Spin-spin correlation functions of the $XXZ-\frac{1}{2}$ Heisenberg chain in a magnetic field

N. Kitanine\textsuperscript{1}, J. M. Maillet\textsuperscript{2}, N. A. Slavnov\textsuperscript{3}, V. Terras\textsuperscript{4}

Abstract

Using algebraic Bethe ansatz and the solution of the quantum inverse scattering problem, we compute compact representations of the spin-spin correlation functions of the $XXZ-\frac{1}{2}$ Heisenberg chain in a magnetic field. At lattice distance $m$, they are typically given as the sum of $m$ terms. Each term $n$ of this sum, $n = 1, \ldots, m$, is represented in the thermodynamic limit as a multiple integral of order $2n + 1$; the integrand depends on the distance as the power $m$ of some simple function. The root of these results is the derivation of a compact formula for the multiple action on a general quantum state of the chain of transfer matrix operators for arbitrary values of their spectral parameters.
1 Introduction

The main challenging problem in the field of quantum integrable models is to compute exact and manageable expressions for their correlation functions. This issue is of great importance, not only from a theoretical viewpoint but also for applications to relevant physical situations.

The archetype of quantum integrable lattice models is provided by the XXZ spin-$\frac{1}{2}$ Heisenberg chain in a magnetic field,

$$H = \sum_{m=1}^{M} \left( \sigma_{m}^{x} \sigma_{m+1}^{x} + \sigma_{m}^{y} \sigma_{m+1}^{y} + \Delta (\sigma_{m}^{z} \sigma_{m+1}^{z} - 1) \right) - h S_{z}, \tag{1.1}$$

where

$$S_{z} = \frac{1}{2} \sum_{m=1}^{M} \sigma_{m}^{z}, \quad [H, S_{z}] = 0. \tag{1.2}$$

Here $\Delta$ is the anisotropy parameter, $h$ an external classical magnetic field, and $\sigma_{m}^{x,y,z}$ denote the usual Pauli matrices acting on the quantum space at site $m$ of the chain. The simultaneous reversal of all spins is equivalent to the change of the sign of the magnetic field, therefore it is enough to consider the case $h \geq 0$. In the thermodynamic limit $M \to \infty$ and at zero magnetic field, the model exhibits three different regimes depending on the value of $\Delta$: for $\Delta \leq -1$, the model is ferromagnetic; for $-1 < \Delta \leq 1$, the model has a non degenerated anti-ferromagnetic ground state, and the spectrum is gapless (massless regime); for $\Delta > 1$, the ground state is twice degenerated with a gap in the spectrum (massive regime).

Although the method to compute eigenstates and energy levels goes back to H. Bethe in 1931 [1, 2, 3, 4], the knowledge of its spin correlation functions has been for a long time restricted to the free fermion point $\Delta = 0$, a case for which nevertheless tremendous works have been necessary to obtain full answers [5, 6, 7, 8, 9, 10].

However, at zero temperature and for zero magnetic field, multiple integral representations of elementary blocks of the correlation functions (see definition below) have been obtained from the $q$-vertex operator approach (inspired from the corner transfer matrix technique) in the massive regime $\Delta > 1$ in 1992 [11], and conjectured in 1996 [12] for the massless regime $-1 < \Delta \leq 1$ (see also [13]). A proof of these results together with their extension to non-zero magnetic field has been obtained in 1999 [14, 15] for both regimes using algebraic Bethe ansatz [16, 17] and the actual resolution of the so-called quantum inverse scattering problem [14, 18].

These elementary blocks for correlation functions are defined in the following way:

$$F_{m}(\{\epsilon_{j}, \epsilon'_{j}\}) = \frac{\langle \psi_{g} | \prod_{j=1}^{m} E_{j}^{\epsilon_{j}, \epsilon'_{j}} | \psi_{g} \rangle}{\langle \psi_{g} | \psi_{g} \rangle}. \tag{1.3}$$
Here $|\psi_g\rangle$ denotes the ground state in the massless regime or any of the two ground states constructed by algebraic Bethe ansatz in the massive regime; $E^{\epsilon_m, \epsilon m}_{m}$ are the elementary operators acting on the quantum space $\mathcal{H}_m$ at site $m$ as the $2 \times 2$ matrices $E^{\epsilon, \epsilon} = \delta_{\epsilon, \epsilon} \delta_{\epsilon, \epsilon}$. Any $n$-point correlation function can be reconstructed as a sum of such elementary blocks.

To compute these elementary blocks, the following successive problems have to be addressed [14, 15]: (i) determination of the ground state $\langle \psi_g |$, (ii) evaluation of the action of the product of the local operators $E^{\epsilon, \epsilon}_{j, j}$ on this ground state, and (iii) computation of the scalar product of the resulting state with $|\psi_g\rangle$.

For the $XXZ$ spin-$\frac{1}{2}$ Heisenberg chain in a magnetic field, these problems have been solved in [14, 15] in the framework of algebraic Bethe ansatz. The central object of this method is the so called quantum monodromy matrix depending on a complex variable $\lambda$ (the spectral parameter). For the $XXZ$ spin-$\frac{1}{2}$ chain, it is a $2 \times 2$ matrix with operator valued entries acting in the quantum space of states $\mathcal{H}$:

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \tag{1.4}$$

The quadratic commutation relations between these four operators are given by the Yang-Baxter algebra. It is governed by a trigonometric $R$ matrix solving the Yang-Baxter equation. The Hamiltonian of the chain is then contained in the commutative family of operators generated by the transfer matrix $T(\lambda) = (A + D)(\lambda)$ for arbitrary values of $\lambda$. The algebraic Bethe ansatz leads to the simultaneous diagonalization of these transfer matrices and of the Hamiltonian.

The ground state $\langle \psi_g |$ (resp. $|\psi_g\rangle$), as the other eigenstates, is given as the successive action of operators $C(\lambda_k)$ (resp. $B(\lambda_k)$) on the ferromagnetic reference state $\langle 0 |$ (resp. $| 0 \rangle$) with all spins up. Namely, we have $\langle \psi_g | = \langle 0 | \prod_k C(\lambda_k)$ and $|\psi_g\rangle = \prod_k B(\lambda_k)| 0 \rangle$ for a particular set of spectral parameters $\{\lambda_k\}$ solving the Bethe equations.

To evaluate the action of local operators on this state, the strategy is to imbed them in the Yang-Baxter algebra of $T$ matrices by solving the quantum inverse scattering problem (see [14, 15] for details) as

$$E^{\epsilon', \epsilon}_{j, j} = \prod_{k=1}^{j-1} (A + D)(\frac{\eta}{2}) \ T_{\epsilon, \epsilon'}^{\epsilon', \epsilon} \prod_{k=1}^{j} (A + D)^{-1}(\frac{\eta}{2}), \tag{1.5}$$

where $\cosh \eta = \Delta$. Then, using the Yang-Baxter algebra, one can reduce any elementary blocks of the correlation functions to multiple sums of scalar products of some states with $|\psi_g\rangle$. Each of these scalar products can be computed as the ratio of two explicit determinants [13, 14]. In the thermodynamic limit, these multiple sums lead to $m$ fold integrals over contours $C^h_j$ which depend on the value of $j$, on the regime considered and also on the value of the magnetic field. The answer can be written generically as [14]

$$F_m(\{\epsilon_j, \epsilon'_j\}) = \prod_{j=1}^{m} \int_{C^h_j} d\lambda_j \ \Omega_m(\{\lambda\}, \{\epsilon_j, \epsilon'_j\}) \ S_h(\{\lambda\}). \tag{1.6}$$

3
Here $\Omega_m(\{\lambda\}, \{\epsilon_j, \epsilon_j'\})$ is a purely algebraic quantity, which in particular does not depend on the regime nor on the magnetic field. In contrast, $S_h(\{\lambda\})$ is a functional of the density function $\rho_h(\lambda)$ solution of the Lieb equation describing the ground state, and hence depends both on the regime and on the value of the magnetic field $h$.

It is remarkable that, for zero magnetic field, the two multiple integral representations resulting from the $q$-vertex operator approach and from Bethe ansatz are identical: contours and integrands coincide. It would be very desirable to understand this intriguing fact directly at the operator level. Note that for non-zero magnetic field, the quantum affine symmetry used in the $q$-vertex operator approach is broken, and no result is known up to now from this method in this case.

In principle, any $n$-point correlation function can be obtained from these elementary building blocks. One should note however that, although these formulas are quite explicit, the actual analytic computation of these multiple integrals is missing up to now. Moreover, the evaluation of correlation functions of physical relevance, as for example the spin-spin correlation functions at distance $m$ on the lattice like $\langle \sigma^+_1 \sigma^-_{m+1} \rangle$, is a priori quite involved. Indeed, the identity

$$
\langle \psi_g | \sigma^+_1 \sigma^-_{m+1} | \psi_g \rangle \equiv \langle \psi_g | E_1^{12} \prod_{j=2}^{m} (E_j^{11} + E_j^{22}) E_{m+1}^{21} | \psi_g \rangle
$$

(1.7)

shows that the corresponding spin-spin correlation function is actually given as a sum of $2^{m-1}$ elementary blocks. So, the number of terms to sum up grows exponentially with $m$, making the problem of asymptotic behavior at large distance extremely difficult to solve in these settings from the present knowledge of the elementary blocks. In the language of algebraic Bethe ansatz, and using the solution of the quantum inverse scattering problem, this question amounts to the computation of the following average value:

$$
\langle \psi_g | C(\frac{\eta}{2}) (A + D)^{m-1}(\frac{\eta}{2}) B(\frac{\eta}{2}) | \psi_g \rangle.
$$

(1.8)

Hence, to obtain manageable (re-summed) formulas for spin-spin correlation functions, and to avoid the computation of the above sum of $2^{m-1}$ terms, we need to derive a compact expression for the action of the shift operator $(A + D)^{m-1}(\frac{\eta}{2})$ (from site 1 to site $m+1$) on arbitrary states.

The main purpose of this paper is to give a solution to this problem in the framework of algebraic Bethe ansatz, and to apply it to the evaluation of spin-spin correlation functions.

In fact, for later use, we will solve an even more general question: the evaluation of a compact formula for the multiple action of transfer matrix operators $(A + D)(x_\alpha)$, for any set of spectral parameters $x_\alpha$, on arbitrary quantum states (a priori not eigenstates) of the XXZ model (see Proposition 4.1). This leads to the evaluation of the spin-spin correlation functions at lattice distance $m$ as the sum of only $m$ terms (instead of $2^{m-1}$), the $n$th term in the sum being expressed in the thermodynamic limit as a multiple integral of order $2n + 1$, for $n = 1, \ldots, m$.
(see Proposition [6.1], Proposition [6.2]). For the two point correlation functions, a typical form of the result is (6.13),

$$\langle \sigma^\alpha_1 \sigma^\beta_{m+1} \rangle = \sum_{n=0}^{m-1} \int_{C_z} d^n z \int_{C_\lambda} d^n \lambda \int_{C_\mu} d^2 \mu \left[ f(\lambda, z) \right]^m \Gamma_n^{\alpha\beta}(\lambda, \mu, z) S_h(\lambda, z).$$  \hspace{1cm} (1.9)

Here, $\alpha, \beta = x, y, z$, and the functions $\Gamma_n^{\alpha\beta}(\lambda, \mu, z)$, $f(\lambda, z)$ are purely algebraic quantities, which do not depend on the regime nor on the magnetic field; the integration contours $C_z, C_\lambda, C_\mu$ and the functional $S_h$ of the density function describing the ground state (evaluated at points $\lambda$ and $z$) depend both on the regime and on the value of the magnetic field.

In this formula, one can interpret the integrals over the $\lambda$ and $z$ variables to be generically associated to the shift operator from site 1 to site $m+1$, while the integrals over the variables $\mu$ correspond to the contribution of the operators $\sigma^\alpha, \sigma^\beta$.

Hence this method provides us with an effective re-summation of the previous $2^{m-1}$ elementary blocks, although we found it more convenient and general to work it out at the operator level and in the algebraic Bethe ansatz framework. In particular, all our considerations are valid for the finite lattice case. We also believe that it can be applied to many other models for which multiple integral representations of elementary blocks of correlation functions are known, like for example the integrable higher spin Heisenberg chains [21].

Moreover, it should be stressed here that, for each term $n$ of this sum, the distance appears now explicitly in the integrand merely as the power $m$ of some function $f(\lambda, z)$ of the integration variables. This feature, which is the result of our re-summation, is obviously of great importance for future asymptotic analysis at large $m$. Let us finally mention that these new representations of the spin-spin correlation functions (valid for arbitrary values of $\Delta > -1$) lead in a simple way to the known results at the free fermion point $\Delta = 0$; the corresponding computations will be presented in a separate publication.

This article is organized as follows. In the next section, we recall some basics about the study of the XXZ spin-$\frac{1}{2}$ Heisenberg chain in the algebraic Bethe ansatz framework. In section 3, we present the list of formulas necessary for the computation of the correlation functions via algebraic Bethe ansatz. In section 4, we derive a compact formula for the multiple action of the transfer matrix operator on an arbitrary quantum state of the chain and for any value of the spectral parameters. It leads in section 5 to the evaluation of the generating functional of the $\sigma^z$ correlation functions. General spin-spin correlation function at lattice distance $m$ are given in section 6. Some perspectives are discussed in the conclusion. Lengthy computations and/or proofs of intermediate results are presented in a set of three appendices.

