Fermionic realization of two-parameter quantum affine algebra $U_{r,s}(C_1^{(1)})$

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Abstract. We construct a Fock space representation and the action of the two-parameter quantum algebra $U_{r,s}(gl_{\infty})$ using extended Young diagrams. In particular, we obtain an integrable representation of the two-parameter quantum affine algebra of type $C_n^{(1)}$ which is a two-parameter generalization of Kang-Misra-Miwa's realization.

Résumé. Nous construisons une représentation de l'espace de Fock et l'action de la 2-paramètre quantique algèbre $U_{r,s}(gl_{\infty})$ en utilisant diagrammes de Young prolongées. Dans particulier, on obtient une représentation intégrable de la 2-paramètres quantique algèbre affine pour le type $C_n^{(1)}$ qui est un 2-paramètres généralisation de la réalisation de Kang-Misra-Miwa.

1. Introduction

Quantum groups, introduced independently by Drinfeld [6] and Jimbo [14], are deformations of the universal enveloping algebras of the Kac-Moody Lie algebras. Among the most important classes of quantum groups, quantum affine algebras have a rich representation theory and broad applications in mathematics and physics. In particular they are expected to provide the mathematical foundation for $q$-conformal field theory.

Two-parameter quantum groups associated to $gl_n$ and $sl_n$ were studied in [3, 4, 5] by Benkart and Witherspoon (see also earlier work by Takeuchi [20]). Other classical types and some exceptional types of two-parameter quantum groups and their representations have been investigated in [1, 2, 10] (see references therein). The two-parameter quantum affine algebras were introduced in [11] and their Drinfeld realization and vertex operator representations were also known with help of Lunden bases for type $A$. More recently these structures have been generalized to all classical untwisted types in [12, 8], which are analog of the basic representations of the quantum affine algebras [7]. The latter builds upon certain quantization of the so-called bosonic fields. From the other angle aimed toward a categorification, [16]
provided a group-theoretic realization of two-parameter quantum toroidal algebras using finite subgroups of $SL_2(\mathbb{C})$ via the McKay correspondence.

It is well known that quantum affine algebras also admit fermionic realizations \cite{9, 19, 17, 18} that have played an important role in integrable systems and representation theory. In \cite{15} such a fermionic realization of the two-parameter quantum affine algebra was constructed for type $A$ using Young diagrams. The combinatorial model gives rise to a natural interpretation of the deforming parameters $r$ and $s$. In this paper, we construct a fermionic realization of the two-parameter quantum affine algebra of type $C$ along the same line. We have taken a slightly different presentation from \cite{15} to use the approach of Kang-Misra-Miwa \cite{17}. We expect that this model will also work for other 2-parameter twisted quantum affine algebras.

2. The Fock Space of $U_{r,s}(gl(\infty))$

In this section, we first define the two-parameter quantum algebra $U_{r,s}(gl(\infty))$, and obtain an irreducible integrable representation using extended Young diagrams.

Let $\{\epsilon_i, | i \in \mathbb{Z}\}$ be an orthonormal basis of a Euclidean space $E$ with an inner product $(,)$. Let $\{\alpha_i | i \in \mathbb{Z}\}$ be the simple roots of the Lie algebra $g = gl(\infty)$.

We assume that the ground field $\mathbb{K}$ is the field $\mathbb{Q}(r,s)$ of rational functions in $r, s$. Similar to the definition of $U_{r,s}(gl(n))$ (cf. \cite{3}), we define $U_{r,s}(gl(\infty))$ as follows.

DEFINITION 2.1. Let $U_{r,s}(gl(\infty))$ be the unital associative algebra over $\mathbb{K}$ generated by the elements $e_i^\infty, f_i^\infty, \omega_i^\infty, \omega'_i^\infty$ for $i \in \mathbb{Z}$ satisfying the following defining relations:

