Classification of multipartite systems featuring only $|W\rangle$ and $|GHZ\rangle$ genuine entangled states

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Abstract

We present several multipartite quantum systems featuring the same type of genuine (tripartite) entanglement. Based on a geometric interpretation of the so-called $|W\rangle$ and $|GHZ\rangle$ states we show that the classification of all multipartite systems featuring those and only those two classes of genuine entanglement can be deduced from earlier work of algebraic geometers. In this paper, non-entangled pure states are identified with the highest weight orbit of the SLOCC group acting on the corresponding Hilbert space.

Keywords: entanglement classification, auxiliary algebraic varieties, projective geometry, simple Lie algebras

1. Introduction

Entanglement is a key resource of quantum information. It corresponds to a form of correlation between subsystems of a given composite system which is stronger than any correlation arising from classical communication [2]. Since the advent of quantum information theory (QIT) a large amount of experimental and theoretical evidence demonstrates that quantum protocols featuring this phenomenon exist that can overperform their classical counterparts. Beyond this, entanglement is of basic importance for obtaining new communication protocols such as quantum teleportation, quantum superdense coding or quantum cryptography. It is...
also acknowledged that quantum entanglement plays a central role in quantum algorithms and quantum computation [32, 49].

Quantum entanglement is a consequence of the superposition principle. Let us illustrate this for a bipartite system. Given two copies of two-state systems (qubits) represented by the vectors $|\psi\rangle_A \in \mathcal{H}_A$ and $|\psi\rangle_B \in \mathcal{H}_B$ with $\mathcal{H}_A \cong \mathcal{H}_B \cong \mathbb{C}^2$, we define the composite system of two qubits as the one represented by the tensor product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Then a canonical basis of $\mathcal{H}$ is $\{ |00\rangle, |01\rangle, |10\rangle, |11\rangle \}$ and the superposition principle tells us that one possible state for the composite system is

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle). \quad (1)$$

However, $|\psi\rangle_{AB}$ cannot be created from an initial state of type $|\psi\rangle_A \otimes |\chi\rangle_B$ by applying only local operations (i.e. operations acting on $|\psi\rangle_A$ and $|\chi\rangle_B$ separately). Such states are called entangled. Since entangled states cannot be generated locally they correspond to a global resource shared by the actors of the protocol. On the other hand, a state is said to be separable if it can be generated locally from a state of the form $|\varphi\rangle_A \otimes |\chi\rangle_B$.

The multipartite generalization of our example provides the basic resource for quantum information. However, as a resource, entanglement needs to be classified. One possible classification scheme is obtained by finding equivalence classes of entangled states under stochastic local operations and classical communications (SLOCC). SLOCC transformations are the mathematical representatives of certain physical manipulations allowed to be performed on our composite system. These manipulations consist of local reversible operations on each component of the multipartite system assisted by classical communication (i.e. the local operations may be coordinated). The word stochastic refers to the local operations which allow the possibility of converting a particular state of the system to another one and vice versa with some (generally different) probability of success (unlike the LOCC class of transformations which imposes the conversion to be achieved with certainty). The mathematical representative of such SLOCC transformations turns out to be a group acting on the Hilbert space of the composite system. The precise form of this group will depend on the observables characterizing this system.

The classification of entanglement classes of multipartite systems under SLOCC has been investigated in the last 15 years by many authors [1, 3, 4, 7, 13, 15–17, 26, 27, 42, 44, 46–48, 50–53, 55, 60–62, 64]. Interestingly, under SLOCC some [4, 7, 15, 17, 42, 55, 62] of these entangled systems feature exactly two genuine types of entanglement. The aim of this paper is to provide, for these systems, a unified approach based on results of algebraic geometry.

The paper is organized as follow. In section 2 we introduce a geometric interpretation [26, 27] of the entangled classes $[W]$ and $[GHZ]$ which correspond to the two genuine entangled classes in the Dür, Vidal and Cirac’s [17] classification of entanglement classes of three qubits. Thanks to this geometric interpretation we can use, in section 3, classical results of invariant theory and algebraic geometry to classify all Hilbert spaces and quantum systems which will feature those two and exactly those two types of genuine entangled classes. In this process, we recover different quantum systems investigated in the quantum information literature and three new cases. The corresponding Hilbert spaces have a similar SLOCC orbit structure (except for the case of three bosonic qubits; see remark 3.5). We also classify quantum systems with two and exactly two entanglement classes (not necessarily of type $[W]$ and $[GHZ]$). The connection of those quantum systems with classification results from algebraic geometry allows us to give a uniform description of those quantum systems and
establish a link between those systems and the simple Lie algebras. In particular, we collect some geometrical information about such systems in appendix A.

**Notations.** In the text $V$ (resp. $\mathcal{H}$) will denote a vector (resp. Hilbert) space over the field of complex numbers $\mathbb{C}$, and $\mathbb{P}(V)$ (resp. $\mathbb{P} (\mathcal{H})$) will denote the corresponding projective space. A vector $v \in V$ will be projectivized to a point $[v] \in \mathbb{P}(V)$. A projective algebraic variety $X \subset \mathbb{P}(V)$ will be defined as the zero locus of a collection of homogeneous polynomials. A point $[x] \in X$ will be said to be smooth if and only if the partial derivatives of the defining equations do not simultaneously vanish at $[x]$. If $[x] \in X$ is smooth, one can define $T_x X \subset \mathbb{P}(V)$ the embedded tangent space of $X$ at $[x]$.

In this article, we only consider pure quantum systems, i.e. a state $\rangle y \langle$ of such a system will always be considered as a (normalized) vector of $\mathbb{H}$.

### 2. The three-qubit classification revisited

Starting from the paper of Dür, Vidal and Cirac [17] three-qubit entanglement has given rise to a number of interesting applications [5, 8, 12, 31, 39]. Let us denote by $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H}_C$ the three Hilbert spaces isomorphic to $\mathbb{C}^2$ corresponding to qubits $A$, $B$ and $C$, then the Hilbert space of the composite system is $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. In this section, for simplicity, let us adopt the notation $|\psi\rangle_A \equiv \psi_A$. If we forget about scalar normalization the relevant SLOCC group turns out to be $GL_2 (\mathbb{C}) \times GL_2 (\mathbb{C}) \times GL_2 (\mathbb{C})$ and the result established in [17] states that under SLOCC action three qubits can be organized into six orbits i.e. SLOCC entanglement classes (table 1). The three-qubit classification features the interesting property of having exactly two classes of genuine entanglement, called the $|W\rangle$ and $|GHZ\rangle$ classes. It should also be clear that, for instance, for the bipartite state $|\psi\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |110\rangle)$, particles $A$ and $C$ are entangled while $B$ is not. Note that from the projective point of view multiplication by a nonzero scalar does not change the nature of the state and thus we can instead consider the $SL_2 (\mathbb{C}) \times SL_2 (\mathbb{C}) \times SL_2 (\mathbb{C})$ orbits of $\mathbb{P}(\mathcal{H})$. It turns out that this classification was known, from a mathematical perspective, since the work of Le Paige [38]. Motivated by this example we can address the basic question of our paper as: which other types of quantum systems have two and only two types of genuine non-equivalent entangled states?

