(a)/\mathcal{I}g^\mathbb{A}y.$$  

**Example 7.7.** Let $\mathcal{A} = \prod_{i \in I} \mathcal{B}_i$, where $\mathcal{B}_i$ are directly indecomposable $BAOs$. Then $\mathfrak{A} = l^2$ and $X(\mathcal{A})$ is the Stone space of this algebra. The stalk $\delta_M(\mathcal{A})$ of $\mathcal{A}$ over $M \in X(\mathcal{A})$ is the ultraproduct $\prod_{i \in I} \mathcal{B}_i/F$ where $F$ is the ultrafilter on $\varphi(I)$ corresponding to $M$.

**Definition 7.8.** Let $\mathcal{A} \in \mathcal{C}_\omega A$, and $x \in A$. The dimension set of $x$, in symbols $\Delta x$, is the set $\{i \in \omega : c_i x \neq x\}$. Let $n \in \omega$. Then the $n$ neat reduc of $\mathcal{A}$ is the cylindric algebra of dimension $n$ consisting only of $n$ dimensional elements (those elements such that $\Delta x \subseteq n$), and with operations indexed up to $n$.

**Example 7.9.**

1. Let $\mathcal{A} \in \mathcal{N}_\omega \mathcal{C}_\omega A$. Then there is a sheaf $X = (X, \delta, \pi)$ such that $\mathcal{A}$ is isomorphic to continuous sections $\Gamma(X; \delta)$ of $X$. Indeed, let $X(\mathcal{A})$ be the Stone space of $\mathfrak{A}$. Then for any maximal ideal $x$ in $\mathfrak{A}$, $\mathcal{I}g^\mathcal{A}(x)$ is maximal in $\mathcal{N}_\omega \mathcal{A}$. Let $\delta(A) = \bigcup G_x$, where $G_x = \mathcal{A}/\mathcal{I}g^\mathcal{A}x$. The projection $\pi : \delta(\mathcal{A}) \rightarrow X(\mathcal{A})$ is defined for $s \in G_x$ by $\pi(s) = x$. For $a \in A$, we define a function $\sigma_a : X(\mathcal{A}) \rightarrow \delta(\mathcal{A})$ by $\sigma_a(x) = a/\mathcal{I}g^\mathcal{A}x \in G_x$. Then $\pi \circ \sigma$ is the identity and $\delta(\mathcal{A})$ has the smallest topology such that these maps are continuous. Then $\eta : \mathcal{A} \rightarrow \Gamma(X(\mathcal{A})), \delta$ defined by $\eta(a) = \sigma_a$ is the desired isomorphism.

2. Let $\mathcal{A} \in \mathcal{N}_\omega \mathcal{C}_\omega A$. For any ultrafilters $\mu$ and $\Gamma$ in $\mathfrak{A}$, the map $\lambda : \mathcal{A}/\mu \rightarrow \mathcal{A}/\Gamma$ defined via, $a/\mu \mapsto a/\Gamma$ maps $\mathfrak{A}$ into $\mathfrak{A}$. (The latter is the set of zero-dimensional elements). The dual morphism is $\lambda^d = (\lambda, \lambda^0) : (X_\Gamma, \delta(\Gamma)) \rightarrow (X_\mu, \delta(\mu))$, is defined by $\lambda(\Delta) = \Delta$ and $\lambda^0(\Delta, (\Delta), a/\Delta) = (\Delta, a/\Delta)$. Thus it is an isomorphism from $(X_\Gamma, \delta(\Gamma))$ onto the restriction of $(X_\mu, \delta(\mu))$ to the closed set $X_\Gamma$. Conversely, every restriction of $(X_\mu, \delta(\mu))$ to a closed subset $Y$ of $X_\mu$ is up to isomorphism the dual space of $\mathcal{N}_\omega \mathcal{A}/F$ for a filter $F$ of $\mathfrak{A}$. For if $\Gamma = \bigcap Y$, then
\[ Y = X_{\Gamma} \text{ since } Y \text{ is closed and the dual space of } \mathfrak{N}_n \mathfrak{A}/\Gamma \text{ is isomorphic to } (Y, \delta(\mu) \upharpoonright Y). \]

For an algebra \( \mathfrak{A} \) and \( X \subseteq \mathfrak{A} \), \( \mathfrak{Ig}^\mathfrak{A} X \) is the ideal generated by \( X \). We write briefly lattice for a BLO; hopefully no confusion is likely to ensue.

**Definition 7.10.**

1. A lattice \( L \) is regular if whenever \( x \) is a prime ideal in \( 3L \), then \( \mathfrak{Ig}^\mathfrak{A} x \) is a prime ideal in \( \mathfrak{A} \).

2. A lattice \( L \) is strongly regular, if whenever \( x \) is a prime ideal in \( 3L \), then \( \mathfrak{Ig}^\mathfrak{A} x \) is a maximal ideal in \( \mathfrak{A} \).

