Chern-Weil Constructions on $\Psi\text{DO}$ Bundles

Sylvie Paycha
Steven Rosenberg

Abstract

We construct Chern-Weil classes on infinite dimensional vector bundles with structure group contained in the algebra $\mathcal{C}_{\leq 0}(M, E)$ of non-positive order classical pseudo-differential operators acting on a finite rank vector bundle $E$ over a closed manifold $M$. Mimicking the finite dimensional Chern-Weil construction, we replace the ordinary trace on matrices by linear functionals on $\mathcal{C}_{\leq 0}(M, E)$ built from the leading symbols of the operators. The corresponding Chern classes vanish on loop groups, but a weighted trace construction yields a non-zero class previously constructed by Freed. For loop spaces, the structure group reduces to a gauge group of bundle automorphisms, and we produce non-vanishing universal Chern classes in all degrees, using a universal connection theorem for these bundles.

1 Introduction

Infinite dimensional vector bundles with connections are frequently encountered in mathematical physics; basic examples include the tangent bundle to loop spaces and the bundle of spinor fields associated to a family of Dirac operators. In this paper, we construct Chern forms and Chern classes for a class of vector bundles including the tangent bundle to loop spaces, and produce examples of non-vanishing classes. Motivation for this work comes from [3, 4, 5, 11, 13], among others.

We should emphasize that we do not construct a unified Chern-Weil theory as free from choices as the finite dimensional theory. Some choices seem inevitable: different choices of model spaces for the vector bundle fibers lead to inequivalent topologies, and the natural choice of the general linear group of a Hilbert space as structure group leads to a trivial theory by [10]. Other choices seem more arbitrary: for the classical groups $GL(n, \mathbb{C}), U(n)$ and (most of) $SO(n)$, Chern-Weil theory depends crucially on properties of the ordinary trace on matrices. In infinite dimensions, there are many possible generalizations of the trace, the operator trace being the most obvious. (It is seldom applicable, since the curvature of the connections we consider are not trace class operators.) Much of the paper is devoted to isolating the key features of the trace, and then constructing useful generalizations to infinite dimensions.

Motivated by the examples mentioned, we concentrate on bundles $E \rightarrow B$ with fibers $E_b$ modeled on either Sobolev spaces of sections or smooth sections of a finite rank bundle

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E over a closed manifold M. The structure group is a group of classical pseudo-differential operators (ΨDOs) on E, and the curvatures of the connections we consider take values in a corresponding Lie algebra of ΨDOs. For loop spaces, the structure group is typically multiplication operators, but the curvatures of the connections we study take values in ΨDOs.

In more detail, in §2 we discuss traces as morphisms \( \lambda : \text{Ad} P \to \mathbb{C} \) from the adjoint bundle of a principal bundle P to the trivial \( \mathbb{C} \) bundle. Here the structure group and the base may be infinite dimensional, and we are thinking of P as the principal bundle associated to \( \mathcal{E} \). We show that such a functional acting on the curvature \( \Omega \) of a connection on P leads to characteristic forms and classes on the base generalizing the usual forms \( \text{tr}(\Omega^k) \) in finite dimensions (Theorem 2.2).

In §3, we introduce two types of traces in infinite dimensions. The first type is the Wodzicki residue, the unique trace on the space of classical ΨDOs. However, we can often restrict the structure group to invertible zeroth order ΨDOs. The corresponding Lie algebra of non-positive order ΨDOs admits a family of traces of the form \( A \mapsto \Lambda(\sigma^A_0) \), where \( \sigma^A_0 \) is the leading symbol of A and \( \Lambda \) is any distribution on the cosphere bundle of M. For both traces, the corresponding Chern classes vanish for loop groups. We show that the associated universal Chern classes are non-zero for the structure group of loop spaces, a gauge group (Theorem 3.3). This result depends on a universal connection theorem proved in §4. Based on Freed’s work \cite{Freed}, we construct a nontrivial characteristic class on loop groups as the most divergent term in the asymptotic expansion of a weighted trace, and give a short, direct proof that this form is closed.

In §4, we prove that the bundle \( E\mathcal{G} \to B\mathcal{G} \) has a universal connection, where \( \mathcal{G} \) is the gauge group of the finite rank bundle E. We hope this result is of independent interest.

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2 Abstract Chern-Weil calculus

We first give a quick review of connections on principal bundles and associated bundles; see \cite{Paycha} for more details. Let \( P \to B \) be a Fréchet, resp. Hilbert principal G-bundle over a manifold \( B \), with \( G \) a Fréchet, resp. Hilbert Lie group, let \( \rho : G \to \text{Aut}(W) \) be a representation of \( G \) on some vector space \( W \) and let \( W := P \times_{\rho} W \to B \) be the associated vector bundle. It is a Fréchet, resp. Hilbert vector bundle over \( B \).

For example, if \( E \to B \) is a finite dimensional vector bundle with structure group \( GL(n, \mathbb{C}) \), then the frame bundle \( P := GL(E) \to B \) is a principal \( GL(n, \mathbb{C}) \)-bundle and \( \text{Ad} P \) is isomorphic to \( \text{End}(E) \), the endomorphism bundle of \( E \). If \( E \) is an infinite dimensional vector bundle with typical fiber \( V \) and structure group \( G \), then the associated
principal $G$-bundle $P^E$ is built by gluing copies of $G$ over a point $b \in U_i \cap U_j$ in overlapping charts by the same $g = g_b \in G$ that glues the copies of $V$ over $b$.

The space $\Omega(B, \mathcal{W})$ of $\mathcal{W}$-valued forms on $B$ can be identified with the space $(\Omega(P) \otimes \mathcal{W})_{\text{basic}}$ of basic forms $\alpha$ on $P$ with values in the trivial bundle $P \times \mathcal{W}$. Recall that a form is basic if it is $G$-invariant ($\rho(g)(g^*\alpha) = \alpha$) and horizontal ($i_X \alpha = \alpha$ for the canonical vector field $\tilde{X}$ generated by $X \in A = \text{Lie}(G)$).

A connection on $P$ given by a one-form $\theta \in \Omega^1(P, A)$ with $\text{Ad}_g(g^*\theta) = \theta$ for $g \in G$ and $i_X \theta = X$ yields a covariant derivative on $\mathcal{W}$ in the following way. The derivative $D\rho : A \to \text{End}(\mathcal{W})$ of $\rho$ induces $D\rho(\theta) \in \Omega^1(P) \otimes \text{End}(\mathcal{W})$ and hence a connection $\nabla = d + D\rho(\theta)$ on the trivial bundle $P \times \mathcal{W}$. The resulting operator $\nabla$ on $\Omega(P) \otimes \mathcal{W}$ descends to an operator on $(\Omega(P) \otimes \mathcal{W})_{\text{basic}}$ and hence a covariant derivative on $\mathcal{W}$ also denoted by $\nabla$, with $\nabla = d + D\rho(\theta)$ locally. Starting with a vector bundle $E \to B$ with connection $\nabla$, we produce a connection one-form $\theta$ on $P^E$, and the curvature of $\nabla$ takes values in $\text{Ad} P$.

If $\mathcal{W} = A$ is the Fréchet Lie algebra of $G$ and if $\rho$ is the adjoint action, then $\mathcal{W}$ is called the adjoint bundle and denoted by $\text{Ad} P$. Recall that the adjoint representation $\text{Ad} : G \to \text{Aut}(A)$ is the differential of conjugation in $G$: $\text{Ad}_g a := (D_e C_g) a$, where $C_g : G \to G$ is $C_g(h) = ghg^{-1}$. The differential of $\text{Ad}$, $\text{ad} = D\text{Ad} : A \to \text{End}(A)$, is given by $\text{ad}_g(a) = [b, a]$, since

$$\text{ad}_g(a) = \frac{d}{dt} \bigg|_{t=0} \text{Ad}_{e^{tb}}(a) = \frac{d}{dt} \bigg|_{t=0} \frac{d}{ds} \bigg|_{s=0} C_{e^{tb}} e^{sa} = [b, a].$$

In particular, a connection one-form $\theta \in \Omega^1(P, A)$ yields a connection $\nabla^{\text{ad}}$ on $\text{Ad} P$, with $\nabla^{\text{ad}} = d + [\theta, \cdot]$. (For this reason, our $\text{Ad} P$ is often denoted $\text{ad} P$.) If $P = \text{GL}(E)$, a connection $\nabla^E$ on $E$ induces a connection one-form on $P$, and the induced connection on $\text{Ad} P$ coincides with the connection $\nabla^{\text{Hom}(E)}$ associated to $\nabla^E$.

A linear form on $A$:

$$\lambda : A \to \mathbb{C}$$

such that $\text{Ad}^* \lambda := \lambda \circ \text{Ad} = \lambda$ induces a bundle morphism

$$\lambda : \text{Ad} P \to B \times \mathbb{C}$$

defined as follows. Given a local trivialization $(U, \Phi)$, where $U \subset B$ is open and $\Phi : \text{Ad} P|_U \to U \times A$ is an isomorphism, and a local section $\sigma \in \Gamma(\text{Ad} P|_U)$, we set

$$\lambda(\sigma) := \lambda(\Phi(\sigma)).$$

This definition is independent of the local trivialization. Indeed, given another local trivialization $(V, \Psi)$, at $b \in U \cap V$ we have

$$\lambda(\Phi(\sigma)) = \lambda(\text{Ad}_g \Psi(\sigma)) = \lambda(\Psi(\sigma)), \quad \text{for some } g = g_b \in G.$$

The connection $\nabla^{\text{ad}}$ on $\text{Ad} P$ induced by a connection $\theta$ on $P$ induces in turn a connection $\nabla^*$ on the dual bundle $\text{Ad} P^*$ (i.e. $(\text{Ad} P)^*$), which is locally described by
\[ \nabla^{\text{ad}} = d + \text{ad}^* \] Since \( \text{Ad}^* \lambda = \lambda \) implies \( \text{ad}^* \lambda = 0 \), we have \( \nabla^{\text{ad}} \lambda = d \lambda = 0 \), since \( \lambda \) is locally constant. Summarizing, we have:

**Lemma 2.1**: Let \( \lambda : \text{Ad} \ P \to B \times C \) be the linear morphism induced by a linear form \( \lambda : A \to C \) with \( \text{Ad}^* \lambda = \lambda \). Let \( \nabla^{\text{ad}} = d + [\theta, \cdot] = d + \text{ad}_\theta \) be a connection on \( \text{Ad} \ P \) induced by a connection \( \theta \) on \( P \). Then

\[ d \circ \lambda = \lambda \circ \nabla^{\text{ad}}. \tag{2.1} \]

**Proof**: Since \( d \lambda = 0 \), we have \( d \circ \lambda = \lambda \circ d \) locally. However, \( \text{ad}^* \lambda = 0 \) implies \( \lambda \circ d = \lambda \circ (d + \text{ad}_\theta) = \lambda \circ \nabla^{\text{ad}} \), so \( d \circ \lambda = \lambda \circ \nabla^{\text{ad}} \) globally. \( \square \)

Abusing notation, we will sometimes denote \( \nabla^{\text{ad}} \alpha \) by \([\nabla, \alpha]\), for \( \alpha \) an \( \text{Ad} \ P \)-valued form, in analogy to the local description \( \nabla^{\text{ad}} = d + [\theta, \cdot] \), with the understanding that \([\nabla, \alpha]\) is a superbracket with respect to the \( \mathbb{Z}_2 \)-grading on differential forms.

