Thermodynamics of doubly charged CGHS model and $D1 - D5 - KK$ black holes of IIB supergravity

Youngjai Kiem(a)*, Chang-Yeong Lee(b)†, and Dahl Park(c)‡

(a) School of Physics, KIAS, Seoul 130-012, Korea
(b) Department of Physics, Sejong University, Seoul 133-747, Korea
(c) Center for Theoretical Physics, Seoul National University, Seoul 151-742, Korea

Abstract

We study the doubly charged Callan-Giddings-Harvey-Strominger (CGHS) model, which has black hole solutions that were found to be $U$-dual to the $D1-D5-KK$ black holes of the IIB supergravity. We derive the action of the model via a spontaneous compactification on $S^3$ of the IIB supergravity on $S^1 \times T^4$ and obtain the general static solutions including black holes corresponding to certain non-asymptotically flat black holes in the IIB supergravity. Thermodynamics of them is established by computing the entropy, temperature, chemical potentials, and mass in the two-dimensional setup, and the first law of thermodynamics is explicitly verified. The entropy is in precise agreement with that of the $D1 - D5 - KK$ black holes, and the mass turns out to be consistent with the infinite Lorentz boost along the $M$ theory circle that is a part of the aforementioned $U$-dual chain.

* ykiem@kiasph.kaist.ac.kr
† leecy@phy.sejong.ac.kr
‡ dpark@ctp.snu.ac.kr
I. INTRODUCTION

The discrete light-cone quantization is increasingly becoming of importance as we gain more understanding of the matrix theory formulation of M theory [1]. If we lift the ten-dimensional type IIA D-particle solutions into the eleven dimensions and take the infinite Lorentz boost along the M theory circle a la Seiberg [2], we get the description of the gravitons traveling along a discrete light-cone [3]. The same consideration for the M theory compactified on a p-torus with $p = 2, 3, 4$ yields an Anti de Sitter space-time tensored with a sphere [4]. In the case of $p = 5$, the main object of interest is D1-D5 bound states [2]. In this context, the infinitely boosted version of the D1-D5-KK black holes of the IIB supergravity on $S^1 \times T^5$ along the M theory circle is expected to play a role [1]. From the point of view of the black hole physics, the same D1-D5-KK black holes of the five-dimensional IIB supergravity is the prime example where we have an analytic handle over their microscopic quantum description [5]. The dimensionally uplifted into eleven dimensions, infinitely boosted, and further dimensionally reduced versions of the original D1-D5-KK solutions, with appropriate U-dual transformations, are the non-asymptotically flat solutions corresponding to the deletion of one in the harmonic function produced by the fivebrane charge [3]. Thus, the space-time geometry is essentially that of the near-horizon part of the D1-D5-KK black holes [3]. Upon further S-dual transformation, these solutions become the doubly charged Callan-Giddings-Harvey-Strominger (CGHS) black holes [7], establishing a U-duality between black holes in differing space-time dimensions [8].

Even at the classical level, the analysis of the thermodynamics for non-asymptotically flat solutions needs a careful treatment, making the thermodynamic analysis of the non-asymptotically flat five-dimensional IIB black holes difficult. However, the aforementioned U-dual relationship implies that we can tackle the same problem in a much simpler setting of the two-dimensional dilaton gravity [8] of the CGHS type, and that is what we achieve in this paper. In Section II, we start from spontaneously compactifying further on $S^3$ the IIB supergravity bosonic action compactified on $S^1 \times T^4$. When there is a non-vanishing
magnetic field produced by NS-fivebranes (D-fivebranes before the $S$-dual transformation) wrapped along the $S^1 \times T^4$, the spontaneous compactification produces a two-dimensional cosmological constant inversely proportional to the NS field strength. The resulting dilaton-gravity sector of the two-dimensional theory precisely becomes that of the CGHS model. In the two-dimensional action, in addition, there are two $U(1)$ gauge fields produced by the fundamental string wrapping along the $S^1$ ($D$-string wrapping number before $S$-dual transformation) and the Kaluza-Klein momentum along the same circle, as well as two scalar fields representing the size of the circle and the four-torus. The two-dimensional action shows the manifest $T$-dual invariance when we take the $T$-dual transformation along the circle. In Section III, we solve the static equations of motion of the two dimensional model in a conformal gauge to get the general static solutions including all black hole and naked singularity solutions. These black holes carry the aforementioned two $U(1)$ charges as well as mass, and we demonstrate the no-scalar hair property for the scalar fields. We note that the original fivebranes charge is traded to become the cosmological constant of the two-dimensional theory under the spontaneous compactification. By transforming these solutions to a radial gauge, we find that the black hole solutions of the two-dimensional theory are the non-asymptotically flat solutions of the IIB supergravity that were found to be $U$-dual to the $D1$-$D5$-$KK$ black hole solutions. From the two-dimensional point of view, however, they are asymptotically flat and the usual thermodynamic analysis applies. In Section IV, we explicitly evaluate the thermodynamic quantities of the black holes such as entropy, mass, temperature, and chemical potentials via the two-dimensional analysis, and explicitly verify the first law of thermodynamics. For the entropy formula, we find an agreement with the entropy of the $D1$-$D5$-$KK$ black holes. This is consistent with the statement that these two black holes are $U$-dual to each other, since the entropy is a $U$-dual invariant quantity. Similarly, we find a (non-extremal) mass formula that is consistent with the infinite Lorentz boost along the $M$ theory circle as prescribed by Seiberg [2] [3]. We also comment on the particularly simple form of the mass formula in this section. In section V, we discuss our results and comment on a related model of McGuigan, Nappi and Yost
The doubly charged CGHS model that we derive in this paper is an interesting model theory of gravity whose black holes have a non-vanishing entropy in the extremal limit.

