Gaps and approximations in the space of growth functions

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Accepted: 11 June 2023 / Published online: 11 August 2023
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Abstract
An important problem in combinatorial noncommutative algebra is to give an asymptotic characterization the growth functions of finitely generated algebras (equivalently, semigroups, or hereditary languages). The growth function of every finitely generated, infinite-dimensional algebra is increasing and submultiplicative. The question of to what extent these natural necessary conditions are also sufficient—and in particular, whether they are sufficient at least for sufficiently rapid functions—was posed and studied by various authors and has attracted a flurry of research. While every increasing and submultiplicative function is realizable as a growth function up to a linear error term, we show that there exist arbitrarily rapid increasing submultiplicative functions which are not equivalent to the growth of any algebra, thus resolving the aforementioned problem and settling a question posed by Zelmanov (and repeated by Alahmadi–Alsulami–Jain–Zelmanov). These can be interpreted as ‘holes’ in the space of growth functions, accumulating to exponential functions in the order topology. We show that there exist monomial algebras and hereditary languages whose growth functions encode the existence of non-prolongable words, and algebras whose growth functions encode the existence of nilpotent ideals (in the graded case). This negatively solves another conjecture of Alahmadi–Alsulami–Jain–Zelmanov in the graded case.

Keywords Growth of algebras · Growth of semigroups · Hereditary languages · Complexity of infinite words · Prolongable languages · Graded algebras

Mathematics Subject Classification 16P90 · 68R15

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1 Introduction

1.1 Growth functions

The question of ‘how do algebras grow?’, or, which functions can be realized as growth functions of algebras, is a major problem in the junction of several mathematical fields including noncommutative algebra, combinatorics of infinite words, symbolic dynamics, formal languages, self-similarity and more. It is considered a basic open problem in combinatorial noncommutative algebra (e.g. [4, 24, 25]).

Consider a finitely generated associative algebra $A$ over an arbitrary field $F$. Suppose that $A$ is infinite-dimensional as an $F$-vector space. Fixing a finite-dimensional generating subspace $A = F \langle V \rangle$, the growth of $A$ with respect to $V$ is defined to be the function:

$$\gamma_{A,V}(n) = \dim_F \left( F + V + V^2 + \cdots + V^n \right).$$

If $1 \in V$ then equivalently $\gamma_{A,V}(n) = \dim_F V^n$. This function evidently depends on the choice of $V$, but might change only up to the following equivalence relation. We say that $f \preceq g$ if $f(n) \leq g(Cn)$ for some $C > 0$ and for all $n \in \mathbb{N}$, and $f \sim g$ (asymptotically equivalent) if $f \preceq g$ and $g \preceq f$. Therefore when talking about ‘the growth of an algebra’ one refers to $\gamma_A(n)$ as the equivalence class of the functions $\gamma_{A,V}(n)$ (for some $V$) under the equivalence relation $\sim$.

The class of growth functions of finitely generated algebras is not only an algebraic entity: it coincides with the classes of growth functions of finitely generated semi-groups and hereditary languages, and (strictly) contains the class of growth functions of finitely generated groups.

There are obvious properties necessarily satisfied by growth functions of algebras; such functions are always:

- **Increasing** (namely, $f(n) < f(n + 1)$); and
- **Submultiplicative** (namely, $f(n + m) \leq f(n)f(m)$).

The main goal in studying the variety of possible growth functions is to investigate to what extent these conditions are in fact sufficient. These properties are referred to as the obvious necessary conditions for growth functions [2, 4, 13, 25].

1.2 The space of growth functions

The first gap result on impossibility of growth functions (apart from the two properties mentioned above—increasing and multiplicative) is Bergman’s gap theorem [6], which asserts that a super-linear growth function must be at least quadratic; for infinite words, this was discovered by Morse and Hedlund [18]. In fact, the discrete derivative $f'(n) = f(n) - f(n - 1)$ is either eventually constant or grows at least linearly. As mentioned in [2], Bergman’s gap is the only gap in the space of growth functions known so far.

Several attempts have been made to realize as wide as possible variety of increasing and submultiplicative functions as growth functions of associative algebras. The first
major step toward this was achieved by Smoktunowicz and Bartholdi [25, Theorem C],
who proved that every increasing and submultiplicative function is equivalent to the
growth of an associative algebra, up to a polynomial factor. Namely, if $f : \mathbb{N} \to \mathbb{N}$
is a submultiplicative and increasing function, then there exists a finitely generated
monomial algebra $B$ whose growth function satisfies:

$$f(n) \leq \gamma_B(n) \leq n^3 f(n).$$

In particular, if $f$ is an increasing submultiplicative function such that $f(n) \sim nf(n)$
then $f$ is equivalent to the growth function of an algebra. This was interpreted in [2,
25] as follows: any ‘sufficiently regular’ function faster than $n^{\ln n}$ is equivalent to the
growth function of an algebra.

To compare, the class of growth functions of groups is much more restricted.
Bartholdi and Erschler [2] realized a broad variety of intermediate functions as growth
rates of finitely generated groups; they proved that any function $f : \mathbb{N} \to \mathbb{N}$ which
grows ‘uniformly faster’ than $\exp(n^\alpha)$ is equivalent to a growth function of some
finitely generated group.\footnote{Here $\alpha = \log 2 / \log \eta \approx 0.7674$, where \eta is the positive root of $X^3 - X^2 - 2X - 4$. ‘Uniformly faster’
here means that $f(2n) \leq f(n)^2 \leq f(n\eta)$ for $n \gg 1$.} Kassabov and Pak [15] constructed groups with oscillating
growth functions, exhibiting a surprising flexibility within the space of growth
functions of groups. The possibility of a parallel pathological behavior for growth of
semigroups and algebras was already known by interesting constructions of Trofimov
[27] and at an extreme by Belov-Borisenko-Latyshev [5], and was recently demon-
strated in the setting of Lie algebras by Petrogradsky [21]. The problem of providing
a complete characterization of growth functions of groups is widely open.

Grigorchuk posed the tantalizing conjecture that there are no groups with super-
polynomial growth which is slower than $\exp(\sqrt{n})$ (see [11]). Grigorchuk proved this
gap for residually-$p$ groups [12], using growth analysis of graded Lie algebras (this
has been extended to residually nilpotent groups in [17]), but the general case is open.
However, there is a gap between polynomial growth functions and a certain quasi-
polynomial function [22].

The space of growth functions of semigroups and algebras is much richer than the
space of growth functions of groups. To what extent are the obvious necessary condi-
tions of being increasing and submultiplicative also sufficient for a (super-quadratic,
to avoid Bergman’s gap) function to be realizable as the growth of an algebra? Bell and
Zelmanov [4] identified an additional condition (on discrete derivatives) satisfied by
all growth functions; their remarkable result is that in fact, every increasing function
satisfying this condition is equivalent to the growth of an associative algebra. They
proved:

**Theorem 1.1** ([4, Theorem 1.1]) A growth function of an algebra is asymptotically
equivalent to a constant function, a linear function, or a non-decreasing function
$f : \mathbb{N} \to \mathbb{N}$ with the following properties:

1. $f'(n) \geq n + 1$ for all $n \in \mathbb{N}$;
2. $f'(m) \leq f'(n)^2$ for all $m \in \{n, \ldots, 2n\}$. 
Conversely, if $f(n)$ is either a constant function, a linear function, or a function with the above properties then it is asymptotically equivalent to the growth function of a finitely generated algebra.

An interesting byproduct of this result, observed in Proposition 3.1, is that every increasing submultiplicative function is realizable as the growth function of an algebra up to a linear error term, thereby improving [25, Theorem C]. This is the best possible approximation, in view of Bergman’s gap theorem.

As the authors suggest, one can interpret this theorem as saying that other than the necessary condition that $f'(m) \leq f'(n)^2$ for all $m \in \{n, \ldots, 2n\}$, which is related to submultiplicativity, the only additional constraints required for being realizable as the growth of an algebra are those arising from Bergman’s gap theorem and the elementary gap that an algebra of sublinear growth is finite-dimensional and hence its growth function is eventually constant.

However, it seems that there is no natural characterization of whether a given function is equivalent to a function satisfying the above condition on discrete derivatives: an increasing, submultiplicative function might be equivalent to a growth function of an algebra, yet need not satisfy condition (2) from [4, Theorem 1.1], which is indeed not preserved under asymptotic equivalence.

Therefore, we remain with the following fundamental question, which was posed on various stages:

**Question 1.2** (Alahmadi-Alsulami-Jain-Zelmanov, 2017 [1]; Zelmanov, 2017 [28])

Let $f : \mathbb{N} \to \mathbb{N}$ be an increasing submultiplicative function such that $f(n) \geq n^2$. Is $f$ asymptotically equivalent to the growth of some finitely generated associative algebra?

It seems that this question had appeared in various other articles in slightly modified versions (e.g. [25, p. 423]). It is reasonable to further weaken the question and ask whether only ‘sufficiently rapid’ increasing and submultiplicative functions (i.e. functions which are faster than a given subexponential function) are realizable [29].

Our first main result answers Question 1.2 in its full generality, refuting even its weaker version. Namely, we prove:

**Theorem 1.3** Let $g : \mathbb{N} \to \mathbb{N}$ be a subexponential function. Then there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that:

- $f$ is submultiplicative and increasing;
- $f \geq g$;
- $f$ is not equivalent to the growth function of any finitely generated algebra.

Moreover, this theorem yields such functions satisfying $f'(n) \geq n + 1$, to emphasize the independence with Bergman’s gap theorem.\(^2\) Notice that a submultiplicative exponentially growing function is equivalent to the growth of a free algebra, so our functions are a fortiori subexponential.

\(^2\) Even though it is a priori possible that any sufficiently rapid, increasing and submultiplicative function is equivalent to a function which additionally satisfies $f'(n) \geq n + 1$. 
Theorem 1.3 exhibits the existence of new ‘holes’ in the space of growth functions. This phenomenon proves the necessity of the error terms in [25, Theorem C] and Proposition 3.1, and hints that there is in fact no characterization in an asymptotic language of growth functions within the class of ‘natural candidates’, namely, increasing and submultiplicative functions. This justifies and emphasizes the importance of using additional characteristics of functions in the attempt to characterize growth functions, such as a submultiplicativity-type condition on discrete derivatives as done in [4, Theorem 1.1].

Theorem 1.3 has the following topological interpretation. Let \( F \) be the set of \( \sim \)-equivalence classes of functions \( f : \mathbb{N} \to \mathbb{N} \) which are increasing and submultiplicative. Then \( F \) is partially ordered by \( \preceq \), and has a top point \( (\exp(n)) \) and a bottom point \( (n) \). This partial order induces a topology (where a basis of open sets is given by open intervals). The main problem now translates to understanding ‘how large’ the subspace \( G \subseteq F \) consisting of equivalence classes of growth functions of algebras is. So, while the top point \( \exp(n) \in G \), Theorem 1.3 tells us that it lies in the closure of \( F \setminus G \).

### 1.3 Realization within restricted classes

Let \( f : \mathbb{N} \to \mathbb{N} \) be (equivalent to) a growth function of an algebra. Can we realize \( f \) as the growth of an algebra with a deeper algebraic/geometric/dynamical structure rather than a pathological combinatorial construction? It was observed by the authors of [4] that their construction (Theorem 1.1 therein) yields monomial algebras with many ‘singular’ monomials, namely, monomials whose product with any other monomial is zero. In fact, the prime radicals of these algebras are so large that the quotients by them are isomorphic to a polynomial ring in one variable.

It is therefore natural to ask whether one can realize growth functions of algebras within the class of monomial algebras whose language of non-zero monomials is prolongable, namely, every non-zero monomial can be extended to a longer non-zero monomial; this direction was proposed to us by Zelmanov. The following construction shows that this is impossible in general. The growth of a hereditary formal language is given by counting the number of its words of length at most \( n \).

