Further on the Aharonov–Bohm Green function: the coincidence limit

J.S. Dowker

Theory Group,
Department of Physics and Astronomy,
The University of Manchester,
Manchester, England

For a scalar CFT with a monodromy defect, a ‘subtracted Green function’ is derived in terms of an Appell $F_1$ function. A conjectured relation of Gimenez-Grau and Liendo is thereby proved and extended and shown to hold for generalised free fields. A possible means of determining the bulk block expansion is outlined. Finite coincidence limits are expressed as combinations of Beta functions.
1. Introduction

In a recent report, [1], it was shown, essentially just by a coordinate transformation, that an old contour integral expression for the Green function, \( G \), of a free conformal scalar, \( \phi \), in the presence of an Aharonov–Bohm flux tube takes the form of an Appell \( F_1 \) function.\(^2\) Expansion of this Green function, or correlator, in a Fourier series effectively provides its defect block expansion, of which several forms can be found, related by hypergeometric transformations.

Useful though it might be, this representation is not the most convenient for computation of those vacuum averages which require a coincidence limit to be taken. In this further short report I take up again a method of determining these limits without encountering the infinities which usually arise and have to be removed.

Only those vacuum averages which result from the action of a differential operator on the correlator are accessible this way and, to keep things concise, I will not discuss any specific examples, apart from the easiest one, \( \langle \phi \phi(x) \rangle \). Furthermore, the situation, being for a free field, is somewhat limited, but this is compensated by the simple outcomes.

The following is, therefore, an extension and refinement of the approach employed in [3] and briefly sketched in [1].

2. The \( \alpha \)-contours

The contour integral introduced in [4], and used in [3] to evaluate the energy–momentum vacuum average, \( \langle T_{00} \rangle \), is a Sommerfeld-Carslaw contour integral, modified to allow for the monodromy phase factor, \( e^{2\pi i \delta} \), undergone as the field point circles the co–dimension 2 defect. It is written out again, if only to set the present notation,

\[
G(x_2, x_1, \delta) \equiv G(\theta, \delta) = \frac{1}{2\pi} \int_A d\alpha \, G(\alpha, 0) P(\alpha + \theta, \delta). \tag{1}
\]

As indicated, the Green function, \( G \), is a function of the polar angles of the arguments through the difference \( \theta_2 - \theta_1 = \theta \). \( P \) is a re–periodising factor that inserts the monodromy, in this case, into the monodromy–free Green function \( (\delta = 0) \) which has the free–field form,

\[
G(\theta, 0) = \frac{\Gamma(\Delta)}{2\pi^{\Delta+1}} \frac{1}{(\sigma^2(\theta))^\Delta}. \tag{2}
\]

\(^2\) This representation has also been derived on the basis of a dispersion relation for the correlator applied to a (generalised) free field by Barrat et al [2].
Setting $\Delta = d/2 - 1$ gives a standard free field. Keeping $\Delta$ general corresponds to a generalised free field.

The interval is

$$\sigma^2(\theta) = r_1^2 + r_2^2 - 2r_1r_2 \cos \theta + (y_1 - y_2)^2,$$

where $r$ is a radial coordinate and $y$ the coordinate along the (flat) defect.

The hyperbolic angle, $\alpha_1$, is defined by,

$$\cosh \alpha_1 = \frac{(y_1 - y_2)^2 + r_1^2 + r_2^2}{2r_1r_2},$$

so that the interval, $\sigma^2(\alpha)$, factorises as,

$$\sigma^2(u) = r_1r_2 u^{-1}(h - u)(u - \frac{1}{h}),$$

where $h$ and $u$ are defined by,

$$h = e^{\alpha_1} > 1, \quad \text{and} \quad u = e^{-i\alpha}.$$}

The re-periodising factor is given in [4] as,

$$P(\alpha, \delta) = \frac{e^{i\delta}}{2i \sin(\alpha/2)}, \quad \delta = \tilde{\delta} - 1/2. \quad (0 \leq \delta \leq 1).$$

The Sommerfeld–Carslaw $\alpha$–contour, $A$, is described in the just cited references and consists of an upper half-plane part and a lower half–plane part which is the reflection of the upper one by $\alpha \rightarrow -\alpha$. The upper one runs in a $\cup$ shape to $i\infty$, going below the light–cone singularity (generally a branch point) at $\alpha = i\alpha_1$ and staying above the real axis.

3. The subtraction

As $x_2$ tends to $x_1$, $G(x_2, x_1, \delta)$ tends to $G(x_2, x_1, 0)$ and will diverge according to (2), as will any derivates of $G$. The divergences can therefore be displayed and removed giving a renormalised, or subtracted, finite coincidence limit. This is the usual procedure.

