First General Zagreb Index of Generalized $F$-sum Graphs

H. M. Awais,$^1$ Muhammad Javaid$^1$, and Akbar Ali$^2$

$^1$Department of Mathematics, School of Science, University of Management and Technology, Lahore, Pakistan
$^2$Department of Mathematics, Faculty of Science, University of Ha’il, Ha’il, Saudi Arabia

Correspondence should be addressed to Akbar Ali; akbarali.maths@gmail.com

Received 14 May 2020; Accepted 10 November 2020; Published 9 December 2020

Academic Editor: Luisa Di Paola

Copyright © 2020 H. M. Awais et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The first general Zagreb (FGZ) index (also known as the general zeroth-order Randić index) of a graph $G$ can be defined as $M^c(G) = \sum_{uv \in E(G)} [d_G^{-1}(u) + d_G^{-1}(v)]$, where $c$ is a real number. As $M^c(G)$ is equal to the order and size of $G$ when $c = 0$ and $c = 1$, respectively, $c$ is usually assumed to be different from 0 to 1. In this paper, for every integer $c \geq 2$, the FGZ index $M^c$ is computed for the generalized $F$-sums graphs which are obtained by applying the different operations of subdivision and Cartesian product. The obtained results can be considered as the generalizations of the results appeared in (IEEE Access 7 (2019) 47494–47502) and (IEEE Access 7 (2019) 105479–105488).

1. Introduction

Graph theory concepts are being utilized to model and study the several problems in different fields of science, including chemistry and computer science. A topological index (TI) of a (molecular) graph is a numeric quantity that remained unchanged under graph isomorphism [1,2]. Many topological indices have found applications in chemistry, especially in the quantitative structure-activity/property relationships studies; for detail, see [3–13].

Wiener index is the first TI introduced by Harry Wiener in 1947, when he was working on the boiling point of paraffin [14]. In 1972, Trinajstić and Gutman [15] obtained a formula concerning the total energy of $\pi$ electrons of molecules where the sum of square of valences of the vertices of a molecular structure was appeared. This sum is nowadays known as the first Zagreb index. In this paper, we are concerned with a generalized version of the first Zagreb index, known as the general first Zagreb index as well as the general zeroth-order Randić index.

There are several operations in graph theory such as product, complement, addition, switching, subdivision, and deletion. In many cases, graph operations may be helpful in finding graph quantities of more complicated graphs by considering the less complicated ones. In chemical graph theory, by using different graph operations, one can develop large molecular structures from the simple and basic structures. Recently, many classes of molecular structures are studied with the assistance of graph operations.

In 2007, Yan et al. [6] listed the five subdivision operations with the help of their vertices and edges. They also discussed the different features of Wiener index of graphs under these operations. After that, Eliasi and Taeri [16] introduced the $F_1$-sum graphs $\Gamma_1 \circ F_i \Gamma_2$ with the assistance of Cartesian product on graphs $F_i (\Gamma_1)$ and $\Gamma_2$, where $F_i (\Gamma_1)$ is obtained by applying the subdivision operations $S_1, R_1, Q_1,$ and $T_1$. They also defined the Wiener indices of these resulting graphs $\Gamma_1 \circ F_i \Gamma_2, \Gamma_1 \circ Q_i \Gamma_2, \Gamma_1 \circ T_i \Gamma_2$, and $\Gamma_1 \circ S_i \Gamma_2$. Later on, Deng et al. [17] calculated the 1st and 2nd Zagreb topological indices, and Imran and Akhtar [18] calculated the forgotten topological index of the $F_1$-sums graph. In 2019, Liu et al. [19] computed the first general Zagreb index of $F_1$-sums graphs.

Recently, Liu et al. [20] introduced the generalized version of the aforesaid subdivided operations of graphs denoted by $S_k, R_k, Q_k,$ and $T_k$, where $k \geq 1$ is counting number. They also defined the generalized $F$-sums graphs using these generalized operations and calculated their 1st and 2nd Zagreb indices. In the present work, we compute the 1st general Zagreb index of the generalized $F$-sums graphs.
\[ M_1(\Gamma) = \sum_{uv \in E(\Gamma)} [d_1(u) + d_1(v)], \]
\[ M_2(\Gamma) = \sum_{uv \in E(\Gamma)} [d_2(u) + d_2(v)]. \]

These two descriptors of the graph were introduced by Trinajstić and Gutman [15]. Such type of TT's have been utilized to discuss the QSAR/QSPR of the different chemical structures such as chirality, complexity, hetero-system, ZE-isomers, π electron energy, and branching [9, 10].

**Definition 2.** If \( R \) is the real number, \( \gamma \in R - \{0, 1\} \), and \( \Gamma \) be a connected graph, then the 1st general Zagreb topological index is given as
\[ M^{\gamma}(\Gamma) = \sum_{uv \in E(\Gamma)} [d_1^{-\gamma}(u) + d_1^{-\gamma}(v)]. \]

**Lemma 1.** For \( F_k \in \{S_k, R_k, Q_k, T_k\} \) and \( (x, y) \in \Gamma_1 + F_k \Gamma_2 \), the degree of \((x, y)\) in \( \Gamma_1 + F_k \Gamma_2 \) is
\begin{align*}
(i) & \quad d_{\Gamma_1 + F_k \Gamma_2}(x, y) = \begin{cases} d_{\Gamma_1}(x) + d_{\Gamma_2}(y), & \text{if } x \in V(\Gamma_1) \wedge y \in V(\Gamma_2), \\ 2, & \text{if } x \in V(S_k(\Gamma_1)) \wedge y \in V(\Gamma_2), \end{cases} \\
(ii) & \quad d_{\Gamma_1 + F_k \Gamma_2}(x, y) = \begin{cases} d_{\Gamma_2}(x) + d_{\Gamma_1}(y), & \text{if } x \in V(\Gamma_1) \wedge y \in V(\Gamma_2), \\ 2, & \text{if } x \in V(S_k(\Gamma_1)) \wedge y \in V(\Gamma_2), \end{cases} \\
(iii) & \quad d_{\Gamma_1 + F_k \Gamma_2}(x, y) = \begin{cases} d_{\Gamma_1}(x) + d_{\Gamma_2}(y), & \text{if } x \in V(\Gamma_1) \wedge y \in V(\Gamma_2), \\ d_{Q_k}(x, y), & \text{if } x \in V(Q_k(\Gamma_1)) \wedge y \in V(\Gamma_2), \end{cases} \\
(iv) & \quad d_{\Gamma_1 + F_k \Gamma_2}(x, y) = \begin{cases} d_{\Gamma_1}(x) + d_{\Gamma_2}(y), & \text{if } x \in V(\Gamma_1) \wedge y \in V(\Gamma_2), \\ d_{T_k}(x, y), & \text{if } x \in V(T_k(\Gamma_1)) \wedge y \in V(\Gamma_2). \end{cases}
\end{align*}
3. Main Results

The main results of FGZ index of the generalized $F$-sum graphs are presented in this section.

