We study the overlaps between eigenvectors of nonnormal matrices. They quantify the stability of the spectrum, and characterize the joint eigenvalues increments under Dyson-type dynamics. Well known work by Chalker and Mehlig calculated the expectation of these overlaps for complex Ginibre matrices. For the same model, we extend their results by deriving the distribution of diagonal overlaps (the condition numbers), and their correlations. We prove:

(i) convergence of condition numbers for bulk eigenvalues to an inverse Gamma distribution; more generally, we decompose the quenched overlap (i.e. conditioned on eigenvalues) as a product of independent random variables;

(ii) asymptotic expectation of off-diagonal overlaps, both for microscopic or mesoscopic separation of the corresponding eigenvalues;

(iii) decorrelation of condition numbers associated to eigenvalues at mesoscopic distance, at polynomial speed in the dimension;

(iv) second moment asymptotics to identify the fluctuations order for off-diagonal overlaps, when the related eigenvalues are separated by any mesoscopic scale;

(v) a new formula for the correlation between overlaps for eigenvalues at microscopic distance, both diagonal and off-diagonal.

These results imply estimates on the extreme condition numbers, the volume of the pseudospectrum and the diffusive evolution of eigenvalues under Dyson-type dynamics, at equilibrium.

We also propose an analogue of (i) for the real Ginibre ensemble, in the bulk of the spectrum (i.e. far from the unit circle and the real axis). Here, the conjectured limiting distribution is given implicitly as an almost sure martingale limit.

**Keywords:** Nonnormal matrices, Ginibre ensemble, Eigenvectors overlaps, condition number, pseudospectrum.
1 Introduction

1.1 The Ginibre ensemble. Throughout this article we will essentially consider a complex Ginibre matrix $G_N = (G_{ij})_{i,j=1}^N$ where the $G_{ij}$’s are independent and identically distributed complex Gaussian random variables, with distribution $\mu = \mu^{(N)}$:

$$G_{ij} \sim \mathcal{N}(0, \frac{1}{2N}\text{Id}), \quad \mu(d\lambda) = \frac{N}{\pi} e^{-N|\lambda|^2} \, dm(\lambda), \quad (1.1)$$

where $m$ is the Lebesgue measure on $\mathbb{C}$. As proved in [26], the eigenvalues of $G_N$ have joint distribution

$$\rho_N(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z_N} \prod_{j<k} |\lambda_j - \lambda_k|^2 \prod_{k=1}^N \mu(d\lambda_k), \quad (1.2)$$

where $Z_N = N^{-N(N-1)/2} \prod_{j=1}^N j!$. The above measure is written $\mathbb{P}_N$, with corresponding expectation $\mathbb{E}_N$. The limiting empirical spectral measure converges to the circular law, i.e. $\frac{1}{N} \sum \delta_{\lambda_i} \rightarrow \frac{1}{\pi} \, 1_{|\lambda|<1} dm(\lambda)$.

We will also sometimes consider the real Ginibre ensemble, defined through $G_{ij} \sim \mathcal{N}(0, \frac{1}{N})$, with independent entries. With this scaling, the limiting circular law also holds.

The statistics of eigenvalues of Ginibre matrices have been studied in great details, and other non-Hermitian matrix models are known to be integrable, see e.g. [20, 34]. Much less is known about the statistical properties of eigenvectors of non-Hermitian ensembles.

1.2 Overlaps. Almost surely, the eigenvalues of a Ginibre matrix are distinct and the limiting empirical spectral measure converges to the circular law, i.e.

$$\lim_{N \to \infty} \rho_N(\lambda_1, \ldots, \lambda_N) = \frac{1}{\pi} \, 1_{|\lambda|<1} \, dm(\lambda) \quad \text{almost surely} \quad \text{as } N \to \infty.$$  

For the Dyson Brownian motion (1.1) at intermediate times, a well known fact in hydrodynamic stability theory [48].

Second, the overlaps also play a fundamental role in non perturbative dynamical settings. First, the large overlaps also play a fundamental role in non perturbative dynamical settings. First, the large
1.3 Overlaps statistics. The statistical study of overlaps started with the seminal work of Chalker and Mehlig [13,14,39]. They estimated the large $N$ limit of the expectation of diagonal and off-diagonal overlaps, for the complex Ginibre ensemble: for any $|z_1|, |z_2| < 1$,

$$E(\theta_{11} \mid \lambda_1 = z_1) \sim N(1 - |z_1|^2),$$  \hspace{1cm} (1.5)

$$E(\theta_{12} \mid \lambda_1 = z_1, \lambda_2 = z_2) \sim \frac{1}{N} - \frac{1 - z_1 \overline{z}_2}{|z_1 - z_2|^2},$$  \hspace{1cm} (1.6)

with (1.6) uniformly in $|z_1 - z_2|$ from the macroscopic up to the microscopic $N^{-1/2}$ scale. In [14], (1.5) and (1.6) were rigorously established for $z_1 = 0$, and convincing heuristics extended them anywhere in the bulk of the spectrum. From (1.5), one readily quantifies the instability of the spectrum, an order $N$ greater than for normal matrices in the bulk, and more stable closer to the edge.

An increasing interest in the statistical properties of overlaps for nonnormal matrices followed in theoretical physics [4,27,30,43,45], often with interest in calculating overlaps averages beyond the Ginibre ensemble. For example, eigenvector overlaps appear to describe resonance shift if one perturbs a scattering theoretical physics [4,27,30,43,45], often with interest in calculating overlaps averages beyond the Ginibre ensemble. Our proof also gives convergence of the expectation, in the complex Ginibre case, so that it extends (1.5).

In the mathematics community, the overlaps were recently studied in [49]. Walters and Starr extended (1.5) to any $z_1$ in the bulk, established asymptotics for $z_1$ at the edge of the spectrum, and suggested an approach towards a proof of (1.6). They also studied the connection between overlaps and mixed matrix moments. Concentration for such moments for more general matrix models was established in [18], together with applications to coupled differential equations with random coefficients. We continue the rigorous analysis of overlaps by deriving the full distribution of the condition numbers for bulk eigenvalues of the complex Ginibre ensembles. We also establish (1.6) and an explicit formula for the correlation between diagonal and off-diagonal overlaps, on any scale including microscopic. These formulas have consequences on the volume of the pseudospectrum and eigenvalues dynamics.

Motivated by our explicit distribution for the overlaps, Fyodorov [22] recently derived the distribution of diagonal overlaps for real eigenvalues of real Ginibre matrices, as well as an alternative proof for the distribution of diagonal overlaps for the complex Ginibre ensemble. His method is completely different. He applies the supersymmetry approach in random matrix theory, while our technique is probabilistic, as described below.

1.4 Main Results. Equation (1.5) suggests that the overlaps have typical size of order $N$. For the complex Ginibre ensemble (like all results below), we confirm that this is indeed the typical behavior, identifying the limiting distribution of $\theta_{11}$. We remind that a Gamma random variable $\gamma_\alpha$ has density $\frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}$ on $\mathbb{R}_+$.\footnote{Our formula (1.6) differs from the analogues in [13,14,39,49] through the additional denominator, due to eigenvalues repulsion: we consider conditional expectation instead of averages.}

**Theorem 1.1** (Limiting distribution of diagonal overlaps). Let $\kappa > 0$ be an arbitrarily small constant. Uniformly in $|z| < 1 - N^{-\frac{\kappa}{2}}$, the following holds. Conditionally on $\lambda_1 = z$, the rescaled diagonal overlap $\theta_{11}$ converges in distribution to an inverse Gamma random variable with parameter 2 as $N \to \infty$, namely

$$\frac{\theta_{11}}{N(1 - |z|^2)} \overset{(d)}{\to} \frac{1}{\gamma_2},$$  \hspace{1cm} (1.7)

Our proof also gives convergence of the expectation, in the complex Ginibre case, so that it extends (1.5). Equation (1.7) means that for any continuous bounded function $f$ we have

$$\mathbb{E} \left( f \left( \frac{\theta_{11}}{N(1 - |z|^2)} \right) \mid \lambda_1 = z \right) \to \int_0^\infty f(t) \frac{e^{-t}}{t^3} dt.$$  \hspace{1cm} (1.8)

This $t^{-3}$ asymptotic density was calculated for $N = 2$ in [14, Section V.A.2], where this heavy tail was suggested to remain in the large $N$ limit.
Remark 1.2. For the real Ginibre ensemble, we believe that uniformly in $\Omega_\kappa = \{|z| < 1 - N^{-\frac{1}{2} + \kappa}\} \cap \{|\text{Im} z| > N^{-\frac{1}{2} + \kappa}\}$, the left hand side of (1.7) also converges in distribution. More precisely, let $(\gamma_k)_{k \geq 1}$ be Gamma random variables with increasing parameter $k$ and, $(\gamma_{1/2})_{k \geq 1}$ be Gamma random variables with parameter $1/2$. Assume they are all independent. Let $\mathcal{G}_n = \sigma(\gamma_k, \gamma_{1/2}, 2 \leq k \leq n)$, and

$$X_n = \frac{1}{n} \prod_{k=2}^{n} \left( 1 + \frac{2\gamma_{(k)}}{\gamma_k} \right).$$

Then $(X_n)_{n \geq 2}$ is a positive $(\mathcal{G}_n)_{n \geq 2}$-martingale, so it converges almost surely to some $X_\infty$. We expect that uniformly in $z \in \Omega_\kappa$, conditionally on $\lambda_1 = z$,

$$\frac{\Theta_{11}}{N(1 - |z|^2)^2} \xrightarrow{(d)} X_\infty. \quad (1.9)$$

Heuristics for (1.9) are given in Subsection 2.4. The following results also probably admit analogues in $\Omega_\kappa$ for real Ginibre matrices, but we will not pursue that direction.

Theorem 1.1 requires integrating over all the randomness of the Ginibre ensemble, in this sense this is an annealed result. It admits a quenched generalization when conditioning on eigenvalues: unexpectedly, the overlap $\Theta_{11}$ can then be decomposed as a product of $N$ independent random variables, see Theorem 2.2.

We observe that the limiting density in (1.8) vanishes exponentially fast at 0, so that it is extremely unlikely to find any bulk overlap of polynomial order smaller than $N$: the spectrum is uniformly unstable. This is confirmed by the following bound on the extremes of condition numbers.

Corollary 1.3 (Bounds on the condition numbers). Let $\kappa, \varepsilon > 0$, $\kappa < \kappa_0 \leq 1/2$ be fixed and $\Omega_N \subset \{1 - N^{-\frac{1}{2} + \kappa_0} \leq |z| < 1 - N^{-\frac{1}{2} + \kappa}\}$ be deterministic, measurable. Then with probability tending to 1 as $N \to \infty$, the following event holds: for any $\lambda_i \in \Omega_N$,

$$N^{\frac{1}{2} + \kappa - \varepsilon} \leq \Theta_{ii} \leq N^{1 + \kappa_0 + \varepsilon} m(\Omega_N)^{1/2}.$$ 

In particular, all bulk overlaps are in $[N^{1-\varepsilon}, N^{3/2+\varepsilon}]$ with large probability. In terms of polynomial scales in $N$, the above lower bound is clearly optimal, and we believe the upper bound is also the best possible.

The next result is a rigorous proof of (1.6) in the bulk of the spectrum. It gives firm grounds to the heuristic arguments of Chalker and Mehlig [14] and answers Conjecture 4.5 in [49].

Theorem 1.4 (Expectation of off-diagonal overlaps, microscopic and mesoscopic scales). For any $\kappa \in (0, 1/2)$ and any small $\varepsilon > 0$ the following holds. Uniformly in $z_1, z_2$ such that $|z_1| < 1 - N^{-\frac{1}{2} + \kappa}$, $\omega = \sqrt{N}|z_1 - z_2| < N^{\kappa - \varepsilon}$, conditionally on $(\lambda_1, \lambda_2) = (z_1, z_2)$, we have

$$\mathbb{E}(\Theta_{12}) = -N \frac{1 - z_1 \bar{z}_2}{|\omega|^2} \frac{1 - (1 + |\omega|^2)e^{-|\omega|^2}}{1 - e^{-|\omega|^2}} \left(1 + O(N^{-2\kappa + \varepsilon})\right). \quad (1.10)$$

In particular, under the same hypothesis, in the mesoscopic regime $|\omega| \to \infty$, equation (1.10) simplifies to

$$\mathbb{E}(\Theta_{12}) = -\frac{1 - z_1 \bar{z}_2}{N|z_1 - z_2|^2} (1 + o(1)), \quad (1.11)$$

showing that the expectation of off-diagonal overlaps decreases with the separation of eigenvalues. Our next result, about second moments at any scale, allows to identify the natural size of off-diagonal overlaps, and gives polynomial decay of correlations between condition numbers.
**Theorem 1.5** (Correlations of overlaps: microscopic and mesoscopic scales). Let $\kappa \in (0, 1/2)$ and $\sigma \in (0, \kappa)$. Let $\varepsilon > 0$ be arbitrarily small. Then uniformly in $z_1, z_2$ such that $|z_1| < 1 - N^{-\frac{1}{2} + \kappa}$, $\omega = \sqrt{N}|z_1 - z_2| \in [N^{-\frac{1}{2} + \kappa}, N^\sigma]$, conditionally on $(\lambda_1, \lambda_2) = (z_1, z_2)$, we have

$$
E(\|\mathcal{O}_{12}\|^2) = \frac{N^2(1 - |z_1|^2)(1 - |z_2|^2)}{|\omega|^4} \left( 1 + O(N^{2(\sigma - \kappa) + \varepsilon}) \right), \quad (1.12)
$$

$$
E(\mathcal{O}_{11}\mathcal{O}_{22}) = \frac{N^2(1 - |z_1|^2)(1 - |z_2|^2)}{|\omega|^4} \frac{1 + |\omega|^4 - e^{-|\omega|^2}}{1 - e^{-|\omega|^2}} \left( 1 + O(N^{2(\sigma - \kappa) + \varepsilon}) \right). \quad (1.13)
$$

For the mesoscopic scales $|\omega| \to \infty$, the above asymptotics become

$$
E(\|\mathcal{O}_{12}\|^2) \sim \frac{(1 - |z_1|^2)(1 - |z_2|^2)}{|z_1 - z_2|^4}, \quad (1.14)
$$

$$
E(\mathcal{O}_{11}\mathcal{O}_{22}) \sim E(\mathcal{O}_{11}) E(\mathcal{O}_{22}). \quad (1.15)
$$

Equations (1.11) and (1.14) show that for any mesoscopic separation, $\mathcal{O}_{12}$ does not concentrate around its mean. Contrary to (1.6), (1.14) therefore identifies the size of off-diagonal overlaps, at mesoscopic scales.

The covariance bounds from Theorem 1.5 yield effective estimates on the volume of the pseudospectrum in any domain $G$ in the bulk, defined through

$$
\sigma_\varepsilon(G) = \{ z : \| z - G \|^{-1} > \varepsilon^{-1} \}. \quad (1.16)
$$

We state the result when $G$ is a mesoscopic ball, although it clearly also holds for $G$ regular enough.

**Corollary 1.6** (Volume of the pseudospectrum). Let $\kappa > a > 0$ be any small constants and $\mathcal{B}_N \subset \{ |z| < 1 - N^{-\frac{1}{2} + \kappa} \}$ be a ball with radius at least $N^{-\frac{1}{2} + a}$, at most $N^{-\frac{1}{2} + \kappa - a}$. Then the volume of the pseudospectrum in $\mathcal{B}_N$ is deterministic at first order: for any $c > 0$, as $N \to \infty$

$$
\lim_{\varepsilon \to 0} \mathbb{P} \left( 1 - c < \frac{m(\sigma_\varepsilon(G) \cap \mathcal{B}_N)}{\varepsilon^2 N^2 \int_{\mathcal{B}_N} (1 - |z|^2)dm(z)} < 1 + c \right) \to 1. \quad (1.17)
$$

Finally, we remark that Theorem 1.5 sheds some light on natural matrix dynamics on nonnormal matrices, the Dyson-type evolution. It shows that the eigenvalues propagate with diffusive scaling, at equilibrium, but slower when close to the boundary. We only state the result for macroscopic domains, although it can also be proved for some mesoscopic scales.

**Corollary 1.7.** Let $\varepsilon, c, a > 0$ be arbitrarily small constants. Consider the matrix dynamics (1.4) with initial condition $G(0)$ distributed as (1.7). Let $\mathcal{B} \subset \{ |z| < 1 - c \}$ be a ball, $N^{-1+c} < t < N^{-1}$, and $A^{(k)}_{t,\varepsilon} = \{ \sup_{0 \leq s \leq t} |\lambda_k(s) - \lambda_k(0)| < N^\varepsilon t^{1/2} \}$. Then as $N \to \infty$ we have

$$
\mathbb{P}(A^{(k)}_{t,\varepsilon}) = 1 - o(1), \quad (1.18)
$$

$$
\mathbb{E}(\lambda_1(t) - \lambda_1(0))^2 1_{A^{(1)}_{t,\varepsilon} \cap \{ \lambda_1(0) \in \mathcal{B} \}} = \int_\mathcal{B} (1 - |z|^2)^2 \frac{dm(z)}{\pi} (1 + o(1)), \quad (1.19)
$$

$$
\mathbb{E} \left( (\lambda_1(t) - \lambda_1(0)) (\lambda_2(t) - \lambda_2(0)) 1_{A^{(1)}_{t,\varepsilon} \cap A^{(2)}_{t,\varepsilon} \cap \{ \lambda_1(0) \in \mathcal{B} \} \cap \{ |\lambda_1(0) - \lambda_2(0)| < N^{-a} \} \right) = o(t N^{-2a}). \quad (1.20)
$$

Given the above scaling exponent in (1.19), we expect that, conditionally on $\{ \lambda_1(0) = z \}$, the process

$$
\frac{(\lambda_1(ts) - z)_{0 \leq s \leq 1}}{\sqrt{t(1 - |z|^2)}}
$$

converges in distribution to a normalized complex Brownian motion as $N \to \infty$ ($t$ is any scale $N^{-1+c} < t < N^{-a}$), i.e. $\frac{1}{\sqrt{t}}(B^{(i)} + i\tilde{B}^{(i)})_{0 \leq s \leq 1}$ with $B^{(i)}$ and $\tilde{B}^{(i)}$ independent standard complex Brownian motions. Moreover, from (1.20), we expect that these limiting processes associated to different eigenvalues are independent.
1.5 About the proofs. Our analysis of the condition numbers starts with the observation of Chalker and Mehlig: the overlaps coincide with those of the Schur form of the original operator, a particularly simple decomposition when the input is a Ginibre matrix, see (2.2).

In Section 2 our method to prove Theorem 1.1 then diverges and follows from a simple, remarkable identity in law: the *quenched* overlap (i.e. conditionally on eigenvalues and integrating only over the complementary randomness of the Ginibre ensemble) is a product of $N-1$ independent random variables, see Theorem 2.2. This identity admits a natural analogue for the real Ginibre ensemble. In the specific case $\lambda_1 = 0$, for the complex Ginibre ensemble, this split of the distribution of $\xi_{11}$ remains in the *annealed* setting, as a consequence of a theorem of Kostlan: the radii of eigenvalues of complex Ginibre matrices are independent random variables. For the Ginibre ensemble with conditioned eigenvalue, we give the extension of Kostlan’s theorem in Section 6. This concludes a short probabilistic proof of Theorem 1.1 in the case $\lambda_1 = 0$.

The extension to general $\lambda_1$ in the bulk proceeds by decomposition of $\xi_{11}$ into a long-range factor, which gives the deterministic $1 - |z|^2$ coefficient, and a short-range factor, responsible for the $\gamma_2^{-1}$ fluctuations. Concentration of the long-range contribution relies on rigidity results for eigenvalues, from [10,11]. To prove that the short-range contribution is independent of the position of $\lambda_1$, we need strong form of invariance for our measure, around the conditioned point $\lambda_1$. While translation invariance for $\mathbb{P}$ follows easily from the explicit form of the Ginibre determinantal kernel, the invariance of the conditioned measure requires more involved tools such as the negative association property for determinantal point processes, see Section 4. Corollary 1.3 directly follows from our estimates on the speed of convergence to the inverse Gamma distribution, as explained in Section 2.

The proof of Theorem 1.5 in Section 3 follows the same scheme: first a quenched identity, then an explicit formula obtained for $z_1 = 0$ and a localization procedure to isolate the short and long-range contributions. The main difference with the proof of Theorem 1.1 concerns the special case $z_1 = 0$, the other step being more robust. Due to conditioning on the second point $z_2$, rotational invariance of the remaining eigenvalues is broken and there is no analogue of Kostlan’s theorem to obtain the explicit joint distribution of $\xi_{12}, \xi_{11}$ and $\xi_{22}$. Despite this lack of rotational invariance, we are surprised that the second moments are explicit (although considerably more involved that expectations) even for finite $N$, see Proposition 3.6. Our correlation estimates on overlaps imply Corollary 1.6 by a second moment method. It is plausible yet unclear that Theorem 1.5 admits generalizations to joint moments with an arbitrary number of conditioned eigenvalues. In fact, our second moment calculation involves a P-recursive sequence (see (3.24)), for which explicit solutions are not expected in general. We hope to address the general study of holonomic sequences obtained from an arbitrary number of points in future work.

In Appendix A, we consider eigenvalues dynamics. After deriving the relevant stochastic differential equations in Proposition A.1, we show that Corollary 1.7 follows from Theorem 1.1 and Theorem 1.4. The diagonal overlaps dictates the eigenvalues quadratic variation, the off-diagonal overlaps their correlations.

Finally, although this article focuses on the eigenvalues condition numbers, the Schur decomposition technique also readily answers the natural question of angles between normalized Ginibre eigenvectors, as explained in Appendix B.

1.6 Numerical test for universality of the distribution of overlaps. Universality of eigenvector statistics recently attracted a lot of attention for random Hermitian matrices. For the Gaussian orthogonal and unitary ensembles, the eigenvectors basis is Haar distributed on the corresponding unitary group. As a consequence, projections of eigenvectors on deterministic directions are asymptotically normal, as the projection of the uniform measure on high dimensional spheres (a result due to Lévy and Borel). These eigenvector statistics are now known to be universal [9,36,46], holding for generalized Wigner matrices.

The situation is quite different for eigenvectors of dense random non Hermitian matrices: orders of magnitude such as delocalization are known [44], but universality seems out of reach with current techniques. In Figure 1 we numerically investigate whether the inverse Gamma distribution from Theorem 1.1 describes the typical behavior of condition numbers. This leads us to conjecture that for any complex random matrix with i.i.d. entries with substantial real and imaginary parts, the normalized condition numbers converge to the inverse Gamma distribution with parameter two.
Notations and conventions. The partial sums of the exponential are written
\[ e_k^{(\ell)}(x) = \sum_{i=1}^{\ell} \frac{x^i}{i!} \]  \hfill (1.21)
and we abbreviate \( e_k = e_k^{(\infty)}, e^{(N)} = e_0^{(N)} \). Moreover, we write \( f = O(\phi) \) if \( |f| < C|\phi| \) for some \( C > 0 \) which does not depend on \( N \), \( f = o(\phi) \) if \( |f|/|\phi| \to 0 \) as \( N \to \infty \).

2 Diagonal overlaps

This section first gives a remarkable identity in law for the diagonal overlap conditioned on the eigenvalues, Theorem 2.2. Eigenvalues are then integrated, first for \( \lambda_1 \) at the center of the spectrum thanks to a variant of Kostlan’s theorem, then anywhere in the bulk.

2.1 The quenched diagonal overlap. For all \( i \neq j \) we denote \( \alpha_{ij} = \frac{1}{\lambda_i - \lambda_j} \), and \( \alpha_{ii} = 0 \). These numbers satisfy
\[ \alpha_{ij} + \alpha_{ji} = 0, \quad \alpha_{ij} + \alpha_{jk} = \frac{\alpha_{ij}\alpha_{jk}}{\alpha_{ik}}. \]  \hfill (2.1)
We first recall the analysis of the overlaps given by Chalker and Mehlig, and include the short proof for completeness.

