ON THE FUNDAMENTAL GROUP OF $\mathbb{R}^3$ MODULO THE CASE-CHAMBERLIN CONTINUUM

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Abstract. It has been known for a long time that the fundamental group of the quotient of $\mathbb{R}^3$ by the Case-Chamberlin continuum is nontrivial. In the present paper we prove that this group is in fact, uncountable.

1. Introduction

In the 1960’s, during the early days of the decomposition theory, the quotient space $X^3$ of the Euclidean 3-space $\mathbb{R}^3$ by the classical Case-Chamberlin continuum $C$ (see [3]) was one of the most interesting examples. One of the most important questions was whether $X^3$ is simply connected. It was settled – in the negative – by Armentrout [1] and Shrikhande [10]. However, it remained an open problem until present day to determine how big is the fundamental group of $X^3$. In this paper we give the solution for this problem – namely, we show that the fundamental group $\pi_1(\mathbb{R}^3/C)$ is uncountable.

Consider the Case-Chamberlin inverse sequence $\mathcal{P}$ (see [3], [5, p.628]):

$$P_0 \xleftarrow{f_0} P_1 \xleftarrow{f_1} P_2 \xleftarrow{f_2} \ldots$$

where $P_0 = \{p_0\}$ is a singleton, $P_i$ is a bouquet of two circles $S^1_{a_i} \vee S^1_{b_i}$, and $p_i$ is the base point of the bouquet $S^1_{a_i} \vee S^1_{b_i}$, for every $i > 0$.

Fix an orientation on each of the circles of the bouquet. Let $f_i : S^1_{a_{i+1}} \vee S^1_{b_{i+1}} \to S^1_{a_i} \vee S^1_{b_i}$ be a piecewise linear mapping which maps the base point $p_{i+1}$ to the base point $p_i$ and maps the natural generators $a_{i+1}$ and $b_{i+1}$ of $\pi_1(S^1_{a_{i+1}} \vee S^1_{b_{i+1}})$ to the commutators $[a_i, b_i]$ and $[a_i^2, b_i^2]$ of $\pi_1(S^1_{a_i} \vee S^1_{b_i})$, respectively.

The Case-Chamberlin continuum $C$ is then defined as the inverse limit $\lim\leftarrow \mathcal{P}$ of the Case-Chamberlin inverse sequence $\mathcal{P}$ (see [3]). Obviously, $C$ is a 1-dimensional continuum and therefore it is embeddable in $\mathbb{R}^3$ (see [4]). It is well-known that the homotopy types of the quotient space $\mathbb{R}^3/f(C)$ are the same for all embeddings $f$ of $C$ into $\mathbb{R}^3$ (see [2]). The main result of our paper is the following theorem:

**Theorem 1.1.** Let $C$ be the Case-Chamberlin continuum embedded in $\mathbb{R}^3$. Then the fundamental group $\pi_1(\mathbb{R}^3/C)$ of the quotient space $\mathbb{R}^3/C$ is uncountable.
2. Preliminaries

Let $G$ be a group. By the commutator of the elements $a$ and $b$ of $G$ we mean the element $[a, b] = a^{-1} b^{-1} ab$ of $G$. Let $G_n$ be the lower central series which is defined inductively (see [9]):

$$G_1 = G, \quad G_{n+1} = [G_n, G],$$

where $[G_n, G]$ is the group generated by the set $\{[a, b] : a \in G_n, b \in G\}$.

Obviously, $G_n \supseteq G_{n+1}$, for every $n$. By the weight $w(g)$ of an element $g \in G$ we mean the maximal number $n$ such that $g \in G_n$ if such a number exists, and $\infty$ otherwise. So the weight of any element of a perfect group is equal to $\infty$. We shall need the following result from [8] Ch. I, Proposition 10.2:

**Proposition 2.1.** : For any free group $F$ the lower central series $F_n$ has trivial intersection, i.e. $\bigcap_{i=1}^{\infty} F_n = \{e\}$.

That is, in any free group the weight of an element $x$ is finite if and only if $x \neq e$. Let

$$C(f_0, f_1, f_2, \ldots)$$

be the infinite mapping cylinder of $\mathcal{P}$ (see e.g. [7] [11]) and let $\tilde{\mathcal{P}}$ be its natural compactification by the Case-Chamberlin continuum $C$. Let $\mathcal{P}^*$ be the quotient space of $\tilde{\mathcal{P}}$ by $C$.

