A Democratically-Optimal Budgeting Algorithm

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Abstract

The budget is the key means for effecting policy in democracies, yet its preparation is typically an opaque and arcane process. Participatory budgeting is making inroads in municipalities, but is usually limited to a small fraction of the total budget as its methods do not scale: They cannot handle quantitative budget items nor hierarchical budget construction.

Here we apply the Condorcet principle to participatory budgeting. We say that a budget is globally-optimal if it is feasible (i.e., within the budget limit) and no other feasible budget is preferred over it by a majority of the voters. A globally-optimal budget does not always exist due to Condorcet cycles; hence, we settle for a democratically-optimal budget, which is globally-optimal up to such cycles. We devise a polynomial-time algorithm that, given a budget proposal, a set of votes (rankings) over it, and a budget limit, produces a democratically-optimal budget. We argue that in practice the resulting budget would most often be either globally-optimal or almost globally-optimal. Our method handles quantitative budget items of arbitrary cost and supports hierarchical budget construction, thus may be applied to entire budgets.

1 Introduction

A basic decision any society (including a multi-agent system) faces is how to distribute its resources. While different members of a society might wish to spend the available resources differently, the amount of available resources is limited, thus there is a need to decide which desires will be funded and which will not. Even in democracies, typically a limited and exclusive group prepares a draft budget, which is then presented to the governing body as a last-minute take-it-or-leave-it proposal. Preferences of the society as a whole are generally not taken into account.

Participatory budgeting [4] aims at rectifying this situation, by offering ways for the whole society to participate in the budget construction. The first to adopt some form of participatory budgeting was Brazil workers’ party [15]. Since then, various societies have adopted similar ideas [7], where usually only
a small fraction of a municipality’s annual budget is decided directly by the residents. After a deliberative phase, participants vote on projects such as schools, bike roads, etc., usually by some variants of $k$-Approval [10], where each resident specifies $k$ projects of highest priority (other types of preference elicitation are discussed later); the approvals of each project are then counted, and, starting from the most-popular project, a set of projects is chosen until the available funds are exhausted. Having its merits, $k$-Approval however does not allow voters to fully state their preferences, and does not guarantee any form of satisfaction of the population from the selected projects.

Here we apply the Condorcet principle to participatory budgeting. We first outline our approach informally; all emphasized terms are defined later. A budget proposal is a set of desired budget items and their quantities, on which voters vote, where each vote is a ranking of the items and the associated quantities; we allow voters to specify weak orders. Budget items have a cost function associated with them, which specifies the cost of any quantity of the item. A budgeting algorithm then takes the votes and a budget limit and produces a feasible budget, whose cost is within limit. We say that a budget is exhaustive if it does not “leave money on the table”. We compare two budgets by considering their symmetric difference; intuitively, a voter would be more disappointed if higher-ranked items are unbudgeted than if lower-ranked items are unbudgeted. Formally, we say that a vote prefers one budget over another if it ranks higher all the items budgeted by the first budget but not the second, compared to all items budgeted by the second budget but not by the first. Given a set of votes on a budget proposal, called a vote profile, we say that one budget strongly dominates another if a majority of the votes prefer it over the other.

Given a vote profile on a proposal and a budget limit, we say that a feasible budget is globally optimal if no other feasible budget strongly dominates it. Unfortunately, due to Condorcet cycles, a globally-optimal budget does not always exist. Hence, we settle for finding a democratically-optimal budget, which is globally-optimal up to cycles. We devise a polynomial-time democratically-optimal budgeting algorithm, which, given a budget proposal, a vote profile, and a budget limit, always produces a democratically-optimal budget. The budgets our algorithm produces are globally optimal in case they have no cycles; otherwise, we argue that in practice they are very close to being globally optimal.

We argue that our budgeting model is quite natural and captures more situations than the usual model of participatory budgeting; that our elicitation method is powerful and allow voters to specify both very simple and very complex preferences; and that the budgets our algorithm computes satisfy the voters’ wishes, and cannot be argued against, since, up to cycles, for any change to them there is no majority of the voters that support the change. Moreover, our budgeting model and method allow for hierarchical budgets, and can be applied to situations where a “high-level” budget (say, a city budget) consists of several “low-level” budgets (say, one for education, another for transportation, etc.). We show how to use our method of participatory budgeting for such hierarchical scenarios, making it applicable to budgets of higher stakes such as national budgets.
Some illustrating examples and proof details are deferred to the appendix.

2 Related Work

Within computational social choice [12], the study of multiwinner elections [6] is receiving more and more attention. Our model of participatory budgeting generalizes the usual model of multiwinner elections since (1) we allow weak orders (as compared to, e.g., [5]; but see also, e.g., [10]), and (2) we allow different costs (as compared to, e.g., [6]; but see also, e.g., [11]). We also mention that, in a way, we generalize the notion of Condorcet’s winners and strong Gehrlein-stable committees [8] and that we use the minmax set extension [14, 1].

Next we compare our work to existing literature on participatory budgeting, according to different criteria.

Preference Elicitation. Our elicitation method, which allows weak orders, is more general than Approval voting (discussed, e.g., in [4]), Knapsack voting (discuss, e.g., in [9]), and Value voting, Value-for-money voting, and Threshold voting (discuss in [2]), which use different variants of Approval voting or use only linear orders.

Modeling Power. Our modeling is stronger than the usual model of participatory budgeting [4, 9], since (1) we allow quantifiable projects of various costs, and (2) we do not require fixing the budget limit beforehand.

Point of View. We study axiomatic properties and efficient algorithms, in contrast to Goel et al. [9] which concentrate on strategic issues and experiments. We follow the Condorcet’s principle in a different way than Goel et al., which uses the Kemeny rule. We also mention the work of Benade et al. [2], which study the distortion [13] of various elicitation methods.

3 Preliminaries

We briefly discuss general preliminaries regarding elections, weak orders, and tournament solutions. Then we formally define our model of participatory budgeting.

3.1 Elections, Weak Orders, and Tournament Solutions

Given a set $A$, a partial order $\succ$ on $A$ is a reflexive, antisymmetric, and transitive relation. A weak order is a total order with ties, and corresponds to an ordered partition: given a set $A$, an ordered partition $\succ$ is a partition of $A$ into linearly ordered and disjoint sets whose union is $A$. Those sets are referred to as components. For example, given the set $A = \{1, 2, 3, 4, 5\}$, the ordered partition $\{2\} \succ \{3, 4\} \succ \{1, 5\}$, whose first, second, and third components are, correspondingly, $\{2\}$, $\{3, 4\}$, and $\{1, 5\}$ defines a weak order where, e.g., 2 is preferred to 4 while 3 and 4 tie. We interchange ordered partition and weak orders.
An election defines a tournament (equivalently, a directed graph) through its majority graph (see, e.g., [3]). A tournament solution is a function $f$ taking a tournament (such as a majority graph) as an input and selecting a nonempty subset of the vertices as its output; that is, $f : T(A) \rightarrow 2^A \setminus \emptyset$, where $T(A)$ is the set of all tournaments over $A$. We would use the tournament solution known as a Schwartz set.

**Definition 1.** A Schwartz component of a tournament $T = (A, E)$ is a minimal set $X \subseteq A$ of vertices of $T$ such that for any $b \in A \setminus X$ there is no $a \in X$ such that $(b, a) \in E$. The Schwartz set is the union of all Schwartz components.

### 3.2 Budget, Budgeting Algorithm

**Budget bag.** The basic mathematical object we analyze and manipulate is a budget bag, which is a multiset of pairs $(b, q)$ of a budget item $b \in B$ and its quantity $q \in \mathbb{N}$. A budget bag $\mathcal{P}$ is unary if $(b, q) \in \mathcal{P}$ implies $q = 1$. We abbreviate and use $b$ instead of $(b, 1)$. A budget bag $\mathcal{P}$ is condensed if it has at most one pair $(b, q)$ for any $b \in B$.

