Vacuum energy, Standard Model physics and the 750 GeV Diphoton Excess at the LHC

Alexander Y. Kamenshchik∗1,2, Alexei A. Starobinsky†2,3, Alessandro Tronconi‡1, Gian Paolo Vacca§1, and Giovanni Venturi¶1

1Dipartimento di Fisica e Astronomia and INFN, Via Irnerio 46,40126 Bologna, Italy
2L.D. Landau Institute for Theoretical Physics of the Russian Academy of Sciences, Kosygin str. 2, 119334 Moscow, Russia
3Bogoliubov Laboratory for Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Moscow Region, 141980, Russia

Abstract

The conditions for the cancellation of one loop contributions to vacuum energy (both U.V. divergent and finite) coming from the known (Standard Model) fields and the field associated with the hypothetical 750 Gev diphoton excess are examined. Depending on the nature of this latter resonance we find that in order to satisfy the constraints (sum rules) different additional, minimal field contributions are required, leading to yet to be observed particles.

∗Alexander.Kamenshchik@bo.infn.it
†alstar@landau.ac.ru
‡Alessandro.Tronconi@bo.infn.it
§Vacca@bo.infn.it
¶Giovanni.Venturi@bo.infn.it
1 Introduction

Almost seventy years ago Pauli [1] suggested that the vacuum (zero-point) energies of all existing fermions and bosons compensate each other. This possibility is based on the fact that the vacuum energy of fermions has a negative sign whereas that of bosons has a positive one. We note that such an idea is realised in a highly constrained way in supersymmetric models, although supersymmetry breaking must be present at probed energies in order to explain the observed data. Subsequently in a series of papers Zeldovich [2] connected the vacuum energy to the cosmological constant, however rather than eliminating the divergences through a boson-fermion cancellation, he suggested a Pauli-Villars regularisation of all divergences by introducing a number of massive regulator fields. Covariant regularisation of all contributions then leads to finite values for both the energy density $\varepsilon$ and (negative) pressure $p$ corresponding to a cosmological constant, i.e. connected by the equation of state $p = -\varepsilon$.

The possibility that such a finite (renormalized) vacuum energy acts as a source for the gravitational dynamics leads to the cosmological constant puzzle of modern physics.

Developing an approach based on the combination of the above two ideas, that is, as Pauli suggested the existing boson and fermion states (fields) should provide the exact cancellation of all the ultraviolet divergences in the vacuum energy while, as Zeldovich suggested, the remaining finite part of the vacuum energy should lead to an effective cosmological constant, is a non-trivial task.

Indeed the original Pauli idea should be extended in a general QFT context where most generally the full structure of all contributions would require solving the dynamics for all interactions.

One possible approach to the problem would be to consider a fundamental QFT setup defined below a UV scale lower than the Planck mass (omitting for the time being quantum gravity effects). This requires choosing a bare action from which to compute the full effective action including all eventual power-like divergences which is a complicated task.

In order to make the problem more tractable, it is better to consider a traditional perturbative approach, that is a loop expansion computed from

---

1 In a free theory context a way to see how and under which assumptions one can derive in a Wilsonian framework the Zeldovich formula in Eq. (1) was discussed in [3] in Sect. 3.2.
the effective action. If we then consider an expansion in powers of $\hbar$, different loop orders can therefore contribute different powers of $\hbar$. In particular the contributions to the vacuum energy (field independent part of the effective action) to the lowest order in $\hbar$ are obtained from the one loop contributions related to the massive fields employing their “physical masses”.

In our specific approximation we shall require that the cancellations, in order to obtain a finite, actually negligible, cosmological constant, should be imposed order by order in $\hbar$ and here we shall limit ourselves to an analysis of the $O(\hbar)$ (lowest order) sum rules. This is in agreement with the fact that in a semiclassical analysis, $\hbar$ being small, gives the dominant corrections. In this framework we shall ignore, as a further approximation, contributions from bound states.

In a previous work we have already examined [4] the problem of the cancellation at order $O(\hbar)$ of the UV divergences of the vacuum energy for both Minkowski and de Sitter space-times and formulated the conditions (sum rules) for the cancellation of all divergences. These conditions led to strong restrictions on the spectra of possible fields. In paper [5] we applied such considerations to the observed particles of the Standard Model (SM) and also studied the finite part of the vacuum energy. In particular, also the possibility of a cancellation for this last contribution, so as to obtain a result compatible with the observed value of the cosmological constant (almost zero with respect to SM particle masses). We showed [5] that it was impossible to construct a minimal extension of the SM by finding a set of boson fields which, besides cancelling the ultraviolet divergencies, could compensate the residual huge contribution of the known fermion and boson fields of the Standard Model to the finite part of the vacuum energy density.

On the other hand we found that the addition of at least one massive fermion field was sufficient for the existence of a suitable set of boson fields which would permit such cancellations and obtained their allowed mass intervals. This result was by itself very suggestive since in extensions of the SM often new extra fermions are considered, independently of any cancellation requirement. Further, on examining one of the simplest SM extensions satisfying the constraints we found that the mass range of the lightest massive boson was compatible with the Higgs mass bounds, which were known at the moment of the publication of the paper [5].

Subsequently some very important results were obtained in experimental particle physics. First of all the Higgs boson was discovered [6]. This long awaited event accomplished the experimental confirmation of the so
called Standard Model. Quite recently another event has excited the physics community: the observed diphoton excess at 750 GeV [7]. This excess, if confirmed, could be interpreted as an indication for the existence of a new heavy elementary or composite particle with a mass of the order of 750 GeV. Not much is known about such a possible state, including if it may have a narrow or large width, in which case other decay channels beyond the SM photon and gluons (typically leading to small width contributions at loop level) could be invoked, eventually related to new particles or composite states, related to dark matter. This has led to a rather frenetic theoretical activity [8].

