Single Level Current and Curvature Distributions
in Mesoscopic Systems.

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Exact analytic results for single level current and curvature distribution functions are derived in a framework of a $2 \times 2$ random matrix model. Current and curvature are defined as the first and second derivatives of energy with respect to a time–reversal symmetry breaking parameter (magnetic flux). The applicability of the obtained distributions for the spectral statistic of disordered metals is discussed. The most surprising feature of our results is the divergence of the second and higher moments of the curvature at zero flux. It is shown that this divergence also appears in the general $N \times N$ random matrix model. Furthermore, we find an unusual logarithmic behavior of the two point current–current correlation function at small flux.

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I. INTRODUCTION

The statistical properties of a single electron energy spectra in disordered mesoscopic systems have been the subject of intensive study during the past decade. Apart from a fundamental interest on its own, the spectral statistic is closely related to such observable characteristics as persistent currents, anomalous magnetization, and conductance fluctuations. After the seminal studies of Efetov [1] and Altshuler and Shklovskii [2] it became clear that the spectral statistic in the diffusive regime, where the mean free path $l$ of the electrons is much smaller than the sample size $L$, may be described by random matrix theory (RMT). The major feature of the spectra, known long ago from RMT [3,4], is a level repulsion. It means that the probability, $P(\epsilon)$, to find two consecutive levels a distance $\epsilon$ apart tends to zero as $\epsilon$ decreases: $P(\epsilon) \xrightarrow{\epsilon \to 0} 0$. Despite great progress in the theory, closed analytical expressions for the distribution function (DF), $P(\epsilon)$, are not available (except small $\epsilon$ behavior). It was demonstrated, however, in a vast amount of numerical experiments [4], that a simple Wigner surmise, obtained for a $2 \times 2$ random Hamiltonian, is an excellent approximation of the $P(\epsilon)$ of large $N \times N$ matrices. Originally the Wigner DF was obtained for three distinct symmetry classes of the Hamiltonian: Gaussian orthogonal, unitary and symplectic ensembles ((GOE),(GUE) and (GSE), respectively) [3]. Subsequently the crossover ensembles from one pure symmetry type to another were introduced and investigated [5]. In the present study we are especially interested in a crossover from GOE to GUE, which corresponds to a gradual breaking of the time–reversal symmetry. To this end it was proposed [5] to study a random Hamiltonian of the following structure:

$$H^{(N)}(\alpha) = H_S^{(N)} + i\alpha H_A^{(N)}, \quad 0 \leq \alpha \leq 1,$$

(1)

where $H_S^{(N)}$ and $H_A^{(N)}$ are, respectively, symmetric and antisymmetric real random $N \times N$ matrices. For $\alpha = 0$ one has GOE, for $\alpha = 1$ the GUE–case. It was shown recently [6] that the time–reversal symmetry breaking parameter $\alpha$ may be uniquely connected to an Aharonov–Bohm flux, $\Phi \equiv \Phi_0/2\pi$, carried by a flux line, which penetrates the system. The
latter, by gauge invariance, may be related to a change of boundary conditions, imposed
on wave functions in the angular direction (around the flux line): $\Psi(2\pi) = \Psi(0)e^{i\phi}$ \[^7\].
According to Ref. \[^6\] the relation between $\alpha$ and $\phi$ has the form
\[
\alpha = \sqrt{\pi E_c \Delta \phi},
\]
where $\Delta$ is the mean level spacing and $E_c$ is the Thouless correlation energy of the system.

Let us now point out that the first and second derivatives of the energy levels with respect
to flux (hence with respect to $\alpha$) have a clear physical interpretation. The first derivative
$\partial \epsilon_n/\partial \Phi$ is exactly a single level current, carried by each energy level in the absence of time
reversal symmetry ($\alpha > 0$). These single level currents manifest themselves, for example,
in persistent currents through a mesoscopic ring. As was conjectured by Thouless \[^8\] and
subsequently discussed by Akkermans and Montambaux \[^9\], a typical single level curvature
(the second derivative with respect to the flux) may be considered as a measure of the
correlation energy, hence it is directly proportional to the dissipative conductance of the
system. The Thouless conjecture is usually written as \[^3,9\]
\[
E_c = \left[ \left\langle \left( \frac{\partial^2 \epsilon_n}{\partial \phi^2} \bigg|_{\phi=0} \right)^2 \right\rangle \right]^{1/2},
\]
where the angular brackets $\langle \ldots \rangle$ denote an averaging over a random ensemble. It certainly
makes sense to ask the question, what are the distributions of the single level current and
curvature, as a functions of flux, in the framework of RMT. Today one knows only the
lowest correlators of these quantities like an average value, variance \[^10,11\] or two point pair
correlator (see e.g. Ref. \[^12\]). This knowledge comes mainly from diagrammatic perturbation
theory, which is usually not applicable to small values of flux.

The aim of the present article is to derive exact analytical results for the single level cur-
current and curvature DF’s, using RMT. We shall also find all the moments of these quantities
and compare them with available perturbative results. Finally we shall calculate the two
point current–current correlation function for a small flux. As was done in the case of the
level spacing distribution, we shall use a simple but exactly solvable $2 \times 2$ random matrix.
model. We prove, however, that some key features of our results are not restricted to this toy model but apply to $N \times N$ random matrices as well.

