Aspects concerning the friction for the motion on an inclined plane of an axisymmetric body

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Abstract. The geometrical intrinsic elements defining the Hertzian point contact between two solid bodies are the common normal and the tangent plane in the contact point. To the relative motions occurring about these elements, forces and moments of friction oppose; these reactions are the elements of the friction torsor from contact. The paper presents a particular case when sliding friction and rolling friction coexist in the tangent plane. The first body is made of two identical bearing balls rigidly fastened and the second body is an inclined plane. The straight line defined by the contact points between the mobile body and the plane differs from the steepest line of the plane. By gradually increasing the slope of the plane, the situation when the body simultaneously rolls and slides with respect to the line defined by the contact points is reached. When the duration of motion, the sliding of the centre of mass and the rotation angle of the body between two marks from the plane are measured, it is concluded that the sliding velocity and the angular rolling velocity are independent. While the rotation angle is constant the sliding and time of motion vary in a relatively wide range thus confirm a significant gradient of the coefficients of sliding and rolling friction.

1. Introduction

In applied mechanics there are frequently met contacting bodies which have different boundary surfaces and therefore the theoretical contact between them takes place in a point [1]. From kinematical point of view, this situation presents the most complex case of relative motions [2]. A kinematical pair of first class is created and from the six simple motions that may occur between two free rigid bodies only the displacement along the normal in contact $n$ is cancelled [3]. The other motions are translations in the tangent plane ($T$) along two perpendicular axes, rotations about these two axes and spinning rotation about the normal and they are impeded by friction forces $F_{sl}$ and friction torques, $M_n$ and $M_s$, respectively, all components of the friction torsor in the contact point, according to Figure 1. The characterization of the components of the torsor of friction is a major tribological topic.
2. Theoretical aspects

For an actual case, the simultaneous incidence of all components of the friction torsor makes very difficult the task of establishing them [4], [5]. A typical example is the thrust ball bearing, where the hypothesis of pure rolling between the ball and the race conducts to the conclusion that spinning and rolling torques are simultaneously present. From this reason, often, considering simpler cases are preferred, where only one component [6-11] of the friction torsor is present and afterwards, for a concrete situation, the results are by some means superposed, with the necessity of stipulating in which manner they influence reciprocally. The inclined plane method is frequently used in tribology as it permits finding the characteristics of sliding and rolling friction. As principle, on an inclined plane of variable angle, initially in horizontal position, a prismatic or body of revolution is set, and then, by progressively increasing the tilting angle of the plane, the body starts moving – translation or rotation, as in Figure 2.

Finding the acceleration of the body for the downward motion and writing the equations of motion one can establish the coefficient of sliding friction (CSF) or rolling friction (CRF). To be mentioned that for the case of plane-parallel motion on the inclined plane of a body of revolution, this may present either sliding or rolling motion; both motions cannot simultaneously occur, Figure 2. But for spatial cases, since in the common tangent plane two perpendicular directions exist, it is possible to appear sliding on one direction and pure rolling on the other one. An illustration and the consequences emerging from this are next presented. The following coordinate systems are considered:
- the $Oxyz$ system, having the $Ox$ and $Oy$ axes perpendicular and contained in a horizontal plane, and the vertical axis, $Oz$;
- the $Ox'y'z'$ system obtained from the $Oxyz$ system by a rotation with the angle $\alpha$ around the $Oz$ axis;
- the $Ox''y''z''$ system obtained from the $Ox'y'z'$ system by a rotation with the angle $\beta$ around the $y'$ axis.

An axisymmetric body is placed on the plane $Ox'y''$ such as the axis of symmetry is parallel to the $Ox'$ axis. The body of revolution is obtained by fastening two identical bearing balls via a cylindrical ring. This solution has the advantage that the points of application of the reaction forces between the body and the plane are well specified, different from the situation when a cylinder is placed on a plane. Another drawback of the cylindrical body consists in the edge effects occurring in the vicinity of the ends of the contact generatrix. The body presented in Figure 3 has the following motions, permitted by the two Hertzian point contacts: translations along the $x''$ and $y''$ axes, pivot rotation around the normal $z''$ and rolling around the $x''$ axis. By conveniently selecting the distance between the centers of the two balls and the tilting angles $\alpha$ and $\beta$ it is seen, by assuming that the reaction forces from the two points of contact are identical, that the spinning motion cannot take place because the spinning torque is zero. For small values of the $\alpha$ angle the body rolls along the $y''$ axis, sliding or resting about the $x''$ direction.

