Exponentially Stable MRAC of MIMO Switched Systems with Matched Uncertainty and Completely Unknown Control Matrix

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Abstract—In this paper an attempt is made to extend the concept of the exponentially stable adaptive control to one class of multi-input-multi-output (MIMO) plants with matched nonlinearity and unknown piecewise constant parameters. Within the intervals between two consecutive parameter switches, the proposed adaptive control system ensures: 1) exponential convergence to zero of the parameter and reference model tracking errors, 2) the monotonicity of the control law adjustable parameters. Both properties are guaranteed in case the regressor is finitely exciting somewhere inside each of such intervals. Compared to the existing methods, the proposed one is applicable to systems with unknown switching signal function and completely unknown control matrix. The theoretical results are supported by the numerical experiments.

Index Terms—adaptive control, closed-loop identification, identification for control, estimation, switched systems, uncertain systems.

1. INTRODUCTION

METHODS of the model reference adaptive control (MRAC) theory are aimed at solving control problems under the conditions of significant parameter uncertainty. Such classical problems are the adjustment of the state- or output-feedback control law parameters under the influence of matched/unmatched, structured/unstructured disturbances and the uncertainty of the state $A \in \mathbb{R}^{n \times n}$ and control $B \in \mathbb{R}^{n \times m}$ matrices [1], [2]. Nowadays, the effective synthesis procedures have been proposed that guarantee the asymptotic convergence of the plant trajectories to reference ones in all stated cases. Efficient robust modifications are developed to ensure boundedness of all signals and certain trajectory tracking quality for a system being affected by external disturbances [3], unmodeled dynamics [4], and delays [5].

However, even under ideal conditions, the classical approaches to adaptive systems synthesis: i) guarantee both the exponential stability of a closed-loop system and the convergence of controller adjustable parameters to their true values only when the strict requirement of the regressor persistent excitation (PE) is met [6], ii) require a priori information about the control matrix [1], [2], [7]. The mentioned disadvantages significantly limit the applicability of adaptive control systems to real-world problems, so that considerable attention of control science community has been attracted to overcome them in recent years.

In [8]–[10], various methods have been developed to relax the PE requirement to ensure parameter error exponential convergence. In [11]–[14], approaches are proposed that simultaneously relax the PE condition and reduce the amount of the required a priori data on the control matrix. The disadvantages of [8]–[10] are that the adaptive laws are insensitive to switches of the plant unknown parameters [11], and only the norm of the unknown parameters identification error is monotonic. In turn, [11] requires the matrix $A$ to be known, the method from [12] demands the known low bound of $\det \{B^TB\}$, whereas [13] – of $\det \{(B^TB_{ref})^{-1}\}$, and $\text{sgn} \{\det \{B^TB_{ref}\}\}$ is needed in [14].

To overcome the shortcomings of the solutions [8]–[14], a new concept of exponentially stable adaptive control has been proposed recently [15]. The adaptive law is derived in two steps. In the first one, considering the plant equations, measurable regressions $z_A(t) = \Delta(t)A$, $z_B(t) = \Delta(t)B$ with a scalar regressor $\Delta \in \mathbb{R}$ with respect to the unknown matrices $A$, $B$ are parameterized using stable minimum-phase filters and the dynamic regressor extension and mixing technique [16]. At the second step, such regressions are substituted into the a priori known equations to calculate the control law ideal parameters (e.g. $A + BK_x = A_{ref}$, $BK_r = B_{ref}$) to obtain measurable regression equations (e.g. $z_{K_r}(t) = \Omega(t)K_r$, $z_{K_r}(t) = \Omega(t)K_r$) again with a scalar regressor $\Omega \in \mathbb{R}$, but now with respect to the mentioned ideal parameters of the control law. Then, on the basis of such a regression, the second Lyapunov method and the results of the Kalman-Yakubovich-Popov Lemma are applied to derive the adaptive law to adjust the controller parameters, which does not require any a priori information about $B$ and ensures the exponential stability of the closed-loop system as well as the monotonic transients of each controller unknown parameter estimate. This approach significantly differs from both conventional indirect and direct MRAC and is the closest ideologically to self-tuning control systems, also some parallels can be found with the Monopoli’s augmented error scheme [1], [2]. The concept of the exponentially stable adaptive control has been successfully applied to several state-feedback problems of: 1) MRAC-based
control of MIMO [17] and SISO systems [15], 2) the direct adaptive pole placement control [18], 3) the adaptive optimal linear quadratic regulation [19].