Following [26] let us rephrase the classification of three-qubit entanglement classes by means of algebraic geometry. In the projectivized Hilbert space $\mathbb{P}(\mathcal{H})$ we consider the image of the following map:

| Name       | Normal form |
|------------|-------------|
| Separable  | $|000\rangle$ |
| Biseparable| $\frac{1}{\sqrt{2}} (|000\rangle + |011\rangle)$ |
| Biseparable| $\frac{1}{\sqrt{2}} (|000\rangle + |101\rangle)$ |
| Biseparable| $\frac{1}{\sqrt{2}} (|000\rangle + |110\rangle)$ |
| $|W\rangle$ | $\frac{1}{\sqrt{3}} (|100\rangle + |010\rangle + |001\rangle)$ |
| $|GHZ\rangle$ | $\frac{1}{\sqrt{2}} (|000\rangle + |111\rangle)$ |

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\[ \phi : \mathbb{P}(\mathcal{H}_A) \times \mathbb{P}(\mathcal{H}_B) \times \mathbb{P}(\mathcal{H}_C) \to \mathbb{P}(\mathcal{H}) \]
\[ (|\psi_A\rangle, |\psi_B\rangle, |\psi_C\rangle) \mapsto [\psi_A \otimes \psi_B \otimes \psi_C]. \]  

(2)

The map \( \phi \) is well-known as the Segre map \([25, \ 29]\). Let \( X = \phi(\mathbb{P}(\mathcal{H}_A) \times \mathbb{P}(\mathcal{H}_B) \times \mathbb{P}(\mathcal{H}_C)) \subseteq \mathbb{P}(\mathcal{H}) \), in what follows we will denote this image simply by \( X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) because the projectivization of the Hilbert space of each particle clearly corresponds to a projective line \( \mathbb{P}(\mathcal{H}) = \mathbb{P}(\mathbb{C}^2) = \mathbb{P}^1 \). It can be easily shown that \( X \) is a smooth projective algebraic variety \([25]\). It is also clear that the variety \( X \) is the \( G = SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \) orbit of any rank 1 tensor in \( \mathbb{P}(\mathcal{H}) \). Indeed given \( |\psi_A\rangle \otimes |\psi_B\rangle \otimes |\psi_C\rangle \), then for any \( |\tilde{\psi}_A\rangle \otimes |\tilde{\psi}_B\rangle \otimes |\tilde{\psi}_C\rangle \), there exists \( g = (g_1, g_2, g_3) \in G \) such that \( |\tilde{\psi}_A\rangle \otimes |\tilde{\psi}_B\rangle \otimes |\tilde{\psi}_C\rangle = [(g_1 \cdot |\psi_A\rangle \otimes (g_2 \cdot |\psi_B\rangle \otimes (g_3 \cdot |\psi_C\rangle)] \). In terms of quantum entanglement it follows from this description that the variety \( X \) represents the set of separable states and can be described as the projectivized orbit of \( \psi = [000] \), i.e. \( X = G \cdot [000] \subseteq \mathbb{P}(\mathcal{H}) \).

Similar to what was done in \([26, \ 27]\) our goal is now to build from the algebraic variety of separable states some auxiliary varieties which will encode different types of entanglement classes. Consider \( Y^n \subseteq \mathbb{P}(V^{n+a+1}) \) a projective algebraic variety of dimension \( n \) embedded in a projective space of dimension \( n + a \), such that \( Y \) is smooth and not contained in a hyperplane. Then, one defines the secant variety \([65]\) of \( Y \), denoted by \( \sigma(Y) \), as the algebraic closure, for the Zariski topology, of the union of secant lines of \( Y \) (equation (3))

\[ \sigma(Y) = \bigcup_{x, y \in Y} \mathbb{P}^1_{x y}. \]

(3)

Another interesting auxiliary variety is the tangential variety of \( Y \), which is defined as the union of embedded tangent spaces, \( \mathcal{T}_Y \mathcal{T}_Y \) of \( Y \) (equation (4))

\[ \tau(Y) = \bigcup_{y \in Y} \mathcal{T}_y Y. \]

(4)

One point of importance is the following: If the variety \( Y \) is \( G \)-invariant for the action of a group \( G \) (i.e. if \( y \in Y \) then \( g \cdot y \in Y \) for all \( g \in G \) then so are the varieties \( \tau(Y) \) and \( \sigma(Y) \). This property follows from the definitions of the two auxiliary varieties \( \sigma(Y) \) and \( \tau(Y) \) which are built from points of \( Y \).

Clearly \( \tau(Y) \subseteq \sigma(Y) \), as tangent lines can be seen as limits of secant lines, and the expected dimensions of those varieties are \( \min\{2n, n + a\} \) for \( \tau(Y) \) and \( \min\{2n + 1, n + a\} \) for \( \sigma(Y) \). A consequence of the Fulton–Hansen connectedness theorem \([20]\) is the following corollary which will be central to what follows.

**Proposition 2.1.** \([\text{Corollary 4 of } [20]]\) One of the following must hold

1. \( \dim(\tau(Y)) = 2n \) and \( \dim(\sigma(Y)) = 2n + 1 \), or
2. \( \tau(Y) = \sigma(Y) \).

If we go back to the case where \( X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^7 \), a standard calculation\(^5\) shows that \( \dim(\sigma(X)) = 7 \), i.e. the secant variety is of the expected dimension and fills the ambient space. Therefore, one automatically knows that \( \tau(X) \) exists and is of codimension one in \( \mathbb{P}^7 \). Moreover, given a general pair of points\(^6\) \((x, y) \in X \times X\) denoted by \( x = [\psi_A \otimes \psi_B \otimes \psi_C] \)

\(^5\) The dimension of secant variety can be calculated via the Terracini’s lemma. The case we are interested in is, for instance, explicitly done in \([34]\) example 5.3.1.5 page 123. Calculations involving Terracini’s lemma in the context of QIT can also be found in \([26, \ 27]\).

\(^6\) In \( \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \) a general pair of rank 1 tensors \((x = e_1 \otimes e_2 \otimes e_3, \ y = f_1 \otimes f_2 \otimes f_3) \in X \times X \) is such that each \((e_i, f_i)\) forms a basis of \( \mathbb{C}^2 \).
and \( y = [\tilde{\psi}_A \otimes \tilde{\psi}_B \otimes \tilde{\psi}_C] \) it is not difficult to see that there exists \( g = (g_1, g_2, g_3) \in SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \) such that \( [g \cdot (000) + (011)] = (g_1, 0) \otimes (g_2, 0) \otimes (g_3, 1)] \otimes (g_2, [1]) \otimes (g_3, 1)] = [\tilde{\psi}_A \otimes \tilde{\psi}_B \otimes \tilde{\psi}_C + \tilde{\psi}_A \otimes \tilde{\psi}_B \otimes \tilde{\psi}_C]. \)

In other words, we have for \( G = SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \)

\[
\sigma(X) = \sigma(G \cdot [000]) = G \cdot [000] + [111]).
\] (5)

Similarly, one can provide an orbit description of \( \tau(X) \): Let \( \gamma(t) = \left[ (\psi_A + t\tilde{\psi}_A) \otimes (\psi_B + t\tilde{\psi}_B) \otimes (\psi_C + t\tilde{\psi}_C) \right] \) be a general curve of \( X \) passing through \([\psi_A \otimes \psi_B \otimes \psi_C]\) such that \( \psi_A \) and \( \tilde{\psi}_A \) are not collinear. Then, a straightforward calculation shows that after differentiation we get \( \gamma'(0) = [\tilde{\psi}_A \otimes \tilde{\psi}_B \otimes \tilde{\psi}_C + \tilde{\psi}_A \otimes \tilde{\psi}_B \otimes \tilde{\psi}_C + \psi_A \otimes \psi_B \otimes \psi_C] \in T_{[\psi_A \otimes \psi_B \otimes \psi_C]}X. \) Again, under the group action \( G \) this calculation tells us that

\[
\tau(X) = \tau(G \cdot [000]) = G \cdot [[010] + [001]).
\] (6)

Equations (5) and (6) say that the SLOCC-orbits of the \(|GHZ\rangle\) and \(|W\rangle\) states form open subsets of the secant and tangential varieties of the set of separable states, respectively. Moreover, the fact that the secant variety has the expected dimension implies that those two states are non-equivalent. Considering the biseparable states, the geometric picture can be completed as follows:

In figure 1 the lines represent inclusions as algebraic varieties and \( \sigma(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \times \mathbb{P}^1 \) is a notation introduced in [26] to denote the variety of the closure of secant lines of \( X \) between points of type \([\tilde{u} \otimes \tilde{v} \otimes \tilde{w}]\) and \([\tilde{u} \otimes v \otimes \tilde{w}]. \)

Based on the previous analysis we see that an alternative way of saying:

‘Three qubits have two non-equivalent classes of genuine entanglement, one of type \(|GHZ\rangle\) and the other of type \(|W\rangle\)’

would be, in geometrical terms,

‘The secant variety of the set of separable states \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) has the expected dimension and fills the ambient space’.

In section 3 we show what this last geometric formulation tells us about other types of multipartite systems.
3. The geometry of tripartite entanglement

Let us now consider a semi-simple complex Lie group $G$ and an irreducible $G$-module $\mathcal{H}$, i.e. a vector space such that $G$ acts on $\mathcal{H}$ and does not contain any nontrivial submodule. We call $\mathcal{H}$ an irreducible representation of $G$.

