CORRECTION AND ADDENDUM TO: THE FOURIER EXPANSION OF HECKE OPERATORS FOR VECTOR-VALUED MODULAR FORMS

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Abstract. We correct a mistake in [St] leading to erroneous formulas in Theorems 5.2 and 5.4. As an immediate corollary of a formula in [BCJ] we give a formula, which relates the Hecke operators $T(p^2) \circ T(p^{2l-2})$, $T(p^{2l})$ and $T(p^{2l-4})$ and comment on it.

1. Introduction

There is a slight error in the calculations which lead to the (unfortunately incorrect) formula (5.3) in Theorem 5.2 in [St] and subsequently to slightly erroneous formulas in Theorem 5.4. I was made aware of this mistake by Bouchar et. al. in their paper [BCJ]. I am grateful to them for pointing this out. All other statements are unaffected. In this note we correct these mistakes and give a formula for algebraic relations the Hecke operators $T(p^{2l})$ satisfy. We also add a remark regarding this formula.

We use the same notation as in [St]. In particular, for integers $a, b$ by $(a, b)$ we mean the greatest common divisor of $a$ and $b$. Additionally, we adopt the some notation from [BCJ]. In particular, for an integer $n$ we use for the scaled quadratic form the symbol $q_n$, that is

\begin{equation}
q_n(\cdot) = nq(\cdot).
\end{equation}

Many calculations of Theorem 5.2 and subsequently of Theorem 5.4 in [St] are based on the identity

\begin{equation}
\sum_{v \in L/p^sL} e\left(\frac{1}{p^s} q(v + \lambda)\right) = e\left(\frac{1}{p^s} q(\lambda)\right) \sum_{v \in L/p^sL} e\left(\frac{1}{p^s} q(v)\right),
\end{equation}

where $s \in \mathbb{Z}$ is positive integer and $\lambda \in \mathcal{L}^{p^s}$. Unfortunately, this equation is incorrect: The left-hand side of (1.2) is independent of the representative of $\lambda$. Passing from $\lambda$ to $\lambda + w$ for some $w \in L$ leaves the sum on the left unchanged. However, the expression $e\left(\frac{1}{p^s} q(\lambda)\right)$ is not independent of the chosen representative. In fact, we have

\begin{equation}
\frac{1}{p^s} q(\lambda + w) = \frac{1}{p^s} q(\lambda) + \frac{1}{p^s} (\lambda, w) + \frac{1}{p^s} q(v).
\end{equation}

The latter summand of (1.3) is not in $\mathbb{Z}$ and therefore

$e\left(\frac{1}{p^s} q(\lambda + w)\right) \neq e\left(\frac{1}{p^s} q(\lambda)\right)$. 

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Later in the proof of Theorem 5.2 it is claimed that the identity
\[
(1.4) \quad \sum_{v \in L/p^sL} e\left(\frac{t}{p^s}q(v + h\lambda - p^l\nu)\right) = e\left(\frac{t}{p^s}q(h\lambda - p^l\nu)\right) \sum_{v \in L/p^sL} e\left(\frac{t}{p^s}q(v)\right)
\]
holds. Here \(\nu\) runs through \(L\) and \(h\lambda - p^l\nu\) is an element of \(L^{pm}\) for each \(\nu\) \((m = (s, l))\), which implies that \(\lambda \in L^{pm}\). The first sum is part of a bigger expression involving a sum over \(L\) parameterized by \(\nu\), see (2.2). The term \(e\left(\frac{t}{p^s}q(h\lambda - p^l\nu)\right)\) needs therefore to be independent of the choice of the representative of \(\lambda\) and \(\nu\). But, as pointed out above, this is not the case. It follows that the sum over \(v\) in (2.2) cannot be replaced by the right-hand side of (1.4).