The most surprising features resulting from the present study are the following: Let us define a single level curvature to be proportional to the second flux derivative of energy

$$c \sim \frac{1}{E_c} \left| \frac{\partial^2 \epsilon_n}{\partial \phi^2} \right|$$

(numerical factors will follow in the body of the article). Then the flux dependent distribution function of the curvature has the form (for $\phi \ll \phi_c \equiv \sqrt{\Delta/(2E_c)} \ll 1$)

$$P_{\phi}(c) \sim \begin{cases} 
  c^{-1/2}, & c \ll 1, \\
  c^{-3}, & 1 \ll c \ll \phi_c/\phi, \\
  \frac{\phi^4}{\phi_c^4} \exp \left( -\frac{\phi^2}{\phi_c^2} c^2 \right), & \phi_c/\phi \ll c 
\end{cases}$$

This distribution implies the following flux dependence of the moments of the curvature

$$\left< \left| \frac{\partial^2 \epsilon_n}{\partial \phi^2} \right|^m \right> \sim \begin{cases} 
  E_c^m, & m = 1, \\
  E_c^2 \ln(\phi_c/\phi)^{-1}, & m = 2, \\
  E_c^m (\phi_c/\phi)^{2-m}, & m \geq 3 
\end{cases}$$

As is shown in Appendix A, this structure of the moments is not an artefact of the simple model, but may be rigorously derived from a general $N \times N$ random Hamiltonian, Eq. (1). Based on this, one may suggest the universality of the large curvature, $\left| \frac{\partial^2 \epsilon_n}{\partial \phi^2} \right| \gg E_c$, behavior of the distribution, Eq. (4). In contrast, the small curvature part, $\left| \frac{\partial^2 \epsilon_n}{\partial \phi^2} \right| \ll E_c$, may be a specific property of our toy model.

Equation (3) definitely contradicts the frequently used expression, Eq. (2), when evaluating the average over the disorder as arithmetical mean. Indeed, according to Eq. (5) the second and higher moments of the curvature diverge at zero flux (or, in other words, in the GOE ensemble). This is a consequence of the absence of the exponential tail in the distribution function at exactly zero flux. In this case the behavior $P_0(c) \sim c^{-3}$ continues up to infinity. As was already mentioned, such a behavior is a common feature of GOE and not a result of an oversimplified model (see Appendix A). Thus when evaluating the
Thouless energy, Eq. (3), one should either use a geometrical mean, as already pointed out by Thouless [8], who assumed a simple Lorentzian distribution of the curvatures, or use another measure of the sensitivity of the spectrum to variations of the boundary conditions:

\[ E_c \sim \left\langle \left| \frac{\partial^2 \epsilon_n}{\partial \phi^2} \right|_{\phi=0} \right\rangle. \]

The same divergence affects dramatically the current–current correlation function. Namely, it will be shown that for \( \phi, \phi' \ll \phi_c \), one has

\[ \left\langle \frac{\partial \epsilon_n(\phi)}{\partial \phi} \frac{\partial \epsilon_n(\phi')}{\partial \phi'} \right\rangle \sim -E_c^2 \phi \phi' \ln \left( \frac{\phi + \phi'}{\phi_c} \right). \]

Differentiating this result with respect to \( \phi \) and \( \phi' \) and then putting \( \phi' = \phi \), one returns back to the just quoted second moment of the curvature. This expression is absolutely unexpected from the point of view of perturbation theory. The latter assumes rather \( E_c^2 \phi \phi' \) (without logarithm) for the above defined correlation function. It would be definitely interesting to see if such a behavior exists in the framework of supersymmetric calculations. In fact, these calculations have already been done in Ref. [16], but, as far as we know, only in GUE, where we do not expect anything unusual.

Curvature distributions have already been investigated in a number of works [13,14,15], however, in a very different context. The considered Hamiltonians had the structure \( H(\lambda) = H_1 + \lambda H_2 \), where \( H(\lambda) \) belongs to the same universality class in the whole range of \( \lambda \) [15]. The curvature is defined as the second derivative of energy with respect to the parameter \( \lambda \). Subsequently, curvature distributions for three pure symmetry classes were studied. Since the above defined parameter \( \lambda \) does not break the time–reversal symmetry, it can not play the role of a magnetic flux. Thus, it has no meaning, for example, to look for a current (defined as a derivative with respect to \( \lambda \)) distribution. For reasons, which become clear in Appendix A, our results for \( \alpha = 0 \) coincide exactly with the GOE results of Refs. [13,14,15]. We recover the \( c^{-3} \) tail of the curvature distribution, first discovered in Ref. [13], and Eq. (28) of the present work may be found in Ref. [15]. However, for any \( \alpha \neq 0 \) our conclusions are very different from those of Refs. [13,14,15]. For example, in GUE (\( \alpha = 1 \)) we have find
a Gaussian tail of the curvature (with respect to flux) distribution. At the same time the distribution function of the curvatures, defined with respect to the parameter $\lambda$, decays only as the fourth power (see Appendix A and Ref. [13]).

The present article has the following structure. In section II we specify the $2 \times 2$ model based on Eq. (4) and re–derive the known results concerning the energy spacing distribution. In sections III and IV single level current and curvature DF’s are derived. Finally in section V we discuss the possible implications of the simple model to real physical systems. In Appendix A the moments of the curvature for a general $N \times N$ random matrix model are considered. A summary of the results for pure ensembles (GOE and GUE) is given in Appendix B.

II. THE MODEL AND ENERGY SPACING DISTRIBUTION

Consider a model based on $H^{(2)}(\alpha)$, as defined by Eq. (4)

$$H^{(2)}(\alpha) = \begin{pmatrix} x_1 + x_2 & x_3 + i\alpha x_4 \\ x_3 - i\alpha x_4 & x_1 - x_2 \end{pmatrix},$$

(6)

where $x_j (j = 1 \ldots 4)$ are real random variables, with a Gaussian distribution law

$$P(x_j) = \frac{1}{\sqrt{2\pi} v} \exp \left( -\frac{x_j^2}{2v^2} \right).$$

(7)

The variance of the distribution $v^2$ and the time–reversal symmetry breaking parameter $\alpha$ are the two free parameters of the model. In the end they should be related to physical observables such as mean level spacing and magnetic flux. Let us, however, postpone this discussion until section V. The spectrum of the Hamiltonian, Eq. (6), is given by

$$\epsilon_{\pm} = x_1 \pm \left( x_2^2 + x_3^2 + \alpha^2 x_4^2 \right)^{1/2},$$

(8)

and the energy spacing $\epsilon(\alpha, x)$ by $\epsilon(\alpha, x) \equiv \epsilon_+ - \epsilon_- = 2 \left( x_2^2 + x_3^2 + \alpha^2 x_4^2 \right)^{1/2}$. Let us consider an energy spacing DF

$$P_\alpha(\epsilon) = \int \delta(\epsilon - \epsilon(\alpha, x)) \prod_{j=1}^{4} P(x_j) dx_j,$$

