Moment and tail estimates and Banach space valued
Non-Central Limit Theorem (NCLT) for sums of
multi-indexed random variables, processes and fields

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Abstract

We derive in this preprint the moment and exponential tail estimates, sufficient conditions for the Non-Central Limit Theorem (NCLT) in the ordinary one-dimensional space as well as in the space of continuous functions for the properly (natural) normalized multi-indexed sums of function of random variables, processes or fields (r.f.), on the other words V - statistics, parametric, in general case.

We construct also some examples in order to show the exactness of obtained estimates.

We will use the theory of the so-called degenerate approximation of the functions of several variables as well as the theory of Grand Lebesgue Spaces (GLS) of measurable functions (random variables).

Key words and phrases: Measure and probability, measurable functions, random variable and vector (r.v.), normalized sum, moment and tail estimates, Non-Central Limit Theorem (NCLT), Banach space of continuous functions, degenerate functions and approximation, triangle inequality, tail of distribution, Lebesgue-Riesz, Orlicz and Grand Lebesgue Spaces (GLS), distance function, compact metric space, metric entropy and entropy integral, cardinal number of the set, Dharmadhikari-Jogdeo-Rosenthal’s constant and inequality, generating function, Lyapunov’s inequality, Young-Orlicz function, conditional expectation, Young-Fenchel transform, white Gaussian measure, rearrangement invariant (r.i.) space.

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1 Definitions. Notations. Previous results.
Statement of problem.

Let \((X, \mathcal{B}, \mu)\) and \((Y, \mathcal{C}, \nu)\) be two probability spaces: \(\mu(X) = \nu(Y) = 1\). We will denote by \(|g|_p = |g|L(p)\) the ordinary Lebesgue-Riesz \(L(p)\) norm of arbitrary
measurable numerical valued function $g : X \to \mathbb{R}$:

$$|g|_p = |g|L_p(X, \mu) := \left[ \int_X |g(x)|^p \, \mu(dx) \right]^{1/p}, \; p \in [1, \infty)$$

analogously for the (measurable) function $h : Y \to \mathbb{R}$

$$|h|_p = |h|L_p(Y, \nu) := \left[ \int_Y |h(y)|^p \, \nu(dy) \right]^{1/p};$$

and for arbitrary integrable function of two variables $f : X \otimes Y \to \mathbb{R}$

$$|f|_p = |f|L_p(X, Y) := \left[ \int_X \int_Y |f(x, y)|^p \, \mu(dx) \, \nu(dy) \right]^{1/p}, \; p \in [1, \infty).$$

Let $Z_+ = \{1, 2, 3, \ldots\}$ and denote $Z_+^d = Z_+ \otimes Z_+, \; Z_+^d = \otimes_{k=1}^d Z_+$. Let also \{\xi(i)\} and \{\eta(j)\}, $i, j = 1, 2, \ldots$, $\xi := \xi(1), \eta := \eta(1)$ be common independent random variables defined on certain probability space $(\Omega, \mathcal{M}, \mathbb{P})$ with distributions correspondingly $\mu, \nu:

$$\mathbb{P}(\xi(i) \in A) = \mu(A), \; A \in \mathcal{B};$$

$$\mathbb{P}(\eta(j) \in F) = \nu(F), \; F \in \mathcal{C}, \tag{1.0}$$

so that

$$\mathbb{E}|g(\xi)|^p = |g|_p^p, \; \mathbb{E}|h(\eta)|^p = |h|_p^p$$

and

$$\mathbb{E}|f(\xi, \eta)|^p = |f|_p^p.$$

Let also $L$ be arbitrary non-empty finite subset of the set $Z_+^d$; denote by $|L|$ a numbers of its elements (cardinal number): $|L| := \text{card}(L)$. It is reasonable to suppose in what follows $|L| \geq 1$.

Define for any centered function $f : X \otimes Y \to \mathbb{R}$, i.e. for which

$$\mathbb{E}f(\xi, \eta) = \int_X \int_Y f(x, y) \, \mu(dx) \, \nu(dy) = 0,$$

the following normalized sum

$$S_L[f] := |L|^{-1/2} \sum_{(k(1), k(2)) \in L} f(\xi(k(1)), \eta(k(2))), \tag{1.1}$$

which is a slight generalization of the classical $U$ and $V$ statistics, see the classical monograph of Korolyuk V.S and Borovskik Yu.V. [24]. Offered here report is the direct generalization of a recent article [33], but we apply here other methods.
The reasonableness of this norming function \(|L|^{-1/2}\) implies that in general, i.e. non-degenerate case \(\text{Var}(S_L) \asymp 1, \ |L| \geq 1\). This proposition holds true still in the multidimensional case.

Our notations and some previous results are borrowed from the works of S.Klesov [20]-[24].

Our claim in this report is to derive the moment and exponential bounds for tail of distribution for the normalized sums of multi-indexed independent random variables from (1.1).

We deduce also as a consequence the sufficient conditions for the weak compactness in the space of continuous function the introduces sums in the case when the function \(f\) dependent still on some parameter (parameters).

Offered here results are generalizations of many ones obtained by S.Klesov in an articles [20]-[24], see also the articles of N.Jenish, and I.R.Prucha [19], and M.J.Wichura [44], where was obtained in particular the CLT for these sums.

The multidimensional case, i.e. when \(\tilde{k} \in \mathbb{Z}^d_+\), will be considered further.

The paper is organized as follows. In the second section we describe and investigate the notion of the degenerate functions and approximation. In the next section we obtain one of the main results: the moment estimates for multi-index sums. We outline in the fourth section the so-called Non-Central Limit Theorem for the multi-indexed sums.

The fifth section contains the multidimensional generalization of obtained results. The sketch of the theory of Grand Lebesgue Spaces (GLS) with some new facts is represented in the next section.

The exponential estimates for tail of multi-index sums is the content of 7th section. The 8th one is devoted to the so-called Banach space valued Non-Central Limit Theorem for multi-index sums.

The last section contains as ordinary some concluding remarks.

## 2 Degenerate functions and approximation.

**Definition 2.1.** The measurable centered function \(f : X \otimes Y \to R\) is said to be *degenerate*, if it has a form

\[
f(x, y) = \sum_{i=1}^{M} \sum_{j=1}^{M} \lambda_{i,j} g_i(x) h_j(y),
\]

where \(\lambda_{i,j} = \text{const} \in \mathbb{R}, \ M = \text{const} = 1, 2, \ldots, \infty\).

The degenerate functions (and kernels) of the form (2.1) are used, e.g., in the approximation theory, in the theory of random processes and fields, in the theory of...
integral equations, in the game theory etc.

A particular application of this notion may be found in the authors article [37].

It will be presumed in this report in addition to the expression (2.1) that all the functions \( \{g_i\}, \{h_j\} \) are centered, such that correspondingly

\[
E g_i(\xi) = \int_X g_i(x) \mu(dx) = 0 \quad (2.2a)
\]

and

\[
E h_j(\eta) = \int_Y h_j(y) \nu(dy) = 0; \; i, j = 1, 2, \ldots, M. \quad (2.2b)
\]

Denotation: \( M = M[f] \overset{\text{def}}{=} \deg(f) \); of course, as a capacity of the value \( M \) one can understood its \textit{constant} minimal value.

Two examples. The equality (2.1) holds true if the function \( f(\cdot, \cdot) \) is trigonometrical or algebraical polynomial.

More complicated example: let \( X \) be compact metrizable space equipped with the non-trivial probability Borelian measure \( \mu \). This imply that an arbitrary non-empty open set has a positive measure.

Let also \( f(x, y), x, y \in X \) be continuous numerical valued non-negative definite function. One can write the famous Karunen-Loev’s decomposition

\[
f(x, y) = \sum_{k=1}^{M} \lambda_k \phi_k(x) \phi_k(y),
\]

where \( \lambda_k, \phi_k(x) \) are correspondingly eigenvalues and orthonormal eigenfunction for the function (kernel) \( f(\cdot, \cdot) \):

\[
\lambda_k \phi_k(x) = \int_{X} f(x, y) \phi_k(y) \mu(dy). \quad (2.3)
\]

We assume without loss of generality

\[
\lambda_1 \geq \lambda_2 \geq \ldots \lambda_k \geq \ldots \geq 0. \quad (2.3a)
\]

Further, let \( B_1, B_2, B_3, \ldots, B_M \) be some rearrangement invariant (r.i.) spaces builded correspondingly over the spaces \( X, Y; Z, W, \ldots \), for instance, \( B_1 = L_p(X), \; B_2 = L_q(Y), 1 \leq p, q \leq \infty \). If \( f(\cdot) \in B_1 \otimes B_2 \), we suppose also in (2.1) \( g_i \in B_1, \; h_j \in B_2 \); and if in addition in (2.1) \( M = \infty \), we suppose that the series in (2.1) converges in the norm \( B_1 \otimes B_2 \)

\[
\lim_{m \to \infty} \| f(\cdot) - \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_{i,j} g_i(\cdot) h_j(\cdot) \|_{B_1 \otimes B_2} = 0. \quad (2.3b)
\]

The condition (2.3b) is satisfied if for example \( ||g_i||_{B_1} = ||h_j||_{B_2} = 1 \) and

\[
\sum_{i,j=1}^{M} |\lambda_{i,j}| < \infty, \quad (2.4)
\]
or more generally when

\[
\sum_{i,j=1}^{M} |\lambda_{i,j}| \cdot ||g_i||_{B_1} \cdot ||h_j||_{B_2} < \infty. \tag{2.4a}
\]

The function of the form (2.1) with \( M = M[f] = \deg(f) < \infty \) is named \textit{degenerate}, notation \( f \in D[M] \); we put also \( D := \bigcup_{M<\infty} D[M] \). Obviously,

\[
B_1 \otimes B_2 = D[\infty].
\]

Define also for each such a function \( f \in D \) the following non-negative quasi-norm

\[
||f||_{D(B_1, B_2)} \overset{def}{=} \inf \left\{ \sum_{i,j=1,2,\ldots,M[f]} |\lambda_{i,j}| \cdot ||g_i||_{B_1} \cdot ||h_j||_{B_2} \right\}, \tag{2.5}\]

where all the arrays \( \{\lambda_{i,j}\}, \{g_i\}, \{h_j\} \) are taking from the representation 2.1.

