SEMILINEAR CAPUTO TIME-FRACTIONAL PSEUDO-PARABOLIC EQUATIONS

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(Communicated by Junping Shi)

\textbf{Abstract.} This paper considers two problems: the initial boundary value problem of nonlinear Caputo time-fractional pseudo-parabolic equations with fractional Laplacian, and the Cauchy problem (initial value problem) of Caputo time-fractional pseudo-parabolic equations. For the first problem with the source term satisfying the globally Lipschitz condition, we establish the local well-posedness theory including existence, uniqueness and regularity of the local solution, and the further local existence theory related to the finite time blow-up are also obtained for the problem with logarithmic nonlinearity. For the second problem with the source term satisfying the globally Lipschitz condition, we prove the global existence theorem.

1. Introduction.

1.1. Statement of the problem. In the present paper, we study fractional pseudo-parabolic equations by considering the following two problems.

- The time fractional pseudo-parabolic equation on a bounded domain with fractional Laplacian

\[
\begin{align*}
&D_t^\alpha (u - m\Delta u) (x, t) + (-\Delta)\sigma u(x, t) = \mathcal{N}(u), \quad x \in \Omega, \ t > 0, \\
&u(x, t) = 0, \quad x \in \partial \Omega, \ t > 0, \\
&u(x, 0) = u_0(x), \quad x \in \Omega, \quad (P_1)
\end{align*}
\]

2020 Mathematics Subject Classification. Primary: 26A33, 33E12, 35B40, 35K70, 44A20.

Key words and phrases. Well-posedness, blow-up, Caputo fractional, pseudo-parabolic equation.

The first and the second author were supported by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2019.09. The third author was supported by National Natural Science Foundation of China (11871017).

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where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded domain with smooth boundary $\partial \Omega$.

- The time fractional pseudo-parabolic equation on $\mathbb{R}^N$
  \[
  \begin{cases}
    D_t^\alpha (u - m\Delta u)(x,t) - \Delta u(x,t) = N(u), & x \in \mathbb{R}^N, t > 0, \\
    u(0,x) = u_0(x), & x \in \mathbb{R}^N.
  \end{cases}
  \]  (P_2)

In problems (P_1) and (P_2), $m > 0$ is a constant; the reaction term $N$ is the source term, which will be specified later. Here the symbol $D_t^\alpha$ ($0 < \alpha < 1$) denotes the Caputo time fractional derivative, which is defined by (see [10, 52, 54])

\[ D_t^\alpha u(t) = \mathcal{P}_{[0,t]}^{1-\alpha} \frac{\partial u}{\partial t}, \quad 0 < \alpha < 1, \]

where

\[ \mathcal{P}_{[0,t]}^\alpha f(t) \overset{\text{def}}{=} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad 0 < \alpha < 1, \quad t > 0, \]

and $\Gamma$ is the Gamma function. In (P_1), the fractional Laplace operator of order $\sigma \in (0,1)$ is denoted by $(-\Delta)^\sigma$ (see the definitions in (2.1) and (2.2) below). Our main purpose in this paper is to consider the local and global well-posedness for Problems (P_1) and (P_2) and give some regularities estimates for solutions.

For $m > 0$ and $\alpha = 1$, the main equations in two problems (P_1) and (P_2) are called pseudo-parabolic equations, which describe many physical processes, for example, the leakage of liquid through cracks in rocks or materials [59, 14], the unidirectional propagation of nonlinear, dispersive, long waves [36] (where $u$ is typically the amplitude or velocity) and the aggregation of populations [44] (where $u$ represents the population density). If $m = 0$ in Problem (P_1), we called the space-time fractional diffusion equations used to model anomalous transport, including finance, semiconductor research, biology and hydrogeology [32, 25, 52]. Recently, fractional PDEs has received significant attentions due to their ability of capturing long-term correlations, for example, material properties and memory effects. Since the fractional dissipation operator $(-\Delta)^\sigma$ is nonlocal and can be regarded as the infinitesimal generators of Lévy stable diffusion processes [51, 38], it describes some physical phenomena more exact than integral differential equations [17, 51, 46, 16, 20, 32]. Many important physical models and practical problems require us to consider the pseudo-parabolic model with fractional derivative rather than classical one, like the physical model considering memory effects [18, 19, 13, 48] and some corresponding engineering problems [4, 5, 7, 44, 46, 50, 56], especially the power-law memory (non-local effects) in time and space [20, 25, 51, 13, 38, 39, 42, 40, 42, 53, 52, 54].

1.2. Previous studies on the pseudo-parabolic equations with nonlinear source. The existing literature on this topic is so separated that the Problems (P_1) and (P_2) generalize different diffusion-parabolic equations according to the values of the parameters $\alpha, m,$ and $\sigma$. If $\alpha = 1, m > 0, \sigma = 1$, Problem (P_1) becomes classical pseudo-parabolic problem as follows [49, 24, 9, 55]. In case of $N(u) = u^p$ with $p \geq 1$, Cao et al. [11] consider the following semilinear pseudo-parabolic equation

\[ u_t - m\Delta u_t - \Delta u = u^p. \]  (1.3)

The authors established the necessary existence, uniqueness for mild solutions. They also investigated the large time behavior of solutions. Later, for $m = 1$, Xu and Su [59] studied the Cauchy problems for equation (1.3), and they introduced a family of potential wells to obtain the invariance of some sets, global existence, nonexistence and asymptotic behavior of solutions with low initial energy. Moreover, they obtain
finite time blow-up with high initial energy by comparison principle. When the right hand side of (1.3) is a logarithmic nonlinear term, i.e. $N(u) = a(t)u \log |u|$, Ji, Yin and Cao [35] established the existence of positive periodic solutions, and discussed the instability of such solutions of initial boundary value problem for equation (1.3). The logarithmic nonlinearity expresses slowly cumulative nonlinearity, thus giving another kind of description of the dynamic process. Studies of logarithmic nonlinearity have a long history in physics as it occurs naturally in inflation cosmology, quantum mechanics, and nuclear physics [8]. PDEs with logarithmic nonlinearity have attracted many authors with many interesting results, for instance, [43, 15, 42, 57, 14, 35, 34] and the references therein.

For the source terms in form of polynomial nonlinearities (without the logarithmic functions), i.e. $N_p(u) = \phi_p(u)$, the model has been considered in [1, 4, 59, 55, 47, 23]. We also refer to the recent papers [36, 44, 12, 31, 33, 60, 62, 37, 41, 21, 6] for the study of asymptotic behavior of semilinear pseudo-parabolic equations. If $\alpha = 1, m > 0$, and $0 < \sigma \neq 1$, Ji et al. [29] considered the Cauchy problem of the following fractional pseudo-parabolic equation

$$u_t - m\Delta u_t + (-\Delta)^\sigma u = u^{p+1}, \quad p > 0.$$  \hfill (1.4)

The authors considered the global existence, time-decay rates and the large time behavior of the solutions. In our knowledge, the results on initial value problems for semilinear time-fractional pseudo-equations are still limited and that is the motivation for us to study the Problems ($P_1$) and ($P_2$).

1.3. Contributions and challenges. Let us discuss the main difficulties and our contribution of this paper.

- For Problem ($P_1$), our approach is to consider directly the mild solutions to Problem ($P_1$) taking the form of the integral equations. These equations contain the nonlinear types of the source functions $N(u)$. Our goal is to achieve properties of the solutions to Problem ($P_1$) in Sobolev spaces $W^{k,p}(\Omega)$. We consider two main results on the Lipschitz properties of the source terms $N(u)$.

  - The first goal is the local well-posedness results for globally Lipschitz nonlinearity which is often trivial and not difficult. However, our study here is new in the sense of considering Sobolev spaces $W^{k,p}(\Omega)$. The key ideas for the existence of solutions in $W^{k,p}(\Omega)$ is to apply the Banach’s fixed point theorem. Based on the conditions of the constants $k, p$ depending on the dimensions $N \geq 1$ and the constant $\sigma > 0$, we set up the Sobolev embeddings $H^{2\sigma}(\Omega) \hookrightarrow W^{k,q}(\Omega) \hookrightarrow L^2(\Omega)$ (see definition of the spaces $H^{2\sigma}(\Omega)$ and $W^{k,q}(\Omega)$ in (2.7) and (2.11) below). Moreover, we also establish the results on the regularity of solutions, derivative of solutions and even regularity of fractional derivatives in $W^{k,p}(\Omega)$.

  - The second task is to consider the model with the logarithmic source terms $N_p(u) = \kappa \phi_p(u) \log |u|$ and $\phi_p(u) = |u|^{p-2}u$, $p \geq 2, \kappa > 0$ (the locally Lipschitz type). For the classical pseudo-parabolic equation, most of its results are on weak solution. However, the fractional derivatives are the nonlocal operators, so we are not able anymore to apply the methods on weak solution, like compact method, Galerkin method, and so on. Hence, we have to find a solution in the sense of mild solution. The limited estimates of Mittag-Leffler functions in the formula of the
existence of the mild solution in \(Z\) for even in case of classical pseudo-parabolic equation are limited.

- For Problem \((P_2)\), as an extension of Problem \((P_1)\), we consider the global existence of the mild solution in \(\mathbb{Z}^{\beta,\mu}((0, T]; L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N))\) (see Definition 1.3 below).

- Now, we end the discussion by a short comparison with fractional diffusion equation, i.e., \(m = 0\). The semigroup operator solution of our problem \(E_{\alpha,1}(-L^\sigma(I + mL)^{-1}t^\alpha)\) brings some difficulties in estimating and analyzing the solution. If \(m = 0\), the semigroup operator \(E_{\alpha,1}(-L^\sigma t^\alpha)\) is available for estimating \(L^p - L^q\). However, some estimates of the form \(L^p - L^q\) is not available on the domain \(\mathbb{R}^N\) when we analyze \(E_{\alpha,1}(-L^\sigma(I + mL)^{-1}t^\alpha)\), which is for the Caputo time-fractional pseudo-parabolic equations.

**Remark 1.** In our work, we use the Caputo fractional derivative since it is more suitable for the initial-value problem compared with the Riemann-Liouville fractional derivative [32]. The Caputo time fractional derivative was used recently to model the fractional diffusion in plasma turbulence. Another advantage of using the Caputo fractional derivative in modeling physical problems is that the fractional derivative of constant functions is zero. And the fractional PDEs include the models using the fractional time derivatives and the ones by using fractional Laplacian.

**Remark 2.** To achieve the properties of the solutions in \(W^{s,q}(\Omega)\), we have the challenges of the Lipschitz properties of the source functions \(N\) (both globally Lipschitz property and locally Lipschitz property). Fortunately, the formula of the mild solutions to the Problems \((P_1)\) and \((P_2)\) (see Lemma 2.6 below) contains the operator \((I - m\Delta)^{-1}\), which helps us use the Lipschitz property of \(N\) to overcome such difficulties.

### 1.4. Main results

We first introduce the following hypothesis for the globally Lipschitz source term and the functional space, which are applied for our theorems.

Assume \(N(0) = 0\) for all \((x, t) \in \Omega \times (0, T)\) and satisfies the global Lipschitz condition:

\[
\|N(u) - N(v)\|_{L^2(\Omega)} \leq K\|u - v\|_{L^2(\Omega)},
\]

with \(K > 0\) independent of \(v_1, v_2\).

For \(\rho > 0\), denote by \(C_\rho([0, T]; B)\) the function space \(C_\rho([0, T]; B)\) equipped with the following weighted norm:

\[
\|v\|_{C_\rho([0, T]; B)} := \sup_{0 \leq t \leq T} \|e^{-\rho t}v(t)\|_{B}, \quad \text{for } v \in C_\rho([0, T]; B).
\]

For Theorem 1.1 we have the following Theorem 1.1 and Theorem 1.2. And Theorem 1.4 is for Problem \((P_2)\).

**Theorem 1.1** (Local well-posedness for Problem \((P_1)\) with globally Lipschitz source term). Let \(N \geq 1\) and \(0 < \alpha, \sigma < 1\).

1. Let \(u_0 \in \mathbb{H}^{2\sigma}(\Omega)\), and assume that \(N\) satisfies (H) and \(k < 2\sigma, 1 \leq p \leq \frac{2N}{N + 2k - 4\sigma}\) such that \(kp < N\) and \(\frac{Np}{N - kp} \geq 2\). Then, there exists a unique mild solution \(u \in C_\rho([0, T]; W^{k,p}(\Omega))\) for some \(\rho > 0\) to Problem \((P_1)\).

2. Let \(u_0 \in \mathbb{H}^{2\sigma}(\Omega)\) and assume that \(N\) satisfy (H) and \(k < 2\sigma, 1 \leq p \leq \frac{2N}{N + 2k - 4\sigma}\) such that \(kp < N\) and \(\frac{Np}{N - kp} \geq 2\) and \(\nu \in (0, 1)\). Suppose that \(u\) is the unique solution of Problem \((P_1)\), we have the following conclusions.
a) If \( u_0 \in \mathbb{H}^{2\nu - 2\sigma + 2}(\Omega) \), then we have
\[
\| u \|_{\mathbb{H}^{\alpha}(0, T; \mathbb{H}^{2\nu - 2\sigma + 2}(\Omega))} \lesssim \| u_0 \|_{\mathbb{H}^{2\nu - 2\sigma + 2}(\Omega)}, \quad \forall t \in (0, T].
\] (1.6)

For \( k < 2\nu \) and \( 1 \leq p \leq \frac{2N}{N + 2k - 4r} \), we have
\[
t^\alpha \| u \|_{W^{k,p}(\Omega)} \lesssim \| u_0 \|_{\mathbb{H}^{2\nu - 2\sigma + 2}(\Omega)}, \quad \forall t \in (0, T].
\] (1.7)

b) Let \( 0 < \alpha < \frac{1}{2}, \gamma > \max\{\frac{1}{p} + \frac{1}{q}; \frac{1}{2}\} \) and \( \nu < \gamma + \sigma - \gamma \), \( k < 2\nu \) and \( 1 \leq p \leq \frac{2N}{N + 2k - 4r} \). If \( u_0 \in \mathbb{H}^{2\nu - 2\sigma + 2}(\Omega) \cap \mathbb{H}^{2 - 2\sigma}(\Omega) \), then \( \partial_t u \in L^q(0, T; W^{k,p}(\Omega)) \) for \( 1 \leq q < \frac{1}{1 - \alpha} \) with the following estimate
\[
t^{1 - \alpha} \| \partial_t u(\cdot, t) \|_{W^{k,p}(\Omega)} \lesssim \| u_0 \|_{\mathbb{H}^{2\nu - 2\sigma + 2}(\Omega)} + t^{\alpha \gamma - 2\alpha} \| u_0 \|_{\mathbb{H}^{2\nu - 2\sigma}(\Omega)}, \quad \forall t \in (0, T].
\] (1.8)

c) For \( \alpha, \nu \in (0, 1) \) such that \( \nu + \sigma < 1 \), \( k < 2\nu \) and \( 1 \leq p \leq \frac{2N}{N + 2k - 4r} \).\( u_0 \in \mathbb{H}^{2\nu + 2\sigma - 2}(\Omega) \cap \mathbb{H}^{2 - 2\sigma}(\Omega) \), then \( \mathbb{D}^\alpha u \in L^r(0, T; W^{k,p}(\Omega)) \) for \( 1 \leq r < \min\{\frac{1}{\alpha}; \frac{1}{1}\} \), \( 0 < \gamma < 1 - \alpha \) with the following estimate
\[
t^{\alpha + \gamma} \| \mathbb{D}^\alpha u(\cdot, t) \|_{W^{k,p}(\Omega)} \lesssim t^{\alpha + \gamma} \| u_0 \|_{\mathbb{H}^{2\nu + 2\sigma - 2}(\Omega)} + (t^\gamma + t^\alpha) \| u_0 \|_{\mathbb{H}^{2\nu - 2\sigma}(\Omega)}, \quad \forall t \in (0, T].
\] (1.9)

(III) Suppose that \( \mathcal{N} \) satisfies (H). For \( u_0 \in \mathbb{H}^2(\Omega) \), then the solution \( u \) of Problem (\( \mathbb{P}_1 \)) depends continuously on the initial data in the following sense. If \( u_{0,n} \to u_0 \) in \( \mathbb{H}^2(\Omega) \) and if \( u_n \) is the corresponding solution with initial data \( u_{0,n} \), then \( u_n \to u \) in \( L^\infty(0, T; \mathbb{H}^2(\Omega)) \) over arbitrary existence interval \((0, T)\).

(IV) Assume that \( u_0(x) > 0 \) a.e. \( x \in \Omega \) and the nonnegative and continuous function \( \mathcal{N} \) satisfies the hypothesis (H), then there exists a unique positive solution \( u \in C((0, \infty); L^2(\Omega)) \) of Problem (\( \mathbb{P}_1 \)).

Next, we consider Problem (\( \mathbb{P}_1 \)) with logarithmic nonlinearities source terms \( \mathcal{N}_p(u) = \kappa \phi_p(u) \log |u| \) and \( \phi_p(u) = |u|^{p - 2}u \) for \( p \geq 2, \kappa > 0 \), and present the following theorem of local existence, continuation and blow-up finite time (or global existence) of the solution in \( W^{s,q}(\Omega) \) for Problem (\( \mathbb{P}_1 \)) with logarithmic nonlinearity source terms (the locally Lipschitz type).

**Theorem 1.2** (Local existence, continuation and blow-up finite time for Problem (\( \mathbb{P}_1 \)) with logarithmic nonlinearity). For \( N \geq 1, \alpha \in (0, 1) \) and \( s \in (0, 2\omega_2), 1 \leq q \leq \frac{2N}{N + 2s - 4\omega_2}, 0 \leq \omega_2 < N/4 \) and \( sq < N \). Let \( u_0 \in \mathbb{H}^{2\omega_2}(\Omega) \cap W^{s,q}(\Omega) \), and for the nonlinearity source \( \mathcal{N}_p(u) = \kappa \phi_p(u) \log |u| \) and \( \phi_p(u) = |u|^{p - 2}u \) for \( p \geq 2, \kappa > 0 \), we have the following results.

(I) There is a constant \( T > 0 \) (depending only on \( u_0 \)) such that the Problem (\( \mathbb{P}_1 \)) admits a unique mild solution belongs to \( C^\infty(\mathbb{R}^+; W^{s,q}(\Omega)) \) for some \( \gamma > 0 \).

(II) The unique solution on \((0, T)\) of Problem (\( \mathbb{P}_1 \)) can be extended to the interval \((0, T^*)\), for some \( T^* > T \), so that, the extended function is also the (unique) weak solution of Problem (\( \mathbb{P}_1 \)) on \((0, T^*)\).

(III) There exists a maximal time \( T_{\max} > 0 \) such that the mild solution of (\( \mathbb{P}_1 \)) \( u \in C^\infty((0, T_{\max}); W^{s,q}(\Omega)) \). Thus, either the Problem (\( \mathbb{P}_1 \)) has a unique global mild solution on \([0, \infty)\) or there exists a maximal time \( T_{\max} < \infty \) such that
\[
\lim \sup_{t \to T_{\max}} \| u(\cdot, t) \|_{W^{s,q}(\Omega)} = \infty.
\]
Remark 3. In Theorem 1.2, we obtain results about the properties of the solution in $W^{s,q}$ for $s \in (0,2\omega_2)$, $1 \leq q \leq \frac{2N}{N+2s-4\omega_2}$, $0 \leq \omega_2 < N/4$ and $sq < N$. Then we infer that $s = 0,1,2,...$. This is an improvement of Theorem 1.1 (which only with $s = 0$ and $s = 1$).

Finally, we consider problem $(P_2)$ with power-type nonlinearity on the unbounded domain $\mathbb{R}^N$. Let us also introduce the following space which is applied for our theorem.

Definition 1.3. Let $\mathcal{Y}$ be a Banach space. Define the Banach space $Z^\beta,\mu((0,T];\mathcal{Y})$ of all Bochner integrable functions $u : [0,\infty) \to \mathcal{Y}$ such that $t^\beta u$ are bounded continuous functions, endowed with the norm

$$\|u\|_{Z^\beta,\mu((0,T];\mathcal{Y})} := \sup_{t \in [0,T]} t^\beta e^{-\mu t} \|u(\cdot, t)\|_{\mathcal{Y}} < \infty.$$ (1.10)

for any $\beta \geq 0, \mu \geq 0$.