The main general drawback of [15], [17]–[19], as well as the whole concept of the exponentially stable adaptive control, is the applicability of the proposed adaptive control systems only to linear plants with time-invariant unknown parameters, as well as the requirement to meet the condition of regressor finite excitation to ensure the asymptotic stability of the reference model states tracking error.

The aim of this study is to extend the concept of the exponentially stable adaptive control [15] to one class of MIMO systems with matched nonlinearity and piecewise constant unknown parameters. The main contribution of the proposed solution with respect to [15], [17]–[19] is threefold:

C1 an adaptive control approach for MIMO systems with matched nonlinearity and piecewise constant unknown parameters, which does not require a priori information about the matrix B, is proposed;

C2 the parameter and reference model tracking errors converge exponentially to zero within the time intervals between consecutive switches of the unknown parameters;

C3 the monotonicity of the control law adjustable parameters is guaranteed within the time ranges stated in (C2).

To achieve such properties, it is proposed to augment the parameterization used in [15], [17]–[19] to obtain \( z_{K_i}(t) \), \( z_{K_r}(t) \) by a recently proposed algorithm to detect time instants of switches of the linear regression equation unknown parameters [20], and to set to zero the outputs of all dynamic filters of the parameterization when such switch is detected. Assuming that the regressor is finitely exciting somewhere inside the time interval between two consecutive switches, such an approach allows one to obtain a measurable regression equation with respect to the current values of the unknown parameters of the control law, which in accordance with [15], [17]–[19] is necessary and sufficient for exponential convergence of both tracking and parameter error within the considered time intervals.

Here we would also like to note that filtering with resetting was previously used in the parameterization of the adaptive control problem in [21]. However, in contrast to this study, the parameterization from [21] requires one to know the matrix \( B \), and the filters are set to a zero state not when the switch of the plant parameters is detected, but when the reference value is changed. Therefore, in comparison with [21], the proposed solution ensures an improved transient quality and can be applied to problems with unknown matrix \( B \).

The structure of the paper is arranged as follows. The generalized problem statement is presented in Section 2. Section 3 describes the proposed adaptive system for MIMO switched systems with matched uncertainty and unknown control matrix. The simulation results are shown in Section 4. The paper is wrapped-up with conclusion in Section 5.

A. Definitions

The definition of the regressor finite excitation and the corollary of the Kalman-Yakubovich-Popov lemma are used in the proofs of theorem and propositions.

**Definition.** A regressor \( \omega(t) \) is finitely exciting \( \omega(t) \in FE \) over a time range \( [t^+_0, t^-_0] \), if there exist \( t^+_0 \geq 0 \), \( t^-_0 \geq t^+_0 \) and \( \alpha \) such that the following inequality holds:

\[
\int_{t^+_0}^{t^-_0} \omega(\tau) \omega^T(\tau) d\tau \geq \alpha I,
\]

where \( \alpha > 0 \) is the excitation level, \( I \) is an identity matrix.

**Corollary 1.** For any matrix \( D > 0 \), scalar \( \mu > 0 \), controllable pair \((A, B)\) with \( B \in \mathbb{R}^{n \times m} \) and Hurwitz matrix \( A \in \mathbb{R}^{n \times n} \) there exist a matrix \( P = P^T > 0 \) and \( Q \in \mathbb{R}^{n \times m} , K \in \mathbb{R}^{n \times m} \), such that:

\[
A^T P + PA = -Q Q^T - \mu P, PB = Q K, \\
K^T K = D + D^T.
\]

II. Problem Statement

A class of linear MIMO switched plants with matched nonlinear uncertainty is considered:

\[
\forall t \geq t^+_0, \dot{x}(t) = \Theta^T(t) \Phi(t) = \begin{cases} \\
A_0 x(t) + B_0 (u(t) + \vartheta_i^T \Psi(x)), t \in [t^+_0; t^-_1) \\
\vdots \\
A_i x(t) + B_i (u(t) + \vartheta_i^T \Psi(x)), t \in [t^+_i; t^-_{i+1}) \end{cases}
\]

\[
Φ(t) = [x^T(t) u^T(t) \Psi^T(x)]^T, \\
Θ_i = [A_i B_i B_i \vartheta_i^T], i \in \mathbb{N},
\]