\begin{align*}
(R1) & \quad (\omega_i^\infty)^{\pm1}, (\omega'_j^\infty)^{\pm1} \text{ all commute with each another and} \\
& \quad \omega_i^\infty (\omega'_i^\infty)^{-1} = (\omega'_i^\infty)(\omega_i^\infty)^{-1} = 1, \\
(R2) & \quad \omega_i^\infty e_j^\infty = r^{(\epsilon_i, \alpha_j)} e_j^\infty \omega_i^\infty \text{ and } \omega_i^\infty f_j^\infty = r^{-(\epsilon_i, \alpha_j)} f_j^\infty \omega_i^\infty, \\
(R3) & \quad \omega_i^\infty e_j^\infty = s^{(\epsilon_i, \alpha_j)} e_j^\infty \omega_i^\infty \text{ and } \omega_i^\infty f_j^\infty = s^{-(\epsilon_i, \alpha_j)} f_j^\infty \omega_i^\infty, \\
(R4) & \quad [e_i^\infty, f_j^\infty] = \frac{\delta_{ij}}{r - s} (\omega_i^\infty \omega_{i+1}^\infty - \omega_{i+1}^\infty \omega_i^\infty), \\
(R5) & \quad [e_i^\infty, e_j^\infty] = [f_i^\infty, f_j^\infty] = 0 \text{ if } |i - j| > 1, \\
(R6) & \quad (e_i^\infty)^2 e_i^\infty = (r + s) e_i^\infty e_{i+1}^\infty e_i^\infty + rs e_{i+1}^\infty (e_i^\infty)^2 = 0, \\
& \quad e_i^\infty (e_{i+1}^\infty)^2 - (r + s) e_{i+1}^\infty e_i^\infty e_{i+1}^\infty + rs (e_{i+1}^\infty)^2 e_i^\infty = 0, \\
(R7) & \quad (f_i^\infty)^2 f_i^\infty - (r^{-1} + s^{-1}) f_i^\infty f_{i+1}^\infty f_i^\infty + (rs)^{-1} f_{i+1}^\infty (f_i^\infty)^2 = 0, \\
& \quad f_i^\infty (f_{i+1}^\infty)^2 - (r^{-1} + s^{-1}) f_{i+1}^\infty f_i^\infty f_{i+1}^\infty + (rs)^{-1} (f_{i+1}^\infty)^2 f_i^\infty = 0.
\end{align*}

Now we construct a Fock space representation for the two-parameter quantum algebra $U_{r,s}(gl_\infty)$, which generalizes the fermionic representation of the usual quantum algebra given in \cite{17}.

We begin with the definition of extended Young diagram given in \cite{13}.

DEFINITION 2.2. An extended Young diagram $Y$ is a sequence $(y_k)_{k \geq 0}$ such that

\begin{enumerate}
  \item[(i)] $y_k \in \mathbb{Z}$, $y_k \leq y_{k+1}$ for all $k$;
  \item[(ii)] there exists fixed integer $y_\infty$ such that $y_k = y_\infty$ for $k \gg 0$.
\end{enumerate}

The integer $y_\infty$ is called the charge of $Y$. 


Another way to identify an extended Young diagram is by specifying the fourth quadrant of the xy-plane with sites \{ (i, j) \in \mathbb{Z} \times \mathbb{Z} | i \geq 0, j \leq 0 \}. Thus an extended Young diagram \( Y = (y_k)_{k \geq 0} \) is an infinite Young diagram drawn on the lattice in the right half plane with sites \{ (i, j) \in \mathbb{Z} \times \mathbb{Z} | i \geq 0, j \leq 0 \}, where \( y_k \) denotes the “depth” of the \( k \)-th column.

Note that if \( y_k \neq y_{k+1} \) for some \( k \), then we will have corners in the extended Young diagram \( Y = (y_k)_{k \geq 0} \). A corner is either “concave” or “convex”. A corner located at site \((i, j)\) is called a \( d \)-diagonal corner (or corner with diagonal number \( d \)), where \( d = i + j \). For more details please see \[13\] and \[17\].

For any fixed integer \( n \), let \( \phi_n \) denote the “empty” diagram \((n, n, n, \cdots)\) of charge \( n \). Let \( Y_n \) denote the set of all extended Young diagrams of charge \( n \). The Fock space of charge \( n \)

\[ \mathcal{F}_n = \bigoplus_{Y \in Y_n} Q(r, s)Y \]

denotes the \( Q(r, s) \)-vector space having all \( Y \in Y_n \) as base vectors.

The algebra \( U_{r,s}(gl(\infty)) \) acts on the Fock space as follows:

**Theorem 2.3.** \( \mathcal{F}_n \) is an irreducible integrable \( U_{r,s}(gl(\infty)) \)-module under the action defined as follows. For \( Y \in Y_n \),

\[ \epsilon_i^\infty Y = Y', \quad \text{if } Y \text{ has an } i \text{-diagonal convex corner then,} \]
\[ Y' \text{ is the same as } Y \text{ except that the } i \text{-diagonal convex corner is replaced by a concave corner,} \]
\[ = 0, \quad \text{otherwise;} \]
\[ f_i^\infty Y = Y'', \quad \text{if } Y \text{ has an } i \text{-diagonal concave corner then,} \]
\[ Y'' \text{ is the same as } Y \text{ except that the } i \text{-diagonal concave corner is replaced by a convex corner,} \]
\[ = 0, \quad \text{otherwise;} \]
\[ \omega_i^\infty Y = s^{-1}Y, \quad \text{if } Y \text{ has an } i \text{-diagonal concave corner}, \]
\[ = r^{-1}, \quad \text{if } Y \text{ has an } i \text{-diagonal convex corner}, \]
\[ = Y, \quad \text{otherwise;} \]
\[ \omega_i'^\infty Y = rY, \quad \text{if } Y \text{ has an } i \text{-diagonal concave corner}, \]
\[ = s, \quad \text{if } Y \text{ has an } i \text{-diagonal convex corner,} \]
\[ = Y, \quad \text{otherwise}. \]

**Proof.** It is straightforward to verify the relations \((R1) – (R7)\) for the action on \( \mathcal{F}_n \) for all generators. We remark that this is very much the same as in type \( A \) situation \[15\].