(9)
The calculation of the integrals is straightforward, finally one obtains

$$P_\alpha(\epsilon) = \frac{\epsilon}{4v^2\sqrt{1-\alpha^2}} \exp\left(-\frac{\epsilon^2}{8v^2}\right) \text{erf}\left(\sqrt{\frac{1-\alpha^2}{2\alpha^2}} \frac{\epsilon}{2v}\right).$$

(10)

In two limiting cases one returns again to the familiar distributions: for $\alpha = 0$

$$P_0(\epsilon) = \frac{\epsilon}{4v^2} \exp\left(-\frac{\epsilon^2}{8v^2}\right); \quad \text{(GOE)},$$

(11)

and for $\alpha = 1$

$$P_1(\epsilon) = \frac{\epsilon^2}{4\sqrt{2\pi}v^3} \exp\left(-\frac{\epsilon^2}{8v^2}\right); \quad \text{(GUE)},$$

(12)

It is a well–known fact that in the case of GOE, $P_0(\epsilon) \propto \epsilon$ for $\epsilon \ll v$, whereas in the absence of time–reversal symmetry (GUE) the level repulsion is stronger: $P_1(\epsilon) \propto \epsilon^2$; ($\epsilon \ll v$). In the intermediate region $0 < \alpha < 1$ one has

$$P_\alpha(\epsilon) \approx \frac{1}{v} \exp\left(-\frac{\epsilon^2}{8v^2}\right) \left\{ \begin{array}{ll}
\left(\frac{\epsilon}{2v}\right)^2 \frac{1}{\sqrt{2\pi} \alpha}; & \epsilon \ll \frac{v\alpha}{\sqrt{1-\alpha^2}}; \\
\left(\frac{\epsilon}{2v}\right)^2 \frac{1}{2\sqrt{1-\alpha^2}}; & \epsilon \gg \frac{v\alpha}{\sqrt{1-\alpha^2}}.
\end{array} \right.$$

(13)

The level repulsion is quadratic for small energy intervals, and becomes linear for larger ones. The moments of the distribution are given by

$$\langle \epsilon^m \rangle_\alpha = v^m \left\{ \begin{array}{ll}
(-1)^n \frac{2^{3n-1/2}}{\sqrt{\pi}} \frac{\alpha^n}{\sqrt{1-\alpha^2}} \left[ t^{-1/2} \arctan \frac{1-\alpha^2}{\alpha^2t} \right]_{t=1}; & m = 2n-1, \\
(-1)^n \frac{2^{3n+1}}{\sqrt{\pi}} \frac{\alpha^n}{\sqrt{1-\alpha^2}} \left[ t^{-1}(\alpha^2t + 1 - \alpha^2)^{-1/2} \right]_{t=1}; & m = 2n.
\end{array} \right.$$  

(14)

In particular for the first moment one has

$$\langle \epsilon \rangle_\alpha = v\sqrt{\frac{8}{\pi}} \left[ \alpha + (1-\alpha^2)^{-1/2} \arctan \frac{1-\alpha^2}{\alpha^2} \right].$$

(15)

This is a smooth monotonous function of $\alpha$, which varies from $\langle \epsilon \rangle_0 = v\sqrt{\frac{8}{\pi}}$ up to $\langle \epsilon \rangle_1 = v\sqrt{\frac{8}{\pi}}2$. The fact that it is almost constant will be useful in section [V], where we shall try to give a physical interpretation of the results.
III. SINGLE LEVEL CURRENT DISTRIBUTION FUNCTION

Define now the single level currents in a $2 \times 2$ model as

$$i_{\pm}(\alpha, x) \equiv \frac{\partial \epsilon_{\pm}}{\partial \alpha} = \pm \frac{2\alpha x^2}{\epsilon(\alpha, x)}.$$  \hspace{1cm} (16)

We will look for a DF, $P_\alpha(i)$ of $i \equiv i_+ \geq 0$. Obviously the distribution of $i_-$ is the same (up to the minus sign of the argument). If one is interested in the DF of $\tilde{i}$, which may have either sign, one simply has $\tilde{P}_\alpha(\tilde{i}) = P_\alpha(|\tilde{i}|)/2$ (the coefficient $1/2$ takes care of the correct normalization). To evaluate $P_\alpha(i)$ let us first calculate the joint (current and energy space) distribution, $P_\alpha(i, \epsilon)$. Besides technical advantages, this way of calculation provides some additional information. Namely, one will be able to identify those energy spacings $\epsilon$, that are mainly responsible for a given current, $i$. The joint DF is defined as

$$P_\alpha(i, \epsilon) = \int_{-\infty}^{\infty} \delta(i - i(\alpha, x))\delta(\epsilon - \epsilon(\alpha, x)) \prod_{j=1}^{4} P(x_j)dx_j.$$ \hspace{1cm}(17)

After some calculations one gets

$$P_\alpha(i, \epsilon) = \theta(\epsilon - 2\alpha i) \frac{\pi}{2(v\sqrt{2\pi})^3} \frac{\epsilon^{3/2}}{\sqrt{2\alpha i}} \exp \left( -\frac{\epsilon^2\alpha^2}{8v^2\alpha^2} \right).$$ \hspace{1cm} (18)

Here $\theta(x)$ is a usual step function (remember that both $i, \epsilon \geq 0$). To get the current distribution, $P_\alpha(i)$, one should, according to Eq. (17), integrate the last expression over $\epsilon$.