---

3. I display arguments, or not, as is convenient at the time.
In [3] an alternative method was used that excised \( G(\theta, 0) \) from \( G(\theta, \delta) \) before taking the coincidence limit. This was achieved by a contour deformation that pushes the upper part of \( A \), say, through the pole (from \( P \)) at \( \alpha = -\theta \) thereafter combining it with the lower part to give, say, two complete vertical lines, oppositely directed, one to the left of the origin and one to the right, denoted by \( C = C_L + C_R \).

The pole contribution is just \( G(\theta, 0) \) which is discarded, a process that constitutes the subtraction recipe and is expressed analytically as,

\[
G_{\text{sub}}(\theta, \delta) = \frac{1}{2\pi} \int_C d\alpha G(\alpha, 0)P(\alpha + \theta, \delta) .
\]  

(5)

Any coincidence limits will then automatically be finite.

Although it is not needed (the integrals suffice for evaluation of coincidence limits) a formula for this subtracted Green function/correlator,\(^4\) will now be derived. For this purpose, it is convenient to change coordinates from \( \alpha \) to \( u = e^{-i\alpha} \), as Carslaw does and as used in [1] when finding the full \( G(\theta, \delta) \).

4. The \( u \)--contours

The exponential map, \( u = e^{-i\alpha} \) transforms vertical straight \( \alpha \)--lines into \( u \)--radii through the \( u = 0 \) origin. Hence the subtracted contour section, \( C_R \) (which can be taken at \( \Re \alpha \sim \pi \)), maps to a radial \( u \)--vector running inwards from \(-\infty \) to \( 0 \) just above the real negative \( u \) axis and \( C_L \) (\( \Re \alpha \sim -\pi \)) maps to an outwards \( u \)--vector from \( 0 \) to \(-\infty \) just below the axis. The combined (disjoint!) \( u \)--contour, \( e^{-iC} \), is denoted by \( C_w \) and the subtracted Green function is,

\[
G_{\text{sub}} = \frac{\Gamma(\Delta)}{(r_1 r_2)^{\Delta} 2\pi i^\Delta + 1} \int_{C_w} du u^{-\delta+\Delta}(u-h^{-1})^{-\Delta}(h-u)^{-\Delta}(e^{i\theta} - u)^{-1} .
\]  

(6)

The five singular points of the integrand in (6) lie at \( u = 0, \infty, h^{-1}, h \) and \( u = e^{i\theta} \), the last being a pole. The cuts attached to \( 0 \) and \( h^{-1} \) run westwards to \(-\infty \) while that from \( h \) runs eastwards to \( \infty \). By contrast, the full Green function, \( G \), results on choosing the contour to be around the eastwards cut which leads to Picard’s form of the Appell \( F_1 \) function, [1]. A similar relation emerges from (6), as will now be shown.

\(^4\) The term ‘Green function’ is actually inappropriate as it satisfies a homogeneous equation of motion.
5. Rewriting the integral

As a first step, it is handy just to change the sign of $u$ defining $z = -u$ to give,

$$G_{sub} = \frac{\Gamma(\Delta)}{(r_1 r_2)^{\Delta + 1} 2\pi^i} \int_{-C_w} dz (-z)^{-\delta + \Delta} (- z - h^{-1})^{-\Delta} (h + z)^{-\Delta} (z + e^{i\theta})^{-1}.$$ \hspace{1cm} (7)

The arguments of both $-z$ and $-z - h^{-1}$ are $-\pi$ on the upper part of the contour and $\pi$ on the lower so that the integrals combine, in a well known fashion, to give a simple line integral,

$$G_{sub} = \frac{\Gamma(\Delta)}{(r_1 r_2)^{\Delta + 1} 2\pi^i} \sin \pi \delta \int_{0}^{\infty} dx x^{-\delta + \Delta} (x + h^{-1})^{-\Delta} (x + h)^{-\Delta} (x + e^{i\theta})^{-1}.$$ \hspace{1cm} (8)

This vanishes, as it should, when $\delta = 0$ and, further, satisfies the conjugation (or parity) relation $G_{sub}(\theta, \delta) = \overline{G_{sub}(\theta, 1 - \delta)} = G_{sub}(-\theta, 1 - \delta)$, which can be checked by sending $x \to 1/x$.

If only coincidence limits (i.e. $h \to 1$ and $\theta \to 0$) are required (perhaps after differentiation with respect to $\theta$ and $h$) then formula (8) can be left alone as it leads directly to a standard Eulerian integral. This is discussed briefly later. However, it is also not without interest to produce an expression for $G_{sub}$ corresponding to that for the full $G$ mentioned in section 1.