In the present paper we wish to apply the methods and ideas, developed in the preceding papers [4, 5] to the analysis of this new experimental situation. The structure of the paper is the following: in the second section we write down the conditions for the cancellations of divergent and finite contributions between fermion and boson fields to the vacuum energy, we briefly present the results of our preceding paper [5] and generalise some formulae by adding more constraints to the model; in the third section we shall take into account different possibilities for the new hypothetical 750 GeV particle and the resulting possible structures of the spectrum for elementary particles beyond the Standard Model. The last section contains some concluding remarks.

2 Vacuum energy and the balance between the fermion and boson fields

As previously discussed we start by considering the contribution of order $O(\hbar)$ to the vacuum energy of the propagating massive particles of a fundamental theory. One knows that the vacuum energy of the harmonic oscillator is equal to $\hbar \omega$. If one has a massive field with mass $m$, then $\omega = \sqrt{k^2c^2 + m^2c^4}$, where $k$ is the wave number. In the following we shall set $\hbar = 1$ and $c = 1$. The energy density of the vacuum energy of a scalar field, treated as oscillators with all possible momenta is given by the divergent integral [2]:

$$\varepsilon = \frac{1}{2} \int d^3k \sqrt{k^2 + m^2} = 2\pi \int_0^\infty dk k^2 \sqrt{k^2 + m^2}. \quad (1)$$
We can regularise this integral by introducing a cutoff $\Lambda$. In this case

$$\varepsilon = 2\pi \int_0^\Lambda dk k^2 \sqrt{k^2 + m^2}$$

$$= 2\pi m^4 \left[ \frac{\Lambda}{8m} \left( \frac{2\Lambda^2 + 1}{m^2} \right) \sqrt{\frac{\Lambda^2}{m^2} + 1} - \frac{1}{8} \ln \left( \frac{\Lambda}{m} + \sqrt{\frac{\Lambda^2}{m^2} + 1} \right) \right]. \quad (2)$$

On expanding this expression with respect to the small parameter $\frac{m}{\Lambda}$, one obtains

$$\varepsilon = \frac{\pi}{2} \frac{\Lambda^4}{4} + \frac{\pi}{2} \Lambda^2 m^2 + \frac{\pi}{16} m^4 (1 - 4 \ln 2) - \frac{\pi}{4} m^4 \ln \frac{\Lambda}{m} + o \left( \frac{m}{\Lambda} \right). \quad (3)$$

The contribution of one fermion degree of freedom coincides with that of Eq. (1) with the opposite sign. It now follows from Eq. (3) that to cancel the quartic ultraviolet divergences, proportional to $\Lambda^4$, one has to have equal numbers of boson and fermion degrees of freedom:

$$N_B = N_F. \quad (4)$$

The conditions for the cancellation of quadratic and logarithmic divergences are

$$\sum m_S^2 + 3 \sum m_V^2 = 2 \sum m_F^2 \quad (5)$$

and

$$\sum m_S^4 + 3 \sum m_V^4 = 2 \sum m_F^4, \quad (6)$$

respectively. Here the subscripts $S$, $V$ and $F$ denote scalar, massive vector and massive spinor Majorana fields respectively (for Dirac fields it is sufficient to put a 4 instead of 2 on the right-hand sides of Eqs. (5) and (6)). For the case such that the conditions (4), (5) and (6) are satisfied the remaining finite part of the vacuum energy density is equal to

$$\varepsilon_{\text{finite}} = \sum m_S^4 \ln m_s + 3 \sum m_V^4 \ln m_V - 2 \sum m_F^4 \ln m_F. \quad (7)$$

Let us now calculate the vacuum pressure, this pressure is given by the formula

$$p = \frac{2\pi}{3} \int_0^\infty dk \frac{k^4}{\sqrt{k^2 + m^2}}. \quad (8)$$
On introducing the cutoff $\Lambda$ we have

$$p = \frac{2\pi}{3} \int_0^\Lambda dk \frac{k^4}{\sqrt{k^2 + m^2}}$$

$$= \frac{2\pi}{3} m^4 \left[ \frac{1}{8} \frac{\Lambda^2}{m^2} \left( \frac{2\Lambda^2}{m^2} \right) \sqrt{\frac{\Lambda^2}{m^2} + 1} - \frac{\Lambda}{m} \sqrt{\frac{\Lambda^2}{m^2} + 1} + \frac{3}{8} \ln \left( \frac{\Lambda}{m} + \sqrt{\frac{\Lambda^2}{m^2} + 1} \right) \right].$$ (9)

On expanding this expression with respect to the small parameter $\frac{m}{\Lambda}$, we obtain

$$p = \frac{\pi}{6} \Lambda^4 - \frac{\pi}{6} \Lambda^2 m^2 - \frac{7\pi}{48} m^4 + \frac{\pi}{4} m^4 \ln m + \frac{\pi}{4} m^4 \ln \Lambda + o\left( \frac{m}{\Lambda} \right).$$ (10)

Then, on comparing the expressions (3) and (10), we see that the quartic divergence satisfies the equation of state for radiation $p = \frac{1}{3} \varepsilon$, the quadratic divergence satisfies the equation of state $p = -\frac{1}{3} \varepsilon$, which sometimes is identified with the so called string gas (see e.g. [9]), while the logarithmic divergence behaves as a cosmological constant with $p = -\varepsilon$. If all these divergences cancel, then the finite part of the pressure is

$$p_{\text{finite}} = -\left( \sum m_S^4 \ln m_s + 3 \sum m_V^4 \ln m_V - 2 \sum m_F^4 \ln m_F \right)$$ (11)

which also behaves like a cosmological constant.

