A note on limiting mean distribution of singular values for products of two Wishart random matrices

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Abstract

The product of $M$ complex random Gaussian matrices of size $N$ has recently been studied by Akemann, Kieburg and Wei. They show that, for fixed $M$ and $N$, the joint probability distribution for the squared singular values of the product matrix forms a determinantal point process with correlation kernel determined by certain biorthogonal polynomials that can be explicitly constructed. We find that, in the case $M = 2$, the relevant biorthogonal polynomials are actually special cases of multiple orthogonal polynomials associated with Macdonald functions (modified Bessel functions of the second kind) which was first introduced by Van Assche and Yakubovich. With known results on asymptotic zero distribution of these polynomials and general theory on multiple orthogonal polynomial ensembles, it is then easy to obtain an explicit expression for the distribution of squared singular values for the products of two complex random Gaussian matrices in the limit of large matrix dimensions.

Keywords: singular values, products of Wishart matrices, determinantal point process, multiple orthogonal polynomial ensembles, Macdonald functions

1 Introduction and statement of the results

Products of random matrices have presently attracted the most interest due to their important applications in statistical physics relating to disordered and chaotic dynamical systems [12], and other fields beyond physics like MIMO (multiple-input and multiple-output) networks in telecommunication [23, 24], etc.. The central issue of the study is to find the distributions of the eigenvalues or singular values for the products in different regimes and...
various methods have been applied to perform the spectral analysis; cf. recent physical literatures \[1, 3, 4, 8, 9, 16\] and mathematical literatures \[6, 14\].

This note originates from a paper by Akemann, Kieburg and Wei \[3\] concerning the correlation functions for the products of \(M\) quadratic random matrices with complex elements and no symmetry. More precisely, let \(X_j, j = 1, \ldots, M\) be independent complex matrices of size \(N\) with identical, independent Gaussian distribution

\[
\exp \left[ -\text{Tr}(X_j^* X_j) \right],
\]

where the superscript \(*\) stands for conjugate transpose. The interest lies in the singular values of the product \(P_M\) defined by

\[
P_M := X_M X_{M-1} \cdots X_1.
\]

Note that for \(M = 1\), it is the well-known chiral Gaussian Unitary Ensemble. By using change of variable and Harish-Chandra-Itzykson-Zuber (HCIZ) integral formula, it is shown that the joint probability density function \(P_{jpdf}(x_1, \ldots, x_N)\) for the squared singular values of \(P_M\) is given by (see \[3, \text{Equation (2.13)}\])

\[
C^{(M)}_{N} \prod_{1 \leq i < j \leq N} (x_j - x_i) \frac{\det}{\det} \left[ G^{M,0}_{0,M} \left( 0, \ldots, -j, 0, j - 1 \mid x_i \right) \right],
\]

where

\[
\left( C^{(M)}_{N} \right)^{-1} = N! \prod_{i=1}^{M} \Gamma(i)^{M+1}
\]

is the normalization constant and \(G^{M,0}_{0,M}\) are the so-called Meijer G-functions which are integrals involving Gamma functions (cf. \[21, \text{Chapter 16}\]). The formula (1.3) is also called “one-matrix” representation in \[3\].

Since the matrix inside the determinant in (1.3) is labeled by indices of the Meijer G-function, it is not convenient for the computation of \(k\)-point correlation function. To overcome this difficulty, they derive an alternative “two-matrix” formulation in the sense of \[2\], which is the joint probability density function for the squared singular values of a single matrix \(X_1\) and the squared singular values of the entire product matrix \(P_M\). This setting leads to another representation of \(P_{jpdf}(x_1, \ldots, x_N)\) with the help of the following biorthogonal polynomials.