We will write for brevity \( ||f||_{D_p} := ||f||_{D(L_p(X), L_p(Y))} \) where all the arrays \( \{\lambda_{i,j}\}, \{g_i\}, \{h_j\} \) are taking from the representation 2.1.

Further, let the function \( f \in B_1 \otimes B_2 \) be given. The error of a degenerate approximation of the function \( f \) by the degenerate ones of the degree \( M \) will be introduced as follows

\[
Q_M[f](B_1 \otimes B_2) \overset{def}{=} \inf_{\tilde{f} \in D[M]} ||f - \tilde{f}||_{B_1 \otimes B_2} = \min_{\tilde{f} \in D[M]} ||f - \tilde{f}||_{B_1 \otimes B_2}. \tag{2.6}\]

Obviously, \( \lim Q_M[f](B_1 \otimes B_2) = 0, \ M \to \infty. \)

For brevity:

\[
Q_M[f]_p \overset{def}{=} Q_M[f](L_p(X) \otimes L_p(Y)). \tag{2.6a}\]

The function \( \tilde{f} \) which realized the minimum in (2.6), not necessary to be unique, will be denoted by \( Z_M[f](B_1 \otimes B_2) : \)

\[
Z_M[f](B_1 \otimes B_2) := \arg\min_{\tilde{f} \in D[M]} ||f - \tilde{f}||_{B_1 \otimes B_2}, \tag{2.7}\]

so that

\[
Q_M[f](B_1 \otimes B_2) = ||f - Z_M[f]||_{B_1 \otimes B_2}. \tag{2.8}\]

For brevity:
\[ Z_M[f]_p := Z_M[f](L_p(X) \otimes L_p(Y)). \]  

(2.9)

Let for instance again \( f(x, y), x, y \in X \) be continuous numerical valued non-negative definite function, see (2.3) and (2.3a). It is easily to calculate

\[ Q_M[f](L_2(X) \otimes L_2(X)) = \sum_{k=M+1}^{\infty} \lambda_k. \]

3 Moment estimates for multi-index sums.

0. Trivial estimate.

The following simple estimate based only on the triangle inequality, may be interpreted as trivial:

\[ |S_L|(L_p(X) \otimes L_p(Y)) \leq |L|^{1/2} |f|L_p(X) \otimes L_p(Y)). \]  

(3.0)

Hereafter \( p \geq 2. \)

1. Let us consider at first the one-dimensional case, namely \( f = f(x) = g(x), x \in X : \)

\[ S_n = S_n[f] := n^{-1/2} \sum_{i=1}^{n} g(\xi_i), \]  

(3.1)

where as before \( \mathbf{E}g(\xi) = 0. \) One can apply the famous Dharmadhikari-Jogdeo-Rosenthal’s inequality

\[ \sup_n |S_n|_p \leq K_R(p) \cdot |g(\xi)|_p, \ p \geq 2, \]  

(3.2)

where the optimal Rosenthal’s function on \( p : p \to K_R(p) \) in (3.2) may be estimated as follows: \( K_R(2) = 1 \) and

\[ K_R(p) \leq C_R \frac{p}{e \cdot \ln p} =: K^0_R(p), \]

and the exact value of Rosenthal’s constant \( C_R \) is following

\[ C_R \approx 1.77638 \ldots \]

and this value is attained when \( p = p_0 \approx 33.4610 \ldots \), see [17], [32].

Analogously

\[ \sup_n \left| n^{-1/2} \sum_{i=1}^{n} g(\xi_i) h(\eta_i) \right|_p \leq K_R(p) |g(\xi)|_p |h(\eta)|_p, \ p \geq 2. \]
2. The two-dimensional case.

In this subsection the kernel-function \( f = f(x, y) \) will be presumed to be degenerate with minimal degree \( M = M[f] = 1 \):

\[
f(x, y) = g(x) \cdot h(y), \quad x \in X, \ y \in Y.
\]

Let us consider the correspondent double sum \( S_L[f] = S_L^{(2)} := \sum_{L} \sum_{i,j \in L} g(\xi_i) h(\eta_j), \quad n = \bar{n} = (n_1, n_2) \in L, \ n_1, n_2 \geq 1, \quad (3.3)\)

where as before \( L \) is arbitrary non-empty, \( |L| \geq 1 \) subset of the integer positive plane \( \mathbb{Z}_+^2 \). O.Klesov in [22], [23] proved the following estimate

\[
\sup_{L:|L|\geq 1} \left| S_L^{(2)} \right|_p \leq K^2_R(p) \sum_{M} \sum_{k_{1, k_2}=1} \lambda_{k_{1, k_2}} \left| g_{k_1}(\xi) \right|_p \left| h_{k_2}(\eta) \right|_p, \quad p \geq 2, \quad (3.4)\]

and analogously for the multi-index sums.

3. Estimation for arbitrary degenerate kernel.

In this subsection \( f(\cdot, \cdot) \) is degenerate:

\[
f(x, y) = \sum_{k_{1, k_2}=1}^M \lambda_{k_{1, k_2}} g_{k_1}(x) h_{k_2}(y), \quad (3.5)\]

where

\[
g_{k_1}(\cdot) \in L_p(X), \ h_{k_2}(\cdot) \in L_p(Y),
\]

and as before \( \mathbb{E}g_{k_1}(\xi) = \mathbb{E}h_{k_2}(\eta) = 0.\)

Let us investigate the introduced before statistics

\[
S_L^{(\lambda)} = S_L^{(\lambda)}[f] := |L|^{-1/2} \sum_{i,j \in L} f(\xi_i, \eta_j) =
\]

\[
|L|^{-1/2} \sum_{i,j \in L} \sum_{k_{1, k_2}=1}^M \lambda_{k_{1, k_2}} g_{k_1}(\xi_i) h_{k_2}(\eta_j) =
\]

\[
|L|^{-1/2} \sum_{k_{1, k_2}=1}^M \lambda_{k_{1, k_2}} \sum_{i,j \in L} g_{k_1}(\xi_i) h_{k_2}(\eta_j), \ M < \infty. \quad (3.6)\]

We have using the triangle inequality and the estimate (3.4)

\[
\sup_{L:|L|\geq 1} \left| S_L^{(\lambda)}[f] \right|_p \leq K^2_R(p) \sum_{k_{1, k_2}=1}^M |\lambda_{k_{1, k_2}}| \left| g_{k_1}(\xi) \right|_p \left| h_{k_2}(\eta) \right|_p. \quad (3.7)\]

This estimate remains true in the case when \( M = \infty \), if of course the right-hand side of (3.7) is finite. Therefore,
\[
\sup_{L : |L| \geq 1} |S_L^{(\lambda)}[f]|_p \leq K_R^2(p) \cdot D_p[f], \quad (3.8)
\]

see (2.5), (2.5a).

4. Main result. Degenerate approximation approach.

**Theorem 3.1.** Let \( f = f(x, y) \) be arbitrary function from the space \( L_p(X) \otimes L_p(Y), \ p \geq 2 \). Then \( \sup_{L : |L| \geq 1} |S_L[f]|_p \leq W[f](p) \), where \( W[f](p) \) is defined as:

\[
\sup_{L : |L| \geq 1} \inf_{M \geq 1} \left[ K_R^2(p) \||Z_M[f]||D_p + |L|^{1/2}Q_M[f]|_p \right]. \quad (3.9)
\]

**Proof** is very simple, on the basis of previous results of this section. Namely, let \( L \) be an arbitrary non-empty set. Consider a splitting

\[
f = Z_M[f] + (f - Z_M[f]) =: \Sigma_1 + \Sigma_2.
\]

We have

\[
|\Sigma_1|_p = |Z_M[S_L[f]]|L_p(X \otimes Y) \leq K_R^2(p) \||Z_M[f]|[D_p.
\]

The member \( |\Sigma_2|_p \) may be estimated by virtue of inequality (3.0):

\[
|\Sigma_2|_p \leq |L|^{1/2} |f - Z_M[f]|_p = |L|^{1/2}Q_M[f]|_p.
\]

It remains to apply the triangle inequality and minimization over \( M \).

**Example 3.1.** We deduce from (3.9) as a particular case

\[
\sup_{L : |L| \geq 1} |S_L^{(\lambda)}[f]|_p \leq K_R^2(p) \cdot ||f||D_p, \quad (3.10)
\]

if of course the right-hand side of (3.10) is finite for some value \( p, \ p \geq 2 \).

Recall that in this section \( d = 2 \).

4 Non-Central Limit Theorem for multi-indexed sums.

We intend to find in this section the limit in distribution as \( |L| \to \infty \) of the random value \( S_L[f] \). The limit distribution coincides with a multiple stochastic integrals relative a white Gaussian measure, alike ones for \( U \) and \( V \) statistics, but applied in this preprint methods are essentially other, namely they based on the degenerate approximation of the kernel \( f(\cdot, \cdot) \).

**Additional assumptions and notations.** We suppose for beginning that the domain \( L \) is rectangle:
\[ L = L(n_1, n_2) = \{(i, j) : 1 \leq i \leq n_1; 1 \leq j \leq n_2\}; \quad (4.1) \]

where both the boundaries \( n_1, n_2 \) tend to infinity: \( \lim n_s = \infty, \quad s = 1, 2 \).

Further, we suppose \( 0 < W[f](2) < \infty \); then the family of distributions \( \text{Law}(S_L) \) is weakly compact on the real line.

Let again the function \( f = f(x, y) \) be degenerate:

\[ f(x, y) = \sum_{k_1, k_2=1}^{M} \lambda_{k_1, k_2} g_{k_1}(x) h_{k_2}(y), \quad M \leq \infty; \quad (4.2) \]

but we impose on the system of a functions \( \{g_{k_1}, h_{k_2}\} \) without loss of generality the following condition of orthonormality:

\[ \mathbb{E}g_k(\xi) g_l(\xi) = \mathbb{E}h_k(\eta) h_l(\eta) = \delta_{k l}^l, \quad (4.3) \]

where \( \delta_{k l}^l \) is the Kronecker’s symbol.

