Higher derivative scalar-tensor theory through a non-dynamical scalar field

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Abstract. We propose a new class of higher derivative scalar-tensor theories without the Ostrogradsky’s ghost instabilities. The construction of our theory is originally motivated by a scalar field with spacelike gradient, which enables us to fix a gauge in which the scalar field appears to be non-dynamical. We dub such a gauge as the spatial gauge. Though the scalar field loses its dynamics, the spatial gauge fixing breaks the time diffeomorphism invariance and thus excites a scalar mode in the gravity sector. We generalize this idea and construct a general class of scalar-tensor theories through a non-dynamical scalar field, which preserves only spatial covariance. We perform a Hamiltonian analysis and confirm that there are at most three (two tensors and one scalar) dynamical degrees of freedom, which ensures the absence of a degree of freedom due to higher derivatives. Our construction opens a new branch of scalar-tensor theories with higher derivatives.

Keywords: gravity, modified gravity

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1 Introduction

A scalar-tensor theory gives an important framework to describe inflation and dark energy. The action of a scalar field with the canonical kinetic term and a potential (together with the Einstein-Hilbert action) has been long considered as the standard one. Later, a more general action consisting of an arbitrary function of a scalar field and its canonical kinetic term was proposed as candidates to describe inflation dubbed as k-inflation [1] and dark energy dubbed as k-essence [2, 3]. This action is the most general single scalar field one consisting of a scalar field and its first derivatives.

Then, it is a natural direction to consider an action including higher derivative terms. Unfortunately, it was shown more than 150 years ago that a non-singular system with higher-than-first time derivatives leads to ghost instabilities [4–6]. Thus, in order to obtain a healthy theory with higher derivatives, we need to consider singular (degenerate) system. Recently, the systematic construction of a healthy point-particle theory with higher time derivative terms [7–12] as well as their field extensions [13, 14] have been given.

One of the important examples of such a ghost-free field theory is the Galileon theory in the Minkowski background [15], which respects the so-called Galilean shift symmetry and gives the most general action whose (Euler-Lagrange) equations of motion are second order even though their action has second order derivative terms. Surprisingly, the scalar-tensor correspondence of this Galileon theory, which is the most general single field scalar-tensor theory with second order (Euler-Lagrangean) equations of motion, had been found by Horndeski more than 40 years ago [16] and this theory was rediscovered in the context of generalized Galileon theory [17], of which the equivalence to the Horndeski theory was shown in [18].
However, the Horndeski theory was shown to be not the most general single field scalar-tensor theory without ghost instabilities, because the requirement to the Horndeski theory, i.e., the equations of motion are of second order in derivatives, is too strong. In fact, Gleyzes et al. explicitly constructed theories beyond the Horndeski scope without ghost instabilities [19, 20]. Langlois and Noui further extended this direction by paying particular attention to the importance of the degeneracy of the kinetic matrix and proposed the degenerate higher order scalar tensor (DHOST) theory [21–23, 23, 24]. Though the non-trivial branch of DHOST theories suffers from gradient instabilities for cosmological perturbations [25, 26], the corresponding vector-tensor theory has stable cosmological solutions [27, 28].

Another interesting yet more powerful approach to obtaining a healthy scalar-tensor theory with higher derivatives is to focus on specific configurations (vacuum expectation values) of the scalar field, which explicitly breaks the local Lorentz symmetry (or general covariance). Theories in this direction include the effective field theory (EFT) of inflation [29–31] and Hořava-Lifshitz theory [32, 33], where only the spatial symmetries are respected. Interestingly, when choosing the so-called unitary gauge by fixing $t = \phi(t, \vec{x})$, the Horndeski Lagrangian can be recast in a form similar to that of the EFT of inflation for cosmological perturbations [34]. In [35, 36], one of the authors in this work proposed a general framework to construct scalar-tensor theories with keeping only spatial symmetries in a systematic way, which manifestly avoids the ghost instabilities associated with higher time derivative terms while arbitrarily higher spatial derivatives can be introduced.\(^1\) Recently, further extension including the time derivative of the lapse function has been made [38]. It should be noticed that this theory is constructed based on the (e.g. constant time) spacelike hypersurface given by a timelike/spacelike scalar field. By timelike/spacelike scalar field, we refer to a scalar field with timelike/spacelike gradient.

In the current work, we will pursue the opposite direction, where the theory is constructed based on a spacelike scalar field originally. In the next section, we first investigate the Horndeski theory in which the scalar field is spacelike. This spacelike scalar field allows us to fix a gauge, which we dub as the spatial gauge, in which the scalar field loses dynamics and appears to be non-dynamical. The most important observation is that, once the action of the theory is given, one can extend our theory by regarding that the scalar field can be not only spacelike but also timelike. As the result, our construction appears to be a novel class of spatially covariant scalar-tensor theories alternative to those proposed in [35, 36, 38]. The crucial ingredient in our construction is the scalar field, which does not have a kinetic term from the construction and hence is non-dynamical. Although this scalar field itself might not have dynamics, its presence (together with the gauge fixing) breaks general covariance and thus induces an alternative dynamical scalar degree of freedom in the gravity sector, which might have novel features.

The idea of having matter fields with time-independent but space-dependent background configurations in the cosmological situation was firstly introduced in the “elastic inflation” [39] and further systematically investigated in “solid inflation” [40], where the scalar fields are Nambu-Goldstone bosons associated with breaking of spatial diffeomorphism. This is similar to the usual EFT of inflation [29–31], where the adiabatic mode of matter perturbations is associated with the breaking of time diffeomorphism.\(^2\) When apparently recovering the general covariance by employing the Stückelberg trick (see e.g., [50–52] for the Hořava-Lifshitz gravity and [53–55] for the massive and bi-gravity), our construction corresponds to a new class of healthy scalar-tensor theories with higher derivatives.

\(^1\)See ref. [37] for the discussions when this theory apparently recovers general covariance.

\(^2\)Effective theories with general spacetime symmetry breaking have also been investigated [41–49].
\( n \mu \nabla \mu \phi \Sigma t \phi = \text{const.} \)

Figure 1. Illustration of the basic picture of the spatial gauge. The vertical and horizontal hypersurfaces represent the constant \( \phi \) hypersurfaces and the spatial sector in our 3+1 decomposition, respectively.