**Figure 3.** Axisymmetric body in sliding and rolling motions performed on two directions

In order to describe the motion of the body, the theorem of the motion of the center of mass:

$$M\vec{\ddot{r}} = N + T_x + T_y - G$$  \hspace{1cm} (1)
\[ J_x \ddot{\phi} = T_y R - M_r \quad (2) \]

are applied. The equation (1) can be written using the projections on the axes:

\[ M\ddot{x}' + M\ddot{y}' - T_x' = -T_y' - T_z' + Nk'' - Gk \quad (3) \]

It is noticed that the unit vector \( \mathbf{k} \) should be expressed as function of the unit vectors of the \( Ox'y'z' \) system. To this end, it is applied the remark made by McCarthy [13], namely:

\[ r_1 = T_{12} r_2 \quad (4) \]

where \( r_1, r_2 \) are the position vectors of the point in the systems "1" and "2" and \( T_{12} \) is the displacement required to superpose the system "1" over "2". Considering the \( Ox'y'z' \) as initial system and \( Oxz \) as final system, the displacement operator \( T_{12} \) is obtained a rotation of angle \( -\beta \) around \( Oy' \) axis followed by a rotation of angle \( \alpha \) around the \( Ox' \) axis.

\[ T_{12} = Y(-\beta)X(\alpha) \quad (5) \]

where [14]:

\[ X(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix} \quad (6) \]

and

\[ Y(\beta) = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix} \quad (7) \]

The following decomposition of the \( \mathbf{k} \) versor is obtained after the calculus is made:

\[ \mathbf{k} = -i'' \cos \alpha \sin \beta - j'' \sin \alpha + k'' \cos \alpha \cos \beta \quad (8) \]

and it results that the projections of the equation(3) have the form:

\[
\begin{align*}
M\ddot{x}' &= Mg \cos \alpha \sin \beta - T_x' \\
M\ddot{y}' &= Mg \sin \alpha - T_y' \\
0 &= N - Mg \cos \alpha \cos \beta
\end{align*}
\]

(9)

The scalar equations of motion of the body have the expressions:

\[
\begin{align*}
M\ddot{x}' &= Mg \cos \alpha \sin \beta - T_x' \\
M\ddot{y}' &= Mg \sin \alpha - T_y' \\
0 &= N - Mg \cos \alpha \cos \beta \\
J_x \ddot{\phi} &= T_y R - M_r
\end{align*}
\]

(10)
The unknowns of the system of equations (10) are the components of the reaction, \( N, T_x, T_y, M_r \) and the laws of motion, \( \varphi = \varphi(t) \) and \( x = (t) \). To the system of equations (10), must be added:

- the kinematics equation characteristic to pure rolling:
  \[ y = R \varphi \]  
  (11)

- the equations characteristic to sliding along \( O_x \) axis:
  \[ T_x = \mu N \]  
  (12)

- and the equation relating the rolling friction torque to the normal reaction force:
  \[ M_r = sN \]  
  (13)

where it was assumed that the torque of rolling friction is proportional to the normal load. Recent theories propose new relations where the rolling friction torque depends on the normal force raised at a power \([15, 16]\). The equations (9-13) complete a system of seven equations with seven unknowns and by solving it, the following expressions are found:

\[ N = Mg \cos \alpha \cos \beta \]  
(14)

\[ T_x = \mu Mg \cos \alpha \cos \beta \]  
(15)

\[ M_r = sMg \cos \alpha \cos \beta \]  
(16)

\[ \ddot{x}^c = (\sin \beta - \mu \cos \beta)g \cos \alpha \]  
(17)

\[ \ddot{\varphi}_2 = Mg \frac{R \sin \alpha - s \cos \alpha \cos \beta}{J_x + MR^2} \]  
(18)

\[ \ddot{y}_2 = Mg \frac{R \sin \alpha - s \cos \alpha \cos \beta}{J_x + MR^2} R \]  
(19)

\[ T_y = gM \frac{J_x \sin \alpha + sRM \cos \alpha \cos \beta}{J_x + MR^2} \]  
(20)