The lemma gives the main result of this section.

**Proposition 2.2**: Let \( P = P^E \) be the principal bundle associated to a vector bundle \( E \to B \) with structure group \( G \) and connection \( \nabla \) and curvature \( \Omega \). Assume that the associated connection \( \nabla^{\text{ad}} \) on \( \text{Ad} \ P \) has \( d \circ \lambda = \lambda \circ \nabla^{\text{ad}} \). Then for any analytic function \( f : \mathbb{C} \to \mathbb{C} \), the form \( \lambda(f(\Omega)) \) is closed, and its de Rham cohomology class in \( H^*(B; \mathbb{C}) \) is independent of the choice of \( \nabla \).

As usual, we mean that the degree \( k \) piece of \( \lambda(f(\Omega)) \) is a closed \( 2k \)-form, for all \( k \in \mathbb{N} \).

**Proof**: The usual finite dimensional proof (see e.g. [2]) runs through, with ordinary traces replaced by \( \lambda \).

In more detail, \( \lambda(f(\Omega)) \) is closed because \( \lambda(\Omega^k) \) is closed for any \( k \in \mathbb{N} \), which we check in a local trivialization of \( \text{Ad} \ P \). We have:

\[ d \lambda(\Omega^k) = \lambda(\nabla^{\text{ad}} \Omega^k) = \sum_{j=1}^{k} \lambda(\Omega^{j-1}(\nabla^{\text{ad}} \Omega) \Omega^{k-j}) = 0 \]

where we have used the Bianchi identity \( \nabla \Omega = 0 \) in the last identity.

To check that the corresponding de Rham class is independent of the choice of connection, we consider a smooth one-parameter family of connections \( \{ \nabla_t, t \in \mathbb{R} \} \) on \( E \) and the induced family of connections \( \nabla^{\text{ad}}_t \) on \( \text{Ad} \ P \). Then

\[ \frac{d}{dt} \lambda(\Omega^k_t) = \sum_{j=1}^{k} \lambda \left( \Omega^{k-j}_t (\nabla_t \nabla_t + \nabla_t \nabla_t) \Omega^{j-1}_t \right) = \sum_{j=1}^{k} \lambda \left( \Omega^{k-j}_t (\nabla^{\text{ad}}_t \nabla_t) \Omega^{j-1}_t \right) \tag{2.2} \]

\[ = \sum_{j=1}^{k} \lambda \left( \nabla^{\text{ad}}_t \Omega^{k-j}_t \nabla_t \Omega^{j-1}_t \right) = d \sum_{j=1}^{k} \lambda \left( \Omega^{k-j}_t \nabla_t \Omega^{j-1}_t \right). \]
In the first equality, we use $\nabla_t^2 = \Omega_t$, in the second we have extended the bracket connection to forms, and in the third we have used the Bianchi identity. (2.2) shows that the dependence on the connection is measured by an exact form and hence vanishes in cohomology.

Remarks: (1) This proof does not use the linearity of $\lambda$, and extends to any map $\lambda: \text{Ad} \ P \to \mathbb{C}$ satisfying (2.1).

(2) Just as in finite dimensions, if $E$ is a trivial bundle, then $[\lambda(f(\Omega))] = 0$, because we can connect $\nabla$ to the trivial connection. In particular, if $M$ is a finite dimensional manifold, and $F \to N$ is a trivial finite rank bundle over a finite dimensional manifold $N$, there is a corresponding infinite dimensional trivial bundle $E$ over $B := \text{Maps}(M, N)$, whose fiber over $\phi$ is the space of sections $\Gamma(\phi^*F)$ for $\phi^*F \to M$. For example, we can take $F = TN$ (if $N$ is parallelizable), in which case $E = TB$. Thus characteristic classes formed by such $\lambda$’s will vanish on $TB$, confirming results in [11]. In particular, for $M = S^1$ and $N = G$, a Lie group, we get the vanishing of the characteristic classes for the tangent bundle to the loop group $LG$.

The previous theorem yields the usual Chern-Weil classes:

**Corollary 2.3:** Let $G \subset GL(n, \mathbb{C})$ be a finite dimensional Lie group, and let $E \to B$ be a vector bundle with structure group $G$. Let $\nabla$ be a connection on $E$ with curvature $\Omega$. For any analytic function $f$, the forms $\text{tr}(f(\Omega)) \in \Omega^*(B, \mathbb{C})$ are closed and their de Rham cohomology classes are independent of the choice of $\nabla$.

This follows from Proposition 2.2 by passing from $E$ to $P^E$ and using $\lambda = \text{tr}$, the ordinary trace on matrices.

**Remark:** For $GL(n, \mathbb{C})$ and $U(n)$, all characteristic classes are generated by $\text{tr}(\Omega^k), k \in \mathbb{N}$. However, we do not capture the Euler class for $SO(n, \mathbb{R})$ by this procedure, as this class is generated by the non-linear, but Ad-invariant function $\sqrt{\text{det}}$. We can treat this case using the identity $\text{det}(1 + A) = \sum_k \text{tr}(A^k A)$, or by the previous remark (1).

**Notation:** Throughout the paper, “ΨDOs” means classical pseudo-differential operators, and $\text{Cl}(M, E)$ denotes the space of all classical ΨDOs acting on smooth sections of the finite dimensional Hermitian bundle $E$ over a closed Riemannian manifold $M$. $\text{Cl}_k(M, E)$ denotes the subspace of ΨDOs of order $k \in \mathbb{R}$, $\text{Cl}_{\leq k}(M, E)$ denotes the space of ΨDOs of order at most $k$. $\text{Cl}_k^0(M, E)$ denotes the set of invertible operators in $\text{Cl}_k(M, E)$. $\text{Ell}_k = \text{Ell}_k^0(M, E)$ denotes the space of elliptic operators of order $k$, and $\text{Ell}_k^*, \text{Ell}_k^+$ denote the spaces of invertible, resp. positive elliptic operators of order $k$.

A bundle $\mathcal{E}$ with fiber modeled on $C^\infty(M, E)$ or on $H^s(M, E)$ is a ΨDO bundle if the transition maps lie in the Lie group $\text{Cl}^0_0(M, E)$. Here $C^\infty(M, E), H^s(M, E)$ are the spaces of smooth and $s$-Sobolev class sections of $E$, respectively. Strictly speaking, the transition maps must be zero order only in the Hilbert (i.e. Sobolev) setting, since there the transition maps and their inverses must be bounded invertible operators on a Hilbert space. In the smooth setting, one can allow non-zero order ΨDOs.
A connection on a $\Psi$DO bundle $\mathcal{E}$ is a $\Psi DO$ connection if its connection one-form takes values in $\mathcal{C}(\mathcal{M}, \mathcal{E})$ in any local trivialization.

**Remark:** If $\theta$ is the locally defined connection one-form of a $\Psi$DO connection on a bundle $\mathcal{E}$ modeled on $\mathcal{C}^\infty(\mathcal{M}, \mathcal{E})$, then under a gauge change $g$, $\theta$ transforms to $g^{-1}\theta g + g^{-1}dg$. Since $g^{-1}dg$ is zeroth order if $g$ is nonconstant, the connection one-form is usually of non-negative order. (For left invariant connections on loop groups, $g^{-1}dg$ vanishes, and $\theta$ can be of any order.) When $\theta$ is non-positive order, it is a bounded operator and hence extends to a connection on the extension of $\mathcal{E}$ to an $H^s(\mathcal{M}, \mathcal{E})$-bundle. We call a connection with connection one-form taking values in $\mathcal{C}l_{\leq 0}(\mathcal{M}, \mathcal{E})$ a $\mathcal{C}l_{\leq 0}$-connection. The curvature of such a connection is a $\mathcal{C}l_{\leq 0}$-valued two-form on the base.

### 3 Traces, weighted traces, and corresponding Chern classes in infinite dimensions

In this section we examine two examples of Proposition 2.2. The trace is furnished by the Wodzicki residue in the first example, and by various traces applied to the leading order symbol of a zeroth order $\Psi$DO in the second. We will focus on the analytical/geometric aspects of this approach, and leave topological aspects to a forthcoming paper.

In each case, we begin with a $\Psi$DO bundle $\mathcal{E}$ over a base space $\mathcal{B}$, with fiber modeled on $\mathcal{C}^\infty(\mathcal{M}, \mathcal{E})$ or $H^s(\mathcal{M}, \mathcal{E})$, equipped with a $\Psi$DO connection. (We will let the connection one-form take values in $\mathcal{C}(\mathcal{M}, \mathcal{E})$ in §3.3). In the language of §2, we pass from $\mathcal{E}$ to the corresponding principal bundle $\mathcal{P} = P\mathcal{E}$ with fiber modeled on $\mathcal{C}l_0^\ast$. $\text{Ad } P$ is then the subbundle of $\text{End}(\mathcal{E})$ consisting of endomorphisms in $\mathcal{C}l_{\leq 0}$. In particular, the curvature $\Omega$ of the connection on $\mathcal{E}$ takes values in $\text{Ad } P$, and we can apply the Chern-Weil machinery of §2 to $\Omega$.