6. The subtracted Green function as an Appell function

To re-express $G_{sub}$ in terms of existing, i.e. named, functions a transformation of integration variable is necessary. This turns out to be of the fractional form,

$$x = \frac{h^{-1} y}{1 - y}, \quad y = \frac{x}{x + h^{-1}},$$

so that $y$ runs from 0 to 1. After making this substitution it is convenient to define new variables, $\lambda$ and $\mu$, by,

$$\lambda = 1 - h^{-2}, \quad \mu = 1 - h^{-1} e^{-i\theta},$$

and $G_{sub}$ becomes,

$$G_{sub} = \frac{\Gamma(\Delta) \sin \pi \delta}{(r_1 r_2)^{\Delta + 1} 2\pi^i} (he^{i\theta})^{-\delta_1 h^{-\Delta}} \int_{0}^{1} dy y^{-\delta} (1 - y)^{-\Delta - \delta} (1 - \lambda y)^{-\Delta} (1 - \mu y)^{-1},$$ \hspace{1cm} (9)
recognised as an Appell function, \( i.e. \),

\[
G_{\text{sub}} = \frac{\Gamma(\Delta) \sin \pi \delta}{(r_1 r_2)^{\Delta^2}} \frac{\Gamma(\Delta - \delta + 1) \Gamma(\Delta + \delta)}{(h e^{i\theta})^{1-\delta} h^{\Delta} \Gamma(2\Delta + 1)} F_1(\Delta-\delta+1, \Delta, 1, 2\Delta+1, \lambda, \mu),
\]

(10)

which has the alternative form,

\[
G_{\text{sub}} = \frac{\Gamma(\Delta) \sin \pi \delta}{(r_1 r_2)^{\Delta^2}} \frac{\Gamma(\Delta - \delta + 1) \Gamma(\Delta + \delta)}{(h e^{-i\theta})^{\delta} h^{\Delta} \Gamma(2\Delta + 1)} F_1(\Delta + \delta, \Delta, 1, 2\Delta + 1, \lambda, \mu),
\]

(11)

obtained by reversing the sign of \( \theta \) and sending \( \delta \to 1 - \delta \).

For calculational purposes, the integral form, (8), is often more convenient than using \( F_1 \) although there exist many transformation properties of \( F_1 \) that could prove useful.

A specific application of the above results is given in the next section.

7. An application to conformal block expansions

In order to extract CFT data via the bulk block expansion, Gimenez–Grau and Liendo, [5], gave an expression for the full correlator expanded about the point \( x = 1, \bar{x} = 1 \), where \( x \) and \( \bar{x} \) are light–cone coordinates orthogonal to the defect. This expression was obtained by guesswork based on special cases such as even dimensions.

It can be observed that the essential part of the formula they give (equn.(2.38) in [5]) is (up to an overall normalisation) just an expansion of \( G_{\text{sub}} \) and should, therefore, follow from the formulae in the previous section.

In order to obtain the requisite expansion, equation (11) will be used. In my conventions, \( x \) and \( \bar{x} \) are related to \( h \) and \( \theta \) by,

\[
x = h^{-1} e^{i\theta}, \quad \bar{x} = h^{-1} e^{-i\theta}, \quad \lambda = 1 - x, \quad \lambda = 1 - x\bar{x},
\]

and so (11) reads, in terms of \( x \) and \( \bar{x} \),

\[
G_{\text{sub}} = \frac{\Gamma(\Delta) \sin \pi \delta}{(r_1 r_2)^{\Delta^2}} x^{\delta} \sqrt{x\bar{x}} \frac{\Gamma(\Delta - \delta + 1) \Gamma(\Delta + \delta)}{\Gamma(2\Delta + 1)} F_1(\Delta + \delta, \Delta, 1, 2\Delta + 1, 1-x\bar{x}, 1-x).
\]

(12)

The variables, \( x \) and \( \bar{x} \) can now be regarded as real and independent. Setting \( x \) to 1, the Appell \( F_1 \) degenerates to a hypergeometric function (best seen from the integral form) and so, \(^5\)

\[
G_{\text{sub}} \bigg|_{x=1} = \frac{\Gamma(\Delta) \sin \pi \delta}{(r_1 r_2)^{\Delta^2}} \sqrt{x\bar{x}} \frac{\Gamma(\Delta - \delta + 1) \Gamma(\Delta + \delta)}{\Gamma(2\Delta + 1)} 2 F_1(\Delta + \delta, \Delta, 2\Delta + 1, 1 - \bar{x}),
\]

(13)

\(^5\) Using (10) gives the same result upon applying one of Kummer’s hypergeometric relations.
which is, after some mild factor reorganisation, precisely the expression conjectured in [5]. Moreover, the derivation here shows that it applies to a *generalised* free field.

