Universality driven analytic structure of QCD crossover: radius of convergence in baryon chemical potential

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(Dated: September 11, 2019)

Recent lattice QCD calculations show strong indications that the chiral crossover of QCD at zero baryon chemical potential (\(\mu_B\)) is a remnant of the second order chiral phase transition. Furthermore, the non-universal parameters needed to map temperature \(T\) and \(\mu_B\) to the universal properties of the second order chiral phase transition have been determined recently. Motivated by these observations, first, we determine the analytic structure of the partition function — the so-called Yang-Lee edge singularity — in the QCD crossover regime, solely based on universal properties. Next, utilizing the lattice QCD results for non-universal parameters we map this singularity to the real \(T\) and complex \(\mu_B\) plane, leading to the determination of the radius of convergence in \(\mu_B\) in the QCD crossover regime. These universality- and QCD-based results provide tight constraints on the range of validity of the lattice QCD calculations at \(\mu_B > 0\). Implication of this result on the location of the conjectured QCD critical point is discussed.

INTRODUCTION

The chiral symmetry of quantum chromodynamics (QCD) is spontaneously broken in the vacuum. First-principle lattice QCD calculations have conclusively shown that the approximate chiral symmetry with physical values of quark masses gets nearly restored at a pseudo-critical temperature \(T_{pc} = 156.5 \pm 1.5\) MeV \(^1\) via a smooth crossover \([2,3]\). Lattice QCD calculations have also shown that similar chiral symmetry restoring crossover takes place at high temperature and for small-to-moderate values of \(\mu_B\); the dependence of \(T_{pc}(\mu_B)\) has been computed \([4]\). It has been conjectured that at some sufficiently large values of \(\mu_B\) the chiral restoration in QCD takes place through a first order transition; the point in the \(T - \mu_B\) phase diagram at which the chiral crossover line turns into a first order phase transition line is known as the QCD critical point (for a review, see Ref. \([4]\)).

While experimental searches to locate this conjectured QCD critical point are on-going at RHIC and SPS, presently, first-principle lattice QCD calculations only provide very limited guidance on the existence and location of the QCD critical point in the \(T - \mu_B\) phase diagram, because a direct lattice QCD calculation at \(\mu_B \neq 0\) is hindered by the fermion sign problem. The present lattice calculations providing information on the QCD thermodynamics — either by carrying out Taylor expansions around \(\mu_B = 0\) \([5]\) or through analytic continuation from purely imaginary values of \(\mu_B\) \([6-8]\) — crucially rely on the assumption that the QCD partition function is an analytic function of complex \(\mu_B\) within a radius of convergence. To what extent these lattice QCD results are trustworthy, and how far in \(\mu_B\) these methods might be extended can be answered only if we have reliable knowledge of the radius of convergence of the QCD partition around \(\mu_B = 0\) in the complex \(\mu_B\) plane. In this work we will provide an estimate for location of the singularity, nearest to \(\mu_B = 0\) and for \(T \sim T_{pc}\), of the QCD partition function in the complex \(\mu_B\) plane, based on the expected universal behavior of the partition function with massless up and down quarks.

ANALYTIC STRUCTURE OF THE QCD FREE ENERGY FOR SMALL QUARK MASS

For massless up and down quarks the chiral symmetry restoration in QCD is expected to happen via a ‘true’ second order chiral phase transition even at \(\mu_B = 0\) (for a review, see Ref. \([4]\)). Recent progresses in lattice QCD calculations have shown that for \(\mu_B = 0\) the chiral phase transition of 2+1-flavor QCD, with massless up and down quarks and a physical strange quark, takes place at the chiral transition temperature \(T^c_\mu = 132^{+3}_{−5}\) MeV \([9]\). The universality class of the chiral phase transition of 2+1-flavor QCD was found to be consistent with that of the three-dimensional \(O(4)\) spin model \([9,12]\).

