Fast transitionless expansions of Gaussian anharmonic traps for cold atoms: bang-singular-bang control

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Combining invariant-based inverse engineering, perturbation theory, and Optimal Control Theory, we design fast, transitionless expansions of cold neutral atoms or ions in Gaussian anharmonic traps. Bounding the possible trap frequencies and using a “bang-singular-bang” control we find fast processes for a continuum of durations up to a minimum time that corresponds to a purely bang-bang (stepwise frequency constant) control.

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I. INTRODUCTION

Cold atoms, neutral or ionized, are manipulated and stored in traps formed by different electromagnetic field configurations. Frequently the atom cloud has to be expanded or compressed, for example to achieve a lower temperature, to decrease the velocity spread, in cooling cycles, or simply to adapt the cloud size and facilitate further operations. Ideally this processes should be fast, to be able to repeat them many times or to avoid decoherence, and should not excite the system, i.e., they should be “frictionless” or “transitionless”.

There is currently much interest in designing fast, transitionless expansions or compressions via “shortcuts to adiabaticity”. They speed up adiabatic processes, reaching the same target states in shorter times. One of these techniques is invariant-based inverse engineering, which has been implemented experimentally for a cold thermal cloud and for a Bose-Einstein condensate in a magnetic trap. In general, it provides families of solutions to accelerate the dynamics. Optimal Control Theory (OCT) can be used to select, from among those families, the ones that optimize some relevant physical variable, such as the transient energy.

Most works devoted to finding shortcuts for trap expansions or compressions considered perfectly harmonic traps but actual confinements are of course anharmonic. In this work we shall design, combining invariant-based inverse engineering, perturbation theory, and OCT, the time dependence of the trap frequency for 1D confinements with (rather common) Gaussian anharmonicities, cigar-shaped optical traps. Here we use a different setting consisting on an effectively 1D trap with a fixed, tight radial confinement, see for example. We assume on the longitudinal direction \( x \) a superposed optical potential of Gaussian form:

\[
V(x,t) = V_0(t) \left( 1 - e^{-2x^2/w_0^2} \right),
\]

where \( V_0(t) \) may be varied by the laser intensity and \( w_0 \) is the waist. We also assume a negligible effect of this potential on the radial confinement and moderate excitations so that the effective 1D potential takes the form:

\[
V(x,t) \approx 2V_0(t) \left( \frac{x^2}{w_0^2} - \frac{x^4}{w_0^4} \right) = \frac{m}{2} \omega^2(t) \left( x^2 - \frac{x^4}{w_0^4} \right),
\]

where \( \omega^2(t) = 4V_0(t)/(mw_0^4) \). The Gaussian function in Eq. \( (1) \) is not unique of optical traps. In particular, the electrostatic potential of an ion trap usually resembles a Gaussian function. If we take as an example the trap geometry presented in, the electrode-voltage configurations given therein yield trapping potentials that can be fitted by Gaussian functions with typical coefficients of determination \( R^2 > 0.98 \). Linearity of the trapping potential with the voltages applied to the trap electrodes ensures that \( w_0 \) is independent of \( V_0 \) for ion traps, too. This makes our results extensible to
ions, where trap expansions and compressions are also of potential interest.

II. MODEL, INVARIANTS AND FIDELITIES

Consider a reduced Hamiltonian of the form $H(t) = \frac{p^2}{2m} + V(x, t)$ for an atom of mass $m$. $H$ can be rewritten as

$$H(t) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2(t)x^2 - \frac{1}{2}m\omega^2(t)\frac{x^4}{4\omega^2}$$

(3)

in terms of an unperturbed harmonic part and an anharmonic perturbation $V_1$. This form is not in the family of Lewis-Leach potentials compatible with quadratic-in-momentum invariants $24, 33$, so invariant-based engineering cannot be applied directly, as would be the case for pure harmonic oscillator expansions. The alternative route followed here is to work out first the family of protocols suitable for the purely harmonic trap; then use perturbation theory to write the fidelity for the perturbed trap; and, finally, maximize the fidelity (alternatively minimize the contribution of the anharmonic term to the potential energy).

