Local solutions to a free boundary problem for the Willmore functional

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Abstract

We consider a free boundary problem for the Willmore functional $W(f) = \frac{1}{4} \int_{\Sigma} H^2 \, d\mu_f$. Given a smooth bounded domain $\Omega \subset \mathbb{R}^3$, we construct Willmore disks which are critical in the class of surfaces meeting $\partial \Omega$ at a right angle along their boundary and having small prescribed area. Using rescaling and the implicit function theorem, we first obtain constrained solutions with prescribed barycenter on $\partial \Omega$. We then study the variation of that barycenter.

Introduction

The Willmore energy of an immersed surface $f : \Sigma \to \mathbb{R}^3$ is given by

$$W(f) = \frac{1}{4} \int_{\Sigma} H^2 \, d\mu_f,$$

for instance $W(S^2) = 4\pi$. Introducing the tracefree second fundamental form by decomposing $h = h^o + \frac{1}{2}Hg$, we can write the (scalar) Euler-Lagrange operator as

$$W[f] = \Delta_g H + |h^o|^2 H.$$

We study a variational problem for the Willmore energy involving a free boundary condition. Let $D = \{ z \in \mathbb{R}^2 : |z| < 1 \}$ and $\Omega \subset \mathbb{R}^3$ be a given smooth, bounded domain. Putting $S = \partial \Omega$ we introduce the class $\mathcal{M}(S)$ of smooth immersions $f : \overline{D} \to \mathbb{R}^3$ meeting $S$ orthogonally from inside along $\partial D$, that is

$$\mathcal{M}(S) = \{ f \in C^\infty(\overline{D}, \mathbb{R}^3) \text{ immersed} : f(\partial D) \subset S, \frac{\partial f}{\partial \eta} = N^S \circ f \text{ on } \partial D \}.$$
Here $\eta, N^S$ are the interior unit normals of $(D, g)$ and $\Omega \subset \mathbb{R}^3$ along the respective boundaries. In the (unbounded) special case $\Omega = \mathbb{R}^3$, the round half-spheres

$$S_+^2(a, \lambda) = a + \lambda S_+^2 \quad (a \in \mathbb{R}^2, \lambda > 0)$$

minimize the Willmore energy in the class $\mathcal{M}(\mathbb{R}^2)$. This follows from Simon's monotonicity formula, see [12], after reflecting across $\mathbb{R}^2$. In particular, the sphere $S_+^2(a, \lambda)$ minimizes in the smaller class of surfaces $f \in \mathcal{M}(S)$ having the same area $A(f) = 2\pi \lambda^2$. For this variational problem we construct critical points in a general domain $\Omega$, provided that the prescribed area is sufficiently small.

**Theorem** Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain, and $S = \partial \Omega$. For each sufficiently small $\lambda > 0$ there exist at least two disk-type surfaces $f : D \to \mathbb{R}^3$ which are critical points for the Willmore functional restricted to the class

$$\mathcal{M}_\lambda(S) = \{ f \in \mathcal{M}(S) : A(f) = 2\pi \lambda^2 \}.$$  

Each critical point in $\mathcal{M}_\lambda(S)$ satisfies, for an appropriate $\alpha \in \mathbb{R}$,

$$\begin{align*}
\Delta_g H + |h^g|^2 H &= \alpha H \quad \text{in } D, \\
\frac{\partial H}{\partial \eta} + h^S(\nu, \nu)H &= 0 \quad \text{on } \partial D.
\end{align*}$$

The proof is based on the implicit function theorem and yields surfaces which are small, almost-round half-spheres, see Corollary 1. We show in addition that as $\lambda \searrow 0$ the constructed surfaces concentrate at critical points $a \in S$ of the function $H^S : S \to \mathbb{R}$ (Corollary 2). Reversely, if $a \in S$ is a nondegenerate critical point of $H^S$, then there is a local family $f_\lambda$ of critical points in $\mathcal{M}_\lambda(S)$ which depends smoothly on $\lambda$ and concentrates at $a$ as $\lambda \searrow 0$; see Theorem 3 for details.

In [18] Nitsche discusses possible boundary conditions for Willmore surfaces on grounds of the boundary terms in the first variation formula. Palmer proves symmetry and uniqueness for Willmore surfaces with boundary moving freely on a plane or round sphere [20], see also Dall'Acqua [5] for related work. It appears that the present variational problem involving the class $\mathcal{M}(S)$ was however not considered in the literature. Our main motivation is the conformal invariance of the class $\mathcal{M}(S)$, which should lead to interesting compactness and regularity issues. We have verified a reflection principle for Willmore surfaces with our boundary condition in the case $\Omega = \mathbb{R}^3_+$. By the work of Bryant [4], all disk-type solutions are then obtained from minimal surfaces with reflectional symmetry, having the type of $S^2$ with finitely many flat ends. Of course one may also consider the variational problem with other prescribed angles. For the one-dimensional Bernoulli elastic energy and for the Willmore energy under rotational symmetry, solutions with Dirichlet or Navier type boundary conditions are constructed by Deckelnick, Grunau et al., see for instance [7, 6]. Existence and regularity results for Willmore minimizers with prescribed curve and tangent plane along the boundary were
proved by Schätzle [22]. A recent paper by Alexakis and Mazzeo considers properly immersed surfaces in hyperbolic 3-space which are (locally) critical points of the $L^2$ energy of the second fundamental form. They show that finite energy surfaces meet the sphere at infinity at a right angle [3, Lemma 2.1].

To prove the existence result we study the problem on $\mathbb{R}^3_+$ with respect to perturbations $\tilde{g}$ of the Euclidean background metric. On the space of variations of $S^2_+$ respecting the boundary condition, the linearized operator has a three-dimensional kernel due to dilations and translations. We arrive at a solvable problem by prescribing the area $A(f, \tilde{g}) = 2\pi$ and a two-dimensional barycenter $C(f, \tilde{g}) = 0 \in \mathbb{R}^2$.

Pulling back the Euclidean metric with a chart near $a \in \partial \Omega$ and rescaling yields a perturbed metric $\tilde{g}^{a, \lambda}$ on $\mathbb{R}^3_+$. Solving the constrained problem for $\tilde{g}^{a, \lambda}$ and transforming back, we get a three-dimensional family $\phi^{a, \lambda}$ of critical points subject to constraints $A(\phi^{a, \lambda}) = 2\pi \lambda^2$ and $C(\phi^{a, \lambda}, S) = a$. In Proposition 1 we prove the expansion

$$|W(\phi^{a, \lambda}) - 2\pi + \pi H^S(a)\lambda| \leq C\lambda^2 \quad \text{where} \quad C = C(\Omega).$$

In particular $\inf_{f \in \mathcal{M}(S)} W(f) < 2\pi$. This indicates that minimizers of $W(f)$ without area constraint are not in the realm of a local approach. In Theorem 2 we show instead the following: for $\lambda \in (0, \lambda_0]$ a constrained solution $\phi^{a, \lambda}$ is critical under prescribed area $A(\phi^{a, \lambda}) = 2\pi \lambda^2$ if and only if the point $a \in S$ is a critical point of the reduced energy function

$$\tilde{W}(\cdot, \lambda) : S \to \mathbb{R}, \quad \tilde{W}(a, \lambda) = W(\phi^{a, \lambda}).$$

In consequence we get at least two critical points in $\mathcal{M}_\lambda(S)$ for $\lambda \in (0, \lambda_0(\Omega)]$, as stated in the theorem.