In this more general setting the irreducible vector space $\mathcal{H}$ will be the Hilbert space of pure quantum states (composite or not, made of distinguishable particles or indistinguishable ones) and the group $G$ will be the complexified dynamical group of the system. In other words, following Klyachko [33], $G = \exp(\mathfrak{g})$ where $\mathfrak{g}$ is the complexified Lie algebra generated by all essential observables of the system. Taking the projective space $\mathbb{P}(\mathcal{H})$ there exists a unique smooth orbit $X = G.[v_\lambda] \subset \mathbb{P}(\mathcal{H})$ called the highest weight orbit\footnote{Highest weight vectors are defined after a choice of an ordering of the roots of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$ which defines for each irreducible representation a unique highest weight $\lambda$ [21]. There is a bijection between the highest weights (up to a choice of an ordering of the root system) and the irreducible representations of $G$.}. Then, the definition of separable states is the following.

**Definition 1.** Let $\mathbb{P}(\mathcal{H})$ be the projectivized Hilbert space of $\mathcal{H}$ with complex dynamical group $G$, and we say that $|\psi\rangle \in \mathcal{H}$ is separable if and only if $|[\psi]\rangle \in X = G.[v_\lambda] \subset \mathbb{P}(\mathcal{H})$ i.e. $|[\psi]\rangle$ is in the $G$-orbit of the highest weight vector of $\mathcal{H}$.

**Remark 3.1.** In the mathematical physics literature, the definition of what entangled/non-entangled states are can be different from one community to another. This is particularly true where indistinguishable particles are concerned. Definition 1 says that all states which are not in the highest weight orbit should be considered as entangled. This is the definition of entanglement of pure states commonly accepted in the high-energy community [3–6, 8, 10, 15, 16, 39, 42, 62] and/or the community using algebraic methods in physics [13, 33, 44]. For multipartite qudits systems this definition of entangled/non-entangled pure states coincides with the definition shared by the quantum information community [19, 28]. For fermionic systems our definition is similar to the definition of [18, 23]. Indeed, in those references non-entangled fermionic states correspond to states of Slater rank 1. But Slater rank for skew symmetric tensors corresponds to tensor rank in the sense of [34] and the highest weight orbit of the vector space of skew symmetric tensors is the orbit of rank 1 (skew-symmetric) tensors. The case of bosonic systems is more subtle. In the quantum information community it is usually considered that for two bosonic systems there exist two distinguished non-entangled classes [23]. Following [23] two bosonic states with Schmidt number 1 are non-entangled and those correspond to the highest weight orbit of the symmetric space (definition 1) but two bosonic states with Schmidt number 2 which correspond to the symmetrization of the product of orthogonal vectors are also considered as non-entangled. In this case two boson states of rank 2 may be considered as non-entangled. Thus, for two bosonic systems our definition 1 is in conflict with [23]. In [18] multipartite bosonic states are said to be non-entangled if they have Slater rank 1 which is equivalent to require that the corresponding symmetric tensors have rank 1. Again, it is in conflict with definition 1 as in the symmetric space one can find rank 1 tensor, i.e. a state of Slater rank 1, which does not belong to the highest weight orbit. However in [24] a definition of non-entangled states of multipartite systems of indistinguishable particles is given in terms of minimal possible $S$-rank which is equivalent to the notion of highest weight orbit for symmetric and skew symmetric tensors i.e. to our definition 1. Thus [24] agrees with definition 1 and contradicts [23]. Those differences between the various definitions of...
entangled/non-entangled pure states do not affect the orbit structure on which this paper is based, but it invites us to take specific care when we talk about bosonic systems.

Remark 3.2. Let $P \subseteq G$ denote the stabilizer of the highest weight $\nu \in \mathcal{H}$, then $X = G/P$ and $X = G/P \subseteq \mathbb{P}(\mathcal{H})$ realize the minimal embedding of the homogeneous variety $G/P$. The subgroup $P \subseteq G$ is called a parabolic subgroup of $G$ [21].

In the case of three qubits we have $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, $G = SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ and $\nu = |0\rangle\langle 0|$, i.e. $X = G \cdot |0\rangle\langle 0|$ = $\mathbb{P}(\{|\psi\rangle\text{ separable}\})$. Let us now introduce others standard examples which fit into this framework.

Example 3.1. Bosons. Consider a multipartite system made of $n$-bosons, i.e. a system of symmetric indistinguishable $k$ single-particle states. The Hilbert space to consider is $\text{Sym}^n(\mathbb{C}^k)$ i.e. the subspace of symmetric tensors of $(\mathbb{C}^k)^\otimes n$. The group of invertible transformations which acts on $\text{Sym}^n(\mathbb{C}^k)$ is $GL_k(\mathbb{C})$. Ignoring multiplicative phase, we can consider the projective space $\mathbb{P}(\mathcal{H}_n)$ with the group $SL_k(\mathbb{C})$. The vector space $\mathcal{H}_n$ is an irreducible representation of $SL_k(\mathbb{C})$. For example let $n = 3$ and $k = 2$, i.e. $\mathcal{H}_3$ is the space of three bosonic qubits and $G = SL_2(\mathbb{C})$. Let us denote by $(|0\rangle, |1\rangle)$ a basis of $\mathbb{C}^2$. Then the highest weight vector can be chosen to be any factorized symmetric three tensor. For instance let $|\psi\rangle = |000\rangle$ be the highest weight vector. The variety of separable states $X$ is the $G$-orbit of $|\psi\rangle$ in $\mathbb{P}(\mathcal{H}_3)$

$$X = G \cdot |\psi\rangle = \mathbb{P}(\{|\psi\rangle\text{ separable}, \nu \in \mathbb{C}^2\}) \subset \mathbb{P}(\mathcal{H}_3).$$

This algebraic variety is well known from geometers as the Veronese embedding of $\mathbb{P}^3$ in $\mathbb{P}^7$ or also as the rational normal curve (see [25]). This embedding can be described in full generality by the following rational map

$$v_k : \mathbb{P}^k \rightarrow \mathbb{P}(\text{Sym}^n(\mathbb{C}^k))$$

$$[v] \mapsto [v_1 v_2 \cdots v_n] \quad n \text{ times}$$

where $\circ$ denotes the symmetric product, i.e. $u_1 \circ \cdots \circ u_m = \frac{1}{\sqrt{m!}} \sum_{\sigma \in S_m} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(m)}$. In the case of three bosonic qubits we get,

$$X = v_3(\mathbb{P}^3).$$

All states $|\psi\rangle \in \mathbb{P}(\mathcal{H}_3) \setminus v_3(\mathbb{P}^3)$ are entangled. The orbit structure of $\mathbb{P}(\mathcal{H}_3)$ under the action of $G$ can be found in many mathematics text books [25, 34]. Its interpretation in the context of QIT as entanglement structure of three bosonic qubits can be found in [7, 62]. The orbit structure in this case is summarized in table 2 with both quantum information and algebraic geometry interpretations.

### Table 2. Three-bosonic-qubit classification.

| Name     | Normal form | Algebraic variety (orbit closure) |
|----------|-------------|----------------------------------|
| Separable | $|000\rangle$ | $v_3(\mathbb{P}^3)$              |
| $|W\rangle$ | $|0\rangle \circ |0\rangle \circ |1\rangle = \frac{1}{\sqrt{3!}} (|001\rangle + |010\rangle + |100\rangle)$ | $\tau(v_3(\mathbb{P}^3))$ |
| $|GHZ\rangle$ | $\frac{1}{\sqrt{2}} (|000\rangle + |111\rangle)$ | $\sigma(v_3(\mathbb{P}^3)) = \mathbb{P}^3$ |

This algebraic variety is well known from geometers as the Veronese embedding of $\mathbb{P}^3$ in $\mathbb{P}^7$ or also as the rational normal curve (see [25]).
Table 3. Three fermions with six single-particle state classification.

| Name        | Normal form                                           | Algebraic variety (orbit closure) |
|-------------|-------------------------------------------------------|-----------------------------------|
| Separable   | $e_1 \wedge e_2 \wedge e_3$                         | $G(3, 6)$                         |
| Biseparable | $\frac{1}{\sqrt{2}}(e_1 \wedge (e_2 \wedge e_3 + e_4 \wedge e_5))$ | $\sigma_e(G(3, 6))$              |
| $|W\rangle$ | $\frac{1}{\sqrt{3}}(e_1 \wedge e_2 \wedge e_4 + e_1 \wedge e_3 \wedge e_5 + e_2 \wedge e_3 \wedge e_6)$ | $\tau(G(3, 6))$                  |
| $|\text{GHZ}\rangle$ | $\frac{1}{\sqrt{2}}(e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6)$ | $\sigma(G(3, 6)) = \mathbb{P}^{19}$ |

Here, if we follow the definition of bosonic entanglement of [19] the $|W\rangle$ state should be considered as non-entangled (see remark 3.1).