where \( x(t) \in \mathbb{R}^n \) is a vector of plant states with unknown initial conditions \( x_0, u(t) \in \mathbb{R}^m \) is a control signal, \( \Psi(x) \in \mathbb{R}^p \) are known basis functions, \( A_i \in \mathbb{R}^{n \times n} \) is an unknown state matrix, \( B_i \in \mathbb{R}^{n \times m} \) is an unknown control matrix, \( \vartheta_i \in \mathbb{R}^{p \times m} \) is a matrix of unknown parameters of the uncertainty, \( t^+_i \) is a known initial time instant. The pair \((A_i, B_i)\) is controllable, \( \forall t > t^+_0 \) the vector \( Θ_i(t) \in \mathbb{R}^{(n+m+p) \times n} \) is measurable, and the matrix \( Θ_i \in \mathbb{R}^{(n+m+p) \times n} \) and the switching time instants \( t^+_i \) are unknown \( \forall i > 0 \).

The required transient quality for the plant is defined by the following reference model:

\[
\forall t \geq t^+_0, \dot{x}_{ref}(t) = A_{ref} x_{ref}(t) + B_{ref} r(t), \\
x_{ref}(t^+_0) = x_{0ref},
\]

where \( x_{ref}(t) \in \mathbb{R}^n \) is a reference model (RM) state vector, \( x_{ref}(t^+_0) \in \mathbb{R}^n \) is a vector of initial conditions, \( r(t) \in \mathbb{R}^m \) is a reference signal, \( A_{ref} \in \mathbb{R}^{n \times n} \) is a Hurwitz state matrix of RM, \( B_{ref} \in \mathbb{R}^{n \times m} \) is a RM control matrix.

Ideal model following conditions are assumed to be met:

**Assumption 1.** There exist \( K^x_i \in \mathbb{R}^{m \times n}, K^r_i \in \mathbb{R}^{m \times m} \) such that:

\[
A_i + B_i K^x_i = A_{ref}, B_i K^r_i = B_{ref}.
\]

Considering Assumption 1, the error equation between (2) and (4) is written as:

\[
\dot{e}_{ref}(t) = A_{ref} e_{ref}(t) + B_i (u(t) + \vartheta_i^T \Psi(x)) - (A_{ref} - A_i) x(t) - B_{ref} r(t) = \\
= A_{ref} e_{ref}(t) + B_i [u(t) + \vartheta_i^T \Psi(x)] - K^x_i x(t) - K^r_i r(t) = \\
= A_{ref} e_{ref}(t) + B_i [u(t) - \vartheta_i^T \omega(t)],
\]
where
\[ e_{ref} (t) = x(t) - x_{ref} (t), \]
\[ \omega (t) = \left[ x^T (t) \quad r^T (t) \quad -\Psi^T (x) \right]^T \in \mathbb{R}^{n+m+p}, \]
\[ \theta_i = \left[ K^T_i \quad K^T_i \quad \theta^T_i \right]^T \in \mathbb{R}^{(n+m+p) \times m}, \]

The equation (6) motivates to apply the control law with adjustable parameters:
\[ u(t) = \hat{\theta}^T (t) \omega (t), \tag{7} \]
where \( \hat{\theta} (t) \in \mathbb{R}^{(n+m+p) \times m} \) is a continuous estimate of the piecewise constant parameters \( \theta_i \).

The equation (7) is substituted into (6) to obtain:
\[ \dot{e}_{ref} (t) = A_{ref} e_{ref} (t) + B_i \left[ \hat{\theta}^T (t) - \theta^T_i \right] \omega (t) = \]
\[ = A_{ref} e_{ref} (t) + B_i \hat{\theta}^T_i (t) \omega (t), \tag{8} \]
where \( \hat{\theta}_i (t) = \hat{\theta} (t) - \theta_i \) is the error of the parameter \( \theta_i \) estimation.

The following assumption about the parameters \( \theta_i \) and the regressor \( \Phi (t) \) excitation is made to state the problem strictly.

