Multigraded Hilbert series of invariants, covariants, and symplectic quotients for some rank 1 Lie groups

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ABSTRACT

We compute univariate and multigraded Hilbert series of invariants and covariants of representations of the circle and orthogonal group $O_2(\mathbb{R})$. The multigradings considered include the maximal grading associated to the decomposition of the representation into irreducibles as well as the bigrading associated to a cotangent-lifted representation, or equivalently, the bigrading associated to the holomorphic and antiholomorphic parts of the real invariants and covariants. This bigrading induces a bigrading on the algebra of on-shell invariants of the symplectic quotient, and the corresponding Hilbert series are computed as well. We also compute the first few Laurent coefficients of the univariate Hilbert series, give sample calculations of the multigraded Laurent coefficients, and give an example to illustrate the extension of these techniques to the semidirect product of the circle by other finite groups. We describe an algorithm to compute each of the associated Hilbert series.

1. Introduction

Let $R = \bigoplus_{d=0}^{\infty} R_d$ be an $\mathbb{N}$-graded commutative algebra over a field $k$ where $\mathbb{N} = \{0, 1, 2, \ldots \}$. The Hilbert series of $R$, also known as the Poincaré series or Hilbert-Poincaré series, is the generating function of the dimensions of the $R_d$,

$$\text{Hilb}_R(t) = \sum_{d=0}^{\infty} t^d \dim_k R_d.$$ 

If $R$ is finitely generated, then $\text{Hilb}_R(t)$ is the power series of a rational function with radius of convergence at least 1; see [21, Section 1.4]. More generally, if $R = \bigoplus_{d \in \mathbb{N}^n} R_d$ is $\mathbb{N}^n$-graded and $M = \bigoplus_{d \in \mathbb{N}^n} M_d$ is an $\mathbb{N}^n$-graded $R$-module, the multigraded Hilbert series of $M$ associated to the $\mathbb{N}^n$-grading is given by

$$\text{Hilb}_M(t) = \sum_{d \in \mathbb{N}^n} t^d \dim_k M_d.$$ 

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where \( t = (t_1, \ldots, t_n) \) and for \( d = (d_1, \ldots, d_n) \), \( t^d = t_1^{d_1} \cdots t_n^{d_n} \); see [51, Chapter 1, Section 2]. Once again, if \( R \) is finitely generated and \( M \) is finitely generated over \( R \), then \( \text{Hilb}_M(t) \) is the power series of a rational function in the \( t_i \); see [51, Chapter 1, Theorem 2.3].

If \( R = \mathbb{k}[V]^G \) is the algebra of invariant polynomials of a finite-dimensional representation \( V \) of a reductive group \( G \) with the usual \( \mathbb{N} \)-grading, then \( \text{Hilb}_R(t) \) can be computed using Molien’s formula when \( G \) is finite or the Molien-Weyl formula in general; see [21, Theorem 3.4.2, Section 4.6.1], [52, Theorem 2.2.1], or Theorem 2.1. Suppose the \( G \)-module \( V \) decomposes as \( V = V_1 \oplus \cdots \oplus V_n \) where we do not necessarily assume that the \( V_i \) are irreducible. Consider the \( \mathbb{N}^n \)-grading of \( \mathbb{k}[V] \) and \( \mathbb{k}[V]^G \) given by assigning to a monomial \( f_1(v_1)f_2(v_2) \cdots f_n(v_n) \) with \( f_i(v_i) \in \mathbb{k}[V_i] \) the degree \( (\deg f_1, \ldots, \deg f_n) \in \mathbb{N}^n \). We refer to this grading as the grading associated to the decomposition \( V_1 \oplus \cdots \oplus V_n \) of \( V \). Similarly, the multigraded Hilbert series \( \text{Hilb}_{\mathbb{k}[V]^G}(t_1, \ldots, t_n) \) is the multigraded Hilbert series associated to the decomposition. More generally, if \( W \) is another finite-dimensional representation of \( G \), then the \( \mathbb{k}[V]^G \)-module of covariants \( \text{Mor}(V, W)^G = (\mathbb{k}[V] \otimes W)^G \) is similarly graded by assigning to the element \( f(v) \otimes w \), where \( f \in \mathbb{k}[V] \) is homogeneous with respect to the grading induced by the decomposition of \( V \) and \( w \in W \), the degree of \( f(v) \). As \( G \) is reductive so that \( \mathbb{k}[V]^G \) is a finitely-generated algebra and \( \text{Mor}(V, W)^G \) is a finitely-generated \( \mathbb{k}[V]^G \)-module by [21, Theorem 4.2.10], [46, Theorem 3.24], the multigraded Hilbert series of invariants or covariants is again the power series of a rational function. If the decomposition of \( V \) is into irreducibles, we refer to the resulting grading as the maximal grading of \( V \). When \( G = S^1 \), then because a polynomial is \( S^1 \)-invariant if and only if each of its monomial terms is invariant, the maximally graded Hilbert series determines the algebra of invariants or covariants; see [51, Chapter 1, Section 3] and Example 3.4 below.

An important case of the above multigrading is the cotangent lift \( V = V_1 \oplus V_1^* \) of the representation \( V_1 \); we refer to the corresponding grading as the cotangent bigrading. If \( \mathbb{k} = \mathbb{C} \), then this grading corresponds to the decomposition of the underlying real representation into holomorphic and antiholomorphic components. Specifically, we can express elements of \( V \) with coordinates \( (z_1, \ldots, z_n, w_1, \ldots, w_n) \) where the \( z_i \) are coordinates for \( V_1 \), the \( w_i \) are dual coordinates for \( V_1^* \), the \( \mathbb{N}^2 \)-degree of each \( z_i \) is \((1, 0)\), and the degree of \( w_i \) is \((0, 1)\). The real polynomial invariants of the real representation underlying \( V_1 \) correspond to the invariants of \( V \) with real coefficients where \( w_i = \overline{z_i} \); see [48, Proposition (5.8)(1)]. The cotangent bigraded Hilbert series was defined and studied in [24].

One may also consider the case of the on-shell invariants of the linear symplectic quotient associated to a unitary representation \( V \) of a compact Lie group \( G \); see [36, Section 2]. Letting \( J : V \to \mathfrak{g}^* \simeq \mathbb{R}^c \) denote the moment map associated to the \( G \)-representation, the on-shell invariants are defined to be \( \mathbb{R}[V]^G/I_f^G \) where \( I_f \) is the ideal of \( \mathbb{R}[V] \) generated by the components of \( J \) and \( I_f^G = I_f \cap \mathbb{R}[V]^G \). Because the components of the moment map have degree \((1, 1)\) with respect to the cotangent bigrading and are not homogeneous with respect to further decompositions of \( V \), the only reasonable gradings for the on-shell invariants are those induced by the standard \( \mathbb{N} \)-grading or the cotangent bigrading on \( V \). In the case that \( I_f \) is a real ideal, the on-shell invariants coincide with the real regular functions on the corresponding symplectic quotient; see [2], [31, Section 2], [22, Section 2.1], [33, Section 4], or [35, Section 2.2]. Note that we will sometimes refer to elements of \( \mathbb{k}[V]^G \) as off-shell invariants to avoid potential confusion with the on-shell invariants.

This paper continues a program of computing Hilbert series which we began in [19, 36] for univariate Hilbert series of on- and off-shell invariants of circle representations, [20, 28] for on- and off-shell invariants associated to \( SL_2 \)- and \( SU_2 \)-modules where multigraded Hilbert series played a minor role, and [30] for the univariate Hilbert series of on-shell invariants of 2-torus modules. See also [11–15, 38, 42], the more recent work [4, 5, 9, 10, 39, 40], and in particular the recent work involving multigraded Hilbert series [6–8], and recall that some authors refer to the Hilbert series as the Poincaré series or the Hilbert-Poincaré series. For applications of Hilbert series and Laurent coefficient computations to identifying properties of and distinguishing between the corresponding singularities, see [16, 22, 27, 29, 32, 35, 37, 45, 49]; for applications in gauge theory, see [3, 23, 26]; and for applications to K-stability of Sasakian manifolds, see [17, 18].
The present paper focuses on the multigraded Hilbert series, which in particular avoids the complications introduced by degeneracies; see [19, 36]. We consider the case of the circle $S^1$ in Section 3 as well as the non-connected orthogonal group $O_2(\mathbb{R})$ in Section 4. In each case, we compute the maximally graded Hilbert series of the off-shell invariants and covariants and the bigraded Hilbert series of the on-shell invariants. We also compute the maximally graded Hilbert series of invariants of the group $S^1 \times \mathbb{Z}/4\mathbb{Z}$ in Section 5 to indicate the extension of these techniques to other non-connected groups.

Specializing to the univariate Hilbert series, we compute the Laurent coefficients of the Laurent expansion of the Hilbert series of circle covariants in Section 3.3, generalizing the results of [19, 36], as well as the Laurent coefficients of covariants and on-shell invariants for $O_2(\mathbb{R})$ in Section 4.4. Though the Laurent coefficients in the multigraded case are ambiguous and less motivated, we give some sample computations of bigraded Laurent coefficients in the case of the on-shell invariants of a circle representation in Section 3.4. In Section 6, we describe an algorithm to compute the Hilbert series in one case which can easily be extended to the other cases computed in this paper.

2. Background

In this section, we collect some background information which we will need throughout this paper. We refer the reader to [21, 46], [36, Section 2], and the explanations in the Introduction for more details.

We begin with the following, which will be our primary tool for computing Hilbert series. For univariate Hilbert series, see [21, Theorem 3.4.2] or [52, Theorem 2.2.1] for the case of invariants of a general reductive group. See [50, equation (13)] and [24, Section IV] for extensions to the multigraded case; the general result stated below is proven similarly.

**Theorem 2.1** (Molien-Weyl Theorem). Let $G$ be a reductive group over $\mathbb{C}$ and $K$ a maximally compact subgroup. Let $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ and $W$ be finite-dimensional rational representations of $G$ over $\mathbb{C}$. Then the multigraded Hilbert series of the module of covariants $\text{Mor}(V, W)^G$ associated to the decomposition $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ is given by

$$\text{Hilb}_{\text{Mor}(V, W)^G}(t) = \int_{z \in K} \frac{\chi_W(z^{-1})}{\prod_{i=1}^n \det_{V_i}(1 - t_i z_i)} \, d\mu,$$

where $t = (t_1, \ldots, t_n)$, $\mu$ is a normalized Haar measure on $K$, $\chi_W$ is the character of the representation $W$, $\det_{V_i}$ denotes the determinant on $\text{End}(V_i)$, and $z_i$ denotes the restriction of the action of $z \in G$ to $V_i$.

Of course, the invariants $\mathbb{C}[V]^G$ correspond to the case when $W$ is the trivial 1-dimensional representation. Because the module of covariants is additive, i.e., $\text{Mor}(V, W_1 \oplus W_2)^G = \text{Mor}(V, W_1)^G \oplus \text{Mor}(V, W_2)^G$, we will often assume that $W$ is irreducible with no loss of generality.

Theorem 2.1 can be applied to the covariants of a compact Lie group $K$ by choosing $G$ to be the complexification of $K$, and to real invariants through the isomorphism $\mathbb{R}[V]^K \otimes \mathbb{C} \cong \mathbb{C}[V]^K = \mathbb{C}[V]^G$ given in [48, Proposition (5.8)(i)]. Note that $V$ and $W$ will always admit Hermitian scalar products with respect to which the representations of $K$ are unitary. Because of these observations, we will state our results for invariants and covariants of unitary representations of compact Lie groups, though the results extend directly to invariants and covariants of rational representations of the complexifications of these groups.

The Laurent coefficients at $t = 1$ of the univariate Hilbert series considered here are most easily described using Schur polynomials and variations thereof, so let us briefly recall these definitions and fix notation. Suppose $x = (x_1, \ldots, x_n)$ is a set of indeterminates. A partition $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ of length $n$ is a set of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. The alternant $\text{Alt}_\lambda(x)$ associated to $\lambda$ in $x$ is the determinant $\text{Alt}_\lambda(x) = |x_\lambda^T|$, it is an alternating polynomial in the $x_i$. Letting $\delta_n = (n-1, n-2, \ldots, 1, 0)$ and $V(x) = \text{Alt}_{\delta_n}(x)$ denote the Vandermonde determinant, any alternant is divisible by $V(x)$, and the
quotient is a symmetric polynomial. The Schur polynomial associated to \( \lambda \) is defined by
\[
s_\lambda(x) = \frac{\text{Alt}_{\lambda+b}(x)}{V(x)}.
\]

For more details and for a combinatorial description of \( s_\lambda(x) \), see [43, Section I.3] and [47, Sections 4.4–6].

If \( x = (x_1, \ldots, x_k) \) and \( y = (y_1, \ldots, y_m) \) are two sets of indeterminates, \( n = k + m \), and \( u \leq n - 2 \) is an integer, then the partial Laurent-Schur polynomial \( sp_u(x, y) \) is defined by
\[
sp_u(x, y) = \frac{1}{V(x)V(y)} \begin{vmatrix}
x_1^u & \cdots & x_k^u & 0 & \cdots & 0 \\
x_1^{n-2} & \cdots & x_k^{n-2} & y_1^{n-2} & \cdots & y_m^{n-2} \\
x_1^{n-3} & \cdots & x_k^{n-3} & y_1^{n-3} & \cdots & y_m^{n-3} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
x_1 & \cdots & x_k & y_1 & \cdots & y_m \\
1 & \cdots & 1 & 1 & \cdots & 1
\end{vmatrix}.
\]

The partial Laurent-Schur polynomial \( sp_u(x, y) \) is a polynomial if \( u \geq 0 \) and a Laurent polynomial if \( u < 0 \); it is symmetric separately in the \( x \) and \( y \). See [19, Section 5] for more details, an expression of \( sp_u(x, y) \) in terms of Schur polynomials in \( x \) and \( y \), and a combinatorial description.

### 3. Invariants and covariants of \( S^1 \)

Let us begin by briefly recalling the irreducible representations of \( S^1 \); as explained in Section 2, the results of this section apply as well to the complexification \( \mathbb{C}^\times \) of \( S^1 \). Because \( S^1 \) is abelian, its irreducible unitary representations are all 1-dimensional. For each \( a \in \mathbb{Z} \), there is a unique irreducible representation \( \epsilon_a: S^1 \to U_1 \) given by \( \epsilon_a(z) = z^a \) acting as multiplication. Hence, a finite-dimensional unitary representation of \( S^1 \) can be described by a weight vector \( a = (a_1, \ldots, a_n) \), indicating the representation \( V = V_a = \bigoplus_{i=1}^n \epsilon_{a_i} \). We will often use the notation \( \alpha_i = |a_i| \) to indicate the absolute values of the weights.

