Deconstructing non-Dirac–Hermitian supersymmetric quantum systems

Pijush K Ghosh

Department of Physics, Siksha-Bhavana, Visva-Bharati University, Santiniketan 731 235, West Bengal, India
E-mail: pijushkanti.ghosh@visva-bharati.ac.in

Received 24 January 2011
Published 3 May 2011
Online at stacks.iop.org/JPhysA/44/215307

Abstract
A method to construct a non-Dirac–Hermitian supersymmetric quantum system that is isospectral with a Dirac–Hermitian Hamiltonian is presented. The general technique involves a realization of the basic canonical (anti-) commutation relations involving both bosonic and fermionic degrees of freedom in terms of non-Dirac–Hermitian operators which are Hermitian in a Hilbert space that is endowed with a pre-determined positive-definite metric. A pseudo-Hermitian realization of the Clifford algebra for a pre-determined positive-definite metric is used to construct supersymmetric systems with one or many degrees of freedom. It is shown that exactly solvable non-Dirac–Hermitian supersymmetric quantum systems can be constructed corresponding to each exactly solvable Dirac–Hermitian system. Examples of non-Dirac–Hermitian (i) non-relativistic Pauli Hamiltonian, (ii) super-conformal quantum system, and (iii) supersymmetric Calogero-type models admitting entirely real spectra are presented.

PACS numbers: 03.65.—w, 11.30.Pb, 03.65.Fd, 02.30.Ik

1. Introduction
The study of non-Dirac–Hermitian quantum systems with unbroken combined parity ($\mathcal{P}$) and time-reversal ($\mathcal{T}$) symmetry has received considerable attention over the last few years [1–17]. A consistent quantum description of such systems, including reality of the entire spectra and unitary time evolution, is admissible with the choice of a new inner product in the Hilbert space [1]. A non-Dirac–Hermitian quantum system admitting entirely real spectra may also be understood in terms of pseudo-Hermitian operators [2, 3]. The existence of a positive-definite metric in the Hilbert space is again crucial in this formalism for showing reality of the entire spectra as well as unitary time evolution. Supersymmetric versions of $\mathcal{PT}$ symmetric and/or pseudo-Hermitian quantum systems have been studied in the literature [4–10, 13].
A general problem encountered in the study of a $\mathcal{PT}$ symmetric non-Dirac–Hermitian quantum system is the construction of the metric in the Hilbert space, which is essential for the calculation of expectation values of physical observables as well as different correlation functions. The description of a quantum system without the knowledge of the metric is thus incomplete, even though the complete spectrum and the associated eigenfunctions may be known explicitly. An approach taken in [11] was to consider Hilbert space with a pre-determined metric so that a non-Dirac–Hermitian quantum system can be constructed from a known Dirac–Hermitian Hamiltonian through isospectral deformation. Although the non-Dirac–Hermitian Hamiltonian constructed in this way is isospectral with the corresponding Dirac–Hermitian Hamiltonian, the difference may appear in the description of different correlation functions of these two quantum systems [14].

Several exactly solvable non-Dirac–Hermitian quantum systems with a complete description were constructed following this approach [11]. It is reassuring that the asymmetric $XXZ$ spin-chain [18], which has been studied extensively in the context of two species reaction-diffusion processes and Kardar–Parisi–Zhang-type growth phenomenon, was shown to be pseudo-Hermitian following this general approach [11]. A non-Dirac–Hermitian transverse-field Ising model appears as a special case of this general class of spin systems [14]. Further, the celebrated Dicke model [19] was shown to admit bound states for previously unexplored range of parameters [13]. These models may be considered as prototype examples of non-Dirac–Hermitian quantum systems with a complete description, which provide the ground for testing any idea related to the subject, including the validity of any approximate or numerical method.

The purpose of this paper is to generalize the approach of [11] to include fermionic degrees of freedom and construct non-Dirac–Hermitian supersymmetric quantum systems with a pre-determined metric in the Hilbert space. The general method involves a realization of the basic canonical (anti-)commutation relations involving both bosonic and fermionic degrees of freedom in terms of non-Dirac–Hermitian operators which are Hermitian with respect to a modified inner product, the Hilbert space. It may be noted that the Hilbert space of a supersymmetric system is $\mathbb{Z}_2$-graded and the metric can be expressed as a direct product of metrics corresponding to bosonic and fermionic degrees of freedom. The method described in [11] is valid for the purely bosonic sector of a supersymmetric system which involves canonical relations involving only bosonic degrees of freedom. A pseudo-Hermitian realization of the Clifford algebra for a pre-determined positive-definite metric is given in this paper. A general construction of pseudo-Hermitian supersymmetric systems with one or many degrees of freedom is presented. Further, it is shown that exactly solvable pseudo-Hermitian supersymmetric quantum systems can be constructed corresponding to each exactly solvable Dirac–Hermitian system. Examples of pseudo-Hermitian (i) non-relativistic Pauli Hamiltonian, (ii) super-conformal quantum system [20–22], and (iii) Calogero-type models [23–26] are also presented.

The paper is organized as follows. A pseudo-Hermitian realization of the Pauli matrices is presented in section 2. A generalization of these results to the Clifford algebra with an arbitrary number of elements is given in section 3. Section 4 contains discussions on pseudo-Hermitian supersymmetric quantum systems with one degree of freedom. It is shown that an exactly solvable pseudo-Hermitian quantum system can be constructed corresponding to each Dirac–Hermitian system with a shape-invariant potential. A general formulation of the pseudo-Hermitian supersymmetric quantum system with many degrees of freedom is presented in section 5. Examples of pseudo-Hermitian Pauli Hamiltonian, super-conformal quantum system and Calogero-type models are given in sections 6.1, 6.2 and 6.3, respectively. Finally, the results are summarized in section 7.
2. Pauli matrices: a non-Dirac–Hermitian realization

The Pauli matrices are Hermitian with the standard inner product $\langle \cdot | \cdot \rangle$ in the space of its eigenvectors $\mathcal{H}_D$. A vector space $\mathcal{H}_{\eta^f}$ that is endowed with the positive-definite metric $\eta^f_{\bullet \bullet}$, $\eta^f_{\bullet \bullet} := \exp(-\vec{\sigma} \cdot \hat{n} \phi)$, with $\hat{n} \cdot \hat{n} = 1$, $\langle \phi, n^a \rangle \in R$, $a = 1, 2, 3$, (1) and the inner product $\langle \langle \cdot | \cdot \rangle \rangle_{\eta^f_{\bullet \bullet}} = \langle \cdot | \eta^f_{\bullet \bullet} \cdot \rangle$ is introduced. The positivity of the metric follows from the fact that the eigenvalues of the matrix $\vec{\sigma} \cdot \hat{n}$ are real. With the introduction of the similarity operator $\rho^f$, $\rho^f := \sqrt{\eta^f_{\bullet \bullet}} = \exp\left(-\frac{\vec{\sigma} \cdot \hat{n} \phi}{2}\right)$, (2) a set of non-Dirac–Hermitian matrices may be introduced in terms of the Pauli matrices $\sigma^a$ as follows:

$$\Sigma^a := (\rho^f)^{-1} \sigma^a \rho^f$$

$$= \sum_{b=1}^{3} R^{ab} \sigma^b$$

$$R^{ab} := n^a n^b \left(1 - \cosh \phi \right) + \delta^{ab} \cosh \phi + i \epsilon^{abc} n^c \sinh \phi.$$ (3)

Note that $R^*_{ij} \neq R_{ij}$ and $\sum_{j=1}^{3} R^2_{ij} = 1$V, where $* \$ denotes complex conjugation. The matrices $\Sigma^a$ obey the same algebra satisfied by the Pauli matrices:

$$[\Sigma^a, \Sigma^b] = 2i \epsilon^{abc} \Sigma^c,$$

$$\{\Sigma^a, \Sigma^b\} = 2 \delta^{ab}$$

$$\Sigma_{\pm} := \frac{1}{2} (\Sigma^1 \pm i \Sigma^2),$$

$$\{\Sigma_-, \Sigma_+\} = 1, \quad \Sigma_\pm^2 = 0.$$ (4)

and are Hermitian in the vector space $\mathcal{H}_{\eta^f}$.