2. Correction

**Theorem 1** (corrected Theorem 5.2). Let \(p\) be an odd prime, \(s, l\) positive integers with \(s < 2l\), \(h \in (\mathbb{Z}/p^s\mathbb{Z})^*\) and \(D = \dim(L)\). Then
\[
(2.1) \quad e_{\lambda} | \beta_{h,s} = \begin{cases} \sum_{v \in L/p^sL} e\left(-\frac{b}{p^{s+\lambda}} q(v + \lambda)\right) e_{p^{s+\lambda}}, & l \geq s, \\ \sum_{v \in L/p^sL} e\left(\frac{-b}{p^s} q(v + \lambda)\right) e_{p^s}, & l < s. \end{cases}
\]

**Proof.** We start with formula (5.9) in the proof of Theorem 5.2 and use subsequently equation (5.12) to obtain
\[
\frac{e(hrq(\lambda))}{\sqrt{|L|\sqrt{|L(p^s)|}}} \sum_{\rho \in L} \sum_{\nu \in L} e(-p^{2l-s}tq(\nu) + b(\nu, -\rho)) \times \sum_{\delta \in L(p^s)} e(p^s\delta + b(\delta, -\rho)) e_\rho
\]
\[
= \frac{e(hrq(\lambda))}{\sqrt{|L|\sqrt{|L(p^s)|}}} \sum_{\rho \in L} \sum_{\nu \in L} e(-p^{2l-s}tq(\nu) + b(\nu, -\rho)) \times e(b(h\lambda - p^l\nu, -\rho) \sum_{v \in L/p^sL} e\left(\frac{t}{p^s}q(v + h\lambda - p^l\nu)\right).
\]

Note that \(p^s\) and \(h\) are related by the equation \(rp^s - ht = 1\). Therefore, \((|L/p^sL|, h) = 1\) and the map \(v \mapsto hv\) is an isomorphism on \(L/p^sL\). Thus
\[
\sum_{v \in L/p^sL} e\left(\frac{t}{p^s}q(v + h\lambda - p^l\nu)\right)
\]
\[
= \sum_{v \in L/p^sL} e\left(\frac{t}{p^s}q(hv + h\lambda - p^l\nu)\right)
\]
\[
= e(p^{2l-s}tq(\nu)) \sum_{v \in L/p^sL} e\left(\frac{th^2}{p^s}q(v + \lambda) - \frac{th}{p^s}b(v + \lambda, p^l\nu)\right).
\]
Note that the last sum over \( \nu \) is independent of the choice of the representative of \( \nu \) and \( \lambda \) since

\[
\sum_{v \in \mathcal{L}/p^s L} e \left( \frac{t \hbar^2}{p^s} q(v + \lambda) - \frac{t \hbar}{p^s} b(v + \lambda, p^j \nu) \right)
\]

(2.4)

\[= e(-p^{2^j-s} q(\nu)) \sum_{v \in \mathcal{L}/p^s L} e \left( \frac{t}{p^s} q(v + h\lambda - p^j \nu) \right)\]

and both, \( e(-p^{2^j-s} q(\nu)) \) and \( \sum_{v \in \mathcal{L}/p^s L} e \left( \frac{t}{p^s} q(v + h\lambda - p^j \nu) \right) \) have this property. Using the relation \( rp^s - ht = 1 \), the right-hand side of (2.4) becomes

\[
e(p^{2^j-s} q(\nu)) e(hrq(\lambda)) e(-rb(\lambda, p^j \nu)) \sum_{v \in \mathcal{L}/p^s L} e \left( \frac{h}{p^s} q(v + \lambda) + \frac{1}{p^s} b(v + \lambda, p^j \nu) \right)
\]

Replacing the sum over \( v \) in (2.2) with (2.5) yields

\[
\frac{1}{\sqrt{|\mathcal{L}|}} \sum_{\rho \in \mathcal{L}} \sum_{\nu \in \mathcal{L}} e(b(\nu, -\rho)) \sum_{v \in \mathcal{L}/p^s L} e \left( \frac{-h}{p^s} q(v + \lambda) + \frac{1}{p^s} b(v + \lambda, p^j \nu) \right) \zeta_{\rho}
\]

(2.6)

\[= \frac{1}{\sqrt{|\mathcal{L}|}} \sum_{\rho \in \mathcal{L}} \sum_{\nu \in \mathcal{L}/p^s L} e \left( b(\nu, \frac{1}{p^s}(v + \lambda) - \rho) \right) \zeta_{\rho}.
\]