#### 3.1. The maximally graded Hilbert series

In this section, we compute the maximally graded Hilbert series of the covariants of unitary \( S^1 \)-modules.

**Remark 3.1.** Assume the representation \( V_a \) is not faithful, i.e., \( g := \gcd(a_1, \ldots, a_n) > 1 \). Then a monomial \( f = x_1^{a_1} \cdots x_n^{a_n} \) in \( \text{Mor}(V_a, W_{-b}) \) is covariant if and only if \( \sum_{i=1}^n p_i a_i = -b \). If \( g \) does not divide \( b \), then there are no solutions to this Diophantine equation and hence no covariants so that the Hilbert series is 0. If \( g \) divides \( b \), then clearly \( f \) is a covariant if and only if it is a covariant for the representations \( V_{a/g} \) with \( a/g = (a_1/g, \ldots, a_n/g) \) and \( W_{-b/g} \), where \( V_{a/g} \) is now faithful. Hence, we may assume that \( V \) is faithful with no loss of generality. In addition, the existence of a trivial subrepresentation \( \epsilon_{b_0} \) with \( a_i = 0 \) has the trivial effect of multiplying the Hilbert series by \( 1/(1-t_i) \), and we may assume with no loss of generality that 0 does not appear as a weight.

**Theorem 3.2.** (Maximally graded Hilbert series of \( S^1 \)-covariants). Let \( V \simeq \mathbb{C}^n \) be a representation of \( S^1 \) with weight vector \( a = (-\alpha_1, \ldots, -\alpha_k, \alpha_{k+1}, \ldots, \alpha_n) \) where \( n \geq 1 \) and each \( \alpha_i > 0 \), and let \( W \simeq \mathbb{C} \) be the irreducible representation of the circle with weight \( -b \). Let \( t = (t_1, \ldots, t_n) \), and let \( \text{Hilb}^{S^1}_{ab}(t) \) denote the Hilbert series for the maximal \( \mathbb{N}^n \)-grading of the covariants \( \text{Mor}(V, W)^{S^1} = (\mathbb{C}[V] \otimes W)^{S^1} \). Then
\[
\text{Hilb}^{S^1}_{ab}(t) = \sum_{i=1}^k \alpha_i \prod_{j=1}^n \left( 1 - \zeta_j t_j^{-a_j/a_i} \right) + (-1)^k \sum_{y \in S_{ab}} \prod_{i=1}^n y_i^{-1} \prod_{i=k+1}^n t_j^{y_j},
\]
where the sum \( \sum_{i=1}^{k} \) is shorthand for \( \sum_{i=1}^{k} \sum_{\xi^{\alpha_i} = 1}^{k} \), summing over \( i = 1, \ldots, k \) and all \( \alpha_i \)th roots of unity \( \zeta \), and

\[
S_{a,b} = \left\{ y \in \mathbb{N}^n : \sum_{j=1}^{n} y_j \alpha_j = -b - \sum_{j=1}^{k} \alpha_j \right\},
\]

which is empty if \( b + \sum_{j=1}^{k} \alpha_j > 0 \).

**Proof.** Using the Molien-Weyl formula,

\[
\text{Hilb}_{b,a}(t) = \frac{1}{2\pi \sqrt{-1}} \int_{\mathbb{S}^1} \frac{z^b \, dz}{z \prod_{j=1}^{k} (1 - t_j z^{-\alpha_j}) \prod_{j=k+1}^{n} (1 - t_j z^{\alpha_j})}
\]

for \( |t_j| < 1 \). If \( b - 1 + \sum_{j=1}^{k} \alpha_j \geq 0 \), i.e., \( -b - \sum_{j=1}^{k} \alpha_j < 0 \), then the integrand is holomorphic at \( z = 0 \); note that in this case \( S_{a,b} = \emptyset \). If \( b - 1 + \sum_{j=1}^{k} \alpha_j < 0 \), then there is a pole at \( z = 0 \). In this case, we compute the residue at \( z = 0 \) as follows. We consider the integrand as the product of factors of the form \( z^{b-1+\sum_{j=1}^{k} \alpha_j} \),

\[
\frac{1}{z^{\alpha_j} - t_j} = \frac{1}{t_j} - \frac{z^{\alpha_j}}{t_j^2} - \frac{z^{2\alpha_j}}{t_j^3} - \cdots = \sum_{d=0}^{\infty} (-1)^{d-1} z^{d \alpha_j},
\]

and

\[
\frac{1}{1 - t_j z^{\alpha_j}} = 1 + t_j z^{\alpha_j} + t_j^2 z^{2\alpha_j} + \cdots = \sum_{d=0}^{\infty} t_j^d z^{d \alpha_j}.
\]

Via the Cauchy product formula, for every collection of nonnegative integers \( y = (y_1, \ldots, y_n) \) such that \( y_1 \alpha_1 + \cdots + y_n \alpha_n + (b - 1 + \sum_{j=1}^{k} \alpha_j) = -1 \), i.e., \( y_1 \alpha_1 + \cdots + y_n \alpha_n = -b - \sum_{j=1}^{k} \alpha_j \), there is a corresponding term contributing to the \( z^{-1} \) term of the product series given by

\[
(-1)^k \prod_{i=1}^{k} t_i^{-y_i-1} \prod_{i=k+1}^{n} t_i^{y_i}.
\]

Summing over all such \( y \), the residue at \( z = 0 \) is given by

\[
(-1)^k \sum_{y \in S_{a,b}} \prod_{i=1}^{k} t_i^{-y_i-1} \prod_{i=k+1}^{n} t_i^{y_i}.
\]

The other poles (and the only poles when \( b - 1 + \sum_{j=1}^{k} \alpha_j \geq 0 \)) occur when \( z^{\alpha_i} = t_i \) for some \( i \in \{1, \ldots, k\} \), i.e., \( z = \zeta t_i^{1/\alpha_i} \), where \( \zeta \) is an \( \alpha_i \)th root of unity and the \( t_i^{1/\alpha_i} \) are defined with respect to a fixed branch of the logarithm. Note that we can choose this branch of log so as to contain all \( t_i \) in its domain. Moreover, each of these poles is simple for generic choice of \( t_i \), e.g., by assuming that the moduli of the \( t_i \) are distinct.
Fixing a specific \( i \leq k \) and \( \alpha_i \)th root \( \zeta_0 \), we express the integrand as

\[
z^{b-1+\sum_{j=1}^{k} \alpha_j} = \frac{(z^{\alpha_i} - t_i) \prod_{j=1}^{n} (z^{\alpha_j} - t_j) \prod_{j=k+1}^{n} (1 - t_j z^{\alpha_j})}{(z - \zeta_0) \prod_{j=1}^{k} (z - \zeta_1^{\alpha_j} - t_j) \prod_{j=k+1}^{n} (1 - t_j z^{\alpha_j})}.
\]

Then the residue at \( \zeta_0^{1/\alpha_i} \) is given by

\[
\frac{\prod_{\zeta^{\alpha_i} = 1}^{\zeta \neq \zeta_0} (\zeta_0^{1/\alpha_i} - \zeta^{1/\alpha_i}) \prod_{\zeta^{\alpha_i} = 1}^{\zeta \neq \zeta_0} ((\zeta_0^{1/\alpha_i} - t_i) \prod_{j=k+1}^{n} (1 - t_j (\zeta_0^{1/\alpha_i} - t_j))}{\zeta_0^{b-1+\sum_{j=1}^{k} \alpha_j} (\zeta_0^{b-1+\sum_{j=1}^{k} \alpha_j} / \alpha_i)} = \frac{\zeta_0^{\alpha_i - 1} (\alpha_i - 1) / \alpha_i \prod_{\zeta^{\alpha_i} = 1}^{\zeta \neq \zeta_0} (1 - \zeta) \prod_{j=1}^{k} (\zeta_0^{\alpha_j} t_i^{\alpha_j} - t_j) \prod_{j=k+1}^{n} (1 - t_j \zeta_0^{\alpha_j} t_i^{\alpha_j})}{\zeta_0^{\sum_{j=1}^{k} \alpha_j} (1 - \zeta_0^{\alpha_i} t_i^{\alpha_j}) \prod_{j=k+1}^{n} (1 - \zeta_0^{\alpha_i} t_i^{\alpha_j})}.
\]

Summing over all \( i \leq k \) and \( \zeta \) completes the proof.

\( \square \)

**Remark 3.3.** Note that Theorem 3.2 applies in the somewhat trivial cases where the weights of \( a \) all have the same sign. In this case, a monomial \( x_1^{p_1} \cdots x_n^{p_n} \) is a covariant if and only if the \( p_i \) are solutions to \( \sum_{i=1}^{n} p_i \alpha_i = \pm b \). As there are clearly only finitely many solutions, the covariants are always a finite-dimensional vector space, which may be 0 or only contain the constants.

If all weights are positive, then the first sum in equation (3.1) is empty and the Hilbert series is given by the second sum. If \( b = 0 \), the second sum is 1, as \( S_{a,0} \) contains only the zero solution, yielding a Hilbert series of 1. If \( b > 0 \), the second sum is 0 as \( S_{a,b} = \emptyset \), yielding a Hilbert series of 0. If \( b < 0 \), then the second sum is (usually) nontrivial but will yield a polynomial with no constant term, as the constants are clearly not covariant.

Now assume all weights are negative. If \( b > -\sum_{i=1}^{n} \alpha_i \), then \( S_{a,b} = \emptyset \) so the Hilbert series is given by the first sum. If \( b = 0 \), then the first sum is nontrivial but will always equal 1, as only the constants are invariant. If \( b > 0 \), then the first sum is nontrivial and will yield a polynomial. If \( b < 0 \), then one easily checks from the definition that there are no covariants; if \( -\sum_{i=1}^{n} \alpha_i < b < 0 \), then the first sum is nontrivial but will always yield 0, and if \( b \leq -\sum_{i=1}^{n} \alpha_i \), then both sums can be nontrivial but the resulting Hilbert series is 0.

We give the following example of a concrete application of Theorem 3.2, in particular to illustrate how the maximally graded Hilbert series completely determines the algebra of invariants.

**Example 3.4.** Consider the invariants (i.e., \( b = 0 \)) of the representation \( V_a \) with weight vector \( a = (-1, -2, 1, 2) \). Applying Theorem 3.2 via the algorithm described in Section 6, the Hilbert series is given by

\[
\text{Hilb}^{(1)}_{(-1,-2,1,2),0}(t_1, t_2, t_3, t_4) = \frac{1 - t_1^2 t_2 t_3^2 t_4}{(1 - t_1 t_3)(1 - t_2 t_3^2)(1 - t_1^2 t_4)(1 - t_2 t_4)}.
\]
Listing all terms with exponents $\leq 4$ in colexicographic order, the Taylor expansion at $t = (0, 0, 0, 0)$ begins

$$1 + t_1 t_3 + t_1^2 t_3^2 + t_2 t_3^2 + t_1^3 t_3^3 + t_1 t_2 t_3^3 + t_1^4 t_3^4 + t_1^2 t_3^2 t_4 + t_2 t_4 + t_1^3 t_3 t_4 + t_1 t_2 t_3 t_4 + t_1^4 t_3^2 t_4$$

$$+ t_1^2 t_3^3 t_4 + t_1^2 t_2 t_3^3 t_4 + t_1 t_2 t_3^4 + t_1^2 t_3^2 t_4 + t_1^2 t_2^2 t_3^2 t_4 + t_1^2 t_3 t_4 + t_1^4 t_2^2 t_3^2 + t_1^2 t_2 t_3^2 t_4$$

$$+ t_1^2 t_2^3 t_4 + t_1^2 t_2^2 t_3^2 + t_1 t_2 t_3^2 t_4 + t_1^2 t_3^2 t_4 + t_1^2 t_2^2 t_3^2 + t_1^2 t_2^2 t_3^2 + t_1 t_2 t_3^2 t_4 + t_1^2 t_3^2 t_4 + t_1^2 t_3^2 t_4$$

$$+ t_1^2 t_3^2 t_4 + t_1^2 t_3^2 t_4 + t_1^2 t_3^2 t_4 + t_1^2 t_3^2 t_4 + t_1^2 t_3^2 t_4 + t_1^2 t_3^2 t_4 + t_1^2 t_3^2 t_4 + t_1^2 t_3^2 t_4 + t_1^2 t_3^2 t_4 + t_1^2 t_3^2 t_4 + t_1^2 t_3^2 t_4 + \cdots.$$ 

Hence, each coefficient is either 0 or 1, and each nonzero term can be thought of as an invariant in the variables $(t_1, t_2, t_3, t_4)$. Via this interpretation, the above expansion yields a complete list of the invariants of $\mathbb{N}^4$-degree $(d_1, d_2, d_3, d_4)$ with each $d_i \leq 4$. That is, the Hilbert series $\text{Hilb}_{(-1,-2,1,2),0}^a (t_1, t_2, t_3, t_4)$ completely determines the algebra of invariants.

Note that in this case, the structure of the algebra of invariants is clear from equation (3.4). In particular, the algebra of invariants in this case is generated by the monomials $t_1 t_2, t_3^2, t_4^2, t_3 t_4$, and $t_2 t_4$ with the single relation $(t_1 t_3)^2 (t_2 t_4) - (t_2 t_3) (t_1 t_4)$, see [51, Chapter I, Theorem 2.3 and Corollary 3.8].

To offer an example where $b \neq 0$, let $V_a$ be as above and $b = 2$. Then Theorem 3.2 and the algorithm in Section 6 yield

$$\text{Hilb}_{(-1,-2,1,2),2}^a (t_1, t_2, t_3, t_4) = \frac{t_1^2 + t_2 - t_1^2 t_2 - t_1^2 t_3^2}{(1 - t_1 t_2)(1 - t_2 t_3)(1 - t_1 t_4)(1 - t_2 t_4)},$$

whose expansion begins

$$t_1^2 + t_2 + t_1^3 t_3 + t_1 t_2 t_3 + t_1^2 t_3^2 + t_2^2 + t_2 t_3^2 + t_1^2 t_3^2 + t_3 t_4 + t_2^2 t_3 t_4 + t_1^2 t_2 t_3 t_4 + t_1 t_2 t_3 t_4 + t_1^2 t_2^2 t_3 t_4 + t_1^2 t_2 t_3^2 t_4$$

$$+ t_1^2 t_3^2 t_4 + t_1^3 t_3^2 t_4 + t_1^2 t_3^2 t_4 + t_1^2 t_3^2 t_4 + t_1^2 t_3^2 t_4 + t_1^2 t_3^2 t_4 + t_1^2 t_3^2 t_4 + t_1^2 t_3^2 t_4 + t_1^2 t_3^2 t_4 + t_1^2 t_3^2 t_4 + t_1^2 t_3^2 t_4 + \cdots.$$ 

Remark 3.5. Note that $\text{Hilb}_{-a,b}^a(t) = \text{Hilb}_{-a,-b}^a (t)$ up to permuting the $t_i$ to maintain the convention that negative weights appear first in $a$. Hence, in applications of Theorem 3.2, one may always reduce to a case where $b \geq 0$ (and, if $b = 0$ and all weights have the same sign, all weights are negative) so that $S_{a,b} = \emptyset$. Therefore, the maximally graded Hilbert series can always be computed using only the first sum of equation (3.1).

