SCATTERING FOR THE CRITICAL 2-D NLS WITH EXPONENTIAL GROWTH

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Abstract. In this article, we establish in the radial framework the $H^1$-scattering for the critical 2-D nonlinear Schrödinger equation with exponential growth. Our strategy relies on both the a priori estimate derived in [10, 23] and the characterization of the lack of compactness of the Sobolev embedding of $H^1_{\text{rad}}(\mathbb{R}^2)$ into the critical Orlicz space $\mathcal{L}(\mathbb{R}^2)$ settled in [4]. The radial setting, and particularly the fact that we deal with bounded functions far away from the origin, occurs in a crucial way in our approach.

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1. Introduction and statement of the results

1.1. Setting of the problem and main result. We are interested in the two dimensional nonlinear Schrödinger equation:

\[
\begin{cases}
  i\partial_t u + \Delta u = f(u), \\
  u|_{t=0} = u_0 \in H^1_{rad}(\mathbb{R}^2),
\end{cases}
\]

where the function \( u \) with complex values depends on \((t, x) \in \mathbb{R} \times \mathbb{R}^2\), and the nonlinearity \( f : \mathbb{C} \to \mathbb{C} \) is defined by

\[
f(u) = \left( e^{4\pi|u|^2} - 1 - 4\pi|u|^2 \right) u.
\]

Let us emphasize that the solutions of the Cauchy problem (1)-(2) formally satisfy the conservation of mass and Hamiltonian

\[
M(u, t) := \int_{\mathbb{R}^2} |u(t, x)|^2 dx \quad \text{and}
\]

\[
H(u, t) := \int_{\mathbb{R}^2} \left( |\nabla u(t, x)|^2 + F(u(t, x)) \right) dx,
\]

where

\[
F(u) = \frac{1}{4\pi} \left( e^{4\pi|u|^2} - 1 - 4\pi|u|^2 - 8\pi^2|u|^4 \right).
\]

The question of the existence of global solutions for the Cauchy problem (1)-(2) was investigated in [11] and subcritical, critical and supercritical regimes in the energy space was identified. This notion of criticality is related to the size of the initial Hamiltonian \( H(u_0) \) with respect to 1. More precisely, the concerned Cauchy problem is said to be subcritical if \( H(u_0) < 1 \), critical if \( H(u_0) = 1 \) and supercritical if \( H(u_0) > 1 \).

In [11], the authors established in both subcritical and critical regimes the existence of global solutions in the functional space \( C(\mathbb{R}, H^1(\mathbb{R}^2)) \cap L^4_{loc}(\mathbb{R}, W^{1,4}(\mathbb{R}^2)) \), and proved that well-posedness fails to hold in the supercritical one. Thereafter in [13], the scattering problem for the concerned nonlinear Schrödinger equation has been solved in the subcritical case.

Note that several works have been devoted to the nonlinear Schrödinger equation (1). In particular, one can mention the following a priori estimate proved independently by Colliander-Grillakis-Tzirakis and by Planchon-Vega in [10, 23]

\[
\|u\|_{L^4(\mathbb{R}, L^8)} \lesssim \|u\|_{L^\infty(\mathbb{R}, L^2)^{3/4}}^{1/4} \|\nabla u\|_{L^\infty(\mathbb{R}, L^2)^{1/4}},
\]

available for any global solution \( u \) in \( L^\infty(\mathbb{R}, H^1) \).
The purpose of this paper is to investigate the critical case \( H(u_0) = 1 \) in the radial framework, and to establish that the \( H^1 \)-scattering also holds in that case. More precisely, our main result states as follows:

**Theorem 1.1.** Let \( u \) be a solution to (1)-(2) satisfying \( H(u) = 1 \), then
\[
 u \in L^4(\mathbb{R}, W^{1,4}(\mathbb{R}^2)),
\]
where
\[
 W^{1,4}(\mathbb{R}^2) := \{ f \in S'(\mathbb{R}^2), \| f \|_{L^4} + \| \nabla f \|_{L^4} < \infty \}.
\]
Besides there exist \( v_\pm \in H^1_{\text{rad}}(\mathbb{R}^2) \) such that
\[
 \| u(t, \cdot) - e^{it\Delta} v_\pm \|_{H^1} \to \mp \infty.
\]

1.2. General scheme of the proof. All along this article, we shall see that the norm \( L^\infty(\mathbb{R}, L^4(\mathbb{R}^2)) \) will play a decisive role in the approach adopted to establish our result. The main difficulty lies in the non-conservation of the \( L^4 \)-norm over time for solutions to the free Schrödinger equation. To investigate the behavior of this norm, we shall resort to the a priori estimate provided in [10, 23] taking advantage of the fact that in view of the radial setting, \( H^1_{\text{rad}}(\mathbb{R}^2) \) embeds on the one hand compactly into \( L^4(\mathbb{R}^2) \) and on the other hand in \( L^\infty(\mathbb{R}^2 \setminus \{0\}) \). It turns out that thanks to this a priori estimate at hand, the proof of our result does not require structure theorems as those obtained in [3, 16, 21] which played a crucial role for instance in the remarkable work of [15].

Roughly speaking, the proof of our main result is done in three steps. In the first step, we establish that for any solution \( u \) of the Cauchy problem (1)-(2) and any positive real sequence \( (t_n)_{n \geq 0} \) tending to \(+\infty\), the evolution of \( u(t_n, \cdot) \) under the flow of the linear Schrödinger equation converges to zero in \( L^\infty(\mathbb{R}^+, L^4) \). This step constitutes the heart of the matter and the key ingredient to achieve it is the a priori estimate derived in [10, 23]. In the second step, we highlight a lack of compactness at infinity making use of the virial identity. Finally, in the third step we complete the proof of our main theorem by distinguishing two cases: a first case where the norm in \( L^\infty(\mathbb{R}^+, \tilde{L}) \) is strictly less than \( \frac{1}{\sqrt{4\pi}} \) and that we will qualify by the case where the whole mass does not concentrate, and a second case where the norm in \( L^\infty(\mathbb{R}^+, \tilde{L}) \) is equal to \( \frac{1}{\sqrt{4\pi}} \) and that we will designate by the case where the whole mass concentrates. The main idea to handle the second case which is the more challenging is the explicit description of that situation by means of the example by Moser settled in [4, 6]. To understand that case, we undertake an analysis depending on whether the whole mass concentrates in small or large times.

1.3. Layout of the paper. The paper is organized as follows: in Section 2 we provide the basic tools which are used in this text, namely critical 2-D Sobolev embeddings, an overview of the lack of compactness of \( H^1(\mathbb{R}^2) \) into the Orlicz space and basic facts about the linear Schrödinger equation. In Section 3, we establish several useful estimates. This includes virial identity and properties of solutions to the nonlinear Schrödinger equation associated to Cauchy data evolving sub-critically.
under the flow of the linear equation. Section 4 is devoted to the proof of our main result. As it is mentioned in Paragraph 1.2, this is achieved in three steps: a first step where the strong convergence to zero of the sequence \((e^{it\Delta} u(t_n, \cdot))\) in \(L^\infty(\mathbb{R}_+, L^4)\) is settled for any real sequence \((t_n)_{n \geq 0}\) tending to \(+\infty\), a second step where a lack of compactness at infinity is emphasized making use of virial identity, and lastly a third step where the proof is complete. Finally, we deal in appendix with various Moser-Trudinger type inequalities which are of constant use all along this article.

Finally, we mention that the letter \(C\) will be used to denote a universal constant which may vary from line to line. We also use \(A \lesssim B\) to denote an estimate of the form \(A \leq CB\) for some constant \(C\). For simplicity, we shall also still denote by \((u_n)\) any subsequence of \((u_n)\) and designate by \(\phi(1)\) any sequence which tends to 0 as \(n\) goes to infinity.

2. Technical tools

2.1. Critical 2-D Sobolev embedding. It is well known that \(H^1(\mathbb{R}^2)\) embeds continuously into \(L^p(\mathbb{R}^2)\) for all \(2 \leq p < \infty\) but not in \(L^\infty(\mathbb{R}^2)\). However, resorting to an interpolation argument, we can estimate the \(L^\infty\) norm of functions in \(H^1(\mathbb{R}^2)\), using a stronger norm but with a weaker growth (namely logarithmic). More precisely, we have the following logarithmic estimate which will be needed in this paper:

**Lemma 2.1 ([12], Theorem 1.3).** Let \(0 < \alpha < 1\). For any \(\lambda > \frac{1}{2\pi \alpha}\) and any \(0 < \mu \leq 1\), a constant \(C_\lambda > 0\) exists such that for any function \(u \in H^1(\mathbb{R}^2) \cap C^\alpha(\mathbb{R}^2)\), we have

\[
\|u\|_{L^\infty}^2 \leq \lambda \|u\|_{H^\mu}^2 \log \left( C_\lambda + \frac{8^\alpha \mu^{-\alpha} \|u\|_{C^\alpha}}{\|u\|_{H^\mu}} \right),
\]

where \(C^\alpha\) denotes the inhomogeneous Hölder space of regularity index \(\alpha\) and \(H^\mu\) the Sobolev space endowed with the norm \(\|u\|_{H^\mu}^2 := \|\nabla u\|^2_{L^2} + \mu^2 \|u\|^2_{L^2}\).

Otherwise in the radial case which is the setting of this article, we have the following estimate which implies the control of the \(L^\infty\)-norm far away from the origin (see for instance [4]):

**Lemma 2.2.** Let \(u \in H^1_{rad}(\mathbb{R}^2)\) and \(1 \leq p < \infty\). Then

\[
|u(x)| \leq \frac{C_p}{r^{\frac{2}{2+p}}} \|u\|_{L^p} \|\nabla u\|_{L^2}^2,
\]

with \(r = |x|\). In particular

\[
|u(x)| \leq \frac{C}{r^{\frac{3}{2}}} \|u\|_{L^2} \|\nabla u\|_{L^2}^{\frac{3}{2}} \quad \text{and}
\]

\[
|u(x)| \leq \frac{C}{r^{\frac{3}{2}}} \|u\|_{L^4} \|\nabla u\|_{L^2}^{\frac{3}{2}}.
\]
Remark 2.3. In the general case, the embedding of $H^1(\mathbb{R}^2)$ into $L^p(\mathbb{R}^2)$ is not compact as it is shown for instance by the example: $u_n(x) = \varphi(x + x_n)$ with $\varphi$ a function belonging to $\mathcal{D}(\mathbb{R}^2)$ and $(x_n)$ a sequence of $\mathbb{R}^2$ satisfying $|x_n| \to \infty$. However, in the radial setting, the following compactness result holds (see for example [8, 14, 25]):

Lemma 2.4. Let $2 < p < \infty$. The embedding of $H^1_{rad}(\mathbb{R}^2)$ into $L^p(\mathbb{R}^2)$ is compact.

For our subject, it will be useful to point out the following refined estimate:

Lemma 2.5. There is a positive constant $C$ such that

$$\|u\|_{L^4} \leq C \|\nabla u\|_{L^2}^{\frac{1}{2}} \|u\|_{H^1}^{\frac{3}{4}},$$

for any $u \in H^1(\mathbb{R}^2)$.

Proof. In view of the continuity of the Fourier transform

$$\mathcal{F} : L^4(\mathbb{R}^2) \longrightarrow L^4(\mathbb{R}^2),$$

it suffices to prove that

$$\|\hat{u}\|_{L^4(\mathbb{R}^2)} \leq C \|\nabla u\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|u\|_{H^1(\mathbb{R}^2)}^{\frac{3}{4}},$$

where $\hat{u}$ denotes the Fourier transform of $u$.

To go to this end, let us begin by observing that

$$\int_{\mathbb{R}^2} |\hat{u}(\xi)|^{\frac{4}{3}} d\xi = \int_{\mathbb{R}^2} |\hat{u}(\xi)| |\xi|^{\frac{1}{3}} |\xi|^{\frac{1}{3}} (|\xi|^{\frac{1}{3}} |\xi|^{\frac{1}{3}})^{-\frac{4}{3}} d\xi$$

$$\leq \left( \int_{\mathbb{R}^2} |\hat{u}(\xi)|^2 |\xi|^{\frac{1}{3}} |\xi|^{\frac{1}{3}} d\xi \right)^{\frac{3}{4}} \left( \int_{\mathbb{R}^2} |\xi|^{\frac{1}{3}} d\xi \right)^{\frac{1}{4}}$$

$$\leq \left( \int_{\mathbb{R}^2} |\hat{u}(\xi)|^2 |\xi|^{\frac{1}{3}} |\hat{u}(\xi)| |\xi|^{\frac{1}{3}} d\xi \right)^{\frac{2}{3}}.$$

Now applying Hölder inequality, we deduce that

$$\int_{\mathbb{R}^2} |\hat{u}(\xi)|^{\frac{4}{3}} d\xi \lesssim \left( \int_{\mathbb{R}^2} |\xi|^2 |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{3}} \left( \int_{\mathbb{R}^2} |\hat{u}(\xi)|^2 |\xi|^2 d\xi \right)^{\frac{1}{3}},$$

which thanks to Fourier-Plancherel formula gives rise to

$$\|\hat{u}\|_{L^4(\mathbb{R}^2)} \lesssim \|\nabla u\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|u\|_{H^1(\mathbb{R}^2)}^{\frac{3}{4}}.$$

This ends the proof of the lemma. \(\square\)

Furthermore

$$H^1(\mathbb{R}^2) \hookrightarrow \mathcal{L},$$

where $\mathcal{L}$ denotes the Orlicz space $L^\phi$ associated to the function $\phi = e^{s^2} - 1$ (see Definition 2.12 below). This embedding stems immediately from the following sharp Moser-Trudinger type inequalities (see [1, 22, 24, 27]):

\footnote{We used the classical notation $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$.}
Proposition 2.6.

