RELATIVE GEOMETRIC INVARIANT THEORY AND UNIVERSAL MODULI SPACES

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0. Introduction

0.1. Motivation: the universal moduli problems. The motivation of this paper is to lay a GIT ground and to apply it to the so-called universal moduli problems such as the following ones.

1. The universal moduli space $\overline{FM}_{g,n} \to M_g$ of Fulton-MacPherson configuration spaces of stable curves. That is, given a stable curve $[C]$ in $M_g$, the fiber in the universal moduli space will be $C[n]/\text{Aut}(C)$, the Fulton-MacPherson configuration space for the curve $C$ modulo the automorphism group of $C$. (See also [21].)

2. The compactified universal Picard $\overline{P_g^d} \to M_g$ of degree $d$ line bundles ([5]).

3. The universal moduli space $\overline{Pg,m(e,r,F,\alpha)} \to M_{g,n}$ of $p$-semistable parabolic sheaves of degree $e$, rank $r$, type $F$, and weight $\alpha$ ([12]).

4. The universal moduli space $M_g(O,P) \to M_g$ of $p$-semistable coherent sheaves of pure dimension 1 with a fixed Hilbert polynomial $P$ such that the fiber over a stable curve $[C]$ is functorially identified with Simpson’s moduli space $M_C(O_C,P)$ of $p$-semistable coherent sheaves over $C$ of the Hilbert polynomial $P$ modulo the automorphism group of $C$.

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The construction of the universal moduli space $M_g(\mathcal{O}, P)$, as described above, bears a straightforward generalization to the universal moduli spaces of higher dimensional varieties (e.g., surfaces of general type, Calabi-Yau 3-folds, etc, see [12]) because Simpson’s construction [28] works for any projective scheme.

5. The universal Hilbert scheme $\text{Hilb}^n_g \to \overline{M}_g$ of 0-dimensional subschemes of length $n$ on the Mumford-Deligne stable curves such that the fiber over a stable curve $[C]$ is canonically the Hilbert scheme of 0-dimensional subschemes of length $n$ on $C$ modulo the automorphism group of $C$. Moreover, when $P(x) = x+n+1-g$ there exists a canonical dominating morphism

$$\psi : \text{Hilb}^n_g \to M_g(\mathcal{O}, P).$$

0.2. Relative GIT. All these universal moduli problems correspond to the following GIT setup which we shall call Relative GIT.

We have a projective map

$$\pi : Y \to X$$

and an epimorphism of two reductive algebraic groups

$$\rho : G' \to G.$$

We assume that $\pi : Y \to X$ is equivariant with respect to the homomorphism $\rho : G' \to G$.

**Question RGIT.** Given a linearization $L$ on $X$ and the GIT quotient $X^{ss}(L)//G$, find a linearization $M$ on $Y$ and the GIT quotient $Y^{ss}(M)//G'$ so that $Y^{ss}(M)//G'$ factors naturally to $X^{ss}(L)//G$.

Using the relation between GIT and moment maps, we solved in this paper the Question RGIT. In the case when $X^{ss} = X^s$ (which is the case for $\overline{M}_g$), the question has a particularly nice solution: the stable locus $X^{ss} = X^s$ downstairs essentially determines the stable locus $Y^{ss}(M) = Y^s(M)$ upstairs. These results are contained in §§3 and 4.

The solution to Question RGIT leads to a unified and easy approach to all those universal moduli problems where the base moduli spaces have satisfactory GIT constructions (e.g., $\overline{M}_g$). In particular, with the aid of Simpson’s approaches to the moduli of coherent sheaves over projective schemes [28], our construction of the universal moduli space

$$M_g(\mathcal{O}, P) \to \overline{M}_g$$

is much shorter than the approaches for the similar moduli problems in [1] and [25]. See §§8 and 9.

0.3. Kollár’s Approaches (as opposed to the traditional GIT [2]). The solution of Question RGIT resembles Kollár’s approaches to algebraic quotient spaces (cf. Conjecture 1.1 of [22]). Now, one should ask how to do all of the above via Kollár’s approaches ([22] and [19]). Theorem 2.14 of [22] sheds
lights on this question. But the insistence on the étaleness of the the equivariant map is too restrictive for universal moduli. There are something more on the relative quotients that one can do with Kollár’s approaches and in many cases Kollár’s approaches are easier to apply than GIT. In light of the fact that many moduli spaces of higher dimensional varieties have no satisfactory GIT constructions, some alternatives are necessary in order to build universal moduli spaces effectively (for example, over the moduli space of surfaces of general type \([21]\)). This quest even includes the case of \(\mathcal{M}_{g,n}\) which so far has no (satisfactory) GIT construction. We will return to these topics in an upcoming paper \([12]\).

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I. Relative GIT

1. Equivariant theory of moment maps

1.1. A moment map for a symplectic action of a compact Lie group \(K\) on a symplectic manifold \((M, \omega)\) with a symplectic form \(\omega\) is a smooth map

\[\Phi : M \rightarrow \mathfrak{k}^*\]

where \(\mathfrak{k}^*\) is the linear dual of the Lie algebra \(\mathfrak{k}\) of \(K\), satisfying the following two properties:

1. \(\Phi\) is equivariant with respect to the given action of \(K\) on \(M\) and the co-adjoint action of \(K\) on \(\mathfrak{k}^*\);
2. for any \(a \in \mathfrak{k}\), \(d\Phi(\xi) \cdot a = \omega(\xi, \xi_a)\) for all vector fields \(\xi\) on \(M\), where \(\xi_a\) is the vector field generated by \(a\).

The equation in (ii) determines the moment map by an additive constant in the set \(\mathfrak{z}^*\) of the central elements of \(\mathfrak{k}^*\) (i.e., \(\mathfrak{z}^* = \) the set of invariants of the \(K\)-coadjoint action = the linear dual of the Lie algebra \(\mathfrak{z}\) of the center group of \(K\)).

When a moment map exists, we call the the symplectic action a Hamiltonian action and the symplectic form \(\omega\) a \((K-)\) Hamiltonian symplectic form.

Fix a maximal torus \(T_k\) of \(K\). We shall use \(T\) to denote the complexification of \(T_k\). Let \(h^*_+\) is a positive Weyl chamber in the linear dual \(h^*\) of the
Lie algebra $\mathfrak{h}$ of the fixed maximal torus $T_k$ of $K$. Since $\mathfrak{h}^*_\pm$ parametrizes the orbit space of the coadjoint action of $K$ on $\mathfrak{k}^*$, we obtain, by quotienting the $K$-adjoint action, the so-called reduced moment map

$$\Phi_{\text{red}} : M \to \mathfrak{h}^*_\pm.$$ 

1.2. Although a Hamiltonian symplectic form $\omega$ only determines the moment map up to an additive central element in $\mathfrak{k}^*$. However, as pointed out by Atiyah [3], there is always a canonical choice, $\Phi_{\omega}$, the one such that

$$\int_M \Phi_{\omega} \omega_{\frac{1}{2} \dim M} = 0.$$ 

Such a moment map will be called the canonical moment map associated to $\omega$. All other moment maps defined by $\omega$ have the form $\Phi_{\bar{\omega}} = \Phi_{\omega} - \mu$ where $\bar{\omega} = (\omega, \mu)$ and $\mu \in \mathfrak{g}$ (consult [2.4] in the sequel).

Remark 1.3. For any invariant closed subset $Z$ in $M$ (possibly singular), one can simply define the moment map for the action of $K$ on $Z$ to be the restriction of the total moment map, for which we shall still use the same notation $\Phi$ if no confusion should emerge.

1.4. Given any value $p$ of a moment map $\Phi$, the orbit space $\Phi^{-1}(O_p)/K$, which is called the symplectic reduction at $p$, carries a natural induced symplectic structure away from singularities (24). Here $O_p$ is the coadjoint orbit through $p$.

1.5. Now we shall consider equivariant maps and the relations between their induced moment maps.

Let $Y$ and $X$ be smooth projective complex algebraic varieties acted on by complex reductive algebraic group $G'$ and $G$ respectively. Consider an algebraic surjection

$$\pi : Y \longrightarrow X$$

that is equivariant with respect to a group epimorphism (surjective homomorphism)

$$\rho : G' \longrightarrow G,$$

that is,

$$\pi(g' \cdot y) = \rho(g') \cdot \pi(y) \text{ for all } y \in Y \text{ and } g' \in G'.$$

One can choose a maximal subgroup $K$ of $G$ and a maximal subgroup $K'$ of $G'$ so that $\rho$ restricts to a group homomorphism (still denoted by $\rho$) $\rho : K' \longrightarrow K$ and $\rho : G' \longrightarrow G$ is the complexification of $\rho : K' \longrightarrow K$.

1.6. Symplectic (Kähler) forms and line bundles on $Y$ will be denoted by letters $\eta$ and $M$, while symplectic (Kähler) forms and line bundles on $X$ will be denoted by $\omega$ and $L$. 
1.7. Let $\omega$ be a symplectic (Kähler) form on $X$. Then $\eta_0 = \pi^* \omega$ is a closed two form on $Y$. We’d like to define a moment map for this degenerated two form $\eta_0$. The trick is to use nearby moment maps and then take the limit. Let $\eta(t), t \in (0, \epsilon]$ be a continuous path of symplectic (Kähler) forms on $Y$ such that $\eta_0 = \lim_{t \to 0} \eta(t)$. Here $\epsilon$ is a small positive number. (This can be done because $\eta_0 = \pi^* \omega$ lies on the boundary of the Kähler cone for $Y$.) Define

$$\Phi^\eta_0(y) = \lim_{t \to 0} \Phi^{\eta(t)}(y)$$

where $\Phi^{\eta(t)}$ are the canonical moment maps for $\eta(t)$.

**Proposition 1.8.** Keep the notations of 1.7. Then we have

1. $\Phi^\eta_0$ is a $K'$-equivariant differentiable map;
2. $\Phi^\eta_0$ satisfies the differential equation

$$d \Phi^\eta_0(\xi) \cdot a = \eta_0(\xi, \xi^Y_a)$$

for every vector field $\xi \in TY$ and $a \in \mathfrak{k}'$, where $\xi^Y_a$ is the vector field on $Y$ generated by the element $a$;
3. $\Phi^\eta_0$ is constant on every fiber of $\pi$.

**Proof.** The equivariancy of the map $\Phi^\eta_0$ is obvious. Now since $X \times [0, \epsilon]$ is compact, we have that $\Phi^{\eta(t)}$ converges uniformly as $t \to 0$. Thus $\Phi^\eta_0 = \lim_{t \to 0} \Phi^{\eta(t)}$ is differentiable. This proves (i). Next, given any fixed vector field $\xi \in TY$ and fixed $a \in \mathfrak{k}'$, we have

$$d \Phi^{\eta(t)}(\xi) \cdot a = \eta(t)(\xi, \xi_a)$$

for all $t \in (0, \epsilon]$. Passing to the limit as $t \to 0$, we obtain

$$d \Phi^\eta_0(\xi) \cdot a = \eta_0(\xi, \xi_a).$$

This proves (ii). To show (iii), picking any $\xi \in T\pi^{-1}(x) \subset TY$ for any $x \in X$. Recall that $\eta_0 = \pi^* \omega$. Now we must have

$$d \Phi^\eta_0(\xi) \cdot a = \eta_0(\xi, \xi_a) = \pi^* \omega(\xi, \xi_a) = \omega(d\pi(\xi), d\pi(\xi_a)) = 0$$

since $d\pi(\xi) = 0$. Because $a$ is an arbitrary element in $\mathfrak{k}'$, we obtain $d \Phi^\eta_0(\xi) = 0$. This implies that $\Phi^\eta_0$ is constant along every fiber of $\pi$. The proposition is thus proved.

**Remark 1.9.** The map $\Phi^\eta_0$ may be considered as a moment map defined by the (pre-symplectic) form $\eta_0$. As the limit of some (true) moment maps, it enjoys a number of properties of the usual momentum mapping such as the convexity. In general, however, a moment map defined by a pre-symplectic form may have much more complicated image that needs not to be convex (see [13]).

1.10. Choose a $K$-equivariant metric on $\mathfrak{k}$ and a $K'$-equivariant metric on $\mathfrak{k}'$ respectively so that $d\rho$ preserves the metrics. Then these two metrics lead two natural isomorphisms $\mathfrak{k} \cong \mathfrak{k}^*$ and $\mathfrak{k}' \cong \mathfrak{k}'^*$, making the following diagram commutes
where \( d\rho \) is the differential of the map \( \rho \) and \((d\rho)^*\) is the codifferential of the map \( \rho \), i.e., the linear dual of the differential \( d\rho \).