In [13, 14] Lamm, Metzger and Schulze study a related perturbation problem for small spheres in Riemannian manifolds. Their solutions are also critical with respect to prescribed area and are called of Willmore type. Another perturbation result, also in a Riemannian manifold, is by Mondino [16].

There is a corresponding analysis for constant mean curvature surfaces. The pioneering work is by Ye [24]. Our approach is close to the work of Pacard and Xu [19] and also Fall [8, 9]. The following difference should however be noted: in the CMC case the orthogonality along the boundary appears as natural boundary condition, whereas here it is imposed as a constraint. Our natural boundary condition is equation (0.3).

We now outline the contents of this paper. In Section 1 we compute the space of admissible variations, that is the tangent space of $\mathcal{M}(S)$, and derive the resulting boundary conditions. One can show that the space $\mathcal{M}(S)$ is a manifold; for the purposes of this paper a graph representation of $\mathcal{M}(S)$ near $S^2_+$ is sufficient (Lemma 3). In Section 2 we solve the constrained perturbation problem with respect to an arbitrary
background Riemannian metric close to the standard metric. Technically we use a two-step procedure where the orthogonality constraint is satisfied first, leading to a certain submanifold on which the other equations are then solved in the second step, see Lemma 6.

This is applied in Section 3 to the local situation around \( a \in S \), pulling back and rescaling as indicated above. Graph coordinates turn out to be sufficient for this purpose. We then prove the main results: the expansion of the energy (Proposition 1), the characterization of critical points using the reduced energy function (Theorem 2) and finally the existence results (Corollary 1 and Theorem 3). In the appendix we review the construction of the two-dimensional barycenter.

1 Constraints and conditions on the boundary

We start by collecting without proof some variational formulae. Let \( f : \Sigma \to (M^3, \tilde{g}) \) be a compact, smoothly immersed surface with boundary \( \partial \Sigma \). We denote by \( \tilde{D} \) the Levi-Civita connection on \( M \) and by \( g = f^*\tilde{g} \) the induced metric on \( \Sigma \). We assume that we have a unit normal \( \nu : \Sigma \to TM \) along \( f \).

**Lemma 1** Let \( f : \Sigma \times I \to (M^3, \tilde{g}) \) be a smooth variation, \( 0 \in I \), with \( \partial_t f = \phi \nu \) at \( t = 0 \). Then at \( t = 0 \) we have the following equations:

\[
\begin{align*}
\tilde{D}_t \partial_k f &= (\partial_k \phi) \nu - \phi g^{ij} h_{jk} \partial_i f \\
\partial_t g_{ij} &= -2h_{ij} \phi \\
\partial_t (d\mu_g) &= -H \phi d\mu_g \\
\tilde{D}_t \nu &= -D_f \cdot \text{grad} \phi \\
\partial_t h_{kl} &= \nabla^2_{kl} \phi - g^{ij} h_{ik} h_{jl} \phi + \tilde{R}(\nu, \partial_k f, \partial_l f, \nu) \phi, \\
\partial_t H &= \Delta_g \phi + (|h|^2 + \tilde{Ric}(\nu, \nu)) \phi, \\
\partial_t \Gamma^k_{ij} &= -g^{kl}(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij}) \phi - g^{kl}((\partial_i \phi) h_{jl} + (\partial_j \phi) h_{il} - (\partial_l \phi) h_{ij}).
\end{align*}
\]

In a space of constant curvature \( \kappa \), the curvature terms simplify to

\[
\tilde{R}(\nu, \partial_k f, \partial_l f, \nu) = \kappa g_{kl} \quad \text{and} \quad \tilde{Ric}(\nu, \nu) = 2\kappa.
\]

Next we derive the wellknown first variation formula for the Willmore energy. A version including boundary terms was stated e.g. in [18].
Theorem 1 For \( f : \Sigma \to (M^3, \tilde{g}) \), the first variation of the Willmore energy in direction of the vector field \( \phi = \varphi \nu + Df \cdot \xi \) is

\[
\frac{d}{dt} W(f)|_{t=0} = \frac{1}{2} \int_{\Sigma} W(f) \varphi \, d\mu_g + \frac{1}{2} \int_{\partial \Sigma} \omega(\eta) \, ds_g =: \delta W(f) \phi,
\]

where \( \eta \) is the interior unit normal with respect to \( g \), and

\[
W(f) = \Delta H + (|h| \nu^2 + \tilde{\text{Ric}}(\nu, \nu)) H, \\
\omega(\eta) = \varphi \frac{\partial H}{\partial \eta} - \frac{\partial H}{\partial \eta} \varphi - \frac{1}{2} H^2 g(\xi, \eta).
\]

Proof. We compute for normal and tangential \( \phi \) separately, starting with the first. In normal coordinates for \( t = 0 \) we get from Lemma 1

\[
\frac{d}{dt} W(f) = \frac{1}{2} \int_{\Sigma} \Delta H \, d\mu_g + \frac{1}{4} \int_{\Sigma} H^2 \frac{\partial}{\partial t} \, d\mu_g \\
= \frac{1}{2} \int_{\Sigma} \left( \Delta \varphi + (|h| \nu^2 + \tilde{\text{Ric}}(\nu, \nu)) \varphi \right) H \, d\mu_g - \frac{1}{4} \int_{\Sigma} H^3 \varphi \, d\mu_g.
\]

Using \( |h|^2 = |h| \nu^2 + \frac{1}{2} H^2 \) and integrating by parts yields

\[
\frac{d}{dt} W(f) = \frac{1}{2} \int_{\Sigma} \left( \Delta \varphi + (|h| \nu^2 + \tilde{\text{Ric}}(\nu, \nu)) \varphi \right) H \, d\mu_g \\
+ \left. \frac{1}{2} \int_{\partial \Sigma} \left( \varphi \frac{\partial H}{\partial \eta} - \frac{\partial \varphi}{\partial \eta} H \right) \right| ds_g.
\]

This proves the claim in the case when \( \phi \) is normal. Now consider a variation of the form \( f \circ \varphi_t \), where \( \varphi_t \) is the flow of the vector field \( \xi \). For \( Q \subset \Sigma \) we get by invariance with respect to reparametrizations

\[
W(f \circ \varphi_t, Q) = W(f, \varphi_t(Q)) \\
= \frac{1}{4} \int_{Q_t} H(y)^2 \, d\mu_g(y) \\
= \frac{1}{4} \int_{Q} H(\varphi_t(x))^2 J\varphi_t(x) \, d\mu_g(x),
\]

where \( J\varphi_t(x) \) is the Jacobian. Differentiating at \( t = 0 \) we get

\[
\frac{d}{dt} W(f \circ \varphi_t, Q) = \frac{1}{4} \int_{Q} \left( \partial_{\xi} H^2 + H^2 \text{div}_g \xi \right) \, d\mu_g \\
= \frac{1}{4} \int_{Q} \text{div}_g(H^2 \xi) \, d\mu_g \\
= -\frac{1}{4} \int_{\partial Q} H^2 g(\xi, \eta) \, ds_g.
\]
Since \( \omega(\eta) = -\frac{1}{2}H^2g(\xi, \eta) \) for \( \phi = Df \cdot \xi \) (i.e. \( \varphi \equiv 0 \)), the formula is proved for all \( \phi \).

