A ternary Relation Algebra of directed lines

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Abstract

We define a ternary Relation Algebra (RA) of relative position relations on two-dimensional directed lines (d-lines for short). A d-line has two degrees of freedom (DFs): a rotational DF (RDF), and a translational DF (TDF). The representation of the RDF of a d-line will be handled by an RA of 2D orientations, CYC, known in the literature. A second algebra, TA, which will handle the TDF of a d-line, will be defined. The two algebras, TA and CYC, will constitute, respectively, the translational and the rotational components of the RA, PA, of relative position relations on d-lines: the PA atoms will consist of those pairs ⟨t,r⟩ of a TA atom and a CYC atom that are compatible. We present in detail the RA PA, with its converse table, its rotation table and its composition tables. We show that a (polynomial) constraint propagation algorithm, known in the literature, is complete for a subset of PA relations including almost all of the atomic relations. We will discuss the application scope of the RA, which includes incidence geometry, GIS (Geographic Information Systems), shape representation, localisation in (multi-)robot navigation, and the representation of motion prepositions in NLP (Natural Language Processing). We then compare the RA to existing ones, such as an algebra for reasoning about rectangles parallel to the axes of an (orthogonal) coordinate system, a “spatial Odyssey” of Allen’s interval algebra, and an algebra for reasoning about 2D segments.

Key words: Relation algebra, Spatial reasoning, Qualitative reasoning, Geometric reasoning, Constraint satisfaction, Knowledge representation.

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1 Introduction

Qualitative Spatial Reasoning (QSR), and more generally Qualitative Reasoning (QR), distinguishes from quantitative reasoning by its particularity of remaining at a description level as high as possible. In other words, QSR sticks at the idea of “making only as many distinctions as necessary” [11,20], idea borrowed to naïve physics [27]. The core motivation behind this is that, whenever the number of distinctions that need to be made is finite, the reasoning issue can get rid of the calculations details of quantitative models, and be transformed into a simple matter of symbols manipulation; in the particular case of constraint-based spatial and temporal reasoning, this means a finite relation algebra (finite RA), with tables recording the results of applying the different operations to the different atoms, and the reasoning issue reduced to a matter of table look-ups: the best illustration to this is certainly Allen’s [1] algebra of time intervals.

The main problem in designing a QSR language is certainly to come up with the right, cognitively adequate, distinctions that need to be made; this problem is often referred to as the qualitative/quantitative dilemma, or the finiteness/density dilemma [25] (how to distinguish between the infinite number of elements of an —infinite— universe using only a finite number of distinctions?): to say it another way, because such a language can make only a finite number of distinctions, it should reflect as good as possible the real world; ideally, such a language would be such that it distinguishes between two situations if and only if Humans, or the agents expected to use the language, do distinguish between the two situations. Qualitative reasoning had to face criticism —examples include Forbus, Nielsen and Faltings’ [18] poverty conjecture, or Habel’s [25] argument that such a language, even when built according to cognitive adequacy criteria, still suffers from not having “the ability to refine discrete structures if necessary”. The tendency has since then changed, due certainly to the success gained by QSR in real applications, such as GIS, robot navigation, or shape description.

QSR has now its place in AI. Its research has focussed for about a decade on aspects such as topology, orientation and distance. The aspect the most developed so far is topology, illustrated by the well-known RCC theory [51], from which derives the RCC-8 calculus [51,16]. The RCC theory, on the other hand, stems from Clarke’s “calculus of individuals” [9], based on a binary “connected with” relation —sharing of a point of the arguments. Clarke’s work, in turn, was developed from classical mereology [42,43] and Whitehead’s “extensionally connected with” relation [60]. The huge interest, the last couple of years, in applications such as robot navigation, illustrated by active and promising RoboCup soccer meetings at the main AI conferences (IJCAI, AAAI: see, for instance, [54] for RoboCup’99), had and still have as a consequence that
relative orientation, and, more generally, relative position, considered as expressing more specific knowledge, are gaining increasing interest from the QSR community.

The research in QSR has reached a point where the integration of different aspects of knowledge, such as relative orientation and relative distance, topology and orientation, or, as in the present work, relative orientation and relative translation, is more than needed, in order to face the new demand of real applications. Such an integration of different aspects of knowledge is seen as position, because a calculus coming from such a combination, if it cannot represent the position of an object as precisely as do quantitative models, yet provides a representation more specific than the ones of the combined calculi. It seems to be the case that all researchers in the area are aware of the problem [11,17,19,20]. When looking at what has really been achieved so far in this direction, apart from the work in [10], and, more recently, the one in [23], not much can be said.

In this work, we consider the geometric element consisting of a (2-dimensional) directed line (d-line for short). Such an element has two degrees of freedom (DFs) [26,38]: a rotational DF (RDF) and a translational DF (TDF). The RDF, on the one hand, constrains the way a d-line can rotate relative to another d-line (relative orientation); the TDF, on the other hand, constrains the way a d-line can translate relative to another, or other, d-lines, so that once the RDF of a d-line, say \( \ell_1 \), has been “absorbed” (i.e., its orientation has been fixed), we know how to translate \( \ell_1 \) (a move parallel to the orientation), so that its desired position gets fixed, and its TDF absorbed (e.g., translate \( \ell_1 \) so that its intersecting point with a second d-line \( \ell_2 \) comes before the intersecting point of a third d-line \( \ell_3 \) with \( \ell_2 \), when we walk along \( \ell_2 \) heading the positive direction; or so that \( \ell_1 \) is parallel to both, and does not lie between, \( \ell_2 \) and \( \ell_3 \)). We can right now notice a point of high importance for the TDF of d-lines, which is that their oriented-ness makes them much richer than undirected lines, or u-lines: contrary to u-lines:

1. when walking along a d-line, we know whether we are heading the positive or the negative direction; and
2. when walking perpendicularly to a d-line, we know whether we are heading towards the right half-plane or towards the left half-plane bounded by the d-line.

We provide a ternary Relation Algebra (RA), \( \mathcal{P}_A_t \), of relative position relations on d-lines; the way we proceed is, somehow, imposed by the two DFs of a d-line:

1. a ternary RA of 2D orientations, \( \mathcal{CYC}_t \), recently known in the literature [35,36], will be the rotational component of \( \mathcal{P}_A_t \); and
(2) a second ternary algebra, $\mathcal{T}A_t$, which will constitute the translational component of $\mathcal{P}A_t$, will be defined.

The $\mathcal{P}A_t$ atoms will consist of those pairs $(t, r)$ of a $\mathcal{T}A_t$ atom and a $\mathcal{Cyc}_t$ atom that are compatible. The work can be seen as a full axiomatisation, given as a ternary RA, with its converse table, its rotation table and its composition tables, of qualitative geometry [4,6], with parallelity and cutting-ness, and with d-lines as the primitive entities. It should be emphasised here that, thanks, again, to the oriented-ness of d-lines:

(1) parallelity, on the one hand, splits into six relations, “parallel to, of same orientation as, and lies in the left half-plane bounded by”, “parallel to, of same orientation as, and coincides with”, “parallel to, of same orientation as, and lies in the right half-plane bounded by”, “parallel to, of opposite orientation than, and lies in the left half-plane bounded by”, “parallel to, of opposite orientation than, and coincides with”, “parallel to, of opposite orientation than, and lies in the right half-plane bounded by”. This allows distinguishing, on the one hand, between equal orientations and opposite orientations, and, on the other hand, between the parallels to a fixed d-line that lie in the left half-plane bounded by, the ones that coincide with, and the ones that lie in the right half-plane bounded by, the d-line. Had we u-lines instead of d-lines, parallelity would split into two relations, “coincides with” and “strictly parallel to”; and

(2) in a similar way, cutting-ness splits into two relations, “cuts, and to the left of” and “cuts, and to the right of”. Cuttingness of u-lines is atomic.

Using the RA $\mathcal{P}A_t$, we can represent knowledge on relative position of d-lines as a CSP (Constraint Satisfaction Problem) [46,47], of which:

(1) the variables range over the set $\mathcal{D}$ of d-lines, and
(2) the constraints consist of $\mathcal{P}A_t$ relations on (triples of) the d-line variables.

In addition to a full definition of the RA $\mathcal{P}A_t$, with its converse table, its rotation table and its composition tables, we show the important result that, a (polynomial) 4-consistency algorithm known in the literature [35,36], is complete for the atomic relations of a coarser version, $c\mathcal{P}A_t$, of $\mathcal{P}A_t$: a CSP such that the constraint on each triple of variables is a $c\mathcal{P}A_t$ atomic relation, can be checked for consistency using the propagation algorithm in [35,36]. Solving a general $c\mathcal{P}A_t$ CSP, in turn, can be achieved using a solution search algorithm, such as Isli and Cohn’s, also in [35,36]. The set of $c\mathcal{P}A_t$ atomic relations includes almost all $\mathcal{P}A_t$ atomic relations.

The proof of the result that the 4-consistency algorithm in [35,36] is complete for CSPs expressed in the set, $\mathcal{S}$, of $c\mathcal{P}A_t$ atomic relations, shows the importance of degrees of freedom [26,38] for this work. If such an algorithm applied
to such a CSP does not derive the empty relation—in which case, the result
says that the CSP is consistent—then, in order to find a spatial scene that is
a model of the CSP, we can proceed as follows:

(1) Start by getting the RDF absorbed for each of the d-line variables in-
volved in the CSP. In other words, start by fixing the orientation for each
of the variables. This can be done using a result in [35,36], stating that
a 4-consistent atomic CSP of 2D orientations is (globally) consistent: the
proof of this result gives a backtrack-free method for the construction of
a solution to such a CSP; the solution can be seen as a set of d-lines all
of which are incident with a fixed point—concurrent d-lines.

(2) Once the RDF has been fixed for each of the d-line variables, the problem
has been brought down to a 1D problem: a simple translational problem.
The TDF, for each of the d-line variables, has to be fixed: how to translate
the d-lines relative to one another, so that the TDF of each of the d-line
variables gets absorbed (i.e., so that all the $\mathcal{T}_A$ constraints get satisfied)
—see the proof of Theorem 2 for details.

In the light of the preceding lines, we can provide a plausible explanation to
the question of why QSR researchers have not, so far, sufficiently tackled the
emerging challenge of integrating different spatial aspects. Combining, for in-
stance, an algebra of relative orientation with an algebra of relative distance,
may lead to an algebra with a high number of atoms (a number that can
go up to the product of the numbers of atoms of the combined algebras). A
high number of atoms, in turn, means, among other things, a big composi-
tion table (which is generally hard to build, sometimes even with the help of
a computer—see the challenge in [52]!). QSR languages known so far, par-
ticularly the constraint-based ones, could be described as all-aspects-at-once
languages, in the sense that the way the different spatial aspects, such as,
for the present work, relative orientation and relative translation, correspond-
ing to the different DFs of the objects—here d-lines—in consideration, are
treated as undecomposable: as a consequence, the composition table is simply
an 2-dimensional table, with $d^2$ entries, $d$ being the number of atoms of the
language. The present work is expected to help changing the tendency, since
the composition of the $\mathcal{P}_A$ relations is brought down to a matter of a cross
product of the composition of the relations of the translational projection, on
the one hand, and the composition of the relations of the rotational projec-
tion, on the other hand. The method could thus be described as a “divide and
conquer” one:

(1) project the knowledge onto the different DFs;
(2) process (compose) the knowledge at the level of each of the projections;
and
(3) perform the cross product of the different results in order to get the
composition of the initial knowledge.
As such, the work can be looked at as answering, at least partly, the challenges in [52] for the particular case of QSR: combining different spatial aspects, in the way it is done in the present work, does not necessarily increase the difficulties related to composition, because each of the combined aspects corresponds to one of the DFs [26,38] of the objects in consideration; the different DFs of an object, in turn, are, in some sense, independent from each other, so that the composition issue can be tackled using the “divide and conquer” method referred to above.

Another point worth mentioning is that the current work illustrates the importance of the work in [35,36], since, as illustrated by the previous lines, the RA $\mathcal{CYC}_t$ in [35,36] is one of the main two components, specifically the rotational component, of the main RA, $\mathcal{PA}_t$, investigated here.

The application scope of the RA is large: we discuss the issue for four application domains, Geographical Information Systems (GIS), shape representation, robot’s panorama description, and the representation of motion prepositions. We also consider incidence geometry and show how to represent with the RA the incidence of a point with a (directed) line, betweenness of three points, and non-collinearity of three points.

We then turn to related work met in the literature, consisting mainly of (relation) algebras for representing and reasoning about polygonally shaped objects: a dipole algebra [48], important for applications such as cognitive robotics [50] and spatial information systems [29]; an algebra of directed intervals [53]; and an algebra of rectangles whose sides are parallel to the axes of an orthogonal coordinate system of the 2-dimensional Euclidean space [3,24,49].

We show that in each case the atomic relations can be expressed in the RA.

Section 2 provides some background on constraint-based, or relational, reasoning; an emphasis is given to relation algebras (RAs). Section 3 is devoted to a quick overview of the ternary RA of 2D orientations in [36]. Section 4 presents in detail the ternary RA of relative position relations on d-lines. Section 5 deals with ternary CSPs of relative position relations on d-lines, expressed in the new RA; and shows the result that, if such a CSP is expressed in the set, $\mathcal{S}$, of atomic relations of a coarser version of the new RA, then a known 4-consistency algorithm [35,36] either detects its inconsistency, or, if the CSP is consistent, makes it globally consistent. Section 6 describes the use of the new RA in incidence geometry, and its applications in domains such as GIS, polygonal shape representation, (self)localisation of a robot, and the representation of motion prepositions in natural language. Section 7 relates the new RA to similar work in the literature: Scivos and Nebel’s work [55] on NP-hardness of Freksa’s calculus [20,61]; Moratz et al.’s dipole algebra [48]; Renz’s spatial Odyssey [53] of Allen’s interval algebra [1]; and the rectangle algebra [3,24,49]. Section 8 summarises the work.
The work to be presented has been inspired by the approach to solving geometric constraint systems, known as Degrees of Freedom Analysis, or DFA for short. For more details on DFA, the reader is referred to [38] (see also [37,39]). For the purpose of this work, we just mention quickly how the inspiration came. “Degrees of freedom analysis employs the notion of incremental assembly as a metaphor for solving geometric constraint systems ...” ([39], page 34). The term incremental refers to the way the method, known as the Metaphor Assembly Plan (MAP) [39], proceeds, by fixing step by step the different degrees of freedom of the object variables involved in the geometric constraint problem, until all of them have been fixed, or absorbed, at which stage all objects have been assigned their right positions, and the problem solved. This inspiration led to the ternary relation algebra of d-lines to be presented, which decomposes into two components, a translational component, handling the translational degrees of freedom of d-lines, and a rotational component, handling the rotational degrees of freedom of d-lines. The atoms of the RA consist of pairs of atoms—an atom of the translational component and an atom of the rotational component. More importantly, the different operations on ternary relations—converse, rotation and composition—applied to the RA’s atoms, reduce to cross products of the operations applied to the atoms of the translational component, on the one hand, and to the atoms of the rotational component, on the other hand. The operations, of which the most important is composition, could thus be parallelised. Furthermore, as will be seen in the proof of Theorem 2, searching for a solution of a problem expressed in the RA can be done in a way which has some similar side with the MAP method used in DFA: we first fix the rotational degrees of freedom of the d-line variables, by searching for a solution of the rotational component; we then fix the translational degrees of freedom by translating, relative to one another, the d-lines of the rotational solution.

2 Constraint satisfaction problems

The aim of this section is to introduce some background on constraint-based reasoning.

A constraint satisfaction problem (CSP) of order \(n\) consists of:

1. a finite set of \(n\) variables, \(x_1, \ldots, x_n\);
2. a set \(U\) (called the universe of the problem); and
3. a set of constraints on values from \(U\) which may be assigned to the variables.
An $m$-ary constraint is of the form $R(x_{i_1}, \ldots, x_{i_m})$, and asserts that the $m$-tuple of values assigned to the variables $x_{i_1}, \ldots, x_{i_m}$ must lie in the $m$-ary relation $R$ (an $m$-ary relation over the universe $U$ is any subset of $U^m$). An $m$-ary CSP is one of which the constraints are $m$-ary constraints.

Composition and converse for binary relations were introduced by De Morgan [12]. Isli and Cohn [35,36] extended the operations of composition and converse to ternary relations, and introduced for ternary relations the operation of rotation, which is not needed for binary relations. For any two ternary relations $R$ and $S$, $R \cap S$ is the intersection of $R$ and $S$, $R \cup S$ is the union of $R$ and $S$, $R \circ S$ is the composition of $R$ and $S$, $R^\sim$ is the converse of $R$, and $R^\downarrow$ is the rotation of $R$:

\begin{align*}
R \cap S &= \{(a, b, c) : (a, b, c) \in R \text{ and } (a, b, c) \in S\} \quad (1) \\
R \cup S &= \{(a, b, c) : (a, b, c) \in R \text{ or } (a, b, c) \in S\} \quad (2) \\
R \circ S &= \{(a, b, c) : \text{for some } d, (a, b, d) \in R \text{ and } (a, d, c) \in S\} \quad (3) \\
R^\sim &= \{(a, b, c) : (a, c, b) \in R\} \quad (4) \\
R^\downarrow &= \{(a, b, c) : (c, a, b) \in R\} \quad (5)
\end{align*}

Three special ternary relations over a universe $U$ are the empty relation $\emptyset$ which contains no triples at all, the identity relation $I_U^t = \{(a, a, a) : a \in U\}$, and the universal relation $\top_U = U \times U \times U$.

### 2.1 Constraint matrices

A ternary constraint matrix of order $n$ over $U$ is an $n \times n \times n$-matrix, say $T$, of ternary relations over $U$ verifying the following:

\begin{align*}
(\forall i \leq n) \quad (T_{iii} \subseteq T_U^t) \quad \text{(the identity property)} \\
(\forall i, j, k \leq n) \quad (T_{ijk} = (T_{ikj})^\sim) \quad \text{(the converse property)} \\
(\forall i, j, k \leq n) \quad (T_{ijk} = (T_{kij})^\downarrow) \quad \text{(the rotation property)}
\end{align*}

Let $P$ be a ternary CSP of order $n$ over a universe $U$. Without loss of generality, we can make the assumption that for any three variables $x_i, x_j, x_k$, there is at most one constraint involving them. $P$ can be associated with the following ternary constraint matrix, denoted $T^P$:

\begin{enumerate}
\item initialise all entries to the universal relation: $(\forall i, j, k \leq n) ((T^P)_{ijk} \leftarrow T_U^t)$;
\end{enumerate}
(2) initialise the diagonal elements to the identity relation: \((\forall i \leq n)((TP)_{iii} \leftarrow I_U)\); and

(3) for all triples \((x_i, x_j, x_k)\) of variables on which a constraint \((x_i, x_j, x_k) \in R\) is specified:

\[
\begin{align*}
(TP)_{ijk} &\leftarrow (TP)_{ijk} \cap R, \\
(TP)_{ikj} &\leftarrow ((TP)_{ijk})^\sim, \\
(TP)_{jki} &\leftarrow ((TP)_{jki})^\sim, \\
(TP)_{kij} &\leftarrow ((TP)_{kji})^\sim, \\
(TP)_{kji} &\leftarrow ((TP)_{kij})^\sim.
\end{align*}
\]

We make the assumption that, unless explicitly specified otherwise, a CSP is given as a constraint matrix.