We deduce from the condition \( 0 < W[f](2) < \infty \) that

\[ \sigma^2 \overset{def}{=} \text{Var}[f(\xi, \eta)] = \sum_{k_1, k_2=1}^{M} \lambda_{k_1, k_2}^2 \in (0, \infty). \quad (4.4) \]

Introduce also two independent series of independent standard normal distributed random variables

\[ \text{Law}(\tau_k) = N(0, 1), \quad \text{Law}(\theta_k) = N(0, 1). \]

**Theorem 4.1.** We assert under formulated above in this section conditions that the sequence of the r.v. \( S_L \) converges in distribution as \( \min(n_1, n_2) \to \infty \) to the multiple stochastic integral \( S_\infty \) relative the white Gaussian random measure with non-random square integrable integrand:

\[ S_L \overset{d}{\to} S_\infty \overset{def}{=} \sum_{k, l=1}^{M} \lambda_{k, l} \tau_k \theta_l, \quad (4.5) \]

where the symbol \( S_L \overset{d}{\to} \nu \) denotes the weak (i.e. in distribution) convergence.

**Proof.** It is sufficient taking into account the weak compactness of \( S_L \) to consider the case when \( M = 1 \), as long as the general case is linear combination of the one-dimensional ones. So, let

\[ f(x, y) = g(x) \cdot h(y), \]

where the functions \( g(\xi), h(\eta) \) are centered and orthonormal. We have as \( n_1, n_2 \to \infty \)

\[ S_L = (n_1, n_2)^{-1/2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} g(\xi_i) h(\eta_j) = \]


\[
\left[ n_{1}^{-1/2} \sum_{i=1}^{n_{1}} g(\xi_{i}) \right] \cdot \left[ n_{2}^{-1/2} \sum_{j=1}^{n_{2}} h(\eta_{j}) \right] \rightarrow d \tau_{1} \cdot \theta_{1},
\]
on the basis of the classical one-dimensional CLT. Therefore, \( S_{L} \rightarrow S_{\infty} \).

In the general case:

\[
S_{L} \rightarrow S_{\infty} = \sum_{k,l=1}^{M} \lambda_{k,l} \tau_{k} \theta_{l},
\]
(4.6)
since

\[
\sum_{k,l=1}^{M} \lambda_{k,l}^{2} < \infty.
\]

Notice that the right-hand side of (4.6) is homogeneous polynomial of degree 2 on the centered Gaussian distributed independent r.v. Let us introduce thence the following white Gaussian measure \( \zeta \) with independent values on the disjoint sets defined on the Borelian sets of positive plane \( \mathbb{R}^{2} \):

\[
\mathbb{E} \zeta(A_{1}, B_{1}) \zeta(A_{2}, B_{2}) = \int_{A_{1} \cap A_{2}}^{1} dx \int_{B_{1} \cap B_{2}}^{1} dy.
\]

All the r.v. \( \{\tau_{k}, \theta_{l}\} \) may be realized as follows.

\[
\tau_{k} := \int_{[k,k+1]}^{1} \zeta(dx, dy); \quad \theta_{l} := \int_{[l,l+1]}^{3} \zeta(dx, dy);
\]
then the limiting r.v. \( S_{\infty} \) may be represented in the form

\[
S_{\infty} = \sum_{k,l=1}^{M} \lambda_{k,l} \tau_{k} \theta_{l} = \sum_{k,l=1}^{M} \lambda_{k,l} \times \int \int_{\mathbb{R}^{2}} [I(x \in [k,k+1]) I(y \in [0,1]) + I(x \in [2,3]) I(y \in [l,l+1])] \zeta(dx, dy) = \int \int_{\mathbb{R}^{2}} r(x,y) \zeta(dx, dy),
\]
(4.7)
where the square integrable on the whole plane deterministic (non-random) function \( r = r(x,y) \) has a form \( r(x,y) = \sum \sum_{k,l=1}^{M} \lambda_{k,l} \times [I(x \in [k,k+1]) I(y \in [0,1]) + I(x \in [2,3]) I(y \in [l,l+1])] \),
(4.8)
where \( I(z \in C) \) denotes the ordinary indicator function of the set \( C \).

So, the limit distribution \( \text{Law}(S_{L}) \) of the sequence \( S_{L} \) coincides with the multiple stochastic integral (4.7) relative the centered Gaussian random white measure.

Let us consider a more general case, i.e. when the sets \( L \) are not rectangle.
We need to introduce some new geometrical notations. Denote by \( \pi_-(L) \) the set of all rectangles which are \textit{inscribed} into the set \( L : \pi_-(L) = \{ L_- \} \), where
\[
L_- = \{ [n(1)_-,n(1)_-] \otimes [n(2)_-,n(2)_-] : L_- \subset L \},
\]
\( 1 \leq n(1)_- \leq n(1)_- < \infty, n(1)_-,n(1)_- \in \mathbb{Z}_+ \),
\( 1 \leq n(2)_- \leq n(2)_- < \infty, n(2)_-,n(2)_- \in \mathbb{Z}_+ \).

We denote analogously by \( \pi^+(L) \) the set of all rectangles which are \textit{circumscribed} about the set \( L : \pi^+(L) = \{ L^+ \} \), where
\[
L^+ = \{ [n(1)^+,n(1)^+] \otimes [n(2)^+,n(2)^+] : L^+ \supset L \},
\]
\( 1 \leq n(1)^+ \leq n(1)^+ < \infty, n(1)^+,n(1)^+ \in \mathbb{Z}_+ \),
\( 1 \leq n(2)^+ \leq n(2)^+ < \infty, n(2)^+,n(2)^+ \in \mathbb{Z}_+ \).

Let now \( \{ L \} = \{ L_\alpha \}, \alpha \to \alpha_0 \) be a sequence or more generally a net of subsets in \( \mathbb{Z}_+^2 \), and let \( \{ L_- \} = \{ L_-,\alpha \}, \alpha \to \alpha_0 \) be arbitrary correspondent sequence of inscribed rectangles:
\[
L_- = L_-(L_\alpha) = \{ [n(1)_-,n(1)_-] \otimes [n(2)_-,n(2)_-] \} : L_- \subset L.
\]
Denote
\[
k_- = \kappa_-(L) = \kappa(L,L_-) \overset{\text{def}}{=} \left| \frac{L \setminus L_-}{|L|^{1/2}} \right|.
\]

We impose the following two conditions on the sequence (net) of the sets \( \{ L_\alpha, L_-(L_\alpha) \} \):
\[
\min [n(1)_-,n(2)_-] \to \infty; \quad (4.12)
\]
\[
k_- = \kappa_-(L_\alpha) = \kappa_-(L_\alpha, L_-,\alpha) \to 0. \quad (4.13)
\]

**Theorem 4.2.** Suppose that both the conditions (4.12) and (4.13) are satisfied. Let also \( 0 < W[f](2) < \infty \). Then the assertion of theorem 4.1 remains true.

**Proof.** It follows from theorem 4.1 that the sequence \( S_{L_-} \) converges in distribution to the multiple stochastic integral \( S_\infty \). Therefore, it is sufficient to ground the convergence
\[
\forall \epsilon > 0 \Rightarrow \mathbf{P} \left( |S_L - S_{L_-}| > \epsilon \right) \to 0, \quad (4.14)
\]
i.e. estimate the remainder term $S_L - S_{L-}$.

Note first of all for this purpose

$$|L|^{1/2} \cdot S_L = \sum_{k \in L} f(\tilde{\xi}) = \sum_{k \in L} f(\tilde{\xi}) + \sum_{k \in L \setminus L^-} f(\tilde{\xi}) \overset{def}{=} \Sigma_1 + \Sigma_2;$$

$$S_{L-} = \left[\frac{|L|}{|S_{L-}|}\right]^{1/2} \cdot S_L \to 1 \cdot S_\infty = S_\infty,$$

as long as $|S_{L-}|/|S_L| \to 1$.

Further,

$$\left| |L|^{-1/2} \Sigma_2 \right|_2 \leq W_2[f] \cdot \frac{|L| - |L^-|}{|L|^{1/2}} \leq W_2[f] \cdot \kappa_-(L_\alpha) \to 0,$$

Q.E.D.

The case of circumscribed rectangle, as well as the multidimensional case $d \geq 3$ may be investigated alike the considered one. Indeed, we impose as above the following two conditions on the sequence (net) of the sets $\{L_\alpha, L^+(L_\alpha)\}$:

$$\min \left[ n(1)^+, n(2)^{++} \right] \to \infty;$$  \hspace{1cm} (4.15)

$$\kappa^+ = \kappa^+(L_\alpha) = \kappa^+(L_\alpha, L^+(L_\alpha)) \to 0,$$  \hspace{1cm} (4.16)

where

$$\kappa^+ = \kappa^+(L) = \kappa(L, L^+) \overset{def}{=} \frac{|L^+ \setminus L|}{|L|^{1/2}}.$$  \hspace{1cm} (4.17)

**Theorem 4.3.** Suppose that both the conditions (4.15) and (4.16) are satisfied. Let also $0 < W[f](2) < \infty$. Then the assertion of theorem 4.1 remains true.

5 Multidimensional generalization.

Let now $(X_m, B_m, \mu_m), m = 1, 2, \ldots, d$, $d \geq 3$ be a family of probability spaces: $\mu_m(X_m) = 1$; $X := \otimes_{m=1}^d X_m$; $\xi(m)$ be independent random variables having the distribution correspondingly $\mu_m : P(\xi(m) \in A_m) = \mu_m(A_m), A_m \in B_m$; $\xi_i(m), i = 1, 2, \ldots, n(m); n(m) = 1, 2, \ldots, n(m) < \infty$ be independent copies of $\xi(m)$ and also independent on the other vectors $\xi_i(s), s \neq m$, so that all the random variables $\{\xi_i(m)\}$ are common independent.

