SCALING INVARIANT SERRIN CRITERION VIA ONE VELOCITY COMPONENT FOR THE NAVIER-STOKES EQUATIONS

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Abstract. In this paper, we prove that the Leray weak solution $u : \mathbb{R}^3 \times (0, T) \to \mathbb{R}^3$ of the Navier-Stokes equations is regular in $\mathbb{R}^3 \times (0, T)$ under the scaling invariant Serrin condition imposed on one component of the velocity $u_3 \in L^{q,1}(0, T; L^p(\mathbb{R}^3))$ with

$$\frac{2}{q} + \frac{3}{p} \leq 1, \quad 3 < p < +\infty.$$ 

This result is an immediate consequence of a new local regularity criterion in terms of one velocity component for suitable weak solutions.

1. Introduction

In this paper, we study the incompressible Navier-Stokes equations

$$(NS) \left\{ \begin{array}{l} \partial_t u - \Delta u + u \cdot \nabla u + \nabla \pi = 0, \\
\text{div} u = 0, \\
u(x, 0) = u_0, \end{array} \right. \tag{1.1}$$

where $(u(x,t), \pi(x,t))$ denote the velocity and the pressure of the fluid respectively.

In the pioneering work [22], Leray introduced the concept of weak solutions to $(NS)$ and proved the global existence for datum $u_0 \in L^2(\mathbb{R}^3)$. Kato [16] initiated the study of $(NS)$ with initial data belonging to the space $L^3(\mathbb{R}^3)$ and obtained global existence in a subspace of $C([0, \infty), L^3(\mathbb{R}^3))$ provided the norm $\|u_0\|_{L^3(\mathbb{R}^3)}$ is small enough. The existence result for initial data small in the Besov space $\dot{B}_{p,q}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ for $p \in [1, \infty)$ and $q \in [1, \infty]$ can be found in [3, 10]. The function spaces $L^3(\mathbb{R}^3)$ and $\dot{B}_{p,q}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ for $(p, q) \in [1, \infty)^2$ both guarantee the existence of local-in-time solution for any initial data. Koch and Tataru [18] showed that global well-posedness holds as well for small initial data in the space $\text{BMO}^{-1}(\mathbb{R}^3)$. On the other hand, it has been shown by Bourgain and Pavlović [2] that the Cauchy problem with initial data in $\dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3)$ is ill-posed no matter how small the initial data is.

In two spatial dimensions, Leray weak solution is unique and regular. In three spatial dimensions, the regularity and uniqueness of weak solution is an outstanding open problem in the mathematical fluid mechanics. It was known that if the weak solution $u$ of (1.1) satisfies so called Ladyzhenskaya-Prodi-Serrin(LPS) type condition

$$u \in L^q(0, T; L^p(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} \leq 1, \quad p \geq 3,$$

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then it is regular in $\mathbb{R}^3 \times (0,T)$, see [8, 13, 29, 30], where the regularity in the class $L^\infty(0,T; L^3(\mathbb{R}^3))$ was proved by Escauriaza, Seregin and Šverák [9]. In [12], based on the work [17], Gallagher, Koch and Planchon gave an alternative proof of the results in [9] by the method of profile decomposition. In [11], they extended the method in [12] to release the space from $L^3$ to the Besov space with negative power. See [1, 35] for further extensions. Recently, Tao [31] proved the blow-up rate of the solution $u$ of (1.1) if the solution $u$ blows up in finite time. We should mention that in the case $\frac{2}{q} + \frac{3}{p} = 1$, the function space $L^q_t L^p_x$ is invariant under the Navier-Stokes scaling:

$$u(x,t) \mapsto u^\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad \forall \lambda > 0,$$

where $u^\lambda$ is still a solution to (1.1) with initial data $u^\lambda_0 := \lambda u_0(\lambda x)$.

Concerning the partial regularity of weak solution satisfying the local energy inequality, initiated by Scheffer [28], Caffarelli, Kohn and Nirenberg [5] showed that one dimensional Hausdorff measure of the possible singular set is zero. One could check Lin [23] and Ladyzhenskaya and Seregin [21] for the simplified proof and improvements. More generalizations could be found from [14, 19, 32–34, 36] and the references therein.

Starting in [26], there are many interesting works devoted to a new LPS type criterion, which only involves one component of the velocity. Neustupa, Novotný and Penel [25] proved the LPS type criterion for one component $u_3 \in L^q(0,T; L^p(\mathbb{R}^3))$ with $\frac{2}{q} + \frac{3}{p} \leq \frac{1}{2}$. Later, this condition was improved by Kukavica and Ziane [20] to

$$\frac{2}{q} + \frac{3}{p} = \frac{5}{8}, \quad p > \frac{24}{5}, \quad \frac{16}{5} \leq q < +\infty;$$

and by Cao and Titi [6] to

$$\frac{2}{q} + \frac{3}{p} \leq \frac{2}{3} + \frac{2}{3p}, \quad p > \frac{7}{2};$$

and then by Pokorný and Zhou [27] up to

$$\frac{2}{q} + \frac{3}{p} \leq \frac{3}{4} + \frac{1}{2p}, \quad p > \frac{10}{3}.$$

However, these conditions are not scaling invariant. Recently, Chemin and Zhang [7] obtained a blow-up criterion via one velocity component in a scaling invariant space $L^p_t(\dot{H}^{\frac{1}{4} + \frac{1}{2}}_x)^{\frac{2}{q} + \frac{3}{p}}$ with $4 < p < 6$. Later, Chemin, Zhang and Zhang [8] released the restriction on $p$ to $4 < p < \infty$ and Han et al. [15] extended the arrange of $p$ to $2 \leq p < +\infty$. However, as stated in [24], the question whether the condition $u_3 \in L^q(0,T; L^p(\mathbb{R}^3))$ for $p$ and $q$, basically satisfying the condition $\frac{2}{q} + \frac{3}{p} \leq 1$, is sufficient for regularity of solution $u$ in $\mathbb{R}^3 \times (t_1,t_2)$, is still open.

Very recently, Chae and Wolf [4] made an important progress and obtained the regularity of solution to (1.1) under the almost LPS type condition

$$u_3 \in L^q(0,T; L^p(\mathbb{R}^3)), \quad \frac{2}{q} + \frac{3}{p} < 1, \quad 3 < p < \infty.$$

The aim of this paper is to obtain LPS criterion via one velocity component with $\frac{2}{q} + \frac{3}{p} \leq 1$. Now let us state our main result.
Theorem 1.1. Let \( u_0 \in L^2(\mathbb{R}^3) \cap L^3(\mathbb{R}^3) \) and \((u, \pi)\) be Leray weak solution of (1.1) in \( \mathbb{R}^3 \times (0, T) \). If \( u \) satisfies the following condition

\[
  u_3 \in L^{q,1}(0, T; L^p(\mathbb{R}^3)), \quad \frac{2}{q} + \frac{3}{p} \leq 1, \quad 3 < p \leq \infty,
\]

then \( u \) is a regular in \( \mathbb{R}^3 \times (0, T) \). Here \( L^{q,1} \) denotes the Lorentz space with respect to variable \( t \).

Theorem 1.1 is a consequence of the following Theorem 1.2 via a similar compactness argument in [4]. Hence we omit the detail.

Remark 1.1. The initial data in \( L^2(\mathbb{R}^3) \cap L^3(\mathbb{R}^3) \) in Theorem 1.1 implies the local-in-time regularity of weak solutions, thus the weak solution is actually suitable weak solution.

Next let us introduce the definition of suitable weak solution.

Definition 1.1. Let \( \Omega \subset \mathbb{R}^3 \) and \( T > 0 \). We say that \((u, \pi)\) is a suitable weak solution of (1.1) in \( \Omega_T = \Omega \times (-T, 0) \) if

1. \( u \in L^\infty(-T, 0; L^2(\Omega)) \cap L^2(-T, 0; H^1(\Omega)) \) and \( \pi \in L^q(\Omega_T); \)
2. (1.1) is satisfied in the sense of distribution;
3. there holds the local energy inequality: for any nonnegative \( \phi \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R}) \) vanishing in a neighborhood of the parabolic boundary of \( \Omega_T \),

\[
  \int_\Omega |u(x,t)|^2 \phi dx + 2 \int_{-T}^t \int_\Omega |\nabla u|^2 \phi dxds \leq \int_{-T}^t \int_\Omega |u|^2 (\partial_s \phi + \Delta \phi) + u \cdot \nabla \phi (|u|^2 + 2\pi)dxds
\]

for any \( t \in [-T, 0] \).

Theorem 1.2. Let \((u, \pi)\) be a suitable weak solution of (1.1) in \( \mathbb{R}^3 \times (-1, 0) \). If \( u \) satisfies the following condition

\[
  u_3 \in L^{q,1}(-1, 0; L^p(\mathbb{R}^3)), \quad \frac{2}{q} + \frac{3}{p} \leq 1, \quad 3 < p \leq \infty,
\]

then it holds that

\[
  r^{-2} \|u\|^3_{L^3(Q_r(z_0))} \leq C,
\]

for any \( 0 < r < \frac{1}{2} \) and any \( z_0 \in \mathbb{R}^3 \times (-\frac{1}{2}, 0) \). Here \( Q_r = B_r(0) \times (-r^2, 0) \).