3. A lattice \( L \) is congruence strongly regular, if whenever \( x \) is a prime ideal in \( 3L \), then \( \mathfrak{Co}^\mathfrak{A} x \) is a maximal congruence of \( \mathfrak{A} \).

If \( L \) is not relatively complemented, then (2) and (3) above are not equivalent; but if it is relatively complemented then they are equivalent. A lattice with the property that every interval is complemented is called a relatively complemented lattice. In other words, a relatively complemented lattice is characterized by the property that for every element \( a \) in an interval \( [c, d] = \{ x : c \leq x \leq d \} \) there is an element \( b \), such that \( a \lor b = d \) and \( a \land b = c \). Such an element is called a complement; it may not be unique, but if the lattice is bounded then relative complements in \( [a, 1] \) are just complements, and in case of distributivity such complements are unique. In arbitrary lattices the lattice of ideas may not be isomorphic to the lattice of congruences, the following theorem gives a sufficient and necessary condition for this to hold. The theorem is a classic due to Gratzer and Schmidt.

**Theorem 7.11.** For the correspondence between congruences and ideals to be an isomorphism it is necessary and sufficient that \( L \) is distributive, relatively complemented with a minimum \( 0 \).

**Proof. Sketch** Clearly the ideal corresponding to the identity relation is the 0 ideal. Since every ideal of \( L \) is a congruence class under some homomorphism, we obtain distributivity. To show relative complementedness, it suffices to show that if \( b < a \), then \( b \) has a complement in the interval \( [0, a] \). Let \( I_{a,b} \) be the ideal which consists of all \( u \) with \( u \equiv 0(\Theta a,b) \). \( V_{a,b} \) is a congruence class under precisely one relation, hence \( a \equiv bmod(\Theta[I_{a,b}]) \). Hence for some \( v \in I_{a,b} \) we have \( b \lor v = a \) and \( b \land v = 0 \). Conversely, we have every ideal is a congruence class under at most one congruence relation, and of course under at least one.

In case of relative complementation, we have

**Theorem 7.12.** the following are equivalent
(1) \( L \) is strongly regular

(2) Every principal ideal of \( L \) is generated by an element in \( \mathfrak{Z}L \)

(3) \( \delta(\mathfrak{A}) \) is semisimple

**Proof.** Easy

We push the duality a step further establishing a correspondence between open (closed) sets of \( BLOs \) and open subsets of its dual. An ideal \( I \) in \( \mathfrak{A} \) is regular if \( \mathfrak{g}^{\mathfrak{A}}(I \cap \mathfrak{Z}\mathfrak{A}) = I \).

**Theorem 7.13.** There is an isomorphism between the set of all regular ideals in \( \Gamma(X,\delta) \) onto the lattice of open subsets of \( X \).

**Proof.** For \( \sigma \in \Gamma(X,\delta) \), let \( [\sigma] = \{ x \in X : \sigma(x) \neq 0_x \} \). For \( U \subseteq X \), let \( J[U] = \{ \sigma \in \Gamma(X,\delta) : [\sigma] \subseteq U \} \). Then \( J \mapsto U[J] \) is an isomorphism, its inverse is \( U[J] = \bigcup [\sigma] : \sigma \in J \).

Note that a simple lattice is necessarily strongly regular (and hence regular), but the converse is not true, even in the case of strong regularity. There are easy examples. As an application to our duality theorem established above, we show that certain properties can extend from simple structures to strongly regular ones. The natural question that bears an answer is how far are strongly regular algebras from simple algebras; and the answer is: pretty far. For example in cylindric algebras any non-complete theory \( T \) in a first order language gives rise to a strongly regular \( \omega \)-dimensional algebra, namely, \( \mathfrak{Sm}_{T} \), that is not simple.

\( ES \) abbreviates that epimorphisms (in the categorial sense) are surjective. Such abstract property is equivalent to the well-known Beth definability property for many abstract logics, including fragments of first order logic, and multi-modal logics.

In fact, it applies to any algebraisable logic (corresponding to a quasi-variety) regarded as a concrete category. This connection was established by Németi. As an application, to our hitherto established duality, we have:

**Theorem 7.14.** Let \( V \) be a class of distributive bounded lattices such that the simple lattices in \( V \) have the amalgamation property (AP). Assume that there exist strongly regular lattices \( \mathfrak{A}, \mathfrak{B} \in V \) and an epimorphism \( f : \mathfrak{A} \to \mathfrak{B} \) that is not onto. Then \( ES \) fails in the class of simple lattices

**Proof.** Suppose, to the contrary that \( ES \) holds for simple algebras. Let \( f^* : \mathfrak{A} \to \mathfrak{B} \) be the given epimorphism that is not onto. We assume that \( \mathfrak{A}^d = (X, \mathcal{L}) \) and \( \mathfrak{B}^d = (Y, \mathcal{G}) \) are the corresponding dual sheaves over the Priestly spaces \( X \) and \( Y \) and by duality that \( (h, k) = H : (Y, \mathcal{G}) \to (X, \mathcal{L}) \) is a monomorphism. Recall that \( X \) is the set of prime ideals in \( Zd\mathfrak{A} \), and similarly for \( Y \). We shall first prove
(i) \( h \) is one to one

(ii) for each \( y \) a maximal ideal in \( \mathfrak{B} \), \( k(y, -) \) is a surjection of the stalk over \( h(y) \) onto the stalk over \( y \).

