ON THE REGULARITY OF THE SOLUTIONS FOR CAUCHY PROBLEM
OF INCOMPRESSIBLE 3D NAVIER-STOKES EQUATION

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Abstract. In this paper we will prove that the vorticity belongs to $L^\infty(0, T; L^2(\mathbb{R}^3))$ for the Cauchy problem of 3D incompressible Navier-Stokes equation, then the existence of a global smooth solution is obtained. Our approach is to construct a set of auxiliary problems to approximate the original one of vorticity equation.

Keywords. Navier-Stokes equation; Regularity; Vorticity.

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1. Introduction

Let $\mathcal{D}(\mathbb{R}^3)$ be the space of $C^\infty$ functions with compact support contained in $\mathbb{R}^3$. Some basic spaces will be used in this paper:

$\mathcal{V} = \{ u \in \mathcal{D}(\mathbb{R}^3), \, \text{div} u = 0 \}$

$V = \text{the closure of } \mathcal{V} \text{ in } H^1(\mathbb{R}^3)$

$H = \text{the closure of } \mathcal{V} \text{ in } L^2(\mathbb{R}^3)$

The velocity-pressure form for Navier-Stokes equation is

\[
\begin{align*}
\partial_t u_1 + u_1 \partial_{x_1} u_1 + u_2 \partial_{x_2} u_1 + u_3 \partial_{x_3} u_1 + \partial_{x_1} p &= \Delta u_1 \\
\partial_t u_2 + u_1 \partial_{x_1} u_2 + u_2 \partial_{x_2} u_2 + u_3 \partial_{x_3} u_2 + \partial_{x_2} p &= \Delta u_2 \\
\partial_t u_3 + u_1 \partial_{x_1} u_3 + u_2 \partial_{x_2} u_3 + u_3 \partial_{x_3} u_3 + \partial_{x_3} p &= \Delta u_3
\end{align*}
\]

(1)

with the initial conditions $(u_1, u_2, u_3)_{|t=0} = (u_{10}, u_{20}, u_{30})(x)$, henceforth we always ignore the assumption of sufficient smoothness of the initial conditions. Moreover, the incompressible condition is

\[\partial_{x_1} u_1 + \partial_{x_2} u_2 + \partial_{x_3} u_3 = 0\]

where $x = (x_1, x_2, x_3)$ is a point of $\mathbb{R}^3$, $u = (u_1, u_2, u_3)$ is velocity, $p$ is pressure, and $\nu > 0$ is viscosity.

The vorticity-velocity form for Navier-Stokes equation is

\[
\begin{align*}
\partial_t \omega_1 + u_1 \partial_{x_1} \omega_1 + u_2 \partial_{x_2} \omega_1 + u_3 \partial_{x_3} \omega_1 - \omega_1 \partial_{x_1} u_1 - \omega_2 \partial_{x_2} u_1 - \omega_3 \partial_{x_3} u_1 &= \Delta \omega_1 \\
\partial_t \omega_2 + u_1 \partial_{x_1} \omega_2 + u_2 \partial_{x_2} \omega_2 + u_3 \partial_{x_3} \omega_2 - \omega_1 \partial_{x_1} u_2 - \omega_2 \partial_{x_2} u_2 - \omega_3 \partial_{x_3} u_2 &= \Delta \omega_2 \\
\partial_t \omega_3 + u_1 \partial_{x_1} \omega_3 + u_2 \partial_{x_2} \omega_3 + u_3 \partial_{x_3} \omega_3 - \omega_1 \partial_{x_1} u_3 - \omega_2 \partial_{x_2} u_3 - \omega_3 \partial_{x_3} u_3 &= \Delta \omega_3
\end{align*}
\]

(2)

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with the initial conditions \((\omega_1, \omega_2, \omega_3)|_{t=0} = (\omega_{10}, \omega_{20}, \omega_{30}) = (\text{curl}u_{10}, \text{curl}u_{20}, \text{curl}u_{30})\), and the incompressible condition:

\[
\begin{align*}
\partial_x u_1 + \partial_x u_2 + \partial_x u_3 &= 0 \\
\partial_x \omega_1 + \partial_x \omega_2 + \partial_x \omega_3 &= 0
\end{align*}
\]

We here recall the global \(L^2\)-estimate from [4] for the Navier-Stokes equation of velocity-pressure form.

In the sequel, it is assumed that the initial value \(u_0\) satisfies the following conditions:

\[
\left| \partial_x^i u_{i0}(x) \right| \leq C_\mu (1 + |x|)^{-\sigma}, \quad i, j = 1, 2, 3
\]

where \(\mu = 0, 1\) and \(\sigma > 0\) is integer.

For the handling the initial value problem, a weighted function is introduced:

\[
\theta_r = \begin{cases} 
  e^{-\frac{|x|^2}{r^2}} & |x| < r \\
  0 & |x| \geq r
\end{cases} \quad (r > 0)
\]

which is of the properties:

\[
\theta_r \to 1, \quad \partial_r \theta_r \to 0, \quad \partial_i \partial_j \theta_r \to 0
\]

as \(r \to +\infty\) for each relatively fixed \(x \in \mathbb{R}^3\).

Moreover, let \(v = \theta_r u\), we still have

\[
\begin{align*}
\partial_t v &= u \partial_t \theta_r + \theta_r \partial_t u \\
\partial_t^2 v &= u \partial_t^2 \theta_r + 2 \partial_r \theta_r \partial_t u + \theta_r \partial_t^2 u \\
\partial_t \partial_j v &= u \partial_t \partial_j \theta_r + \partial_j \theta_r \partial_t u + \partial_t \theta_r \partial_j u + \theta_r \partial_j \partial_j u
\end{align*}
\]

Since

\[
\int_{\mathbb{R}^3} \theta_r u_i (u_1 \partial_x u_i + u_2 \partial_x u_i + u_3 \partial_x u_i) = \frac{1}{2} \int_{\mathbb{R}^3} \theta_r (u_1 \partial_x u_i^2 + u_2 \partial_x u_i^2 + u_3 \partial_x u_i^2)
\]

\[
= -\frac{1}{2} \int_{\mathbb{R}^3} u_i^2 (\partial_x u_i \theta_r + \partial_x u_i \theta_r + \partial_x u_i \theta_r)
\]

\[
= -\frac{1}{2} \int_{\mathbb{R}^3} u_i^2 \left( \partial_x u_i + \partial_x u_i + \partial_x u_i \right) = \frac{1}{2} \int_{\mathbb{R}^3} u_i^2 (u_1 \partial_x u_i \theta_r + u_2 \partial_x u_i \theta_r + u_3 \partial_x u_i \theta_r)
\]

Taking \(r \to +\infty\) we get

\[
\int_{\mathbb{R}^3} u_i (u_1 \partial_x u_i + u_2 \partial_x u_i + u_3 \partial_x u_i) = 0, \quad i = 1, 2, 3
\]
in the same way, 
\[ \int_{\mathbb{R}^3} (u_1 \partial_{x_1} p + u_2 \partial_{x_2} p + u_3 \partial_{x_3} p) = 0 \]
and
\[ \int_{\mathbb{R}^3} u_i \Delta u_i = \int_{\mathbb{R}^3} u_i (\partial_{x_1}^2 u_i + \partial_{x_2}^2 u_i + \partial_{x_3}^2 u_i) = - \int_{\mathbb{R}^3} ((\partial_{x_1} u_i)^2 + (\partial_{x_2} u_i)^2 + (\partial_{x_3} u_i)^2) \]
then,
\[ \int_{\mathbb{R}^3} u_1 \partial_t u_1 + \int_{\mathbb{R}^3} u_1 (u_1 \partial_{x_1} u_1 + u_2 \partial_{x_2} u_1 + u_3 \partial_{x_3} u_1) + \int_{\mathbb{R}^3} u_1 \partial_{x_1} p = \int_{\mathbb{R}^3} u_1 \Delta u_1 \]
\[ \int_{\mathbb{R}^3} u_2 \partial_t u_2 + \int_{\mathbb{R}^3} u_2 (u_1 \partial_{x_1} u_2 + u_2 \partial_{x_2} u_2 + u_3 \partial_{x_3} u_2) + \int_{\mathbb{R}^3} u_2 \partial_{x_2} p = \int_{\mathbb{R}^3} u_2 \Delta u_2 \]
\[ \int_{\mathbb{R}^3} u_3 \partial_t u_3 + \int_{\mathbb{R}^3} u_3 (u_1 \partial_{x_1} u_3 + u_2 \partial_{x_2} u_3 + u_3 \partial_{x_3} u_3) + \int_{\mathbb{R}^3} u_3 \partial_{x_3} p = \int_{\mathbb{R}^3} u_3 \Delta u_3 \]
so that
\[ \frac{1}{2} \partial_t \int_{\mathbb{R}^3} (u_1^2 + u_2^2 + u_3^2) + \int_{\mathbb{R}^3} ((\partial_{x_1} u_1)^2 + (\partial_{x_2} u_1)^2 + (\partial_{x_3} u_1)^2 + 
(\partial_{x_1} u_2)^2 + (\partial_{x_2} u_2)^2 + (\partial_{x_3} u_2)^2 + (\partial_{x_1} u_3)^2 + (\partial_{x_2} u_3)^2 + (\partial_{x_3} u_3)^2) = 0 \]
it follows that
\[ \int_{\mathbb{R}^3} (u_1^2 + u_2^2 + u_3^2) + 2 \int_0^T (\|\nabla u_1\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla u_2\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla u_3\|_{L^2(\mathbb{R}^3)}^2) = \int_{\mathbb{R}^3} (u_{10}^2 + u_{20}^2 + u_{30}^2) \]
Hence from (3) we have
\[ \sup_{t \in (0, T)} \int_{\mathbb{R}^3} (u_1^2 + u_2^2 + u_3^2) < +\infty \quad (6) \]
\[ \int_0^T (\|\nabla u_1\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla u_2\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla u_3\|_{L^2(\mathbb{R}^3)}^2) < +\infty \]
Above u can be interpreted as the Galerkin approximation of the solution, but (6) are also true for the solution of problem (1).

The rest of sections are arranged as follows : In section 2 and 3, we introduce a set of auxiliary problems and prove the uniform boundedness and the existence of their solutions in \( L^\infty (0, T; L^2(\mathbb{R}^3)) \). Then it is shown that the solutions of the auxiliary problems converge to that of Naiver-Stokes equation with vorticity-velocity form, which also belongs to \( L^\infty (0, T; L^2(\mathbb{R}^3)) \). Final section will present the solution of Navier-Stokes equation with velocity-pressure form belongs to \( L^\infty (0, T; H^2(\mathbb{R}^3)) \).
2. Auxiliary Problems

For the 3D regularity, we only need to prove that the vorticity in (2) belongs to $L^\infty(0, T; L^2(\mathbb{R}^3))$.

Given a partition with respect to $t$ as follows:

$$0 = t_0 < t_1 < t_2 < \cdots < t_{k-1} < t_k < \cdots < t_N = T$$

On each $t \in (t_{k-1}, t_k)$, we introduce an auxiliary problem:

\[
\begin{align*}
\partial_t \tilde{\omega}_1 + \frac{1}{\Delta t_k} \int_{t_{k-1}}^{t_k} \partial_t \tilde{\omega}_i(x, t)dt = & 
\frac{1}{\Delta t_k} \int_{t_{k-1}}^{t_k} u_i(x, t)dt \\
\tilde{\omega}_i(x, t) = & \tilde{\omega}_i^{k-1}(x), \quad \tilde{\omega}_i(x, 0) = \omega_{i0}(x), \quad i = 1, 2, 3
\end{align*}
\]

and

\[
\frac{1}{\Delta t_k} \int_{t_{k-1}}^{t_k} \tilde{\omega}_i(x, t)dt = \tilde{\omega}_i^{k}(x), \quad i = 1, 2, 3
\]

In addition, let $\varepsilon > 0$, we construct a mollifier $J_\varepsilon \in C_0^\infty(\mathbb{R}^3)$ such that

i) $J_\varepsilon(x) \geq 0$, $x \in \mathbb{R}^3$,

ii) $J_\varepsilon(x) = 0$ if $|x| \geq \varepsilon$, and

iii) $\int_{\mathbb{R}^3} J_\varepsilon(x) dx = 1$.

then a convolution is defined as

\[
\tilde{\omega}_i^k(x) = J_\varepsilon \ast \tilde{\omega}_i^k(x) = \int_{\mathbb{R}^3} J_\varepsilon(x-y) \tilde{\omega}_i^k(y) dy
\]

Similarly we can set the incompressible condition:

$$\begin{align*}
\partial_x u_1 + \partial_x u_2 + \partial_x u_3 &= 0 \\
\partial_t \omega_1 + \partial_x \omega_2 + \partial_x \omega_3 &= 0
\end{align*}$$

It is easy to check that

$$\begin{align*}
\partial_x u_1 + \partial_x u_2 + \partial_x u_3 &= 0 \quad \Rightarrow \quad \partial_x \omega_1^k + \partial_x \omega_2^k + \partial_x \omega_3^k = 0 \\
\partial_x \omega_1 + \partial_x \omega_2 + \partial_x \omega_3 &= 0 \quad \Rightarrow \quad \partial_x \omega_1^k + \partial_x \omega_2^k + \partial_x \omega_3^k = 0
\end{align*}$$
In the section 3, by means of the Galerkin method and the compactness imbedding theorem, we can prove the local existences of the weak solutions of these systems for each \((t_{k-1}, t_k)\) being small enough. Below we also interpret \(\hat{\omega}\) as the Galerkin approximation of the solution of the problems (7), and first prove that \(\hat{\omega}, t \in (0, T)\) belong to \(L^\infty(0, T; L^2(\mathbb{R}^3))\). In section 4, an approach of approximation is used to assert that the solution of (2) also belongs to \(L^\infty(0, T; L^2(\mathbb{R}^3))\).

Since

\[
\int_{\mathbb{R}^3} \theta_r \left[ \frac{\partial}{\partial t} \left( \begin{array}{c}
\omega_1 (u_1^k x_1) + u_2^k x_1 + u_3^k x_1 \\
\omega_2 (u_1^k x_2) + u_2^k x_2 + u_3^k x_2 \\
\omega_3 (u_1^k x_3) + u_2^k x_3 + u_3^k x_3
\end{array} \right) \right] \\
+ \omega_2 (u_1^k x_2) + u_2^k x_2 + u_3^k x_2 \\
+ \omega_3 (u_1^k x_3) + u_2^k x_3 + u_3^k x_3 \right] \\
= - \int_{\mathbb{R}^3} \left[ \frac{\partial}{\partial t} \left( \begin{array}{c}
\omega_1 (x_1) + \omega_2 (x_2) + \omega_3 (x_3) \end{array} \right) \right] \\
+ \omega_1 (x_1) + \omega_2 (x_2) + \omega_3 (x_3) \right] \\
= - \int_{\mathbb{R}^3} \left[ \theta_r (x_1) \right] \\
+ \left( \begin{array}{c}
\omega_1 (x_1) + \omega_2 (x_2) + \omega_3 (x_3)
\end{array} \right] \right] \\
= - \int_{\mathbb{R}^3} \left[ \theta_r (x_1) \right] \\
+ \left( \begin{array}{c}
\omega_1 (x_1) + \omega_2 (x_2) + \omega_3 (x_3)
\end{array} \right] \right].
\]
Let $r \to +\infty$ we get

\[
\int_{\mathbb{R}^3} \left[ \hat{\omega}_1 (\bar{w}_1^1 \partial_x \bar{w}_1^k + \bar{w}_2^1 \partial_x \bar{w}_2^k + \bar{w}_3^1 \partial_x \bar{w}_3^k) \\
+ \hat{\omega}_2 (\bar{w}_1^2 \partial_x \bar{w}_1^k + \bar{w}_2^2 \partial_x \bar{w}_2^k + \bar{w}_3^2 \partial_x \bar{w}_3^k) \\
+ \hat{\omega}_3 (\bar{w}_1^3 \partial_x \bar{w}_1^k + \bar{w}_2^3 \partial_x \bar{w}_2^k + \bar{w}_3^3 \partial_x \bar{w}_3^k) \right] \\
= - \int_{\mathbb{R}^3} \left[ (\bar{w}_1^k \partial_x \bar{w}_1^1 + \bar{w}_2^k \partial_x \bar{w}_2^1 + \bar{w}_3^k \partial_x \bar{w}_3^1) \\
+ \bar{w}_2^k \partial_x \bar{w}_2^1 + \bar{w}_3^k \partial_x \bar{w}_3^1 \\
+ \bar{w}_3^k \partial_x \bar{w}_3^1 \right] \\
+ \int_{\mathbb{R}^3} (\bar{w}_1^k \partial_x \bar{w}_1^1 + \bar{w}_2^k \partial_x \bar{w}_2^1 + \bar{w}_3^k \partial_x \bar{w}_3^1) \\
+ \bar{w}_2^k \partial_x \bar{w}_2^1 + \bar{w}_3^k \partial_x \bar{w}_3^1 \\
+ \bar{w}_3^k \partial_x \bar{w}_3^1 \\
\right] \\
+ \int_{\mathbb{R}^3} (\bar{w}_1^k \partial_x \bar{w}_1^1 + \bar{w}_2^k \partial_x \bar{w}_2^1 + \bar{w}_3^k \partial_x \bar{w}_3^1) \\
+ \bar{w}_2^k \partial_x \bar{w}_2^1 + \bar{w}_3^k \partial_x \bar{w}_3^1 \\
+ \bar{w}_3^k \partial_x \bar{w}_3^1 \\
= 0
\]