Now let \( \Omega \subset \mathbb{R}^3 \) be a domain with smooth boundary. We put \( S = \partial\Omega \) and denote by \( N^S : S \to S^2 \) the interior unit normal. Then for a smooth compact surface \( \Sigma = \Sigma \cup \partial\Sigma \) we consider the class of immersions

\[
\mathcal{M}(S) = \{ f \in C^\infty(\Sigma, \mathbb{R}^3) \text{ immersed: } f(\partial\Sigma) \subset S, \frac{\partial f}{\partial \eta} = N^S \circ f \}. \tag{1.9}
\]

Let \( h \) and \( h^S \) be the second fundamental forms of \( f \) and \( S \), respectively. We calculate, using that \( \tilde{D}^2 f(\tau, \tau) = h(\tau, \tau)\nu \) is normal to \( N^S \circ f \), for \( \tau \) the unit tangent along \( \partial\Sigma \),

\[
0 = \frac{\partial}{\partial \tau} \tilde{g} \left( \frac{\partial f}{\partial \tau}, N^S \circ f \right) = \tilde{g} \left( Df \cdot \nabla_\tau \tau, N^S \circ f \right) + \tilde{g} \left( \frac{\partial f}{\partial \tau}, \left( DN^S \circ f \right) \cdot \frac{\partial f}{\partial \tau} \right).
\]

The geodesic curvature of \( \partial\Sigma \) with respect to the induced metric \( g \) is defined by

\[
\nabla_\tau \tau = \kappa_g \eta \quad \Leftrightarrow \quad \nabla_\tau \eta = -\kappa_g \tau.
\]

Thus \( \kappa_g = +1 \) for the standard disk. We have

\[
\kappa_g = h^S \left( \frac{\partial f}{\partial \tau}, \frac{\partial f}{\partial \tau} \right). \tag{1.10}
\]

Taking the derivative of \( \tilde{g}(\nu, N^S) = 0 \) in the direction of \( \tau \) yields

\[
h(\tau, \eta) + h^S(\nu, \frac{\partial f}{\partial \tau}) = 0. \tag{1.11}
\]

A further tangential derivative implies

\[
\nabla_\tau h(\tau, \eta) + \kappa_g(h(\eta, \eta) - h(\tau, \tau)) + \frac{\partial}{\partial \tau} \left[ h^S(\nu, \frac{\partial f}{\partial \tau}) \right] = 0. \tag{1.12}
\]

Next we linearize the constraints. Let \( f = f(p, t) \in \mathcal{M}(S) \) and put

\[
\frac{\partial f}{\partial t}\big|_{t=0} = \phi = \varphi \nu + Df \cdot \xi.
\]

Differentiating the equation \( f(\partial\Sigma, t) \in S \) yields

\[
0 = \tilde{g} \left( \phi, N^S \circ f \right) = \tilde{g} \left( Df \cdot \xi, Df \cdot \eta \right) = g(\xi, \eta) \quad \text{along } \partial\Sigma. \tag{1.13}
\]

For the variation of the normal we have the standard formula

\[
\frac{\tilde{D} \nu}{\partial t} = Df \cdot \left( -\text{grad}_g \varphi + W\xi \right) \quad \text{on } \Sigma. \tag{1.14}
\]
where $W$ is the Weingarten map given by $h(X, Y) = -g(WX, Y)$ or $\tilde{D} \nu = Df \cdot W$.

The first variation of the orthogonality relation gives

$$0 = \frac{\partial}{\partial t} \tilde{g}(\nu, N^S \circ f) = \tilde{g}(Df \cdot (-\text{grad}_g \varphi + W\xi), N^S \circ f) + \tilde{g}(\nu, (W^S \circ f) \phi).$$

In this calculation we used $f(\partial \Sigma, t) \subset S$ so that $\phi \in T_f S$ and $(W^S \circ f) \phi$ makes sense.

Now from $N^S \circ f = \frac{\partial f}{\partial \eta}$, we have

$$\tilde{g}(Df \cdot (-\text{grad}_g \varphi + W\xi), N^S \circ f) = g(-\text{grad}_g \varphi + W\xi, \eta) = -\frac{\partial \varphi}{\partial \eta} - h(\xi, \eta).$$

Using further $\tilde{g}(\nu, (W^S \circ f) \phi) = -h^S(\nu, \varphi \nu + Df \cdot \xi)$ we arrive at the following two linearized equations, for the variation vectorfield $\phi = \varphi \nu + Df \cdot \xi$,

(1.15) \hspace{1cm} g(\xi, \eta) = 0 \quad \text{on} \ \partial \Sigma,

(1.16) \hspace{1cm} \frac{\partial \varphi}{\partial \eta} + h(\xi, \eta) + \varphi h^S(\nu, \nu) + h^S(\nu, Df \cdot \xi) = 0 \quad \text{on} \ \partial \Sigma.

Equation (1.15) holds if and only if $\xi = \mu \tau$ for some function $\mu$ on $\partial \Sigma$. Then (1.16) simplifies using (1.11) and we are left with

(1.17) \hspace{1cm} \frac{\partial \varphi}{\partial \eta} + \varphi h^S(\nu, \nu) = 0 \quad \text{where} \ \phi = \varphi \nu + \mu \frac{\partial f}{\partial \tau} \quad \text{on} \ \partial \Sigma.

The variation vector fields $\phi$ with (1.17) are called admissible for $f$ and are denoted by $T_f M(S)$. Any function $\varphi$ given on $\partial \Sigma$ admits an extension to $\Sigma$ such that (1.17) holds, and for any $\mu$ on $\partial \Sigma$ there exists a vector field $\xi$ on $\Sigma$ such that $\xi|_{\partial \Sigma} = \mu \tau$. Then the variation $\phi = \varphi \nu + Df \cdot \xi$ is admissible.

Now assume that $f \in M(S)$ satisfies

(1.18) \hspace{1cm} \delta W(f) \phi = 0 \quad \text{for all} \ \phi \in T_f M(S).

Then clearly $W(f) = 0$, and the definition of $T_f M$ as in (1.17) implies further

$$0 = \frac{1}{2} \int_{\partial \Sigma} \varphi \left( \frac{\partial H}{\partial \eta} + H h^S(\nu, \nu) \right) ds_g \quad \text{for all} \ \varphi \in C^\infty(\partial \Sigma).$$

So we arrive at the two boundary conditions

(1.19) \hspace{1cm} \tilde{g}(\nu, N^S \circ f) = 0 \quad \text{on} \ \partial \Sigma,

(1.20) \hspace{1cm} \frac{\partial H}{\partial \eta} + H h^S(\nu, \nu) = 0 \quad \text{on} \ \partial \Sigma.