If the light up/down quark mass, \(m_l\), is small enough then universality of chiral phase transition dictates the behavior of the order parameter (chiral condensate) \([12]\)

\[
M(T, m_l, \mu_B) = \left(\frac{m_l}{m_s^{\text{phys}}}\right)^{\frac{1}{2}} f_G(z) + F_{\text{reg}}(T, m_l, \mu_B),
\]

where the so-called scaling variable, \(z\), is given by

\[
z = z_0 \left(\frac{m_l}{m_s^{\text{phys}}}\right)^{-\frac{1}{2}} \times \left[\frac{T - T^c_\mu}{T^c_\mu} + \kappa^2_2 \left(\frac{\mu_B}{T^c_\mu}\right)^2 + \kappa^4_4 \left(\frac{\mu_B}{T^c_\mu}\right)^4 + \ldots\right].
\]
Here, \( f_G, \beta \) and \( \delta \) are the relevant scaling function and critical exponents; \( z_0 \) and the curvature parameters of \( T_{p\ell}(\mu_B, \alpha^B_{2\Delta}) \), are non-universal but unique definite for QCD. Current lattice QCD calculations obtained that \( \kappa^B \) is consistent with zero within the precision of the calculation [1]; based on this we set \( \kappa^B \) and possible higher order corrections denoted by ellipses to zero in what follows. The physical strange quark mass is denoted by \( m_{B_{phys}} \). The function \( F_{reg} \) characterizes small deviations, if any, from the scaling behavior, and is analytic in all its arguments. Lattice calculations have provided evidence that chiral observables of QCD with physical values up/down quark masses are well-described by Eq. (1) by including only small corrections from \( F_{reg} \) [3,14,15]. Since \( F_{reg} \) is not expected to have any singularities close to zero chemical potential, the analyticity of \( M \) in the complex-\( \mu_B \) plane is governed by the analytic structure of the universal function \( f_G \). The corresponding universal chiral behavior of the QCD free energy is given by (\( m_{B_{phys}}^{1+2/\beta} f(z) \)), where the scaling function \( f(z) \) is related to \( f_G \) through: \( f_G = z f'(z)/(\beta \delta) - (1 + 1/\delta) f(z) \); the prime denotes the derivative with respect to \( z \).

It is well-known that in the complex-\( z \) plane the function \( f_G \) has a singularity, the so-called Yang-Lee edge singularity (and its complex conjugate, which is implicitly assumed in the remaining text), of the form \( f_G \sim (z - z_c)^\sigma \) [14]. From this the behavior of the free energy around \( z_c \) can be easily deduced by focusing on the most singular contribution: \( f(z) \sim (z - z_c)^{1+\sigma} \). Note that, the singularity of the free energy, \( 1 + \sigma \) is not related to the \( O(N) \) specific heat critical exponent, \( 2 - \alpha \), as one naively might assume. Additionally, the critical exponent \( \phi \) introduced in Ref. [15] for the crossover singularities is nothing but \( \sigma \). The Yang-Lee edge singularity can be treated as an ordinary critical point, belonging to the \( Z(2) \) universality class of \( \phi^3 \) theory in three spatial dimension, and with an imaginary coupling [13]. For all finite values of \( N \) of three-dimensional \( O(N) \) universality class the value of the critical exponent is \( \sigma = 0.085(1) \) [13,16]. The argument that \( z_c \) is known in terms of the \( O(N) \) critical exponents [17,18]: \( z_c = |z_c| e^{i \pi \sigma} \). Owing to the universality of \( f_G \) as a function of (complex) \( z \) the value of \( |z_c| \) also is universal, but its value for \( O(4) \), or for a general \( O(N) \), is not known. However, \( z_c \) can be analytically calculated in two limits—mean-field and \( N \to \infty \).

### ANALYTIC STRUCTURE OF \( g \) IN THE MEAN-FIELD AND \( N \to \infty \) LIMITS

In both limits, the mean-field and the \( N \to \infty \), \( f_G \) can be represented in the general form [19]

\[
f_G^{\text{mean}} \left[ z + f_G^z \right] = 1. \quad (3)
\]

The specific cases can be obtained by plugging in the corresponding critical exponents — \( \beta = 1/2 \) (note that this is partially accounted in Eq. (3)), \( \delta = 3 \) for mean-field, and \( \beta = 1/2, \delta = 5 \) for \( N \to \infty \). For the mean-field case this equation can also be obtained straightforwardly from the free energy of the \( \phi^4 \) Landau-Ginzburg theory [20]. To determine the Yang-Lee edge singularity branch-point, \( z_c \), from \( f_G \) we consider the inverse function \( z(f_G) \). The branch-point can be obtained from the condition \((dz/df_G)|_{z_c} = 0 \), leading to

\[
\frac{1}{\delta - 1} \left[ z_c + f_G^{1/\beta}{(z_c)} \right] + f_G^{1/\beta}{(z_c)} = 0. \quad (4)
\]

