Wonderful resolutions and categorical crepant resolutions of singularities

Roland Abuaf *

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Abstract

Let $X$ be an algebraic variety with Gorenstein singularities. We define the notion of a wonderful resolution of singularities of $X$ by analogy with the theory of wonderful compactifications of semi-simple linear algebraic groups. We prove that if $X$ has rational singularities and has a wonderful resolution of singularities, then $X$ admits a categorical crepant resolution of singularities. As an immediate corollary, we get that all determinantal varieties defined by the minors of a generic square/symmetric/skew-symmetric matrix admit categorical crepant resolution of singularities.

*Institut Fourier, 100 rue des maths, 38402, Saint Martin d’Hères, France. E-mail: roland.abuaf@ujf-grenoble.fr
1 Introduction

Let $X$ be an algebraic variety over $\mathbb{C}$. Hironaka proved in [Hir64] that one can find a proper birational morphism $\tilde{X} \to X$, with $\tilde{X}$ smooth. Such a $\tilde{X}$ is called a resolution of singularities of $X$. Unfortunately, given an algebraic variety $X$, there is, in general, no minimal resolution of singularities of $X$. In case $X$ is Gorenstein, a crepant resolution of $X$ (that is a resolution $\pi : \tilde{X} \to X$ such that $\pi^*K_X = K_{\tilde{X}}$) is often considered to be minimal. The conjecture of Bondal-Orlov (see [BO02]) gives a precise meaning to that notion of minimality:

**Conjecture 1.0.1** Let $X$ be an algebraic variety with canonical Gorenstein singularities. Assume that $X$ has a crepant resolution of singularities $\tilde{X} \to X$. Then, for any other resolution of singularities $Y \to X$, there exists a fully faithful embedding:

$$D^b(\tilde{X}) \hookrightarrow D^b(Y).$$

Varieties admitting a crepant resolution of singularities are quite rare. For instance, non-smooth Gorenstein $\mathbb{Q}$-factorial terminal singularities (e.g. a cone over $v_2(\mathbb{P}^m) \subset \mathbb{P}^{\frac{m(m+1)}{2}}$, for even $m$, see [Kuz08], section 7) never admit
crepant resolution of singularities. Thus, it seems natural to look for minimal resolutions among categorical ones. Kuznetsov has given the following definition ([Kuz08]):

**Definition 1.0.2** Let $X$ be an algebraic variety with Gorenstein and rational singularities. A categorical resolution of singularities of $X$ is a triangulated category $\mathcal{T}$ with a functor $R_{\pi_*}: \mathcal{T} \to \text{D}^b(X)$ such that:

- there exists a resolution of singularities $\pi: \tilde{X} \to X$ such that $\delta: \mathcal{T} \hookrightarrow \text{D}^b(\tilde{X})$ is admissible and $R_{\pi_*} = R_{\pi*} \circ \delta$,
- we have $L_{\pi^*}\text{D}^\text{perf}(X) \subset \mathcal{T}$ and for all $\mathcal{F} \in \text{D}^\text{perf}(X)$:
  \[ R_{\pi_*}L_{\pi^*}\mathcal{F} \cong \mathcal{F}, \]
  where $L_{\pi^*}$ is the left adjoint to $R_{\pi_*}$.

If for all $\mathcal{F} \in \text{D}^\text{perf}(X)$, there is a quasi-isomorphism:

\[ L_{\pi^*}\mathcal{F} \cong L_{\pi^*}\mathcal{T}, \]

where $L_{\pi^*}$ is the right adjoint of $R_{\pi_*}$, we say that $\mathcal{T}$ is weakly crepant.

Finally, if $\mathcal{T}$ has a structure of module category over $\text{D}^\text{perf}(X)$ and the identity is a relative Serre functor for $\mathcal{T}$ with respect to $\text{D}^b(X)$, then $\mathcal{T}$ is said to be strongly crepant.

Obviously, if $\mathcal{T}$ is a strongly crepant resolution of $X$, then it is also a weakly crepant resolution of $X$. The converse is false, as shown is section 8 of [Kuz08]. If $\pi: \tilde{X} \to X$ is a crepant resolution of $X$, the one easily shows that $\text{D}^b(\tilde{X}) \to \text{D}^b(X)$ is a strongly crepant categorical resolution. The main result of [Kuz08] is the:

**Theorem 1.0.3** Let $X$ be an algebraic variety with Gorenstein rational singularities. Let $\pi: \tilde{X} \to X$ be a resolution of singularities with a positive integer $m$ such that $K_{\tilde{X}} = \pi^*K_X \otimes \mathcal{O}_{\tilde{X}}(mE)$, where $E$ is the scheme-theoretic exceptional divisor of $\pi$. Assume moreover that we have a semi-orthogonal decomposition:

\[ \text{D}^b(E) = \langle \mathcal{O}_E(mE) \otimes B_m, \ldots, \mathcal{O}_E(E) \otimes B_1, B_0 \rangle, \]

with:

\[ L_{\pi^*}\text{D}^b(\pi(E)) \subset B_m \subset \cdots \subset B_1 \subset B_0, \]
then $X$ admits a categorical weakly crepant resolution of singularities.

Assume moreover that $B_m = \cdots = B_1 = B_0$, then $X$ admits a categorical strongly crepant resolution of singularities.

As a consequence, Kuznetsov obtains (see [Kuz08], sections 7 and 8) the:

**Corollary 1.0.4** The following varieties admit a categorical strongly crepant resolution of singularities:

- a cone over $v_2(\mathbb{P}^n) \subset \mathbb{P}(S^2\mathbb{C}^{n+1})$ (odd $n$),
- a cone over $\mathbb{P}^n \times \mathbb{P}^n \subset \mathbb{P}(\mathbb{C}^{n+1} \otimes \mathbb{C}^{n+1})$ (any $n$),
- the Pfaffian variety: $\mathbb{P}f_4(n) := \mathbb{P}\{\omega \in \wedge^2 \mathbb{C}^n, \text{ such that } \text{rk}(\omega) \leq 4\}$ (odd $n$).

The following varieties admit categorical weakly crepant resolution of singularities:

- a cone over a smooth Fano variety in its anti-canonical embedding,
- the Pfaffian variety $\mathbb{P}f_4(n)$ (even $n$).

Of course, one would like to generalize Kuznetsov’s result, to apply it to higher corank determinantal varieties for instance. Using Kodaira relative vanishing theorem and some adjunction formulae, it is not difficult to prove the following (this is the case $n = 1$ of Proposition 3.4.1):

**Proposition 1.0.5** Let $X$ be an algebraic variety with Gorenstein rational singularities. Let $\pi : \tilde{X} \to X$ be a resolution of singularities such that the exceptional divisor $E$ of $\pi$ is irreducible, smooth and flat over $\pi(E)$. Then there exists a positive integer $m$ such that:

$$K_{\tilde{X}} = K_X \otimes \mathcal{O}_{\tilde{X}}(mE),$$

and we have a semi-orthogonal decomposition:

$$\mathcal{D}^b(E) = \langle \mathcal{O}_E(mE) \otimes B_m, \ldots, \mathcal{O}_E(E) \otimes B_1, B_0 \rangle,$$

with $\mathcal{L}^{\pi}\mathcal{D}^{\text{perf}}(\pi(E)) \subset B_m \subset \cdots \subset B_1 \subset B_0$.

As a consequence of this proposition, we get a first mild generalization of the first part of Kuznetsov’s theorem:
Theorem 1.0.6 Let $X$ be an algebraic variety with Gorenstein rational singularities. Let $\pi : \tilde{X} \to X$ be a resolution of singularities such that the exceptional divisor $E$ of $\pi$ is irreducible, smooth and flat over $\pi(E)$. Then $X$ admits a categorical weakly crepant resolution of singularities.

Note that all examples in Corollary 1.0.4 satisfy the hypotheses of Theorem 1.0.6.

Now, one remembers the strong version of Hironaka’s theorem ([Hir64]) : any variety can be desingularized by a sequence of blow-ups such that every exceptional divisor is flat over its center of blowing-up and the total exceptional divisor of the resolution has (with its reduced structure) simple normal crossings. So, one could hope to get a very far-reaching generalization of Kuznetsov’s result : any Gorenstein variety with rational singularities admits a categorical weakly crepant resolution of singularities.

Unfortunately, things are not so simple. Indeed, in order to construct a categorical crepant resolution starting from a sequence of blow-ups which desingularizes our variety, one needs strong compatibility conditions between the semi-orthogonal decompositions of the derived categories of the various exceptional divisors. Those compatibility conditions can be formulated at the categorical level (see Proposition 3.1.1). But one would rather like to know geometric situations where these compatibility conditions are satisfied. I have thus formalized the notion of wonderful resolution of singularities (see definition 2.1.2 of the present paper), which applies to a sequence of blow-ups:

$$X_n \rightarrow \cdots \rightarrow X_1 \rightarrow X,$$

giving a resolution of singularities of $X$. It is remarkable that this definition, which I made up to describe geometrically some compatibility conditions among the derived categories of the exceptional divisors of a resolution of singularities, happens to be the one which perfectly identifies the resolution process of the boundary divisor for the most basic wonderful compactifications of semi-simple linear algebraic groups (see [DCP83] for details). With this notion in hand, quite technical but predictable computations yields the:

Theorem 1.0.7 (Main Theorem) Let $X$ be an algebraic variety with Gorenstein rational singularities. Assume that $X$ has a wonderful resolution of singularities. Then $X$ admits a categorical weakly crepant resolution of singularities.

At first glance, the notion of wonderful resolution of singularities seems to be quite restrictive and one could naively guess that there are too few examples
of such resolutions. However, some reformulations of the work of Vainsencher \cite{Vai84} and Thaddeus \cite{Tha99} show that it is not the case. Indeed, we have the:

**Theorem 1.0.8 (\cite{Vai84}, \cite{Tha99})** All determinantal varieties (square as well as symmetric and skew-symmetric) admit wonderful resolution of singularities.

As a consequence, we get the:

**Corollary 1.0.9** All Gorenstein determinantal varieties (square as well as symmetric and skew-symmetric) admit categorical weakly crepant resolutions of singularities.

Let us now briefly indicate the plan of the paper. In section 2, we give the definition of a wonderful resolution of singularities and study its basic cohomological properties. We also exhibit some examples of varieties which have a wonderful resolution of singularities. In section 3, we prove the main theorem. This is the technical core of the paper. In section 4, we discuss some minimality properties for categorical crepant resolutions of singularities and some existence problems related to prehomogeneous spaces.

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# 2 Wonderful resolutions of singularities

We work over $\mathbb{C}$ the field of complex numbers. An algebraic variety is a reduced algebraic scheme of finite type over $\mathbb{C}$ (in particular it may be reducible). For any proper morphism $f : X \to Y$ of schemes of finite type over $\mathbb{C}$, we denote by $f_*$ the total derived functor $Rf_* : D^b(X) \to D^b(Y)$, by $f^*$ the total derived functor $Lf^* : D^-(Y) \to D^-(X)$ and by $f!$ the right adjoint functor to $Rf_*$. In case we need to use specific homology sheaves of these functors, we will denote them by $R^if_*$, $L^jf^*$ and $L^jf!$.
2.1 Wonderful resolutions

Let $Y \subset X$ be a closed irreducible subvariety of $X$. We say that $Y$ is a normally flat center in $X$ if the natural map:

$$E \rightarrow Y$$

is flat, where $E$ is the exceptional divisor of the blow up of $X$ along $Y$. Hironaka proved in [Hir64] that any algebraic variety can be desingularized by a finite sequence of blow-ups along smooth normally flat centers.