2.2 (Strong) k-consistency, refinement

Let \(P\) be a ternary CSP of order \(n\), \(V\) its set of variables and \(U\) its universe. An instantiation of \(P\) is any \(n\)-tuple \((a_1, a_2, \ldots, a_n)\) of \(U^n\), representing an assignment of a value to each variable. A consistent instantiation, or solution, of \(P\) is an instantiation satisfying all the constraints. \(P\) is consistent if it has at least one solution; it is inconsistent otherwise. The consistency problem of \(P\) is the problem of verifying whether \(P\) is consistent.

Let \(V' = \{x_{i_1}, \ldots, x_{i_k}\}\) be a subset of \(V\). The sub-CSP of \(P\) generated by \(V'\), denoted \(P_{\mid V'}\), is the CSP with set of variables \(V'\) and whose constraint matrix is obtained by projecting the constraint matrix of \(P\) onto \(V'\). \(P\) is k-consistent [21] if for any subset \(V'\) of \(V\) containing \(k - 1\) variables, and for any variable \(X \in V\), every solution to \(P_{\mid V'}\) can be extended to a solution to \(P_{\mid V' \cup \{X\}}\).

\(P\) is strongly \(k\)-consistent if it is \(j\)-consistent, for all \(j \leq k\). 1-consistency, 2-consistency and 3-consistency correspond to node-consistency, arc-consistency and path-consistency, respectively [46,47]. Strong \(n\)-consistency of \(P\) corresponds to what is called global consistency in [13]. Global consistency facilitates the important task of searching for a solution, which can be done, when the property is met, without backtracking [21].

A refinement of \(P\) is a CSP \(P'\) with the same set of variables, and such that \((\forall i, j, k)((TP')_{ijk} \subseteq (TP)_{ijk})\).

2.3 Relation algebras

We recall some basic notions on relation algebras (RAs). For more details, the reader is referred to [57,14,15,40] for binary RAs, as first introduced by Tarski [57], who was mainly interested in formalising the theory of binary relations; and to [36] for ternary RAs, motivated by the authors with the fact that binary RAs are not sufficient for the representation of spatial knowledge,
such as cyclic ordering of three points of the plane, known to be of primary importance for applications such as robot localisation (how to represent the knowledge that, seen from the robot’s position, three landmarks, say \( L_1, L_2 \) and \( L_3 \), are met in that order, when we scan, say in the anticlockwise direction, a circle centred at the robot’s position, starting from \( L_1 \)) —cyclic ordering can be looked at as the cyclic time counterpart of linear time betweenness.\(^1\)

A Boolean algebra (BA) with universe \( A \) is an algebra of the form \( \langle A, \oplus, \odot, -, \bot, \top \rangle \) which satisfies the following properties, for all \( R, S, T \in A \):

\[
\begin{align*}
R \oplus (S \oplus T) &= (R \oplus S) \oplus T \quad (6) \\
R \oplus S &= S \oplus R \quad (7) \\
R \odot S \odot R &= R \quad (8) \\
R \odot S \odot T &= (R \oplus T) \odot (S \oplus T) \quad (9) \\
R \oplus \overline{R} &= \top \quad (10)
\end{align*}
\]

\( \mathcal{R} \) is a binary RA with universe \( A \) [57,40] if:

1. \( \mathcal{A} \) is a set of binary relations; and
2. \( \mathcal{R} = \langle \mathcal{A}, \oplus, \odot, -, \bot, \top, I \rangle \), where \( \langle \mathcal{A}, \oplus, \odot, -, \bot, \top \rangle \) is a BA (called the Boolean part, or reduct, of \( \mathcal{R} \)), \( \circ \) is a binary operation, \( \sim \) is a unary operation, \( I \in \mathcal{A} \), and the following identities hold for all \( R, S, T \in \mathcal{A} \):

\[
\begin{align*}
(R \circ S) \circ T &= R \circ (S \circ T) \quad (11) \\
(R \oplus S) \circ T &= R \circ T \oplus S \circ T \quad (12) \\
R \circ I &= I \circ R = R \quad (13) \\
(R \sim)^\sim &= R \quad (14) \\
(R \oplus S)^\sim &= R \sim \oplus S \sim \quad (15) \\
(R \circ S)^\sim &= S \sim \circ R \sim \quad (16) \\
R \sim \circ \overline{R} \circ S \odot S &= \bot \quad (17)
\end{align*}
\]

Ternary RAs [36] need a (unary) operation called rotation, in addition to an adaptation to the ternary case of the operations of composition and converse, first introduced by De Morgan for binary relations [12]. \( \mathcal{R} \) is a ternary RA with universe \( A \) [36] if:

1. \( \mathcal{A} \) is a set of ternary relations; and
2. \( \mathcal{R} = \langle \mathcal{A}, \oplus, \odot, -, \bot, \top, I \rangle \) where \( \langle \mathcal{A}, \oplus, \odot, -, \bot, \top \rangle \) is a BA (called the Boolean part, or reduct, of \( \mathcal{R} \)), \( \circ \) is a binary operation, \( \sim \) and \( \overline{\sim} \) are unary operations, \( I \in \mathcal{A} \), and the following identities hold for all \( R, S, T \in \mathcal{A} \):

\[
\begin{align*}
(R \circ S) \circ T &= R \circ (S \circ T) \quad (11) \\
(R \oplus S) \circ T &= R \circ T \oplus S \circ T \quad (12) \\
R \circ I &= I \circ R = R \quad (13) \\
(R \sim)^\sim &= R \quad (14) \\
(R \oplus S)^\sim &= R \sim \oplus S \sim \quad (15) \\
(R \circ S)^\sim &= S \sim \circ R \sim \quad (16) \\
R \sim \circ \overline{R} \circ S \odot S &= \bot \quad (17)
\end{align*}
\]

\(^1\) The work in [32] shows how to solve a CSP of cyclic time intervals [5,30] using results on cyclic ordering of 2D orientations [35,36], which emphasises the link between cyclic time and cyclic ordering of 2D orientations.
(R ∪ S) ⊙ T = R ∪ (S ⊙ T)  \tag{18} \\
(R ⊕ S) ⊙ T = R ⊕ T ⊕ S ⊙ T 
\tag{19} \\
R ∩ I = I ∩ R = R 
\tag{20} \\
(R^−)^− = R 
\tag{21} \\
(R ∪ S)^− = R^− ⊕ S^− 
\tag{22} \\
(R ∩ S)^− = S^− ∩ R^− 
\tag{23} \\
R^− (R ∪ S ∩ S) = ∅ 
\tag{24} \\
((R^−)^−)^− = R 
\tag{25} \\
(R ∪ S)^− = R^− ⊕ S^− 
\tag{26} \\

Let \( \mathcal{R} \) be an RA. The elements of \( \mathcal{R} \) are just the relations in its universe. An atom of \( \mathcal{R} \) is a minimal nonzero element, i.e., \( R \) is an atom if \( R \neq \perp \) and for every \( S \in \mathcal{A} \), either \( R ⊗ S = \perp \) or \( R ⊗ S = \perp \). \( \mathcal{R} \) is atomic if every nonzero element has an atom below it; i.e., if for all nonzero elements \( R \), there exists an atom \( A \) such that \( A ⊗ R = A \). \( \mathcal{R} \) is finite if its universe has finitely many elements. A finite RA is atomic, and its Boolean part is completely determined by its atoms. Furthermore, in an atomic RA, the result of applying any of the operations of the RA to any elements can be obtained from the results of applying the different operations to the atoms. Specifying a finite, thus atomic, RA reduces thus to specifying the identity element and the results of applying the different operations to the different atoms.

The full binary (resp. ternary) RA over a set \( U \) is the RA \( \mathcal{F}B_U = (\text{binRel}(U), \cup, \cap, −, \emptyset, \top_U, ⊙, ^−, \mathcal{I}_U^b) \) (resp. \( \mathcal{F}T_U = (\text{terRel}(U), \cup, \cap, −, \emptyset, \top_U, ⊙, ^−, \mathcal{I}_U^t) \)), where:

1. the universe \( \text{binRel}(U) \) (resp. \( \text{terRel}(U) \)) is the set of all binary (resp. ternary) relations over \( U \);
2. \( \cup, \cap \) and \( − \) are, respectively, the usual set-theoretic operations of union, intersection and complement;
3. \( \emptyset \) is the empty relation;
4. \( \top_U^b \) (resp. \( \top_U^t \)) is the universal binary (resp. ternary) relation over \( U 
\\n\top_U^b = U × U \) (resp. \( \top_U^t = U × U × U \));
5. \( ⊙ \) and \( ^− \) are, respectively, the operations of composition and converse of binary (resp. ternary) relations;
6. \( ^− \) is the operation of rotation of ternary relations; and
7. \( \mathcal{I}_U^b \) (resp. \( \mathcal{I}_U^t \)) is the binary (resp. ternary) identity relation over \( U 
\\n\mathcal{I}_U^b = \{(a, a) | a ∈ U \} \) (resp. \( \mathcal{I}_U^t = \{(a, a, a) | a ∈ U \} \)).

A binary (resp. ternary) RA over a set \( U \) is an RA \( \mathcal{R} = (\mathcal{A}, \cup, \cap, −, \emptyset, \top_U^b, ⊙, ^−, \mathcal{I}_U^b) \) (resp. \( \mathcal{R} = (\mathcal{A}, \cup, \cap, ^−, \emptyset, \top_U^t, ⊙, ^−, \mathcal{I}_U^t) \)), with universe \( \mathcal{A} \subseteq \text{binRel}(U) \) (resp. \( \mathcal{A} \subseteq \text{terRel}(U) \)), such that:

1. \( \mathcal{A} \) is closed under the distinguished operations of \( \text{binRel}(U) \) (resp. \( \text{terRel}(U) \)), namely, under the operations \( \cup, \cap, −, ⊙ \) and \( ^− \) (resp. the operations \( \cup, \cap, −, ⊙ \) and \( ^− \)); and
(2) \( \mathcal{A} \) contains the distinguished constants, namely, the relations \( \emptyset, \top \) and \( \mathcal{I}_b^U \) (resp. the relations \( \emptyset, \top \) and \( \mathcal{I}_t^U \)).

Such a binary (resp. ternary) RA is a subalgebra of the full RA \( \mathcal{F} \mathcal{B}_U \) (resp. \( \mathcal{F} \mathcal{T}_U \)).

Let \( \{ R_i : i \in I \} \subseteq \text{binRel}(U) \) (resp. \( \{ R_i : i \in I \} \subseteq \text{terRel}(U) \)). The binary (resp. ternary) RA generated by \( \{ R_i : i \in I \} \), denoted by \( \langle R_i : i \in I \rangle \) (resp. \( \langle R_i : i \in I \rangle \)), such that \( \mathcal{A} \) is the smallest subset of \( \text{binRel}(U) \) (resp. \( \text{terRel}(U) \)) closed under the distinguished operations of \( \text{binRel}(U) \) (resp. \( \text{terRel}(U) \)). We refer to \( \{ R_i : i \in I \} \) as a base of \( \langle R_i : i \in I \rangle \).

Of particular interest to this work are atomic, finite ternary RAs over a set \( U \), of the form \( \langle 2^A, \cup, \cap, \neg, \emptyset, \top, \circ, \ulcorner, \urcorner, \mathcal{I}_t^U \rangle \), where \( A \) is a nonempty finite set of atoms that are Jointly Exhaustive and Pairwise Disjoint (JEPD): for all triples \( (x, y, z) \in U^3 \), there exists one and only one atom \( t \) from \( A \) such that \( t(x, y, z) \). Such a set \( A \) of atoms corresponds to the finite partitioning, \( \bigcup_{t \in A} t \), of the universal ternary relation over \( U \), \( \mathcal{I}_t^U \). Such an RA is nothing else than the RA \( \langle t : t \in A \rangle \) generated by \( A \). The universe \( U \) will be, unless otherwise specified, the set \( \longrightarrow \mathcal{L} \) of 2D directed lines.

Throughout the rest of the paper, given an \( n \)-ary algebra \( \mathcal{R} \), with atoms \( r_1, \ldots, r_m \), and universe \( U \), we shall use the notation \( \mathcal{U}\text{-at} \) to refer to the set \( \{ r_1, \ldots, r_m \} \) of all atoms; an \( \mathcal{R} \) relation, say \( R \), is any subset of \( \mathcal{U}\text{-at} \), interpreted as follows:

\[
(\forall x_1, \ldots, x_n \in U)(R(x_1, \ldots, x_n) \iff \bigvee_{r \in R} r(x_1, \ldots, x_n))
\]

An \( \mathcal{R} \) atomic relation is an \( \mathcal{R} \) relation consisting of one single atom (singleton set).

3 Isli and Cohn’s ternary RA of 2D orientations

We use \( \mathbb{R}^2 \) as a model of the plane, and assume that \( \mathbb{R}^2 \) is associated with a Cartesian coordinate system \((x, O, y)\). We refer to the set of 2D orientations as \( 2DO \); to the circle centred at \( O \) and of unit radius, as \( \mathcal{C}_{O,1} \); to the set of directed lines of the plane as \( \overrightarrow{\mathcal{L}} \); to the set of undirected lines of the plane as \( \overleftarrow{\mathcal{L}} \); to the union \( \overrightarrow{\mathcal{L}} \cup \overleftarrow{\mathcal{L}} \) as \( \mathcal{L} \); and to the set, subset of \( \overrightarrow{\mathcal{L}} \), of directed lines of the plane containing (incident with) \( O \), as \( \overrightarrow{\mathcal{L}}_O \). Throughout the rest of the paper, we use d-line and u-line as abbreviations for “directed line” and
undirected line”, respectively. Given two distinct points \(x\) and \(y\) of the plane \(\mathbb{R}^2\), we denote by \(\overrightarrow{xy}\) the d-line containing \(x\) and \(y\) and oriented from \(x\) to \(y\); given a set \(A\), \(|A|\) denotes the cardinality (i.e., the number of elements) of \(A\); given \(\ell \in \overrightarrow{L}\), \(O(\ell)\) refers to the orientation of \(\ell\); given \(\ell \in L\), \(pts(\ell)\) refers to the set of points of the plane belonging to \(\ell\).

It is common in geometry to consider a line as a set of points, so that one can write, for a line \(\ell\), that \(\ell = pts(\ell)\); this is possible as long as we are concerned only with u-lines, i.e., with the set \(\overrightarrow{L}\); when the space in consideration is \(L\), or its superset \(L\), this is not possible any longer, for \(pts(\ell)\) does not contain the information of whether \(\ell\) is a d-line or a u-line.

**Definition 1** The isomorphisms \(I_1\) and \(I_2\) are defined as follows:

(1) \(I_1 : 2DO \rightarrow C_{O,1}; I_1(z)\) is the point \(P_z \in C_{O,1}\) such that the orientation of the d-line \(\overrightarrow{OP_z}\) is \(z\).

(2) \(I_2 : 2DO \rightarrow \overrightarrow{O}; I_2(z)\) is the line \(\ell_{O,z} \in \overrightarrow{O}\) of orientation \(z\).

**Definition 2** The angle determined by two d-lines \(D_1\) and \(D_2\), denoted \(\prec D_1, D_2 \succ\), is the one corresponding to the move in an anticlockwise direction from \(D_1\) to \(D_2\). The angle \(\prec z_1, z_2 \succ\) determined by orientations \(z_1\) and \(z_2\) is the angle \(\prec I_2(z_1), I_2(z_2) \succ\).

The set \(2DO\) can thus be viewed as the set of points of \(C_{O,1}\) (or of any fixed circle), or as the set of d-lines containing \(O\) (or any fixed point). Isli and Cohn [35,36] have defined a binary RA of 2D orientations, \(\mathcal{CYC}_b\), that contains four atoms: \(e\) (equal), \(l\) (left), \(o\) (opposite) and \(r\) (right). For all \(x, y \in 2DO\):

\[
\begin{align*}
\text{e}(y, x) & \iff \prec x, y \succ = 0 \\
\text{l}(y, x) & \iff \prec x, y \succ \in (0, \pi) \\
\text{o}(y, x) & \iff \prec x, y \succ = \pi \\
\text{r}(y, x) & \iff \prec x, y \succ \in (\pi, 2\pi)
\end{align*}
\]

Based on \(\mathcal{CYC}_b\), Isli and Cohn [35,36] have built a ternary RA, \(\mathcal{CYC}_t\), for cyclic ordering of 2D orientations: \(\mathcal{CYC}_t\) has 24 atoms, thus \(2^{24}\) relations. The atoms of \(\mathcal{CYC}_t\) are written as \(b_1b_2b_3\), where \(b_1, b_2, b_3\) are atoms of \(\mathcal{CYC}_b\), and such an atom is interpreted as follows: \((\forall x, y, z \in 2DO) (b_1b_2b_3(x, y, z) \iff b_1(y, x) \land b_2(z, y) \land b_3(z, x))\). Figure 1 reproduces the \(\mathcal{CYC}_t\) converse and rotation table. Figure 2 illustrates the 24 \(\mathcal{CYC}_t\) atoms, and the angle determined by two d-lines. The reader is referred to [35,36] for the \(\mathcal{CYC}_t\) composition tables.
We define in this section our algebra of ternary relations on d-lines. The knowledge the algebra can express, consists of a combination of translational knowledge and rotational knowledge. The translational component records as ternary relations knowledge such as, the order in which two d-lines cut a third one, or the order in which come three parallel d-lines, when we move from the left half-plane towards the right half-plane bounded by one of the d-lines.

The rotational component, on the other hand, records, also as ternary relations, knowledge on the relative angles of the three d-line arguments; specifically, on the angles determined by pairs of the three arguments.

4.1 The translational component

We start by defining three binary relations, cuts, coinc-with (coincides with) and s-par-to (strictly parallel to), over the set $\overline{\mathcal{L}}$ of d-lines, and the derived relation par-to (parallel to) of parallelity. For all $x, y \in \overline{\mathcal{L}}$:

\[
\begin{align*}
cuts(x, y) &\iff |\text{pts}(x) \cap \text{pts}(y)| = 1 \\
\text{coinc-with}(x, y) &\iff \text{pts}(x) = \text{pts}(y) \\
s\text{-par-to}(x, y) &\iff \text{pts}(x) \cap \text{pts}(y) = \emptyset \\
\text{par-to}(x, y) &\iff \text{coinc-with}(x, y) \lor s\text{-par-to}(x, y)
\end{align*}
\]

The first three relations are symmetric, in the sense that for all $r \in \{\text{cuts, coinc-with, s-par-to}\}$, and for all $x, y \in \overline{\mathcal{L}}$, if $r(x, y)$ then $r(y, x)$. They define a partition
Fig. 2. (a) Graphical illustration of the 24 \(\text{CYC}_t\) atoms: from top to bottom, left to right, the atoms are \(lrl, lel, lll, llr, lor, lrr, rll, rol, rrl, rrr, rer, rlr, lre, llo, rle, rro, ell, err, orl, olr, eee, eeo, ooe, oeo\); (b) The angle \(\langle D_1, D_2 \rangle\) determined by two d-lines \(D_1\) and \(D_2\) is the one corresponding to the move in an anticlockwise direction from \(D_1\) to \(D_2\).

of \(\overrightarrow{L} \times \overrightarrow{L}\); in other words, using a terminology now common in Qualitative Spatial Reasoning (QSR), the three relations \(\text{cuts}, \text{coinc-with}\) and \(\text{s-par-to}\) are Jointly Exhaustive and Pairwise Disjoint (JEPD).