Now, since \( \nu \mapsto e \left( b(\nu, \frac{1}{p^s}(v + \lambda) - \rho) \right) \) is character of \( \mathcal{L} \), we have

\[
\sum_{\nu \in \mathcal{L}} e \left( b(\nu, \frac{1}{p^s}(v + \lambda) - \rho) \right) = \begin{cases} |\mathcal{L}|, & \text{if } \rho = p^j \left( \frac{1}{p^s}(v + \lambda) \right), \\ 0, & \text{otherwise} \end{cases}
\]

(2.7)

At this point we need to distinguish the cases \( s \leq l \) and \( s > l \).

\( s \leq l \): In this case \( p^j \left( \frac{1}{p^s}(v + \lambda) \right) \) is equal to \( p^{j-s}(v + \lambda) \in \mathcal{L} \) and we obtain for the last expression in (2.4)

\[
p^{-sD/2} \sum_{v \in \mathcal{L}/p^s L} e \left( \frac{-h}{p^s} q(v + \lambda) \right) \zeta_{p^{j-s} \lambda}.
\]

(2.8)

\( s > l \): As in [St], (5.9), we write the sum over \( v \) on the right-hand side of (2.6) in the form

\[
\sum_{\delta \in \mathcal{L}(p^s)} e(-hp^s q(\delta)).
\]

Since \( \rho \) is an element of \( \mathcal{L} \), the sum in (2.4) is non-zero if and only if the multiplication of \( \delta = \frac{1}{p^s}(v + \lambda) \) with \( p^j \) yields an element of \( \mathcal{L} \). This is the case if and only if \( \delta \in \mathcal{L}(p^j) \). Thus, we may replace \( \mathcal{L}(p^s) \) in (2.9) with \( \mathcal{L}(p^j) \). Taking (2.9) and the
thoughts before into account, we can replace the right-hand side of (2.10) with

$$p^{-s D/2} \sum_{\delta \in \mathcal{L}(p^l)} e(-hp^s q(\delta)) \epsilon_{\mu, \delta} = p^{-s D/2} \sum_{\mu \in \mathcal{L}} \sum_{\delta \in \mathcal{L}(p^l)} e(-hp^s q(\delta)) \epsilon_{\mu} \sum_{p^{-s-l} \mu = \lambda} e\left(-\frac{hp^s-1}{p^l} q(v + \mu)\right) \epsilon_{\mu}.$$  

\[ \square \]

**Remark 2.** Note that both sums in (2.11) are zero unless \( \lambda \in \mathcal{L}(p^l) \) as remarked in [St], p. 245, for the same type of sum. Compared to the original, but false formula, we don’t have to add the separate symbol \( \delta(\lambda, \cdot) \) to highlight this fact.

Also note that the formulas (2.11) coincide with the corresponding formulas in [BCJ].

The slightly different formulas for \( \rho_L(\beta_{h,s}) \) lead to slightly different formulas of the Fourier expansions of \( T(p^{2l}) \) compared to [St]. To state the corrected theorem, we introduce the following notation. According to the decomposition (5.1) in [St] we may write

$$f |_{k,L} T(p^{2l}) (\tau) = g_{\alpha}(\tau) + \sum_{s = 1}^{2l-1} \sum_{h \in (\mathbb{Z}/p^l\mathbb{Z})^*} g_{\delta_{h,s}}(\tau) + \sum_{b \in \mathbb{Z}/p^{2l}\mathbb{Z}} g_b(\tau).$$  

To lighten the formulas of the following theorem, we introduce some notation. For positive integers \( s, l \) with \( s < 2l \) let \( \lambda \in \mathcal{L}(p^{l-s}), \lambda' \in \mathcal{L}, n \in \mathbb{Z} + q(\lambda) \) and put

$$\mu(\lambda, \lambda') = \lambda/p^{l-s} + \lambda' \in \mathcal{L},$$

$$n(\lambda, \lambda') = \frac{n - p^{2(s-l)} q(\mu(\lambda, \lambda'))}{p^{2(l-s)}} + q(\mu(\lambda, \lambda')) \in \mathbb{Z} + q(\mu(\lambda, \lambda')).$$

