K-FRAMES IN HILBERT $C^*$-MODULES

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Abstract. In this paper, firstly we investigate conditions under which the action of an operator on a $K$-frame, remain again a $K$-frame for Hilbert module $E$. We also give a generalization of Douglas Theorem and we shall use it to prove the sum of two $K$-frame under certain condition is again a $K$-frame. Finally, we characterize the $K$-frame generators for a unitary system in terms of operators.

1. Introduction

Frames were first introduced in 1952 by Duffin and Schaeffer [6]. Frames can be viewed as redundant bases which are generalization of orthonormal bases. Many generalizations of frames were introduced, e.g., frames of subspaces [4], Pseudo-frames [1], G-frames [17], and fusion frames [3]. Recently, L. Gavruta introduced the concept of $K$-frame for a given bounded operator $K$ on Hilbert space in [9]. Hilbert $C^*$-modules arose as generalizations of the notion Hilbert space. The basic idea was to consider modules over $C^*$-algebras instead of linear spaces and to allow the inner product to take values in the $C^*$-algebra of coefficients being $C^*$-(anti-)linear in its arguments [13]. In [10] authors generalized frame concept for operators in Hilbert $C^*$-modules. The paper is organized as follows. In Section 2, some notations and preliminary results of Hilbert Modules, their frames and $K$-frames are given. In Section 3, we study the action of operators on $K$-frames and under certain conditions, we shall show that it is again a $K$-frame. The next section is devoted to sum of $K$-frames, to show that the sum of two $K$-frames under certain conditions is again a $K$-frame we need to say a generalization of the Douglas Theorem [18], which may interest by its own. Finally, in the last section, we consider a unitary system.

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of operators and characterize the $K$-frame generators in terms of operators. We also look forward to sum of two $K$-frame generators to be a $K$-frame generator.

2. Preliminaries

In this section we give preliminaries about frames, $K$-frames for Hilbert space and Hilbert module and related operators which we need in the sections following. A finite or countable sequence $\{f_k\}_{k \in J}$ is called a frame for a separable Hilbert space $H$ if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B\|f\|^2, \quad \forall f \in H.$$ 

Frank and Larson [10] introduced the notion of frames in Hilbert $C^*$-modules as a generalization of frames in Hilbert spaces. A (left) Hilbert $C^*$-module over the $C^*$-algebra $A$ is a left $A$-module $E$ equipped with an $A$-valued inner product satisfy the following conditions:

1. $\langle x, x \rangle \geq 0$ for every $x \in E$ and $\langle x, x \rangle = 0$ if and only if $x = 0$,
2. $\langle x, y \rangle = \langle y, x \rangle^*$ for every $x, y \in E$,
3. $\langle \cdot, \cdot \rangle$ is $A$-linear in the first argument,
4. $E$ is complete with respect to the norm $\|x\|^2 = \|\langle x, x \rangle\|_A$.

Given Hilbert $C^*$-modules $E$ and $F$, we denote by $L_A(E, F)$ or $L(E, F)$ the set of all adjointable operators from $E$ to $F$ i.e. the set of all maps $T : E \to F$ such that there exists $T^* : F \to E$ with the property

$$\langle Tx, y \rangle = \langle x, T^*y \rangle$$

for all $x \in E$, $y \in F$. It is well-known that each adjointable operator is necessarily bounded $A$-linear in the sense $T(ax) = aT(x)$, for all $a \in A, x \in E$. We denote $L(E)$ for $L(E, E)$. In fact $L(E)$ is a $C^*$-algebra. Let $A$ be a $C^*$-algebra and consider

$$\ell^2(A) := \{\{a_n\}_n \subseteq A : \sum_n a_n a_n^* \text{ converges in norm in } A\}.$$ 

It is easy too see that $\ell^2(A)$ with pointwise operations and the inner product

$$\langle \{a_n\}, \{b_n\} \rangle = \sum_n a_n b_n^*,$$

becomes a Hilbert $C^*$-module which is called the standard Hilbert $C^*$-module over $A$. Throughout this paper, we suppose $E$ is a Hilbert $A$-module, $J$ a countable index set. Also we denote the range of $T \in L(E)$ by $R(T)$, and kernel of $T$ by $N(T)$. A Hilbert $A$-module $E$
is called finitely generated (countably generated) if there exist a finite subset \( \{ x_1, ..., x_n \} \) (countable set \( \{ x_n \}_{n \in \mathbb{J}} \)) of \( E \) such that \( E \) equals the closed \( \mathcal{A} \)-linear hull of this set. The basic theory of Hilbert \( C^* \)-modules can be found in [13].