**Theorem 1.** Let $\Gamma_1$ and $\Gamma_2$ be two simple graphs and $\gamma \in N - \{0, 1\}$. The FGZ index of the generalized $S$-sum graph $\Gamma_1 + S_{\gamma} \Gamma_2$ is

\[
M^r(\Gamma_1 + S_{\gamma} \Gamma_2) = \sum_{i=0}^{\alpha} \binom{\alpha}{i} (M^{\alpha-i}_{\Gamma_1})(M^{\gamma}_{\Gamma_2}) + n_{\Gamma_1} M^{\gamma}_{S_{\gamma}}(\Gamma_1) + \sum_{i=1}^{\alpha} \binom{\alpha}{i} M^i_{\Gamma_1} M^{\gamma-i}_{\Gamma_2} + 2^{\gamma+1}(k-1)n_{\Gamma_2} e_{\Gamma_1},
\]

where $N$ is the set of natural numbers and $\alpha = \gamma - 1$. 

![Figure 1](image1.png)

**Figure 1:** (a) $\Gamma_1$, (b) $S_2(\Gamma)$, (c) $R_2(\Gamma)$, (d) $Q_2(\Gamma)$, and (e) $T_2(\Gamma)$.

![Figure 2](image2.png)

**Figure 2:** (a) $\Gamma_1 \cong P_3$, (b) $\Gamma_2 \cong P_2$. (c) $\Gamma_1 + S_2 \Gamma_2$, (d) $\Gamma_1 + R_2 \Gamma_2$. 

![Figure 3](image3.png)
Proof. Let

\[ M'(\Gamma_1 + \Gamma_2) = \sum_{(a,b) \in V(\Gamma_1 + \Gamma_2)} d_{1+\gamma}^2(a,b). \]  

(7)

\[ M'(\Gamma_1 + \Gamma_2) = \sum_{a \in V(\Gamma_1)} \sum_{b \in V(\Gamma_2)} \left[ d_{1+\gamma}^a(a,b) + d_{1+\gamma}^a(b,c) \right] + \sum_{b \in V(\Gamma_2)} \sum_{c \in V(\Gamma_2)} \left[ d_{1+\gamma}^b(a,b) + d_{1+\gamma}^b(b,c) \right]  

+ \sum_{b \in V(\Gamma_1)} \sum_{c \in V(\Gamma_2)} \sum_{d \in V(\Gamma_2)} \left[ d_{1+\gamma}^c(a,b) + d_{1+\gamma}^c(b,c) \right]. \]  

(8)

For every vertex \( a \in V(\Gamma_1) \) and edge \( b,d \in E(\Gamma_2) \), then 1st term of (8) will be

\[ \sum_{a \in V(\Gamma_1)} \sum_{b \in V(\Gamma_2)} \sum_{c \in V(\Gamma_2)} \left[ d_{1+\gamma}^a(a,b) + d_{1+\gamma}^a(b,c) \right] = \sum_{a \in V(\Gamma_1)} \sum_{b \in V(\Gamma_2)} \left[ \sum_{i=0}^{\alpha} \binom{\alpha}{i} d_{1+\gamma}^{\alpha-i}(a) d_{1+\gamma}^{\alpha}(b) \right] + \sum_{b \in V(\Gamma_2)} \sum_{c \in V(\Gamma_2)} \left[ d_{1+\gamma}^{\alpha}(a) + d_{1+\gamma}^{\alpha}(b) \right]  

+ \sum_{b \in V(\Gamma_1)} \sum_{c \in V(\Gamma_2)} \sum_{d \in V(\Gamma_2)} \left[ d_{1+\gamma}^{\alpha}(a) + d_{1+\gamma}^{\alpha}(b) \right] \]  

(9)
Theorem 2. Let \( \Gamma_1 \) and \( \Gamma_2 \) be two simple graphs and \( \gamma \in N - \{0, 1\} \). The FGZ index of the generalized R-sum \( \Gamma_1 + R_\gamma \Gamma_2 \) graph is

\[
M^\gamma(\Gamma_1 + R_\gamma \Gamma_2) = \sum_{i=0}^{\frac{\alpha}{i}} \binom{\alpha}{i} \left( M_i^{\alpha-i} \right) \left( M_i^{1+i} \right) + n_\Gamma M_2^{\beta} + \sum_{i=1}^{\frac{\alpha}{i}} \binom{\alpha}{i} M_i^{(1-\gamma)M_i^{\alpha-i}} + 2^{\alpha+1}(k-1)\eta_1 e_{\Gamma_1}.
\]

where \( N \) is the set of natural numbers and \( \alpha = \gamma - 1 \).

Proof. Then by definition, we have,

\[
M^\gamma(\Gamma_1 + R_\gamma \Gamma_2) = \sum_{(a,b)\in (\Gamma_1 + R_\gamma \Gamma_2)} d_{1+a,1+b}^{\alpha}.
\]

For \( \alpha = \gamma - 1 \), the above equation is consider as
\[ M^T(\Gamma_1 + R_k \Gamma_2) = \sum_{(a,b) \in \mathcal{E}(\Gamma_1 + R_k \Gamma_2)} \left[ d^e_{r_{\Gamma_1} + R_k \Gamma_2}(a,b) + d^e_{r_{\Gamma_1} + R_k \Gamma_2}(c,d) \right] \]

\[ = \sum_{a \in V(\Gamma_1)} \sum_{b \in E(\Gamma_1)} \left[ d^a_{r_{\Gamma_1} + R_k \Gamma_2}(a,b) + d^a_{r_{\Gamma_1} + R_k \Gamma_2}(a,d) \right] + \sum_{b \in V(\Gamma_1)} \sum_{a \in \mathcal{E}(\Gamma_1)} \left[ d^a_{r_{\Gamma_1} + R_k \Gamma_2}(a,b) + d^a_{r_{\Gamma_1} + R_k \Gamma_2}(b,c) \right] \]

\[ = \sum_{a \in V(\Gamma_1)} \sum_{b \in \mathcal{E}(\Gamma_1)} \left[ d^a_{r_{\Gamma_1} + R_k \Gamma_2}(a,b) + d^a_{r_{\Gamma_1} + R_k \Gamma_2}(a,d) \right] + \sum_{b \in \mathcal{E}(\Gamma_1)} \sum_{a \in V(\Gamma_1)} \left[ d^a_{r_{\Gamma_1} + R_k \Gamma_2}(a,b) + d^a_{r_{\Gamma_1} + R_k \Gamma_2}(b,c) \right] \]

\[ + \sum_{b \in \mathcal{E}(\Gamma_1)} \sum_{a \in V(\Gamma_1)} \left[ d^a_{r_{\Gamma_1} + R_k \Gamma_2}(a,b) + d^a_{r_{\Gamma_1} + R_k \Gamma_2}(b,c) \right] \sum_{b \in \mathcal{E}(\Gamma_1)} \left[ d^e_{r_{\Gamma_1} + R_k \Gamma_2}(a,b) + d^e_{r_{\Gamma_1} + R_k \Gamma_2}(c,b) \right]. \]