Any budget bag $\mathcal{P}$ corresponds naturally to a unary budget bag, referred to as the corresponding unary budget bag, obtained by splitting each pair $(b, q) \in \mathcal{P}$ with $q > 1$ into $q$ items of the form $(b, 1)$ (e.g., $\{(b_1, 2), (b_2, 1), (b_2, 3)\}$ corresponds to the unary budget bag $\{b_1, b_1, b_2, b_2, b_2\}$). Notice that, for convenience, we usually abbreviate $(b, 1)$ to $b$. Any budget bag $\mathcal{P}$ also corresponds naturally to a condensed budget bag, referred to as the corresponding condensed budget bag, obtained by exhaustively replacing $(b, q)$ and $(b, q')$ by $(b, q + q')$ for any pair $(b, q), (b, q') \in \mathcal{P}$. (e.g., $\{(b_1, 2), (b_2, 1), (b_2, 3)\}$ corresponds to the condensed budget bag $\{(b_1, 2), (b_2, 4)\}$). Importantly, budget bags are semantically equivalent to their corresponding unary and condensed budget bags, thus we view those as alternative representations of the same mathematical object and switch between them at our convenience.

We consider relations between budget bags, e.g., a budget bag $T_1$ is a subset of a budget bag $T_2$ (denoted by $T_1 \subseteq T_2$) if this relation holds for their corresponding unary bags.

**Budget proposal.** The output of the budget deliberation phase is a budget bag called a budget proposal.

**Budget ranking.** A ranking of a budget proposal is an ordered partition over the corresponding unary budget proposal. For example, $\{b_3\} \succ \{b_1, b_1, b_2\} \succ \{b_1\}$ is a ranking of the budget proposal $\mathcal{B} = \{(b_1, 2), (b_2, 3)\}$. Observe that, while the same budget item $b$ may occur in multiple bags in a ranking, with potentially different quantities in each bag, the sum of those quantities, however, shall be equal to the quantity of $b$ in the condensed budget proposal.

**Vote.** A vote on a budget proposal is a ranking of the budget proposal. A set of votes on the same budget proposal $\mathcal{P}$ is called a vote profile (over $\mathcal{P}$) and is usually denoted by $\mathcal{V}$. 

4
Budget. A budget $B$ is a subset of a budget proposal $P$ (with both unary). It is empty if $B = \emptyset$ and full if $B = P$. Members of $B$ are referred to as budgeted pairs while members of $P \setminus B$ are unbudgeted pairs. Note that a budget item $b$ may be partially budgeted, when $(b, q) \in B$ and $(b, q') \in P, q < q'$ (with both condensed).

Budget cost. With each budget item $b \in B$ we associate a cost function $F_b : \mathbb{N} \to \mathbb{R}$, where $F_b(q)$ is the cost of $q$ copies of $b$. The cost of $(b, q)$ is $F_b(q)$, also written as cost($\{(b, q)\}$). We generalize the definition of cost to condensed budget bags, so that cost($\{(b, q) \cup P\}) = F_b(q) + \text{cost}(P)$ if $(b, q) \cup P$ is a condensed budget bag (implying that $b$ does not occur in $P$); cost($\emptyset$) = 0.

Budget limit. A budget limit is a non-negative number. Given a budget limit $l$, a budget $B$ is feasible if \text{cost}(B) \leq l$. The empty budget is always feasible.

Next we describe what does it mean for a budget to be exhaustive (intuitively it means that it exhaustively uses the funds within limit) and define a budgeting algorithm.

Definition 2. Given a proposal $P$ and a limit $l$, a budget $B$ is exhaustive if it is feasible but for any unbudgeted pair $(b, q) \in P \setminus B$, the budget $B \cup \{b\}$ is not feasible.

Definition 3. A budgeting algorithm takes as input a budget proposal $P$, a vote profile $V$ over $P$, and a budget limit $l$ and outputs a feasible budget.

A budgeting algorithm is exhaustive if it only produces exhaustive budgets. Here is an example of a budgeting algorithm that is exhaustive: the algorithm prepares a budget $B$ of $P$ given a limit $l$ by starting with $B$ empty. It then repeatedly transfers an arbitrary unbudgeted item from (the unary) budget bag of so-far unbudgeted items $P \setminus B$ to $B$ as long as cost($B$) \leq l, until no more items can be transferred without the cost exceeding the budget limit. This simple budgeting algorithm is indeed exhaustive, albeit definitely not useful; in particular, it disregards the democratic vote altogether.

4 Democratically-Optimal Budgets

To apply the Condorcet principle to budgets, we derive a notion of voter preference among budgets from voter rankings.

We naturally extend the minmax set extension \cite{14} to our scenario. Since our scenario involved quantities, we develop the concepts of remainders and ranked difference. This section assumes unary budget bags.

Definition 4 (Remainder). Let $P$ be a proposal, $B$ a budget of $P$, and $v : C_1 > C_2 > \ldots > C_z$ a ranking over $P$. The unbudgeted remainder of $B$ with respect to $v$ is denoted by $Rem_B(v)$ and is computed (and defined) as follows ($i \in [z]$): $Rem_B(C_i > C_{i+1} > \ldots > C_z) = C'_i > Rem_B(C_{i+1} > \ldots > C_z)$, where $C'_i = C_i \setminus B$, $B' = B \setminus C_i$, and $Rem_B(\emptyset) = \emptyset$.  

5
Intuitively, the $i^{th}$ component of the remainder of some budget $B$ of $P$ with respect to $v$ contains exactly the elements of the $i^{th}$ component of $v$ which are unbudgeted by $B$. The remainder of a full budget is empty. Further, given a vote $v$ and a budget $B$ over $P$, the union of the elements of $Rem_B(v)$ equals $P \setminus B$, which are all items unbudgeted by $B$. We use remainders to compute the ranking of the elements in the symmetric difference between two budgets.

**Definition 5** ( Ranked difference between budgets). Let $P$ be a proposal, $v$ a ranking over $P$, and $B$ and $B'$ budgets over $P$. The ranked difference between the unbudgeted remainders of the two budgets with respect to $v$ is $Diff_{B,B'}(v) = Rem_{B'}(v) \setminus Rem_B(v)$. Further, we define Ind$_{B,B'}(v)$ as the set of indices of the components for which $Diff_{B,B'}(v)$ is non-empty; that is, denoting $Diff_{B,B'}(v)$ as $C_1 \succ \ldots \succ C_z$, we define $Ind_{B,B'}(v) = \{i : C_i \neq \emptyset\}$.

That is, while $Rem_B(v)$ collects and ranks items unbudgeted by $B$ according to their ranking by $v$, $Diff_{B,B'}(v)$ collects and ranks items budgeted by $B$ but not by $B'$ according to their ranking, and each index in $Ind_{B,B'}(v)$ names a component of $v$ that has at least one item budgeted by $B$ but not by $B'$. We are ready to define what it means for a vote to prefer one budget over another. Intuitively, a vote prefers one budget over another if it ranks higher all the items budgeted by the first budget but not the second, compared to all items budgeted by the second budget but not by the first.

**Definition 6** (Prefers). Let $P$ be a proposal, $v$ a ranking over $P$, and $B$ and $B'$ budgets over $P$. We say that $v$ prefers $B$ over $B'$ if it holds that:

$$\max(Ind_{B,B'}(v)) < \min(Ind_{B',B}(v)),$$

where we set $\max(\emptyset) = \min(\emptyset) = \infty$.