The following lemma found the relation between the range of an operator and kernel of its adjoint operator.

**Lemma 2.1.** ([19], Lemma 15.3.5; [13], Theorem 3.2) Let \( T \in L(E, F) \), then

1. \( N(T) = N(|T|) \), \( N(T^*) = R(T) \perp \), \( N(T^*) \perp = R(T) \perp \supseteq \overline{R(T)} \);
2. \( R(T) \) is closed if and only if \( R(T^*) \) is closed, and in this case \( R(T) \) and \( R(T^*) \) are orthogonally complemented with \( \overline{R(T)} = N(T^*) \perp \) and \( \overline{R(T^*)} = N(T) \perp \).

The following theorem is extended Douglas theorem [7] for Hilbert modules.

**Theorem 2.2.** [18] Let \( T' \in L(G, F) \) and \( T \in L(E, F) \) with \( \overline{R(T^*)} \) orthogonally complemented. The following statements are equivalent:

1. \( T'T'^* \leq \lambda TT^* \) for some \( \lambda > 0 \);
2. There exists \( \mu > 0 \) such that \( \| T^*z \| \leq \mu \| Tz \| \) for all \( z \in F \);
3. There exists \( D \in L(G, E) \) such that \( T' = TD \), i.e. the equation \( TX = T'X \) has a solution;
4. \( R(T') \subseteq R(T) \).

Here, we recall the concept of frame in Hilbert \( C^* \)-modules which is defined in [10]. Let \( E \) be a countably generated Hilbert module over a unital \( C^* \)-algebra \( \mathcal{A} \). A sequence \( \{ x_n \} \subset E \) is said to be a frame if there exist two constant \( C, D > 0 \) such that

\[
C \langle x, x \rangle \leq \sum_n \langle x, x_n \rangle \langle x_n, x \rangle \leq D \langle x, x \rangle \quad \text{for all } x \in E. \tag{2.1}
\]

The optimal constants (i.e. maximal for \( C \) and minimal for \( D \)) are called frame bounds. If the sum in (2.1) converges in norm, the frame is called standard frame. In this paper all frames consider standard frames. The sequence \( \{ x_n \} \) is called a Bessel sequence with bound \( D \) if the upper inequality in (2.1) holds for every \( x \in E \). Let \( \{ x_j \}_{j \in \mathbb{J}} \) be a Bessel sequence for Hilbert module \( E \) over \( \mathcal{A} \). The operator \( T : E \to \ell^2(\mathcal{A}) \) defined by \( Tx = \{ \langle x, x_j \rangle \}_{j \in \mathbb{J}} \) is called the analysis operator. The adjoint operator \( T^* : \ell^2(\mathcal{A}) \to E \) is given by

\[
T^*\{ c_j \}_{j \in \mathbb{J}} = \sum_{j \in \mathbb{J}} c_j x_j,
\]
where is called the pre-frame operator or the synthesis operator. By composing \( T \) and \( T^\ast \), we obtain the frame operator \( S : E \to E \) given by

\[
Sx = T^*Tx = \sum_{j \in J} \langle x, x_j \rangle x_j, \quad x \in E.
\]

In the case where \( \{x_j\}_{j \in \mathbb{J}} \) is a frame, the frame operator is positive and invertible, also it is the unique operator in \( L(E) \) such that the reconstruction formula

\[
x = \sum_{j \in \mathbb{J}} \langle x, S^{-1} x_j \rangle x_j = \sum_{j \in \mathbb{J}} \langle x, x_j \rangle S^{-1} x_j, \quad x \in E,
\]

holds. It is easy to see that the sequence \( \{S^{-1} x_j\}_{j \in \mathbb{J}} \) is a frame for \( E \).

The frame \( \{S^{-1} x_j\}_{j \in \mathbb{J}} \) is said to be the canonical dual frame of \( \{x_j\}_{j \in \mathbb{J}} \).

**Theorem 2.3.** [ see [14], proposition 2.2 ] Let \( \{x_n\}_{n \in \mathbb{J}} \) be a sequence in \( E \) such that \( \sum_{n \in \mathbb{J}} c_n x_n \) converges for all \( c = \{c_n\}_{n \in \mathbb{J}} \in \ell^2(A) \). Then \( \{x_n\}_{n \in \mathbb{J}} \) is a Bessel sequence in \( E \).