The requirement that the finite part of the vacuum energy, i.e. of the observable effective cosmological constant, is very small compared with SM masses suggests that we also need a compensation between the finite parts of fermion and boson vacuum energies, thus obtaining

$$\sum m_S^4 \ln m_s + 3 \sum m_V^4 \ln m_V - 2 \sum m_F^4 \ln m_F = 0.$$ (12)

As is known the observed number of fermion degrees of freedom in the Standard Model is much bigger than the number of boson degrees of freedom [10]. Indeed $N_F$ is equal to 96 (if we consider the neutrinos as massive particles) while the number of boson degrees of freedom, carried by the photon, the gluons, $W^\pm$ and $Z^0$ bosons and by the Higgs boson is equal to 28. Thus we need an additional 68 boson degrees of freedom. At this point it is natural
to search for some minimal extension of the Standard Model, which does not modify the fermion degrees of freedom while just adding some hypothetical bosons.

The main result of paper \[5\] was a proof that, within the given framework, such an extension did not exist. In other words, we showed that on introducing new boson fields, which provide the cancellation of the ultraviolet divergences in the vacuum energy density, the finite part of the effective cosmological constant is always positive and of order of the mass of the top quark to the fourth power, which is much bigger than the value of the effective cosmological constant compatible with cosmological observations. This led to the necessity of introducing new heavy fermions. Indeed we found explicit realisations with zero finite energy by introducing at least one fermion with a suitable mass.

Let us now sketch briefly the approach of the preceding paper \[5\] which will also be used in the present paper.

After a general analysis, for the sake of simplicity, we considered an explicit minimal extension of the SM with a few massive bosons and weakly coupled, practically massless others so as to satisfy the requirement \(N_B = N_F\). Such a possibility is viable in effective action approaches and, for example, has been considered recently in scenarios such as unparticle physics \[11\]. In this minimal framework we analysed the boson masses allowed by the cancellation constraints.

Let us suppose that the contributions of fermions (or bosons) to the constraint equations \(5\), \(6\) and \(12\) are bigger than those of bosons (or fermions). In the Standard Model before the observation of the 750 GeV Diphoton Excess at the LHC the fermions dominated. However, with this last observation, the situation in inverted: it is now bosons that dominate. For definiteness lets talk about the dominance of the fermions, although it is not essential for the formalism. Thus, we take the differences between the contributions of fermions and bosons into the constraint equations to be positive and call them \(R^2\), \(h\) and \(L\) respectively. Then, if we call \(x_i\) the masses squared of the boson degrees of freedom which we wish to add to create the balance, we can write down the constraint equations as

\[
\sum_{i=1}^{n} x_i^2 = R^2,
\]  

(13)
\[ \sum_{i=1}^{n} x_i = h, \]  \hspace{1cm} (14) \\
\[ \phi \equiv \sum_{i=1}^{n} x_i^2 \ln x_i = L, \]  \hspace{1cm} (15)

where \( n \) is the number of the boson degrees of freedom.

The constraints (13), (14) have a simple geometrical sense [4]: they describe a sphere and a plane in an \( n \)-dimensional space and their intersection \( S \) is an \((n - 2)\)-dimensional sphere, eventually to be sliced on the positivity boundaries of the \( x_i \). The distance of the plane from the origin of the coordinates is \( h/\sqrt{n} \). In order to have an intersection between the sphere of radius \( R \) and the plane it is then necessary to have

\[ n > \frac{h^2}{R^2}. \]  \hspace{1cm} (16)

In general it is convenient to introduce the integer value \( n_0 \) for such a threshold

\[ n_0 = \left\lceil \frac{h^2}{R^2} + 1 \right\rceil, \]  \hspace{1cm} (17)

so that \( n \geq n_0 \) is the requirement to have a non empty \( S \) (here the square brackets denote the integer part of a number). Further, the sphere \( S \) should contain points such that all the coordinates \( x_i \) are positive. This condition is given by the inequality

\[ h^2 > R^2 \]  \hspace{1cm} (18)

and geometrically is equivalent to the requirement that the hyperplane (14) intersect the axis of the \( n \)-dimensional space outside the hypersphere (13).

The conditions (16,18) for the existence of physical solutions to the geometrical constraint equations (13,14) can be reduced to the following inequality

\[ R^2 < h^2 < nR^2. \]  \hspace{1cm} (19)

Then, in principle, if (18) is verified, one can find a multiplet of massive bosonic particles satisfying the geometrical constraints.

In [4] we calculated the maximum and minimum values for the \( x_i \)'s satisfying the geometrical equations. When the constraint (15) is added such values also give an estimate of the interval on which \( x_i \)'s vary. When more general model building is considered additional hypothesis are added to the above
constraints (13–15). For example one may ask for cancellations among particle multiplets with different multiplicities $w_i$’s. In such a case the above formulae should be modified accordingly. In particular the geometrical equations become

$$\sum_{i=1}^{n} w_i x_i^2 = R^2, \quad \sum_{i=1}^{n} w_i x_i = h$$

(20)

where $n$ is the number of particles in the multiplet and, by redefining $\sqrt{w_i x_1} \equiv y_i$ one is led to slightly modified geometrical constraints with the same hypersphere but a different hyperplane. The distance between the plane and the origin $y_i = 0$ is now $d = h/\sqrt{\sum_{i=1}^{n} w_i}$ and the condition (16) becomes

$$\tilde{n} > \frac{h^2}{R^2}$$

(21)

where $\tilde{n}$ is the number of degrees of freedom. In order to have positive solutions from the intersection between the sphere and the plane we now have

$$h^2 > w_{\text{min}} R^2$$

(22)

where $w_{\text{min}}$ is the minimum multiplicity we consider in the multiplet. Then, if the condition

$$w_{\text{min}} R^2 < h^2 < \tilde{n} R^2$$

(23)

is satisfied we have physically acceptable solutions from the intersection (20).

