Eccentricities and the Stability of Closely-Spaced Five-Planet Systems

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Abstract

We investigate the stability of idealized planetary systems consisting of five planets, each equal in mass to the Earth, orbiting a one solar mass star. All planets orbit in the same plane and in the same direction, and the planets are uniformly spaced in units of mutual Hill Sphere radii. However, in contrast to analogous studies by [Smith and Lissauer, 2009] and [Obertas et al., 2017], we integrate systems where one or more planets begin on eccentric orbits, with eccentricities, $e$, as large as $e = 0.05$ being considered. For a given initial orbital separation, larger initial eccentricity of a single planet generally leads to shorter system lifetime, regardless of which planet is initially on an eccentric orbit. The approximate trend of instability times increasing exponentially with initial orbital separation of the planets found previously for planets with initially circular orbits is also present for systems with initially eccentric orbits. Mean motion resonances also tend to destabilize these systems, although the reductions in system lifetimes are not as large as for initially circular orbits. Systems with all planets having initial $e = 0.05$ and aligned periapse angles typically survive far longer than systems with the same spacing in initial semi-major axis and one planet with $e = 0.05$, but they have slightly shorter lifetimes than those with planets initially on circular orbits.

1 Introduction

NASA’s Kepler mission has discovered hundreds of multiple planet systems [Fabrycky et al., 2014; Rowe et al., 2014]. Many of these multi-planet systems are quite tightly packed, where adjacent planets orbit much closer to one another than do those in the Solar System [Lissauer et al., 2011a]. For instance, the Kepler-11 system harbors six known planets, five of which have orbital periods between 10 and 47 days [Lissauer et al., 2011b]. Studying the orbital stability of such tightly-packed planetary systems provides clues to how they form and how long they survive. The present study can be seen as an extension of [Obertas et al., 2017] (henceforth OVT17), who investigated the...
stability of closely-spaced five-planet systems with the same masses and relative spacings as we use herein and all planets on initially circular orbits. Here, we expand on those findings and look at systems for which one planet starts out with a nonzero initial eccentricity. We also consider a restricted range of systems in which all planets begin on eccentric orbits.

The structure of the paper is as follows: We start with describing the setup of our simulations in Section 2. In Section 3 the focus is on circular systems and the well-known log-linear relationship between initial planetary separation and time to close encounter, which we reproduce with almost identical regression parameters. We then move on to the core part of this work. In Section 4 we study systems that start out with the innermost planet eccentric. We consider several different eccentricities and compare the systems’ lifetimes. In Section 5 we perform similar integrations, giving the outermost planet nonzero eccentricity, and in Section 6 we conduct a more limited study with one of the intermediate planets given nonzero eccentricity. Section 7 considers systems with all planets initially having eccentricity of 0.05: we study systems in which all planets initially have the same longitude of periapse, as well as systems in which the longitudes of periapse are chosen randomly. Our conclusions are summarized in Section 8.

2 Methodology

Our systems consist of five identical, 1 M⊕ (Earth-mass) planets, orbiting a 1 M⊙ (solar mass) star in the same direction, and initially separated by the same multiple of their mutual Hill radii (equation 2). We restrict ourselves to a two dimensional study by setting all inclinations to zero. We investigate the effects of nonzero initial eccentricities on these five-planet systems.

In what follows, time is measured in units of the initial orbital period of the innermost planet, i.e., Earth-years if the innermost planet begins at 1 AU. Our simulations were performed with REBOUND, the N-body simulation code by [Rein and Liu, 2012]. In particular, we use the symplectic Wisdom-Holman integrator (WHFast) provided by REBOUND [Rein and Tamayo, 2015]. WHFast is fast and unbiased, but ill-suited for resolving collisions and close encounters. Thus, we stop a simulation as soon as a close encounter occurs, and compare lifetimes computed in this manner.