**Theorem 1.11.** The map \( \Phi^{\rho_0} \) descends to the moment map \( \Phi^\omega \) for the action of \( K \) on \( X \) with respect to the symplectic form \( \omega \). In particular, we have the following commutative diagram:

\[
\begin{array}{ccc}
Y & \xrightarrow{\Phi^{\rho_0}} & \mathfrak{t}^* \\
\downarrow{\pi} & \uparrow{(d\rho)^*} & \downarrow{d\rho} \\
X & \xrightarrow{\Phi^\omega} & \mathfrak{t}^*
\end{array}
\]

Proof. By Proposition 1.8 (3), the map \( \Phi^{\rho_0} \) descends to a well-defined differentiable map \( \Psi^{\rho_0} \) from \( X \) to \( \mathfrak{t}^* \) such that \( \Phi^{\rho_0} = \Psi^{\rho_0} \circ \pi \). Define \( \Phi^\omega \) to be the composition:

\[
\Phi^\omega : X \xrightarrow{\Psi^{\rho_0}} \mathfrak{t}^* \xrightarrow{(d\rho)^*} \mathfrak{t}^* \xrightarrow{d\rho} \mathfrak{t} \xrightarrow{\pi} \mathfrak{t}^.
\]

Clearly, this is a differentiable map, and we have that \( \Psi^{\rho_0} = (d\rho)^* \Phi^\omega \).

To show that \( \Phi^\omega \) is equivariant with respect to the given \( K \)-action on \( X \) and the coadjoint action on \( \mathfrak{t}^* \), take any element \( k' \in K \) and \( y \in Y \), we then have

\[
\Phi^\omega(k' \cdot y) = \Phi^\omega(\pi(k' \cdot y)) = d\rho \circ \Psi^{\rho_0}(\pi(k' \cdot y)) = d\rho \circ (d\rho)^* \Phi^\omega(y) = \text{Ad}(\rho(k')) \Phi^\omega(y).
\]

Here we have used the identity \( d\rho \circ \text{Ad}(k') = \text{Ad}(\rho(k')) d\rho \) coming from the equivariance of the homomorphism \( \rho \). Using the fact that both \( \pi \) and \( \rho \) are surjective, we obtain

\[
\Phi^\omega(k \cdot x) = \text{Ad}(k) \Phi^\omega(x), \text{ for all } k \in K \text{ and } x \in X.
\]

To check that it satisfies the differential equation in the definition of a moment map (1.1 (2)), notice that \( d\pi : TY \to TX \) is surjective everywhere but on a lower dimensional locus and \( d\pi(\xi_{a'}^X) = \xi_{(d\rho(a'))}^X \) for any \( a' \in \mathfrak{t}' \) because \( \pi \) is \( \rho \)-equivariant, where again, \( \xi_{a'}^X \) is the vector field on \( Y \) generated by \( a' \in \mathfrak{t}' \), while \( \xi_{(d\rho(a'))}^X \) is the vector field on \( X \) generated by \( d\rho(a') \in \mathfrak{t} \). Now for any \( \xi \in TY \) and \( a' \in \mathfrak{t}' \), we have

\[
\omega(d\pi \xi, \xi_{(d\rho(a'))}^X) = \omega(d\pi \xi, d\pi(\xi_{a'}^X)) = \pi^* \omega(\xi, \xi_{a'}^X) = \eta_0(\xi, \xi_{a'}^X)
\]

\[
= d\Phi^{\rho_0}(\xi) \cdot a' \quad \text{(because of 1.8 (2))}
\]

\[
= d\Psi^{\rho_0} \circ \pi(a') \quad \text{(because \( \Phi^{\rho_0} = \Psi^{\rho_0} \circ \pi \))}
\]

\[
= (d\rho)^* d\Phi^\omega \circ \pi(a') \quad \text{(because \( \Psi^{\rho_0} = (d\rho)^* \Phi^\omega \))}
\]

\[
(\text{because of 1.8 (2))}
\]

\[
= (d\pi)^* d\Phi^\omega(a') \quad \text{(because \( \Phi^{\rho_0} = \Psi^{\rho_0} \circ \pi \))}
\]

\[
(\text{because \( \Psi^{\rho_0} = (d\rho)^* \Phi^\omega \))}
\]

\[
(\text{because \( \Phi^{\rho_0} = \Psi^{\rho_0} \circ \pi \))}
\]

\[
(\text{because \( \Psi^{\rho_0} = (d\rho)^* \Phi^\omega \))}
\]

\[
\begin{array}{ccc}
Y & \xrightarrow{\Phi^{\rho_0}} & \mathfrak{t}^* \\
\downarrow{\pi} & \uparrow{(d\rho)^*} & \downarrow{d\rho} \\
X & \xrightarrow{\Phi^\omega} & \mathfrak{t}^*
\end{array}
\]
By the surjectivity of $d\rho$, this means that the differential equation
\[ d\Phi^\omega(\xi^X) \cdot a = \omega(\xi^X, \xi_a^X) \]
holds for vector fields $\xi^X \in TX$ almost everywhere. Thus by continuity, it holds for all $\xi^X$ in $TX$ everywhere. That is, $\Phi^\omega$ is a moment map for the action of $K$ on $X$ with respect to the symplectic form $\omega$. Since $\Phi^{\eta(t)}$ are the canonical moment maps for $\eta(t)$, it is easy to check that $\Phi^\omega$, as the limit of $\Phi^{\eta(t)}$, is the canonical moment map for $\omega$.

1.12. Finally, we remark that when $G' = G$, the above implies there is a deformation of moment maps
\[ \Phi^{\eta(t)} : Y \to \mathfrak{k}^* \]
from $\Phi^{\eta(\epsilon)} : Y \to \mathfrak{k}^*$ to $\Phi^{\omega} : X \to \mathfrak{k}^*$, where $\Phi^{\omega}(X) = \Phi^{\eta_0}(Y)$. When $\dim G' > \dim G$, the dimension of $\Phi^{\omega}(X)$ is less than those of $\Phi^{\eta(t)}(Y) (t \in (0, \epsilon))$. In this case, we say that $\Phi^\omega$ is a degeneration of $\Phi^{\eta(t)} (t \in (0, \epsilon))$.

2. G-Effective ample cone

Much of what follows is taken from [6] and [11].

2.1. In this section we assume that $X$ is a smooth projective complex algebraic variety\(^1\) over $\mathbb{C}$ acted on by a reductive group $G$ with a fixed maximal compact form $K$.

**Theorem 2.2.** (cf. Theorem 2.3.6, [6].) Let $X$ be a smooth projective variety acted on by a reductive complex algebraic group $G$. Let $\omega$ and $\omega'$ be two $K$-equivariant Kähler forms. Suppose that $\omega$ is cohomological equivalent to $\omega'$. Then $\Phi^\omega = \Phi^{\omega'}$.

**Proof.** The proof is the same as that for Theorem 2.3.6, [6]. □

2.3. Let $X$ be a smooth projective variety. We set $\mathcal{R}^G(X)$ to be the collection of all $K$-equivariant Kähler forms that are compatible with the algebraic $G$-action modulo cohomological equivalence. Theorem 2.3 says that there is a well-defined canonical moment map $\Phi^{[\omega]}$ for each element $[\omega]$ of $\mathcal{R}^G(X)$. For notational simplicity, we shall omit the use the bracket $"[\ ]"$ when it is not likely to cause confusion.

\(^1\)We point out that much of results in this section can be extended to Kähler category (i.e., $X$ being Kähler manifolds only). Since our primary applications will be algebraic moduli spaces, we are content with working in the category of algebraic varieties.
2.4. When $X$ is smooth, set $\mathcal{M}^G(X)$ to be the collection of all pairs consisting of an element in $\mathcal{K}^G(X)$ and a moment map defined by it. Thus \[ \mathcal{M}^G(X) \cong \mathcal{K}^G(X) \times \mathfrak{z}^*. \]

We shall adopt the following conventional scheme: $\tilde{\omega}$ denotes an element in $\mathcal{M}^G(X)$ with its underlying (Hamiltonian) Kähler form symbolized by $\omega \in \mathcal{K}^G(X)$. Such a symbol $\tilde{\omega}$ will be referred as an enriched symplectic Kähler form. Thus an enriched symplectic (Kähler) form $\tilde{\omega}$ in $\mathcal{M}^G(X)$ has the form $(\omega, \Phi_\omega - \mu)$ or simply $(\omega, \mu)$, where $\mu \in \mathfrak{z}^*$.

**Remark 2.5.** All of the results in §1 are valid without modification if the symplectic (Kähler) forms $(\omega$ and $\eta(t)$, etc.) are replaced by the enriched symplectic (Kähler) forms $(\tilde{\omega}$ and $\tilde{\eta}(t)$, etc.)

**Remark 2.6.** If $X$ is (possibly) singular, we can set $\mathcal{M}^G(X)$ to be the cone spanned by the images of all linearized ample line bundles in $\text{NS}^G(X) \otimes_{\mathbb{Z}} \mathbb{R}$ (\[3\]).

**Definition 2.7.** Assume that $X$ is smooth. The $G$-effective ample cone is a subcone of $\mathcal{M}^G(X)$ defined as follows:

\[ E^G(X) = \{ (\omega, \mu) \in \mathcal{M}^G(X) \mid \mu \in \Phi_\omega(X) \}; \]

**Remark 2.8.** When $X$ is (possibly) singular, we define $E^G(X)$ to be the subcone of $\mathcal{M}^G(X)$ that is spanned by the images of $G$-effective ample line bundles (that is, spanned by the ones such that $X^{ss}(L) \neq \emptyset$, see \[3\]). When $X$ is actually smooth, the two definitions are equivalent (\[11\]).

**2.9.** $E^G(X)$ projects to $\mathcal{K}^G(X)$ whose fiber at $\omega \in \mathcal{K}^G(X)$ is the intersection of the moment map image $\Phi_\omega(X)$ with $\mathfrak{z}^*$. This fiber $\Phi_\omega(X) \cap \mathfrak{z}^* = \Phi_{\omega,\text{red}}(X) \cap \mathfrak{z}^*$ is a convex compact polytope. In fact, it is not hard to see that $\Phi_\omega(X) \cap \mathfrak{z}^*$ is the image of $X$ under the canonical moment map attached to the induced action of the center group of $K$.

**Remark 2.10.** $E^G(X)$ may be an empty subset of $\mathcal{M}^G(X)$ when $G$ is semisimple. For example, when $X = G/P$ is a generalized flag variety. However, $E^G(X \times G/B)$ is never empty. In particular, $E^G(X)$ is never empty when $G$ is a torus.

**2.11.** Recall from \[11\] that for any point $x \in X$, we can define a (generalized Hilbert-Mumford) numerical function \[ M^*(x) : E^G(X) \to \mathbb{R} \]
whose value $M^{\tilde{\omega}}(x)$ at $\tilde{\omega} \in E^G(X)$ is defined as the signed distance from the origin to the boundary of $\Phi_{\tilde{\omega}}(G \cdot x)$: it takes a positive value if $0$ is outside
of $\Phi_{\bar{\omega}}(G \cdot x)$; it takes a nonpositive value otherwise. Using this numerical function we have the following criteria for (Kähler) semistabilities,

1. $X_{ss}(\bar{\omega}) = \{ x \in X | M_{\bar{\omega}}(x) \leq 0 \}$;
2. $X^s(\bar{\omega}) = \{ x \in X | M_{\bar{\omega}}(x) < 0 \}$;
3. $X^{us}(\bar{\omega}) = \{ x \in X | M_{\bar{\omega}}(x) > 0 \}$;
4. $(iv) \ X_{sss}^{ss}(\bar{\omega}) = \{ x \in X | M_{\bar{\omega}}(x) < 0 \}$.

Remark 2.12. When a group is needed to be specified, we will write $X_{ss}^G(\bar{\omega})$, $X^s_G(\bar{\omega})$, etc. This applies especially when there is a reductive subgroup $H$ of $G$. In this case, the restriction map $\mathcal{M}^G(X) \rightarrow \mathcal{M}^H(X)$ (the $H$-moment map is obtained from the $G$-moment map by the orthogonal projection $\mathfrak{k}^* \rightarrow \mathfrak{h}^*$ where $\mathfrak{h}$ is the Lie algebra of a suitable compact form of $H$) induces a linearization in $\mathcal{M}^H(X)$ for each linearization $\bar{\omega}$ in $\mathcal{M}^G(X)$. To specify this effect, we will write $X_{ss}^{ss}(\bar{\omega})$ ($X^s_H(\bar{\omega})$, etc) for the set of semistable (stable, etc) points for the action of $H$.

By the works of Kempf-Ness (for the algebro-geometric cases [16]) and Kirwan (for the Kähler generalizations [17]), there is a Hausdorff quotient topology on $X_{ss}(\bar{\omega})/G$ such that it contains the orbit space $X^s(\bar{\omega})/G$ as a dense open subset. It is a Kähler space and has a Kähler form induced from $\omega$ away from the singularities. Moreover, it is homeomorphic to the symplectic reduction $(\Phi_{\bar{\omega}})^{-1}(0)/K$.