For every vertex \( a \in V(\Gamma_1) \) & edge \( b \in E(\Gamma_2) \), then the 1st term of (16) is

\[ \sum_{a \in V(\Gamma_1)} \sum_{b \in \mathcal{E}(\Gamma_1)} \left[ d^a_{r_{\Gamma_1} + R_k \Gamma_2}(a,b) + d^a_{r_{\Gamma_1} + R_k \Gamma_2}(a,d) \right] = \sum_{a \in V(\Gamma_1)} \sum_{b \in \mathcal{E}(\Gamma_1)} \left[ \sum_{i=0}^{a} \binom{\alpha}{i} d^{\alpha-i}_{r_{\Gamma_1} + R_k \Gamma_2}(a), d^i_{\Gamma_2}(b) \right] \]

\[ = \sum_{i=0}^{a} \binom{\alpha}{i} \sum_{a \in V(\Gamma_1)} \sum_{b \in \mathcal{E}(\Gamma_1)} d^{\alpha-i}_{r_{\Gamma_1} + R_k \Gamma_2}(a), d^i_{\Gamma_2}(b) \]

\[ = \sum_{i=0}^{a} \binom{\alpha}{i} \sum_{a \in V(\Gamma_1)} \sum_{b \in \mathcal{E}(\Gamma_1)} (2d_{\Gamma_1}(a))^{\alpha-i} d^i_{\Gamma_2}(b) \]

\[ = \sum_{i=0}^{a} \binom{\alpha}{i} \sum_{b \in \mathcal{E}(\Gamma_1)} (2d^{\alpha-i}_{\Gamma_1} M_{\Gamma_1}^{\alpha-i} \sum_{b \in \mathcal{E}(\Gamma_1)} d^i_{\Gamma_2}(b) + d^i_{\Gamma_2}(d)) \]

\[ = \sum_{i=0}^{a} \binom{\alpha}{i} (2^{\alpha-i} M_{\Gamma_1}^{\alpha-i} M_{\Gamma_2}^{\alpha-i+1}). \]
For every vertex $b \in V(\Gamma_2) \& \text{ edge } ac \in E(R_k(\Gamma_1))a,c \in V(\Gamma_1)$, then the 2nd term of (16) will be

$$\sum_{b \in V(\Gamma_2)} \sum_{ac \in E(R_k(\Gamma_1))} \left[ d_{11}^{\alpha} + d_{12}^{\alpha} (a, b) + d_{21}^{\alpha} + d_{22}^{\alpha} (b, c) \right]$$

$$= \sum_{b \in V(\Gamma_2)} \sum_{ac \in E(R_k(\Gamma_1))} \left[ \sum_{i=0}^{\alpha} \left( \begin{array}{c} \alpha \\ i \end{array} \right) d_{R_k(\Gamma_1)}^{\alpha-i} (a) . d_i^{\alpha-i} (b) \right] + \sum_{i=0}^{\alpha} \left( \begin{array}{c} \alpha \\ i \end{array} \right) d_{R_k(\Gamma_1)}^{\alpha-i} (c) . d_i^{\alpha-i} (b) \right]$$

$$= \sum_{ac \in E(R_k(\Gamma_1))} \sum_{b \in V(\Gamma_2)} \left[ \sum_{i=0}^{\alpha} \left( \begin{array}{c} \alpha \\ i \end{array} \right) d_i^{\alpha-i} (a) \left[ d_{R_k(\Gamma_1)}^{\alpha-i} (a) + d_{R_k(\Gamma_1)}^{\alpha-i} (c) \right] \right]$$

$$= \sum_{ac \in E(R_k(\Gamma_1))} \sum_{b \in V(\Gamma_2)} \left[ \sum_{i=0}^{\alpha} \left( \begin{array}{c} \alpha \\ i \end{array} \right) M_i^{\alpha-i} \left[ (2d_{11}^{\alpha-i} (a))^{\alpha-i} + (2d_{12}^{\alpha-i} (c))^{\alpha-i} \right] \right]$$

$$= \sum_{ac \in E(R_k(\Gamma_1))} \sum_{b \in V(\Gamma_2)} \left[ \sum_{i=0}^{\alpha} \left( \begin{array}{c} \alpha \\ i \end{array} \right) M_i^{\alpha-i} \left[ 2^{\alpha-i} (d_{11}^{\alpha-i} (a)) + 2^{\alpha-i} (d_{12}^{\alpha-i} (c)) \right] \right]$$

$$= \sum_{ac \in E(R_k(\Gamma_1))} \sum_{b \in V(\Gamma_2)} \left[ \sum_{i=0}^{\alpha} \left( \begin{array}{c} \alpha \\ i \end{array} \right) M_i^{\alpha-i} \left[ 2^{\alpha-i} (d_{11}^{\alpha-i} (a)) + 2^{\alpha-i} (d_{12}^{\alpha-i} (c)) \right] \right]$$

$$= \sum_{ac \in E(R_k(\Gamma_1))} \sum_{b \in V(\Gamma_2)} \left[ \sum_{i=0}^{\alpha} \left( \begin{array}{c} \alpha \\ i \end{array} \right) M_i^{\alpha-i} \left[ (2^{\alpha-i} (d_{11}^{\alpha-i} (a)) + 2^{\alpha-i} (d_{12}^{\alpha-i} (c)) \right] \right]$$

$$= \sum_{ac \in E(R_k(\Gamma_1))} \sum_{b \in V(\Gamma_2)} \left[ \sum_{i=0}^{\alpha} \left( \begin{array}{c} \alpha \\ i \end{array} \right) M_i^{\alpha-i} \left[ (2^{\alpha-i} (d_{11}^{\alpha-i} (a)) + 2^{\alpha-i} (d_{12}^{\alpha-i} (c)) \right] \right]$$

$$= \sum_{ac \in E(R_k(\Gamma_1))} \sum_{b \in V(\Gamma_2)} \left[ \sum_{i=0}^{\alpha} \left( \begin{array}{c} \alpha \\ i \end{array} \right) M_i^{\alpha-i} \left[ (2^{\alpha-i} (d_{11}^{\alpha-i} (a)) + 2^{\alpha-i} (d_{12}^{\alpha-i} (c)) \right] \right]$$

