Quantized octupole acoustic topological insulator

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The Berry phase\(^1\) associated with energy bands in crystals can lead to quantized quantities, such as the quantization of electric dipole polarization\(^2,3\) in an insulator, known as a one-dimensional (1D) topological insulator (TI) phase\(^4\). Recent theories\(^5,6\) have generalized such quantization from dipole to higher multipole moments, giving rise to the discovery of multipole TIs, which exhibit a cascade hierarchy of multipole topology at boundaries of boundaries: A quantized octupole moment in the three-dimensional (3D) bulk can induce quantized quadrupole moments on its two-dimensional (2D) surfaces, which then produce quantized dipole moments along 1D hinges. The model of 2D quadrupole TI has been realized in various classical structures\(^7-12\), exhibiting zero-dimensional (0D) in-gap corner states. Here we report on the realization of a quantized octupole TI on the platform of a 3D acoustic metamaterial. By direct acoustic measurement, we observe 0D corner states, 1D hinge states, 2D surface states, and 3D bulk states, as a consequence of the topological hierarchy from octupole moment to quadrupole and dipole moment. The critical conditions of forming a nontrivial octupole moment are further demonstrated by comparing with another two samples possessing a trivial octupole moment. Our work thus establishes the multipole topology and its full cascade hierarchy in 3D geometries.

Topological phases of matter are generally formulated by quantized quantities expressed through Berry phase\(^1,13\). In 1D, the quantization of electric dipole moment, which is associated with the Berry phase of a parallel transport of ground state in momentum space\(^2,3\), leads to a 1D TI\(^4\). The mathematical generalization of the formulation of electric dipole moment based on Berry phase has produced topological invariants such as the celebrated Thouless-Kohmoto-Nightingale-den Nijs invariant\(^14\), or the so-called Chern number. Yet over decades, the relationship between Berry phase and multipole moments, as well as whether multipole moments can give rise to
topological phases, has remained as an open question until recent seminal works on quantized electric multipole TIs\textsuperscript{5,6}. Take the quantized quadrupole as an example. It has been shown in various classical systems\textsuperscript{7-12} that a 2D quantized quadrupole TI exhibits topological states at “boundaries of boundaries”: it lacks gapless 1D topological edge states, as in conventional 2D topological insulators, but instead hosts topologically protected 0D corner states. Such a generalized bulk-boundary correspondence has opened the door to the pursuit of higher-order TIs\textsuperscript{5-12,15-18} that can arise from both quantized dipole and multipole moments.

The most striking feature of multipole TIs is their cascade hierarchy of topology. Fig. 1a schematically illustrates such a three-level hierarchy. At the lowest level, the corner states ($Q$) are a direct result of quantized dipoles ($p_x$, $p_y$, $p_z$), similar to the well-known 1D TIs\textsuperscript{4}. At the intermediate level, the quantized dipoles are induced along 1D edges by the nontrivial quantized quadrupole ($q_{xy}$, $q_{yz}$, $q_{zx}$) in a 2D bulk, as demonstrated in previous works\textsuperscript{7-12}. At the highest level, the quantized quadrupoles arise on 2D surfaces of a 3D bulk which exhibit a nontrivial quantized octupole ($o_{xyz}$). From the perspective of the cascade hierarchy of multipole topology, the previously observed quadrupole TI phases\textsuperscript{7-12} can be treated as the 2D projection of a 3D octupole TI, and the 1D TI phases appear as 1D projection along the hinges. To our knowledge, the 3D octupole TI as well as its cascade hierarchy of multipole topology has never been observed.

Here we report on the observation of an octupole TI in an acoustic metamaterial. This metamaterial consists of coupled acoustic resonators, whose local resonances serve as artificial atomic orbitals and thus fulfill the description of tight-binding models\textsuperscript{5,6}. The positive and negative couplings\textsuperscript{7,8} between resonators are achieved by connecting the resonators with thin waveguides at different sides of resonance’s nodal line. By direct local acoustic measurements sweeping over all resonators of a finite sample, we are able to observe the in-gap 0D corner states, gapped 1D
hinge states, gapped 2D surface states, and gapped 3D bulk states, all of which arise from the nontrivial octupole moment ($\sigma_{xyz} = 1/2$) in the bulk. Furthermore, we corroborate our results by measuring another two samples with trivial octupole moments ($\sigma_{xyz} = 0$). These trivial samples exhibit completely different boundary signatures from those in the nontrivial sample. Our results demonstrate a full hierarchy of multipole topology from octupole moment to quadruple and dipole moment, which is unique to a quantized octupole TI.