Theorem 2.13. (Kempf-Ness-Kirwan) Let $\bar{\omega} = (\omega, \mu)$ be an enriched Kähler form in $\mathcal{E}^G(X)$. Then $(\Phi_{\bar{\omega}})^{-1}(0) \subset X_{ss}(\bar{\omega})$ and the inclusion induces a homeomorphism

$$(\Phi_{\bar{\omega}})^{-1}(0)/K \xrightarrow{\sim} X_{ss}(\bar{\omega})/G.$$ 

In case that $\bar{\omega}$ is integral (i.e., coming from a linearized ample line bundle), then $X_{ss}(\bar{\omega})/G$ carries a projective structure.

Remark 2.14. The $G$-effective ample cone takes care of symplectic reductions at central values of $\mathfrak{k}^*$ and identifies them with Kähler quotients of $X$ by the complex reductive group $G$. To include symplectic reductions at non-central values, one has to consider the so-called enlarged moment cone (see [1]) and use the so-called shifting trick to identify them with the Kähler quotients on $X \times G/P$ by the diagonal action of $G$. The relative GIT for the morphisms $X \times G/B \rightarrow X \times G/P$ has been studied and linked to degenerated quotients of $X \times G/B$ in [11]. So, in this paper, we stick with just the $G$-effective ample cone.
2.15. The union of the zero sets of $M^\bullet(x) : \mathfrak{e}^G(X) \to \mathbb{R}$ for all $x$ with isotropy subgroups of positive dimensions equals the union $\mathcal{W}$ of all walls in $\mathfrak{e}^G(X)$ (see \[6\] and \[11\]). A connected component of $\mathfrak{e}^G(X) \setminus \mathcal{W}$ is a chamber. Linearizations in the same chamber define the same notion of stabilities.

We shall need the following results in the sequel.

2.16. Given any element $\tilde{\omega} \in \mathfrak{e}^G(X)$, hence a unique moment map $\Phi \tilde{\omega} = \Phi$. We have a stratification, the momentum Morse stratification with respect to $\Phi$, $X = \bigcup_{\beta \in \mathcal{B}} S_\beta$ induced by the norm square $|\Phi|^2$ of the moment map (\[17\]). The strata $S_\beta$ and their indexes $\beta$ can be described as follows:

$$S_\beta = \{ x \in X | \beta \text{ is the unique closest point to 0 of } \Phi_{\text{red}}(G \cdot x) \}.$$

Theorem 2.17. (\[6\], \[11\]) There are only finitely many momentum Morse stratifications.

Definition 2.18. (\[11\]) For any momentum Morse stratification of $X$, we choose precisely one stratum from it. Then the intersection of all the chosen strata is called a thin momentum Morse stratum provided that it is not empty. There are only finitely many such strata. The thin momentum Morse strata form a stratification of $X$.

Proposition 2.19. (\[11\]) Two points in the same thin momentum Morse stratum give rise to the same numerical function $M^\bullet(x) : \mathfrak{e}^G(X) \to \mathbb{R}$.

Proof. The proof is short. So we repeat it here (see \[11\]). Let $x$ and $y$ be two points of a thin momentum Morse stratum. Then by definition, for any $\tilde{\omega} \in \mathfrak{e}^G(X)$, $x, y$ belong to the same momentum Morse stratum $S_{\beta(\tilde{\omega})}$ for some index $\beta(\tilde{\omega})$. That is, $M_{\tilde{\omega}}(x) = M_{\tilde{\omega}}(y) = |\beta(\tilde{\omega})|$ for all $\tilde{\omega} \in \mathfrak{e}^G(X)$. Hence

$$M^\bullet(x) = M^\bullet(y) : \mathfrak{e}^G(X) \to \mathbb{R}.$$

Corollary 2.20. There are only finitely many numerical functions

$$M^\bullet(x) : \mathfrak{e}^G(X) \to \mathbb{R}.$$

(And the points in the same thin momentum Morse stratum give rise to the same numerical function $M^\bullet(x)$.)

Remark 2.21. To close this section we make the following useful remark. We view the integral points in $\mathcal{R}^G(X)$ ($\mathfrak{e}^G(X)$) as being induced from the Hodge metrics of (linearized) ample line bundles. We shall think of these integral points and (linearized) ample line bundles interchangably. By default, the Kähler quotiens and maps associated to these integral points will be projective as in the traditional GIT cases.
3. A relative GIT: first cases

3.1. In the rest of the paper, unless specified otherwise, we shall assume that \( \pi: Y \to X \) is a projective morphism between two (possibly singular) quasi-projective algebraic varieties that is equivariant with respect to an epimorphism \( \rho: G' \to G \) between two reductive complex algebraic groups having fixed maximal compact subgroups \( K' \) and \( K \), respectively.

3.2. Let \( G_0 \) be the kernel of \( \rho \). Then we have
\[
\{1\} \to G_0 \to G' \to G \to \{1\}.
\]
That is, up to a finite central extension, we may think of \( G' = G_0 \times G \).

Because GIT problems for a finite group is trivial (i.e., the orbit space is the natural solution to the quotient problems for a finite group action), to simplify the exposition, one may well assume that \( G \cong G'/G_0 \) is a subgroup of \( G' \) and \( G' = G_0 G \).

Due to the above comments, the relative GIT problem for this case can be divided into two steps:

1. The fiberwise GIT problem: \( G_0 \) acts only on the fibers of \( \pi: Y \to X \); (One may look at this from a slightly different point of view: \( Y/X \) is projective over the base scheme \( X \). \( G_0/X \) is the trivial group scheme over \( X \). \( G_0/X \) acts on \( Y/X \). This viewpoint is helpful for some universal moduli problems.)

2. The \( G \)-equivariancy GIT problem: treat \( \pi: Y \to X \) as a \( G \)-equivariant map alone. Here we identify the quotient group \( G'/G_0 \) with the group \( G \) by the isomorphism induced by the epimorphism \( \rho \) and \( G \) acts on \( Y \) via this identification.

After those have been done, we can then put the two together and relate some properly chosen \( Y \)-quotient by \( G' \) to a given \( X \)-quotients by \( G \).

**Proposition 3.3.** Let \( M \) be a \( G_0 \)-linearized relatively ample line bundle with respect to the morphism \( \pi: Y \to X \). Let \( Y_x \) denote the fiber of \( \pi \) over a point \( x \in X \) and \( M_x \) the restriction of \( M \) to \( Y_x \). Then we have
\[
Y_x^{ss}(M_x) = Y_x \cap Y_{G_0}^{ss}(M).
\]
In particular \( Y_{G_0}^{ss}(M) = \bigcup_{x \in X} Y_x^{ss}(M_x) \).

**Proof.** It follows from Proposition 1.19 of [2]. See also [28]. \( \square \)

Consequently, since the categorical quotient is universal, one sees that the morphism \( \pi : Y_{G_0}^{ss}(M) \to X \) descends to a map
\[
\tilde{\pi} : Y_{G_0}^{ss}(M)/G_0 \to X
\]
with fibers \( Y_x^{ss}(M_x)/G_0 \).

**Proposition 3.4.** If \( M \) is a \( G' \)-linearized relatively ample line bundle with respect to the morphism \( \pi: Y \to X \), then the \( G' \)-action on \( Y \) descends to a \( G'/G_0 = G \)-action on \( Y_{G_0}^{ss}(M)/G_0 \) making the map \( \tilde{\pi} : Y_{G_0}^{ss}(M)/G_0 \to X \) \( G \)-equivariant.
Proof. First we need to show that $Y^ss_{G_0}(M)$ is $G'/G_0$-invariant (recall that the quotient group $G'/G_0$ is identified with the group $G$ by the isomorphism induced by the epimorphism $\rho$). Take a point $y \in Y^ss_{G_0}(M)$ and an element $g \in G$. Let $s \in \Gamma(Y,M_{\Theta})$ be an $G_0$-invariant section such that $Y_s := \{y' \in Y|s(y') \neq 0\}$ is affine and contains $y$ (see Definition 1.7 in §4 of [2]). Set

$$gs(y') = g\hat{s}(g^{-1}y')$$

for all $y' \in Y$. We claim that $gs \in \Gamma(Y,M_{\Theta})$. To see this, pick any element $g_0 \in G_0$. We have $g_0(gs)(y') = g_0(g\hat{s})(y') = ggs(g^{-1}g_0^{-1}y') = g\hat{g}_0s(g^{-1}g_0^{-1}y') = g\hat{g}_0s(\hat{g}_0^{-1}g^{-1}y') = g(g_0)s(g^{-1}y') = (gs)(y')$ where $\hat{g}_0$ is an element in $G_0$ such that $g_0g = g\hat{g}_0$ (note that $G_0$ is a normal subgroup). This implies that $g_0(gs) = g_s$, i.e., $gs \in \Gamma(Y,M_{\Theta})$. Clearly $Y_s = gY_s$ is affine and contains $gy$. That is, $gy \in Y^ss_{G_0}(M)$.

Since $Y^ss_{G_0}(M) \subset Y$ factors to $\tilde{X}$ in a $G_0$-equivariant way ($G_0$ acts trivially on $X$), by the universality of categorical quotient, we see that there is a naturally induced morphism $\tilde{\pi} : Y^ss_{G_0}(M)/G_0 \to \tilde{X}$. Next it is straightforward to verify that the $G$-action on $Y^ss_{G_0}(M)$ passes naturally to the quotient and the $G$-equivariance of $\pi : Y^ss_{G_0}(M) \to X$ implies the $G$-equivariance of $\tilde{\pi} : Y^ss_{G_0}(M)/G_0 \to \tilde{X}$.

3.5. Now let us consider the $G$-equivariance GIT problems. That is, we are going to treat morphism $\pi : Y \to X$ as a $G$-equivariant map only. Our finest results in this section lie in the case when $X^{ss}(L) = X^s(L)$. This is what we shall sometimes refer as “good cases”.

3.6. We begin with assuming that $\pi : Y \to X$ is a projective morphism between two smooth projective varieties. Let $\tilde{\omega} = (\omega, \mu)$ be an enriched $K$-equivariant Kähler form on $X$ and $\tilde{\eta}_0 = \pi^*\tilde{\omega} = (\pi^*\omega, \mu)$ be an enriched closed $K$-equivariant two form on $Y$. Let $\tilde{\eta} : (0, \epsilon] \to \mathfrak{E}^G(Y)$ be a continuous path in $\mathfrak{E}^G(Y)$ such that $\tilde{\eta}_0 = \lim_{t \to 0} \tilde{\eta}(t)$. As in 1.7, define

$$\Phi_{\tilde{\eta}_0}(y) = \lim_{t \to 0} \Phi_{\tilde{\eta}(t)}(y).$$

Then we have

Theorem 3.7. Let $Y \to X$ be a $G$-equivariant projective morphism between two smooth projective varieties. Then there exists $\delta > 0$ such that

1. If $x = \pi(y)$ is stable in $X$ w.r.t $\tilde{\omega}$, then $y$ is stable in $Y$ w.r.t. $\tilde{\eta}(t), t \in (0, \delta]$;
2. If $x = \pi(y)$ is non-semistable in $X$ w.r.t $\tilde{\omega}$ then $y$ is non-semistable in $Y$ w.r.t. $\tilde{\eta}(t), t \in (0, \delta]$;
3. $Y^s(\tilde{\eta}(t)) \supset \pi^{-1}(X^s(\tilde{\omega}))$ and $Y^{ss}(\tilde{\eta}(t)) \subset \pi^{-1}(X^{ss}(\tilde{\omega}))$ for $t \in (0, \delta]$.

Proof. Let $Y = \bigsqcup_{i=1}^d S_i$ be the thin momentum Morse stratification of $Y$ (see Definition 2.18). By Corollary 2.20, it suffices to consider some fixed
representing elements in $S_i, i = 1, \cdots, d$. For any $1 \leq i \leq d$, if $\pi(S_i) \subset X^{ss}(\omega)$, set $\delta_i = 1$.

Otherwise, pick up some representing elements $y_i \in S_i, 1 \leq i \leq d$.

If $x_i = \pi(y_i)$ is stable in $X$, then $0 \in \text{int}(\Phi^0(G \cdot x_i))$. By the remarks in 1.12, a small deformation of $\Phi^0(G \cdot x_i)$ should still contain the origin in its interior. That is, there exists $\delta_i > 0$ such that $0 \in \text{int}(\Phi^0(t)(G \cdot y))$ for $t \leq \delta_i$. Thus $y_i$ is stable in $Y$ w.r.t. $\eta(t)$ for $t \leq \delta_i$.

If $x_i = \pi(y_i)$ is non-semistable in $X$, then $0$ is outside of $\Phi^0(G \cdot x_i)$. By the similar deformation argument as above, the same is true for its small deformations. This is, there exists $\delta_i > 0$ such that $0 \notin \text{int}(\Phi^0(t)(G \cdot y))$ for $t \leq \delta_i$. In other words, $y$ is non-semistable in $Y$ w.r.t. $\eta(t)$ for $t \leq \delta_i$.