Furthermore, attached to \( \nu, \rho \in \mathcal{L} \) and \( m \in \mathbb{Z} + q(\nu) \), \( r \in \mathbb{Z} + p^{2(s-l)} q(\rho) \) (assuming \( s > l \)) we define the representation numbers modulo \( a \)

$$N_{\nu,m}(a) = \{ v \in L/\mathbb{Z} | q(v + \nu) - m \equiv 0 \text{ mod } a \},$$

$$\tilde{N}_{\rho,r}(a) = \{ v \in L/\mathbb{Z} | q_{p^{2(s-l)}}(v + \rho) - r \equiv 0 \text{ mod } a \}.$$  

and associated to them the sums

$$G_{\nu,m}(a) = \sum_{a | p^s} \mu \left( \frac{p^s}{a} \right) a^{1-D} N_{\nu,m}(a),$$

$$\tilde{G}_{\rho,r}(a) = \sum_{a | p^s} \mu \left( \frac{p^s}{a} \right) a^{(s-l)-D} \tilde{N}_{\rho,r}(a).$$

Note that the numbers in (2.12) are well known. For example, they appear as a part of the Fourier expansion of vector valued Eisenstein series, cf. [BK]. The number \( \tilde{N}_{\rho,n}(a) \) can be interpreted as a variant of the representation number \( N_{\lambda,n}(a) \) with respect to the scaled quadratic form \( q_{p^{2(s-l)}} \).

In terms of these quantities we have
**Theorem 3** (corrected Theorem 5.4). Let $D = \dim(L)$, $p$ an odd prime, $s, l$ positive integers with $s < 2l$ and

$$K_p = p^{2l(k-1)+s(D/2-k)}, \quad \tilde{K}_p = p^{2l(k-1)-sk+(2l-s)D/2}. $$

Let $f \in M_{k,L}$ with Fourier expansion

$$f(\tau) = \sum_{\lambda \in \mathcal{L}} \sum_{n \in \mathbb{Z}+q(\lambda), n \geq 0} c(\lambda, n)e(n\tau)\varepsilon_\lambda$$

and

$$\sum_{h \in \mathbb{Z}/p^s\mathbb{Z}^*} g_{\beta_{h,s}}(\tau) = \sum_{\lambda \in \mathcal{L}} \sum_{n \in \mathbb{Z}+q(\lambda), n \geq 0} b_s(\lambda, n)e(n\tau)\varepsilon_\lambda.$$ 

Then for $s = 1, \ldots, l$,

$$b_s(\lambda, n) = \sum_{\lambda' \in \mathcal{L}_{l-l-s_{\mathcal{L}}} - \mathcal{L}_{l-l-s_{\mathcal{L}}}} c(\mu(\lambda, \lambda'), n(\lambda, \lambda')) G_{\mu(\lambda, \lambda'), n(\lambda, \lambda')} (s),$$

if $\lambda \in \mathcal{L}^{l-s}$ and zero otherwise. For $s = l+1, \ldots, 2l-1$

$$b_s(\lambda, n) = \tilde{K}_p c(p^{l-s-l}, p^{2s-l-n}) \bar{G}_{\lambda, p^{2s-l-n}}(s).$$

**Proof.** In view of section 11 we only need to adjust those parts of the Fourier expansion of $f \mid_{k,L} T(p^{2l})$ which involve the terms $\rho_L(\beta_{h,s})$, that is, the Fourier expansions of $\sum_{s=1}^{2l-1} \sum_{h \in \mathbb{Z}/p^s\mathbb{Z}^*} g_{\beta_{h,s}}(\tau)$. According to the formulas of $\rho_L(\beta_{h,s})$, we distinguish the cases $s \leq l$ and $s > l$.