Example 2.1.1 Let $W$ be a vector space of dimension at least 6 and let $X = \mathbb{P}(\{\omega \in \bigwedge^2 W, \text{rank}(\omega) \leq 4\}$ be the 2nd Pfaffian variety in $\bigwedge^2 W$. Then $X$ is singular exactly along $\text{Gr}(2, W) = \mathbb{P}(\{\omega \in \bigwedge^2 W, \text{rank}(\omega) \leq 2\}$. Here $\text{Gr}(2, W)$ is a smooth normally flat center for $X$. Indeed, if $E$ is the exceptional divisor of the blow-up of $X$ along $\text{Gr}(2, W)$, then $E$ is the flag variety $\text{Fl}(2, 4, W)$ and the natural map onto $\text{Gr}(2, W)$ is given by the second projection. Obviously it is flat. See [Kuz08], section 8 for more details.

Given $G$ a semi-simple affine algebraic group, one often wants to find a good equivariant compactification of $G$. Equivariant compactifications of $G$ for which the boundary divisors have simple normal crossings are called wonderful in the literature (see [Hur11], chapter 3 for instance). One notices that the most basic wonderful compactifications we know are obtained by the following procedure. Take $\overline{G}$ be a naive $G$-equivariant compactification of $G$. Then find an embedded resolution of the boundary divisor in $\overline{G}$ such that it’s smooth model has simple normal crossings with the exceptional divisors of the modification of $\overline{G}$. This embedded resolution is obtained by a succession of blow-ups along smooth centers which satisfy nice intersection properties. The following definition captures the most essential features of this sequence of blow-ups.

Definition 2.1.2 (Wonderful resolutions) Let $X$ be an algebraic variety with Gorenstein singularities. For all $n \geq 1$, we define a $n$-step wonderful resolution of singularities in the following recursive way:

- A 1-step wonderful resolution is a single blow up:

$$\pi : \tilde{X} \rightarrow X,$$

over a smooth normally flat center $Y \subset X$, such that $\tilde{X}$ and the exceptional divisor $E \subset \tilde{X}$ are smooth.

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1For instance, let $V$ be a linear representation of $G$ and consider the closure of $G$ in $\mathbb{P}($End($V$)).
For $n \geq 2$, a $n$-step wonderful resolution of $X$ is a sequence of blow-ups:

$$X_n \xrightarrow{\pi_2} X_{n-1} \cdots X_1 \xrightarrow{\pi_1} X_0 = X$$

over smooth normally flat centers $Y_{k+1} \subset X_k$ such that:

1. all the $X_k$ are Gorenstein,
2. the map $\pi_2 \cdots \pi_n : X_n \to X_1$ is a $(n - 1)$-step wonderful resolution of $X_1$,
3. the intersection of $Y_{k+1}$ with $E_1^{(k)}$ is proper and smooth (where $E_1^{(k)}$ is the total transform of $E_1$, the exceptional divisor of $\pi_1$, with respect to $\pi_2 \cdots \pi_k$, for $k > 1$),
4. the map $\pi_2 \cdots \pi_n|_{E_1^{(n)}} : E_1^{(n)} \to E_1$ is a $(n - 1)$-step wonderful resolution of singularities.

As far as I know, the term wonderful resolution first appeared in [CF07] where it was used to describe the resolution of indeterminacies of a stratified Mukai flop.

**Example 2.1.3 (Determinantal varieties)** Let $E$ a vector space of dimension $n$. Let $2 \leq r \leq n$ and let $X$ be the subvariety of $\mathbb{P}(\text{End}(E))$ defined by the vanishing of the minors of size $r$. It is well known that $X$ is Gorenstein with rational singularities (see [Wey03], corollary 6.1.5). We define $X_1$ to be the blow up:

$$\pi_1 : X_1 \to X,$$

of $X$ along $Y_1$, where $Y_1$ is the subvariety of $\mathbb{P}(\text{End}(E))$ defined by the minors of size 2. For $3 \leq k \leq r - 1$, we define recursively $X_k$ to be the blow-up:

$$\pi_k : X_k \to X_{k-1},$$

of $X_{k-1}$ along $Y_k$, where $Y_k$ is the strict transform through $\pi_1 \cdots \pi_{k-1}$ of the subvariety of $\mathbb{P}(\text{End}(E))$ defined by the vanishing of the minors of rank $k$. By theorem 1 of [Vai84], we know that $X_{r-1}$ is smooth and that all $Y_k$ for $1 \leq k \leq r - 1$ are smooth. Moreover, theorem 2.4 of [Vai84] shows that the $Y_k$ are normally flat centers and that item 1, 2, 3 and 4 in the definition of a wonderful resolution are satisfied for the following resolution of $X$:

$$X_{r-1} \xrightarrow{\pi_{r-1}} \cdots X_1 \xrightarrow{\pi_1} X$$

Thus, $X$ has Gorenstein rational singularities and admits a wonderful resolution of singularities.
Let $X^{[\text{sym}]}_r \subset \mathbb{P}(S^2(E))$ (resp. $X^{[\text{skew}]}_r \subset \mathbb{P}(\wedge^2(E))$) denote the determinantal variety defined by the vanishing of the minors of size $r$ of the generic $n \times n$ symmetric (resp. skew-symmetric) matrix with linear entries. If $n - r$ is even (resp. no conditions), we know by [Wey03], corollary 6.3.7 (resp. proposition 6.4.3) that $X^{[\text{sym}]}_r$ (resp. $X^{[\text{skew}]}_r$) is Gorenstein with rational singularities. Moreover, the appendices 10 and 11 of [Tha99] show that $X^{[\text{sym}]}_r$ and $X^{[\text{skew}]}_r$ also admit wonderful resolutions of singularities.

Example 2.1.4 (Secant variety of $\mathbb{OP}^2$) Let $X = \mathbb{OP}^2 = E_6 / P_{\alpha_1} \subset \mathbb{P}^{26}$ be the Cayley plane into its highest weight embedding, where $P_{\alpha_1}$ is the maximal parabolic associated to the root $\alpha_1$ of the $E_6$ root system. The Cayley plane can also be recovered as the scheme defined by the $2 \times 2$ minors of the generic hermitian $3 \times 3$ octonionic matrix:

$$M = \begin{pmatrix} a_1 & a_2 & a_3 \\ \overline{a_2} & a_4 & a_5 \\ a_3 & \overline{a_5} & a_6 \end{pmatrix},$$

where $\overline{a_i}$ is the octonionic conjugate of $a_i \in \mathbb{O}$.

Let $S(X)$ be the secant variety of $X$ inside $\mathbb{P}^{26}$. It can be seen as the scheme defined by the determinant of the above matrix $M$. The variety $S(X)$ is a cubic hypersurface (so it is Gorenstein) which is singular exactly along $\mathbb{OP}^2$. Let

$$\pi : \widetilde{S(X)} \rightarrow S(X)$$

be the blow up of $S(X)$ along $X$. It is smooth and the exceptional divisor $E$ is isomorphic to $E_6 / Q_{\alpha_1,\alpha_5}$ (where $Q_{\alpha_1,\alpha_5}$ is the parabolic associated to the roots $\alpha_1$ and $\alpha_5$). It is a fibration into smooth 8-dimensional quadrics over $X$. As a consequence, the map $\pi : \widetilde{S(X)} \rightarrow S(X)$ is a wonderful resolution of singularities. We refer to [Zak93] ch. III and [FM12] for more details on the beautiful geometric and categorical features of the Cayley plane.

2.2 Wonderful resolutions and singularities of the intermediate divisors

The above examples also suggest that the definition of a wonderful resolution imposes strong conditions on the singularities of the exceptional divisor $E_1^{(k)}$. Indeed, the following three propositions show that they must be similar to the singularities of $X$.

**Proposition 2.2.1** Let $X$ be an algebraic variety with Gorenstein and rational singularities. Let $X_n \xrightarrow{\pi_n} X_{n-1} \ldots X_1 \xrightarrow{\pi_1} X_0 = X$ be a wonderful resolution
of singularities of $X$. Then, for all $1 \leq k \leq n$, the varieties $X_k$ and the exceptional divisors $E_1^{(k)}$ have Gorenstein rational singularities.

**Proof:**

◮ The fact that the $X_k$ are Gorenstein is in the definition of a wonderful resolution of singularities. Let $\pi_1 : X_1 \to X$ be the blow-up along $X_1$ with exceptional divisor $E_1$ and let $E_{\alpha_1}, \ldots, E_{\alpha_d}$ be the irreducible components of $E_1$. We also denote by $E_{\alpha_i}^{(k)}$ the total transform of $E_{\alpha_i}$ under $\pi_2 \ldots \pi_k$. Since the $Y_k$ meet transversally the $E_{\alpha_i}^{(k)}$ for all $k \geq 2$, the coefficients appearing in front of the $E_{\alpha_i}^{(k)}$ in the expression of $K_{X_k}$ are the same as the coefficients in front of the $E_{\alpha_i}$ in the expression of $K_{X_1}$. Now $X$ is Gorenstein with rational singularities, hence it has canonical singularities by [Ko97] corollary 11.13. Thus, the coefficients in front of the $E_{\alpha_i}^{(k)}$ in the expression of $K_{X_k}$ are positive, and so are the coefficient in front of the $E_{\alpha_i}$ in the expression of $K_{X_1}$. As a consequence, we have $\pi_1_* \mathcal{O}_{X_1} = \mathcal{O}_X$. Since we also have $\pi_1 \ldots \pi_{n*} \mathcal{O}_{X_n} = \mathcal{O}_X$, we find that $X_1$ has rational singularities. An obvious induction shows that all $X_k$ have rational singularities.

The divisors $E_1^{(k)}$ are obviously Gorenstein, as they are Cartier divisors inside Gorenstein varieties. The main point of the proposition is thus to show that the $E_1^{(k)}$ have rational singularities for any $k \geq 1$.

Item 3 of definition 2.1.2 implies that for any $k \geq 2$ the map

$$\pi_k|_{E_1^{(k)}} : E_1^{(k)} \to E_1^{(k-1)}$$

is the blow-up along $Y_k \cap E_1^{(k-1)}$ (we recall that $E_1^{(k)}$ is the total transform of $E_1$ through $\pi_2 \ldots \pi_k$). Thus, we deduce from item 4 of definition 2.1.2 that for any $k \geq 1$, the map

$$\Pi_k = \pi_{k+1} \ldots \pi_n|_{E_1^{(n)}} : E_1^{(n)} \to E_1^{(k)}$$

is a resolution of singularities. Hence, to prove that $E_1^{(k)}$ has rational singularities we only have to compute $(\Pi_k)_* \mathcal{O}_{E_1^{(n)}}$.

