Space Efficient Breadth-First and Level Traversals of Consistent Global States of Parallel Programs

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Abstract. Enumerating consistent global states of a computation is a fundamental problem in parallel computing with applications to debugging, testing and runtime verification of parallel programs. Breadth-first search (BFS) enumeration is especially useful for these applications as it finds an erroneous consistent global state with the least number of events possible. The total number of executed events in a global state is called its rank. BFS also allows enumeration of all global states of a given rank or within a range of ranks. If a computation on \( n \) processes has \( m \) events per process on average, then the traditional BFS (Cooper-Marzullo and its variants) requires \( O(m n - 1) \) space in the worst case, whereas our algorithm performs the BFS requires \( O(m^2 n^2) \) space. Thus, we reduce the space complexity for BFS enumeration of consistent global states exponentially, and give the first polynomial space algorithm for this task. In our experimental evaluation of seven benchmarks, traditional BFS fails in many cases by exhausting the 2 GB heap space allowed to the JVM. In contrast, our implementation uses less than 60 MB memory and is also faster in many cases.

1 Introduction

Parallel programs are not only difficult to design and implement, but once implemented are also difficult to debug and verify. The technique of predicate detection \([16,11]\) is helpful in verification of these implementations as it allows inference based analysis to check many possible system states based on one execution trace. The technique involves execution of the program, and modeling of its trace as a partial order. Then all possible states of the model that are consistent with the partial order are visited and evaluated for violation of any constraints/invariants. A large body of work uses this approach to verify distributed applications, as well as to detect data-races and other concurrency related bugs in shared memory parallel programs \([10,13,18,22]\). Finding consistent global states of an execution also has critical applications in snapshotting of modern distributed file systems \([11,20]\).

A fundamental requirement for this approach is the traversal of all possible consistent global states, or consistent cuts, of a parallel execution. Let us call the execution of a parallel program a computation. The set of all consistent cuts of a computation can be represented as a directed acyclic graph in which each
vertex represents a consistent cut, and the edges mark the transition from one
global state to another by executing one operation. Moreover, this graph has a
special structure: it is a distributive lattice [23]. Multiple algorithms have been
proposed to traverse the lattice of consistent cuts of a parallel execution. Cooper
and Marzullo’s algorithm [11] starts from the source — a consistent cut in which
no operation has been executed by any process — and performs a breadth-
first-search (BFS) visiting the lattice level by level. Alagar and Venkatesan’s
algorithm [2] performs a depth-first-search (DFS) traversal of the lattice, and
Ganter’s algorithm [14] enumerates global states in lexical order.

The BFS traversal of the lattice is particularly useful in solving two key
problems. First, suppose a programmer is debugging a parallel program to find
a concurrency related bug. The global state in which this bug occurs is a counter-
example to the programmer’s understanding of a correct execution, and we want
to halt the execution of the program on reaching the first state where the bug
occurs. Naturally, finding a small counter example is quite useful in such cases.
The second problem is to check all consistent cuts of given rank(s). For example,
a programmer may observe that her program crashes only after $k$ events have
been executed, or while debugging an implementation of Paxos [21] algorithm,
she might only be interested in analyzing the system when all processes have
sent their promises to the leader. Among the existing traversal algorithms, the
BFS algorithm provides a straightforward solution to these two problems. It is
guaranteed to traverse the lattice of consistent cuts in a level by level manner
where each level corresponds to the total number of events executed in the
computation. This traversal, however, requires space proportional to the size of
the biggest level of the lattice which, in general, is exponential in the size of the
computation.

In this paper, we present a new algorithm to perform BFS traversal of the
lattice in space that is polynomial in the size of the computation. In short, the
contribution of this paper are:

- For a computation on $n$ processes such that each process has $m$ events on
  average, our algorithm requires $O(m^2 n^2)$ space in the worst case, whereas
  the traditional BFS algorithm requires $O(\frac{m^2 n^2}{n-1})$ space (exponential in $n$).
- Our evaluation on seven benchmark computations shows the traditional BFS
  runs out of the maximum allowed 2 GB memory for three of them, whereas
  our implementation can traverse the lattices by using less than 60 MB mem-
  ory for each benchmark.

The exponential reduction in space is sometimes at the cost of a loss in time
required to perform the BFS traversal. Our analysis in experimental results

2 Background

We model a computation $P = (E, \rightarrow)$ on $n$ processes $\{P_1, P_2, \ldots, P_n\}$ as a partial
order on the set of events, $E$. The events are ordered by Lampport’s happened-
before ($\rightarrow$) relation [20]. This partially ordered set (poset) of events is partitioned
into chains:
Definition 1 (Chain Partition) A chain partition of a poset places every element of the poset on a chain that is totally ordered. Formally, if \( \alpha \) is a chain partition of poset \( P = (E, \rightarrow) \) then \( \alpha \) maps every event to a natural number such that
\[
\forall x, y \in E : \alpha(x) = \alpha(y) \Rightarrow (x \rightarrow y) \lor (y \rightarrow x).
\]

Generally, a computation on \( n \) processes is partitioned into \( n \) chains such that the events executed by process \( P_i \) \((1 \leq i \leq n)\) are placed on \( i \)th chain.

Mattern \[23\] and Fidge \[12\] proposed vector clocks, an approach for timestamping events in a computation such that the happened-before relation can be tracked. For a program on \( n \) processes, each event’s vector clock is a \( n \)-length vector of integers. Note that vector clocks are dependent on chain partition of the poset that models the computation. For an event \( e \), we denote \( e.V \) as its vector clock.

Throughout this paper, we use the following representation for interpreting chain partitions and vector clocks: if there are \( n \) chains in the chain partition of the computation, then the lowest chain (process) is always numbered 1, and the highest chain being numbered \( n \). A vector clock on \( n \) chains is represented as a \( n \)-length vector: \( [c_n, c_{n-1}, ..., c_i, ..., c_2, c_1] \) such that \( c_i \) denotes the number of events executed on process \( P_i \).

Hence, if event \( e \) was executed on process \( P_i \), then \( e.V[i] \) is \( e \)'s index (starting from 1) on \( P_i \). Also, for any event \( f \) in the computation: \( e \rightarrow f \Leftrightarrow \forall j : e.V[j] \leq f.V[j] \land \exists k : e.V[k] < f.V[k] \). A pair of events, \( e \) and \( f \), is concurrent iff \( e \not\rightarrow f \lor f \not\rightarrow e \). We denote this relation by \( e || f \). Fig. 1a shows a sample computation with six events and their corresponding vector clocks. Event \( b \) is the second event on process \( P_1 \), and its vector clock is \([0, 2]\). Event \( g \) is the third event on \( P_2 \), but it is preceded by \( f \), which in turn is causally dependent on \( b \) on \( P_1 \), and thus the vector clock of \( g \) is \([3, 2]\).