Suppose that \( h(x) = h(y) \) for some \( x, y \in Y \). Then \( G_x, G_y \) and \( L_{hx} \) are simple algebra, so there exists a simple \( D \in V \) and monomorphism \( f_x : G_x \to D \) and \( f_y : G_y \to D \) such that

\[
f_x \circ k_x = f_y \circ k_y.
\]

Here we are using that the algebras considered are strongly regular, and that the simple algebras have \( AP \). Consider the sheaf \( (1, D) \) over the one point space \( \{0\} = 1 \) and sheaf morphisms \( H_x : (\lambda_x, \mu) : (1, D) \to (Y, G) \) and \( H_y = (\lambda_y, \nu) : (1, D) \to (Y, G) \) where \( \lambda_x(0) = x \lambda_y(0) = y \mu_0 = f_x \) and \( \nu_0 = f_y \).

The sheaf \( (1, D) \) is the space dual to \( D \in V \) and we have \( H \circ H_x = H \circ H_y \).

Since \( H \) is a monomorphism \( H_x = H_y \) that is \( x = y \). We have shown that \( h \) is one to one. Fix \( x \in Y \). Since, we are assuming that \( ES \) holds for simple algebras of \( V \), in order to show that \( k_x : L_{hx} \to G_x \) is onto, it suffices to show that \( k_x \) is an epimorphism. Hence suppose that \( f_0 : G_x \to D \) and \( f_1 : G_x \to D \) for some simple \( D \) such that \( f_0 \circ k_x = f_1 \circ k_x \).

Introduce sheaf morphisms \( H_0 : (\lambda, \mu) : (1, D) \to (Y, G) \) and \( H_1 = (\lambda, \nu) : (1, D) \to (Y, G) \) where \( \lambda(0) = x, \mu_0 = f_0 \) and \( \nu_0 = f_1 \). Then \( H \circ H_0 = H \circ H_1 \), but \( H \) is a monomorphism, so we have \( H_0 = H_1 \) from which we infer that \( f_0 = f_1 \).

We now show that (i) and (ii) implies that \( f^* \) is onto, which is a contradiction. Let \( \mathfrak{A}^d = (X, \mathfrak{L}) \) and \( \mathfrak{B}^d = (Y, G) \). It suffices to show that \( \Gamma((f^*)^d) \) is onto (Here we are taking a double dual). So suppose \( \sigma \in \Gamma(Y, G) \). For each \( x \in Y \), \( k(x, -) \) is onto so \( k(x, t) = \sigma(x) \) for some \( t \in L_{hx} \). That is \( t = \tau_x(h(x)) \) for some \( \tau_x \in \Gamma(X, G) \). Hence there is a clopen neighborhood \( N_x \) of \( x \) such that \( \Gamma(f^*)((\tau)_x(y)) = \sigma(y) \) for all \( y \in N_x \). Since \( h \) is one to one and \( X, Y \) are Boolean spaces, we get that \( h(N_x) \) is clopen in \( h(Y) \) and there is a clopen set \( M_x \) in \( X \) such that \( h(N_x) = M_x \cap h(Y) \). Using compactness, there exists a partition of \( X \) into clopen subsets \( M_0 \ldots M_{k-1} \) and sections \( \tau_i \in \Gamma(M_i, L) \) such that

\[
k(y, \tau_i(h(y)) = \sigma(y)
\]

wherever \( h(x) \in M_i \) for \( i < k \). Defining \( \tau \) by \( \tau(z) = \tau_i(z) \) whenever \( z \in M_i \) \( i < k \), it follows that \( \tau \in \Gamma(X, \mathfrak{L}) \) and \( \Gamma((f^*)^d) \tau = \sigma \). Thus \( \Gamma((f^*)^d) \) is onto \( \Gamma(\mathfrak{B}^d) \), and we are done.

8 Omitting types in non classical logics, topologically

From now on \( \mathfrak{L} \) is a core fuzzy logic. We set up the context where omitting types apply.
Definition 8.1. (i) An $n$ type, or simply a type, is a set $\Gamma$ whose formulas have free variables among the first $n$ variables. Fix a theory $T$ and an $n$ type $\Gamma$. $\mathcal{M} = (M, L)$ realizes $\Gamma$ if there is $s \in {}^n M$ such that $||\phi(s)||^L_{\mathcal{M}} = 1$ for all $\phi \in \Gamma$. $\mathcal{M}$ omits $\Gamma$ if $\mathcal{M}$ does not realize $\Gamma$.

(ii) $\Gamma$ is a principal type in $T$ if there is a formula $\phi(\bar{x})$ such that for all $M |= T$ for all $v \in {}^n M$, $||\phi(v) \implies \psi(v)||^L_{\mathcal{M}} = 1$ for all $\psi \in \Gamma$. Otherwise $\Gamma$ is non-principal.