For all $1 \leq k \leq n$, we have a fibered diagram:
The maps $i_1^{(n)}$ and $i_1^{(k)}$ are locally complete intersection embeddings and codim $E_1^{(k)} \subset X_k = \text{codim} E_1^{(n)} \subset X_n$ so that we have the commutation of derived functors (see [Kuz06], corollary 2.27):

$$(i_1^{(k)})^*(\pi_{q+1} \cdots \pi_n)* \mathcal{O}_{X_n} = (\Pi_k)^*(i_1^{(n)})^* \mathcal{O}_{X_n},$$

that is:

$$(\Pi_k)^* \mathcal{O}_{E_1^{(n)}} = \mathcal{O}_{E_1^{(k)}}.$$ 

So $E_1^{(k)}$ has rational singularities.

The next proposition (2.2.2) is important, because for any wonderful resolution of a variety $X$ with rational Gorenstein singularities:

$$X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \cdots X_1 \xrightarrow{\pi_1} X_0 = X,$$

it allows to produce a simple formula relating $K_{X_k}, \pi_k^* K_{X_{k-1}}$ and $E_k$ for any $1 \leq k \leq n$. The (positive) coefficient appearing in front of $E_k$ in the expression of $K_{X_k}$ can be interpreted as a kind of multiplicity of $Y_k$ in $X_{k-1}$. Proposition 2.2.3 then shows that the multiplicity of $Y_{k+1}$ in $X_k$ is equal to the multiplicity of $Y_k \cap E_1^{(k)}$ in $E_1^{(k)}$ for any $k \geq 1$.

**Proposition 2.2.2** Let $X$ be an algebraic variety with Gorenstein and rational singularities. Let $X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \cdots X_1 \xrightarrow{\pi_1} X_0 = X$ be a wonderful resolution of singularities of $X$. Then the exceptional divisor $E_1$ of $\pi_1 : X_1 \to X_0$ is irreducible.

**Proof:**

For $y \in Y_1$, denote by $\mathcal{C}_y(X)$ the tangent cone to $X$ at $y$ and let $\rho_Y : \mathcal{C}_{Y_1}(X) \to Y_1$ be the tangent cone to $X$ along $Y_1$. Since $Y_1$ is a smooth normally flat center in $X$, we know by [Hir64], theorem 2, p.195, that $\mathcal{C}_y(X)$ is a cone with vertex $T_{Y_1,y}$ over $\rho_{Y_1}^{-1}(y)$. Since $X$ is Cohen-Macaulay, the main result of [Sch77] implies that the cone $\mathcal{C}_y(X)$ is connected in codimension 1 (which means that one needs to subtract from $\mathcal{C}_y(X)$ a variety of codimension at most 1 to disconnect it). As $\dim \rho_{Y_1}^{-1}(y) \geq 1$ for all $y \in Y$, we get that $\rho_{Y_1}^{-1}(y)$ is connected in codimension 1 for all $y \in Y$. The map $\rho_{Y_1}$ being flat, this finally implies that $E_1 = \mathbb{P}(\mathcal{C}_{Y_1}(X))$ is connected in codimension 1.

Assume that $E_1$ is reducible. By the above discussion, we know that each irreducible component of $E_1$ meets another component in codimension 1. Since $\dim E_1 \geq 2$, we get that $E_1$ is not normal, which contradicts the fact that it has rational singularities (see 2.2.1). ◯
In the following, for $2 \leq k \leq n$, we denote by $E^{(k)}_{1k}$ the intersection of $E^{(k)}_1$ with $E_k$, the exceptional divisor of $\pi_k : X_k \to X_{k-1}$. By item 3 of definition 2.1.2, $E^{(k)}_{1k}$ is also the exceptional divisor of the blow-up $E^{(k)}_1 \to E^{(k-1)}_1$.

**Proposition 2.2.3** Let $2 \leq k \leq n$. If $m_k$ is the non-negative integer such that $K_{X_k} = \pi_k^* K_{X_{k-1}} \otimes \mathcal{O}_{X_k}(m_k E_k)$, then we have:

$$K_{E^{(k)}_1} = \pi_k|_{E^{(k)}_1}^* \mathcal{O}_{E^{(k-1)}_1} \otimes \mathcal{O}_{E^{(k)}_1}(m_k E_{1k}).$$

Note that the integer $m_k$ is well defined because $X_k \to X_{k-1}$ is a blow-up whose exceptional divisor is irreducible. Moreover, it is non-negative because Gorenstein rational singularities are canonical (see [Kol97], corollary 11.13).

**Proof:**

The adjunction formula implies that $K_{E^{(k)}_1} = K_{X_k}|_{E^{(k)}_1} \otimes \mathcal{O}_{E^{(k)}_1}(E^{(k)}_1)$ and that $K_{E^{(k-1)}_1} = K_{X_{k-1}}|_{E^{(k-1)}_1} \otimes \mathcal{O}_{E^{(k-1)}_1}(E^{(k-1)}_1)$.

Now, we tensor the formula $K_{X_k} = \pi_k^* K_{X_{k-1}} \otimes \mathcal{O}_{X_k}(m_k E_k)$ by $\mathcal{O}_{E^{(k)}_1}$ and taking into account the fact that $E_k|_{E^{(k)}_1} = E_{1k}$ and that $\pi_k|_{E^{(k)}_1}^* \mathcal{O}_{E^{(k-1)}_1}(E^{(k-1)}_1) = \mathcal{O}_{E^{(k)}_1}(E^{(k)}_1)$, we get the announced formula.

**3 Categorical crepant resolutions of singularities and wonderful resolutions**

**3.1 Categorical crepant resolution of singularities**

Now we come back to the notion of categorical resolutions of singularities. Let us recall some basic facts about derived categories of coherent sheaves on an algebraic variety.

Let $X$ be an algebraic variety, then $D^b(X)$ denotes the derived category of bounded complexes of coherent sheaves on $X$. The subcategory of bounded complexes of locally free sheaves is denoted by $D^{perf}(X)$. Recall that if $X$ is smooth, then these two categories are equivalent. Let $A$ be a full subcategory of $D^b(X)$ and $\alpha : A \to D^b(X)$ the embedding functor. We say that $A$ is admissible if $\alpha$ has a left and a right adjoint. This is equivalent to asking that there exist semi-orthogonal decompositions:

$$D^b(X) = \langle A, \bot A \rangle$$

and $D^b(X) = \langle A^\bot, A \rangle$.
where:

\[ \perp A = \{ F \in \text{D}^b(X), \, \text{Hom}(F, a) = 0, \text{ for all } a \in A \}, \]

and

\[ A \perp = \{ F \in \text{D}^b(X), \, \text{Hom}(a, F) = 0, \text{ for all } a \in A \}. \]

We refer to [Kuz08], section 2 for more details on semi-orthogonal decompositions. Recall the definition of a categorical crepant resolution of singularities we gave in the introduction (see also [Kuz08]):

**Definition 3.1.1** Let \( X \) be an algebraic variety with Gorenstein and rational singularities. A categorical resolution of singularities of \( X \) is a triangulated category \( \mathcal{T} \) with a functor \( R\pi_\mathcal{T}^* : \mathcal{T} \to \text{D}^b(X) \) such that:

- there exists a resolution of singularities \( \pi : \tilde{X} \to X \) such that \( \delta : \mathcal{T} \hookrightarrow \text{D}^b(\tilde{X}) \) is admissible and \( R\pi_\mathcal{T}^* = R\pi_* \circ \delta \),

- we have \( L\pi^*\text{D}^{\text{perf}}(X) \subset \mathcal{T} \) and for all \( \mathcal{F} \in \text{D}^{\text{perf}}(X) \):

\[ R\pi_\mathcal{T}^* L\pi^*_\mathcal{T} \mathcal{F} \cong \mathcal{F}, \]

where \( L\pi^*_\mathcal{T} \) is the left adjoint to \( R\pi_\mathcal{T}^* \).

If for all \( \mathcal{F} \in \text{D}^{\text{perf}}(X) \), there is a quasi-isomorphism:

\[ L\pi^*_\mathcal{T} \mathcal{F} \cong L\pi^1_\mathcal{T} \mathcal{F}, \]

where \( L\pi^1_\mathcal{T} \) is the right adjoint of \( R\pi_\mathcal{T}^* \), we say that \( \mathcal{T} \) is weakly crepant.

Now we can state our main theorem.

**Theorem 3.1.2 (Existence of categorical weakly crepant resolutions)** Let \( X \) be an algebraic variety with Gorenstein and rational singularities. Assume that \( X \) has a wonderful resolution of singularities. Then \( X \) admits a categorical weakly crepant resolution of singularities.

An analogous notion of "non-commutative" crepant resolution of singularities was also studied for determinantal varieties in [BLVdB10], [BLVdB11] and [WZ12]. In these works, non-commutative crepant resolutions are proved to exist for some determinantal varieties which already admit "geometric" crepant resolution of singularities.
3.2 Lefschetz semi-orthogonal decompositions

A key point in the proof of the theorem is the notion of dual Lefschetz decomposition which was introduced in [Kuz07].

**Definition 3.2.1** Let $X$ be an algebraic variety and $L$ a line bundle on $X$. Let $\mathcal{T} \subset D^b(X)$ be a full admissible subcategory such that for all $T \in \mathcal{T}$, we have $T \otimes L \in \mathcal{T}$. We say that $\mathcal{T}$ admits a **dual Lefschetz decomposition** with respect to $L$, if there exists a semi-orthogonal decomposition:

$$\mathcal{T} = \langle B_m \otimes L^m, B_{m-1} \otimes L^{m-1}, \ldots, B_0 \rangle,$$

where $B_m \subset B_{m-1} \subset \ldots \subset B_1 \subset B_0$ are full admissible subcategories of $\mathcal{T}$.

**Example 3.2.2** The following semi-orthogonal decomposition:

$$D^b(\mathbb{P}^n) = \langle \mathcal{O}_{\mathbb{P}^n}(-n), \mathcal{O}_{\mathbb{P}^n}(-n + 1), \ldots, \mathcal{O}_{\mathbb{P}^n} \rangle,$$

is a dual Lefschetz decomposition of $D^b(\mathbb{P}^n)$ with respect to $\mathcal{O}_{\mathbb{P}^n}(-1)$.

The following lemma is proposition 4.1 of [Kuz08].

**Lemma 3.2.3** Let $X$ be a smooth algebraic variety and let $i : E \hookrightarrow X$ a Cartier divisor such that we have a dual Lefschetz decomposition:

$$D^b(E) = \langle B_m \otimes \mathcal{O}_E(mE), B_{m-1} \otimes \mathcal{O}_E(E), \ldots, B_0 \rangle,$$

with $B_m \subset \ldots \subset B_1 \subset B_0$. Then we have a semi-orthogonal decomposition:

$$D^b(X) = \langle i_* (B_m \otimes \mathcal{O}_E(mE)) \ldots i_*(B_1 \otimes \mathcal{O}_E(E)), A_0(X) \rangle,$$

where $A_0(X) = \{ F \in D^b(X), i^* F \in B_0 \}$.

