Sublinear-Time Language Recognition and Decision by One-Dimensional Cellular Automata

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Abstract
After an apparent hiatus of roughly 30 years, we revisit a seemingly neglected subject in the theory of (one-dimensional) cellular automata: sublinear-time computation. The cellular automata model considered is that of ACAs, which are language acceptors whose acceptance condition depends on the states of all cells in the automaton. We prove a time hierarchy theorem for sublinear-time ACA classes, analyze their intersection with the regular languages, and, finally, establish strict inclusions respective to the parallel computation classes \( SC \) and (uniform) \( AC \). As an addendum, we introduce and investigate the concept of a strong ACA (SACA) as the decider counterpart of a (weak) ACA, show the class of languages decidable in constant time by SACAs equals the locally testable languages, and determine \( \Omega(\sqrt{n}) \) as the (tight) time complexity threshold for SACAs up to which no advantage respective to constant time is possible.

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1 Introduction
While there have been several works on linear- and real-time language recognition by cellular automata over the years (see, e.g., [15, 27] for an overview), interest on the sublinear-time case has been scanty at best. We can only speculate this has been due to a certain obstinacy concerning what is now the established acceptance condition for cellular automata, namely that the first cell determines the automaton’s response (despite alternatives being long known [19]). The implication (for sublinear-time computation) is that then only a constant-size prefix ever influences the automaton’s decision, effectively dooming sublinear time to be but a trivial case just as it is for Turing machines, for example. Nevertheless, in the case of Turing machines, this restriction was readily circumvented by adapting the classical model with random access, thus sparking rich theories on parallel computation \([6, 21]\), probabilistically checkable proofs \([25]\), and property testing \([8, 20]\).

In the case of cellular automata, the adaptation needed is an alternate acceptance condition, covered in Section 2. Interestingly, in the resulting model, called ACA, parallelism and local behavior seem to be more marked features, taking priority over cell communication and synchronization (which seems to be the dominant themes of the the linear- and real-time constructions). As mentioned above, the body of theory on sublinear-time ACAs is very small and, to the best of our knowledge, resumes itself to \([11, 14, 22]\). \([11]\) shows sublinear-time ACAs are capable of recognizing non-regular languages and also determines a threshold (namely \( \Omega(\log n) \)) up to which no advantage over constant time is possible. Meanwhile, \([14, 22]\) analyze the constant-time case subject to different acceptance conditions and characterize it based on the locally testable languages, a subclass of the regular languages.
Indeed, as we shall see in Section 3, the defining property of the locally testable languages, that is, that words which locally appear to be the same are equivalent with respect to membership in the language at hand, effectively translates into an inherent property of acceptance by sublinear-time ACAs. In Section 4 we prove a time hierarchy theorem for sublinear-time ACAs as well as further relate the language classes they define to the regular languages and the parallel computation classes $\text{SC}$ and (uniform) $\text{AC}$. Finally, in Section 5 we elevate ACAs to language deciders, that is, machines which must not only must accept words in the target language but also reject those which do not. Section 6 concludes.

2 Definitions

We assume the reader is familiar with the theory of formal languages and cellular automata as well as with complexity theory. The goal of this section is to review the basic concepts and establish the notion of an ACA.

$\mathbb{Z}$ denotes the set of integers, $\mathbb{N}_+$ that of (strictly) positive integers, and $\mathbb{N}_0 = \mathbb{N}_+ \cup \{0\}$. For a map $f : A \rightarrow B$ and $A' \subseteq A$, $f|_{A'}$ indicates the restriction of $f$ to $A'$. For a word $w \in \Sigma^*$ over an alphabet $\Sigma$, $w(i)$ is the $i$-th symbol of $w$, and $|w|_k$ indicates the number of occurrences of $x \in \Sigma$ in $w$. For $k \in \mathbb{N}_0$, $p_k(w)$, $s_k(w)$, and $I_k(w)$ denote the prefix, suffix and set of infixes of length $k$ of $w$, respectively, and $\Sigma^{\leq k}$ is the set of words over $\Sigma^*$ of length less than or equal to $k$. Unless otherwise noted, $n$ is the input length.

2.1 (Strictly) Locally Testable Languages

The class $\text{REG}$ of regular languages is defined in terms of (deterministic) automata with finite memory and which read their input in a single direction (i.e., from left to right), one symbol at a time; once all symbols have been read, the machine outputs a single bit representing its decision. In contrast, a scanner is a memoryless machine which reads a span of $k \in \mathbb{N}_+$ symbols at a time of an input provided with start and end markers (so it can handle prefixes and suffixes separately); the scanner validates the substrings it reads using the same Boolean predicate over and over again, and it accepts if and only if all these validations are successful. The languages accepted by these machines are the strictly locally testable languages.

$\square$ Definition 1 (strictly locally testable). Let $\Sigma$ be an alphabet. A language $L \subseteq \Sigma^*$ is strictly locally testable if there is some $k \in \mathbb{N}_+$ and sets $\pi, \sigma \subseteq \Sigma^{\leq k}$ and $\mu \subseteq \Sigma^k$ such that, for every word $w \in \Sigma^*$, $w \in L$ if and only if $p_k(w) \in \pi$, $I_k(w) \subseteq \mu$, and $s_k(w) \in \sigma$. $\text{SLT}$ denotes the class of strictly locally testable languages.

A more general notion of locality is provided by the locally testable languages. Intuitively, $L$ is locally testable if a word $w$ being in $L$ or not is entirely dependent on a property of the substrings of $w$ of some constant length $k \in \mathbb{N}_+$. Thus, if any two words have the same set of substrings of length $k$, then they are equivalent with respect to being in $L$.

$\square$ Definition 2 (locally testable). Let $\Sigma$ be an alphabet. A language $L \subseteq \Sigma^*$ is said to be locally testable if there is some $k \in \mathbb{N}_+$ such that, for every $w_1, w_2 \in \Sigma^*$ with $p_k(w_1) = p_k(w_2)$, $I_k(w_1) = I_k(w_2)$, and $s_k(w_1) = s_k(w_2)$ we have that $w_1 \in L$ if and only if $w_2 \in L$. $\text{LT}$ denotes the class of locally testable languages.

$\text{LT}$ is the Boolean closure of $\text{SLT}$, that is, its closure under union, intersection, and complement \cite{17}. In particular, $\text{SLT} \subseteq \text{LT}$ (i.e., the inclusion is proper \cite{16}).

$\square$\footnote{The terminology “(locally) testable in the strict sense” ((L)TSS) is also common \cite{14, 16, 17}.}
2.2 Cellular Automata

In this paper, we are strictly interested in one-dimensional cellular automata with the standard neighborhood. For \( r \in \mathbb{N}_0 \), let \( N_r(z) = \{ z' \in \mathbb{Z} \mid |z - z'| \leq r \} \) denote the extended neighborhood of radius \( r \) of the cell \( z \in \mathbb{Z} \).

