Abstract. We study a free boundary problem modeling multi-layer tumor growth with a small time delay \(\tau\), representing the time needed for the cell to complete the replication process. The model consists of two elliptic equations which describe the concentration of nutrient and the tumor tissue pressure, respectively, an ordinary differential equation describing the cell location characterizing the time delay and a partial differential equation for the free boundary. In this paper we establish the well-posedness of the problem, namely, first we prove that there exists a unique flat stationary solution \((\sigma^*, p^*, \rho^*, \xi^*)\) for all \(\mu > 0\). The stability of this stationary solution should depend on the tumor aggressiveness constant \(\mu\). It is also unrealistic to expect the perturbation to be flat. We show that, under non-flat perturbations, there exists a threshold \(\mu^* > 0\) such that \((\sigma^*, p^*, \rho^*, \xi^*)\) is linearly stable if \(\mu < \mu^*\) and linearly unstable if \(\mu > \mu^*\). Furthermore, the time delay increases the stationary tumor size. These are interesting results with mathematical and biological implications.

Keywords. Free boundary problem; Tumor model; Stability; Time-delay

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1. Introduction

There is a variety of shapes of tumors in tissue cultures. It is known that three-dimensional tumors grown in tissue culture are likely to take the shape of spheroids; a large number of partial differential equation (PDE) sphere-shaped tumors models have been developed, and a variety of properties including well-posedness, asymptotic stability, bifurcation, the impact of a variety of biological relevant parameters, etc., are studied. For example, the first model of free boundary problem for a solid tumor growth is proposed and analyzed by Greenspan in [12] and [13]. In [11], Friedman and Reitich considered global well-posedness and global asymptotically stability for radially symmetric solutions. For the non-symmetric case, Bazaliy and Friedman established the local well-posedness and asymptotic behavior under non-radial perturbations for the time-dependent problem in [2] and [1]. In particular, Friedman and Hu extended the work by giving a precise threshold in [8]. For more details, we refer to the papers [9,10,18] and the references therein.

Medico-biologists have recently developed that cellular aggregates gather on permeable membranes, causing them to form multilayered tumor cell. Because multilayered tumor cells are grown on permeable membranes which can separate two reservoirs of the diffusion apparatus directly, it is an important task to study the fluidity of drug and metabolism of tumor tissue. See [14–17] for the study of multilayered tumor cells.

Following the works of Cui and Escher [5] and Zhou, Escher and Cui [24], we consider in this paper the following 3-dimensional multilayered tumor region of the flat-shaped form

\[
\Omega(t) \triangleq \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}; \ 0 < y < \rho(t,x)\}, \quad x = (x_1, x_2, y),
\]

where \(\rho(t,x)\) is an unknown positive function. Denote by \(\Gamma(t)\) the upper boundary \(\{y = \rho(t,x)\}\) of \(\Omega(t)\) (the free boundary).
Through the upper boundary $\Gamma(t)$, a multi-layer tumor acquires nutrients (denoted by $\sigma$), mostly oxygen or glucose, enabling tumor cells to grow and proliferate. The nutrient $\sigma$ satisfies the diffusion equation $\lambda \sigma_t - \Delta \sigma + \sigma = 0$, where $\lambda$ is the ratio of the rate for nutrients diffusion to the rate for the cell proliferation, so it is small, and in this paper we assume a quasi-steady state approximation by taking $\lambda = 0$.

For simplicity, we assume that the tumor is immersed in an environment with nutrient concentration $\sigma$. Let $\Gamma_0$ denote the lower boundary $\{ y = 0 \}$, which is assumed to be an impermeable layer, i.e., there is no nutrient flux through $\Gamma_0$:

$$
-\Delta \sigma + \sigma = 0, \quad (x_1, x_2, y) \in \Omega(t), \quad t > 0,
$$

$$
\sigma = \sigma, \quad (x_1, x_2, y) \in \Gamma(t), \quad t > 0,
$$

$$
\frac{\partial \sigma}{\partial y} = 0, \quad (x_1, x_2, y) \in \Gamma_0, \quad t > 0.
$$

If the tumor is assumed to be of porous medium type where Darcy’s law (i.e., $\vec{V} = -\nabla p$, where $p$ is the pressure, here we consider extracellular matrix as “porous medium” in which cell moves) can be used, then the conversation of mass $\text{div} \vec{V} = S$ (where $S$ is the proliferation rate) implies

$$
-\Delta p = S,
$$

The proliferation rate $S$ is proportional to $\sigma - \bar{\sigma}$, where $\bar{\sigma}$ is the threshold concentration that is needed by the tissue to maintain itself. Since the cells need time (say $\tau$) to replicate and proliferate, it is assumed that $S = \mu[\sigma(\xi(t - \tau; x_1, x_2, y, t), t - \tau) - \bar{\sigma}]$, where $\mu$ is the tumor aggressiveness constant and $\xi(s; x, t)$ tracks the cell location at time $s$ which reaches the location $x = (x_1, x_2, y)$ at time $t$, and moves with the velocity field $\vec{V} = -\nabla p$:

$$
\frac{d\xi(s; x, t)}{ds} = -\nabla p(\xi(s; x, t), s), \quad t - \tau \leq s \leq t,
$$

$$
\xi(s; x_1, x_2, y, t) = (x_1, x_2, y), \quad s = t.
$$

Combining the expression of $S$ and the Darcy’s law, we derive

$$
-\Delta p = \mu[\sigma(\xi(t - \tau; x_1, x_2, y, t), t - \tau) - \bar{\sigma}], \quad (x_1, x_2, y) \in \Omega(t), \quad t > 0,
$$

and assuming the velocity field is continuous up to the boundary, the normal velocity of the moving boundary $\Gamma(t)$ is

$$
V_n = -\nabla p \cdot n = -\frac{\partial p}{\partial n}, \quad (x_1, x_2, y) \in \Gamma(t), \quad t > 0.
$$

Because most of the proteins and lipids that make up the cell membrane are held together with the cell-to-cell adhesiveness, we have the boundary condition, see [4],

$$
p = \kappa, \quad (x_1, x_2, y) \in \Gamma(t), \quad t > 0,
$$

where $\kappa$ is the mean curvature. And

$$
\frac{\partial p}{\partial y} = 0, \quad (x_1, x_2, y) \in \Gamma_0, \quad t > 0.
$$

For convenience of our discussion, we shall also impose the $2\pi$-periodic condition in the $x_1$ and $x_2$ directions.
We finally prescribe initial conditions. For simplicity we assume initial data are time independent
on the interval \([-\tau, 0]\):
\[
\begin{align*}
\Omega(t) &= \Omega_0, \quad -\tau \leq t \leq 0, \\
p(x_1, x_2, y, t) &= p_0(x_1, x_2, y), \quad (x_1, x_2, y) \in \Omega_0, \quad -\tau \leq t \leq 0,
\end{align*}
\]
where we assume the compatibility condition \(\frac{\partial \Omega}{\partial n} = 0\) on \(\partial \Omega_0\). The \(p\) and \(\xi\) are interdependent on
the interval \([t - \tau, t]\); the value of \(\xi\) at 0, for example, depends on the value of \(p\) at \([-\tau, 0]\). Once the initial
data for \(p\) is available on \([-\tau, 0]\), we can solve \(\xi\). So we only assume initial data for \(p_0\).

The idea of adding time delay on the tumor model was initiated by Byrne [3], and recently, the
radially symmetric version has drawn considerable attention of other researchers, see [6, 7, 19–21].
The time delay represents the time taken for cells to undergo replication (approximately 24 hours).
The non-radially symmetric model was established by Zhao and Hu [22, 23], a radially symmetric
stationary solution was found, stability with respect to non-radially symmetric perturbation was
studied, and bifurcation branches were established. In this paper we shall extend the linear stability
results to the flat domains with non-flat perturbations. We begin with the existence and uniqueness
of the stationary solution. In contrast to the results in [22], our domain is different, resulting various
distinct estimates need in order to carry out the proofs. The stationary solution \((\sigma_*, p_*, \rho_*, \xi_*)\) is said to be flat if \(\sigma_*, p_*, \rho_*\) are independent of the variables \(x_1, x_2\) and \(\xi_*(s; x_1, x_2, y) = (x_1, x_2, \xi_30(s; y))\); roughly speaking, here \(s_*\) represents the limit of the variable \(s - t\) as \(t \to \infty\) and therefore \(-\tau \leq s_* \leq 0\): this is the amount of time needed to replace the dead cells by the same amount of new born cells to
make the tumor stationary.

Theorem 1.1. For all \(\mu > 0\), there exists a unique flat stationary solution \((\sigma_*, p_*, \rho_*, \xi_*)\) to the
problem (1.1)-(1.11) for sufficiently small \(\tau\).

In order to obtain the linear stability results, we first linearize the system at the flat stationary
solution \((\sigma_*, p_*, \rho_*, \xi_*)\).

Assume the initial conditions are perturbed from the stationary solution:
\[
\begin{align*}
\partial \Omega(t) : y &= \rho_* + \varepsilon \rho_0(x_1, x_2), \quad -\tau \leq t \leq 0, \\
p(x_1, x_2, y, t) &= p_*(y) + \varepsilon q_0(x_1, x_2, y), \quad -\tau \leq t \leq 0.
\end{align*}
\]
Substituting
\[
\begin{align*}
\partial \Omega(t) : y &= \rho_* + \varepsilon \rho(x_1, x_2, t) + O(\varepsilon^2), \\
\sigma(x_1, x_2, y, t) &= \sigma_*(y) + \varepsilon w(x_1, x_2, y, t) + O(\varepsilon^2), \\
p(x_1, x_2, y, t) &= p_*(y) + \varepsilon q(x_1, x_2, y, t) + O(\varepsilon^2), \\
\xi(s; x_1, x_2, y, t) &= \xi_*(s - t; x_1, x_2, y) + \varepsilon (\xi_{11}, \xi_{21}, \xi_{31}) + O(\varepsilon^2)
\end{align*}
\]
into (1.1)-(1.11) and collecting the \(\varepsilon\)-order terms, we get the linearized system for \((\partial \Omega, \sigma, p, \xi)\) at the
flat stationary solution \((\sigma_*, p_*, \rho_*, \xi_*)\). We define
\[
\begin{align*}
\mu_j(\rho_*^0) &= \frac{1}{\delta} \frac{\rho_*^{j/2} \tanh(\sqrt{j} \rho_*^0)}{k_1(j, \rho_*^0)} \quad \text{for } j > j_0, \\
k_1(j, \rho_*^0) &= 1 - \frac{\tanh \rho_*^0}{\rho_*^0} - \tanh \rho_*^0 \cdot \left[ \sqrt{1 + j} \tanh(\sqrt{1 + j} \rho_*^0) - \sqrt{j} \tanh(\sqrt{j} \rho_*^0) \right],
\end{align*}
\]
where \(\rho_*^0\) is the zeroth-order terms in \(\tau\) of \(\rho_*\) and \(j_0\) is the unique zero of \(k_1(\cdot, \rho_*^0)\). Setting
\[
\mu_j(\rho_*^0) = +\infty \quad \text{for } 0 \leq j \leq j_0, \quad \mu_*(\rho_*^0) = \min_{j > j_0} \mu_j(\rho_*^0).
\]
We now state the linear stability result of the flat stationary solution \((\sigma_*, p_*, \rho_*, \xi_*)\).

