A systematic study of the radion in the compact Randall-Sundrum model

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We systematically study the question of identification and consistent inclusion of the radion, within the Lagrangian approach, in a two brane Randall-Sundrum model. Exploiting the symmetry properties of the theory, we show how the radion can be identified unambiguously and give the action to all orders in the radion field and the metric. Using the background field method, we expand the theory to quadratic orders in the fields. We show that the most general classical solutions, for the induced metric on the branes in the case of a constant radion and a factorizable 4D metric, correspond to Einstein spaces. We discuss extensively the diagonalization of the quadratic action. Furthermore, we obtain the 4-dimensional effective theory from this and study the question of the spectrum as well as the couplings in these theories.

I. INTRODUCTION

The idea that we may live in a world of more than four space-time dimensions is certainly not new [1], but in recent years it has received a lot of attention because of the observation that the extra compact dimensions need not all be of the order of the Planck length scale. A large extra dimension of the order of a $(TeV)^{-1}$ [2] provides an alternate solution to the hierarchy problem (the desert scenario), namely, it explains in a natural manner the large ratio for the Planck scale to the electro-weak scale. There are basically two interesting approaches to solving this problem. The first, due to Arkani-Hamed, Dimopoulos and Dvali [2], is based on the standard Kaluza-Klein approach, where the higher dimensional space is a direct product of the 4 dimensional space-time and a compact manifold. The second, due to Randall and Sundrum (RS) [3,4], uses an alternate and interesting scenario, where the higher dimensional space-time is not factorizable.

The simplest RS model can be envisaged as follows. Let us consider a five dimensional space-time manifold, with the extra coordinate taking values on an orbifold $S^1/Z_2$, with two fixed points, which are chosen to be at, say, 0 and $\pi$. At each of the two fixed points of the orbifold, there is a singular source - a 3-brane - which carries a constant, nonzero tension. In addition, a negative five dimensional cosmological constant is also assumed to be present in the bulk, which leads to a five-dimensional metric that is a slice of the AdS space [3]. Since the warp factor turns out to be an exponential function depending on the extra coordinate, the ratio of the values of the warp factor at the two branes, for such a metric, naturally leads to the ratio of the Planck to $TeV$ scales without the need for an extremely large extra dimension. We can think of our physical world as described by one of the two branes, conventionally chosen to be the one at $\pi$. In the original RS model, the values of the five dimensional cosmological term and the tensions of the two branes are chosen in such a way, that the effective four dimensional cosmological term is zero and the induced metric is just the Minkowski metric. In this model, the physical distance between the two branes is an arbitrary constant $r_c$, whose value cannot be determined from the classical analysis. The stabilization of the size of the extra dimension requires additional mechanisms [3]. Even when the five dimensional cosmological constant and the brane tensions are not fine tuned to cancel (as is the case in the original RS model), it was shown in [3] that the model admits solutions with induced metrics of dS or AdS type. In such a case, the distance between the two branes can be determined in terms of the parameters of the model.

The classical solutions of the RS model, as discussed above, correspond only to the vacuum sector of the theory when there is no matter present. One can, of course, add matter to the theory, which is assumed to be located on the physical brane. In principle, one can also add bulk matter. In addition, one can also analyze the fluctuations of the five dimensional gravity around the vacuum solution, as well as the fluctuation of the distance between the branes. In this paper, we will not consider the inclusion of matter, and will investigate only the contributions arising from the fluctuations of the metric and the geometry. To fix the terminology, let us note that the four dimensional scalar field, which describes the distance between the two branes, is called the radion. The vacuum expectation value of this
field is expected to determine the constant $r_c$, although we will not worry about the mechanism that leads to this vacuum expectation value. The radion was first introduced in [3,4], where fluctuations of gravity and the radion were simply added to the vacuum values of the metric and $r_c$ respectively. Although this is quite natural and is standard in quantum field theory, it was later pointed out in [7] that the ansatz of [3,4] does not solve the linearized Einstein equations. Instead, the authors of [7] used an alternate form of the metric, that is correct to linear order in the radion fluctuations.

The inclusion of the radion field was of great importance in the construction of the supersymmetric version of the RS model [8,9], where the bosonic sector of the theory, in addition to other fields, contains the radion and the graviton (but not its KK tower) to all orders in these fields. The ansatz used in [3] coincides to linear order with that of [7], while that in [8] corresponds to [3,4]. The two papers contain similar results, which are valid up to second order in the space-time derivatives. The inclusion of the radion in the case of dS and AdS branes has also been discussed in [10], using an ansatz of the form of [3,4].

We believe that, at present, we do not have a complete understanding of the radion in the RS model and it is the proper identification and the consistent inclusion of the radion field in an arbitrary two brane model which is the purpose of this paper. We derive the complete five dimensional Lagrangian for the RS model, including the radion and the graviton (with its KK tower), and study systematically the question of the spectrum of the theory by restricting the Lagrangian up to second order in these fields. We follow the geometrical approach of [7], where the authors identified the radion field from an analysis of the symmetries of the Einstein equations in the bulk, and derived the form of the five dimensional metric up to linear order in the radion. The work of [7] emphasizes the importance of the junction conditions which, as we will see, are of utmost importance in our investigation as well and are quite relevant for understanding the mixing between the radion and the graviton and its KK tower. However, in our opinion, there are some features in the analysis of [7] that remain obscure and we study this problem systematically from the conventional Lagrangian approach. We formulate the RS model as the model of five dimensional space-time with two 3-branes at the fixed points of an orbifold, whose action is manifestly invariant under arbitrary five dimensional transformations, and we carefully elucidate the symmetry properties of the model. Such an approach makes the identification of the radion field unambiguous and free of assumptions. Subsequently, employing the background field method, we are able to make a very general and systematic investigation of the possible vacuum solutions in the RS model as well as give a satisfactory 5D Lagrangian description (that holds for any vacuum solution) of the radion and the graviton and its KK tower, up to second order in the fields, without mixing between them.

The paper is organized as follows. In section 2, we describe the model, both in an interval as well as on the orbifold. Taking advantage of the symmetries of the theory, we parameterize the metric in a manner that makes the identification of the radion natural and is convenient for our subsequent discussions. The Lagrangian density of the theory is then expressed completely in terms of this parameterization to all orders in the field variables. In section 3, we use the background field method to expand the action as well as the junction conditions to quadratic order in the field variables. The classical equations are solved in a unified manner and we show that a factorizable background metric, in general, defines an Einstein space multiplied by a warp factor. We then discuss the question of diagonalization of the quadratic action as well as the boundary conditions in a systematic manner. In section 4, we rewrite this action as an effective action in 4-dimensions and discuss further the properties of this diagonalized theory. In section 5, we present a brief summary of our results. In appendix A, we describe some relations that are useful in the background field expansion of the theory while in appendix B, we discuss the Kaluza-Klein decomposition of the metric as well as some of the properties of the basis functions.

II. THE MODEL

Let us consider a 5-dimensional space-time manifold with signature $(-,+,+,+,+)$, which is parameterized by the usual 4-dimensional space-time coordinates as well as a fifth coordinate that is bounded by the locations of two 4-dimensional hypersurfaces. The hypersurfaces can be specified by equations of the forms

$$f_L(x, z) = 0, \quad f_R(x, z) = 0$$

The functions $f_{L,R}$ are, in general, arbitrary except for the condition that everywhere on the boundary hypersurfaces, the normal fields, given by $n = df_{L,R}(x, z)|_{f=0}$, are space-like. The locations of the two hypersurfaces can be determined by inverting (1), namely, $z_L = \phi_L(x), \quad z_R = \phi_R(x)$.
where $\phi_{L,R}(x)$ are not necessarily small. Therefore, $z \in I_z := [\phi_L(x), \phi_R(x)]$. From 12 it is clear that under arbitrary 4D transformations, $z' = z$, $x' = x'(x)$, the functions $\phi_{L,R}$ transform as 4D scalars.

On this 5D space-time manifold, one can define a theory which is invariant under arbitrary 5D transformations. The action has the form

$$ S = S_{\text{bulk}} + S_L + S_R $$

where

$$ S_{\text{bulk}} \propto \int_{\text{bulk}} d^5 V \sqrt{-G} \left( -R^{(5)} - \Lambda + \ldots \right) $$

$$ S_i \propto \int_{I_i=0} d^4 V \sqrt{-g_i} ( -V_i + \ldots ), \quad i = L, R $$

Here, $G_{MN}$ and $G$ are the 5D metric and its determinant respectively, while $g_{i}^{\mu\nu}$ is the 4D induced metric on the respective boundary hypersurface. In this paper, we use the convention that capital roman letters denote 5D indices, while lower case roman letters represent 4D indices. We also use the notations and definitions of 11 throughout. $\Lambda$ is a 5D cosmological constant, $V_i$ are the tensions on each of the branes and the dots denote possible terms representing 5D as well as 4D matter, which we ignore in the present study.