![Diagram](image)

**Fig. 1:** A computation with vector clocks of events, and its consistent cuts

Definition 2 (Consistent Cut) Given a computation \( (E, \rightarrow) \), a subset of events \( C \subseteq E \) forms a consistent cut if \( C \) contains an event \( e \) only if it contains all events that happened-before \( e \). Formally, \((e \in C) \land (f \rightarrow e) \implies (f \in C)\).

A consistent cut captures the notion of a possible global state of the system at some point during its execution \[6\]. Consider the computation shown in Fig. 1a. The subset of events \( \{a, b, e\} \) forms a consistent cut, whereas the subset \( \{a, e, f\} \) does not; because \( b \rightarrow f \) (\( b \) happened-before \( f \)) but \( b \) is not included in...
the subset.

**Vector Clock Notation of Cuts:** So far we have described how vector clocks can be used to time-stamp events in the computation. We also use them to represent cuts of the computation. If the computation is partitioned into \( n \) chains, then for any cut \( G \), its vector clock is an \( n \)-length vector such that \( G[i] \) denotes the number of events from \( P_i \) included in \( G \). Note that in our vector clock representation the events from \( P_i \) are at the \( i^{th} \) index from the right.

For example, consider the state of the computation in Fig. 1a when \( P_1 \) has executed events \( a \) and \( b \), and \( P_2 \) has only executed event \( e \). The consistent cut for this state, \( \{a, b, e\} \), is represented by \([1, 2]\). Note that cut \([2, 1]\) is not consistent, as it indicates execution of \( f \) on \( P_2 \) without \( b \) being executed on \( P_1 \).

The computation in Fig. 1a has twelve consistent cuts; and the lattice of these consistent cuts (in their vector clock representation) is shown in Fig. 1b.

**Rank of a Cut:** Given a cut \( G \), we define \( \text{rank}(G) = \sum G[i] \). The rank of a cut corresponds to the total number of events, across all processes, that have been executed to reach the cut.

In Fig. 1b there is one source cut \([0, 0]\) with rank 0, then there are two cuts each of ranks 1 to 5, and finally there is one cut \([3, 3]\) has rank 6.

### 2.1 Breadth-First Traversal of Lattice of Consistent Cuts

Consider a parallel computation \( P = (E, \rightarrow) \). The lattice of consistent cuts, \( C(E) \), of \( P \) is a DAG whose vertices are the consistent cuts of \( (E, \rightarrow) \), and there is a directed edge from vertex \( u \) to vertex \( v \) if state represented by \( v \) can be reached by executing one event on \( u \); hence we also have \( \text{rank}(v) = \text{rank}(u) + 1 \). The source of \( C(E) \) is the empty set: a consistent cut in which no events have been executed on any process. The sink of this DAG is \( E \): the consistent cut in which all the events of the computation have been executed. Breadth-first search (BFS) of this lattice starts from the source vertex and visits all the cuts of rank 1; it then visits all the cuts of rank 2 and continues in this manner till reaching the last consistent cut of rank \( |E| \). For example, in Fig. 1b the BFS algorithm will traverse cuts in the following order: \([0, 0], [0, 1], [1, 0], [0, 2], [1, 1], [0, 3], [1, 2], [1, 3], [2, 2], [2, 3], [3, 2], [3, 3] \).

The standard BFS on a graph needs to store the vertices at distance \( d \) from the source to be able to visit the vertices at distance \( d + 1 \) (from the source). Hence, in performing a BFS on \( C(E) \) we are required to store the cuts of rank \( r \) in order to visit the cuts of rank \( r + 1 \). Observe that in a parallel computation there may be exponentially many cuts of rank \( r \). Thus, traversing the lattice \( C(E) \) requires space which is exponential in the size of input. The optimized vector clock based BFS traversal takes \( O(n^2) \) time per cut [15], where \( n \) is the number of processes in the computation.

### 2.2 Related Work

Cooper and Marzullo [11] gave the first algorithm for global states enumeration which is based on breadth first search (BFS). Let \( i(P) \) denote the total number
of consistent cuts of a poset \( P \). Cooper-Marzullo algorithm requires \( O(n^2 \cdot i(P)) \) time, and exponential space in the size of the input computation. The exponential space requirement is due to the standard BFS approach in which consistent cuts of rank \( r \) must be stored to traverse the cuts of rank \( r + 1 \).

There is also a body of work on enumeration of consistent cuts in order different than BFS. Alagar and Venkatesan [3] presented a depth first algorithm using the notion of global interval which reduces the space complexity to \( O(|E|) \). Steiner [28] gave an algorithm that uses \( O(|E| \cdot i(P)) \) time, and Squire [27] further improved the computation time to \( O(\log |E| \cdot i(P)) \). Pruesse and Ruskey [25] gave the first algorithm that generates global states in a combinatorial Gray code manner. The algorithm uses \( O(|E| \cdot i(P)) \) time and can be reduced to \( O(\Delta(P) \cdot i(P)) \) time, where \( \Delta(P) \) is the in-degree of an event; however, the space grows exponentially in \( |E| \). Later, Jegou et al. [19] and Habib et al. [17] improved the space complexity to \( O(n \cdot |E|) \).

Ganter [14] presented an algorithm, which uses the notion of lexical order, and Garg [15] gave the implementation using vector clocks. The lexical algorithm requires \( O(n^2 \cdot i(P)) \) time but the algorithm itself is stateless and hence requires no additional space besides the poset. Paramount [8] gave a parallel algorithm to traverse this lattice in lexical order, and QuickLex [7] provides an improved implementation for lexical traversal that takes \( O(n \cdot \Delta(P) \cdot i(P)) \) time, and \( O(n^2) \) space overall.

3 Uniflow Chain Partition

A uniflow partition of a computation’s poset \( P = (E, \rightarrow) \) is its partition into \( n_u \) chains \( \{P_i \mid 1 \leq i \leq n_u \} \) such that no element (event of \( E \)) in a higher numbered chain is smaller than any element in lower numbered chain; that is if any event \( e \) is placed on a chain \( i \) then all causal dependencies of \( e \) must be placed on chains numbered lower than \( i \). For poset \( P = (E, \rightarrow) \), chain partition \( \mu \) is uniflow if

\[
\forall x, y \in P : \mu(x) < \mu(y) \Rightarrow \neg(y \not\rightarrow x)
\]

Visually, in a uniflow chain partition all the edges, capturing happened-before relation, between separate chains always point upwards because their dependencies — elements of poset that are smaller — are always placed on lower chains. Fig. 2 shows two posets with uniflow partition. Whereas Fig. 3 shows two posets with partitions that do not satisfy the uniflow property. The poset in Fig. 3(a) can be transformed into a uniflow partition of three chains as shown in Fig. 3(b). Similarly, Fig. 3(c) can be transformed into a uniflow partition of two chains shown in Fig. 3(d). Observe that:

**Lemma 1** Every poset has at least one uniflow chain partition.
Fig. 3: Posets in (a) and (c) are not in uniflow partition: but (b) and (d) respectively are their equivalent uniflow partitions.