Theorem 1.2. For sufficiently small \(\tau\), there exists a threshold value \(\mu_*(\rho_*^0) > 0\) such that the
stationary solution \((\sigma_*, p_*, \rho_*, \xi_*)\) is linearly stable if \(\mu < \mu_*(\rho_*^0)\), i.e., there exist \(C > 0\) and \(\delta > 0\)
such that for the problem linearized in both \(\varepsilon\)-perturbation terms and in time-delay \(\tau\), respectively,
\[
|p(t)| \leq Ce^{-\delta t} \quad \text{for all } t > 0,
\]
the stationary solution \((\sigma_*, p_*, \rho_*, \xi_*)\) is linearly unstable if \(\mu > \mu_*(\rho_*)\).

The structure of this article is as follows. In section 2, we collect some properties of hyperbolic function which will be useful later. We prove the existence and uniqueness of a flat stationary solution by using the contraction mapping principle in section 3. In section 4, we obtain the linearized system of (1.1)-(1.11) and establish the linear stability results. We show the impact of time delay for tumor growth in section 5 and present mathematical and biological implications of our results in section 6.

2. Preliminaries

For convenience, we collect some elementary properties for special functions which are needed later on.

The following are easy to verify:

\[
\begin{align*}
(2.1) \quad & \frac{d}{dp} \frac{\tanh \rho}{\rho} = \frac{1}{\rho} \left( 1 - \frac{\tanh \rho}{\rho} - \tanh^2 \rho \right) = \frac{\rho - \sinh \rho \cosh \rho}{\rho^2 \cosh^2 \rho} < 0, \quad \rho > 0, \\
(2.2) \quad & \lim_{\rho \to 0^+} \frac{\tanh \rho}{\rho} = 1, \quad \lim_{\rho \to +\infty} \frac{\tanh \rho}{\rho} = 0, \\
(2.3) \quad & \int e^{\sqrt{jx}} \cosh(\sqrt{1+jx}) \, dx = \sqrt{1+jx} \sinh(\sqrt{1+jx}) - \sqrt{jx}e^{\sqrt{jx}} \cosh(\sqrt{1+jx}), \\
(2.4) \quad & \int e^{-\sqrt{jx}} \cosh(\sqrt{1+jx}) \, dx = \sqrt{1+jx} \sinh(\sqrt{1+jx}) + \sqrt{jx}e^{-\sqrt{jx}} \cosh(\sqrt{1+jx}), \\
(2.5) \quad & \int e^{-\sqrt{jx}} \sinh(\sqrt{1+jx}) \, dx = \sqrt{1+jx} e^{-\sqrt{jx}} \cosh(\sqrt{1+jx}) + \sqrt{jx}e^{-\sqrt{jx}} \sinh(\sqrt{1+jx}), \\
\end{align*}
\]

and

\[
\begin{align*}
(2.6) \quad & \int_0^\rho \sinh^2 y \, dy = \frac{1}{2} \sinh \rho \cosh \rho - \frac{1}{2} \rho, \\
(2.7) \quad & \int_0^\rho y \sinh y \, dy = \rho \cosh \rho - \sinh \rho.
\end{align*}
\]

It is also derived in [5, section 4],

\[
(2.8) \quad \frac{\partial^2}{\partial x^2} [\sqrt{x} \tanh(\sqrt{x} \rho)] = \frac{\rho}{2} \frac{\partial}{\partial x} \frac{\sinh(\sqrt{x} \rho) \cosh(\sqrt{x} \rho) + \sqrt{x} \rho}{(\rho \sqrt{x}) \cosh^2(\sqrt{x} \rho)} < 0,
\]

for \(x > 0\) and \(\rho > 0\).

3. Flat Stationary Solution

In this section, we prove that there exists a unique flat stationary solution \((\sigma_*, p_*, \rho_*, \xi_*)\) of the system (1.1)-(1.11) for all \(\mu > 0\). Whenever there is no confusion, it is customary to let \(C\) to denote various positive constants in our estimates, although it may change from one line to another. Letting the \(t\)-derivatives to be zero in (1.1)-(1.11), we find that the stationary problem is of the form

\[
\begin{align*}
(3.1) \quad & \begin{cases}
-\sigma''(y) + \sigma(y) = 0, \quad 0 < y < \rho, \\
\sigma(\rho) = \bar{\sigma}, \quad \frac{\partial \sigma}{\partial y} \bigg|_{y=0} = 0,
\end{cases} \\
(3.2) \quad & \begin{cases}
-\rho''(y) = \mu [\sigma(\xi_{30}(-\tau; y)) - \bar{\sigma}], \quad 0 < y < \rho, \\
\rho(\rho) = 0, \quad \frac{\partial \rho}{\partial y} \bigg|_{y=0} = 0,
\end{cases} \\
(3.3) \quad & \begin{cases}
\frac{d\xi_{30}}{ds_*}(s_*; y) = -\frac{\partial \rho}{\partial y}(\xi_{30}(s_*; y)), \quad -\tau \leq s_* \leq 0, \\
\xi_{30}(s_*; y) = y, \quad s_* = 0,
\end{cases} \\
(3.4) \quad & \int_0^\rho \left( \sigma(\xi_{30}(-\tau; y)) - \bar{\sigma} \right) \, dy = 0.
\end{align*}
\]
The equation (3.1) admits an explicit solution:
\[ \sigma_*(y) = \frac{\cosh y}{\cosh \rho} \]

We now proceed to establish the existence of a unique flat stationary solution \((\sigma_*, p_*, \rho_*, \xi_*)\) to the problem (1.1)-(1.11).

**Proof of Theorem 1.1.** Taking \(\hat{\gamma} = \frac{y}{\rho}, \hat{\sigma}(\hat{\gamma}) = \sigma(y), \hat{p}(\hat{\gamma}) = \rho p(y)\) and \(\hat{\xi}_{30}(s_*; \hat{y}) = \frac{\xi_{30}(s_*; y)}{\rho}\) into (3.1)-(3.4), dropping the "\(\hat{\cdot}\)" for notational convenience, we get

\[
\begin{cases}
\sigma''(y) = \rho^2 \sigma(y), & 0 < y < 1, \\
\sigma(1) = \sigma, & \frac{\partial \sigma}{\partial y}|_{y=0} = 0,
\end{cases}
\]

(3.5)\[
\begin{cases}
-p''(y) = \mu \rho^3 \left[ \sigma \left( y + \frac{1}{\rho^3} \int_{-\tau}^{0} \frac{\partial p}{\partial y}(\xi_{30}(s_*; y)) \, ds \right) - \hat{\sigma} \right], & 0 < y < 1, \\
p(1) = 0, & \frac{\partial p}{\partial y}|_{y=0} = 0,
\end{cases}
\]

(3.6)\[
\begin{cases}
\frac{d\xi_{30}}{ds_*}(s_*; y) = -\frac{1}{\rho^3} \frac{\partial p}{\partial y}(\xi_{30}(s_*; y)), & -\tau \leq s_* \leq 0, \\
\xi_{30}(s_*; y) = y, & s_* = 0,
\end{cases}
\]

(3.7)\[
\int_{0}^{1} \left[ \sigma \left( y + \frac{1}{\rho^3} \int_{-\tau}^{0} \frac{\partial p}{\partial y}(\xi_{30}(s_*; y)) \, ds \right) - \hat{\sigma} \right] \, dy = 0.
\]

(3.8)

Equation (3.5) is solved explicitly. For convenience, we also extend the solution outside \([0, 1] : \]

\[
\sigma_*(y; \rho) = \frac{\cosh(\rho y)}{\cosh \rho}, \quad 0 \leq y \leq 1, \quad \sigma_*(y; \rho) = \sigma, \quad 1 < y \leq 2.
\]

Assume that \(\rho_*\) exists and will be in the range of its maximum value \(\rho_{\text{max}}\) and minimum value \(\rho_{\text{min}}\) which will be determined later on. By integrating the first equation of (3.6), we have

\[
p(y) = \int_{y}^{1} \int_{0}^{\rho} \mu \rho^3 \sigma_* \left( z + \frac{1}{\rho_*^3} \int_{-\tau}^{0} \frac{\partial p}{\partial y}(\xi_{30}(s_*; z)) \, ds \right) - \hat{\sigma} \right) \, dz \, d\eta.
\]

(3.10)

Next we prove the existence and uniqueness of \(p\) by using the contraction mapping principle. Obviously, \(0\) is the lower solution of (3.7). But there is no guarantee that the solution of (3.7) stays below the upper boundary \(\{y = 1\}\). So we shall proceed as in [22] to extend \(p\) beyond \(y = 1\). Let

\[X = \{ p \in W^{2, \infty}[0, 2]; ||p||_{W^{2, \infty}[0, 2]} \leq 3 \mu \rho_{\text{max}}^3(\sigma + \hat{\sigma}) \} .\]

For each \(p \in X\), we first solve \(\xi_{30}\) from the ODE (3.7), and substitute it into (3.10) to define a mapping \(T\):

\[
Tp(y) = \int_{y}^{1} \int_{0}^{\rho} \mu \rho^3 \sigma_* \left( z + \frac{1}{\rho_*^3} \int_{-\tau}^{0} \frac{\partial p}{\partial y}(\xi_{30}(s_*; z)) \, ds \right) - \hat{\sigma} \right) \, dz \, d\eta, \quad 0 \leq y \leq 1.
\]

(3.11)

Clearly, \(Tp(1) = 0, \frac{\partial (Tp)}{\partial y} |_{y=0} = 0\). We now extend \(Tp\) to the interval \([0, 2]\) by defining

\[
Tp(y) = \begin{cases}
Tp(y), & 0 \leq y \leq 1, \\
Tp'(1)(y - 1), & 1 < y \leq 2.
\end{cases}
\]

(3.12)

It is clear with this extension, \(Tp\) is continuous with continuous derivative across \(y = 1\), and \(Tp \in W^{2, \infty}[0, 2]\).