One can calculate, for each $\bar{x}$ solution of (20), how their maximum and minimum ($\bar{x}_M$ and $\bar{x}_m$ respectively) vary as a function of the corresponding multiplicity $\bar{w}$. We find:

$$\bar{x}_{M,m} = \frac{h}{\tilde{n}} \pm \sqrt{\frac{\tilde{n} R^2 - h^2}{\tilde{n}^2} \sqrt{\frac{\tilde{n}}{\bar{w}}} - 1}$$

(24)

and we observe that, correspondingly, the other masses $x_i$ in the multiplet are equal.

For a given value of $h, R$ and a suitable choice of $\tilde{n}$ we find that a multiplicity larger than one leads to a smaller maximum and a bigger minimum. Let us note that the minimum is positive if

$$n < \frac{h^2}{R^2} + w \Rightarrow n \leq \left[ \frac{h^2}{R^2} \right] + w = n_0 + w - 1,$$

(25)
and then in order to find very massive particles in the intersection between
the sphere and the plane one must restrict the analysis to the case \( n = n_0 \).
The problem of finding the extrema of the \( x_i \)'s can be generalised and the
whole set of three constraints (13-15) taken into account. For such a case, let
us suppose we need the extrema for \( x_1 \), then one can introduce the auxiliary
function

\[
f(\{x_i\}) \equiv x_1 + \alpha \left( \sum_{i=1}^{n} w_i x_i^2 \ln x_i - l \right) - \lambda \left( \sum_{i=1}^{n} w_i x_i^2 - R^2 \right) - \mu \left( \sum_{i=1}^{n} w_i x_i - h \right)
\]

where \( \alpha, \lambda \) and \( \mu \) are three Lagrange multipliers. The condition \( \partial f/\partial x_i = 0 \) takes the form

\[
\frac{\delta_{1i}}{w_1} + \alpha (2 x_i \ln x_i + x_i) + 2 \lambda x_i + \mu = 0
\]

and is independent on \( w_i \) for \( i \neq 1 \). One then observes that the max/min values for \( x_1 \) are in the multiplets formed by 3 different masses: \( \bar{x}_1, \bar{x}_2, \bar{x}_3 \) satisfying the constraints (13-15):

\[
\begin{align*}
w_1 \bar{x}_1 + \tilde{n}_2 \bar{x}_2 + (\tilde{n} - w_1 - \tilde{n}_2) \bar{x}_3 &= h \\
w_1 \bar{x}_1^2 + \tilde{n}_2 \bar{x}_2^2 + (\tilde{n} - w_1 - \tilde{n}_2) \bar{x}_3^2 &= R^2 \\
w_1 \bar{x}_1^2 \ln \bar{x}_1 + \tilde{n}_2 \bar{x}_2^2 \ln \bar{x}_2 + (\tilde{n} - w_1 - \tilde{n}_2) \bar{x}_3^2 \ln \bar{x}_3 &= L
\end{align*}
\]

where \( w_1 \) is the multiplicity associated with \( x_1 \), \( \tilde{n} \) is the total number of d.o.f. chosen to satisfy the constraints and \( \tilde{n}_2 \) is number of d.o.f. with mass \( \bar{x}_2 \). Depending on the multiplicities \( w_i, \tilde{n}_2 \) may take a few possible values between 1 and \( \tilde{n} - w_1 - 1 \). The solutions of the system (28) can be found numerically on varying \( \tilde{n}_2 \) and then the max/min values for \( x_1 \) can be determined. Let us note that the above approach can be extended to the case of hybrid particle multiplets containing both boson and fermion d.o.f. on switching the relative sign of the \( w_i \)'s. In this latter case eq. (27) for \( i \neq 1 \) is unchanged and one can still conclude that the multiplet consists of 3 masses.

In order to now see when the condition (15) can also be satisfied, it is
convenient to calculate the minimum value of the function \( \phi = \frac{1}{2} \sum_{i=1}^{n} x_i^2 \ln x_i^2 \)
on the constraint surface S. Let us consider an auxiliary function

\[
F(\{x_i\}) = \frac{1}{2} \sum_{i=1}^{n} x_i^2 \ln x_i^2 - \lambda \left( \sum_{i=1}^{n} x_i^2 - R^2 \right) - \mu \left( \sum_{i=1}^{n} x_i - h \right)
\]

(29)
where \( \lambda \) and \( \mu \) are the Lagrange multipliers. The search for the extrema of the function \( F \) implies we should equate its derivatives with respect to \( x_i, \lambda \) and \( \mu \) to zero. These last two conditions \( \partial F/\partial \lambda \) and \( \partial F/\partial \mu \) again give the constraints (13), (14). Differentiation with respect to \( x_i \) gives the system of equations:

\[
x_i \ln x_i^2 + x_i - 2\lambda x_i - \mu = 0, \quad i = 1, \ldots, n.
\]

(30)

Without loss of generality we can choose \( x_1 \neq x_2 \). Indeed it is possible to have \( x_1 = x_2 = \cdots = x_n \) if and only if \( h^2/n^2 = R^2 \), but this is a degenerate case, when the sphere and the plane touch each other in only one point.

On substituting the values of \( x_1 \) and \( x_2 \) into the first two equations of the system (30), one obtains \( \bar{\lambda} \) and \( \bar{\mu} \) as functions of \( x_1 \) and \( x_2 \):

\[
\bar{\lambda} = \frac{1}{2} + \frac{x_1 \ln x_1^2 - x_2 \ln x_2^2}{2(x_1 - x_2)}, \quad \bar{\mu} = \frac{x_1 x_2(\ln x_2^2 - \ln x_1^2)}{x_1 - x_2}.
\]

(31)