**Proof.** Any total order derived from the poset is a uniflow chain partition in which each element is a chain by itself. In this trivial uniflow chain partition the number of chains is equal to the number of elements in the poset.

The structure of uniflow chain partitions can be used for efficiently obtaining consistent cuts of larger ranks.

**Lemma 2 (Uniflow Cuts Lemma)** Let \( P \) be a poset with a uniflow chain partition \( \{ P_i \mid 1 \leq i \leq n_u \} \), and \( G \) be a consistent cut of \( P \). Then any \( H_k \subseteq P \) for \( 1 \leq k \leq n_u \) is also a consistent cut of \( P \) if it satisfies:

\[
\forall i: k < i \leq n_u : H_k[i] = G[i], \text{ and }
\forall i : 1 \leq i \leq k : H_k[i] = |P_i|.
\]

**Proof.** Using Equation 1, we exploit the structure of uniflow chain partitions: the causal dependencies of any element \( e \) lie only on chains that are lower than \( e \)'s chain. As \( G \) is consistent, and \( H_k \) contains the same elements as \( G \) for the top \( n_u - k \) chains, all the causal dependencies that need to be satisfied to make \( H_k \) have to be on chain \( k \) or lower. Hence, including all the elements from all of the lower chains will naturally satisfy all the causal dependencies, and make \( H_k \) consistent.

For example, in Fig. 2(b), consider the cut \( G = [1, 2, 1] \) that is a consistent cut of the poset. Then, picking \( k = 1 \), and using Lemma 2 gives us the cut \([1, 2, 3]\) which is consistent; similarly choosing \( k = 2 \) gives us \([1, 3, 3]\) that is also consistent. Note that the claim may not hold if the chain partition does not have uniflow property. For example, in Fig. 3(c), \( G = [2, 2] \) is a consistent cut. The chain partition, however, is not uniflow and thus applying the Lemma with \( k = 1 \) gives us \([2, 3]\) which is not a consistent cut as it includes the third event on \( P_1 \), but not its causal dependency — the third event on \( P_2 \).

### 3.1 Finding a Uniflow Partition

The problem of finding a uniflow chain partition is a direct extension of finding the *jump number* of a poset [9, 29]. Multiple algorithms have been proposed to find the jump number of a poset; which in turn arrange the poset in a uniflow chain partition. Finding an optimal (smallest number of chains) uniflow chain partition of a poset is a hard problem [9, 5]. Bianco et al. [5] present a heuristic
algorithm to find a uniflow partition, and show in their experimental evaluation that in most of the cases the resulting partitions are relatively close to optimal. We use a vector clock based online algorithm to find a uniflow partition for a computation. The details of this algorithm are given in Appendix B. Note that we need to re-generate vector clocks of the events for the uniflow partition. This is a simple task using existing vector clock implementation techniques, and we omit these details.

4 Polynomial Space Breadth-First Traversal of Consistent Cuts

BFS traversal of the lattice of consistent cuts of any poset can be performed in space that is polynomial in the size of the poset. We do so by first obtaining the poset’s uniflow chain partition, and then using this partition for traversal of cuts in increasing order of ranks. We start from the empty cut, and then traverse all consistent cuts of rank 1, then all consistent cuts of rank 2 and so on. For rank \( r, 1 \leq r \leq |E| \), we traverse the consistent cuts in the following lexical order:

\[
\text{Definition 3 (Lexical Order on Consistent Cuts)} \quad \text{Given any chain partition of poset } P \text{ that partitions it into } n \text{ chains, we define a total order called lexical order on all consistent cuts of } P \text{ as follows. Let } G \text{ and } H \text{ be any two consistent cuts of } P. \text{ Then, } G <_l H \equiv \exists k : (G[k] < H[k]) \wedge (\forall i : n \geq i > k : G[i] = H[i])
\]

Recall from our vector clock notation (Section 2) that the right most entry in the vector clock is for the least significant (lowest) chain. Consider the poset with a non-uniflow chain partition in Fig. 4(a). The vector clocks of its events are shown against the four events. The lexical order on the consistent cuts of this chain partition is: \([0,0] <_l [0,1] <_l [1,0] <_l [1,1] <_l [1,2] <_l [2,1] <_l [2,2]\). For the same poset, Fig. 4(b) shows the equivalent uniflow partition, and the corresponding vector clocks. The lexical order on the consistent cuts for this uniflow chain partition is: \([0,0,0] <_l [0,0,1] <_l [0,1,0] <_l [0,1,1] <_l [0,2,1] <_l [1,1,1] <_l [1,2,1]\).

Note that the number of consistent cuts remains same for both of these chain partitions, and there is a one-to-one mapping between the consistent cuts in the two partitions. Algorithm 1 shows the steps of our BFS traversal using a computation in a uniflow chain partition. From Lemma 1, we know that every poset has a uniflow chain partition. Recall that the vector clocks of the events depend on the chain partition of the poset. Thus, in generating this input we need two pre-processing steps: (a) finding a uniflow partition, and (b) regenerating vector clocks for this partition. For example, given a computation on two
Algorithm 1 TraverseBFSUniflow($P$)

Input: A poset $P = (E, \rightarrow)$ that has been partitioned into a uniflow chain partition of $n_u$ chains, and the vector clock of the events have been regenerated for this partition.

1: $G = \text{new int}[n_u]$  // initial consistent cut
2: enumerate($G$)  // evaluate the predicate on empty cut $G$.
3: for ($r = 1; r \leq |E|; r++$) do
4:   // make $G$ lexically smallest cut of given rank
5:   $G = \text{GetMinCut}(G, r)$
6: while $G \neq \text{null}$ do
7:   enumerate($G$)  // evaluate the predicate on $G$.
8:   // find the next bigger lexical cut of same rank
9:   $G = \text{GetSuccessor}(G, r)$

processes shown in Fig. 4(a), we will first convert it to the computation shown in Fig. 4(b). These steps are performed only once for a computation, and are relatively inexpensive in comparison to the traversal of lattice. For each rank $r$, $1 \leq r \leq |E|$, Algorithm 1 first finds the lexically smallest consistent cut at of rank $r$. This is done by the GetMinCut (shown in Algorithm 2) routine that returns the lexically smallest consistent cut of $P$ bigger than $G$ of rank $r$. For example, in Fig. 5, GetMinCut([0, 0, 0], 4) returns [0, 1, 3]. Given a consistent cut $G$ of rank $r$, we repeatedly find the next lexically bigger consistent cut of rank $r$ using the routine GetSuccessor given in Algorithm 3. For example, in Fig. 5, GetSuccessor([0, 0, 3], 3) returns the next lexically smallest consistent cut [0, 1, 2].