II. LOWER DIMENSIONAL SETUP OF THE PROBLEM

In this section, we reformulate the problem of five-dimensional IIB black holes in terms of the two-dimensional black holes. Let us consider a IIB black hole on $M_5 \times S^1 \times T^4$, where $D$ five-branes wrap the five-torus $S^1 \times T^4 Q_5$ times, $D$-strings wrap the circle $S^1$ $Q_2$ times, and we have the Kaluza-Klein (KK) momentum $Q$ along the circle $S_1$. Here $M_5$ is the non-compact five-dimensional space-time. This black hole has been shown in Ref. [3] to be $U$-dual to the doubly charged CGHS black hole. The precise chain of the dual transformations are as follows. We take an appropriate number of $T$-duals to turn the $D$ five-branes into $D$-particles. We lift this solution into eleven dimensions and take an infinite boost along the $M$-theory circle following Seiberg [2]. After dimensionally reducing it to ten dimensions and taking $T$-duals back, we further take an $S$-dual transformation. In the resulting space time geometry, $M_5$ gets spontaneously compactified to yield $M_2 \times S^3$ where the three-sphere $S^3$ has a constant radius and $M_2$ is a two dimensional manifold. At the same time, the size of the $T^4$ becomes a constant. Now, the $(2 + 1)$-dimensional part of the space-time $M_2 \times S^1$ turns out to be a black string solution, which, upon the compactification along the $S^1$, becomes a doubly charged CGHS black hole.

Technically speaking, the aforementioned analysis was done at the level of classical solutions. What we want to do in this section is to follow the same prescription at the level of the classical action. The ten-dimensional IIB supergravity action with the non-vanishing RR two-form gauge field $B$, graviton $g^{(10)}_{\mu\nu}$, and the dilaton $\phi^{(10)}$ is given by

$$I = \frac{1}{16\pi(8\pi^6)} \int d^{10}x \sqrt{-g^{(10)}} \left( e^{-2\phi^{(10)}} R^{(10)} + 4e^{-2\phi^{(10)}} (D\phi^{(10)})^2 - \frac{1}{12} H^{(10)^2} \right),$$

where the field strength $H^{(10)} = dB$. We use a unit where $\alpha' = 1$, the string coupling $g \equiv \exp(\phi^{(10)})$, and the ten-dimensional Newton’s constant $G_N^{(10)} = 8\pi^6$. After the taking an $S$-dual transformation, we get
\[ I = \frac{1}{16\pi (8\pi^6)} \int d^{10}x \sqrt{-g^{(10)}e^{2\phi^{(10)}}} \left( R^{(10)} + 4(D\phi^{(10)})^2 - \frac{1}{12} H^{(10)2} \right), \]  

(2)

where the original RR two-form gauge field changes to the NS-NS two-form gauge field. We consider the case when NS (D before the S-duality) five-branes are wrapped along the \( S^1 \times T^4 \), and we allow the fundamental string (D-string) wrapping and the momentum along the circle \( S^1 \). We thus choose the metric of the form

\[ ds^2 = g^{(6)}_{\mu\nu} dx^\mu dx^\nu + e^{2\psi} dx^m dx^m \]

where the index \( m \) ranges from six to nine and the six-dimensional metric \( g^{(6)}_{\mu\nu} \) is given by

\[ g^{(6)}_{\mu\nu} = \begin{pmatrix} g^{(5)}_{\alpha\beta} + e^{\psi_1} A_\alpha A_\beta & e^{\psi_1} A_\alpha \\ e^{\psi_1} A_\beta & e^{\psi_1} \end{pmatrix}, \]

to get the five-dimensional action

\[ I = \frac{1}{4\pi^2} \int d^5x \sqrt{-g^{(5)}e^{2\phi_1}} \left( R^{(5)} + 4(D\phi_1)^2 - (D\psi)^2 - \frac{1}{4}(D\psi_1)^2 - \frac{1}{12} H^2 - \frac{1}{4} e^{-\psi_1} F_2^2 - \frac{1}{4} e^{\psi_1} F^2 \right), \]

(3)

where the five-dimensional dilaton \( 2\phi_1 = 2\phi^{(10)} + 2\psi + \psi_1/2 \). The volume of the unit five-torus, \( (2\pi)^5 \), was multiplied to the numerical factor in front of the action. The field strength \( F = dA \) has the Kaluza-Klein momentum as its charge, the field strength \( F_{2\alpha\beta} = H_{\alpha\beta x^5} \) comes from the string wrapping along the \( S^1 \), and \( H' = H - A \wedge F_2 \) originates from the five-brane wrapping along \( S^1 \times T^4 \). The s-wave dynamics of the above five-dimensional system are summarized by the following two-dimensional action [11]

\[ I = \frac{1}{2} \int d^2x \sqrt{-g} e^{-2\phi} \left( R + 6e^{2\psi_2} + 4(D\phi)^2 - (D\psi_1)^2 - \frac{1}{4}(D\psi_1)^2 - 3(D\psi_2)^2 \right) - \frac{1}{2} e^{6\psi_2} H_0^2 - \frac{1}{4} e^{-\psi_1} F_2^2 - \frac{1}{4} e^{\psi_1} F^2, \]

(4)

where we set the five-dimensional metric as

\[ ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta + e^{-2\psi_2} d\Omega_{(3)} \]

5
and we require that all fields are independent of the angular coordinates. The volume of the unit three-sphere, \(2\pi^2\), was multiplied to the numerical factor in front of the action. Here, \(d\Omega_3\) is the metric on the standard unit three-sphere \(S^3\) and \(g_{\alpha\beta}\) is the two dimensional metric. In fact, \(H'_{ijk} = H_0\epsilon_{ijk}\) and \(H_0\) is a zero-form field strength on the two-dimensional space time, since \(H'\) is proportional to the volume form \(\epsilon_{ijk}\) of the three-sphere. As such, it satisfies the Bianchi identity \(\partial_{\alpha}H_0 = 0\) which forces it to be a constant. This number is proportional to the five-brane wrapping number along \(S^1 \times T^4\). The two dimensional dilaton field \(\phi\) is given by \(2\phi = -2\phi_1 + 3\psi_2\).