(iii) Call a formula $\exists x\phi$ containing free variables $y_1 \ldots y_n$ witnessed in $(\mathcal{M}, L)$ if for each $a_1 \ldots a_n \in M$, there is an element $b \in M$ such that $||\exists x\phi(x, a_1 \ldots a_n)||^L_{\mathcal{M}} = ||\phi(b, a_1 \ldots a_n)||^L_{\mathcal{M}}$; similarly for $\forall x\phi$. Call $(\mathcal{M}, \Sigma)$ witnessed if each formula beginning with a quantifier is witnessed.

Let $\kappa$ be a cardinal. Consider the following statement:

$OTT(\kappa)$. If $\Sigma$ is a countable theory and $\Gamma_i$, $i \in \kappa$, are non-principal types, then there is a witnessed safe model of $T$ omitting these types.

Theorem 8.2. (i) The statement $(\forall \kappa \leq \omega)OTT(\kappa)$ is provable in ZFC.

(ii) The statement $(\forall \kappa < \omega^2)OTT(\kappa)$ is independent of ZFC.

Proof. We only prove (i). A similar statement is proved in [25] but our proof is completely different. (ii) will be proved later. Let $\Sigma$ be a countable theory, and $\{\Gamma_i : i < \omega\}$ be a family of non-principal types. Add infinitely many countably many constants, let the added constants be $C$. Now $\Sigma$ can be expanded to Henkin complete theory $T$ which is a countable union $\bigcup_{m \in \omega} T_m$ where $T_0 = T \subseteq T_1 \subseteq T_2 \subseteq \ldots T_m \ldots$ such that

(1) each $T_m$ is consistent and is obtained from $T$ by adding finitely many axioms using only finitely many constants,

(2) for every pair of sentences in the expanded language we have $T_m \vdash \phi \implies \psi$ or $T_m \vdash \psi \implies \phi$,

(3) If $T_m \not\vdash \forall y \phi(y)$, then $T_m \not\vdash \phi(c)$.

Consider the algebra of sentences $\mathfrak{B} = \mathfrak{M}_T$. We write $[\phi]$ for $[\phi]_T$ the equivalence class containing $\phi$. Let $X = \text{Max}(\mathfrak{B})$, be the compact Hausdorff space with countable basis $D_M(a)$ for $a \in B$. Then in $\mathfrak{B}$, since the extension $T$ is Henkin, we have

$$[\exists x \alpha(x)] = \bigvee_{c \in C} [\alpha(c)] \quad (1)$$
\[ [\forall x \alpha(x)] = \bigwedge_{c \in C} [\alpha(c)] \]  
(2)

Now since the types considered are non-principal, they remain non-principal in the expanded language. For if not, then \( T \vdash \phi \rightarrow \Gamma_i(\bar{c}) \) for some \( i \), then \( T_m \vdash \phi \rightarrow \Gamma_i(\bar{c}) \). Now \( T_m \) and \( \phi \) contain only finitely many constants, replacing those by new variables (distinct variables for distinct contents) such that substitutions are free, avoiding collisions, we obtain that \( \Sigma \vdash \phi' \rightarrow \Gamma_i \) which is a contradiction. It therefore follows that

\[ \forall c_1, \ldots, c_n, \forall i \in \omega, \bigwedge_{\phi \in \Gamma_i} [\phi(c_1, \ldots, c_n)] = 0 \]  
(3)

Then for every variable \( x \) and formula with one free variable \( \alpha \) we have

\[ H_{\alpha, x} = V_M(\exists x \alpha(x)) \sim \bigcup_{c \in C} V_M(\alpha(c)) \]  
(4)

\[ J_{\alpha, x} = \bigcap_{c \in C} V_M(\alpha(c)) \sim V_M(\forall x \alpha x) \]  
(5)

and for every \( i \),

\[ K_{(c_1, \ldots, c_n, \Gamma_i)} = \bigcap_{\phi \in \Gamma_i} V_M(\phi(c_1, \ldots, c_n)) \]  
(6)

are, by theorem 7.4, nowhere dense sets. Let

\[ H = \bigcup_{\alpha} \bigcup_x H_{\alpha, x}, \]

\[ J = \bigcup_{\alpha} \bigcup_x J_{\alpha, x}, \]

and

\[ K = \bigcup_{\bar{c}} \bigcup_{i \in \omega} K(\bar{c}, \Gamma_i). \]

Then each of these sets is a countable union of nowhere dense sets. Given any \( a = [\psi] \in \mathcal{B} \), let \( F \) be a maximal filter in the complement of the union of these sets and in \( D_{Ma} \). this is possible since by theorem 7.4, \( \text{Max}(\mathcal{B}) \) is a Polish space, and the Baire category theorem holds, so that the complement of a countable collection of nowhere dense sets is dense. Let \( T_1 = \cup F = \{ \phi : [\phi]_T \in F \} \). Then we have the following:

1. \( F \notin \bigcup_{\alpha, x} H_{\alpha, x} \) implies that for any \( \alpha \), for any \( x \), if \( \exists x \alpha(x) \in T_1 \) then \( \alpha(c) \) is in \( T \) for some \( c \).
2. $F \notin \bigcup_{\alpha,x} J_{\alpha,x}$, implies that for $\alpha$ for any $x$, if not $T_1 \vdash \forall x \alpha(x)$ then not $T_1 \vdash \alpha(c)$. This means that $T_1$ is Henkin.