Proposition 2.1 (from [13,14,39]). The following joint equality in distribution holds:
\[ \mathcal{O}_{11} = \sum_{i=1}^{N} |b_i|^2, \quad \mathcal{O}_{12} = -b_2 \sum_{i=2}^{N} b_i \overline{a_i}, \quad \mathcal{O}_{22} = (1 + |b_2|^2) \sum_{i=2}^{N} |d_i|^2, \]
where \( b_1 = d_2 = 1, d_1 = 0, \) and the next terms are defined by the induction
\[ b_i = \alpha_{1i} \sum_{k=1}^{i-1} b_k T_{ki} \text{ for } i \geq 2, \quad d_i = \alpha_{2i} \sum_{k=1}^{i-1} d_k T_{ki} \text{ for } i \geq 3. \]
The \( T_{ij} \) are independent complex Gaussian variables, centered and with variance \( N^{-1} \) \((T_{ij}^{(d)} = G_{ij})\). The same result holds for the real Ginibre ensemble, with \( T_{ij}^{(d)} \sim \mathcal{N}(0, N^{-1}) \).
Proof. We defined $X$ and $Y = X^{-1}$ such that $G = XAY$ (see subsection 2.2). The QR algorithm applied to $X$ yields $X = UR$, $Y = R^{-1}U^*$, and thus the Schur form $G = UTU^*$ with $T = R\Delta R^{-1}$. It is straightforward to check that $U$ is independent of $T$ and uniformly distributed on the unitary group. The overlaps are unchanged by an unitary change of basis, and therefore we can study directly the overlaps of the matrix $T$. As proved in [40, Appendix 35], this matrix $T$ has the eigenvalues of $G$ as diagonal entries, and independently the upper triangle consists in uncorrelated Gaussian random variables, $T_{ij} \overset{\text{d}}{=} G_{ij}$:

$$T = \begin{pmatrix} \lambda_1 & T_{12} & \ldots & T_{1N} \\ 0 & \lambda_2 & \ldots & T_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & \lambda_N \end{pmatrix}.$$  \hspace{1cm} (2.2)

For $T$, the right-eigenvectors are of type

$$R_1 = (1,0,\ldots,0), \quad R_2 = (a,1,0,\ldots,0),$$

and the left eigenvectors are denoted

$$L_1 = (b_1,\ldots,b_N), \quad L_2 = (d_1,\ldots,d_N).$$

The equalities $\langle L_i|R_j \rangle = \delta_{ij}$ give $b_1 = 1$, $d_1 = 0$, $d_2 = 1$ and $a = -\bar{b}_2$. The formulas given for the overlaps follow, and the recurrence formulas proceed from the definition of the eigenvectors, i.e. $L_j T = \lambda_j L_j$. \hfill \Box

Proposition 2.1 shows that the eigenvectors, and thus the overlaps, are obtained according to a very straightforward random process. Indeed, let us consider the sequences of column vectors:

$$B_k = (1,b_2,\ldots,b_k)^t \quad \text{so that } L_1 = B_N,$$

$$D_k = (0,1,d_3,\ldots,d_k)^t \quad \text{so that } L_2 = D_N,$$

$$T_k = (T_{1,k+1},\ldots,T_{k,k+1})^t \quad \text{(subset of the } k+1 \text{ th column of } T).$$

For any $k$, $T_k$ is a $k$-dimensional centered Gaussian vector with independent coordinates and variance $1/N$. We denote the corresponding $\sigma$-algebras

$$\mathcal{F}_n = \sigma(T_k, 1 \leq k \leq n) = \sigma(T_{i,j+1}, 1 \leq i \leq j \leq n).$$

In particular, $b_2 = \frac{T_{12}}{\lambda_1 - \lambda_2} \in \mathcal{F}_1$. The recurrence formula from Proposition 2.1 becomes

$$b_{n+1} = \alpha_{1,n+1} B_n^t T_n, \quad d_{n+1} = \alpha_{2,n+1} D_n^t T_n, \quad n \geq 1. \hspace{1cm} (2.3)$$

Theorem 2.2. The following equality in law holds conditionally on $\{\lambda_1,\ldots,\lambda_N\}$:

$$\mathcal{O}_{11} \overset{\text{d}}{=} \prod_{n=2}^N \left( 1 + \frac{|X_n|^2}{N|\lambda_n|^2} \right)$$

where the $X_n$’s are independent standard complex Gaussian random variables $X_n \overset{\text{d}}{=} \frac{1}{\sqrt{2}}(\mathcal{N}_1 + i\mathcal{N}_2)$ with standard real Gaussians $\mathcal{N}_1$, $\mathcal{N}_2$. In the case of the real Ginibre ensemble, the same identity in law holds with the $X_n$’s being real, independent standard Gaussian random variables.

Remark 2.3. In particular

$$\mathbb{E}_T(\mathcal{O}_{11}) = \prod_{n=2}^N \left( 1 + \frac{1}{N|\lambda_1 - \lambda_n|^2} \right). \hspace{1cm} (2.4)$$

where $\mathbb{E}_T$ is partial integration in the upper-diagonal variables $(T_{ij})_{j>i}$. We therefore recover the result by Chalker and Mehlig [13,14,39].
Proof. For fixed $N$ and $1 \leq n \leq N$ we shall use the notations
\[ \psi_{11}^{(n)} = \|B_n\|^2, \quad \psi_{12}^{(n)} = -\frac{B_2^1 B_2^1 D_n}{\|B_2\|^2}, \quad \psi_{22}^{(n)} = (1 + |b_2|^2)\|D_n\|^2. \]
Note that
\[ \psi_{11}^{(1)} = 1, \quad \psi_{11}^{(2)} = 1 + |b_2|^2 = 1 + \left| \frac{T_{12}}{\lambda_1 - \lambda_2} \right|^2 \]
and, with (2.3),
\[ \psi_{11}^{(n+1)} = \|B_{n+1}\|^2 = \|B_n\|^2 + |b_{n+1}|^2 = \|B_n\|^2 \left(1 + \frac{\alpha_{n+1} \lambda_2^2}{\|B_n\|^2}\right) = \psi_{11}^{(n)} \left(1 + \frac{|\lambda_{n+1}|^2}{N|\lambda_1 - \lambda_{n+1}|^2}\right), \]
where $X_{n+1} := \sqrt{\frac{B_2^1 T_{n}}{\|B_n\|^2}}$ is a $\mathcal{F}_n$-measurable Gaussian with variance 1, independent of $\mathcal{F}_{n-1}$. We have therefore proved the expected factorization with independent $X_n$'s, by an immediate induction.

2.2 The annealed diagonal overlap at the origin. We remind that a Gamma random variable $\gamma_\alpha$ has density $\frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}$ on $\mathbb{R}_+$, and a Beta random variable $\beta_{a,b}$ has density $\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} \mathbb{I}_{[0,1]}(x)$.

**Proposition 2.4.** Conditionally on $\{\lambda_1 = 0\}$, the following equality in distribution holds:
\[ \psi_{11}^{(d)} = \frac{1}{\beta_{2,N-1}^1}. \]
In particular, $\mathbb{E}(\psi_{11}^1 \mid \lambda_1 = 0) = N$ and $N^{-1}\psi_{11}^1$ converges weakly to $\gamma_2^{-1}$.

**Proof.** With the notations from Theorem 2.2, we have $(|X_2|^2, \ldots, |X_N|^2) \overset{(d)}{=} (\gamma_1^{(2)}, \ldots, \gamma_1^{(N)})$, a collection of $N-1$ independent Gamma random variables with parameter 1. Moreover, still conditionally on $\lambda_1 = 0$, from Corollary 5.6 we have $\{|\lambda_2|^2, \ldots, |\lambda_N|^2\} \overset{(d)}{=} \{\gamma_2, \ldots, \gamma_N\}$, a set of independent Gamma random variables with corresponding parameters. Theorem 2.2 therefore yields, conditionally on $\lambda_1 = 0$,
\[ \psi_{11}^1 \overset{(d)}{=} \prod_{j=2}^N \left(1 + \frac{\gamma_j^{(1)}}{\gamma_j}\right), \]
where all random variables are independent. Equation (2.6) then follows immediately from Lemma 2.5 below. This readily implies
\[ \mathbb{E}(\psi_{11}^1 \mid \lambda_1 = 0) = \mathbb{E}\left(\frac{1}{\beta_{2,N-1}^1}\right) = \frac{\Gamma(1)\Gamma(N+1)}{\Gamma(2)\Gamma(N)} = N. \]
The convergence in distribution follows from a simple change of variables: for any bounded test function $f$,
\[ \mathbb{E}\left(f\left(\frac{1}{N\beta_{2,N-1}^1}\right)\right) = \int_0^1 f\left(\frac{1}{N x}\right) \frac{N!}{(N-2)!} x(1-x)^{N-2} dx = \int_{N-1}^\infty f\left(\frac{N-1}{N t}\right) \left(1 - \frac{1}{N t}\right)^{N-2} \frac{dt}{t^2}, \]
which clearly converges to the right hand side of (1.8) as $N \to \infty$.

**Lemma 2.5.** The following equalities in distribution hold, where all random variables with different indexes are independent:
\[ \frac{\gamma_a}{\gamma_a + \gamma_b} \overset{(d)}{=} \beta_{a,b}, \]
\[ \prod_{j=2}^N \beta_{j,1}^{(d)} = \beta_{2,N-1}. \]

**Proof.** Equation (2.9) is standard, see e.g. [31, Chapter 25]. The equality (2.10) follows by immediate induction from the following property [16]: if $\beta_{p,q}$ and $\beta_{p+q,r}$ are independent, their product has distribution $\beta_{p,p+r}$.
2.3 The annealed diagonal overlap in the bulk. With the following theorem, we first remind how the expectation of diagonal overlaps is accessible, following [13, 14, 39, 49]. We then prove Theorem 1.1.

The following was proved by Chalker and Mehlig for $z = 0$. They gave convincing arguments for any $z$ in the bulk, a result then proved by Walters and Starr. We include this short proof for the sake of completeness.

**Theorem 2.6** (from [13, 14, 39, 49]). For any $z \in \mathbb{D}$, we have

$$E(\mathcal{O}_{11} | \lambda_1 = z) = N(1 - |z|^2) + O\left(\sqrt{N} \frac{e^{-c(z)N}}{1 - |z|^2}\right)$$

where $c(z) = |z|^2 - 1 - \log(|z|^2) > 0$.

**Proof.** From (2.11), we can write

$$E(\mathcal{O}_{11} | \lambda_1 = z) = E\left(\prod_{k=2}^{N} \left(1 + \frac{1}{N|z - \lambda_k|^2}\right) | \lambda_1 = z\right).$$

Theorem 5.3 with $g(\lambda) = 1 + \frac{1}{N|z - \lambda|^2}$ then gives

$$E\left(\prod_{k=2}^{N} g(\lambda_k) | \lambda_1 = z\right) = \frac{\det(f_{ij})}{Z_N^{(z)}} \text{ where } f_{ij} = \frac{1}{i!} \int \lambda^{i-1} \bar{\lambda}^{j-1} \left(\frac{1}{N} + |z - \lambda|^2\right) \mu(d\lambda).$$

This is the determinant of a tridiagonal matrix, with entries (we use (5.1))

$$f_{ii} = \frac{1}{N^i} + \frac{N^{-1} + |z|^2}{iN^{i-1}}, \quad f_{i,i+1} = -\frac{i}{N^i}, \quad f_{i,i-1} = -\frac{z}{iN^{i-1}}.$$

Denoting $x = N|z|^2$ and $d_k = \det((M_{ij})_{1 \leq i,j \leq k})$, with the convention $d_0 = 1$ we have

$$d_1 = \frac{x + 2}{N^2},$$

$$d_k = \left(1 + \frac{x + 1}{k}\right) \frac{1}{N^k} d_{k-1} - \frac{x}{kN^{2k-1}} d_{k-2},$$

so that $a_k = d_k N^{-\frac{k(k+1)}{2}}$ satisfies $a_0 = 1, a_1 = x + 2$,

$$a_k = \left(1 + \frac{x + 1}{k}\right) a_{k-1} - \frac{x}{k} a_{k-2}.$$

This gives $a_k = (k+1) e^{(k+1)}(x) - x e^{(k)}(x)$ by an immediate induction. Thus, we conclude

$$E(\mathcal{O}_{11} | \lambda_1 = z) = N \frac{e^{(N)}(x)}{e^{(N-1)}(x)} - x.$$ (2.11)

From the asymptotics

$$e^{(N)}(x) = e^x - \sum_{\ell > N} \frac{x^\ell}{\ell!} = e^x + O\left(\frac{x^N}{N!(1 - |z|^2)}\right) = e^x \left(1 + O\left(\frac{e^{N(1-|z|^2) \log |z|^2}}{\sqrt{2\pi N}(1 - |z|^2)}\right)\right),$$

the expected formula follows.\qed

The following proofs make use of special integrability when the conditioned particle is at the center, together with a separation of the short-range and long-range eigenvalues. This separation of scales idea is already present in [14], though not rigorous. To illustrate the main ideas, we first give an alternative proof of Theorem 2.6 (with deteriorated error estimate) which does not rely on explicit formulas, but rather on rigidity and translation invariance. We then prove the main result of this section, Theorem 1.1.
Alternative proof of Theorem 2.6. We denote $\nu_z$ the measure \eqref{1.2} conditioned to $\lambda_1 = z$. Note that $\nu_z$ is a determinantal measure as it has density $\prod_{2 \leq i < j \leq N} |\lambda_i - \lambda_j|^2 e^{-N^2 \nu(z, \delta)}$ for some external potential $V$ (which depends on $z$). With a slight abuse of language, we will abbreviate $\mathbb{E}_{\nu_z}(X) = \mathbb{E}(X \mid \lambda_1 = z)$ even for $X$ a function of the overlaps.

The proof consists in three steps: we first show that we can afford a small cutoff of our test function around the singularity, then we decompose our product into smooth long-range and a short-range parts. The long range concentrates, and the short range is invariant by translation.

First step: small cutoff. Let $g_z(\lambda) = (1 + 1/(N|z - \lambda|^2))$. Remember that from \eqref{2.4}, $\mathbb{E}_{\nu_z}(g_{11}) = \mathbb{E}_{\nu_z}(\prod_{i=2}^N g_z(\lambda_i))$. We denote $h_z(\lambda) = (1 + 1/(N|z - \lambda|^2)\mathbb{1}_{|z - \lambda| > N^{-A}}$, with $A = A(\kappa)$ a large enough chosen constant, and first prove the following elementary equation:

$$
\mathbb{E}_{\nu_z}\left(\prod_{i=2}^N g_z(\lambda_i)\right) = \mathbb{E}_{\nu_z}\left(\prod_{i=2}^N h_z(\lambda_i)\right) + O(N^{-3}). \tag{2.12}
$$

Note that the exponent $N^{-3}$ here is just chosen for the sake of concreteness. The left hand size coincides with $\mathbb{E}_{N-1}\left(\prod_{i=2}^N (|z - \lambda_i| + N^{-1})\right) \left(\mathbb{E}_{N-1}\left(\prod_{i=2}^N (|z - \lambda_i|)\right)\right)^{-1}$, so that \eqref{2.12} follows if we can prove

$$
\mathbb{E}_{N-1}\left(\prod_{i=2}^N (|z - \lambda_i| + N^{-1})e^{-N(|z|^2 - 1)}\right) - \mathbb{E}_{N-1}\left(\prod_{i=2}^N (|z - \lambda_i| + N^{-1})\mathbb{1}_{|z - \lambda_i| > N^{-A}}e^{-N(|z|^2 - 1)}\right) = O(N^{-B}), \tag{2.13}
$$

$$
\mathbb{E}_{N-1}\left(\prod_{i=2}^N (|z - \lambda_i| + N^{-1})e^{-N(|z|^2 - 1)}\right) \geq N^{-C}, \tag{2.14}
$$

for a constant $B$ much larger than $C$. Equation \eqref{2.14} follows from Lemma \ref{2.11}. For equation \eqref{2.13}, note that the left hand side has size order

$$
\mathbb{E}_{N-1}\left(\prod_{i=2}^N (|z - \lambda_i| + N^{-1})e^{-N(|z|^2 - 1)}\mathbb{1}_{|z_i| - \lambda_i < N^{-A}}\right) = O(N^{-A/10})\mathbb{E}_{N-1}\left(\prod_{i=2}^N (|z - \lambda_i| + N^{-1})^2 e^{-2N(|z|^2 - 1)}\right)^{1/2},
$$

by the Cauchy-Schwarz inequality. This last expectation is bounded by Lemma \ref{2.11}, which concludes the proof of \eqref{2.12} by choosing $A$ large enough.

Second step: the long-range contribution concentrates. We smoothly separate the short-range from a long-range contributions on the right hand side of \eqref{2.12}:

$$
h_z(\lambda) = e^{h_z^s(\lambda) + h_z^l(\lambda)},
$$

$$
h_z^s(\lambda) = \log \left(1 + \frac{1}{N|z - \lambda|^2}\mathbb{1}_{|z - \lambda| > N^{-A}}\right) \chi_{z,\delta}(\lambda), \tag{2.15}
$$

$$
h_z^l(\lambda) = \log \left(1 + \frac{1}{N|z - \lambda|^2}\right)(1 - \chi_{z,\delta}(\lambda)),
$$

and we remind $\chi_{z,\delta}(\lambda) = \chi\left(\frac{1}{N^{1/2} - \delta} |z - \lambda|\right)$ with $\delta \in (0, \kappa)$. We define $f_{z}^s(\lambda) = \frac{1}{N|z - \lambda|^2}(1 - \chi_{z,\delta}(\lambda))$ and $\tilde{f}_z = (N - 1) \int_D 1 - \chi_{z,\delta}(\lambda) \frac{dm(\lambda)}{\pi}$. Note that

$$
\left|\sum_{i=2}^N h_z^l(\lambda_i) - \tilde{f}_z\right| \leq \left|\sum_{i=2}^N f_z^s(\lambda_i) - \tilde{f}_z\right| + \sum_{i=2}^N \frac{1}{N|z - \lambda_i|^4}(1 - \chi_{z,\delta}(\lambda_i)). \tag{2.16}
$$
To bound the first term on the right hand side, we rely on [11, Lemma 3.2]: for any $\alpha$ such that $\alpha \| f^\ell_z \| \infty < 1/3$ (in practice $\| f^\ell_z \| \infty < N^{-2\delta}$ so that we will choose $\alpha = cN^2\delta$ for some fixed small $c$), we have

$$E_{\nu_z} \left( e^{\alpha(\sum_{i=2}^N f^\ell_z(\lambda_i) - \nu_z(\sum_{i=2}^N f^\ell_z(\lambda_i)))} \right) \leq e^{C\alpha^2 \text{Var}_{\nu_z}(\sum_{i=2}^N f^\ell_z(\lambda_i))}$$

(2.17)

for some $C$ which does not depend on $N$. We first bound the above variance. Introduce a partition of type $1 = \chi + \sum_{k \geq 1} \xi(2^{-k} x)$ for any $x > 0$, with $\xi$ smooth, compactly supported. Let $f^\ell_z, k(\lambda) = f^\ell_z(\lambda)\xi(2^{-k} N^{1/2-\delta}|z - \lambda|)$ and $K = \min\{k : 2^{-k} N^{1/2-\delta} > 10\}$. Then $\sum_i f^\ell_z(\lambda_i) = \sum_{k=1}^K \sum_i f^\ell_z, k(\lambda_i)$ with probability $1 - e^{-cN}$ (here we use that there are no eigenvalues $|\lambda_k| > 1 + \varepsilon$ with probability $1 - e^{-c(\varepsilon)N}$, thanks to the Corollary 5.5). Moreover, from [10, Theorem 1.2], for any small $\varepsilon > 0$ and large $D > 0$, there exists $N_0 > 0$ such that for any $N \geq N_0$ and $1 \leq k \leq K$ we have

$$P_{N-1} \left( \left| \sum_{i=1}^{N} f^\ell_z, k(\lambda_i) - (N - 1) \int_{|z| < 1} f^\ell_z, k \right| > N^{-2\delta + \varepsilon} \right) \leq N^{-D}.$$ 

This implies the same estimate for the conditioned measure by a simple Cauchy Schwarz inequality:

$$P_{\nu_z} \left( \left| \sum_{i=1}^{N} f^\ell_z(\lambda_i) - (N - 1) \int_{|z| < 1} f^\ell_z \right| > N^{-2\delta + \varepsilon} \right) \leq N^C P_{N-1} \left( \left| \sum_{i=1}^{N} f^\ell_z(\lambda_i) - (N - 1) \int_{|z| < 1} f^\ell_z \right| > N^{-2\delta + \varepsilon} \right)^{1/2} \text{Var}_{\nu_z}(\sum_{i=2}^N f^\ell_z(\lambda_i))^{1/2} \leq N^{-D + 2C}.$$ 

where we used Lemma 2.11. We conclude that for any small $\varepsilon > 0$ and large $D$ we have

$$P_{\nu_z} \left( \left| \sum_{i=1}^{N} f^\ell_z(\lambda_i) - \nu_z \right| > N^{-2\delta + \varepsilon} \right) \leq N^{-D},$$

so that $\text{Var}_{\nu_z}(\sum_{i=2}^N f^\ell_z(\lambda_i)) = O(N^{-4\delta + \varepsilon})$ and $E_{\nu_z}(\sum_{i=2}^N f^\ell_z(\lambda_i)) = \nu_z + O(N^{-2\delta + \varepsilon})$. As a consequence, (2.17) becomes

$$E_{\nu_z} \left( e^{\alpha(\sum_{i=2}^N f^\ell_z(\lambda_i) - \nu_z)} \right) \leq e^{C\alpha^2 N^{-4\delta + \varepsilon} + C\alpha N^{-2\delta + \varepsilon}}.$$ 

(2.18)

The same reasoning yields

$$E_{\nu_z} \left( e^{\alpha \sum_{i=2}^N h^\ell_z(\lambda_i) - \nu_z(1-\chi_s, \delta(\lambda_i))} \right) \leq e^{C\alpha^2 N^{-4\delta + \varepsilon} + C\alpha N^{-2\delta + \varepsilon}}.$$ 

(2.19)

The choice $\alpha = cN^2\delta$ in (2.18), (2.19) together with (2.16) implies

$$P_{\nu_z}(A) < e^{-cN^s}, \quad \text{where} \quad A = \left\{ \sum_{i=2}^N h^\ell_z(\lambda_i) - \nu_z > N^{-2\delta + \varepsilon} \right\}.$$ 

This yields, for some $p = 1 + c(\kappa), c(\kappa) > 0$ and some $q, r > 1$,

$$E_{\nu_z} \left( e^{\sum_{i=2}^N (h^\ell_z(\lambda_i) + h^\ell_z(\lambda_i))} \right) \leq E_{\nu_z} \left( e^{p \sum_{i=2}^N h^\ell_z(\lambda_i)} \right)^{1/p} \leq e^{cN^{s}},$$ 

(2.20)

Here we used that the third term has size order $e^{-cN^s}$, the second one is of order $e^{q\nu_z} = O(N^C)$ from (2.18), (2.19), and so is the first one from Lemma 2.9 (we needed the initial small cutoff changing $g$ into $h$ in order to apply this Lemma). Moreover,

$$E_{\nu_z} \left( e^{\sum_{i=2}^N (h^\ell_z(\lambda_i) + h^\ell_z(\lambda_i))} \right) = (1 + O(N^{-2\delta + \varepsilon}))E_{\nu_z} \left( e^{\sum_{i=2}^N h^\ell_z(\lambda_i)} \right) - (1 + O(N^{-2\delta + \varepsilon}))E_{\nu_z} \left( e^{\sum_{i=2}^N h^\ell_z(\lambda_i) + \nu_z} \right),$$ 

(2.21)
and this last expectation is of order $e^{-cN^\varepsilon}$ for the same reason as (2.20). To summarize, with the previous two equations we have proved (up to exponentially small error terms)

$$
\mathbb{E}_{\nu_z} \left( \prod_{i=2}^N h_z(\lambda_i) \right) = (1 + O(N^{-2\delta + \varepsilon})) e^{\bar{f}_z} \mathbb{E}_{\nu_z} \left( e^{\sum_i h_z^*(\lambda_i)} \right).
$$

**Third step: the local part is invariant.** With $p = 1$ in (2.29), we have

$$
\mathbb{E}_{\nu_z} \left( e^{\sum_i h_z^*(\lambda_i)} \right) = \mathbb{E}_{\nu_0} \left( e^{\sum_i h_0^*(\lambda_i)} \right) + O \left( e^{-cN^{2\varepsilon}} \right).
$$

This yields

$$
\mathbb{E}_{\nu_z} \left( \prod_{i=2}^N h_z(\lambda_i) \right) = (1 + O(N^{-2\delta + \varepsilon})) e^{\bar{f}_z - f_0} \mathbb{E}_{\nu_0} \left( \prod_{i=2}^N h_0(\lambda_i) \right).
$$

From Lemma 2.8, $e^{\bar{f}_z - f_0} = 1 - |z|^2$, and from (2.7), (2.12) we have $\mathbb{E}_{\nu_0} \left( \prod_{i=2}^N h_0(\lambda_i) \right) = N + O(N^{-2})$. This concludes the proof.

**Proof of Theorem 1.1.** We follow the same method as in the previous proof, except that we won’t need a small a priori cutoff: we are interested in convergence in distribution, not in $L^1$.