Let \(p_j^{(M)}(x)\) and \(q_k^{(M)}(y)\) be two sequences of monic polynomials (i.e., with leading coefficient one) of degree \(j\) and \(k\) respectively, and satisfy the biorthogonality relations

\[
\int_{0}^{\infty} \int_{0}^{\infty} \omega^{(M)}(x,y) p_j^{(M)}(x) q_k^{(M)}(y) \, dx \, dy = \delta_{j,k} h_j^{(M)}, \quad j, k = 0, 1, 2, \ldots,
\]

(1.4)
with squared norms $h_j^{(M)} = (j!)^{M+1}$ and the weight function as

$$w^{(M)}(x, y) = y^{-1} e^{-y G^{M-1.0}_{0,M-1}} \left( - \frac{x}{y} \right), \quad M > 1,$$

(1.5)

see [3] Equations (3.2) and (3.3)]. In particular, one has

$$w^{(2)}(x, y) = y^{-1} e^{-y G^{1.0}_{0,1}} \left( - \frac{x}{y} \right) = y^{-1} e^{-\frac{x}{y} - y}.$$  

(1.6)

The formulas for $p_j^{(M)}(x)$ and $q_k^{(M)}(y)$ can be derived explicitly. Indeed, we have

$$q_k^{(M)}(x) = (-1)^k k! L_k(x)$$  

(1.7)

is the monic Laguerre polynomials associated with weight $e^{-t}$, where $L_k$ is the standard Laguerre polynomials of degree $k$ (cf. [21, Chapter 18]), and

$$p_k^{(M)}(x) = \sum_{j=0}^{k} \frac{(-1)^{k-j}}{(k-j)!} \left( \frac{k!}{j!} \right)^{M+1} x^j;$$  

(1.8)

see [3] Equations (3.19) and (3.21)].

The squared singular values of $P_M$ then forms a determinantal point process of the form

$$P_{jpdf}(x_1, \ldots, x_N) = \frac{1}{N!} \det_{1 \leq i,j \leq N} \left[ K_N^{(M)}(x_i, x_j) \right],$$

(1.9)

where $K_N^{(M)}$ is the correlation kernel given by

$$K_N^{(M)}(x, y) = \sum_{j=0}^{N-1} \frac{1}{h_j^{(M)}} \left( \int_0^\infty w^{(M)}(x, s) q_j^{(M)}(s) \, ds \right) p_j^{(M)}(y)$$  

(1.10)

with $h_j^{(M)}$ as in (1.4). It is shown in [3] that the two representations (1.3) and (1.9)–(1.10) are equivalent.

A natural question now is to establish the limiting mean distribution of the squared singular values for $P_M$. It is observed in [3] that, after proper scaling, the macroscopic limits of the correlation kernel $K_N^{(M)}$ exist for each $M$. For $M = 1$, it is given by the famous Marchenko-Pastur density, as expected. For $M \geq 2$, however, it is mentioned there the explicit formula is difficult to derive. Alternatively, they give the algebraic equation satisfied by the Stieltjes transform of the limits, following the approach in [8]. The relevant solution of the equation for the case $M = 2$ is presented as well. It is the aim of this note to point out that for $M = 2$, we still have an explicit formula for the limit. Our main theorem is stated as follows.
Theorem 1.1. Let $X_1$ and $X_2$ be two independent complex matrices of size $N$ with Gaussian distribution as in (1.1). If we denote by $\lambda_i$, $i = 1, \ldots, N$ the squared singular values of $X_1X_2$ (i.e., $M = 2$ in (1.2)), then, almost surely, the associated scaled empirical measure

$$
\mu_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i/N^2},
$$

(1.11)

where $\delta_x$ stands for the Dirac point mass at $x$, converges weakly and in moments to a probability measure $\mu$ on the positive real axis with density

$$
\frac{d\mu}{dx}(x) = \begin{cases} 
\frac{4}{27} h\left(\frac{4x}{27}\right), & x \in (0, \frac{27}{4}), \\
0, & \text{elsewhere},
\end{cases}
$$

(1.12)

where

$$
h(y) = \frac{3\sqrt{3}}{4\pi} \frac{(1 + \sqrt{1 - y})^{1/3} - (1 - \sqrt{1 - y})^{1/3}}{y^{2/3}}, \quad 0 < y < 1,
$$

(1.13)

as $N \to \infty$.