Using the expression in (3.11) and the extension (3.12), estimating respectively on the interval \([0, 1]\) and \([1, 2]\), we find that

\[
||Tp||_{W^{2, \infty}[0, 2]} \leq 3 \mu \rho_{\text{max}}^3(\sigma + \hat{\sigma}),
\]

(3.13)

and therefore \(T\) maps \(X\) into itself.

We shall establish that \(T\) is a contraction, namely, for some \(M < 1\),
\begin{equation}
\|T\tilde{p} - Tp\|_X \leq M \|\tilde{p} - p\|_X, \quad \forall \tilde{p}, p \in X.
\end{equation}

Next, we prove (3.14). Let \( \xi_{30} \) and \( \tilde{\xi}_{30} \) be the corresponding solutions. By integrating the first equation of (3.7), we have

\[
\max_{-\tau \leq s \leq 0, 0 \leq y \leq 1} |\tilde{\xi}_{30}(s; y, 0) - \xi_{30}(s; y, 0)| = \max_{-\tau \leq s \leq 0} \left| \frac{1}{\rho^*_s} \int_{s}^{0} \left[ \frac{\partial \tilde{p}}{\partial y}(\tilde{\xi}_{30}(s; y)) - \frac{\partial p}{\partial y}(\xi_{30}(s; y)) \right] ds \right| \\
\leq \frac{\tau}{\rho^*_s} \left[ \|\tilde{p} - p\|_{W^2,\infty[0,2]} + \|p\|_{W^2,\infty[0,2]} \max_{-\tau \leq s \leq 0} |\tilde{\xi}_{30} - \xi_{30}| \right] \\
\leq \frac{\tau}{\mu \rho^*_s - \tau C} \left[ \|\tilde{p} - p\|_{W^2,\infty[0,2]} \right],
\]

where by the choice of our \( X \), \( \|p\|_{W^2,\infty[0,2]} \leq 3 \mu \rho^*_s \|\sigma + \tilde{\sigma}\| \leq C \frac{\tau}{\tau} \) if \( \tau \) is small. Thus

\begin{equation}
\max_{-\tau \leq s \leq 0, 0 \leq y \leq 1} |\tilde{\xi}_{30}(s; y, 0) - \xi_{30}(s; y, 0)| \leq \frac{\tau}{\rho^*_s - \tau C} \|\tilde{p} - p\|_{W^2,\infty[0,2]}.
\end{equation}

From (3.15), (3.12) and (3.11),

\begin{equation}
\|Tq - Tp\| \leq \frac{\tau}{\mu \rho^*_s - \tau C} \|q - p\|_{W^2,\infty[0,2]}.
\end{equation}

Since \( Tp(1) = T\tilde{p}(1) = 0 \) and \( (Tp)'(0) = (T\tilde{p})'(0) = 0 \), the above estimates imply

\begin{equation}
\|T\tilde{p} - Tp\|_{W^2,\infty[0,2]} \leq C \mu \rho^*_s \left[ \frac{\tau}{\mu \rho^*_s - \tau C} \|\tilde{p} - p\|_{W^2,\infty[0,2]} \right].
\end{equation}

If \( \tau \) is suitably small, then \( M \leq C \mu \rho^*_s \left[ \frac{\tau}{\mu \rho^*_s - \tau C} \right] < 1 \), therefore we established (3.14) and \( T \) is a contraction, which admits a unique fixed point \( p_* \). Substituting \( p_* \) into (3.7) and from ODE theory, we obtain \( \xi_* \).

To complete the proof, it suffices to show that there exists a unique solution \( \rho_* \in [\rho_{\text{min}}, \rho_{\text{max}}] \) satisfying (3.8). After substituting (3.9) into (3.8), we find that this is equivalent to solving the following equation for \( \rho \):

\[
F(\rho, \tau) \triangleq \int_{0}^{1} \left\{ \frac{\cosh \left[ \rho \left( y + \frac{1}{\rho^*_s} \int_{-\tau}^{0} \frac{\partial \tilde{p}}{\partial y}(\xi_{30}(s; y)) ds \right) \right] - \tilde{\sigma}}{\cosh \rho} \right\} dy = 0.
\]

Clearly,

\[
F(\rho, 0) = \int_{0}^{1} \left( \frac{\cosh \left( \frac{\cosh \rho}{\rho} \right) - \tilde{\sigma}}{\cosh \rho} \right) dy = \frac{\tilde{\sigma}}{\rho} \tanh \rho - \tilde{\sigma},
\]

and from (2.1) and (2.2),

\[
\lim_{\rho \to 0} F(\rho, 0) = \tilde{\sigma} - \tilde{\sigma} > 0, \quad \lim_{\rho \to \infty} F(\rho, 0) = -\tilde{\sigma} < 0.
\]

Notice that (2.1) also implies that \( F(\rho, 0) \) is monotone decreasing in \( \rho \), so that the equation \( F(\rho, 0) = 0 \) admits a unique solution (denoting by \( \rho_S \)) and

\begin{equation}
F\left( \frac{1}{2} \rho_S, 0 \right) > 0, \quad F\left( \frac{3}{2} \rho_S, 0 \right) < 0.
\end{equation}
The mean value theorem implies, for some $0 \leq \eta \leq \tau$,
\[
\frac{\partial F(\rho, \tau)}{\partial \rho} - \frac{\partial F(\rho, 0)}{\partial \rho} = \frac{\partial^2 F}{\partial \rho \partial \tau}(\rho, \eta) \tau = O(\tau).
\]
It follows that $\frac{\partial F(\rho, \tau)}{\partial \rho} < 0$ when $\tau$ is small enough. In a similar argument, we also have
\[
F\left(\frac{1}{2} \rho_S, \tau\right) > 0, \quad F\left(\frac{3}{2} \rho_S, \tau\right) < 0.
\]
Therefore, when $\tau$ is small enough, the equation (3.8) admits a unique solution $\rho_*$ satisfying $F(\rho_*, \tau) = 0$ and $\frac{1}{2} \rho_S < \rho_* < \frac{3}{2} \rho_S$. The proof is complete with $\rho_{\min} = \frac{1}{2} \rho_S$ and $\rho_{\max} = \frac{3}{2} \rho_S$. □

4. LINEAR STABILITY

In this section, we consider the linear stability of the unique flat stationary solution $(\sigma_*, p_*, \rho_*, \xi_*)$ obtained in section 3 under non-flat perturbations. We begin by taking some small non-flat perturbations on the initial conditions:

(4.1) \[ \partial \Omega(t) : y = \rho_* + \varepsilon \rho_0(x_1, x_2), \quad -\tau \leq t \leq 0, \]

(4.2) \[ p(x_1, x_2, y, t) = p_*(y) + \varepsilon \rho_0(x_1, x_2, y), \quad (x_1, x_2, y) \in \Omega_0, \quad -\tau \leq t \leq 0. \]

Then for $t > 0$, we expect to have formal expansion:

(4.3) \[
\begin{align*}
\sigma(x_1, x_2, y, t) &= \sigma_*(y) + \varepsilon \sigma_0(x_1, x_2, y, t) + O(\varepsilon^2), \\
p(x_1, x_2, y, t) &= p_*(y) + \varepsilon p_0(x_1, x_2, y, t) + O(\varepsilon^2), \\
\xi(s; x_1, x_2, y, t) &= \xi_*(s - t; x_1, x_2, y) + \varepsilon(\xi_{11}, \xi_{21}, \xi_{31}) + O(\varepsilon^2).
\end{align*}
\]

Writing in Cartesian coordinates,

(4.4) \[ \xi(s; x_1, x_2, y, t) = \xi_1(s; x_1, x_2, y, t) \hat{i} + \xi_2(s; x_1, x_2, y, t) \hat{j} + \xi_3(s; x_1, x_2, y, t) \hat{k}, \]

we obtain from (1.4)–(1.5) that

(4.5) \[
\begin{align*}
\frac{d\xi_1(s; x_1, x_2, y, t)}{ds} &= -\frac{\partial p}{\partial x_1}(\xi_1, \xi_2, \xi_3, s), & t - \tau \leq s \leq t, \\
\xi_1(s; x_1, x_2, y, t)\big|_{s=t} &= x_1;
\end{align*}
\]

(4.6) \[
\begin{align*}
\frac{d\xi_2(s; x_1, x_2, y, t)}{ds} &= -\frac{\partial p}{\partial x_2}(\xi_1, \xi_2, \xi_3, s), & t - \tau \leq s \leq t, \\
\xi_2(s; x_1, x_2, y, t)\big|_{s=t} &= x_2;
\end{align*}
\]

(4.7) \[
\begin{align*}
\frac{d\xi_3(s; x_1, x_2, y, t)}{ds} &= -\frac{\partial p}{\partial y}(\xi_1, \xi_2, \xi_3, s), & t - \tau \leq s \leq t, \\
\xi_3(s; x_1, x_2, y, t)\big|_{s=t} &= y.
\end{align*}
\]

We then expand $\xi_1, \xi_2, \xi_3$ in $\varepsilon$ as

(4.8) \[
\begin{align*}
\xi_1(s; x_1, x_2, y, t) &= x_1 + \varepsilon \xi_{11}(s; x_1, x_2, y, t) + O(\varepsilon^2), \\
\xi_2(s; x_1, x_2, y, t) &= x_2 + \varepsilon \xi_{21}(s; x_1, x_2, y, t) + O(\varepsilon^2), \\
\xi_3(s; x_1, x_2, y, t) &= \xi_{30}(s - t; y) + \varepsilon \xi_{31}(s; x_1, x_2, y, t) + O(\varepsilon^2).
\end{align*}
\]