**First step: the long-range contribution concentrates.** We smoothly separate the short-range from a long-range contributions in Theorem 2.2

$$
\mathcal{G}_{11}^{(d)} = e^{\sum_z g_z^*(\lambda_i, X_i) + \sum_z g_z^*(\lambda_i, X_i)},
$$

$$
g_z^*(\lambda, x) = \log \left( 1 + \frac{|x|^2}{N|z - \lambda|^2} \right) \chi_{z,\delta}(\lambda),
$$

$$
g_z^*(\lambda, x) = \log \left( 1 + \frac{|x|^2}{N|z - \lambda|^2} \right) (1 - \chi_{z,\delta}(\lambda)),
$$

and we remind $\chi_{z,\delta}(\lambda) = \chi \left( N^{\frac{\varepsilon}{2} - \delta} |z - \lambda| \right)$ with $\delta \in (0, \kappa)$. We define $f_z^l(\lambda, x) = \frac{|x|^2}{N|z - \lambda|^2} (1 - \chi_{z,\delta}(\lambda))$ and, as previously, $\bar{f}_z = (N - 1) \int_{\mathbb{R}} \frac{1 - \chi_{z,\delta}(\lambda)}{N|z - \lambda|^2} \frac{d\nu_\lambda}{\pi}$. Let $\mathcal{G}$ be the distribution of $(X_2, \ldots, X_N)$. For any $\varepsilon > 0$, by Gaussian tail we have

$$
\mathcal{G}(B) \geq 1 - e^{-cN^{\varepsilon/10}}, \text{ where } B = \{|X_i|^2 \leq N^{\varepsilon/10} \text{ for all } 2 \leq i \leq N\}.
$$

Moreover,

$$
\left| \sum_{i=2}^N g_z^l(\lambda_i, X_i) - \bar{f}_z |1_B \right| \leq \sum_{i=2}^N f_z^l(\lambda_i, X_i) - \bar{f}_z | + 10N^{\varepsilon/2} \sum_{i=2}^N \frac{1}{N^2 |z - \lambda_i|^2} (1 - \chi_{z,\delta}(\lambda_i)).
$$

(2.22)

To bound the first term on the right hand side, we first integrate over the Gaussian variables:

$$
\mathbb{E}_{\nu_z} \chi_{\mathcal{G}} \left( e^{\alpha \left( \sum_{i=2}^N f_z^l(\lambda_i, X_i) - \bar{f}_z \right)} \right) = \mathbb{E}_{\nu_z} \left( \prod_{i=2}^N \frac{1 - \chi_{z,\delta}(\lambda_i)}{N|z - \lambda_i|^2} e^{-\alpha f_z} \right)
$$

$$
\leq \mathbb{E}_{\nu_z} \left( e^{\alpha \left( \sum_{i=2}^N f_z^l(\lambda_i) - \bar{f}_z \right) + 10N^\varepsilon/2 \sum_{i=2}^N \frac{1}{N^2 |z - \lambda_i|^2} (1 - \chi_{z,\delta}(\lambda_i))^2} \right) \leq e^{C\alpha^2 N^{-2\delta + \varepsilon} + C\alpha N^{-2\delta + \varepsilon}}.
$$
where, for the last inequality, we used (2.18) and (2.19) together with the Cauchy-Schwarz inequality. With $\alpha = cN^{2\delta}$ and (2.22), we obtain
\[
\mathbb{P}_{\nu_x \times G}(A) < e^{-cN^\varepsilon}, \quad \text{where } A = \left\{ \left| \sum_{i=2}^N g_z^i(\lambda_i, X_i) - f_x \right| > N^{-2\delta + \varepsilon} \right\}.
\]  
(2.23)

Let $\xi \in \mathbb{R}$ be fixed. From the above bound we have
\[
\mathbb{E} \left( \left( \frac{\Theta_{11}}{\sqrt{N}} \right)^i | \lambda_1 = z \right) = \mathbb{E}_{\nu_x \times G} \left( \left( \frac{\sum_{i=2}^N g_z^i(\lambda_i, X_i) + \sum_2^N g_z^f(\lambda_i, X_i)}{N(1 - |z|^2)} \right)^i \right) + O(\varepsilon N^\varepsilon) \\
= \mathbb{E}_{\nu_x \times G} \left( \left( \frac{\sum_{i=2}^N g_z^f(\lambda_i, X_i) + f_x}{N(1 - |z|^2)} \right)^i \right) + O(\varepsilon N^{-2\delta + \varepsilon}) \\
\]
where we used $e^{f_x}/(1 - |z|^2) = e^{f_0}$, from Lemma 2.8.

We now define the function $a_z$ (omitting the dependence in $\xi$ in the notation) through
\[
e^{a_z(\lambda)} = \mathbb{E}_G \left( e^{\xi g_z^f(\lambda, X_i)} \right).
\]
Note that $a_z$ does not depend on $i$ because the $X_i$'s are identically distributed. We want to apply Lemma 4.3. Note that $a_z$ is supported on $|z - \lambda| < C N^{-\frac{1}{2} + \delta}$ and $\text{Re}(a_z) \leq 0$, so that (4.3) is automatically satisfied and $(N||\nu||)^\varepsilon \leq C N^{2\delta\varepsilon}$, hence (4.2) holds for the choice $\delta = \kappa/10$. For this choice of $\delta$, we therefore have
\[
\mathbb{E}_{\nu_x} \left( e^{\sum_{i=2}^N a_z(\lambda_i)} \right) = \mathbb{E}_{\nu_0} \left( e^{\sum_{i=2}^N a_0(\lambda_i)} \right) + O\left( e^{-cN^{2\varepsilon}} \right)
\]  
uniformly in $\xi$. This proves
\[
\mathbb{E} \left( \left( \frac{\Theta_{11}}{\sqrt{N}} \right)^i | \lambda_1 = z \right) = \mathbb{E}_{\nu_0 \times G} \left( \left( \frac{\sum_{i=2}^N g_0^f(\lambda_i, X_i) + f_0}{N} \right)^i \right) + O(\varepsilon N^{-2\delta + \varepsilon}) \\
= \mathbb{E} \left( \left( \frac{\Theta_{11}}{\sqrt{N}} \right)^i | \lambda_1 = 0 \right) + O(\varepsilon N^{-2\delta + \varepsilon}).
\]  
(2.25)

Together with Proposition 2.4, this concludes the proof of Theorem 1.1.

Proof of Corollary 1.3 We start with the lower bound. From (2.23) we have
\[
\mathbb{P} \left( \Theta_{11} > N^{1-\varepsilon} | \lambda_1 = z \right) \leq \mathbb{P}_{\nu_x \times G} \left( \sum_{i=2}^N g_z^f(\lambda_i, X_i) + f_0 < N^{1-\varepsilon} \right) + O(e^{-cN^\varepsilon}).
\]
From (2.24) and Lemma 2.10 with $R = T = e^{N^{\varepsilon/10}}$ and $F$ smooth bounded equal to 1 on the interval $(-\infty, (1 - \varepsilon/2)\log N - f_0)$, 0 on $[(1 - \varepsilon/2)\log N - f_0 + 1, \infty)$, we have
\[
\mathbb{P}_{\nu_x \times G} \left( \sum_{i=2}^N g_z^f(\lambda_i, X_i) + f_0 < N^{1-\varepsilon} \right) \leq \mathbb{P}_{\nu_0 \times G} \left( \sum_{i=2}^N g_0^f(\lambda_i, X_i) + f_0 < N^{1-\varepsilon} \right) + O(e^{-N^{\varepsilon/10}}) \\
\leq \mathbb{P} \left( \Theta_{11} < N^{1-\varepsilon/8} | \lambda_1 = 0 \right) + O(e^{-N^{\varepsilon/10}}) = O(e^{-N^{\varepsilon/10}}),
\]  
(2.26)

where this last probability was estimated thanks to Proposition 2.4. For $|z| < 1 - N^{-\frac{1}{2} + \varepsilon}$, this yields
\[
\mathbb{P} \left( \Theta_{11} < N^{-\frac{1}{2} + \varepsilon} N^{1-\varepsilon} | \lambda_1 = z \right) = O(e^{-N^{\varepsilon/10}}),
\]  

and we conclude by a union bound. For the upper estimate, in the same way as previously, for any $x > 0$ we obtain
\[
\mathbb{P}\left(\frac{\Theta_{11}}{1 - |z|^2} > x \mid \lambda_1 = z\right) \leq \mathbb{P}\left(\Theta_{11} > x/2 \mid \lambda_1 = 0\right) + O(e^{-N^{2/3}}). \tag{2.27}
\]
For $x \gg N$, the following is easy to justify:
\[
\mathbb{P}\left(\Theta_{11} > x \mid \lambda_1 = 0\right) = N(N-1) \int_0^{1/x} u(1-u)^{N-2}du = \frac{N-1}{N} \int_{x/N}^{\infty} \left(1 - \frac{1}{Nt}\right)^{N-2} \frac{dt}{t^3} \sim \int_{x/N}^{\infty} \frac{dt}{t^3} = \frac{N^2}{2x^2}.
\]

We obtained
\[
\sum_{i=1}^{N} \mathbb{P}(\Theta_{ii} \geq x) \lesssim N \int_{\mathbb{D}} \frac{N^2(1 - |z|^2)^2}{2x^2} \frac{dm(z)}{\pi} = \frac{N^3}{6x^2}.
\]
If the overlaps are sufficiently independent (a fact suggested by (1.12)), this hints towards convergence of the maximum to a Fréchet distribution: for any fixed $y > 0$, as $N \to \infty$
\[
\mathbb{P}\left(\max_{1 \leq i \leq N} \frac{\Theta_{ii}}{N^{3/2}} < y\right) \to e^{-\frac{1}{\pi y^2}}.
\]
Remember that $0 \leq \chi \leq 1$ is a smooth cut-off function on $\mathbb{R}_+$ such that $\chi(x) = 1$ for $x < 1/2$ and 0 for $x > 1$, and we denote
\[
\chi_{\varepsilon, \delta}(\lambda) = \chi\left(N^{1/2} - \varepsilon \cdot |\lambda - z|\right).
\]
The following three lemmas were used in the previous proofs.

**Lemma 2.8.** There exists a constant $c(\chi)$ such that for any $|z| < 1 - N^{-\frac{1}{2} + \delta}$ we have
\[
\frac{1}{\pi} \int_{\mathbb{D}} \frac{1 - \chi_{\varepsilon, \delta}(\lambda)}{|z - \lambda|^2} \, dm(\lambda) = (1 - 2\delta) \log(N) + \log(1 - |z|^2) + c(\chi).
\]

**Proof.** To compute this integral, we partition the unit disk into the domains below, where $\rho = N^{-1/2+\delta}$:
\[
\begin{align*}
D_0 &= D(0, |z| - \rho), \\
D_1 &= D(z, \rho), \\
R_1 &= D(0, |z| + \rho) - D(0, |z| - \rho), \\
R_2 &= D(0, 1) - D(0, |z| + \rho).
\end{align*}
\]
On $D_1 \subset \mathbb{D}$, we have
\[
\frac{1}{\pi} \int_{D_1} \frac{1 - \chi_{\varepsilon, \delta}(\lambda)}{|z - \lambda|^2} \, dm(\lambda) = 2 \int_0^\rho \frac{1 - \chi(N^{1/2} - \delta \rho)}{r^2} \, dr = 2 \int_0^1 \frac{1 - \chi(\rho)}{r} \, dr = c_1(\chi).
\]
Outside $D_1$, the integrand equals $\frac{1}{|\lambda - z|^2}$ so that the corresponding calculations are explicit. For example, on the central disk $D_0$, we expand

$$\frac{1}{\pi} \int_{D_0} \frac{1}{|\lambda - z|^2} \, dm(\lambda) = \frac{1}{\pi} \int_{D_0} \frac{1}{|z|^2} \left(1 - \frac{1}{z} + \frac{1}{z^2}\right) \, dm(\lambda) = \frac{1}{\pi} \int_{r=0}^{1} \frac{1}{|z|^2} \sum_{j,k} \left(\frac{Re^{i\theta}}{z}\right)^j \frac{Re^{i\theta}}{z} \, r \, dr \, d\theta$$

$$= \int_{r=0}^{1} \frac{2r}{|z|^2 - r^2} \, dr = \log(|z|^2) - \log(2|z| - \rho) - \log(\rho).$$

On the outer ring $R_2$, the same type of expansion gives

$$\frac{1}{\pi} \int_{R_2} \frac{1}{|\lambda - z|^2} \, dm(\lambda) = \log(1 - |z|^2) - \log(2|z| + \rho) - \log(\rho),$$

while the contribution from $R_1$ converges to $\Omega = \{|z| > 1\} \cap \{|\Re(z)| < 1\}$ around $z$, up to a homothecy, so that $c_2 = \int_{\Omega} dm(\lambda)/|\lambda|^2$.

Bringing these four estimates together gives

$$\frac{1}{\pi} \int_{D} \frac{1}{|z - \lambda|^2} \, dm(\lambda) = (1 - 2\delta) \log(N) + \log(|z|^2) - \log(4) + \log(1 - |z|^2) + c(\varepsilon) + c(1) + o(1).$$

This completes the proof. \qed

**Lemma 2.9.** Let $h^s_\varepsilon$ be given by (2.15). For any $\kappa \in (0, 1/2)$, there exists $c(\kappa)$, $C(\kappa) > 0$ such that for any $|z| < 1 - N^{-1/2+\kappa}$ and $p \in [1, 1+c(\kappa)]$, we have

$$\mathbb{E}_{\nu_\varepsilon} \left( e^{p \sum_{s=2}^{N} h^s_\varepsilon(\lambda)} \right) \leq N^{C(\kappa)}. $$

**Proof.** First, the result is true for $z = 0$. Indeed,

$$\mathbb{E}_{\nu_0} \left( e^{p \sum_{s=2}^{N} h^s_\nu(\lambda)} \right) \leq \mathbb{E}_{\nu_0} \left( \prod_{s=2}^{N} \left(1 + \frac{1}{N|\lambda|}\right)^p \right) = \prod_{s=2}^{N} \mathbb{E} \left( \left(1 + \frac{1}{\gamma_k}\right)^p \right) = O(N^C), \quad (2.28)$$

where we used Corollary 5.6.

We want to apply Lemma 4.3 to conclude the proof. For this we need to check conditions (4.2) and (4.3) for our function $f = ph^s_\varepsilon$, and $\nu = e^{-1}$. First note that $|\nu|_1 \leq C \int_{N^{-10} < |\lambda| < 1} (|N|^{1/2})^p \leq C N^{5/2} N^{10(2p-2)}$, so that (4.2) holds by choosing $p = 1 + c(\kappa)$ with $c(\kappa)$ small enough. To prove (4.3), we rely on Lemma 4.2

$$\mathbb{E}_{\nu_\varepsilon} \left( e^{p \sum_{s=2}^{N} h^s_\varepsilon(\lambda)} \right) = \mathbb{E}_{\nu_\varepsilon} \left( e^{p \sum_{s=2}^{N} h^s_\varepsilon(\lambda)} \right) \left(1+o(1)\right) \leq \mathbb{E} \left( \left(1 + \frac{1}{\gamma_k} \right)^p \right) \prod_{k=2}^{N} \mathbb{E} \left( \left(1 + \frac{1}{\gamma_k} \right)^p \right) \leq N^C$$

where we used (4.2) for the first equation, and the calculation $\int_{N^{-10} < |\lambda| < 1} \frac{1}{|\lambda|} e^{-x} \leq N^C$. We therefore obtained

$$\mathbb{E}_{\nu_\varepsilon} \left( e^{p \sum_{s=2}^{N} h^s_\varepsilon(\lambda)} \right) = \mathbb{E}_{\nu_0} \left( e^{p \sum_{s=2}^{N} h^s_\varepsilon(\lambda)} \right) + O \left( e^{-cN^{2\kappa}} \right) \quad (2.29)$$

for any $1 \leq p \leq 1 + c(\kappa)$ with $c(\kappa)$ small enough. Equations (2.28) and (2.29) conclude the proof. \qed

To quantitatively invert the Fourier transform, we use the following crude bound, see [3, Lemma 2.6] following from [5, Corollary 11.15].

**Lemma 2.10.** There exists a constant $c$ such that if $\mu$ and $\nu$ are probability measures on $\mathbb{R}$ with Fourier transforms $\hat{\mu}(t) = \int e^{it\mu} (dx)$ and $\hat{\nu}(t) = \int e^{it\nu} (dx)$, then for any $R, T > 0$ and any function $f : \mathbb{R} \to \mathbb{R}$ with Lipschitz constant $C$,

$$|\mu(f) - \nu(f)| \leq c \frac{C}{T} + c\|f\|_{\infty} \{RT\|1_{(-T,T)}(\hat{\mu} - \hat{\nu})\|_{\infty} + \mu([-R, R]) + \nu([-R, R])\}. \quad (2.30)$$
The following crude a priori estimates are used in this paper. Note that for \( z \) strictly in the bulk of the spectrum (\(|z| < 1 - \varepsilon\) for fixed \( \varepsilon > 0 \)), the first statement is a simple consequence of the main result in [50]. This lemma is proved in [8].

**Lemma 2.11.** For any \( p, \kappa > 0 \), there exists \( C > 0 \) such that, for large enough \( N \), uniformly in \(|z_1|, |z_2| < N^{-\frac{1}{2} + \kappa}\) we have

\[
N^{-C} \leq \mathbb{E}_N \left( \prod_{i=1}^{N} |z_1 - \lambda_i|^2 e^{-pN(|z_1|^2 - 1)} \right) \leq NC,
\]

\[
N^{-C} \leq \mathbb{E}_N \left( \prod_{i=1}^{N} \left( |z_1 - \lambda_i|^2 + \frac{1}{N} \right)^p e^{-pN(|z_1|^2 - 1)} \right) \leq NC,
\]

\[
N^{-C} \leq \mathbb{E}_N \left( \prod_{i=1}^{N} |z_1 - \lambda_i|^2 |z_2 - \lambda_i|^2 e^{-N(|z_1|^2 - 1)} e^{-N(|z_2|^2 - 1)} \right) \leq NC.
\]

### 2.4 The real Ginibre ensemble: heuristics behind Remark 1.2

In this paragraph, we propose a strategy to prove (1.9). The main ideas mimic our proof for the complex Ginibre ensemble.

The first step is strictly similar to the complex Ginibre case: as mentioned at the end of Theorem 2.2 an annealed remarkable identity in law also holds for the real Ginibre point process, which can be written with the notations of Remark 1.2 as

\[
\vartheta^{(d)}_{11} = \prod_{n=2}^{N} \left( 1 + \frac{2\gamma_{1/2}^{(k)}}{N|\lambda_1 - \lambda_n|^2} \right).
\]

We now need to integrate over the eigenvalues \( \lambda_n \)'s, \( 2 \leq n \leq N \), which are independent of the \( \gamma_{1/2}^{(k)} \)'s.

A decomposition into long and short ranges can be performed, similarly to (2.21). The long range concentrates around \( f_z \) thanks to the rigidity of eigenvalues [10] (for convergence in distribution we only need that the long range concentrates, not need for the exponential concentration: [11, Lemma 3.2] is only necessary for convergence of the expectation).

For the short range part, again we define (remember we omit the dependence in \( \xi \) in the notation)

\[
e^{a_z(\lambda)} = \mathbb{E}_G \left( e^{\xi a_z(\lambda, X_i)} \right),
\]

where \( X_i \) is now a real standard Gaussian random variable. Assume the following analogue of (2.24) holds:

\[
\mathbb{E}_{\text{Gin}(\mathbb{R})} \left( e^{\sum_{i=2}^{N} a_z(\lambda_i)} \mid \lambda_1 = z \right) = \mathbb{E}_{\text{Gin}(\mathbb{C})} \left( e^{\sum_{i=2}^{N} a_0(\lambda_i)} \mid \lambda_1 = 0 \right) (1 + o(1)),
\]

for any \( z \in \Omega_\kappa \), i.e. a translation invariance result similar to Lemma 4.3 now to compare local products of the real Ginibre ensemble in \( \Omega_\kappa \) and the complex Ginibre ensemble around 0. Then, similarly to the first line of (2.25), we obtain that \( \vartheta^{(d)}_{11}/(N(1 - |z|^2) \) (overlap for real Ginibre, \( z \in \Omega_\kappa \)) and \( \prod_{k=2}^{N} \left( 1 + \frac{2\gamma_{1/2}^{(k)}}{N|\lambda_1|^2} \right) \)

(where the \( \lambda_i \)'s are distributed as in the complex Ginibre ensemble) have the same limit in distribution.

Together with Corollary 5.6, this justifies (1.9).

We expect the missing piece (2.31) to hold because the correlation functions of the real Ginibre ensemble in \( \Omega_\kappa \) are very close to those of the complex Ginibre ensemble in the bulk [7, 21]: the convergence in [7, Theorem 11] can be shown to be exponentially fast when in \( \Omega_\kappa \). A complete proof of (2.31) would require substitution of determinantal formulas with Pfaffian ones, and an asymptotic negative association property (used in the proof of Lemma 4.3) for Pfaffian point processes, beyond the scope of this article.
3 OFF-DIAGONAL OVERLAPS

In this section we consider the distribution of \(N\) Ginibre points conditioned to \(\{\lambda_1 = z_1, \lambda_2 = z_2\}\). We will successively prove identities for the quenched off-diagonal overlaps, for all \(z_1, z_2\), and then get explicit relations for \(z_1 = 0\) in the annealed setting. Finally, these new correlation identities are extended to any \(z_1, z_2\) in the bulk of the spectrum by a decomposition of short and long range contributions.

3.1 The quenched off-diagonal overlap. Contrary to the diagonal overlap, the factorization here doesn’t involve independent variables.

**Proposition 3.1.** With the notation (2.5), the following equality in law holds, for fixed eigenvalues:

\[
\Theta_{12} \overset{(d)}{=} -\left| \frac{T_{12}}{\lambda_1 - \lambda_2} \right|^2 \prod_{n=3}^{N} \left( 1 + \frac{Z_n}{N(\lambda_1 - \lambda_n)(\lambda_2 - \lambda_n)} \right),
\]

where each \(Z_n\) is a product of two (correlated) complex Gaussian random variables, and \(\mathbb{E}(Z_n \mid F_{n-2}) = 1\).

**Proof.** As for the diagonal overlap, we simply compute \(\Theta_{12}^{(2)} = -|b_2|^2 = -\left| \frac{T_{12}}{\lambda_1 - \lambda_2} \right|^2\) and

\[
\Theta_{12}^{(n+1)} = -b_2 B_n D_{n+1} = -b_2 (B_n^T \bar{D}_n + b_{n+1} d_{n+1}) = -b_2 (B_n^T \bar{D}_n + \alpha_{1,n+1} B_n T_n \bar{D}_{n+1} + D_{n+1} T_n)
\]

\[
= -b_2 B_n^T \bar{D}_n \left( 1 + \alpha_{1,n+1} \frac{B_n^T D_n}{B_n^T \bar{D}_n} \right) = \Theta_{12}^{(n)} \left( 1 + \frac{Z_{n+1}}{N(\lambda_1 - \lambda_n)(\lambda_2 - \lambda_n)} \right),
\]

where

\[
Z_{n+1} = N \frac{B_n^T T_n \bar{D}_n D_n}{B_n^T \bar{D}_n}.
\]

Clearly, \(Z_{n+1}\) is a product of two complex Gaussian random variables, a distribution which depends on \(\Theta_{12}^{(n)}\). Moreover, \(B_n, D_n \in \mathcal{F}_{n-1}\) and \(T_n\) is independent of \(\mathcal{F}_{n-1}\), so that \(\mathbb{E}(Z_{n+1} \mid \mathcal{F}_{n-1}) = 1\).

**Remark 3.2.** By successive conditional expectations with respect to \(\mathcal{F}_{n-2}, \ldots, \mathcal{F}_1\), Proposition 3.1 implies

\[
\mathbb{E}(\Theta_{12}) = -\frac{1}{N(\lambda_1 - \lambda_2)^2} \prod_{k=3}^{N} \left( 1 + \frac{1}{N(\lambda_1 - \lambda_k)(\lambda_2 - \lambda_k)} \right),
\]

an important fact already proved in [13,14,39].

3.2 The annealed off-diagonal overlap: expectation. Remarkably, the works [13,14,39] also explicitly integrated the random variable (3.2) over \(\lambda_3, \ldots, \lambda_N\), in the specific case \(\lambda_1 = 0\). We state the resulting asymptotics and add the proof from Chalker and Mehlig, for completeness.

**Corollary 3.3** (Chalker, Mehlig [13,14,39]). For any \(\varepsilon > 0\), there exists \(c > 0\) such that uniformly in \(|z| < N^{-\varepsilon}\),

\[
\mathbb{E}(\Theta_{12} \mid \lambda_1 = 0, \lambda_2 = z) = -\frac{1}{N|z|^4} \frac{1 - (1 + N|z|^2) e^{-N|z|^2}}{1 - e^{-N|z|^2}} + O(e^{-cN}).
\]

**Proof.** From (3.2), we want to evaluate

\[
\mathbb{E}(\Theta_{12} \mid \lambda_1 = 0, \lambda_2 = z) = -\frac{1}{N|z|^2} \mathbb{E}\left( \prod_{k=3}^{N} \left( 1 - \frac{1}{N\lambda_k (z - \lambda_k)} \right) \mid \lambda_1 = 0, \lambda_2 = z \right)
\]
From Theorem [5.4] with \( g(\lambda) = 1 - \frac{1}{N\lambda(z - \lambda)} \), we find that
\[
\mathbb{E} \left( \prod_{k=2}^{N} g(\lambda_k) \mid \lambda_1 = 0, \lambda_2 = z \right) = \frac{1}{Z_N^{0,z}} \det(f_{i,j})_{i,j=1}^{N-2}
\]
where
\[
f_{i,j} = \frac{1}{(i+1)!} \int \lambda^{i-1} \lambda^{j-1} |\lambda|^2 \| z - \lambda \|^2 g(\lambda) \mu(d\lambda) = \frac{1}{(i+1)!} \int \lambda^{i-1} \lambda^{j-1} \left( |\lambda|^2 \| z - \lambda \|^2 - \frac{1}{N} \lambda(z - \lambda) \right) \mu(d\lambda).
\]
This matrix is tridiagonal with entries
\[
f_{ii} = \frac{1}{(i+1)!} \cdot \frac{|z|^2}{(i+1)N^i} + \frac{1}{(i+1)N^{i+1}}, \quad f_{i,i+1} = - \frac{2}{N^{i+1}}, \quad f_{i,i-1} = - \frac{z}{i N^i}.
\]
Let \( d_k = \det(f_{i,j})_{i,j=1}^{k} \) and \( x = N|z|^2 \). With the convention \( d_0 = 1 \) we have
\[
d_1 = \frac{1}{N^2} \left( \frac{3}{2} + \frac{x}{2} \right),
\]
\[
d_k = \left( 1 + \frac{x+1}{k+1} \right) \frac{1}{N^{k+1}} d_{k-1} - \frac{x}{k} \frac{1}{N^{2k+1}} d_{k-2}.
\]
so that \( a_k = d_k N^{-\frac{x(k+3)}{2}} \) satisfies \( a_0 = 1, a_1 = 3/2 + x/2, \)
\[
a_k = \left( 1 + \frac{x+1}{k+1} \right) a_{k-1} - \frac{x}{k} a_{k-2}.
\]
An immediate induction gives \( a_k = (k+2)x^{-\frac{2}{k}} e_{2}^{(k+2)}(x) \). Thus, we conclude
\[
\mathbb{E} (\Theta_{12} \mid \lambda_1 = 0, \lambda_2 = z) = - \frac{N e_2^{(N)}(x)}{x^2 e_1^{(N)}(x)}.
\]
The proof is then concluded by standard asymptotics.