A few comments are in order at this point.

(i) The matrices $\Sigma^a$ depend on three real independent parameters. A general $2 \times 2$ non-Dirac–Hermitian matrix may be constructed in terms of $\Sigma^a$’s and the $2 \times 2$ identity matrix $I$,

$$\Sigma = p^0 I + \sum_{a=1}^{3} p^a \Sigma^a, \quad (p^0, p^a) \in R,$$ (5)

where $p^0$ and $p^a$ are four real parameters. The matrix depends on seven independent real parameters and is Hermitian in the vector space $\mathcal{H}_D$. The real eigenvalues $\lambda_{\pm}$ and the associated eigenvectors $v_{\pm}$ of $\Sigma^a$’s are

$$\lambda_{\pm} = p^0 \pm \sqrt{p^2}, \quad p \equiv \sum_{a=1}^{3} (p^a)^2,$$

$$v_{\pm} = N_{\pm} (\rho^f)^{-1} \left(\lambda_{\pm} + p^3 - p^0 \right) p^1 + i p^2.$$ (6)

The most general $2 \times 2$ matrix with complex elements depends on eight real parameters. The complex matrix $\Sigma$ admitting entirely real eigenvalues depends on seven real parameters and thus is more general than the one presented in [27]. Moreover, the method in constructing $\Sigma$ is completely different from the one followed previously [27]. The matrix $\Sigma$ may be used as the toy model for studying different ideas related to $\mathcal{T}$ symmetric and pseudo-Hermitian quantum systems.
(ii) Pseudo-Hermitian spin chain systems and Dicke models have been constructed previously \([11, 13, 14]\) by using \(\Sigma^a\)’s with the choice \(n^1 = n^2 = 0, n^3 = 1\). More general pseudo-Hermitian spin chain systems and Dicke models may be constructed by using \(\Sigma^a\)’s with arbitrary \(n^1, n^2, n^3\).

(iii) Any Dirac–Hermitian representation of the Pauli matrices is unitary equivalent to the standard representation, where \(\sigma^3\) is taken to be diagonal. The unitary transformations that relate different representations correspond to gauge transformations and within the formalism of Hermitian matrix models, real gauge potentials are constructed using the unitary matrix. On the other hand, the non-Dirac–Hermitian matrices \(\Sigma^a\)’s are equivalent to the Pauli matrices through a non-unitary similarity transformation. Such transformations can again be interpreted as gauge transformations with complex gauge potentials.

(iv) The pseudo-Hermitian matrix \(\Sigma\) may be used to construct a \(2 \times 2\) pseudo-unitary matrix \(D\) and its inverse \(D^{-1}\) in terms of seven real parameters as

\[
D = e^{\mathbf{i} \mathbf{E}}, \quad D^{-1} = e^{-\mathbf{i} \mathbf{E}}. \tag{7}
\]

The inner product \(\langle\langle v, u \rangle\rangle_{\eta^f}\) involving two arbitrary vectors \(u, v\) remains invariant under the pseudo-unitary transformation \(u \rightarrow u' = Du, v \rightarrow v' = Dv\), which can be shown using the relation \([28]\)

\[
D^\dagger = \eta^f D (\eta^f)^{-1}. \tag{8}
\]

Such pseudo-unitary matrices may have applications in the study of pseudo-Hermitian random matrix models \([28]\).

(v) An anti-linear \(\mathcal{PT}\) transformation may be introduced with the actions of the anti-linear operator \(T\) and the discrete symmetry operator \(\mathcal{P}\) on \(\sigma^1, \sigma^2, \sigma^3\) as follows:

\[
T : \quad i \rightarrow -i, \quad \sigma^a \rightarrow \sigma^a; \quad \mathcal{P} : \quad \sigma^1 \rightarrow \sigma^1 = \sigma^1 \cos \beta + \sigma^2 \sin \beta, \\
\sigma^2 \rightarrow \sigma^2 = \sigma^1 \sin \beta - \sigma^2 \cos \beta, \\
\sigma^3 \rightarrow \sigma^3 = \sigma^3, \quad 0 \leq \beta \leq 2\pi. \tag{9}
\]

It may be checked that \(\Sigma^3\), which appears in the description of a single-particle supersymmetric non-Dirac–Hermitian Hamiltonian in section 4, is \(\mathcal{PT}\) symmetric for \(\beta = 2 \tan^{-1} \frac{u^2}{\eta^2}\).

3. A pseudo-Hermitian realization of Clifford algebra

The real Clifford algebra with \(2N\) elements is described by the relations

\[
\{\xi_p, \xi_q\} = 2 \delta_{pq}, \quad p, q = 1, 2, \ldots, 2N. \tag{10}
\]

The complexification of the algebra can be achieved by introducing the fermionic variables \(\psi_i\) and their conjugates \(\psi^\dagger_i\) in \(\mathcal{H}_D\):

\[
\psi_i := \frac{1}{2} (\xi_i - i \xi_{N+i}), \quad \psi^\dagger_i := \frac{1}{2} (\xi_i + i \xi_{N+i}), \quad i, j = 1, 2, \ldots, N. \tag{11}
\]

These fermionic variables satisfy the complex Clifford algebra

\[
\{\psi_i, \psi_j\} = 0 = \{\psi^\dagger_i, \psi^\dagger_j\}, \quad \{\psi_i, \psi^\dagger_j\} = \delta_{ij}, \quad \{\psi_i, \psi_j\} = \delta_{ij}. \tag{12}
\]

and either \(\xi_p\) or \(\psi_i, \psi^\dagger_j\) may be used to construct Dirac–Hermitian supersymmetric systems.
A realization of the \( N(2N - 1) \) number of generators of the \( O(2N) \) group is given in terms of elements of the Clifford algebra \( \xi_p \) as follows:

\[
J_{pq} := \frac{1}{4}[\xi_p, \xi_q].
\]

These generators may be used to obtain a multi-parameter-dependent pseudo-Hermitian realization of the Clifford algebra, much akin to the case of Pauli matrices. A ‘complex rotation’ in the space of elements \( \xi_p \) is described in terms of the Hermitian operator \( \eta_p \),

\[
\eta_p^\pm := e^{-T}, \quad T := \frac{1}{2} \sum_{p,q=1}^{2N} t_{pq} J_{pq}, \quad t_{pq} = -t_{qp}.
\]

where \( t_{pq} \) are \( N(2N - 1) \) real parameters. The positive-definiteness of \( \eta_p^+ \) follows from the fact that the eigenvalues of \( T \) are real in the whole of the parameter space. It may be noted that the operator \( T \) can be expressed as a quadratic form of fermionic operators \( \psi_i (\psi_i^\dagger) \) which is known [29, 30] to admit entirely real spectra. Without loss of any generality, a particular form of \( \eta_p^+ \) is chosen in this paper for its simplicity,

\[
\eta_p^+ := \prod_{i=1}^{N} e^{\gamma_i J_i N + \gamma_i}, \quad \rho_p^+ := \sqrt{\eta_p^+} = \prod_{i=1}^{N} e^{\gamma_i J_i N - \gamma_i}, \quad \gamma_i \in R \forall i.
\]