Theorem 1.4 (Global existence for Problem $(P_2)$). Let $\frac{1}{2} < \alpha < 1$, $0 < \theta < 1 - \frac{1}{2\alpha}$, and $\beta$ satisfy

$$1 + \alpha \theta - \alpha < \beta < \alpha - \alpha \theta,$$ (1.11)

and

$$p > \max\left(1, \frac{1}{1 - \frac{2\alpha - 2\beta}{\alpha N}}, \frac{1}{1 - \frac{2\alpha + 2\beta - 2}{\alpha N}}\right).$$ (1.12)

Suppose that $u_0 \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$, and the function $\mathcal{N} : \mathbb{R} \to \mathbb{R}$ satisfies $\mathcal{N}(0) = 0$ and

$$|\mathcal{N}(v_1) - \mathcal{N}(v_2)| \leq K_d|v_1 - v_2|, \quad v_1, v_2 \in \mathbb{R},$$ (1.14)

where $K_d > 0$ independent of $v_1, v_2$. Then for any $T > 0$, there exists a number $\mu_0$ enough small such that Problem $(P_2)$ admits a unique mild solution in $Z^\beta,\mu((0,T];L^p(\mathbb{R}^N)) \cap L^q(\mathbb{R}^N))$.

Remark 4. Different from Problem $(P_1)$, when we study Problem $(P_2)$, the Sobolev embeddings on $\mathbb{R}^N$ are not available. We overcome this difficulty by considering the weighted Sobolev spaces as in Definition 1.1.

2. Notation and preliminaries.

2.1. Relevant notations. Let us recall that the spectral problem

$$\begin{cases}
(-\Delta)^p \varphi_j(x) = \lambda_j^p \varphi_j(x), & x \in \Omega, \quad \sigma \in (0,1], \\
\varphi_j(x) = 0, & x \in \partial \Omega,
\end{cases}$$ (2.1)

admits a family of eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ... \leq \lambda_j \leq ... \not\to \infty.$$ If the domain under consideration is the whole space $\mathbb{R}^N$,

$$(-\Delta)^p u = \mathcal{F}^{-1}(|\xi|^{2\sigma} \mathcal{F} u),$$ (2.2)

where $\mathcal{F}$ denotes the usual Fourier transform and $\mathcal{F}^{-1}$ represents the inverse Fourier transform in $L^2(\mathbb{R}^N)$. This operator approximate the usual Laplacian as $\sigma \to 1$ and
the identity as $σ → 0$. The operator $(-Δ)^σ$ with $0 < σ < 1$ can also be defined as [17, 22]

$$(-Δ)^σ u(x, t) = C_{N, σ} P.V. \int_{\mathbb{R}^N} \frac{u(x, t) - u(z, t)}{|x - z|^{N + 2σ}} \, dz, \quad σ \in (0, 1), \quad (2.3)$$

with

$$C_{N, σ} = \frac{4^N Γ(N/2 + σ)}{π^{N/2} |Γ(-σ)|} \cdot$$

which is taken in order to be consistent with definition (2.2).

The notation $∥·∥_B$ stands for the norm in the Banach space $B$. For $1 ≤ p ≤ ∞$, $T > 0$, the Banach space of real-valued measurable functions $v : (0, T) → B$ can be well defined with norms

$$∥v∥_{L^p(0, T; B)} = \left( ∫_0^T ∥v(t)∥_B^p \, dt \right)^{1/p}, \quad \text{for } 1 ≤ p < ∞, \quad (2.4)$$

and

$$∥v∥_{L^∞(0, T; B)} = \text{ess sup}_{t \in (0, T)} ∥v(t)∥_B, \quad \text{for } p = ∞. \quad (2.5)$$

The norm of the function space $C^k([0, T]; B)$, for $0 ≤ k ≤ ∞$ is denoted by

$$∥v∥_{C^k([0, T]; B)} = \sup_{i = 0}^k ∥v^{(i)}(t)∥_B < ∞. \quad (2.6)$$

For any real numbers $k > 0$ and $1 ≤ p < ∞$, we recall the fractional Sobolev-type spaces $W^{k, p}(Ω)$ via the Gagliardo approach (also called Aronszajn or Slobodeckij spaces). Fixed a number $k ∈ (0, 1)$ and for any $p ∈ [1, ∞)$, let us define $W^{k, p}(Ω)$ as follows (see [17, 22])

$$W^{k, p}(Ω) = \left\{ v ∈ L^p(Ω) \text{ s.t. } \frac{|v(x) - v(y)|}{|x - y|^{N + kp}} ∈ L^p(Ω × Ω) \right\}. \quad (2.7)$$

For $0 < k < 1$, we can regard the $W^{k, p}(Ω)$ as an intermediate Banach space between $L^p(Ω)$ and $W^{1, p}(Ω)$, endowed the corresponding norm

$$∥v∥_{W^{k, p}(Ω)} = \left( ∫_Ω |v|^p \, dx + ∫_Ω ∫_Ω \frac{|v(x) - v(y)|^p}{|x - y|^{N + kp}} \, dx \, dy \right)^{1/p}, \quad (2.8)$$

where the seminorm

$$|v|_{W^{k, p}(Ω)} = \left( ∫_Ω ∫_Ω \frac{|v(x) - v(y)|^p}{|x - y|^{N + kp}} \, dx \, dy \right)^{1/p}, \quad (2.9)$$

denotes the Gagliardo (semi) norm of $v$. For $p = 2$ in (2.7), together with the norm $∥∥_{W^{k, 2}(Ω)}$, $W^{k, 2}(Ω)$ becomes a Hilbert space. Let us also set $W^{0, 2}(Ω) = C^∞_c(Ω)^{W^{k, 2}(Ω)}$. It is well known that if $Ω$ is bounded then we have the following
continuous embeddings (see [22])

\[
W_0^{k,2}(\Omega) \hookrightarrow \begin{cases} L^{\frac{2N}{N-2}}(\Omega), & \text{if } k < \frac{N}{2}, \\ L^p(\Omega), & \text{if } k = \frac{N}{2}, \\ C^{0,k-\frac{N}{2}}(\Omega), & \text{if } k > \frac{N}{2}. \end{cases}
\] (2.10)

More details on fractional Sobolev spaces can be found in [39, 1] and the references therein.

For any \( \sigma \geq 0 \), we define the Hilbert space

\[
\mathbb{H}^\sigma(\Omega) = \left\{ v = \sum_{j=1}^{\infty} (v, \varphi_j) \varphi_j(x) \in L^2(\Omega) : \|v\|_{\mathbb{H}^\sigma(\Omega)}^2 = \sum_{j=1}^{\infty} (v, \varphi_j)^2 \lambda_j^\sigma \right\}.
\] (2.11)

We denote by \( \mathbb{H}^{-\sigma}(\Omega) \) the dual space of \( \mathbb{H}^\sigma(\Omega) \) provided that the dual space of \( L^2(\Omega) \) is identified with itself. The space \( \mathbb{H}^{-\sigma}(\Omega) \) is a Hilbert space with the norm

\[
\|v\|_{\mathbb{H}^{-\sigma}(\Omega)} = \left( \sum_{j=1}^{\infty} -\sigma(v, \varphi_j)^2 \lambda_j^{-\sigma} \right)^{\frac{1}{2}},
\] (2.12)

for \( v \in \mathbb{H}^{-\sigma}(\Omega) \) where \( -\sigma(\cdot, \cdot) \) is the dual product between \( \mathbb{H}^{-\sigma}(\Omega) \) and \( \mathbb{H}^\sigma(\Omega) \).

**Remark 5.** For \( \sigma \geq 0 \), from the definitions of the spaces \( \mathbb{H}^\sigma(\Omega) \) and \( \mathbb{H}^{-\sigma}(\Omega) \), we observe that

\[
\|v\|_{\mathbb{H}^{-\sigma}(\Omega)} \leq A \|v\|_{L^2(\Omega)}, \quad \text{and} \quad \|v\|_{L^2(\Omega)} \leq A \|v\|_{\mathbb{H}^\sigma(\Omega)}, \quad \text{for } A > 0.
\]

Given a Banach space \( B \), let \( C([0,T];B) \) be the set of all continuous functions which map \([0,T]\) into \( B \). For \( \eta > 0 \), we define the following Banach space (see [54])

\[
\mathcal{C}^{\eta}([0,T];B) = \left\{ v \in C([0,T];B) : \|v\|_{\mathcal{C}^{\eta}([0,T];B)} = \sup_{t \in [0,T]} t^\eta \|v(t)\|_B < \infty \right\}.
\] (2.13)

### 2.2. Properties of Mittag-Leffler functions and some related results.

The Mittag-Leffler function is defined by [27]

\[
E_{\alpha,\alpha'}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \alpha')}, \quad z \in \mathbb{C},
\] (2.14)

where \( \alpha > 0 \) and \( \alpha' \in \mathbb{R} \) are arbitrary constants, \( \Gamma \) is the usual Gamma function. Especially, for \( \alpha' \in \mathbb{R} \), and \( \alpha \in (0,2) \), we have

\[
|E_{\alpha,\alpha'}(-z)| \leq \frac{C}{1 + |z|}, \quad \mu \leq \arg(z) \leq \pi,
\]

where \( C > 0 \) depends on \( \alpha, \alpha', \mu \) and \( \frac{\pi \alpha'}{2} < \mu < \min\{\pi, \pi \alpha'\} \).

Next, we give some properties of the Mittag-Leffler functions and related results.
Lemma 2.1. (See [27]) For $0 < \alpha_1 < \alpha_2 < 1$ and $\alpha \in [\alpha_1, \alpha_2]$, there exist the positive constants $C, \overline{C}$, such that

\[ a) \quad E_{\alpha,1}(-z) > 0, \quad \text{for any} \quad z > 0; \quad (2.15a) \]
\[ b) \quad \frac{C}{1 + z} \leq E_{\alpha,\alpha'}(-z) \leq \overline{C} \quad \text{for} \quad \alpha' \in \mathbb{R}, \quad z > 0. \quad (2.15b) \]

Lemma 2.2. (See [30]) Let $\alpha, \lambda, \gamma$ be positive constants. For every $t > 0, n \in \mathbb{N}$, we have

\[ a) \quad \frac{d^n}{dt^n} [E_{\alpha,1}(-\lambda t^n)] = -\lambda t^{\alpha-n} E_{\alpha,\alpha-n+1}(-\lambda t^n); \quad (2.16a) \]
\[ b) \quad \frac{d}{dt} \left[ t^{n-1} E_{\alpha,\alpha}(-\lambda t^n) \right] = t^{n-2} E_{\alpha,\alpha-1}(-\lambda t^n); \quad (2.16b) \]
\[ c) \quad \mathbb{E}_t^\alpha E_{\alpha,1}(-\lambda t^n) = -\lambda E_{\alpha,1}(-\lambda t^n); \quad (2.16c) \]
\[ d) \quad \mathbb{E}_t^\gamma (t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^n)) = -\lambda E_{\alpha,\alpha}(-\lambda t^n); \quad (2.16d) \]
\[ e) \quad |t^{1-\gamma} E_{\alpha,\alpha}(-\lambda t^n)| \leq C t^{\gamma-2}. \quad (2.16e) \]

Lemma 2.3. The following equality holds (for proof, see [26])

\[ E_{\alpha,1}(-z) = \int_0^\infty \mathcal{M}_\alpha(s) e^{-zs} ds, \quad \text{for} \quad z \in \mathbb{C}, \quad (2.17) \]

where we recall the definition of the Wright type function

\[ \mathcal{M}_\alpha(s) := \sum_{j=0}^{\infty} \frac{s^j}{j! \Gamma(-\alpha j + 1 - \alpha)}, \quad 0 < \alpha < 1. \quad (2.18) \]

Moreover, $\mathcal{M}_\alpha(s)$ is a probability density function, that is,

\[ \mathcal{M}_\alpha(s) \geq 0, \quad \text{for} \quad s > 0 \quad \text{and} \quad \int_0^\infty \mathcal{M}_\alpha(s) ds = 1. \quad (2.19) \]

Lemma 2.4. (See [16]) For $\alpha \in (0, 1)$ and $\theta > -1$, the following properties hold

\[ \int_0^{\infty} y^{\theta} \mathcal{M}_\alpha(y) dy = \frac{\Gamma(\theta + 1)}{\Gamma(\theta + \alpha + 1)}, \quad \forall \theta > -1. \quad (2.20) \]

Lemma 2.5. (See [45]) For two real numbers $a, b$ we have

\[ \int_a^b (b - \tau)^{m-1}(\tau - a)^{n-1} d\tau = (b - a)^{m+n-1} \mathbb{B}(m, n), \]

where $\mathbb{B}(m, n)$ is Beta function and

\[ \mathbb{B}(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m + n)}. \]

Lemma 2.6. As shown in [11], a function $u$ is a mild solution of $(P_1)$ if $u \in C([0, T]; L^2(\Omega)))$ and satisfies the following integral equation (set $L = -\Delta$)

\[ u(t) = E_{\alpha,1} \left( -\mathbb{L}^\alpha (I + m\mathbb{L})^{-1} \right) u_0 \]
\[ + (I + m\mathbb{L})^{-1} \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha} \left( -\mathbb{L}^\sigma (I + m\mathbb{L})^{-1} (t - \tau)^\alpha \right) \mathcal{N}(u)(\tau) d\tau, \quad (2.21) \]

for all $t < T$, $0 < \sigma < 1$, and $\alpha \in (0, 1)$. 
Lemma 2.7 (Weakly singular Grönwall’s inequality). (See [58]) Let \( A, B, \delta, \delta' \) be non-negative constants and \( \delta, \delta'' < 1 \). Assume that \( u \in L^1[0, T] \) satisfy

\[
 u(t) \leq At^{-\delta} + B \int_0^t (t-\tau)^{-\delta'} u(\tau) d\tau, \quad \text{for a.e. } t \in (0, T]. \tag{2.22}
\]

Then there exists a constant \( C(B, \delta', T) \) such that

\[
 u(t) \leq C(B, \delta', T) \frac{At^{-\delta}}{1 - \delta'}, \quad \text{for a.e. } t \in (0, T]. \tag{2.23}
\]

Lemma 2.8. We have the following useful inequalities:

a) For \( \chi > 0 \), there exists a constant \( C > 0 \) depending on \( \chi \) such that the following inequality holds

\[
 \begin{cases}
 \log z \leq Cz^{-\chi}, & \text{for } \chi > 0, \ 0 < z < 1, \\
 \log z \leq Cz^\chi, & \text{for } \chi > 0, \ z \geq 1.
\end{cases}
\tag{2.24}
\]

b) Hölder’s inequality for negative exponents] (See [28], page 191) Let \( m < 0 \), and \( k \in \mathbb{R} \) be such that: \( \frac{1}{m} + \frac{1}{k} = 1 \). Let \( f(x), g(x) \geq 0, \ \forall x \in \Omega \) be Lebesgue measurable functions. Then

\[
 \int_{\Omega} fg dx \geq \left( \int_{\Omega} |f|^m dx \right)^{\frac{1}{m}} \left( \int_{\Omega} |g|^k dx \right)^{\frac{1}{k}}. \tag{2.25}
\]

Lemma 2.9. (See [39, 1]) For \( \Omega \subset \mathbb{R}^N \), \( k, s \in \mathbb{N} \) with \( k \geq s \) satisfies \( (k-s)p < N \) and \( 1 \leq p < \infty \), it holds

\[
 \begin{cases}
 (SE1): W^{k,p}(\Omega) \hookrightarrow W^{s,q}(\Omega), & \text{for } 1 \leq q < \frac{pN}{N - (k-s)p}, \\
 (SE2): \mathbb{H}^s(\Omega) \hookrightarrow W^{s,2}(\Omega), & \text{for } s > 0, \\
 (SE3): L^p(\Omega) \hookrightarrow \mathbb{H}^s(\Omega), & \text{for } -\frac{N}{2} < s \leq 0, \ p \geq \frac{2N}{N - 2s}, \\
 (SE4): \mathbb{H}^s(\Omega) \hookrightarrow L^p(\Omega), & \text{for } 0 \leq s < \frac{N}{2}, \ p \leq \frac{2N}{N - 2s}.
\end{cases}
\tag{2.26}
\]

3. Proof of Theorem 1.1. First, from the properties of Mittag-Leffler functions, we give the following lemma in order to finish the proof of conclusion (II) of Theorem 1.1.

Lemma 3.1. For \( \alpha, \sigma \in (0, 1), \alpha' \in \mathbb{R}, 0 \leq \nu < 1 \) and \( 0 < z \leq T \), there holds

\[
 \|E_{\alpha,\alpha'}(-L^\sigma(I + mL)^{-1}z^\alpha)v\|_{\mathbb{H}^{2\nu}(\Omega)} \leq Cz^{-\alpha} \|v\|_{\mathbb{H}^{2\nu-2\sigma+2}(\Omega)}. \tag{3.1}
\]

Proof. For \( v \in \mathbb{H}^{2\nu-2\sigma+2}(\Omega) \), using Lemma 2.1, we get

\[
 \|E_{\alpha,\alpha'}(-L^\sigma(I + mL)^{-1}z^\alpha)v\|_{\mathbb{H}^{2\nu}(\Omega)}^2 = \sum_{j=1}^{\infty} (v, \varphi_j)^2 E_{\alpha,\alpha'} \left( -\frac{\lambda_j^\nu z^\alpha}{1 + m\lambda_j} \right) \lambda_j^{2\nu}
\]

\[
 \leq \sum_{j=1}^{\infty} (v, \varphi_j)^2 \left( \frac{C(1 + m\lambda_j)}{\lambda_j^\nu z^\alpha} \right)^2 \lambda_j^{2\nu} \leq 2C^2 z^{-2\alpha} \sum_{j=1}^{\infty} (v, \varphi_j)^2 \left( \lambda_j^{2\nu-2\sigma} + m\lambda_j^{2\nu-2\sigma+2} \right)
\]

\[
 \leq 2C^2 z^{-2\alpha} \left( \|v\|_{\mathbb{H}^{2\nu-2\sigma}(\Omega)}^2 + m \|v\|_{\mathbb{H}^{2\nu-2\sigma+2}(\Omega)}^2 \right) \leq Cz^{-2\alpha} \|v\|_{\mathbb{H}^{2\nu-2\sigma+2}(\Omega)}^2, \tag{3.2}
\]

where we have used \( \|v\|_{\mathbb{H}^{2\nu-2\sigma}(\Omega)} \leq C \|v\|_{\mathbb{H}^{2\nu-2\sigma+2}(\Omega)} \). Taking the square root implies (3.1). The proof of the lemma is completed. \( \square \)
And the following Lemma will be used for the proof of (IV) of Theorem 1.1.

**Lemma 3.2** (See [2]). Let $A$ be a Hausdorff locally convex linear topological space, $U$ be a convex subset of $A$, $V$ be an open subset of $U$, and $\mu \in V$. Suppose that $\mathcal{F} : V \to U$ is a continuous, compact map. Then, either

a) The map $\mathcal{F}$ has a fixed point in $V$; or

b) there are $u \in \partial V$ (the boundary of $V$ in $U$) and $\theta \in (0,1)$ with $u = \theta \mathcal{F}u + (1 - \theta)\mu$.

Next we prove the main conclusions of Theorem 1.1.