- **Definition 3 (cellular automaton).** A cellular automaton (CA) \( C \) is a triple \( (Q, \delta, \Sigma) \) where \( Q \) is a finite, non-empty set of states, \( \delta : Q^3 \to Q \) is the local transition function, and \( \Sigma \subseteq Q \) is the input alphabet. An element of \( Q^3 \) (resp., \( Q^2 \)) is called a local (resp., global) configuration of \( C \). \( \delta \) induces the global transition function \( \Delta : Q^2 \to Q^2 \) on the configuration space \( Q^2 \) by \( \Delta(c)(z) = \delta(c(z - 1), c(z), c(z + 1)) \), where \( z \in \mathbb{Z} \) is a cell and \( c \in Q^2 \).

We now view a CA as a machine which receives an input and processes it until a final state is reached. The input is provided from left to right, with one cell for each input symbol. The surrounding cells are inactive and remain so for the entirety of the computation (i.e., the CA is bounded). In addition, it is customary for CAs to have a distinguished cell, usually cell zero, which communicates the machine’s output. As mentioned in the introduction, this convention is inadequate for computation in sublinear time; thus, we require the finality condition to depend on the entire (global) configuration (modulo inactive cells):

- **Definition 4 (CA computation).** There is a distinguished state \( q \in Q \setminus \Sigma \), called the inactive state, which, for every \( q_1, q_2, q_3 \in Q \), satisfies \( \delta(q_1, q_2, q_3) = q \) if and only if \( q_2 = q \). A cell not in state \( q \) is said to be active. For an input \( w \in \Sigma^* \), the initial configuration \( c_0 \in Q^2 \) of \( C \) for \( w \) is \( c(i) = w(i) \) for \( i \in \{0, \ldots, |w| - 1\} \) and \( c(i) = q \) otherwise. For \( F \subseteq Q \setminus \{q\} \), a configuration \( c \in Q^2 \) is \( F \)-final (for \( w \)) if there is \( \tau \in \mathbb{N}_0 \) such that \( \Delta^\tau(c_0) \) contains only states in \( F \cup \{q\} \) and \( \tau \) is minimal with this property. In this context, the sequence \( c_0, \ldots, \Delta^\tau(c_0) \) is the trace of \( w \), and \( \tau \) is the time complexity of \( C \) (with respect to \( F \) and \( w \)).

Because we effectively consider only bounded CAs, the computation of an input \( w \) involves exactly \(|w| \) active cells. The surrounding inactive cells are irrelevant except for marking where the input starts and ends. As a side effect, the empty word \( \varepsilon \) does not admit a final configuration (for any choice of final states), and so we disregard it for the rest of this paper.

The final notion needed is relating final configurations to computation results. We adopt an acceptance condition as in \([22, 19]\) and obtain a so-called ACA; here, the “A” of “ACA” refers to the property that all (active) cells are relevant for finality of computation.

- **Definition 5 (ACA).** An ACA is a CA \( C \) which has a non-empty subset \( A \subseteq Q \setminus \{q\} \) of accept states. \( C \) accepts an input \( w \in \Sigma^* \) if there is an \( A \)-final configuration for \( w \). \( L(C) \) denotes the set of words accepted by \( C \). Given a function \( t : \mathbb{N}_+ \to \mathbb{N}_0 \), we write \( \text{ACA}(t) \) for the class of languages which are accepted by an ACA with time complexity bounded by \( t \), that is, for which the time complexity of each input \( w \) does not exceed \( t(|w|) \).

\( \text{ACA}(t_1) \subseteq \text{ACA}(t_2) \) is immediate for \( t_1, t_2 : \mathbb{N}_+ \to \mathbb{N}_0 \) with \( t_1(n) \leq t_2(n) \) for every \( n \in \mathbb{N}_+ \). In the rest of this paper, when we say a cell is accepting, we intend to say the cell keeps track of its former contents while marking itself as such. This is possible because Definition 5 allows multiple accept states.

Figure 1 illustrates the computation of an ACA with input alphabet \( \Sigma = \{0, 1\} \) and which accepts the language \( \{0\}^* \) with time complexity equal to 1. The local transition function of the ACA is such that \( \delta(0) = a \), \( a \) being its accept state, and \( \delta(z) = z \) for any state \( z \neq a \).
4 Sublinear-Time Language Recognition and Decision by One-Dimensional Cellular Automata

Figure 1 Computation of an ACA which recognizes \( \{0\}^\ast \). The input words are 000000 and 001010, respectively.

3 First Observations

This section recalls relevant results on sublinear-time ACA computation (i.e., ACA\((t)\) where \( t \in o(n) \)) from the related work \([14, 22, 11]\) while providing some additional remarks. The highlight is Lemmas 9 and 11, which characterize the locality of ACA computation.

We start with the constant-time case (i.e., ACA\((O(1))\)). Here, the connection between scanners and ACAs is apparent: If an ACA accepts an input \( w \) in time \( \tau = \tau(w) \), then \( w \) can be verified by a scanner with an input span of \( 2\tau + 1 \) symbols and whose Boolean predicate is induced by the local transition function of the ACA (i.e., the predicate is true if and only if the symbols read correspond to \( N_{\tau}(z) \) for some cell \( z \) and \( z \) is accepting after \( \tau \) steps).

Constant-time ACA computation has been studied in \([14, 22]\). Although \([14]\) gives a characterization based on SLT, their acceptance condition differs slightly from that in Definition 5 in particular, they insist the automaton runs for a number of steps which is fixed (for each automaton), and the outcome is evaluated (only) in the final step. In contrast, the earlier work \([22]\) proves the following, where SLT\(\vee\) denotes the closure of SLT under union:

\[ \text{Theorem 6 ([22])} \quad \text{ACA}(O(1)) = \text{SLT}\vee. \]

It follows ACA\((O(1))\) is closed under union. We extend this to the sublinear-time case:

\[ \text{Proposition 7.} \quad \text{For } t \in o(n), \text{ ACA}(O(t)) \text{ is closed under union.} \]

\[ \text{Proof.} \quad \text{Let } L_1 \text{ and } L_2 \text{ be languages accepted by the ACAs } C_1 \text{ and } C_2, \text{ respectively, in time } O(t). \text{ Furthermore, let } Q_i \ (\text{resp., } Q^a_i) \text{ denote the set of states (resp., accepting states) of } C_i. \text{ We construct an ACA } C \text{ which accepts } L_1 \cup L_2 \text{ as follows: } C \text{ simulates } C_1 \text{ and } C_2 \text{ and switches between the two simulations at every step. Each cell maintains components } q_1 \in Q_1, q_2 \in Q_2, \text{ and } r \in \{1, 2\}, \text{ where } r \text{ indicates which of the two simulations is to be updated next; that is, at each step, a cell next updates } q_r \text{ according to the local transition function of } C_r \text{ and the } q_r \text{ states of its neighbors. At the start of the computation, all cells simultaneously initialize } r = 1. \text{ The accept states of } C \text{ are } Q_1 \times Q^a_2 \times \{1\} \cup Q^a_1 \times Q_2 \times \{2\}. \text{ Thus, } L(C) = L_1 \cup L_2 \text{ and } C \text{ has time complexity } O(2t) = O(t). \quad \blacksquare \]

\[ \text{[22]} \text{ show ACA}(O(1)) \text{ is closed under intersection. It is an open question whether ACA}(O(t)) \text{ is also closed under intersection for every } t \in o(n). \]