It may be noted that ordering of the generators \( J_i N \) is not required in equation (15), since the commutators \([J_i N, J_j N]\) = 0 for any \( i \) and \( j \). A set of non-Dirac–Hermitian elements of the real Clifford algebra is introduced as follows:

\[
\Gamma_1^p := (\rho_p^+)^{-1}\xi_p\rho_p^+, \quad \Gamma_i := \xi_i \cosh \gamma_i + i\xi_{N+i} \sinh \gamma_i, \quad \Gamma_{N+i} := -i\xi_i \sinh \gamma_i + \xi_{N+i} \cosh \gamma_i.
\]

The elements \( \Gamma_p \) are Hermitian in \( \mathcal{H}_{\eta_p}^+ \). A pseudo-Hermitian realization of the generators of the group \( O(2N + 1) \) is facilitated by the introduction of the element \( \Gamma_{2N+1} \):

\[
\Gamma_{2N+1} := \Gamma_1 \Gamma_2 \ldots \Gamma_{2N-1} \Gamma_{2N} = \xi_1 \xi_2 \ldots \xi_{2N-1} \xi_{2N},
\]

which anti-commutes with all the \( \Gamma_p/\xi_p \)'s and squares to unity.

A set of fermionic operators \( \Psi_i \)'s and their adjoints \( \Psi_i^\dagger \) in \( \mathcal{H}_{\eta_p}^+ \) may be defined in terms of \( \Gamma_p \) as

\[
\Psi_i := \frac{1}{2} (\Gamma_i - i\Gamma_{N+i}) = e^{-\gamma_i} \psi_i, \quad \Psi_i^\dagger := \frac{1}{2} (\Gamma_i + i\Gamma_{N+i}) = e^{\gamma_i} \psi_i^\dagger,
\]

which satisfy the basic canonical anti-commutation relations:

\[
[\Psi_i, \Psi_j] = 0 = \{ \Psi_i^\dagger, \Psi_j^\dagger \}, \quad \{ \Psi_i, \Psi_j^\dagger \} = \delta_{ij}.
\]

The metric \( \eta_p^+ \) and the similarity operator \( \rho_p^+ \) are expressed in terms of \( \psi_i, \psi_i^\dagger \) as

\[
\eta_p^+ = \prod_{i=1}^{N} e^{-2\gamma_i} \psi_i \sum_{i=1}^{N} e^{-2\gamma_i} \psi_i^\dagger.
\]

The total fermion number operator \( N_f \) has identical expression

\[
N_f = \sum_{i=1}^{N} \psi_i^\dagger \psi_i = \sum_{i=1}^{N} \Psi_i^\dagger \Psi_i.
\]

\[
\sum_{i=1}^{N} \psi_i^\dagger \psi_i = \sum_{i=1}^{N} \Psi_i^\dagger \Psi_i. \]
in \( \mathcal{H}_D \) as well as in \( \mathcal{H}_{\eta'} \). However, a general eigenstate \( |f_1, \ldots, f_i, \ldots, f_N \rangle_{\mathcal{H}_{\eta'}} \) of \( N \) in \( \mathcal{H}_D \) is related to the corresponding state \( |f_1, \ldots, f_i, \ldots, f_N \rangle_{\mathcal{H}_{\eta'}} \) in the Hilbert space \( \mathcal{H}_{\eta'} \) through the following relation:

\[
|f_1, \ldots, f_i, \ldots, f_N \rangle_{\mathcal{H}_{\eta'}} = \prod_{k=1}^{N} e^{i k f_i} |f_1, \ldots, f_i, \ldots, f_N \rangle_{\mathcal{H}_D}, \quad f_i = 0, 1 \forall i. \tag{22}
\]

The \( 2^N \) states \( |f_1, \ldots, f_i, \ldots, f_N \rangle_{\mathcal{H}_{\eta'}} \) form a complete set of orthonormal states in \( \mathcal{H}_{\eta'} \), while \( |f_1, \ldots, f_i, \ldots, f_N \rangle_{\mathcal{H}_D} \) constitute a complete set of orthonormal states in \( \mathcal{H}_D \). The action of \( \Psi_i (\Psi_i^\dagger) \) on \( |f_1, \ldots, f_i, \ldots, f_N \rangle_{\mathcal{H}_{\eta'}} \) is identical to that of \( \psi_i (\psi_i^\dagger) \) on \( |f_1, \ldots, f_i, \ldots, f_N \rangle_{\mathcal{H}_D} \). In particular,

\[
\Psi_i |f_1, \ldots, f_i, \ldots, f_N \rangle_{\mathcal{H}_{\eta'}} = 0, \quad \text{if } f_i = 0,
= |f_1, \ldots, 0, \ldots, f_N \rangle_{\mathcal{H}_{\eta'}}, \quad \text{if } f_i = 1,
\Psi_i^\dagger |f_1, \ldots, f_i, \ldots, f_N \rangle_{\mathcal{H}_{\eta'}} = 0, \quad \text{if } f_i = 1,
= |f_1, \ldots, 1, \ldots, f_N \rangle_{\mathcal{H}_{\eta'}}, \quad \text{if } f_i = 0. \tag{23}
\]

Either \( \Gamma_i \) or \( \Psi_i, \Psi_i^\dagger \) may be used to construct pseudo-Hermitian supersymmetric quantum systems.

The known representation [31] of the elements \( \xi_p \) in terms of \( 2^N \times 2^N \) matrices can be used to find the corresponding representation for \( \Gamma_p, \Psi_i, \Psi_i^\dagger \). In general, the matrices \( \Gamma_p \) depend on \( N \) real parameters \( \gamma_p \). A \( 2^N \times 2^N \) pseudo-Hermitian matrix depending on \( 2^N \times N \) real parameters may be constructed in terms of \( \Gamma_p \)'s and the \( 2^N \times 2^N \) identity matrix \( I \) as

\[
\Gamma = a^0 I + \sum_{p=1}^{2^N} a_p^\dagger \Gamma_p + \sum_{p_1 < p_2}^{2^N} a_{p_1 p_2} \Gamma_{p_1} \Gamma_{p_2} + \cdots
+ \sum_{p_1 = p_2 < p_3}^{2^N} a_{p_1 p_2 p_3} \Gamma_{p_1} \Gamma_{p_2} \Gamma_{p_3} + \cdots + a^{2^N} \Gamma_1 \Gamma_2 \cdots \Gamma_{2^N}, \tag{24}
\]

where \( a^0, a_p^\dagger, a_{p_1 p_2} \ldots, a^{2^N} \) are \( 2^N \) real parameters. The eigenvalues of the matrix \( \Gamma \) are real by construction. A complete set of orthonormal eigenvectors can also be constructed in \( \mathcal{H}_{\eta'} \).