To obtain the equations of motion, we need to vary the action (3). As long as the positions of the boundaries are not specified (i.e. $f_{L,R}$ are arbitrary), it is not enough to consider the variation of the metric alone. We must also extremize the action with respect to the “volume” where the theory is defined. To do so, it will be more convenient to make use of the 5D coordinate invariance of the model and choose a special coordinate system, where the volume does not depend on dynamical quantities such as $f_{L,R}$ and, consequently, is not subject to variations.

Such a transformation can be done in two steps. First, we adopt Gaussian Normal (GN) coordinates with respect to the brane $\phi_L$. In this case, we have $\phi_L = \text{constant}$, which we choose to be zero, namely, $\phi_L = 0$. In this coordinate system, the second boundary is located at $\phi_R = \hat{\phi}(x)$ and the 5D metric takes the form, $G_{m5} = 0$, $G_{55} = 1$. Clearly, the function $\hat{\phi}(x)$ represents the physical distance between the two boundaries, and can be related to the radion. Next, we rescale the fifth coordinate $z = \hat{\phi}(x)t/\pi$, $t \in I_t := [0, \pi]$ ($t$ does not correspond to time which is denoted by $t^0$). Introducing a new function $\phi(x) := \hat{\phi}(x)/\pi$, we have the two boundaries at the fixed end points $t = 0$ and $t = \pi$ respectively and the 5D volume takes the form: $M^4 \times I_t$. (A more general transformation of this type has already been discussed in 12.) Note that neither the above transformations nor the field $\phi(x)$, which we call radion, have to be small. It is worth noting here that $\phi(x)$ is a four dimensional scalar field as the radion should be and it has no dependence on the extra coordinate, a reflection of the fact that there is no Kaluza-Klein tower for the radion.

In these coordinates, the 5D metric takes the form

$$ G_{MN} = \begin{pmatrix} g_{mn}(x, t) & N_n \\ N_m & \phi^2(x) \end{pmatrix} $$

and the inverse metric is determined to be

$$ G^{MN} = \begin{pmatrix} g^{mn}(x, t) + \frac{N^m N^n}{N} & -\frac{N^n}{N} \\ -\frac{N^m}{N} & \frac{1}{N} \end{pmatrix} $$

where the raising and the lowering of the 4D indices, $m, n, \ldots$, is done with the metric $g_{mn}(x, t)$ satisfying $g_{mn}(x, t)g^{nk}(x, t) = \delta^k_n$. Furthermore, $N_m(x, t) = \frac{1}{2}((\phi^2(x))_m$ and $N(x, t) = \phi^2(x) - N_m(x, t)N^m(x, t)$ where a comma denotes a derivative. We see that the metric in these coordinates has a form similar to that of Arnowitt-Deser-Misner 13. Let us note that, in these coordinates, the induced metric on each of the boundaries is given by

$$ g_{mn}^i(x) = g_{mn}(x, t)|_{t=t_i}. $$

Furthermore, it is easy to show that

$$ \det (G_{MN}(x, t)) = N \det (g_{mn}(x, t)). $$

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This defines the theory on an interval $t \in I = [0, \pi]$. Let us next replace the interval, $I_t$, by the orbifold $S^3/Z_2$. In order to do that, we extend the variable $0 \leq t \leq \pi$ to the interval $[-\pi, \pi]$, imposing the additional symmetry $t \rightarrow -t$. As a result, we need to replace $t$ by $\tau = |t|$ in the metric (6) so that $g_{mn}(x, \tau)$ is a symmetric and non-degenerate, but otherwise arbitrary tensor and,

$$N_m(x, \tau) = \frac{\tau}{2} (\phi'^2(x).m)$$

$$N(x, \tau) = \phi'^2(x) - N_m(x, \tau)N^m(x, \tau)$$

Throughout the paper, we will denote by a dot (e.g. $\dot{a}$) differentiation with respect to $\tau$, while a prime (e.g. $a'$) denotes differentiation with respect to $t$. We denote the 4D partial derivatives by a comma (e.g. $a_{,m}$), while a covariant derivative is represented by a semicolon.

The action (8), on the orbifold, can be written as

$$S = \sum_{i=0, \pi} S_i$$

$$S_{\text{bulk}} = \int d^4x \int_{-\pi}^{\pi} dt \sqrt{-G} \left(-2M^3 R^{(5)} - \Lambda \right)$$

$$S_i = \int d^4x \int_{-\pi}^{\pi} dt \sqrt{-g^i} \left(-\nabla_i \delta(t - t_i) \right), \quad t_i = 0, \pi$$

where $R^{(5)}$ is the Ricci scalar constructed from the 5D metric $G_{MN}$ given in (8) and generalized to the orbifold, as discussed above. In studying the solutions of this theory, the standard approach would be to solve the Einstein equations following from the action (11). However, we note that, since all components of the metric, $R_{MN}$, are even functions of the special coordinate choice we have made, it is natural to recast the action first in terms of the 4D metric, $g_{mn}(x, \tau)$, and the radion field, $\phi(x)$, using this parameterization. After some lengthy algebra, we obtain

$$S = -2M^3 \int d^4x \int_{-\pi}^{\pi} dt \sqrt{-g} \left(\mathcal{L} + \sum_{i=0, \pi} \mathcal{L}_i \right)$$

where

$$\mathcal{L} = \sqrt{N} \left( R^{(4)} + \frac{\Lambda}{2M^3} + \frac{N^mN^n}{N} R^{(4)}_{mn} \right) - \frac{1}{4N} \left[ g_{mn}(g^{mn})' + (g^{mn})'(g_{mn})' \right] + \mathcal{L}_{\text{high}}$$

$$\mathcal{L}_{\text{high}} = \frac{\sqrt{N}}{2N^2} \left( g^{mn}N^sN_t(\phi'^2)_{,m}N_{,n} - N^s_{,m}N_t^m(\phi'^2)_{,n} \right) + 2N^mN^n \left[ N^k_{,k}N_{m,n} - N^k_{,m}N_{k,n} \right]$$

$$\mathcal{L}_i = \frac{\nabla_i}{2M^3} \delta(t - t_i)$$

In the above expression, $R^{(4)}_{mn}$ is the Ricci tensor constructed from the metric $g_{mn}$, and a semicolon denotes the covariant derivative with respect to this metric. Because of the dependence on the fifth coordinate, all these operations are to be carried out for fixed values of $t$. The tensor nature of each of the terms, with respect to 4D coordinate transformations, can be read off easily.

Here, we would like to emphasize the fact that, although our starting theory was manifestly invariant under 5D coordinate transformations, only a residual 4D symmetry survives in (12) because of the special choice of coordinates made. Thus, action (12) is invariant under arbitrary 4D-coordinate reparameterizations, $x \rightarrow x'(x)$. Let us also note that, since all components of the metric, $G_{MN}$, are even functions of the extra coordinate, it follows from the chain rule, $F'(\tau) = \dot{F}(\tau)\tau'$, that expressions linear in $\frac{d}{d\tau}$ derivative are odd and, therefore, do not contribute to the action. We have omitted such expressions in (13). Furthermore, we have also dropped surface terms arising from integration by parts. This, therefore, gives the complete action involving the graviton (and its KK tower) and the radion.
III. BACKGROUND METHOD

To study the solutions of this theory systematically, we use the background field method [14,15] (see in particular section 4 in [13]). We write each of the field variables as a sum of a classical (possibly large) background and a small fluctuation, and expand the action up to quadratic terms in the fluctuations. For the metric we write

\[ g_{mn}(x, \tau) = \tilde{g}_{mn}(x, \tau) + h_{mn}(x, \tau) \]
\[ g^{mn}(x, \tau) = \tilde{g}^{mn}(x, \tau) - h^{mn}(x, \tau) + \delta h_{k}(x, \tau)h^{kn}(x, \tau) \tag{16} \]

while, for the radion, we use the decomposition

\[ \phi(x) = r(x) + r(x)d(x) \tag{17} \]

where \( \tilde{g}_{mn}(x, \tau) \) and \( r(x) \) are the background fields which are assumed to satisfy the classical field equations, while \( h_{mn} \) and \( d \) are the corresponding fluctuations (note that we have introduced \( d \) as a dimensionless field). The new (background) metric satisfies \( \tilde{g}_{mk}\tilde{g}^{kn} = \delta_m^n \), and from now on all indices are lowered and raised with the background metric \( \tilde{g}_{mn} \) and its inverse. For simplicity, we will refer to this as the metric.