We dedicate this paper to the memory of our friend and colleague A. Izergin. When we began this work two years ago, he was about to join us, but unfortunately these plans were suddenly stopped.
2 The XXZ spin-$\frac{1}{2}$ Heisenberg chain

The Hamiltonian of the cyclic XXZ chain with $M$ sites is given by (1.1). In the framework of algebraic Bethe ansatz, it can be obtained from the monodromy matrix $T(\lambda)$, which is in turn completely defined by the $R$-matrix. The $R$-matrix of the XXZ chain acts in the space $\mathbb{C}^2 \otimes \mathbb{C}^2$ and is equal to

$$R(\lambda) = \frac{1}{\sinh(\lambda + \eta)} \begin{pmatrix} \sinh(\lambda + \eta) & 0 & 0 & 0 \\ 0 & \sinh \lambda & \sinh \eta & 0 \\ 0 & \sinh \eta & \sinh \lambda & 0 \\ 0 & 0 & 0 & \sinh(\lambda + \eta) \end{pmatrix}, \quad \cosh \eta = \Delta. \quad (2.1)$$

It is a trigonometric solution of the Yang-Baxter equation. Identifying one of the two vector spaces of the $R$-matrix with the quantum space $\mathcal{H}_m$, one defines the quantum $L$-operator at site $m$ by

$$L_m(\lambda) = R_{0m}(\lambda - \eta/2). \quad (2.2)$$

Here $R_{0m}$ acts in $\mathbb{C}^2 \otimes \mathcal{H}_m$. The monodromy matrix $T(\lambda)$ is then constructed as an ordered product of the $L$-operators with respect to all the sites of the chain:

$$T(\lambda) = \left( \begin{array}{cc} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{array} \right) = L_M(\lambda) \ldots L_2(\lambda)L_1(\lambda). \quad (2.3)$$

The Hamiltonian (1.1) at $h = 0$ can be obtained from $T(\lambda)$ by the trace identity

$$H = 2 \sinh \eta \frac{\partial}{\partial \lambda} \log T(\lambda) \bigg|_{\lambda = \frac{\eta}{2}} + \text{const.} \quad (2.4)$$

Here, the transfer matrix

$$T(\lambda) = \text{tr} T(\lambda) = A(\lambda) + D(\lambda) \quad (2.5)$$

generates a continuous set of commuting conserved quantities. For technical reasons it is convenient to consider the inhomogeneous XXZ model, where

$$L_m(\lambda) = L_m(\lambda, \xi_m) = R_{0m}(\lambda - \xi_m), \quad T(\lambda) = L_M(\lambda, \xi_M) \ldots L_2(\lambda, \xi_2)L_1(\lambda, \xi_1), \quad (2.6)$$

and $\xi_m$ are arbitrary complex numbers attached to each lattice site that are called inhomogeneity parameters. In the homogeneous limit $\xi_m = \eta/2$, we come back to the original model (1.1). The commutation relations between the entries of the monodromy matrix are given by the Yang-Baxter quadratic relation,

$$R_{12}(\lambda_1 - \lambda_2)T_1(\lambda_1)T_2(\lambda_2) = T_2(\lambda_2)T_1(\lambda_1)R_{12}(\lambda_1 - \lambda_2). \quad (2.7)$$

The equation (2.7) holds in the space $V_1 \otimes V_2 \otimes \mathcal{H}$ (where $V_j \sim \mathbb{C}^2$). The matrix $T_j(\lambda)$ acts in a nontrivial way in the space $V_j \otimes \mathcal{H}$, while the $R$-matrix $R_{12}$ is nontrivial in $V_1 \otimes V_2$. 

6
The space of states is generated by the action of creation operators $B(\lambda)$ and annihilation operators $C(\lambda)$ on the reference state $|0\rangle$ with all spins up. In the following, we will consider general states of the form

$$|\psi\rangle = \prod_{j=1}^{N} B(\lambda_j)|0\rangle, \quad N = 0, 1, \ldots, M,$$

which are eigenstates of the transfer matrix $T(\mu)$ (and thus of the Hamiltonian in the homogeneous case) when the parameters $\lambda_j$ satisfy the system of Bethe equations

$$\prod_{m=1}^{M} \frac{\sinh(\lambda_j - \xi_m)}{\sinh(\lambda_j - \xi_m + \eta)} \cdot \prod_{k=1}^{N} \frac{\sinh(\lambda_j - \lambda_k + \eta)}{\sinh(\lambda_j - \lambda_k - \eta)} = 1, \quad j = 1, \ldots, N. \quad (2.9)$$

The corresponding eigenvalue $\tau(\mu, \{\lambda_j\})$ of the operator $T(\mu)$ (containing the energy level following (2.1)) is

$$\tau(\mu, \{\lambda_j\}) = a(\mu) \prod_{j=1}^{N} f(\lambda_j, \mu) + d(\mu) \prod_{j=1}^{N} f(\mu, \lambda_j). \quad (2.10)$$

Here and further we use abbreviated notations for certain combinations of hyperbolic functions:

$$f(\lambda, \mu) = \frac{\sinh(\lambda - \mu + \eta)}{\sinh(\lambda - \mu)}, \quad t(\lambda, \mu) = \frac{\sinh \eta}{\sinh(\lambda - \mu) \sinh(\lambda - \mu + \eta)}. \quad (2.11)$$

The functions $d(\mu)$ and $a(\mu)$ are eigenvalues of the operators $D(\mu)$ and $A(\mu)$ on the reference state:

$$d(\mu) = \prod_{m=1}^{M} f^{-1}(\mu, \xi_m), \quad a(\mu) = 1. \quad (2.12)$$

Following the paper [15], we consider below the action of the monodromy matrix elements on the dual state which can be constructed similarly to (2.8) via the operators $C(\lambda)$ as

$$\langle \psi | = \langle 0 | \prod_{j=1}^{N} C(\lambda_j), \quad N = 0, 1, \ldots, M. \quad (2.13)$$

Here $\langle 0 | = |0\rangle^+$, and (2.13) defines a dual eigenstate if the parameters $\lambda_j$ satisfy the same system of Bethe equations (2.9).

Our final goal is to compute the correlation functions in the ground state in the thermodynamic limit $M \to \infty$. The thermodynamics of the XXZ chain was studied in [5, 4, 20]. Here we merely recall the formulas we need for our study.

The ground state $|\psi_g\rangle$ of the infinite chain can be constructed as the limit of the finite chain eigenstate $\prod_{j=1}^{N} B(\lambda_j)|0\rangle$ for $M \to \infty$, $N \to \infty$ and $N/M$ equal to some constant whose value
depends on the magnetic field $h$. In this limit, the Bethe equations for the set of parameters $\{\lambda_j\}$ reduce to the integral Lieb equation for the ground state spectral density $\rho_{\text{tot}}(\lambda)$:

$$-2\pi i \rho_{\text{tot}}(\lambda) + \int_C K(\lambda - \mu) \rho_{\text{tot}}(\mu) \, d\mu = t(\lambda, \frac{\eta}{2}),$$

(2.14)

where

$$K(\lambda) = \frac{\sinh 2\eta}{\sinh(\lambda + \eta) \sinh(\lambda - \eta)}.$$  

(2.15)

The integration contour $C = [-\Lambda_h, \Lambda_h]$ in (2.14) depends on the regime considered. In the massless case $-1 < \Delta \leq 1$, the contour $C$ is an interval of the real axis and the parameter $\eta$ is imaginary: $\eta = -i\zeta$, $\zeta > 0$. In particular, at $h \to 0$, $\Lambda_h \to \infty$, and the Lieb equation can be solved explicitly (see (2.17)). For $\Delta > 1$ ($\eta < 0$) the limits $\pm \Lambda_h$ are imaginary, which means that the integral in (2.14) is taken over an interval of the imaginary axis. At $h = 0$, $\Lambda_h = -i\pi/2$ and the solution of the Lieb equation is given in terms of theta-functions (see (2.17)).

For technical purposes, we also introduce the inhomogeneous density $\rho(\lambda, \xi)$ as the solution of the integral equation

$$-2\pi i \rho(\lambda, \xi) + \int_C K(\lambda - \mu) \rho(\mu, \xi) \, d\mu = t(\lambda, \xi).$$

(2.16)

It coincides with $\rho_{\text{tot}}(\lambda)$ at $\xi = \eta/2$. For our goals it is enough to consider $-\zeta < \text{Im}(\xi) < 0$ for $-1 < \Delta \leq 1$ and $\eta < \text{Re}(\xi) < 0$ for $\Delta > 1$. For zero magnetic field, one has

$$\rho(\lambda, \xi) = \begin{cases} 
\frac{i}{2\zeta \sinh \frac{\pi}{2}(\lambda - \xi)}, & |\Delta| < 1, \ \zeta = i\eta, \\
\frac{i}{2\pi} \prod_{n=1}^{\infty} \left( \frac{1 - q^{2n}}{1 + q^{2n}} \right)^2 \frac{\vartheta_3(i(\lambda - \xi), q)}{\vartheta_4(i(\lambda - \xi), q)}}, & \Delta > 1, \ q = e^{\eta}.
\end{cases}$$

(2.17)

In the presence of the magnetic field, the equation (2.16) cannot be solved explicitly in terms of known elementary or special functions. Some properties of $\rho(\lambda, \xi)$ can nevertheless be established: in particular, it is not difficult to see that $\rho(\lambda, \xi)$ has a simple pole at $\lambda = \xi$ with the residue $2\pi i \text{Res} \rho(\lambda, \xi)|_{\lambda = \xi} = -1$. This property was used in [13] for the computation of the elementary blocks, and we shall also use it in Section 3.

3 From quantum inverse scattering problem to correlation functions via algebraic Bethe ansatz

In this section, we review the main steps of the method proposed in [14, 15] for the computation of the correlation functions in the framework of algebraic Bethe ansatz using the solution of the quantum inverse scattering problem: we first recall briefly how to obtain a multiple integral
representation for the elementary building blocks (1.3) (see [14, 15] for details), then we discuss in this context the case of the spin-spin correlation functions at lattice distance \( m \).

The elementary blocks of the correlation functions are defined as the normalized expectation values of products of local matrices \( E_{\epsilon j}^{\epsilon_j'j} \) from site \( j = 1 \) to site \( j = m \) with respect to some eigenstate of the transfer matrix (for which the set of spectral parameters \( \lambda_k \) satisfy the Bethe equations):

\[
F_m(\{\epsilon_j, \epsilon_j'\}) = \frac{\langle 0 | \prod_{k=1}^N C(\lambda_k) \left( \prod_{j=1}^m E_{\epsilon_j}^{\epsilon_j'j} \right) \prod_{k=1}^N B(\lambda_k) | 0 \rangle}{\langle 0 | \prod_{k=1}^N C(\lambda_k) \prod_{k=1}^N B(\lambda_k) | 0 \rangle}. \tag{3.1}
\]

For technical reasons, it is convenient to achieve the calculation in the generic inhomogeneous case (2.6); it is easy at the end to particularize the result to the homogeneous chain (1.1).

To compute the expectation values (3.1), or more generally any kind of correlation functions, one has first to express the elementary local operators \( E_{\epsilon j}^{\epsilon_j'j} \) (or equivalently the local spin operators) in terms of the entries of the quantum monodromy matrix. Such a representation is given by the solution of the quantum inverse scattering problem [14, 18]:

**Theorem 3.1.** [14, 18] Let us consider the inhomogeneous XXZ model (2.4) with arbitrary inhomogeneity parameters \( \xi_k, 1 \leq k \leq M \). The local spin operators at any site \( j \) of the chain can be expressed in terms of the elements of the quantum monodromy matrix as

\[
\sigma_j^- = \prod_{k=1}^{j-1} T(\xi_k) \cdot B(\xi_j) \cdot \prod_{k=1}^j T^{-1}(\xi_k),
\]

\[
\sigma_j^+ = \prod_{k=1}^{j-1} T(\xi_k) \cdot C(\xi_j) \cdot \prod_{k=1}^j T^{-1}(\xi_k), \tag{3.2}
\]

\[
\sigma_j^z = \prod_{k=1}^{j-1} T(\xi_k) \cdot (A - D)(\xi_j) \cdot \prod_{k=1}^j T^{-1}(\xi_k).
\]

In particular, these formulas apply for the homogeneous model (1.4) where \( \xi_k = \eta/2, 1 \leq k \leq M \).

**Remark 3.1.** The identity operator in the site \( j \) can also be written in a form similar to (3.2):

\[
1_j = \prod_{k=1}^{j-1} T(\xi_k) \cdot (A + D)(\xi_j) \cdot \prod_{k=1}^j T^{-1}(\xi_k). \tag{3.3}
\]

**Remark 3.2.** The resolution (3.2)–(3.3) of the quantum inverse scattering problem can also be expressed in terms of the elementary matrices \( E_{\epsilon j}^{\epsilon_j'j} \) in the site \( j \). In the homogeneous case, the corresponding reconstruction formulas are given by (1.3).
From Theorem 3.1 and Remark 3.2, one obtains

\[ F_m(\{\epsilon_j, \epsilon'_j\}) = \frac{\langle \psi_g | T_{\epsilon_1, \epsilon'_1}(\xi_1) \cdots T_{\epsilon_m, \epsilon'_m}(\xi_m) | \psi_g \rangle}{\langle \psi_g | \psi_g \rangle}, \]  

(3.4)

where \( \Phi_m(\{\lambda\}) \) is the ground state eigenvalue of the corresponding product of the transfer matrices:

\[ \Phi_m(\{\lambda\}) = \prod_{j=1}^{m} \prod_{a=1}^{N} \frac{\sinh(\lambda_a - \xi_j)}{\sinh(\lambda_a - \xi_j + \eta)}. \]  

(3.5)

For the computation of these expectation values, one needs then to act successively on the left with all the elements \( T_{\epsilon_m, \epsilon'_m}(\xi_m) \) of the monodromy matrix. One thus has to use the expressions of the action of the operators \( A, B, C, D \) on an arbitrary state \( \langle \psi = \prod_{j=1}^{N} C(\lambda_j) \). In the case of \( A \) and \( D \), they are given by [10]:

\[ \langle 0 | \prod_{j=1}^{N} C(\lambda_j) A(\lambda_{N+1}) = \sum_{b=1}^{N+1} a(\lambda_b) \prod_{j=1}^{N+1} \frac{\sinh(\lambda_j - \lambda_b + \eta)}{\sinh(\lambda_j - \lambda_b)} \langle 0 | \prod_{j=1}^{N+1} C(\lambda_j), \]  

(3.6)

\[ \langle 0 | \prod_{j=1}^{N} C(\lambda_j) D(\lambda_{N+1}) = \sum_{a=1}^{N+1} d(\lambda_a) \prod_{j=1}^{N+1} \frac{\sinh(\lambda_a - \lambda_j + \eta)}{\sinh(\lambda_a - \lambda_j)} \langle 0 | \prod_{j=1}^{N+1} C(\lambda_j). \]  

(3.7)

The action of the operator \( B \) is more complicated, and it is similar to the successive action of \( A \) and \( D \):

\[ \langle 0 | \prod_{j=1}^{N} C(\lambda_j) B(\lambda_{N+1}) = \sum_{a=1}^{N+1} d(\lambda_a) \prod_{k=1}^{N+1} \frac{\sinh(\lambda_a - \lambda_k + \eta)}{\sinh(\lambda_a - \lambda_k)} \]  

\[ \times \sum_{a'=1}^{N+1} \frac{a(\lambda_{a'})}{\sinh(\lambda_{N+1} - \lambda_{a'} + \eta)} \prod_{j=1}^{N+1} \frac{\sinh(\lambda_j - \lambda_{a'} + \eta)}{\sinh(\lambda_j - \lambda_{a'})} \langle 0 | \prod_{j=1}^{N+1} C(\lambda_j). \]  

(3.8)

Finally, the action of the operator \( C \) is free. Recall once more that in the formulas (3.6)–(3.8) the parameters \{\lambda\} are arbitrary complex numbers (which are not necessarily solutions of Bethe equations). In the above sums, the terms containing \( a(\lambda_{N+1}) \) or \( d(\lambda_{N+1}) \) are usually called direct terms, while the others are called indirect terms.
The action of an arbitrary monomial \( T_{\epsilon_1, \epsilon'_1}(\xi_1) \ldots T_{\epsilon_m, \epsilon'_m}(\xi_m) \) on the state \( \langle \psi | \) can be obtained by applying recursively these formulas. This leads to a linear combination of states, 

\[
\langle 0 | \prod_{k=1}^{N} C(\lambda_k) \cdot T_{\epsilon_1, \epsilon'_1}(\xi_1) \ldots T_{\epsilon_m, \epsilon'_m}(\xi_m) = \sum_{i \in I} \alpha_i \langle 0 | \prod_{k \in K_i} C(\mu_k),
\]

with some (computable) coefficients \( \alpha_i \). Here, sums and products are taken over (multiple) sets \( I \) and \( K_i \), where the \( K_i, i \in I \), are subsets of \( 1, \ldots, m + N \) with \( (\mu_1, \ldots, \mu_{m+N}) = (\lambda_1, \ldots, \lambda_N, \xi_1, \ldots, \xi_m) \).