Obviously, $\mathcal{P}^*$ is homeomorphic to the one-point compactification of an infinite 2-dimensional polyhedron $C(f_0, f_1, f_2, \ldots)$. Let

$$C(f_k, f_{k+1}, f_{k+2}, \ldots)$$

be the mapping cylinder of the inverse sequence:

$$P_k \xrightarrow{f_k} P_{k+1} \xrightarrow{f_{k+1}} P_{k+2} \xrightarrow{f_{k+2}} \ldots.$$

We shall denote the corresponding one-point compactification by

$$C(f_k, f_{k+1}, f_{k+2}, \ldots)^*.$$

We shall consider $C(f_k, f_{k+1}, f_{k+2}, \ldots)^*$ as a subspace of $\mathcal{P}^*$ and we shall denote the compactification point by $p^*$.

We consider $P_i$, for $i \geq 0$, as a subspace of $C(f_0, f_1, \ldots)$ and we consider $C(f_k, f_{k+1}, f_{k+2}, \ldots)$, for $k \geq 0$, as a subspace of $\tilde{\mathcal{P}}$. Obviously, $P_1$ is a strong deformation retract of $C(f_1, f_2, \ldots)$. We have the following homomorphism

$$\varphi_{i+1} = (f_1 \cdots f_i )_\#: \pi_1(P_{i+1}) \rightarrow \pi_1(P_1)$$

which is a monomorphism, since it is the composition of monomorphisms $(f_i)_\#: \pi_1(P_{i+1}) \rightarrow \pi_1(P_i)$. Note that for a fixed $i$, the elements $[a_i, b_i]$ and $[a_i^2, b_i^2]$ are free generators of a subgroup $(f_i)_\#(\pi_1(P_{i+1}))$ of $\pi_1(P_1)$ (see Exercise 12 on p.119 of [9]).

Since $\varphi_1$ is a monomorphism, we can consider the group $\pi_1(P_1)$ as a subgroup of $\pi_1(P_i) = F$, where $F$ is a free group on two generators $a_1$ and $b_1$. In particular, by identification, we have

$$a_2 = [a_1, b_1], \quad a_3 = [a_2, b_2] = [[a_1, b_1], [a_1^2, b_1^2]], \ldots$$

Since $a_i \neq e$, the weight $w(a_i)$ is a finite number (cf. Proposition 2.1 above). It follows by definition of $a_i$ that $w(a_i) \geq i$, for every $i$. 
Choose an increasing sequence of natural numbers \( \{n_i\} \) as follows: Let \( n_0 = 1 \) and \( n_1 = 2 \). If \( n_k \) is already defined, then let \( n_{k+1} \) be any natural number such that \( n_{k+1} > w(a_{n_k}) \) for \( k \geq 1 \). Then we have \( a_{n_k} \not\in F_{n_{k+1}} \).

Let \( I_i \) be the unit segment which connects the points \( p_{i+1} \) and \( p_i \) and which corresponds to the mapping cylinder of the mapping \( f_i|_{\{p_{i+1}\}} \) of the one-point set \( \{p_{i+1}\} \) to the one-point set \( \{p_i\} \), for \( i \geq 0 \).

To define a certain kind of loops we need a new notion. For two paths \( f, g : \mathbb{I} \to X \) satisfying \( f(1) = g(0) \), let \( fg : \mathbb{I} \to X \) be the path defined by:

\[
fg(s) = \begin{cases} 
    f(2s) & \text{if } 0 \leq s \leq 1/2, \\
    g(2s - 1) & \text{if } 1/2 \leq s \leq 1.
\end{cases}
\]

We also let

\[
\overline{f}(s) = f(1 - s) \text{ for } 0 \leq s \leq 1.
\]

Two paths are simply said to be homotopic, if they are homotopic relative to the end points. A loop in \( X \) is a path \( f : \mathbb{I} \to X \), satisfying \( f(0) = f(1) \). For a sequence of units and zeros

\[
\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \cdots), \quad \varepsilon_i \in \{0, 1\}
\]

define a path \( g_\varepsilon : \mathbb{I} \to \mathcal{P}^* \) so that the following properties hold:

1. \( g_\varepsilon(0) = p_1 \) and \( g_\varepsilon(1) = p^* \),
2. \( g_\varepsilon \) maps \( [(2k - 2)/(2k - 1), (2k - 1)/2k] \) homeomorphically onto \( \bigcup_{i=1}^{n} I_i \) starting from \( p_{n_k-1} \) to \( p_{n_k} \) for \( k \geq 1 \), and
3. \( g_\varepsilon \) maps \( [(2k-1)/2k, 2k/(2k+1)] \) onto \( S^1_{a_{n_k}} \) as a winding in the positive direction, if \( \varepsilon_k = 1 \), and \( g_\varepsilon \) maps \( [(2k - 1)/2k, 2k/(2k + 1)] \) to the point set \( \{p_{n_k}\} \) constantly otherwise, for \( k \geq 1 \).