For a generic non-degerate \(\psi_2\), i.e., \(d\psi_2 \neq 0\), the action Eq. (4) simply describes the bosonic s-wave sector of the five non-compact dimensional IIB supergravity. Our main interest in this paper however is to concentrate on the degenerate case \((d\psi_2 = 0)\) where we have a spontaneous compactification on the three-sphere \(S^3\) of the five-manifold, as explained earlier in this section. For this purpose, we consider the equation of motion for \(\psi_2\) field

\[
2e^{2\phi}D^\alpha(e^{-2\phi}D_\alpha \psi_2) + 4e^{2\psi_2} - H_0^2e^{6\psi_2} = 0.
\]

(5)

For a constant \(\psi_2\), Eq. (5) implies

\[
H_0^2 = 4e^{-4\psi_2}.
\]

(6)

In this case, other equations of motion are summarized by the following two-dimensional action

\[
I = \frac{1}{2} \int d^2x \sqrt{-g}e^{-2\phi}(R + 4(D\phi)^2 + 4\Lambda - (D\psi)^2 - \frac{1}{4}(D\psi_1)^2 - \frac{1}{4}e^{-\psi_1}F_2^2 - \frac{1}{4}e^{\psi_1}F_2^2)
\]

(7)

which can also be obtained from Eq. (4) by plugging Eq. (6) in. The constant radius \(R_{S^3}\) of the three-sphere \(S^3\) and the two-dimensional cosmological constant \(\Lambda\) are related to \(H_0\) as follows.

\[
\Lambda = \frac{1}{R_{S^3}^2} = \frac{2}{|H_0|} = e^{2\psi_2}.
\]

(8)

Now we note that the action Eq.(7) is precisely that of the two-dimensional CGHS model coupled with two \(U(1)\) gauge fields, the doubly charged CGHS model. We note that the
exponential of the scalar field $\psi_1$ measures the radius of the circle $S^1$. We have a manifest $T$-dual invariance; under the $T$-dual transformation along the circle $S^1$, that is implemented in our context by the transformation $\psi_1 \to -\psi_1$, the winding number of the fundamental string (which produces the non-vanishing gauge field strength $F$) gets interchanged with the Kaluza-Klein momentum number (which produces the non-vanishing gauge field strength $F$).

As far as the static solutions are concerned, the four-torus size, measured by the exponential of the scalar field $\psi$, becomes a constant if we try to avoid naked singularity solutions as we will show in the next section. This is consistent with the two-dimensional version of the no-scalar-hair theorem.

### III. GENERAL STATIC SOLUTIONS

The classical static equations of motion of the two-dimensional action Eq. (7) can be completely solved by resorting to the method of [11]. After the rescaling of the metric $g_{\alpha\beta} \to e^{2\phi} g_{\alpha\beta}$, the action Eq. (7) becomes

$$I = \frac{1}{2} \int d^2 x \sqrt{-g} e^{-2\phi} \left( R + 4\Lambda e^{2\phi} - (D\psi)^2 - \frac{1}{4} e^{-\psi_1 - 2\phi} F_2^2 - \frac{1}{4} e^{\psi_1 - 2\phi} F_2^2 \right). \tag{9}$$

To follow the method of [11], we choose a conformal gauge

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -\exp(2\rho) dx^+ dx^- \tag{10}$$

The equations of motion from the action Eq. (9) are as follows; by varying Eq. (9) with respect to the two-dimensional dilaton field $\phi$, we have

$$4\partial_+ \partial_- \rho + 2\partial_+ \psi \partial_- \psi + \frac{1}{2} \partial_+ \psi_1 \partial_- \psi_1 + 2e^{-\psi_1 - 2\phi} \Omega F_2^2 + 2e^{\psi_1 - 2\phi} \Omega F_2^2 = 0 \tag{11}$$

where $\Omega = e^{-2\phi}$. From the variation with respect to the conformal factor $\rho$ of the metric, we have

$$2\partial_+ \partial_- \Omega + 2\Lambda e^{2\rho} - e^{-\psi_1 - 2\phi} \Omega^2 F_2^{2+} - e^{\psi_1 - 2\phi} \Omega^2 F_2^{2-} = 0 \tag{12}$$

and the equations of motion for the scalar fields become
\[ \partial_+ (\Omega \partial_- \psi) + \partial_- (\Omega \partial_+ \psi) = 0, \]  
\[ (13) \]

and
\[ \partial_+ (\Omega \partial_- \psi_1) + \partial_- (\Omega \partial_+ \psi_1) + 2e^{-\psi_1-2\rho} \Omega^2 F_{2+}^2 - 2e^{\psi_1-2\rho} \Omega^2 F_{2-}^2 = 0. \]  
\[ (14) \]

We also have the equations for the two U(1) gauge fields
\[ \partial_+ (e^{-\psi_1-2\rho} \Omega^2 F_{2+}) = \partial_- (e^{-\psi_1-2\rho} \Omega^2 F_{2-}) = 0, \]  
\[ (15) \]

and
\[ \partial_+ (e^{\psi_1-2\rho} \Omega^2 F_{2-}) = \partial_- (e^{\psi_1-2\rho} \Omega^2 F_{2+}) = 0. \]  
\[ (16) \]

The equations of motion, Eqs \((11)-(16)\), should be supplemented with the gauge constraints resulting from the choice of a conformal gauge
\[ T_{++} = T_{--} = 0, \]  
\[ (17) \]

where \(T_{\pm \pm}\) is the \(\pm \pm\) components of the stress-energy tensor.