The GetMinCut routine on poset $P$ assumes that the rank of $G$ is at most $r$ and that $G$ is a consistent cut of the $P$. It first computes $d$ as the difference between $r$ and the rank of $G$. We need to add $d$ elements to $G$ to find the smallest consistent cut of rank $r$. We exploit the Uniflow Cut Lemma (Lemma 2) by adding as many elements from the lowest chain as possible. If all the elements from the lowest chain are already in $G$, then we continue with the second lowest chain, and so on. For example in Fig. 5 consider finding smallest consistent cut of rank 5 starting from $G = [0, 0, 2]$. In this case, we add all three elements from $P_1$ to reach [0, 0, 3], and then add first two elements from $P_2$ to get the answer as [0, 2, 3].

Algorithm 2 GetMinCut($G, r$)

Input: $G$: a consistent cut of poset $P$ from Algorithm 1
Output: Smallest consistent cut of rank $r$ that is lexically greater than or equal to $G$.
1: $d = r - \text{rank}(G)$  // difference in ranks
2: for ($j = 1; j \leq n_u; j = j + 1$) do
3:   if $d \leq |P_j| - G[j]$ then
4:     $G[j] = G[j] + d$
5:     return $G$
6: else  // take all the elements from chain $j$
7:     $G[j] = G[j] + |P_j|$
8:     $d = d - |P_j|$
The GetSuccessor routine (Algorithm 3) finds the lexical successor of \( G \) at rank \( r \). The approach for finding a lexical successor is similar to counting numbers in a decimal system: if we are looking for successor of 2199, then we can’t increment the two 9s (as we are only allowed digits 0-9), and hence the first possible increment is for entry 1. We increment it to 2, but we must now reset the entries at lesser significant digits. Hence, we reset the two 9s to 0s, and get the successor as 2200.

In our GetSuccessor routine, we start at the second lowest chain in a uniflow poset, and if possible increment the cut by one event on this chain. We then reset the entries on lower chains, and then make the cut consistent by satisfying all the causal dependencies. If the rank of the resulting cut is less than or equal to \( r \), then calling the GetMinCut routine gives us the lexical successor of \( G \) at rank \( r \).

Line 1 copies cut \( G \) in \( K \). The for loop covering lines 2–13 searches for an appropriate element not in \( G \) such that adding this element makes the resulting consistent cut lexically greater than \( G \). We start the search from chain 2, instead of chain 1, because for a non-empty cut \( G \) adding any event from the lowest chain to \( G \) will only increase \( G \)’s rank as there are no lower chains to reset. Line 3 checks if there is any possible element to add in \( P_i \). If yes, then lines 4–6 increment \( K \) at chain \( i \), and then set all its values for lower chains to 0. To ensure that \( K \) is a consistent cut, for every element in \( K \), we add its causal dependencies to \( K \) in lines 7–11. Line 12 checks whether the resulting consistent cut is of rank \( \leq r \). If \( rank(K) \) is at most \( r \), then we have found a suitable cut that can be used to find the next lexically bigger consistent cut and we call GetMinCut routine to find it. If we have tried all values of \( i \) and did not find a suitable cut, then \( G \) is the largest consistent cut of rank \( r \) and we return null.

In Fig. 5, consider the call of GetSuccessor ([1, 2, 3], 6). As there is no next element in \( P_1 \), we consider the next element in \( P_2 \). After line 5, the value of \( K \) is [1, 3, 0], which is not consistent. Lines 7–10 make \( K \) a consistent cut, now \( K = [1, 3, 1] \). Since \( rank(K) \) is 5, we call GetMinCut at line 13 to find

![Fig. 5: Illustration: GetSuccessor](image-url)
the smallest consistent cut of rank 6 that is lexically bigger than \([1, 3, 1]\). This consistent cut is \([1, 3, 2]\).

We present the proof of correctness of in Appendix A.

### 4.1 Optimization for Time Complexity

We can find the lexical successor of any consistent cut in \(O(n^2 u)\) time, instead of \(O(n^3 u)\) time taken in GetSuccessor, by using additional \(O(n^2 u)\) space.

Observe that GetSuccessor routine iterates over \(n_u - 1\) chains in the outer loop at line 2, and the two inner loops at lines 8 and 9 perform \(O(n^2 u)\) work in the worst case. When we cannot find a suitable cut of rank less than or equal to \(r\) (check performed at line 12), we move to a higher chain (with the outer loop at line 2). Thus, we repeat a large fraction of the \(O(n^2 u)\) work in the two inner loops at lines 8 and 9 for this higher chain. We can avoid this repetition by storing the combined causal dependencies from higher chains on each lower chain.

Let us illustrate this with an example. Consider the uniflow computation shown in Fig. 6. Suppose we want the lexical successor of \(G = [1, 3, 2]\). Then, for each chain, starting from the top we compute the projection of events included in \(G\) on lower chains. For example, \(G[3] = 1\), and thus on the top-most chain, the projection is only the vector clock of the first event on \(P_3\), which is \([1, 0, 0]\). Thus \(proj[3] = [1, 0, 0]\). On \(P_2\), the projection must include the combined vector clocks of \(G[3]\) and \(G[2]\) — the events from top two chains. As \(G[2] = 3\), we use the vector clock of third event on \(P_2\), which is \([0, 3, 1]\) as that event is causally dependent on first event on \(P_1\). Combining the two vectors gives us the projection on \(P_2\) as \(proj[2] = [1, 3, 1]\).

Algorithm 4 shows the steps involved in computing the projections of a cut on each chain. We create an auxiliary matrix, \(proj\), of size \(n_u \times n_u\), to store these projections. In GetSuccessor routine, once we have computed a new successor by using some event on chain \(i\), we need to update the stored projections on chains lower than \(i\); and not all \(n_u\) chains. This is because the projections for unchanged entries in \(G\) above chain \(i\) will not change on chain \(i\), or any chain above it. Hence, we only update the relevant rows and columns — rows and

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**Algorithm 4 ComputeProjections(G)**

**Input:** \(G\): a consistent cut of rank \(r\)

1: for \((i = n_u; i > 1; i--)\) do // go top to bottom
2: \(val = G[i]\) // event number in \(G\) on chain \(i\)
3: \(vc = \text{vector clock of event num } val \text{ on chain } i\)
4: if \(i == n_u\) then // on highest chain
5: \(proj[i] = vc\)
6: else // process relevant entries in vector
7: for \((j = i; j > 0; j--)\) do
8: \(//\text{projection on chain } i:\)
9: \(proj[i][j] = \text{MAX}(vc[j], proj[i+1][j])\)

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Fig. 6: Projections of a cut on chains
columns with number $i$ or lower — in $proj$; i.e. only the upper triangular part of the matrix $proj$. We keep track of the chain that gave us the successor cut, and pass it as an additional argument to Algorithm 4. We read and update $n_u^2/2$ entries in the matrix, and not all $n_u^2$ of them.