**Theorem 1.4** There exists a hereditary language whose growth is not equivalent to the growth of any prolongable hereditary language.

Equivalently, there exists a monomial algebra whose growth function is not equivalent to the growth of any monomial algebra without singular monomials. Hence the phenomenon that the monomial algebras constructed by Bell and Zelmanov [4] contain many singular monomials is, in fact, inevitable.

Therefore, one may ask which growth functions of algebras are realizable within algebras having a deeper algebraic structure, or a more natural mathematical origin rather than a pathological construction. By analogy with geometric group theory, while groups of intermediate growth were known for decades, only a few years ago a simple group of intermediate growth was constructed [20]; simple algebras (and simple Lie algebras) of prescribed intermediate growth were constructed by the author in [8, 9].
Within the class of monomial algebras, important algebraic properties (prolongability, primeness, just-infiniteness) reflect dynamical properties of the underlying subshift from which the algebra originates.

A ring is prime if the product of non-zero ideals is non-zero; this is a non-commutative extension of being an integral domain. In particular, prime rings contain no non-zero nilpotent ideals. A ring is primitive if it acts faithfully on an irreducible module. This is an important representation-theoretic notion, which implies primeness. For a comprehensive survey of primitive rings, see [3].

Conjecture 1.5 (Alahmadi–Alsulami–Jain–Zelmanov, 2017 [1]; Zelmanov, 2017 [28])

The following are equivalent for a function \( f : \mathbb{N} \to \mathbb{N} \):

- \( f \) is equivalent to the growth of a finitely generated algebra;
- \( f \) is equivalent to the growth of a finitely generated primitive algebra;
- \( f \) is equivalent to the growth of a finitely generated nil algebra.

(Excluding linear functions, since by [23] algebras of linear growth satisfy a polynomial identity and are therefore neither primitive nor nil [14]; an algebra is nil if all of its elements are nilpotent.) Let us concentrate on primitive algebras. The above conjecture has a partial affirmative solution up to a quadratic error term:

Proposition 1.6

Let \( f : \mathbb{N} \to \mathbb{N} \) be an increasing, submultiplicative function. Then there exists a finitely generated primitive algebra \( A \) such that:

\[
 f(n) \leq \gamma_A(n) \leq n^2 f(n).
\]

(This was done up to a weaker error term by the author in [10, Section 3]). Surprisingly, there is also a partial negative answer, if we restrict to graded algebras. 3

Theorem 1.7

There exists a finitely generated algebra \( A \) such that:

1. \( \gamma_A(n) \) is not equivalent to the growth of any graded algebra without non-zero nilpotent ideals. In particular, it is not equivalent to the growth of any prime graded algebra.
2. \( \gamma_A(n) \) is not equivalent to the growth of any graded algebra generated in degree 1 with a regular homogeneous element.

In particular, there exist infinite-dimensional monomial algebras all of whose graded deformations contain non-zero nilpotent ideals. If \( A \) is a graded algebra then \( A[t] \) obviously has a regular homogeneous element, so up to a linear error term, all growth functions of algebras are equivalent to growth functions of graded algebras with regular homogeneous elements. It should be mentioned that there exist graded prime algebras without regular homogeneous elements, and graded algebras with regular homogeneous elements which are not prime.

We start with an introductory section (Sect. 2), illustrating the hierarchy of the main classes of growth functions of algebras, languages and subshifts, followed by a short section (Sect. 3) presenting positive realization results which emphasize the significance of our main results.

3 The growth of any algebra is equivalent to the growth of some graded algebra; it is potentially possible, however, that a function is equivalent to the growth of a primitive algebra, but not equivalent to the growth of any primitive graded algebra.
2 A road map of growth: algebra, combinatorics and dynamics

2.1 Algebras

All algebras in this paper are associative and finitely generated over an arbitrary base field $F$. Let $A$ be an $F$-algebra, generated by a finite-dimensional subspace $V$. We say that $A$ is graded if it decomposes into a direct sum of homogeneous components $A = \bigoplus_{n=0}^{\infty} A_n$ such that $A_n A_m \subseteq A_{n+m}$. Throughout the paper, graded algebras are assumed to have finite-dimensional homogeneous components. An algebra is monomial if it can be presented as a quotient of a free algebra modulo monomial relations; a monomial algebra is graded by any assignment of (positive integer) degrees to its generators in the monomial presentation. Monomial algebras are prime if and only if for any two non-zero monomials $u, v$ there exists a monomial $w$ such that $uwv$ is non-zero. For more on monomial algebras, see [5].

Let $A = F \langle x_1, \ldots, x_d \rangle / I$ be an arbitrary algebra. Consider the ideal in $(I) \triangleleft F \langle x_1, \ldots, x_d \rangle$ generated by the initial monomials of elements in $I$, with respect to the order $x_1 \prec \cdots \prec x_d$. The growth of $A$ is equal to the growth of $F \langle x_1, \ldots, x_d \rangle / \text{in}(I)$, the latter being a monomial (hence graded) algebra. Therefore, while the class of graded algebras is very restricted compared to the whole class of algebras, the classes of growth functions of algebras, graded algebras and monomial algebras coincide. We remark that $F \langle x_1, \ldots, x_d \rangle / \text{in}(I)$ might fail to be prime even when $F \langle x_1, \ldots, x_d \rangle / I$ is. The classical comprehensive reference for growth of algebras is [16].

2.2 Languages and subshifts

Let $\Sigma = \{x_1, \ldots, x_d\}$ be a finite alphabet. A language over $\Sigma$ is a subset $L \subseteq \Sigma^* = \bigcup_{i=1}^{\infty} \Sigma^i$, and its elements are called words. Following [4], the growth function of $L$ is $\gamma_L(n) = \#\{\text{length } \leq n \text{ words in } L\}$. A language is hereditary if it is closed under taking subwords. The set of non-zero monomials in a finitely generated monomial algebra forms a hereditary language, and conversely, any hereditary language gives rise to a monomial algebra spanned by its words: $F \langle x_1, \ldots, x_d \rangle / \langle w \text{ monomial} \mid w \notin L \rangle$. Hence the class of growth functions of algebras coincides with the class of growth functions of hereditary languages. We say that a language $L$ is prolongable if for every $w \in L$ there exist $u, v$ such that $uwv \in L$.

2.2.1 Subshifts

The set $\Sigma^\mathbb{Z}$, endowed with the product topology, is a dynamical system with respect to the shift operator $T$. A subshift is a non-empty, closed, shift-invariant subset $X \subseteq \Sigma^\mathbb{Z}$. With any subshift $X \subseteq \Sigma^\mathbb{Z}$ a hereditary language $L(X)$ over $\Sigma$ is associated, namely, the language of its finite factors. The factor complexity of $X$ is the function $p_X(n) = \#\{\text{length-}n \text{ factors of } X\}$. To align with the framework of hereditary languages, let us define the growth function of $X$ to be $P_X(n) = \#\{\text{length } \leq n \text{ factors of } X\}$; thus $P_X(n) = \gamma_{L(X)}(n)$. We call a subshift $X$ transitive if it has
a dense orbit\(^4\): \(X = \{ T^i w \}_{i \in \mathbb{Z}} \) for some (bi-)infinite word \( w \in \Sigma^\mathbb{Z} \). In this case we refer to the complexity (resp. growth) function of the subshift as of the bi-infinite word \( w \), denoted \( p_X(n) = p_w(n) \) (resp. \( P_X(n) = P_w(n) \)). For a survey on complexity functions of infinite words, see [7].

An infinite word in which every factor appears infinitely many times in both sides is called recurrent, and a subshift which has a dense orbit of a recurrent word is irreducible: alternatively, for any two factors \( u, v \in \mathcal{L}(X) \) there exists \( w \) such that \( u w v \in \mathcal{L}(X) \). A transitive aperiodic subshift has no isolated points if and only if it is irreducible.

2.2.2 Groupoids and convolution algebras

Corresponding to any (aperiodic) subshift \( X \subseteq \Sigma^\mathbb{Z} \) one can endow the topological space \( \mathbb{Z} \times X \) with a (topological) groupoid structure, denoted \( \mathfrak{G}_X \), where partial multiplication is given by:

\[(n, T^m u) \cdot (m, u) = (n + m, u).\]

Units are elements of the form \( g g^{-1} \), so in the above case units are of the form \((0, x)\) for \( x \in X \). The convolution algebra over a (discrete) field \( F \) of \( \mathfrak{G}_X \) is the ring of compactly supported continuous functions \( \varphi: \mathfrak{G}_X \to F \) equipped with the pointwise addition and convolution operation. For more on convolution algebras of groupoids associated with subshifts, see [19].

2.2.3 Equivalence of growth functions of languages and subshifts

The equivalence relation of growth functions \( f \sim g \), which is natural in the context of geometric group theory and growth of algebras (where one is not interested in specifying a generating set) has the following interpretation for languages, and hence for subshifts.

Let \( \mathcal{L} \) be a subshift over an alphabet \( \Sigma = \{ x_1, \ldots, x_d \} \). To each letter one can assign a ‘weight’: \( w(x_1), \ldots, w(x_d) \in \mathbb{R}_{>0} \). Accordingly, one can define the weighted growth, measuring the number of words of total weight \( \leq n \) in \( \mathcal{L} \). Then the various weighted complexity functions \( P_{\mathcal{L}, w}(n) \) resulting from any such weighting might be distinct, but are always equivalent: \( \gamma_{\mathcal{L}, w}(n) \sim \gamma_{\mathcal{L}}(n) \). Similarly, for a subshift \( X \subseteq \Sigma^\mathbb{Z} \), all weighted growth functions are equivalent to its standard growth function.

2.2.4 Subshifts correspond to hereditary prolongable languages

The set of all factors of a given subshift forms a hereditary prolongable language; conversely, let \( \mathcal{L} \) be a hereditary prolongable language over an alphabet \( \Sigma = \{ x_1, \ldots, x_d \} \). Enumerate its words: \( w_1, w_2, \ldots \) and notice that since \( \mathcal{L} \) is prolongable we can find bi-infinite words \( W_1, W_2, \ldots \) whose factors belong to \( \mathcal{L} \) and \( w_i \) factors \( W_i \) for each

\(^4\) In the more general setting of dynamical systems, transitivity may refer to having any non-empty open subset intersect any other non-empty open subset, after a suitable shift.
\( i \in \mathbb{N} \). Then the closure of the shift-orbits of \( \{W_1, W_2, \ldots\} \) in \( \Sigma^\mathbb{Z} \) forms a subshift whose set of factors is exactly \( \mathcal{L} \). Therefore, the class of growth functions of subshifts coincides with the class of growth functions of hereditary prolongable languages.

### 2.3 Discrete derivatives

Let \( f : \mathbb{N} \to \mathbb{N} \) be a function. Its \textit{discrete derivative} is the function \( f' : \mathbb{N} \to \mathbb{N} \) given by \( f'(n) = f(n) - f(n - 1) \) (we set \( f'(1) = f(1) \)).

#### 2.3.1 Graded algebras

Let \( A \) be an algebra. If \( A = \bigoplus_{n=0}^{\infty} A_n \) is (connected) graded, generated in degree 1 then the growth function corresponding to the generating subspace \( F + A_1 \) is \( \gamma_A(n) = \dim_F \bigoplus_{i=0}^{n} A_i \) and \( \gamma'_A(n) = \dim_F A_n \).