3. Finally, $F \notin K$ means that for all $i \in \omega$ for all $\phi \in \Gamma_i$ there exists $\bar{c}$ such that $\phi(\bar{c}) \notin T_1$.

Then $T_1$ is a Henkin theory, $T_1 \not\models \psi$ and its canonical model is as desired. That is the canonical model of $T_1$ is safe, witnessed and omits the given types.

Theories considered remain countable. However, we now ask for the omission of possibly uncountably many non-principal types. We shall prove (ii). We will see that we are actually touching upon somewhat deep issues in set theory here. We need two lemmas. The first is a known consequence of Martin’s axiom (henceforth $MA$).

**Lemma 8.3.** The statement $(\forall \kappa < \omega^2)OTT(\kappa)$ is provable in $\text{ZFC} + MA$

**Proof.** By Martin’s axiom the union of $< \omega^2$ nowhere dense sets is equal to a countable union.

**Theorem 8.4.** The statement $(\forall k < \omega^2)(OTT(\kappa))$ is independent of $\text{ZFC} + \neg \text{CH}$.

**Proof.** We have proved consistency since $MA$ implies the required statement. Now we prove independence. Let $covK$ be the least cardinal $\kappa$ such that the real line can be covered by $\kappa$ many closed disjoint nowhere dense sets. It is known that $\omega < covK \leq 2^\omega$. In any Polish space the intersection of $< covK$ dense sets is dense [?]. But then if $\kappa < covK$, then $OTT(\kappa)$ is true. The independence can be proved using standard iterated forcing to show that it is consistent that $covK$ could be literally anything greater than $\omega$ and $\leq \omega^2$. 

**Theorem 8.5.** The statement $(\forall \kappa < covK)OTT(\kappa)$ is provable in $\text{ZFC}$.

Note that Martin’s axiom implies that $covK = 2^\omega$ which reproves [8.3]. This is mentioned in [?]. The connection between omitting types and combinatorics of the real line was first discovered by Newelski [23].

### 8.1 Some model theoretic consequences of the omitting types theorem

In classical first order logic, the omitting types theorem is used to construct what is known as atomic and prime models, which are minimal models. In this section we define atomic, prime and saturated models in the fuzzy context and work out some connections between them. All languages considered are countable.
Now, having an omitting types theorem at our disposal for fuzzy logic, such investigations can be carried out in this more general context. We give a sample by studying the so called atomic models. We define atomic models. Let $L$ be a core fuzzy predicate logic.

**Definition 8.6.** Let $T$ be a theory.

(i) A formula $\phi(x_1 \ldots x_n)$ is said to be complete in $T$ if $\phi$ is consistent with $T$ and whenever $\psi$ is consistent with $T$ and $T \models \psi \implies \phi$ then $T \models \phi \implies \psi$.

(ii) A formula $\theta$ is completable if there is a complete formula $\psi$ such that $T \models \psi \implies \theta$.

(iii) $T$ is atomic if every formula consistent with $T$ is completable.

(iv) For a model $\mathfrak{M}$, let $Th(\mathfrak{M})$ be the set of all sentences in the language of $\mathfrak{M}$ that are valid. That is $\phi \in Th(\mathfrak{M})$ iff $||\phi||_\mathfrak{M} = 1$. A model $\mathfrak{M} = (M, L)$ is an atomic model, if for every $a_1, \ldots, a_n \in M$, there exists a complete formula $\phi(x_1 \ldots x_n)$, with respect to $Th(\mathfrak{M})$, such that $||\phi(a_1 \ldots a_n)||^L_\mathfrak{M} = 1$.

**Lemma 8.7.** Let $\mathfrak{B}$ be a BL algebra. Let $X \subseteq B$, be such that for all non-zero $b \in \mathfrak{B}$, there exists a non-zero $x \in X$, such that $x \leq a$. Then $\bigvee X = 1$.

**Proof.** Let $\bigvee X = b$. Assume that $b < 1$. Then $1 \cap -b \neq 0$. Let $x \in X$ be non-zero below $1 \cap -b$. Then $x \leq -b$ and $x \leq b$, hence $x \leq b \cap -b = 0$ which is a contradiction. \[ \square \]

**Theorem 8.8.** Let $\Sigma$ be a theory. Then $\Sigma$ has a countable safe witnessed atomic model if and only if $\Sigma$ is atomic.