Remark 1.2. Compared with the result in [4], our main contribution is that the condition (1.3) with the equality is invariant under the Navier-Stokes scaling. Due to the inclusion \( L^{q,1} \subseteq L^q \) for \( q > 1 \), the regularity of weak solution under the condition \( u_3 \in L^q(0, T; L^p(\mathbb{R}^3)) \) with \( \frac{2}{q} + \frac{3}{p} = 1 \) is still open.
2. An intuitive argument

In this section, let us present an intuitive argument to show the regularity condition via one velocity component. We introduce

\[ U(t, x_3) = \int_{\mathbb{R}^2} |u(x_h, x_3, t)|^2 dx_h. \]

Then \( U(t, x_3) \) satisfies

\[
\frac{1}{2} \partial_t U - \frac{1}{2} \partial_{x_3}^2 U + \int_{\mathbb{R}^2} |\nabla u(x, t)|^2 dx_h = - \int_{\mathbb{R}^2} u \cdot \nabla u \cdot u dx_h - \int_{\mathbb{R}^2} \nabla \pi \cdot u dx_h. 
\]

By integration by parts and \( \nabla \cdot u = 0 \), we get

\[
\frac{1}{2} \partial_t U - \frac{1}{2} \partial_{x_3}^2 U + \int_{\mathbb{R}^2} |\nabla u(x, t)|^2 dx_h = - \int_{\mathbb{R}^2} \partial_{x_3} u_3 \frac{1}{2} |u|^2 dx_h - \int_{\mathbb{R}^2} u_3 \partial_{x_3} \pi u_3 dx_h 
\]

Since there is a velocity component \( u_3 \) for each nonlinear term on the right hand side, this simple argument shows the reason that the regularity criterion via one component is reasonable.

Next let us motivate our result via the following toy equation

\[
\partial_t U - \partial_{x_3}^2 U = - \partial_{x_3} \int_{\mathbb{R}^2} u_3 |u|^2 dx_h. 
\]

Then we have

\[
U(t, x_3) = e^{\partial_{x_3}^2 t} U_0 - \int_0^t e^{(t-s)\partial_{x_3}^2} \partial_{x_3} \left( \int_{\mathbb{R}^2} u_3 |u|^2 dx_h \right) ds. 
\]

Using the estimate of heat kernel, we obtain

\[
\|U(t)\|_{L^\infty} \leq \|U(t - \delta)\|_{L^\infty} + C \int_{t-\delta}^t (t-s)^{-\frac{1}{2}} \|u_3(s)\|_{L^\infty} \|U(s)\|_{L^\infty} ds, 
\]

which gives

\[
\sup_{s \in [t-\delta, t]} \|U(s)\|_{L^\infty} \leq \|U(t - \delta)\|_{L^\infty} + C \int_{t-\delta}^t (t-s)^{-\frac{1}{2}} \|u_3(s)\|_{L^\infty} ds \sup_{s \in [t-\delta, t]} \|U(s)\|_{L^\infty}. 
\]

Therefore, if \( u_3 \in L^{2,1}((0, T); L^\infty) \), then we have

\[
\int_{t-\delta}^t (t-s)^{-\frac{1}{2}} \|u_3(s)\|_{L^\infty} ds \leq \|(t-s)^{-\frac{1}{2}}\|_{L^{2,1}} \|u_3(s)\|_{L^\infty} \|_{L^{2,1}} \leq C \|u_3(s)\|_{L^\infty} \|_{L^{2,1}(t-\delta, t)}. 
\]
This shows that
\[ \sup_{s \in [t-\delta,t]} \|U(s)\|_{L^\infty} \leq \|U(t-\delta)\|_{L^\infty} + C \|\nabla u_3(s)\|_{L^\infty} \sup_{s \in [t-\delta,t]} \|U(s)\|_{L^\infty}. \]
Thus, if \( \delta \) is small enough, we conclude that
\[ \sup_{s \in [t-\delta,t]} \|U(s)\|_{L^\infty} \leq 2 \|U(t-\delta)\|_{L^\infty}. \]
In particular, this argument implies that \( u \in L^\infty_{t \rightarrow x_3}(L^2_{x_\delta}) \), which is a scaling invariant estimate under the Navier-Stokes scaling.

Regarding to nonlocal pressure, this argument seems difficult to apply the original Navier-Stokes equations.

3. Local energy estimates

In this section, we apply a similar argument as in the proof of Caffarelli-Kohn-Nirenberg theorem [5] or Chae-Wolf [4], inserting \( \phi = \Phi_n \zeta \) in (1.4), where \( \zeta \) denotes a cut-off function, while \( \Phi_n \) stands for the shifted fundamental solution to the backward heat equation in one spatial dimension, i.e.,
\[ \Phi_n(x,t) = \frac{1}{\sqrt{4\pi(-t + r_n^2)}} e^{-\frac{x^2}{4(-t + r_n^2)}}, \quad (x,t) \in \mathbb{R}^3 \times (-\infty,0), \]
where \( r_n = 2^{-n}, \quad n \in \mathbb{N}. \)
Following the notations in [4], for \( 0 < R < \infty \), we set
\[ U_n(R) = U(0,R,r_n) = B'(R) \times (-r_n,r_n), \]
\[ Q_n(R) = U_n \times (-r_n^2,0), \]
\[ A_n(R) = B'(R) \times A^*_n, \]
where
\[ A^*_n = Q^*_n \setminus Q^*_{n+1}, \quad Q^*_n = (-r_n,r_n) \times (-r_n^2,0). \]
Clearly, there exist absolute constants \( c_1, c_2 > 0 \) such that for all \( 0 < R < +\infty, \quad n \in \mathbb{N} \) and \( j = 1, \ldots, n \), it holds
\[ \Phi_n \leq c_2 r^{-1}_n, \quad |\partial^3 \Phi_n| \leq c_2 r^{-2}_n \quad \text{in } A_j(R); \tag{3.1} \]
\[ c_1 r^{-1} \leq \Phi_n \leq c_2 r^{-1}_n, \quad |\partial^3 \Phi_n| \leq c_2 r^{-2}_n \quad \text{in } Q_n(R). \tag{3.2} \]
Given \( R > 0 \) and \( n \in \mathbb{N}_0 \), the following notation will be used in what follows:
\[ E_n(R) = \sup_{t \in (-r_n^2,0)} \int_{U_n(R)} |u(t)|^2 dx + \int_{-r_n^2}^{0} \int_{U_n(R)} |
abla u|^2 dx ds, \]
\[ E = \sup_{t \in (-1,0)} \int_{\mathbb{R}^3} |u(t)|^2 dx + \int_{-1}^{0} \int_{\mathbb{R}^3} |
abla u|^2 dx ds. \]
By Sobolev’s embedding theorem and a standard interpolation argument, it follows that

$$
\|u\|_{L^m(-r_0^2,0;L^t(U_n(R)))} \leq CE_n(R), \quad \forall \ 2 \leq m \leq \infty, \quad \frac{2}{m} + \frac{3}{t} = \frac{3}{2}
$$

(3.3)

(see also Lemma 3.1 in [4]).

Let \(\eta(x_3,t) \in C_c^\infty((-1,1) \times (-1,0])\) denote a cut-off function such that \(0 \leq \eta \leq 1\) in \(\mathbb{R} \times (-1,0]\), \(\eta = 1\) on \(Q_1^1(-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2},0)\). In addition, let \(\frac{1}{2} \leq \rho < R\) be arbitrarily chosen, but \(|R - \rho| < \frac{1}{2}\). Let \(\psi = \psi(x') \in C^\infty(\mathbb{R}^2)\) with \(0 \leq \psi \leq 1\) in \(B'(R)\) satisfying

$$
\psi(x) = \psi(|x|) = \begin{cases} 
1 & \text{in } B'(\rho); \\
0 & \text{in } \mathbb{R}^2 \setminus B'(\frac{R+\rho}{2}),
\end{cases}
$$

(4.4)

and

$$
|D\psi| \leq \frac{C}{R-\rho}, \quad |D^2\psi| \leq \frac{C}{(R-\rho)^2}.
$$

Direct energy estimates yield that

$$
\frac{1}{2} \int_{U_0(R)} |u(\cdot,t)|^2 \phi_n(\cdot,t) \eta(\cdot,t) \psi dx + \int_{-1}^t \int_{U_0(R)} |\nabla u|^2 \phi_n \eta \psi dxds
\leq \frac{1}{2} \int_{-1}^t \int_{U_0(R)} |u|^2 (\partial_t + \Delta)(\phi_n \eta \psi) dxds + \frac{1}{2} \int_{-1}^t \int_{U_0(R)} |u|^2 u \cdot \nabla (\phi_n \eta \psi) dxds
+ \int_{-1}^t \int_{U_0(R)} \pi u \cdot \nabla (\phi_n \eta \psi) dxds.
$$

(3.5)

Our goal of rest of this section is to control the three terms on the right hand side of the above estimate.

3.1. Estimates for nonlinear terms.

**Lemma 3.1.** Let \((u, \pi)\) be a suitable weak solution of (1.7) in \(\mathbb{R}^3 \times (-1,0)\). Suppose that \((u, \pi)\) satisfies the same assumption of Theorem 1.2. Then we have

$$
\int_{-1}^t \int_{U_0(R)} |u|^2 (\partial_t + \Delta)(\phi_n \eta \psi) dxds \leq C \frac{C}{(R-\rho)^2},
$$

(3.6)

and there exists a positive series \(\{B_i\}_{i \in \mathbb{N}}\) with \(\sum_{i=0}^\infty B_i \leq \|u_3\|_{L^1_t L^p_x}\) such that for any \(n \in \mathbb{N}\) we have

$$
\int_{-1}^t \int_{U_0(R)} |u|^2 u \cdot \nabla (\phi_n \eta \psi) dxds
\leq C \sum_{i=0}^n (r^{-1}_i E_i(R)) B_i + C(R-\rho)^{-1} E^{\frac{1}{2}} \sum_{i=0}^n r^{\frac{1}{2}}_i (r^{-1}_i E_i(R)) + C E^{\frac{3}{2}}.
$$

(3.7)

**Proof.** Let \((u, \pi)\) be the solution satisfying the condition in Lemma 3.1. The proof of (3.6) is quite similar with [4]. In details, we notice that

$$
\int_{-1}^t \int_{U_0(R)} |u|^2 (\partial_t + \Delta)(\phi_n \eta \psi) dxds
$$
We first notice that the last term $I$ implies that

$$I = \int_{-1}^{t} \int_{U_0(R)} |u|^2 (\partial_t + \triangle)(\Phi_n \eta \psi) dx ds,$$

which along with (3.1) and (3.2) implies that

$$\int_{-1}^{t} \int_{U_0(R)} |u|^2 (\partial_t + \triangle)(\Phi_n \eta \psi) dx ds \leq C \int_{A_0(R)} |u|^2 dx ds + C \frac{1}{(R - \rho)^2} \int_{Q_0(R)} |u|^2 \Phi_n dx ds \leq C(R - \rho)^{-2} \sum_{j=0}^{n} r_j^{-1} \int_{Q_j(R)} |u|^2 dx ds.$$

By taking the $L^\infty$ in time and the definition of $Q_j(R)$, we obtain

$$\int_{-1}^{t} \int_{U_0(R)} |u|^2 (\partial_t + \triangle)(\Phi_n \eta \psi) dx ds \leq C(R - \rho)^{-2} \sum_{j=0}^{n} r_j \mathcal{E} \leq C(R - \rho)^{-2} \mathcal{E}.$$ 

This proves (3.6).

