Coherent states for PT-/non-PT-symmetric and non-Hermitian Morse potentials via the path integral method

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Abstract
We discuss the coherent states for PT-/non-PT-symmetric and non-Hermitian generalized Morse potentials obtained by using path integral formalism over the holomorphic coordinates. We transform the action of generalized Morse potentials into two harmonic oscillators with a new parametric time to establish the parametric time coherent states. We calculate the energy eigenvalues and the corresponding wave functions in parabolic coordinates.

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1. Introduction

PT-symmetric quantum mechanics has been widely investigated recently. In standard formalism of quantum mechanics, the Hamiltonian that defines the dynamics and symmetries of the system is Hermitian. In the PT-symmetric case, the Hamiltonian has real spectra although it is not Hermitian. Bender and Boettcher [1] studied PT-symmetric Hamiltonians that have real eigenvalues. Following these works, many authors studied PT-symmetric and non-Hermitian Hamiltonians having real and/or complex eigenvalues [2, 3, 5] by applying analytic methods and numerical techniques. In this direction, several methods have been developed to calculate energy eigenvalues and the wave functions [4]. One of the methods providing exact solutions is the path integral method. This method is also employed to find the transition amplitudes among the configuration space eigenstates [6]. The energy spectra and eigenfunctions of PT-/non-PT- and non-Hermitian Morse potentials are obtained [7].

Morse potential, which has an important role in molecular physics, is used for the description of the interaction between the atoms in diatomic molecules [8–10]. The bound-state energy spectrum and the wave functions of Morse potentials are obtained by Duru [11] using the path integral method. Later Unal [9] derived parametric time coherent states using path integrals over holomorphic coordinates. Coherent states, which were first constructed by Schrödinger, are defined as eigenstates of lowering operators for harmonic oscillators [14]. They are minimum uncertainty states and do not disperse during the evolution of the system [9, 15, 16]. Both PT-symmetric and non-Hermitian generalized Morse potentials have been studied exactly [7, 13]. In the present work, coherent states of PT-/non-PT- and non-Hermitian Morse potentials are discussed by using the path integral method. We obtain the classical dynamics in terms of the holomorphic coordinates because holomorphic coordinates are the classical analogues of the rising and decreasing operators of the harmonic oscillators. Energy eigenvalues and the corresponding wave functions are obtained by transforming the problem into the two oscillators in the parametric time and quantizing these oscillators using the path integration over the holomorphic coordinates. The path integral technique can be applied to any potential if it is transformed into the harmonic oscillator potential.

The paper is organized as follows: In section 2, we introduce coherent states of the generalized Morse potential. We obtain the energy eigenvalues and wave functions in parabolic coordinates. In sections 3 and 4, we obtain the coherent states, energy eigenvalues and corresponding
eigenfunctions of the PT-symmetric and non-PT-symmetric non-Hermitian forms of the generalized Morse potential. Finally, the results are discussed in section 5.

2. Generalized Morse potential

We consider a particle with mass $m$ moving in the generalized Morse potential

$$V(x) = V_1 e^{-2\alpha x} - V_2 e^{-\alpha x},$$  \hspace{1cm} (1)

where $V_1$ and $V_2$ are constants and $\alpha$ is the width of the potential. The action of the one-dimensional generalized Morse potential is

$$A = \int dt \left[ p_x \frac{dx}{dt} - \left( \frac{p_x^2}{2m} + V_1 e^{-2\alpha x} - V_2 e^{-\alpha x} \right) \right].$$  \hspace{1cm} (2)

Let us define a new coordinate $u(t)$

$$x = -2\alpha \ln u$$  \hspace{1cm} (3)

and the canonical conjugate momentum

$$p_x = -\frac{\alpha u}{2} p_u.$$  \hspace{1cm} (4)

After this transformation, the action takes the form

$$A = \int dt \left[ p_u u \left( \frac{\alpha u^2}{4} \frac{p_u^2}{2m} + V_1 u^4 - V_2 u^2 \right) \right].$$  \hspace{1cm} (5)

In order to annihilate the factorization $u^2$ in kinetic energy terms, we define a new time parameter as

$$\frac{dt}{du} = \frac{1}{u^2}.$$  \hspace{1cm} (6)