With Corollary [3.3] the expectation of \( \Theta_{12} \) is known for \( \lambda_1 = 0 \). To extend the result to anywhere in the bulk of the spectrum, we mimic the alternative proof of Theorem [2.6] from Subsection [2.3].

**Proof of Theorem [1.4].** We denote \( \nu_{z_1,z_2} \) the measure \( (1.2) \) conditioned to \( \lambda_1 = z_1, \lambda_2 = z_2 \). Note that \( \nu_{z_1,z_2} \) is a determinantal measure. With a slight abuse of language, we will abbreviate \( E_{\nu_{z_1,z_2}}(X) = E(X \mid \lambda_1 = z_1, \lambda_2 = z_2) \) even for \( X \) a function of the overlaps.

We follow the same three steps as in the alternative proof of Theorem [2.6]. Strictly speaking, if we were to impose \( |z_1 - z_2| > N^{-C} \) for some fixed \( C \), we would not need the first step below, as the singularity \( 1/|z| \) is integrable, contrary to our previous singularity \( 1/|z|^2 \). However, in Theorem [1.4] we allow \( z_1 \) and \( z_2 \) to be arbitrarily close, so we first perform an initial small cutoff.

**First step: small cutoff.** Let \( g_{z_1,z_2}(\lambda) = (1 + 1/(N(z_1 - \lambda)(z_2 - \lambda))) \). Remember that, from [3.2]
\[
E_{\nu_{z_1,z_2}}(\Theta_{12}) = - \frac{1}{N|z_1 - z_2|^2} E_{\nu_{z_1,z_2}} \left( \prod_{n=3}^{N} g_{z_1,z_2}(\lambda_n) \right).
\]
We denote \( h_{z_1,z_2}(\lambda) = (1 + 1/(N(z_1 - \lambda)(z_2 - \lambda))) 1_{\lambda \notin B} \) where \( B = \{ |\lambda - z_1| < N^{-A} \} \cup \{ |\lambda - z_2| < N^{-A} \} \).
For \( A \) a large enough constant, the analogue of [2.12] holds:
\[
E_{\nu_{z_1,z_2}} \left( \prod_{i=3}^{N} g_{z_1,z_2}(\lambda_i) \right) = E_{\nu_{z_1,z_2}} \left( \prod_{i=3}^{N} h_{z_1,z_2}(\lambda_i) \right) + O(N^{-3}). \tag{3.3}
\]
Indeed, by making explicit the above conditional measures, (3.3) follows from
\begin{equation}
\mathbb{E}_{N^{-2}} \left( \prod_{i=3}^{N} (|z_1 - \lambda_i|^2 |z_2 - \lambda_i|^2 + N^{-1} (|z_1 - \lambda_i| (z_2 - \lambda_i)) e^{-N(|z_1|^2 - 1) - N(|z_2|^2 - 1)} \right) = O(N^{-B}),
\end{equation}
(3.4)
\begin{equation}
\mathbb{E}_{N^{-2}} \left( \prod_{i=3}^{N} (|z_1 - \lambda_i|^2 |z_2 - \lambda_i|^2 + N^{-1} (|z_1 - \lambda_i| (z_2 - \lambda_i)) \mathbb{1}_{|\lambda_i| \notin \mathbb{R}} e^{-N(|z_1|^2 - 1) - N(|z_2|^2 - 1)} \right) = O(N^{-B}),
\end{equation}
(3.5)
\begin{equation}
\mathbb{E}_{N^{-2}} \left( \prod_{i=3}^{N} (|z_1 - \lambda_i|^2 |z_2 - \lambda_i|^2) e^{-N(|z_1|^2 - 1) - N(|z_2|^2 - 1)} \right) \geq N^{-C_1}.
\end{equation}
(3.6)

with $B$ much larger than $C_1$. Lemma 2.11 gives (3.6). The left hand side of (3.5) has size order
\begin{equation}
\mathbb{E}_{N^{-2}} \left( \prod_{i=3}^{N} (|z_1 - \lambda_i|^2 + N^{-1}(|z_2 - \lambda_i|^2 + N^{-1}) e^{-N(|z_1|^2 - 1) - N(|z_2|^2 - 1)} \mathbb{1}_{|\lambda_i| \notin \mathbb{R}} \right) = O(N^{-A+C_2})
\end{equation}
by the Cauchy-Schwarz inequality and Lemma 2.11 for some $C_2$ which does not depend on $A$. This concludes the proof of (3.3) by choosing $A$ large enough.

**Second step: the long-range contribution concentrates.** We smoothly separate the short-range from a long-range contributions on the right hand side of (3.3):
\begin{align*}
h_{z_1,z_2}(\lambda) &= e^{h_{z_1,z_2}(\lambda)} + h^{s}_{z_1,z_2}(\lambda), \\
h^{s}_{z_1,z_2}(\lambda) &= \log \left( 1 + \frac{1}{N(z_1 - \lambda)(z_2 - \lambda)} \mathbb{1}_{z \notin \mathbb{R}} \right) \chi_{z,\delta}(\lambda), \\
h^{f}_{z_1,z_2}(\lambda) &= \log \left( 1 + \frac{1}{N(z_1 - \lambda)(z_2 - \lambda)} \right) (1 - \chi_{z,\delta}(\lambda)),
\end{align*}
and we denote $z = (z_1 + z_2)/2$, remind $|z_1 - z_2| < N^{-\frac{1}{2} + \varepsilon}$, $\chi_{z,\delta}(\lambda) = \chi \left( N^{\frac{1}{2} - \delta} |z - \lambda| \right)$, and choose $\delta \in (\kappa - \varepsilon, \kappa)$. In the definition of $h^{s}_{z_1,z_2}$, we can choose any branch for the logarithm, this won’t have any impact on the rest of the proof. In the long-range contribution $h^{f}_{z_1,z_2}$, the logarithm is defined by continuity from $\log(1) = 0$. Let $f^f_{z_1,z_2}(\lambda) = \frac{1}{N(z_1 - \lambda)(z_2 - \lambda)} \chi_{z,\delta}(\lambda)$ and $f^{s}_{z_1,z_2} = \frac{N-2}{N} \mathbb{1} \int_0^{N-2} \frac{1}{1+\mathbb{1}_{|\lambda_i| \notin \mathbb{R}}} d\mu(\lambda)$. Note that
\begin{equation}
\left| \sum_{i=3}^{N} h^{f}_{z_1,z_2}(\lambda_i) - f^{s}_{z_1,z_2} \right| \leq \left| \sum_{i=3}^{N} f^{s}_{z_1,z_2}(\lambda_i) - f^{s}_{z_1,z_2} \right| + \frac{1}{2} \sum_{i=3}^{N} \frac{1}{N^2 |z_i - \lambda_i|^4} (1 - \chi_{z,\delta}(\lambda_i)) + \sum_{i=3}^{N} \frac{1}{N^2 |z_2 - \lambda_i|^4} (1 - \chi_{z,\delta}(\lambda_i)).
\end{equation}
(3.7)

The last two sums are bounded as in (2.19). For the first term on the right hand side, we bound the real and imaginary parts separately: similarly to (2.17), we have
\begin{equation}
\mathbb{E}_{N^{-2}} \left( e^{\alpha (\sum_{i=3}^{N} \text{Re} f^{s}_{z_1,z_2}(\lambda_i)) - \text{Im} f^{s}_{z_1,z_2} (\sum_{i=3}^{N} \text{Re} f^{f}_{z_1,z_2}(\lambda_i))} \right) \leq e^{C \alpha^2 \text{Var}_{z_1,z_2} (\sum_{i=3}^{N} \text{Re} f^{f}_{z_1,z_2}(\lambda_i))}
\end{equation}
(3.8)
where $\alpha = c N^{2\delta}$ for some fixed small $c$ and $C$ which does not depend on $N$. We first bound the above variance. Remember we have a partition of type $1 = \chi + \sum_{k>1} \xi(2^{-k}x)$ for any $x > 0$, with $\xi$ smooth, compactly supported. Let $f^{f}_{z_1,z_2,k}(\lambda) = f^{f}_{z_1,z_2}(\lambda) \xi(2^{-k}N^{1/2} \delta |z - \lambda|)$ and $K = \min\{k \geq 1 : 2^{-k}N^{-1/2} \delta > 10\}$. Then $\sum_{k=1}^{K} f^{f}_{z_1,z_2,k}(\lambda_i)$ by probability $1 - e^{-cN}$, and with [10, Theorem 1.2], for any small $\varepsilon > 0$ and large $D > 0$, there exists $N_0 > 0$ such that for any $N \geq N_0$ and $1 \leq k \leq K$ we have (we now omit to write the real part, being understand that $f$ is either $\text{Re} f$ or $\text{Im} f$)
\begin{equation}
\mathbb{P}_{N^{-2}} \left( \sum_{k=1}^{K} f^{f}_{z_1,z_2,k}(\lambda_i) - (N-2) \int_{|z|<1} f^{f}_{z_1,z_2,k} \right) > N^{-2\delta + \varepsilon} \right) \leq N^{-D}.
\end{equation}
(3.9)
The same estimate holds for the conditioned measure by the Cauchy-Schwarz inequality:

\[ P_{\nu_{z_1,z_2}} \left( \left| \sum_{i=2}^{N} f^\ell_{z_1,z_2,k}(\lambda_i) - (N-2) \int_{|z|<1} f^\ell_{z_1,z_2,k} \right| > N^{-2\delta+\varepsilon} \right) \]

\[ \leq P_{N-2} \left( \left| \sum_{i=2}^{N} f^\ell_{z_1,z_2,k}(\lambda_i) - (N-2) \int_{|z|<1} f^\ell_{z_1,z_2,k} \right| > N^{-2\delta+\varepsilon} \right)^{1/2} N^C \]

\[ \mathbb{E}_{N-2} \left( \prod_{i=2}^{N} |z_i - \lambda_i|^4 e^{-2N(|z| - 1)} \prod_{i=2}^{N} |z_i - \lambda_i|^4 e^{-2N(|z| - 1)} \right)^{1/2} \leq N^{-D+2C}. \]

for some \( C \) which only depends on \( \kappa \), where we used Lemma \[2.11\]. We conclude that for any small \( \varepsilon > 0 \) and large \( D \) we have

\[ P_{\nu_{z_1,z_2}} \left( \left| \sum_{i=2}^{N} f^\ell_{z_1,z_2}(\lambda_i) - \bar{f}_{z_1,z_2} \right| > N^{-2\delta+\varepsilon} \right) \leq N^{-D}, \]

so that \( \text{Var}_{\nu_{z_1,z_2}}(\sum_{i=3}^{N} f^\ell_{z_1,z_2}(\lambda_i)) = O(N^{-4\delta+\varepsilon}) \) and \( \mathbb{E}_{\nu_{z_1,z_2}}(\sum_{i=3}^{N} f^\ell_{z_1,z_2}(\lambda_i)) = \bar{f}_{z_1,z_2} + O(N^{-2\delta+\varepsilon}) \). As a consequence, \[3.8\] becomes

\[ \mathbb{E}_{\nu_{z_1,z_2}} \left( e^{\alpha(\sum_{i=3}^{N} f^\ell_{z_1,z_2}(\lambda_i) - \bar{f}_{z_1,z_2})} \right) \leq e^{C\alpha^2 N^{-4\delta+\varepsilon}+C\alpha N^{-2\delta+\varepsilon}}. \] (3.9)

With \( \alpha = cN^{2\delta} \), we obtain

\[ P_{\nu_{z_1,z_2}}(A) < e^{-cN^\varepsilon}, \text{ where } A = \left\{ \left| \sum_{i=3}^{N} h^\ell_{z_1,z_2}(\lambda_i) - \bar{f}_{z_1,z_2} \right| > N^{-2\delta+\varepsilon} \right\}. \]

This yields, for some \( p = 1 + c(\kappa), c(\kappa) > 0 \) and some \( q, r > 1, \)

\[ \mathbb{E}_{\nu_{z_1,z_2}} \left( e^{\sum_{i=3}^{N} (h^\ell_{z_1,z_2}(\lambda_i) + h^\ell_{z_1,z_2}(\lambda_i))} 1_A \right) \leq \mathbb{E}_{\nu_{z_1,z_2}} \left( e^{p \sum_{i=3}^{N} h^\ell_{z_1,z_2}(\lambda_i)} \right)^{1/p} \mathbb{E}_{\nu_{z_1,z_2}} \left( e^{q \sum_{i=3}^{N} h^\ell_{z_1,z_2}(\lambda_i)} \right)^{1/q} P(A)^{1/r} \leq e^{-cN^\varepsilon}. \] (3.10)

Here we used that the third term has size order \( e^{-cN^\varepsilon} \), the second one is of order \( e^{q\bar{f}_{z_1,z_2}} = O(N^C) \), and so is the first one from Lemma \[3.8\] and \( 1 + \frac{1}{(z_1-\lambda)(z_2-\lambda)} \leq (1 + \frac{1}{N(z_1-\lambda^2)})(1 + \frac{1}{N(z_2-\lambda^2)}). \) Moreover,

\[ \mathbb{E}_{\nu_{z_1,z_2}} \left( e^{\sum_{i=3}^{N} (h^\ell_{z_1,z_2}(\lambda_i) + h^\ell_{z_1,z_2}(\lambda_i))} 1_{A^c} \right) = (1 + O(N^{-2\delta+\varepsilon})) \mathbb{E}_{\nu_{z_1,z_2}} \left( e^{\sum_{i=3}^{N} h^\ell_{z_1,z_2}(\lambda_i) + \bar{f}_{z_1,z_2}} \right) \]

\[ - (1 + O(N^{-2\delta+\varepsilon})) \mathbb{E}_{\nu_{z_1,z_2}} \left( \sum_{i=3}^{N} h^\ell_{z_1,z_2}(\lambda_i) + \bar{f}_{z_1,z_2} 1_{A^c} \right), \] (3.11)

and this last expectation is of order \( e^{-cN^\varepsilon} \) for the same reason as \(3.10\). To summarize, with the previous two equations we have proved (up to exponentially small additive error terms)

\[ \mathbb{E}_{\nu_{z_1,z_2}} \left( \prod_{i=3}^{N} h^\ell_{z_1,z_2}(\lambda_i) \right) = (1 + O(N^{-2\delta+\varepsilon})) e^{\bar{f}_{z_1,z_2}} \mathbb{E}_{\nu_{z_1,z_2}} \left( e^{\sum_{i=3}^{N} h^\ell_{z_1,z_2}(\lambda_i)} \right). \]

Third step: the local part is invariant. For our test function \( h^\ell_{z_1,z_2} \), the reader can easily check the conditions of Lemma \[4.4\] so that

\[ \mathbb{E}_{\nu_{z_1,z_2}} \left( e^{\sum_{i=3}^{N} h^\ell_{z_1,z_2}(\lambda_i)} \right) = \mathbb{E}_{\nu_{0,z_2-z_1}} \left( e^{\sum_{i=3}^{N} h^\ell_{0,z_2-z_1}(\lambda_i)} \right) + O \left( e^{-cN^{2\delta}} \right). \]

This yields

\[ \mathbb{E}_{\nu_{z_1,z_2}} \left( \prod_{i=3}^{N} h^\ell_{z_1,z_2}(\lambda_i) \right) = (1 + O(N^{-2\delta+\varepsilon})) e^{\bar{f}_{z_1,z_2} - \bar{f}_{0,z_2-z_1}} \mathbb{E}_{\nu_{0,z_2-z_1}} \left( \prod_{i=3}^{N} h^\ell_{0,z_2-z_1}(\lambda_i) \right). \]

From Lemma \[3.4\] \( e^{\bar{f}_{z_1,z_2} - \bar{f}_{0,z_2-z_1}} = 1 - z_1 \overline{z_2} \). Together with Corollary \[3.3\], this concludes the proof. \( \square \)
Lemma 3.4. For any $\lambda_1, \lambda_2 \in \mathbb{D}$,
\[
\frac{1}{\pi} \int_0 \frac{1}{(\lambda_1 - z)(\lambda_2 - z)} \, dm(z) = \log \left( \frac{1 - \lambda_1 \bar{\lambda}_2}{|\lambda_1 - \lambda_2|^2} \right).
\]

Proof. We consider the following domains, assuming $0 < |\lambda_1| < |\lambda_2| < 1$ and $\varepsilon > 0$ is small enough. The following computation still holds if $|\lambda_1| = |\lambda_2|$, as long as $\lambda_1 \neq \lambda_2$. Integrability is clear, as the poles are simple and isolated. Moreover, under these conditions, the integral cancels on the disks $D_1$ and $D_2$.

Integration over the domain $D_0$ yields
\[
\frac{1}{\pi} \int_{D_0} \frac{1}{(\lambda_1 - z)(\lambda_2 - z)} \, dm(z) = \frac{1}{\pi} \int_{r=0}^{r=1} \frac{r \, dr \, d\theta}{\lambda_1 \lambda_2 (1 - \text{re}^{i\theta})(1 - \text{re}^{i\theta})} = 2 \int_{r=0}^{r=1} \frac{1}{\lambda_1 \lambda_2} \sum_k \left( \frac{r^2}{\lambda_1 \lambda_2} \right)^k \, r \, dr
\]
\[
= \int_{r=0}^{r=1} \frac{2r}{\lambda_1 \lambda_2 - r^2} \, dr = \log(\lambda_1 \bar{\lambda}_2) - \log(\lambda_1 \bar{\lambda}_2 - (|\lambda_1| - \varepsilon)^2). \quad (3.12)
\]
The same type of expansion shows that the integral over $R_2$ vanishes and the contribution from $R_4$ is
\[
\frac{1}{\pi} \int_{R_4} \frac{1}{(\lambda_1 - z)(\lambda_2 - z)} \, dm(z) = \log(1 - \lambda_1 \bar{\lambda}_2) - \log((|\lambda_2| + \varepsilon)^2 - \lambda_1 \bar{\lambda}_2). \quad (3.13)
\]
As the expression is integrable, on the domains $R_1, R_3$ there is no contribution as $\varepsilon \to 0$. Summing (3.12) and (3.13) gives the result.

3.3 The quenched off-diagonal overlap: second moments. The main result of this subsection is the following lemma, which gives the expectation of second moments of overlaps conditionally on the eigenvalues positions. For this, we define
\[
X_n = \begin{pmatrix} [B_n D_n]^2 / \|B_n\|^2 \|D_n\|^2 \end{pmatrix}, \quad \gamma_{ij} = \frac{a_{ij}}{\sqrt{N}} \quad A_n = \begin{pmatrix} |1 + \gamma_{1,n} \bar{\gamma}_{2,n}|^2 & |\gamma_{1,n} \bar{\gamma}_{2,n}|^2 \\ |\gamma_{1,n} \gamma_{2,n}|^2 & (1 + |\gamma_{1,n}|^2)(1 + |\gamma_{2,n}|^2) \end{pmatrix}. \quad (3.14)
\]

Lemma 3.5. For any $2 \leq n \leq N - 1$ we have
\[
E(X_{n+1} | F_{n-1}) = A_{n+1} X_n. \quad (3.15)
\]
In particular,
\[
\begin{pmatrix} E_T(|\theta_{12}|^2 | F_1) \\ E_T(\theta_{11}^2 \theta_{22}^2 | F_1) \end{pmatrix} = \begin{pmatrix} |b_2|^2 & 0 \\ 0 & 1 + |b_2|^2 \end{pmatrix} \left( \prod_{i=3}^n A_i \right) \begin{pmatrix} |b_2|^2 & 0 \\ 0 & 1 + |b_2|^2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (3.16)
\]

Proof. We remind the notation (3.11) and the property $E(Z_{n+1} | F_{n-1}) = 1$. A short calculation also gives $E(|Z_{n+1}|^2 | F_{n-1}) = 1 + \|B_n D_n\|^2 / \|B_n D_n\|^4$. Abbreviating $\gamma_k = \gamma_{k,n+1}$, this gives
\[
E(|B_n D_n|^2 | F_{n-1}) = E(|B_n D_n|^2 [1 + \gamma_1 \bar{\gamma}_2 Z_{n+1}]^2 | F_{n-1})
\]
\[
= |B_n D_n|^2 E(1 + \gamma_1 \bar{\gamma}_2 Z_{n+1} + \gamma_1 \bar{\gamma}_2 Z_{n+1}^2 + |\gamma_1 \bar{\gamma}_2 Z_{n+1}|^2 | F_{n-1})
\]
\[
= |1 + \gamma_1 \bar{\gamma}_2|^2 |B_n D_n|^2 + |\gamma_1 \bar{\gamma}_2|^2 \|B_n\|^2 \|D_n\|^2. \quad (3.17)
\]
We now denote $X_{n+1} = \sqrt{N} \frac{B_n}{\|B_n\|} T_n$, $Y_{n+1} = \sqrt{N} \frac{D_n}{\|D_n\|} T_n$, so that $\mathbb{E}(|X_{n+1}|^2 \mid \mathcal{F}_{n-1}) = \mathbb{E}(|Y_{n+1}|^2 \mid \mathcal{F}_{n-1}) = 1$ and $\mathbb{E}(|X_{n+1}Y_{n+1}|^2 \mid \mathcal{F}_{n-1}) = 1 + \frac{\|B_n\| \|D_n\|}{\|B_n\| \|D_n\|}^2$. This yields

$$\mathbb{E} (\|B_{n+1}\|^2 \|D_{n+1}\|^2 \mid \mathcal{F}_{n-1}) = \mathbb{E} (\|B_n\|^2 \|D_n\|^2 (1 + |\gamma_1 X|^2) (1 + |\gamma_2 Y|^2) \mid \mathcal{F}_{n-1})$$

$$= \|B_n\|^2 \|D_n\|^2 \mathbb{E} (1 + |\gamma_1 X|^2 + |\gamma_2 Y|^2 + |\gamma_1 \gamma_2 X Y|^2 \mid \mathcal{F}_{n-1})$$

$$= (1 + |\gamma_1|^2)(1 + |\gamma_2|^2) \|B_n\|^2 \|D_n\|^2 + |\gamma_1 \gamma_2|^2 \|B_n^2 \|D_n\|^2. \tag{3.18}$$

Equations (3.17) and (3.18) together conclude the proof of (3.15). Denoting $Y_n = \xi T(X_n \mid \mathcal{F}_1)$, we obtain

$$Y_n = \left( \prod_{i=3}^n A_i \right) Y_2 = \left( \prod_{i=3}^n A_i \right) \mathbb{E} \left( \left( \frac{|B_2 D_2|^2}{\|B_2\|^2 \|D_2\|^2} \right) \mid \mathcal{F}_1 \right) = \left( \prod_{i=3}^n A_i \right) \left( \frac{|b|^2}{1 + |b|^2} \right)$$

and (3.16) immediately follows. \hfill \Box

### 3.4 The annealed off-diagonal overlap: second moments for $\lambda_1 = 0$

We now want to integrate (3.16) over the eigenvalues $\lambda_3, \ldots, \lambda_N$, first in the special case $\lambda_1 = 0$. This requires some new notations.

We abbreviate $\delta = N|\lambda_1 - \lambda_2|^2$, $a = \frac{\delta}{2} + \sqrt{1 + \frac{\delta^2}{4}}$, $b = -a^{-1} = \frac{\delta}{2} - \sqrt{1 + \frac{\delta^2}{4}}$ and will often use the property

$$\delta = a - \frac{1}{a} = b - \frac{1}{b}. \tag{3.19}$$

We also define the following rational fractions of $x$

$$u_k(x) = 1 - \frac{1 - x^{-1}}{k + 3}, \tag{3.20}$$

$$d_k(x, \delta) = \frac{(k+2)(k+3)}{c_1^{(k+1)}(\delta)} \left( u_k(x) \frac{1}{(x-1)^2} c_3(k)(\delta) - \frac{\delta u_k(x)}{2x} + \frac{1}{(x-1)^2} x^2 + (k+2)x + (k+1)(k+3) \right) \frac{\delta^{k+1}}{(k+3)!}.$$

We can now state the main proposition on which Theorem 1.5 depends. The reason why such formulas exist relies on two algebraic facts.

(i) The $A_n$'s from (3.14) are diagonalizable in the same basis, see (3.24). Their commutation was expected, our choice of eigenvalues ordering being arbitrary, while the left hand side of (3.16) is intrinsic.

(ii) More surprisingly, the obtained holonomic sequence (3.24) is exactly solvable.