This paper studies a perturbed boundary value problem with respect to Riemannian metrics $\tilde{g}$ which are close to the Euclidean metric $\delta$, aiming at immersions close to the standard $S^2_+$. We now collect some formulae for radial graphs

$$f : S^2_+ \to \mathbb{R}^3, \ f(\omega) = (1 + w(\omega))\omega.$$
For a tangent vector \( \tau \in T_\omega S^2 \) we have
\[
\partial_\tau f(\omega) = (1 + w(\omega))\tau + (\partial_\tau w)(\omega)\omega.
\]
In an orthonormal frame \( \tau_1, \tau_2 \) on \( S^2 \) the metric \( g = f^*\tilde{g} \) is given by
\[
g(\tau_\alpha, \tau_\beta) = (1 + w)^2 \tilde{g}(\tau_\alpha, \tau_\beta) + (1 + w)(\partial_\tau w)\tilde{g}(\omega, \tau_\beta) + (\partial_\tau w)(\partial_\tau w)\tilde{g}(\omega, \omega).
\]
Here \( \tilde{g} \) is always evaluated at \( f(\omega) \). The area of \( f \) with respect to \( \tilde{g} \) is
\[
A(f, \tilde{g}) = \int_{S^2} \sqrt{\det g(\tau_\alpha, \tau_\beta)} d\mu_{S^2}.
\]
Let \( \tilde{v}_{R^2} \) be the upper unit normal along \( R^2 \) with respect to \( \tilde{g} \). We compute
\[
\text{grad}_{\tilde{g}} x^3 = \sum_{i,j=1}^3 \tilde{g}^{ij} \partial_i x^3 e_j = \sum_{j=1}^3 \tilde{g}^{3j} e_j.
\]
Further \( \tilde{g}(\text{grad}_{\tilde{g}} x^3, \text{grad}_{\tilde{g}} x^3) = \tilde{g}_{jk} \tilde{g}^{3j} \tilde{g}^{3k} = \tilde{g}^{33} \). Thus we have
\[
\tilde{v}_{R^2} = \frac{1}{\sqrt{\tilde{g}^{33}}} \sum_{j=1}^3 \tilde{g}^{3j} e_j.
\]
Now let \( \nu : S^2_+ \to R^3, \nu = \nu[f, \tilde{g}] \), be the unit normal along \( f \) with respect to \( \tilde{g} \), such that \( \nu(\omega) = -\omega \) for \( u = 0, \tilde{g} = \delta \). Then
\[
\tilde{g}(\nu, \tilde{v}_{R^2}) = \frac{1}{\sqrt{\tilde{g}^{33}}} \tilde{g}(\nu, \text{grad}_{\tilde{g}} x^3)) = \frac{1}{\sqrt{\tilde{g}^{33}}} (\nu, e_3).
\]
With respect to \( \tilde{g}(f(\omega)) \), the component of \( \omega \) which is tangential along \( f \) is
\[
\omega^\top = g^{\alpha\beta} \tilde{g}(\omega, \partial_\tau f)\partial_\tau f.
\]
Here \( \tilde{g} \) is always evaluated at \( f(\omega) \). Then \( \omega^\perp = \omega - \omega^\top \) has the norm
\[
\tilde{g}(\omega^\perp, \omega^\perp) = \tilde{g}(\omega, \omega - \omega^\top) = \tilde{g}(\omega, \omega) - g^{\alpha\beta} \tilde{g}(\omega, \partial_\tau f)\tilde{g}(\omega, \partial_\tau f).
\]
Dividing we obtain the formula
\[
\nu(\omega) = -\frac{\omega - g^{\alpha\beta} \tilde{g}(\omega, \partial_\tau f)\partial_\tau f}{\sqrt{\tilde{g}(\omega, \omega) - g^{\alpha\beta} \tilde{g}(\omega, \partial_\tau f)\tilde{g}(\omega, \partial_\tau f)}}.
\]
The following two lemmas show that the constraint of orthogonality is nondegenerate at the standard \( S^2_+ \).
Lemma 2 We have $W^{2,2}(S^2_+) = X_0 \oplus Y_0$ as topological direct sum, where

$$X_0 = \{ u \in W^{2,2}(S^2_+) : \frac{\partial u}{\partial \eta} = 0 \ \text{on} \ \partial S_+^2 \}$$

$$Y_0 = \{ v \in W^{2,2}(S^2_+) : \Delta_{S^2} v = \text{const. on} \ S_+^2, \int_{S_+^2} v d\nu_g = 0 \}.$$ 

Moreover $C^{k,\alpha}(S^2_+) = (X_0 \cap C^{k,\alpha}(S^2_+)) \oplus (Y_0 \cap C^{k,\alpha}(S^2_+))$ for any $k \geq 2, \ \alpha \in (0, 1)$.

**Proof.** $X_0$ and $Y_0$ are closed subspaces of $W^{2,2}(S^2_+)$ with $X_0 \cap Y_0 = \{0\}$. Any $w \in W^{2,2}(S^2_+)$ decomposes uniquely as $w = u + v$, where $u \in X_0, \ v \in Y_0$ are chosen with

$$\Delta_{S^2} v = -\frac{1}{2\pi} \int_{\partial S^2_+} \frac{\partial w}{\partial \eta} ds_g \ \text{in} \ S^2_+, \ \frac{\partial v}{\partial \eta} = \frac{\partial w}{\partial \eta} \ \text{on} \ \partial S^2_+$$

$$u = w - v.$$ 

Using Sobolev trace theory [1, 17] we have the a priori estimates

$$\|u\|_{W^{2,2}(S^2_+)} + \|v\|_{W^{2,2}(S^2_+)} \leq C \|w\|_{W^{2,2}(S^2_+)}.$$

Therefore the map $X_0 \oplus Y_0 \to W^{2,2}(S^2_+), (u, v) \mapsto u + v,$ is an isomorphism of Banach spaces. Moreover by Schauder regularity [10, 17] for the Neumann problem

$$\|u\|_{C^{k,\alpha}(S^2_+)} + \|v\|_{C^{k,\alpha}(S^2_+)} \leq C \|w\|_{C^{k,\alpha}(S^2_+)}.$$

This proves the second statement. \qed

In the following calculations we assume the background metric $\tilde{g}$ to be given on the cylinder $Z_2 = D_2(0) \times [-2, 2]$, which compactly contains the ball $B_1(0)$.