Next we turn to give the proof of (3.7). We first notice that

$$\int_{-1}^{t} \int_{U_0(R)} |u|^2 u \cdot \nabla (\Phi_n \eta \psi) dx ds \leq \sum_{i=0}^{n-1} \int_{A_i(R)} |u|^2 |u_3| \partial_3 \Phi_n \eta \psi dx ds + \int_{Q_n(R)} |u|^2 |u_3| \partial_3 \Phi_n \eta \psi dx ds + \int_{Q_0(R)} |u|^2 |\nabla \Phi_n \eta \psi dx ds + \int_{Q_0(R)} |u|^2 |\partial_3 \eta \Phi_n \psi dx ds \leq C \sum_{i=0}^{n-1} \int_{A_i(R)} |u|^2 |u_3| \frac{|x_3|}{(\sqrt{-s + r_n^2})^3} e^{-\frac{x_3^2}{4(-s + r_n^2)}} \eta \psi dx ds + C \int_{Q_0(R)} |u|^2 |u_3| \frac{|x_3|}{(\sqrt{-s + r_n^2})^3} e^{-\frac{x_3^2}{(s + r_n^2)}} \eta \psi dx ds + \int_{Q_0(R)} |u|^2 |\nabla \Phi_n \eta \psi dx ds + \int_{Q_0(R)} |u|^2 |\partial_3 \eta \Phi_n \psi dx ds \equiv I_{21} + \cdots + I_{24}$$

We first notice that the last term $I_{24}$ on the right hand side can by easily controlled as

$$I_{24} \leq \int_{Q_0(R)} |u|^2 |u_3| dx ds \leq C \mathcal{E}^{\frac{2}{3}}.$$ 

Before presenting the details of estimate about $I_{21}$ and $I_{22}$, we introduce $B_i$ as

$$B_i = \sum_{k=1}^{\infty} \left( \int_{-r_k^2}^{0} \|u_3(\cdot, s)\|_{L^p}^{2p} ds \right)^{\frac{2p-2}{2p}} r_k^{-1} e^{-\frac{r_k^2}{32k^2}}$$
\[ + r_{i+1}^{-1} \left( \int_{-r_{i+1}^2}^0 \| u_3 (\cdot, s) \|_{L^p}^{2p-3} ds \right)^{2p-3/2p} \].

About \( I_{21} \), we first notice that
\[
I_{21} \leq C \sum_{i=0}^{n-1} \int_{A_i(R) \cap \{ r_{i+1} \leq |x| \leq r_i, -r_{i+1}^2 \leq s \leq 0 \}} |u|^2 |u_3| \left( \frac{|x_3|}{\sqrt{(-s + r_i^2)^3}} \right)^3 e^{-x_i^2 \eta \psi ds} x_i^2 ds \]
\[ + C \sum_{i=0}^{n-1} \int_{A_i(R) \cap \{ -r_i \leq |x| \leq r_i, -r_i^2 \leq s \leq -r_{i+1}^2 \}} |u|^2 |u_3| \left( \frac{|x_3|}{\sqrt{(-s + r_i^2)^3}} \right)^3 e^{-x_i^2 \eta \psi ds} \]
\[ = I_{211} + I_{212}. \]

Firstly, we observe that by (3.3)
\[ \| u \|^2_{L^{2p}(-r_i^2, 0; L^{2p'}(U_i(R)))} \leq CE_i(R), \]
as \[ \frac{2p-3}{2p} + \frac{2p}{2p} = \frac{3}{2} \]. For the first term \( I_{211} \), we observe that
\[
I_{211} \leq C \sum_{i=0}^{n-1} \int_{r_{i+1}^2}^0 \| u (\cdot, t) \|_{L^{2p'}(U_i(R))}^{2p} \| u_3 (\cdot, s) \|_{L^p} \frac{1}{(-s + r_i^2)} e^{- \frac{x_i^2}{32(-s + r_i^2)}} ds \]
\[ \leq C \sum_{i=0}^{n-1} E_i(R) \left( \int_{r_{i+1}^2}^0 \| u_3 (\cdot, s) \|_{L^p}^{2p} \frac{1}{(-s + r_i^2)} e^{- \frac{pr_i^2}{32(-s + r_i^2)} ds} \right)^{2p-3/2p} \]
\[ \leq C \sum_{i=0}^{n-1} E_i(R) \left( \int_{r_{i+1}^2}^0 \| u_3 (\cdot, s) \|_{L^p}^{2p} \frac{1}{(-s + r_i^2)} e^{- \frac{pr_i^2}{32(-s + r_i^2)} ds} \right)^{2p-3/2p}. \]

On the other hand, we notice the following fact that for any \( s \in [-r_i^2, 0] \)
\[ \frac{1}{(-s + r_i^2)^{2p-3}} e^{- \frac{pr_i^2}{32(-s + r_i^2)} + \frac{r_i^2}{32(-s + r_i^2)}} \leq C r_i^{-\frac{2p}{2p-3}}. \]

Gathering the above two estimates, we obtain
\[
I_{211} \leq C \sum_{i=0}^{n-1} \left( r_i^{-1} E_i(R) \right) \left( \int_{r_{i+1}^2}^0 \| u_3 (\cdot, s) \|_{L^p}^{2p} \frac{1}{(-s + r_i^2)^{2p-3}} e^{- \frac{r_i^2}{32(-s + r_i^2)} ds} \right)^{2p-3/2p}. \]

Before going further, we pay our attention to the term:
\[
\left( \int_{-r_i^2}^0 \| u_3 (\cdot, s) \|_{L^p}^{2p} \frac{1}{(-s + r_i^2)^{2p-3}} e^{- \frac{r_i^2}{32(-s + r_i^2)} ds} \right)^{2p-3/2p}, \]
which is actually controlled by \( B_i \). Indeed, we notice that for any \( 0 \leq i \leq n - 1, \)
\[
\left( \int_{-r_i^2}^0 \| u_3 (\cdot, s) \|_{L^p}^{2p} \frac{1}{(-s + r_i^2)^{2p-3}} e^{- \frac{r_i^2}{32(-s + r_i^2)} ds} \right)^{2p-3/2p} \]
\[ = \left( \int_{-r_{i+1}^2}^{-r_i^2} \| u_3 (\cdot, s + r_i^2) \|_{L^p}^{2p} \frac{1}{(-s)^{2p-3}} e^{- \frac{r_i^2}{32s} ds} \right)^{2p-3/2p}. \]
On the other hand, we notice that for any $i$ and for any $k$, which along with (3.9) and (3.10) implies

$$\left( \int_{-r_i^2}^{0} \| \chi_n(s) u_3(\cdot, s + r_n^2) \|_{L^{p-3}}^{2p} \frac{1}{(s - r_i^2)^{2p-3}} e^{-\frac{r_i^2}{32r} ds} \right)^{\frac{2p-3}{2p}},$$

where $\chi_n(s) = 1 - 1_{(-r_k^2, 0)}(s)$. Now we denote $J_k = (-r_k^2, -r_{k+1}^2]$, then

$$\left( \int_{-r_i^2}^{0} \| \chi_n(s) u_3(\cdot, s + r_n^2) \|_{L^{p-3}}^{2p} \frac{1}{(s - r_i^2)^{2p-3}} e^{-\frac{r_i^2}{32r} ds} \right)^{\frac{2p-3}{2p}} \left( \int_{J_k} \| \chi_n(s) u_3(\cdot, s + r_n^2) \|_{L^{p-3}}^{2p} \frac{1}{(s - r_i^2)^{2p-3}} e^{-\frac{r_i^2}{32r} ds} \right)^{\frac{2p-3}{2p}},$$

which along with (3.9) and (3.10) implies

$$\left( \int_{-r_i^2}^{0} \| u_3(\cdot, s) \|_{L^{p-3}}^{2p} \frac{1}{(s + r_k^2)^{2p-3}} e^{-\frac{r_i^2}{32(s + r_k^2)^{2p-3}}} ds \right)^{\frac{2p-3}{2p}} \leq 2 \sum_{k=1}^{\infty} \left( \int_{-r_i^2}^{0} \| u_3(\cdot, s) \|_{L^{p-3}}^{2p} \frac{1}{(s + r_k^2)^{2p-3}} e^{-\frac{r_i^2}{32(s + r_k^2)^{2p-3}}} ds \right)^{\frac{2p-3}{2p}} r_k^{-1} e^{-\frac{r_i^2}{32r}} \leq C B_i.$$

Therefore, we obtain

$$I_{211} \leq C \sum_{i=0}^{N-1} (r_i^{-1} E_i(R)) B_i.$$  (3.11)