For every vertex $b \in V(\Gamma_2) \& \text{ edge } ac \in E(R_k(\Gamma_1))a \in V(\Gamma_1), c \in v(R_k(\Gamma_1)) = V(\Gamma_1)$. Since we have $d_{R_k(\Gamma_1)}^{\alpha-i} (a) = 2d_{11}^{\alpha-i} (a) \forall a \in V(\Gamma_1)$ also $d_{R_k(\Gamma_1)}^{\alpha-i} (c) = 2 \forall c \in V(R_k(\Gamma_1)) - V(\Gamma_1)$. So the 3rd term of (16) will be

$$= \sum_{ac \in E(R_k(\Gamma_1))} \sum_{b \in V(\Gamma_2)} \left[ d_{11}^{\alpha} + d_{12}^{\alpha} (a, b) + d_{21}^{\alpha} + d_{22}^{\alpha} (b, c) \right]$$

$$= \sum_{ac \in E(R_k(\Gamma_1))} \sum_{b \in V(\Gamma_2)} \left[ \sum_{i=0}^{\alpha} \left( \begin{array}{c} \alpha \\ i \end{array} \right) d_{R_k(\Gamma_1)}^{\alpha-i} (a) . d_i^{\alpha-i} (b) + d_{R_k(\Gamma_1)}^{\alpha-i} (c) . d_i^{\alpha-i} (b) \right]$$

(19)
Here $d_{R_k}(\Gamma_1) = 2^x$ and $d_{R_k}(\Gamma_1) \cdot (a) = (2d_{\Gamma_1}(a))^\alpha$: 

\[ d_{R_k}(\Gamma_1) = \sum_{b \in V(\Gamma_2) \setminus a \in V(\Gamma_2)} \sum_{c \in V(\Gamma_2)} \left[ \sum_{i=0}^{\alpha} \binom{\alpha}{i} (2d_{\Gamma_1}(a))^{\alpha-i} d^{\alpha-i}_{\Gamma_2}(b) + 2^x \right] \]

\[ = \sum_{b \in V(\Gamma_2) \setminus a \in V(\Gamma_2)} \sum_{c \in V(\Gamma_2)} \left[ \sum_{i=0}^{\alpha} \binom{\alpha}{i} (2d_{\Gamma_1}(a))^{\alpha-i} d^{\alpha-i}_{\Gamma_2}(b) + 2^x \right] \]

\[ = \sum_{i=0}^{\alpha} \binom{\alpha}{i} \left[ \sum_{b \in V(\Gamma_2) \setminus a \in V(\Gamma_2)} \sum_{c \in V(\Gamma_2)} (2d_{\Gamma_1}(a))^{\alpha-i} d^{\alpha-i}_{\Gamma_2}(b) + 2^x \right] \]

\[ = \sum_{i=0}^{\alpha} \binom{\alpha}{i} \left[ \sum_{b \in V(\Gamma_2) \setminus a \in V(\Gamma_2)} \sum_{c \in V(\Gamma_2)} (2d_{\Gamma_1}(a))^{\alpha-i} d^{\alpha-i}_{\Gamma_2}(b) + 2^x \right] \]

\[ = \sum_{i=0}^{\alpha} \binom{\alpha}{i} (2d_{\Gamma_1}(a))^{\alpha-i}(d^{\alpha-i}_{\Gamma_2}(b))_{\Gamma_2} + 2^x \epsilon_{\Gamma_2} \eta_{\Gamma_2} \]

\[ = \sum_{i=0}^{\alpha} \binom{\alpha}{i} (2d_{\Gamma_1}(a))^{\alpha-i}(d^{\alpha-i}_{\Gamma_2}(b))_{\Gamma_2} + 2^x \epsilon_{\Gamma_2} \eta_{\Gamma_2} , \]

and the 4th term of (5) is

\[ \sum_{b \in V(\Gamma_2) \setminus a \in V(\Gamma_2)} \sum_{c \in V(\Gamma_2)} \left[ d^{\alpha}_{\Gamma_1 + \gamma R_k \Gamma_2}(a, b) + d^{\alpha}_{\Gamma_1 + \gamma R_k \Gamma_2}(c, b) \right] = \sum_{b \in V(\Gamma_2) \setminus a \in V(\Gamma_2)} \sum_{c \in V(\Gamma_2)} [2^x + 2^x]. \] (21)

Since in this case $|E(S_k(\Gamma_1))| = (k-1)|\epsilon_{\Gamma_1}$, we have

\[ = 2^{\alpha+1}(k-1)n_{\Gamma_2} \epsilon_{\Gamma_1}. \] (22)

Using (17), (18), (20), and (22) in (16), then we have

\[ M(\Gamma_1 + \gamma R_k \Gamma_2) = \sum_{i=0}^{\alpha} \binom{\alpha}{i} 2^{\alpha-i} M^{\alpha-i}_{\Gamma_1} M^{\alpha+1}_{\Gamma_2} + 2 \sum_{i=0}^{\alpha} \binom{\alpha}{i} 2^{\alpha-i} M^{\alpha-i}_{\Gamma_1} M^{\alpha}_{\Gamma_2} + 2^x \epsilon_{\Gamma_2} \eta_{\Gamma_2} + 2^{\alpha+1}(k-1)n_{\Gamma_2} \epsilon_{\Gamma_1}. \] (23)

Theorem 3. Let $\Gamma_1$ and $\Gamma_2$ be two simple graphs and $\gamma \in N$ – $\{0, 1\}$. The FGZ index of the generalized Q-sum $\Gamma_1 + \gamma R_k \Gamma_2$ graph is

\[ M^{\gamma}(\Gamma_1 + \gamma R_k \Gamma_2) = \sum_{i=0}^{\alpha} \binom{\alpha}{i} (M^{\alpha-i}_{\Gamma_1})(M^{\alpha}_{\Gamma_2}) + \sum_{i=0}^{\alpha} \binom{\alpha}{i} (M^{\alpha-i}_{\Gamma_1})(M^{\alpha}_{\Gamma_2}) + 2n_{\Gamma_2} \sum_{i=0}^{\alpha} \binom{\alpha}{i} (d^{\alpha-i}_{\Gamma_1}(u), d^{\alpha}_{\Gamma_2}(v)) \]

\[ + n_{\Gamma_1} \sum_{uv \in E(\Gamma_1) \cup E(\Gamma_2) \cup E(\Gamma_2)} \left[ \sum_{i=0}^{\alpha} \binom{\alpha}{i} d^{\alpha-i}_{\Gamma_1}(u), d^{\alpha}_{\Gamma_2}(v) \right] + \sum_{i=0}^{\alpha} \binom{\alpha}{i} d^{\alpha-i}_{\Gamma_1}(v), d^{\alpha}_{\Gamma_2}(u) \]

\[ + 2(k-1)n_{\Gamma_2} \sum_{uv \in E(\Gamma_1)} [d^{\alpha}_{\Gamma_1}(u) + d^{\alpha}_{\Gamma_1}(v)], \] (24)

where $N$ is the set of natural numbers and $\alpha = \gamma - 1$.

