ON THE SPLITTING OF POLYNOMIAL FUNCTORS

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Abstract. We develop methods for proving that certain extensions of polynomial functors do not split naturally. As an application we give a functorial description of the third and the fourth stable homotopy groups of the classifying spaces of free abelian groups.

1. Introduction

There are different important spaces whose homotopy type depends naturally on an abelian group $A$. For example, Eilenberg-MacLane spaces $K(A,n)$, $n \geq 1$, suspensions $\Sigma^m K(A,n)$ of these, $m \geq 1$, wedges $K(A,n) \vee K(A,m)$, etc. The homology and homotopy groups of these spaces can be viewed as functors on the category of abelian groups. The problem of describing these as functors is more difficult than an abstract description. Indeed, an abstract description of homology groups $H_n(A)$ of abelian groups is simple, as it follows from the Künneth formula, whereas the functors $A \mapsto H_n(A)$ are complicated [5]. This paper continues the research started in [5], [6]. It is standard that the functorial description of different homological or homotopical functors follows from certain spectral sequences. As a rule, the result of the convergence of a spectral sequence gives not a functor as a whole, but only a filtration on it. To solve the extension problem, i.e. to glue the real functor from different pieces, one needs methods for the control of functorial extensions. Such methods are developed in this paper. Note that all the functors considered in this paper are over $\mathbb{Z}$, they are defined on the category of (free) abelian groups; observe that the analogous results over fields can be obtained more easily.

To give some examples which illustrate the spirit of questions considered in this paper, let us start with two complexes of abelian groups $C^\ast$, $D^\ast$, one can compute the homology of their tensor product $H(C^\ast \otimes D^\ast)$ in terms of $H(C^\ast)$ and $H(D^\ast)$ using the well-known Künneth formula. Now consider three abelian groups $A, B, C$. The Künneth formula gives the following exact sequence

$$0 \to \text{Tor}(A, B) \otimes C \to H_1 \left( A^L \otimes B^L \otimes C \right) \to \text{Tor}(A \otimes B, C) \to 0$$

(1.1)

which splits as a sequence of abelian groups. The middle term of this sequence is the functor $\text{Trip}(A, B, C)$ of MacLane [18] which is simply the first homology group of the iterated tensor product in the derived category. S. MacLane proved in [18] that the sequence (1.1) does not split naturally as a sequence of multi-functors. On the other hand, let us fix the two groups $B = C = \mathbb{Z}/2$. In this case the sequence (1.1) has the

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form

\[ 0 \to Tor(A, \mathbb{Z}/2) \to H_1 \left( A \otimes \mathbb{Z}/2 \otimes \mathbb{Z}/2 \right) \to A \otimes \mathbb{Z}/2 \to 0, \]

and this sequence splits as a sequence of functors. This simply follows from the fact that we can choose a splitting \( A \otimes \mathbb{Z}/2 \otimes \mathbb{Z}/2 \simeq A \otimes (\mathbb{Z}/2 \oplus \mathbb{Z}/2)[1] \) in the derived category functorially in \( A \).

One more example is the following. Let \( A \) be an abelian group. A description of the third homology of \( A \) as a functor is given in [5]. There is the following natural exact sequence

\[ 0 \to \Lambda^3(A) \to H_3(A) \to \Omega^2(A) \to 0 \]  \hspace{1cm} (1.2)

\[ \text{where } \Lambda^3 \text{ is the third exterior power and } \Omega^2 \text{ is the quadratic functor due to Eilenberg-MacLane, which is in fact the first derived functor of the exterior square. The sequence (1.2) splits as a sequence of abelian groups. We prove (see corollary 3.1) that (1.2) is not the case. More generally, we will prove that, for all } n \geq 3, \text{ the natural injection } \Lambda^n(A) \hookrightarrow H_n(A) \text{ induced by Pontryagin product in homology, does not split naturally (see proposition 3.1).} \]

The paper is organized as follows. We recall in section 2 the description of the polynomial functors on the category of free abelian groups in terms of maps between cross-effects from [3]. The language of polynomial \( \mathbb{Z} \)-modules developed in [3] and [1] is useful for the description of \( \text{Hom} \) and \( \text{Ext} \)-groups for polynomial functors in the category of free abelian groups. We use this language for proving that certain exact sequences do not split.

We describe the third stable homotopy group of \( K(A, 1) \) as a functor in section 4. We show that, for a free abelian group \( A \), there is a short exact sequence

\[ 0 \to S^2(A) \otimes \mathbb{Z}/2 \to \pi_3^S K(A, 1) \to \Lambda^3(A) \to 0 \]

which does not split naturally. Moreover, the functor \( \pi_3^S K(A, 1) \) represents the unique non-trivial element in the group of functorial extensions \( \text{Ext}(\Lambda^3, S^2 \otimes \mathbb{Z}/2) = \mathbb{Z}/2 \). In section 5 we give a functorial description of the fourth stable homotopy group of \( K(A, 1) \) for a free abelian group \( A \).

On the other hand, there are some cases in which the existence of a functorial splitting can be proved. For an object \( C \) of the derived category of abelian groups concentrated in non-positive dimensions \( \text{DAb}_{\leq 0} \), we show that the exact triangle in \( \text{DAb}_{\leq 0} \)

\[ \text{LS}^2(C[1]) \to \text{LG}_2(C[1]) \to C \otimes \mathbb{Z}/2[1] \to \text{LS}^2(C[1])[1] \]

induces the functorial splitting (theorem 6.1)

\[ \pi_i(\text{LG}_2(C[1])) \simeq \pi_i(\text{LS}^2(C[1])) \oplus \pi_i \left( C \otimes \mathbb{Z}/2[1] \right), \quad i \geq 1 \]  \hspace{1cm} (1.3)

While the well-known theorem of Dold [11] implies that there is a splitting on the level of complexes which induces the splitting (1.3) on homotopy, we show that this splitting is not functorial, i.e.

\[ \text{LG}_2(C[1]) \neq \text{LS}^2(C[1]) \oplus C \otimes \mathbb{Z}/2[1] \]

in the derived category \( \text{DAb}_{\leq 0} \).

---

1. We will always use the traditional notation \([n]\) for the shift of degree \( n \) in the derived category.

2. After posting the paper the author was informed by N.Kuhn that this result follows from example 7.6 [16].
2. Polynomial functors

Denote by $\text{Ab}$ (resp. $\text{fAb}$) the category of finitely generated abelian (resp. f.g. free abelian) groups. For a small category $C$, let $\text{Fun}(C, \text{Ab})$ be the category of functors from $C$ to $\text{Ab}$. Morphisms in $\text{Fun}(C, \text{Ab})$ are natural transformations between functors. It is well-known that $\text{Fun}(C, \text{Ab})$ is an abelian category with enough projectives and injectives. By $D\text{Ab}_{\leq 0}$ we mean the derived category of abelian groups living in non-negative degrees which is equivalent to the homotopy category of simplicial abelian groups via the Dold-Kan correspondence [12].

The main functors which we will consider are the following ($n \geq 1$):

- Tensor powers $\otimes^n : \text{Ab} \to \text{Ab}$
- Symmetric powers $S^n : \text{Ab} \to \text{Ab}$
- Exterior powers $\Lambda^n : \text{Ab} \to \text{Ab}$
- Divided powers $\Gamma^n : \text{Ab} \to \text{Ab}$
- Antisymmetric square $\sim^2 : \text{Ab} \to \text{Ab}$, defined as $\sim^2(A) := A \otimes A/\{a \otimes b + b \otimes a, \ a, b \in A\}$

We will use the same notation for functors on $\text{Ab}$ and for their restriction on $\text{fAb}$. To distinguish $\text{Hom}$ and $\text{Ext}$ groups for functors on $\text{Ab}$ and $\text{fAb}$, we will use the notation $\text{Hom}(F, G)$ (resp. $\text{Ext}(F, G)$) for ordinal natural transformations (resp. extensions) of functors $F, G : \text{Ab} \to \text{Ab}$ and $\text{Hom}_f(F, G)$ (resp. $\text{Ext}_f(F, G)$) for functors $F, G : \text{fAb} \to \text{Ab}$.

Let $F : \text{Ab} \to \text{Ab}$ be a functor. Recall that the cross-effects of $F$ are multi-functors defined as

$$F(X_1|\ldots|X_n) = \ker\{F(X_1 \oplus \ldots \oplus X_n) \to \oplus_{i=1}^n F(X_1 \oplus \ldots \hat{X}_i \ldots \oplus X_n)\}, \ X_i \in \text{Ab}, \ n \geq 2 \quad (2.1)$$

where the maps $F(X_1 \oplus \ldots \oplus X_n) \to F(X_1 \oplus \ldots \hat{X}_i \ldots \oplus X_n)$ are induced by natural retractions. The functor $F$ is polynomial of degree $d$ ($d \geq 1$) if $F(0) = 0$ and $F(X_1|\ldots|X_d)$ is linear in each variable $X_i, \ i = 1, \ldots, d$.

