A PACKAGE FOR COMPUTATIONS WITH CLASSICAL RESULTANTS

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ABSTRACT. We present the Macaulay2 package Resultants, which provides commands for the effective computation of multivariate resultants, discriminants, and Chow forms. We provide some background for the algorithms implemented and show, with a few examples, how the package works.

INTRODUCTION

The resultant characterizes the existence of nontrivial solutions for a square system of homogeneous polynomial equations as a condition on the coefficients. One of its important features is that it can be used to compute elimination ideals and to solve polynomial equations. Indeed, it provides one of the two main tools in elimination theory, along with Gröbner bases. The resultant of the system of equations given by the partial derivatives of a complex homogeneous polynomial $F$ is called (up to a constant factor) the discriminant of $F$. It characterizes the existence of singular points in the projective hypersurface $V(F)$ as a condition on the coefficients of $F$. In this special case, all polynomial equations have the same total degree. Every time the system of equations consists of $n+1$ polynomial equations of the same total degree $d$, the resultant has a further interesting property: it can be expressed as a polynomial of degree $d^n$ in the $(n+1) \times (n+1)$ minors of a $(n+1) \times \binom{n+d}{n}$ matrix, the coefficient matrix of the system of equations. This allows us to write down a generic resultant in a more compact form. The polynomial of degree $d^n$ so obtained is geometrically interpreted as the Chow form of the $d$-th Veronese embedding of $\mathbb{P}^n$.

The package Resultants, included with Macaulay2 [GS16], provides commands for the explicit computation of resultants and discriminants. The main algorithm used is based on the so-called Poisson formula, which reduces the computation of the resultant of $n+1$ equations to the product of the resultant of $n$ equations with the determinant of an appropriate matrix. This algorithm requires a certain genericity condition on the input polynomials, achievable with a generic change of coordinates. The package also includes tools for working with Chow forms and more generally with tangential Chow forms.

In Section 1, from a more computational point of view, we give some background information on the general theory of resultants, discriminants, and Chow forms. In Section 2, we briefly illustrate how to use the package with the help of some examples; more detailed information and examples can be found in its documentation.
1. Overview on Classical Resultants

In this section, we present an overview of some classically well-known facts on the theory of resultants for forms in several variables. For details and proofs, we refer mainly to [GKZ94] and [CLO05]; other references are [Jou91, Jou97, dW50, Dem12, EM99, BGW88, BJ14], and [CLO07] for the case of two bivariate polynomials.

1.1. Resultants. Suppose we are given \( n + 1 \) homogeneous polynomials \( F_0, \ldots, F_n \) in \( n + 1 \) variables \( x_0, \ldots, x_n \) over the complex field \( \mathbb{C} \). For \( i = 0, \ldots, n \), let \( d_i \) denote the total degree of \( F_i \) so that we can write \( F_i = \sum_{|\alpha| = d_i} c_{i, \alpha} x^\alpha \), where \( x^\alpha \) denotes \( x_0^{\alpha_0} \cdots x_n^{\alpha_n} \). For each pair of indices \( i, \alpha \), we introduce a variable \( u_{i, \alpha} \) and form the universal ring of coefficients \( \mathbb{U} = \mathbb{Z}[u_{i, \alpha} : i = 0, \ldots, n, |\alpha| = d_i] \). If \( F \in \mathbb{U} \), we denote by \( P(F_0, \ldots, F_n) \) the element in \( \mathbb{C} \) obtained by replacing each variable \( u_{i, \alpha} \) with the corresponding coefficient \( c_{i, \alpha} \).

**Theorem 1.1** ([GKZ94, CLO05]). If we fix positive degrees \( d_0, \ldots, d_n \), then there is a unique polynomial \( \text{Res} = \text{Res}_{d_0, \ldots, d_n} \in \mathbb{U}_{d_0, \ldots, d_n} \) which has the following properties:

1. If \( F_0, \ldots, F_n \in \mathbb{C}[x_0, \ldots, x_n] \) are homogeneous of degrees \( d_0, \ldots, d_n \), then the equations
   \[
   F_0 = 0, \ldots, F_n = 0
   \]
   have a nontrivial solution over \( \mathbb{C} \) (i.e., \( \emptyset \neq V(F_0, \ldots, F_n) \subseteq \mathbb{P}^{n+1} \)) if and only if \( \text{Res}(F_0, \ldots, F_n) = 0 \);
2. \( \text{Res} \) is irreducible, even when regarded as a polynomial over \( \mathbb{C} \);
3. \( \text{Res}(x_0^{d_0}, \ldots, x_n^{d_n}) = 1 \).