Now choose $\delta = \min\{\delta_1, \cdots, \delta_d, \epsilon\}$, the above implies both (1) and (2).

(3) follows from (2). \hfill \Box

Remark 3.8. When $x = \pi(y)$ is strictly semistable, i.e., the origin is contained on the boundary of $\Phi^0(G \cdot x)$, the stability of $y$ depends on the direction of deformation. So in general, all three possible situations (stable, strictly semistable, non-semistable) may happen.

3.9. In keeping with the theme of the traditional GIT, consider the case that $\omega$ is an integral form induced from an ample linearized line bundle $L$ on $X$. The pullback $\pi^*L$ is only a nef linearized line bundle on $Y$ inducing the 2-form $\eta_0 = \pi^*\omega$. To get an ample linearized line bundle on $Y$, we need to choose an arbitrary relatively ample linearized line bundle $M$ on $Y$ and take a sufficiently large tensor power of $\pi^*L$. That is, $\pi^*L^n \otimes M$ is ample for $n \gg 0$ ([1]). Set fractional linearizations $M_n = \frac{1}{n}(\pi^*L^n \otimes M)$. Clearly, $\lim_{n \to \infty} M_n = \pi^*L$. The purpose of this scheme is two fold: to get (fractional multiples of) ample linearized line bundles $M_n$ on $Y$; and to make $\pi^*L$ and $M_n = \frac{1}{n}(\pi^*L^n \otimes M)$ sufficiently close for sufficiently large $n$.

We need the following easy lemma

Lemma 3.10. Let $G$ act on $X$ whose action is linearized by $L$ and $i : Z \hookrightarrow X$ be a $G$-invariant closed embedding linearized by $i^*L$. Then $Z^{ss}(i^*L) = Z \cap X^{ss}(L)$ and $Z^s(i^*L) = Z \cap X^s(L)$.

Proof. This follows from Proposition 1.19 of [2]. \hfill \Box

Theorem 3.11. Let $\pi : Y \to X$ be a $G$-equivariant projective morphism between two (possibly singular) quasi-projective varieties. Given any linearized ample line bundle on $L$ on $X$, choose a relatively ample linearized line bundle $M$ on $Y$. Then there exists $n_0$ such that when $n \geq n_0$, we have

1. $Y^{ss}(\pi^*L^n \otimes M) \subset \pi^{-1}(X^{ss}(L))$;
2. $Y^s(\pi^*L^n \otimes M) \supset \pi^{-1}(X^s(L))$

If in addition, $X^{ss}(L) = X^s(L)$, then
3. \( Y^{ss}(\pi^* L^n \otimes M) = Y^s(\pi^* L^n \otimes M) = \pi^{-1}(X^s(L)) = \pi^{-1}(X^{ss}(L)) \). In particular, \( \pi^* L^n \otimes M \) lie in the same chamber of \( \mathfrak{E}^G(Y) \) for all \( n \geq n_0 \).

**Proof.** Given a linearized ample line bundle \( L \) on \( X \), and a relatively linearized ample line bundle \( M \) for \( \pi : Y \to X \), by \([\ref{remark:ample}]](remark:ample)\), there exists \( m_0 \) such that when \( m \geq m_0 \), \( L^m \) and \( \pi^* L^m \otimes M \) are very ample. Consider some equivariant projective embeddings induced from \( L^m \) and \( \pi^* L^m \otimes M \) (for some \( m \geq m_0 \))

\[
Y \leftrightarrow X \times \mathbb{P}^{N'} \leftrightarrow \mathbb{P}^N \times \mathbb{P}^{N'} \\
\downarrow \pi \quad \quad \quad \downarrow \text{proj} \\
X \quad \leftrightarrow \quad \mathbb{P}^N
\]

We may think \( L^m \) as the pull back of a very ample line bundle \( O_{\mathbb{P}^N}(1) \). By \([\ref{remark:ample}]](remark:ample)\), there exists \( d_0 > 0 \) such that when \( d \geq d_0 \), Theorem \([\ref{theorem:Ampleness}]\) can be applied to the \( G \)-equivariant projection \( \mathbb{P}^N \times \mathbb{P}^{N'} \to \mathbb{P}^N \) with respect to the linearizations \( O_{\mathbb{P}^N}(1) \) on \( \mathbb{P}^N \) and \( \frac{1}{d}(O_{\mathbb{P}^N}(d) \otimes O_{\mathbb{P}^N'}(1)) \) on \( \mathbb{P}^N \times \mathbb{P}^{N'} \), respectively. Now restricting everything to \( \pi : Y \to X \) (see Lemma \([\ref{lemma:pushforward}]](lemma:pushforward)\), we see that \( \pi : Y \to X \) shares all the properties as in Theorem \([\ref{theorem:Ampleness}]\). This proves (1) and (2).

All of (3) follows immediately from (1) and (2).

**Remark 3.12.** In the case that \( M \) is ample and \( X \) and \( Y \) are projective, Theorem \([\ref{theorem:Ampleness}]\) is largely due to Z. Reichstein \([\ref{reichstein}]\) whose proof is completely different. But we only need to assume that \( M \) is relatively ample and \( X \) and \( Y \) are quasi-projective. We emphasize the practical convenience that can result from the relative ampleness of the line bundle \( M \), because in constructing moduli spaces, relative projective embeddings of relative Hilbert schemes have been constructed very explicitly by Grothendieck (hence are ready to use), whereas absolute projective embeddings will not be offhand.

**Theorem 3.13.** Keep the assumption as in Theorem \([\ref{theorem:Ampleness}]\). Then the inclusion \( Y^{ss}(\pi^* L^n \otimes M) \subset \pi^{-1}(X^{ss}(L)) \) induces a projective morphism

\[
\hat{\pi} : Y^{ss}(\pi^* L^n \otimes M)/G \to X^{ss}(L)/G
\]

such that

1. \( \hat{\pi}^{-1}([G \cdot x]) \cong \pi^{-1}(x)/G_x \) for any \( x \in X^s(L) \);
2. if \( \pi \) is a fibration, \( X^{ss}(L) = X^s(L) \), and \( G \) acts freely on \( X^s(L) \), then \( \hat{\pi} \) is also a fibration with the same fibers.

**Proof.** The morphism \( \pi \) restricts to give a morphism \( Y^{ss}(\pi^* L^n \otimes M) \to X^{ss}(L) \). This in turn induces an obvious morphism

\[
Y^{ss}(\pi^* L^n \otimes M) \to X^{ss}(L) \to X^{ss}(L)/G.
\]

Now by the universality of the categorical quotient, we get our desired induced morphism

\[
\hat{\pi} : Y^{ss}(\pi^* L^n \otimes M)/G \to X^{ss}(L)/G.
\]
4. A relative GIT: general cases

Now we begin to investigate the relation between $G'$-stability on $Y$ and the $G$-stability on $X$.

**Definition 4.1.** A linearized pair $(L, M)$ for the morphism $\pi : Y \to X$ consists of a $G$-linearized ample line bundle $L$ over $X$ and a $G'$-linearized $\pi$-ample line bundle $M$ over $Y$.

Out of a linearized pair $(L, M)$, we can have two Zariski open subsets $Y_{G_0}^{ss}(M)$ and $\pi^{-1}(X^{ss}(L))$. Proposition 3.4 says that $Y_{G_0}^{ss}(M)$ is $G$-invariant and hence $G'$-invariant. Obviously $\pi^{-1}(X^{ss}(L))$ is $G_0$-invariant, hence also $G'$-invariant.

**Lemma 4.2.** Let $(L, M)$ be a linearized pair for the morphism $\pi : Y \to X$. Then

1. $Y_{G_0}^{ss}(\pi^*L^n \otimes M) = Y_{G_0}^{ss}(M)$;
2. $Y_{G_0}^s(\pi^*L^n \otimes M) = Y_{G_0}^s(M)$;

**Proof.** Notice that $G_0$ preserves the fibers of the morphism $\pi$. Thus we are in the position to apply Proposition 3.3. Given any point $x$ in $X$. Since the restricted line bundle $\pi^*L^n|_{Y_x}$ is trivial, we obtain $(Y_x)_{G_0}^{ss}((\pi^*L^n \otimes M)|_{Y_x}) = (Y_x)_{G_0}^s(M)$. Now Proposition 3.3 implies that $Y_{G_0}^{ss}(\pi^*L^n \otimes M) = Y_{G_0}^s(M)$.

(2) follows from (1) because of the fact: a point is stable if and only if its isotropy is finite and its orbit is closed in the semistable locus.

Again by 4.6.13 (ii) of [1], there exists an positive integer $n_0$ such that $\pi^*L^n \otimes M$ is ample when $n \geq n_0$.

**Lemma 4.3.** Assume that $n \geq n_0$. Then

1. $Y^{ss}(\pi^*L^n \otimes M) = Y_{G_0}^{ss}(\pi^*L^n \otimes M) \cap Y_G^{ss}(\pi^*L^n \otimes M)$.
2. $Y^s(\pi^*L^n \otimes M) = Y_{G_0}^s(\pi^*L^n \otimes M) \cap Y_G^s(\pi^*L^n \otimes M)$.

**Proof.** By general nonsense, we have

$Y^{ss}(\pi^*L^n \otimes M) \subset Y_{G_0}^{ss}(\pi^*L^n \otimes M) \cap Y_G^{ss}(\pi^*L^n \otimes M)$.

To show the other way inclusion, notice that $G' = G_0G$ and any 1-PS of $G'$ can be written as $\lambda \lambda_0$ where $\lambda$ is a 1-PS of $G$ and $\lambda_0$ is a 1-PS of $G_0$. Take
Proof. From Lemma 4.3 (1),
\[ (4) \text{ follow directly from the combination of Lemma 4.2, Lemma 4.3, and Theorem 3.11.} \]
we have
\[ \text{Then by Lemma 4.2 (1) and Theorem 3.11 (1) (see also Theorem 3.7 (3)),} \]
\[ \text{a homomorphism } G \rightarrow L,M \text{ is linearized.} \]
\[ \text{Lemma 4.2 (2), and Theorem 3.11 (2).} \]
\[ \text{bundle } M \text{ 4.6.} \]
\[ \text{The structure of the quotient of } G \text{ shall take a unified approach as follows.} \]

Remark 4.5. When \( \pi : Y \rightarrow \{ \text{point} \} \) is the total contraction and \( \rho \) is the homomorphism \( G \rightarrow \{ \text{point} \} \), we have \( L = O_{\{ \text{point} \}} \). In this case, the relative \( \pi \)-ampleness is equivalent to the absolute ampleness and the effect of the a linearized pair \( (L, M) \) is equivalent to that of the linearized ample line bundle \( M \). Thus we recover the traditional (i.e., the absolute) GIT.

4.6. The structure of the quotient of \( Y^{ss}(\pi^* L^n \otimes M) \) by the large group \( G' \) may be divided in two ways: mod out by the group \( G_0 \) first and then by the group \( G \); or the other way around. In the end, we would have to prove that the quotient is independent of the order of the two procedures. But we shall take a unified approach as follows.
Theorem 4.7. Keep the assumption as in Theorem 4.4. Then the categorical quotient \( Y^{ss}(\pi^*L^n \otimes M)/G' \) exists and there is a naturally induced morphism
\[
\hat{\pi} : Y^{ss}(\pi^*L^n \otimes M)/G' \rightarrow X^s(L)/G.
\]
Moreover, given an orbit \( G \cdot x \) in \( X^s(L) \), the fiber \( \hat{\pi}^{-1}(G \cdot x) \) can be identified with the quotient \((Y_x)^{ss}_{G0}(M|Y_x)/G_0)G_x = (Y_x)^{ss}_{G0}(M|Y_x)/G_0G_x\) where \( G_x \) is the (finite) isotropy subgroup at the point \( x \).

Proof. The idea of proof is similar to that of Theorem 3.13. We shall present the detail as follows.

By [2], the categorical quotient \( Y^{ss}(\pi^*L^n \otimes M)/G' \) exists as a quasiprojective variety. By Theorem 4.4 (1), the morphism \( \pi \) restricts to a morphism (still denoted by \( \pi \))
\[
\pi : Y^{ss}(\pi^*L^n \otimes M) \rightarrow X^{ss}(L).
\]
Thus we get an obvious induced morphism
\[
\hat{\pi} : Y^{ss}(\pi^*L^n \otimes M)/G' \rightarrow X^s(L)/G.
\]
Since a categorical quotient is a universal quotient [2], the above morphism passes to the quotient to give us the desired morphism
\[
\hat{\pi} : Y^{ss}(\pi^*L^n \otimes M)/G' \rightarrow X^s(L)/G.
\]
To prove the rest of the statement, let \( [G' \cdot y_1] \) and \( [G' \cdot y_2] \) be arbitrary two points in \( Y^{ss}(\pi^*L^n \otimes M)/G' \) that are mapped down to \( [G \cdot x] \in X^s(L)/G \). Then we must have that \( \pi(y_1), \pi(y_2) \in X^s(L) \). Hence \( G \cdot \pi(y_1) = G \cdot \pi(y_2) = G \cdot x \). Thus we may well assume that \( y_1, y_2 \in \pi^{-1}(x) \) by applying some elements of \( G \) to \( y_1 \) and \( y_2 \) if necessary. But, in this case, \( [G' \cdot y_1] = [G' \cdot y_2] \) if and only if \( G_x[G_0 \cdot y_1] = G_x[G_0 \cdot y_2] \) where \( G_x \) acts naturally on \((Y_x)^{ss}_{G0}(M_x)/G_0\) and \([G_0 \cdot y_1], [G_0 \cdot y_2]\) are regarded as points in \((Y_x)^{ss}_{G0}(M_x)/G_0\). This implies that \( \hat{\pi}^{-1}(G \cdot x) \cong (Y_x)^{ss}_{G0}(M_x)/G_0G_x \).