\( \sup_{\|u\|_{H^1} \leq 1} \int_{\mathbb{R}^2} \left( e^{4\pi|u(x)|^2} - 1 \right) dx := \kappa < \infty, \)

and states as follows

\( \|u\|_{L^\infty} \leq \frac{1}{\sqrt{4\pi}} \|u\|_{H^1}, \)

when the Orlicz space \( L \) is endowed with the norm \( \| \cdot \|_L \) where the number 1 in Definition 2.12 is replaced by the above constant \( \kappa \).

In this article we are rather interested in the Sobolev embedding

\( H^1(\mathbb{R}^2) \hookrightarrow \tilde{L}, \)

where \( \tilde{L} \) is the Orlicz space \( L^\phi \) associated to the function \( \phi = e^{s^2} - 1 - s^2 \), and which arises naturally in the study of the nonlinear Schrödinger equation with exponential growth (1)-(2). It is obvious that

\( \|u\|_{\tilde{L}} \leq \frac{1}{\sqrt{4\pi}} \|u\|_{H^1}, \)

where \( \| \cdot \|_{\tilde{L}} \) is the Orlicz norm introduced in Definition 2.12, with the constant \( \kappa \) appearing in Identity (14) instead of the number 1. Besides, as it can be shown by the example by Moser \( f_\alpha \) given by (23), the Sobolev constant appearing in (17) is optimal.

For our purpose, we shall resort to the following Moser-Trudinger type inequalities and the resulting corollaries that will be demonstrated in Appendix A:

**Proposition 2.7.** Let \( \alpha \in [0, 4\pi] \) and \( p \) be a nonnegative real larger than 2. A constant \( C(\alpha, p) \) exists such that

\[ \int_{\mathbb{R}^2} e^{\alpha|u(x)|^2} |u(x)|^p dx \leq C(\alpha, p) \int_{\mathbb{R}^2} |u(x)|^p dx, \]

for all \( u \) in \( H^1(\mathbb{R}^2) \) satisfying \( \|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1. \)

A byproduct of Proposition 2.7 is the following useful result.

**Proposition 2.8.** Let \( \alpha \in [0, 4\pi]. \) There is a constant \( c_\alpha \) such that

\[ \int_{\mathbb{R}^2} \left( e^{\alpha|u(x)|^2} - 1 - \alpha|u(x)|^2 \right) dx \leq c_\alpha \|u\|^4_{L^4}, \]

for all \( u \) in \( H^1(\mathbb{R}^2) \) satisfying \( \|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1. \)

From (19), it is easy to deduce the following consequence.

**Corollary 2.9.** Let \( (u_n) \) be a bounded sequence in \( H^1(\mathbb{R}^2) \) such that

\[ \|u_n\|_{L^1} \xrightarrow{n \to \infty} 0, \]
then
\begin{equation}
\|u_n\|_{\tilde{L}} \leq \frac{1}{\sqrt{4\pi}} \|\nabla u_n\|_{L^2} + o(1), \quad n \to \infty.
\end{equation}

Remark 2.10. Let us point out that estimate (15) (respectively (17)) fails if we replace in the right hand side \(\|u\|_{H^1}\) by \(\|\nabla u\|_{L^2}\). To be convinced, just consider the sequence \((u_n)_{n \geq 0}\) defined by \(u_n(x) := \frac{1}{n} e^{-\frac{|x|^2}{n^2}}\) (respectively \(u_n(x) := \frac{1}{\sqrt{n}} e^{-\frac{|x|^2}{n}}\)).

Inequality (18) fails for \(\alpha = 4\pi\) as it can be shown by the example by Moser defined by (23). However, the following estimate needed in the sequel occurs:

Corollary 2.11. For any \(\delta > 0\), there exist \(c_\delta\) and \(\varepsilon_0\) such that for all \(0 < \varepsilon \leq \varepsilon_0\) and all nonnegative real \(p \geq 2\), there is a positive constant \(C(\delta, \varepsilon, p)\) such that for \(r = 1 - \varepsilon c_\delta\) the following estimate holds
\begin{equation}
\int_{\mathbb{R}^2} e^{4\pi(1+\varepsilon)|u(x)|^2} |u(x)|^p dx \leq C(\delta, \varepsilon, p) \left(\|u\|_{L^p(\mathbb{R}^2)}^p + \|u\|_{L^{p^*}(\mathbb{R}^2)}^p\right),
\end{equation}
for all \(u\) in \(H^1(\mathbb{R}^2)\) satisfying \(\|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1\) and \(\|u\|_{\tilde{L}} \leq \frac{1}{\sqrt{4\pi(1+2\delta)}}\).

Let us close this section by introducing the definition of the so-called Orlicz spaces on \(\mathbb{R}^d\).

Definition 2.12. Let \(\phi : \mathbb{R}^+ \to \mathbb{R}^+\) be a convex increasing function such that
\(\phi(0) = 0 = \lim_{s \to 0^+} \phi(s), \quad \lim_{s \to \infty} \phi(s) = \infty\).

We say that a measurable function \(u : \mathbb{R}^d \to \mathbb{C}\) belongs to \(L^\phi\) if there exists \(\lambda > 0\) such that
\begin{equation}
\int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\lambda}\right) dx < \infty.
\end{equation}

We denote then
\begin{equation}
\|u\|_{L^\phi} = \inf \left\{ \lambda > 0, \quad \int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1 \right\}.
\end{equation}

2.2. Development on the lack of compactness of Sobolev embedding in the Orlicz space. The Sobolev embeddings (15) and (17) are non compact at least for two reasons. The first reason is the lack of compactness at infinity that we can highlight through the sequence \(u_n(x) = \varphi(x + x_n)\) where \(0 \neq \varphi \in \mathcal{D}\) and \(|x_n| \to \infty\). The second reason is of concentration-type derived by J. Moser in [22] and by P.-L. Lions in [19, 20] and is illustrated by the following fundamental sequence \((f_{\alpha_n})_{n \geq 0}\), when \((\alpha_n)_{n \geq 0}\) is a sequence of positive reals tending to infinity:
\begin{equation}
f_{\alpha_n}(x) = \begin{cases}
\sqrt{\frac{2\alpha_n}{2\pi}} & \text{if } |x| \leq e^{-\alpha_n},
\frac{-\log |x|}{\sqrt{2\alpha_n x}} & \text{if } e^{-\alpha_n} \leq |x| \leq 1,
0 & \text{if } |x| \geq 1.
\end{cases}
\end{equation}
Indeed, one can prove by straightforward computations (detailed for instance in [4]) that \( \|f_\alpha\|_L \xrightarrow{n \to \infty} \frac{1}{\sqrt{4\pi}} \) and \( \|f_\alpha\|_{\tilde{L}} \xrightarrow{n \to \infty} \frac{1}{\sqrt{4\pi}} \).

In [4, 5, 6], the lack of compactness of the critical Sobolev embedding 

\[ H^1(\mathbb{R}^2) \hookrightarrow L(\mathbb{R}^2) \]

was described in terms of an asymptotic decomposition by means of generalization of the above example by Moser. To state this characterization in a clear way, let us recall some definitions.

**Definition 2.13.** We shall designate by a scale any sequence \( \alpha := (\alpha_n) \) of positive real numbers going to infinity and by a profile any function \( \psi \) belonging to the set

\[ P := \{ \psi \in L^2(\mathbb{R}, e^{-2s} ds); \; \psi' \in L^2(\mathbb{R}) \; \text{and} \; \psi|_{[-\infty,0]} = 0 \}. \]

Two scales \( \alpha, \beta \) are said orthogonal (in short \( \alpha \perp \beta \)) if

\[ \left| \log \left( \frac{\beta_n}{\alpha_n} \right) \right| \xrightarrow{n \to \infty} \infty. \]

The asymptotically orthogonal decomposition derived in [4] is formulated in the following terms:

**Theorem 2.14.** Let \( (u_n)_{n \geq 0} \) be a bounded sequence in \( H^1_{rad}(\mathbb{R}^2) \) such that

\[ u_n \rightharpoonup 0, \quad \limsup_{n \to \infty} \|u_n\|_L = A_0 > 0 \quad \text{and} \quad \lim R \to \infty \limsup_{n \to \infty} \int_{|x| > R} |u_n(x)|^2 dx = 0. \]

Then, there exist a sequence \( (\alpha(j)) \) of pairwise orthogonal scales and a sequence of profiles \( (\psi(j)) \) in \( P \) such that, up to a subsequence extraction, we have for all \( \ell \geq 1, \)

\[ u_n(x) = \sum_{j=1}^\ell \sqrt{\frac{\alpha(j)}{2\pi}} \psi(j) \left( \frac{-\log |x|}{\alpha(j)} \right) + r_\ell^{(n)}(x), \quad \limsup_{n \to \infty} \|r_\ell^{(n)}\|_L \xrightarrow{\ell \to \infty} 0. \]

Moreover, we have the following stability estimates

\[ \|\nabla u_n\|_{L^2(\mathbb{R}^2)}^2 = \sum_{j=1}^\ell \|\psi(j)'\|_{L^2(\mathbb{R})}^2 + \|\nabla r_\ell^{(n)}\|_{L^2(\mathbb{R}^2)}^2 + o(1), \quad n \to \infty. \]

**Remarks 2.15.**

- The example by Moser can be written as

\[ f_\alpha(x) = \sqrt{\frac{\alpha_n}{2\pi}} L \left( \frac{-\log |x|}{\alpha_n} \right). \]
where

\[ L(s) = \begin{cases} 
0 & \text{if } s \leq 0, \\
 s & \text{if } 0 \leq s \leq 1, \\
 1 & \text{if } s \geq 1.
\end{cases} \tag{27} \]

- Let us emphasize that it was proved in [4] that

\[ \|u_n\|_L \xrightarrow{n \to \infty} \sup_{j \geq 1} \left( \lim_{n \to \infty} \|g_n^{(j)}\|_L \right), \tag{28} \]

where \( g_n^{(j)}(x) = \sqrt{\frac{\alpha_n^{(j)}}{2\pi}} \psi^{(j)} \left( -\frac{\log |x|}{\alpha_n^{(j)}} \right) \), and

\[ \lim_{n \to \infty} \|g_n^{(j)}\|_L = \frac{1}{\sqrt{4\pi}} \max_{s > 0} \frac{|\psi^{(j)}(s)|}{\sqrt{s}}. \tag{29} \]

- Therefore (see [6] for a detailed proof), if in addition to the assumptions of Theorem 2.14 the sequence \((u_n)_{n \geq 0}\) satisfies

\[ \|u_n\|_{\tilde{L}} = 1 \sqrt{\frac{4\pi}{\|\nabla u_n\|_{L^2} + o(1)}, \tag{30} \]

then we have necessary

\[ u_n(x) = \sqrt{\frac{\alpha_n}{2\pi}} L \left( -\frac{\log |x|}{\alpha_n} \right) + r_n(x), \quad \|\nabla r_n\|_{L^2} \to 0 \quad \text{as} \quad n \to \infty, \]

with \( L \) the Lions profile given by (27).

- Taking advantage of the above remark, we infer that if a bounded sequence \((u_n)_{n \geq 0}\) in \( H^1_{rad}(\mathbb{R}^2) \) converges to zero in \( L^4(\mathbb{R}^2) \) and satisfies

\[ \|u_n\|_{\tilde{L}} = 1 \sqrt{\frac{4\pi}{\|\nabla u_n\|_{L^2} + o(1)}, \tag{31} \]

then

\[ u_n(x) = \sqrt{\frac{\alpha_n}{2\pi}} L \left( -\frac{\log |x|}{\alpha_n} \right) + r_n(x), \quad \|\nabla r_n\|_{L^2} \to 0 \quad \text{as} \quad n \to \infty. \tag{32} \]

Indeed, write

\[ u_n = \chi u_n + (1 - \chi) u_n, \]

where \( \chi \) is a radial function in \( D(\mathbb{R}^2) \) equal to one in the unit ball and valued in \([0, 1]\). Thus by virtue of the radial estimate (11), we have

\[ \|\tilde{v}_n\|_{L^\infty} \xrightarrow{n \to \infty} 0, \]

where \( \tilde{v}_n := (1 - \chi) u_n \), which implies that

\[ \|\tilde{v}_n\|_{\tilde{L}} \xrightarrow{n \to \infty} 0. \]

Therefore (31) also reads

\[ \|v_n\|_{\tilde{L}} = 1 \sqrt{\frac{4\pi}{\|\nabla u_n\|_{L^2} + o(1)}, \tag{33} \]

with \( v_n := \chi u_n \).
By Hölder inequality
\begin{equation}
\|v_n\|_{L^2} \leq \|v_n\|_{L^4} \|\chi\|_{L^4} \xrightarrow{n \to \infty} 0,
\end{equation}
which gives rise in view of the Sobolev embedding (17) to
\begin{equation}
\|v_n\|_{\mathcal{L}} \leq \frac{1}{\sqrt{4\pi}} \|\nabla v_n\|_{L^2} + o(1).
\end{equation}
Taking advantage of the fact that the function $\chi$ takes its values in $[0, 1]$, we get
\[
\|\nabla v_n\|_{L^2}^2 + \|\nabla \tilde{v}_n\|_{L^2}^2 = \|\chi \nabla u_n\|_{L^2}^2 + \|(1 - \chi) \nabla u_n\|_{L^2}^2 + o(1)
\leq \|\nabla u_n\|_{L^2}^2 + o(1).
\]
Thus by virtue of (33), (34) and (35)
\[
\|v_n\|_{\mathcal{L}} = \frac{1}{\sqrt{4\pi}} \|\nabla v_n\|_{L^2} + o(1),
\]
and $\|\nabla \tilde{v}_n\|_{L^2}^2 = o(1)$.