**Theorem 2.4.** [12] Let \( E \) be a finitely or countably generated Hilbert module over a unital \( C^* \)-algebra \( A \), and \( \{x_n\}_{n \in \mathbb{J}} \) be a sequence in \( E \). Then \( \{x_n\}_{n \in \mathbb{J}} \) is a frame for \( E \) with bounds \( C \) and \( D \) if and only if

\[
C \|x\|^2 \leq \| \sum_n \langle x, x_n \rangle \langle x_n, x \rangle \| \leq D \|x\|^2.
\]

In compare with to \( K \)-frames on Hilbert space, Najati in [14] define atomic system and a \( K \)-frame on Hilbert module.

**Definition 2.5.** A sequence \( \{x_n\}_{n \in \mathbb{J}} \) of \( E \) is called an atomic system for \( K \in L(E) \) if the following statement hold:

1. The series \( \sum_{n \in \mathbb{J}} c_n x_n \) converges for all \( c = \{c_n\}_{n \in \mathbb{J}} \in \ell^2(A) \);
2. There exists \( C > 0 \) such that for every \( x \in E \) there exists \( \{a_{n,x}\}_{n \in \mathbb{J}} \in \ell^2(A) \) such that \( \sum_{n \in \mathbb{J}} a_{n,x} a^*_{n,x} \leq C \langle x, x \rangle \) and

\[
Kx = \sum_{n \in \mathbb{J}} a_{n,x} x_n.
\]

By Theorem 2.3, the condition (1), in the above definition, actually says that \( \{x_n\}_{n \in \mathbb{J}} \) is a Bessel sequence.

**Theorem 2.6.** [14] If \( K \in L(E) \), then there exists an atomic system for \( K \).

**Theorem 2.7.** [14] Let \( \{x_n\}_{n \in \mathbb{J}} \) be a Bessel sequence for \( E \) and \( K \in L(E) \). Suppose that \( T \in L(E, \ell^2(A)) \) is given by \( T(x) = \{\langle x, x_n \rangle\}_{n \in \mathbb{J}} \) and \( R(T) \) is orthogonally complemented. Then the following statements are equivalent:
(1) \( \{x_n\}_{n \in J} \) is an atomic system for \( K \);
(2) There exist \( C, B > 0 \) such that
\[
B\|K^*x\|^2 \leq \sum_n \langle x, x_n \rangle \langle x_n, x \rangle \leq C\|x\|^2;
\]
(3) There exist \( D \in L(E, \ell^2(A)) \) such that \( K = T^* D \).

**Definition 2.8.** Let \( E \) be a Hilbert \( \mathcal{A} \)-module, \( \{x_n\}_{n \in J} \subset E \) and \( K \in L(E) \). The sequence \( \{x_n\}_{n \in J} \) is said to be a \( K \)-frame if there exist constant \( C, D > 0 \) such that
\[
C\langle K^*x, K^*x \rangle \leq \sum_n \langle x, x_n \rangle \langle x_n, x \rangle \leq D\langle x, x \rangle, \ x \in E. \quad (2.2)
\]

The following theorem gives a characterization of \( K \)-frames using linear adjointable operators.

**Theorem 2.9.** [14] Let \( K \in L(E) \) and \( \{x_n\}_{n \in J} \) be a Bessel sequence for \( E \). Suppose that \( T \in L(E, \ell^2(A)) \) is given by \( T(x) = \{\langle x, x_n \rangle\}_{n \in J} \) and \( R(T) \) is orthogonally complemented. Then \( \{x_n\}_{n \in J} \) is a \( K \)-frame for \( E \) if and only if there exist a linear bounded operator \( L : \ell^2(A) \to E \) such that \( Le_n = x_n \) and \( R(K) \subseteq R(L) \), where \( \{e_n\} \) is the canonical orthonormal basis for \( \ell^2(A) \).

### 3. Operators On \( K \)-frames

In this section we study the action of an operator on a \( K \)-frame. The following lemma shows that the action of an adjointable operator on a Bessel sequence is again a Bessel sequence.

**Lemma 3.1.** Let \( E \) be a Hilbert \( \mathcal{A} \)-module and \( \{x_n\}_{n \in J} \) be a Bessel sequence, then \( \{Mx_n\}_{n \in J} \) is a Bessel sequence for every \( M \in L(E) \).