Besides the above equations, the inequality characteristic to pure rolling around the axis passing through the contact points, which is parallel to \( O_x \) axis, should be verified:

\[ T_y < \mu N \]  
(21)

An analysis upon the relation (14-20) shows that all the right members of these equations are constant values and this proves that all motions should be uniformly accelerated translation and rotation, respectively. Additionally, all the forces and moments from the contact points must be constant. Moreover, when sliding and rolling occur simultaneously with respect to the same axis \( O_x \), from the tangent plane, in the present situation, from the relations (15) and (16) a proportionality between the acceleration on the sliding direction and the angular rolling acceleration should exist.

\[ \frac{\ddot{x}^c}{\ddot{\varphi}_2} = \frac{(\sin \beta - \mu \cos \beta) \cos \alpha}{M \frac{R \sin \alpha - s \cos \alpha \cos \beta}{J_x + MR^2} } \]  
(22)
If the equation (22) proves to be correct, then finding the coefficient of friction would be easier since the coefficient of rolling friction may be found by measuring the coefficient of sliding friction $\mu$, the angular acceleration $\dot{\phi}_2$ and the sliding acceleration $\ddot{x}$. Furthermore, in the two ratios from equation (22), both the numerator and the denominator are constant, and from here the conclusion that a necessary condition is:

$$\frac{\ddot{x}}{\dot{\phi}_2} = \text{const} \quad (23)$$

3. Experimental set-up

A device was designed and assembled in order to test the validity of the hypothesis expressed by the equation (23). The experimental set-up, based on the schematics form Figure 3, consists in a 10 mm thick aluminum plate 1, placed on another plate made of laminated wood 2, that at its turn is placed on the top of an horizontal table. Between the table and the wood plate, on one side, and between the wood plate and the aluminum plate, on the other side, feeler gauges are introduced to accomplish the intended tilt of the aluminum plate, according to the scheme form Figure 3. The rolling body 3 is made from two identical bearing balls jointed together with the purpose to block the spinning motion and to define precisely the position of the rolling axis (the axis passing through the contact points between the balls and the plate). The rolling body is set into contact with two plates 4 made from electronic circuit bare board, insulated one from each other, but both attached firmly to a metallic prism 5. The prism 5 is on the top of aluminum plate with edges parallel to the ones of the plate. The launching position assumes that each of the balls contacts one of the electric boards and thus an electrical circuit having a source S and a small light emitting diode is closed. Therefore, when the body is set to motion, the circuit opens and the moment of start of the motion can be precisely identified. At the other end of the plate, a rubber prism 6 is positioned to stop the body and damp the motion. A ruler 7 is fixed to the rubber prism in order to observe the transversal sliding of the body. An auxiliary prism 8 was used so as to ensure the launch of the body from the same position.

Figure 4. General view of the experimental set-up

It was concluded that the sliding motion can be estimated with a sufficient accuracy but for the rolling motion the errors in estimations are appreciable. To improve this aspect of the device, another alternative for the rolling body was selected, as presented in Figure 5. It consists in two identical bearing balls connected by a metallic cylindrical part. On the side surface of the cylinder an adhesive leaf was glued and the tracks of an axial symmetry plane 1 and normal symmetry plane 2 respectively, were traced on it. These tracks are used in finding the rotation displacement and sliding of the body, respectively.
The new option for the rolling body is presented in Figure 6 and Figure 7, respectively.

The body for the launching position and for the final position is presented in Figure 6 and Figure 7, respectively.