Note that in this section, we are treating the structure group $\mathcal{C}l_0^\ast$ as a generalization of $GL(n, \mathbb{C})$. As in finite dimensions, we focus only on invariant polynomials on the Lie algebras given by traces. We do not discuss the interesting question of whether all such polynomials on $\mathcal{C}l_{\leq 0}$ are generated by these traces.

Exactly how these examples generalize the finite dimensional situation is open to interpretation. When the manifold is reduced to a point, the leading symbol of an endomorphism in the fiber, a “zeroth order $\Psi$DO,” is just the endomorphism itself, and the only trace, up to normalization, is the ordinary trace on a vector space. In contrast, in finite dimensions the Wodzicki residue vanishes. So in this interpretation, the Wodzicki residue is a purely infinite dimensional phenomenon, while the symbol trace generalizes the finite dimensional theory (since the leading symbol of a finite dimensional linear transformation $A$ is just $A$).

On the other hand, both the Wodzicki residue and the symbol trace appear in the most divergent term in asymptotic expansions: the Wodzicki residue of an operator $A$ is the residue of the pole of the zeta function regularization $\text{Tr}(AQ^{-s})$ at $s = 0$ (for any positive elliptic operator $Q$), and the symbol trace is related to the coefficient of the most divergent term in the heat operator regularization $\text{Tr}(Ae^{-\varepsilon Q})$ as $\varepsilon \to 0$. (The last statement is proved...
in Proposition 3.4.) Since the two corresponding “regularizations” in finite dimensions using a positive definite matrix $Q$ simply reduce to $\text{Tr}(A)$, we can alternatively view both examples as proper generalizations of the finite dimensional Chern-Weil theory.

There is a third approach to generalizing from finite dimensions, namely by taking the finite part of a regularized trace. In the zeta and heat regularizations above, this amounts to picking a different term in the Laurent/asymptotic expansion, one that is more difficult to calculate in general. This approach was taken in [13] with limited success. In §3.3, we use weighted traces to produce closed forms from the most divergent term in an asymptotic series. This is not an example of Proposition 2.2, as the weighted trace does not satisfy (2.1).

In summary, not only is there no canonical generalization of finite dimensional Chern-Weil theory, there is no canonical interpretation of whether a specific method is indeed a proper generalization. The particular choice of regularization depends on a combination of physical motivation, computability, and perhaps aesthetics.

### 3.1 The Wodzicki residue

Recall that the Wodzicki residue $\text{res}_w A$ of a $\Psi$DO $A$ acting on sections of a bundle $E$ over a closed manifold $M$ is defined to be the residue of the pole term of $\text{Tr}(AQ^{-s})$ at $s = 0$, for an elliptic operator $Q$ with certain technical conditions. Alternatively, $\text{res}_w A$ is proportional to the coefficient of $\log \varepsilon$ in the asymptotic expansion of $\text{Tr}(Ae^{-\varepsilon Q})$ as $\varepsilon \to 0$, for $Q \in \mathcal{E}l^+ (M, E)$. The strengths of the Wodzicki residue are (i) its local nature:

$$\text{res}_w A = \frac{1}{(2\pi)^n} \int_{S^* M} \text{tr} \sigma^A_{-n}(x, \xi) \, d\xi \, dx,$$

where $n = \dim(M)$, $S^* M$ is the unit cosphere bundle of $M$, and $\sigma^A_{-n}$ is the $(-n)^{th}$ homogeneous piece of the symbol of $A$; and (ii) the fact that is it the unique trace on $\text{Cl} = \text{Cl}(M, E)$, up to normalization. Its drawback is its vanishing on all differential and multiplication operators, all trace class operators (and so all $\Psi$DOs of order less than $-n$) and all operators of non-integral order.

Given an infinite dimensional bundle $\mathcal{E}$ over a base $B$ with fibers modeled either on $H^s(M, E)$ $E$ (with $s \gg 0$) or on $C^\infty(M, E)$, and a connection on $\mathcal{E}$ with curvature $\Omega \in \Omega^2(B, \text{Cl})$, we can form the $k^{th}$ Wodzicki-Chern form by setting

$$c^w_k(\Omega) = \text{res}_w \Omega^k \in \Omega^{2k}(B).$$

By Proposition 2.2, $c^w_k(\Omega)$ is closed and independent of the connection.

By the remark after Proposition 2.2, the Wodzicki-Chern classes vanish for current groups. For completeness, we give a direct proof; see [14] for a more general discussion. For simplicity, we only consider Hilbert current groups, $\mathcal{C} = H^s(M, G)$, the space of $H^s$-maps from a closed Riemannian manifold $M$ to a semi-simple Lie group $G$ of compact type. (These assumptions ensure that the Killing form is nondegenerate and that the adjoint representation is antisymmetric for this form). $\mathcal{C}$ is a Hilbert Lie group with Lie
algebra $H^s(M, A)$, the space of $H^s$ sections of the trivial bundle $M \times A$, where $A = \text{Lie}(G)$. Thus the tangent bundle $TC$ is a $\Psi$DO bundle with fibers modeled on $H^s(M, A)$. For $\Delta$ the Laplacian on functions on $M$, we set $Q_0 := \Delta \otimes 1_A$, a second order elliptic operator acting densely on $H^s(M, A)$. $Q_0$ is non-negative for the scalar product $\langle \cdot, \cdot \rangle_0 := \int_M \text{dvol}(x)(\cdot, \cdot)$, where $(\cdot, \cdot)$ is minus the Killing form. $TC$ has a left-invariant weight $Q_\gamma = L_\gamma Q_0 L^{-1}_\gamma$ (i.e. a family of elliptic operators on the fibers), where $L_\gamma$ is left translation by $\gamma \in C$.

$C$ has a left-invariant Sobolev $s$-metric defined by

$$\langle \cdot, \cdot \rangle^s := \langle Q_0^{\frac{s}{2}} \cdot, Q_0^{\frac{s}{2}} \cdot \rangle_0,$$

where $Q_0$ is really $Q_0 + P$, for $P$ the orthogonal projection of $Q_0$ onto its kernel. The corresponding left-invariant Levi-Civita connection has the global expression $\nabla^s = d + \theta^s$, with $\theta^s$ a left-invariant $\text{End}(TC)$-valued one-form on $C$ induced by the $\text{End}(H^s(M, A))$-valued one-form on $H^s(M, A)

$$\theta_0^s(U) = \frac{1}{2} (\text{ad}_U + Q_0^{-s} \text{ad}_U Q_0^s - Q_0^{-s} \text{ad}_{Q_0 U}),$$

(3.1)

for $U \in H^s(M, A)$ (see [7] (1.9)] up to a sign error). $\theta^s$ takes values in $\text{Cl}_{\leq 0}(M, M \times A)$.

There is another natural left-invariant connection on $C$ given by

$$\tilde{\nabla}^s := d + \tilde{\theta}^s,$$

(3.2)

defined by the corresponding $\tilde{\theta}_0^s(U) := Q_0^{-s} \text{ad}_U Q_0^s$. $\tilde{\theta}^s$ also takes values in $\text{Cl}_{\leq 0}(M, M \times A)$, and so $\nabla^s$ and $\tilde{\nabla}^s$ can be joined by a line of such connections.

We now show that the Wodzicki-Chern forms vanish for $\tilde{\nabla}^s$, from which the vanishing of the Wodzicki-Chern classes of $\nabla^s$ follows. The connection $\tilde{\nabla}^s$ has curvature

$$\tilde{\Omega}^s(U, V) = [\tilde{\theta}_0^s(U), \tilde{\theta}_0^s(V)] - \tilde{\theta}_0^s([U, V]) = Q_0^{-s}[\text{ad}_U, \text{ad}_V] Q_0^s - Q_0^{-s} \text{ad}_{[U, V]} Q_0^s = Q_0^{-s}(\Omega^0(U, V)) Q_0^s,$$

for $U, V \in H^s(M, A)$, and where $\Omega^0(U, V) = [\text{ad}_U, \text{ad}_V] - \text{ad}_{[U, V]}$ is a multiplication operator. As before, applying the Wodzicki residue to any analytic function $f$ yields

$$\text{res}_w(f(\tilde{\Omega}^s))(U, V) = \text{res}_w \left( Q_0^{-s} f \left( \Omega^0(U, V) \right) Q_0^s \right) = \text{res}_w \left( f \left( \Omega^0(U, V) \right) \right) = 0,$$

since $\Omega^0(U, V)$ is a multiplication operator.

In general, no examples seem to be known where the Wodzicki-Chern classes are non-zero; see [11] for examples on spaces of maps where the forms vanish.

### 3.2 Leading symbol traces

The uniqueness of the trace on $C_l$ defined by the Wodzicki residue does not rule out the existence of other traces on subalgebras of $C_l$. Indeed, the ordinary operator trace on $C_{l < -n}$ is an example. In this subsection, we will introduce a family of traces on $C_{l < 0}$ and show
that they produce non-vanishing Chern classes on the universal bundle associated to the
gauge group for the $\mathcal{E} = TLM$, the tangent bundle to the free loop space of a Riemannian
manifold $M$. To our knowledge, this is the first example of non-vanishing Chern classes of
infinite dimensional bundles above $c_1$.

We first produce a “trace” on $\mathcal{C}l_{\leq 0}$ with values in $S^*M$, and an associated family of
true traces. Let $\mathcal{D}(X)$ denoted the space of complex valued distributions on a compact
topological space $X$.

**Lemma 3.1:** Let $\sigma^A_0$ be the zeroth order symbol of $A \in \mathcal{C}l_{\leq 0}$. The map $\text{Tr} = \text{Tr}_0 : \mathcal{C}l_{\leq 0}(M,E) \rightarrow C^\infty(S^*M)$ defined by

$$\text{Tr}(A) = \text{tr}_x(\sigma^A_0(x,\xi)) \text{ has } \text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B), \text{ Tr}(\lambda A) = \lambda \text{Tr}(A), \text{ and } \text{Tr}(AB) = \text{Tr}(BA).$$

For any $f \in \mathcal{D}(S^*M)$, the map $\text{Tr}^f : \mathcal{C}l_{\leq 0} \rightarrow \mathbb{C}$ given by $\text{Tr}^f(A) = f(\text{Tr}(A))$ is a trace.