For example, if $B' \subset B$, then $B$ is preferred over $B'$ (by any vote $v$) since $Ind_{B,B'}(v) = \emptyset$ while $Ind_{B',B}(v)$ is not. In addition, prefers is irreflexive since $\infty \not< \infty$. When a vote neither prefers $B$ over $B'$ nor $B'$ over $B$ we say that it is indifferent to the choice between those budgets. To illustrate the notions defined above, consider the following example.

**Example 1.** Let $P = \{a, a, a, b\}$ be a proposal, where the cost of all items is 1. Let $V = \{v\}$ be a profile, where $v : a \succ a \succ b \succ a$. Let $B = \{a, a, b\}$ and $B' = \{a, a, a\}$ be two budgets. Then, $Rem_B(v) = \{\} \succ \{\} \succ \{a\}$, $Rem_{B'}(v) = \{\} \succ \{b\} \succ \{\}$, $Ind_{B,B'}(v) = \{3\}$, and $Ind_{B',B}(v) = \{4\}$. Thus, $v$ prefers $B$ over $B'$.

Our definition is conservative as it may refrain from judging one budget as preferable to another even in cases when one might be inclined intuitively to make such a judgment. We chose it as, one the one hand, it is restrictive and thus makes only solid judgments, and on the other hand it is powerful enough to provide a foundation for the following definitions and algorithm.
Definition 7 (Dominance). Let $\mathcal{V}$ be a profile and $B$ and $B'$ budgets over a proposal $\mathcal{P}$. We say that (1) $B$ strongly dominates $B'$ if a majority of the votes $v \in \mathcal{V}$ prefer $B$ over $B'$; (2) $B$ and $B'$ are tied if neither one dominates the other; and (3) $B$ (weakly) dominates $B'$, denoted by $B \triangleright B'$, if $B$ strongly dominates $B'$ or $B$ and $B'$ are tied.

The weak dominance relation $\triangleright$ is reflexive ($B \triangleright B$ for all $B$) and total ($B \triangleright B'$ or $B' \triangleright B$ or both for all $B$ and $B'$). It is, however, not transitive (see Example 2).

The following crucial definition is inspired by the Condorcet principle. Informally, a budget is globally optimal if it weakly dominates all other budgets; thus it cannot be argued against by the voters, as no proposed change in it has a majority that supports the change. For convenience, we consider unary budgets in the next definition.

Definition 8 (Globally-optimal budget). Given a profile $\mathcal{V}$ over a proposal $\mathcal{P}$ and a limit $l$, a feasible budget $B$ of $\mathcal{P}$ is globally optimal if $B \triangleright B'$ holds for any feasible budget $B'$ of $\mathcal{P}$.

We observe the following property. (We also observe an interesting local property of globally optimal budgets; see the Appendix.)

Observation 1. A globally-optimal budget is exhaustive.

It seems natural to define a budgeting algorithm to be globally optimal if it always produces a globally-optimal budget, however such budgets do not always exist.

Example 2. Consider the proposal $\mathcal{P} = \{b_1, b_2, b_3\}$, where the cost of each item is 1, and the following profile over it (the following profile is perhaps the instantiation of the Condorcet paradox in participatory budgeting):

$$
\begin{align*}
v_1 : b_1 & \succ b_2 \succ b_3 \\
v_2 : b_2 & \succ b_3 \succ b_1 \\
v_3 : b_3 & \succ b_1 \succ b_2
\end{align*}
$$

Following the symmetry of the above profile, we can assume, without loss of generality, that for the budget limit $l = 1$, any budget $B$ which is within limit $l$ equals to one of the following two options: (1) $B = \emptyset$; or (2) $B = \{b_1\}$.

Budget (1) is not exhaustive, since, e.g., $b_1$ can fit within limit, and hence not optimal via Observation 1. Regarding budget (2), both $v_2$ and $v_3$ prefer the budget $B' = \{b_3\}$ over the budget $B = \{b_1\}$ and hence $B$ is not globally optimal.

Example 2 illustrates a Condorcet-like cycle; specifically, let $B_i = \{b_i\}$ ($i \in [3]$) and notice that $B_1$ dominates $B_2$, $B_2$ dominates $B_3$, and $B_3$ dominates $B_1$. Below we formalize such budget cycles.
Definition 9 (Weak domination path, cycle). There is a weak domination path (path for short) from a budget $B$ to a budget $B'$, denoted by $B \Rightarrow B'$, if there is a sequence of budgets $(B_1, B_2, \ldots, B_k)$, $k \geq 2$, such that $B = B_1$, $B_k = B'$, and $B_i \Rightarrow B_{i+1}$ for each $i \in [k-1]$. Such a path is a cycle if, in addition, $k > 2$ and $B_1 = B_k$. A budget $B$ is a member of a cycle if there is a cycle $C$ that includes $B$, in which case $C$ is a cycle of $B$.

As exemplified before, globally-optimal budgets do not always exist, therefore we define the following practical alternative.

Definition 10 (Democratically-optimal budget). Given a profile $V$ over a proposal $P$ and a budget limit $l$, a feasible budget $B$ is democratically optimal if $B \Rightarrow B'$ holds for any feasible budget $B'$ of $P$.

Observe that: (1) A globally-optimal budget is democratically-optimal, since $B \Rightarrow B'$ implies $B \Rightarrow B'$ for any two budgets; (2) $\Rightarrow$ is transitive, as paths can be concatenated; (3) Any budget $B$ on a cycle with a democratically-optimal budget $B'$ is also democratically optimal, since one can combine $B \Rightarrow B' \Rightarrow B''$ for any feasible budget $B''$. (4) A democratically-optimal budget $B$ that has no cycles is globally optimal; if it is not globally optimal it must have a cycle: there exists a budget $B'$ such that $B' \Rightarrow B$; but $B \Rightarrow B'$ hence the cycle $B' \Rightarrow B \Rightarrow B'$.

Definition 11 (Democratically optimal budgeting algorithm). A budgeting algorithm is democratically optimal if it always produces a democratically-optimal budget.

5 Democratically-Optimal Budgeting Algorithm

We show, constructively, that, unlike globally-optimal budgets, democratically-optimal budgets always exist. We describe a Democratically-Optimal Budgeting algorithm, termed DOB, prove that it is indeed democratically optimal, and show that it runs in polynomial time.

A pseudo-code of DOB is given in Algorithm 1 below we describe its overall operation. DOB is composed of the following two procedures. (1) ranking: Given a proposal $P$ and a profile $V$ over it, the ranking procedure aggregates the rankings in $V$ into a single ordered partition (over $P$) and outputs it. (2) pruning: Given an ordered partition (specifically, the output of the ranking procedure) and a limit, the pruning procedure outputs a feasible budget.