Suppose that \( A \) has a regular (namely, a non zero-divisor) homogeneous element \( a \in A_1 \). Notice that since \( A \) is generated in degree 1 then \( A \) is a finitely generated module over the Veronese subring \( A(k) = \bigoplus_{i=0}^{\infty} A_{kn} \), which is again generated in degree 1 with respect to a modified grading \( (\deg(r) = \frac{1}{k} \deg(r)) \). Now \( \gamma_{A(k)}(n) = \dim_F \bigoplus_{i=0}^{n} A_{ki} \leq \dim_F \bigoplus_{i=0}^{kn} A_i = \gamma_A(kn) \) and:

\[
\gamma_A(n) = \dim_F \bigoplus_{i=0}^{n} A_i \leq C \cdot \dim_F \bigoplus_{i=0}^{n} A_{ki} \leq \dim_F \bigoplus_{i=0}^{cn} A_{ki} = \gamma_{A(k)}(cn),
\]

where the last inequality follows (for suitable \( c \) depending on \( C \)) since \( A(k) \) contains a regular element. Therefore, we may assume without loss of generality that \( A \) itself contains a regular element \( a \) of degree 1. Then \( \dim_F A_n = \dim_F a A_n \leq \dim_F A_{n+1} \), so \( \gamma'_A(n) \) is non-decreasing, or equivalently \( \gamma''_{A}(n) \geq 0 \). Notice that having a non-decreasing discrete derivative is not invariant under equivalence, namely, it is possible that growth functions of the same algebra with respect to distinct generating subspaces will not share this property.

As observed by Bell and Zelmanov [4], the growth function of any finitely generated algebra with respect to any generating subspace has a submultiplicative discrete derivative. However, a function might be equivalent to a growth function of an algebra, yet have a non-submultiplicative discrete derivative.

#### 2.3.2 Languages

Let \( \mathcal{L} \) be a hereditary language over an alphabet \( \Sigma = \{x_1, \ldots, x_d\} \). Since the (standard) growth function \( \gamma_{\mathcal{L}}(n) \) counts words of length \( \leq n \), its discrete derivative \( \gamma'_{\mathcal{L}}(n) \) counts words of length \( n \). If moreover \( \mathcal{L} \) is prolongable then \( \gamma'_{\mathcal{L}}(n) \) naturally coincides with the complexity function of the underlying subshift. Since any word of length \( n \) can be extended to the right by some letter, and it can be recovered from this extension, we get that the complexity function is non-decreasing. In other words, \( \gamma'_{\mathcal{L}}(n) \geq 0 \). We make the following observation, which we will freely use in the sequel.
Lemma 2.1 Let \( f : \mathbb{N} \rightarrow \mathbb{N} \). If \( f'' \geq 0 \) then \( f(n) \sim nf'(n) \).

Proof Indeed,

\[
   f(n) = \sum_{i=1}^{n} f'(i) \leq \sum_{i=1}^{n} f'(n) = nf'(n)
\]

and on the other hand:

\[
   nf'(n) \leq \sum_{i=n+1}^{2n} f'(i) \leq f(2n),
\]

so \( f(n) \sim nf'(n) \).

In particular, this applies when \( f(n) \) is the (standard) growth function of a prolongable language.

2.4 Hierarchy of growth functions

We close this section with an overview of the hierarchy of classes of growth functions, as presented in Fig. 1. Let us prove the non-trivial relations among it. The arrow \((vi)\) is valid only for super-linear growth functions.

Proof \((i), (ii)\) are shown in Sect. 2.1 and 2.2.

\((iii)\): Let \( \mathcal{L} \) be a prolongable language over a finite alphabet \( \Sigma = \{x_1, \ldots, x_d\} \). Consider the quotient algebra of the free \( F \)-algebra generated by \( \Sigma \) modulo the ideal generated by all monomials which do not belong to \( \mathcal{L} \), say, \( A \). Then \( A \) is a monomial algebra and thus graded by \( \deg(x_1) = \cdots = \deg(x_d) = 1 \) and \( a = x_1 + \cdots + x_d \) is a homogeneous element of degree 1. Moreover, it is regular: suppose that \( f = \sum_{i=1}^{m} c_i u_i \) is a linear combination of distinct monomials such that \( af = 0 \); we may assume that \( u_1, \ldots, u_m \) have an equal degree. Thus:

\[
   \sum_{i=1}^{m} \sum_{j=1}^{d} c_i x_j u_i = 0.
\]

Since \( A \) is a monomial algebra and \( \{x_j u_i\}_{i=1}^{m}, j=1 \) are distinct, and for each \( i \) one of them is non-zero, the coefficients must vanish, so \( f = 0 \). The claim for \( fa = 0 \) is similar.

Conversely, suppose that \( A \) is a monomial algebra with a regular homogeneous element \( a = \sum_{i=1}^{m} c_i u_i \in A \), where \( u_1, \ldots, u_m \) are monomials of the same length and their coefficients \( c_1, \ldots, c_m \) are non-zero. We claim that the language constituting of all non-zero monomials in \( A \) (in the monomial presentation) is a hereditary prolongable language. It is clearly hereditary; to see that it is prolongable, let \( u \in A \) be a non-zero
monomial. Then, since $a$ is regular, we know that:

$$\sum_{i=1}^{m} c_i u_i u = a u \neq 0,$$

and it follows that at least one of the monomials $u_1 u, \ldots, u_m u$ is non-zero, so $u$ is prolongable to the left. Extension to the right is similar, as $ua \neq 0$.

$(iv)$ is shown in Sect. 2.2.4.

$(v)$ follows from $(iii), (vii)$ and the other relations.

$(vi)$: Let $w \in \Sigma^\mathbb{Z}$ and $X = \{T^i w\}_{i \in \mathbb{Z}} \subseteq \Sigma^\mathbb{Z}$. Let $P(n)$ be the corresponding growth function and $p(n) = P'(n)$ the complexity function. Denote the groupoid of the action $\mathbb{Z} \curvearrowright X$ by $\mathcal{G}$. If $P(n)$ is linear then it is not equivalent to the growth of any finitely generated primitive algebra, since finitely generated algebras of linear growth are PI [23], and finitely generated primitive PI algebras are finite-dimensional [14]. Now, if $P(n)$ is super-linear then $w$ is not periodic. Notice that $\mathcal{G}$ is a Hausdorff groupoid, which is ample and effective (see relevant definitions and their validity in our case in [19, 26]) since $w$ is aperiodic. Let $A = F[\mathcal{G}]$ be the convolution algebra of $\mathcal{G}$. Therefore, by [26, Theorem 4.10], it follows that the convolution algebra $F[\mathcal{G}]$ is primitive (clearly $\mathcal{G}$ has a dense orbit; an explicit faithful simple module can be obtained by the bi-infinite matrix representation presented in [19]). Besides, the growth
of the convolution algebra is:

\[ \gamma_{F[\emptyset]}(n) \sim np(n) \]

(see e.g. [19, Proposition 4.5].) But since \( P(n) \) is the growth function of a prolongable language, we have:

\[ np(n) \sim P(n). \]

(vii), (viii) are immediate from the definitions together with the observation that a monomial algebra is prime if and only if for any two non-zero monomials \( u, v \) there exists \( w \) such that \( uwv \) is non-zero, and an irreducible subshift has a dense orbit generated by a recurrent word.

\[ \square \]

3 Realizing and approximating growth functions

3.1 Submultiplicativity suffices up to a linear error term

In [25], it is shown that if \( f : \mathbb{N} \to \mathbb{N} \) is an increasing, submultiplicative function then there exists a finitely generated graded algebra \( A \) such that:

\[ f(n) \leq \dim_F A_n \leq n^2 f(n) \]

and consequently:

\[ f(n) \leq \gamma_A(n) \leq n^3 f(n). \]

Building on [4], we can realize increasing submultiplicative functions up to a linear error term, which is the best possible approximation.

**Proposition 3.1** Let \( f : \mathbb{N} \to \mathbb{N} \) be an increasing, submultiplicative function. Then there exists a finitely generated algebra \( A \) whose growth function satisfies:

\[ f(n) \leq \gamma_A(n) \leq nf(n). \]

**Proof** Let \( f : \mathbb{N} \to \mathbb{N} \) be an increasing, submultiplicative function. Let:

\[ F(n) = \sum_{i=1}^{n} f(i) \]

be its ‘discrete integral’. Then, since \( f = F' \) is increasing and submultiplicative, in particular \( F \) satisfies the conditions of [4, Theorem 1.1] and therefore there exists a finitely generated algebra \( A \) such that \( \gamma_A \sim F \). But \( f(n) \leq F(n) \leq nf(n) \), and the claim follows. \[ \square \]
Remark 3.2 The above approximation is best possible: for any \( \varepsilon > 0 \) one can find a submultiplicative and increasing function \( f : \mathbb{N} \to \mathbb{N} \) such that \( f(n) = \Theta \left( n^{1+\frac{\varepsilon}{2}} \right) \). Thus any function squeezed between \( f(n) \) and \( n^{1-\varepsilon} f(n) \) is super-linear and sub-quadratic, so cannot be equivalent to the growth of any algebra by Bergman’s gap theorem.

3.2 Prime monomial algebras

We start with the following construction, which originates from [25] and modified in [10], brought here in an improved version.

Let \( f : \mathbb{N} \to \mathbb{N} \) be a non-decreasing function satisfying \( f(2^{n+1}) \leq f(2^n)^2 \). Let \( X \) be a set of cardinality \( f(1) \). Define a sequence of integers \( c_{2^n} \) for \( n \geq 0 \) as follows. First, \( c_1 = \lceil \frac{f(2)}{f(1)} \rceil \). For \( n \geq 1 \):

\[
c_{2^n} = \begin{cases} 
\lceil \frac{f(2^{n+1})}{f(2^n)} \rceil & \text{if } f(1)c_1c_2 \cdots c_{2^{n-1}} < 2f(2^n) \\
\lfloor \frac{f(2^{n+1})}{f(2^n)} \rfloor & \text{if } f(1)c_1c_2 \cdots c_{2^{n-1}} \geq 2f(2^n)
\end{cases}
\]

Lemma 3.3 We have:

\[
f(2^{n+1}) \leq f(1)c_1c_2 \cdots c_{2^n} \leq 4f(2^{n+1})
\]

Proof We prove this by induction. For \( n = 0 \), notice that \( f(2) \leq f(1)c_1 \) and \( f(1)c_1 \leq f(2) + f(1) \leq 2f(2) \leq 4f(2) \).

Now for the induction step, assume that \( f(2^n) \leq f(1)c_1c_2 \cdots c_{2^{n-1}} \leq 4f(2^n) \). If \( f(1)c_1c_2 \cdots c_{2^{n-1}} < 2f(2^n) \) then:

\[
f(1)c_1c_2 \cdots c_{2^{n-1}}c_{2^n} < 2f(2^n) \left( \frac{f(2^{n+1})}{f(2^n)} + 1 \right) = 2f(2^{n+1}) + 2f(2^n) \leq 4f(2^{n+1})
\]

and:

\[
f(2^{n+1}) = f(2^n) \cdot \frac{f(2^{n+1})}{f(2^n)} \leq f(1)c_1c_2 \cdots c_{2^{n-1}}c_{2^n}
\]

whereas if \( f(1)c_1c_2 \cdots c_{2^{n-1}} \geq 2f(2^n) \) then:

\[
f(1)c_1c_2 \cdots c_{2^{n-1}}c_{2^n} \leq 4f(2^n) \cdot \frac{f(2^{n+1})}{f(2^n)} = 4f(2^{n+1})
\]

and, if furthermore \( f(2^{n+1}) \geq 2f(2^n) \):

\[
f(1)c_1c_2 \cdots c_{2^{n-1}}c_{2^n} \geq f(1)c_1c_2 \cdots c_{2^{n-1}} \left( \frac{f(2^{n+1})}{f(2^n)} - 1 \right)
\]
\[
\geq 2 f(2^n) \left( \frac{f(2^{n+1})}{f(2^n)} - 1 \right) \\
= 2 f(2^{n+1}) - 2 f(2^n) \geq f(2^{n+1}).
\]

Otherwise if \( f(2^{n+1}) < 2 f(2^n) \) then:

\[
f(1)c_1c_2\cdots c_{2^n-1}c_{2^n} \geq f(1)c_1c_2\cdots c_{2^n-1} \geq 2 f(2^n) \geq f(2^{n+1})
\]

as claimed. \( \square \)

Define a sequence of non-empty sets \( \{W(2^n), C(2^n)\}_{n=0}^{\infty} \) of finite words over \( X \), along with a sequence of finite (ordered) sequences of words \( U(2^n) \) over \( X \) such that:

- \( C(2^n) \subseteq W(2^n) \subseteq X^{2^n} \) and \( |C(2^n)| = c_{2^n} \)
- \( W(2^{n+1}) = W(2^n)C(2^n) \); in particular, every element in \( W(2^j) \) is a prefix of some element in \( W(2^i) \) whenever \( i \leq j \)
- The elements of \( U(2^n) \) come from \( \bigcup_{i=0}^{n} W(2^i) \).

