Thermodynamics of Black Holes from Equipartition of Energy and Holography

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Abstract

A gravitational potential in the relativistic case is introduced as an alternative to Wald’s potential used by Verlinde, which reproduces the familiar entropy/area relation $S = A/4$ (in the natural units) when Verlinde’s idea is applied to the black hole case. Upon using the equipartition rule, the correct form of the Komar mass (energy) can also be obtained, which leads to the Einstein equations. It is explicitly shown that our entropy formula agrees with Verlinde’s entropy variation formula in spherical cases. The stationary space-times, especially the Kerr-Newman black hole, are then discussed, where it is shown that the equipartition rule involves the reduced mass, instead of the ADM mass, on the horizon of the black hole.

Key words: Black hole thermodynamics, holographic principle

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I. INTRODUCTION

The discovery of black hole entropy and thermodynamics \cite{1} reveals a rather general and profound relation between gravity and thermodynamics. Later, based on the area law of entropy for all local acceleration horizons, the Einstein equations were derived from the first law of thermodynamics \cite{2}. Recently, Padmanabhan reinterpreted the relation $E = 2TS$ \cite{3} between the Komar energy, temperature and entropy as the equipartition rule of energy \cite{4}, and Verlinde derived the Einstein equations \cite{5} from the equipartition rule of energy and the holographic principle \cite{6}. Some related works can be seen in \cite{7–24}.

A well-accepted fact is the entropy of black hole horizon satisfies $S = A/4$, i.e. the Bekenstein-Hawking entropy. It is interesting to ask whether Verlinde’s idea will match this well-known fact. If the answer is positive, it will strongly support Verlinde’s idea. However, in Verlinde’s proposal and following works, only the change of entropy are concentrated on, while the entropy itself is not clearly discussed. In this paper, we try to consider this problem. Based on the linear superposition of the gravitational potential in the nonrelativistic case, we find that the entropy associated to the nonrelativistic (Laplace) horizon naturally satisfies the familiar entropy/area relation $S = A/4$. Then we generalize the nonrelativistic entropy formula to the relativistic case, but it is shown that we must use a new potential, instead of Wald’s potential used by Verlinde, in order to obtain the correct entropy/area relation $S = A/4$ on the black hole horizon. Because the potential plays a critical role in Verlinde’s derivation in the relativistic case, we need to check whether Verlinde’s proposal works for our new potential. First, we show that this new potential can lead to the correct form of the Komar mass (energy) and so the Einstein equations, as well, upon using the equipartition rule. Furthermore, We also verify our relativistic entropy formula by showing that it matches Verlinde’s formula for the change of entropy in two spherical cases. That evidence means that our potential and entropy formula maybe correct.

In the stationary case, it is obvious that there is ambiguity on the choice of the Killing vector, which gives no contribution in the static case. This ambiguity can be largely avoided. To be explicit, we take the Kerr-Newman black hole as a rather general example, where we introduce a canonical choice of the Killing vector and show that the equipartition rule involves the reduced mass, instead of the ADM mass, on the horizon of the black hole.
II. NONRELATIVISTIC ENTROPY

According to Verlinde’s discussion \[5\], if the change of gravitational potential on the holographic screen \(S\) induced by the movement of a particle (with mass \(m\)) outside the screen\(^1\) is \(\delta \Phi\), then the corresponding change of entropy density \(\delta s\) is determined by\(^2\)

\[
\delta s dA = -k_B \frac{\delta \Phi}{2c^2} dN,
\]

where \(dN\) is the number of degrees of freedom (bits) on the infinitesimal area element \(dA\), given by

\[
dN = \frac{c^3}{G\hbar} dA.
\]

So we have

\[
\delta S = \int_S \delta s dA = -\int_S k_B \frac{c}{2G\hbar} \delta \Phi dA.
\]

It is easy to check that, given the Poisson equation satisfied by the gravitational potential, a normal shift \(\delta x\) of the particle approaching the screen will induce the correct change of entropy

\[
\delta S = -2\pi k_B \frac{mc}{\hbar} \delta x,
\]

if the screen is an equipotential surface.