2.1 Mutual Hill radius

Before describing the initial conditions, we recall the definition of the Hill radius, which is our unit of distance between initial planetary orbits. The Hill radius of an astronomical object, here a planet, orbiting a much more massive object, here a star, is the radius of the zone in which the planet’s gravitational pull is the primary force affecting motion relative to the planet. For small eccentricities, it can be approximated as

\[
R_H \approx a \left( \frac{m}{3M_\star} \right)^{1/3},
\]

where \(M_\star\) and \(m\) are the masses of the star and planet, respectively, and \(a\) is the semi-major axis. The Hill radius is then proportional to the semi-major axis. In this paper, following convention for this type of simulations, we use a slightly modified version: the mutual Hill radius between planet \(i + 1\) and its immediate neighbor orbiting closer to the host star, planet \(i\):

\[
R_{H_{i,i+1}} \approx \left( \frac{m_{i+1} + m_i}{3M_\star} \right)^{1/3} \left( \frac{a_i + a_{i+1}}{2} \right).
\]

Here the denominator of the first ratio continues to be the host star’s mass, as in OVT17, but [Quarles and Lissauer, 2018] use the more accurate formula \(M_\star + (i - 1)M_⊕\). The initial separations
between the planets are expressed in terms of their mutual Hill radii, with the innermost planet placed at 1 AU.

The stability of systems of two planets has been characterized analytically; see Hill, 1878 and Gladman, 1993. In contrast, stability of more than two planet systems cannot be derived analytically. Such an investigation therefore requires extensive numerical integrations, for example Chambers et al., 1996; Marzari and Weidenschilling, 2002; Smith and Lissauer, 2009; Pu and Wu, 2015; Morrison and Kratter, 2016; Obertas et al., 2017.

2.2 Initial and stopping conditions

We start by stating the initial conditions and describing the parameters used for our simulations.

2.2.1 Initial positions

The initial semi-major axes of the five planets are always chosen to be separated by the same number (not necessarily integer), \( \beta \), times their mutual Hill radii (we follow the notation of Smith and Lissauer, 2009, while OVT17 use \( \Delta \) instead of \( \beta \)):

\[
a_{i+1} - a_i = \beta R_{H_{i,i+1}}. \tag{3}
\]

From Equations (2) and (3), we have:

\[
a_{i+1} = a_i \left[ 1 + \beta \left( \frac{m_i + m_{i+1}}{3M_*} \right)^{1/3} \right] \left[ 1 - \beta \left( \frac{m_i + m_{i+1}}{3M_*} \right)^{1/3} \right]^{-1}. \tag{4}
\]

Systems of two planets on initially circular orbits are stable if \( \beta > 2\sqrt{3} \approx 3.4641 \), so we begin our numerical studies at \( \beta = 3.47 \). We perform batches of integrations in which we increment the initial \( \beta \) by 0.01 for each new simulation, while keeping everything else fixed at some specified nonzero initial value.

The true longitude is defined as

\[
\theta = \Omega + \omega + \nu, \tag{5}
\]

where \( \Omega \) is the longitude of ascending node, \( \omega \) the argument of pericenter, and \( \nu \) the true anomaly. Since we are working in a co-planar setting, the longitude of ascending node \( \Omega \) is zero, and for convenience we also set the argument of pericenter \( \omega = 0 \). This leaves us with the initial true anomaly \( \nu \), which is the angle from the pericenter to the planet’s position, and the only nonzero parameter, so the true longitude is \( \theta = \nu \). In this work, we choose it as \( \theta_i = 2\pi i \lambda \) radians, where \( i \) is the \( i \)-th planet starting from the innermost one with \( i = 1 \), and \( \lambda = 1 + \sqrt{5} \approx 1.618 \) is the golden ratio. This choice avoids any pair of planets starting near conjunction.

Finally, our time step has been chosen to be 0.049281314 years \( \approx 18 \) days.