Moving beyond constant time, \([11]\) show the following:

\[ \text{Theorem 8 ([11])} \quad \text{For } t \in o(\log n), \text{ ACA}(t) \subseteq \text{REG.} \]

\[ \text{[11]} \text{ also show ACA}(O(\log n)) \text{ contains non-regular languages. In particular, they give an example which is essentially a variation of the language } \]

\[ \text{BIN} = \{\text{bin}_k(0)\# \text{bin}_k(1)\# \cdots \# \text{bin}_k(2^k - 1) \mid k \in \mathbb{N}_+\} \]
where bin\(_k(m)\) is the \(k\)-digit binary representation of \(m \in \{0, \ldots, 2^k - 1\}\)\(^2\).

To illustrate the ideas involved, we present a construction for a language related to BIN (though it results in a different time complexity); the same example shall also be useful for later discussions in Section 5. Let \(w_k(i) = 0^i10^{k-i-1}\) and consider the language

\[
\text{IDMAT} = \{w_k(0)\#w_k(1)\# \cdots \#w_k(k-1) \mid k \in \mathbb{N}_+\}
\]

of line-for-line representations (separated by \('#') of identity matrices of arbitrary size.

Denote each group of cells initially containing a (maximally long) \(\{0,1\}^+\) substring of \(w \in \text{IDMAT}\) by a block. We obtain an ACA for \(\text{IDMAT}\) as follows: Each block of size \(b\) propagates its contents to the neighboring blocks (in separate registers); using a textbook CA technique, this requires exactly \(2b\) steps. Once the strings align, a block initially containing \(w_k(i)\) verifies it has received \(w_k(i-1)\) and \(w_k(i+1)\) from its left and right neighbor blocks (if either exists), respectively. The cells of a block as well as its delimiters become accepting if and only if the comparison is successful and there is a single \('#') between the block and its neighbors. The process just described takes time linear in \(b\); since any \(w \in \text{IDMAT}\) has \(O(\sqrt{|w|})\) many blocks, each with \(b \in O(\sqrt{|w|})\) cells, it follows \(\text{IDMAT} \in \text{ACA}(O(\sqrt{n}))\).

To show the above construction is time-optimal, we use the following observation, which is also central in proving several other results in this paper:

\[\textbf{Lemma 9.}\] Let \(C\) be an ACA and let \(w\) be an input which \(C\) accepts with time complexity \(\tau = \tau(w)\) (i.e., in exactly \(\tau\) steps). Then, for every input \(w'\) such that \(p_{2\tau}(w) = p_{2\tau}(w')\), \(I_{2\tau+1}(w') \subseteq I_{2\tau+1}(w)\), and \(s_{2\tau}(w) = s_{2\tau}(w')\), \(C\) accepts \(w'\) with time complexity \(\tau\).

The lemma is intended to be used with \(\tau < \frac{|w|}{2}\) since otherwise \(w = w'\). It can be used, for instance, to show that \(\text{SOMEONE} = \{w \in \{0,1\}^+ \mid |w|_1 \geq 1\}\) is not in \(\text{ACA}(t)\) for any \(t \in o(n)\) (e.g., set \(w = 0^k10^k\) and \(w' = 0^{2k+1}\) for \(k \in \mathbb{N}_+\) large enough). It follows \(\text{REG} \not\subseteq \text{ACA}(t)\) for \(t \in o(n)\).

**Proof.** Assume \(C\) does not accept \(w'\) in strictly less than \(\tau\) steps. Let \(A\) be the set of accept states of \(C\), and let \(c_0\) and \(c'_0\) denote the initial configurations for \(w\) and \(w'\), respectively. Given \(w \in L(C)\), we prove \(c'_\tau = \Delta^\tau(c'_0)\) is \(A\)-final.

Let \(i \in \mathbb{Z}\). If all of \(c'_0|_{N_\tau}(i)\) is inactive, then cell \(i\) is also inactive in \(c'_\tau\) (i.e., \(c'_\tau(i) = q\)). If \(c'_0|_{N_\tau}(i)\) contains both inactive and active states, then \(i < \tau\) or \(i \geq |w| - \tau\), in which case \(p_{2\tau}(w) = p_{2\tau}(w')\) and \(s_{2\tau}(w) = s_{2\tau}(w')\) imply \(c'_0|_{N\tau}(i) = c_0|_{N\tau}(i)\). Finally, if \(c'_0|_{N\tau}(i)\) is purely active, \(c'_0|_{N\tau}(i)\) (seen as a word over the input alphabet of \(C\)) is in \(I_{2\tau+1}(w') \subseteq I_{2\tau+1}(w)\); as a result, there is \(j \in \mathbb{Z}\) with \(t \leq j < |w| - t\) such that \(c'_0|_{N\tau}(i) = c_0|_{N\tau}(j)\). In both the previous cases, because \(c_\tau\) is \(A\)-final, it follows \(c'_\tau(i) \in A\), implying \(c'_\tau\) is \(A\)-final. \(\blacksquare\)

Since the complement of \(\text{SOMEONE}\) (respective to \(\{0,1\}^+\)) is \(\{0\}^+ \in \text{ACA}(O(1))\) (see Section 2), \(\text{ACA}(t)\) is not closed under complement for any \(t \in o(n)\). Also, \(\text{SOMEONE}\) is a regular language and \(\text{BIN} \in \text{ACA}(O(\log n))\) is not, so we have:

\[\textbf{Proposition 10.}\] For \(t \in o(n)\) and \(t \in \Omega(\log n)\), \(\text{ACA}(t)\) and \(\text{REG}\) are incomparable.

If the inclusion of infixes in Lemma 9 is strengthened to an equality, one may apply it in both directions and obtain the following stronger statement:

\[\text{REG} \not\subseteq \text{ACA}(t)\text{ for any } t \in o(n).\]

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Lemma 11. Let $C$ be an ACA with time complexity bounded by $t : \mathbb{N}_+ \to \mathbb{N}_0$ (i.e., if $C$ accepts any input of length $n$, then it does so in at most $t(n)$ steps). Then, for any two inputs $w$ and $w'$ such that $p_{2\tau}(w) = p_{2\tau}(w')$, $I_{2\tau+1}(w) = I_{2\tau+1}(w')$, and $s_{2\tau}(w) = s_{2\tau}(w')$ where $\tau = \max\{t(|w|), t(|w'|)\}$, we have that $w \in L(C)$ if and only if $w' \in L(C)$.

Finally, to conclude our discussion of IDMAT, we prove our construction is time-optimal:

Proposition 12. For any $t \in o(\sqrt{n})$, IDMAT $\not\subseteq$ ACA($t$).