**Proposition 3.6.** Conditionally on $\lambda_1 = 0, \lambda_2 = z$, we have

$$\mathbb{E}(|\xi_{12}|^2) = \frac{1}{1 + a^2} \left( \frac{a^2}{(a+1)^2} d_{N-2}(a, \delta) + \frac{a^2}{(a-1)^2} d_{N-2}(b, \delta) \right),$$

$$\mathbb{E}(\xi_{11} \xi_{22}) = \frac{1}{1 + a^2} \left( \frac{a^4}{(a+1)^2} d_{N-2}(a, \delta) + \frac{a^4}{(a-1)^2} d_{N-2}(b, \delta) \right).$$

**Proof.** Importantly, the matrices $A_n, 3 \leq n \leq N$, can be diagonalized in the same basis: from (3.14) an elementary calculation based on (2.11) gives

$$A_n = (1 + |\gamma_1 n^2|(1 + |\gamma_2 n^2|) I_2 + |\gamma_1 n \gamma_2 n|^2 \left( \begin{array}{cc} -N|\lambda_1 - \lambda_2|^2 & 1 \\ 1 & 0 \end{array} \right), \tag{3.21}$$

so that its eigenvectors clearly do not depend on $n$. With these notations, the eigenvalues of $A_n$ are

$$\lambda_+(n) = (1 + |\gamma_1 n|^2)(1 + |\gamma_2 n|^2) - |\gamma_1 n \gamma_2 n|^2 a,$$

$$\lambda_-(n) = (1 + |\gamma_1 n|^2)(1 + |\gamma_2 n|^2) - |\gamma_1 n \gamma_2 n|^2 b.$$
and the orthogonal basis
\[ U = \frac{1}{\sqrt{1 + a^2}} \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix} \]
diagonalizes all \( A_n \)'s simultaneously: \( U A_n U^T = \text{diag}(\lambda_+(n), \lambda_-(n)) \). Then \( \{3.16\} \), together with the codiagonalization of the \( A_n \)'s, \( E(T)^2 = 1 \) and \( E(T)^4 = 2 \), yields the following simple expression:
\[
\begin{pmatrix}
E_T(\Theta^{12}_{12}) \\
E_T(\Theta^{11}_{11} \Theta^{22}_{22})
\end{pmatrix} = \frac{1}{1 + a^2} \begin{pmatrix}
\frac{a^2}{(a+1)^2} d_+ + \frac{a^4}{(a-1)^2} d_- \\
\frac{a^2}{(a+1)^2} d_- + \frac{a^4}{(a-1)^2} d_-
\end{pmatrix},
\tag{3.22}
\]
\[
d_+ = \prod_{n=3}^{N} \lambda_+(n), \quad d_- = \prod_{i=3}^{N} \lambda_-(n).
\tag{3.23}
\]

Note that we have not yet used \( z_1 = 0 \); the above formula holds for any given \( z_1, z_2 \)

The remarkable fact is that in the specific case \( z_1 = 0 \), \( E(d_+) \) and \( E(d_-) \) can be calculated, as shown below (here \( E \) denotes integration over all variables except \( \lambda_1, \lambda_2 \)). We start with \( E(d_+) \). From Theorem \[ \ref{5.4} \] the following holds:
\[
E \left( \prod_{n=3}^{N} \lambda_+(n) \mid \lambda_1 = 0, \lambda_2 = z \right) = \frac{\det(f_{i,j})_{1 \leq i, j \leq N-2}}{Z_N^{(0,z)}}
\]
where \( Z_N^{(0,z)} \) is given by \( \{5.2\} \) and
\[
f_{i,j} = \frac{1}{(i+1)!} \int \lambda^{i-1} \bar{\lambda}^{j-1} |\lambda|^2 |z - \lambda|^2 \mu(d\lambda)
= \frac{1}{(i+1)!} \int \lambda^{i-1} \bar{\lambda}^{j-1} (|\lambda|^2 |z - \lambda|^2 + N^{-1} |\lambda|^2 + N^{-1} |z - \lambda|^2 - N^{-2} (a-1)) \mu(d\lambda).
\]

We expand \( |\lambda - z|^2 = |\lambda|^2 + |z|^2 - \bar{\lambda} \lambda - \bar{\lambda} z \). The resulting matrix is thus tridiagonal with
\[
f_{i,i-1} = -\frac{z}{iN^i},
\quad f_{i,i} = \frac{1}{N^{i+1}} + \frac{2N^{-1} + |z|^2}{(i+1)N^i} + \frac{N^{-1} |\lambda|^2 - N^{-2} (a-1)}{i(i+1)N^{-1}},
\quad f_{i,i+1} = -\frac{z}{(i+1)N^{i+1}}.
\]

With the notation \( d_k = \det(f_{i,j})_{1 \leq i, j \leq k} \), the induction \( d_k = f_{k,k} d_{k-1} - f_{k-1,k} f_{k-1,k} d_{k-2} \) holds, so that defining \( a_k = d_k N^{\frac{k(k+3)}{2}} \) we have
\[
a_k = \left( 1 + 2 + \frac{\delta}{k+1} + \frac{1-a^{-1}}{k(k+1)} \right) a_{k-1} - 2\delta \frac{i+1}{i^2} a_{k-2},
\]
with the convention \( a_0 = 1 \) and \( a_1 = 2 + \frac{\delta}{2} + \frac{1-a^{-1}}{2} \). Note that for \( z = 0 \) we have \( \delta = 0 \) and \( a = 1 \), hence
\[
a_k^0 := a_k(z = 0) = 2 \prod_{j=2}^{k} \left( 1 + \frac{2}{j+1} \right) = \frac{(k+2)(k+3)}{6}.
\]

As a consequence, \( g_k = \frac{a_k^0}{a_k^1} \) satisfies \( g_0 = 1, g_1 = 1 + \frac{\delta}{2} + \frac{1-a^{-1}}{4} \), and
\[
g_k = m_1(k,a) g_{k-1} - m_2(k,a) g_{k-2},
\tag{3.24}
\]
\[
m_1(k,a) = 1 + \frac{\delta}{k+3} + \frac{1-a^{-1}}{k(k+3)},
\]
\[
m_2(k,a) = \delta \frac{(k+1)^2}{k(k+2)(k+3)}.
\]
We cannot see an a priori reason why this equation could be solved, but it can be. We remark that the function $u_k = u_k(a)$ from (3.20) also satisfies the induction

$$u_k = m_1(k,a)u_{k-1} - m_2(k,a)u_{k-2}.$$  

(3.25)

The reader who would like to check the above equation can substitute $\delta = a - \frac{1}{a}$ and verify that the Laurent series in $a$ on both sides of (3.25) coincide. This equation implies

$$g_{k-1} - \frac{u_k}{u_{k-1}} = m_1(k,a)g_{k-1} - m_2(k,a)g_{k-1} - \frac{u_{k-2}}{u_{k-1}}$$

(3.26)

Subtracting (3.26) from (3.24) gives

$$\delta_k := g_k - g_{k-1} \frac{u_k}{u_{k-1}} = -m_2(k,a)\left(g_{k-2}(x) - g_{k-1} \frac{u_{k-2}}{u_{k-1}} \right) = m_2(k,a) \frac{u_{k-2}}{u_{k-1}} \delta_{k-1},$$

which simplifies (3.28) into

$$\delta_k(x) = \left( \prod_{j=2}^k m_2(j,a) \frac{u_{j-2}}{u_{j-1}} \right) \delta_1 = 36 \frac{1}{(k+3)!} \frac{k+1}{k+2} \delta^{k-1} \frac{u_0}{u_{k-1}} \delta_1.$$  

(3.27)

Together with $\frac{g_k}{u_k} = \frac{g_{k-1}}{u_{k-1}} + \frac{\delta_k}{u_k}$ from (3.27) we obtain

$$\frac{g_k}{u_k} = 36 \sum_{j=1}^k \frac{1}{(j+3)!} \frac{j+1}{j+2} \delta^{j-1} \frac{u_0}{u_{j-1}u_j} \delta_1 + \frac{g_0}{u_0}.$$  

A calculation gives $u_0 \delta_1 = \frac{1}{6} (a - a^{-1})(1 + a^{-1})$, so that

$$\frac{g_k}{u_k} = 6(1 + a^{-1}) \sum_{j=1}^k \frac{1}{(j+3)!} \frac{j+1}{j+2} \delta^{j-1} \frac{1}{u_{j-1}u_j} + 1.$$  

(3.28)

Note that

$$\frac{1}{u_{j-1}u_j} = \frac{(j+2)(j+3)}{1-a^{-1}} \left( \frac{1}{u_{j-1}} - \frac{1}{u_j} \right),$$

which simplifies (3.28) into

$$\frac{g_k}{u_k} = 6 \frac{1 + a^{-1}}{1-a^{-1}} \sum_{j=1}^k \frac{1}{j!(j+2)} \left( \frac{1}{u_{j-1}} - \frac{1}{u_j} \right) \delta^j + \frac{1}{u_0}$$

$$= 6 \frac{1 + a^{-1}}{1-a^{-1}} \left( \sum_{j=1}^k \frac{1}{u_j} \left( \frac{\delta}{(j+1)!} - \frac{1}{j!(j+2)} \right) \delta^j \right) + \frac{1}{u_0}$$

$$= 6 \frac{a+1}{a-1} \left( \sum_{j=1}^k \frac{1}{a(j+1)!} - \frac{1}{(j+1)!} \right) \delta^j \frac{1}{k!(k+2)} \delta^k + \frac{1}{u_0}$$

$$= 6 \frac{a+1}{a-1} \left( \sum_{j=3}^k \frac{\delta^j}{j!} + \frac{a \delta^{k-1}}{(k+1)!} \right) \frac{1}{k!(k+2)} \delta^k + \frac{1}{u_0}$$

where we used $\delta = a - a^{-1}$ and (3.20) at several steps. The above formula can be written in terms of (1.21) and further simplified as

$$g_k = 6u_k \frac{a+1}{a-1} \frac{1}{a \delta^2} \mathcal{E}_3^{(k)}(\delta) - \frac{3u_k}{a} + \frac{a+1}{a-1} \left( \frac{a(k+2)}{a} + \frac{a^{-1}(k+1)(k+3)}{(k+3)!} \right) \delta^{k-1}.$$  

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By definition of $g_k$, $a_k = g_k a_k^0$, that is
\[ a_k = u_k \frac{1}{(a-1)^2 \delta} \frac{k + 2}{k + 3} e_3^{(k)}(\delta) - \frac{u_k}{2a} \frac{(k + 2)(k + 3)}{a - 1} \frac{a + 1 + a + (k + 2) + a^{-1}(k + 1)(k + 3)}{(k + 1)!} \delta^{k-1}. \]

Then, using the normalizing constant (5.2), we obtain
\[
\mathbb{E}(d_+) = \frac{N(N+1)}{e_1^{(N-1)}(\delta)} \left( \frac{u_{N-2}(a)}{(a-1)^2} e_3^{(N-2)}(\delta) - \frac{\delta u_{N-2}(a)}{2a} \frac{a^2 + N \lambda + (N-1)(N+1) \delta}{(N+1)!} \right) \tag{3.29}
\]
as $\delta = N|z|^2$. We find
\[
\mathbb{E}(d) = \frac{N(N+1)}{e_1^{(N-1)}(\delta)} \left( \frac{u_{N-2}(a)}{(a-1)^2} e_3^{(N-2)}(\delta) - \frac{\delta u_{N-2}(a)}{2a} \frac{a^2 + N \lambda + (N-1)(N+1) \delta}{(N+1)!} \right) \tag{3.29}
\]
which has been defined as $d_{N-2}(a, \delta)$. The formula for $\mathbb{E}(d_-)$ is obtained in the exact same way, with the only difference that $a$ is replaced by $b$. Finally, conditionally on $\{\lambda_1 = 0, \lambda_2 = z\}$, (3.22) gives
\[
\left( \begin{array}{c}
\mathbb{E}(|\mathcal{O}_{12}^2|) \\
\mathbb{E}(\mathcal{O}_{11} \mathcal{O}_{22})
\end{array} \right) = \frac{1}{1 + a^2} \left( \begin{array}{c}
\frac{a^2}{(a+1)^2} \mathbb{E}(d) + \frac{a}{(a-1)^2} \mathbb{E}(d_-)
\frac{1}{(a+1)^2} \mathbb{E}(d) + \frac{a}{(a-1)^2} \mathbb{E}(d_-)
\end{array} \right) .
\]

We can replace $\mathbb{E}(d_+)$ and $\mathbb{E}(d_-)$ by their exact expressions to obtain the claimed formula. \qed

**Proposition 3.7.** Let $\sigma \in (0, 1/2)$. Denoting $\omega = \sqrt{N}|z|$, uniformly in $|z| \in [0, N^{-\frac{1}{2} + \sigma}]$ we have
\[
\mathbb{E}(|\mathcal{O}_{12}^2|) = \frac{N^2}{|\omega|^4} \left( 1 + O(N^{2\sigma-1}) \right) , \tag{3.30}
\]
\[
\mathbb{E}(\mathcal{O}_{11} \mathcal{O}_{22}) = \frac{N^2}{|\omega|^4} \left( 1 + O(N^{2\sigma-1}) \right) . \tag{3.31}
\]

**Proof.** We consider asymptotics in Proposition 3.6. First, the term $\frac{\delta}{(N+1)}$ is obviously negligible. Second, we always have $a \geq 1$ and $|b| \geq c/\delta \geq N^{-2\sigma}$, so that $u_{N-2}(a) = 1 + O(N^{-1})$, $u_{N-2}(b) = 1 + O(N^{2\sigma-1})$. Moreover,
\[
e_1^{(N-1)}(\delta) = (e^\delta - 1) (1 + O(N^{-1})), \quad e_3^{(N-2)}(\delta) = (e^\delta - 1 - \delta - \frac{\delta^2}{2}) (1 + O(N^{-1})).
\]
From (3.19), we have $\frac{a^2}{(a+1)^2} - \frac{b^2}{(a-b)^2} = \frac{1}{a^2}$, so that (3.32) and its analogue for $d_-$ give
\[
\frac{a^2}{(a+1)^2} \mathbb{E}(d_+) = \frac{N^2}{\delta^2(e^\delta - 1)} \left( e^\delta - 1 - \delta - \frac{\delta^2}{2} \right) \left( 1 + O(N^{-1}) \right) , \tag{3.32}
\]
\[
\frac{a^2}{(a-1)^2} \mathbb{E}(d_-) = \frac{N^2}{\delta^2(e^\delta - 1)} \left( a^2 \left( e^\delta - 1 - \delta - \frac{\delta^2}{2} \right) + \frac{a}{2} (a+1)^2 \right) \left( 1 + O(N^{2\sigma-1}) \right) . \tag{3.33}
\]
We observe that
\[
\frac{1}{a^2 + 1} \left( -\frac{\delta (a-1)^2}{2a} + \frac{\delta a}{2} (a+1)^2 \right) = \delta + \frac{\delta^2}{2} ,
\]
and the previous three equations give
\[
\mathbb{E}(|\mathcal{O}_{12}^2|) = \frac{1}{1 + a^2} \left( \frac{a^2}{(a+1)^2} \mathbb{E}(d) + \frac{a}{(a-1)^2} \mathbb{E}(d_-) \right) = \frac{N^2}{\delta^2(e^\delta - 1)} \left( \left( e^\delta - 1 - \delta - \frac{\delta^2}{2} + \delta + \frac{\delta^2}{2} \right) \left( 1 + O(N^{2\sigma-1}) \right) \right) .
\]
The similar computation for $\mathbb{E}(\theta_{11}\theta_{22} \mid \lambda_1 = 0, \lambda_2 = z)$ involves the two terms

$$\begin{align*}
\frac{1}{a^4} \mathbb{E}(d_-) &= \frac{N^2}{e^3 - 1} \left( \frac{1}{(a-1)^2(a+1)^2} \left( e^{\delta} - 1 - \delta - \frac{\delta^2}{2} \right) + \frac{\delta}{2(a+1)^2} \right) \left( 1 + O(N^{-1}) \right), \\
\frac{a^4}{(a-1)^3} \mathbb{E}(d_-) &= \frac{N^2}{e^3 - 1} \left( \frac{a^6}{(a-1)^2(a+1)^2} \left( e^{\delta} - 1 - \delta - \frac{\delta^2}{2} \right) + \frac{\delta^2}{2(a+1)^2} \right) \left( 1 + O(N^{2\sigma-1}) \right) .
\end{align*}$$

Moreover, some algebra gives

$$\begin{align*}
&\frac{1}{1 + a^2} \left( \frac{1}{(a-1)^2(a+1)^2} + \frac{a^6}{(a-1)^2(a+1)^2} \right) = \frac{1 + \delta^2}{\delta^2} , \\
&\frac{1}{1 + a^2} \left( \frac{\delta a^5}{2(a-1)^2} - \frac{\delta}{2(a+1)^2} \right) = \frac{(\delta + 1)(\delta^2 + \delta + 2)}{2\delta} .
\end{align*}$$

Once combined, these four equations yield

$$\begin{align*}
\mathbb{E}(\theta_{11}\theta_{22} \mid \lambda_1 = 0, \lambda_2 = z) &= \frac{N^2}{e^3 - 1} \left( \frac{1 + \delta^2}{\delta^2} \left( e^{\delta} - 1 - \delta - \frac{\delta^2}{2} \right) + \frac{(\delta + 1)(\delta^2 + \delta + 2)}{2\delta} \right) \left( 1 + O(N^{-2\sigma}) \right) = \frac{N^2}{\sigma^2} \frac{1 + |\omega|^4 - e^{-|\omega|^2}}{1 - e^{-|\omega|^2}} \left( 1 + O(N^{2\sigma-1}) \right) ,
\end{align*}$$

which concludes the proof. \(\square\)

### 3.5 The annealed off-diagonal overlap: second moments in the general case.

We can now prove Theorem 1.5. We closely follow the method developed first in our alternative proof of Theorem 2.6 in Subsection 2.3, then in our proof of Theorem 1.4 in Subsection 3.2. In particular, following the proof of Theorem 1.4, we denote $\nu_{z_1, z_2}$ the measure (1.2) conditioned to $\lambda_1 = z_1, \lambda_2 = z_2$. 

**Proof of Theorem 1.5.** Remember the notation (3.23). Assume we can prove (under the hypothesis of Theorem 1.5) that

$$\begin{align*}
\mathbb{E}_{\nu_{z_1, z_2}}(d_+) &= (1 - |z_1|^2)(1 - |z_2|^2)\mathbb{E}_{\nu_{0, z_2 - z_1}}(d_+) \left( 1 + O(N^{-2\nu + \epsilon}) \right) , \\
\mathbb{E}_{\nu_{z_1, z_2}}(d_-) &= (1 - |z_1|^2)(1 - |z_2|^2)\mathbb{E}_{\nu_{0, z_2 - z_1}}(d_-) \left( 1 + O(N^{-2\nu + \epsilon}) \right) .
\end{align*}$$

From (3.32) and (3.33), a calculation gives $\mathbb{E}_{\nu_{0, z_2 - z_1}}(d_+) > 0$, $\mathbb{E}_{\nu_{0, z_2 - z_1}}(d_-) > 0$, so that (3.34), (3.35) together with (3.22) give

$$\begin{align*}
\left( \frac{\mathbb{E}_{\nu_{z_1, z_2}}(|\theta_{12}|^2)}{\mathbb{E}_{\nu_{z_1, z_2}}(\theta_{11}\theta_{22})} \right) &= (1 - |z_1|^2)(1 - |z_2|^2) \left( \frac{\mathbb{E}_{\nu_{0, z_2 - z_1}}(|\theta_{12}|^2)}{\mathbb{E}_{\nu_{0, z_2 - z_1}}(\theta_{11}\theta_{22})} \right) \left( 1 + O(N^{-2\nu + \epsilon}) \right) .
\end{align*}$$

Together with Proposition 4.1, this concludes the proof. We therefore only need to show (3.34). The proof for (3.35) is identical up to trivial adjustments.

**First step: small cutoff.** Our test function of interest and its short-range cut version are

$$\begin{align*}
g_{z_1, z_2}(\lambda) &= \left( 1 + \frac{1}{N|z_1 - \lambda|^2} \right) \left( 1 + \frac{1}{N|z_2 - \lambda|^2} \right) - \frac{a}{N^2} \frac{1}{|z_1 - \lambda|^2} \frac{1}{|z_2 - \lambda|^2} , \\
h_{z_1, z_2}(\lambda) &= \left( 1 + \frac{1}{N|z_1 - \lambda|^2} \mathbb{I}_{\lambda \not\in B} \right) \left( 1 + \frac{1}{N|z_2 - \lambda|^2} \mathbb{I}_{\lambda \not\in B} \right) - \frac{a}{N^2} \frac{1}{|z_1 - \lambda|^2} \frac{1}{|z_2 - \lambda|^2} \mathbb{I}_{\lambda \not\in B} .
\end{align*}$$
where \( \mathcal{B} = \{|\lambda - z_1| < N^{-A}\} \cup \{|\lambda - z_2| < N^{-A}\} \). We first prove (3.3) for our new definition of \( g_{z_1,z_2}, h_{z_1,z_2} \). It follows from

\[
E_{N-2} \left( \prod_{i=3}^{N} \left( (|z_1 - \lambda_i|^2 + \frac{1}{N})(|z_2 - \lambda_i|^2 + \frac{1}{N}) - \frac{a}{N^2} \right) e^{-N(|z_1|^2-1)-N(|z_2|^2-1)} \right) - E_{N-2} \left( \prod_{i=3}^{N} (|z_1 - \lambda_i|^2 + \frac{1}{N})(|z_2 - \lambda_i|^2 + \frac{1}{N}) - \frac{a}{N^2} \right) e^{-N(|z_1|^2-1)-N(|z_2|^2-1)} \right) = O(N^{-A/2+C}),
\]

(3.38)

and (3.6), for some \( C \) which does not depend on \( A \). Equation (3.39) holds as the left hand side is bounded by

\[
E_{N-2} \left( \prod_{i=3}^{N} \left( (|z_1 - \lambda_i|^2 + \frac{1}{N})(|z_2 - \lambda_i|^2 + \frac{1}{N}) - \frac{a}{N^2} \right) e^{-N(|z_1|^2-1)-N(|z_2|^2-1)} \right)
\]

\[
= O(N^{-A/2})E_{N-2} \left( \prod_{i=3}^{N} \left( (|z_1 - \lambda_i|^2 + \frac{1}{N})(|z_2 - \lambda_i|^2 + \frac{1}{N}) - \frac{a}{N^2} \right) e^{-2N(|z_1|^2-1)-2N(|z_2|^2-1)} \right)^{1/2},
\]

(3.40)

where we used the Cauchy–Schwarz inequality and

\[
\left| (|z_1 - \lambda_i|^2 + \frac{1}{N})(|z_2 - \lambda_i|^2 + \frac{1}{N}) - \frac{a}{N^2} \right| \leq (|z_1 - \lambda_i|^2 + \frac{1}{N})(|z_2 - \lambda_i|^2 + \frac{1}{N}).
\]

Indeed, after rescaling and shifting and introducing \( z \) so that \(|z|^2 = \delta\), the above bound follows from

\[
2(1 + |\lambda|^2)(1 + |z - \lambda|^2) - a = ((1 + |\lambda|^2)(1 + |z - \lambda|^2) - \delta) + (1 + |\lambda|^2)(1 + |z - \lambda|^2) + (\delta - a) \geq 1 + \delta - a = 1 - \frac{1}{a} \geq 0.
\]

We used \((1 + |\lambda|^2)(1 + |z - \lambda|^2) \geq \delta\), as proved by a simple optimization. The last expectation in (3.40) has size order at most \( N^C \) from Lemma 2.11 which concludes the proof of our initial short-range cutoff by choosing \( A \) large enough.

**Second step: the long-range contribution concentrates.** We smoothly separate the short-range from a long-range contributions in (3.37):

\[
h_{z_1,z_2}(\lambda) = \epsilon ^{h_{z_1,z_2}(\lambda)} h_{z_1,z_2}(\lambda),
\]

\[
h^\delta_{z_1,z_2}(\lambda) = (\log h_{z_1,z_2}(\lambda)) \chi_{x,\delta}(\lambda),
\]

\[
h^\delta_{z_1,z_2}(\lambda) = (\log h_{z_1,z_2}(\lambda))(1 - \chi_{x,\delta}(\lambda)),
\]

and, as earlier in this article, we denote \( z = (z_1 + z_2)/2 \), remind \(|z_1 - z_2| < N^{-\frac{1}{2} + \sigma}, \chi_{x,\delta}(\lambda) = \chi \left( N^{\frac{1}{2} - \delta} |z - \lambda| \right) \), and choose \( \delta \in (\sigma, \kappa) \). We define \( f_{z_1,z_2}(\lambda) = (\frac{1}{N |z_1 - \lambda|^2} + \frac{1}{N |z_2 - \lambda|^2})(1 - \chi_{x,\delta}(\lambda)) \) and \( \tilde{f}_{z_1,z_2} = (N - 2)^{1/2} \int_{\mathbb{B}} f_{z_1,z_2}(\lambda) dm(\lambda) \). Note that

\[
\left| \sum_{i=3}^{N} h_{z_1,z_2}(\lambda_i) - \tilde{f}_{z_1,z_2} \right| \leq \left| \sum_{i=3}^{N} f_{z_1,z_2}(\lambda_i) - \tilde{f}_{z_1,z_2} \right| + \left( \sum_{i=3}^{N} \frac{N^{2\sigma}}{N^2 |z_1 - \lambda_i|^4} (1 - \chi_{x,\delta}(\lambda_i)) + \sum_{i=3}^{N} \frac{N^{2\sigma}}{N^2 |z_2 - \lambda_i|^4} (1 - \chi_{x,\delta}(\lambda_i)) \right),
\]

where we used \( a \leq N^{2\sigma} \).