Restricting our attention only to the static solutions, we require that all fields depend only on a space-like coordinate \(x \equiv x_+ - x_-\). We also make a convenient gauge choice for the U(1) gauge fields by introducing two functions \(A(x)\) and \(A_2(x)\)
\[ A_\pm = \frac{1}{2} A(x), \quad A_{2\pm} = \frac{1}{2} A_2(x). \]  
\[ (18) \]

The static equations of motion are then summarized by a one-dimensional action
\[ I = \int dx \left( 4\Omega \rho' + 2\Lambda e^{2\rho} - 2\Omega \psi'^2 - \frac{1}{2} \Omega \psi'^2 + e^{-\psi_1-2\rho} \Omega^2 A_2' + e^{\psi_1-2\rho} \Omega^2 A_2' \right) \]  
\[ (19) \]

where the prime represents the differentiation with respect to \(x\). The gauge constraints Eq.\((17)\) become a single condition
\[ \Omega'' - 2\rho' \Omega' + \Omega \psi'^2 + \frac{1}{4} \Omega \psi'^2 = 0. \]  
\[ (20) \]
To integrate the equations from the static action Eq. (19) under the gauge constraint Eq. (20) for six unknown functions, we observe that there exist the following six symmetries of the action Eq. (19)

(a) $\psi \rightarrow \psi + \alpha,$
(b) $\psi_1 \rightarrow \psi_1 + \alpha, \ A_2 \rightarrow e^{\alpha/2}A_2, \ A \rightarrow e^{-\alpha/2}A,$
(c) $A_2 \rightarrow A_2 + \alpha,$
(d) $A \rightarrow A + \alpha,$
(e) $x \rightarrow x + \alpha,$
(f) $x \rightarrow e^{\alpha}x, \ \Omega \rightarrow e^{\alpha}\Omega, \ e^{2\rho} \rightarrow e^{-\alpha}e^{2\rho}, \ A_2 \rightarrow e^{-\alpha}A_2, \ A \rightarrow e^{-\alpha}A,$

where $\alpha$ is an arbitrary real parameter of each continuous transformation. Therefore, the integration of the second order differential equations once to get the first order differential equations is straightforward. The results are summarized by the following Noether charge expressions.

(a) $\psi_0 = \Omega \psi'$
(b) $\psi_{10} = \Omega \psi_1' - e^{-\psi_1 - 2\rho} \Omega^2 A_2' A_2 + e^{\psi_1 - 2\rho} \Omega^2 A' A,$
(c) $Q_2 = e^{-\psi_1 - 2\rho} \Omega^2 A_2',$
(d) $Q = e^{\psi_1 - 2\rho} \Omega^2 A',$
(e) $c_0 = \Omega' \rho' - \frac{1}{2} \Omega \psi'' - \frac{1}{8} \Omega \psi_1'' - \frac{1}{2} \Lambda e^{2\rho} + \frac{1}{4} e^{-\psi_1 - 2\rho} \Omega^2 A_2'^2 + \frac{1}{4} e^{\psi_1 - 2\rho} \Omega^2 A'^2,$
(f) $s + c_0 x = -\frac{1}{2} \Omega' + \rho' \Omega - \frac{1}{2} e^{-\psi_1 - 2\rho} \Omega^2 A_2' A_2 - \frac{1}{2} e^{\psi_1 - 2\rho} \Omega^2 A' A.$

The gauge constraint Eq. (20) and the equation of motion for $\rho$ Eq. (12) give $c_0 = 0,$ thereby reducing the number of constants of motion from six to five. The derivation of the solutions is most transparent under the introduction of a following set of field redefinitions and a coordinate change

$$\bar{\Omega} = e^{\psi_1/2} \Omega, \ e^{2\bar{\rho}} = e^{-\psi_1/2} e^{2\rho}, \ d\bar{x} = e^{\psi_1/2} dx.$$
In terms of these redefined field, we can rewrite Eqs. (21) - (24) and Eq. (26) as follows.

\[ \psi_0 = \bar{\Omega} \dot{\psi} \] (28)

\[ \psi_{10} = \bar{\Omega} \dot{\psi}_1 - Q_2 A_2 + QA, \] (29)

\[ Q_2 \dot{A} = Q e^{-2\psi_1} \dot{A}_2, \] (30)

\[ Q = e^{-2\bar{\Omega}^2} \dot{A}, \] (31)

\[ \bar{s} = -\frac{1}{2} \ddot{\bar{\Omega}} + \ddot{\bar{\rho}} \bar{\Omega} - QA, \] (32)

where \( \bar{s} = s - \psi_{10}/2 \) and the overdot represents the differentiation with respect to \( \bar{x} \). Combining Eqs. (31) and (32), we find

\[ (2\bar{s} + 2QA) \dot{A} = \frac{d}{d\bar{x}}(Q e^{2\bar{\rho} \bar{\Omega}^{-1}}), \]

which, upon integration, becomes

\[ Q e^{2\bar{\rho} \bar{\Omega}^{-1}} = QA^2 + 2sA + c \equiv P(A), \] (33)

where we introduce a function \( P(A) \) and \( c \) is a constant of integration. Putting Eq. (33) into Eq. (31), we get

\[ \frac{\ddot{\bar{\Omega}} dA}{d\bar{x}} = P(A) \quad \rightarrow \quad \frac{\ddot{\bar{\Omega}} dA}{dA} = P(A) \frac{d}{dA}, \] (34)

By changing the differentiation variable from \( \bar{x} \) to \( A \) with the help of Eq. (34), we immediately find that Eq. (28) can be integrated to yield

\[ \psi = \psi_0 \int \frac{dA}{P(A)} + \psi_c \] (35)

where \( \psi_c \) is the constant of integration. In a similar way, we can rewrite Eq. (29) and Eq. (30) as

\[ P(A) \frac{d\psi_1}{dA} + QA - Q_2 A_2 = \psi_{10} \] (36)

and
\[ Q \frac{dA_2}{dA} = Q_2 e^{2\psi_1}. \]  