Substitution of this new time parameter into equation (5) gives us

$$A = \int du \left[ p_u \frac{du}{dt} - \left( \frac{\alpha u^2}{4} \frac{p_u^2}{2m} + V_1 u^4 - V_2 u^2 \right) \right] + \frac{1}{2} \left( \frac{dp_u}{du} - \frac{1}{u^2} \right)^2.$$  \hspace{1cm} (7)

Here an additional Lagrange multiplier $p_0$ is used owing to the presence of a new parametric time. Defining an auxiliary momentum $p_0$ as $d\phi/d\tau$, the action becomes

$$A = \int d\tau \left[ p_u \frac{du}{d\tau} - \frac{1}{2M} \left( p_u^2 + 2MV_1 u^2 - 2MV_2 + \frac{p_0^2}{u^2} \right) \right] + \frac{1}{2} \left( \frac{dp_u}{d\tau} - \frac{1}{u^2} \right)^2,$$  \hspace{1cm} (8)

where $M = 4m/\alpha^2$. Let us define $\omega = \sqrt{2V_1/M}$ as the frequency of the two oscillators. So equation (8) can be rewritten as

$$A = \int d\tau \left[ p_u \frac{du}{d\tau} - \frac{1}{2M} \left( p_u^2 + \frac{1}{2} M^2 \omega^2 u^2 + \frac{p_0^2}{u^2} - 2MV_2 \right) \right] + \frac{1}{2} \left( \frac{dp_u}{d\tau} - \frac{1}{u^2} \right)^2 + \frac{1}{2} \left( p_0 - \sqrt{-2M p_0} \right)^2.$$  \hspace{1cm} (9)

We represent the position vector of the particle in the two-dimensional polar coordinates as

$$\ddot{u} = (u_1, u_2) = \sqrt{u}(\cos \phi, \sin \phi).$$  \hspace{1cm} (10)

The action in terms of $(u_1, u_2)$ becomes

$$A = \int d\tau \left[ p_{u_1} \frac{du_1}{d\tau} + p_{u_2} \frac{du_2}{d\tau} - \omega \left( p_{u_1}^2 + p_{u_2}^2 \right) + \frac{1}{2} \frac{2MV_2}{\omega^2} \right] + \frac{1}{2} \frac{2MV_2}{\omega^2}. \hspace{1cm} (11)

Here one can define $A'$ as an action of the two-oscillator system with mass $M$, frequency $\omega$ and energy $V_2$. Thus, we have

$$A' = \int d\tau \left[ p_{u_1} \frac{du_1}{d\tau} + p_{u_2} \frac{du_2}{d\tau} \right] - \omega \left[ \frac{p_{u_1}^2 + p_{u_2}^2}{2M \omega} + \frac{1}{2} \frac{2MV_2}{\omega^2} \right] - V_2.$$  \hspace{1cm} (12)

The term $\left( \frac{p_{u_1}^2 + p_{u_2}^2}{2M \omega} + \frac{1}{2} \frac{2MV_2}{\omega^2} \right) - V_2$ corresponds to the Hamiltonian of two oscillators.

The holomorphic coordinates are defined as

$$a = \frac{a_1}{a_2} = \frac{1}{\sqrt{2}} \left( \sqrt{\omega} \sin \phi_1 + i \sqrt{\frac{1}{M \omega}} p_{u_1} \right)$$  \hspace{1cm} (13)

and also conjugate dynamical variables as

$$a^\dagger = (a^\dagger_1, a^\dagger_2) = \frac{1}{\sqrt{2}} \left( \sqrt{\omega} \sin \phi_j - i \sqrt{\frac{1}{M \omega}} p_{u_j} \right), \hspace{1cm} j = 1, 2.$$  \hspace{1cm} (14)

The action in equation (12) is again rewritten in terms of the holomorphic coordinates

$$A'(a^\dagger_1, \phi_1; a_1, \tau_1) = V_2 (\tau_2 - \tau_1) + \int d\tau \left[ \frac{1}{2i} \left( \frac{da^\dagger}{d\tau} a - a^\dagger \frac{da}{d\tau} \right) - \omega(a^\dagger a) \right].$$  \hspace{1cm} (15)