**Lemma 3** Let $v = \nu[w, \tilde{g}]$ denote the unit normal of the graph of $w \in C^{k,\alpha}(S^2_+)$ with respect to the Riemannian metric $\tilde{g} \in C^l(Z_2, \mathbb{R}^{3 \times 3})$. For $1 \leq k \leq l$ the map

$$B[w, \tilde{g}] = \tilde{g}(\nu, \nu_{\mathbb{R}^2}) = \frac{1}{\tilde{g}^{3/2}}(\nu, e_3)|_{\partial S^2_+} \in C^{k-1,\alpha}(\partial S^2_+)$$

is well-defined and of class $C^{l-k}$. For $2 \leq k < l$ there exist open neighborhoods $U \subset X_0 \cap C^{k,\alpha}(S^2_+), \ V \subset Y_0 \cap C^{k,\alpha}(S^2_+)$ and $G \subset C^l(Z_2, \mathbb{R}^{3 \times 3})$ of $\nu \equiv 0, \ v \equiv 0$ and $\tilde{g} \equiv \delta$, and a $C^{l-k}$ map $\Psi : U \times G \to V$ such that for all $u \in U, \ v \in V, \ \tilde{g} \in G$ we have

$$B[u + v, \tilde{g}] = 0 \iff v = \Psi[u, \tilde{g}].$$

We have $D_u \Psi[0, \delta] = 0$, and $h = D_\tilde{g} \Psi[0, \delta] q \in Y_0 \cap C^{k,\alpha}(S^2_+)$ is the unique solution of

$$(1.21) \quad -\Delta_{S^2} h = \frac{1}{2\pi} \int_{\partial S^2_+} q(\nu, e_3) ds \ \text{in} \ S^2_+, \ \frac{\partial h}{\partial \eta} = q(\nu, e_3) \ \text{on} \ \partial S^2_+.$$
Proof. The map $B : C^{k,\alpha}(S^2_+) \times C^l(Z_2, \mathbb{R}^{3 \times 3}) \to C^{k-1,\alpha}(\partial S^2_+)$ is well-defined and of class $C^{l-k}$ near $w \equiv 0, \tilde{g} \equiv \delta$, and has the derivative

$$D_w B[0, \delta] \varphi = -\frac{\partial \varphi}{\partial \eta},$$

thus $\ker D_w B[0, \delta] = X_0 \cap C^{k,\alpha}(S^2_+)$. The operator $D_w B[0, \delta]|_{Y_0} : Y_0 \cap C^{k,\alpha}(S^2_+) \to C^{k-1,\alpha}(\partial S^2_+)$ is an isomorphism: for any $\beta \in C^{k-1,\alpha}(\partial S^2_+)$ there is a unique $v \in Y_0 \cap C^{k,\alpha}(S^2_+)$ with $D_w B[0, \delta] v = \beta$, in other words

$$-\Delta_{S^2} v = \frac{1}{2\pi} \int_{\partial S^2_+} \beta ds_g, \quad \int_{S^2_+} v d\mu_g = 0, \quad \frac{\partial v}{\partial \eta} = \beta.$$

Moreover that solution $v$ satisfies the estimate $\|v\|_{C^{k,\alpha}(S^2_+)} \leq C \|\beta\|_{C^{k-1,\alpha}(\partial S^2_+)}$, which means that $D_w B[0, \delta]|_{Y_0}$ has a bounded inverse. Existence and uniqueness of $\Psi[u, \tilde{g}]$ follows from the implicit function theorem. Now $\Psi[0, \delta] = 0$, and we have for any $\varphi \in X_0 \cap C^{k,\alpha}(S^2_+)$

$$0 = \frac{d}{dt} B [t \varphi + \Psi(t \varphi, \delta, \delta)]|_{t=0} = \underbrace{D_w B[0, \delta] \varphi}_{=0} + D_w B[0, \delta] D_u \Psi[0, \delta] \varphi.$$

This shows $D_u \Psi[0, \delta] = 0$. We have further for $\tilde{g} = \delta + tq$ and $\nu = \nu[0, \tilde{g}]$

$$D_{\tilde{g}} B[0, \delta] \cdot q = \langle \frac{\partial \nu}{\partial t}|_{t=0}, e_3 \rangle = \frac{\partial}{\partial t} \tilde{g}(\nu, e_3)|_{t=0} - \frac{\partial \tilde{g}}{\partial t}(\nu, e_3)|_{t=0} = -q(\nu, e_3),$$

which yields the remaining claim, namely

$$0 = \frac{d}{dt} B [\Psi[0, \delta + tq], \delta + tq]|_{t=0} = D_w B[0, \delta] D_{\tilde{g}} \Psi[0, \delta] \cdot q - q(\nu, e_3).$$

\[\square\]

2 The Riemannian perturbation problem

Using reflection and Simon’s monotonicity formula, it is easy to see that the standard half-sphere $S^2_+$ minimizes the Willmore functional among surfaces meeting $\mathbb{R}^2$ orthogonally. One might hope to get corresponding Willmore surfaces for perturbed background metrics $\tilde{g}$ using the implicit function theorem. However the linearized problem has a kernel $K_0$. For any $\lambda > 0$ the dilated sphere $\lambda S^2_+$ also minimizes, and is represented as graph of $w^\lambda(\omega) \equiv \lambda - 1$ over $S^2_+$. Hence $K_0$ contains the function

$$\frac{\partial}{\partial \lambda} w^\lambda|_{\lambda = 1} \equiv 1.$$
Likewise for any \( a \in \mathbb{R}^2, |a| < 1 \), the translated halfspheres \( S_2^2(a) \) admit the graph representations \( w^a(\omega) = \langle \omega, a \rangle - 1 + \sqrt{1 - |a|^2 + \langle \omega, a \rangle^2} \) over \( S_2^2 \), hence \( K_0 \) also contains the functions
\[
\frac{\partial}{\partial \varepsilon} w^{\varepsilon a}(\omega)|_{\varepsilon=0} = \langle \omega, a \rangle.
\]

We get a solvable problem by prescribing the Riemannian area and two-dimensional barycenter. For these constrained solutions the Willmore operator is in the space \( K(\tilde{g}) \) spanned by the \( L^2 \) gradients of the constraints, and we have \( K(\delta) = K_0 \). In the next section we will study the Willmore energy as a function on the manifold of constrained solutions.

**Lemma 4** Let \( K_0 = \text{Span} \{ 1, \langle \omega, e_1 \rangle, \langle \omega, e_2 \rangle \} \), and define the Hilbert space
\[
W^{2,2}_{0,\perp}(S_2^2) = \{ u \in W^{2,2}(S_2^2) : \frac{\partial u}{\partial \eta} = 0 \text{ on } \partial S_2^2, u \perp_{L^2} K_0 \}.
\]

Then the linear operator
\[
L : W^{2,2}_{0,\perp}(S_2^2) \to W^{2,2}_{0,\perp}(S_2^2)', \langle Lu, v \rangle = \int_{S_2^2} (\Delta_{S^2} u \Delta_{S^2} v - 2 \langle \nabla u, \nabla v \rangle) \, d\mu_{S^2}
\]
is an isomorphism.