This is done as now follows. Let \( W(1) = X \) (recall that \( |X| = f(1) \)) and \( U(1) = W(1) \). Suppose that \( W(2^n) \subseteq X^{2^n} \) and \( U(2^n) = \{u_1, \ldots, u_l\} \) are given and let us show how to construct \( C(2^n) \subseteq W(2^n) \) and \( W(2^{n+1}), U(2^{n+1}) \) adhering to the above requirements.

We pick a subset \( C(2^n) \subseteq W(2^n) \) of cardinality \( c_{2^n} \) such that \( C(2^n) \) contains at least one word with \( u_1 \) as a prefix. This is indeed possible since \( u_1 \in \bigcup_{i=0}^{n} W(2^i) \), and is therefore a prefix of some word from \( W(2^n) \). Let \( W(2^{n+1}) = W(2^n)C(2^n) \). We define \( U(2^{n+1}) \) via modifying \( U(2^n) \) as follows: append to (the end of) \( U(2^n) \) all of the words of \( W(2^{n+1}) \) in some arbitrary order, and delete \( u_1 \) (from the beginning); thus \( U(2^{n+1}) \) looks like \( \{u_2, \ldots, u_l, w_1, \ldots, w_r\} \) where \( W(2^n) = \{w_1, \ldots, w_r\} \). Rename the words of \( U(2^{n+1}) = \{u_1, \ldots, u_l\} \). This process is indeed possible by Lemma 3.3, since:

\[
c_{2^n} \leq \left[ \frac{f(2^{n+1})}{f(2^n)} \right] \leq f(2^n) \leq f(1)c_1c_2\cdots c_{2^n-1}
\]

\[
= |W(1)C(1)C(2)C(2^2)\cdots C(2^{n-1})| = |W(2^n)|.
\]

Consider the following set of right-infinite words:

\[
\mathcal{S} = W(1)C(1)C(2)C(2^2)C(2^3)C(2^4)\cdots \subseteq X^\infty
\]

and let \( A_\mathcal{S} \) be the quotient of the free algebra \( F\langle X \rangle \) modulo the ideal generated by all monomials which do not occur as factors of any word from \( \mathcal{S} \). Recall that a monomial algebra is prime if and only if the product of any two non-zero monomial ideals in it is non-zero.

**Lemma 3.4** The algebra \( A_\mathcal{S} \) is prime.
Proof By construction, every word \( u \in W(2^n) \) appears in \( U(2^n) \), and therefore for some \( N > n \) we have that the first word in \( U(2^N) \) is \( u \); consequently, there is a word in \( C(2^N) \) with \( u \) as a prefix. It follows that \( u \) factors words from infinitely many \( C(2^{n_i}) \)’s.

Now let \( w_1, w_2 \in A_S \) be non-zero monomials. By the above argument, each one of them factors words from infinitely many \( C(2^{n_i}) \)’s, so let \( d_1 < d_2 \) be natural numbers such that \( w_1 \) factors a word from \( C(2^{d_1}) \), say \( w'_1 \), and \( w_2 \) factors a word from \( C(2^{d_2}) \), say \( w'_2 \). Pick an arbitrary word \( v \in C(2^{d_1+1}) \cdots C(2^{d_2-1}) \) (if \( d_2 = d_1 + 1 \) set \( v \) to be the empty word). Now \( w'_1vw'_2 \in C(2^{d_1}) \cdots C(2^{d_2}) \) and is therefore non-zero in \( A_S \).

Since \( w'_1vw'_2 \in \langle w_1 \rangle \cdot \langle w_2 \rangle \), it follows that the product of these ideals is non-zero and \( A_S \) is prime. \( \square \)

Let \( \gamma_S(n) \) be the number of words of length at most \( n \) which occur as factors of some infinite word in \( S \). This function coincides with the growth function of \( A_S \) with respect to the standard set of generators \( X \cup \{1\} \).

Lemma 3.5 Under the above notations, there exist \( C, D > 0 \) such that:

\[
\gamma'_S(n) \leq Cnf(Dn), \quad \gamma_S(n) \leq Cn^2f(Dn)
\]

and \( f(n) \leq \gamma'_S(2n) \).

Proof By [25, Lemma 6.3], every length-\( 2^m \) word which factors \( S \) is a subword of \( W(2^m)C(2^m) \cup C(2^m)W(2^m) \). Each word in \( W(2^m)C(2^m) \cup C(2^m)W(2^m) \) has length \( 2^m+1 \), so the number of length-\( 2^m \) words, which is equal to \( \gamma'_S(2^m) = \gamma_S(2^m) \), is at least \( |W(2^m)| \) and at most:

\[
(2^m + 1) \cdot \left| \left( W(2^m)C(2^m) \cup C(2^m)W(2^m) \right) \right|
\]

Now since \( f(2^m) \leq |W(2^m)| \leq 4f(2^m) \), we have:

\[
f(2^m) \leq \gamma'_S(2^m) \leq (2^m + 1) \cdot \left( 2 \cdot 4f(2^m) \cdot \left[ \frac{f(2^{m+1})}{f(2^m)} \right] \right) \leq 2^{m+5}f(2^{m+1})
\]

Since every monomial in \( S \) is extendable, \( \gamma'_S \) is non-decreasing. Since \( f \) is non-decreasing as well, for every \( n \), if we take \( 2^m \leq n \leq 2^{m+1} \) then we obtain that \( f(n) \leq \gamma'_S(2n) \), \( \gamma'_S(n) \leq 64nf(4n) \) and \( \gamma_S(n) \leq 64n^2f(4n) \). \( \square \)

Proposition 3.6 Let \( f : \mathbb{N} \rightarrow \mathbb{N} \) be an increasing submultiplicative function. Then there exists a finitely generated prime monomial algebra \( A \) whose growth satisfies:

\[
f(n) \leq \gamma_A(n) \leq n^2f(n)
\]

Proof This immediately follows from Lemma 3.4 and Lemma 3.5. \( \square \)

Prime monomial algebras give rise to hereditary prolongable languages and correspond to recurrent bi-infinite words or equivalently irreducible subshifts (see Sect. 2).
**Corollary 3.7** Let $f : \mathbb{N} \to \mathbb{N}$ be an increasing submultiplicative function. Then there exists an irreducible subshift $X \subseteq \Sigma^\mathbb{Z}$ whose complexity function satisfies:

$$f(n) \preceq p_X(n) \preceq nf(n)$$

Consequently, for the growth function $\gamma_X(n)$ of the subshift $X$, we have:

$$f(n) \preceq \gamma_X(n) \preceq n^2 f(n).$$

**Proof** This follows from Proposition 3.6 and Lemma 3.4, since prime monomial algebras give rise to recurrent bi-infinite words (see Sect. 2.4), and hence to irreducible subshifts with the same growth functions; recall that $\gamma'_S = p_X$ for $X$ being the subshift corresponding to the prime monomial algebra $A_S$. \qed

### 3.3 Primitive algebras

**Proposition 3.8** (Proposition 1.6) Let $f : \mathbb{N} \to \mathbb{N}$ be an increasing submultiplicative function. Then there exists a finitely generated primitive algebra $B$ such that:

$$f(n) \preceq \gamma_B(n) \preceq n^2 f(n)$$

In particular, this applies to $f = \gamma_A$ the growth function of an arbitrary algebra.

**Proof** By Proposition 3.6 there exists a finitely generated, prime monomial algebra $A$ such that $f(n) \preceq \gamma_A(n) \preceq n^2 f(n)$. As proven in Sect. 2.4, there exists a finitely generated primitive algebra $B$ such that $\gamma_A \sim \gamma_B$ (we may obviously assume that $A$ does not have linear growth). The claim follows. \qed

### 4 Arbitrarily rapid holes in the space of growth functions

#### 4.1 Asymptotic properties of growth functions

By a result of Bell and Zelmanov [4, Proposition 2.1], if $\gamma$ is a growth function of a finitely generated algebra then $\gamma'(m) \leq \gamma'(n)^2$ for every $m \in \{n, \ldots, 2n\}$. Their proof technique can be modified to yield:

**Proposition 4.1** Let $\gamma$ be the growth function of a finitely generated algebra with respect to some generating subspace. Let $d \in \mathbb{N}$. Then $\gamma'(m) \leq \gamma'(n)^d$ for every $m \in \{n, \ldots, dn\}$.

**Proof** We may assume the algebra is monomial, so $\gamma'(n)$ is the number of (non-zero) words of length $n$ in the generators. But if $n \leq m \leq dn$ then every word $u$ of length $m$ can be recovered from the following ordered list of length-$n$ factors:

$$u[1, n], u[n + 1, 2n], \ldots, u[(r - 1)n + 1, rn]$$

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where \( r = \lfloor m/n \rfloor \) and in addition, if \( m < dn \) (hence \( r \leq d - 1 \)), the factor:

\[
u[m - n + 1, m].\]

Since these are at most \( d \) factors of length \( n \) each, it follows that \( \gamma'(m) \leq \gamma'(n)^d \). \( \square \)

This is utilized to construct an obstruction for a function to be realizable, up to equivalence, as a growth function of an algebra.

**Proposition 4.2** Suppose that \( f : \mathbb{N} \rightarrow \mathbb{N} \) is equivalent to the growth function \( \gamma : \mathbb{N} \rightarrow \mathbb{N} \) of a finitely generated algebra with respect to a generating subspace. Then there exists \( C \in \mathbb{N} \) such that for all \( D \geq C^2 \), for all \( n > 1 \) we have:

\[
f(2CDn) - f(2CDn - C) \leq 2D^2n(f(CDn) - f(Cn - C))^{2D}.
\]

**Proof** Since \( f \sim \gamma \), there exists \( C > 0 \) such that \( f(n) \leq \gamma(Cn) \) and \( \gamma(n) \leq f(Cn) \). In particular, for every \( D \geq C^2 \), we have \( \gamma(n) \leq f(Cn) \leq \gamma(Dn) \). Set \( h(n) = f(Cn) \) and \( \varphi(n) = \gamma(Dn) \). Then \( h(n) \leq \varphi(n) \leq h(Dn) \). Observe that:

\[
h'(n) = f(Cn) - f(Cn - C) \leq \gamma(Dn) - \gamma(n - 1) = \sum_{k=n}^{Dn} \gamma'(k) \leq Dn\gamma'(n)^D.
\]

Note also that:

\[
\gamma'(Dn) = \gamma(Dn) - \gamma(Dn - 1) \\
\leq \gamma(Dn) - \gamma(Dn - D) \\
= \varphi(n) - \varphi(n - 1) \\
\leq h(Dn) - h(n - 1).
\]

Putting these together, we get that:

\[
f(2CDn) - f(2CDn - C) = h'(2Dn) \\
\leq 2D^2n\gamma'(2Dn)^D \\
\leq 2D^2n\gamma'(Dn)^{2D} \\
\leq 2D^2n(h(Dn) - h(n - 1))^{2D} \\
= 2D^2n(f(CDn) - f(Cn - C))^{2D},
\]

as desired. \( \square \)

**4.2 Constructing a submultiplicative function**

Let \( 2 < d_1 < d_2 < \cdots \) be an increasing sequence of integers, and \( n_1, n_2, \ldots \) a sequence of positive integers such that:

\[
n_1 < d_1n_1 < n_2 < d_2n_2 < n_3 < \cdots.
\]
Both sequences are to be restricted in the sequel by conditions of the form “$d_k$ is greater than a function of $\{d_i, n_i\}_{i=1}^{k-1}$” and “$n_k$ is greater than a function of $\{d_i, n_i\}_{i=1}^{k-1}$ and $d_k$”.