Similarly, for the second term $I_{212}$, we have

$$I_{212} \leq C \sum_{i=0}^{N-1} \left( \int_{-r_i^2}^{0} \| u_3(\cdot, s) \|_{L^{p-3}}^{2p} \frac{1}{(s + r_n^2)^{2p-3}} ds \right)^{\frac{2p-3}{2p}} \| u \|_{L^{p}}^2 \left( \int_{-r_i^2}^{-r_{i+1}^2} (U_i(R)) \right)^{\frac{2p-3}{2p}} \leq C \sum_{i=0}^{N-1} \left( r_i^{-1} E_i(R) \right) \left( \int_{-r_i^2}^{0} \| u_3(\cdot, s) \|_{L^{p-3}}^{2p} \frac{1}{(s + r_n^2)^{2p-3}} ds \right)^{\frac{2p-3}{2p}} \| u \|_{L^{p}}^2 \left( \int_{-r_i^2}^{-r_{i+1}^2} (U_i(R)) \right)^{\frac{2p-3}{2p}} \leq C \sum_{i=0}^{N-1} \left( r_i^{-1} E_i(R) \right) \left( \int_{-r_i^2}^{0} \| u_3(\cdot, s) \|_{L^{p-3}}^{2p} \frac{1}{(s + r_n^2)^{2p-3}} ds \right)^{\frac{2p-3}{2p}} \| u \|_{L^{p}}^2 \left( \int_{-r_i^2}^{-r_{i+1}^2} (U_i(R)) \right)^{\frac{2p-3}{2p}} \leq C \sum_{i=0}^{N-1} \left( r_i^{-1} E_i(R) \right) \left( \int_{-r_i^2}^{0} \| u_3(\cdot, s) \|_{L^{p-3}}^{2p} \frac{1}{(s + r_n^2)^{2p-3}} ds \right)^{\frac{2p-3}{2p}} \| u \|_{L^{p}}^2 \left( \int_{-r_i^2}^{-r_{i+1}^2} (U_i(R)) \right)^{\frac{2p-3}{2p}} \leq C \sum_{i=0}^{N-1} (r_i^{-1} E_i(R)) B_i.$$
Due to $s \in [-r_i^2, -r_{i+1}^2]$, we then have

$$\frac{r_i^{2p-3}}{(-s + r_i^2)^{2p-3}} \leq C \frac{1}{(-s + r_i^2)^{2p-3}},$$

which along the above estimate about $I_{212}$ implies

$$I_{212} \leq C \sum_{i=0}^{n-1} (r_i^{-1} E_i(R)) \left( \int_{-r_i^2}^{-r_{i+1}^2} \|u_3(\cdot, s)\|_{L^p}^{2p-3} \frac{1}{(-s + r_i^2)^{2p-3}} ds \right)^{2p-3} 2p \leq C \sum_{i=0}^{n-1} (r_i^{-1} E_i(R)) B_i. \quad (3.12)$$

Therefore, we obtain

$$I_{21} \leq C \sum_{i=0}^{n-1} (r_i^{-1} E_i(R)) B_i. \quad (3.13)$$

Similarly, we have

$$I_{22} = \int_{Q_n(R)} |u|^2 |u_3| |\partial_3 \Phi_n| |\eta\psi| dx ds \leq Cr_n^{-1} E_n(R) \left( \int_{-r_i^2}^0 \|u_3(\cdot, s)\|_{L^p}^{2p-3} \frac{1}{(-s + r_i^2)^{2p-3}} ds \right)^{2p-3} 2p \leq C r_n^{-1} E_n(R) B_n. \quad (3.14)$$

About $I_{23}$, we have

$$I_{23} = \int_{Q_n(R)} |u|^3 |\nabla \psi| |\Phi_n\eta| dx ds \leq C \frac{1}{(R - \rho)} \sum_{i=0}^{n} r_i^{-1} \int_{Q_i(R)} |u|^3 dx ds \leq C \frac{1}{(R - \rho)} \sum_{i=0}^{n} r_i^{-1} r_i^{1/2} \|u\|_{L^4(-r_i^2, 0; L^4(U_i(R)))}^2 \leq C (R - \rho)^{-1} \varepsilon^{\frac{3}{2}} \sum_{i=0}^{n} r_i^{1/2} (r_i^{-1} E_i(R)). \quad (3.15)$$

Combining (3.13) to (3.15), we finally have

$$\int_{-1}^{t} \int_{U_0(R)} |u|^2 u \cdot \nabla (\Phi_n \eta \psi) dx ds \leq C \sum_{i=0}^{n} (r_i^{-1} E_i(R)) B_i + C (R - \rho)^{-1} \varepsilon^{\frac{3}{2}} \sum_{i=0}^{n} r_i^{1/2} (r_i^{-1} E_i(R)) + C \varepsilon^{\frac{3}{2}}. \quad (3.16)$$
This finishes the proof of (3.7). We are left with the proof of
\[ \sum_{i=0}^{\infty} B_i \leq \|u_3(\cdot, s)\|_{L_t^q L_x^p}. \]

We notice that
\begin{equation}
\sum_{i=0}^{\infty} B_i \leq \sum_{i=0}^{\infty} \sum_{k=i}^{\infty} \left( \int_{-r_k^2}^{0} \|u_3(\cdot, s)\|_{L_t^p}^{2p} ds \right)^{\frac{2p-3}{2p}} r_k^{-1} e^{-\frac{r_k^2}{32k}} \quad (3.17)
\end{equation}

About the first term on the right hand side, we have
\begin{align*}
\sum_{i=0}^{\infty} \sum_{k=i}^{\infty} \left( \int_{-r_k^2}^{0} \|u_3(\cdot, s)\|_{L_t^p}^{2p} ds \right)^{\frac{2p-3}{2p}} r_k^{-1} e^{-\frac{r_k^2}{32k}} &\leq \sum_{k=0}^{\infty} \left( \int_{-r_k^2}^{0} \|u_3(\cdot, s)\|_{L_t^p}^{2p} ds \right)^{\frac{2p-3}{2p}} \sum_{i=0}^{k} r_i^{-1} e^{-\frac{r_i^2}{32i}} \\
&\leq C \sum_{k=0}^{\infty} \frac{1}{r_k} \left( \int_{-r_k^2}^{0} \|u_3(\cdot, s)\|_{L_t^p}^{2p} ds \right)^{\frac{2p-3}{2p}} \leq C \sum_{k=0}^{\infty} r_k^{-\frac{2}{p} - \frac{2}{q}} \left( \int_{-r_k^2}^{0} \|u_3(\cdot, s)\|_{L_t^q}^{2} ds \right)^{\frac{1}{q}},
\end{align*}
where
\[ \frac{2p}{2p - 3} \leq q = \frac{2p}{p - 3}, \quad 3 < p < \infty. \]

By Lemma [A.3] we obtain
\begin{equation}
\sum_{i=0}^{\infty} \sum_{k=i}^{\infty} \left( \int_{-r_k^2}^{0} \|u_3(\cdot, s)\|_{L_t^p}^{2p} ds \right)^{\frac{2p-3}{2p}} r_k^{-1} e^{-\frac{r_k^2}{32k}} \leq C \|u_3(\cdot, s)\|_{L_t^{q,1} L_x^p}. \quad (3.18)
\end{equation}

On the other hand, the control of the second term of (3.17) is quite obvious, since
\[ \sum_{i=0}^{\infty} r_i^{-1} \left( \int_{-r_i^2}^{0} \|u_3(\cdot, s)\|_{L_t^p}^{2p} ds \right)^{\frac{2p-3}{2p}} \leq C \|u_3(\cdot, s)\|_{L_t^{q,1} L_x^p} \]
by using Hölder inequality and Lemma [A.3] again.
Hence, we obtain
\[ \sum_{i=0}^{\infty} B_i \leq C \|u_3(\cdot, s)\|_{L_t^{q,1} L_x^p}. \]

The proof of this lemma is completed. □
3.2. Estimate for the pressure. This part is devoted to show the estimates about the third term on the right side of (3.5), which is related to the control of the pressure \( \pi \). We first decompose the pressure \( \pi \) as \( \pi = \pi_0 + \pi_h \), where

\[
- \Delta \pi_0 = \partial_i \partial_j (u_i u_j \chi_{Q_0(R)}) \quad \text{in} \quad \mathbb{R}^3 \times (-1, 0).
\]

Hence \( \pi_h \) is harmonic in \( Q_0(R) \). Then we have

\[
\int_{-1}^{t} \int_{U_0(R)} \pi u \cdot \nabla (\Phi_n \eta \psi) dx ds
= \int_{-1}^{t} \int_{U_0(R)} \pi_0 u \cdot \nabla (\Phi_n \eta \psi) dx ds + \int_{-1}^{t} \int_{U_0(R)} \pi_h u \cdot \nabla (\Phi_n \eta \psi) dx ds
= \int_{-1}^{t} \int_{U_0(R)} \pi_0 u_3 \partial_3 \Phi_n (\eta \psi) dx ds + \int_{-1}^{t} \int_{U_0(R)} \pi_0 \Phi_n u \cdot \nabla (\eta \psi) dx ds
- \int_{-1}^{t} \int_{U_0(R)} \nabla \pi_h \cdot u (\Phi_n \eta \psi) dx ds.
\]

The purpose of rest of this part is to show the controls of the three terms on the right side of the above equation.

**Lemma 3.2.** Let \((u, \pi)\) be a suitable weak solution of (1.1) in \( \mathbb{R}^3 \times (-1, 0) \). Suppose that \((u, \pi)\) satisfies the same assumption of Theorem 1.2. Then there exists a positive series \( \{C_i\}_{i \in \mathbb{N}} \) with \( \sum_{i=0}^{\infty} C_i \leq \|u_3(\cdot, s)\|_{L^1_tL^6_x} \) such that for any \( n \in \mathbb{N} \) we have

\[
\int_{-1}^{t} \int_{U_0(R)} u_3 \pi_0 (\partial_3 \Phi_n \eta \psi) dx ds \leq C \sum_{i=0}^{n} (r_1^{-1} E_i(R)) C_i.
\]

We can also represent \( \pi_0 \) in the following way. For any \( f_{ij} \in L^p(Q_0(R)) \) with \( 1 < p < \infty \) and \( i, j = 1, 2, 3 \), we define

\[
T(f)(x, t) = \text{P.V.} \int_{\mathbb{R}^3} K(x, y) : f(y, t) \chi_{U_0(R)}(y) dy, \quad (x, t) \in \mathbb{R}^3 \times (-1, 0),
\]

with the kernel

\[
K_{ij} = \partial_i \partial_j \left( \frac{1}{4\pi|x|} \right), \quad i, j = 1, 2, 3.
\]

Then we have \( \pi_0 = T(u_i u_j \chi_{Q_0(R)}) \).