Hence, the optimized implementation of finding the lexical successor of $G$ requires two changes. First, every call of GetSuccessor $(G, r)$ starts with first computing the projections of $G$ using Algorithm 4. Second, we replace the two inner for loops at lines 8 and 9 in GetSuccessor by one $O(n_u)$ loop to compute the max of the two vector clocks: vector clock of $K[i]$, and $proj[i]$. In interest of space, we show the optimized routine for GetSuccessor with these changes in Algorithm 7 in Appendix C.

4.2 Re-mapping Consistent Cuts to Original Chain Partition

The number of consistent cuts of a computation is independent of the chain partition used. Their vector clock representation, however, varies with chain partitions as the vector clocks of events in the computation depend on the chain partition used to compute them. There is a one-to-one mapping between a consistent cut in the original chain partition of the computation on $n$ chains (processes), and its uniflow chain partition on $n_u$ chains. We now show how to map a consistent cut in a uniflow chain partition to its equivalent cut in the original chain partition of the computation. Let $P = (E, \rightarrow)$ be a computation on $n$ processes, and let $n_u$ be the number of chains in its uniflow chain partition. If $G_u$ is a consistent cut in the uniflow chain partition, then its equivalent consistent cut $G$ for the original chain partition (of $n$ chains) can be found in $O(n_u + n^2)$ time.

We do so by mapping two additional entries with the new vector clock of each event for uniflow chain partition: the chain number $c$, and event number $e$ from the original chain partition over $n$ chains. For example, in Fig. 4(b), for uniflow vector clock $[1, 1, 1]$, its chain number in original poset is 1, and its event number on that chain is 2. When generating the uniflow vector clocks, we populate these entries in a map. Given a uniflow vector clock $uvc$, the call to OriginalChain($uvc$) returns $c$, and OriginalEvent($uvc$) returns $e$. To compute $G$ from $G_u$, we use these two values from the corresponding event for each entry in $G_u$. We start with $I$ as an all-zero vector of length $n$. Now, we iterate over $G_u$, and we update $I$ by setting $I[c] = max(I[c], e)$. As vector $G_u$ has length $n_u$, this step takes $O(n_u)$ time. We now initiate $G$ as an all-zero vector clock of length $n$, and for each entry $I[k]$, $1 \leq k \leq n$, we get the vector clock, $vce$, of event $I[k]$ on chain $k$ in the original computation. We then set $G$ to the component-wise maximum of $G$ and $vce$. As there are $n$ entries in $I$, and for each non-zero entry we perform $O(n)$ work in updating $G$ (in lines 11–14 in Algorithm 5) the total work in this step is $O(n^2)$.

4.3 Traversing consistent cuts of a given rank

A key benefit of our algorithm is that it can traverse all the consistent cuts of a given rank, or within a range of ranks, without traversing the cuts of lower ranks.
**Algorithm 5 REMAP($G_u, n_u, n$)**

**Input:** $G_u$: a consistent cut in uniflow chain partition on $n_u$ chains  
**Output:** $G$: equivalent consistent cut in original chain partition on $n$ chains

1: $G = \text{new int}[n]$ // allocate memory for $G$
2: $I = \text{new int}[n_u]$ // reduction vector
3: for ($i = n_u; i \geq 1; i --$) do // go over all the uniflow chains  
   4: $uvc = \text{event number } G_u[i]'s \text{ vector-clock on uniflow chain } i$
   5: // chain of this event in original poset  
   6: $c = \text{OriginalChain}(uvc)$
   7: // $uvc$'s event number on chain $c$ in original poset  
   8: $e = \text{OriginalEvent}(uvc)$
   9: if $I[c] < e$ then // update indicator with $e$
      10: $I[c] = e$
11: for ($j = n; i \geq 1; i --$) do // go over chains in original poset  
   12: $vce = \text{event number } I[j]'s \text{ vector-clock on chain } j$ in original poset
13: for ($k = n; k \geq 1; k --$) do // update $G$ entries  
      14: $G[k] = \text{MAX}(G[k], vce[k])$
15: return $G$

In contrast, the traditional BFS traversal must traverse, and store, consistent cuts of rank $R - 1$ to traverse cuts of rank $R$, which in turn requires it to traverse cuts of rank $R - 2$ and so on.

To traverse all the cuts of rank $R$, we only need to change the loop bounds at line 3 in Algorithm 1 to for ($r = R; r \leq R; r + +$). Thus, starting with an empty cut we can find the lexically smallest consistent cut of rank $r$ in $O(n_u)$ time with the GETMINCUT routine. Then we repeatedly find its lexical successor of the same rank, until we have traversed the lexically biggest cut of rank $R$. Similarly, consistent cuts between the ranks of $R_1$ and $R_2$ can be traversed by changing the loop at line 3 in Algorithm 1 to: for ($r = R_1; r \leq R_2; r + +$).

**Lemma 3** Let $L_k$ denote the number of consistent cuts of rank $k$ for a computation $(E, \rightarrow)$. Then, traversing consistent cuts of rank $r$ takes $O(n_u^2 L_r)$ time with Algorithm 1. For the same traversal, the traditional BFS algorithm requires $O(n^2 \sum_{k=1}^{r} L_k)$ time, and Lex algorithm takes $O(n^2 \sum_{k=1}^{r} L_k)$ time.

## 5 Time and Space Complexity

Algorithm 1 requires a computation in its uniflow chain partition. Multiple polynomial time algorithms exist to find a non-trivial uniflow chain partition of a poset, and we give a vector clock based online algorithm to find one in Appendix sec:partition that takes $O(n)$ time per event. We analyze the worst case time and space complexities of our algorithms.

Given any computation on $n$ processes and $E$ events, we can find its trivial uniflow chain partition in $O(n|E|\log|E|)$ time by lexically ordering the vector clocks of all the events.
Suppose the number of chains in the uniflow partition is $n_u$, then the step of computing new vector clocks takes $\mathcal{O}(n_u|E| \cdot \Delta)$ time where $\Delta$ is the maximum in-degree of any event in the computation; note that $\Delta \leq n$. The GetMINCUT sub-routine has only one for loop that iterates over the chains of the uniflow partition. Hence, it takes $\mathcal{O}(n_u)$ time in the worst case. The optimized version of finding the successor, sub-routine GETSUCCESSOROPTIMIZED, takes $\mathcal{O}(n^2_u)$ time in the worst case due to the two nested for loops at lines 3, and 10. Hence, for any rank, our algorithm requires $\mathcal{O}(n^2_u)$ time per consistent cut in the uniflow partition. Re-mapping this cut to the original computation takes $\mathcal{O}(n_u + n^2)$ time. Thus, we take $\mathcal{O}(n^2_u + n^2)$ time per consistent cut.

