A family of measures associated with iterated function systems

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Abstract

Let \((X, d)\) be a compact metric space, and let an iterated function system (IFS) be given on \(X\), i.e., a finite set of continuous maps \(\sigma_i: X \rightarrow X, i = 0, 1, \cdots, N-1\). The maps \(\sigma_i\) transform the measures \(\mu\) on \(X\) into new measures \(\mu_i\). If the diameter of \(\sigma_1 \circ \cdots \circ \sigma_k(X)\) tends to zero as \(k \rightarrow \infty\), and if \(p_i > 0\) satisfies \(\sum i p_i = 1\), then it is known that there is a unique Borel probability measure \(\mu\) on \(X\) such that

\[
\mu = \sum_i p_i \mu_i \tag{*}
\]

In this paper, we consider the case when the \(p_i\)'s are replaced with a certain system of sequilinear functionals. This allows us to study the variable coefficient case of (*), and moreover to understand the analog of (*) which is needed in the theory of wavelets.

1 Introduction

A finite system of continuous functions \(\sigma_i: X \rightarrow X\) in a compact metric space \(X\) is said to be an iterated function system (IFS) if there is a mapping \(\sigma: X \rightarrow X\), onto \(X\), such that

\[
\sigma \circ \sigma_i = id_X \tag{1.1}
\]

If there is a constant \(0 < c < 1\) such that

\[
d(\sigma_i(x), \sigma_i(y)) \leq c \ d(x, y), \ x, y \in X, \tag{1.2}
\]

then we say that the IFS is contractive. In that case, there is, for every configuration \(p_i > 0, \sum_i p_i = 1\), a unique Borel probability measure \(\mu, \mu = \mu(p)\) on \(X\) such that

\[
\mu = \sum_i p_i \mu \circ \sigma_i^{-1}. \tag{1.3}
\]

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This follows from a theorem of Hutchinson [4]. The mappings \( \sigma_i \) might be defined initially on some Euclidean space \( E \). If the contractivity (1.2) is assumed, then there is a unique compact subset \( X \subset E \) such that

\[
X = \bigcup_i \sigma_i(X),
\]

and this set \( X \) is the support of \( \mu \).

**Example 1.1** Let \( E = \mathbb{R} \), \( \sigma_0(x) = \frac{x}{3}, \sigma_1(x) = \frac{x+1}{3} \), \( p_0 = p_1 = \frac{1}{2} \). In this case, \( X \) is the familiar middle-third Cantor set, and \( \mu \) is the Cantor measure supported in \( X \) with Hausdorff dimension \( d = \frac{\ln 2}{\ln 3} \). But at the same time \( X \) may be identified with the compact Cartesian product \( X \sim = \prod \mathbb{Z}^2 = \mathbb{Z}^\mathbb{N} \), where \( \mathbb{Z}^2 = \mathbb{Z} \setminus 2\mathbb{Z} = \{0,1\} \), and \( N = \{0,1,2,\ldots\} \), and \( \mu \) is the infinite product measure on \( \mathbb{Z}^2 \times \mathbb{Z}^2 \times \cdots \) with weights \( \left( \frac{1}{2}, \frac{1}{2} \right) \) on each factor.

**Example 1.2** Let \( E = \mathbb{R} \), \( \sigma_0(x) = \frac{x}{2}, \sigma_1(x) = \frac{x+1}{2} \), \( p_0 = p_1 = \frac{1}{2} \). In this case, \( X = [0,1] \), i.e., the compact unit interval, and \( \mu \) is the restriction to \([0,1]\) of the usual Lebesgue measure \( dt \) on \( \mathbb{R} \).

Let \( T = \mathbb{R}/2\pi \mathbb{Z} = \{ z \in \mathbb{C} \mid |z| = 1 \} \) be the usual torus. Let \( N \in \mathbb{N}, N \geq 2 \), and let \( m_i: T \to \mathbb{C}, i = 0,1,\ldots,N-1 \) be a system of \( L^\infty \)-functions such that the \( N \times N \) matrix

\[
\frac{1}{\sqrt{N}} \left( m_j \left( e^{i2\pi \frac{k}{N}} \right) \right)_{j,k=0}^{N-1}, z \in T
\]

is unitary. Set

\[
S_j f(z) = m_j(z)f(z^N), z \in T, f \in L^2(T). \tag{1.6}
\]

Then it is well known [7] that the operators \( S_j \) satisfy the following two relations,

\[
S_j^* S_k = \delta_{j,k} I \tag{1.7}
\]

\[
\sum_j S_j S_j^* = I \tag{1.8}
\]

where \( I \) denotes the identity operator in the Hilbert space \( \mathcal{H} = L^2(T) \). The converse implication also holds, see [2]. Systems of isometries satisfying (1.7) – (1.8) are called representations of the Cuntz algebra \( O_N \), see [3], and the particular representations (1.6) are well known to correspond to multiresolution wavelets; the functions \( m_j \) are denoted wavelet filters. These same functions are used in subband filters in signal processing, see [2].