Now in view of the previous remark, we deduce that
\[
v_n(x) = \sqrt{\frac{\alpha_n}{2\pi}} L \left(\frac{-\log |x|}{\alpha_n}\right) + r_n(x), \quad \|\nabla r_n\|_{L^2} \to 0 \quad \text{as} \quad n \to \infty,
\]
which ensures the explicit description (32) since
\[
v_n = u_n + \tilde{r}_n, \quad \text{with} \quad \|\nabla \tilde{r}_n\|_{L^2} \xrightarrow{n \to \infty} 0.
\]
• It was shown in [7] that the sequence $(f_{\alpha_n})$ also writes under the form:
\[
f_{\alpha_n}(x) = \tilde{f}_{\alpha_n}(x) + \tilde{r}_n(x),
\]
with
\begin{equation}
\tilde{f}_{\alpha_n}(x) = \frac{1}{(2\pi)^2} \sqrt{\frac{2\pi}{\alpha_n}} \int_{1 \leq |\xi| \leq e^{\alpha_n}} e^{i x \cdot \xi} \frac{1}{|\xi|^2} d\xi,
\end{equation}
and $\|\tilde{r}_n\|_{H^1} \xrightarrow{n \to \infty} 0$.

2.3. Linear Schrödinger equation. It is well-known that the solutions of the linear Schrödinger equation:
\begin{equation}
\begin{cases}
  i \partial_t v + \Delta v = 0, \\
  v|_{t=0} = v_0 \in H^1(\mathbb{R}^2),
\end{cases}
\end{equation}
satisfy the conservation laws
\begin{equation}
E_0(v, t) = \|\nabla v(t, \cdot)\|_{L^2}^2 = \|\nabla v(0, \cdot)\|_{L^2}^2 = E_0(v),
\end{equation}
\begin{equation}
\|v(t, \cdot)\|_{L^2}^2 = \|v(0, \cdot)\|_{L^2}^2,
\end{equation}
and for \( t \neq 0 \) the dispersive inequality

\[
\|v(t, \cdot)\|_{L^\infty} \lesssim \frac{\|v(0, \cdot)\|_{L^1}}{|t|}.
\]

Combining (39), (40) together with the interpolation between \( L^p \) spaces imply that

\[
\forall t \in \mathbb{R} \setminus \{0\}, \forall p \in [2, \infty], \|v(t, \cdot)\|_{L^p} \lesssim \frac{1}{|t|^{(1-\frac{2}{p})}} \|v(0, \cdot)\|_{L'^p}.
\]

Thanks to the so-called \( TT^* \) argument which is the standard method for converting the dispersive estimates into inequalities involving suitable space-time Lebesgue norms of the solutions, we get the following estimates known by Strichartz estimates which will be of constant use in this paper (see [9]):

**Proposition 2.16.** Let \( I \subset \mathbb{R} \) be a time slab, \( t_0 \in I \) and \((q,r), (\tilde{q}, \tilde{r})\) two \( L^2 \)-admissible Strichartz pairs, i.e.,

\[
2 \leq r, \tilde{r} < \infty \quad \text{and} \quad \frac{1}{q} + \frac{1}{r} = \frac{1}{\tilde{q}} + \frac{1}{\tilde{r}} = \frac{1}{2}.
\]

There exists a positive constant \( C \) such that if \( u \) is the solution of the Cauchy problem

\[
\begin{cases}
i \partial_t v + \Delta v = G(t, x), \\
v|_{t=t_0} = u_0 \in H^1(\mathbb{R}^2),
\end{cases}
\]

then for \( j \in \{0, 1\} \)

\[
\|\nabla^j v\|_{L^q(I,L^r)} \leq C \left( \|\nabla^j v(t_0)\|_{L^2} + \|\nabla^j G\|_{L_{\tilde{r}}(I,L')'} \right),
\]

where \( p' \) denotes the conjugate exponent of \( p \), defined by:

\[
\frac{1}{p} + \frac{1}{p'} = 1, \text{ with the rule that } \frac{1}{\infty} = 0.
\]

Note in particular that \((q,r) = (4,4)\) is an admissible Strichartz pair and that (see [2] for instance)

\[
W^{1,4}(\mathbb{R}^2) \hookrightarrow C^{1/2}(\mathbb{R}^2).
\]

Now, for any time slab \( I \subset \mathbb{R} \), we shall denote

\[
\|v\|_{ST(I)} := \sup_{j \in \{0,1\}} \left( \|\nabla^j v\|_{L^4(I,L^4)} + \|\nabla^j v\|_{L^\infty(I,L^2)} \right) \quad \text{and} \quad \|v\|_{ST^*(I)} := \sup_{j \in \{0,1\}} \|\nabla^j v\|_{L^\infty(I,L^4)}.
\]

3. **Virial identity and scattering under smallness conditions**

This section is devoted to the proof of basic estimates needed to develop the proof of Theorem 1.1, namely virial identity and the \( H^1 \)-scattering under smallness conditions.
3.1. **Virial identity.** The aim of this paragraph is to present virial identity in the framework of the two dimensional nonlinear Schrödinger equation (1). For that purpose, let us introduce a smooth and radial function \( \Phi \) satisfying \( 0 \leq \Phi \leq 1 \), \( \Phi(r) = r \), for all \( r \leq 1 \), and \( \Phi(r) = 0 \) for all \( r \geq 2 \). For any positive real \( R \) and any function \( u(x,t) \), we define

\[
V_R(t) := \int_{\mathbb{R}^2} \Phi_R(x)|u(x,t)|^2 \, dx,
\]

where \( \Phi_R(x) := R^2\Phi(|x|^2/R^2) \).

As usual integrating by parts, we get in view of (1)-(2) and properties of \( \Phi_R \) the following useful estimate known by virial identity:

**Lemma 3.1.** If \( u \) is a solution to (1)-(2), then \( V_R \) satisfies

\[
\frac{d}{dt} V_R(t) = 2\mathcal{I} \int_{\mathbb{R}^2} (\nabla \Phi_R(x) \cdot \nabla u(t,x)) \bar{u}(t,x) \, dx,
\]

where for \( z \in \mathbb{C} \), \( \mathcal{I}(z) \) denotes the imaginary part of \( z \), and

\[
\frac{d^2}{dt^2} V_R(t) = 8 \int_{\mathbb{R}^2} \Phi' \frac{|x|^2}{R^2} |\nabla u(t,x)|^2 \, dx + 16 \int_{\mathbb{R}^2} \Phi'' \left( \frac{|x|^2}{R^2} \right) \frac{|x \cdot \nabla u(t,x)|^2}{R^2} \, dx - \int_{\mathbb{R}^2} |u(t,x)|^2 \Delta^2 \Phi_R(x) \, dx + 2 \int_{\mathbb{R}^2} \Delta \Phi_R(x) (|u(t,x)|^2 \tilde{f}(|u(t,x)|^2) - g(|u(t,x)|^2)) \, dx,
\]

where \( \tilde{f}(s) = e^{4\pi s} - 1 - 4\pi s \) and \( g(s) = \int_0^s \tilde{f}(\rho) \, d\rho \).

**Proof.** According to the fact that \( u \bar{f} = \bar{u} f \), we infer that if \( u \) is a solution to (1)-(2) then we have

\[
\frac{d}{dt} V_R(t) = i \int_{\mathbb{R}^2} \Phi_R(x) (\bar{u} \Delta u - u \Delta \bar{u})(t,x) \, dx.
\]

This gives rise to (45) by integration by parts.

Let us now go to the proof of (46). As \( u \) solves (1), we deduce from (45) that

\[
\frac{d^2}{dt^2} V_R(t) = 2\mathcal{I} \int_{\mathbb{R}^2} \nabla \Phi_R(x) \cdot \nabla (i u - i f(u))(t,x) \bar{u}(t,x) \, dx + 2\mathcal{I} \int_{\mathbb{R}^2} \nabla \Phi_R(x) \cdot \nabla u(t,x) (-i \Delta \bar{u} + i f(\bar{u}))(t,x).
\]

Integrating by parts the first term of the right hand side of the above identity gives

\[
\frac{d^2}{dt^2} V_R(t) = 2\mathcal{R} \int_{\mathbb{R}^2} \Delta \Phi_R(x) f(u(t,x)) \bar{u}(t,x) + (f(u) \nabla \bar{u} + f(\bar{u}) \nabla u)(t,x) \cdot \nabla \Phi_R(x) \, dx - 2\mathcal{R} \int_{\mathbb{R}^2} \Delta \Phi_R(x) \bar{u}(t,x) \Delta u(t,x) + \nabla \Phi_R(x) \cdot \left( \nabla \bar{u} \Delta u + \nabla u \Delta \bar{u} \right)(t,x) \, dx,
\]
where for $z \in \mathbb{C}$, $\mathcal{R}(z)$ denotes the real part of $z$.

Besides straightforward computations lead to
\[
J(t) = \int_{\mathbb{R}^2} \Delta \Phi_R(x)f(u(t,x))\bar{u}(t,x) + \bar{f}(|u(t,x)|^2)\nabla|u(t,x)|^2 \cdot \nabla \Phi_R(x) \, dx
\]
\[
\quad = \int_{\mathbb{R}^2} \Delta \Phi_R(x)(|u|^2\bar{f}(|u|^2) - g(|u|^2))(t,x) \, dx,
\]
where
\[
J(t) := \int_{\mathbb{R}^2} \Delta \Phi_R(x)f(u(t,x))\bar{u}(t,x) + (f(u)\nabla \bar{u} + f(\bar{u})\nabla u)(t,x) \cdot \nabla \Phi_R(x) \, dx.
\]
Along the same lines, we obtain
\[
2\mathcal{R} \int_{\mathbb{R}^2} \Delta \Phi_R(x)\bar{u}(t,x)\Delta u(t,x) = \int_{\mathbb{R}^2} |u(t,x)|^2\Delta^2 \Phi_R(x) - 2\Delta \Phi_R(x)|\nabla u(t,x)|^2,
\]
and
\[
4\mathcal{R} \int_{\mathbb{R}^2} \nabla \Phi_R(x) \cdot \nabla u(t,x)\Delta \bar{u}(t,x) \, dx = 2\int_{\mathbb{R}^2} \Delta \Phi_R(x)|\nabla u(t,x)|^2 \, dx
\]
\[
\quad - 8\int_{\mathbb{R}^2} \Phi'(\frac{|x|^2}{R^2})|\nabla u(t,x)|^2 \, dx - 16\int_{\mathbb{R}^2} \Phi''(\frac{|x|^2}{R^2})\frac{|x \cdot \nabla u(t,x)|^2}{R^2} \, dx.
\]
This easily ensures the result. \hfill \Box

As a byproduct of virial identity, we get the following useful result:

**Corollary 3.2.** Let $u$ be a solution to (1)-(2) of mass $M$ and Hamiltonian $H \leq 1$. Then, there is a constant $C(M,H)$ such that for any positive real $\tau$, we have
\[
\int_t^{t+\tau} \int_{|x| \leq 1} G(u(s,x)) \, dx \, ds \leq C(M,H)(\tau),
\]
for all $t \in \mathbb{R}$, where $G(u) = \left( e^{4\pi|u|^2} - 1 - 4\pi|u|^2 \right)|u|^2$.

**Proof.** We shall proceed as in the proof of Lemma 3.1 setting
\[
V(t) := \int_{\mathbb{R}^2} \Phi_1(x)|u(x,t)|^2 \, dx,
\]
where $\Phi_1(x) := \Phi(|x|^2)$, with $\Phi$ the function introduced above. Thus thanks to (45), we get
\[
|\frac{d}{dt}V(t)| \lesssim H^{\frac{1}{2}}M^{\frac{1}{2}}.
\]
Moreover, in light of (46)
\[
\frac{d^2}{dt^2}V(t) = 8\int_{\mathbb{R}^2} \Phi'(|x|^2)|\nabla u(t,x)|^2 \, dx + 16\int_{\mathbb{R}^2} \Phi''(|x|^2)|x \cdot \nabla u(t,x)|^2 \, dx
\]
\[
\quad - \int_{\mathbb{R}^2} |u(t,x)|^2\Delta^2 \Phi_1(x) \, dx + 2\int_{\mathbb{R}^2} \Delta \Phi_1(x)\psi(u(t,x)) \, dx,
\]
where, with the notations of Lemma 3.1,  
\[ \psi(u) := (|u|^2 \tilde{f}(|u|^2) - g(|u|^2)). \]

Straightforward computations lead to
\[ 2 \int_{|x| \leq 1} \Delta \Phi_1(x) \psi(u(t, x)) \, dx = 8 \int_{|x| \leq 1} \psi(u(t, x)) \, dx = \frac{d^2}{dt^2} V(t) + R(t), \]
where the remainder term \( R(t) \) satisfies
\[ |R(t)| \lesssim \|u(t, \cdot)\|_{L^2}^2 + \|\nabla u(t, \cdot)\|_{L^2}^2 + \int_{|x| \geq 1} \psi(u(t, x)) \, dx. \]

Besides by virtue of the radial estimate (10) and the conservation laws (3)-(4), we have
\[ \|u(t, \cdot)\|_{L^\infty(\mathbb{R}, L^\infty(|x| \geq 1))} \lesssim C(M, H), \]
which gives rise to
\[ \int_{|x| \geq 1} \psi(u(t, x)) \, dx \lesssim \int_{|x| \geq 1} e^{4\pi |u(t, x)|^2} |u(t, x)|^6 \, dx \lesssim C(M, H) \int_{|x| \geq 1} |u(t, x)|^2 \, dx \lesssim C(M, H). \]