The lattice model to realize an octupole TI\textsuperscript{5,6} is shown in Fig. 1b, where intra-cell (left panel) and inter-cell (right panel) couplings has strength of $\gamma$ and $\lambda$. Solid and dashed lines denote positive and negative couplings, respectively. The unit cell can be regarded as two coupled unit cells of quadrupole TIs with opposite settings. Due to the negative couplings, a $\pi$ flux exists on each facet, which not only opens a spectrum gap at half-filling, but also maintains the mirror symmetries (up to a gauge transformation). In the presence of mirror and inversion symmetries, the bulk octupole $O_{xyz}$ moment can only take quantized values of 0 or $1/2$, depending on the relative strength of $\gamma$ and $\lambda$\textsuperscript{5,6}.

We adopt coupled acoustic resonators\textsuperscript{19-24} to implement this model. The unit cell design is illustrated in Fig. 1c. Here each lattice site is a hard-wall cuboid resonator filled with air of size $80 \text{ mm} \times 40 \text{ mm} \times 10 \text{ mm}$. The dimensions are carefully designed so that a single resonator supports a dipole resonance mode (Fig. 1d, left panel) at $f_0 = 2162.5$ Hz, which is far away from other modes (Fig. 5a). The nearest neighbor coupling is realized by connecting resonators with thin waveguides. As shown in Fig. 1d, it is well known that when two resonators are coupled together with a positive coupling coefficient $\gamma$, the resulted eigenmodes will exhibit symmetric and antisymmetric phase relations with split eigenfrequencies of $f_0 + \gamma$ and $f_0 - \gamma$. If the sign of coupling is flipped, then the eigenmodes also exchange\textsuperscript{25}. To achieve that, we simply relocate the connecting waveguide to the
other side of the dipole mode’s nodal line. As plotted in the right panel of Fig. 1d, the two eigenmodes switch their eigenfrequencies. In addition, the amplitude of coupling is controlled by tuning the width of connecting waveguide. Following an optimization approach which takes into account effects such as the resonance frequency shift and the coupling between the dipole resonance mode and other resonance modes (Methods and Fig. 5), we design the acoustic octupole TI structure as in Fig. 1c. The connecting waveguides enabling positive (negative) couplings are colored in orange (blue) for illustration. According to the effective tight-binding model of our acoustic design obtained from numerical simulations via Schrieffer-Wolff (SW) transformation\textsuperscript{7,26}, the unwanted effects caused by the connecting waveguides such as resonance frequency shifts and couplings with other modes are neglectable in our design (Methods and Fig. 5). We have numerically retrieved the couplings as $\gamma = 9.8$ Hz and $\lambda = 53.3$ Hz.

To demonstrate the physics of the octupole TI, three samples are constructed. Nested Wilson loops are used to reveal the hierarchy topology\textsuperscript{5,6} (Methods and Fig. 6). Firstly, Wilson loop along one direction shows a gapped surface spectrum. Secondly, a nested Wilson loop along the second direction uncovers a gapped hinge spectrum. Finally, one more nested Wilson loop along the third direction gives the Wannier sector polarization $p_{\alpha}^\nu$, where $\nu$ refers to a certain Wannier sector and $\alpha = x, y, z$. We calculate the Wannier sector polarizations and eigenmodes of finite lattices with parameters obtained from simulations. The first sample with intra-cell couplings smaller than inter-cell couplings ($\gamma = 9.8 \text{ Hz} < \lambda = 53.3 \text{ Hz}$) has a quantized bulk octupole moment. Calculated topological indices are close to $\{p_x^\nu, p_y^\nu, p_z^\nu\} = \{1/2, 1/2, 1/2\}$, indicating that the hinges have quantized dipole moments, which are induced by quantized surface quadrupole moments that arise from a quantized octupole moment in the bulk. Calculations of a finite $5 \times 5 \times 5$ lattice consisting of
1000 sites show the observable effects of the bulk octupole moment: gapped bulk states, gapped surface states, gapped hinge states and in-gap corner states (Fig. 2a, d).