It is easy to see that there is a unique Borel measure \( P \) on \([0,1]\) taking values in the orthogonal projections of \( \mathcal{H} \) such that

\[
P \left( \left[ \frac{i_l}{N} + \cdots + \frac{i_k}{N^k}, \frac{i_l}{N} + \cdots + \frac{i_k}{N^k} + \frac{1}{N^k} \right] \right) = S_{i_1} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_1}^*. \tag{1.9}
\]

Let

\[
e_n(z) = z^n, z \in T, n \in \mathbb{Z}. \tag{1.10}
\]
Example 1.3 Let $N \in \mathbb{N}$, $N \geq 2$, and set $m_j(z) = z^j$, $0 \leq j < N$. Then (1.2) is satisfied, and we have

\[
\begin{cases}
S_0^*e_0 = e_0 \\
S_j^*e_0 = 0, \quad 0 < j < N.
\end{cases}
\]

It follows easily that the corresponding measure

\[
\mu_0(\cdot) = \|P(\cdot)e_0\|^2
\]
on $[0,1)$ is the Dirac measure $\delta_0$ at $x = 0$, i.e.,

\[
\delta_0(E) = \begin{cases}
1 & \text{if } 0 \in E \\
0 & \text{if } 0 \notin E.
\end{cases}
\]

Here $P(\cdot)$ refers to the projection valued measure determined by (1.9) when the representation of $\mathcal{O}_N$ is specified by the system $m_j = e_j$, $0 \leq j < N$.

Example 1.4 Let $N \in \mathbb{N}$, $N \geq 2$, and set

\[
m_j(z) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{i2\pi \frac{k}{N}} z^k.
\]

Again the condition (1.5) is satisfied, and one checks that

\[
S_j^*e_0 = \frac{1}{\sqrt{N}} e_0, \quad 0 \leq j < N;
\]

and now the measure $\mu_0(\cdot) = \|P(\cdot)e_0\|^2$ on $[0,1)$ is the restriction to $[0,1)$ of the Lebesgue measure on $\mathbb{R}$. It is well known that the wavelet corresponding to (1.13) is the familiar Haar wavelet corresponding to $N$-adic subdivision, see [2]. It is also known that generally, for wavelets other than the Haar systems, the corresponding representations (1.6) of $\mathcal{O}_N$ does not admit a simultaneous eigenvector $f$, i.e., there is no solution $f \in \mathcal{H} \setminus \{0\}$, $\lambda_j \in \mathbb{C}$ to the joint eigenvalue problem

\[
S_j^*f = \lambda_j f, \quad 0 \leq j < N.
\]

Proposition 1.5 Let $N \in \mathbb{N}$, $N \geq 2$, and let $(S_j)_{0 \leq j < N}$ be a representation of $\mathcal{O}_N$ on a Hilbert space $\mathcal{H}$. Suppose there is a solution $f \in \mathcal{H}$, $\|f\| = 1$, to the eigenvalue problem (1.15) for some $\lambda_j \in \mathbb{C}$. Then $\sum |\lambda_j|^2 = 1$, and the measure $\mu: = \|P(\cdot)f\|^2$ satisfies

\[
\mu = \sum_{j=0}^{N-1} |\lambda_j|^2 \mu \circ \sigma_j^{-1}
\]

where $\sigma_j(x) = \frac{x + j}{N}$, $\mu \circ \sigma_j^{-1}(E) = \mu(\sigma_j^{-1}(E))$ for Borel sets $E \subset [0,1)$, and $\sigma_j^{-1}(E) = \{x \mid \sigma_j(x) \in E\}$.

Proof. The reader may prove the proposition directly from the definitions, but the conclusion may also be obtained as a special case of the theorem in the next section. □
2 Measures And Iterated Function Systems

Let \((X, d)\) be a compact metric space, and let \((\sigma_j)_{0 \leq j < N}\) be an \(N\)-adic iterated function system (IFS). We say that the system is complete if

\[
\lim_{k \to \infty} \text{diameter} (\sigma_{i_1} \circ \cdots \circ \sigma_{i_k}(X)) = 0, \tag{2.1}
\]

We say that the IFS is non-overlapping if for each \(k\) the sets

\[
A_k(i_1, \cdots, i_k) := \sigma_{i_1} \circ \cdots \circ \sigma_{i_k}(X) \tag{2.2}
\]

are disjoint, i.e., for every \(k\), the sets \(A_k(i_1, \cdots, i_k)\) are mutually disjoint for different multi-indices, i.e., different points in

\[
\mathbb{Z}_N \times \cdots \times \mathbb{Z}_N
\]
k-times

where \(\mathbb{Z}_N := \{0, 1, \cdots, N - 1\}\).

**Remark 2.1** It is immediate that, if a given IFS \((\sigma_j)_{0 \leq j < N}\) arises as a system of distinct branches of the inverse of a single mapping \(\sigma: X \to X\), i.e., if \(\sigma(\sigma_i(x)) = x\) for \(x \in X\), and \(0 \leq i < N\), then the partition system \(\sigma_{i_1} \circ \cdots \circ \sigma_{i_k}(X)\) is non-overlapping.

**Theorem 2.2** Let \(N \in \mathbb{N}, N \geq 2\), and a Hilbert space \(\mathcal{H}\). Let \((\sigma_j)_{0 \leq j < N}\) be an IFS which is complete and non-overlapping. Then there is a unique projection valued measure \(P\) defined on the Borel subsets of \(X\) such that

\[
P(A_k(i_1, \cdots, i_k)) = S_{i_1} \cdots S_{i_k}^* S_{i_k}^* \cdots S_{i_1}^*, \tag{2.3}
\]

This measure satisfies:

(a) \(P(E) = P(E)^* = P(E)^2, E \in \mathcal{B}(X) = \) the Borel subsets of \(X\).

(b) \(\int_X dP(x) = 1_{\mathcal{H}}\)

(c) \(P(E)P(F) = 0\) if \(E, F \in \mathcal{B}(X)\) and \(E \cap F = \emptyset\).

(d) \(\sum_{j=0}^{N-1} S_j P(\sigma_j^{-1}(E)) S_j^* = P(E), E \in \mathcal{B}(X)\).