**Proof:** Certainly taking the $0^{th}$ order symbol is linear. Since $\sigma_0$ is multiplicative on $\mathcal{C}l_{\leq 0}$, we have

$$\text{tr}_x \sigma^A_0 = \text{tr}_x(\sigma^A_0 \cdot \sigma^B_0) = \text{tr}_x(\sigma^B_0 \cdot \sigma^A_0) = \text{tr}_x \sigma^{BA}_0.$$ 

The proof of the second statement is immediate. \hfill \Box

**Remarks:** (1) There are corresponding traces on $\mathcal{C}l_{<p}$ for any $p < 0$. In fact, for $r \in [2p, p]$, $\text{Tr}_r A = \text{tr}_x(\sigma^A_r(x,\xi))$ is a trace, as $\text{Tr}_r(AB)$ trivially vanishes for $r > 2p$. The proof of the lemma covers the case $r = 2p$.

(2) Let $Q \in \mathcal{E}ll^+(M,E)$ have scalar leading symbol $\sigma^Q_l(x,\xi) = f(x,\xi)\text{Id}$. Define $f \in \mathcal{D}(S^*M)$ by

$$f(\phi) = \frac{\Gamma\left(\frac{2}{q}\right) \text{dim}(E)}{q^2(2\pi)^n} \int_{S^*M} \phi \cdot f(x,\xi)^{-\frac{n}{q}},$$

where $n = \text{dim}(M), q = \text{ord}(Q)$. Then $\text{Tr}^f(A)$ is the leading term in the asymptotics of $\text{Tr}(A e^{-\varepsilon Q})$ (see Proposition 3.3).

Recall that the ring of characteristic classes for e.g. $U(n)$ bundles is generated by the
Chern classes $c_k = [\text{Tr}(\Lambda^k)]$, or equivalently by the components $\nu_k = [\text{Tr}(\Omega^k)]$ of
the Chern character. Note we are momentarily distinguishing between $\text{Tr}(\Lambda^k A)$ and $\text{Tr}(A^k)$
for a matrix $A$. We will concentrate on Chern forms, and abuse notation by writing
$c_k = [\text{Tr}(\Omega^k)]$.

**Definition:** Let $\mathcal{E}$ be a bundle over $B$ modeled on $H^*(M,E)$ or $C^\infty(M,E)$, and let $\nabla$ be a connection on $\mathcal{E}$ with curvature $\Omega$ in $\mathcal{C}l_{\leq 0}(M,E)$. The $k^{th}$ Chern class of $\nabla$ with respect to $f \in \mathcal{D}(S^*M)$ is defined to be the de Rham cohomology class

$$[c^f_k(\Omega)] = [f(\text{tr}_x \sigma^0_0(x,\xi))] \in H^{2k}(B;\mathbb{C}). \quad (3.3)$$

**Remarks:** (1) As an example, if $f \equiv 1 \in C^\infty(S^*M)$, then

$$c^1_k(\Omega) = \int_{S^*M} \text{tr}_x \sigma^0_0(x,\xi).$$
At another extreme, if \( f = \delta_{(x_0, \xi_0)} \) is a delta function, then
\[
c_k^f(\Omega) = \text{tr}_x \sigma_0^{\Omega_k}(x_0, \xi_0).
\]

(2) As in the previous remark, we can define Chern classes for connections with curvature forms taking values in \( Cl_{\leq p} \) for any \( r \in [2p, p] \), \( p < 0 \). Note that for \( r < p \) and e.g. \( f \equiv 1 \), these classes are defined only after a choice of coordinates on \( M, E \) and a partition of unity on \( M \), since integrals of non-leading order symbols depend on such choices.

Proposition 2.2 immediately gives the following:

**Theorem 3.2:** Let \( \nabla \) be a \( Cl_{\leq 0} \) connection on a \( \Psi \text{DO} \) bundle \( \mathcal{E} \). Then the differential forms \( c_k^f(\Omega) \) are closed, and their cohomology classes are independent of the choice of \( Cl_{\leq 0} \) connection.

**Proof:** By Lemma 3.1, \( \text{Tr}^f \) is a trace on \( Cl_{\leq 0}(M, E) \), so we can apply Lemma 2.1 to the principal bundle \( P = P^\mathcal{E} \) built from \( \mathcal{E} \) to get the relation \( d \circ \text{Tr}^f = \text{Tr}^f \circ \nabla_{\text{ad}} \). We then apply Proposition 2.2 to get the corresponding Chern classes \( [c_k^f(\Omega)] \).

*Remark:* For the linear functionals \( \text{Tr}_p \) on \( Cl_{\leq p} \), the proof that the Chern forms are closed goes through, but the proof of their independence of choice of connection breaks down. The point is that the line of \( Cl_{\leq 0} \) connections joining two connections with curvature forms lying in \( Cl_{\leq p} \) might not stay in this class.

When the structure group reduces to a gauge group, we can construct an example of non-zero Chern classes \( [c_k^f(\Omega)] \). Fix \( n > k \) and consider the Grassmanian \( BU(n) = G(n, \infty) \) with its universal vector bundle \( \gamma_n \). We consider the pullback bundle \( E = \pi^*\gamma_n \) over \( S^1 \times BU(n) \), with \( \pi \) the projection onto \( BU(n) \). We now “loopify” to form \( B = L(S^1 \times BU(n)) \), the free loop space of \( S^1 \times BU(n) \), with bundle \( \mathcal{E} \) whose fiber over a loop \( \gamma \) is the space of smooth sections of \( \gamma^*E \) over \( S^1 \). \( \mathcal{E}_1 \) is the space of loops in \( E \) lying over \( \gamma \), suitably interpreted at self-intersection points of \( \gamma \), so we will write \( \mathcal{E} = L\pi^*\gamma_n \). Since \( \gamma^*E \) is (non-canonically) isomorphic to the trivial bundle \( S^1 \times \mathbb{C}^n \) over \( S^1 \), it is easily checked that the structure group for \( \mathcal{E} \) is the gauge group \( G \) of this trivial bundle.

Take a hermitian connection \( \nabla \) on \( \gamma_n \) (e.g. the universal connection \( PdP \), where \( P_x \) is the projection of \( \mathbb{C}^\infty \) onto the \( n \)-plane \( x \)) and its pullback connection \( \pi^*\nabla \) on \( S^1 \times BU(n) \). As in the case of the tangent bundle to a loop space, we can take an \( L^2 \) or pointwise connection \( \nabla^0 \) on \( \mathcal{E} \) by setting
\[
\nabla^0_X Y(\gamma)(\theta) = \pi^*\nabla_{X(\theta)} Y(\theta),
\]
for \( X \) a vector field along \( \gamma \) (i.e. a tangent vector in \( B \) at \( \gamma \)) and \( Y \) a local section of \( \mathcal{E} \). The curvature \( \Omega^0 \) acts pointwise and hence is a multiplication operator. In particular, its symbol is independent of \( \xi \).

Pick the distribution \( \delta = (1, +) \) on \( C^\infty(S^*S^1) = C^\infty(S^1 \times \{ \pm \partial_\theta \}) \): i.e.
\[
\delta(f(\theta, \partial_\theta), g(\theta, -\partial_\theta)) = \frac{1}{2\pi} \int_{S^1} f(\theta, \partial_\theta) d\theta.
\]
We claim that $[c^k_*(\Omega^0)]$ is nonzero in $H^{2k}(B; \mathbb{C})$. To see this, let $a = a_{2k} \in H_{2k}(BU(n), \mathbb{C})$ be such that $\langle c_k(\gamma_n), a \rangle = 1$. Define $c \in H^{2k}(B)$ to be $c = \beta_*a$, where $\beta : BU(n) \to L(S^1 \times BU(n))$ is given by $\beta(x)(\theta) = (\theta, x)$. Now

$$\langle [c^j_*(\Omega^0)], c \rangle = \langle [c^j_*(\Omega^0)], \beta_*a \rangle = \langle [\beta^*c^j_*(\Omega^0)], a \rangle,$$

(3.4)

since $\beta$ has degree one. For $\gamma \in L(S^1 \times BU(n))$, we have

$$c^j_*(\Omega^0)(\gamma) = \frac{1}{2\pi} \int_{S^1} \text{tr}(\sigma_0^{(\Omega^0)^k}(\gamma(\theta), \partial_x(\gamma(\theta)))) \ d\theta = \frac{1}{2\pi} \int_{S^1} \text{tr}(\Omega^k_{\gamma(\theta)}) \ d\theta. \quad (3.5)$$

For a tangent vector $X \in T_B BU(n)$, it is immediate that $\beta_*X \in T_{\gamma_*B}$ has $\beta_*X(\theta, x) = (0, X)$. Thus by (3.4),

$$\beta^*c^j_*(\Omega^0)(X_1, \ldots, X_{2k}) = \frac{1}{2\pi} \int_{S^1} \text{tr}(\Omega^k)(X_1, \ldots, X_{2k}) = \text{tr}(\Omega^k)(X_1, \ldots, X_{2k}). \quad (3.6)$$

Combining (3.5) and (3.6), we get

$$\langle [c^j_*(\Omega^0)], c \rangle = \langle [\text{tr}(\Omega^k)], a \rangle = 1.$$

In particular, the class $[c^j_*(\Omega^0)]$ must be non-zero.

**Theorem 3.3:** The cohomology classes $[c^j_*(\Omega)]$ are non-zero in general. In particular, the corresponding classes for the universal bundle $EG$ are nonzero in the cohomology of the classifying space $BG$, where $G$ is the gauge group of the trivial bundle $S^1 \times \mathbb{C}^n$ over $S^1$.

We have shown the first statement. To explain the second statement, note that although the structure group of $L\pi^*\gamma_n$ is the gauge group of the trivial bundle $S^1 \times \mathbb{C}^n$ over $S^1$, the curvature of the connection will take values in $Cl_0(S^1 \times \mathbb{C}^n)$ of this bundle. As a result, the classifying space is really $BCl_0^*$. However, the principal symbol map gives a retraction of $Cl_0^*$ onto the gauge group of the trivial bundle over $S^1 \times S^1$, which is just two copies of $G$. Thus $BCl_0^*$ is homotopy equivalent to $BG \coprod BG$, and each $[c_k^{(1,\pm)}]$ is non-zero in one copy of $BG$. The proof of the second statement depends on the existence of a universal connection on $EG$ over $BG$, and will be given in §4.