Proof. Then by definition, we have
\[ M^r(\Gamma_1 + \Omega_a \Gamma_2) = \sum_{(a,b) \in E(\Gamma_1 + \Omega_a \Gamma_2)} d_{\Gamma_1 + \Omega_a \Gamma_2}^r (a, b) = \sum_{(a,b) \in \Omega_a \Gamma_2} \left[ d_{\Gamma_1 + \Omega_a \Gamma_2}^r (a, b) + d_{\Gamma_1 + \Omega_a \Gamma_2}^r (c, d) \right] \]
\[ = \sum_{a \in V(\Gamma_1)} \sum_{b \in E(\Gamma_1)} \left[ d_{\Gamma_1 + \Omega_a \Gamma_2}^r (a, b) + d_{\Gamma_1 + \Omega_a \Gamma_2}^r (a, d) \right] + \sum_{b \in V(\Gamma_2)} \sum_{a \in \Omega_a \Gamma_2} \left[ d_{\Gamma_1 + \Omega_a \Gamma_2}^r (a, b) + d_{\Gamma_1 + \Omega_a \Gamma_2}^r (b, c) \right]. \] (25)

For every vertex \( a \in V(\Gamma_1) \) & edge \( b \in E(\Gamma_2) \), then the 1st term of (25) will be

\[
\sum_{a \in V(\Gamma_1)} \sum_{b \in E(\Gamma_1)} \left[ d_{\Gamma_1 + \Omega_a \Gamma_2}^r (a, b) + d_{\Gamma_1 + \Omega_a \Gamma_2}^r (a, d) \right] \\
= \sum_{a \in V(\Gamma_1)} \sum_{b \in E(\Gamma_1)} \left[ \sum_{i=0}^{\alpha} \binom{\alpha}{i} d_{\Omega_a \Gamma_2}^r (a) d_{\Gamma_1}^r (b) + \sum_{j=0}^{\alpha} \binom{\alpha}{j} d_{\Omega_a \Gamma_2}^r (a) d_{\Gamma_1}^r (d) \right] \\
= \sum_{a \in V(\Gamma_1)} \sum_{b \in E(\Gamma_1)} \left[ \sum_{i=0}^{\alpha} \binom{\alpha}{i} d_{\Gamma_1}^r (a) \left[ d_{\Gamma_1}^r (b) + d_{\Gamma_1}^r (d) \right] \right] \\
= \sum_{a \in V(\Gamma_1)} \left[ \sum_{i=0}^{\alpha} \binom{\alpha}{i} d_{\Gamma_1}^r (a) \right] \left( M_1^{r+1} \right) \\
= \sum_{i=1}^{\alpha} \binom{\alpha}{i} (M_1^{r+1}) (M_1^{r+1}). \] (26)

For every vertex \( b \in V(\Gamma_2) \) & edge \( ac \in E(\Omega_a \Gamma_1) \) \( a, c \in V(\Gamma_1) \), then the 2nd term of equation (25) will be

\[
\sum_{b \in V(\Gamma_2)} \sum_{a \in \Omega_a \Gamma_2} \left[ d_{\Gamma_1 + \Omega_a \Gamma_2}^r (a, b) + d_{\Gamma_1 + \Omega_a \Gamma_2}^r (b, c) \right] \\
= \sum_{b \in V(\Gamma_2)} \sum_{a \in \Omega_a \Gamma_2} \sum_{c \in V(\Gamma_1)} \left[ d_{\Gamma_1 + \Omega_a \Gamma_2}^r (a, b) + d_{\Gamma_1 + \Omega_a \Gamma_2}^r (b, c) \right] \\
+ \sum_{b \in V(\Gamma_2)} \sum_{a \in \Omega_a \Gamma_2} \sum_{c \in V(\Gamma_1)} \left[ d_{\Gamma_1 + \Omega_a \Gamma_2}^r (a, b) + d_{\Gamma_1 + \Omega_a \Gamma_2}^r (b, c) \right]. \] (27)
Now $\forall b \in V(\Gamma_2)$, $ac \in E(Q_k(\Gamma_1))$ if $a \in V(\Gamma_1)$ and $c \in V(Q_k(\Gamma_1)) - V(\Gamma_1)$; the 1st term of (27) will be

\[
\sum_{b \in V(\Gamma_2)} \sum_{ac \in E(Q_k(\Gamma_1)) \setminus V(\Gamma_1) \setminus V(\Gamma_1)} \left[ a^{d_{\Gamma_1}^{a} + d_{\Gamma_2}^{c}} (a,b) \right] + \left[ d_{\Gamma_1}^{a} + d_{\Gamma_2}^{c} (b,c) \right]
\]

\[
= \sum_{b \in V(\Gamma_2)} \sum_{ac \in E(Q_k(\Gamma_1)) \setminus V(\Gamma_1) \setminus V(\Gamma_1)} \left[ d_{\Gamma_1}^{a} (a) + d_{\Gamma_2}^{c} (b) \right] + \left[ d_{\Gamma_1}^{a} (c) \right]
\]

\[
= \sum_{b \in V(\Gamma_2)} \sum_{ac \in E(Q_k(\Gamma_1)) \setminus V(\Gamma_1) \setminus V(\Gamma_1)} \left[ \sum_{i=0}^{\alpha} \binom{\alpha}{i} d_{\Gamma_1}^{a-i} (a) d_{\Gamma_2}^{c} (b) \right] + \left[ d_{\Gamma_1}^{a} (c) \right]
\]

\[
= \sum_{b \in V(\Gamma_2)} \sum_{ac \in E(Q_k(\Gamma_1)) \setminus V(\Gamma_1) \setminus V(\Gamma_1)} \left[ \sum_{i=0}^{\alpha} \binom{\alpha}{i} d_{\Gamma_1}^{a-i} (a) d_{\Gamma_2}^{c} (b) \right] + \left[ d_{\Gamma_1}^{a} (c) \right]
\]

\[
= \sum_{b \in V(\Gamma_2)} \sum_{ac \in E(Q_k(\Gamma_1)) \setminus V(\Gamma_1) \setminus V(\Gamma_1)} \left[ \sum_{i=0}^{\alpha} \binom{\alpha}{i} M_{\Gamma_1}^{a-i} M_{\Gamma_2}^{c} + n_{\Gamma_2} \sum_{ac \in E(Q_k(\Gamma_1)) \setminus V(\Gamma_1) \setminus V(\Gamma_1)} d_{\Gamma_1}^{a} (c) \right]
\]

\[
= \sum_{b \in V(\Gamma_2)} \sum_{ac \in E(Q_k(\Gamma_1)) \setminus V(\Gamma_1) \setminus V(\Gamma_1)} \left[ \sum_{i=0}^{\alpha} \binom{\alpha}{i} M_{\Gamma_1}^{a-i} M_{\Gamma_2}^{c} + 2 \sum_{ac \in E(\Gamma_1)} \left( d_{\Gamma_1}^{a} (u) + d_{\Gamma_1}^{a} (v) \right) \right] + \left[ 2 \sum_{i=0}^{\alpha} \binom{\alpha}{i} d_{\Gamma_1}^{a-i} (u) d_{\Gamma_1}^{c} (v) \right]
\]