We ignore the total derivative $\frac{1}{2} \sum_{j=1}^2 \frac{dp_{u_j}}{dt}$ in the action, equation (15). The kernel of two oscillators with parametric time $\tau$ in holomorphic coordinates is

$$K'(a^\dagger_1, \phi_1; a_1, \tau_1) = e^{i(2V_2 - \omega)(\tau_2 - \tau_1)} \int Da^\dagger_1 D\phi \exp \left[ i \int_{\tau_1}^{\tau_2} \frac{1}{2i} \left( \frac{da^\dagger}{d\tau} a - a^\dagger \frac{da}{d\tau} \right) - \omega(a^\dagger a) \right].$$  \hspace{1cm} (16)

Here we take $\hbar = 1$ and the last term $\omega$ appears from the quantum ordering terms between the operators $a^\dagger$ and $\hat{a}$. After the integrations over $a$ and $a^\dagger$ [9], we obtain

$$K'(a^\dagger_1, \phi_1; a_1, \tau_1) = e^{i(2V_2 - \omega)(\tau_2 - \tau_1)} \exp \left[ a^\dagger_1 a_1 e^{-i\omega(\tau_2 - \tau_1)} \right].$$  \hspace{1cm} (17)
Since the transformation in equation (16) has double value, the physical kernel becomes

\[ K'(a^{\dagger}_b, \tau_b; a_a, \tau_a) = e^{i(V_2 - \omega)(\tau_b - \tau_a)} \left\{ \exp \left[ a^{\dagger}_b a_a e^{-i\omega(t_0 - \tau_a)} \right] + \exp \left[ -a^{\dagger}_b a_a e^{-i\omega(t_0 - \tau_a)} \right] \right\}, \]

(18)

If the exponential term is expanded into a power series of \( a^{\dagger}_b \), we have

\[ K'(a^{\dagger}_b, \tau_b; a_a, \tau_a) = e^{\tau_b - \tau_a} \sum_{n_1, n_2=0}^{\infty} \left[ 1 + (-1)^{n_1+n_2} \right] \times e^{-i[(n_1+n_2+1)\omega - V_2](\tau_b - \tau_a)} \left( \sum_{j=1}^{2} \frac{(a^{\dagger}_b a_a)^{n_j}}{n_j + 1} \right), \]

(19)

where \( n_1 \) and \( n_2 \) are quantum numbers. They can be expressed in terms of the radial and angular quantum numbers \( n_r \) and \( m \) of the two oscillators as

\[ n_1 = n_r + \frac{|m + m|}{2}, \quad n_2 = n_r + \frac{|m - m|}{2}. \]

(20)

and let us take \( a^\pm \) final eigenstates

\[ a^\pm = \frac{a_0 \mp ia_{2\nu}}{\sqrt{2}} \]

(21)

and \( \lambda^\pm \) initial eigenstates

\[ \lambda^\pm = (a_{14} \pm ia_{24}) \]

(22)

Thus, we rewrite the kernel in equation (19) as

\[ K'(a^\pm, \tau; \lambda^\pm; \tau_a) = \sum_{n, m=0}^{\infty} \sum_{m=-\infty}^{\infty} e^{-i[(n_1+n_2+1)\omega - V_2](\tau_b - \tau_a)} \times \left[ 1 + (-1)^{2|m|} \frac{(a^\pm_0 a_\lambda)^{n_1} (a^\pm_\lambda a^\pm_\lambda)^{n_2}}{(m + 1) / 2(n_2 + 1)} \right]. \]

(23)

We suppose that \( \phi \) varies as \(-\pi < \phi < \pi\) for the derivation in equation (23). However, the generalized Morse potential is not periodic in this interval. If we let \( \phi \to 2\pi \phi / 2L \) and taking the limit \( L \to \infty \), equation (23) becomes

\[ K'(a^\pm, \tau; \lambda^\pm; \tau_a) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} d\xi e^{-i[(n_1+n_2+1)\omega - V_2](\tau_b - \tau_a)} \times \left[ 1 + (-1)^{2|m|} \frac{(a_0^\pm a_\lambda)^{n_1} (a^\pm_\lambda a^\pm_\lambda)^{n_2}}{(m + 1) / 2(n_2 + 1)} \right]. \]

(24)