Finally, to evaluate the expectation value (3.1), it remains to compute scalar products of the type

\[
\langle 0 | \prod_{j=1}^{N} C(\mu_j) \prod_{j=1}^{N} B(\lambda_j)|0\rangle,
\]

where \( \prod_{j=1}^{N} B(\lambda_j)|0\rangle \) is an eigenstate of the transfer matrix, while the parameters \( \{\mu_j\}_{1 \leq j \leq N} \) are arbitrary. The result for (3.10) is given by [19, 22] (see [14] for another proof):

**Proposition 3.1.** [19, 22, 14] The scalar product of a Bethe state with an arbitrary state of the form (2.8) can be expressed in the following way:

\[
\langle 0 | \prod_{j=1}^{N} C(\mu_j) \prod_{j=1}^{N} B(\lambda_j)|0\rangle = \frac{\prod_{a,b=1}^{N} \sinh(\lambda_b - \mu_a + \eta)}{\prod_{a>b} \sinh(\mu_a - \mu_b) \sinh(\lambda_b - \lambda_a)} \det_N \Psi'(\{\mu\}|\{\lambda\}),
\]

for \( \{\lambda_j\}_{1 \leq j \leq N} \) solution of the Bethe equation, and for any set of complex parameters \( \{\mu_j\}_{1 \leq j \leq N} \). The \( N \times N \) matrix \( \Psi'(\{\mu\}|\{\lambda\}) \) is defined by

\[
\Psi'_{jk}(\{\mu\}|\{\lambda\}) = t(\lambda_j, \mu_k) - d(\mu_k) t(\mu_k, \lambda_j) \prod_{a=1}^{N} \frac{\sinh(\mu_k - \lambda_a + \eta)}{\sinh(\mu_k - \lambda_a - \eta)}.
\]

Here and further \( \det_N \) denotes the determinant of an \( N \times N \) matrix. Setting \( \{\mu\} = \{\lambda\} \) in (3.11), one obtains the square of the norm of the corresponding eigenstate [23]:

\[
\langle 0 | \prod_{j=1}^{N} C(\lambda_j) \prod_{j=1}^{N} B(\lambda_j)|0\rangle = \sinh^N \eta \prod_{a,b=1}^{N} \frac{\sinh(\lambda_a - \lambda_b + \eta)}{\sinh(\lambda_a - \lambda_b)} \det_N \Phi'(\{\lambda\}).
\]

where

\[
\Phi'_{jk}(\{\lambda\}) = \delta_{jk} \left[ \frac{d'(\lambda_j)}{d(\lambda_j)} - \sum_{a=1}^{N} K(\lambda_j - \lambda_a) \right] + K(\lambda_j - \lambda_k),
\]

with \( K(\lambda) \) given in (2.13). Note that both matrices \( \Psi' \) and \( \Phi' \) can be written in the form of Jacobians [14, 23]. We also would like to point out that the entries of the matrix \( \Psi' \) are linear
combinations of $t(\lambda_j, \mu_k)$ and $t(\mu_k, \lambda_j)$. In the next section we also deal with determinants of matrices possessing similar structure. In fact all these determinants are various deformations of $\det t(\lambda_j, \mu_k)$ describing the partition function of the six-vertex model with domain wall boundary conditions [24]. An explanation of this deformation was given in [14].

Using the formulas (3.2)–(3.14), one can compute the normalized average value (3.1) on the finite lattice. The remaining step is to proceed to the thermodynamic limit. Observe that the actions (3.6)–(3.8) produce sums with respect to parameters $\{\lambda\}$. The successive action of several $E_{j^j, j'j}$ (i.e. successive action of the entries of the monodromy matrix) gives multiple sums. Let the parameters $\lambda_j$ describe the ground state; then, in the thermodynamic limit, each of these sums turns into the integral

$$\frac{1}{M} \sum_{\{\lambda\}} f(\lambda) \longrightarrow \int_C f(\lambda) \rho_{tot}(\lambda) \, d\lambda,$$

(3.15)

where $\rho_{tot}(\lambda)$ is the solution of Lieb equation (2.14). It was shown also in [13] that in the thermodynamic limit the ratio of the determinants of $\Psi'$ and $\Phi'$ can be evaluated as a determinant of inhomogeneous densities (2.16). Namely, if in (3.11) we have $\{\mu\} = \{\xi_1, \ldots, \xi_n\} \cup \{\lambda_{n+1}, \ldots, \lambda_N\}$,

$$\frac{\det_N \Psi'}{\det_N \Phi'} = \prod_{a=1}^n (M \rho_{tot}(\lambda_a))^{-1} \det_n \rho(\lambda_j, \xi_k).$$

(3.16)

In [18], it has also been shown that, when acting with the monodromy matrix elements, one can get rid of all direct type terms in the thermodynamic limit: the procedure is just to shift properly the integration contour of the corresponding $\lambda$ variables. In this way, the expectation value of $T_{\xi_1, \xi'_1}(\xi_1) \ldots T_{\xi_m, \xi'_m}(\xi_m)$ in the ground state is represented as a multiple integral in which the number of integrals coincides with the number of operators in the product. This quantity is called an elementary block.

This method gives generic answers for the computation of the expectation values of the monomials of the type $T_{\xi_1, \xi'_1}(\xi_1) \ldots T_{\xi_m, \xi'_m}(\xi_m)$ corresponding to elementary blocks of the correlation functions [12, 15]. The problem of the evaluation of the spin-spin correlation functions is more involved. Let us consider, for example, the correlation function $\langle \sigma_1^+ \sigma_{m+1}^- \rangle$. From the solution of the inverse scattering problem (Theorem 3.1), one obtains the identity

$$\langle \psi | \sigma_1^+ \sigma_{m+1}^- | \psi \rangle \equiv \langle \psi | C(\xi_1) \cdot \prod_{a=2}^m (A + D)(\xi_a) \cdot B(\xi_{m+1}) \cdot \prod_{b=1}^{m+1} (A + D)^{-1}(\xi_b) | \psi \rangle.$$  

(3.17)

Here $|\psi\rangle$ is an eigenstate of the transfer matrix $(A + D)$, and therefore it is straightforward to act on the right with $\prod_{b=1}^{m+1} (A + D)^{-1}(\xi_b)$. However, the determination of the action of $\prod_{a=2}^m (A + D)(\xi_a)$ is more involved: clearly after acting with $C(\xi_1)$ on $|\psi\rangle$ (or equivalently with $B(\xi_{m+1})$ on $|\psi\rangle$), one obtains a sum of states which are no longer Bethe states; therefore the multiple action of $(A + D)$ on these states is not simple. In fact, in the framework of the above
approach, the product $\prod_{a=2}^{m}(A+D)(\xi_a)$ would be computed as a sum of $2^{m-1}$ monomials, which eventually leads us to the sum of $2^{m-1}$ elementary blocks. As already discussed in Introduction, this form of the result is not suitable, in particular at large distance $m$. Therefore, to obtain manageable (re-summed) expressions for spin-spin correlation functions, it is essential to obtain an alternative and compact evaluation of the multiple action of the transfer matrix on arbitrary states. This is the subject of the next section.

4 The multiple action of the transfer matrix $(A+D)(x)$

The new results obtained in this section play a central role in the computation of spin-spin correlation functions at distance $m$: we evaluate here the multiple action of the transfer matrix $A+D$ on an arbitrary state, which enables us to solve the problem mentioned at the end of the previous section.

More precisely, let us consider the product

$$\prod_{a=1}^{m} (A + e^{\beta D})(x_a),$$

where $x_1, \ldots, x_m$ and $\beta$ are arbitrary complex. Our aim is to find a compact formula for the action of this operator on a state $|\psi\rangle = |0\rangle \prod_{j=1}^{N} C(\lambda_j)$, where parameters $\lambda_j$ are also arbitrary numbers. For simplicity, we first consider the case $m \leq N$.

Due to (3.6), (3.7), the action of single operator $(A + e^{\beta D})(x)$ on the state $|\psi\rangle$ can be written in the form:

$$|0\rangle \prod_{j=1}^{N} C(\lambda_j) (A + e^{\beta D})(x) = \Lambda |0\rangle \prod_{j=1}^{N} C(\lambda_j) + \sum_{k=1}^{N} \Lambda_k |0\rangle \prod_{j=1, j \neq k}^{N} C(\lambda_j) \cdot C(x).$$

Here

$$\Lambda = a(x) \prod_{j=1}^{N} f(\lambda_j, x) + e^{\beta} d(x) \prod_{j=1}^{N} f(x, \lambda_j),$$

and

$$\Lambda_k = a(\lambda_k) \frac{\sinh \eta}{\sinh(x - \lambda_k)} \prod_{j=1, j \neq k}^{N} f(\lambda_j, \lambda_k) + e^{\beta} d(\lambda_k) \frac{\sinh \eta}{\sinh(\lambda_k - x)} \prod_{j=1, j \neq k}^{N} f(\lambda_k, \lambda_j).$$

Recall that $a(\lambda) \equiv 1$; however, up to the end of this section, we do not use this property, nor the explicit form of $d(\lambda)$. All our derivations are based on the use of the equations (3.6), (3.7), which in turn are direct corollary of the intertwining relation (2.7). Thus, the result (4.9), (4.10) below for the action of the operator (4.1) on an arbitrary state is valid for any model with an $R$-matrix of the form (2.1).

Let us denote the first and the second type of action in the r.h.s. of (4.2) by ‘direct’ and ‘indirect’ actions respectively. The corresponding coefficients (4.3) and (4.4) are called direct
and indirect terms. The direct action of \((A + e^\beta D)\) preserves the state \(\langle \psi \rangle\), as if this state was an eigenstate of the operator. The indirect action leads to the remaining terms, where the resulting states differ from the original one: in these states, the argument \(x\) of the operator \((A + e^\beta D)\) becomes the argument of one of the operators \(C\).

One can similarly define direct and indirect action for the operator \((4.1)\). Namely, the direct action of this operator corresponds to the term where the resulting state coincides with the original one, which gives

\[
\langle \psi \rvert \prod_{a=1}^{m} \left( A + e^\beta D \right) (x_a) \rvert (\text{dir.}) \rangle = \prod_{a=1}^{m} \left\{ a(x_a) \prod_{j=1}^{N} f(\lambda_j, x_a) + e^\beta d(x_a) \prod_{j=1}^{N} f(x_a, \lambda_j) \right\} \langle \psi \rangle. \tag{4.5}
\]

All the remaining terms result from the indirect action. For \(m \leq N\), a particular kind of indirect terms corresponds to the case where the whole set of arguments \(\{x\}\) of the operators \((A + e^\beta D)\) enters the resulting states. The corresponding action is called ‘completely indirect’. It can be written in the form

\[
\langle \psi \rvert \prod_{a=1}^{m} \left( A + e^\beta D \right) (x_a) \rvert (\text{c.-ind.}) \rangle = \sum_{\{\lambda\}=(\lambda_{a+}) \cup (\lambda_{a-})} S_m(\{x\}|\{\lambda_{a+}\}|\{\lambda_{a-}\}) \langle 0 \rvert \prod_{a=1}^{m} C(x_a) \prod_{b \in \alpha_-} C(\lambda_b). \tag{4.6}
\]

Here the set of the parameters \(\{\lambda\}\) is divided into two subsets: \(\{\lambda\} = \{\lambda_{a+}\} \cup \{\lambda_{a-}\}\). The subset \(\{\lambda_{a+}\}\) is replaced with \(\{x\}\) in the resulting states, and therefore the number of elements in this subset is equal \(m\) (we have indicated this fact in \((4.4)\) by \(|\alpha_+| = m\)). The subset \(\{\lambda_{a-}\}\) remains in the resulting states. The sum in \((4.6)\) is taken with respect to all such partitions of the parameters \(\{\lambda\}\). Following the tradition, we call the corresponding factor \(S_m(\{x\}|\{\lambda_{a+}\}|\{\lambda_{a-}\})\) the highest coefficient.