**Proof.** Since \( \{x_n\}_{n \in J} \) is a Bessel sequence then there exists constant \( D \) such that
\[
\sum_n \langle x, x_n \rangle \langle x_n, x \rangle \leq D\langle x, x \rangle
\]
for every \( x \in E \). So
\[
\sum_n \langle x, Mx_n \rangle \langle Mx_n, x \rangle = \sum_n \langle M^*x, x_n \rangle \langle x_n, M^*x \rangle \leq D\langle M^*x, M^*x \rangle = D\langle MM^*x, x \rangle \leq D\|M\|^2\langle x, x \rangle
\]
for every \( x \in E \). \( \square \)
Theorem 3.2. Let $E$ be a Hilbert $A$-module, $K \in L(E)$ and $\{x_n\}_{n \in \mathbb{J}}$ be a $K$-frame for $E$. Let $M \in L(E)$ with $R(M) \subset R(K)$ and $R(K^*)$ orthogonally complemented. Then $\{x_n\}_{n \in \mathbb{J}}$ is an $M$-frame for $E$.

Proof. Since $\{x_n\}_{n \in \mathbb{J}}$ is a $K$-frame then there exist positive numbers $\mu$ and $\lambda$ such that

$$\lambda \langle K^* x, K^* x \rangle \leq \sum_n \langle x, x_n \rangle \langle x_n, x \rangle \leq \mu \langle x, x \rangle \tag{3.1}$$

Using the theorem 2.2 by the fact that $R(M) \subset R(K)$ shows that, $MM^* \leq \lambda' KK^*$ for some $\lambda' > 0$. So

$$\frac{\lambda}{\lambda'} \langle MM^* x, x \rangle \leq \lambda \langle K^* x, K^* x \rangle$$

From (3.1), we have

$$\frac{\lambda}{\lambda'} \langle MM^* x, x \rangle \leq \sum_n \langle x, x_n \rangle \langle x_n, x \rangle \leq \mu \langle x, x \rangle.$$

Therefor $\{x_n\}_{n \in \mathbb{J}}$ is an $M$-frame with bound $\frac{\lambda}{\lambda'}$ and $\mu$ for $E$. \qed

Theorem 3.3. Let $E$ be a Hilbert $A$-module and $K \in L(E)$ with the dense range. Let $\{x_n\}_{n \in \mathbb{J}}$ be a $K$-frame for $E$ and $T \in L(E)$ has closed range. If $\{Tx_n\}_{n \in \mathbb{J}}$ is a $K$-frame for $E$, then $T$ is surjective.

Proof. Suppose that $K^* x = 0$ for $x \in E$, then for each $y \in E$, $\langle Ky, x \rangle = \langle y, K^* x \rangle = 0$. So $\langle z, x \rangle = 0$ for each $z \in E$, since $R(K)$ is dense in $E$. Thus $x = 0$ and $K^*$ is injective. We shall show that $T^*$
is injective too. Note that \( \{Tx_n\}_{n \in J} \) is a \( K \)-frame for \( E \) with bounds \( \lambda \) and \( \mu \), hence
\[
\lambda \|K^*x\|^2 \leq \left\| \sum_n \langle x, Tx_n \rangle \langle Tx_n, x \rangle \right\| \leq \mu \|x\|^2.
\]
for \( T^*x \in E \) and therefore,
\[
\lambda \|K^*x\|^2 \leq \left\| \sum_n \langle T^*x, x_n \rangle \langle x_n, T^*x \rangle \right\| \leq \mu \|x\|^2.
\]
If \( x \in N(T^*) \) then \( T^*x = 0 \) so \( \langle T^*x, x_n \rangle = 0 \) for each \( n \in \mathbb{N} \), and so \( K^*x = 0 \) by the last inequality. On the other hand \( K^* \) is injective, so \( x = 0 \), and so \( T^* \) is injective. Therefore
\[
E = N(T^*) + R(T) = R(T) = R(T),
\]
and this complete the proof. \( \square \)

**Theorem 3.5.** Let \( K \in L(E) \) and \( \{x_n\}_{n \in J} \) be a \( K \)-frame for \( E \). If \( T \in L(E) \) with closed range such that \( R(TK) \) is orthogonal complemented and \( KT = TK \). Then \( \{Tx_n\}_{n \in J} \) is a \( K \)-frame for \( R(T) \).