It follows in particular that, for every \(f \in \mathcal{H}\), the measure \(\mu_f(\cdot) := \|P(\cdot)f\|^2\) satisfies

\[
\sum_{j=0}^{N-1} \mu_{S_j^* f \circ \sigma_j^{-1}} = \mu_f, \tag{2.4}
\]

or equivalently

\[
\sum_{j=0}^{N-1} \int_X \psi \circ \sigma_j d\mu_{S_j^* f} = \int_X \psi d\mu_f \tag{2.5}
\]

for all bounded Borel functions \(\psi\) on \(X\).
Corollary 2.3 Let $N \in \mathbb{N}$, $N \geq 2$, be given, and consider a representation $(S_j)_{0 \leq j < N}$ of $O_N$, and an associated IFS which is complete and non-overlapping. Let $P(\cdot)$ be the corresponding projection valued measure, i.e.,

$$P(\sigma_{i_1} \circ \cdots \circ \sigma_{i_k}(X)) = S_{i_1} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_1}^*.$$  \hspace{1cm} (2.6)

For $f \in \mathcal{H}$, $\|f\| = 1$, set $\mu_f(\cdot) := \|P(\cdot) f\|^2$. Let $\mathfrak{A}$ be the abelian $C^*$-algebra generated by the projections $S_{i_1} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_1}^*$ and let $\mathcal{H}_f$ be the closure of $\mathfrak{A} f$. Then there is a unique isometry $V_f : L^2(\mu_f) \to \mathcal{H}_f$ of $L^2(\mu_f)$ onto $\mathcal{H}_f$ such that

$$V_f(1) = f,$$  \hspace{1cm} (2.7)

and

$$V_f M_{X_{A_k(i)}} V_f^* = S_{i_1} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_1}^*$$  \hspace{1cm} (2.8)

where $M_{X_{A_k(i)}}$ is the operator which multiplies by the indicator function of $A_k(i) := \sigma_{i_1} \circ \cdots \circ \sigma_{i_k}(X).$ \hspace{1cm} (2.9)

Proof. (Theorem 2.2) We refer to the paper [6] for a more complete discussion. With the assumptions, we note that for every $k$, and every multi-index $i = (i_1, \cdots, i_k)$ we have an abelian algebra of functions $F_k$ spanned by the indicator functions $\chi_{A_k(i)}$ where $A_k(i) := \sigma_{i_1} \circ \cdots \circ \sigma_{i_k}(X)$. Since, for every $k$, we have the non-overlapping unions

$$\bigcup_i A_{k+1}(i_1, i_2, \cdots, i_k, i) = A_k(i_1, \cdots, i_k),$$  \hspace{1cm} (2.10)

there is a natural embedding $F_k \subset F_{k+1}$. We wish to define the projection valued measure $P$ as an operator valued map on functions on $X$ in such a way that $\int_X \chi_{A_k(i)} dP = S_{i_1} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_1}^*$. This is possible since the projections $S_{i_1} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_1}^*$ are mutually orthogonal when $k$ is fixed, and $(i_1, \cdots, i_k)$ varies over $(\mathbb{Z}_N)^k$. In view of the inclusions

$$F_k \subset F_{k+1},$$  \hspace{1cm} (2.11)

it follows that $\bigcup_k F_k$ is an algebra of functions on $X$. Since the $N$-adic subdivision system $\{A_k(i) \mid k \in \mathbb{N}, i \in (\mathbb{Z}_N)^k\}$ is complete, it follows that every continuous function on $X$ is the uniform limit of a sequence of functions in $\bigcup_k F_k$. Using now a standard extension procedure for measures, we conclude that the projection valued measure $P(\cdot)$ exists, and that it has the properties listed in the theorem. The reader is referred to [5] for additional details on the extension from $\bigcup_k F_k$ to the Borel function on $X$. [Q.E.D.]

Proof of Corollary 2.3 Let the systems $(S_j)$ and $(\sigma_j)$ be as in the statement of the theorem. To define $V_f : L^2(\mu_f) \to \mathcal{H}_f$, we set

$$V_f(1) = f,$$

and $V_f \chi_{A_k(i)} = S_{i_1} \cdots S_{i_k} S_{i_k}^* \cdots S_{i_1}^* f.$  \hspace{1cm} (2.12)
It is clear from the theorem that $V_f$ defined this way extends to an isometry of $L^2(\mu_f)$ onto $H_f$, and a direct verification reveals that the covariance relation is satisfied.

It remains to prove (d) in theorem 2.2 or equivalently to prove (2.4) for every $f \in H$. The argument is based on the same approximation procedure as we used above, starting with the algebra $\bigcup_k F_k$. Note that

$$\int_X \chi_{A_k(\alpha_1,\ldots,\alpha_k)}(\sigma_i(x)) \, d\mu_{f^i}(x) = \delta_{i,\alpha_1} \int_X \chi_{A_k(\alpha_1,\ldots,\alpha_k)}(x) \, d\mu_{f^i}(x) = \delta_{i,\alpha_1} \|S_{\alpha_k}^* \cdots S_{\alpha_2}^* S_{\alpha_1}^* f\|^2 = \delta_{i,\alpha_1} \int_X \chi_{A_k(\alpha_1,\ldots,\alpha_k)}(x) \, d\mu_f(x).$$

Summing over $i$, we get

$$\sum_i \int_X \chi_{A_k(\alpha_1,\ldots,\alpha_k)}(x) \, d\mu_{f^i} = \int_X \chi_{A_k(\alpha_1,\ldots,\alpha_k)}(x) \, d\mu_f = \|S_{\alpha_k}^* \cdots S_{\alpha_2}^* S_{\alpha_1}^* f\|^2.$$

The desired identity now follows by yet another application of the standard approximation argument which was used in the proof of the first part of the theorem.