A. Invariant-based inverse engineering for the harmonic oscillator

A harmonic oscillator with time dependent frequency has the following invariant $34$:

$$I = \frac{1}{2}\left(\frac{1}{b^2}m\omega^2(t)x^2 + \frac{1}{m}\Pi^2\right),$$

(4)

where $\Pi = pb - m\dot{x}$ plays the role of a momentum conjugate to the scaled position $x/b$. The scaling factor $b = b(t)$ satisfies the Ermakov equation $34$

$$\ddot{b} + \omega^2(t)b = \frac{K^2}{b^3},$$

(5)

and $K$ is in principle an arbitrary constant, which we fix as the initial frequency $K = \omega(0)$. To have $[H_0(t), I(t)] = 0$ at $t = 0$ and $t_f$, so that these operators share common eigenfunctions at the boundary times, we impose the boundary conditions $[7]

$$b(0) = 1, \quad \dot{b}(0) = 0, \quad \ddot{b}(0) = 0,$$

$$b(t_f) = \gamma, \quad \dot{b}(t_f) = 0, \quad \ddot{b}(t_f) = 0,$$

(6)

where $\gamma = (\omega(0)/\omega(t_f))^{1/2}$. The dynamical modes are eigenfunctions of $I(t)$ multiplied by Lewis-Riesenfeld phase factors $34$ and have the form

$$(x|\psi_n(t)) = \frac{1}{(2\pi)^{1/2}n!^{1/2}} e^{-\sum(n+1/2)\omega(0)f_n'/\sqrt{m\omega(0)}} \times \exp \left(\frac{m\omega(0)}{2\hbar} \left(\frac{b}{b} + \frac{\dot{\omega}(0)}{b^2}\right)^2 H_n \left(\frac{m\omega(0)}{\hbar} \right)^{1/2} \frac{x}{b}\right).$$

(7)

The solution of the time dependent Schrödinger equation with the Hamiltonian $H_0(t)$ can be expressed as $|\Psi(t)\rangle = \sum_n c_n|\psi_n(t)\rangle$, where $n = 0, 1, \ldots$ and the $c_n$ are time-independent amplitudes $34$.

B. Perturbation theory

Let us evaluate the fidelity $F = |\langle \psi_n(t_f)|\Psi(t_f)\rangle|$, where $\Psi(t) = U(t, 0)|\psi_n(0)\rangle$ and $U(t, 0)$ is the evolution operator for the Hamiltonian $H_0$. Using time-dependent perturbation theory, we approximate

$$|\Psi(t_f)\rangle = |\psi_n(t_f)\rangle - i\hbar \int_0^{t_f} dt U_0(t_1, t)|\psi_n(t_1)\rangle$$

$$- \frac{1}{\hbar^2} \int_0^{t_f} dt \int_0^{t_f} dt' U_0(t_1, t)|V_1(t)U_0(t, t')|\psi_n(t')\rangle + \ldots,$$

(8)

where $U_0$ is the unperturbed propagator for the invariant-based trajectory corresponding to the harmonic trap. Using $U_0(t, t') = \sum_j |\psi_j(t')\rangle\langle\psi_j(t)|$ we may write, after cancellation of some terms,

$$F = \sqrt{1 - \sum_{n\neq n'} |f_{n,n'}^{(1)}|^2},$$

(9)

where the first-order transition amplitudes are

$$f_{n,n'}^{(1)} = -\frac{i}{\hbar} \int_0^{t_f} dt' \langle\psi_n(t')|V_1(t')|\psi_n(t')\rangle$$

$$= \frac{i\hbar}{2m\omega(0)^2} \sqrt{\pi 2n+1} n!$$

$$\alpha_{n,n'} = \int_{-\infty}^{\infty} dy e^{-y^2} H_n(y) H_{n'}(y),$$

(10)

$H_n$ is the Hermite polynomial, and

$$\beta_{n,n'} = \int_0^{t_f} dt_1 b^4(t_1)\omega^2(t_1) e^{-i(n-n')\omega(0)\int_0^{t_f} \frac{dt_2}{\sqrt{\omega(t_2)}}}.$$

(12)