The technical tool for the proof of the main theorem is the following proposition. We first need some notation. Let $X$ be an algebraic variety with Gorenstein and canonical singularities. Let

$$X_n \xrightarrow{\pi_n} X_{n-1} \ldots X_1 \xrightarrow{\pi_1} X_0 = X,$$

be a wonderful resolution of $X$, which is a succession of blow-ups along the smooth normally flat centers $Y_t \subset X_{t-1}$ for $t = 1 \ldots n$. For any such $t$, we
have a diagram:

\[
\begin{array}{c}
E^{(n)}_t \\
\downarrow q_t \\
Y_t
\end{array} \quad \xymatrix{ \ar[r]^{i^{(n)}_j} & X_n } \quad \begin{array}{c}
\pi_{t+1} \ldots \pi_n \\
\downarrow \\
X_{t-1}
\end{array}
\]

We recall that \( E^{(n)}_t \) is the total transform of \( E_t \subset X_t \) through \( \pi_{t+1} \ldots \pi_n \) and that \( q_t \) is a flat projection with smooth fibers. We also have the formula \( K_{X_t} = \pi^*_t K_{X_{t-1}} + m_t E_t \). For any \( 1 \leq j \leq n \) and for any \( 0 \leq k_j \leq m_j - 1 \), we define the subcategories of \( \text{D}^b(X_n) \):

\[ A_{j,k_j} = (i^{(n)}_j)_* \left[ q_j^! \mathcal{D}^b(Y_j) \otimes \mathcal{O}_{E^{(n)}_j} \left( (m_j - k_j) E^{(n)}_j + \sum_{t=j+1}^n m_t E^{(n)}_t \right) \right]. \]

**Proposition 3.2.4** With the above hypotheses and notations, we have a semi-orthogonal decomposition:

\[ \text{D}^b(X_n) = \langle A_{1,0}, \ldots, A_{1,m_1-1}, \ldots, A_{j,k_j}, \ldots, A_{n,m_n-1}, D_{X_n} \rangle, \]

where \( D_{X_n} \) is the left orthogonal to the full admissible subcategory generated by the \( A_{j,k_j} \) for \( 0 \leq k_j \leq m_j - 1 \) and \( 1 \leq j \leq n \). Moreover, we have the inclusion \( (\pi_1 \ldots \pi_n)^* \text{D}^\text{perf}(X) \subset D_{X_n} \).

We postpone the proof of the proposition and we will show that it implies that \( D_{X_n} \) is a categorical weakly crepant resolution of singularities of \( X \).

Let \( \delta : D_{X_n} \to \text{D}^b(X_n) \) be the fully faithful admissible embedding and denote by \( \pi \) the resolution \( X_n \to X \). Since \( \pi^* \text{D}^\text{perf}(X) \subset D_{X_n} \), the only thing left to prove is the crepancy of \( \delta \pi^* \), that is \( \delta^* \pi^*(F) = \delta^! \pi^!(F) \), for all \( F \in \text{D}^\text{perf}(X) \). Recall that

\[ \pi^!(F) = \pi^*(F) \otimes \pi^*(K_X^{-1}) \otimes K_{X_n} = \pi^*(F) \otimes \mathcal{O}_{X_n} \left( \sum_{t=1}^n m_t E^{(n)}_t \right), \]

for any \( F \in \text{D}^\text{perf}(X) \).

Now, since the functor \( \delta \) is fully faithful, the equality \( \delta^* \pi^*(F) = \delta^! \pi^!(F) \) is equivalent to \( \delta(\delta^* \pi^*(F)) = \delta(\delta^! \pi^!(F)) \), for any \( F \in \text{D}^\text{perf}(X) \). As \( \pi^* \text{D}^\text{perf}(X) \subset D_{X_n} \), we have \( \delta(\delta^* \pi^*(F)) = \pi^*(F) \), for all \( F \in \text{D}^\text{perf}(X) \). We are going to show that \( \delta(\delta^! \pi^!(F)) = \pi^!(F) \) for any \( F \in \text{D}^\text{perf}(X) \). For \( 1 \leq j \leq n \) and \( 0 \leq k_j \leq m_j - 1 \), we have exact sequences:
\[0 \to \mathcal{O}_{X_n} \left( (m_j - k_j - 1)E_1^{(n)} + \sum_{t=j+1}^{n} m_tE_t \right) \to \mathcal{O}_{X_n} \left( (m_j - k_j)E_j^{(n)} + \sum_{t=j+1}^{n} m_tE_t \right) \to \]

\[(i_j^{(n)})_* \mathcal{O}_{E_j}^{(n)} \left( (m_j - k_j)E_j^{(n)} + \sum_{t=j+1}^{n} m_tE_t \right) \to 0,
\]

So for each \( F \in D^{\text{perf}}(X) \), we deduce a long sequence of triangles:

\[
\begin{aligned}
\pi^*(F) \to F_{n,m_{n-1}} & \to \ldots \to F_{j,k_j} \to \ldots \to F_{1,1} \to F_{1,0} \\
\mathcal{F}_{n,m_{n-1}} & \to \mathcal{F}_{j,k_j} \to \ldots \to \mathcal{F}_{1,0}
\end{aligned}
\]

where:

\[
F_{j,k_j} = \pi^*F \otimes \mathcal{O}_{X_n} \left( (m_j - k_j)E_j^{(n)} + \sum_{t=j+1}^{n} m_tE_t^{(n)} \right),
\]

and

\[
\mathcal{F}_{j,k_j} = (i_j^{(n)})_*(i_j^{(n)})^* \left[ \pi^*F \otimes \mathcal{O}_{X_n} \left( (m_j - k_j)E_j^{(n)} + \sum_{t=j+1}^{n} m_tE_t^{(n)} \right) \right],
\]

for \( 0 \leq k_j \leq m_j - 1 \) and \( 1 \leq j \leq n \).

Since \( \delta : D_{X_n} \to D^b(X_n) \) is fully faithful and admissible, it is well known that \( \delta(\delta^!\pi^*(F)) \) is the \( D_{X_n} \)-component of \( \pi^!(F) = \pi^*(F) \otimes \mathcal{O}_{X_n}(\sum_{t=1}^{n} m_tE_t^{(n)}) \) in the semi-orthogonal decomposition of \( D^b(X_n) \) given by proposition 3.2.4 (see [Kuz08] section 2 for more details on semi-orthogonal decompositions).
Consider the fibered diagram:

\[
\begin{array}{c}
E_j^{(n)} \xrightarrow{\ i_j^{(n)} \ } X_n \\
\downarrow q_j \quad \quad \quad \quad \quad \quad \downarrow \pi_{j...n} \\
Y_j \xrightarrow{\ i_j \ } X_{j-1} \\
\downarrow \pi_{1...j-1} \quad \quad \quad \quad \quad \downarrow \pi \\
X
\end{array}
\]

Since \((i_j^{(n)})^*(\pi^*(F)) = q_j^*((\pi_1...\pi_j\cdot ij)^*(F)) \in q_j^*(\mathcal{D}^b(Y_j))\), we have \(\mathcal{F}_{j,k} \in A_{j,k_j}\), for \(1 \leq j \leq n\) and \(0 \leq k_j \leq m_j - 1\). As a consequence, the above sequence of triangles shows that \(\delta(\delta^!\pi^!(F)) = \pi^*(F)\) and we are done.

### 3.3 Some vanishing lemmas

Before diving into the proof of proposition 3.2.4, we need a vanishing result which will be very useful.

**Lemma 3.3.1** Let \(f : Z \to S\) be a flat and projective morphism such that \(Z\) has Gorenstein rational singularities. Assume that there exists a relatively anti-ample divisor \(E \subset Z\), a line bundle \(F\) on \(S\) and a positive integer \(r \geq 1\) such that:

\[
K_Z = f^*F \otimes \mathcal{O}_Z((r + 1)E).
\]

Then we have the vanishing:

\[
R^i f_* \mathcal{O}_Z(kE) = 0
\]

for any \(i > 0\) if \(k \leq r\) and for any \(i < \dim Z - \dim S\) if \(k \geq 1\). As a consequence, we have:

\[
H^i(X, f^*A \otimes \mathcal{O}_X(kE)) = 0
\]

for any vector bundle \(A\) on \(S\) for any \(i \geq 0\) if \(1 \leq k \leq r\).

**Proof:**

- By hypotheses, we have \(K_Z = f^*F \otimes \mathcal{O}_Z((r + 1)E)\), the variety \(Z\) has
rational singularities and $E$ is relatively anti-ample so that by Kawamata-Viehweg relative vanishing (see [KMM87], theorem 1.2.5) we have

$$R^if_*\mathcal{O}_Z(kE) = 0$$

for any $i > 0$ if $k \leq r$.

Again by hypotheses, we know that $Z$ is Gorenstein and that $f$ is flat. This implies that all the fibers are pure $d$-dimensional Gorenstein schemes, where $d = \dim Z - \dim S$. As a consequence, we can apply the relative duality for a flat morphism with Gorenstein fibers (see [Kle80], theorem 20) and we find

$$R^if_* (\mathcal{O}_Z(-kE) \otimes K_{Z/S}) = \mathcal{H}om_{\mathcal{O}_S}(R^{d-i}f_* \mathcal{O}_Z(kE), \mathcal{O}_S).$$

As a consequence, we have

$$R^if_* \mathcal{O}_Z(kE) = 0,$$

for any $i < d$ if $k \geq 1$.

The last vanishing

$$H^i(X, f^*A \otimes \mathcal{O}_X(kE)) = 0$$

for any vector bundle $A$ on $S$ and any $i \geq 0$ if $1 \leq k \leq r$ is then a direct consequence of the projection formula and Leray’s spectral sequence. △

Thus, we get the following corollary:

**Corollary 3.3.2** Let $X$ be an algebraic variety with Gorenstein singularities and let $f : \tilde{X} \to X$ be the blow-up of $X$ along the smooth normally flat center $Y \subset X$ with exceptional divisor $E$. Assume that $E$ has Gorenstein rational singularities and that $K_{\tilde{X}} = f^*K_X + rE$ for a positive integer $r$. Then we have:

$$R^if_* \mathcal{O}_{\tilde{X}}(kE) = 0$$

for all $i > 0$ if $k \leq r$ and

$$R^0f_* \mathcal{O}_{\tilde{X}}(kE) = \mathcal{O}_X$$

if $k \geq 0$.

**Proof**: We will proceed by induction. The divisor $-E$ is relatively ample so by Grothendieck’s vanishing theorem we have $R^if_* \mathcal{O}_{\tilde{X}}(kE) = 0$ for all $i > 0$, if...
Now, let $k \leq r - 1$ be an integer such that $R^i f_* \mathcal{O}_X(kE) = 0$ for all $i > 0$. We have the exact sequence:

$$0 \to \mathcal{O}_X(kE) \to \mathcal{O}_X((k+1)E) \to \mathcal{O}_E((k+1)E) \to 0,$$

and by the adjunction formula: $K_E = f^*|_E K_X + (r + 1)E$. So, the long exact sequence in cohomology associated to $f_*$ and lemma 3.3.1 applied to $f : E \to Y$ show that $R^i f_* \mathcal{O}_X((k+1)E) = 0$ for any $i > 0$.

The variety $X$ is Gorenstein, hence Cohen Macaulay, so by Schaub’s theorem [Sch77] (see the proof of proposition 2.2.2), we know that $R^0 f_* \mathcal{O}_X = \mathcal{O}_X$. Using the above exact sequence and lemma 3.3.1, we prove by induction that $R^0 f_* \mathcal{O}_X(kE) = \mathcal{O}_X$ for any $k \geq 0$.