In contrast, the second sample possesses the intra-cell couplings larger than the inter-cell couplings ($\gamma = 53.3$ Hz $> \lambda = 9.8$ Hz). In such a case, $\{p^x_x, p^y_y, p^z_z\} = \{0, 0, 0\}$, meaning that only bulk states emerge and no boundary signatures can be found (Figs. 2b, e). The third sample has a more subtle feature. The intra-cell couplings are smaller than the inter-cell ones along $x$ and $y$ directions ($\gamma_{x,y} = 9.8$ Hz $< \lambda_{x,y} = 53.3$ Hz). In the $z$ direction, however, intra-cell couplings are larger than the inter-cell ones ($\gamma_z = 53.3$ Hz $> \lambda_z = 9.8$ Hz). In such a case, the Wilson loops yield the topological indices of $\{p^x_x, p^y_y, p^z_z\} = \{0.5, 0.5, 0\}$, leading to hinge states along $z$ and surface states on surfaces normal to $x$ and $y$ directions, apart from the bulk states (Figs. 2d, f).

We then probe above signatures of the octupole TI experimentally. We measure the acoustic response upon a local excitation at each site of the first sample (Fig. 3a) consisting of 1000 sites. Left panel of Fig. 3b shows measured results at an arbitrary frequency (2000 Hz). We divide the whole sample into four regions, as schematically shown in the right panel of Fig. 3b. The “corner” region consists of 8 sites at corners. The “hinge” region refers to 12 hinges, each of which covers 8 sites. The “surface” region has 6 surfaces, each of which is composed of 64 sites. The other 512 sites constitute the “bulk” region. The resulted average intensity spectra (see Methods for details) for different regions are shown in Fig. 3c. Peak frequencies of the spectra are in agreement with the calculation in Fig. 2a. We further plot the integrated intensity spectra over frequencies (integration regions are indicated in Fig. 3c) around peaks of corresponding spectra. As shown in Figs. 3d-g, the corner, hinge, surface and bulk states are successfully identified. Note that the measured intensities on surfaces are larger than those in the bulk in Fig. 3f, but are smaller than in the bulk in Fig. 3g. The most pronounced feature here is the in-gap corner states, which is away
from other states and thus can be clearly observed (Fig. 3d). These corner states are consequences of nontrivial bulk and feature unique robustness (Fig. 8), which have great potential for applications. The hinge, surface and bulk states are closer in frequency, leading to overlaps in the integrated intensity spectra (Figs. 3e-g).

Similar measurements are also conducted on the second and third samples. For the second sample, since there are no boundary states, the measured average intensity spectra for different regions have identical peaks (Fig. 4a). For the third sample, because of the anisotropy, we further treat hinges and surfaces along different directions separately. Because there is no corner state, the measured spectra in the corner region shows overlapped peaks as those along $z$ direction hinges (Fig. 4b, upper panel); they correspond to the hinge states (Fig. 4c). Moreover, spectra peaks along $x$ and $y$ direction hinges coincide with those in $x$ and $y$ normal surfaces (middle panel in Fig. 4b, where the spectra along $y$ direction hinge and in $y$ normal surface are not shown for brevity), which are features of the surface states (Fig. 4d). The remaining sites are dominated by bulk states which spread over $z$ normal surfaces and the bulk region (the lower panel in Figs. 4b, and Fig. 4e). These results suggest the tunable nature of the hinge and surface states, with the potential of constructing reconfigurable devices.

In conclusion, an acoustic octupole TI is designed and demonstrated through direct local acoustic measurements which reveal the boundary states induced by the nontrivial bulk octupole moment. Our results identify a new class of TIs with a quantized octupole moment, paving the way to studying various exciting phenomena such as topological multipole moment pumping\textsuperscript{5,6}, the effects of non-Hermicity\textsuperscript{27-29}, nonlinearity\textsuperscript{30} and disorder\textsuperscript{31} on multipole TIs in 3D geometries. The design principle can also be used to construct multipole TIs in photonic and mechanical
systems. In practice, the corner states of octupole TIs can be used as robust 0D cavities in 3D devices for energy trapping, sensing, enhanced light-matter interaction and lasing.