**4.2.1 The interval $[1, n_2]$**

We will define a function $f : \mathbb{N} \to \mathbb{N}$, first by defining it on the domain $[1, n_2]$:

- For $x \leq n_1$, take $f(x) = 2^x$;
- For $n_1 < x \leq d_1 n_1$, take $f(x) = f(x-1) + x + 1$;
- For $d_1 n_1 < x \leq n_2$, take $f(x) = \lceil 2^{1/2d_1} f(x-1) \rceil$.

Denote $\alpha_1 = f(d_1 n_1) - f(n_1) < d_1^2 n_1^2$. Since $\alpha_1$ is polynomial with respect to $n_1$ (assuming $d_1$ was fixed), if we take $n_1 \gg 1$ then we may assume that $f(d_1 n_1) = 2^{n_1} + \alpha_1 \leq 2^{n_1 + \frac{1}{3}}$ and $f'(x) \geq x + 1$ for $x > 2$ under the above definition. We also need the following fact:

**Lemma 4.3** Given $c > 1$ and $\varepsilon > 0$, for all $a_0 \gg 1$ the sequence $a_{k+1} = \lfloor c a_k \rfloor$ satisfies $c^{k-\varepsilon} a_0 \leq a_k \leq c^k a_0$.

**Proof** Obviously $a_{k+1} \leq c a_k$ so $a_k \leq c^k a_0$. By induction $a_k \geq c^k a_0 - \frac{c^k - 1}{c - 1}$, so:

$$a_k - c^{k-\varepsilon} a_0 \geq (c^k - c^{k-\varepsilon}) a_0 - \frac{c^k - 1}{c - 1} \xrightarrow{a_0 \to \infty} \infty,$$

uniformly with $k$.

Using Lemma 4.3 (taking $c = 2^{\frac{1}{2d_1}}, \varepsilon = 2^{-3}$), we can take $n_1 \gg 1$ so that if $x \geq d_1 n_1$ then $f(x) \geq f(d_1 n_1) \cdot 2^{\frac{x - d_1 n_1}{2d_1} - 2^{-3}}$. If we take $n_1 \gg 1$ we may assume that $f$ is increasing in $[1, n_2]$. We now turn to prove that $f$ is submultiplicative.

**Proposition 4.4** If $p, q \in \mathbb{N}$ satisfy $p + q \leq n_2$ then $f(p + q) \leq f(p) f(q)$.

**Proof** We first take care of the interval $[d_1 n_1 + 1, n_2]$. Pick $p + q > d_1 n_1$ with $p \leq q$ and we must show that $f(p + q) \leq f(p) f(q)$. We assume that $d_1 > 2$ and $n_1 \gg 1$ (by means to be specified along the proof) and compute that:

$$f(p + q) \leq f(d_1 n_1) \cdot 2^{p + q - d_1 n_1} = 2^{n_1 + \frac{1}{3} + p + q - d_1 n_1} = 2^{\frac{1}{3} n_1 + \frac{p + q}{2d_1} + 1}.$$

Assume that $p \leq n_1$, then $q > n_1$. Whether or not $q \leq d_1 n_1$, we have that:

$$\frac{f(p + q)}{f(q)} \leq 2^{\frac{p}{2d_1}} \leq 2^p = f(p).$$
the first inequality follows since the ratio between two successive values of $f$ over $[n_1, n_2]$ is $\leq 2^\frac{1}{2d_1^2}$ if we only take $n_1 \gg 1$. Thus we suppose $n_1 < p$ (so in particular $f(q) \geq f(p) \geq 2^{n_1}$).

- If $d_1n_1 \leq p$ then (assuming $n_1 > 1$):
  \[
  f(p)f(q) \geq 2^{n_1+\frac{p-d_1n_1}{2d_1^2}} 2^{n_1+\frac{q-d_1n_1}{2d_1^2}} = 2^{n_1+\frac{p+q}{2d_1^2}} 2^{-3} \\
  \geq 2^{\frac{1}{2}n_1+\frac{p+q}{2d_1^2}+\frac{1}{3}} \geq f(p+q).
  \]

- If $q \leq d_1n_1$ then:
  \[
  f(p)f(q) \geq 2^{2n_1} \geq 2^{\frac{1}{2}n_1+n_1+\frac{1}{3}} \geq f(p+q),
  \]
  the latter inequality follows since $p+q \leq 2d_1n_1$.

- If $p < d_1n_1 < q$ then:
  \[
  f(p)f(q) \geq 2^{n_1+\frac{q-d_1n_1}{2d_1^2}} 2^{-3} = 2^{\frac{3}{2}n_1+\frac{q}{2d_1^2}} 2^{-3} \\
  \geq 2^{\frac{3}{2}n_1+\frac{p+q}{2d_1^2}+\frac{1}{2}n_1-\frac{2}{3}} > f(p+q).
  \]

(The penultimate inequality follows since $p < d_1n_1$.) As for submultiplicativity in the interval $[n_1, d_1n_1]$ (note that the case $p+q \in [1, n_1]$ is trivial as the function $x \mapsto 2^x$ is submultiplicative), assume that $n_1 \leq p+q \leq d_1n_1$.

- If $p > n_1$ then:
  \[
  f(p)f(q) \geq f(n_1)^2 = 2^{2n_1} \geq 2^{n_1+\frac{1}{3}} \geq f(d_1n_1) \geq f(p+q).
  \]

- If $q < n_1$ then:
  \[
  f(p)f(q) = 2^{p+q} \geq f(p+q).
  \]

- If $p \leq n_1 \leq q$ then:
  \[
  f(p+q) \leq f(d_1n_1) \\
  \leq 2^{n_1+\frac{1}{3}} \leq 2^p 2^{n_1} = f(p)f(n_1) \leq f(p)f(q).
  \]

And the claim follows.
4.2.2 Extending $f$ to $\mathbb{N}$

We now extend $f$ to $\mathbb{N}$ as follows. Suppose that $d_1, \ldots, d_{k-1}, n_1, \ldots, n_{k-1}$ are fixed and $f$ is defined on the domain $[1, n_k]$ (we choose $n_k$ only after $\{d_i, n_i\}_{i=1}^{k-1}, d_k$ were fixed). Assumptions on $d_k, n_k$ will be made explicitly during the proof of submultiplicativity, in order to clarify where these assumptions originate from. We assume $d_k \geq n_k - 1$ (for $k = 2$ simply $d_2 \geq n_1/2$). Define:

- For $n_k < x \leq d_k n_k$, take $f(x) = f(x-1) + x + 1$;
- For $d_k n_k < x \leq n_{k+1}$, take $f(x) = \lfloor \frac{x - d_k n_k}{2} \rfloor$.

Note that by taking $n_k$ to be large enough we can make sure that $f'(x) \geq x + 1$.

**Condition (I).** We pick $n_k$ large enough such that for all $d_k n_k \leq x \leq y \leq n_{k+1}$ we have that:

$$f(y) \geq f(x) \cdot 2^\frac{y-x}{d_k} - 2^{-k-2}$$

(This is possible by Lemma 4.3 applied to $c = 2^{\frac{1}{d_1 \cdots d_k}}$ and $\epsilon = 2^{-k-2}$.) In particular,

$$f(x) \geq f(d_k n_k) \cdot 2^\frac{x - d_k n_k}{d_k} - 2^{-k-2}$$

**Lemma 4.5** We can choose $\{n_k\}_{k=1}^\infty$ to be sufficiently sparse such that for every $2 \leq x \leq d_k n_k$ we have that:

$$f(x) \geq 2^\frac{x}{d_k} + 1 + 2^{-k-1}$$

**Proof** We prove the assertion by induction on $k$. For $k = 1$, if $2 \leq x \leq n_1$ then:

$$f(x) = 2^x \geq 2^{\frac{1}{d_1} + 1 + 2^{-2}}$$

and if $n_1 < x \leq d_1 n_1$ then:

$$f(x) \geq 2^n_1 \geq 2^{\frac{d_1 n_1}{d_1} + 1 + 2^{-2}} \geq 2^{\frac{x}{d_1} + 1 + 2^{-2}}$$

so the assertion holds (indeed, we take $n_1 \geq 3$). Suppose the claim holds for $2 \leq x \leq d_k n_k$ and let us prove it for $2 \leq x \leq d_k + 1 n_{k+1}$; if $x \leq d_k n_k$ this is immediate from the induction hypothesis. If $d_k n_k < x \leq n_{k+1}$ then by Condition (I):

$$f(x) \geq f(d_k n_k) \cdot 2^\frac{x - d_k n_k}{d_k} - 2^{-k-2}$$

We can bound the latter term from below (using what we already know about $d_k n_k$):

$$f(d_k n_k) \cdot 2^\frac{x - d_k n_k}{d_k} - 2^{-k-2} \geq 2^\frac{d_k n_k}{d_k} + 1 + 2^{-k-1} \cdot 2^\frac{x - d_k n_k}{d_k} - 2^{-k-2}$$
If \( n_{k+1} < x \leq d_{k+1}n_{k+1} \) then (using what we already know about \( n_{k+1} \)):

\[
\begin{align*}
  f(x) &\geq f(n_{k+1}) \\
  &\geq 2^{\frac{n_{k+1}}{2d_1 - d_k} + 1 + 2^{-k-2}} \\
  &\geq 2^{\frac{d_{k+1}n_{k+1} + 1 + 2^{-(k+1) - 1}}{2d_1 - d_{k+1}}} \\
  &\geq 2^{\frac{x}{2d_1 - d_{k+1}} + 1 + 2^{-(k+1) - 1} - (k + 1) - 1},
\end{align*}
\]

as desired. \( \square \)

We will use the following freely:

**Lemma 4.6** Given \( \varepsilon > 0 \) we can choose \( \{d_k, n_k\}_{k=1}^{\infty} \) such that \( f(d_k n_k) \leq f(n_k) \cdot 2^\varepsilon \).

**Proof** Using Lemma 4.5 we can make sure that:

\[
(2^\varepsilon - 1) f(n_k) \geq (2^\varepsilon - 1) 2^{\frac{n_k}{2d_1 - d_k}} \geq d_k^2 n_k^2 \geq f(d_k n_k) - f(n_k).
\]

where the middle inequality follows since as \( d_1, \ldots, d_k \) are fixed, the left hand side grows more rapidly than the right hand side. The last inequality follows from the definition of \( f \) on \([n_k, d_k n_k]\), and the claim follows. \( \square \)

We take \( n_k \) large enough such that \( f(d_k n_k) \leq f(n_k) \cdot 2^{\frac{1}{3}} \).

**Proposition 4.7** The function \( f : \mathbb{N} \to \mathbb{N} \) constructed above is submultiplicative.

**Proof** Let \( p, q \in \mathbb{N} \) be such that \( n_k + 1 \leq p + q \leq n_{k+1} \) and let us prove that \( f(p + q) \leq f(p) f(q) \). We work by induction, where the base case \( k \leq 1 \) follows from Proposition 4.4.

We begin with \( p + q \in [d_k n_k + 1, n_{k+1}] \) (without limiting \( n_{k+1} \), which can be thought of as infinity). Without loss of generality, \( p \leq q \). Denote \( \beta = f(d_{k-1} n_{k-1}) \).