We use the relations \(\text{cuts}\) and \(\text{par-to}\) to define four ternary relations, \(cc, cp, pc\) and \(pp\), over \(\overrightarrow{L}\). For all \(x, y, z \in \overrightarrow{L}\):

\[
cc(x, y, z) \iff \text{cuts}(y, x) \land \text{cuts}(z, x)
\]
\[
cp(x, y, z) \iff \text{cuts}(y, x) \land \text{par-to}(z, x)
\]
\[
pc(x, y, z) \iff \text{par-to}(y, x) \land \text{cuts}(z, x)
\]
\[
pp(x, y, z) \iff \text{par-to}(y, x) \land \text{par-to}(z, x)
\]

The relations \(cp\) and \(pc\) are the converses of each other: \(cp^\sim = pc\) and \(pc^\sim = cp\); each of the other two relations, \(cc\) and \(pp\), is its own converse: \(cc^\sim = cc\) and \(pp^\sim = pp\). The relations \(cc, cp, pc\) and \(pp\) provide for each of their last two arguments the knowledge of whether it cuts, or is parallel to, the first argument.
In order for the translational component of our algebra to be expressively interesting, we want it to express as well knowledge such as the following:

1. when the last two arguments both cut the first, which of them comes first when we walk along the first argument heading the positive direction;
2. when one of the last two arguments is parallel to the first, which side of the first argument (the left half-plane, the d-line itself, or the right half-plane) it belongs to; and
3. when all three arguments are parallel to each other, in what order do they appear when we walk perpendicularly to, from the left half-plane and heading towards the right half-plane bounded by, the first argument.

**Definition 3** Let $\ell \in \mathcal{L}$. The relations $<_{\ell}$, $=_{\ell}$ and $>_{\ell}$ are defined as follows. For all $x, y \in \mathbb{R}^2$:

\[
\begin{align*}
  x <_{\ell} y &\iff x \in \text{pts}(\ell) \land y \in \text{pts}(\ell) \land x \neq y \land < \ell, \vec{x}y >= 0 \\
  x =_{\ell} y &\iff x \in \text{pts}(\ell) \land y \in \text{pts}(\ell) \land x = y \\
  x >_{\ell} y &\iff y <_{\ell} x
\end{align*}
\]

Readers familiar with Vilain and Kautz’s [59] linear time point algebra, $\mathcal{P}A$, can easily notice a similarity between the relations in Definition 3, $<_{\ell}$, $=_{\ell}$ and $>_{\ell}$, and the $\mathcal{P}A$ atoms, $<$, $=$ and $>$; the latter uses the time line as the reference directed line, which, because it is a global reference line, does not need to appear as a subscript in the relations. As argued in Appendix A, the fact that $\mathcal{P}A_t$ is an RA is a direct consequence of the conjunction of the two facts that (1) $\mathcal{P}A$ is an RA [40], and (2) $\mathcal{C}YC_t$ is an RA [36].

We make use of the relations $<_{\ell}$, $=_{\ell}$ and $>_{\ell}$ of Definition 3 to refine the relation $cc$ into three relations $cc_{<}$, $cc_{=}$ and $cc_{>}$, which add to the knowledge already contained in $cc$, the order in which the last two arguments are met in the walk along the first argument heading the positive direction. For all $\ell_1, \ell_2, \ell_3 \in \mathcal{L}$:

\[
\begin{align*}
  cc_{<}(\ell_1, \ell_2, \ell_3) &\iff cc(\ell_1, \ell_2, \ell_3) \land \\
  &\land (\forall x \in \text{pts}(\ell_2) \cap \text{pts}(\ell_1))(\forall y \in \text{pts}(\ell_3) \cap \text{pts}(\ell_1))(x <_{\ell_1} y) \\
  cc_{=}(\ell_1, \ell_2, \ell_3) &\iff cc(\ell_1, \ell_2, \ell_3) \land (\text{pts}(\ell_2) \cap \text{pts}(\ell_1) = \text{pts}(\ell_3) \cap \text{pts}(\ell_1)) \\
  cc_{>}(\ell_1, \ell_2, \ell_3) &\iff cc_{<}(\ell_1, \ell_3, \ell_2)
\end{align*}
\]

**Definition 4 (plane partition determined by a d-line)** A d-line $\ell$ defines the obvious partition of the plane illustrated in Figure 3(a). We refer to the set of all regions of the partition as $p$-partition($\ell$), and to each region in $p$-partition($\ell$) as $pp$-region$_x(\ell)$, where $x$ is the label associated with the region in Figure 3(a).
Given a d-line $\ell$, we will also refer to $pp-region_L(\ell)$, $pp-region_R(\ell)$ and $pp-region_c(\ell)$ as $lhp(\ell)$ (the open left half-plane bounded by $\ell$), $pts(\ell)$ (the set of points of $\ell$) and $rhp(\ell)$ (the open right half-plane bounded by $\ell$), respectively.

We now split the relation $s$-par-to into two obvious (finer) relations, $l$-par-to ($l$ for left) and $r$-par-to ($r$ for right). For all $\ell, \ell' \in \overline{\mathcal{L}}$:

\[
\begin{align*}
    l\text{-par-to}(\ell', \ell) &\iff \text{s-par-to}(\ell', \ell) \land (\forall x \in pts(\ell'))(x \in lhp(\ell)) \\
    r\text{-par-to}(\ell', \ell) &\iff \text{s-par-to}(\ell', \ell) \land \neg l\text{-par-to}(\ell', \ell)
\end{align*}
\]

In other words, we have the following, for all d-lines $\ell$ and $\ell'$:

\[
\begin{align*}
    l\text{-par-to}(\ell', \ell) &\iff pts(\ell') \subset lhp(\ell) \\
    \text{coinc-with}(\ell', \ell) &\iff pts(\ell') = pts(\ell) \\
    r\text{-par-to}(\ell', \ell) &\iff pts(\ell') \subset rhp(\ell)
\end{align*}
\]

Readers familiar with Vilain and Kautz’s point algebra $\mathcal{PA}$ [59] can, again, easily notice a similarity between the relations $l$-par-to, coinc-with and $r$-par-to, on the one hand, and the $\mathcal{PA}$ atoms $<, = \text{ and } >$, on the other hand.

We make use of the relations $l$-par-to, coinc-with and $r$-par-to to refine the relation $cp$ into three relations, $cp_l$, $cp_c$ and $cp_r$; the relation $pc$ into three relations, $pc_l$, $pc_c$ and $pc_r$; and the relation $pp$ into three relations, $pp_l$, $pp_c$ and $pp_r$. For all $\ell_1, \ell_2, \ell_3 \in \overline{\mathcal{L}}$:

\[
\begin{align*}
    cp_l(\ell_1, \ell_2, \ell_3) &\iff cp(\ell_1, \ell_2, \ell_3) \land l\text{-par-to}(\ell_3, \ell_1) \\
    cp_c(\ell_1, \ell_2, \ell_3) &\iff cp(\ell_1, \ell_2, \ell_3) \land \text{coinc-with}(\ell_3, \ell_1) \\
    cp_r(\ell_1, \ell_2, \ell_3) &\iff cp(\ell_1, \ell_2, \ell_3) \land r\text{-par-to}(\ell_3, \ell_1) \\
    pc_l(\ell_1, \ell_2, \ell_3) &\iff pc(\ell_1, \ell_2, \ell_3) \land l\text{-par-to}(\ell_2, \ell_1) \\
    pc_c(\ell_1, \ell_2, \ell_3) &\iff pc(\ell_1, \ell_2, \ell_3) \land \text{coinc-with}(\ell_2, \ell_1) \\
    pc_r(\ell_1, \ell_2, \ell_3) &\iff pc(\ell_1, \ell_2, \ell_3) \land r\text{-par-to}(\ell_2, \ell_1) \\
    pp_l(\ell_1, \ell_2, \ell_3) &\iff pp(\ell_1, \ell_2, \ell_3) \land l\text{-par-to}(\ell_2, \ell_1) \\
    pp_c(\ell_1, \ell_2, \ell_3) &\iff pp(\ell_1, \ell_2, \ell_3) \land \text{coinc-with}(\ell_2, \ell_1) \\
    pp_r(\ell_1, \ell_2, \ell_3) &\iff pp(\ell_1, \ell_2, \ell_3) \land r\text{-par-to}(\ell_2, \ell_1)
\end{align*}
\]

Again, readers familiar with Vilain and Kautz’s algebra $\mathcal{PA}$ [59] can easily notice a similarity between the relations $cp_l$, $cp_c$ and $cp_r$ and the $\mathcal{PA}$ atoms, $<, = \text{ and } >$; between the relations $pc_l$, $pc_c$ and $pc_r$ and the $\mathcal{PA}$ atoms; and between the relations $pp_l$, $pp_c$ and $pp_r$ and the $\mathcal{PA}$ atoms.

**Definition 5 (line partition)** Let $\ell_1$ and $\ell_2$ be two cutting d-lines —i.e., such that $\text{cuts}(\ell_1, \ell_2)$. $\ell_2$ defines a partition of $\ell_1$ as illustrated in Figure 3(b).

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The three regions of the partition, labelled $<$, $=$ and $>$ in Figure 3(b), correspond, respectively, to the open left half-line bounded by the intersecting point of $\ell_1$ and $\ell_2$, the intersecting point of $\ell_1$ and $\ell_2$, and the open right half-line bounded by the intersecting point of $\ell_1$ and $\ell_2$. We refer to the set of all regions of the partition as line-partition($\ell_1, \ell_2$), and to each region in line-partition($\ell_1, \ell_2$) as lp-region$_x(\ell_1, \ell_2)$, where $x$ is the label associated with the region in Figure 3(b).

Using the line partition of Definition 5, we have the following, for all d-lines $\ell_1$, $\ell_2$ and $\ell_3$ verifying $cc(\ell_1, \ell_2, \ell_3)$: $cc_<(\ell_1, \ell_2, \ell_3)$ iff $pts(\ell_2) \cap pts(\ell_1) \subset lp-region_<(\ell_1, \ell_3)$; $cc= (\ell_1, \ell_2, \ell_3)$ iff $pts(\ell_2) \cap pts(\ell_1) = lp-region_=(\ell_1, \ell_3)$; and $cc>(\ell_1, \ell_2, \ell_3)$ iff $pts(\ell_2) \cap pts(\ell_1) \subset lp-region_>(\ell_1, \ell_3)$.

**Definition 6 (plane partition determined by two parallel d-lines)** Two parallel d-lines $\ell_1$ and $\ell_2$ define a partition of the plane as illustrated in Figure 3(c) for the case l-par-to($\ell_2, \ell_1$), in Figure 3(d) for the case coinc-with($\ell_2, \ell_1$), and in Figure 3(e) for the case r-par-to($\ell_2, \ell_1$). Each region of the partition is an open half-plane bounded by either $\ell_1$ or $\ell_2$, a line ($\ell_1$ or $\ell_2$), or the intersection of two open half-planes bounded by $\ell_1$ and $\ell_2$. We refer to the set of all regions of the partition as p-partition($\ell_1, \ell_2$), and to each region in p-partition($\ell_1, \ell_2$) as pp-region$_x(\ell_1, \ell_2)$, where $x$ is the label associated with the region in Figures 3(c-d-e).

The partition of the plane determined by two parallel d-lines —Definition 6— is now used to refine the relation $pp_i$ into $pp_0$, $pp_1$, $pp_2$, $pp_3$ and $pp_4$; the relation $pp_c$ into $pp_{c0}$, $pp_{c1}$ and $pp_{c2}$; and the relation $pp_r$ into $pp_{r0}$, $pp_{r1}$, $pp_{r2}$, $pp_{r3}$ and $pp_{r4}$. For all $\ell_1, \ell_2, \ell_3 \in \mathcal{L}$:

$$(\forall i \leq 4)(pp_i(\ell_1, \ell_2, \ell_3) \iff pp_i(\ell_1, \ell_2, \ell_3) \land pts(\ell_3) \subseteq pp-region_i(\ell_1, \ell_2))$$

$$(\forall i \leq 2)(pp_c(\ell_1, \ell_2, \ell_3) \iff pp_c(\ell_1, \ell_2, \ell_3) \land pts(\ell_3) \subseteq pp-region_i(\ell_1, \ell_2))$$

$$(\forall i \leq 4)(pp_r(\ell_1, \ell_2, \ell_3) \iff pp_r(\ell_1, \ell_2, \ell_3) \land pts(\ell_3) \subseteq pp-region_i(\ell_1, \ell_2))$$

Readers familiar with Ligozat’s $(p, q)$-relations [44] can easily notice a similarity between $(1, 2)$-relations and the relations $pp_i$, $i \in \{0, \ldots, 4\}$, on the one hand, and between $(1, 2)$-relations and the relations $pp_i$, $i \in \{0, \ldots, 4\}$, on the other hand. Ligozat’s $(1, 2)$-relations are called point-interval relations in [58]. Again, readers familiar with Vilain and Kautz’s algebra $\mathcal{P} \mathcal{A}$ [59] can easily notice a similarity between the relations $pp_{r0}$, $pp_{c1}$ and $pp_{c2}$ and the $\mathcal{P} \mathcal{A}$ atoms, $<$, $=$ and $>$.

From now on, we refer to the translational component as $\mathcal{T} \mathcal{A}_t$ (Translational Algebra of ternary relations —over $\mathcal{L}$); to the set of all $\mathcal{T} \mathcal{A}_t$ atoms as $\mathcal{T} \mathcal{A}_t$-at:

$$cc = \{cc_<, cc_-, cc_>\}$$
Fig. 3. (a) The plane partition determined by a d-line; (b) the line partition determined by a d-line $\ell_2$ on a d-line $\ell_1$; (c) the plane partition determined by two d-lines $\ell_1$ and $\ell_2$ verifying $l$-par-to($\ell_2, \ell_1$); (d) the plane partition determined by two d-lines $\ell_1$ and $\ell_2$ verifying coinc-with($\ell_2, \ell_1$); (e) the plane partition determined by two d-lines $\ell_1$ and $\ell_2$ verifying $r$-par-to($\ell_2, \ell_1$).

\[
\begin{align*}
cp &= \{ cp_l, cp_c, cp_r \} \\
p_c &= \{ pc_l, pc_c, pc_r \} \\
pp_l &= \{ pp_{l0}, pp_{l1}, pp_{l2}, pp_{l3}, pp_{l4} \} \\
pp_c &= \{ pp_{c0}, pp_{c1}, pp_{c2} \} \\
pp_r &= \{ pp_{r0}, pp_{r1}, pp_{r2}, pp_{r3}, pp_{r4} \} \\
pp &= pp_l \cup pp_c \cup pp_r \\
\mathcal{T A}_l-at &= cc \cup cp \cup pc \cup pp
\end{align*}
\]

The $\mathcal{T A}_l$ composition tables. Given four d-lines $x, y, z, t$ and two $\mathcal{T A}_l$ atoms $r$ and $s$, the conjunction $r(x, y, z) \land s(x, z, t)$ is inconsistent if the most specific binary relation, $r_{31}(z, x)$, implied by $r(x, y, z)$ on the pair $(z, x)$, is different from the most specific binary relation, $s_{21}(z, x)$, on the same pair $(z, x)$, implied by $s(x, z, t)$ (see Figure 4 for illustration). Each of $r_{31}$ and $s_{21}$ can be
either of the four binary relations \( \text{cuts}, \ l\text{-par-to}, \ \text{coinc-with} \) or \( \text{r-par-to} \); these four binary relations are Jointly Exhaustive and Pairwise Disjoint (JEPD), which means that any two d-lines are related by one and only one of the four relations. Stated otherwise, when \( r_{31} \neq s_{21} \), we have \( r \circ s = \emptyset \). Thus composition splits into four composition tables, corresponding to the following four cases:

(1) **Case 1**: \( r_{31} = s_{21} = \text{cuts} \). This corresponds to \( r \in \text{cuts}_{31} \) and \( s \in \text{cuts}_{21} \), with \( \text{cuts}_{31} = \{ cc_<, ce=, cc_>, pc_l, pc_c, pc_r \} \) and \( \text{cuts}_{21} = \{ cc_<, ce=, cc_>, cp_l, cp_c, cp_r \} \);  

(2) **Case 2**: \( r_{31} = s_{21} = \text{l-par-to} \). This corresponds to \( r \in \text{l-par-to}_{31} \) and \( s \in \text{l-par-to}_{21} \), with \( \text{l-par-to}_{31} = \{ cp_l, pp_{l0}, pp_{l1}, pp_{l2}, pp_{c0}, pp_{r0} \} \) and \( \text{l-par-to}_{21} = \{ pc_l, pp_{l0}, pp_{l1}, pp_{l3}, pp_{l4} \} \);  

(3) **Case 3**: \( r_{31} = s_{21} = \text{coinc-with} \). This corresponds to \( r \in \text{coinc-with}_{31} \) and \( s \in \text{coinc-with}_{21} \), with \( \text{coinc-with}_{31} = \{ cp_c, pp_{l3}, pp_{c1}, pp_{r1} \} \) and \( \text{coinc-with}_{21} = \{ pc_c, pp_{c0}, pp_{c1}, pp_{c2} \} \); and  

(4) **Case 4**: \( r_{31} = s_{21} = \text{r-par-to} \). This corresponds to \( r \in \text{r-par-to}_{31} \) and \( s \in \text{r-par-to}_{21} \), with \( \text{r-par-to}_{31} = \{ cp_r, pp_{r4}, pp_{c2}, pp_{r2}, pp_{r3}, pp_{r4} \} \) and \( \text{r-par-to}_{21} = \{ pc_r, pp_{r0}, pp_{r1}, pp_{r2}, pp_{r3}, pp_{r4} \} \).

Figure 5 presents the four composition tables.  

4.2 The rotational component

It is important to insist at this point on the importance, for the translational component \( T.A_t \), of the oriented-ness of d-lines: if the objects we are dealing with were simple u-lines, we would not be able, when two lines both cut a third line, to say more than whether they cut it at the same point or at distinct points (specifically, when the cutting points are distinct, saying that one of the lines comes before the other would make no sense); similarly, we would only be able to say, when two lines are parallel, whether they coincide or not.