Given a functor $F$ and an abelian group $A$, consider the system of abelian groups:

$$F_1 = F(A), \ F_2 = F(A|A), \ldots, F_n = F(A|\ldots|A) \ (n \text{ copies of } A)$$

together with the homomorphisms

$$H^m_n : F_n(A) \to F_{n+1}(A), \ P^m_n : F_{n+1}(A) \to F_n(A), \ m = 1, 2, \ldots, m < n$$

which are defined as composite maps

$$H^m_{n+1} : F_n(A) \hookrightarrow F(A^\oplus n) \to F(A^\oplus n+1) \to F_{n+1}(A)$$
$$P^m_{n+1} : F_{n+1}(A) \to F(A^{\oplus n+1}) \to F(A^{\oplus n}) \to F_n(A),$$
induced by natural maps $A^\oplus n \to A^\oplus n+1$, $A^\oplus n+1 \to A^\oplus n$ given by
\[
(a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_m, a_m, \ldots, a_n)
\]
\[
(a_1, \ldots, a_{n+1}) \mapsto (a_1, \ldots, a_{m-1}, a_m + a_{m+1}, a_{m+2}, \ldots, a_{n+1}).
\]
Denote the system of these abelian groups and maps by $J_F(A)$:

These maps satisfy certain standard relations [3], which do not depend on $F$ and $A$. For a polynomial functor $F$ of degree $d$ and an abstract collection of $d$ abelian groups \{$F_i(\mathbb{Z})\}_{i=1,\ldots,d}$ together with corresponding maps which satisfy these relations is known as $d$-polynomial $\mathbb{Z}$-module. Polynomial functors from free abelian groups to abelian groups can be described in terms of polynomial $\mathbb{Z}$-modules [3]. We now consider some particular cases.

1. Quadratic functors. In the case of quadratic functors (see [1]), the required relations are simple: 
\[
A_1 \xrightarrow{H_1^2} A_2 \xrightarrow{P_1^2} A_1
\] 
(2.2) 

Such a diagram of abelian groups is called a quadratic $\mathbb{Z}$-module. It is easy to compute the quadratic $\mathbb{Z}$-modules, which correspond to the classical quadratic functors mentioned above and to $- \otimes \mathbb{Z}/2$. Here they are:

\[
\mathbb{Z}^{\otimes 2} = (\mathbb{Z} \xrightarrow{1} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{1} \mathbb{Z})
\]
\[
\mathbb{Z}^{A^2} = (0 \to \mathbb{Z} \to 0)
\]
\[
\mathbb{Z}^{P^2} = (\mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{2} \mathbb{Z})
\]
\[
\mathbb{Z}^{SP^2} = (\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{1} \mathbb{Z})
\]
\[
\mathbb{Z}^{\mathbb{Z}/2^2} = (\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z}/2)
\]
\[
\mathbb{Z}^{\mathbb{Z}/2} = (\mathbb{Z}/2 \to 0 \to \mathbb{Z}/2)
\]

Given a quadratic $\mathbb{Z}$-module $M$ (2.2), one can define a quadratic functor on the category of abelian groups as follows (see 6.13 [1]): for an abelian group $A$, $A \otimes M$ is generated by the symbols $a \otimes m$, $\{a, b\} \otimes n$, $a, b \in A$, $m \in A_1$, $n \in A_2$ with the relations

\[
(a + b) \otimes m = a \otimes m + b \otimes m + \{a, b\} \otimes H_1^2(m),
\]
\[
\{a, a\} \otimes n = a \otimes P_1^2(n),
\]
\[
a \otimes m \text{ is linear in } m,
\]
\[
\{a, b\} \otimes n \text{ is linear in } a, b, n.
\]

The correspondence $A \mapsto A \otimes M$ defines a quadratic functor and an equivalence between categories of quadratic $\mathbb{Z}$-modules and quadratic functors $\text{fAb} \to \text{Ab}$. 
2. Cubical functors. The cubical $\mathbb{Z}$-module is given by the diagram

\[
\begin{array}{c}
A_1 \xleftarrow{H_2} \xrightarrow{P_2^2} A_2 \xleftarrow{P_3^2} A_3
\end{array}
\]

with the following relations (see [3], [13]):

\[
\begin{align*}
H_3^1H_1^2 &= H_3^2H_1^1, \quad P_2^2P_3^3 = P_2^3P_3^2, \quad H_2^3P_1^3 = 0, \\
H_3^3P_3^3 &= 0, \quad H_3^1P_1^3H_1^1 = 2H_3^1, \quad P_3^3H_3^1P_1^3 = 2P_3^3, \\
H_3^2P_3^2H_2^2 &= 2H_3^2, \quad P_3^2H_3^2P_3^2 = 2P_3^3, \\
H_3^2P_2^2H_1^2 &= 2H_3^2 + 2(P_1^3 + P_3^3)H_1^3H_1^3, \\
P_2^2H_2^1P_1^2 &= 2P_2^2 + 2P_2^3P_3^3(H_3^1 + H_3^1), \\
H_3^3H_1^3P_1^2 + H_3^1 + H_3^2 &= H_3^2P_2^3H_3^1P_3^3H_1^2 + H_3^3P_3^3H_3^2P_2^2H_1^3, \\
H_3^2P_2^2P_3^3 + P_3^3 + P_3^3 &= P_3^3H_3^1P_3^3H_2^2P_2^2 + P_3^3H_3^2P_2^3H_1^3.
\end{align*}
\]

The simplest examples of the cubical $\mathbb{Z}$-modules, which correspond to the exterior and symmetric cubes are the following:

\[
\Lambda^3 \leadsto 0 \xleftarrow{0} 0 \xrightarrow{Z} \quad (3.3)
\]

\[
S^3 \leadsto \mathbb{Z} \xleftarrow{(3,3)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(0,2)} \mathbb{Z} \quad (3.4)
\]

3. $\Delta$-properties. For any functor $F$, the sequence

\[
\mathcal{F}(\mathbb{Z}) : \quad F_1(\mathbb{Z}) \xleftarrow{P_2^2} F_2(\mathbb{Z}) \xrightarrow{P_3^3} F_3(\mathbb{Z}) \xleftarrow{\cdots} \]

is a $\Delta$-group, that is, the standard simplicial relations for the face maps are satisfied. Taking the homology of this complex, we obtain the values of the derived functors in the sense of Dold-Puppe [12]:

\[
H_i(\mathcal{F}(\mathbb{Z})) \simeq L_{i+1}F(\mathbb{Z}, 1)
\]

This follows from the fact that the cross-effect spectral sequence from [12] degenerates to the complex $\mathcal{F}(\mathbb{Z})$.

2.1. Natural transformations between functors. All natural transformations between quadratic functors $\text{fAb} \to \text{Ab}$ are given as morphisms of corresponding quadratic $\mathbb{Z}$-modules. One can therefore use quadratic $\mathbb{Z}$-modules for the computation of the group of natural transformations between given quadratic functors.
**Examples.** 1. A natural map $S^2(A) \to \Gamma_2(A)$ is given by the following diagram:

\[
\begin{array}{ccc}
\mathbb{Z} & \overset{2}{\to} & \mathbb{Z} \\
\downarrow^{2k} & & \downarrow^{2k} \\
\mathbb{Z} & \overset{1}{\to} & \mathbb{Z}
\end{array}
\]

for $k \in \mathbb{Z}$, and $\text{Hom}_f(S^2, \Gamma_2) = \mathbb{Z}$.