\( K' \) can be written in terms of oscillator energy eigenstates \([n_r, m]\) as

\[ K'(a^\pm, \tau; \lambda^\pm; \tau_a) = \sum_{n, m=0}^{\infty} \int_{-\infty}^{\infty} d\xi \left\{ n_r, m \left| a^\pm_r \right|^2 \left| n_r, m \right| \right\} \times U(\tau_b - \tau_a) \left| \lambda^\pm \right>. \]

(25)

where \( U(\tau_b - \tau_a) \) is the parametric time evolution operator between initial coherent states of the two oscillators. We can denote

\[ |a^\pm_r> = e^{(\Delta \tau_0)} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} d\xi \left\{ 1 + (-1)^{n_r+m} \right\} \frac{(a^\pm_0)^{n_1} (a^\pm_\lambda)^{n_2}}{\sqrt{\Gamma(n_1+1) \Gamma(n_2+1)}} \]

(26)

and

\[ U(\tau_b - \tau_a) |\lambda^\pm> = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} d\xi e^{-2i[(n_1+n_2+1)\omega - V_2](\tau_b - \tau_a)} \times \left[ 1 + (-1)^{n_r+m} \right] \frac{(a^\pm_0)^{n_1} (a^\pm_\lambda)^{n_2}}{\sqrt{\Gamma(n_1+1) \Gamma(n_2+1)}}. \]

(27)

To obtain the coherent states as a function of physical coordinates, Green’s function of the system is obtained as

\[ G'(u_b, \tau_b; \lambda^\pm, \tau_a) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} d\xi e^{-i[(n_1+n_2+1)\omega - V_2](\tau_b - \tau_a)} \times \left[ 1 + (-1)^{n_r+m} \right] \frac{(a^\pm_0)^{n_1} (a^\pm_\lambda)^{n_2}}{\sqrt{\Gamma(n_1+1) \Gamma(n_2+1)}} \].

(28)

The poles of Green’s function give the energy eigenvalues of the system. In order to obtain the Green’s function of the physical problem, the dummy coordinates should be eliminated. One of the methods to eradicate a variable in the path integration formalism is to integrate over it. The other method is to take the physical eigenvalues of the corresponding conjugate momenta in the wave function formalism. To eliminate the dummy coordinate \( \phi \), we prefer the method in the wave function formalism [16]. We take \( m \) as an azimuthal quantum number corresponding to the operator \( \hat{p}_\phi \). Thus, the Green’s function is converted into

\[ G'(a^\pm, \lambda^\pm) \sum_{n=0}^{\infty} \left[ 1 + (-1)^{n_r+m} \right] \frac{(a^\pm_0)^{n_1} (a^\pm_\lambda)^{n_2}}{\sqrt{\Gamma(n_1+1) \Gamma(n_2+1)}} \].

(29)

and the physical coherent states become

\[ |a^\pm> = \sum_{n=0}^{\infty} \left[ 1 + (-1)^{n_r+m} \right] \frac{(a^\pm_0)^{n_1} (a^\pm)^{n_2}}{\sqrt{\Gamma(n_1+1) \Gamma(n_2+1)}}. \]

(30)

As a function of physical coordinates \( u \) and \( t \), the kernel of the system is written as

\[ K'(u, \phi; \lambda^\pm(t)) = \int \frac{da^\pm_0 d\xi}{(2\pi)^2} e^{-a^\pm_0 a^\pm_0} \langle u, \phi | a^\pm_0 \rangle \times K'(a^\pm_0, \tau; \lambda^\pm_0, \tau_a) , \]

(31)

where \( K'(u, \phi; \lambda^\pm(t)) \) is

\[ \langle u, \phi | a^\pm_0 \rangle = N e^{-a^2/2} \sqrt{\frac{1}{2}} M \omega^2 + 2M\omega^2 (a_+ e^{i\phi} + a_- e^{-i\phi}) \frac{a^2}{2}. \]

(32)

where \( a^2 = a_+ a_- + a_- a_+ \) and \( N = (\frac{\omega}{2})^2 e^{-a^2/2} \). If we take integrations over \( a^\pm_0 \) and \( a^\pm_0 \) in equation (31),
it will be

\[
K(u, \phi; \lambda_{+}) = N \exp \left[ -\frac{1}{2} \hbar \omega u^2 + 2 \sqrt{\hbar \omega} e^{-i \omega t} \times \left( \lambda_{+} e^{i \phi} + \lambda_{-} e^{-i \phi} \right) - \frac{\lambda_{+}^2}{2} \right].
\]  