One of course notes that part of corollary 3.3.2 is a direct consequence of Kawamata-Viehweg relative vanishing. But in any case we will need both proposition 3.3.1 and corollary 3.3.2 in the proof of the main theorem.

### 3.4 The technical induction

The proof of proposition 3.2.4 will be done by induction, but we will have to prove a more precise statement. We begin with some more notation. Let $X$ be an algebraic variety with Gorenstein and canonical singularities. Let

$$X_n \xrightarrow{\pi_n} X_{n-1} \ldots X_1 \xrightarrow{\pi_1} X_0 = X,$$

be a wonderful resolution of $X$. We introduced the following subcategories of $D^b(X_n)$:

$$A_{j,k} = i_{j}^{(n)} \left[ q^*_j D^b(Y_j) \otimes \mathcal{O}_{E_j^{(n)}} \left( (m_j - k_j) E_j^{(n)} + \sum_{t=j+1}^{n} m_t E_t^{(n)} \right) \right],$$

for all $0 \leq k_j \leq m_j - 1$ and $1 \leq j \leq n$.

Now, let $1 \leq k \leq n$, and let $k+1 \leq p \leq n$. We denote by $j_{k,p}^{(n)} : E_{k,p}^{(n)} \hookrightarrow E_k^{(n)}$ the embedding of the intersection $E_{k,p}^{(n)} = E_p^{(n)} \cap E_k^{(n)}$ which is also the total transform of $E_{k,p}^{(p)} = E_p^{(p)} \cap E_k^{(p)}$ through $\pi_{p+1} \ldots \pi_n$, by item 3 and 4 of definition 2.1.2. Hence we have a fibered diagram:
We denote by \( q_{k,p} \) the flat map \( q_{k,p} : E_{k,p}^{(n)} \rightarrow Y_{k,p} \), where \( Y_{k,p} \) is the smooth and proper intersection \( Y_{k,p} = Y_p \cap E_{k}^{(p-1)} \). Notice that, again by item 3 and 4 of definition 2.1.2, the intersection \( E_p \cap E_{k}^{(p)} \) is the exceptional divisor of the blow up of \( E_{k}^{(p-1)} \) along \( Y_{k,p} \). Thus, we have another fibered diagram:

\[
\begin{array}{ccc}
E_{k,p}^{(n)} & \xrightarrow{j_{k,p}^{(n)}} & E_k^{(n)} \\
\downarrow & & \downarrow \\
E_{k,p}^{(p)} & \xrightarrow{j_{k,p}^{(p)}} & E_k^{(p)} \\
\downarrow & & \downarrow \\
Y_{k,p} & \xrightarrow{\pi_{p+1} \cdots \pi_n} & E_{k}^{(p-1)} \\
\end{array}
\]

For any \( k + 1 \leq p \leq n \), let \( m_p \) be the positive integer such that \( K_{X_p} = \pi_p^*K_{X_{p-1}} + m_pE_p \). By proposition 2.2.3, this is also the integer such that:

\[
K_{E_{k}^{(p)}} = \pi_p^*|E_k^{(p-1)}|K_{E_{k}^{(p-1)}} + m_pE_{k,p}^{(p)}.
\] (1)

We define:

\[
B_{p,k}^k = j_{k,p}^{(n)} \left[ q_{k,p}^*D^b(Y_{k,p}) \otimes \mathcal{O}_{E_{k}^{(n)}}^\ast \left( (m_p - k_p)E_{k,p}^{(n)} + \sum_{i=p+1}^{n} m_iE_{k,i}^{(n)} \right) \right],
\]

for all \( 0 \leq k_p \leq m_p - 1 \).

Finally we let:

\[
C_{k,r}^k = q_k^*D^b(Y_k) \otimes \mathcal{O}_{E_k^{(n)}}^\ast \left( (m_k - r_k)E_k^{(n)} + \sum_{i=k+1}^{n} m_iE_i^{(n)} \right),
\]
for all $0 \leq r_k \leq m_k - 1$.

The following result is the more precise version of proposition 3.2.4 that we will prove:

**Proposition 3.4.1** With the above hypothesis and notations, we have a semi-orthogonal decomposition:

$$D^b(X) = \langle A_{1,0}, \ldots, A_{1,m_1-1}, \ldots, A_{j,k}, \ldots, A_{n,m_n-1}, D_{X_n} \rangle,$$

where $D_{X_n}$ is the left orthogonal to the full admissible subcategory generated by the $A_{j,k}$. Moreover, we have the inclusion $(\pi_1 \ldots \pi_n)^* D_{\text{perf}}(X) \subset D_{X_n}$.

For each $1 \leq k \leq n$, we also have a semi-orthogonal decomposition:

$$D^b(E^{(n)}_k) = \langle C_{k,0}, \ldots, C_{k,m_k-1}, B_{k+1,0}, \ldots, B_{p,k_p}, \ldots, B_{n,m_n-1}, D_{E^{(n)}_k} \rangle,$$

where $D_{E^{(n)}_k}$ is the left orthogonal to the subcategory generated by the $C_{k,r_k}$ and the $B_{p,k_p}$. Moreover, we have the inclusion $q_*^k D^b(Y_k) \subset D_{E^{(n)}_k}$.

**Proof:**

Assume that $n = 1$. The map $\pi_1 : X_1 \to X$ is the blow-up of $X$ along $Y_1$. By definition of a wonderful resolution, the exceptional divisor $E_1$ is smooth, the variety $X_1$ is smooth, the map $\pi_1 : E_1 \to Y_1$ is flat and $Y_1$ is smooth. Moreover, by definition of $m_1$ and by the adjunction formula, we have:

$$K_{E_1} = q^* K_X|_{Y_1} + (m_1 + 1)E_1.$$

So by lemma 3.3.1 for all $k \in \mathbb{Z}$, we have:

$$Rq_1^* R\mathcal{H}om_{E_1}(\mathcal{O}_{E_1}(kE_1), \mathcal{O}_{E_1}(kE_1)) = \mathcal{O}_{E_1}.$$

Therefore, the categories $\mathcal{O}_{E_1}(kE_1) \otimes q_1^* D^b(Y_1) = C_{1,m_1-k}^1$ are full admissible subcategories of $D^b(E_1)$, for all $k \in \mathbb{Z}$. Moreover, again by lemma 3.3.1 we have a semi-orthogonal decomposition:

$$D^b(E_1) = \langle C_{1,0}^1, \ldots, C_{1,m_1-1}, D_{E_1} \rangle,$$

with the property $q_1^* D^b(Y_1) \subset D_{E_1}$.

Now, we can apply lemma 3.2.3 to the embedding $i_1 : E_1 \hookrightarrow X_1$, and we find a semi-orthogonal decomposition:
$$D^b(X_1) = \langle A_{1,0}, \ldots, A_{1,m_1-1}, D_{X_1} \rangle,$$

with $D_{X_1} = \{ F \in D^b(X), i^*_1 F \in D_{E_1} \}$. Let $G \in D^{perf}(X)$. We have $(\pi_1 i_1)^* G = q_1^*(G|_{Y_1})$. But $G|_{Y_1} \in D^b(Y_1)$, so that $q_1^*(G|_{Y_1}) \in D_{E_1}$, that is $i^*_1(G) \in D_{E_1}$. Thus $\pi_1^*(D^{perf}(X)) \subset D_{X_1}$, which settles the proof of the proposition in the case $n = 1$.

Let $n \geq 2$ and assume that the proposition is true if $X$ admits a $n-1$ step resolution of singularities. We will prove that the proposition is true for a $n$-step wonderful resolution of singularities. Namely, let $X$ be an algebraic variety with Gorenstein and canonical singularities and let:

$$X_n \xrightarrow{\pi_n} X_{n-1} \ldots X_1 \xrightarrow{\pi_1} X_0 = X,$$

be a $n$-step wonderful resolution of $X$. Let $1 \leq k \leq n$. By item 2 and 4 of definition [2.1.2] we know that $E_k$ admits a $(n-k)$-step wonderful resolution:

$$E^{(n)}_k \xrightarrow{\Pi_{n,n-1}} \ldots E^{(k+1)}_k \xrightarrow{\Pi_{k+1,k}} E^{(k)}_k = E_k,$$

which is a succession of blow-ups along the smooth normally flat centers $Y_{k,k+1}, \ldots, Y_{k,n}$. Thus by the recursion hypothesis and by formula [1] we have a semi-orthogonal decomposition:

$$D^b(E^{(n)}_k) = \langle B^{k}_{k+1,0}, \ldots, B^{k}_{k+1,m_{k+1}-1}, \ldots, B^{k}_{p,k_p}, \ldots, B^{k}_{n,m_n-1}, \tilde{D}_{E^{(n)}_k} \rangle,$$

with the inclusion $(\Pi_{k+1,k} \ldots \Pi_{n,n-1})^* D^{perf}(E_k) \subset \tilde{D}_{E^{(n)}_k}$.

**Step 1**

Now we want to prove the following:

**Claim 3.4.2** For $0 \leq r_k \leq m_k - 1$, the categories:

$$C^{k}_{r_k} = q_k^* D^b(Y_k) \otimes \mathcal{O}_{E^{(n)}_k} \left( (m_k - r_k)E^{(n)}_k + \sum_{i=k+1}^{n} m_i E^{(n)}_i \right)$$

are full admissible subcategories of $D^b(E^{(n)}_k)$, right orthogonal to one another and right orthogonal to the subcategories $B^{k}_{p,k_p}$ for $k+1 \leq p \leq n$ and $0 \leq k_p \leq m_p - 1$. Moreover, the category $q_k^* D^b(Y_k)$ is left orthogonal to the subcategories $C^{k}_{r_k}$ and $B^{k}_{p,k_p}$.
We start with the first vanishing, that is we want to prove that:

\[ \text{Hom}(B_{p,k_r}^k, C_{k,r_k}^k) = 0 \]
\[ \text{Hom}(q_k^*D^b(Y_k), C_{k,r_k}^k) = 0. \]
\[ \text{Hom}(q_k^*D^b(Y_k), B_{p,k_r}^k) = 0 \]

We start with the first vanishing, that is we want to prove that:

\[
\text{Hom}(j_{k,p,*}^{(n)} (q_{k,p}^*D^b(Y_{k,p}) \otimes \mathcal{O}_{E_{k,p}^{(n)}}((m_p - k_p)E_{k,p}^{(n)} + \sum_{i=p+1}^n m_iE_{k,i}^{(n)})),
\]
\[
q_k^*D^b(Y_k) \otimes \mathcal{O}_{E_{k}^{(n)}}((m_k - r_k)E_{k}^{(n)} + \sum_{i=k+1}^n m_iE_{k,i}^{(n)})) = 0.
\]

By formula \[ and the adjunction formula we have:

\[ K_{E_{k}^{(n)}} = (m_k + 1)E_{k}^{(n)}|_{E_{k}^{(n)}} + \sum_{i=k+1}^n m_iE_{k,i}^{(n)} + q_{k,n}^*K_{X_i}|_{Y_{k+1}}. \]