Let us now suppose that \( \bar{x}_1, \cdots, \bar{x}_n, \bar{\lambda}, \bar{\mu} \) are a solution of the system (30) on S, i.e. with the constraints (13) and (14) already satisfied. On then substituting these values of \( \bar{\lambda} \) and \( \bar{\mu} \) into the \( n - 2 \) remaining equations of the system (30) one can easily see that a solution is given by \( x_1 = x_3 = x_4 = \cdots = x_{k+1} \) and \( x_2 = x_n = x_{n-1} = \cdots = x_{k+2} \). This solution is a stationary point of the function \( F \), or in other words the conditional stationary point of the function \( \phi \). Such a solution, with \( k \) coordinates having the value \( \bar{x}_1 = x \) and the remaining \( n - k \) coordinates the other value \( \bar{x}_2 = y \), is given, as function of \( k \) and \( n \), by

\[
x = x(k, n) = \frac{h}{n} + \sqrt{\frac{R^2(n-k)}{nk} - \frac{h^2(n-k)}{n^2k}},
\]

(32)

\[
y = y(k, n) = \frac{h}{n} - \sqrt{\frac{R^2k}{n(n-k)} - \frac{h^2k}{n^2(n-k)}},
\]

(33)

where \( 1 \leq k \leq n - 1 \).

The values of \( x \) given by Eq. (32) are always positive, while the values of \( y \) can be negative. It is easy to show that the condition for the positivity of \( y \) is

\[
k < n_0 \leq n.
\]

(34)

We have seen that points of the type described above always satisfy the stationarity conditions (30) on the constraint surface. This does not mean
that stationary points of other types cannot exist. Indeed, the analysis of the structure of Eq. (30) shows that, in principle, stationary points whose coordinates $x_i$ have three different values can exist. However, if such points exist, at least one of these three values is negative and, hence, is of no interest to us. Thus, the minimum of the function $\phi$ can be reached only for the stationary points having the coordinates (32), (33) or on the boundary of the positivity region, where at least one of the coordinates $x_i$ is equal to zero. For this last case the problem is reduced to one with a lower dimensionality than $n$.

If $n = n_0$ (the smallest possible value for the dimensionality of $n$) we notice that on the surface $S$ all the $x_i$ have positive values. Thus, the maximum and minimum values of the function $\phi$ on the constraint surface are obtained only for one of the pairs of points with the coordinates $x$ and $y$ (see formulae (32), (33)).

Furthermore the following more general statement is true: for a given $n \geq n_0$ the maximum value of the function $\phi$ corresponds to $k = 1$ while its minimum value corresponds to $k = n - 1$. To prove this one may compute the derivatives of the function

$$\phi_1(k, n) = \frac{k}{2} x^2 \ln x^2 + \frac{(n - k)}{2} y^2 \ln y^2,$$

(35)

with respect to $k$ and $n$. It can be shown that $d\phi_1/dk < 0$ and $d\phi_1/dn > 0$ for the range of possible physical values of $k$ and $n$. In particular this means that the function $\phi_1(k, n)$ decreases with increasing $k$ and has its minimum value at $k = n_0 - 1$ and $n = n_0$. Once one chooses $n$ such that it satisfies the geometrical constraints then the minimum and the maximum values for $\phi_1(k, n)$ are respectively

$$\phi_{1\text{min}} = \phi_1(n - 1, n)$$

(36)

where one has to use Eqs. (32) and (33) to express $x$ and $y$, and

$$\phi_{1\text{max}} = \phi_1(1, n).$$

(37)

The solution of the equation

$$\sum_{i=1}^{n} x_i^2 \ln x_i = L$$

(38)
exists on the constraint surface $S$ only if

$$\phi_{1 \text{ min}} < L < \phi_{1 \text{ max}}.$$  

(39)

This constraint will be then used in the next section as a criterium for building the extensions of the SM containing a 750 GeV boson. Let us note that this last result is also valid when multiplicities $w_i \neq 1$ are considered. For such a case equations (30,35) are the same as $w_i$ simplifies. Depending on the particle content, however, $k$ may not take all the integers values in the interval $[1, n - 1]$ and the resulting (36,37) should be modified accordingly.

In paper [5] by direct calculation it was shown that it was impossible to to satisfy the conditions (5), (6), (12) by adding some hypothetical bosons. The extension of the Standard Model had to also include some new fermions. In the next section we apply the technique sketched in this section to various hypothesis concerning the particle responsible for the 750 GeV puzzle.

### 3 Beyond the Standard Model: 750 GeV puzzle and the constraint equations

The Standard Model particles which give a significant contribution to our equalities have the following particle masses: the top quark mass $m_t = 173$ GeV, the Higgs mass $m_H = 125$ GeV, the Z boson mass $m_Z = 91$ GeV and the W boson mass $m_W = 80$ GeV. If we normalize $x_i$ w.r.t. the top mass we then have $x_t = 1, x_H = 0.52, x_Z = 0.28, x_W = 0.21$.

Let us suppose that a scalar boson exists with mass $m_B = 750$ GeV and $x_B = 18.8$. In this case the bosons dominate the constraint equations and

$$\begin{align*}
(R^{(1)})^2 &= x_B^2 + x_H^2 + 3x_Z^2 + 6x_W^2 - 12x_t^2 \\
h^{(1)} &= x_B + x_H + 3x_Z + 6x_W - 12x_t \\
L^{(1)} &= x_B^2 \ln x_B + x_H^2 \ln x_H + 3x_Z^2 \ln x_Z + 6x_W^2 \ln x_W - 12x_t^2 \ln x_t
\end{align*}$$

(40)

with $R^{(1)} = 18.5, h^{(1)} = 9.43$ and $L^{(1)} = 1035$. Hence we wish to add fermions which provide the necessary cancellations. However $h^2 < R^2$ and the condition (18) is not satisfied hence it is impossible to find such a set of fermions without also adding some bosons. One can then follow two strategies. One can first add a fermion $x_f$ which is massive enough to invert the
Figure 1: The above figures plot $n_0(x_f)$ (large, light grey band) and the constraint (39) on varying $x_f$ and for different choices of $n$ (small, dark grey band) when a 750 GeV scalar and a fermion with mass $x_f$ are added to the SM. In the figure on the l.h.s. the case for a quark is plotted, in the figure on the centre the case with a colourless Dirac fermion is shown and in the figure on the r.h.s. the case for a Majorana fermion is shown.

balance, satisfy the conditions (18) and (39) and then search for a set of bosons satisfying our constraint equations which is the approach previously followed ([5]). Conversely one can add a massive boson $x_b$ with $n_b$ degrees of freedom such that $(h + n_b x_b)^2 > R^2 + n_b x_b^2$ and then search for a set of fermions cancelling vacuum energy divergencies and finite part. In the latter case many massless boson degrees of freedom must be added to in order to compensate for the fermion ones, hence we then prefer to follow the former approach, whenever possible. For both cases we assume that the massive particle which must be added is heavier than $x_B$ and is thus still unobserved.