**Proof.** Let \((u, \pi)\) be the solution satisfying the condition in Lemma 3.2. We first introduce the following notations and definition. For \( j \in \mathbb{N}_0 \) let \( \chi_j = \chi_{Q_j(R)} \). Moreover, we set

\[
\phi_j = \left\{ \begin{array}{ll}
\chi_j - \chi_{j+1}, & \text{if} \quad j = 0, 1, \ldots, n - 1; \\
\chi_n, & \text{if} \quad j = n.
\end{array} \right.
\]

It is clear that

\[
\sum_{j=0}^{n} \phi_j = (\chi_0 - \chi_1) + \cdots + \chi_n = 1 \Rightarrow f = \sum_{j=0}^{n} f \phi_j \quad \text{in} \quad Q_0(R).
\]
Taking $f = u_i u_j \chi_{Q_0(R)}$, it holds that
\[ \pi_0 = T(f) = \sum_{j=0}^{n} T(f \phi_j) = \sum_{j=0}^{n} \pi_{0,j}. \]

Then we have
\[
\int_{-1}^{t} \int_{U_0(R)} u_3 \pi_0 (\partial_3 \Phi_n \eta \psi) dx \, ds = \sum_{j=0}^{n} \int_{-1}^{t} \int_{U_0(R)} \pi_{0,j} u_3 \partial_3 \Phi_n \phi_k \eta \psi dx \, ds
\]
\[
= \sum_{j=0}^{n} \sum_{k=0}^{n} \int_{1}^{t} \int_{U_0(R)} \pi_{0,j} u_3 \partial_3 \Phi_n \phi_k \eta \psi dx \, ds
\]
\[= \Pi_1 + \Pi_2. \]

We now deal with the term $\Pi_1$. By the definitions of cut-off function $\phi_i$ and singular operator $T$, $\Pi_1$ can be written as
\[ \Pi_1 = \sum_{k=0}^{n} \int_{1}^{t} \int_{U_0(R)} \Pi_{0,k} u_3 \partial_3 \Phi_n \phi_k \eta \psi dx \, ds \]
with
\[ \Pi_{0,k} = \begin{cases} \pi_0, & \text{if } k = 0; \\ T(\chi_k f), & \text{if } k = 1, \ldots, n. \end{cases} \]

We first notice that
\[
\Pi_1 \leq C \sum_{i=0}^{n-1} \int_{A_i(R) \cap \{ r_{i+1} \leq |x_i| \leq r_{i}, -r^2_{i+1} \leq s \leq 0 \}} \Pi_{0,i} \| u_3 \| \frac{|x_3|}{(\sqrt{-s + r^2_n})^3} e^{-\frac{s^2}{4(-s + r^2_n)}} \eta dx \, ds
\]
\[+ C \sum_{i=0}^{n-1} \int_{A_i(R) \cap \{ |x_i| \leq r_i, -r^2_i \leq s \leq -r^2_{i+1} \}} \Pi_{0,i} \| u_3 \| \frac{|x_3|}{(\sqrt{-s + r^2_n})^3} e^{-\frac{s^2}{4(-s + r^2_n)}} \eta dx \, ds
\]
\[+ C \int_{Q_n(R)} \| \Pi_{0,n} \| u_3 |r_n^{-2} dx \, ds
\]
\[
\leq C \sum_{i=0}^{n-1} \int_{-r^2_{i+1}}^{0} \| u(\cdot, t_l) \|_{L^p(U_i(R))}^2 \| u_3(\cdot, s) \|_{L^p} \frac{1}{|s - r^2_n|} e^{-\frac{r^2}{32(-s + r^2_n)}} ds
\]
\[+ C \sum_{i=0}^{n-1} r^2_i \left( \int_{-r^2_i}^{0} \| u_3(\cdot, s) \|_{L^p}^{2p-3} ds \right)^{\frac{2p-3}{2p}} \| u \|_{L^p}^{\frac{2p}{2p-3}} \| u \|_{L^{2p'}(-r^2_{i+1}; L^{2p'}(U_i(R)))}
\]
\[+ Cr^2_n \left( \int_{-r^2_n}^{0} \| u_3(\cdot, s) \|_{L^p}^{2p-3} ds \right)^{\frac{2p-3}{2p}} \| u \|_{L^p}^{\frac{2p}{2p-3}} \| u \|_{L^{2p}(-r^2_n; 0; L^{2p'}(U_n(R)))}
\]
provided that the kernel of $T$ is a Calderón-Zygmund kernel such that for any $t \in (-1, 0)$,
\[ \| \Pi_{0,k}(\cdot, t) \|_{L^{p'}(R^3)} \leq C \| u_i u_j(\cdot, t) \chi_k(\cdot, t) \|_{L^{p'}} \leq \| u(\cdot, t) \|_{L^{2p'}}^2. \]
By a similar argument leading to (3.11) and (3.12), we obtain

\[ II_1 \leq C \sum_{i=0}^{n} (r_i^{-1} E_i(R)) B_i. \] (3.19)

Now we turn to show the control the \( II_2 \), which is much more complicated. We first have

\[
II_2 = \sum_{j=n-2}^{n} \sum_{k=j}^{n-3} \int_{-1}^{t} \int_{U_0(R)} \pi_{0,j} u_3 \partial_3 \Phi_n \phi_k \eta \psi dx ds \\
+ \sum_{j=n-2}^{n} \sum_{k=j}^{n-3} \int_{-1}^{t} \int_{U_0(R)} \pi_{0,j} u_3 \partial_3 \Phi_n \phi_k \eta \psi dx ds \\
+ \sum_{j=n-2}^{n} \sum_{k=j+4}^{n} \int_{-1}^{t} \int_{U_0(R)} \pi_{0,j} u_3 \partial_3 \Phi_n \phi_k \eta \psi dx ds = II_{21} + II_{22} + II_{23}.
\]

Using the property of singular operator \( T \) and a similar argument as above, we get

\[
II_{21} \leq C \sum_{i=n-2}^{n} \sum_{k=i}^{n} \int_{Q_k(R)} |\pi_{0,i}||u_3| \frac{|x_3|}{(\sqrt{(-s + r_i^2)} \eta \psi)} dx ds \\
\leq C \sum_{i=n-2}^{n} \sum_{k=i}^{n} \left( \int_{-r_i^2}^{0} \left( \frac{2p}{2p-1} \right)^{\frac{2p-1}{2p}} ds \right) r_i^{-2} \|u\|_{L^{2p}(-r_i^2,0;L^{2p}(U_i(R)))} R_i \\
\leq C \sum_{i=0}^{n} (r_i^{-1} E_i(R)) B_i,
\]

and

\[
II_{22} \leq C \sum_{i=0}^{n} \sum_{k=i}^{n} \int_{A_k(R) \cap \{ r_{k+1} \leq |x_3| \leq r_k, \leq r_{k+1}^2 \leq s \leq 0 \}} |\pi_{0,i}||u_3| \frac{|x_3|}{(\sqrt{(-s + r_i^2)} \eta \psi)} dx ds \\
+ C \sum_{i=0}^{n} \sum_{k=i}^{n} \int_{A_k(R) \cap \{ |x_3| \leq r_k, r_{k+1}^2 \leq s \leq r_k^2 \}} |\pi_{0,i}||u_3| \frac{|x_3|}{(\sqrt{(-s + r_i^2)} \eta \psi)} dx ds \\
\leq C \sum_{i=0}^{n} \sum_{k=i}^{n} \int_{-r_{k+1}^2}^{0} \left( \frac{2p}{2p-1} \right)^{\frac{2p-1}{2p}} ds \|u\|_{L^{2p}(U_i(R))} R_i \\
+ C \sum_{i=0}^{n} \sum_{k=i}^{n} \left( \int_{-r_k^2}^{-r_{k+1}^2} \|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-1}} \frac{1}{(-s)^{\frac{2p}{2p-1}}} ds \right) R_i^{\frac{2p-3}{2p}} \|u\|_{L^{2p}((-r_k^2,-r_{k+1}^2;L^{2p}(U_i(R))))} R_i \\
\leq C \sum_{i=0}^{n} (r_i^{-1} E_i(R)) \left( \int_{-r_k^2}^{-r_{k+1}^2} \|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-1}} \frac{1}{(-s)^{\frac{2p}{2p-1}}} ds \right) R_i^{\frac{2p-3}{2p}} \|u\|_{L^{2p}((-r_k^2,-r_{k+1}^2;L^{2p}(U_i(R))))} R_i \\
+ C \sum_{i=0}^{n} (r_i^{-1} E_i(R)) \left( \int_{-r_k^2}^{-r_{k+1}^2} \|u_3(\cdot,s)\|_{L^p}^{\frac{2p}{2p-1}} \frac{1}{(-s)^{\frac{2p}{2p-1}}} ds \right) R_i^{\frac{2p-3}{2p}} \|u\|_{L^{2p}((-r_k^2,-r_{k+1}^2;L^{2p}(U_i(R))))} R_i.\]
Hence, as in (3.11) and (3.12) we obtain
\[ II_{I1} + II_{I2} \leq C \sum_{i=0}^{n} (r_i^{-1} E_i(R)) B_i. \] (3.20)

At last, we estimate the term $II_{I3}$ as
\[ II_{I3} = \sum_{j=0}^{n-3} \sum_{k=j+4}^{n} \int_{t-1}^{t} \int_{U_0(R)} \pi_{0,j} u_3 \partial_3 \Phi_i \phi_{k} \eta \psi dx ds. \]

By the definition of $\pi_{0,j}$, which is harmonic in $\mathbb{R}^2 \times (-r_{j+2}, r_{j+2}) \times (-r_{j+2}^2, 0)$, with the help of Lemma 3.3, we get
\[ \| \pi_{0,j}(\cdot, s) \|_{L^p(U_k(R))} \leq C r_k^{\frac{2}{3}} r_j^{\frac{2-\ell}{3}} \| \pi_{0,j}(\cdot, s) \|_{L^p(\mathbb{R}^3)}. \]