If we consider the rotational knowledge present in the \( T.A_t \) relations, i.e., the

\[ 2 \] Alternatively, one could define one single composition table for \( T.A_t \). Such a table would have \( 22 \times 22 \) entries, most of which (i.e., \( 22 \times 22 - (6 \times 6 + 6 \times 4 + 4 \times 6 + 6 \times 6) \)) would be the empty relation.
Fig. 5. (Top) the converse \( t^- \) and the rotation \( t^- \) for each \( \mathcal{T}_A \) atom \( t \); (Middle and Bottom) the \( \mathcal{T}_A \) composition tables (case 1, case 2, case 3 and case 4, respectively). \( pp_t \) and \( pp_{rr} \) stand, respectively, for \( \{pp_t, pp_{t1}, pp_{t2}\} \) and \( \{pp_{r2}, pp_{r3}, pp_{r4}\} \).
knowledge on the relative angles of the three arguments, we realise that this consists, for pairs \((x, y)\) of the three arguments, of knowledge of the form \(\langle x, y \rangle \in (0, \pi) \cup (\pi, 2\pi)\), inferable from \(x\) and \(y\) being cutting d-lines, or of the form \(\langle x, y \rangle \in \{\pi, 2\pi\}\), inferable from \(x\) and \(y\) being parallel d-lines. However, so restricting the rotational expressiveness would mean that we are not exploiting the oriented-ness of the d-line arguments. In other words, this would mean that we are using d-lines as if they were simple u-lines. The oriented-ness of d-lines, again, makes them much richer than u-lines, so that we can, for instance, say that a d-line is to the left of, or opposite to, another d-line; a level of relation granularity which cannot be reached using the universe of u-lines.

It should be clear that the relations cuts and par-to relate to the \(\text{CYC}_6\) relations in the following way. For all \(x, y \in \mathcal{L}\):

\[
\text{cuts}(x, y) \Leftrightarrow \{l, r\}(\mathcal{O}(x), \mathcal{O}(y))
\]

\[
\text{par-to}(x, y) \Leftrightarrow \{e, o\}(\mathcal{O}(x), \mathcal{O}(y))
\]

On the other hand, it is easy to see that the rotational information recorded by the four relations \(cc, cp, pc\) and \(pp\) can be expressed using the RA \(\text{CYC}_t\). Namely, for all \(x, y, z \in \mathcal{L}\):

\[
cc(x, y, z) \Leftrightarrow \phi_1(\mathcal{O}(x), \mathcal{O}(y), \mathcal{O}(z))
\]

\[
up(x, y, z) \Leftrightarrow \phi_2(\mathcal{O}(x), \mathcal{O}(y), \mathcal{O}(z))
\]

\[
pp(x, y, z) \Leftrightarrow \phi_3(\mathcal{O}(x), \mathcal{O}(y), \mathcal{O}(z))
\]

\[
pp(x, y, z) \Leftrightarrow \phi_4(\mathcal{O}(x), \mathcal{O}(y), \mathcal{O}(z))
\]

where \(\phi_1, \phi_2, \phi_3\) and \(\phi_4\) are the following \(\text{CYC}_t\) relations, defining a partition of the set \(\text{CYC}_t\)-at of all \(\text{CYC}_t\) atoms:

\[
\phi_1 = \{brl, kel, ill, llr, lor, brr, rrl, rol, rrr, rer, rlr\}
\]

\[
\phi_2 = \{brel, llo, rle, rro\}
\]

\[
\phi_3 = \{ell, err, orl, obl\}
\]

\[
\phi_4 = \{ee, eso, oeo, eeo\}
\]

The first two rows in Figure 2(a) illustrate the \(\text{CYC}_t\) atoms in \(\phi_1\), the third row illustrates the atoms in \(\phi_2\), the fourth row illustrates the atoms in \(\phi_3\), and the bottom row illustrates the atoms in \(\phi_4\).

In other words, the rotational expressiveness of what we have defined so far reduces to the four \(\text{CYC}_t\) relations \(\phi_1, \phi_2, \phi_3\) and \(\phi_4\) above. We augment the rotational component by using the whole RA \(\text{CYC}_t\). From now on, given a \(\text{CYC}_t\) relation \(R\) and three d-lines \(x, y\) and \(z\), we use the notation \(R(x, y, z)\) as a synonym to \(R(\mathcal{O}(x), \mathcal{O}(y), \mathcal{O}(z))\):

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\[(\forall R \in \text{Cyc})(\forall x, y, z \in \overrightarrow{L})(R(x, y, z) \Leftrightarrow R(O(x), O(y), O(z)))\]

4.3 The final algebra

From now on, we refer to the final algebra as \(\mathcal{P}A_t\) (Positional Algebra of ternary relations — over \(\overrightarrow{L}\)).

**Definition 7 (the \(\mathcal{P}A_t\) atoms)** (1) A \(\mathcal{TA}_t\) atom \(t\) is compatible with a \(\text{Cyc}_t\) atom \(r\), denoted by \(\text{comp}(t, r)\), if and only if there exists a configuration of three d-lines \(x, y\) and \(z\) such that both \(t(x, y, z)\) and \(r(x, y, z)\) hold; (2) a \(\mathcal{TA}_t\) atom \(t\) and a \(\text{Cyc}_t\) atom \(r\) such that \(\text{comp}(t, r)\) define a \(\mathcal{P}A_t\) atom, which we refer to as \(\langle t, r \rangle\).

Figure 6 considers one atom \(r\) for each of the four \(\text{Cyc}_t\) disjunctive relations \(\phi_1, \phi_2, \phi_3\) and \(\phi_4\), and illustrates all \(\mathcal{P}A_t\) atoms \(\langle t, r \rangle\) by considering all \(\mathcal{TA}_t\) atoms \(t\) that are compatible with \(r\). For each such \(r\):

1. the figure provides a spatial scene of three d-lines \(x, y\) and \(z\) satisfying \(r\); i.e., such that \(r(x, y, z)\); and
2. for each \(\mathcal{TA}_t\) atom \(t\) that is compatible with \(r\) — \(\langle t, r \rangle\) being therefore a \(\mathcal{P}A_t\) atom — the figure provides a spatial scene of three d-lines \(x, y\) and \(z\) satisfying \(\langle t, r \rangle\); i.e., such that \(\langle t, r \rangle(x, y, z)\).

More generally, each \(\mathcal{TA}_t\) atom in \(cc = \{cc_-, cc_-, cc_+\}\) (resp. \(cp = \{cp_1, cp_2, cp_3\}\), \(pp = \{pp_0, pp_1, pp_2, pp_3, pp_4, pp_5, pp_6, pp_7, pp_8, pp_9, pp_{10}, pp_{11}\}\)) is compatible with each \(\text{Cyc}_t\) atom in \(\phi_1\) (resp. \(\phi_2, \phi_3, \phi_4\)). Thus the set of all \(\mathcal{P}A_t\) atoms is

\[\mathcal{P}A_t-\text{at}=\{\langle t, r \rangle|\langle t \in cc \land r \in \phi_1 \rangle \lor \langle t \in cp \land r \in \phi_2 \rangle \lor \langle t \in pc \land r \in \phi_3 \rangle \lor \langle t \in pp \land r \in \phi_4 \rangle\}\]

The total number of \(\mathcal{P}A_t\) atoms is \(3 \times 12 + 3 \times 4 + 3 \times 4 + 13 \times 4 = 112\).

**Definition 8 (projection and cross product)** Let \(T\) be a \(\mathcal{TA}_t\) relation, \(R\) a \(\text{Cyc}_t\) relation, and \(S\) a \(\mathcal{P}A_t\) relation:

1. The translational projection, \(\nabla_t(S)\), and the rotational projection, \(\nabla_r(S)\), of \(S\) are the \(\mathcal{TA}_t\) relation and the \(\text{Cyc}_t\) relation, respectively, defined as follows:

\[\nabla_t(S) = \{t \in \mathcal{TA}_t-\text{at}|(\exists r \in \text{Cyc}_t-\text{at})(\langle t, r \rangle \in S)\}\]
\[\nabla_r(S) = \{r \in \text{Cyc}_t-\text{at}|(\exists t \in \mathcal{TA}_t-\text{at})(\langle t, r \rangle \in S)\}\]

2. The cross product, \(\Pi(T, R)\), of \(T\) and \(R\) is the \(\mathcal{P}A_t\) relation defined as follows:
Fig. 6. Each $\mathcal{T}A_t$ atom in $cc = \{cc_<, cc_=, cc_>\}$ is compatible with each $\mathcal{CYC}_t$ atom $r$ in $\phi_1 = \{lrl, lll, llr, lor, lrt, rll, rol, rrl, rrr, rer, rlr\}$ (see the top row for $r = lrl$); each $\mathcal{T}A_t$ atom in $cp = \{cp_l, cp_c, cp_r\}$ is compatible with each $\mathcal{CYC}_t$ atom $r$ in $\phi_2 = \{lre, llo, rle, rlo\}$ (see the second row from the top for $r = llo$); each $\mathcal{T}A_t$ atom in $pc = \{pc_l, pc_c, pc_r\}$ is compatible with each $\mathcal{CYC}_t$ atom $r$ in $\phi_3 = \{ell, err, orl, obr\}$ (see the third row from the top for $r = ell$); and each $\mathcal{T}A_t$ atom in $pp = \{pp_{l0}, pp_{l1}, pp_{l2}, pp_{l3}, pp_{l4}, pp_{c0}, pp_{c1}, pp_{c2}, pp_{r0}, pp_{r1}, pp_{r2}, pp_{r3}, pp_{r4}\}$ is compatible with each $\mathcal{CYC}_t$ atom $r$ in $\phi_4 = \{eee, eoo, oeo, oeo\}$ (see the last three rows from the top for $r = eoo$).
Fig. 7. Each $\mathcal{TA}_t$ atom in $cc$ is compatible with each $\mathcal{CYC}_t$ atom $r$ in $\phi_1$ (see the top pair of boxes for $t = cc$); each $\mathcal{TA}_t$ atom in $cp$ is compatible with each $\mathcal{CYC}_t$ atom $r$ in $\phi_2$ (see the second pair of boxes from the top for $t = cp$); each $\mathcal{TA}_t$ atom in $pc$ is compatible with each $\mathcal{CYC}_t$ atom $r$ in $\phi_3$ (see the third pair of boxes from the top for $t = pc$); and each $\mathcal{TA}_t$ atom in $pp$ is compatible with each $\mathcal{CYC}_t$ atom $r$ in $\phi_4$ (see the last pair of boxes from the top for $t = pp$).

$$\Pi(T, S) = \{\langle t, r \rangle \in \mathcal{PA}_{-at} | (t \in T) \land (r \in R)\}$$

The notation $\langle T, S \rangle$ will be used synonymously to $\Pi(T, S)$.

(3) $S$ is projectable if it is equal to the cross product of its translational projection and its rotational projection; i.e., if $S = \Pi(\nabla_t(S), \nabla_r(S))$. 
4.4 RDFs and TDFs of d-lines: their independance

Consider a CYC atom, say $r$, and three d-lines $X, Y$ and $Z$ such that $r(X,Y,Y)$. For all $TA$ atoms $t$ that are compatible with $r$, one can find d-lines $X', Y'$ and $Z'$ that are translations of $X, Y$ and $Z$, respectively, thus verifying $r(X',Y',Y')$, such that $t(X',Y',Y')$:

$$r(X',Y',Y') \land t(X',Y',Y')$$

This is illustrated in Figure 6 for each CYC atom $r$ in $\{brl, llo, ell, eoo\}$.

In a similar way, consider a TA atom, say $t$, and three d-lines $X, Y$ and $Z$ such that $t(X,Y,Y)$. For all CYC atoms $r$ that are compatible with $t$, one can find d-lines $Y'$ and $Z'$ that are rotations of $Y$ and $Z$, respectively, each about its intersecting point with $X$ thus verifying $r(X',Y',Y')$, such that $t(X',Y',Y')$:

$$r(X',Y',Y') \land t(X',Y',Y')$$

This is illustrated in Figure 7 for each CYC atom $t$ in $\{cc<, cp_t, pc_t, pp_{00}\}$.

4.5 The operations applied to the $PA$ atoms

The converse table, the rotation table and the composition tables of $CYC$ can be found in [35,36] (the converse table and the rotation table are reproduced in Figure 1). For $TA$, Figure 5 provides the converse table and the composition tables. Thus, thanks to the independence property discussed above, the converse and the composition of $PA$ atoms can be obtained from the converse and the composition of the atoms of the translational component, $TA$, and the converse and the composition of the rotational component, $CYC$. Namely, if $s_1 = \langle t_1, r_1 \rangle$ and $s_2 = \langle t_2, r_2 \rangle$ are two $PA$ atoms, then:

$$(s_1)\sim = \Pi((t_1)\sim, (r_1)\sim) = \langle (t_1)\sim, (r_1)\sim \rangle$$

$$s_1 \circ s_2 = \Pi(t_1 \circ t_2, r_1 \circ r_2) = \langle t_1 \circ t_2, r_1 \circ r_2 \rangle$$

As an example:

1. from $(pp_{00})\sim = pp_{t2}$ and $(ooe)\sim = eoo$, we get $\langle pp_{00}, ooe \rangle\sim = \langle pp_{t2}, eoo \rangle$;
2. from $cc_\prec \circ cc_\prec = cc_\prec$ and $rrl \circ lrr = lrr$, we get $\langle cc_\prec, rrl \rangle \circ \langle cc_\prec, lrr \rangle = \langle cc_\prec, lrr \rangle$; and
3. from $cc_\prec \circ cp_t = cp_t$ and $rrl \circ llo = rro$, we get $\langle cc_\prec, rrl \rangle \circ \langle cp_t, llo \rangle = \langle cp_t, rro \rangle$. 

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The $\mathcal{T}A_t$ relations take into account only the orientation of the first argument, which is sufficient, given the knowledge, summarised below, the algebra is supposed to represent:

(1) If the last two arguments both cut the first, the algebra is supposed to represent the order of the cutting points, in the walk along the first argument heading the positive direction.

(2) If one of the last two arguments is parallel to the first, the algebra is supposed to represent the side of the first argument (the left half-plane, the argument itself, or the right half-plane) the parallel d-line lies in.

(3) If the last two arguments are both parallel to the first, the algebra is supposed to represent the order in which the three arguments are encountered, when we walk perpendicularly to, from the left half-plane and heading towards the right half-plane bounded by, the first argument.

The orientations of the last two arguments are ignored by the $\mathcal{T}A_t$ relations. This has an effect on the computation of the rotations of the $\mathcal{T}A_t$ atoms, as explained below.

Composition records the relation one can infer on the triple $(x, y, t)$, given a relation $r_1$ on a triple $(x, y, z)$ and a relation $r_2$ on a triple $(x, z, t)$. For the particular case of $\mathcal{T}A_t$, since $r_1$ and $r_2$ hold on the triples $(x, y, z)$ and $(x, z, t)$, respectively, this means that the only orientation that is taken into consideration in the two relations is that of the common first argument, $x$; and since the relation $R$ we want to infer is on the triple $(x, y, t)$, which also has $x$ as the first argument, we can, just from the way $y$ and $z$, on the one hand, and $z$ and $t$, on the other hand, compare relative to $x$, easily infer how the extreme variables, $y$ and $t$, compare relative to the same variable $x$ (again, this is very similar to work done so far on temporal relations, such as point-point relations [59], interval-interval relations [1], and point-interval and interval-point relations [44,58]).

Similarly to composition, from a $\mathcal{T}A_t$ atom $r$ on a triple $(x, y, z)$, which, again, takes into consideration only the orientation of the first argument, $x$, the converse operation needs to find a relation $r^\sim$ on the triple $(x, z, y)$, which needs to take into consideration only the orientation of the first argument, which happens to be also $x$ (i.e., the same argument as the one $r$ takes into consideration).

Computing the composition and the converse for the $\mathcal{T}A_t$ atoms poses thus no problem. This is however not the case when considering the rotation operation. From a $\mathcal{T}A_t$ atom $r$ on $(x, y, z)$, which takes the orientation of $x$ into account, the operation needs to infer the relation $r^\frown$ on $(y, z, x)$, which needs, but cannot get from what is known, the orientation of the first argument, $y$. Instead of showing how to get the rotation of the $\mathcal{P}A_t$ atoms from the rotation of the
In a similar way, a \( \mathcal{TA} \) element rotation table recording for each \( \mathcal{TA} \) element of a line is a \( \mathcal{PA} \) relation of the \( \mathcal{PA} \), and the entry at the intersection of the line and the column records the rotation of the \( \mathcal{PA} \) atom \( \langle t, r \rangle \).

\( \mathcal{TA} \) atoms and the rotation of the \( \mathcal{CYC} \) atoms, which is possible but not as straightforward as for composition and converse, we preferred to draw a 122-element rotation table recording for each \( \mathcal{PA} \) atom \( \langle t, r \rangle \) its rotation \( \langle t, r \rangle \) (see Figure 8).

**Remark 1** A \( \mathcal{CYC} \) relation \( R \) is equivalent to the \( \mathcal{PA} \) relation consisting of all \( \mathcal{PA} \) atoms \( \langle r, t \rangle \) verifying \( r \in R \):

\[
R \equiv \{ \langle r, t \rangle \in \mathcal{PA} \text{ at}\mid r \in R \}
\]

In a similar way, a \( \mathcal{TA} \) relation \( T \) is equivalent to the \( \mathcal{PA} \) relation consisting...
of all $\mathcal{PA}_t$ atoms $\langle r, t \rangle$ verifying $t \in T$:

$$T \equiv \{\langle r, t \rangle \in \mathcal{PA}_t \text{-at} | t \in T\}$$

5 Reasoning about $\mathcal{PA}_t$ relations: $\mathcal{PA}_t$-CSPs

A $\mathcal{PA}_t$-CSP (resp. $\mathcal{CYC}_t$-CSP, $\mathcal{T}_A_t$-CSP) is a CSP \cite{46,47} of ternary constraints (ternary CSP), of which

(1) the variables range over the set $\mathcal{L}$ of d-lines; and
(2) the constraints consist of $\mathcal{PA}_t$ (resp. $\mathcal{CYC}_t$, $\mathcal{T}_A_t$) relations on (triples of) the variables.

A CSP of either of the three forms is said to be atomic if the entries of its constraint matrix all consist of atomic relations. A scenario is a refinement which is atomic. The translational (resp. rotational) projection, $\nabla_t(P)$ (resp. $\nabla_r(P)$), of a $\mathcal{PA}_t$-CSP $P$ is the $\mathcal{T}_A_t$-CSP (resp. $\mathcal{CYC}_t$-CSP) defined as follows:

(1) the variables are the same as the ones of $P$; and
(2) the constraint matrix of the projection is obtained by projecting the constraint matrix of $P$ onto $\mathcal{T}_A_t$ (resp. $\mathcal{CYC}_t$):

$$(\forall i, j, k)[(T^{\nabla_t(P)})_{ijk} = \nabla_t([T^P]_{ijk})]$$

$$(\text{resp. } (\forall i, j, k)[[T^{\nabla_r(P)}]_{ijk} = \nabla_r([T^P]_{ijk})])$$

The solution search algorithm in \cite{35,36} for $\mathcal{CYC}_t$-CSPs, which we refer to as $IC$-$sa$ algorithm, can be easily adapted so that it searches for a 4-consistent scenario of a $\mathcal{PA}_t$-CSP, if any, or otherwise reports inconsistency. The $IC$-$sa$ algorithm differs from Ladkin and Reinefeld’s \cite{41} in that:

(1) it refines the relation on a triple of variables at each node of the search tree, instead of the relation on a pair of variables; and
(2) it makes use of a procedure achieving 4-consistency, in the preprocessing step and as the filtering method during the search, instead of a path consistency procedure.