It appears that \( \Gamma \) is not the most general \( 2^N \times 2^N \) pseudo-Hermitian matrix having entirely real eigenvalues. For example, a non-Dirac–Hermitian matrix \( \tilde{\Gamma} \) depending on \( 2^N + 3N \) real parameters may be constructed by replacing \( \Gamma_p \rightarrow \tilde{\Gamma}_p \) in equation (24), where

\[
\tilde{\Gamma}_p := (\hat{\rho}^f)^{-1} \xi_p \hat{\rho}^f
\]

\[
\hat{\rho}^f := \prod_{i=1}^{N} \exp \left( -\frac{\sigma \cdot \hat{n}_i \phi_i}{2} \right), \quad \hat{n}_i \cdot \hat{n}_i = 1 \forall i, \tag{25}
\]

and it is understood that \( \xi_p \)'s are realized in terms of Pauli matrices [31]. The matrix \( \tilde{\Gamma} \) thus constructed will have entirely real eigenvalues with a complete set of orthonormal vectors in \( \mathcal{H}_{\eta'} \). Other possibilities including a more general operator \( \Gamma \) appearing in the definition of \( \eta_i \), and with more number of real parameters also exist, which will not be pursued in this paper.

A comment regarding the \( \mathcal{PT} \) symmetry in the space of the elements of the Clifford algebra is in order before the end of this section. An anti-linear \( \mathcal{PT} \) transformation may be introduced with the action of the anti-linear operator \( \mathcal{T} \) and the discrete symmetry \( \mathcal{P} \) on the
elements $\xi_p$ as
\[
T : i \rightarrow -i, \quad \xi_p \rightarrow \xi_p \forall p;
\]
\[
P : \xi_i \rightarrow \tilde{\xi}_i = \xi_i \cos \beta + \xi_{N+i} \sin \beta, \\
\xi_{N+i} \rightarrow \tilde{\xi}_{N+i} = \xi_i \sin \beta - \xi_{N+i} \cos \beta, \quad 0 \leq \beta \leq 2\pi.
\]

The action of the $\mathcal{PT}$ transformation on the fermionic variables $\psi_i$ is as follows:
\[
P : \psi_i \rightarrow e^{-i\beta}\psi_i, \quad \psi_i^\dagger \rightarrow e^{i\beta}\psi_i^\dagger,
\]
\[
T : \psi_i \rightarrow \psi_i^\dagger, \quad \psi_i^\dagger \rightarrow \psi_i,
\]
\[
\mathcal{PT} : \psi_i \rightarrow e^{i\beta}\psi_i, \quad \psi_i^\dagger \rightarrow e^{-i\beta}\psi_i^\dagger,
\]
where $\beta$ appears as a phase which may be fixed at some specific value depending on the physical requirements. A supersymmetric Hamiltonian in the linear realization of the super-algebra contains bilinear terms involving fermionic variables. It may be noted that the bilinear terms of the form $\psi_i^\dagger\psi_j$ are $\mathcal{PT}$ invariant for any $\beta$. However, bilinear terms of the form $\psi_i\psi_j, \psi_i^\dagger\psi_j^\dagger$ are $\mathcal{PT}$ invariant for $\beta = 0$. A particular nonlinear realization [25] of the super-algebra involves $\Gamma_{2N+1}$ which is also $\mathcal{PT}$ invariant for any $\beta$. This can be checked by expressing $\Gamma_{2N+1}$ in terms of fermionic variables as
\[
\Gamma_{2N+1} = (-1)^N \prod_{i=1}^{N} (2\psi_i^\dagger \psi_i - 1).
\]
This provides a framework for constructing a $\mathcal{PT}$ symmetric supersymmetric non-Dirac–Hermitian Hamiltonian.

4. Single-particle non-Dirac–Hermitian SUSY

The supercharges are introduced as follows:
\[
Q_1 = [p + iW_1(x)]\Sigma_-, \quad Q_2 = [p - iW_2(x)]\Sigma_+,
\]
\[
p = -\frac{d}{dx}, \quad W_1(x) = \frac{dW_1(x)}{dx}, \quad W_2(x) = \frac{dW_2(x)}{dx},
\]
where $W_1, W_2$ are two arbitrary functions. It may be noted that $Q_1$ is not the adjoint of $Q_2$ in $\mathcal{H}_D$. The Hamiltonian that may be constructed in terms of a quadratic form of these supercharges is also non-Hermitian in $\mathcal{H}_D$:
\[
H = \{Q_1, Q_2\} = \Pi^2 + (W'_+)^2 + W''_+ \sum_{b=1}^{3} R^{3b}\sigma^b
\]
\[
W_\pm = \frac{1}{2} [W_1(x) \pm W_2(x)], \quad W'_\pm = \frac{dW_\pm}{dx}, \quad \Pi = p + iW'_+.
\]
The generalized momentum operator $\Pi$ includes an imaginary gauge potential corresponding to the real part of $W'$. It may be noted that such imaginary gauge potentials are relevant in the context of metal-insulator transitions or depinning of flux lines from extended defects in type-II superconductors [32]. The imaginary gauge potential also appears in the study of unzipping of DNA [33]. Apart from the imaginary gauge potential containing in $W'_+$, the non-Dirac-hermiticity in $H$ is also introduced through the complex parameters $R^{3b}$ and complex functions $W_{1,2}$. The appearance of $R^{3b}$ is due to a non-standard non-Dirac–Hermitian representation.
of the Pauli matrices which may be interpreted as arising due to a gauge-transformation with complex gauge potentials.

The Hamiltonian is not in the diagonal form due to a non-standard representation of the Pauli matrices. The eigenvalue equation of the non-Dirac–Hermitian $H$ is thus given in terms of a set of two coupled second-order differential equations. Such coupled differential equations appear in a variety of physical situations [34]. The Hamiltonian can be brought to a diagonal form by defining

$$
\eta_+ := \eta_b^b \otimes \eta_f^f, \quad \eta_- := e^{-2\text{Re}(W_+)} \otimes \eta_f^f,
$$

$$
\rho := \rho_b^b \otimes \rho_f^f, \quad \rho^- := e^{-\text{Re}(W_-)}, \quad U := e^{-i\text{Im}(W_-)},
$$

(31)

and using the similarity transformation

$$
h := (U\rho) H (U\rho)^{-1}
$$

$$
= \rho^3 + (W_+^\prime + W_-^\prime)^2 + W_{+}^{\prime\prime}\sigma_3,
$$

(32)

where $\text{Re}(f)/\text{Im}(f)$ corresponds to the real/imaginary part of $f$. The complex gauge potential has also been removed by the combined use of the similarity operator $\eta_b^b$ and the unitary operator $U$. In general, the Hamiltonian $h$ is non-Dirac–Hermitian due to the appearance of complex $W_+^\prime$ and $W_-^\prime$. A number of supersymmetric systems with complex superpotentials $W_+^\prime$ have been shown to admit entirely real spectra [4–10, 13]. However, the metric or the inner product in the Hilbert space is not known for most of the cases. There may exist similarity transformations which map $h$ to a Dirac–Hermitian Hamiltonian for specific choices of $W_1$ and $W_2$. However, such an investigation is beyond the scope of this paper and henceforth, $W_+$ is considered to be real so that $h$ is Dirac–Hermitian. It may be noted that real $W_+$ can be obtained even for complex $W_1$ and $W_2$. In particular, $W_1$ and $W_2$ may be chosen as

$$
W_1(x) = W(x) + \chi(x) + i\theta_1(x), \quad W_2(x) = W(x) - \chi(x) + i\theta_2(x),
$$

(33)

where $W(x)$, $\chi(x)$, $\theta_1(x)$ and $\theta_2(x)$ are four independent real functions of their argument. A real $W_+$ is obtained for $\theta_1 = -\theta_2 \equiv \theta$. Further, with the choice of $W$ as any shape-invariant superpotential of the Dirac–Hermitian quantum system [35], the non-Dirac–Hermitian Hamiltonian $H$ becomes exactly solvable with entirely real spectra and unitary time evolution in $H_{\eta_+}$. It may be noted that the metric $\eta_b^b = e^{-2\chi}$ solely depends on the choice of $\chi$ and does not change for different choices of $W, \theta_1, \theta_2$. Consequently, appropriate choices of $\chi$ would result in a positive-definite and bounded metric.