**Assumption 2.** Let \( \exists \Delta \theta > 0, \) \( T_{\min} > \min_{\forall i \in \mathbb{N}} T_i > 0 \) such that
\[ \forall i \in \mathbb{N} \text{ simultaneously:} \]
\[ 1) \quad t^+_{i+1} - t^+_i \geq T_{\min}, \| \theta_i - \theta_{i-1} \| = \| \Delta \theta \| \leq \Delta \theta, \]
\[ 2) \quad \Phi (t) \in FE \text{ over } [t^+_i; t^+_i + T_i], \quad \text{with excitation level } \alpha_i, \]
\[ 3) \quad \Phi (t) \in FE \text{ over } [t^+_i; t^+_i + T_i], \quad \alpha_i > \pi_i > 0, \theta_i \in [t^+_i; t^+_i + T_i), \]
\[ 4) \quad \Phi (t) \text{ is a measurable regression output for the plant } (3). \]

**Goal.** It is required to ensure that the following inequality holds \( \forall i \in [t^+_i; t^+_i + T_i] \) for the plant under the conditions that Assumptions 1 and 2 are met:
\[ \| \xi (t) \| \leq c_1 e^{-c_2(t-t^+_i-T_i)}, \tag{9} \]
where \( \xi (t) = e^T_{ref} (t) \quad vec^T \left( \theta_i (t) \right) \)^T is an augmented tracking error.

**III. MAIN RESULT**

Extending the concept of exponentially stable adaptive control [15] to the class of systems with piecewise constant unknown parameters, it is required to generate a measurable linear regression equation with respect to the unknown parameters \( \theta_i \) of the control law over time intervals \( [t^+_i; t^+_i + T_i] \) to achieve the goal (9):
\[ \forall i \in [t^+_i; t^+_i + T_i], \quad \Omega (t) = \Omega (t) \theta_i, \tag{10} \]
where \( \Omega (t) \in \mathbb{R}^{(n+m+p) \times m} \) is a measurable regression output (regrressand). \( \Omega (t) \in \mathbb{R} \) is a measurable scalar regressor, such that \( \forall i \in [t^+_i + T_i; t^+_i + T_i] \), \( \Omega (t) \geq \Omega_{LB} > 0. \)

To derive the equation (10), first of all, the estimates of \( \dot{\theta}_i \) are required. The detection algorithm, which has been proposed in [20], is applied to obtain them.

**Proposition 1.** On the basis of the filter states:
\[ \Phi (t) = -\Phi (t), \quad \Phi (t) = 0_{m+n+p}, \quad l > 0, \tag{11} \]
normalized signals:
\[ \varphi_n (t) = n_s (t) \varphi (t), \quad \varphi (t) = \left[ \mathbb{M}^T (t) \quad e^{-i(t-t^+_i)} \right]^T, \]
\[ n_s (t) = \frac{1}{1 + \varphi (t)^T \varphi (t)}, \]
\[ \varphi_n (t) = n_s (t) \left[ x(t) - r(t) \right], \]
\[ \varphi (t) = \left[ I_{n \times n} \quad 0_{n \times m} \quad 0_{n \times p} \right] \varphi (t), \]
and the dynamic regressor extension and mixing procedure:
\[ z (t) = \text{adj} \{ \varphi (t) \} \varphi (t), \quad \Delta (t) = \text{det} \{ \varphi (t) \}, \]
\[ \varphi (t) = \int^t_\tau e^{-\varphi (\tau)} \varphi (\tau) d\tau, \]
\[ \varphi (t) = \int^t_\tau e^{-\varphi (\tau)} \varphi (\tau) d\tau, \tag{13} \]
using the following indicator:
\[ \epsilon (t) = \Delta (t) \varphi (t) \varphi_n (t) - \varphi (t) \varphi_n (t) z (t), \tag{14} \]
the algorithm:

initialize: \( i \leftarrow 1, \quad t_{up} = t^+_{i-1} \)
\[ \text{IF } t - t_{up} \geq \Delta pr \text{ AND } \| \varphi (t) \| > 0 \]
\[ \text{THEN } t^+_i = t + \Delta pr, \quad t_{up} = t, \quad i \leftarrow i + 1, \]
ensure that \( \forall i \in \mathbb{N} \) the following inequalities holds: \( t^+_i > t^+_i, \)
\[ t^+_i = \Delta pr < T_i \text{ if the regressor excitation is propagated } \]
\[ \text{Proposition of 1.. is postponed to Appendix.} \]

Having at hand the estimate \( \hat{\dot{\theta}} \) and signals \( z (t), \Delta (t), \) which are obtained using the filtering (13) with resetting at time instants \( t^+_i \), the equation (10) can be derived.