We shall need the following technical theorem in our later study of universal moduli problems.

Theorem 4.8. Let \( \pi : Y \rightarrow X \) be a projective morphism between two algebraic varieties and equivariant with respect to \( G \)-actions on both \( X \) and \( Y \). Assume that with respect to some \( G \)-equivariant projective embedding, \( X \) is
1. contained in the stable locus; and
2. is closed in the semistable locus.
Then there exists in addition a \( G \)-equivariant relative projective embedding for \( \pi : Y \rightarrow X \) and an induced \( G \)-linearization such that \( \tilde{Y} \) is
1. contained in the stable locus; and
2. is closed in the semistable locus. Moreover
3. \( Y/G \) factors naturally to \( X/G \) with the fiber over \( [G \cdot x] \in X/G \) isomorphic to \( \pi^{-1}(x)/G_x \) for \( x \in X \).
Proof. It follows from Theorem 3.13 by restricting everything to $\pi : Y \to X$.

A more general technical theorem which we shall quote in studying moduli problems later is as follows:

**Theorem 4.9.** Let $\pi : Y \to X$ be a projective morphism between two algebraic varieties and equivariant with respect to a homomorphism $\rho : G' \to G$ where $G'$ acts $Y$ and $G$ acts on $X$. Assume that with respect to some $G$-equivariant projective embedding, $X$ is

1. contained in the stable locus; and
2. is closed in the semistable locus.

Then there exists in addition a $\rho$-equivariant relative projective embedding for $\pi : Y \to X$ and an induced $G'$-linearization such that $Y$ is

1. contained and closed in the $G_0$-semistable locus;
2. contained and closed in the $G$-stable locus;
3. contained and closed in the $G'$-semistable locus;
4. $Y//G'$ factors naturally to $X/G$ with the fiber over $[G \cdot x] \in X/G$ isomorphic to $\pi^{-1}(x)/G_0G_x$ for $x \in X$.

Proof. It follows from Theorem 4.7 by restricting everything to $\pi : Y \to X$.

**Remark 4.10.** In studying moduli problems, the above two theorems will allow us to avoid actually writing down a relative projective embedding (for a universal curve or a relative Quot scheme) which is sometimes technical and time consuming.

**4.11.** To close this section, let’s look at an interesting special case. Assume that $G$ acts on two projective varieties $X$ and $X_0$ respectively and $G_0$ acts on $X_0$ only (acts on $X$ trivially). Set $Y = X_0 \times X$ acted on by $G' = G_0 \times G$ where $G_0$ operates on the first factor only, while $G$ acts diagonally. Thus the projection onto the first factor $\pi : Y \to X$ is equivariant with respect to the projection onto the second factor $G' \to G$. (This example has been worked out by R. Pandharipande in the case that $X_0 = \mathbb{P}(V)$ and $X = \mathbb{P}(W)$.) We begin with considering the diagonal action of $G$ on $Y = X_0 \times X$. The GIT problem for $G_0$ on $Y$ is the same as the GIT problem for $G_0$ on $X_0$.

We have the following inclusion

$$\operatorname{Pic}^G(X) \otimes \operatorname{Pic}^G(X_0) \subset \operatorname{Pic}^G(X_0 \times X).$$

We shall concentrate on the linearizations coming from $\operatorname{Pic}^G(X) \otimes \operatorname{Pic}^G(X_0)$. Take $L \in \operatorname{Pic}^G(X)$ and $L_0 \in \operatorname{Pic}^G(X_0)$. By Theorem 3.11, we have that there exists $m_0$ such that when $m \geq m_0$,

1. $Y^{ss}_G(L_0 \otimes L \otimes m) \subset \pi^{-1}(X^{ss}(L))$.
2. $Y^s_G(L_0 \otimes L \otimes m) \supset \pi^{-1}(X^s(L))$. 
In case that \( X^{ss}(L) = X^s(L) \), we have
\[
Y^{ss}_G(L_0 \otimes L^{\otimes m}) = Y^{ss}_G(L_0 \otimes L^{\otimes m}) = \pi^{-1}(X^s(L)) = X_0 \times X^s(L).
\]
Clearly, this implies that when \( L_0 \in \operatorname{Pic}^G(X_0) \)
\[
Y^{ss}(L_0 \otimes L^{\otimes m}) = (X_0)^{ss}_{G_0}(L_0) \times X^s(L).
\]
So we get the induced morphism
\[
\hat{\pi} : Y^{ss}(L_0 \otimes L^{\otimes m})//G' \to X^s(L)//G
\]
with the fiber isomorphic to \((X_0)^{ss}_{G_0}(L_0)//G_0G_x\) at the point \([G:x] \in X^s(L)//G\). This induced morphism \(\hat{\pi}\) needs not to be a trivial fibration.

5. Generalized Kempf-Ness’s theorem

5.1. Again, we treat \(\pi : Y \to X\) as a \(G\)-equivariant morphism alone. Place ourselves in the situation of 3.6. According to Theorem 3.7, in the case that \(X^{ss}(\tilde{\omega}) = \emptyset\), \(\tilde{\omega}\) determines uniquely a semistable locus upstairs. This motivates:

**Definition 5.2.** Let \(\tilde{\eta}_0\) be a polarization on the boundary of \(\mathfrak{C}^G(Y)\) which is the pullback of \(\tilde{\omega}\) by the morphism \(\pi\). Assume that \(X^{ss}(\tilde{\omega}) = \emptyset\). Then we set \(Y^{ss}(\tilde{\eta}_0) = Y^s(\tilde{\eta}_0) = \pi^{-1}(X^{ss}(\tilde{\omega}))\).

**Lemma 5.3.** If \(y \in (\Phi^{\tilde{\eta}_0})^{-1}(0)\), then \(G \cdot y \cap (\Phi^{\tilde{\eta}_0})^{-1}(0) = K \cdot y\).

**Proof.** First observe that \(y \in (\Phi^{\tilde{\eta}_0})^{-1}(0)\) implies \(\Phi^{\tilde{\omega}}(\pi(y)) = 0\). That is, \(\pi(y) \in (\Phi^{\tilde{\omega}})^{-1}(0)\).

We shall adopt the proof of Lemma 7.2 of [17]. Suppose that \(g \in G\) is such that \(g \cdot y \in (\Phi^{\tilde{\eta}_0})^{-1}(0)\). We want to show that there exists an element \(k \in K\) such that \(g \cdot y = k \cdot y\). Since \(\Phi^{\tilde{\eta}_0}\) is \(K\)-equivariant and \(G = K\exp(it)\), we may assume that \(g = \exp(it)\), \(a \in \mathfrak{g}\). Let \(h(t) = \Phi^{\tilde{\eta}_0}(\exp(it) \cdot y) \cdot a\).

Then
\[
h(t) = \Phi^{\tilde{\omega}} \circ \pi(\exp(it) \cdot y) \cdot a.
\]
\(h(t)\) vanishes at \(t = 0\) and \(t = 1\) because both \(y\) and \(\exp(it) \cdot y\) belong to \((\Phi^{\tilde{\eta}_0})^{-1}(0)\). Thus there must be a point \(t \in (0, 1)\) such that
\[
0 = h'(t) = d\Phi^{\tilde{\omega}} \circ d\pi(i(\xi^Y_a)_z) \cdot a = \omega(d\pi(i(\xi^Y_a)_z), (\xi^X_a)_{\pi(z)})
\]
\[
= \omega(i(\xi^X_a)_{\pi(z)}, (\xi^X_a)_{\pi(z)}) = <(\xi^X_a)_{\pi(z)}, (\xi^X_a)_{\pi(z)}> = \]
where \(z = \exp(it) \cdot y\). Hence \((\xi^X_a)_{\pi(z)} = 0\). Or, \(a \in \operatorname{Lie}(G_{\pi(z)})\). But
\[
\pi(z) = \exp(it) \cdot \pi(y) \in G(\Phi^{\tilde{\omega}})^{-1}(0).
\]
So \(a = 0\) because \(G_{\pi(z)}\) must be a finite group. This completes the proof. 

Now we have the following theorem which extends the Kempf-Ness theorem to the degenerated polarizations.
Theorem 5.4. (Generalized Kempf-Ness’s Theorem) Assume that $X^{ss}(\overline{\omega}) = X^s(\overline{\omega})$. Then

1. $Y^{ss}(\overline{\eta}_0) = G(\Phi_{0\overline{\eta}})^{-1}(0)$;
2. the topological quotient $Y^{ss}(\overline{\eta}_0)/G$ is Hausdorff;
3. the inclusion $(\Phi_{0\overline{\eta}})^{-1}(0) \hookrightarrow Y^{ss}(\overline{\eta}_0)$ induces a homeomorphism between $(\Phi_{0\overline{\eta}})^{-1}(0)/K$ and $Y^{ss}(\overline{\eta}_0)/G$.
4. $Y^{ss}(\overline{\eta}_0)/G = (\Phi_{0\overline{\eta}})^{-1}(0)/K$ inherits from $\eta_0$ a closed 2-form (which may be degenerated somewhere) away from singularities.

Proof. (1) By definition,

$$Y^{ss}(\overline{\eta}_0) = \pi^{-1}(X^s(\overline{\omega}) = \pi^{-1}(G(\Phi_{0}\overline{\omega})^{-1}(0) = G\pi^{-1}(\Phi_{0\overline{\omega}})^{-1}(0)$$

$$= G(\Phi_{0\overline{\omega}} \circ \pi)^{-1}(0) = G(\Phi_{0\overline{\eta}})^{-1}(0).$$

(2) This follows from the fact that $Y^{ss}(\overline{\eta}_0) = Y^{ss}(\overline{\eta}(t))$ for sufficiently small positive numbers $t$.

(3) (1) implies that the induced map $(\Phi_{0\overline{\eta}})^{-1}(0)/K \rightarrow Y^{ss}(\overline{\eta}_0)/G$ is surjective. Lemma 5.3 implies that it is injective. Now as a continuous bijection between Hausdorff spaces, it must be a homeomorphism.

(4) The proof is the same as the one for non-degenerated 2-forms (24).

Remark 5.5. In fact by Theorem 5.7, the quotient $Y^{ss}(\overline{\eta}_0)/G$ admits a complex structure and many other non-degenerated 2-forms induced from $\overline{\eta}(t), t \in (0, \delta]$ such that the 2-form induced from $\overline{\eta}_0$ is the limit of the above.

Remark 5.6. When $X^{ss}(\overline{\omega}) \neq \emptyset$, there is a difficulty in defining that

$$Y^{ss}(\overline{\eta}_0) := \{ y \in Y | 0 \in \Phi_{0\overline{\eta}}(G \cdot y) \} = \{ y \in Y | 0 \in \Phi_{0\overline{\omega}}(G \cdot \pi(y)) \} = \pi^{-1}(X^{ss}(\overline{\omega})).$$

The problem is as follows. Let $x \in X^{ss}(\overline{\omega})$ be a point such that $G \cdot x$ is closed in $X^s(\overline{\omega})$. Then $G_x$ is reductive and acts on the fiber $\pi^{-1}(x) \subset Y^{ss}(\overline{\eta}_0)$. Since dim$G_x > 0$, $\pi^{-1}(x)/G_x$ is non-Hausdorff. Thus one would like to exclude some points in $\pi^{-1}(x)$ from $Y^{ss}(\overline{\eta}_0)$, presumably by using a non-degenerate closed 2-form on $\pi^{-1}(x)$. That is, one would like to pick up some semistability on $\pi^{-1}(x)$ for the action of $G_x$. The form $\eta_0$ is helpless in this regard since the fiber $\pi^{-1}(x)$ is exactly where $\eta_0$ vanishes. Hence among many choices of the semistabilities on $\pi^{-1}(x)$ for the action of $G_x$, we do not know, a priori, which to choose. Notice that the same ambiguity does not happen when $X^{ss}(\overline{\omega}) = \emptyset$. In this case, the isotropy subgroup $G_x$ for every $x \in X^{ss}(\overline{\omega})$ is a finite group. $\pi^{-1}(x)/G_x$ is always a good quotient.