The organization of this paper is as follows. In the next section, after briefly giving our idea, we will give concrete expressions of the Horndeski theory in the spatial gauge as an example illuminating our idea. A complicated part of this result and another interesting example with higher order derivative terms are given in appendix A and B. In the section 3, the general framework of our theory will be given. In the section 4, we will make Hamiltonian analysis for our theory and count the number of the degrees of freedom, which yields only three (two tensors and one scalar) dynamical degree of freedom without Ostrogradsky’s ghosts. Final section is devoted to the conclusion.

2 Spatial gauge

2.1 Spatial gauge

In order to clarify our idea, we assume that the scalar field \( \phi \) possesses a spacelike gradient \( \nabla \mu \phi \) that is non-vanishing everywhere in spacetime. The basic picture is depicted in figure 1. Since \( \nabla \mu \phi \) is spacelike, it is possible to choose a set of timelike curves on the constant \( \phi \) hypersurfaces, of which the tangent vector is \( n^\mu \). By definition the Lie derivative of \( \phi \) along \( n^\mu \) is vanishing

\[ \mathcal{L}_{n^\mu} \phi \equiv 0. \]  

Without loss of generality, we can normalize \( n^\mu \) so that \( n_\mu n^\mu = -1 \). We assume there are spacelike hypersurfaces \( \Sigma_t \) which are orthogonal to \( n^\mu \). We now fix the coordinates. We use \( t \) as a time coordinate and \( x^i \) as arbitrary spatial coordinates on each foliation \( \Sigma_t \). Thus, fixing on the time coordinate through \( n^\mu \) effectively breaks time diffeomorphism invariance, which leaves only spatial diffeomorphism invariance in our theory. We refer the induced metric on \( \Sigma_t \) to \( h_{\mu\nu} \) which is defined through

\[ h_{\mu\nu} := g_{\mu\nu} + n_\mu n_\nu. \]  

As usual, we define the lapse and shift functions through the decomposition of the time vector \( t^\mu := (\partial_t)^\mu \),

\[ t^\mu := N n^\mu + N^\mu. \]
Due to the ambiguity of the choice of spatial coordinates, the time vector $t^\mu$ has not yet been specified at this moment, which may or may not be parallel to the constant $\phi$ hypersurfaces depending on the shift functions.

We refer to the above choice of time slices $\Sigma_t$ and in particular the condition (2.1) as the \textit{spatial gauge}. Note the gradient of the scalar field $\nabla_\mu \phi$ is split into two parts: the timelike part $\mathcal{L}_n \phi$ and the spacelike part $D_\mu \phi := h_\mu^\nu \nabla_\nu \phi$. If the scalar field is timelike, one is able to fix $D_\mu \phi \equiv 0$, which is dubbed as the unitary gauge in the literature. Thus the spatial gauge is the natural counterpart to the unitary gauge when the scalar field is spacelike.

2.2 Horndeski theory in the spatial gauge

In order to find the general form of healthy theories in the spatial gauge, it is useful to derive the gauge-fixed form of a healthy covariant theory in the spatial gauge. The starting point is the Horndeski theory [16], which describes the most general single-field scalar-tensor theory with second-order equations of motion in four-dimensional spacetime. The action is reformulated in [17, 18]

$$\int d^4x \sqrt{-g} L^H = \int d^4x \sqrt{-g} \left( L_2^H + L_3^H + L_4^H + L_5^H \right),$$

with

$$L_2^H = G_2 (X, \phi),$$

$$L_3^H = G_3 (X, \phi) \Box \phi,$$

$$L_4^H = G_4 (X, \phi) ^4R + \frac{\partial G_4}{\partial X} \left[ (\Box \phi)^2 - (\nabla_\mu \nabla_\nu \phi)^2 \right],$$

$$L_5^H = G_5 (X, \phi) \psi_{\mu \nu} \nabla_\mu \nabla_\nu \phi - \frac{1}{6} \frac{\partial G_5}{\partial X} \left[ (\Box \phi)^3 - 3 \Box \phi (\nabla_\mu \nabla_\nu \phi)^2 + 2 (\nabla_\mu \nabla_\nu \phi)^3 \right],$$

where

$$X \equiv - \frac{1}{2} (\nabla \phi)^2,$$

and $\Box \phi \equiv \nabla_\mu \nabla^\mu \phi$, $^4R$ and $\psi_{\mu \nu}$ are the four-dimensional Ricci scalar and Einstein tensor, respectively.

Our procedure composes of two steps. First we perform the standard 3+1 decomposition with spatial hypersurfaces specified by $n^\mu$. The basic relations are

$$\nabla_\mu \phi = -n_\mu \mathcal{L}_n \phi + D_\mu \phi,$$

and

$$\nabla_\mu \nabla_\nu \phi = n_\mu n_\nu \left( \mathcal{L}_n^2 \phi - a^\rho \mathcal{D}_\rho \phi \right) - 2 n_\mu \left( \mathcal{D}_\nu \mathcal{L}_n \phi - K_\nu^\rho \mathcal{D}_\rho \phi \right) - \mathcal{L}_n \phi K_{\mu \nu} + D_\mu D_\nu \phi,$$

as well as the Gauss-Codazzi relations. Here $\mathcal{D}_\mu$ stands for the covariant derivative with respect to the induced metric $h_{\mu \nu}$. In (2.11), $D_\mu \phi \equiv h_\mu^\nu \nabla_\nu \phi$, $\mathcal{L}_n \phi \equiv n^\mu \nabla_\mu \phi$, the acceleration $a_\mu$, and the extrinsic curvature $K_{\mu \nu}$ are introduced by

$$\nabla_\mu n_\nu = -n_\mu a_\nu + K_{\mu \nu},$$
where \( a_\mu = n^\nu \nabla_\nu n_\mu \). Then we take the spatial gauge (i.e., coordinates) by choosing \( \mathcal{L}_n \phi = 0 \). It immediately follows that in the spatial gauge, terms involving \( \mathcal{L}_n \phi \) and its spatial derivatives drop out and we are left with

\[
\nabla_\mu \phi \rightarrow \delta_\mu^i D_i \phi, \tag{2.13}
\]

and

\[
\nabla_\mu \nabla_\nu \phi \rightarrow -N^2 \delta_\mu^0 \delta_\nu^0 \nabla_\nu \phi - 2N \delta_\mu^\nu \delta_\nu^i K_i \phi + \delta_\mu^\nu \nabla_\nu \phi D_i \phi. \tag{2.14}
\]

As the result, the canonical kinetic term \( X \) defined in (2.9) reduces to

\[
X \rightarrow -\frac{1}{2} D_i \phi D^i \phi. \tag{2.15}
\]