This ensures that
\[ R(t) \leq C(M, H). \]

Thus taking into account of (48), we deduce that
\[ 8 \int_t^{t+\tau} \int_{|x| \leq 1} \psi(u(s, x)) \, dx \, ds = \int_t^{t+\tau} \frac{d^2}{ds^2} V(s) \, ds + \int_t^{t+\tau} R(s) \, ds \leq \left[ \frac{d}{ds} V(s) \right]_t^{t+\tau} + \tau C(M, H) \leq C(M, H) \langle \tau \rangle. \]

This ends the proof of the Corollary under the fact that \( G(u) \lesssim \psi(u). \)

3.2. \( H^1 \)-scattering under smallness conditions. The purpose of this paragraph is to establish that solutions to (1)-(2) scatter on \( \mathbb{R}_\pm \), provided that the evolution of the Cauchy data under the flow of the linear Schrödinger equation is sufficiently small in \( L^\infty(\mathbb{R}_\pm, L^1) \) and strictly less than \( \frac{1}{\sqrt{4\pi}} \) in \( L^\infty(\mathbb{R}_\pm, \tilde{L}) \). More precisely, we have the following lemma which turns out to be essential in our strategy:

**Lemma 3.3.** For all \( M > 0 \) and all \( \delta > 0 \), there is \( \beta_0 = \beta_0(M, \delta) > 0 \) such that if \( u \) is the solution to the Cauchy problem
\[ \begin{cases} 
    i\partial_t u + \Delta u = f(u), \\
    u|_{t=0} = u_0 \in H^1(\mathbb{R}^2),
\end{cases} \]
with \( f(u) \) given by (2), and \( u_0 \) satisfying
\[ H(u_0) \leq 1, \quad \|u_0\|_{L^2} = M, \]
\[ \|e^{it\Delta} u_0\|_{L^\infty(\mathbb{R}_\pm, \tilde{L})} \leq \frac{1}{\sqrt{4\pi(1+\delta)}} \quad \text{and} \quad \|e^{it\Delta} u_0\|_{L^\infty(\mathbb{R}_\pm, L^1)} \leq \beta_0, \]
then
\[ \|u\|_{L^4(\mathbb{R}_\pm, W^{1,4})} < \infty \quad \text{and} \quad \|f(u)\|_{ST^*(\mathbb{R}_\pm)} < \infty. \]
Moreover there is a positive constant $C$ such that \(^2\)
\[
\|u(t, \cdot) - e^{it\Delta} u_0\|_{ST(\mathbb{R}^\pm)} \leq C \beta_0^{\frac{3}{2}}.
\]

**Remark 3.4.** Note that (49) implies in a standard way the existence of $v_\pm$ in $H^1_{\text{rad}}(\mathbb{R}^2)$ such that
\[
\|u(t, \cdot) - e^{it\Delta} v_\pm\|_{H^1} \to t \to \pm \infty 0.
\]

**Proof.** The proof of Lemma 3.3 is based on the following bootstrap result.

**Lemma 3.5.** Under the assumptions of Lemma 3.3, there exist $\beta_0 = \beta_0(M, \delta) > 0$ and $C = C(M, \delta) > 0$ such that for any $\beta \leq \beta_0$ and any real $T$, if
\[
\|u(t, \cdot) - e^{it\Delta} u_0\|_{ST(I_T)} \leq \beta,
\]
then
\[
\|u(t, \cdot) - e^{it\Delta} u_0\|_{ST(I_T)} \leq C \beta_0^{\frac{3}{2}},
\]
where $I_T := [0, T]$ if $T \geq 0$ and $I_T := [T, 0]$ if $T \leq 0$.

**Proof of Lemma 3.5.** Let us denote $\|e^{it\Delta} u_0\|_{L^\infty(\mathbb{R}^\pm, L^4)} = \beta$. In view of the triangle inequality, Estimate (50) ensures that
\[
\|u\|_{L^4(I_T, W^{1,4})} \leq C(M) + \beta \quad \text{and} \quad \|u\|_{L^\infty(I_T, L^4)} \leq \beta + C\|u(t, \cdot) - e^{it\Delta} u_0\|_{L^\infty(I_T, H^1)} \leq C\beta,
\]
which by virtue of the triangle and Sobolev inequalities leads to
\[
\|u\|_{L^\infty(I_T, \tilde{L})} \leq \frac{1}{\sqrt{4\pi(1 + \delta)}} + C\beta \leq \frac{\Theta}{\sqrt{4\pi}},
\]
with $\Theta < 1$ provided that $\beta$ is sufficiently small.

Let us also notice that the conservation laws (3)-(4) imply that
\[
\|\nabla u(t, \cdot)\|_{L^2} \leq 1 \quad \text{and} \quad \|u(t, \cdot)\|_{L^2} = M,
\]
and as $u$ solves the Cauchy problem (1)-(2), we have
\[
u(t, \cdot) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-s)\Delta} f(u(s, \cdot)) ds.
\]
Thus thanks to Strichartz estimate (43), we get
\[
\|u(t, \cdot) - e^{it\Delta} u_0\|_{ST(I_T)} \lesssim \|f(u)\|_{ST^*(I_T)}.
\]

---

\(^2\)In fact, one can prove that for all $0 < r < 1$ there is a positive constant $C_r$ such that
\[
\|u(t, \cdot) - e^{it\Delta} u_0\|_{ST(\mathbb{R}^\pm)} \leq C_r \beta_0^{1+r}.
\]
But we fixed $r = \frac{1}{2}$ in (49) to avoid heaviness.
But on the one hand \( |f(u)| \lesssim |u|^5 e^{4\pi |u|^2} \), thus for any \( 0 < \epsilon < 1 \)
\[
\|f(u(t, \cdot))\|_{L^\frac{5}{4}(\mathbb{R}^2)}^{\frac{3}{5}} \lesssim \int_{\mathbb{R}^2} e^{\frac{16\pi}{5}|u(t,x)|^2} |u(t, x)|^{\frac{20}{3}} \, dx
\lesssim e^{4\pi(1-\epsilon)|u(t, \cdot)|^2} \int_{\mathbb{R}^2} e^{4\pi(\frac{1}{3}+\epsilon)|u(t,x)|^2} |u(t, x)|^{\frac{20}{3}} \, dx.
\]
Choosing \( \epsilon < \frac{2}{3} \), we obtain in view of Proposition 2.7
\[
\|f(u(t, \cdot))\|_{L^\frac{5}{4}(\mathbb{R}^2)}^{\frac{3}{5}} \lesssim e^{4\pi(1-\epsilon)|u(t, \cdot)|^2} \int_{\mathbb{R}^2} |u(t, x)|^{\frac{20}{3}} \, dx
\]
(56)
\[
\lesssim e^{4\pi(1-\epsilon)|u(t, \cdot)|^2} \|u(t, \cdot)\|_{L^\infty}^{\frac{5}{4}} \|u(t, \cdot)\|_{L^4(\mathbb{R}^2)}^4,
\]
which leads to
\[
\|f(u(t, \cdot))\|_{L^\frac{5}{4}(\mathbb{R}^2)}^{\frac{3}{5}} \lesssim \|u(t, \cdot)\|_{L^\infty}^{\frac{8}{5}} \|u(t, \cdot)\|_{L^4(\mathbb{R}^2)}^4
\]
(57)
Indeed in the case when \( \|u(t, \cdot)\|_{L^\infty} \leq 1 \), the estimate (56) writes
\[
\|f(u(t, \cdot))\|_{L^\frac{5}{4}(\mathbb{R}^2)}^{\frac{3}{5}} \lesssim \|u(t, \cdot)\|_{L^\infty}^{\frac{8}{5}} \|u(t, \cdot)\|_{L^4(\mathbb{R}^2)}^4
\]
which thanks to (7) and (44) entails that
\[
\|f(u(t, \cdot))\|_{L^\frac{5}{4}(\mathbb{R}^2)}^{\frac{3}{5}} \lesssim \|u(t, \cdot)\|_{W^{1,4}(\mathbb{R}^2)}^{\frac{8}{5}} \|u(t, \cdot)\|_{L^4(\mathbb{R}^2)}^4
\]
Furthermore in the case when \( \|u(t, \cdot)\|_{L^\infty} \geq 1 \), it follows from (56) that
\[
\|f(u(t, \cdot))\|_{L^\frac{5}{4}(\mathbb{R}^2)}^{\frac{3}{5}} \lesssim e^{4\pi(1-\frac{\lambda}{2})\|u(t, \cdot)\|_{L^\infty}^{\frac{2}{5}}} \|u(t, \cdot)\|_{L^4(\mathbb{R}^2)}^4
\]
which in light of the logarithmic inequality (8) implies for any fixed \( \lambda > \frac{1}{\pi} \)
\[
\|f(u(t, \cdot))\|_{L^\frac{5}{4}(\mathbb{R}^2)}^{\frac{3}{5}} \lesssim \|u(t, \cdot)\|_{L^4(\mathbb{R}^2)}^4 \left( 1 + \|u(t, \cdot)\|_{H^\mu} \right)^{4\pi(1-\frac{\lambda}{2})\|u(t, \cdot)\|_{H^\mu}^{\frac{2}{5}}}
\]
(58)
Recalling that
\[
\|u(t, \cdot)\|_{H^\mu}^2 = \|\nabla u(t, \cdot)\|_{L^2}^2 + \mu^2 \|u(t, \cdot)\|_{L^2}^2 \leq 1 + \mu^2 M^2,
\]
we infer in view of (44) that \( \mu > 0 \) and \( \lambda > \frac{1}{\pi} \) can be fixed so that
\[
\|f(u(t, \cdot))\|_{L^\frac{5}{4}(\mathbb{R}^2)}^{\frac{3}{5}} \lesssim \|u(t, \cdot)\|_{L^4(\mathbb{R}^2)}^4 \left( 1 + \|u(t, \cdot)\|_{L^4(\mathbb{R}^2)}^4 \right).
\]
Since we are dealing with the case \( \|u(t, \cdot)\|_{L^\infty} \geq 1 \), this ensures that
\[
\|f(u(t, \cdot))\|_{L^\frac{5}{4}(\mathbb{R}^2)}^{\frac{3}{5}} \lesssim \|u(t, \cdot)\|_{L^4(\mathbb{R}^2)}^4 \|u(t, \cdot)\|_{W^{1,4}(\mathbb{R}^2)}^4
\]
which ends the proof of (57) and leads by a time integration to
\[
\|f(u)\|_{L^\frac{5}{4}(L^\infty, L^\frac{5}{4})}^{\frac{3}{5}} \lesssim \|u\|_{L^\infty(I_T, L^4)}^2 \|u\|_{L^4(I_T, W^{1,4})}^3
\]
(59)
On the other hand observing that $|\nabla f(u)| \lesssim e^{4\pi|u|^2} |\nabla u| |u|^4$ and applying Hölder inequality, we get
\[
\|\nabla f(u)\|_{L^4(T, L^4)}^2 \lesssim \|\nabla u\|_{L^4(I_T, L^4)}^2 \left( \int_{\mathbb{R} \times \mathbb{R}^2} e^{8\pi|u(t,x)|^2} |u(t,x)|^8 \, dx \, dt \right)^{\frac{1}{4}}.
\]
Arguing as above, we infer that if $\|u(t,\cdot)\|_{L^\infty} \leq 1$ then
\[
\int_{\mathbb{R}^2} e^{8\pi|u(t,x)|^2} |u(t,x)|^8 \, dx \lesssim \int_{\mathbb{R}^2} |u(t,x)|^8 \, dx
\]
(60)
Moreover if $\|u(t,\cdot)\|_{L^\infty} \geq 1$, we obtain for any $0 < \epsilon < 1$
\[
\int_{\mathbb{R}^2} e^{8\pi|u(t,x)|^2} |u(t,x)|^8 \, dx \lesssim e^{4\pi(1-\epsilon)|u(t,\cdot)|^2_{L^{\infty}}} \int_{\mathbb{R}^2} e^{4\pi(1+\epsilon)|u(t,x)|^2} |u(t,x)|^8 \, dx.
\]
Taking advantage of Corollary 2.11, we get
\[
\int_{\mathbb{R}^2} e^{8\pi|u(t,x)|^2} |u(t,x)|^8 \, dx \lesssim e^{4\pi(1-\epsilon)|u(t,\cdot)|^2_{L^{\infty}}} \left( \int_{\mathbb{R}^2} |u(t,x)|^8 \, dx + \left( \int_{\mathbb{R}^2} |u(t,x)|^{8p} \, dx \right)^{\frac{1}{p}} \right)^{\frac{1}{8}}
\]
\[
\lesssim e^{4\pi(1-\epsilon)|u(t,\cdot)|^2_{L^{\infty}}} \left( \|u(t,\cdot)\|_{L^4}^4 \|u(t,\cdot)\|_{L^{\infty}} + \|u(t,\cdot)\|_{L^4}^4 \|u(t,\cdot)\|_{L^{\infty}}^{\frac{8}{2}} \right)^{\frac{1}{8}}
\]
\[
\lesssim e^{4\pi(1-\epsilon)|u(t,\cdot)|^2_{L^{\infty}}} \|u(t,\cdot)\|_{L^4}^4,
\]
provided that $\epsilon$ is chosen sufficiently small.