**Proof.** Assume that $\Sigma$ has an atomic model $\mathfrak{M}$. Let $\phi(x_1, \ldots, x_n)$ be a consistent with $\Sigma$. Then $\psi = \exists x_1 \ldots \exists x_n \phi(x_1, \ldots, x_n) \in \Sigma$. For if not, then since $\Sigma$ is complete, then $\Sigma \cup \psi \vdash \bot$ and this is impossible since $\phi$ is consistent with $\Sigma$. Let $a_1, \ldots, a_n \in M$ be satisfied by $\phi$. Let $\theta$ be a complete formula satisfiable by $a_1, \ldots, a_n$. Then we have by completeness of $\Sigma$ that $\Sigma \vdash \theta \implies \phi$ or $\Sigma \vdash \phi \implies \theta$, but in the second case we will also have $\Sigma \vdash \theta \implies \phi$ since $\theta$ is complete.

Now for the converse. In the classical case, the omitting types theorem is used, but the proof depends on negation. Here we give a direct proof, which is very similar to that of the omitting types theorem. Let $\Sigma$ be the given atomic theory. Like in the proof of the omitting types theorem add a set $C$ of countably many constants, form a Henkin complete extension $T$ of $\Sigma$, and let $\mathfrak{B} = \mathfrak{Fm}_T$. 

60
Now in this present case, we consider the following meets and joins: Then in $\mathcal{B}$ we have

$$[\exists x \alpha(x)] = \bigvee_{c \in C} [\alpha(c)]$$  \hfill (7)

$$[\forall x \alpha(x)] = \bigwedge_{c \in C} [\alpha(c)]$$  \hfill (8)

Let $Fm_n$ denote the set of formulas where at most the $n$ first variables can be free. Now since the theory is atomic, if we let $\Gamma_n = \{ \phi \in Fm_n : \phi \text{ is complete} \}$, then we have from lemma 8.7

$$\forall c_1, \ldots, c_n, \forall i \in \omega, \bigvee_{\phi \in \Gamma_i} [\phi(c_1, \ldots, c_n)] = 1$$ \hfill (9)

Then for every variable $x$ and formula with one free variable $\alpha$ we have

$$H_{\alpha,x} = V_M(\exists x \alpha(x)) \sim \bigcup_{c \in C} V_M(\alpha(c))$$

$$J_{\alpha,x} = \bigcap_{c \in C} V_M(\alpha(c)) \sim V_M(\forall x \alpha x)$$

and for every $i$,

$$K_{(c_1, \ldots, c_n, \Gamma_i)} = \text{Max}(\mathcal{B}) \sim \bigcup_{\phi \in \Gamma_i} V_M(\phi(c_1 \ldots c_n))$$

are nowhere dense sets. As in the omitting types theorem, define $H$, $J$ and $K$. Then each of these sets is a countable union of nowhere dense sets in $\text{Max}(\mathcal{B})$. Let $F$ be a maximal filter in the complement of these sets. Let $T = \bigcup F$. Then $T$ is a Henkin theory and its canonical model is as desired. Lets check this. The first two conditions imply that $T$ is a Henkin theory. Finally, $F \notin K$ means that for all $i \in \omega$ for all $\phi \in \Gamma_i$ there exists $\bar{c}$ such that $\phi(\bar{c}) \in T$.

\[\blacksquare\]

**Definition 8.9.** Let $S$ be a signature.

1. Two $S$ structures $(\mathfrak{M}_1, L_1)$ and $(\mathfrak{M}_2, L_2)$ are elementary equivalent, if for each sentence $\phi$, we have $\mathfrak{M} \models \phi$ if and only if $\mathfrak{M} \models \phi$.

2. An elementary embedding of $(\mathfrak{M}_1, L_1)$ into $(\mathfrak{M}_2, L_2)$ is a pair $(f, g)$ such that
   1. $f$ is an injection from $M_1$ into $M_2$,
   2. $g$ is an embedding of $L_1$ into $L_2$,
   3. $g(||\phi(a_1 \ldots a_n)||^{L_1}_{\mathfrak{M}_1} = ||\phi(f(a_1) \ldots f(a_n)||^{L_2}_{\mathfrak{M}_2}$.
(3) \((\mathfrak{M}_1, L_1)\) and \((\mathfrak{M}_2, L_2)\) are isomorphic if there is an elementary embedding \((f, g)\) such that \(f\) is a bijection and \(g\) is an isomorphism.

From now on, we fix one algebra \(L\), so that all models are of the form \((\mathfrak{M}, L)\), which we sometimes write as \(\mathfrak{M}\). Furthermore we assume that all models are witnessed.

**Definition 8.10.** Let \(T\) be a theory.

1. A set \(\Gamma\) in the variables \(x_1 \ldots x_n\) is consistent with \(T\) if there exists a model \((\mathfrak{M}, L)\) of \(T\) and \(s \in {}^n M\) such that \(||\phi(s)||_M^n = 1\) for all \(\phi \in \Gamma\).