2. The natural map $\Gamma_2(A) \to A \otimes \mathbb{Z}/2$ is given by the following diagram:

\[
\begin{array}{ccc}
\mathbb{Z} & \overset{1}{\to} & \mathbb{Z} \\
\downarrow & & \downarrow \\
\mathbb{Z}/2 & \to & 0 & \to & \mathbb{Z}/2
\end{array}
\]

3. Let us now prove that there do not exist non-zero natural maps

\[
\Lambda^2(A) \to S^2(A), \quad S^2(A) \to A \otimes \mathbb{Z}/2
\]

(2.5)

Since the functors in (2.5) are right exact, it is enough to consider these functors on the category of free abelian groups. Hence we can look at morphisms between the corresponding quadratic $\mathbb{Z}$-modules. To every map $\Lambda^2 \to SP^2$ corresponds a morphism of quadratic $\mathbb{Z}$-modules:

\[
\begin{array}{ccc}
0 & \to & \mathbb{Z} \\
\downarrow & & \downarrow \\
\mathbb{Z} & \overset{2}{\to} & \mathbb{Z} \\
\downarrow & & \downarrow \\
\mathbb{Z} & \overset{1}{\to} & \mathbb{Z}
\end{array}
\]

We see that the middle vertical map must be zero, hence the result. Same reasoning applies to the natural transformation $S^2(A) \to A \otimes \mathbb{Z}/2$:

\[
\begin{array}{ccc}
\mathbb{Z} & \overset{2}{\to} & \mathbb{Z} \\
\downarrow & & \downarrow \\
\mathbb{Z}/2 & \to & 0 & \to & \mathbb{Z}/2
\end{array}
\]

We see that any such vertical map is zero. Hence, there is no any non-zero natural transformations (2.5).

4. Consider the case of cubical functors. The functors $S^3$ and $\Lambda^3$ are right exact, so that in order to prove that $\text{Hom}(S^3, \Lambda^3) = 0$ it is enough to show that $\text{Hom}_f(S^3, \Lambda^3) = 0$, i.e. that any map between cubical $\mathbb{Z}$-modules 2.3 and 2.3 is zero. It is easy to see that all vertical maps in the following commutative diagrams are zero:

---

For a map between cyclic groups $f : A \to B$, we will use the notation $A \to^m B$ if $f(a) = nb$, where $a$ and $b$ are some given generators of $A$ and $B$. Analogously we describe the maps between finitely-generated abelian groups by integral matrices.
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\( \mathbb{Z}^{(3,3)} \xrightarrow{(2,0)} \mathbb{Z}^{(1,1)} \xrightarrow{(0,1)} \mathbb{Z}^{(1,0)} \)

\( \mathbb{Z}^{(0,2)} \xrightarrow{(2,0)} \mathbb{Z}^{(1,1)} \xrightarrow{(0,1)} \mathbb{Z}^{(1,0)} \)

This case is very simple due to the structure of cross-effects of \( \Lambda^3 \) and can be easily extended to high dimensional symmetric and exterior powers.

We collect the \( \text{Hom} \)-functors between main quadratic functors in the category of free abelian groups in the following table:

| \( G \) \( \setminus \) \( F \) | \( \Gamma_2 \) | \( \otimes^2 \) | \( \tilde{\otimes}^2 \) | \( \mathbb{Z}/2 \) | \( S^2 \) | \( \Lambda^2 \) | \( \Lambda^2 \otimes \mathbb{Z}/2 \) |
|---|---|---|---|---|---|---|---|
| \( \Gamma_2 \) | \( \mathbb{Z} \) | \( \mathbb{Z} \) | 0 | 0 | \( \mathbb{Z} \) | 0 | 0 |
| \( \otimes^2 \) | \( \mathbb{Z} \) | \( \mathbb{Z} \oplus \mathbb{Z} \) | \( \mathbb{Z} \) | 0 | \( \mathbb{Z} \) | \( \mathbb{Z} \) | 0 |
| \( \tilde{\otimes}^2 \) | \( \mathbb{Z}/2 \) | \( \mathbb{Z} \) | \( \mathbb{Z} \) | \( \mathbb{Z}/2 \) | 0 | \( \mathbb{Z} \) | 0 |
| \( \mathbb{Z}/2 \) | \( \mathbb{Z}/2 \) | 0 | 0 | \( \mathbb{Z}/2 \) | 0 | 0 | 0 |
| \( S^2 \) | \( \mathbb{Z} \) | \( \mathbb{Z} \) | 0 | 0 | \( \mathbb{Z} \) | 0 | 0 |
| \( \Lambda^2 \) | 0 | \( \mathbb{Z} \) | \( \mathbb{Z} \) | 0 | 0 | \( \mathbb{Z} \) | 0 |
| \( \Lambda^2 \otimes \mathbb{Z}/2 \) | 0 | \( \mathbb{Z}/2 \) | \( \mathbb{Z}/2 \) | 0 | \( \mathbb{Z}/2 \) | \( \mathbb{Z}/2 \) | \( \mathbb{Z}/2 \) |

Table 1. \( \text{Hom}_f(F, G) \)

Now observe that some of the polynomial functors \( F \) of degree \( n \) have the property that \( \text{Hom}(F, G) = \text{Hom}(G, F) = 0 \) for any functor \( G \) of degree less than \( n \). Let us now consider examples of functors which do not satisfy this property.

1. For an abelian group \( A \), we have a natural map

\( \Gamma_2(A) \to A \otimes \mathbb{Z}/2 \)

The kernel \( K(A) \) of the natural map

\( \Gamma_2(A) \to A \otimes A \)

defines a functor \( K \) in the category of abelian groups (see [4] for the description of this functor). A simple analysis shows that \( K \) is a linear functor.

2. There are natural transformations

\[ \text{Tor}(A, \mathbb{Z}/2) \to S^2(A), \quad a \mapsto a^2, \quad a \in A, \quad 2a = 0 \]

and

\[ A \otimes \text{Tor}(A, \mathbb{Z}/2) \to S^3(A), \quad a \otimes b \mapsto ab^2, \quad a, b \in A, \quad 2b = 0. \]
Now observe, that for any functor $F$ from the set $\{\Lambda^n, \Lambda^n \otimes \mathbb{Z}/p, \otimes^n, \otimes^n \otimes \mathbb{Z}/p \ (p \text{ is a prime})\}$, the natural map

$$F(A) \rightarrow F(A|\ldots|A) \ (n - \text{th cross effect})$$

induced by the diagonal embedding $A \hookrightarrow A \oplus \cdots \oplus A \ (n \text{ copies of } A)$ is injective. It follows that $\text{Hom}(G, F) = 0$ for every functor $G$ of degree less than $n$. Similarly, the natural projection

$$A \oplus \cdots \oplus A \mapsto A, \ (a_1, \ldots, a_n) \mapsto a_1 + \cdots + a_n, \ a_i \in A,$$

induces a natural epimorphism

$$F(A|\ldots|A) \rightarrow F(A)$$

hence $\text{Hom}(F, G) = 0$ for every functor $G$ of degree less than $n$.

2.2. Extensions between functors. A natural short exact sequence

$$0 \rightarrow S^2(A) \otimes \mathbb{Z}/2 \rightarrow \Gamma_2(A \otimes \mathbb{Z}/4) \rightarrow \Gamma_2(A \otimes \mathbb{Z}/2) \rightarrow 0$$

is given by the following short exact sequence of the quadratic $\mathbb{Z}$-modules$^4$:

\begin{align*}
\mathbb{Z}/2 & \quad \rightarrow \quad \mathbb{Z}/2 \quad \rightarrow \quad \mathbb{Z}/2 \\
\uparrow & \quad & \uparrow & \quad \uparrow \\
\mathbb{Z}/8 & \quad \rightarrow \quad \mathbb{Z}/4 & \quad \rightarrow \quad \mathbb{Z}/8 \\
\uparrow & \quad & \uparrow & \quad \uparrow \\
\mathbb{Z}/4 & \quad \rightarrow \quad \mathbb{Z}/2 & \quad \rightarrow \quad \mathbb{Z}/4 \\
\end{align*}

(2.6)

Similarly, the natural exact sequence

$$0 \rightarrow \Lambda^2(A) \otimes \mathbb{Z}/2 \rightarrow \Gamma_2(A) \otimes \mathbb{Z}/2 \rightarrow A \otimes \mathbb{Z}/2 \rightarrow 0$$

is given by

\begin{align*}
0 & \quad \rightarrow \quad \mathbb{Z}/2 & \quad \rightarrow \quad 0 \\
\uparrow & \quad & \uparrow & \quad \uparrow \\
\mathbb{Z}/2 & \quad \rightarrow \quad \mathbb{Z}/2 & \quad \rightarrow \quad 0 & \quad \rightarrow \quad \mathbb{Z}/2 \\
\uparrow & \quad & \uparrow & \quad \uparrow & \quad \uparrow \\
\mathbb{Z}/2 & \quad \rightarrow \quad 0 & \quad \rightarrow \quad \mathbb{Z}/2 \\
\end{align*}

The following proposition follows directly from the structure of polynomial $\mathbb{Z}$-module which corresponds to the exterior power.

**Proposition 2.1.** Let $F$ be a functor of degree $d$, then $\text{Ext}_f(F, \Lambda^{d+2}) = \text{Ext}_f(\Lambda^{d+2}, F) = 0$.