Example 3.2. Fermions. We consider $G = SL_6(\mathbb{C})$ and $H_2 = \Lambda^3 \mathbb{C}^6$. The vector space $H_2$ is an irreducible representation of $G$ (more generally $\Lambda^V$ are irreducible representations of $SL(V)$ called the fundamental representations, see [21] page 221). Given $(e_1, e_2, ..., e_6)$ a basis of $\mathbb{C}^6$, a highest weight vector can be chosen to be $|\psi_2\rangle = e_1 \wedge e_2 \wedge e_3$. Then, in this case $X = SL_6(\mathbb{C}) \cdot \{e_1 \wedge e_2 \wedge e_3\} \subset \mathbb{P}(\Lambda^3 \mathbb{C}^6) = \mathbb{P}^{19}$. The variety $X$ represents the set of 3-dimensional planes in $\mathbb{C}^6$ also known as the Grassmannian $G(3, 6)$. Given a three plane of $\mathbb{C}^6$ spanned by $u, v$ and $w$ we can always find $g \in SL_6(\mathbb{C})$ such that $[g \cdot (e_1 \wedge e_2 \wedge e_3)] = [g \cdot e_1 \wedge g \cdot e_2 \wedge g \cdot e_3] = [u \wedge v \wedge w]$. In terms of skew-symmetric tensors, the variety $X$ is the set of rank 1 tensors in $\mathcal{H}$. Now let us recall that, in quantum information theory, $H_2 = \Lambda^3 \mathbb{C}^6$ is the Hilbert space describing systems made of three fermions with six single-particle states. The tensor rank for skew symmetric tensors is what is called the Slater rank in the quantum information literature (see [23] for two-partite systems and [42] for tripartite ones). The projectivization of the highest weight orbit for the representation $H_2 = \Lambda^3 \mathbb{C}^6$ and group action $G = SL_6(\mathbb{C})$ is the set of pure tripartite fermions with six single-particles states and Slater rank 1. Then, from the above description, it follows that for three fermions with six single-particle states, $X = G(3, 6)$ is the set of separable states and $G = SL_6(\mathbb{C})$ is the corresponding SLOCC group. The orbit structure under the action of $G$ is known from algebraic geometry perspective since the work of Donagi [14] and was understood in terms of QIT in [42]. We recall in both languages this stratification of the ambient space by $G$-orbits in table 3.

Remark 3.3. The notation $\sigma_e(X)$ in table 3 will be explained in remark 3.5.

Example 3.3. A qubit and two fermions. Let us describe the situation corresponding to a system of three particles composed of a qubit and two fermions with four single-particle states. The Hilbert space corresponding to this composite situation is $H_3 = \mathbb{C}^2 \otimes \Lambda^2 \mathbb{C}^4$. The SLOCC group acting on this vector space is $G = SL_2(\mathbb{C}) \times SL_4(\mathbb{C})$ and $H_3$ is an irreducible representation of $G$. Let us denote by $(e_1, e_2)$ a basis of $\mathbb{C}^2$ and $(f_1, ..., f_4)$ a basis of $\mathbb{C}^4$, then a highest weight vector of $H_3$ can be chosen to be $|\psi_3\rangle = e_1 \otimes (f_1 \wedge f_2)$. In the projective space $\mathbb{P}(H_3)$ the smooth variety $X = G \cdot [|\psi_3\rangle]$ corresponds to the set of separable states. Indeed any separable states $|\psi\rangle$ of $H_3$ can be written in the form $u \otimes (v \wedge w)$ with $u \in \mathbb{C}^2$, $v, w \in \mathbb{C}^4$ with $v$ and $w$ linearly independent. Therefore $[|\psi\rangle]$ is a point of $X$ as we can always find $g = (g_1, g_2) \in G$ such that $[g \cdot |\psi\rangle] = [|\psi\rangle]$. The variety $X$ is the Segre embedding of the product of $\mathbb{P}^1$ and $G(2, 4)$, i.e. $X = \mathbb{P}^1 \times G(2, 4)$. It follows that entangled states correspond to points of $\mathbb{P}(H_3) \setminus \mathbb{P}^1 \times G(2, 4)$. The orbit structure of $\mathbb{P}(H_3)$ under the action of $G$ is
Table 4. One qubit and two fermions with four single-particle state classification.

| Name       | Normal form                                                                 | Algebraic variety (orbit closure) |
|------------|-----------------------------------------------------------------------------|-----------------------------------|
| Separable  | $e_1 \otimes (f_1 \wedge f_2)$                                              | $\mathbb{P}^1 \times G(2, 4)$     |
| Biseparable| $\frac{1}{\sqrt{2}}(e_1 \otimes (f_1 \wedge f_2) + f_1 \wedge f_2)$        | $\mathbb{P}^1 \times \mathbb{P}^5$|
| $|W\rangle$ | $\frac{1}{\sqrt{3}}(e_1 \otimes (f_1 \wedge f_2) + e_1 \otimes (f_4 \wedge f_2) + e_2 \otimes (f_1 \wedge f_2))$ | $\sigma(\mathbb{P}^1 \times G(2, 4))$ |
| $|GHZ\rangle$ | $\frac{1}{\sqrt{2}}(e_1 \otimes (f_1 \wedge f_2) + e_2 \otimes (f_1 \wedge f_2))$ | $\sigma(\mathbb{P}^1 \times G(2, 4)) = \mathbb{P}^{11}$ |

well known [62] and we recall it in table 4 with both QIT and algebraic geometry interpretations.

Regarding the existence of two classes of genuine entangled states, the similarity of the classifications provided by tables 2–4 and the three qubits case (table 1) can be rephrased geometrically: they all correspond to tripartite systems such that the secant variety of the set of separable states is of the expected dimension and fills the ambient space.

In order to determine all other systems featuring the same classification (i.e. with only $|W\rangle$ and $|GHZ\rangle$ like states as genuine entangled ones) let us ask a geometrically equivalent question: what are the semi-simple Lie groups $G$ and the corresponding irreducible representations $\mathcal{H}$ such that

$$
\tau(X) \subseteq \sigma(X) = \mathbb{P}(\mathcal{H})
$$

where $X$ is the projectivization of the highest weight vector.

First, it should be noted that $\sigma(X) = \sigma(G/P) = \mathbb{P}(\mathcal{H})$ and $\tau(X)$ is a hypersurface of $\mathbb{P}(\mathcal{H})$ which imply that the ring of $G$-invariant polynomials on $\mathcal{H}$ is generated by the $G$-invariant irreducible polynomial vanishing on $\tau(X)$, i.e. $\mathbb{C}[\mathcal{H}]^G = \mathbb{C}[F]$ where $F$ is the irreducible (up to scale) homogeneous polynomial defining $\tau(X)$. Indeed, the fact that $\sigma(G/P) = \mathbb{P}(\mathcal{H})$ says there is a dense orbit (because the secant variety is always the closure of the orbit $G.[u + v]$ where $(u, v)$ is a general pair of points of $X$, see [65]). Therefore, there are either no invariants or the ring of invariants is generated by a single polynomial. The fact that $\tau(X)$ is a $G$-invariant hypersurface tells us that we are in the second case. The representations such that $\mathbb{C}[\mathcal{H}]^G = \mathbb{C}[F]$ have been classified by Kac, Popov and Vinberg [30]. It can be deduced from this classification which representations satisfy $\sigma(G.[v]) = \mathbb{P}(\mathcal{H}) = \mathbb{P}^{|2n+1|}$ where the highest weight orbit $G.[v]$ has dimension $n$. This is in fact done explicitly in the book of Zak [65], pages 51 and 53, where the author studies in detail homogeneous varieties of small codimension in order to understand a special class of them called the Severi varieties. We summarize the result of Zak’s studies in table 5 and we put in perspective the corresponding systems in quantum information theory as well as the references where those cases have been separately investigated.