Since the nonrelativistic gravitational potential satisfies linear superposition, it is reasonable to choose the entropy associated to the screen as

\[
S = -\int_S k_B \frac{c}{2G\hbar} \Phi dA,
\]

where we have omitted a possible integration constant. If we take a point source \(M\) with the potential \(\Phi = -GM/r\), then we have

\[
S = \int_S k_B \frac{c}{2\hbar} \frac{M}{r} dA = \frac{2\pi k_B c}{\hbar} Mr.
\]

On the Laplace horizon \(\Phi = c^2/2\),\(^3\) the entropy becomes

\[
S = \frac{k_B c^3}{4G\hbar} A,
\]

which is just the familiar entropy/area relation for the (relativistic) black hole horizon, but now in the nonrelativistic context.

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1. By this term we mean that the particle is in the already emerged space.
2. The area element \(dA\) is missing in eq.(4.22) of Verlinde’s original article \[5\].
3. The nonrelativistic horizon was taken at \(\Phi = 2c^2\) in Figure 4 of Verlinde’s original article \[5\].
III. RELATIVISTIC ENTROPY

In the relativistic case, the explicit form of the change of entropy density has not been discussed in [5]. A natural choice is to directly generalize eq. (1) to the relativistic case, i.e.

$$\delta s dA = -k_B \frac{\delta \Phi}{2c^2} dN,$$

so eq. (5) becomes

$$S = -\int_S \frac{k_B c}{2G\hbar} \Phi dA,$$

(9)

where $\Phi$ is the relativistic counterpart of the nonrelativistic potential $\Phi$. However, if we let

$$\Phi = \frac{c^2}{2} \ln(-\xi^2)$$

(10)
as proposed by Verlinde, where $\xi^a$ is a time-like Killing vector, it is easy to see that the above entropy is divergent when the screen approaches the black hole horizon.

The above problem can be avoided by choosing an alternative relativistic potential

$$\Phi = -\frac{c^2}{2} (1 + \xi^2),$$

(11)

where $\xi^a$ is also the time-like Killing vector normalized at spacial infinity, i.e. $\xi^2 = -1$ at spacial infinity. This potential has the same asymptotic behavior as Verlinde's potential (10) when the screen goes towards the spacial infinity (near the spacial infinity, the behavior of the norm square of $\xi^a$ is $\xi^2 \to -1 + \frac{2M}{r} + o(r^{-1})$ [25]), but correctly gives

$$S = \frac{k_B c^3}{4G\hbar} A$$

(12)

when the screen approaches the black hole horizon $\xi^2 \to 0$.

In Verlinde's proposal, the gravitational potential plays a central role. Using his gravitationnal potential and the equipartition rule on the screen, Verlinde finds that the quasi-local energy contained in the screen is just the Komar energy. This agrees with the standard result of general relativity. Based on this result, he also gets the Einstein equation. In order to check whether our potential is correct, we must show that our potential (11) can also give the correct quasi-local energy (Komar mass) inside the screen via the equipartition rule.
First, the temperature

\[ k_B T = \frac{\hbar}{2\pi c} N^a \nabla_a \Phi \]

\[ = -\frac{\hbar c}{2\pi} N^a \xi^b \nabla_a \xi_b \]

\[ = \frac{\hbar c}{2\pi} N^a \xi^b \nabla_b \xi_a \]

\[ = \frac{\hbar c}{2\pi} (-\xi^2) N^a \frac{\xi^b}{|\xi|^2} \nabla_b \xi_a \]