**3. Fock Space Representations of \( U_{r,s}(C_1^{(1)}) \)**

Having constructed the Fock space representation of the two-parameter quantum affine algebra \( U_{r,s}(gl(\infty)) \), we can build the Fock space representation of \( U_{r,s}(C_1^{(1)}) \) by generalizing the well-known embedding of the latter inside \( U_{r,s}(gl(\infty)) \).

First let us recall the definition of the two-parameter quantum affine algebra \( U_{r,s}(C_1^{(1)}) \) from \[8\].
Let \( I_0 = \{0, 1, 2, \cdots, n\} \), and \((\alpha_{ij}), i, j \in I_0\) be the Cartan matrix of type \( C^{(1)}_l \). We take the normalization \((\alpha_0, \alpha_0) = (\alpha_l, \alpha_l) = 1\) and \((\omega_i, \alpha_i) = \frac{1}{2}\) for \(1 \leq i \leq l - 1\). Let \( r_i = r^{(\alpha_i, \alpha_i)}\) and \( s_i = s^{(\alpha_i, \alpha_i)}\). Denote by \( c \) the canonical central element of \( g(C^{(1)}_l) \) and let \( \delta_{ij} \) denote the Kronecker symbol.

**Definition 3.1.** The two-parameter quantum affine algebra \( U_{r,s}(C^{(1)}_n) \) is the unital associative algebra over \( \mathbb{K} \) generated by the elements \( e_j, f_j, \omega_j^{\pm 1}, \omega_j'^{\pm 1} (j \in I_0) \), \( \gamma^{\pm 1}, \gamma'^{\pm 1}, D^{\pm 1}, D'^{\pm 1} \), satisfying the following relations:

\[ e_j e_i - e_i e_j - (\delta_{ij}) e_i = 0, \]

\[ e_j f_i - f_i e_j = (\delta_{ij} r_i - s_i)(\omega_j - \omega_j), \]

\[ f_j e_i - e_i f_j + (\delta_{ij} s_i - r_i)(\omega_j - \omega_j) = 0, \]

\[ f_j f_i - f_i f_j = (\delta_{ij} c r_i - s_i)(\omega_j - \omega_j) \]

**Definition 3.2.** For all \( \gamma = \gamma' = 0, \gamma' = (r,s)^\epsilon \), such that \( \omega_i \omega_i = \omega_i' \omega_i' = 0, \) the following relations:

\[ [\omega_i^{\pm 1}, \omega_j^{\pm 1}] = [\omega_i^{\pm 1}, D^{\pm 1}] = [\omega_i^{\pm 1}, D'^{\pm 1}] = 0, \]

\[ [\omega_i^{\pm 1}, \omega_j'^{\pm 1}] = [\omega_i'^{\pm 1}, \omega_j'^{\pm 1}] = [D^{\pm 1}, D'^{\pm 1}] = 1 = DD^{-1} = D'D'^{-1}, \]

**Definition 3.3.** For \( 0 \leq i, j \leq l \),

\[ D e_i D^{-1} = r_i^{\delta_{ii}} e_i, \quad D f_i D^{-1} = r_i^{-\delta_{ii}} f_i, \]

\[ D e_i D^{-1} = (i, j) e_i, \quad D f_i D^{-1} = (j, i) f_i. \]

**Definition 3.4.** For \( 0 \leq i, j \leq l \),

\[ D' e_i D'^{-1} = s_i^{\delta_{ii}} e_i, \quad D' f_i D'^{-1} = s_i^{-\delta_{ii}} f_i, \]

\[ D' e_i D'^{-1} = (i, j) e_i, \quad D' e_i D'^{-1} = (j, i) f_i. \]

**Definition 3.5.** For all \( 1 \leq i \neq j \leq l \) such that \( a_{ij} = 0 \),

\[ e_i e_j = [f_i, f_j] = 0, \]

\[ e_i e_0 = rs e_0 e_i, \quad f_0 f_1 = rs f_1 f_0. \]

**Definition 3.6.** For \( 1 \leq i \leq l - 2 \), the \((r, s)\)-Serre relations for \( e_i \):

\[ e_0'^3 e_i - (r + s) e_0 e_1 e_i + rs e_1 e_0'^2 = 0, \]

\[ e_i'^3 e_i' + (r + s) e_i e_i' e_i + (r + s) e_i e_i'^2 = 0, \]

\[ e_n e_i + (r + s) e_i e_i' e_i + (r + s) e_i e_i'^2 = 0, \]

\[ e_i'^2 e_{i' - 1} - (r^2 + s^2) e_{i - 1} e_{i' - 1} + (r^2 + s^2) e_{i' - 1} e_{i - 1} = 0, \]

\[ e_i'^3 e_i - (r + s) e_{i - 1} e_i e_{i' - 1} = 0, \]

\[ + (rs) e_{i - 1} e_i e_{i' - 1} = 0, \]

\[ e_i'^3 e_0 = (r^2 + s^2) e_i e_0, \]

\[ + (rs) e_{i - 1} e_i e_{i' - 1} e_0 e_i e_{i' - 1} = 0. \]