For $s \leq l$, by replacing the $f$ with its Fourier expansion and $\rho_L(\beta_{h,s})$ with (2.8), we have

$$\sum_{h \in \mathbb{Z}/p^s\mathbb{Z}^*} (f_L \mid_k \beta_{h,s}) \rho_{L}^{-1}(\beta_{h,s}) \varepsilon_\lambda$$

$$= p^{k(1-s)-sD/2} \sum_{\lambda \in \mathcal{L}} \sum_{n \in \mathbb{Z}+q(\lambda), n \geq 0} c(\lambda, n)e\left(\frac{np^{2l-s-\tau}}{p^s}\right) \times$$

$$\sum_{v \in \mathcal{L}/p^sL, h \in \mathbb{Z}/p^s\mathbb{Z}^*} e\left(\frac{h(q(v + \lambda) - n)}{p^s}\right) \varepsilon_{p^l-s}. $$

The sum over $\mathbb{Z}/p^s\mathbb{Z}^*$ is a Ramanujan sum, which can be evaluated in terms of the Moebius function $\mu$. Using the same steps as in the proof of [BK], Proposition 3, we obtain

$$p^{k(1-s)-sD/2} \sum_{\lambda \in \mathcal{L}} \sum_{n \in \mathbb{Z}+q(\lambda), n \geq 0} c(\lambda, n)G_{\lambda, n}(s)e\left(\frac{np^{2(l-s)}}{p^s}\right) \varepsilon_{p^l-s}. $$

Inserting the right-hand side of (2.17) into the Fourier expansion (2.16), yields

$$p^{k(1-s)+sD/2} \sum_{\lambda \in \mathcal{L}} \sum_{n \in \mathbb{Z}+q(\lambda), n \geq 0} c(\lambda, n)G_{\lambda, n}(s)e\left(\frac{np^{2(l-s)}}{p^s}\right) \varepsilon_{p^l-s}.$$
At this point the proof in [St] remains unchanged. We merely have to replace the Gauss sum \( g(p^s, \chi_{f_n}, n-q(\lambda)) \) with the sum \( G_{\lambda,n}(s) \) and follow the subsequent steps to obtain the claimed result.

\( s > l \): We proceed in the same way as before using the formula (2.10):

\[
(2.18) \quad \sum_{h \in (\mathbb{Z}/p^s\mathbb{Z})^\times} \sum_{\lambda \in L} (f_\lambda | k \beta_{h,s}) \rho^{-1}_{L}(\beta_{h,s}) \epsilon_\lambda \\
= \sum_{\lambda \in L^{p^s-i}} f_\lambda \left( \frac{p^{2l-s}T + h}{p^s} \right) \times \frac{p^k(1-s-D/2)}{\sum_{\lambda \in L^{p^s-i}} \sum_{v \in L/p^sL} h \in (\mathbb{Z}/p^s\mathbb{Z})^\times} e \left( -h \frac{p^{s-l}}{p^s} q(v + \lambda/p^s + \lambda') \right) \epsilon_{\lambda/p^s-i+\lambda'}
\]

To evaluate the latter two sums, we perform the same steps as in the proof of Proposition 3, [BK], with a slight modification including the fact that the first sum runs over \( L/p^sL \) and not over \( L/p^sL \). We obtain

\[
(2.19) \quad \sum_{v \in L/p^sL} h \in (\mathbb{Z}/p^s\mathbb{Z})^\times \sum \frac{e \left( h(q(p^{s-l}v + p^{s-l}p) - n) \right)}{p^s} \epsilon_{\rho}. \\
= p^D \sum_{a \mid p^s, q(p^{s-l}v + p^{s-l}p) - n} \mu \left( \frac{p^s}{a} \right) a(a, p^s)^{-D} \tilde{N}_{p,n}(a).
\]