Another notations, conditions, restrictions and definitions. $L \subset Z_+^d, |L| = \text{card}(L) > 1; j = \vec{j} \in L; k = \vec{k} = (k(1), k(2), \ldots, k(d)) \in Z_+^d; N(\vec{k}) := \max_{j=1,2,\ldots,d} k(j); \quad (5.0)$
\( \vec{\xi} := \{\xi(1), \xi(2), \ldots, \xi(n(m))\}; \quad \vec{\xi}_i := \{\xi_i(1), \xi_i(2), \ldots, \xi_i(n(m))\}; \quad X := \otimes_{i=1}^d X_i, \quad f : X \to R \) be measurable centered function, i.e. such that \( \mathbf{E} f(\vec{\xi}) = 0; \)

\[
S_L[f] := |L|^{-1/2} \sum_{k \in L} f(\vec{\xi}_k).
\] (5.1)

The following simple estimate is named as before trivial:

\[
|S_L[f]|_p \leq |L|^{1/2} |f|_{L^p}.
\] (3.0a)

Recall that by-still hereafter \( p \geq 2. \)

By definition, as above, the function \( f : X \to R \) is said to be degenerate, iff it has the form

\[
f(\vec{x}) = \sum_{\vec{k} \in \mathbb{Z}_d^d, \ N(\vec{k}) \leq M} \lambda(\vec{k}) \prod_{s=1}^d g^{(s)}_{k(s)}(x(s)),
\] (5.2)

for some integer constant value \( M, \) finite or not, where all the functions \( g^{(s)}_{k(s)}(\cdot) \) are in turn centered: \( \mathbf{E} g^{(s)}_{k(s)}(\xi(s)) = 0. \) Denotation: \( M = \text{deg}[f]. \)

Define also as in the two-dimensional case for each such a function \( f \in D \) the following non-negative quasi-norm

\[
||f||_{D_p} \overset{def}{=} \inf \left\{ \sum_{\vec{k} \in \mathbb{Z}_d^d, \ N(\vec{k}) \leq M[f]} |\lambda(\vec{k})| \cdot \prod_{s=1}^d |g^{(s)}_{k(s)}(\xi(s))|_p \right\},
\] (5.3)

where all the arrays \( \{\lambda(\vec{k})\}, \{g_j\}, \) are taking from the representation 5.2.

The last assertion allows a simple estimate: \( ||f||_{D_p} \leq ||f||_{D^0_p}, \) where

\[
||f||_{D^0_p} \overset{def}{=} \sum_{\vec{k} \in \mathbb{Z}_d^d, \ N(\vec{k}) \leq M[f]} |\lambda(\vec{k})| \cdot \prod_{s=1}^d |g^{(s)}_{k(s)}(\xi(s))|_p,
\] (5.3a)

and if we denote

\[
G(p) := \prod_{j=1}^d |g^{(j)}_{k_j}(\xi_j)|_p, \quad p \geq 1; \quad ||\lambda||_1 := \sum_{\vec{k} \in \mathbb{Z}_d^d} |\lambda(\vec{k})|,
\]

then

\[
||f||_{D_p} \leq ||f||_{D^0_p} \leq G(p) \cdot ||\lambda||_1.
\] (5.3b)

Further, let the function \( f \in B_1 \otimes B_2 \otimes \ldots \otimes B_d \) be given. The error of a degenerate approximation of the function \( f \) by the degenerate ones of the degree \( M \) will be introduced as before.
\[ Q_M[f](B_1 \otimes B_2 \otimes \ldots \otimes B_d) \overset{\text{def}}{=} \inf_{\tilde{f} \in D[M]} \| f - \tilde{f} \| \| B_1 \otimes B_2 \otimes \ldots \otimes B_d. \]  

(5.4)

Obviously, \( \lim Q_M[f](B_1 \otimes B_2 \otimes \ldots \otimes B_d) = 0, \quad M \to \infty. \)

For brevity:

\[ Q_M[f]_p \overset{\text{def}}{=} Q_M[f](L_p(X_1) \otimes L_p(X_2) \otimes \ldots \otimes L_p(X_d)). \]  

(5.5)

The function \( \tilde{f} \) which realized the minimum in (5.4), not necessary to be unique, will be denoted by \( Z_M[f](B_1 \otimes B_2 \otimes \ldots \otimes B_d) : \)

\[ Z_M[f](B_1 \otimes B_2 \otimes \ldots \otimes B_d) := \text{argmin}_{\tilde{f} \in D[M]} \| f - \tilde{f} \| \| B_1 \otimes B_2 \otimes \ldots \otimes B_d. \]  

(5.6)

so that

\[ Q_M[f](B_1 \otimes B_2 \otimes \ldots \otimes B_d) = \| f - Z_M[f] \| \| B_1 \otimes B_2 \otimes \ldots \otimes B_d. \]  

(5.7)

For brevity:

\[ Z_M[f]_p := Z_M[f](L_p(X_1) \otimes L_p(X_2) \otimes \ldots \otimes L_p(X_d)). \]  

(5.8)

We deduce analogously to the third section

**Theorem 5.1.** Let \( f = f(x) = f(\bar{x}), \quad x \in X \) be arbitrary function from the space \( L_p(X_1) \otimes L_p(X_2) \otimes \ldots \otimes L_p(X_d), \quad p \geq 2. \) Then

\[ \sup_{L|L| \geq 1} |S_L[f]|_p \leq W_d[f](p), \]  

where \( W_d[f](p) \overset{\text{def}}{=} \)

\[ \sup_{L|L| \geq 1} \inf_{M \geq 1} \left[ K_R^d(p) \| Z_M[f]\| D_p + |L|^{1/2} Q_M[f]_p \right]. \]  

(5.9)

**Example 5.1.** We deduce alike the example 3.1 as a particular case

\[ \sup_{L|L| \geq 1} |S_L[f]|_p \leq K_R^d(p) \cdot |f| \cdot D_p, \]  

(5.9a)

if of course the right-hand side of (5.9a) is finite for some value \( p, \quad p \geq 2. \)

As a slight consequence:

\[ \sup_{L|L| \geq 1} |S_L[f]|_p \leq K_R^d(p) \cdot |f| \cdot D_p^\alpha \leq K_R^d(p) \cdot G(p) \cdot ||\lambda||_1. \]  

(5.9b)
Remark 5.1. Notice that the last estimates (5.9), (5.9a), and (5.9b) are essentially non-improvable. Indeed, it is known still in the one-dimensional case \( d = 1 \); for the multidimensional one it is sufficient to take as a trial factorizable function; say, when \( d = 2 \), one can choose

\[
f_0(x, y) := g_0(x) \ h_0(y), \quad x \in X, \ y \in Y.
\]

Let us investigate now the Non-Central Limit Theorem for the multi-indexed sums in the case when \( d \geq 3 \). We suppose first of all without loss of generality that in the expression (5.2) the function \( g_j(\cdot) \) are (common) orthonormal:

\[
\mathbb{E} g_j^{(s)}(\xi(j)) \ g_t^{(t)}(\xi(l)) = \delta_k^l \cdot \delta_s^t, \ s, t = 1, 2, \ldots, d; \tag{5.10}
\]

\( \delta_k^l \) is Kronecker’s symbol.

Further, we suppose that the set \( L, \ L \subset \mathbb{Z}^d_+ \) is a parallelepiped:

\[
L = [1, n(1)] \otimes [1, n(2)] \otimes \ldots \otimes [1, n(d)], \tag{5.11}
\]

then \( |L| = \prod_{m=1}^d n(m) \); and that

\[
\min_l n(l) \to \infty. \tag{5.12}
\]

Let us introduce the following array \( \beta = \{\beta^{(s)}(\vec{k})\}, \ s = 1, 2, \ldots, d; \ \vec{k} \in L \) consisting on the (common) independent standard normal (Gaussian) distributed random variables, defined on certain sufficiently rich probability space.

**Theorem 5.2.** We assert under conditions (5.12) and \( W_2[f](2) < \infty \), that the generalized sequence of the r.v. \( S_L \) converges in distribution as \( \min_l (n(l)) \to \infty \) to the multiple stochastic integral \( S_\infty \) relative the white Gaussian random measure with non-random square integrable integrand:

\[
S_L \stackrel{d}{\to} S_\infty \overset{\text{def}}{=} \sum_{\vec{k} \in L} \lambda(\vec{k}) \cdot \prod_{s=1}^d \beta^{(s)}(k_s), \tag{5.13}
\]

where the symbol \( S_L \stackrel{d}{\to} \nu \) denotes the weak (i.e. in distribution) convergence.

**Proof** is at the same as in theorem 4.1 and may be omitted.

### 6 Grand Lebesgue Spaces (GLS).

We intend to derive in this section the uniform relative the amount of summand \( |L| \) exponential bounds for tail of distribution of the r.v. \( S_L \), based in turn on the moments bound obtained above as well as on the theory of the so-called
Grand Lebesgue Spaces (GLS). We recall now some facts about these spaces and supplement more.

Let \((\Omega, \mathcal{M}, \mathcal{P})\) be certain probability space. Let also \(\psi = \psi(p), p \in [1, b), b = \text{const} \in (1, \infty]\) be some bounded from below: \(\inf \psi(p) > 0\) continuous inside the semi-open interval \(p \in [1, b)\) numerical valued function. We can and will suppose without loss of generality

\[
\inf_{p \in [1, b)} \psi(p) = 1
\]  

(6.0)

and \(b = \sup \{p, \psi(p) < \infty\}\), so that \(\text{supp } \psi = [1, b)\) or \(\text{supp } \psi = [1, b]\). The set of all such a functions will be denoted by \(\Psi(b) = \{\psi(\cdot)\}; \Psi := \Psi(\infty)\).

By definition, the (Banach) Grand Lebesgue Space (GLS) \(G\psi = G\psi(b)\) consists on all the real (or complex) numerical valued measurable functions (random variables, r.v.) \(f : X \to R\) defined on our probability space and having a finite norm

\[
|| f || = ||f||G\psi \overset{\text{def}}{=} \sup_{p \in [1, b)} \left[\frac{|f|_p}{\psi(p)}\right].
\]  

(6.1)

The function \(\psi = \psi(p)\) in the definition (6.1) is said to be the generating function for this space.