Similarly,

\[
\int_{\mathbb{R}^3} \left[ \hat{\omega}_1 (\bar{w}_1^1 \partial_x \bar{w}_1^k + \bar{w}_2^1 \partial_x \bar{w}_2^k + \bar{w}_3^1 \partial_x \bar{w}_3^k) \\
+ \hat{\omega}_2 (\bar{w}_1^2 \partial_x \bar{w}_1^k + \bar{w}_2^2 \partial_x \bar{w}_2^k + \bar{w}_3^2 \partial_x \bar{w}_3^k) \\
+ \hat{\omega}_3 (\bar{w}_1^3 \partial_x \bar{w}_1^k + \bar{w}_2^3 \partial_x \bar{w}_2^k + \bar{w}_3^3 \partial_x \bar{w}_3^k) \right] \\
= - \int_{\mathbb{R}^3} \left[ (\bar{w}_1^k \partial_x \bar{w}_1^1 + \bar{w}_2^k \partial_x \bar{w}_2^1 + \bar{w}_3^k \partial_x \bar{w}_3^1) \\
+ \bar{w}_2^k \partial_x \bar{w}_2^1 + \bar{w}_3^k \partial_x \bar{w}_3^1 \\
+ \bar{w}_3^k \partial_x \bar{w}_3^1 \right] \\
+ \int_{\mathbb{R}^3} (\bar{w}_1^k \partial_x \bar{w}_1^1 + \bar{w}_2^k \partial_x \bar{w}_2^1 + \bar{w}_3^k \partial_x \bar{w}_3^1) \\
+ \bar{w}_2^k \partial_x \bar{w}_2^1 + \bar{w}_3^k \partial_x \bar{w}_3^1 \\
+ \bar{w}_3^k \partial_x \bar{w}_3^1 \\
= 0
\]

and

\[
\int_{\mathbb{R}^3} (\hat{\omega}_1 \partial_x q + \hat{\omega}_2 \partial_x q + \hat{\omega}_3 \partial_x q) = 0
\]

furthermore,

\[
\int_{\mathbb{R}^3} \hat{\omega}_1 \Delta \hat{\omega}_1 = \int_{\mathbb{R}^3} (\partial_{x_1} \hat{\omega}_1 + \partial_{x_2} \hat{\omega}_1 + \partial_{x_3} \hat{\omega}_1) = - \int_{\mathbb{R}^3} \left( (\partial_{x_1} \hat{\omega}_1)^2 + (\partial_{x_2} \hat{\omega}_1)^2 + (\partial_{x_3} \hat{\omega}_1)^2 \right)
\]

Thus from (7) we have

\[
\int_{\mathbb{R}^3} \hat{\omega}_1 \partial_t \hat{\omega}_1 + \int_{\mathbb{R}^3} \hat{\omega}_2 (\bar{w}_1^1 \partial_x \bar{w}_1^k + \bar{w}_2^1 \partial_x \bar{w}_2^k + \bar{w}_3^1 \partial_x \bar{w}_3^k) \\
- \int_{\mathbb{R}^3} \hat{\omega}_2 (\bar{w}_1^2 \partial_x \bar{w}_1^k + \bar{w}_2^2 \partial_x \bar{w}_2^k + \bar{w}_3^2 \partial_x \bar{w}_3^k) \\
+ \int_{\mathbb{R}^3} \hat{\omega}_3 (\bar{w}_1^3 \partial_x \bar{w}_1^k + \bar{w}_2^3 \partial_x \bar{w}_2^k + \bar{w}_3^3 \partial_x \bar{w}_3^k) \\
- \int_{\mathbb{R}^3} \hat{\omega}_3 (\bar{w}_1^3 \partial_x \bar{w}_1^k + \bar{w}_2^3 \partial_x \bar{w}_2^k + \bar{w}_3^3 \partial_x \bar{w}_3^k) \\
= \int_{\mathbb{R}^3} \hat{\omega}_1 \Delta \hat{\omega}_1 \\
\int_{\mathbb{R}^3} \hat{\omega}_2 \partial_t \hat{\omega}_2 + \int_{\mathbb{R}^3} \hat{\omega}_2 (\bar{w}_1^2 \partial_x \bar{w}_1^k + \bar{w}_2^2 \partial_x \bar{w}_2^k + \bar{w}_3^2 \partial_x \bar{w}_3^k) \\
- \int_{\mathbb{R}^3} \hat{\omega}_2 (\bar{w}_1^2 \partial_x \bar{w}_1^k + \bar{w}_2^2 \partial_x \bar{w}_2^k + \bar{w}_3^2 \partial_x \bar{w}_3^k) \\
+ \int_{\mathbb{R}^3} \hat{\omega}_2 \partial_x q = \int_{\mathbb{R}^3} \hat{\omega}_2 \Delta \hat{\omega}_2
\]
\[
\int_{\mathbb{R}^3} \tilde{\omega}_3 \partial_t \tilde{\omega}_3 + \int_{\mathbb{R}^3} \tilde{\omega}_3 (\overrightarrow{\nabla}^1 \partial_x \tilde{\omega}^1 + \overrightarrow{\nabla}^2 \partial_x \tilde{\omega}^2 + \overrightarrow{\nabla}^3 \partial_x \tilde{\omega}^3) \\
- \int_{\mathbb{R}^3} \tilde{\omega}_3 (\overrightarrow{\nabla}^1 \partial_x \tilde{\omega}^1 + \overrightarrow{\nabla}^2 \partial_x \tilde{\omega}^2 + \overrightarrow{\nabla}^3 \partial_x \tilde{\omega}^3) + \int_{\mathbb{R}^3} \tilde{\omega}_3 \partial_x \tilde{\omega}_3 q = \int_{\mathbb{R}^3} \tilde{\omega}_3 \Delta \tilde{\omega}_3
\]
so that
\[
\frac{1}{2} \partial_t \int_{\mathbb{R}^3} (\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{\omega}_3^2) + \int_{\mathbb{R}^3} \left[ (\partial_x \tilde{\omega}_1)^2 + (\partial_x \tilde{\omega}_1)^2 + (\partial_x \tilde{\omega}_1)^2 \\
+ (\partial_x \tilde{\omega}_2)^2 + (\partial_x \tilde{\omega}_2)^2 + (\partial_x \tilde{\omega}_2)^2 \\
+ (\partial_x \tilde{\omega}_3)^2 + (\partial_x \tilde{\omega}_3)^2 + (\partial_x \tilde{\omega}_3)^2 \right] \\
- \int_{\mathbb{R}^3} (\overrightarrow{\nabla}^1 \partial_x \tilde{\omega}_1 + \overrightarrow{\nabla}^2 \partial_x \tilde{\omega}_1 + \overrightarrow{\nabla}^3 \partial_x \tilde{\omega}_1 \\
+ \overrightarrow{\nabla}^2 \partial_x \tilde{\omega}_2 + \overrightarrow{\nabla}^2 \partial_x \tilde{\omega}_2 + \overrightarrow{\nabla}^2 \partial_x \tilde{\omega}_2 \\
+ \overrightarrow{\nabla}^3 \partial_x \tilde{\omega}_3 + \overrightarrow{\nabla}^3 \partial_x \tilde{\omega}_3 + \overrightarrow{\nabla}^3 \partial_x \tilde{\omega}_3) \\
+ \int_{\mathbb{R}^3} \left( \overrightarrow{\nabla}^1 \partial_x \tilde{\omega}_1 + \overrightarrow{\nabla}^2 \partial_x \tilde{\omega}_2 + \overrightarrow{\nabla}^3 \partial_x \tilde{\omega}_3 \right) = 0
\]

By using Young inequality: \( uv \leq \frac{1}{4} u^2 + v^2 \), it follows that
\[
\partial_t \int_{\mathbb{R}^3} (\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{\omega}_3^2) + 2 \int_{\mathbb{R}^3} \left[ (\partial_x \tilde{\omega}_1)^2 + (\partial_x \tilde{\omega}_1)^2 + (\partial_x \tilde{\omega}_1)^2 \\
+ (\partial_x \tilde{\omega}_2)^2 + (\partial_x \tilde{\omega}_2)^2 + (\partial_x \tilde{\omega}_2)^2 \\
+ (\partial_x \tilde{\omega}_3)^2 + (\partial_x \tilde{\omega}_3)^2 + (\partial_x \tilde{\omega}_3)^2 \right] \\
\leq 2 \int_{\mathbb{R}^3} \left( \overrightarrow{\nabla}^1 \partial_x \tilde{\omega}_1 + \overrightarrow{\nabla}^2 \partial_x \tilde{\omega}_1 + \overrightarrow{\nabla}^3 \partial_x \tilde{\omega}_1 \\
+ \overrightarrow{\nabla}^2 \partial_x \tilde{\omega}_2 + \overrightarrow{\nabla}^2 \partial_x \tilde{\omega}_2 + \overrightarrow{\nabla}^2 \partial_x \tilde{\omega}_2 \\
+ \overrightarrow{\nabla}^3 \partial_x \tilde{\omega}_3 + \overrightarrow{\nabla}^3 \partial_x \tilde{\omega}_3 + \overrightarrow{\nabla}^3 \partial_x \tilde{\omega}_3 \right) \\
+ 2 \int_{\mathbb{R}^3} \left( \overrightarrow{\nabla}^1 \partial_x \tilde{\omega}_1 + \overrightarrow{\nabla}^1 \partial_x \tilde{\omega}_1 + \overrightarrow{\nabla}^1 \partial_x \tilde{\omega}_1 \\
+ \overrightarrow{\nabla}^2 \partial_x \tilde{\omega}_2 + \overrightarrow{\nabla}^2 \partial_x \tilde{\omega}_2 + \overrightarrow{\nabla}^2 \partial_x \tilde{\omega}_2 \\
+ \overrightarrow{\nabla}^3 \partial_x \tilde{\omega}_3 + \overrightarrow{\nabla}^3 \partial_x \tilde{\omega}_3 + \overrightarrow{\nabla}^3 \partial_x \tilde{\omega}_3 \right) \\
+ \int_{\mathbb{R}^3} \left[ (\partial_x \tilde{\omega}_1)^2 + (\partial_x \tilde{\omega}_1)^2 + (\partial_x \tilde{\omega}_1)^2 \\
+ (\partial_x \tilde{\omega}_2)^2 + (\partial_x \tilde{\omega}_2)^2 + (\partial_x \tilde{\omega}_2)^2 \\
+ (\partial_x \tilde{\omega}_3)^2 + (\partial_x \tilde{\omega}_3)^2 + (\partial_x \tilde{\omega}_3)^2 \right]
\]
Thus,
\[
\frac{\partial}{\partial t} \int_{\mathbb{R}^3} \left( \dot{\omega}_1^2 + \dot{\omega}_2^2 + \dot{\omega}_3^2 \right) + \int_{\mathbb{R}^3} \left[ (\partial_x \omega_1)^2 + (\partial_x \omega_1)^2 + (\partial_x \omega_1)^2 \right. \\
+ (\partial_x \omega_2)^2 + (\partial_x \omega_2)^2 + (\partial_x \omega_2)^2 \\
+ (\partial_x \omega_3)^2 + (\partial_x \omega_3)^2 + (\partial_x \omega_3)^2 \bigg] \\
\leq 4 \left\{ \left\| \mathcal{F}_i^k \right\|^2_{L^2(\mathbb{R}^3)} \left( \left\| \mathcal{F}_1^k \right\|^2_{L^\infty(\mathbb{R}^3)} + \left\| \mathcal{F}_2^k \right\|^2_{L^\infty(\mathbb{R}^3)} + \left\| \mathcal{F}_3^k \right\|^2_{L^\infty(\mathbb{R}^3)} \right) \\
+ \left\| \mathcal{F}_1^k \right\|^2_{L^2(\mathbb{R}^3)} \left( \left\| \mathcal{F}_2^k \right\|^2_{L^\infty(\mathbb{R}^3)} + \left\| \mathcal{F}_3^k \right\|^2_{L^\infty(\mathbb{R}^3)} + \left\| \mathcal{F}_1^k \right\|^2_{L^\infty(\mathbb{R}^3)} \right) \\
+ \left\| \mathcal{F}_2^k \right\|^2_{L^2(\mathbb{R}^3)} \left( \left\| \mathcal{F}_1^k \right\|^2_{L^\infty(\mathbb{R}^3)} + \left\| \mathcal{F}_3^k \right\|^2_{L^\infty(\mathbb{R}^3)} + \left\| \mathcal{F}_2^k \right\|^2_{L^\infty(\mathbb{R}^3)} \right) \right\} \\
= 4 \left( \left\| \mathcal{F}_1^k \right\|^2_{L^2(\mathbb{R}^3)} + \left\| \mathcal{F}_2^k \right\|^2_{L^2(\mathbb{R}^3)} + \left\| \mathcal{F}_3^k \right\|^2_{L^2(\mathbb{R}^3)} \right) \left( \left\| \mathcal{F}_1^k \right\|^2_{L^\infty(\mathbb{R}^3)} + \left\| \mathcal{F}_2^k \right\|^2_{L^\infty(\mathbb{R}^3)} + \left\| \mathcal{F}_3^k \right\|^2_{L^\infty(\mathbb{R}^3)} \right)
\]

Note that
\[
\left\| \mathcal{F}_i^k \right\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \left( \frac{1}{\Delta t_k} \int_{t_{k-1}}^{t_k} u_i(x, t) dt \right)^2 \leq \frac{1}{\Delta t_k^2} \int_{\mathbb{R}^3} \Delta t_k \int_{t_{k-1}}^{t_k} u_i^2(x, t) dt \\
= \frac{1}{\Delta t_k} \int_{t_{k-1}}^{t_k} \left\| u_i \right\|^2_{L^2(\mathbb{R}^3)} \leq \sup_{(t_{k-1}, t_k)} \left\| u_i \right\|^2_{L^2(\mathbb{R}^3)}
\]

and similarly
\[
\left\| \mathcal{F}_i^k \right\|^2_{L^2(\mathbb{R}^3)} \leq \frac{1}{\Delta t_k} \int_{t_{k-1}}^{t_k} \left\| \dot{\omega}_i \right\|^2_{L^2(\mathbb{R}^3)}, \quad i = 1, 2, 3
\]

In addition, a convolution inequality in [1] is applied to get
\[
\left\| \mathcal{F}_i^k \right\|^2_{L^\infty(\mathbb{R}^3)} = \left\| J_\varepsilon \ast \mathcal{F}_i^k(x) \right\|^2_{L^\infty(\mathbb{R}^3)} \\
\leq \left\| J_\varepsilon \right\|^2_{L^2(\mathbb{R}^3)} \left\| \mathcal{F}_i^k \right\|^2_{L^2(\mathbb{R}^3)} \leq \frac{1}{\mu_\varepsilon} \sup_{(t_{k-1}, t_k)} \left\| \dot{\omega}_i \right\|^2_{L^2(\mathbb{R}^3)}
\]

where the quantity \( \mu_\varepsilon \to 0 \) as \( \varepsilon \to 0 \). We need further assuming that \( \varepsilon \to 0 \) and
\[
\frac{\Delta t_k}{\mu_\varepsilon} \to 0 \quad \text{as} \quad k \to \infty \quad \text{or} \quad \Delta t_k \to 0
\]