Let us illustrate how Remark 3.5 can be used to simplify computations.

Example 3.6. Consider the representation $V_a$ with weight vector $a = (-2, 2, 5)$ and $b = -4$. Applying Theorem 3.2 directly, the first sum in equation (3.1) is equal to

$$\frac{1}{t_1^2(1 - t_1 t_2)(1 - t_2 t_3)},$$

the set $S_{(-2,2,5),-4} = \{(1, 0, 0); (0, 1, 0)\}$, and so the second sum is equal to

$$\frac{-1}{t_1^2} - \frac{t_2}{t_1}.$$ 

This yields a Hilbert series of

$$\text{Hilb}_{(-2,2,5),-4}^a (t_1, t_2, t_3) = \frac{t_2^2 + t_1^2 t_3^2}{(1 - t_1 t_2)(1 - t_1 t_3)}.$$
To avoid consideration of $S_{(-2,2,5),-4}$, one may instead consider $-a$ and $-b$ (permuting coordinates to conform to our sign conventions). In this case, the second sum in equation (3.1) is empty and the first yields

$$\text{Hilb}^{G_1}_{(-2, -5, 2), 4}(t_1, t_2, t_3) = \frac{t_1^2 + t_2^2 t_3 - t_1 t_2^2 t_3^5}{(1 - t_1 t_3)(1 - t_1^2 t_3^2)},$$

which is the same expression up to permuting variables.

**Remark 3.7.** The Hilbert series of covariants can be computed in terms of the Hilbert series of invariants by observing that the covariants $\text{Mor}(V_a, W_{-b})^{G_1}$ are precisely the invariants of the representation with weight $(a, b)$ that are linear in the variable corresponding to the weight $b$. Hence, up to permuting the $t_i$ to maintain our convention about ordering the weights, $\text{Hilb}^{G_1}_{a, b}(t)$ is the degree 1 coefficient of the series expansion of $\text{Hilb}^{G_1}_{(a, b), 0}(t)$ at $t_n+1 = 0$ where $t_n+1$ is the variable associated to the last weight $b$. In practice, adding the additional weight $b$ increases the computation time so that applying Theorem 3.2 directly is faster.

**Example 3.8.** To illustrate Remark 3.7, let $a = (-1, 2, 3)$ and $b = 3$. The second sum in equation (3.1) is empty, and the first yields

$$\text{Hilb}^{G_1}_{(-1,2,3), 3}(t_1, t_2, t_3) = \frac{t_1^3}{(1 - t_1^2 t_2)(1 - t_1^3 t_3)}.$$

Alternatively, one can compute the Hilbert series of the invariants of the representation with weight matrix $(-1, 2, 3, 3)$, yielding

$$\text{Hilb}^{G_1}_{(-1,2,3,3), 0}(t_1, t_2, t_3, t_4) = \frac{1}{(1 - t_1^2 t_2)(1 - t_1^3 t_3)(1 - t_1^4 t_4)}.$$

Expanding at $t_4 = 0$ and extracting the coefficient of $t_4$ recovers $\text{Hilb}^{G_1}_{(-1,2,3), 3}(t_1, t_2, t_3)$.

Substituting $t_i = t$ in the expression in Theorem 3.2 and applying the analytic continuation argument of [30, Section 3.3] in the case when the $\alpha_i$ for $1 \leq i \leq k$ are not distinct yields the univariate Hilbert series of the covariants. In particular, setting $b = 0$, this recovers the expression for the Hilbert series of the invariants in [19, Theorem 3.3].

**Corollary 3.9 (Univariate Hilbert series of $S^1$-covariants).** Let $V \simeq \mathbb{C}^n$ be a representation of $S^1$ with weight vector $a = (-\alpha_1, \ldots, -\alpha_k, \alpha_{k+1}, \ldots, \alpha_n)$ where each $\alpha_i > 0$, and let $W \simeq \mathbb{C}$ be the irreducible representation of the circle with weight $-b$. The univariate Hilbert series $\text{Hilb}^{G_1}_{a, b}(t)$ of the module of covariants $\text{Mor}(V, W)^{G_1} = (\mathbb{C}[V] \otimes W)^{G_1}$ is given by

$$\text{Hilb}^{G_1}_{a, b}(t) = \lim_{(c_1, \ldots, c_n) \to a} \sum_{\xi_i \geq 1} \frac{t^{b/\alpha_i}}{\xi_1 \prod_{j=1}^{n} (1 - \xi^{a_j (\alpha_i - \xi_j) / c_i})} + (-1)^k \sum_{\eta \in S_{a, b}} t^{-\sum_{i=1}^{k}(y_i+1)+\sum_{i=k+1}^{n}y_i},$$

where $S_{a, b}$ is as in Theorem 3.2.

### 3.2. The $\mathbb{N}^2$-grading of a cotangent-lifted representation and the symplectic quotient

Suppose $V \simeq \mathbb{C}^{2n}$ is a cotangent-lifted representation of $S^1$, meaning that the weight vector is of the form $a^c = (a, -a)$ for a weight vector $a \in \mathbb{Z}^n$. In this case, we use the notation

$$a^c = (a_1, \ldots, a_n, -a_1, \ldots, -a_n) = (-\alpha_1, \ldots, -\alpha_k, \alpha_{k+1}, \ldots, \alpha_n, \alpha_1, \ldots, \alpha_k, -\alpha_{k+1}, \ldots, -\alpha_n),$$
continuing the convention that each $\alpha_j = |a_j|$. We again assume that no weight is zero, i.e., that $V$ has no trivial factors, to avoid trivialities.

We consider the decomposition $V_{a^*} = V_1 \oplus V_2$ where $V_1$ has weight vector $a$ and $V_2$ has weight vector $-a$ and let $(d_1, d_2)$ denote the corresponding grading with formal variables $(s, t)$. Then we have the following.

**Corollary 3.10** (Bivariate Hilbert series of $S^1$-covariants of a cotangent-lifted representation). Let $V \simeq \mathbb{C}^n$ be a representation of $S^1$ with weight vector $a = (-\alpha_1, \ldots, -\alpha_k, \alpha_{k+1}, \ldots, \alpha_n)$ where each $\alpha_i > 0$, let $a^* = (a, -a)$ denote the weight vector of the cotangent lift of $V$, and let $W \simeq \mathbb{C}$ be the irreducible representation of the circle with weight $-b$. Let $\text{Hilb}_{a^*:b}^{S^1}(s, t)$ denote the bivariate Hilbert series of the covariants $\text{Mor}(V \oplus V^*, W)^{S^1} = \langle \mathbb{C}[V \oplus V^*] \otimes W \rangle^{S^1}$. Then

$$\text{Hilb}_{a^*:b}^{S^1}(s, t) = \lim_{(\epsilon_1, \ldots, \epsilon_n) \to a} \frac{1}{1-st} \left( \sum_{i=1}^{k} \frac{\zeta^{b_{i}/a_i}}{\alpha_i \prod_{j=1}^{n} (1 - \zeta^{a_j t(c_j/c_i)})(1 - \zeta^{-a_j s(c_j/c_i)t})} + \sum_{i=k+1}^{n} \frac{\zeta^{b_{i}/a_i}}{\alpha_i \prod_{j=1}^{n} (1 - \zeta^{a_j t(c_j/c_i)})(1 - \zeta^{-a_j s(c_j/c_i)t})} \right) + \sum_{(y, y') \in S_{a^*,b}} (-1)^n \prod_{i=1}^{k} s^{-y_i} t^{y'_i} \prod_{i=k+1}^{n} s^{y_i} t^{-y'_i} \] (3.6)

where

$$S_{a^*,b} = \left\{ (y, y') \in \mathbb{N}^2 : \sum_{j=1}^{n} (y_j + y'_j)\alpha_j = -b - \sum_{j=1}^{n} \alpha_j \right\},$$

(3.7)

which is empty if $b + \sum_{j=1}^{n} \alpha_j > 0$. If the $\alpha_i$ are distinct, then the limit is unnecessary and equation (3.6) becomes

$$\text{Hilb}_{a^*:b}^{S^1}(s, t) = \frac{1}{1-st} \left( \sum_{i=1}^{k} \frac{\zeta^{b_{i}/a_i}}{\alpha_i \prod_{j=1}^{n} (1 - \zeta^{a_j t(c_j/c_i)})(1 - \zeta^{-a_j s(c_j/c_i)t})} + \sum_{i=k+1}^{n} \frac{\zeta^{b_{i}/a_i}}{\alpha_i \prod_{j=1}^{n} (1 - \zeta^{a_j t(c_j/c_i)})(1 - \zeta^{-a_j s(c_j/c_i)t})} \right) + \sum_{(y, y') \in S_{a^*,b}} (-1)^n \prod_{i=1}^{k} s^{-y_i} t^{y'_i} \prod_{i=k+1}^{n} s^{y_i} t^{-y'_i} \]
Proof. Using the weight vector \( \mathbf{a}^c \) and substituting \( t_i = s \) for \( 1 \leq i \leq n \) and \( t_i = t \) for \( n + 1 \leq i \leq 2n \), the first sum in equation (3.1) becomes

\[
\sum_{i=1}^{k} \frac{(1 - st)^{-1} \zeta^{b_i} b_i/\alpha_i}{1 - j \neq i} \alpha_i \prod_{j=1}^{n} \left( 1 - \zeta^{-a_j} s^{-1} \frac{1}{a_j} \right) (1 - \zeta^{-a_j} a_j/\alpha_j) t_i = 1
\]

Note that if the \( a_i \) are not distinct, then the factors of the form \( (1 - \zeta^{-a_j} s^{-1} \frac{1}{a_j}) \) introduce singularities when \( a_i = a_j \); in this case, we apply the analytic continuation argument of [30, Section 3.3] and take the limit as parameters \( c_i \) approach the \( a_i \). If \( b + \sum_{j=1}^{n} a_j \leq 0 \), then we specialize equation (3.2) to the case at hand as equation (3.7), and the second sum in equation (3.1) becomes

\[
(-1)^n \sum_{(y, y')} \prod_{i=1}^{k} s^{-y_i - 1} \prod_{i=k+1}^{n} t^{-y_i - 1} \prod_{i=1}^{n} \frac{a_i}{b_i} \prod_{i=k+1}^{n} \frac{a_i}{c_i}
\]

completing the proof.

Remark 3.11. Using the observation of Remark 3.5 and the fact that \( \mathbf{a}^c = -\mathbf{a}^c \) up to permuting weights, we have \( \text{Hilb}_{\mathbf{a}^c; b}^{\text{Gl}}(s, t) = \text{Hilb}_{\mathbf{a}^c; -b}^{\text{Gl}}(t, s) \). Hence, applications of Corollary 3.10 can always be reduced to a case where \( \mathbf{S}_{\mathbf{a}^c; b} = \emptyset \).

Taking the limit as \( s \to t \) and, for \( 1 \leq i \leq k \), relabeling \( \zeta \) as \( \zeta^{-1} \) to permute terms and express them uniformly, we obtain the following.

Corollary 3.12 (Univariate Hilbert series of \( S^1 \)-covariants of a cotangent-lifted representation). Let \( V \simeq \mathbb{C}^n \) be a representation of \( S^1 \) with weight vector \( \mathbf{a} = (-\alpha_1, \ldots, -\alpha_s, \alpha_{s+1}, \ldots, \alpha_n) \) where each \( \alpha_i > 0 \), let \( \mathbf{a}^c = (\mathbf{a}, -\mathbf{a}) \) denote the weight vector of the cotangent lift of \( V \), and let \( W \simeq \mathbb{C} \) be the irreducible representation of the circle with weight \( -b \). Let \( \text{Hilb}_{\mathbf{a}^c; b}^{\text{Gl}}(t) \) denote the univariate Hilbert series of the covariants \( \text{Mor}(V \oplus V^*, W) \mathbb{C}^{\text{Gl}} = (\mathbb{C}[V \oplus V^*] \otimes W) \mathbb{C}^{\text{Gl}} \). Then

\[
\text{Hilb}_{\mathbf{a}^c; b}^{\text{Gl}}(t) = \lim_{(\epsilon_1, \ldots, \epsilon_n) \to \mathbf{a}} \frac{1}{1 - t^2} \sum_{\epsilon^a = 1}^{n} \frac{\zeta^{a_i b_i / \alpha_i} t^{b_i / \alpha_i}}{1 - j \neq i} \alpha_i \prod_{j=1}^{n} \left( 1 - \zeta^{-a_j} t^{(1 - \epsilon_j - \epsilon_j)/\epsilon_j} (1 - \zeta^{-a_j} \epsilon_j + \epsilon_j) / \epsilon_j \right)
\]

\[
+ (-1)^n \sum_{(y, y') \in \mathbf{S}_{\mathbf{a}^c; b}} \prod_{i=1}^{k} \frac{t^{y_i - 1}}{y_i - 1} \prod_{i=k+1}^{n} \frac{t^{y_i - 1}}{y_i - 1}, \quad (3.8)
\]

where \( \mathbf{S}_{\mathbf{a}^c; b} \) is as defined in equation (3.7). If the \( \alpha_i \) are distinct, then the limit is unnecessary and the right-hand side of equation (3.8) becomes

\[
\frac{1}{1 - t^2} \sum_{\epsilon^a = 1}^{n} \frac{\zeta^{a_i b_i / \alpha_i} t^{b_i / \alpha_i}}{1 - j \neq i} \alpha_i \prod_{j=1}^{n} \left( 1 - \zeta^{-a_j} t^{(1 - \epsilon_j - \epsilon_j)/\epsilon_j} (1 - \zeta^{-a_j} \epsilon_j + \epsilon_j) / \epsilon_j \right) + (-1)^n \sum_{(y, y') \in \mathbf{S}_{\mathbf{a}^c; b}} \prod_{i=1}^{k} \frac{t^{y_i - 1}}{y_i - 1} \prod_{i=k+1}^{n} \frac{t^{y_i - 1}}{y_i - 1}.
\]
In the case of the on-shell invariants associated to the symplectic quotient, the moment map associated to \( a \) in complex coordinates \( x = (x_1, \ldots, x_n) \) for \( V \) is given by
\[
J(x) = \sum_{i=1}^{n} a_i x_i \bar{x}_i
\]
and hence has degree \((1, 1)\). Then \( \text{Hilb}_{a}^{S^1, \text{on}}(s, t) = (1 - st) \text{Hilb}_{a}^{S^1, \text{off}}(s, t) \), where \( \text{Hilb}_{a}^{S^1, \text{off}}(s, t) = \text{Hilb}_{a,0}^{S^1}(s, t) \) denotes the corresponding Hilbert series of off-shell invariants; see [36, Proposition 2.1]. We therefore have the following.