2.2.2 Stopping conditions

We stopped any given simulation if a close encounter occurred, which we define as the distance between any two objects at the end of any time step being below 0.01 AU. We did not encounter a case where a planet has been ejected. We also set two more stopping conditions: one, due to limited computational resources, for any given simulation, namely a maximum run time of \( 10^{10} \) orbits of the innermost planet’s initial orbit (i.e., Earth-years, years hereafter); and one for a given batch of simulations running over a \( \beta \)-range, namely five of them exceeding \( 2 \times 10^9 \approx 10^{9.3} \) years. For systems that did not have a close encounter within ten billion years, we used ten billion years.
as the lifetime in the statistics, which produces a smaller bias in our fits than ignoring those runs would.

3 Initially Circular Orbits

For the circular case, we ran our simulations in the region $\beta \in [3.47, 8.73]$, the upper bound representing the initial separation of the fifth system that exceeded two billion years before a close encounter occurred. System lifetimes are shown as a function of $\beta$ in Figure 1. The general patterns that we find, with lifetimes increasing roughly exponentially with initial separation apart from dips near mean motion resonances, are similar to those noted by SL09 and OVT17. Our results show somewhat less scatter for similar values of $\beta$ than OVT17, presumably because they performed far more simulations and randomly selected initial planetary longitudes (with the latter choice being especially important for small values of $\beta$).

Figure 1: Close encounter times for systems whose planets all start out with zero eccentricity. On the vertical axis, $t_e$ stands for “time to encounter”. The blue line corresponds to the best-fitting power law dependence of system lifetime as a function of initial orbital separation; the orange line shows the fit for $5.32 \leq \beta \leq 8.73$, interval for which we have good stability lifetime values for all sets of runs with the outer four planets beginning on circular orbits; the green line shows the best fit including only systems with $\beta \leq 8.4$; its slope can be compared to previous work by Smith and Lissauer (2009) and OVT17 (see Table 1). The lines differ only slightly over these $\beta$-ranges, with the largest slope (in orange) belonging to the highest considered lower bound, where $\beta \in [5.32, 8.73]$.

The exponential increase in system lifetimes with separation is generally quantified using a log-linear relationship between the close encounter time, $t_e$, and initial separation of the planetary
orbital semi-major axes in units of the mutual Hill radius \( \beta \):

\[
\log \left( \frac{t_c}{t_0} \right) \approx b\beta + c, \tag{6}
\]

where \( t_0 \) is the initial orbital period of the innermost planet (one year), and \( b \) and \( c \) are constants. As noted in OVT17, however, this power-law fit does not capture the variations in close encounter times, which are up to two orders of magnitude off from the regression line.

We compare our coefficients with those found for the nonzero eccentricity cases in Section 4. A full tabulation of all computed coefficients and their standard errors \( \sigma \) can be found in Table 3. There, however, we considered new parameters \( b' \) and \( c' \), computed from a new variable \( \beta' \), shifted from \( \beta \) by \( \beta' \equiv \beta - 2\sqrt{3} \), following [Quarles and Lissauer, 2018].

| Reference                  | \( b \)   | \( c \)    | Range            |
|----------------------------|-----------|------------|------------------|
| Smith and Lissauer (2009)  | 1.012     | -1.686     | \( 3.4 \leq \beta \leq 8.4 \) |
| Obertas et al. (2017)      | 0.951     | -1.202     | \( 2\sqrt{3} < \beta < 8.4 \) |
| Obertas et al. (2017)      | 1.086     | -1.881     | \( 2\sqrt{3} < \beta < 10 \)  |
| This work                  | 0.972 ± 0.018 | -1.333 ± 0.107 | \( 2\sqrt{3} < \beta < 8.4 \)  |
| This work                  | 1.080 ± 0.019 | -1.895 ± 0.119 | \( 2\sqrt{3} < \beta \leq 8.73 \)  |

Table 1: Comparison of slopes and intercepts of power law fits to lifetimes of initially circular five planet systems in our simulations with those of previous studies.
4 Innermost planet eccentric

We now turn to the investigation of systems with initial conditions analogous to those considered in Section 3, with the exception that the innermost planet starts out with a nonzero eccentricity. In our first batch of simulations, we set $e_1 = 0.01$. The resulting lifetimes are plotted in Figure 2. Predictably, compared to the circular case, the close encounter times of those systems tend to be shorter for given $\beta$; hence we needed to go to higher values of $\beta$ to find five systems that survived for $> 2 \times 10^9$ years.