Proof. Let $n \in \mathbb{N}_+$ be such that $t(n) < \frac{1}{5} \sqrt{n}$, and let $w \in$ IDMAT with be given with $|w| = n$. In particular, $|w| = k^2 + k - 1$ for some $k \in \mathbb{N}_+$ and $w = w_k(0)\# \cdots \# w_k(k - 1)$; without restriction, we also assume $k$ is even. Let $w'$ be the word obtained from $w$ by replacing its $(\frac{k}{2} - 1)$-th block with $w_k(\frac{k}{2})$. Then, $p_{2\tau(n)}(w) = p_{2\tau(n)}(w')$, $s_{2\tau(n)}(w) = s_{2\tau(n)}(w')$, and $I_{2\tau(n)+1}(w) \subseteq I_{2\tau(n)+1}(w')$, where the latter follows from $w_k(\frac{k}{2}) = \Theta(\sqrt{k})$. Thus, by Lemma 9, if an ACA with time complexity bounded by $t$ accepts $w$, so does it accept $w' \not\in$ IDMAT.

4 Main Results

In this section, we state and prove various results regarding ACA($t$) where $t \in o(n)$. First, we obtain a time hierarchy theorem, that is, for $t' \in o(t)$ and under plausible conditions (e.g., $t \in \Omega(\log n)$; see also Theorem 5), ACA($t'$) $\subseteq$ ACA($t$). Next, we show ACA($t$) $\cap$ REG is (strictly) contained in LT. Finally, we study inclusion relations between ACA($t$) and the classes in the SC and (uniform) AC hierarchies. Save for the definitions and results covered up to this point, all three subsections stand out independently from one another.

4.1 Time Hierarchy

We say a function $f : \mathbb{N}_+ \to \mathbb{N}_0$ is time-constructible by CAs in time $t : \mathbb{N}_+ \to \mathbb{N}_0$ if there is a CA $C$ which, on input $1^n$, produces the value $f(n)$ (binary-encoded) in at most $t(n)$ steps.

Note that, since CAs can simulate (one-tape) Turing machines in real-time, any function constructible by Turing machines (in the corresponding sense) is also constructible by CAs.

Theorem 13. Let $f \in \omega(n)$ with $f(n) \leq 2^n$, $g(n) = 2^{n - \lfloor \log f(n) \rfloor}$, and let $f$ and $g$ be time-constructible (by CAs) in time $t$. Furthermore, let $t : \mathbb{N}_+ \to \mathbb{N}_0$ be such that $3f(k) \leq t(f(k)g(k)) \leq cf(k)$ for some constant $c \geq 3$ and all but finitely many $k \in \mathbb{N}_+$. Then, for every $t' \in o(t)$, ACA($t'$) $\not\subseteq$ ACA($t$).

Given $a > 1$, this can be used, for instance, with any time-constructible $f \in \Theta(n^a)$ (resp., $f \in \Theta(2^{n/a})$, in which case $a = 1$ is also possible) and $t \in \Theta((\log n)^a)$ (resp., $t \in \Theta(n^{1/a})$).

The proof idea is to construct a language $L$ similar to BIN (see Section 3) in which every $w \in L$ has length exponential in the size of its blocks while the distance between any two blocks is $\Theta(t(|w|))$. Due to Lemma 9, the latter implies $L$ is not recognizable in time $o(t(|w|))$.

Proof. For simplicity, we assume $f(n) > n$. Consider the language $L = \{w_k \mid k \in \mathbb{N}_+\}$ where $w_k = \text{bin}_k(0)\#^k \text{bin}_k(1)\#^k \cdots \text{bin}_k(g(k) - 1)\#^k$

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3 Just as is the case for Turing machines, there is not a single definition for time-constructibility by CAs (see, e.g., [13] for an alternative). Here, we opt for a plausible variant which has the benefit of simplifying the ensuing line of argument.
and note $|w_k| = f(k)g(k)$. Because $t(|w_k|) \in O(f(k))$ and $f(k) \in \omega(k)$, given any $t' \in o(t)$, setting $w = w_k$, $w' = \theta^k\#^{\#w_k-k}$, and $\tau = t'(|w_k|)$ and applying Lemma \[\] for sufficiently large $k$ yields $L \not\subseteq \text{ACA}(t')$.

By assumption it suffices to show $w = w_k \in L$ is accepted by an ACA $C$ in time at most $3f(k) \leq t(|w|)$ for sufficiently large $k \in \mathbb{N}^+$. The cells of $C$ perform two procedures simultaneously: The first of them is as in the ACA for BIN (see Section 3) and ensures that the blocks of $w$ have the same length, that the respective binary encodings are valid, and that the last value is correct (i.e., equal to $g(k) - 1$). For the second procedure, each block computes $f(k)$ as a function of its block length $k$. Subsequently, the value $f(k)$ is decreased using a real-time counter (see, e.g., [13] for a construction). Every time the counter is decremented, a signal starts from the block’s leftmost cell and is propagated to the right. This allows every group of cells of the form $bs$ with $b \in \{0, 1\}^+$ and $s \in \{\#\}^+$ to assert there are precisely $f(k)$ symbols in total (i.e., $|bs| = f(k)$). A cell becomes accepting if both of the procedures it has participated in are successful. The proof is complete by noticing either procedure takes a maximum of $3f(k)$ steps (again, for sufficiently large $k$).

### 4.2 Intersection with the Regular Languages

In light of Proposition \[\] we now consider the intersection $\text{ACA}(t) \cap \text{REG}$ for $t \in o(n)$ (in the same spirit as a conjecture by Straubing [24]). For this section, we assume the reader is familiar with the theory of syntactic semigroups (see, e.g., [7] for an in-depth treatment).

Given a language $L$, let $SS(L)$ denote the syntactic semigroup of $L$. It is well-known that $SS(L)$ is finite if and only if $L$ is regular. A semigroup $S$ is a semilattice if $x^2 = x$ and $xy = yx$ for every $x, y \in S$. Additionally, $S$ is locally semilattice if $eSe$ is a semilattice for every idempotent $e \in S$, that is, $e^2 = e$. We use the following characterization of locally testable languages:

► **Theorem 14** ([2] [16]). $L \in \text{LT}$ if and only if $SS(L)$ is finite and locally semilattice.

In conjunction with Lemma \[\] this allows us to prove the following, where the strict inclusion is due to SOMEONE $\not\in \text{ACA}(t)$ (since SOMEONE $\in \text{LT}$; see Section 3):

► **Theorem 15.** For every $t \in o(n)$, $\text{ACA}(t) \cap \text{REG} \subseteq \text{LT}$.

**Proof.** Let $L \in \text{ACA}(t)$ be a language over the alphabet $\Sigma$ and, in addition, let $L \in \text{REG}$, that is, $S = SS(L)$ is finite. By Theorem 14 it suffices to show $S$ is locally semilattice. To that end, let $e \in S$ be idempotent, and let $x, y \in S$.