Our next task is to decompose the action up to second order in \( h \) and \( d \). The zeroth order action leads to the classical equations for the background fields. The linear terms vanish, since the backgrounds satisfy the classical field equations and, therefore, the first nontrivial term in the expansion of the action corresponds to the quadratic part of the action in the \( h \) and the \( d \) fields. We note here that our interest lies in the case where the classical solutions lead to \( r(x) = r_c \) = constant primarily for two reasons. Such a case would correspond to have the highest symmetry and the calculations will be much simpler. When \( r(x) = r_c \) is a constant, it is easy to see that the terms in (14) would contain terms that are at least third order in \( d(x) \) and, therefore, will not be relevant for our analysis. It is for this reason that we have separated out these terms in (13) and we will neglect this term in the rest of our discussions. On the other hand, if non-constant solutions for \( r(x) \) are of interest, then, the term in (14) will contribute to the expansion up to quadratic order.

Before carrying out the expansion, however, let us discuss the Israel junction conditions [16] that are crucial in analyzing solutions in this theory. Let us note that we are considering a theory with singular sources and, in such a case, boundary conditions on the branes are extremely important in determining solutions. The junction conditions for the metric \( g_{mn} \), can be derived following [17], and in the present case of an orbifold read as

\[ \dot{g}_{mn}(x, \tau)|_{t=t_i} = -\xi(t) \frac{V_i}{12M^3} \sqrt{N(x, \tau)}g_{mn}(x, \tau)|_{t=t_i} \tag{18} \]

where

\[ \xi(t) = \begin{cases} 1 & \text{when } t = 0, \\ -1 & \text{when } t = \pi. \end{cases} \tag{19} \]

Decomposing the metric and the radion as in (16), (17) and using (10) as well as the results from appendix A, we get order by order in the fluctuations

\[ \dot{\tilde{g}}_{mn}(x, \tau)|_{t=t_i} = -\xi(t) \frac{V_i r_c}{12M^3} \tilde{g}_{mn}(x, \tau)|_{t=t_i} \tag{20} \]
\[ \dot{h}_{mn}(x, \tau)|_{t=t_i} = -\xi(t) \frac{V_i r_c}{12M^3} (h_{mn}(x, \tau) + d(x)\tilde{g}_{mn}(x, \tau)) |_{t=t_i} \tag{21} \]

and so on. The equations following from the action (12) need to be solved subject to these boundary conditions.

A. Classical Action

The zeroth order action (which does not contain \( d \) or \( h \)) is obtained from (12) with \( g \) and \( \phi \) replaced by \( \tilde{g} \) and \( r \) respectively. The equations of motion for the two fields can be obtained by varying with respect to \( \tilde{g} \) and \( r \) which leads to two equations - one tensor and one scalar. Let us note parenthetically that \( \delta G_{m5} \) (see (6)) is not independent and, consequently, the \( m5 \) Einstein equation would not be independent.

The zeroth order action or the classical action is easily obtained from (12) to have the form
\[ S^{(0)} = -2M^3 \int d^4x \int_{-\pi}^{\pi} dt \sqrt{-g} \left( r\bar{R} + r\frac{\Lambda}{2M^3} \right. \]
\[ \left. - \frac{1}{4r} \left[ \dot{g}_{mn}(\ddot{g}^{mn})' + (\ddot{g}^{mn}\dot{g}_{mn})^2 \right] + \sum_{i=0, \pi} \frac{\mathcal{V}_i}{2M^3} \delta(t - t_i) \right) \]

(22)

The Euler-Lagrange equations, following from this action, have the form (upon setting \( r(x) = r_c \))

\[ 0 = \bar{R} + \frac{\Lambda}{2M^3} + \frac{1}{4r_c^2} \left[ \dot{g}_{mn}(\ddot{g}^{mn})' + (\ddot{g}^{mn}\dot{g}_{mn})^2 \right] \]

(23)

\[ 0 = \bar{R}_{mk} - \frac{1}{2} \bar{R} \bar{g}_{mk} - \frac{\Lambda}{4M^3} \bar{g}_{mk} + \frac{1}{8r_c^2} \left[ 4\ddot{g}_{mk} - 4\dot{g}^{ab}\dot{g}_{ab}\ddot{g}_{mk} \right. \]
\[ - \left. 4\dot{g}^{ab}\dot{g}_{am}\dot{g}_{bk} - 3(\dot{g}^{ab})'\dot{g}_{ab}\ddot{g}_{mk} + 2\dot{g}^{ab}\dot{g}_{ab}\ddot{g}_{mk} - (\dot{g}^{ab}\dot{g}_{ab})^2 \ddot{g}_{mk} \right] \]
\[ - \sum_{i=0, \pi} \frac{\mathcal{V}_i}{4r_cM^3} \delta(t - t_i) \bar{g}_{mk} \]

(24)

As expected, equation (24) contains singular terms proportional to \( \delta \)-functions. However, let us note that these equations, which are written in terms of derivatives with respect to \( t \in [-\pi, \pi] \), can be simplified if we express them in terms of the variable \( \tau \in [0, \pi] \) (Recall that all the field variables depend on \( \tau \)). We note that

\[ \ddot{g}_{mn}(\tau) = \dot{g}_{mn}\tau', \quad \dot{g}''_{mn}(\tau) = \ddot{g}_{mn} + \dot{g}_{mn}\tau'' \]

Since \( \tau'^2 = 1 \), in the classical equations, we can simply replace terms quadratic in single \( t \) derivatives (primes) by those with \( \tau \) derivatives (dots). However, in terms with double \( t \) derivatives, the change to \( \tau \) derivatives would introduce new delta function singularities because of the \( \tau'' \) term. On the other hand, we note from (24) that \( \dot{g}_{mn} \), on both the branes, is proportional to \( \bar{g}_{mn} \) so that these new singular terms cancel precisely the singular delta function terms already present in the equation (24). In fact, this is a general result, namely, singular terms arising from changing “primes” to “dots” exactly cancel the singular boundary terms that appear explicitly in the equations of motion, because of the junction conditions. We would, however, like to stress that, although the dot derivatives appear to be the natural ones in the equations of motion, it is more useful to have the prime derivatives in the action to avoid subtleties in integration by parts.

Rewritten in terms of the \( \tau \) (dot) derivatives, the equations of motion have the form

\[ 0 = \bar{R} + \frac{\Lambda}{2M^3} + \frac{1}{4r_c^2} \left[ \dot{g}_{mn}\ddot{g}^{mn} + (\ddot{g}^{mn}\dot{g}_{mn})^2 \right] \]

(25)

\[ 0 = \bar{R}_{mk} - \frac{1}{2} \bar{R} \bar{g}_{mk} - \frac{\Lambda}{4M^3} \bar{g}_{mk} + \frac{1}{8r_c^2} \left[ 4\ddot{g}_{mk} - 4\dot{g}^{ab}\dot{g}_{ab}\ddot{g}_{mk} \right. \]
\[ - \left. 4\dot{g}^{ab}\dot{g}_{am}\dot{g}_{bk} - 3(\dot{g}^{ab})'\dot{g}_{ab}\ddot{g}_{mk} + 2\dot{g}^{ab}\dot{g}_{ab}\ddot{g}_{mk} - (\dot{g}^{ab}\dot{g}_{ab})^2 \ddot{g}_{mk} \right] \]
\[ - \sum_{i=0, \pi} \frac{\mathcal{V}_i}{4r_cM^3} \delta(t - t_i) \bar{g}_{mk} \]

(26)

without any singular terms involving \( \delta \)-functions. These are highly nonlinear equations which are clearly nontrivial to solve in general. Therefore, we look for a solution of the metric, \( \bar{g}_{mn} \), in the factorizable form,

\[ \bar{g}_{mk}(x, \tau) = a(\tau) \bar{g}_{mk}(x) \]

(27)

Requiring \( \bar{g}_{mk}(x) \bar{g}^{kn}(x) = \delta^m_k \), we have

\[ \bar{g}^{mk} = \frac{1}{a} \bar{g}^{mk}, \quad \bar{R}_{mk} = \bar{R}_{mk}, \quad \bar{R} = \frac{1}{a} \bar{R} \]

(28)

Using (27) and (28), equations (25) and (26) respectively take the forms

\[ 0 = \bar{R} + a \left[ \frac{\Lambda}{2M^3} + \frac{3}{r_c^2} \left( \frac{\dot{a}}{a} \right)^2 \right] \]

(29)

\[ 0 = \bar{R}_{mk} - \frac{1}{2} \bar{R} \bar{g}_{mk} + \frac{\Lambda}{4} \bar{g}_{mk} \]

(30)
where we have defined

$$\lambda = -a \left( \frac{\Lambda}{M^3} + \frac{6}{r_c^2} \frac{\dot{a}}{a} \right)$$  \hspace{1cm} (31)$$

Contracting (30) with $g^{mk}$, we obtain

$$R = \lambda$$  \hspace{1cm} (32)$$

The left-hand-side of (32) is a function of $x$ only, while the right-hand side depends only on $\tau$. It follows, therefore, that $\lambda = \text{constant} = R$, which, in turn, implies that the metric $g_{mk}$ defines an Einstein space

$$R_{mk} = \frac{\lambda}{4} g_{mk}$$  \hspace{1cm} (33)$$

with constant scalar curvature $\lambda$. It follows now from (3) and (28) that the induced metric on each of the branes will define an Einstein space as well, with curvature $\tilde{R}(t = t_i) = \frac{\lambda}{a(t = t_i)}$. We also note that every solution, $g_{mk}$, of (33) corresponding to a given $\lambda$, will have the same warp factor $a$ determined from (29) and (31).