**Proof (I).** For $v \in C_\rho([0,T]; W^{k,p}(\Omega))$, we consider the following function

$$Bv(t) := E_{\alpha,1} \left( -\mathbb{L}^\alpha (I + m\mathbb{L})^{-1} I^\alpha \right) u_0$$

$$+ (I + m\mathbb{L})^{-1} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha} \left( -\mathbb{L}^\alpha (I + m\mathbb{L})^{-1} (t-\tau)^{\alpha-1} \right) \mathcal{N}(v)(\tau) d\tau.$$  \hspace{1cm} (3.3)

For $u_0 \in \mathbb{H}^{2\alpha}(\Omega)$, we have

$$e^{-2\rho t} \| E_{\alpha,1} \left( -\mathbb{L}^\alpha (I + m\mathbb{L})^{-1} I^\alpha \right) u_0 \|^2_{\mathbb{H}^{2\alpha}(\Omega)}$$

$$= e^{-2\rho \sum_{j=1}^\infty (u_0, \varphi_j)^2 E_{\alpha,1} \left( -\frac{\lambda_j^\alpha}{1 + m\lambda_j} \right) \lambda_j^{2\alpha} \leq C^2 \sum_{j=1}^\infty (u_0, \varphi_j)^2 \lambda_j^{2\alpha} \leq C^2 \| u_0 \|^2_{\mathbb{H}^{2\alpha}(\Omega)}.$$

From $k \leq 2\sigma$ and $1 \leq p \leq \frac{2N}{N+2k-4\sigma}$, we have $\mathbb{H}^{2\alpha}(\Omega) \hookrightarrow W^{k,p}(\Omega)$, which gives

$$\| E_{\alpha,1} \left( -\mathbb{L}^\alpha (I + m\mathbb{L})^{-1} I^\alpha \right) u_0 \|_{C_\rho([0,T]; W^{k,p}(\Omega))} \leq C \| u_0 \|_{\mathbb{H}^{2\alpha}(\Omega)}.$$  \hspace{1cm} (3.5)

We aim to show that the map $B : C_\rho([0,T]; W^{k,p}(\Omega)) \to C_\rho([0,T]; W^{k,p}(\Omega))$, for $\rho > 0$ has a unique fixed point $u$ when we imply $u$ is a local solution of (2.21). In fact, we will prove that for any $v_1, v_2 \in C_\rho([0,T]; W^{k,p}(\Omega))$, using Lemma 2.1b) and Remark 5 we have

$$\| e^{-\rho t} (Bv_1(t) - Bv_2(t)) \|_{\mathbb{H}^{2\alpha}(\Omega)}$$

$$= e^{-\rho t} \| F(t)(\mathcal{N}(v_1) - \mathcal{N}(v_2)) \|_{\mathbb{H}^{2\alpha}(\Omega)}$$

$$\leq e^{-\rho t} \int_0^t (t-\tau)^{\alpha-1}$$

$$\times \left\| (I + m\mathbb{L})^{-1} E_{\alpha,\alpha} \left( -\mathbb{L}^\alpha (I + m\mathbb{L})^{-1} (t-\tau)^{\alpha-1} \right) \left( \mathcal{N}(v_1) - \mathcal{N}(v_2) \right) \right\|_{\mathbb{H}^{2\alpha}(\Omega)} d\tau$$

$$\leq e^{-\rho t} \int_0^t (t-\tau)^{\alpha-1}$$

$$\times \left( \sum_{j=1}^\infty \left( \frac{1}{1 + m\lambda_j} \right)^2 E_{\alpha,\alpha}^2 \left( -\frac{\lambda_j^\alpha (t-\tau)^{\alpha-1}}{1 + m\lambda_j} \right) \left( \mathcal{N}(v_1) - \mathcal{N}(v_2), \varphi_j \right)^2 \lambda_j^{2\alpha} \right)^{1/2} d\tau$$

$$\leq \frac{Ce^{-\rho t}}{m} \int_0^t (t-\tau)^{\alpha-1} \| \mathcal{N}(v_1)(\cdot, \tau) - \mathcal{N}(v_2)(\cdot, \tau) \|_{\mathbb{H}^{2\alpha-2}(\Omega)} d\tau$$

$$\leq Ce^{-\rho t} A^{1-\sigma} \int_0^t (t-\tau)^{\alpha-1} \| \mathcal{N}(v_1)(\cdot, \tau) - \mathcal{N}(v_2)(\cdot, \tau) \|_{L^2(\Omega)} d\tau.$$  \hspace{1cm} (3.6)
Using the hypothesis (H) and for the constants \(k, p\) satisfy \(k \geq \frac{2N-Np}{2p}\), we have the Sobolev embedding \(W^{k,p}(\Omega) \hookrightarrow L^2(\Omega)\), which implies that

\[
\left\| e^{-\rho t} (Bv_1(t) - Bv_2(t)) \right\|_{L^2(\Omega)}^2 \\
\leq C K A^{1-\sigma} \int_0^t (t-\tau)^{\alpha+1} e^{-\rho(t-\tau)} \sup_{\tau \in [0, T]} \left\| e^{-\rho \tau} (v_1(\cdot, \tau) - v_2(\cdot, \tau)) \right\|_{W^{k,p}(\Omega)} \, d\tau \\
\leq C K A^{1-\sigma} \int_0^t (t-\tau)^{\alpha+1} e^{-\rho(t-\tau)} \, d\tau \| v_1 - v_2 \|_{C_p([0, T]; W^{k,p}(\Omega))}.
\] (3.3)

From (3.7), if \(v_2 = 0\), then we have \(Bv_2 = E_{\alpha,1} \left(-\mathcal{L}^\alpha (I+m\mathcal{L})^{-1} t^\alpha\right) u_0\). Thanks to (3.9), we claim that if \(v \in C_\rho([0, T]; W^{k,p}(\Omega))\), then we get

\[
\left\| B v - E_{\alpha,1} \left(-\mathcal{L}^\alpha (I+m\mathcal{L})^{-1} t^\alpha\right) u_0 \right\|_{C_\rho([0, T]; W^{k,p}(\Omega))} \leq C \left(\rho\right)^{\frac{\alpha}{\xi}} \left\| v \right\|_{C_\rho([0, T]; W^{k,p}(\Omega))},
\] (3.10)

which combined with (3.5), implies

\[
\left\| Bv \right\|_{C_\rho([0, T]; W^{k,p}(\Omega))} \\
\leq \left\| E_{\alpha,1} \left(-\mathcal{L}^\alpha (I+m\mathcal{L})^{-1} t^\alpha\right) u_0 \right\|_{C_\rho([0, T]; W^{k,p}(\Omega))} + C \left(\rho\right)^{\frac{\alpha}{\xi}} \left\| v \right\|_{C_\rho([0, T]; W^{k,p}(\Omega))} \\
\leq C \left\| u_0 \right\|_{L^2(\Omega)} + C \left(\rho\right)^{\frac{\alpha}{\xi}} \left\| v \right\|_{C_\rho([0, T]; W^{k,p}(\Omega))}.
\] (3.11)
From (3.1), we obtain for $C > 0$
\[
\|E_{a,1} \left(-L^\mu (I + mL)^{-1} t^\alpha \right) u_0\|_{\mathbb{H}^{2\nu+2}(\Omega)} \leq Ct^{-\alpha} \|u_0\|_{\mathbb{H}^{2\nu-2\sigma+2}(\Omega)}.
\] (3.12)
To estimate the last term of (3.11), based on Lemma 2.1b, Remark 5 and the hypothesis (H) (for $v = 0$), we get that
\[
\|F(t)(N(u))\|_{\mathbb{H}^{2\nu}(\Omega)} \leq \frac{C}{m} \int_0^t (t-\tau)^{\alpha-1} \|N(u)(\cdot, \tau)\|_{\mathbb{H}^{2\nu-2}(\Omega)} d\tau
\leq \frac{C\Lambda^{1-\nu}}{m} \int_0^t (t-\tau)^{\alpha-1} \|N(u)(\cdot, \tau)\|_{L^2(\Omega)} d\tau
\leq \frac{K\Lambda^{1-\nu}}{m} \int_0^t (t-\tau)^{\alpha-1} \|u(\cdot, \tau)\|_{L^2(\Omega)} d\tau
\leq \frac{K\Lambda}{m} \int_0^t (t-\tau)^{\alpha-1} \|u(\cdot, \tau)\|_{\mathbb{H}^{2\nu}(\Omega)} d\tau.
\] (3.13)
From (3.11), (3.12) and (3.13), we deduce that
\[
\|u(\cdot, t)\|_{\mathbb{H}^{2\nu}(\Omega)} \leq Ct^{-\alpha} \|u_0\|_{\mathbb{H}^{2\nu-2\sigma+2}(\Omega)} + \frac{K\Lambda}{m} \int_0^t (t-\tau)^{\alpha-1} \|u(\cdot, \tau)\|_{\mathbb{H}^{2\nu}(\Omega)} d\tau.
\] (3.14)
Thanks to Lemma 2.7 and noticing that $C(K,\alpha,T,\overline{C},A,m)$ is a positive constant dependent on $K,\alpha,\overline{C},A,m$ and $T$ we have
\[
\|u(\cdot, t)\|_{\mathbb{H}^{2\nu}(\Omega)} \leq \frac{C(K,\alpha,T,\overline{C},A,m)}{1-\alpha} t^{-\alpha} \|u_0\|_{\mathbb{H}^{2\nu-2\sigma+2}(\Omega)} \leq Ct^{-\alpha} \|u_0\|_{\mathbb{H}^{2\nu-2\sigma+2}(\Omega)}.
\] (3.15)
Multiplying the both sides of (3.15) by $t^\alpha$ we get
\[
t^\alpha \|u(\cdot, t)\|_{\mathbb{H}^{2\nu}(\Omega)} \leq C \|u_0\|_{\mathbb{H}^{2\nu-2\sigma+2}(\Omega)}, \quad \text{for } t \in (0,T),
\] (3.16)
which implies (1.6). Moreover, for $k \leq 2\nu$ and $1 \leq p \leq \frac{2N}{N+2k-4\nu}$, the Sobolev embedding $\mathbb{H}^{2\nu}(\Omega) \hookrightarrow W^{k,p}(\Omega)$ implies (1.7).

Proof (IIb). Next, we show that $\partial_t u \in L^q(0,T;W^{k,p}(\Omega))$. For $t \in (0,T]$, we have
\[
\partial_t u(t)
= \sum_{j=1}^\infty \left( (u_0, \varphi_j) \frac{\lambda_j^2 \mu}{1+m\lambda_j} t^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\lambda_j^\alpha \mu}{1+m\lambda_j} \right) \right) \varphi_j(x)
+ \sum_{j=1}^\infty \int_0^t \left( N(u)(\cdot, \tau), \varphi_j \right) (t-\tau)^{\alpha-2} E_{\alpha,\alpha-1} \left(-\frac{\lambda_j^\alpha (t-\tau)^\alpha}{1+m\lambda_j} \right) d\tau \varphi_j(x)
:= A_1(t)u_0 + A_2(t)\mathcal{N}(u), \quad \text{respectively}.
\] (3.17)
From Lemma 2.1b) and Parseval’s relation, we obtain
\[
\|A_1(t)u_0\|^2_{\mathbb{H}^{2\nu}(\Omega)}
= \sum_{j=1}^\infty \lambda_j^{2\nu} \left( (u_0, \varphi_j) \frac{\lambda_j^2 \mu}{1+m\lambda_j} t^{\alpha-1} E_{\alpha,\alpha} \left(-\frac{\lambda_j^\alpha \mu}{1+m\lambda_j} \right) \right)^2
\leq \frac{C_2}{m^2} \sum_{j=1}^\infty \lambda_j^{2\nu} \|u_0\|^2_{\mathbb{H}^{2\nu+2\sigma-2}(\Omega)} \leq Ct^{2\alpha-2} \|u_0\|^2_{\mathbb{H}^{2\nu-2\sigma+2}(\Omega)}, \quad \forall t \in (0,T].
\] (3.18)
And using (2.16e) we conclude that
\[
\|A_2(t)V(u)\|_{\mathbb{H}^2_{-2}(\Omega)}^2 \\
= \sum_{j=1}^{\infty} \left( \frac{\lambda_j^2}{1+m\lambda_j} \right)^{1-\gamma} \int_0^t (N(u)(\cdot, \tau), \varphi_j) (t-\tau)^{\alpha-2} E_{\alpha, \alpha-1} \left( \frac{-\lambda_j^2 (t-\tau)^\alpha}{1+m\lambda_j} \right) d\tau \right)^2 \times \frac{\lambda_j^{2\sigma\gamma+2\nu-2\sigma}}{(1+m\lambda_j)^{2\gamma}} \\
\leq C \sum_{j=1}^{\infty} \left( \int_0^t (t-\tau)^{\alpha\gamma-2} (N(u)(\cdot, \tau), \varphi_j) d\tau \right)^2 \lambda_j^{2\sigma\gamma+2\nu-2\gamma-2\sigma} \\
\leq C t \int_0^t (t-\tau)^{2\alpha\gamma-4} \sum_{j=1}^{\infty} \lambda_j^{2\sigma\gamma+2\nu-2\gamma-2\sigma} (N(u)(\cdot, \tau), \varphi_j)^2 d\tau \\
\leq C t \int_0^t (t-\tau)^{2\alpha\gamma-4} \|N(u)(\cdot, \tau)\|_{\mathbb{H}^{2\sigma\gamma+2\nu-2\gamma-2\sigma}(\Omega)}^2 d\tau, \quad \forall \tau \in (0, T]. (3.19)
\]

By choosing \( \nu \leq \gamma + \sigma - \sigma\gamma \), using Remark 5, we have
\[
\|N(u)\|_{\mathbb{H}^{2\sigma\gamma+2\nu-2\gamma-2\sigma}(\Omega)} \leq A^{\gamma+\sigma-\sigma\gamma-\nu} \|N(u)\|_{L^2(\Omega)}.
\]

Using the hypothesis (H) (for \( v = 0 \)), it follows readily from these estimates for all \( t \in (0, T] \)
\[
\|A_2(t)V(u)\|_{\mathbb{H}^2_{-2}(\Omega)}^2 \leq C t A^{\gamma+\sigma-\sigma\gamma-\nu} \int_0^t (t-\tau)^{2\alpha\gamma-4} \|N(u)(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau \\
\leq C K t A^{\gamma+\sigma-\sigma\gamma-\nu} \int_0^t (t-\tau)^{2\alpha\gamma-4} \|u(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau. (3.20)
\]

From (3.16), we choose \( \nu = 0 \) to get \( \|u(\cdot, t)\|_{L^2(\Omega)} \leq C t^{-\alpha} \|u_0\|_{\mathbb{H}^{2\sigma-2\gamma}(\Omega)} \) for a.e. \( t \in (0, T] \), which leads to
\[
\|A_2(t)V(u)\|_{\mathbb{H}^2_{-2}(\Omega)}^2 \leq C t \int_0^t (t-\tau)^{2\alpha\gamma-4} \tau^{-2\alpha} \|u_0\|_{\mathbb{H}^{2\sigma-2\gamma}(\Omega)}^2 d\tau. (3.21)
\]

Using Lemma 2.5 we deduce for the constants \( \alpha \in (0, \frac{1}{2}) \) and \( \gamma > \frac{3}{2\alpha} \) that
\[
\int_0^t (t-\tau)^{2\alpha\gamma-4} \tau^{-2\alpha} d\tau \\
= \int_0^t (t-\tau)(2\alpha\gamma-3)-1 \tau^{-(1-2\alpha)-1} d\tau \\
\leq t^{2\alpha\gamma-3-2\alpha} \frac{\Gamma(2\alpha\gamma-3)\Gamma(1-2\alpha)}{\Gamma(2\alpha\gamma-2\alpha-2)} \leq \mathbb{B}(2\alpha\gamma-3, 1-2\alpha) t^{2\alpha\gamma-3-2\alpha}, (3.22)
\]
where \( \mathbb{B} \) is Beta function. From (3.18), (3.21) and (3.22), we conclude that
\[
\|\partial_t u(\cdot, t)\|_{\mathbb{H}^{2\sigma}(\Omega)} \leq C t^{\alpha-1} \|u_0\|_{\mathbb{H}^{2\sigma-2\gamma}(\Omega)} + \mathbb{B}(2\alpha\gamma-3, 1-2\alpha) t^{\alpha\gamma-1-\alpha} \|u_0\|_{\mathbb{H}^{2\sigma-2\gamma}(\Omega)}.
\]

For \( k \leq 2\nu \) and \( 1 \leq p \leq \frac{2N}{N+2k-4\nu} \), the Sobolev embedding \( \mathbb{H}^{\nu}(\Omega) \hookrightarrow W^{k,p}(\Omega) \) implies that
\[
\|\partial_t u(\cdot, t)\|_{W^{k,p}(\Omega)} \\
\leq C t^{\alpha-1} \|u_0\|_{\mathbb{H}^{2\sigma+2\nu-2\gamma}(\Omega)} + \mathbb{B}(2\alpha\gamma-3, 1-2\alpha) t^{\alpha\gamma-1-\alpha} \|u_0\|_{\mathbb{H}^{2\sigma-2\gamma}(\Omega)}, (3.23)
\]
where, from $\gamma > \frac{1+\alpha}{\alpha}$ we have $\gamma \alpha - 1 - \alpha > 0$, which tells $\partial_t u \in L^q((0, T; W^{k,p}(\Omega))$ for $1 < q < \frac{1}{1-\alpha}$.

Proof. Next, we shall prove that $D_t^\alpha u \in L'(0, T; W^{k,p}(\Omega))$. It follows readily from (2.21) that

\begin{equation}
D_t^\alpha u(t) = \sum_{j=1}^{\infty} \left( u_0, \varphi_j \right) \frac{\lambda_j^\alpha}{1 + m \lambda_j} E_{\alpha, 1} \left( -\frac{\lambda_j^\alpha t^\alpha}{1 + m \lambda_j} \right) \varphi_j(x)
\end{equation}

\begin{equation}
+ \sum_{j=1}^{\infty} \left( \frac{\lambda_j^\alpha}{1 + m \lambda_j} \right)^2 \int_0^t \left( \mathcal{N}(u)(\cdot, \tau), \varphi_j \right) (t-\tau)^{\alpha-1} E_{\alpha, \alpha} \left( -\frac{\lambda_j^\alpha (t-\tau)^\alpha}{1 + m \lambda_j} \right) d\tau \varphi_j(x)
\end{equation}

\begin{equation}
+ \sum_{j=1}^{\infty} \frac{\lambda_j^\alpha}{1 + m \lambda_j} \left( \mathcal{N}(u)(\cdot, \tau), \varphi_j \right) \varphi_j(x) =: A_3(t) u_0 + A_4(t) \mathcal{N}(u) + A_5(t) \mathcal{N}(u).
\end{equation}

By an argument analogous to the previous one, we get for every $t \in [0, T]$

\begin{equation}
\|A_3(t) u_0\|_{H^{2\alpha}(\Omega)} \leq C \|u_0\|_{H^{2\alpha+2\sigma-2}(\Omega)}
\end{equation}

From (3.16), choosing $\nu = 0$ to get $|u(\cdot, t)|_{L^2(\Omega)} \leq C \|u_0\|_{H^{2\sigma-2}(\Omega)}$ for $t \in (0, T]$, using hypothesis (H) (for $\nu = 0$) and Remark 5 (for $\nu + \sigma < 1$), we see

\begin{equation}
\|A_5(t) \mathcal{N}(u)\|_{H^{2\alpha}(\Omega)} = \left\| \sum_{j=1}^{\infty} \frac{\lambda_j^\alpha}{1 + m \lambda_j} \left( \mathcal{N}(u)(\cdot, \tau), \varphi_j \right) \varphi_j(x) \right\|_{H^{2\alpha}(\Omega)}
\end{equation}

\begin{equation}
\leq C \|\mathcal{N}(u)\|_{H^{2\alpha+2\sigma-2}(\Omega)} \leq A_{1-\nu-\sigma} \|\mathcal{N}(u)\|_{L^2(\Omega)}
\end{equation}

\begin{equation}
\leq K A_{1-\nu-\sigma} \|u(\cdot, t)\|_{L^2(\Omega)} \leq C t^{-\alpha} \|u_0\|_{H^{2\sigma-2}(\Omega)}.
\end{equation}