To show $(exe)(eye) = (eye)(exe)$, let $a, b \in \Sigma^*$ and consider the words $u = a(exe)(eye)b$ and $v = a(eye)(exe)b$. For $m \in \mathbb{N}_+$, let $u'_m = a(e^mxe^m)(e^mye^m)b$, and let $r \in \mathbb{N}_+$ be such that $r > \max\{|x|, |y|, |a|, |b|\}$ and $t(|u'_{2r+1}|) < \frac{1}{\beta|\tau|}\|u'_{2r+1}\| < r$. Since $e$ is idempotent, $u' = u'_{2r+1}$ and $u$ belong to the same class in $S$, that is, $u' \in L$ if and only if $u \in L$; the same is true for $v' = a(e^{2r+1}ye^{2r+1})(e^{2r+1}xe^{2r+1})b$ and $v$. Furthermore, we have $p_{2r}(u') = p_{2r}(v')$, $I_{2r+1}(u') = I_{2r+1}(v')$, and $s_{2r}(u') = s_{2r}(v')$. Since $L \in \text{ACA}(t)$, Lemma \[\] applies.

The proof of $(exe)(exe) = exe$ is analogous. Simply consider the words $a(e^mxe^m)b$ and $a(e^mxe^m)(e^mxe^m)b$ for sufficiently large $m \in \mathbb{N}_+$ and use, again, Lemma \[\] and the fact that $e$ is idempotent.

### 4.3 Relation to Parallel Complexity Classes

Intuitively, ACAs are parallel computers in which every processor receives a small portion of the input, with possibly some overlaps in the portions provided to neighboring processors.
In this section, we relate ACA(t) to other classes which characterize parallel computation, namely those in the SC and (uniform) AC hierarchies.

In this context, SC^k is the class of problems decidable by Turing machines in space \(O((\log n)^k)\) and polynomial time, whereas AC^k is that decidable by Boolean circuits with depth \(O((\log n)^k)\) and gates with unbounded fan-in. The class SC (resp., AC) is the union of all SC^k (resp., AC^k) for \(k \in \mathbb{N}_0\). Naturally, we consider only uniform versions of AC; when relevant, we state the respective uniformity condition. Although SC^1 = L \subseteq AC^1 is known, it is unclear whether any other containment holds between SC and AC.

One should not expect to include (subclasses of) SC or AC in ACA(t) for any \(t \in o(n)\). Conceptually speaking, whereas said classes correspond to models capable of random access to their input, ACAs are inherently local (as evinced by Lemmas 9 and 11). There are, in fact, explicit counterexamples for such inclusions, namely the unary languages: For any fixed \(m \in \mathbb{N}_+\) and \(w_1, w_2 \in \{1\}^+\) with \(|w_1|, |w_2| \geq m\), we trivially have \(p_{m-1}(w_1) = p_{m-1}(w_2)\), \(I_m(w_1) = I_m(w_2)\), and \(s_{m-1}(w_1) = s_{m-1}(w_2)\). Therefore, by Lemma 9 if an ACA accepts \(w \in \{1\}^+\) in time bounded by \(t \in o(n)\) and \(|w|\) is large enough (e.g., \(|w| > 4t(|w|)\)), then the same ACA must accept any \(w' \in \{1\}^+\) with \(|w'| \geq |w|\). Hence, extending a result from 22:

\[\text{Proposition 16. If } t \in o(n) \text{ and } L \in ACA(t) \text{ is a unary language (i.e., } L \subseteq \Sigma^* \text{ and } |\Sigma| = 1\), then } L \text{ is either finite or co-finite.}\]

In light of the above, the rest of this section is concerned with the converse inclusion (i.e., establishing SC or AC classes as supersets of ACA(t)). For \(f, s, t : \mathbb{N}_+ \to \mathbb{N}_0\) with \(f(n) \leq s(n)\) we say \(f\) is constructible (by a Turing machine) in space \(s\) and time \(t\) if there is a Turing machine \(T\) which, given \(1^n\) as input, outputs \(f(n)\) using at most \(s(n)\) space and \(t(n)\) time. (Note that, due to \(f(n) \leq s(n)\), it is irrelevant whether the output \(f(n)\) is binary- or unary-encoded.) Additionally, recall a Turing machine can simulate a CA with \(m\) (active) cells in \(O(m)\) space and \(O(m^2)\) time.

\[\text{Proposition 17. Let } C \text{ be an ACA with time complexity bounded by } t \in o(n), t(n) \geq \log n, \text{ and let } t \text{ be constructible in space } s \text{ and time } t. \text{ Then there is a Turing machine which decides } L(C) \text{ in } O(t) \text{ space and polynomial time.}\]

\[\text{Proof. We construct a machine } T \text{ with the purported property. Given an input } w, T \text{ first determines the input length } |w| \text{ and computes the value } t(|w|) \text{ in time bounded by a polynomial } p : \mathbb{N}_+ \to \mathbb{N}_0, \text{ thus requiring } O(t(|w|)+\log n) = O(t(|w|)) \text{ space and } O(|w|+p(|w|)) \text{ time. The rest of the computation of } T \text{ takes place in stages, where stage } i \text{ corresponds to step } i \text{ of } C. \text{ } T \text{ maintains a counter for } i, \text{ incrementing it after each stage, and rejects whenever } i > t(|w|). \text{ In stage } i, T \text{ iterates over every (active) cell } z \in \{0, \ldots, |w| - 1\} \text{ of } C \text{ and, by iteratively applying the local transition function of } C \text{ on } N_i(z), \text{ determines the state of } z \text{ in step } i \text{ of } C; \text{ since } |N_i(z)| = 2i + 1, \text{ this requires space } O(i) \text{ and time } O(i^2) \text{ for each } z. \text{ If all states computed in the same stage are accept states, then } T \text{ accepts. Since } T \text{ runs for at most } t(|w|) \text{ stages, it uses space } O(t(|w|)) \text{ and time } O(|w| \cdot t(|w|)^3 + p(|w|)) \text{ in total.}\]

Thus, for the case of polylogarithmic \(t\) (where the strict inclusion is due to Proposition 16):

\[\text{Corollary 18. For } k \in \mathbb{N}_+, \text{ ACA}(O((\log n)^k)) \subseteq SC^k.\]

Moving on to the AC classes, we employ some notions from descriptive complexity theory (see, e.g., 12 for an introduction). Let FO_L[t] be the class of languages describable by first-order formulas with numeric relations in \(L\) (i.e., logarithmic space) and quantifier block iterations bounded by \(t : \mathbb{N}_+ \to \mathbb{N}_0\). For \(C, Q, \delta, \text{ and } \Delta \text{ as in Definition 3, we extend } \Delta\) so
\[ \Delta(c) \] is also defined for \( c \in Q^{2r+1} \) with \( r \in \mathbb{N}^+ \) by \( \Delta(c)(i) = \delta(c(i), c(i+1), c(i+2)) \), where \( i \in \{0, \ldots, 2r-1\} \); in particular, \( \Delta(c) \in Q^{2r-1} \). Additionally, for \( s \in Q \), let \( \text{DELTAC}(c, s) \) be the relation which is true if and only if \( \Delta^*(c) = s \). Note \( \text{DELTAC} \) is computable by a Turing machine in \( O(\tau) \) space and \( O(\tau^2) \) time.

