Realization of Posets

Patrice Ossona de Mendez
CNRS UMR 8557
E.H.E.S.S.
54 Bd Raspail, 75006 Paris, France
http://www.ehess.fr/centres/cams/person/pom/index.html
pom@ehess.fr

Abstract.
We prove a very general representation theorem for posets and, as a corollary, deduce that any abstract simplicial complex has a geometric realization in the Euclidean space of dimension $\dim P(\Delta) - 1$, where $\dim P(\Delta)$ is the Dushnik-Miller dimension of the face order of $\Delta$. 
1 Introduction

Schnyder proved in [3] that a graph is planar if and only if its incidence poset (that is: the poset where \( x < y \) iff \( x \) is a vertex, \( y \) is an edge and \( y \) is incident to \( x \)) has dimension at most 3. That an incidence poset has dimension at most 3 implies that the corresponding graph is planar has been extended to abstract simplicial complexes in [2]: if the face order of an abstract simplicial complex \( \Delta \) is bounded by \( d + 1 \), then \( \Delta \) has a geometric realization in \( \mathbb{R}^d \). We prove here a more general result on poset representation which implies this last result straightforwardly.

We shall first recall some basic definitions from poset theory: A partially ordered set (or poset) \( P \) is a pair \((X, P)\) where \( X \) is a set and \( P \) a reflexive, antisymmetric, and transitive binary relation on \( X \). A poset is \( P = (X, P) \) is finite if its ground set \( X \) is finite. We shall write \( x \leq y \) in \( P \) or \( x \leq_P y \) if \((x, y) \in P \). Two elements \( x, y \in X \) such that \( x \leq y \) in \( P \) or \( y \leq x \) in \( P \) are said to be comparable; otherwise, they are said to be incomparable.

If \( P \) and \( Q \) are partial orders on the same set \( X \), \( Q \) is said to be an extension of \( P \) if \( x \leq y \) in \( P \) implies \( x \leq y \) in \( Q \), for all \( x, y \in X \).

A partial order (that is: a partial order in which every pair of elements are comparable) then it is a linear extension of \( P \). The dimension \( \dim P = (X, P) \) is the least positive integer \( t \) for which there exists a family \( R = (<_1, <_2, \ldots, <_t> \) of linear extensions of \( P \) so that \( P = \bigcap R = \bigcap_{i=1}^{t} <_i \). This concept has been introduced by Dushnik and Miller in [1]. A family \( R = (<_1, <_2, \ldots, <_t> \) of linear orders on \( X \) is called a realizer of \( P \) on \( X \) if \( P = \bigcap R \).

For an extended study of partially ordered sets, we refer the reader to [4].

We shall further introduce the following notation: the down-set (or filter) of a poset \( P = (X, P) \) induced by a set \( A \subseteq X \) is the set

\[
\text{Inf}(A) = \bigcap_{a \in A} \text{Inf}\{a\} = \{x \in X, \forall a \in A, x \leq a \text{ in } P\}
\]

2 The Poset Representation Theorem

Definition 2.1 Let \( P = (X, P) \) be a finite poset, \( n \) an integer and \( f : X \to \mathbb{R}^n \) a mapping from \( X \) to the \( n \)-dimensional space \( \mathbb{R}^n \).

Then \( f \) is said to have the separation property for \( P \) if, for any \( A, B \subseteq X \), there exists a hyperplane of \( \mathbb{R}^n \) which separates the points of \( f(\text{Inf}(A) \setminus \text{Inf}(B)) \) and the ones of \( f(\text{Inf}(B) \setminus \text{Inf}(A)) \), where \( \text{Inf}(Z) = \{x \in X, \forall z \in Z, x \leq_P z\} \) for any \( Z \subseteq X \).

Theorem 2.1 Let \( P = (X, P) \) be a finite poset and let \( d = \dim P \) be its dimension. Then, there exists a function \( f : X \to \mathbb{R}^{d-1} \), which satisfies the separation property for \( P \).