![Figure 2: Close encounter times for systems whose innermost planet starts out with eccentricity $e_1 = 0.01$; the upward-pointing triangle in the upper right represents a system that survived for the entire $10^{10}$ year interval simulated. In blue the corresponding regression line; in orange, the regression line for the $\beta$ interval for which we have values for all batches with the outer four planets beginning on circular orbits, which is [5.32, 8.73]. All slopes and intercept values can be found in Table 3.](image)

The next four figures show a visualization of the persistence of a log-linear relationship between initial separation and close encounter times, with various slopes. The lifetimes of systems with initial $e_1 = 0.02$ are displayed in Figure 3, those with $e_1 = 0.03$ are shown in Figure 4, and those with $e_1 = 0.05$ are shown in Figure 5. Note that several systems (one for $e_1 = 0.02$, eight for $e_1 = 0.03$, and twenty-two for $e_1 = 0.05$), suffer a close approach extremely rapidly, with lifetimes less than ten years. Those encounters are between planets 1 and 2 and occur during their first synodic period: given our choices of initial longitudes and periapse angle of the inner planet, the first passage occurs when the inner planet is near apoapse. Those excessively short lifetimes are thus direct consequences of our initial longitudes. Finally, Figure 6 combines all runs in one plot.
Figure 3: Close encounter times for systems whose innermost planet starts out with eccentricity $e_1 = 0.02$. In blue the corresponding regression line with the very first point removed (that system had a close encounter during the first synodic period); in orange, the regression line for $\beta \in [5.32, 8.73]$.

Figure 4: Close encounter times for systems whose innermost planet starts out with eccentricity $e_1 = 0.03$. In blue the corresponding regression line, starting after the last invalid point; in orange the regression line for the $\beta$ interval for which we have values for all $e_1$ eccentricities we considered.
Figure 5: Close encounter times for systems whose innermost planet starts out with eccentricity $e_1 = 0.05$. In blue the corresponding regression line, starting after the last invalid point; in orange, the regression line for the largest $\beta$ interval wherein simulations that survived more than one synodic period.
Figure 6: Close encounter times for initial $e_1 = 0, 0.01, 0.02, 0.03,$ and $0.05$, respectively.

We conclude that for any given value of $e_1$ simulated, system lifetime tends to roughly follow the log-linear relationship with $\beta$ of the form given in Eq. 6, although the coefficients differ from those of initially circular orbits.

We now quantify the differences in close encounter times by computing their average over a range of three Hill radii, ending with the last point available for the circular case (i.e., the fifth initially circular system exceeding two billion years of stability), which amounts to the range $\beta \in [5.74, 8.73]$.

Thus in Figure 7 each point represents the log-difference in close encounter time between one of the four eccentricity cases studied, and the initially circular system. Not surprisingly, the higher the eccentricity, the larger the close encounter time differences. The four averages over the whole range are recorded in Table 2.
Figure 7: Close encounter time differences between simulations with $e_1 = 0$, $e_1 = 0.01$, $e_1 = 0.02$, and $e_1 = 0.05$. In the legend and below, $[e_1 = 0]$ is a time unit and means $\log T_{e_1=0}$; and so on. The average time difference between the circular system and the corresponding (same $\beta$) eccentric system is given in Table 2. We omit similar plots for other comparisons of averages – the ratios follow the same pattern for comparisons of different eccentricities for the same planet. However, an interesting pattern is found for the comparison of the average close encounter time between different planets starting out with the same eccentricity.