In fact, $BG$ equals $L_0 BU(n)$, the space of contractible loops on $BU(n) \coprod$. It is known that $H^*(BG, \mathbb{C})$ is a super-polynomial (i.e. super-commutative) algebra with one generator in each degree $k \in \{1, \ldots, 2n\}$. In analogy with finite dimensions, we conjecture that $[c^j_k]$ is a nonzero multiple of the generator in degree $2k$. For $k = 1$, this is clear.

We now outline a conjectured construction of geometric representatives of the odd generators in $H^*(BG)$. The tangent bundle $TLM$ of any loop space splits off a trivial line bundle, namely the span of $\gamma$ at the loop $\gamma$. Note that for any connection on a bundle over $LM$, we have

$$di^*_\gamma c^j_k(\Omega) = di^*_\gamma c^j_k(\Omega) + i^*_\gamma dc^j_k(\Omega) = L^*_\gamma c^j_k(\Omega), \quad (3.7)$$

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where $i$ is interior product and $L$ is Lie derivative. We state without (the elementary) proof that (3.7) implies
\[ di\dot{\gamma}_c f_k(\Omega) = \partial f_k(\Omega), \] (3.8)
where $\partial f$ is the derivative of $f$ as a distribution on $S^*S^1$. In particular, we see that $i\dot{\gamma}_c f_k(\Omega)$ are closed forms. It remains to be seen if $[i\dot{\gamma}_c f_k(\Omega)] \in H^{2k-1}(BG \bigsqcup BG)$ are non-zero.

We can also use (3.8) to understand the dependence of $[c f_k(\Omega)]$ on $f$. Since $f$ is a zero current on $S^*S^1$ and hence is trivially closed, and since exact zero currents $\partial f$ produce vanishing Chern classes by (3.8), we see that the space of classes
\[ \{ [c f_k(\Omega)] : f \in \mathcal{D}(S^*S^1) \} \in H^{2k}(BG) \]
is isomorphic to the zeroth cohomology group of complex currents on $S^*S^1$. (Here we are extending the usual confusion of functions $f$ and one-forms $fd\theta$ on $S^1$ to a confusion of zero- and one-currents.) This cohomology group is isomorphic to $H_0(S^*S^1)$, and is spanned by $(1, \pm)$. (The reader may wish to check directly from (3.8) that all $\delta$-function currents on one copy of $S^1$ produce the same cohomology class. A more interesting exercise is to show that these delta functions produce the same cohomology class as one of $(1, \pm)$ using Fourier series.)

**Remarks:** (1) The general case, where the gauge group is associated to the bundle $E$ over a closed manifold $M$, is more complicated. The cohomology of $BG$ is known, and in general has odd dimensional cohomology $[\Pi]$. We do not know at present which part of $H^*(BG)$ is spanned by $[c f_k(\Omega^E)]$, where we use the universal connection mentioned above. We also do not know how to produce geometric representatives of odd dimensional classes in $H^*(BG)$, nor do we know how the Chern classes depend on the distribution.

(2) In the loop group case, Freed showed [7] that the curvature $\Omega^s$ of the $H^s$ Levi-Civita connection is a $\Psi$DO of order $-1$ for $s > 1/2$. It is for this reason that we did not use this Levi-Civita connection in our construction; the Chern forms built from $\sigma_0$ trivially vanish.

### 3.3 Weighted first Chern forms on loop groups

In this subsection we treat the weighted first Chern forms of [13] for loop groups. In contrast to the traces in previous sections, the weighted traces are associated to the finite part of an asymptotic expansion, and do not produce closed characteristic forms in general. There is an important exception: for the $H^{1/2}$ metric on a loop group, Freed [7] showed that a conditional trace (recalled below) of the curvature form converges and equals the Kähler form. Thus this conditional first Chern form is certainly closed with non-zero cohomology class. In [8], it was shown that this curvature form is a weighted first Chern form.

We first give an alternative proof that the conditional first Chern form is closed (Proposition 3.5). We then show more generally that forms occurring as the most divergent term in an asymptotic expansion associated to weighted traces are closed under certain hypotheses.
(Theorem 3.4). In particular, we recognize the most divergent term in Proposition 3.4 as a symbol trace from §3.2, which we know leads to closed characteristic forms. The conditional first Chern form case is special precisely because the most divergent term and the finite part coincide on LG. Thus, Freed’s computation fits into the framework of weighted traces, which is strengthened in [14].

We first recall the definition of weighted traces. These are functionals on $Cl(M, E)$ defined for $Q \in \mathcal{E}(M, E)$ by $\text{tr}^Q(A) := \text{Tr}(Ae^{-\varepsilon Q})$ for any $\varepsilon > 0$. Recall that a $\Psi DO$ bundle $\mathcal{E}$ with structure group $Cl_0(M, E)$ has an associated bundle of algebras $Cl_{\leq 0}(\mathcal{E}) = \text{Ad } P^\varepsilon$ with fibers modeled on $Cl_{\leq 0}(M, E)$. A weight is a section $Q \in \Gamma(Cl(\mathcal{E}))$ with $Q$ elliptic with positive definite leading symbol and of constant order. These conditions are independent of local chart, since the transition maps are $\Psi DO$s. In particular, if $\{g_b\}$ is the transition map between two trivializations of $\mathcal{E}$ over $b$, then $Q_b$ transforms into $g_b^{-1}Q_bg_b$; the same holds for sections of $Cl(\mathcal{E})$. For $A \in \Gamma(Cl(\mathcal{E}))$, $\text{tr}^Q(A)$ is well-defined, since

$$\text{tr}^Q(g^{-1}A) = \text{Tr}(g^{-1}Ag^{-\varepsilon^{-1}Q}g) = \text{Tr}(g^{-1}Ag^{-\varepsilon^{-1}Q}g) = \text{tr}^Q(A).$$

(3.9)

We set $\text{tr}^Q(A)$ to be the finite part of $\text{tr}^Q(A)$ as $\varepsilon \to 0$. In other words, $\text{tr}^Q(A)$ is the coefficient of $\varepsilon^0$ in the asymptotic expansion (3.10). This is equivalent to taking the zeta function regularization $\text{Tr}(AQ^{-\varepsilon})|_{\varepsilon=0}$, provided $Q$ is invertible and (3.10) contains no log terms.

As a preliminary result, we show that the most divergent term in the asymptotics of $\text{tr}(Ae^{-\varepsilon Q})$ is given by an integral of the leading order symbol $\sigma_L(A)$ of $Q$ has leading order symbol $\sigma_L(Q)(x, \xi) = ||\xi||^k$ for some $k$. Here $A$ is a zeroth order $\Psi DO$ and $Q$ is a Laplacian-type operator. This folklore result follows from the analysis developed in [8], while the analysis in the following proof is hidden in the local nature of the Wodzicki residue.

**Proposition 3.4:** Let $A \in Cl_{\leq 0}(M, E)$ have integral order $a \geq -n, n = \dim(M)$, and let $Q$ be an elliptic $\Psi DO$ of order $q$ with positive scalar leading symbol $\sigma_L(Q)(x, \xi) = f(x, \xi)\text{Id}$. Let $c = c(n, a, q)$ be

$$c = \frac{\Gamma(n+a)}{q(2\pi)^n}, \text{ if } a \neq -n; \quad c = \frac{\dim(E)(n-1)!}{(2\pi)^n}, \text{ if } a = -n.\]$$

Then as $\varepsilon \to 0$,

$$\text{tr}(Ae^{-\varepsilon Q}) = c \int_{S^*M} \text{tr}((\sigma_a(A))(f(x, \xi))^{-\frac{a+a}{q}} \cdot \varepsilon^{-\frac{a+a}{q}} + o(\varepsilon^{-\frac{n+a}{q}})).$$

In particular, if $\sigma_L(Q)(x, \xi) = ||\xi||^k$ for some $k$, then

$$\text{tr}(Ae^{-\varepsilon Q}) = c \int_{S^*M} \text{tr}((\sigma_a(A)) \cdot \varepsilon^{-\frac{a+a}{q}} + o(\varepsilon^{-\frac{n+a}{q}})).$$

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Proof: We want to compute the coefficient $a_0(A, Q)$ in the known asymptotic expansion

$$\text{tr}(A e^{-\varepsilon Q}) = \sum_{j=0}^{a+n} a_j(A, Q)\varepsilon^{\frac{j-a-n}{q}} + b_0(A, Q) \log \varepsilon + O(1),$$

for general $A \in \text{Cl}(M, E)$ of order $a$, and with $a_j(A, Q), b_0(A, Q) \in \mathbb{C}$. The coefficient $b_0(A, Q)$ satisfies $b_0(A, Q) = -\frac{1}{q} \text{res}_w(A)$.

A Mellin transform yields

$$a_0(A, Q) = \text{Res}_{z=n+a} \Gamma\left(\frac{n+a}{q}\right) \text{tr}(AQ^{-z})$$

(the case $A = 1$ considered in [9, (12)] easily extends to a general $\Psi$DO $A$). Thus, for $A$ as in the hypothesis,

$$a_0(A, Q) = \frac{\Gamma\left(\frac{n+a}{q}\right)}{q \Gamma(n+a)} a_0(A, Q^{\frac{1}{q}}), \text{ if } a \neq -n; \quad a_0(A, Q) = a_0(A, Q^{\frac{1}{q}}), \text{ if } a = -n.$$

Thus it suffices to prove the formula for $Q_1$ of order one. Since $\text{ord}(AQ_1^{-(n+a)}) = -n$, (3.10) becomes

$$\text{tr}(AQ_1^{-(n+a)} e^{-\varepsilon Q_1}) = -\text{res}_w(AQ_1^{-(n+a)}) \log \varepsilon + O(1).$$

Differentiating this expansion $n + a$ times (valid since $n + a \geq 0$) with respect to $\varepsilon$, we get

$$\text{tr}(A e^{-\varepsilon Q_1}) \sim (n + a - 1)! \text{res}_w(AQ_1^{-(n+a)}) \varepsilon^{-(n+a)}.$$

The local formula for the Wodzicki residue yields:

$$\text{res}_w(AQ_1^{-(n+a)}) = \frac{1}{(2\pi)^n} \int_{S^\ast M} \text{tr}(\sigma_{-n}(AQ_1^{-(n+a)}))$$

$$= \frac{1}{(2\pi)^n} \int_{S^\ast M} \text{tr}\left(\sigma_a(A)\sigma_{-(n+a)}(Q_1^{-(n+a)})\right)$$

$$= \frac{\dim(E)}{(2\pi)^n} \int_{S^\ast M} \text{tr}(\sigma_a(A)) f(x, \xi)^{-(n+a)}.$$

Hence our original $Q$ has

$$a_0(A, Q) = \frac{\Gamma\left(\frac{n+a}{q}\right)}{q \Gamma(n+a)} a_0(A, Q^{\frac{1}{q}}) = c \int_{S^\ast M} \text{tr}(\sigma_a(A)) (f(x, \xi))^{-(a+n)/q},$$

if $a \neq -n$, and there is a similar formula for $a = -n$. \qed

We now show that the weighted first Chern forms are closed. We use the notation of the previous section.