$F$ in Eq. (9) is correct to the second order but, since the resulting expression is complicated to optimize we shall use a simpler, approximate approach. Instead of Eq. (9) we may write the fidelity in terms of diagonal amplitudes as $|1 + f_{n,n}^{(1)} + f_{n,n}^{(2)} + \cdots|$, where

$$f_{n,n}^{(2)} = -\frac{1}{2} \sum_{n,j} |f_{n,j}^{(1)}|^2.$$
Using the triangular inequality $|x + y| \geq ||x| - |y||$, $F \geq 1 - |f_{n,n}^{(1)} + f_{n,n}^{(2)} + \cdots|$ and assuming that the perturbative corrections satisfy $|f_{n,n}^{(1)}| \gg |f_{n,n}^{(2)}|$, then

$$F \geq 1 - |f_{n,n}^{(1)}|. \quad (14)$$

In general the right hand side does not provide an accurate approximation for the fidelity but an approximate bound $F_b = 1 - |f_{n,n}^{(1)}|$, where, using Eq. (110) and Ermakov’s equation,

$$|f_{n,n}^{(1)}| = \frac{3\hbar}{8mw_0^2}[(n+1)^2 + n^2] \left(t_f - \frac{1}{\omega_0} \int_0^{t_f} \dot{b}b^3 dt \right). \quad (15)$$

The bound may thus be rewritten as

$$F_b = 1 - \lambda \int_0^{t_f} (\omega(0)^2 - \dot{b}b^2)dt, \quad (16)$$

where $\lambda = (3\hbar/4m\omega(0)^2w_0^2)(n^2 + n + 1/2)$. Integrating by parts, Eq. (110) is finally

$$F_b = 1 - \lambda \omega(0)^2 t_f - 3\lambda \int_0^{t_f} \dot{b}b^2 dt. \quad (17)$$

A more rigorous justification for the use of $F_b$ relies on rewriting it as $F_b = 1 - \nabla_1 t_f/\hbar$, where

$$\nabla_1 = \frac{1}{t_f} \int_0^{t_f} \langle \psi_n(t)|V_1|\psi_n(t)\rangle dt,$$

is the time-averaged perturbation energy. Thus, maximizing $F_b$ amounts to minimize the time average of the anharmonic perturbation. Next, we will use OCT to maximize $F_b$ for the ground state $n = 0$.

III. OPTIMAL CONTROL THEORY

If we set $x_1 = b$, $x_2 = \dot{b}/\omega_0$, $u(t) = \omega^2(t)/\omega_0^2$ and rescale time according to $t = \omega_0 t$ (from now on dots are derivatives with respect to $\tau$), the Ermakov equation can be replaced by the system

$$\dot{x}_1 = x_2, \quad (18)$$
$$\dot{x}_2 = -ux_1 + \frac{1}{x_1}. \quad (19)$$

The optimization goal is to find $u(t)$, constrained by $|u(t)| \leq \delta$, with $u(0) = 1$ and $u(t_f) = 1/\gamma^4$, see Eq. (25), that minimizes a cost function $J$. Generally, to minimize the cost function

$$J(u) = \int_0^{t_f} g(x(\tau))d\tau, \quad (20)$$

the maximum principle states that for the dynamical system $x = f(x, u)$, the coordinates of the extremal vector $x(\tau)$ and of the adjoint state $p(\tau)$ formed by Lagrange multipliers, fulfill Hamilton’s equations for a control Hamiltonian $H_c$: \[27\]

$$\dot{x} = \frac{\partial H_c}{\partial p}, \quad \dot{p} = -\frac{\partial H_c}{\partial x}. \quad (21)$$

where

$$H_c(p, x, u) = p_0g(x(\tau)) + p^T \cdot f(x(\tau), u). \quad (22)$$

The superscript $T$ here denotes “transpose”. For $0 \leq \tau \leq \tau_f$, the function $H_c(p, x, u)$ attains its maximum at $u = u(\tau)$, and $H_c(p, x, u) = c$, where $c$ is constant.

In the anharmonic trap expansion we define the cost function

$$J = \int_0^{t_f} x_1^2 x_2^2 d\tau, \quad (23)$$

to maximize $F_b$, see Eq. (17), and the control Hamiltonian is

$$H_c = p_0x_1^2 x_2^2 + p_1x_2 + p_2 \left(-ux_1 + \frac{1}{x_1} \right). \quad (24)$$

With the control Hamiltonian, Eq. (24) gives the following costate equations:

$$\dot{p}_1 = -2p_0x_1^2 x_2^2 + p_2u + p_2^2 \frac{3}{x_1^2}, \quad (25)$$
$$\dot{p}_2 = -2p_0x_1 x_2 - p_2.$$