The eigenfunctions $\Phi_n$ of $h$ with associated real eigenvalues $E_n$, satisfying the eigenvalue equation,

$$
h \Phi_n = E_n \Phi_n, \quad \Phi_n \equiv \begin{pmatrix} \Phi^+_n \\ \Phi^-_n \end{pmatrix},
$$

(34)

may be used to construct the eigenfunctions of $H$. In particular, the eigenfunctions $\chi_n$ of $H$ with eigenvalues $E_n$ are

$$
\chi_n = (U\rho)^{-1} \Phi_n
$$

$$
= e^{\chi(x)+i\theta(x)} \begin{pmatrix} \Delta^+ \Phi^+_n + n^+ \sinh \left( \frac{\phi}{2} \right) \Phi^+_n \\ n^+ \sinh \left( \frac{\phi}{2} \right) \Phi^+_n + \Delta^- \Phi^-_n \end{pmatrix},
$$

$$
\Delta^\pm = \cosh \left( \frac{\phi}{2} \right) \pm n^3 \sinh \left( \frac{\phi}{2} \right), \quad n^\pm = n^1 \pm in^2.
$$

(35)
It may be noted that $\Phi_n$ constitute a complete set of orthonormal eigenfunctions in $\mathcal{H}_D$, while $\chi_n$ are a complete set of orthonormal eigenstates in $\mathcal{H}_\eta$.

5. Many-particle non-Dirac–Hermitian SUSY

The metric $\eta_\xi$ is chosen as

\[ \eta^b_\xi := e^{-2(\delta B + \text{Re}(W_-))}, \quad \eta^f_\xi := \eta^b_\xi \otimes \eta^f_\xi, \quad \delta \in \mathbb{R}, \quad (36) \]

where $\eta^f_\xi$ is given by equation (20) and $W_\pm$ is as defined in equation (30) with the understanding that $W_1, W_2$ are now functions of the $N$ bosonic coordinates. The operator $\hat{B}$ acts on the bosonic coordinates only. A specific choice of $\hat{B}$ will be made in the next section while presenting a few examples. It is assumed that the bosonic coordinates $x_i$ and the momenta $p_i$ are not Hermitian in $\mathcal{H}_\eta$. For the type of operator $\hat{B}$ that will be considered in this paper, A set of Hermitian coordinates $X_i$ and momenta $P_i$ in $\mathcal{H}_\eta$ may be introduced as follows:

\[ X_i = \rho^{-1} x_i \rho, \quad P_i = \rho^{-1} p_i \rho, \quad \rho := \sqrt{\eta^e}. \quad (37) \]

Although the operators $X_i, P_i$ are non-Dirac–Hermitian, they satisfy the basic canonical commutation relations $[X_i, P_j] = \delta_{ij}$. Further, the length in the momentum space as well as in the coordinate space remains invariant under the transformations defined by equation (37).

The supercharges are introduced as follows:

\[ \tilde{Q}_1 = \sum_{i=1}^{N} e^{-\gamma_i} \psi_i (P_i + i W_{1,i}), \]

\[ \tilde{Q}_2 = \sum_{i=1}^{N} e^{\gamma_i} \psi^\dagger_i (P_i - i W_{2,i}), \]

\[ P_i = -i \frac{\partial}{\partial X_i}, \quad W_{1,i} = \frac{\partial W_1}{\partial X_i}, \quad W_{2,i} = \frac{\partial W_2}{\partial X_i}, \quad (38) \]

which are not adjoint of each other in $\mathcal{H}_D$. The supersymmetric Hamiltonian that may be constructed by using these supercharges reads

\[ \tilde{H} := \{ Q_1, Q_2 \} \]

\[ = \sum_{i=1}^{N} [\Pi_i^2 + (W_{e,i})^2] + 2 \sum_{i,j=1}^{N} e^{\gamma_i - \gamma_j} W_{e,i,j} \psi_i^\dagger \psi_j, \]

\[ W_{\pm,i,j} := \frac{1}{2} (W_{1,i,j} \pm W_{2,i,j}), \quad \Pi_i := P_i + i W_{e,-i}, \quad W_{e,i,j} := \frac{\partial^2 W_e}{\partial X_i \partial X_j}. \quad (39) \]

In general, the Hamiltonian $\tilde{H}$ is non-Dirac–Hermitian. Apart from the complex superpotentials $W_1, W_2$, the non-Dirac–Hermitian interactions are introduced in $\tilde{H}$ through imaginary gauge potentials containing in the generalized momentum operators $\Pi_i$ and fermion operators $\Psi_i$. It is worth re-emphasizing that imaginary gauge potentials appear in the study of a diverse branch of physics including metal–insulator transitions or depinning of flux lines from extended defects in type-II superconductors [32] and unzipping of DNA [33]. The non-Dirac–Hermitian bosonic potentials may appear in $\tilde{H}$ depending on specific physical situations and a few such explicit examples will be discussed in the next section.

The decomposition of $W_1, W_2$ in equation (33) is used in this section with the understanding that $W, \chi, \theta_1, \theta_2$ are real functions of the $N$ bosonic coordinates. The metric
The operators $\Pi_i$ and the functions $W_{+,i}$ may be rewritten as
\[
\eta^b_+ = e^{-2(\hat{B}=\chi)} , \quad \Pi_i = P_i + i \chi_i - \frac{1}{2}(\theta_{1,i} - \theta_{2,i}),
\]
\[
W_{+,i} = W_i + \frac{i}{2}(\theta_{1,i} + \theta_{2,i}) , \quad F_i = \frac{\partial F}{\partial X_i} , \quad F \equiv \{W, \chi, \theta_1, \theta_2\}.
\]

The Hamiltonian $\hat{H}$ becomes Hermitian in the Hilbert space $\mathcal{H}_{\eta^b_+}$ provided the following condition is satisfied:
\[
\theta_1 = -\theta_2 \equiv \theta.
\]

The hermiticity can be checked by re-expressing $\hat{H}$ as
\[
\hat{H} = \sum_{i=1}^N \left[ \Pi_i^2 + (W_i)^2 - W_i \right] + 2 \sum_{i,j=1}^N W_{ij} \psi_i^\dagger \psi_j.
\]

Specific choices of $W$ for which exactly solvable many-particle supersymmetric quantum systems are known may be used to construct iso-spectral non-Dirac–Hermitian quantum systems $\hat{H}$. A set of orthonormal eigenfunctions $\chi_n$ of $\hat{H}$ in $\mathcal{H}_{\eta^b_+}$ may be constructed from the orthonormal eigenfunctions $\Phi_n$ of $H$ in $\mathcal{H}_D$ by using the relation $\chi_n = (U\rho)^{-1} \Phi_n$.