Furthermore, let now \(\eta = \eta(z), z \in S\) be arbitrary family of random variables defined on any set \(z \in S\) such that

\[
\exists b = \text{const} \in (1, \infty], \forall p \in [1, b) \Rightarrow \psi_S(p) := \sup_{z \in S} |\eta(z)|_p < \infty.
\]

The function \(p \to \psi_S(p)\) is named as a natural function for the family of random variables \(S\). Obviously,

\[
\sup_{z \in S} ||\eta(z)||G\Psi_S = 1.
\]

The family \(S\) may consists on the unique r.v., say \(\Delta:\)

\[
\psi_\Delta(p) := |\Delta|_p,
\]

if of course the last function is finite for some value \(p = p_0 > 1\).

Note that the last condition is satisfied if for instance the r.v. \(\Delta\) satisfies the so-called Cramer’s condition; the inverse proposition is not true.

The generating \(\psi(\cdot)\) function in (6.1) may be introduced for instance as natural one for some family of functions.

These spaces are Banach functional space, are complete, and rearrangement invariant in the classical sense, see [4], chapters 1, 2; and were investigated in particular in many works, see e.g. [5], [11]-[12], [25]-[26], [28]-[32], [36] etc. We refer here some used in the sequel facts about these spaces and supplement more.
The so-called tail function $T_f(y), \ y \geq 0$ for arbitrary (measurable) numerical valued function (random variable, r.v.) $f$ is defined as usually

$$T_f(y) \overset{\text{def}}{=} \max(\mathbb{P}(f \geq y), \ \mathbb{P}(f \leq -y)), \ y \geq 0.$$  

It is known that

$$|f|^p = \int_X |f|^p(\omega) \mathbb{P}(d\omega) = p \int_0^\infty y^{p-1} T_f(y) \ dy$$

and if $f \in G\psi, f \neq 0$, then

$$T_f(y) \leq \exp \left( -v_\psi^*(\ln(y/ ||f||_{G\psi})) \right), \ y \geq e ||f||_{G\psi},$$

(6.2)

where

$$v(p) = v_\psi(p) := p \ \ln \psi(p).$$

Here and in the sequel the operator (non-linear) $f \to f^\ast$ will denote the famous Young-Fenchel, or Legendre transform

$$f^*(u) \overset{\text{def}}{=} \sup_{x \in \text{Dom}(f)} (x u - f(x)).$$

Conversely, the last inequality may be reversed in the following version: if

$$T_\zeta(y) \leq \exp \left( -v_\psi^*(\ln(u/K)) \right), \ u \geq e K,$$

and if the auxiliary function $v(p) = v_\psi(p)$ is positive, finite for all the values $p \in [1, \infty)$, continuous, convex and such that

$$\lim_{p \to \infty} \psi(p) = \infty,$$

then $\zeta \in G(\psi)$ and besides $||\zeta|| \leq C(\psi) \cdot K.$

Let us consider the so-called exponential Orlicz space $\Phi(M)$ equipped with an ordinary Luxemburg norm built over source probability space with correspondent Young-Orlicz function

$$M(y) = M[\psi](y) = \exp \left( v_\psi^*(\ln |y|) \right), \ |y| \geq e; \ M(y) = Cy^2, \ |y| < e. \quad (6.2a)$$

Of course, $Ce^2 = \exp \left( v_\psi^*(1) \right).$

The exponentiality implies in particular that the Orlicz space $\Phi(M)$ is not separable in general case as long as the correspondent Young - Orlicz function $M(y) = M[\psi](y)$ does not satisfy the $\Delta_2$ condition.

The Orlicz-Luxemburg $|| \cdot ||_{\Phi(M)} = || \cdot ||_{\Phi(M[\psi](\cdot)))}$ and $|| \cdot ||_{G\psi}$ norms are quite equivalent:

$$||f||_{G\psi} \leq C_1 ||f||_{\Phi(M)} \leq C_2 ||f||_{G\psi},$$
0 < C_1 = C_1(\psi) < C_2 = C_2(\psi) < \infty. \quad (6.3)

**Example 6.0.** Let us consider also the so-called extremal \( \Psi \) – function \( \psi_{(r)}(p) \), where \( r = \text{const} \in [1, \infty) \):

\[
\psi_{(r)}(p) \overset{\text{def}}{=} 1, \; p \in [1, r];
\]
so that the correspondent value \( b = b(r) \) is equal to \( r \). One can extrapolate formally this function onto the whole semi-axis \( R_+^1 \):

\[
\psi_{(r)}(p) := \infty, \; p > r.
\]

The classical Lebesgue-Riesz \( L_r \) norm for the r.v. \( \eta \) is quite equal to the GLS norm
\[
||| \eta |||_{G\psi_{(r)}}:
\]

\[
|| \eta ||_{r} = ||| \eta |||_{G\psi_{(r)}}.
\]

Thus, the ordinary Lebesgue-Riesz spaces are particular, more precisely, extremal cases of the Grand-Lebesgue ones.

**Example 6.1.** For instance, let \( \psi \) function has a form

\[
\psi(p) = \psi_m(p) = p^{1/m}, \; p \in [1, \infty), \; m = \text{const} > 0. \quad (6.4)
\]

The function \( f : X \to R \) belongs to the space \( G\psi_m \):

\[
||f||_{G\psi_m} = \sup_{p \geq 1} \left\{ \frac{|f|_p}{p^{1/m}} \right\} < \infty
\]

if and only if the correspondent tail estimate is follow:

\[
\exists V = V(m) > 0 \Rightarrow T_f(y) \leq \exp \left\{ -\left( y/V(m) \right)^m \right\}, \; y \geq 0. \quad (6.5)
\]

The correspondent Young-Orlicz function for the space \( G\psi_m \) has a form

\[
M_m(y) = \exp \left( |y|^m \right), \; |y| > 1; \; M_m(y) = e \; y^2, \; |y| \leq 1.
\]

There holds for arbitrary function \( f \)

\[
||f||_{G\psi_m} \asymp ||f||\Phi(M_m) \asymp V(m),
\]

if of course as a capacity of the value \( V = V(m) \) we understand its minimal positive value from the relation (6.5).

The case \( m = 2 \) correspondent to the so-called subgaussian case, i.e. when

\[
T_f(y) \leq \exp \left\{ -\left( y/V(2) \right)^2 \right\}, \; y > 0.
\]
It is presumed as a rule in addition that the function \( f(\cdot) \) has a mean zero:
\[
\int_X f(x) \, \mu(dx) = 0.
\]
More examples may be found in [6], [25], [26], [28], [29], [30], [31], [36] etc.

We bring a more general example, see [26]. Let \( m = \text{const} > 1 \) and define \( q = m' = m/(m-1) \). Let also \( R = R(y), \ y > 0 \) be positive continuous differentiable slowly varying at infinity function such that
\[
\lim_{\lambda \to \infty} \frac{R(y/R)}{R(y)} = 1. \quad (6.6)
\]

Introduce a following \( \psi \) - function
\[
\psi_{m,L}(p) \overset{\text{def}}{=} p^{1/m} R^{-1/(m-1)} \left( p^{(m-1)^2/m} \right), \ p \geq 1, m = \text{const} > 1, \quad (6.7a)
\]
and the correspondent exponential tail function
\[
T^{(m,R)}(y) \overset{\text{def}}{=} \exp \left\{- y^m R^{m-1} \left( y^{m-1} \right) \right\}, \ y > 0. \quad (6.7b)
\]

The following implication holds true:
\[
0 \neq f \in G_{\psi_{m,L}} \iff \exists C = \text{const} \in (0, \infty), \ T_f(y) \leq T^{(m,R)}(y/C). \quad (6.8)
\]

A particular cases: \( R(y) = \ln^r(y + c), \ r = \text{const}, \ y \geq 0 \); then the correspondent generating functions has a form
\[
\psi_{m,r}(p) = p^{1/m} \ln^{-r}(p), \ p \in [2, \infty), \quad (6.9a)
\]
and the correspondent tail function
\[
T^{m,r}(y) = \exp \left\{- y^m (\ln y)^r \right\}, \ y \geq e. \quad (6.9b)
\]

For instance, for the Poisson distribution
\[
\psi(p) = \frac{C_R p}{e \ln p}, \ p \geq 2,
\]
and
\[
T(y) = \exp(-y \ln y), \ y \geq e.
\]

For the exponential distribution \( r = 0 \); therefore the correspondent \( \psi \) - function has a form \( \psi(p) = p \). For Gaussian distribution or more generally subgaussian one again \( r = 0 \) and \( m = 2 \).

More precisely, if
\[
||\xi||_{G_{\psi_{m,0}}} \leq 1, \iff |\xi|_p \leq p^{1/m}, \ p \geq 1,
\]
then
\[ T_\xi(y) \leq \exp \left\{ -(me)^{-1}y^m \right\}, \ y \geq 0. \]

The inverse conclusion is also true up to multiplicative constant.

**Example 6.2.** Bounded support of generating function.

Introduce the following tail function

\[ T^{<b,\gamma,R>}(x) \overset{def}{=} x^{-b} (\ln x)^\gamma R(\ln x), \ x \geq e, \quad (6.10) \]

where as before \( R = R(x), \ x \geq 1 \) is positive continuous slowly varying function as \( x \to \infty \), and

\[ b = \text{const} \in (1, \infty), \ \gamma = \text{const} > -1. \]

Introduce also the following (correspondent!) \( \Psi(b) \) function

\[ \psi^{<b,\gamma,R>}(p) \overset{def}{=} C_1(b, \gamma, R) (b - p)^{-\gamma - 1/b} R^{1/b} \left( \frac{1}{b - p} \right), \ 1 \leq p < b. \quad (6.11) \]

Let the measurable function (r.v.) \( f(\cdot) \) be such that

\[ T_f(y) \leq T^{<b,\gamma,R>}(y), \ y \geq e, \]

then

\[ |f|_p \leq C_2(b, \gamma, R) \ \psi^{<b,\gamma,R>}(p), \ p \in [1, b) \quad (6.12) \]

or equivalently

\[ ||f|| \in G^{\psi^{<b,\gamma,R>}} \iff ||f||G^{\psi^{<b,\gamma,R>}} < \infty. \quad (6.13) \]

Conversely, if the estimate (6.12) holds true, then

\[ T_f(y) \leq C_3(b, \gamma, L) \ y^{-b} (\ln y)^{\gamma + 1} R(\ln y), \ y \geq e \quad (6.14a) \]

or equally

\[ T_f(y) \leq T^{<b,\gamma+1,R>}(y/C_4), \ y \geq C_4 e. \quad (6.14b) \]

Notice that there is a logarithmic “gap” as \( y \to \infty \) between the estimations (6.13) and (6.14). The cause of this effect is following: the support on these \( \psi \) function is bounded, in contradiction to the other examples.