\[ \textbf{Theorem 19.} \text{ Let } t: \mathbb{N}^+ \rightarrow \mathbb{N}_0 \text{ with } t(n) \geq \log n \text{ be constructible in logarithmic space (and arbitrary time). For any ACA } C \text{ whose time complexity is bounded by } t, \text{ } L(C) \in \text{FO} \{O\left(\frac{\log n}{n}\right)\}. \]

In the following proof, we let “\( \equiv \)” denote the equality relation inside a formula.

\[ \textbf{Proof.} \text{ Let } Q \text{ be the state set of } C \text{ and let } A \subseteq Q \text{ be the set of accepting states of } C; \text{ without restriction, we may assume } |Q| \geq 2. \text{ In addition, let } w \text{ be an input for } C \text{ and set } r = \log |Q| |w|, \text{ which (again without restriction) we assume to be odd.} \]

Assuming we have a predicate \( \text{STATE}_{C,w}(z, s, t') \) which is true if and only if cell \( z \in C \), given input \( w \), is in state \( s \) after \( t' \) steps, we may express whether \( C \) accepts \( w \) or not by the following formula:

\[ \varphi_{C,w} = (\exists t' \leq t(|w|))(\forall z)(\exists s \in A) \text{STATE}_{C,w}(z, s, t'). \]

Note \( \varphi_{C,w} \) is true if and only if the initial configuration \( c_0 = c_0(w) \) of \( w \) is A-final (i.e., \( C \) accepts \( w \)) and an A-final configuration is reached in at most \( t' \leq t(|w|) \) steps of \( C \). In turn, \( \text{STATE}_{C,w} \) may be expressed as follows:

\[ \text{STATE}_{C,w}(z, s, t') = (t' \leq r \land \text{DELTAC}(c_0|N_{t'}(z), s)) \lor (\exists r' \leq r) [B]^{t'/r} c \equiv c_0|N_{r'}(z) \]

where \( B \) is a quantifier block in which the state \( s \) of \( z \) is traced back \( r \) steps to a former subconfiguration \( c \) (of size \( 2r+1 \)), followed by checking that the state of every cell \( z' \) in \( c \) is consistent (with the computation of \( C \)):

\[ B = (\exists c \in Q^{2r+1} \text{DELTAC}(c, s))(\forall z') |z-z'| \leq r(\exists z \equiv c(z'-z+r))(\exists z \equiv z'). \]

Note all numeric predicates in \( \varphi_{C,w} \) are computable in logarithmic space, including \( \text{DELTAC} \). Since \( w \in L(C) \) if and only if \( w \models \varphi_{C,w} \) holds, the claim follows. \[ \square \]

Recalling \( \text{FO} \{O((\log n)^k)\} \) equals L-uniform \( \text{AC}^k \) \[12\] and using Proposition \[16\] we obtain:

\[ \textbf{Corollary 20.} \text{ For } k \in \mathbb{N}_+, \text{ } \text{ACA}(O((\log n)^k)) \text{ is strictly contained in } L\text{-uniform } \text{AC}^{k-1}. \]

Because \( \text{SC}^1 \not\subseteq \text{AC}^0 \) (regardless of non-uniformity) \[9\], this improves Corollary \[18\] at least when \( k = 1 \). Nevertheless, note the usual uniformity condition for \( \text{AC}^0 \) is not \( L \)- but the more restrictive \( \text{DLOGTIME}\)-uniformity \[28\], and there is good evidence that the two classes are distinct \[5\]. Using methods from \[1\], Corollary \[20\] may be rephrased for \( \text{AC}^0 \) in terms of \( \text{TIME}(\text{poly}(\log n)) \)- or even \( \text{TIME}(\text{log}^*(\text{log} n)) \)-uniformity (since \( \text{DELTAC} \) is computable in quadratic time by a Turing machine), but the \( \text{DLOGTIME}\)-uniformity case remains unclear.

\section{Strong ACA}

So far, we have considered ACAs strictly as language acceptors. As such, their time complexity for inputs not in the target language (i.e., those which are not accepted) is essentially ignored. In this section, we investigate ACAs as \textit{deciders}, that is, as machines which must also reject invalid inputs (under the same complexity condition as accepting). Inspired by \[26\], we name these machines \textit{strong} ACAs and refer to the ACAs hitherto considered as \textit{weak} ACAs. A strong ACA signals rejection via the same mechanism as acceptance (i.e., all cells are simultaneously in a final state) but using a distinct set of final states:
Sublinear-Time Language Recognition and Decision by One-Dimensional Cellular Automata

We stress the above does not necessarily imply $ACA(t) \subseteq SACA(t)$; there may well be an ACA which recognizes its language in time, say, $O(\sqrt{n})$, while rejecting words not in the language (with an SACA) requires $\Omega(n)$ time. Nevertheless, $SACA(t) \not\subseteq ACA(t)$ holds for $t \in o(n)$ since $SOMEONE \notin ACA(O(1))$ (see Section 3) but $SOMEONE \in SACA(O(1))$. For example, the local transition function $\delta$ of the SACA may be such that $\delta(q_1,0,q_2) = r$, $\delta(q_1,1,q_2) = a$, $\delta(q_1,r,q_2) = a$, and $\delta(q_1,a,q_2) = a$, where $q_1$ and $q_2$ are arbitrary states, and $a$ and $r$ are the accept and reject states, respectively; see Figure 2 for an example. We also remark $ACA(O(n)) = SACA(O(n))$ and that Lemma 9 generalizes to strong ACAs:

Definition 21 (SACA). A strong ACA (SACA) is an ACA $C$ which, in addition to its set $A$ of accept states, also has a non-empty subset $R \subseteq Q \setminus \{q\}$ of reject states that is disjoint from $A$ (i.e., $A \cap R = \emptyset$). Every input $w \in \Sigma^*$ of $C$ admits (either) an $A$- or an $R$-final configuration (but not both); in the latter case, $C$ is said to reject $w$. The time complexity of $C$ (with respect to $w$) is the number of steps elapsed until $C$, given input $w$, reaches an $A$- or $R$-final configuration (depending on whether $w \in L(C)$ or $w \notin L(C)$, respectively). $SACA(t)$ denotes the SACA analogue of $ACA(t)$.

Lemma 22. Let $C$ be an ACA and let $w \in \{0,1\}^*$ be a word which $C$ decides with time complexity $\tau = \tau(w)$ (i.e., in exactly $\tau$ steps). Then, for every word $w' \in \{0,1\}^*$ such that $p_{2\tau}(w) = p_{2\tau}(w')$, $I_{2\tau+1}(w') = I_{2\tau+1}(w)$, and $s_{2\tau}(w) = s_{2\tau}(w')$, $C$ decides $w'$ with time complexity $\tau$, and $w \in L(C)$ holds if and only if $w' \in L(C)$.