For $\sigma, \nu \in (0, 1)$, i.e., $2\nu + 2\sigma - 4 < 0$, using Remark 5, hypothesis (H) (for $\nu = 0$), and (3.16), choosing $\nu = 0, \gamma > 0$, we get $t^{\nu + \gamma} |u(\cdot, t)|_{L^2(\Omega)} \leq C T^\gamma t^{-\alpha} \|u_0\|_{H^{2\sigma-2}(\Omega)}$ for $t \in (0, T]$. Then we estimate the term $A_4(t) \mathcal{N}(u)$ as follows

\begin{equation}
\|A_4(t) \mathcal{N}(u)\|_{H^{2\alpha}(\Omega)} \leq C \int_0^t (t-\tau)^{\alpha-1} \|\mathcal{N}(u)(\cdot, \tau)\|_{H^{2\alpha+2\sigma-4}(\Omega)} d\tau
\end{equation}

\begin{equation}
\leq C A_{2-\nu-\sigma} \int_0^t (t-\tau)^{\alpha-1} \|\mathcal{N}(u)(\cdot, \tau)\|_{L^2(\Omega)} d\tau
\end{equation}

\begin{equation}
\leq K C A_{2-\nu-\sigma} \|u(\cdot, \tau)\|_{L^2(\Omega)} \leq C t^{-\alpha} \|u_0\|_{H^{2\sigma-2}(\Omega)}
\end{equation}

\begin{equation}
\leq K C A_{2-\nu-\sigma} T^\gamma \|u_0\|_{H^{2\sigma-2}(\Omega)} \int_0^t (t-\tau)^{\alpha-1} \tau^{-\alpha-\gamma} d\tau
\end{equation}

\begin{equation}
\leq \frac{\Gamma(\alpha) \Gamma(1 - \alpha - \gamma)}{\Gamma(1 - \gamma)} t^{-\gamma} \|u_0\|_{H^{2\sigma-2}(\Omega)} \leq C t^{-\gamma} \|u_0\|_{H^{2\sigma-2}(\Omega)},
\end{equation}

where we have used Lemma 2.5 for the constants $\gamma < 1 - \alpha$ and $C > 0$ as

\begin{equation}
\int_0^t (t-\tau)^{\alpha-1} \tau^{-\alpha-\gamma} d\tau = \int_0^t (t-\tau)^{\alpha-1} \tau^{(1-\alpha-\gamma)-1} d\tau
\end{equation}

\begin{equation}
= \frac{\Gamma(\alpha) \Gamma(1 - \alpha - \gamma)}{\Gamma(1 - \gamma)} t^{-\gamma} = \mathbb{B}(\alpha, 1 - \alpha - \gamma) t^{-\gamma},
\end{equation}
From (3.25), (3.26), (3.27) and the Sobolev embedding $H^{2\nu}(\Omega) \hookrightarrow W^{k,p}(\Omega)$ we infer that
\[
\|D^n u(\cdot, t)\|_{W^{k,p}(\Omega)} \leq C \|D^n u(\cdot, t)\|_{H^{2\nu}(\Omega)} \tag{3.28}
\]
\[
\leq C \|u_0\|_{H^{2\nu+2\nu-2}(\Omega)} + C t^{-\alpha} \|u_0\|_{H^{2\nu-2\alpha}(\Omega)} + B(\alpha, 1 - \alpha - \gamma) t^{-\gamma} \|u_0\|_{H^{2\nu-2\alpha}(\Omega)},
\]
which implies that $D^n u \in L^r(0, T; W^{k,p}(\Omega))$ for $1 \leq r < \min\{\frac{1}{\alpha}, \frac{1}{\gamma}\}$. The proof of Part (II) is completed.

Proof(III). Let $u_0 \in H^2(\Omega)$ and consider $\{u_{0,n}\}_{n \in \mathbb{N}^*} \subset H^2(\Omega)$ such that
\[
u_n \to u_0, \quad \text{as } n \to \infty.
\]
For $n$ sufficiently large we have that
\[
\|u(\cdot, t) - u_n(\cdot, t)\|_{H^{2\nu}(\Omega)} \tag{3.29}
\]
\[
\leq \left\|E_{\alpha,1} \left(-L^\nu(I + mL)^{-1}t^\alpha\right)(u_0 - u_{0,n})\right\|_{H^{2\nu}(\Omega)} + \int_0^t (t - \tau)^{\alpha-1} \cdot \left\|E_{\alpha,1} \left(-L^\nu(I + mL)^{-1}t^\alpha\right)(N(u(\cdot, \tau) - N(u_n(\cdot, \tau)))\right\|_{H^{2\nu}(\Omega)} d\tau.
\]
Using (3.1) for $\nu = \sigma$, we get
\[
\|E_{\alpha,1} \left(-L^\sigma(I + mL)^{-1}t^\alpha\right)(u_0 - u_{0,n})\|_{H^{2\nu}(\Omega)} \leq Ct^{-\alpha} \|u_0 - u_{0,n}\|_{H^2(\Omega)}, \quad \forall t \in (0, T]. \tag{3.30}
\]
From Lemma 2.1b), hypothesis (H) and Remark 5, we have
The second term on the (RHS) of (3.29)
\[
\leq \frac{C}{m} \int_0^t (t - \tau)^{\alpha-1} \|N(u(\cdot, \tau) - N(u_n(\cdot, \tau))\|_{L^{2\nu-2\alpha}(\Omega)} d\tau
\]
\[
\leq \frac{C\Lambda^{1-\alpha}}{m} \int_0^t (t - \tau)^{\alpha-1} \|N(u(\cdot, \tau) - N(u_n(\cdot, \tau))\|_{L^2(\Omega)} d\tau
\]
\[
\leq \frac{K\Lambda}{m} \int_0^t (t - \tau)^{\alpha-1} \|u(\cdot, \tau) - u_n(\cdot, \tau)\|_{H^{2\nu}(\Omega)} d\tau. \tag{3.31}
\]
Combining (3.29)-(3.31), we deduce that for $t \in (0, T]$ 
\[
\|u(\cdot, t) - u_n(\cdot, t)\|_{H^{2\nu}(\Omega)} \leq Ct^{-\alpha} \|u_0 - u_{0,n}\|_{H^2(\Omega)} + \frac{K\Lambda}{m} \int_0^t (t - \tau)^{\alpha-1} \|u(\cdot, \tau) - u_n(\cdot, \tau)\|_{H^{2\nu}(\Omega)} d\tau. \tag{3.32}
\]
From Lemma 2.7 (Grönnwall’s inequality) and by an argument analogous to (3.15), we see that
\[
\|u(\cdot, t) - u_n(\cdot, t)\|_{H^{2\nu}(\Omega)} \lesssim t^{-\alpha} \|u_0 - u_{0,n}\|_{H^2(\Omega)}, \quad t \in (0, T]. \tag{3.33}
\]
Let $n \to \infty$ and we have $u_{0,n} \to u_0$, then $u_n \to u$ in $H^{2\nu}(\Omega)$.

Remark 6. If $u_0(x) > 0$ a.e. $x \in \Omega$ and the continuous function $N(u)$ is nonnegative, then the explicit solution of Problem $(P_1)$ presented in (2.21) is positive.

Proof(IV). Step I: Existence of a positive solution $u \in C((0, \infty); L^2(\Omega))$. For $\ell > 0$, let us set
\[
U := \{v \in C((0, \infty); L^2(\Omega)) \mid u(\cdot, \ell) \geq \ell, \ a.e \ (x, t) \in \Omega \times (0, T), \ \text{for } T \in (0, \infty)\}. \tag{3.34}
\]
Consider the operator $\mathcal{F} : U \to U$ defined by
\[
\mathcal{F} v(t) = E_{\alpha,1} \left( -\mathbb{L}^\alpha (I + m\mathbb{L})^{-1}t^\alpha \right) u_0 + F(t)(\mathcal{N}(v)),
\]
where $F(t)(\mathcal{N}(u))$ is defined in (3.3). From Remark 6, since $u_0 > 0$ a.e. in $\Omega$ and $\mathcal{N}(u)$ is nonnegative, then $\mathcal{F} u$ is nonnegative. By the proof of Part (I) (of this theorem), we know that the operator $\mathcal{F}$ has a unique fixed point. Let
\[
V := \left\{ u \in C((0,T]; L^2(\Omega)) : \|u(\cdot,t)\|_{L^2(\Omega)} \lesssim t^{-\alpha} \|u_0\|_{H^{2-2\alpha}(\Omega)}, \quad t > 0 \right\}.
\]
Then, we can show that $\mathcal{F} : \bar{V} \to U$ is continuous and compact by the usual techniques (see e.g. [61]). Moreover, for $\theta \in (0,1)$, if $u \in U$ is any solution of the equation (3.34) then we get
\[
u(t) = \theta \mathcal{F} u(t) + (1 - \theta) E_{\alpha,1} \left( -\mathbb{L}^\alpha (I + m\mathbb{L})^{-1}t^\alpha \right) u_0
\]
\[
= E_{\alpha,1} \left( -\mathbb{L}^\alpha (I + m\mathbb{L})^{-1}t^\alpha \right) u_0 + \theta F(t)(\mathcal{N}(u)).
\]
By an argument analogous to that used for the proof of Part (II) (of this theorem for setting $\nu = 0$), one obtains
\[
\|u(\cdot,t)\|_{L^2(\Omega)} \lesssim t^{-\alpha} \|u_0\|_{H^{2-2\alpha}(\Omega)}.
\]
We invoke Lemma 3.2 to deduce that $\mathcal{F}$ has a fixed point in $\bar{V}$. Then, this fixed point is a positive solution of Problem ($P_1$). Since the arbitrariness of $T \in (0, +\infty)$, then we claim that there exists a positive solution $u \in C((0,\infty); L^2(\Omega))$ of Problem ($P_1$).

Step II: Uniqueness of positive solution in $C((0,\infty); L^2(\Omega))$. By Step I, we suppose that $u, v \in C((0,\infty); L^2(\Omega))$ are two positive solutions of Problem ($P_1$). Then, we conclude that (the same argument as (3.31) for $\sigma = 0$)
\[
\|u(\cdot,t) - v(\cdot,t)\|_{L^2(\Omega)} \leq \frac{KCA}{m} \int_0^t (t - \tau)^{-\alpha - 1} \|u(\cdot,\tau) - v(\cdot,\tau)\|_{L^2(\Omega)} d\tau.
\]
Applying Lemma 2.7 (Grönwall inequality), we derive $u = v$. This implies the uniqueness of the solution. The Theorem 1.1 has been proved.

Example 1. In Theorem 1.1, for $N \geq 1$ and $0 < \alpha, \sigma, \nu < 1$, we choose $\alpha = \sigma = \nu = \frac{1}{2}$ and $N = 3$. From the conditions
\[
\begin{cases}
kp < N, k < 2\sigma, k < 2\nu, k \in \mathbb{N}, \\
1 \leq p \leq \frac{2N}{N + 2k - 4\sigma}, \\
\frac{Np}{N - kp} \geq 2,
\end{cases}
\]
we have
\[
\begin{cases}
k = 0, \\
1 \leq p \leq 6, \\
\frac{Np}{N - kp} \geq 2
\end{cases}
\]
is fulfilled for $2 \leq p \leq 6$.

Then, the integral equation (2.21) has a unique mild solution $u \in C_p([0,T]; L^p(\Omega))$, for $2 \leq p \leq 6$, and some $p > 0$. Moreover, the estimates in (1.7), (1.8) and (1.9) become (respectively)
\[
t^{\frac{1}{2}} \|u\|_{L^p(\Omega)} \lesssim \|u_0\|_{H^{2}(\Omega)}, \quad \forall t \in (0,T];
\]
\[
t^{\frac{1}{2}} \|\partial_t u(\cdot,t)\|_{L^p(\Omega)} \lesssim \|u_0\|_{L^2(\Omega)} + t^{-1/2} \|u_0\|_{H^{1}(\Omega)}, \quad \forall t \in (0,T], \gamma > 3;
\]
\[
t \left\| D^\gamma_t u(\cdot,t) \right\|_{L^p(\Omega)} \lesssim t^{\gamma/2} \|u_0\|_{L^2(\Omega)} + (t^{\gamma} + t^{\frac{1}{2}}) \|u_0\|_{H^{1}(\Omega)}, \forall t \in (0,T], \gamma > 0.
\]
Remark 7. From Part (I) and (II) in Theorem 1.1, for $\alpha, \sigma, \nu \in (0, 1)$, the Sobolev spaces $W^{k,p}(\Omega)$ used in above clearly can be only $W^{k-p}(\Omega)$ for $k = 1$ and $1 \leq p \leq \frac{N+2k(1-2\sigma)}{2}$ or $L^p(\Omega)$ for $k = 0$ and $1 \leq q \leq \frac{2N}{4\sigma}$ considering the constant $k \in \mathbb{N}$ and $k < 2\sigma$ (or $k \leq 2\nu$). This restriction will be released in the proof of Theorem 1.2 when the constant $k$ satisfies $0 \leq k < 2\omega_2$ for $0 \leq \omega_2 < \frac{N}{4}$.

4. Proof of Theorem 1.2. First we need the following lemma.

Lemma 4.1. For $N_p(u)(x, t) = \kappa|\phi_p(u) \log |u| \in L^\infty(\Omega \times (0, T) \times \mathbb{R})$, $p \geq 2$, $\kappa > 0$, there exists a positive constant $C$ such that

$$|N_p(u) - N_p(v)| \leq C \left( (1 + |u|^{p-2} + |v|^{p-2}) \log |u| + |v|^{p-2} \right) |u - v|, \quad (4.1)$$

for all $(x, t) \in \Omega \times (0, T)$, $\forall u, v \in \mathbb{R}$.

Proof of the Lemma 4.1. For $(x, t) \in \Omega \times (0, T)$ and $u, v \in \mathbb{R}$, we have

$$|N_p(u) - N_p(v)| \leq \kappa \left| \phi_p(u) \log |u| - \phi_p(v) \log |v| \right| \leq \kappa \left( |\phi_p(u) - \phi_p(v)| \log |u| + |\phi_p(v)| \left| \log |u| - \log |v| \right| \right). \quad (4.2)$$

Thanks to the results in [1], we have the fact that

$$|\phi_p(u) - \phi_p(v)| = |u|^{p-2}u - |v|^{p-2}v \leq C(1 + |u|^{p-2} + |v|^{p-2})|u - v|. \quad (4.3)$$

Using the basic inequality $\log(1 + z) < z$ for $z > 0$, we have

$$\left| \log |u| - \log |v| \right| \leq \log \left( 1 + \frac{|u| - |v|}{|v|} \right) \leq \frac{|u - v|}{|v|}. \quad (4.4)$$

From (4.2)-(4.4), we finish the proof of Lemma 4.1. \qed

Now we are ready to prove Theorem 1.2.

Proof (I). For $N \geq 1, p \geq 2$, we set

$$0 \leq \omega_2 < \min \left\{ 1; \frac{(p-1)N}{4p} \right\}, \quad (4.5)$$

$$\max \{ p\omega_2; Q_i \} < \omega^* < \min \left\{ p\omega_2 + \frac{N}{4}; 1 + (p-1)\omega_2 \right\}, \quad i = 1, ..., 6, \quad (4.6)$$

$$\omega_1 = p\omega_2 - \omega^*, \quad (4.7)$$

$s \in \mathbb{N}$ satisfies $s < 2\omega_2$, \quad (4.8)

$$1 \leq q \leq \min \left\{ \frac{2N}{N + 2s - 4\omega_2}; \frac{\chi N}{N + \chi s} \right\}, \quad sq < N, \quad (4.9)$$
with $0 < \chi < p - 2$ (the constant $\chi$ is given in Lemma 2.8a)) and

\[
\begin{align*}
Q_1 & := \frac{N(2p - q) + 2pq(2\omega_2 - s)}{4q}, \\
Q_2 & := \frac{N(2p\chi - q) + 2pq(2\omega_2 - \chi s)}{4q}, \\
Q_3 & := \frac{N(2p(p - 2) - q) + 2pq(2\omega_2 - (p - 2)s)}{4q}, \\
Q_4 & := \frac{4q}{N(p - 1)} \left( N(2p\chi - q(p - 1)) + 2pq(2(p - 1)\omega_2 - \chi s) \right), \\
Q_5 & := \frac{4q(p - 1)}{N(p - 1)} \left( N(2p\chi - q(p - 2)) + 2pq(2(p - 2)\omega_2 - \chi s) \right), \\
Q_6 & := \frac{N(2p(p - 2) - q(p - 1)) + 2pq(2(p - 1)\omega_2 - (p - 2)s)}{4q(p - 1)}.
\end{align*}
\]

Let $T > 0$ and $d > 0$ to be chosen later, and we consider the following space

\[
\mathcal{B} := \left\{ u \in C^{\alpha\gamma}((0, T]; W^s, q(\Omega)) : u(\cdot, 0) = u_0, \|u - u_0\|_{C^{\alpha\gamma}((0, T]; W^s, q(\Omega))} \leq d \right\},
\]

for $0 < \alpha < 1$ and $0 < \gamma < \min\{1; \frac{1}{2\omega}; \frac{1}{\alpha(p-1+\chi)}\}$ satisfying $\alpha - \alpha\gamma(p - 1 + \chi) > 0$.

We define the mapping $J$ on $\mathcal{B}$ by

\[
J u(t) := E_{\alpha, 1} \left( -\mathbb{L}^\sigma (I + m\mathbb{L})^{-1}t^\alpha \right) u_0
+ (I + m\mathbb{L})^{-1} \int_0^t (t - \tau)^{-\alpha - 1} E_{\alpha, 1} \left( -\mathbb{L}^\sigma (I + m\mathbb{L})^{-1}(t - \tau)^\alpha \right) \mathcal{N}_\mu(u)(\tau) d\tau.
\]

We show that $J$ is invariant in $\mathcal{B}$ and $J$ is a contraction.

- Claim 1: If $u_0 \in H^{2\omega_2}(\Omega) \cap W^{s, q}(\Omega)$, then $J$ is $\mathcal{B}$-invariant.