(37)

Differentiating Eq. (36) with respect to \( A \) and using Eq. (37), we get

\[ \frac{d}{dA} \left( P(A) \frac{d\psi_1}{dA} \right) + Q - \frac{Q_2^2}{Q} e^{2\psi_1} = 0. \]  

(38)

By setting

\[ \psi_1 = -\frac{1}{2} \ln |P(A)| + \hat{\psi}_1, \]

and introducing a new variable

\[ \hat{A} = \int \frac{dA}{P(A)} \]

we find that Eq. (38) is a one-dimensional classical Liouville equation

\[ \frac{d^2}{d\hat{A}^2} \hat{\psi}_1 - \frac{Q_2^2}{|Q|} e^{2\hat{\psi}_1} = 0 \]

which can be exactly solved to give

\[ e^{2\psi_1} = \frac{|c_1|}{Q_2^2 P(A)} \frac{1}{\sinh^2 \left[ \sqrt{c_1} \left( \int \frac{dA}{P(A)} + \tilde{c}_1 \right) \right]} \]  

(39)

where \( c_1 \) and \( \tilde{c}_1 \) are constants of integration. We can find \( A_2 \) using Eq. (36) and Eq. (39)

\[ Q_2 A_2 = -\psi_{10} - \bar{s} - \sqrt{c_1} \coth \left[ \sqrt{c_1} \left( \int \frac{dA}{P(A)} + \tilde{c}_1 \right) \right]. \]  

(40)

We express Eq. (25) in terms of the redefined fields Eq. (27), change variable from \( x \) to \( \hat{A} \), and plug in Eq. (35), Eq. (39) and Eq. (40) to get

\[ \left( \frac{d\hat{\phi}}{d\hat{A}} \right)^2 - \frac{\Lambda}{|Q|} e^{2\hat{\phi}} - D_2 = 0 \]  

(41)

where \( e^{\hat{\phi}} = \frac{\hat{\Omega}}{P} \frac{1}{|Q|} \frac{1}{\sinh^2 \left[ \sqrt{D_2} \left( \int \frac{dA}{P(A)} + \tilde{c} \right) \right]} \). This equation Eq. (41) is the first integration of the one-dimensional classical Liouville equation where \( D_2 \) plays the role of the constant of integration. By integrating Eq. (41) we get

\[ \bar{\Omega}^2 = \frac{|D_2|}{\Lambda} \frac{Q}{P(A)} \frac{1}{\sinh^2 \left[ \sqrt{D_2} \left( \int \frac{dA}{P(A)} + \tilde{c} \right) \right]} \]  

(42)
where $\tilde{c}$ is a constant of integration. Finally, we can determine $A$ in terms of $x$ using Eq. (34)

$$\begin{align*}
x - x_0 &= \int \frac{\Omega}{P(A)} dA.
\end{align*}$$

Eqs. (33), (35), (39), (40), (42) and (43) are the general static solutions of the equations of motion. We have eleven parameters for the general static solutions of the action Eq. (7).

Among possible choices of the parameters we restrict our attention only to the case when $c_1 > 0$, $\bar{s}^2 - Qc \geq 0$, $c/Q > 0$, and $\bar{s} > 0$, which, as we will see shortly, corresponds to the case considered in [3]. Under these conditions, we further choose the following radial coordinate $r$ and the time coordinate $t$.

$$r^2 = -\frac{r_0^2}{2} \left[ \coth(\sqrt{D} \hat{A}(A) + \tilde{c}) - 1 \right], \quad t = x^+ + x^-$$

where $r_0^2 = 2\sqrt{D}Q/c$. In terms of these $(t, r)$ coordinates, the ten-dimensional metric in the radial gauge and the dilaton field can be written as follows.

$$ds^2 = \frac{f_1(r)}{Z_1(r)} \left[ -\beta^2 dt^2 + d\theta^2 + f_2(r)(\beta \cosh \sigma dt \pm \sinh \sigma d\theta)^2 \right] + \frac{\Lambda^{-1} dr^2}{r^2 Z_0(r)} + \Lambda^{-1} d\Omega^2 + Z_\psi dx^m dx^m,$$

$$e^{-2\phi} = \frac{\Lambda^{-1}}{r^2 Z_1(r)}$$

where we introduce five functions

$$f_1(r) = \frac{\sqrt{Q}\c}{s + \sqrt{s^2 - Qc}} e^{\sqrt{s^2 - Qc}} \left( 1 - \frac{r_0^2}{r^2} \right)^{(1 - \sqrt{Q}/D_2)D_2/2},$$

$$f_2(r) = 1 - \frac{s + \sqrt{s^2 - Qc}}{s - \sqrt{s^2 - Qc}} e^{-2\sqrt{s^2 - Qc}} \left( 1 - \frac{r_0^2}{r^2} \right)^{(s^2 - Qc)/D_2},$$

$$Z_0(r) = 1 - \frac{r_0^2}{r^2},$$

$$Z_1(r) = \sqrt{\frac{cQ^2}{4c_1Q}} e^{\sqrt{c_1} (\tilde{c} - \tilde{\tilde{c}})} \left( 1 - \frac{r_0^2}{r^2} \right)^{(1 - \sqrt{c_1}/D_2)D_2/2} - e^{-\sqrt{c_1} (\tilde{c} - \tilde{\tilde{c}})} \left( 1 - \frac{r_0^2}{r^2} \right)^{(1 + \sqrt{c_1}/D_2)D_2/2}.$$
\[
Z_\psi = e^{\psi_0 - \psi_0^2} \left( 1 - \frac{r_0^2}{r^2} \right)^{\psi_0/(2D_2)},
\]

and two parameters
\[
\beta = \sqrt{\frac{c}{4Q}}, \quad \tanh \sigma = -\frac{c}{|c|} \frac{\bar{s} - \sqrt{s^2 - Qc}}{\sqrt{Qc}}.
\]