On the other hand, the 4-consistency procedure in \cite{35,36} for $\mathcal{CYC}_t$-CSPs, which we refer to as $IC$-$pa$ algorithm, can be adapted so that it achieves 4-consistency for a $\mathcal{PA}_t$-CSP. Such an adaptation would repeat the following steps until either the empty relation is detected (indicating inconsistency), or a fixed point is reached, indicating that the CSP has been made 4-consistent:
Consider a quadruple \((X_i, X_j, X_k, X_l)\) of variables verifying 
\((T^P)_{ijkl} \not\subseteq (T^P)_{ijjl} \circ (T^P)_{ijkl}\); 
(2) \((T^P)_{ijl} \leftarrow (T^P)_{ijl} \cap (T^P)_{ijk} \circ (B^P)_{ikl};\)
(3) if \((T^P)_{ijl} = \emptyset\) then exit (the CSP is inconsistent).

The reader is referred to [35,36] for more details on the IC-sa and IC-pa algorithms. IC-pa achieves 4-consistency for \(PA_t\)-CSPs.

**Theorem 1** Let \(P\) be a \(PA_t\)-CSP. Applied to \(P\), the IC-pa algorithm either detects that \(P\) is inconsistent, or achieves strong 4-consistency for \(P\).

**Proof.** Suppose that IC-pa [35,36] applied to \(P\) does not detect any inconsistency: we show that \(P\) has then been made strongly 4-consistent. The definition of composition for ternary relations implies that, if \(P\) is closed under the 4-consistency operation, \((T^P)_{ijl} \leftarrow (T^P)_{ijl} \cap (T^P)_{ijk} \circ (B^P)_{ikl},\) which is the case if \(P\) is 4-consistent, then any solution to any 3-variable sub-CSP extends to any fourth variable, as long as the composition computed from the composition tables matches the exact composition; i.e., as long as, given any two \(PA_t\) atoms, say \(r\) and \(s\), it is the case that \(r \circ s = T[r, s]\), where \(T[r, s]\) is the computed composition of \(r\) and \(s\) (computed, as we have seen, as the cross product of the composition of the translational projections of \(r\) and \(s\), on the one hand, and the composition of the rotational projections of \(r\) and \(s\), on the other hand). But this is the case since \(PA_t\) is an RA — see Appendix A. 

The important question now is whether the IC-pa algorithm in [35,36] is complete for atomic \(PA_t\)-CSPs. A positive answer would imply, on the one hand, that we can check complete knowledge, expressed in \(PA_t\) as an atomic CSP, in polynomial time; and, on the other hand, that a general CSP expressed in \(PA_t\) can be checked for consistency using the IC-sa solution search algorithm in [35,36]. We show that the answer is almost in the affirmative: completeness holds for a set \(S\), defined below, of \(PA_t\) JEPD relations including almost all of the \(PA_t\) atomic relations.

**Definition 9** \((S = S_1 \cup S_2)\) The set \(S\) of \(PA_t\) relations is defined as \(S = S_1 \cup S_2\), with the subsets \(S_1\) and \(S_2\) defined as follows:

1. \(S_1\) is the set of all \(PA_t\) atomic relations \(\{(t, s)\}\) holding on triples of d-lines involving at least two arguments that are parallel to each other —strictly parallel or coincide.
2. \(S_2\) is the set of all \(PA_t\) relations of the form \(\{(cc_<, s), (cc_=, s), (cc_>, s)\}\), where \(s\) is any CYC \(t\) atom from the set PairwiseCutting = \{lll, lrl, lrr, rll, rrl, rrr\}. Each element of PairwiseCutting has the property that it is compatible with and only with each \(TA_t\) atom from the set 2and3cut1 = \{cc_<, cc_=, cc_>\}. Therefore the set \(S_2\) can be written either as \(\{(cc_<, s), (cc_=, s), (cc_>, s)\}|s \in\) PairwiseCutting\}, or as a set of CYC \(t\) atomic relations, \(\{\{s\}|s \in\) PairwiseCutting\}. 

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The relations in $\mathcal{S}$ are JEPD; their particularity is that they can represent the knowledge consisting of the order in which two d-lines cut a third d-line only if the first two d-lines are parallel to each other. For triples of d-lines that are pairwise cutting (no two arguments are parallel to each other), the relations in $\mathcal{S}$ can represent only their rotational knowledge —namely, they cannot represent the order in which any two of the three arguments cut the third. To summarise, we have $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, with $\mathcal{S}_1 = \{\langle t, s \rangle | \langle t, s \rangle \in \mathcal{PA}_t \text{-at} \land s \notin \text{PairwiseCutting}\}$ and $\mathcal{S}_2 = \{\{cc_\prec, s\}, \{cc_\succ, s\}, \{cc_\succ_\prec, s\}\} | s \in \text{PairwiseCutting}\}$.

We refer to the subalgebra of $\mathcal{PA}_t$ generated by $\mathcal{S}$ as $c\mathcal{PA}_t$ (coarse $\mathcal{PA}_t$). Each $\mathcal{PA}_t$ atomic relation $\{\langle t, s \rangle\}$ from the set $\mathcal{S}_1$ gives rise to an atom of the RA $c\mathcal{PA}_t$: the atom $\langle t, s \rangle$, which is also an atom of $\mathcal{PA}_t$. Each $\mathcal{PA}_t$ relation $\{\{cc_\prec, s\}, \{cc_\succ, s\}, \{cc_\succ_\prec, s\}\}$ from the set $\mathcal{S}_2$ ($s \in \text{PairwiseCutting}$) gives rise to an atom of the RA $c\mathcal{PA}_t$: the atom $\langle *, s \rangle$, which is semantically the same as the $\text{CYC}_t$ atom $s$ (intuitively, the * symbol in the translational part of the atom says that the atom records no translational information —to say it differently, the information recorded by the atom is the same as that recorded by the $\text{CYC}_t$ atom appearing in the rotational part). The set $c\mathcal{PA}_t$-at of $c\mathcal{PA}_t$ atoms is thus $c\mathcal{PA}_t$-at $= \{\langle t, s \rangle | \langle t, s \rangle \in \mathcal{PA}_t$-at $\land s \notin \text{PairwiseCutting}\} \cup \{\langle *, s \rangle | s \in \text{PairwiseCutting}\}$.

The next theorem states tractability of the set $c\mathcal{PA}_t$-ar $= \{r | r \in c\mathcal{PA}_t$-at\}$ of $c\mathcal{PA}_t$ atomic relations.

**Theorem 2** Let $P$ be a $c\mathcal{PA}_t$-CSP expressed in the set $c\mathcal{PA}_t$-ar of atomic relations. If $P$ is 4-consistent then it is globally consistent.

The proof uses Helly’s convexity theorem [8].

**Theorem 3** (Helly’s theorem [8]) Let $\Gamma$ be a set of convex regions of the $m$-dimensional space $\mathbb{R}^m$. If every $m + 1$ elements of $\Gamma$ have a non empty intersection then the intersection of all elements of $\Gamma$ is non empty.

We now prove Theorem 2.

**Proof.** Let $P$ be a 4-consistent atomic $c\mathcal{PA}_t$-CSP, $P' = \nabla_t(P)$ and $P'' = \nabla_r(P)$. From Theorem 1, we get strong 4-consistency of $P$. From strong 4-consistency of $P$, we get strong 4-consistency of $P'$ and strong 4-consistency of $P''$. $P''$ being atomic and strongly 4-consistent, it is globally consistent [35,36]. Let $S = (d_1, \ldots, d_i, \ldots, d_n)$ be an instantiation of the variable $n$-tuple $(X_1, \ldots, X_i, \ldots, X_n)$ that is solution to $P''$. We can, and do, suppose that the $d_i$’s are d-lines through $O$ (see Definition 1). We now show that we can move the $d_i$’s around, so that the new instantiation of the variables is still solution to $P''$, and is solution to $P'$ (thus to $P$): one way, which is the way followed, to make the solution to $P''$ remain solution to $P'$ is to use only d-lines translation.
translating a d-line does not modify its orientation. So all that is needed, is to find the right translation, one that makes the solution to \( P' \) also solution to \( P \). Thanks to the property of independence of the translational component and the rotational component of a \( \mathcal{P}_t \) atom (see Subsection 4.4), \( \langle i, r \rangle \), it is the case that for all configurations \( \langle \ell_1, \ell_2, \ell_3 \rangle \) of three d-lines, such that \( r(\ell_1, \ell_2, \ell_3) \) holds, we can translate the \( \ell_i \)'s, \( i = 1 \ldots 3 \), relative to one another (again, a translation does not alter the rotational knowledge on the triple), so that the \( \mathcal{T}_t \) atom \( t \) holds on the triple \( \langle \ell'_1, \ell'_2, \ell'_3 \rangle \), where \( \ell'_1, \ell'_2 \) and \( \ell'_3 \) are, respectively, the translations of \( \ell_1, \ell_2 \) and \( \ell_3 \).

We go back to our rotational solution \( S = (d_1, \ldots, d_i, \ldots, d_n) \). We suppose that we have successfully translated \( d_1, \ldots, d_i \) \( i \geq 3 \), so that the new instantiation \( (d'_1, \ldots, d'_i) \) of the variable \( i \)-tuple \( (X_1, \ldots, X_i) \) is solution to the sub-CSP \( P'_{\{X_1, \ldots, X_i\}} \), thus to \( P_{\{X_1, \ldots, X_i\}} \). We show that we can translate \( d_{i+1} \), so that \( (d'_1, \ldots, d'_i, d''_{i+1}) \), where \( d''_{i+1} \) is the new instantiation of \( X_{i+1} \) resulting from the translation of \( d_{i+1} \), is solution to \( P'_{\{X_1, \ldots, X_i, X_{i+1}\}} \), thus to \( P_{\{X_1, \ldots, X_i, X_{i+1}\}} \). For this purpose, we suppose that the 2D space is associated with a system \((x, O, y)\) of coordinates. Without loss of generality, we assume that the \( x \)-axis, \( Ox \), (is parallel to, and) has the same orientation as the d-line \( d_{i+1} \).

As a consequence, all d-lines \( d'_j \), \( j \in \{1, \ldots, i\} \), that are parallel to \( d_{i+1} \), are curves of equation of the form \( y = q_j \), where \( q_j \) is a constant; furthermore, the equation of the d-line \( d''_{i+1} \) we are looking for, should be of the form \( y = q_{i+1} \), where \( q_{i+1} \) is a constant. The problem thus is to show that such a constant \( q_{i+1} \) can be found.

Given this, we can write as follows the constraints relating the d-line \( d''_{i+1} \) we are looking for, to the d-lines \( d'_1, \ldots, d'_i \), constituting the assignments of the variables already consistently instantiated, both rotationally and translationally. Initialise \( S \) to the empty set; then for all \( j, k \in \{1, \ldots, i\} \), with \( j \leq k \):

1. if \( (\mathcal{T}_r)_{(i+1)jk} = \{\langle *, s \rangle\} \) (with \( s \in PairwiseCutting \)). According to the definition of the \( c\mathcal{P}_t \) atoms, the * symbol refers to the \( \mathcal{T}_t \) relation \( cc = \{cc_<, cc_=, cc_>\} \), consisting of all \( \mathcal{T}_t \) atoms that are compatible with each of the \( \mathcal{C}Y\mathcal{C}_t \) atoms from the set \( PairwiseCutting \): \( cc(d'_{i+1}, d'_j, d'_k) \) (see the illustration of Figure 9(a)). All translations \( d''_{i+1} \) of \( d_{i+1} \) satisfy the constraint \( cc(d'_{i+1}, d'_j, d'_k) \), thus \( (\mathcal{T}_r)_{(i+1)jk}(d''_{i+1}, d'_j, d'_k) \), since they already satisfy the rotational constraint \( s(d''_{i+1}, d'_j, d'_k) \) — nothing is added to \( S \);
2. if \( (\mathcal{T}_r)_{(i+1)jk} = \{\langle cp_m, s \rangle\} \), with \( m \in \{l, c, r\} \), \( s \notin PairwiseCutting \) (see the illustration of Figure 9(b) for \( m = l \)), then the d-line \( d''_{i+1} \) we are looking for, should be so that \( d''_k \) is parallel to \( d''_{i+1} \), and lies within the left open half-plane bounded by \( d''_{i+1} \) if \( m = l \), coincides with \( d''_{i+1} \) if \( m = c \), or lies within the right open half-plane bounded by \( d''_{i+1} \) if \( m = r \). The translational sub-constraint \( cp_m(d''_{i+1}, d'_j, d'_k) \) is equivalent to the constraint that the equation of \( d''_{i+1} \) should be of the form \( y = q_{i+1} \), with
Fig. 9. Illustration of the proof of Theorem 2: (a) all translations $d_{i+1}^+$ of $d_{i+1}$ satisfy the constraint $cc(d_{i+1}^+, d_j^+, d_k^+)$; (b) the constraint $cp(d_{i+1}^+, d_j^+, d_k^+)$ is equivalent to the constraint that the equation of $d_{i+1}^+$ should be of the form $y < q_k$, $y = q_k$ or $y > q_k$, depending on whether $m = l$, $m = c$ or $m = r$, respectively.

$q_{i+1}$ being a constant, and $q_{i+1} < q_k$, $q_{i+1} = q_k$ or $q_{i+1} > q_k$, depending on whether $m = l$, $m = c$ or $m = r$, respectively. Add to $S$ the corresponding equivalent linear inequality:

$$S \leftarrow \begin{cases} S \cup \{q_{i+1} \in (-\infty, q_k)\} & \text{if } m = l, \\ S \cup \{q_{i+1} \in \{q_k\}\} & \text{if } m = c, \\ S \cup \{q_{i+1} \in (q_k, +\infty)\} & \text{if } m = r. \end{cases}$$

(3) if $(T^P)_{(i+1)jk} = \{pc_m, s\}$, with $m \in \{l, c, r\}$, $s \notin PairwiseCutting$. The translational sub-constraint $pc_m(d_{i+1}^+, d_j^+, d_k^+)$ is equivalent to $cp_m(d_{i+1}^+, d_j^+, d_k^+)$, obtained by swapping $d_j^+$ and $d_k^+$ and replacing the atom $pc_m$ by its converse, $cp_m$. In a similar way as in the previous point, we add to $S$ the equivalent linear inequality:
Consider now two elements $q_{i+1} \in S_1$ and $q_{i+1} \in S_2$ of $S$, and suppose that the sets $S_1$ and $S_2$ have an empty intersection. There would then exist $j, k \in \{1, \ldots, i\}$ such that $S_1 \in \{(-\infty, q_j), \{q_j\}, (q_j, +\infty)\}$, $S_2 \in \{(-\infty, q_k), \{q_k\}, (q_k, +\infty)\}$ and $S_1 \cap S_2 = \emptyset$. From the construction of $S$, we get that there exist $j_1, k_1 \in \{1, \ldots, i\}$ such that $(T^p)_{(i+1)j_1}(d^+_{j_1}, d'_j, d_k') \Rightarrow q_{i+1} \in S_1$ and $(T^p)_{(i+1)k_1}(d^+_{k_1}, d'_k, d'_k) \Rightarrow q_{i+1} \in S_2$. Because the elements of $\{(-\infty, q_j), \{q_j\}, (q_j, +\infty)\}$, on the one hand, and the elements of $\{(-\infty, q_k), \{q_k\}, (q_k, +\infty)\}$, on the other hand, are jointly exhaustive (their union gives the whole set of real numbers) and pairwise disjoint, and the CSP $P'$ strongly 4-consistent, it must be the case that for all $l \in \{1, \ldots, i\}$, $(T^p)_{(i+1)j_1}(d^+_{j_1}, d'_j, d'_l) \Rightarrow q_{i+1} \in S_1$ and $(T^p)_{(i+1)k_1}(d^+_{k_1}, d'_k, d'_l) \Rightarrow q_{i+1} \in S_2$. Strong 4-consistency of the CSP implies that it must be the case that $S_1$ and $S_2$ have a nonempty intersection.
Let $S'$ be the set of all elements of the form $A$ such that $q_{i+1} \in A$ is element of $S$:

$$S' = \{ A | \{q_{i+1} \in A \} \subseteq S \}$$

The point now is that the elements of $S'$ are convex subsets of the set $\mathbb{R}$ of real numbers. The elements of $S'$ being pairwise intersecting, Helly's theorem [8] specialised to $m = 2$ (see Theorem 3) implies that the intersection of all elements in $S'$ is non empty. Any translation $d_{i+1}'$ of $d_{i+1}$ of equation $y = q_{i+1}$, with $q_{i+1}$ being a constant from $S'$, would make the tuple $(x_i', \ldots, x_i', x_{i+1}'$) solution to $P'(\{x_1', \ldots, x_i, x_{i+1}\})$, thus to $P(\{x_1', \ldots, x_i, x_{i+1}\})$. \[\qed\]

Given that the $c\mathcal{P}A_t$ atomic relations are tractable, and that an atomic $c\mathcal{P}A_t$-CSP can be be solved using the $IC$-$pa$ propagation algorithm in [35,36], it follows that a general $c\mathcal{P}A_t$-CSP can be solved using the $IC$-$sa$ search algorithm also in [35,36], alluded to before.

**Corollary 1** Let $P$ be a $c\mathcal{P}A_t$-CSP. The consistency problem of $P$ can be solved using the $IC$-$sa$ search algorithm in [35,36].

### 6 Use of the RA $\mathcal{P}A_t$

The RA $\mathcal{P}A_t$ is clearly well suited for applications such as robot localisation and navigation, Geographical Information Systems (GIS), and shape description. First, thanks to the objects the RA deals with, namely, d-lines: such an object is much richer than a simple u-line, because it does not consist only of a support (e.g., “the support of the current robot’s motion is the line University-TrainStation”), but also of the important feature of orientation, which allows, in the particular case of robot navigation, of representing, in addition of the motion’s support, the motion’s direction (e.g., “the current robot’s motion is supported by, and is of the same orientation as, the d-line University-TrainStation”). Then, thanks to the kind of relations on the handled objects; the strength of the relations comes from their two features, a rotational feature and a translational feature: the former handles the RDFs of the represented objects, the latter their TDFs:

1. The rotational feature allows for the representation of statements such as “parallel to, and of same/opposite orientation as”, or “cuts, and to the left/right of”. As we saw, this feature corresponds to what Isli and Cohm’s RA $C\mathcal{CYC}_t$ [35,36] can express.
2. The translational feature allows for the representation of statements such as “$\ell_1$ is parallel to, and lies strictly to the left of”, or “$\ell_2$ cuts $\ell_1$ before $\ell_3$ does”, or (and this is an important disjunctive relation!) “$\ell_3$ is parallel
to both, and does not lie between, $\ell_1$ and $\ell_2$”. This last statement is represented as follows:\footnote{The representation of the other statements is left to the reader.} $\footnote{\ell_1$ and $\ell_2$ may coincide, in which case the statement “$\ell_3$ is parallel to both, and does not lie between, $\ell_1$ and $\ell_2$” is synonymous of “$\ell_3$ is parallel to both, and does not coincide with, $\ell_1$ and $\ell_2$”, represented by the subformula $\{pp_{c0}, pp_{c2}\}(\ell_1, \ell_2, \ell_3)$.}$  

$$\{pp_{l0}, pp_{l4}, pp_{c0}, pp_{c2}, pp_{r0}, pp_{r4}\}(\ell_1, \ell_2, \ell_3)$$

We discuss below some of the potential application areas of the presented work.