It is well known (see e. g. [McCl]) that the sum over \( (\mathbb{Z}/p^s\mathbb{Z})^\times \) on the left-hand side of (2.19) is zero unless

\[
\frac{p^s}{(p^s, q(p^{s-l}v + p^{s-l}p) - n)}
\]

is square-free. Equivalently, this sum is non-zero if and only if \( p^{s-1} \) or \( p^s \) divides \( p^{2(s-l)}q(v + p) - n \). Since \( s - 1 \geq 2(s-l) \), we may conclude that in the Fourier expansion of \( \sum_{h \in (\mathbb{Z}/p^s\mathbb{Z})^\times} g_{\beta_{h,s}} \) only those \( n \) with \( n - q_{p^{2(s-l)}}(\rho) \in p^{2(s-l)}\mathbb{Z} \) appear. This fact allows us to replace \( n \) with \( p^{2(s-l)}m \), where \( m \in \mathbb{Z} + q(\rho) \). We finally obtain for the Fourier expansion in this case

\[
p^k(1-s)+(2l-s)D/2 \sum_{\rho \in L} m \in (\mathbb{Z} + q(\rho)) \sum_{m \geq 0} c(p^{s-l} \rho, p^{2(s-l)}m) \tilde{G}_{\rho,p^{2(s-l)}m}(s) e(m\tau) \epsilon_{\rho}.
\]

\[\square\]

2.1. **Algebraic relations of the Hecke operators** \( T(p^{2l}) \). Bouchard, Creutzig and Joshi (see [BCJ]) introduced a Hecke operator \( H_n \) on the space \( M_{k, L} \) of vector valued modular forms transforming with the Weil representation, however on a completely different way than in [BS]. Their approach is more general and allows
them to deduce algebraic relations for the operators $H_n$ for any $n \in \mathbb{N}$, in particular for all $n \in \mathbb{N}$ with $(n, |L|) > 1$. Moreover, Bouchard et al. were able to prove that their Hecke operators coincide with the ones we defined in [BS] and considered in [St] and the present paper, that is, they proved $H_{p^m} = T(p^{2l})$ for all primes $p$. Based on the equivalence of these two constructions we can carry over the algebraic relations of the operators $H_{p^m}$ to the operators $T(p^{2l})$.

Below we will briefly recall the definition of the Hecke operators $H_n$ and the algebraic relations they satisfy.

The definition of $H_{n^2}$ involves the two different operators $T_n$ and $P_n$ (see Def. 3.8 and 4.2 in [BCJ]). The statement of the algebraic relations depends on a third operator $U_n$ (Def. 3.13 in [BCJ]). The operators $U_n$ and $P_n$ can be interpreted as special instances of the operators $g_A^U H$ and $g_A^D H$ as for example studied in [Br], Chapter 3.

For $n, k \in \mathbb{N}$, let $M_{k,L}$ and $M_{k,L(n^2)}$ be the space of vector valued modular forms transforming with the Weil representation associated to $L$ and the scaled lattice $L(n^2)$, respectively. Also, for any $\mu \in L(r)$ and $k, l \in \mathbb{N}$ with $kl = r$ we define

$$\Delta_r(\mu, k) = \begin{cases} 1, & \text{if } \mu \in L(l) \subset L(r), \\ 0, & \text{otherwise}. \end{cases}$$

Then

i)

(2.20)

$$T_{n^2} : M_{k,L} \rightarrow M_{k,L(n^2)}, \quad F = \sum_{\lambda \in \mathcal{L}} f_{\lambda} \varepsilon_{\lambda} \mapsto T_{n^2}(F) \text{ with}$$

$$T_{n^2}(F)(\tau) = n^{2(k-1)} \sum_{\mu \in \mathcal{L}(n^2)} \left( \sum_{r,s > 0, rs = n^2} \sum_{t=0}^{s-1} \frac{1}{s^k} \Delta_{n^2}(\mu, r) \Delta_{n^2}(\mu, s) e \left( -\frac{t}{r} q_{n^2}(\mu) \right) f_{s\mu} \left( \frac{r\tau + t}{s} \right) \varepsilon_{\mu} \right).$$

ii)

(2.21)

$$U_{n^2} : M_{k,L} \rightarrow M_{k,L(n^2)}, \quad F = \sum_{\lambda \in \mathcal{L}} f_{\lambda} \varepsilon_{\lambda} \mapsto U_{n^2}(F) \text{ with}$$

$$U_{n^2}(F)(\tau) = \sum_{\mu \in \mathcal{L}(n^2)} \Delta(n, \mu) f_{n\mu}(\tau) \varepsilon_{\mu},$$

iii)