Then:

\[
\begin{align*}
  f(p + q) &\leq 2^{\frac{p+q-d_k n_k}{2d_1 - d_k}} f(d_k n_k) \\
  &\leq 2^{\frac{p+q-d_k n_k}{2d_1 - d_k}} f(n_k) \\
  &\leq 2^{\frac{p+q-d_k n_k}{2d_1 - d_k}} \cdot \beta \cdot 2^{\frac{n_k - d_k - 1}{2d_1 - d_k}} \\
  &\geq \beta \cdot 2^{\frac{p+q-d_k n_k + 1}{2d_1 - d_k} + \frac{1}{3}}.
\end{align*}
\]

We divide into cases:

- **Suppose that** \( p \leq d_{k-1} n_{k-1} \). Notice that for any \( t \in [n_k, d_k n_k - 1] \) we have that:

\[
\frac{f(t + 1)}{f(t)} = 1 + \frac{t + 2}{f(t)} \leq 1 + \frac{d_k n_k + 1}{2^{n_k/d_k}} \quad (#)
\]
which we can take to be smaller than $2^{d_1^{-\frac{1}{d_2}}}$ by taking $n_k$ to be large enough. Thus, if in addition we take $d_k \geq \frac{n_{k-1} - d_k}{2d_1 \cdots d_{k-2}}$ then:

$$\frac{f(p + q)}{f(q)} \leq 2^{\frac{p}{d_1^{-\frac{1}{d_2}}}} \leq 2^{\frac{d_k - 1}{2d_1^{-\frac{1}{d_2}}}} \leq 2 \leq f(p).$$

(Note that the first inequality is evident if $q \geq d_k n_k$, and otherwise follows from the argument in the beginning of this case.)

- **If** $d_{k-1} n_{k-1} < p \leq n_k$ **then** $q \geq n_k$ (as $d_k > 2$), **and assume in addition that** $q \leq d_k n_k$. Note also that we can choose $d_k > d_{k-1} n_{k-1} + 1$ so:

$$d_{k-1} d_k n_{k-1} \leq (d_{k-1} n_{k-1} + 1)(d_k - 1) \leq p(d_k - 1)$$

and thus (recalling that $q \leq d_k n_k$):

$$pd_k + d_k n_k - 2d_{k-1} d_k n_{k-1} \geq pd_k - p(d_k - 1) - d_{k-1} d_k n_{k-1} + d_k n_k$$

$$\geq p + q - d_{k-1} d_k n_{k-1}. \quad \text{(⋆)}$$

Then, using Condition (I):

$$f(p) f(q) \geq f(p) f(n_k)$$

$$\geq \beta^2 \cdot 2^{\frac{p - d_k - 1}{2d_1^{-\frac{1}{d_2}}}} - 2^{-(k-1)-2} + \frac{n_k - d_k - 1}{2d_1^{-\frac{1}{d_2}} - 2} - 2^{-(k-1)-2}$$

$$= \beta^2 \cdot 2^{\frac{p + q - d_k - 1}{2d_1^{-\frac{1}{d_2}}}} + \frac{1}{2}$$

$$\geq \beta \cdot 2^{\frac{p + q - d_k - 1}{2d_1^{-\frac{1}{d_2}}}}$$

$$\geq f(p + q).$$

(The one before last inequality follows from (⋆) combined with the fact that $\beta \geq 2$.)

- **If** $d_{k-1} n_{k-1} < p \leq n_k$ **then** $q \geq n_k$, **and now assume that moreover** $q > d_k n_k$. By Lemma 4.5 we have:

$$\frac{f(p + q)}{f(q)} \leq 2^{\frac{p}{d_1^{-\frac{1}{d_2}}}} \leq f(p).$$

For the remaining cases, let $p > n_k$.

- **If** $p \geq d_k n_k$ then by Condition (I):

$$f(p) f(q) \geq f(d_k n_k)^2 \cdot 2^{\frac{p + q - 2d_k n_k}{2d_1^{-\frac{1}{d_2}}}} - 2^{k-2}$$

$$f(p + q) \leq f(d_k n_k) \cdot 2^{\frac{p + q - d_k n_k}{2d_1^{-\frac{1}{d_2}}}}.$$}

Therefore:

$$\frac{f(p) f(q)}{f(p + q)} \geq f(d_k n_k) \cdot 2^{\frac{d_k n_k}{2d_1^{-\frac{1}{d_2}}}} - 2^{k-1}$$
\[
\geq 2^{\frac{d_k n_k}{d_{k-1}-d_k}+1+2^{-k-1}} \cdot 2^{\frac{d_k n_k}{2d_{k-1}-d_k}-2^{-k-1}} > 1.
\]

(The middle inequality follows by Lemma 4.5.)

- If \( n_k \leq p < d_k n_k \) and \( q < d_k n_k \) then:

\[
f(p + q) \leq \beta \cdot 2^{\frac{p+q-d_k n_k - 1}{d_{k-1}-d_k} + 1} \leq \beta \cdot 2^{\frac{2d_k n_k-d_k d_{k-1} n_{k-1}}{d_{k-1}-d_k} + 1}
\]

and by Condition (I) specified for \( x = n_k \):

\[
f(p) f(q) \geq f(n_k)^2 \geq \beta^2 \cdot 2^{\frac{2n_k-2d_k n_k - 1}{d_{k-1}-d_k} - 2 - (k-1)-2} = \beta^2 \cdot 2^{\frac{2d_k n_k-d_k d_{k-1} n_{k-1}}{d_{k-1}-d_k} - 2-k},
\]

and by Lemma 4.5 applied for \( x = d_k n_{k-1} \):

\[
\beta \geq 2^{\frac{d_k n_{k-1} n_{k-1}}{d_{k-1}-d_k} + 1} = 2^{\frac{d_k n_{k-1} n_{k-1}}{d_{k-1}-d_k} + 1}
\]

so:

\[
f(p) f(q) \geq \beta \cdot 2^{\frac{2d_k n_k-d_k d_{k-1} n_{k-1}}{d_{k-1}-d_k} + 1-2-k} \geq f(p + q).
\]

- If \( n_k < p < d_k n_k < q \) then, applying Lemma 4.5 to \( x = p \), and Condition (I) to \( x = q \):

\[
f(p) f(q) \geq 2^{\frac{p}{2d_{k-1}-d_k}+1} f(q) \geq 2^{\frac{q}{2d_{k-1}-d_k}+1} \cdot f(d_k n_k) \cdot 2^{\frac{q-d_k n_k}{2d_{k-1}-d_k} - 2-k-2} \geq f(d_k n_k) \cdot 2^{\frac{p+q-d_k n_k}{2d_{k-1}-d_k}} \geq f(p + q).
\]

We thus proved that \( f(p + q) \leq f(p) f(q) \) whenever \( p + q \in [d_k n_k + 1, n_{k+1}] \).

It remains to show that \( f(p + q) \leq f(p) f(q) \) for \( p + q \in [n_k + 1, d_k n_k] \).

- If \( p > n_k \) then, applying Lemma 4.5 to \( x = n_k \):

\[
f(p) f(q) \geq f(n_k)^2 \geq f(n_k) \cdot 2^{\frac{n_k}{2d_{k-1}-d_k}+1} \geq f(n_k) + d_k^2 n_k^2 \geq f(d_k n_k) \geq f(p + q),
\]

where the inequality

\[
f(n_k) \cdot 2^{\frac{n_k}{2d_{k-1}-d_k}+1} \geq f(n_k) + d_k^2 n_k^2
\]
follows since if $d_k$ is fixed then the left hand side grows, as a function of $n_k$, more rapidly than the right hand side, so in particular we can take $n_k$ to be large enough such that this inequality holds.

- **If** $p \leq q < n_k$ **then**:

$$f(p + q) \leq f(n_k) + (p + q)^2$$
$$\leq f(n_k) + 4n_k^2$$
$$\leq f(n_k) \cdot 2^{\frac{1}{2}}$$
$$\leq \beta \cdot 2^{\frac{n_k - d_k - 1}{2d_1 - d_k - 1} + \frac{1}{2}}.$$  

(The penultimate inequality follows since $f$ grows exponentially in the interval $[d_k - 1 n_k - 1, n_k]$, so in particular we can take $n_k$ to be large enough such that $f(n_k) \gg n_k^2$.)

**If in addition** $d_k - 1 n_k - 1 < p$ **then**, using Condition (I):

$$f(p) f(q) \geq \beta^2 \cdot 2^{\frac{p + q - 2d_k - 1 n_k - 1}{2d_1 - d_k - 1} - 2^{-(k-1)-2}}$$
$$= \beta^2 \cdot 2^{\frac{p + q - d_k - 1 n_k - 1}{2d_1 - d_k - 1} \cdot 2^{\frac{-d_k - 1 n_k - 1}{2d_1 - d_k - 1} - 2^{-k}}}.$$  

But applying Lemma 4.5 to $x = d_k - 1 n_k - 1$ we get $\beta \geq 2^{\frac{d_k - 1 n_k - 1}{2d_1 - d_k - 1} + 1}$ so:

$$f(p) f(q) \geq \beta \cdot 2^{\frac{p + q - 2d_k - 1 n_k - 1}{2d_1 - d_k - 1} + 1 - 2^{-k}}$$
$$\geq \beta \cdot 2^{\frac{n_k - d_k - 1 n_k - 1}{2d_1 - d_k - 1} + 1 - 2^{-k}}$$
$$\geq \beta \cdot 2^{\frac{n_k - d_k - 1 n_k - 1}{2d_1 - d_k - 1} + \frac{1}{2}}$$
$$\geq f(p + q).$$

- **If** $p \leq d_k - 1 n_k - 1$ (no restrictions on $q$, except that $q \leq p + q \leq d_k n_k$), then, taking $n_k > 2d_k - 1 n_k - 1$, it follows that $q > d_k - 1 n_k - 1$. Now by (#):

$$\frac{f(p + q)}{f(n_k)} \leq 2^{\frac{p + q - n_k}{2d_1 - d_k}} \leq 2^{\frac{p + q - n_k}{2d_1 - d_k}}$$

and if $q \leq n_k$ then by the definition of $f$:

$$\frac{f(n_k)}{f(q)} \leq 2^{\frac{n_k - q}{2d_1 - d_k}}$$

so, whether or not $q \leq n_k$:

$$\frac{f(p + q)}{f(q)} \leq 2^{\frac{p}{2d_1 - d_k}} \leq f(p)$$
where the last inequality follows from Lemma 4.5 for \( x = p \) (keeping in mind that \( p \leq d_{k-1}n_{k-1} \)).

- **It remains to take care of the case** \( d_{k-1}n_{k-1} \leq p \leq n_k \leq q \). But notice that:

\[
f(p + q) \leq f(d_kn_k) \leq f(n_k) + d_k^2n_k^2 \leq 2f(n_k)
\]

since \( f \) grows exponentially in the interval \([d_{k-1}n_{k-1} + 1, n_k]\), so in particular we can take \( n_k \) to be large enough such that \( f(n_k) \gg d_k^2n_k^2 \) (note that \( d_k \) is already fixed when we choose \( n_k \)). Now:

\[
f(p)f(q) \geq 2f(q) \geq 2f(n_k) \geq f(p + q).
\]

We thus proved that \( f : \mathbb{N} \to \mathbb{N} \) is a submultiplicative function. \( \Box \)

### 4.3 Our construction is not equivalent to any growth function

Let \( f : \mathbb{N} \to \mathbb{N} \) be an increasing, submultiplicative function as constructed in Sect. 4.2, with respect to the sequences \( \{d_k, n_k\}_{k=1}^{\infty} \).

**Proposition 4.8** We can choose \( \{d_k, n_k\}_{k=1}^{\infty} \) such that the resulting function \( f \) is not equivalent to the growth function of any finitely generated algebra.

**Proof** Since the conditions on \( d_k \) and \( n_k \) in Sect. 4.2 are of the form:

\[
d_k \geq \mu_k (d_1, n_1, \ldots, d_{k-1}, n_{k-1}), \quad n_k \geq v_k (d_1, n_1, \ldots, d_{k-1}, n_{k-1}, d_k)
\]

for suitable functions \( \mu_k : \mathbb{N}^{2^k-2} \to \mathbb{N} \), \( v_k : \mathbb{N}^{2^k-1} \to \mathbb{N} \), we may further assume that \( n_k = km_k \) for \( m_k > 1 \) to be determined in the sequel.