**Definition 3.7.** For \( 1 \leq i \leq l - 2 \), the \((r, s)\)-Serre relations for \( f_i \) are obtained from (C6) by replacing \( e_i \) for \( f_i \) and \( rs \) by \( r^{-1}, s^{-1} \) respectively.
In the above \((i, j)\) are the matrix entries of the two-parameter quantum Cartan matrix for type \(C_1^{(1)}\):

\[
\begin{pmatrix}
rs^{-1} & r^{-1} & 1 & \cdots & 1 & rs \\
rs & r^{s-1} & r^{-1} & \cdots & 1 & 1 \\
1 & 1 & 1 & \cdots & rs & rs^{-1} \\
\end{pmatrix}
\]

We now describe the integrable representation of the two-parameter quantum affine algebra \(U_{r,s}(C_1^{(1)})\). We start with the folding map

\[\pi : \{0, 1, \cdots, 2l-1\} \rightarrow \{0, 1, \cdots, l\}\]

where \(\pi(0) = 0, \pi(l) = l\) and \(\pi(i) = \pi(2l - i) = i\) for \(1 \leq i \leq l-1\). Extend \(\pi\) to a map from \(Z\) into \(\{0, 1, \cdots, l\}\) by periodicity \(2l\).

For any \(Y = (y_k)_{k \geq 0} \in F_n\) define the operators:

\[t_k Y = r^a Y, \quad t_k' = s^a Y\]

where \(a = |\{ p \in Z | y_k < p \leq n \text{ and } \pi(p + k) = 0\}|\) which depends on \(k\).

As we still act on the space \(\mathcal{F}_n\), so we continue to use the same notation for the new Fock space representation. The following theorem is proved by direct verification (see [13]).

**Theorem 3.2.** For \(k = 0, 1, \cdots, l\), the algebra \(U_{r,s}(C_1^{(1)})\) acts on \(\mathcal{F}_n\) by the equations:

\[
e_i = \sum_{\pi(j) = i} \left( \prod_{k \geq j} \omega_k^{(\alpha, \alpha)} \right)^{e_j^{\infty}}_j,
\]

\[
f_i = \sum_{\pi(j) = i} f_j^{\infty} \left( \prod_{k < j} \omega_k^{(\alpha, \alpha)} \right)^{f_j^{\infty}}_j,
\]

\[
\omega_i = \left( \prod_{\pi(j) = i} \omega_j^{\infty} \right)^{\omega_i^{\infty}}_i,
\]

\[
\omega'_i = \left( \prod_{\pi(j) = i} \omega'_j^{\infty} \right)^{\omega'_i^{\infty}}_i,
\]

\[
D = \prod_{k \geq 0} t_k, \quad D' = \prod_{k \geq 0} t'_k.
\]

Under the above action \(\mathcal{F}_n\) is an integrable \(U_{r,s}(C_1^{(1)})\)-module.

**Proof.** We proceed in the same way. First we have

\[
\omega'_i e_i \omega'^{-1}_j = \left( \prod_{k \geq j} \omega'_k^{(\alpha, \alpha)} \right)^{\omega'_i^{(\alpha, \alpha)}}_i \sum_{\pi(j) = i} \left( \prod_{j > j'} \omega_{j'}^{\infty} \right)^{\omega'^{-1}_j}_{j'} e_j^{\infty} \left( \prod_{\pi(k) = j} \omega_k^{\infty} \right)^{-(\alpha, \alpha)}_j.
\]

We don’t have to prove anything for \(|i - j| \geq 2\) due to \(\omega'^{-1}_i e_j = e_j \omega'^{-1}_i\).

For \(i = j\), we have \(e_m^{\infty} \omega'_m^{(\alpha, \alpha)} e_m^{\infty} = e_m^{\infty} \omega'_m^{(\alpha, \alpha)}\). For \(0 \leq i = j - 1 \leq l - 1\), applying \(e_m^{\infty} \omega_m^{(\alpha, \alpha)} e_m^{\infty} = e_m^{\infty} \omega_m^{(\alpha, \alpha)}\).
and $\langle i + 1, i \rangle^{-1} = s^{-1}(\alpha_{i+1}, \alpha_{i+1})$, we arrive at the required relation. Finally, when $1 \leq i = j + 1 \leq l$, we have $e_{m}^{\infty}(\omega_{m}^{\infty})^{-1} = r(\omega_{m-1}^{\infty})^{-1} e_{m}^{\infty}$ and $\langle i - 1, i \rangle^{-1} = r^{-1}(\alpha_{i-1}, \alpha_{i-1})$, and this implies the conclusion.