$$P_\alpha(i) = \frac{\pi}{2(v\sqrt{2\pi})^3} (2\alpha i)^2 \int_0^{\infty} (1 + t)^{3/2} \exp \left( -\frac{t^2}{2v^2}(t^2\alpha^2 + t(1 + \alpha^2) + 1) \right) dt,$$ \hspace{1cm} (19)

where a variable $t$ was introduced as $\epsilon = 2\alpha i(t+1)$. The last integral is not known in special functions, except in the two limiting cases $\alpha = 0$ and $\alpha = 1$ (see below). However one can work out its asymptotic behavior in various regions

$$P_\alpha(i) \approx \frac{1}{v} \begin{cases} \Gamma(1/4) \sqrt{\frac{v}{i\alpha}}; & i \ll v\alpha; \quad (\epsilon \approx v) \\ 3\alpha^2 \left( \frac{v}{i} \right)^3; & v\alpha \ll i \ll v; \quad (\epsilon \approx \alpha i) \\ \sqrt{\frac{2}{\pi}} \frac{\alpha^2}{1 + \alpha^2} \exp \left( -\frac{i^2}{2v^2} \right); & v \ll i; \quad (\epsilon \approx \alpha v). \end{cases}$$ \hspace{1cm} (20)
The values of energy spacings $\epsilon$, which provide the main contribution in each case are designated in brackets. Equation (20) shows that the single level current DF has an integrable square root singularity at small currents. Realizations with energy spacings of the order of an average one are mostly responsible for this singularity. In the intermediate region, which exists only if $\alpha \ll 1$, the current DF decreases as $i^{-3}$. Finally, for large currents the distribution has a Gaussian tail. For $\alpha \ll 1$ the last two regions arise due to realizations with extremely small energy spacings.

For the two pure cases one can calculate the distributions analytically: for $\alpha = 0$

$$P_0(i) = \frac{1}{v} 2\delta\left(\frac{v}{i}\right), \quad \text{(GOE)}$$

(21)

which is evident without any calculations, and for $\alpha = 1$

$$P_1(i) = \frac{1}{v} \frac{1}{\sqrt{2\pi i}} \sqrt{\frac{2v}{i}} \Gamma\left(\frac{5}{4}, \left(\frac{v}{\sqrt{2}i}\right)^2\right), \quad \text{(GUE)}$$

(22)

where $\Gamma(a, x)$ is an incomplete gamma function. The asymptotic behavior of $P_1(i)$ is given by the first and the third lines of Eq. (20) (with $\alpha = 1$). The moments of the current distribution are given by

$$\langle i^m \rangle_{\alpha \ll 1} \approx v^m \begin{cases} \sqrt{\pi/2} \alpha; & m = 1, \\ -3\alpha^2 \ln \alpha; & m = 2, \\ \frac{2^{m/2+1}}{\sqrt{\pi}} \alpha^2 \Gamma\left(\frac{m+3}{2}\right) \frac{1}{m-2}; & m \geq 3, \end{cases}$$

(23)

for $\alpha \ll 1$ in leading order in $\alpha$, and

$$\langle i^m \rangle_{\alpha \approx 1} \approx v^m \frac{2^{m/2}}{\sqrt{\pi}} \Gamma\left(\frac{m+3}{2}\right) \frac{1}{m+1/2},$$

(24)

for $\alpha \approx 1$ in leading order in $(1 - \alpha^2)$. Let us mention particularly that the second moment of a single level current is given by $\langle i^2 \rangle_{\alpha \ll 1} \approx -3v^2\alpha^2 \ln \alpha$. This can be seen directly from Eq. (20). The leading term comes from the intermediate region of the currents ($v\alpha \ll i \ll v$). The main contribution to $\langle i^2 \rangle_{\alpha \ll 1}$ arises from realizations with very small spacings $\epsilon \approx \alpha^2 v \ll \langle \epsilon \rangle$. Starting from the third one, all the moments are determined by a Gaussian tail of the distribution ($v \ll i$), whereas the average current $\langle i \rangle_{\alpha \ll 1}$ comes from the opposite region $i \ll v$. 

9
IV. CURVATURE DISTRIBUTION FUNCTION

Following the same scheme as in the previous section, consider now the single level curvature DF.

\[ c_\pm(\alpha, x) \equiv \frac{\partial^2 \epsilon_\pm}{\partial \alpha^2} = \pm \left( \frac{2x_4^2}{\epsilon(\alpha, x)} - \frac{8\alpha^2 x_4^2}{e^4(\alpha, x)} \right). \]  

(25)

Again we consider a distribution of a positive defined quantity \( c \equiv c_+ \geq 0 \) (it is indeed positive as \( \epsilon \geq 2\alpha x_4 \), according to Eq. (8)). If one wants to include also \( c_- \), one should again make the distribution symmetrical with a proper normalization. The joint (curvature, energy spacing) DF, \( P_\alpha(c, \epsilon) \), may be found in elementary functions in the closed form

\[
P_\alpha(c, \epsilon) = \theta(\epsilon - 8\alpha^2 c) \frac{\pi}{4 (v/\sqrt{2\pi})^3} \frac{\epsilon^{3/2}}{\sqrt{c}} \exp \left( -\frac{\epsilon^2}{8v^2} \frac{1 + \alpha^2}{2\alpha^2} \right) \times B^{-1} \left[ \sqrt{1 + B} \exp \left( \frac{\epsilon^2}{8v^2} \frac{1 - \alpha^2}{2\alpha^2} B \right) + \sqrt{1 - B} \exp \left( -\frac{\epsilon^2}{8v^2} \frac{1 - \alpha^2}{2\alpha^2} B \right) \right],
\]

(26)

where \( B = \sqrt{1 - 8\alpha^2 c/\epsilon} \). One now integrates the last expression over \( \epsilon \) and obtains the following asymptotics for the curvature DF:

\[
P_\alpha(c) \approx \begin{cases} \frac{\Gamma(1/4)}{2^{9/4} \sqrt{\pi}} \frac{v}{c} & ; \quad \epsilon \approx v \end{cases}
\]

\[
\begin{cases} \frac{3}{\sqrt{1 + \alpha^2 v}} c \exp \left( -\frac{\alpha^2 c^2}{2v^2} g_\alpha \right) & ; \quad v \ll c \ll \frac{v}{\alpha} \ll \frac{v}{\alpha(1 - \alpha^2)} \ll c, \\
1; & ; \quad c \ll v, \\
2; & ; \quad v \ll c \ll \frac{v}{\alpha} \ll \frac{v}{\alpha(1 - \alpha^2)} \ll c, \end{cases}
\]