### 3.1 An extra coloured quark

Let us first consider the case of a coloured quark, which has 12 degrees of freedom, with mass $x_f$. One can then plot $n_0$ and the constraint (39) as functions of $x_f$ (see Fig. 1 on the left). Thus the intervals for the quark mass which satisfy the constraint (39) are $x_f \in [34.9, 47.8]$ ($m_f \in [1022, 1196]$)
with 12 d.o.f., $x_f \in [15.1, 30.8]$ and $x_f \in [11.6, 26.9]$ with 13 and 14 d.o.f. respectively. We shall not consider a larger number of degrees of freedom even if it is possible, in principle, to satisfy (39) with $n > 14$ and $x_f > x_B$. The first case given above – adding 12 boson degrees of freedom $x_i$ – is relevant because $n = n_0$ and from (25) we know, independently of the multiplicities $w_i$ of the boson particle content, for $x_f \in [34.9, 47.8]$, that the solutions to the constraints must have $x_i > x_B$. Conversely if $n > n_0$ the geometrical minimum $x_m$ is less than $x_B$ and we expect solutions with a few masses below $x_B$.

We first study a few cases with the minimal particle content. Such cases can be investigated numerically and the entire set of solutions to the constraints can be easily plotted.

1a) Consider the addition of 4 vector bosons with two independent masses $x_1$ and $x_2$. For such a case we satisfy the three constraints for $x_f = 32.9, x_1 = 32.9$ and $x_2 = 44.3$ and these 3 particles are heavier than 750 GeV.

2a) Consider the addition of 4 vector bosons with 3 independent masses $x_1, x_2$ and $x_3$ where $x_1$ is the common mass of 2 vector bosons. One then finds a one-parameter set of solutions shown on the left-hand side of Fig. 2 where: the dotted line represents the coincident masses of two vector bosons, while the solid line represents the masses of the remaining bosons. All the boson and fermion masses are heavier than 750 GeV.

More general cases with at least 12 bosonic degrees of freedom can also be discussed as the minimum masses in the multiplet always appear in patterns made of 3 distinct $\bar{x}_i$ (as discussed in the previous section). More features of such cases can be extracted numerically by using a Monte Carlo (MC) inspired technique which samples the hyperspace spanned by the masses in the boson sector $x_i$. If we consider a set of bosons containing 3 new massive vectors and 4 scalars and fix $x_f = 45$ ($m_f = 1160$ GeV), we can then calculate the lightest mass of the scalar sector $x_s = 24.4$ of the lightest vector $x_v = 41.3$ and with the MC we find their respective mass interval $m_s \in [854, 867]$ GeV, $m_v \in [1112, 1167]$ GeV. If two of these three vector bosons have the same mass (like the $W^\pm$ bosons of the Standard Model), then the lightest scalar particle has a mass in the interval $m_s \in [854, 871]$ GeV, while the lightest vector particle has a mass in
3.2 An extra Dirac fermion

Let us now consider the case of an additional Dirac fermion with 4 degrees of freedom. On plotting $n_0$ and the constraint (39) as functions of $x_f$ (see Fig. 1 in the middle) one observes that at least 4 boson degrees of freedom are needed (with $x_f \in [37.7, 46.4]$), 5 if $x_f \in [18.8, 19.3]$. Starting with the cases with a minimal particle content we find the following.

1b) On adding one vector boson ($x_v$) and one scalar ($x_s$) : for this case we satisfy the constraints with $x_f = 37.7$, $x_v = 29.9$ and $x_s = 51.4$ with the three particles heavier than 750GeV.

2b) On adding a vector boson and 2 scalars ($x_1$, $x_2$ and $x_3$ respectively), we find a one-parameter set of solutions illustrated in the Fig. 2. For
Figure 3: The above figures plot $n_0(x_f)$ (large, light grey band) and the constraint \[^{[39]}\] on varying $x_f$ and for different choices of $n$ (small, dark grey band) when a 375 GeV fermion and a fermion with mass $x_f$ are added to the SM. In the figure on the l.h.s. the case with a two Dirac fermions is plotted, i and in the figure on the r.h.s. the case with two quark fermions is shown.

such a case light particles are always present in the spectrum.

3b) On adding a Dirac fermion with mass $x_f = 45$ and four scalar particles with different masses, one has that the lightest mass in the scalar sector is $m_s = 854$ GeV and the boson multiplet is made of four particles particles heavier than 750GeV.

3.3 An extra Majorana fermion

On adding a Majorana fermion with 2 degrees of freedom, we find that it is necessary to have at least two additional boson d.o.f. to satisfy \[^{[18]}\]. The cases which we considered are the following:

1c) On adding two scalar fields $x_1$ and $x_2$ we find the solution $x_f = 43.9, x_1 = 24.4, x_2 = 54.1$ with the three particles heavier than 750GeV.