**Corollary 3.13** (Bivariate Hilbert series of the on-shell invariants of an \( S^1 \)-symplectic quotient). Let \( V \cong \mathbb{C}^n \) be a representation of \( S^1 \) with weight vector \( a = (-\alpha_1, \ldots, -\alpha_k, \alpha_{k+1}, \ldots, \alpha_n) \) where each \( \alpha_i > 0 \). Let \( \text{Hilb}_{a}^{S^1, \text{on}}(s, t) \) denote the bivariate Hilbert series of the on-shell invariants of the corresponding symplectic quotient. Then

\[
\text{Hilb}_{a}^{S^1, \text{on}}(s, t) = \lim_{(c_1, \ldots, c_n) \to a} \left( \sum_{i=1}^{k} \frac{1}{\alpha_i} \frac{1}{\prod_{j=1}^{n} (1 - \zeta^{a_i s (c_i - \epsilon_j) / c_j} (1 - \zeta^{-a_i s (c_i - \epsilon_j) / c_j} t)} + \sum_{i=1}^{k+1} \frac{1}{\alpha_i} \frac{1}{\prod_{j=1}^{n} (1 - \zeta^{a_i s (c_i - \epsilon_j) / c_j} (1 - \zeta^{-a_i s (c_i - \epsilon_j) / c_j} t)} \right). \tag{3.9}
\]

If the \( a_i \) are distinct, then \( \text{Hilb}_{a}^{S^1, \text{on}}(s, t) \) is given by

\[
\sum_{i=1}^{k} \frac{1}{\alpha_i} \frac{1}{\prod_{j=1}^{n} (1 - \zeta^{a_i s (a_i - a_j) / a_i} (1 - \zeta^{-a_i s (a_i - a_j) / a_i} t) + \sum_{i=1}^{k+1} \frac{1}{\alpha_i} \frac{1}{\prod_{j=1}^{n} (1 - \zeta^{a_i s (a_i - a_j) / a_i} (1 - \zeta^{-a_i s (a_i - a_j) / a_i} t)} \tag{3.10}
\]

Taking the limit as \( s \to t \) and again relabeling \( \zeta \) as \( \zeta^{-1} \) for \( 1 \leq i \leq k \), we recover the univariate Hilbert series of [36, Theorem 3.1].

### 3.3. Laurent coefficients of the univariate Hilbert series

Let \( V = V_a \) be a representation of \( S^1 \) with \( a \in \mathbb{Z}^n \), let \( W = W_{-b} \) be an irreducible representation of \( S^1 \), and let \( \text{Hilb}_{a,b}^{S^1}(t) \) denote the univariate Hilbert series as computed in Corollary 3.9. We consider the Laurent expansion at \( t = 1 \) of the form

\[
\text{Hilb}_{a,b}^{S^1}(t) = \sum_{m=0}^{\infty} \gamma_m^{S^1}(a; b)(1 - t)^{m-n+1}.
\]

We will use the simplified notation \( \gamma_m = \gamma_m^{S^1}(a; b) \) when the representations are clear from the context. Note that we index the \( \gamma_m \) so that \( \gamma_0 \) is the coefficient of degree \( 1 - n \); the pole order of \( \text{Hilb}_{a,b}^{S^1}(t) \) is at most \( n - 1 \) and usually obtains this value, but it is possible that \( \gamma_0 = 0 \), e.g., if all weights have the
same sign. Our goal in this section is to compute \( \gamma_0 \) and \( \gamma_1 \) in general as well as when \( V \) is a cotangent lift. When \( b = 0 \), these coefficients along with \( \gamma_2 \) and \( \gamma_3 \) were computed in [36, Section 5] and [19, Section 6]; as we will see, the coefficients in the case of \( b \neq 0 \) are not substantially different.

It is difficult to uniformly treat the cases where all weights have the same sign, so let us first consider these cases separately; see Remark 3.3. If all weights have the same sign and \( b = 0 \), then \( \text{Hilb}^{\odot 1}_{a,b}(t) = 1 \). Hence \( \gamma_{n-1} = 1 \) and all other \( \gamma_m = 0 \). If all weights are positive and \( b > 0 \), then \( \text{Hilb}^{\odot 1}_{a,b}(t) = 0 \), so all \( \gamma_m \) vanish. If all weights are positive and \( b < 0 \), then \( \text{Hilb}^{\odot 1}_{a,b}(t) \) is a polynomial. First assume \( n = 1 \). By Remark 3.1, \( \text{Hilb}^{\odot 1}_{(a_i),b}(t) = 0 \) if the single weight \( a_1 \) does not divide \( b \) and otherwise \( \text{Hilb}^{\odot 1}_{(a_i),b}(t) = \text{Hilb}^{\odot 1}_{1b/ai}(t) = t^{-b/a_1} \). In the former case, all \( \gamma_m \) vanish, while in the latter case, \( \gamma_0 = 1 \) and \( \gamma_1 = b/a_1 \). If \( n = 2 \), then \( \gamma_0 = 0 \) and \( \gamma_1 \) is a nonnegative integer that counts the number of solutions to \( y_1 \alpha_1 + y_2 \alpha_2 = -b \). If \( n > 2 \), then \( \gamma_0 = \gamma_1 = 0 \). Cases with all weights negative can be reduced to those above using the fact that \( \text{Hilb}^{\odot 1}_{a,b}(t) = \text{Hilb}^{\odot 1}_{-a,-b}(t) \); see Remark 3.5.

We now assume that \( a \) contains at least one weight of each sign so that \( n \geq 2 \). We assume without loss of generality that \( V \) is faithful so that \( \gcd(a_1, \ldots, a_n) = 1 \), see Remark 3.1, and \( b \geq 0 \), see Remark 3.5.

By \( \text{sp}_u(a) \), we mean the partial Schur-Laurent polynomial, see equation (2.2), where the first set of variables is the set of negative weights and the second is the set of positive weights. We let \( \mathbf{e}_1(a) = \sum_{i=1}^n a_i \) denote the degree 1 elementary symmetric polynomial in the weights.

**Theorem 3.14 (\( \gamma_0 \) and \( \gamma_1 \) for \( \mathbb{S}^1 \)-covariants).** Let \( V \cong \mathbb{C}^n \) be a faithful representation of \( \mathbb{S}^1 \) with weight vector \( a = (-\alpha_1, \ldots, -\alpha_k, \alpha_{k+1}, \ldots, \alpha_n) \) where each \( \alpha_i > 0 \), \( n \geq 2 \), and \( 1 \leq k < n \), and let \( W \cong \mathbb{C} \) be the irreducible representation of the circle with weight \( -b \) where \( b \geq 0 \). Then the first two Laurent coefficients \( \gamma_0^{\mathbb{S}^1}(a;b) \) and \( \gamma_1^{\mathbb{S}^1}(a;b) \) of the Hilbert series \( \text{Hilb}^{\odot 1}_{a,b}(t) \) of covariants \( \text{Mor}(V, W)^{\mathbb{S}^1} = (\mathbb{C}[V] \otimes W)^{\mathbb{S}^1} \) are given by

\[
\gamma_0^{\mathbb{S}^1}(a;b) = \frac{-\text{sp}_{n-2}(a)}{\prod_{p=1}^k \prod_{q=k+1}^n (a_p - a_q)}, \quad \text{and} \\
\gamma_1^{\mathbb{S}^1}(a;b) = \frac{(\mathbf{e}_1(a) - 2b)\text{sp}_{n-3}(a) - \text{sp}_{n-2}(a)}{2 \prod_{p=1}^k \prod_{q=k+1}^n (a_p - a_q)} + \sum_{j=1}^n \left( \frac{2[b\cdot g_j^{-1} b_j]_j - g_j - 1}{2} \right) \gamma_0^{\mathbb{S}^1}(a_j;b),
\]

where \( g_j = \gcd(a_i : i \neq j) \), \( a_i \in \mathbb{Z}^{n-1} \) is the weight vector formed by removing \( a_j \) from \( a \), \( [r]_s^{-1} \) denotes the multiplicative inverse of \( r \) mod \( s \), and \( [r]_s \) is the representative of the equivalence class of \( r \) mod \( s \) such that \( 1 \leq |r|_s \leq s \). When the negative weights in \( a \) are distinct, the coefficients can be expressed as

\[
\gamma_0^{\mathbb{S}^1}(a;b) = \sum_{i=1}^k \frac{-a_i^{n-2}}{\prod_{j=1}^n (a_i - a_j),} \quad \text{and} \\
\gamma_1^{\mathbb{S}^1}(a;b) = \sum_{i=1}^k \frac{a_i^{n-3} \left( -2b + \sum_{j=1}^n a_j \right)}{2 \prod_{\ell=1, \ell \neq i}^n (a_i - a_\ell)} + \sum_{i=1}^k \sum_{j=1}^n \sum_{\ell=1, \ell \neq i,j} \left( \frac{2[b\cdot g_j^{-1} b_j]_j - g_j - 1}{2} \right) \frac{-a_i^{n-3}}{\prod_{\ell=1, \ell \neq i,j}^n (a_i - a_\ell)}.
\]

(3.11)

In particular, \( \gamma_0^{\mathbb{S}^1}(a;b) = \gamma_0^{\mathbb{S}^1}(a;0) \), where \( \gamma_0^{\mathbb{S}^1}(a;0) \) was computed in [19, Theorems 6.2 and 6.3].
Proof. The hypotheses ensure that the second sum in equation (3.5) is empty, so we consider the Laurent expansion of the first sum. The maximal pole order of a term is $n - 1$ and occurs when $\zeta^{a_j} = 1$ for each $j$; as $\gcd(a_1, \ldots, a_n) = 1$, this implies $\zeta = 1$. Such a term is of the form

$$ t^{b/\alpha_i} \frac{n}{\prod_{\ell=1}^{n} (1 - t^{(c_i - c_\ell)/c_i})}, $$

(3.12)

and because the Laurent expansion of $t^{b/\alpha_i}$ begins

$$ t^{b/\alpha_i} = 1 - \frac{b}{\alpha_i} (1 - t) + \frac{b(b - \alpha_i)}{2\alpha_i^2} (1 - t)^2 \pm \cdots, $$

(3.13)

applying the Cauchy product formula, the degree $1 - n$ Laurent coefficient of the term in equation (3.12) is that of

$$ \frac{1}{\alpha_i \prod_{\ell=1}^{n} (1 - t^{(c_i - c_\ell)/c_i})}. $$

In particular, this implies $\gamma_0^1(a; b) = \gamma_0^1(a; 0)$, where the latter was computed in [19, Theorem 6.2].

The computation of $\gamma_1^1(a; b)$ is similar to [19, Theorem 6.3]. We first consider the contribution of terms with $\zeta = 1$ for the given equation (3.12). For simplicity, assume the negative weights of $a$ are distinct and let $c_j = a_j$ for each $j$. Using the Cauchy product formula, the degree $2 - n$ coefficient of such a term is

$$ a_i^{-3} \left( -2b + \sum_{j=1}^{n} a_j \right), $$

and for each $g_j$th root of unity $\zeta \neq 1$, the degree $2 - n$ coefficient of such a term is given by

$$ \zeta^{a_i} \frac{a_i^{-3} \left( -2b + \sum_{j=1}^{n} a_j \right)}{\alpha_i (1 - \zeta^{a_j} \prod_{\ell=1}^{n} (1 - t^{(c_i - c_\ell)/c_i}))}. $$

(3.14)

The only other contributions to $\gamma_1^1(a; b)$ are from terms where $\zeta \neq 1$ is an $\alpha_i$th root of unity such that $\zeta^{\alpha_i} = 1$ for all $\ell$ except for one, say $\zeta^{a_j} \neq 1$. Such a term is then of the form

$$ \zeta^{b/\alpha_i} \frac{a_i^{-3} \left( -2b + \sum_{j=1}^{n} a_j \right)}{\alpha_i (1 - \zeta^{a_j} \prod_{\ell=1}^{n} (1 - t^{(c_i - c_\ell)/c_i}))}. $$

(3.15)

and occurs for each $g_j$th root of unity $\zeta \neq 1$. The degree $2 - n$ coefficient of such a term is given by

$$ \sum_{\zeta^{a_j} = 1} \frac{\zeta^{b/\alpha_i} a_i^{-3} \left( -2b + \sum_{j=1}^{n} a_j \right)}{\alpha_i (1 - \zeta^{a_j} \prod_{\ell=1}^{n} (1 - t^{(c_i - c_\ell)/c_i}))} = \sum_{\zeta^{a_j} = 1} \frac{\zeta^{b/\alpha_i} a_i^{-3} \left( -2b + \sum_{j=1}^{n} a_j \right)}{\alpha_i (1 - \zeta^{a_j} \prod_{\ell=1}^{n} (1 - t^{(c_i - c_\ell)/c_i}))}. $$

(3.16)

Noting that $g_j$ and $a_j$ are relatively prime by construction, we have

$$ \sum_{\zeta^{g_j} = 1} \frac{\zeta^{b/\alpha_i} a_i^{-3} \left( -2b + \sum_{j=1}^{n} a_j \right)}{\alpha_i (1 - \zeta^{g_j} \prod_{\ell=1}^{n} (1 - t^{(c_i - c_\ell)/c_i}))} = \sum_{\zeta^{g_j} = 1} \frac{\zeta^{b/\alpha_i} a_i^{-3} \left( -2b + \sum_{j=1}^{n} a_j \right)}{\alpha_i (1 - \zeta^{g_j} \prod_{\ell=1}^{n} (1 - t^{(c_i - c_\ell)/c_i}))}. $$

(3.17)