A. Bang-bang control

According to the maximum principle, the control $u(t)$ maximizes the control Hamiltonian at each time. Note that $H_c$ is a linear function of the control variable $u$. Since $u$ is bounded, $|u| \leq \delta$, the optimal control that maximizes $H_c$ is determined by the sign of the coefficient of $u$, which is $-p_2x_1$. Since $x_1 > 0$, when $p_2 \neq 0$, the optimal control in $(0, \tau_f)$ is given by

$$u(\tau) = \begin{cases} -\delta, & p_2 > 0 \\ \delta, & p_2 < 0 \end{cases}, \quad (26)$$

so the control sequence is a “bang-bang” process with piecewise constant frequencies,

$$u(\tau) = \begin{cases} 1, & \tau \leq 0 \\ -\delta, & 0 < \tau < \tau_1 \\ \delta, & \tau_1 < \tau < \tau_1 + \tau_2 \\ 1/\gamma^4, & \tau \geq \tau_f. \end{cases} \quad (27)$$

At the time boundaries they are given by the boundary conditions, and in between they saturate the imposed bound in two segments with imaginary and real values. Since imaginary frequencies are allowed the bound limits the curvature of the potential which becomes an anti-trap (repulsive) in the first time segment. When $u$ is constant and Eqs. (27) and (10) are satisfied, we have

$$x_1^2 + ux_1^2 + \frac{1}{x_1} = c, \quad (28)$$
where $c$ is a constant. Using Eq. (28) and the boundary conditions, $x_1(\tau)$ can be solved as

$$x_1(\tau) = \begin{cases} \delta - \frac{1}{\delta} + \frac{2}{2\delta} \cosh 2\sqrt{\delta} \tau, & \tau \in [0, \tau_1], \\ \delta^2 - \frac{4}{2\delta} \cosh \frac{1}{\delta}(\tau_f - \tau), & \tau \in [\tau_1, \tau_1 + \tau_2], \end{cases}$$

where $\tau_f = \tau_1 + \tau_2$ and

$$\tau_1 = \frac{1}{2\sqrt{\delta}} \cosh^{-1} \left[ \frac{\delta + 1}{\gamma^2 (\delta + 1)} \right],$$

$$\tau_2 = \frac{1}{2\sqrt{\delta}} \cos^{-1} \left[ \frac{\gamma^2 (\delta - 1)}{\gamma^4 - 1} \right].$$

We calculate $\tau_f$ (bang-bang) = $\tau_1 + \tau_2$ from the parameters $\delta$ and $\gamma$, so $\tau_f$ (bang-bang) is not arbitrary if they are fixed, as is usually the case. In the next section we shall see that arbitrary times larger than $\tau_f$ (bang-bang) are possible.

**B. Bang-singular-bang control**

1. Constrained frequency

When $p_2 = 0$ in some time interval, the maximum principle provides a priori no information about the optimal (“singular”) control in this interval [20, 33]. Suppose that $p_2 = 0$ for $\tau \in [\tau_1, \tau_1 + \tau_2]$, then it follows from Eq. (25)

$$\dot{p}_1 = -2p_0 x_1 x_2^2, \quad p_1 = -2p_0 x_2^2 x_2.$$  (31)

Therefore

$$\dot{x}_2 x_1 + x_2^2 = 0.$$  (32)

Integrating the above equation, we have

$$x_2 = \frac{c_1}{x_1}.$$  (33)

Then $x_1(\tau)$ takes the form

$$x_1(\tau) = \sqrt{c_1 \tau + c_2}, \quad \tau \in [\tau_1, \tau_1 + \tau_2].$$  (34)

Using Eq. (19), the control on the singular point $p_2 = 0$ is given by

$$u_s = \frac{1 + x_1^2 x_2^2}{x_1^4}.$$  (35)

The “bang-singular-bang” control sequence [20, 33] with two intermediate switchings at $\tau = \tau_1$ and $\tau = \tau_1 + \tau_2$ is

$$u(t) = \begin{cases} 1, & \tau \leq 0, \\ \delta, & 0 < \tau < \tau_1, \\ u_s, & \tau_1 < \tau < \tau_1 + \tau_2, \\ \delta, & \tau_1 + \tau_2 < \tau < \tau_f, \\ \frac{1}{\gamma^4}, & \tau \geq \tau_f. \end{cases}$$  (36)

where $\tau_f = \tau_1 + \tau_2 + \tau_3$.