(27)

where \( g_\alpha = (7\alpha^4 + 18\alpha^2 + 7)/(1 + \alpha^2) \). The first two lines in Eq. (27) look very similar to those of the current distribution (cf. Eq. (20)). The reason is that for \( c \ll v/\alpha \) the last term in the expression for the curvature, Eq. (25), may be omitted, thus one has a trivial relationship between current and curvature, \( i = \alpha c \). The large curvature (\( c \gg v/\alpha \)) tail of the distribution is affected by the last term of Eq. (25), resulting in a complicated form of the tail of the DF, Eq. (27). For \( \alpha = 0 \) (GOE) the tail disappears completely, in this case one can get an exact result for the curvature distribution (see also Ref. [15])

\[
P_0(c) = \frac{3}{8v} \sqrt{\frac{2v}{c}} \exp \left( \frac{1}{4} \left( \frac{c}{2v} \right)^2 \right) D_{-5/2} \left( \frac{c}{2v} \right), \quad \text{GOE}
\]

(28)

10
where \( D_a(x) \) is a Whittaker parabolic cylinder function. The asymptotic behavior of this DF is given by the first two lines of the general expression, Eq. (27). Due to the absence of an exponential tail in a GOE curvature distribution, Eq. (28), all the moments of it, starting from the second one, diverge. Indeed, a direct evaluation of the moments in leading order in \( \alpha \) results in

\[
\langle c^m \rangle_{\alpha \ll 1} \approx v^m \begin{cases} 
\sqrt{\pi/2}; & m = 1, \\
-3 \ln \alpha; & m = 2, \\
\frac{2^{m/2+1}}{\sqrt{\pi}} \alpha^{2-m} \Gamma \left( \frac{m+3}{2} \right) \sum_{p=0}^{m-2} \left( -1 \right)^p \binom{m}{p} \frac{m-2+2p}{m}; & m \geq 3,
\end{cases}
\]  

for \( \alpha \ll 1 \) (cf. with Eq. (23)), and

\[
\langle c^m \rangle_{\alpha \approx 1} \approx v^m \frac{2^{m/2}}{\sqrt{\pi}} \Gamma \left( \frac{m+3}{2} \right) \sum_{p=0}^{m+1/2} \left( -1 \right)^p \binom{m}{p} \frac{m+1/2+p}{m+1/2}. 
\]  

for \( \alpha \approx 1 \) (cf. with Eq. (24)); \( \binom{m}{p} \) is a binomial coefficient. As in the case of the moments of the current, all the moments of the curvature, starting from the second one arise from realizations with small energy spacings \( \epsilon \ll \langle \epsilon \rangle \) (if \( \alpha \ll 1 \)). At \( \alpha = 0 \) only the first moment exists, whereas all higher moments diverge. This unexpected fact will be discussed in more detail in the next section.

V. DISCUSSION OF THE RESULTS

As already mentioned in section I, the Wigner surmise obtained for \( 2 \times 2 \) matrices, works extremely well also for a large \( N \). To establish this connection one should relate the phenomenological parameter \( v^2 \) – the variance of the distribution – to an average level spacing \( \Delta \). One simply demands that

\[
\langle \epsilon \rangle_\alpha = \Delta.
\]

Strictly speaking the average spacing \( \langle \epsilon \rangle_\alpha \) is a function of \( \alpha \) (magnetic flux), although the mean level spacing \( \Delta \) is presumably a constant, independent of external parameters.
However, as we noticed after Eq. (15), the dependence on \( \alpha \) is very week (especially for small \( \alpha \)). Using this fact we shall disregard its \( \alpha \)-dependence and just admit

\[
\langle \epsilon \rangle_0 = \Delta = v \sqrt{2\pi},
\]

where we have used Eq. (15). Being honest, one should re-identify the parameters for each value of \( \alpha \) separately. This procedure, although trivial, is not transparent enough for our illustrative purposes.

Having an energy spacing DF as an example, one may hope that the single level current and curvature DF’s, derived for a \( 2 \times 2 \) system, may be suitable for larger systems as well. To support the last statement let us put forward the following arguments. As we have seen in previous sections, all the moments of a single level current and curvature DF’s, starting from the second one, arise mainly from realizations of a random Hamiltonian with very small gaps. For these realizations the \( 2 \times 2 \) ansatz is supposed to be essentially correct, because for the close pair of levels only their mutual interaction appears to be important. In Appendix A we prove that the small flux behavior of the moments is indeed observed in the general \( N \times N \) model as well. The first moment, however, is determined by realizations with an energy gap of the order of the average one. In this case the \( 2 \times 2 \) scheme need not be precise. Thus one should not trust the value of the first moment, but rather connect it phenomenologically with the microscopic characteristics of a system. Following Thouless, one may relate a typical second derivative (not r.m.s !) with respect to flux at zero flux to a correlation energy

\[
E_c = \left\langle \left| \frac{\partial^2 \epsilon^\pm}{\partial \alpha^2} \right|_{\alpha=0} \right\rangle,
\]

cf. with Eq. (3). On the other hand, we had (see Eq’s. (25), (29))

\[
\left\langle \left| \frac{\partial^2 \epsilon^\pm}{\partial \alpha^2} \right|_{\alpha=0} \right\rangle = v \sqrt{\frac{\pi}{2}}.
\]

Using the definition of a mean level spacing, Eq. (31), one obtains

\[
\alpha = \sqrt{\frac{2E_c}{\Delta}} \phi,
\]
This should be compared with the conjecture of Dupuis and Montambaux [1], Eq. (2) (with 
$N = 2$). As one sees, the agreement is extremely good, the slight discrepancy may be 
attributed to the fact that Eq. (2) was obtained for the large $N$ limit. We conjecture thus, 
that with the identifications, Eq.’s. (31), (33), the tails of the distributions obtained for a 
$2 \times 2$ model are applicable for larger systems as well. Let us discuss the further consequences 
of this rather strong assumption.