Recalling that we already obtained the zeroth order equation $\frac{d\xi_{30}}{dt}(s, y) = -\frac{\partial p}{\partial y}(\xi_{30}(s, y))$; $\xi_{30}\big|_{s=0} = y$ (cf. (3.3)). By substituting (4.8) into (4.5)–(4.7) and dropping the higher order terms, we obtain
the first order equations for $\xi$:

\[
\begin{cases}
\frac{d\xi_{11}}{ds} = -\frac{\partial q}{\partial x_1}(x_1, x_2, \xi_{30}, s), & t - \tau \leq s \leq t, \\
\xi_{11}|_{s=t} = 0;
\end{cases}
\]

\[
\begin{cases}
\frac{d\xi_{12}}{ds} = -\frac{\partial q}{\partial x_2}(x_1, x_2, \xi_{30}, s), & t - \tau \leq s \leq t, \\
\xi_{12}|_{s=t} = 0;
\end{cases}
\]

\[
\begin{cases}
\frac{d\xi_{13}}{ds} = -\frac{\partial^2 p_s}{\partial y^2}(\xi_{30}(s-t; y))\xi_{31}(s; x_1, x_2, y, t) - \frac{\partial q}{\partial y}(x_1, x_2, \xi_{30}, s), & t - \tau \leq s \leq t, \\
\xi_{13}|_{s=t} = 0.
\end{cases}
\]

By substituting (4.3) and (4.9)–(4.11) into (1.1)–(1.11), applying the following mean-curvature formula in the 3-dimensional case for $y = f(x_1, x_2)$:

\[
\kappa = -\frac{(1 + f_{x_2}^2)_{x_1} + (1 + f_{x_1}^2)_{x_2} - 2f_{x_1}f_{x_2}f_{x_1x_2}}{2(1 + f_{x_1}^2 + f_{x_2}^2)^{3/2}},
\]

and collecting the $\varepsilon$-order terms, we get the linearized system of (1.1)–(111):

\[
\begin{cases}
\Delta w(x_1, x_2, y, t) = w(x_1, x_2, y, t), & 0 < y < \rho, \quad t > 0, \\
w(x_1, x_2, y, t)\bigg|_{y=\rho} = -\frac{\partial \sigma_s}{\partial y}\bigg|_{y=\rho}(x_1, x_2, t), \quad \frac{\partial w}{\partial y}(x_1, x_2, y, t)\bigg|_{y=0} = 0,
\end{cases}
\]

\[
\begin{cases}
-\Delta q(x_1, x_2, y, t) = \mu \frac{\partial \sigma_s}{\partial y}(\xi_{30}(s-t; y))\xi_{31}(s; x_1, x_2, y, t) + \mu w(x_1, x_2, \xi_{30}(s-t; y), t - \tau), & 0 < y < \rho, \quad t > 0, \\
q(x_1, x_2, y, t)\bigg|_{y=\rho} = -\frac{1}{2} \left(\rho_{x_1x_1} + \rho_{x_2x_2}\right), \quad \frac{\partial q}{\partial y}(x_1, x_2, y, t)\bigg|_{y=0} = 0,
\end{cases}
\]

\[
\begin{cases}
\frac{d\rho}{dt}(x_1, x_2, t) = -\frac{\partial^2 p_s}{\partial y^2}\bigg|_{y=\rho}(x_1, x_2, t) - \frac{\partial q}{\partial y}\bigg|_{y=\rho}(x_1, x_2, y, t).
\end{cases}
\]

Together with (111), we obtain a linearized system. We look for solutions of the form:

\[
\begin{aligned}
w(x_1, x_2, y, t) &= w_{n,m}(y, t) \cos(nx_1) \cos(mx_2), \\
q(x_1, x_2, y, t) &= q_{n,m}(y, t) \cos(nx_1) \cos(mx_2), \\
\rho(x_1, x_2, t) &= \rho_{n,m}(t) \cos(nx_1) \cos(mx_2), \\
\xi_{31}(s; x_1, x_2, y, t) &= \varphi_{n,m}(s; y, t) \cos(nx_1) \cos(mx_2).
\end{aligned}
\]

As we shall easily verify that the equations for $w_{n,m}, q_{n,m}, \rho_{n,m}, \varphi_{n,m}$ will not change if we replace $\cos(nx_1) \cos(mx_2)$ by any of the following

\[
\cos(nx_1) \sin(mx_2), \quad \sin(nx_1) \cos(mx_2), \quad \sin(nx_1) \sin(mx_2).
\]

These constitute a base for the Fourier series $2\pi$ periodic in $x_1$ and $x_2$. From (4.12)–(4.14), we derive

\[
\begin{cases}
-\frac{\partial^2 w_{n,m}}{\partial y^2}(y, t) + (n^2 + m^2 + 1)w_{n,m}(y, t) = 0, \\
w_{n,m}(\rho, t) = -\frac{\partial \sigma_s}{\partial y}\bigg|_{y=\rho}(\rho_{n,m}(t), \frac{\partial w_{n,m}}{\partial y}(0, t) = 0,
\end{cases}
\]

\[
\begin{cases}
-\frac{\partial^2 q_{n,m}}{\partial y^2}(y, t) + (n^2 + m^2)q_{n,m}(y, t) = \mu w_{n,m}(\xi_{30}(s-t; y), t - \tau) + \mu \frac{\partial \sigma_s}{\partial y}(\xi_{30}(s-t; y))\varphi_{n,m}(t - \tau; y, t), \\
q_{n,m}(\rho, t) = \frac{1}{2} (n^2 + m^2) \rho_{n,m}(t), \quad \frac{\partial q_{n,m}}{\partial y}(0, t) = 0,
\end{cases}
\]
The boundary conditions in (3.2) are expanded as follows:

\begin{equation}
\frac{\partial \varphi_{n,m}}{\partial s}(s; y, t) = - \frac{\partial^2 p_s}{\partial y^2}(\xi_{30}(s - t; y)) \varphi_{n,m}(s; y, t) - \frac{\partial q_{n,m}}{\partial y}(\xi_{30}(s - t; y), s), \quad t - \tau \leq s \leq t,
\end{equation}

\begin{equation}
\varphi_{n,m}(s; y, t) \big|_{s=t} = 0.
\end{equation}

Solving (4.15), we obtain

\begin{equation}
w_{n,m} = - \frac{\partial \sigma_s}{\partial y}(\rho_s) \rho_{n,m}(t) \frac{\cosh(\sqrt{n^2 + m^2 + 1}y)}{\cosh(\sqrt{n^2 + m^2 + 1} \rho_s)}.
\end{equation}

### 4.1. Expansion in $\tau$

As $\tau$ is small, we seek expansion in $\tau$ of the form:

\begin{align*}
\rho_s &= \rho_s^0 + \tau \rho_s^1 + O(\tau^2), \\
\sigma_s(y) &= \sigma_s^0(y) + \tau \sigma_s^1(y) + O(\tau^2), \\
p_s(y) &= p_s^0(y) + \tau p_s^1(y) + O(\tau^2), \\
w_{n,m}(y, t) &= w_{n,m}^0(y, t) + \tau w_{n,m}^1(y, t) + O(\tau^2), \\
q_{n,m}(y, t) &= q_{n,m}^0(y, t) + \tau q_{n,m}^1(y, t) + O(\tau^2), \\
\rho_{n,m}(y, t) &= \rho_{n,m}^0(y, t) + \tau \rho_{n,m}^1(y, t) + O(\tau^2).
\end{align*}

#### 4.1.1. Expansion of (3.1)

From (3.1), we find

\[ \sigma_s^0 + \tau \sigma_s^1 + O(\tau^2) = \frac{\sigma_s \cosh y}{\cosh \rho_s} = \sigma \left( \frac{\cosh y}{\cosh \rho_s} - \tau \frac{\cosh y \sinh \rho_s}{\cosh^2 \rho_s} \rho_s^1 + O(\tau^2) \right), \]

therefore,

\begin{equation}
\sigma_s^0(y) = \sigma \frac{\cosh y}{\cosh \rho_s}, \quad \sigma_s^1(y) = - \frac{\sigma \rho_s^1 \sinh \rho_s}{\cosh^2 \rho_s} \cosh y.
\end{equation}

The boundary conditions in (3.1) are expanded as follows:

\[ \sigma_s^0 (\rho_s^0) + \tau \frac{\partial \sigma_s^0}{\partial y} (\rho_s^0) \rho_s^1 + \tau \sigma_s^1 (\rho_s^0) + O(\tau^2) = \sigma, \]

thus,

\begin{align*}
\sigma_s^0 (\rho_s^0) &= \sigma, & \left. \frac{\partial \sigma_s^0}{\partial y} (\rho_s^0) \right|_{y=0} &= 0, \\
\sigma_s^1 (\rho_s^0) &= - \left. \frac{\partial \sigma_s^1}{\partial y} (\rho_s^0) \rho_s^1 \right|_{y=0} &= 0.
\end{align*}

#### 4.1.2. Expansion of (3.2)

Integrating equation (3.3) over the interval $(-\tau, 0)$, we derive

\begin{equation}
\xi_{30}(-\tau; y) = y + \int_{-\tau}^0 \frac{\partial p_s}{\partial y}(\xi_{30}(s; y)) ds = y + \tau \frac{\partial p_s}{\partial y}(y) + O(\tau^2).
\end{equation}

It follows that

\[ \sigma_s(\xi_{30}(-\tau; y)) = \sigma_s \left( y + \tau \frac{\partial p_s^0}{\partial y}(y) + O(\tau^2) \right) = \sigma_s^0(y) + \tau \left( \frac{\partial \sigma_s^0}{\partial y}(y) \frac{\partial p_s^0}{\partial y}(y) + \sigma_s^1(y) \right) + O(\tau^2). \]

Substituting the above expression into the right-hand side of (3.2), we get

\[ - \frac{\partial^2 p_s^0}{\partial y^2} = \mu (\sigma_s^0 - \bar{\sigma}), \quad - \frac{\partial^2 p_s^1}{\partial y^2} = \mu \frac{\partial \sigma_s^0}{\partial y} \frac{\partial p_s^0}{\partial y} + \mu \sigma_s^1. \]

The boundary conditions in (3.2) are expanded as follows:

\[ p_s^0 (\rho_s^0) + \tau \frac{\partial p_s^0}{\partial y} (\rho_s^0) \rho_s^1 + \tau \rho_s^1 (\rho_s^0) + O(\tau^2) = 0, \]
Therefore
\[ p_s^0(\rho_s^0) = 0, \quad \frac{\partial p_s^0}{\partial y} \bigg|_{y=0} = 0, \quad p_s^1(\rho_s^0) = \frac{\partial p_s^0}{\partial y}(\rho_s^0)\rho_s^1, \quad \frac{\partial p_s^1}{\partial y} \bigg|_{y=0} = 0. \]