The simplest subdivision system is the one where the subdivisions are given by the $N$-adic fractions $\frac{\alpha_1}{N} + \frac{\alpha_2}{N^2} + \cdots + \frac{\alpha_k}{N^k}$ where $\alpha_i \in \mathbb{Z}_N = \{0, 1, \ldots, N-1\}$. Setting $\sigma_j(x) = \frac{x+j}{N}$, $j \in \mathbb{Z}_N$, we note that

$$\sigma_{\alpha_1} \circ \cdots \circ \sigma_{\alpha_k}([0,1)) = \left[\frac{\alpha_1}{N} + \cdots + \frac{\alpha_k}{N^k} \frac{\alpha_1}{N} + \cdots + \frac{\alpha_k}{N^k} + \frac{1}{N^k}\right].$$

As a result both the projection valued measure $P(\cdot)$ and the individual measures $\mu_f(\cdot) = \|P(\cdot)f\|^2$ are defined on the Borel subsets of $[0,1)$. If $\hat{P}(t) := \int_0^1 e^{itx} \, dP(x)$, then $\langle f | \hat{P}(t)f \rangle = \hat{\mu}_f(t)$ is the usual Fourier transform of the measure $\mu_f$ for $f \in H$.

Moreover, by the Spectral theorem, there is a selfadjoint operator $D$ with spectrum contained in $[0,1]$ such that

$$\hat{P}(t) = e^{itD},$$

see [8]. In fact, the spectrum of $D$ is equal to the support of the projection valued measure $P(\cdot)$.

**Corollary 2.4** Suppose the $N$-adic partition system used in the theorem is given by the $N$-adic fractions as in 2.10. Then the Fourier transform

$$\hat{\mu}_f(t) = \int_0^1 e^{itx} \, d\mu_f(x), \quad t \in \mathbb{R} \quad (2.13)$$
of the measure $\mu_f(\cdot) = \|P(\cdot)f\|^2$ satisfies

$$\hat{\mu}_f(t) = \sum_{k=0}^{N-1} e^{it\frac{k}{N}} \hat{\mu}_{S^*_\alpha f}(t/N).$$

(2.14)

**Proof.** With the $N$-adic subdivisions of the unit interval, the maps $\sigma_k$ are $\sigma_k(x) = \frac{x+k}{N}$ for $k \in \mathbb{Z}_N = \{0, 1, \cdots, N-1\}$. Setting $\psi_t(x) = e^{itx}$ in (2.4), the desired result (2.14) follows immediately.

In the next result we show that for every $k \in \mathbb{N}$, there is an approximation formula for the Fourier transform $\hat{\mu}_f$ of the measure $\mu_f(\cdot) = \|P(\cdot)f\|^2$ involving the numbers $\|S^*_{\alpha_k} \cdots S^*_{\alpha_1} f\|^2$ as the multi-index $\alpha = (\alpha_1, \cdots, \alpha_k)$ ranges over $(\mathbb{Z}_N)^k$.

**Corollary 2.5** Let $N \in \mathbb{N}$, $N \geq 2$ be given. Let $(S_j)_{0 \leq j < N}$ be a representation of $O_N$ on a Hilbert space $\mathcal{H}$, and let $P(\cdot)$ be the corresponding projection-valued measure defined on $\mathcal{B}([0,1])$. Let $f \in \mathcal{H}$, $\|f\| = 1$, and set $\mu_f(\cdot) = \|P(\cdot)f\|^2$.

Then, for every $k$, we have the approximation

$$\left| \hat{\mu}_f(t) - \sum_{\alpha_1, \cdots, \alpha_k} e^{it\left(\frac{\alpha_1}{N} + \cdots + \frac{\alpha_k}{N^k}\right)} \|S^*_\alpha f\|^2 \right| \leq |t| N^{-k}$$

(2.15)

where the summation is over multi-indices from $(\mathbb{Z}_N)^k$, and $S^*_\alpha := S^*_{\alpha_k} \cdots S^*_{\alpha_1}$.

**Proof.** A $k$-fold iteration of formula (2.14) from the previous corollary yields,

$$\hat{\mu}_f(t) = \sum_{\alpha_1, \cdots, \alpha_k} e^{it\left(\frac{\alpha_1}{N} + \cdots + \frac{\alpha_k}{N^k}\right)} \hat{\mu}_{S^*_\alpha f}(t/N^k)$$

and

$$\hat{\mu}_{S^*_\alpha f}(t/N^k) - \|S^*_\alpha f\|^2 = \int_0^1 (e^{itN^{-k}x} - 1) \, d\mu_{S^*_\alpha f}(x);$$

and therefore

$$\left| \hat{\mu}_{S^*_\alpha f}(tN^{-k}) - \|S^*_\alpha f\|^2 \right| \leq |t| N^{-k} \int_0^1 x \, d\mu_{S^*_\alpha f}(x)$$

$$\leq |t| N^{-k} \int_0^1 d\mu_{S^*_\alpha f}(x)$$

$$= |t| N^{-k} \|S^*_\alpha f\|^2.$$

It follows that the difference on the left-hand side in (2.14) is estimated above.
in absolute value by
\[
\sum_{\alpha_1, \ldots, \alpha_k} |e^{it \left( \frac{\alpha_1}{N} + \cdots + \frac{\alpha_k}{N} \right)}| |t| N^{-k} \| S_\alpha^* f \|^2
\]
\[
= |t| N^{-k} \sum_{\alpha_1, \ldots, \alpha_k} \| S_\alpha^* f \|^2
\]
\[
= |t| N^{-k} \left( f \left| \sum_{\alpha_1, \ldots, \alpha_k} S_\alpha S_\alpha^* f \right| \right)
\]
\[
= |t| N^{-k} \| f \|
\]
\[
= |t| N^{-k}.
\]