The computation of $\gamma_0^1(a; b)$ is similar to [19, Theorem 6.3]. We first consider the contribution of terms with $\zeta = 1$ for the given equation (3.12). For simplicity, assume the negative weights of $a$ are distinct and let $c_j = a_j$ for each $j$. Using the Cauchy product formula, the degree $2 - n$ coefficient of such a term is

$$ a_i^{-3} \left( -2b + \sum_{j=1}^{n} a_j \right), $$

and for each $g_j$th root of unity $\zeta \neq 1$, the degree $2 - n$ coefficient of such a term is given by

$$ \zeta^{b/\alpha_i} \frac{a_i^{-3} \left( -2b + \sum_{j=1}^{n} a_j \right)}{\alpha_i (1 - \zeta^{a_j} \prod_{\ell=1}^{n} (1 - t^{(c_i - c_\ell)/c_i}))}. $$

(3.18)

The only other contributions to $\gamma_0^1(a; b)$ are from terms where $\zeta \neq 1$ is an $\alpha_i$th root of unity such that $\zeta^{\alpha_i} = 1$ for all $\ell$ except for one, say $\zeta^{a_j} \neq 1$. Such a term is then of the form

$$ \zeta^{b/\alpha_i} \frac{a_i^{-3} \left( -2b + \sum_{j=1}^{n} a_j \right)}{\alpha_i (1 - \zeta^{a_j} \prod_{\ell=1}^{n} (1 - t^{(c_i - c_\ell)/c_i}))}. $$

(3.19)

and occurs for each $g_j$th root of unity $\zeta \neq 1$. The degree $2 - n$ coefficient of such a term is given by

$$ \sum_{\zeta^{a_j} = 1} \frac{\zeta^{b/\alpha_i} a_i^{-3} \left( -2b + \sum_{j=1}^{n} a_j \right)}{\alpha_i (1 - \zeta^{a_j} \prod_{\ell=1}^{n} (1 - t^{(c_i - c_\ell)/c_i}))} = \sum_{\zeta^{a_j} = 1} \frac{\zeta^{b/\alpha_i} a_i^{-3} \left( -2b + \sum_{j=1}^{n} a_j \right)}{\alpha_i (1 - \zeta^{a_j} \prod_{\ell=1}^{n} (1 - t^{(c_i - c_\ell)/c_i}))}. $$

(3.20)

Noting that $g_j$ and $a_j$ are relatively prime by construction, we have

$$ \sum_{\zeta^{g_j} = 1} \frac{\zeta^{b/\alpha_i} a_i^{-3} \left( -2b + \sum_{j=1}^{n} a_j \right)}{\alpha_i (1 - \zeta^{g_j} \prod_{\ell=1}^{n} (1 - t^{(c_i - c_\ell)/c_i}))} = \sum_{\zeta^{g_j} = 1} \frac{\zeta^{b/\alpha_i} a_i^{-3} \left( -2b + \sum_{j=1}^{n} a_j \right)}{\alpha_i (1 - \zeta^{g_j} \prod_{\ell=1}^{n} (1 - t^{(c_i - c_\ell)/c_i}))}. $$

(3.21)
where the last equation follows from [25, Corollary 3.2]. Hence the resulting contribution to \( \gamma_1^{S^1}(a; b) \) is

\[
\frac{-a_i^{n-3}}{\prod_{\ell=1}^{n} (a_i - a_{\ell})} \left( \frac{2 \left| a_j \right|_{g_j}^{-1} b_{g_j} - g_j - 1}{2} \right).
\]

Summing over \( i \) and \( j \) yields equation (3.11).

To express \( \gamma_1^{S^1}(a; b) \) in terms of partial Schur-Laurent polynomials, first note that

\[
\sum_{i=1}^{k} \frac{a_i^{n}}{\prod_{\ell=1}^{n} (a_i - a_{\ell})} = \frac{sp_{n+1}(a)}{\prod_{p=1}^{n} \prod_{q=k+1}^{n} (a_p - a_q)},
\]

see [19, equation (5.2)]. We compute

\[
\sum_{i=1}^{k} \frac{a_i^{n-3}}{2 \prod_{\ell=1}^{n} (a_i - a_{\ell})} \left( -2b + \sum_{j=1}^{n} a_j \right) + \sum_{i=1}^{k} \sum_{j=1}^{n} \left( \frac{2 \left| a_j \right|_{g_j}^{-1} b_{g_j} - g_j - 1}{2} \right) \frac{-a_i^{n-3}}{\prod_{\ell=1}^{n} (a_i - a_{\ell})} \\
= \sum_{i=1}^{k} \left( \frac{e_1(a)}{2 \prod_{\ell=1}^{n} (a_i - a_{\ell})} - \frac{2b}{2 \prod_{\ell=1}^{n} (a_i - a_{\ell})} \right) + \sum_{j=1}^{n} \left( \frac{2 \left| a_j \right|_{g_j}^{-1} b_{g_j} - g_j - 1}{2} \right) \gamma_0^{S^1}(a; b),
\]

\[
= \sum_{i=1}^{k} \left( e_1(a) - 2b \right) sp_{n-3}(a) - sp_{n-2}(a) + \sum_{j=1}^{n} \left( \frac{2 \left| a_j \right|_{g_j}^{-1} b_{g_j} - g_j - 1}{2} \right) \gamma_0^{S^1}(a; b).
\]

We now consider the case of a cotangent-lifted representation and compute \( \gamma_0^{S^1}(a'; b) \) and \( \gamma_1^{S^1}(a'; b) \) where \( \gamma_0^{S^1}(a'; b) \) occurs in degree 2n - 1. In this case, we do not need to restrict \( n \) nor the signs of the weights. We continue to assume without loss of generality that \( b \geq 0 \) by Remark 3.5.

**Theorem 3.15** (\( \gamma_0 \) and \( \gamma_1 \) for \( S^1 \)-covariants of a cotangent-lifted representation). Let \( V \cong \mathbb{C}^n \) be a faithful representation of \( S^1 \) with weight vector \( a = (-\alpha_1, \ldots, -\alpha_k, \alpha_{k+1}, \ldots, \alpha_n) \) where each \( \alpha_i > 0 \), let \( a' = (a, -a) \) denote the weight vector of the cotangent lift of \( V \), let \( a = (a_1, \ldots, a_n) \), and let \( W \cong \mathbb{C} \) be the irreducible representation of the circle with weight \( -b \) where \( b \geq 0 \). Then the first two Laurent coefficients \( \gamma_0^{S^1}(a'; b) \) and \( \gamma_1^{S^1}(a'; b) \) of the Hilbert series \( \text{Hilb}_{a'; b}^{S^1}(t) \) of covariants \( \text{Mor}(V \oplus V^*, W)^{S^1} = (\mathbb{C}[V \oplus V^*] \otimes W)^{S^1} \) are given by

\[
\gamma_0^{S^1}(a'; b) = \frac{s(n-2, n-2, n-3, \ldots, 1, 0)(\alpha)}{2s(n-1, n-2, n-3, \ldots, 1, 0)(\alpha)}, \quad \text{and}
\]

\[
\gamma_1^{S^1}(a'; b) = \frac{\gamma_1^{S^1}(a'; b)}{2}.
\]
When the $\alpha_i$ are distinct, $\gamma_0^{\oplus_1}(a^i; b)$ can be expressed as

$$\gamma_0^{\oplus_1}(a^i; b) = \sum_{i=1}^{n} \frac{\alpha_i^{2n-3}}{2 \prod_{j=1}^{n} (\alpha_i^2 - \alpha_j^2)}.$$ 

In particular, $\gamma_m^{\oplus_1}(a^i; b) = \gamma_m^{\oplus_1}(a^i; 0)$ for $m = 1, 2$, where the $\gamma_m^{\oplus_1}(a^i; 0)$ were computed in [36, Theorem 5.1 and Remark 5.3].

**Proof.** We consider the Laurent expansion of equation (3.8), where our hypotheses imply that the second sum vanishes. The maximum pole order of a term

$$\frac{c_i^{2n-2}}{(1 - t^2)\alpha_i \prod_{j=1}^{n} (1 - t^{\alpha_j^2})},$$

is $2n - 1$ and occurs when each $\alpha_j^2 = 1$; this implies by the faithfulness of $V$ that $\zeta = 1$. Note that if $\zeta \neq 1$, then the pole order is at most $2n - 3$, so the only terms that contribute to $\gamma_0$ and $\gamma_1$ are of the form

$$\frac{c_i^{2n-2}}{(1 - t^2)\alpha_i \prod_{j=1}^{n} (1 - t^{(\zeta c_i^2 + \zeta^2 c_j^2)})}.$$ 

Applying the Cauchy product formula to such a term and summing over $i$, we obtain

$$\gamma_0^{\oplus_1}(a^i; b) = \lim_{(c_1, \ldots, c_n) \to a} \sum_{i=1}^{n} \frac{c_i^{2n-2}}{2\alpha_i \prod_{j=1}^{n} (c_i^2 - c_j^2)} = \lim_{(c_1, \ldots, c_n) \to a} \sum_{i=1}^{n} \frac{|c_i|^{2n-3}}{2 \prod_{j=1}^{n} (c_i^2 - c_j^2)},$$

which is shown to be equal to the expression in equation (3.14) in [36, Section 5.2].

To compute $\gamma_1^{\oplus_1}(a^i; b)$, we apply the Cauchy product formula to the term in equation (3.16) for each $i$, yielding

$$\gamma_1^{\oplus_1}(a^i; b) = \lim_{(c_1, \ldots, c_n) \to a} \sum_{i=1}^{n} \left[ \sum_{j=1}^{n} \frac{c_i^{2n-4} \left( c_i c_j - c_i c_j \right)}{2(\alpha_i^2 - \alpha_j^2)} + \frac{c_i^{2n-2}}{(\alpha_i c_i^2 + \alpha_j^2 c_j^2)} \right]$$

$$= \lim_{(c_1, \ldots, c_n) \to a} \sum_{i=1}^{n} \frac{|c_i|^{2n-3}}{4 \prod_{j=1}^{n} (c_i^2 - c_j^2)} - \frac{b}{2} \sum_{i=1}^{n} \frac{n c_i^{2n-4}}{\prod_{j=1}^{n} (c_i^2 - c_j^2)}.$$ 

The first of the resulting sums is equal to $\gamma_0^{\oplus_1}(a^i; b)/2$. To see that the second sum vanishes, rewrite

$$\sum_{i=1}^{n} \frac{n c_i^{2n-4}}{\prod_{j=1}^{n} (c_i^2 - c_j^2)} = \frac{\sum_{i=1}^{n} (-1)^{i-1} c_i^{2n-4} \prod_{1 \leq j < k \leq n} (c_j^2 - c_k^2)}{\prod_{1 \leq j < k \leq n} (c_j^2 - c_k^2)}.$$
and observe that the numerator is the cofactor expansion along the first row of a matrix whose first two rows are identical, both listing powers $c_i^{2n-1}$.

### 3.4. Laurent coefficients of multigraded Hilbert series

Using the approach of Section 3.3, we can also consider the Laurent coefficients of the multigraded Hilbert series. There are several approaches to Laurent expansions of multivariate functions, see [1] for a careful exposition, and the meaning of the coefficients is in this context less clear. Hence, we give some sample calculations of the first few iterated Laurent series coefficients of $\text{Hilb}_{a}^{G_{1},on}(s, t)$ computed in Corollary 3.13. For simplicity, we assume that $n \geq 3$ and the $a_i$ are distinct and hence consider equation (3.10); we furthermore assume that the weights are pairwise relatively prime, i.e., $\gcd(a_i, a_j) = 1$ for $i \neq j$, which in particular implies that $V$ is faithful. We let $\gamma_{i,j}^{G_{1},on,s,t}(a)$ denote the $j$th coefficient of the expansion at $t = 1$ of the $i$th coefficient of the expansion at $s = 1$, where $i = 0$ and $j = 0$ indicate the first nonzero coefficient of the corresponding expansion. Note that $\text{Hilb}_{a}^{G_{1},on}(s, t) = \text{Hilb}_{a}^{G_{1},on}(t, s)$ by Remark 3.11 so that in this instance, $\gamma_{i,j}^{G_{1},on,s,t}(a) = \gamma_{i,j}^{G_{1},on,t,s}(a)$ for all $i, j$. However, changing the expansion order yields alternate formulas for these coefficients.

We first expand in $s = 1$ and then $t = 1$. Choose $i \leq k$, and hence consider a term in the first sum of equation (3.10). Such a term has a pole of order at most $n - 1$ at $s = 1$, with this pole order obtained if and only if $\gamma_{n,i} = 1$ for each $j$. As $V$ is faithful, this implies that $\gamma = 1$. Note that if $\gamma \neq 1$, then there is no pole at $s = 1$ so that, as $n \geq 3$, the corresponding term will not contribute to the first two Laurent coefficients at $s = 1$.