Hence, it follows that
\[ II_{I3} \leq C \sum_{i=0}^{n-3} \sum_{k=i+4}^{n-1} \int_{A_k(R) \cap \{ |x| \leq s \leq -r_{k+1}^2 \}} |\pi_{0,i}| u_3 \left| \frac{|x_i|}{(\sqrt{-s + r_{k+1}^2})^{3/2}} e^{-\frac{s^2}{4(-s+r_{k+1}^2)}} \eta dx ds \right. \]
\[ + C \sum_{i=0}^{n-3} \sum_{k=i+4}^{n-1} \int_{A_k(R) \cap \{ |x| \leq r_k, -r_{k+1}^2 \leq s \leq -r_{k+1}^2 \}} |\pi_{0,i}| u_3 \left| \frac{|x_i|}{(\sqrt{-s + r_{k+1}^2})^{3/2}} e^{-\frac{s^2}{4(-s+r_{k+1}^2)}} \eta dx ds \right. \]
\[ + C \sum_{i=0}^{n-3} \int_{Q_{n}(R)} |\pi_{0,i}| u_3 \left| \frac{|x_i|}{(\sqrt{-s + r_{n+1}^2})^{3/2}} \eta dx ds \right. \]
\[ \leq C \sum_{i=0}^{n-3} \sum_{k=i+4}^{n-1} \int_{-r_{k+1}^2}^{0} \left( r_k^{\frac{1}{3}} r_i^{\frac{2}{3}} \right) \left( \int_{-r_k^2}^{0} \left( u_i(s) \right)^{\frac{2}{p-3}dx ds} \right)^{\frac{2p-3}{2p}} r_k^{\frac{1}{p-2}} r_i^{\frac{2-\ell}{3}} \eta dx ds \]
\[ + C \sum_{i=0}^{n-3} \sum_{k=i+4}^{n-1} \left( \int_{-r_k^2}^{0} \left( u_i(s) \right)^{\frac{2}{p-3}dx ds} \right)^{\frac{2p-3}{2p}} r_k^{\frac{1}{p-2}} r_i^{\frac{2-\ell}{3}} \eta dx ds \]
\[ = II'_{I2}, \]

where $1 < \ell < p'$. Note that
\[ II'_{I2} \leq C \sum_{i=0}^{n-3} \sum_{k=i+4}^{n-1} \left( \int_{-r_k^2}^{0} \left( u_i(s) \right)^{\frac{2}{p-3}dx ds} \right)^{\frac{2p-3}{2p}} r_k^{\frac{1}{p-2}} r_i^{\frac{2-\ell}{3}} \left( \sum_{k=i+4}^{n} \frac{r_k^{\frac{2}{p-3}}}{\frac{2}{p-3}} \right) \left( \frac{2}{p-3} \right) \left( \sum_{k=i+4}^{n} r_k^{\frac{2}{p-3}} \right) E_i(R). \]
where we choose $\tilde{q}$ close to $q$ and $\ell$ close to 1 such that

$$2 - \frac{3}{p} - \frac{2}{q} + 1 - 2 + \frac{3}{p} - 3 + \frac{3}{\ell} > 0.$$ 

Consequently, we have

$$II_2' \leq C \sum_{i=0}^{n-3} r_i^{1 - \frac{3}{p} - \frac{2}{q} - 1} \left( \int_{-r_{i+4}^2}^0 \| u_3(\cdot, s) \|_{L_p}^q ds \right) \frac{1}{q} E_i(R).$$

Thanks to

$$\frac{1}{(-s + r_n^2)^{\frac{1}{2}}} e^{-\frac{r_n^2}{32(s + r_n^4)}} \leq C r_k^{-2},$$

it follows from Hölder inequality that

$$II_1' \leq C \sum_{i=0}^{n-3} \sum_{k=i+4}^{n} \left( \int_{-r_k^2}^0 \| u_3(\cdot, s) \|_{L_p}^{2p} ds \right) \frac{2p}{2p-3} r_k^{\frac{1}{2}} r_i^{-2} r_i^{\frac{1}{2}} r_k^{-\frac{2}{7} - \frac{3}{4}} E_i(R)$$

$$\leq C \sum_{i=0}^{n-3} r_i^{\frac{2p}{2p-3}} \left( \int_{-r_{i+4}^2}^0 \| u_3(\cdot, s) \|_{L_p}^{2p} ds \right) \frac{1}{q} \sum_{k=i+4}^{n} r_k^{1 - \frac{3}{p} - \frac{2}{q} + 1} r_k^{\frac{1}{2} - 2} r_k^{\frac{1}{2} - \frac{3}{4}} E_i(R).$$

Hence, we obtain

$$II_1' \leq C \sum_{i=0}^{n-3} r_i^{1 - \frac{3}{p} - \frac{2}{q} - 1} \left( \int_{-r_{i+4}^2}^0 \| u_3(\cdot, s) \|_{L_p}^{q} ds \right) \frac{1}{q} E_i(R).$$

Therefore, from above two estimates, we deduce that

$$II_{23} \leq C \sum_{i=0}^{n-3} r_i^{1 - \frac{3}{p} - \frac{2}{q} - 1} \left( \int_{-r_{i+4}^2}^0 \| u_3(\cdot, s) \|_{L_p}^{q} ds \right) \frac{1}{q} E_i(R). \quad (3.21)$$

We denote

$$C_i = B_i + r_i^{1 - \frac{3}{p} - \frac{2}{q}} \left( \int_{-r_i^2}^0 \| u_3(\cdot, s) \|_{L_p}^{q} ds \right)^{\frac{1}{q}}, \quad (3.22)$$

which combined with (3.19), (3.20) and (3.21) implies that

$$\int_{-1}^t \int_{U_0(R)} u_3 \pi_0 (\partial_3 \Phi \eta \psi) dx ds \leq C \sum_{i=0}^{n} (r_i^{-1} E_i(R))C_i.$$ 

The bounds of $C_i$ is guaranteed by $B_i$ and Lemma A.3. \qed
Lemma 3.3. Let \((u, \pi)\) be a suitable weak solution of \((1.1)\) in \(\mathbb{R}^3 \times (-1, 0)\). Suppose that \((u, \pi)\) satisfies the same assumption of Theorem 1.2. Then we have

\[
\left| \int_{-1}^{t} \int_{U_0(R)} \pi_0 \Phi_n u \cdot \nabla(\eta \psi) dx ds \right| \leq C \mathcal{E}^{\frac{2}{5}} + C(R - \rho)^{-1} \mathcal{E}^{\frac{1}{2}} \sum_{i=0}^{n} r_i^{-1} E_i(R),
\]

and

\[
\left| \int_{-1}^{t} \int_{U_0(R)} \nabla \pi \cdot u(\Phi_n \eta \psi) dx ds \right| \leq \frac{C}{R - \rho} \mathcal{E}^{\frac{2}{5}}.
\]

The proof of this lemma is similar as in [4], and the only difference is that we need to deal with a large value of \(R\).

Proof. We first notice that

\[
\left| \int_{-1}^{t} \int_{U_0(R)} \pi_0 \Phi_n u \cdot \nabla(\eta \psi) dx ds \right| = \left| \int_{-1}^{t} \int_{U_0(R)} \pi_0 \Phi_n u \cdot \nabla \eta dx ds + \int_{-1}^{t} \int_{U_0(R)} \pi_0 \Phi_n u \cdot \nabla(\psi \eta) dx ds \right|.
\]

Here we give the estimates about the second term on the right side of the above inequality. We find that

\[
\int_{-1}^{t} \int_{U_0(R)} \pi_0 \Phi_n \eta u \cdot \nabla \psi dx ds
\]

\[
= \sum_{k=0}^{n} \int_{-1}^{t} \int_{U_0(R)} \pi_0 \Phi_n \eta u \cdot \nabla \psi_k dx ds = \sum_{j=0}^{n} \sum_{i=0}^{n} \int_{-1}^{t} \int_{U_0(R)} \pi_0 \Phi_n \eta u \cdot \nabla \psi_k dx ds
\]

\[
= \sum_{j=0}^{n} \sum_{k=j+1}^{n} \int_{-1}^{t} \int_{U_0(R)} \pi_0 \Phi_n \eta u \cdot \nabla \psi_k dx ds + \sum_{j=0}^{n} \sum_{i=0}^{n} \int_{-1}^{t} \int_{U_0(R)} \pi_0 \Phi_n \eta u \cdot \nabla \psi_k dx ds
\]

\[
= I' + II'.
\]

As in the proof of Lemma 3.2 we know that \(I'\) can be written as

\[
I' = \sum_{k=0}^{n} \int_{-1}^{t} \int_{U_0(R)} \Pi_{0,k} \Phi_n \eta u \cdot \nabla \psi_k dx ds
\]

with

\[
\Pi_{0,k} = \begin{cases} 
\pi_0, & \text{if } k = 0; \\
T(\chi_k f), & \text{if } k = 1, \ldots, n.
\end{cases}
\]

Then we get

\[
I' \leq \frac{C}{R - \rho} \sum_{i=0}^{n} \int_{A_i(R)} |\Pi_{0,i}| |u| \frac{1}{\sqrt{(-s + r_i^2)}} e^{-\frac{|x|^2}{4(-s + r_i^2)}} \eta dx ds
\]

\[
\leq \frac{C}{R - \rho} \sum_{i=0}^{n} r_i^{-1} \left\| u \right\|_{L^\infty((-r_i^2, 0; L^2(U_i(R)))} \left\| u \right\|_{L^2((-r_i^2, 0; L^2(U_i(R)))}
\]

\[
\leq \frac{C}{R - \rho} \sum_{i=0}^{n} r_i^{-1} \left\| u \right\|_{L^\infty((-r_i^2, 0; L^2(U_i(R)))} \left\| u \right\|_{L^2((-r_i^2, 0; L^2(U_i(R)))}
\]
\begin{align*}
&\leq C(R - \rho)^{-1} \mathcal{E}^{\frac{1}{2}} \sum_{i=0}^{n} r_i^{\frac{1}{2}} (r_i^{-1} E_i(R)).
\end{align*}