$^4$we will always display quadratic $\mathbb{Z}$-modules horizontally
Using the language of quadratic \( \mathbb{Z} \)-modules, one can compute the values for \( \text{Ext} \)-functors for main quadratic functors \( F, G : \text{fAb} \to \text{Ab} \). For example,

\[
\begin{align*}
\text{Ext}_f(\Lambda^2 \otimes \mathbb{Z}/2, \Gamma_2) &= \mathbb{Z}/2 \\
\text{Ext}_f(- \otimes \mathbb{Z}/2, S^2) &= \mathbb{Z}/2 \\
\text{Ext}_f(\Lambda^2 \otimes \mathbb{Z}/2, - \otimes \mathbb{Z}/2) &= \mathbb{Z}/2 \\
\text{Ext}_f(\Gamma_2, S^2) &= 0 \\
\text{Ext}_f(- \otimes \mathbb{Z}/2, \Lambda^2 \otimes \mathbb{Z}/2) &= \mathbb{Z}/2 \\
\text{Ext}_f(\Gamma_2, \Gamma_2) &= 0 \\
\text{Ext}_f(\Gamma_2, \Lambda^2 \otimes \mathbb{Z}/2) &= 0
\end{align*}
\]

The proofs are direct, they follow from computations of the extensions between quadratic \( \mathbb{Z} \)-modules which correspond to the quadratic functors.

The generators of the Ext-groups \( \text{Ext}_f(\Lambda^2 \otimes \mathbb{Z}/2, \Gamma_2), \text{Ext}_f(- \otimes \mathbb{Z}/2, S^2), \text{Ext}_f(\Lambda^2 \otimes \mathbb{Z}/2, - \otimes \mathbb{Z}/2) \) one can find in the following diagram

\[
\begin{array}{ccc}
S^2(A) & \longrightarrow & S^2(A) \\
\downarrow & & \downarrow \gamma \\
\Gamma_2(A) & \longrightarrow & \Lambda^2(A) \otimes \mathbb{Z}/2 \\
\downarrow & & \downarrow \\
A \otimes \mathbb{Z}/2 & \longrightarrow & S^2(A) \otimes \mathbb{Z}/2 & \longrightarrow & \Lambda^2(A) \otimes \mathbb{Z}/2
\end{array}
\]

where the map \( \gamma \) is given by setting \( \gamma(a) = a^2, \ a \in A \). The generators of the groups \( \text{Ext}_f(\Gamma_2 \otimes \mathbb{Z}/2, - \otimes \mathbb{Z}/2), \text{Ext}_f(- \otimes \mathbb{Z}/2, \Lambda^2 \otimes \mathbb{Z}/2) \) on can find in the following diagram

\[
\begin{array}{ccc}
A \otimes \mathbb{Z}/2 & \longrightarrow & S^2(A) \otimes \mathbb{Z}/2 & \longrightarrow & \Lambda^2(A) \otimes \mathbb{Z}/2 \\
\downarrow & & \downarrow & & \downarrow \\
A \otimes \mathbb{Z}/2 & \longrightarrow & \Gamma_2(A \otimes \mathbb{Z}/2) & \longrightarrow & \Gamma_2(A) \otimes \mathbb{Z}/2 \\
\downarrow & & \downarrow & & \downarrow \\
A \otimes \mathbb{Z}/2 & \longrightarrow & A \otimes \mathbb{Z}/2
\end{array}
\]

3. Homology of abelian groups

We will now show that \( (1.2) \) does not split naturally. For an abelian group \( A \), recall the bar-resolution

\[
\mathcal{B}(A) : \quad \cdots \longrightarrow \mathbb{Z}[A \oplus A \oplus A] \overset{d_4}{\longrightarrow} \mathbb{Z}[A \oplus A] \overset{d_3}{\longrightarrow} \mathbb{Z}[A] \overset{d_2}{\longrightarrow} \mathbb{Z}
\]
where the differential
\[ d_i : \mathbb{Z}[A^\oplus i] \rightarrow \mathbb{Z}[A^\oplus (i-1)] \]
is given by
\[ d_i : (a_1, \ldots, a_i) \mapsto (a_1, \ldots, a_{i-1}) + \sum_{j=1}^{i-1} (-1)^j (a_1, \ldots, a_j + a_{j+1}, \ldots, a_i). \]

There is a natural isomorphism in the derived category
\[ \mathbb{Z}[A[1]] \simeq \mathcal{B}(A) \]
and, in particular, an isomorphism
\[ H_i(A) \simeq H_i(\mathcal{B}(A)), \ i \geq 0. \]

Some generators of the homology groups \( H_i(A) \) can be easily described in terms of \( \mathcal{B}(A) \), for example, the map
\[ H_2(A) \simeq \Lambda^2(A) \rightarrow \ker(d_2) \]
is given by
\[ a \wedge b \mapsto (a, b) - (b, a), \ a, b \in A. \]

Consider the map \( f : H_2(A; \mathbb{Z}/2) \rightarrow H_2(A; \mathbb{Z}/2) \rightarrow Tor(A, \mathbb{Z}/2) \rightarrow 0 \)

\[ (3.1) \]

Consider the map \( f : H_2(A; \mathbb{Z}/2) \rightarrow H_2(A; \mathbb{Z}/2) = A \otimes A \otimes \mathbb{Z}/2 \) induced by the diagonal map \( A \rightarrow A \oplus A \). Suppose that the sequence \( (3.1) \) splits naturally, i.e. \( H_2(A; \mathbb{Z}/2) = \Lambda^2(A) \otimes \mathbb{Z}/2 \oplus Tor(A, \mathbb{Z}/2) \). Then the composition map \( Tor(A, \mathbb{Z}/2) \rightarrow H_2(A; \mathbb{Z}/2) \xrightarrow{f} H_2(A; \mathbb{Z}/2) \) is zero, since we have seen that there is no non-trivial natural transformation between a linear functor and \( A \otimes A \otimes \mathbb{Z}/2 \).

In particular, for \( A = \mathbb{Z}/2 \), the map
\[ f : H_2(\mathbb{Z}/2; \mathbb{Z}/2) \rightarrow H_2(\mathbb{Z}/2; \mathbb{Z}/2) \]
is zero. Let \( a \) be a generator of \( A = \mathbb{Z}/2 \). One has \( H_2(A; \mathbb{Z}/2) = \mathbb{Z}/2 \) and the generator of this \( \mathbb{Z}/2 \) in the bar-resolution can be chosen as \( (a, a) \in \mathbb{Z}/2[A \oplus A] \). This follows from the fact that \( (a, a) \in ker(d_2) \setminus \ker(d_3) \), since \( \ker(d_3) \) lies in the augmentation ideal of \( \mathbb{Z}/2[A \oplus A] \). Taking \( B = A = \mathbb{Z}/2 \) and \( b \) as a generator of \( B \), we now consider the map of bar resolutions \( \mathcal{B}(A) \rightarrow \mathcal{B}(A \oplus B) \), induced by the diagonal map \( A \rightarrow A \oplus B \). We see that the image of the map \( f \) is generated by an element \( (a + b + a + b) \in \mathbb{Z}/2[(A \oplus B) \oplus (A \oplus B)] \).

It is easy to verify that
\[ (a + b + a + b) \equiv (a, b) + (b, a) \mod \ker(d_3) \]
in \( \mathbb{Z}/2[(A \oplus B) \oplus (A \oplus B)] \). This implies that the image of the element \( (a + b, a + b) \) under the map \( H_2(A \oplus B; \mathbb{Z}/2) \rightarrow H_2(A; \mathbb{Z}/2) = A \otimes B \otimes \mathbb{Z}/2 \) is the same as the image of the element \( (a, b) + (b, a) \). However, the image of the element \( (a, b) + (b, a) \) in \( A \otimes B \otimes \mathbb{Z}/2 \) is exactly \( a \otimes b \otimes 1 \), which is the generator of \( A \otimes B \otimes \mathbb{Z}/2 \). This proves that the sequence \( (3.1) \) does not split functorially.