2. \(\Gamma\) is a complete type if it a maximal consistent set, that is \(\alpha \not\in \Gamma\), then \(T \cup \alpha \vdash \bot\). \(S_n(T)\) denotes the set of complete \(n\)-types, that is types using only \(n\) variables.

3. For a model \((\mathfrak{M}, L)\) and \(a_1 \ldots a_n \in M\), \(tp^{\mathfrak{M}}(a_1 \ldots a_n) = \{\phi \in Fm_n : ||\phi(a_1 \ldots a_n)|| = 1\}\). We may write \(tp^{\mathfrak{M}}(\bar{a})\). We omit the superscript \(\mathfrak{M}\) when clear from context.

4. For a model \(\mathfrak{M} = (M, L)\) and \(Y \subseteq M\), \(\mathfrak{M}_Y\) is the model obtained by adding a constant \(c_a\) for each \(a \in Y\) and interpreting \(c_a\) as \(a\).

5. A model \((\mathfrak{M}, L)\) is \(\omega\) or countably saturated if for each finite set \(Y \subseteq M\), every set \(\Gamma(x)\) consistent with \(Th(\mathfrak{M}_Y)\) is realized in \(\mathfrak{M}_Y\).

Notice that \(S_n(T)\) is a compact Hausdorff space, since it is the set of maximal filters in the algebra \(\mathfrak{Fm}_n/T\) where \(\mathfrak{Fm}_n\) is the set of formulas which has only at most \(n\) free variables.

**Theorem 8.11.** Let \(T\) be a complete theory. Then \(T\) has a countably saturated model if and only if for each \(n < \omega\), \(T\) has countably many complete \(n\) types in \(n\) variables

**Proof.** We prove the harder direction. Add a countable list \(\{c_1, c_2 \ldots\}\) of new constants forming \(\mathfrak{L}\). For each finite subset \(Y \subseteq C\), the types \(\Gamma(x)\) of in \(L_Y\) are countable. Let

\[
\Gamma_1(x), \ldots, \Gamma_n(x), \ldots
\]

be an enumeration of all types of \(T\) in all expansions \(L_Y\), \(Y\) a finite subset of \(C\). Let

\[
\phi_1, \ldots, \phi_n, \ldots
\]

be an enumeration of all sentences of \(\mathfrak{L}\). Define inductively an increasing sequence

\[
T = T_0 \subseteq T_1 \subseteq T_2 \subseteq \ldots
\]

of theories of \(\mathfrak{L}\) such that for each \(m < \omega\):
(1) each $T_m$ is consistent and is obtained from $T$ by adding finitely many axioms using only finitely many constants,

(2) If $\phi_m = \alpha \rightarrow \beta$, then either $\alpha \rightarrow \beta \in T_m$ or $\beta \rightarrow \alpha \in T_m$,

(3) If $\phi_m = \forall x \psi$ is not in $T_{m+1}$ then $\psi(c)$ is not in $T_{m+1}$,

(4) if $\Gamma_m(x)$ is consistent with $T_{m+1}$ then $\Gamma_m(d) \subseteq T_{m+1}$ for some $d \in C$.

The first three items are like the proof of completeness forming a complete Henkin extension. It is clear that the last task can be implemented without interfering with the first three. For assume inductively that $T_{m+1}'$ have been constructed satisfying (1), (2) and (3). Then if $\Gamma_m(x)$ is consistent with $T_{m+1}'$, then one chooses a constant $d$ not occurring in $T_m$ nor $\Gamma_m(x)$, this is possible, since only finitely many constants are in use and puts $T_{m+1} = T_{m+1}' \cup \Gamma_m(d)$. Else he puts $T_{m+1} = T_{m+1}'$. The union $T_\omega$ is a Henkin complete theory, and its canonical model is as required. Let $Y \subseteq M$ be finite and let $\Sigma(x)$ be consistent with $Th(\mathcal{M}_Y)$. Then extend $\Sigma(x)$ to a type $\Gamma(x)$ in $Th(\mathcal{M}_Y)$. Then for some $m$, $\Gamma(x) = \Gamma_m(x)$, and the latter is consistent with $T_{m+1}$, Hence $\Gamma_m(c) \subseteq T_{m+1}$, then $c$ realizes $\Gamma(x)$ in $\mathcal{M}_Y$.

Now saturated models are the large models. Now we investigate their dual, the small models.

**Definition 8.12.** (1) A model $\mathcal{M} \models T$ is a prime model, if it is elementary embeddable in every model of $T$.

(2) A model $\mathcal{M} \models T$ is atomic, if for every $n \in \omega$, for every consistent formula $\psi$ using $n$ free variables, there exists a minimal formula $\psi'$, also using only $n$ free variables such that $T \models \psi \rightarrow \phi$. Here minimal means that for any formula $\xi$ with only $n$ free variables, whenever $T \models \xi \rightarrow \psi$, then $\xi \rightarrow \bot$ or $T \models \psi \rightarrow \xi$.

It is easy to see that countable atomic models are prime. The proof goes like the classical case.