With the exact same reasoning as from (2.17) to (2.18) (with \( \nu_z \) replaced by \( \nu_{z_1,z_2} \)), we obtain that \( \sum_{i=3}^{N} f_{z_1,z_2}(\lambda_i) - \tilde{f}_{z_1,z_2} \) is exponentially concentrated on scale \( N^{-2\delta + \epsilon} \). Moreover, similarly, to (2.19), we now have

\[
E_{\nu_{z_1,z_2}} \left( e^{\alpha \sum_{i=2}^{N} \frac{N^{2\sigma}}{N^2 |z_1 - \lambda_i|^4} (1 - \chi_{x,\delta}(\lambda_i))} \right) \leq e^{C_\alpha \nu_{z_1,z_2} N^{2\sigma - 8\delta + \epsilon} + C_\alpha N^{2\sigma - 2\delta + \epsilon}}.
\]

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With \( \alpha = cN^{2\delta} \), we therefore obtain
\[
\mathbb{P}_{\nu_1,\nu_2}(A) < e^{-cN^\varepsilon}, \text{ where } A = \left\{ \left. \sum_{i=2}^{N} h_{z_1,z_2}(\lambda_i) - \bar{f}_{z_1,z_2} \right| N^{2\sigma-2\delta+\varepsilon} \right\}.
\]

This yields, for some \( p = 1 + c(\kappa), c(\kappa) > 0 \) and some \( q, r > 1, \)
\[
\mathbb{E}_{\nu_1,\nu_2} \left( e^{\nu_1 \sum_{i=3}^{N} h_{z_1,z_2}(\lambda_i) + h_{z_1,z_2}(\lambda_i)} \right) \leq e^{\nu_1 \sum_{i=3}^{N} h_{z_1,z_2}(\lambda_i) + f_{z_1,z_2}} (1 + O(N^{2\sigma-2\delta+\varepsilon})) \mathbb{E}_{\nu_1,\nu_2} \left( e^{\nu_1 \sum_{i=3}^{N} h_{z_1,z_2}(\lambda_i) + f_{z_1,z_2}} \right)
\]
\[
= (1 + O(N^{2\sigma-2\delta+\varepsilon})) \mathbb{E}_{\nu_1,\nu_2} \left( e^{\nu_1 \sum_{i=3}^{N} h_{z_1,z_2}(\lambda_i) + f_{z_1,z_2}} \right)
\]

Here we used that the third term has size order \( e^{-cN^\varepsilon} \), the second one is of order \( e^{\nu f_{z_1,z_2}} = O(N^C) \), and so is the first one from Lemma 3.8. Moreover,
\[
\mathbb{E}_{\nu_1,\nu_2} \left( \prod_{i=3}^{N} h_{z_1,z_2}(\lambda_i) \right) = (1 + O(N^{2\sigma-2\delta+\varepsilon})) \mathbb{E}_{\nu_1,\nu_2} \left( e^{\nu_1 \sum_{i=3}^{N} h_{z_1,z_2}(\lambda_i) + f_{z_1,z_2}} \right)
\]

Following similar arguments as (3.10), (3.11), still relying on Lemma 3.8, we finally obtain
\[
\mathbb{E}_{\nu_1,\nu_2} \left( \prod_{i=3}^{N} h_{z_1,z_2}(\lambda_i) \right) = (1 + O(N^{2\sigma-2\delta+\varepsilon})) e^{f_{z_1,z_2}} \mathbb{E}_{\nu_1,\nu_2} \left( e^{\nu_1 \sum_{i=3}^{N} h_{z_1,z_2}(\lambda_i)} \right).
\]

**Third step: the local part is invariant.** For our test function \( h_{z_1,z_2}^\varepsilon \), the reader can easily check the conditions of Lemma 4.4. The only new ingredient is
\[
\int d\nu \frac{dm(\lambda)}{|z_1 - \lambda|^2 |z_2 - \lambda|^2} \leq \log N \left| \frac{1}{|z_1 - z_2|^2} \right|
\]
so that in this setting, the existence of \( r > 2, d < \kappa \) such that \( (N ||\nu||_1)^r \leq N^d \) means that there exists \( \varepsilon > 0 \) such that \( \left( \frac{\log N}{N |z_1 - z_2|^2} \right)^{2+\varepsilon} \leq N^{2\kappa - \varepsilon} \), i.e. \( |z_1 - z_2| \geq N^{-\frac{\kappa}{2} + \frac{\varepsilon}{4}} \), as we assumed by hypothesis. This gives
\[
\mathbb{E}_{\nu_1,\nu_2} \left( e^{\nu_1 \sum_{i=3}^{N} h_{z_1,z_2}(\lambda_i)} \right) = \mathbb{E}_{\nu_0,\nu_2-z_1} \left( e^{\nu_0 \sum_{i=3}^{N} h_{0,z_2-i}(\lambda_i)} \right) + O \left( e^{-cN^{2\kappa}} \right).
\]

This yields
\[
\mathbb{E}_{\nu_1,\nu_2} \left( \prod_{i=3}^{N} h_{z_1,z_2}(\lambda_i) \right) = (1 + O(N^{2\sigma-2\delta+\varepsilon})) e^f_{z_1,z_2} \mathbb{E}_{\nu_0,\nu_2-z_1} \left( \prod_{i=3}^{N} h_{0,z_2-i}(\lambda_i) \right).
\]

From Lemma 2.8, \( e^f_{z_1,z_2} = (1 - |z_1|^2)(1 - |z_2|^2) \). With Proposition 3.7, this concludes the proof. \( \square \)

**Lemma 3.8.** For any \( 0 < \sigma < d < \kappa < 1/2, \) there exists \( p > 1, C > 0 \) such that for any \( |z_1| < 1 - N^{-1/2+\kappa}, |z_2| < N^{-1/2+\kappa}, \) \( z = (z_1 + z_2)/2, \) we have
\[
\mathbb{E}_{\nu_1,\nu_2} \left( \prod_{k=3}^{N} \left( \frac{1}{N |\lambda_k - z_1|^2} \left( 1 + \frac{1}{N |\lambda_k - z_2|^2} \right) \right)^{p_{\nu_1,\nu_2}(\lambda_k)} \right) \leq N^{C}.\]

**Proof.** The above left hand side is at most
\[
\frac{E \left( \prod_{k=3}^{N} \left( \frac{1}{N |\lambda_k - z_1|^2} \left( 1 + \frac{1}{N |\lambda_k - z_2|^2} \right) \right)^{(p-1)\chi_{\nu_1,\nu_2}(\lambda_k)} \right) }{E \left( \prod_{k=3}^{N} |z_1 - \lambda_k|^2 |z_2 - \lambda_k|^2 - \lambda_k^2 e^{-N(|z_1|^2-1)-N(|z_2|^2-1)} \right)}
\]
With Lemma 2.11 and the Cauchy-Schwarz inequality, we therefore just need to prove that
\[ \mathbb{E}_{N^{-2}} \left( \sum_{k=3}^{N} \left( 1 + \frac{1}{N|\lambda_k - z|^2} \right)^{4(p-1)\chi_{x.z}} \right) \leq N^{C}. \]

We can apply Lemma 4.2 (for \( p \) small enough we have \( N\|\nu\|_1 = O(1) \)) to compare it to the case \( z_1 = 0 \), which is easily shown to be \( O(N^{C'}) \) by Corollary 5.6.

### 3.6 Proof of Corollary 1.6

Following a notation from [47], let \( \kappa(\lambda) = \Theta_{jj}^{1/2} \) be the condition number associated to \( \lambda_j \). As the spectrum of a \( G \) is almost surely simple, from [47, Equation (52.11)] we know that
\[ ||(z - G)^{-1}|| = \left( \frac{\kappa(\lambda_j)}{|z - \lambda_j|} + O \left( \sum_{k\neq j} \frac{\kappa(\lambda_k)}{|z - \lambda_k|} \right) \right) \]
as \( z \to \lambda_j \). Together with (1.16), this gives the following almost sure asymptotics:
\[ m(\sigma_x(G) \cap \mathbb{B}_N) \sim \sum_{\lambda_j \in \mathcal{B}_N} \pi(\kappa(\lambda_j)\varepsilon)^2. \]

Denoting \( c_N = N^2 \int_{\mathbb{H}_N} (1 - |z|^2)^{\text{dim}(z)} \frac{\text{d}m(z)}{\pi} \), we therefore have
\[ \lim_{\varepsilon \to 0} \mathbb{P} \left( 1 - c < \frac{m(\sigma_x(G) \cap \mathbb{B}_N)}{\pi \varepsilon^2 c_N} < 1 + c \right) = \mathbb{P} \left( \left| \sum_{\lambda_j \in \mathcal{B}_N} \Theta_{jj} \right| \frac{c_N}{c_N} - 1 < c \right). \]

From Corollary 1.3
\[ \mathbb{P}(\exists i : \lambda_i \in \mathcal{B}_N : |\Theta_{ii}| > N^{-10}) = o(1), \]
hence we only need to prove
\[ \mathbb{P} \left( \left| \sum_{\lambda_j \in \mathcal{B}_N} \Theta_{jj} \right| c_N - 1 < c, \Theta_{ii} \leq N^{10} \text{ for all } \lambda_i \in \mathcal{B}_N \right) \to 1. \] (3.42)

We proceed by bounding the second moment
\[ \mathbb{E} \left( \left( \sum_{\lambda_j \in \mathcal{B}_N} \Theta_{jj} - c_N \right)^2 \prod_{i=1}^{N} \mathbb{1}_{\Theta_{ii} \leq N^{10}} \right) \leq \mathbb{E} \left( \left( \sum_{\lambda_j \in \mathcal{B}_N} \Theta_{jj}^2 \mathbb{1}_{\Theta_{jj} \leq N^{10}} \right) \right) \]
\[ + \mathbb{E} \left( \sum_{\lambda_i, \lambda_j \in \mathcal{B}_N, i \neq j} \Theta_{ii} \Theta_{jj} \mathbb{1}_{\Theta_{ii}, \Theta_{jj} \leq N^{10}} \right) \]
\[ + c_N^2 - 2c_N \mathbb{E} \left( \sum_{\lambda_i \in \mathcal{B}_N} \Theta_{ii} \right) + 2c_N \mathbb{E} \left( \sum_{\lambda_i \in \mathcal{B}_N} \Theta_{ii} \mathbb{1}_{\exists \lambda_j \in \mathcal{B}_N : \Theta_{jj} \geq N^{10}} \right) \] (3.44)

To bound the first term, note that
\[ \mathbb{E} (\Theta_{jj}^2 \mathbb{1}_{\lambda_j \in \mathcal{B}_N, \Theta_{jj} \leq N^{10}}) \leq C \int_{\mathbb{B}_N} \int_{0}^{N^{10}} x \mathbb{P}(\Theta_{jj} \geq x \mid \lambda_j = z) \text{d}z \text{d}m(z). \]

Following the same reasoning as (2.26), we have
\[ \mathbb{P}(\Theta_{jj} \geq x \mid \lambda_j = z) \leq \mathbb{P}(\Theta_{jj} \geq x N^{-\varepsilon}(1 - |z|^2)^{-1} \mid \lambda_j = 0) + O(e^{-N^*}) \leq \frac{CN^{2+2\varepsilon}(1 - |z|^2)^2}{x^2} + O(e^{-N^*}), \]
where we used Proposition 2.4. Denoting $a$ the center of $\mathcal{B}_N$, we therefore bounded the right hand side of (3.43) by $N^{3+2\varepsilon}(1-|a|^2)N^2m(\mathcal{B}) \leq c_N^2$ because $m(\mathcal{B}) \geq N^{-1+2a}$ and $\varepsilon$ is arbitrarily small.

To bound (3.44), we first consider close eigenvalues and bound $\mathcal{O}_i\mathcal{O}_j \leq \frac{1}{2}(\mathcal{O}_i^2 + \mathcal{O}_j^2)$:

\[
\mathcal{E}\left( \sum_{\lambda_i, \lambda_j \in \mathcal{B}_N, N^{-\frac{1}{2}+\varepsilon} < |\lambda_i - \lambda_j|} \mathcal{O}_i \mathcal{O}_j \right) \leq C \mathcal{E}\left( \mathcal{O}_1^2 \mathcal{I}_{\mathcal{B}_N, \lambda_1, \lambda_2 \in \mathcal{B}_N, N^{-\frac{1}{2}+\varepsilon} < 1} \right)
\leq CN^{1+2\varepsilon}\mathcal{E}\left( \mathcal{O}_1^2 \mathcal{I}_{\mathcal{B}_N, \lambda_1, \lambda_2 \in \mathcal{B}_N} \right) m(\mathcal{B}) \leq CN^{3+4\varepsilon}(1-|a|^2)^2 \leq c_N^2.
\]

In the second estimate, we used rigidity from [10]; the above number of close eigenvalues is at most $CN^{2\varepsilon}$ for some large $C$, with probability at least $1 - N^{-D}$ for arbitrarily large $D$. The third estimate was obtained in the same way we bounded (3.43).

For eigenvalues at mesoscopic distance in $[N^{-\frac{1}{2}+\varepsilon}, N^{-\frac{1}{2}+\kappa-a}]$, the contribution of (3.44) is obtained thanks to (1.13):

\[
\mathcal{E}\left( \sum_{\lambda_i, \lambda_j \in \mathcal{B}_N, N^{-\frac{1}{2}+\varepsilon} < |\lambda_i - \lambda_j|} \mathcal{O}_i \mathcal{O}_j \right) = N(N-1)\mathcal{E}\left( \mathcal{O}_1 \mathcal{O}_2 \mathcal{I}_{\mathcal{B}_N, \lambda_1, \lambda_2 \in \mathcal{B}_N, N^{-\frac{1}{2}+\varepsilon} < 1} \right)
\leq CN \left( 1 - (1 - N^{-\frac{1}{2}+\varepsilon})^2 \right) = c_N^2(1 + O(N^{-\varepsilon})).
\]

Finally, the line (3.45) is easily shown to be of order $c_N^2$ thanks to (1.15) and (2.27). We conclude that the left hand side of (3.43) is at most $N^{-\varepsilon}c_N^2$, which concludes the proof of (3.42) by Markov’s inequality.

4 Translation invariance for conditioned measures

Recall that the Ginibre kernel is

\[
K_N(z, w) = \frac{N^2}{\pi} e^{-N(|z|^2 + |w|^2)} \sum_{k=0}^{N-1} \frac{(Nz\bar{w})^k}{k!}.
\]

We also denote its bulk limit as $k_N(z, w) = \frac{N^2}{\pi} e^{-N(|z|^2 + |w|^2) - zw}$.

**Lemma 4.1.** Let $\kappa > 0$. There exists $c = c(\kappa) > 0$ such that for any $|z|, |w| \leq 1 - N^{-\frac{1}{2}+\kappa}$, we have

\[
K_N(z, w) = k_N(z, w) + O(e^{-cN^{2\kappa}})
\]

**Proof.** This is a straightforward adaptation of the proof of [10, Lemma 4.2].

We denote $\mathcal{B}_{a, \delta}$ the ball with center $a$ and radius $N^{-\frac{1}{2}+\delta}$.

**Lemma 4.2.** Let $0 < \delta < \kappa < 1/2$ be fixed constants. Consider any $\mathbb{C}$-valued measurable function $f$ supported on $\mathcal{B}_{a, \delta}$, $|a| \leq 1 - N^{-\frac{1}{2}+\kappa}$, and $\nu(z) = e^{f(z)} - 1$. For any small $r > 2$ there exists $c, C > 0$ depending only on $\kappa, \delta, r$ such that

\[
\mathcal{E}\left( e^{\sum_{i=1}^{N} f(\lambda_i - a)} \right) = \mathcal{E}\left( e^{\sum_{i=1}^{N} f(\lambda_i)} \right) + O\left( e^{-cN^{2\kappa}} e^{C(N\nu(1))^{1-r}} \right).
\]
Proof. Let $K_N^{(a)}(z,w) = K_N(z-a,w-a)$. We define $\|K\| = \sup_{z,w \in \text{supp}(\nu)} |K(z,w)|$. We successively compare linear statistics for $K_N^{(a)}, k_N^{(a)}, k_N$ and $K_N$. First note that $k_N$ is the kernel of a translation invariant point process, so that comparison between $k_N^{(a)}$ and $k_N$ is trivial. For the other steps, we use [1, Lemma 3.4.5] and obtain

$$
\left| \mathbb{E} \left( e^{\sum_{i=1}^{N} f(\lambda_{i}-a)} \right) - \mathbb{E} \left( e^{\sum_{i=1}^{N} f(\lambda_{i})} \right) \right| \leq \sum_{n=1}^{\infty} \frac{n^{1+\frac{d}{2}}}{n!} \|\nu\|_{1}^{n} \left( \max(\|K_N^{(a)}\|, \|k_N^{(a)}\|) \right)^{-1} \|K_N^{(a)} - k_N^{(a)}\| + \max(\|K_N\|, \|k_N\|) \|K_N - k_N\| \right). \quad (4.1)
$$

Clearly, $\|K_N\| \leq \frac{N}{\pi}$ and we bound $\|K_N^{(a)} - K_N\|$ with Lemma 4.1. We conclude that for a universal large enough $C$, and $1/r + 1/s = 1$, we have

$$
\left| \mathbb{E} \left( e^{\sum_{i=1}^{N} f(\lambda_{i}-a)} \right) - \mathbb{E} \left( e^{\sum_{i=1}^{N} f(\lambda_{i})} \right) \right| \leq e^{-CN^{2\kappa}} \left( \frac{N}{\pi} \|\nu\|_{1} \right)^{n} \leq Ce^{-CN^{2\kappa}} e^{C(\|\nu\|_{1})^{r}},
$$

where we used Hölder’s inequality.

\[\square\]

Lemma 4.3. Let $0 < \delta < \kappa < 1/2$ be fixed constants. Consider any $\mathbb{C}$-valued measurable function $f$ supported on $\mathcal{B}_{0, \delta}$, $|a| \leq 1 - N^{-\frac{1}{2}+\kappa}$. Assume also that either (i) or (ii) below holds:

(i) $\Re(f) = 0$;

(ii) there exists $d < 2\kappa$, $p > 1$ and $r > 2$ such that $f = 0$ on $|z| < e^{-N^{d}}$, and $(f_{+} = \max(\Re f, 0))$

$$
(N\|\nu\|_{1})^{r} \leq N^{d}, \quad \max \left( \log \|e^{\sum_{i=2}^{N} f_{+}(\lambda_{i}-a)}\|_{L_{p}}, \log \|e^{\sum_{i=2}^{N} f_{+}(\lambda_{i})}\|_{L_{p}} \right) \leq N^{d}. \quad (4.3)
$$

Then

$$
\mathbb{E}_{N} \left( e^{\sum_{i=2}^{N} f(\lambda_{i}-a) | \lambda_{1} = a} \right) = \mathbb{E}_{N} \left( e^{\sum_{i=2}^{N} f(\lambda_{i}) | \lambda_{1} = 0} \right) + O \left( e^{-CN^{2\kappa}} \right).
$$

Proof. In this proof we first consider the most difficult case (ii), and we will finally mention the simple modifications required for (i). We start with

$$
\mathbb{E}_{N} \left( e^{\sum_{i=2}^{N} f(\lambda_{i}-a) | \lambda_{1} = a} \right) = \frac{\mathbb{E}_{N-1} \left( e^{\sum_{i=2}^{N} f(\lambda_{i}-a) \prod_{i=2}^{N} \lambda_{i} - a \|\lambda_{i} - a\|^{2} e^{-N(|a|^{2}-1)} \right)}{\mathbb{E}_{N-1} \left( \prod_{i=2}^{N} \lambda_{i} - a \|\lambda_{i} - a\|^{2} e^{-N(|a|^{2}-1)} \right)}. \quad (4.4)
$$

Fix some constants $\kappa_1, \kappa_2$ such that $d < \kappa_2 < \kappa_1 < 2\kappa$ and define $\chi_{j}^{a}(\lambda) = 1_{|\lambda-a| < e^{-N^{\kappa j}}}, j = 1, 2$. We first show we can afford imposing $\chi_{2}^{a}(\lambda_2) = 0$: for some positive $q, r$ such that $p^{-1} + q^{-1} + r^{-1} = 1$, we have

$$
\mathbb{E}_{N-1} \left( e^{\sum_{i=2}^{N} f(\lambda_{i}-a) \left( 1 - \prod_{i=2}^{N} (1 - \chi_{2}^{a}(\lambda_{i})) \right) \prod_{i=2}^{N} \lambda_{i} - a \|\lambda_{i} - a\|^{2} e^{-N(|a|^{2}-1)} \right) \leq \left\| e^{\sum_{i=2}^{N} f(\lambda_{i}-a)} \right\|_{L_{p}} \left( 1 - \prod_{i=2}^{N} (1 - \chi_{2}^{a}(\lambda_{i})) \right) \left\| \prod_{i=2}^{N} \lambda_{i} - a \|\lambda_{i} - a\|^{2} e^{-N(|a|^{2}-1)} \right\|_{L_{r}}. \quad (4.5)
$$

where $L_{p} = L_{p}(\mathbb{F}_{N-1})$. By hypothesis (4.3), the first norm is at most $e^{CN^{d}}$. The second is at most $NP_{N-1}(|\lambda_2| \leq e^{-N^{\kappa_2}}) \leq e^{-CN^{\kappa_2}}$. The third norm is at most $NC$, as a simple consequence of Lemma

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2.11 These estimates also hold for \( f = 0 \), so that we proved
\[
\mathbb{E}_N \left( e^{\sum_{i=2}^{N} f(\lambda_i - a) \mid \lambda_1 = a} \right) = \frac{\mathbb{E}_{N-1} \left( e^{\sum_{i=2}^{N} f(\lambda_i - a) \prod_{i=2}^{N} (1 - \chi_2^a(\lambda_i)) |\lambda_i - a|^2 e^{-N(|\lambda_i|^2 - 1)} \right) + O(e^{-cN^{c^2}})}{\mathbb{E}_{N-1} \left( \prod_{i=2}^{N} (1 - \chi_2^a(\lambda_i)) |\lambda_i - a|^2 e^{-N(|\lambda_i|^2 - 1)} \right) + O(e^{-cN^{c^2}})}.
\]
(4.6)
If \(|\lambda_1 - a| < e^{-N^{c_1}}\) and \(|\lambda_i - a| > e^{-N^{c_2}}, 2 \leq i \leq N\), we have
\[
\prod_{i=2}^{N} |\lambda_i - a| = \left( 1 + O(e^{-cN^{c_1}}) \right) \prod_{i=2}^{N} |\lambda_i - \lambda_1|, \quad e^{-N|a|^2} = \left( 1 + O(e^{-cN^{c_1}}) \right) e^{-N|\lambda_1|^2}.
\]
The expectation in the numerator of (4.6) is therefore (in the first equation below \( \lambda_1 \) has distribution \( U \), the uniform measure on the unit disk with center \( a \) and radius \( e^{-N^{c_1}} \), with volume \( b_N = \pi(e^{-2N^{c_1}}) \):
\[
\mathbb{E}_{\mathbb{P}_{N-1} \times U} \left( e^{\sum_{i=2}^{N} f(\lambda_i - a) \prod_{i=2}^{N} (1 - \chi_2^a(\lambda_i)) |\lambda_i - \lambda_1|^2 e^{-N(|\lambda_1|^2 - 1)} \right) \left( 1 + O(e^{-cN^{c_1}}) \right)
\]
\[
= e^N \frac{Z_N}{Z_{N-1}} \frac{1}{b_N} \mathbb{E}_N \left( e^{\sum_{i=2}^{N} f(\lambda_i - a) \prod_{i=2}^{N} (1 - \chi_2^a(\lambda_i)) \chi_1^a(\lambda_1)} \left( 1 + O(e^{-cN^{c_1}}) \right) \right)
\]
\[
= e^N \frac{Z_N}{Z_{N-1}} \frac{1}{b_N} \mathbb{E}_N \left( e^{\sum_{i=2}^{N} f(\lambda_i - a) \prod_{i=2}^{N} (1 - \chi_2^a(\lambda_i)) \chi_1^a(\lambda_1)} \right) + O(e^{-cN^{c_1}})
\]
(4.7)
where for the last equation we used (4.3).