**Lemma 3.1.** There is a natural isomorphism
\[ H_2(A; \mathbb{Z}/2) \simeq H_3(A; \mathbb{Z}/2). \]
Proof. We have
\[ B(A \oplus \mathbb{Z}/2) \cong B(A) \otimes B(\mathbb{Z}/2) \]
Since
\[ B(\mathbb{Z}/2) \cong \mathbb{Z} \oplus \bigoplus_{n \geq 0} \mathbb{Z}/2[2n + 1], \]
we have a natural isomorphism
\[ H_3(A \oplus \mathbb{Z}/2) \cong H_3(B(A) \oplus (B(A) \otimes \mathbb{Z}/2[1]) \oplus (B(A) \otimes \mathbb{Z}/2[3])) \cong H_3(A) \oplus H_2(B(A) \otimes \mathbb{Z}/2) \oplus H_3(\mathbb{Z}/2), \quad (3.2) \]
where the natural maps \( H_3(A) \to H_3(A \oplus \mathbb{Z}/2) \) and \( H_3(\mathbb{Z}/2) \to H_3(A \oplus \mathbb{Z}/2) \) are splitting monomorphisms on the direct summands in (3.2). It follows that
\[ H_3(A|\mathbb{Z}/2) \cong H_2(B(A) \otimes \mathbb{Z}/2) \cong H_2(A; \mathbb{Z}/2). \]
\[ \square \]

Corollary 3.1. The natural sequence (1.2)
\[ 0 \to \Lambda^3(A) \to H_3(A) \to \Omega_2(A) \to 0 \quad (3.3) \]
does not split functorially.

Proof. Suppose that the sequence (3.3) splits naturally, i.e. there is a natural isomorphism
\[ H_3(A) \cong \Lambda^3(A) \oplus \Omega_2(A). \]
This induces the following natural decomposition for the cross-effect functor
\[ H_3(A|\mathbb{Z}/2) \cong \Lambda^2(A) \otimes \mathbb{Z}/2 \oplus \text{Tor}(A, \mathbb{Z}/2). \]
Lemma 3.1 implies that there is a natural isomorphism
\[ H_2(A; \mathbb{Z}/2) \cong \Lambda^2(A) \otimes \mathbb{Z}/2 \oplus \text{Tor}(A, \mathbb{Z}/2), \]
however this contradicts the fact that the sequence (3.3) does not split functorially. \[ \square \]

Observe that, for any free abelian group, there is a natural isomorphism
\[ H_2(A \otimes \mathbb{Z}/2; \mathbb{Z}/2) \cong \Gamma_2(A) \otimes \mathbb{Z}/2 \]
and the functor \( H_2(A \otimes \mathbb{Z}/2; \mathbb{Z}/2) \) represents the non-trivial element of
\[ \text{Ext}_f(- \otimes \mathbb{Z}/2, \Lambda^2 \otimes \mathbb{Z}/2) = \mathbb{Z}/2. \]
As a consequence of corollary 3.1, for a free abelian group \( A \), the functor
\[ A \mapsto H_3(A \otimes \mathbb{Z}/2) \]
lives in the following short exact sequence
\[ 0 \to \Lambda^3(A) \otimes \mathbb{Z}/2 \to H_3(A \otimes \mathbb{Z}/2) \to \Gamma_2(A) \otimes \mathbb{Z}/2 \to 0 \]
and is represented by the following cubical \( \mathbb{Z} \)-module:
\[
\begin{array}{cccc}
\mathbb{Z}/2 & \xrightarrow{1} & \mathbb{Z}/2 & \xrightarrow{1} \\
\downarrow & & \downarrow & \\
\mathbb{Z}/2 & & \mathbb{Z}/2 & \xrightarrow{0}
\end{array}
\]
In particular, this functor represents the non-trivial element in the corresponding $Ext$-group:
\[ H_3(A \otimes \mathbb{Z}/2) \neq 0 \in Ext_f(\Gamma_2 \otimes \mathbb{Z}/2, \Lambda^3 \otimes \mathbb{Z}/2) = \mathbb{Z}/2. \] (3.4)

We are now ready to generalize corollary 3.1 to the case of higher homology functors.

**Proposition 3.1.** Let $A$ be an abelian group. For $n \geq 3$, the natural monomorphism induced by Pontryagin product
\[ \Lambda^n(A) \hookrightarrow H_n(A) \]
does not split naturally.

**Proof.** For $n = 3$ this is corollary 3.1. Now the result follows by induction on $n$, observing that there is a natural isomorphism
\[ H_n(A|\mathbb{Z}) \simeq H_{n-1}(A) \]
which follows from the Künneth formula. Indeed, assuming that the monomorphism $\Lambda^n(A) \hookrightarrow H_n(A)$ splits naturally, we get the natural splitting of the cross-effects
\[
\begin{array}{ccc}
\Lambda^n(A|\mathbb{Z}) & \longrightarrow & H_n(A|\mathbb{Z}) \\
\uparrow & \simeq & \uparrow \\
\Lambda^{n-1}(A) & \longrightarrow & H_{n-1}(A)
\end{array}
\]
but the lower map is not split by induction hypothesis. \qed

**Remark 3.1.** For an odd prime $p$, the functor
\[ A \mapsto H_3(A \otimes \mathbb{Z}/p), \ A \in \mathfrak{Ab} \]
splits as
\[ H_3(A \otimes \mathbb{Z}/p) = \Lambda^3(A) \otimes \mathbb{Z}/p \oplus \Gamma_2(A) \otimes \mathbb{Z}/p. \] (3.5)

**Proof.** First we prove that $Ext_f(\Gamma_2 \otimes \mathbb{Z}/p, \Lambda^3 \otimes \mathbb{Z}/p) = 0$. Every element of this Ext-group can be presented as a diagram of the form
\[
\begin{array}{ccc}
0 & \leftrightarrow & 0 \\
\downarrow & \downarrow & \downarrow \\
\mathbb{Z}/p & \xrightarrow{1} & \mathbb{Z}/p & \xrightarrow{h_1} & \mathbb{Z}/p & \xrightarrow{h_2} & \mathbb{Z}/p \\
\downarrow & \downarrow & \downarrow \\
\mathbb{Z}/p & \xrightarrow{1} & \mathbb{Z}/p & \xrightarrow{\mathbb{Z}/p} & 0
\end{array}
\]
It follows immediately that $p_1$ and $p_2$ are zero maps. The relations
\[ h_1p_1h_1 = 2h_1, \ h_2p_2h_2 = 2h_2 \]
imply that $h_1$ and $h_2$ are zero map. Hence
\[ Ext_f(\Gamma_2 \otimes \mathbb{Z}/p, \Lambda^3 \otimes \mathbb{Z}/p) = 0. \]
The splitting (3.5) follows from the fact that, for a free abelian \( A \), the sequence (3.3) has the form
\[
0 \to \Lambda^3(A) \otimes \mathbb{Z}/p \to H_3(A \otimes \mathbb{Z}/p) \to \Gamma_2(A) \otimes \mathbb{Z}/p \to 0.
\]

\[ \square \]

4. THE THIRD STABLE HOMOTOPY GROUP OF \( K(A,1) \)

4.1. **Whitehead’s exact sequence.** Let \( X \) be a \((r-1)\)-connected CW-complex, \( r \geq 2 \). Consider the following long exact sequence of abelian groups [23]:
\[
\ldots H_{n+1}X \to \Gamma_nX \to \pi_nX \xrightarrow{h_n} H_nX \to \Gamma_{n-1}X \to \ldots,
\]
where \( \Gamma_nX = im\{\pi_nX^{n-1} \to \pi_nX^n\} \) (here \( X^i \) is the \( i \)-th skeleton of \( X \)), \( h_n \) is the \( n \)th Hurewicz homomorphism. The Hurewicz theorem is equivalent to the statement \( \Gamma_1X = 0 \), \( i \leq r \). J.H.C. Whitehead computed the term \( \Gamma_{r+1}X \) (see [23] [5].

\[
\Gamma_{r+1}X = \begin{cases} 
\Gamma_2(\pi_2X), & r = 2 \\
\pi_rX \otimes \mathbb{Z}/2, & r > 2 
\end{cases}
\]

where \( \Gamma_2 : \text{Ab} \to \text{Ab} \) is the universal quadratic functor (or equivalently the divided square).

Consider the stable analog of the Whitehead exact sequence in low degrees. Here we recall the description of functors \( \Gamma_i \), \( i = r+1, r+2, r+3 \) from [2]. Assume that \( X \) is \((r-1)\)-connected complex, \( r \geq 6 \). In this case, we have the following:

\[
\eta_1 : \pi_r(X) \otimes \mathbb{Z}/2 \to \pi_{r+1}(X)
\]

is induced by the Hopf map \( \eta_r \in \pi_{r+1}(S^r) \), i.e. \( \eta^i(\alpha \otimes 1) = \alpha \eta_r \) and there is a natural exact sequence
\[
0 \to \pi_{r+1}(X) \otimes \mathbb{Z}/2 \to \Gamma_{r+2}X \to \text{Tor}(\pi_r(X), \mathbb{Z}/2) \to 0
\]
where the composite map \( \pi_{r+1}(X) \otimes \mathbb{Z}/2 \to \pi_{r+2}(X) \) is induced by the Hopf map \( \eta_{r+1} \in \pi_{r+1}(S^r) \).