**Proposition 3.5:** Let $\Omega = \Omega^{(1/2)}$ be the curvature of the Levi-Civita connection $\nabla = d + \theta^{(1/2)}$ for the $H^{1/2}$ metric on the loop group $LG$. Then the weighted first Chern form $\text{tr}^Q(\Omega)$ is closed.
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**Proof:** The conditional trace of the curvature in $[\mathcal{H}]$ is $\text{tr}(\text{tr}_{\text{Lie}}(\Omega))$, where $\text{tr}_{\text{Lie}}$ denotes the trace with respect to the Killing form in the Lie algebra of $G$, and the outer trace is the ordinary operator trace. In particular, $\text{tr}_{\text{Lie}}(\Omega)$ is a trace class $\Psi DO$ on the trivial $\mathbb{C}^{\dim(G)}$ bundle over $S^1$. It is shown in $[\mathfrak{M}]$ that $\text{tr}^Q(\Omega) = \text{tr}(\text{tr}_{\text{Lie}}(\Omega))$. Since $\text{tr}_{\text{Lie}}(\Omega)$ is trace class, we have

$$\text{tr}(\text{tr}_{\text{Lie}}(\Omega)) = \lim_{\varepsilon \to 0} \text{tr}(\text{tr}_{\text{Lie}}(\Omega)e^{-\varepsilon Q}) = \lim_{\varepsilon \to 0} \text{tr}(\Omega e^{-\varepsilon Q}).$$

Since $\Omega$ is a $\Psi DO$ of order $-1$ $[\mathfrak{M}$, Thm. 1.11], the previous proposition gives

$$\text{tr}^Q(\Omega) = \lim_{\varepsilon \to 0} \text{tr}(\Omega e^{-\varepsilon Q}) = a_0(\Omega, Q) = \frac{\dim(G)}{2\pi} \int_{S^*S^1} \text{tr}_{\text{Lie}}\sigma_{-1}(\Omega). \quad (3.11)$$

Thus

$$d\text{tr}^Q(\Omega) = \frac{\dim(G)}{2\pi} \int_{S^*S^1} \text{tr}_{\text{Lie}}\sigma_{-1}(d\Omega).$$

Since the connection one-form $\theta = \theta^{(1/2)}$ has order zero, $\sigma_{-1}([\theta, \Omega])$ equals the leading symbol $\sigma_L([\theta, \Omega])$, which as usual vanishes. Therefore

$$d\text{tr}^Q(\Omega) = \frac{\dim(G)}{2\pi} \int_{S^*S^1} \text{tr}_{\text{Lie}}\sigma_{-1}(\nabla\Omega) = 0.$$

**Remark:** It is not true that the Kähler class $[\text{tr}^Q(\Omega^{(1/2)})]$ is independent of the connection on $LG$. Indeed, replacing $\theta^{(1/2)}$ by $t\theta^{(1/2)}$, $t \in [0, 1]$, joins $\nabla$ to the trivial connection $d$, which of course has $[\text{tr}^Q(\Omega)] = 0$. The difficulty is that the order of the corresponding curvatures $\Omega_t$ jumps from $-1$ at $t = 1$ to zero for $t < 1$. This prevents one from fixing a symbol term as in the proof of Proposition 3.5 for all $t$.

We now generalize Proposition 3.5 to show that the most divergent term in an asymptotic series associated to a weighted trace is closed.

**Theorem 3.6:** Let $E \to B$ be a $\Psi DO$ bundle with a $\mathcal{C}^{\leq 0}$ connection $\nabla$. Let $Q \in \mathcal{C}^l(E)$ be a smooth family of weights of constant order $q$ and with scalar leading symbol $\sigma_L(Q_b)$ independent of $b \in B$. Let $A \in \Omega^k(B, \mathcal{C}^l(E))$ be a $\mathcal{C}^l(E)$-valued form whose order $a \in \mathbb{Z}$, $a \geq -n, n = \dim(M)$ is independent of $b \in B$, and such that $\nabla A = 0$. Write

$$\text{tr}(A e^{-\varepsilon Q}) = \sum_{j=0}^{a+n} a_j(A, Q) \varepsilon^{j-a-n} + b_0(A, Q) \log \varepsilon + O(1).$$

Then as elements of $\Omega^k(B, \mathcal{C})$,

(i) $\text{res}_w(A)$ is closed;

(ii) $a_0(A, Q)$ is closed, if $a > -n$;

(iii) $a_0(A, Q)$ is closed, if $a = -n$ and $\text{res}_w(A) = 0$.

Note that the condition on the leading symbol is independent of trivialization of $E$.
Proof: (i) This was shown in §3.1.

(ii) and (iii) Under the hypotheses, $a_0(A, Q)$ is the most divergent term in the asymptotic expansion. For $c$ as in Proposition 3.4, we have

$$
d a_0(A, Q) = d \left[ c \int_{S^*M} \text{tr} \left( \sigma_a(A) \right) \left( f(x, \xi) \right)^{- (n + a)/q} \right]
= c \int_{S^*M} \text{tr} \left( d \sigma_a(A) \right) \left( f(x, \xi) \right)^{- (n + a)/q}
= c \int_{S^*M} \text{tr} \left( \sigma_a(dA) \right) \left( f(x, \xi) \right)^{- (n + a)/q}
= c \int_{S^*M} \text{tr} \left( \sigma_a(dA + [\theta, A]) \right) \left( f(x, \xi) \right)^{- (n + a)/q}
= c \int_{S^*M} \text{tr} \left( \sigma_a(\nabla A) \right) \left( f(x, \xi) \right)^{- (n + a)/q}
= 0.
$$

Here we use the fact that $\nabla$ has non-positive order, so that

$$
\text{tr} \left( \sigma_a([\theta, A]) \right) = \text{tr} \sigma_L([\theta, A]) = \text{tr} \left( [\sigma_0(\theta), \sigma_a(A)] \right) = 0,
$$

if $[\theta, A]$ has expected order $a$. Finally, $\sigma_a([\theta, A]) = 0$ trivially if the order of $[\theta, A]$ is less than $a$.

We now apply Theorem 3.6 to $LG$ to recover Proposition 3.5. Let $\Omega = \Omega^{(1/2)}$, $\theta = \theta^{(1/2)}$. Note that by (3.11), $\text{tr} Q(\Omega)$ is the most divergent term $\alpha_0(\Omega, Q)$ (and the finite part) of $\text{tr} (Q e^{-\epsilon Q})$. For $s = 1/2$, $a = -n$ and by case (iii), we must verify that $	ext{res}_w(\Omega(X, Y)) = 0$, for tangent vectors $X, Y$ to $LG$. Now

$$
\text{res}_w(\Omega(X, Y)) = \text{res}_w([\theta(X), \theta(Y)]) - \text{res}_w(\theta[X, Y]) = -\text{res}_w(\theta[X, Y]).
$$

Setting $U = [X, Y]$, we have by (3.11)

$$
\text{res}_w(\theta(U)) = \frac{1}{2} \left( \text{res}_w(\text{ad}_U) + \text{res}_w(Q^{-1/2} \text{ad}_U Q^{1/2}) - \text{res}_w(Q^{-1/2} \text{ad}_{Q^{1/2} U}) \right) = -\frac{1}{2} \left( \text{res}_w(Q^{-1/2} \text{ad}_{Q^{1/2} U}) \right),
$$

since $\text{res}_w(\text{ad}_U) = \text{res}_w(Q^{-1/2} \text{ad}_{U} Q^{1/2})$, and the Wodzicki residue of this multiplication operator vanishes. Moreover,

$$
\text{res}_w(Q^{-1/2} \text{ad}_{Q^{1/2} U}) = \frac{1}{2\pi} \int_{S^*S^1} \text{tr}_{\text{Lie}} \left( \sigma_{-1}(Q^{-1/2} \text{ad}_{Q^{1/2} U}) \right) = \frac{1}{2\pi} \int_{S^*S^1} \sigma_{-1}(Q^{-1/2}) \text{tr}_{\text{Lie}}(\text{ad}_{Q^{1/2} U}) = 0.
$$

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Characteristic Classes on $\Psi DO$ bundles

Here we have used that $Q^{-1/2}$ has order $-1$, $\text{ad}_{Q^{1/2}U}$ is a multiplication operator, and $\text{tr}_{\text{Lie}}(\text{ad}_Z) = 0$ for all $Z$ in the Lie algebra.

Remark: If the order of $A$ is zero, Proposition 3.4 implies that the most divergent term $a_0(A, Q)$ is a symbol trace in the sense of §3.2. Thus Theorem 3.6 gives another proof that symbol traces produce closed forms.

4 Universal connections

In this section, we prove a Narasimhan-Ramanan theorem for universal connections on bundles with structure group given by the gauge group $G$ of a fixed Hermitian bundle $E$ over a closed manifold $M$. As explained in the last section, this justifies the second statement in Theorem 3.3.