Applying the above lines of reasoning, we get by virtue of the logarithmic inequality (8) still again in the case when $\|u(t,\cdot)\|_{L^\infty} \geq 1$
\[
(61) \quad \int_{\mathbb{R}^2} e^{8\pi|u(t,x)|^2} |u(t,x)|^8 \, dx \lesssim \|u(t,\cdot)\|_{W^{1,4}}^4 \|u(t,\cdot)\|_{L^4}^4.
\]
Combining (60) and (61) we find that for any $p > 1$, with $p - 1$ sufficiently small
\[
\|\nabla f(u)\|_{L^4(T, L^4)} \lesssim \|u\|_{L^\infty(I_T, L^4)}^4 \|u\|_{W^{1,4}(I_T, L^4)}^4.
\]
Therefore, there is a positive constant $C = C(M, \delta)$ so that
\[
(62) \quad \|f(u)\|_{ST^*(I_T)} \leq C \|u\|_{L^4(W^{1,4})}^3 \|u\|_{L^\infty(I_T, L^4)}^4.
\]
Thus taking into account of (55), we deduce that
\[
\|u(t, \cdot) - e^{it\Delta} u_0\|_{ST(I_T)} \leq C \|u\|_{L^4(W^{1,4})}^3 \|u\|_{L^\infty(I_T, L^4)}^3,
\]
which together with (52) and (53) lead to (50), provided that $\beta$ is sufficiently small. This ends the proof of the lemma. \(\square\)

By the standard continuity arguments, we deduce from Lemma 3.5 that for $\beta_0$ sufficiently small $\|u\|_{ST(R^\pm)} \lesssim \beta_0$. This implies in light of (62) that
\[
\|f(u)\|_{ST^*(R^\pm)} \leq C\beta_0^3,
\]
which achieves the proof of Lemma 3.3. □

4. Proof of Theorem 1.1

In this section, we shall demonstrate in the critical case that any solution $u$ to the Cauchy problem (1)-(2) belongs to $L^4(\mathbb{R}, W^{1,4}(\mathbb{R}^2))$. By considering the conjugate $\overline{u}$ of $u$, one can reduce the proof of (6) to $\mathbb{R}^+$. As it is mentioned in Section 1.2, this is achieved in three steps. A first step where for any positive real sequence $(t_n)_{n \geq 0}$ tending to $+\infty$, the strong convergence to zero of the sequence $(e^{it \Delta} u(t_n, \cdot))_{n \geq 0}$ in $L^\infty(\mathbb{R}^+, L^4(\mathbb{R}^2))$ is settled. A second step where a lack of compactness at infinity is derived, and a third step where the proof of the result is completed.

4.1. First step: strong convergence to zero in $L^\infty(\mathbb{R}^+, L^4(\mathbb{R}^2))$.

**Proposition 4.1.** Let $u$ be a solution to (1)-(2) and $(t_n)_{n \geq 0}$ be a positive real sequence tending to $+\infty$. Then

$$\|e^{it \Delta} u(t_n, \cdot)\|_{L^\infty(\mathbb{R}^+, L^p(\mathbb{R}^2))} \xrightarrow{n \to \infty} 0,$$

for all $2 < p < \infty$.

**Proof.** Proposition 4.1 stems from the following lemmas that we admit for a while.

**Lemma 4.2.** If $u$ is a solution to (1)-(2) and $(t_n)_{n \geq 0}$ is a positive real sequence tending to $+\infty$, then

$$u(t_n, \cdot) \xrightarrow{n \to \infty} 0, \text{ in } H^1(\mathbb{R}^2).$$

**Lemma 4.3.** Let $u$ be a solution to (1)-(2), $(t_n)$ and $(\tau_n)$ two positive real sequences tending to $+\infty$, then

$$\|e^{i\tau_n \Delta} u(t_n, \cdot)\|_{L^p(\mathbb{R}^2)} \xrightarrow{n \to \infty} 0,$$

for all $2 < p < \infty$.

Before going into the proof of these fundamental results, let us show how they lead to Property (63). For that purpose, we shall proceed by contradiction assuming that the sequence $(t_n)_{n \geq 0}$ admits a subsequence $(t_{n_k})_{k \geq 0}$ and that there exist a positive real sequence $(\tau_k)_{k \geq 0}$ and a positive real $\alpha_0$ such that for all $k \in \mathbb{N}$

$$\|e^{i\tau_k \Delta} u(t_{n_k}, \cdot)\|_{L^p(\mathbb{R}^2)} \geq \alpha_0.$$

There are two possibilities up to extraction

1. $\tau_k \xrightarrow{k \to \infty} \tau_0 < \infty$ or else
2. $\tau_k \xrightarrow{k \to \infty} +\infty$.

In the first case, the continuity of the flow implies that

$$\|e^{i\tau_0 \Delta} u(t_{n_k}, \cdot)\|_{L^p(\mathbb{R}^2)} \gtrsim \alpha_0,$$

which contradicts Lemma 4.2 in light of Lemma 2.4, and the second case can not occur in view of Lemma 4.3.

Now the heart of the matter consists to establish Lemmas 4.2 and 4.3. Let us begin by demonstrating the first lemma.
Proof of Lemma 4.2. To go to the proof of the result, we shall proceed by contradiction. Assume that there is a subsequence \((t_{n_k})_{k \geq 0}\) such that
\[
 u(t_{n_k}, \cdot) \xrightarrow{k \to \infty} \varphi, \quad \text{in } H^1(\mathbb{R}^2),
\]
with \(\varphi \neq 0\) and for the sake of simplicity denote \((t_{n_k})\) by \((t_n)\). This allows us to write
\[
 u(t_n, \cdot) = \varphi + v_n,
\]
with \(v_n \xrightarrow{n \to \infty} 0\) in \(H^1(\mathbb{R}^2)\), which implies that
\[
 \left\| \nabla v_n \right\|_{L^2}^2 = \left\| \nabla u(t_n, \cdot) \right\|_{L^2}^2 - \left\| \nabla \varphi \right\|_{L^2}^2 + o(1), \quad n \to \infty.
\]
This ensures the existence of a positive constant \(\delta\) such that \(\|\nabla v_n\|_{L^2} < 1 - 10\delta\).

By virtue of Lemma 2.4, one has
\[
 \sup_{t \in [-1,1]} \left\| e^{it\Delta} v_n \right\|_{L^p(\mathbb{R}^2)} \xrightarrow{n \to \infty} 0,
\]
for all \(2 < p < \infty\), which in view of Corollary 2.9 gives rise to
\[
 \sup_{t \in [-1,1]} \left\| e^{it\Delta} v_n \right\|_{\tilde{L}^p} \leq \frac{1}{\sqrt{4\pi}} \left\| \nabla v_n \right\|_{L^2} + o(1).
\]
Thus for \(n\) sufficiently large
\[
 \sup_{t \in [-1,1]} \left\| e^{it\Delta} v_n \right\|_{\tilde{L}^p} \leq \frac{1 - 8\delta}{\sqrt{4\pi}}.
\]
Besides, by density arguments one can decompose \(\varphi\) as follows:
\[
 \varphi := \varphi_0 + \varphi_1,
\]
where \(\varphi_0 \in D\) and \(\|\varphi_1\|_{H^1} \leq \delta\), which implies according to Sobolev embedding (16) that
\[
 \sup_{t \in [-1,1]} \left\| e^{it\Delta} \varphi_1 \right\|_{\tilde{L}^p} \leq \frac{\delta}{\sqrt{4\pi}}.
\]
Now our aim is to prove the existence of \(T(\delta) > 0, \alpha(\delta) > 0\) and \(c(\delta) > 0\) such that for any \(n\) large enough, we have
\[
 u(t, \cdot) = e^{i(t-t_n)\Delta} u(t_n, \cdot) + r_n(t, \cdot),
\]
with
\[
 \|r_n\|_{ST([t_n, t_n+T])} \leq c(\delta) T^{\alpha(\delta)},
\]
for any \(0 \leq T \leq T(\delta)\).

We shall prove Claim (69) by bootstrap argument, assuming that for some \(T > 0\)
\[
 \|r_n\|_{ST(I_T^n)} \leq \delta,
\]
where \(I_T^n := [t_n, t_n+T]\). Actually under Assumption (69), we have on the one hand
\[
 \|u\|_{ST(I_T^n)} \leq \|u(t_n, \cdot)\|_{H^1(\mathbb{R}^2)} + \|r_n\|_{ST(I_T^n)} \leq C(M, H) + \|r_n\|_{ST(I_T^n)} \leq C(M, H, \delta),
\]
for any \(0 \leq T \leq T(\delta)\).
and on the other hand we may decompose $u$ as follows
\[
u(t, \cdot) = u_0(t, \cdot) + \tilde{u}(t, \cdot),
\]
where $u_0(t, \cdot) = e^{i(t-t_n)\Delta} \varphi_0$ and
\[
\|\tilde{u}\|_{L^\infty(I_n^0, \mathbb{L})} \leq \frac{1 - 6\delta}{\sqrt{4\pi}}.
\]
Indeed, by hypothesis
\[
\tilde{u}(t, \cdot) = e^{i(t-t_n)\Delta} (\varphi_1 + v_n) + r_n(t, \cdot),
\]
which clearly ensures the result thanks to (67), (68) and (69) provided that $T \leq 1$.

In order to establish (69), let us recall that since $u$ solves the Cauchy problem
\[
\begin{cases}
    i\partial_t u + \Delta u = f(u), \\
    u|_{t=t_n} = u(t_n, \cdot),
\end{cases}
\]
the remainder term $r_n$ writes
\[
r_n(t, \cdot) = -i \int_{t_n}^t e^{i(t-s)\Delta} f(u(s, \cdot)) ds.
\]
Thus taking advantage of Strichartz estimates (43), we deduce that
\[
\|r_n\|_{ST(I_n^0)} \lesssim \|f(u)\|_{L^\frac{4}{3}(I_n^0, L^\frac{4}{3})} + \|
abla f(u)\|_{L^\frac{4}{3}(I_n^0, L^\frac{4}{3})}.
\]
Firstly using the fact that $|f(u)| \lesssim |u|^5 e^{4\pi|u|^2}$, we obtain
\[
\|f(u(t, \cdot))\|_{L^\frac{4}{3}(\mathbb{R}^2)} \lesssim \int_{\mathbb{R}^2} e^{\frac{16\pi}{3}|u(t,x)|^2} |u(t, x)|^{\frac{20}{3}} dx
\]
\[
\lesssim e^{\frac{4\pi}{3}(1 - \frac{2\pi}{15})|u(t, \cdot)|^2_{L^\infty}} \int_{\mathbb{R}^2} e^{4\pi(1 + \frac{16}{15})|u(t, x)|^2} |u(t, x)|^{\frac{20}{3}} dx
\]
\[
\lesssim e^{\frac{4\pi}{3}(1 - \frac{2\pi}{15})|u(t, \cdot)|^2_{L^\infty}} \int_{\mathbb{R}^2} e^{4\pi(1 + \frac{16}{15})|u_0(t, x) + \tilde{u}(t, x)|^2} |u(t, x)|^{\frac{20}{3}} dx.
\]
Remembering that $\varphi_0 \in \mathcal{D}$ and thus $\|u_0\|_{L^\infty(\mathbb{R}, L^\infty(\mathbb{R}^2))} \lesssim \|\varphi_0\|_{H^2}$, we infer that
\[
\|f(u(t, \cdot))\|_{L^\frac{4}{3}(\mathbb{R}^2)} \lesssim e^{\frac{4\pi}{3}(1 - \frac{2\pi}{15})|u(t, \cdot)|^2_{L^\infty}} e^{c_3|u_0|^2_{L^\infty}} \int_{\mathbb{R}^2} e^{4\pi(1 + \frac{22}{15})|\tilde{u}(t,x)|^2} |u(t, x)|^{\frac{20}{3}} dx
\]
\[
\lesssim e^{\frac{4\pi}{3}(1 - \frac{2\pi}{15})|u(t, \cdot)|^2_{L^\infty}} \int_{\mathbb{R}^2} e^{4\pi(1 + \frac{22}{15})|\tilde{u}(t,x)|^2} |u(t, x)|^{\frac{20}{3}} dx.
\]
Actually, in view of the continuous embedding of $H^1(\mathbb{R}^2)$ into $L^\frac{20}{3}(\mathbb{R}^2)$ and the conservation laws (3)-(4), we deduce that
\[
\int_{|\tilde{u}| \leq 1} e^{4\pi(1 + \frac{22}{15})|\tilde{u}(t,x)|^2} |u(t, x)|^{\frac{20}{3}} dx \leq \int_{\mathbb{R}^2} |u(t, x)|^{\frac{20}{3}} dx \lesssim C(M, H).
Besides δ being fixed small enough, we get by virtue of (72)
\[
\int_{|\tilde{u}| \geq 1} e^{4\pi(1 + \frac{1}{10})|\tilde{u}(t,x)|^2} |u(t, x)|^{\frac{20}{3}} dx \lesssim \int_{\mathbb{R}^2} (e^{4\pi(1 + \frac{1}{10})|\tilde{u}(t,x)|^2} - 1 - 4\pi(1 + \frac{3}{10})|\tilde{u}(t, x)|^2) dx 
\lesssim \kappa.
\]
Therefore
\[
\|f(u(t, \cdot))\|_{L^3_\delta(\mathbb{R}^2)}^4 \lesssim e^{\frac{4\pi}{3}(1 - \frac{4}{10})\|u(t, \cdot)\|_{H_\delta}}.
\]
The logarithmic inequality (8) yields for any fixed \(\lambda > \frac{1}{\pi}\),
\[
e^{\frac{4\pi}{3}(1 - \frac{4}{10})\|u(t, \cdot)\|_{H_\delta}} \lesssim \left( 1 + \frac{\|u(t, \cdot)\|_{H_\delta}}{c_2} \right)^{\frac{4\pi}{3}(1 - \frac{4}{10})\|u(t, \cdot)\|_{H_\delta}}.
\]
Arguing as in the proof of Lemma 3.3, we infer that \(\mu > 0\) and \(\lambda > \frac{1}{\pi}\) can be fixed so that
\[
\|f(u(t, \cdot))\|_{L^3_\delta(\mathbb{R}^2)}^4 \lesssim \left( 1 + \|u(t, \cdot)\|_{H_\delta}\right)^{\frac{4\pi}{3}(1 - \frac{4}{10})\|u(t, \cdot)\|_{H_\delta}}.
\]
Along the same lines, using the fact that \(|\nabla f(u)| \lesssim e^{4\pi|u|^2} |\nabla u| |u|^4\), we obtain
\[
\|\nabla f(u(t, \cdot))\|_{L^3_\delta(\mathbb{R}^2)}^4 \lesssim \|\nabla u(t, \cdot)\|_{L^3(\mathbb{R}^2)}^4 \left( \int_{\mathbb{R}^2} e^{8\pi|u(t,x)|^2} |u(t, x)|^8 \ dx \right)^{\frac{4}{3}} 
\lesssim \|\nabla u(t, \cdot)\|_{L^3(\mathbb{R}^2)}^4 e^{\frac{8\pi}{3}(1 - \frac{4}{10})\|u(t, \cdot)\|_{H_\delta}} \left( \int_{\mathbb{R}^2} e^{4\pi(1 + \frac{1}{10})|u(t,x)|^2} |u(t, x)|^8 \ dx \right)^{\frac{4}{3}} 
\lesssim \|u(t, \cdot)\|_{W^{1,4}}^4 \left( 1 + \|u(t, \cdot)\|_{W^{1,4}}^{\frac{8}{3}(1 - \frac{4}{10})}\right). 
\]
To summarize, we proved that for any fixed \(\delta\) sufficiently small there is a positive constant \(c_3\) so that
\[
\|f(u(t, \cdot))\|_{L^3_\delta(\mathbb{R}^2)}^4 + \|\nabla f(u(t, \cdot))\|_{L^3_\delta(\mathbb{R}^2)}^4 \leq c_3 \left( 1 + \|u(t, \cdot)\|_{W^{1,4}}^{\frac{8}{3}(1 - \frac{4}{10})}\right),
\]
which in view of (71) and (73) gives rise to
\[
\|r_n\|_{ST(L^6)} \leq c_3 \left(T + T^{\frac{4}{10}}\|u\|_{ST(L^6)}^{\frac{4}{3}(1 - \frac{4}{10})}\right)^{\frac{3}{4}} \lesssim c_3 T^{\alpha(\delta)},
\]
with \(\alpha(\delta) = \frac{3\delta^2}{100}\). By the standard continuity argument, this gives the required result for \(T\) sufficiently small.