**Theorem 1** Given a computation $P = (E, \rightarrow)$ on $n$ processes, Algorithm 1 performs breadth-first traversal of its lattice of consistent cuts using $\mathcal{O}((n_u + n)|E|)$ space which is polynomial in the size of the computation.

**Proof.** Storing the original computation requires $\mathcal{O}(n|E|)$ space — each event’s vector clock having at most $n$ integers. Vector clocks for the uniflow chain partition with $n_u$ chains takes $\mathcal{O}(n_u)$ space per event. Thus, we require $\mathcal{O}(n_u|E|)$ additional space overall to store the computation in its uniflow form. Traversing the lattice as per Algorithm 1 only requires $\mathcal{O}(n^2_u)$ space as at most two vectors of length $n_u$ are stored/created during this traversal, and we use the auxiliary matrix of $n_u \times n_u$ size in the optimized implementation of GETSUCCESSOR. From Lemma 1 we know that $n_u \leq |E|$. Thus, the worst case space complexity of is $\mathcal{O}(|E|^2 + n|E|)$ which is polynomial in the size of the input.

### 6 Experimental Evaluation

We conduct an experimental evaluation to compare the space and time required by BFS, Lex, and our uniflow based traversal algorithm to traverse consistent cuts of specific ranks, as well as all consistent cuts up to a given rank. We do not evaluate DFS implementation as previous studies have shown that Lex implementation outperforms DFS based traversals in both time and space [15,8,7]. Lexical enumeration is significantly better for enumerating all possible consistent cuts of a computation [8,7]. However, it is not well suited for only traversing cuts of a specified rank, or finding the smallest counter example. For these tasks, BFS traversal remains the algorithm of choice. We optimize the traditional BFS implementation as per [15] to enumerate every global state exactly once. We use seven benchmark computations from recent literature on traversal of consistent cuts [8,7]. The details of these benchmarks are shown in first four columns of Table 1. Benchmarks $d-100$, $d-300$ and $d-500$ are randomly generated posets for modeling distributed computations. The benchmarks $bank$ and $hedc$ are computations obtained from real-world concurrent programs that are used by [10,13,30] for evaluating their predicate detection algorithms. The benchmark $bank$ contains a typical error pattern in concurrent programs, and $hedc$ is a web-crawler. Benchmarks $w-4$ and $w-8$ have 480 events distributed over 4 and 8 processes respectively, and help to highlight the influence of degree of parallelism on the performance of enumeration algorithms.
Table 1: Benchmark details, heap-space consumed (in MB) and runtimes (in seconds) for two BFS implementations to traverse the full lattice of consistent cuts. $T_{\text{part}}=\text{time (seconds) to find uniflow partition; } \times = \text{out-of-memory error}$

| Name | $n$ | $|E|$ | $n_u$ | $T_{\text{part}}$ | Traditional BFS | Uniflow BFS |
|------|-----|-----|------|-------------------|-----------------|--------------|
|      |     |     |      | Space  | Time  | Space  | Time  |
| d-100 | 10  | 100 | $1.2 \times 10^6$ | 26     | 0.030 | 108    | 0.48  | 31     | 0.37  |
| d-300 | 10  | 300 | $4.3 \times 10^7$ | 68     | 0.031 | 842    | 16.84 | 33     | 46.20 |
| d-500 | 10  | 500 | $4.9 \times 10^9$ | 112    | 0.033 | 893    | 108.07| 34     | 607.55|
| bank | 8   | 96  | $8.2 \times 10^8$ | 8      | 0.023 | $\times$ | $\times$ | 59     | 73.2  |
| hede | 12  | 216 | $4.5 \times 10^9$ | 26     | 0.028 | $\times$ | $\times$ | 56     | 1129  |
| w-4  | 4   | 480 | $9.3 \times 10^6$ | 121    | 0.036 | 258    | 0.99  | 25     | 8.59  |
| w-8  | 8   | 480 | $7.3 \times 10^9$ | 63     | 0.032 | $\times$ | $\times$ | 40     | 1445.57|

Table 2: Runtimes (in seconds) for `tbfs`: Traditional BFS, `lex`: Lexical, and `uni`: Uniflow BFS implementations to traverse cuts of given ranks

| Name | $r = \frac{|E|}{4}$ | $r = \frac{|E|}{2}$ | $r = \frac{3|E|}{4}$ | $r \leq 32$ |
|------|-------------------|-------------------|-----------------|-----------|
|      | tbfs | lex | uni | tbfs | lex | uni | tbfs | lex | uni | tbfs | lex | uni |
| d-100 | 0.12 | 0.10 | 0.04 | 0.22 | 0.11 | 0.05 | 0.20 | 0.89 | 0.04 | 0.19 | 0.93 | 0.12 |
| d-300 | 0.39 | 1.23 | 0.05 | 2.70 | 1.15 | 0.07 | 6.33 | 1.25 | 0.13 | 0.20 | 1.22 | 0.14 |
| d-500 | 2.29 | 5.73 | 0.11 | 7.83 | 6.52 | 0.33 | 67.59 | 6.86 | 1.48 | 0.19 | 4.93 | 0.19 |
| bank | 3.36 | 16.80 | 0.27 | $\times$ | 16.34 | 3.07 | $\times$ | 17.02 | 0.32 | 45.43 | 16.87 | 5.70 |
| hede | 4.72 | 16.50 | 0.40 | $\times$ | 152.76 | 15.70 | $\times$ | 153.54 | 0.51 | 0.23 | 128.60 | 0.12 |
| w-4  | 0.09 | 0.13 | 0.07 | 0.53 | 0.18 | 0.10 | 0.93 | 0.19 | 0.09 | 0.01 | 0.13 | 0.05 |
| w-8  | 26.39 | 143.08 | 0.72 | $\times$ | 171.23 | 120.27 | $\times$ | 169.21 | 3.09 | 0.02 | 196.21 | 0.05 |

We conduct two sets of experiments: (a) complete traversal of lattice of consistent cuts (of the computation) in BFS manner, and (b) traversal of cuts of specific ranks. We conduct all the experiments on a Linux machine with an Intel Core i7 3.4GHz CPU, with L1, L2 and L3 caches of size 32KB, 256KB, and 8192KB respectively. We compile and run the programs on Oracle Java 1.7, and limit the maximum heap size for Java virtual machine (JVM) to 2GB. For each run of our traversal algorithm, we use Algorithm 6 (in Appendix B) to find the uniflow chain partition of the poset. The runtimes and space reported for our uniflow traversal implementation include the time and space needed for finding and storing the uniflow chain partition of the poset.