Now $\forall b \in V(\Gamma_2)$ & edge $ac \in E(Q_k(\Gamma_1))$ if the vertex $a, c \in V(Q_k(\Gamma_1)) - V(\Gamma_1)$. Then the 2nd term of equation (27) splits into two parts for the vertices $a$ and $c$, then the equation will be
Using (26), (28), (29), and (30) in (25), we get the required result:

\[
\sum_{\text{be } V(G_2) \text{ acc } E(Q_0, (G_2))} \sum_{\text{ac } V(Q_0, (G_2)) - V(G_1)} \left[ d_{(G_1 + G_2)}^{\alpha-i}(a, b)^{a} + d_{(G_1 + G_2)}^{\alpha-i}(b, c)^{a} \right] = \sum_{\text{be } V(G_2) \text{ acc } E(Q_0, (G_2))} \sum_{\text{ac } V(Q_0, (G_2)) - V(G_1)} \left[ d_{(G_1 + G_2)}^{\alpha-i}(a)^{a} + d_{(G_1 + G_2)}^{\alpha-i}(c)^{a} \right]
\]

\[
= \sum_{\text{be } V(G_2) \text{ acc } E(Q_0, (G_2))} \sum_{\text{ac } V(Q_0, (G_2)) - V(G_1)} \left[ \left( \sum_{i=0}^{\alpha} (\alpha, i) \right) d_{(G_1 + G_2)}^{\alpha-i}(i) \cdot d_{(G_1)}^{\alpha-i}(i) \right] + \left( \sum_{i=0}^{\alpha} d_{(G_1 + G_2)}^{\alpha-i}(u) \cdot d_{(G_1)}^{\alpha-i}(u) \right) + \left( \sum_{i=0}^{\alpha} d_{(G_1 + G_2)}^{\alpha-i}(v) \cdot d_{(G_1)}^{\alpha-i}(v) \right)
\]

\[
= c_{(G_1 + G_2)} \sum_{\text{be } V(G_2) \text{ acc } E(Q_0, (G_2))} \sum_{\text{ac } V(Q_0, (G_2)) - V(G_1)} \left[ \left( \sum_{i=0}^{\alpha} (\alpha, i) \right) d_{(G_1 + G_2)}^{\alpha-i}(i) \cdot d_{(G_1)}^{\alpha-i}(i) \right] + \left( \sum_{i=0}^{\alpha} d_{(G_1 + G_2)}^{\alpha-i}(u) \cdot d_{(G_1)}^{\alpha-i}(u) \right) + \left( \sum_{i=0}^{\alpha} d_{(G_1 + G_2)}^{\alpha-i}(v) \cdot d_{(G_1)}^{\alpha-i}(v) \right)
\]

\[
= 2(k - 1) \sum_{\text{be } V(G_2) \text{ acc } E(Q_0, (G_2))} \sum_{\text{ac } V(Q_0, (G_2)) - V(G_1)} \left[ d_{(G_1 + G_2)}^{\alpha-i}(i) \cdot d_{(G_1)}^{\alpha-i}(i) \right] + \left( \sum_{i=0}^{\alpha} d_{(G_1 + G_2)}^{\alpha-i}(u) \cdot d_{(G_1)}^{\alpha-i}(u) \right) + \left( \sum_{i=0}^{\alpha} d_{(G_1 + G_2)}^{\alpha-i}(v) \cdot d_{(G_1)}^{\alpha-i}(v) \right)
\]

(29)

(30)

**Theorem 4.** Let \( G_1 \) and \( G_2 \) be two simple graphs. The FGZ index of the generalized T-sum graph \( G_1 + T_1 G_2 \) is

\[
M^\alpha(G_1 + T_1 G_2) = \sum_{i=0}^{\alpha} \left( \begin{array}{c} \alpha \\ i \end{array} \right) (2)^{\alpha-i} M_{(G_1 + T_1 G_2)}^{\alpha-i} + \sum_{i=0}^{\alpha} \left( \begin{array}{c} \alpha \\ i \end{array} \right) M_{(G_1 + T_1 G_2)}^{\alpha-i} (2)^{\alpha-i} M_{(G_1 + T_1 G_2)}^{\alpha-i}
\]

\[
+ 2n_{(G_1 + T_1 G_2)} \sum_{i=0}^{\alpha} \left( \begin{array}{c} \alpha \\ i \end{array} \right) \sum_{\text{be } E(G_1) \text{ acc } E(G_2)} \left( d_{(G_1 + T_1 G_2)}^{\alpha-i}(u) \cdot d_{(G_1 + T_1 G_2)}^{\alpha-i}(v) \right) + n_{(G_1 + T_1 G_2)}
\]

\[
\sum_{\text{be } E(G_1) \text{ acc } E(G_2)} \left[ \left( \sum_{i=0}^{\alpha} \left( \begin{array}{c} \alpha \\ i \end{array} \right) d_{(G_1 + T_1 G_2)}^{\alpha-i}(u) \cdot d_{(G_1 + T_1 G_2)}^{\alpha-i}(v) \right) + \left( \sum_{i=0}^{\alpha} \left( \begin{array}{c} \alpha \\ i \end{array} \right) d_{(G_1 + T_1 G_2)}^{\alpha-i}(v) + d_{(G_1 + T_1 G_2)}^{\alpha-i}(u) \right) \right]
\]

\[
+ 2(k - 1)n_{(G_1 + T_1 G_2)} \sum_{\text{be } E(G_1) \text{ acc } E(G_2)} \left[ d_{(G_1 + T_1 G_2)}^{\alpha-i}(u) + d_{(G_1 + T_1 G_2)}^{\alpha-i}(v) \right].
\]

(31)
where \( \gamma \in \mathbb{N}^+ - \{0, 1\} \) and \( \alpha = \gamma - 1 \).