\[ = \frac{\hbar c}{2\pi} (-\xi^2) N^a \frac{\xi^b}{|\xi|} \nabla_b \xi_a - \frac{\hbar c}{2\pi c} (-\xi^2)(N \cdot \xi) \frac{\xi^b}{|\xi|} \nabla_b \frac{1}{|\xi|} \]

\[ = \frac{\hbar c}{2\pi} (-\xi^2) N^a \frac{\xi^b}{|\xi|} \nabla_b \xi_a - \frac{\hbar c}{2\pi c} (-\xi^2)(N \cdot \xi) \frac{\xi^b}{|\xi|} \frac{1}{|\xi|^3} \nabla_b (-\xi_a \xi^a) \]

\[ = \frac{\hbar c}{2\pi} (-\xi^2) N^a \frac{\xi^b}{|\xi|} \nabla_b \xi_a - \frac{\hbar c}{2\pi c} \xi^2(N \cdot \xi) \frac{\xi^b}{|\xi|^4} \nabla_b \xi_a \]

\[ = \frac{\hbar c}{2\pi} (-\xi^2) N^a \frac{\xi^b}{|\xi|} \nabla_b \xi_a - \frac{\hbar c}{2\pi c} \xi^2(N \cdot \xi) \frac{\xi^b}{|\xi|^4} \nabla_b \xi_a \]

\[ = \frac{\hbar c}{2\pi c} (-\xi^2) N^a \xi^b \]

is defined following Verlinde’s proposal (but using our potential), where \( N^b \) is an outward unit vector normal to the screen, and

\[ a^b = c^2 u^a \nabla_a u^b = c^2 \frac{\xi^a}{|\xi|} \nabla_a \frac{\xi^b}{|\xi|} \]

the proper acceleration. We interpret this temperature as measured with respect to the reference point at spatial infinity, since the potential \( \Phi \) is defined with respect to that reference point. But when employing the equipartition rule at the screen, we must use the local temperature \( T_S \) just measured there, which is determined by the well-known Tolman relation

\[ T = \sqrt{-\xi^2 T_S}. \]
Then, the (generalized) equipartition rule reads

\[
E = \frac{1}{2} \int_S k_B T_S dN = \frac{\hbar}{4\pi c} \int_S \sqrt{-\xi^2 N^b a_b} dN = \frac{c^2}{4\pi G} \int_S \sqrt{-\xi^2 N^b a_b} dA = \frac{c^2}{4\pi G} \int_S 2 \epsilon N^a \xi^b \nabla_b \xi_a = \frac{c^2}{8\pi G} \int_S *d\xi = M_{\text{Komar}}(S, \xi),
\]

so it is clear that the energy got from holographic principle is indeed the Komar energy inside the screen. Then the strategy given by Verlinde, from the expression of the Komar mass (energy) to the Einstein equations, can be similarly followed. So we have shown that Verlinde’s proposal also works for our new potential.

In [5], the relativistic generalization of eq. (14) is taken to be

\[
\nabla_a S = -2\pi k_B \frac{mc}{\hbar} N_a.
\]

Now we want to check the above relation from our assumptions (9) and (11) in some simple settings.

First, let us consider a Schwarzschild black hole surrounded by a spherical thin shell. The mass of black hole is \(M_i\), the (proper) mass of the thin shell is \(m\), and the radius of the shell is \(R_m\). The mass \(m\) distributes uniformly on the shell. The screen \(S\) can be located between the horizon of the black hole and the thin shell, with radius \(R\), i.e. \(2M_i < R < R_m\). The shell is formed by dusts. In fact, each particle of the shell is held by some extra force which keeps the shell staying at its position stationarily. Such a configuration can be realized by the following way: we splice a Schwarzschild solution with mass \(M_i\) into another Schwarzschild solution with mass \(M_o\), where \(\Delta M = M_o - M_i\) can be determined later. The boundary between these two solutions is at \(r = R_m\), i.e. the thin shell.