From (8) we have
\[
\int_{\mathbb{R}^3} \left( \dot{\omega}_1^2 + \dot{\omega}_2^2 + \dot{\omega}_3^2 \right) + \int_{t_{k-1}}^{t_k} \left( \left\| \nabla \dot{\omega}_1 \right\|^2_{L^2(\mathbb{R}^3)} + \left\| \nabla \dot{\omega}_2 \right\|^2_{L^2(\mathbb{R}^3)} + \left\| \nabla \dot{\omega}_3 \right\|^2_{L^2(\mathbb{R}^3)} \right) \\
\leq \int_{\mathbb{R}^3} \left( \dot{\omega}_1^{k-1} + \dot{\omega}_2^{k-1} + \dot{\omega}_3^{k-1} \right) + \\
+ \frac{4\Delta t_k}{\mu_\varepsilon} \sup_{(t_{k-1}, t_k)} \left\{ \left\| u_1 \right\|^2_{L^2(\mathbb{R}^3)} + \left\| u_2 \right\|^2_{L^2(\mathbb{R}^3)} + \left\| u_3 \right\|^2_{L^2(\mathbb{R}^3)} \right\} \cdot \sup_{(t_{k-1}, t)} \int_{\mathbb{R}^3} \left( \dot{\omega}_1^2 + \dot{\omega}_2^2 + \dot{\omega}_3^2 \right)
\]
By (6) we have
\[
\sup_{(t_{k-1}, t_k)} \left\{ \|u_1\|_{L^2_2(\mathbb{R}^3)}^2 + \|u_2\|_{L^2_2(\mathbb{R}^3)}^2 + \|u_3\|_{L^2_2(\mathbb{R}^3)}^2 \right\}
\leq K_0 = \sup_{t \in (0, T)} \int_{\mathbb{R}^3} \left( u_1^2 + u_2^2 + u_3^2 \right) + \int_0^T \left( \|\nabla u_1\|_{L^2_2(\mathbb{R}^3)}^2 + \|\nabla u_2\|_{L^2_2(\mathbb{R}^3)}^2 + \|\nabla u_3\|_{L^2_2(\mathbb{R}^3)}^2 \right) < +\infty
\]

Thus,
\[
\left( 1 - 4K_0 \frac{\Delta t_k}{\mu \varepsilon} \right) \sup_{t \in (t_{k-1}, t_k)} \int_{\mathbb{R}^3} \left( \tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{\omega}_3^2 \right) + \int_{t_{k-1}}^{t_k} \left( \|\nabla \tilde{\omega}_1\|_{L^2_2(\mathbb{R}^3)}^2 + \|\nabla \tilde{\omega}_2\|_{L^2_2(\mathbb{R}^3)}^2 + \|\nabla \tilde{\omega}_3\|_{L^2_2(\mathbb{R}^3)}^2 \right) \leq \int_{\mathbb{R}^3} \left( \tilde{\omega}_{1}^{k-1} + \tilde{\omega}_2^{k-1} + \tilde{\omega}_3^{k-1} \right)
\]

Now we set
\[
M_0 = \int_{\mathbb{R}^3} (\omega_{10}^2 + \omega_{20}^2 + \omega_{30}^2)
\]
\[
M_k = \sup_{t \in (t_{k-1}, t_k)} \int_{\mathbb{R}^3} (\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{\omega}_3^2)
\]
\[
\delta_k = \int_{t_{k-1}}^{t_k} \left( \|\nabla \tilde{\omega}_1\|_{L^2_2(\mathbb{R}^3)}^2 + \|\nabla \tilde{\omega}_2\|_{L^2_2(\mathbb{R}^3)}^2 + \|\nabla \tilde{\omega}_3\|_{L^2_2(\mathbb{R}^3)}^2 \right)
\]

then we have
\[
\left( 1 - 4K_0 \frac{\Delta t_k}{\mu \varepsilon} \right) M_k + \delta_k \leq M_{k-1}
\]

The partition is assumed to be fine enough. Because of the local existence of Galerkin solution in section 3 and the absolute continuity of integration with respect to \( t \), it is valid that \( \delta_k \to 0 \) as \( \Delta t_k \to 0 \).

We may first consider the case that
\[
M_{k-1} \frac{\Delta t_k}{\delta_k} \to 0, \quad \text{as} \quad \Delta t_k \to 0
\]

which may be a subsequence \( k' \), still denoted \( k \). At this time, we can choose \( \varepsilon_k \) on each \( (t_{k-1}, t_k) \) such that
\[
\mu \varepsilon_k = 4K_0 M_{k-1} \frac{\Delta t_k}{\delta_k} \quad \text{and} \quad 1 - 4K_0 \frac{\Delta t_k}{\mu \varepsilon_k} \geq \frac{1}{2}
\]

\[ \varepsilon = \max_k \{ \varepsilon_k \} . \]

Then we obtain
\[
\left( 1 - 4K_0 \frac{\Delta t_k}{\mu \varepsilon_k} \right) M_k + \delta_k = \left( 1 - \frac{\delta_k}{M_{k-1}} \right) M_k + \delta_k \leq M_{k-1}
\]
it follows that $M_k \leq M_{k-1}$.

Otherwise, $\delta_k \leq O(\Delta t_k) M_{k-1}$. In this case, a convolution inequality in [1] is applied to get

$$\left\| \nabla \omega_i^* \right\|^2_{L^2(\mathbb{R}^3)} \leq \left\| J \ast \omega_i \right\|^2_{L^2(\mathbb{R}^3)} \leq \left\| J \right\|^2_{L^1(\mathbb{R}^3)} \left\| \nabla \omega_i^* \right\|^2_{L^2(\mathbb{R}^3)}$$

where $J = \omega_i \ast 1$ is the convolution of $\omega_i$ with the indicator function $1$. This leads to that

$$\left\| \nabla \omega_i^* \right\|^2_{L^2(\mathbb{R}^3)} \leq \sup_{(t_{k-1}, t_k)} \left\| \omega_i \right\|^2_{L^2(\mathbb{R}^3)}$$

This leads to that

$$\{ \left\| \nabla \omega_1^* \right\|^2_{L^2(\mathbb{R}^3)} + \left\| \nabla \omega_2^* \right\|^2_{L^2(\mathbb{R}^3)} + \left\| \nabla \omega_3^* \right\|^2_{L^2(\mathbb{R}^3)} \} \leq \frac{1}{\Delta t_k} \int_{t_{k-1}}^{t_k} \left( \left\| \nabla \omega_1 \right\|^2_{L^2(\mathbb{R}^3)} + \left\| \nabla \omega_2 \right\|^2_{L^2(\mathbb{R}^3)} + \left\| \nabla \omega_3 \right\|^2_{L^2(\mathbb{R}^3)} \right) \leq O(1) M_{k-1}$$

Since these $(t_{k-1}, t_k)$ are of finite length, the number of them is finite. According to Cauchy-Schwartz inequality, similar to (8), we have

$$\partial_t \int_{\mathbb{R}^3} (\omega_1^2 + \omega_2^2 + \omega_3^2) + \int_{\mathbb{R}^3} \left[ (\partial_x \omega_1)^2 + (\partial_x \omega_2)^2 + (\partial_x \omega_3)^2 \right]$$

$$\leq 4 \left\{ \left( \int_{\mathbb{R}^3} \nabla \omega_1^4 \right) \frac{1}{4} \left( \int_{\mathbb{R}^3} \nabla \omega_1^4 \right) \frac{1}{4} \left( \int_{\mathbb{R}^3} \nabla \omega_2^4 \right) \frac{1}{4} + \left( \int_{\mathbb{R}^3} \nabla \omega_1^4 \right) \frac{1}{4} \left( \int_{\mathbb{R}^3} \nabla \omega_3^4 \right) \frac{1}{4} \left( \int_{\mathbb{R}^3} \nabla \omega_1^4 \right) \frac{1}{4} \left( \int_{\mathbb{R}^3} \nabla \omega_3^4 \right) \frac{1}{4} \right\}$$

$$= 4 \left\{ \left\| \nabla \omega_1 \right\|^2_{L^4(\mathbb{R}^3)} \left\| \nabla \omega_2 \right\|^2_{L^4(\mathbb{R}^3)} + \left\| \nabla \omega_3 \right\|^2_{L^4(\mathbb{R}^3)} \left\| \nabla \omega_1 \right\|^2_{L^4(\mathbb{R}^3)} + \left\| \nabla \omega_3 \right\|^2_{L^4(\mathbb{R}^3)} \left\| \nabla \omega_1 \right\|^2_{L^4(\mathbb{R}^3)} \right\}$$

From Sobolev imbedding theorem in [1], there exists a constant $C_1 > 0$ independent of $\omega$ such that

$$\left\| \nabla \omega_i \right\|^2_{L^4(\mathbb{R}^3)} \leq C_1 \left\{ \left\| \nabla \omega_i \right\|^2_{L^2(\mathbb{R}^3)} + \left\| \nabla \omega_i^* \right\|^2_{L^2(\mathbb{R}^3)} \right\}, \quad i = 1, 2, 3$$
Therefore,
\[
\int_{\mathbb{R}^3} (\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{\omega}_3^2) + \int_{t_{k-1}}^t \left( \|\nabla \tilde{\omega}_1\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \tilde{\omega}_2\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \tilde{\omega}_3\|_{L^2(\mathbb{R}^3)}^2 \right)
\leq \int_{\mathbb{R}^3} (\tilde{\omega}_1^{k-1})^2 + \tilde{\omega}_2^{k-1} + \tilde{\omega}_3^{k-1}^2 + C_2 \left( \|\tilde{\omega}_1^1\|_{L^2(\mathbb{R}^3)}^2 + \|\tilde{\omega}_2^1\|_{L^2(\mathbb{R}^3)}^2 + \|\tilde{\omega}_3^1\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \tilde{\omega}_1\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \tilde{\omega}_2\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \tilde{\omega}_3\|_{L^2(\mathbb{R}^3)}^2 \right)
\times \int_{t_{k-1}}^t \int_{\mathbb{R}^3} (\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{\omega}_3^2) +
\int_{t_{k-1}}^t \left( \|\nabla \tilde{\omega}_1\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \tilde{\omega}_2\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \tilde{\omega}_3\|_{L^2(\mathbb{R}^3)}^2 \right)
\times \left( \|\nabla \tilde{\omega}_1\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \tilde{\omega}_2\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \tilde{\omega}_3\|_{L^2(\mathbb{R}^3)}^2 \right)
\]

Thus we have
\[
\int_{\mathbb{R}^3} (\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{\omega}_3^2) + \int_{t_{k-1}}^t \left( \|\nabla \tilde{\omega}_1\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \tilde{\omega}_2\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla \tilde{\omega}_3\|_{L^2(\mathbb{R}^3)}^2 \right)
\leq \int_{\mathbb{R}^3} (\tilde{\omega}_1^{k-1})^2 + \tilde{\omega}_2^{k-1} + \tilde{\omega}_3^{k-1}^2 + C_2 \left( \sup_{(t_{k-1}, t_k)} \{ \|u_1\|_{L^2(\mathbb{R}^3)}^2 + \|u_2\|_{L^2(\mathbb{R}^3)}^2 + \|u_3\|_{L^2(\mathbb{R}^3)}^2 \} \right)
\times \int_{t_{k-1}}^t \int_{\mathbb{R}^3} (\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{\omega}_3^2) +
+ \int_{t_{k-1}}^t \left( \|\nabla u_1\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla u_2\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla u_3\|_{L^2(\mathbb{R}^3)}^2 \right)
\times \left( \|\nabla u_1\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla u_2\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla u_3\|_{L^2(\mathbb{R}^3)}^2 \right)
\times \left( \|\nabla u_1\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla u_2\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla u_3\|_{L^2(\mathbb{R}^3)}^2 \right)
\]

where $C_2, C_3 > 0$ are constants independent of $k$. Set
\[
K^*_k = \Delta t_k \sup_{(t_{k-1}, t_k)} \{ \|u_1\|_{L^2(\mathbb{R}^3)}^2 + \|u_2\|_{L^2(\mathbb{R}^3)}^2 + \|u_3\|_{L^2(\mathbb{R}^3)}^2 \} +
+ \int_{t_{k-1}}^t \left( \|\nabla u_1\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla u_2\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla u_3\|_{L^2(\mathbb{R}^3)}^2 \right)
\]

and
\[
f_k(t) = \sup_{(t_{k-1}, t)} \int_{\mathbb{R}^3} (\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{\omega}_3^2)
\]
Then we arrive at
\[ f_k(t) \leq M_{k-1} + C_2 \frac{1}{\Delta t_k} K_k^* \int_{t_{k-1}}^t f_k(t) + C_3 K_k^* M_{k-1} \]

By using Gronwall inequality it follows that
\[ M_k \leq (1 + C_3 K_k^*) \exp(C_2 K_k^*) M_{k-1} \]

Note that
\[
\sum_{k=1}^N K_k^* \leq T \sup_{t \in (0,T)} \int_{\mathbb{R}^3} (u_1^2 + u_2^2 + u_3^2) + \int_0^T (\|\nabla u_1\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla u_2\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla u_3\|_{L^2(\mathbb{R}^3)}^2) \\
\leq (T + 1) K_0 < +\infty
\]

Hence, combining above two cases, we obtain
\[ M_1 \leq (1 + C_3 K_1^*) \exp(C_2 K_1^*) M_0 \]
\[ M_2 \leq (1 + C_3 K_1^*) (1 + C_3 K_2^*) \exp\left(C_2 \sum_{k=1}^2 K_k^*\right) M_0 \]
\[ \vdots \]
\[ M_N \leq \prod_{k=1}^N (1 + C_3 K_k^*) \exp\left(C_2 \sum_{k=1}^N K_k^*\right) M_0 \]

Note that
\[
\prod_{k=1}^N (1 + C_3 K_k^*) = \exp\left(\ln \prod_{k=1}^N (1 + C_3 K_k^*)\right) \\
= \exp\left(\sum_{k=1}^N \ln(1 + C_3 K_k^*)\right) \leq \exp\left(C_3 \sum_{k=1}^N K_k^*\right) = \exp(C_3(T + 1)K_0)
\]

These mean that
\[ M_k \leq M_0 \exp((C_2 + C_3)(T + 1)K_0) \quad k = 1, \cdots, N \]

Finally we get
\[
\sup_{t \in (0,T)} \int_{\mathbb{R}^3} (\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{\omega}_3^2) \leq \max_k \{M_k\} \\
\leq M_0 \exp((C_2 + C_3)(T + 1)K_0)
\]
From condition (3) it is found that $M_0$ is bounded, namely,
\[
\int_{\mathbb{R}^3} \left( \omega_{10}^2(x) + \omega_{20}^2(x) + \omega_{30}^2(x) \right) \leq \int_{|x| \leq R} \left( \omega_{10}^2(x) + \omega_{20}^2(x) + \omega_{30}^2(x) \right) + 12 C^2_\mu \int_{|x| > R} \frac{1}{(1 + |x|)^{2\sigma}} < +\infty
\]
and $R$ is a large constant.

This conclusion is also true for the weak solution of problem (7), by means of the result of section 3 and the lower limit of Galerkin sequence according to the page 196 of [4].