Table 1 compares the size of JVM heap and runtimes for traditional BFS and our uniflow based BFS traversal of lattice of consistent cuts of the benchmarks. The traditional BFS implementations runs out of memory on `hede`, `bank`, and `w-8`. Our implementation requires significantly less memory, and even though it is slower, it enables us to do BFS traversal on large computations — something that is impossible with traditional BFS due to its memory requirement.
Table 2 highlights the strength of our algorithm in traversing consistent cuts of specific ranks. We compare our implementation with traditional BFS as well as the implementation of Lexical traversal. For traversing consistent cuts of three specified ranks (equal to quarter, half, and three-quarter of number of events) our algorithm is consistently and significantly faster than both traditional BFS, as well as Lex algorithm. Thus, it can be extremely helpful in quickly analyzing traces when the programmer has knowledge of the conditions when an error/bug occurs.

In addition, there are many cases when we are not interested in checking all consistent cuts of a computation. It has been argued that most concurrency related bugs can be found relatively early in execution traces [24,4]. We also perform well in visiting all consistent cuts of rank less than or equal to 32. Hence, our implementation is faster on most benchmarks for smaller ranks, and requires much less memory (memory consumption details for this experiment are in Appendix D). These results emphasize that our algorithm is useful for practical debugging tasks while consuming less resources.

7 Future Work & Conclusion

Algorithm 1 can perform the BFS traversal without regenerating the vector clocks for uniflow chain partitions. This is particularly beneficial for the computations in which \(|E| >> n\), and hence the \(O(|E|^2)\) space needed to regenerate the vector clocks is expensive. Observe that any chain partition, including a uniflow chain partition, of a computation is only an arrangement of its graph. Hence, we can implement Algorithm 1 without regenerating new vector clocks, and by only finding the positions of the events in the uniflow chain partition. To do so, we assign a unique id to each event, and then place this event id on its corresponding uniflow chain. We also store a mapping of original vector clocks against the event ids. The space requirement for our algorithm will reduce to \(O(n_u \cdot n)\) as we do not regenerate vector clock, and computation of projections can be performed using \(n_u \times n\) space instead of \(n_u \times n_u\) space. As a future work, we plan to implement and evaluate this strategy.

It is easy to parallelize Algorithm 1 as it traverses cuts of rank \(r + 1\) independently of those of rank \(r\). We can perform a parallel traversal easily using a parallel-for loop at line 3 of Algorithm 1. We intend to implement this parallel approach and compare its performance against parallel traversal algorithms such as Paramount [8].

The ubiquity of multicore and cloud computing has significantly increased the degree of parallelism in programs. This change has in turn made verification and analysis of large parallel programs even more challenging. For such verification and analysis tasks, breadth-first-search based traversal of global states of parallel programs is a crucial routine. We have reduced the space complexity of this routine from exponential to quadratic in the size of input computation. This reduction in space complexity allows us to analyze computation with high degree of parallelism with relatively small memory footprint — a task that is practically impossible with traditional BFS implementations.
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Lemma 4 Let $G$ be any consistent cut of rank at most $r$. Then, $H = \text{GetMinCut}$ is the lexically smallest consistent cut of rank $r$ greater than or equal to $G$.

Proof. We first show that $H$ is a consistent cut. Initially, $H$ is equal to $G$ which is a consistent cut. We show that $H$ continues to be a consistent cut after every iteration of the for loop. At iteration $j$, we add elements from the $j^{th}$ chain from the bottom to $H$. Since all elements from higher numbered chains are already part of $H$, and all elements from lower numbered chains cannot be smaller than any of the newly added element, we get that $H$ continues to be a consistent cut.

By construction of our algorithm it is clear that rank of $H$ is exactly $r$. We now show that $H$ is the lexically smallest consistent cut of rank $r$ greater than
or equal to \( G \). Suppose not, and let \( W \prec H \) be the lexically smallest consistent cut of rank \( r \) greater than or equal to \( G \). Since \( W \prec H \), let \( k \) be the smallest index such that \( W[k] < H[k] \). Since \( G \leq W \), \( k \) is one of the indices for which we have added at least one event to \( G \). Because rank of \( W \) equals rank of \( H \), there must be an index \( k' \) lower than \( k \) such that \( W[k'] > H[k'] \). However, our algorithm forces that for \( H \) for any index \( k' \) lower than \( k \), \( H[k'] \) equals \( |P_{w'}| \). Hence, \( W[k'] \) cannot be greater than \( H[k'] \).

**Lemma 5** Let \( G \) be any consistent cut of rank at most \( r \), then \textbf{GetSuccessor} returns the least consistent cut of rank \( r \) that is lexically greater than \( G \).

**Proof.** Let \( W \) be the cut returned by \textbf{GetSuccessor}. We consider two cases. Suppose that \( W \) is null. This means that for all values of \( i \), either all elements in chain \( P_i \) are already included in \( G \), or on inclusion of the next element in \( P_i \), \( z \), the smallest consistent cut that includes \( z \) has rank greater than \( r \). Hence, \( G \) is lexically biggest consistent cut of rank \( r \).

Now consider the case when \( W \) is the consistent cut returned at line 16 by \textbf{GetMinCut}(\( K, r \)). We first observe that after executing line 11, \( K \) is the next lexical consistent cut (of any rank) after \( G \). If \( rank(K) \) is at most \( r \), then by Lemma 4 we know that \textbf{GetMinCut}(\( K, r \)) returns the smallest lexical consistent cut greater than or equal to \( G \) of rank \( r \). If \( rank(K) \) is greater than \( r \), then there is no consistent cut of rank \( r \) such that \( \forall k : i + 1 \leq k \leq n_u : K[k] = G[k] \) and \( K[i] > G[i] \) and \( rank(K) \leq r \). Thus, at line 16 we use the largest possible value of \( i \) for which there exists a lexically bigger consistent cut than \( G \) of rank \( r \).

**B Uniflow Partitioning Algorithm**

Lemma 1 establishes that every poset has a uniflow chain partition. The trivial uniflow partition, however, will result in number of chains \( n_u = |P| \). Given that our traversal technique takes \( \mathcal{O}(n_u^2) \) time per consistent cut in the worst case, it is beneficial to find a uniflow chain partition that has fewer number of chains. We now discuss an online chain partitioning algorithm to find such non-trivial uniflow chain partitions of posets.

**B.1 Online Algorithm**

This algorithm processes events of the computation \( P = (E, \rightarrow) \) in an online manner: when a process \( P_i \) executes event \( e \) it sends the event information to this partitioning algorithm. Algorithm 6 shows the steps of finding an appropriate chain for \( e \) in the uniflow partition.