**Definition 1.2.** We call \( \text{Res}(F_0, \ldots, F_n) \) the resultant of \( F_0, \ldots, F_n \).

**Remark 1.3.** If \( A \) is any commutative ring, we define the resultant of \( n + 1 \) homogeneous polynomials \( F_0, \ldots, F_n \in A[x_0, \ldots, x_n] \) again as \( \text{Res}(F_0, \ldots, F_n) \in A \), i.e., by specializing the coefficients of the integer polynomial \( \text{Res} \). Thus, the formation of resultants commutes with specialization.

**Example 1.4.** The resultant is a direct generalization of the determinant. Indeed, if \( d_0 = \cdots = d_n = 1 \), then \( \text{Res}(F_0, \ldots, F_n) \) equals the determinant of the \( (n + 1) \times (n + 1) \) coefficient matrix.

**Proposition 1.5** ([Jou91, Jou97]). The following hold:

1. (Homogeneity) For a fixed \( j \) between 0 and \( n \), \( \text{Res} \) is homogeneous in the variables \( x_{i, \alpha} \); \( |\alpha| = d_j \), of degree \( d_0 \cdot d_{j+1} \cdots d_n \); hence its total degree is \( \sum_{j=0}^n d_0 \cdots d_{j-1} d_{j+1} \cdots d_n \).
2. (Symmetry) If \( \sigma \) is a permutation of \( \{0, \ldots, n\} \), then
   \[
   \text{Res}(F_{\sigma(0)}, \ldots, F_{\sigma(n)}) = \text{sign}(\sigma)^{d_0 - d_n} \text{Res}(F_0, \ldots, F_n).
   \]
3. ( Multiplicativity) If \( F_j = F_j' F_j'' \), then we have
   \[
   \text{Res}(F_0, \ldots, F_j, \ldots, F_n) = \text{Res}(F_0, \ldots, F_j', \ldots, F_n) \text{Res}(F_0, \ldots, F_j'', \ldots, F_n).
   \]
4. (SL\((n+1)\)-invariance) For each \( (n+1) \times (n+1) \) matrix \( A \) over \( \mathbb{C} \), we have
   \[
   \text{Res}(F_0(Ax), \ldots, F_n(Ax)) = \text{det}(A)^{d_0 - d_n} \text{Res}(F_0(x), \ldots, F_n(x))
   \]
   where \( Ax \) denotes the product of \( A \) with the column vector \( (x_0, \ldots, x_n)' \).
Theorem 1.7

If \( H_i \) is homogeneous of degree \( d_i - d_i \), then
\[
\text{Res}(F_0, \ldots, F_j + \sum_{i \neq j} H_i F_1, \ldots, F_n) = \text{Res}(F_0, \ldots, F_j, \ldots, F_n).
\]

Remark 1.6

On the product \( \mathbb{C}^M \times \mathbb{P}^n = \text{Spec}(\mathbb{C}[u_i, \alpha_i]) \times \text{Proj}(\mathbb{C}[x_0, \ldots, x_n]) \), where \( M = \sum_{i=0}^{n} \binom{n+1}{n} \), we have an incidence variety
\[
W := \{(c_i, \alpha_i), p) \in \mathbb{C}^M \times \mathbb{P}^n : p \in V(\sum_{|\alpha| = d_0} c_0, \alpha, \alpha \alpha^\alpha, \ldots, \sum_{|\alpha| = d_n} c_n, \alpha, \alpha^\alpha)\}.
\]

The first projection \( \pi_1 : W \to \mathbb{C}^M \) is birational onto its image, whereas all the fibers of the second projection \( \pi_2 : W \to \mathbb{P}^n \) are linear subspaces of dimension \( M - n - 1 \). It follows that \( W \) is a smooth irreducible variety which is birational to \( \pi_1(W) = \pi_1(V) = \pi_1(\text{Res}_{d_0, \ldots, d_n}) \subset \mathbb{C}^M \).