2c) On adding a vector boson and a scalar ($x_v$ and $x_s$ respectively), we find $x_f = 25.2, x_s = 29.8, x_v = 3.7$, and a light boson in the spectrum.
3.4 Composite particle

Instead of the 750 GeV scalar particle one can suppose that the 750 GeV diphoton excess at the LHC is associated with the existence of a composite boson made of a Dirac fermion-antifermion pair with a total mass 750 GeV and negligible binding energy. For such a case the vacuum energy cancellation constraints are

\[
\begin{align*}
(R^{(c)})^2 &= -w_f x_f^2 + x_H^2 + 3 x_Z^2 + 6 x_W^2 - 12 x_t^2 \\
h^{(c)} &= -w_f x_f + x_H + 3 x_Z + 6 x_W - 12 x_t \\
L^{(c)} &= -w_f x_f^2 \ln x_f + x_H^2 \ln x_H + 3 x_Z^2 \ln x_Z + 6 x_W^2 \ln x_W - 12 x_t^2 \ln x_t
\end{align*}
\]

with \(R^{(c)} = 16.6\), \(h^{(c)} = 65.7\) and \(L^{(c)} = 410.8\) for the case of a coloured Dirac fermion and \(R^{(c)} = 10\), \(h^{(c)} = 28.2\) and \(L^{(c)} = 137.5\) for the case of a colourless Dirac fermion. In both cases the condition \((39)\) is not satisfied for \(n = n_0\) and one must add one more fermion in order to keep the masses of the boson d.o.f. higher than 750 GeV. Let us consider the case of colourless fermions. In this case it is necessary to add one more Dirac fermion. For such a case one also needs at least 5 massive boson d.o.f. (see Fig. 3)). Starting with just a few particles we have

1c) Addition of two vector bosons \((x_1\) and \(x_2)\). In this case we satisfy the constraints with \(x_f = 73.1\), \(x_1 = 26.6\) and \(x_2 = 81.2\).

2c) For a vector boson and 2 scalars \((x_1\), \(x_2\) and \(x_3)\) respectively), we find a one-parameter set of solutions illustrated in the Fig. (4). For such a case the particles needed are always heavier than 750 GeV.

The cases with more particles involved can be studied separately. Let us take a fermion of mass \(x_f = 110\). Then, one has to add at least 5 boson degrees of freedom. On considering 5 scalar fields, we find that the lightest one has a mass in the interval \(m_s \in [1359, 1469]\) GeV.

The composite boson can also be made of a coloured quark-antiquark pair. This case is similar to the colourless case but now at least 13 d.o.f. are involved.
3c) If we consider the case of 4 vector bosons with 2 different masses $x_1$ and $x_2$ plus a scalar $x_3$, we find a one-parameter set of solutions illustrated in Fig. 4.

Let us consider a quark with mass $x_f = 120$ and 14 scalars. For this case the lightest scalar in the multiplet has a mass $m_s = 1268$ GeV. The same lightest scalar appears in more complicated multiplets with, at least, 2 scalars. For example let us consider a multiplet formed by three vector fields (two of them with the same mass) and 5 scalars. In this case the lowest scalar mass lies in the interval $m_s \in [1268, 1777]$ GeV while the lowest vector mass belongs to the interval $m_v \in [1483, 1815]$ GeV, where the first value in the last two intervals is calculated analytically and the second is estimated by MC techniques.

### 3.5 Spin two particle

Finally we consider a spin two particle - a massive graviton, which has 5 degrees of freedom (see e.g. [12]). The equations (13-15) take the following
form:

\[
\begin{align*}
(R^{(2)})^2 &= 5x_B^2 + x_H^2 + 3x_Z^2 + 6x_W^2 - 12x_t^2 \\
h^{(2)} &= 5x_B + x_H + 3x_Z + 6x_W - 12x_t \\
L^{(2)} &= 5x_B \ln x_B + x_H \ln x_H + 3x_Z \ln x_Z + 6x_W \ln x_W - 12x_t \ln x_t
\end{align*}
\] (42)

with \(R^{(2)} = 41.9\), \(h^{(2)} = 84.6\) and \(L^{(2)} = 5180\). To balance its big contribution to the vacuum energy, one has add a fermion with a mass greater than that of the massive graviton. For such a minimal case we could not find any interesting solution (with masses bigger than 750 GeV) on only adding one or two massive fermions, which does appear satisfactory from the observations point of view.

4 Conclusions

Let us begin by briefly summarizing our results. We first considered the case for which the 750 GeV particle is a scalar and found, in order to satisfy the constraints, the following minimal particle contents:

a) an extra coloured quark with mass in the interval \([1022, 1196]\) GeV and correspondingly 12 bosonic d.o.f.;

b) an extra Dirac fermion with mass in the interval \([1062, 1178]\) GeV and correspondingly 4 bosonic d.o.f.;

c) an extra Majorana fermion with a mass of 1146 GeV and correspondingly 2 scalar d.o.f..

Going beyond such minimal cases one generally finds unsatisfactory (light) masses for either the fermion or the bosons.

We then considered the case for which the 750 GeV particle is a composite object consisting of a lightly bound fermion-antifermion pair, each fermion having a mass of 375 GeV. In order to satisfy the constraints we found the following minimal particle contents:

d) for the 375 GeV Dirac fermion, one extra colourless fermions and at least 5 bosonic d.o.f.;
e) for the 375 GeV coloured fermion, one extra coloured quark and at least 13 bosonic d.o.f..

Lastly we considered a spin 2 750 GeV particle. We could not compensate such a large contribution on only adding one or two extra massive fermions plus any massive boson (heavier than 750 GeV) multiplet.

We have thus seen that on adding to the Standard Model different hypothetical particles associated with the 750 GeV diphoton excess and requiring the cancellation of its vacuum energy, one can find solutions in which various new fields appear, both bosons and fermions with masses compatible with present data.