The general formula, (12), for the subtracted correlator is not, however, immediately well suited algebraically to explore the bulk block expansion which is a double power series in \((1 - x)\) and \((1 - \overline{x})\) although it does go further than the limiting case of (13) and must contain this expansion in one form or another.

In this connection, I remark that the bulk block expansion involves the quotient (cf eqn.(2.40) in [5]),

\[
\frac{G_{\text{sub}}(\theta, \delta)}{G(\theta, 0)},
\]

where, in light cone coordinates,

\[
G(\theta, 0) = \left( \frac{\sqrt{x_{\overline{x}}} \Delta}{(1 - x)(1 - \overline{x})} \right),
\]

and \(G_{\text{sub}}\) is here given by (12).

A possible algebraic approach is to note that \((1 - x\overline{x})\) can be written in terms of \(\mu \equiv 1 - \overline{x}\) as \(1 - x\overline{x} = \overline{\mu} + \mu - \mu\overline{\mu}\) so that a power is,

\[
(1 - x\overline{x})^a = \mu^a \left( 1 + \frac{1 - \mu}{\mu} \right)^a,
\]

and, using (12), a systematic expansion in \(\overline{\mu}\) should allow the coefficients in the bulk block expansion to be determined, one by one, as, in general, sums of hypergeometric functions.

8. Coincidence limits

The application of a differential operator to \(G_{\text{sub}}\) and the subsequent coincidence limit \((h \to 1, \theta \to 0)^6\), will produce, from (8), a sum of integrals of the type,

\[
I(\nu, \delta, \Delta) = \int_0^\infty dx \, x^{\Delta - \delta} (x + 1)^{-2\Delta - 1 - \nu},
\]

where \(\nu\) is related to the derivative order.

This Eulerian integral is none other than the original definition of the Beta function, \(^7\) therefore,

\[
I(\nu, \delta, \Delta) = B(\Delta - \delta + 1, \nu + \Delta + \delta).
\]

---

\(^6\) \(h = 1\) entails \(r_1 = r_2\) and \(y_1 = y_2\).

\(^7\) For those interested, historical remarks and information can be found in the little book by Graf, [6], and also in Meyer, [7].
For $\nu = 0$ one has the simplest average, $\langle \phi \bar{\phi}(x) \rangle$, which will not be written out as it exists elsewhere. It is invariant under $\delta \to 1 - \delta$. This symmetry will generally be the case when there are an even number of $\theta$-derivatives, as for $\langle T^\mu_\nu \rangle$. For an odd number, as when calculating the current average, $\langle J_\mu \rangle$, the result will be antisymmetric under $\delta \to 1 - \delta$ and so will vanish at the midpoint, $\delta = 1/2$, as is known.

9. Even dimensions

Even dimension $d$, is a somewhat special case in that the light–cone singularity is a pole and all calculations can be reduced to residues, yielding closed formulae. For example, in the coincidence limit, in the $u$–plane instead of computing a difference integral along the westward cut, the lower contour, now there is no $h^{-1}$ cut, can be rotated, picking up a residue as it passes through the multiple pole at $u = 1$ and then brought into coincidence with the upper contour against which it cancels.

In the $\alpha$–plane, this was explicitly performed in [3] for $\langle T^0_0 \rangle$ in the case when there was also a conical singularity and the residues were presented in terms of generalised Bernoulli polynomials.

10. Comments and conclusion

The methods outlined here allow any coincidence limit to be derived quickly in terms of Beta functions.

Their extension to the Dirac field should present no difficulty.

The extraction of CFT data from a bulk block expansion remains to be properly investigated.

Added note: Further work has revealed that it is possible to devise an efficient scheme to calculate the coefficients in the bulk block expansion, e.g. eqn. (2.18) in [5]. This will be exposed at a later time. Although these coefficients can already be determined (by recursion), a direct evaluation might be welcome.
References.

1. Dowker, J.S. *On the Green function of an Aharonov–Bohm flux tube*, arXiv: 2205.08477.
2. Barrat, J., Gimenez–Grau, A. and Liendo, P. *A dispersion relation for defect CFT*, arXiv:2205.09765.
3. Dowker, J.S. *Casimir effect around a cone*, Phys. Rev. **D36** (1987) 3095.
4. Dowker, J.S. *Quantum field theory on a cone*, J. Phys. **A10** (1977) 115.
5. Gimenez–Grau, A. and Liendo, P. *Bootstrapping Monodromy Defects in the Wess–Zumino Model*, arXiv:2108.05107.
6. Graf, J.H. *Einleitung in die Theorie der Gammafunktion und der Euler’sche Integrale*, (Wyss, Bern, 1894).
7. Meyer, G.F. *Vorlesungen über die Theorie der bestimmten Integrale* (Teubner, Leipzig, 1871).