4.1.3. Expansion of (3.4). From (4.21), we have
\[ 0 = \int_0^{\rho_s^*} [\sigma_s(\xi_{30}(-\tau; y)) - \bar{\sigma}]dy \]
\[ = \int_0^{\rho_s^*} [\sigma_s^0(y) - \bar{\sigma}]dy + \tau \int_0^{\rho_s^0} \left[ \frac{\partial \sigma_s^0}{\partial y}(y) \frac{\partial p_s^0}{\partial y}(y) + \sigma_s^1(y) \right] dy + O(\tau^2). \]
The first part of (4.22) can be integrated as follows:
\[ \int_0^{\rho_s^0} [\sigma_s^0(y) - \bar{\sigma}]dy = \int_0^{\rho_s^0} \left( \frac{\sigma_s^0(y)}{\cosh \rho_s^0} - \bar{\sigma} \right) dy = \sigma \sinh \rho_s^0 - \bar{\sigma} \rho_s^0 \]
\[ = \sigma \tanh \rho_s^0 - \bar{\sigma} \rho_s^0 + \tau (\sigma \rho_s^1 - \bar{\sigma} \rho_s^1) + O(\tau^2). \]
By combining (4.22) and (4.23), we deduce
\[ \sigma \tanh \rho_s^0 - \bar{\sigma} \rho_s^0 = 0, \]
\[ (\sigma - \bar{\sigma})\rho_s^1 + \int_0^{\rho_s^0} \left[ \frac{\partial \sigma_s^0}{\partial y}(y) \frac{\partial p_s^0}{\partial y}(y) + \sigma_s^1(y) \right] dy = 0. \]
Recalling the definitions of \( \rho_s \) in (3.18) and \( F(\rho,0) \) in the proof of Theorem 1.1, we must have \( \rho_s^0 = \rho_s \).

4.1.4. Expansion of (4.15). The expansion of the first equation in (4.15) is straightforward we omit it. The boundary conditions are evaluated as follows:
\[ w_{n,m}^0(\rho_s^0, t) + \tau \frac{\partial w_{n,m}^0}{\partial y}(\rho_s^0)\rho_s^1 + \tau w_{n,m}^1(\rho_s^0) \]
\[ = - \left( \frac{\partial \sigma_s^0}{\partial y}(\rho_s^0) + \tau \frac{\partial^2 \sigma_s^0}{\partial y^2}(\rho_s^0)\rho_s^1 + \tau \frac{\partial \sigma_s^1}{\partial y}(\rho_s^0) \right) \left[ \rho_{n,m}^0(t) + \tau \rho_{n,m}^1(t) \right] + O(\tau^2), \]
which implies
\[ w_{n,m}^0(\rho_s^0, t) = - \frac{\partial \sigma_s^0}{\partial y}(\rho_s^0)\rho_{n,m}^0(t), \]
\[ w_{n,m}^1(\rho_s^0, t) = - \frac{\partial w_{n,m}^0}{\partial y}(\rho_s^0)\rho_s^1 - \frac{\partial \sigma_s^0}{\partial y}(\rho_s^0)\rho_{n,m}^0(t) - \frac{\partial^2 \sigma_s^0}{\partial y^2}(\rho_s^0)\rho_s^1\rho_{n,m}^0(t) - \frac{\partial \sigma_s^1}{\partial y}(\rho_s^0)\rho_{n,m}^0(t). \]
Similarly, the boundary condition at \( \{ y = 0 \} \) is evaluated:
\[ \frac{\partial w_{n,m}^0}{\partial y} \bigg|_{y=0} + \tau \frac{\partial w_{n,m}^1}{\partial y} \bigg|_{y=0} + O(\tau^2) = 0, \]
which gives
\[ \frac{\partial w_{n,m}^0}{\partial y} \bigg|_{y=0} = 0, \quad \frac{\partial w_{n,m}^1}{\partial y} \bigg|_{y=0} = 0. \]

4.1.5. Expansion of (4.16). To find the expansion of (4.16), we first compute, by (4.17):
\[ \varphi_{n,m}(t - \tau; y, t) = \varphi_{n,m}(t; y, t) - \tau \frac{\partial \varphi_{n,m}}{\partial s}(t; y, t) + O(\tau^2) \]
\[ = 0 - \tau \left( - \frac{\partial^2 p_s}{\partial y^2}(y)\varphi_{n,m}(t; y, t) - \frac{\partial q_{n,m}}{\partial y}(y, t) \right) + O(\tau^2) \]
\[ = \tau \frac{\partial q_{n,m}}{\partial y}(y, t) + O(\tau^2); \]
it follows that
\[
\frac{\partial \sigma}{\partial y} (\xi_30(-\tau; y)) \varphi_{n,m}(t - \tau; y, t) = \left( \frac{\partial \sigma^0}{\partial y} (y) + O(\tau) \right) \left( \frac{\partial q^0_{n,m}}{\partial y} (y, t) + O(\tau^2) \right) \\
= \tau \frac{\partial \sigma^0}{\partial y} (y) \frac{\partial q^0_{n,m}}{\partial y} (y, t) + O(\tau^2).
\]

On the other hand,
\[
w_{n,m}(\xi_30(-\tau; y), t - \tau) \\
= w_{n,m} (\xi_30(0; y) - \tau \frac{\partial \xi_30}{\partial s} (0; y) + O(\tau^2), t - \tau) \\
= w_{n,m} \left( y + \tau \frac{\partial p^0}{\partial y} (\xi_30(0; y)) + O(\tau^2), t - \tau \right) \quad \text{(by (3.3))}
\]
\[(4.25)\]
\[
w_{n,m}(y, t) + \tau \frac{\partial w_{n,m}}{\partial y} (y, t) \frac{\partial p^0}{\partial y} (y) - \tau \frac{\partial w_{n,m}}{\partial t} (y, t) + O(\tau^2) \\
= w^0_{n,m}(y, t) + \tau \left[ \frac{\partial w^0_{n,m}}{\partial y} (y, t) \frac{\partial p^0}{\partial y} (y) - \frac{\partial w^0_{n,m}}{\partial t} (y, t) + w^1_{n,m}(y, t) \right] + O(\tau^2).
\]

By substituting (4.24)–(4.25) into (4.16), we obtain
\[
- \frac{\partial^2 q^0_{n,m}}{\partial y^2} + (n^2 + m^2)q^0_{n,m} = \mu w^0_{n,m},
\]
\[
- \frac{\partial^2 q^1_{n,m}}{\partial y^2} + (n^2 + m^2)q^1_{n,m} = \mu \frac{\partial \sigma^0}{\partial y} \frac{\partial q^0_{n,m}}{\partial y} + \mu \frac{\partial w^0_{n,m}}{\partial y} \frac{\partial p^0}{\partial y} - \mu \frac{\partial w^0_{n,m}}{\partial t} + \mu w^1_{n,m}.
\]

The first boundary condition in (4.16) is given by
\[
q^0_{n,m}(\rho^0_*, t) + \tau \frac{\partial q^0_{n,m}}{\partial y} (\rho^0_*, t) \rho^1_* + \tau q^1_{n,m}(\rho^0_*, t) = \frac{1}{2} (n^2 + m^2) [\rho^0_{n,m}(t) + \tau \rho^1_{n,m}(t)] + O(\tau^2),
\]
which implies
\[
q^0_{n,m}(\rho^0_*, t) = \frac{1}{2} (n^2 + m^2) \rho^0_{n,m}(t),
\]
\[
q^1_{n,m}(\rho^0_*, t) = - \frac{\partial q^0_{n,m}}{\partial y} (\rho^0_*, t) \rho^1_* + \frac{1}{2} (n^2 + m^2) \rho^1_{n,m}(t).
\]

Likewise, the second boundary condition is evaluated as:
\[
\left. \frac{\partial q^0_{n,m}}{\partial y} \right|_{y=0} + \tau \left. \frac{\partial q^1_{n,m}}{\partial y} \right|_{y=0} + O(\tau^2) = 0,
\]
which gives
\[
\left. \frac{\partial q^0_{n,m}}{\partial y} \right|_{y=0} = 0, \quad \left. \frac{\partial q^1_{n,m}}{\partial y} \right|_{y=0} = 0.
\]

4.1.6. Expansion of (4.18). The following equation
\[
\frac{d}{dt} [\rho^0_{n,m}(t) + \tau \rho^1_{n,m}(t)] = - \left( \frac{\partial^2 \rho^0_*}{\partial y^2} (\rho^0_* + \tau \rho^1_*) + \tau \frac{\partial^2 \rho^1_*}{\partial y^2} (\rho^0_*) \right) [\rho^0_{n,m}(t) + \tau \rho^1_{n,m}(t)] \\
- \frac{\partial (q^0_{n,m} + \tau q^1_{n,m})}{\partial y} (\rho^0_* + \tau \rho^1_*) + O(\tau^2)
\]
implies
\[
\frac{d \rho^0_{n,m}(t)}{dt} = - \frac{\partial^2 \rho^0}{\partial y^2} (\rho^0_*) \rho^0_{n,m}(t) - \frac{\partial q^0_{n,m}}{\partial y} (\rho^0_*, t),
\]
\[
\frac{d\rho^0_{n,m}(t)}{dt} = -\frac{\partial^2 p_0^0(\rho^0_\ast)}{\partial y^2}(\rho^1_{n,m}(t)) - \frac{\partial^2 p_0^0(\rho^0_\ast)}{\partial y^2}(\rho^1_{n,m}(t)) - \frac{\partial^2 q_{n,m}^0(\rho^0_\ast)}{\partial y^2}(\rho^0_\ast, t) - \frac{\partial q_{n,m}^0}{\partial y}(\rho^0_\ast, t).
\]