For further reference, we need a few useful identities.

**Lemma 3.3.** By direct calculations, we get the actions on $\mathcal{F}_{n},$

\[
\begin{align*}
f_{m}^{\infty}(\omega_{m}^{\infty})^{-1} &= \langle m, m' \rangle_{\infty}^{-1}(\omega_{m}^{\infty})^{-1} f_{m}^{\infty}, \\
e_{k}^{\infty}(\omega_{m}^{\infty})^{-1} &= \langle k, m' \rangle_{\infty}(\omega_{m}^{\infty})^{-1} e_{k}^{\infty}, \\
f_{m}^{\infty}(\omega_{k'}^{\infty})^{-1} &= \langle m, k' \rangle_{\infty}(\omega_{k'}^{\infty})^{-1} f_{m}^{\infty},
\end{align*}
\]

where $\langle i, j \rangle_{\infty}$ is defined as follows:

\[
\langle i, j \rangle_{\infty} = \begin{cases} 
rs^{-1}, & i = j; \\
r^{-1}, & i = j - 1; \\
s, & i = j + 1; \\
1, & otherwise.
\end{cases}
\]

Now we turn to the relation (\hat{C}4). From definition and Lemma 3.3, it follows that

\[
e_{i}f_{j} - f_{j}e_{i}
= \sum_{k} \left( \prod_{\pi(k) = i} \omega_{k'}^{\infty} \right)^{(\alpha_{i}, \alpha_{i})} e_{k}^{\infty} \sum_{m} f_{m}^{\infty} \left( \prod_{\pi(m) = j} \omega_{m'}^{\infty} \right)^{(\alpha_{j}, \alpha_{j})}
- \sum_{m} f_{m}^{\infty} \left( \prod_{\pi(m') = j} \omega_{m}^{\infty} \right)^{(\alpha_{j}, \alpha_{j})} \sum_{k} \left( \prod_{\pi(k') = i} \omega_{k'}^{\infty} \right)^{(\alpha_{i}, \alpha_{i})} e_{k}^{\infty}
\]

\[
= \sum_{k, m} \left[ \left( \prod_{\pi(k') = i} \omega_{k'}^{\infty} \right)^{(\alpha_{i}, \alpha_{i})} e_{k}^{\infty} f_{m}^{\infty} \left( \prod_{\pi(m') = j} \omega_{m'}^{\infty} \right)^{(\alpha_{j}, \alpha_{j})}
- f_{m}^{\infty} \left( \prod_{\pi(m') = j} \omega_{m}^{\infty} \right)^{(\alpha_{j}, \alpha_{j})} \left( \prod_{\pi(k') = i} \omega_{k'}^{\infty} \right)^{(\alpha_{i}, \alpha_{i})} e_{k}^{\infty} \right]
\]

\[
= \sum_{k > m} \left( \prod_{\pi(k') = i} \omega_{k'}^{\infty} \right)^{(\alpha_{i}, \alpha_{i})} \left( \prod_{\pi(m') = j} \omega_{m'}^{\infty} \right)^{(\alpha_{j}, \alpha_{j})} \left( e_{k}^{\infty} f_{m}^{\infty} - f_{m}^{\infty} e_{k}^{\infty} \right)
+ \delta_{i,j} \sum_{k} \left( \prod_{\pi(k') = i} \omega_{k'}^{\infty} \right)^{(\alpha_{i}, \alpha_{i})} \left( \prod_{\pi(k')} \omega_{k'}^{\infty} \right)^{(\alpha_{j}, \alpha_{j})} \left( e_{k}^{\infty} f_{m}^{\infty} - f_{m}^{\infty} e_{k}^{\infty} \right)
+ \sum_{k < m} \left( \prod_{\pi(k') = i} \omega_{k'}^{\infty} \right)^{(\alpha_{i}, \alpha_{i})} \left( \prod_{\pi(m') = j} \omega_{m'}^{\infty} \right)^{(\alpha_{j}, \alpha_{j})}
\]
Note that if \( m = k + 1 \), then we have \( e_k^\infty f_m^\infty = 0 = f_m^\infty e_k^\infty \), and if \( m > k + 1 \), then

\[
\sum_{\substack{m' < m \\ \pi(k) = i \\ \pi(m) = j = \pi(m')}} \langle k, m' \rangle_{\infty}^{(\alpha_i, \alpha_j)} = \sum_{\substack{k' > k \\ \pi(m) = j \\ \pi(k) = i = \pi(k')}} \langle m, k' \rangle_{\infty}^{(\alpha_i, \alpha_j)}
\]