Suppose that \( C \in \mathbb{N} \) is given. Consider \( k = C, n = m_k + 1 \) and \( D = [d_k(1 - \frac{1}{m_k+1})] \), and observe that \( \frac{1}{2}d_k \leq D \leq d_k \). If we take \( d_k \geq 2k^2 \) then it follows that \( D \geq \frac{1}{2}d_k \geq k^2 = C^2 \). We aim to contradict the property stated in Proposition 4.2 with respect to these parameters, namely, we aim to show that:

\[
f(2CDn) - f(2CDn - C) > 2D^2n (f(CDn) - f(Cn - C))^{2D}.
\]

Notice that \( 2CDn \leq 2k \left( d_k \frac{m_k}{m_k+1} \right) (m_k + 1) = 2d_kn_k \) and we have:

\[
2CDn - C \geq 2CDn - CD \geq 2kDm_k \geq kd_km_k = d_kn_k
\]

so, if we assume that \( n_{k+1} > 2d_kn_k \) then we can make sure that:

\[
f(2CDn) - f(2CDn - C) \geq f(2CDn - C + 1) - f(2CDn - C)
\]

\[
= \left[ 2^{\frac{1}{2d_k}} f(2CDn - C) \right] - f(2CDn - C)
\]

\[
\geq \left( 2^{\frac{1}{2d_k}} - 1 \right) f(2CDn - C) - 1
\]

\[\Box\]
\[
\geq \frac{1}{2} \left( 2^{\frac{1}{2d_k - 1}} - 1 \right) f(2CDn - C)
\]

where the last inequality can be assumed since \( f(2CDn - C) \geq f(d_k n_k) \) which can be arbitrarily large compared to \( 2^{\frac{1}{2d_k - 1}} - 1 \), which depends only on \( d_k \) and not on \( n_k \).

Denote \( \Delta_k = \frac{1}{2} \left( 2^{\frac{1}{2d_k - 1}} - 1 \right) \) and observe that \( \Delta_k > 0 \) and does not depend on \( n_k \).

We have:

\[
f(2CDn) - f(2CDn - C) \geq f(2CDn - C) \cdot \Delta_k.
\]

Now, using Lemma 4.5 for \( x = d_k n_k \):

\[
f(2CDn - C) \geq f(d_k n_k) \geq 2^{\frac{d_k n_k}{2d_k - 1}} = d_k^{n_k} = d_k^{km_k}
\]

where \( q_k = 2^{\frac{1}{2d_k - 1}} > 1 \). So, \( f(2CDn) - f(2CDn - C) \geq q_k^{km_k} \cdot \Delta_k \). On the other hand,

\[
n_k = km_k = Cn - C < CDn
\]

\[
\leq k \left( d_k \frac{m_k}{m_k + 1} \right) (m_k + 1) = d_k n_k
\]

so (by the definition of \( f \) on intervals of this type):

\[
f(CDn) - f(Cn - C) \leq (CDn)^2 \leq (kd_k (m_k + 1))^2
\]

hence:

\[
2D^2 n(f(CDn) - f(Cn - C))^{2D} \leq 2d_k^2 (m_k + 1)(kd_k (m_k + 1))^{4d_k}
\]

\[
\leq (m_k + 1)^{4d_k + 1} \cdot \Gamma_k
\]

where \( \Gamma_k = 2d_k^2 (kd_k)^{4d_k} \) depends only on \( k, d_k \) (but not on \( m_k \), or equivalently, on \( n_k \)). Finally, note that as we fix \( k, d_1, \ldots, d_k \) and let \( m_k \to \infty \) we get:

\[
f(2CDn) - f(2CDn - C) \geq (q_k^k m_k \cdot \Delta_k)
\]

\[
\gg (m_k + 1)^{4d_k + 1} \cdot \Gamma_k
\]

\[
\geq 2D^2 n(f(CDn) - f(Cn - C))^{2D}
\]

contradicting the property ensured by Proposition 4.2 for functions which are equivalent to growth functions of algebras. Thus, \( f \) is not equivalent to the growth of any finitely generated algebra. \( \square \)
4.4 Arbitrarily rapid holes: Proof of Theorem 1.3

Proposition 4.9 Let \( g : \mathbb{N} \to \mathbb{N} \) be an arbitrary subexponential function. Then we can choose \( \{d_k, n_k\}_{k=1}^\infty \) such that the resulting function in our construction \( f \) satisfies \( f \geq g \).

Proof Since \( g \) is assumed to be subexponential, there exists \( \omega : \mathbb{N} \to \mathbb{R} \) such that \( \omega(n) \xrightarrow{n \to \infty} 0 \) and \( g(n) \leq 2^{n\omega(n)} \).

We can take \( n_k > \max\{m \mid \omega(m) \geq \frac{1}{2d_1 \cdots d_k}\} \) for all \( k \geq 1 \). We claim that for all \( x \geq n_1 \) we have that \( f(x) \geq 2^{x\omega(x)} \). There are two possibilities: either \( x \in [n_j, d_j n_j] \) or \( x \in [d_j n_j, n_{j+1}] \) for some \( j \geq 1 \). Let us consider the first case. Then \( x \geq n_j \) so by the way we picked \( n_j \) we have that \( \omega(x) < \frac{1}{2d_1 \cdots d_j} \). By Lemma 4.5 we have that:

\[
f(x) \geq 2^{\frac{x}{2d_1 \cdots d_j}} \geq 2^{x\omega(x)}.
\]

As for the second case, if \( x \in [d_j n_j, n_{j+1}] \) then by Condition (I),

\[
f(x) \geq 2^{\frac{x-d_j n_j}{2d_1 \cdots d_j}} - 2^{-j-2} f(d_j n_j)
\]

Now by Lemma 4.5 for \( d_j n_j \),

\[
f(d_j n_j) \geq 2^{\frac{d_j n_j}{2d_1 \cdots d_j} + 1 + 2^{-j-1}}
\]

so:

\[
f(x) \geq 2^{\frac{x-d_j n_j}{2d_1 \cdots d_j} - 2^{-j-2} \cdot 2^{\frac{d_j n_j}{2d_1 \cdots d_j} + 1 + 2^{-j-1}}}
\]

\[
\geq 2^{\frac{x}{2d_1 \cdots d_j}}
\]

\[
\geq 2^{x\omega(x)}.
\]

Thus \( f(x) \geq 2^{x\omega(x)} \geq g(x) \) for all \( x \gg 1 \).

Finally, we have:

Proof (Proof of Theorem 1.3) The theorem follows since we can take \( \{d_k, n_k\}_{k=1}^\infty \) satisfying all conditions required in Propositions 4.7, 4.8 and 4.9.

Remark 4.10 If \( f : \mathbb{N} \to \mathbb{N} \) is an increasing and submultiplicative function and \( \limsup_{n \to \infty} \sqrt[n]{f(n)} > 1 \) then \( f \sim \exp(n) \), thus equivalent to the growth of (any) noncommutative free algebra.

Indeed, if \( f(t_i) > a^i \) for some \( t_1 < t_2 < \cdots \) pick \( s_i \) such that \( t_i < 2^{s_i} \leq 2t_i \) and then \( f(2^{s_i}) > \sqrt[a]{2^{s_i}} \); set \( b = \sqrt[a]{a} \). But if \( f(2^s) > b^{2^s} \) for some \( s \) then we have \( f(2^{s-1}) \geq \sqrt[a]{f(2^s)} > b^{2^{s-1}} \), so since \( s_1 < s_2 < \cdots \) are unbounded then for all \( s \) we
have $f(2^s) \geq b^{2^s}$. And then for any $t$, take $s$ such that $2^s \leq t < 2^{s+1}$ and we have $f(t) \geq f(2^s) \geq b^{2^s} \geq \sqrt[4]{b^t}$.

Thus, even though we did not explicitly prove it, it follows that $f$ from Theorem 1.3 is always subexponential.

5 Growth rates encoding nilpotent ideals and non-prolongable words

5.1 Asymptotically invariant conditions on higher discrete derivatives

We begin with a somewhat technical definition, which would be useful in the sequel:

**Definition 5.1** We say that an increasing function $f : \mathbb{N} \to \mathbb{N}$ satisfies Condition (II) if there exists $C \in \mathbb{N}$ such that for all $N, m \geq 1$:

$$f'((C^2 + 1)m) \leq \left( f(2C^2N) - f(N) \right) \left( f \left( 2C^2N + (C^4 + C^2)m \right) - f(N + m) \right).$$

We are now ready to translate certain algebraic properties of algebras to validity of Condition (II) for functions lying in the equivalence classes of their growth functions.

**Lemma 5.2** Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a graded algebra generated by $F + A_1 + \cdots + A_p$. Then for any $r \in \mathbb{N}$ we have that $\bigoplus_{i=r}^{\infty} A_i$ is generated as a left ideal by $\bigoplus_{i=0}^{p-1} A_{r+i}$.

**Proof** Let $L \leq A$ be the left ideal generated by $A_r + \cdots + A_{r+(p-1)}$. We prove by induction that every homogeneous element of degree $\geq r$ belongs to $L$. If $a \in A$ is homogeneous of degree $\text{deg}(a) \in [r, r+p-1]$ the claim is evident. If $\text{deg}(a) \geq r+p$, write $a = \sum f_{i_1} \cdots f_{i_k}$ where all $f_{i_j}$ are homogeneous of degrees $1 \leq \text{deg}(f_{i_j}) \leq p$. Then each summand has the form $f_{i_1}w$ where $r \leq \text{deg}(w) < \text{deg}(a)$, and so by the induction hypothesis $w \in L$ and hence each of these summands belongs to $L$, and consequently $a$ itself belongs to $L$ as well. \qed

**Proposition 5.3** Let $f : \mathbb{N} \to \mathbb{N}$ be an increasing function which is equivalent to the growth of a finitely generated graded semiprime algebra. Then $f$ satisfies Condition (II).

**Proof** Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a finitely generated semiprime algebra. Then it can be assumed that $A$ is generated by $V = A_0 + A_1 + \cdots + A_p$ for some $p \geq 1$. Since $V^u \subseteq \bigoplus_{i=0}^{pu} A_i$ and $\bigoplus_{i=0}^{u} A_i \subseteq V^u$, it follows that the growth of $A$ with respect to $V$ is equivalent to the function $\gamma(n) := \dim_F \bigoplus_{i=0}^{pn} A_i$. Notice that $\gamma(v) - \gamma(u) = \dim_F \bigoplus_{i=pu+1}^{pv} A_i$ for all $u < v$. Assume that $f \sim \gamma$, namely, there exists some $C \geq 1$ such that $f(n) \leq C\gamma(Cn)$ and $\gamma(n) \leq f(Cn)$ for all $n \in \mathbb{N}$.