![Graph showing close encounter time differences](image)

Table 2: The “average multiplicative factor” in close encounter time difference for each batch of systems compared to the circular systems. We compute this factor by exponentiating the average of the difference of the logarithms of the encounter times of simulations with the same value of $\beta$ within a specified range in $\beta$. The range consists of the last 300 overlapping points $\beta \in [5.74, 8.73]$. The batch of simulations labeled $e_{\text{align}} = 0.05$ has all planets on eccentric and aligned orbits; see Section 7 for details.
5 Outermost planet eccentric

In this section, we present simulations analogous to those in Section 4 except that the initial eccentricity is given to the outer planet. We expect results qualitatively similar to those in which the inner planet is eccentric, but quantitative differences may exist.

We introduce here the Angular Momentum Deficit (AMD). For the planar problem that we are studying herein, the AMD of a planet is defined as the difference between the angular momentum it would have on a circular orbit of radius equal to its semi-major axis and its actual orbital angular momentum:

$$\text{AMD} = m\sqrt{GMa}\left(1 - \sqrt{1 - e^2}\right).$$  (7)

The total AMD of a system is not changed by secular interactions among the planets, and thus the close encounter times of our systems may be strongly correlated with their initial AMD. Comparing the AMD of a system with inner planet eccentric with one having planets at the same semi-major axes but the outer planet having the same eccentricity, we see that because the outer planet has larger $a$, the system with the outer planet eccentric has larger AMD.

What role does angular momentum deficit play in the destabilization of the closely-packed planetary systems that we are investigating? We look to answer this question in the following way. By giving nonzero eccentricity not to the innermost planet, but to a different one instead, we can calculate the expected outcome if AMD is at play. In particular, if AMD is key, then there should be no systematic difference in lifetimes if the nonzero initial eccentricity of the new planet is adjusted relative to its respective initial semi-major axis such that it matches the initial AMD of the innermost planet.

We performed six batches of simulations with only the outer planet starting on an eccentric orbit. Three of these used fixed values of $e_5$. The other three adjusted $e_5$ to match the AMD of the corresponding system where the innermost planet starts out with nonzero initial eccentricity $e_1$. For each matched AMD simulation (i.e., each value of $\beta$), the initial eccentricity, $e_5$, is adjusted such that the initial angular momentum deficit of that system is the same as that for the $e_1 \neq 0$ system:

$$m_1\sqrt{GM_\odot a_1}\left(1 - \sqrt{1 - e_1^2}\right) = m_5\sqrt{GM_\odot a_5}\left(1 - \sqrt{1 - e_5^2}\right).$$  (8)

This equation can be solved for $e_5$. Taking into account equal planetary masses $m_1 = m_5$, we have

$$e_5 = \left[1 - \left(1 - \frac{a_1}{a_5}\left(1 - \sqrt{1 - e_1^2}\right)\right)^2\right]^{1/2}.$$  (9)

We use Equation (4) to get the semi-major axes ratio, and for simplicity define

$$X = \frac{1}{2} \left(\frac{2M_\oplus}{3M_\odot}\right)^{1/3} \approx 0.0063.$$  (10)

We recursively apply Equation (4) giving us $a_5$ in terms of $a_1$:

$$a_5 = \left(1 + \beta X\right)^4 a_1.$$  (11)

Plugging this back into Equation (9), we finally have

$$e_5 = \left[1 - \left(1 - \left(1 + \beta X\right)^{-2} \left(1 - \sqrt{1 - e_1^2}\right)\right)^2\right]^{1/2}.$$  (12)

The choice of initial $e_5$ thus depends on both the initial separation $\beta$ and the corresponding system’s initial eccentricity $e_1$ of the innermost planet.
Figure 8: Close encounter times for initial $e_5 = 0, 0.01, 0.02, 0.05$, and the AMD-adjusted $e_5$. The AMD-adjusted systems tend to be slightly shorter-lived than those starting out at $e_5 = 0.05$ for all $\beta$, which is to be expected, since the adjustment puts the initial $e_5$ eccentricity on an interval going from 0.048 for $\beta = 3.47$ down to 0.043 for $\beta = 12.32$, thus always remaining smaller than but close to 0.05 eccentricity. However, the interesting part is to compare those systems with those whose innermost planet begins at $e_1 = 0.05$, since their initial AMD is identical by construction, while the initial eccentricity is in different planets. This figure is the counterpart to Figure 6, where we plotted the lifetimes of runs with different eccentricities for the innermost planet.