Because of the Birkhoff’s theorem, we know that the metric should be in natural units

\[
ds^2 = \begin{cases} 
-(1 - \frac{2M_o}{r})dt^2 + (1 - \frac{2M_i}{r})^{-1}dr^2 + r^2 d\Omega^2, & r > R_m; \\
-C(1 - \frac{2M_i}{r})dt^2 + (1 - \frac{2M_i}{r})^{-1}dr^2 + r^2 d\Omega^2, & R_m > r > R.
\end{cases}
\]
Here $C$ is some constant. Based on the standard requirements of constructing thin shell solutions \[26\], the tangent components of the metric on the thin shell should be continuous, so we obtain the metric besides the thin shell as

\[
ds^2 = \begin{cases} 
-(1 - \frac{2M_o}{r})dt^2 + (1 - \frac{2M_o}{r})^{-1}dr^2 + r^2d\Omega^2, & r > R_m; \\
\frac{R_m-2M_o}{R_m-2M_i}(1 - \frac{2M_i}{r})dt^2 + (1 - \frac{2M_i}{r})^{-1}dr^2 + r^2d\Omega^2, & R_m > r > R.
\end{cases}
\] (19)

Like an ordinary thermodynamic process, let the thin shell collapse towards the screen quasi-statically. The radius of the shell is $R_{mo}$ at the beginning of this process and is $R_{mi}$ at the end. Because the process is quasi-static, each middle state in the process can be described by the metric (19) with different $R_m$, where $R_{mi} < R_m < R_{mo}$. If we consider an infinitesimal quasi-static collapsing process ($R_{mo} - R_{mi} \ll R_{mi}$), the change of the entropy at the screen is

\[
\delta S = -\int_S \frac{k_B}{2} \delta \Phi dA \\
= -\int_S \frac{k_B}{2} \frac{d\Phi}{dR_m} \delta R_m dA \\
= -\frac{k_B}{2} \frac{\Delta M}{4\pi R^2} \frac{1}{(R_m - 2M_i)^2} \delta R_m. 
\] (20)

The energy-momentum tensor of the shell can be evaluated, by substituting the metric (19) into the Einstein equations, as

\[
T_{ab} = \frac{\Delta M}{4\pi r^2} \delta (r - R_m) u_a u_b. 
\] (21)

Comparing the above equation with the standard expression

\[
T_{ab} = \frac{m}{4\pi r^2 \sqrt{g_{rr}}} \delta (r - R_m) u_a u_b 
\] (22)

in terms of the mass $m$, we see that for $\Delta M \ll M_i$

\[
\Delta M = (1 - \frac{2M_i}{R_m})^{1/2} m.
\] (23)

Thus, the change of entropy is

\[
\delta S = -2\pi k_B \Delta M \frac{R^2}{R_m^2} (1 - \frac{2M_i}{R_m})^{-2} (1 - \frac{2M_i}{R}) \delta R_m \\
= -2\pi k_B m \frac{R^2}{R_m^2} (1 - \frac{2M_i}{R_m})^{-1} (1 - \frac{2M_i}{R}) \delta l,
\] (24)
where \( l \) is the proper length in the normal direction of the shell. When the thin shell is very close to the screen \((R_m \rightarrow R)\), we just obtain eq. (17).

Second, we consider another example, a charged thin shell surrounds a Schwarzschild black hole. In this case, the gravity is balanced by the electric force. Such solution has been studied very carefully [27]. Because of the Birkhoff’s theorem, the metric is

\[
\begin{align*}
 ds^2 &= \begin{cases} 
 -(1 - \frac{2M_o}{r} + \frac{q^2}{r^2})dt^2 + (1 - \frac{2M_o}{r} + \frac{q^2}{r^2})^{-1}dr^2 + r^2d\Omega, & r > R_m; \\
 -C(1 - \frac{2M_i}{r})dt^2 + (1 - \frac{2M_i}{r})^{-1}dr^2 + r^2d\Omega, & R < r < R_m;
\end{cases} \\
 C &= \frac{1 - \frac{2M_o}{R_m} + \frac{q^2}{R_m^2}}{1 - \frac{2M_i}{R_m}}. 
\end{align*}
\]