(1) RANKING. Given a budget proposal $P = \{(b_j, q_j) : j \in [m]\}$ and a vote profile $V$ over it, uniquely tag each item of the unary budget proposal corresponding to $P$: define $P' = \{(b'_j, 1) : j \in [m], i \in [q_j]\}$, and, correspondingly, define $V' = \{v' : v \in V\}$ where, for each $v \in V$, construct $v' : C_1 \Rightarrow \ldots \Rightarrow C_z$ so that item tags are monotonically increasing with the indices of the components they are members of, in the following way: first, consider the unary vote corresponding to $v$. Second, change each occurrence of some $(b_j, 1)$ to some $(b'_j, 1)$ such that, for each $i_1 < i_2$, it holds that if $(b'_j, 1) \in C_{i_1}$ and $(b'_j, 1) \in C_{i_2}$ then
Algorithm 1 A democratically-optimal algorithm (DOB)

1: procedure RANKING \(\triangleright\) Aggregates votes into \(V\)
2: \(\text{input:}\) a budget proposal \(\mathcal{P}\)
3: \(\text{input:}\) a vote profile \(V\)
4: \(\mathcal{P}', V' \leftarrow \text{tagged-unary}(\mathcal{P}), \text{tagged-unary}(V)\)
5: Construct a budgeting graph \(G\) \(\triangleright\) Majority graph
6: \(V \leftarrow \emptyset\) \(\triangleright\) Initialize the aggregated vote
7: \(\text{while } G \neq \emptyset \text{ do} \triangleright\) Repeat while vertices remain in \(G\)
8: \(S \leftarrow \text{Schwartz-set}(G)\)
9: \(V \leftarrow V \searrow S; G \leftarrow G \setminus S\) \(\triangleright\) Move Schwartz set
10: return \(V\) \(\triangleright\) An aggregated vote

11: procedure PRUNING \(\triangleright\) Prunes an ordered partition
12: \(\text{input:}\) a budget proposal \(\mathcal{P}\)
13: \(\text{input:}\) an aggregated vote \(V : C_1 \succ \ldots \succ C_z\)
14: \(\text{input:}\) a budget limit \(l\)
15: \(B \leftarrow \emptyset; \triangleright\) Initialize the budget
16: \(\text{for } i \in [z] \text{ do} \triangleright\) Update budget
17: \(B_i \leftarrow \text{a maximal subset of } C_i \text{ with cost}(B \cup B_i) \leq l\)
18: \(B \leftarrow B \cup B_i\)
19: return \(B\)

20: procedure DOB \(\triangleright\) Produces a democratically-optimal budget
21: \(\text{input:}\) a budget proposal \(\mathcal{P}\)
22: \(\text{input:}\) a vote profile \(V\)
23: \(\text{input:}\) a budget limit \(l\)
24: \(V \leftarrow \text{RANKING}(\mathcal{P}, V)\)
25: \(B \leftarrow \text{PRUNING}(\mathcal{P}, V, l)\)
26: return \(B\)

\(t_1 \leq t_2\). (One way of achieving this is to tag items according to the order of the vote’s components, arbitrarily breaking ties within each component.)

Operating on the (tagged) vote profile \(V'\) over the (tagged) budget proposal \(\mathcal{P}'\), create the budgeting graph. The vertices of this directed graph are the uniquely-tagged budget items \(b_{i,j}\). For each pair of vertices \(b_{i,j}\) and \(b_{j,j'}\) (corresponding to the budget items \(b_{i,j}\) and \(b_{j,j'}\), respectively), add an arc from \(b_{i,j}\) to \(b_{j,j'}\); if a majority of the votes prefer \(b_{i,j}\) to \(b_{j,j'}\); call such arcs as majority arcs.

Having constructed the budgeting graph as explained above, iteratively construct the aggregated vote \(V\). Initially the vote \(V\) is empty. Then, in each iteration, identify the Schwartz set (see the Preliminaries section) and add its vertices as a (next) component of \(V\); further, remove those vertices from the budgeting graph. When no vertices remain in the budgeting graph, halt; at this point the aggregated vote \(V\) is indeed an ordered partition of \(\mathcal{P}\).
Notice that there could be ties between budget items (and thus, so-far no arcs between some pairs of vertices in the budgeting graph) either because votes have opposing preferences and/or since some voters provide weak orders. We discuss tie breaking schemes in the Discussion section.

(2) Pruning. Given the aggregated vote $V : C_1 \succ C_2 \succ \ldots \succ C_z$ (given in unary) computed by the ranking procedure, gradually populate the set of budgeted items $B$ as follows. Iterate over the components of $V$, starting from $C_1$, where in the $i^{th}$ iteration consider the $i^{th}$ component $C_i$. Choose an arbitrary maximal subset $B_i$ of $C_i$ such that the cost of $B \cup B_i$ remains within limit. Notice that if cost($B \cup C_i$) ≤ $l$ then simply add all the items of the current component $C_i$ to $B$ (namely $B_i = C_i$). If this is not the case, then some items of $C_i$ will remain unbudgeted, but budgeting any of them would cause $B$ to go over the limit. Then consider the next component of $V$. After considering all components of $V$, output the budget $B$.

5.1 The Optimality of DOB

**Definition 12** (Weak budgeting graph). Let $G = (V_G, E_G)$ be a budgeting graph computed by DOB. We define the corresponding weak budgeting graph to be $G' = (V_{G'}, E_{G'})$ where $V_{G'} = V_G$ and $E_{G'} = E_G \cup \{(u, v), (v, u) : u, v \in V \land (u, v) \notin E_G \land (v, u) \notin E_G\}$.

**Definition 13** (Weak budgeting path). Let $G$ be a budgeting graph computed by DOB and let $b$ and $b'$ be two vertices in it. We say that there is a weak budgeting path from $b$ to $b'$, and denote it by $b \sim b'$, if there is a path from $b$ to $b'$ in the weak budgeting graph $G'$ corresponding to $G$.

The following lemmas relate weak budgeting paths to weak domination paths. Notably, the first denotes a path in the weak budgeting graph while the latter a path in the (implicit) graph connecting budgets via weak domination, which contains a vertex for each possible budget and an arc from one budget to another if the first budget weakly dominates the other.

**Lemma 1** (Weak budgeting arc implies domination). Let $b$ and $b'$ be two vertices in the budgeting graph computed by DOB for some vote profile $V$ over some budget proposal $P$. Let $B = \{b\}$ and $B' = \{b'\}$. If the arc $(b, b')$ is present in the weak budgeting graph then $B \triangleright B'$.

**Lemma 2** (Weak budgeting path implies domination path). Let $b$ and $b'$ be two vertices in the budgeting graph computed by DOB for some vote profile $V$ over some budget proposal $P$. Let $B = \{b\}$ and $B' = \{b'\}$. If $b \sim b'$ then $B \triangleright B'$.

**Lemma 3** (Crucial lemma). Let $B$ and $B'$ be two feasible budgets, with budget limit $l$ over a budget proposal $P$. Let $V$ be a vote profile. If $b \sim b'$ for some $b \in B \setminus B'$ and $b' \in B' \setminus B$, then $B \triangleright B'$.

Next is our main theorem.

**Theorem 4.** DOB is democratically optimal.
Proof. Consider the aggregated vote $V = C_1 \succ \ldots \succ C_z$ produced by the ranking procedure of DOB. Consider the budget $B$ produced by the pruning procedure of DOB, based on the aggregated vote $V$ and let $B_i = B \cap C_i, i \in [z]$; note that the $B_i$’s are exactly the maximal subsets selected for budgeting by the pruning procedure. Let $B'$ be a budget within limit and let $B'_i = B' \cap C_i, i \in [z]$. Below we show that $B \succ B'$ which will finish the proof.

If $B' \setminus B = \emptyset$ then $B' \subseteq B$ and thus $B'$ cannot strongly dominate $B$, thus we assume that $B' \setminus B \neq \emptyset$. Consider the smallest index $i$ ($i \in [z]$) for which $B'_i \setminus B_i \neq \emptyset$ and consider some $b' \in B'_i \setminus B_i$. By construction, $b' \in C_i \setminus B_i$. Since $B_i$ is a maximal subset of $C_i$ such that the budget $(B \cap \{C_1, \ldots, C_{i-1}\}) \cup B_i$ is within limit (refer to Line $18$ in Algorithm $1$), it follows that $(B \setminus B') \cap \{C_1, \ldots, C_i\} \neq \emptyset$. Let $b \in (B \setminus B') \cap \{C_1, \ldots, C_i\}$, let $B_b = \{b\}$, and notice that $B_b$ is a budget within limit since $B_b \subseteq B$ and $B$ is within limit. Further, notice that $B \triangleright B_b$ since $B_b \subseteq B$. It remains to show that $B_b \triangleright B'$ which would finish the proof.