In the above and the following, \( D_i \phi \) is the covariant derivative adapted to \( h_{ij} \). The indices \( i, j, \ldots \) stand for the components with respect to the basis

\[
\omega^i := N^i dt + dx^i, \tag{2.16}
\]

and these indices are raised and lowered by \( h_{ij} \) and \( h^{ij} \). For example, \( D_i \phi \) is defined by

\[
D_\mu \phi dx^\mu = D_i \phi \omega^i.
\]

In the spatial gauge, \( \mathcal{L}_2^{H(s.g.)} \) takes the same functional form as (2.5) but with \( X \) replaced by (2.15). After some manipulations, we find

\[
\mathcal{L}_3^{H(s.g.)} = \frac{\partial G_3}{\partial X} D_i D_j \phi D^i \phi D^j \phi - \frac{\partial G_3}{\partial \phi} D_i \phi D^i \phi, \tag{2.17}
\]

and

\[
\mathcal{L}_4^{H(s.g.)} = G_4 \left( R + K_{ij} K^{ij} - K^2 \right) + 2 \frac{\partial G_4}{\partial X} \left( R^{ij} + K^{ik} K^{jk} - K K^{ij} \right) D_i \phi D_j \phi
\]

\[
- \frac{\partial G_4}{\partial X} \left( (D^2 \phi)^2 - D_i \phi D^i \phi D^j \phi \right)
\]

\[
+ 2 \frac{\partial^2 G_4}{\partial X \partial \phi} \left( D_i D_j \phi D^i \phi D^j \phi - D^i \phi D^j \phi D_i \phi D_j \phi \right)
\]

\[
+ 2 \frac{\partial^2 G_4}{\partial \phi^2} \left( 2 D_i D_j \phi D^i \phi D^j \phi - D^i \phi D_i \phi D^2 \phi \right)
\]

\[
- 2 \frac{\partial^2 G_4}{\partial \phi^2} D_i \phi D^i \phi - 2 \frac{\partial G_4}{\partial \phi} D^2 \phi. \tag{2.18}
\]

The cubic Horndeski Lagrangian in the spatial gauge \( \mathcal{L}_5^{H(s.g.)} \) can be found in (A.1) due to its length.

When the scalar field possesses a timelike gradient, the 3+1 decomposition of the Horndeski theory in the unitary gauge has been given in [34] (see also [56]). It is clear that the above expressions are complementary to the expressions in the unitary gauge.

3 General framework

The Horndeski theory is a generally covariant scalar-tensor theory, which is healthy no matter the scalar field is timelike or spacelike. In other words, the Lagrangians in the spatial gauge \( \mathcal{L}_2^{H(s.g.)} \), \( \mathcal{L}_3^{H(s.g.)} \), \( \mathcal{L}_4^{H(s.g.)} \) and \( \mathcal{L}_5^{H(s.g.)} \) represent explicit examples for healthy scalar-tensor
theories in the spatial gauge. We have several observations from the expressions of $\mathcal{L}_2^{H,(s.g.)}$, $\mathcal{L}_3^{H,(s.g.)}$, $\mathcal{L}_4^{H,(s.g.)}$ and $\mathcal{L}_5^{H,(s.g.)}$. First, in the spatial gauge, instead of an overall factor, the lapse function $N$ enters the Lagrangians only in terms of the extrinsic curvature. Second, and most importantly, the action explicitly depends on a non-dynamical scalar field $\phi$, which has no time derivative terms, while spatial derivative terms such as $D_i \phi$, $D_i D_j \phi$ etc are generally allowed.

Inspired by the above observations, we consider the following general action

$$S^{(s.g.)} = \int dt d^3x N \sqrt{h} \mathcal{L} (h_{ij}, K_{ij}, R_{ij}, N, \phi, D_i) . \quad (3.1)$$

By inclusion of the nonlinear dependence of $N$, (3.1) includes a very general class of higher derivative scalar-tensor theories, which is beyond the DHOST theory [21, 22, 24] in the spatial gauge (see appendix B). One may worry that our theory possibly includes 4 degrees of freedom, one of which is the Ostrogradsky’s ghost due to the higher derivatives. One of the main purposes of this work is to show that the model (3.1) propagates up to 3 degrees of freedom, through a detailed Hamiltonian constraint analysis.

We have three comments on the action (3.1). First, once we write down this action, one can extend the theory by regarding that the scalar field can be not only spacelike but also timelike. We have only to write down an action on some spacelike hypersurface. Second, since the action does not have a kinetic term (no time derivative) for the scalar field, the scalar field itself is non-dynamical. Nevertheless, an alternative dynamical scalar degree of freedom arises due to the explicit breaking of general covariance in the action (3.1). Finally, the action (3.1) is invariant under spatial diffeomorphism as explained in the previous section. The timelike vector $n^\mu$ orthogonal to some spacelike hypersurface fixes time slice and breaks time diffeomorphism invariance while keeping spatial one. In this sense, it can be viewed as a novel class of spatially covariant theories of gravity, and can be compared with action in [35, 36] (see also [38]) generally given by in the comoving gauge $\phi = t$ (a special case of the unitary gauge),

$$S^{(c.g.)} = \int dt d^3x N \sqrt{h} \mathcal{L} (t, h_{ij}, K_{ij}, R_{ij}, N, D_i) . \quad (3.2)$$

We would like to emphasize that our theory (3.1) does not reduce to (3.2) even if one allows timelike $\phi$ and imposes an ansatz $\phi(t) = t$, since the constraint equation from the $\phi$ variation is not imposed in (3.2). To be precise, even if the scalar field $\phi$ is timelike, since the action in (3.1) has no general covariance, it does not admit gauge freedom to choose $\phi = \phi(t)$. One may also worry that our theory (3.1) reduces to (3.2), except for explicit $t$ dependence, after integrating $\phi$ out. However this is not true because our theory (3.1) generally include spatial derivative of $\phi$ and inverse of spatial derivatives must be included after eliminating $\phi$.

For our purpose, it is convenient to rewrite (3.1) in an equivalent form

$$\tilde{S} := \int dt d^3x N \sqrt{h} \tilde{\mathcal{L}} := S + \int dt d^3x \frac{\delta S}{\delta B_{ij}} (K_{ij} - B_{ij}) , \quad (3.3)$$

where $S$ is the same action as (3.1) with $K_{ij}$ replaced by $B_{ij}$, i.e.,

$$S := \int dt d^3x N \sqrt{h} \mathcal{L} (h_{ij}, B_{ij}, R_{ij}, N, \phi, D_i) . \quad (3.4)$$

Note that the lapse function $N$ appears \textit{not} merely as the overall factor in the above action.
At this point, note $\tilde{S}$ depends on 17 variables:

$$\Phi_I := \{ N^i, \phi, B_{ij}, N, h_{ij} \},$$

where the indices $I, J, \cdots$ formally denote different kinds of variables as well as their tensorial indices.