This theorem also implies the statement

$$
\lim_{N \to \infty} NK_N^{(2)}(N^2x, N^2x) = \frac{d\mu}{dx}(x), \quad x > 0,
$$

where $K_N^{(2)}(x, y)$ is defined in (1.10).

The rest of this note is mainly devoted to the proof of Theorem 1.1. The essential issue is the observation that the biorthogonal polynomials $p_k^{(2)}$ in (1.8) are actually certain multiple orthogonal polynomials (cf. Nikishin and Sorokin [20, Chapter 4, §3], Ismail [17, Chapter 23]) that have already been studied by the community of orthogonal polynomials. In view of the natural appearance of multiple orthogonal polynomials in certain random models from mathematical physics over the past few years, including random matrix theory, non-intersecting paths, etc. (cf. [18, 19] and references therein), this note provides another example that bridges two different groups. We end this paper with some concluding remarks.

2 Proof of the main theorem

2.1 Multiple orthogonal polynomials associated with Macdonald functions

Multiple orthogonal polynomials are polynomials of one variable which are defined by orthogonality relations with respect to $r$ different measures, where
$r \geq 1$ is a positive integer. They can be viewed as a generalization of orthogonal polynomials, which have important applications in approximation theory and number theory; cf. [5, 20, 25].

For our purpose, let us define a function $\rho_\gamma$ by

$$\rho_\gamma(x) = 2x^{\gamma/2} K_\gamma(2\sqrt{x}), \quad x > 0,$$

where $K_\gamma$ is the Macdonald function (modified Bessel function of the second kind; see [21, Chapter 10]) and consider two weights

$$d\mu_1(x) = x^\kappa \rho_\gamma(x) \, dx, \quad d\mu_2(x) = x^\kappa \rho_{\gamma+1}(x) \, dx, \quad \kappa > -1, \quad \gamma \geq 0,$$

on the positive real line. For any vector $(k, m) \in \mathbb{N}^2$, the type II multiple orthogonal polynomials $p^{(\gamma,\kappa)}_{k,m}$ for the system of weights $(\mu_1, \mu_2)$ are such that $p^{(\gamma,\kappa)}_{k,m}$ is a monic polynomial of degree $k + m$ and satisfies the following multiple orthogonality conditions:

$$\int_0^\infty p^{(\gamma,\kappa)}_{k,m}(x)x^j d\mu_1(x) = 0, \quad j = 0, 1, ..., k - 1,$$  

(2.3)

$$\int_0^\infty p^{(\gamma,\kappa)}_{k,m}(x)x^j d\mu_2(x) = 0, \quad j = 0, 1, ..., m - 1.$$  

(2.4)

By taking $m = k$, we set

$$P^{(\gamma,\kappa)}_k(x) = p^{(\gamma,\kappa)}_{k,k}(x), \quad P^{(\gamma,\kappa)}_{2k+1}(x) = p^{(\gamma,\kappa)}_{k+1,k}(x).$$

An explicit formula for $P^{(\gamma,\kappa)}_k$ is given by

$$P^{(\gamma,\kappa)}_k(x) = \sum_{j=0}^k a_k(j)x^{k-j},$$  

(2.5)

where

$$a_k(j) = (-1)^j \binom{k}{j} \frac{(\kappa + 1)_k(\kappa + \gamma + 1)_k}{(\kappa + 1)_k(\kappa + \gamma + 1)_k j}, \quad 0 \leq j \leq k;$$  