Recall that $b' \in C_j$ and consider the following two cases. The cases differentiate in whether $b$ is in the same component $C_j$ as $b'$ (Second case) or in an earlier component $C_j$, $j < i$ (First case). In each case we show that $B_b \triangleright B'$ and thus conclude that indeed $B \triangleright B'$ and thus DOB is democratically optimal.

**First case.** If $b \in C_j$ for some $j < i$, then the arc $(b', b)$ cannot be present in the budgeting graph, since $C_j$ is a Schwartz set in the budgeting graph not containing the vertices in $C_1, \ldots, C_{j-1}$ but containing $b'$. Therefore, the arc $(b, b')$ is present in the weak budgeting graph, thus $b \sim b'$. Applying Lemma $2$ we conclude that $B_b \triangleright B'$.

**Second case.** If $b \in C_i$, then $b$ and $b'$ are in the same Schwartz set $C_i$. If $b$ and $b'$ are in different Schwartz components, then the arcs $(b, b')$ and $(b', b)$ are not present in the budgeting graph, thus they are present in the weak budgeting graph. In particular, $b \sim b'$, and by applying Lemma $2$ we conclude that $B_b \triangleright B'$. Otherwise, if $b$ and $b'$ are in the same Schwartz component then $b \sim b'$ since each Schwartz component is a cycle in the budgeting graph. Again, applying $2$ we conclude that $B_b \triangleright B'$.

### 5.2 The Computational Efficiency of DOB

First we observe that, as is, DOB is pseudo-polynomial.

**Observation 2.** DOB is pseudo-polynomial.

*Proof.* The budgeting graph created by DOB has pseudo-polynomial many vertices, specifically $\sum_{j \in [m]} q_j$. For a given graph, computing a Schwartz set can be done in polynomial time with respect to the number of its vertices (Schwartz components correspond to strongly connected components; this result is folklore).

Observation $2$ means that, on the one hand, if the input is given in unary, or, equivalently, if the sum of the quantities in the proposed budget items is polynomially-bounded in the input size, then DOB would run in polynomial
time; on the other hand, if the input is given in binary, or, equivalently, if some budget items come in huge quantities, then DOB would need super-polynomial time.

While in some real life budgeting scenarios, it seems plausible that the total quantities $\sum_{j \in [m]} Q_j$ of the proposed budget items are not too large, rendering DOB efficient by Observation 2. Next we show how to modify DOB to account for other situations where the total quantities are huge. To explain our modifications to DOB, it is useful to identify that DOB is currently inefficient because it operates on a tagged version of the corresponding unary budget proposal, and thus constructs a budgeting graph with pseudo-polynomial number of vertices. To fix this inefficiency, it seems natural to consider a tagged version of the corresponding condensed budget proposal: indeed, the resulting budgeting graph would have polynomial number of vertices. We would not, however, be able to represent all majority relations between the budget items, since, informally, the condensed budget proposal is too “coarse”. To overcome this difficulty, we will indeed initiate the budgeting graph with the condensed budget proposal, but then split its vertices according to the way they are split by the voters.

**Theorem 5.** There is a polynomial-time democratically-optimal budgeting algorithm.

### 6 Distance to Global Optimality

Here we attempt to ease the possible discomfort with DOB, having its budgets be democratically optimal but not necessarily globally optimal. First we point again to the fact that globally optimal budgets do not always exists (see Example 2). Second, we mention that in a subsequent paper (not yet published) we show that identifying globally optimal budgets is NP-hard. Third, we introduce the concept of a broken component; to this end, recall that DOB finds a maximal subset $B_i$ of each Schwartz set $C_i$ it considers and that $C_i$ consists of one or more Schwartz components. We say that $B_i$ breaks a Schwartz component if there is a Schwartz component $D_j \in C_i$ such that DOB funds some items in $D_j$ but not all of them (formally, $\emptyset \subset D_j \setminus B_i \subset D_j$).

**Observation 3.** If DOB does not break any Schwartz component then it outputs a globally-optimal budget.

Fourth, we provide preliminary experimental evidence showing that the globally-optimal fraction of the budget DOB produces is usually quite high. Specifically, in the simulations we ran, we generated several elections in the following way. In each election we have $m$ items, where the cost of each item is drawn according to a Gaussian distribution. We have $n$ votes, each drawn according to the Impartial Culture assumption (i.e., each vote is a linear order and we uniformly at random select a linear order for each vote). In Figure 1 (and Figure 2 in the Appendix) we show simulations for certain values of $m$, $n$, and limit $l$. In each simulation we first ran DOB, and then consider all of
its subsets; denoting by $l_{DOB}$ the total cost of the (democratically optimal, and thus also exhaustive) budget $B$ computed by DOB, and by $l_{OPTIMAL}$ the total cost of the subset of $B$ which is globally optimal and has the highest total cost among these, each cell in Figure 1 shows the average fraction $l_{OPTIMAL}/l_{DOB}$ over a certain number of simulation instances. E.g., a value of 0.9 means that, even though sometimes DOB does not identify a globally optimal budget, on the average, it has a globally optimal subset consisting of 90% of its total cost.

7 Hierarchical Budgeting

In reality, budgets of complex organizations and societies are constructed hierarchically; here we use our approach to achieve that. Such budgets are composed of several sections, with each section having its independent budgeting process that decides upon the priorities within the section, and a consolidating budgeting process that prioritizes the allocation of funds to the different sections. E.g., consider a government with several ministries: each ministry conducts its own budgeting process and then the government conducts its overall budgeting process to prioritize between the ministries.

An important feature of our algorithm is that it does not require fixing an a priori budget limit (specifically, the ranking procedure does not depend on the limit); this flexibility turns out to be crucial for constructing hierarchical budgets. Our hierarchical budgeting method can be applied in two scenarios: One, where each section budget is created by votes of experts/stakeholders, and then the consolidated budget is created by the votes of the sovereign body. Another, when the same body decides on both section budgets and the consolidated bud-
get. This method helps the sovereign body to focus in turn on each section budget independently, and then consolidate the results.

Given \( n \) sections, we begin by creating \( n \) budget section proposals \( P_i, i \in [n] \). Then, the voters (of each section) vote independently on each section proposal. We separately apply only the ranking procedure to the votes of each section, to produce the section rankings \( V_i, i \in [n] \). We arbitrary linearize the unary rankings; denote the resulting sequences \( V^L_i, i \in [n] \) and define \( V^L_i[k] \) to be the set of the first \( k \) items of \( V^L_i \). We then define \( n \) “artificial” derived budget items \( s_i, i \in [n] \), one for each section, and associated cost functions \( F_i(n) = \text{cost}(V^L_i[n]) - \text{cost}(V^L_i[n-1]) \). The consolidated budget proposal is then \( P = \bigcup_{i \in [n]} (s_i, n_i) \), where \( n_i \) is the number of of items in \( V^L_i \). Then, each vote on the consolidated budget proposal is a ranking of the consolidated proposal \( P \), which consists of only \( s_i \)'s. (Practically, however, when a voter chooses, say, to rank first six \( s_3 \)'s, this would be after the voter has looked “under the hood” and knows what are the first six items of the budget of the \( s_3 \) section.) Given a limit \( l \) on the consolidated budget, all votes on the consolidated budget proposal \( P \) are then treated as usual: we apply the ranking procedure and then the pruning procedure with the limit \( l \), and produce a (high-level) democratically-optimal budget.