(33)

The parametric time dependence of the eigenvalues of the operators \( \hat{a}_j \) is the form of \( a_j(t) = a_j e^{-i \omega t - \tau} \). Let us rewrite it in terms of trigonometric functions

\[
K(u, \phi; \lambda_{+}(\tau)) = N e^{\frac{1}{2} \hbar \omega u^2 - i \frac{\lambda_{+}}{2}} \times \exp \left[ -2i \sqrt{\frac{\hbar \omega}{2}} (\lambda_{+}^2 + 1) \left( e^{-i(\phi + \delta)} + e^{i(\phi - \delta)} \right) \right].
\]

(34)

Here \( \delta \) is the complex phase and is determined as

\[
e^{-i \delta} = \frac{\lambda_{+} - i \lambda_{-}}{\sqrt{\lambda_{+} \lambda_{-} + \lambda_{-} \lambda_{+}}}.\]

(35)

To obtain coherent states in the parabolic coordinates, we express the kernel in terms of Bessel functions [17]

\[
K(u, \phi; \lambda_{+}(\tau)) = N e^{\frac{1}{2} \hbar \omega u^2 - i \frac{\lambda_{+}}{2}} \sum_{m = -\infty}^{\infty} \left[ 1 + (-1)^{|m|} \right] \times J_m \left( \sqrt{\frac{\hbar \omega}{2}} (\lambda_{+}^2 + 1) \right) e^{-i m (\phi + \delta)}.
\]

(36)

Integrating the kernel over the parametric time \( \tau \), Green’s function can be written as

\[
G(u; \lambda_{+}) = \int \frac{dp_{\tau}}{2 \pi} e^{-ip_{\tau}(t - t_u)} \sum_{n_e = 0}^{\infty} \frac{1}{\omega} \left[ \frac{1 + (-1)^{|m|}}{n_i + |m| + 1} + \frac{2}{\lambda_{+}^2 + 1} - \frac{V_2}{2} \right] \Phi_{n, m}(u) \times e^{i |m|^2} \frac{\gamma^{(m)/2}}{\Gamma (n_i + |m| + 1)} \cos m \left( \delta + \frac{\pi}{2} \right),
\]

(37)

where \( \Phi_{n, m}(u) \) is

\[
\Phi_{n, m}(u) = e^{-\frac{1}{2} \hbar \omega u^2} \left( \frac{\sqrt{2 \hbar \omega \pi}}{2} \right)^{1/4} \frac{1}{\sqrt{\Gamma (n_i + |m| + 1)}} \times J_{n_e \pm |m|} \left( \frac{\hbar \omega u^2}{2} \right).
\]

(38)

Thus, the Green’s function of physical time in the parabolic coordinates can be obtained as

\[
G(u, \phi; \lambda_{+}, t_u) = \int \frac{dp_{\tau}}{2 \pi} e^{-ip_{\tau}(t - t_u)} \sum_{n_e = 0}^{\infty} \frac{1}{\omega} \left[ \frac{1 + (-1)^{|m|}}{n_i + |m| + 1} + \frac{2}{\lambda_{+}^2 + 1} - \frac{V_2}{2} \right] \Phi_{n, m}(u) \times e^{i |m|^2} \frac{\gamma^{(m)/2}}{\Gamma (n_i + |m| + 1)} \cos m \left( \delta + \frac{\pi}{2} \right),
\]

(39)

where \( n = n_i + |m| / 2 \). From equation (39), energy eigenstates become

\[
\psi_{n, m}(u) = \sqrt{\frac{2M}{n}} \frac{\sqrt{MV_2 / 2n + 1}}{\pi^{1/4} \Gamma (n_i + |m| + 1)} \times \frac{L_{n_e \pm |m|} (\hbar \omega u^2 / 2)}{\sqrt{2}}.
\]

(40)

Using the residue of Green’s function, the energy eigenvalues of the system become

\[
E = -V_2 \left[ 1 - \frac{\omega}{V_2} (n_i + 1/2) \right]^2.
\]

(41)

The above result agrees with the one available in the literature [13].