### 6. Examples

A few specific examples of non-Dirac–Hermitian supersymmetric systems with complex bosonic potentials are considered in this section. The operator $\hat{B}$ appearing in the metric $\eta^b_+$ in equation (36) is chosen as
\[
\hat{B} = L_{12} = x_1 p_2 - x_2 p_1.
\]

The presence of $L_{12}$ in the metric allows the non-Dirac–Hermitian bosonic potential in the Hamiltonian [11]. In particular, the co-ordinates $x_1, x_2$ and the momenta $p_1, p_2$ are not Hermitian in $\mathcal{H}_{\eta^b_+}$. A new set of canonical conjugate operators which are Hermitian in the Hilbert space $\mathcal{H}_{\eta^b_+}$ may be introduced by using relation (37) as follows [11, 12]:
\[
X_1 = x_1 \cosh \delta + i x_2 \sinh \delta , \quad X_2 = -i x_1 \sinh \delta + x_2 \cosh \delta , \quad X_i = x_i \text{ for } i > 2
\]
\[
P_1 = p_1 \cosh \delta + i p_2 \sinh \delta , \quad P_2 = -i p_1 \sinh \delta + p_2 \cosh \delta , \quad P_i = p_i \text{ for } i > 2.
\]

It may be noted that $L_{12} = X_1 P_2 - X_2 P_1 = L_{12}$ is Hermitian both in $\mathcal{H}_D$ and $\mathcal{H}_{\eta^b_+}$. This ensures that $\eta^b_+$ defined in equation (36) is positive-definite. Without loss of any generality, $\theta$
is chosen as zero, since it can always be rotated away by using the unitary operator \( U = e^{-i\theta} \).

The generalized momentum operators \( \Pi_i \) now reads

\[
\Pi_i = P_i + i\chi_i.
\] (46)

The imaginary gauge potentials \( \chi_i \) can be removed from \( \Pi_i \) by using a non-unitary similarity transformation. However, it should be noted that \( P_i \) and \( \Pi_i \) act on different Hilbert spaces.

An anti-linear \( \mathcal{P} T \) transformation for the bosonic coordinates may be introduced as follows [12]:

\[
\mathcal{P} : x_1 \leftrightarrow x_2, \quad p_1 \leftrightarrow p_2, \quad (x_i, p_i) \rightarrow (x_i, p_i) \quad \forall i > 2;
\]

\[
T : i \rightarrow -i, \quad x_i \rightarrow x_i, \quad p_i \rightarrow -p_i;
\]

\[
\mathcal{P} T : X_1 \leftrightarrow X_2, \quad P_1 \leftrightarrow -P_2, \quad (X_i, P_i) \rightarrow (X_i, -P_i) \quad \forall i > 2;
\]

\[
\mathcal{P} T : \Pi_1 \leftrightarrow -\Pi_2, \quad \Pi_i \rightarrow -\Pi_i \quad \forall i > 2.
\] (47)

The transformations in the last two lines of equation (47) are derived from the defining relations in the first two lines. Further, the transformation of \( \Pi_i \) under \( \mathcal{P} T \), as stated in the last line of (47), is only valid for \( P \)-symmetric \( \chi \). It may be noted that although \( P_2 \Pi_2^2 \) or \( P_2^2 \Pi_2 \) are not \( \mathcal{P} T \)-symmetric individually, the combinations \( P_2^2 + P_2^2 \) and \( \Pi_2^2 + \Pi_2^2 \) are always \( \mathcal{P} T \)-symmetric.

6.1. Pauli Hamiltonian

The supercharge \( Q \) is introduced as

\[
Q = \sum_{a=1}^{3} \Pi_a \Sigma^a, \quad \Pi_a = \Pi_a - A_a,
\] (48)

where the vector potential \( \vec{A} \) with the components \( A_1, A_2, A_3 \) is given by

\[
A_1 = \frac{B}{2} x_2 = \frac{B}{2} (ix_1 \sinh \delta - x_2 \cosh \delta),
\]

\[
A_2 = \frac{B}{2} x_1 = \frac{B}{2} (x_1 \cosh \delta + ix_2 \sinh \delta), \quad A_3 = 0.
\] (49)

Although the vector potential \( \vec{A} \) is non-Dirac–Hermitian and contains an imaginary part, the magnitude \( B \) of the corresponding magnetic field is a real constant and points in the \( z \)-direction. The operators \( \Pi_a, \Sigma^a \) and the supercharge \( Q \) are Hermitian in \( \mathcal{H}_\eta \). A non-Dirac–Hermitian Hamiltonian is introduced as

\[
\hat{H} := Q^2 = \sum_{a=1}^{3} \Pi_a^2 - iB \Sigma^3,
\] (50)

which is also \( \mathcal{P} T \) symmetric with the transformation under \( \mathcal{P} \) as given in equations (9) and (47), while the action of \( T \) is defined as \( T : i \rightarrow -i, x_1 \leftrightarrow -x_2, p_{1,2} \rightarrow -p_{1,2}, \sigma^{1,2,3} \rightarrow -\sigma^{1,2,3} \). The Hamiltonian \( \hat{H} \) is isospectral with the non-relativistic Dirac–Hermitian Pauli Hamiltonian [36], which can be shown by using a similarity transformation with \( \rho \) as the similarity operator.

6.2. Super-conformal quantum system

The superpotential \( W \) to describe a super-conformal quantum system is chosen as

\[
W = -\lambda \ln r, \quad r = \left( \sum_{i=1}^{N} x_i^2 \right)^{\frac{1}{2}} = \left( \sum_{j=1}^{N} X_j^2 \right)^{\frac{1}{2}},
\] (51)
which is Dirac–Hermitian as well as Hermitian in $\mathcal{H}_\eta$. With this choice of the superpotential,

$$W_i = -\frac{\lambda X_i}{r^2}, \quad W_{ij} = \frac{\lambda}{r^2} (2X_iX_j - \delta_{ij}r^2), \quad \sum_{i=1}^N W_{ii} = -\frac{\lambda(N-2)}{r^2},$$  \hspace{1cm} (52)

and the non-Dirac–Hermitian supersymmetric Hamiltonian reads

$$\hat{H} = \sum_{i=1}^N \Pi_i^2 + \frac{\lambda(\lambda + N - 2)}{r^2} + \frac{2\lambda}{r^4} \sum_{i,j=1}^N e^{\gamma_i \gamma_j} (2X_iX_j - \delta_{ij}r^2) \psi_i^\dagger \psi_j.$$  \hspace{1cm} (53)

The purely bosonic Hamiltonian is obtained by projecting $\hat{H}$ in the fermionic vacuum $|0\rangle_{\eta}$.

The resulting Hamiltonian with a further choice of $\chi = 0$ is Dirac–Hermitian and has been studied [20] extensively as a model of conformal quantum system. The Hamiltonian $\hat{H}$ reduces to the super-conformal quantum Hamiltonian [21] in the limit $\phi = \chi = 0, \gamma_i = 0 \forall i$. The Hamiltonian $\hat{H}$ is invariant under a combined $\mathcal{PT}$ operation as defined in equations (27) and (47).