Table 3 presents coefficients of the power-law fits to all six batches of systems with just the outer planet eccentric. Figure 8 shows the close encounter times, again as a function of $\beta$, of all three batches with constant values of the outermost planet’s initial eccentricity, $e_5$, as well as the batch with $e_5$ chosen to match the AMD of the run with $e_1 = 0.05$. General trends are, as expected, similar to those seen for analogous batches with the inner planet eccentric, but close comparison reveals some key differences.
Figure 9: Close encounter times of systems with initial $e_1 = 0.05$, $e_5 = 0.05$, and $e_5 = f_{AMD}(e_1 = 0.05, \beta)$.

Figure 9 compares lifetimes for batches of systems with either the inner or outer planet beginning with eccentricity $e = 0.05$, and Figure 10 shows difference in close encounter time between innermost and outermost planet for both the standard $e_5$ batch and the AMD adjusted batch, complementing the comparison of different eccentricities for the same planet in Figure 7. There, we focused on the change in close encounter times for different initial eccentricities of the same (innermost) planet, whereas here our focus is on the change in close encounter times for different planets with the same initial eccentricity (we choose the largest eccentricities, $e \approx 0.05$).

Figure 10 shows that the systems with $e_5 \approx 0.05$ and those with $e_1 = 0.05$ innermost planet have similar lifetimes. The $e_5 = 0.05$ systems on average have a somewhat shorter close encounter time than those with $e_1 = 0.05$, but the difference is small, with the average multiplicative factor being 0.881. On the other hand, the red points for the AMD-adjusted batch lie significantly below the line of zero difference: the AMD-adjusted systems live much longer than the $e_1 = 0.05$ systems, with an average ratio of 2.393. This is mostly due to the smaller initial eccentricities of those systems: they range from $e_5 = 0.0444$ at $\beta = 9.33$ to $e_5 = 0.0428$ at $\beta = 12.32$. 
Figure 10: Close encounter differences between simulations with initial $e_1 = 0.05$, $e_5 = 0.05$, and $e_5 = f_{amd}(e_1 = 0.05, \beta)$.

Finally, we recorded the slopes and intercepts of all our runs in Table 3, using a separation shifted by the critical separation for Hill stability of two-planet systems, $\beta = 2\sqrt{3}$; $\beta' \equiv \beta - 2\sqrt{3}$. In other words, the critical two-planet separation is $\beta' = 0$, and $b'$, $\sigma_b'$ and $c'$, $\sigma_c'$ are the slopes and intercepts and their standard errors.
Finally, the last comparison looks at the batches with one planet having eccentricity $\approx \beta$. Most batches, we considered multiple intervals in $\beta$. We then record the data for all batches, starting from the first point beyond which there are no more invalid points, until the fifth system within the batch reaching two billion years of lifetime. Next, we compare innermost planet eccentricities including $e_i = 0.05$, before comparing all batches except any single $e_i = 0.05$ (but including $e_{align} = 0.05$). Finally, the last comparison looks at the batches with one planet having eccentricity $\approx 0.05$.