Wherein all the estimates (6.14) and (6.13) are non-improvable, see [26], [28], [32].

**Example 6.3.**

Let us consider the following \( \psi_{\beta,C}(p) \) function
\[ \psi_{\beta,C}(p) := \exp \left( C p^\beta \right), \quad C, \beta = \text{const} > 0. \] (6.15)

Obviously, the r.v. \( \tau \) for which

\[ \forall p \geq 1 \Rightarrow |\tau|^p \geq \psi_{\beta,C}(p) \]

does not satisfy the Cramer’s condition.

Let \( \xi \) be a r.v. belongs to the \( G_{\psi_{\beta,C}(\cdot)} \) space:

\[ ||\xi||_{G_{\psi_{\beta,C}}} = 1, \] (6.16a)

or equally

\[ |\xi|^p \leq \exp \left\{ C p^\beta \right\}, \quad p \in [1, \infty). \] (6.16b)

The last restriction is quite equivalent to the following tail estimate

\[ T_{\xi}(y) \leq \exp \left( -C_1(C, \beta) \left[ \ln(1 + y) \right]^{1+1/\beta} \right), \quad y > 0, \] (6.17)

and the following Orlicz norm finiteness

\[ ||\xi||_{\Phi(N_{\beta})} \leq C_2(C, \beta) < \infty, \]

where

\[ N_{\beta}(u) := \exp \left( C_3(C, \beta) \left[ \ln(1 + |u|) \right]^{1+1/\beta} \right), \quad |u| \geq 1. \] (6.18)

Remark 6.1. These GLS spaces are used, for example, for obtaining of an exponential estimates for sums of independent and dependent random variables and fields, estimations for non-linear functionals from random fields, theory of Fourier series and transform, theory of operators etc., see e.g. [5], [18], [25], [26], [29], sections 1.6, 2.1-2.5.

7 Exponential estimates for tail of multi-index sums.

Suppose in this section that the conclusions (5.9) and (5.9a) of theorem 5.1 holds true for all the values \( p \) belonging to some non-trivial interval \( p \in [1, b) \), finite or not: \( b = \text{const} \in (1, \infty] \):

\[ W_d[f](p) < \infty, \quad p \in [1, b) \] (7.1)

or as a particular case

\[ K^d_{R}(p) ||f||D_p < \infty, \quad p \in [1, b). \] (7.2)
Denote correspondingly

\[ v_{d,W}(f)(p) := p \ln [W_d(f)(p)] ; \quad v_{K,d}(f)(p) := p \ln [K_R^d(p) \|f\|D_p] , \quad (7.3) \]

so that

\[
\sup_{L:|L|\geq 1} |S_L|^p \leq v_{d,W}(f)(p), \quad \sup_{L:|L|\geq 1} |S_L|^p \leq v_{K,d}(f)(p)
\]

or equivalently

\[
\sup_{L:|L|\geq 1} ||S_L||Gv_{d,W} \leq 1, \quad \sup_{L:|L|\geq 1} ||S_L||Gv_{K,d} \leq 1. \quad (7.3a)
\]

**Proposition 7.1.**

It follows immediately from the theory of Grand Lebesgue Spaces, namely, from the estimate (6.2), the following exponential tail estimate

\[
\sup_{L:|L|\geq 1} T_{S_L[f]}(u) \leq \exp \left(-v_{d,W}^*(f)(\ln u) \right), \quad u \geq e, \quad (7.4)
\]

and following as a particular case

\[
\sup_{L:|L|\geq 1} T_{S_L[f]}(u) \leq \exp \left(-v_{K,d}^*(f)(\ln u) \right), \quad u \geq e. \quad (7.5)
\]

The relations (7.4) and (7.5) may be rewritten on the terms of Orlicz spaces, in accordance with (6.2), (6.2a), (6.3), as follows. Introduce the following correspondent Young-Orlicz functions

\[
N_1(y) = \exp \left(v_{d,W}^*(\ln |y|) \right), \quad |y| \geq e; \quad N_1(y) = C_1y^2, \quad |y| < e;
\]

\[
N_2(y) = \exp \left(v_{K,d}^*(\ln |y|) \right), \quad |y| \geq e; \quad N_2(y) = C_2y^2, \quad |y| < e.
\]

and correspondent Orlicz’s norms \( ||f||\Phi_{N_j}, \; j = 1, 2 \) defined for the measurable functions (random variables) with support on the source probability space.

The inequalities (7.4) and (7.5) are completely correspondingly equivalent up to multiplicative constants under at the same conditions to the Orlicz norms estimates

\[
\sup_{L:|L|\geq 1} ||S_L||\Phi(N_1) \leq C_3 < \infty,
\]

\[
\sup_{L:|L|\geq 1} ||S_L||\Phi(N_2) \leq C_4 < \infty.
\]

Let us bring some examples.
Example 7.0. Let the function $f : X \rightarrow R$ be from the representation (5.2):

$$f(\vec{x}) = \sum_{\vec{k} \in \mathbb{Z}_+^d, N(\vec{k}) \leq M} \lambda(\vec{k}) \prod_{j=1}^d g^{(j)}_{k_j}(x_j),$$

(5.2a)

for some constant integer value $M$, finite or not, where all the functions $g^{(j)}(\cdot)$ are in turn centered: $E g^{(j)}_k(\xi(k)) = 0$. Recall the denotation: $M = \deg[f]$.

Suppose here and in what follows in this section that

$$\sum_{\vec{k} \in \mathbb{Z}_+^d, N(\vec{k}) \leq M} |\lambda(\vec{k})| \leq 1$$

and that each the (centered) r.v. $g^{(j)}_k(\xi(k))$ belongs to some $G\psi_k$ – space uniformly relative the index $j$:

$$\sup_j |g^{(j)}_k(\xi(k))|_p \leq \psi_k(p).$$

Of course, as a capacity of these functions may be picked the natural functions for the r.v. $g_k(\xi(k))$:

$$\psi_k(p) \overset{def}{=} \sup_j |g^{(j)}_k(\xi(k))|_p,$$

if the last function is finite for some non-trivial interval $[2, a(k))$, where $a(k) \in (2, \infty]$.

Obviously,

$$|f(\vec{\xi})|_p \leq \prod_{k=1}^d \psi_k(p),$$

and the last inequality is exact if for instance $M = 1$ and all the functions $\psi_k(p)$ are natural for the family of the r.v. $g^{(j)}_k(\xi(k))$.

Define the following $\Psi$ – function

$$\zeta(p) = \zeta_{d[\vec{\xi}]}(p) \overset{def}{=} K^d_{\Psi}(p) \cdot \prod_{k=1}^d \psi_k(p).$$

The assertion of proposition (7.1) gives us the estimations

$$\sup_{L : |L| \geq 1} ||S_L[f]||G\zeta \leq 1$$

(7.6a)

and hence

$$\sup_{L : |L| \geq 1} T_{S_L[f]}(u) \leq \exp \left(-v^*_\zeta(\ln u)\right), \quad u \geq e,$$

(7.6b)

with correspondent Orlicz norm estimate.
Example 7.1.
Suppose again that
\[ \sum_{\vec{k} \in \mathbb{Z}^d_+, N(\vec{k}) \leq M} |\lambda(\vec{k})| \leq 1 \] (7.7)
and that the arbitrary r.v. \( g_k^{(j)}(\xi(k)) \) belongs uniformly relative the index \( j \) to the correspondent \( G_{\psi_{m(k),\gamma(k)}} \) space:
\[
\sup_j | g_k^{(j)}(\xi(k)) |_p \leq p^{1/m(k)} [\ln p]^{\gamma(k)}, \quad p \geq 2, \quad m(k) > 0, \quad \gamma(k) \in \mathbb{R},
\]
(7.8a)
or equally
\[
\sup_j T_{g_k^{(j)}(\xi(k))}(u) \leq \exp \left( -C(k) \ u^{m(k)} [\ln u]^{-\gamma(k)} \right), \quad u > e.
\]
(7.8b)
Define the following variables:
\[
m_0 := \left[ d + \sum_{k=1}^d 1/m(k) \right]^{-1}, \quad \gamma_0 := \sum_{k=1}^d \gamma(k) - d,
\]
(7.9)
\[
\hat{S}_L = \hat{S}_L[f] := e^d C_R^{-d} S_L.
\]
(7.10)
We conclude by means of the proposition 7.1
\[
\sup_{L: |L| \geq 1} \| \hat{S}_L \|_{G_{\psi_{m_0,\gamma_0}}} \leq 1
\]
(7.11a)
and therefore
\[
\sup_{L: |L| \geq 1} T_{\hat{S}_L}(u) \leq \exp \left\{ -C(d, m_0, \gamma_0) \ u^{m_0} (\ln u)^{-\gamma_0} \right\}, \quad u > e,
\]
(7.11b)
or equally
\[
\sup_{L: |L| \geq 1} \| \hat{S}_L[f] \|_{\Phi_{m_0,\gamma_0}} \leq C_1(d, m_0, \gamma_0) < \infty,
\]
(7.11c)
where the Young-Orlicz function \( \Phi_{m,\gamma}(\cdot) \) is defined as follows
\[
\Phi_{m,\gamma}(u) \overset{def}{=} \exp \left( u^m [\ln |u|]^{-\gamma} \right), \quad |u| \geq e.
\]
Example 7.2.
Let us consider as above the following \( \psi_{\beta}(p) \) function
\[
\psi_{\beta,C}(p) := \exp \left( C p^\beta \right), \quad C, \quad \beta = \text{const} > 0,
\]
(7.12)
see example 6.3, (6.15) - (6.17).
Let \( g_j^{(j)}(\xi(k)) \) be centered independent random variables belonging to the certain \( \mathcal{G}_{\psi,C}^\beta,\cdot \) space uniformly relative the indexes \( k,j \):

\[
\sup_j \sup_k ||g_j^{(j)}(\xi(k))||_{\mathcal{G}_{\psi,C}^\beta} = 1, \tag{7.13a}
\]
or equally

\[
\sup_j \sup_k T_{g_j^{(j)}(\xi(k))}(y) \leq \exp \left( -C_1(C, \beta) \left[ \ln(1 + y) \right]^{1+1/\beta} \right), \quad y > 0. \tag{7.13b}
\]