The corresponding two-dimensional metric and two-dimensional dilaton field are
\[
d s^2 = -\frac{Z_0(r)}{Z_1(r)Z_2(r)} \beta^2 dt^2 + \frac{\Lambda^{-1}}{r^2} \frac{dr^2}{Z_0(r)} \tag{47}
\]
\[
e^{-2\phi} = \Lambda^{-1/2} r^2 \sqrt{Z_1(r)Z_2(r)} \tag{48}
\]

where
\[
Z_2(r) = \sqrt{\frac{Qc}{4(s^2 - Qc)}} \left| e^{\tilde{c} \sqrt{s^2 - Qc}} \left( 1 - \frac{r_0^2}{r^2} \right)^{(1 - \sqrt{(s^2 - Qc)/D_2})/2}
\right.
\]
\[
- e^{-\tilde{c} \sqrt{s^2 - Qc}} \left( 1 - \frac{r_0^2}{r^2} \right)^{(1 + \sqrt{(s^2 - Qc)/D_2})/2}
\]

As was shown in [11], we have the black hole solutions when following two conditions are met
\[
\psi_0 = 0, \quad D_2 = c_1 \rightarrow D_2 = c_1 = s^2 - Qc, \tag{49}
\]

and, otherwise, the solutions have a naked singularity. Therefore, for the black hole solutions, the size of the four-torus is a fixed value (zero charge) and the scalar charge of the circle size field \(\psi_1\) is determined by other constants of motion. This is basically the no-scalar-hair theorem [11].

To see that our black hole solutions indeed represent the solutions of Ref. [3], we make further restrictions on parameters such as
\[
\frac{\bar{s} + \sqrt{s^2 - Qc}}{\bar{s} - \sqrt{s^2 - Qc}} = e^{2\tilde{c} \sqrt{s^2 - Qc}} \tag{50}
\]
\[
2 \frac{|Q_2|}{r_0^2} \sinh (\sqrt{c_1}(\tilde{c} - \tilde{c}_1)) = 1.
\]
These two conditions set the radius of the asymptotic circle $S^1$ to be one and, as shown in Eq. (52), set the asymptotic value of the potential $A_t$ at the spatial infinity to be zero. In addition, we also set the radii of the four-torus as one by setting $\psi_c = 0$, which gives $Z_\psi = 1$ under the black hole conditions Eq. (49). Under the conditions Eqs. (49) and (50), we can also verify that $f_1(r) = 1$, $f_2(r) = r_0^2/r^2$ and

$$Z_1(r) = 1 + \frac{r_1^2}{r^2}, \quad Z_2(r) = 1 + \frac{r_2^2}{r^2}$$

where

$$r_1^2 = \sqrt{Q_2^2 + \frac{r_0^4}{4} - \frac{r_0^2}{2}}, \quad r_2^2 = \sqrt{Q^2 + \frac{r_0^4}{4} - \frac{r_0^2}{2}}.$$ (51)

The two $U(1)$ gauge fields become

$$A_t = -\frac{Q\beta}{r^2 + r_2^2}, \quad A_{2t} = -\frac{Q_2\beta}{r^2 + r_1^2}$$ (52)

where we have set $\psi_{10} = 2\beta(r_1^2 - r_2^2)$, which makes the asymptotic value of the gauge field $A_{2t}$ at the spatial infinity zero. Of the original eleven constants of integration, we imposed two conditions to avoid naked singularities, two conditions to set the asymptotic value of the potentials to be zero, and two conditions to set the asymptotic radii of $S^1$ and $T^4$ to be one, resulting $11 - 6 = 5$ parameters. Our black hole solutions are thus characterized by four parameters $(r_0, r_1, r_2, \beta)$ that will be related to the two $U(1)$ charges, the mass and the time scale choice. We note that the time translational invariance hides one extra parameter. The black hole solutions, Eqs. (45), (46) and (52), are identical to the non-asymptotically flat solutions of [3].

**IV. THERMODYNAMICS**

The analysis of the thermodynamics in the two-dimensional dilaton gravity was performed in Ref. [12]. In that reference, the mass $M$ and the entropy $S$ are given by

$$M = \frac{1}{2} \left[ e^{-2\phi} \partial_\alpha g \right]_\infty, \quad S = 2\pi \left[ e^{-2\phi} \right]_{\text{horizon}}$$ (53)
from their Eq. (2.13) in a gauge where the metric is $ds^2 = -g dt^2 + g^{-1} dx^2$. The subscript $\infty$ in the mass formula means that we have to evaluate the expression at the spatial infinity. The subscript horizon in the entropy formula means that we evaluate the expression at the black hole horizon. Under a gauge choice where the metric expression is $ds^2 = -g_1 dt^2 + g_2^{-1} dr^2$, Eq. (53) changes into

$$M = \frac{1}{2} \left[ e^{-2\phi} \sqrt{g_2 g_1} \right]_{\infty}, \quad S = 2\pi \left[ e^{-2\phi} \right]_{\text{horizon}}.$$  (54)

Under the same gauge choice, by computing the inverse of the period of the Euclidean time coordinate for the euclideanized metric, the temperature is given by

$$T = \frac{1}{4\pi} \left[ \sqrt{g_2 g_1} \right]_{\text{horizon}}.$$  (55)

As derived in Section III, the metric Eq. (47) and the dilaton field Eq. (48) of the two-dimensional doubly charged CGHS black holes are

$$ds^2 = -\left(1 - \frac{r_0^2}{r^2}\right) \left(1 + \frac{r_1^2}{r^2}\right)^{-1} \left(1 + \frac{r_2^2}{r^2}\right)^{-1} \beta^2 dt^2 + \frac{\Lambda^{-1}}{r^2} \left(1 - \frac{r_0^2}{r^2}\right)^{-1} dr^2$$  (56)

$$e^{-2\phi} = \sqrt{\Lambda^{-1}(r^2 + r_1^2)(r^2 + r_2^2)}.$$  (57)