3. Existence

In this section we have to consider the existence of solutions of the auxiliary problems. We just need considering the following system on $(0, \delta)$:
\[
\begin{align*}
\partial_t \tilde{\omega}_1 + \nabla_x \cdot \mathbb{P}_1 &= \nabla_x \cdot \tilde{\mathbb{P}}_1 + \nabla_x \cdot \tilde{\mathbb{P}}_2 + \nabla_x \cdot \tilde{\mathbb{P}}_3 - \nabla_x \cdot \omega_{10} - \nabla_x \cdot \omega_{20} - \nabla_x \cdot \omega_{30} + \partial_x q = \Delta \tilde{\omega}_1 \\
\partial_t \tilde{\omega}_2 + \nabla_x \cdot \mathbb{P}_2 &= \nabla_x \cdot \tilde{\mathbb{P}}_2 + \nabla_x \cdot \tilde{\mathbb{P}}_3 - \nabla_x \cdot \omega_{10} - \nabla_x \cdot \omega_{20} - \nabla_x \cdot \omega_{30} + \partial_x q = \Delta \tilde{\omega}_2 \\
\partial_t \tilde{\omega}_3 + \nabla_x \cdot \mathbb{P}_3 &= \nabla_x \cdot \tilde{\mathbb{P}}_3 - \nabla_x \cdot \omega_{10} - \nabla_x \cdot \omega_{20} - \nabla_x \cdot \omega_{30} + \partial_x q = \Delta \tilde{\omega}_3
\end{align*}
\]

with the initial value $\tilde{\omega}_i(x,0) = \omega_{i0}$ $(i = 1, 2, 3)$ and
\[
\mathbb{P}(x) = \frac{1}{\delta} \int_0^{\delta} u_i(x,t)dt
\]
and
\[
\mathbb{I}(x) = \frac{1}{\delta} \int_0^{\delta} \tilde{\omega}_i(x,t)dt,
\]
as well as the incompressible conditions:
\[
\begin{align*}
\partial_x u_1 + \partial_x u_2 + \partial_x u_3 &= 0 \quad \Rightarrow \quad \partial_x \mathbb{P}_1 + \partial_x \mathbb{P}_2 + \partial_x \mathbb{P}_3 = 0 \\
\partial_x \tilde{\omega}_1 + \partial_x \tilde{\omega}_2 + \partial_x \tilde{\omega}_3 &= 0 \quad \Rightarrow \quad \partial_x \tilde{\mathbb{P}}_1 + \partial_x \tilde{\mathbb{P}}_2 + \partial_x \tilde{\mathbb{P}}_3 = 0
\end{align*}
\]

(i) The Galerkin procedure is applied. For each $m$ and $i = 1, 2, 3$ we define an approximate solution $(\tilde{\omega}_{1m}, \tilde{\omega}_{2m}, \tilde{\omega}_{3m})$ as follows:
\[
\tilde{\omega}_{im} = \sum_{j=1}^{m} g_{ij}(t)w_{ij}
\]
where $\{w_{11}, \ldots, w_{im}, \ldots\}$ is the basis of $W$, and $W = \text{the closure of } \mathcal{V}$ in the Sobolev space $W^{2,q}(\mathbb{R}^3)$, which is separable and is dense in $V$. Thus by means
of weighted function $\theta_r$ introduced in Section 1,

\[
(\theta_r \partial_t \tilde{\omega}_{im}, w_{il}) + (\theta_r \nabla \tilde{\omega}_{im}, \nabla w_{il}) + (\nabla \tilde{\omega}_{im}, w_{il} \nabla \theta_r) +
(\theta_r \langle \nabla \tilde{\omega}_{im}, \nabla \tilde{\omega}_{im} \rangle, w_{il}) = 0
\]  

(10)

let $r \to +\infty$ we get

\[
(\partial_t \tilde{\omega}_{im}, w_{il}) + (\nabla \tilde{\omega}_{im}, \nabla w_{il}) + (\langle \nabla \tilde{\omega}_{im}, \nabla \tilde{\omega}_{im} \rangle, w_{il}) = 0
\]

(11)

\[ t \in (0, \delta), \quad \tilde{\omega}_{im}(0) = \omega_{im}^0, \quad l = 1, \cdots, m \]

where $\omega_{im}^0$ is the orthogonal projection in $H$ of $\omega_{i0}$ onto the space spanned by $w_{i1}, \cdots, w_{im}$. Therefore,

\[
\sum_{j=1}^{m} (w_{ij}, w_{il}) g_{ij}^0(t) + \sum_{j=1}^{m} (\nabla w_{ij}, \nabla w_{il}) g_{ij}(t) +
\sum_{j=1}^{m} \{ (\langle \nabla \omega \rangle w_{ij}^*, w_{il}) - (\langle \nabla \omega \rangle w_{il}, \omega_i(t)) \} \bar{g}_{ij}(t) = 0
\]

where $w_{ij}^* = J_{x} * w_{ij}$, $w_{ij}^* = J_{x} * w_{ij}$, $\bar{g}_{ij}(t) = \frac{1}{\delta} \int_{0}^{\delta} g_{ij}(t) dt$ and $u_i \in L^\infty(0, T; H)$ from Section 1 which are determined by equations (1). Inverting the nonsingular matrix with elements $(w_{ij}, w_{il})$, $1 \leq j, l \leq m$, we can write above system in the following form

\[
g_{ij}(t) + \sum_{l=1}^{m} \alpha_{ijl} g_{il}(t) + \sum_{l=1}^{m} \beta_{ijl} \bar{g}_{il}(t) = 0
\]

(12)

where $\alpha_{ijl}$, $\beta_{ijl}$ are constants.

The initial conditions are equivalent to

\[
g_{ij}(0) = \tilde{g}_{ij}^0 = \text{the } j^{th} \text{ component of } \omega_{i0}^m
\]

We construct a sequence $\{g_{ij}^k\}$ by using a successive approximation:

\[
g_{ij}^1 = - \sum_{l=1}^{m} \alpha_{ijl} g_{il}^0 - \sum_{l=1}^{m} \beta_{ijl} \bar{g}_{il}^0 \Rightarrow \quad g_{ij}^1 = g_{ij}^0 - \int_{0}^{t} \left( \sum_{l=1}^{m} \alpha_{ijl} g_{il}^0 + \sum_{l=1}^{m} \beta_{ijl} \bar{g}_{il}^0 \right)
\]

\[
g_{ij}^2 = - \sum_{l=1}^{m} \alpha_{ijl} g_{il}^0 - \sum_{l=1}^{m} \beta_{ijl} \bar{g}_{il}^0 \Rightarrow \quad g_{ij}^2 = g_{ij}^0 - \int_{0}^{t} \left( \sum_{l=1}^{m} \alpha_{ijl} g_{il}^1 + \sum_{l=1}^{m} \beta_{ijl} \bar{g}_{il}^0 \right)
\]

\[
\cdots \cdots 
\]

\[
g_{ij}^k = - \sum_{l=1}^{m} \alpha_{ijl} g_{il}^{k-1} - \sum_{l=1}^{m} \beta_{ijl} \bar{g}_{il}^{k-1} \Rightarrow \quad g_{ij}^k = g_{ij}^0 - \int_{0}^{t} \left( \sum_{l=1}^{m} \alpha_{ijl} g_{il}^{k-1} + \sum_{l=1}^{m} \beta_{ijl} \bar{g}_{il}^{k-1} \right)
\]

so that

\[
|g_{ij}^k(t) - g_{ij}^{k-1}(t)| \leq \int_{0}^{t} \left( \sum_{l=1}^{m} |\alpha_{ijl}| \left| g_{il}^{k-1}(t) - g_{il}^{k-2}(t) \right| + \sum_{l=1}^{m} |\beta_{ijl}| \left| \bar{g}_{il}^{k-1}(t) - \bar{g}_{il}^{k-2}(t) \right| \right)
\]

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It follows that
\[
\max_{i,j} \sup_t |g_{ij}^k(t) - g_{ij}^{k-1}(t)| \leq \max_{i,j} \sum_{l=1}^m (|\alpha_{ijl}| + |\beta_{ijl}|) \cdot t \cdot \max_{i,j} |g_{ij}^{k-1}(t) - g_{ij}^{k-2}(t)|
\]
Taking \(\delta := \frac{1}{\max_{i,j} \sum_{l=1}^m (|\alpha_{ijl}| + 2|\beta_{ijl}|)}\), as \(t \to 0\), then choosing \(\delta^*\):
\[
0 < \delta^* = \frac{\max_{i,j} \sum_{l=1}^m (|\alpha_{ijl}| + |\beta_{ijl}|)}{\max_{i,j} \sum_{l=1}^m (|\alpha_{ijl}| + 2|\beta_{ijl}|)} < 1
\]
we have
\[
\max_{i,j} \|g_{ij}^k - g_{ij}^{k-1}\|_{\infty} \leq \delta^* \max_{i,j} \|g_{ij}^{k-1} - g_{ij}^{k-2}\|_{\infty} \leq \cdots \leq (\delta^*)^{k-1} \max_{i,j} \|g_{ij}^1 - g_{ij}^0\|_{\infty}
\]
For any \(n, k\) (we can set \(n > k\) without loss of generality), we get
\[
\max_{i,j} \|g_{ij}^n - g_{ij}^k\|_{\infty} \leq \max_{i,j} \|g_{ij}^n - g_{ij}^{n-1}\|_{\infty} + \cdots + \max_{i,j} \|g_{ij}^{k+1} - g_{ij}^k\|_{\infty}
\]
\[
\leq ((\delta^*)^{n-1} + \cdots + (\delta^*)^k) \max_{i,j} \|g_{ij}^n - g_{ij}^0\|_{\infty} = (\delta^*)^k \frac{1 - (\delta^*)^{n-k}}{1 - \delta^*} \max_{i,j} \|g_{ij}^1 - g_{ij}^0\|_{\infty}
\]
\[
\to 0 \quad (k \to \infty)
\]
Thus, for every \(i = 1, 2, 3; \ j = 1, \cdots, m\), \(\{g_{ij}^n\}\) is a Cauchy sequence in \(L^\infty(0, \delta)\). Since \(L^\infty(0, \delta)\) is complete, then there exists a function \(g_{ij}^* \in L^\infty(0, \delta)\) such that
\[
\|g_{ij}^k - g_{ij}^*\|_{\infty} \to 0 \quad \text{as} \quad k \to \infty.
\]
From
\[
g_{ij}^k(t) = g_{ij}^0 - \int_0^t \left( \sum_{l=1}^m \alpha_{ijl} g_{il}^{k-1}(t) + \sum_{l=1}^m \beta_{ijl} \bar{g}_{il}^{k-1}(t) \right)
\]
let \(k \to \infty\), it follows that
\[
g_{ij}^*(t) = g_{ij}^0 - \int_0^t \left( \sum_{l=1}^m \alpha_{ijl} g_{il}^*(t) + \sum_{l=1}^m \beta_{ijl} \bar{g}_{il}(t) \right)
\]
i.e., \(g_{ij}^*\) is a solution of the system (12) on \((0, \delta)\) for which \(g_{ij}^*(0) = g_{ij}^0\), \(i = 1, 2, 3; \ j = 1, \cdots, m\).

(ii) By means of the weighted function \(\theta_r\):
\[
\sum_{i=1}^3 (\theta_r \partial_h \bar{\omega}_{im}, \bar{\omega}_{im}) + \sum_{i=1}^3 (\theta_r \nabla \bar{\omega}_{im}, \nabla \bar{\omega}_{im}) + \sum_{i=1}^3 (\nabla \bar{\omega}_{im}, \bar{\omega}_{im} \nabla \theta_r) + \sum_{i=1}^3 (\theta_r (\bar{\rho} \cdot \nabla) \bar{\omega}_{im}, \bar{\omega}_{im}) - \sum_{i=1}^3 (\theta_r (\bar{\rho} \cdot \nabla) \bar{\omega}_{im}, \bar{\omega}_{im}) = 0
\]
Let \( r \to +\infty \) we get
\[
3 \sum_{i=1}^{3} (\partial_t \tilde{\omega}_{im}, \tilde{\omega}_{im}) + \sum_{i=1}^{3} (\nabla \tilde{\omega}_{im}, \nabla \tilde{\omega}_{im}) + \sum_{i=1}^{3} ((\vec{\pi} \cdot \nabla)\vec{\pi}_{im}, \tilde{\omega}_{im}) - \sum_{i=1}^{3} ((\vec{\pi} \cdot \nabla)\pi_{i}, \tilde{\omega}_{im}) = 0
\]
Then we write
\[
\frac{1}{2} \frac{d}{dt} \left( \sum_{i=1}^{3} \|\tilde{\omega}_{im}\|_{L^2(\mathbb{R}^3)}^2 \right) + \sum_{i=1}^{3} \|\nabla \tilde{\omega}_{im}\|_{L^2(\mathbb{R}^3)}^2 - \sum_{i=1}^{3} ((\vec{\pi} \cdot \nabla)\tilde{\omega}_{im}, \vec{\pi}_{im}) + \sum_{i=1}^{3} ((\vec{\pi} \cdot \nabla)\tilde{\omega}_{im}, \pi_{i}) = 0
\]
Similar to those in the section 2, and \( \eta \) is chosen to be small enough, we have
\[
3 \sum_{i=1}^{3} \|\tilde{\omega}_{im}\|_{L^2(\mathbb{R}^3)}^2 + \int_{0}^{\eta} \left( \sum_{i=1}^{3} \|\nabla \tilde{\omega}_{im}\|_{L^2(\mathbb{R}^3)}^2 \right) \leq 2 \left( \sum_{i=1}^{3} \|\omega_{i0}\|_{L^2(\mathbb{R}^3)}^2 \right)
\]
as \( 1 - 4K_0 \eta/\mu_z \geq 1/2 \). Hence,
\[
\sup_{t \in (0, \eta)} \left( \sum_{i=1}^{3} \|\tilde{\omega}_{im}\|_{L^2(\mathbb{R}^3)}^2 \right) \leq 2 \left( \sum_{i=1}^{3} \|\omega_{i0}\|_{L^2(\mathbb{R}^3)}^2 \right)
\]
and
\[
3 \sum_{i=1}^{3} \|\tilde{\omega}_{im}(\eta)\|_{L^2(\mathbb{R}^3)}^2 + \int_{0}^{\eta} \left( \sum_{i=1}^{3} \|\nabla \tilde{\omega}_{im}\|_{L^2(\mathbb{R}^3)}^2 \right) \leq 2 \left( \sum_{i=1}^{3} \|\omega_{i0}\|_{L^2(\mathbb{R}^3)}^2 \right)
\]
The inequalities (13) and (14) are valid for any fixed \( \delta \leq \eta \).

(iii) Let \( \tilde{\omega}_m \) denote the function from \( \mathbb{R} \) into \( V \), which is equal to \( \tilde{\omega}_m \) on \((0, \delta)\) and to 0 on the complement of this interval. The Fourier transform of \( \tilde{\omega}_m \) is denoted by \( \hat{\omega}_m \). We want to show that
\[
\int_{-\infty}^{+\infty} |\tau|^{2\gamma} \left( \sum_{i=1}^{3} \|\hat{\omega}_{im}\|_{L^2(\Omega)}^2 \right) d\tau < +\infty, \quad \forall \Omega \subset \mathbb{R}^3
\]
For some \( \gamma > 0 \). Along with (14) this will imply that
\[
\tilde{\omega}_m \text{ belongs to a bounded set of } H^\gamma(\mathbb{R}, H^1(\Omega), L^2(\Omega)), \quad \forall \Omega
\]
and will enable us to apply the result of compactness.

We observe that (10) can be written as
\[
\frac{d}{dt} \left( \sum_{i=1}^{3} (\theta_r \tilde{\omega}_{im}, w_{ij}) \right) = \sum_{i=1}^{3} (\theta_r \tilde{f}_{im}, w_{ij}) + \sum_{i=1}^{3} (\theta_r \omega_{i0}^m, w_{ij}) \eta_0 - \sum_{i=1}^{3} (\theta_r \tilde{\omega}_{im}(\delta), w_{ij}) \eta_\delta
\]
}\]
where \( \eta_0, \eta_\delta \) are Dirac distributions at 0 and \( \delta \), and

\[
f_{im} = -\Delta \hat{\omega}_{im} + (\bar{\omega} \cdot \nabla) \bar{\bar{\omega}}_{im} - (\bar{\omega}_m \cdot \nabla) \bar{u}_i
\]

\( \hat{f}_{im} = f_{im} \) on \((0, \delta)\), 0 outside this interval.

By the Fourier transform,

\[
2i\pi \tau \sum_{i=1}^{3} (\theta_r \hat{\omega}_{im}, w_{ij}) = \sum_{i=1}^{3} (\theta_r \hat{f}_{im}, w_{ij}) + \sum_{i=1}^{3} (\theta_r \omega_{m0}^i, w_{ij}) - \sum_{i=1}^{3} (\theta_r \hat{\omega}_{im}(\delta), w_{ij}) \exp(-2i\pi \delta) 
\]

where \( \hat{\omega}_{im} \) and \( \hat{f}_{im} \) denote the Fourier transforms of \( \omega_{im} \) and \( f_{im} \) respectively.