We first show that the structure group of a $\Psi DO$ bundle reduces to a gauge group. We have the exact sequence at the (Lie) algebra level

$$0 \rightarrow \mathfrak{cl}_{<0} \rightarrow \mathfrak{cl}_{\leq 0} \xrightarrow{\sigma_0} C^\infty(S^*M, \text{End}(\pi^*E)) \rightarrow 0.$$ 

$\sigma_0$ is the zeroth order symbol map; because zero order symbols have homogeneity zero in the cotangent variable, these symbols are in fact functions on the unit cosphere bundle $S^*M$ with values in $\text{End}(E)$. This function space is the Lie algebra of the gauge group $C^\infty(S^*M, \text{Aut}(\pi^*E))$ of the pullback bundle $\pi^*E$ over $S^*M$, where $\pi : T^*M \rightarrow M$ is the projection. We exponentiate this sequence to obtain

$$1 \rightarrow 1 + \mathfrak{cl}_{<0} \rightarrow \mathcal{E}l^*_0 \xrightarrow{\alpha} C^\infty(S^*M, \text{Aut}(\pi^*E)) \rightarrow 1,$$

where $\alpha$ is a non-canonical splitting recalled below. Since $\mathfrak{cl}_{<0}$ is a vector space and hence contractible, and since it is easy to verify that this exact sequence is a fibration, we have a retraction of $\mathcal{E}l^*_0 = \mathfrak{cl}_{<0}$ onto $\alpha(C^\infty(S^*M, \text{Aut}(\pi^*E)))$.

We now show that in fact $\mathcal{E}l^*_0$ deformation retracts onto this image. This implies that the corresponding classifying spaces satisfy $B\mathcal{E}l^*_0 \sim BC^\infty(S^*M, \text{Aut}(\pi^*E))$. Thus we can take a gauge group as structure group, although it is the gauge group for a bundle over $S^*M$.

For the deformation retraction, it is standard that for $A \in \mathfrak{cl}^*_0$, we can construct $\tilde{B} \in \mathfrak{cl}^*_0$ with the total symbol of $\tilde{B}$ satisfying $\sigma_{\text{tot}}(\tilde{B}) = \sigma_0(\tilde{B}) = \sigma_0(A)$. In fact, if we fix a smooth function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ with $\psi \geq 0, \psi(0) = 0$, and $\psi(|\xi|) = 1$ for $|\xi| \geq 1$, then we can set $\tilde{B} = \alpha(A)$, where $\alpha : C^\infty(S^*M, \text{Aut}(\pi^*E)) \rightarrow \mathcal{B}$ is defined by

$$\alpha(\sigma)(f)(x) = \int_M e^{ix\cdot \xi} \psi(\xi) \sigma(x, \xi) \hat{f}(\xi) d\xi.$$

Here, as usual, we have omitted notation for the trivialization of $E$ and partitions of unity. This is the splitting map $\alpha$ in (4.1).
Set $B = \tilde{B} + P_B$, where $P_B$ is the orthogonal projection onto the kernel of $\tilde{B}$. Note that $\sigma_{\text{tot}}(B) = \sigma_0(B) = \sigma_0(A)$. $B$ is invertible, because it is an injective operator with index$(B) = \text{index}(A) = 0$. Thus we can write $A = B(1 + K)$, where $K$ is the compact operator $B^{-1}(A - B)$. The set of $\lambda \in \mathbb{C}$ for which $B(1 + \lambda K)$ is not injective is discrete, accumulates only at infinity, and does not contain 0 or 1. Since the same is true for $(1 + \lambda K^*)B^*$, the union of these two sets is a discrete set $S \subset \mathbb{C} \setminus \{0, 1\}$, and for $\mu \in \mathbb{C} \setminus S$, $B(1 + \mu K)$ is invertible. Fix a continuous curve $\gamma : [0, 1] \to \mathbb{C} \setminus S, \gamma(0) = 0, \gamma(1) = 1$, chosen so that $\gamma(t) = t$ except near points of $S$, where it avoids points of $S$ via e.g. a canonical small semicircle in the upper half plane. Then the family $A_t = B(1 + \gamma(t)K)$ provides a deformation retraction of $\mathcal{E}l^0_{\text{tot}}$ to $\mathcal{B} = \{B \in \mathcal{E}l^0_{\text{tot}} : \sigma_{\text{tot}}(B) = \sigma_0(B)\}$. Since elements of $\mathcal{B}$ are determined by a zeroth order symbol and a smoothing operator, or equivalently by such a symbol and a smooth kernel, we have

$$
\mathcal{B} = \alpha(C^\infty(S^*M, \text{Aut}(\pi^*E))) \times C^\infty(M \times M, E \otimes E).
$$

Since the set of smooth sections of a vector bundle is contractible, we see that $\mathcal{E}l^0_{\text{tot}}$ deformations retracts onto $\alpha(C^\infty(S^*M, \text{Aut}(\pi^*E)))$.

As a result, we may assume that our bundles have the gauge group $C^\infty(S^*M, \text{Aut}(\pi^*E))$ as structure group. We now begin the demonstration that the universal bundle for the structure group $\mathcal{G} = C^\infty(N, \text{Aut}(F))$ has a universal connection. Here the automorphism group of a fiber is $G = U(n)$, but the argument applies to the other classical groups. The reader should keep in mind our case $S^*M = N, \pi^*E = F$.

Let $X$ be a (for us always locally trivial) principal $\mathcal{G}$-bundle over a topological space $B$. $X$ induces a $G$ principal bundle $X'$ over $B \times N$ as follows. We first set up a 1-1 correspondence of local sections $\Gamma_{\text{loc}}(X) \leftrightarrow \Gamma_{\text{loc}}(X')$ by picking $U_i \subset B$ over which $X'|_{U_i} \approx U_i \times \mathcal{G}$. For $s \in \Gamma(U_i)$, we map $s \leftrightarrow u_s$, with $u_s(b, x) = s(b)(x)$. This agrees on overlaps: on $U_i \cap U_j$, we have $s_i(b) = g_{ij}(b)s_j(b), g_{ij}(b) \in \mathcal{G}$, and so

$$
u_{s,i}(b, x) = g_{ij}(b, x)u_{s,j}(b, x),$$

with $g_{ij}(b, x) \equiv g_{ij}(b)(x)$. Now we define $X'$ over $B \times N$ to be the $G$ bundle with transition functions $g_{ij}(b, x)$ over $(U_i \cap U_j) \times N$. In particular, we may associate a $G$ bundle $EG'$ over $BG \times N$ to the universal bundle $E\mathcal{G} \to BG$. Note that $E\mathcal{G} = (\pi_1)_*EG'$ for the projection $\pi_1 : BG \times N \to BG$.

We will use the following diagram to move from the universal connection on $EG$ to a universal connection on $E\mathcal{G}$.

$$
\begin{array}{ccc}
E\mathcal{G} & \simeq & (\pi_1)_*EG' \\
\downarrow & & \downarrow \\
BG & \overset{\pi_1}{\to} & BG \times N \\
& & \overset{\text{ev}}{\to} BG
\end{array}
$$

The middle isomorphism is contained in the proof of Lemma 4.1.

As a preliminary step, we show how to classify connections on $G$ bundles over $B \times N$. Recall that $BG = \text{Maps}_F(N, BG)$, the space of maps which induce $F$. This gives the
evaluation map $B\mathcal{G} \times N \to BG$, $ev(f, x) = f(x)$. Note that $ev^*EG$ restricted to $\{f\} \times N$ is precisely $f^*EG$, which is isomorphic to the unitary frame bundle $Fr(F)$.

Remark: By $\square_1$, $EG = Maps^G(Fr(F), EG)$, with the fiber over $f$ consisting of bundle maps lying over $f$. Here the superscript $G$ indicates that the maps are $G$-equivariant. As a result, $E\mathcal{G}$ consists of all isomorphisms of $Fr(F)$ with pullback bundles $f^*EG$. Since every connection on $Fr(F)$ is isomorphic to the pullback $f^*\nabla^u$ of the universal connection $\nabla^u$ on $EG$ for some $f \in B\mathcal{G}$ $\square_2$, we see that $E\mathcal{G}$ contains all connections on $Fr(F)$. Connections occur with repetitions if $\mathcal{G}$ does not act freely on $\mathcal{A}$, the space of all connections on $F$. (This is the difference between using $\mathcal{G}$ and the set $\mathcal{G}_0$ of gauge transformations which are the identity at a fixed point; $\mathcal{G}_0$ acts freely on $\mathcal{A}$, and so $E\mathcal{G}_0 = \mathcal{A}$.)

Let $(X', \nabla')$ be a $G$ bundle with connection over $B \times N$, and let $\beta : B \times N \to BG$ be a geometric classifying map – i.e. $\beta^*\nabla^u = \nabla'$, where we suppress the isomorphism between $X'$ and $\beta^*EG$. Set

$$
\beta_1 : B \to B\mathcal{G} = Maps_F(M, BG), \quad \beta_1(b)(x) = \beta(b, x).
$$

Thus $(\beta_1, \text{id}) : B \times N \to B\mathcal{G} \times N$. Since $ev \circ (\beta_1, \text{id}) = \beta$, we have

$$
X' = \beta^*EG = (\beta_1, \text{id})^*ev^*EG, \quad \nabla' = \beta^*\nabla^u = (\beta_1, \text{id})^*(ev^*\nabla^u).
$$

Thus the pair $(ev^*EG, ev^*\nabla^u)$ is a universal bundle with connection for these bundles.

We will need some topological converses. First, if $X \to B$ is a $\mathcal{G}$-bundle and if $\beta : B \times N \to BG$ classifies $X'$, then $\beta_1$ classifies $X$; i.e. $X \simeq \beta_1^*EG$. The proof is an easy check. Conversely, if $X \to B$ is classified by a map $\beta_1 : B \to B\mathcal{G}$, then $(\beta_1, \text{id}) : B \times N \to B\mathcal{G} \times N$ has $(\beta_1, \text{id})^*E\mathcal{G}' \simeq X'$. Indeed,

$$
(\beta_1, \text{id})^*E\mathcal{G}'|_{(b, x)} = E\mathcal{G}'|_{(\beta_1(b), x)} = Maps^G(Fr(F_x), EG|_{\beta_1(b)(x)})
= Aut(F_x, \gamma_{\beta_1(b)}^\infty|_{\beta_1(b)(x)}) = \beta_1^*EG|_b \text{ at } x.
$$

This last expression equals $X'|_b$ at $x$, which equals $X'|_{(b, x)}$.