Thus, by Serre duality we have the equality (note that since all intersections are proper, \( \mathcal{O}_{E_{k}^{(n)}}(E_{k,i}^{(n)}) = \mathcal{O}_{E_{k}^{(n)}}(E_{k,i}^{(n)})):

\[
\text{Hom}(j_{k,p,*}^{(n)} (q_{k,p}^*D^b(Y_{k,p}) \otimes \mathcal{O}_{E_{k,p}^{(n)}}((m_p - k_p)E_{k,p}^{(n)} + \sum_{i=p+1}^n m_iE_{k,i}^{(n)})),
\]
\[
q_k^*D^b(Y_k) \otimes \mathcal{O}_{E_{k}^{(n)}}((m_k - r_k)E_{k}^{(n)} + \sum_{i=k+1}^n m_iE_{k,i}^{(n)}))
\]
\[= \text{Hom}(q_k^*D^b(Y_k) \otimes \mathcal{O}_{E_{k}^{(n)}}((-1 - r_k)E_{k}^{(n)}),
\]
\[
j_{k,p,*}^{(n)} (q_{k,p}^*D^b(Y_{k,p}) \otimes \mathcal{O}_{E_{k,p}^{(n)}}((m_p - k_p)E_{k,p}^{(n)} + \sum_{i=p+1}^n m_iE_{k,i}^{(n)}))^{\vee}.
\]

Note that we used that

\[ q_k^*D^b(Y_k) \otimes \mathcal{O}_{E_{k}^{(n)}}((-1 - r_k)E_{k}^{(n)}) \otimes K_{E_{k}^{(n)}} \]

which are right orthogonal to one another is a simple variation of what we proved for the case \( n = 1 \) of the proposition, the details are left to the reader. In order to prove claim 3.4.2 we are left to prove (with an obvious abuse of notation) that:

\[ \text{Hom}(B_{p,k_r}^k, C_{k,r_k}^k) = 0 \]
\[ \text{Hom}(q_k^*D^b(Y_k), C_{k,r_k}^k) = 0. \]
\[ \text{Hom}(q_k^*D^b(Y_k), B_{p,k_r}^k) = 0 \]
in the above equality. By adjunction we also have:

\[
\begin{align*}
\text{Hom}(q^*_k \mathcal{D}^b(Y_k) \otimes \mathcal{O}_{E_k^{(n)}}((-1 - r_k)E_k^{(n)})), \\
\quad j_{k,p}^*(q^*_k \mathcal{D}^b(Y_{k,p}) \otimes \mathcal{O}_{E_{k,p}^{(n)}}((m_p - k_p)E_{k,p}^{(n)} + \sum_{i=p+1}^n m_i E_{k,i}^{(n)}))), \\
= \text{Hom}((j_{k,p}^*)^*(q^*_k \mathcal{D}^b(Y_k) \otimes \mathcal{O}_{E_k^{(n)}}((-1 - r_k)E_k^{(n)})), \\
q_{k,p}^* \mathcal{D}^b(Y_{k,p}) \otimes \mathcal{O}_{E_{k,p}^{(n)}}((m_p - k_p)E_{k,p}^{(n)} + \sum_{i=p+1}^n m_i E_{k,i}^{(n)}))).
\end{align*}
\]

But \((j_{k,p}^*)^*(q^*_k \mathcal{D}^b(Y_k)) \subset q_{k,p}^* \mathcal{D}^b(Y_{k,p})\) and \(j_p^* \mathcal{O}_{E_k^{(n)}}(E_k^{(n)}) = q_{k,p}^* \mathcal{O}_{Y_{k,p}}(E_{k,p}^{(p-1)})\) because we have a fibred diagram:

\[
\begin{array}{ccc}
E_{k,p}^{(n)} & \xrightarrow{j_{k,p}^*} & E_k^{(n)} \\
\downarrow q_{k,p} & & \downarrow \pi_{p,\ldots,n} |_{E_k^{(n)}} \\
Y_{k,p} & \xrightarrow{j} & E_{k,p}^{(p-1)}
\end{array}
\]

As a consequence, to prove that:

\[
\begin{align*}
\text{Hom}((j_{k,p}^*)^*(q^*_k \mathcal{D}^b(Y_k) \otimes \mathcal{O}_{E_k^{(n)}}((-1 - r_k)E_k^{(n)})), \\
q_{k,p}^* \mathcal{D}^b(Y_{k,p}) \otimes \mathcal{O}_{E_{k,p}^{(n)}}((m_p - k_p)E_{k,p}^{(n)} + \sum_{i=p+1}^n m_i E_{k,i}^{(n)}))) = 0,
\end{align*}
\]

we only have to prove that:

\[
q_{k,p}^* \left( \mathcal{O}_{E_{k,p}^{(n)}}((m_p - k_p)E_{k,p}^{(n)} + \sum_{i=p+1}^n m_i E_{k,i}^{(n)})) = 0. \tag{2}\right.
\]

The map \(q_{k,p}^* : E_{k,p}^{(n)} \to Y_{k,p}\) factors as \(\theta_{k,p}^n : E_{k,p}^{(n)} \to E_{k,p}^{(p)}\) followed by the projection \(\Pi_{p,p-1} : E_{k,p}^{(p)} \to Y_{k,p}\). But the map \(\theta_{k,p}^n\) is a succession of blow-ups along the smooth normally flat centers \(Y_{i} \cap E_{k,i}^{(i-1)}\), which exceptional divisors on \(E_{k,i}^{(n)}\) are the \(E_{k,i}^{(n)}|_{E_{k,p}^{(n)}}\), for \(i = p + 1 \ldots n\). Moreover, by item 3 and 4 of definition 2.1.2 and by proposition 2.2.3 we know that:
\[ K_{E_k^{(n)}} = (\theta_{k,p}^n)^* K_{E_k^{(p)}} + \sum_{i=p+1}^n m_i E_{k,i}^{(n)} |_{E_k^{(n)}}. \]

We apply corollary 3.3.2 to the morphism \( \theta_{k,p}^n : E_k^{(n)} \to E_k^{(p)} \) to get:

\[ \theta_{k,p}^n \left( \mathcal{O}_{E_k^{(n)}} \left( (m_p - k_p) E_k^{(n)} + \sum_{i=p+1}^n m_i E_{k,i}^{(n)} \right) \right) = \mathcal{O}_{E_k^{(p)}} \left( (m_p - k_p) E_k^{(p)} \right), \]

so that

\[ q_{k,p}^* \left( \mathcal{O}_{E_k^{(n)}} \left( (m_p - k_p) E_k^{(n)} + \sum_{i=p+1}^n m_i E_{k,i}^{(n)} \right) \right) = \prod_{p,p-1} \mathcal{O}_{E_k^{(p)}} \left( (m_p - k_p) E_k^{(p)} \right). \]

Now, by formula \( \ref{formula1} \) and the adjunction formula:

\[ K_{E_k^{(p)}} = \prod_{p,p-1} \mathcal{O}_{E_k^{(p)}} \left( (m_p - k_p) E_k^{(p)} \right) = 0, \]

for all \( 0 \leq k_p \leq m_p - 1 \), which is precisely what we wanted.

To conclude step 1, we must show that:

\[ q_k^* D^b(Y_k) \subset \perp C_{k,r_k} \text{ and } q_k^* D^b(Y_k) \subset \perp B^k_{p,k_p} \]

for all \( k+1 \leq p \leq n \). We have:

\[ \text{Hom} \left[ q_k^* D^b(Y_k), (j_{k,p}^{(n)})_* \left( q_{k,p}^* D^b(Y_k) \otimes \mathcal{O}_{E_k^{(n)}} \left( (m_p - k_p) E_k^{(n)} + \sum_{i=p+1}^n m_i E_{k,i}^{(n)} \right) \right) \right] \]

\[ = \text{Hom} \left[ (j_{k,p}^{(n)})_* q_k^* D^b(Y_k), q_{k,p}^* D^b(Y_k) \otimes \mathcal{O}_{E_k^{(n)}} \left( (m_p - k_p) E_k^{(n)} + \sum_{i=p+1}^n m_i E_{k,i}^{(n)} \right) \right], \]

\[ \subset \text{Hom} \left[ q_{k,p}^* D^b(Y_k), q_{k,p}^* D^b(Y_k) \otimes \mathcal{O}_{E_k^{(n)}} \left( (m_p - k_p) E_k^{(n)} + \sum_{i=p+1}^n m_i E_{k,i}^{(n)} \right) \right], \]

since \( (j_{k,p}^{(n)})_* q_k^* D^b(Y_k) \subset q_{k,p}^* D^b(Y_k). \)
But we have already proved in formula (2) that:

\[ q_{k,p}^* E_{k,p}^{(n)} \left( (m_p - k_p) E_{k,p}^{(n)} + \sum_{i=p+1}^{n} m_i E_{k,i}^{(n)} \right) = 0, \]

so that:

\[
\text{Hom} \left[ q_{k,p}^* D^b(Y_{k,p}), q_{k,p}^* D^b(Y_{k,p}) \otimes O_{E(n)}(m_p - k_p) E_{k,p}^{(n)} + \sum_{i=p+1}^{n} m_i E_{k,i}^{(n)} \right] = 0.
\]

This proves that \( q_{k,p}^* D^b(Y_{k,p}) \subset \perp B_{p,k} \), for all \( k + 1 \leq p \leq n \). The fact that \( q_{k,p}^* D^b(Y_{k,p}) \subset \perp C_{k,r} \) is done in the same fashion. Step 1 is thus complete, that is we have a semi-orthogonal decomposition:

\[
D^b(E_k^{(n)}) = \langle C_{k,0}^k, \ldots, C_{k,m_k-1}^k, B_{k+1,0}^k, \ldots, B_{p,k_p}^k, \ldots, B_{n,m_n-1}^k, D_{E(n)} \rangle,
\]

with the property : \( q_{k,p}^* D^b(Y_{k,p}) \subset D_{E(n)} \).

**Step 2**

From the above semi-orthogonal decomposition of \( D^b(E_k^{(n)}) \) for all \( 1 \leq k \leq n \), we want to deduce a semi-orthogonal decomposition:

\[
D^b(X_n) = \langle A_{1,0}, \ldots, A_{1,m_1-1}, \ldots, A_{j,k_j}, \ldots, A_{n,m_n-1}, D_{X_n} \rangle,
\]

with the property : \((\pi_1 \ldots \pi_n)^* D_{\text{perf}}(X) \subset D_{X_n}\), where \( D_{X_n} \) is the left orthogonal to the subcategory generated by the \( A_{j,k_j} \).

Let \( j \in [1, \ldots, n] \). We note that \( A_{j,k_j} = (i_j^{(n)})^* C_{j,k_j}^{j_n} \). Lemma 3.2.3 applied to the semi-orthogonal decomposition we found for \( D^b(E_j^{(n)}) \) proves that for all \( 0 \leq k_j \leq m_j - 1 \), the subcategories \( A_{j,k_j} \) are admissible full subcategories of \( D^b(X_n) \) which are left orthogonal to one another.