Definition 2.6 Let \( k \in \mathbb{N} \), and set
\[
\text{x}_k(\alpha) := \frac{\alpha_1}{N} + \cdots + \frac{\alpha_k}{N^k} \quad \text{for} \quad \alpha_i \in \{0,1,\ldots,N-1\}. \tag{2.16}
\]

Let \((S_i)\) and \((\sigma_i)\) be as in Corollary 2.4 and let \( f \in \mathcal{H}, \|f\| = 1 \). We set
\[
\mu_f^{(k)} = \sum_{\alpha_1, \ldots, \alpha_k} \| S_\alpha^* f \|^2 \delta_{\text{x}_k(\alpha)}. \tag{2.17}
\]

These measures form the sequence of measures which we use in the Riemann sum approximation of Corollary 2.5, and we are still viewing the measures \( \mu_f \) and \( \mu_f^{(k)} \) as measures on the unit-interval \([0,1)\).

Corollary 2.7 Let \( N \in \mathbb{N}, N \geq 2 \). Let \((S_i)\) and \((\sigma_i)\) be as in corollary 2.4, i.e., \((S_i)\) is in \( \text{Rep}(O_N, \mathcal{H}) \) for some Hilbert space \( \mathcal{H} \), and \( \sigma_i(x) = \frac{x + j}{N} \) for \( x \in [0,1) \) and \( j \in \{0,1,\ldots,N-1\} \). Let \( \psi \) be a continuous function on \([0,1)\), and let \( k \in \mathbb{N} \). Then
\[
\left| \int_0^1 \psi \; d\mu_f - \int_0^1 \psi \; d\mu_f^{(k)} \right| \leq N^{-k} \int_\mathbb{R} |\hat{\psi}(t)| \; dt \tag{2.18}
\]
where
\[
\hat{\psi}(t) = \int_0^1 \psi(x) e^{-itx} \; dx \tag{2.19}
\]
is the usual Fourier transform; and we are assuming further that
\[
\int_\mathbb{R} |\hat{\psi}(t)| \; dt < \infty.
\]
Proof. By the Fourier inversion formula, \( \psi(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\psi}(t) e^{itx} \, dt \); and we get the following formula by a change of variables, and by the use of Fubini’s theorem:

\[
\int \psi \, d\mu_f - \int \psi \, d\mu_f^{(k)} = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\psi}(t) \left( \hat{\mu}_f(t) - \hat{\mu}_f^{(k)}(t) \right) \, dt. \tag{2.20}
\]

Since

\[
\hat{\mu}_f^{(k)}(t) = \sum_{\alpha_1, \ldots, \alpha_k} e^{itx_k(\alpha)} \| S_{\alpha} f \|^2, \tag{2.21}
\]

the estimate (2.18) from Corollary 2.4 applies. An estimation of the differences in (2.20) now yields:

\[
\left| \int \psi \, d\mu_f - \int \psi \, d\mu_f^{(k)} \right| \leq \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{\psi}(t)| |t| \cdot N^{-k} \, dt
\]

which is the desired result.

Remark 2.8 In general, a sequence of probability measures on a compact Hausdorff space \( X \), \( (\mu_k) \) is said to converge weakly to the limit \( \mu \) if

\[
\lim_{k \to \infty} \int_X \psi \, d\mu_k = \int_X \psi \, d\mu \text{ for all } \psi \in C(X). \tag{2.22}
\]

However, the conclusion of Corollary 2.9 for the convergence \( \lim_{k \to \infty} \mu_f^{(k)} = \mu_f \) is in fact stronger than weak convergence as we will show. The notion of weak convergence of measures is significant in probability theory, see e.g., [1].

Since the measures \( \mu_f \) and \( \mu_f^{(k)} \) are defined on \( B([0,1]) \), the corresponding distribution functions \( F_f \) and \( F_f^{(k)} \) are defined for \( x \in [0,1] \) as follows

\[
F_f(x) = \mu_f([0,x]) \text{ and } F_f^{(k)}(x) = \mu_f^{(k)}([0,x]). \tag{2.23}
\]

Corollary 2.9 Let \( (S_i) \) and \( (\sigma_i) \) be as in Corollary 2.7. Let \( f \in H^1, \|f\| = 1 \), be given, and let \( \mu_f \) resp., \( \mu_f^{(k)} \) be the corresponding measures, with distribution functions \( F_f \) and \( F_f^{(k)} \), respectively. Then

\[
\lim_{k \to \infty} F_f^{(k)}(x) = F_f(x). \tag{2.24}
\]

Proof. We already proved that the sequence of measures \( \mu_f^{(k)} \) converges weakly to \( \mu_f \) as \( k \to \infty \). Furthermore, it is known that weak convergence \( \mu_f^{(k)} \to \mu_f \) implies that (2.24) holds whenever \( x \) is a point of continuity for \( F_f \), see [1] Theorem 2.3, p.5. (For the case of the wavelet representations, it is known that every \( x \) is a point of continuity, but Example [1.3 shows that the measures \( \mu_f \) are not continuous in general.) The argument from the proof of Corollary 2.4 shows
that, in general, the points of discontinuity of \( F_f(\cdot) \) must lie in the set \( \text{(2.16)} \) of \( N \)-adic fractions. Using Theorem \( \text{(2.2)} \) and formula \( \text{(2.17)}, \) we conclude that if \( x_k(\alpha) \) is a point of discontinuity of \( F_f(\cdot) \), then \( |F_f^{(k+n)}(x_k(\alpha)) - F_f(x_k(\alpha))| \leq N^{-k-n} \), and therefore
\[
\lim_{n \to \infty} F_f^{(k+n)}(x_k(\alpha)) = F_f(x_k(\alpha))
\]
which is the desired conclusion, see \( \text{(2.22)} \).