First of all one notices that the $\alpha = 1$ (GUE) case corresponds to the value of a flux 
$\phi_c = \sqrt{\Delta/(2E_c)}$. This value is well–known as a correlation flux. Up to this flux a typical 
level may change parabolically, without crossing other levels. At $\phi = \phi_c$ the first avoiding 
crossing event usually happens, and the simple $2 \times 2$ scheme obviously breaks down. It was 
demonstrated numerically [9], that at $\phi \approx \phi_c$ the crossover to GUE is indeed practically 
completed. This shows that the applicability of a $2 \times 2$ model for $0 \leq \phi \leq \phi_c$ ($0 \leq \alpha \leq 1$) 
is quite reasonable, as well as the identification of the $\phi = \phi_c$ point with GUE.

Consider now the second moment of the single level current in GUE (or, the same, at 
$\phi = \phi_c$). Using Eq. (24), one obtains $(\langle i^2 \rangle_{\alpha = 1})^{1/2} = v\sqrt{3/5}$, or in physical parameters (using 
Eq’s. (31), (33)) 

$$
\left( \langle \frac{\partial \epsilon}{\partial \phi} \bigg|_{\phi = \phi_c} \rangle^2 \right)^{1/2} = \sqrt{\frac{3}{5\pi}} \sqrt{\Delta E_c}.
$$

This result is also well-known from perturbation theory (up to the numerical coefficient) 
[11,10].

Being thus convinced that the obtained results lead to reasonable predictions for real 
physical systems, let us discuss the most surprising feature of the considered DF’s: At zero 
flux all the moments of the curvature, starting from the second one, diverge. Thus, when 
calculating the correlation energy from the curvatures, one has to use another measure for 
their typical value than just the root mean square, like Eq. (32) or the geometrical mean 
proposed by Thouless [8].

The consequences of the discussed divergence are, however, deeper than just the neccessity 
of a more careful definition of the correlation energy. One also should reconsider the
universal relationship between dissipative and correlation conductances, derived by Akkermans and Montambaux [9]. Mathematically this relation was expressed as [9]

\[
\langle \left[ \frac{\partial \epsilon_n}{\partial \phi} \right]^2 \rangle = a \Delta \langle \left[ \frac{\partial^2 \epsilon_n}{\partial \phi'^2} \right]_{\phi=0}^2 \rangle^{1/2},
\]

where bar denotes integration with respect to flux, and a is a universal numerical factor. According to the present results this relation can not hold, when the typical curvature is calculated as an arithmetical mean. Indeed, using Eq. (19) and Eq’s. (31), (33), one obtains for the l.h.s. of the last expression \(\sqrt{\Delta E_c}\) (up to a coefficient of the order of unity), whereas the r.h.s. diverges.

To understand the reason for this phenomena let us consider the two point current–current correlation function

\[
\mathbf{C}(\alpha, \alpha') \equiv \langle i(\alpha) i(\alpha') \rangle.
\]

(34)

A very similar object was recently considered in Ref. [12]. One can explicitly perform the averaging in a \(2 \times 2\) model by integrating over \(d\mathbf{x}\) with the corresponding weight, precisely as one did in the previous sections. The general answer is cumbersome, but one needs only the behavior for small flux. In this case one easily gets

\[
\mathbf{C}(\alpha, \alpha') \approx -3v^2 \alpha \alpha' \ln(\alpha + \alpha') ; \quad \alpha, \alpha' \ll 1.
\]

(35)

Putting here \(\alpha = \alpha'\), one returns again to the expression for the second moment of the current (the second line in Eq. (23)). On the other hand, differentiating Eq. (33) with respect to \(\alpha\) and \(\alpha'\) and then putting \(\alpha = \alpha'\), one recognizes the second line of Eq. (29). In physical parameters Eq. (35) may be rewritten as

\[
\mathbf{C}(\phi, \phi') \equiv \langle \frac{\partial \epsilon_n(\phi)}{\partial \phi} \frac{\partial \epsilon_n(\phi')}{\partial \phi'} \rangle \approx -6\pi^{-1} E_c^2 \phi \phi' \ln\left( \frac{\phi + \phi'}{\phi_c} \right) ; \quad \phi, \phi' \ll \phi_c.
\]

(36)

This should be compared with the corresponding result of the perturbative calculations

\[
\tilde{\mathbf{C}}(\phi, \phi') \approx 12\pi^{-2} E_c^2 \phi \phi' \left( \frac{\Delta}{\gamma} \right)^2 ; \quad \phi, \phi' \ll \phi_c.
\]

(37)
where $\gamma$ is a cut off in perturbation theory, which is usually supposed to be of the order of $\Delta [6]$. The discrepancy between the two results is rather dramatic. Whereas Eq. (37) leads to a finite second moment of the curvature ($\approx E^2_c$), Eq. (36) results in a divergent second moment. Let us also point out that the perturbative result obviously may be expressed in a form $\tilde{C}(\phi, \phi') = f(\phi + \phi') - f(\phi - \phi')$, which may be traced back to Diffuson and Cooperon channels in the diagrammatic expansion. The Eq. (36) does not allow such a decomposition. This might be a point where the present scheme contradicts the derivation of Ref. [9]. Indeed, it was assumed explicitly [9], that the Diffuson–Cooperon decomposition (which is certainly correct for a large flux $\phi \gg \phi_c$) is also valid in the vicinity of zero flux. According to the present consideration this is not the case.

It is not clear at the moment whether the discussed divergence has a real physical meaning, but if so, it might cause difficulties in numerical calculations of correlation functions in GOE. We conclude that further analytical (both RMT–like and supersymmetric) calculations and numerical work are necessary to clarify this unexpectedly controversial issue.