The solutions for the warp factor $a$ are already known. The flat case, $g_{mk} = \eta_{mk}$, was analyzed in [3], while the dS and the AdS solutions were derived in [6]. It is clear from the preceding discussion that these solutions exhaust all possible cases for the warp factor and here we will briefly describe an alternate and unified derivation of these solutions for completeness.

The equations (29) and (31) determining the warp factor can be rewritten in a more convenient form

$$\left( \frac{\dot{a}}{a} \right)^2 = -\frac{r_c^2}{3} \left( \frac{\Lambda}{2M^3} + \frac{\lambda}{a} \right)$$  \hspace{1cm} (34)$$

$$\ddot{a} - (\dot{a})^2 = \frac{\lambda r_c^2}{6} a$$  \hspace{1cm} (35)$$

We note from the above equations that a constant solution for $a(\tau)$ exists only for the physically uninteresting case when $\Lambda = 0 = \mathcal{V}_i$, which we will not consider. When $a \neq \text{constant}$, equation (35) follows from (34), and, therefore, this is the only equation that we need to analyze. Furthermore, this equation must be supplemented by the boundary conditions (see (20))

$$\dot{a}|_{t = t_i} = -\xi(t) \frac{\mathcal{V}_i r_c}{12M^3} a|_{t = t_i}$$  \hspace{1cm} (36)$$

The general solution of (34) is of trigonometric type for $\Lambda > 0$, of exponential type for $\Lambda < 0$ and of polynomial type for $\Lambda = 0$. Keeping in mind the spirit of the RS model, where the aim is to solve the hierarchy problem without introducing large numbers, we will consider only the case $\Lambda < 0$ here. Following [3], we define

$$\frac{\Lambda}{24M^3} = -k^2$$  \hspace{1cm} (37)$$

and parameterize the tensions on the branes, for convenience, as

$$\frac{\mathcal{V}_0}{24M^3} = \alpha k, \quad \frac{\mathcal{V}_\pi}{24M^3} = -\beta k$$  \hspace{1cm} (38)$$

where $\alpha, \beta$ are arbitrary parameters. In terms of these parameters, the equation for $a$ reads

$$\left( \frac{\dot{a}}{a} \right)^2 = 4k^2 r_c^2 - \frac{\lambda r_c^2}{3a}$$  \hspace{1cm} (39)$$

which needs to be solved subject to the boundary conditions

$$\frac{\dot{a}}{a}(\tau = 0) = -2\alpha kr_c, \quad \frac{\dot{a}}{a}(\tau = \pi) = -2\beta kr_c$$  \hspace{1cm} (40)$$

The solutions of (39) subject to (40) are easily determined to be
\[ a(\tau) = \begin{cases} \frac{\lambda}{12\pi^2} \cosh^2(kr_c(p-\tau)) & \text{when } \lambda > 0, \\ \exp(-2k\tau) & \text{when } \lambda = 0, \\ \frac{1}{12\pi^2} \sinh^2(kr_c(p-\tau)) & \text{when } \lambda < 0, \end{cases} \] (41)

When \( \lambda \neq 0 \), \( r_c \) is determined completely in terms of \( \alpha \) and \( \beta \) as

\[ kr_c = \frac{1}{\pi} \text{arctanh} \left( \frac{\alpha - \beta}{1 - \alpha \beta} \right) \] (42)

and the parameter \( p \) is defined through

\[ e^{2kr_c p} = \begin{cases} \frac{1 + \alpha}{1 - \alpha} & \text{when } \lambda > 0, \\ \frac{\alpha + 1}{\alpha - 1} & \text{when } \lambda < 0, \end{cases} \] (43)

\[ e^{2kr_c p} = \frac{1 + \alpha}{1 - \alpha} \] (44)

The range of the parameters, \( \alpha \) and \( \beta \), can be obtained from three consistency relations. We note that \( r_c > 0 \), which through \((12)\) implies that \( 0 < \frac{\alpha - \beta}{1 - \alpha \beta} < 1 \). Furthermore, the right-hand-side of \((13)\) must be positive, so that \( |\alpha| < 1 \) for \( \lambda > 0 \) and \( |\alpha| > 1 \) for \( \lambda < 0 \). Finally, we have \( a(\tau) > 0 \) for any \( \tau \) and it follows from \((11)\) that \( p \notin [0, \pi] \) when \( \lambda < 0 \). It is easy to show that, for the solutions in \((41)\), the right-hand-side of \((39)\) is non-negative for any \( \lambda < 0 \) and, consequently, this does not introduce any further restriction on the parameters \( \alpha \) and \( \beta \). The limiting case \( \lambda = 0 \) cannot be obtained from these and a solution exists only if the two tensions satisfy \( \alpha = \beta = 1 \), as can be seen from \((11)\) and \((40)\). In this case, the value of \( r_c \) remains undetermined.

### B. Quadratic Action

If the background fields satisfy the classical equations, then, the linear order terms in the expansion of the action vanish, which can be explicitly checked. The leading nontrivial correction to the action comes at the quadratic order. After some algebra, the part of the action \((12)\), quadratic in \( h_{mn} \) and \( d \), can be determined to be (we give some useful relations for this expansion in appendix A)

\[ S^{(2)} = -2M^3 r_c \int d^4 x \sqrt{-g} \int_{\tau}^{\tau} dt \left( L(d) + L(dh) + L(h) \right) \] (45)

where

\[ L(d) = \frac{\lambda}{4} r_c^2 g^{2m} d_m, \quad \frac{3}{r_c^2} (a')^2 d'^2 \] (46)

\[ L(dh) = a \left( h_{mk}^{am; \alpha} - \frac{3}{4} a \right) d_{mn} - \frac{3}{4} a a' \] (47)

\[ L(h) = \frac{1}{4} \frac{2}{4r_c^2} \left( h_{mk}^{am; \alpha} - \frac{3}{4} a a' \right) \left( h_{mk}^{am; \alpha} - \frac{3}{4} a a' \right) + \frac{3}{4} a a' \left( 2h_{mk} h_{mk} - h' h \right) \] (48)

\[ + \frac{1}{4} \left( \hat{V}_i \right) \left( 2h_{mk} h_{mk} - h^2 \right) \] (49)

\[ - \sum_{i=0, \pi} \frac{a^2}{r_c^2} \left( 2h_{mk} h_{mk} - h^2 \right) \delta (t - t_i) \] (50)

Keeping the factorization of the background metric in mind, we have introduced a new field \( h_{mk}(x, \tau) \) through

\[ h_{mk} = ah_{mk} ; \quad h_{mk}^{\prime} = ah_{mk}^{\prime} ; \quad h = h_{mk} * h_{mk} \] (52)

All indices are raised and lowered with the background metric \( g_{mn} \), which also defines the covariant derivatives. As a result, let us note that 4D covariant derivatives simply commute with derivatives with respect to \( t \) (or \( \tau \)).

The boundary condition satisfied by \( h_{mk} \) is determined from \((21)\) to be
\[ \bar{h}_{mk}(x, \tau) \big|_{t=t_i} = -\xi(t) \frac{\mathcal{V}_r r_c}{12 M^4} \int d \frac{g_{mn}}{g_{mk}} |_{t=t_i} \] 

The most important feature to note from (45) is the mixing between the fields \( \bar{h}_{mk} \) and \( d \). In order to understand the physical spectrum of the theory, we must, of course, diagonalize the second order Lagrangian density. Unfortunately, this is not the complete story since there is mixing between the two fields arising from the boundary conditions (53) as well. Therefore, we need to diagonalize simultaneously the quadratic Lagrangian density, (53), as well as the junction conditions to linear order, (54). We note that higher order mixing, both in the action as well as in the junction conditions, would simply correspond to higher order interactions and is not of interest for our analysis.