**Lemma 4.1.** *The highest coefficient \(S_m\) in \((4.6)\) can be expressed in the following form:*

\[
S_m(\{x\}|\{\lambda_{a+}\}|\{\lambda_{a-}\}) = \prod_{b \in \alpha_+} \prod_{a=1}^{m} \sinh(x_a - \lambda_b + \eta) \prod_{a \geq b, a, b \in \alpha_+} \sinh(\lambda_a - \lambda_b) \prod_{a < b} \sinh(x_a - x_b) \det_{j \in \alpha_+} M_{jk}, \tag{4.7}
\]

where the \(m \times m\) matrix \(M\) is given by

\[
M_{jk} = a(\lambda_j) t(x_k, \lambda_j) \prod_{a \in \alpha_-} f(\lambda_a, \lambda_j) - e^\beta d(\lambda_j) t(\lambda_j, x_k) \prod_{a \in \alpha_-} f(\lambda_j, \lambda_a) \prod_{b=1}^{m} \frac{\sinh(\lambda_j - x_b + \eta)}{\sinh(\lambda_j - x_b - \eta)}, \tag{4.8}
\]

\[14\]
The coefficient \( R \) written as
\[
\text{and completely indirect actions respectively.}
\]
in the final state, and the number of elements in these subsets is equal to possible partitions of such type. The limiting cases
\[
\text{Proposition 4.1. Let } p = \min(m, N). \text{ The action of } \prod_{a=1}^{m} (A + e^\beta D) (x_a) \text{ on a general state } \langle \psi \rangle = \langle 0 | \prod_{j=1}^{N} C(\lambda_j) \rangle, \text{ for any sets of complex parameters } \{x_a\}_{1 \leq a \leq m} \text{ and } \{\lambda_j\}_{1 \leq j \leq N}, \text{ can be written as}
\]
\[
\langle \psi \rangle \prod_{a=1}^{m} (A + e^\beta D) (x_a)
\]
\[
= \sum_{n=0}^{p} \sum_{(\lambda) = \{\lambda_{a+}\} \cup \{\lambda_{a-}\}} \sum_{(x) = \{x_{\gamma+}\} \cup \{x_{\gamma-}\}} R_n(\{x_{\gamma+}\}|\{x_{\gamma-}\}|\{\lambda_{a+}\}|\{\lambda_{a-}\})(0 | \prod_{a \in \gamma+} C(x_a) \prod_{b \in \alpha-} C(\lambda_b). \quad (4.9)
\]
The coefficient \( R_n \) in (4.3) is given by
\[
R_n(\{x_{\gamma+}\}|\{x_{\gamma-}\}|\{\lambda_{a+}\}|\{\lambda_{a-}\}) = S_n(\{x_{\gamma+}\}|\{\lambda_{a+}\}|\{\lambda_{a-}\})
\]
\[
\times \prod_{a \in \gamma-} \left\{ a(x_a) \prod_{b \in \gamma+} f(x_b, x_a) \prod_{b \in \alpha-} f(\lambda_b, x_a) + e^\beta d(x_a) \prod_{b \in \gamma+} f(x_a, x_b) \prod_{b \in \alpha-} f(x_a, \lambda_b) \right\}. \quad (4.10)
\]
Proof. Let us start with the case \( p = m \). Note first that, in this formula, we deal with two partitions of the sets \{\lambda\} and \{x\}: the subset \{\lambda_{a+}\} is replaced with the parameters \{x_{\gamma+}\} in the final state, and the number of elements in these subsets is equal to \( n \); the remaining variables \{x_{\gamma-}\} do not enter the final state. Once again, the sum is taken with respect to all possible partitions of such type. The limiting cases \( n = 0 \) and \( n = m \) correspond to the direct and completely indirect actions respectively.

One can now use standard considerations of algebraic Bethe ansatz. Namely, let us fix a certain partition in (4.9) and try to obtain only the state \( \langle 0 | \prod_{a \in \gamma+} C(x_a) \prod_{b \in \alpha-} C(\lambda_b) \rangle \), applying the operator (4.1) to the original state \( \langle \psi \rangle \). Due to their commutativity, one can re-order the operators in the product (4.1) as
\[
\prod_{a=1}^{m} (A + e^\beta D) (x_a) = \prod_{a \in \gamma+} (A + e^\beta D) (x_a) \cdot \prod_{a \in \gamma-} (A + e^\beta D) (x_a). \quad (4.11)
\]
Then it is easy to see that the action of the first group of operators must be completely indirect, otherwise one of the parameters \( x_a, a \in \gamma+ \), is missing in the set of arguments of the final state. Moreover, the subset of parameters replaced in the original state \( \langle \psi \rangle \) must be exactly equal to
\{\lambda_{\alpha_i}\}: otherwise, at least one of the elements of \{\lambda_{\alpha_i}\} or of \{x_{\gamma_-}\} belongs to the final state. Therefore, the action of the first product of operators contributes as

\[
\langle \psi | \prod_{a \in \gamma_+} \left( A + e^\beta D \right) (x_a) \prod_{a \in \gamma_-} \left( A + e^\beta D \right) (x_a) \longrightarrow S_n(\{x_{\gamma_+}\}|\{\lambda_{\alpha_+}\}|\{\lambda_{\alpha_-}\}) \langle 0 | \prod_{a \in \gamma_+} C(x_a) \prod_{b \in \alpha_-} C(\lambda_b) \prod_{a \in \gamma_-} \left( A + e^\beta D \right) (x_a). \quad (4.12)
\]

In its turn, the action of the remaining group of operators \((A + e^\beta D)(x_a)\) in (4.12) must be direct, otherwise one of the elements of \{x_{\gamma_-}\} should appear in the final state. Using (4.5) we immediately arrive at (4.10).

This proof obviously extends immediately to the case \(m \geq N\). We just have in this case additional direct type actions, and hence no completely indirect term. Thus, the action of the operator (4.1) on the state \(\langle \psi \rangle\) is given by (4.9) with the coefficients \(R_n\) defined in (4.10). \(\square\)

For our purposes, we will need to specify the arguments \(\{x\}\) of the operator (4.1). In the inhomogeneous model, we should set them equal to the inhomogeneity parameters: \(x_j = \xi_j\). Then, since \(d(\xi_j) = 0\), the part of \(R_n\) corresponding to the direct action simplifies (the operator \(D(\xi_j)\) has no direct action). Thus, the equation (4.9) takes the form \((p = \min(m, N)):\)

\[
\langle \psi | \prod_{a=1}^{m} \left( A + e^\beta D \right) (\xi_a) = \sum_{n=0}^{p} \sum_{\{\xi_{\gamma_+}\}|\{\lambda_{\alpha_+}\}|\{\lambda_{\alpha_-}\}} S_n(\{\xi_{\gamma_+}\}|\{\lambda_{\alpha_+}\}|\{\lambda_{\alpha_-}\})
\times \prod_{a \in \gamma_-} \left\{ a(\xi_a) \prod_{b \in \gamma_+} f(\xi_b, \xi_a) \prod_{b \in \alpha_-} f(\lambda_b, \xi_a) \right\} \langle 0 | \prod_{a \in \gamma_+} C(\xi_a) \prod_{b \in \alpha_-} C(\lambda_b). \quad (4.13)
\]

Finally, one can also consider the particular case when \(\langle \psi \rangle\) is an eigenstate of the transfer matrix. Since the parameters \(\{\lambda\}\) satisfy the system of Bethe equations (2.9), one obtains:

**Corollary 4.1.** If \(\langle \psi \rangle\) is an eigenstate of the transfer matrix, that is if the parameters \(\{\lambda\}\) satisfy the system of Bethe equations (2.3), the multiple action of \((A + e^\beta D)(\xi_j)\) is given by

\[
\langle \psi | \prod_{a=1}^{m} \left( A + e^\beta D \right) (\xi_a) = \sum_{n=0}^{p} \sum_{\{\xi_{\gamma_+}\}|\{\lambda_{\alpha_+}\}|\{\lambda_{\alpha_-}\}} S_n(\{\xi_{\gamma_+}\}|\{\lambda_{\alpha_+}\}|\{\lambda_{\alpha_-}\})
\times \prod_{a \in \gamma_-} \prod_{b \in \gamma_+} f(\xi_b, \xi_a) \prod_{a \in \gamma_-} \prod_{b \in \alpha_-} f(\lambda_a, \lambda_b) \langle 0 | \prod_{a \in \gamma_+} C(\xi_a) \prod_{b \in \alpha_-} C(\lambda_b). \quad (4.14)
\]
Here \( p = \min(m, N) \), and
\[
\tilde{S}_n(\xi_1, \ldots, \xi_n | \lambda_1, \ldots, \lambda_n) = \frac{\prod_{a=1}^{n} \prod_{b=1}^{n} \sinh(\xi_a - \lambda_b + \eta)}{\prod_{a > b} \sinh(\lambda_a - \lambda_b) \prod_{a < b} \sinh(\xi_a - \xi_b)} \cdot \det \tilde{M}_{jk},
\]
where the \( n \times n \) matrix \( \tilde{M}_{jk} \) is
\[
\tilde{M}_{jk} = t(\xi_k, \lambda_j) + e^\beta t(\lambda_j, \xi_k) \prod_{a=1}^{n} \frac{\sinh(\lambda_a - \lambda_j + \eta)}{\sinh(\lambda_j - \lambda_a + \eta)} \prod_{a=1}^{n} \frac{\sinh(\lambda_j - \xi_a + \eta)}{\sinh(\xi_a - \lambda_j + \eta)}.
\]

Here we have set \( a(\lambda) = 1 \). The remarkable property of \( \tilde{S}_n \) is that \( \tilde{S}_n = \delta_{n,0} \) at \( \beta = 0 \) (see Appendix B). Thus, in this case, all the terms in (4.14) with \( n \geq 1 \) vanish. The subsets \( \{\lambda_{\alpha+}\} \) and \( \{\xi_{\gamma+}\} \) are empty, and we arrive at
\[
\langle \psi | m \prod_{a=1}^{N} (A + D)(\xi_a) = \prod_{a=1}^{m} \prod_{b=1}^{N} f(\lambda_b, \xi_a) \langle \psi |,
\]
as it should be for the eigenstates of the transfer matrix.

5 Generating functional for the \( \sigma^z \) correlation function

In the present and in the next sections, we use the above results to compute the spin-spin correlation functions \( \langle \sigma^z_1 \sigma^z_{m+1} \rangle \). As we are in the framework of algebraic Bethe ansatz, we begin with the evaluation of normalized expectation values on the finite lattice with respect to an arbitrary eigenstate \( |\psi\rangle = \prod_{j=1}^{N} B(\lambda_j)|0\rangle \). In this case, the relationship between \( m \) and \( N \) is not fixed. However, for the ground state in the thermodynamic limit, \( N \) tends to infinity while \( m \) remains finite. Therefore it is clear that we can consider the case \( m < N \) only.

One of the simplest applications of the formulas obtained in the previous section is the computation of the generating functional for the correlation function of the third components of spins \( \langle \sigma^z_1 \sigma^z_{m+1} \rangle \). Following [27, 30], we define an operator \( Q_{1,m} \) as
\[
Q_{1,m} = \frac{1}{2} \sum_{k=1}^{m} (1 - \sigma^z_k).
\]
The generating functional is equal to the expectation value \( \langle \exp(\beta Q_{1,m}) \rangle \), where \( \beta \) is some complex number. In particular, taking the second derivative of this quantity with respect to \( \beta \) at \( \beta = 0 \), we obtain \( \langle \beta Q_{1,m} \rangle \). Taking then the second lattice derivative we have:
\[
\frac{1}{2} \langle (1 - \sigma^z_1)(1 - \sigma^z_{m+1}) \rangle = \langle Q^2_{1,m+1} \rangle - \langle Q^2_{1,m} \rangle - \langle Q^2_{2,m+1} \rangle + \langle Q^2_{2,m} \rangle.
\]
Thus, the two-point correlation function of the third components of the local spins can be easily extracted from the expectation value of the operator \( \exp(\beta Q_{1,m}) \) in the homogeneous limit.
Proposition 5.1. The correlation function \( \langle \sigma\sigma_{m+1}^z \rangle \) in the homogeneous model is given by

\[
\langle \sigma\sigma_{m+1}^z \rangle = \left( 2D_m^2 \frac{\partial^2}{\partial \beta^2} - 4D_m \frac{\partial}{\partial \beta} + 1 \right) \langle \exp(\beta Q_{1,m}) \rangle \bigg|_{\beta=0}, \tag{5.3}
\]

where the symbols \( D_m \) and \( D^2_m \) mean the first and the second lattice derivative respectively:

\[
D_m f(m) \equiv f(m + 1) - f(m), \quad D^2_m f(m) \equiv f(m + 1) + f(m - 1) - 2f(m). \tag{5.4}
\]

The expression of the ground state expectation value of the operator \( \exp(\beta Q_{1,m}) \) in the inhomogeneous case is

\[
\langle \exp(\beta Q_{1,m}) \rangle = \sum_{n=0}^{m} \frac{1}{(n!)^2} \oint \frac{dz_j}{2\pi i} \int_C d^n \lambda \prod_{a=1}^{n} \prod_{b=1}^{m} \frac{f(z_b, \xi_a)}{f(\lambda_b, \xi_a)} 
\times W_n(\{\lambda\}, \{z\}) \cdot \det_n \left[ \tilde{M}_{jk}(\{\lambda\} | \{z\}) \right] \cdot \det_n \left[ \rho(\lambda_j, z_k) \right]. \tag{5.5}
\]

Here

\[
W_n(\{\lambda\}, \{z\}) = \prod_{a=1}^{n} \prod_{b=1}^{m} \frac{\sinh(\lambda_a - z_b + \eta) \sinh(z_b - \lambda_a + \eta)}{\sinh(\lambda_a - \lambda_b + \eta) \sinh(z_a - z_b + \eta)}, \tag{5.6}
\]

and

\[
\tilde{M}_{jk}(\{\lambda\} | \{z\}) = t(z_k, \lambda_j) + e^\beta t(\lambda_j, z_k) \prod_{a=1}^{n} \frac{\sinh(\lambda_a - \lambda_j + \eta) \sinh(\lambda_j - z_a + \eta)}{\sinh(\lambda_j - \lambda_a + \eta) \sinh(z_a - \lambda_j + \eta)}. \tag{5.7}
\]

The contour \( C \), as in the Lieb equation \( (2.14) \), depends on the regime and on the value of the magnetic field, and \( \Gamma \) surrounds only the singularities at the inhomogeneity parameters \( \xi_k \), \( 1 \leq k \leq m \).