3. The PT-symmetric and non-Hermitian generalized Morse case

If \( V_1 \) and \( V_2 \) are real and \( a = i a \) then the generalized Morse potential has the form

\[
V(x) = V_1 e^{-2i\alpha x} - V_2 e^{-i\alpha x}.
\]

(42)

We can obtain the parametric time coherent states and energy spectra for this potential following the steps of section 2 by using the same variables and then the action becomes

\[
A = \int \frac{d\tau}{\sqrt{2M}} \left[ p_{\alpha} \frac{d\alpha}{d\tau} + p_{\alpha^*} \frac{d\alpha^*}{d\tau} \right] + \omega \left[ p_{\alpha^*}^2 + p_{\alpha}^2 + \frac{1}{2} M \omega (u_{\alpha}^2 + u_{\alpha^*}^2) + \frac{V_2}{\omega} \right] + \frac{( - p_0 )}{d\tau} \frac{d\phi}{d\tau} \left( \frac{\phi - \sqrt{2MP_0}}{d\tau} \right).
\]

(43)

The frequency of the two-oscillator system is again \( \omega = \sqrt{-2V_1} \) and energy is \( V_2 \). Here, we see that the kinetic energy term is negative. By holomorphic coordinate transformations the action becomes

\[
A'(a_{\alpha}^*, a_{\alpha^*}, a_{\alpha}, a_{\alpha^*}) = \int \frac{d\tau}{\sqrt{2M}} \left[ \frac{1}{2i} \left( \frac{d\alpha^*}{d\tau} a - \frac{d\alpha}{d\tau} a^* \right) - \omega (a a^*) \right].
\]

(44)

The action of the system becomes as in equation (15) again. Equation (16) takes the form

\[
K'(a_{\alpha}^*, a_{\alpha^*}, a_{\alpha}, a_{\alpha^*}) = e^{-i \omega t} e^{-i \delta_t} \int \frac{d\tau}{\sqrt{2M}} \left[ \frac{1}{2i} \left( \frac{d\alpha^*}{d\tau} a - \frac{d\alpha}{d\tau} a^* \right) - \omega (a a^*) \right].
\]

(45)
and Green’s function as a function of physical coordinates is written as

\[ G^r(u_b, t_b; \lambda, \tau) = \sum_{n=0}^{\infty} \frac{i [1 + (-1)^{2|m|}]}{V_2/2 - \omega (n + |m| + 1/2)} \]
\[ \times \left( a^*_n \lambda \right)^{n-i} (a^*_n \lambda)^{\frac{m}{2}} \frac{(\omega \tau)^m}{\Gamma(n + 1) \Gamma(n_2 + 1)}. \]  

(46)

Hence parametric time coherent states for PT-symmetric and non-Hermitian generalized Morse can be written as

\[ |a_{\pm}\rangle = \sum_{n=0}^{\infty} \left[ 1 + (-1)^{|m|} \right] \frac{(a_n a_{-n})^{\frac{m}{2}}}{\sqrt{\Gamma(n + 1) \Gamma(n_2 + 1)}}. \]  

(47)

After performing integrations again, Green’s function of the generalized Morse potential takes the form

\[ G(u; \lambda) = \int \frac{dp_0}{2\pi i} e^{-i p_0(t-t_0)} \]
\[ \times \sum_{n=0}^{\infty} \frac{i [1 + (-1)^{|m|}]}{V_2/2 - \omega (n + |m| + 1/2)} \Phi_{n,m}(u) \]
\[ \times e^{\frac{|m|^2}{2}} \frac{\varepsilon^{-\lambda M/2} (\Omega M \omega t^2)^{|m|}}{\Gamma(n + 1 + 1/2)} \cos \left( \delta + \frac{\pi}{2} \right). \]  

(48)

where \( \Phi_{n,m}(u) \) is

\[ \Phi_{n,m}(u) = \frac{e^{-\frac{\lambda M}{2} M \omega t^2}}{\Gamma(n + 1 + 1/2)} L_{n+m}^{|m|} \left( \frac{M \omega t^2}{2} \right). \]  

(49)