(2.6)

see [11, Theorem 2]. Since the measures $(d\mu_1, d\mu_2)$ from (2.2) form an AT system (cf. [20]), it is known that all the zeros of $P^{(\gamma,\kappa)}_k$ are simple, lie in $(0, +\infty)$, and satisfy the interlacing property [10]. Furthermore, it is shown in [27] that $P^{(\gamma,\kappa)}_k$ satisfies the following four-term recurrence relation

$$xP^{(\gamma,\kappa)}_k(x) = P^{(\gamma,\kappa)}_{k+1}(x) + b_kP^{(\gamma,\kappa)}_k(x) + c_kP^{(\gamma,\kappa)}_{k-1}(x) + d_kP^{(\gamma,\kappa)}_{k-2}(x)$$  

(2.7)

with recurrence coefficients

$$b_k = (k + \kappa + 1)(3k + \kappa + 2\gamma) - (\kappa + 1)(\gamma - 1),$$  

$$c_k = k(k + \kappa)(k + \kappa + \gamma)(3k + 2\kappa + \gamma),$$  

$$d_k = k(k - 1)(k + \kappa - 1)(k + \kappa)(k + \kappa + \gamma - 1)(k + \kappa + \gamma).$$  

(2.8)
These polynomials constitute one of few examples of multiple orthogonal polynomials that are not related to the classical orthogonal polynomials. They are first introduced by Van Assche and Yakubovich in [27], which solve an open problem posed by Prunikov [22].

We now introduce a new parameter \( n \in \mathbb{N} \) and put

\[
P^{(\gamma,\kappa)}_{k,n}(x) := \frac{P^{(\gamma,\kappa)}_k(n^2x)}{n^{2k}}.
\] (2.9)

Clearly, \( P^{(\gamma,\kappa)}_{k,n}(x) \) is a monic polynomial of degree \( k \) for each \( n \). For each \( P^{(\gamma,\kappa)}_{k,n} \), we can associate the normalized counting zero measure defined by

\[
\nu(P^{(\gamma,\kappa)}_{k,n}) = \frac{1}{k} \sum_{P^{(\gamma,\kappa)}_{k,n}(x)=0} \delta_x.
\] (2.10)

A measure \( \nu^\xi \) is called the asymptotic zero distribution of \( \{P^{(\gamma,\kappa)}_{k,n}\} \) if

\[
\lim_{k/n \to \xi} \int f \, d\nu(P^{(\gamma,\kappa)}_{k,n}) = \int f \, d\nu^\xi
\] (2.11)

for every bounded continuous function \( f \) on \( \mathbb{R} \), i.e., it is the weak limit of the measures \( \nu(P^{(\gamma,\kappa)}_{k,n}) \). Here the notation \( \lim_{k/n \to \xi} \) means that both \( k, n \to \infty \) with \( k/n \to \xi > 0 \).

The explicit formula for \( \nu^\xi \) is established by Coussement, Coussement and Van Assche as stated in the following proposition (see [10, Theorem 2.7]).

**Proposition 2.1.** The asymptotic zero distribution of the multiple orthogonal polynomials associated with Macdonald functions (2.9) exists and has the density

\[
\frac{d\nu^\xi}{dx}(x) = \begin{cases} 
\frac{4}{27\pi} h\left(\frac{4y}{27\xi^2}\right), & x \in (0, \frac{27\xi^2}{4}), \\
0, & \text{elsewhere},
\end{cases}
\] (2.12)

where

\[
h(y) = \frac{3\sqrt{3}}{4\pi} \frac{(1 + \sqrt{1-y})^{1/3} - (1 - \sqrt{1-y})^{1/3}}{y^{2/3}}, \quad 0 < y < 1.
\] (2.13)

**Remark 2.2.** In practice, one can also scale the parameters \( (\kappa, \gamma) \), say, putting \( \kappa \mapsto pn \) and \( \gamma \mapsto qn \) with \( p, q > 0 \). The case when \( \kappa \) and \( \gamma \) are fixed corresponds to taking the limits \( p, q \to 0 \), respectively. Under this general setting, the normalized zero counting measure (2.10) converges weakly to the first component of a vector of two measures which satisfies a vector equilibrium problem with two external fields and the explicit formula for the equilibrium vector is given in terms of solutions of an algebraic equation; see [28] for details. The vector equilibrium problem that characterizes \( \nu^1 \) is also presented in Section 3 below.