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**Author contributions**

H.X., Y.C. and B.Z. conceived the idea. H.X. designed the sample and performed theoretical analysis. S.-Q.Y. and H.-X.S. designed the experiments. Y.G., D.J., Y.-J.G. and H.-X.S. conducted the experiments. H.X., Y.C. and B.Z. wrote the manuscript with input from all authors. Y.C. and B.Z. supervised the whole project.

**Competing interests**

The authors declare no competing interests.

**Data availability**

The data that support the findings of this study are available from the corresponding authors on reasonable request.

**Methods**

**Design of the acoustic octupole topological insulator.** To build an acoustic metamaterial that can be mapped to the desired tight-binding model, we make use of two kinds of building blocks: resonators and coupling waveguides. Here a resonator is a hard-wall cavity filled with air, which supports various modes at different frequencies. The mode of our interest is a dipole mode which has a nodal line that can be used to achieve negative couplings. The cavity is designed to have a size of 80 mm × 40 mm × 10 mm to achieve a large separation between the dipole mode of our interest and other modes (Fig. 5a). Then we couple the resonators with small coupling waveguides, which is also hard-wall and filled with air. The strategy to realize altering sign of the coupling is just to tune the position of waveguide so that different parts of the nodal line are connected (see
Fig. 1d in the main text). The coupling strength is controlled by the width of the waveguide. With the building blocks and construction strategies mentioned above, we proceed to implement the octupole TI model. Noticing the tight-binding model for a quantized octupole TI (Fig. 1b in the main text) is constructed from two quadrupole TIs with opposite settings coupled along z direction, we firstly design the two layers as shown in Figs. 5b, c. Then these two layers are coupled along the third direction with dimerized coupling strength to build the acoustic octupole TI. Note along z direction we use two coupling waveguides to connect adjacent resonators in order to firmly hold the structure.

With the preliminary design in hand, we now proceed to the optimization process. There are several issues we consider. Firstly, the design is generally anisotropic. Especially the coupling strengths along z do not equal to those in the xy plane. Secondly, the coupling waveguides will introduce resonance frequency shifts, which may be different at different sites. Thirdly, the coupling waveguides will not only couple the dipole modes of different sites, but also couple all other modes. Although the first issue will not destroy the quantization of octupole moment, the last two will and may even lead to topological phase transition if their effects are sufficiently strong. Fortunately, recent studies\textsuperscript{7,26} have introduced the method of SW transformation\textsuperscript{32,33} to design metamaterials from tight-binding models. We extract the effective tight-binding model of our acoustic design by following steps. Firstly, we solve the lowest 5 eigenmodes of a single resonator using COMSOL Multiphysics, eigenmode solver. Then we also solve a sample containing eight unit cells (64 sites) for the lowest 320 modes (5 modes × 64 sites, excluding modes introduced by coupling waveguides). Next, we grid the data from simulations, resulting in matrices $U_j$ whose $i$th column contain $i$th eigenmode at site $j$. Subsequently, the prepared data is projected onto the basis of single resonator eigenmodes, given by $P_j=(A^T A)^{-1} A^T U_j$ where columns of matrix A contain the
eigenmodes of a single resonator. The coupling matrix $V$ then can be calculated through $V=PD P^{-1}$-$H_0$. Here $P = (P_1 \ldots P_{64})$, $D$ is a square matrix with eigenfrequencies of the 8-unit-cell sample contained in its diagonal elements, and $H_0$ is also a square matrix whose diagonal elements contain eigenfrequencies of single resonator and repeated 64 times. To get an effective description within the space of the mode of our interest, we perform SW transformation perturbatively\textsuperscript{25,33} to account for the couplings between the mode of our interest and other modes as long range couplings among the mode of our interest. With the effective tight-binding model in hand, we then tune the widths of coupling waveguides and distances between resonators so that the nearest coupling strengths along $z$ is almost the same as those in the $xy$ plane, and the effects of resonate frequency shifts and long ranges couplings are small while the frequency gap is still large enough for experiments. Besides, small holes are introduced to resonators located at the boundaries to shift their resonance frequencies to be the same as those in the bulk. In the final design, the lattice constants along $x$, $y$, $z$ directions are 200 mm, 200 mm and 100 mm, respectively, and the widths for intra-cell (inter-cell) coupling waveguides are 1.6 mm (4 mm) in the $xy$ plane and 1.34 mm (3.08 mm) along $z$ direction. The final design has a ratio of $\gamma/\lambda \approx 0.18$ with $\lambda = 9.8 \times (1 \pm 4\%)$ Hz and $\gamma = 53.3 \times (1 \pm 4\%)$ Hz, and the resonance frequencies of eight resonators in the unit cell are $f_0 = 2145.1 \times (1 \pm 0.05\%)$ Hz. The dispersion calculated from effective tight-binding model matches well with the simulated one (Fig. 5d). Due to the large separation between the target mode and other modes, the next-nearest couplings in the extracted tight-binding model are quite small (Figs. 5e, f). Thus, the designed acoustic lattice deviates almost negligibly from the ideal tight-binding model.