By a similar argument as \( II_2 \) in the proof of Lemma 3.2, we can obtain the estimate of \( II' \) by using the norm of \( \|u\|_{L_t^\infty L^p_x} \) instead of the norm of \( u_3 \). Then
\[
\left| \int_{-1}^{t} \int_{U_0(R)} \pi_0 \Phi_n u \cdot \nabla (\eta \psi) dx ds \right| \leq C \mathcal{E}^{\frac{1}{2}} + C(R - \rho)^{-1} \mathcal{E}^{\frac{1}{2}} \sum_{i=0}^{n} r_i^{\frac{1}{2}} (r_i^{-1} E_i(R)).
\]

Now we turn to show the proof of the second result of the lemma. We first choose a cut-off function \( \zeta(x_3, t) \in C^\infty_c \left( (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{4}, 0) \right) \) satisfying \( \zeta(x_3, t) = 1 \) in \( (-\frac{1}{4}, \frac{1}{4}) \times (-\frac{1}{16}, 0] \) and
\[
|\partial_3 \zeta| \leq C.
\]

Then
\[
\left| \int_{Q_0(R)} \nabla \pi_h \cdot u(\Phi_n \eta \psi) (1 - \zeta) dx ds \right| 
\leq \left| \int_{Q_0(R)} \pi_h u \cdot \nabla (\Phi_n \eta \psi (1 - \zeta)) dx ds \right|
\leq \frac{C}{R - \rho} \|\pi_h\|_{L^2(Q_0(R))} \|u\|_{L^2(Q_0(R))} \leq \frac{C}{R - \rho} \mathcal{E}^{\frac{1}{2}}.
\]

Moreover, for fixed \( R \) and \( \rho \), there exist finite balls centered at \( x_j' \in B'(\frac{R + \rho}{2}) \) with \( j = 1, \ldots, J \), whose radius is \( \frac{R - \rho}{4} \), and there hold
\[
\bigcup_{j=1}^{J} \left\{ x', |x' - x_j'| < \frac{R - \rho}{4} \right\} \supset B'(\frac{R + \rho}{2}),
\]
\[
\sum_{j=1}^{J} \left| \left\{ x', |x' - x_j'| < \frac{R - \rho}{2} \right\} \right| \leq C |B'(R)|.
\]

Then
\[
\left| \int_{-1}^{t} \int_{U_0(R)} \nabla \pi_h \cdot u(\Phi_n \eta \psi) \zeta dx ds \right|
\leq C \sum_{k=1}^{n} \sum_{j=1}^{J} r_k^{-1} \int_{Q_k(R) \cap \{|x' - x_j'| < \frac{R - \rho}{4}\}} |\nabla \pi_h| |u| dx ds
\leq C \sum_{k=1}^{n} \sum_{j=1}^{J} r_k^{3} \|\nabla \pi_h\|_{L^2(-r_k^2, 0; L^\infty(U_k(R) \cap \{|x' - x_j'| < \frac{R - \rho}{4}\}))} \|u\|_{L^1(-r_k^2, 0; L^1(U_k(R) \cap \{|x' - x_j'| < \frac{R - \rho}{4}\}))}
\leq C(R - \rho)^{\frac{1}{2}} \sum_{k=1}^{n} \sum_{j=1}^{J} r_k^{-1/3} \|\nabla \pi_h\|_{L^2(-r_k^2, 0; L^\infty(U_k(R) \cap \{|x' - x_j'| < \frac{R - \rho}{4}\}))}.
\[\cdot \|u\|_{L^3(-r_k^2,0;L^3(U_k(R) \cap \{|x'-x_j'| < \frac{R-\rho}{4}\})}^3 \leq C(R-\rho)^2 \sum_{k=1}^{n} \sum_{j=1}^{J} r_k^{-1/3} \|\nabla \pi_h\|_{L^\infty(-r_k^2,0;L^\infty(U_k(R) \cap \{|x'-x_j'| < \frac{R-\rho}{4}\})}^{\frac{3}{2}}
\]

\[+ C \sum_{k=1}^{n} r_k^{-1/3} \|u\|_{L^3(-r_k^2,0;L^3(U_k(R)))}^3.\]

For any \(x^* \in U_k(R) \cap \{x = (x',x_3) ; |x'-x_j'| < \frac{R-\rho}{4}\}\), we have

\[d(x^*, \partial U_0(R)) > \min\left\{\frac{1}{2}, \frac{R-\rho}{4}\right\} = \frac{R-\rho}{4}\]

due to \(k \geq 1\) and \(|R-\rho| \leq \frac{1}{2}\). Thus, there exists \(x_3^* \in (\frac{3}{2}, \frac{5}{2})\) such that

\[x^* \in B\left((x_j^*, x_3^*), \frac{R-\rho}{2}\right) \subset U_0(R) \cap \{ |x'-x_j'| < \frac{R-\rho}{2}\} .\]

Since \(\pi_h\) is harmonic in \(U_0(R)\), there holds

\[|\nabla \pi_h|(x^*) \leq \frac{\|\nabla \pi_h\|_{L^\infty(U_k(R) \cap \{|x'-x_j'| < \frac{R-\rho}{4}\})}^{\frac{3}{2}} \int_{B((x_j^*, x_3^*), \frac{R-\rho}{2})} |\pi_h| dx}{R-\rho |R-\rho|^\frac{1}{3}} \leq \frac{C}{(R-\rho)^{1/3}} \|\nabla \pi_h\|_{L^\infty(U_0(R) \cap \{|x'-x_j'| < \frac{R-\rho}{4}\})}^{\frac{3}{2}},\]

which implies

\[\|\nabla \pi_h\|_{L^\infty(-r_k^2,0;L^\infty(U_k(R) \cap \{|x'-x_j'| < \frac{R-\rho}{4}\})}^{\frac{3}{2}} \leq \int_{-r_k^2}^{0} \|\nabla \pi_h(\cdot, s)\|_{L^\infty(U_k(R) \cap \{|x'-x_j'| < \frac{R-\rho}{4}\})}^{\frac{3}{2}} ds \leq \frac{C}{(R-\rho)^{1/3}} \int_{-r_k^2}^{0} \int_{U_0(R) \cap \{|x'-x_j'| < \frac{R-\rho}{4}\}} |\pi_h|^{\frac{3}{2}} ds .\]

Hence,

\[I_{33} \leq \frac{C}{R-\rho} \sum_{k=1}^{n} r_k^{-1/3}\left(\|\pi_h\|_{L^\infty(-r_k^2,0;L^\infty(U_k(R)))}^{\frac{3}{2}} + \|u\|_{L^3(-r_k^2,0;L^3(U_k(R)))}^3 \right) \leq \frac{C}{R-\rho} \sum_{k=1}^{n} r_k^{1/6}\left(\|\pi_h\|_{L^2(-r_k^2,0;L^2(U_k(R)))}^{\frac{3}{2}} + \|u\|_{L^3(-r_k^2,0;L^3(U_k(R)))}^3 \right) \leq \frac{C}{R-\rho} \left(\|\pi\|_{L^2(-1,0;L^2(U_0(R)))}^{\frac{3}{2}} + \|E\|_{L^\infty}^3 \right).\]
Applying Calderón-Zygmund estimates, there holds
\[ \left| \int_{-1}^{t} \int_{U_0(R)} \nabla \pi_h \cdot u(\Phi_n \eta \psi) \, dx \, ds \right| \leq \frac{C}{R - \rho} \varepsilon_{\frac{3}{2}}. \]

The proof is completed. \( \square \)

4. **Proof of Theorem 1.2**

This section is devoted to the proof of Theorem 1.2 by using Lemma A.1 and the interpolation inequality.

**Proof.** Gathering the estimates in Lemma 3.1, 3.2 and 3.3, we have
\[ r^{-1} E_n(\rho) \leq C \frac{\mathcal{E}}{(R - \rho)^2} + \frac{C}{R - \rho} \varepsilon_{\frac{3}{2}} + C \sum_{i=0}^{n} (r^{-1} E_i(R))(B_i + C_i) \]
\[ + C(R - \rho)^{-1} \mathcal{E}_{\frac{3}{2}} \sum_{i=0}^{n} r_i^{1/2} (r^{-1} E_i(R)). \]

Note that \( B_i \leq C_i \) and \( r^{-1} E_n(R) \leq 2r^{-1} E_{n-1}(R) \). Then we have
\[ r^{-1} E_n(\rho) \leq C \frac{\mathcal{E}}{(R - \rho)^2} + \frac{C}{R - \rho} \varepsilon_{\frac{3}{2}} + C \sum_{i=0}^{n-1} (r^{-1} E_i(R)) C_i \]
\[ + C(R - \rho)^{-1} \mathcal{E}_{\frac{3}{2}} \sum_{i=0}^{n-1} r_i^{1/2} (r^{-1} E_i(R)). \]

Choosing \( \rho = R - \frac{1}{2} \), we have
\[ r^{-1} E_n(\rho) \leq C \mathcal{E} + C \mathcal{E}_{\frac{3}{2}} + C \sum_{i=0}^{n-1} (r^{-1} E_i(R)) C_i \]
\[ + C \mathcal{E}_{\frac{3}{2}} \sum_{i=0}^{n-1} r_i^{1/2} (r^{-1} E_i(R)). \]

Then let \( R, \rho \to \infty \), which deduces that
\[ r^{-1} E_n(\infty) \leq C \mathcal{E} + C \mathcal{E}_{\frac{3}{2}} + C \sum_{i=0}^{n-1} (r^{-1} E_i(\infty)) C_i + C \mathcal{E}_{\frac{3}{2}} \sum_{i=0}^{n-1} r_i^{1/2} (r^{-1} E_i(\infty)) \]

At last, due to \( \sum_{i \geq 0} C_i \leq C \| u_3 \|_{L^q(B)} \) and Lemma A.1, we have for any \( n \in \mathbb{N} \),
\[ r^{-1} E_n(\infty) \leq C(\mathcal{E} + \mathcal{E}_{\frac{3}{2}}) e^{\sum_{i=0}^{\infty} (C_i + \mathcal{E}_{\frac{3}{2}} r_i^{1/2})} \leq C(\mathcal{E} + \mathcal{E}_{\frac{3}{2}}) e^{C r^{1/2}}, \]
which yields that
\[ r^{-2} \| u \|_{L^3(Q)} \leq C r^{-2} \| u \|_{L^3(-r^2, 0; L^3(B(r)))} \leq C(r^{-1} E(r))^{1/2} \leq C. \]

The proof is completed. \( \square \)
Appendix A.