Then

\[
\sup_{L: |L| \geq 1} T_{S_L}(y) \leq \exp \left( -C_2(C, \beta) \left[ \ln(1 + y) \right]^{1+1/\beta} \right), \quad y > 0, \tag{7.14a}
\]
or equally

\[
\sup_{L: |L| \geq 1} ||S_L[f]||_{\mathcal{G}_{\psi,C}^\beta} < \infty, \tag{7.14b}
\]
or equally

\[
\sup_{L: |L| \geq 1} ||S_L[f]||_{\Theta(C, \beta)} \leq C_5(C, \beta) < \infty, \tag{7.14c}
\]

where (we recall)

\[
\Phi_\beta(u) = \exp \left( C_6(C, \beta) \left[ \ln(1 + |u|) \right]^{1+1/\beta} \right), \quad |u| \geq 1.
\]

**Example 7.3.** Suppose now that the each centered random variable \( g_k^{(j)}(\xi(k)) \) belongs uniformly relative the index \( j \) to certain \( \mathcal{G}_{\psi,b(k),\theta(k)}^\beta \) space, where \( b(k) \in (2, \infty), \theta(k) \in \mathbb{R} \):

\[
\sup_j ||g_k^{(j)}(\xi(k))||_{\mathcal{G}_{\psi,b(k),\theta(k)}^\beta} < \infty, \tag{7.15}
\]

where (we recall)

\[
\psi^{b(k),\theta(k)}(p) \overset{\text{def}}{=} C_1(b(k), \theta)(b(k) - p)^{-(\theta(k)+1)/b(k)}, \quad 1 \leq p < b(k). \tag{7.16}
\]

This case is more complicates than considered before.

Note that if the r.v. \( \eta \) satisfies the inequality

\[
T_\eta(y) \leq C \ y^{-b(k)} \left[ \ln y \right]^{\theta(k)}, \quad y \geq e,
\]

then \( \eta \in \mathcal{G}_{\psi,b(k),\theta(k)}^\beta \), see the example 6.2.

One can assume without loss of generality

\[
b(1) \leq b(2) \leq b(3) \leq \ldots b(d). \tag{7.16}
\]
Denote
\[ \nu_k(p) := \psi_{<b(k), \theta(k)>}(p), \quad b(0) := \min_k b(k), \] (7.17)
so that \( b(0) = b(1) = \)
\[ b(2) = \ldots = b(k(0)) < b(k(0) + 1) \leq \ldots \leq b(d), \quad 1 \leq k(0) \leq d; \] (7.18)
\[ \Theta := \sum_{k=1}^{k(0)} (\theta(k) + 1)/b(0), \] (7.18a)
\[ v(p) = v_\xi[f](p) \overset{\text{def}}{=} \prod_{t=1}^{k(0)} \nu_t(p) = C \cdot [b(0) - p]^{-\Theta}, \quad 2 \leq p < b(0). \] (7.19)

Obviously,
\[ K^d_R(p) \prod_{k=1}^{d} \nu_k(p) \leq C \cdot v(p) = C \cdot [b(0) - p]^{-\Theta}, \quad C = C_d(\xi, b, \tilde{\theta}, k(0)). \] (7.20)

Thus, we obtained under formulated above conditions
\[ \sup_{L: |L| \geq 1} |S_L|_p \leq C_2 (b(0) - p)^{-\Theta}, \quad p \in [2, b(0)) \] (7.21)
with correspondent tail estimate
\[ \sup_{L: |L| \geq 1} T_{S_L}(y) \leq C_3 y^{-b(0)} \cdot [\ln y]^{b(0) \Theta}, \quad y \geq e. \] (7.22)

8 Banach space valued Non-Central Limit Theorem for multi-index sums.

Let \( V = \{ v \} \) be certain compact metrizable space; the concrete choosing of the distance function will be clarified below. We consider in this section the case when the “kernel” centered function \( f \) in (5.1) dependent continuously also on the additional (deterministic!) parameter \( v : v \in V : f = f(v, \xi) \) and when the (generalized) sequence \( S_L[f] \) converges in distribution as \( \min n(k) \to \infty \) in the space of continuous functions \( C(V) \) to the non-degenerate parametric multiple stochastic integral relative the Gaussian white measure, alike the fourth section for the ordinary one-dimensional axis.

The one-dimensional parametric case \( d = 1 \) correspondent to the classical Central Limit Theorem in the space of continuous functions, i.e. when the sequence
$S_L$ converges weakly to the Gaussian continuous centered random field, [27], [28], [29].

See also the previous articles: [19], [34].

More details. $X := \otimes_{i=1}^d X_i$, $f : V \otimes X \to R$ be again measurable (r.v.) for each value $v \in V$ centered function, i.e. random process (field) such that $\forall v \in V \; \mathbb{E}f(v, \xi) = 0$;

\[
Q_L[f](v) := |L|^{-1/2} \sum_{i \in L} f\left(v, \overline{\xi}_i\right).	ag{8.1}
\]

In what follows the kernel function $f : V \times X \to R$ will be some degenerate parametric generalization of the the representation (5.2):

\[
f(x) = \sum_{k \in \mathbb{Z}_+^d, N(k) \leq M} \lambda(v, \overline{k}) \prod_{j=1}^d g_k^{(j)}(x_j),	ag{8.1a}
\]

for some constant integer value $M$, finite or not, where all the functions $g_k^{(j)}(\cdot)$ are in turn centered: $\mathbb{E}g_k^{(j)}(\xi(k)) = 0$. Retain the denotation: $M = \text{deg}[f]$.

We impose again without loss of generality that in the expression (8.1a) the function $g_k^{(s)}(\cdot)$ are (common) orthonormal:

\[
\mathbb{E}g_k^{(s)}(\xi(j)) g_l^{(t)}(\xi(l)) = \delta_k^l \cdot \delta_s^t, \; s, t = 1, 2, \ldots, d; \tag{8.2a}
\]

$\delta_k^l$ is Kronecker’s symbol.

Further, we suppose that the set $L, L \subset \mathbb{Z}_+^d$ is a parallelepiped:

\[
L = [1, n(1)] \otimes [1, n(2)] \otimes \ldots \otimes [1, n(d)],	ag{8.2b}
\]

then $|L| = \prod_{m=1}^d n(m)$; and that

\[
\min_l n(l) \to \infty. \tag{8.2c}
\]

Let us introduce again the following array $\beta = \{\beta^{(s)}(\overline{k})\}, \; s = 1, 2, \ldots, d; \; \overline{k} \in L$ consisting on the (common) independent standard normal (Gaussian) distributed random variables, defined on certain sufficiently rich probability space. Define the following multiple parametric stochastic integral relative the Gaussian random white measure with non-random square integrable parametric integrand:

\[
Q_\infty(v) \overset{\text{def}}{=} \sum_{\overline{k} \in L} \lambda(v, \overline{k}) \cdot \prod_{s=1}^d \beta^{(s)}(k_s),	ag{8.2}
\]

We intend to ground the weak convergence in the space of continuous functions $C(V)$ the random fields $Q_L[f](\cdot)$ to one $Q_\infty[f](\cdot)$, write $f = f(v, \xi) \in NCLT$, presuming of course its continuity. More detail: for arbitrary bounded continuous functional $F : C(V) \to R$

\[
\lim_{\min_k n(k) \to \infty} \mathbb{E}F(Q_L[f](\cdot)) = \mathbb{E}F(Q_\infty[f](\cdot)). \tag{8.3}
\]
The term (notion) NCLT was introduced for the first time in the article [7]. See also [35].

For the parametric $U-$statistics this problem was investigated, e.g., in [33], [34] by means of martingale representations. We will use here some other methods: degenerate approximation approach, metric entropy technique etc.

We will investigate as a capacity of such a functions $\{f\}$ again the degenerate ones. By definition, as above, the function $f: V \otimes X \to R$ is said to be parametric degenerate, iff it has the form

$$f(v, \bar{x}) = \sum_{k \in \mathbb{Z}^d_+, N(k) \leq M} \lambda(v, \bar{k}) \prod_{j=1}^d g^{(j)}_{k_j}(x_j), \quad (8.4)$$

for some integer constant value $M$, finite or not, where all the functions $g^{(j)}_{k}(\cdot)$ are in turn centered: $E g^{(j)}_{k}(k) = 0$.

Some new notation:

$$\sigma = \sigma_\lambda := \sup_{v \in V} \sum_{k \in \mathbb{Z}^d_+} |\lambda(v, \bar{k})|; \quad (8.5)$$

$$\rho(v_1, v_2) = \rho_\lambda(v_1, v_2) := \sum_{k \in \mathbb{Z}^d_+} \left| \lambda(v_1, \bar{k}) - \lambda(v_2, \bar{k}) \right|, \quad v_1, v_2 \in V. \quad (8.6)$$

The function $(v_1, v_2) \to \rho(v_1, v_2)$ is quasi - distance function on the set $V$. The metric entropy function $H(V, r, \epsilon)$ for arbitrary such a distance - function $r = r(v_1, v_2)$ is defined as ordinary as a logarithm of a minimal set of points $\{v(i)\}, \ v(i) \in V$, which forms an epsilon set in the whole space $(V, r)$. Denote also

$$N(V, r, \epsilon) = \exp H(V, r, \epsilon).$$

It entails from the Haussdorf’s theorem that the value $N(V, r, \epsilon)$ is finite for any $\epsilon > 0$ if the space $(V, r)$ is pre-compact set. One can suppose in the sequel that the set $V$ relative the distance $\rho_\lambda$ will be complete, therefore will be compact set.