We multiply above equalities by \( \hat{g}_{ij}(\tau) = \text{Fourier transform of} \ g_{ij} \) and add the resulting equations for \( j = 1, \cdots, m \), we get

\[
2i\pi \tau \sum_{i=1}^{3} \left\| \theta_r^{1/2} \hat{\omega}_{im}(\tau) \right\|_{L^2(\mathbb{R}^3)}^2 = \sum_{i=1}^{3} (\theta_r \hat{f}_{im}(\tau), \hat{\omega}_{im}(\tau)) + \sum_{i=1}^{3} (\theta_r \omega_{m0}^i, \hat{\omega}_{im}(\tau)) - \sum_{i=1}^{3} (\theta_r \hat{\omega}_{im}(\delta), \hat{\omega}_{im}(\tau)) \exp(-2i\pi \delta) 
\]

For some \( \varphi_i \in V \),

\[
\int_0^{\delta} \sum_{i=1}^{3} (\theta_r f_{im}, \varphi_i) = \int_0^{\delta} \sum_{i=1}^{3} (-\theta_r \Delta \hat{\omega}_{im}, \varphi_i) + \int_0^{\delta} \sum_{i=1}^{3} (\theta_r (\bar{\omega} \cdot \nabla) \bar{\bar{\omega}}_{im}, \varphi_i) - \int_0^{\delta} \sum_{i=1}^{3} (\theta_r (\bar{\omega}_m \cdot \nabla) \bar{u}_i, \varphi_i) 
\]

\[
= \int_0^{\delta} \sum_{i=1}^{3} (\theta_r \nabla \hat{\omega}_{im}, \nabla \varphi_i) + \int_0^{\delta} \sum_{i=1}^{3} (\nabla \hat{\omega}_{im}, \varphi_i \nabla \theta_r) - \int_0^{\delta} \sum_{i=1}^{3} (\varphi_i (\bar{\omega} \cdot \nabla) \theta_r, \bar{\omega}_m) + \int_0^{\delta} \sum_{i=1}^{3} (\varphi_i (\bar{\omega}_m \cdot \nabla) \theta_r, \bar{u}_i) 
\]

Let \( r \rightarrow +\infty \) we get

\[
\int_0^{\delta} \sum_{i=1}^{3} (f_{im}, \varphi_i) = \int_0^{\delta} \sum_{i=1}^{3} (\nabla \hat{\omega}_{im}, \nabla \varphi_i) - \int_0^{\delta} \sum_{i=1}^{3} ((\bar{\omega} \cdot \nabla) \varphi_i, \bar{\omega}_m) + \int_0^{\delta} \sum_{i=1}^{3} ((\bar{\omega}_m \cdot \nabla) \varphi_i, \bar{u}_i) 
\]

\[
\leq \int_0^{\delta} \sum_{i=1}^{3} \left\| \nabla \hat{\omega}_{im} \right\|_{L^2(\mathbb{R}^3)} \left\| \nabla \varphi_i \right\|_{L^2(\mathbb{R}^3)} + 
\]
\[ + 2 \int_0^\delta \left( \sum_{i=1}^3 \| \nabla^i \|_{L^2(\mathbb{R}^3)}^2 \right)^{1/2} \left( \sum_{i=1}^3 \| \nabla \varphi \|_{L^2(\mathbb{R}^3)}^2 \right)^{1/2} \left( \sum_{i=1}^3 \| \nabla \varphi \|_{L^2(\mathbb{R}^3)}^2 \right)^{1/2} \]

\[ \leq \int_0^\delta \left( \sum_{i=1}^3 \| \nabla \omega \|_{L^2(\mathbb{R}^3)}^2 \right)^{1/2} \left( \sum_{i=1}^3 \| \nabla \varphi \|_{L^2(\mathbb{R}^3)}^2 \right)^{1/2} \left( \sum_{i=1}^3 \| \nabla \varphi \|_{L^2(\mathbb{R}^3)}^2 \right)^{1/2} + 
\]

\[ + 2C \delta \left( \sum_{i=1}^3 \| \nabla \omega \|_{L^2(\mathbb{R}^3)}^2 + \| \nabla \varphi \|_{L^2(\mathbb{R}^3)}^2 \right)^{1/2} \]

\[ \times \left( \sum_{i=1}^3 \| \nabla \omega \|_{L^2(\mathbb{R}^3)}^2 + \| \nabla \varphi \|_{L^2(\mathbb{R}^3)}^2 \right)^{1/2} \prod_{i=1}^3 \| \nabla \varphi \|_{L^2(\mathbb{R}^3)} \]

this remains bounded according to (6) and (13), (14). Therefore,

\[ \int_0^\delta \| \hat{f}_{im}(t) \|_V \, dt = \int_0^\delta \max_{\| \varphi \| \leq 1} \sum_{i=1}^3 \langle f_{im}, \varphi_i \rangle < +\infty \]

it follows that

\[ \sup_{\tau \in \mathbb{R}} \| \hat{f}_{im}(\tau) \|_V < +\infty, \quad \forall m \]

Due to (13) we have

\[ \| \omega_{im}(0) \|_{L^2(\mathbb{R}^3)} < +\infty, \quad \| \omega_{im}(\delta) \|_{L^2(\mathbb{R}^3)} < +\infty \]

then by Poincare inequality,

\[ \sum_{i=1}^3 \left( \int \| \theta^{1/2} \omega_{im}(\tau) \|_{L^2(\mathbb{R}^3)}^2 \right)^{1/2} \leq c_1 \sum_{i=1}^3 \| \hat{f}_{im}(\tau) \|_V \| \theta \omega_{im}(\tau) \|_V \]

\[ + c_2 \sum_{i=1}^3 \| \theta \omega_{im}(\tau) \|_{L^2(\mathbb{R}^3)} \]

\[ \leq c_3 \sum_{i=1}^3 \| \nabla (\theta \omega_{im}(\tau)) \|_{L^2(\mathbb{R}^3)} \]

\[ \leq c_4 \sum_{i=1}^3 \left( \| \omega_{im} \nabla \theta \|_{L^2(\mathbb{R}^3)} + \| \theta \omega_{im} \|_{L^2(\mathbb{R}^3)} \right) \]

Using $x^2 e^{-\kappa x} \leq C_1$ ($\kappa > 0$) and assuming that $r$ is sufficiently large, we get
\[
|\tau| \sum_{i=1}^{3} \left\| \theta_r^{1/2} \hat{\omega}_{im}(\tau) \right\|_{L^2(\mathbb{R}^3)}^2 \leq c_5 \sum_{i=1}^{3} \left\| \theta_r^{1/2} \hat{\omega}_{im} \right\|_{L^2(\mathbb{R}^3)} + c_6 \sum_{i=1}^{3} \left\| \theta_r \nabla \hat{\omega}_{im} \right\|_{L^2(\mathbb{R}^3)}
\]
(15)

For $\gamma$ fixed, $\gamma < 1/4$, we observe that
\[
|\tau|^{2\gamma} \leq c_7(\gamma) \frac{1 + |\tau|}{1 + |\tau|^{1-2\gamma}}, \quad \forall \tau \in \mathbb{R}
\]
Thus by (15),
\[
\int_{-\infty}^{+\infty} |\tau|^{2\gamma} \left( \sum_{i=1}^{3} \left\| \theta_r^{1/2} \hat{\omega}_{im}(\tau) \right\|_{L^2(\mathbb{R}^3)}^2 \right) d\tau \leq c_8 \int_{-\infty}^{+\infty} \frac{1 + |\tau|}{1 + |\tau|^{1-2\gamma}} \left( \sum_{i=1}^{3} \left\| \theta_r^{1/2} \hat{\omega}_{im}(\tau) \right\|_{L^2(\mathbb{R}^3)}^2 \right) d\tau + c_9 \int_{-\infty}^{+\infty} \sum_{i=1}^{3} \left\| \theta_r \nabla \hat{\omega}_{im}(\tau) \right\|_{L^2(\mathbb{R}^3)} d\tau + c_{10} \int_{-\infty}^{+\infty} \sum_{i=1}^{3} \left\| \theta_r^{1/2} \hat{\omega}_{im}(\tau) \right\|_{L^2(\mathbb{R}^3)}^2 d\tau
\]
Because of the Parseval equality,
\[
\int_{-\infty}^{+\infty} \sum_{i=1}^{3} \left\| \theta_r \omega_{im}(\tau) \right\|_{L^2(\mathbb{R}^3)} d\tau = \int_{0}^{\delta} \sum_{i=1}^{3} \left\| \theta_r \hat{\omega}_{im}(t) \right\|_{L^2(\mathbb{R}^3)} dt
\]
\[
\leq C_3 \delta \sup_{(0,\delta)} \sum_{i=1}^{3} \left\| \hat{\omega}_{im} \right\|_{L^2(\mathbb{R}^3)} < +\infty
\]
\[
\int_{-\infty}^{+\infty} \sum_{i=1}^{3} \left\| \theta_r \nabla \hat{\omega}_{im}(\tau) \right\|_{L^2(\mathbb{R}^3)} d\tau = \int_{0}^{\delta} \sum_{i=1}^{3} \left\| \theta_r \nabla \hat{\omega}_{im}(t) \right\|_{L^2(\mathbb{R}^3)} dt
\]
\[
\leq C_4 \int_{0}^{\delta} \sum_{i=1}^{3} \left\| \nabla \hat{\omega}_{im} \right\|_{L^2(\mathbb{R}^3)}^2 < +\infty
\]
as $m \to \infty$. By Cauchy-Schwarz inequality and the Parseval equality,
\[
\int_{-\infty}^{+\infty} \frac{1}{1 + |\tau|^{1-2\gamma}} \sum_{i=1}^{3} \left\| \theta_r^{1/2} \hat{\omega}_{im}(\tau) \right\|_{L^2(\mathbb{R}^3)} d\tau
\]
\[
\leq \sqrt{3} \left( \int_{-\infty}^{+\infty} \frac{1}{(1 + |\tau|^{1-2\gamma})^2} d\tau \right)^{1/2} \left( \int_{0}^{\delta} \sum_{i=1}^{3} \left\| \theta_r^{1/2} \hat{\omega}_{im}(t) \right\|_{L^2(\mathbb{R}^3)}^2 dt \right)^{1/2} < +\infty
\]
\[
\int_{-\infty}^{+\infty} \frac{1}{1 + |\tau|^{1-2\gamma}} \sum_{i=1}^{3} \| \theta_i \nabla \tilde{\omega}_{im}(\tau) \|_{L^2(\mathbb{R}^3)} d\tau \\
\leq \sqrt{3} \left( \int_{-\infty}^{+\infty} \frac{1}{(1 + |\tau|^{1-2\gamma})^2} d\tau \right)^{1/2} \left( \int_{0}^{\delta} \sum_{i=1}^{3} \| \theta_i \nabla \tilde{\omega}_{im}(t) \|_{L^2(\mathbb{R}^3)}^2 dt \right)^{1/2} < +\infty
\]
as \(m \to \infty\) by \(\gamma < 1/4\) and (14).

(iv) The estimates (13) and (14) enable us to assert the existence of an element \(\tilde{\omega}^* \in L^2(0, \delta; H^1(\Omega)) \cap L^\infty(0, \delta; L^2(\Omega)), \quad \forall \Omega \subset \mathbb{R}^3\), and a subsequence \(\tilde{\omega}_{m'}\) such that

\[\tilde{\omega}_{m'} \to \tilde{\omega}^* \text{ in } L^2(0, \delta; H^1(\Omega)) \text{ weakly, and in } L^\infty(0, \delta; L^2(\Omega)) \text{ weak-star,}
\]
as \(m' \to \infty\), for any \(\Omega \subset \mathbb{R}^3\)

Due to (iii) we also have

\[\tilde{\omega}_{m'} \to \tilde{\omega}^* \text{ in } L^2(0, \delta; L^2(\Omega)) \text{ strongly as } m' \to \infty, \text{ for any } \Omega \subset \mathbb{R}^3\]

which means

\[\tilde{\omega}_{m'} \to \tilde{\omega}^* \text{ in } L^2(0, \delta; L^2_{\text{loc}}(\Omega)) \text{ strongly}\]

In particular, for a fixed \(j\)

\[\tilde{\omega}_{m'}|_{\Omega'} \to \tilde{\omega}^*|_{\Omega'} \text{ in } L^2(0, \delta; L^2(\Omega')) \text{ strongly}\]

where \(\Omega'\) denotes the support of \(w_{ij}\). This convergence result enable us to pass to the limit.

Let \(\psi_i\) be a continuously differentiable function on \((0, \delta)\) with \(\psi_i(\delta) = 0\). We multiply (11) by \(\psi_i(t)\) then integrate by parts. This leads to the equation

\[- \int_{0}^{\delta} \sum_{i=1}^{3} (\tilde{\omega}_{im}(t), \partial_t \psi_i(t) w_{ij}) dt + \int_{0}^{\delta} \sum_{i=1}^{3} (\nabla \tilde{\omega}_{im}, \psi_i(t) \nabla w_{ij}) dt \\
+ \int_{0}^{\delta} \sum_{i=1}^{3} ((\nabla \cdot \nabla) \tilde{\omega}_{im}, w_{ij} \psi_i(t)) - \int_{0}^{\delta} \sum_{i=1}^{3} ((\nabla \cdot \nabla) \tilde{\omega}_{im}, w_{ij} \psi_i(t)) = \sum_{i=1}^{3} (\tilde{\omega}^*_w, w_{ij}) \psi_i(0)\]

Since \(\tilde{\omega}_{im'}\) converges to \(\tilde{\omega}^*_w\) in \(L^2(0, \delta; L^2(\Omega))\) strongly as \(m' \to \infty\), then \(\tilde{\omega}_{im'}\) also converges strongly to \(\tilde{\omega}_{im'}\), and

\[\int_{0}^{\delta} \sum_{i=1}^{3} (\tilde{\omega}_{im'}, \partial_t \psi_i(t) w_{ij}) dt \to \int_{0}^{\delta} \sum_{i=1}^{3} (\tilde{\omega}^*_w, \partial_t \psi_i(t) w_{ij}) dt \\
\int_{0}^{\delta} \sum_{i=1}^{3} (\nabla \tilde{\omega}_{im'}, \psi_i(t) \nabla w_{ij}) dt = - \int_{0}^{\delta} \sum_{i=1}^{3} (\tilde{\omega}_{im'}, \psi_i(t) \Delta w_{ij}) dt\]

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\[ -\int_0^\delta \sum_{i=1}^3 (\tilde{\omega}_i^\ast, \psi_i(t)\Delta w_{ij}) = \int_0^\delta \sum_{i=1}^3 (\nabla \tilde{\omega}_i^\ast, \psi_i(t)\nabla w_{ij}) \, dt \]

\[ \int_0^\delta \sum_{i=1}^3 ((\overline{u} \cdot \nabla)\overline{\omega}_{im'}, w_{ij}\psi_i(t)) = -\int_0^\delta \sum_{i=1}^3 ((\overline{u} \cdot \nabla)w_{ij}\psi_i(t), \overline{\omega}_{im'}) \]

\[ \rightarrow \int_0^\delta \sum_{i=1}^3 ((\overline{u} \cdot \nabla)w_{ij}\psi_i(t), \overline{\omega}_i^\ast) = \int_0^\delta \sum_{i=1}^3 ((\overline{u} \cdot \nabla)\overline{\omega}_i^\ast, w_{ij}\psi_i(t)) \]

\[ \int_0^\delta \sum_{i=1}^3 ((\overline{\omega}_{im'} \cdot \nabla)\overline{u}_i, w_{ij}\psi_i(t)) \rightarrow \int_0^\delta \sum_{i=1}^3 ((\overline{\omega} \cdot \nabla)\overline{u}_i, w_{ij}\psi_i(t)) \]

\[ \sum_{i=1}^3 (\omega_{i0}^m, w_{ij})\psi_i(0) \rightarrow \sum_{i=1}^3 (\omega_{i0}, w_{ij})\psi_i(0) \]

Thus, in the limit we find

\[ -\int_0^\delta \sum_{i=1}^3 (\tilde{\omega}_i^\ast, \partial_t\psi_i(t)v_i) \, dt + \int_0^\delta \sum_{i=1}^3 (\nabla \tilde{\omega}_i^\ast, \psi_i(t)\nabla v_i) \, dt \]

\[ + \int_0^\delta \sum_{i=1}^3 ((\overline{u} \cdot \nabla)\overline{\omega}_i^\ast, v_i\psi_i(t)) \rightarrow -\int_0^\delta \sum_{i=1}^3 ((\overline{\omega} \cdot \nabla)\overline{u}_i, v_i\psi_i(t)) = \sum_{i=1}^3 (\omega_{i0}, v_i)\psi_i(0) \]

holds for \( v_i = w_{i1}, w_{i2}, \ldots \); by this equation holds for \( v_i = \) any finite linear combination of the \( w_{ij} \), and by a continuity argument above equation is still true for any \( v_i \in V \). Hence we find that \( \tilde{\omega}_i^\ast (i = 1, 2, 3) \) is a Leray-Hopf weak solution of the system (9).