In the homogeneous limit \( \xi_j = \eta/2 \), the generating functional of the two-point function can be written as

\[
\langle \exp(\beta Q_{1,m}) \rangle = \sum_{n=0}^{m} \frac{1}{(n!)^2} \oint \frac{dz_j}{2\pi i} \int_C d^n \lambda \prod_{a=1}^{n} \left( \frac{\sinh(z_a + \eta/2) \sinh(\lambda_a - \eta/2)}{\sinh(z_a - \eta/2) \sinh(\lambda_a + \eta/2)} \right)^m 
\times W_n(\{\lambda\}, \{z\}) \cdot \det_n \left[ \tilde{M}_{jk}(\{\lambda\} | \{z\}) \right] \cdot \det_n \left[ \rho(\lambda_j, z_k) \right], \tag{5.8}
\]

where \( \Gamma \) surrounds the point \( \eta/2 \).

Proof. The operator \( \exp(\beta Q_{1,m}) \) has a very simple representation in terms of the entries of the monodromy matrix. Due to \( (3.2) \) we have

\[
\exp(\beta Q_{1,m}) = \prod_{a=1}^{m} \left( A + e^\beta D \right) (\xi_a) \prod_{b=1}^{m} (A + D)^{-1} (\xi_b). \tag{5.9}
\]
Hence, we can directly apply the equation (1.14) for the computation of \( \langle \exp(\beta Q_{1,m}) \rangle \). We begin with the derivation of the normalized expectation value of \( \exp(\beta Q_{1,m}) \) on the finite lattice with respect to an arbitrary eigenstate of the transfer matrix, \( |\psi\rangle = \prod_{j=1}^{N} B(\lambda_j)|0\rangle \):

\[
\frac{\langle \psi | \exp(\beta Q_{1,m}) | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{\langle \psi | \prod_{a=1}^{m} (A + e^\beta D) (\xi_a) \prod_{b=1}^{m} (A + D)^{-1}(\xi_b) | \psi \rangle}{\langle \psi | \psi \rangle}.
\] (5.10)

The action of the operators \( (A + D)^{-1}(\xi_b) \) to the right is trivial:

\[
\prod_{b=1}^{m} (A + D)^{-1}(\xi_b)| \psi \rangle = \prod_{b=1}^{m} N f^{-1}(\lambda_a, \xi_b)| \psi \rangle.
\] (5.11)

The action of the operators \( (A + e^\beta D) (\xi_a) \) to the left is given by \( (1.14) - (1.16) \). We obtain

\[
\frac{\langle \psi | \exp(\beta Q_{1,m}) | \psi \rangle}{\langle \psi | \psi \rangle} = \prod_{b=1}^{m} \prod_{a=1}^{N} f^{-1}(\lambda_a, \xi_b) \sum_{n=0}^{m} \sum_{(\lambda)=(\lambda_{a+}) \cup (\lambda_{a-})} \sum_{(\xi)=\{\xi_{a+} \cup \xi_{a-}\}} \sum_{|\alpha_+|=|\gamma_+|=n} \tilde{S}_n(\{\xi_{a+}\}|\{\alpha_+\})
\]

\[
\times \prod_{a \in \gamma_-} \prod_{b \in \gamma_+} f(\xi_b, \xi_a) \prod_{a \in \gamma_-} \prod_{b \in \alpha_-} f(\lambda_b, \xi_a) \prod_{a \in \alpha_-} \prod_{b \in \gamma_+} f(\lambda_a, \lambda_b)
\]

\[
\times \frac{\langle 0 | \prod_{a \in \gamma_+} C(\xi_a) \prod_{b \in \alpha_-} C(\lambda_b) \prod_{j=1}^{N} B(\lambda_j)|0 \rangle}{\langle 0 | \prod_{j=1}^{N} C(\lambda_j) \prod_{j=1}^{N} B(\lambda_j)|0 \rangle}.
\] (5.12)

In fact, for the remaining part of the calculations, we can use the formulas of Section 4. Using (1.11) - (1.14) we find the ratio of the scalar products in the r.h.s. of (5.12) and finally arrive at

\[
\frac{\langle \psi | \exp(\beta Q_{1,m}) | \psi \rangle}{\langle \psi | \psi \rangle} = \sum_{n=0}^{m} \sum_{(\lambda)=(\lambda_{a+}) \cup (\lambda_{a-})} \sum_{(\xi)=\{\xi_{a+} \cup \xi_{a-}\}} \sum_{|\alpha_+|=|\gamma_+|=n} \tilde{M}_n.
\]

\[
\times \prod_{a \in \gamma_+} \prod_{b \in \alpha_-} \sinh(\xi_a - \lambda_b + \eta) \sinh(\lambda_a - \xi_a + \eta)
\]

\[
\times \prod_{a, b \in \gamma_+ \cap \alpha_-} \sinh(\xi_a - \xi_b) \prod_{a, b \in \alpha_-} \sinh(\lambda_a - \lambda_b + \eta) \cdot \det \tilde{M}_n \cdot \frac{\det \Psi'\{\xi_{a+}\} \cup \{\lambda_{a-}\}|\{\lambda\}]}{\det \Psi'\{\lambda\}}.
\] (5.13)

The representation (5.13) gives us the expectation value of the operator \( \exp(\beta Q_{1,m}) \) on the finite lattice.

Up to this stage, our derivation was purely algebraic. Now we should proceed to the thermodynamic limit. In spite of the fact that this limit strongly depends on the phase of the
model, one can present the final result in a quite general form. First, using (3.16), we have in the thermodynamic limit

\[
\frac{\det_N \Psi'((\xi_{\gamma_+}) \cup \{\lambda_\alpha\} | \{\lambda\})}{\det_N \Phi'(|\lambda\rangle)} = \prod_{a \in \alpha_+} (M \rho_{\text{tot}}(\lambda_a))^{-1} \det_n \rho(\lambda_j, \xi_k),
\]

where \(\lambda_j \in \{\lambda_{\alpha_+}\}\) and \(\xi_k \in \{\xi_{\gamma_+}\}\). Second, we should get rid of the sum with respect to partitions in (5.13), replacing them with integrals.

Consider first the partitions of the set \(\{\xi\}\). Let \(F(z_1, \ldots, z_n)\) be a symmetric function of \(n\) variables, analytical with respect to each argument in the vicinities of \(\{\xi\}\). Then

\[
\sum_{|\gamma_+|=n} \prod_{a \in \gamma_+} f(\xi_a, \xi) \cdot \det(\{\xi_{\gamma_+}\}) = \frac{1}{n!} \int \cdots \int \prod_{a=1}^n \frac{dz_a}{2\pi i} f(z_a, \xi) \frac{\prod_{b=1}^n \sinh(z_a - z_b)}{\prod_{a=1}^n \sinh(z_a - z_b + \eta)} F(\{z\}).
\]

Here the contour \(\Gamma\) surrounds the points \(\xi_1, \ldots, \xi_m\) and does not contain any other singularities of the integrand. In particular, when all \(\xi_j \to \eta/2\) (homogeneous limit), one can chose \(\Gamma\) as an enough small circle around \(\eta/2\). Observe also that one can easily take the homogeneous limit in the r.h.s. of (5.15), while the existence of such limit in the l.h.s. is not so obvious. Thus, we can replace the sum with respect to the partitions of the set \(\{\xi\}\) with the set of contour integrals. In fact, we could do this already on the finite lattice.

As for the sum of the partitions of the set \(\{\lambda\}\), we essentially use the properties of the thermodynamic limit. Let now \(F(\lambda_1, \ldots, \lambda_n)\) be a symmetric function of \(n\) variables, vanishing at \(\lambda_j = \lambda_k, j, k = 1, \ldots, n\). Then,

\[
\frac{1}{M^n} \sum_{|\alpha_+|=n} F(\{\lambda_{\alpha_+}\}) = \frac{1}{n!} \int_C \cdots \int_C \prod_{j=1}^n \rho_{\text{tot}}(\lambda_j) F(\lambda_1, \ldots, \lambda_n),
\]

when \(C\) is the contour involved in the Lieb equation (2.14).

Thus, the sum with respect to partitions in the thermodynamic limit transforms into \(2n\) integrals, and we obtain the expectation value (5.5) of the operator \(\exp(\beta Q_{1, m})\) in the ground state.

We would like to mention that, as for the elementary blocks of correlation functions obtained in [15], the integrand in (5.8) consists of two parts: thermodynamic and algebraic. The
thermodynamic part depends on the phase of the model ($\Delta$ and $h$), but not on the considered set of local operators; it includes the determinant of densities and the integration contour $C$ for the variables $\{\lambda\}$. On the contrary, the algebraic part, which includes the remaining factors of the integrand, does not depend on the phase of the model, but only on the particular set of operators the correlation functions of which one wants to compute.

Nevertheless, there exists a principal difference between the representation (5.8) and the multiple integrals for the elementary blocks:

Each of the elementary blocks involved in the construction of $\prod_{a=1}^{m} (A + e^\beta D) (\xi_a)$ contains exactly $m$ integrals. However, some of these integrals, which correspond to the action of $D(\xi_a)$, are taken over the contours $C$ (see (2.14)), while the other integration contours (corresponding to $A$ type actions) are shifted. The reason of this difference is that, due to the fact that $d(\xi_a) = 0$, the operator $D(\xi_a)$ has no direct action, while $A(\xi_a)$ does have. The shift of the contours allows us to get rid of the terms produced by the direct action of $A(\xi_a)$ (see [15] for more details).

On the contrary, in the equation (5.8), the integrals with respect to all $\{\lambda\}$ are taken over the same contour $C$. Moreover, the number of these integrals is not fixed, but varies from 0 to $m$: the term $n = m$ in (5.8) corresponds to completely indirect actions of $A$ and $D$; the terms with $n < m$ contain direct actions of $A$, and in particular the term $n = 0$ describes the direct action of the whole product $A(\xi_1) \ldots A(\xi_m)$.

From this observation, one can easily understand how to obtain the representation (5.8) from the formulas for the elementary blocks. First of all, one should move back all the shifted contours in each of the $2^m$ constituent elementary blocks of the product $\prod_{a=1}^{m} (A + e^\beta D) (\xi_a)$. Hereby one crosses the poles of the densities $\rho(\lambda, \xi)$, which produces the terms where the number of integrals is less than $m$. Then, one needs to gather the terms with the same number of integrals and symmetrize all the obtained integrands with respect to $\{\lambda\}$. The symmetrization produces the sum over the partitions of inhomogeneities $\{\xi\}$, which can be effectively taken into account by the set of auxiliary $z$-integrals. Generically, this way meets extremely serious practical difficulties. However, for some simple particular cases, it can be successfully applied. In Appendix C, we illustrate this method by considering the limit $\beta \to \infty$ in the equation (5.8). Then we have

$\lim_{\beta \to \infty} e^{-\beta m} \exp(\beta Q_{1,m}) = \prod_{a=1}^{m} D(\xi_a) \prod_{b=1}^{m} (A + D)^{-1}(\xi_b). \quad (5.17)$

Hence, we obtain the correlation function corresponding to the emptiness formation probability which was considered in Section 4 of [15]. On the other hand, in the equation (5.8), only the term with $n = m$ survives, and the determinant $\tilde{M}_{jk}$ simplifies:

$\lim_{\beta \to \infty} e^{-\beta m} \det_m \tilde{M}_{jk} = \prod_{a,b=1}^{m} \frac{\sinh(\lambda_b - z_a + \eta)}{\sinh(z_a - \lambda_b + \eta)} \det_m [t(\lambda_j, z_k)]. \quad (5.18)$

In Appendix C, we obtain this result by symmetrization of the corresponding multiple integral obtained in [15]. Finally we observe that, due to the fact that $\det_n \tilde{M}_{jk} = \delta_{n0}$ at $\beta = 0$, the
expectation value of the identity operator is equal to 1. This obvious fact would be highly nontrivial to prove from the multiple integral representations for elementary blocks.

6 Spin-spin correlation functions

In the previous section, we have obtained a multiple integral representation of the correlation function \( \langle \sigma_z^1 \sigma_z^{m+1} \rangle \) from the generating functional \( \langle \exp(\beta Q_{1,m}) \rangle \). It has been computed from the average value in the ground state of the product of \( m \) commuting operators \( (A + e^\beta D)(\xi_\alpha) \) for \( \alpha = 1, \ldots, m \). However, for general \( n \)-spin correlation functions, we have to deal with ground state average values of products of non-commuting operators. Indeed, a generic \( k \)-point correlation function can be written as

\[
\langle \psi | \prod_{j=1}^k E_{m_j}^{\epsilon_j^\prime, \epsilon_j} | \psi \rangle, \quad (6.1)
\]

where \( m_1 < m_2 < \ldots < m_k \) is an ordered set of \( k \) sites on the lattice. Using the solution of the quantum inverse scattering problem, it can be reduced to the evaluation of the following average value:

\[
\langle \psi | T_{\epsilon_1, \epsilon_1'}(\xi_{m_1}) \cdot \prod_{\alpha=m_1+1}^{m_2-1} (A + D)(\xi_\alpha) \cdot T_{\epsilon_2, \epsilon_2'}(\xi_{m_2}) \cdot \prod_{\alpha=m_2+1}^{m_3-1} (A + D)(\xi_\alpha) \ldots T_{\epsilon_k, \epsilon_k'}(\xi_{m_k}) | \psi \rangle, \quad (6.2)
\]

To compute such a correlation function, the strategy is the following. As usual, we act with the operators to the left. The action of the shift operators (products of transfer matrices) is given as in (4.9). The action of the isolated elements of the monodromy matrix \( T_{\epsilon_j, \epsilon_j'}(\xi_{m_j}) \) for \( j = 1, \ldots, k \) is computed as in the elementary blocks: in particular, in the thermodynamic limit, we get rid of all direct type terms by shifting the corresponding integration contours (see [15]). This helps the final expression to be as compact as possible. We apply below this procedure to the case of the spin-spin correlation functions.

Let us start with \( \langle \sigma_z^1 \sigma_z^{m+1} \rangle \).