Let \( h : \mathbb{I} \to \mathcal{P}^* \) be a path from \( p^* \) to \( p_1 \) which maps \( \mathbb{I} \) homeomorphically onto \( \bigcup_{i=1}^{\infty} I_i \cup \{p^*\} \). Finally, let \( f_\varepsilon = g_\varepsilon h \). Then \( f_\varepsilon \) is a loop with base point \( p_1 \) corresponding to \( a_\varepsilon = a_{n_1}^{\varepsilon_1}a_{n_2}^{\varepsilon_2}a_{n_3}^{\varepsilon_3} \cdots \).

3. **Proof of Theorem 1.1**

For our proof of Theorem 1.1 we shall need the following two lemmata:

**Lemma 3.1.** Let \( C \) be the Case-Chamberlin continuum embedded in \( \mathbb{R}^3 \). Then the quotient space \( \mathbb{R}^3/C \) is homotopy equivalent to the 2-dimensional compactum \( \mathcal{P}^* \).

**Proof.** The proof is completely analogous to the proof of the first assertion of Theorem 1.1 of [1] and therefore we shall omit it. \( \square \)

**Lemma 3.2.** Let \( p_0, p_1, p^* \) be distinct points in a Hausdorff space \( X \) and let \( f \) be a loop with base point \( p_1 \) such that \( f^{-1}(\{p_0\}) \) is empty and \( f^{-1}(\{p^*\}) \) is a singleton. If \( f \) is null-homotopic, then there exists a loop \( f' \) in \( X \setminus \{p_0, p^*\} \) such that \( f \) and \( f' \) are homotopic in \( X \setminus \{p_0\} \).

**Proof.** Since \( f \) is null-homotopic, we have a homotopy \( F : \mathbb{I} \times I \to X \) from \( f \) to the constant mapping to \( p_1 \), i.e.

\[
F(s, 0) = f(s), F(s, 1) = F(0, t) = F(1, t) = p_1 \text{ for } s, t \in I.
\]
Let \( \{s_0\} \) be the singleton \( f^{-1}(\{p^*\}) \). Let \( M \) be the connectedness component of \( F^{-1}(\{p^*\}) \) containing \( (s_0,0) \), and \( O \) the connectedness component of \( \mathbb{I} \times \mathbb{I} \setminus M \) containing \( \mathbb{I} \times \{1\} \). Define \( G : \mathbb{I} \times \mathbb{I} \to X \) by:

\[
G(s,t) = \begin{cases} 
F(s,t) & \text{if } (s,t) \in O, \\
p^* & \text{otherwise.}
\end{cases}
\]

Then \( G \) is also a homotopy from \( f \) to the constant mapping to \( p_1 \) and \( G^{-1}(\{p_0\}) \) is contained in \( O \).

Consider \( G^{-1}(\{p^*,p_0\}) \cap O \) and \( \mathbb{I} \times \mathbb{I} \setminus O \). By definition of \( M \), \( G^{-1}(\{p^*,p_0\}) \cap O \) is compact and disjoint from \( (\mathbb{I} \times \mathbb{I} \setminus O) \cup \mathbb{I} \times \{0\} \). Using a polygonal neighborhood of \( (\mathbb{I} \times \mathbb{I} \setminus O) \cup \mathbb{I} \times \{0\} \) whose closure is disjoint from \( G^{-1}(\{p^*,p_0\}) \cap O \), we get a piecewise linear injective path \( g : \mathbb{I} \to \mathbb{I} \times \mathbb{I} \) such that

\[
\text{Im}(G \circ g) \subseteq X \setminus \{p_0,p^*\}, \quad g(0) \in \{0\} \times \mathbb{I}, \quad \text{and } g(1) \in \{1\} \times \mathbb{I}
\]

and \( \text{Im}(g) \) divides \( \mathbb{I} \times \mathbb{I} \) into two components, one of which contains \( G^{-1}(\{p_0\}) \) and the other contains \( M \cup \mathbb{I} \times \{0\} \). We now see that \( G \circ g \) is the desired loop \( f' \).