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APPENDIX A: MOMENTS OF THE CURVATURE IN A $N \times N$ MODEL

Let us show now that the flux dependence of the moments of curvature, derived for a $2 \times 2$ model, Eq. (29), and summarized in Eq. (3), may be obtained from a general $N \times N$ model. Consider an $N \times N$ random matrix Hamiltonian, given by Eq. (1). Without loss of
generality one may assume that its spectrum is not degenerate. Then, using second order perturbation theory, one obtains the following exact relationship

\[
\frac{\partial^2 \epsilon_n(\alpha)}{\partial \alpha^2} = 2 \sum_{k \neq n}^{N} \frac{< \alpha, k | H_A^{(N)} | n, \alpha >^2}{\epsilon_n(\alpha) - \epsilon_k(\alpha)}, \quad (A1)
\]

where \(\epsilon_k(\alpha)\) and \(|k, \alpha >\) are eigenvalues and eigenfunctions of the full Hamiltonian, \(H^{(N)}(\alpha)\). As it is well-known from RMT \([4]\), statistics of eigenvalues and statistics of eigenfunctions are completely independent of each other.

Let us first consider the case of exactly zero flux \((\alpha = 0, \text{GOE})\). In this case the energies in the denominator on the r.h.s of Eq. (A1) are eigenvalues of \(H_{S}^{(N)}\), whereas in the numerator one has matrix elements of \(H_A^{(N)}\). Hence, matrix elements, \(|< 0, k | H_A^{(N)} | n, 0 >|^2\), and eigenvalues, \(\epsilon_k \equiv \epsilon_k(0)\), may be considered as independent random variables. The statistic of the eigenvalues is given by a Wigner–Dyson (GOE) distribution \([4]\)

\[
P_N(\epsilon_1, \ldots, \epsilon_N) = \text{const} \times \exp \left( -\frac{1}{2} \sum_{k=1}^{N} \epsilon_k^2 \right) \prod_{1 \leq k \leq n \leq N} |\epsilon_n - \epsilon_k|, \quad (A2)
\]

whereas an exact form of the matrix element distribution is not important for our purposes. One is now in a position to consider the moments of the random variable \(c = |\partial^2 \epsilon_n(\alpha)/\partial \alpha^2|_{\alpha=0}\). Doing this, one will be interested only in the manner of divergence, omitting all the prefactors as well as less divergent terms. Raising Eq. (A1) to the \(m^{th}\) power and averaging, one obtains

\[
\langle c^m \rangle \sim N^{-1} \int \ldots \int \frac{R_2(\epsilon_1, \epsilon_2)}{[\epsilon_1 - \epsilon_2]^m} d\epsilon_1 d\epsilon_2 + \ldots , \quad (A3)
\]

where averaging over matrix elements, \(|< 0, k | H_A^{(N)} | n, 0 >|^2\), leads to some omitted constant prefactor, and “…” denotes less divergent terms, arising from the non–diagonal contributions, like

\[
\int \ldots \int \frac{R_3(\epsilon_1, \epsilon_2, \epsilon_3)}{[\epsilon_1 - \epsilon_2]^{m-1}[\epsilon_2 - \epsilon_3]} d\epsilon_1 d\epsilon_2 d\epsilon_3 ,
\]

etc. Here \(R_i(\epsilon_1, \ldots, \epsilon_i)\) is an \(i^{th}\)–point correlation function \([4]\), for example

\[
R_2(\epsilon_1, \epsilon_2) \equiv \frac{1}{N(N-1)} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} P_N(\epsilon_1, \ldots, \epsilon_N) d\epsilon_3 \ldots d\epsilon_N. \quad (A4)
\]
In the limit of large $N$, the correlation functions $R_i$ depend only on differences of the eigenvalues. Then the integral over $\epsilon_1 + \epsilon_2$ in Eq. (A3) leads to some constant of the order of $N$, and one finally obtains

$$\langle c^m \rangle \sim \int_0^\infty \frac{R_2(\epsilon)}{\epsilon^m} d\epsilon,$$

(A5)

where $\epsilon = |\epsilon_1 - \epsilon_2|$. Taking into account the well-known GOE result, $R_2(\epsilon) \sim \epsilon (\epsilon \ll \Delta)$, one notices that all the moments of the curvature, starting from the second one, diverge. This is in exact agreement with the result for a 2 $\times$ 2 model, but now one has demonstrated the validity of this statement for a general $N \times N$ model.

The above developed scheme is applicable without any changes for the case where $\alpha$ is not a time–reversal symmetry breaking parameter, hence $H^{(N)}(\alpha)$ belongs to the same universality class in the whole range of $\alpha$ \cite{13,15}. Indeed, in this case the matrix elements of the perturbation (numerator on r.h.s of Eq. (A1)) and the eigenenergies (denominator) can always be considered as independent random variables. One can, for example, repeat exactly the same arguments for GUE with a single modification: $R_2(\epsilon) \sim \epsilon^2$. Then one obtains the divergence of all the moments of the curvature starting from the third one. This perfectly agrees with the result for the tail of the curvature distribution, proved in Refs. \cite{13,15}: $P(c) \sim c^{-4}$. In the same way for GSE, using $R_2(\epsilon) \sim \epsilon^4$, one obtains the divergence of the moments starting from the fifth one, in agreement with $P(c) \sim c^{-6}$ \cite{13,15}. This is the reason, why the result for the GOE curvature distribution, Eq. (28), coincides exactly with that of Ref. \cite{13}.

However, when $\alpha$ plays the role of a magnetic flux, it does change the symmetry class of the Hamiltonian. Then the energies $\epsilon_k(\alpha)$ are eigenvalues of the sum $H^{(N)}_S + i\alpha H^{(N)}_A$, and thus no longer statistically independent of the matrix elements, $< \alpha, k | H^{(N)}_A | n, \alpha >$. We are going to show that in this case, for $\alpha \neq 0$, all the moments converge. This in turn means that, contrary to the case of Refs. \cite{13,15}, the curvature DF has an exponential tail. For small $\alpha$ and small $|\epsilon_1 - \epsilon_2|$ one can show that

$$|\epsilon_1(\alpha) - \epsilon_2(\alpha)| \approx \left((\epsilon_1 - \epsilon_2)^2 + \alpha^2 X^2\right)^{1/2},$$

where

$$X = \left|\frac{1}{2} \left(\epsilon_1 - \epsilon_2 + \sqrt{(\epsilon_1 - \epsilon_2)^2 + \alpha^2 X^2}\right)\right|.$$
where the quantity $X$ depends only on the matrix elements (and not on the energies), hence it is statistically independent of the zero–flux energies, $\epsilon_k$. Repeating again the steps, leading to Eq. (A5) one obtains

$$\langle c^m \rangle \sim \int f(X) dX \int_0^\infty \frac{R_2(\epsilon)}{(\epsilon^2 + \alpha^2 X^2)^{m/2}} d\epsilon \sim \begin{cases} \text{const}; & m = 1, \\ \ln \alpha^{-1}; & m = 2, \\ \alpha^{2-m}; & m \geq 3, \end{cases}$$