3 The Measures \( \mu_f \)

In the previous section, we showed that the decomposition theory for representations of the Cuntz algebra \( \mathcal{O}_N \) may be analyzed by the use of projection valued measures on a class of iterated function systems (IFS). It is known that there is a simple \( C^* \)-algebra \( \mathcal{O}_N \) for each \( N \in \mathbb{N}, N \geq 2 \), such that the representations of \( \mathcal{O}_N \) are in one-one correspondence with systems of isometries \( \{S_i\} \) which satisfy the two relations \( \text{[1.7, 1.8]} \). The \( C^* \)-algebra \( \mathcal{O}_N \) is defined abstractly on \( N \) generators \( s_i \) which satisfy
\[
\sum_{i=0}^{N-1} s_i s_i^* = 1 \quad \text{and} \quad s_i s_j = \delta_{i,j}1. \tag{3.1}
\]
A representation \( \rho \) of \( \mathcal{O}_N \) on a Hilbert space \( \mathcal{H} \) is a *-homomorphism from \( \mathcal{O}_N \) into \( B(\mathcal{H}) = \) the algebra of all bounded operators on \( \mathcal{H} \). The set of representations acting on \( \mathcal{H} \) is denoted \( \text{Rep}(\mathcal{O}_N, \mathcal{H}) \). The connection between \( \rho \) and the corresponding \( \{S_i\} \)-system is fixed by \( \rho(s_i) = S_i \). While the subalgebra \( \mathcal{C} \) in \( \mathcal{O}_N \) generated by the monomials \( s_i \cdots s_k s_k^* \cdots s_1^* \) is maximal abelian in \( \mathcal{O}_N \), the von Neumann algebra \( \mathfrak{A} \) generated by \( \rho(\mathcal{C}) \) may not be maximally abelian in \( B(\mathcal{H}) \). Whether it is, or not, depends on the representation. It is known to be maximally abelian if the operators \( S_i = \rho(s_i) \) are given by \( \text{(1.6)}, \) and if the functions \( m_i \) satisfy the usual subband conditions from wavelet theory. For details, see \( \text{[5]} \) and \( \text{[2]} \). For these representations, \( \mathcal{H} = L^2(\mathbb{T}) \); and the representations define wavelets
\[
\psi_{i,j,k}(x) = N^{\frac{j}{2}} \psi_i(N^j x - k), \quad i = 1, \cdots, N-1, \quad j, k \in \mathbb{Z} \tag{3.2}
\]
in \( L^2(\mathbb{R}) \). Such wavelets are specified by the functions \( \psi_1, \cdots, \psi_{N-1} \in L^2(\mathbb{R}) \). These representations \( \rho \) are called wavelet representations. The assertion is that, if \( \mathfrak{A} \) is defined by a wavelet representation, then \( \mathfrak{A} \) is maximally abelian in \( B(L^2(\mathbb{T})) \). The operators commuting with \( \mathfrak{A} \) are denoted \( \mathfrak{A}' \), and it is easy to see that \( \mathfrak{A} \) is maximally abelian if and only if \( \mathfrak{A}' \) is abelian. An abelian von Neumann algebra \( \mathfrak{A} \subset B(\mathcal{H}) \) is said to have a cyclic vector \( f \) if the closure of \( \mathfrak{A} f \) is \( \mathcal{H} \). For \( f \in \mathcal{H} \), the closure of \( \mathfrak{A} f \) is denoted \( \mathcal{H}_f \). It is known that \( \mathfrak{A} \) has a cyclic vector if and only if it is maximally abelian. Clearly, if \( f \) is a cyclic vector, then the measure \( \mu_f(\cdot) = \|P(\cdot)f\|^2 \) determines the other measures \( \{\mu_g \mid g \in \mathcal{H}\} \).
Lemma 3.1 Let \( \mathfrak{A} \subset B(\mathcal{H}) \) be an abelian \( C^* \)-algebra, and let \( \rho: C(X) \cong \mathfrak{A} \) be the Gelfand representation, \( X \) a compact Hausdorff space. Let \( f \in \mathcal{H}, \|f\| = 1 \).

(a) Then there is a unique Borel measure \( \mu \) on \( X \), and an isometry \( V_f: L^2(\mu) \to \mathcal{H} \), such that

\[
V_f(1) = f,
\]

\[
V_f(\psi) = \rho(\psi)f, \quad \psi \in C(X),
\]

and

\[
V_f(L^2(\mu)) = \mathcal{H}f.
\]

(b) Let \( f_i \in \mathcal{H}, \|f_i\| = 1, i = 1, 2 \), and suppose \( \mu_1 << \mu_2 \). Setting \( k = \frac{d\mu_1}{d\mu_2} \), where \( \mu_i = \mu_{f_i}, i = 1, 2 \), then \( U\psi = \sqrt{k}\psi \) defines an isometry \( U: L^2(\mu_1) \to L^2(\mu_2) \), and \( W = V_{f_2}UV_{f_1}^* : \mathcal{H}f_1 \to \mathcal{H}f_2 \) is in the commutant of \( \mathfrak{A} \).