The notations of table 5 are as follow:

- $\text{Sym}^n V$ and $\Lambda^n V$ denote respectively the symmetric and skew-symmetric parts of $V^\otimes n$.
- $\nu_k : \mathbb{P}(V) \rightarrow \mathbb{P}(\text{Sym}^k V)$ is the Veronese map defined by $\nu_k([v]) = [v \circ v \circ \ldots \circ v]$ and $\nu_2(\mathbb{P}^1)$ and $\nu_3(\mathbb{P}^3)$ are curves corresponding to the images of $\mathbb{P}^1$ by $\nu_2$ and $\nu_3$ also known as the conic and the twisted cubic [25] see example 3.1.
| $G$                      | $\mathcal{H}$ | Highest weight orbit | QIT interpretation                                      | References                        |
|-------------------------|---------------|----------------------|--------------------------------------------------------|-----------------------------------|
| $SL_2(\mathbb{C})$      | $Sym^3(\mathbb{C}^2)$ | $V_3(\mathbb{P}^1) \subset \mathbb{P}^3$ | 3 bosonic qubits                                        | Brody, Gustavsson, Hughston [7]   |
| $SL_2(\mathbb{C}) \times SO(m)$ | $\mathbb{C}^2 \otimes Sym^2(\mathbb{C}^2)$ | $\mathbb{P}^1 \times V_2(\mathbb{P}^1) \subset \mathbb{P}^5$ | 1 distinguished qubit and 2 bosonic qubits | Vrana and Lévay [62]             |
| $m = 4$                 | $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ | $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ | 3 bosonic qubits                      | Dür, Vidal, Cirac [17]           |
| $m = 5$                 | $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^4$ | $\mathbb{P}^1 \times LG(2, 4) \subset \mathbb{P}^{9}$ | 1 distinguished qubit and 2 fermions with 4 single-particle states and a symplectic form condition | New                               |
| $m > 6$                 | $\mathbb{C}^2 \otimes \mathbb{C}^m$ | $\mathbb{P}^1 \times Q^{m-2} \subset \mathbb{P}^{2m-1}$ | 1 qubit and 1 isotropic $(m-1)$-dits, with 6 single-particle states and a symplectic form condition | Vrana and Lévay [62]             |
| $Sp_6(\mathbb{C})$      | $\mathbb{C}^{12}$ | $LG(3, 6) \subset \mathbb{P}^{13}$ | 3 fermions                                              | New                               |
| $SL_6(\mathbb{C})$      | $\mathbb{C}^{21}$ | $G(3, 6) \subset \mathbb{P}^{19}$ | 3 fermions with 6 single-particle states                | Levay and Vrana [42]             |
| $Spin_{12}$             | $\Delta_{12}$ | $S_6 \subset \mathbb{P}^{31}$ | Particles in fermionic Fock spaces                      | Sàrosi and Lévay [55]            |
| $E_7$                   | $V_{56}$ | $E_7/P_1 \subset \mathbb{P}^{55}$ | Tripartite entanglement of 7 qubits                     | Duff and Ferrara [15]            |
is the standard representation of the Lie group $E_{7}$.

- The variety $LG(k, n) \subset \mathbb{P}(\Lambda^{k}\mathbb{C}^{n})$ is the so-called Lagrangian Grassmannian. Given a non-degenerate symplectic form $\omega$ on $\mathbb{C}^{n}$, $LG(k, n)$ is the variety of isotropic $k$-planes in $\mathbb{C}^{n}$ with respect to $\omega$. The vector space $\Lambda^{k}\mathbb{C}^{n}$ is the space of skew-symmetric tensors that are isotropic for $\omega$. Those vector spaces correspond to the standard representations of the symplectic group $Sp_{n}$.

- As already mentioned in example 3.2, the variety $G(k, n) \subset \mathbb{P}(\Lambda^{k}\mathbb{C}^{n})$ is the Grassmannian variety of $k$-planes in $\mathbb{C}^{n}$.

- The vector space $\Delta_{12}$ is the standard representation of the group $Spin_{12}$, i.e. the double covering of $SO(12)$, see [21]. The variety $\mathcal{S}_{6} \subset \mathbb{P}(\Delta_{12})$ is the corresponding highest weight orbit, called the spinor variety. It is the variety of pure spinors [9]. See [40, 55] for an introduction of spinor representations in the context of entanglement classification.

- The vector space $V_{56}$ is the standard representation of the Lie group $E_{7}$ and $E_{7}/P_{1}$ denotes the corresponding highest weight orbit (in terms of parabolic groups $P_{1}$ corresponds to the parabolic group defined by the root $\alpha_{1}$).

Table 5 provides a classification of quantum systems featuring two and only two classes of genuine entanglement of types $|W\rangle$ and $|GHZ\rangle$. Although most of these systems have been studied independently by various authors of the quantum information theory community, it is interesting to point out here that thanks to the work of F. Zak now a purely geometric approach allows us to present them in a unique classification scheme. As will be discussed in appendix A this classification also corresponds to the classification of Freudenthal varieties. The role of Freudenthal construction in the study of these quantum systems, in particular the role of Freudenthal triple systems (FTS) has been already understood and used by different authors [4, 6, 62]. The Hilbert spaces and quantum systems of our table 5 obtained by geometric arguments are the same Hilbert spaces and SLOCC groups of table II of [4] built from FTS. However, the FTS construction does not show that this table provides a complete classification of quantum systems featuring this peculiar entanglement behavior.

Let us also point out that three new types of quantum systems with entanglement classes similar to the three-qubit systems appear in this classification. Their set of separable states corresponds to the following three algebraic varieties:

- $X = \mathbb{P}^{1} \times LG(2, 4) \subset \mathbb{P}^{9}$,
- $X = LG(3, 6) \subset \mathbb{P}^{13}$
- $X = \mathbb{P}^{1} \times O^{m-2} \subset \mathbb{P}^{2} \times \mathbb{P}^{m-1}, m > 6$.

As mentioned in table 5, the first system is made of a distinguished qubit and two fermions with four single-particle states satisfying a symplectic condition and the second system corresponds to three fermions with six single-particle states satisfying a symplectic condition.

The last new case corresponds to a system made of a qubit and a $m - 1$-dits ($m > 6$) which satisfies an isotropic condition given by a quadratic form.

**Remark 3.4.** Even if symplectic geometry has been previously used to describe the dynamics of quantum entanglement [57], to the best of our knowledge, the entanglement of systems of fermions satisfying a symplectic condition has not been investigated so far in the context of QIT. However, these ‘symplectic entangled systems’ can easily be implemented in the formalism of the coupled-cluster method [11, 58] (CCM) used by quantum chemists, a method which has recently been related to quantum entanglement [43]. The argument goes as follows. Usually, the expansion of the state vector $\Psi$ of an $N$ fermionic system is given as a
linear combination of Slater determinants (the so-called CI expansion). In a dual formalism
similar to ours the Slater determinants simply correspond to separable states like $e_1 \otimes e_2 \otimes e_3$
in the $N = 3$ case. In the CCM this CI expansion is replaced by a new one. The CCM
expansion is constructed via the action of a set of commuting cluster operators $e^{\hat{h}_i}$, $I = 1, 2, ...$
for a special Slater determinant $\Psi_0$ singled out by physical considerations. The single-particle states (modes) comprising $\Psi_0$ are called the occupied ones. For example, in
the case of three fermions with six modes one can take $\Psi_0 = e_1 \otimes e_2 \otimes e_3$, hence in this
case the modes $1, 2, 3$ are occupied and the ones $4, 5, 6$ are not. Let us denote by $
\hat{n}_a, i = 1, 2, 3$ interior multiplication by the basis vector $e_i$ and $\hat{p}^a, a = 4, 5, 6$ (or
$a = I, 2, 3$) exterior multiplication by the one $e^a$. Then, in this case $\hat{T}_I \equiv T_I^a \hat{p}^a \hat{n}_a$ i.e. it is an
operator annihilating the occupied mode ‘$i$’ and creating an unoccupied one ‘$a$’, and $T_I^a$ is a
$3 \times 3$ matrix. Similarly, the cluster operators $\hat{T}_I$ describe single, double, and triple transitions from
the occupied modes to the non-occupied ones [43]. Then, in the CCM picture the state is expanded as $\Psi \equiv e^{\hat{h}_1 + \hat{h}_2 + \hat{h}_3} \Psi_0$ and it is regarded as a deviation from the distinguished one $\Psi_0$. This deviation is effectuated by multiple transitions from occupied states to non-occupied
ones. Now it can be shown [43] that the transformation effectuated by the operator $e^{\hat{h}_i}$ is a
determinant one SLOCC transformation. Moreover, writing our state in the form $\Psi = e^{\hat{h}_I} \Psi'$
where $\Psi' = e^{\hat{h}_1 + \hat{h}_2 + \hat{h}_3} \Psi_0$ reveals that the cluster operators $\hat{T}_I$, $I \neq 1$ encode information on the
SLOCC classes. Now it is easy to check that if the $3 \times 3$ matrix $T_{I'}$ is a symmetric one, then
the state $\Psi$ is on the $Sp_6$ orbit of the state $\Psi'$. Hence, restricting our attention from ordinary
SLOCC transformations to ‘symplectic SLOCC’ ones from the physical point of view amounts to ensuring that amplitudes describing transitions like $1 \leftrightarrow 2$ and $2 \leftrightarrow 3$ are the
same. We remark that the matrix $T_{I'}$ taking care of the transition from the representative $\Psi$ to $\Psi'$
belonging to the same entanglement class is one frequently used by quantum chemists.
Within this context (provided the range of indices $i$ and $a$ are the same) it would be interesting to
explore the physical meaning of our symplectic constraint.