The parameters of the system satisfy [27]

\[
m = R_m \left( \sqrt{1 - \frac{2M_i}{R_m}} - \sqrt{1 - \frac{2M_o}{R_m} + \frac{q^2}{R_m^2}} \right). 
\]

If this shell experiences an infinitesimal quasi-static process, the change of entropy on the screen is

\[
\begin{align*}
 \delta S &= -\int_S \frac{k_B}{2} \delta \Phi dA \\
 &= -\int_S \frac{k_B}{2} \frac{1}{2} \delta C \left( 1 - \frac{2M_i}{r} \right) \delta R_m dA \\
 &= -\frac{k_B}{2} 4\pi R^2 \frac{1}{2} \frac{\partial C}{\partial R} \left( 1 - \frac{2M_i}{R} \right) \delta R_m \\
 &= -\frac{k_B}{2} 4\pi R^2 \frac{1}{2} \left( 1 - \frac{2M_i}{R} \right) \left( \frac{2M_o}{R_m} - \frac{2q^2}{R_m^2} \right) \left( 1 - \frac{2M_o}{R_m} + \frac{q^2}{R_m^2} \right) \delta R_m \\
 &= -\frac{k_B}{2} 4\pi R^2 \frac{1}{2} \left( 1 - \frac{2M_i}{R} \right) \left( 1 - \frac{2M_i}{R_m} \right)^{-2} 2 \frac{R^2}{R_m^2} \left( \Delta M - \frac{q^2}{R_m} + \frac{M_i q^2}{R_m R_m} \right) \delta R_m \\
 &= -2\pi k_B \left( 1 - \frac{2M_i}{R} \right) \left( 1 - \frac{2M_i}{R_m} \right)^{-2} \frac{R^2}{R_m^2} \left[ \left( \Delta M - \frac{q^2}{2R_m} \right) - \left( 1 - \frac{2M_i}{R_m} \right) \frac{q^2}{2R_m} \right] \delta R_m. 
\end{align*}
\]

In our case, the mass and charge of the shell is very small, so we have from eq. (26)

\[
m = \frac{1}{\sqrt{1 - \frac{2M_i}{R_m}}} \left( \Delta M - \frac{q^2}{2R_m} \right). 
\]

Substituting the above result into eq. (27), the change of entropy on the screen is

\[
\delta S = -2\pi k_B \frac{R^2 - 2M_i R}{R_m^2 - 2M_i R_m} \left( m - \sqrt{1 - \frac{2M_i}{R_m}} \right) l. 
\]
When the shell is very close to the screen ($R_m \to R$), the change of entropy becomes

$$\delta S = -2\pi k_B (m - \sqrt{1 - \frac{2M_i}{R} \frac{q^2}{2R}}) \delta l. \quad (30)$$

Comparing this result with eq. (17), the second part in the bracket can be regarded as the contribution of the electric field of the shell, which is from the electric self-interaction.

IV. STATIONARY SPACE-TIMES AND BLACK HOLES

In [5], Verlinde only considers the static case, but in fact, similar considerations can be applied to the stationary case, as can be seen from the previous section. As we have emphasized, the potential plays a central role in Verlinde’s proposal, which depends on the Killing vector. In fact, Verlinde’s derivation of the Komar integral is valid for any Killing vector $\xi^a$. If there exist more than one Killing vector, there will be ambiguity on the choice of the Killing vector $\xi^a$. In the static case, such ambiguity gives no contribution to the value of the Komar integral because of the static condition. But in the stationary case, things will be a little complex. The famous “No-Hair” conjecture [28] tells us that the general non-extremal electric-vacuum stationary space-time should be the Kerr-Newman space-time. So we focus on the Kerr-Newman black hole. Because there is an additional Killing vector $\partial_\phi$ besides $\partial_t$, there is ambiguity in choosing the Killing vector $\xi^a$, which should be time-like at least in a neighborhood of the screen.