**Nested Wilson loops and topological invariants.** From the effective tight-binding model obtained by the method described in previous section, we can calculate the topological numbers from the nested Wilson loop method\textsuperscript{5,6}. Firstly, we calculate a Wilson loop within the four bands
below the bandgap along $z$ direction, yielding 2D Wannier bands (Fig. 6a). This procedure splits the original four bands which are almost degenerate in frequency into two Wannier sectors (labelled as $v_z^+$ and $v_z^-$ in Fig. 6b) that are spatially separated along $z$. Being gapped, the two Wannier sectors can carry their topological invariants. Next, we calculate a nested Wilson loop within one of the Wannier sectors (here we choose $v_z^-$) along $y$, which again yields two separated Wannier bands, denoted as $v_y^+$ and $v_y^-$ in Fig. 6c. One more nested Wilson loop over one of the sectors (again we choose $v_y^-$) gives the Wannier sector polarization along $x$ as $p_x^y = 0.49$. Similar procedures can be done for all directions (here the order of the (nested) Wilson loops makes negligible difference) to obtain all three polarizations $\{p_x^v, p_y^v, p_z^v\} = \{0.49, 0.48, 0.49\}$. Thus, the effects of couplings between the mode of our interest and other modes, and the resonance frequency shifts can be ignored.

**Numerical and experimental details.** Numerical simulations presented in this work are performed by commercial software COMSOL Multiphysics within the Pressure Acoustics module. The photosensitive rein boundaries are considered as sound rigid walls due to the large impedance match with air ($\rho = 1.18$ kg/m$^3$ and $v = 346$ m/s).

All samples are fabricated via a stereo lithography apparatus with the thickness of the rein to be 6 mm. The samples consisting of 1000 resonators (see Fig. 3a in the main text) have sizes around 1 m $\times$ 1 m $\times$ 0.5 m, exceeding the fabrication limit. Thus, they are divided into 8 parts which are fabricated separately and then assembled together. There are two small holes ($r = 2$ mm) located at two sides of each resonator, allowing for excitation and detection. When not in use, they are blocked with plugs.

In the experiments, the acoustic signal is launched from a balanced armature speaker, guided into the samples through a narrow tube ($r = 1.5$ mm) and collected by a microphone (Brüel&Kjær Type
Then the measured data is processed by Brüel&Kjær 3160-A-022 module to get the frequency domain spectrum.