Lemma A.1. Let \( \{b_j\}_{j \in \mathbb{N}}, \{y_j\}_{j \in \mathbb{N}} \) be non-negative series and satisfy the following inequality

\[
y_n \leq C_0 + \sum_{j=0}^{n-1} b_j y_j, n \geq 1 \text{ and } y_0 \leq C_0.
\]

Then we have for any \( n \in \mathbb{N} \),

\[
y_n \leq C_0 e^{\sum_{j=0}^{n-1} b_j}.
\]

Proof. We first define the following non-negative series \( \{x_j\} \)

\[
x_0 = C_0, \quad x_n = C_0 + \sum_{j=0}^{n-1} b_j x_j, \quad n \geq 1.
\]

It is easy to check that for any \( j \in \mathbb{N}, x_j \geq y_j \). On the other hand, by the definition of \( \{x_j\} \), it can be represented as for any \( n \geq 1 \)

\[
x_n = C_0 \prod_{i=0}^{n-1} (1 + b_i) \leq C_0 e^{\sum_{i=0}^{n-1} b_i}.
\]

Hence, we obtain that for any \( n \geq 1 \)

\[
y_n \leq x_n \leq C_0 e^{\sum_{i=0}^{n-1} b_i},
\]

which along with the condition that \( y_0 \leq C_0 \) completes the proof of this lemma. □

Lemma A.2. Let \( 0 < p, q < \infty \). Then for any \( f \in L^{p,q}(\mathbb{R}) \), there exists a sequence \( \{c_n\}_{n \in \mathbb{Z}} \in \ell^q \) and sequence of functions \( \{f_n\}_{n \in \mathbb{Z}} \) with each \( f_n \) bounded by \( 2^{-n/p} \) and supported on a set of measure \( 2^n \) such that

\[
f = \sum_{n \in \mathbb{Z}} c_n f_n,
\]

and

\[
c(p,q)\|\{c_n\}\|_{\ell^q} \leq \|f\|_{L^{p,q}} \leq C(p,q)\|\{c_n\}\|_{\ell^q},
\]

where the constant \( c(p,q) \) and \( C(p,q) \) only depend on \( p,q \).

Proof. Let \( f \in L^{p,q}(\mathbb{R}) \). We denote \( f^* \) as the corresponding decreasing rearrangement of \( f \). We let

\[
c_n := 2^{n/p} f^*(2^n), \quad A_n := \{x : f^*(2^{n+1}) < |f(x)| \leq f^*(2^n)\} \quad \text{and} \quad f_n := c_n^{-1} f 1_{A_n}. \quad (A.1)
\]

By direct calculation, it is easy to check that

\[
f = \sum_{n \in \mathbb{Z}} c_n f_n.
\]
Now we start to prove the second statement. We notice that by the definition of Lorentz space, we have

\[
\|f\|_{L^p,q}^q = \sum_{n \in \mathbb{Z}} \int_{2^n}^{2^{n+1}} (s^{1/p} f^*(s))^q s^{-1} ds 
\]

\[
\leq \sum_{n \in \mathbb{Z}} (f^*(2^n))^{2q^2q/p} \int_{2^n}^{2^{n+1}} s^{\frac{2q^2q}{p}-1} ds 
\]

\[
\leq \frac{p}{q} (2q/p - 1) \sum_{n \in \mathbb{Z}} (f^*(2^n))^{2q^2q/p} = \frac{p}{q} (2q/p - 1) \|(c_n)\|_{\ell^q}^q 
\]

and

\[
\|f\|_{L^p,q}^q = \sum_{n \in \mathbb{Z}} \int_{2^n}^{2^{n+1}} (s^{1/p} f^*(s))^q s^{-1} ds 
\]

\[
\geq \sum_{n \in \mathbb{Z}} 2q(n+1/p) (f^*(2^n+1))^{2q^2q/(n+1)p} \int_{2^n}^{2^{n+1}} s^{q/p - 1} ds 
\]

\[
= \frac{p}{q} (1 - 2^{-q/p}) \|(c_n)\|_{\ell^q}^q. 
\]

\[\square\]

**Lemma A.3.** For any

\[
\frac{2p}{2p - 3} \leq \tilde{q} < q = \frac{2p}{p - 3}, \quad 3 < p < \infty, 
\]

we have

\[
\sum_{k=0}^{\infty} r_k 1 - \frac{3}{p} - \frac{2}{q} \left( \int_{-r_k^2}^{0} \|u_3(\cdot, s)\|_{L^p} ds \right)^\frac{1}{q} \leq C \|u_3\|_{L^{1,1}_t L^{p}_{x}(-r_0, 0; \mathbb{R}^3)} 
\]

**Proof.** Let \( f(s) = \|u_3(\cdot, s)\|_{L^p} \). By Lemma [A.2] we know that

\[
f = \sum_{\ell=0}^{+\infty} c_{\ell} f_{\ell}, \quad \|f\|_{L^{q,1}} \approx \sum_{\ell=0}^{\infty} |c_{\ell}|, 
\]

where

\[
|f_{\ell}| \leq 2^\ell, \quad |D_{\ell} = \text{supp } f_{\ell}| \approx 2^{-\ell}. 
\]

Then we have

\[
\sum_{k=0}^{\infty} r_k 1 - \frac{3}{p} - \frac{2}{q} \left( \int_{-r_k^2}^{0} \|u_3(\cdot, s)\|_{L^p} ds \right)^\frac{1}{q} \leq \sum_{k=0}^{\infty} r_k 1 - \frac{3}{p} - \frac{2}{q} \left( \int_{I_k} |f|^q ds \right)^\frac{1}{q} 
\]

\[
\leq \sum_{k=0}^{\infty} r_k 1 - \frac{3}{p} - \frac{2}{q} \sum_{\ell} |c_{\ell}| 2^{\ell} \sum_{k=0}^{\infty} |D_{\ell} \cap I_k|^{\frac{1}{q}} \leq \sum_{\ell} |c_{\ell}| 2^{\ell} \sum_{k=0}^{\infty} r_k 1 - \frac{3}{p} - \frac{2}{q} |D_{\ell} \cap I_k|^{\frac{1}{q}} 
\]
\[ \leq \sum_{\ell} |c_\ell| 2^\ell \sum_{k=0}^{\ell/2} r_k^{1-\frac{3}{p} - \frac{2}{q}} |D_\ell \cap I_k|^{\frac{1}{q}} + \sum_{\ell} |c_\ell| 2^\ell \sum_{k=\ell/2}^{\infty} r_k^{1-\frac{3}{p} - \frac{2}{q}} |D_\ell \cap I_k|^{\frac{1}{q}}, \]

where \( I_k = (-r_k^2, 0) \). On the other hand, we notice that for any \( k \leq \ell/2 \),
\[ r_k^{1-\frac{3}{p} - \frac{2}{q}} |D_\ell \cap I_k|^{\frac{1}{q}} \leq C 2^{-\frac{k}{4}} 2^{-\frac{k}{2} (1-\frac{3}{p} - \frac{2}{q})}, \]
and for any \( \ell/2 \leq k < \infty \),
\[ r_k^{1-\frac{3}{p} - \frac{2}{q}} |D_\ell \cap I_k|^{\frac{1}{q}} \leq C 2^{-\frac{k}{4}} 2^{-k (1-\frac{3}{p} - \frac{2}{q})} = C 2^{-k (1-\frac{3}{p})}. \]

Then we have
\[
\sum_{k=0}^{\infty} r_k^{1-\frac{3}{p} - \frac{2}{q}} \left( \int_{-r_k^2}^{0} \|u_3(\cdot, s)\|_{L^p}^q ds \right)^{\frac{1}{q}} 
\leq \sum_{\ell=0}^{\infty} |c_\ell| 2^{\ell/2} \sum_{k=0}^{\ell/2} 2^{-k (1-\frac{3}{p} - \frac{2}{q})} + \sum_{\ell=0}^{\infty} |c_\ell| 2^{\ell/2} \sum_{k=\ell/2}^{\infty} 2^{-k (1-\frac{3}{p})},
\]
which along with the restriction on \( p, q \) implies
\[
\sum_{k=0}^{\infty} r_k^{1-\frac{3}{p} - \frac{2}{q}} \left( \int_{-r_k^2}^{0} \|u_3(\cdot, s)\|_{L^p}^q ds \right)^{\frac{1}{q}} \leq C \sum_{\ell=0}^{\infty} |c_\ell| \leq C \|u_3\|_{L_t^4 L_x^{1,1} (\mathbb{R}^3)}.
\]

The proof is completed. \( \square \)

Finally let us recall the lemma about the harmonic function in [4].

**Lemma A.4.** Let \( 0 < r \leq R < \infty \) and \( h : B'(2R) \times (-r, r) \to \mathbb{R} \) be harmonic. Then for all \( 0 < \rho \leq \frac{r}{4} \) and \( 1 \leq \ell \leq p < \infty \),
\[ \|h\|_{L_p(B'(R) \times (-\rho, \rho))}^p \leq c \rho^{2-\frac{3p}{2}} \|h\|_{L_t^\infty(B'(2R) \times (-r, r))}^p. \]

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