In fact, from Lemma 2.1b), we have

\[
\begin{align*}
& t^{\alpha\gamma} \left\| E_{\alpha, 1} \left( -\mathbb{L}^\sigma (I + m\mathbb{L})^{-1}t^\alpha \right) u_0 - u_0 \right\|_{H^{2\omega_2}(\Omega)} \\
& = t^{\alpha\gamma} \sum_{j=1}^\infty \left( u_{0, \phi_j} \right)^2 \left( E_{\alpha, 1} \left( -\frac{\lambda_j^2 t^\alpha}{1 + m\lambda_j} \right) - 1 \right)^2 \lambda_j^{2\omega_2} \\
& \leq 2(C^2 + 1)T^{\alpha\gamma} \sum_{j=1}^\infty \left( u_{0, \phi_j} \right)^2 \lambda_j^{2\omega_2} \leq CT^{\alpha\gamma} \left\| u_0 \right\|_{H^{2\omega_2}(\Omega)}, \quad \forall t \in (0, T).
\end{align*}
\]

For the constants $s, q$ satisfying (4.8), we see that $H^{2\omega_2}(\Omega) \hookrightarrow W^{s, q}(\Omega)$, then we conclude from (4.13) that

\[
\begin{align*}
t^{\alpha\gamma} \left\| E_{\alpha, 1} \left( -\mathbb{L}^\sigma (I + m\mathbb{L})^{-1}t^\alpha \right) u_0 - u_0 \right\|_{W^{s, q}(\Omega)} \\
& \leq CT^{\alpha\gamma} \left\| u_0 \right\|_{H^{2\omega_2}(\Omega)}, \quad t \in (0, T).
\end{align*}
\]

From (4.6), we have $p\omega_2 < \omega^* < 1 + (p - 1)\omega_2$, which implies that $0 < \omega_2 - \omega_1 < 1$ for $p\omega_2 < \omega^* < p\omega_2 + \frac{2N}{3\omega}$, or $-\frac{N}{3} < p\omega_2 - \omega^* < 0$ thus $-\frac{N}{3} < \omega_1 < 0$. Taking $p_\omega^* := \frac{2N}{N - 4\omega_2}$, and combining with Lemma 2.9, we obtain $L^{p_\omega^*}(\Omega) \hookrightarrow H^{2\omega_2}(\Omega)$, which leads to $\| \cdot \|_{H^{2\omega_2}(\Omega)} \leq C \| \cdot \|_{L^{p_\omega^*}(\Omega)}$. Using Lemma 2.1b) we have for $t \in (0, T)$

\[
\left\| J_1(u)(t) \right\|_{H^{2\omega_2}(\Omega)}.
\]
where we have chosen

\[ C \leq \frac{\sqrt{T}}{m \lambda_j^{1+\omega_2-w_1}} \int_0^t (t-\tau)^{-\alpha} \|N_p(u)(\cdot, \tau)\|_{H^{2\omega_1}(\Omega)} d\tau \]

\[ \leq C \int_0^t (t-\tau)^{-\alpha} \|N_p(u)(\cdot, \tau)\|_{L^{p^*}_{\dot{\chi}}(\Omega)} d\tau. \]

Next we first set \( \Omega_1 := \{ x \in \Omega : |u(x)| < 1 \} \) and \( \Omega_2 := \{ x \in \Omega : |u(x)| \geq 1 \}. \) Using the Hölder's inequality, we have

\[ \int_{\Omega_1} |N_p(u)|^{p^*} dx \]
\[ \leq \mathcal{N} \int_{\Omega_1} |u|^{(p-1)p^*} |\log |u|| dx \]
\[ \leq C \left( \int_{\Omega_1} |\log |u||^{pp^*} dx \right)^{\frac{1}{p}} \left( \int_{\Omega_1} |u|^{pp^*} dx \right)^{\frac{p-1}{p}} \]
\[ \leq C \left( \int_{\Omega_1} |\log |u||^{pp^*} dx \right)^{\frac{1}{p}} + \left( \int_{\Omega_2} |\log |u||^{pp^*} dx \right)^{\frac{1}{p}} \left( \int_{\Omega_1} |u|^{pp^*} dx \right)^{\frac{p-1}{p}}, \]

where we have used the elementary inequality \((a + b)^q \leq a^q + b^q,\) for \(0 < q < 1.\) From the inequality (2.24) for \(|u(x)| < 1, \forall x \in \Omega_1\), we have

\[ \left( \int_{\Omega_1} |\log |u||^{pp^*} dx \right)^{\frac{1}{p}} \]
\[ \leq C \left( \int_{\Omega_1} |u(x)|^{-p \chi p^*} dx \right)^{\frac{1}{p}} \leq \left( \int_{\Omega_1} |u(x)|^{-p \chi p^*} dx \right)^{-\frac{1}{p^*}} \]
\[ \leq C \left( \int_{\Omega_1} |u(x)|^{\chi} dx \right) \left( \int_{\Omega_1} 1 dx \right)^{-\frac{1}{p^*}} \leq C \|u\|^{-\chi p^*}_{L^{\chi p^*}(\Omega)} \]

where we have chosen \( m = -\frac{1}{pp^*} < 0 \) in the inequality (2.25). For \(|u(x)| \geq 1, \forall x \in \Omega_2\), we have

\[ \left( \int_{\Omega_2} |\log |u||^{pp^*} dx \right)^{\frac{1}{p}} \leq C \left( \int_{\Omega_2} |u(x)|^{p \chi p^*} dx \right)^{\frac{1}{p}} \leq C \|u\|^{\chi p^*}_{L^{p \chi p^*}(\Omega)} \]

From (4.16), (4.17) and (4.18), we conclude that

\[ \|N_p(u)\|_{L^{p^*}_{\dot{\chi}}(\Omega)} \leq C \left( \|u\|^{-\chi}_{L^\chi(\Omega)} + \|u\|_{L^{\chi p^*}(\Omega)} \right) \|u\|^{(p-1)p^*}_{L^{p^*}_{\dot{\chi}}(\Omega)}. \]

From (4.9), \( q \leq \frac{Nq}{N-sq} \) and \( sq < N \), we imply that \( \frac{Nq}{N-sq} \leq \chi. \) By using Lemma 2.9 we have \( L^\chi(\Omega) \hookrightarrow W^{s,q}(\Omega) \) implies \( \|u\|_{W^{s,q}(\Omega)} \leq C\|u\|_{L^\chi(\Omega)}. \) Then we get

\[ \|u\|_{L^\chi(\Omega)} \leq C \|u\|_{W^{s,q}(\Omega)} \]
for \( \chi > 0. \)
From (4.10), the fact \( \omega^s > d_1 = \frac{N(2p-q)+2pq(2\omega_2-s)}{4q} \), and
\[
pp^*_{\omega_1} = \frac{2Np}{N-4\omega_1} = \frac{2Np}{N + 4\omega - 4\omega_2} < \frac{2Np}{N + 4d_1 - 4\omega_2} = \frac{Nq}{N - sq},
\]
we can also obtain \( W^{s,s}(\Omega) \hookrightarrow LP^{p_2}(\Omega) \). For \( \omega^s > Q_2 \) (\( Q_2 \) is defined in (4.10)), we infer that \( p \chi p^*_{\omega_1} \leq \frac{Nq}{N - sq} \), which implies \( W^{s,s}(\Omega) \hookrightarrow LP^{p_2}(\Omega) \). Then we get
\[
\|\mathcal{N}_p(u)\|_{L^{p^*_{\omega_1}}(\Omega)} \leq C \left( \|u\|_{W^{s,s}(\Omega)}^{\frac{1}{p}} + \|u\|_{W^{s,s}(\Omega)} \right) \|u\|_{\tilde{W}^{s,s}(\Omega)}^{p-1}, \tag{4.20}
\]
The right hand side of (4.15) becomes
\[
\text{The (RHS) of (4.15) becomes:} \tag{4.21}
\]
\[
\leq C \int_0^t (t - \tau)^{\alpha-1} \left( \|u(\cdot, \tau)\|_{W^{s,s}(\Omega)}^{p-1} + \|u(\cdot, \tau)\|_{W^{s,s}(\Omega)}^{\frac{1}{p} + \chi} \right) d\tau \\
\leq C \int_0^t (t - \tau)^{\alpha-1} \tau^{-\alpha\gamma(p-1-\chi)} \left( \tau^{\alpha\gamma} \|u(\cdot, \tau)\|_{W^{s,s}(\Omega)} \right)^{p-1-\chi} d\tau \\
+ C \int_0^t (t - \tau)^{\alpha-1} \tau^{-\alpha\gamma(p-1+\chi)} \left( \tau^{\alpha\gamma} \|u(\cdot, \tau)\|_{W^{s,s}(\Omega)} \right)^{p-1+\chi} d\tau \\
\leq C \left( d + T^{\alpha\gamma} \|u_0\|_{W^{s,s}(\Omega)} \right)^{p-1-\chi} \int_0^t (t - \tau)^{\alpha-1} \tau^{-\alpha\gamma(p-1-\chi)} d\tau \\
+ C \left( d + T^{\alpha\gamma} \|u_0\|_{W^{s,s}(\Omega)} \right)^{p-1+\chi} \int_0^t (t - \tau)^{\alpha-1} \tau^{-\alpha\gamma(p-1+\chi)} d\tau, \tag{4.22}
\]
where we have used the fact that \( \tau^{\alpha\gamma} \|u(\cdot, \tau)\|_{W^{s,s}(\Omega)} \leq d + T^{\alpha\gamma} \|u_0\|_{W^{s,s}(\Omega)} \) from (4.11). For \( \gamma < \frac{1}{\alpha(p-1+\chi)} \), then \( \gamma < \frac{1}{\alpha(p-1-\chi)} \), using Lemma 2.5, we get
\[
\int_0^t (t - \tau)^{\alpha-1} \tau^{-\alpha\gamma(p-1-\chi)} d\tau \\
= \int_0^t (t - \tau)^{\alpha-1} \tau^{(1-\alpha\gamma(p-1-\chi))} d\tau = \frac{\Gamma(\alpha) \Gamma(1 - \alpha\gamma(p-1-\chi))}{\Gamma(\alpha + 1 - \alpha\gamma(p-1-\chi))} \rho^{\alpha - \alpha\gamma(p-1-\chi)} \\
\leq B(a, 1 - \alpha\gamma(p-1-\chi)) t^{\alpha - \alpha\gamma(p-1-\chi)}, \tag{4.23}
\]
and
\[
\int_0^t (t - \tau)^{\alpha-1} \tau^{-\alpha\gamma(p-1+\chi)} d\tau \\
= \int_0^t (t - \tau)^{\alpha-1} \tau^{(1-\alpha\gamma(p-1+\chi))} d\tau = \frac{\Gamma(\alpha) \Gamma(1 - \alpha\gamma(p-1+\chi))}{\Gamma(\alpha + 1 - \alpha\gamma(p-1+\chi))} \rho^{\alpha - \alpha\gamma(p-1+\chi)} \\
\leq B(a, 1 - \alpha\gamma(p-1+\chi)) t^{\alpha - \alpha\gamma(p-1+\chi)}, \tag{4.24}
\]
where \( B \) is Beta function. From (4.15), (4.21) and (4.23), we obtain for \( t \in (0, T] \)
\[
\|J_1(u)(t)\|_{\tilde{W}^{s,s}(\Omega)} \leq C \left( d + T^{\alpha\gamma} \|u_0\|_{W^{s,s}(\Omega)} \right)^{p-1-\chi} B(a, 1 - \alpha\gamma(p-1-\chi)) \rho^{\alpha - \alpha\gamma(p-1-\chi)} \\
+ C \left( d + T^{\alpha\gamma} \|u_0\|_{W^{s,s}(\Omega)} \right)^{p-1+\chi} B(a, 1 - \alpha\gamma(p-1+\chi)) \rho^{\alpha - \alpha\gamma(p-1+\chi)}. \tag{4.25}
\]
For the constants $s, q$ satisfying \( (4.8) \), we imply that \( \mathbb{H}^{2s_2}(\Omega) \hookrightarrow W^{s,q}(\Omega) \). For \( \chi < p - 2 \), implies that \( p - 1 - \chi > 0 \) and \( p - 1 + \chi > 0 \). Choosing \( \gamma \in (0, \frac{1}{\alpha(p - 1 + \chi)}) \) such that \( \alpha - \alpha\gamma(p - 1 + \chi) > 0 \), we get for \( t \in (0, T] \)
\[
t^{\alpha\gamma} \| J_1(u)(t) \|_{W^{s,q}(\Omega)} \leq C t^{\alpha\gamma} \| J_1(u)(t) \|_{\mathbb{H}^{2s_2}(\Omega)} \tag{4.26}
\]
\[
\leq C \left( d + T^{\alpha\gamma} \| u_0 \|_{W^{s,q}(\Omega)} \right)^{p - 1 - \chi} \mathbb{B}(\alpha, 1 - \alpha\gamma(p - 1 - \chi))T^{\alpha + \alpha\gamma(p - 1 - \chi)}
+ C \left( d + T^{\alpha\gamma} \| u_0 \|_{W^{s,q}(\Omega)} \right)^{p - 1 + \chi} \mathbb{B}(\alpha, 1 - \alpha\gamma(p - 1 + \chi))T^{\alpha + \alpha\gamma(p - 1 + \chi)}.
\]

Hence, from \((4.14)\) and \((4.26)\), for every \( t \in (0, T] \), it follows
\[
\| J_u - u_0 \|_{\mathbb{H}^{2s_2}(\Omega)} \leq \| E_{\alpha, 1} (t + m\mathbb{L})^{-1} \| u_0 - u_0 \|_{\mathbb{H}^{2s_2}(\Omega)} + \| J_1(u)(t) \|_{\mathbb{H}^{2s_2}(\Omega)}
\leq C t^{\alpha\gamma} \| u_0 \|_{\mathbb{H}^{2s_2}(\Omega)}
+ C \left( d + T^{\alpha\gamma} \| u_0 \|_{W^{s,q}(\Omega)} \right)^{p - 1 - \chi} \mathbb{B}(\alpha, 1 - \alpha\gamma(p - 1 - \chi))T^{\alpha + \alpha\gamma(p - 1 - \chi)}
+ C \left( d + T^{\alpha\gamma} \| u_0 \|_{W^{s,q}(\Omega)} \right)^{p - 1 + \chi} \mathbb{B}(\alpha, 1 - \alpha\gamma(p - 1 + \chi))T^{\alpha + \alpha\gamma(p - 1 + \chi)}. \tag{4.27}
\]
Therefore we see that if \( d = 2C t^{\alpha\gamma} \| u_0 \|_{\mathbb{H}^{2s_2}(\Omega)} \) and
\[
d \geq 4C \left( d + T^{\alpha\gamma} \| u_0 \|_{W^{s,q}(\Omega)} \right)^{p - 1 - \chi} \mathbb{B}(\alpha, 1 - \alpha\gamma(p - 1 - \chi))T^{\alpha + \alpha\gamma(p - 1 - \chi)},
\]
and
\[
d \geq 4C \left( d + T^{\alpha\gamma} \| u_0 \|_{W^{s,q}(\Omega)} \right)^{p - 1 + \chi} \mathbb{B}(\alpha, 1 - \alpha\gamma(p - 1 + \chi))T^{\alpha + \alpha\gamma(p - 1 + \chi)},
\]
then \( J \) is invariant in \( \mathcal{B} \).

- **Claim 2**: \( J : \mathcal{B} \to \mathcal{B} \) is a contraction map. Let \( u, v \in \mathcal{B} \), similar to \((4.15)\) and using Lemma \( 2.1b \), for every \( t \in (0, T] \), we have
\[
\| J_u(t) - J_v(t) \|_{\mathbb{H}^{2s_2}(\Omega)} \leq \frac{C \sqrt{T}}{m \lambda_1^{1 - \omega_2 + \omega_1}} \int_0^t (t - \tau)^{\alpha - 1} \| N_p(u)(\cdot, \tau) - N_p(v)(\cdot, \tau) \|_{\mathbb{H}^{2s_1}(\Omega)} d\tau
\leq \frac{C \sqrt{T}}{m \lambda_1^{1 - \omega_2 + \omega_1}} \int_0^t (t - \tau)^{\alpha - 1} \| N_p(u)(\cdot, \tau) - N_p(v)(\cdot, \tau) \|_{L^{2s_1}(\Omega)} d\tau, \tag{4.28}
\]
in which we used the Sobolev embedding \( L^{2s_1}(\Omega) \hookrightarrow \mathbb{H}^{2s_1}(\Omega) \), for \( p_{s_1}^* = \frac{2N}{N - 4\omega_1} \) and \(-\frac{N}{4} < \omega_1 \leq 0\). By recalling Lemma \( 4.1 \), we arrive at
\[
| N_p(u) - N_p(v) | \leq C \left( (1 + |u|^{p - 2} + |v|^{p - 2}) \log |u| + |u|^{p - 2} \right) |u - v|. \tag{4.29}
\]
For \( p_{s_1}^* \geq 1 \), using Hölder’s inequality we get
\[
\int_\Omega (| \log |u|| |u - v||^2 )^{s_1} dx \tag{4.30}
= \int_\Omega | \log |u||^{p_{s_1}} |u - v||^{p_{s_1}} dx \leq \left( \int_\Omega | \log |u||^{p_{s_1}^*} dx \right)^{\frac{p_{s_1}^* - 1}{p_{s_1}}} \left( \int_\Omega |u - v||^{p_{s_1}^*} dx \right)^{\frac{1}{p_{s_1}}}
\leq \left( \int_{\Omega_1} | \log |u||^{p_{s_1}^*} dx + \int_{\Omega_2} | \log |u||^{p_{s_1}^*} dx \right)^{\frac{p_{s_1}^* - 1}{p_{s_1}}} \left( \int_\Omega |u - v||^{p_{s_1}^*} dx \right)^{\frac{1}{p_{s_1}}},
\]
Combining (4.34) and (4.35) we get that

\[
\begin{aligned}
&\left(\int_{\Omega_1} |\log |u|| \frac{\rho x}{p+1} \, dx\right)^{\frac{p-1}{p}} + \left(\int_{\Omega_2} |\log |u|| \frac{\rho x}{p+1} \, dx\right)^{\frac{p-1}{p}} \left(\int_{\Omega} |u-v|^{|\rho x|} \, dx\right)^{\frac{1}{p}}.
\end{aligned}
\]

From the inequality (2.24) for $|u(x)| < 1$, \(\forall x \in \Omega\), we have

\[
\left(\int_{\Omega_1} |u(x)|^{\frac{p x}{p+1}} \, dx\right)^{\frac{p-1}{p}} \leq C \left(\int_{\Omega_1} |u(x)|^{\frac{p x}{p+1}} \, dx\right)^{\frac{p-1}{p}} \leq \left(\int_{\Omega_1} |u(x)|^{-\frac{p-1 + \rho x}{p+1}} \, dx\right)^{-\rho x}.
\]

\[
\leq C \left(\left(\int_{\Omega_1} |u(x)|^{\lambda} \, dx\right) \left(\int_{\Omega_1} 1 \, dx\right) \right)^{-\rho x} \leq C \|u\|_{L^\lambda(\Omega)}^{\rho x} \|u\|_{L^{p+1}(\Omega)}^{-\rho x},
\]

where we have chosen \(m = -\frac{p-1}{p+1} < 0\) in the inequality (2.25). For $|u(x)| \geq 1$, we have

\[
\left(\int_{\Omega_2} |\log |u|| \frac{\rho x}{p+1} \, dx\right)^{\frac{p-1}{p}} \leq C \left(\int_{\Omega_2} |u(x)|^{\frac{p x}{p+1}} \, dx\right)^{\frac{p-1}{p}} \leq C \|u\|_{L^\lambda(\Omega)}^{\rho x} \|u\|_{L^{p+1}(\Omega)}^{-\rho x}.
\]

From (4.30), (4.31) and (4.32), we conclude that

\[
\|\log |u||u-v\|_{L^p(\Omega)} \leq C \left(\|u\|_{L^\lambda(\Omega)}^{\rho x} + \|u\|_{L^\lambda(\Omega)}^{\rho x} \right) \|u-v\|_{L^p(\Omega)}.
\]

Thanks to Hölder’s inequality, we get

\[
\int_{\Omega} (|u|^{p-2} \log |u||u-v|) \, dx
= \int_{\Omega} |u|^{p-2} \frac{p x}{p+1} \, dx \int_{\Omega} |u|^{p x} \, dx \, dx
\leq \left(\int_{\Omega} |\log |u|| \frac{p x}{p+1} \, dx\right)^{\frac{p-2}{p}} \left(\int_{\Omega} |u|^{p-2} \, dx\right) \left(\int_{\Omega} |u-v|^{p x} \, dx\right)^{\frac{1}{p}}.
\]

Similar to (4.31) and (4.32), we have

\[
\left(\int_{\Omega} |\log |u|| \frac{p x}{p+1} \, dx\right)^{\frac{p-2}{p}} \leq C \left(\|u\|_{L^\lambda(\Omega)}^{\rho x} + \|u\|_{L^\lambda(\Omega)}^{\rho x} \right).
\]

Combining (4.34) and (4.35) we get that

\[
\|u|^{p-2} \log |u||u-v\|_{L^p(\Omega)}
\leq C \left(\|u\|_{L^\lambda(\Omega)}^{\rho x} + \|u\|_{L^\lambda(\Omega)}^{\rho x} \right) \|u|^{p-2} \log |u||u-v\|_{L^p(\Omega)}.
\]

Similarly, we also have

\[
\|v|^{p-2} \log |u||u-v\|_{L^p(\Omega)}
\]
\[
\begin{align*}
\leq & C \left( \|u\|_{L^\chi(\Omega)}^\chi + \|u\|_{L^{p,\omega_1, p \omega_1 \over p - 1}(\Omega)}^{\chi \left( p - 2 \right) p \omega_1} \right) \left( \|u - v\|_{L^{p,\omega_1, p \omega_1 \over p - 1}(\Omega)}^{p - 2} \right) \|u - v\|_{L^{p,\omega_1, p \omega_1 \over p - 1}(\Omega)} \quad \text{(4.37)}
\end{align*}
\]

We apply again Hölder’s inequality to obtain that
\[
\begin{align*}
\|u - v\|_{L^{p,\omega_1, p \omega_1 \over p - 1}(\Omega)}^{p - 2} & \leq \int_{\Omega} (|u - v|)^{p - 2} \, dx \left( \int_{\Omega} |u - v|^{p \omega_1 \over p - 1} \, dx \right)^{p - 1 \over p} \\
& \leq \left( \int_{\Omega} |u - v|^{p \omega_1 \over p - 1} \, dx \right)^{p - 1 \over p} \|u - v\|_{L^{p,\omega_1, p \omega_1 \over p - 1}(\Omega)}^{p - 2} \quad \text{(4.38)}
\end{align*}
\]

Combining (4.29), (4.33), (4.36), (4.37) and (4.38), we deduce that
\[
\begin{align*}
\|N_p(u)(\cdot, t) - N_p(v)(\cdot, t)\|_{L^{p,\omega_1, p \omega_1 \over p - 1}(\Omega)} & \leq C \left( \|u(\cdot, t)\|_{L^\chi(\Omega)} + \|u(\cdot, t)\|_{L^{p,\omega_1, p \omega_1 \over p - 1}(\Omega)}^{\chi \left( p - 2 \right) p \omega_1} \right) \left( \|u(\cdot, t) - v(\cdot, t)\|_{L^{p,\omega_1, p \omega_1 \over p - 1}(\Omega)}\right) \\
& \quad + C \left( \|u(\cdot, t)\|_{L^\chi(\Omega)} + \|u(\cdot, t)\|_{L^{p,\omega_1, p \omega_1 \over p - 1}(\Omega)}^{\chi \left( p - 2 \right) p \omega_1} \right) \left( \|u(\cdot, t) - v(\cdot, t)\|_{L^{p,\omega_1, p \omega_1 \over p - 1}(\Omega)}\right) \\
& \quad \quad \times \left( \|u(\cdot, t)\|_{L^{p,\omega_1, p \omega_1 \over p - 1}(\Omega)}^{p - 2} \right) \left( \|u(\cdot, t) - v(\cdot, t)\|_{L^{p,\omega_1, p \omega_1 \over p - 1}(\Omega)}\right) \\
& \quad + C \left( \|u(\cdot, t)\|_{L^\chi(\Omega)} + \|u(\cdot, t)\|_{L^{p,\omega_1, p \omega_1 \over p - 1}(\Omega)}^{\chi \left( p - 2 \right) p \omega_1} \right) \left( \|u(\cdot, t) - v(\cdot, t)\|_{L^{p,\omega_1, p \omega_1 \over p - 1}(\Omega)}\right) \\
& \quad \quad \times \left( \|v(\cdot, t)\|_{L^{p,\omega_1, p \omega_1 \over p - 1}(\Omega)}^{p - 2} \right) \left( \|u(\cdot, t) - v(\cdot, t)\|_{L^{p,\omega_1, p \omega_1 \over p - 1}(\Omega)}\right) \\
& \quad + C \|v(\cdot, t)\|_{L^{p,\omega_1, p \omega_1 \over p - 1}(\Omega)}^{p - 2} \left( \|u(\cdot, t) - v(\cdot, t)\|_{L^{p,\omega_1, p \omega_1 \over p - 1}(\Omega)}\right) \quad \text{(4.39)}
\end{align*}
\]

From (4.5)-(4.10) we have the following.