Under the Boyer-Lindquist coordinates, the standard (1+3)-decomposed form of the Kerr-Newman metric is

$$ds^2 = -\frac{\rho^2}{Q} \frac{\Delta}{\Delta} dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{Q}{\rho^2} \sin^2 \theta (d\phi - a \frac{2Mr - q^2}{Q} dt)^2, \quad (31)$$

where $\rho^2 = r^2 + a^2 \cos^2 \theta$ with $a = J/M$ the angular momentum per unit (ADM) mass, $\Delta = r^2 + a^2 - 2Mr + q^2$ with $q$ the electric charge, and $Q = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta$. Up to a constant overall factor, the most general form of the Killing vector $\xi$ is $\partial_t + \omega \partial_\phi$, with $\omega$ some arbitrary “angular velocity”. The horizon is at $\Delta = 0$, i.e.

$$r_{\pm} = M \pm \sqrt{M^2 - a^2 - q^2}. \quad (32)$$

But we see that in this case, when concerning fixed $(r_S, \theta_S)$ outside the outer horizon, there is a natural choice of the Killing vector $\xi$ as

$$\xi = \partial_t + a \frac{2Mr_S - q^2}{Q_S} \partial_\phi, \quad (33)$$
which is automatically time-like at the given \((r_S, \theta_S)\) and whose norm square is just the square of the lapse function

\[
f^2 = \frac{\rho^2 \Delta}{Q}.
\]  

(34)

We call this \(\xi\) the canonical Killing vector with respect to \((r_S, \theta_S)\). The corresponding acceleration is

\[
a^\mu = u^\nu \nabla_\nu u^\mu = f^{-2} \xi^\nu \nabla_\nu \xi^\mu
\]

\[
= \frac{1}{2} f^{-2} g^{\mu \nu} \partial_\nu f^2,
\]

so we have

\[
a^r = f^{-2} g^{rr} \partial_r \frac{\rho^2 \Delta}{Q} = f^{-2} g^{rr} \left( r \Delta + \rho^2 (r - M) - \frac{\rho^2 \Delta [2r(r^2 + a^2) - (r - M)a^2 \sin^2 \theta]}{Q^2} \right),
\]

(36)

\[
a^\theta = \frac{1}{2} f^{-2} g^{\theta \theta} \partial_\theta \rho^2 \Delta = f^{-2} g^{\theta \theta} \left( -a^2 \sin \theta \cos \theta \Delta - \frac{\rho^2 \Delta (-\Delta a^2 \sin \theta \cos \theta)}{Q^2} \right)
\]

\[
= -f^{-2} g^{\theta \theta} \Delta a^2 \sin \theta \cos \theta \frac{(2Mr - q^2)(r^2 + a^2)}{Q^2},
\]

(37)

\[
a^\phi = 0.
\]

(38)

In order to calculate the Komar energy (16), we must assign a screen \(S\). The key point here is that the Killing vector \(\xi\) is globally defined, so it must be the same everywhere on the screen. Thus we see that a screen adapted to the canonical Killing vector (33) is which satisfies

\[
b = 2Mr - q^2 = \text{const.}
\]

(39)

When \(r\) ranges from \(r_+\) to \(\infty\), \(b\) ranges from \((r_+^2 + a^2)^{-1}\) to 0. Then the Komar energy (16) can be expressed as a function of \(b\), but the explicit form is rather complicated and is not necessary to present here, so we only discuss two important cases: the infinity and the horizon. In both these cases, the screens are spherical \((N^b \partial_b \sim \partial_r)\) and the corresponding expressions simplify drastically.