**Theorem 8.1.** (Power level). Suppose that for some certain value $p \in [2, \infty) \Rightarrow G(p) < \infty$ and that the following so-called entropic integral converges

$$I := \int_0^1 N^{1/p}(V, K^{d}_R(p) \ G(p) \ \rho_\lambda, \epsilon) \ d\epsilon < \infty. \quad (8.7)$$

Then the r.v. $f = f(v, \bar{x})$ satisfies the NCLT in the space of continuous functions $C(V, \rho)$ and wherein

$$\sup_{L: |L| \geq 1} \ E \sup_{v \in V} |Q_L(v)|^p < \infty. \quad (8.7a)$$

**Proof.** The convergence of all the finite - dimensional distributions for the sequence of r.f. $Q_L(v)$ to ones for $Q_\infty(v)$ follows immediately from theorem 5.2. The continuity a.e. of the limiting r.f. $Q_\infty(v)$ follows from the main result of an
article [35]; continuity almost everywhere of the r.f. \( Q_L(v) \), as well as its weak compactness its distributions in the space of continuous functions \( C(V, \rho) \) may be simple deduced from theorem of Pizier [38], as long as

\[
\sup_{L} \sup_{v \in V} | Q_L(v) |_p \leq K^d_R(p) G(p) \sigma \lambda < \infty,
\]

\[
\sup_{L} | Q_L(v_1) - Q_L(v_2) |_p \leq K^d_R(p) G(p) \rho \lambda(v_1, v_2).
\]

In order to obtain the so-called exponential level for NCLT, we need to introduce some new notations and assumptions.

Suppose that each the r.v. \( g_k^{(s)}(\xi(k)) \) belongs to some \( G\psi \) space:

\[
\max_{s} | g_k^{(s)}(\xi(k)) |_p \leq v_k(p), \ 2 \leq p < a(k).
\]

Then

\[
G(p) \leq \prod_{k=1}^{d} v_k(p), \ 2 \leq p < a \overset{def}{=} \min_k a(k) \in (2, \infty].
\]

Therefore

\[
\sup_{L} \sup_{v \in V} | Q_L(v) |_p \leq K^d_R(p) G(p) \sigma \lambda,
\]

and analogously

\[
\sup_{L} | Q_L(v_1) - Q_L(v_2) |_p \leq K^d_R(p) G(p) \cdot \rho \lambda(v_1, v_2). \tag{8.9b}
\]

Let us introduce a new \( G\psi \) – function \( \tau = \tau(p) \) as follows

\[
\tau(p) = \tau_{[f]}(p) := K^d_R(p) G(p), \ p \in [2, a). \tag{8.10}
\]

Evidently, \( \tau(\cdot) \in \Psi \). Both the estimates (8.9a) and (8.9b) may be rewritten as follows.

\[
\sup_{L} \sup_{v \in V} \| Q_L(v) \| G\tau \leq \sigma \lambda, \tag{8.10a}
\]

and analogously

\[
\sup_{L} \| Q_L(v_1) - Q_L(v_2) \| G\tau \leq \rho \lambda(v_1, v_2). \tag{8.10b}
\]

Denote

\[
z(y) := \ln \tau(1/y), \ w(x) := \inf_{y>0} (xy + z(y)),
\]

\[
J := \int_{0}^{1} \exp(w(H(V, \rho, \epsilon))) \ d\epsilon - \tag{8.11}
\]
be again the (generalized) entropic integral.

**Theorem 8.2.** (Exponential level). Suppose that for some value $a \in (2, \infty]$ \( \forall p \in [2, a) \Rightarrow G(p) < \infty \) and that \( \sigma(\lambda) < \infty, J < \infty \). Then the r.f. \( f = f(v, \xi) \) satisfies the NCLT in the space of continuous functions \( C(V, \rho) \) and wherein

\[
\sup_{L: |L| \geq 1} \left| \sup_{v \in V} Q_L(v) \right| G_\tau < \infty.
\]

(8.12)

**Proof** is at the same as above in theorem 8.1; the more general than Pisier's sufficient condition for weak compactness of the family of the continuous random fields defined on some metrizable compact space is obtained, e.g., in the monograph [29], chapter 4, section 4.3.

**Remark 8.1.** The condition \( J < \infty \) is satisfied if for instance the set \( V \) is bounded closure of convex open set in the Euclidean space \( R^l \) equipped with ordinary norm \( |v| \) and the distance function \( \rho(\cdot, \cdot) \) is such that

\[
\rho(v_1, v_2) \leq C |v_1 - v_2|^{\alpha}, \quad \alpha > 0.
\]

In this case

\[
N(V, \rho, \epsilon) \leq C \epsilon^{l/\alpha}, \quad \epsilon \in (0, 1).
\]

**Example 8.2.** Suppose in (8.11) that the function \( \tau = \tau(p) \) coincides with one \( \psi_{m,r}(p), \quad m = \text{const} > 0, \quad r = \text{const} \in R \). The condition \( J < \infty \) in (8.11) takes a form

\[
\int_0^1 H^{1/m}(V, \rho, \epsilon) \left[ \ln H(V, \rho, \epsilon) \right]^{-r} d\epsilon < \infty.
\]

(8.13)

9 Upper bounds for these statistics.

**A.** A simple lower estimate in the Klesov’s (3.4) inequality may has a form

\[
\sup_{L: |L| \geq 1} \left| S_L^{(2)} \right|_p \geq \left| S_1^{(2)} \right|_p = |g(\xi)|_p |h(\eta)|_p, \quad p \geq 2,
\]

(9.1)

as long as the r.v. \( g(\xi), h(\eta) \) are independent.

Suppose now that \( g(\xi) \in G\psi_1 \) and \( h(\eta) \in G\psi_2 \), where \( \psi_i \in \Psi(b), \quad b = \text{const} \in (2, \infty] \); for instance \( \psi_j, \quad j = 1, 2 \) must be the natural functions for these r.v. Put \( \nu(p) = \psi_1(p) \psi_2(p) \); then

\[
\nu(p) \leq \sup_{L: |L| \geq 1} \left| S_L^{(2)} \right|_p \leq K_R^2(p) \nu(p).
\]

(9.2)
Assume in addition that \( b < \infty \); then \( K_R^2(p) \leq C(b) < \infty \). We get to the following assertion.

**Proposition 9.1.** We deduce under formulated above in this section conditions

\[
1 \leq \frac{\sup_{L: |L| \geq 1} |S_L|}{\nu(p)} \leq C(b) < \infty, \quad p \in [2, b).
\] (9.3)

**B. Tail approach.** We will use the example 7.2 (and notations therein.) Suppose in addition that all the (independent) r.v. \( \xi(k) \) have the following tail of distribution

\[
T_{\xi(k)}(y) = \exp \left( -[\ln(1 + y)]^{1+1/\beta} \right), \quad y \geq 0, \quad \beta = \text{const} > 0,
\]

i.e. an unbounded support. As we knew,

\[
\sup_{L: |L| \geq 1} T_{S_L}(y) \leq \exp \left( -C_5(\beta, d) [\ln(1 + y)]^{1+1/\beta} \right), \quad y > 0,
\]

see (7.14a). On the other hand,

\[
\sup_{L: |L| \geq 1} T_{S_L}(y) \geq T_{S_1}(y) \geq \exp \left( -C_6(\beta, d) [\ln(1 + y)]^{1+1/\beta} \right), \quad y > 0.
\] (9.4)

**C. An example.** Suppose as in the example 7.1 that the independent centered r.v. \( g^{(j)}(\xi(k)) \) have the standard Poisson compensated distribution: \( \text{Law}(\xi(k) + 1) = \text{Poisson}(1) \), \( k = 1, 2, \ldots, d \). Assume also that in the representation (5.2a) \( M = 1 \) (a limiting degenerate case). As long as

\[
|g^{(j)}(\xi(k))|_p \asymp \frac{p}{\ln p}, \quad p \geq 2,
\]

we conclude by virtue of theorem 5.1

\[
\sup_{L: |L| \geq 1} |S_L| \leq C_2^d \frac{p^{2d}}{[\ln p]^{2d}}, \quad p \geq 2,
\] (9.5)

therefore

\[
\sup_{L: |L| \geq 1} T_{S_L}(y) \leq \exp \left( -C_1(d) y^{1/(2d)} [\ln y]^{2d} \right), \quad y \geq e.
\] (9.6)

On the other hand,

\[
\sup_{L: |L| \geq 1} |S_L| \geq |S_1| \geq C_3(d) \frac{p^d}{[\ln p]^d},
\]

and following

\[
\sup_{L: |L| \geq 1} T_{S_L}(y) \geq \exp \left( -C_4(d) y^{1/d} [\ln y]^d \right), \quad y \geq e.
\] (9.7)
10 Concluding remarks.

A. It is interesting by our opinion to generalize obtained in this report results onto the mixing sequences or onto martingales, as well as onto the multiple integrals instead of sums.

B. Perhaps, a more general results may be obtained by means of the so-called method of majorizing measures, see [1]-[3], [8]-[10], [16], [27], [36], [39]-[43].

C. Possible applications: statistics and Monte-Carlo method, alike [13], [15] etc.

D. It is interesting perhaps to generalize the assertions of theorems 4.2 and 4.3 onto the sequences of domains \( \{ L \} \) tending to “infinity” in the van Hove sense, in the spirit of an article [6].

E. A simple qualitative analysis of limit theorems (proposition 6.4) shows us that the speed of convergence in regular case is equal to \( O[\min_s n(s)]^{-1/2} \).

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