Thus for PT-symmetric and non-Hermitian generalized Morse potentials, the Green’s functions of physical time in the parabolic coorindates can be obtained as

\[ G(u, \phi, \tau; \lambda, \tau_0) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} e^{-\lambda V_2 (1 + \frac{1}{\sqrt{2}} (n+\frac{1}{2}))^2 (t-t_0)} \]
\[ \times \frac{e^{i \phi}}{2\pi} \left( \lambda + \lambda_{-n+m} \right)^{n|m} \left( \frac{\omega}{\sqrt{\Gamma(n + 1) \Gamma(n_2 + 1)} \right) \]
\[ \times \frac{\varepsilon^{-\lambda M \omega t^2} (\Omega M \omega t^2)^{|m|/2}}{\pi^{1/4} \Gamma(n + |m| + 1)} L_{n+m}^{|m|} \left( \frac{M \omega t^2}{2} \right). \]  

(50)

The eigenstates of PT-symmetric and non-Hermitian generalized Morse potentials are

\[ \psi_{n,m}(u) = \sqrt{\frac{2M}{n \pi}} \frac{\varepsilon^{-\lambda M \omega t^2} (\Omega M \omega t^2)^{|m|/2}}{\pi^{1/4} \Gamma(n + |m| + 1)} L_{n+m}^{|m|} \left( \frac{M \omega t^2}{2} \right). \]  

(51)

Since the energy of the system is a residue of Green’s function, using the residue of equation (47), we obtain

\[ E = -V_2 \left[ 1 + \frac{\omega}{V_2} \left( n + \frac{1}{2} \right)^2 \right]. \]  

(52)

From the above equation, one can easily see that there are only real spectra for the PT-symmetric and non-Hermitian Morse case [4, 7, 10, 13, 18].

4. The non-PT-symmetric and non-Hermitian generalized Morse case

If we take \( V_1 = (A + i B)^2 \), \( V_2 = (2C + 1) (A + i B) \) and \( \alpha = 1 \), the generalized Morse potential can be rewritten as

\[ V(x) = (A + i B)^2 e^{2ix} - (2C + 1) (A + i B) e^{-ix}. \]  

(53)

Here \( A, B \) and \( C \) are arbitrary parameters. This potential is non-PT-symmetric and non-Hermitian, but it has real spectra. If \( V_1 \) is real, \( V_2 = A + i B \) and \( \alpha = i \omega \) are taken, then the Morse potential is transformed into the form

\[ V(x) = V_{1} e^{-2i\omega x} - (A + i B) e^{-i\omega x}. \]  

(54)

Now we can derive coherent states and energy spectra following the same steps as in section 2 and we obtain the action as

\[ A = \int dx \left\{ (-p_0) \frac{dt}{d\tau} + \sqrt{2m_0} \frac{d\phi}{d\tau} + A' \right\}. \]  

(55)

where \( \phi \) is a dummy coordinate. The Lagrange multiplier \( \sqrt{2m_0} \) is added. Hence, the new action becomes

\[ A' = \int d\tau \left\{ p_{1} \frac{du_1}{d\tau} + p_{2} \frac{du_2}{d\tau} - \omega \left( p_{1}^2 + p_{2}^2 + \frac{1}{2} M \omega (u_1^2 + u_2^2) - \frac{A + i B}{\omega} \right) \right\}. \]  

(56)

Here \( A' \) is the two oscillator action, and the frequency \( \omega = \sqrt{-V_1/M} \) and the energy is \( A + i B \). Here, we note that the kinetic energy term is negative. The action can be written by holomorphic coordinate transformation as

\[ A'(u_{1}^a, \tau_0; u_a, \tau_a) = \int d\tau \left[ \frac{1}{2i} \left( \frac{da^a}{d\tau} - a^a \frac{da}{d\tau} \right) - \omega (a^a a) - (A + i B) \right]. \]  

(57)

The kernel of the system becomes

\[ K'(a_{1}^a, \tau_0; a_a, \tau_a) = \int d\tau_0 \left[ \frac{1}{2i} \frac{d\tau}{d\tau_0} \exp \left\{ i \int d\tau_0 \left[ \frac{1}{2i} \left( \frac{d\tau}{d\tau_0} - a \frac{da}{d\tau_0} \right) - \omega (a a) - (A + i B) \right] \right\}. \]  

(58)

So, Green’s function of the physical system can be obtained as follows:

\[ G'(u_b, t_b; \lambda, \tau_0) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dm \frac{i [1 + (-1)^{|m|}]}{A + i B - \omega (n + |m| + 1/2)} \]
\[ \times \left( a^*_n \lambda \right)^{n-i} (a^*_n \lambda)^{\frac{m}{2}} \frac{\varepsilon^{-\lambda M/2} (\Omega M \omega t^2)^{|m|/2}}{\pi^{1/4} \Gamma(n + 1) \Gamma(n_2 + 1)}. \]  