![Diagram](image1.png)

**Fig. 10.** Illustration of the use of the RA $\mathcal{P}\mathcal{A}_t$ — incidence geometry.

### 6.1 Incidence geometry

2D incidence geometry \cite{4} deals with the universe of points and (directed) lines. Incidence (of a point with a line), betweenness (of three points, but also betweenness of three parallel d-lines), and non-collinearity (of three points) are easily representable in the RA $\mathcal{P}\mathcal{A}_t$: 

1. A point $P$ will be considered as the intersection of two d-lines $\ell_1$ and $\ell_2$, such that $\text{cuts}(\ell_2, \ell_1)$ and $l(\ell_2, \ell_1) = \ell_1$ and $\ell_2$ are cutting d-lines and $\ell_2$ is to the left of $\ell_1$ (see Figure 10(a)). Transforming the conjunction $\ell_3$ is parallel to both, and does not coincide with, $\ell_1$ and $\ell_2$", represented by the subformula $\{pp_{c0}, pp_{c2}\}(\ell_1, \ell_2, \ell_3)$.

$\footnote{A d-line $\ell_2$ is between d-lines $\ell_1$ and $\ell_3$ if and only if $\ell_2$ is parallel to both, and lies between, $\ell_1$ and $\ell_3$.}$
cuts(ℓ₂, ℓ₁) ∧ l(ℓ₂, ℓ₁) into the RA PA, we get ⟨cp_c, br⟩(ℓ₁, ℓ₂, ℓ₁). We refer to the pair (ℓ₁, ℓ₂) as the PA representation of P, and denote it by ψ(P): ψ(P) = (ℓ₁, ℓ₂).

(2) Let P be a point such that ψ(P) = (ℓ₁, ℓ₂), and ℓ a d-line. Incidence of P with ℓ, inc-with(P, ℓ), is represented in PA as {cc, cp_c, pc_c}(ℓ₁, ℓ₂), saying that the three d-lines ℓ, ℓ₁ and ℓ₂ are concurrent.

(3) Let P₁, P₂ and P₃ be three points such that ψ(P₁) = (ℓ₁, ℓ₂), ψ(P₂) = (ℓ₃, ℓ₄) and ψ(P₃) = (ℓ₅, ℓ₆). P₂ is between P₁ and P₃ can be represented using four d-lines ℓₐ, ℓₖ, ℓₐ, ℓ₆ on which we impose the constraints that (see Figure 10(b) for illustration):⁶

(a) ℓₖ is parallel to both, and lies between, ℓₐ and ℓ₆; and
(b) ℓ₆ cuts ℓₐ at P₁, ℓₖ at P₂ and ℓ₆ at P₃.

Statement 3a defines betweenness of parallel d-lines, and is represented as btw_dd(ℓₐ, ℓₖ, ℓₖ) = {pp₁₀, pp₁₁, pp₆₀, pp₆₁, pp₆₂, pp₆₃, pp₆₄}(ℓₐ, ℓₖ, ℓₖ), which splits into:

• {pp₁₀, pp₁₁}(ℓₐ, ℓₖ, ℓₖ), corresponding to ℓₖ being strictly to the left of ℓₐ;
• {pp₆₀, pp₆₁, pp₆₂}(ℓₐ, ℓₖ, ℓₖ), corresponding to ℓₖ coinciding with ℓₐ; and
• {pp₆₃, pp₆₄}(ℓₐ, ℓₖ, ℓₖ), corresponding to ℓₖ being strictly to the right of ℓₐ.

Statement 3b is represented using the previous point on incidence of a point with a line. Namely:

- the substatement “ℓ₆ cuts ℓₐ at P₁” is represented as

  \[
  \text{inc-with}(P₁, ℓₐ) ∧ \text{inc-with}(P₁, ℓ₆) \equiv \{cc, cp_c, pc_c\}(ℓₐ, ℓ₁, ℓ₂) ∧ \{cc, cp_c, pc_c\}(ℓ₆, ℓ₁, ℓ₂);
  \]

- the substatement “ℓ₆ cuts ℓₖ at P₂” as

  \[
  \text{inc-with}(P₂, ℓₖ) ∧ \text{inc-with}(P₂, ℓ₆) \equiv \{cc, cp_c, pc_c\}(ℓₖ, ℓ₃, ℓ₄) ∧ \{cc, cp_c, pc_c\}(ℓ₆, ℓ₃, ℓ₄);
  \]

and

- the substatement “ℓ₆ cuts ℓₖ at P₃” as

  \[
  \text{inc-with}(P₃, ℓₖ) ∧ \text{inc-with}(P₃, ℓ₆) \equiv \{cc, cp_c, pc_c\}(ℓₖ, ℓ₅, ℓ₆) ∧ \{cc, cp_c, pc_c\}(ℓ₆, ℓ₅, ℓ₆).
  \]

Putting everything together, betweenness of P₁, P₂ and P₃, btw_p(P₁, P₂, P₃), is represented as follows:

\[
\text{btw_p}(P₁, P₂, P₃) \equiv \{pp₁₀, pp₁₁, pp₆₀, pp₆₁, pp₆₂, pp₆₃, pp₆₄\}(ℓₐ, ℓₖ, ℓₖ) ∧ \]

⁶ We represent here large betweenness of P₁, P₂ and P₃, in the sense that P₂ coincides with P₁, lies strictly between P₁ and P₃, or coincides with P₃.
\{cc_\prec, cc_\succ\}(\ell_a, \ell_b, \ell_c) \land \{cc_\prec, cc_\succ\}(\ell_a, \ell_1, \ell_2) \land \\
\{cc_\prec, cp_c, pc_c\}(\ell_a, \ell_1, \ell_2) \land \{cc_\prec, cp_c, pc_c\}(\ell_1, \ell_1, \ell_2) \land \\
\{cc_\prec, cp_c, pc_c\}(\ell_a, \ell_3, \ell_4) \land \{cc_\prec, cp_c, pc_c\}(\ell_b, \ell_3, \ell_4) \land \\
\{cc_\prec, cp_c, pc_c\}(\ell_c, \ell_3, \ell_4) \land \{cc_\prec, cp_c, pc_c\}(\ell_3, \ell_3, \ell_4) \land \\
\{cc_\prec, cp_c, pc_c\}(\ell_b, \ell_5, \ell_6) \land \{cc_\prec, cp_c, pc_c\}(\ell_c, \ell_5, \ell_6)

(4) Let \(P_1, P_2\) and \(P_3\) as in the previous point: \(\psi(P_1) = (\ell_1, \ell_2), \psi(P_2) = (\ell_3, \ell_4)\) and \(\psi(P_3) = (\ell_5, \ell_6)\). Non-collinearity of the three points, \(\text{non-coll}(P_1, P_2, P_3)\), can be represented using three \(d\)-lines \(\ell_a, \ell_b, \ell_c\) on which we impose the constraints that (see Figure 10(c) for illustration):
(a) \(\ell_b\) and \(\ell_c\) both cut \(\ell_a\), but at distinct points;
(b) \(P_1\) is incident with each of \(\ell_a\) and \(\ell_b\), \(P_2\) with each of \(\ell_a\) and \(\ell_c\), and \(P_3\) with each of \(\ell_b\) and \(\ell_c\).
We get:
\[
\text{non-coll}(P_1, P_2, P_3) \equiv \{cc_\prec, cc_\succ\}(\ell_a, \ell_b, \ell_c) \land \\
\text{inc-with}(P_1, \ell_a) \land \text{inc-with}(P_1, \ell_b) \land \\
\text{inc-with}(P_2, \ell_a) \land \text{inc-with}(P_2, \ell_c) \land \\
\text{inc-with}(P_3, \ell_b) \land \text{inc-with}(P_3, \ell_c)
\]
Translating the incidence relation into the RA \(\mathcal{PA}\), we get:
\[
\text{non-coll}(P_1, P_2, P_3) \equiv \{cc_\prec, cc_\succ\}(\ell_a, \ell_b, \ell_c) \land \\
\{cc_\prec, cp_c, pc_c\}(\ell_a, \ell_1, \ell_2) \land \{cc_\prec, cp_c, pc_c\}(\ell_1, \ell_1, \ell_2) \land \\
\{cc_\prec, cp_c, pc_c\}(\ell_a, \ell_3, \ell_4) \land \{cc_\prec, cp_c, pc_c\}(\ell_b, \ell_3, \ell_4) \land \\
\{cc_\prec, cp_c, pc_c\}(\ell_c, \ell_3, \ell_4) \land \{cc_\prec, cp_c, pc_c\}(\ell_3, \ell_3, \ell_4) \land \\
\{cc_\prec, cp_c, pc_c\}(\ell_b, \ell_5, \ell_6) \land \{cc_\prec, cp_c, pc_c\}(\ell_c, \ell_5, \ell_6)
\]

6.2 Geographical Information Systems

The objects manipulated by GIS applications are mainly points, segments and proper polygons (more than two sides) of the 2-dimensional space. A general (for instance, concave) polygon can always be decomposed into a union of conex polygons —see, for instance, the work in [7], where a system answering queries on the RCC-8 [51] relation between two input (polygonal) regions of a geographical database is defined.

In order to use the RA \(\mathcal{PA}\) to reason about polygons, we need to provide a representation of convex polygons, not in terms of an ordered, say anticlockwise, list of vertices, of the form \((X_1, \ldots, X_n)\), \(n \geq 1\), but in terms of an ordered, clockwise, list of \(d\)-lines, of the form \((\ell_1, \ldots, \ell_n)\), \(n \geq 1\). A ordered, anticlockwise, list of \(d\)-lines is a list \((\ell_1, \ldots, \ell_n)\), \(n \geq 1\), of \(d\)-lines, such that the list \((\ell'_1, \ldots, \ell'_n)\), obtained by translating each of the \(\ell_i\)'s so that it contains a fixed point \(O\), verifies the following: the positive half-lines of \(\ell'_1, \ldots, \ell'_n\) bounded by \(O\) are met in that order when scanning a circle, say \(C\), centered at \(O\), starting from the intersecting point of \(C\) with the positive half-line of \(\ell'_1\) bounded by \(O\).
soning about a collection of convex polygons transforms then into reasoning about the d-lines in the representations of the different convex polygons in the collection.

Points and (directed) line segments are special cases of convex polygons:

1. We have already seen how to represent a point \( P \) as the intersection of two d-lines \( \ell_1 \) and \( \ell_2 \), such that \( \text{cuts}(\ell_2, \ell_1) \) and \( l(\ell_2, \ell_1) \), which transforms into the RA \( \mathcal{PA}_t \) as \( \{(\text{cp}_{\ell_2, \ell_1})\}(\ell_1, \ell_2, \ell_1) \). The pair \((\ell_1, \ell_2)\) is referred to as the RA \( \mathcal{PA}_t \) representation of \( P \), denoted \( \psi(P) = (\ell_1, \ell_2) \) —see Figure 10(a).

2. A segment \( S = (X_1, X_2) \) will be represented using three d-lines \( \ell_1, \ell_2 \), and \( \ell_3 \), such that \( \ell_2 \) and \( \ell_3 \) are parallel to each other; \( \ell_3 \) lies within the left half-plane bounded by \( \ell_2 \); and \( \ell_2 \) is to the left of, and \( \ell_3 \) to the right of, \( \ell_1 \) (the segment is then the part of \( \ell_1 \) between the intersecting points with the other two d-lines, oriented from the intersecting point with \( \ell_3 \) to the intersecting point with \( \ell_2 \)). An illustration is provided in Figure 11(a). We get the following: \( \{(\text{cc}_{\ell_2, \ell_1})\}(\ell_1, \ell_2, \ell_3) \). We refer to the triple \((\ell_1, \ell_2, \ell_3)\) as the RA \( \mathcal{PA}_t \) representation of \( S \), which we denote by \( \psi(S) = (\ell_1, \ell_2, \ell_3) \).

A convex polygon with \( p \) vertices, with \( p \geq 3 \), will be represented as the \( p \)-tuple \((\ell_1, \ell_2, \ldots, \ell_p)\) of d-lines, such that, for all \( i = 1 \ldots p \), the d-lines \( \ell_i, \ell_{i+p+1} \) and \( \ell_{i+p+2} \) verify the following:

1. \( \ell_i \) and \( \ell_{i+p+2} \) both cut \( \ell_{i+p+1} \), but \( \ell_i \) does it before \( \ell_{i+p+2} \);
2. \( \ell_{i+p+1} \) is to the left of \( \ell_i \); and
3. \( \ell_{i+p+2} \) is to the left of \( \ell_{i+p+1} \),

where \(+p\) is cyclic addition over the set \( \{1, \ldots, p\} \); i.e.

\[
i + p \begin{cases} i + 1 & \text{if } i \leq p - 1, \\ 1 & \text{if } i = p \end{cases}, \quad i + p \begin{cases} i + 2 & \text{if } i \leq p - 2, \\ 1 & \text{if } i = p - 1, \\ 2 & \text{if } i = p \end{cases}
\]

The polygon is then the contour of the surface consisting of the intersection of the \( p \) left half-planes delimited by the d-lines \( \ell_1, \ldots, \ell_p \). The conjunction of the three points 1, 2 and 3 just above translates into the RA \( \mathcal{PA}_t \) as follows:

\[\{(\text{cc}_{\ell_1, \ell_2}), (\text{cc}_{\ell_1, \ell_3}), (\text{cc}_{\ell_2, \ell_3})\}(\ell_{i+p+1}, \ell_i, \ell_{i+p+2})\]

—see the illustration of Figure 11(b-c-d).
Fig. 11. (Left) a (directed) line segment $S = (X_1, X_2)$ is represented in the RA $\mathcal{PA}_t$ using three d-lines $\ell_1$, $\ell_2$ and $\ell_3$, such that $\ell_2$ is to the left of, and $\ell_3$ to the right of, $\ell_1$; and $\ell_3$ is parallel to, and lies within the left half-plane bounded by, $\ell_2$ ($X_1$ is then the intersecting point of $\ell_1$ with $\ell_3$; and $X_2$ the intersecting point of $\ell_1$ with $\ell_2$): $\langle \langle cc_>, l0r \rangle \rangle (\ell_1, \ell_2, \ell_3)$. (Right) a $p$-vertex convex polygon, given as an ordered, anticlockwise, list $(X_1, X_2, X_3, \ldots, X_p)$ of $p$ vertices, $p \geq 3$, is represented in the RA $\mathcal{PA}_t$ as an ordered, anticlockwise, list $(\ell_1, \ell_2, \ell_3, \ldots, \ell_p)$ of $p$ d-lines, such that every three consecutive d-lines $\ell_i$, $\ell_{i+1}$ and $\ell_{i+2}$ verify $\langle \langle cc_>, rll \rangle \rangle (\ell_{i+1}, \ell_i, \ell_{i+2})$ (see illustration (b) for $i = 1$), $\langle \langle cc_>, rol \rangle \rangle (\ell_{i+1}, \ell_i, \ell_{i+2})$ (see illustration (c) for $i = 1$) or $\langle \langle cc_>, rrl \rangle \rangle (\ell_{i+1}, \ell_i, \ell_{i+2})$ (see illustration (d) for $i = 1$): $\langle \langle cc_>, rll \rangle, \langle cc_>, rol \rangle, \langle cc_>, rrl \rangle \rangle (\ell_{i+1}, \ell_i, \ell_{i+2})$.

6.3 (Polygonal) shape representation

In shape representation, the shapes dealt with are mostly polygonal; when they are not, they are generally given polygonal approximations (a circle, for instance, can be so approximated).

Example 1

To illustrate the use of the RA $\mathcal{PA}_t$ for polygonal shape representation, we consider a first example illustrated by the shape of Figure 12(top), representing a table composed of three parallelogram-like parts, $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$. The parts $\mathcal{P}_2$ and $\mathcal{P}_3$ constitute the base of the table, i.e., the part reposing on the ground, and holding the top part, represented by $\mathcal{P}_1$. The side AB of the upper part
is collinear with the diagonal EG of part $\mathcal{P}_2$. The vertex C of part $\mathcal{P}_1$ comes strictly inside the side IL of part $\mathcal{P}_3$. Finally, the non-horizontal sides of parts $\mathcal{P}_2$ and $\mathcal{P}_3$ are pairwise parallel.

With each side XY of the three table parts we associate a d-line $L_{xy}$, as indicated in Figure 12(bottom). Part $\mathcal{P}_1$ of the table is then the surface consisting of the intersection of the left half-planes bounded by the d-lines $\ell_{ab}$, $\ell_{bc}$, $\ell_{cd}$ and $\ell_{da}$; Part $\mathcal{P}_2$ is the surface consisting of the intersection of the left half-planes bounded by the d-lines $\ell_{ef}$, $\ell_{fg}$, $\ell_{gh}$ and $\ell_{he}$; and Part $\mathcal{P}_3$ is the surface consisting of the intersection of the left half-planes bounded by the d-lines $\ell_{ij}$, $\ell_{jk}$, $\ell_{kl}$ and $\ell_{li}$.