Using the boundary conditions, Eq. (36), the function $x_1(\tau)$ can be solved as

$$x_1(\tau) = \begin{cases} \frac{\delta}{\sqrt{\delta}} - \frac{1}{\delta} + \frac{2}{2\sqrt{\delta}} \cosh 2\sqrt{\delta} \tau, & \tau \in [0, \tau_1], \\ \frac{\delta^2}{\sqrt{\delta}} - \frac{4}{2\sqrt{\delta}} \cosh \frac{1}{\delta}(\tau_f - \tau), & \tau \in [\tau_1, \tau_1 + \tau_2]. \end{cases}$$

From the boundary conditions, Eqs. (33) and (38), we find the trajectory

$$x_2^2 - \delta x_1^2 + \frac{1}{x_1^4} = 1 - \delta,$$  (38)

for $0 \leq \tau \leq \tau_1$ and

$$x_2^2 + \delta x_1^2 + \frac{1}{x_1^4} = \delta^2 + \frac{1}{\gamma^4},$$  (39)

for $\tau_1 + \tau_2 \leq \tau \leq \tau_f$. Then we solve the first junction point, using Eqs. (38) and (39), as

$$x_1^2(\tau_1) = \frac{\delta - 1 + \sqrt{\delta^2 + (4\delta^2 + 2)\delta + 1}}{2\delta},$$  (40)

and at the second junction we get from Eqs. (33) and (39)

$$x_1^2(\tau_1 + \tau_2) = \frac{\delta^2 + 1 + \sqrt{\delta^2 - 4\delta^2 + 2})\delta - 1 + \delta}{2\delta\gamma^2}. $$  (41)

Because of the continuity of the function $x_1(\tau)$, the interval times $\tau_{1,2,3}$ can be found from Eq. (37),

$$\tau_1(c_1) = \frac{1}{2\sqrt{\delta}} \cosh^{-1} \left[ \frac{\delta x_2^2(\tau_1) - \delta - 1}{\delta + 1} \right],$$  (42)

$$\tau_2(c_1) = \frac{1}{2\sqrt{\delta}} \left[ x_1^2(\tau_1 + \tau_2) - x_1^2(\tau_1) \right],$$  (43)

$$\tau_3(c_1) = \frac{1}{2\sqrt{\delta}} \cos^{-1} \left[ \frac{2\delta^2 - x_1^2(\tau_1 + \tau_2) - \delta - 1}{\delta + 1} \right].$$  (44)

The final time $\tau_f$ is

$$\tau_f = \tau_1(c_1) + \tau_2(c_1) + \tau_3(c_1),$$  (45)

which determines the constant $c_1$, and thus $c_2 = x_1^2(\tau_1) - 2c_1 \tau_1$.

Fig. 2 depicts the optimal trajectory of $x_1$ and $x_2$ for different final times and specific values of $\gamma$ and $\delta$. The optimal bang-singular-bang solution, $\tau_f$ (bang-bang). With increasing $\tau_f$, the first and third step times, $\tau_1$ and $\tau_3$, decrease, and the intermediate step becomes dominant. Fig. 2 shows the control function $u(t)$ and corresponding scaling factor $b$ for bang-singular-bang control when $\tau_f = 5$. 
At this point the complication of a bang-singular-bang protocol may appear unnecessary, as the bang-bang control is simpler and takes less time. However, we shall see in the next section that the bang-bang protocol is also less stable with respect to the anharmonic perturbation than the bang-singular-bang ones.

The frequency is unconstrained. In this case, Eq. (42) becomes

\[ \tau_1 = 0, \quad \tau_2(c_1) = (\gamma^2 - 1)/2c_1, \quad \tau_3 = 0, \]

so that the final time is \(\tau_f = (\gamma^2 - 1)/2c_1, c_2 = 1\), and the trajectory is

\[ x_1(\tau) = \sqrt{\frac{\gamma^2 - 1}{\tau_f}} \tau + 1, \quad \tau \in [0, \tau_f], \]

with a scaling factor of the form

\[ b(t) = \sqrt{\frac{\gamma^2 - 1}{\tau_f}} t + 1, \quad t \in [0, \tau_f], \]

which was found independently using the Euler-Lagrange equation [10] (the boundary conditions [9] are not completely fulfilled). The bound for the fidelity, see Eq. (16), becomes

\[ F_{EL} = 1 - \frac{3\hbar}{8\mu v^2} \left[ t_f + \frac{3(\gamma^2 - 1)^2}{4t_f \omega_0^2} \right]. \]

Here and in the next section we rescale time again as \( t = \tau/\omega(0) \).