On \( F_n \), it is clear that

\[
e_k^\infty f_m^\infty - f_m^\infty e_k^\infty = \delta_{k,m} \left( \frac{\left( \prod_{k' \geq k} \omega_{k'}^\infty \right)^{(\alpha_i, \alpha_j)}}{r_i - s_i} - \left( \prod_{k' < k} \omega_{k'}^\infty \right)^{(\alpha_i, \alpha_j)} \right)
\]

Consequently, it follows that on \( F_n \),

\[
e_i f_j - f_j e_i = \delta_{i,j} (r_i - s_i)^{-1} \sum_{\pi(k) = i} \left\{ \left( \prod_{k' \geq k} \omega_{k'}^\infty \right)^{(\alpha_i, \alpha_j)} - \left( \prod_{k' < k} \omega_{k'}^\infty \right)^{(\alpha_i, \alpha_j)} \right\}
\]

\[
= \delta_{i,j} \left( \frac{\omega_i - \omega_i'}{r_i - s_i} \right)
\]

It is straightforward to check the relation (\( \hat{C}5 \)),

\[
e_l e_0 = \sum_{\pi(k) = l} \left( \prod_{k' \geq k} \omega_{k'}^\infty \right)^{(\alpha_i, \alpha_j)} e_k^\infty \sum_{\pi(m) = 0} \left( \prod_{m' > m} \omega_{m'}^\infty \right)^{(\alpha_0, \alpha_0)} e_m^\infty = r \sum_{\pi(k) = l} \left( \prod_{k' \geq k} \omega_{k'}^\infty \right)^{(\alpha_i, \alpha_j)} e_k^\infty \sum_{\pi(m) = 0} \left( \prod_{m' > m} \omega_{m'}^\infty \right)^{(\alpha_0, \alpha_0)} e_m^\infty
\]

\[
= rs \sum_{k,m} \left( \prod_{k' > k} \omega_{k'}^\infty \right)^{(\alpha_0, \alpha_0)} e_m^\infty \left( \prod_{k' > k} \omega_{k'}^\infty \right)^{(\alpha_i, \alpha_j)} e_k^\infty
\]

The others relations can be proved similarly.

The last task is to verify the Serre relations (\( \hat{C}6 \)) and (\( \hat{C}7 \)). Here we only check the relation (\( \hat{C}6 \)) as the other relations are proved exactly in the same way.
To show the Serre relations ($\hat{C}6$), let us begin with the following notation to save space.

\[ p_j = \prod_{j > j, \pi(j') = \pi(j)} \omega_j^\infty \]

\[ p'_j = \prod_{j > j, \pi(j') = \pi(j)} \omega_j'^\infty. \]

Let us write $i \gg j$ if $i - j > 2$. The following lemmas can be checked directly.

**Lemma 3.4.** For all $j$ and $k$, on $F_n$ then we obtain,

\[ e_k^\infty e_k^\infty = 0, \]

\[ e_k^\infty e_j^\infty e_k^\infty = 0. \]

**Lemma 3.5.** If $\pi(k) = 0 = \pi(j)$, then it holds

\[ e_j^\infty p_k = \left\{ \begin{array}{ll}
p_k e_j^\infty, & \text{for } j \leq k; \\
r^{-1} s p_k e_j^\infty, & \text{for } j > k.
\end{array} \right. \]

\[ e_j^\infty p'_k = \left\{ \begin{array}{ll}
p'_k e_j^\infty, & \text{for } j \leq k; \\
s p'_k e_j^\infty, & \text{for } j > k.
\end{array} \right. \]

**Lemma 3.6.** If $\pi(j) = 0, \pi(k) = 1$, then it follows that

\[ e_j^\infty p_k = \left\{ \begin{array}{ll}
p_k e_j^\infty, & \text{for } j \leq k; \\
r^2 p_k e_j^\infty, & \text{for } j \gg k.
\end{array} \right. \]

\[ e_j^\infty p'_k = \left\{ \begin{array}{ll}
p'_k e_j^\infty, & \text{for } j \leq k; \\
s^2 p'_k e_j^\infty, & \text{for } j \gg k.
\end{array} \right. \]

**Lemma 3.7.** If $\pi(j) = 1, \pi(k) = 0$, then we have

\[ e_j^\infty p_k = \left\{ \begin{array}{ll}
p_k e_j^\infty, & \text{for } j \leq k \text{ or } j = k + 1; \\
s^{-1} p_k e_j^\infty, & \text{for } j \gg k.
\end{array} \right. \]

\[ e_j^\infty p'_k = \left\{ \begin{array}{ll}
p'_k e_j^\infty, & \text{for } j \leq k \text{ or } j = k + 1; \\
r^{-1} p'_k e_j^\infty, & \text{for } j \gg k.
\end{array} \right. \]