4.2. Zeroth-order terms in \(\tau\). Collecting the zeroth-order terms from subsection 4.1, we obtain

\[
\sigma^0(y) = \frac{\sigma}{\cosh \rho^0_\ast} \cosh y, \\
\left\{\begin{aligned}
- (p_0^0)^{\prime\prime}(y) &= \mu[\sigma^0(y) - \bar{\sigma}], \quad 0 < y < \rho, \\
p_0^0(\rho^0_\ast) &= 0, \quad \frac{\partial p_0^0}{\partial y}(0) = 0,
\end{aligned}\right.
\]

\[
\tanh \frac{\rho^0_\ast}{\rho^0_\ast} = \frac{\bar{\sigma}}{\sigma}, \quad \text{i.e.,} \quad \rho^0_\ast = \rho_S,
\]

\[
\left\{\begin{aligned}
- \frac{\partial^2 w_{n,m}^0(y, t)}{\partial y^2}(y, t) &= (n^2 + m^2 + 1)w_{n,m}^0(y, t) = 0, \\
w_{n,m}^0(\rho^0_\ast, t) &= - \frac{\partial \sigma^0}{\partial y} \big|_{y=\rho^0_\ast} \rho_{n,m}(t), \quad \frac{\partial w_{n,m}^0}{\partial y}(0, t) = 0,
\end{aligned}\right.
\]

\[
\left\{\begin{aligned}
- \frac{\partial^2 q_{n,m}^0(y, t)}{\partial y^2}(y, t) &= (n^2 + m^2) q_{n,m}^0(y, t) = \mu w_{n,m}^0(y, t), \\
q_{n,m}^0(\rho^0_\ast, t) &= \frac{1}{2} (n^2 + m^2) \rho_{n,m}(t), \quad \frac{\partial q_{n,m}^0}{\partial y}(0, t) = 0,
\end{aligned}\right.
\]

\[
\frac{d\rho^0_{n,m}(t)}{dt} = -\frac{\partial^2 p_0^0(\rho^0_\ast)}{\partial y^2}(\rho^0_{n,m}(t)) - \frac{\partial q_{n,m}^0}{\partial y}(\rho^0_\ast, t).
\]

From (4.26), we obtain

\[
\frac{\partial \sigma^0}{\partial y}(y) = \frac{\sigma}{\cosh \rho^0_\ast} \sinh y, \quad \frac{\partial^2 \sigma^0}{\partial y^2}(y) = \frac{\sigma}{\cosh \rho^0_\ast} \cosh y.
\]

Substituting (4.26) into (4.27), we solve \(p^0_\ast\) explicitly:

\[
p^0_\ast(y) = \frac{1}{2} \mu \bar{\sigma} y^2 + \mu \bar{\sigma} - \frac{1}{2} \mu \bar{\sigma}(\rho^0_\ast)^2 - \mu \sigma y^2 - \mu \sigma \cosh y.
\]

it follows that

\[
\frac{\partial p^0_\ast}{\partial y}(y) = \mu \bar{\sigma} y - \mu \bar{\sigma} \frac{\sinh y}{\cosh \rho^0_\ast}, \quad \frac{\partial^2 p^0_\ast}{\partial y^2}(y) = \mu \bar{\sigma} - \mu \sigma \frac{\cosh y}{\cosh \rho^0_\ast}.
\]

Similarly, (4.29) is solved explicitly:

\[
w_{n,m}^0(y, t) = - \frac{\partial \sigma^0}{\partial y}(\rho^0_\ast) \frac{\cosh(\sqrt{1 + n^2 + m^2} y)}{\cosh(\sqrt{1 + n^2 + m^2} \rho^0_\ast)} \rho_{n,m}(t).
\]

To solve \(q_{n,m}^0\), we let

\[
z(y, t) = \frac{\partial q_{n,m}^0}{\partial y}(y, t) - \sqrt{n^2 + m^2} q_{n,m}^0(y, t).
\]

Then

\[
\frac{\partial z}{\partial y}(y, t) + \sqrt{n^2 + m^2} z(y, t) = - \mu w_{n,m}^0(y, t) = \frac{\partial \sigma^0}{\partial y}(\rho^0_\ast) \frac{\cosh(\sqrt{1 + n^2 + m^2} y)}{\cosh(\sqrt{1 + n^2 + m^2} \rho^0_\ast)} \rho_{n,m}(t).
\]
Using the integration formula (2.3), we obtain

\[
z(y, t) = e^{-\sqrt{n^2 + m^2}y} \left[ \int \mu \frac{\partial \sigma^0}{\partial y}(\rho^0)\rho^0_{n,m}(t) \frac{\cosh(\sqrt{1 + n^2 + m^2}y)}{\cosh(\sqrt{1 + n^2 + m^2}\rho^0_{s})} \cosh(\sqrt{1 + n^2 + m^2}y)dy + C_1 \right]
\]

(4.37)

where \( C_1 \) is independent of \( y \).

From (4.36) and integration formula (2.4)–(2.5), we get

\[
q^0_{n,m} = e^{\sqrt{n^2 + m^2}y} \left( \int e^{-\sqrt{n^2 + m^2}y} zdy + C_2 \right)
\]

(4.38)

\[
= \frac{\mu \frac{\partial \sigma^0}{\partial y}(\rho^0)\rho^0_{n,m}(t)}{\cosh(\sqrt{1 + n^2 + m^2}\rho^0_{s})} \frac{\cosh(\sqrt{1 + n^2 + m^2}y)}{\cosh(\sqrt{1 + n^2 + m^2}\rho^0_{s})} \left( n^2 + m^2 \right) - \frac{C_1}{2\sqrt{n^2 + m^2}} e^{-\sqrt{n^2 + m^2}y} + C_2 e^{\sqrt{n^2 + m^2}y}.
\]

Applying the boundary conditions in (4.30), we find

\[
C_1 = \left[ \frac{2n\frac{\partial \sigma^0}{\partial y}(\rho^0) - (n^2 + m^2)}{2\cosh(\sqrt{n^2 + m^2}\rho^0_{s})} \right] \rho^0_{n,m}(t), \quad C_2 = \frac{1}{2} \left( n^2 + m^2 \right) - \frac{\mu \frac{\partial \sigma^0}{\partial y}(\rho^0)\rho^0_{n,m}(t)}{2\cosh(\sqrt{n^2 + m^2}\rho^0_{s})} e^{\sqrt{n^2 + m^2}y}.
\]

(4.39)

Substituting the expressions \( C_1 \) and \( C_2 \) into (4.38), we derive

\[
q^0_{n,m} = \rho^0_{n,m}(t) \left\{ \mu \frac{\partial \sigma^0}{\partial y}(\rho^0) \frac{\cosh(\sqrt{1 + n^2 + m^2}y)}{\cosh(\sqrt{1 + n^2 + m^2}\rho^0_{s})} \right\} + \frac{(n^2 + m^2) - \mu \frac{\partial \sigma^0}{\partial y}(\rho^0)}{2\cosh(\sqrt{n^2 + m^2}\rho^0_{s})} \cosh(\sqrt{n^2 + m^2}y).
\]

Hence

\[
\frac{\partial q^0_{n,m}(\rho^0)}{\partial y} = \frac{\partial \sigma^0}{\partial y}(\rho^0)\rho^0_{n,m}(t) \left( n^2 + m^2 \right) - \frac{2n\frac{\partial \sigma^0}{\partial y}(\rho^0)}{2\cosh(\sqrt{n^2 + m^2}\rho^0_{s})} \left( n^2 + m^2 \right) \frac{\tanh(\sqrt{n^2 + m^2}\rho^0_{s})}{\cosh(\sqrt{n^2 + m^2}\rho^0_{s})} + \frac{1}{2}\rho^0_{n,m}(t) \left( n^2 + m^2 \right). \quad \frac{\partial \sigma^0}{\partial y}(\rho^0)
\]

(4.40)

Combining (4.34) and (4.40) with (4.31), we get

\[
\frac{d\rho^0_{n,m}(t)}{dt} = h_{n,m}(\mu, \rho^0_{s})\rho^0_{n,m}(t),
\]

where, after substituting \( \frac{\partial \sigma^0}{\partial y}(\rho^0) = \sigma \tanh \rho^0_s \) and \( \frac{\partial \sigma^0}{\partial y}(\rho^0) = \tau = \frac{\partial \sigma^0}{\partial y}(\rho^0) \) into the expressions,

\[
h_{n,m}(\mu, \rho^0_{s}) = \mu \left[ \frac{\sigma - \sigma}{\cosh(\sqrt{n^2 + m^2}\rho^0_{s})} + \frac{\tanh(\sqrt{n^2 + m^2}\rho^0_{s})}{\cosh(\sqrt{n^2 + m^2}\rho^0_{s})} \right] - \frac{1}{2}(n^2 + m^2)\tanh(\sqrt{n^2 + m^2}\rho^0_{s})
\]

(4.42)

4.3. first-order terms in \( \tau \). Collecting first-order terms from subsection 4.1, we obtain the following system:

\[
\sigma^1_s(y) = -\sigma^1_s \frac{\sinh \rho^0_s}{\cosh(\sqrt{n^2 + m^2}\rho^0_{s})} \cosh y.
\]

(4.43)
Now we need to compute the terms \( \frac{\partial^2 w_{n,m}^1}{\partial y^2} \) and \( \frac{\partial q_{n,m}^1}{\partial y} \) in the above equation. The method for solving the ODEs (4.44)–(4.47) is more or less the same as the equations for the zeroth-order terms in \( \tau \), except that the expressions are more complex. We shall omit the detailed computations and summarize the final results here. One can, of course, simply verify these results by substituting into the equations.

Here are the solutions. For the zeroth order terms that are needed in the computation of first order terms:

\[
\frac{\partial \rho_0^1}{\partial y} = \frac{\sigma \sinh y}{\cosh \rho_0^0}, \quad \frac{\partial \sigma_0^0}{\partial y}(\rho_0^0) = \sigma \tanh \rho_0^0, \quad \bar{\sigma} = \frac{\sigma \tanh \rho_0^0}{\rho_0^1},
\]

\[
\frac{\partial p_0^0}{\partial y} = \mu \bar{\sigma} y - \mu \sigma \frac{\sinh y}{\cosh \rho_0^0} \frac{\partial p_0^0}{\partial y}(\rho_0^0) = 0,
\]

\[
\frac{\partial^2 p_0^0}{\partial y^2} = \mu \bar{\sigma} - \mu \sigma \frac{\cosh y}{\cosh \rho_0^0} \frac{\partial^2 p_0^0}{\partial y^2}(\rho_0^0) = \mu \bar{\sigma} - \mu \sigma \left( 1 - \frac{\tanh \rho_0^0}{\rho_0^1} \right),
\]

\[
\frac{\partial^2 q_{n,m}^0}{\partial y^2} = -\mu \sigma \frac{\sinh y}{\cosh \rho_0^0} \frac{\partial^2 q_{n,m}^0}{\partial y^2}(\rho_0^0), \quad \frac{\partial^3 p_{n,m}^0}{\partial y^3}(\rho_0^0) = -\mu \sigma \tanh \rho_0^0,
\]