The presence of mixing, in the quadratic action, simply corresponds to the fact that \( \bar{h}_{mk} \) and \( d \) are not the appropriate field variables and the physical fields will be, in general, a linear combination of the two. However, we note that, unlike \( \bar{h}_{mk} \), the field \( d \), in some sense, carries a direct physical meaning - it labels the deviation of the distance between the two branes from the classical value \( r_c \). Consequently, we prefer to treat the field \( d \) as physical and look for a general redefinition of the metric fluctuation. The most general linear redefinition of \( \bar{h}_{mk} \) can be seen to be

\[ \bar{h}_{mk}(x, \tau) = X_{mk}(x, \tau) + (\ln a + f(\tau))d(x) \frac{g_{mn}}{g_{mk}} + S(\tau)d_{\ell m} + Z(\tau)d_{\ell m}(x) \frac{g_{mn}}{g_{mk}} \] 

where compatibility with (53) requires (see also (36))

\[ \dot{X}_{mk}(x, \tau) |_{t=t_i} = 0 \quad \dot{f}(\tau) |_{t=t_i} = \dot{S}(\tau) |_{t=t_i} = \dot{Z}(\tau) |_{t=t_i} \]  

Note that the term \( L(h) \) is form-invariant under the transformation (54). However, \( L(d) \) and \( L(dh) \) do transform and it is easy to see that if \( Z \neq 0 \), such a transformation will generate mixing between \( X_{mk} \) and \( d \) with fourth order derivatives. Since this does not help in diagonalization, we set \( Z = 0 \). Furthermore, we find that there is no such \( f \) and \( S \) obeying (56) which set the mixing terms, in the Lagrangian density, to zero.

There is another possible solution to the mixing problem. We note here that it is the action that we need to diagonalize (subject to the boundary conditions). Therefore, if we know the KK decomposition for the field \( X_{mk} \), then the action (45) would be a 5D action for 4D fields that depend only on the coordinate \( x \), with \( t \)-dependent coefficients. It would, therefore, be sufficient to arrange the coefficient functions in front of the mixing terms to vanish when integrated over the extra dimension so that such terms can be removed from the action.

To find a set of basis functions, in terms of which we can decompose the field \( X_{mk} \), we proceed as follows. Let us assume that the change (54) leads to a diagonalization of the action (it also diagonalizes the boundary conditions because of (53)). Since the transformation (54) is linear and leaves the part of the action depending only on \( \bar{h}_{mk} \) form-invariant, the other two terms in the action can only lead to terms that are quadratic in the field \( d \). As a result, the KK decomposition of \( X_{mk} \) can be easily studied by analyzing the \( h_{mn} \) part of the action and we can simply set \( d = 0 \) in (43) and (53) for this purpose. Having the basis functions determined in this way, \( X_{mk} \) can be expanded in this basis (\( d(x) \) does not have a KK tower) in the action (43), and the resulting action diagonalized at each level of the KK tower. As we will see, this procedure works quite nicely. We discuss the actual solutions of (45) and (33) for \( d = 0 \) in appendix B.

The most convenient way to implement the program discussed above, is to first make the change of variables

\[ \bar{h}_{mk}(x, \tau) = X_{mk}(x, \tau) + (\ln a + f(\tau))d(x) \frac{g_{mn}}{g_{mk}} \] 

with

\[ \dot{X}_{mk}(x, \tau) |_{t=t_i} = \dot{f}(\tau) |_{t=t_i} = 0 \] 

The function \( f(\tau) \) is otherwise arbitrary. Since the new field \( X_{mk} \) satisfies (58), we can decompose it in the system of basis functions \( \Phi^\alpha \) (whose derivative vanishes at the boundaries, see appendix B for details) as

\[ X_{mk}(x, \tau) = \sum_\alpha (\alpha) X_{mk}(x) \Phi^\alpha(\tau) \] 

The action (45) can now be written in the form

\[ S^{(2)} = -2M^3 r_c \int d^4 x \sqrt{-g} \int_{-\pi}^{\pi} dt (L(d) + L(dX) + L(X)) \]
Using (B10) and (B12), we can rewrite the mixing terms, (62), in the action in the form
\[
L(d) = K(\tau) \, d^m d_m + \frac{M(\tau)}{r_c^2} \, d^2
\]  
(61)
\[
L(dX) = a(1 + \ln a + f) \left[ (X^{mk}_{\lambda k} - X^{\alpha \cdot m}) d_m - \frac{\lambda}{4} d X \right] + \frac{3a^2}{2r_c^2} \left( \dot{f} + 2\dot{a} \dot{f} \right) dX
\]  
(62)
\[
L(X) = \frac{1}{2} \sum_{\alpha} a(\Phi_{\alpha})^2 \, L_{PF}^{(\alpha)} (X^{mk}_{\lambda k}; m_{\alpha}^2)
\]  
(63)
Here we have defined \(K\) and \(M\) as
\[
K(\tau) = \frac{\lambda}{4} r_c^2 \tau^2 - \frac{3 a}{2} (\ln a + f)(\ln a + f + 2)
\]  
(64)
\[
M(\tau) = -6a\dot{\alpha}(\ln a + f)(\ln a + f + 1) - 3a\ddot{f}(a\dot{f} + 2\dot{a}(\ln a + f)) + \sum_{i=0}^{\infty} \frac{a^2}{2 M i^2}(\ln a + f)^2 \delta(t - t_i)
\]  
(65)
and \(L_{PF}\) represents the Pauli-Fierz term that we will discuss later (it is also discussed in appendix \textbf{B}).

Because of (58), we can also decompose \(f\) in terms of \(\Phi^\alpha\) as
\[
f(\tau) = \sum_{\alpha} f_{\alpha} \Phi^\alpha(\tau)
\]  
(66)
Using (B11) and (B13), we can rewrite the mixing terms, (62), in the action in the form
\[
L(dX) = a(1 + \ln a + \rho) \Phi^0 \left[ \left( (0) \, X^{mk}_{\lambda k} - (0) \, X^{\alpha \cdot m} \right) d_m - \frac{\lambda}{4} d (0) \, X \right] + \sum_{\alpha \geq 1} a(\Phi^\alpha)^2 \left[ (I^\alpha + f_{\alpha}) \left( \left( (\alpha) \, X^{mk}_{\lambda k} - (\alpha) \, X^{\alpha \cdot m} \right) d_m - \frac{\lambda}{4} d (\alpha) \, X \right) - \frac{3}{2} f_{\alpha} m_{\alpha}^2 d (\alpha) \, X \right]
\]  
(67)
where we have introduced the notations
\[
\rho = f_0 \Phi^0 = \text{constant}, \quad I^\alpha = \int_{-\pi}^{\pi} dt \, a \ln a \, \Phi^\alpha, \quad \alpha \geq 1
\]  
(68)
and have used the equality \[\int_{-\pi}^{\pi} dt \, a \ln a \Phi^\alpha = \int_{-\pi}^{\pi} dt \, a(\Phi^\alpha)^2 I^\alpha.\]

It is impossible to find a set of constants, \(f_{\alpha}\), which will make the mixing terms in the action vanish. As a result, we are forced to make a second transformation involving the second derivative of \(d\), which is done at each level of the KK tower
\[
(\alpha) \, X^{mk}_{\lambda k}(x) = (\alpha) \, H^{mk}_{\lambda k}(x) + \sigma_{\alpha} \, d^{mk}_{\lambda k}, \quad \alpha \geq 0
\]  
(69)
The numbers \(\sigma_{\alpha}\) can be thought of as the coefficients in the decomposition of the function \(S(\tau)\), in (54), in the basis functions \(\Phi^\alpha\), namely,
\[
S(\tau) = \sum_{\alpha} \sigma_{\alpha} \Phi^\alpha(\tau)
\]  
(70)
Under the transformation (53), the Pauli-Fierz Lagrangian (see (B13)) becomes
\[
L_{PF}^{(\alpha)}(X, m_{\alpha}^2) = L_{PF}^{(\alpha)}(H, m_{\alpha}^2) - m_{\alpha}^2 \sigma_{\alpha} \left( (\alpha) \, H^{mk}_{\lambda k} - (\alpha) \, H^{\alpha \cdot m} \right) d_m
\]  
(71)
\[
- \frac{\lambda \sigma_{\alpha}^2}{8} \left( d^{m}_{\alpha} \right)^2 + \frac{\lambda \sigma_{\alpha}^2 m_{\alpha}^2}{8} \, d^m d_m
\]
and the mixing term (67) takes the form
\[ L(dX) = a(1 + \ln a + \rho)\Phi^0 \left[ \left( ^{(0)}H^m_{mk} - ^{(0)}H^m \right) d_m - \frac{\lambda}{4} d ^{(0)}H \right] \\
+ \sum_{\alpha \geq 1} a(\Phi^\alpha)^2 \left[ \left( I^\alpha + f_\alpha - \frac{m_\alpha^2 \sigma_\alpha}{2} \right) \left( ^{(\alpha)}H^m_{mk} - ^{(\alpha)}H^m \right) d_m \\
+ \left( -\frac{\lambda}{4} (I^\alpha + f_\alpha) - \frac{3}{2} f_\alpha m_\alpha^2 \right) d^{(\alpha)}H \right] \] (72)