**Proof.** Xu and Sheng in [20] show that if \( T \) has closed range then \( T \) has Moore-Penrose inverse operator \( T^\dagger \) such that \( TT^\dagger T = T \) and \( T^\dagger TT^\dagger = T^\dagger \). So \( TT^\dagger \big|_{R(T)} = I_{R(T)} \) and \( (TT^\dagger)^* = I^* = I = TT^\dagger \). For every \( x \in R(T) \) we have
\[
\langle K^*x, K^*x \rangle = \langle (TT^\dagger)^*K^*x, (TT^\dagger)^*K^*x \rangle
\]
\[
= \langle T^\dagger T^*K^*x, T^\dagger T^*K^*x \rangle
\]
\[
\leq \|T^\dagger\|^2 \langle T^*K^*x, T^*K^*x \rangle
\]
and so
\[
\|T^\dagger\|^2 \langle K^*x, K^*x \rangle \leq \langle T^*K^*x, T^*K^*x \rangle.
\]
Since \( \{x_n\}_{n \in J} \) is a \( K \)-frame and \( R(T^*K^*) \subset R(K^*T^*) \), if \( \lambda \) is a lower bound then by using Theorem 2.2, there exists some \( \lambda' > 0 \) such that
\[
\sum_n \langle x, Tx_n \rangle \langle Tx_n, x \rangle = \sum_n \langle T^*x, x_n \rangle \langle x_n, T^*x \rangle
\]
\[
\geq \lambda \langle K^*T^*x, K^*T^*x \rangle
\]
\[
\geq \lambda' \lambda \langle T^*K^*x, T^*K^*x \rangle
\]
\[
\geq \lambda' \lambda \|T^\dagger\|^2 \langle K^*x, K^*x \rangle.
\]
This is the lower inequality for \( \{Tx_n\}_{n \in J} \). On the other hand by Lemma 3.1, \( \{Tx_n\}_{n \in J} \) is a Bessel sequence, so \( \{Tx_n\}_{n \in J} \) is a \( K \)-frame for Hilbert module \( R(T) \).

**Theorem 3.6.** Let \( E \) be a Hilbert \( A \)-module, \( K \in L(E) \) and \( \{x_n\}_{n \in J} \) be a \( K \)-frame for \( E \), and \( T \in L(E) \) is a co-isometry such that \( R(T^*K^*) \subset R(K^*T^*) \) with \( R(TK) \) orthogonal complemented. Then \( \{Tx_n\}_{n \in J} \) is a \( K \)-frame for \( E \).

**Proof.** Using Lemma 3.1 \( \{Tx_n\}_{n \in J} \) is a Bessel sequence. By Theorem 2.2, there exist \( \lambda' > 0 \) such that

\[
\|T^*K^*x\|^2 \leq \lambda'\|K^*T^*x\|^2,
\]

for each \( x \in E \). Suppose \( \lambda \) is a lower bound for the \( K \)-frame \( \{x_n\}_{n \in J} \). Since \( T \) is a co-isometry, then

\[
\frac{\lambda}{\lambda'}\|K^*x\|^2 = \frac{\lambda}{\lambda'}\|T^*K^*x\|^2
\leq \lambda\|K^*T^*x\|^2
\leq \sum_n \langle T^*x_n, x_n \rangle \langle x_n, T^*x \rangle
= \sum_n \langle x, Tx_n \rangle \langle Tx_n, x \rangle.
\]

which implies that \( \{Tx_n\}_{n \in J} \) is a \( K \)-frame for \( E \).

**Remark 3.7.** If \( K \in L(E) \) with dense range, \( T \in L(E) \) with closed range such that \( TK = KT \) and \( \{x_n\}_{n \in J} \) is a \( K \)-frame for \( E \). Then \( \{Tx_n\}_{n \in J} \) is a \( K \)-frame for \( E \) if and only if \( T \) is surjective.

**Theorem 3.8.** Let \( K \in L(E) \) with dense range and \( \{x_n\}_{n \in J} \) is a \( K \)-frame for \( E \). Let \( T \in L(E) \) with closed range. If \( \{Tx_n\}_{n \in J} \) and \( \{T^*x_n\}_{n \in J} \) are \( K \)-frames for \( E \) then \( T \) is invertible.