**Proof of Theorem 1.1** By Lemma 5.1 it clearly suffices to consider \( \pi_1(\mathcal{P}^*) \) instead of \( \pi_1(\mathbb{R}^3/C) \). Suppose therefore, that the group \( \pi_1(\mathcal{P}^*) \) were at most countable. We can assume that \( p_1 \) is the base point of the space \( \mathcal{P}^* \) and all of its subspaces considered below. Since the set of all sequences of units and zeros is uncountable, there would then exist an uncountable set \( E \), such that for every \( \varepsilon, \varepsilon' \) from \( E \), the loops \( f_\varepsilon \) and \( f_{\varepsilon'} \) with the base point \( p_1 \) would be homotopy equivalent. Fix a loop \( f_{\varepsilon_0} (\varepsilon_0 \in E) \).

Then every loop \( f_\varepsilon \) is null-homotopic for every \( \varepsilon \in E \). Since \( \{s : g_\varepsilon g_{\varepsilon_0}(s) = p^*\} \) is a singleton, we can apply Lemma 3.2 to \( g_\varepsilon g_{\varepsilon_0} \). Since \( f_\varepsilon f_{\varepsilon_0} \) is homotopic to \( g_\varepsilon g_{\varepsilon_0} \in \mathcal{P}^* \setminus \{p_0,p^*\} \), we conclude that \( f_\varepsilon f_{\varepsilon_0} \) is homotopic to a loop \( f'_{\varepsilon} \) in \( \mathcal{P}^* \setminus \{p_0,p^*\} \), where the homotopy is in \( \mathcal{P}^* \setminus P_0 \).

Since \( E \) is uncountable and \( \mathcal{P}^* \setminus \{p_0,p^*\} \) is homotopy equivalent to the bouquet of two circles \( S^1 \sqcup S^1 \), that is, \( \pi_1(\mathcal{P}^* \setminus \{p_0,p^*\}) \) is countable, there exist distinct \( \varepsilon \) and \( \varepsilon' \) in \( E \) such that \( f'_{\varepsilon} \) is homotopic to \( f'_{\varepsilon'} \) in \( \mathcal{P}^* \setminus \{p_0,p^*\} \) and hence in \( \mathcal{P}^* \setminus P_0 \). It follows that \( f_{\varepsilon} f_{\varepsilon_0} \) is homotopic to \( f_{\varepsilon'} f_{\varepsilon_0} \), and hence \( f_{\varepsilon} \) is homotopic to \( f_{\varepsilon'} \) in \( \mathcal{P}^* \setminus P_0 \). Let \( k \) be the minimal number such that \( \varepsilon_k \neq \varepsilon_k' \), say \( \varepsilon_k = 1 \) and \( \varepsilon_k' = 0 \). Let \( Y_k \) be the quotient space of \( \mathcal{P}^* \setminus P_0 \) by the the closed subspace \( C(f_{k+1},f_{k+2},f_{k+3},\ldots) \).

Consider the projection

\[
q : \pi_1(\mathcal{P}^* \setminus P_0) \to \pi_1(Y_{n+1})
\]

and let \( [f_{\varepsilon}] \) and \( [f_{\varepsilon'}] \) be the homotopy classes containing \( f_{\varepsilon} \) and \( f_{\varepsilon'} \) respectively. Since \( a_{n_k+1}, b_{n_k+1} \in F_{n_k+1}, F/F_{n_k+1} \) is a quotient group of \( \pi_1(Y_{n+1}) \). Then,

\[
q([f_{\varepsilon}]) = q(a_{n_k}^{\varepsilon_1} \cdots a_{n_k}^{\varepsilon_{k-1}})q(a_{n_k}) \text{ and } q([f_{\varepsilon'}]) = q(a_{n_k}^{\varepsilon'_1} \cdots a_{n_k}^{\varepsilon'_{k-1}}). \]

Since \( a_{n_k} \notin F_{n_k+1}, \) it follows that \( q(a_{n_k}) \) is non-trivial and hence \( f_{\varepsilon} \) is not homotopic to \( f_{\varepsilon'} \) in \( \mathcal{P}^* \setminus P_0 \). This contradiction shows that our initial assumption was false and therefore \( \pi_1(\mathcal{P}^*) \cong \pi_1(\mathbb{R}^3/C) \) is indeed an uncountable group, as asserted.

**Question 3.3.** Let \( C \) be the Case-Chamberlin continuum embedded in \( \mathbb{R}^3 \). Is the first singular homology group with integer coefficients \( H_1(\mathbb{R}^3/C;\mathbb{Z}) \) of the quotient space \( \mathbb{R}^3/C \) also uncountable?
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