**Proof.** Let \( E_k \subset L^2(S^2), k \in \mathbb{N}_0 \), be the space of even eigenfunctions of \(-\Delta_{S^2}\) on the 2-sphere, with eigenvalue \( \lambda_k = k(k+1) \) (even means \( u(x, z) = u(x, -z) \)). We have
\[
2 \langle Lu_k, u_l \rangle = \int_{S^2} (\Delta_{S^2} u_k \Delta_{S^2} u_l - 2 \langle \nabla u_k, \nabla u_l \rangle) \, d\mu_{S^2}
\]
\[
= \int_{S^2} \Delta_{S^2} u_k (\Delta_{S^2} + 2) u_l \, d\mu_{S^2}
\]
\[
= \lambda_k (\lambda_l - 2) \langle u_k, u_l \rangle_{L^2(S^2)}.
\]

Now \( \lambda_k \geq 6 \) for \( k \geq 2 \), thus for a finite sum \( u = \sum_{k=2}^N u_k \) we see
\[
\int_{S^2} ( (\Delta_{S^2} u)^2 + u^2 ) \, d\mu_{S^2} = \sum_{k=2}^N (\lambda_k^2 + 1) \| u_k \|^2_{L^2(S^2)} \leq \frac{37}{24} \sum_{k=2}^N \lambda_k (\lambda_k - 2) \| u_k \|^2_{L^2(S^2)} \leq \frac{37}{12} \langle Lu, u \rangle.
\]

Applying the Bochner Formula on \( S^2 \) we conclude that
\[
\int_{S^2} ( |\nabla^2 u|^2 + |\nabla u|^2 + u^2 ) \, d\mu_{S^2} = \int_{S^2} ( (\Delta_{S^2} u)^2 + u^2 ) \, d\mu_{S^2} \leq \frac{37}{12} \langle Lu, u \rangle.
\]

Extending functions \( u \in W^{2,2}_{0,\perp}(S_2^2) \) by even reflection across \( \partial S_2^2 \) yields \( W^{2,2} \)-functions on the sphere. It is then easy to see that the algebraic sum \( \bigoplus_{k=2}^\infty E_k \) is \( W^{2,2} \)-dense in \( W^{2,2}_{0,\perp}(S_2^2) \). The coercivity of \( L \) and hence the claim of the lemma follows. \( \square \)
Lemma 5  For $k \geq 4$ and $\alpha \in (0,1)$ the linear operator

$$
\mathcal{L} : C_{0,1}^{k,\alpha}(S^2_+) \to C_{1}^{k-4,\alpha}(S^2_+) \times C^{k-3,\alpha}(\partial S^2_+), \quad \mathcal{L}u = (\Delta_{S^2}(\Delta_{S^2} + 2)u, \frac{\partial(\Delta_{S^2}u)}{\partial \eta}),
$$

is an isomorphism.

Proof. We have a commutative diagram

$$
\begin{array}{ccc}
C_{0,1}^{k,\alpha}(S^2_+) & \xrightarrow{\mathcal{L}} & C_{1}^{k-4,\alpha}(S^2_+) \times C^{k-3,\alpha}(\partial S^2_+) \\
\cap & & \cap \\
W_{0,1}^{2,2}(S^2_+) & \xrightarrow{\mathcal{L}} & W_{0,1}^{2,2}(S^2_+)'.
\end{array}
$$

Here for $(f_0, f_1) \in C_{1}^{k-4,\alpha}(S^2_+) \times C^{k-3,\alpha}(\partial S^2_+)$ the inclusion on the right is given by

$$
\Lambda(v) = \int_{S^2_+} f_0 v \, d\mu_{S^2} + \int_{\partial S^2_+} f_1 v \, ds.
$$

The injectivity of $\mathcal{L}$ follows from Lemma 4. Moreover for given $(f_0, f_1) \in C_{1}^{k-4,\alpha}(S^2_+) \times C^{k-3,\alpha}(\partial S^2_+)$ there exists $u \in W_{0,1}^{2,2}(S^2_+)$ such that

$$
\int_{S^2_+} (\Delta_{S^2}u \Delta_{S^2}v - 2(\nabla u, \nabla v)) \, d\mu_{S^2} = \int_{S^2_+} f_0 v \, d\mu_{S^2} + \int_{\partial S^2_+} f_1 v \, ds \quad \text{for all } v \in W_{0,1}^{2,2}(S^2_+).
$$

This means that $u \in W_{0,1}^{2,2}(S^2_+)$ solves the equations

$$
\Delta_{S^2}(\Delta_{S^2} + 2)u = f_0 \text{ in } S^2_+, \quad \frac{\partial(\Delta_{S^2}u)}{\partial \eta} = f_1 \text{ on } \partial S^2_+.
$$

In fact, integrating by parts for functions $u, v \in C^4(\overline{S^2_+})$ yields

$$
\begin{align*}
\int_{S^2_+} & \Delta_{S^2}(\Delta_{S^2} + 2)u \cdot v \, d\mu_{S^2} \\
= & \int_{S^2_+} \left( \div (\nabla(\Delta_{S^2}u + 2u) \cdot v) \, d\mu_{S^2} - \int_{S^2_+} (\nabla(\Delta_{S^2}u + 2u), \nabla v) \right) \, d\mu_{S^2} \\
= & \int_{S^2_+} \left( \div [\nabla(\Delta_{S^2}u + 2u) \cdot v] \, d\mu_{S^2} - \int_{S^2_+} \div [(\Delta_{S^2}u + 2u)\nabla v] \, d\mu_{S^2} \\
& + \int_{S^2_+} (\Delta_{S^2}u + 2u)\Delta_{S^2}v \, d\mu_{S^2} \\
= & \int_{S^2_+} \left( \Delta_{S^2}u\Delta_{S^2}v - 2(\nabla u, \nabla v) \right) \, d\mu_{S^2} - \int_{\partial S^2_+} u \frac{\partial v}{\partial \eta} \, ds \\
& - \int_{\partial S^2_+} \frac{\partial(\Delta_{S^2}u + 2u)}{\partial \eta} \cdot v \, ds + \int_{\partial S^2_+} (\Delta_{S^2}u + 2u) \frac{\partial v}{\partial \eta} \, ds.
\end{align*}
$$
Schauder theory, see [2, 17], implies \( u \in C^{k,\alpha}_{0,\perp}(S^2_+) \) and
\[
\|u\|_{C^{k,\alpha}(S^2_+)} \leq C(\|f_0\|_{C^{k-4,\alpha}(S^2_+)} + \|f_1\|_{C^{k-3,\alpha}(\partial S^2_+)}).
\]
This proves the lemma.