(59)

and the parametric time coherent states for non-PT-symmetric and non-Hermitian generalized Morse potentials become the
same as the ones given in equation (30). After performing integrations, wave function of the coherent states are

\[ G(u; \lambda_{\mp}) = \int \frac{dp_{0}}{2\pi i} e^{-ip_{0}t_{0}} \times \sum_{n_{0},m_{0}} \frac{e^{i[(A+iB)n_{0}]}[1+(A+iB)(n_{0}+\frac{1}{2})]}{\sqrt{\Gamma[(n_{0}+m_{0})+1]} \cdot \Phi_{n_{0},m_{0}}(u)} \times e^{\frac{-i}{M_{0}}\sqrt{\frac{M_{0}}{2}}[\frac{2}{\sqrt{\Gamma[(n_{0}+m_{0})+1]} \cdot \Phi_{n_{0},m_{0}}(u)}]} \times \cos m \left( \delta + \frac{\pi}{2} \right), \]  

(60)

where \( \Phi_{n_{0},m_{0}}(u) \) is

\[ \Phi_{n_{0},m_{0}}(u) = \frac{e^{-iM_{0}u^{2}/2} \sqrt{\Gamma[(n_{0}+m_{0})+1]} \cdot L_{n_{0}+m_{0}+1}^{(1)} \left( \frac{M_{0}u^{2}}{2} \right)}{\sqrt{\Gamma[(n_{0}+m_{0})+1]} \cdot \Phi_{n_{0},m_{0}}(u)}. \]  

(61)

Therefore for non-PT-symmetric and non-Hermitian Morse potentials, the wave function of physical time in the parabolic coordinates can be obtained as

\[ G(u, \phi, t; \lambda_{\mp}, t_{0}) = \sum_{n_{0},m_{0}} \sum_{m_{0}} e^{-i[(A+iB)n_{0}]}[1+(A+iB)(n_{0}+\frac{1}{2})] \times \frac{e^{i\lambda_{\mp}m_{0}} \left( \frac{L_{m_{0}}^{(1)}(\frac{M_{0}u^{2}}{2})}{\sqrt{\Gamma[(n_{0}+m_{0})+1]} \cdot \Phi_{n_{0},m_{0}}(u)} \right)}{\sqrt{\Gamma[(n_{0}+m_{0})+1]} \cdot \Phi_{n_{0},m_{0}}(u)}} \times \sqrt{M(A+iB)(n_{0}+1)} \times e^{-iM_{0}u^{2}/2} \sqrt{\Gamma[(n_{0}+m_{0})+1]} \cdot \Phi_{n_{0},m_{0}}(u)}. \]  

(62)

The eigenstates of the non-PT-symmetric and non-Hermitian generalized Morse case are

\[ \Psi_{n_{0},m_{0}}(u) = \sqrt{\frac{2M_{0}u^{2}}{\sqrt{\Gamma[(n_{0}+m_{0})+1]} \cdot \Phi_{n_{0},m_{0}}(u)}} \times L_{n_{0}+m_{0}+1}^{(1)} \left( \frac{M_{0}u^{2}}{2} \right). \]  

(63)

Since the energy of the system is a residue of Green’s function, from the residue of equation (58), we obtain energy eigenvalues as

\[ E = -(A+iB) \left[ 1 + \frac{\omega}{A+iB} \left( n + \frac{1}{2} \right) \right]^{2}. \]  

(64)

It is clear that energy spectra are real in cases \( V_{1} > 0 \) if and only if \( \text{Re}(V_{2}) = 0 \) and \( V_{1} < 0 \) if and only if \( \text{Im}(V_{2}) = 0 \).

5. Conclusion

We have studied parametric time coherent states for PT-/non-PT-symmetric and non-Hermitian generalized Morse potentials by using path integral formalism. Energy eigenvalues and the corresponding wave functions are calculated. We have discussed the negative energy coherent states. The wave functions of the potential in sections 3 and 4 are physical like in section 2. Energy eigenvalues are positive, contrary to expectation. The negative energy coherent states can be obtained by analytic continuation like in [9].

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