The Hamiltonian $\hat{H}$ along with $D$ and $K$,

$$D = -\frac{1}{4} \sum_{i=1}^N (X_i \Pi_i + \Pi_i X_i), \quad K = \frac{1}{4} \sum_{i=1}^N X_i^2,$$  \hspace{1cm} (54)

satisfies the $O(2, 1)$ algebra which appears as a bosonic sub-algebra of the $OSp(2|2)$ supergroup. The complete algebra of the $OSp(2|2)$ can be realized by defining

$$S_1 = \frac{1}{2} \sum_{i=1}^N e^{-\gamma_i} \psi_i X_i, \quad S_2 = \frac{1}{2} \sum_{i=1}^N e^{\gamma_i} \psi_i X_i, \quad Y = \frac{1}{4} \sum_{i=1}^N [\psi_i^\dagger, \psi_i].$$  \hspace{1cm} (55)

The non-Dirac–Hermitian generators $\hat{H}, D, S_1$ and $S_2$ of $OSp(2|2)$ are Hermitian in $\mathcal{H}_\eta$. The Dirac–Hermitian generators $K$ and $Y$ are also Hermitian in $\mathcal{H}_\eta$.

The zero energy ground-state wavefunction of $\hat{H}$,

$$\psi_0 = e^{\lambda r^2} |0\rangle_{\eta \eta},$$  \hspace{1cm} (56)

is not even plane-wave normalizable in $\mathcal{H}_\eta$, for any choices of $\chi$. Following the prescription [20, 21], a compact operator of the sub-group $O(2, 1) \times U(1)$ of $OSp(2|2)$ may be chosen to study the time evolution of the system. The relevant non-Dirac–Hermitian Hamiltonian,

$$\hat{H}_\eta = \hat{H} + K \pm Y,$$  \hspace{1cm} (57)

has a complete description including entirely real spectra and unitary time evolution in $\mathcal{H}_\eta$. In fact, the Hamiltonian $\hat{H}$ is isospectral with the supersymmetric quantum system with inverse-square and harmonic potentials [21]. It is worth mentioning here that the dynamical supersymmetry of $\hat{H}$ is $SU(1, 1|2)$ for $N = 2$ with the following non-Dirac–Hermitian realization of the $SU(2)$ generators

$$J_1 := e^{-\gamma_i \gamma_j} \psi_i \psi_j, \quad J_2 := e^{\gamma_i \gamma_j} \psi_i^\dagger \psi_j^\dagger, \quad J_3 := \psi_i \psi_j + \psi_j \psi_i - 1.$$  \hspace{1cm} (58)

The relevant discussions for a Dirac–Hermitian system [21, 22] may be generalized for the non-Dirac–Hermitian Hamiltonian $\hat{H}$ in a straightforward way.

A comment is in order before the end of this section. Normalizable zero energy eigenfunctions of $\hat{H}$ in $\mathcal{H}_\eta$ exist for specific choices of $\chi$. For example, with the choice of $\chi = -\frac{\lambda}{r^2}, \kappa \in \mathbb{R}$, the normalizable zero energy eigenfunctions of $\hat{H}$ in $\mathcal{H}_\eta$ are

$$\langle \psi_1 = e^{\frac{\lambda}{r^2} r^k} |0\rangle_{\eta \eta}, \quad \lambda > 0,$$

$$\langle \psi_2 = e^{\frac{\lambda}{r^2} r^{-\kappa}} |N\rangle_{\eta \eta}, \quad \lambda < 0,$$  \hspace{1cm} (59)
where $|N\rangle_{H_D}$ is the conjugate vacuum satisfying $\psi_i^{\dagger}|N\rangle_{H_D} = 0 \forall i$. The supersymmetry is preserved for the entire range of the parameter $\lambda$. It may be noted that a scale is introduced in the system for $\kappa \neq 0$.

6.3. Calogero-type systems

The superpotential $W$ is chosen as

$$W(X_1, X_2, \ldots, X_N) = -\ln G(X_1, X_2, \ldots, X_N) + \frac{1}{2} \sum_{i=1}^{N} X_i^2,$$

(60)

where $G$ is a homogeneous function of degree $d$:

$$\sum_{i=1}^{N} X_i \frac{\partial G(X_1, \ldots, X_N)}{\partial X_i} = dG(X_1, \ldots, X_N),$$

(61)

$$\sum_{i=1}^{N} x_i \frac{\partial G(x_1, \ldots, x_N)}{\partial x_i} = dG(x_1, \ldots, x_N).$$

The homogeneity condition on $G$ is to ensure that the many-body interaction scales inverse-squarely [25]. Rational Calogero-type models corresponding to different root systems may be introduced for specific choices of $G$ [25]. The $A_{N+1}$ Calogero-type model is obtained for the choice

$$G = \prod_{i<j}^{N} (X_i - X_j)^{\lambda}.$$  

(62)

The non-Dirac–Hermitian Hamiltonian reads

$$\tilde{H} = \sum_{i=1}^{N} p_i^2 + \lambda(\lambda - 1) \sum_{i \neq j}^{N} x_i^2 + \sum_{i=1}^{N} x_i^2$$

$$+ 2\lambda \sum_{i \neq j}^{N} x_i^{\lambda-2} (\psi_i^{\dagger}\psi_i - e^{\gamma} - e^{-\gamma}\psi_i^{\dagger}\psi_j) + 2 \sum_{i=1}^{N} \psi_i^{\dagger}\psi_i - N - \lambda N(N-1),$$

(63)

$$X_{12} = (x_1 - x_2) \cosh \delta + i(x_1 + x_2) \sinh \delta,$$

$$X_{1j} = x_1 \cosh \delta + i x_2 \sinh \delta - x_j, \quad j > 2,$$

$$X_{2j} = -i x_1 \sinh \delta + x_1 \cosh \delta - x_j, \quad j > 2,$$

$$X_{ij} = x_i - x_j, \quad (i, j) > 2.$$  

Unlike the rational Calogero model [23], the many-body inverse-square interaction term in $\tilde{H}$ is neither invariant under translation nor singular for $x_1 = x_i, i > 1$ and $x_2 = x_i, i > 2$. Further, the permutation of the bosonic and fermionic coordinates $x_i \leftrightarrow x_j, \psi_i \leftrightarrow \psi_j, \psi_i^{\dagger} \leftrightarrow \psi_j^{\dagger}$ does not keep $\tilde{H}$ invariant. However, the Hamiltonian is invariant under a combined $PT$ operation as defined in equations (27) and (47).

The Hamiltonian $\tilde{H}$ can be mapped to the Dirac–Hermitian rational $A_{N+1}$ Calogero model through a similarity transformation and thus, these models are isospectral. A word of caution is in order at this point. The rational $A_{N+1}$ Calogero model has been solved for boundary conditions by both excluding [23] and including [26] the singular points from the configuration space. The Hamiltonian $\tilde{H}$ has $(N - 3)(N - 2)$ number of less singular points compared to the standard Calogero model [23] due to the non-singular points at $x_1 = x_i, i > 1$ and $x_2 = x_i, i > 2$. Thus, identical boundary conditions should be used for both the non-Dirac–Hermitian and Dirac–Hermitian systems in order to claim that these systems are isospectral.
7. Conclusions

A general prescription to construct non-Dirac–Hermitian supersymmetric quantum system that is isospectral with a Dirac–Hermitian Hamiltonian has been given. The basic canonical (anti-)commutation relations defining the supersymmetric system have been realized in terms of non-Dirac–Hermitian operators which are Hermitian in a Hilbert space that is endowed with a pre-determined positive-definite metric. The canonical relations involving bosonic degrees of freedom have been realized following the method described in [11]. A pseudo-Hermitian realization of the Clifford algebra has been given which has been used to construct supersymmetric quantum systems. It has been shown that exactly solvable non-Dirac–Hermitian supersymmetric quantum systems which are isospectral with a known exactly solvable Dirac–Hermitian system can always be constructed. Specific examples of the non-Dirac–Hermitian nonrelativistic Pauli Hamiltonian, superconformal quantum system and supersymmetric Calogero-type models have been presented.