We now want to remove the constraint on \((\lambda_i)^N_{i=2}\), i.e. prove
\[
\frac{1}{b_N} \mathbb{E}_N \left( e^{\sum_{i=2}^{N} f(\lambda_i - a) \prod_{i=2}^{N} (1 - \chi_2^a(\lambda_i)) \chi_1^a(\lambda_1)} \right) = \frac{1}{b_N} \mathbb{E}_N \left( e^{\sum_{i=2}^{N} f(\lambda_i - a) \chi_1^a(\lambda_1)} \right) + O(e^{-cN^{c_2}}).
\]
(4.8)
This requires a longer argument. Let \( \mathcal{B}_i^a = \{ |z - a| \leq e^{-N^{c_1}} \}, i = 1 \) or 2, \( \xi = \sum_1^N \delta_{\lambda_i}, \bar{\xi} = \sum_2^N \delta_{\lambda_i} \). Then,
\[
\left| e^{\sum_{i=2}^{N} f(\lambda_i - a)} \left( 1 - \prod_{i=2}^{N} (1 - \chi_2^a(\lambda_i)) \chi_1^a(\lambda_1) \right) \right| \leq e^{\sum_{i=2}^{N} \text{Re}(\lambda_i)^a} \bar{\xi}(\mathcal{B}_2^a) \chi_1^a(\lambda_1)
\]
\[
\leq e^{\sum_{i=2}^{N} \text{Re}(\lambda_i)^a} \bar{\xi}(\mathcal{B}_2 - \mathcal{B}_1^a) \chi_1^a(\lambda_1) + e^{\sum_{i=2}^{N} \text{Re}(\lambda_i)^a} \bar{\xi}(\mathcal{B}_2^a) \chi_1^a(\lambda_1).
\]
(4.9)
To bound the first term, we use the negative association property of determinantal point processes for disjoint sets (see e.g. [38]), using \( \text{Re} f_+ \geq 0 \) and \( f = 0 \) on \( \mathcal{B}_2^a \):
\[
\mathbb{E}_N \left( e^{\sum_{i=2}^{N} \text{Re}(\lambda_i)^a} \bar{\xi}(\mathcal{B}_2 - \mathcal{B}_1^a) \chi_1^a(\lambda_1) \right) \leq \mathbb{E}_N \left( e^{\sum_{i=2}^{N} \text{Re}(\lambda_i)^a} \right) \mathbb{E}_N \left( \bar{\xi}(\mathcal{B}_2^a) \chi_1^a(\lambda_1) \right) + O(b_N e^{-cN^{c_2}}).
\]
(4.10)
By (4.3), the first expectation above has size order at most \( e^{cN^{c_2}} \). The second is of order \( e^{-cN^{c_2}} \) and the third one is bounded by \( Nb_N(1 + o(1)) \), so that the first term in (4.9) gives an error \( O(b_N e^{-cN^{c_2}}) \).

For the second term in (4.9), we also use the negative association property and \( f = 0 \) on \( \mathcal{B}_2^a \):
\[
\mathbb{E}_N \left( e^{\sum_{i=2}^{N} \text{Re}(\lambda_i)^a} \bar{\xi}(\mathcal{B}_1^a) \chi_1^a(\lambda_1) \right) \leq \mathbb{E}_N \left( e^{\sum_{i=2}^{N} \text{Re}(\lambda_i)^a} \right) \mathbb{E}_N \left( \bar{\xi}(\mathcal{B}_1^a) \chi_1^a(\lambda_1) \right) \leq e^{N^2} \mathbb{E}_N \left( \bar{\xi}(\mathcal{B}_1^a) \chi_1^a(\lambda_1) \right).
\]
(4.9)
Together with
\[
\mathbb{E} \left( \bar{\xi}(\mathcal{B}_1^a) \chi_1^a(\lambda_1) \right) \leq \mathbb{E} (\bar{\xi}(\mathcal{B}_1^a)(\bar{\xi}(\mathcal{B}_1^a) - 1)) = \int_{(\mathcal{B}_1^a)^2} |K_N(z_1, z_2)|^2 \leq N^2 b_N^2,
\]
(4.11)
we have proved that the second term in (4.9) gives an error \(O(b_N e^{-c N^{2}})\). This concludes the proof of (4.8), so that the numerator in (4.6) is

\[
eN Z_N \left( \frac{1}{b_N} \mathbb{E}_N \left( e^{\sum_{i=2}^{N} f(\lambda_i-a)\chi_1^a(\lambda_1)} \right) + O(e^{-c N^{2}}) \right) + O(e^{-c N^{2}})
\]

\[
= eN Z_N \left( \frac{1}{b_N} \mathbb{E}_N \left( e^{\sum_{i=2}^{N} f(\lambda_i-a)\chi_1^a(\lambda_1)} \right) + O(e^{-c N^{2}}) \right),
\]

where we used \(\frac{e^{N}Z_N}{Z_{N-1}} \sim c_1 N^{c_2}\) for some \(c_1, c_2\), as obtained from (1.2). In the same way, the denominator in (4.6) is \(\frac{e^{N}Z_N}{Z_{N-1}} \sim O(e^{-c N^{2}})\), so that we obtained

\[
\mathbb{E}_N \left( e^{\sum_{i=2}^{N} f(\lambda_i-a)\chi_1^a(\lambda_1)} \right) = 1 + O(e^{-c N^{2}}),
\]

so that the numerator in (4.6) is

\[
\sum_{i=1}^{N} |\sum_{b} e^{\sum_{i=2}^{N} f(\lambda_i-a)\chi_1^a(\lambda_1)}| \leq 1 + O(e^{-c N^{2}}),
\]

where we successively used that fact that \(f\) vanishes on \(B_1^a\) and symmetrized.

To conclude the proof, we therefore just need

\[
f_a'(0) = f_0'(0) + O(e^{-c N^{2}}), \text{ where } f_a(w) = \frac{1}{N b_N} \mathbb{E}_N \left( e^{\sum_{i=1}^{N} f(\lambda_i-a+w)\xi(B_1^a)} \right).
\]

From Lemma 4.2 we know that uniformly on \(|w| < 1\) we have

\[
f_a(w) = f_0(w) + O(e^{-c N^{2}}),
\]

which proves (4.13) by Cauchy’s theorem, and therefore the lemma in case (ii).

Under the assumption (i), up to (4.12) the results hold and the reasoning is simplified as all \(L^p\) norms related to \(f\) can be bounded by 1. To justify an analogue of (4.12) and the end of the reasoning, we first replace \(f\) by \(\tilde{f} = f1_{(B_1^a)^c}\) and note that

\[
\left| \frac{1}{b_N} \mathbb{E}_N \left( e^{\sum_{i=2}^{N} f(\lambda_i-a)\chi_1^a(\lambda_1)} \right) \right| \leq \frac{2}{b_N} \mathbb{E}_N(\xi(B_1^a)(\xi(B_1^a) - 1)) = O(N^2 b_N),
\]

so that by symmetrizing we now obtain

\[
\mathbb{E}_N \left( e^{\sum_{i=2}^{N} f(\lambda_i-a)\chi_1^a(\lambda_1)} \right) = \frac{1}{N b_N} \mathbb{E}_N \left( e^{\sum_{i=1}^{N} \tilde{f}(\lambda_i-a)\xi(B_1^a)} \right) + O(e^{-c N^{2}}).
\]

The rest of the proof is identical to case (i).

We now state and proof an analogue of Lemma 4.3 when conditioning on two points. We will only need case (ii), as we are interested in expectations in Section 3 not in convergence in distribution.

**Lemma 4.4.** Let \(0 < \delta < \kappa < 1/2\) be fixed constants and \(C > 0\) fixed, arbitrarily large. Consider any \(\mathbb{C}\)-valued measurable function \(f\) supported on \(B_{0,\delta}\), \(|a|, |b| \leq 1 - N^{-\frac{2}{3}+\delta}\), and \(N < |b-a| < N^{-1/2+\delta}\).

Assume that there exists \(d < 2\kappa\), \(p > 1\) and \(r > 1\) such that \(f = 0\) on \(|z| < e^{-N^d}\), \(|z-(b-a)| < e^{-N^d}\), and (4.2) and (4.3) hold. Then

\[
\mathbb{E}_N \left( e^{\sum_{i=3}^{N} f(\lambda_i-a)\chi_1^a(\lambda_1)} \right) = \mathbb{E}_N \left( e^{\sum_{i=3}^{N} f(\lambda_i-a)\chi_1^a(\lambda_1)} \right) + O\left(e^{-c N^{2\kappa}}\right).
\]
Proof. We start similarly to the proof of Lemma 4.3, by writing

\[
\mathbb{E}_N \left( e^{\sum_{i=2}^N f(\lambda_i - a)} \mid \lambda_1 = a, \lambda_2 = b \right) = \frac{\mathbb{E}_{N-2} \left( e^{\sum_{i=3}^N f(\lambda_i - a)} \prod_{i=3}^N |\lambda_i - a|^2 e^{-N(|\lambda_i|^2 - 1)} \prod_{i=3}^N |\lambda_i - b|^2 e^{-N(|\lambda_i|^2 - 1)} \right)}{\mathbb{E}_{N-2} \left( \prod_{i=3}^N |\lambda_i - a|^2 e^{-N(|\lambda_i|^2 - 1)} \prod_{i=3}^N |\lambda_i - b|^2 e^{-N(|\lambda_i|^2 - 1)} \right)}. \tag{4.14}
\]

Again, we fix some constants \(\kappa_1, \kappa_2\) such that \(d < \kappa_2 < \kappa_1 < 2\kappa\) and define \(\chi^2(\lambda) = 1_{|\lambda-a|<e^{-N^\kappa_2}}, j = 1, 2, x = a, b\). The strict analogue of (4.5) holds, so that the left hand side of (4.14) can be written

\[
\mathbb{E}_{N-2} \left( e^{\sum_{i=3}^N f(\lambda_i - a)} \prod_{i=3}^N (1 - \chi^2(\lambda_i)) |\lambda_i - a|^2 e^{-N(|\lambda_i|^2 - 1)} \prod_{i=3}^N (1 - \chi^2(\lambda_i)) |\lambda_i - b|^2 e^{-N(|\lambda_i|^2 - 1)} \right) + O(e^{-cN^{\kappa_2}}) \tag{4.15}
\]

The analogue of (4.7) then holds exactly in the same way: the expectation in the numerator of (4.15) is

\[
\frac{e^{2N} Z_N}{|a-b|^2 Z_{N-2}} \left( \frac{1}{b_N^2} \mathbb{E}_N \left( e^{\sum_{i=3}^N f(\lambda_i - a)} \prod_{i=3}^N (1 - \chi^2(\lambda_i))(1 - \chi^2(\lambda_i)) \chi^1(\lambda_i) \chi^1(\lambda_2) \right) + O \left( e^{-cN^{\kappa_1}} \right) \right).
\]

Again, we want to remove the constraint on \((\lambda_i)_{i=3}^N\), i.e. prove

\[
\frac{e^{2N} Z_N}{|a-b|^2 Z_{N-2}} \left( \frac{1}{b_N^2} \mathbb{E}_N \left( e^{\sum_{i=3}^N f(\lambda_i - a)} \prod_{i=3}^N (1 - \chi^2(\lambda_i))(1 - \chi^2(\lambda_i)) \chi^1(\lambda_1) \chi^1(\lambda_2) \right) + O \left( e^{-cN^{\kappa_1}} \right) \right).
\]

With the negative association property, the strict analogues of equation (4.9), (4.10) and (4.11) hold, so that the numerator in (4.15) is

\[
\frac{e^{2N} Z_N}{|a-b|^2 Z_{N-2}} \left( \frac{1}{b_N^2} \mathbb{E}_N \left( e^{\sum_{i=3}^N f(\lambda_i - a)} \chi^1(\lambda_1) \chi^1(\lambda_2) \right) + O \left( e^{-cN^{\kappa_1}} \right) \right).
\]

where we used \(\frac{e^{2N} Z_N}{|a-b|^2 Z_{N-2}} \sim c_1 N^{c_2}\) for some \(c_1, c_2\), as obtained from (4.2). In the same way, the denominator in (4.15) is \(\frac{e^{2N} Z_N}{|a-b|^2 Z_{N-2}} (1 + O \left( e^{-cN^{\kappa_2}} \right))\), so that we obtained

\[
\mathbb{E}_N \left( e^{\sum_{i=3}^N f(\lambda_i - a)} \right) = \frac{\mathbb{E}_N \left( e^{\sum_{i=3}^N f(\lambda_i - a)} \chi^1(\lambda_1) \chi^1(\lambda_2) \right)}{1/(N-1) b_N^2} \mathbb{E}_N \left( e^{\sum_{i=1}^N f(\lambda_i - a)} \chi^1(\lambda_1) \chi^1(\lambda_2) \right) + O(e^{-cN^{\kappa_1}})
\]

where we successively used that fact that \(f\) vanishes on \(B_1^0 \cup B_1, B_1 \cap B_1^0 = \emptyset\) and symmetrized. To conclude the proof, we therefore just need \(\partial_{z_1 z_2} f_{a,b}(0,0) = \partial_{z_1 z_2} f_{0,b-a}(0,0) + O(e^{-cN^{\kappa_2}})\), where

\[
f_{a,b}(z_1, z_2) = \frac{1}{N(N-1) b_N^2} \mathbb{E}_N \left( e^{\sum_{i=1}^N f(\lambda_i - a) + z_1 \xi(\mathcal{B}_1^0) + z_2 \xi(\mathcal{B}_1^0_1)} \right) \tag{4.17}
\]

This follows from Lemma 4.2 and Cauchy’s Theorem, similarly to the end of the proof of Lemma 4.3.
This section gives applications of Andréeif’s identity to the conditioned measures of interest in this work. In particular, it proves some slight extensions of Kostlan’s theorem (Corollary 5.5), following a method from [17]. The common main tool will be the following classical Lemma, by Andréeif [2] (see [15] for a short proof). Note that the original proof of Kostlan’s theorem [35] and some of its extensions [29] were based on different arguments.

**Lemma 5.1** (Andréeif’s identity). On a measured space \((E, \mathcal{E}, \mu)\) For any functions \((\phi_i, \psi_i)_{i=1}^N \in L_2(\mu)^{2N}\),

\[
\frac{1}{N!} \int_{E^N} \det(\phi_i(\lambda_j)) \det(\psi_i(\lambda_j)) \mu(d\lambda_1) \cdots \mu(d\lambda_N) = \det(f_{i,j}) \text{ where } f_{i,j} = \int_E \phi_i(\lambda) \psi_j(\lambda) \mu(d\lambda).
\]

**Theorem 5.2.** Let \(E = \mathbb{C}, g \in L_2(\mu), \) and \(\{\lambda_1, \ldots, \lambda_N\}\) eigenvalues from the Ginibre ensemble. Then

\[
\mathbb{E} \left( \prod_{k=1}^N g(\lambda_k) \right) = N^{N(N-1)/2} \det(f_{i,j})_{i,j=1}^N \text{ where } f_{i,j} = \frac{1}{(j-1)!} \int \lambda^{i-1} \bar{\lambda}^{j-1} g(\lambda) \mu(d\lambda).
\]

**Proof.** The following is elementary:

\[
\int |\lambda|^2 d\mu(\lambda) = \frac{i!}{N!}.
\]

The proof then follows from Andréeif’s identity. \(\square\)

**Theorem 5.3.** We have (remember \(\mu = \mu(\mathcal{N})\))

\[
\mathbb{E} \left( \prod_{k=2}^N g(\lambda_k) \mid \lambda_1 = z \right) = \frac{1}{Z_2^{(z)}} \det(f_{i,j})_{i,j=1}^{N-1} \text{ where } f_{i,j} = \frac{1}{j!} \int \lambda^{i-1} \bar{\lambda}^{j-1} |z - \lambda|^2 g(\lambda) \mu(d\lambda)
\]

and

\[
Z_2^{(z)} = N^{-\frac{N(N-1)}{2}} e^{(N-1)(N|z|^2)}.
\]

**Proof.** Using Andréeif’s identity with \(\phi_i(\lambda) = \lambda^{i-1} g(\lambda) |z - \lambda|^2, \psi_j(\lambda) = \lambda^{j-1}\), we find

\[
\mathbb{E} \left( \prod_{k=2}^N g(\lambda_k) \mid \lambda_1 = z \right) = \frac{1}{Z_2^{(z)}} \det(f_{i,j})_{i,j=1}^{N-1}
\]

where

\[
Z_2^{(z)} = \det(M_{ij})_{i,j=1}^{N-1}, \quad M_{ij} = \frac{1}{i!} \int \lambda^{i-1} \bar{\lambda}^{j-1} |z - \lambda|^2 \mu(d\lambda).
\]

By expanding \(|z - \lambda|^2 = |z_1|^2 + |\lambda|^2 - \bar{z}_1 \lambda - \bar{\lambda} \lambda\), we see that that \(M\) is tridiagonal, with entries (remember \([5.1]\))

\[
M_{ii} = \frac{1}{N} + \frac{|z|^2}{iN^2 - 1}, \quad M_{i,i+1} = -\frac{z}{N}, \quad M_{i,i-1} = -\frac{z}{iN^2 - 1}.
\]

Denoting \(x = N|z|^2\) and \(d_k = \det((M_{ij})_{1 \leq i,j \leq k})\), with the convention \(d_0 = 1\) we have

\[
d_1 = 1 + x, \\
d_k = \left(1 + \frac{x}{k}\right) \frac{1}{N^k} d_{k-1} - \frac{x}{k} \frac{1}{N^{2k-1}} d_{k-2}
\]

so that \(a_k = d_k N^\frac{k(k+1)}{2}\) satisfies \(a_0 = 1, a_1 = 1 + x, a_k = \left(1 + \frac{x}{k}\right) a_{k-1} - \frac{x}{k} a_{k-2}\).

This gives \(a_k = e^{(k)}(x)\) by an immediate induction. \(\square\)
Theorem 5.4. We have
\[
\mathbb{E} \left( \prod_{k=3}^{N} g(\lambda_k) \mid \lambda_1 = 0, \lambda_2 = z \right) = \frac{1}{Z_N^{(0,z)}} \det(f_{i,j})_{i,j=1}^{N-2} \quad \text{where} \quad f_{i,j} = \frac{1}{(i+1)!} \int \lambda^{i-1} \bar{\lambda}^{j-1} |\lambda|^2 |z-\lambda|^2 g(\lambda) \mu(d\lambda)
\]
and
\[
Z_N^{(0,z)} = N^{-(N-2)(N+1)} e_1^{(N-1)}(N|z|^2) |N|^{-2}.
\]

Proof. By Andréief’s identity, the result holds with
\[
Z_N^{(0,z)} = \det(M_{ij})_{i,j=1}^{N-2}, \quad M_{ij} = \frac{1}{(i+1)!} \int \lambda^{i-1} \bar{\lambda}^{j-1} |\lambda|^2 |z-\lambda|^2 \mu(d\lambda).
\]
Expanding $|z-\lambda|^2 = |z|^2 + |\lambda|^2 - z\bar{\lambda} - \bar{z}\lambda$, we see that $M$ is tridiagonal with entries
\[
M_{ii} = \frac{1}{N^{i+1}} + \frac{|z|^2}{(i+1)N^i}, \quad M_{i,i+1} = -\frac{\bar{z}}{N^{i+1}}, \quad M_{i,i-1} = -\frac{z}{(i+1)N^i}.
\]
Denoting $x = N|z|^2$ and $d_k = det((M_{ij})_{1 \leq i,j \leq k})$, with the convention $d_0 = 1$ we have
\[
d_1 = \frac{2+x}{2N^2},
\]
\[
d_k = \left(1 + \frac{x}{k+1}\right) \frac{1}{N^{k+1}} d_{k-1} - \frac{x}{k+1} \frac{1}{N^{2k+1}} d_{k-2}.
\]
so that $a_k = d_k N^{-\frac{k(k+1)}{2}}$ satisfies $a_0 = 1$, $a_1 = 1 + x/2$, \[
a_k = \left(1 + \frac{x}{k+1}\right) a_{k-1} - \frac{x}{k+1} a_{k-2}.
\]
This gives the expected result by an immediate induction.

Kostlan’s theorem now comes as a corollary, as well as a similar property for the Ginibre ensemble conditioned on $\lambda_1 = 0$.

Corollary 5.5 (Kostlan). The set $N\{\lambda_1^2, \ldots, \lambda_N^2\}$ is distributed as $\{\gamma_1, \ldots, \gamma_N\}$, a set of (unordered) independent Gamma variables of parameters $1, 2, \ldots, N$.

Proof. Let $g \in \mathbb{C}[X]$ and use Theorem 5.2 with the radially symmetric function $g(|\cdot|^2)$. The relevant matrix is then diagonal, with coefficients
\[
f_{i,i} = \frac{1}{(i-1)!} \int \lambda^{2i-2} g(|\lambda|^2) \mu(d\lambda) = \frac{N^{-(i-1)}}{(i-1)!} \int_0^{\infty} r^{i-1} g(r/N) e^{-r} dr = N^{-(i-1)} \mathbb{E}(g(\gamma_i/N)).
\]
In other words,
\[
\mathbb{E} \left( \prod_{i=1}^{N} g(|\lambda_i|^2) \right) = \mathbb{E} \left( \prod_{i=1}^{N} g(\gamma_i/N) \right).
\]
Note that these statistics characterize the distribution of a set of unordered points, as such expressions with polynomial $g$ generate all symmetric polynomials, as shown in Lemma 5.7 and the gamma distributions are characterized by their moments. For more details, see [17]. We conclude that $N\{\lambda_1^2, \ldots, \lambda_N^2\} \overset{d}{=} \{\gamma_1, \ldots, \gamma_N\}$.

Corollary 5.6. Conditioned on $\{\lambda_1 = 0\}$, $N\{\lambda_2^2, \ldots, \lambda_N^2\}$ is distributed as $\{\gamma_2, \ldots, \gamma_N\}$, a set of (unordered) independent Gamma variables of parameters $2, 3, \ldots, N$.  

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Complex Ginibre dynamics. Let the eigenvalues evolution is more intricate.

for eigenvalues repulsion.

drift term shows no interaction between eigenvalues: only the correlation of martingale terms is responsible

consider the Ornstein-Uhlenbeck version so that the equilibrium measure is the (real or complex) Ginibre en-

This Appendix derives the Dyson-type dynamics for eigenvalues of nonnormal matrices. More precisely, we

YGX = Δ = Diag(λ₁, ..., λₙ), where X, Y are the matrices of the right- and left-eigenvectors of G(0).

we assume that G(0) has simple spectrum, and X, Y invertible. The right eigenvectors (xᵢ) are the column

and the left-eigenvectors (yᵢ) are the rows of Y. They are chosen such that XY = I and, for any

1 ≤ k ≤ N, Xkk = 1.

We now consider the complex Dyson-type dynamics: for any 1 ≤ i, j ≤ N,

\[ \frac{dG_{ij}(t)}{\sqrt{N}} - \frac{1}{2} G_{ij}(t) dt, \tag{A.1} \]

where the Bᵢⱼ’s are independent standard complex Brownian motions: \( \sqrt{2} \text{Re}(B_{ij}) \) and \( \sqrt{2} \text{Im}(B_{ij}) \) are standard real Brownian motions. One can easily check that G(t) converges to the Ginibre ensemble as t → ∞, with normalization (1.1).

In the following, the bracket of two complex martingales M, N is defined by bilinearity: \( \langle M, N \rangle = \langle \text{Re} M, \text{Re} N \rangle - \langle \text{Im} M, \text{Im} N \rangle + i(\text{Re} M, \text{Im} N) + i(\text{Im} M, \text{Re} N) \).

Proposition A.1. The spectrum (λ₁(t), ..., λₙ(t)) is a semimartingale satisfying the system of equations

\[ dλ_k(t) = dM_k(t) - \frac{1}{2} λ_k(t) dt \]

where the distribution of the martingales \( (M_k)_{1 \leq k \leq N} \) is characterized by the brackets \( \langle M_i, M_j \rangle = 0 \) and

\[ d(M_i, M_j)_t = \mathcal{O}_{ij}(t) \frac{dt}{N}. \]
Remark A.2. As explained below, this equation (in particular the off-diagonal brackets) is coherent with the eigenvalues repulsion observed in \((1.2)\). Contrary to the Hermitian Dyson Brownian motion, all eigenvalues are martingales (up to the Ornstein-Uhlenbeck drift term), so that their repulsion is not due to direct mutual interaction, but to correlations between these martingales at the microscopic scale.

For example, assume that \(G(0)\) is already at equilibrium. Using physics conventions, for any bulk eigenvalues \(\lambda_1, \lambda_2\) satisfying \(w = O(1)\) (remember \(w = \sqrt{N}(\lambda_1 - \lambda_2)\)), Proposition A.1 and Theorem 1.5 imply

\[
E (d\lambda_1 d\lambda_2 | \lambda_1 = z_1, \lambda_2 = z_2) \sim E(\Omega_{12} | \lambda_1 = z_1, \lambda_2 = z_2) \frac{dt}{N} \sim -(1 - |z_1|^2) \frac{1 - (1 + |w|^2)e^{-|w|^2}}{1 - e^{-|w|^2}} dt
\]

in the bulk. By considering the real part in this equation and denoting \(d\lambda_1 = dx_1 + idy_1, d\lambda_2 = dx_2 + idy_2\), we have in particular \(E(dx_1 dx_2 + dy_1 dy_2) < 0\), and this negative correlation is responsible for repulsion: the eigenvalues tend to move in opposite directions. Moreover, as eigenvalues get closer on the microscopic scale, \(w \to 0\) and the repulsion gets stronger:

\[
E (d\lambda_1 d\lambda_2 | \lambda_1 = z_1, \lambda_2 = z_2) \sim -\frac{1 - |z_1|^2}{|w|^2} dt.
\]

On the other hand, for mesoscopic scale \(N^{-1/2} \ll |\lambda_1 - \lambda_2|\), Proposition A.1 and Theorem 1.5 give

\[
E (d\lambda_1 d\lambda_2) \sim -\frac{(1 - |\lambda|^2)}{N^2 |\lambda|^2} dt = o(dt), \quad \text{so that increments are uncorrelated for large } N.
\]

For a given differential operator \(f \mapsto f'\), we introduce the matrix \(C = X^{-1}X'\). Along the following lemmas, all eigenvalues are assumed to be distinct. In our application, this spectrum simplicity will hold almost surely for any \(t \geq 0\) as \(G(0)\) has simple spectrum.

Lemma A.3. We have \(X' = XC\) and \(Y' = -CY\).