This consolidated budget has the same properties as non-hierarchical budgets produced by our algorithm, however it refers only to the allocation between sections. Votes on the consolidated proposal can prioritize items among the different sections, and know what are they prioritizing, but cannot affect, at the consolidation stage, the relative priority of the underlying “true” items within each section. Indeed, applying our hierarchical method might hurt optimality.

**Example 3.** Let \( a, b \) be items of section \( s_1 \) and \( c \) and item of section \( s_2 \). Let all items be of unit cost and define the budget limit \( l = 1 \). Let \( v_1 : c \succ b \succ a \) and \( v_2 : a \succ c \succ b \) be votes over all items. Applying DOB over the non-hierarchical profile gives either \( \{a\} \) or \( \{c\} \), both being globally optimal budgets. Applying our hierarchical method might give also \( \{b\} \), which is not globally optimal.

However, we conjecture the following.

**Conjecture 1.** Our hierarchical method outputs democratically optimal budgets.

Finally, notice that, while the description above has only two levels of hierarchy, the process naturally generalizes to further levels.

### 8 Discussion

We developed a novel model for participatory budgeting based on Condorcet’s principle. We defined democratically-optimal budgets and described a polynomial-time algorithm that produces such budgets; our method generalizes to hierarchical budgeting processes. Below we discuss some avenues for further research.

**User Interface Considerations.** As a vote is a weak order, a convenient user interface is needed to specify it. Two simple user interfaces are a sequence
of pairs (item, quantity) and a pie chart. We interpret the former as an odered partition where each partition consist of one pair. A pie chart might be interpreted as one component with multiple pairs, but we propose to interpret it differently, as follows: If not all items can be funded, then items should be funded proportionally according to the cost ratios specified by the chart. As our budgeting model refers to discrete items, each with a specific cost, such proportional allocation can only be approximated. To exploit the full power of weak orders, a user interface should cater for a sequence of pie charts.

BREAKING TIES. Recall that DOB picks an arbitrary maximal subset of each Schwartz set. As some subsets might be considered better, we suggest (1) sorting by cost, thus favoring cheaper items, resulting in budgets with more items; and (2) sorting by indices, thus favoring an item which was never budgeted to an a further item of a type which was already budgeted several times, resulting in more diverse budgets.

MODEL VARIANTS. We require a democratically optimal budget $B$ to have a weak domination path to any other budget $B'$. Requiring strong domination paths is not always within reach: consider the proposal $P = \{a, b\}$ where the cost of each item is 1, and the profile over it with $v_1 : a \succ b$ and $v_2 : b \succ a$. Somehow related, we used the least decisive set extension (recall Definition 6), since we did not want to make finer judgments in the presence of costs and quantities. Indeed, one may consider other set extensions (we conjecture, however, that specifically Lemma 3 breaks for more decisive set extensions). Further, our model and algorithm naturally generalize to voter that specify partial order.

PROPORTIONALITY. Our hierarchical budgeting process seems especially useful for situations where the electorate and the sections are orthogonal. In other cases, a more representative method of voting is needed, lest the majority (e.g. the big cities) will hog the budget from minorities (e.g. smaller municipalities). Borrowing ideas from multiwinner voting rules aiming at diversity or proportional representation seems promising [6].

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9 Appendix

We provide some proof details missing from the main text, discuss a local property of globally optimal budgets, and provide a few illustrating examples and further simulation results.

9.1 Missing Proofs

We provide proofs missing from the main text.

9.1.1 Proof of Observation 1

Proof. Let \( V \) be a vote profile over a budget proposal \( P \), \( l \) a budget limit, and \( B \) a budget over \( P \) within limit \( l \).

By way of contradiction, assume \( B \) is not exhaustive. Hence, there is a budget item \( b \) such that \((b,g) \in U, B' = B \cup \{(b,1)\} \), and \( \text{cost}(B') \leq l \). Since \( B \subseteq B' \) then, by definition, all votes \( v \in V \) prefer \( B' \) over \( B \). Hence \( B' \) is within the limit \( l \) and strongly dominates \( B \). Hence \( B \) is not globally optimal. \( \square \)

9.1.2 Proof of Lemma 1

Proof. Towards a contradiction, assume that \( B \) does not weakly dominate \( B' \), thus \( B' \) strongly dominates \( B \). Let \( M \) be a set of more than half of the voters such that each voter \( v \in M \) prefers \( B' \) to \( B \). By definition, this means that \( \max(\text{Ind}_{B''_v,B'_v}(v)) < \min(\text{Ind}_{B_v,B'_v}(v)) \) holds for each \( v \in M \), thus each \( v \in M \) ranks \( b' \) strictly before it ranks \( b \); more formally, recalling that \( v : C_1 \succ \ldots \succ C_z \) is an ordered partition, we conclude that \( b' \in C_i, b \in C_j \) for some \( i < j \). Thus, arc \((b',b)\) is in the budgeting graph, in contradiction to the assumption that \((b,b')\) is in the weak budgeting graph. \( \square \)

9.1.3 Proof of Lemma 2

Proof. Since \( b \sim b' \) it follows that there is a set of vertices \( b_1, \ldots , b_t \) such that \( b = b_1 \), \( b_t = b' \), and the arc \((b_i,b_{i+1})\) is present in the weak budgeting graph for each \( i \in [t-1] \). From Lemma 1 it follows that \( B_{b_i} \triangleright B_{b_{i+1}} \) holds for each \( i \in [t-1] \), where \( B_{b_i} = \{b_i\} \) and \( B_{b_{i+1}} = \{b_{i+1}\} \). We conclude that \( B \triangleright B' \). \( \square \)

9.1.4 Proof of Lemma 3

Proof. Let \( b \in B \setminus B' \) and \( b' \in B' \setminus B \) such that \( b \sim b' \). Let \( B_b = \{b\} \) and notice that \( B_b \) is a budget within the limit \( l \) since \( B_b \subseteq B \) and \( B \) is within the limit \( l \). Further, \( B \triangleright B_b \) since in particular \( B \triangleright B_b \) as \( B_b \subseteq B \). In order to show that \( B \triangleright B' \) it remains to show that \( B_b \triangleright B' \).

Since \( b \sim b' \) there is a set of vertices \( b_1, \ldots , b_t \) such that \( b = b_1, b_t = b' \), and \((b_i,b_{i+1})\) is present in the weak budgeting graph for each \( i \in [t-1] \). Notice that \( b = b_1 \notin B' \) and that \( b' = b_t \in B' \). Let \( \bar{j} \) \((\bar{j} \in [t-1])\) be the smallest index for which \( b_{\bar{j}} \notin B' \) but \( b_{\bar{j}+1} \in B' \). Let \( e = b_{\bar{j}} \) and \( e'' = b_{\bar{j}+1} \). (Indeed, it might be
that $b' = b''$; this would happen if the weak budgeting path from $b$ to $b'$ does not go through any vertex in $B'$.) Using Lemma 2 and since $b \sim e$ it follows that $B_b \triangleright B_e$ where $B_e = \{e\}$, thus it remains to show that $B_c \triangleright B'$.

Using Lemma 1 and since the arc $(e, b'')$ is present in the weak budgeting graph it follows that $B_c \triangleright B_{b''}$ where $B_{b''} = \{b''\}$. Thus, for each set $M$ of more than half of the voters there exists at least one voter $v \in M$ which does not prefer $B_{b''}$ to $B_c$; i.e., $\max(\text{Ind}_{B_{b''}, B_c}(v)) \geq \min(\text{Ind}_{B_c, B_{b''}}(v))$. In particular, $v$ does not rank $b''$ before it ranks $e$; more formally, recalling that $v : C_1 \succ \ldots \succ C_z$ is an ordered partition, it must hold that $b'' \in C_{i_{b''}}$, $e \in C_{i_e}$ for some $i_{b''} \geq i_e$.