Hence we consider a term of the form

$$1 \alpha_i \prod_{j=1}^{n} (1 - s^{(a_i - a_j)/a_i})(1 - s^{a_j/a_i t})$$

By the Cauchy product formula, the expansion at $s = 1$ of such a term begins

$$\frac{a_i^{n-1}}{\alpha_i (1 - t)^{n-1} \prod_{j=1, j \neq i}^{n} (a_i - a_j)}$$

$$+ \left( \sum_{j=1, j \neq i}^{n} \frac{-a_i^{n-2} a_j}{2\alpha_i (a_i - a_j)(1 - t)^{n-1} \prod_{\ell=1, \ell \neq i, j}^{n} (a_i - a_\ell)} \right) + \left( \sum_{j=1, j \neq i}^{n} \frac{-a_i^{n-2} a_j t}{\alpha_i (1 - t) \prod_{\ell=1, \ell \neq i}^{n} (a_i - a_\ell)} \right) (1 - s)^{2-n} + \cdots$$

$$= \frac{-a_i^{n-2}}{(1 - t)^{n-1} \prod_{j=1, j \neq i}^{n} (a_i - a_j)} (1 - s)^{1-n} + \sum_{j=1, j \neq i}^{n} \frac{a_i^{n-3} a_j (1 + t)}{2(1 - t)^{n} \prod_{\ell=1, \ell \neq i}^{n} (a_i - a_\ell)} (1 - s)^{2-n} + \cdots$$

Summing over the corresponding $i$ and expanding at $t = 1$ yields for the first term

$$(1 - s)^{-n} (1 - t)^{1-n} \sum_{i=1}^{k} \frac{-a_i^{n-2}}{\prod_{j=1, j \neq i}^{n} (a_i - a_j)}$$
and for the second term

\[(1 - s)^{2-n}(1 - t)^{-n} \sum_{i=1}^{k} \sum_{j=1}^{n} \frac{a_i^{n-3} a_j}{n \prod (a_i - a_\ell)} + (1 - s)^{2-n}(1 - t)^{1-n} \sum_{i=1}^{k} \sum_{j \neq i}^{n} \frac{-a_i^{n-3} a_j}{2 \prod (a_i - a_\ell)}.\]

When \(i > k\), the corresponding term is of the form

\[1 \prod_{j=1}^{n} (1 - \zeta^{a_i s t^{a_j} / a_i}) \prod_{j=1}^{n} (1 - \zeta^{-a_i t^{a_j} / a_i})\]

Such a term does not have a pole at \(s = 1\) regardless of the value of \(\zeta\) so, as \(n \geq 3\), will not contribute to the first two Laurent coefficients at \(s = 1\). Hence, we have

\[\gamma^S_{0,0,1,0}(a) = \sum_{i=1}^{k} \frac{-a_i^{n-2}}{n \prod_{j=1}^{n} (a_i - a_j)},\]

which is equal to \(\gamma^S_0(a; 0)\); see Theorem 3.14. Similarly,

\[\gamma^S_{1,0,1,0}(a) = \sum_{i=1}^{k} \sum_{j \neq i}^{n} \frac{a_i^{n-3} a_j}{n \prod (a_i - a_\ell)}, \quad \text{and} \quad \gamma^S_{1,1,1,0}(a) = \sum_{i=1}^{k} \sum_{j \neq i}^{n} \frac{-a_i^{n-3} a_j}{2 \prod (a_i - a_\ell)}.
\]

Expanding at \(t = 1\) and then \(s = 1\) is similar. If \(i \leq k\), then we have a term of the form

\[1 \prod_{j=1}^{n} (1 - \zeta^{a_i s^{a_j} / a_i}) (1 - \zeta^{-a_i t^{a_j} / a_i})\]

which does not have a pole at \(t = 1\) and hence, as \(n \geq 3\), will not contribute to the first two Laurent coefficients at \(t = 1\). If \(i \geq k + 1\), we again have a pole at \(t = 1\) only if \(\zeta = 1\), in which case we have a term of the form

\[1 \prod_{j=1}^{n} (1 - \zeta^{a_i s^{a_j} / a_i} t^{a_j} / a_i)\]

This term has a pole of order \(n - 1\) at \(t = 1\), and the expansion at \(t = 1\) begins

\[\frac{a_i^{n-1}}{\alpha_i (1 - s)^{n-1} \prod_{j=1}^{n} (a_i - a_j)} (1 - t)^{1-n} \]

\[+ \sum_{j=1}^{n} \frac{-a_i^{n-2} a_j}{2 \alpha_i (1 - s)^{n-1} \prod_{j=1}^{n} (a_i - a_\ell)} + \sum_{j=1}^{n} \frac{-a_i^{n-2} a_j s}{\alpha_i (1 - s)^{n} \prod_{\ell=1}^{n} (a_i - a_\ell)} (1 - t)^{2-n} + \cdots\]
In this section, we consider the Hilbert series of invariants and covariants of representations of $O_2$. Let here, we briefly recall the representation theory of $O_2$. For each root of unity, it is isomorphic to $O_2$.

4. Invariants and covariants of $O_2(\mathbb{R})$

In this section, we consider the Hilbert series of invariants and covariants of representations of $O_2(\mathbb{R})$. We consider $O_2(\mathbb{R})$ as the set of $2 \times 2$ real matrices of the form

$$r_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad \text{and} \quad s_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}.$$ 

4.1. The irreducible representations of $O_2(\mathbb{R})$

Here, we briefly recall the representation theory of $O_2(\mathbb{R})$ and refer the reader to [44, Theorem 7.2.1] or [41, Section 11.2] for more details.

For each $a \in \mathbb{Z}$, let $\tau_a : O_2(\mathbb{R}) \to U_2$ denote the representation given by

$$\tau_a : \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \mapsto \begin{pmatrix} e^{a\theta} \sqrt{-1} & 0 \\ 0 & e^{-a\theta} \sqrt{-1} \end{pmatrix}, \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \mapsto \begin{pmatrix} 0 & e^{a\theta} \sqrt{-1} \\ e^{-a\theta} \sqrt{-1} & 0 \end{pmatrix}.$$ 

Identifying $S^1$ with $SO_2(\mathbb{R}) \leq O_2(\mathbb{R})$ as above $\tau_a$ is the representation of $O_2(\mathbb{R})$ induced by the $S^1$-representation $\epsilon_a$ defined at the beginning of Section 3. The representation $\tau_0$ splits into a 1-dimensional trivial representation and the 1-dimensional representation det, and for $a > 0$, $\tau_a$ is equivalent to $\tau_{-a}$. Hence, every non-trivial finite-dimensional irreducible unitary representation of $O_2(\mathbb{R})$ is isomorphic to det or $\tau_a$ for an integer $a > 0$. Note that det has kernel $SO_2(\mathbb{R})$, and the kernel of $\tau_a$ for $a \geq 1$ is the set of $a$th roots of unity in $S^1 \simeq SO_2(\mathbb{R})$. In the latter case, the quotient of $O_2(\mathbb{R})$ by the finite group of $a$th roots of unity is isomorphic to $O_2(\mathbb{R})$, and the resulting faithful representation is $\tau_1$.

4.2. The Hilbert series of covariants of $O_2(\mathbb{R})$

Let $V$ be a finite-dimensional unitary representation of $O_2(\mathbb{R})$. As in Remark 3.1, we assume that $V$ has no trivial subrepresentations. Then there are integers $\alpha_1, \ldots, \alpha_n, d$ with $n \geq 0$ and each $\alpha_i > 0$ such that $V$ is of the form

$$V = \bigoplus_{i=1}^n \tau_{\alpha_i} \oplus d \det.$$
We will refer to this representation as $V_{\alpha,d}$ where $\alpha = (\alpha_1, \ldots, \alpha_n)$. Note that the corresponding weight vector for the action of the torus $SO_2(\mathbb{R}) \leq O_2(\mathbb{R})$ is $\alpha^c$ concatenated with $d$ zero entries. If $n = 0$, then $V$ is simply copies of det and $SO_2(\mathbb{R})$ acts trivially. If $n \geq 1$, then the kernel of the action is the set of $\gcd(\alpha_1, \ldots, \alpha_n)$th roots of unity. As in Remark 3.1, we will assume that $V$ is faithful, i.e., $n \geq 1$ and $\gcd(\alpha_1, \ldots, \alpha_n) = 1$.

**Theorem 4.1** (Maximally graded Hilbert series of $O_2(\mathbb{R})$-covariants). Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ where $n \geq 1$, each $\alpha_i > 0$, and $d \geq 0$; let $V_{\alpha,d} \simeq \mathbb{C}^{2n+d}$ be the corresponding faithful representation of $O_2(\mathbb{R})$; and let $W$ be an irreducible representation of $O_2(\mathbb{R})$ with character $\chi = \chi_W$. Then either $W = \tau_\beta$ for some $\beta > 0$ or $W$ is the trivial representation or det, where in the latter two cases we set $\beta = 0$. Let $t = (t_1, \ldots, t_{n+d})$, and let $\text{Hill}_{\alpha,d,W}(t)$ denote the Hilbert series for the maximal $\mathbb{N}^n$-grading of the covariants $\text{Mor}(V, W)^{O_2(\mathbb{R})} = (\mathbb{C}[V] \otimes W)^{O_2(\mathbb{R})}$. Then

\[
\text{Hill}_{\alpha,d,W}(t) = \frac{C_1}{2 \prod_{j=1}^d (1 - t_{n+j})} \sum_{i=1}^n \frac{\alpha_i (1 - t_i^2) \prod_{j=1}^d (1 - \zeta^{-\alpha_j} t_j^{-\alpha_j/\alpha_i})(1 - \zeta^{\alpha_j} t_j^{\alpha_j/\alpha_i})}{\zeta^{\beta_i^{\beta/\alpha_i}}} + \frac{C_2}{2 \prod_{j=1}^d (1 - t_j^2) \prod_{j=1}^d (1 + t_{n+j})},
\]

where

\[
C_1 = \begin{cases} 
1, & \text{if } W = \text{det or the trivial representation}, \\
2, & \text{if } W = \tau_\beta \text{ for } \beta > 0,
\end{cases}
\]

and

\[
C_2 = \begin{cases} 
-1, & \text{if } W = \text{det}, \\
0, & \text{if } W = \tau_\beta \text{ for } \beta > 0, \\
1, & \text{if } W \text{ is the trivial representation}.
\end{cases}
\]

**Proof.** Because $O_2(\mathbb{R})$ has two connected components, the Molien-Weyl Theorem yields an integral over each connected component with a prefactor of 1/2 for each; see [26, Section 4.1]. We first consider the case of a rotation $z \in O_2(\mathbb{R})^\circ = SO_2(\mathbb{R})$, which we consider as an element of $\mathbb{S}^1 \subset \mathbb{C}$ as above. If $W = \tau_\beta$ for some $\beta > 0$, then $\chi(z) = z^\beta + z^{-\beta}$; if $W$ is trivial or det, then $\chi(z) = 1$ is constant. Hence, the integral (with prefactor 1/2) over $SO_2(\mathbb{R})$ is given by

\[
\frac{1}{4\pi \sqrt{-1}} \int_{\mathbb{S}^1} \chi(z) \frac{dz}{z \prod_{j=1}^d (1 - t_j z^{\alpha_j})(1 - t_j z^{-\alpha_j}) \prod_{j=1}^d (1 - t_{n+j})} = \frac{1}{2 \prod_{j=1}^d (1 - t_{n+j})} \left( \frac{1}{2\pi \sqrt{-1}} \int_{\mathbb{S}^1} \chi(z) \frac{dz}{z \prod_{j=1}^d (1 - t_j z^{\alpha_j})(1 - t_j z^{-\alpha_j})} \right).
\]

On the connected component of reflections, $\chi(z)$ is constant; specifically, $\chi(z) = 0$ if $W = \tau_\beta$ for some $\beta > 0$, $\chi(z) = -1$ if $W = \text{det}$, and $\chi(z) = 1$ if $W$ is the trivial representation. Hence, denoting the
constant $\chi(z)$ simply as $\chi$, the integral (with prefactor $1/2$) is
\[
\frac{1}{4\pi \sqrt{-1}} \int_{S^1} \frac{\chi}{z \prod_{j=1}^{n} (1 - t_j^2) \prod_{j=1}^{d} (1 + t_{n+j})} \, dz = \frac{\chi}{4\pi \sqrt{-1} \prod_{j=1}^{n} (1 - t_j^2) \prod_{j=1}^{d} (1 + t_{n+j})} \int_{S^1} \frac{dz}{z}.
\]

The integral in the second line of equation (4.2) can be interpreted as a specialization of that of Theorem 3.2 with the weight vector $a = (-\alpha_1, \ldots, -\alpha_n, \alpha_1, \ldots, \alpha_n)$ and the substitution $t_{n+i} := t_i$.

When $W = \tau_{\beta}$, we apply the theorem with $b = \pm \beta$ and sum the results; when $W$ is the trivial representation or det, we use $b = 0$.

When $b \geq 0$, the assumption that each $\alpha_i > 0$ implies $b + \sum_{i=1}^{n} \alpha_i > 0$ so that the integral (along with the prefactor in parentheses) is
\[
\sum_{i=1}^{n} \frac{\alpha_i (1 - t_i^2) \prod_{j=1}^{n} (1 - \zeta^{-\alpha_i t_i^j \alpha_j / \alpha_i}) (1 - \zeta^{-\alpha_i t_i \alpha_j / \alpha_i})}{\prod_{j=1}^{d} (1 + t_{n+j})}.
\]

When $b < 0$, we can use the fact that $-a = a$, up to permuting the weights; see Remarks 3.5 and 3.11.

Then as the permutation of weights corresponds to transposing $t_i$ with $t_{n+i}$ for $1 \leq i \leq n$, after applying the substitution $t_{n+i} := t_i$, we obtain equation (4.4) again. Combining equations (4.2), (4.3), and (4.4) completes the proof.

Substituting $t_i = t$ in $\text{Hilb}_{O_2^{(\mathbb{R})}}^j(t)$ and, in the case that the $\alpha_i$ are not distinct, applying the analytic continuation argument of [30, Section 3.3], we obtain the univariate Hilbert series for the covariants of a unitary representation of $O_2(\mathbb{R})$.

**Corollary 4.2** (Univariate Hilbert series of $O_2(\mathbb{R})$-covariants). Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ where $n \geq 1$, each $\alpha_i > 0$, and $d \geq 0$, let $V_{\alpha,d} \simeq \mathbb{C}^{2n+d}$ be the corresponding faithful representation of $O_2(\mathbb{R})$, and let $W$ be an irreducible representation of $O_2(\mathbb{R})$. The univariate Hilbert series $\text{Hilb}_{O_2^{(\mathbb{R})}}(t)$ of the algebra of covariants $\text{Mor}(V, W)^{O_2(\mathbb{R})} = (\mathbb{C}[V] \otimes W)^{O_2(\mathbb{R})}$ is given by
\[
\lim_{(\alpha_1, \ldots, \alpha_n) \to \alpha} \frac{C_1}{2(1 - t)^d} \sum_{i=1}^{n} \frac{\zeta^{\beta i \alpha_i}}{\alpha_i (1 - t_i^2) \prod_{j=1}^{n} (1 - \zeta^{-\alpha_i t_i^j (\alpha_i - \alpha_j) / \alpha_i}) (1 - \zeta^{-\alpha_i t_i^j (\alpha_i + \alpha_j) / \alpha_i})} + \frac{C_2}{2(1 - t^2)^n(1 + t)^d},
\]

where $\beta$, $C_1$, and $C_2$ are as in Theorem 4.1. If the $\alpha_i$ are distinct, then equation (4.5) simplifies to
\[
\frac{C_1}{2(1 - t)^d} \sum_{i=1}^{n} \frac{\zeta^{\beta i \alpha_i}}{\alpha_i (1 - t_i^2) \prod_{j=1}^{n} (1 - \zeta^{-\alpha_i t_i^j (\alpha_i - \alpha_j) / \alpha_i}) (1 - \zeta^{-\alpha_i t_i^j (\alpha_i + \alpha_j) / \alpha_i})} + \frac{C_2}{2(1 - t^2)^n(1 + t)^d}.
\]

**Remark 4.3.** Note that we can express $\text{Hilb}_{O_2^{(\mathbb{R})}}^j(t)$ in terms of Hilbert series associated to the restricted $S^1$-representation as follows. That is, equation (4.1) can be written
\[
\text{Hilb}_{O_2^{(\mathbb{R})}}^j(t) = \frac{C_1}{2 \prod_{j=1}^{d} (1 - t_{n+j})} \text{Hilb}_{O_2^{(\mathbb{R})}}^j(t_1, \ldots, t_n, t_i, \ldots, t_n) + \frac{C_2}{2 \prod_{j=1}^{d} (1 - t_{n+j})},
\]

where $j$ is the index of the covariant and $t_i$ are the variables of the representation.
where \( \beta = 0 \) when \( W \) is det or the trivial representation. In particular, let us temporarily relax the hypothesis that \( V \) is faithful for this remark. If \( V \) contains at least one \( \tau_{\alpha_i} \) summand, \( W = \tau_{\beta} \), and 
\[ g := \gcd(\alpha_1, \ldots, \alpha_n) > 1 \]
do not divide \( \beta \), then 
\[ \text{Hilb}^{\beta}(-\alpha, \alpha_{\beta})(t_1, \ldots, t_n, t_1, \ldots, t_n) = C_2 = 0 \]
so that the Hilbert series is 0 and there are no covariants; see Remark 3.1. If \( g \) divides \( \beta \), we can without loss of generality take the quotient of \( O_2(\mathbb{R}) \) by the kernel of the action on \( V \), resulting in the faithful representation \( V_{\alpha/g, d} \) and \( W = \tau_{\beta/g} \).