Eqs. (56) and (57) are manifestly symmetric under the exchange $r_1 \leftrightarrow r_2$ (momentum-winding exchange) reflecting the underlying $T$-dual invariance. In the case of the doubly charged CGHS black holes, the location of the horizon is $r = r_0$. The spatial infinity is $r = \infty$ where the two-dimensional coupling $\exp(\phi)$ vanishes. Thus, via Eqs. (54) and (55), the mass, entropy and the temperature of the doubly charged CGHS black holes are computed to be

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1We note that our action Eq. (6) has a factor $1/2$ in front of it. In other words, unlike the convention of Ref. [12] where the two-dimensional Newton’s constant was set to one, the two-dimensional Newton’s constant in our case is two. Therefore, the mass and the entropy expressions in Eq. (53) are divided by two compared to Eq. (2.13) of [12].
\[ M = r_0^2 + r_1^2 + r_2^2, \quad (58) \]

\[ S = 2\pi \sqrt{\Lambda^{-1}(r_0^2 + r_1^2)(r_0^2 + r_2^2)} \quad (59) \]

and

\[ T = \frac{1}{2\pi} \frac{r_0^2}{r_0^2 + r_1^2 + r_2^2}. \quad (60) \]

Similar to the metric and the dilaton expressions, Eqs. (58) - (60) are invariant under \( r_1 \leftrightarrow r_2 \). Using Eqs. (58) - (60) and Eq. (51), we compute

\[ dM - TdS = -\left[\beta^{-1}A_1\right]_{\text{horizon}} dQ - \left[\beta^{-1}A_2t\right]_{\text{horizon}} dQ_2 \quad (61) \]

where the value of the electric potentials at the horizon are given by

\[ [A_1]_{\text{horizon}} = -\frac{Q\beta}{r_0^2 + r_2^2} \quad (62) \]
\[ [A_2t]_{\text{horizon}} = -\frac{Q_2\beta}{r_0^2 + r_1^2}. \quad (63) \]

Eq. (61) is the first law of thermodynamics for the doubly charged black holes. We note that we already set the value of the electric potentials at the spatial infinity as zero. Therefore, the coefficients of \(-dQ\) and \(-dQ_2\) in Eq. (61) are chemical potentials. We observe that the scale factor \( \beta \), which is not a gauge-invariant constant of motion, actually drops out in the chemical potentials. One interesting observation is that the Noether charges for the \( U(1) \) charges, \( Q \) and \( Q_2 \) in Eqs. (23) and (24), are the physical charges of the black holes as shown in the first law of thermodynamics Eq. (61). This two-dimensional consideration is consistent with the IIB supergravity side consideration\(^2\) in [13] where the RR electric charge is obtained by performing the integration of the gauge field strength over the three-sphere \( S^3 \) and it also turns out to be \( Q_2 \). In the same reference, the quantized Kaluza-Klein charge is also given by \( Q \).

\(^2\)Due to our choice of the constants of the integration, we have to set \( R = V = \alpha' = g = 1 \) in [13].
When there are no $U(1)$ gauge fields, the thermodynamic quantities we computed in this section reduce to the well-known results in the CGHS model [12], where the temperature is a strict constant. With the inclusion of two additional $U(1)$ gauge fields, we have an extremal limit ($r_0 \to 0$) where the temperature Eq. (60) vanishes but the entropy Eq. (59) becomes finite

$$S = 2\pi \Lambda^{-1/2} r_1 r_2 = 2\pi \sqrt{Q_5 Q_2 Q},$$

(64)

where we used Eq. (8) to relate the inverse cosmological constant to the five-brane wrapping number ($\Lambda = 2/H_0 = 1/Q_5$) [13]. This is precisely the extremal entropy of the $D1$-$D5$-$KK$ black holes of the IIB supergravity [5].

The mass formula Eq. (58), which is generically non-extremal, is very simple; it is the sum of three terms, $r_0^2$, $r_1^2$ and $r_2^2$, reminiscent of the BPS mass formula for the asymptotically flat $D1$-$D5$-$KK$ black holes of the IIB supergravity $M_{BPS} = Q + Q_2 + Q_5$. In the extremal ($r_0 \to 0$) limit, Eq. (58) reduces to $M = Q + Q_2 = M_{BPS} - Q_5$. The mass $M$, which was calculated in the two-dimensional framework, actually represents the mass of the non-asymptotically flat IIB black holes of [3]. The absence of the $Q_5$ contribution to $M$ can be understood if we consider the relationship between the asymptotically flat and non-asymptotically flat IIB black holes. Starting from the asymptotically flat $D1$-$D5$-$KK$ black holes, we take the $T$-duals that turn the $D$-fivebranes to the $D$-particles. We uplift this ten-dimensional solutions to the eleven dimensional supergravity and, in this process, the $D$-particles translate to the gravitons moving along the spatial $M$ theory circle. The infinite boost procedure of Seiberg along the $M$ theory circle maps this spatial $M$ theory circle toward a (asymptotically) light-like circle [3]. Therefore, the original time coordinate gets mapped into the (asymptotically) light-like time coordinate. Thus the energy ($E = p$) of gravitons conjugate to the original time gets mapped into the energy conjugate to the (asymptotically) light-cone time, and this new energy vanishes ($E_+ = E - p = 0$). Upon the dimensional reduction along the (asymptotically) light-like circle, we return to the ten dimensions and further $T$ duals back turn the $D$-particle charges to the $D$-fivebrane charges.
The overall effect of this process is that the original asymptotically-flat $D1$-$D5$-$KK$ black holes gets mapped into the non-asymptotically flat IIB black holes that we analyzed in terms of the doubly charged CGHS black holes (after an additional $S$-dual transformation). Thus, the contribution to the mass $M$ from the fivebranes should be absent. Indeed, this is what we found from the purely two-dimensional analysis leading to Eq. (58). An interesting point is that the mass itself is conventionally evaluated near the spatial infinity (see Eq. (53)). As far as the near-horizon geometry is concerned, the non-asymptotically flat IIB black holes and the their dual asymptotically flat IIB black holes are indistinguishable. Therefore, the agreement of the entropy expression, that is a typical near-horizon quantity, for both cases is intuitively clear. However, since two types of black holes have radically different asymptotic geometries, it is surprising that the mass formulas agree (up to the aforementioned contribution from the fivebranes) especially from the point of view of the ten-dimensional physics. In a similar vein, it was observed by Hyun that the greybody factor calculations for the doubly charged CGHS black holes and for the $D1$-$D5$-$KK$ black holes agree [14]. Considering the intuitive fact that the greybody factor encodes the accumulated contribution from the spatial infinity to the black hole horizon, it is also surprising.