4 Hamiltonian analysis

4.1 Primary constraints

The conjugate momenta corresponding to the variables (3.5) are defined as

$$\Pi_I := \frac{\delta \tilde{S}}{\delta \dot{\Phi}_I},$$

which are explicitly given by

$$\pi_i := \frac{\delta \tilde{S}}{\delta \dot{N}^i} = 0,$$

$$p := \frac{\delta \tilde{S}}{\delta \dot{\phi}} = 0,$$

$$p^{ij} := \frac{\delta \tilde{S}}{\delta \dot{B}_{ij}} = 0,$$

$$\pi := \frac{\delta \tilde{S}}{\delta \dot{N}} = 0,$$

$$\tilde{\pi}^{ij} := \frac{\delta \tilde{S}}{\delta \dot{h}_{ij}} = \frac{1}{2N} \frac{\delta S}{\delta B_{ij}},$$

According to (4.2)–(4.4), there are in total 17 primary constraints

$$\pi_i \approx 0, p \approx 0, p^{ij} \approx 0, \pi \approx 0, \tilde{\pi}^{ij} := \pi^{ij} - \frac{1}{2N} \frac{\delta S}{\delta B_{ij}} \approx 0,$$

where and throughout this work “$\approx$” represents the so-called “weak equality” that holds when the primary constraints are taken into account. For later convenience, we denote

$$\Pi^I := \{ \pi_i, p, p^{ij}, \pi, \tilde{\pi}^{ij} \},$$

for the set of momenta, and

$$\varphi^I := \{ \pi_i, p, p^{ij}, \pi, \tilde{\pi}^{ij} \},$$

for the set of primary constraints.

4.2 Canonical Hamiltonian

The canonical Hamiltonian is obtained by

$$H_C := \int d^3 x \left( \sum_I \Pi_I \dot{\Phi}_I - N \sqrt{\hbar} \mathcal{L} \right) \approx \int d^3 x \left( NC + \pi^{ij} \mathcal{L}_N h_{ij} \right),$$

with

$$C := 2 \pi^{ij} B_{ij} - \sqrt{\hbar} \mathcal{L}.$$
One of the crucial properties of our model (3.1) is that it possesses spatial covariance. As a result, techniques that are used in the Hamiltonian analysis for the spatially covariant gravity in [38] can be also applied. Especially, inspired by the discussion in [38], it is convenient to define the “proper” canonical Hamiltonian to be

\[ H_{\mathcal{C}} := \int d^3x (NC) + X[\vec{\xi}], \]  

(4.10)

where \( C \) is given in (4.9), and for a general spatial vector field \( \vec{\xi} \), \( X[\vec{\xi}] \) is defined to be

\[ X[\vec{\xi}] := \int d^3x \sum I \Pi^I \mathcal{L}_{\vec{\xi} \Phi^I} \simeq \int d^3x \xi^I C_i, \]  

(4.11)

with

\[ C_i = \pi D_i N - 2\sqrt{\hbar} D_j \left( \frac{1}{\sqrt{\hbar}} \pi^j \right) + pD_i\phi + p^{kl}D_iB_{kl} - 2\sqrt{\hbar} D_j \left( \frac{p^{jk}}{\sqrt{\hbar}} B_{ik} \right) \]
\[ + \pi_j D_i N^j + \sqrt{\hbar} D_j \left( \frac{1}{\sqrt{\hbar}} \pi_i N^j \right). \]  

(4.12)

The canonical Hamiltonian \( H_{\mathcal{C}} \) defined in (4.10) weakly equals to (4.8). We prefer to use (4.10) due to the following property of \( X[\vec{\xi}] \). For any functional on the phase space \( \mathcal{F} = \mathcal{F}[\Phi^I, \Pi^I] \) that is invariant under the time-independent spatial diffeomorphism, the following equality holds

\[ \left[ \int d^3x \sum I \Pi^I \mathcal{L}_{\vec{\xi} \Phi^I}, \mathcal{F} \right] = \int d^3x \sum I \Pi^I \mathcal{L}_{\vec{\xi}[\mathcal{F}] \Phi^I}, \]  

or compactly

\[ \left[ X[\vec{\xi}], \mathcal{F} \right] = X \left[ \vec{\xi}, \mathcal{F} \right], \]  

(4.13)

(4.14)

up to a boundary term. We refer to [38] for more details and for the general proof. As we shall see, defining the canonical Hamiltonian as (4.10) significantly simplifies the calculation of Poisson brackets.

Due to the presence of primary constraints, the time-evolution is determined by the so-called total Hamiltonian, which is given by

\[ H_T := H_{\mathcal{C}} + \int d^3y \sum I \lambda_I (\vec{y}) \varphi^I \left( \vec{y} \right), \]  

(4.15)

where \( \lambda_I := \{ v^i, v, v_{ij}, \lambda, \lambda_{ij} \} \) are undetermined Lagrange multipliers which are associated with \( \varphi^I = \{ \pi_i, p, p^{ij}, \pi, \tilde{\pi}^{ij} \} \).

4.3 Constraint algebra

The matrix composed of Poisson brackets of the primary constraints takes the following form

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \left[ p (\vec{x}), \tilde{\pi}^{kl} (\vec{y}) \right] \\
0 & 0 & 0 & 0 & \left[ p^{ij} (\vec{x}), \tilde{\pi}^{kl} (\vec{y}) \right] \\
0 & \left[ \tilde{\pi}^{ij} (\vec{x}), p (\vec{y}) \right] & \left[ \pi^{ij} (\vec{x}), p^{kl} (\vec{y}) \right] & \left[ \tilde{\pi}^{ij} (\vec{x}), \pi (\vec{y}) \right] & \left[ \tilde{\pi}^{ij} (\vec{x}), \tilde{\pi}^{kl} (\vec{y}) \right] \\
0 & \left[ \pi (\vec{x}), \tilde{\pi}^{ij} (\vec{y}) \right] & 0 & 0 & 0
\end{pmatrix},
\]