Similarly, equation (4.5) can be written
\[
\text{Hilb}^{\alpha}_{\alpha, d; W}(t) = \frac{C_1}{2(1 - t)^d} \text{Hilb}^{\beta}(-\alpha, \alpha_{\beta})(t) + \frac{C_2}{2(1 - t^2)n(1 + t)^d}.
\]

In addition, we will need the \( \mathbb{N}^2 \)-graded Hilbert series of the covariants of a cotangent-lifted representation in the next section. A representation \( V = V_{\alpha, d} \simeq \mathbb{C}^{2n+d} \) is self-dual so that the cotangent lift of \( V \) is isomorphic to \( 2V \), i.e., \( V_{2\alpha, 2d} \simeq \mathbb{C}^{4n+2d} \) where \( 2\alpha \) indicates \( \alpha \) concatenated with itself. As in Section 3.2, we use the grading \((d_1, d_2)\) induced by the decomposition \( 2V \) with formal variables \((s, t)\). Adapting Theorem 4.1 to the representation \( V_{2\alpha, 2d} \simeq \mathbb{C}^{4n+2d} \), applying the substitutions \( t_i = s \) for \( 1 \leq i \leq n \) and \( t_i = t \) for \( n < i \leq 2n \) to compute as in the proof of Corollary 3.10, and using analytic continuation [30, Section 3.3] when the \( \alpha_i \) are not distinct, we obtain the following.

**Corollary 4.4** (Bivariate Hilbert series of cotangent-lifted \( O_2(\mathbb{R}) \)-covariants). Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) where \( n \geq 1 \), each \( \alpha_i > 0 \), and \( d \geq 0 \), let \( V = V_{\alpha, d} \simeq \mathbb{C}^{2n+d} \) be the corresponding faithful representation of \( O_2(\mathbb{R}) \), and let \( W \) be an irreducible representation of \( O_2(\mathbb{R}) \). The \( \mathbb{N}^2 \)-graded Hilbert series \( \text{Hilb}_{\alpha, d; W}^{O_2(\mathbb{R}), \alpha}((s, t)) \) associated to the cotangent lift of \( V \) is given by
\[
\lim_{(c_1, \ldots, c_{2n}) \to 2\alpha} \frac{C_1}{2(1 - s)^d(1 - t)^d(1 - st)} \sum_{\ell = 1}^{\infty} \left( \frac{\zeta^\beta s^{\beta/\alpha_i}(1 - s^{-c_i/\epsilon_i})}{\alpha_i(1 - s^2)} \right) \prod_{j=1, j \neq i}^{n} F_1(\zeta, i, j) \frac{1}{\alpha_i(1 - t^2)} \prod_{j=1, j \neq i}^{n} F_2(\zeta, i, j) + \frac{C_2}{2(1 - s^2)^n(1 - t^2)^n(1 + s)^d(1 + t)^d},
\]
where \( \beta, C_1, \text{and } C_2 \) are as in Theorem 4.1,
\[
F_1(\zeta, i, j) = (1 - \zeta^{-\alpha_j s^{-c_i/\epsilon_i}})(1 - \zeta^{-\alpha_i s^{-c_j/\epsilon_j}})(1 - \zeta^{-\alpha_j s^{-c_{i+j}/\epsilon_i}})(1 - \zeta^{-\alpha_i s^{-c_{j+i}/\epsilon_j}}),
\]
and
\[
F_2(\zeta, i, j) = (1 - \zeta^{-\alpha_i st^{-c_i/\epsilon_i}})(1 - \zeta^{-\alpha_j st^{-c_j/\epsilon_j}})(1 - \zeta^{-\alpha_i t^{-c_{i+j}/\epsilon_i}})(1 - \zeta^{-\alpha_j t^{-c_{j+i}/\epsilon_j}}).
\]
If the \( \alpha_i \) are distinct, then the limit is unnecessary and we may substitute \( c_i = \alpha_i \) and \( c_{n+i} = \alpha_i \) for \( 1 \leq i \leq n \).

### 4.3. Symplectic quotients by \( O_2(\mathbb{R}) \)

Let \( V = V_{\alpha, d} \simeq \mathbb{C}^{2n+d} \) be a representation of \( O_2(\mathbb{R}) \) as in the previous section. We consider coordinates \((x, y) := (x_{1,1}, x_{1,2}, x_{2,1}, \ldots, x_{n,2}, y_1, \ldots, y_d)\) for \( V \) where \((x_{i,1}, x_{i,2})\) are coordinates for \( \tau_{\alpha_i} \) and the \( y_i \) are coordinates for each copy of det. As the moment map \( J: V \to \mathfrak{g}^* \) depends only on the connected component of the identity of the group, it coincides with the moment map of the underlying circle action,
\[
J(x, y) = \sum_{i=1}^{n} \alpha_i(x_{i,1}x_{i,1} - x_{i,2}x_{i,2}).
\]

It follows from this expression that \( J(r_0(x, y)) = J(x, y) \) for any \( r_0 \in SO_2(\mathbb{R}) \), i.e., \( J \) is invariant under the action of \( SO_2(\mathbb{R}) \leq O_2(\mathbb{R}) \). However, if \( s_0 \in O_2(\mathbb{R}) \setminus SO_2(\mathbb{R}) \), then \( s_0 \) can be expressed as an element of
SO\(_2(\mathbb{R})\) followed by swapping each \(x_{i,1}\) with \(x_{i,2}\), and therefore we have \(f(s_0(x, y)) = -f(x, y)\). Hence, the moment map is never an \(O_2(\mathbb{R})\)-invariant, and the relationship between the on- and off-shell invariants is not as simple as in the case of the circle.

However, the Hilbert series of the on-shell invariants of the symplectic quotient can be computed using the techniques of [35, Section 6.3] as applied to the case SU\(_2\) in [28, Proposition 2.1]. Note that the shell only depends on the action of the Lie algebra of \(O_2(\mathbb{R})\) and hence on the corresponding action of \(SO_2(\mathbb{R})\). Therefore, if \(V = d\, \text{det}\), then the shell is trivially equal to \(V\). Otherwise, \(V\) has at least one \(\tau_\alpha\) summand with \(\alpha > 0\). As a representation of \(SO_2(\mathbb{R}) \cong S^1\), \(\tau_\alpha\) has weight vector \((-\alpha, \alpha)\) which by [33, Theorem 3.2] is \(l\)-large; see that reference for the definition. It follows that \(V\) is \(l\)-large by [33, Theorem 3.1] so that the complexification of the shell is a reduced, irreducible complete intersection.

Let \(\mu = f \otimes \mathbb{C}\) denote the complexification of \(f\). The Koszul complex of \(\mu\) is a free resolution of \(\mathbb{C}[V \oplus V^*]/(\mu)\) yielding an exact sequence

\[
0 \rightarrow \mathbb{C}[V \oplus V^*] \otimes \mathfrak{o}_2 \rightarrow \mathbb{C}[V \oplus V^*] \rightarrow \mathbb{C}[V \oplus V^*]/(\mu) \rightarrow 0.
\]

Note that \(\mathfrak{o}_2 \cong \text{det}\) as an \(O_2(\mathbb{C})\)-module and the elements of \(\mathfrak{o}_2\) are in degree \((1, 1)\) with respect to the bigrading. Taking invariants and using the fact that \((\mathbb{C}[V \oplus V^*] \otimes \text{det})O_2(\mathbb{C})\) is the module of covariants with \(W = \text{det}\), it follows that the Hilbert series \(Hilb_{\alpha, d}^{O_2(\mathbb{R}), \text{on}}(s, t)\) of the on-shell invariants of the symplectic quotient associated to \(V\) is

\[
Hilb_{\alpha, d}^{O_2(\mathbb{R}), \text{on}}(s, t) = Hilb_{\alpha, d}^{O_2(\mathbb{R}), \text{off}}(s, t) - st \, Hilb_{\alpha, d}^{O_2(\mathbb{R}), \text{off}}(s, t).
\]

Combining this with Corollary 4.4 yields the following.

**Corollary 4.5 (Bivariate Hilbert series of on-shell invariants of an \(O_2(\mathbb{R})\)-symplectic quotient).** Let \(\alpha = (\alpha_1, \ldots, \alpha_n)\) where \(n \geq 1\), each \(\alpha_i > 0\), and \(d \geq 0\), and let \(V = V_{\alpha, d} \cong \mathbb{C}^{2n+2d}\) be the corresponding faithful representation of \(O_2(\mathbb{R})\). Then the bivariate Hilbert series \(Hilb_{\alpha, d}^{O_2(\mathbb{R}), \text{on}}(s, t)\) of the real on-shell invariants of the corresponding symplectic quotient is given by

\[
\lim_{(c_1, \ldots, c_{2n}) \rightarrow 2\alpha} \frac{1}{2(1-s)^d(1-t)^d} \sum_{\xi, j=1}^{n} \frac{1}{\alpha_i(1-s^2)} \prod_{j=1 \atop j \neq i}^{n} F_1(\xi, i, j) \frac{1}{\alpha_i(1-t^2)} \prod_{j=1 \atop j \neq i}^{n} F_2(\xi, i, j) + \frac{1+st}{2(1-s^2)^n(1-t^2)^n(1+s)^d(1+t)^d}, \tag{4.9}
\]

where \(F_1(\xi, i, j)\) and \(F_2(\xi, i, j)\) are as defined in Corollary 4.4.

Adopting the convention that \(\alpha_{n+1} := \alpha_1\) for \(1 \leq i \leq n\), the univariate Hilbert series \(Hilb_{\alpha, d}^{O_2(\mathbb{R}), \text{on}}(t)\) of the real on-shell invariants of the symplectic quotient is given by

\[
\lim_{(c_1, \ldots, c_{2n}) \rightarrow 2\alpha} \frac{1}{2(1-t)^{2d}} \sum_{\xi, j=1}^{2n} \frac{1}{\alpha_i \prod_{j=1 \atop j \neq i}^{n} (1-\xi^{-\alpha_j t(c_i-c_j)/\alpha_i})(1-\xi^{-\alpha_j t(c_i+\alpha_j)/\alpha_j})} + \frac{1+t^2}{2(1-t^2)^{2n}(1+t)^{2d}}, \tag{4.10}
\]

The univariate Hilbert series of on-shell invariants can be obtained from the bivariate Hilbert series by taking the limit as \(s \rightarrow t\). In particular, the singularities at \(s = t\) are removable.

**Remark 4.6.** It will be helpful in Section 4.4 to observe the following. Using equation (4.6) in Remark 4.3, equation (4.9) can also be written

\[
Hilb_{\alpha, d}^{O_2(\mathbb{R}), \text{on}}(s, t) = \frac{1}{2(1-s)^d(1-t)^d} Hilb_{\alpha, d}^{\mathbb{R}_0, \text{on}}(s, t) + \frac{1+st}{2(1-s^2)^n(1-t^2)^n(1+s)^d(1+t)^d}, \tag{4.11}
\]
where $\text{Hilb}^{s_1, o_n}_{(-\alpha, \alpha)}(s, t)$ is the Hilbert series of the on-shell invariants of the symplectic quotient with weight vector $(-\alpha, \alpha)$ as in Corollary 3.13. Similarly, equation (4.10) can be written

$$\text{Hilb}^{O_2(\mathbb{R}), o_n}_{\alpha, d}(t) = \frac{1}{2(1-t)} \text{Hilb}^{s_1, o_n}_{(-\alpha, \alpha)}(t) + \frac{1 + t^2}{2(1-t^2)^2(1+t)^2}.$$  \tag{4.12}

### 4.4. The Laurent coefficients for symplectic quotients by $O_2(\mathbb{R})$

Let $V = V_{\alpha, d} \simeq \mathbb{C}^{2n+d}$ and $W$ be representations of $O_2(\mathbb{R})$ with $W$ irreducible as above. As in Section 3.3, we consider the Laurent coefficients $\gamma_m^{O_2(\mathbb{R})}(\alpha, d; W)$ of $\text{Hilb}^{O_2(\mathbb{R})}_{\alpha, d; W}(t)$ at $t = 1$ where $\gamma_m^{O_2(\mathbb{R})}(\alpha, d; W)$ occurs in degree $1 - 2n - d$. If $V$ contains no $\tau_{\alpha, i}$ summands, i.e., $\alpha$ is empty, then $V = d \det$. Following the proof of Theorem 4.1, equations (4.2) and (4.3), the Hilbert series is simply

$$\text{Hilb}^{O_2(\mathbb{R})}_{\alpha, d; W}(t) = \frac{2 - C_1}{2(1-t)} + \frac{C_2}{2(1+t)}$$

with $C_1$ and $C_2$ as defined in Theorem 4.1, and the Laurent expansion is given by

$$\text{Hilb}^{O_2(\mathbb{R})}_{\alpha, d; W}(t) = \frac{2 - C_1}{2(1-t)} + \sum_{m=0}^{\infty} \frac{C_2}{2^{d+m}} \left( \frac{d + m - 1}{m} \right)(1-t)^m.$$

Hence, we will hereafter ignore this case and assume that $V$ contains at least one $\tau_{\alpha, i}$ summand, in which case we can assume by Remark 4.3 that $V$ is faithful.

Using equation (4.7) and the fact that the second term has pole order $n$, only the first term contributes to the first two Laurent coefficients of $\text{Hilb}^{O_2(\mathbb{R})}_{\alpha, d; W}(t)$ except for small values of $n$. Considering these values by inspection, we have the following.