We now use the fact that in this notation

$$
X' \simeq (\beta_1, \text{id})^*E\mathcal{G}' \simeq (\beta_1, \text{id})^*ev^*EG.
$$

to show that there exists a non-canonical isomorphism $\alpha : E\mathcal{G}' \to ev^*EG$. As a result, we get:

Lemma 4.1: Let $X$ be a $\mathcal{G}$ bundle over $B$, and let $X'$ be the associated $G$ bundle over $B \times N$. Then the pair $(E\mathcal{G}', \alpha^*ev^*\nabla^u)$ is geometrically classifying for $(X', \nabla')$, for any connection $\nabla'$ on $X'$.

Here $\alpha : B \times N \to B\mathcal{G} \times N$ is the map under $\alpha$.

Proof: Let $a : B\mathcal{G} \times N \to BG$ classify $E\mathcal{G}'$, and let $\beta : B \times N \to BG$ classify $X'$. Then as above

$$
X' \simeq (\beta_1, \text{id})^*ev^*EG.
$$

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By the work above, we also have $X' \simeq (\beta_1, \text{id})^*E\mathcal{G}'$ and $E\mathcal{G}' \simeq (\alpha_1, \text{id})^*\text{ev}^*EG$, so

$$X' \simeq (\beta_1, \text{id})^*(\alpha_1, \text{id})^*\text{ev}^*EG.$$  

The last two indented equations imply that

$$\text{ev} \circ (\beta_1, \text{id}) \sim \text{ev} \circ (\alpha_1 \beta_1, \text{id}).$$

Now if $f_0, f_1 : B \to B\mathcal{G}$ have $\text{ev} \circ (f_0, \text{id}) \sim \text{ev} \circ (f_1, \text{id})$, then $f_0 \sim f_1$. For we have $F_t = F(\cdot, t) : B \times N \times [0, 1] \to B\mathcal{G}$ with $F_0(b, x) = \text{ev} \circ (f_0, \text{id})(b, x) = f_0(b)(x)$, and $F_1(b, x) = f_1(b)(x)$. Define $\tilde{f}_t : B \to B\mathcal{G}$ by $\tilde{f}_t(b)(x) = F_t(b, x)$. Then $\tilde{f}_0 = f_0, \tilde{f}_1 = f_1$.

Thus $\beta_1 \sim a_1 \beta_1$ for all maps $\beta_1 : B \to B\mathcal{G}$. Setting $\beta_1 = \text{id}$ (which corresponds to taking $\beta = \text{ev}$), we get $\text{id} \sim a_1$. Thus $E\mathcal{G}' \simeq (\alpha_1, \text{id})^*\text{ev}^*EG \simeq \text{ev}^*EG$, and we set $\alpha = \text{id}$ to be the composition of these isomorphisms.

By this lemma, we expect that $EG$ will also have a universal connection for $\mathcal{G}$ bundles with connection. We need to be able to go back and forth between connections on a $\mathcal{G}$ bundle $X \to B$ and connections on the associated $X' \to B \times N$. One direction is clear: given a connection $\nabla'$ on $X'$, we define a connection $\nabla = \nabla'$ on $X$ by

$$\nabla_{\mathbf{Z}s}(b)(x) = \nabla'_{(\mathbf{Z}, 0)}u_s(b, x),$$

(4.2)

for $\mathbf{Z} \in TB$.

The only difficulty in going from $\nabla$ to $\nabla'$ is defining differentiations in $N$ directions. So given a connection $\nabla$ on $X$, we use (4.2) to define $\nabla'_{(\mathbf{Z}, 0)}$, and for $\mathbf{Y} \in TN$, we set

$$\nabla'_{(0, \mathbf{Y})}u_s(b, x) = \nabla^b_{\mathbf{Y}}u_s(b, x),$$

(4.3)

where $\nabla^b$ is an arbitrary (but smoothly varying in $b$) connection on $X'|_{\{b\} \times N}$. Note that $\nabla \mapsto \nabla' \mapsto \nabla$ is the identity, but $\nabla' \mapsto \nabla \mapsto \nabla'$ is not defined, due to this arbitrariness.

As $\beta = (\beta_1, \text{id}) : B \times N \to B\mathcal{G}$ runs through all maps classifying $X'$ (equivalently, as $\beta_1 : B \to B\mathcal{G}$ runs through all maps classifying $X$), and as $\alpha$ runs through all isomorphisms of $X'$ and $\beta^*\text{ev}^*EG$, we know that $\alpha^{-1}\beta^*\text{ev}^*\nabla^u$ runs through all connections on $X'$. Here $\alpha^{-1}\beta^*\text{ev}^*\nabla^u u_s$ means $\alpha^{-1}(\beta^*\text{ev}^*\nabla^u (\alpha \circ u_s))$.

We claim that $\overline{\nabla} \equiv (\text{ev}^*\nabla^u)_{\text{ext}}$ is a universal connection on $E\mathcal{G} \to B\mathcal{G}$. Given a connection $\nabla$ on $X$, construct $\nabla'$ as in (4.2), (4.3), and write $\nabla' = \alpha^{-1}\beta^*\text{ev}^*\nabla^u$ for some $\alpha, \beta$. Then

$$\nabla'_{(\mathbf{Z}, 0)}u_s(b, x) = \alpha^{-1}(\beta^*\text{ev}^*\nabla^u_{(\mathbf{Z}, 0)}(\alpha \circ u_s))(b, x).$$

For $\overline{\beta_1} : \beta_1^*E\mathcal{G} \to E\mathcal{G}$ the bundle map corresponding to $\beta_1$, we have

$$\alpha^{-1}(\beta_1^*\overline{\nabla})Z(\alpha \circ s)(b)(x) = \alpha^{-1}(\beta_1^*\overline{\nabla})(\beta_1 \circ \alpha \circ s)(b)(x)$$

$$= \alpha^{-1}(\text{ev}^*\nabla^u )((\beta_1 \circ \alpha \circ s)u_{\beta_1 \circ \alpha \circ s} )(b, x)$$

$$= \alpha^{-1}(\text{ev}^*\nabla^u )_{\beta_1 \circ \alpha \circ s} (b, x)$$

$$= \alpha^{-1}(\beta_1^*\text{ev}^*\nabla^u )_{\beta_1 \circ \alpha \circ s} (b, x)$$

$$= \alpha^{-1}(\beta_1^*\text{ev}^*\nabla^u )_{\beta_1 \circ \alpha \circ s} (b, x).$$

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In the last two lines, we have used \( u_{\alpha_0 \circ s} = \beta \circ u_{\alpha_0} \), \( u_{\alpha_0} = \alpha \circ u_s \), for \( \alpha \circ s \in \Gamma(B, B_1^*E\mathcal{G}) \), which we leave as annoying exercises. As a result, we have:

**Theorem 4.2:** The natural extension of \( ev^*\nabla^u \) to a connection on \( E\mathcal{G} \to B\mathcal{G} \) is a universal connection for \( \mathcal{G} \) bundles.

As a final remark, we show that the bundle \( E\mathcal{G}_0' \) is isomorphic to the universal bundle \( \tilde{P} \) in gauge theory [6, Ch. 5]. Here \( \mathcal{G}_0 \) is the group of gauge transformations which are the identity at a fixed point of \( N \), and \( \tilde{P} = (A \times P)/\mathcal{G} \) as a \( \mathcal{G} \)-bundle over \( (A \times N)/\mathcal{G} = B\mathcal{G} \times N \) with the action \( g \cdot (\mathcal{V}, v) = (g \cdot \mathcal{V}, g^{-1}xv) \) for \( v \in P_x \). Recall that \( A \) is the space of connections on the principal bundle \( P = \text{Fr}(F) \).

**Lemma 4.3:** \( E\mathcal{G}_0' \sim \tilde{P} \) as bundles over \( B\mathcal{G}_0 \times N \).

**Proof:** We have shown that \( E\mathcal{G} \simeq ev^*E\mathcal{G} \), and the same argument works for \( \mathcal{G}_0' \).

We drop the subscript 0. Since \( E\mathcal{G} = \text{Maps}^G(P, E\mathcal{G}) \), every element of which covers some \( f : N \to B\mathcal{G} \), we can write

\[
E\mathcal{G} = \{ (\alpha, f) : \alpha \in \text{Iso}(P, f^*E\mathcal{G}) \}.
\]

There is a map \( \Psi : E\mathcal{G} \to \mathcal{A} \), thought of as the space of all connections on \( E \), given by \( \Psi(\alpha, f) = \alpha \cdot (f^*\nabla^u) = \alpha^{-1} f^*\nabla^u \alpha \). This map is surjective by the finite dimensional universal connection theorem. We define the trivial equivalence relation \( (\alpha, f) \sim (\beta, h) \Leftrightarrow \Psi(\alpha, f) = \Psi(\beta, h) \). Thus the space of all connections \( \mathcal{A} \) is in one-to-one correspondance with \( E\mathcal{G}/\sim \).

From its definition,

\[
\tilde{P} = \frac{E\mathcal{G}}{(\alpha, f) \sim (\beta, h)} \times P, \\
\tilde{P}|_{(f, x)} = \{ ([\alpha_x, f], v) : \alpha_x \text{ is the restriction of } \alpha \in \text{Iso}(P, f^*E\mathcal{G}) \text{ to } x \},
\]

where the equivalence class \([\alpha_x, f], v\) is defined by \( ([\alpha_x, f], v) \sim ([\alpha_x g_x, f], g_x^{-1}v) \).

We define a map \( \tilde{P} \to ev^*E\mathcal{G} \) by \( ([\alpha_x, f], v) \mapsto \alpha_x(v)|_f \). This map is well defined and surjective. For injectivity, say \( \alpha_x(v) = \alpha'_x(v') \). There exists \( g \) with \( \alpha'_x = \alpha_x g \), so \( \alpha'_x(v') = (\alpha_x g)(v') = \alpha_x(gv') \). Since \( \alpha_x \) is an isomorphism, \( v = gv' \). Thus

\[
[\alpha'_x, f], v'] = [\alpha_x g, f], g^{-1}v] = [\alpha_x, f], v].
\]

\[\square\]
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