Proof: Let \( R = \{<_1, \ldots, <_d\} \) be a realizer of \( P \) and denote \( \min(X, <_i) \) the minimum element of set \( X \) with respect to linear order \( <_i \). Let \( F_1, \ldots, F_d \) be
functions from $X$ to $[1;+\infty]$, each $F_i$ being fast increasing with respect to $<_i$, which means that
\[ \forall x <_i y, \quad F_i(x) < dF_i(y). \]
We define the function $F : X \mapsto \mathbb{R}^d$ by $F(x) = (F_1(x), \ldots, F_d(x))$.

For any $A, B \subseteq X$ such that $\text{Inf}(B) \not\subseteq \text{Inf}(A)$, define the linear form $L_{A,B} : \mathbb{R}^{d-1} \mapsto \mathbb{R}$, as:
\[ \forall \pi = (\pi_1, \ldots, \pi_d) \in \mathbb{R}^d, \quad L_{A,B}(\pi) = \sum_{1 \leq i \leq d} \frac{\pi_i}{\min(a \in A) F_i(a)}. \]

On one hand, for any $z \in \text{Inf}(B) \setminus \text{Inf}(A)$, there exists $a \in A$ and $1 \leq i_0 \leq d$, with $z >_{i_0} a$. Then, we get $F_{i_0}(z) > dF_{i_0}(a)$. As $\min(B, <_{i_0}) \geq i_0$, $z >_{i_0} \min(A, <_{i_0})$, we obtain: $L_{A,B}(F(z)) > d$.

On the other hand, for any $z \in \text{Inf}(A)$, we have $F_i(z) \leq F_i(a)$ for every $i \in [d]$ and every $a \in A$. Thus, $L_{A,B}(F(z)) \leq d$.

Altogether, for any $A, B \subseteq X$ such that none is included in the other, the hyperplane $H_{A,B}$ with equation $L_{A,B}(\pi) - L_{B,A}(\pi) = 0$ separates the points from $F(\text{Inf}(B) \setminus \text{Inf}(A))$ (for which $L_{A,B}(F(z)) > d \geq L_{B,A}(F(z))$) and those from $F(\text{Inf}(A) \setminus \text{Inf}(B))$ (for which $L_{A,B}(F(z)) \leq d < L_{B,A}(F(z))$). Notice that the origin $O$ belongs to all the so-constructed hyperplanes.

Now, consider a hyperplane $H_0$ with equation $\sum_{1 \leq i \leq d} \pi_i = 1$, which separates the origin $O$ and the set of the images of $X$ by $F$. To each element $z$ of $X$, we associate the point $f(z)$ of $H_0$ which is the intersection of $H_0$ with the line $(O, F(z))$.

Now, for any $A, B \subseteq X$ (such that none is included in the other), as $H_{A,B}$ includes $O$, the hyperplane $H_{A,B} \cap H_0$ of $H_0$ separates the points from $F(\text{Inf}(B) \setminus \text{Inf}(A))$ and those from $F(\text{Inf}(A) \setminus \text{Inf}(B))$. As $H_0 \simeq \mathbb{R}^{d-1}$ and as the separation property would be obviously true if $A \subseteq B$ or conversely, the theorem follows.

The preceding theorem is sharp, as proved here using the standard example $S_n$ of poset of dimension $n$ (introduced in [1]):

**Theorem 2.2** For any $n \geq 3$, there exists no function $f : [n] \mapsto \mathbb{R}^{n-2}$ which satisfies the separation property for the standard example $S_n$ of poset of dimension $n$, which is the height two poset on $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$, with minima $\{a_1, \ldots, a_n\}$, maxima $\{b_1, \ldots, b_n\}$ and such that $\forall i,j, \quad (a_i < b_j) \iff (i \neq j)$.

**Proof:** Assume there exists a function $f : \{a_1, \ldots, a_n, b_1, \ldots, b_n\} \mapsto \mathbb{R}^{n-2}$ having the separation property for $S_n$.