So, we are left to prove that for \( 1 \leq j < p \leq n \), for all \( 0 \leq k_j \leq m_j - 1 \) and for all \( 0 \leq k_p \leq m_p - 1 \), we have:

\[
\text{Hom}(i_j^{(n)}(q_{p}^* D^b(Y_p)) \otimes O_{E_j^{(n)}}((m_p - k_p) E_{p}^{(n)} + \sum_{t=p+1}^{n} m_t E_{t}^{(n)})),
\]

\[
i_j^{(n)}(q_{j}^* D^b(Y_j) \otimes O_{E_j^{(n)}}((m_j - k_j) E_{j}^{(n)} + \sum_{t=j+1}^{n} m_t E_{t}^{(n)})) = 0,
\]

26
that is:

\[
\text{Hom}(i_j^{(n)\ast}i_p^{(n)\ast}(q_p^b\mathcal{D}^b(Y_p) \otimes \mathcal{O}_{E_p^{(n)}}((m_p - k_p)E_p^{(n)} + \sum_{t=p+1}^n m_tE_t^{(n)})],
\]

\[
q_j^b\mathcal{D}^b(Y_j) \otimes \mathcal{O}_{E_j^{(n)}}((m_j - k_j)E_j^{(n)} + \sum_{t=j+1}^n m_tE_t^{(n)})) = 0.
\]

or with our notations:

\[
\text{Hom}\left[ j_j^{(n)\ast}i_p^{(n)\ast}\left(q_p^b\mathcal{D}^b(Y_p) \otimes \mathcal{O}_{E_p^{(n)}}((m_p - k_p)E_p^{(n)} + \sum_{t=p+1}^n m_tE_t^{(n)})\right), C_{j,k}^{j} \right] = 0.
\]

But we have a fibered diagram:

\[
\begin{array}{ccc}
E_j^{(n)} & \xrightarrow{j_j^{(n)}} & E_j^{(n)} \\
\downarrow j_p^{(n)} & & \downarrow i_j^{(n)} \\
E_p^{(n)} & \xrightarrow{i_p^{(n)}} & X_n
\end{array}
\]

The intersection \(E_j^{(n)} \cap E_p^{(n)} = E_{j,p}^{(n)}\) is proper and all varieties appearing in this fibered diagram are smooth so that this diagram is exact cartesian, that is:

\[
i_j^{(n)\ast}i_p^{(n)\ast}(F) = j_j^{(n)\ast}j_p^{(n)\ast}(F),
\]

for all \(F \in \text{D}^b(E_p^{(n)})\). In particular we have:

\[
i_j^{(n)\ast}i_p^{(n)\ast}\left(q_p^b\mathcal{D}^b(Y_p) \otimes \mathcal{O}_{E_p^{(n)}}((m_p - k_p)E_p^{(n)} + \sum_{t=p+1}^n m_tE_t^{(n)})\right)
= j_{j,p}^{(n)\ast}j_p^{(n)\ast}\left(q_p^b\mathcal{D}^b(Y_p) \otimes \mathcal{O}_{E_p^{(n)}}((m_p - k_p)E_p^{(n)} + \sum_{t=p+1}^n m_tE_t^{(n)})\right)
= j_{j,p}^{(n)}\left(q_p^b\mathcal{D}^b(Y_j,p) \otimes \mathcal{O}_{E_j^{(n)}}((m_p - k_p)E_j^{(n)} + \sum_{t=j+1}^n m_tE_t^{(n)})\right)
= B_{j,p,k}^{j,p,k}
\]

27
The semi-orthogonal decomposition we found for $D^b(E^{(n)}_j)$ in step 1 shows that for $1 \leq j < p \leq n$, for all $0 \leq k_j \leq m_j - 1$ and for all $0 \leq k_p \leq m_p - 1$:

$$\text{Hom}(B_{p,k_p}^j, C_{j,k_j}^j) = 0,$$

which is the vanishing we wanted. As a consequence, we have a semi-orthogonal decomposition:

$$D^b(X_n) = \langle A_{1,0}, \ldots, A_{1,m_1-1}, \ldots, A_{j,k_j}, \ldots, A_{n,m_n-1}, D_{X_n} \rangle.$$

The fact that $(\pi_1 \ldots \pi_n)^*D^{\text{perf}}(X) \subset D_{X_n}$ is proved easily, if one notices that for all $1 \leq j \leq n$:

$$i_j^{(n)*}((\pi_1 \ldots \pi_n)^*D^{\text{perf}}(X)) \subset q_j^*D^b(Y_k),$$

and that for all $0 \leq k_j \leq m_j - 1$:

$$\text{Hom} \left[ q_j^*D^b(Y_j), q_j^*D^b(Y_j) \otimes O_{E_j^{(n)}} \left( (m_j - k_j)E_j^{(n)} + \sum_{t=j+1}^n m_tE_t^{(n)} \right) \right] = 0,$$

since $q_j^*D^b(Y_j) \subset D_{E_j^{(n)}}$. This concludes Step 2 and the recursive proof of the proposition. \[\blacktriangle\]
4 Conclusion: Minimality and further existence results

4.1 Minimality for categorical resolutions

In this section we will discuss minimality of categorical crepant resolutions of singularities in some special settings. In [Kuz08], Kuznetsov conjectures the following:

**Conjecture 4.1.1** Let $X$ be an algebraic variety with rational Gorenstein singularities. Let $\mathcal{I} \to D^b(X)$ be a categorical strongly crepant resolution of $X$. Then, for any other categorical resolution $\mathcal{I}' \to D^b(X)$, there exists a fully faithful functor:

$$\mathcal{I} \hookrightarrow \mathcal{I}'$$

This is indeed a generalization of the Bondal-Orlov conjecture we mentioned in the introduction. Hence, it seems very interesting to look for categorical strongly crepant resolutions of singularities.

If $X$ admits a 1-step wonderful resolution of singularities $\pi: \tilde{X} \to X$, Kuznetsov relates the existence of a strongly crepant categorical resolution of singularities to the existence of a rectangular Lefschetz decomposition on the exceptional divisor of $\pi$ (see Theorem 1.0.3). It would be interesting to see if his techniques can be pushed in our context and to find more examples of varieties admitting strongly crepant categorical resolution of singularities.

**Question 4.1.2** What are the determinantal varieties which can be proved to have strongly crepant categorical resolutions of singularities?

Kuznetsov answers positively to the question for the Pfaffian $\text{Pf}_4 = \mathbb{P}(\{\omega \in \wedge^2 V, \text{rank}(\omega) \leq 4\}$, when $\dim V$ is odd (see [Kuz08], section 8).

Conjecture 4.1.1 shows that strongly crepant resolution of singularities are expected to enjoy very strong minimality properties. But this conjecture seems to be highly non trivial and very difficult to check, even with the most basic examples. Nevertheless, there is a slightly different, certainly easier to check, point of view on minimality for a resolution of singularities.

\footnote{The hard point being that, in the setting of conjecture 4.1.1 we have absolutely no clue how to construct the functor $\mathcal{I} \hookrightarrow \mathcal{I}'$.}
Definition 4.1.3 Let $X$ be an algebraic variety with Gorenstein and rational singularities. Let $\pi_1 : \tilde{X}_1 \to X$ be a resolution of singularities of $X$ and let $\delta_1 : \mathcal{F}_1 \hookrightarrow D^b(\tilde{X}_1)$ be a categorical resolutions of singularities of $X$. We say that $\mathcal{F}_1$ is weakly minimal if for any other resolution of singularities $\pi_2 : \tilde{X}_2 \to X$ with a morphism $\pi_{12} : \tilde{X}_1 \to \tilde{X}_2$ and a commutative diagram:

$$
\begin{array}{ccc}
D^b(\tilde{X}_1) \\
\pi_{12}^* \\
\downarrow \\
D^b(\tilde{X}_2) \\
\pi_1^* \\
\downarrow \\
D^b(X)
\end{array}
$$

and any other categorical resolution $\delta_2 : \mathcal{F}_2 \hookrightarrow D^b(\tilde{X}_2)$ with a commutative diagram:

$$
\begin{array}{ccc}
\mathcal{F}_1 \\
\pi_{\mathcal{F}_1, \mathcal{F}_2} \\
\downarrow \\
\mathcal{F}_2 \\
\pi_{\mathcal{F}_1} \\
\downarrow \\
D^b(X)
\end{array}
$$

with $\pi_{\mathcal{F}_1} = \pi_1 \delta_1$, $\pi_{\mathcal{F}_1, \mathcal{F}_2} = \delta_2^! \pi_{12} \delta_1$ and such that $\mathcal{F}_1$ and $\mathcal{F}_2$ are $D^{\text{perf}}(X)$-module categories, we have the implication:

$$\pi_{\mathcal{F}_1, \mathcal{F}_2}^* \text{is fully faithful} \implies \mathcal{F}_1 = \mathcal{F}_2.$$ 

Recall that given any triangulated category $\mathcal{A}$, we say that it is connected if it does not split as a sum $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$, where the $\mathcal{A}_i$ are both non trivial and totally orthogonal to each other.

Proposition 4.1.4 Let $X$ be a projective algebraic variety with Gorenstein and rational singularities. Let $\pi_1 : \tilde{X}_1 \to X$ be a resolution of $X$ and let $\delta_1 : \mathcal{F}_1 \hookrightarrow D^b(\tilde{X}_1)$ be a categorical resolution of singularities of $X$. If $\mathcal{F}_1$ is strongly crepant and connected, then $\mathcal{F}_1$ is weakly minimal.

This is a categorical generalization of the following result (which follows from the ramification formulas). Let $X$ be an algebraic variety with Gorenstein rational singularities. Let $\pi_1 : \tilde{X} \to X$ be a crepant resolution of singularities of $X$ and let $\pi_2 : Y \to X$ be any other resolution of singularities of $X$.
Assume that there is a morphism \( \pi_{12} : \tilde{X} \to Y \) which makes the following triangle commutative:

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\pi_{12}} & Y \\
\pi_1 & \downarrow & \\
X & \xleftarrow{\pi_2}
\end{array}
\]

then \( \tilde{X} = Y \).

Note that if \( T \to \text{D}^b(X) \) is a categorical strongly crepant resolution of \( X \), then there exists a full admissible connected subcategory \( T' \subset T \), such that \( T' \to \text{D}^b(X) \) is a strongly crepant resolution (see Lemme 1.3.9 of \[Abu13b\]). Finally, we do expect that the projectivity assumption should be removed, but the proof in the non-projective case seems to be quite annoying. One would need to introduce semi-orthogonal decompositions of a derived category with respect to a base scheme and prove all the basic results concerning semi-orthogonal decomposition in that setting. We leave it as an open question for the reader.