In the measurements of site-resolved local response (Fig. 3 and Fig. 4 in the main text), the source and probe are always located at the same site. At each site $i$, we obtain the intensity of acoustic field normalized by the intensity of the source over a range of frequencies (1900 Hz – 2400 Hz), denoted as $P_i(w)$. This procedure is repeated over all the 1000 sites. To eliminate the influence of variations of excitation efficiency over different sites, we normalize the data by the sum of intensity over all frequencies to get normalized spectra, $N_i(w) = P_i(w)/\sum w P_i(w)$. Then the average intensity spectra for four different regions (bulk, surface, hinge and corner) are obtained by calculating average normalized intensity within each area respectively: $A_\alpha(w) = \sum_{i \in \alpha} N_i(w)/N_\alpha$, where $\alpha$ denotes calculated region (i.e., bulk, surface, hinge or corner), the summation is taken over the sites within the calculated region and $N_\alpha$ is the number of sites within the calculated region.

In all figures where arbitrary units are used, the data is normalized to the maximum value in each figure.

**Verification of $\pi$ flux.** Although the most remarkable feature of an octupole TI is the topological corner states located at corners of a finite sample, such corner states in general can also be found in systems without an octupole moment$^{34-36}$. Thus, it is important to verify the bulk topology to ensure the topological corner states come from the bulk octupole moment. Here we verify the $\pi$ flux in the limit of $\lambda \to 0$ (Fig. 7b) and $\gamma \to 0$ (Fig. 7c) to ensure our design can be mapped to the tight-binding lattice shown in Fig. 1b. As shown in Figs. 7d and f, the measured spectra when the source and microphone are placed at the same site (R1 or R7) features two peaks, corresponding to the two branches of eigenmodes (denoted by black circle on the horizontal axis). Furthermore,
when we fix the source at R1 (or R7) and measure the acoustic field over all eight sites, features of π flux on each facet of the cubic arise\textsuperscript{8,25}. As plotted in Figs. 7, e and g, intensity on the resonators located at diagonal positions of the excitation is almost zero and phases on the resonators adjacent to the excitation are 0 (π) if the coupling is positive (negative).