**Proposition 2.** Using (5), on the basis of (13) the functions \( \upsilon (t) \) and \( \Omega (t) \) are defined as:
\[ \upsilon (t) = \left[ \begin{array}{c}
\text{adj} \{ z^B (t) z_B (t) \} z^B (t) \Delta (t) A_{ref} - z_A (t) \\
\text{adj} \{ z^B (t) z_B (t) \} z^B (t) \Delta (t) B_{ref} \\
\text{adj} \{ z^B (t) z_B (t) \} z^B (t) \nu (t)
\end{array} \right]^T, \]
\[ \Omega (t) = \text{adj} \{ z^B (t) z_B (t) \} z^B (t) \Delta (t) B_i = \]
\[ = \text{adj} \{ z^B (t) z_B (t) \} z^B (t) z_B (t) = \]
\[ = \text{det} \{ z^B (t) z_B (t) \}. \tag{16} \]
with the following notation:
\[ z_A (t) = z^T (t) L_A, \quad z_B (t) = z^T (t) L_B, \]
\[ z_B \nu (t) = z^T (t) L_B \nu, \]
\[ L_A = \left[ I_{n \times n} \quad 0_{n \times (m+p+1)} \right]^T, \tag{17} \]
\[ L_B = \left[ 0_{m \times n} \quad 1_{m \times m} \quad 0_{m \times (p+1)} \right]^T, \]
\[ L_B \nu = \left[ 0_{m \times (n+m)} \quad 1_{m \times p} \quad 0_{m \times 1} \right]^T, \]
and, if the implication \( \Phi (t) \in FE \Rightarrow \varphi (t) \in FE \) holds, then \( \forall i \in [t^+_i + T_i; t^+_{i+1}] \Omega (t) \geq \Omega (t) \geq \Omega_{LB} > 0. \)

**Proof of Proposition 2 is presented in Appendix.**

Having obtained a regression equation with a scalar regressor \( \Omega (t) \) that is a non-zero \( \forall i \in [t^+_i + T_i; t^+_{i+1}] \), in
Transients of the error (10), and adaptive law (18) were set as:

\[
A_{t} = A_{2} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad B_{0} = B_{2} = \begin{bmatrix} 0.8 & 0.8 \\ 0 & 0.8 \end{bmatrix},
\]

\[
\vartheta_{0} = \vartheta_{2} = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}, \quad \vartheta_{1} = \begin{bmatrix} -0.2 \\ 0 \end{bmatrix}, \quad \vartheta_{1} = \begin{bmatrix} 0.8 \\ -0.8 \end{bmatrix}.
\]

The parameters of the reference model (4), filters (11), (13), detection algorithm (15) and adaptive law (18) were set as:

\[
A_{ref} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad B_{ref} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \quad r (t) = \begin{bmatrix} 1 \\ e^{-t - 1} \end{bmatrix},
\]

\[
\hat{\theta} (0) = [0_{2 \times 2} I_{2 \times 2} 0_{2 \times 2}]^{T}, l = 10, \quad \sigma = 5,
\]

\[
\Delta_{pr} = 0.1, \quad \rho = 10^{25}, \quad \gamma_{0} = 1, \quad \gamma_{1} = 1.
\]

Figure 1 presents the transients of \( \hat{\tau}^{+} \) and \( \Omega (t) \).

The curves that are presented in Fig. 1 fully validated the conclusions made in Propositions 1 and 2. The detection algorithm (15) was capable to detect switches of the system (3) parameters, and the regressor \( \Omega (t) \) was nonzero after each such switch starting from some time instant if Assumption 2 was met.

Transients of \( \| \text{vec} \hat{\theta} (t) \| \) and \( \| e_{ref} (t) \| \) are shown in Figure 2.

The simulation results confirmed the Theorem. The augmented tracking error \( \xi (t) \) did exponentially converge to zero over the time ranges between two consecutive changes of the unknown parameters.

### V. Conclusion

The concept of the exponentially stable adaptive control [15] was extended to the class of MIMO switched systems
The equation (A2) is substituted into definition of $z(t)$ to obtain \((i > 0, \forall t \in \left[t^*_i; t^*_i+T_i\right]):\)

$z(t) = \Delta(t) \Omega(t) + z_1(t), \quad z_1(t) = \text{adj} \left\{ \omega(t) \right\} \left( \int_{t^*_i}^{t^*_i+T_i} e^{-l(t-\tau)} \varphi_n(\tau) \varphi_0(\tau) d\tau + \right.$

\[
\left. + \int_{t^*_i}^{t^*_i+T_i+T_1} e^{-l(t-\tau)} \varphi_n(\tau) \varphi_0(\tau) d\varphi_n(\tau) (\varphi_{i-1} - \varphi_i) \right). \tag{A3}
\]