Proof. Let $A$ and $R$ be the set of accept and reject states of $C$, respectively, and let $c_0$ and $c'_0$ denote the initial configurations for $w$ and $w'$, respectively. Given $w \in L(C)$ (resp., $w \notin L(C)$), it suffices to prove that, on input $w'$, the following holds: (A) $c'_r = \Delta^\tau(c'_0)$ is $A$-final (resp., $R$-final); and (B) $c'_r = \Delta^\tau(c'_0)$ is neither $A$- nor $R$-final for $\tau' < \tau$.

The proof of (A) is essentially the same as that of Lemma 9. For (B), it suffices to prove that, in every such $c'_r$, there is a cell $z_1$ which is not accepting as well as a cell $z_2$ which is not rejecting. Since the trace of $C$ for $w$ is such that only the last configuration $c_r$ can be $A$- or $R$-final, there is $z_1$ which is not accepting as well as $z_2$ which is not rejecting in $\Delta^\tau(c'_0)$. An argument as in the proof of Lemma 9 yields this is also the case for $c'_r$.

It might appear tempting to relax the requirements of the lemma to $I_{2\tau+1}(w') \subseteq I_{2\tau+1}(w)$ (as in Lemma 9). We stress, however, the equality $I_{2\tau+1}(w) = I_{2\tau+1}(w')$ is crucial towards establishing claim (B); otherwise, it might be the case that $C$ takes strictly less than $\tau$ steps to decide $w'$ and, hence, $w \in L(C)$ may not be equivalent to $w' \in L(C)$.

We note that, in addition to Lemmas 9 and 11, the results from Section 4 are extendable from weak to strong ACAs; we leave a more systematic treatment as a topic for future work. The remainder of this section is concerned with characterizing constant-time SACA computation (as a parallel to Theorem 6), as well as establishing the minimal time bound for SACAs to decide languages other than those in $SACA(O(1))$ (as Theorem 5 and the result $BIN \in ACA(O(\log n))$ do for weak ACAs; see Section 3).
5.1 The Constant-Time Case

First, notice that, for any SACA $C$, swapping the accept and reject states yields an SACA with the same time complexity and which decides the complement of $L(C)$. Hence:

**Proposition 23.** For any function $t : \mathbb{N}_+ \to \mathbb{N}_0$, SACA$(t)$ is closed under complement.

Using this, we prove the following, which characterizes constant-time SACA computation as a parallel to Theorem 6 (and from which also $\text{ACA}(O(1)) \subseteq \text{SACA}(O(1))$ follows):

**Theorem 24.** $\text{SACA}(O(1)) = \text{LT}$.  

**Proof.** The inclusion $\text{SACA}(O(1)) \subseteq \text{LT}$ is obtained by using the locally testable property (see Definition 2) together with Lemma 22. For the converse inclusion, we use that, for every set $F$ in the Boolean closure over a collection $\mathcal{S}$, there are $S_1, \ldots, S_m \in \mathcal{S}$ such that

$$F = S_1 \cup (\overline{S_2} \cap (S_3 \cap (S_4 \cap \cdots (S_{m-1} \cup \overline{S_m})))))$$

where $\overline{S}$ denotes the complement of $S \in \mathcal{S}$. Using that LT is the Boolean closure of $\mathcal{S} = \text{SLT}$, it suffices to show that, for any such $F$, there is an SACA $C$ with $L(C) = F$.

First, we prove $\text{ACA}(O(1)) \subseteq \text{SACA}(O(1))$; due to Theorem 6, $\text{SLT} \subseteq \text{SACA}(O(1))$ follows. Let $C'$ be an ACA with time complexity bounded by $t \in \mathbb{N}_0$. Since $C'$ must accept any input $w$ in at most $t$ steps, if $t + 1$ steps elapse without $C'$ accepting, then necessarily $w \notin L(C')$. Hence, $C'$ can be transformed into an SACA $C''$ by having all cells unconditionally become rejecting in step $t + 1$. We have then $L(C') = L(C'')$, and $\text{ACA}(O(1)) \subseteq \text{SACA}(O(1))$ follows.

For the construction of $C$, fix $F$ and use induction on the number $r \in \mathbb{N}_0$ of set operations in $F$. The induction basis is given by $\text{SLT} \subseteq \text{SACA}(O(1))$. For the induction step, let $F' = \overline{S_2} \cap (S_3 \cup (S_4 \cup \cdots))$ and note the complement of $F'$ is $F'' = S_2 \cup (\overline{S_3} \cap (S_4 \cup \cdots))$; hence, applying the induction hypothesis we obtain an SACA for $F''$ and, using Proposition 23, we also obtain an SACA $C'$ with $L(C') = F'$. In addition, let $C_1$ be an ACA for $S_1$ with time complexity $t \in \mathbb{N}_0$. Now we may describe the operation of $C$: $C$ simulates $C_1$ on its input $w$ (while saving $w$) for $t$ steps and accepts if $C_1$ does; otherwise, if $t + 1$ steps elapse without $C$ accepting, then it simulates $C''$ (including its acceptance or rejection behavior). Thus, if $w \in S_1$, then $w \in L(C)$; otherwise, $w \in L(C)$ if and only if $w \in F'$. Since $C_1$ and $C'$ both run in constant time, so does $C$. ▶

5.2 Beyond Constant Time

Theorem 8 establishes a logarithmic minimal time bound for (weak) ACAs to have an advantage over their constant-time counterparts. We now turn to obtaining a similar result for SACAs. As it turns out, the bound for SACAs is considerably larger:

**Theorem 25.** For any $t \in o(\sqrt{n})$, SACA$(t) \not\subseteq \text{REG}$.  

The inequality is due to Theorem 24. The proof idea is that any SACA whose time complexity is not constant admits an infinite sequence of words with increasing time complexity: however, the time complexity of each such word can be traced back to a critical set of cells which prevent the automaton from either accepting or rejecting. By contracting the words while keeping the extended neighborhoods of these cells intact, we obtain a new infinite sequence of words which the SACA necessarily takes $\Omega(\sqrt{n})$ time to decide:

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4 This is due to Hausdorff [10]. Interestingly, the same resource found application, for instance, not only complexity theory [3, 4] but also in characterizing related variants of cellular automata [18].
Proof. Let \( C \) be an SACA with time complexity bounded by \( t \). Due to Theorem 24, we may assume \( C \) does not have constant time complexity, that is, for every \( k \in \mathbb{N}_0 \) there is an input \( w \) which \( C \) takes strictly more than \( k \) steps to decide. Then, for every \( i \in \mathbb{N}_0 \), there is \( w_i \) such that \( C \) takes strictly more than \( i \) steps to decide \( w_i \); in particular, when \( C \) receives \( w_i \) as input, there are cells \( x_j^i \) and \( y_j^i \) for \( j \in \{0, \ldots, i\} \) such that \( x_j^i \) (resp., \( y_j^i \)) is not accepting (resp., rejecting) in step \( j \). Let \( J_i \) be the set of all \( z \in \{0, \ldots, |w_i| - 1\} \) with \( \min\{|z - x_j^i|, |z - y_j^i|\} \leq j \), that is, \( z \in N_j(x_j^i) \cup N_j(y_j^i) \) for some \( j \). Consider the restriction \( w_i' \) of \( w_i \) to the symbols having index in \( J_i \) (i.e., \( w_i'(k) = w_i(j_k) \) for \( J_i = \{j_0, \ldots, j_{m-1}\} \) and \( j_0 < \cdots < j_{m-1} \) and notice \( w_i' \) has the same property as \( w_i \) (i.e., \( C \) takes strictly more than \( i \) steps to decide \( w_i \)) and also \( |w_i'| = |J_i| \leq 2(i + 1)^2 \). Therefore, \( C \) has time complexity \( \Omega(\sqrt{n}) \) on the infinite set \( \{w_i' \mid i \in \mathbb{N}_0\} \), and the claim follows.

Using the language \( \text{IDMAT} \) from Section 3, we show the bound in Theorem 25 is optimal:

\begin{proposition}
\( \text{IDMAT} \in \text{SACA}(O(\sqrt{n})) \).
\end{proposition}

Proof. We construct an ACA \( A \) which, given an input \( w \), decides \( \text{IDMAT} \) in \( O(\sqrt{|w|}) \) time.

As a warm-up, first consider the case in which every block \( B \) has the same length \( b \in O(\sqrt{|w|}) \) and that every neighboring pair of blocks is separated by a single \# . The leftmost cell in \( B \) creates a special marker symbol \( m \). During this first procedure, every cell which does not contain such an \( m \) is rejecting. At each step, if \( m \) is on a cell containing a 0 or it determines the string it has read so far does not satisfy the regular expression \( 0^*10^* \), then it marks the cell as rejecting; otherwise, it does nothing (i.e., the cell remains not rejecting). \( m \) propagates itself to the right with speed 1, the result being that \( A \) does not reject in \( i > 0 \) steps if and only if for every \( p \in \{1, 01, \ldots, 0^{i-1}1\} \) there is a block in \( w \) for which \( p \) is a prefix. It follows that \( |w| \geq \frac{1}{2}i(i + 1) \) and, in particular, if \( b \) steps have elapsed, then \( |w| > \frac{1}{2}b^2 \). Thus, if \( A \) rejects a word during this procedure, then it does so in \( O(\sqrt{|w|}) \) time. Once \( m \) encounters \# , it triggers a block comparison procedure as in the ACA \( A' \) which accepts \( \text{IDMAT} \). This requires \( O(b) \) time. If a violation is detected, \( B \) becomes rejecting and maintains that state. Finally, if a number of steps have elapsed such that \( A' \) would already have accepted (which by construction of \( A' \) can be determined in \( O(b) \) time as a function of \( b \)), \( B \) becomes rejecting and maintains that result, even if it contains cells which had been previously marked as accepting. Thus, \( A \) accepts if and only if \( A' \) does (and rejects otherwise).

For the general case in which the block lengths vary, we let the two procedures run in parallel, with the cells of \( A \) switching between the two back and forth. More precisely, the computation of \( A \) is subdivided into rounds, with each round consisting of two phases, both taking constant time each; the time complexity of \( A \), then, is directly proportional to the number of rounds elapsed until a final configuration is reached. The two phases correspond to the two aforementioned procedures, that is, phase one \( (P_1) \) checks that the blocks satisfy the regular expression \( 0^*10^* \) as well as ensures the presence of the \( 0^*1 \) prefixes; phase two \( (P_2) \)
checks the blocks are of the same length and have valid contents. $P_1$ advances its procedure one step at a time, while $P_2$ advances two steps (as in the ACA construction; see Section 3), and we separate the two so as to not interfere with each other; namely, if a cell is accepting (resp., rejecting) in one of the two phases, then it is not necessarily so in the other one; it is only so if the procedure corresponding to the latter phase mandates it to be so. If $w \in IDMAT$, the two phases end simultaneously after $3b \in O(\sqrt{|w|})$ steps, and $A$ accepts. Conversely, if $A$ rejects, then it also does so in at most $3b' \in O(\sqrt{|w|})$ steps where $b' \in \mathbb{N}_+$ is maximal such that $#1, #01, \ldots, #0^{b'-1}1$ are all substrings of $w \not\in IDMAT$.

6 Conclusion and Open Problems

Following the definition of ACAs in Section 2, Section 3 reviewed existing results concerning the languages accepted by these machines in sublinear time, and we also observed that these ACAs operate in an inherently local manner (Lemmas 9 and 11). In Section 4, we proved a time hierarchy theorem (Theorem 13), narrowed down the languages in $ACA(t) \cap \text{REG}$ (Theorem 15), and obtained (strict) inclusions in the parallel computation classes $SC$ and $AC$ (Corollaries 18 and 20, respectively). The existence of a hierarchy theorem for ACAs is particularly interesting because obtaining an equivalent result for $NC$ and $AC$ (as alternating Turing machine classes) is an open problem in complexity theory. Also of note is that our proof of the hierarchy theorem does not rely on diagonalization (which is the prevalent technique for most computational models) but, rather, on a quintessential property of sublinear-time ACA computation (i.e., locality, as in the sense of Lemmas 9 and 11).

In Section 5, we considered ACAs as language deciders as opposed to simply acceptors, obtaining strong ACAs. The respective constant-time class is $LT$ (Theorem 24), a (strict) superset of $ACA(O(1)) = \text{SLT}_v$, while $\Omega(\sqrt{n})$ is the time complexity threshold for deciding languages other than those in $LT$ (Theorem 25 and Proposition 26).

There seems to be quite a few topics for future work. The primary concern is extending the results of Section 4 to SACAs. Naturally, one would also like to improve the inclusion of Theorem 15 to an equality (i.e., $ACA(t) \cap \text{REG} = \text{SLT}_v$ for $t \in o(n)$), which, together with Theorem 8, would imply $ACA(t) = ACA(O(1))$ for $t \in o(\log n)$; however, we are unaware of a characterization of $\text{SLT}_v$ in terms of syntactic subgroups, so a different approach may be needed. $SACA(O(1)) = LT$ is closed under union and intersection and we saw that $SACA(t)$ is closed under complement for any $t \in o(n)$; a further question would be whether $SACA(t)$ is also closed under union and intersection. Finally, the inclusion $ACA(t) \subseteq SACA(t)$ holds for $t \in o(\log n)$ as well as for $t \in \Omega(n)$ but not for $t \in \Omega(\log n) \cap o(\sqrt{n})$ (due to Theorem 25 and $\text{BIN} \in ACA(\log n)$); it remains open whether it holds or not for $t \in \Omega(\sqrt{n}) \cap o(n)$.

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