Now consider on \( Z_2 = \{(x, z) \in \mathbb{R}^2 \times \mathbb{R} : |x|, |z| < 2\} \) a given Riemannian metric \( \tilde{g} \in C^4(Z_2, \mathbb{R}^{3\times 3}) \). We want to find a function \( w \in C^{k,\alpha}(S^2_+) \), resp. the surface \( f(\omega) = \omega + w(\omega) \omega \), satisfying the orthogonality constraint
\[
B[w, \tilde{g}] = \tilde{g}(\nu, \nu_{\tilde{g}^2}) = \frac{1}{\sqrt{\tilde{g}^{33}(\nu, e_3)}} = 0,
\]
and such that \( Q[w, \tilde{g}] = 0 \) where \( Q = Q_1, \ldots, Q_4 \) is as follows:
\[
Q_1[w, \tilde{g}] = P^\perp W[f, \tilde{g}],
\]
\[
Q_2[w, \tilde{g}] = \frac{\partial H}{\partial \eta} + \bar{h}^{32}(\nu, \nu)H,
\]
\[
Q_3[w, \tilde{g}] = A[f, \tilde{g}] - 2\pi,
\]
\[
Q_4[w, \tilde{g}] = C[f, \tilde{g}] \in \mathbb{R}^2.
\]
See Lemma 12 in the appendix for the definition of the twodimensional barycenter \( C[f, \tilde{g}] \). We denote by \( K = K[w, \tilde{g}] \) the space spanned by the functions
\[
\psi_0 = \frac{1}{\sqrt{8\pi}} H[w, \tilde{g}], \quad \psi_i = -\sqrt{\frac{2\pi}{3}} \text{grad}_{L^2} C^i[w, \tilde{g}] \quad (i = 1, 2).
\]
A formula for \( \psi_{1,2} \) is derived in (4.3). For \( w = 0, \tilde{g} = \delta \) the functions form an orthonormal basis of \( K(0, \delta) = K_0 \subset L^2(S^2_+) \), in fact by (4.4)
\[
\psi_0(\omega) = \frac{1}{\sqrt{2\pi}}, \quad \psi_i(\omega) = \sqrt{\frac{3}{2\pi}} \langle \omega, e_i \rangle.
\]
\( P^\perp \) is the \( L^2 \) projection, with respect to \( g \), onto the orthogonal complement of \( K[w, \tilde{g}] \), thus \( Q_1[w, \tilde{g}] = 0 \) means \( W[f, \tilde{g}] \in K[w, \tilde{g}] \). A function \( w \) satisfying (2.1) – (2.5) is called a constrained solution for the given metric \( \tilde{g} \). The Frechet derivative \( D_w Q[0, \delta] \) is the linear operator
\[
L : C^{k,\alpha}(S^2_+) \longrightarrow C^{k-4,\alpha}(S^2_+) \times C^{k-1,\alpha}(\partial S^2_+) \times \mathbb{R} \times \mathbb{R}^2
\]
having the following components:
\[
L_1 \varphi = \Delta_{g^2}(\Delta_{g^2} + 2) \varphi,
\]
\[
L_2 \varphi = \frac{\partial}{\partial \eta}(\Delta_{g^2} + 2) \varphi,
\]
\[
L_3 \varphi = -2 \int_{S^2_+} \varphi \mu_{g^2},
\]
\[
L_i \varphi = \frac{3}{2\pi} \int_{S^2_+} \varphi(\omega) \langle \omega, e_i \rangle \mu_{g^2}(\omega) \quad \text{for } i = 1, 2.
\]
See (4.4) for the derivation of (2.10), and note that \( \varphi(\omega) = -\varphi(\omega) \mu(\omega) \).
Lemma 6 Let \( k \geq 4 \) and \( l \geq k+1 \). Then there exist open neighborhoods \( W \subset C^{k,\alpha}(\mathbb{S}^2_+) \) of \( w \equiv 0 \) and \( G \subset C^l(Z_2, \mathbb{R}^{3x3}_{\text{sym}}) \) of \( \bar{g} = \delta \), and a \( C^{l-k} \) function \( w : G \to W \) such that for \( w \in W, \bar{g} \in G \)

\[
(2.11) \quad B[w, \bar{g}] = 0, \quad Q[w, \bar{g}] = 0 \iff w = w[\bar{g}].
\]

Moreover for \( \|\bar{g} - \delta\|_{C^l(Z_2)} \) sufficiently small and \( C = C(k, \alpha) < \infty \) we have the estimate

\[
(2.12) \quad \|w[\bar{g}]\|_{C^{k,\alpha}(\mathbb{S}^2_+)} \leq C\|\bar{g} - \delta\|_{C^l(Z_2)}.
\]

Proof. By the coordinate expressions and the results of the appendix, we see that \( Q[w, \bar{g}] \) is well-defined as a map from \( W \times G \) into \( C^{k-4,\alpha}(\mathbb{S}^2_+) \times C^{k-3,\alpha}(\partial\mathbb{S}^2_+ \times \mathbb{R} \times \mathbb{R}^2 \), and is of class \( C^{l-k} \) under the assumptions. To construct the solution \( w[\bar{g}] \) we make the ansatz \( w = u + \Psi[u, \bar{g}] \), where \( u \in U, \Psi[u, \bar{g}] \in V \) are as in Lemma 3, in particular \( \Psi[u, \bar{g}] \) is also of class \( C^{l-k} \). The condition (2.1) is then fulfilled, and we must solve the equation

\[
(2.13) \quad \overline{Q}[u, \bar{g}] := Q[u + \Psi[u, \bar{g}], \bar{g}] = 0.
\]

We know from Lemma 3 that \( D_u \Psi[0, \delta] = 0 \). Linearizing with respect to \( u \) yields the operator \( L : C^{k,\alpha}(\mathbb{S}^2_+) \to C^{k-4,\alpha} \times C^{k-3,\alpha}(\partial\mathbb{S}^2_+ \times \mathbb{R} \times \mathbb{R}^2) \), where

\[
L\varphi = \begin{pmatrix}
\Delta_{\mathbb{S}^2}(\varphi + 2)\varphi \\
d_\varphi \\
\sigma_{\bar{g}} \\
-2\int_{\mathbb{S}^2_+} \varphi \langle \omega, e_i \rangle \, d\mu_{\mathbb{S}^2}
\end{pmatrix}.
\]

Using Lemma 5 it is immediate that \( L \) is an isomorphism. By the implicit function theorem there is a solution \( u = u[\bar{g}] \) of (2.13). The \( C^{l-k} \) function \( w[\bar{g}] = u[\bar{g}] + \Psi[u[\bar{g}], \bar{g}] \) then solves (2.11).

Now assume that \( w, \bar{g} \) satisfy \( B[w, \bar{g}] = 0 \) and \( Q[w, \bar{g}] = 0 \). By uniqueness in Lemma 3, we then have \( w = u + \Psi[u, \bar{g}] \) for some \( u \in U \), and uniqueness for (2.13) implies further \( u = u[\bar{g}] \). This proves the reverse implication in (2.11).

For \( \|u\|_{C^{k,\alpha}(\mathbb{S}^2_+)} + \|\bar{g} - \delta\|_{C^l(Z_2)} \) small, we have writing \( \|\cdot\| \) for operator norms

\[
\|D_u \overline{Q}[u, \bar{g}] - D_u \overline{Q}[0, \delta]\| + \|D_{\bar{g}} \overline{Q}[u, \bar{g}] - D_{\bar{g}} \overline{Q}[0, \delta]\| < \varepsilon.
\]

We have by the fundamental theorem of calculus

\[
\overline{Q}[u, \bar{g}] = D_u \overline{Q}[0, \delta]u + D_{\bar{g}} \overline{Q}[0, \delta](\bar{g} - \delta)
\]

\[
+ \int_0^1 \left( D_u \overline{Q}[tu, (1-t)\delta + t\bar{g}] - D_u \overline{Q}[0, \delta] \right) u \, dt
\]

\[
+ \int_0^1 \left( D_{\bar{g}} \overline{Q}[tu, (1-t)\delta + t\bar{g}] - D_{\bar{g}} \overline{Q}[0, \delta] \right) (\bar{g} - \delta) \, dt.
\]
Using $\|L^{-1}\| \leq C$ and $Q[u, \tilde{g}] = 0$ for $u = u[\tilde{g}]$, we obtain after absorbing

$$\|u[\tilde{g}]\|_{C^{k,\alpha}(S^2)} \leq C\|\tilde{g} - \delta\|_{C^l(Z_2)}.$$  