**Corollary 4.7** ($\gamma_0$ and $\gamma_1$ for $O_2(\mathbb{R})$-covariants). Let $V = V_{\alpha, d} \simeq \mathbb{C}^{2n+d}$ and $W$ be representations of $O_2(\mathbb{R})$ with $V$ faithful, $n \geq 1$, and $W$ irreducible. Let $\beta = 0$ if $W$ is det or the trivial representation and otherwise let $W = \tau_\beta$. Then

$$\gamma_0^{O_2(\mathbb{R})}(\alpha, d; W) = \begin{cases} \frac{C_1 \gamma_0^{s_1}_{(-\alpha, \alpha), \beta}}{2} + \frac{C_2}{4}, & \text{if } n = 1 \text{ and } d = 0, \\
\frac{C_1 \gamma_0^{s_1}_{(-\alpha, \alpha), \beta}}{2}, & \text{otherwise}, \end{cases}$$

$$\gamma_1^{O_2(\mathbb{R})}(\alpha, d; W) = \begin{cases} \frac{C_1 \gamma_1^{s_1}_{(-\alpha, \alpha), \beta}}{2} + \frac{C_2}{8}, & \text{if } n + d \leq 2, \\
\frac{C_1 \gamma_1^{s_1}_{(-\alpha, \alpha), \beta}}{2}, & \text{otherwise}, \end{cases}$$

where $C_1$ and $C_2$ are as defined in Theorem 4.1.

In the same way, the following is a consequence of equation (4.12) and inspection of low-dimensional cases.

**Corollary 4.8** ($\gamma_0, \gamma_1, \gamma_2,$ and $\gamma_3$ for on-shell invariants of $O_2(\mathbb{R})$-symplectic quotients). Let $V = V_{\alpha, d} \simeq \mathbb{C}^{2n+d}$ be a faithful representation of $O_2(\mathbb{R})$ with $n \geq 1$. Then

$$\gamma_0^{O_2(\mathbb{R}), o_n}(\alpha, d) = \begin{cases} \frac{\gamma_0^{s_1}_{o_n}(\alpha, \alpha)}{2} + \frac{1}{4}, & \text{if } n = 1 \text{ and } d = 0, \\
\frac{\gamma_0^{s_1}_{o_n}(\alpha, \alpha)}{2}, & \text{otherwise}, \end{cases}$$

$$\gamma_1^{O_2(\mathbb{R}), o_n}(\alpha, d) = 0,$$ and

$$\gamma_2^{O_2(\mathbb{R}), o_n}(\alpha, d) = \gamma_3^{O_2(\mathbb{R}), o_n}(\alpha, d) = \begin{cases} \frac{\gamma_2^{s_1}_{o_n}(\alpha, \alpha)}{2} + \frac{1}{16}, & \text{if } n + d \leq 2, \\
\frac{\gamma_2^{s_1}_{o_n}(\alpha, \alpha)}{2}, & \text{otherwise}. \end{cases}$$
For $m \leq 3$, the coefficients $\gamma_m^{S^1,\text{on}}((-\alpha, \alpha))$ for $S^1$-symplectic quotients were computed in [34, Theorem 5.1].

5. Other semidirect products of $S^1$ by finite groups

The approach of Section 4 can be applied to other extensions of the circle by a finite group. As an example, consider the extension $S^1 \rtimes \mathbb{Z}/4\mathbb{Z}$ where the elements of $\mathbb{Z}/4\mathbb{Z}$ of order 4 act on $S^1$ via $t \mapsto t^{-1}$ and the other elements act trivially. That is, if $\gamma$ is a generator of $\mathbb{Z}/4\mathbb{Z}$, then the multiplication of $S^1 \rtimes \mathbb{Z}/4\mathbb{Z}$ is given by

$$(z_1, \gamma^j)(z_2, \gamma^k) = (z_1z_2^{(-1)^j}, \gamma^{j+k})$$

for $z_1, z_2 \in S^1$. The representation of $S^1 \rtimes \mathbb{Z}/4\mathbb{Z}$ induced by the representation $\epsilon_\alpha$ of the normal subgroup $S^1$ is denoted $\nu_\alpha$: $S^1 \rtimes \mathbb{Z}/4\mathbb{Z} \rightarrow \text{U}_4$ and given by

$$(z, 1) \mapsto \begin{pmatrix} z^a & 0 & 0 & 0 \\ 0 & z^{-a} & 0 & 0 \\ 0 & 0 & z^a & 0 \\ 0 & 0 & 0 & z^{-a} \end{pmatrix}, \quad (z, \gamma) \mapsto \begin{pmatrix} 0 & z^a & 0 & 0 \\ 0 & 0 & z^{-a} & 0 \\ 0 & 0 & 0 & z^a \\ z^{-a} & 0 & 0 & 0 \end{pmatrix},$$

$$(z, \gamma^2) \mapsto \begin{pmatrix} 0 & 0 & z^a & 0 \\ 0 & 0 & 0 & z^{-a} \\ z^a & 0 & 0 & 0 \\ 0 & z^{-a} & 0 & 0 \end{pmatrix}, \quad (z, \gamma^3) \mapsto \begin{pmatrix} 0 & 0 & z^a & 0 \\ z^{-a} & 0 & 0 & 0 \\ 0 & z^a & 0 & 0 \\ 0 & 0 & z^{-a} & 0 \end{pmatrix},$$

where $z \in S^1$.

For simplicity, we consider a representation of the form $V = \bigoplus_{i=1}^n \nu_{a_i}$ where each $a_i > 0$. The maximally graded Hilbert series of the invariants can be computed using the same methods as in Theorem 4.1. Specifically, for the connected components corresponding to elements of the form $(z, \gamma)$ and $(z, \gamma^3)$, each integral in the Molien-Weyl Theorem is simply

$$\frac{1}{8\pi} \int_{S^1} dz \frac{1}{z \prod_{i=1}^{n} (1 - t_i^4)} \frac{1}{4 \prod_{i=1}^{n} (1 - t_i^4)}.$$ 

For the connected component associated to elements of the form $(z, 1)$, the integral is

$$\frac{1}{8\pi} \int_{S^1} dz \frac{1}{z \prod_{i=1}^{n} (1 - t_i z^{a_i})^2 (1 - t_i z^{-a_i})^2}.$$ 

Choosing an $i$ and an $a_i$th root of unity $\zeta_0$ and rewriting the integrand as

$$(1 - t_i z^{a_i})^2 (z - \zeta_0 t_i^{1/a_i})^2 \prod_{\zeta^{a_i} = 1}^{\zeta \neq \zeta_0} (z - \zeta t_i^{1/a_i})^2 \prod_{j=1}^{n} (1 - t_j z^{a_j})^2 (1 - t_j z^{-a_j})^2,$$

the residue at $z = \zeta_0 t_i^{1/a_i}$ is given by

$$\frac{\partial}{\partial z} \left( z^{2a_i - 1} \prod_{\zeta^{a_i} = 1}^{\zeta \neq \zeta_0} (z - \zeta t_i^{1/a_i})^2 \prod_{j=1}^{n} (1 - t_j z^{a_j})^2 (1 - t_j z^{-a_j})^2 \right)_{z = \zeta_0 t_i^{1/a_i}}.$$
For the connected component associated to elements of the form \((z, \gamma^2)\), the integral is
\[
\frac{1}{8\pi \sqrt{-1}} \int \frac{dz}{z \prod_{i=1}^{n} (1 - t_i z^{-a_i})(1 + t_i z^{-a_i})(1 - t_i z^{a_i})(1 + t_i z^{a_i})}.
\]

Rewriting the integrand as
\[
(z^{a_i} - t_i)(z^{a_i} + t_i)(1 - t_i z^{a_i})(1 + t_i z^{a_i}) \prod_{j=1, j \neq i}^{n} (1 - t_j z^{-a_j})(1 + t_j z^{-a_j})(1 - t_j z^{a_j})(1 + t_j z^{a_j})
\]
and choosing an \(a_i\)th root of unity \(\zeta_0\), the residue at a pole of the form \(z = \zeta_0 t_i^{1/a_i}\) is given by
\[
\frac{1}{2 a_i (1 - t_i^d) \prod_{j=1, j \neq i}^{n} (1 - \zeta_0^{-a_j} t_i^{-a_j/a_i} t_j)(1 + \zeta_0^{-a_j} t_i^{-a_j/a_i} t_j)(1 - \zeta_0^{a_j} (-t_i)^{a_j/a_i} t_j)(1 + \zeta_0^{a_j} (-t_i)^{a_j/a_i} t_j)}.
\]

and the residue at a pole of the form \(z = \zeta_0 (-t_i)^{1/a_i}\) is given by
\[
\frac{1}{2 a_i (1 - t_i^d) \prod_{j=1, j \neq i}^{n} (1 - \zeta_0^{-a_j} (-t_i)^{-a_j/a_i} t_j)(1 + \zeta_0^{-a_j} (-t_i)^{-a_j/a_i} t_j)(1 - \zeta_0^{a_j} (-t_i)^{a_j/a_i} t_j)(1 + \zeta_0^{a_j} (-t_i)^{a_j/a_i} t_j)}.
\]

Hence the maximally graded Hilbert series of the invariants of the representation \(V = \bigoplus_{i=1}^{n} v_{a_i}\) is given by
\[
\frac{1}{2 \prod_{i=1}^{n} (1 - t_i^d)} + \frac{1}{4} \sum_{i=1}^{n} a_i^4 (1 - t_i^2)^4 \sum_{\zeta_0^{-a_j} t_i^{-a_j/a_i} t_j} \prod_{j=1, j \neq i}^{n} (1 - \zeta_0^{-a_j} t_i^{-a_j/a_i} t_j)(1 + \zeta_0^{-a_j} t_i^{-a_j/a_i} t_j)(1 - \zeta_0^{a_j} (-t_i)^{a_j/a_i} t_j)(1 + \zeta_0^{a_j} (-t_i)^{a_j/a_i} t_j)
\]
\[
+ \sum_{i=1}^{n} \frac{1}{8 a_i (1 - t_i^d) \prod_{j=1, j \neq i}^{n}} \left[ \prod_{j=1}^{n} (1 - \zeta_0^{-a_j} (-t_i)^{-a_j/a_i} t_j)(1 + \zeta_0^{-a_j} (-t_i)^{-a_j/a_i} t_j)(1 - \zeta_0^{a_j} (-t_i)^{a_j/a_i} t_j)(1 + \zeta_0^{a_j} (-t_i)^{a_j/a_i} t_j) \right]
\]
\[
+ \left. \prod_{j=1, j \neq i}^{n} (1 - \zeta_0^{-a_j} (-t_i)^{-a_j/a_i} t_j)(1 + \zeta_0^{-a_j} (-t_i)^{-a_j/a_i} t_j)(1 - \zeta_0^{a_j} (-t_i)^{a_j/a_i} t_j)(1 + \zeta_0^{a_j} (-t_i)^{a_j/a_i} t_j) \right]^{1}
\]
where
\[
D_i = a_i^2 (1 - t_i^2)^2 \prod_{j=1, j \neq i}^{n} (1 - \zeta_0^{a_j} t_i^{a_j/a_i} t_j)^2 (1 - \zeta_0^{-a_j} t_i^{-a_j/a_i} t_j)^2,
\]
and
\[
D_i' = \frac{d}{dz} \left. \left( (1 - t_i z^{a_i})^2 \prod_{\eta \neq i}^{n} (z - \eta t_i^{1/a_i})^2 \prod_{j=1, j \neq i}^{n} (1 - t_j z^{a_j})^2 (1 - t_j z^{-a_j})^2 \right) \right|_{z = t_i^{1/a_i}}.
\]
One could also compute the first few Laurent coefficients of the corresponding univariate Hilbert series using the techniques of Sections 3.3 and 4.4, though it is clear that the level of complexity increases considerably.

6. Algorithms to compute the Hilbert series

The formulas given in Theorems 3.2 and 4.1 and Corollaries 3.9, 3.10, 3.12, 3.13, 4.2, 4.4, and 4.5 each indicate an algorithm for the computation of the corresponding Hilbert series very similar to the algorithms described in [36, Section 4] and [19, Section 4]; see also [20, Section 6] and [30, Section 3.3]. We give a brief description of this algorithm for the case of Theorem 3.2, as the others are the same with minor modifications, and refer the reader to the above references for more details.

Let \(a \in \mathbb{Z}, a > 0\). For a formal power series \(F(t)\) in the variables \(t = (t_1, \ldots, t_n)\), let \(U_{a,i}F(t)\) denote the operator

\[
U_{a,i}F(t) = \frac{1}{a} \sum_{\xi^a = 1} F(t_1, \ldots, \xi t_i^{1/a}, \ldots, t_n).
\]

Let \(t_i = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)\), and note that if \(F(t) = \sum_{d=0}^{\infty} c_d(t_i)t_i^d\), then

\[
U_{a,i}F(t) = \sum_{d=0}^{\infty} c_d(t_i)t_i^d.
\]  

Using this operator, the first sum in equation (3.1) can be rewritten as \(\sum_{i=1}^{k} U_{a,i}F_1(t)\) where

\[
F_1(t) = \prod_{j=1, j \neq i}^{n} \frac{t_i^b}{1 - t_j^a t_i^{aj}}.
\]

The function \(F_1(t)\) can be written as \(P(t)/Q(t)\) where \(P(t)\) is a Laurent monomial in \(t\) and \(Q(t)\) is a product of factors of the form \((1 - t_j^a t_i^{aj})\) with \(q > 0\). Then \(U_{a,i}(P(t)/Q(t))\) is a rational function with denominator given by the replacement rule

\[
(1 - t_j^a t_i^{aj}) \mapsto (1 - t_j^{\alpha_j \gcd(\alpha_i, q)} t_i^{\gcd(\alpha_i, q)} q^\gcd(\alpha_i, q)}
\]

applied to \(Q(t)\). The degrees of the numerator and denominator of \(U_{a,i}(P(t)/Q(t))\) can be computed using [30, equation (16)], so that knowing the denominator and the degree of the numerator, the numerator of \(U_{a,i}(P(t)/Q(t))\) can be computed using equation (6.1). This yields a computation of the first sum in equation (3.1), and the second sum can be computed directly by searching through the finite set of possible elements of \(S_{a,b}\) as defined in equation (3.2), e.g., by using the FrobeniusSolve function on Mathematica [53]. However, as noted in Remark 3.5, one can always reduce to a case where \(S_{a,b}\) is empty so that the second sum vanishes.

These algorithms have been implemented on Mathematica and are available from the authors upon request.

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The authors report there are no competing interests to declare.

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