Fix $m, t, N, s \in \mathbb{N}$. Fix a basis $\mathcal{B}$ of homogeneous elements for $\bigoplus_{i=pN+1}^{p(N+s)} A_i$. Consider the linear map:

$$T : \bigoplus_{i=pm+1}^{p(m+t)} A_i \to \left( \bigoplus_{i=pN+1}^{p(N+s)} A_i \right) \otimes \left( \bigoplus_{i=p(m+N)+2}^{p(m+t+N+s)} A_i \right).$$
given by:

$$T : f \mapsto \sum_{u \in \mathcal{G}} u \otimes u f.$$  

Then every element $f \in \text{Ker}(T)$ satisfies $A_i f = 0$ for all $pN + 1 \leq i \leq pN + ps$ and therefore by Lemma 5.2, also $A_i f = 0$ for all $i \gg 1$. Hence $(f) \prec A$ is nilpotent, so by semiprimeness necessarily $f = 0$. Therefore $T$ is injective, and thus:

$$\gamma(m + t) - \gamma(m) = \text{dim}_F \bigoplus_{i=pN+1}^{p(m+t)} A_i \leq \left( \text{dim}_F \bigoplus_{i=pN+1}^{p(N+s)} A_i \right) \cdot \left( \text{dim}_F \bigoplus_{i=p(m+N)+2}^{p(m+t+N+s)} A_i \right) \leq \left( \text{dim}_F \bigoplus_{i=pN+1}^{p(N+s)} A_i \right) \cdot \left( \text{dim}_F \bigoplus_{i=p(m+N)+1}^{p(m+t+N+s)} A_i \right) = (\gamma(N + s) - \gamma(N)) (\gamma(N + m + t + s) - \gamma(N + m)).$$

Scaling this by $C$ we get:

$$\gamma(Cm + Ct) - \gamma(Cm) \leq (\gamma(CN + Cs) - \gamma(CN)) \cdot (\gamma(CN + Cm + Ct + Cs) - \gamma(CN + Cm)).$$

Now taking $t = C^2 m$, $s = N$ and utilizing the connection between $f$ and $\gamma$, we deduce:

$$f'((C^2 + 1)m) \leq f((C^2 + 1)m) - f(C^2 m) = f(m + t) - f(C^2 m) \leq \gamma(Cm + Ct) - \gamma(Cm) \leq (\gamma(CN + Cs) - \gamma(CN)) (\gamma(CN + Cm + Ct + Cs) - \gamma(CN + Cm)) \leq (f(C^2 N + C^2 s) - f(N)) (f(C^2 N + C^2 m + C^2 t + C^2 s) - f(N + m)) = (f(2C^2 N) - f(N)) (f(2C^2 N + (C^4 + C^2)m) - f(N + m)).$$

Thus $f$ satisfies Condition (II). \hfill \Box

We also have:

**Proposition 5.4** Let $f : \mathbb{N} \to \mathbb{N}$ be an increasing function which is equivalent to an increasing function $g : \mathbb{N} \to \mathbb{N}$ such that $g''(n) \geq 0$. Then $f$ satisfies Condition (II).

**Proof** Let $f$, $g$ be as in the statement of the proposition (in particular, $g'(n) \leq g'(n + t)$ for all $t \geq 0$). Then there exists $C \in \mathbb{N}$ such that $f(n) \leq g(Cn)$ and $g(n) \leq f(Cn)$...
for all \( n \in \mathbb{N} \). We also know that for every \( n' \geq n \) and \( t > 0 \):

\[
g(n + t) - g(n) = \sum_{k=n+1}^{n+t} g'(k) \leq \sum_{k=n+1}^{n+t} g'(k + n' - n) = \sum_{k=n'+1}^{n'+t} g'(k) = g(n' + t) - g(n').
\]

Now taking \( n = Cm, \ t = C^3m, \ n' = C(N + m) \) we obtain:

\[
f'((C^2 + 1)m) \leq f((C^2 + 1)m) - f(C^2m) \\
\leq g((C^3 + C)m) - g(Cm) \\
= g(n + t) - g(n) \\
\leq g(n' + t) - g(n') \\
= g(CN + Cm + C^3m) - g(CN + Cm) \\
\leq f(C^2N + C^2m + C^4m) - f(N + m) \\
\leq f \left( 2C^2N + (C^4 + C^2)m \right) - f(N + m) \\
\leq \left( f(2C^2N) - f(N) \right) \left( f \left( 2C^2N + (C^4 + C^2)m \right) - f(N + m) \right).
\]

Thus \( f \) satisfies Condition (II). \( \square \)

5.2 Constructing an algebra

We construct a function \( f : \mathbb{N} \to \mathbb{N} \) inductively, then show it satisfies the conditions of [4, Theorem 1.1], and deduce that it is equivalent to the growth function of a finitely generated monomial algebra.

5.2.1 The growth function

Suppose that a sequence of positive integers \( n_1 < n_2 < \cdots \) is given, satisfying

\( n_{i+1} > n_i + i \) for all \( i \in \mathbb{N} \) (along with further constraints to be made next). Let \( f(k) = 1 + 2^1 + \cdots + 2^k \) for all \( k \leq 2^n_1 \). For \( k > 2^n_1 \) proceed as follows:

- If \( 2^n_i < k \leq 2^{n_i+i} \) let \( f(k) = f(k-1) + k + 1 \);
- If \( 2^{n_i+i} < k \leq 2^{n_i+1} \) let \( f(k) = f(k-1) + \min\{f'(d)^2|k/2 \leq d < k\} \).

By [4, Theorem 1.1], in order to prove that \( f \) is equivalent to the growth function of some finitely generated monomial algebra, it suffices to show:

1. \( f'(k) \geq k + 1 \) for all \( k \geq 1 \);
2. \( f'(k) \leq f'(n)^2 \) for all \( n \leq k \leq 2n \).

It is evident that \( f \) is increasing. We first show by induction that \( f'(k) \geq k + 1 \) for all \( k > 1 \). For \( k = 1, 2, 3, \ldots, 2^n_1 \) this is clear (since \( f(k) = 1 + 2^1 + \cdots + 2^k \) in
this segment). Suppose that \( k > 2^n_i \) is given. If \( 2^n_i < k \leq 2^{n_i+i} \) for some \( i \) then this evidently follows from the definition of \( f \), and otherwise, \( f'(k) = f'(d)^2 \) for some \( \frac{k}{2} \leq d < k \) so by the induction hypothesis:

\[
    f'(k) = f'(d)^2 \geq (d + 1)^2 \geq \left( \frac{k}{2} + 1 \right)^2 \geq k + 1.
\]

We now show that \( f'(k) \leq f'(n)^2 \) for all \( k \in \{n, n+1, \ldots, 2n\} \). For \( k = 1, 2, 3, \ldots, 2^n \) this is clear since \( f(k) = 1 + 2^1 + \cdots + 2^k \) in this segment. If \( 2^{n_i+i} < k \leq 2^{n_i+1} \) for some \( i \) then this is evident from the definition of \( f \). Otherwise, suppose that \( 2^n_i < k \leq 2^{n_i+i} \). If \( k/2 \leq n < k \) then:

\[
    f'(n)^2 \geq (n + 1)^2 \geq \left( \frac{k}{2} + 1 \right)^2 \geq k + 1 = f'(k).
\]

It follows from [4, Theorem 1.1] that there exists a finitely generated graded algebra such that \( \gamma_A(n) \sim f(n) \).

### 5.2.2 Super-polynomial segments

We now make some assumptions on \( \{n_k\}_{k=1}^{\infty} \): we assume that for each \( i \in \mathbb{N} \) we have \( n_{i+1} > 2(n_i + i + 1) \) and \( n_1 > 2 \) (so \( n_i > 2^i \)), and derive a technical property of \( f \) which will be utilized in the proofs of Theorems 1.4 and 1.7.

**Proposition 5.5** For all \( 2^{n_i+1-1} < k \leq 2^{n_i+1} \) we have:

\[
    f'(k) \geq 2^{2^{n_i+1}}.
\]

**Proof** Observe that if \( 2^n_i < k \leq 2^{n_i+1} \) then \( f'(k) \geq k + 1 > 2^n_i \). If in addition \( k > 2^{n_i+i} \) then \( f'(k) = f'(d)^2 \) for some \( \frac{k}{2} \leq d < k \). In particular, \( f'(d) \geq d + 1 > 2^n_i \) so \( f'(k) \geq 2^{2n_i} \). By induction, it follows that for every \( t > 0 \) such that:

\[
    2^{n_i+i+t} < k \leq 2^{n_i+i+t+1} \leq 2^{n_i+1}
\]

we have that:

\[
    f'(k) \geq 2^{2^t n_i}
\]

Taking \( t = n_{i+1} - n_i - i - 1 > \frac{1}{2} n_{i+1} \) we get that for \( 2^{n_i+1-1} < k \leq 2^{n_i+1} \):

\[
    f'(k) \geq 2^{2^t n_i} \geq 2^{2^{1/2} n_{i+1}},
\]

as claimed. \( \square \)
5.3 Non-primeness and non-prolongability

We are now ready to prove Theorems 1.4 and 1.7.

Proof (Proof of Theorems 1.4 and 1.7) Let $f$ be constructed as above, and $A$ a monomial algebra from [4] whose growth function is equivalent to $f$.

- If $f \sim \gamma : \mathbb{N} \to \mathbb{N}$ being the growth function of a graded, finitely generated semiprime algebra then by Proposition 5.3 it is guaranteed that $f$ satisfies Condition (II).

- If $f$ is equivalent to the growth of a finitely generated graded algebra generated in degree 1 with a homogeneous regular element then $f \sim f_0$ for $f_0$ increasing with a non-decreasing derivative (as in Sect. 2.3.1, by transferring to a suitable Veronese subring). Thus by Proposition 5.4, $f$ satisfies Condition (II).

- If $f$ is equivalent to the growth of a prolongable hereditary language then it is equivalent to the growth of a graded algebra with a regular homogeneous element, see Sect. 2.4, so again $f$ satisfies Condition (II).

We thus assume on the contrary that $f$ satisfies Condition (II), aiming for a contradiction. In particular, for $m = \left\lfloor \frac{2^ni+1}{C^2+1} \right\rfloor$ and $N = 2^{ni+1} + 1$ (where $i \gg 1$ in a sense that will be made clear next) we get:

$$f'(\left( C^2 + 1 \right) \left\lfloor \frac{2^ni+1}{C^2+1} \right\rfloor) \leq \left( f(2C^2 2^{ni+1}) - f(2^{ni+1}) \right) \cdot \left( f\left( 2C^2 2^{ni+1} + (C^4 + C^2) \left\lfloor \frac{2^ni+1}{C^2+1} \right\rfloor \right) - f\left( 2^{ni+1} + \left\lfloor \frac{2^ni+1}{C^2+1} \right\rfloor \right) \right).$$

We now make two observations:

- Notice that, assuming $i \gg C^1$, the left hand side is $f'(k)$ for some $2^{ni+1-1} < k \leq 2^{ni+1}$, hence by Proposition 5.5 it is bounded from below by:

$$2^{\frac{1}{2} ni+1}$$

- Notice that the right hand side is bounded from above by:

$$\left( f(2C^2 2^{ni+1}) - f(2^{ni+1}) \right) \left( f(3C^2 2^{ni+1}) - f(2^{ni+1}) \right) \leq \left( f(3C^2 2^{ni+1}) - f(2^{ni+1}) \right)^2 = \left( \sum_{u=2^{ni+1} + 1}^{3C^2 2^{ni+1}} f'(u) \right)^2$$

For $i \gg C^1$ we get that $3C^2 2^{ni+1} < 2^{ni+1+i+1}$ and hence by construction, the right hand side is bounded from above by:

$$\left( \sum_{u=2^{ni+1} + 1}^{3C^2 2^{ni+1}} u + 1 \right)^2$$
which is bounded from above by:

\[(3C^2 2^{n_i+1} + 1)^4 \leq C' 2^{4n_i+1}\]

for suitable \(C' \in \mathbb{N}\).

Thus, we obtain:

\[2^{\frac{1}{2} n_i+1} \leq C' 2^{4n_i+1}\]

which is false for \(i \gg 1\), contradicting the assumption that \(f\) satisfies Condition (II). Theorems 1.4 and 1.7 now follow.

Let \(A\) be a monomial algebra. Let us say that a graded algebra \(\tilde{A}\) is a deformation of \(A\) if its Gröbner basis with respect to some monomial ordering coincides with that of \(A\). Since \(\gamma_A(n) = \gamma_{\tilde{A}}(n)\), we immediately obtain from Theorem 1.7 the following:

**Corollary 5.6** There exist finitely generated, infinite-dimensional monomial algebras all of whose graded deformations contain non-zero nilpotent ideals.

**Acknowledgements** The author thanks Jason Bell and Efim Zelmanov for related inspiring discussions and Guy Blachar for helpful suggestions regarding the paper. The author is thankful to the referee for their careful reading of the paper and useful comments.

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