Given an event \( e \), we set its uniflow chain, \( uid \), to the id of the chain (process) on which it was executed. Then, we go over all its causal dependencies, and in case any of the dependencies were placed on higher numbered chains, we update the \( uid \) (lines 3–4). We know that to maintain the uniflow chain partitioning, \( e \) must be placed either on a chain with id \( uid \), or above it. Lines 6–8 check if
Algorithm 6 \textsc{FindUniflowChain}(e)

\begin{algorithm}
\begin{algorithmic}
\State \textbf{Input:} An event $e$ of the computation $P = (E, \rightarrow)$ on $n$ processes
\State \textbf{Output:} $e$ is placed on a chain in the uniflow chain partition of $P$
\State 1: $\text{maxid}$: id of highest uniflow chain till now
\State 2: $\text{uid} = e\text{.procid}$ // start with chain that executed $e$
\State 3: \textbf{for} each causal dependency $\text{dep} \text{ of } e$ \textbf{do}
\State 4: \hspace{1em} $\text{uid} = \text{MAX}(\text{dep\text{.procid}}, \text{uid})$
\State 5: \hspace{1em} // now check if there exists any chain with the same id
\State 6: \hspace{1em} \textbf{if} $\exists$ a chain with $id = \text{uid}$ \textbf{then}
\State 7: \hspace{2em} $f$ = last event on this chain
\State 8: \hspace{2em} \textbf{if} $e || f$ then // $e$ is concurrent with $f$
\State 9: \hspace{3em} $\text{maxid} + +$ // increment max used chain id
\State 10: \hspace{3em} create new chain with id $= \text{maxid}$
\State 11: \hspace{3em} add $e$ at the end of this chain
\State 12: \hspace{2em} \textbf{else} // $e$ not concurrent with $f$
\State 13: \hspace{3em} add $e$ at the end of chain (with $id = \text{uid}$)
\State 14: \hspace{1em} \textbf{else}
\State 15: \hspace{2em} create new chain with id $\text{uid}$
\State 16: \hspace{2em} add $e$ at the end of this chain
\State 17: $\text{maxid} = \text{uid}$ // increment max assigned chain id
\end{algorithmic}
\end{algorithm}

there already exists a chain with that id, and if the last event on this chain is concurrent with $e$. If so, we cannot place $e$ on this chain and must put it on a chain above — possibly by creating a new chain (lines 9–11). Otherwise, we can place $e$ at the end of this chain, and do so at line 12. If no chain has been created with $\text{uid}$ as its number, that means $e$ is the first event on some chain (process) in $P$ and we must create a new chain in our uniflow partition for it. This is done in lines 15–17.

Let us illustrate the execution of the algorithm on the poset of Fig.\textsuperscript{3(a)}. Let $\mu$ denote the uniflow chain partition, which is initially empty. Suppose the first event on process $P_1$ is the very first event sent to the this algorithm. As there is no event in $\mu$, this event will be placed on chain id 1. As we assume an online setting, the first event on $P_2$ must going to be presented next. As this event also has no causal dependencies, the $\text{uid}$ value for it at line 6 will be 2 — the id of the process that executed it. As there is no chain with id 2, we execute lines 14–17 to create a new chain and place this event on chain 2 in $\mu$. Suppose the next event to arrive is the second event on $P_2$. Even though is causally dependent on the first event on $P_1$, its $\text{uid}$ value after the loop of line 3–4 is still 2 as we take the maximum of the ids. There is a chain with id 2 in $\mu$, and this event is not concurrent with the last event on this chain. Hence, we execute lines 12–13, and place this event as the last event on chain 2. The last event to arrive will be the second event on $P_1$. After executing lines 3–4, the $\text{uid}$ value for this event will be 2. As there is a chain with id 2 in $\mu$, we will compare this event with the last event on chain 2. However, the last event on chain 2 and this event are...
concurrent (line 8). Hence, we will be forced to create a new chain, with id 3, and place the event on this chain as per lines 9–11.

Let $\Delta$ be the maximum in-degree of any event in the computation. Then lines 3–4 of Algorithm 6 perform $O(\Delta)$ work. Checking for existence of a chain id at line 6 is a constant time operation as we use a hash-table for storing the chains against their ids. The check for concurrency of two events is $O(n)$ as we can use the original vector clocks of the two events. Lines 9–11 then perform constant work. If the events are not concurrent, and we execute line 13, we still perform constant work in appending the event at the end of a chain. Lines 15–17 also perform constant work. Hence, the total work performed by the algorithm is $O(n + \Delta)$ per event.

C Optimized Implementation of GetSuccessor

Algorithm 7 GetSuccessorOptimized($G, r$)

Input: $G$: a consistent cut of rank $r$

Output: $K$: lexical successor of $G$ with rank $r$

1: ComputeProjections($G$) // $G$’s projections
2: $K = G$ // Create a copy of $G$ in $K$
3: for ($i = 2; i \leq n_u; i++$) do
4: if next element on $P_i$ exists then
5: $K[i] = K[i] + 1$ // increment cut in $P_i$
6: // fix dependencies using projections
7: $vc =$ vector clock of event number $K[i]$ on $P_i$
8: // take component-wise max
9: for ($k = i - 1; k > 0; k -= 1$) do
10: $K[k] = \text{MAX}(vc[k], proj[i][k])$
11: if rank($K$) $\leq r$ then
12: return GetMinCut($K, r$) // make $K$’s rank equal to $r$
13: return null // could not find a candidate cut

D Memory Consumption for Traversing Specific Ranks

Table 3 shows the memory consumed by the three algorithms (traditional BFS, Lex, and Uniflow based BFS) in traversing consistent cuts of specific ranks, as well as all cuts of rank less than or equal to 32.
| Name | $r = \frac{|E|}{d}$ | $r = \frac{|E|}{3d}$ | $r = \frac{3|E|}{d}$ | $r \leq 32$ |
|------|------------------|------------------|------------------|------------|
|      | tbfs  | lex  | uni | tbfs  | lex  | uni | tbfs  | lex  | uni | tbfs  | lex  | uni |
| d-100| 95    | 32   | 41  | 121   | 29   | 41  | 134   | 32   | 42  | 112   | 32   | 42  |
| d-300| 107   | 33   | 53  | 342   | 32   | 54  | 583   | 32   | 54  | 113   | 31   | 42  |
| d-500| 299   | 33   | 56  | 695   | 32   | 55  | 1604  | 34   | 55  | 112   | 32   | 41  |
| bank | 1014  | 21   | 52  | ×     | 22   | 54  | ×     | 21   | 54  | 1312  | 22   | 54  |
| hedc | 934   | 33   | 61  | ×     | 34   | 62  | ×     | 34   | 62  | 602   | 31   | 60  |
| w-4  | 83    | 21   | 49  | 313   | 22   | 49  | 301   | 21   | 51  | 36    | 20   | 49  |
| w-8  | 1786  | 27   | 44  | ×     | 28   | 43  | ×     | 28   | 45  | 1240  | 28   | 43  |

Table 3: Heap Memory Consumed (in MB) for tbfs: Traditional BFS, lex: Lexical, and uni: Uniflow BFS implementations to traverse cuts of given ranks. × = out-of-memory error