- For \( q \leq \chi N \) and \( sq < N \), ensuring \( \frac{Nq}{N - sq} \leq \chi \), we deduce from Lemma 2.9 that \( L^\chi(\Omega) \hookrightarrow W^{s,q}(\Omega) \). Then we get \( \|u\|_{W^{s,q}(\Omega)} \leq C \|u\|_{L^\chi(\Omega)} \), implies that
  \[ \|u\|_{L^\chi(\Omega)} \leq C \|u\|_{W^{s,q}(\Omega)}, \quad \text{for } \chi > 0. \]

- For \( \omega^* > Q_1 \), we observe that
  \[ np_{\omega_1} = \frac{2Np}{\chi - 4\omega_1} = \frac{2Np}{N + 4\omega^* - 4p\omega_2} < \frac{2Np}{N + 4d_1 - 4p\omega_2} = \frac{Nq}{N - sq} =: p^*_{s,q}, \]
  and then we deduce from Lemma 2.9 that \( W^{s,q}(\Omega) \hookrightarrow L^{p^*_{s,q}}(\Omega) \).

- For \( \omega^* > Q_3 \), implies that \( p(p - 2)p_{\omega_1} < p^*_{s,q} \) then \( W^{s,q}(\Omega) \hookrightarrow L^{p\left(p - 2\right)p_{\omega_1}}(\Omega) \).

- For \( \omega^* > Q_4 \), implies that \( p_{\omega_1}^{p_{\omega_1} - 1} < p^*_{s,q} \) then \( W^{s,q}(\Omega) \hookrightarrow L^p(\Omega) \).

- For \( \omega^* > Q_5 \), implies that \( p_{\omega_1}^{p - 2} < p^*_{s,q} \) then \( W^{s,q}(\Omega) \hookrightarrow L^p(\Omega) \).

Combining all these above results together with (4.39) we deduce
\[
\|N_p(u)(\cdot, t) - N_p(v)(\cdot, t)\|_{L^{\omega_1, p \omega_1 \over p - 1}(\Omega)} \leq K(\alpha t)^{-\alpha^*} \|u(\cdot, t) - v(\cdot, t)\|_{W^{s,q}(\Omega)}, \quad \text{(4.40)}
\]
where we have used the fact that \( \max \{ t^{\alpha \gamma} \| u \|_{W^{s,q}(\Omega)}; t^{\alpha \gamma} \| v \|_{W^{s,q}(\Omega)} \} \leq \alpha + T^{\alpha \gamma} \| u_0 \|_{W^{s,q}(\Omega)}, \forall t \in (0, T) \), and \( K(u_0) := K(N, p, \chi, \omega_1, \alpha, \gamma, T, d, \| u_0 \|_{W^{s,q}(\Omega)}) \) independent of \( t \). From (4.40), we have

\[
\text{The (RHS) of (4.28)} \quad (4.41)
\]

\[
\begin{align*}
&\leq CK(u_0) \int_0^t (t - \tau)^{\alpha - 1 - \alpha \gamma} \| u(\cdot, \tau) - v(\cdot, \tau) \|_{W^{s,q}(\Omega)} \, d\tau \\
&\leq CK(u_0) \int_0^t (t - \tau)^{\alpha - 1 - \alpha \gamma} \left( \tau^{\alpha \gamma} \| u(\cdot, \tau) - v(\cdot, \tau) \|_{W^{s,q}(\Omega)} \right) \, d\tau \\
&\leq CK(u_0) \sup_{0 < \tau \leq T} \left( \tau^{\alpha \gamma} \| u(\cdot, \tau) - v(\cdot, \tau) \|_{W^{s,q}(\Omega)} \right) \int_0^t (t - \tau)^{\alpha - 1 - 2\alpha \gamma} \, d\tau \\
&\leq CK(u_0) \sup_{0 < \tau \leq T} \left( \tau^{\alpha \gamma} \| u(\cdot, \tau) - v(\cdot, \tau) \|_{W^{s,q}(\Omega)} \right) B(\alpha, 1 - 2\alpha \gamma) t^{\alpha - 2\alpha \gamma},
\end{align*}
\]

where we have applied Lemma 2.5 for \( \gamma \leq \frac{1}{2\alpha} \).

\[
\int_0^t (t - \tau)^{\alpha - 1 - 2\alpha \gamma} \, d\tau = \int_0^t (t - \tau)^{\alpha - 1} \left( 1 - 2\alpha \gamma \right) \, d\tau = \frac{\Gamma(\alpha) \Gamma(1 - 2\alpha \gamma)}{\Gamma(\alpha + 1 - 2\alpha \gamma)} t^{\alpha - 2\alpha \gamma}.
\]

Inserting the result of (4.41) into (4.28), we obtain that

\[
\begin{align*}
&\| J u(t) - J v(t) \|_{\mathcal{H}^{2s-2}(\Omega)} \\
&\leq CK(u_0) B(\alpha, 1 - 2\alpha \gamma) t^{\alpha - 2\alpha \gamma} \sup_{0 < \tau \leq T} \left( \tau^{\alpha \gamma} \| u(\cdot, \tau) - v(\cdot, \tau) \|_{W^{s,q}(\Omega)} \right).
\end{align*}
\]

For the constants \( s, q \) satisfying (4.8), we imply that \( \mathcal{H}^{2s-2}(\Omega) \hookrightarrow W^{s,q}(\Omega) \). Then the following estimate holds for every \( t \in (0, T] \) and \( 0 < \gamma < \min\{ \frac{1}{2\alpha}, 1 \} \)

\[
\begin{align*}
&\| J u - J v \|_{\mathcal{C}^{\alpha \gamma}((0,T];W^{s,q}(\Omega))} \\
&\leq CK(u_0) B(\alpha, 1 - 2\alpha \gamma) T^{\alpha - \alpha \gamma} \| u - v \|_{\mathcal{C}^{\alpha \gamma}((0,T];W^{s,q}(\Omega))}.
\end{align*}
\]

Choosing \( T \) and \( K(u_0) \) small enough such that \( CK(u_0) B(\alpha, 1 - 2\alpha \gamma) T^{\alpha + \alpha \gamma - 2\alpha \gamma} < 1 \), it follows that \( J \) is a contraction map on \( \mathcal{B} \). Hence, we can invoke the contraction mapping principle to conclude that the map \( J \) has a unique fixed point \( u \) in \( \mathcal{B} \).

**Proof (II).** Since we already know that the local mild solution of (\( \mathcal{P}_1 \)) does exist. Next, we prove this solution is a continuation to a bigger interval of existence. Firsts, we consider the following definition.

**Definition 4.2.** Given a mild solution \( u \in \mathcal{C}^{\alpha \gamma}((0,T];W^{s,q}(\Omega)) \) of (\( \mathcal{P}_1 \)) for some \( \gamma > 0 \), we say that \( u^* \) is a **continuation** of \( u \) if \( u^* \in \mathcal{C}^{\alpha \gamma}((0,T^*];W^{s,q}(\Omega)) \) is a mild solution for \( T^* > T \) and \( u^*(\cdot,t) = u(\cdot,t) \) whenever \( t \in (0, T] \).

Let \( u : [0, T] \to W^{s,q}(\Omega) \) be a mild solution of Problem (\( \mathcal{P}_1 \)), and \( T \) be the time defined in Part (I). Fix \( d^* > 0 \), and for \( T^* > T \), \( (T^* \) depending on \( d^*) \), we shall prove that \( u^* : [0, T^*] \to W^{s,q}(\Omega) \) is a mild solution of Problem (\( \mathcal{P}_1 \)). This should be well dealt with \( 0 < \gamma < \min\{ 1, \frac{1}{2\alpha} ; \frac{1}{\alpha(p-1+\chi)} \} \) satisfying \( \alpha - \alpha \gamma(p-1+\chi) > 0 \) and

\[
T^{-\alpha} (T^*)^{\alpha \gamma \alpha} \| u_0 \|_{\mathcal{H}^{2s-2}(\Omega)} \leq \frac{d^*}{6},
\]

\[
C \left( d^* + (T^*)^{\alpha \gamma} \| u(\cdot,T) \|_{W^{s,q}(\Omega)} \right)^{p-1-\chi}
\]
\[ \times \mathbb{B}(\alpha, 1 - \alpha \gamma(p - 1 - \chi))(T^*)^{\alpha + \alpha \gamma - \alpha \gamma(p - 1 - \chi)} \leq \frac{d^*}{6}, \quad (4.45) \]

\[ C \left( d^* + (T^*)^{\alpha \gamma} \| u(\cdot, T) \|_{W^{\alpha, q}(\Omega)} \right)^{p - 1 + \chi} \]

\[ \times \mathbb{B}(\alpha, 1 - \alpha \gamma(p - 1 + \chi))(T^*)^{\alpha + \alpha \gamma - \alpha \gamma(p - 1 + \chi)} \leq \frac{d^*}{6}, \quad (4.46) \]

\[ C \left( d + (T^*)^{\alpha \gamma} \| u_0 \|_{W^{\alpha, q}(\Omega)} \right)^{p - 1 - \chi} \]

\[ \times \mathbb{B}(\alpha, 1 - \alpha \gamma(p - 1 - \chi))(T^*)^{\alpha + \alpha \gamma - \alpha \gamma(p - 1 - \chi)} \leq \frac{d^*}{6}, \quad (4.47) \]

\[ C \left( d + (T^*)^{\alpha \gamma} \| u_0 \|_{W^{\alpha, q}(\Omega)} \right)^{p - 1 + \chi} \]

\[ \times \mathbb{B}(\alpha, 1 - \alpha \gamma(p - 1 + \chi))(T^*)^{\alpha + \alpha \gamma - \alpha \gamma(p - 1 + \chi)} \leq \frac{d^*}{6}, \quad (4.48) \]

\[ CK(u_0)\mathbb{B}(\alpha, 1 - 2\alpha \gamma)(T^*)^{\alpha - \alpha \gamma} \leq \frac{d^*}{6}, \quad (4.49) \]

where \( K(u_0) \) be defined in the proof of Part (I) and \( \mathbb{B} \) is Beta function. For \( T^* \geq T > 0 \) and \( d^* > 0 \), let us define

\[ \mathcal{B}^* \]

\[ := \left\{ u^* \in \mathcal{C}^{\alpha \gamma}(0, T^*]; W^{\alpha, q}(\Omega) : \| u^*(\cdot, t) - u(\cdot, t) \|_{W^{\alpha, q}(\Omega)} \leq d^*, \forall t \in [T, T^*] \right\} \]

\[
\bullet \text{ Step I. We show that } J \text{ defined as in (4.12) is an operator on } \mathcal{B}^*. \text{ Indeed, let } u^* \in \mathcal{B}^*, \text{ we consider the following two cases.}
\]

* If \( t \in (0, T] \), then by virtue of the Part (I), we know that Problem \((P_1)\) admits a unique solution and \( u^*(\cdot, t) = u(\cdot, t) \), thus \( \| Ju^* - J u \|_{\mathcal{C}^{\alpha \gamma}(0, T]; W^{\alpha, q}(\Omega)} = 0 \) in \( \mathcal{B}^* \) for all \( t \in (0, T] \).

* If \( t \in [T, T^*] \), we have

\[ \| Ju^*(t) - u(\cdot, T) \|_{W^{\alpha, q}(\Omega)} \leq \left\| (E_{\alpha, 1}(-\partial^\alpha(I + mL)^{-1}) - E_{\alpha, 1}(-\partial^\alpha(I + mL)^{-1}T^\alpha)) u_0 \right\|_{W^{\alpha, q}(\Omega)} \]

\[ + \left\| (I + mL)^{-1} \int_0^T (t - \tau)^{\alpha - 1} E_{\alpha, 1}((t - \tau)^{\alpha}N_p(u^*) d\tau \right\|_{W^{\alpha, q}(\Omega)} \]

\[ + \left\| (I + mL)^{-1} \int_0^T (t - \tau)^{\alpha - 1} E_{\alpha, 1}((t - \tau)^{\alpha} - (T - \tau)^{\alpha - 1}E_{\alpha, 1}((T - \tau)^{\alpha}) N_p(u^*) d\tau \right\|_{W^{\alpha, q}(\Omega)} \]

\[ =: \| J_2(u_0)(t) \|_{W^{\alpha, q}(\Omega)} + \| J_3(u^*)(t) \|_{W^{\alpha, q}(\Omega)} + \| J_4(u^*)(t) \|_{W^{\alpha, q}(\Omega)}, \quad (4.51) \]

where \( E_{\alpha, 1}(\cdot) := E_{\alpha, 1}(-\partial^\alpha(I + mL)^{-1}) \), for simplicity of exposition.

To estimate the term \( \| J_2(u_0)(t) \|_{W^{\alpha, q}(\Omega)} \) in (4.51), using Lemma 2.3, for all \( t \in [T, T^*] \) we have

\[ \| J_2(u_0)(t) \|_{W^{\alpha, q}(\Omega)}^2 \]

\[ = \sum_{j=1}^{\infty} \lambda_j^{2\alpha} (u_0, \varphi_j)^2 \left( E_{\alpha, 1} \left(-\frac{\lambda_j^\alpha t^\alpha}{1 + mL_j} \right) - E_{\alpha, 1} \left(-\frac{\lambda_j^\alpha T^\alpha}{1 + mL_j} \right) \right)^2 \]
we can choose\( T \) from (4.50) to give that for all

Using (4.45) and (4.46), we infer that

By (4.44), the following estimate holds

For the constants \( s, q \) satisfying (4.8), we imply that \( H^{2\alpha}(\Omega) \hookrightarrow W^{s, q}(\Omega) \). Hence, we get

By (4.44), the following estimate holds

From (4.50) to give that for all \( t \in [T, T^*] \)

and similar to (4.25), we have the following estimate for all \( t \in [T, T^*] \) (note that we can choose \( T^* > T \) and close enough to \( T \))

Using (4.45) and (4.46), we infer that

\[
\sum_{j=1}^{\infty} \lambda_j^{2\alpha} (u_0, \varphi_j)^2 \left( \int_0^\infty M_\alpha(z) \left| \exp \left( -z \frac{\lambda_j^\sigma t^\alpha}{1 + m\lambda_j} \right) - \exp \left( -z \frac{\lambda_j^\sigma T^\alpha}{1 + m\lambda_j} \right) \right| \, dz \right)^2
\leq \sum_{j=1}^{\infty} \lambda_j^{2\alpha} (u_0, \varphi_j)^2 \left( \int_0^\infty M_\alpha(z) \exp \left( -z \frac{\lambda_j^\sigma T^\alpha}{1 + m\lambda_j} \right) \exp \left( -z \frac{\lambda_j^\sigma (t^\alpha - T^\alpha)}{1 + m\lambda_j} \right) - 1 \, dz \right)^2.
\]

For \( z > 0 \), using the inequality \( 1 - e^{-z} \leq z \), and \( ze^{-z} \leq 1 \), we obtain

\[
\| J_2(u_0)(t) \|_{H^{2\alpha}(\Omega)}^2 
\leq \sum_{j=1}^{\infty} \lambda_j^{2\alpha} (u_0, \varphi_j)^2 \left( \int_0^\infty M_\alpha(z) \left| \frac{\lambda_j^\sigma T^\alpha}{1 + m\lambda_j} \right| \, dz \right)^2
\leq \sum_{j=1}^{\infty} \lambda_j^{2\alpha} (u_0, \varphi_j)^2 \left( (t^\alpha - T^\alpha) T^{-\alpha} \int_0^\infty M_\alpha(z) \, dz \right)^2
\leq (t - T)^{2\alpha} T^{-2\alpha} \| u_0 \|_{H^{2\alpha}(\Omega)}^2.
\]

(4.52)

where, we have used the inequality

\[ a^c - b^c \leq (a - b)^c, \quad \text{for } a > b > 0, c \in (0, 1), \quad \text{and } \int_0^\infty M_\alpha(z) \, dz = 1. \]

For the constants \( s, q \) satisfying (4.8), we imply that \( H^{2\alpha}(\Omega) \hookrightarrow W^{s, q}(\Omega) \). Hence, we get

\[
C (t - T)^\alpha T^{-\alpha} (T^*)^{\gamma + \alpha} \| u_0 \|_{H^{2\alpha}(\Omega)} \leq CT^{-\alpha} (T^*)^{\gamma + \alpha} \| u_0 \|_{H^{2\alpha}(\Omega)}. \]

(4.53)

and

\[
\| J_2(u_0) \|_{W^{s, q}(\Omega)} \leq T^{-\alpha} (T^*)^{\gamma + \alpha} \| u_0 \|_{H^{2\alpha}(\Omega)} \leq \frac{d^*}{6}. \]

(4.54)
\[ + C \left( d^* + (T^*)^{\alpha \gamma} \| u(\cdot, T) \|_{W^{s,q}(\Omega)} \right)^{p-1+\chi} \times \mathbb{B}(\alpha, 1 - \alpha\gamma(p - 1 + \chi))(T^*)^{\alpha + \alpha\gamma - \alpha\gamma(p - 1 + \chi)} \leq \frac{d^*}{6}, \frac{\alpha\gamma(p - 1 + \chi)}{6} = \frac{d^*}{3}. \quad (4.56) \]