One of ways to solve (actually to pass) the problem of $X^{ss}(\overline{\omega}) \neq \emptyset$ is as follows. Assume that dim $C^G(X) > 1$ and it has at least one top chamber. Choose a linearization $\omega'$ in a top chamber that contains $\overline{\omega}$ in its closure. We then have $X^{ss}(\omega') = X^s(\omega') \subset X^{ss}(\overline{\omega})$. Now $\pi^{-1}(X^{ss}(\omega'))$ has a good complete quotient and one has natural maps

$$\pi^{-1}(X^{ss}(\omega'))/G \rightarrow X^{ss}(\omega')/G \rightarrow X^{ss}(\overline{\omega})/G.$$


This helps to reduce the case when \( X^{ss}(\omega) \neq \emptyset \) to the nicer case when \( X^{ss}(\omega) = \emptyset \).

**Remark 5.7.** In general, the semistable set \( Y^{ss}(\pi^*L^n \otimes M)(n \gg 0) \) may be recovered as follows: Let \( X^{ss}_c(L) \) be the set of closed orbits in \( X^{ss}(L) \). Then

1. \( Y^{ss}_c(\pi^*L^n \otimes M) = \{ y \in Y | y \in \pi^{-1}(X^{ss}_c(L)) \cap (Y_{\pi(y)})^{ss}(M_{\pi(y)}) \} \).
2. \( Y^{ss}(\pi^*L^n \otimes M) = \{ y \in Y | \text{Gr} \cdot y \cap Y^{ss}_c(\pi^*L^n \otimes M) \neq \emptyset \} \).
3. \( Y^{ss}(\pi^*L^n \otimes M) = \{ y \in Y | y \in Y^{ss}_c(\pi^*L^n \otimes M), G_y \text{ is finite} \} \).

**Remark 5.8.** There are nef linearized line bundles that are not the pull-backs for any algebraic contraction maps. In this case, we do not know yet how to make sense of RGIT in the algebraic category. One possible alternative is to allow the contractions to be just complex analytic and work in the complex analytic category. The price paid is then the loss of algebraicity and the possible validity over positive characteristics.

### 6. Configuration spaces and Grassmannians

In this section, we shall present an application of our various RGIT theorems to some elementary finitely dimensional settings (as opposed to the later applications to the moduli problems).

**6.1.** Consider the Grassmannian of \( n \)-subspaces in \( \mathbb{C}^m \), \( \text{Gr}(n, \mathbb{C}^m) \), acted on by the maximal torus \( T = (\mathbb{C}^*)^{m-1} \). Since \( \text{Pic Gr}(n, \mathbb{C}^m) \cong \mathbb{Z} \) and is generated by an ample line bundle, the stabilities of linearized actions are determined by the characters of \( T = (\mathbb{C}^*)^{m-1} \). In fact, let \( \Phi : \text{Gr}(n, \mathbb{C}^m) \to \mathbb{R}^m \) be the standard moment map induced by the Plücker embedding. Then the moment map image is the so called hypersimplex \( \Delta^m_n \)

\[
\Delta^m_n = \{ (\alpha_1, \cdots, \alpha_m) | 0 \leq \alpha_i \leq 1, \sum_j \alpha_j = n \}.
\]

Thus we have that the \( T \)-effective ample cone \( \mathcal{E}^T(\text{Gr}(n, \mathbb{C}^m)) \) can be identified with the cone over \( \Phi(\text{Gr}(n, \mathbb{C}^m)) = \Delta^m_n \).

**6.2.** On the other hand, consider the diagonal action of \( G = \text{SL}(n) \) on \( X = (\mathbb{P}^{n-1})^m \)

\[
\text{SL}(n) \times (\mathbb{P}^{n-1})^m \to (\mathbb{P}^{n-1})^m.
\]

We have \( \text{Pic}^G(X) \cong \text{Pic}(X) \cong \mathbb{Z}^m \). A nef line bundle

\[
L \cong \mathcal{O}_{\mathbb{P}^{n-1}}(k_1) \otimes \cdots \otimes \mathcal{O}_{\mathbb{P}^{n-1}}(k_m), \quad k_i \geq 0
\]

is \( G \)-effective (i.e., \( X^{ss}(L) \neq \emptyset \)) if and only if \( nk_i \leq \sum_j k_j \) for any \( i = 1, \cdots, m \). If we set \( \alpha_i = nk_i / \sum_j k_j \), we can express this condition by the inequalities \( 0 \leq \alpha_i \leq 1, \sum \alpha_i = n \). This shows that the \( G \)-effective ample cone \( \mathcal{E}^G(X) \) is equivalent to the cone over the polytope

\[
\Delta^m_n = \{ (\alpha_1, \cdots, \alpha_m) \in \mathbb{R}^m : 0 \leq \alpha_i \leq 1, \sum \alpha_i = n \}.
\]
6.3. Hence we obtain the identification $\mathcal{E}^T(\text{Gr}(n, \mathbb{C}^m)) = \mathcal{E}^G(\mathbb{P}^{n-1})^m)$. Applying the 1-1 correspondence between the $T$-orbits on $\text{Gr}(n, \mathbb{C}^m)$ and the $G$-orbits on $(\mathbb{P}^{n-1})^m$, we can further obtain the identification between the $T$-GIT quotients on the Grassmannian $\text{Gr}(n, \mathbb{C}^m)$ and the $G$-GIT quotients on $(\mathbb{P}^{n-1})^m$.

6.4. Notice that the underlying line bundle of any element in $\mathcal{E}^T(\text{Gr}(n, \mathbb{C}^m))$ is ample. Thus the boundary of $\mathcal{E}^T(\text{Gr}(n, \mathbb{C}^m))$ consists of degenerating characters but with ample underlying line bundles. However, on the contrast, notice that the group $\text{SL}(n)$ has no characters and the boundary of $\mathcal{E}^G(X)$ consists of only nef line bundles. This shows an interesting phenomenon that the two sorts of degenerations can sometimes be harmoniously linked.

6.5. In the following, we shall use the above identifications freely in the rest of paper. One will see that one point of view sometimes has advantage over the other (and vice versa). We will use the above to exhibit our theorems in §§3 and 4. To simplify exhibition, we only consider the case when $n = 2$, i.e., the action of $T = (\mathbb{C}^*)^{m-1}$ on $\text{Gr}(2, \mathbb{C}^m)$ and the action of $G = \text{SL}(2)$ on $((\mathbb{P}^1)^m$, leaving out some possible generalizations for $n > 2$ to a future paper.

6.6. From 6.3, we have

$$\mathcal{E}^T(\text{Gr}(2, \mathbb{C}^m)) = \mathcal{E}^G((\mathbb{P}^1)^m) = \Delta_2^m.$$ 

The walls of $\Delta_2^m$ are of the form

$$W_J = \{(\alpha_1, \cdots, \alpha_m) \in \Delta_2^m | \sum_{i \in J} \alpha_i = 1\}$$

where $J \subset \{1, \cdots, m\}$ is any proper subset. The faces of $\Delta_2^m$ but vertices can be obtained by setting some of the coordinates to be 0 or one coordinate to be 1. (If two coordinates are 1, we will get a vertex.) Thus they are divided into two types: the ones that are obtained by setting $k$ coordinates to be 0 are again hypersimplexes $\Delta_2^{m-k}$; the ones that are obtained by setting $k-1$ coordinates to be 0 but one coordinate to be 1 become simplexes $\Delta_1^{m-k}$.

In particular, there are in general two different types of facets (faces of codimension 1): $\Delta_2^{m-1}$ and $\Delta_1^{m-1}$. Precisely, they are

$$\Delta_2^{m-1}[i] = \{(\alpha_1, \cdots, \alpha_m) \in \Delta_2^m | \alpha_i = 0, \sum_{j \neq i} \alpha_j = 2\}, \text{ for all } 1 \leq i \leq m.$$ 

$$\Delta_1^{m-1}[i] = \{(\alpha_1, \cdots, \alpha_m) \in \Delta_2^m | \alpha_i = 1, \sum_{j \neq i} \alpha_j = 1\}, \text{ for all } 1 \leq i \leq m.$$ 

6.7. Given an element $\alpha = (\alpha_1, \cdots, \alpha_m) \in \Delta_2^m$, we use $\mathcal{M}^m_\alpha$ to denote the corresponding (GIT or symplectic) quotient of either $(\mathbb{P}^1)^m$ by $G = \text{SL}(2)$ or $\text{Gr}(2, \mathbb{C}^m)$ by $T = (\mathbb{C}^*)^{m-1}$.
Remark 6.8. We point out that the (GIT or symplectic) quotients of these two actions admit some other interesting interpretations. First of all, they appear as the arithmetic quotients of complex balls (Deligne-Mostow). Secondly, they are also the moduli spaces of spatial polygons modulo orientation-preserving Euclidean motions ([18], [14]). Moreover, it is easy to see that the real points of these quotients are the trivial double covers (disjoint unions) of the moduli spaces of linkages in Euclidean plane modulo orientation-preserving Euclidean motions ([13] and [10]). Thus, as the sets of the real points of the symplectic quotients $\mathcal{M}_m^α$, the topological changes of the moduli spaces of linkages with prescribed side lengths in the Euclidean plane when crossing walls are governed by the changes of $\mathcal{M}_m^α$ (cf. [10]). This is to say that two moduli spaces of linkages (without zero side lengths) in the Euclidean plane are related by a sequence of real blowup and blowdowns.

6.9. There are $m$ many forgetful morphisms from $(\mathbb{P}^1)^m$ to $(\mathbb{P}^1)^{m-1}$. Use $f_i$ to denote the forgetful map obtained by forgetting the $i$th factor from $(\mathbb{P}^1)^m$. These are $G$-equivariant trivial fibrations with fiber $\mathbb{P}^1$. In terms of $G$-effective cones, $f_i$ corresponds to the inclusion of the facet

$$\Delta_{2}^{m-1}[i] \subset \Delta_{2}^{m}$$

where $\Delta_{2}^{m-1}[i] \cong \Delta_{2}^{m-1}$ is obtained by setting the $i$th coordinate to be zero. Noting that $\mathcal{M}_m^α = \mathcal{M}_m^{α-1} \times \mathbb{P}^1$, we have

**Theorem 6.10.** (cf. Theorem 2.1, [11]) Let $α = (α_1, \cdots, α_i-1, α_i+1, \cdots, α_m) \in \Delta_{2}^{m-1}$ do not lie on any wall and set $\bar{α}_ε = (α_1 - \frac{ε}{m-1}, \cdots, α_i-1 - \frac{ε}{m-1}, ε, α_i+1 - \frac{ε}{m-1}, \cdots, α_m - \frac{ε}{m-1}) \in \Delta_{2}^{m}$ where $ε$ is a sufficiently small positive number. Then $\mathcal{M}_m^α = \mathcal{M}_m^{α-1} × \mathbb{P}^1$.

**Proof.** Applying Theorem 3.13 (2) to the forgetful map $f_i : (\mathbb{P}^1)^m \to (\mathbb{P}^1)^{m-1}$, one sees immediately that $\mathcal{M}_m^α$ is a fibration over $\mathcal{M}_m^{α-1}$ with fibers $\mathbb{P}^1$. The triviality follows from the fact that $f_i$ is equivariantly trivial and $\text{SL}(2)$ acts freely on any stable locus. □

**Remark 6.11.** Using the remark in 6.8 about the real loci of the quotients $\mathcal{M}_m^α$, Theorem 6.10 gives an alternative justification for Corollary 15 of [13].

**Remark 6.12.** In terms of Grassmannians, the forgetting map $f_i$ corresponds to the (rational) projection from $\text{Gr}(2, \mathbb{C}^m)$ to $\text{Gr}(2, \mathbb{C}^{m-1})$ by projecting a 2-plane in $\mathbb{C}^m$ to the $i$th coordinate $(m-1)$-hyperplane (the hyperplane is obtained by setting the $i$th coordinate to be zero).

6.13. To study the inclusion

$$\Delta_{1}^{m-1}[i] \subset \Delta_{2}^{m}$$
where $\Delta^{m-1}_i$ is obtained by setting the $i$th coordinate to be 1, we switch our point of view from configuration spaces of points on $\mathbb{P}^1$ to Grassmannians. There are $m$ many rational facet maps $f_i$ from $\text{Gr}(2, \mathbb{C}^m)$ to $\text{Gr}(1, \mathbb{C}^{m-1}) = \mathbb{P}^{m-2}$ by taking the intersection of a 2-plane in $\mathbb{C}^m$ with the $i$th coordinate $(m-1)$-hyperplane (the hyperplane is obtained by setting the $i$th coordinate to be zero). These are truly rational maps but are equivariant with respect to the projection $\rho$ from the maximal torus $T = (\mathbb{C}^*)^m = (\mathbb{C}^*)^m/\text{diagonal}$ to the quotient group $T/T_1$ where the $T_1$ acts trivially on $\text{Gr}(1, \mathbb{C}^{m-1}) = \mathbb{P}^{m-2}$. Any element of $\Delta^{m-1}_i[i]$ is of the form $\alpha = (\alpha_1, \ldots, \alpha_{i-1}, 1, \alpha_{i+1}, \ldots, \alpha_m)$. Since $T/T_1 = (\mathbb{C}^*)^{m-2}$ acts on $\text{Gr}(1, \mathbb{C}^{m-1}) = \mathbb{P}^{m-2}$ with a dense open orbit, we see that $\mathcal{M}_m^\alpha$ is a point for such an element $\alpha$.