Recall that we assumed that \(\varphi \neq 0\), thus there is a positive constant \(\eta\) such that
\[
\sup_{t \in [-1, 1]} \|e^{it\Delta} \varphi\|_{L^8} \geq \eta > 0.
\]
Taking advantage of (69), there exists \(T > 0\) so that for any \(n\) large enough we have
\[
\sup_{t \in L^6_T} \|r_n(t, \cdot)\|_{L^8} \leq C \sup_{t \in L^6_T} \|r_n(t, \cdot)\|_{H^1} \leq \frac{\eta}{2}.
\]
Consequently taking advantage of (66), we deduce that for any \( t \in I^n_T \)
\[
\|u(t, \cdot)\|_{L^8} \geq \|e^{(t-t_n)\Delta} \varphi\|_{L^8} - \|\eta_n(t, \cdot)\|_{L^8} - \|e^{(t-t_n)\Delta} v_n\|_{L^8} \geq \eta - \frac{\eta}{2} + o(1),
\]
which ensures that
\[
\|u\|_{L^8(I^n_T, L^8)} \geq \frac{\eta}{2} T^\frac{1}{4} + o(1).
\]
But in view of a priori estimate (5), we necessarily have \( \|u\|_{L^8(I^n_T, L^8)} = o(1) \) which yields a contradiction since \( (u(t_n, \cdot))_{n \in \mathbb{N}} \) is bounded in \( H^1(\mathbb{R}^2) \). This ends the proof of the Lemma.

**Proof of Lemma 4.3.** According to Lemma 2.4, it suffices to prove that
\[
e^{i\tau_n \Delta} u(t_n, \cdot) \xrightarrow{n \to \infty} 0, \quad \text{in} \quad H^1(\mathbb{R}^2).
\]
But since \( (e^{i\tau_n \Delta} u(t_n, \cdot)) \) is bounded in \( H^1(\mathbb{R}^2) \), it is enough to prove (77) in \( L^2(\mathbb{R}^2) \).
Thus by density arguments, we are reduced to prove that for any function \( \varphi \) in \( \mathcal{S}(\mathbb{R}^2) \) whose Fourier transform \( \widehat{\varphi} \) belongs to \( \mathcal{D}(\mathbb{R}^2 \setminus \{0\}) \), we have
\[
\left( e^{i\tau_n \Delta} u(t_n, \cdot), \varphi \right)_{L^2(\mathbb{R}^2)} \xrightarrow{n \to \infty} 0.
\]
Since \( u \) solves the Cauchy problem (1)-(2), \( e^{i\tau_n \Delta} u(t_n, \cdot) \) may be decomposed as follows:
\[
e^{i\tau_n \Delta} u(t_n, \cdot) = v_n^{(0)} + v_n^{(1)} + v_n^{(2)},
\]
where
\[
v_n^{(0)} := e^{i(\tau_n + t_n) \Delta} u_0,
\]
and
\[
v_n^{(1)}(x) := -i \int_0^{t_n} e^{i(\tau_n + t_n - s) \Delta} \left[ (1 - \chi(x)) f(u(s, x)) + 8\pi^2 \chi(x) |u(s, x)|^4 u(s, x) \right] ds,
\]
with \( \chi \) a radial function in \( \mathcal{D}(\mathbb{R}^2) \) valued in \([0,1]\) satisfying
\[
\chi(x) = \begin{cases} 
1 & \text{if } |x| \leq \frac{1}{2}, \\
0 & \text{if } |x| \geq 1.
\end{cases}
\]
Obviously
\[
v_n^{(2)}(x) = -i \int_0^{t_n} e^{i(\tau_n + t_n - s) \Delta} \chi(x) G_2(u(s, x)) ds,
\]
where \( G_2(u) = \left( e^{4\pi|u|^2} - 1 - 4\pi|u|^2 - 8\pi^2 |u|^4 \right) u. \)

Now, we shall treat differently the three parts in (78). Firstly thanks to the dispersive estimate (40)
\[
\|v_n^{(0)}\|_{L^p(\mathbb{R}^2)} \xrightarrow{n \to \infty} 0,
\]
for all \( 2 < p < \infty \).

Secondly,
\[
\|v_n^{(1)}\|_{L^\infty(\mathbb{R}^2)} \xrightarrow{n \to \infty} 0.
\]
Indeed, we know that
\[ v_n^{(1)}(x) := -i \int_0^{t_n} e^{i(\tau_n + t_n - s)} G_1(u(s, x)) \, ds, \]
where
\[ G_1(u(s, x)) = \left[ (1 - \chi(x)) f(u(s, x)) + 8\pi^2 \chi(x) |u(s, x)|^4 u(s, x) \right]. \]

Besides the radial estimate (10) and the conservation laws (3)-(4) ensure that the function \( u \) is bounded away from the origin uniformly on \( s \in \mathbb{R} \). Thus, there is a positive constant \( C \) such that for any \( (s, x) \in \mathbb{R} \times \mathbb{R}^2 \), we have
\[ |G_1(u(s, x))| \leq C |u(s, x)|^5. \]

Consequently
\[ \|G_1(s, \cdot)|L^1(\mathbb{R}^2) \lesssim \|u(s, \cdot)|L^5(\mathbb{R}^2) \lesssim \|u(s, \cdot)|L^2(\mathbb{R}^2) \|u(s, \cdot)|L^4(\mathbb{R}^2) \lesssim \|u(s, \cdot)|L^8(\mathbb{R}^2) \]
and then under the dispersive estimate (40)
\[ \|v_n^{(1)}|L^\infty(\mathbb{R}^2) \lesssim\int_0^{t_n} \frac{1}{(\tau_n + t_n - s)} \|u(s, \cdot)|L^8(\mathbb{R}^2) \, ds \lesssim \frac{1}{\tau_n} \|u|L^4(\mathbb{R}, L^8(\mathbb{R}^2)) \]
which provides the result in view of the a priori estimate (5).

Finally, for any \( \varphi \) in \( \mathcal{S}(\mathbb{R}^2) \) whose Fourier transform \( \hat{\varphi} \) belongs to \( \mathcal{D}(\mathbb{R}^2 \setminus \{0\}) \), we infer that
\[ \left( v_n^{(2)}; \varphi \right)_{L^2(\mathbb{R}^2)} \rightarrow 0. \]

Indeed, by definition
\[ \left( v_n^{(2)}; \varphi \right)_{L^2(\mathbb{R}^2)} = i \int_0^{t_n} \left( \chi \, G_2(u(s, \cdot)), e^{-i(\tau_n + t_n - s)} \varphi \right)_{L^2(\mathbb{R}^2)} \, ds. \]

But by repeated integration by parts, we obtain for any nonnegative integer \( k \)
\[ \left( e^{-i(\tau_n + t_n - s)} \varphi \right)(s, x) \leq C_k \left| \frac{\langle x \rangle}{\tau_n + t_n - s} \right|^k, \]
which in the particular case where \( k = 2 \) implies that
\[ \left| \left( v_n^{(2)}; \varphi \right)_{L^2(\mathbb{R}^2)} \right| \lesssim \int_{\mathbb{R}^2} \int_0^{t_n} \frac{\langle x \rangle^2 \chi(x) G_2(u(s, x))}{(\tau_n + t_n - s)^2} \, ds \, dx \]
\[ \lesssim \int_0^{t_n} \frac{1}{(\tau_n + t_n - s)^2} \int_{|x| \leq 1} G_2(u(s, x)) \, ds \, dx \]
\[ \lesssim \sum_{m=0}^{[t_n]} \frac{1}{(\tau_n + t_n - (m + 1))^2} \int_{m}^{m+1} \int_{|x| \leq 1} G_2(u(s, x)) \, ds \, dx. \]

But in view of Corollary 3.2 we have
\[ \int_{m}^{m+1} \int_{|x| \leq 1} G_2(u(s, x)) \, ds \, dx \lesssim \int_{m}^{m+1} \int_{|x| \leq 1} G(u(s, x)) \, ds \, dx \lesssim 1. \]
This implies that
\[
\left| \left( v_n^{(2)}, \varphi \right)_{L^2(\mathbb{R}^2)} \right| \lesssim \sum_{m=0}^{[t_n]} \frac{1}{(\tau_n + t_n - (m + 1))^2} \lesssim \frac{1}{\tau_n} \rightarrow 0,
\]
which achieves the proof of the lemma.

\[\square\]

4.2. **Second step: lack of compactness at infinity.** The approach that we shall adopt to establish Theorem 1.1 relies on virial identity and uses in a crucial way the radial setting and particularly the fact that we deal with bounded functions far away from the origin. Roughly speaking, virial identity asserts that any critical solution to (1)-(2) displays necessarily a lack of compactness at infinity. More precisely, we have the following lemma:

**Lemma 4.4.** Let \( u \) be a solution to the nonlinear Schrödinger equation (1)-(2) satisfying
\[ H(u_0) = 1. \]
Then there exist \( \varepsilon_0 > 0 \), \( t_n \xrightarrow{n \to \infty} \infty \) and \( R_n \xrightarrow{n \to \infty} \infty \) such that
\[ \int_{|x| \geq R_n} |\nabla u(t_n, x)|^2 \, dx \geq \varepsilon_0. \]

**Proof.** Clearly if \( u \) is a solution of the concerned nonlinear Schrödinger equation, then in view of (45) we have
\[ \left| \frac{d}{dt} V_R(t) \right| \lesssim R, \]
where
\[ V_R(t) = \int_{\mathbb{R}^2} \Phi_R(x)|u(t, x)|^2 \, dx, \]
with \( \Phi_R(x) = R^2 \Phi \left( \frac{\rho}{R^2} \right) \), \( \Phi \) being a smooth and radial function satisfying \( 0 \leq \Phi \leq 1 \), \( \Phi(r) = r \), for all \( r \leq 1 \), and \( \Phi(r) = 0 \) for all \( r \geq 2 \).