**Proof.** By Theorem 3.4, \( T \) is surjective. Since \( \{T^*x_n\}_{n \in J} \) is a \( K \)-frame for \( E \) then there exist positive numbers \( \mu \) and \( \lambda \) such that for every \( x \in E \)

\[
\lambda\|K^*x\|^2 \leq \|\sum_n \langle x, T^*x_n \rangle \langle T^*x_n, x \rangle\| \leq \mu\|x\|^2
\]

So for \( x \in N(T) \) we have

\[
\lambda\|K^*x\|^2 \leq \sum_n \langle x, T^*x_n \rangle \langle T^*x_n, x \rangle = 0
\]

Then \( \|K^*x\|^2 = 0 \), so \( x \in N(K^*) \). On the other hand \( K \in L(E) \) has dense range so \( K^* \) is injective and so \( T \) is injective.
4. Sums of $K$-frames

In this section we shall show that the sum of two $K$-frames in a Hilbert $C^*$-module under certain conditions is again a $K$-frame. It is proved, in Hilbert space case, by Ramu and Johnson [15]. In the proof of Theorem 3.2 of [13] indicates that if $T$ has closed range then $R(T^*T)$ is closed and $R(T) = R(T^*T)$. The following theorem says that this result still holds for adjointable operators between Hilbert $C^*$-modules (even though $R(T^*)$ may not be complemented).

**Theorem 4.1.** [13] For $T$ in $L(E,F)$, the sub-spaces $R(T^*)$ and $R(T^*T)$ have the same closure.

In [16], Sharifi show that the conversely of the above theorem is also true.

**Theorem 4.2** (Lemma 1.1, [16]). Suppose $T \in L(E)$, then the operator $T$ has closed range if and only if $R(TT^*)$ has closed rang. In this case, $R(T) = R(TT^*)$.

**Corollary 4.3.** Suppose $T \in L(E^+)$, then $R(T)$ is closed if and only if $R(T^{1/2})$ is closed. In this case, $R(T) = R(T^{1/2})$.

**Proof.** The proof is immediately consequence of replacement $T$ by $T^{1/2}$ in the above theorem. □

**Theorem 4.4.** Let $E$ be a Hilbert module and $A,B \in L(E)$ such that $R(A) + R(B)$ is closed. Then

$$R(A) + R(B) = R((AA^* + BB^*)^{1/2})$$

**Proof.** Define $T \in L(E \oplus E)$ by $T := \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$ then $T^* = \begin{bmatrix} A^* & 0 \\ B^* & 0 \end{bmatrix}$ and

$$TT^* = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A^* & 0 \\ B^* & 0 \end{bmatrix} = \begin{bmatrix} AA^* + BB & 0 \\ 0 & 0 \end{bmatrix}.$$

So we have

$$(TT^*)^{1/2} = \begin{bmatrix} (AA^* + BB^*)^{1/2} & 0 \\ 0 & 0 \end{bmatrix}.$$

On the other hand

$$T \begin{bmatrix} E \\ E \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E \\ E \end{bmatrix}$$

thus

$$R(T) = R(A) + R(B) \oplus \{0\}.$$
Since $R(T) = (R(A) + R(B))$ is closed then by Theorem 4.2, $R(T) = R(TT^*)$, but by the Corollary 4.3, $R(TT^*) = R((TT^*)^{1/2})$. So we have

$$R(A) + R(B) = R((AA^* + BB^*)^{1/2}).$$

□

The following theorem is a generalization of Douglas theorem [Theorem 1.1, [18] ], for Hilbert modules.

**Theorem 4.5.** Let $A, B_1, B_2 \in L(E)$ and $R(B_1) + R(B_2)$ is closed. The following statements are equivalent.

1. $R(A) \subset R(B_1) + R(B_2)$;
2. $AA^* \leq \lambda(B_1B_1^* + B_2B_2^*)$ for some $\lambda > 0$;
3. There exist $X, Y \in L(E)$ such that $A = B_1X + B_2Y$.

**Proof.** (1) $\implies$ (2): Suppose $R(A) \subset R(B_1) + R(B_2)$ then by Theorem 4.4, we have

$$R(A) \subset R(B_1) + R(B_2) = R((B_1B_1^* + B_2B_2^*)^{1/2})$$

thus by Theorem 2.2, $AA^* \leq \lambda(B_1B_1^* + B_2B_2^*)$ for some $\lambda > 0$.

(2) $\implies$ (1): By Theorems 2.2, and 4.5, it is clear.