**Theorem 6.14.** (cf. Theorem 2.1, [10]) Let $\alpha = (\alpha_1, \ldots, \alpha_{i-1}, 1, \alpha_{i+1}, \ldots, \alpha_m)$ be any element in the interior of $\Delta^{m-1}_i[i]$ and set

$$\bar{\alpha}_{1-\epsilon} = (\alpha_1 + \frac{\epsilon}{m-1}, \ldots, \alpha_{i-1} + \frac{\epsilon}{m-1}, 1 - \epsilon, \alpha_{i+1} + \frac{\epsilon}{m-1}, \ldots, \alpha_m + \frac{\epsilon}{m-1})$$

where $\epsilon$ is a sufficiently small positive number. Then $\mathcal{M}_{\bar{\alpha}_{1-\epsilon}}$ is isomorphic to $\mathbb{P}^{m-3}$.

**Proof.** By Bialynicki-Birula’s decomposition theorem [4], the fibers of the map $f_i$ have the structures of $T_1$-modules where $T_1$ acts with positive weights. Since $T$ acts quasi-freely (no finite isotropy subgroups except for the identity group), all of these weights are 1. The theorem now follows from Theorem [1,7] and the fact that $\mathcal{M}_m^\alpha$ is a point.

**Remark 6.15.** In terms of configurations of points on $\mathbb{P}^1$, the facet map $f_i$ correspond to the projection onto the $i$th factor ($1 \leq i \leq m$). The reason that we hesitated working over $(\mathbb{P}^1)^m$ is that the $i$th factor $(\mathbb{P}^1)$ has no stable points with respect to the action of $\text{SL}(2)$ although a quotient trivially exists.
II. Universal Moduli

7. Quest for universal moduli spaces

7.1. Recall that one has the following coarse moduli spaces in the case of curves.

1. $M_g$, parameterizing nonsingular curves of genus $g \geq 2$ and its compactification $\overline{M}_g$, parameterizing Mumford-Deligne stable curves.
2. $M_{g,m}$, $2g - 2 + m > 0$, parameterizing nonsingular $m$-pointed curves of genus $g \geq 2$, and its compactification $\overline{M}_{g,m}$, parameterizing stable $m$-pointed curves.

It is well known that $\overline{M}_g$ and $\overline{M}_{g,m}$ are projective.

Precisely, the Mumford-Deligne stable curves (resp. $m$-pointed curves) are defined as follows:

Definition 7.2. Let $S$ be a base scheme. A stable (resp. semistable) curve of genus $g \geq 2$ over $S$ is a proper flat morphism $f : C \to S$ such that for all $s \in S$ the geometric fiber $C_s$ of $f$ over $s$, satisfies:

1. $C_s$ is reduced, connected scheme of dimension 1 with $h^1(C_s, \mathcal{O}_{C_s}) = g$;
2. every singular point of $C_s$ is an ordinary double point;
3. if $C_1$ is an irreducible rational component of $C_s$ then $C_1$ meets the rest of $C_s$ in at least 3 points (resp. 2 points).

Definition 7.3. An $m$-pointed curve stable (resp. semistable) curve of genus $g \geq 2$ over $S$ is a proper flat morphism $f : C \to S$ together with $m$-distinct sections $s_i : S \to C$ such that for all $s \in S$ the geometric fiber $C_s$ of $f$ over $s$, satisfies:

1. $C_s$ is reduced, connected scheme of dimension 1 with $h^1(C_s, \mathcal{O}_{C_s}) = g$;
2. every singular point of $C_s$ is an ordinary double point;
3. all $s_i(s)$ are smooth points of $C_s$ and $s_i(s) \neq s_j(s)$ for $i \neq j$;
4. if $C_1$ is an irreducible rational component of $C$ then the number of points where $C_1$ meets the rest of $C_s$ plus the number of points $s_i(s)$ on $C_1$ is at least 3 (resp. 2 points).

Rather than going through some abstract definitions of universal moduli spaces, let us go over (briefly) the concrete universal moduli problems listed in 0.1 of the introduction.

7.4. 1. The universal moduli space $\overline{FM}_{g,n} \to \overline{M}_g$ of Fulton-MacPherson configuration spaces of stable curves. (see also [26])
2. The compactified universal Picard $\overline{P_d}_g \to \overline{M}_g$ of degree $d$ line bundles ([3]).
3. The universal moduli space $\overline{P}_{g,m}(e, r, F, \alpha) \to \overline{M}_{g,n}$ of $p$-semistable parabolic sheaves of degree $e$, rank $r$, type $F$, and weight $\alpha$ ([12]).
4. The universal moduli space $M_g(O, P) \rightarrow \overline{M_g}$ of $p$-semistable coherent sheaves with a fixed Hilbert polynomial $P$. To be more precise, for each $[C] \in \overline{M_g}$, one has a natural projective variety, $M_C(O_C, P)$, parameterizing (Simpson’s) $p$-semistable coherent sheaves of a fixed Hilbert polynomial $P$. Our aim is to construct the universal moduli space $M_g(O, P) \rightarrow \overline{M_g}$ parametrizing the set of equivalence classes of pairs $(C, E)$ where $[C] \in \overline{M_g}$ and $[E] \in M_C(O_C, P)$ such that the fiber over the stable curve $C$ is $M_C(O_C, P)/\text{Aut}(C)$.

5. The universal Hilbert scheme $\text{Hilb}^n_{g,m} \rightarrow \overline{M_g}$ of 0-dimensional schemes of length $n$ on the Mumford-Deligne stable curves. There exists a natural dominating morphism $\psi : \text{Hilb}^n_{g,m} \rightarrow M_g(O, P)$ when $P(x) = x + n + 1 - g$.

Remark 7.5. The last two cases in the above list are what we shall considered seriously in the sequel. The construction of the universal moduli space $M_g(O, P) \rightarrow \overline{M_g}$ will be done by using our RGIT.

Remark 7.6. The moduli problem in (3) can be specified as follows. For each $[(C, p_1, \cdots, p_m)] \in M_{g,m}$, one has a natural projective variety, $P_C(e, r, F, \alpha)$, parameterizing $\alpha$-semistable vector bundles of degree $e$ and rank $r$ with quasi-parabolic structures of type $F$ at points $p_i$. Let $P_{g,m}(e, r, F, \alpha)$ be the set of equivalence classes of pairs $((C, p_1, \cdots, p_m), E)$ where $[(C, p_1, \cdots, p_m)] \in M_{g,m}$ and $[E] \in P_C(e, r, F, \alpha)$. Now it is natural to ask for a compactification $\overline{P}_{g,m}(e, r, F, \alpha)$ of $P_{g,m}(e, r, F, \alpha)$ with the following desired properties (cf. [25]):

1. $\overline{P}_{g,m}(e, r, F, \alpha)$ is a projective variety parameterizing equivalence classes of algebro-geometric objects.
2. $\overline{P}_{g,m}(e, r, F, \alpha)$ contains $P_{g,m}(e, r, F, \alpha)$ as an open dense subset.
3. There is a natural morphism $\eta : \overline{P}_{g,m}(e, r, F, \alpha) \rightarrow \overline{M_{g,m}}$ such that the following natural diagram commutes:

\[
\begin{array}{ccc}
P_{g,m}(e, r, F, \alpha) & \longrightarrow & \overline{P}_{g,m}(e, r, F, \alpha) \\
\downarrow & & \downarrow \eta \\
M_{g,m} & \longrightarrow & \overline{M_{g,m}}
\end{array}
\]

4. For each $[(C, p_1, \cdots, p_m)] \in M_{g,m}$, there is an isomorphism $\eta^{-1}([(C, p_1, \cdots, p_m)]) \cong P_C(e, r, F, \alpha)/\text{Aut}(C, p_1, \cdots, p_m)$.

Because of the lack of satisfactory GIT construction of $\overline{M_{g,m}}$, we postpone treating this problem in a later publication [12].
8. The universal moduli space $M_g(\mathcal{O}, P) \to \overline{M_g}$

8.1. Simpson’s construction of the moduli space of $p$-semistable coherent sheaves. Let $X$ be a projective scheme over $\mathbb{C}$ with a very ample invertible sheaf $\mathcal{O}_X(1)$. For any coherent sheaf $\mathcal{E}$ over $X$, let $p(\mathcal{E}, n)$ be the Hilbert polynomial of $\mathcal{E}$ with $p(\mathcal{E}, n) = \dim H^0(X, \mathcal{E}(n))$ for $n \gg 0$. Let $d = d(\mathcal{E})$ be the dimension of the support of $\mathcal{E}$ which is also the degree of the Hilbert polynomial of $\mathcal{E}$. The leading coefficient is $r/d!$ where $r = r(\mathcal{E})$ is an integer which is called the rank of $\mathcal{E}$. A coherent sheaf $\mathcal{E}$ is of pure dimension $d = d(\mathcal{E})$ if for any non-zero subsheaf $\mathcal{F} \subset \mathcal{E}$, we have that $d(\mathcal{F}) = d(\mathcal{E})$.

Definition 8.2. A coherent sheaf $\mathcal{E}$ is $p$-semistable (resp. $p$-stable) if it is of pure dimension, and if for any subsheaf $\mathcal{F} \subset \mathcal{E}$, there exists $N$ such that for $n \geq N$

$$\frac{p(\mathcal{F}, n)}{r(\mathcal{F})} \leq \frac{p(\mathcal{E}, n)}{r(\mathcal{E})}.$$

As usual, a $p$-semistable sheaf $\mathcal{E}$ admits a filtration by subsheaves

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_k = \mathcal{E}$$

such that the quotient sheaves $\mathcal{E}_i/\mathcal{E}_{i-1}$ are $p$-stable. This filtration is not unique, but $gr(\mathcal{E}) = \oplus \mathcal{E}_i/\mathcal{E}_{i-1}$ is. Two $p$-semistable sheaves $\mathcal{E}$ and $\mathcal{E}'$ are $s$-equivalent if $gr(\mathcal{E}) = gr(\mathcal{E}')$.

Simpson also extends the above definition to the following relative version.

8.3. Let $S$ be a base scheme of finite type over $\mathbb{C}$ and $X \to S$ a projective scheme over $S$. Fix a (Hilbert) polynomial $P$ of degree $d$. A $p$-semistable (resp. stable) sheaf $\mathcal{E}$ on $X/S$ with Hilbert polynomial $P$ is a coherent sheaf $\mathcal{E}$ on $X$, flat over $S$, such that for each closed point $s \to S$, $\mathcal{E}_s$ is a $p$-semistable (resp. stable) sheaf of pure dimension $d$ and Hilbert polynomial $P$ on the fiber $X_s$.

8.4. Hilbert schemes and Grassmannians. As above, let $X \to S$ be a projective scheme over $S$ with a relatively very ample invertible sheaf $\mathcal{O}_X(1)$. Fix a (Hilbert) polynomial $P(n)$. Suppose that $\mathcal{W}$ is a coherent sheaf on $X$ flat over $S$. The Hilbert scheme $\text{Hilb}(\mathcal{W}, P)$ parametrizing quotients

$$\mathcal{W} \to \mathcal{F} \to 0$$

with Hilbert polynomial $P$. The fiber of $\text{Hilb}(\mathcal{W}, P)$ over a closed point $s \in S$ is $\text{Hilb}(\mathcal{W}_s, P)$.

Grothendieck gives some very explicit relative projective embeddings of $\text{Hilb}(\mathcal{W}, P)$ over $S$. There is an $M > 0$ such that for any $m \geq M$ we get a closed embedding

$$\psi_m : \text{Hilb}(\mathcal{W}, P) \to \text{Grass}(H^0(X/S, \mathcal{W}(m)), P(m)).$$

There is a canonical invertible sheaf $\mathcal{L}_m$ on the relative Grassmannian by the embedding $\psi_m$. Over any point in the Grassmannian represented by the quotient $\mathcal{W} \to \mathcal{F}$, the restriction of the invertible sheaf $\mathcal{L}_m$ is canonically
identified with the $\bigwedge^{P(m)} H^0(X/S, F(m))$. These invertible sheaves are the ones we shall use in place of the relatively ample line bundle $M$ in our various theorems in RGIT (§§3 and 4).