According to Radon’s lemma, for any family of $n$ point in $\mathbb{R}^{n-2}$, there exists a bipartition $V, W$ of them, such that the convex hulls of $V$ and $W$ intersects and thus such that $V$ and $W$ cannot be separated by an hyperplane of $\mathbb{R}^{n-2}$. Let $A = \{b_i, f(a_i) \not\in V\}$ and $B = \{b_i, f(a_i) \not\in W\}$. Then, $V \subseteq f(\text{Inf}(A))$ and $W \subseteq f(\text{Inf}(B))$. Hence, the separation property fails for $A, B$. \(\square\)
From Theorem 2.1, one derives a sufficient condition for a graph to be planar, which is that its incidence poset shall be of dimension at most 3 and this condition is actually also a necessary condition:

**Theorem 2.3 (Schnyder [3])** The incidence poset \( \text{Incid}(G) \) of a graph \( G \) has dimension at most 3 if and only if \( G \) is planar, that is: if and only if there exists a mapping \( f \) from \( V(G) \cup E(G) \) to \( \mathbb{R}^2 \) having the separation property for \( \text{Incid}(G) \).

\[ \square \]

## 3 Applications

**Corollary 3.1** Let \( U \) be a finite set, and \( \mathcal{F} \) a family of subsets of \( U \) such that:

\[ \forall x, y \in U, \exists X \in \mathcal{F}, \; x \in X \text{ and } y \notin X. \quad (1) \]

Let \( d \) be the Dushnik-Miller dimension of the inclusion order \( \subset \) on \( \mathcal{F} \). Then, there exists a function \( f : U \mapsto \mathbb{R}^{d-1} \) such that (denoting \( f(A) \) the set \( \{ f(z), z \in A \} \), for \( A \subseteq U \)):

\[ \forall X \in \mathcal{F}, \; \text{Conv}(f(X)) \cap f(U) = f(X), \quad (2) \]

\[ \forall X \neq Y \in \mathcal{F}, \; \text{Conv}(f(X \setminus Y)) \cap \text{Conv}(f(Y \setminus X)) = \emptyset. \quad (3) \]

**Proof:** Equation (3) is a direct consequence of Theorem 2.1. For (2), consider successively all the elements \( z \notin X \): According to (1), the intersection of all the sets in \( \mathcal{F} \) including \( z \) does not intersect \( X \). Hence, setting \( A = \{ X \} \) and \( B = \{ Y \in \mathcal{F}, z \in Y \} \), it follows from Theorem 2.1 that \( z \) does not belong to \( \text{Conv}(f(X)) \).

**An abstract simplicial complex** \( \Delta \) is a family of finite sets such that any subset of a set in \( \Delta \) belongs to \( \Delta \): \( \forall X \in \Delta, \forall Y \subseteq X, \; Y \in \Delta \). The **face order** of \( \Delta \) is the partial ordering of the elements of \( \Delta \) by \( \subseteq \). A **geometric realization** of \( \Delta \) is an injective mapping \( f \) of the ground set \( | \Delta | = \bigcup_{X \in \Delta} X \) to some Euclidean space \( \mathbb{R}^d \), such that, for any two elements (or faces) \( X, Y \) of \( \Delta \), the convex hulls of the images of \( X \) and \( Y \) have the convex hull of their intersection: \( \text{Conv}(f(X)) \cap \text{Conv}(f(Y)) = \text{Conv}(f(X \cap Y)) \). It is a folklore lemma that a mapping from \( \Delta \) to \( \mathbb{R}^d \) is a geometric realization of \( \Delta \) if and only if disjoint faces of \( \Delta \) are mapped to point sets with disjoint convex hulls.

It is well known that an abstract simplicial complex has a geometric realization in \( \mathbb{R}^d \) when \( d > 2(\max_{X \in \Delta} |X| - 1) \) and that, obviously, it has no geometric realization in \( \mathbb{R}^d \) if \( d < \max_{X \in \Delta} |X| - 1 \).

**Theorem 3.2 (Ossona de Mendez [2])** Let \( \Delta \) be an abstract simplicial complex, and let \( d \) be the dimension of the face order of \( \Delta \). Then, \( \Delta \) has a geometric realization in \( \mathbb{R}^{d-1} \).

**Proof:** Consider the mapping from the ground set \( | \Delta | \) of \( \Delta \) to \( \mathbb{R}^{d-1} \), whose existence is ensured by Corollary 3.1. Then, for any disjoint faces \( F, F' \) of \( \Delta \), we get \( \text{Conv}(f(F)) \cap \text{Conv}(f(F')) = \emptyset \), that is: \( f \) induces a geometric realization of \( \Delta \) in \( \mathbb{R}^{d-1} \).

\[ \square \]
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