It is possible to evaluate this quantity by computing directly the expectation value

\[
\langle \psi | \sigma_z^1 \sigma_z^{m+1} | \psi \rangle = \langle \psi | (A - D)(\xi_1) \cdot \prod_{a=2}^m (A + D)(\xi_a) \cdot (A - D)(\xi_{m+1}) \cdot \prod_{b=1}^{m+1} (A + D)^{-1}(\xi_b) | \psi \rangle. \quad (6.3)
\]

First, we have to act with \( (A - D)(\xi_1) \) on the eigenstate \( | \psi \rangle \) using (3.6), (3.7); then, we can apply (4.13) for the action of the product \( \prod_{a=2}^m (A + D)(\xi_a) \), and finally again use (3.6), (3.7) for the action of \( (A - D)(\xi_{m+1}) \). This method does not meet some new principal obstacles, but the final answer has a bit more complicated form. In fact, in this case, one obtains a sum similar to (5.8), but in which the \( n^{th} \) term consists of four summands corresponding to the action of the operators \( A \) and \( D \) in the points \( \xi_1 \) and \( \xi_{m+1} \). Therefore the evaluation of \( \langle \sigma_z^1 \sigma_z^{m+1} \rangle \) via
the generating functional seems to be the most preferable, at least from the standpoint of its compactness.

However, in particular cases, other representations for the correlation function \( \langle \sigma_1^z \sigma_{m+1}^z \rangle \) may present certain advantages, even if these representations have slightly more complicated forms. Note that generically the correlation function \( \langle \sigma_1^z \sigma_{m+1}^z \rangle \) contains the term \( \langle \sigma^z \rangle^2 \) which, due to the translation invariance, does not depend on the distance \( m \). However, at zero magnetic field in the massless regime, the magnetization \( \langle \sigma^z \rangle \) is zero and the correlations of the third components of spin should be a decreasing function of \( m \). On the other hand, each term of the sum (5.8), even after taking the lattice and \( \beta \)-derivatives, still contains a part which does not depend on the distance \( z \).

However, in particular cases, other representations for the correlation function \( \langle \sigma_1^z \sigma_{m+1}^z \rangle \) may present certain advantages, even if these representations have slightly more complicated forms. Note that generically the correlation function \( \langle \sigma_1^z \sigma_{m+1}^z \rangle \) contains the term \( \langle \sigma^z \rangle^2 \) which, due to the translation invariance, does not depend on the distance \( m \). However, at zero magnetic field in the massless regime, the magnetization \( \langle \sigma^z \rangle \) is zero and the correlations of the third components of spin should be a decreasing function of \( m \). On the other hand, each term of the sum (5.8), even after taking the lattice and \( \beta \)-derivatives, still contains a part which does not depend on the distance \( m \). It follows from (5.2) that eventually these constant terms give 1/2. For the purposes of the forthcoming asymptotic analysis, it would be desirable to obtain a representation of \( \langle \sigma_1^z \sigma_{m+1}^z \rangle \) without any constant contribution at zero magnetic field. Such a representation, of course, is provided by (5.3), but we can also consider a certain modification of the generating functional \( \langle \exp(\beta Q_{1,m}) \rangle \), namely \( \langle \exp(\beta Q_{1,m}) \sigma_{m+1}^z \rangle \). Indeed, it is easy to see that

\[
\frac{1}{2} \langle (1 - \sigma_1^z) \sigma_{m+1}^z \rangle = \frac{\partial}{\partial \beta} \left[ \langle \exp(\beta Q_{1,m}) - \exp(\beta Q_{2,m}) \rangle \sigma_{m+1}^z \right] \bigg|_{\beta=0},
\]

and, since the magnetization is zero at \( h = 0 \) in the massless regime, we obtain

\[
\langle \sigma_1^z \sigma_{m+1}^z \rangle = -2D_m \frac{\partial}{\partial \beta} \langle \exp(\beta Q_{1,m}) \sigma_{m+1}^z \rangle \bigg|_{\beta=0}, \quad h = 0, \quad |\Delta| < 1.
\]

**Proposition 6.1.** The ground state expectation value \( \langle \exp(\beta Q_{1,m}) \sigma_{m+1}^z \rangle \) for the homogeneous case is given as

\[
\langle \exp(\beta Q_{1,m}) \sigma_{m+1}^z \rangle = \sum_{n=0}^{m} \frac{-1}{n!^2} \prod_{j=1}^{n} \frac{dz_j}{2\pi i} \int_{\Gamma} d\lambda \cdot \prod_{a=1}^{n} \frac{\sinh(z_a + \eta/2 \lambda) \sinh(\lambda_a - \eta/2 \lambda)}{\sinh(z_a - \eta/2 \lambda) \sinh(\lambda_a + \eta/2 \lambda)} \cdot W_n(\{|\lambda\}|\{z\}) \det_n \left[ M_{jk}(\{|\lambda\}|\{z\}) \right] \prod_{a=1}^{n} \frac{\sinh(\lambda_a - \eta)}{\sinh(z_a - \eta)}
\]

\[
\times \left( \int_{\hat{C}} d\lambda_{n+1} \prod_{a=1}^{n} \frac{\sinh(\lambda_{n+1} - z_a - \eta)}{\sinh(\lambda_{n+1} - \lambda_a - \eta)} + \int_{\hat{C}} d\lambda_{n+1} \prod_{a=1}^{n} \frac{\sinh(\lambda_{n+1} - z_a + \eta)}{\sinh(\lambda_{n+1} - \lambda_a + \eta)} \right) \det_{n+1} \left[ \rho(\lambda_j, z_k) \right]
\]

Here one should set \( z_{n+1} = \eta/2 \) in the last column of the determinant of the densities. \( C \) and \( \Gamma \) are as in Proposition 5.4, while the shifted contour \( \hat{C} \) for \( \lambda_{n+1} \) is such that \( \hat{C} \cup (-C) \) surrounds the points \( \{z\} \), in which the functions \( \rho(\lambda_{n+1}, z_k) \) have simple poles, but it does not contain other singularities. For zero magnetic field one has

\[
\langle \exp(\beta Q_{1,m}) \sigma_{m+1}^z \rangle = \sum_{n=0}^{m} \frac{1}{(n!)^2} \prod_{j=1}^{n} \frac{dz_j}{2\pi i} \int_{C} d\lambda_{n+1} \cdot \prod_{a=1}^{n} \frac{\sinh(z_a + \eta/2 \lambda) \sinh(\lambda_a - \eta/2 \lambda)}{\sinh(z_a - \eta/2 \lambda) \sinh(\lambda_a + \eta/2 \lambda)} \cdot W_n(\{|\lambda\}|\{z\}) \det_n \left[ M_{jk}(\{|\lambda\}|\{z\}) \right]
\]

\[
\times \left( \int_{C} d\lambda_{n+1} \prod_{a=1}^{n} \frac{\sinh(\lambda_{n+1} - z_a - \eta)}{\sinh(\lambda_{n+1} - \lambda_a - \eta)} + \int_{C} d\lambda_{n+1} \prod_{a=1}^{n} \frac{\sinh(\lambda_{n+1} - z_a + \eta)}{\sinh(\lambda_{n+1} - \lambda_a + \eta)} \right) \det_{n+1} \left[ \rho(\lambda_j, z_k) \right]
\]
After the calculation of the actions of the product \( \prod_{a=1}^{n} \left( \frac{\sinh(\lambda_a - n \frac{\pi}{2})}{\sinh(z_a - n \frac{\pi}{2})} \right) \left( \prod_{a=1}^{n} \frac{\sinh(\lambda_{a+1} - z_a)}{\sinh(\lambda_{a+1} - \lambda_a)} - \prod_{a=1}^{n} \frac{\sinh(\lambda_{a+1} - z_a + \eta)}{\sinh(\lambda_{a+1} - \lambda_a + \eta)} \right) \cdot W_n(\{\lambda\}|\{z\}) \)

\[
\times \det_n \left[ \bar{M}_{jk}(\{\lambda\}|\{z\}) \right] \det_{n+1} \left[ \rho(\lambda_j, z_1), \ldots, \rho(\lambda_j, z_n), \rho(\lambda_j, \frac{\pi}{2}) \right].
\] (6.7)

**Proof.** We omit parts of the proof which coincide with parts of the proof of Proposition 5.1. Instead, we focus our attention on certain peculiarities.

The normalized expectation value of \( \exp(\beta Q_{1,m})\sigma^z_{m+1} \) on the finite lattice is given by

\[
\langle \psi | \exp(\beta Q_{1,m})\sigma^z_{m+1} | \psi \rangle = \frac{\langle \psi | \prod_{a=1}^{m+1} (A + e^\beta D)(\xi_a) \cdot (A - D)(\xi_{m+1}) \cdot \prod_{b=1}^{m+1} (A + D)^{-1}(\xi_b) | \psi \rangle}{\langle \psi | \psi \rangle}. \] (6.8)

After the calculation of the actions of the product \( \prod_{b=1}^{m+1} (A + D)^{-1}(\xi_b) \) on the state \( | \psi \rangle \) and of \( \prod_{a=1}^{m} (A + e^\beta D)(\xi_a) \) on the state \( \langle \psi | \) one has to act with \( (A - D)(\xi_{m+1}) \) on the resulting states \( \langle 0 | \prod_{a \in \gamma_+} C(\xi_a) \prod_{b \in \alpha_-} C(\lambda_b) \). The action of \( D(\xi_{m+1}) \) gives

\[
\langle 0 | \prod_{a \in \gamma_+} C(\xi_a) \prod_{b \in \alpha_-} C(\lambda_b) \cdot D(\xi_{m+1}) = \sum_{\ell \in \alpha_-} \frac{\sinh \eta}{\sinh(\lambda_{\ell} - \xi_{m+1})} \prod_{a \in \gamma_+} f(\lambda_{\ell}, \xi_a) \prod_{\alpha_- \neq \lambda_{\ell}} f(\lambda_a, \lambda_{\ell})
\]

\[
\times \prod_{a \in \alpha_+} f(\lambda_a, \lambda_{\ell}) \cdot \langle 0 | C(\xi_{m+1}) \prod_{a \in \gamma_+} C(\xi_a) \prod_{b \in \alpha_- \neq \lambda_{\ell}} C(\lambda_b). \] (6.9)

Here we have used that \( \{\lambda\} \) satisfy the system of Bethe equations. Then, one has to compute the normalized scalar products of the obtained states with \( | \psi \rangle \) and proceed to the thermodynamic limit. We obtain:

\[
\langle \exp(\beta Q_{1,m}) D(\xi_{m+1}) \rangle = \sum_{n=0}^{m} \frac{1}{(n)!^2} \int \prod_{j=1}^{n} \frac{dz_j}{2\pi i} \int_{C} d^n \lambda \cdot \prod_{b=1}^{m} \prod_{a=1}^{n} f(z_b, \xi_a)
\]

\[
\times W_n(\{\lambda\}|\{z\}) \det_n \left[ \bar{M}_{jk}(\{\lambda\}|\{z\}) \right] \prod_{a=1}^{n} \frac{\sinh(\lambda_a - \xi_{m+1})}{\sinh(z_a - \xi_{m+1})}
\]

\[
\times \int_{C} d\lambda_{n+1} \cdot \prod_{a=1}^{n} \frac{\sinh(\lambda_{n+1} - z_a + \eta)}{\sinh(\lambda_{n+1} - \lambda_a + \eta)} \cdot \det_{n+1} \left[ \rho(\lambda_j, z_1), \ldots, \rho(\lambda_j, z_n), \rho(\lambda_j, \xi_{m+1}) \right]. \] (6.10)

Observe that, in comparison with (5.8), the integrand in (6.10) contains additional factors. Moreover, we see that the sum over \( \ell \) in (6.9) produces one more integral with respect to \( \lambda_{n+1} \) in the thermodynamic limit.
The action of the operator $A(\xi_{m+1})$ is more complicated. First, it contains terms similar to (5.8):

$$
\sum_{\ell \in \gamma_+} \frac{\sinh \eta}{\sinh(\xi_{m+1} - \xi_\ell)} \prod_{a \in \gamma_+} f(\xi_a, \lambda_\ell) \prod_{a \in \gamma_-} f(\lambda_a, \lambda_\ell) \cdot \langle 0 | C(\xi_{m+1}) \prod_{a \in \gamma_+} C(\xi_a) \prod_{b \in \gamma_-} C(\lambda_b) | \psi \rangle.
$$

In the thermodynamic limit, the contribution of these terms turns into integrals of the type (6.10), where in the last line one should make the replacement

$$
\sum_{a=1}^n \frac{\sinh(\lambda_{n+1} - z_a + \eta)}{\sinh(\lambda_{n+1} - \lambda_a + \eta)} \to - \sum_{a=1}^n \frac{\sinh(\lambda_{n+1} - z_a - \eta)}{\sinh(\lambda_{n+1} - \lambda_a - \eta)}
$$

However, the action of the operator $A(\xi_{m+1})$ contains also the direct term

$$
\prod_{a \in \gamma_+} f(\xi_a, \xi_{m+1}) \prod_{a \in \gamma_-} f(\lambda_a, \xi_{m+1}) \cdot \langle 0 | C(\xi_a) \prod_{b \in \gamma_-} C(\lambda_b),
$$

and terms where the argument $\xi_{m+1}$ exchanges with one of $\{\xi_a\}$:

$$
\sum_{\ell \in \gamma_+} \frac{\sinh \eta}{\sinh(\xi_{m+1} - \xi_\ell)} \prod_{a \in \gamma_+} f(\xi_a, \xi_\ell) \prod_{a \in \gamma_-} f(\lambda_a, \xi_\ell) \cdot \langle 0 | C(\xi_{m+1}) \prod_{a \in \gamma_+} C(\xi_a) \prod_{b \in \gamma_-} C(\lambda_b).
$$

These terms give also non-vanishing contributions to the expectation value $\langle \exp(\beta Q_{1,m})\sigma_{m+1}^z \rangle$. However, in the thermodynamic limit, one can withdraw them by shifting the integration contour for $\lambda_{n+1}$ in complete analogy with the method used in [15] for the elementary blocks. This gives us (6.9). This representation is valid in arbitrary regime of the model. For zero magnetic field, the original integration contour $C$ is the real axis for $|\Delta| < 1$ and the interval $[i\pi/2, -i\pi/2]$ for $\Delta > 1$. In both cases we can choose $\bar{C} = C + \eta$ (recall that in the massless regime $\eta = -i\zeta, \zeta > 0$). Changing then $\lambda_{n+1}$ with $\lambda_{n+1} + \eta$, we arrive at (6.7). □

The representation (6.7), as it was expected, has a slightly more complicated form than (6.8). However, after the lattice and $\beta$-derivations, the sum (6.7) does not contain any constant contribution for $|\Delta| < 1$. This fact may play an important role for the asymptotic analysis of the correlation function $\langle \sigma_i^z \sigma_{m+1}^z \rangle$.