We also remark that symplectic entanglement can be found in the context of the black
hole/qubit correspondence (BHQC). For instance, Moore’s construction [41, 45] of the
attractor mechanism in black hole physics makes use of the vector space $\mathbb{C}^3 \otimes \mathbb{C}^6$ of isotropic
3-forms endowed with the action of the symplectic group $Sp_6$. Similarly, the Hilbert space
$\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^m$ appears in Calabi Yau compactifications in string theory where the first
factor takes care of the $S$-duality and the second of the so-called $T$-duality.

**Remark 3.5.** The orbit structure of the projectivized Hilbert spaces $\mathbb{P}(\mathcal{H})$ with the SLOCC

group $G$ of table 5 is fully provided by [35]. In particular, the authors show that, except for
$G = SL_2(\mathbb{C})$ and $\mathcal{H} = Sym^3(\mathbb{C}^2)$ (three bosonic qubits), there are exactly four orbits. The
Zariski closures of those orbits can be described as follows:

$$\begin{align*}
X &\subset \sigma_0(X) \subset \tau(X) \subset \sigma(X) = \mathbb{P}(\mathcal{H})
\end{align*}$$

(11)

The variety $\sigma(X)$ is the closure of points of type $|\psi\rangle + |\chi\rangle$ where $|\psi\rangle$ and $|\chi\rangle$ are two
separable states which do not form a generic pair (see [35] for the description of the isotropic
condition satisfied by this pair $(|\psi\rangle, |\chi\rangle)$). The smooth points of $\sigma(X)$ are therefore identified
with biseparable states. This variety is irreducible except in the case of three qubits where
$\sigma_\tau(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ splits into three irreducible components (see figure 1).

For three bosonic qubits, $X = v_3(\mathbb{P}^3)$, the orbit structure is slightly different. It is only
made of three orbits as there is no variety such as $\sigma_\tau(X)$ see table 2.
We conclude this section with a variation of our initial problem. Instead of classifying systems with two and only two classes of genuine entanglement of type $|W\rangle$ and $|GHZ\rangle$, let us consider systems having two and only two types of genuine entanglement (but not necessarily featuring $|W\rangle$ and $|GHZ\rangle$).

**Example 3.4.** Let $\mathcal{H} = \mathbb{C}^3 \otimes \mathbb{C}^3$, $G = SL_3(\mathbb{C}) \times SL_3(\mathbb{C})$, and $X = G.\{|\psi\rangle\} = \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8 = \mathbb{P}(\mathcal{H})$, with $|\psi\rangle = |00\rangle$ i.e. $X$ is the set of separable states of two qutrits. The variety $X$ can also be identified with the projectivization of the rank $1$ $3 \times 3$ matrices and $\mathbb{P}(\mathcal{H})$ is the projectivization of the space of $3 \times 3$ matrices. Under the action of the SLOCC group $G$, it is well known that we have only three orbits:

$$X = \mathbb{P}\{\text{Matrices of rank } 1\} \subset \mathbb{P}\{\text{Matrices of rank } \leq 2\} \subset \mathbb{P}\{\text{Matrices of rank } \leq 3\} = \mathbb{P}^8$$

(12)

The variety of rank less than two matrices is the secant variety of $X$ and general points of this variety correspond to the orbit of the state $|\psi\rangle_2 = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The complement of $\sigma(\mathbb{P}^2 \times \mathbb{P}^2)$, i.e. $\mathbb{P}^8 \setminus \sigma(\mathbb{P}^2 \times \mathbb{P}^2)$ is the projectivization of the orbit of $|\psi\rangle_3 = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$. But in this example there is no tangential variety because of the dimension condition. Indeed dim$(\sigma(X)) = 7 < 2 \times 4 + 1$ and by proposition 2.1 we have $\sigma(X) = \tau(X)$. Thus, this is an example of a multipartite system with two and only two types of non-equivalent entangled states but there is no entangled class of type $|W\rangle$.

**Example 3.5.** Let $\mathcal{H} = \text{Sym}^2\mathbb{C}^3$, $G = SL_3(\mathbb{C})$, and $X = G.\{|[00]\rangle\} = v_2(\mathbb{P}^2) \subset \mathbb{P}^5 = \mathbb{P}(\mathcal{H})$. In this example, the Hilbert space $\mathcal{H}$ is the Hilbert space of two bosonic qutrits, i.e. two bosons with three single-particle states. This space can clearly be identified with the space of symmetric $3 \times 3$ matrices and the separable states correspond to rank $1$ symmetric matrices, i.e. states of type $|\psi\rangle_1 = |uu\rangle_b$ with $u \in \mathbb{C}^3$ (the subscript $b$ means we consider symmetric product i.e. $|uv\rangle_b = |u\rangle \circ |v\rangle = \frac{1}{\sqrt{2}}(|uv\rangle + |vu\rangle)$). In this Hilbert space, according to definition 1, entangled states are of two types, either SLOCC equivalent to $|\psi\rangle_2 = |01\rangle_b = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ or SLOCC equivalent to $|\psi\rangle_3 = \frac{1}{\sqrt{3}}(|01\rangle_b + |22\rangle_b)$. The secant variety $\sigma(X)$ corresponds to the projectivization of the SLOCC orbit closure of the state $|\psi\rangle_2$ and is of dimension $4 < 2 \times 2 + 1$ (the equation defining the secant variety is given by the vanishing of the determinant which allows us to distinguish the states SLOCC equivalent to $|\psi\rangle_2$ from the ones SLOCC equivalent to $|\psi\rangle_2$). Therefore, there is no tangential variety (more precisely $\tau(X) = \sigma(X)$) and no states of type $|W\rangle$.

If, instead of definition 1 we consider the usual definition of entanglement for bosons as given in [19, 23], then the state $|\psi\rangle_2$ should be considered as non-entangled.

It is clear from the previous examples that quantum systems with only two types of genuine entangled classes which are not considered in table 5 should correspond to systems whose set of separable states $X \subset \mathbb{P}(\mathcal{H})$ satisfies the following geometric conditions:

$$\text{dim}(\sigma(X)) < 2\text{dim}(X) + 1$$

and there is a SLOCC orbit corresponding to $\mathbb{P}(\mathcal{H}) \setminus \sigma(X)$

(13)

In turns out that the classification of homogeneous varieties $X = G/P$ under the conditions of equation (13) can also be deduced from Zak’s work (See [65] pages 54 and 59). We summarize this result in table 6.