(1) Part $\mathcal{P}_1$ reposes on parts $\mathcal{P}_2$ and $\mathcal{P}_3$, which means that (the supports of) the three d-lines $L_{bc}$, $L_{he}$ and $L_{li}$ coincide:

$$\{\langle pp_{e1}, oeo \rangle \}(L_{bc}, L_{he}, L_{li})$$

(2) The d-lines $L_{fg}$ and $L_{jk}$, which constitute the base of the table, coincide; and are both parallel to, and lie within the right open half-plane bounded by, $L_{bc}$:

$$\{\langle pp_{r3}, eee \rangle \}(L_{bc}, L_{fg}, L_{jk})$$

(3) The d-line $L_{da}$ is parallel to, lies within the left open half-plane bounded by, and is of opposite orientation than, the d-line $L_{bc}$:

$$\{\langle pp_{l1}, oeo \rangle \}(L_{bc}, L_{da}, L_{da})$$

Fig. 12. Shape representation —example 1.
(4) Parallelity of the non-horizontal sides of part $\mathcal{P}_1$ can be expressed thus:

\[
\{\langle pp_{l1}, oeo \rangle\}(L_{ab}, L_{cd}, L_{cd})
\]

(5) Pairwise parallelity of the non-horizontal sides of parts $\mathcal{P}_2$ and $\mathcal{P}_3$ can be expressed thus:

\[
\{\langle pp_{l0}, ooe \rangle\}(L_{ef}, L_{gh}, L_{ij}) \land \{\langle pp_{r4}, ooe \rangle\}(L_{gh}, L_{ij}, L_{kl})
\]

(6) Collinearity of side $AB$ with diagonal $EG$ of part $\mathcal{P}_2$ can be expressed by concurrency of d-lines $L_{ab}$, $L_{bc}$ and $L_{ef}$, on the one hand, and concurrency of d-lines $L_{ab}$, $L_{fg}$ and $L_{gh}$, on the other hand:

\[
\{\langle cc_-, lrr \rangle\}(L_{ab}, L_{bc}, L_{ef}) \land \{\langle cc_-, lll \rangle\}(L_{ab}, L_{fg}, L_{gh})
\]

(7) Finally, strict betweenness of vertices $I$, $C$ and $L$ ($C$ strictly between $I$ and $L$) is expressed by the conjunction of “$L_{cd}$ cuts $L_{li}$ before $L_{ij}$ does” and “$L_{cd}$ cuts $L_{li}$ after $L_{kl}$ does”. This translates into the RA $\mathcal{PA}_t$ as follows:

\[
\{\langle cc_-, rll \rangle\}(L_{li}, L_{cd}, L_{ij}) \land \{\langle cc_-, rrr \rangle\}(L_{li}, L_{cd}, L_{kl})
\]

**Example 2**

As a second example, we consider the three polygonal shapes of Figure 13. Similarly to the previous example, we associate with each side, say $XY$, a d-line $L_{xy}$. The $\mathcal{PA}_t$ algebra is able to distinguish between the three shapes: for all triples $(\ell_1, \ell_2, \ell_3)$ of d-lines not involving the d-line $L_{CD}$, the $\mathcal{PA}_t$ relation on $(\ell_1, \ell_2, \ell_3)$ is the same for all three shapes; the relation on any of the triples involving the d-line $L_{CD}$, however, differs from any of the three shapes to any of the other two. For instance, if we consider the triple $(L_{ab}, L_{bc}, L_{cd})$, we get the relation $\{\langle cc_>, rll \rangle\}$ for the shape of Figure 13(a); the relation $\{\langle cp_r, rle \rangle\}$ for the shape of Figure 13(b); and the relation $\{\langle cc_-, rll \rangle\}$ for the shape of Figure 13(c).
6.4 Localisation in multi-robot navigation

Self-localisation of a robot, embedded in an environment with \( n \) landmarks, consists of describing the panorama of the robot w.r.t. the landmarks; i.e., how the different landmarks are situated relative to one another, as viewed from the current robot’s position (the robot is supposed equipped with a camera). The standard way of representing such a panorama is to give the (cyclic) order in which the landmarks appear in a 360-degrees anticlockwise turn, starting, say, from landmark 1 (the landmarks are supposed numbered from 1 to \( n \)). If we use the d-lines relating the robot to the different landmarks, then the problem can be represented using the RA \( \mathcal{CYC}_t \) [36], by providing for each triple \((\ell_1, \ell_2, \ell_3)\) of the d-lines the \( \mathcal{CYC}_t \) relation it satisfies. For instance, if the three d-lines \( \ell_1, \ell_2 \) and \( \ell_3 \) appear in that order in a 360-degrees anticlockwise turn about the robot’s location, the situation can be described using the \( \mathcal{CYC}_t \) relation

\[
\text{cyc} = \{lrl, orl, rll, rol, rrl, rro, rrr\},
\]

expressing anticlockwise betweenness: \( \text{cyc}(\ell_1, \ell_2, \ell_3) \).\(^8\)

Consider now the situation depicted in Figure 14, with two robots, \( R_a \) and \( R_b \), embedded in a three-landmark environment. As long as we are only concerned with the panorama of one of the two robots, say \( R_a \), we can use the RA \( \mathcal{CYC}_t \)

---

\(^8\) This can be easily checked using the illustrations of the different \( \mathcal{CYC}_t \) atoms, given in Figure 2(a).
to represent it, by providing for each triple of the four d-lines \( \ell_{a1}, \ell_{a2}, \ell_{a3} \) and \( \ell_{ab} \), connecting, respectively, the robot \( R_a \) to the three landmarks \( L_1, L_2 \) and \( L_3 \) and to the robot \( R_b \), the order in which they appear in a 360-degrees anticlockwise turn about \( R_a \)'s location. Because the two robots are embedded in a same environment, it is clearly unrealistic to consider only the panorama of one of them. The knowledge should thus consist of the conjuction of the panoramas of both robots, providing thus the way each of the two robots sees “its environment” (which includes the other robot). We thus need to consider the d-lines connecting each of the two robots to each of the three landmarks, and to the other robot. The involved d-lines are thus not concurrent, as is the case with a one-robot panorama. The RA \( \mathcal{CYC}_t \), which handles 2D orientations, which can be viewed as d-lines through a fixed point (see Definition 1, isomorphism \( \mathcal{I}_2 \)), is therefore not sufficient to represent the knowledge at hand: the RA \( \mathcal{PA}_t \) is, however, well-suited for the purpose, as we show below.

(1) The panorama of robot \( R_a \), which provides the \( \mathcal{CYC}_t \) relation on each triple of the d-lines joining \( R_a \)'s position to the different landmarks and to the other robot, is given by the conjunction \( rlr(l_{a1}, l_{a2}, l_{a3}) \land rll(l_{a1}, l_{a2}, l_{ab}) \). The \( \mathcal{CYC}_t \) relation on the triple \( (l_{a2}, l_{a3}, l_{ab}) \) is not provided explicitly, but it is implicitly present in the knowledge, and can be inferred by propagation as follows:

(a) Using the rotation operation, we get from \( rlr(l_{a1}, l_{a2}, l_{a3}) \) and \( rll(l_{a1}, l_{a2}, l_{ab}) \) the relations \( lll(l_{a2}, l_{a3}, l_{a1}) \) and \( lll(l_{a2}, l_{ab}, l_{a1}) \), respectively. From \( lll(l_{a2}, l_{ab}, l_{a1}) \), we get, using the converse operation, the relation \( brl(l_{a2}, l_{a1}, l_{ab}) \). Finally, from the conjunction \( lll(l_{a2}, l_{a3}, l_{a1}) \land brl(l_{a2}, l_{a1}, l_{ab}) \), we get, using the composition operation, a first relation on the triple \( (l_{a2}, l_{a3}, l_{ab}) \):

\[ \{lll, lll, brl\}(l_{a2}, l_{a3}, l_{ab}) \].

(b) Using rotation and then converse, we get from \( lll(l_{a2}, l_{a3}, l_{a1}) \) the relation \( rll(l_{a3}, l_{a2}, l_{a1}) \), and from \( rrr(l_{a1}, l_{a3}, l_{ab}) \) the relation \( brr(l_{a3}, l_{a1}, l_{ab}) \).

Using composition, we infer from the conjunction \( rll(l_{a3}, l_{a2}, l_{a1}) \land brr(l_{a3}, l_{a1}, l_{ab}) \) the relation \( \{rre, lrl, brr\} \) on the triple \( (l_{a3}, l_{a2}, l_{ab}) \):

\[ \{rre, lrl, brr\}(l_{a3}, l_{a2}, l_{ab}) \].

Using, again, rotation and then converse, we get a second relation on the triple \( (l_{a2}, l_{a3}, l_{ab}) \):

\[ \{re, lrl, brr\}(l_{a2}, l_{a3}, l_{ab}) \].

(c) Intersecting the results of the last two points, we get the final relation on the triple \( (l_{a2}, l_{a3}, l_{ab}) \):

\[ \{brl\}(l_{a2}, l_{a3}, l_{ab}) \].

(2) Similarly, the panorama of robot \( R_b \) is given by the conjunction \( \{lor\}(l_{b1}, l_{b2}, l_{b3}) \land \{brl\}(l_{b1}, l_{b2}, l_{ba}) \). The \( \mathcal{CYC}_t \) relation on the triple \( (l_{b1}, l_{b3}, l_{ba}) \) as well as the one on the triple \( (l_{b2}, l_{b3}, l_{ba}) \) are not provided explicitly, but they are implicitly present in the knowledge.

It is important to note that, if we want to combine the knowledge consisting of \( R_a \)'s panorama, on the one hand, and \( R_b \)'s panorama, on the other hand, we cannot any longer use orientations, but d-lines: simply because the d-line variables involved in \( R_a \)'s panorama and the ones involved in \( R_b \)'s panorama are not all concurrent. As a consequence, we have to leave the realm of 2D
orientations and enter the one of d-lines: we transform the above knowledge into the RA $\mathcal{P}_A$. Basically, we need to add to the relations in the previous enumeration the fact that the arguments of each triple consist of concurrent d-lines: we need to use the $\mathcal{T}_A$ relation $cc_\prec$. The main two conjunctions in Items 1 and 2 become, respectively, as follows: \[\{(cc_\prec, rlr)\}(l_{a1}, l_{a2}, l_{a3}) \land \{(cc_\prec, rlr)\}(l_{a1}, l_{a2}, l_{ab}) \land \{(cc_\prec, rrr)\}(l_{a1}, l_{a3}, l_{ab}) \land \{(cc_\prec, lrl)\}(l_{b1}, l_{b2}, l_{b3})\].

What has been done so far expresses only relations whose arguments consist of d-lines that are (1) all incident with $R_a$'s position, or (2) all incident with $R_b$'s position. Knowledge combining the two kinds of d-lines needs also to be expressed; examples include the following:

(1) The d-lines $l_{ab}$ and $l_{ba}$ coincide and are of opposite orientations; this can be expressed thus: \[\{(pp_{c1}, oeo)\}(l_{ab}, l_{ba}, l_{ba})\].

(2) the d-line $l_{a3}$ cuts the d-line $l_{ab}$ before $l_{b3}$ does: \[cc_\prec\}(l_{ab}, l_{a3}, l_{b3})\}; the orientational knowledge on the same triple is \[l_{a3}\}(l_{ab}, l_{a3}, l_{b3})\}; the positional knowledge is thus the $\mathcal{P}_A$ relation \[cc_\prec, l_{a3}\}(l_{ab}, l_{a3}, l_{b3})\].

6.5 Natural language processing: representation of motion prepositions

According to Herskovits [28], every motion preposition fits in a syntactic frame

$NP \ [activity \ verb] \ Preposition \ NP$

Examples include:

(1) The ball rolled across the street.
(2) The ball rolled along the street.
(3) The ball rolled toward the boy.

The moving object is referred to as the Figure; the referent of the object of the preposition (the reference object) is referred to as the Ground [56]. The preposition constrains the trajectory, or path of the Figure.

“In conclusion, a motion preposition defines a field of directed lines w.r.t. the Ground” [28].

The examples above on the use of motion prepositions concern perception. Herskovits [28] also discusses the use of motion prepositions in motion planning, as well as in navigation and cognitive maps:

(1) “The linguistic representation of objects’ paths as lines is fundamental to motion planning” ([28], page 174).
(2) “Navigation in large-scale spaces is guided by cognitive maps whose major components are landmarks and routes, represented, respectively, as points and lines. Moreover, in the context of a cognitive map, a moving Figure is conceptualised as a point, and its trajectory as a line” ([28], page 174).

This shows the importance of d-lines, and points, for the representation of motion prepositions. As we have already seen, points can be represented in our RA $\mathcal{P}A_t$ as a pair of cutting d-lines. To relate Herskovits work to ours, we show how to represent in the RA $\mathcal{P}A_t$ Talmy’s schema for “across” (Figure 15(a)), as well as the third sentence in the examples’ list above, “the ball rolled toward the boy”.

Talmy [56] has provided a list of conditions defining “across” (see also [28], page 182). We consider here a more general definition for “across”, given by the following conditions (F = the Figure object; G = the Ground object):

a. F is linear and bounded at both ends (a line segment)
b. G is ribbonal — the part of the plane between two parallel lines
d. The axes of F and G are strictly cutting; i.e., they have a single-point intersection
e. F and G are coplanar
f. F’s length is at least as great as G’s width
h. F touches both of G’s edges (G’s edges are here the lines bounding it)

The items in the enumeration are numbered alphabetically, and the letters match the ones of the corresponding conditions in the list given by Herskovits ([28], page 182).

The Figure F has a directionality and is considered as a directed line segment $F = (P_1, P_3)$. Therefore there are two possibilities for F to be across G (see Figure 15(b-c)). According to what we have seen in Subsection 6.2 (Item 2), the $\mathcal{P}A_t$ representation of F is a triple $\psi(F) = (\ell_3, \ell_4, \ell_5)$ of d-lines verifying $\{\langle cc_>, lor\}\{\ell_3, \ell_4, \ell_5\}$.

The Ground G can be represented as a pair $(\ell_1, \ell_2)$ of d-lines such that $\ell_2$ coincides with, or is parallel to, and lies within the left half-plane bounded by $\ell_1$. In other words, $\ell_1$ and $\ell_2$ are such that $l-par-to(\ell_2, \ell_1) \lor coinc-with(\ell_2, \ell_1)$. We refer to the pair $(\ell_1, \ell_2)$ as the $\mathcal{P}A_t$ representation of G, and denote it by $\psi(G)$: $\psi(G) = (\ell_1, \ell_2)$.

Given the representations $\psi(F)$ and $\psi(G)$, of F and G, respectively, the condition for F to be across G can now be stated in terms of $\mathcal{P}A_t$ relations as follows:

(1) $\ell_2$ is parallel to, and lies within the left half-plane bounded by, $\ell_1$; and $\ell_3$ cuts $\ell_1$: $\{pc\}\{\ell_1, \ell_2, \ell_3\}$. 46
Fig. 15. Diagrammatic representation of Talmy’s schema for “across”.

Fig. 16. The partition of the plane on which is based the relative orientation calculus in [20,61].

(2) $\ell_5$ cuts $\ell_3$ before $\ell_1$ does; $\ell_5$ cuts $\ell_3$ before $\ell_2$ does; $\ell_1$ cuts $\ell_3$ before $\ell_4$ does; and $\ell_2$ cuts $\ell_3$ before $\ell_4$ does: \{cc$_<$\}(\ell_3, \ell_5, \ell_1) \land \{cc$_<$\}(\ell_3, \ell_5, \ell_2) \land \{cc$_<$\}(\ell_3, \ell_1, \ell_4) \land \{cc$_<$\}(\ell_3, \ell_2, \ell_4).

7 Related work

We now discuss the most related work in the literature.
7.1 Scivos and Nebel’s NP-hardness result of Freksa’s calculus

A well-known model of relative orientation of 2D points is the calculus, often referred to as the Double-Cross Calculus, defined in [20], and developed further in [61]. The calculus corresponds to a specific partition, into 15 regions, of the plane, determined by a parent object, say \( A \), and a reference object, say \( B \) (Figure 16(d)). The partition is based on the following:

(1) the left/straight/right partition of the plane determined by an observer placed at the parent object and looking in the direction of the reference object (Figure 16(a));
(2) the front/neutral/back partition of the plane determined by the same observer (Figure 16(b)); and
(3) the similar front/neutral/back partition of the plane obtained when we swap the roles of the parent object and the reference object (Figure 16(c)).

Combining the three partitions (a), (b) and (c) of Figure 16 leads to the partition of the plane on which is based the calculus in [20,61] (Figure 16(d)). The region numbered \( n, n \in \{1, \ldots, 15\} \), in the partition is referred to as \( \text{reg}(A,B,n) \), and gives rise to a basic relation, or atom, of the calculus, which we refer to as \( f_n \):

\[
(\forall n \in \{1, \ldots, 15\})(\forall C)(f_n(A,B,C) \Leftrightarrow C \in \text{reg}(A,B,n))
\]

Scivos and Nebel [55] have shown that the subset \( \{\{f_{10}\},T\} \), where \( T \) is the universal relation, is NP-hard; the proof uses a reduction of the betweenness problem ([22], page 279). We consider here a coarser version of Freksa’s calculus, which does not distinguish between \( f_1, f_2, f_3, f_4, f_5 \), on the one hand, and between \( f_{11}, f_{12}, f_{13}, f_{14}, f_{15} \), on the other hand. We show that a CSP, \( P \), expressed in \( FC = \{f_\ell, \{f_6\}, \{f_7\}, \{f_8\}, \{f_9\}, \{f_{10}\}, f_r, T\} \), where \( f_\ell = \{f_1, f_2, f_3, f_4, f_5\} \) and \( f_r = \{f_{11}, f_{12}, f_{13}, f_{14}, f_{15}\} \), can be translated into an equivalent \( PA_t \)-CSP, \( P' \). The idea is to first eliminate the relations \( \{f_7\} \) and \( \{f_9\} \) from \( P \), which involve necessarily equal variables (see Definition 10 below). We then show how to translate each constraint of the resulting problem, expressed in \( FC \setminus \{\{f_7\}, \{f_9\}\} = \{f_\ell, \{f_6\}, \{f_8\}, \{f_{10}\}, f_r, T\} \), into the RA \( PA_t \).

**Definition 10** Two variables of a CSP are necessarily equal if they receive the same instantiation in all models of the CSP.

(1) Let \( X_i, X_j \) and \( X_k \) be three variables such that \( \{f_7\} \{X_i, X_j, X_k\} \). Then \( X_i \) and \( X_k \) are necessarily equal. In such a case, we perform the following, for all variables \( X_l \) and \( X_m \):

(a) \( (T^P)_{lmi} \leftarrow (T^P)_{lmi} \cap (T^P)_{lmk} \)
(b) \((T^P)_{ilm} \leftarrow (T^P)_{ilm} \cap (T^P)_{ikm}\)
(c) \((T^P)_{ilm} \leftarrow (T^P)_{ilm} \cap (T^P)_{klm}\)

If in any of the above replacement operations, the empty relation is detected then the CSP is clearly inconsistent. Otherwise, the variable \(X_k\) can be removed from the CSP: replacement the CSP by the sub-CSP \(P_{V \setminus \{X_k\}}\), where \(V\) is the set of all variables.

(2) If there exist variables \(X_i, X_j\) and \(X_k\) such that \(\{f_7\}(X_i, X_j, X_k)\) then repeat the process from 1.

(3) If there exist three variables \(X_i, X_j\) and \(X_k\) such that \(\{f_9\}(X_i, X_j, X_k)\) then \(X_j\) and \(X_k\) are necessarily equal. The constraint \(\{f_9\}(X_i, X_j, X_k)\) is equivalent to the constraint \(\{f_7\}(X_j, X_i, X_k)\):
(a) \((T^P)_{ijk} \leftarrow (T^P)_{ijk} \cap \{f_7\}\)
(b) If \((T^P)_{ijk} = \emptyset\) then exit (the CSP is inconsistent)
(c) Repeat the process from Item 1

(4) If there exist no three variables \(X_i, X_j\) and \(X_k\) such that \(\{f_9\}(X_i, X_j, X_k)\) then the process has been achieved: the resulting CSP has no two variables that are necessarily equal; in other words, it is expressed in \(FC \setminus \{\{f_7\}, \{f_9\}\}\).

The second step is to show how to translate a CSP, \(P\), expressed in \(FC \setminus \{\{f_7\}, \{f_9\}\}\) into a CSP \(P'\) expressed in the RA \(P_A\).