The following result is called Poisson formula and allows one to compute resultants inductively.

Theorem 1.7 ([Jou91]; see also [CLO05]). Let \( f_i(x_0, \ldots, x_n) := F_i(x_0, \ldots, x_n, 1) \) and \( \overline{F}_i(x_0, \ldots, x_n) := F_i(x_0, \ldots, x_n, 0) \). If \( \text{Res}(\overline{F}_0, \ldots, \overline{F}_{n-1}) \neq 0 \), then the quotient ring \( A = \mathbb{C}[x_0, \ldots, x_n]/(f_0, \ldots, f_{n-1}) \) has dimension \( d_0 \cdots d_{n-1} - 1 \) as a vector space over \( \mathbb{C} \), and
\[
\text{Res}(F_0, \ldots, F_n) = \text{Res}(\overline{F}_0, \ldots, \overline{F}_{n-1})^{d_n} \det(m_{f_n} : A \to A),
\]
where \( m_{f_n} : A \to A \) is the linear map given by multiplication by \( f_n \).

With the same hypotheses as Theorem 1.7, a monomial basis for \( A \) over \( \mathbb{C} \) (useful in the implementation) can be constructed as explained in [CLO05, Chapter 2, §2]. Note also that we have
\[
\det(m_{f_n} : A \to A) = \prod_{p \in V} f_n(p)^{\text{mult}_p(V)},
\]
where \( V = V(f_0, \ldots, f_{n-1}) \).

We now describe the most popular way to compute resultants, which is due to Macaulay [Mac02]. Let \( \delta = \sum_{i=0}^{n} d_i - n \) and \( N = \binom{n+\delta}{n} \). We can divide the monomials \( x^\alpha \) of total degree \( \delta \) into the \( n + 1 \) mutually disjoint sets
\[
S_i := \{x^\alpha : |\alpha| = \delta, \min\{j : x_j^{d_j} | x^\alpha| = i\} \}, \text{ for } i = 0, \ldots, n.
\]
A monomial \( x^\alpha \) of total degree \( \delta \) is called reduced if \( x_i^{d_i} \) divides \( x^\alpha \) for exactly one \( i \). Consider the following \( N \) homogeneous polynomials of degree \( \delta \):
\[
x^\alpha / x_i^{d_i} F_i, \text{ for } i = 0, \ldots, n \text{ and } x^\alpha \in S_i.
\]

By regarding the monomials of total degree \( \delta \) as unknowns, the polynomials in (1.3) form a system of \( N \) linear equations in \( N \) unknowns. Let \( \mathbb{D} = \mathbb{D}(F_0, \ldots, F_n) \) denote the coefficient matrix of this linear system, and let \( \mathbb{D}'(F_0, \ldots, F_n) \) denote the submatrix of \( \mathbb{D} \) obtained by deleting all rows and columns corresponding to reduced monomials. The following result is called Macaulay formula and allows one to compute the resultant as a quotient of two determinants.

Theorem 1.8 ([Mac02]; see also [Jou97, CLO05]). The following formula holds:
\[
\det(\mathbb{D}(F_0, \ldots, F_n)) = \text{Res}(F_0, \ldots, F_n) \det(\mathbb{D}'(F_0, \ldots, F_n)).
\]
1.2. Discriminants. Let $F = \sum_{|\alpha|=d} c_{\alpha} x^\alpha \in \mathbb{C}[x_0, \ldots, x_n]$ be a homogeneous polynomial of a certain degree $d$. As above, for each index $\alpha$ we introduce a variable $u_{\alpha}$ and form the universal ring of coefficients $\mathbb{U}_d := \mathbb{C}[u_{\alpha} : |\alpha| = d]$. Then one can show that, up to sign, there is a unique polynomial $\text{Dis} = \text{Dis}_d \in \mathbb{U}_d$ which has the following properties:

1. if $F \in \mathbb{C}[x_0, \ldots, x_n]$ is homogeneous of degrees $d$, then the equations
   \[ \frac{\partial F}{\partial x_0} = 0, \ldots, \frac{\partial F}{\partial x_n} = 0 \]
   have a nontrivial solution over $\mathbb{C}$ (i.e., the hypersurface defined by $F$ is singular) if and only if $\text{Dis}(F) = 0$;

2. $\text{Dis}$ is irreducible, even when regarded as a polynomial over $\mathbb{C}$.