**Proof.** Since we have \( d_{\Gamma_1 + \Gamma_2} (a,b) = d_{\Gamma_1 + \Gamma_2} (a,b) \) for every vertex \( a \in V (\Gamma_1) \) and \( b \in V (\Gamma_2) \), also \( d_{\Gamma_1 + \Gamma_2} (a,b) = d_{\Gamma_1 + \Gamma_2} (a,b) \) for every vertex \( a \in V (T_k (\Gamma_1)) - V (\Gamma_1) \) and \( b \in V (\Gamma_2) \), the result follows by the proof of Theorems 2 and 3.

\[ \begin{align*}
(i)\ M^\gamma (\Gamma_1 + \Gamma_2) &= \sum_{i=0}^{\infty} \binom{\gamma}{i} (M_{\Gamma_1}^i)(M_{\Gamma_2}^i) + n_{\Gamma_1} \cdot M_{\Gamma_1}^0 (\Gamma_1) + \sum_{i=1}^{\infty} \binom{\gamma}{i} M_{\Gamma_1}^{\alpha-1} M_{\Gamma_2}^i + 2^{\gamma+1} (k-1)n_{\Gamma_2} e_{\Gamma_1} \\
(ii)\ M^\gamma (\Gamma_1 + \Gamma_2) &= \sum_{i=0}^{\infty} \binom{\gamma}{i} 2^i M_{\Gamma_1}^i M_{\Gamma_2}^{\alpha-i} + 2 \sum_{i=1}^{\infty} \left( \binom{\gamma}{i} 2^i M_{\Gamma_1}^{\alpha-1} M_{\Gamma_2}^i \right) + 2^\gamma n_{\Gamma_1} n_{\Gamma_2} + 2^{\gamma+1} (k-1)n_{\Gamma_2} e_{\Gamma_1} \\
(iii)\ M^\gamma (\Gamma_1 + \Gamma_2) &= \sum_{i=0}^{\infty} \binom{\gamma}{i} (M_{\Gamma_1}^i)(M_{\Gamma_2}^i) + \sum_{i=1}^{\infty} \binom{\gamma}{i} (M_{\Gamma_1}^{\alpha-1})(M_{\Gamma_2}^i) + 2n_{\Gamma_1} \left[ \sum_{i=0}^{\infty} \binom{\gamma}{i} (d_{\Gamma_1}^i (u) \cdot d_{\Gamma_1}^{\alpha-i} (v)) \right] \\
&+ \sum_{i=0}^{\infty} \binom{\gamma}{i} (d_{\Gamma_1}^i (u) \cdot d_{\Gamma_1}^{\alpha-i} (v)) \\
&+ 2^{\gamma+1} (k-1)n_{\Gamma_2} \sum_{uv \in E (\Gamma_1), uv \in E (\Gamma_1)} [d_{\Gamma_1}^i (u) + d_{\Gamma_1}^i (v)] \\
(iv)\ M^\gamma (\Gamma_1 + \Gamma_2) &= \sum_{i=0}^{\infty} \binom{\gamma}{i} 2^i M_{\Gamma_1}^i M_{\Gamma_2}^{\alpha-i} + \sum_{i=1}^{\infty} \binom{\gamma}{i} 2^i M_{\Gamma_1}^{\alpha-1} M_{\Gamma_2}^i \\
&+ \sum_{i=0}^{\infty} \binom{\gamma}{i} (M_{\Gamma_1}^{\alpha+1})(M_{\Gamma_2}^i) + 2n_{\Gamma_1} \sum_{i=0}^{\infty} \binom{\gamma}{i} \sum_{uv \in E (\Gamma_1)} (d_{\Gamma_1}^i (u) \cdot d_{\Gamma_1}^{\alpha-i} (v)) \\
&+ \sum_{i=0}^{\infty} \binom{\gamma}{i} (d_{\Gamma_1}^i (u) \cdot d_{\Gamma_1}^{\alpha-i} (v)) \\
&+ 2^{\gamma+1} (k-1)n_{\Gamma_2} \sum_{uv \in E (\Gamma_1), uv \in E (\Gamma_1)} [d_{\Gamma_1}^i (u) + d_{\Gamma_1}^i (v)] \end{align*} \]
Proof. The above proof is similar as of Theorems 1–4. Let \( \Gamma_1 \) be a negative integer, so from Theorem 5, Corollary 1 is obtained.

\[
\begin{align*}
\text{(i)} M^t(\Gamma_1+\varsigma_2 \Gamma_2) &= \sum_{i=0}^{\infty} (-1)^i \binom{\alpha + i - 1}{i} (M_{i+1}^t)(M_{\alpha+i+1}^{t+1}) + n_{\tau_1} \sum_{i=0}^{\infty} (-1)^i \binom{\alpha + i - 1}{i} (M_{i+1}^{t+1})(M_{\alpha+i+1}^{t+1}) + 2^{\gamma+1} (k-1)n_{\tau_1} e_t, \\
\text{(ii)} M^t(\Gamma_1+\varsigma_3 \Gamma_2) &= \sum_{i=0}^{\infty} (-1)^i \binom{\alpha + i - 1}{i} 2^{i} M_{i+1}^t M_{\alpha+i+1}^{t+1} + 2 \sum_{i=0}^{\infty} (-1)^i \binom{\alpha + i - 1}{i} 2^{i} M_{i+1}^{t+1} M_{\alpha+i+1}^{t+1} + 2^{\gamma+1} (k-1)n_{\tau_1} e_t, \\
\text{(iii)} M^t(\Gamma_1+\varsigma_1 \Gamma_2) &= \sum_{i=0}^{\infty} (-1)^i \binom{\alpha + i - 1}{i} \sum_{\omega \in E(\Gamma_1)} (d_{\tau_1}^t(u))M_{\alpha+i+1}^{t+1}(v) + n_{\tau_1} \sum_{\omega \in E(\Gamma_1)} \binom{\alpha + i - 1}{i} (M_{i+1}^{t+1})(M_{\alpha+i+1}^{t+1}) + 2 (k-1)n_{\tau_1} [d_{\tau_1}^t(u) + d_{\tau_1}^t(v)], \\
\text{(iv)} M^t(\Gamma_1+\varsigma_4 \Gamma_2) &= \sum_{i=0}^{\infty} (-1)^i \binom{\alpha + i - 1}{i} 2^{i} M_{i+1}^t M_{\alpha+i+1}^{t+1} + 2 \sum_{i=0}^{\infty} (-1)^i \binom{\alpha + i - 1}{i} 2^{i} M_{i+1}^{t+1} M_{\alpha+i+1}^{t+1} + 2^{\gamma+1} (k-1)n_{\tau_1} e_t.
\end{align*}
\]

(34)

4. Applications

Now, we present some examples as applications of the obtained results Theorems 1–4. Also the numerical comparisons are presented in Tables 1–4, and the graphical representations are depicted in Figures 4–7.

\[
1. M^t(P_{m+n} \beta_n) = \sum_{i=0}^{\infty} C_i^\gamma \left[ 2^{\gamma-i} (m-2) + 2 \right] \left[ 2^{i-2} (n-2) + 2 \right] + \sum_{i=0}^{\infty} C_i^\gamma \left[ 2^{\gamma-i} (m-2) + 2 \right] \left[ 2^i (n-2) + 2 \right] + n (2^\gamma (2m - 3) + 2) + 4 (k-1) n (m-1).
\]

(35)

Example 1. Let \( P_m \) and \( P_n \) be two simple graphs with \( m \geq 2 \) and \( n \geq 2 \). Then, we have