**Theorem 8.13.** Let $\mathcal{L}$ be a countable language and let $T$ be a complete theory. Then the following are equivalent

(i) $T$ has a prime model

(ii) $T$ has an atomic model

(iii) The principal types in $S_n(T)$ are dense for all $n$

**Proof.** The proof is like the classical case, see [9], so we will be sketchy:
1. (i) \(\rightarrow\) (ii) Here we use the omiting types theorem. Assume that \(\mathfrak{M}\) is countably prime. Let \(a_1 \ldots a_n \in M\) and let \(\Gamma(x_1 \ldots x_n)\) be the set of formulas \(\mu(x_1 \ldots x_n)\) such that

\[||\mu(a_1 \ldots a_n)||_{\mathfrak{M}} = 1.\]

For any countable model \(\mathfrak{B}\) of \(T\) we have an elementary embedding \(f : \mathfrak{M} \rightarrow \mathfrak{B}\) whence \(f(a_1), \ldots f(a_n)\) satisfy \(\Gamma\). Therefore \(\Gamma\) is realized in every model of \(T\). By the omitting types theorem, there is a formula \(\phi\) that isolates \(\Gamma\). Then \(\phi\) is complete and is satisfied by \(a_1 \ldots a_n\).

2. (ii) \(\rightarrow\) (iii) Let \(\phi\) be an \(L\) formula such that \([\phi]\) is a non empty open set in \(S_n(T)\). Let \(M \models T\) be atomic. Then, as above since \(T\) is complete, we have \(T \models \exists \bar{v} \phi(\bar{v})\). There is an \(\bar{a} \in M^n\) such that \(\mathfrak{M} \models \phi(\bar{a})\). Then \(\text{tp}^\mathfrak{M}(\bar{a}) \in [\phi]\) and is isolated since \(M\) is atomic.

3. (iii) \(\rightarrow\) (i) Suppose that the isolated types in \(T\) are dense. Add a countable set of constants to form a Henkin extension of \(T\). Then \(\bigvee_{\phi \in \Gamma_n}[\phi(\bar{c})] = 1\) where \(\Gamma_n = \{\phi \in Fm_n : \phi\) is complete \}. This follows from the fact that every \([\phi]\) contains a principal type, and principal types are generated by complete formulas, since they are maximal. In other words every formula is completable. Next proceed as in the proof of Theorem 8.8 constructing an atomic hence prime model.

\[\beth\]

**Corollary 8.14.** The following are equivalent for a theory \(T\).

(i) Every formula is completable.

(ii) The isolated types are dense in \(S_n(T)\) for every \(n\).

**Definition 8.15.** A complete theory \(T\) is \(\omega\) categorial iff it has up to isomorphism only one countable model.

**Theorem 8.16.** Let \(T\) be a complete theory. Then the following are equivalent:

(i) \(T\) is \(\omega\) categorial.

(ii) \(T\) has a model which is both atomic and saturated.

(iii) Every type \(\Gamma(x_1 \ldots x_n)\) is principal.

(iv) All countable models of \(T\) are atomic.

**Proof.**
(1) \((i) \rightarrow (ii)\). Let \(\mathcal{M}\) be the unique countable model of \(T\). Then \(\mathcal{M}\) is countable prime and so is atomic. Since \(T\) has only one countable model, it has a countably saturated model. Hence \(\mathcal{M}\) is countably saturated.

(2) \((ii) \rightarrow (iii)\) Since \(\mathcal{M}\) is \(\omega\) saturated, the type \(\Gamma\) is realized by some \(n\) tuple \(a_1 \ldots a_n\). Since \(\mathcal{M}\) is atomic, \(a_1 \ldots a_n\) satisfies an atomic formula \(\phi\). Clearly \(\phi \in \Gamma\).

(3) \((v) \rightarrow (vi)\) Direct

(4) \((vi) \rightarrow (i)\) We show that any two models that are atomic and elementary equivalent are isomorphic. But this follows from a back and forth argument as in [9].

In the classical case the two more equivalences can be added. That the number of types in \(Fm_n/T\) is finite, and that there are finitely many formulas modulo \(T\) for each \(n\). This follows from the algebraic property of Boolean algebras that if in an algebra all maximal filters are principal, then both the algebra and hence the set of maximal filters are finite. The above does not work for any \(BL\) algebra. Consider for example the Heyting algebra, which is an infinite linear order. Then the algebra has one maximal filter but it is not finite. However these statements imply the other 4 formulas, but are not equivalent to any of them.

The following results follows like the classical case:

**Theorem 8.17.** Any complete theory \(T\) which has a countably saturated model, has a countable atomic model.

**Proof.** Using a binary tree argument and lemma ???. Assume that \(T\) has no atomic model. Then \(T\) has a consistent formula that is not completable. For each incompletable formula we can choose two formulas below it that are incompatible. This can be done infinitely many times giving a tree of incompletable formulas. Each branch gives a consistent set of formulas and there are \(\omega^2\) branches, which can be extended to obtain \(\omega^2\) types contrary to the assumption that it has a saturated model.

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