(4.16)
where the generally non-vanishing entries are

\[
\begin{align*}
[p (\vec{x}), \tilde{\pi}^{kl} (\vec{y})] &= \frac{1}{2N (\vec{y})} \frac{\delta^2 S}{\delta \phi (\vec{x}) \delta B_{kl} (\vec{y})}, \\
[p^{ij} (\vec{x}), \tilde{\pi}^{kl} (\vec{y})] &= \frac{1}{2N (\vec{y})} \frac{\delta^2 S}{\delta B_{ij} (\vec{x}) \delta B_{kl} (\vec{y})}, \\
[\pi (\vec{x}), \tilde{\pi}^{ij} (\vec{y})] &= -\frac{1}{2} \delta^{3} (\vec{x} - \vec{y}) \frac{1}{N^2 (\vec{y})} \frac{\delta S}{\delta B_{ij} (\vec{x})} - \frac{1}{2} \frac{1}{N (\vec{x})} \frac{\partial^2 S}{\partial B_{ij} (\vec{x}) \partial B_{kl} (\vec{y})}, \\
[\tilde{\pi}^{ij} (\vec{x}), \tilde{\pi}^{kl} (\vec{y})] &= \frac{1}{2N (\vec{y})} \frac{\partial^2 S}{\delta h_{ij} (\vec{x}) \delta B_{kl} (\vec{y})} - \frac{1}{2N (\vec{x})} \frac{\partial^2 S}{\delta B_{ij} (\vec{x}) \delta h_{kl} (\vec{y})}.
\end{align*}
\]  

Before we proceed, note if the lapse function \( N \) enters (3.4) linearly as in the case of Horndeski theory, we may evaluate (4.19) explicitly and find

\[
[\pi (\vec{x}), \tilde{\pi}^{ij} (\vec{y})] = -\frac{1}{2} \delta^{3} (\vec{x} - \vec{y}) \frac{1}{N} \sqrt{h} \sum_{n=1} (-1)^n \partial_{k_1} \cdots \partial_{k_n} \frac{\partial L}{\partial (\partial_{k_1} \cdots \partial_{k_n} B_{ij})}
\]

\[
+ \frac{1}{2} \frac{1}{\sqrt{N (\vec{y})}} \sum_{n=1} (-1)^n \partial_{y^1} \cdots \partial_{y^n} \delta^{3} (\vec{x} - \vec{y}) \frac{\partial L (\vec{x})}{\partial (\partial_{x^1} \cdots \partial_{x^n} B_{ij})}.
\]

Then \([\pi (\vec{x}), \tilde{\pi}^{ij} (\vec{y})] \neq 0\) only if our Lagrangian in (3.4) depends on spatial derivatives of \( B_{ij} \), i.e., the extrinsic curvature \( K_{ij} \) enters the original action (3.1) with spatial derivatives. Fortunately, the following counting number of degrees of freedom is not subject to this subtlety.

Constraints must be preserved in time. For the primary constraints \( \varphi^I \) in (4.7), we must require that

\[
\frac{d\varphi^I (\vec{x})}{dt} = \int d^3 y \sum_J [\varphi^I (\vec{x}), \varphi^J (\vec{y})] \lambda_J (\vec{y}) + [\varphi^I (\vec{x}), H_C],
\]

where \([\varphi^I (\vec{x}), \varphi^J (\vec{y})]\) is given in (4.16). After some manipulations, we find

\[
[\varphi^I (\vec{x}), H_C] = \begin{pmatrix}
-C_i (\vec{x}) \\
\frac{\delta S}{\delta \phi (\vec{x})} \\
0 \\
-C (\vec{x}) \\
\tilde{\pi}^{ij} (\vec{x}), H_C
\end{pmatrix},
\]

where \(C_i\) is defined in (4.12), \( C \) is given by

\[
C (\vec{x}) = C (\vec{x}) + \int d^3 y N (\vec{y}) \frac{\delta C (\vec{y})}{\delta N (\vec{x})},
\]

\[
= 2 \pi^{ij} (\vec{x}) B_{ij} (\vec{x}) - \frac{\delta S}{\delta N (\vec{x})},
\]

and

\[
[\tilde{\pi}^{ij} (\vec{x}), H_C] = \frac{\delta S}{\delta B_{ij} (\vec{x})} - \frac{1}{N (\vec{x})} \int d^3 z \frac{\delta^2 S}{\delta B_{ij} (\vec{x}) \delta h_{kl} (\vec{z})} N (\vec{z}) B_{kl} (\vec{z}).
\]
The consistency conditions (4.22) yield the following 5 equations:

\[-C_i (\vec{x}) = 0, \tag{4.26}\]
\[\int d^3y \left[ p(\vec{x}) , \tilde{\pi}^{kl}(\vec{y}) \right] \lambda_{kl}(\vec{y}) + \frac{\delta S}{\delta \phi(\vec{x})} = 0, \tag{4.27}\]
\[\int d^3y \left[ p^{ij}(\vec{x}) , \tilde{\pi}^{kl}(\vec{y}) \right] \lambda_{kl}(\vec{y}) = 0, \tag{4.28}\]
\[\int d^3y \left[ \pi(\vec{x}) , \tilde{\pi}^{kl}(\vec{y}) \right] \lambda_{kl}(\vec{y}) - C(\vec{x}) = 0, \tag{4.29}\]
\[\int d^3y \left\{ \left[ \tilde{\pi}^{ij}(\vec{x}) , p(\vec{y}) \right] v_{ij}(\vec{y}) + \left[ \tilde{\pi}^{ij}(\vec{x}) , p^{kl}(\vec{y}) \right] v_{kl}(\vec{y}) + \left[ \tilde{\pi}^{ij}(\vec{x}) , \pi(\vec{y}) \right] \lambda(\vec{y}) + \left[ \tilde{\pi}^{ij}(\vec{x}) , \tilde{\pi}^{kl}(\vec{y}) \right] \lambda_{kl}(\vec{y}) \right\} + \left[ \tilde{\pi}^{ij}(\vec{x}) , H_C \right] = 0, \tag{4.30}\]

where Poisson brackets among the primary constraints are given in (4.17)–(4.20). It immediately follows from (4.26) that

\[C_i \approx 0, \tag{4.31}\]

are 3 secondary constraints, as expected. Since we assume \(\frac{\delta^2 S}{\delta B_{ij}(\vec{x}) \delta B_{kl}(\vec{y})} \) is not degenerate, we are able to fix the Lagrange multipliers from (4.28) to be

\[\lambda_{ij} = 0. \tag{4.32}\]

Then (4.27) and (4.29) yield another two secondary constraints:

\[\frac{\delta S}{\delta \phi} \approx 0, \quad C \approx 0. \tag{4.33}\]

Note since \(\phi\) itself has no dynamics, its equation of motion \(\frac{\delta^2 S}{\delta \phi} = 0\) must be a constraint, as expected. The last equation (4.30) simply further fixes the Lagrange multiplier \(v_{ij}\) instead of generating new constraint. To summarize, the time consistency equations for the 17 primary constraints reduces to 5 secondary constraints \(C_i, C\) and \(\delta S/\delta \phi\) and 12 equations which fix the Lagrange multipliers \(\lambda_{ij}\) and \(v_{ij}\).