| Eccentricity | Range  | $b'$ | $\sigma_{b'}$ | $c'$ | $\sigma_{c'}$ |
|--------------|--------|------|---------------|------|---------------|
| 0            | $3.47 \leq \beta \leq 8.73$ | 1.080 | 0.019 | 1.85 | 0.06 |
| $e_i = 0.01$ | $3.47 \leq \beta \leq 8.73$ | 0.999 | 0.016 | 1.87 | 0.05 |
| $e_i = 0.01$ | $3.47 \leq \beta \leq 10.05$ | 1.030 | 0.014 | 1.82 | 0.05 |
| $e_i = 0.02$ | $3.48 \leq \beta \leq 10.34$ | 0.984 | 0.010 | 1.61 | 0.04 |
| $e_i = 0.03$ | $4.19 \leq \beta \leq 11.21$ | 0.867 | 0.011 | 1.58 | 0.05 |
| $e_i = 0.05$ | $5.32 \leq \beta \leq 12.61$ | 0.785 | 0.010 | 1.02 | 0.06 |
| $e_i = 0.05$ | $5.49 \leq \beta \leq 12.99$ | 0.827 | 0.010 | 0.52 | 0.06 |
| $e_i = 0.01$ | $3.47 \leq \beta \leq 10.09$ | 1.017 | 0.012 | 1.82 | 0.05 |
| $e_i = 0.02$ | $4.00 \leq \beta \leq 11.09$ | 0.921 | 0.012 | 1.64 | 0.06 |
| $e_i = 0.05$ | $5.07 \leq \beta \leq 12.99$ | 0.810 | 0.009 | 0.56 | 0.06 |
| $e_i = 0.01$ | $3.47 \leq \beta \leq 9.98$ | 1.016 | 0.014 | 1.84 | 0.05 |
| $e_i = 0.02$ | $3.47 \leq \beta \leq 10.80$ | 0.911 | 0.012 | 1.79 | 0.05 |
| $e_i = 0.05$ | $4.86 \leq \beta \leq 12.38$ | 0.759 | 0.010 | 1.13 | 0.06 |
| $e_i = e_{align} = 0.01$ | $3.47 \leq \beta \leq 10.05$ | 0.981 | 0.013 | 1.93 | 0.05 |
| $e_i = e_{align} = 0.02$ | $3.47 \leq \beta \leq 10.30$ | 0.983 | 0.011 | 1.68 | 0.05 |
| $e_i = e_{align} = 0.05$ | $3.47 \leq \beta \leq 12.32$ | 0.826 | 0.011 | 1.02 | 0.06 |
| $e_i = 0.05$ | $3.47 \leq \beta \leq 10.08$ | 0.837 | 0.016 | 2.18 | 0.06 |

Table 3: Slopes and intercepts for power-law fits to lifetimes of simulated batches of systems. For most batches, we considered multiple intervals in $\beta$. We begin by comparing our numbers with circular initial orbits (lines one and two). We then record the data for all batches, starting from the first point beyond which there are no more invalid points, until the fifth system within the batch reaching two billion years of lifetime. Next, we compare innermost planet eccentricities including $e_i = 0.05$, before comparing all batches except any single $e_i = 0.05$ (but including $e_{align} = 0.05$). Finally, the last comparison looks at the batches with one planet having eccentricity $\approx 0.05$. 
6 One intermediate planet eccentric

We close the main part of this investigation by considering systems in which one of the intermediate planets is initially eccentric. For most of these runs we choose the middle (third) planet initially eccentric, and also included a run with $e_2 = 0.05$. Figure 11 confirms the log-linear relationship for those planetary systems.

We compare, one more time, and this time including eccentric intermediate planets, the $0.05$ systems in Figure 12. We find slightly lower close encounter times for eccentric intermediate planets, as shown in Figure 13. We thus confirm that a log-linear relationship continues to hold for intermediate planets eccentric, and find the close encounter times decrease on average by roughly a factor of two when the initially eccentric planet has two neighbors. The slopes ($b'$) and intercepts ($c'$) fit for this relationship in the overlap range $\beta \in [5.49, 12.32]$ are similar for all four runs, with the systems with the initially eccentric planet in the middle having slightly steeper slopes but somewhat smaller intercept values than those with the inner or outer planet eccentric. The $e_3 = e_{amid}^{0.05}$ batch has a steeper slope because the eccentricity drops (albeit slowly) as $\beta$ is increased.

Figure 11: Comparison of close encounter times for $e_3 = 0.01, e_3 = 0.02,$ and $e_3 = 0.05$, and, in purple, $e_2 = 0.05$. 