**Infinity** When \(S \to S_\infty\), the leading-order behavior of \(a^r\) is \(a^r \approx M/r^2\), so eq.(16) gives

\[
E = M, \text{ the ADM mass of the black hole, as expected.}
\]
**Horizon** When $S \to S_H$, we have $Q \to (r_+^2 + a^2)^2$ and

$$\sqrt{-\xi^2 N^b a_b} \to \frac{r_+ - M}{r_+^2 + a^2},$$

so eq. (16) gives $E = M_0 = \sqrt{M^2 - a^2 - q^2}$, the reduced mass of the black hole. Note that when tending to the horizon, the canonical Killing vector (33) is the only possible choice to be a time-like one. Thus we see clearly that when applying the equipartition rule to the black hole horizon, it is the reduced mass $M_0$ that takes the place of the ADM mass $M$.

Finally, we mention briefly two special cases where there are simple, explicit forms of the Komar energy for generic $b$.

- The $a = 0$ case, i.e. the Reissner-Nordstrom black hole. In this case, the canonical Killing vector (33) just becomes $\partial_t$ and there is no restriction on the shape of the screen, so the well-known result

$$E = M - \frac{q^2}{r}$$

is recovered for general spherical screens, which tends to the reduced mass $M_0 = \sqrt{M^2 - q^2}$ when $r \to r_+ = M + M_0$.

- The $q = 0$ case, i.e. the Kerr black hole. Since the Komar energy (16) is linear in $\xi$, we have from eq. (33)

$$E = M + abM_\phi,$$

where $M$ and $M_\phi$ are the Komar integrals corresponding to $\partial_t$ and $\partial_\phi$, respectively. In fact, $M$ and $M_\phi = -2aM$ are independent of the screen in this vacuum case, so we obtain the (canonical) Komar energy

$$E = M - 2a^2 b M.$$
V. CONCLUDING REMARKS

In this paper, we introduce a new gravitational potential, instead of Wald’s potential used by Verlinde, in the relativistic case to obtain the correct entropy/area relation \( S = A/4 \) on the black hole horizon. Upon using the equipartition rule of energy, we also obtain the correct form of the Komar mass (energy), which leads to the Einstein equations. We then discuss the stationary space-times, especially the Kerr-Newman black hole, where it is shown that a canonical choice of the Killing vector can be defined, and the equipartition rule involves the reduced mass, instead of the ADM mass, on the horizon of the black hole.

In [5], Verlinde gives the formula of the change of screen entropy caused by movement of a massive particle near the screen. In order to check our relativistic entropy formula, we consider two simple examples and find that our entropy formula agrees with Verlinde’s formula in both cases. However, the general case is much more difficult, due to the high nonlinearity of the Einstein equations. How to prove that relation in the general case is still an open problem.

The recovery of the correct entropy/area relation in the relativistic case is certainly helpful to clarify some subtle points in this case in Verlinde’s discussion, and to further investigate the microscopic, or statistical, meaning of the gravitational thermodynamics, which is left for future works.

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[1] J.M. Bardeen, B. Carter and S.W. Hawking, Commun. Math. Phys. 31, 161 (1973); J.D. Bekenstein, Phys. Rev. D 7, 949 (1973); J.D. Bekenstein, Phys. Rev. D 7, 2333 (1973); S.W. Hawking, Commun. Math. Phys. 43, 199 (1975) [Erratum-ibid. 46, 206 (1976)].
[2] T. Jacobson, Phys. Rev. Lett. 75, 1260 (1995).
[3] T. Padmanabhan, Class. Quantum Grav. 21, 4485 (2004).

[4] T. Padmanabhan, Equipartition of energy in the horizon degrees of freedom and the emergence of gravity, arXiv:0912.3165.

[5] E.P. Verlinde, On the Origin of Gravity and the Laws of Newton, arXiv:1001.0785 [hep-th].