The notations for table 6 are as follows:
| G               | \(\mathcal{H}\)       | Highest weight orbit | QIT interpretation                          |
|-----------------|------------------------|----------------------|---------------------------------------------|
| \(SL_2(\mathbb{C})\) | \(\text{Sym}^2(\mathbb{C}^1)\) | \(v_2(\mathbb{P}^2) \subset \mathbb{P}^3\) | 2 bosons with 3 single-particle states       |
| \(SL_3(\mathbb{C}) \times SL_1(\mathbb{C})\) | \(\mathbb{C}^1 \otimes \mathbb{C}^1\) | \(\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8\) | 2 qutrits                                    |
| \(SL_3(\mathbb{C})\) | \(\mathbb{C}^3 \otimes \mathbb{C}^1\) | \(G(2, 5) \subset \mathbb{P}^{14}\) | 2 fermions with 5 single-particle states     |
| \(E_6\)       | \(V_{27}\)            | \(E_6/P_1 \subset \mathbb{P}^{26}\) | Bipartite entanglement of 3 qutrits [16]     |
| \(SL_3(\mathbb{C}) \times SL_4(\mathbb{C})\) | \(\mathbb{C}^3 \otimes \mathbb{C}^4\) | \(\mathbb{P}^2 \times \mathbb{P}^3 \subset \mathbb{P}^{11}\) | 1 qutrit and one 4-qudit                    |
| \(SL_7(\mathbb{C})\) | \(\mathbb{C}^7 \otimes \mathbb{C}^1\) | \(G(2, 7) \subset \mathbb{P}^{20}\) | 2 fermions with 7 single-particle states     |
• $V_{27}$ is the standard representation of $E_6$ and $E_6/P_1$ is the highest weight orbit.

The first four varieties of table 6 are the so-called Severi varieties studied by Zak [65]. In terms of entanglement, tables 5 and 6 lead to the following result.

**Theorem 1.** The pure quantum systems having two and only two types of genuine entanglement classes are classified by tables 5 and 6.

**Remark 3.6.** Theorem 1 should be understood according to definition 1 where non-entangled states are identified with the highest weight orbit as is also the case in [24]. However, if we consider the definition of entanglement by means of Slater rank, definition 1 does not provide the correct notion of non-entangled states for bosonic systems. In this case, theorem 1 remains correct if we consider all multipartite systems but the bosonic ones. However, table 5 does provide the classification of pure quantum systems featuring $|W\rangle$ and $|GHZ\rangle$ states with the SLOCC orbit of $|GHZ\rangle$ being dense no matter what definition of entangled/non-entangled states we take into consideration. In other words, table 5 does classify the pure quantum systems having a similar SLOCC-orbit structure as the three-qubit case.

**Remark 3.7.** It should be pointed out that the composite quantum systems of table 5 are all tripartite systems (except in the case of $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^m$ with $G = SL_2(\mathbb{C}) \times SO(m)$ for $m > 6$), while the composite systems of table 6 are all bipartite systems. This will be emphasized in appendix A.2 when we refer to a uniform geometric parametrization [35] of the varieties of separable states given by tables 5 and 6.

### 4. Conclusion

By means of algebraic geometry in this paper we intended to provide a uniform description of pure quantum systems featuring a classification of entanglement types similar to the famous case of three qubits. More precisely, we explained how a geometric interpretation of what the $|W\rangle$ and $|GHZ\rangle$ states are, allows us to use results of algebraic geometry and invariant theory to give an explicit list (table 5) of all Hilbert spaces, with the corresponding SLOCC group, such that the only types of genuine entangled states are the exact analogues of the $|W\rangle$ and $|GHZ\rangle$ states. Depending on the definition of non-entangled states we consider (see remark 3.1), this result should be moderated for multipartite bosonic systems. Indeed, according to the definition of entanglement we choose, the $|W\rangle$ state may or may not be considered as entangled for bosons. However, the orbit stratification of the Hilbert spaces of table 5, similar to the three-qubit case, does not depend on the definition of entangled/non-entangled state we work with. It turns out that the list of separable states, in the sense of definition 1, for those Hilbert spaces corresponds to the list of Freudenthal varieties. Those varieties have a strong connection with exceptional simple Lie algebras (fundamental sub-adjoint varieties). They also admit a uniform description as an image of the same rational map (Plücker embedding) over different composition algebras. This map found by [35] is described in appendix A.2. The translation of the work of algebraic geometers [35–37, 65] to quantum information theory language could be summarized in the following sentence: ‘Three fermions with six single-particle states over composition algebras can be entangled in two different ways’. This sentence includes all known cases of tripartite systems having a similar orbit structure as the three-qubit case.
Acknowledgments

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Appendix A. The tripartite entanglement and the Freudenthal varieties

The algebraic varieties of Table 5 have been studied in the mathematics literature as the fundamental sub-adjoint varieties or the Freudenthal varieties. In the early 2000s, Landsberg and Manivel investigated the geometry of those varieties in a series of papers [35–37]. Their goal was to establish new connections between representation theory and algebraic geometry. In this appendix, we collect some results and descriptions of this sequence of varieties which we believe to be relevant for quantum information theory.

A.1. The sub-adjoint varieties

Let us consider a complex simple Lie algebra of type $B_n$, $D_n$, $G_2$, $F_4$, $E_6$, $E_7$ and $E_8$ (i.e. all complex simple Lie algebras except those of type $A_n$ and $C_n$). They correspond to the fundamental simple Lie algebras, i.e. Lie algebras whose adjoint representation is fundamental [35]. Then, let $X_G \subset \mathbb{P}(g)$ be the highest weight orbit for the adjoint representation of the corresponding Lie group $G$. Consider $T_x X_G$ the embedded tangent space at any point $x$ of the homogeneous variety $X_G$. Then $Y = X_G \cap T_x X_G$ is an homogeneous variety. Table 7 gives the correspondence between the Lie algebras $g$ and the homogeneous varieties $Y$.

The sequence of algebraic varieties corresponding to quantum multipartite systems featuring only the two types of genuine entanglement $W$ and $GHZ$ are connected to fundamental adjoint representations of Lie algebras by this construction. Moreover, in [36] Landsberg and Manivel prove the existence of a rational map of degree 4 which allows from the knowledge of $Y$ to reconstruct the adjoint variety $X_G \subset \mathbb{P}(g)$ and thus to recover the structure of the Lie algebra $g$.

To illustrate the construction of this rational map, let us detail one example.

**Example A.1.** Let $Y = G(3, 6) \subset \mathbb{P}^{19} = \mathbb{P}(V)$ be the variety of separable states for a system made of three fermions with six single-particle states. Let us denote by $\{x_1, ..., x_{20}\}$ a dual basis of $V$. Then embedded linearly $\mathbb{P}(V) = \mathbb{P}^{19} \subset \mathbb{P}^{20} \subset \mathbb{P}^{21}$ and consider the rational map $\phi : \mathbb{P}^{21} \rightarrow \mathbb{P}^{27}$ defined by

$$
\phi(\{x_0, ..., x_{21}\}) = \{x_0^4, x_0 x_1, x_0^3 x_1, x_0^2 I_2(Y), x_0 x_1 x_2, x_0 I_3(\tau(Y)), x_0^2 x_1^2, I_4(\tau(Y))\}
$$
where $1 \leq i \leq 20$, $I_d(Z)$ denotes a set of generators of the ideal of degree $k$ polynomials defining $Z$ and $\tau(Y)_{\text{sing}}$ is the subvariety of singular points of $\tau(Y)$. Then $\phi(G(3, 6)) = X_{E_6}$, i.e., $\phi$ maps the set of separable three fermions with six single-particle states to the $E_6$ adjoint variety. The $E_6$ adjoint variety contains the information defining the Lie algebra $e_6$ as we have $X_{E_6} = L(\e_6)$, i.e., the linear span fills the full space and the algebraic structure can be recovered from the geometry of $X_{E_6}$.

One sees from the previous example that the Lie algebra $e_6$ can be reconstructed from the defining equation of $\tau(G(3, 6))$, i.e., the unique (up to a multiplication by a scalar) SLOCC quartic invariant on $H = N(\mathbb{C}^6)$. Indeed the ideal of degree three polynomials vanishing on the singular locus of $\tau(G(3, 6))$ is generated by the derivatives of the quartic invariant and the ideal of degree two polynomials defining $G(3, 6)$ is spanned by the second derivative of the quartic invariant. But this quartic invariant is known in the context of entanglement as the analogue for three fermions of the 3-tangle [62].