Figure 12: Comparison of close encounter times for $e_1 = 0.05$, $e_2 = 0.05$, $e_3 = 0.05$, and $e_5 = 0.05$. 
Figure 13: Close encounter differences between simulations with initial $e_1 = 0.05$, $e_3 = 0.05$, and $e_5 = 0.05$. This plot represents the quantitative comparison between the systems in Figure 12 and complements Figure 10. Here, the average multiplicative factor in close encounter time compared to the $e_1 = 0.05$ system are, in the plotting order, 0.525, 0.436, and, as mentioned in Section 5, 0.881. Thus, the systems with intermediate planet eccentric are about half as long lived on average as those without two planetary neighbors.

7 All planets on eccentric orbits

In the previous sections, we found that for eccentricities up to 0.05, systems with one planet eccentric follow a log-linear behavior as a function of initial separation, and this is true no matter which planet starts out with the nonzero eccentricity. Here, we consider two kinds of systems with all planets initially eccentric with $e = 0.05$, one of which has all periapses aligned, $e_{\text{align}} = 0.05$, and the other has randomly-drawn periapse angles, $e_{\text{random}} = 0.05$.

Figure 14 compares our results with previously-shown results for circular orbits and for $e_1 = 0.05$. We immediately recognize that the aligned systems behave much more similarly to circular systems than to the single eccentricity ones. The distribution of close encounter times are only slightly larger than those for the circular case. The initial alignment of their periapses act as a stabilizing feature. In other words, the relative eccentricity between the planets is zero for both the circular and the all-eccentric cases, yielding longer close encounter times. The lifetimes of runs in the circular and the all-eccentric batches are especially close for small separations; the all-eccentric runs are
systematically lower for wider separations, reaching the stopping point of five runs lasting longer
than $2 \times 10^9$ years at separation similar to the batches with one planet having initial $e = 0.01$.

The difference between the systems with just one planet eccentric and all of them eccentric
is stark: when all planets start out aligned and at the same eccentricity of $e_{\text{align}} = 0.05$, the
system lifetimes are considerably longer than those with just one planet having $e = 0.05$. In
contrast, randomized initial angular parameters dramatically reduce the close encounter times, and
no resonance-related structure is appearing up until at least $\beta = 15$. Nonetheless, all batches
approximately follow a log-linear relationship between initial spacing and encounter time. It is
clear that the initial relative eccentricities of its planets is more important than individual planetary
eccentricity values.

![Figure 14: Superposition of close encounter times for the circular system; all planets eccentric and
aligned; all planets eccentric and randomized initial angular positions; and the $e_1 = 0.05$ system.](image)

8 Conclusions

We investigated the orbital stability of idealized systems of five identical, closely-spaced planets. In
most cases considered, one planet was initially on an eccentric orbit and the other four planets began
on circular orbits. As expected, system lifetimes generally increased for larger orbital separations

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and dropped for larger initial eccentricity. For a given initial eccentricity, we found that systems approximately obeyed a log-linear relationship between initial separation and close encounter time, in agreement with previous studies of planets starting with circular orbits, but the slopes dropped as the eccentricity increased. Mean motion resonances reduced system stability, but by smaller amounts as initial eccentricities increased.

Lifetimes of systems with the outer planet initially eccentric were on average similar to those with the inner planet starting with the same eccentricity; giving the specified initial eccentricity to an intermediate planet resulted, on average, to slightly shorter system lifetimes. The dependence of the difference in system lifetime on which planet is initially eccentric do not scale with total system angular momentum deficit, suggesting that mean motion resonances dominate over secular resonances in destabilizing the planetary systems studied.

Additional studies in which all planets began with an eccentricity $e = 0.05$ produced interesting results. If the planetary periapses were selected randomly, system lifetimes were, as expected, far shorter than comparable systems with only one planet initially eccentric. However, if all the of the planets’ periapses were initial aligned, systems were generally much longer-lived than systems in which only one planet had initial $e = 0.05$, with planetary spacing required for systems to survive $> 10^9$ years being similar to those for which one planet began with $e = 0.01$.

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