The remaining two-point functions are $\langle \sigma_i^- \sigma_{m+1}^+ \rangle$ and $\langle \sigma_i^+ \sigma_{m+1}^- \rangle$. On the finite lattice these two quantities are given by

$$
\frac{\langle \psi | \sigma_i^- \sigma_{m+1}^+ | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{\langle \psi | B(\xi_1) \cdot \prod_{a=2}^m (A + D)(\xi_a) \cdot C(\xi_{m+1}) \cdot \prod_{b=1}^{m+1} (A + D)^{-1}(\xi_b) | \psi \rangle}{\langle \psi | \psi \rangle},
$$

(6.11)

$$
\frac{\langle \psi | \sigma_i^+ \sigma_{m+1}^- | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{\langle \psi | C(\xi_1) \cdot \prod_{a=2}^m (A + D)(\xi_a) \cdot B(\xi_{m+1}) \cdot \prod_{b=1}^{m+1} (A + D)^{-1}(\xi_b) | \psi \rangle}{\langle \psi | \psi \rangle}
$$

(6.12)

25
It is clear that our method can be applied without significant changes for the calculation of (6.11), (6.12) as well. Therefore we present here only the final result. Moreover, for simplicity, we first consider the case of zero magnetic field (for both massive and massless phases), when the correlation functions \( \langle \sigma_1^- \sigma_{m+1}^+ \rangle \) and \( \langle \sigma_1^+ \sigma_{m+1}^- \rangle \) coincide.

**Proposition 6.2.** The ground state expectation value \( \langle \sigma_1^+ \sigma_{m+1}^- \rangle \) for the homogeneous case at zero magnetic field can be expressed as

\[
\langle \sigma_1^+ \sigma_{m+1}^- \rangle = \sum_{n=0}^{m-1} \frac{1}{n!(n+1)!} \oint \prod_{j=1}^{n+1} \frac{dz_j}{2\pi i} \int_C \prod_{a=1}^{n+2} \lambda \prod_{a=1}^{n+1} \left( \frac{\sinh(z_a + \eta)}{z_a - \eta} \right) \prod_{a=1}^{n} \left( \frac{\sinh(\lambda_a - \eta)}{\sinh(\lambda_a + \eta)} \right)
\]

\[
\times \frac{1}{\sinh(\lambda_{n+1} - \lambda_{n+2})} \cdot \left( \prod_{a=1}^{n+1} \frac{\sinh(\lambda_{n+1} - z_a + \eta) \sinh(\lambda_{n+2} - z_a)}{\sinh(\lambda_{n+1} - \lambda_a + \eta) \sinh(\lambda_{n+2} - \lambda_a)} \right) \cdot \hat{W}_n(\{\lambda\}, \{z\})
\]

\[
\times \det_{n+1} \hat{M}_{jk} \cdot \det_{n+2} \left[ \rho(\lambda_j, z_1), \ldots, \rho(\lambda_j, z_{n+1}), \rho(\lambda_j, \frac{\eta}{2}) \right],
\]

where the contours \( C \) and \( \Gamma \) are defined as in Proposition 5.1. Here the analog of the function \( W_n(\{\lambda\}, \{z\}) \) is

\[
\hat{W}_n(\{\lambda\}, \{z\}) = \prod_{a=1}^{n+1} \frac{\prod_{b=1}^{n+1} \sinh(\lambda_a - z_b + \eta) \sinh(z_b - \lambda_a + \eta)}{\prod_{a=1}^{n+1} \sinh(\lambda_a - \lambda_{n+2} + \eta) \prod_{b=1}^{n+1} \sinh(\lambda_a - z_b + \eta)},
\]

and the \((n+1) \times (n+1)\) matrix \( \hat{M} \) has the entries

\[
\hat{M}_{jk} = t(z_k, \lambda_j) - t(\lambda_j, z_k) \prod_{a=1}^{n} \frac{\sinh(\lambda_a - \lambda_j + \eta)}{\sinh(\lambda_a - \lambda_j + \eta)} \prod_{b=1}^{n+1} \frac{\sinh(\lambda_j - z_b + \eta)}{\sinh(\lambda_j - z_b + \eta)}, \quad j \leq n,
\]

and \( \hat{M}_{n+1,k} = t(z_k, \frac{\eta}{2}) \) for \( j = n+1 \).

**Proof.** The only difference between (6.12) and the expectation values considered above is that now we deal with the operators \( C \) and \( B \), and thus use (3.8) for the action of \( B \) (recall that the action of \( C \) is free). The rest of the computations is mostly the same. In analogy with (6.7), the integrals with respect to \( \lambda_{n+1} \) and \( \lambda_{n+2} \) describe the action of the operator \( B(\xi_{m+1}) \) in (6.12). The direct action of \( B(\xi_{m+1}) \) is taken into account by the shift of the integration contour for \( \lambda_{n+2} \). \( \square \)

It is clear that this result can be easily generalized for the case of non-zero magnetic field in complete analogy with (6.6). In particular, for the correlation function \( \langle \sigma_1^+ \sigma_{m+1}^- \rangle \), one should replace the variable \( \lambda_{n+2} \) in (6.13) with \( \lambda_{n+2} + \eta \), and choose for this variable the shifted integration contour \( \hat{C} \).
Conclusion

The main result of this paper is a new multiple integral representation for the spin-spin correlation functions at lattice distance $m$ of the $XXZ$ Heisenberg chain in a magnetic field. In particular, it gives generically an effective re-summation of the corresponding $2^m$ elementary blocks as the sum of only $m$ terms, each containing the distance as the power $m$ of some simple function. Hence, our method opens the possibility of the asymptotic analysis of the spin-spin correlation functions at large distance. It will be shown in a separate publication that it also leads in a direct way to the known answers at the free fermion point $\Delta = 0$.

It should also be noted that the compact formula for the multiple action of the transfer matrix operator, for any values of the spectral parameter and on arbitrary quantum state, is central in our result. It can be used to compute multi-spins correlation functions. It contains in particular the possibility to act on any quantum state with generic conserved quantities responsible for the quantum integrability of the $XXZ$ Heisenberg chain, for example with the Hamiltonian itself. Therefore, this formula is also the key to the dynamical correlation functions. Note finally that this result depends only on the general structure of the $R$-matrix, and thus can be generalized to other models admitting quantum inverse scattering problem solution [18], like the integrable Heisenberg higher spin chains [21].

Acknowledgments

N. K. would like to thank the University of York, the SPhT in Saclay and JSPS for financial support. N. S. is supported by the grants INTAS-99-1782, RFBR-99-01-00151, Leading Scientific Schools 00-15-96046, the Program Nonlinear Dynamics and Solitons and CNRS. J.M. M. is supported by CNRS. V. T is supported by DOE grant DE-FG02-96ER40959 and by CNRS. N. K, N. S. and V. T. would like to thank the Theoretical Physics group of the Laboratory of Physics at ENS Lyon for hospitality, which makes this collaboration possible.

A The highest coefficient

Let

$$S_n(x_1, \ldots, x_n | \mu_1, \ldots, \mu_n | \mu_{n+1}, \ldots, \mu_N) = \frac{\prod_{b=1}^{n} \prod_{a=1}^{n} \sinh(x_a - \mu_b + \eta)}{\prod_{a>b} \sinh(\mu_a - \mu_b) \sinh(x_b - x_a)} \cdot \det_n M_{jk}, \quad (A.1)$$

where the $n \times n$ matrix $M_{jk}$ is

$$M_{jk} = a(\mu_j)t(x_k, \mu_j) \prod_{a=n+1}^{N} f(\mu_a, \mu_j) - e^\beta d(\mu_j)t(\mu_j, x_k) \prod_{a=n+1}^{N} f(\mu_j, \mu_a) \prod_{b=1}^{n} \frac{\sinh(\mu_j - x_b + \eta)}{\sinh(\mu_j - x_b - \eta)} \quad (A.2)$$
First of all we prove an auxiliary lemma, establishing the recursion property of the function \( S_n \).

**Lemma A.1.**

\[
S_n(x_1, \ldots, x_n|\mu_1, \ldots, \mu_n|\mu_{n+1}, \ldots, \mu_N)
\]

\[
= \sum_{i=1}^{n} S_{n-1}(x_1, \ldots, x_{n-1}|\mu_1, \ldots, \mu_i, \ldots, \mu_n|\mu_{n+1}, \ldots, \mu_N, \mu_i) \quad (A.3)
\]

\[
\times \left( a(\mu_1)g(x_n, \mu_1) \prod_{a=1}^{n-1} f(x_a, \mu_a) \prod_{a=n+1}^{N} f(\mu_a, \mu_1) + e^\beta d(\mu_1)g(\mu_1, x_n) \prod_{a=1}^{n-1} f(\mu_1, x_a) \prod_{a=n+1}^{N} f(\mu_a, \mu_1) \right),
\]

where the symbol \( \mu_1 \) means that the corresponding parameter is omitted in the set \( \mu_1, \ldots, \mu_n \).

**Proof.** Consider an auxiliary contour integral:

\[
I = \frac{1}{2\pi i} \int \frac{d\omega}{\sinh(x_n - \omega)} S_n(x_1, \ldots, x_{n-1}, \omega|\mu_1, \ldots, \mu_n|\mu_{n+1}, \ldots, \mu_N). \quad (A.4)
\]

The integral is taken with respect to the boundaries of a horizontal strip of the width \( i\pi \). For instance, one can take for the lower boundary \( \Im(\omega) = \omega_0 \), and for the upper boundary \( \Im(\omega) = \omega_0 + i\pi \). Hereby \( \omega_0 \) is an arbitrary real number satisfying the conditions \( \omega_0 \neq \Im(x_n + i\pi k) \) and \( \omega_0 \neq \Im(\mu_j + i\pi k) \), where \( j = 1, \ldots, n, k \in \mathbb{Z} \). Obviously, the integrand decreases as \( \exp(-2|\omega|) \) at \( \omega \to \pm\infty \). Moreover, the integrand is a periodic function of \( \omega \) with the period \( i\pi \). Thus, \( I = 0 \) and, hence, the sum of the residues inside the contour vanishes. The pole at \( \sinh(\omega - x_n) = 0 \) gives us the term in the l.h.s. of \( (A.3) \). On the other hand, the only singularities of the function \( S_n \) are simple poles at \( \sinh(\omega - \mu_j) = 0 \). The residues in these poles give us the r.h.s. of \( (A.3) \). Thus, the lemma is proved. \( \square \)

**Proposition A.1.** The highest coefficient of the completely indirect action is equal to the function \( S_n \) at \( \mu_1, \ldots, \mu_n = \{\lambda_{\alpha_+}\} \) and \( \mu_{n+1}, \ldots, \mu_N = \{\lambda_{\alpha_-}\} \).

**Proof.** One can use the induction with respect to \( n \). For \( n = 1 \), the equations (4.7), (4.8) give us exactly (4.12). Let the highest coefficient have the form (4.7), (4.8) for \( n - 1 \). Then we have

\[
\langle \psi \rangle \prod_{a=1}^{n} \left( A + e^\beta D \right)(x_a) \bigg| (c.-\text{ind.}) \bigg| \sum_{\{\lambda\} = \{(\lambda_{\alpha_+})\cup(\lambda_{\alpha_-})\}} S_{n-1}(x_1, \ldots, x_{n-1} | \{\lambda_{\alpha_+}\} | \{\lambda_{\alpha_-}\}) \times \langle 0 \rangle \prod_{a=1}^{n-1} C(x_a) \prod_{b \in \alpha_-} C(\lambda_b) \cdot \left( A + e^\beta D \right)(x_n) \bigg| (c.-\text{ind.}) \bigg|
\]

\[
\quad \times \langle \psi \rangle \prod_{a=1}^{n} \left( A + e^\beta D \right)(x_a) \bigg| (c.-\text{ind.}) \bigg| \sum_{\{\lambda\} = \{(\lambda_{\alpha_+})\cup(\lambda_{\alpha_-})\}} S_{n-1}(x_1, \ldots, x_{n-1} | \{\lambda_{\alpha_+}\} | \{\lambda_{\alpha_-}\}) \times \langle 0 \rangle \prod_{a=1}^{n-1} C(x_a) \prod_{b \in \alpha_-} C(\lambda_b) \cdot \left( A + e^\beta D \right)(x_n) \bigg| (c.-\text{ind.}) \bigg|
\]

(4.5)
Now, acting with the last operator, we need to exchange $x_n$ with one of the $\lambda \in \{\lambda_{\alpha_-}\}$. This gives us

$$\langle \psi | \prod_{a=1}^{n} (A + e^\beta D) (x_a) \right|_{(c.-ind.)} = \sum_{(\lambda) = (\lambda_{\alpha_+}) \cup (\lambda_{\alpha_-})} \sum_{l \in \alpha_-} S_{n-1}(x_1, \ldots, x_{n-1}|\{\lambda_{\alpha_+}\}|\{\lambda_{\alpha_-}\})$$

$$\times \left( a(\lambda_l) g(x_n, \lambda_l) \prod_{a=1}^{n-1} f(x_a, \lambda_l) \prod_{a \in \alpha_-} f(\lambda_a, \lambda_l) + e^\beta d(\lambda_l) g(\lambda_l, x_n) \prod_{a=1}^{n-1} f(\lambda_l, x_a) \prod_{a \in \alpha_-} f(\lambda_l, \lambda_a) \right)$$

$$\times \langle 0 | \prod_{a=1}^{n} C(x_a) \prod_{b \in \alpha_-} C(\lambda_b) \right). \quad (A.6)$$

For each fixed partition, one can define the new sets

$$\{\lambda_{\alpha'_+}\} = \{\lambda_{\alpha_+}\} \cup \lambda_l,$$

$$\{\lambda_{\alpha'_-}\} = \{\lambda_{\alpha_-}\} \setminus \lambda_l. \quad (A.7)$$

Then, (A.6) takes the form

$$\langle \psi | \prod_{a=1}^{n} (A + e^\beta D) (x_a) \right|_{(c.-ind.)} = \sum_{(\lambda) = (\lambda_{\alpha'_+}) \cup (\lambda_{\alpha'_-})} \sum_{l \in \alpha'_+} S_{n-1}(x_1, \ldots, x_{n-1}|\{\lambda_{\alpha'_+}\}|\lambda_l|\{\lambda_{\alpha'_-}\} \cup \lambda_l)$$

$$\times \left( a(\lambda_l) g(x_n, \lambda_l) \prod_{a=1}^{n-1} f(x_a, \lambda_l) \prod_{a \in \alpha'_-} f(\lambda_a, \lambda_l) + e^\beta d(\lambda_l) g(\lambda_l, x_n) \prod_{a=1}^{n-1} f(\lambda_l, x_a) \prod_{a \in \alpha'_-} f(\lambda_l, \lambda_a) \right)$$

$$\times \langle 0 | \prod_{a=1}^{n} C(x_a) \prod_{b \in \alpha'_-} C(\lambda_b) \right). \quad (A.8)$$

Identifying in (A.8) and (A.3) $\lambda_l = \mu_l$, $\{\lambda_{\alpha'_+}\} = \mu_1, \ldots, \mu_n$ and $\{\lambda_{\alpha'_-}\} = \mu_{n+1}, \ldots, \mu_N$, we come to the conclusion that the coefficient at the state $\langle 0 | \prod_{a=1}^{n} C(x_a) \prod_{b \in \alpha'_-} C(\lambda_b)$ is exactly $S_n(x_1, \ldots, x_n|\{\lambda_{\alpha'_+}\}|\{\lambda_{\alpha'_-}\})$, what ends the proof. \qed
The properties of the function \( \tilde{S}_n \)

**Proposition B.1.** The matrix \( \tilde{M}_{jk}(\xi_1, \ldots, \xi_n|\lambda_1, \ldots, \lambda_n) \) at \( \beta = 0 \) possesses the eigenvector

\[
\theta_j = \prod_{a=1}^{n} \sinh(\xi_j - \lambda_a) \left( \prod_{\substack{a=1 \atop a \neq j}}^{n} \sinh(\xi_j - \xi_a) \right)^{-1} \tag{B.1}
\]

with zero eigenvalue.