With the above preparations, we are ready to prove Theorem 1.1.
2.2 Proof of Theorem 1.1

A comparison of (1.8) and (2.5)–(2.6) immediately gives that

\[ p^{(2)}_k(x) = P^{(0,0)}_k(x), \tag{2.14} \]

thus, \( p^{(2)}_k \) are multiple orthogonal polynomials with respect to two weight functions \( 2K_0(2\sqrt{x}), 2K_1(2\sqrt{x}) \) for \( x > 0 \). This, together with (2.10) and Proposition 2.1 implies that, as \( N \to \infty \), the measure

\[ \nu_N := \frac{1}{N} \sum_{p^{(2)}_N(x)=0} \delta_{x/N^2} = \nu(P^{(0,0)}_{N,N}) \tag{2.15} \]

converges weakly to \( \nu^1 \), which agrees with the measure \( \mu \) defined in Theorem 1.1.

To show the measure \( \mu \) is also the almost sure weak convergence of the empirical measure (1.11), we note that \( p^{(2)}_N \) is actually the average characteristic polynomials with respect to the multiple orthogonal polynomial ensemble (1.9), i.e.,

\[ p^{(2)}_N(x) = \mathbb{E} \left[ \prod_{i=1}^N (z - x_i) \right], \quad z \in \mathbb{C}, \]

where \( \mathbb{E} \) refers to (1.9) with \( M = 2 \); cf. [18, Proposition 2.2]. For a determinantal point process on \( \mathbb{R} \) that belongs to biorthogonal ensembles [7], a sufficient condition that the limiting zero distribution of the average characteristic polynomials coincides with the almost sure weak convergence of the empirical measure of this determinantal point process is recently established by Hardy [15], which in particular includes multiple orthogonal polynomial ensembles as a special case. By [15, Theorem 1.3 and Corollary 1.5], it suffices to show that

- the recurrence coefficients for \( p^{(2)}_N(n^2x)/n^{2N} = P^{(0,0)}_{N,n}(x) \) are bounded for every \( N, n \) and certain \( \varepsilon > 0 \) such that \( |\frac{N}{n} - 1| \leq \varepsilon \);
- \( \nu(P^{(0,0)}_{N,n}) \) defined in (2.10) converges to \( \nu^\xi \) in moments, i.e., for any \( l \in \mathbb{N} \),

\[ \lim_{N/n \to \xi} \int x^l \, d\nu(P^{(0,0)}_{N,n}) = \int x^l \, d\nu^\xi, \tag{2.16} \]

where \( \xi \in [0, 1 + \varepsilon) \).

These two conditions are easily verified in our case. Indeed, from (2.7) and (2.8), one has

\[ xP^{(0,0)}_{N,n}(x) = P^{(0,0)}_{N+1,n}(x) + b_{N,n}P^{(0,0)}_{N,n}(x) \]
\[ + c_{N,n}P^{(0,0)}_{N-1,n}(x) + d_{N,n}P^{(0,0)}_{N-2,n}(x), \tag{2.17} \]
with recurrence coefficients given by

\[ b_{N,n} = \frac{3N(N+1) - 1}{n^2} \to 3\xi^2, \]
\[ c_{N,n} = \frac{3N^4}{n^4} \to 3\xi^4, \quad (2.18) \]
\[ d_{N,n} = \frac{N^3(N-1)}{n^6} \to \xi^6, \]

if \( N/n \to \xi \). Thus, there exists \( \varepsilon > 0 \), such that

\[ \max_{|\frac{N}{n} - 1| \leq \varepsilon} (|b_{N,n}|, |c_{N,n}|, |d_{N,n}|) < + \infty. \quad (2.19) \]