**Lemma 3.8.** If $\pi(j) = 1 = \pi(k)$, it is easy to see that

\[ e_j^\infty p_k = \left\{ \begin{array}{ll}
p_k e_j^\infty, & \text{for } j \leq k; \\
r^{-1} s p_k e_j^\infty, & \text{for } j > k.
\end{array} \right. \]

\[ e_j^\infty p'_k = \left\{ \begin{array}{ll}
p'_k e_j^\infty, & \text{for } j \leq k; \\
r s^{-1} p'_k e_j^\infty, & \text{for } j > k.
\end{array} \right. \]
We now prove the following Serre relation:

\[(3.20) \quad e_0^2 e_1 + (r + s)e_0 e_1 e_0 + rse_1 e_0^2 = 0\]

We first use definition to simply the left hand side (LHS) of (3.20).

\[\text{LHS} = \sum_{j,k,m} \left[ \left( \prod_{j' > j} \omega_{k'} \right) e_j \left( \prod_{k' > k} \omega_{j'} \right) e_k \left( \prod_{m' > m} \omega_{m'} \right) \right] e_m \]

\[-\left( r + s \right) \left( \prod_{j' > j} \omega_{k'} \right) e_j \left( \prod_{m' > m} \omega_{m'} \right) + \left( \prod_{m' > m} \omega_{m'} \right) e_m \left( \prod_{k' > k} \omega_{j'} \right) e_k \]

\[= \left( \sum_{m > j > k} + \sum_{m = j+1 > k} + \sum_{j > m > k} + \sum_{j > m = k+1} + \sum_{j > k > m} + \sum_{j = k+1 > m} \right) \]

\[\times \left\{ p_j e_j \prod_{k'} \omega_{k'} p_m e_m + p_k e_k \prod_{j'} \omega_{j'} p_j e_j \prod_{m'} \omega_{m'} e_m \right\} \]

Using Lemma 3.4 through Lemma 3.8, we would like to show that each summand is actually 0. Taking the second summand for example, we get immediately,

\[
\sum_{m = j+1 > k} \left\{ p_j e_j \prod_{k'} \omega_{k'} p_m e_m + p_k e_k \prod_{j'} \omega_{j'} p_j e_j \prod_{m'} \omega_{m'} e_m \right\} \\
\left\{-\left( r + s \right) \left( p_j e_j \prod_{k'} \omega_{k'} p_m e_m + p_k e_k \prod_{j'} \omega_{j'} p_j e_j \prod_{m'} \omega_{m'} e_m \right) \right\} \\
+ \left\{ p_j e_j \prod_{k'} \omega_{k'} p_m e_m + p_k e_k \prod_{j'} \omega_{j'} p_j e_j \prod_{m'} \omega_{m'} e_m \right\} \\
= 0.
\]

The other summands are seen as zero by the same method. Subsequently Relation (3.20) has been verified.

Next we turn to the relation

\[(3.21) \quad e_1^2 e_0 - (r^{-1} + (r)^{-\frac{1}{2}} + s^{-1}) e_0^2 e_1 + (r)^{-\frac{1}{2}} \times \\
(r^{-1} + (r)^{-\frac{1}{2}} + s^{-1}) e_1 e_0 e_1^2 - (r)^{-\frac{1}{2}} e_0 e_1^3 = 0.
\]
Note that by definition, the left hand side (LHS) of (3.21) is equal to

\[ LHS = \sum_{i,j,k,m} \left[ p_i^j e_i^\infty p_j^k e_j^\infty p_k^l e_k^\infty p_m e_m^\infty \right. \]

\[-(r^{-1}+(rs)^{-\frac{1}{2}}+s^{-1}) p_i^j e_i^\infty p_j^k e_j^\infty p_m e_m^\infty p_k^l e_k^\infty \]

\[+(rs)^{-\frac{1}{2}} (r^{-1}+(rs)^{-\frac{1}{2}}+s^{-1}) p_i^j e_i^\infty p_m e_m^\infty p_j^k e_j^\infty p_k^l e_k^\infty \]

\[-(rs)^{-\frac{1}{2}} p_m e_m^\infty p_i^j e_i^\infty p_j^k e_j^\infty p_k^l e_k^\infty \left] \right. \]

Applying Lemma 3.4 through Lemma 3.8, the last relation becomes,

\[ LHS = \left( \sum_{m \gg j \gg i} \sum_{m=j+1 \gg k \gg i} \sum_{j \gg m \gg k \gg i} \sum_{j \gg m = k+1 \gg i} + \sum_{j = m+1 \gg k \gg i} \sum_{j > k \gg m \gg i} \sum_{j > k \gg m = i+1} \sum_{j > k \gg m + 1 \gg i} \right) \left( p_i^j e_i^\infty p_j^k e_j^\infty p_k^l e_k^\infty p_m e_m^\infty \right) \]

\[+p_i^j e_i^\infty p_k^l e_k^\infty p_j^k e_j^\infty p_m e_m^\infty + p_j^k e_j^\infty p_k^l e_k^\infty p_i^j e_i^\infty p_m e_m^\infty \]