The Pauli matrices appear in diverse branches of physics. The pseudo-Hermitian realization of these matrices may be used to construct non-Dirac–Hermitian quantum systems admitting entirely real spectra. Some of the possibilities include more general pseudo-Hermitian spin-chains, Dicke models, random matrix models, etc. Further, the fermionic operators constructed in this paper may be interpreted as standard fermionic operators with imaginary gauge potentials and may have applications in condensed matter systems.

References

[1] Bender C M 2005 Contemp. Phys. 46 277
Bender C M and Boettcher S 1998 Phys. Rev. Lett. 80 5243
[2] Mostafazadeh A 2010 Int. J. Geom. Meth. Mod. Phys. 7 1191
Mostafazadeh A 2002 J. Math Phys. 43 205
Mostafazadeh A 2002 J. Math Phys. 43 2814
Mostafazadeh A 2002 J. Math Phys. 43 3944
[3] Scholtz F G, Geyer H B and Hahne F J W 1992 Ann. Phys. 213 74
[4] Mostafazadeh A 2002 Nucl. Phys. B 640 419
Mostafazadeh A 2004 J. Phys. A: Math. Gen. 37 10193
[5] Quesne C 2008 J. Phys. A: Math. Theor. 41 244022
[6] Bagchi B, Banerjee A, Caliceti E, Cannata F, Geyer H B, Quesne C and Znojil M 2005 Int. J. Mod. Phys. A 20 7107
[7] Bagchi B and Quesne C 2010 J. Phys. A: Math. Theor. 43 305301
Bagchi B, Mallik S and Quesne C 2002 Mod. Phys. Lett. A 17 1651
Bagchi B, Mallik S and Quesne C 2001 Int. J. Mod. Phys. A 16 2859
Quesne C, Bagchi B, Mallik S, Bila H, Jakubsky V and Znojil M 2005 Czech. J. Phys. 55 1161
[8] Abhinav K and Panigrahi P K 2010 Ann. Phys. 325 1198
[9] Andrianov A A, Cannata F and Sokolov A V 2007 Nucl. Phys. B 773 107
[10] Bazeia D, Das Ashok, Greenwood L and Losano L 2009 Phys. Lett. B 673 283
Das A and Greenwood L 2009 Phys. Lett. B 678 504
[11] Ghosh P K 2010 J. Phys. A: Math. Theor. 43 125203
[12] Ghosh P K 2011 Int. J. Theor. Phys. 50 1143
[13] Ghosh P K 2005 J. Phys. A: Math. Gen. 38 7313
Deguchi T and Ghosh P K 2009 Phys. Rev. E 80 021107
Deguchi T, Ghosh P K and Kudo K 2009 Phys. Rev. E 80 026213
[14] Deguchi T and Ghosh P K 2009 J. Phys. A: Math. Theor. 42 475208
[15] Basu-Mallick B and Kundu A 2000 Phys. Rev. B 62 9927
Basu-Mallick B, Bhattacharya T and Mandal B P 2005 Mod. Phys. Lett. A 20 543
Basu-Mallick B and Mandal B P 2001 Phys. Lett. A 284 231
[16] Ghosh P K and Gupta K S 2004 Phys. Lett. A 323 29
[17] Znojil M and Tater M 2001 J. Phys. A: Math. Gen. 34 1793
Fring A 2006 *Mod. Phys. Lett.* A 21 691
Fring A and Znojil M 2008 *J. Phys. A: Math. Theor.* 41 194010
Assis P E G and Fring A 2009 *J. Phys. A: Math. Theor.* 42 425206
Fring A and Smith M 2010 *J. Phys. A: Math. Theor.* 43 325201

[18] Alcaraz F C, Droz M, Henkel M and Rittenberg V 1994 *Ann. Phys.*, NY 230 250
Derrida B, Evans M R, Hakim V and Pasquier V 1993 *J. Phys. A: Math. Gen.* 26 1493

[19] Jaynes E T and Cummings F W 1963 *Proc. IEEE* 51 89
Tavis M and Cummings F W 1968 *Phys. Rev.* 170 379
Golmire R and Bowdon C M 1976 *J. Math. Phys.* 17 1617

[20] de Alfaro V, Fubini S and Furlan G 1976 *Nuovo Cimento* A 34 569
Fubini S and Rabinovici E 1984 *Nucl. Phys.* B 245 17

[21] Ghosh P K 2001 *J. Phys. A: Math. Gen.* 34 5583
Calogero F 1969 *J. Math. Phys.* 10 2191
Calogero F 1971 *J. Math. Phys.* 12 419

[22] Oshlanetsky M A and Perelomov A M 1981 *Phys. Rep.* 71 314
Oshlanetsky M A and Perelomov A M 1983 *Phys. Rep.* 94 6
Polychronakos A P 1998 Les Houches Lectures arXiv:hep-th/9902157

[23] Ghosh P K 2001 *Nucl. Phys.* B 595 519
Ghosh P K 2004 *Nucl. Phys.* B 681 359

[24] Basu-Mallick B, Ghosh P K and Gupta K S 2003 *Phys. Lett.* A 311 87
Basu-Mallick B, Ghosh P K and Gupta K S 2003 *Nucl. Phys.* B 659 437
Feher L, Tsutsui I and Fuolop T 2005 *Nucl. Phys.* B 715 713
Yonezawa N and Tsutsui I 2006 *J. Math. Phys.* 47 012104

[25] Wang Q-h, Chia S-z and Zhang J-h 2010 arXiv:1002.2676

[26] Feher L, Tsutsui I and Fuolop T 2005 *Nucl. Phys.* B 715 713

[27] Ahmed Z and Jain S R 2003 *Phys. Rev.* E 67 045106
Ahmed Z and Jain S R 2003 *J. Phys. A: Math. Gen.* 36 3349
Ahmed Z and Jain S R 2006 *Mod. Phys. Lett.* A 21 331
Jain S R and Srivastava S C L 2008 *Phys. Rev.* E 78 036213

[28] Lieb E, Schultz T and Mattis D 1961 *Ann. Phys.* 16 407

[29] van Hemmen J L 1980 *Z. Phys.* B 38 271

[30] Coqueaux R 1982 *Phys. Lett.* B 115 389
De Crombrugge M and Rittenberg V 1983 *Ann. Phys.* 151 99

[31] Hatano N and Nelson D R 1997 *Phys. Rev.* Lett. 77 570
Hatano N and Nelson D R 1997 *Phys. Rev.* B 56 8651

[32] Bhattacherjee S M 2000 *J. Phys. A: Math. Gen.* 33 L423
Bhattacherjee S M 2000 *J. Phys. A: Math. Gen.* 33 9003 (erratum)

[33] Das T K and Chakrabarty B 1999 *J. Phys. A: Math. Gen.* 32 2387

[34] Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* 251 267

[35] D’Hoker E and Vinet L 1985 *Commun. Math. Phys.* 97 391