**Proof.** The action of \( \tilde{M}_{jk}(\xi_1, \ldots, \xi_n|\lambda_1, \ldots, \lambda_n) \) at \( \beta = 0 \) on the vector (B.1) can be written in the form

\[
\sum_{k=1}^{n} \tilde{M}_{jk} \theta_k = G_j^{(+)} + G_j^{(-)} \prod_{a=1}^{n} \frac{\sinh(\lambda_a - \lambda_j + \eta) \sinh(\xi_j - \xi_a + \eta)}{\sinh(\lambda_j - \lambda_a + \eta) \sinh(\lambda_a - \lambda_j + \eta)}, \tag{B.2}
\]

where

\[
G_j^{(\pm)} = \sum_{k=1}^{n} \frac{\sinh \eta}{\sinh(\xi_k - \lambda_j) \sinh(\xi_k - \lambda_j \pm \eta)} \cdot \frac{\prod_{a=1}^{n} \sinh(\xi_k - \lambda_a)}{\prod_{\substack{a=1 \atop a \neq k}}^{n} \sinh(\xi_k - \xi_a)}. \tag{B.3}
\]

To find \( G_j^{(\pm)} \) we consider a contour integral similar to the integral in Lemma A.1

\[
I_j^{(\pm)} = \frac{1}{2\pi i} \int \frac{\sinh \eta}{\sinh(\omega - \lambda_j) \sinh(\omega - \lambda_j \pm \eta)} \cdot \frac{\prod_{a=1}^{n} \sinh(\omega - \lambda_a)}{\prod_{\substack{a=1 \atop a \neq k}}^{n} \sinh(\omega - \xi_a)} \, d\omega. \tag{B.4}
\]

Just like in the Lemma A.1, the integral is taken with respect to the boundaries of a horizontal strip of the width \( i\pi \). Due to the periodicity of the integrand and its vanishing at \( \omega \to \pm \infty \), we conclude that \( I_j^{(\pm)} = 0 \), and thus that the sum of the residues within the contour vanishes. The sum of the residues at \( \sinh(\omega - \xi_k) = 0 \) gives \( G_j^{(\pm)} \). In addition, we have one more pole at \( \sinh(\omega - \lambda_j \pm \eta) = 0 \). Combining all together we find

\[
G_j^{(\pm)} = \pm \prod_{a=1}^{n} \frac{\sinh(\lambda_a - \lambda_j \pm \eta)}{\sinh(\xi_a - \lambda_j \pm \eta)}. \tag{B.5}
\]

Substituting this into (B.2) we obtain \( \sum_{k=1}^{n} \tilde{M}_{jk} \theta_k = 0 \), and the Proposition is proved. \( \square \)

Thus, at \( \beta = 0 \), the determinant of the matrix \( \tilde{M}_{jk} \) vanishes for \( n \geq 1 \) and, hence, \( \tilde{S}_n = \delta_{n0} \).
C Symmetrization of an elementary block

The multiple integral obtained in [15] for the emptiness formation probability $\tau(m)$ on the inhomogeneous lattice has the form

$$\tau(m) = \int_C I(\{\lambda\}, \{\xi\}) \prod_{a<b} \sinh(\xi_a - \xi_b) d^m \lambda,$$

where

$$I(\{\lambda\}, \{\xi\}) = \prod_{j=1}^m \left\{ \prod_{k=1}^{j-1} \sinh(\lambda_j - \xi_k + \eta) \prod_{k=j+1}^m \sinh(\lambda_j - \xi_k) \right\} \prod_{a>b} \sinh(\lambda_a - \lambda_b + \eta).$$

Clearly, due to the factor $\det_m[\rho(\lambda_j, \xi_k)]$, the symmetrization of the integrand with respect to all $\{\lambda\}$ is equivalent to the alternating sum of $I(\{\lambda\}, \{\xi\})$ with respect to the permutations $\sigma : \lambda_1, \ldots, \lambda_m \to \lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(m)}$.

**Proposition C.1.**

$$\sum_\sigma (-1)^{p(\sigma)} I(\{\lambda_{\sigma}\}, \{\xi\}) = Z_m(\{\lambda\}, \{\xi\}),$$

where

$$Z_m(\{\lambda\}, \{\xi\}) = \prod_{\alpha=1}^m \prod_{\beta=1}^m \frac{\sinh(\lambda_\alpha - \xi_\beta) \sinh(\lambda_\alpha - \xi_\beta + \eta)}{\sinh(\lambda_\alpha - \lambda_\beta + \eta)} \cdot \frac{\det_m t(\lambda_j, \xi_k)}{\prod_{\alpha>\beta} \sinh(\xi_\alpha - \xi_\beta)}.$$

**Proof.** It is convenient to introduce new variables $x_j = e^{2\lambda_j}$, $y_j = e^{2\xi_j}$, $q = e^\eta$. Then (C.3) takes the form

$$\sum_\sigma (-1)^{p(\sigma)} \prod_{j=1}^m \left\{ \prod_{k=1}^{j-1} (q^{-1} x_{\sigma(j)} - q y_k) \prod_{k=j+1}^m (x_{\sigma(j)} - y_k) \right\} \prod_{m \geq a > b \geq 1} (q^{-1} x_{\sigma(a)} - q x_{\sigma(b)}) = \tilde{Z}_m(\{x\}, \{y\}),$$

and

$$\tilde{Z}_m(\{x\}, \{y\}) = \left( \prod_{a=1}^m x_a \right) \left( \prod_{m \geq a > b \geq 1} (y_a - y_b) \right)^{-1} \prod_{a=1}^m \prod_{b=1}^m \left( \frac{(x_a - y_b)(q^{-1} x_a - q y_b)}{(q^{-1} x_a - q x_b)} \right) \times \det_m \left[ \frac{q^{-1} - q}{(x_j - y_k)(q^{-1} x_j - q y_k)} \right].$$

Observe that (C.5) holds for $m = 1$. Suppose it is valid for $m - 1$. Let us consider the properties of the both sides of (C.5) as functions of $y_m$. Obviously, these functions are polynomials
of $y_m$ of $m - 1$ degree (the poles in $y_m = y_a$ in the r.h.s. disappear due to the zeros of the determinant in the same points). The coefficients of these polynomials are antisymmetric functions of parameters $\{x\}$. Thus, in order to prove \((C.3)\), it is enough to compare the values of both sides of this equality in $y_m = x_a$, $a = 1, \ldots, m$. Moreover, due to the antisymmetry of the coefficients with respect to $\{x\}$, it is sufficient to consider the case $y_m = x_m$. For $y_m = x_m$ the determinant in \((C.6)\) reduces to the product of the last diagonal element by the corresponding minor. Extracting all the dependency on $x_m$ and $y_m$, we obtain for $\tilde{Z}_m$:

$$
\tilde{Z}_m(\{x\}, \{y\})\bigg|_{y_m = x_m} = \prod_{a=1}^{m-1} \frac{(x_a - y_m)(q^{-1}x_m - qy_a)}{(q^{-1}x_m - qx_a)} \tilde{Z}_{m-1}(\{x \neq x_m\}, \{y \neq y_m\}). \tag{C.7}
$$

Consider now the l.h.s. of \((C.5)\). Each term of this sum contains the product $\prod_{j=1}^{m-1} (x_{\sigma(j)} - y_m)$. For $y_m = x_m$, this product does not vanish if and only if $x_{\sigma(m)} = x_m$. Hence, in this case, we need to sum up only with respect to the permutations of the $m - 1$ variables $x_1, \ldots, x_{m-1}$, while $x_m$ remains fixed. Denoting these permutation as $\sigma'$, and extracting again the dependency on $x_m$ and $y_m$, we obtain for the l.h.s. \((C.5)\):

$$
\prod_{a=1}^{m-1} \frac{(x_a - y_m)(q^{-1}x_m - qy_a)}{(q^{-1}x_m - qx_a)} \sum_{\sigma'} (-1)^{p(\sigma')} \prod_{j=1}^{m-1} \left\{ \prod_{k=1}^{j-1} (q^{-1}x_{\sigma'(j)} - qy_k) \prod_{k=j+1}^{m-1} (x_{\sigma'(j)} - y_k) \right\} \prod_{m-1 \geq a > b \geq 1} (q^{-1}x_{\sigma'(a)} - qx_{\sigma'(b)}) \tag{C.8}
$$

Due to the assumption of the induction, the sum with respect to the permutations of the variables $x_1, \ldots, x_{m-1}$ gives $\tilde{Z}_{m-1}(\{x \neq x_m\}, \{y \neq y_m\})$. Then, comparison of \((C.8)\) and \((C.7)\) completes the proof. \hfill \Box

Thus, after symmetrization of the integrand, the multiple integral \((C.1)\) for the emptiness formation probability takes the form

$$
\tau(m) = \frac{1}{m!} \int_C \frac{Z_m(\{\lambda\}, \{\xi\})}{\prod_{a < b} \sinh(\xi_a - \xi_b)} \det_m[\rho(\lambda_j, \xi_k)] \, d^m \lambda. \tag{C.9}
$$

On the other hand, taking the limit $\beta \to \infty$ in \((5.7)\), we obtain

$$
\lim_{\beta \to \infty} e^{-\beta m} \langle \exp(\beta Q_{1,m}) \rangle = \frac{1}{(m!)^2} \left( \prod_{j=1}^n \frac{dz_j}{2\pi i} \int_C \frac{d^m \lambda \prod_{b=1}^m \prod_{a=1}^m f(z_b, \xi_a)}{f(\lambda_b, \xi_a)} \right) \times W_m(\{\lambda\}, \{z\}) \cdot \det_m[t(\lambda_j, z_k)] \cdot \det_m[\rho(\lambda_j, z_k)]. \tag{C.10}
$$

Taking the contour integrals with respect to all $z_j$, we immediately arrive at \((C.9)\).

32
References

[1] H. Bethe, Zeitschrift für Physik, 71 (1931) 205.
[2] R. Orbach, Phys. Rev., 112 (1958) 309.
[3] L. R. Walker, Phys. Rev., 116 (1959) 1089.
[4] C. N. Yang and C. P. Yang, Phys. Rev., 150 (1966) 321 - 339.
[5] E. Lieb, T. Shultz and D. Mattis, Ann. Phys., 16 (1961) 407.
[6] B. M. McCoy, Phys. Rev., 173 (1968) 531.
[7] T. T. Wu, B. M. McCoy, C. A. Tracy, E. Barouch, Phys. Rev. 13, (1976) 316.
[8] B. M. McCoy, C. A. Tracy and T. T. Wu, Phys. Rev. Lett., 38 (1977) 793.
[9] M. Sato, T. Miwa, M. Jimbo, Pub RIMS 14 (1978) 223; 15 (1979) 201; 15 (1979) 871; 16 (1980) 531.
[10] F. Colomo, A. G. Izergin and V. E. Korepin and V. Tognetti, Theor. Math. Phys. 94 (1993) 11.
[11] M. Jimbo, K. Miki, T. Miwa and A. Nakayashiki, Phys. Lett. A 168 (1992) 256.
[12] M. Jimbo and T. Miwa, Journ. Phys. A: Math. Gen., 29 (1996) 2923.
[13] M. Jimbo and T. Miwa, Algebraic analysis of solvable lattice models (AMS, 1995).
[14] N. Kitanine, J. M. Maillet and V. Terras, Nucl. Phys. B, 554 [FS] (1999) 647, math-ph/9807020.
[15] N. Kitanine, J. M. Maillet and V. Terras, Nucl. Phys. B, 567 [FS] 554. (2000), math-ph/9907019
[16] L. D. Faddeev, E. K. Sklyanin and L. A. Takhtajan, Theor. Math. Phys. 40 (1980) 688.
[17] L. A. Takhtajan and L. D. Faddeev, Russ. Math. Surveys. 34 (1979) 11.
[18] J. M. Maillet and V. Terras, Nucl. Phys. B 575 (2000) 627, hep-th/9911030.
[19] N. A. Slavnov, Theor. Math. Phis. 79 (1989) 502.
[20] E. Lieb and W. Liniger, Phys. Rev., 130 (1963) 1605.
[21] N. Kitanine, J. Phys. A 34 (2001) 8151-8169, math-ph/0104016.
[22] N. A. Slavnov, Zap. Nauchn. Sem. POMI 245 (1997) 270.
[23] V. E. Korepin, Commun. Math. Phys. 86 (1982) 391.

[24] A. G. Izergin, Sov. Phys. Dokl., 32 (1987) 878

[25] A. G. Izergin and V. E. Korepin, Commun. Math. Phys. 99 (1985) 271.