(A6)

where $f(X)$ denotes the (unspecified) distribution of matrix elements. In Eq. (A6) we used again $R_2(\epsilon) \sim \epsilon$. Thus we have shown that the small flux behavior of the moments of the curvature, initially derived for a $2 \times 2$ model, Eq. (5), is valid for large $N$ as well.

**APPENDIX B: LIST OF DISTRIBUTIONS FOR PURE ENSEMBLES**

*GOE; $\alpha = 0$*

Spacing distribution:

$$P_0(\epsilon) = \frac{\epsilon}{4v^2} \exp \left( -\frac{\epsilon^2}{8v^2} \right),$$

(B1)

$$\langle \epsilon^m \rangle_0 = v^m 2^{3m/2} \Gamma \left( \frac{m+2}{2} \right).$$

(B2)

Current distribution

$$P_0(i) = \frac{1}{v} 2\delta \left( \frac{i}{v} \right),$$

(B3)

$$\langle i^m \rangle_0 = 0.$$  

(B4)

Curvature distribution

$$P_0(c) = \frac{3}{8v} \sqrt{\frac{2v}{c}} \exp \left( \frac{1}{4} \left( \frac{c}{2v} \right)^2 \right) D_{-5/2} \left( \frac{c}{2v} \right),$$

(B5)

$$\langle c \rangle_0 = v \sqrt{\frac{\pi}{2}}; \quad \langle c^m \rangle_0 = \infty; \quad m \geq 2.$$  

(B6)
\[ GUE; \alpha = 1 \]

Spacing distribution:
\[
P_1(\epsilon) = \frac{\epsilon^2}{4\sqrt{2\pi v^3}} \exp\left(-\frac{\epsilon^2}{8v^2}\right), \quad (B7)
\]
\[
\langle \epsilon^m \rangle_1 = v^m \frac{2^{3m/2+1}}{\sqrt{\pi}} \Gamma\left(\frac{m+3}{2}\right). \quad (B8)
\]

Current distribution
\[
P_1(i) = \frac{1}{v} \frac{1}{\sqrt{2\pi}} \frac{1}{i} \sqrt{2v} \Gamma\left(\frac{5}{4}, \left(\frac{i}{\sqrt{2v}}\right)^2\right), \quad (B9)
\]
\[
\langle i^m \rangle_1 = v^m \frac{2^{m/2}}{\sqrt{\pi}} \Gamma\left(\frac{m+3}{2}\right) \frac{1}{m+1/2}. \quad (B10)
\]

Curvature distribution
\[
P_1(c) \approx \frac{1}{v} \begin{cases} 
\frac{\Gamma(1/4)}{2^{9/4}\sqrt{\pi}} \sqrt{\frac{v}{c}}; & c \ll v, \\
\frac{4c}{v} \exp\left(-8\left(\frac{c}{v}\right)^2\right); & v \ll c,
\end{cases} \quad (B11)
\]
\[
\langle c^m \rangle_1 = v^m \frac{2^{m/2}}{\sqrt{\pi}} \Gamma\left(\frac{m+3}{2}\right) \sum_{p=0}^{m} \frac{(-1)^p (m^m)}{m + 1/2 + p}. \quad (B12)
\]
REFERENCES

[1] K. B. Efetov, Adv. Phys. 32, 53 (1983).

[2] B. L. Altshuler, and B. I. Shklovskii, Zh. Eksp. Teor. Fiz. 91, 220 (1986) [Sov. Phys. JETP 64, 127 (1986)].

[3] T. A. Brody et all, Rev. Mod. Phys. 53, 385, (1981), and references there.

[4] M. L. Mehta, “Random Matrices” ,second edition, Academic Press, 1991.

[5] A. Pandey, and M. L. Mehta, Commun. Math. Phys. 87, 449 (1983).

[6] N. Dupuis, and G. Montambaux, Phys. Rev. B 43, 14390 (1991).

[7] N. Byers, and C. N. Yang, Phys. Rev. Lett 7, 46 (1961).

[8] J.T. Edwards, D.J. Thouless, J. Phys. C 5, 807 (1972); D.J. Thouless, Phys. Rep. 13 C, 95 (1974); D.J. Thouless, Phys. Rev. Lett. 39, 1167 (1977).

[9] E. Akkermans, and G. Montambaux, Phys. Rev. Lett. 68, 642 (1992).

[10] F. von Oppen, and E. K. Riedel, Phys. Rev. Lett. 66, 84 (1991).

[11] Y. Gefen, B. Reulet, and H. Bouchiat, Phys. Rev. B 46, 15922 (1992).

[12] A. Szafer, and B. Altshuler, Phys. Rev. Lett 70, 587, (1993).

[13] P. Gaspard, S.A. Rice, and K. Nakamura, Phys. Rev. Lett. 63, 930, (1989); P. Gaspard, S.A. Rice, H.J. Mikeska, and K. Nakamura, Phys. Rev. A 42, 4015, (1990).

[14] T. Takami, and H. Hasegawa, Phys. Rev. Lett. 68, 419, (1992)

[15] J. Zakrzewski, and D. Delande, Phys. Rev. E 47, 1650, (1993).

[16] B.D. Simons, A. Szafer, and B. Altshuler, Pis’ma Zh. Eksp. Teor. Fis. 57, 268, (1993) [JETP Letters 57, 276, (1993)].

[17] J. B. French, et all, Ann. Phys. 181, 198, (1988).