**Proof:**

Let \( \pi_2 : \tilde{X}_2 \to X \) and \( \delta_2 \hookrightarrow \text{D}^b(\tilde{X}_2) \) be another categorical resolution of singularities of \( X \) for which we have the commutative diagrams of definition 4.1.3. Since \( \pi_{\mathcal{T}_1, \mathcal{T}_2}^* \) is fully faithful and has a left adjoint, we have a semi-orthogonal decomposition:

\[
\mathcal{T}_1 = \langle (\pi_{\mathcal{T}_1, \mathcal{T}_2}^* \mathcal{T}_2) ^\perp, \pi_{\mathcal{T}_1, \mathcal{T}_2}^* \mathcal{T}_2 \rangle.
\]

Since \( \mathcal{T}_1 \) is connected, to prove that \( \mathcal{T}_1 = \mathcal{T}_2 \), we only have to prove that the above decomposition is totally orthogonal. This means that we have to prove that for all \( a \in (\pi_{\mathcal{T}_1, \mathcal{T}_2}^* \mathcal{T}_2) ^\perp \) and for all \( b \in \pi_{\mathcal{T}_1, \mathcal{T}_2}^* \mathcal{T}_2 \):

\[
\text{Hom}(a, \pi_{\mathcal{T}_1, \mathcal{T}_2}^* b) = 0.
\]
But
\[
\text{Hom}(a, \pi_{\mathcal{F}_1, \mathcal{F}_2}^* b) \\
= \text{Hom}(\pi_{\mathcal{F}_1, \mathcal{F}_2}^* b, S_{\mathcal{F}_1}(a))^\vee, \text{ by Serre duality,} \\
= \text{Hom}(\delta_1^* \pi_{\mathcal{F}_1, \mathcal{F}_2}^* b, S_{\mathcal{F}_1}(a))^\vee, \text{ by definition of } \pi_{\mathcal{F}_1, \mathcal{F}_2}^*, \\
= \text{Hom}(\pi_{\mathcal{F}_1, \mathcal{F}_2}^* b, \delta_1 S_{\mathcal{F}_1}(a))^\vee, \text{ by adjunction,} \\
= \text{Hom}(\pi_{\mathcal{F}_1, \mathcal{F}_2}^* b, \pi_{\mathcal{F}_1, \mathcal{F}_2}^* K_X \otimes \delta_1 a)^\vee, \text{ because } \mathcal{F}_1 \text{ is a strongly crepant resolution of } X, \\
= \text{Hom}(\pi_{\mathcal{F}_1, \mathcal{F}_2}^* b, \delta_1 a)^\vee, \text{ with } b' \in \mathcal{F}_2, \text{ because } \mathcal{F}_2 \text{ is a } \text{D}^{\text{perf}}(X)-\text{module category,} \\
= \text{Hom}(\pi_{\mathcal{F}_1, \mathcal{F}_2}^* b', a)^\vee, \text{ by adjunction,} \\
= 0, \text{ as } a \in (\pi_{\mathcal{F}_1, \mathcal{F}_2}^* \mathcal{F}_2)^\perp.
\]

\[\blacksquare\]

### 4.2 Existence results for prehomogeneous spaces

In section 2.1 of the present paper, we describe some varieties with Gorenstein rational singularities which have a wonderful resolution of singularities. As a consequence of our main theorem, they admit a categorical weakly crepant resolution of singularities. These examples fit very well into the theory of reductive prehomogeneous vector spaces. We recall that a reductive prehomogeneous vector space is the data \((G, V)\) of a reductive linear group \(G\) and a finite dimensional vector space \(V\) such that \(G\) acts on \(V\) with a dense orbit. For instance, the determinantal varieties defined by the minors of the generic square (resp. symmetric, resp. skew-symmetric) \(n \times n\) matrix are the orbit closures of the action of \(\text{GL}_n \times \text{GL}_n\) (resp. \(\text{GL}_n\), resp. \(\text{GL}_n\)) on \(V \otimes V\) (resp. \(S^2(V)\), resp. \(\Lambda^2 V\)). As for the affine cone over \(\mathbb{P}^2 = E_6/P_1\) and its secant variety, they are the orbit closures of the action of \(\mathbb{C}^* \times E_6\) on \(V_{\omega_1}\), where \(\omega_1\) is the weight associated to \(P_1\). So one is tempted to make the following conjecture:

**Conjecture 4.2.1** Let \((G, V)\) be a prehomogeneous vector space. Let \(Z \subset V\) be the closure of an orbit of \(G\). Assume that \(Z\) has Gorenstein rational singularities. Then \(Z\) admits a categorical weakly crepant resolution of singularities.

To investigate this conjecture in more details, one can ask the following:

**Question 4.2.2** Let \((G, V)\) be a prehomogeneous vector space. Let \(Z \subset V\) be the closure of an orbit of \(G\). When does \(Z\) admit a wonderful resolution of singularities?
The following example shows that the answer to the above question cannot be "always".

**Example 4.2.3** (Tangent variety of $\text{Gr}(3,6)$) Let $V$ be a vector space with $\dim V = 6$ and let $W = \text{Gr}(3, V) \subset \mathbb{P}(\bigwedge^3 V)$ be the Grassmannian of $\mathbb{C}^3 \subset V$ inside its Plücker embedding. We can decompose $\bigwedge^3 V$ as:

$$\mathbb{C} \oplus U \oplus U^* \oplus \mathbb{C},$$

where $U$ is identified with the space of $3 \times 3$ matrices, (see [LM01] section 5 for more details). We denote by $C$ the determinant on $U$, which can be seen as a map $S^3 U \to \mathbb{C}$ or as a map $S^2 U \to U^*$. We also denote by $C^*$ the determinant on $U^*$.

Let $Z$ denote the tangent variety to $W$. It is shown in [LM01] that an equation (up to an automorphism of $\mathbb{P}(\bigwedge^3 V)$) of $Z$ is:

$$Q(x, X, Y, y) = (3xy - \frac{1}{2}(X, Y))^2 + \frac{1}{3}(yC(X \otimes^3) + xC^*(Y \otimes^3)) - \frac{1}{6}(C^*(Y \otimes^2), C(X \otimes^2))$$

where $(.,.)$ is the standard pairing between $U$ and $U^*$. The partial derivatives of $Q$ give the equations of the variety of "stationary secants" to $W$, which we denote by $\sigma^+(W)$. A simple computation of the Taylor expansion of $Q$ shows that the variety $\sigma^+(W)$ is singular precisely along $W$, but contrary to what is claimed in proposition 5.10 of [LM01], $W$ is not defined by all the second derivatives of $Q$. The orbit closures structure of the action of $\text{SL}_6$ on $\mathbb{P}(\bigwedge^3 V)$ is the following:

$$W \subset \sigma^+(W) \subset Z \subset \mathbb{P}(\bigwedge^3 V).$$

As a consequence, the only "natural" procedure to get a wonderful resolution of singularities of $Z$ would be to consider the blow-up of $Z$ along $W$:

$$\pi_1 : Z_1 \to Z$$

and then the blow-up of $Z_1$ along the strict transform of $\sigma^+(W)$:

$$\pi_2 : Z_2 \to Z_1.$$ 

One can check that the proper transform of $\sigma^+(W)$ under $\pi_1$, the exceptional divisor of $\pi_2$ and the variety $Z_2$ are smooth. But an easy computation shows that the tangent cone of $Z$ along any point of $W$ is a double hyperplane, so

\[\text{One would of course like this resolution to be also } \text{SL}_6\text{-equivariant. So we must start with the blow-up of a } \text{SL}_6\text{-invariant subvariety.} \]
that the exceptional divisor $E_1$ of the blow-up of $Z$ along $W$ is globally non reduced. The strict transform of $E_1$ after any normally flat blow-up of $Z_1$ will still be globally non reduced. As a consequence the above sequence of blow-ups will never produce a wonderful resolution of singularities of $Z$.

Now, consider $p = (p_0, P_0, P_1, p_1)$ a generic point of $\mathbb{P}(\bigwedge^3 V)$ and let $\mathcal{P}(Q, p)$ be the polar equation of $Q$ with respect to $p$, that is:

$$\mathcal{P}(Q, p) = p_0 \frac{\partial Q}{\partial x} + P_0 \frac{\partial Q}{\partial X} + P_1 \frac{\partial Q}{\partial Y} + p_1 \frac{\partial Q}{\partial y}$$

It is easily noticed that the cubic hypersurface (which we also denote by $\mathcal{P}(Q, p)$) defined by this equation is smooth and it contains $\sigma_+(W) = Z_{\text{sing}}$. For any $w \in \sigma_+(W) - W$, the tangent space of $\mathcal{P}(Q, p)$ at $w$ is transverse to the tangent cone to $Z$ at $w$, so that the tangent cone to $\mathcal{P}(Z, p) = \mathcal{P}(Q, p) \cap Z$ at $w$ is a cone over a smooth quadric of dimension 4 with vertex the embedded tangent space to $\sigma_+(W)$ at $w$. For any $w \in W = \text{Gr}(3, 6)$, the tangent space of $\mathcal{P}(Q, p)$ at $w$ is equal to the reduced tangent cone to $Z$ at $w$. Thus, looking at the Taylor expansion of $Q$ at $w$, one can prove that the tangent cone to $\mathcal{P}(Z, p)$ at $w$ is the secant variety of a cone over $\mathbb{P}^2 \times \mathbb{P}^2$ (this cone over $\mathbb{P}^2 \times \mathbb{P}^2$ being the set of $C^3 \subset V$ which intersect $w$ in dimension at least 2). The vertex of this cone is the embedded tangent space to $\text{Gr}(3, 6)$ at $w$ and this cone is singular precisely along the cone over $\mathbb{P}^2 \times \mathbb{P}^2$ (see [LM01] for instance).

Note that the tangent cone to $\mathcal{P}(Z, p)$ at any point $w \in \text{Gr}(3, 6)$ does not depend on the choice of a generic $p \in \mathbb{P}(\bigwedge^3 V)$, as predicted by the theory of Lê-Teissier (see [Abu11], section 2.2 for some recollections on the theory of Lê-Teissier in the setting of projective geometry or [LT88] and [Tei82] for the theory in its general setting). The above description of the tangent cones of $\mathcal{P}(Z, p)$ along its various strata shows that if one considers the blow-up of $\mathcal{P}(Z, p)$ along $W$:

$$\pi_1: \mathcal{P}_1 \to \mathcal{P}(Z, p),$$

and then the blow-up of $\mathcal{P}_1$ along the strict transform of $\sigma_+(W)$ through $\pi_1$:

$$\pi_2: \mathcal{P}_2 \to \mathcal{P}_1,$$

then one gets a wonderful resolution of singularities of $\mathcal{P}(Z, p)$.

The above example suggests the following conjecture.

**Conjecture 4.2.4** Let $(G, V)$ be a prehomogeneous vector space. Let $Z \subset V$ be the closure of an orbit of $G$. There is an integer $d \leq \dim V$ such that for a generic $L \in G(d, \dim V)$, the polar $\mathcal{P}(Z, L)$ contains $Z_{\text{sing}}$ and admits a wonderful resolution of singularities.
Finally, let us mention that [Del11] undergoes a thorough study of a possible homological projective dual of $\text{Gr}(3, V) \subset \mathbb{P}(\Lambda^3 V)$ for $\dim V = 6$. Such a homological dual is expected to be a categorical crepant resolution of singularities of the double cover of $\mathbb{P}(\Lambda^3 V^*)$ ramified along the projective dual of $\text{Gr}(3, V)$ (which is equal to the tangent variety of $\text{Gr}(3, V^*) \subset \mathbb{P}(\Lambda^3 V^*)$). However, from [Del11], it is not clear that a categorical crepant resolution of this double cover does exist. Nevertheless, we strongly believe that the existence of such a categorical crepant resolution should be linked to the existence of a categorical crepant resolution of the dual variety of $\text{Gr}(3, V)$, which in turn should be linked to the wonderful resolution of its generic polar. We come back to this circle of questions in [Abu13a].
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