The description of the term \( \Gamma_{r+3}X \) is given as follows. There is a natural exact sequence
\[
L_2\Gamma^2(\eta_1) \to \Gamma^3(\eta_1, \eta_2) \to \Gamma_{r+3}X \to L_1\Gamma^2(\eta^1) \to 0,
\]
where the functors in this sequence can be described in the following way:
\[
L_1\Gamma^2(\eta_1) = coker\{\pi_r(X) \otimes \mathbb{Z}/2 \xrightarrow{h_3} \text{Tor}(\pi_{r+1}(X), \mathbb{Z}/2)\}
\]
\[
L_2\Gamma^2(\eta_1) = ker(\eta_1)
\]
and \( \Gamma^3(\eta_1, \eta_2) = \pi_r(X) \otimes \mathbb{Z}/3 \oplus P \), where \( P \) is given by the pushout
\[
\begin{array}{ccc}
\pi_r(X) \otimes \mathbb{Z}/2 & \xrightarrow{\pi_r(X) \otimes 4} & \pi_{r+2} \otimes \mathbb{Z}/2 \\
\downarrow & & \downarrow \\
\pi_r(X) \otimes \mathbb{Z}/8 & \longrightarrow & P
\end{array}
\]

\(^5\)Care should be taken to distinguish between Whitehead’s functors \( \Gamma_nX \) and the divided power functors \( \Gamma_i(A) \).
where the upper horizontal map induced by the map $S^{r+2} \to S^r$, which defines a generator of $\pi_2^s = \mathbb{Z}/2$.

4.2. Spectral sequence. Recall the spectral sequence from [20]. Consider an abelian group $A$ and its two-step flat resolution

$$0 \to A_1 \to A_0 \to A \to 0.$$ By Dold-Kan correspondence, one obtains the following free abelian simplicial resolution of $A$:

$$N^{-1}(A_1 \hookrightarrow A_0) : \quad \ldots \longmapsto A_1 \oplus s_0(A_0) \longmapsto A_0.$$

Applying Carlsson construction (see [8] or [20] for the detailed description of this construction) to the resolution $N^{-1}(A_1 \hookrightarrow A_0)$, we obtain the following bisimplicial group:

$$E^{N^{-1}(A_1 \hookrightarrow A_0)_2}(S^n)_3 \longmapsto E^{N^{-1}(A_1 \hookrightarrow A_0)_2}(S^n)_2 \longmapsto N^{-1}(A_1 \hookrightarrow A_0)_2 \longmapsto A_1 \oplus s_0(A_0) \longmapsto A_0.$$ Here the $m$th horizontal simplicial group is Carlsson construction $E^{N^{-1}(A_1 \hookrightarrow A_0)_m}(S^n)$. By the result of Quillen [21], we obtain the following spectral sequence:

$$E^2_{p,q} = \pi_q(\pi_p^s \Sigma^n K(N^{-1}(A_1 \hookrightarrow A_0), 1)) \Longrightarrow \pi_{p+q}^s \Sigma^n K(A, 1). \quad (4.4)$$

In particular, for $n$ sufficiently large, the spectral sequence becomes

$$E^2_{p,q} = \pi_q(\pi_p^s K(N^{-1}(A_1 \hookrightarrow A_0), 1)) \Longrightarrow \pi_{p+q}^s K(A, 1). \quad (4.5)$$

where $\pi_n^s$ is the $n$th stable homotopy group.

4.3. $\pi_3^s K(A, 1)$. We now apply the above results for the description of the stable homotopy groups of $K(A, 1)$ in low degrees. Given an abelian group $A$, the homotopy functors $\pi_n^s : \text{Ab} \to \text{Ab}, \ A \mapsto \pi_n^s K(A, 1)$ can be viewed as parts of the Whitehead exact sequence, which functorially depends on $A$. Recall that, for $r \geq 2$, $\pi_2^s K(A, 1) = \pi_{r+2}^s \Sigma^r K(A, 1)$ is the antisymmetric square, and the Whitehead sequence has the form [7]:

$$\begin{align*}
\Gamma_{r+2}^s \Sigma^r K(A, 1) & : \quad \pi_{r+2}^s \Sigma^r K(A, 1) \longrightarrow H_2 K(A, 1) \\
A \otimes \mathbb{Z}/2 & \longrightarrow A \langle s \rangle A \longrightarrow \Lambda^2(A)
\end{align*}$$

Now consider the next step, the functor $\pi_3^s K(A, 1) = \pi_{r+3}^s \Sigma^r K(A, 1)$ for $r \geq 4$. First consider the case of a free abelian group $A$. Observe that, for a free abelian $A$, one has a natural isomorphism

$$A \langle s \rangle A \otimes \mathbb{Z}/2 \cong S^2(A) \otimes \mathbb{Z}/2.$$
We have the following exact sequence

\[
\begin{array}{cccccccc}
H_4(A) & \longrightarrow & \Gamma_{r+3} K(A, 1) & \longrightarrow & \pi_3^S K(A, 1) & \longrightarrow & H_3(A) \\
\Lambda^4(A) & \longrightarrow & S^2(A) \otimes \mathbb{Z}/2 & \longrightarrow & \pi_3^S K(A, 1) & \longrightarrow & \Lambda^3(A)
\end{array}
\]

Now observe that any natural transformation \( \Lambda^4(A) \to S^2(A) \otimes \mathbb{Z}/2 \) is zero, since, for all \( n \geq 2 \), there is no non-trivial transformations between \( \Lambda^n(A) \) and any functor of degree less than \( n \). Therefore, the functor \( \pi_3^S : \text{fAb} \to \text{Ab} \) lives in the following exact sequence

\[
0 \to S^2(A) \otimes \mathbb{Z}/2 \to \pi_3^S K(A, 1) \to \Lambda^3(A) \to 0 \quad (4.6)
\]

It follows from a simple analysis of the extensions of the cubical \( \mathbb{Z} \)-modules which correspond to the functors \( S^2 \otimes \mathbb{Z}/2 \) and \( \Lambda^3 \) that any nontrivial extension between these functors can be given by a diagram of the form

\[
\begin{array}{cccccccc}
\mathbb{Z}/2 & \xrightarrow{0} & \mathbb{Z}/2 & \xrightarrow{=} & 0 \\
\downarrow & = & \downarrow & = & \downarrow \\
\mathbb{Z}/2 & \xrightarrow{0} & \mathbb{Z}/2 & \xrightarrow{0} & \mathbb{Z} \\
\downarrow & = & \downarrow & = & \downarrow \\
0 & \xleftarrow{=} & 0 & \xrightarrow{=} & \mathbb{Z}
\end{array}
\]

and

\[
\text{Ext}_{f}(\Lambda^3, S^2 \otimes \mathbb{Z}/2) = \mathbb{Z}/2. \quad (4.7)
\]

We will show now that the extension (4.6) presents a non-trivial element of (4.7).

**Theorem 4.1.** The functor

\[
\pi_3^S : \text{fAb} \to \text{Ab}, \quad A \mapsto \pi_3^S K(A, 1)
\]

is given by the following cubical module:

\[
\begin{array}{cccccccc}
\mathbb{Z}/2 & \xrightarrow{0} & \mathbb{Z}/2 & \xrightarrow{0} & \mathbb{Z} \\
\downarrow & = & \downarrow & = & \downarrow \\
0 & \xleftarrow{=} & 0 & \xrightarrow{=} & \mathbb{Z}
\end{array}
\]

**Proof.** Assume that, for a free abelian group \( A \), the functor \( \pi_3^S K(A, 1) \) presents the zero element in (4.7), i.e. \( \pi_3^S K(A, 1) = S^2(A) \otimes \mathbb{Z}/2 \oplus \Lambda^3(A) \), and let \( B \) be a non-free abelian group. The spectral sequence (4.5) implies that there is a natural exact sequence

\[
0 \to S^2(B) \otimes \mathbb{Z}/2 \oplus \Lambda^3(B) \to \pi_3^S K(B, 1) \to L_1 \hat{\otimes}^2 (B) \to 0 \quad (4.8)
\]

Consider now the functor

\[
\pi_3^S : \text{fAb} \to \text{Ab}, \quad A \mapsto \pi_3^S K(A \otimes \mathbb{Z}/2, 1)
\]