Finally it remains to prove that \( \tilde{\omega}_i^\ast \) satisfy the initial conditions. For this we multiply (9) by \( v_i\psi_i(t) \), after integrating some terms by parts, we get in the same way,

\[ -\int_0^\delta \sum_{i=1}^3 (\tilde{\omega}_i^\ast, \partial_t\psi_i(t)v_i) \, dt + \int_0^\delta \sum_{i=1}^3 (\nabla \tilde{\omega}_i^\ast, \psi_i(t)\nabla v_i) \, dt \]

\[ + \int_0^\delta \sum_{i=1}^3 ((\overline{u} \cdot \nabla)\overline{\omega}_i^\ast, v_i\psi_i(t)) \rightarrow -\int_0^\delta \sum_{i=1}^3 ((\overline{\omega} \cdot \nabla)\overline{u}_i, v_i\psi_i(t)) = \sum_{i=1}^3 (\omega_{i0}^*, v_i)\psi_i(0) \]

By comparison with (16),

\[ \sum_{i=1}^3 (\tilde{\omega}_i^\ast(0) - \omega_{i0}, v_i)\psi_i(0) = 0 \]

Therefore we can choose \( \psi_i \) particularly such that

\( (\tilde{\omega}_i^\ast(0) - \omega_{i0}, v_i) = 0, \quad \forall \ v_i \in V \)
4. Convergence

Now the partition is refined infinitely and \( \varepsilon \) becomes sufficiently small, we will prove that there exists some subsequence of the solutions of auxiliary problems which converges to a weak solution of (2).

Since

\[
\sup_{t \in (0, T)} \int_{\mathbb{R}^3} (\tilde{\omega}^2_1 + \tilde{\omega}^2_2 + \tilde{\omega}^2_3) < +\infty
\]

the family \((\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3)\) is uniformly bounded in \(L^2(0, T; H) \cap L^\infty(0, T; H)\), then we can choose \(k' \to \infty\) or \(\Delta t'_k \to 0\) (in this case \(\varepsilon' \to 0\) and \(m'\) has to tend to \(\infty\)), such that there exists a subsequence \((\tilde{\omega}'_1, \tilde{\omega}'_2, \tilde{\omega}'_3)\) converging weakly in \(L^2(0, T; H)\) and weak-star in \(L^\infty(0, T; H)\) to some element \((\omega'^*_1, \omega'^*_2, \omega'^*_3)\). On the other hand, because \(\tilde{\omega}_i (i = 1, 2, 3)\) belong to \(L^2(0, T; H)\), we can verify that

\[
\bar{u}_i(x, t) = \left\{ \frac{1}{\Delta t_k} \int_{t_{k-1}}^{t_k} \tilde{\omega}_i(x, t) dt, \ t \in (t_{k-1}, t_k) \subset (0, T) \right\}
\]

also belongs to \(L^2(0, T; H)\). In fact,

\[
\int_0^T \int_{\mathbb{R}^3} \bar{u}_i^2(x, t) = \sum_k \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}^3} \left( \frac{1}{\Delta t_k} \int_{t_{k-1}}^{t_k} \tilde{\omega}_i(x, t) \right)^2 dt \leq \sum_k \frac{1}{\Delta t_k} \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}^3} \tilde{\omega}_i^2(x, t) dt \leq \sum_k \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}^3} \tilde{\omega}_i^2(x, t) dt = \int_0^T \int_{\mathbb{R}^3} \tilde{\omega}_i^2(x, t) dt < +\infty
\]

In the same way, we know from (6) that the function

\[
\bar{u}_i(x, t) = \left\{ \frac{1}{\Delta t_k} \int_{t_{k-1}}^{t_k} u_i(x, t) dt, \ t \in (t_{k-1}, t_k) \subset (0, T) \right\}
\]

belongs to \(L^2(0, T; H)\).

Finally we will prove that \((\omega'^*_1, \omega'^*_2, \omega'^*_3)\) is a solution of the vorticity-velocity form of Navier-Stokes equation (2).

Taking \(\varphi_i \in C^\infty((0, T) \times \mathbb{R}^3) \ (i = 1, 2, 3)\), and

\[
\partial_{x_1} \varphi_1 + \partial_{x_2} \varphi_2 + \partial_{x_3} \varphi_3 = 0
\]

we have

\[
\sum_{k=1}^N \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}^3} \theta_r \varphi_1 (\partial_i \tilde{\omega}_1 + \bar{u}^k_1 \partial_{x_1} \bar{u}^k_1 + \bar{u}^k_2 \partial_{x_2} \bar{u}^k_1 + \bar{u}^k_3 \partial_{x_3} \bar{u}^k_1) -
\]

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\[-\bar{\omega}_1^t \partial_{x_1} \bar{u}_1^t - \bar{\omega}_2^t \partial_{x_2} \bar{u}_1^t - \bar{\omega}_3^t \partial_{x_3} \bar{u}_1^t + \partial_{x_1} q - \Delta \bar{\omega}_1) = 0 \]

\[\sum_{k=1}^N \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}^3} \theta_r \varphi_2 (\partial_t \bar{\omega}_2 + \bar{\omega}_1^k \partial_{x_1} \bar{\omega}_3^k + \bar{\omega}_2^k \partial_{x_2} \bar{\omega}_3^k + \bar{\omega}_3^k \partial_{x_3} \bar{\omega}_3^k) - \]

\[-\bar{\omega}_1^t \partial_{x_1} \bar{u}_1^t - \bar{\omega}_2^t \partial_{x_2} \bar{u}_1^t - \bar{\omega}_3^t \partial_{x_3} \bar{u}_1^t + \partial_{x_2} q - \Delta \bar{\omega}_2) = 0 \]

\[\sum_{k=1}^N \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}^3} \theta_r \varphi_3 (\partial_t \bar{\omega}_3 + \bar{\omega}_1^k \partial_{x_1} \bar{\omega}_3^k + \bar{\omega}_2^k \partial_{x_2} \bar{\omega}_3^k + \bar{\omega}_3^k \partial_{x_3} \bar{\omega}_3^k) - \]

\[-\bar{\omega}_1^t \partial_{x_1} \bar{u}_1^t - \bar{\omega}_2^t \partial_{x_2} \bar{u}_1^t - \bar{\omega}_3^t \partial_{x_3} \bar{u}_1^t + \partial_{x_3} q - \Delta \bar{\omega}_3) = 0 \]

Here $\bar{\omega}_i$ ($i = 1, 2, 3$) denote the collection of those solutions of problem (7) defined on every $(t_{k-1}, t_k)$. Integrating by parts we get

\[\sum_{k=1}^N \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}^3} \theta_r (\partial_t \bar{\omega}_1^t \partial_{x_1} \varphi_1 + \bar{\omega}_1^k ((\bar{\omega}_1^k \partial_{x_1} \varphi_1 + \varphi_1 \partial_{x_1} \bar{\omega}_1^k) + (\bar{\omega}_2^k \partial_{x_2} \varphi_1 + \varphi_1 \partial_{x_2} \bar{\omega}_2^k) +
\]

\[+ (\bar{\omega}_3^k \partial_{x_3} \varphi_1 + \varphi_1 \partial_{x_3} \bar{\omega}_3^k)) - \bar{\omega}_1^k ((\bar{\omega}_1^k \partial_{x_1} \varphi_1 + \varphi_1 \partial_{x_1} \bar{\omega}_1^k) + (\bar{\omega}_2^k \partial_{x_2} \varphi_1 + \varphi_1 \partial_{x_2} \bar{\omega}_2^k) +
\]

\[+ (\bar{\omega}_3^k \partial_{x_3} \varphi_1 + \varphi_1 \partial_{x_3} \bar{\omega}_3^k)) - \bar{\omega}_1^k ((\bar{\omega}_1^k \partial_{x_1} \varphi_1 + \varphi_1 \partial_{x_1} \bar{\omega}_1^k) + (\bar{\omega}_2^k \partial_{x_2} \varphi_1 + \varphi_1 \partial_{x_2} \bar{\omega}_2^k) +
\]

\[+ (\bar{\omega}_3^k \partial_{x_3} \varphi_1 + \varphi_1 \partial_{x_3} \bar{\omega}_3^k)) - \bar{\omega}_1^k \partial_{x_1} \varphi_1 + \bar{\omega}_1^k \partial_{x_1} \varphi_1 + \bar{\omega}_2^k \partial_{x_2} \varphi_1 + \bar{\omega}_3^k \partial_{x_3} \varphi_1) +
\]

\[+ \theta_r (\varphi_1 (x, t_k) \bar{\omega}_1 (x, t_k) - \varphi_1 (x, t_{k-1}) \bar{\omega}_1 (x, t_{k-1}))) +
\]

\[\sum_{k=1}^N \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}^3} \theta_r (\partial_t \bar{\omega}_2^k \partial_{x_2} \varphi_2 + \bar{\omega}_2^k ((\bar{\omega}_1^k \partial_{x_1} \varphi_2 + \varphi_2 \partial_{x_1} \bar{\omega}_1^k) + (\bar{\omega}_2^k \partial_{x_2} \varphi_2 + \varphi_2 \partial_{x_2} \bar{\omega}_2^k) +
\]

\[+ (\bar{\omega}_3^k \partial_{x_3} \varphi_2 + \varphi_2 \partial_{x_3} \bar{\omega}_3^k)) - \bar{\omega}_2^k ((\bar{\omega}_1^k \partial_{x_1} \varphi_2 + \varphi_2 \partial_{x_1} \bar{\omega}_1^k) + (\bar{\omega}_2^k \partial_{x_2} \varphi_2 + \varphi_2 \partial_{x_2} \bar{\omega}_2^k) +
\]

\[+ (\bar{\omega}_3^k \partial_{x_3} \varphi_2 + \varphi_2 \partial_{x_3} \bar{\omega}_3^k)) - \bar{\omega}_2^k \partial_{x_2} \varphi_2 + \bar{\omega}_2^k \partial_{x_2} \varphi_2 + \bar{\omega}_3^k \partial_{x_3} \varphi_2 + \bar{\omega}_3^k \partial_{x_3} \varphi_2) +
\]

\[+ \theta_r (\varphi_2 (x, t_k) \bar{\omega}_2 (x, t_k) - \varphi_2 (x, t_{k-1}) \bar{\omega}_2 (x, t_{k-1}))) +
\]

\[\sum_{k=1}^N \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}^3} \theta_r (\partial_t \bar{\omega}_3^k \partial_{x_3} \varphi_3 + \bar{\omega}_3^k ((\bar{\omega}_1^k \partial_{x_1} \varphi_3 + \varphi_3 \partial_{x_1} \bar{\omega}_1^k) + (\bar{\omega}_2^k \partial_{x_2} \varphi_3 + \varphi_3 \partial_{x_2} \bar{\omega}_2^k) +
\]

\[+ (\bar{\omega}_3^k \partial_{x_3} \varphi_3 + \varphi_3 \partial_{x_3} \bar{\omega}_3^k)) - \bar{\omega}_3^k ((\bar{\omega}_1^k \partial_{x_1} \varphi_3 + \varphi_3 \partial_{x_1} \bar{\omega}_1^k) + (\bar{\omega}_2^k \partial_{x_2} \varphi_3 + \varphi_3 \partial_{x_2} \bar{\omega}_2^k) +
\]

\[\]
\[ \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}^3} \left( \bar{\omega}_3^k (\varphi_3 \tilde{\omega}_1 \partial_x \varphi_3 + \varphi_3 \tilde{\omega}_2 \partial_x \varphi_3 + \varphi_3 \tilde{\omega}_3 \partial_x \varphi_3) \right) + \tilde{\omega}_3 (\varphi_3 \tilde{\omega}_1 \partial_x \varphi_3 + \varphi_3 \tilde{\omega}_2 \partial_x \varphi_3 + \varphi_3 \tilde{\omega}_3 \partial_x \varphi_3) \] 

\[ + \sum_{k=1}^{N} \sum_{l=1}^{t_k} \int_{\mathbb{R}^3} \left( \bar{\omega}_3^k (\varphi_3 \tilde{\omega}_1 \partial_x \varphi_3 + \varphi_3 \tilde{\omega}_2 \partial_x \varphi_3 + \varphi_3 \tilde{\omega}_3 \partial_x \varphi_3) \right) - \] 

\[ - \bar{\omega}_3^k (\varphi_3 \tilde{\omega}_1 \partial_x \varphi_3 + \varphi_3 \tilde{\omega}_2 \partial_x \varphi_3 + \varphi_3 \tilde{\omega}_3 \partial_x \varphi_3) + q \varphi_3 \partial_x \varphi_3 + \tilde{\omega}_3 \varphi_3 \Delta \varphi_3 + 2 \tilde{\omega}_3 (\partial_x \varphi_3 + \Delta \varphi_3) \] 

\[ = \sum_{k=1}^{N} \int_{\mathbb{R}^3} \varphi_3 (x, t_k) \tilde{\omega}_3 (x, t_k) - \varphi_3 (x, t_{k-1}) \tilde{\omega}_3 (x, t_{k-1}) \] 

Let \( r \to +\infty \),

\[ \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \int_{\mathbb{R}^3} \left( \tilde{\omega}_1 \partial_x \varphi_1 + \tilde{\omega}_1 (\varphi_3 \tilde{\omega}_1 + \varphi_1 \tilde{\omega}_1) \right) + \left( \tilde{\omega}_2 \partial_x \varphi_2 + \tilde{\omega}_2 (\varphi_3 \tilde{\omega}_2 + \varphi_2 \tilde{\omega}_2) \right) + \] 

\[ + \left( \tilde{\omega}_3 \partial_x \varphi_3 + \tilde{\omega}_3 (\varphi_3 \tilde{\omega}_3 + \varphi_3 \tilde{\omega}_3) \right) + q \partial_x \varphi_3 + \tilde{\omega}_3 \Delta \varphi_3 \]

\[ = \sum_{k=1}^{N} \int_{\mathbb{R}^3} \varphi_3 (x, t_k) \tilde{\omega}_3 (x, t_k) - \varphi_3 (x, t_{k-1}) \tilde{\omega}_3 (x, t_{k-1}) \]

For a certain solution \( u \) of (1), we can prove due to (6) that

\[ \bar{\omega}_i \to u_i \text{ in } L^2(0, T; H) \text{ strongly} \]

as \( k \to \infty \), or \( \Delta t_k \to 0 \).
In fact, set \( Q = (0, T) \times \mathbb{R}^3, \Delta t = \max_k \{ \Delta t_k \}, \forall \varepsilon > 0, \) and \( u_i \in L^2(0, T; L^2(\mathbb{R}^3)) \), there exists a \( v_i \in C^\infty(0, T; L^2(\mathbb{R}^3)) \) such that
\[
\| u_i - v_i \|_{L^2(Q)} < \varepsilon
\]
By means of the same partition as that for \( u_i \) to construct \( v_i \), since there exists a constant \( C > 0 \) such that \( \| \partial_t u_i \|_{L^2(\mathbb{R}^3)} \leq C \), and \( \max_t \| u_i - v_i \|_{L^2(\mathbb{R}^3)} \leq C \Delta t \), it follows that
\[
\| v_i - v_i \|_{L^2(Q)} = \left( \int_0^T \| v_i - v_i \|^2_{L^2(\mathbb{R}^3)} \right)^{1/2} \leq C T^{1/2} \Delta t
\]
Thus
\[
v_i \to v_i \quad (L^\infty(0, T; L^2(\mathbb{R}^3))) \text{, as } \Delta t \to 0.
\]
Take \( \Delta t \) such that \( \| v_i - v_i \|_{L^2(Q)} < \varepsilon \). Moreover,
\[
\int_0^T \| \bar{u}_i - v_i \|^2_{L^2(\mathbb{R}^3)} = \sum_{k=1}^N \left\| \frac{1}{\Delta t_k} \int_{t_{k-1}}^{t_k} (u_i - v_i) \right\|^2_{L^2(\mathbb{R}^3)} \Delta t_k
\]
so that \( \| \bar{u}_i - v_i \|_{L^2(Q)} \leq \| u_i - v_i \|_{L^2(Q)} < \varepsilon \). Therefore,
\[
\| \bar{u}_i - u_i \|_{L^2(Q)} \leq \| u_i - v_i \|_{L^2(Q)} + \| v_i - \bar{u}_i \|_{L^2(Q)} + \| \bar{u}_i - v_i \|_{L^2(Q)} < 3\varepsilon
\]
Hence as \( \Delta t \to 0 \), we have \( \| \bar{u}_i - u_i \|_{L^2(Q)} \to 0 \).

On the other hand, from section 2 we have the following conclusions:
\( \tilde{\omega}_i \to \omega_* \) in \( L^2(0, T; H) \) weakly, and in \( L^\infty(0, T; H) \) weak-star
for a subsequence as \( k' \to \infty \), or \( \Delta t_k \to 0 \).