Considering only the branch closest to \( z = 0 \) on the physical Riemann sheet, i.e., the one connected to \( z = 0 \) as defined by the standard normalization condition \( f_G(0) = 1 \), Eqs. (3) and (4) completely determine

\[
f_G(z_c) = \left( \frac{1}{1 - \delta} \right)^{2(\sigma - 1)/\delta}, \quad z_c = (1 - \delta)^{-\frac{1}{\delta - 1}} \left( \frac{1}{\delta - 1} \right)^{\frac{2\sigma}{2\delta - 1}}.
\]

More explicitly,

\[
\begin{align*}
  z_c &= \frac{3}{2^{2/\phi}} e^{\frac{\pi}{4}}, \quad \text{mean-field}, \\
  z_c &= \frac{5}{2^{2/\phi}} e^{\frac{\pi}{4}}, \quad N \to \infty. \tag{5}
\end{align*}
\]

To obtain the value of \( \sigma \) one needs to expand \( f_G \) around \( z_c \): \( f_G(z) - f_G(z_c) \sim (z - z_c)\sigma \). Substituting this expansion into Eq. (3) it is easy to see \( \sigma = 1/2 \) for both the cases, independent of the critical exponents \( \beta, \delta \). Also, the value of \( \sigma \) for the mean-field approximation and in the \( N \to \infty \) limit are different from the \( N\)-independent value of \( O(N) \) models with a finite \( N \). Recent analysis of the lattice data [21] based on the mean-field value of the critical exponent \( \sigma \) thus might be improved by using the actual value instead.

### RADIUS OF CONVERGENCE IN THE COMPLEX-\( \mu_B \) PLANE

Following Eqs. (1) and (2), for a fixed value of \( m_l > 0 \), such as in QCD, the derivatives of \( M \) with respect to \( \mu_B \) are proportional to the derivatives \( f_G(z) \) with respect to \( z \). Thus, the convergence of the Taylor expansion in \( \mu_B \) around \( \mu_B = 0 \), as well as analytic continuation in the complex-\( \mu_B \) plane are bounded by the value of \( z_c \). Specifically, for a given \( T \) the Taylor series about zero chemical potential will have the radius of convergence given by

\[
\left| \frac{z_c}{z_0} \left( \frac{m_l^{\text{phys}}}{m_l^{\text{phys}}} \right)^{1/\beta} \right|^{1/2} - \frac{T - T_G}{T_G^{1/2}} < \frac{T}{\sqrt{m_l^{\text{phys}}}}. \tag{6}
\]

At \( T = T_G^{0} \), the radius of convergence is directly proportional to \( |z_c| \).

The previous discussion clearly demonstrate that, in the vicinity of the chiral scaling regime, validity of the Taylor expansions in \( \mu_B \) and the analytic continuations in
complex-μB plane of the QCD free energy is determined by the value of |zc|. As mentioned before, 2+1-flavor lattice QCD calculations show that the chiral condensate, M0, for the physical value of light up/down quark mass, m1phys = m̄sphys/27, are well-described by the 3-dimensional O(4) scaling function fG, with inclusion of small corrections from the analytic function Freg [9, 11–13]. Obviously, the contributions of Freg will unavoidably modify the convergence of the low-order Taylor coefficients, however, these analytic contributions will not change the radius of convergence. Thus, also for physical QCD the singularity nearest to μB = 0 in the complex-μB plane will be dictated by zc. If zc is known then Eq. (2) can be used to translate this singularity to the complex-μB plane and, thereby, determine the corresponding radius of convergence. The rest of the universal and non-universal parameters entering Eq. (2) are known—(i) The critical exponents of the O(4) universality class β = 0.380, δ = 4.824 [22], (ii) Both m̄sphys and T are purely real, (iii) Tc0 = 132.3 ± 6 MeV [9], (iv) The curvature of the pseudo-critical temperature Te(μB), κ2μB = 0.012(2) [1]. (v) Based on the lattice QCD results of Ref. [9] on Te(m1) the scale factor is estimated to be z0 ≃ 1–2 [23]. For the 3-dimensional O(N) universality class |zc| is not known. Currently the best estimate for |zc| is available from preliminary Functional Renormalization Group studies [24]; they show that |zc| is close to the value obtained in the large N limit. We suspect that this is partially because the critical exponent δ of O(4) is close to the one in large N limit. Due to the lack of a better estimate for O(4) we approximate |zc| with the corresponding value in this limit. The difference between a mean-field approximation and the N → ∞ result is within 15%. Therefore, to demonstrate systematic uncertainty we vary the N → ∞ value of |zc| by 15%.