We also need to check the consistency conditions for the secondary constraints got so far. To this end, first we evaluate the Poisson brackets of the secondary constraints with the primary constraints as well as with themselves. Thanks to the property (4.14), one can show that \(C_i\) has vanishing Poisson bracket with all the constraints (see [38] as well as appendix C for details). The Poisson brackets of \(C \approx 0\) with the primary constraints are:

\[[\pi_i(\vec{x}) , C(\vec{y})] = 0, \tag{4.34}\]

and

\[[\pi(\vec{x}) , C(\vec{y})] = \frac{\delta^2 S}{\delta N(\vec{x}) \delta N(\vec{y})}, \tag{4.35}\]
\[[p(\vec{x}) , C(\vec{y})] = \frac{\delta^2 S}{\delta \phi(\vec{x}) \delta N(\vec{y})}, \tag{4.36}\]
\[[p^{ij}(\vec{x}) , C(\vec{y})] = -2\delta^3(\vec{x} - \vec{y}) \pi^{ij}(\vec{y}) + \frac{\delta^2 S}{\delta B_{ij}(\vec{x}) \delta N(\vec{y})}, \tag{4.37}\]
\[[\tilde{\pi}^{ij}(\vec{x}) , C(\vec{y})] = \frac{\delta^2 S}{\delta h_{ij}(\vec{x}) \delta N(\vec{y})} - \frac{1}{N(\vec{x})} \frac{\delta^2 S}{\delta B_{ij}(\vec{x}) \delta h_{kl}(\vec{y})} B_{kl}(\vec{y}). \tag{4.38}\]
The Poisson brackets of \( \frac{\delta S}{\delta \phi} \approx 0 \) with the primary constraints are:

\[
\left[ \pi_i (\vec{x}), \frac{\delta S}{\delta \phi (\vec{y})} \right] = 0,
\]

and

\[
\left[ p_i (\vec{x}), \frac{\delta S}{\delta \phi (\vec{y})} \right] = -\frac{\delta^2 S}{\delta \phi (\vec{x}) \delta \phi (\vec{y})},
\]

\[
\left[ p^{ij} (\vec{x}), \frac{\delta S}{\delta \phi (\vec{y})} \right] = -\frac{\delta^2 S}{\delta B_{ij} (\vec{x}) \delta \phi (\vec{y})},
\]

\[
\left[ \pi (\vec{x}), \frac{\delta S}{\delta \phi (\vec{y})} \right] = -\frac{\delta^2 S}{\delta \delta h_{ij} (\vec{x}) \delta \phi (\vec{y})},
\]

\[
\left[ \tilde{\pi}^{ij} (\vec{x}), \frac{\delta S}{\delta \phi (\vec{y})} \right] = -\frac{\delta^2 S}{\delta h_{ij} (\vec{x}) \delta \phi (\vec{y})}.
\]

Finally, the Poisson brackets between \( \mathcal{C} \) and \( \frac{\delta S}{\delta \phi} \) are

\[
[\mathcal{C}(\vec{x}), \mathcal{C}(\vec{y})] = -\frac{\delta^2 S}{\delta h_{ij} (\vec{y}) \delta N (\vec{x})} 2B_{ij} (\vec{y}) + 2B_{ij} (\vec{x}) \frac{\delta^2 S}{\delta h_{ij} (\vec{x}) \delta N (\vec{y})},
\]

\[
\left[ \frac{\delta S}{\delta \phi (\vec{x})}, \mathcal{C}(\vec{y}) \right] = 0,
\]

\[
\left[ \frac{\delta S}{\delta \phi(\vec{x})}, \mathcal{C}(\vec{y}) \right] = \frac{\delta^2 S}{\delta \delta h_{ij} (\vec{y})} 2B_{ij} (\vec{y}).
\]

We are now ready to check the consistency conditions for the secondary constraints. Since \( \mathcal{C}_i \) has vanishing Poisson brackets with all the constraints got so far as well as \( H_C \), its consistency condition is automatically satisfied. On the other hand, the consistency condition for \( \mathcal{C} \approx 0 \) and \( \frac{\delta S}{\delta \phi} \approx 0 \) yield two equations:

\[
0 \equiv \frac{d \mathcal{C}(\vec{x})}{dt} = \int d^3 y \left[ \mathcal{C}(\vec{x}), \pi (\vec{y}) \right] \lambda (\vec{y}) + \int d^3 y \left[ \mathcal{C}(\vec{x}), p (\vec{y}) \right] v (\vec{y})
\]

\[
+ \int d^3 y \left[ \mathcal{C}(\vec{x}), p^{ij} (\vec{y}) \right] v_{ij} (\vec{y}) + [\mathcal{C}(\vec{x}), H_C],
\]

and

\[
0 \equiv \frac{d}{dt} \left( \frac{\delta S}{\delta \phi(\vec{x})} \right) = \int d^3 y \left[ \frac{\delta S}{\delta \phi(\vec{x})}, \pi (\vec{y}) \right] \lambda (\vec{y}) + \int d^3 y \left[ \frac{\delta S}{\delta \phi(\vec{x})}, p (\vec{y}) \right] v (\vec{y})
\]

\[
+ \int d^3 y \left[ \frac{\delta S}{\delta \phi(\vec{x})}, p^{ij} (\vec{y}) \right] v_{ij} (\vec{y}) + \left[ \frac{\delta S}{\delta \phi(\vec{x})}, H_C \right],
\]

where various Poisson brackets are given in (4.36)–(4.43), and

\[
[\mathcal{C}(\vec{x}), H_C] = 2B_{ij} (\vec{x}) \frac{\delta S}{\delta h_{ij} (\vec{x})} - \int d^3 z \frac{\delta S}{\delta N(\vec{x}) \delta h_{ij} (\vec{z})} N (\vec{z}) 2B_{ij} (\vec{z}),
\]