The mixing for \( \alpha = 0 \) can be removed if we choose \( \rho \) to satisfy

\[ \int_{-\pi}^{\pi} dt \ a(1 + \ln a + \rho) = 0 \] (73)

For \( \alpha \geq 1 \), there are two conditions that must hold for the mixing to disappear, namely,

\[ I^\alpha + f_\alpha = \frac{m_\alpha^2 \sigma_\alpha}{2} \]
\[ \frac{\lambda}{4} (I^\alpha + f_\alpha) + \frac{3}{2} f_\alpha m_\alpha^2 = 0 \]

which can be easily solved to give

\[ f_\alpha = -\frac{\lambda}{\lambda + 6 m_\alpha^2} I^\alpha \] (74)
\[ \sigma_\alpha = \frac{12}{\lambda + 6 m_\alpha^2} I^\alpha \] (75)

(Note that the above expressions are free from singularities in view of (B16).)

Thus, we see that the transformation (71) together with (79) diagonalizes the quadratic action as well as the boundary conditions, with \( f(\tau) \) determined from (73) and (74). We note that, while \( \sigma_\alpha \), for \( \alpha \geq 1 \), is determined from (75) the coefficient \( \sigma_0 \) remains arbitrary. (This arbitrariness is related to the invariance of the theory under a gauge transformation of the graviton of the form (B13).)

The diagonalized action now takes the form

\[ S^{(2)} = -2 M^3 r_c \int d^4x \sqrt{-g} \int_{-\pi}^{\pi} dt \ (L(d) + L(H)) \] (76)

where

\[ L(d) = -\frac{\lambda}{16} \left( \sum_{\alpha > 0} a(\Phi^\alpha)^2 \sigma_\alpha \right) (d^{m}_{m})^2 \]
\[ + \left( K(\tau) - \frac{\lambda}{16} \sum_{\alpha > 1} a(\Phi^\alpha)^2 m_\alpha^2 \sigma_\alpha \right) d^{m} d^{m} + \frac{M(\tau)}{r_c^2} d^{2} \] (77)
\[ L(H) = \frac{1}{2} \sum_{\alpha > 0} a(\Phi^\alpha)^2 L_{PF}(^{(\alpha)}H_{mk}; m_\alpha^2) \] (78)

and the functions \( K \) and \( M \) are defined in (64) and (65). The properties of the diagonalized theory will be discussed in more detail in the next section.

Let us present here the final structure of the metric \( g_{mk}(x, \tau) \) after all the transformations. Combining (16), (27), (52), (57) and (69) we have

\[ g_{mk}(x, \tau) = a(\tau) \left[ (1 + d(x)[\ln a + f(\tau)]) g_{mk}(x) + \right. \]
\[ + \left. ^{(0)}H_{mk}(x)\Phi^0 + S(\tau)d_{mk}(x) + \sum_{\alpha \geq 1} ^{(\alpha)}H_{mk}(x)\Phi^\alpha(\tau) \right] \] (79)
The functions \( f(\tau) \) and \( S(\tau) \) are given in terms of their coefficients (73), (74) and (75) respectively. As we noted earlier, the function \( S(\tau) \) is determined up to an additive constant. The reason is quite clear. As we will discuss in appendix B, the massless field \( (0)^H_{mk} \) is determined only up to a gauge transformation (B15) and, therefore, the term proportional to \( d_{mk} \) in (54) can be gauged away at any fixed point \( t = \text{constant} \), in particular, on any one of the two branes, but cannot be gauged away in the whole bulk.

IV. 4D EFFECTIVE THEORY

To study further the properties as well as the physical implications of the diagonalized theory, we need to know the values of the coefficients present in (77) and (78) after integration over the fifth coordinate. Although so far we have treated both the cases, \( \lambda = 0 \) and \( \lambda \neq 0 \), on the same footing, it will be more convenient, in this section, to consider these two cases separately.

\( \lambda = 0 \):

Let us note, at the outset, that \( \lambda = 0 \) does not automatically imply \( g_{mk} = \eta_{mk} \). For an arbitrary Einstein space, we have [11]

\[
R_{mkab} = -\frac{\lambda}{12} (g_{mk}g_{ab} - g_{mb}g_{ak}) + C_{mkab}
\]

where \( C_{mkab} \) is the Weyl conformal tensor, and the solutions for any given \( \lambda \) are completely degenerate with respect to different choices of \( C_{mkab} \).

When \( \lambda = 0 \), it is clear from (74) that \( f_\alpha = 0 \) for \( \alpha \geq 1 \), so that \( f(\tau) = \rho \) and the warp factor is given in (41). With these, a direct evaluation of (73) determines

\[
\rho = -\frac{2\pi kr_c}{e^{2\pi kr_c} - 1} < 0
\]

The coefficient in front of the kinetic term of the radion can now be determined from (77) and (64) to be

\[
\kappa_d = \int_{-\pi}^{\pi} dt K(\tau) = -\frac{3}{2} \int_{-\pi}^{\pi} a(\ln a + \rho)(\ln a + \rho + 2) dt = -3\pi\rho > 0
\]

Similarly, we can obtain the radion mass from (77) and (65) and the explicit evaluation gives

\[
\int_{-\pi}^{\pi} a^2(\ln a + \rho)(\ln a + \rho + 1) dt = \int_{-\pi}^{\pi} \sum_{i=0,\pi} a^2 \frac{V_i r_c}{2M3}(\ln a + \rho)^2 \delta(t-t_i) dt = 0
\]

Therefore, the radion is massless, since

\[
\int_{-\pi}^{\pi} \frac{M(\tau)}{r_c^2} dt = 0
\]

The quadratic action, therefore, takes the form

\[
S_{eff} = \int d^4x \sqrt{-g} \left( -6\pi|\rho| M_3 r_c d^m d_m - M_3 r_c \sum_{\alpha \geq 0} L_{PP}(^{(\alpha)}H_{mk}; m_{\alpha})^2 \right)
\]

Let us next normalize the fields \( d \) and \( ^{(\alpha)}H_{mk} \) in the following way. Let us assume that \( \exp(-2\pi kr_c) \ll 1 \) and express \( M \) in terms of the 4D effective Planck mass \( M_{PL}^2 = M^3/k \). Furthermore, let us define

\[
d(x) = \frac{e^{kr_c \pi}}{\sqrt{24\pi kr_c M_{PL}}} D(x)
\]

\[
^{(\alpha)}H_{mk}(x) = \frac{1}{\sqrt{kr_c M_{PL}}} ^{(\alpha)}X_{mk}(x)
\]
In terms of these fields, the quadratic action becomes:

$$S_{\text{eff}} = \int d^3x \sqrt{-g} \left( -\frac{1}{2} D^m D_m - \sum_{\alpha \geq 0} L_{PF}(\chi_{mk}; m_{\alpha}^2) \right)$$  \hfill (86)

The content of the theory, at the quadratic level, is now obvious - there is a massless scalar field (the radion), a massless spin-2 field (graviton) and an infinite tower of massive spin-2 fields with masses $m_{\alpha}^2$. We see that no ghost fields are present in this theory, since the kinetic term of the radion has the right sign, and the fields $\chi_{mk}$ carry precisely spin-2 content. We can, therefore, think of these fields as the physical fields. The non-presence of (ghost) scalar component in the fields $\chi_{mk}$ (for $\alpha \geq 1$) is due to the conditions (B14) which these fields satisfy. Note that in our framework, the conditions (B14) arise naturally from the equations of motion for these fields (for more details see the appendix in [19]), and do not have to be imposed by hand.