There is the following short exact sequence (see [20]):

\[
0 \to A \otimes \mathbb{Z}/2 \to L_1 \hat{\otimes}^2 (A \otimes \mathbb{Z}/2) \to \Gamma_2 (A) \otimes \mathbb{Z}/2 \to 0
\]
and $L_1 \otimes^2 \mathbb{Z}/2 = \mathbb{Z}/4$. Hence $L_1 \otimes^2 (A \otimes \mathbb{Z}/2)$ describes a nontrivial element of 

$$\text{Ext}(\Gamma_2(A) \otimes \mathbb{Z}/2, A \otimes \mathbb{Z}/2) = \mathbb{Z}/2$$

and, therefore,

$$L_1 \otimes^2 (A \otimes \mathbb{Z}/2) \simeq \Gamma_2(A \otimes \mathbb{Z}/2).$$

Therefore, the sequence (4.8) can be rewritten for $B = A \otimes \mathbb{Z}/2$ as

$$0 \to S^2(A) \otimes \mathbb{Z}/2 \oplus \Lambda^3(A) \otimes \mathbb{Z}/2 \to \pi^S_3 K(A \otimes \mathbb{Z}/2, 1) \to \Gamma_2(A \otimes \mathbb{Z}/2) \to 0 \quad (4.9)$$

We know from [17] that $\pi^S_3 K(\mathbb{Z}/2, 1) = \mathbb{Z}/8$. The diagram of cubical $\mathbb{Z}$-modules which correspond to the extension (4.9) has the following form

$$
\begin{array}{ccc}
\mathbb{Z}/2 & \overset{0}{\longrightarrow} & \mathbb{Z}/2 \\
\downarrow & & \downarrow \\
\mathbb{Z}/8 & \overset{1}{\longrightarrow} & \mathbb{Z}/2 \\
\downarrow & & \downarrow \\
\mathbb{Z}/4 & \overset{1}{\longrightarrow} & 0
\end{array}
$$

One verifies that the above extension is unique and therefore,

$$\pi^S_3 (A \otimes \mathbb{Z}/2, 1) \simeq \Gamma_2(A \otimes \mathbb{Z}/4) \oplus \Lambda^3(A) \otimes \mathbb{Z}/2$$

(see (2.6)). Observe also that the Whitehead sequence implies that the Hurewicz map

$$\pi^S_3 K(A \otimes \mathbb{Z}/2, 1) \to H_3(A \otimes \mathbb{Z}/2)$$

is a natural surjection, which induces isomorphism on the triple cross-effects. However, it is not possible to construct a commutative diagram of the form

$$
\begin{array}{ccc}
\mathbb{Z}/8 & \overset{1}{\longrightarrow} & \mathbb{Z}/4 \\
\downarrow & & \downarrow \\
\mathbb{Z}/2 & \overset{1}{\longrightarrow} & \mathbb{Z}/2
\end{array}
$$

This gives a contradiction. Therefore, the functor $\pi^S_3 K(A, 1)$ describes a non-trivial element of (4.7).

[17]

Theorem 4.1 implies that the functor $\pi^S_3 K(-\mathbb{Z}/2, 1) : \text{fAb} \to \text{Ab}$ is represented by the cubical $\mathbb{Z}$-module

$$
\begin{array}{ccc}
\mathbb{Z}/8 & \overset{1}{\longrightarrow} & \mathbb{Z}/2
\end{array}
$$
The portion of the Whitehead sequence which contains the natural transformation $\pi^S_3 \to H_3$ has the following form

$$S^2(A) \otimes \mathbb{Z}/2 \to F(A) \to \Lambda^3(A) \otimes \mathbb{Z}/2$$ (4.10)

where the functor $F : \text{fAb} \to \text{Ab}$ is given by the cubical $\mathbb{Z}$-module

$$\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xleftarrow{1} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2$$

The spectral sequence (4.5) therefore implies the following

**Proposition 4.1.** For an abelian group $A$, there is a natural exact sequence

$$0 \to L_0 F(A) \to \pi^S_3 K(A, 1) \to L_1 \otimes^2 (A) \to 0$$

which does not split.

5. The fourth stable homotopy group of $K(A, 1)$

Consider first the case of a free abelian group $A$. We have the following diagram

$$H_5(A) \to \Gamma_{r+4} \Sigma^r K(A, 1) \to \pi^S_3 K(A, 1) \to H_4(A)$$

$$\Lambda^5(A) \to A \otimes \mathbb{Z}/3 \oplus P \to \pi^S_3 K(A, 1) \to \Lambda^4(A)$$

where the term $P$ was defined in (4.3). The functor $A \otimes \mathbb{Z}/3 \oplus P$ is cubical. The natural transformation $\Lambda^5(A) \to A \otimes \mathbb{Z}/3 \oplus P$ is zero and we have a natural short exact sequence

$$0 \to A \otimes \mathbb{Z}/3 \oplus P \to \pi^S_4 K(A, 1) \to \Lambda^4(A) \to 0.$$ (5.1)

Here, by (4.3), the functor $P$ is given by the pushout diagram

$$A \otimes \mathbb{Z}/2 \to \pi^S_3 K(A, 1)$$

$$\xrightarrow{A \otimes 4}$$

$$A \otimes \mathbb{Z}/8 \to P$$
It follows that the functor $P$ can be described by the cubical $\mathbb{Z}$-module:

\[
\begin{array}{ccc}
\mathbb{Z}/8 & \xrightarrow{0} & \mathbb{Z}/2 \\
\xleftarrow{0} & & \xrightarrow{0} \\
& \mathbb{Z}/2 & \xleftarrow{1}
\end{array}
\]

**Theorem 5.1.** The functor $\pi_i^S : f\text{Ab} \to \text{Ab}$, $A \mapsto \pi_i^S K(A, 1)$

is described by the following quartic $\mathbb{Z}$-module

\[
\begin{array}{ccc}
\mathbb{Z}/24 & \xrightarrow{0} & \mathbb{Z}/2 \\
\xleftarrow{0} & & \xrightarrow{0} \\
& \mathbb{Z}/2 & \xleftarrow{1}
\end{array}
\] (5.2)

**Proof.** The proof is similar to that of theorem 4.1. Since the quartic $\mathbb{Z}$-module which corresponds to the functor $\Lambda^4$ has a simple form, it is easy to see that

\[
\text{Ext}_f(\Lambda^4, - \otimes \mathbb{Z}/3 \oplus P) = \mathbb{Z}/2
\] (5.3)

and a nontrivial element in (5.3) is given by the quartic $\mathbb{Z}$-module (5.2). It remains to show that the sequence (5.1) does not split naturally.

Let us assume that the sequence (5.1) does split. Recall that

\[
\text{Hom}(\Lambda^4 \otimes \mathbb{Z}/2, G) = \text{Hom}(G, \Lambda^4 \otimes \mathbb{Z}/2) = 0
\]

for every cubical functor $G$. The spectral sequence (4.5) implies that the functor $\pi_i^S (\otimes \mathbb{Z}/2, 1) : f\text{Ab} \to \text{Ab}$, $A \mapsto \pi_i^S (A \otimes \mathbb{Z}/2, 1)$

can be represented as a direct sum

\[
\pi_i^S (A \otimes \mathbb{Z}/2, 1) \simeq Q \oplus \Lambda^4(A) \otimes \mathbb{Z}/2
\]

for some cubical functor $Q : f\text{Ab} \to \text{Ab}$. The description (4.2) of $\Gamma_{r+4} \Sigma^4 K(A \otimes \mathbb{Z}/2, 1)$ implies that it is a cubical functor for all $r$. It follows that the image of the Hurewicz map

\[
\pi_i^S (A \otimes \mathbb{Z}/2, 1) \to H_4(A \otimes \mathbb{Z}/2)
\]

also has the form $\bar{Q} \oplus \Lambda^4(A) \otimes \mathbb{Z}/2$ for some cubical functor $\bar{Q}$. The diagram (4.10) implies that the Hurewicz map is an epimorphism, hence we obtain that there is a natural isomorphism

\[
H_4(A \otimes \mathbb{Z}/2) \simeq \bar{Q} \oplus \Lambda^4(A) \otimes \mathbb{Z}/2.
\]