Proposition 1.10 ([GKZ94]). Up to sign, we have the formula

\[ \text{Dis}(F) = c_{d,n} \text{Res}(\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n}), \]

where $c_{d,n} = d^{\frac{1}{2}(n+1)(d-1)^n}.$

Definition 1.11. We call the polynomial defined by (1.5) the discriminant of $F$.

Proposition 1.12 ([GKZ94]). The following hold:

1. The polynomial $\text{Dis}$ is homogeneous of degree $(n+1)(d-1)^n$.

2. For each $(n+1) \times (n+1)$ matrix $A$ over $\mathbb{C}$, we have
   \[ \text{Dis}(F(Ax)) = \det(A)^{d(d-1)^n} \text{Dis}(F(x)), \]
   where $Ax$ denotes the product of $A$ with the column vector $(x_0, \ldots, x_n)'$.

Geometrically, we have the following interpretation.

Proposition 1.13 ([GKZ94]). The discriminant hypersurface $V(\text{Dis}_d)$ in the space of forms of degree $d$ on $\mathbb{P}^n$ coincides with the dual variety of the $d$-th Veronese embedding of $\mathbb{P}^n$. 
1.3. Chow forms. Let \( X \subseteq \mathbb{P}^n \) be an irreducible subvariety of dimension \( k \) and degree \( d \). Consider the subvariety \( Z(X) \) in the Grassmannian \( \mathbb{G}(n-k-1, \mathbb{P}^n) \) of all \((n-k-1)\)-dimensional projective subspaces of \( \mathbb{P}^n \) that intersect \( X \). It turns out that \( Z(X) \) is an irreducible hypersurface of degree \( d \); thus \( Z(X) \) is defined by the vanishing of some element \( R_X \), unique up to a constant factor, in the homogeneous component of degree \( d \) of the coordinate ring of the Grassmannian \( \mathbb{G}(n-k-1, \mathbb{P}^n) \) in the Plücker embedding. This element is called Chow form of \( X \). It is notable that \( X \) can be recovered from its Chow form. See [GKZ94, Chapter 3, § 2] for details.

Consider the product \( \mathbb{P}^k \times X \) as a subvariety of \( \mathbb{P}^{(k+1)(n+1)-1} \) via the Segre embedding. Identify \( \mathbb{P}^{(k+1)(n+1)-1} \) with the projectivization \( \mathbb{P}(\text{Mat}(k+1, n+1)) \) of the space of \((k+1) \times (n+1)\) matrices and consider the natural projection \( \rho : \mathbb{P}(\text{Mat}(k+1, n+1)) \to \mathbb{G}(k,n) \simeq \mathbb{G}(n-k-1,n) \). The following result is called Cayley trick.

**Theorem 1.14** ([GKZ94]; see also [WZ94]). The dual variety of \( \mathbb{P}^k \times X \) coincides with the closure \( \overline{\rho^{-1}(Z(X))} \), where \( Z(X) \subseteq \mathbb{G}(n-k-1,n) \) is the hypersurface defined by the Chow form of \( X \).

The defining polynomial of the hypersurface \( \overline{\rho^{-1}(Z(X))} \subseteq \mathbb{P}(\text{Mat}(k+1, n+1)) \) is called \( X \)-resultant; it provides another way of writing the Chow form of \( X \).

Now, let \( F_0, \ldots, F_n \) be \( n+1 \) generic homogeneous polynomial on \( \mathbb{P}^n \) of the same degree \( d \geq 0 \), and let \( \mathbb{M} = \mathbb{M}(F_0, \ldots, F_n) \) be the \((n+1) \times N\) matrix of the coefficients of these polynomials, \( N = \binom{n+d}{n} \). We consider the projection \( \rho_{n,d} : \mathbb{P}(\text{Mat}(n+1, N)) \to \mathbb{G}(n,N-1) \simeq \mathbb{G}(N-n-2,N-1) \) defined by the maximal minors of \( \mathbb{M} \).