Finally, the equations (A2) and (A3) are substituted into (A4) \((i > 0, \forall t \in \left[t^*_i; t^*_i+T_1\right]):\)

$\epsilon(t) = \Delta(t) \varphi_n(t) \varphi_0(t) - \varphi_n(t) \varphi_0(t) \varphi_1(t). \tag{A4}$

By conducting similar to (A2), (A3), (A4) reasoning for the time range \([t^*_i; t^*_i+T_1]\), we have $\varphi_0(t) = 0, \quad \varphi_1(t) = 0$, and hence the interval-based definition of $\epsilon(t)$ is valid:

$\forall i \in \mathbb{N} i \epsilon(t) = \left\{ \begin{array}{ll} \text{(A4)}, & i > 0, \forall t \in \left[t^*_i; t^*_i+T_1\right] \\
0, & \forall t \in \left[t^*_i; t^*_i+T_1\right] \end{array} \right. \tag{A5}$

from which it follows that the function $\epsilon(t)$ is indeed an indicator of the plant parameters switches is (nonzero only over the detection delay interval $[t^*_i; t^*_i+T_1]$). So, according to the proof from [20] together with the fact that Assumption 2 is met, the inequalities $\forall t \in \mathbb{N} i \epsilon_i \geq t^*_i, \quad \epsilon_i = \Delta_{pr} \leq T_i$ hold when the algorithm (A5) is applied.

Proof of Proposition 2. In accordance with the proof of Proposition 1, as far as the time range $[t^*_i; t^*_i+T_1]$ is concerned, we have $\varphi_1(t) = 0$. Then the following definitions hold:

$z_A(t) = z^T(t) \Omega_A = \Delta(t) A_i, \quad z_B(t) = z^T(t) \Omega_B = \Delta(t) B_i, \quad z_B(t) = z^T(t) \Omega_{B\tau} = \Delta(t) B_i \varphi_1^T = z_B(t) \varphi_1^T, \tag{A6}$

where $z_A(t) \in \mathbb{R}^{n \times n}, \quad z_B(t) \in \mathbb{R}^{n \times m}, \quad z_{B\tau}(t) \in \mathbb{R}^{n \times p}$ are measurable functions.

The third equation from (A6) is left-multiplied by $\text{adj} \left\{ \frac{z_B(t) z_B(t)}{z_B(t) \varphi_1^T} \right\} z_B(t)$ and each equation from (5) - $\text{adj} \left\{ z_B(t) z_B(t) \right\} z_B(t) \Delta(t)$. As a result, the definition (16) is obtained.

When $\frac{\Psi(t)}{\varphi(t)}$ is in $\mathbb{F} \left(t^*_i; t^*_i+T_1\right), \quad \left[t^*_i; t^*_i+T_1\right], \quad \varphi(t) \in \mathbb{F}(t)$, thus, if Assumption 2 is met, then according to [20] $\forall t \in \left[t^*_i+T_1; t^*_i+T_1+T_2\right] \Delta_{UB} \geq \Delta(t) \geq \Delta_{LB} > 0$.

The following equation holds $\varphi(t) \Omega(t)$:

$\Omega(t) = \det \left\{ z_B(t) z_B(t) \right\} = \Delta^0(t) \det \left\{ B_i^T B_i \right\}, \tag{A7}$

Owing to the controllability, we have $\det \left\{ B_i^T B_i \right\} \neq 0$. As a result, when $\frac{\Psi(t)}{\varphi(t)}$ is in $\mathbb{F} \left(t^*_i; t^*_i+T_1\right), \quad \left[t^*_i; t^*_i+T_1\right], \quad \varphi(t) \in \mathbb{F}(t)$ and excitation is propagated $\varphi(t) \in \mathbb{F} \Rightarrow \varphi(t) \in \mathbb{F}$, then $\forall t \in \left[t^*_i+T_1; t^*_i+T_1\right] \Omega_{UB} \geq \Omega(t) \geq \Omega_{LB} > 0$, which completes the proof.

Proof of Theorem. When $\rho \in \left[0, \Omega_{LB}\right]$, the differential equation with respect to $\hat{\varphi}(t)$ $\forall t \in \left[t^*_i; t^*_i+T_1\right]$ is written as: $
\hat{\varphi}(t) = -\left(\gamma_0 \lambda_{\text{max}} \left(\omega(t) \omega^T(t) + \gamma_1\right) \hat{\varphi}(t), \tag{A8}$

from which it immediately follows that the first statement of Theorem holds.