and

\[
\left[ \frac{\delta S}{\delta \phi(\vec{x})}, H_C \right] = \int d^3 z \frac{\delta^2 S}{\delta h_{ij} (\vec{x}) \delta \phi(\vec{z})} N (\vec{z}) 2B_{ij} (\vec{z}).
\]
The consistency conditions (4.47) and (4.48) (together with (4.30)) merely fix the Lagrange multipliers \( v \) and \( \lambda \), provided that the coefficient matrix of \( v \) and \( \lambda \) is non-singular, i.e.,

\[
\det \begin{pmatrix}
\frac{\delta^2 S}{\delta N(y)} & \frac{\delta^2 S}{\delta \phi(y)} \\
\frac{\delta N(x) \delta N(y)}{\delta \phi(x) \delta \phi(y)} & \frac{\delta N(x) \delta \phi(y)}{\delta \phi(x) \delta \phi(y)}
\end{pmatrix} \neq 0,
\]

(4.51)
on the constraint surface.\(^3\) As a result, we do not have any further secondary (that is, tertiary) constraint and 3 Lagrange multipliers \( v^i \) are undetermined.

Finally, the Poisson brackets among all the constraints can be summarized in the following table:

| \([\cdot, \cdot]\) | \(\pi_k(y)\) | \(p(y)\) | \(p^{ij}(y)\) | \(\pi(y)\) | \(\tilde{\pi}^{kl}(y)\) | \(C_k(y)\) | \(C(y)\) | \(\frac{\delta S}{\delta \phi(y)}\) |
|---|---|---|---|---|---|---|---|---|
| \(\pi_i(x)\) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \(p(x)\) | 0 | 0 | 0 | 0 | X | 0 | X | X |
| \(p^{ij}(x)\) | 0 | 0 | 0 | 0 | X | 0 | X | X |
| \(\pi(x)\) | 0 | 0 | 0 | 0 | X | 0 | X | X |
| \(\tilde{\pi}^{ij}(x)\) | 0 | X | X | X | X | 0 | X | X |
| \(C_i(x)\) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| \(C(x)\) | 0 | X | X | X | X | 0 | X | X |
| \(\frac{\delta S}{\delta \phi(x)}\) | 0 | X | X | X | X | 0 | X | 0 |

where “X” stands for generally non-vanishing entries. It is thus transparent that in our theory there are 22 = 17 + 3 + 2 constraints which can be divided into two classes:

6 first-class: \( \pi_i, \ C_i, \)

16 second-class: \( p, \ p^{ij}, \ \pi, \ \tilde{\pi}^{ij}, \ \frac{\delta S}{\delta \phi}, \ C. \)

The number of physical degrees of freedom is calculated to be

\[
\#_{dof} = \frac{1}{2} (2 \times \#_{\text{var}} - 2 \times \#_{\text{1st}} - \#_{\text{2nd}}) \\
= \frac{1}{2} (2 \times 17 - 2 \times 6 - 16) \\
= 3.
\]

(4.52)

Thus the degree of freedom due to the higher derivatives, that is the Ostrogradsky’s ghost, is absent. We note that if the condition (4.51) is not satisfied, additional constraints or gauge degrees of freedom are expected to appear. In such a case, the number of degrees of freedom is less than 3.

5 Conclusion

We have proposed a new class of higher derivative scalar-tensor theories which break the general covariance. The action is given by (3.1), which can be also viewed as a new type

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\(^3\)In the case where \( S \) linearly depends on \( N \), this condition reduces to \( \delta \mathcal{L}(y)/\delta \phi(x) \neq 0 \) on the constraint surface. We note that, though this expression is similar to a secondary constraint \( \delta S/\delta \phi(x) \), \( \delta \mathcal{L}(y)/\delta \phi(x) \) does not need to vanish on the constraint surface when \( \mathcal{L} \) includes spatial derivative of \( \phi \) as in the case of Horndeski theory.
of spatially covariant gravity theories alternative to those proposed in [35, 36, 38]. The construction of our theory is originally motivated from a scalar field with spacelike gradient while the usual spatially covariant theory in [35, 36, 38] is constructed based on a scalar field with timelike gradient. As an illuminating example, we first gave concrete expressions for the Horndeski action in the spatial gauge. Motivated with these expressions, we gave a generic action for our new theory (3.1). Once the action of the theory is given, it does not rely on that the scalar field gradient is spacelike. That is, the scalar field in (3.1) can be both spacelike and timelike. The crucial point is that the scalar field has no kinetic term and hence is non-dynamical. Another important point is the spatial diffeomorphism invariance of the action (3.1). The timelike vector $n^\mu$ orthogonal to some spacelike hypersurface fixes time slice and breaks time diffeomorphism invariance while keeping the spatial one. A dynamical scalar degree of freedom from the gravity sector gets excited due to the lack of general covariance. While the presence of the non-dynamical scalar field changes the behaviour of the dynamical scalar mode comparing with that in [35, 36, 38]. We also make the Hamiltonian analysis for our action (3.1) and confirm that there are only three (two tensors and one scalar) dynamical degree of freedom, which ensures the absence of the Ostrogradsky’s ghosts.

We notice that a very large class of covariant scalar-tensor theories with higher derivatives fall into (3.1) in the spatial gauge. Our analysis suggests that such theories propagate upto 3 degrees of freedom when we impose the spatial gauge condition, although more degrees of freedom would arise in general. This is simply because, when the spatial gauge is accessible, the kinetic term of the scalar field disappears and the kinetic matrix becomes degenerate automatically, and a single scalar degree of freedom arises due to the breaking of general covariance.

We have a lot of remaining tasks to investigate concerning various aspects of this new theory, which will be done in the future publications soon. For example, we are going to make the analysis of cosmological perturbations. Though the construction was initially inspired by assuming the presence of a spacelike scalar field, it is interesting to examine whether this theory still admits cosmological background solutions, which yield new classes of inflation and/or dark energy models with new phenomenological features. Moreover, by construction our theory (3.1) describes a class of scalar-tensor theories wider than DHOST theory in the spatial gauge. It is, however, important to investigate the relationship between our theory and the spatially covariant theory in [35, 36, 38]. Another interesting question is to investigate what happens if we apparently recover the general covariance of this theory by use of the Stückelberg trick. As for the usual spatially covariant theories, it was argued that the corresponding generally covariant scalar-tensor theories are healthy at least when the scalar field gradient is timelike [37]. Yet another interesting question is to consider theories with multiple scalar fields configurations, though, in this paper, we introduce only one non-dynamical scalar field. All of these topics will be discussed in the future publications.