The equations (2.17) and (2.18) also implies that \( P_{N,n}^{(0,0)}(\lambda) = 0 \) if and only if \( \lambda \) is an eigenvalue of the banded Toeplitz matrix \( T_N \) defined by

\[
T_N = \begin{pmatrix}
    b_{N,n} & 1 & 0 & \ldots & \ldots & 0 \\
    c_{N,n} & b_{N,n} & 1 & \ddots & \vdots & \\
    d_{N,n} & c_{N,n} & b_{N,n} & 1 & \ddots & \\
    0 & d_{N,n} & c_{N,n} & b_{N,n} & \ddots & 0 \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    0 & \ldots & 0 & d_{N,n} & c_{N,n} & b_{N,n}
\end{pmatrix}_{N \times N}.
\]

(2.20)

By Gershgorin circle theorem, it follows that the spectral radius \( \rho_N \) of \( T_N \) is bounded by

\[ \sup_{N \leq n} |b_{N,n}| + \sup_{N \leq n} |c_{N,n}| + \sup_{N \leq n} |d_{N,n}| + 1. \]

This, together with (2.19), implies \( \sup_N \rho_N < + \infty \). We then have that the support of \( \nu(P_{N,n}^{(0,0)}) \) is uniformly bounded. Since it is already known that \( \nu(P_{N,n}^{(0,0)}) \) converges weakly to \( \nu^\xi \), we obtain the convergence of \( \nu(P_{N,n}^{(0,0)}) \) to \( \nu^\xi \) in moments.

This completes the proof of our main theorem.

3 Concluding remarks

In this note, we have given the explicit formula for the limiting distribution of squared singular values for the products of two complex random Gaussian matrices. Our approach is based on the finite-\( N \) density derived in [3] and identifying the relevant biorthogonal polynomials with the special cases of multiple orthogonal polynomials associated with Macdonald functions. It
would be interesting to see whether the general multiple orthogonal polynomials associated with Macdonald functions appear in other random matrix models. A possible candidate might be the products of rectangular matrices [8].

The next challenge is to establish the microscopic limit of the correlation kernel [1,10], which will lead to universality results for the products of Wishart random matrices. From Theorem [1,1] it is readily seen that, for $M = 2$, the density function of the limiting mean distribution vanishes like a squared root at the soft edge $27/4$, and blows up of order $x^{-2/3}$ at the hard edge. This observation implies the expectation to, as also pointed out in [3], the universal sin and Airy kernels for the bulk and soft edge of the spectrum, but new universality classes are required for the description of the local behavior at the hard edge. Due to the appearance of multiple orthogonal polynomials, a possible way to tackle this problem is to perform Deift/Zhou steepest descent analysis [13] for the associated Riemann-Hilbert problem [26]. To this end, it is worthwhile to mention an alternative characterization of the measure $\mu$ given in Theorem [1,1]. Consider the energy functional defined by

$$
\iint \log \frac{1}{|x - y|} d\nu_1(x) d\nu_1(y) + \iint \log \frac{1}{|x - y|} d\nu_2(x) d\nu_2(y) - \iint \log \frac{1}{|x - y|} d\nu_1(x) d\nu_2(y) + \int \sqrt{4x} d\nu_1(x).
$$

We want to minimize this functional over all vectors of measures $(\nu_1, \nu_2)$ such that $\text{supp}(\nu_1) \subset [0, \infty)$, $\int d\nu_1 = 1$ and $\text{supp}(\nu_2) \subset (-\infty, 0]$, $\int d\nu_2 = 1/2$. It is shown by Román and the author in [28] that the measure $\mu$ is the first component of the unique minimizer. We believe such kind of characterization will play a important role in the nonlinear steepest descent analysis of the Riemann-Hilbert problem.

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