Since we know that there exists a natural exact sequence (see [5])

\[
0 \to \Lambda^4(A) \otimes \mathbb{Z}/2 \to H_4(A \otimes \mathbb{Z}/2) \to L_1 \Lambda^3(A \otimes \mathbb{Z}/2) \to 0,
\] (5.4)

we have an isomorphism $\bar{Q} \simeq L_1 \Lambda^3(A \otimes \mathbb{Z}/2)$ and the sequence (5.4) splits. To prove this rigorously, one considers a quartic $\mathbb{Z}$-module associated to $\bar{Q} \oplus \Lambda^4(A) \otimes \mathbb{Z}/2$ and compares it with any possible extension of type (5.4). It remains to observe that

\[
H_4(A \otimes \mathbb{Z}/2) \simeq H_3(A \otimes \mathbb{Z}/2)
\]

so that splitting of (5.4) implies the splitting of

\[
0 \to \Lambda^3(A) \otimes \mathbb{Z}/2 \to H_3(A \otimes \mathbb{Z}/2) \to \Gamma_2(A) \otimes \mathbb{Z}/2 \to 0.
\]
6. The Splitting of the derived functors

6.1. Derived functors. Let $A$ be an abelian group, and $F$ an endofunctor on the category of abelian groups. Recall that for every $n \geq 0$ the derived functor of $F$ in the sense of Dold-Puppe [12] are defined by

$$L_i F(A, n) = \pi_i(FKP_\ast[n]), \quad i \geq 0$$

where $P_\ast \to A$ is a projective resolution of $A$, and $K$ is the Dold-Kan transform, the inverse to the Moore normalization functor $N : \text{Simpl}(\text{Ab}) \to \text{Chain}(\text{Ab})$ from simplicial abelian groups to chain complexes. We denote by $L_i F(A, n)$ the object in the homotopy category of simplicial abelian groups determined by the simplicial abelian group $FK(P_\ast[n])$, so that $L_i F(A, n) = \pi_i(LF(A, n))$.

We set $LF(A) := LF(A, 0)$ and $L_i F(A) := L_i F(A, 0)$ for any $i \geq 0$.

6.2. The splitting of the derived functors of $\Gamma_2$. The natural exact sequence

$$0 \to S^2(A) \to \Gamma_2(A) \to A \otimes \mathbb{Z}/2 \to 0$$

implies that, for an object $C \in \text{DAb}_{\leq 0}$, one has a distinguished triangle

$$LS^2(C) \to L\Gamma_2(C) \to C \otimes \mathbb{Z}/2 \to LSP^2(C)[1]$$

Theorem 6.1. For any $C \in \text{DAb}_{\leq 0}$, there are natural isomorphisms

$$\pi_i(L\Gamma_2(C[1])) \simeq \pi_i(LS^2(C[1])) \oplus \pi_i\left(C \otimes \mathbb{Z}/2[1]\right)$$

for all $i \geq 0$.

The following lemma follows from (Satz 12.1 [12]).

Lemma 6.1. Let $C \in \text{DAb}_{\leq 0}$ be such that $H_0(C) = 0$, then one has $\pi_1(S^2(C)) = 0$. If $H_i(C) = 0$ for $i \leq m$ ($m \geq 1$), then

$$\pi_i(LS^2(C)) = 0, \quad i \leq m + 2$$

Lemma 6.2. For every $C \in \text{DAb}_{\leq 0}$, the suspension homomorphism

$$\pi_1(LS^2(C)) \to \pi_2(LS^2(C[1]))$$

is the zero map.
Proof. We have the following natural diagram
\[
\begin{array}{ccc}
\pi_1(LS^2(C)) & \simeq & L_1S^2(H_0(C)) \\
\downarrow \text{susp} & & \downarrow \\
\pi_2(LS^2(C[1])) & \simeq & \Lambda^2(H_0(C))
\end{array}
\] (6.3)

The right-hand vertical map is zero by (Corollary 6.6, [12]). Another way to see why this map is trivial is to write the cross-effect spectral sequence for \(\pi_*(LS^2(C[1]))\) from [12].

The first page of this spectral sequence implies that there is an exact sequence
\[
0 \to L_1\Lambda^2(H_0(C)) \to \text{Tor}(H_0(C), H_0(C)) \to L_1S^2(H_0(C)) \to \Lambda^2(H_0(C)) \to H_0(C) \otimes H_0(C) \to S^2(H_0(C)) \to 0
\]

where the middle map is the map from (6.3). It is zero map since the natural transformation \(\Lambda^2(H_0(C)) \to H_0(C) \otimes H_0(C)\) is injective. \(\square\)

Proof of theorem 6.1
The proof is by induction on \(i\). Lemma 6.1 implies that there is a natural isomorphism
\[
\pi_1(L\Gamma_2(C[1])) \simeq \pi_1\left(\frac{C}{C \otimes \mathbb{Z}/2[1]}\right)
\]

which is induced by the map \(L\Gamma_2(C[1]) \to C \otimes \mathbb{Z}/2[1]\) from (6.1).

Let us consider separately the case \(i = 2\). The assertion follows from the suspension diagram
\[
\begin{array}{ccccccc}
\pi_3\left(\frac{C}{C \otimes \mathbb{Z}/2[1]}\right) & \to & \pi_2(LS^2(C[1])) & \to & \pi_2(L\Gamma_2(C[1])) & \to & \pi_2\left(\frac{C}{C \otimes \mathbb{Z}/2[1]}\right) \\
\pi_2\left(\frac{C}{C \otimes \mathbb{Z}/2}\right) & \to & \pi_1(LS^2(C)) & \to & \pi_1(L\Gamma_2(C)) & \to & \pi_1\left(\frac{C}{C \otimes \mathbb{Z}/p}\right)
\end{array}
\]

where the left hand vertical homomorphism is zero by lemma 6.2.

Now assume by induction, that, for some \(j \geq 2\) and for all \(i \geq j\), there are natural isomorphisms (6.2), induced by (6.1). Representing the object \(C\) as
\[
\cdots \to C_i \xrightarrow{\partial_i} C_{i+1} \to \cdots,
\]

consider the subcomplex \(Z\) defined by
\[
Z_i = C_i, \quad i \geq j - 1, \\
Z_{j-2} = \text{im}(\partial_{j-1}), \\
Z_i = 0, \quad i < j - 2
\]
The complex $Z$ has the following properties:

1) the natural map $Z \to C$ induces isomorphisms
   $$\pi_i\left(Z \otimes \mathbb{Z}/2\right) \simeq \pi_i\left(C \otimes \mathbb{Z}/2\right), \quad i \geq j;$$

2) $H_i(Z) = 0$, $i \leq j - 2$.

Consider the natural diagram

$$\begin{array}{cccc}
\pi_{j+2}\left(C \otimes \mathbb{Z}/2[1]\right) & \longrightarrow & \pi_{j+1}(LS^2(C[1])) & \longrightarrow & \pi_{j+1}(L\Gamma_2(C[1])) & \longrightarrow & \pi_{j+1}\left(C \otimes \mathbb{Z}/2[1]\right) \\
\pi_{j+2}\left(Z \otimes \mathbb{Z}/2[1]\right) & \longrightarrow & \pi_{j+1}(LS^2(Z[1])) & \longrightarrow & \pi_{j+1}(L\Gamma_2(Z[1])) & \longrightarrow & \pi_{j+1}\left(Z \otimes \mathbb{Z}/2[1]\right)
\end{array}$$

(6.4)

Lemma 6.1 implies that $\pi_{j+1}(LS^2(Z[1])) = 0$. The required splitting now follows from diagram (6.4). The inductive step is complete so that the splitting (6.2) proved for all $i$. □

**Proposition 6.1.** The sequence

$$LS^2(C[1]) \to L\Gamma_2(C[1]) \to C \otimes Z/2[1]$$

(6.5)

does not split in the category $\text{DAb}_{\leq 0}$.

**Proof.** We will prove the statement for the simplest case, when $C$ is a free abelian group. Suppose that $L\Gamma_2(C[1]) \simeq LS^2(C[1]) \oplus C \otimes Z/2[1]$. Then

$$\pi_2\left(L\Gamma_2(C[1]) \otimes Z/2\right) \simeq \pi_2\left(LS^2(C[1]) \otimes Z/2 \oplus C \otimes Z/2 \oplus Z/2[1]\right) \simeq$$

$$\Lambda^2(C) \otimes Z/2 \oplus C \otimes Z/2$$

However, $L\Gamma_2(C[1])$ can be represented by complex

$$(C \otimes C \otimes Z/2 \to \Gamma_2(C) \otimes Z/2)[1]$$

with the obvious map, and

$$\pi_2\left(L\Gamma_2(C[1]) \otimes Z/2\right) = \ker\{C \otimes C \otimes Z/2 \to \Gamma_2(C) \otimes Z/2\}.$$

However, there are no non-trivial natural transformation $C \otimes Z/2 \to C \otimes C \otimes Z/2$. This contradicts to the splitting of (6.5). □

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