**Proposition 1.15** ([GKZ94, CLO05]). The hypersurface of degree \((n+1)d^n\) in \( \mathbb{P}(\text{Mat}(n+1, N)) \) defined by the resultant \( \text{Res}(F_0, \ldots, F_n) \) coincides with the closure \( \overline{\rho_{n,d}^{-1}(V(R_{n,d}))} \), where \( R_{n,d} \) denotes the Chow form of the \( d \)-th Veronese embedding of \( \mathbb{P}^n \). In particular, \( \text{Res}(F_0, \ldots, F_n) \) is a polynomial in the maximal minors of \( \mathbb{M} \).

2. Implementation

In this section, we illustrate briefly some of the methods available in the package \texttt{Resultants}, included with Macaulay2 [GS16]. We refer to the package documentation (which can be viewed with \texttt{viewHelp Resultants}) for more details and examples.

One of the main methods is \texttt{Resultant}, which accepts as input a list of \( n+1 \) homogeneous polynomials in \( n+1 \) variables with coefficients in some commutative ring \( A \) and returns an element of \( A \), the resultant of the polynomials. There are no limitations on the ring \( A \) because of Remark 1.3. The algorithm implemented are the Poisson formula (Theorem 1.7) and the Macaulay formula (Theorem 1.8). The former is used by default since it is typically faster, while for the latter one has to set the Algorithm option: \texttt{Resultant(..., Algorithm=>Macaulay)}.