APPENDIX

Proof of Proposition 1. In accordance with Lemma 6.8 in [1], if $\Phi(t) \in \mathbb{F}$, then $\Phi(t) \in \mathbb{F}$, and as, following the statement of Proposition 1, $\Phi(t) \in \mathbb{F} \Rightarrow \varphi(t) \in \mathbb{F}$ is also true, then $\Phi(t) \in \mathbb{F} \Rightarrow \varphi(t) \in \mathbb{F}$. In [20] it has been proved that the implication $\varphi(t) \in \mathbb{F} \Rightarrow \Delta(t) \in \mathbb{F}$ holds. According to Assumption 2, $\Phi(t) \in \mathbb{F}$ over $t^*_i; t^*_i+T_1, \quad \left[t^*_i; t^*_i+T_1\right], \quad \varphi(t) \in \mathbb{F}$ over the time ranges.

As it was proved in [20], the algorithm (15) $\forall i \in \mathbb{N}$ ensures that $t^*_i \geq t^*_i, \quad t^*_i = \Delta_{pr} \leq T_i$ if $\varphi(t) \in \mathbb{F}, \quad \Delta(t) \in \mathbb{F}$ over the time intervals $t^*_i; t^*_i+T_1, \quad \left[t^*_i; t^*_i+T_1\right]$ and $\epsilon(t)$ is an indicator of the plant parameters switches.

Then, the only thing we need to prove is that $\epsilon(t)$ is such an indicator. To this end, the equation $\chi(t) = x(t) - \tau(t)$ is differentiated with respect to $t$:

\[
\dot{\chi}(t) = \dot{x}(t) - \tau(t) = \theta^T(t) \Phi(t) + \tau(t) t - t \tau(t) = \]

\[
= -\chi(t) + \theta^T(t) \Phi(t), \quad \chi(t) = x(t) \tag{A1}
\]

Considering the time range $t^*_i; t^*_i$, the solution of (A1) is substituted into the definition of $\varphi(t)$ $\forall t \in \left[t^*_i; t^*_i\right]$:

\[
\varphi(t) = n_s(t) \left( e^{-l(t-t^*_i-1)} x(t^*_i-1) + \right.
\]

\[
+ \int_{t^*_i}^{t^*_i+T} e^{-l(t-\tau)} \Phi(\tau) d\tau \left. \right) = \]

\[
= n_s(t) \left( e^{-l(t-t^*_i-1)} x(t^*_i-1) + \right.
\]

\[
+ \theta^T(t) e^{-l(t-\tau)} \Phi(\tau) d\tau + \right)
\]

\[
+ \theta^T \int_{t^*_i}^{t^*_i+T} e^{-l(t-\tau)} \Phi(\tau) d\tau + \right)
\]

\[
+ \theta^T \int_{t^*_i}^{t^*_i+T} e^{-l(t-\tau)} \Phi(\tau) d\tau = \left. \right) = \theta^T \left( \varphi(t) \right) + \right.
\]

\[
\left. + n_s(t) \left( \theta^T - \theta^T \right) \int_{t^*_i}^{t^*_i+T} e^{-l(t-\tau)} \Phi(\tau) d\tau \right) \tag{A2}
\]

with matched uncertainty and completely unknown control matrix. The solution was obtained by way of augmentation of the earlier-proposed parameterization with dynamic filters that reset their states at time instants, which were identified by a recently proposed algorithm to detect switching of the unknown parameters of the linear regression equations [20]. The proposed adaptive control system does not require the plant control matrix to be known, ensures the aperiodic transients of the adjustable parameters, and guarantees exponential convergence of the augmented tracking error over the time range between two consecutive switches in case the regressor is finitely exciting somewhere inside such interval. However, it can only be applied to systems without finite escape time over time range $[t^*_i; t^*_i+T_1]$.