\[+p_j^k e_j^\infty p_i^j e_i^\infty p_k^l e_k^\infty p_m e_m^\infty + p_k^l e_k^\infty p_i^j e_i^\infty p_j^k e_j^\infty p_m e_m^\infty \]

\[+p_k^l e_k^\infty p_j^k e_j^\infty p_i^j e_i^\infty p_m e_m^\infty \right) - (r^{-1}+(rs)^{-\frac{1}{2}}+s^{-1}) \]

\[ \left( p_i^j e_i^\infty p_j^k e_j^\infty p_m e_m^\infty p_k^l e_k^\infty + p_i^j e_i^\infty p_k^l e_k^\infty p_j^k e_j^\infty p_m e_m^\infty \right) \]

\[+p_j^k e_j^\infty p_k^l e_k^\infty p_m e_m^\infty p_i^j e_i^\infty + p_j^k e_j^\infty p_i^j e_i^\infty p_k^l e_k^\infty p_m e_m^\infty \]

\[+p_k^l e_k^\infty p_i^j e_i^\infty p_j^k e_j^\infty p_m e_m^\infty + p_k^l e_k^\infty p_j^k e_j^\infty p_i^j e_i^\infty \right) \]

\[+(rs)^{-\frac{1}{2}} \left( r^{-1}+(rs)^{-\frac{1}{2}}+s^{-1} \right) \left( p_i^j e_i^\infty p_m e_m^\infty p_j^k e_j^\infty p_k^l e_k^\infty \right) \]

\[+p_i^j e_i^\infty p_m e_m^\infty p_k^l e_k^\infty p_j^k e_j^\infty + p_j^k e_j^\infty p_m e_m^\infty p_i^j e_i^\infty p_k^l e_k^\infty \]

\[+p_j^k e_j^\infty p_m e_m^\infty p_i^j e_i^\infty p_k^l e_k^\infty + p_j^k e_j^\infty p_k^l e_k^\infty p_i^j e_i^\infty p_m e_m^\infty \]

\[+p_k^l e_k^\infty p_m e_m^\infty p_j^k e_j^\infty p_i^j e_i^\infty \right) - (rs)^{-\frac{1}{2}} \left( p_m e_m^\infty p_i^j e_i^\infty p_j^k e_j^\infty p_k^l e_k^\infty \right) \]

\[+p_m e_m^\infty p_i^j e_i^\infty p_k^l e_k^\infty p_j^k e_j^\infty + p_m e_m^\infty p_j^k e_j^\infty p_i^j e_i^\infty p_k^l e_k^\infty \]

\[+p_m e_m^\infty p_k^l e_k^\infty p_i^j e_i^\infty p_j^k e_j^\infty + p_m e_m^\infty p_k^l e_k^\infty p_j^k e_j^\infty p_i^j e_i^\infty \]

\[+p_m e_m^\infty p_k^l e_k^\infty p_j^k e_j^\infty p_i^j e_i^\infty \right) = 0. \]

Every summand of the last relation can be shown to be 0 as before.
Finally we check that Serre relation involving in $i$ and $i+1$ holds in the Fock space. For $1 \leq i \leq l-2$, we compute that
\[
e_i^2 e_{i+1} - (r_i + s_i) e_i e_{i+1} e_i + (r_is_i) e_{i+1} e_i e_i = \sum_{\pi(j) \neq i \neq \pi(k)} \sum_{\pi(m) = i+1} \{ p_j e_j \infty p_k e_k \infty p_m e_m \infty + p_k e_k \infty p_j e_j \infty p_m e_m \infty \\
- (r_i + s_i) (p_j e_j \infty p_m e_m \infty p_k e_k \infty p_j e_j \infty + p_k e_k \infty p_m e_m \infty p_j e_j \infty ) \\
+ (r_is_i) (p_m e_m \infty p_j e_j \infty p_k e_k \infty p_j e_j \infty + p_j e_j \infty p_m e_m \infty p_k e_k \infty p_j e_j \infty ) \}
\]
\[= \left( \sum_{m \gg j > k} + \sum_{j \gg m \gg k} + \sum_{j > m \gg k} + \sum_{j = m+1} \right) \left( p_j e_j \infty p_k e_k \infty p_m e_m \infty + p_k e_k \infty p_j e_j \infty p_m e_m \infty \\
- (r_i + s_i) (p_j e_j \infty p_m e_m \infty p_k e_k \infty p_j e_j \infty + p_k e_k \infty p_m e_m \infty p_j e_j \infty ) \\
+ (r_is_i) (p_m e_m \infty p_j e_j \infty p_k e_k \infty p_j e_j \infty + p_j e_j \infty p_m e_m \infty p_k e_k \infty p_j e_j \infty ) \right) = 0.
\]
Therefore we have finished the proof of Theroem 3.2.

\[\square\]

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