Let us now consider $B'$ and recall that $e \notin B'$. Following the above discussion, we conclude that $i_e \in \text{Ind}_{B_b, B_c}(v)$ and $i_{b''} \in \text{Ind}_{B_{b''}, B_c}(v)$. But since $i_{b''} \geq i_e$ it means in particular that $\max(\text{Ind}_{B', B_e}(v)) \geq \min(\text{Ind}_{B_e, B'}(v))$ therefore $v$ does not prefer $B'$ over $B_e$. Since such a voter $v$ exists in every set $M$ containing more than half of the voters, it follows that $B_c \triangleright B'$.

9.1.5 Proof of Theorem 5

Proof. We describe a modification to DOB, which we denote as EDOB. EDOB is similar to DOB and only differs by the way it constructs the budgeting graph. Specifically, we first construct a budgeting graph $G$ which corresponds to the consolidated budget proposal; $G$ would have a vertex $(b, q)$ for each item $(b, q)$ in the consolidated budget proposal. Notice that, so far, the size of the budgeting graph is polynomial in the (binary) encoding of the input. Next we explain how to split the vertices of $G$.

We iteratively process each budget item $(b, q)$, initially corresponding to a vertex $(b, q)$ in $G$, as follows. For each vote $v : C_1 \succ \ldots \succ C_z$, represented in consolidated form, and where $b$ appears in the components $C = \{C_{i_1}, C_{i_2}, \ldots, C_{i_l}\}$ ($i_j \in [z]$, $j \in [l]$) with corresponding quantities $q_{i_1}, q_{i_2}, \ldots, q_{i_l}$, such that $(b, q_{i_j}) \in C_{i_j}$ ($i_j \in [z]$, $j \in [l]$) we notice that $\sum_{j \in [l]} q_{i_j} = q$, and create the following sequence of numbers: $[q_{i_1}, q_{i_1} + q_{i_2}, \ldots, \sum_{j \in [l]} q_{i_l}]$; this sequence it denoted by split$(v, b)$. Intuitively, split$_v(b)$ collects the $v$’s splitting points of $b$. For example, recall Example 8 and consider $v_3 : (a, 2) \succ (b, 2) \succ (a, 1)$. Notice that $v$ split the 3 occurrences of $a$ into two components, where in the first component $v$ has two of its occurrences while in the second component $v$ has the third occurrence of $b$; correspondingly, split$_{v_3}(a) = [2, 3]$.

After we compute split$_v(b)$ for each vote $v$, we merge those sequences into a single sequence denoted by split$(b)$; that is, split$(b) = \cup_{v \in \mathcal{V}}$split$_v(b)$, where $\mathcal{V}$ is the given vote profile. Intuitively, split$(b)$ collects the splitting points of $b$ across all voters. Recall that currently the budgeting graph has one vertex $(b, q)$ corresponding to the budget item $b$; first, we delete $(b, q)$ from the budgeting graph. Then, denote split$(b) = [q_{i_1}, \ldots, q_{i_x}]$ and create the following $x$ vertices $\{(b, [1, q_{i_1}]), (b, [q_{i_1} + 1, q_{i_2}])), \ldots, (b, [q_{x-1} + 1, q_{i_x}])\}$. Intuitively, a vertex $(b, [l, r])$ stands for the $l^{th}$, $(l + 1)^{th}$, \ldots, $(r - 1)^{th}$, and $r^{th}$ occurrence of $b$.

The above process is done for each budget item $b$. Importantly, the number of vertices in the budgeting graph is polynomially bounded in the input, since any splitting point in split$(b)$, and thus each vertex in the budgeting graph can
be charged to at least one voter. Further, the constructed budgeting graph can represent all majority relations since no voter splits a budget item \( b \) “finer” than \( \text{split}(b) \).

### 9.1.6 Proof of Observation 3

**Proof.** Let \( B \) be an output of DOB which does not break any Schwartz component and let \( B' \) be a budget which strongly dominates \( B \), denote \( B = B_1 \cup \cdots \cup B_z \) and \( B' = B'_1 \cup \cdots \cup B'_z \), and since \( B' \not\subseteq B \), let \( i \in [z] \) be the first index such that \( B' \) budgets an item in \( C_i \) which \( B \) does not, and let \( b_i \in B'_i \setminus B_i \).

Since \( B \) is exhaustive, and DOB picks a maximal subset from each Schwartz set, let \( j \leq i \) be the first index such that \( B \) budgets an item in \( C_j \) which \( B' \) does not (notice that \( j \leq i \)). If \( j < i \) then there is no voter majority preferring \( B' \) over \( B \), while if \( j = i \), since \( B \) does not break any Schwartz component, then \( b_i \) and \( b'_i \) are in different Schwartz components, and thus again there is no voter majority preferring \( B' \) over \( B \). \( \square \)

### 9.2 A Local Property of Globally Optimal Budgets

Indeed, the property of being globally optimal is, well, global, as it is in comparison to any feasible budget. Here we consider a seemingly weaker property, of a local optimum, which is built upon the next definition of a budget *neighbor*; later we show that it entails global optimum.

**Definition 14 (Neighbor).** Let \( P \) be a budget proposal, \( l \) a budget limit and \( B \) a feasible unary budget over \( P \). A budget \( B' \) over \( P \) is a neighbor of \( B \) if there is an unbudgeted pair \( u \in P \setminus B \) and a set of budgeted pairs \( T \subseteq B \) such that \( B' = B \cup \{u\} \setminus T \) and \( \text{cost}(B') \leq l \).

That is, a neighbor of a budget \( B \) is constructed from \( B \) by budgeting a new item and unbudgeting other items to ensure feasibility of the revised budget. Informally, a budget is locally optimal if it weakly dominates all its neighbors.

**Definition 15 (Locally-optimal budget).** Given a vote profile \( V \) over a budget proposal \( P \) and a budget limit \( l \), a feasible budget \( B \) of \( P \) is *locally optimal* if \( B \triangleright B' \) for any feasible neighbor \( B' \) of \( B \).

It turns out that our model has the following somewhat surprising property.

**Lemma 6.** A budget that is locally optimal is globally optimal.

**Proof.** Let \( V \) be a vote profile over a budget proposal \( P \), \( l \) a budget limit, and \( B \) a feasible budget over \( P \).

By way of contradiction, assume that \( B \) is not globally optimal. Hence there is a feasible budget \( B' \) that strongly dominates \( B \), meaning there is a majority of votes \( V \subseteq V \) that prefer \( B' \) over \( B \). Let \( u \) be any element in \( B' \setminus B \) and \( T = B \setminus B' \). Let \( B'' = B \cup \{u\} \setminus T \).

Since \( B'' \subseteq B' \) it follows that \( \text{cost}(B'') \leq \text{cost}(B') \leq l \). Furthermore, for each \( v \in V \) it holds that \( \max(\text{Ind}_{B',B}(v)) < \min(\text{Ind}_{B,B'}(v)) \), since each \( v \in V \).
prefers $B'$ to $B$. Since $B^u \subseteq B'$ it also follows that $\Ind_{B^u, B}(v) \subseteq \Ind_{B', B}(v)$ and hence $\max(\Ind_{B^u, B}(v)) \leq \max(\Ind_{B', B}(v))$.