Let us further note that to treat the finite contribution of quantum fields to the vacuum energy it is reasonable to combine the cancellation mechanism proposed by W. Pauli [1] which was applied in [4, 5] and in the present paper together with the standard renormalization procedure for the ultraviolet divergences in the traditional quantum field theory of the scattering matrix (see e.g. [13, 14]). Indeed, the cancellation of the contributions of boson and fermion fields to the vacuum energy is useful only to treat the one-loop divergences coming from propagators of free fields. As is well-known these divergences, which in standard quantum field theory, in the absence of gravity, are eliminated by the normal ordering procedure, create problems, when gravity is included. All the other ultraviolet divergences need not be cancelled, but rather renormalized. It is worth remembering that even in the majority of models possessing an exact supersymmetry not all the ultraviolet divergences are cancelled. For example, in the Wess-Zumino model [15] the loop-contributions to the vertices are ultraviolet finite, while those to the propagators are divergent and must be renormalized.

Thus, the standard procedure of the elimination of ultraviolet divergences, results in the renormalization of masses, coupling constants and kinetic terms of the fields under consideration. The last (kinetic terms) can be fixed in such a way that they have standard form, i.e. the corresponding renormalization coefficients are equal to one [13]. The renormalization of the masses reduces to the appearance of the physically measurable masses - these are the ones which we used for the sum rules in [4, 5] and in the present paper.

Let us conclude by noting that we have assumed some suitable breaking of supersymmetry which breaks equality of boson and fermion masses (existing before breaking even in the presence of interactions), but still keeps
the relations (5), (6), (12). It could be that this is possible in the context of supergravity only. This is because we know that even in renormalizable theories, only "measurable" quantities participating in some interactions can be renormalized, quantities which we cannot measure using some interaction vertices need not be finite. Since gravity is needed to measure the total vacuum energy density, not its difference between different quantum states, gravity has to be included, and then supersymmetry transforms to supergravity. However this is a scope for further work.

Acknowledgements

The work of A.K. and A.S. was partially supported by the RFBR grant 14-02-00894.

References

[1] W. Pauli, Selected Topics in Field Quantization, V. 6 of Pauli Lecture on Physics, Dover, Mineola, New York, 2000, Lectures delivered in 1950-1951 at the Swiss Federal Institute of Technology.

[2] Ya.B. Zeldovich, Cosmological Constant and Elementary Particles, JETP Lett. 6 (1967) 316; Ya.B. Zeldovich, The Cosmological Constant and the Theory of Elementary Particles Sov. Phys. - Uspekhi 11 (1968) 381.

[3] G. P. Vacca and L. Zambelli, Functional RG flow equation: regularization and coarse-graining in phase space, Phys. Rev. D 83 (2011) 125024 arXiv:1103.2219 [hep-th].

[4] A.Y. Kamenshchik, A. Tronconi, G.P. Vacca, G. Venturi, Vacuum energy and spectral function sum rules, Phys. Rev. D 75 (2007) 083514, hep-th/0612206.

[5] G. L. Alberghi, A. Y. Kamenshchik, A. Tronconi, G. P. Vacca, G. Venturi, Vacuum energy, cosmological constant and standard model physics, JETP Lett. 88 (2008) 705.
[6] G. Aad et al., [ATLAS Collaboration], *Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC*, Phys. Lett. B 716 (2012) 1; S. Chatrchyan et al., [CMS Collaboration], *Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC*, Phys. Lett. B 716 (2012) 30.

[7] The ATLAS collaboration, ATLAS-CONF-2015-081; CMS Collaboration [CMS Collaboration], collisions at 1 3TeV, CMS-PAS-EXO-15-004.

[8] K. Harigaya and Y. Nomura, *Composite Models for the 750 GeV Diphoton Excess*, Phys. Lett. B 754 (2016) 151; R. Franceschini et al., *What is the gamma gamma resonance at 750 GeV?*, [arXiv:1512.04933 [hep-ph]]; S. Di Chiara, L. Marzola and M. Raidal, *First interpretation of the 750 GeV di-photon resonance at the LHC*, [arXiv:1512.04939 [hep-ph]]; J. Ellis, S. A. R. Ellis, J. Quevillon, V. Sanz and T. You, *On the Interpretation of a Possible ~ 750 GeV Particle Decaying into $\gamma\gamma$*, [arXiv:1512.05327 [hep-ph]]; A. Ahmed, B. M. Dillon, B. Grzadkowski, J. F. Gunion and Y. Jiang, *Higgs-radion interpretation of 750 GeV diphoton excess at the LHC*, [arXiv:1512.05771 [hep-ph]], M. Redi, A. Strumia, A. Tesi and E. Vigiani, *Di-photon resonance and Dark Matter as heavy pions*, [arXiv:1602.07297 [hep-ph]].

[9] A. Y. Kamenshchik and I. M. Khalatnikov, *Some properties of the ‘String gas’ with the equation of state $p = -\rho/3$*, Int. J. Mod. Phys. D 21 (2012) 1250004; I. M. Khalatnikov, A. Y. Kamenshchik and A. A. Starobinsky, *Quasi-isotropic expansion for a two-fluid cosmological model containing radiation and stringy gas*, [arXiv:1312.0237 [astro-ph.CO]].

[10] K.A. Olive et al. (Particle Data Group), *REVIEW OF PARTICLE PHYSICS*, Chin. Phys. C 38 (2014) 090001.

[11] H. Georgi, *Unparticle Physics*, Phys. Rev. Lett. 98, 221601 (2007), [hep-ph/0703260].

[12] K. Hinterbichler, *Theoretical Aspects of Massive Gravity*, Rev. Mod. Phys. 84 (2012) 671.

[13] N.N. Bogoliubov and D.V. Shirkov, *Introduction to the theory of quantized fields*, New York, Wiley-Interscience, 1980.
[14] Y. Nambu, *S Matrix in semiclassical approximation*, *Phys. Lett. B* 26 (1968) 626.

[15] J. Wess and B. Zumino, *A lagrangian model invariant under supergauge transformations*, *Phys. Lett. B* 49 (1974) 52.