To understand the coupling of the fields $D(x)$ and $\chi_{mk}$ to matter, we need to express the metric $g_{mk}(x, \tau)$ in terms of these physical fields. To that end, let us recall the expression for the normalized functions $\Phi^\alpha(\tau)$ \hfill [21]

$$\Phi^\alpha(\tau) = \frac{\sqrt{kr_c} e^{kr_c \pi}}{J_2(x_{\alpha})} e^{2kr_c(\tau-\pi)} \left[ J_2(x_{\alpha} e^{kr_c(\tau-\pi)}) + \frac{\pi}{4} x_{\alpha} e^{-2kr_c \pi} Y_2(x_{\alpha} e^{kr_c(\tau-\pi)}) \right]$$ \hfill (87)

for $\alpha \neq 0$, and $\Phi^0 = \sqrt{kr_c}$. Here, $x_{\alpha}$’s represent the positive zeroes of the Bessel function $J_1(x)$. Furthermore, the masses, $m_{\alpha}^2$, of the fields $\chi_{mk}$ are given by \hfill [21]

$$m_{\alpha}^2 = k^2 x_{\alpha}^2 e^{-2kr_c \pi}$$ \hfill (88)

On the visible brane, we have

$$\Phi^\alpha(\tau = \pi) = \sqrt{kr_c} e^{kr_c \pi}, \quad \alpha \geq 1$$ \hfill (89)

while on the hidden brane they are rescaled roughly by a factor ($\alpha$-dependent) of the order of $\exp(-2kr_c \pi)$. Collecting earlier results, we can now write the metric $g_{mk}(x, \tau)$ in terms of the physical fields (83) as

$$g_{mk}(x, \tau) = e^{-2kr_c \tau} \left[ \left( 1 - \frac{(\pi e^{-2kr_c \pi} + \tau) e^{kr_c \pi}}{\sqrt{6\pi} M_{PL}} D(x) \right) g_{mk}(x) + \frac{S(\tau) e^{kr_c \pi}}{\sqrt{24\pi} kr_c M_{PL}} D_{,mk}(x) \right. \right.$$ \hfill (90)

$$\left. + \frac{1}{M_{PL}} (0) \chi_{mk}(x) + \frac{1}{\sqrt{kr_c} M_{PL}} \sum_{\alpha \geq 1} (\alpha) \chi_{mk}(x) \Phi^\alpha(\tau) \right]$$

Of particular interest to us is the restriction of (84) to the visible brane, which would determine the coupling of the fields $D$ and $\chi_{mk}$ to matter located there.

$$g_{mk}^{\text{vis}}(x) = e^{-2kr_c \pi} \left[ \left( 1 - \frac{e^{kr_c \pi}}{\sqrt{6\pi} M_{PL}} D(x) \right) g_{mk}(x) \right. \right.$$ \hfill (91)

$$\left. + \frac{1}{\sqrt{6\pi} kr_c k^2 M_{PL}} \left( e^{kr_c \pi} \sum_{\alpha \geq 1} \frac{r_{\alpha}}{x_{\alpha}^2} \right) D_{,mk}(x) \right. \right.$$ \hfill (92)

$$\left. + \frac{1}{M_{PL}} (0) \chi_{mk}(x) + \frac{e^{kr_c \pi}}{M_{PL}} \sum_{\alpha \geq 1} (\alpha) \chi_{mk}(x) \right]$$ \hfill (93)

Some comments are in order. The couplings of the spin-2 states (83) were already discussed in [21]. The radion has two types of couplings to matter. The non-derivative one, on the visible brane, has strength comparable to that of the massive spin-2 states in (83) which is of the order of $(\text{TeV})^{-1}$. This coupling is nonzero on the hidden brane as well, although much weaker (as also discussed in [7]). The coupling on the hidden brane has its origin in the mixing of the radion with the graviton. The expression for the derivative coupling of the radion is nontrivial, due to the fact that we account for the presence of the massive KK tower of states. As we have noted earlier, the mixing of the radion, involving derivatives, with the graviton in (54) involves only the coefficient $a_0$, which is a constant and can, therefore, be gauged away in the whole bulk. However, the mixing of the radion involving derivatives with the massive KK tower
in (54) involves coefficients that depend on \( \tau \), namely, the function \( S(\tau) \) in (90). As a result, in general, its value will be different at different points of the fifth dimension. The derivative coupling term can be arranged to vanish on the visible brane, but unless the function \( S(\tau) \) takes special values, the term \( D_{m\kappa} \) cannot be simultaneously gauged away on both the branes. Let us estimate the coefficient of this term in (92). Since \( k \lesssim M_{PL} \) (see (21)), the number outside the parenthesis is of the order of \((T_eV)^{-3}\). The term inside the parenthesis needs a more careful consideration. We use \( kr_c = 12 \) and take the values for the first fifty roots, \( x_\alpha \), with eight digit precision from (22). A direct numerical evaluation shows that the first term of the series is \(-3.5593\). The subsequent terms, in the series, are of alternating sign and tend to have very slowly decreasing magnitude. We believe that the series is absolutely convergent. The numerical evaluation of the series including the first fifty terms yields the value \(0.74\) times the first term. This value essentially stabilizes after approximately the twentieth term. The correction to this value from the remainder of the series is of the order of \(10^{-3}\). This is roughly the order of magnitude of each of the terms in the series for the next fifty or so terms. The 200-th term, in the series, for example, has a value of the order of \(10^{-4}\). Therefore, we estimate the term in the parenthesis to be of the order of unity.

\[ \lambda \neq 0: \]

When \( \lambda \neq 0 \), the diagonalized quadratic action is of the form

\[ S_{eff} = \int d^4x \sqrt{-g} \mathcal{L}(d) + \sum_{\alpha \geq 0} \int d^4x \sqrt{-g} \mathcal{L}^\alpha \]  

(94)

where

\[
\begin{align*}
\mathcal{L}(d) &= \zeta_1 (d^{m;m})^2 + \zeta_2 d^m d_m + \zeta_m d^2 \\
\mathcal{L}^\alpha &= -M^3 r_c L_{PF}(\langle (\alpha) H; m_\alpha^2 \rangle)
\end{align*}
\]

and

\[
\begin{align*}
\zeta_1 &= \frac{M^3 r_c \lambda}{8} \left( \sum_{\alpha \geq 0} \sigma_\alpha^2 \right) \\
\zeta_2 &= -2M^3 r_c \left( \int_{-\pi}^{\pi} dt K(\tau) - \frac{\lambda}{16} \sum_{\alpha \geq 1} m_\alpha^2 \sigma_\alpha^2 \right) \\
\zeta_m &= -2\frac{M^3}{r_c} \int_{-\pi}^{\pi} dt M(\tau)
\end{align*}
\]

(95)

The spin-2 content of the theory has properties similar to the case \( \lambda = 0 \) and does not need any further discussion. However, the radion now exhibits new features. The kinetic term has a fourth order derivative term whose coefficient has the sign of \( \lambda \). Although in principle the \( \lambda \neq 0 \) case can be analyzed in a fashion similar to the \( \lambda = 0 \) case, there are some technical complications which make the analysis quite difficult. For example, we do not know of a complete system of functions \( \Phi^\alpha \) that is of a simple form, although equation (B11) can be solved in terms of associated Legendre functions or equivalently in terms of hypergeometric functions. Since the functions \( K(\tau) \) and \( M(\tau) \) (see (64), (65)) are given in terms of the function \( f(\tau) \), which in turn can only be determined from its series expansion, we do not have much knowledge about these functions either. Therefore, we are unable to discuss the values of the coefficients any further. Nonetheless, it is easy to see that the coefficient \( \zeta_1 \) is unlikely to be zero and has the same sign as \( \lambda \) (It will be zero if all the \( I^\alpha \)'s are to vanish which is highly unlikely although not impossible.). This is sufficient to indicate that higher derivative ghosts will be associated with the radion field in this case.

V. CONCLUSIONS

In this paper, we have systematically studied, within the Lagrangian approach, the question of identification and consistent inclusion of the radion field in the two brane Randall-Sundrum model. We have exploited the symmetries of the theory to identify the radion in an unambiguous manner. Using the background field method, we have given a unified derivation of the classical solutions as well as the form of the action at quadratic order in the fields. We
have shown that the background metric, in general, corresponds to that of an Einstein space with a warp factor. We have discussed in detail the question of diagonalization of this quadratic action which is essential in understanding the spectrum of the theory. We have studied further the effective 4-dimensional action following from this diagonalized action and it is shown that the radion has no Kaluza-Klein tower and is massless for backgrounds with $\lambda = 0$. In this case, the graviton as well as the infinite tower of states truly describe spin-2 particles and the theory is free of ghosts. For the case $\lambda \neq 0$, however, the situation is less clear and higher derivative ghost terms, involving the radion, appear in the action. The question of matter coupling is also discussed. We would like to emphasize here that there are only two assumptions that we have made, i) the background $r(x) = r_c = \text{constant}$ and ii) a factorizable form of the background metric, $\tilde{g}_{mn}(x, \tau) = a(\tau) g_{mn}(x)$. The entire analysis, otherwise, is quite general.