Proof. Part (a) follows from the spectral theorem applied to abelian \( C^* \)-algebras, see e.g., [8]. To prove (b), let \( f_i \) be the two vectors in \( \mathcal{H} \), and set \( \mu_i = \mu_{f_i}, i = 1, 2 \). Since \( \mu_1 << \mu_2 \), the Radon-Nikodym derivative \( k = \frac{d\mu_1}{d\mu_2} \) is well defined. Clearly then

\[
\|U\psi\|_{L^2(\mu_2)}^2 = \int_X |\psi|^2 k \, d\mu_2 = \int_X |\psi|^2 \, d\mu_1 = \|\psi\|_{L^2(\mu_1)}^2.
\]

As a result \( W = V_{f_2}UV_{f_1}^* \) is a well defined partial isometry in \( \mathcal{H} \). For \( \psi \in C(X) \), we compute

\[
W\rho(\psi) = V_{f_2}UV_{f_1}^*\rho(\psi)
\]

\[
= V_{f_2}UM_\psi V_{f_1}
\]

\[
= V_{f_2}M_\psi UV_{f_1}^*
\]

\[
= \rho(\psi)V_{f_2}UV_{f_1}^*
\]

\[
= \rho(\psi)W,
\]

and we conclude that \( W \in \mathfrak{A}' \). The prime stands for commutant.

Theorem 3.2 Let \( N \in \mathbb{N}, N \geq 2 \); let \( \mathcal{H} \) be a Hilbert space, and \( (S_i) \) a representation of \( \mathcal{O}_N \) in \( \mathcal{H} \). Let \( (X, d) \) be a compact metric space, and \( (\sigma_i)_{0 \leq i < N} \) an iterated function system which is complete and non-overlapping. Let \( P: B(X) \to B(\mathcal{H}) \) the corresponding projection valued measure. Suppose the von Neumann algebra \( \mathfrak{A} \) generated by \( \{S_\alpha S_\alpha^* \mid k \in \mathbb{N}, \alpha \in (\mathbb{Z}_N)^k\} \) is maximally abelian. For \( f \in \mathcal{H}, \|f\| = 1 \), set \( \mu_f(\cdot) = \|P(\cdot)f\|^2 \). Then the following two conditions are equivalent:

(i) \( f \) is a cyclic vector for \( \mathfrak{A} \).

(ii) \( \mu_f \circ \sigma_i^{-1} << \mu_f, i = 0, 1, \ldots, N - 1 \).
Proof. We first claim that

\[ \mu_f \circ \sigma_i^{-1} = \mu_{S_i f}. \]  

(3.6)

To see this, we apply \( \mathcal{A} \) to \( S_i f \). Then \( \mu_{S_i f} = \sum_j \mu_{S_j^*} \circ \sigma_j^{-1} = \mu_f \circ \sigma_i^{-1} \) since \( S_j^* S_i f = \delta_{i,j} f \). This is the desired identity (3.6).

Secondly, let \( i \neq j \). Then naturally \( S_i f \perp S_j f \). But we claim that \( \mathcal{H}_{S_i f} \perp \mathcal{H}_{S_j f} \):

\[ \mathcal{H}_{S_i f} \perp \mathcal{H}_{S_j f}; \]  

(3.7)

i.e., for all \( A \in \mathcal{A} \), \( \langle S_i f \mid AS_j f \rangle = 0 \). Since \( \mathcal{A} \) is generated by the projections \( S_\alpha S_\alpha^* \), it is enough to show that \( S_i^* S_\alpha S_\alpha^* S_j = 0 \) for \( \alpha = (\alpha_1, \cdots, \alpha_k) \). But \( S_i^* S_\alpha S_\alpha^* S_j = \delta_{i\alpha,\alpha_1} \delta_{j\alpha,\alpha_2} \cdots \delta_{i\alpha,\alpha_k} S_{\alpha_\alpha}^* \cdots S_{\alpha_\alpha}^* = 0 \) since \( i = j \). The orthogonality relation (3.7) follows.

We first prove (i)\( \Rightarrow \) (ii); in fact we prove that \( \mu_g << \mu_f \) for all \( g \in \mathcal{H} \), if \( f \) is assumed cyclic. If \( f \) is cyclic, and \( g \in \mathcal{H} \), \( \|g\| = 1 \), then clearly \( \mathcal{H}_g \subset \mathcal{H}_f \). By the argument in Lemma 3.1(b), we conclude that \( W = V_f^* V_g \); \( L^2(\mu_g) \to L^2(\mu_f) \) commutes with the multiplication operators. Setting \( k = W(1) \), we have \( \int x |\psi|^2 d\mu_g = \int x |W \psi|^2 d\mu_f = \int |\psi|^2 |k|^2 d\mu_f \), or equivalently, \( d\mu_g = |k|^2 d\mu_f \). The conclusion \( d\mu_g < d\mu_f \) follows, and \( \frac{d\mu_g}{d\mu_f} = |k|^2 \).

To prove (ii)\( \Rightarrow \) (i); let \( i, j \in \mathbb{Z}_N \), and suppose \( i \neq j \). We saw that then \( \mathcal{H}_{S_i f} \perp \mathcal{H}_{S_j f} \). Suppose \( f \) is not cyclic. Since by (3.6) \( \mu_{S_i f} = \mu_f \circ \sigma_i^{-1} \), we get the two isometries \( V_{S_i f} \): \( L^2(\mu_f \circ \sigma_i) \to \mathcal{H}_{S_i f} \) with orthogonal ranges. Let

\[ k_i = \frac{d\mu_f \circ \sigma_i^{-1}}{d\mu_f}, \]  

and

\[ k_j = \frac{d\mu_f \circ \sigma_j^{-1}}{d\mu_f}. \]

Set

\[ U_i \psi = \psi \sqrt{k_i} \]  

and

\[ U_j \psi = \psi \sqrt{k_j}. \]

Then the following operator

\[ W = V_{S_i f} U_i^* U_j V_{S_j f} \]  