\[
\frac{\partial q_{n,m}^1}{\partial y} = -\frac{\partial \rho_{n,m}^0}{\partial y}(\rho_0^0) \sqrt{1 + n^2 + m^2} \frac{\sinh(\sqrt{1 + n^2 + m^2} y)}{\cosh(\sqrt{1 + n^2 + m^2} y)} \rho_{n,m}^0(t),
\]

\[
\frac{\partial \rho_{n,m}^0}{\partial t} = -\frac{\partial \rho_{n,m}^0}{\partial y}(\rho_0^0) \cosh(\sqrt{1 + n^2 + m^2} y) \frac{d \rho_{n,m}^0(t)}{dt},
\]

\[
\frac{\partial \rho_{n,m}^0}{\partial y} = \rho_{n,m}^0(t) \left\{ \mu \frac{\partial \sigma_{n,m}^0}{\partial y}(\rho_0^0) \sqrt{1 + n^2 + m^2} \frac{\sinh(\sqrt{1 + n^2 + m^2} y)}{\cosh(\sqrt{1 + n^2 + m^2} y)} \right\},
\]
\begin{equation}
(n^2 + m^2) - 2\mu \frac{\partial \sigma_0}{\partial y}(\rho_*^0) \\sqrt{\frac{2}{n^2 + m^2 + \rho_*^0}} \ \cosh(\sqrt{n^2 + m^2}) \ \sinh(\sqrt{n^2 + m^2}) \bigg) \bigg) \bigg). 
\end{equation}

\[ \frac{\partial^2 \rho_{n,m}}{\partial y^2} = \rho_{n,m}^{0}(t) \left\{ \mu \frac{\partial \rho_0}{\partial y}(\rho_*^0)(1 + n^2 + m^2) \frac{\cos(1 + n^2 + m^2)}{\cos(1 + n^2 + m^2)} \right. 
\]
\[ + \frac{2}{n^2 + m^2 + \rho_*^0} (n^2 + m^2) \cosh(\sqrt{n^2 + m^2}) \bigg) \bigg). 
\end{equation}

Now we proceed to compute the first order. For \( \rho_*^1 \), we substitute various expressions into (4.45) and evaluate the integral to obtain:

\[ \rho_*^1 = \mu \sigma - (\rho_*^0)^2 - \rho_*^0 \sinh \rho_*^0 \cosh \rho_*^0 + 2 \sinh^2 \rho_*^0, \]

where the denominator is negative, by (2.1). For \( p_*^1 \), we only need \( \frac{\partial p_*^0}{\partial \sigma_*}(\rho_*^0) \), which can be derived from (4.44):

\[ \frac{\partial^2 p_*^1}{\partial y^2}(\rho_*^0) = -\mu \frac{\partial \rho_*}{\partial y}(\rho_*^0) \frac{\partial \rho_0}{\partial y}(\rho_*^0) - \mu \sigma_1(\rho_*^0) = -\mu \sigma_1(\rho_*^0) = \mu \sigma_1 \tan \rho_*^0. \]

The computation of \( \frac{\partial^2 \rho_{n,m}}{\partial y^2}(\rho_*^0, t) \) in (4.48) is much more involved, but fortunately the right-hand side, the integrals and the ODEs can all be evaluated explicitly. From (4.46),

\[ w_{n,m}^1 = \left[ - \frac{\partial u_{n,m}}{\partial y}(\rho_*^0, t) \rho_*^1 - \frac{\partial \rho_*^0}{\partial y}(\rho_*^0, t) \rho_*^1 - \frac{\partial \rho_*^0}{\partial y}(\rho_*^0, t) \rho_*^1 \right] \cos(\sqrt{1 + n^2 + m^2}), \]

The right-hand side of (4.48) is then consolidated into the form

\[ c_2(t) \sinh y \cdot \sinh(\sqrt{1 + n^2 + m^2}) + c_3(t) \sinh y \cdot \sinh(\sqrt{n^2 + m^2}), \]

with

\[ c_2(t) = -\frac{2 \mu^2 \sigma \sqrt{1 + n^2 + m^2}}{\cosh \rho_*^0 \cdot \cosh(1 + n^2 + m^2)} \frac{\partial \rho_0}{\partial y}(\rho_*^0, t) \rho_*^0, \]

\[ c_3(t) = -\frac{\mu \sigma \sqrt{n^2 + m^2}}{2 \cosh \rho_*^0} \frac{\partial \rho_0}{\partial y}(\rho_*^0, t) \rho_*^0, \]

\[ c_4(t) = \frac{\mu \sigma \sqrt{1 + n^2 + m^2}}{2 \cosh(1 + n^2 + m^2)} \frac{\partial \rho_0}{\partial y}(\rho_*^0, t) \rho_*^0, \]

\[ c_5(t) = -\frac{\mu}{\cosh(1 + n^2 + m^2)} \left\{ \frac{\partial \rho_0}{\partial y}(\rho_*^0, t) \right\} \frac{\partial \rho_0}{\partial y}(\rho_*^0, t) \right\} \frac{\partial \rho_0}{\partial y}(\rho_*^0, t) \rho_*^0, \]

\[ = -\frac{\mu}{\cosh(1 + n^2 + m^2)} \left\{ \frac{\partial \rho_0}{\partial y}(\rho_*^0, t) h_{n,m}(\mu, \rho_*^0, \rho_*^0, \rho_*^0, t) \right\} \frac{\partial \rho_0}{\partial y}(\rho_*^0, t) \right\} \frac{\partial \rho_0}{\partial y}(\rho_*^0, t) \rho_*^0, \]
so that

\[
q_{n,m}^1 = \left[ c_1(t) - Q_{n,m}^1(\rho^0_*, t) \right] \frac{\cosh(\sqrt{n^2 + m^2}y)}{\cosh(\sqrt{n^2 + m^2} \rho^0_*)} + Q_{n,m}^1(y, t),
\]

where

\[
Q_{n,m}^1 = c_2(t) \left[ - \frac{\sinh y \cdot \sinh(\sqrt{1 + n^2 + m^2}y)}{2(n^2 + m^2)} + \frac{\sqrt{1 + n^2 + m^2} \cosh y \cdot \cosh(\sqrt{1 + n^2 + m^2}y)}{2(n^2 + m^2)} \right] \\
+ c_3(t) \left[ - \frac{\sinh y \cdot \sinh(\sqrt{n^2 + m^2}y)}{4(n^2 + m^2) - 1} + \frac{2\sqrt{n^2 + m^2} \cosh y \cdot \cosh(\sqrt{n^2 + m^2}y)}{4(n^2 + m^2) - 1} \right] \\
+ c_4(t) \left[ y \cdot \sinh(\sqrt{1 + n^2 + m^2}y) - 2\sqrt{1 + n^2 + m^2} \cosh(\sqrt{1 + n^2 + m^2}y) \right] \\
+ c_5(t) \cosh(\sqrt{1 + n^2 + m^2}y),
\]

\[c_1(t) = q_{n,m}^1(\rho^0_*, t).\]

It follows that

\[
\frac{\partial Q_{n,m}^1}{\partial y}(\rho^0_*, t) = c_2(t) \frac{\cosh \rho^0_* \cdot \sinh(\sqrt{1 + n^2 + m^2} \rho^0_*)}{2} \\
+ c_3(t) \left[ \frac{\sqrt{n^2 + m^2} \sinh \rho^0_* \cdot \cosh(\sqrt{n^2 + m^2} \rho^0_*)}{4(n^2 + m^2) - 1} + \frac{(2n^2 + 2m^2 - 1) \cosh \rho^0_* \cdot \sinh(\sqrt{n^2 + m^2} \rho^0_*)}{4(n^2 + m^2) - 1} \right] \\
+ c_4(t) \left[ \sqrt{1 + n^2 + m^2} \rho^0_* \cdot \cosh(\sqrt{1 + n^2 + m^2} \rho^0_*) - (1 + 2n^2 + 2m^2) \sinh(\sqrt{1 + n^2 + m^2} \rho^0_*) \right] \\
+ c_5(t) \sqrt{1 + n^2 + m^2} \sinh(\sqrt{1 + n^2 + m^2} \rho^0_*).
\]

Substituting the expressions of \(c_1(t)\) into the equations, we find

\[
\frac{\partial q_{n,m}^1}{\partial y}(\rho^0_*) = \left[ c_1(t) - Q_{n,m}^1(\rho^0_*, t) \right] \sqrt{n^2 + m^2} \tanh(\sqrt{n^2 + m^2} \rho^0_*) + \frac{\partial Q_{n,m}^1}{\partial y}(\rho^0_*, t) \\
= K_{n,m}(\mu, \rho^0_*, \rho^1_*, \rho^0_*) \rho^0_{n,m}(t) + \left\{ \mu \sigma \sqrt{1 + n^2 + m^2} \tanh(\sqrt{1 + n^2 + m^2} \rho^0_*) \right. \\
\left. + \left( \frac{1}{2} (n^2 + m^2) - \mu \sigma \tanh \rho^0_* \right) \sqrt{n^2 + m^2} \tanh(\sqrt{n^2 + m^2} \rho^0_*) \right\} \rho^1_{n,m}(t),
\]

where

\[|K_{n,m}(\mu, \rho^0_*, \rho^1_*)| \leq C(n^2 + m^2 + 1)^{5/2}.\]

Substituting these expressions into (4.48), we derive

\[
\frac{d\rho^1_{n,m}(t)}{dt} = h^1_{n,m}(\mu, \rho^0_*, \rho^1_*, n, m) \rho^1_{n,m}(t) + k^1_{n,m}(\mu, \rho^0_*, \rho^1_*, n, m) \rho^0_{n,m}(t),
\]

where

\[
h^1_{n,m}(\mu, \rho^0_*, \rho^1_*, n, m) = - \frac{\partial^2 \rho^0_*}{\partial y^2}(\rho^0_*) - \mu \sigma \sqrt{1 + n^2 + m^2} \tanh(\sqrt{1 + n^2 + m^2} \rho^0_*) \\
- \left( \frac{1}{2} (n^2 + m^2) - \mu \sigma \tanh \rho^0_* \right) \sqrt{n^2 + m^2} \tanh(\sqrt{n^2 + m^2} \rho^0_*),
\]

\[|k^1_{n,m}(\mu, \rho^0_*, \rho^1_*, n, m)| \leq C(n^2 + m^2 + 1)^{5/2}.
\]