Besides, in light of (46)
\[ \frac{d^2}{dt^2} V_R(t) = 8 \int_{|x| \leq R} |\nabla u(t, x)|^2 \, dx + 8 \int_{|x| \leq R} (|u|^2 \tilde{f}(|u|^2) - g(|u|^2))(t, x) \, dx + \tilde{V}_R(t), \]
where as it was proved in Section 3.1, \( \tilde{f}(s) = e^{4\pi s} - 1 - 4\pi s \) and \( g(s) = \int_{0}^{s} \tilde{f}(\rho) \, d\rho \), and where the remainder term \( \tilde{V}_R(t) \) satisfies
\[
\left| \tilde{V}_R(t) \right| \lesssim \int_{R \leq |x| \leq 2R} |\nabla u(t, x)|^2 \, dx + \int_{R \leq |x| \leq 2R} (|u|^2 \tilde{f}(|u|^2) - g(|u|^2))(t, x) \, dx
+ \frac{1}{R^2} \int_{R \leq |x| \leq 2R} |u(t, x)|^2 \, dx.
\]
Now taking into account the expressions of \( f \) and \( g \), we infer in view of (9) that
\[
\int_{R \leq |x| \leq 2R} (|u|^2 f(|u|^2) - g(|u|^2))(t, x) \, dx \lesssim \|u(t, \cdot)\|^6_{L^6(|x| \geq R)}
\lesssim \|u(t, \cdot)\|^2_{L^2(|x| \geq R)} \|u(t, \cdot)\|^4_{L^\infty(|x| \geq R)}
\lesssim \frac{1}{R^2} \|u(t, \cdot)\|^4_{L^2(|x| \geq R)} \|\nabla u(t, \cdot)\|^2_{L^2(|x| \geq R)}.
\]

Therefore, we deduce in light of the conservation laws (3)-(4) that
\[
|\tilde{V}_R(t)| \lesssim \|\nabla u(t, \cdot)\|^2_{L^2(|x| \geq R)} + \frac{1}{R^2}.
\]

Finally the fact that
\[
\frac{2}{3} g(|u|^2) \leq |u|^2 f(|u|^2) - g(|u|^2),
\]
ensures the existence of positive constants \( C_1 \) and \( C_2 \) such that
\[
\frac{d^2}{dt^2} V_R(t) \geq C_1 \int_{|x| \leq R} (|\nabla u(t, x)|^2 + F(u(t, x))) \, dx - C_2 \|\nabla u(t, \cdot)\|^2_{L^2(|x| \geq R)} - \frac{C_2}{R^2}.
\]

This gives rise to
\[
\frac{d^2}{dt^2} V_R(t) \geq C_1 H(u_0) - C_1 \int_{|x| \geq R} (|\nabla u(t, x)|^2 + F(u(t, x))) \, dx - C_2 \|\nabla u(t, \cdot)\|^2_{L^2(|x| \geq R)} - \frac{C_2}{R^2} \geq C_1 - (C_1 + C_2) \int_{|x| \geq R} |\nabla u(t, x)|^2 \, dx - \frac{C_2}{R^2} - C_1 \int_{|x| \geq R} F(u(t, x)) \, dx.
\]

Now the solution \( u \) is radial, so thanks to (9) it is bounded on \( \{|x| \geq R\} \). Therefore
\[
\int_{|x| \geq R} F(u(t, x)) \, dx \lesssim \|u\|^6_{L^\infty(\mathbb{R}, L^6(|x| \geq R))} \lesssim \|u\|^2_{L^\infty(\mathbb{R}, L^2(|x| \geq R))} \|u\|^4_{L^\infty(\mathbb{R}, L^\infty(|x| \geq R))}.
\]

Taking advantage of (9), we deduce that
\[
\int_{|x| \geq R} F(u(t, x)) \, dx \lesssim \frac{1}{R^2} \|u\|^4_{L^\infty(\mathbb{R}, L^2(|x| \geq R))} \|\nabla u\|^2_{L^\infty(\mathbb{R}, L^2(|x| \geq R))},
\]
which enables us due to the conservation laws (3)-(4) to infer that
\[
\frac{d^2}{dt^2} V_R(t) \geq C_1 - (C_1 + C_2) \int_{|x| \geq R} |\nabla u(t, x)|^2 \, dx - \frac{C_3}{R^2},
\]
which easily ensures (81) taking into account of (82).

\[\square\]

4.3. **Third step: study of the sequence** \( (e^{it\Delta} u(t_n, \cdot)) \). In order to complete the proof of Theorem 1.1, we shall investigate the solution to the linear Schrödinger equation
\[
v_n := e^{it\Delta} u(t_n, \cdot),
\]
where \((t_n)_{n \geq 0}\) is the sequence given by Lemma 4.4. In view of the first step, the sequence \((v_n)_{n \in \mathbb{N}}\) converges strongly to zero in \(L^\infty(\mathbb{R}^+, L^4(\mathbb{R}^2))\). For that purpose, we have to distinguish two sub-cases depending on whether the sequence \(v_n\) satisfies

\[
\lim_{n \to \infty} \|v_n\|_{L^\infty(\mathbb{R}^+, \tilde{L}^* \mathbb{R}^2)} < \frac{1}{\sqrt{4\pi}} \quad \text{or} \quad \lim_{n \to \infty} \|v_n\|_{L^\infty(\mathbb{R}^+, \tilde{L}^* \mathbb{R}^2)} = \frac{1}{\sqrt{4\pi}}.
\]

In the first situation where we have \(\lim \inf_{n \to \infty} \|v_n\|_{L^\infty(\mathbb{R}^+, \tilde{L}^* \mathbb{R}^2)} < \frac{1}{\sqrt{4\pi}}\), Theorem 1.1 derives immediately from Lemma 3.3.

Let us at present consider the more challenging situation where we are dealing with a sequence \((v_n)_{n \in \mathbb{N}}\) satisfying \(\|v_n\|_{L^\infty(\mathbb{R}^+, L^4(\mathbb{R}^2))} \to 0\) and \(\|v_n\|_{L^\infty(\mathbb{R}^+, \tilde{L}^* \mathbb{R}^2)} \to \frac{1}{\sqrt{4\pi}}\). This in particular means that there exists a sequence \((\tau_n)_{n \in \mathbb{N}}\) of positive reals such that

\[
\|e^{i\tau_n \Delta u(t_n, \cdot)}\|_{\tilde{L}} \xrightarrow{n \to \infty} \frac{1}{\sqrt{4\pi}}.
\]

Thus in view of (32)

\[
e^{i\tau_n \Delta u(t_n, \cdot)} = \sqrt{\frac{\alpha_n}{2\pi}} L \left( -\log |\cdot| \right) + r_n,
\]

where \(\alpha_n\) is a scale in the sense of Definition 2.13, \(L\) is the Lions profile given by (27) and \(\|\nabla r\|_{L^2} \xrightarrow{n \to \infty} 0\).

There are two possibilities up to extraction

1. \(\frac{\tau_n \alpha_n}{\sqrt{\alpha_n}} \xrightarrow{n \to \infty} +\infty\) or else
2. \(\frac{\tau_n \alpha_n}{\sqrt{\alpha_n}} \lesssim 1\).

To investigate the case (1), we shall decompose \(v_n\) as follows

\[
v_n(t, \cdot) := v_n^{(1)}(t, \cdot) + v_n^{(2)}(t, \cdot) + e^{i(t-\tau_n)\Delta} r_n,
\]

where for \(j \in \{1, 2\}\)

\[
v_n^{(j)}(t, \cdot) := e^{i(t-\tau_n)\Delta} \sqrt{\frac{\alpha_n}{2\pi}} \psi^{(j)} \left( -\log |\cdot| \right),
\]

with, for fixed \(0 < \delta < \frac{1}{2}\),

\[
\psi^{(1)}(s) = \begin{cases} 
0 & \text{if } \ s \leq 0, \\
s & \text{if } \ 0 \leq s \leq 1 - \delta, \\
1 - \delta & \text{if } \ s \geq 1 - \delta,
\end{cases}
\]

and

\[
\psi^{(2)}(s) = \begin{cases} 
0 & \text{if } \ s \leq 1 - \delta, \\
s - (1 - \delta) & \text{if } \ 1 - \delta \leq s \leq 1, \\
\delta & \text{if } \ s \geq 1 - \delta.
\end{cases}
\]
Note that $\|v_n^{(i)}\|_{L^\infty(\mathbb{R}, L^2)} = o(1)$, for $i \in \{1, 2\}$. Moreover, by virtue of the Sobolev embedding (17) and the conservation laws (38)-(39)

$$
\|v_n^{(1)}\|_{L^\infty(\mathbb{R}, \tilde{L}^2)} \leq \frac{1}{\sqrt{4\pi}} \left( \|\nabla v_n^{(1)}\|_{L^\infty(\mathbb{R}, L^2)}^2 + \|v_n^{(1)}\|_{L^\infty(\mathbb{R}, L^2)}^2 \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{4\pi}} \left( 1 - \delta + o(1) \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{4\pi}} \left( 1 - \frac{\delta}{2} \right)^{\frac{1}{2}},
$$

for $n$ large enough.

Besides if $\frac{r_n e^{\alpha_n}}{\sqrt{\alpha_n}} \to +\infty$, then in view of the dispersive estimate (40)

$$
\sup_{t \in \mathbb{R}-} \|v_n^{(2)}(t, \cdot)\|_{L^\infty} \leq \frac{1}{r_n} \left\| \sqrt{\frac{\alpha_n}{2\pi}} \psi(2) \left( -\log \frac{|\cdot|}{\alpha_n} \right) \right\|_{L^1}.
$$

But

$$
\left\| \sqrt{\frac{\alpha_n}{2\pi}} \psi(2) \left( -\log \frac{|\cdot|}{\alpha_n} \right) \right\|_{L^1} \leq \frac{1}{\sqrt{\alpha_n}} e^{-2\alpha_n(1-\delta)},
$$

which implies that

$$
\sup_{t \in \mathbb{R}-} \|v_n^{(2)}(t, \cdot)\|_{L^\infty} \lesssim e^{-\alpha_n(1-2\delta)} n \to 0.
$$

This gives rise to

$$
\|v_n^{(2)}\|_{L^\infty(\mathbb{R}_-, \tilde{L}^2)} n \to 0.
$$

Finally knowing that $\|\nabla e^{i(t-r_n)\Delta} r_n\|_{L^\infty(\mathbb{R}, L^2)} \to 0$ and $\|e^{i(t-r_n)\Delta} r_n\|_{L^\infty(\mathbb{R}, L^2)} \lesssim 1$, we infer in view of the refined estimate (12) that

$$
\|e^{i(t-r_n)\Delta} r_n\|_{L^\infty(\mathbb{R}, L^4)} n \to 0,
$$

and therefore by Corollary 2.9

$$
\|e^{i(t-r_n)\Delta} r_n\|_{L^\infty(\mathbb{R}, \tilde{L}^2)} n \to 0.
$$

Combining (83), (84) and (85), we deduce that for any $n$ large enough

$$
\|e^{it\Delta} u(t_n, \cdot)\|_{L^\infty(\mathbb{R}_-, \tilde{L}^2)} < \frac{1}{\sqrt{4\pi}} \left( 1 - \frac{\delta}{4} \right)^{\frac{1}{2}},
$$

which implies in view of Lemma 3.3 that there is a positive constant C such that

$$
\|u(\cdot + t_n, \cdot)\|_{ST(\mathbb{R}_-)} \leq C \quad \text{and} \quad \|f(u(\cdot + t_n, \cdot))\|_{ST^*(\mathbb{R}_-)} \leq C.
$$

Since $t_n \to +\infty$, this means that

$$
\|u\|_{ST(\mathbb{R})} + \|f(u)\|_{ST^*(\mathbb{R})} < \infty,
$$

which ends the proof of the result in that case.
To handle the case (2), we shall make use of the Fourier approximation of the example by Moser, namely (36) which gives rise to

\begin{equation}
(86)
\begin{aligned}
u(t_n, x) &= \frac{1}{(2\pi)^2} \sqrt{\frac{2\pi}{\alpha_n}} \int_{1 \leq |\xi| \leq e^{\alpha_n}} e^{i x \cdot \xi} e^{i \tau_n |\xi|^2} \frac{1}{|\xi|^2} d\xi + \tilde{r}_n(x),
\end{aligned}
\end{equation}

with \(\|\nabla \tilde{r}_n\|_{L^2} \xrightarrow{n \to \infty} 0\).

One can easily check that if \(\theta\) denotes a regular function valued in \([0, 1]\) and satisfying

\[
\begin{align*}
\theta(\rho) &= \begin{cases} 
0 & \text{if } \rho \leq 2, \\
1 & \text{if } \rho \geq 3,
\end{cases}
\end{align*}
\]

then the above identity (86) writes

\[
u(t_n, x) = \frac{1}{(2\pi)^2} \sqrt{\frac{2\pi}{\alpha_n}} \int_{1 \leq |\xi| \leq e^{\alpha_n}} e^{i x \cdot \xi} e^{i \tau_n |\xi|^2} \theta(|\xi|) \theta(e^{\alpha_n} - |\xi|) \frac{1}{|\xi|^2} d\xi
\]

\[+ \tilde{r}_n(x) + r^n#, n(x),\]

where

\[
\begin{align*}
r^n#(x) &= \frac{1}{(2\pi)^2} \sqrt{\frac{2\pi}{\alpha_n}} \int_{1 \leq |\xi| \leq 3} e^{i x \cdot \xi} e^{i \tau_n |\xi|^2} (1 - \theta(|\xi|)) \frac{1}{|\xi|^2} d\xi
\end{align*}
\]

\[+ \frac{1}{(2\pi)^2} \sqrt{\frac{2\pi}{\alpha_n}} \int_{e^{\alpha_n} - 3 \leq |\xi| \leq e^{\alpha_n}} e^{i x \cdot \xi} e^{i \tau_n |\xi|^2} (1 - \theta(e^{\alpha_n} - |\xi|)) \frac{1}{|\xi|^2} d\xi,
\]

and verifies \(\|r^n#\|_{H^1} \xrightarrow{n \to \infty} 0\).

Now let us denote

\[
u_{0,n}(x) := \frac{1}{(2\pi)^2} \sqrt{\frac{2\pi}{\alpha_n}} \int_{1 \leq |\xi| \leq e^{\alpha_n}} e^{i x \cdot \xi} \theta(|\xi|) \theta(e^{\alpha_n} - |\xi|) \frac{1}{|\xi|^2} d\xi.
\]

Using the classical estimate

\[
||| \cdot | e^{-i \tau_n \Delta} \nabla \nu_{0,n} |||_{L^2} \lesssim ||| \cdot | \nabla \nu_{0,n} |||_{L^2} + || \tau_n | \Delta \nu_{0,n} |||_{L^2},
\]

we deduce that

\begin{equation}
(87)
||| \cdot | e^{-i \tau_n \Delta} \nabla \nu_{0,n} |||_{L^2} \lesssim 1.
\end{equation}

Indeed, on the one hand

\[
\|\Delta \nu_{0,n}\|^2_{L^2} \lesssim \frac{1}{\alpha_n} \int_{1 \leq |\xi| \leq e^{\alpha_n}} d\xi \lesssim \frac{e^{2\alpha_n}}{\alpha_n},
\]

thus \(\tau_n \|\Delta \nu_{0,n}\|_{L^2} \lesssim 1\) since by assumption \(\frac{\tau_n e^{\alpha_n}}{\sqrt{\alpha_n}} \lesssim 1\).