We continue with the estimate of the third term of (4.51), using (2.16b) and Lemma 2.1b), for all \( t \in [T, T^*] \), to obtain

\[
\mathcal{D}_j(t, \tau) = \left| (t - \tau)^{\alpha - 1} E_{a,a} \left( -\frac{\lambda^a_j (t - \tau)^{\alpha}}{1 + m\lambda_j} \right) - (T - \tau)^{\alpha - 1} E_{a,a} \left( -\frac{\lambda^a_j (T - \tau)^{\alpha}}{1 + m\lambda_j} \right) \right|
\[
= \left| \int_{T-\tau}^{T-\tau} z^{\alpha - 2} E_{a,a-1} \left( -\frac{\lambda^a_j z^{\alpha}}{1 + m\lambda_j} \right) dz \right| \leq C \int_{T-\tau}^{T-\tau} z^{\alpha - 2} dz \leq C \left( (T - \tau)^{\alpha - 1} - (t - \tau)^{\alpha - 1} \right) \leq C (T - \tau)^{\alpha - 1}. \quad (4.57) \]

Hence, we deduce that

\[
\| J_4(u^*)(t) \|_{\mathcal{H}^2_{\omega}(\Omega)} \leq \int_0^T \left( \sum_{j=1}^{\infty} \frac{\lambda^{\omega \omega}_j}{(1 + m\lambda_j)^2} \left( \mathcal{N}_{\rho}(u^*)(\cdot, \tau), \varphi_j \right)^2 |\mathcal{D}_j(t, \tau)|^2 \right)^{\frac{1}{2}} d\tau \leq C \int_0^T \frac{T}{m\lambda^{1+\omega}_j} \left( \sum_{j=1}^{\infty} \left( \mathcal{N}_{\rho}(u^*)(\cdot, \tau), \varphi_j \right)^2 \lambda^{2\omega\omega}_j \right)^{\frac{1}{2}} d\tau \leq C \int_0^T (T - \tau)^{\alpha - 1} \left( \mathcal{N}_{\rho}(u^*)(\cdot, \tau) \right)_{L^p(\Omega)} d\tau \leq C \int_0^T (T - \tau)^{\alpha - 1} \left( \mathcal{N}_{\rho}(u^*)(\cdot, \tau) \right)_{L^p(\Omega)} d\tau, \quad (4.58) \]

where we have used the Sobolev embedding \( L^{p^*}_{\omega}(\Omega) \hookrightarrow \mathcal{H}^{2\omega\omega}(\Omega) \) with \( \omega_1 \) satisfying \( -\frac{N}{2} < \omega_1 \leq 0 \) and \( p^*_\omega = \frac{2N}{N - 2 \omega_1} \). In the same way as in (4.21), (4.23) and (4.24), we have

The (RHS) of (4.58)

\[
\leq C \int_0^T (T - \tau)^{\alpha - 1} \left( \| u^*(\cdot, \tau) \|_{W^{s,q}(\Omega)}^{p - 1 - \chi} + \| u^*(\cdot, \tau) \|_{W^{s,q}(\Omega)}^{p - 1 + \chi} \right) d\tau \leq C \left( d + (T^*)^{\alpha \gamma} \| u_0 \|_{W^{s,q}(\Omega)} \right)^{p - 1 - \chi} \int_0^T (T - \tau)^{\alpha - 1 - \alpha\gamma(p - 1 - \chi)} d\tau + C \left( d + (T^*)^{\alpha \gamma} \| u_0 \|_{W^{s,q}(\Omega)} \right)^{p - 1 + \chi} \int_0^T (T - \tau)^{\alpha - 1 - \alpha\gamma(p - 1 + \chi)} d\tau \leq C \left( d + (T^*)^{\alpha \gamma} \| u_0 \|_{W^{s,q}(\Omega)} \right)^{p - 1 - \chi} \mathbb{B}(\alpha, 1 - \alpha\gamma(p - 1 - \chi))T^{\alpha - \alpha\gamma(p - 1 - \chi)} + C \left( d + (T^*)^{\alpha \gamma} \| u_0 \|_{W^{s,q}(\Omega)} \right)^{p - 1 + \chi} \mathbb{B}(\alpha, 1 - \alpha\gamma(p - 1 + \chi))T^{\alpha - \alpha\gamma(p - 1 + \chi)}.
\]

For the constants \( s, q \) satisfying (4.8), we have that \( \mathcal{H}^{2\omega\omega}(\Omega) \hookrightarrow W^{s,q}(\Omega) \) and obtain

\[
\| J_4(u^*)(t) \|_{W^{s,q}(\Omega)} \leq C \left( d + (T^*)^{\alpha \gamma} \| u_0 \|_{W^{s,q}(\Omega)} \right)^{p - 1 - \chi} \mathbb{B}(\alpha, 1 - \alpha\gamma(p - 1 - \chi))T^{\alpha - \alpha\gamma(p - 1 - \chi)} \quad (4.60) \]
From (4.47), we obtain
\[
J(t) = J_{0} + \int_{0}^{t} J(\tau) \, d\tau.
\]
Thus, for \( t \in [T, T^*] \), we deduce that
\[
\begin{align*}
J(t) &\leq C \left( d + (T^*)^{\alpha} \| u_0 \|_{W^{s,q}(\Omega)} \right)^{p - 1 + \chi} \mathcal{B}(\alpha, 1 - \alpha\gamma(p - 1 + \chi)) T^{\alpha - \alpha\gamma(p - 1 + \chi)}.) \\
&\leq C \left( d + (T^*)^{\alpha} \| u_0 \|_{W^{s,q}(\Omega)} \right)^{p - 1 + \chi} \mathcal{B}(\alpha, 1 - \alpha\gamma(p - 1 + \chi)) (T^*)^{\alpha - \alpha\gamma(p - 1 + \chi)} \\
&\leq C \left( d + (T^*)^{\alpha} \| u_0 \|_{W^{s,q}(\Omega)} \right)^{p - 1 + \chi} \mathcal{B}(\alpha, 1 - \alpha\gamma(p - 1 + \chi)) (T^*)^{\alpha - \alpha\gamma(p - 1 + \chi)}.
\end{align*}
\]
From (4.47), we get
\[
\begin{align*}
&\| J(u^*) \|_{\mathcal{S}^{\alpha,\gamma}(0,T^*; W^{s,q}(\Omega))} \\
&\leq C \left( d + (T^*)^{\alpha} \| u_0 \|_{W^{s,q}(\Omega)} \right)^{p - 1 + \chi} \mathcal{B}(\alpha, 1 - \alpha\gamma(p - 1 + \chi)) (T^*)^{\alpha - \alpha\gamma(p - 1 + \chi)} \\
&\quad + C \left( d + (T^*)^{\alpha} \| u_0 \|_{W^{s,q}(\Omega)} \right)^{p - 1 + \chi} \mathcal{B}(\alpha, 1 - \alpha\gamma(p - 1 + \chi)) (T^*)^{\alpha - \alpha\gamma(p - 1 + \chi)}.
\end{align*}
\]
It follows from (4.54), (4.56), (4.62) that for every \( t \in [T, T^*] \)
\[
\begin{align*}
&\| J(u^*) - u(\cdot, T) \|_{\mathcal{S}^{\alpha,\gamma}(0,T^*; W^{s,q}(\Omega))} \\
&\leq C \left( d + (T^*)^{\alpha} \| u_0 \|_{W^{s,q}(\Omega)} \right)^{p - 1 + \chi} \mathcal{B}(\alpha, 1 - \alpha\gamma(p - 1 + \chi)) (T^*)^{\alpha - \alpha\gamma(p - 1 + \chi)} \\
&\quad + C \left( d + (T^*)^{\alpha} \| u_0 \|_{W^{s,q}(\Omega)} \right)^{p - 1 + \chi} \mathcal{B}(\alpha, 1 - \alpha\gamma(p - 1 + \chi)) (T^*)^{\alpha - \alpha\gamma(p - 1 + \chi)}.
\end{align*}
\]
We have shown that \( J \) is a map \( \mathcal{B}^* \rightarrow \mathcal{B}^* \).

- **Step II.** We show that \( J \) is a contraction on \( \mathcal{B}^* \). Let \( u, v \in \mathcal{B}^* \), and we have that for \( 0 \leq t \leq T^* \)
\[
J(u)(t) - J(v)(t) = (I + mL)^{-1} \int_{0}^{t} (t - \tau)^{\alpha - 1} E_{\alpha,\alpha} \left( -\mathcal{L}^s (I + mL)^{-1} (t - \tau)^{\alpha} \right) (N_p(u)(\tau) - N_p(v)(\tau)) \, d\tau,
\]
where we note that \( J(u)(t) - J(v)(t) = 0, \) for all \( t \in (0, T) \). Then, for all \( t \in [0, T^*] \), proceeding as in the proof of Claim 2 of Part (1), we have
\[
\begin{align*}
&\| J(u)(t) - J(v)(t) \|_{\mathbb{H}^{2s-2}(\Omega)} \\
&\leq CK(u_0) \mathcal{B}(\alpha, 1 - 2\alpha\gamma) T^{\alpha - 2\alpha\gamma} \sup_{0 < \tau \leq T} \left( \tau^{\alpha\gamma} \| u(\cdot, \tau) - v(\cdot, \tau) \|_{W^{s,q}(\Omega)} \right).
\end{align*}
\]
Hence, from (4.49), we deduce that
\[
\begin{align*}
&\| J(u) - J(v) \|_{\mathbb{H}^{2s-2}(\Omega)} \leq CK(u_0) \mathcal{B}(\alpha, 1 - 2\alpha\gamma) (T^*)^{\alpha - 2\alpha\gamma} \| u - v \|_{\mathcal{S}^{\alpha,\gamma}(0,T^*; W^{s,q}(\Omega))} \\
&\quad \leq \frac{d^*}{6} \| u - v \|_{\mathcal{S}^{\alpha,\gamma}(0,T^*; W^{s,q}(\Omega))}.
\end{align*}
\]
This implies that \( J \) is a \( \frac{d^*}{6} \)-contraction. By the Banach contraction principle follows that \( J \) has a unique fixed point \( u^* \) of \( J \) in \( \mathcal{B}^* \), which is a continuation of \( u \).
Proof (III). As an immediate consequence of Part (II) of this theorem, we guarantee the existence of a maximal time. Next, we prove the results which are on global existence or non-continuation by a blow-up.

**Definition 4.3 (Maximal existence time).** Let \( u(x,t) \) be a weak solution of \((P_1)\). We define the maximal existence time \( T_{max} \) of \( u(x,t) \) as follows:

(i) If \( u(x,t) \) exists for all \( 0 \leq t < \infty \), then \( T_{max} = \infty \).

(ii) If there exists \( T \in (0,\infty) \) such that \( u(x,t) \) exists for \( 0 \leq t < T \), but does not exist at \( t = T \), then \( T_{max} = T \).

**Definition 4.4 (Finite time blow-up).** Let \( u(x,t) \) be a weak solution of \((P_1)\). We say \( u(x,t) \) blows up in finite time if the maximal existence time \( T_{max} \) is finite and

\[
\lim_{t \to T_{max}^-} \| u(\cdot, t) \|_{W^{s,q}(\Omega)} = \infty. 
\]  

(4.65)

Let \( u_0 \in H^{2s,2}(\Omega) \cap W^{s,q}(\Omega) \) and define

\[
T_{max} := \sup \{ T > 0 : \text{there exits a solution on } (0, T) \}.
\]

Assume that \( T_{max} < \infty \), and \( \| u(\cdot, t) \|_{W^{s,q}(\Omega)} \leq d_0 \), for some \( d_0 > 0 \), \( \forall t \in (0, T_{max}) \).

Now suppose that there exists a sequence \( \{t_n\} \in [0, T_{max}) \) such that \( t_n \to T_{max} \).

Let us show that \( \{u(\cdot, t_n)\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( W^{s,q}(\Omega) \). Indeed, given \( \epsilon > 0 \) fix \( N \in \mathbb{N} \) such that for all \( n, h > N, 0 < t_n \leq t_n < h < T_{max} \), we have

\[
\| u(\cdot, t_n) - u(\cdot, t_n) \|_{W^{s,q}(\Omega)}
\]

\[
\leq \left\| (E_{\alpha,1} - \mathbb{I}) (I + mL)^{-1} t_n^\alpha - E_{\alpha,1} (I + mL)^{-1} t_n^\alpha \right\|_{W^{s,q}(\Omega)}
\]  

\[+
\int_{t_n}^{t_n} (t_n - \tau)^{\alpha-1} \| (I + mL)^{-1} E_{\alpha} ((t_n - \tau)^\alpha) \|_{W^{s,q}(\Omega)} d\tau
\]  

\[+
\int_{t_n}^{t_n} (t_n - \tau)^{\alpha-1} E_{\alpha} ((t_n - \tau)^\alpha) - (T_{max} - T_{max})^{\alpha-1} E_{\alpha} ((T_{max} - T_{max})^\alpha) \|_{W^{s,q}(\Omega)} \|_{W^{s,q}(\Omega)} d\tau
\]  

(4.66)

Similarly to (4.52), and using the Sobolev embedding \( H^{2s,2}(\Omega) \hookrightarrow W^{s,q}(\Omega) \), we have

\[
\| J_s(u_0) \|_{W^{s,q}(\Omega)} \leq C \| J_s(u_0) \|_{H^{2s,2}(\Omega)} \leq C |t_n - t_n|^\alpha \| u_0 \|_{H^{2s,2}(\Omega)}.
\]  

(4.67)

In the same way as (4.55), we get

\[
\| J_s(u) \|_{W^{s,q}(\Omega)}
\]

\[
\leq C \left( d_0 + T_{max}^\alpha \| u_0 \|_{W^{s,q}(\Omega)} \right)^{p-1-\chi} \| B(\alpha, 1 - \alpha \gamma(p-1-\chi)) |t_n - t_n|^\alpha \gamma(p-1-\chi)
\]  

\[+
C \left( d_0 + T_{max}^\alpha \| u_0 \|_{W^{s,q}(\Omega)} \right)^{p-1+\chi} \| B(\alpha, 1 - \alpha \gamma(p-1+\chi)) |t_n - t_n|^\alpha \gamma(p-1+\chi).
\]  

(4.68)

Similarly to (4.60), we have

\[
\| J_\gamma(u) \|_{W^{s,q}(\Omega)}
\]

\[
\leq C \left( d_0 + T_{max}^\alpha \| u_0 \|_{W^{s,q}(\Omega)} \right)^{p-1-\chi} \| B(\alpha, 1 - \alpha \gamma(p-1-\chi)) |T_{max} - t_n|^\alpha \gamma(p-1-\chi)
\]  

\[+
C \left( d_0 + T_{max}^\alpha \| u_0 \|_{W^{s,q}(\Omega)} \right)^{p-1+\chi} \| B(\alpha, 1 - \alpha \gamma(p-1+\chi)) |T_{max} - t_n|^\alpha \gamma(p-1+\chi),
\]  

(4.69)
Thus, since \( \{ A \} \) and \( \{ H \} \),

we can invoke Part (II) to deduce that the solution can be extended to some larger time interval \((0, T_{\text{max}})\). This means

\[
\lim_{t \to T_{\text{max}}^-} u(t) = 0.
\]

Hence, given \( \epsilon > 0 \) there exists \( N \in \mathbb{N} \) such that

\[
\| u(\cdot, t) - u(\cdot, t_n) \|_{W^{s,q}(\Omega)} < \epsilon, \quad \text{for } h, n \geq N,
\]

which means \( \{ u(\cdot, t_n) \}_{n \in \mathbb{N}} \in W^{s,q}(\Omega) \) is a Cauchy sequences. Arguing by contradiction, we suppose that for \( \{ t_n \}_{n \in \mathbb{N}} \) is arbitrary we have the existence of the limit

\[
\lim_{t \to T_{\text{max}}^-} u(\cdot, t) = 0.
\]

We can invoke Part (II) to deduce that the solution can be extended to some larger time interval \((0, T_{\text{max}})\), and this contradicts the definition of \( T_{\text{max}} \). Thus, either \( T_{\text{max}} = \infty \) or if \( T_{\text{max}} < \infty \) then \( \lim_{t \to T_{\text{max}}^-} u(\cdot, t) \|_{W^{s,q}(\Omega)} = \infty \).

The proof of the Theorem 1.2 is finished.

5. **Proof of Theorem 1.4.** First we introduce the Mittag-Leffler operators. Let \( A = \Delta(m\Delta - I)^{-1} \) be the infinitesimal generator of an analytic semigroup \( \{ T(t) \}, t \geq 0 \). Then, for each \( \alpha \in (0, 1) \), we define the Mittag-Leffler families as follows

\[
E_{\alpha, 1}(-\Delta(m\Delta - I)^{-1}t^\alpha) = \int_0^\infty M_\alpha(r)T(rt^\alpha)dr
\]

and

\[
E_{\alpha, \alpha}(-\Delta(m\Delta - I)^{-1}t^\alpha) = \int_0^\infty \alpha r M_\alpha(r)T(rt^\alpha)dr
\]

and set \( \mathcal{G} = -(m\Delta - I)^{-1} \).

In order to state the main results, we need the following lemmas.
Lemma 5.1. (See [11], expressions (2.12)-(2.13), page 4) If $1 \leq p \leq q$ and $f \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ then for constant $M > 0$
\begin{align*}
\|T(t)f\|_{L^q(\mathbb{R}^N)} & \leq M(1 + t)^{\frac{N(\frac{1}{2} - \frac{1}{p})}{2}} \|f\|_{L^p(\mathbb{R}^N)} + Me^{-t}\|f\|_{L^q(\mathbb{R}^N)}, \quad t \geq 0, \quad (5.3) \\
\|\mathcal{G}f\|_{L^q(\mathbb{R}^N)} & \leq M\|f\|_{L^q(\mathbb{R}^N)}. \quad (5.4)
\end{align*}

Lemma 5.2. As shown in [11], a mild solution of Problem $(\mathbb{P}_2)$ is given by the following
\begin{align*}
u(t) = & E_{\alpha,1}(\Delta(m\Delta - I)^{-1}t^\alpha)u_0 \\
& + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\Delta(m\Delta - I)^{-1}(t-s)^\alpha) \mathcal{G}N(u)ds. \quad (5.5)
\end{align*}

Lemma 5.3. (See [16], Lemma 8, page 9) Let $\omega_1 > -1$, $\omega_2 > -1$ such that $\omega_1 + \omega_2 \geq -1$, $h > 0$ and $t \in [0,T]$. For $\mu > 0$, the following limit holds
\begin{align*}
\lim_{\mu \to \infty} \left(\sup_{t \in [0,T]} t^h \int_0^1 s^{\omega_1}(1-s)^{\omega_2}e^{-\mu t(1-s)}ds\right) = 0.
\end{align*}