IV. EXACT FIDELITIES AND ANHARMONIC PERTURBATION ENERGY

So far we have maximized an approximate bound [17] for the fidelity of a trap expansion with anharmonic terms. Now we shall calculate for comparison the actual fidelity, \( F = |\langle \psi_0(t_f) | \Psi(t_f) \rangle|\), of the resulting protocol, solving the time-dependent Schrödinger equation with the split-operator method. For further comparison we also consider the protocol for the pure harmonic oscillator based on the simple polynomial ansatz \( b(t) = \sum_{j=0}^{5} a_j t^j \)

Solving for the coefficients with the boundary conditions [18], we get \( b(t) = 6(\gamma - 1)s^5 - 15(\gamma - 1)s^4 + 10(\gamma - 1)s^3 + 1 \), where \( s = t/t_f \). The corresponding frequency is found from the Ermakov equation [19].

Fig. 3 (a) shows the (approximate) bound \( F_b \) for the polynomial ansatz, and for the optimized bang-singular-bang protocol versus the waist \( w_0 \) for a fixed \( t_f \). The third (upper) curve is the bound \( F_{EL} \) in Eq. (17) which, as the frequency is not constrained for it, is above the others. The actual fidelities, corresponding to the numerical solution of the time-dependent Schrödinger equation with the designed protocols, are above these bounds, see Fig. 3 (b), which shows the high fidelity achieved by the bang-singular-bang control.

Fig. 4 depicts the bounds and actual fidelities with respect to \( t_f \) for a fixed waist \( w_0 \). Along the curve for the optimized protocols (red dashed line), the bang-bang control (the point at the left extreme) takes the minimal time but is also more sensitive to the anharmonicity,
as it gives the worst fidelity, see Fig. 4 (b). To achieve higher fidelities larger times and thus bang-singular-bang control are necessary. The fidelity of the optimized protocol is higher than that for the polynomial ansatz as long as $|\omega(t)|$ for the polynomial ansatz stays below the imposed frequency bound for all $t$. The time $t_f$ below which this condition does not hold is also marked by a dot on the polynomial curve. The fidelity bound $F_b$ decreases after a maximum, see Fig. 5 (a). This behavior is not reproduced by the actual fidelity, which tends to one as $t_f$ increases. It may be understood by noting that $F_b = 1 - \sqrt{V_1 t_f/\hbar}$, where $V_1$ is the time-averaged perturbation energy. From Fig. 4 $V_1 \propto t_f^{-2}$ for small $t_f$ but it tends to a constant value for larger times as no transient excitations are produced. Correspondingly $F_b$ shows two asymptotic regimes

$$F_b = \begin{cases} 1 - a_1/t_f, & t_f \ll 2\pi/\omega(0) \\ 1 - a_2 t_f, & t_f \gg 2\pi/\omega(0) \\ \end{cases}$$

where $a_1$ and $a_2$ are constants.

**V. CONCLUSION**

In this work, we have combined invariant-based inverse engineering, perturbation theory, and OCT to design fast and transitionless expansions of cold atoms in an anharmonic Gaussian trap. We find that the optimal protocol obtained from an approximate fidelity bound $F_b$ is a bang-singular-bang solution. This protocol minimizes the contribution of the anharmonicity to the potential energy.

Even though we have specifically treated a one dimensional trap with the quartic anharmonicity resulting from a Gaussian beam the results could be applied or generalized to several other systems presenting anharmonic deviations from a harmonic confinement in optomechanics [4], mechanical resonators [36], or trapped ions [37, 38]. Ion traps in particular may offer soon the technological possibility to change the trapping potential on time scales much shorter than the ion oscillation frequencies facilitating the practical application of bang-bang or bang-singular-bang protocols [32].
FIG. 6: (Color online) Time average of the anharmonic potential energy $\langle V(t)/\hbar \rangle$ as a function of the final expansion time $t_f$. $k$ is the scaling exponent. The optimal protocols with constrained frequency (dashed red) and unconstrained frequency (dotted black), and the protocol designed with the polynomial ansatz (solid blue) are compared. Parameters: $\omega_0 = 20\lambda$, $\lambda = 1060 \text{ nm}$, $\omega(0) = 2\pi \times 2500 \text{ Hz}$, $\omega(t_f) = 2\pi \times 25 \text{ Hz}$, and $\delta = 1$.

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