8.5. The construction of the moduli spaces of $p$-semistable coherent sheaves. Fix a large number $N$. Let $\mathcal{W} = \mathcal{O}_X(-N)$ and $V = \mathbb{C}^{P(N)}$. Let $Q_1 \subset \text{Hilb}(V \otimes \mathcal{W}, P)$ denote the open subset of $p$-semistable sheaves of pure dimension $d$. We can assume that $N$ is chosen large enough so that: every $p$-semistable coherent sheaf with Hilbert polynomial $P$ appears as a quotient corresponding to a point of $Q_1$. Now set $Q_2$ equal to the open subset in $Q_1$ such that $\alpha : V \otimes \mathcal{O}_S \to H^0(X/S, E(N))$ is isomorphism where $\alpha$ is a morphism such that the sections in the image of $\alpha$ generate $E(N)$. The group $\text{SL}(V)$ acts on $\text{Hilb}(V \otimes \mathcal{W}, P)$ and the line bundle $\mathcal{L}_m$. The open subset $Q_2$ is invariant under this action.

Let $M_X^s(\mathcal{O}_X, P)$ be the functor for the moduli problem of $s$-equivalence classes of $p$-semistable coherent sheaves on $X$ of pure dimension $d$, Hilbert polynomial $P$, and flat over $S$.

Theorem 8.6. (C. Simpson. [28]) $Q_2$ is contained in the semistable locus $\text{Hilb}(V \otimes \mathcal{W}, P)^{ss}(\mathcal{L}_m)$ with respect to the action of $\text{SL}(V)$ and the linearized line bundle $\mathcal{L}_m$. And the categorical quotient $M_X^s(\mathcal{O}_X, P) = Q_2/\text{SL}(V)$ is a projective scheme over $S$ which coarsely represents the moduli functor $M_X^s(\mathcal{O}_X, P)$.

We also need to recall Gieseker’s construction of $\overline{M}_g$.

8.7. Gieseker’s construction of $\overline{M}_g$. Fix $g \geq 2$, $e = n(2g - 2)$ $(n \geq 10)$, $I = e - g$, and a polynomial in $x$, $p(x) = ex - g + 1$. Set the following Hilbert scheme of subschemes in $\mathbb{P}^I:

\text{Hilb}^{p(x)}_I := \{\text{subschemes in } \mathbb{P}^I \text{ with Hilbert polynomial } p(x)\}.

The group of projective linear transformations $\text{PGL}(I + 1)$ acts on $\text{Hilb}^{p(x)}_I$ naturally. For the reason of lifting to a linear action, we take $G = \text{SL}(I + 1)$. Now consider the locus $H_g$ of n-canonical stable curves in $\text{Hilb}^{p(x)}_I$, that is,

$$H_g = \{[i_{\omega^n}(C)] \in \text{Hilb}^{p(x)}_I\}$$

where $C$ is a DM-stable curve, $i_{\omega^n} : C \to \mathbb{P}^I$ is the embedding induced by the $n$th power of the canonical line bundle $\omega$ over $C$, and $[i_{\omega^n}(C)]$ is the corresponding Hilbert point of the n-canonical curve $i_{\omega^n}(C)$. $H_g$ is a $G$-invariant, irreducible, nonsingular subscheme of $\text{Hilb}^{p(x)}_I$.

By [11], there can be chosen a $\text{SL}(I + 1)$-linearization on the Hilbert scheme $\text{Hilb}^{p(x)}_I$ such that

1. $H_g$ is contained in the stable locus;
2. $H_g$ is closed in the semistable locus; and
3. the GIT quotient $H_g/\text{SL}(I + 1)$ is the moduli space $\overline{M}_g$. 

For preciseness, we take \( e = 10(2g - 2) \) and \( I = 10(2g - 2) - g \), once and for all.

**8.8. The construction of the universal moduli space** \( \mathbf{M}_g(\mathcal{O}, P) \to \mathcal{M}_g \). Let \( \hat{U}_g \) be the universal curve over \( H_g \). Consider \( X = \hat{U}_g \to H_g = S \) as a projective scheme over the base scheme \( S = H_g \). Fix a Hilbert polynomial \( P(x) = rx + d + r(1 - g) \). By Simpson, we get the coarse moduli space \( \mathbf{M}_X(\mathcal{O}_X, P) \) over the base scheme \( S = H_g \) of \( p \)-semistable coherent sheaves (of pure dimension 1) with the Hilbert polynomial \( P(x) = rx + d + r(1 - g) \). The moduli space \( \mathbf{M}(\mathcal{O}_X, P) \) (as a projective scheme over \( S = H_g \)) is constructed as the GIT quotient of \( Q_2/S \) by the group \( SL(V) = SL(P(N)) \) (see 8.5 and Theorem 8.6).

**Theorem 8.9.** Fix the Hilbert polynomial \( P(x) = rx + d + r(1 - g) \). The projective categorical quotient \( \mathbf{M}_g(\mathcal{O}, P) = (Q_2/H_g)/(SL(V) \times SL(I + 1)) \) exists and factors naturally to \( \mathcal{M}_g \) such that over each point \( [C] \in \mathcal{M}_g \) the fiber is canonically identified with the moduli space \( \mathbf{M}_C(\mathcal{O}_C, P) \) of \( p \)-semistable coherent sheaves (of pure dimension 1) with the Hilbert polynomial \( P \) modulo the automorphism group of \( C \).

**Proof.** Consider the map \( \pi : Q_2/H_g \to H_g \) equivariant with respect to the projection \( \rho : SL(V) \times SL(I + 1) \to SL(I + 1) \). For \( H_g \) we use a linearization \( L \) as found by Gieseker (see 8.7), while for \( Q_2/H_g \) we use the linearization \( \pi^*L^k \otimes L_m \) (for \( k \gg 0 \) and some sufficiently large \( m \), see 8.4). Now the theorem follows from Theorem 8.7 or Theorem 8.8. \( \square \)

**Remark 8.10.** That is, every point of the moduli space \( \mathbf{M}_g(\mathcal{O}, P) \) represents an equivalence class of pairs \((C, \mathcal{E})\) up to automorphism group of \( C \), where \( C \) is a Mumford-Deligne stable curve and \( \mathcal{E} \) is a \( p \)-semistable coherent sheaf of pure dimension 1 over \( C \) with the Hilbert polynomial \( P(x) = rx + d + r(1 - g) \). When \( P(x) = x + n + 1 - g \) the moduli space \( \mathbf{M}_g(\mathcal{O}, P) \) is a compactification of the universal Picard \( P^n_g \). It would be interesting to compare our moduli spaces \( \mathbf{M}_g(\mathcal{O}, P) \) with those in [7] and [23].

**9. The universal Hilbert scheme** \( \text{Hilb}^n_g \)

In this last section, we will give a GIT construction of the universal Hilbert scheme \( \text{Hilb}^n_g \) over \( \mathcal{M}_g \) of 0-dimensional subschemes of length \( n \) on Mumford-Deligne stable curves and a canonical morphism from \( \text{Hilb}^n_g \) to the compactified universal Picard \( M_g(\mathcal{O}, P) \) where \( P(x) = x + n + 1 - g \).

**9.1. Let** \( U_g \to \mathcal{M}_g \) be the (fake) universal curve of genus \( g \geq 2 \) over \( \mathcal{M}_g \). \( U_g \) has an obvious GIT construction as the quotient \( \hat{U}_g/SL(I + 1) \) by our theorems for \( G \)-equivarancy RGIT (see 8.8 for the definition of \( \hat{U}_g \)). Set \( \text{Hilb}^n_g \to \mathcal{M}_g \) to be the relative Hilbert scheme over \( \mathcal{M}_g \) of relatively 0-dimensional subschemes in \( U_g \) of length \( n \). Then the fiber of \( \text{Hilb}^n_g \) over a
point \([C] \in \overline{M}_g\) is the Hilbert scheme \(\text{Hilb}^n_{C}\) of 0-dimensional subschemes in \(C\) of length \(n\) modulo the automorphism group \(\text{Aut}(C)\) (we will give a GIT construction of \(\text{Hilb}^n_{g} \rightarrow \overline{M}_g\) in the sequel).

**Theorem 9.2.** Let \(P(x) = x + n + 1 - g\). Then there exists a natural dominating morphism \(\psi\) from \(\text{Hilb}^n_{g}\) to \(M_g(\mathcal{O}, P)\) such that the following diagram is commutative

\[
\begin{array}{ccc}
\text{Hilb}^n_{g} & \xrightarrow{\psi} & M_g(\mathcal{O}, P) \\
\downarrow & & \downarrow \\
\overline{M}_g & \xrightarrow{\text{id}} & \overline{M}_g
\end{array}
\]

**Proof.** To construct this morphism scheme-theoretically, we first need to give a GIT construction of \(\text{Hilb}^n_{g}\) using our theory on \(G\)-equivariancy RGIT.

Recall that \(\overline{M}_g\) is constructed as a GIT quotient of a smooth irreducible scheme \(H_g\) by the linear transformation \(SL(I + 1)\) \((8.7)\). Let \(\widehat{U}_g\) be the universal family over \(H_g\). Set \(\widehat{\text{Hilb}}^n_{g} \rightarrow H_g\) to be the relative Hilbert scheme over \(H_g\) of relatively 0-dimensional subschemes in \(\widehat{U}_g\) of length \(n\). The group \(SL(I + 1)\) operates on \(\widehat{\text{Hilb}}^n_{g}\) by moving the subschemes. Theorems 3.11 and 3.13 imply that the GIT quotient \(\widehat{\text{Hilb}}^n_{g}/SL(I + 1)\) exists and factors naturally to \(H_g/SL(I + 1) = \overline{M}_g\) with the fiber at a point \([C] \in \overline{M}_g\) isomorphic to \(\text{Hilb}^n_{C}/\text{Aut}(C)\). The quotient \(\widehat{\text{Hilb}}^n_{g}/SL(I + 1)\) is our universal Hilbert scheme \(\text{Hilb}^n_{g} \rightarrow \overline{M}_g\).

Now given any relatively 0-dimensional subschemes \(Z\) in \(\widehat{U}_g\) of length \(n\). We can get a coherent sheave of rank 1 and pure dimension 1, \(\mathcal{O}_{\widehat{U}_g/\overline{H}_g}(Z) = \mathcal{O}_{\widehat{U}_g/\overline{H}_g}(-Z)^* = (I_Z)^*\). This leads to a morphism

\[
\varphi : \widehat{\text{Hilb}}^n_{g} \rightarrow Q_2/H_g
\]

\[
\varphi : Z \rightarrow \mathcal{O}_{\widehat{U}_g/\overline{H}_g}(Z)
\]

(see \(\S 5\) for the definition of \(Q_2\)). By passing the the quotient we get

\[
\varphi' : \widehat{\text{Hilb}}^n_{g} \rightarrow Q_2/H_g \rightarrow (Q_2/H_g)/(SL(V) \times SL(I + 1)) = M_g(\mathcal{O}, P)
\]

where \(P = x + n + 1 - g\). One checks that \(\varphi'\) is constant on the \(SL(I + 1)\)-orbits. By the universality of categorical quotient, we obtain a canonical morphism

\[
\psi : \text{Hilb}^n_{g} = \widehat{\text{Hilb}}^n_{g}/SL(I + 1) \rightarrow M_g(\mathcal{O}, P).
\]

Both \(\text{Hilb}^n_{g}\) and \(M_g(\mathcal{O}, P)\) are projective and \(\psi\) maps surjectively to the universal Picard over nonsingular curves of genus \(g\). Hence \(\psi\) is dominating. \(\square\)
Remark 9.3. We do not know if the morphism $\psi : \text{Hilb}_g^n \to M_g(O, P)$ (or its variants) can be useful in the study of limit linear series for stable curves (cf. [7]). We do not know if there exists a canonical morphism from the universal FM configuration space over $M_g$ to $M_g(O, P)$ so that Theorem 9.2 holds. The universal FM configuration space $FM_{g,n} \to M_g$ can be constructed by RGIT as follows. Again let $\widehat{U}_g \to H_g$ be the universal family of curves of genus $g$. Let $\widehat{U}_g[n] \to H_g$ be the relative FM configuration space over $H_g$ (cf. [24]). Then the action of $SL(I+1)$ lifts to a canonical action on $\widehat{U}_g[n]$. Now applying Theorem 4.8 to $\widehat{U}_g[n] \to H_g$ and taking the quotients by the group $G = SL(I+1)$, we obtain the universal FM configuration space $FM_{g,n} = \widehat{U}_g[n]/SL(I+1) \to H_g/SL(I+1) = M_g$ whose fiber at a stable curve $[C]$ is isomorphic to $C[n]/\text{Aut}(C)$.

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Abstract. We expose in detail the principle that the relative geometric invariant theory of equivariant morphisms is related to the GIT for linearizations near the boundary of the $G$-effective ample cone. We then apply this principle to construct and reconstruct various universal moduli spaces. In particular, we constructed the universal moduli space over $\overline{M}_g$ of Simpson’s $p$-semistable coherent sheaves and a canonical dominating morphism from the universal Hilbert scheme over $\overline{M}_g$ to a compactified universal Picard.

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