(3) $\implies$ (1): It is obvious.

(1) $\implies$ (3): Define $S, T \in L(E \oplus E)$

$$S = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} B_1 & B_2 \\ 0 & 0 \end{bmatrix}$$

. Then $R(S) \subset R(T)$, by Theorem 2.2, suppose

$$X = \begin{bmatrix} X_1 & X_3 \\ X_2 & X_4 \end{bmatrix}$$

is the solution of $S = TX$, so we have $A = B_1X_1 + B_2X_2$. This completes the proof.

□

Now we want to show that under certain conditions the sum of two $K$-frame, is a $K$-frame. Firstly suppose $\{x_n\}_{n \in J}$ and $\{y_n\}_{n \in J}$ are two Bessel sequences in Hilbert module $E$, then by the Minkowski’s inequality it is easy to see that $\{x_n + y_n\}_{n \in J}$ is also a Bessel sequence for $E$.

**Theorem 4.6.** Let $\{x_n\}_{n \in J}$ and $\{y_n\}_{n \in J}$ be two $K$-frames for $E$ and also let the corresponding operators in Theorem 2.9, be $L_1$ and $L_2$ respectively. If $L_1L_2^*$ and $L_2L_1^*$ are positive operators and $R(L_1) + R(L_2)$ is closed, then $\{x_n + y_n\}_{n \in J}$ is a $K$-frame for $E$. 
Proof. By the hypothesis we have

\[ L_1 e_n = x_n, \quad L_2 e_n = y_n, \quad R(K) \subset R(L_1), R(K) \subset R(L_2), \]

where \( \{e_n\}_{n \in J} \) is the canonical orthonormal basis of \( \ell^2(A) \). So \( R(K) \subset R(L_1) + R(L_2) \), by Theorem 4.5, \( KK* \leq \lambda (L_1 L_1^* + L_2 L_2^*) \) for some \( \lambda > 0 \). On the other hand for each \( x \in E \),

\[
\sum_{n=1}^{\infty} \langle x, x_n + y_n \rangle \langle x_n + y_n, x \rangle = \sum_{n=1}^{\infty} \langle (L_1^* + L_2^*)x, e_n \rangle \langle e_n, (L_1 + L_2)^*x \rangle \\
= \sum_{n=1}^{\infty} \langle (L_1 + L_2)^*x, e_n \rangle \langle e_n, (L_1 + L_2)^*x \rangle \\
= \| (L_1 + L_2)^*x \|_{\ell^2(A)} \\
= \langle (L_1^* + L_2^*)x, L_1^*x \rangle \\
= \langle L_1^*x, L_1^*x \rangle + \langle L_1^*x, L_2^*x \rangle \\
+ \langle L_2^*x, L_1^*x \rangle + \langle L_2^*x, L_2^*x \rangle \\
\geq \| (L_1 L_1^* + L_2 L_2^*)x \| \\
\geq \frac{1}{\lambda} \langle KK^*x, x \rangle \\
\geq \frac{1}{\lambda} \langle K^*x, K^*x \rangle.
\]

Thus \( \{x_n + y_n\}_{n \in J} \) is a \( K \)-frame. \( \Box \)

5. \( K \)-frame vectors for unitary systems

A unitary system is a set of unitary operators contains the identity operator. A vector \( \psi \) in \( E \) is called a \textit{complete} \( K \)-frame vector for a unitary system \( U \) on \( E \) if \( U \psi = \{ U \psi \mid U \in U \} \) is a \( K \)-frame for \( E \). If \( U \psi \) is an orthonormal basis for \( E \), then \( \psi \) is called a \textit{complete wandering} vector for \( U \). The set of all complete \( K \)-frame vectors and complete wandering vectors for \( U \) is denoted by \( F_K(U) \) and \( \omega(U) \), respectively. In this section we characterize \( F_K(U) \) in terms of operators and elements of \( \omega(U) \). Also we give conditions under which a linear operation on given elements of \( F_K(U) \) remain an element of \( F_K(U) \).

**Definition 5.1.** For unitary system \( U \) on Hilbert module \( E \) and \( \psi \in E \), the local commutant \( C_\psi(U) \) of \( U \) at \( \psi \) is defined by

\[ C_\psi(U) = \{ T \in L(E) \mid TU \psi = UT \psi, \quad U \in U \}. \]

Also let \( \ell^2(U) \) be the Hilbert \( A \)-module defined by

\[ \ell^2(U) = \{ \{ a_U \} \subset A : \sum a_U a_U^* \text{ converges in } \| \cdot \| \}. \]
The following theorem characterizes complete $K$-frame vectors in terms of operators on complete wandering vectors.