After a careful comparison with (4.42), we find

\[
h^1_{n,m}(\mu, \rho^0_*, \rho^1_*, n, m) = h_{n,m}(\mu, \rho^0_*).
\]
This is significant, since the first order term in $\tau$ does not change the leading coefficient. If we expand the ODE for $\rho_{n,m}(t)$ in $\tau$, then
\[
\frac{d\rho_{n,m}(t)}{dt} - \frac{d\rho^0_{n,m}(t)}{dt} + \tau \frac{d\rho^1_{n,m}(t)}{dt} + O(\tau^2) = h_{n,m}(\mu, \rho^0_{n,m}(t)) + \tau h_{n,m}(\mu, \rho^1_{n,m}(t)) + \tau k^1_{n,m}\rho^0_{n,m}(t) + O(\tau^2) = h_{n,m}(\mu, \rho^0_{n,m}(t)) + \tau k^1_{n,m}\rho^0_{n,m}(t) + O(\tau^2).
\]

Thus the leading order stability up to the order of $O(\tau)$ also depends on the sign of $h_{n,m}(\mu, \rho^0_{n,m})$. It is not difficult to see that
\[
\lim_{n^2+m^2\to\infty} \frac{h_{n,m}(\mu, \rho^0_{n,m})}{(n^2 + m^2)^{3/2}} = -\frac{1}{2}.
\]

We now start to investigate the sign of $h_{n,m}(\mu, \rho^0_{n,m})$. For convenience we write (c.f., (4.42)),
\[
h_{n,m}(\mu, \rho^0_{n,m}) = \mu \sigma k_1(n^2 + m^2, \rho^0_{n,m}) - k_2(n^2 + m^2, \rho^0_{n,m}),
\]
where
\[
k_1(j, \rho^0_{\ast}) = 1 - \frac{\tanh \rho^0_{\ast}}{\rho^0_{\ast}} - \tanh \rho^0_{\ast} \left[ \sqrt{1 + j} \tanh(\sqrt{1 + j}\rho^0_{\ast}) - \sqrt{j} \tanh(\sqrt{j}\rho^0_{\ast}) \right],
\]
\[
k_2(j, \rho^0_{\ast}) = \frac{1}{2} j^{3/2} \tanh(\sqrt{j}\rho^0_{\ast}).
\]

From (2.8), we deduce that $k_1(j, \rho^0_{\ast})$ is monotonically increasing in $j$, and by (2.1), we have
\[
k_1(0, \rho^0_{\ast}) = 1 - \frac{\tanh \rho^0_{\ast}}{\rho^0_{\ast}} - \tanh^2 \rho^0_{\ast} < 0, \quad \lim_{j \to +\infty} k_1(j, \rho^0_{\ast}) = 1 - \frac{\tanh \rho^0_{\ast}}{\rho^0_{\ast}} > 0.
\]

Hence there exists a unique $j_0 = j_0(\rho^0_{\ast}) > 0$ (not necessarily an integer) such that $k_1(j_0, \rho^0_{\ast}) = 0$. It follows that
\[
k_1(j, \rho^0_{\ast}) < 0 \quad \text{for } 0 \leq j < j_0, \quad k_1(j, \rho^0_{\ast}) > 0 \quad \text{for } j > j_0.
\]

Define
\[
\mu_j(\rho^0_{\ast}) = \frac{k_2(j, \rho^0_{\ast})}{\sigma k_1(j, \rho^0_{\ast})} \quad \text{for } j > j_0.
\]

With the structure of $h_{n,m}$ given by (4.56), we established:

**Lemma 4.1.** The following assertions (i)–(ii) hold:

(i) if $0 \leq n^2 + m^2 \leq j_0$, then $h_{n,m}(\mu, \rho^0_{n,m}) < 0$ for any $\mu > 0$;

(ii) if $j = n^2 + m^2 > j_0$, then

(1) $h_{n,m}(\mu, \rho^0_{n,m}) < 0$ for $0 < \mu < \mu_j(\rho^0_{\ast})$;

(2) $h_{n,m}(\mu, \rho^0_{n,m}) = 0$ for $\mu = \mu_j(\rho^0_{\ast})$;

(3) $h_{n,m}(\mu, \rho^0_{n,m}) > 0$ for $\mu > \mu_j(\rho^0_{\ast})$.

For convenience we define
\[
\mu_j(\rho^0_{\ast}) = +\infty \quad \text{for } 0 \leq j \leq j_0, \quad \mu_\ast(\rho^0_{\ast}) = \min_{j = n^2 + m^2 > j_0} \mu_j(\rho^0_{\ast}).
\]

We now proceed to find $\mu_\ast(\rho^0_{\ast})$.

In the following figures, we let $\overline{\sigma} = 1$ and have plotted $\mu_j(\rho^0_{\ast})$ for several $\rho^0_{\ast}$ in the range from $\rho^0_{\ast} = 0.25$ to 2.

Take $\rho^0_{\ast} = 1$, for example, then $2 < j_0(\rho^0_{\ast}) < 3$, and we have plotted $\mu_j(\rho^0_{\ast})$ for $j \geq 3$ (see Figure 3). The function $\mu_j(\rho^0_{\ast})$ at non-integer values of $j$ are not plotted since they are not needed. The minimum is reached at $j = 5$ and $\mu_5(1) \approx 84.054$. Since $5 = 2^2 + 1^2$, it is admissible. Other values of $\rho^0_{\ast}$ are listed in the Table 1 and plotted in Figures 1–2, 4.

**Remark 4.1.** Numerical evidence shows that $\mu_\ast(\rho^0_{\ast}) = \mu_1(\rho^0_{\ast})$ when $\rho^0_{\ast} > \overline{\sigma} \approx 1.8471$. 

Lemma 4.2. If \( \mu < \mu_*(\rho_0) \), then for sufficiently small \( \tau \), there exists \( \delta > 0 \) such that

\[
\frac{h_{n,m}(\mu, \rho_0)}{n,m} \leq -\delta (n^2 + m^2 + 1)^{3/2} \quad \text{for all} \quad n = 0, 1, 2, \ldots, m = 0, 1, 2, \ldots.
\]

Proof. By (4.55), the estimate (4.59) is certainly valid for \( n^2 + m^2 > N \) for some \( N \) sufficiently large. For \( 0 \leq n^2 + m^2 \leq N \), this estimate follows from the definition of \( \mu_*(\rho_0) \) if \( \delta \) is taken to be small enough.

Proof of Theorem 1.2. From the above lemma and (4.41) we find that, for \( \mu < \mu_* \),

\[
|\rho_{n,m}^0(t)| = |\rho_{n,m}^0(0)|e^{h_{n,m}(\mu, \rho_0)t} \leq |\rho_{n,m}^0(0)|e^{-\delta (n^2 + m^2 + 1)^{3/2}t}.
\]

Similarly, (4.52) and (4.53) imply

\[
|\rho_{n,m}^1(t)| \leq e^{-\delta (n^2 + m^2 + 1)^{3/2}t} \left[ |\rho_{n,m}^1(0)| + C|\rho_{n,m}^0(0)| (n^2 + m^2 + 1)^{5/2}t \right].
\]

Taking Fourier series in \((n,m)\) we find that \( |\rho^0(t) + \tau \rho^1(t)| \leq Ce^{-\delta t} \) for \( t > 0 \), i.e., the system linearized both in perturbation and \( \tau \) is stable.
For $\mu > \mu_*(\rho^0_*)$, the mode corresponding to the mode $(n, m)$ which gives the minimum of $\mu_j$'s is clearly unstable.

5. Impact of Time Delay

In this section, we shall show the impact of time delay $\tau$ on the tumor growth.

Theorem 5.1. $\rho_1^1 > 0$, and $\rho_1^1$ is monotonically increasing in $\mu$.

Proof. By (4.49),

$$\rho_1^1 = \mu\sigma - (\rho^0_*)^2 - \rho^0_* \sinh \rho^0_* \cosh \rho^0_* + 2 \sinh^2 \rho^0_* \rho^0_*.$$

The denominator in the above expression is negative, by (2.1). If we show that the numerator is also negative, then clearly $\rho_1^1 > 0$ and $\rho_1^1$ is monotonically increasing in $\mu$.

Clearly, $(\sinh \rho - \rho \cosh \rho)' = -\rho \sinh \rho < 0$ and $\sinh 0 - 0 \cosh 0 = 0$. Hence

$$f''(\rho) \equiv [-(\rho)^2 - \rho \sinh \rho \cosh \rho + 2 \sinh^2 \rho]'' = 4 \sinh(\sinh \rho - \rho \cosh \rho) < 0.$$  

Since we clearly have $f'(0) = f(0) = 0$, we must have $f(\rho) < 0$ for $\rho > 0$. The proof is complete.

6. Conclusion

In this paper we have investigated the impact of time delay on a tumor model in a flat domain. The existence, uniqueness, stability of the stationary problem are studied. In addition, here are some interesting observations on the impact of time delay $\tau$, the tumor aggressiveness constant $\mu$, the nutrient supply $\sigma$, and the tumor size (order 0 is $\rho^0_*$, order 1 is $\rho^0_* + \tau \rho^1_*$).

1) Adding the time delay (at $O(\tau)$) to the system would not alter the threshold value $\mu_*$ for which the stability of the stationary solution changes (section 4).

2) Some other properties remain the same as in the case without time delay (see table 1): the bigger the size (measured by $\rho^0_*$) of the tumor, the smaller the value of $\mu_*$, that is to say that smaller stationary tumor is much more likely to be stable than its larger counter part. When the thickness is reduced from 0.5 to 0.25, the tumor aggressiveness would have to increase more than 30 fold to cause problems (Table 1). In other word, a small sized tumor is less likely to proliferate than a large sized tumor. Implication: treat the small tumor before it grows larger. Since a long time is needed for it to grow (stable), there is plenty of time to treat it.

3) Flat perturbation (mode $(0, 0)$) is always stable, regardless of the value of $\mu$.

4) As we increase $\mu$ across the threshold of stability, the instability comes from the modes $(0, 1)$ or $(1, 0)$ for larger tumors (larger $\rho^0_*$). It can come for other modes for smaller tumors, e.g., for $\rho^0_1 = 1$, it comes from the modes $(2, 1)$ or $(1, 2)$ (see table 1).

5) Adding the time delay would result in a larger stationary tumor (at $O(\tau)$) when compared to the same system without delay (Theorem 5.1). The bigger the tumor proliferation intensity $\mu$ is, the greater impact that time delay has on the size of the stationary tumor (Theorem 5.1).

6) Since the nutrient supply $\sigma$ appears together with $\mu$ as a product in the definition of the threshold, increasing the nutrient supply would promote instability and proliferation.

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