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A \( \mathcal{L}_5^H \) in the spatial gauge

In the spatial gauge, the cubic Horndeski Lagrangian (2.8) reduces to

\[
\mathcal{L}_{5,(s.g.)} = \frac{\partial G_5}{\partial X} G_i D^i \phi D^j \phi D^k D_k \phi + \frac{1}{2} \frac{\partial G_5}{\partial X} D^i \phi D_i \phi D_j \phi \left( K^2 - K_{kl} K^{kl} \right) \\
- \frac{\partial G_5}{\partial X} \left( K_{ik} K_{kj} D^2 \phi - K_{kij} D_i \phi D_j \phi \right) D_l \phi D_l \phi \\
+ 2 \frac{\partial G_5}{\partial X} \left( K_{i} K_j K_k - K K_{ij} K_k \right) D_i D_l \phi D^l \phi D_j \phi \\
- \frac{\partial G_5}{\partial X} \left( K_{kij} K_{k} - K K_{klkl} K_k \right) D_i D_j \phi D_k \phi D_l \phi \\
+ \frac{1}{3} \frac{\partial G_5}{\partial X} \left[ (D^2 \phi)^3 - 3 D_i D_j \phi D^i \phi D^j \phi D^2 \phi + 2 D_i D_j \phi D^i \phi D^j \phi D^k D_k \phi D^l \phi \right] \\
+ \frac{1}{2} \frac{\partial G_5}{\partial \phi} R D_i \phi D_i \phi - 2 \frac{\partial G_5}{\partial \phi} R_{ij} D_i \phi D^j \phi - \frac{1}{2} \frac{\partial G_5}{\partial \phi} \left( K^2 - K_{kl} K^{kl} \right) D_i \phi D^j \phi \\
- 2 \frac{\partial^2 G_5}{\partial X \partial \phi} D_i D_j \phi D^i \phi D^j \phi \left( D^i \phi D^j \phi D^2 \phi - D_i D_j \phi D^i \phi D^j \phi \right) \\
- \frac{1}{2} \frac{\partial^2 G_5}{\partial X \partial \phi} D_i D_j \phi D^i \phi D^j \phi \left( (D^2 \phi)^2 - D_i D_j \phi D^i \phi D^j \phi \right) \\
- 2 \frac{\partial^2 G_5}{\partial X \partial \phi} D_i \phi D_i \phi \left( (D^2 \phi)^2 - D_k D_i \phi D^k D^l \phi \right) \\
+ \frac{1}{2} \frac{\partial^2 G_5}{\partial \phi^2} D_i \phi D^i \phi D^2 \phi - D_i \phi D^i \phi D^j \phi D^l \phi \right). \tag{A.1}
\]

B Higher derivative interactions beyond DHOST theory

In this appendix, we show that our general action (3.1) includes the theory which can be obtained by imposing the “spatial gauge” from higher derivative scalar-tensor theories beyond DHOST theory. Here, as a simple example, let us consider a scalar-tensor theory which non-linearly depends on \( \Box \phi \),

\[
S^B = \int d^4 x \sqrt{-g} F(X, \phi, \Box \phi). \tag{B.1}
\]

By using (2.11) and imposing spatial gauge condition (2.1), we obtain

\[
S_{B,(s.g.)} = \int d^4 x N \sqrt{\kappa} F \left( -\frac{1}{2} D_i \phi D^i \phi, \phi, D_i D^i \phi + D_i \phi D^i \log N \right), \tag{B.2}
\]

where the acceleration \( a_i \) is expressed by \( N \) through \( a_i = D_i \log N \). Clearly the action (B.2) is included by our general action (3.1).
C Poisson bracket with $C_i$

Let us calculate Poisson bracket between $C_i$ and an arbitrary function of the phase space variables $Q(\Phi_I, \pi_I)$. By introducing test functions $\xi^i$ and $f$, let us evaluate the following quantity,

$$\int d^3x d^3y \xi^i(x) f(y) [C_i(x), Q(y)] = [X[\xi^i], F],$$

where $F$ is a functional of $f$ and phase space variables $\Phi_I$ and $\pi_I$, which is given by

$$F[f, \Phi_I, \pi_I] = \int d^3y f(y) Q(y).$$

Here we define $f(y)$ so that $F$ is invariant under the time independent spatial diffeomorphism,

$$\delta \xi F = \int d^3x \left( \frac{\delta F}{\delta f} \xi^i f + \frac{\delta F}{\delta \phi_I} \xi^i \phi_I + \frac{\delta F}{\delta \pi_I} \xi^i \pi_I \right) = 0. \quad (C.3)$$

For example, if $Q$ is a spatial vector $Q^i$, $f$ also has a suffix and $f_i/\sqrt{h}$ is assumed to transform as a covariant vector. As similar way to the derivation of (4.14) is given in [38], we obtain

$$[X[\xi^i], F] = X[[\xi^i, F]] + \int d^3x \frac{\delta F}{\delta f} \xi^i f.$$ \quad (C.4)

The first term vanishes because $\xi^i$ is not a phase space variable. Then, by plugging (C.2) in the second term, we obtain,

$$[X[\xi^i], F] = \int d^3x Q(\bar{x}) \xi^i f(\bar{x}). \quad (C.5)$$

The right hand side vanishes on the constraint surface when $Q(x)$ is a constraint. Thus (C.5) guarantees that the Poisson bracket of $C_i$ with any constraint vanishes on the constraint surface.

As a consistency check, let us derive the Poisson bracket between $C_i$ and $N$.

$$\int d^3x d^3y \xi^i(\bar{x}) f(\bar{y}) [C_i(\bar{x}), N(\bar{y})] = \int d^3x N(\bar{x}) \xi^i f(\bar{x}),$$

$$= - \int d^3x \xi^i N(\bar{x}) f(\bar{x}) = - \int d^3x \xi^i N(\bar{x}) \frac{\partial^3}{\partial x_i} N(\bar{x}).$$

By comparing the expression in the first and last line, we obtain

$$[C_i(\bar{x}), N(\bar{y})] = - \delta^3(\bar{x} - \bar{y}) \partial_{x_i} N(\bar{x}). \quad (C.7)$$

The same expression can be obtained by directly using the definition of $C_i$, (4.12).
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