**Theorem 5.2.** Suppose $U$ is a unitary system of $E$, $K \in L(E)$, $\psi \in \omega(U)$, $\eta \in E$, and suppose that $\psi_\eta \in L(E, \ell^2_U(A))$ is given by $T_\eta(x) = \{\langle x, U_\eta \rangle \}_{U \in U}$ and $R(T_\eta^*)$ is orthogonal complemented. Then $\eta \in F_K(U)$ if and only if there exist an operator $A \in C_\psi(U)$ with $R(K) \subset R(A)$ such that $\eta = A\psi$.

**Proof.** ($\Rightarrow$) Suppose $\{e_U\}_{U \in U}$ denote the standard orthonormal basis of $\ell^2_U(A)$, where $e_U$ takes value $1_A$ at $U$ and $0_A$ at every where else. Now suppose $\eta \in F_K(U)$, define operator $T_\psi$ from $E$ to $\ell^2_U(A)$ by $T_\psi x = \sum_{U \in U} \langle x, U\psi \rangle U_\eta e_U$. It is easy to check that $T_\psi$ is well defined, adjointable and invertible. Let $A = T_\eta^* T_\psi$. Then for any $x \in E$, we have $Ax = \sum_{U \in U} \langle x, U\psi \rangle U_\eta$ and $A^* x = \sum_{U \in U} \langle x, U_\eta \rangle U\psi$, also

$$\langle A^* x, A^* x \rangle = \left\langle \sum_{U \in U} \langle x, U\eta \rangle U_\psi, \sum_{U \in U} \langle x, U\eta \rangle U_\psi \right\rangle$$

$$= \sum_{U \in U} \langle x, U\eta \rangle \langle U\eta, x \rangle \geq c \langle K^* x, K^* x \rangle,$$

where $c > 0$ is the lower bound for $K$-frame $\{U_\eta \mid U \in U\}$. On the other hand $R(A) = R(T_\eta^*)$ and so by Theorem 2.2, we have $R(K) \subset R(A)$. To complete the proof, it remains to prove that $\eta = A\psi$ and $A \in C_\psi(U)$. For any $U$ and $V$ in $U$

$$\langle V_\eta, AU_\psi \rangle = \langle V_\eta, \sum_{U \in U} \langle U_\psi, W_\psi \rangle W_\eta \rangle$$

$$= \sum_{U \in U} \langle V_\eta, W_\eta \rangle \langle W_\psi, U_\psi \rangle \geq c \langle V_\eta, AU_\psi \rangle = \langle V_\psi, U_\psi \rangle.$$

This implies that $AU_\psi = U_\eta$, so $A\psi = \eta$. Also $AU_\psi = U_\eta = UA_\psi$, hence $A \in C_\psi(U)$ and this completes the proof of this part.

($\Leftarrow$): Suppose that there exists an operator $A \in C_\psi(U)$ with $R(K) \subset R(A)$ such that $\eta = A\psi$. Then for any $x \in E$, we have
\[
\sum_{U \in \mathcal{U}} \langle x, U\eta \rangle \langle U\eta, x \rangle = \sum_{U \in \mathcal{U}} \langle x, UA\psi \rangle \langle UA\psi, x \rangle = \sum_{U \in \mathcal{U}} \langle A^*x, U\psi \rangle \langle U\psi, A^*x \rangle = \langle A^*x, A^*x \rangle \leq \|A^*\|^2 \|x\|^2 \tag{5.3}
\]

So \( \{U\eta \mid U \in \mathcal{U}\} \) is a Bessel sequence for \( E \). Now let \( T_\eta \) and \( T_\psi \) be the operators as we defined in the first part of the proof, since \( \eta = A\psi \) so we have \( T_\eta = T_\psi A^* \). Since \( \psi \in w(\mathcal{U}) \), it is easy to see that \( T_\psi^* \) is invertible and hence \( R(T_\eta^*) = R(A) \). So \( R(K) \subset R(T_\eta^*) \). Therefore by using Theorem 3.2 of [10] \( \eta \in \mathcal{U}_K(\mathcal{U}) \).

\[\Box\]

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