Proof of the Theorem 1.4. By (5.1) we get
\begin{align*}
\left\|E_{\alpha,1}(\Delta(m\Delta - I)^{-1}t^\alpha) f\right\|_{L^q(\mathbb{R}^N)} \leq \int_0^\infty \mathcal{M}_\alpha(r) \left\|T(rt^\alpha)f\right\|_{L^q(\mathbb{R}^N)} dr. \quad (5.6)
\end{align*}
Using the following inequality
\begin{align*}
\|T(t)f\|_{L^q(\mathbb{R}^N)} & \leq M(1 + t)^{\frac{N(\frac{1}{2} - \frac{1}{p})}{2}} \|f\|_{L^p(\mathbb{R}^N)} + Me^{-t}\|f\|_{L^q(\mathbb{R}^N)}, \quad t \geq 0, \quad (5.7)
\end{align*}
and the fact that $e^{-z} \leq \theta^\alpha e^{-\theta z}z^{-\theta}$ for any $\theta \in (0,1)$ we find that
\begin{align*}
\int_0^\infty \mathcal{M}_\alpha(r) \left\|T(rt^\alpha)f\right\|_{L^q(\mathbb{R}^N)} dr & \leq M\int_0^\infty \mathcal{M}_\alpha(r) \left(1 + (rt^\alpha)^{\frac{N(\frac{1}{2} - \frac{1}{p})}{2}}\right)^{\frac{1}{2}} \|f\|_{L^p(\mathbb{R}^N)} dr + M\int_0^\infty \mathcal{M}_\alpha(r) e^{-rt^\alpha}\|f\|_{L^q(\mathbb{R}^N)} dr \\
& = Mt^{\frac{N(\frac{1}{2} - \frac{1}{p})}{2}}\|f\|_{L^p(\mathbb{R}^N)} \int_0^\infty \mathcal{M}_\alpha(r) r^{\frac{N(\frac{1}{2} - \frac{1}{p})}{2}} dr \\
& \quad + M\theta^\alpha t^{-\alpha\theta}\|f\|_{L^q(\mathbb{R}^N)} \int_0^\infty r^{-\theta}\mathcal{M}_\alpha(r) dr \\
& = \frac{Mt^{\frac{N(\frac{1}{2} - \frac{1}{p})}{2}}}{\Gamma\left(\frac{\alpha N(N(\frac{1}{2} - \frac{1}{p})}{2} + 1\right)} \|f\|_{L^p(\mathbb{R}^N)} + \frac{M\theta^\alpha t^{-\alpha\theta}\Gamma(1-\theta)}{\Gamma(1-\theta\alpha)}\|f\|_{L^q(\mathbb{R}^N)} \\
& = \tilde{A}_1 t^{\frac{N(\frac{1}{2} - \frac{1}{p})}{2}}\|f\|_{L^p(\mathbb{R}^N)} + \tilde{A}_2 t^{-\alpha\theta}\|f\|_{L^q(\mathbb{R}^N)}, \quad (5.8)
\end{align*}
where $-\theta > -1$ and $\frac{N(\frac{1}{2} - \frac{1}{p})}{2} > -1$ and
\begin{align*}
\tilde{A}_1 = \frac{M_1 \Gamma\left(\frac{N(\frac{1}{2} - \frac{1}{p})}{2} + 1\right)}{\Gamma\left(\frac{\alpha N(N(\frac{1}{2} - \frac{1}{p})}{2} + 1\right)}, \quad \tilde{A}_2 = \frac{M_2 \theta^\alpha \Gamma(1-\theta)}{\Gamma(1-\theta\alpha)}.
\end{align*}
So, we deduce that
\begin{align*}
\left\|E_{\alpha,1}(\Delta(m\Delta - I)^{-1}t^\alpha) f\right\|_{L^q(\mathbb{R}^N)} \leq \tilde{A}_1 t^{\frac{N(\frac{1}{2} - \frac{1}{p})}{2}}\|f\|_{L^p(\mathbb{R}^N)} + \tilde{A}_2 t^{-\alpha\theta}\|f\|_{L^q(\mathbb{R}^N)}.
\end{align*}
Using (5.2) and the similar argument as above, we have the following estimate

\[ \left\| E_{\alpha,\alpha}(-\Delta(m\Delta - I)^{-1}t^\alpha) f \right\|_{L^q(\mathbb{R}^N)} \leq \alpha \int_0^\infty r M_\alpha(r) \left\| \mathcal{T}(rt^\alpha) f \right\|_{L^q(\mathbb{R}^N)} dr \]

\[ \leq \alpha M \int_0^\infty r M_\alpha(r) \left( 1 + (rt^\alpha) \right)^{N(\frac{1}{2} + \frac{1}{q})} \left\| f \right\|_{L^p(\mathbb{R}^N)} dr + M \alpha \int_0^\infty r M_\alpha(r) e^{-rt^\alpha} \left\| f \right\|_{L^q(\mathbb{R}^N)} dr \]

\[ = M \alpha t^{-\frac{N(1 + \frac{1}{2})}{2}} \left\| f \right\|_{L^p(\mathbb{R}^N)} \int_0^\infty M_\alpha(r) r^{1 + \frac{N(1 + \frac{1}{2})}{2}} dr \]

\[ + \alpha M \theta^\alpha e^{-\theta t^{-\alpha \theta}} \left\| f \right\|_{L^q(\mathbb{R}^N)} \int_0^\infty r^{1-\theta} M_\alpha(r) dr \]

\[ = M \alpha \Gamma \left( \frac{N(1 + \frac{1}{2})}{2} + 2 \right) t^{-\frac{N(1 + \frac{1}{2})}{2}} \left\| f \right\|_{L^p(\mathbb{R}^N)} + \frac{M \alpha \theta^\alpha e^{-\theta t^{-\alpha \theta}} \Gamma(2 - \theta)}{\Gamma(1 + \alpha - \theta)} \left\| f \right\|_{L^q(\mathbb{R}^N)} \]

\[ = \widetilde{A}_3 t^{-\frac{N(1 + \frac{1}{2})}{2}} \left\| f \right\|_{L^p(\mathbb{R}^N)} + \widetilde{A}_4 t^{-\alpha \theta} \left\| f \right\|_{L^q(\mathbb{R}^N)}, \]

where

\[ \widetilde{A}_3 = \frac{M \alpha \Gamma \left( \frac{N(1 + \frac{1}{2})}{2} + 2 \right)}{\Gamma \left( \frac{N(1 + \frac{1}{2})}{2} + 1 + \alpha \right)}, \quad \widetilde{A}_4 = \frac{M \alpha \theta^\alpha e^{-\theta t^{-\alpha \theta}} \Gamma(2 - \theta)}{\Gamma(1 + \alpha - \theta)}. \]

Consider the following function

\[ \Psi(t) := E_{\alpha,\alpha}(-\Delta(m\Delta - I)^{-1}t^\alpha) u_0 \]

\[ + \int_0^t (t - s)^{\alpha - 1} E_{\alpha,\alpha}(-\Delta(m\Delta - I)^{-1}t^\alpha) \mathcal{G}(v) ds. \]

(5.11)

First, we treat the term \( E_{\alpha,\alpha}(-\Delta(m\Delta - I)^{-1}t^\alpha) u_0 \). Since \( u_0 \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N) \), we know that

\[ \left\| E_{\alpha,\alpha}(-\Delta(m\Delta - I)^{-1}t^\alpha) u_0 \right\|_{L^q(\mathbb{R}^N)} \leq \tilde{A}_1 t^{-\frac{N(1 + \frac{1}{2})}{2}} \| u_0 \|_{L^p(\mathbb{R}^N)} + \tilde{A}_2 t^{-\alpha \theta} \| u_0 \|_{L^q(\mathbb{R}^N)}, \]

(5.12)

and

\[ \left\| E_{\alpha,\alpha}(-\Delta(m\Delta - I)^{-1}t^\alpha) u_0 \right\|_{L^p(\mathbb{R}^N)} \leq \tilde{A}_1 \| u_0 \|_{L^p(\mathbb{R}^N)} + \tilde{A}_2 t^{-\alpha \theta} \| u_0 \|_{L^q(\mathbb{R}^N)}. \]

(5.13)

From (5.12) and (5.13) yields that

\[ t^\beta e^{-\mu t} \left\| E_{\alpha,\alpha}(-\Delta(m\Delta - I)^{-1}t^\alpha) u_0 \right\|_{L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)} \]

\[ \leq e^{-\beta t} \left( \tilde{A}_1 t^\beta + \frac{\alpha N(1 + \frac{1}{2})}{2} \tilde{A}_1 t^\beta + \tilde{A}_2 t^\beta - \alpha \theta \right) \| u_0 \|_{L^p(\mathbb{R}^N)} + e^{-\beta t} \tilde{A}_1 t^\beta - \alpha \theta \| u_0 \|_{L^q(\mathbb{R}^N)} \]

(5.14)

From (1.12), it follows that \( \frac{1}{q} - \frac{1}{p} \leq \frac{2\alpha + 2\beta - 2}{\alpha N} \leq \frac{2\beta}{\alpha N} \), which implies \( \beta + \frac{\alpha N(1 + \frac{1}{2})}{2} \geq 0 \). Then by (1.11), we know

\[ \beta > 1 + \alpha \theta - \alpha > \alpha \theta. \]
Hence, from (5.14), we deduce that
\[
t^\beta e^{-\mu t} \left| E_{\alpha,1} \left( -\Delta (m\Delta - I)^{-1} t^\alpha \right) u_0 \right|_{L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)} \leq \left( A_1 T^{\beta + \frac{\alpha N(\frac{1}{2} - \frac{1}{q})}{2}} + A_2 T^{\beta - \alpha \theta} \right) \| u_0 \|_{L^p(\mathbb{R}^N)} + \| T \alpha T^{\beta - \alpha \theta} \| u_0 \|_{L^q(\mathbb{R}^N)} \tag{5.15}
\]
From (5.15), we deduce that \( E_{\alpha,1} \left( -\Delta (m\Delta - I)^{-1} t^\alpha \right) u_0 \) belongs to the space \( Z^{\beta,\mu_0} \left( (0, T]; L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N) \right) \) for any \( \mu > 0 \) and \( \beta \) satisfies (1.11). From (5.5) and \( N(0) = 0 \), it means that if \( v = 0 \) then \( \Psi v = E_{\alpha,1} \left( -\Delta (m\Delta - I)^{-1} t^\alpha \right) u_0 \in Z^{\beta,\mu_0} \left( (0, T]; L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N) \right) \). We also continue to show that there exists a \( \mu_0 \) such that
\[
\| \Psi v_1 - \Psi v_2 \|_{Z^{\beta,\mu_0} \left( (0, T]; L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N) \right)} \leq \varrho \| v_1 - v_2 \|_{Z^{\beta,\mu_0} \left( (0, T]; L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N) \right)}, \tag{5.16}
\]
for any \( v_1, v_2 \in Z^{\beta,\mu_0} \left( (0, T]; L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N) \right) \) and the constant \( \varrho \) is independent of \( t \). Indeed, we have
\[
t^\beta e^{-\mu t} \left| \Psi v_1 - \Psi v_2 \right|_{L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)} = t^\beta e^{-\mu t} \left| \int_0^t \left( t - s \right)^{\alpha - 1} E_{\alpha,\alpha} \left( -\Delta (m\Delta - I)^{-1} \left( t - s \right)^\alpha \right) \times \mathcal{G} \left( N(v_1)(s) - N(v_2)(s) \right) ds \right|_{L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)}
\]
\[
= t^\beta e^{-\mu t} \left| \int_0^t \left( t - s \right)^{\alpha - 1} E_{\alpha,\alpha} \left( -\Delta (m\Delta - I)^{-1} \left( t - s \right)^\alpha \right) \mathcal{G} \left( N(v_1)(s) - N(v_2)(s) \right) ds \right|_{L^p(\mathbb{R}^N)}
\]
\[
+ t^\beta e^{-\mu t} \left| \int_0^t \left( t - s \right)^{\alpha - 1} E_{\alpha,\alpha} \left( -\Delta (m\Delta - I)^{-1} \left( t - s \right)^\alpha \right) \mathcal{G} \left( N(v_1)(s) - N(v_2)(s) \right) ds \right|_{L^q(\mathbb{R}^N)} = (I) + (II). \tag{5.17}
\]
By the assumptions (1.14) we find that
\[
(I) \leq t^\beta e^{-\mu t} A_3 \int_0^t \left( t - s \right)^{\alpha - 1} \left| \mathcal{G} \left( N(v_1)(s) - N(v_2)(s) \right) \right|_{L^p(\mathbb{R}^N)} ds
\]
\[
+ t^\beta e^{-\mu t} A_4 \int_0^t \left( t - s \right)^{\alpha - 1 - \alpha \theta} \left| \mathcal{G} \left( N(v_1)(s) - N(v_2)(s) \right) \right|_{L^p(\mathbb{R}^N)} ds
\]
\[
\leq K_d (A_3 + A_4) t^\beta e^{-\mu t} \int_0^t \left( t - s \right)^{\alpha - 1} \left| v_1(s) - v_2(s) \right|_{L^p(\mathbb{R}^N)} ds
\]
\[
+ K_d (A_3 + A_4) t^\beta e^{-\mu t} \int_0^t \left( t - s \right)^{\alpha - 1 - \alpha \theta} \left| v_1(s) - v_2(s) \right|_{L^p(\mathbb{R}^N)} ds, \tag{5.18}
\]
(IV)
Combining some above observations, we find

\[
(I) \leq t^\beta e^{-\mu t} \tilde{A}_3 \int_0^t (t-s)^{\alpha-1}(t-s)^{\frac{\alpha N(t^\beta - 1)}{2}} \left\| \mathcal{G} (\mathcal{N}(v_1)(s) - \mathcal{N}(v_2)(s)) \right\|_{L^p(\mathbb{R}^N)} ds \\
+ t^\beta e^{-\mu t} \tilde{A}_4 \int_0^t (t-s)^{\alpha-1-\alpha \delta} \left\| \mathcal{G} (\mathcal{N}(v_1)(s) - \mathcal{N}(v_2)(s)) \right\|_{L^p(\mathbb{R}^N)} ds \\
\leq K_d(\tilde{A}_3 + \tilde{A}_4) t^\beta e^{-\mu t} \int_0^t (t-s)^{\alpha-1+\frac{\alpha N(t^\beta - 1)}{2}} \left\| v_1(s) - v_2(s) \right\|_{L^p(\mathbb{R}^N)} ds \\
+ K_d(\tilde{A}_3 + \tilde{A}_4) t^\beta e^{-\mu t} \int_0^t (t-s)^{\alpha-1-\alpha \delta} \left\| v_1(s) - v_2(s) \right\|_{L^p(\mathbb{R}^N)} ds.
\]

(5.19)

We treat the term \((III)\) as follows

\[
(III) = K_d(\tilde{A}_3 + \tilde{A}_4) t^\beta \int_0^t e^{-\mu(t-s)} s^{-\beta}(t-s)^{\alpha-1}s^{-\beta} e^{-\mu s} \left\| v_1(s) - v_2(s) \right\|_{L^p(\mathbb{R}^N)} ds \\
\leq K_d(\tilde{A}_3 + \tilde{A}_4) t^\beta \left( \int_0^t e^{-\mu(t-s)} s^{-\beta}(t-s)^{\alpha-1} ds \right) \sup_{0 \leq s \leq T} s^{-\beta} e^{-\mu s} \left\| v_1(s) - v_2(s) \right\|_{L^p(\mathbb{R}^N)} \\
= K_d(\tilde{A}_3 + \tilde{A}_4) t^\beta L_{1,\mu}(t, \alpha, \beta) \left\| v_1 - v_2 \right\|_{Z^{\beta,\mu}((0,T];L^p(\mathbb{R}^N))}. \tag{5.20}
\]

By a similar argument, we also get some following estimates

\[
(IV) \leq K_d(\tilde{A}_3 + \tilde{A}_4) t^\beta L_{2,\mu}(t, \alpha, \beta) \left\| v_1 - v_2 \right\|_{Z^{\beta,\mu}((0,T];L^p(\mathbb{R}^N))}; \\
(V) \leq K_d(\tilde{A}_3 + \tilde{A}_4) t^\beta L_{3,\mu}(t, \alpha, \beta) \left\| v_1 - v_2 \right\|_{Z^{\beta,\mu}((0,T];L^p(\mathbb{R}^N))}; \\
(VI) \leq K_d(\tilde{A}_3 + \tilde{A}_4) t^\beta L_{2,\mu}(t, \alpha, \beta) \left\| v_1 - v_2 \right\|_{Z^{\beta,\mu}((0,T];L^p(\mathbb{R}^N))}. \tag{5.21}
\]

where

\[
L_{2,\mu}(t, \alpha, \beta) = \int_0^t e^{-\mu(t-s)} s^{-\beta}(t-s)^{\alpha-1-\alpha \delta} ds, \\
L_{3,\mu}(t, \alpha, \beta) = \int_0^t e^{-\mu(t-s)} s^{-\beta}(t-s)^{\alpha-1+\frac{\alpha N(t^\beta - 1)}{2}} ds.
\]

Combining some above observations, we find

\[
\left\| \Psi v_1 - \Psi v_2 \right\|_{Z^{\beta,\mu}((0,T];L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N))} \\
\leq \sup_{0 \leq t \leq T} t^\beta e^{-\mu t} \left\| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left( -\Delta (m \Delta - I)^{-1} (t-s) \right) \right\|_{L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)} \\
\times \left\| \mathcal{G} (\mathcal{N}(v_1)(s) - \mathcal{N}(v_2)(s)) \right\|_{L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)} \\
\leq 2K_d(\tilde{A}_3 + \tilde{A}_4) \sup_{0 \leq t \leq T} \left( t^\beta L_{1,\mu}(t, \alpha, \beta) + t^\beta L_{2,\mu}(t, \alpha, \beta) + t^\beta L_{3,\mu}(t, \alpha, \beta) \right) \\
\times \left\| v_1 - v_2 \right\|_{Z^{\beta,\mu}((0,T];L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N))} \\
\leq \theta \left\| v_1 - v_2 \right\|_{Z^{\beta,\mu}((0,T];L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N))}. \tag{5.22}
\]
We first consider the term $L_{2,\mu}(t, \alpha, \beta)$. Indeed, by changing variable $s = ts'$, we get
\[
L_{2,\mu}(t, \alpha, \beta) = t^{\beta + \alpha - 1 - \alpha \theta} \int_0^1 e^{-\mu t(1-s)} s^{-\beta} (1-s)^{\alpha - 1 - \alpha \theta} \, ds.
\] (5.23)

By (1.11), we can easily verify that the following conditions hold
\[
\beta + \alpha - 1 - \alpha \theta > 0, \quad -\beta > -1, \quad \alpha - 1 - \alpha \theta > -1, \quad \alpha - 1 - \alpha \theta - \beta > -1,
\]
which combining Lemma 5.3 gives
\[
\lim_{\mu \to \infty} t^{\beta} L_{2,\mu}(t, \alpha, \beta) = \lim_{\mu \to \infty} \left( \sup_{t \in [0,T]} t^{\beta + \alpha - 1 - \alpha \theta} \int_0^1 e^{-\mu t(1-s)} s^{-\beta} (1-s)^{\alpha - 1 - \alpha \theta} \, ds \right) = 0.
\] (5.24)

Noting that $L_{1,\mu}(t, \alpha, \beta) \leq T^{\alpha \theta} L_{2,\mu}(t, \alpha, \beta)$, we deduce that
\[
\lim_{\mu \to \infty} t^{\beta} L_{2,\mu}(t, \alpha, \beta) = 0.
\] (5.25)

By (1.11)-(1.13), we can easily verify that the following conditions hold
\[
\beta + \alpha - 1 + \frac{\alpha N(\frac{1}{q} - \frac{1}{p})}{2} > 0, \quad -\beta > -1, \quad \alpha - 1 + \frac{\alpha N(\frac{1}{q} - \frac{1}{p})}{2} > -1,
\]
\[
\alpha - 1 + \frac{\alpha N(\frac{1}{q} - \frac{1}{p})}{2} - \beta > -1,
\]
which combining Lemma 5.3 gives
\[
\lim_{\mu \to \infty} t^{\beta} L_{3,\mu}(t, \alpha, \beta) = \lim_{\mu \to \infty} \left( \sup_{t \in [0,T]} t^{\beta + \alpha - 1 + \frac{\alpha N(\frac{1}{q} - \frac{1}{p})}{2}} \int_0^1 e^{-\mu t(1-s)} s^{-\beta} (1-s)^{\alpha - 1 + \frac{\alpha N(\frac{1}{q} - \frac{1}{p})}{2}} \, ds \right) = 0.
\] (5.26)

From (5.24), (5.25) and (5.26), we know that there exists a $\mu_0$ enough small such that
\[
\vartheta := 2K_d(\tilde{A}_3 + \bar{A}_4) \sup_{0 \leq t \leq T} \left( t^{\beta} L_{1,\mu_0}(t, \alpha, \beta) + t^{\beta} L_{2,\mu_0}(t, \alpha, \beta) + t^{\beta} L_{3,\mu_0}(t, \alpha, \beta) \right) < 1.
\] (5.27)

Combining (5.22) and (5.27), we find that $\Psi$ is a contraction in the space $Z^2,\mu_0((0,T]; L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N))$. It means that there exists a unique solution $u$ satisfies (5.5). Hence, the proof is completed.

**Acknowledgement.** The authors wish to express their gratitude to the anonymous referees and the editor for their valuable comments.

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Received July 2020; revised September 2020.

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