For a \( \omega_i \in L^2(0, T; H) \), similar to above we know that
\( \bar{\omega}_i \to \omega \) in \( L^2(0, T; H) \) strongly
as \( k \to \infty \) or \( \Delta t_k \to 0 \).

Moreover, set \( B_\varepsilon = \{ x : |x| < \varepsilon \} \), then
\[
\| \bar{\omega}_i - \omega_i \|_{L^2(\mathbb{R}^3)} = \left\| \int_{|y| \leq \varepsilon} J_\varepsilon(y) \left[ \bar{\omega}_i(x - y) - \omega_i(x) \right] dy \right\|_{L^2(\mathbb{R}^3)}
\]
as \( \varepsilon \to 0 \), and

\[
\| \overline{\omega}_i - \omega_i \|_{L^2(Q)} \leq \left( \int_0^T \| \overline{\omega}_i - \omega_i \|_{L^2(\mathbb{R}^3)}^2 \right)^{1/2} + \| \overline{\omega}_i - \omega_i \|_{L^2(Q)} \to 0
\]
as \( k \to \infty \) or \( \Delta t_k \to 0 \).

Thus we finally we obtain

\[
\overline{\omega}_i^k \to \omega_i^s \quad \text{in} \quad L^2(0, T, H) \quad \text{weakly}
\]
as \( k' \to \infty \), or \( \Delta t_k' \to 0 \).

These convergence results enable us to pass the limit. That is,

\[
\sum_{k'} \int_{t_{k'-1}}^{t_{k'}} \int_{\mathbb{R}^3} \left( \tilde{\omega}_1 \partial_t \varphi_1 + \overline{\omega}_1^k \partial_{x_1} \varphi_1 + \overline{\omega}_2^k \partial_{x_2} \varphi_1 + \overline{\omega}_3^k \partial_{x_3} \varphi_1 \right) -
- \overline{\omega}_1^k \partial_{x_1} \varphi_1 \partial_{x_2} \varphi_1 \partial_{x_3} \varphi_1 \partial_{x_3} \varphi_1 + q \partial_t \varphi_1 + \tilde{\omega}_1 \Delta \varphi_1 \right) = \int_{\mathbb{R}^3} \left( \varphi_1(x, T) \tilde{\omega}_1(x, T) - \varphi_1(x, 0) \tilde{\omega}_1(x, 0) \right)
\]

\[
\sum_{k'} \int_{t_{k'-1}}^{t_{k'}} \int_{\mathbb{R}^3} \left( \tilde{\omega}_2 \partial_t \varphi_2 + \overline{\omega}_1^k \partial_{x_1} \varphi_2 + \overline{\omega}_2^k \partial_{x_2} \varphi_2 + \overline{\omega}_3^k \partial_{x_3} \varphi_2 \right) -
- \overline{\omega}_2^k \partial_{x_1} \varphi_2 \partial_{x_2} \varphi_2 \partial_{x_3} \varphi_2 + q \partial_t \varphi_2 + \tilde{\omega}_2 \Delta \varphi_2 \right) = \int_{\mathbb{R}^3} \left( \varphi_2(x, T) \tilde{\omega}_2(x, T) - \varphi_2(x, 0) \tilde{\omega}_2(x, 0) \right)
\]

\[
\sum_{k'} \int_{t_{k'-1}}^{t_{k'}} \int_{\mathbb{R}^3} \left( \tilde{\omega}_3 \partial_t \varphi_3 + \overline{\omega}_1^k \partial_{x_1} \varphi_3 + \overline{\omega}_2^k \partial_{x_2} \varphi_3 + \overline{\omega}_3^k \partial_{x_3} \varphi_3 \right) -
- \overline{\omega}_3^k \partial_{x_1} \varphi_3 \partial_{x_2} \varphi_3 \partial_{x_3} \varphi_3 + q \partial_t \varphi_3 + \tilde{\omega}_3 \Delta \varphi_3 \right) = \int_{\mathbb{R}^3} \left( \varphi_3(x, T) \tilde{\omega}_3(x, T) - \varphi_3(x, 0) \tilde{\omega}_3(x, 0) \right)
\]

This is equivalent to

\[
\int_0^T \int_{\mathbb{R}^3} \left\{ (\omega_1 \partial_t \varphi_1 + \omega_2 \partial_t \varphi_2 + \omega_3 \partial_t \varphi_3) + \right. \]

\[
\left. (\omega_1 \Delta \varphi_1 + \omega_2 \Delta \varphi_2 + \omega_3 \Delta \varphi_3) \right. \]
\[ + \omega^i_1(u_1 \partial_{x_1} \varphi_1 + u_2 \partial_{x_2} \varphi_1 + u_3 \partial_{x_3} \varphi_1) + \omega^i_2(u_1 \partial_{x_1} \varphi_2 + u_2 \partial_{x_2} \varphi_2 + u_3 \partial_{x_3} \varphi_2) + \\
+ \omega^i_3(u_1 \partial_{x_1} \varphi_3 + u_2 \partial_{x_2} \varphi_3 + u_3 \partial_{x_3} \varphi_3) - u_1(\omega^i_1 \partial_{x_1} \varphi_1 + \omega^i_2 \partial_{x_2} \varphi_1 + \omega^i_3 \partial_{x_3} \varphi_1) - u_2(\omega^i_1 \partial_{x_1} \varphi_2 + \omega^i_2 \partial_{x_2} \varphi_2 + \omega^i_3 \partial_{x_3} \varphi_2) - \\
- u_3(\omega^i_1 \partial_{x_1} \varphi_3 + \omega^i_2 \partial_{x_2} \varphi_3 + \omega^i_3 \partial_{x_3} \varphi_3) \]

\[ = \int_{\mathbb{R}^3} \{ (\varphi_1(x,T)\omega^i_1(x,T) + \varphi_2(x,T)\omega^i_2(x,T) + \varphi_3(x,T)\omega^i_3(x,T)) - \\
- (\varphi_{10}(x)\omega_{10}(x) + \varphi_{20}(x)\omega_{20}(x) + \varphi_{30}(x)\omega_{30}(x)) \} \]

Here we also have
\[ \omega^i_1(x,0) = \omega_{i0}(x), \quad \varphi_i(x,0) = \varphi_{i0}(x), \quad i = 1, 2, 3 \]

Hence we know that there exists some \( \omega^*_i \) which belongs to \( L^\infty(0,T;L^2(\mathbb{R}^3)) \) and is a Leray-Hopf weak solution of (2).

Note that a weak formulation of the following equations:
\[ \omega = \text{curl} u \]
\[ \int_0^T \int_{\mathbb{R}^3} \varphi \cdot [\partial_t \omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u - \nu \Delta \omega] = 0 \]

is equivalent to
\[ \int_0^T \int_{\mathbb{R}^3} \tilde{\varphi} \cdot [\partial_t u + (u \cdot \nabla)u + \nabla p - \nu \Delta u] = 0 \]

for any \( \varphi \in C^\infty((0,T) \times \mathbb{R}^3) \) with \( \varphi_i \in C^\infty_0(\Omega) \) and zero extension outside \( \Omega, \forall \Omega \subset \mathbb{R}^3 \), and \( \tilde{\varphi} = \text{curl} \varphi \), in some distribution sense.

5. Regularity

We can still use Galerkin procedure as in Section 2 and 3. Since \( V \) is separable there exists a sequence of linearly independent elements \( w_{11}, \cdots, w_{im}, \cdots \) which is total in \( V \). For each \( m \) we define an approximate solution \( u_{im} \) of (1) as follows:
\[ u_{im} = \sum_{j=1}^{m} g_{ij}(t) w_{ij} \]

and by means of weighted function \( \theta_r \)
\[ \int_{\mathbb{R}^3} \theta_r w_{1j} \partial_t u_{1m} + \int_{\mathbb{R}^3} \theta_r (u_{1m} \partial_{x_1} u_{1m} + u_{2m} \partial_{x_2} u_{1m} + u_{3m} \partial_{x_3} u_{1m})w_{1j} + \\
+ \int_{\mathbb{R}^3} \theta_r w_{1j} \partial_{x_1} p = \int_{\mathbb{R}^3} \theta_r w_{1j} \Delta u_{1m} \]
\[ \int_{\mathbb{R}^3} \theta_r w_{2j} \partial_t u_{2m} + \int_{\mathbb{R}^3} \theta_r (u_{1m} \partial_t u_{2m} + u_{2m} \partial_t u_{2m} + u_{3m} \partial_t u_{2m}) w_{2j} + \]
\[+ \int_{\mathbb{R}^3} \theta_r w_{2j} \partial_{x2} p = \int_{\mathbb{R}^3} \theta_r w_{2j} \Delta u_{2m} \]
\[\int_{\mathbb{R}^3} \theta_r w_{3j} \partial_t u_{3m} + \int_{\mathbb{R}^3} \theta_r (u_{1m} \partial_t u_{3m} + u_{2m} \partial_t u_{3m} + u_{3m} \partial_t u_{3m}) w_{3j} + \]
\[+ \int_{\mathbb{R}^3} \theta_r w_{3j} \partial_{x3} p = \int_{\mathbb{R}^3} \theta_r w_{3j} \Delta u_{3m} \]
\[u_{im}(0) = u_{im}^0, \quad j = 1, \ldots, m \]

where \(u_{im}^0\) is the orthogonal projection in \(H\) of \(u_{i0}\) on the space spanned by \(w_1, \ldots, w_m\).

We now are allowed to differentiate (17) in the \(t\), we get
\[ \int_{\mathbb{R}^3} \theta_r w_{1j} \partial_t^2 u_{1m} + \int_{\mathbb{R}^3} \theta_r (\partial_t u_{1m} \partial_t u_{1m} + \partial_t u_{2m} \partial_t u_{2m} + \partial_t u_{3m} \partial_t u_{3m}) w_{1j} + \]
\[+ \int_{\mathbb{R}^3} \theta_r (u_{1m} \partial_t u_{1m} + u_{2m} \partial_t u_{2m} + u_{3m} \partial_t u_{3m}) w_{1j} + \]
\[+ \int_{\mathbb{R}^3} \theta_r w_{1j} \partial_{x1} \partial_t u_{1m} = \int_{\mathbb{R}^3} \theta_r w_{1j} \Delta \partial_t u_{1m} \]
\[\int_{\mathbb{R}^3} \theta_r w_{2j} \partial_t^2 u_{2m} + \int_{\mathbb{R}^3} \theta_r (\partial_t u_{2m} \partial_t u_{2m} + \partial_t u_{2m} \partial_t u_{2m} + \partial_t u_{3m} \partial_t u_{3m}) w_{2j} + \]
\[+ \int_{\mathbb{R}^3} \theta_r (u_{2m} \partial_t u_{2m} + u_{3m} \partial_t u_{2m}) w_{2j} + \]
\[+ \int_{\mathbb{R}^3} \theta_r w_{2j} \partial_{x2} \partial_t u_{2m} = \int_{\mathbb{R}^3} \theta_r w_{2j} \Delta \partial_t u_{2m} \]
\[\int_{\mathbb{R}^3} \theta_r w_{3j} \partial_t^2 u_{3m} + \int_{\mathbb{R}^3} \theta_r (\partial_t u_{3m} \partial_t u_{3m} + \partial_t u_{3m} \partial_t u_{3m} + \partial_t u_{3m} \partial_t u_{3m}) w_{3j} + \]
\[+ \int_{\mathbb{R}^3} \theta_r (u_{3m} \partial_t u_{3m}) w_{3j} + \]
\[+ \int_{\mathbb{R}^3} \theta_r w_{3j} \partial_{x3} \partial_t u_{3m} = \int_{\mathbb{R}^3} \theta_r w_{3j} \Delta \partial_t u_{3m} \]
\[j = 1, \ldots, m \]

We multiply (18) by \(g_{ij}(t)\) and add the resulting equations for \(j = 1, \ldots, m\), we find
\[ \frac{1}{2} \partial_t \int_{\mathbb{R}^3} \theta_r (\partial_t u_{1m})^2 + \int_{\mathbb{R}^3} \theta_r \partial_t u_{1m} (\partial_t u_{1m} \partial_{x1} u_{1m} + \partial_t u_{2m} \partial_{x2} u_{1m} + \partial_t u_{3m} \partial_{x3} u_{1m}) + \]
\[+ \int_{\mathbb{R}^3} \theta_r \partial_t u_{1m} (u_{1m} \partial_{x1} \partial_t u_{1m} + u_{2m} \partial_{x2} \partial_t u_{1m} + u_{3m} \partial_{x3} \partial_t u_{1m}) + \]
\[+ \int_{\mathbb{R}^3} \theta_r \partial_t u_{1m} \partial_{x1} \partial_t p = \int_{\mathbb{R}^3} \theta_r \partial_t u_{1m} \Delta \partial_t u_{1m} \]

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\[
\frac{1}{2} \frac{\partial}{\partial t} \int_{\mathbb{R}^3} \theta_r \left( \partial_t u_{2m} \right)^2 + \int_{\mathbb{R}^3} \theta_r \partial_t u_{2m} \left( \partial_t u_{1m} \partial_{x_1} u_{2m} + \partial_t u_{2m} \partial_{x_2} u_{2m} + \partial_t u_{3m} \partial_{x_3} u_{2m} \right) + \\
+ \int_{\mathbb{R}^3} \theta_r \partial_t u_{2m} \left( u_{1m} \partial_{x_1} u_{2m} + u_{2m} \partial_{x_2} u_{2m} + u_{3m} \partial_{x_3} u_{2m} \right) + \\
+ \int_{\mathbb{R}^3} \theta_r \partial_t u_{2m} \partial_{x_2} \partial_{x_2} p = \int_{\mathbb{R}^3} \theta_r \partial_t u_{2m} \Delta \partial_t u_{2m}
\]

\[
\frac{1}{2} \frac{\partial}{\partial t} \int_{\mathbb{R}^3} \theta_r \left( \partial_t u_{3m} \right)^2 + \int_{\mathbb{R}^3} \theta_r \partial_t u_{3m} \left( \partial_t u_{1m} \partial_{x_1} u_{3m} + \partial_t u_{2m} \partial_{x_2} u_{3m} + \partial_t u_{3m} \partial_{x_3} u_{3m} \right) + \\
+ \int_{\mathbb{R}^3} \theta_r \partial_t u_{3m} \left( u_{1m} \partial_{x_1} u_{3m} + u_{2m} \partial_{x_2} u_{3m} + u_{3m} \partial_{x_3} u_{3m} \right) + \\
+ \int_{\mathbb{R}^3} \theta_r \partial_t u_{3m} \partial_{x_2} \partial_{x_2} p = \int_{\mathbb{R}^3} \theta_r \partial_t u_{3m} \Delta \partial_t u_{3m}
\]

Let \( r \rightarrow +\infty \).

\[
\frac{1}{2} \frac{\partial}{\partial t} \int_{\mathbb{R}^3} \left( \partial_t u_{1m} \right)^2 + \int_{\mathbb{R}^3} \partial_t u_{1m} \left( \partial_t u_{1m} \partial_{x_1} u_{1m} + \partial_t u_{2m} \partial_{x_2} u_{1m} + \partial_t u_{3m} \partial_{x_3} u_{1m} \right) + \\
+ \int_{\mathbb{R}^3} \partial_t u_{1m} \left( u_{1m} \partial_{x_1} u_{1m} + u_{2m} \partial_{x_2} u_{1m} + u_{3m} \partial_{x_3} u_{1m} \right) + \\
+ \int_{\mathbb{R}^3} \partial_t u_{1m} \partial_{x_1} \partial_{x_1} p = \int_{\mathbb{R}^3} \partial_t u_{1m} \Delta \partial_t u_{1m}
\]

\[
\frac{1}{2} \frac{\partial}{\partial t} \int_{\mathbb{R}^3} \left( \partial_t u_{2m} \right)^2 + \int_{\mathbb{R}^3} \partial_t u_{2m} \left( \partial_t u_{1m} \partial_{x_1} u_{2m} + \partial_t u_{2m} \partial_{x_2} u_{2m} + \partial_t u_{3m} \partial_{x_3} u_{2m} \right) + \\
+ \int_{\mathbb{R}^3} \partial_t u_{2m} \left( u_{1m} \partial_{x_1} u_{2m} + u_{2m} \partial_{x_2} u_{2m} + u_{3m} \partial_{x_3} u_{2m} \right) + \\
+ \int_{\mathbb{R}^3} \partial_t u_{2m} \partial_{x_2} \partial_{x_2} p = \int_{\mathbb{R}^3} \partial_t u_{2m} \Delta \partial_t u_{2m}
\]

\[
\frac{1}{2} \frac{\partial}{\partial t} \int_{\mathbb{R}^3} \left( \partial_t u_{3m} \right)^2 + \int_{\mathbb{R}^3} \partial_t u_{3m} \left( \partial_t u_{1m} \partial_{x_1} u_{3m} + \partial_t u_{2m} \partial_{x_2} u_{3m} + \partial_t u_{3m} \partial_{x_3} u_{3m} \right) + \\
+ \int_{\mathbb{R}^3} \partial_t u_{3m} \left( u_{1m} \partial_{x_1} u_{3m} + u_{2m} \partial_{x_2} u_{3m} + u_{3m} \partial_{x_3} u_{3m} \right) + \\
+ \int_{\mathbb{R}^3} \partial_t u_{3m} \partial_{x_2} \partial_{x_2} p = \int_{\mathbb{R}^3} \partial_t u_{3m} \Delta \partial_t u_{3m}
\]