(3.8)

is well defined. It is a partial isometry in \( \mathcal{H} \) with initial space \( \mathcal{H}_{S_i f} \) and final space \( \mathcal{H}_{S_j f} \); i.e., \( W^* W = \text{proj}(\mathcal{H}_{S_i f}) = p_i \), and \( WW^* = \text{proj}(\mathcal{H}_{S_j f}) = p_j \). By the lemma, \( W \) is in the commutant of \( \mathcal{A} \). But the two projections \( p_i \) and \( p_j \) are orthogonal by the lemma, i.e., \( p_i p_j = 0 \). Relative to the decomposition \( p_i \mathcal{H} \oplus p_j \mathcal{H} \), we now consider the following two block matrix operators

\[
\begin{pmatrix}
0 & 0 \\
W & 0
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
0 & W^* \\
0 & 0
\end{pmatrix};
\]

and note that

\[
\begin{pmatrix}
0 & 0 \\
W & 0
\end{pmatrix} \begin{pmatrix}
0 & W^* \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
0 & p_j
\end{pmatrix},
\]

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while
\[
\begin{pmatrix}
0 & W^* \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
W & 0
\end{pmatrix}
= 
\begin{pmatrix}
p_i & 0 \\
0 & 0
\end{pmatrix}.
\]

Since the two non-commuting operators are in \( \mathfrak{A}' \), it follows that \( \mathfrak{A}' \) is non-abelian, and as a result that \( \mathfrak{A} \) is not maximally abelian.

Two Examples: (a) Let \( \mathcal{H} = L^2(\mathbb{T}) \) where as usual \( \mathbb{T} \) denotes the torus, equipped with Haar measure. Set \( e_n(z) = z^n, z \in \mathbb{T}, n \in \mathbb{Z} \), and define
\[
\begin{cases}
S_0 f(z) = f(z^2) & \text{for } f \in \mathcal{H}, \text{ and } z \in \mathbb{T} \\
S_1 f(z) = z f(z^2)
\end{cases}
\] (3.9)

As noted in Section 1, this system is in \( \text{Rep} (\mathcal{O}_2, \mathcal{H}) \). By Theorem 2.2, there is a unique projection valued measure \( P(\cdot) \) on \( B([0, 1)) \) such that
\[
P \left( \left[ \frac{\alpha_1}{2} + \cdots + \frac{\alpha_k}{2^k}, \frac{\alpha_1}{2} + \cdots + \frac{\alpha_k}{2^k} + \frac{1}{2^k} \right] \right) = S_\alpha S^*_\alpha^\prime (3.10)
\]

where \( S_\alpha = S_{\alpha_1} \cdots S_{\alpha_k} \).

It is easy to check that the range of the projection \( S_\alpha S^*_\alpha \) is the closed subspace in \( \mathcal{H} \) spanned by
\[
\{ e_n | n = \alpha_1 + 2 \alpha_2 + \cdots + 2^{k-1} \alpha_k + 2^k p, p \in \mathbb{Z} \},
\]
and it follows that
\[
S_\alpha S^*_\alpha e_0 = \begin{cases}
e_0 & \text{if } \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0 \\
0 & \text{otherwise}.
\end{cases}
\]

Hence
\[
\mathcal{H} e_0 = [\mathfrak{A} e_0] = \mathbb{C} e_0
\]
is one-dimensional, and
\[
\mu_{e_0}(\cdot) = ||P(\cdot)e_0||^2 = \delta_0
\]

where \( \delta_0 \) is the Dirac measure on \( [0, 1) \) at \( x = 0 \). With the IFS \( \sigma_0(x) = \frac{x}{2}, \sigma_1(x) = \frac{x+1}{2} \) on the unit-interval, we get
\[
\begin{bmatrix}
\mu_{e_0} \circ \sigma_0^{-1} = \delta_0 \\
\mu_{e_0} \circ \sigma_1^{-1} = \delta_{\frac{1}{2}}
\end{bmatrix}
\] (3.11)

making it clear that condition (ii) in Theorem 3.2 is not satisfied.

(b) We now modify (3.9) as follows:

Set
\[
\begin{cases}
S_0 f(z) = \frac{1}{\sqrt{2}} (1 + z) f(z^2) & \text{for } f \in L^2(\mathbb{T}), \text{ and } z \in \mathbb{T} \\
S_1 f(z) = \frac{1}{\sqrt{2}} (1 - z) f(z^2)
\end{cases}
\] (3.12)
This system \((S_t)\) is in \(\text{Rep}(\mathcal{O}_2, L^2(\mathbb{T}))\), and \(\mu_{e_0}(\cdot) = \|P(\cdot)e_0\|^2 = \) Lebesgue measure \(dt\) on \([0,1)\), where \(P(\cdot)\) is again determined by \((3.10)\). This is the representation of \(O_2\) which corresponds to the usual Haar wavelet, i.e., to

\[
\psi(x) = \begin{cases} 
1 & \text{if } 0 \leq x < \frac{1}{2} \\
-1 & \text{if } \frac{1}{2} \leq x < 1
\end{cases}
\]

and

\[
\psi_{j,k}(x) = 2^j \psi(2^j x - k) \quad \text{for } j, k \in \mathbb{Z}
\]

is then the standard Haar basis for \(L^2(\mathbb{R})\); compare this with \((3.13)\). For this representation \(\mathfrak{A}\) can be checked to be maximally abelian, but it also follows from the theorem, since now the analog of \((3.11)\) is

\[
\left\{ \begin{array}{l}
\mu_{e_0} \circ \sigma_0^{-1} = 2dt \text{ restricted to } [0, \frac{1}{2}) \\
\mu_{e_0} \circ \sigma_1^{-1} = 2dt \text{ restricted to } [\frac{1}{2}, 1).
\end{array} \right.
\]

Since \(\mu_{e_0} = dt\) restricted to \([0,1)\), it is clear that now condition (ii) in Theorem \((3.2)\) is satisfied.

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