Next we show that $\min(\Ind_{B, B'}(v)) \leq \min(\Ind_{B^u, B}(v))$, which will finish the proof. To this end, assume that some $i < \min(\Ind_{B, B'}(v))$ exists such that the $i^{th}$ component of $\Diff_{B, B'}(v)$ is not empty. But this means that there is some item $t$ in the $i^{th}$ component of $v$ which is budgeted by $B$ but unbudgeted by $B'$; specifically, it must hold that $t \in T$. But $t$ is also unbudgeted by $B'$ violating the fact that the $i^{th}$ component of $\Ind_{B, B'}(v)$ is empty.

Thus, we have that $\max(\Ind_{B^u, B}(v)) \leq \max(\Ind_{B', B}(v)) < \min(\Ind_{B, B'}(v)) \leq \min(\Ind_{B^u, B}(v))$. Hence, $B^u$ strongly dominates $B$, implying that $B$ is not locally optimal.

**Example 4.** Here is an example illustrating the definitions and proof above. Let $B = \{b_1, ..., b_n, t_1, t_2\}$, $B' = \{b_1, ..., b_n, t'_1, t'_2\}$, and let $v$ be the vote $\{t'_1\}, \{t'_2\}, \{b_1, ..., b_n\}, \{t_1\}, \{t_2\}$.

Then $\Rem_{B', B}(v) = \{t'_1\}, \{t'_2\}, \emptyset, \emptyset, \emptyset$ and $\Rem_{B, B'}(v) = \emptyset, \emptyset, \{t_1\}, \{t_2\}$. $B'$ dominates $B$ since $\max(\Ind_{B', B}) = \max(\{1, 2\}) = 2 < 4 = \min(\{4, 5\}) = \min(\Ind_{B, B'})$ hence $B$ is not a global optimum.

Let $T = B \setminus B' = \{t_1, t_2\}$, $u = t'_1$ and $B^u = B \cup \{u\} \setminus T = \{b_1, ..., b_n, t'_1\}$. Notice that, by definition, $B^u$ is a neighbor of $B$. Further, $\Rem_{B^u, B}(v) = \{t'_1\}, \emptyset, \emptyset, \emptyset$ and $\Rem_{B, B^u}(v) = \emptyset, \emptyset, \{t_1\}, \{t_2\}$. Finally, $B^u$ dominates $B$ since $\max(\Ind_{B^u, B}) = \max(\{1\}) = 1 < 4 = \min(\{4, 5\}) = \min(\Ind_{B, B^u})$. Hence $B$ is not a local optimum either.

![Figure 2: Globally-Optimal Fraction of DOB budgets.](image)
9.3 Further Examples

We provide some examples to illustrate the notions defined in this paper and demonstrating the operations of the procedures used within.

Example 5 (Democratically-optimal budgets). Consider the budget proposal $\mathcal{P} = \{a, b_1, b_2, b_3\}$, where the cost of each item is 1, with a vote profile $\mathcal{V}$ containing the following votes:

$v_1 : a \succ b_1 \succ b_2 \succ b_3$
$v_2 : a \succ b_2 \succ b_3 \succ b_1$
$v_3 : a \succ b_3 \succ b_1 \succ b_2$,

and let the budget limit be 2.

Similarly to Example 2, no globally-optimal budget exists here as well, however, e.g., the budget $B_1 = \{a, b_1\}$ is democratically optimal, as the only budget within limit which $B_1$ does not dominate is $B_3 = \{a, b_3\}$, but, nevertheless, $B_1 \triangleright B_3$, since $B_1 \triangleright B_2 \triangleleft B_3$, where $B_2 = \{a, b_2\}$.

Example 6 (Operation of the ranking procedure). Consider a budget proposal $\mathcal{B} = \{(b_1, 2), (b_2, 1), (b_3, 1), (b_4, 1)\}$ where each budget item costs 1 (that is, $F_j(i) = i$), and a vote profile with the following three voters:

$v_1 : (b_1, 2) \succ (b_2, 1) \succ (b_3, 1) \succ (b_4, 1)$
$v_2 : (b_1, 2) \succ (b_3, 1) \succ (b_4, 1) \succ (b_2, 1)$
$v_3 : (b_1, 2) \succ (b_4, 1) \succ (b_2, 1) \succ (b_3, 1)$

The ranking procedure of DOB begins by creating a budgeting graph containing the vertices

$\{b_1^1, b_2^1, b_3^1, b_4^1\}$

and the arcs

$\{(b_1^1, b_2^1), (b_1^1, b_3^1), (b_1^1, b_4^1), (b_2^1, b_3^1), (b_2^1, b_4^1), (b_3^1, b_4^1)\}$,

$\{(b_2^1, b_3^1), (b_2^1, b_4^1), (b_3^1, b_1^1), (b_3^1, b_4^1), (b_4^1, b_1^1)\}$.

In the first iteration, $b_1^1$ is the unique vertex in the Schwartz set (specifically, it is the Condorcet winner), thus we have the temporary ordered partition $P : \{b_1^1\}$. In the second iteration, $b_2^1$ is the unique vertex in the Schwartz set, thus we have the temporary ordered partition $P : \{b_1^1\} \triangleright \{b_2^1\}$.

Now we are left with a cycle, whose vertices are the Schwartz set, thus resulting in the aggregated vote $V : \{b_1^1\} \triangleright \{b_2^1\} \triangleright \{b_3^1, b_4^1\}$, which corresponds to the following aggregated vote:

$V : b_1^1 \succ b_2^1 \succ b_3^1, b_4^1$,

which is the output of the ranking procedure of DOB.
Example 7 (Operation of the pruning procedure). Consider the vote profile \( V \) over the budget proposal \( P \) which is discussed in Example 6 the aggregated ordered partition
\[
P : b_1^1 \succ b_1^2 \succ \{b_2^1, b_3^1, b_4^1\},
\]
which is computed by the ranking procedure, and the budget limit 3.

We initialize \( B = \emptyset \). Then, in the first iteration, we consider the first component \( \{b_1^1\} \) and since all of it fits within limit, we set \( B = \{b_1^1\} \). In the second iteration, we consider the second component \( \{b_2^1\} \) and since, again, all of it fits within limit, we set \( B = \{b_1^1, b_2^1\} \). Then, in the third iteration we consider the third component \( \{b_1^2, b_3^1, b_4^1\} \) and notice that not all of it fits within limit. We select, say, \( \{b_1^2\} \) as a maximal set which still fits within limit, and set \( B = \{b_1^1, b_2^1, b_3^2\} \).

Example 8 (The inefficiency of DOB). Consider the budget proposal \( P = \{(a, 3), (b, 2)\} \), where the cost of each item is 1, and the following vote profile over it:

\[
\begin{align*}
v_1 &: (a, 3) \succ (b, 2) \\
v_2 &: (a, 2) \succ (b, 2) \succ (a, 1) \\
v_3 &: (a, 2) \succ (b, 2) \succ (a, 1)
\end{align*}
\]

The tagged version corresponding to the unary budget proposal would be:

\[
\begin{align*}
v_1 &: a^1 \succ a^2 \succ a^3 \succ b^1 \succ b^2 \\
v_2 &: a^1 \succ a^2 \succ b^1 \succ b^2 \succ a^3 \\
v_3 &: a^1 \succ a^2 \succ b^1 \succ b^2 \succ a^3
\end{align*}
\]

Notice that while all voters rank \( a^1 \) and \( a^2 \) higher than \( b^1 \) and \( b^2 \), there is a majority (namely \( v_2 \) and \( v_3 \)) which rank \( b^1 \) and \( b^2 \) before \( a^3 \). The problem is that a budgeting graph corresponding to the consolidated budget proposal would have only two vertices, namely \((a, 3)\) and \((b, 2)\) and would not be able to distinguish \( a^1 \) and \( a^2 \) to \( a^3 \).