Moreover,

\[
\int_{\mathbb{R}^3} \theta_r \left( \partial_t u_{1m} \partial_{x_1} \partial_{x_1} p + \partial_t u_{2m} \partial_{x_2} \partial_{x_2} p + \partial_t u_{3m} \partial_{x_3} \partial_{x_3} p \right) \\
= -\int_{\mathbb{R}^3} \theta_r \partial_t p \ \partial_t \left( \partial_{x_1} u_{1m} + \partial_{x_2} u_{2m} + \partial_{x_3} u_{3m} \right) \\
- \int_{\mathbb{R}^3} \partial_t p \left( \partial_t u_{1m} \partial_{x_1} \theta_r + \partial_t u_{2m} \partial_{x_2} \theta_r + \partial_t u_{3m} \partial_{x_3} \theta_r \right)
\]
let \( r \to +\infty \) we get

\[
\int_{\mathbb{R}^3} (\partial_t u_{1m} \partial_{x_1} \partial_t p + \partial_t u_{2m} \partial_{x_2} \partial_t p + \partial_t u_{3m} \partial_{x_3} \partial_t p) = 0
\]

and

\[
\int_{\mathbb{R}^3} \theta_r \partial_t u_{im}(u_{1m} \partial_{x_1} \partial_t u_{im} + u_{2m} \partial_{x_2} \partial_t u_{im} + u_{3m} \partial_{x_3} \partial_t u_{im})
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^3} \theta_r (u_{1m} \partial_{x_1} (\partial_t u_{im})^2 + u_{2m} \partial_{x_2} (\partial_t u_{im})^2 + u_{3m} (\partial_t u_{im})^2)
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^3} \theta_r (\partial_t u_{im})^2 (\partial_{x_1} u_{1m} + \partial_{x_2} u_{2m} + \partial_{x_3} u_{3m})
\]

\[
- \frac{1}{2} \int_{\mathbb{R}^3} (\partial_t u_{im})^2 (u_{1m} \partial_{x_1} \theta_r + u_{2m} \partial_{x_2} \theta_r + u_{3m} \partial_{x_3} \theta_r)
\]

let \( r \to +\infty \) we get

\[
\int_{\mathbb{R}^3} \partial_t u_{im}(u_{1m} \partial_{x_1} \partial_t u_{im} + u_{2m} \partial_{x_2} \partial_t u_{im} + u_{3m} \partial_{x_3} \partial_t u_{im}) = 0, \quad i = 1, 2, 3
\]

as well as

\[
\int_{\mathbb{R}^3} \theta_r \partial_t u_{im} \Delta \partial_t u_{im} = \int_{\mathbb{R}^3} \theta_r \partial_t u_{im} (\partial^2_{x_1} \partial_t u_{im} + \partial^2_{x_2} \partial_t u_{im} + \partial^2_{x_3} \partial_t u_{im})
\]

\[
= - \int_{\mathbb{R}^3} \theta_r ((\partial_{x_1} \partial_t u_{im})^2 + (\partial_{x_2} \partial_t u_{im})^2 + (\partial_{x_3} \partial_t u_{im})^2)
\]

\[
- \int_{\mathbb{R}^3} \partial_t u_{im}(\partial_{x_1} \theta_r \partial_{x_1} \partial_t u_{im} + \partial_{x_2} \theta_r \partial_{x_2} \partial_t u_{im} + \partial_{x_3} \theta_r \partial_{x_3} \partial_t u_{im})
\]

let \( r \to +\infty \) we get

\[
\int_{\mathbb{R}^3} \partial_t u_{im} \Delta \partial_t u_{im} = - \int_{\mathbb{R}^3} ((\partial_{x_1} \partial_t u_{im})^2 + (\partial_{x_2} \partial_t u_{im})^2 + (\partial_{x_3} \partial_t u_{im})^2), \quad i = 1, 2, 3
\]

it follows from (19) and above conclusions that

\[
\frac{1}{2} \partial_t \int_{\mathbb{R}^3} ((\partial_t u_{1m})^2 + (\partial_t u_{2m})^2 + (\partial_t u_{3m})^2) +
\]

\[+ \left\| \nabla \partial_t u_{1m} \right\|_{L^2(\mathbb{R}^3)} + \left\| \nabla \partial_t u_{2m} \right\|_{L^2(\mathbb{R}^3)} + \left\| \nabla \partial_t u_{3m} \right\|_{L^2(\mathbb{R}^3)}
\]

\[
\leq \left\| \partial_t u_{1m} \right\|_{L^4(\mathbb{R}^3)} \left\| \partial_{x_1} u_{1m} \right\|_{L^2(\mathbb{R}^3)} + \left\| \partial_t u_{2m} \right\|_{L^4(\mathbb{R}^3)} \left\| \partial_{x_2} u_{1m} \right\|_{L^2(\mathbb{R}^3)} + \left\| \partial_t u_{3m} \right\|_{L^4(\mathbb{R}^3)} \left\| \partial_{x_3} u_{1m} \right\|_{L^2(\mathbb{R}^3)}
\]

\[
+ \left\| \partial_t u_{2m} \right\|_{L^4(\mathbb{R}^3)} \left\| \partial_{x_1} u_{2m} \right\|_{L^2(\mathbb{R}^3)} + \left\| \partial_t u_{2m} \right\|_{L^4(\mathbb{R}^3)} \left\| \partial_{x_2} u_{2m} \right\|_{L^2(\mathbb{R}^3)} + \left\| \partial_t u_{3m} \right\|_{L^4(\mathbb{R}^3)} \left\| \partial_{x_3} u_{2m} \right\|_{L^2(\mathbb{R}^3)}
\]
\[
+ \| \partial_t u_{3m} \|_{L^4(\mathbb{R}^3)} \left( \| \partial_t u_{1m} \|_{L^4(\mathbb{R}^3)} \| \partial_{x_1} u_{3m} \|_{L^2(\mathbb{R}^3)} + \| \partial_t u_{2m} \|_{L^4(\mathbb{R}^3)} \| \partial_{x_2} u_{3m} \|_{L^2(\mathbb{R}^3)} + \\
+ \| \partial_t u_{3m} \|_{L^4(\mathbb{R}^3)} \| \partial_{x_3} u_{3m} \|_{L^2(\mathbb{R}^3)} \right) \\
\leq \left( \sum_{i=1}^3 \| \partial_t u_{im} \|_{L^4(\mathbb{R}^3)} \right)^{1/2} \left( \sum_{j=1}^3 \| \partial_t u_{jm} \|_{L^4(\mathbb{R}^3)} \right)^{1/2} \left( \sum_{i,j=1}^3 \| \partial_{x_i} u_{jm} \|_{L^2(\mathbb{R}^3)} \right)^{1/2}
\]

Since
\[
\sum_{i=1}^3 \| \partial_t u_{im} \|_{L^4(\mathbb{R}^3)}^2 \leq 2 \sum_{i=1}^3 \left( \| \partial_t u_{im} \|_{L^2(\mathbb{R}^3)}^{1/2} \| \nabla \partial_t u_{im} \|_{L^2(\mathbb{R}^3)}^{3/2} \right) \\
\leq 2 \left( \sum_{i=1}^3 \| \partial_t u_{im} \|_{L^2(\mathbb{R}^3)} \right) \left( \sum_{i=1}^3 \| \nabla \partial_t u_{im} \|_{L^2(\mathbb{R}^3)} \right)^{3/4}
\]
then
\[
\partial_t \left( \sum_{i=1}^3 \| \partial_t u_{im} \|_{L^2(\mathbb{R}^3)}^2 \right) + 2 \left( \sum_{i=1}^3 \| \nabla \partial_t u_{im} \|_{L^2(\mathbb{R}^3)}^2 \right) \\
\leq 2^2 \left( \sum_{i=1}^3 \| \partial_t u_{im} \|_{L^2(\mathbb{R}^3)} \right)^{1/4} \left( \sum_{i=1}^3 \| \nabla \partial_t u_{im} \|_{L^2(\mathbb{R}^3)} \right)^{3/4} \left( \sum_{i=1}^3 \| \nabla u_{im} \|_{L^2(\mathbb{R}^3)}^2 \right)^{1/2} \\
\leq 3^3 \left( \sum_{i=1}^3 \| \partial_t u_{im} \|_{L^2(\mathbb{R}^3)} \right) \left( \sum_{i=1}^3 \| \nabla u_{im} \|_{L^2(\mathbb{R}^3)} \right)^2 + \left( \sum_{i=1}^3 \| \nabla \partial_t u_{im} \|_{L^2(\mathbb{R}^3)} \right)
\]
it follows that
\[
\partial_t \left( \sum_{i=1}^3 \| \partial_t u_{im} \|_{L^2(\mathbb{R}^3)}^2 \right) + \left( \sum_{i=1}^3 \| \nabla \partial_t u_{im} \|_{L^2(\mathbb{R}^3)}^2 \right) \leq \phi_m(t) \left( \sum_{i=1}^3 \| \partial_t u_{im} \|_{L^2(\mathbb{R}^3)}^2 \right)
\]
where
\[
\phi_m(t) = 1 + 3^3 \left( \sum_{i=1}^3 \| \nabla u_{im} \|_{L^2(\mathbb{R}^3)} \right)^2
\]

Introducing a stream function: \( \psi = (\psi_2, \psi_2, \psi_3) \),
\[
\text{curl} \psi = (\partial_{x_2} \psi_3 - \partial_{x_3} \psi_2, \ \partial_{x_3} \psi_1 - \partial_{x_1} \psi_3, \ \partial_{x_1} \psi_2 - \partial_{x_2} \psi_1)
\]
According to \( \omega = \text{curl} u, \ u = \text{curl} \psi \) and \( \text{div} \psi = 0 \), we have
\[
\text{curl} \text{curl} \psi = -\Delta \psi = \omega, \ \ -\Delta \text{curl} \psi = \text{curl} \omega
\]
Hence, \(-\Delta u = \text{curl} \omega\). Then \((-\Delta u, u) = (\text{curl} \omega, u)\), where

\[
(-\Delta u, \theta_r u) = \sum_{i=1}^{3} (-\Delta u_i, \theta_r u_i) = \sum_{i=1}^{3} (\nabla u_i, \theta_r \nabla u_i) + \sum_{i=1}^{3} (\nabla u_i, u_i \nabla \theta_r)
\]

let \(r \to +\infty\) we get

\[
(-\Delta u, u) = \sum_{i=1}^{3} (\nabla u_i, \nabla u_i) = \sum_{i=1}^{3} \|\nabla u_i\|_{L^2(\mathbb{R}^3)}^2
\]

In addition,

\[
(\text{curl} \omega, \theta_r u) = (\partial_{x_3} \omega_3 - \partial_{x_2} \omega_2, \theta_r u_1) + (\partial_{x_3} \omega_1 - \partial_{x_1} \omega_3, \theta_r u_2) + (\partial_{x_1} \omega_2 - \partial_{x_2} \omega_1, \theta_r u_3)
\]

\[
= -(\omega_3, \theta_r \partial_{x_2} u_1) + (\omega_2, \theta_r \partial_{x_3} u_1) - (\omega_1, \theta_r \partial_{x_3} u_2) + (\omega_3, \theta_r \partial_{x_1} u_3) - (\omega_2, \theta_r \partial_{x_2} u_3) - (\omega_1, \theta_r \partial_{x_2} u_2) + (\omega_3, \theta_r \partial_{x_1} u_2) - (\omega_2, \theta_r \partial_{x_1} u_3) + (\omega_1, \theta_r \partial_{x_1} u_3)
\]

let \(r \to +\infty\) we get

\[
(\text{curl} \omega, u) = -(\omega_3, \partial_{x_2} u_1) + (\omega_2, \partial_{x_3} u_1) - (\omega_1, \partial_{x_3} u_2) + (\omega_3, \partial_{x_1} u_2)
\]

\[
= (\omega_1, \partial_{x_2} u_3 - \partial_{x_3} u_2) + (\omega_2, \partial_{x_2} u_1 - \partial_{x_1} u_3) + (\omega_3, \partial_{x_1} u_2 - \partial_{x_2} u_1)
\]

\[
= (\omega, \text{curl} u) = (\omega, \omega) = \sum_{i=1}^{3} \|\omega_i\|_{L^2(\mathbb{R}^3)}^2
\]

Hence

\[
\left( \sum_{i=1}^{3} \|\nabla u_i\|_{L^2(\mathbb{R}^3)}^2 \right)^{1/2} = \left( \sum_{i=1}^{3} \|\omega_i\|_{L^2(\mathbb{R}^3)}^2 \right)^{1/2}
\]

it follows that

\[
\phi_m(t) = 1 + 3^3 \left( \sum_{i=1}^{3} \|\omega_{im}\|_{L^2(\mathbb{R}^3)}^2 \right)^2 < +\infty
\]

By the Gronwall inequality,

\[
\frac{d}{dt} \left( \sum_{i=1}^{3} \|\partial_t u_{im}\|_{L^2(\mathbb{R}^3)}^2 \right) \exp \left( - \int_0^t \phi_m(s) ds \right) \leq 0
\]

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whence
\[
\sup_{t \in (0, T)} \left( \sum_{i=1}^{3} \| \partial_t u_{im}(t) \|^2_{L^2(\mathbb{R}^3)} \right) \leq \exp \left( \int_0^T \phi_m(s) \, ds \right) \left( \sum_{i=1}^{3} \| \partial_t u_{im}(0) \|^2_{L^2(\mathbb{R}^3)} \right)
\]

Therefore
\[
\partial_t u_{im} \in L^\infty(0, T; H) \cap L^\infty(0, T; V), \quad i = 1, 2, 3
\]

Finally we write (1) in the form
\[
\sum_{i=1}^{3} (-\Delta (\theta_r u_i), v_i) = \sum_{i=1}^{3} (-\theta_r \partial_t u_i - \theta_r (u \cdot \nabla) u_i + g_i, v_i), \quad v_i \in V
\]

where
\[
g_i = -u_i \Delta \theta_r - 2 (\nabla \theta_r, \nabla u_i) + p \partial_{x_i} \theta_r
\]

That is,
\[
\sum_{i=1}^{3} (\nabla (\theta_r u_i), \nabla v_i) = \sum_{i=1}^{3} (-\theta_r \partial_t u_i - \theta_r (u \cdot \nabla) u_i + g_i, v_i)
\]

Since
\[
\partial_t u_i \in L^\infty(0, T; H), \quad (u \cdot \nabla) u_i \in L^\infty(0, T; H)
\]

Similar to the Theorem 3.8 in Chapter 3 of [4], and let \( r \to +\infty \), we obtain
\[
u_i \in L^\infty(0, T; H^2(\mathbb{R}^3)), \quad i = 1, 2, 3
\]

**Remark 1.** Noting that \((-\Delta u, v) = (-\partial_t u - (u \cdot \nabla) u, v)\). Since \( \partial_t u \) and \((u \cdot \nabla) u\) are of some degree of continuity, then \( u \) can reach a higher degree of continuity, based upon the smoothing effect of inverse elliptic operator \( \Delta^{-1} \).

By repeated application of this process one can prove that the solution \( u \) is in \( C^\infty((0, T) \times \mathbb{R}^3) \).

**Remark 2.** Based on problems separated and potential theory of fluid flow, we may keep the same result for the general initial-boundary value problems of 3D Navier-Stokes equation under the assumptions of regularity on the boundary and data.
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