[6] L. Susskind, J. Math. Phys. 36, 6377 (1995). G. t Hooft, gr-qc/9310026. E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998).

[7] F.-W. Shu and Y.-G. Gong, Equipartition of energy and the first law of thermodynamics at the apparent horizon, arXiv:1001.3237 [gr-qc].

[8] R.-G. Cai, L.-M. Cao and N. Ohta, Phys. Rev. D 81, 061501 (2010).

[9] L. Smolin, Newtonian gravity in loop quantum gravity, arXiv:1001.3668 [gr-qc].

[10] J. Makela, Notes Concerning On the Origin of Gravity and the Laws of Newton by E. Verlinde (arXiv:1001.0785), arXiv:1001.3808 [gr-qc].

[11] F. Caravelli and L. Modesto, Holographic actions from black hole entropy, arXiv:1001.4364 [gr-qc].

[12] M. Li and Y. Wang, Quantum UV/IR Relations and Holographic Dark Energy from Entropic Force, arXiv:1001.4466 [hep-th].

[13] C.-J. Gao, Modified Entropic Force, arXiv:1001.4585 [hep-th].

[14] Y. Zhang, Y.-G. Gong and Z.-H. Zhu, Modified gravity emerging from thermodynamics and holographic principle, arXiv:1001.4677 [hep-th].

[15] H. Culetu, Boundary stress tensors for spherically symmetric conformal Rindler observers, arXiv:1001.4740 [hep-th].

[16] Y. Wang, Towards a Holographic Description of Inflation and Generation of Fluctuations from Thermodynamics, arXiv:1001.4786 [hep-th].

[17] T. Wang, The Coulomb Force as an Entropic Force, arXiv:1001.4965 [hep-th].

[18] S.-W. Wei, Y.-X. Liu and Y.-Q. Wang, Friedmann equation of FRW universe in deformed Horava-Lifshitz gravity from entropic force, arXiv:1001.5238 [hep-th].

[19] Y. Ling and J.-P. Wu, A note on entropic force and brane cosmology, arXiv:1001.5324 [hep-th].

[20] J.-W. Lee, H.-C. Kim and J. Lee, Gravity from Quantum Information, arXiv:1001.5445 [hep-th].

[21] L. Zhao, Hidden symmetries for thermodynamics and emergence of relativity, arXiv:1002.0488 [hep-th].
[22] Y.-S. Myung, Entropic force in the presence of black hole, arXiv:1002.0871 [hep-th].

[23] J. Kowalski-Glikman, A note on gravity, entropy, and BF topological field theory, arXiv:1002.1035 [hep-th].

[24] Y.-X. Liu, Y.-Q. Wang and S.-W. Wei, Temperature and Energy of 4-dimensional Black Holes from Entropic Force, arXiv:1002.1062 [hep-th].

[25] R. Bartnik, Commun. Pure Appl. Math. 39 (1986) 661.

[26] W. Isreal, Nuovo Cimento 44 B (1966) 1; V. de la Cruz and W. Isreal, Nuovo Cimento 51 A (1967) 744; K. Kucar, J. Czech, J. Phys. B 18 (1968) 435; K. Lake, Phys. Rev. D 19 (1979) 2847; W.A. Hiscock, J. Math. Phys. 22 (1981) 215.

[27] T. Dray, Class. Quant. Grav. 7 (1990) L131; J.P.S. Lemos and V.T. Zanchin, J. Math. Phys. 47 (2006) 042504; J.P.S. Lemos and O.B. Zaslavskii, Phys. Rev. D 76 (2007) 084030; S. Gao and J.P.S. Lemos, Inter. J. Modern Phys. A, 23 (2008) 2943.

[28] P. T. Chrusciel, Helv. Phys. Acta 69 (1996) 529; P. T. Chrusciel, Contemp. Math. 170 (1994) 23; V. P. Frolov and I. D. Novikov, Black Hole Physics, Kluwer Academic Publishers, 1998.