Therefore in the context of entanglement Landsberg and Manivel’s construction tells us that table 7 can be read as follows: Consider a fundamental Lie algebra $\mathfrak{g}$ and the corresponding multipartite quantum system $Y$. Then $\mathfrak{g}$ can be reconstructed from the knowledge of the unique irreducible SLOCC invariant of degree 4 (i.e., the generalization of the 3-tangle). This is another approach to construct Lie algebra from qubits [10].

### A.2. The Freudenthal varieties

The sub-adjoint varieties also appeared in the work of Landsberg and Manivel in their geometric investigation of the so-called Freudenthal magic square. Let us recall that the Freudenthal magic square is a square of semi-simple Lie algebras due to Freudenthal [22] and Tits [59] obtained from a pair of composition algebras $(\mathbb{A}, \mathbb{B})$ (where $\mathbb{A}$ and $\mathbb{B}$ are the complexification of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, the quaternions or $\mathbb{O}$, the octonions) by the following construction:

$$\mathfrak{g} = \text{Der}(\mathbb{A}) \oplus (\mathbb{A}_0 \otimes J_3(\mathbb{B})_0) \oplus \text{Der}(J_3(\mathbb{B}))$$  \hspace{1cm} (A1)$$

where $\mathbb{A}_0$ denotes the space of imaginary elements, $J_3(\mathbb{B})$ denotes the Jordan algebra of $3 \times 3$ Hermitian matrices over $\mathbb{B}$ and $J_3(\mathbb{B})_0$ is the subspace of traceless matrices of $J_3(\mathbb{B})$. For an algebra $A$, $\text{Der}(A)$ is the derivation of $A$, i.e., the Lie algebra of the automorphism group of $A$. We reproduce the Freudenthal magic square in table 8.

|  | $\mathbb{R}$ | $\mathbb{C}$ | $\mathbb{H}$ | $\mathbb{O}$ |
|---|---|---|---|---|
| $\mathfrak{g}$ | $\frak{so}_3$ | $\frak{sl}_3$ | $\frak{sp}_6$ | $\frak{f}_4$ |
| $\mathfrak{g}$ | $\frak{sl}_2 \times \frak{sl}_2$ | $\frak{sp}_4$ | $\frak{so}_{10}$ | $\frak{e}_6$ |
| $\mathfrak{g}$ | $\frak{sp}_6$ | $\frak{sl}_6$ | $\frak{so}_{12}$ | $\frak{e}_7$ |
| $\mathfrak{g}$ | $\frak{f}_4$ | $\frak{e}_6$ | $\frak{e}_7$ | $\frak{e}_8$ |

The relevance of Freudenthal construction to the study of entanglement has been pointed out by various authors [4, 62]. However, the geometric contribution has not been completely explained so far in the context of quantum information theory. The geometric version of the Freudenthal magic square of [36, 37] is a square of homogeneous varieties given in table 9.

The geometric magic square has the property that each homogeneous variety of the square is homogeneous for the corresponding Lie group in the Freudenthal magic square.
Moreover, each variety of a given row can be recovered as a section (tangential or linear) of the next one. The connection with composition algebras leads Landsberg and Manivel to formulate a geometrical uniform description of the varieties of the third row (the one relevant for the classification of table 5) as Grassmannians over the composition algebras.

It is well known that the variety $G(3, 6)$ of examples 3.2 and A.1 can be parametrized by the so-called Plücker map [25]. Let $v_1$, $v_2$ and $v_3$ be three complex vectors defining a three plane in $\mathbb{C}^6$. The coordinates can be chosen so that $v_1 = [1 : 0 : 0 : 0 : 0 : 0]$, $v_2 = [0 : 1 : 0 : 0 : 0 : 0]$ and $v_3 = [0 : 0 : 1 : 0 : 0 : 0]$. Let $[v_1 \wedge v_2 \wedge v_3]$ be a three plane in the neighborhood of $[v_1 \wedge v_2 \wedge v_3]$. One can choose $v_1 = [1 : 0 : 0 : a_1 : a_2 : a_3]$, $v_2 = [0 : 1 : 0 : a_{21} : a_{22} : a_{23}]$ and $v_3 = [0 : 0 : 1 : a_{31} : a_{32} : a_{33}]$. Locally, the variety $G(3, 6)$ is parametrized in the neighborhood of $[v_1 \wedge v_2 \wedge v_3]$ by

$$\phi(1, P) = (1, P, \text{com}(P), \text{det}(P))$$ (A2)

where $P$ is the matrix $P = (a_{ij})$ and $\text{com}(P)$ is its comatrix. The map $\phi$ is the Plücker map.

An alternative description of $G(3, 6)$ can be given by considering $P \in J_3(\mathbb{A})$ where

$$\mathbb{A} = \mathbb{C} \oplus \mathbb{C}$$

is the complexification of $\mathbb{C}$, i.e. $P = \left( \begin{array}{ccc} \alpha & x_1 & x_2 \\ \beta & x_3 & \gamma \end{array} \right)$ with $\alpha, \beta, \gamma \in \mathbb{C}$ and $x_1, x_2, x_3 \in \mathbb{C} \oplus \mathbb{C}$. Then, to recover the same parametrization one requires that the three row vectors defining the matrix $(I_3|P)$ are orthogonal with respect to the symplectic form $\omega = \left( \begin{array}{cc} 0 & I_3 \\ -I_3 & 0 \end{array} \right)$, i.e. the corresponding 3-plane is isotropic for $\omega$.

Under the symplectic condition, one has $G(3, 6) = LG_{\mathbb{C} \oplus \mathbb{C}}(3, 6)$. Similarly, the Plücker map of equation (A2) can be defined for $P \in J_3(\mathbb{A})$, with $\mathbb{A}$ one of the three other complex composition algebras. Then, if we denote by $\mathbb{A} = \mathbb{C}$, $M_2(\mathbb{C})$, $\mathbb{O}_\mathbb{C}$ the complexifications of $\mathbb{R}$, $\mathbb{H}$, $\mathbb{O}$, Landsberg and Manivel proved [36] that the varieties of the third row can all be interpreted as $LG_{\mathbb{A}}(3, 6)$, i.e.

$$\begin{align*}
LG(3, 6) &= LG_{\mathbb{C} \oplus \mathbb{C}}(3, 6) \\
G(3, 6) &= LG_{\mathbb{C} \oplus \mathbb{C}}(3, 6) \\
S_6 &= LG_{M_2(\mathbb{C})}(3, 6) \\
E_7/P_3 &= LG_{\mathbb{O}_\mathbb{C}}(3, 6)
\end{align*}$$

Table 9. The geometric magic square.

| R | C | H | O |
|---|---|---|---|
| $v_2(Q^1)$ | $\mathbb{P}(\mathbb{P}(2^2))$ | $LG(2, 6)$ | $E_6/P_1 \cap H$ |
| $v_1(\mathbb{P}^2)$ | $\mathbb{P}^2 \times \mathbb{P}^2$ | $G(2, 6)$ | $E_6/P_1$ |
| $LG(3, 6)$ | $G(3, 6)$ | $S_6$ | $E_7/P_3$ |
| $X_{fi}$ | $X_{E_6}$ | $X_{E_6}$ | $X_{E_6}$ |

Moreover, if we consider the case $P \in J_3(\mathbb{A}) = \{ \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} \mid \alpha \in \mathbb{C} \}$ (notations of [36]) and the case $P \in J_3(\mathbb{O}) = \{ \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} \mid \alpha, \beta, \gamma \in \mathbb{C} \}$, one obtains a similar Plücker parametrization of $v_3(P^1) = LG_{\mathbb{A} \oplus \mathbb{O}}(3, 6)$ and $P^1 \times P^1 \times P^1 = LG_{\mathbb{O}}(3, 6)$. 

\[18\]
An important consequence for quantum information theory is that this geometric interpretation of the varieties of the extended third row, as Lagrangian Grassmannians over $\mathbb{A}$, says that all quantum systems, which feature only the states $|W\rangle$ and $|\text{GHZ}\rangle$ as their genuine entangled classes, are tripartite systems of indistinguishable particles with six single-particle states with coefficients in a complex composition algebra satisfying a symplectic condition.

Remark A.1. Similarly, a description of the first four varieties of separable states of table 6 as Lagrangian Grassmannians $LG(\mathbb{h}^2, \mathbb{h}^d)$ is given in [36]. Those varieties correspond to the second row of the geometric magic square.

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