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**APPENDIX A: SOME USEFUL RELATIONS**

In this appendix, we collect some relations that are useful in the decomposition of the action as well as the junction conditions. For $N$ defined in (10), and with (16), (17), we have the following decompositions up to second order in the fluctuations $h_{mn}$ and $d$.

\[
N = r_c^2 \left( 1 + 2d + d^2 - \frac{\tau^2 r_c^2 \tilde{g}^{ab} d_a d_b}{2} \right)
\]
\[
\sqrt{N} = r_c \left( 1 + d + d^2 - \frac{\tau^2 r_c^2 \tilde{g}^{ab} d_a d_b}{2} \right)
\]
\[
\frac{1}{N} = \frac{1}{r_c^2} \left( 1 - 2d + 3d^2 + \tau^2 r_c^2 \tilde{g}^{ab} d_a d_b \right)
\]
\[
\frac{1}{\sqrt{N}} = \frac{1}{r_c} \left( 1 - d + d^2 + \frac{\tau^2 r_c^2 \tilde{g}^{ab} d_a d_b}{2} \right)
\]  

(A1)

The determinant of the metric $g_{mn}$ can also be decomposed up to quadratic order as (see [15])

\[
\sqrt{-g} = \sqrt{-\tilde{g}} \left( 1 + \frac{1}{2} h - \frac{1}{4} h^a b^b + \frac{1}{8} h^2 \right)
\]  

(A2)

where $h = \tilde{g}^{mk} h_{mk}$, and we have, to quadratic order in the fluctuations,

\[
\sqrt{N} \sqrt{-g} = r_c \sqrt{-\tilde{g}} \left( 1 + d + \frac{1}{2} h + \frac{1}{2} dh + \frac{1}{8} h^2 - \frac{1}{4} h^a b^b - \frac{\tau^2 r_c^2 \tilde{g}^{ab} d_a d_b}{2} \right)
\]  

(A3)

The decomposition of $R^{(4)}_{mn}$ and $R^{(4)}$ can be found in [13] (note that the notations there differ slightly from ours).

**APPENDIX B: KALUZA-KLEIN DECOMPOSITION**

In this appendix, we describe the Kaluza-Klein decomposition for the fluctuation of the metric, $H_{mk}$, and discuss some of the properties of the basis functions. Let us consider the action (45). When $d = 0$, it is easy to check from eqs. (57) and (69) that

\[
h_{mk} = X_{mk} = H_{mk}
\]

so that the quadratic action takes the form

\[
S(H) = -2 M^3 r_c \int d^4 x \sqrt{-g} \int_{-\pi}^{\pi} dt \, L(H)
\]  

(B1)

where
\[
L(H) = \frac{a}{2} \left( \frac{1}{2} H_{ak;m} H^{ak;m} - H_{ak;m} H^{am;k} + H^{mk}_{;i} H_{m} - \frac{1}{2} H^{m} H_{,m} \right) \\
+ \frac{a^2}{4 r_c^2} (H'_{mk} (H'^{mk})' - (H')^2) + \frac{3 a a'}{4} \left( 2 H'_{mk} H^{mk} - H'H \right) \\
+ \left( \frac{a \lambda}{8} + \frac{3 (a')^2}{4 r_c^2} \right) (2H_{mk} H^{mk} - H^2) \\
- \sum_{i=0,\pi} \frac{a^2}{r_c} \frac{\nu_i}{16 M^2} (2H_{mk} H^{mk} - H^2) \delta(t - t_i)
\]

Here, as usual, \( H = H^{mk}_{,m} \) and the field \( H_{mk} \) has to satisfy the boundary condition (see (53))
\[
\dot{H}_{mk}(x, \tau)|_{t=t_i} = 0
\]

Varying (B1) with respect to \( H_{mk} \), we obtain
\[
H_{mk;\alpha}^{;\alpha} - H^{;\alpha}_{,\alpha} g_{mk} + H^{ab;\alpha} g_{mk} - H_{ma;k}^{;\alpha} - H_{ka;m}^{;\alpha} + \frac{a}{r_c} \left( \ddot{H}_{mk} - \dot{H} g_{mk} \right) + \frac{2 a}{r_c} \left( H_{mk} - H g_{mk} \right) - \frac{\lambda}{4} \left( 2H_{mk} - H g_{mk} \right) = 0
\]

As we have seen from the discussion following (24), all singular terms from the equations of motion disappear when we use dot \((\tau)\) derivatives. The above equation admits separation of variables and let us introduce a system of functions \( \Phi^{\alpha}(\tau) \) such that:
\[
H_{mk}(x, \tau) = \sum_{\alpha} (\alpha) H_{mk}(x) \Phi^{\alpha}(\tau)
\]

Then, (B7) can be rewritten as
\[
(\alpha) H_{mk;\alpha}^{;\alpha} - (\alpha) H^{;\alpha}_{,\alpha} g_{mk} + (\alpha) H_{mk} + (\alpha) H^{ab;\alpha} g_{mk} - (\alpha) H_{ma;k}^{;\alpha} - (\alpha) H_{ka;m}^{;\alpha} + \frac{a}{r_c} \left( \ddot{\Phi}^{\alpha} - \dot{\Phi}^{\alpha} \right) + \frac{m_{\alpha}^2 r_c^2}{a} \Phi^{\alpha} = 0
\]

where \( m_{\alpha}^2 \) represent the separation constants.

Let us first analyze the system of functions \( \Phi^{\alpha}(\tau) \), defined through equation (B10). These functions satisfy the boundary condition
\[
\dot{\Phi}^{\alpha}(\tau)|_{t=t_i} = 0
\]
as a consequence of the junction conditions (B6) and (B8). They define an orthonormal basis with respect to the scalar product:
\[
\int_{-\pi}^{\pi} a(\tau) \Phi^{\alpha}(\tau) \Phi^{\beta}(\tau) d\tau = \delta^{\alpha\beta}
\]

The explicit forms for these functions, when \( \lambda = 0 \), have already been obtained in \([4]\) and \([20]\) (See also \([21]\)).

The equations (B9) are simply the equations of motion for spin-2 fields with mass \( m_{\alpha} \) on a background which is an Einstein-space \([18]\) (for a detailed discussion of the flat case see also \([19]\)). In \([18]\) it is shown that, when \( m_{\alpha}^2 > 0 \),
\[
(\alpha) H^{;\alpha}_{mk} - 2 R_{ambk}^{(\alpha)} H_{ab} - m_{\alpha}^2 (\alpha) H_{mk} = 0
\]
\[
(\alpha) H = (\alpha) H^{;k}_{mk} = 0
\]

and, therefore, we have five propagating degrees of freedom. Here, \( R_{ambk} \) is the Riemann tensor constructed from the metric \( g_{mk} \). On the other hand, when \( m_{\alpha} = 0 \), the theory is invariant under the gauge transformation
\[ H_{mk} \rightarrow H_{mk} + \xi_{m;k} + \xi_{k;m} \]  

(B15)

and describes a massless spin-2 field. In [18] it was also demonstrated that the masses \( m_\alpha \) of the states in the KK tower have to satisfy

\[ \lambda + 6m_\alpha^2 \neq 0 \]  

(B16)

Using (B8) and (B10), (B4), (B5), (B11) and (B12), we can rewrite the action (B1) in terms of the KK states \( ^{(a)}H_{mk} \). After some algebra, we obtain

\[
S^{(2)}(d=0) = -2M^3r_c \sum_\alpha \int d^4x \sqrt{-g} \int_\pi^\pi \frac{a}{2} (\Phi^a)^2 \ L_{PF}^{(a)}(H_{mk};m_\alpha^2) 
\]

(B17)

where \( L_{PF} \) is the generalization of the Pauli-Fierz Lagrangian for a background which is an Einstein space [8], namely,

\[
L_{PF}(H,m^2) = \frac{1}{2}H_{ak;m}H^{ak;m} - H_{ak;m}H^{ak;m} + H_{mk}H_{m} - \frac{1}{2}H^mH_m \\
+ \frac{m^2}{2} (H_{mk}H_{mk} - H^2) + \frac{\lambda}{8} (2H_{mk}H_{mk} - H^2)
\]

(B18)

It is easy to check that the equations of motion following from (B18) are precisely the same as (B9).

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