In Figure 1 we show the radius of convergence in μB in the T − μB plane for different values of m1 = 0 − m̄sphys in the N → ∞ limit value for zc, z0 = 2, O(4) critical exponents, and other lattice QCD-determined non-universal parameters described above. Note that, in the chiral limit, QCD free energy is essentially singular at T = Te0, μB = 0 and, therefore, the radius of convergence at this point is zero.

Figure 2 provides a more realistic estimate for the radius of convergence in μB in the T − μB plane for m̄sphys by varying |zc| around its N → ∞ limit value and z0 = 1–2. While the variation of |zc| leads to a limited uncertainty of the radius of convergence, more precise lattice QCD result for z0 is needed to improve this estimate.

CONCLUSIONS

Relying only on the universal behavior of QCD in the chiral crossover region we investigated the analytic behavior of the free energy. We argued that if the chiral behavior of QCD is well-described by the universal scaling, as borne out in recent the lattice QCD calculations, then analyticity of the free energy will be completely governed by the analytic structure of the corresponding universal scaling function in the complex scaling variable. We estimated the relevant singularity of the scaling function based on the two extreme limits of mean-field and N → ∞. We showed how these estimates can be translated to the singularity in the complex-μB plane to determine the radius of convergence in μB, solely based on the universal critical exponents and well-determined non-universal parameters from lattice QCD calculations. Figure 2 summarizes our universality- and QCD-based estimate for the radius of convergence in μB for temperatures in the vicinity of the QCD chiral crossover. This shows that the radius of convergence is larger than |μB| ≥ 400 MeV, implying that the present lattice QCD calculations based on Taylor expansions in μB and analytic continuations from imaginary values of μB can be
reliable below this region, as suggested also by recent lattice QCD calculations [15, 24].

The present state-of-the-art lattice QCD calculations do not find any evidence for an additional singularity for \( \mu_B \lesssim 400 \text{ MeV} \) [11, 15, 24]. Our result on the radius of convergence \( |\mu_B| \gtrsim 400 \text{ MeV} \), coupled with these lattice QCD results, suggest that QCD critical point, if one exists, will most likely be located at \( \mu_B \gtrsim 400 \text{ MeV} \).

Such conclusion will potentially have an important impact on the on-going beam energy scan experiments at RHIC and SPS, as well as on the future experiments, such as at FAIR and NICA, planned to search for the QCD critical point. In fact, in the temperature-complex chemical potential space, the Yang-Lee singularity is smoothly connected to critical point. As shown in Refs. [17], a critical point is located where the Yang-Lee singularity and its complex conjugate pinch the real chemical potential axis. Thus, the curves in Figure 2 may also help understand how to map critical Ising direction, \( t \), to QCD, which is of relevance not only for static but also dynamic properties [26] near a possible critical point.

The curvature of the transition surface is also known for charge and strange chemical potentials, \( \mu_Q, S \). For the latter, \( \kappa^S_2 \) is nearly the same as \( \kappa^B_2 \); thus we expect to get the same radius of convergence in \( \mu_S \) as in \( \mu_B \). With our estimate that the radius of convergence is about 400-500 MeV in the relevant temperature range, the Yang-Lee singularity may obstruct the one corresponding to the kaon condensation; this entails that we might not be able to probe it with Taylor series expansion/analytic continuation. For the charge chemical potential, \( \kappa^Q_2 \) (along with \( \kappa^B_2 \)) is about factor of two larger than \( \kappa^B_2 \); thus the radius of convergence in charge chemical potential is about factor \( \sqrt{2} \) smaller that the one for the baryon chemical potential; it, however, is larger than the pion mass. Under the assumption that the pion mass has only weak temperature dependence, we expect that the pion condensation would be possible to probe through Taylor series expansion at zero chemical potential.

ACKNOWLEDGMENTS

This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of Nuclear Physics: (i) Through the Contract No. DE-SC0012704; (ii) Through the contract No. DE-SC0020081; (iii) Within the framework of the Beam Energy Scan Theory (BEST) Topical Collaboration. V.S. also thanks the ExtreMe Matter Institute EMMI (GSI Helmholtzzentrum für Schwerionenforschung, Darmstadt, Germany) for partial support and their hospitality.

We thank Bengt Friman, Frithjof Karsch, Lex Kemper, Robert Pisarski, Krzysztof Redlich, Thomas Schaefer, and Mithat Unsal for illuminating discussions.

We thank the organizers of EMMI Workshop “Probing the Phase Structure of Strongly Interacting Matter: Theory and Experiment”, which inspired us to work together on this project.

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We thank Anirban Lahiri for providing us this input on behalf of the HotQCD collaboration.

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