The scope of further research is to ensure the global exponential stability of the closed-loop adaptive control system instead of the interval exponential boundedness [2].
The time interval \([t_i^+; t_i^+ + T_i]\) is divided into two ranges: \([t_i^+; t_i^+ + 1]\) and \([t_i^+ + 1; t_i^+ + T_i]\). Considering \([t_i^+; t_i^+ + 1]\), according to (A3), the equation (13) is affected by the disturbance \(\tilde{\varepsilon}_i(t)\), which is bounded owing to the definition (A3) (more details can be found in the proof of Proposition 3 in [20]). Such perturbation influences the function \(\mathcal{C}(t)\) through the parameterization (16) in a multiplicative way:

\[
\forall t \in [t_i^+; t_i^+ + 1] \quad \mathcal{C}(t) = \Omega(t) \tilde{\theta}_i + d(t),
\]

where \(d(t)\) is a new bounded disturbance.

Then the adaptive law (18) \(\forall t \in [t_i^+; t_i^+ + 1]\) is written as:

\[
\dot{\tilde{\theta}}_i(t) = - \left( \gamma_0 \max(\omega(t), \omega^2(t) + \gamma_1) \left( \tilde{\theta}_i(t) - \frac{d(t)}{1\mu t_0} \right) \right).
\]

Therefore, it follows that \(\forall t \in [t_i^+; t_i^+ + 1]\)

\[
\frac{\|d(t)\|_M}{1\mu t_0} \leq \left\| \frac{\|d(t)\|_M}{1\mu t_0} \right\|
\]

Considering the time range \([t_i^+; t_i^+ + T_i]\) and taking into account that \(\forall t \in [t_i^+; t_i^+ + T_i] \Omega(t) < \Omega_{LB} \Rightarrow \dot{\tilde{\theta}}_i(t) = 0\), the conclusion is made that the parameter error is bounded \(\forall t \in [t_i^+; t_i^+ + T_i]\)

\[
\left\| \hat{\theta}_i(t) \right\| \leq \frac{\|d(t)\|_M}{1\mu t_0}.
\]

According to the statement 4 of Assumption 2, the system (3) has no finite escape time over the time range \([t_i^+; t_i^+ + T_i]\), so, considering the conservative case, the error \(e_{ref}(t)\) tends to infinity with exponential rate only and bounded by its arbitrarily large but finite value at the right-hand-bound of the time interval \(\|e_{ref}(t)\| \leq \|e_{ref}(t_i^+ + T_i)\|\). So, then \(\forall t \in [t_i^+; t_i^+ + T_i]\)

\[
\text{the inequality } \left\| \xi(t) \right\| \leq \xi_{UB}\text{ holds.}
\]

To prove the third statement of Theorem, the function is introduced over \([t_i^+ + T_i; t_i^+ + 1]\):

\[
V = e_{ref}^T P e_{ref} + tr \left\{ \tilde{\theta}_i^T \tilde{\theta}_i \right\},
\]

where \(P\) is from (2) with \(D = 0.5 I_{n \times n}, B := B_i, A := A_{ref}\).

Applying \(tr(AB) = BA\), the derivative of (10) with respect to (11) is written as:

\[
\dot{V} = e_{ref}^T \left( A_{ref}^T P + PA_{ref} e_{ref}^T + 2e_{ref}^T P B_i \tilde{\theta}_i^T \omega - tr \left( \tilde{\theta}_i^T \tilde{\theta}_i \omega \right) \right) - e_{ref}^T P e_{ref} - & \quad \dot{\tilde{\theta}}_i^T \tilde{\theta}_i^T - \tilde{\theta}_i^T \tilde{\theta}_i^T \omega.
\]

Then, completing the square in (A11), it is obtained:

\[
\dot{V} = -\mu e_{ref}^T P e_{ref} + tr \left( -K_{TQ}^T e_{ref}^T Q K + 2\tilde{\theta}_i^T \omega \tilde{\theta}_i^T - \tilde{\theta}_i^T \omega \tilde{\theta}_i^T \Omega \tilde{\theta}_i \right) - e_{ref}^T P e_{ref} + tr \left( -\left( \omega \tilde{\theta}_i^T + \tilde{\theta}_i^T \omega \right)事\right) \leq \eta \min (\mu_{max}(P), \gamma_1 + \kappa) \tilde{\theta}_i^T \leq \eta V\]

where \(0 < \kappa \leq \gamma_0\max(\omega(t), \omega^2(t) + \gamma_1)\).

\[
\eta = \min (\mu_{max}(P), \gamma_1 + \kappa).
\]

Therefore, it immediately follows that:

\[
\|\xi(t)\| \leq \sqrt{\frac{\lambda}{\eta}} e^{-\|\xi(t)\|_2} \left\| \xi(t_i^+ + T_i) \right\|.
\]