To better understand the mass formula Eq. (58), we rewrite it in terms of $Q$ and $Q_2$.

$$M = \sqrt{Q_2^2 + \frac{r_0^4}{4}} + \sqrt{Q^2 + \frac{r_0^4}{4}}$$  \hspace{1cm} (65)$$

Eq. (65) formally looks like a threshold bound state energy of two particles with mass $r_0^2/2$ with the momentum $Q$ and $Q_2$, respectively. In fact, for the massive black holes ($r_0^2 \gg Q, Q_2$), we get

$$M \approx 2(\frac{r_0^2}{2}) + \frac{Q_2^2}{2(\frac{r_0^2}{2})} + \frac{Q^2}{2(\frac{r_0^2}{2})} + \cdots$$

while for the extremal black holes where $r_0$ is vanishingly small, we have $M = Q_2 + Q$, an ultrarelativistic energy-momentum relation. In fact, if the string wrapping number $Q_2 = 0$, the natural energy produced by a threshold bound state of a particle of mass $r_0^2/2$ at rest and a particle of the same rest mass moving along the circle $S^1$ with the momentum $Q$ will
be $E = (r_0^2/2) + \sqrt{Q^2 + r_0^2/4}$. Now the $T$-duality between the momentum and the winding will give Eq. (33) if there is no interaction energy (that will be symmetric under the exchange of $Q$ and $Q_2$, and vanish when one of them becomes zero), since the mass expression should be symmetric under the exchange of $Q$ and $Q_2$. Similar to the BPS mass formula where the supersymmetry ensures the absence of such interaction energy contribution, the generically non-extremal (thus, non-supersymmetric) mass expression Eq. (33) apparently has no interaction energy contribution.

V. DISCUSSIONS

In this paper, we established the thermodynamics for non-asymptotically flat five-dimensional black holes via the simpler analysis in the two-dimensional system. This way, by reliably computing the entropy, we were able to independently provide the support for the statement of [3] that $D1$-$D5$-$KK$ black holes are $U$-dual to the doubly charged CGHS black holes. Furthermore, we showed that the mass formula computed from the two-dimensional perspective reflects the infinite Lorentz boost process along the $M$ theory circle, which is a part of the $U$-dual chain [3].

As can be seen from our thermodynamic analysis, the black holes of the doubly charged CGHS model have rich but fairly simple structures that capture the essentials of the $D1$-$D5$-$KK$ black holes of the IIB supergravity. As such, it provides us with a simple toy model to better understand the space-time physics of string theory. Intimately related to the doubly charged CGHS model is the two-dimensional model of McGuigan, Nappi, and Yost

\[
I = \frac{1}{2} \int dx^2 \sqrt{-g} e^{-2\phi} \left( R + 4(D\phi)^2 - \frac{1}{4} F^2 + c \right)
\]

which contains a single $U(1)$ gauge field (see also [12]) and can be considered as the heterotic string target space effective action [10]. The thermodynamics of this model was explicitly worked out in [12], and the resulting entropy was found to be identical to the $D1$-$D5$-$KK$ black holes of the IIB supergravity in [16] when we require that the Kaluza-Klein charge
and the fundamental string winding number are the same, thereby effectively reducing the number of $U(1)$ gauge fields to one. Starting from the action of the doubly charged CGHS model, Eq. (7), we consider the case when $\psi_1 = 0$ and $F_1^2 = F^2 = \tilde{F}^2 / 2$. The imposition of these conditions are consistent with the equations of motion, and we end up recovering the action Eq.(66) with $c = 4A$. From this point of view, the McGuigan-Nappi-Yost model can also be regarded as the self-dual sector of the doubly charged CGHS model (or the type IIB supergravity) under the $T$ duality along the circle where the strings are wrapped.

By further chains of $U$-dual transformations, we can map the doubly charged CGHS black holes to the three-dimensional charged BTZ black holes [17], that is asymptotically $AdS_3$ [8] [18]. Considering this fact, the doubly charged CGHS model provides yet another way to understand the physics of the $AdS$ supergravity [19]. In fact, a qualitative similarity in this regard was noted in [20]. Just like the analysis of the uncharged CGHS model, it is sensible to introduce a time-like boundary in our two-dimensional model [21]. Furthermore, since our model is a dimensionally reduced one from the IIB supergravity, we have a better understanding of the internal dynamics of the boundary point. The dynamical investigation of the doubly charged CGHS black holes, for example by including the fundamental string probes, should give us a better understanding of the $AdS$ supergravity and five-dimensional $D1$-$D5$-$KK$ black holes in a much simpler setting of the two-dimensional gravity. We plan to address this issue in future publication.

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