The motion of the body was filmed from the beginning to the moment of the collision with the rubber prism, in order to measure the time of motion. The movie was subsequently split into frames. The initial moment of the motion is precisely identified as the instant when the LED is off but the final moment required supplementary actions, for accurate timing. To this end, it was necessary that the longitudinal track is observed in the vicinity of the instant of impact with the rubber prism. To this purpose, the hypothesis of pure rolling (which was experimentally validated later on) was accepted. The distance between the starting prism and the final rubber prism was adjusted to be equal to the sum between the diameter of the ball and an integer number of circumferences and thus the centre of the ball runs the same integer number of diameters and the axial track should attain in the final position the same as in the launching one.

Concerning the accuracy of finding the final instant of the motion, in Figure 8 there are presented three successive frames from a movie: one before the impact with the rubber, one at the end and the third after impact.
It can be noticed that before impact, due to high rotation velocity, the track cannot be identified, Figure 8a. Quite after impact the rotation velocity decreases significantly and the track can be easily observed, Figure 8c, the body being away from the rubber prism. Therefore, the impact instant should be discovered between the two images form Figure 8a-8b, the error being equal to the velocity of frame acquisition, namely 1/30 frames/s. The position of the axial track at the end of the motion is also attested in Figure 8, being identical to the track from the launching moment.

The final positions of the body are presented for a number of nine identical launchings of the body, with the longitudinal track placed initially in the vertical plane of symmetry. It can be noticed that, practically, the final position of this track is the same and this confirms the correctness of the pure rolling hypothesis (there was no relative displacement between the plate and the body in the contact points). Concerning the sliding, a variation of it can be observed.

![Figure 9. The final positions of the body for a series of successive launchings](image)

For the cases from Figure 9, the variation of the running times is presented in Figure 10 and the sliding distance is presented in Figure 11.

As it can be remarked, the running times present variation within quite a wide time range. According to the above considerations, both torques $M_r$ and $T_s$ must be constant. Moreover, for the case of rolling motion, because the angle of rotation is practically constant for all launchings, the constant angular acceleration assumes the necessity of constant running times. This aspect directs to the conclusion that the parameters characteristic to rolling friction present a high gradient (appreciable variation from one point to another).
Finding the indexes of the frames corresponding the start and the end for a series of successive launchings

More precisely, the motion of the body is performed in a manner that ensures that the trajectories of the points of contact ball-plane are very close for two launchings, but the moments of rolling friction vary noticeably from a trajectory to another. Considering that the rolling accelerations and sliding accelerations are inversely proportional to the time of motion squared, the condition (22) requires that for $x_2$ of constant value, then $x_2$ should be constant on its turn. But it can be easily observed that this condition is not fulfilled. From Figure 11 it is noticed that the sliding distances vary in a narrow range and therefore the relations (22) and (23) ought to be reconsidered and regarded from statistical point of view. In this situation, considering that for one launching, sliding and rolling motions take place during the same time and that the rotation is practically constant, the relation (22) can be re-written as:

$$I_{med} = \frac{(\sin \beta - \mu \cos \beta) \cos \alpha}{M \left( R \sin \alpha - s \cos \alpha \cos \beta \right) \phi_{2\max}} \frac{\phi_2}{\phi_2 \max}$$

(24)

where $L$ is the length of sliding distance. Now, from the above relation, the coefficient of rolling friction can be found.
4. Conclusions
The motion of an axisymmetric body obtained by two jointed bearing balls on an inclined plane is analyzed. The tilt of the plane is set in a manner that ensures pure rolling of the body with respect to the straight line passing through the contact points and sliding with respect to this line.

The equations of motion of the body are obtained and the conclusion that both rotation and translation should take place with constant accelerations is reached. This fact allows for writing a relation between the rotation acceleration and the sliding acceleration. If this relation is correct, finding the coefficient of rolling friction would be much simpler.

The experimental tests performed on a test rig designed and assembled in the laboratory revealed that for all launchings, the rotation angle was the same but the sliding distance presents noticeable variations. This fact contradicts the hypothesis of proportionality between the sliding and rolling accelerations.

The paper requires further researches in which the proportionality between the two accelerations and the parameters involved in this relation, including the coefficient of sliding friction and the coefficient of rolling friction, should be statistically considered.

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