**Robustness of corner state.** Here we present tight-binding calculations as well as experimental investigations on the robustness of the corner states. Two types of perturbation are considered (see inset of Fig. 8c): resonance frequency shifts on the three sites next to the corner (perturbation 1) and resonance frequency shift on the corner site (perturbation 2). For perturbation 1, we tune the resonant frequencies of the three sites next to one of the corners to be $f_0(1 + df_i)$, where $f_0$ is the unperturbed resonant frequency and $df_i$ are random numbers uniformly distributed from -δ to δ. Fig. 8a shows calculated eigenfrequencies with different perturbation strength δ. As can be seen, the corner state stays almost unaffected. For perturbation 2, the perturbation is added on one of the corner sites. This perturbation will lead to the frequency shift of one of the corner states (Fig. 8b). To test the robustness of the corner states experimentally, we deliberately introduce on-site perturbations by putting metal balls into resonators which will shift the resonant frequency of a single resonator about 10 Hz. As shown in Fig. 8c, the spectra measured at the corner features a single peak both for the unperturbed case (grey curve) and the two perturbed cases (blue and orange curves). Perturbation 1 at the surrounding sites does not cause frequency shift, while perturbation 2 at the corner does. Then we map out the field distributions for the three cases at corresponding peak frequencies by fixing the source at the corner and measuring the acoustic pressure over all sites. The results near the corner (fields at positions out of the plots are neglectable and thus are not shown) are plotted in Figs. 8d-f. In particular, even with frequency shift in perturbation 2, the corner state maintains its stable distribution.
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Figure 1 | Octupole topological insulator and its acoustic metamaterial realization. **a,** Schematic of a bulk octupole moment, which induces surface quadrupole moments, edge dipole moments and corner charges. **b,** Tight-binding model of a quantized octupole TI. Left (right) panel illustrates intra-cell (inter-cell) coupling configuration. Solid (dashed) lines represent positive (negative) couplings. **c,** Acoustic metamaterial realization of the model in **b.** Here one unit cell that consists of eight resonators coupled by thin waveguides (orange and blue colors denote positive and negative couplings, respectively) is shown. **d,** Implementation of negative coupling with coupled acoustic resonators. Left panel shows the dipole mode of a single resonator. Right panel illustrates the eigenmodes of two coupled-resonator systems with coupling waveguides located at different sides of the resonance’s nodal line, which correspond to opposite coupling sign.
Figure 2 Calculations on finite lattices in topological and trivial phases. a-c, Calculated eigenvalues of finite 5×5×5 lattices with different coupling configurations. d-f, Summation of probabilities for different types of states in a-c, respectively. In a, there are four types of states (bulk, surface, hinge, and corner), which are plotted in d. In b, all the states are bulk states (e). In c, there are three types of states (bulk, surface and hinge), which are plotted in f.
Figure 3 Experimental demonstration of an acoustic octupole topological insulator. a, Photo of the fabricated sample with five unit cells along each direction. b, Left panel: measured sound intensity of all sites at an arbitrary frequency (2000 Hz). Right panel: illustration of the four regions (corner, hinge, surface and bulk) that are used to calculate average intensity spectra. c, Measured average intensity spectra for a finite lattice in the topological phase. d-g, Integrated intensity maps over frequencies around the peaks of corner (d), hinge (e), surface (f) and bulk (g) spectra in c.
Figure 4 Experimental demonstration of two samples with trivial bulk octupole moment. a, Measured average intensity spectra for the sample with \( \gamma > \lambda \). b, Measured average intensity spectra for the sample with \( \gamma_{x,y} < \lambda_{x,y} \) and \( \gamma_{z} > \lambda_{z} \). Spectra with peaks located at similar frequencies are plotted in the same figure. Shaded regions denote the frequency regions for intensity integration. c-e, Integrated intensity maps over frequencies around peaks of spectra in b, which reveal that the peaks in b correspond to hinge, surface and bulk states, respectively.
Figure 5 | Design of the acoustic octupole topological insulator. a, Lowest five eigenmodes of an isolated acoustic resonator with size 80 mm × 40 mm × 10 mm. Dashed box denotes the mode of our interest. b-c, Two layers of the designed acoustic lattice. Grey rectangles represent resonators and orange (blue) ones denotes coupling waveguides that realize positive (negative) couplings. d, Bulk dispersions of the designed acoustic lattice along high symmetry lines calculated from effective tight-binding model (blue lines) and COMSOL simulation (red dots). e-f, Illustrations of the effective tight-binding model extracted from simulations. e and f show the in-plane configurations for the bottom and top layers, respectively. Here the dots denote the sites, and width and color of the lines correspond to strength and sign of the couplings, respectively. The nearest coupling along z have the same strength as those in the xy plane, and the next-nearest couplings along z are neglectable.
Figure 6 | Nested Wilson loops and Wannier bands. a, Illustration of a Wilson loop along $z$ over the four bands below the bandgap. b, Resulted 2D Wannier bands (left) which are used to calculate the nested Wilson loop along $y$ (right). c, Resulted 1D Wannier bands whose sectors carry nontrivial polarizations along $x$. 
Figure 7 | Identification of π flux in the limits of vanishing inter-cell and intra-cell couplings.

a, Photo of a sample with vanishing intra-cell couplings (γ = 0). b, Schematic of the isolated cubic in the limit of λ→0. c, Schematic of the isolated cubic in the limit of γ→0. d, Measured spectra of the structure shown in b. Result for excitation and detection at R1 (R7) is plotted as blue (orange) curve. Black circles on the horizontal axis indicate simulated eigenfrequencies. e, Left panel: theoretically calculated eigenmodes. Right panel: experimentally measured field distributions at peak frequencies (denoted by vertical dashed lines in d) when source is placed at R1 (blue dashed box) and R7 (orange dashed box). f-g, Similar to d-e but measured for structure shown in c. In e-g, radius of the balls indicates amplitude and color denotes phase.
Figure 8 | Robustness of corner states. a, Eigenfrequencies of a finite lattice with resonant frequency shifts on the three sites next to one of the corners as a function of perturbation strength $\delta$. b, Eigenfrequencies of a finite lattice with resonant frequency shifts on one of the corners as a function of perturbation strength $\delta$. In the calculations, we have taken a lattice consisting of five unit cells along each direction with $f_0 = 2145.1$ Hz, $\lambda = 9.8$ Hz and $\gamma = 53.3$ Hz. c, Experimentally measured spectra at the corner site for different cases suggested in the inset. d-f, Experimentally measured field distributions for the three cases.