On the other hand

\[
||| \cdot | \nabla \nu_{0,n} |||_{L^2} \lesssim ||| \cdot | \nabla \nu_{0,n} |||_{L^2(|x| \leq 1)} + ||| \cdot | \nabla \nu_{0,n} |||_{L^2(|x| \geq 1)}
\]

\[\leq \|\nabla \nu_{0,n}\|_{L^2} + ||| \cdot | \nabla \nu_{0,n} |||_{L^2(|x| \geq 1)}.
\]
To achieve the proof of (87), it suffices then to bound \( \| | \cdot | \nabla u_{0,n} \|_{L^2(|x| \geq 1)} \). To go to this end, we shall perform integration by parts with respect to the vector fields 

\[
\mathcal{X} := -i \sum_{i=1}^{2} \frac{x_i \partial_{\xi_i}}{|x|^2},
\]

which satisfies \( \mathcal{X}(e^{ix\xi}) = e^{ix\xi} \).

More precisely, taking advantage of the fact that \( \theta(|\xi|) \theta(e^{\alpha_n} - |\xi|) \) is compactly supported, we get for any nonnegative integer \( N \) and \( i \in \{1, 2\} \)

\[
(\partial_x, u_{0,n})(x) = \frac{(-1)^N}{(2\pi)^2} \sqrt{\frac{2\pi}{\alpha_n}} \int_{1 \leq |\xi| \leq e^{\alpha_n}} e^{ix\xi} \mathcal{X}^N \left( \frac{i \xi_i \theta(|\xi|) \theta(e^{\alpha_n} - |\xi|)}{|\xi|^2} \right) d\xi.
\]

Taking advantage of the fact that \( \theta(|\xi|) \theta(e^{\alpha_n} - |\xi|) \equiv 1 \) for \( 3 \leq |\xi| \leq e^{\alpha_n} - 3 \), we obtain

\[
| |x| \nabla u_{0,n}(x)| \lesssim \frac{1}{\sqrt{a_n |x|^{N-1}}},
\]

uniformly for \( |x| \geq 1 \) and achieves the proof of the fact that \( \| | \cdot | \nabla u_{0,n} \|_{L^2} \lesssim 1 \).

In conclusion, we have

\[
u(t_n, \cdot) = e^{-ir_n \Delta} u_{0,n} + r_n^3,
\]

with \( \| | \cdot | e^{-ir_n \Delta} \nabla u_{0,n} \|_{L^2} \lesssim 1 \) and \( \| \nabla r_n^3 \|_{L^2} \stackrel{n \to \infty}{\longrightarrow} 0 \). Thus, for all \( R > 0 \)

\[
\left( \int_{|x| \geq R} |\nabla u(t_n, x)|^2 \, dx \right)^{\frac{1}{2}} \lesssim \frac{1}{R} \| | \cdot | e^{-ir_n \Delta} \nabla u_{0,n} \|_{L^2} + \| \nabla r_n^3 \|_{L^2} \lesssim \frac{1}{R} + o(1).
\]

This implies that for \( n \) and \( R \) large enough

\[
\left( \int_{|x| \geq R} |\nabla u(t_n, x)|^2 \, dx \right)^{\frac{1}{2}} \leq \frac{\varepsilon_0}{2},
\]

which contradicts (81) and shows that this case cannot occur.

**Appendix A. A useful Moser-Trudinger inequality**

**A.1. Notion of rearrangement.** Before proving Proposition 2.7 which plays a key role in our approach, let us start with an overview of the notion of rearrangement of functions involving in its proof. This notion consists in associating to any measurable function vanishing at infinity, a nonnegative decreasing radially symmetric function. To begin with, let us first define the symmetric rearrangement of a measurable set.
Definition A.1. Let $A \subset \mathbb{R}^d$ be a Borel set of finite Lebesgue measure. We define $A^*$, the symmetric rearrangement of $A$, to be the open ball centered at the origin whose volume is that of $A$. Thus,

$$A^* = \{x : |x| < R\} \text{ with } \left( |S^{d-1}|/d \right) R^d = |A|,$$

where $|S^{d-1}|$ is the surface area of the unit sphere $S^{d-1}$.

This definition allows us to define in an obvious way the symmetric-decreasing rearrangement of a characteristic function of a set, namely

$$\chi_A^* := \chi_{A^*}.$$

More generally, if $f : \mathbb{R}^d \to \mathbb{C}$ is a measurable function vanishing at infinity i.e.

$$\forall t > 0, \quad \left| \{ x : |f(x)| > t \} \right| < \infty,$$

then we define the symmetric decreasing rearrangement, $f^*$, of $f$ as

$$(88) \quad f^*(x) = \int_0^\infty \chi_{|f|>t}^*(x) \, dt.$$

In the following proposition, we collect without proofs some features of the rearrangement $f^*$ (for a complete presentation and more details, we refer the reader to [17, 18, 26] and the references therein), namely that the process of Schwarz symmetrization minimize the energy and preserves Lebesgue and Orlicz norms:

**Proposition A.2.** Let $f \in H^1(\mathbb{R}^2)$ then

$$\| \nabla f \|_{L^2} \geq \| \nabla f^* \|_{L^2},$$

$$\| f \|_{L^p} = \| f^* \|_{L^p},$$

$$\| f \|_{L^\infty} = \| f^* \|_{L^\infty}.$$

A.2. Proofs of Proposition 2.7 and Corollaries 2.8, 2.9 and 2.11.

A.2.1. Proof of Proposition 2.7. The proof of inequalities (18) uses in a crucial way the rearrangement of functions defined above. By virtue of density arguments and Proposition A.2, one can reduce to the case of a nonnegative radially symmetric and nonincreasing function $u$ belonging to $\mathcal{D}(\mathbb{R}^2)$. With this choice, let us introduce the function

$$w(t) = \sqrt{4\pi} u(|x|), \quad \text{where} \quad |x| = e^{-\frac{t}{4}}.$$

It is then obvious that the functions $w(t)$ and $\dot{w}(t)$ are nonnegative and

$$\int_{\mathbb{R}^2} |\nabla u(x)|^2 \, dx = \int_{-\infty}^{+\infty} |w'(t)|^2 \, dt,$$

$$\int_{\mathbb{R}^2} |u(x)|^p \, dx = \frac{\pi}{(4\sqrt{\pi})^p} \int_{-\infty}^{+\infty} |w(t)|^p \, e^{-t} \, dt \quad \text{and}$$

$$\int_{\mathbb{R}^2} e^{\alpha \rho(x)} |u(x)|^p \, dx = \frac{\pi}{(4\sqrt{\pi})^p} \int_{-\infty}^{+\infty} e^{\frac{\alpha}{4\pi} |w(t)|^2} |w(t)|^p \, e^{-t} \, dt.$$
Besides since $u \in \mathcal{D}(\mathbb{R}^2)$, there exists $t_0 \in \mathbb{R}$ such that

$$w(t) = 0, \quad \text{for } t \leq t_0.$$  

So we are reduced to prove that for all $\beta \in [0, 1]$, there exists $C_\beta \geq 0$ so that

$$\int_{-\infty}^{+\infty} e^{\beta|w(t)|^2} |w(t)|^p e^{-t} dt \leq C_\beta \int_{-\infty}^{+\infty} |w(t)|^p e^{-t} dt, \quad \forall \beta \in [0, 1],$$

provided that

$$\int_{-\infty}^{+\infty} |w'(t)|^2 dt \leq 1.$$  

For that purpose, let us set

$$T_0 := \sup \{ t \in \mathbb{R}, \ w(t) \leq 1 \}.$$  

Knowing that $w$ is nonnegative and increasing function, we deduce that

$$w : [-\infty, T_0] \rightarrow [0, 1].$$

It is then obvious that

$$\int_{-\infty}^{T_0} e^{\beta|w(t)|^2} |w(t)|^p e^{-t} dt \leq e^{\beta} \int_{-\infty}^{T_0} |w(t)|^p e^{-t} dt.$$  

To estimate the integral on $[T_0, +\infty]$, let us first notice that in view of (90), we have

$$\int_{T_0}^{\infty} e^{\beta(w(t))^2} |w(t)|^p e^{-t} dt \leq e^{\beta} \int_{-\infty}^{T_0} |w(t)|^p e^{-t} dt.$$  

Thus, using the fact that for any $\varepsilon > 0$ and any $s \geq 0$, we have

$$(1 + s^2)^2 \leq (1 + \varepsilon) s + 1 + \frac{1}{\varepsilon} = (1 + \varepsilon) s + C_\varepsilon,$$

we infer that for any $\varepsilon > 0$ and all $t \geq T_0$

$$|w(t)|^2 \leq (1 + \varepsilon)(t - T_0) + C_\varepsilon.$$  

Now $\beta$ being fixed in $[0, 1]$, let us choose $\varepsilon > 0$ so that $\beta(1 + \varepsilon)^2 < 1$. Therefore in light of (91), we obtain

$$\int_{T_0}^{+\infty} e^{\beta|w(t)|^2} |w(t)|^p e^{-t} dt \leq \int_{T_0}^{+\infty} e^{\beta((1+\varepsilon)(t-T_0)+C_\varepsilon)} \left( (1 + \varepsilon)(t - T_0) + C_\varepsilon \right)^{\frac{p}{2}} e^{-t} dt$$

$$\leq C(p, \varepsilon) \int_{T_0}^{+\infty} e^{\beta(1+\varepsilon)((1+\varepsilon)(t-T_0)+C_\varepsilon)} e^{-t} dt$$

$$\leq C(p, \varepsilon) \frac{e^{\beta(1+\varepsilon)C_\varepsilon-T_0}}{1 - \beta(1+\varepsilon)^2}. $$
Finally observing that
\[ e^{-T_0} = \int_{T_0}^{+\infty} e^{-t} \, dt \leq \int_{T_0}^{+\infty} |w(t)|^p e^{-t} \, dt, \]
we end up with the result.

A.2.2. Proof Corollary 2.8. Corollary 2.8 derives immediately from Proposition 2.7 once we have observed that
\[ \left( e^{\alpha |u(x)|^2} - 1 - \alpha |u(x)|^2 \right) \lesssim e^{\alpha |u(x)|^2} |u(x)|^4. \]

A.2.3. Proof Corollary 2.9. Corollary 2.9 follows easily from Proposition 2.8 by applying the estimate (19) to the sequence \((v_n)\) defined by
\[ v_n := u_n / \|\nabla u_n\|_{L^2} + \sqrt{\|u_n\|_{L^4}}. \]

A.2.4. Proof Corollary 2.11. Firstly it is obvious that
\[ \int_{|u| \leq 1} e^{4\pi(1+\epsilon)|u(x)|^2} |u(x)|^p \, dx \lesssim \int_{\mathbb{R}^2} |u(x)|^p \, dx. \]

Secondly making use of Hölder inequality, we infer that for any real \( r \geq 1 \), we have
\[ \int_{|u| \geq 1} e^{4\pi(1+\epsilon)|u(x)|^2} |u(x)|^p \, dx \lesssim \left( \int_{|u| \geq 1} e^{4\pi r(1+\epsilon-\epsilon(2+\frac{1}{\delta}))|u(x)|^2} |u(x)|^{pr} \, dx \right)^{\frac{1}{r}} \times \left( \int_{|u| \geq 1} e^{4\pi r'(2+\frac{1}{\delta})|u(x)|^2} \, dx \right)^{\frac{1}{r'}}, \]
where \( r' \) denotes the conjugate exponent of \( r \).

The choice \( r' = \frac{1 + \delta}{\epsilon(2 + \frac{1}{\delta})} \) gives rise to
\[ \int_{|u| \geq 1} e^{4\pi r'(2+\frac{1}{\delta})|u(x)|^2} \, dx = \int_{|u| \geq 1} e^{4\pi(1+\delta)|u(x)|^2} \, dx \leq C(\delta) \int_{|u| \geq 1} \left( e^{4\pi(1+\delta)|u(x)|^2} - 1 - 4\pi(1 + \delta)|u(x)|^2 \right) \, dx \leq C(\delta) \kappa, \]
in view of the assumption \( \|u\|_{\tilde{L}^\infty} \leq \frac{1}{\sqrt{4\pi(1+2\delta)}} \).

Consequently
\[ \int_{|u| \geq 1} e^{4\pi(1+\epsilon)|u(x)|^2} |u(x)|^p \, dx \leq (C(\delta) \kappa)^{\frac{1}{p}} \left( \int_{|u| \geq 1} e^{4\pi r(1+\epsilon-\epsilon(2+\frac{1}{\delta}))|u(x)|^2} |u(x)|^{pr} \, dx \right)^{\frac{1}{r}}. \]

But
\[ r(1 + \epsilon - \epsilon(2 + \frac{1}{\delta})) = 1 - \frac{\delta \epsilon}{1 + \delta} + O(\epsilon^2), \]
which by virtue of Proposition 2.7 leads to
\[ \int_{|u| \geq 1} e^{4\pi(1+\varepsilon)|u(x)|^2} |u(x)|^p \, dx \leq \left( \int_{|u| \geq 1} |u(x)|^{pr} \, dx \right)^{\frac{1}{r}}, \]

provided that \( \varepsilon \) is sufficiently small. This ends the proof of the corollary.

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