2. \( M^t(P_{m+n} \beta_n) = \sum_{i=0}^{\infty} C_i^\gamma \left[ 2^{\gamma-i} (m-2) + 2 \right] \left[ 2^{i-2} (n-2) + 2 \right] + \sum_{i=0}^{\infty} C_i^\gamma \left[ 2^{\gamma-i} (m-2) + 2 \right] \left[ 2^i (n-2) + 2 \right] + 2 (m-1) n + 4 (k-1) n (m-1).
\]

(36)
From Figure 5, it is clear that the behavior of FGZ index of the generalized \( R \)-sum graph \( \Gamma_1^+ R_\Gamma \Gamma_2 \) at \( t = 0 \) is more better than \( t = 1 \) and \( t = 2 \).
From Figure 6, it is clear that the behavior of FGZ index of the generalized $Q$-sum graph $\Gamma_1 + Q_k \Gamma_2$ at $t = 2$ is more better than $t = 0$ and $t = 1$:

\[
M_v'(P_mQ_kP_n) = \sum_{t=0}^{k} C_{T}^{t-1} 2^{t-1-t} (m-2)\left[2^{t-1} (n-2) + 2\right] + 2 \sum_{t=0}^{k} C_{T}^{t-1} 2^{t-1-t} \left[2^{t-1} (m-2) + 2\right] 2((m-2)^t + 2n) + n \sum_{t=0}^{k} C_{T}^{t-1} \left[2^t + 2(m-1)2^{t-1} + 2^{t-1-t}\right] 2(k-1)n 2^{t+1} (m-2) + 2. \tag{38}
\]

From Figure 7, it is clear that the behavior of FGZ index of the generalized $T'$-sum graph $\Gamma_1 + T_k \Gamma_2$ at $t = 0$ is more better than $t = 1$ and $t = 2$.

5. Conclusions

Now, we close our discussion with the following remarks:

(i) For positive integer $k$ and two graphs $\Gamma_1$ & $\Gamma_2$, we have computed FGZ index of the generalized $F$-sums graphs $\Gamma_1 + F_k \Gamma_2$, where generalized $F$-sums graphs are obtained by the different operations of subdivision and Cartesian product on $\Gamma_1$ & $\Gamma_2$.

(ii) The obtained results are also verified and illustrated for the particular classes of graphs.

(iii) The behavior of FGZ index is also analyzed with the help of numerical and graphical presentations.

(iv) However, the problem is still open to compute the different topological indices (degree and distance based) for the generalized $F$-sum graphs.

Data Availability

All the data are included within this paper. However, the reader may contact the corresponding author for more details of the data.

Conflicts of Interest

The authors have no conflicts of interest.

Acknowledgments

The authors are also thankful to Dr. Muhammad Kamran Siddiqui who helped in the graphical analysis. The University of Hail, Saudi Arabia, partially supported the study.

References

[1] H. Gonzalez-Diaz, S. Vilar, L. Santana, and E. Uriarte, "Medicinal chemistry and bioinformatics-current trends in drugs discovery with networks topological indices," Current Topics in Medicinal Chemistry, vol. 7, pp. 1025–1039, 2007.

[2] R. Gozalbes, J. Doucet, and F. Derouin, "Application of topological descriptors in QSAR and drug design: history and
new trends,” Current Drug Target-Infectious Disorders, vol. 2, no. 1, pp. 93–102, 2002.

[3] G. Rücker and C. Rücker, “On topological indices, boiling points, and cycloalkanes,” Journal of Chemical Information and Computer Sciences, vol. 39, no. 5, pp. 788–802, 1999.

[4] M. Randic, “On characterization of molecular branching,” Journal of the American Chemical Society, vol. 97, pp. 6609–6615, 1975.

[5] A. R. Matamala and E. Estrada, “Generalised topological indices: optimisation methodology and physico-chemical interpretation,” Chemical Physics Letters, vol. 410, no. 4-6, pp. 343–347, 2005.

[6] W. Yan, B.-Y. Yang, and Y.-N. Yeh, “The behavior of Wiener indices and polynomials of graphs under five graph decorations,” Applied Mathematics Letters, vol. 20, no. 3, pp. 290–295, 2007.

[7] H. González-Díaz, S. Vilar, L. Santana, and E. Uriarte, “Medicinal chemistry and bioinformatics-current trends in drugs discovery with networks topological indices,” Current Topics in Medicinal Chemistry, vol. 7, no. 10, pp. 1015–1029, 2007.

[8] L. H. Hall and L. B. Kier, Molecular Connectivity in Chemistry and Drug Research, Academic Press, Boston, MA, USA, 1976.

[9] M. V. Diudea, QSAR/QSAR Studies by Molecular Descriptors, NOVA, New York, NY, USA, 2001.

[10] J. Devillers and A. T. Balaban, Topological Indices and Related Descriptors in QSAR and QSAR, Gordon & Breach, Amsterdam, Netherlands, 1999.

[11] R. Todeschini, V. Consonni, R. Mannhold, H. Kubinyi, and H. Timmerman, Handbook of Molecular Descriptors, Wiley-VCH, Weinheim, Germany, 2002.

[12] I. Gutman and O. Polansky, Mathematical Concepts in Organic Chemistry, Springer-Verlag, Berlin, Germany, 1986.

[13] I. Gutman, ”Degree-based topological indices,” Croatica Chemica Acta, vol. 86, no. 4, pp. 351–361, 2013.

[14] H. Wiener, ”Structural determination of paraffin boiling points,” Journal of the American Chemical Society, vol. 69, no. 1, pp. 17–20, 1947.

[15] N. Trinajstić and I. Gutman, ”Graph theory and molecular orbitals. Total ϕ-electron energy of alternant hydrocarbons,” Chemical Physics Letters, vol. 17, no. 4, pp. 535–538, 1972.

[16] M. Eliasi and B. Taeri, ”Four new sums of graphs and their Wiener indices,” Discrete Applied Mathematics, vol. 157, no. 4, pp. 794–803, 2009.

[17] H. Deng, D. Sarala, S. K. Ayyaswamy, and S. Balachandran, ”The Zagreb indices of four operations on graphs,” Applied Mathematics and Computation, vol. 275, pp. 422–431, 2016.

[18] S. Akhter and M. Imran, ”Computing the forgotten topological index of four operations on graphs,” AKCE International Journal of Graphs and Combinatorics, vol. 14, no. 1, pp. 70–79, 2017.

[19] J.-B. Liu, S. Javed, M. Javaid, and K. Shabbir, ”Computing first general Zagreb index of operations on graphs,” IEEE Access, vol. 7, pp. 47494–47502, 2019.

[20] J.-B. Liu, M. Javaid, and H. M. Awais, ”Computing Zagreb indices of the subdivision-related generalized opeations of graphs,” IEEE Access, vol. 7, pp. 105479–105488, 2019.