The method can also be configured to involve interpolation of multivariate polynomials (see [CM93]), i.e. it can reconstruct the polynomial resultant from its values at a sufficiently large number of points, which in turn are evaluated using the same formulas. The main derived method is Discriminant, which applies (1.5) to compute discriminants of homogeneous polynomials.
Example 2.1. In the following code, we take two forms $F,G$ of degree 6 on $\mathbb{P}^3$. We first verify that $\text{Dis}(F) = 0$ and $\text{Dis}(G) \neq 0$ and then we compute the intersection of the pencil generated by $F$ and $G$ with the discriminant hypersurface in the space of forms of degree 6 on $\mathbb{P}^3$, which is a hypersurface of degree 500 in $\mathbb{P}^8^3$. (The algorithm behind is the Poisson formula; this is one of the cases where the Macaulay formula is much slower).

```
Macaulay2, version 1.9.2.1
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : loadPackage "Resultants";
i2 : ZZ[w,x,y,z]; (F,G) = (w^6+x^6+y^6+w*x*y^4,w^6+x^6+y^6+z^6)
6 6 4 6 6 6 6 6
o3 = (w + x + w*x*y + y , w + x + y + z )
o3 : Sequence
i4 : time Discriminant F
-- used 0.0316056 seconds
o4 = 0
i5 : time Discriminant G
-- used 0.0468577 seconds
o5 = 1405708114831691994706380179327883585442821877717397846563248267679562160278
476323640662415208552366768116974888824357602177140783996641050196723813
397482285763888016904232984135762316136175977862425221732448345919411204360
2458289920741512289591637737144663616815976480976575307073983449978648683
601657856
i6 : R:=ZZ[t,u][w,x,y,z]; pencil = t*sub(F,R) + u*sub(G,R)
6 6 4 6 6
o7 = (t + u)w + (t + u)x + t*w*x*y + (t + u)y + u*z
o7 : ZZ[t, u][w, x, y, z]
i8 : time D = Discriminant pencil
-- used 8.50332 seconds
375 125 374 126 ...
10091123920621172...000t u + 37941930001954874...000t u ...
o8 : ZZ[t, u]
i9 : time factor D
-- used 0.0705482 seconds
125 195 3 2 2 2 2 3 3 2 3 3
o9 = (u) (t + u) (25t + 81t u + 81t*u + 27u) (29t + 81t u + 81t*u + 27u) (15624255880587450170437507046708633270299798826595329963898110452369305869880160529524
7551216082478521015733247707536054199723946299843256347645863521276292845401973114044
08565582286130499837676997663301332329790198932824457503901180704530266391399673715314
825491157575353906490653026526751205470651399776)
o9 : Expression of class Product
In particular, we deduce that the pencil $\langle F,G \rangle$ intersects the discriminant hypersurface in $F$ with multiplicity 125, in $F - G$ with multiplicity 195, and in other 6 distinct points with multiplicity 30.

The package also provides methods for working with Chow forms and more generally tangential Chow forms of projective varieties (see [GKZ94, p. 104], [GM86, Koh16]). In the following example, we apply some of these methods.

Example 2.2. Take $C \subset \mathbb{P}^3$ to be the twisted cubic curve.
```
i10 : QQ[a_0..a_3]; C = trim minors(2,matrix{{a_0,a_1,a_2},{a_1,a_2,a_3}})
2 2
o11 = ideal (a - a a , a a - a a , a - a a )
2 1 3 1 2 0 3 1 0 2
The Chow form of $C$ in $\mathbb{G}(1,3)$ can be obtained as follows:

```plaintext
i12 : time w = ChowForm C
  -- used 0.0644871 seconds
3 2 2
o12 = a - a a a + a a + a a - 2a a a - a a a
  1,2 0,2 1,2 1,3 0,1 1,3 0,2 2,3 0,1 1,2 2,3 0,1 0,3 2,3
QQ[a , a , a , a , a , a ]
  0,1 0,2 1,2 0,3 1,3 2,3
o12 : ------------------------------------------------------
  a a - a a + a a
  1,2 0,3 0,2 1,3 0,1 2,3
We can recover $C$ from its Chow form by taking the so-called Chow equations ([GKZ94], p. 102], [Cat92]).

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The latter can compute different resultant matrices; in particular, it contains an implementation of the Macaulay formula.

REFERENCES

[BGW88] C. Bajaj, T. Garrity, and J. Warren, On the applications of multi-equational resultants, Tech. Report 88-826, Department of Computer Science, Purdue University, 1988.

[BJ14] L. Busé and J.-P. Jouanolou, On the discriminant scheme of homogeneous polynomials, Math. Comput. Sci. 8 (2014), no. 2, 175–234.

[Cat92] F. Catanese, Chow varieties, Hilbert schemes and moduli spaces of surfaces of general type, J. Algebraic Geom. 1 (1992), 561–596.

[CLO05] D. Cox, J. Little, and D. O’Shea, Using algebraic geometry, second ed., Grad. Texts in Math., vol. 185, Springer, 2005.

[CM93] J. Canny and D. Manocha, Multipolynomial resultant algorithms, J. Symbolic Comput. 15 (1993), 99–122.

[Dem12] M. Demazure, Résultant, discriminant, Enseign. Math. (2) 58 (2012), no. 3-4, 333–373, an unpublished manuscript dated 1969.

[dW50] B. Van der Waerden, Modern algebra, Volume II, F. Ungar Publishing Co., New York, 1950.

[EM98] M. Elkadi and B. Mourrain, Some applications of Bezoutians in effective algebraic geometry, Tech. Report 3572, INRIA, 1998.

[EM99] I. Emiris and B. Mourrain, Matrices in elimination theory, J. Symbolic Comput. 28 (1999), no. 1, 3–44.

[GKZ94] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, Discriminants, resultants, and multidimensional determinants, Mathematics: Theory & Applications, Birkhäuser Boston, 1994.

[GM86] M. Green and I. Morrison, The equations defining Chow varieties, Duke Math. J. 53 (1986), no. 3, 733–747.

[GS16] D. R. Grayson and M. E. Stillman, MACAULAY2 — A software system for research in algebraic geometry (version 1.9.2), available at http://www.math.uiuc.edu/Macaulay2/, 2016.

[Jou91] J.-P. Jouanolou, Le formalisme du résultat, Adv. Math. 90 (1991), no. 2, 117–263.

[Jou97] J.-P. Jouanolou, Formes d’inertie et résultat: un formulaire, Adv. Math. 126 (1997), no. 2, 119–250.

[Koh16] K. Kohn, Coisotropic hypersurfaces in the Grassmannian, available at https://arxiv.org/abs/1607.05932, 2016.

[KSY94] D. Kapur, T. Saxena, and L. Yang, Algebraic and geometric reasoning using Dixon resultants, Proceedings of International Symposium on Symbolic and Algebraic Computation (New York), ACM Press, 1994, pp. 99–107.

[Mac02] F. Macaulay, On some formulas in elimination, Proc. Lond. Math. Soc. 3 (1902), 3–27.

[WZ94] J. Weyman and A. Zelevinsky, Multiplicative properties of projectively dual varieties, Manuscripta Math. 82 (1994), no. 1, 139–148.

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