(2.22)

$$P_{n^2} : M_{k,L(n^2)} \rightarrow M_{k,L}, \quad F = \sum_{\mu \in \mathcal{L}(n^2)} f_{\mu} \varepsilon_{\mu} \mapsto P_{n^2}(F) \text{ with}$$

$$P_{n^2}(F)(\tau) = \sum_{\lambda \in \mathcal{L}} \left( \sum_{\mu \in \mathcal{L}(n)} f_{\mu}(\tau) \varepsilon_{\lambda} \right) \varepsilon_{\lambda} \text{ and}$$

iv)

(2.23)

$$H_{n^2} : M_{k,L} \rightarrow M_{k,L}, \quad H_{n^2} = P_{n^2} \circ T_{n^2}.$$
It is proved in \cite{BCJ} that $\mathcal{P}_{n^2}$ is the left inverse operator to $\mathcal{U}_{n^2}$, that is
\begin{equation}
\mathcal{P}_{n^2} \circ \mathcal{U}_{n^2} = \text{id}.
\end{equation}

However, the relation $(\mathcal{U}_{n^2} \circ \mathcal{P}_{n^2})(F) = F$ is only valid if $F$ supported on $\mathcal{L}(n)$ (i. e. $f_\lambda = 0$ if $\lambda \notin \mathcal{L}(n)$). As already mentioned, Bouchard et. al compared their Hecke operator with the Hecke operator $T(p^{2l})$ constructed in \cite{BS} and obtained
\begin{equation}
T(p^{2l}) = \mathcal{H}_{p^{2l}}
\end{equation}
for all primes $p$ and all $l \in \mathbb{N}$. The following theorem summarizes the algebraic relations the Hecke operators $\mathcal{H}_{n^2}$ satisfy.

**Theorem 4** (\cite{BCJ}, Theorem 4.12). i) For $m, n \in \mathbb{N}$ with $(m, n) = 1$ we have
$$\mathcal{H}_{m^2} \circ \mathcal{H}_{n^2} = \mathcal{H}_{m^2n^2}.$$ ii) For any prime $p$ and any $l \in \mathbb{N}$, $l \geq 2$ the relation
\begin{equation}
\mathcal{H}_{p^{2l}} = \mathcal{P}_{p^{2l-1}} \circ \mathcal{H}_{p^{2l-2}} \circ \mathcal{U}_{p^{2l-2}} - p^{k-1}\mathcal{H}_{p^{2l-2}} - p^{2(k-1)}\mathcal{H}_{p^{2l-4}}
\end{equation}
holds.

The identities (2.25) and (2.26) immediately yield to the corresponding relation for the operators $T(p^{2l})$ as a corollary.

**Corollary 5.** For any prime $p$ and any $l \in \mathbb{N}$ the relation
\begin{equation}
T(p^{2l}) = \mathcal{P}_{p^{2l-2}} \circ T(p^2) \circ T(p^{2l-2}) \circ \mathcal{U}_{p^{2l-2}} - p^{k-1}T(p^{2l-2}) - p^{2(k-1)}T(p^{2l-4})
\end{equation}
holds.

**Remark 6.** The formula (2.27) is not as definitive as the corresponding formula for the classical scalar valued Hecke operators due to the appearance of the operators $\mathcal{P}_{p^{2l-2}}$ and $\mathcal{U}_{p^{2l-2}}$. For example, it is unclear to me how to derive a recursion formula for the eigenvalues of a common Hecke eigenform from (2.27) (if it is even possible). Also, it does not provide the basis for the proof that the Hecke operators are commutative. It would be desirable to have a formula expressing $T(p^2)\circ T(p^{2l-2})$ (without the operators $\mathcal{P}_{p^{2l-2}}$ and $\mathcal{U}_{p^{2l-2}}$) in terms of the Hecke operators $T(p^{2l})$ and $T(p^{2l-4})$ for all primes $p$. However, such a formula does seem not to exist. Even under quite strong restrictions on the Jordan block decomposition of the discriminant form $\mathcal{L}$ (implying in particular that $\mathcal{L}$ is anisotropic) a formula of this type cannot be deduced. It would be interesting to know under what conditions this is possible.

**References**

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