Proof. The first equality is the definition of \(C\). For the second one, \(XY = I\) gives \(XY' + X'Y = 0\), hence \(Y' = -X^{-1}X'Y = -CY\).

Lemma A.4. The first order perturbation of eigenvalues is given by \(\lambda'_k = y_k G' x_k\).

Proof. We have \(\Delta' = (YG'X)' = Y'GX + YG'X + YGX' = YG'X + YG'X + CYG'X = YG'X + [\Delta, C]\). Therefore \(\lambda'_k = (YG'X)_{kk} + [\Delta, C]_{kk} = y_k G' x_k\).

Lemma A.5. For any \(i \neq j\), \(C_{ij} = \frac{y_j G' x_j}{\lambda_j - \lambda_i}\).

Proof. For such \(i, j\), \(\Delta_{ij} = 0\). With the same computation as in the previous lemma, this gives \((YG'X)_{ij} + [\Delta, C]_{ij} = 0\). Thus \((\lambda_i - \lambda_j)C_{ij} = -(YG'X)_{ij} = -y_i G' x_j\), from which the result follows.

Lemma A.6. For any \(1 \leq k \leq N\), \(C_{kk} = -\sum_{l \neq k} X_{kl} \frac{y_l G' x_l}{\lambda_k - \lambda_l}\).

Proof. We use the assumption \(X_{kk} = 1\). From this, and the definition of \(C\), we get

\[
X_{kk}' = 0 = (XC)_{kk} = \sum_{l=1}^{n} X_{kl} C_{lk} = X_{kk} C_{kk} + \sum_{l \neq k} X_{kl} C_{lk}.
\]

As a consequence, \(C_{kk} = -\sum_{l \neq k} X_{kl} C_{lk}\) and we obtain the result thanks to the previous lemma.

From now on the differential operator will be either \(\partial_{Re G_{ab}} (G' = E_{ab} = \{\delta_{ab}\}_{1 \leq i, j \leq N})\), or \(\partial_{Im G_{ab}}, (G' = i E_{ab})\). In both cases, \(G'' = 0\). We denote \(C_{Re}\) and \(C_{Im}\) accordingly. In particular, for any \(k\) and \(i \neq j\) the following holds:

\[
\begin{align*}
\partial_{Re G_{ab}} \lambda_k &= Y_{ka} X_{b,k}, \quad \partial_{Im G_{ab}} \lambda_k = i Y_{ka} X_{b,k} \\
C_{ij}^{Re} &= \frac{Y_{ia} X_{b,j}}{\lambda_j - \lambda_i}, \quad C_{kk}^{Re} = -\sum_{l \neq k} X_{kl} \frac{Y_{la} X_{b,k}}{\lambda_k - \lambda_l}, \quad C_{ij}^{Im} = \frac{i Y_{ia} X_{b,j}}{\lambda_j - \lambda_i}, \quad C_{kk}^{Im} = -\sum_{l \neq k} X_{kl} \frac{Y_{la} X_{b,k}}{\lambda_k - \lambda_l}. \quad \text{(A.2)}
\end{align*}
\]
Lemma A.7. We have

\[ \partial_{\text{Re } G_{ab}} X_{ij} = \sum_{l \neq j} (X_{il} - X_{ij} X_{jl}) \frac{Y_{la} X_{b,l}}{\lambda_j - \lambda_l}, \quad \partial_{\text{Im } G_{ab}} X_{ij} = \sum_{l \neq j} (X_{il} - X_{ij} X_{jl}) \frac{Y_{la} X_{b,l}}{\lambda_j - \lambda_l}. \]

\[ \partial_{\text{Re } G_{ab}} Y_{ij} = \sum_{l \neq i} \frac{1}{\lambda_i - \lambda_l} (X_{il} Y_{a,l} X_{b,i} Y_{ij} + Y_{la} X_{b,l} Y_{ij}), \quad \partial_{\text{Im } G_{ab}} Y_{ij} = \sum_{l \neq i} \frac{1}{\lambda_i - \lambda_l} (X_{il} Y_{a,l} X_{b,i} Y_{ij} + Y_{la} X_{b,l} Y_{ij}). \]

Proof. Below is the computation for \( \partial_{\text{Re } G_{ab}} X_{ij} \). We use \( X' = X C \) and (A.2):

\[ X_{ij}' = (XC)_{ij} = \sum_{l=1}^{n} X_{il} C_{lj} = \sum_{l \neq j} (X_{il} Y_{a,l} X_{b,l} - X_{ij} X_{jl} Y_{la} X_{b,j}) = \sum_{l \neq j} (X_{il} - X_{ij} X_{jl}) \frac{Y_{la} X_{b,l}}{\lambda_j - \lambda_l}. \]

The case \( \partial_{\text{Im } G_{ab}} X_{ij} \) is obtained similarly, as are the formulas for \( Y \).

Lemma A.8. The second order perturbation of eigenvalues is given by

\[ \partial_{\text{Re } G_{ab}}^2 \lambda_k = 2 \sum_{l \neq k} \frac{Y_{ka} X_{b,l} Y_{la} X_{b,k}}{\lambda_k - \lambda_l}, \quad \partial_{\text{Im } G_{ab}}^2 \lambda_k = -2 \sum_{l \neq k} \frac{Y_{ka} X_{b,l} Y_{la} X_{b,k}}{\lambda_k - \lambda_l}. \]

Proof. We compute the perturbation for \( \partial_{\text{Re } G_{ab}} \). Differentiating \( \lambda \) a second time gives

\[ \lambda_k'' = y_k G' x_k + y_k G'' x_k + y_k G' x_k' = Y_{ka} X_{b,k} + Y_{ka} X_{b,k}. \]

Replacing \( X' \) and \( Y' \) with their expressions yields

\[ \lambda_k'' = \sum_{l \neq k} \frac{1}{\lambda_k - \lambda_l} (X_{kl} Y_{la} X_{b,k} Y_{ka} + Y_{ka} X_{b,l} Y_{la} X_{b,k}) + \sum_{l \neq j} \frac{1}{\lambda_j - \lambda_l} (X_{il} Y_{a,l} X_{b,i} Y_{ja} X_{b,j} + Y_{ka} X_{b,l} Y_{la} X_{b,k} - Y_{ka} X_{b,k} X_{kl} Y_{la} X_{b,k}) \]

which concludes the proof, the other cases being similar.

For the proof of Proposition [A.1] we need the following elementary lemma.

Lemma A.9. Let \( \tau = \inf \{ t \geq 0 : \exists i \neq j, \lambda_i(t) = \lambda_j(t) \} \). Then \( \tau = \infty \) almost surely.

Proof. The set of matrices \( G \) with Jordan form of type

\[ \lambda_1 \oplus \cdots \oplus \lambda_{N-2} \oplus \left( \begin{array}{cc} \lambda_{N-1} & 1 \\ 0 & \lambda_{N-1} \end{array} \right) \]

(respectively \( \lambda_1 \oplus \cdots \oplus \lambda_{N-2} \oplus \lambda_{N-1} \oplus \lambda_{N-1} \))

is a submanifold \( M_1 \) (resp. \( M_2 \)) of \( \mathbb{C}^{N^2} \) with complex codimension 1 (resp. 3), see e.g. [33, 42]. Therefore, almost surely, a Brownian motion in \( \mathbb{C}^{N^2} \) starting from a diagonalizable matrix with simple spectrum will not hit \( M_1 \) or \( M_2 \). This concludes the proof.

All derivatives can therefore be calculated, as eigenvalues and eigenvectors are analytic functions of the matrix entries (see [32]).
Proof of Proposition [A.1] In our context, the Itô formula will take the following form: for a function $f$ from $\mathbb{C}^n$ to $\mathbb{C}$ of class $C^2$, where $B_i = (B^i_1, \ldots, B^i_n)$ is made of independent standard complex Brownian motions, we have

$$df(B_t) = \sum_{i=1}^n \left( \frac{\partial f}{\partial \text{Re } z_i} \text{d Re } B^i_t + \frac{\partial f}{\partial \text{Im } z_i} \text{d Im } B^i_t \right) + \frac{1}{2} \left( \sum_{i=1}^n \frac{\partial^2 f}{\partial \text{Re } z_i^2} + \frac{\partial^2 f}{\partial \text{Im } z_i^2} \right) \text{d}t. \quad (A.3)$$

For any given $0 < \varepsilon < \min \{ |\lambda_i(0) - \lambda_j(0)|, i \neq j \}$, let

$$\tau_\varepsilon = \inf \{ t > 0 : \exists i \neq j, |\lambda_i(t) - \lambda_j(t)| < \varepsilon \}. \quad (A.4)$$

Eigenvalues are smooth functions of the matrix coefficients on the domain $\cap_{i,j} \{ |\lambda_i - \lambda_j| > \varepsilon \}$, so that equation $(A.3)$ together with Lemmas [A.4] and [A.8] gives the following equality of stochastic integrals, with substantial cancellations of the drift term:

$$d\lambda_k(t \wedge \tau_\varepsilon) = \sum_{i,j=1}^n Y_{ki} X_{jk} \left( \frac{dB_{ij}(t \wedge \tau_\varepsilon)}{\sqrt{N}} - \frac{G_{ij}}{2} d(t \wedge \tau_\varepsilon) \right) + \sum_{i,j,l \neq k} \left( \frac{Y_{ki} Y_{jl} X_{jk}}{\lambda_k - \lambda_l} - \frac{Y_{ki} Y_{jl} Y_{li} X_{jk}}{\lambda_k - \lambda_l} \right) d(t \wedge \tau_\varepsilon)$$

$$= \sum_{i,j=1}^n Y_{ki} X_{jk} \frac{dB_{ij}(t \wedge \tau_\varepsilon)}{\sqrt{N}} - \frac{1}{2} \sum_{i,j=1}^n Y_{ki} G_{ij} X_{jk} d(t \wedge \tau_\varepsilon) = \sum_{i,j=1}^n Y_{ki} X_{jk} \frac{dB_{ij}(t \wedge \tau_\varepsilon)}{\sqrt{N}} - \frac{1}{2} \lambda_k d(t \wedge \tau_\varepsilon).$$

Taking $\varepsilon \to 0$ in the above equation together with Lemma [A.9] yields

$$d\lambda_k(t) = \sum_{i,j=1}^n Y_{ki} X_{jk} \frac{dB_{ij}(t)}{\sqrt{N}} - \frac{1}{2} \lambda_k d t.$$ 

The eigenvalues martingales terms are correlated. Their brackets are

$$d\langle \lambda_i, \lambda_j \rangle_t = \frac{1}{N} \sum_{a,b,c,d=1}^n Y_{ia} X_{b,i} Y_{j,c} X_{d,j} d(B_{ab}, dB_{cd})_t = (X^t X)_{ij} (Y^t Y^*)_{ij} \frac{dt}{N} = 0. \quad (A.5)$$

$$d\langle \lambda_i, \lambda_j \rangle_t = 0. \quad (A.6)$$

This concludes the proof.

**Real Ginibre dynamics.** We now consider $G(0)$ a real matrix of size $N$, again assumed to be diagonalized as $YGX = \Delta = \text{Diag}(\lambda_1, \ldots, \lambda_N)$, where $X, Y$ are the matrices of the right- and left-eigenvectors of $G(0)$. We also assume that $G(0)$ has simple spectrum, and $X, Y$ invertible. We keep the same notations for the right eigenvectors ($x_i$), columns of $X$, and the left-eigenvectors ($y_j$), rows of $Y$. They are again chosen such that $XY = I$ and, for any $1 \leq k \leq N$, $X_{kk} = 1$.

In this subsection, the real Dyson-type dynamics are $(1 \leq i, j \leq N)$,

$$dG_{ij}(t) = \frac{dB_{ij}(t)}{\sqrt{N}} - \frac{1}{2} G_{ij}(t) dt, \quad (A.7)$$

where the $B_{ij}$’s are independent standard Brownian motions. One can easily check that $G(t)$ converges to the real Ginibre ensemble as $t \to \infty$.

Note that the real analogue of Lemma [A.9] gives weaker repulsion: the set of real matrices with Jordan form of type

$$\lambda_1 \oplus \cdots \oplus \lambda_{N-2} \oplus \begin{pmatrix} \lambda_{N-1} & 1 \\ 0 & \lambda_{N-1} \end{pmatrix}$$

is a submanifold $\mathcal{M}_1$ of $\mathbb{R}^{N^2}$, supported on $\lambda_{N-1} \in \mathbb{R}$, with real codimension 1 (as proved by a straightforward adaptation of [33, Theorem 7]). Denoting $\tau = \inf \{ t \geq 0 : \exists i \neq j, \lambda_i(t) = \lambda_j(t) \}$, under the dynamics $(A.7)$ for any $t > 0$ we therefore have

$$\mathbb{P}(\tau < t) > 0,$$
Proposition A.10. The spectrum \((\lambda_1(t), \ldots, \lambda_n(t))\) evolves according to the following stochastic equations, up to the first collision:

\[
\mathrm{d}\lambda_k(t \wedge \tau) = \mathrm{d}M_k(t \wedge \tau) + \left( \sum_{i \neq k} \frac{\mathcal{O}_{k\ell}}{\lambda_k - \lambda_\ell} - \frac{1}{2} \lambda_k \right) \mathrm{d}(t \wedge \tau)
\]

where the distribution of the martingales \((M_k)_{1 \leq k \leq N}\) is characterized by the brackets

\[
\mathrm{d} \langle M_i, M_j \rangle_{t \wedge \tau} = \mathcal{O}_{ij}(t) \frac{\mathrm{d}(t \wedge \tau)}{N}, \quad \mathrm{d} \langle M_i, M_j \rangle_{t \wedge \tau} = \mathcal{O}_{ij}(t) \frac{\mathrm{d}(t \wedge \tau)}{N}.
\]

Note that the real eigenvalues have associated real eigenvectors. For those, \(\mathcal{O}_{k\ell} = \mathcal{O}_{\ell k}\), and the variation is real: real eigenvalues remain real as long as they do not collide.

Remark A.11. Proposition A.10 is coherent with the attraction between conjugate eigenvalues exhibited in [41]. In fact, if \(\eta = \text{Im}(\lambda_k) > 0\), the drift interaction term with \(\hat{\lambda}_k\) is \(\mathcal{O}_{kk}/(\lambda_k - \hat{\lambda}_k) = -i\mathcal{O}_{kk}/(2\eta)\), so that these eigenvalues attract each other stronger as they approach the real axis.

For the proof, we omit the details and only mention the differences with respect to Proposition A.1. We apply the Itô formula for a \(\mathcal{C}^2\) function \(f\) from \(\mathbb{R}^n\) to \(\mathbb{C}\), with argument \(U_t = (U_{1t}, \ldots, U_{nt})\) is made of independent Ornstein-Uhlenbeck processes. Together with the perturbation formulas for \(\lambda_k\), Lemmas A.4 and A.8, we obtain (remember the notation A.4)

\[
\mathrm{d}\lambda_k(t \wedge \tau) = \sum_{i,j=1}^n Y_{ki}X_{jk} \left( \frac{\mathrm{d}B_{ij}(t \wedge \tau)}{\sqrt{N}} \right) \frac{Y_{kj}X_{ij}}{\lambda_k - \lambda_\ell} \mathrm{d}(t \wedge \tau) + \left( \sum_{i,j} \sum_{i \neq k} \frac{Y_{ki}X_{ij}Y_{kj}X_{ij}}{\lambda_k - \lambda_\ell} - \frac{1}{2} \lambda_k \right) \mathrm{d}(t \wedge \tau).
\]

We can take \(\varepsilon \to 0\) in the above formulas and the brackets are calculated as follows, concluding the proof:

\[
\mathrm{d}\langle \lambda_i, \bar{\lambda}_j \rangle_{t \wedge \tau} = \frac{1}{N} \sum_{a,b,c,d=1}^n Y_{ia}X_{bi}Y_{jc}X_{d,j} \mathrm{d}(B_{ab}, dB_{cd})_{t \wedge \tau} = (X^t X)_{ij} (YY^*)_{ij} \frac{\mathrm{d}(t \wedge \tau)}{N} = \mathcal{O}_{ij}(t) \frac{\mathrm{d}(t \wedge \tau)}{N},
\]

\[
\mathrm{d}\langle \lambda_i, \lambda_j \rangle_{t \wedge \tau} = \frac{1}{N} \sum_{a,b,c,d=1}^n Y_{ia}X_{bi}Y_{jc}X_{d,j} \mathrm{d}(B_{ab}, dB_{cd})_{t \wedge \tau} = (X^t X)_{ij} (YY^*)_{ij} \frac{\mathrm{d}(t \wedge \tau)}{N} = \mathcal{O}_{ij}(t) \frac{\mathrm{d}(t \wedge \tau)}{N}.
\]
**Proof of Corollary 1.7.** We start with (1.18). From Proposition A.1 and Itô’s formula, we have

\[
|\lambda_k(t) - \lambda_k(0)|^2 = 2\text{Re} \int_0^t \lambda_k(s) - \lambda_k(0) \left( dM_k(s) - \frac{1}{2} \lambda_k(s) ds \right) + \int_0^t \frac{\mathcal{O}_{kk}(s)}{N} ds \tag{A.10}
\]

The above stochastic integral is a martingale because \( \mathbb{E} \left( \int_0^t |\lambda_k(s) - \lambda_k(0)|^2 d\langle M_k \rangle_s \right) < \infty \), so that

\[
\mathbb{E} \left( |\lambda_k(t) - \lambda_k(0)|^2 \right) = O(t) + \int_0^t \mathbb{E} \left( \frac{\mathcal{O}_{kk}(s)}{N} \right) ds = O(t),
\]

where in the last equality we used \( \mathbb{E}(\mathcal{O}_{kk}(s)) = O(N) \), which follows from (2.11). The estimate (1.18) follows by Doob’s and Markov’s inequalities.

For (1.19), we start again with (A.10) and obtain

\[
\mathbb{E} \left( |\lambda_1(t) - \lambda_1(0)|^2 \mathbbm{1}_{A_{1,0}^{(1)} \cap \{ \lambda_1(0) \in \mathscr{B} \}} \right) = \int_0^t \mathbb{E} \left( \frac{\mathcal{O}_{11}(s)}{N} \mathbbm{1}_{A_{1,0}^{(1)} \cap \{ \lambda_1(0) \in \mathscr{B} \}} \right) ds + o(t), \tag{A.11}
\]

where we used \( |\lambda_1(s) - \lambda_1(0)| < N^{1/2} \leq 1 \) on \( A_{1,0}^{(1)} \). Moreover,

\[
\left| \mathbb{E} \left( \frac{\mathcal{O}_{11}(s)}{N} \mathbbm{1}_{A_{1,0}^{(1)} \cap \{ \lambda_1(0) \in \mathscr{B} \}} - \mathbb{E} \left( \frac{\mathcal{O}_{11}(s)}{N} \mathbbm{1}_{\{ \lambda_1(s) \in \mathscr{B} \}} \right) \right) \right| \leq \mathbb{E} \left( \frac{\mathcal{O}_{11}(s)}{N} \mathbbm{1}_{\text{dist}(\lambda_1(s), \partial \mathscr{B}) \leq N^{1/2}} \right) = O(N^{1/2}),
\]

where for the last inequality we used (2.11) again. Equation (A.11) therefore becomes

\[
\mathbb{E} \left( |\lambda_1(t) - \lambda_1(0)|^2 \mathbbm{1}_{A_{1,0}^{(1)} \cap \{ \lambda_1(0) \in \mathscr{B} \}} \right) = \int_0^t \mathbb{E} \left( \frac{\mathcal{O}_{11}(s)}{N} \mathbbm{1}_{A_{1,0}^{(1)} \cap \{ \lambda_1(0) \in \mathscr{B} \}} \right) ds + o(t).
\]

In this last expectation, we now can remove the event \( A_{1,0}^{(1)} \) because for \( 1/p + 1/q = 1 \) with \( p < 2 \),

\[
\mathbb{E} \left( \frac{\mathcal{O}_{11}(s)}{N} \mathbbm{1}_{(A_{1,0}^{(1)})^c \cap \{ \lambda_1(s) \in \mathscr{B} \}} \right) \leq \mathbb{E} \left( \left( \frac{\mathcal{O}_{11}(s)}{N} \right)^p \mathbbm{1}_{\{ \lambda_1(s) \in \mathscr{B} \}} \right)^{1/p} \mathbb{P} \left( (A_{1,0}^{(1)})^c \right)^{1/q} = o(1),
\]

where we used (1.18) and that on \( \mathscr{B} \) and uniformly in \( N \), \( \mathcal{O}_{11}/N \) has finite moment of order \( p < 2 \). Finally, we have proved

\[
\mathbb{E} \left( |\lambda_1(t) - \lambda_1(0)|^2 \mathbbm{1}_{A_{1,0}^{(1)} \cap \{ \lambda_1(0) \in \mathscr{B} \}} \right) = \int_0^t \mathbb{E} \left( \frac{\mathcal{O}_{11}(s)}{N} \mathbbm{1}_{\{ \lambda_1(s) \in \mathscr{B} \}} \right) ds + o(t).
\]

and one concludes with (2.11).

The proof of (1.20) is identical, except that we rely on the off-diagonal bracket \( d\langle \lambda_1, \bar{\lambda}_2 \rangle_s = \mathcal{O}_{12}(s) \frac{ds}{N} \) and the estimate (1.10).

**APPENDIX B** **NORMALIZED EIGENVECTORS**

This paper focuses on the condition numbers and off-diagonal overlaps, but the Schur decomposition also easily gives information about other statistics such as the angles between eigenvectors. We include these results for the sake of completeness. We denote the complex angle as

\[
\text{arg}(\lambda_1, \lambda_2) = \frac{R_1^* R_2}{\|R_1\| \|R_2\|},
\]

where the phases of \( R_1(1) \) and \( R_2(1) \) can be chosen independent uniform on \([0, 2\pi)\). We also define

\[
\Phi(z) = \frac{z}{\sqrt{1 + |z|^2}}.
\]
Proposition B.1. Conditionally on \( \lambda_1 = z_1, \lambda_2 = z_2 \), we have
\[
\arg(\lambda_1, \lambda_2) \xrightarrow{(d)} \Phi \left( \frac{X}{\sqrt{N|z_1 - z_2|}} \right)
\]
where \( X \sim \mathcal{N}(0, \frac{1}{2} \text{Id}) \).

In particular, for \( \lambda_1, \lambda_2 \) at mesoscopic distance, the complex angle converges in distribution to a Dirac mass at 0. Therefore in such a setting eigenvectors strongly tend to be orthogonal: matrices sampled from the Ginibre ensemble are not far from normal, when only considering eigenvectors angles. The limit distribution becomes non trivial in the microscopic scaling \( |\lambda_1 - \lambda_2| \sim N^{-1/2} \). It is the pushforward of a complex Gaussian measure by \( \Phi \), for which we do not know any specific interesting property.

Proof. From Proposition 2.1 we know that \( R_T^* R_T = R_{T, 1} R_{T, 2} \), \( \|R_1\| = \|R_{T, 1}\| \) and \( \|R_2\| = \|R_{T, 2}\| \), where \( R_{T, 1} \) (and \( L_{T, i} \)) are the normalized bi-orthogonal bases of right and left eigenvectors for \( T \), defined as \( \{2, 2\} \).

The first eigenvectors are written \( R_{T, 1} = (1, 0, \ldots, 0) \) and \( R_{T, 2} = (a, 1, 0, \ldots, 0) \) where \( a = -b_2 - \lambda_1 - \lambda_2 \), with \( T_{12} \) complex Gaussian \( \mathcal{N}(0, \frac{1}{2} \text{Id}) \), independent of \( \lambda_1 \) and \( \lambda_2 \). This gives
\[
\arg(\lambda_1, \lambda_2) = -\frac{b_2}{\sqrt{1 + |b_2|^2}}
\]
and concludes the proof.

From Proposition B.1 the distribution of the angle for fixed \( \lambda_1 \) and random \( \lambda_2 \) can easily be inferred. For example, if \( \lambda_2 \) is chosen uniformly among eigenvalues in a macroscopic domain \( \Omega \subset \{|z| < 1\} \) with nonempty interior, we obtain the convergence in distribution \( (X_{\Omega}) \)
\[
N|\arg(\lambda_1, \lambda_2)|^2 \xrightarrow{\alpha \to \infty} \frac{|\mathcal{N}|^2}{|z_1 - X_{\Omega}|^2}.
\]
When \( z_1 = 0 \) and \( z_2 \) is free, the following gives a more precise distribution, for finite \( N \) and in the limit.

Corollary B.2. Conditionally on \( \{\lambda_1 = 0\} \) we have
\[
N|\arg(\lambda_1, \lambda_2)|^2 \xrightarrow{(d)} N\beta_{1, U_N} \xrightarrow{(d)} X
\]
where \( U_N \) is an independent random variable uniform on \( \{2, \ldots, N\} \), and \( X \) has density \( \frac{1 - (1 + t)e^{-t}}{t^2} \mathbf{1}_{R_+}(t) \).

Proof. From Corollary 5.6 \( N|\lambda_2|^2 \sim \gamma_U \). Together with Lemma 2.5 this gives
\[
|\arg(\lambda_1, \lambda_2)|^2 = \frac{\mathcal{N} |\lambda_2|^2}{1 + \mathcal{N} |\lambda_2|^2} \xrightarrow{(d)} \frac{\gamma_1}{\gamma_1 + \gamma_U} \xrightarrow{(d)} \beta_{1, U_N}.
\]
The limiting density then follows from the explicit distribution of \( \beta \) random variables.

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