(1) Initialise the set \(V'\) of variables and the set \(C'\) of constraints of \(P'\) to the empty set:
\(V' \leftarrow \emptyset, C' \leftarrow \emptyset\)

(2) For each pair \((X_i, X_j), i < j\), of variables from \(V\), the set of variables of \(P\), we create a d-line variable \(X_{ij}\):
\(V' \leftarrow V' \cup \{X_{ij}\}\)

(3) For each variable \(X_i\) from \(V\), we create two d-line variables \(X_{i1}\) and \(X_{i2}\) such that \(\psi(X_i) = (X_{i1}, X_{i2})\); i.e., such that \(\{< c_{cp}, \le >\}(X_{i1}, X_{i2}, X_i)\):
\(V' \leftarrow V' \cup \{X_{i1}, X_{i2}\}\)

(4) For all distinct variables \(X_i\) and \(X_j\) of \(P\), we have \(X_i \in X_{ij}\) and \(X_j \in X_{ji}\):
\(C' \leftarrow C' \cup \{\{cc_l, c_{cp}, c_{pc}\}(X_{ij}, X_{i1}, X_{i2}), \{cc_l, c_{cp}, c_{pc}\}(X_{ij}, X_{j1}, X_{j2})\}\)

(5) \(X_{ij}\) is oriented from \(X_i\) to \(X_j\); i.e., \(X_i\) is met before \(X_j\) in the positive walk along \(X_{ij}\):\(^9\)
\(C' \leftarrow C' \cup \{\{cc_l, c_{cp}, c_{pc}, c_{pp}\}(X_{ij}, X_{i1}, X_{i2}, X_{j1}), \{cc_l, c_{cp}, c_{pc}, c_{pp}\}(X_{ij}, X_{i1}, X_{i2}, X_{j2})\}\)

(6) For each constraint of \(P\) of the form \(\{f_{\ell}\}(X_i, X_j, X_k)\), we add the constraint \(\{\ell_{\ell}\}(X_{ij}, X_{ik}, X_{ik})\) to \(P'\):
\(C' \leftarrow C' \cup \{\{\ell_{\ell}\}(X_{ij}, X_{ik}, X_{ik})\}\)

\(^9\) The conditions \(\psi(X_i) = (X_{i1}, X_{i2})\) and \(\psi(X_j) = (X_{j1}, X_{j2})\) imply that in all solutions to the CSP \(P'\) in construction, the instantiations \(X_{ij} = \ell_{ij}, X_{i1} = \ell_{i1}, X_{i2} = \ell_{i2}, X_{j1} = \ell_{j1}, X_{j2} = \ell_{j2}\) of the variables in \(\{X_{ij}, X_{i1}, X_{i2}, X_{j1}, X_{j2}\}\) are such that \(\ell_{i1}\) and \(\ell_{i2}\), on the one hand, and \(\ell_{j1}\) and \(\ell_{j2}\), on the other hand, cannot be both parallel to \(\ell_{ij}\).
(7) For each constraint of $P$ of the form $\{f_6\}(X_i, X_j, X_k)$, we add the constraint $\{oeo\}(X_{ij}, X_{ik}, X_{ik})$ to $P'$:
$$C' \leftarrow C'' \cup \{\{oeo\}(X_{ij}, X_{ik}, X_{ik})\}.$$  

(8) For each constraint of $P$ of the form $\{f_8\}(X_i, X_j, X_k)$, we add the constraints $\{eee\}(X_{ij}, X_{ik}, X_{ik})$ and $\{oeo\}(X_{ij}, X_{jk}, X_{jk})$ to $P'$:
$$C' \leftarrow C'' \cup \{\{eee\}(X_{ij}, X_{ik}, X_{ik}), \{oeo\}(X_{ij}, X_{jk}, X_{jk})\}.$$  

(9) For each constraint of $P$ of the form $\{f_{10}\}(X_i, X_j, X_k)$, transform it into the equivalent constraint $\{f_6\}(X_j, X_i, X_k)$ and apply Step 7

(10) For each constraint of $P$ of the form $\{f_r\}(X_i, X_j, X_k)$, transform it into the equivalent constraint $\{f_\ell\}(X_j, X_i, X_k)$ and apply Step 6

7.2 Moratz et al.’s dipole algebra and Renz’s spatial Odyssey of Allen’s interval algebra

A dipole is an oriented line segment. We follow here the notation in [48], and denote dipoles by the letters A, B, C, ..., the starting endpoint and the ending endpoint of a dipole $A$ by $s_A$ and $e_A$, respectively. The simple version of the dipole algebra, denoted $D_{24}$, presents 24 atoms, which are characterised by the fact that they cannot represent a configuration of two dipoles with at least three of the four endpoints collinear and pairwise distinct. The reason for presenting a simple version of the algebra is that it has the advantage of being a relation algebra\footnote{As far as we can say, the authors did not check, for instance, that the entries of the composition table record the exact composition of the corresponding atoms; i.e., whether, given any two atoms, say $r$ and $s$, it is the case that $r \circ s = T[r, s]$, where $T[r, s]$ is the entry at row $r$ and column $s$ of the composition table. If this is not the case, the calculus would not be an RA.} [57,40,36], and of presenting a relatively small number of atoms, offering thus a manageable composition table, and a wide application domain. The complete version, denoted $D_{69}$, contains 69 atoms that are Jointly Exhaustive and Pairwise Disjoint: any spatial configuration of two dipoles is described by one and only one of the 69 atoms. We focus on the more general version, $D_{69}$. The description of $D_{69}$ atoms is based on seven dipole-point relations, $l$ (left), $b$ (behind), $s$ (starts), $i$ (inside), $e$ (ends), $f$ (front) and $r$ (right): given a dipole $A = (s_A, e_A)$, a point $P$ and a dipole-point relation $R \in \{l, b, s, i, e, f, r\}$, we have $A R P$ if and only if the point $P$ belongs to the region labelled $R$ in the partition of the plane determined by $A$, illustrated in Figure 17 (see [48] for details).

A dipole-dipole relation is denoted in [48] by a word of length 4 over the alphabet $\{l, b, s, i, e, f, r\}$, of the form $R^1 R^2 R^3 R^4$. Such a relation on two dipoles $A$ and $B$, denoted by $A R^1 R^2 R^3 R^4 B$, is interpreted as follows:

$$A R^1 R^2 R^3 R^4 B \Leftrightarrow (A R^1 s_B) \land (A R^2 e_B) \land (B R^3 s_A) \land (B R^4 e_A)$$ (31)
In order to get a \( \mathcal{PA} \) we need to provide a relations and a point \( P \). We associate with \( A \) consisting of its \( \mathcal{PA} \) set of

\[
\ell(b) \text{ From hand: (7.1, third enumeration, Items 6-10), thanks to the following equivalences: (see Subsection 7.1, third enumeration, Item 4):}
\]

\[
\{ \{ cc, cp, pc \} (\ell_1, \ell_{s_A}, \ell_{e_A}), \{ cc, cp, pc \} (\ell_A, \ell_{e_A}, \ell_{e_A}) \}
\]

(2) We associate with \( A \) a d-line \( \ell_A \) oriented from \( s_A \) to \( e_A \):

(a) From \( s_A \in \ell_A \) and \( e_A \in \ell_A \), we get the following set of constraints (see Subsection 7.1, third enumeration, Item 4):

\[
\{ \{ cc, cp, pc \} (\ell_A, \ell_{s_A}, \ell_{e_A}), \{ cc, cp, pc \} (\ell_A, \ell_{s_A}, \ell_{e_A}) \}
\]

(b) From \( \ell_A \) oriented from \( s_A \) to \( e_A \), we get the following set of constraints (see Subsection 7.1, third enumeration, Item 5):

\[
\{ \{ cc, cp, pc \} (\ell_A, \ell_{s_A}, \ell_{e_A}), \{ cc, cp, pc \} (\ell_A, \ell_{s_A}, \ell_{e_A}) \}
\]

(3) The relations \( l, b, i, f \) and \( r \) are translated into \( \mathcal{PA} \) in a similar way as the relations \( f_l, f_0, f_s, f_{10} \) and \( f_r \) of Freksa's calculus (see Subsection 7.1, third enumeration, Items 6-10), thanks to the following equivalences:

\( A l P \) iff \( f_l(s_A, e_A, P) \); \( A b P \) iff \( f_0(s_A, e_A, P) \); \( A i P \) iff \( f_8(s_A, e_A, P) \); \( A f P \) iff \( f_10(s_A, e_A, P) \); and \( A r P \) iff \( f_r(s_A, e_A, P) \).

(4) A s \( P \) (\( P \) coincides with \( s_A \)) is expressed by concurrence of \( \ell_1, \ell_2, \) and \( \ell_{s_A} \), on the one hand, and concurrence of \( \ell_1, \ell_2, \) and \( \ell_{s_A} \), on the other hand:

\[
\{ \{ cc, cp, pc \} (\ell_1, \ell_2, \ell_{s_A}), \{ cc, cp, pc \} (\ell_1, \ell_2, \ell_{s_A}) \}
\]

(5) In a similar way, we translate A e \( P \) (\( P \) coincides with \( e_A \)) using a double concurrence:

\[
\{ \{ cc, cp, pc \} (\ell_1, \ell_2, \ell_{e_A}), \{ cc, cp, pc \} (\ell_1, \ell_2, \ell_{e_A}) \}
\]
Renz’s work [53] was motivated by applications such as traffic scenarios, where cars and their regions of influence can be represented as directed intervals of an underlying line representing the road. We refer to the underlying line as \( \ell_{\text{Renz}} \). The 26 atomic relations of Renz’s algebra of directed intervals [53] can be seen as particular relations of Moratz et al.’s \( D_{69} \) algebra [48], as long as we have a mean of constraining all involved directed intervals to belong to the underlying line \( \ell_{\text{Renz}} \).

Consider two directed intervals \( x \) and \( y \). \( x \) and \( y \) are particular oriented segments \( x = (s_x, e_x) \) and \( y = (s_y, e_y) \). As in the above discussion of Moratz et al.’s \( D_{69} \) algebra, we associate with each point \( P \) in \( \{ s_x, e_x, s_y, e_y \} \) its \( \mathcal{PA} \) representation \( \psi(P) = (\ell^1_P, \ell^2_P) \), and with each oriented segments \( S \) in \( \{ x, y \} \) a d-line \( \ell_S \) oriented from the left endpoint to the right endpoint of the segment.

1. Constraining a directed interval, for instance \( x \), to be part of the underlying line \( \ell_{\text{Renz}} \) can be easily done with the RA \( \mathcal{PA} \), by saying that the line \( \ell_x \) coincides with \( \ell_{\text{Renz}} \):
   \[
   \{ \langle pp_{c1}, eec \rangle, \langle pp_{c1}, oeo \rangle \}(\ell_{\text{Renz}}, \ell_x, \ell_x)
   \]

2. Once each of the directed intervals is constrained to be part of \( \ell_{\text{Renz}} \), the atomic relations can be translated into \( \mathcal{PA} \) using what has been done above for Moratz et al.’s \( D_{69} \) dipole algebra, thanks to the equivalences of Figure 18 between directed intervals base relations and \( D_{69} \) base relations.

### 7.3 Reasoning about parallel-to-the-axes parallelograms

Approaches to reasoning about parallelograms of the 2D space, whose sides are parallel to the axes of some system \((x, O, y)\) of coordinates, can be found in the literature [3,24,49]. These approaches are straightforward extensions of Allen’s algebra of temporal intervals [1]. Given such a parallelogram, say \( P \), we denote by \( P^x \) and \( P^y \) the intervals consisting of the projections of \( P \) on the \( x \)- and \( y \)-axes, respectively. The atoms of the corresponding rectangle algebra, \( \mathcal{RgA} \), are of the form \((r_1, r_2)\), where \( r_1 \) and \( r_2 \) are Allen’s atoms [1] (the Allen’s atoms are \(<\) (before), \(m\) (meets), \(o\) (overlaps), \(s\) (starts), \(d\) (during), \(f\) (finishes); their respective converses \(>\) (after), \(mi\) (met-by), \(oi\) (overlapped-by), \(si\) (started-by), \(di\) (contains), \(fi\) (finished-by); and \(eq\) (equals), which is its proper converse). If \( P_1 \) and \( P_2 \) are two parallelograms as described, then:

\[
(r_1, r_2)(P_1, P_2) \Leftrightarrow r_1(P^x_1, P^x_2) \land r_2(P^y_1, P^y_2)
\]

The translation of the atoms of the rectangle algebra into the RA \( \mathcal{PA} \), can thus be obtained from the procedure already described on translating into \( \mathcal{PA} \) Moratz et al.’s relations [48], on the one hand, and Renz’s relations [53], on the other hand (see Figure 18).
| Directed Intervals Base Relation | Symbol | Pictorial Example | $D_{69}$ Base Relation |
|----------------------------------|--------|------------------|------------------------|
| $x$ behind $y$                   | $b_m$  | $-x->$           | $x f f b b y$          |
| $y$ in-front-of $x$              | $f_m$  | $-y->$           | $y b b f f x$          |
| $x$ behind $\neq y$             | $b_{\neq}$ | $<-x-$         | $x b b b y$            |
| $x$ in-front-of $\neq y$        | $f_{\neq}$ | $-x->$         | $<e y$                |
| $x$ meets-from-behind $y$        | $m_{b_m}$ | $-x->$           | $x e f b s y$          |
| $y$ meets-in-the-front $x$       | $m_{f_m}$ | $-y->$           | $x b s e f x$          |
| $x$ meets-from-behind $\neq y$  | $m_{b_{\neq}}$ | $<-x-$        | $x s b b y$            |
| $x$ meets-in-the-front $\neq y$ | $m_{f_{\neq}}$ | $-x->$         | $<e y$                |
| $x$ overlaps-from-behind $y$     | $o_{b_m}$ | $-x->$           | $x i f b i y$          |
| $y$ overlaps-in-the-front $x$    | $o_{f_m}$ | $-y->$           | $y b i f x$            |
| $x$ overlaps-from-behind $\neq y$ | $o_{b_{\neq}}$ | $<-x-$         | $x i b b y$            |
| $x$ overlaps-in-the-front $\neq y$ | $o_{f_{\neq}}$ | $-x->$         | $<e y$                |
| $x$ contained-in $y$             | $c_m$  | $-x->$           | $x b f i i y$          |
| $y$ extends $x$                  | $e_m$  | $-y->$           | $y i b f x$            |
| $x$ contained-in $\neq y$       | $c_{\neq}$ | $<-x-$         | $x f b i i y$          |
| $y$ extends $\neq y$             | $e_{\neq}$ | $-y->$         | $y i f b x$            |
| $x$ contained-in-the-back-of $y$ | $c_{b_m}$ | $-x->$           | $x s f s i y$          |
| $y$ extends-the-back-of $x$      | $e_{b_m}$ | $-y->$           | $y s i s f x$          |
| $x$ contained-in-the-back-of $\neq y$ | $c_{b_{\neq}}$ | $<-x-$       | $x e b i y$            |
| $y$ extends-the-back-of $\neq y$ | $e_{b_{\neq}}$ | $-y->$       | $y i s e b x$          |
| $x$ contained-in-the-front $y$   | $c_{f_m}$ | $-x->$           | $x b e i e y$          |
| $y$ extends-the-front $x$        | $e_{f_m}$ | $-y->$           | $y i e b e x$          |
| $x$ contained-in-the-front $\neq y$ | $c_{f_{\neq}}$ | $<-x-$       | $x f s e i y$          |
| $y$ extends-the-front $\neq y$   | $e_{f_{\neq}}$ | $-y->$       | $y e f s x$            |
| $x$ equals $y$                   | $e_{q_m}$ | $-x->$           | $x s e s e y$          |
| $y$ equals $\neq y$              | $e_{q_{\neq}}$ | $-x->$       | $<e y$                |

Fig. 18. The 26 base relations of the directed intervals algebra, and their translation into the RA $P_A t$.  

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8 Conclusion and further work

We have presented a Relation Algebra (RA) [57,40,36] of relative position relations on 2-dimensional directed lines (d-lines). The converse table, the rotation table and the composition tables of the RA have been provided. Furthermore, thanks to its inspiration from the theory of degrees of freedom analysis [26,38], the work can be seen as answering, at least partly, the challenges in [52] for the particular case of qualitative spatial reasoning: computing, for instance, the composition of two relations is derived from:

(1) the composition of the rotational projections of the two relations, on the one hand; and
(2) the composition of the translational projections, on the other hand.

More importantly, current research shows clearly the importance of developing spatial RAs: specialising an $\mathcal{ALC}(\mathcal{D})$-like Description Logic (DL) [2], so that the roles are temporal immediate-successor (accessibility) relations, and the concrete domain is generated by a decidable spatial RA in the style of the well-known Region-Connection Calculus RCC-8 [51], such as the RA $c\mathcal{PA}_t$ defined in this paper, leads to a computationally well-behaving family of languages for spatial change in general, and for motion of spatial scenes in particular:

(1) Deciding satisfiability of an $\mathcal{ALC}(\mathcal{D})$ concept w.r.t. to a cyclic TBox is, in general, undecidable (see, for instance, [45]).
(2) In the case of the spatio-temporalisation, however, if we use what is called weakly cyclic TBoxes in [31] (see also the related work [34]), then satisfiability of a concept w.r.t. such a TBox is decidable. The axioms of a weakly cyclic TBox capture the properties of modal temporal operators. The reader is referred to [31] for details.\(^\text{11}\)

Extending the presented RA to 3D would allow, for instance, for the representation of 3D shapes, such as polyhedra, which are the 3D counterpart of polygons in the 2D space, which are themselves the 2D counterpart of (convex) intervals in the 1D space. This could be achieved thus:

(1) extend Isli and Cohn’s RA $c\mathcal{CYC}_t$ [35,36] to orientations of the 3D space;
(2) extend the $\mathcal{T}\mathcal{A}_t$ algebra to directed lines of the 3D space; and
(3) combine the two calculi, as done in this work for the 2D counterparts, to get a calculus for reasoning about relative position of directed lines of the 3D space.

\(^{11}\) A full version of [31], including the decidability proof, will be made downloadable soon [33].
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A Checking that $\mathcal{P}A_t$ satisfies the ternary RA properties

$\mathcal{P}A_t$ is the structure $\mathcal{P}A_t = \langle 2^{\mathcal{P}A_t-\text{at}}, \cup, \cap, -, \emptyset, \mathcal{P}A_t-\text{at}, \circ, \neg, \nabla, \mathcal{I}_{23} \rangle$, where:

1. the set $2^{\mathcal{P}A_t-\text{at}}$ of subsets of $\mathcal{P}A_t-\text{at}$ is the set of elements, or relations, of $\mathcal{P}A_t$;
2. the Boolean operations of addition, product and complement are given by the set-theoretic operations of union ($\cup$), intersection ($\cap$) and complement ($-)$;
3. the empty set provides the empty, or bottom, element of $\mathcal{P}A_t$;
4. the set $\mathcal{P}A_t-\text{at}$ of atoms provides the universal, or top, element of $\mathcal{P}A_t$;
5. the composition, converse and rotation of elements of $\mathcal{P}A_t$ are, respectively, the operations $\circ$, $\neg$ and $\nabla$, as defined in [36]—see also Section 2, Equations (3)-(4)-(5); and
6. the identity element is given by $\mathcal{I}_{23} = \{ (a, b, b) : a, b \in \mathcal{L} \}$.

The verification that $\mathcal{P}A_t$ satisfies the nine ternary RA properties (18)−⋯−(26) is done in a similar way as for $\mathcal{C}YC_t$-see [36], Appendix B. The only thing that remains to be checked is that the converse table, the rotation table and the composition tables record, respectively, the exact converses, the exact rotations and the exact compositions. But this follows straightforwardly from the facts:

1. that $\mathcal{C}YC_t$ is an RA [36], and
2. that Vilain and Kautz’s calculus of time points [59] is an RA [40].

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