Now $D_u\Psi[u, \tilde{g}] + D_{\tilde{g}}\Psi[u, \tilde{g}] \leq C$ for $\|u\|_{C^{k,\alpha}(S^2)} + \|\tilde{g} - \delta\|_{C^l(Z_2)}$ small, thus

$$\|\Psi[u, \tilde{g}]\|_{C^{k,\alpha}(S^2)} \leq C(\|u\|_{C^{k,\alpha}(S^2)} + \|\tilde{g} - \delta\|_{C^l(Z_2)}).$$

Combining yields the inequality (2.12). \(\square\)

**Lemma 7** For radial graphs $f(\omega) = \omega + w(\omega)\omega$ and $l \geq 1$, consider

$$W : C^2(S^2_+) \times C^l(Z_2) \to \mathbb{R}, W[w, \tilde{g}] = \frac{1}{4} \int_{S^2_+} H^2 d\mu_g.$$  

The functional is well-defined and of class $C^{l-1}$ on the set $\|w\|_{C^1(S^2_+)} + \|\tilde{g} - \delta\|_{C^0(Z_2)} < \varepsilon_0$. It has the derivatives, choosing $\nu(\omega) = -\omega$,

$$D_wW(0, \delta)\varphi = -\int_{S^2_+} \frac{\partial\varphi}{\partial \eta} ds,$$

$$D_{\tilde{g}}W(0, \delta)q = \int_{S^2_+} \left( -\frac{1}{2} \text{tr}_{S^2} q + q(\nu, \nu) + \text{tr}_{S^2} \nabla_q (\cdot, \nu) - \frac{1}{2} \text{tr}_{S^2} \nabla_\nu q \right) d\mu_{S^2}.$$  

**Proof.** The first formula follows from Theorem 1. Let $\tilde{g} = \tilde{g}(\varepsilon)$ be a family with $\tilde{g}(0) = \langle \cdot, \cdot \rangle|_{S^2}$ and $\frac{\partial \tilde{g}}{\partial \varepsilon}|_{\varepsilon=0} = q$.

Let $\varphi : U \to S^2_+$ be a parametrization. For the derivative of the normal we compute

$$0 = \frac{\partial}{\partial \varepsilon} \tilde{g}(\nu, \partial_\alpha \varphi)|_{\varepsilon=0} = q(\nu, \partial_\alpha \varphi) + \left( \frac{\partial \nu}{\partial \varepsilon} |_{\varepsilon=0}, \partial_\alpha \varphi \right),$$

$$0 = \frac{\partial}{\partial \varepsilon} \tilde{g}(\nu, \nu)|_{\varepsilon=0} = q(\nu, \nu) + 2\left( \frac{\partial \nu}{\partial \varepsilon} |_{\varepsilon=0}, \nu \right).$$

Thus we have

$$\frac{\partial \nu}{\partial \varepsilon} |_{\varepsilon=0} = -g^{\alpha\beta} q(\nu, \partial_\alpha \varphi) \partial_\beta \varphi - \frac{1}{2} q(\nu, \nu).$$

The derivative of the background connection (the Christoffel symbols) is denoted by $\gamma(X, Y) = \frac{\partial}{\partial \varepsilon} \tilde{D}_X Y |_{\varepsilon=0}$, in coordinates

$$\gamma^k_{ij} = \frac{1}{2} (\partial_i q_{jk} + \partial_j q_{ik} - \partial_k q_{ij}).$$

We obtain for the second fundamental form

$$\frac{\partial h_{\alpha\beta}}{\partial \varepsilon} |_{\varepsilon=0} = \frac{\partial}{\partial \varepsilon} \tilde{g}(\tilde{D}_\alpha \partial_\beta \varphi, \nu)|_{\varepsilon=0} = \frac{1}{2} q(\nu, \nu) h_{\alpha\beta} + \langle \gamma(\partial_\alpha \varphi, \partial_\beta \varphi), \nu \rangle.$$
Contracting yields for the mean curvature, using $h_{\alpha\beta} = g_{\alpha\beta}$ and $H = 2$,
\[
\frac{\partial H}{\partial \varepsilon}\bigg|_{\varepsilon=0} = -g^{\alpha\beta} q(\partial_\alpha \varphi, \partial_\beta \varphi) + q(\nu, \nu) + g^{\alpha\beta} \langle \gamma(\partial_\alpha \varphi, \partial_\beta \varphi), \nu \rangle.
\]
We have further
\[
\frac{\partial}{\partial \varepsilon} d\mu_g\bigg|_{\varepsilon=0} = \frac{1}{2} g^{\alpha\beta} q(\partial_\alpha \varphi, \partial_\beta \varphi) d\mu_g.
\]
Collecting terms we find
\[
\frac{\partial}{\partial \varepsilon} \mathcal{W}[0, \tilde{g}]\bigg|_{\varepsilon=0} = \int_{S^2_+} \left( -\frac{1}{2} \text{tr}_{S^2} q + q(\nu, \nu) + \text{tr}_{S^2} \langle \gamma, \nu \rangle \right) d\mu_{S^2}.
\]
Finally for vectors $\tau_1, \tau_2 \in T_\omega S^2$ we have
\[
\langle \gamma(\tau_1, \tau_2), \nu \rangle = \frac{1}{2} \left( \nabla_{\tau_1} q(\tau_2, \nu) + \nabla_{\tau_2} q(\tau_1, \nu) - \nabla_\nu q(\tau_1, \tau_2) \right).
\]
Inserting proves the second formula.

**Lemma 8** For $l \geq 6$ the function $w[\tilde{g}]$ from Lemma 6 satisfies, putting $q = \tilde{g} - \delta$, 
\[
\left| \mathcal{W}[w[\tilde{g}], \tilde{g}] - 2\pi - \int_{S^2_+} \left( -\frac{1}{2} \text{tr}_{S^2} q + q(\nu, \nu) + \text{tr}_{S^2} \nabla. q(\cdot, \nu) - \frac{1}{2} \text{tr}_{S^2} \nabla_\nu q \right) d\mu_{S^2} \right| + \int_{\partial S^2_+} q(\nu, e_3) ds \leq C \|q\|^2_{C^6(S^2)}.
\]

**Proof.** Putting $\varphi = Dw[\delta]q$ we compute
\[
D_w \mathcal{W}[0, \delta] \varphi + D_{\tilde{g}} \mathcal{W}[0, \delta]q = \int_{S^2_+} \left( -\frac{1}{2} \text{tr}_{S^2} q + q(\nu, \nu) + \text{tr}_{S^2} \nabla. q(\cdot, \nu) - \frac{1}{2} \text{tr}_{S^2} \nabla_\nu q \right) d\mu_{S^2} - \int_{S^2_+} \frac{\partial \varphi}{\partial \eta} ds.
\]
On the other hand we had in Lemma 3
\[
0 = D_w B[0, \delta] \varphi + D_{\tilde{g}} B[0, \delta]q = \frac{\partial \varphi}{\partial \eta} - q(\nu, e_3).
\]
The claim follows by Taylor’s formula, taking $k = 4, l = 6$ in Lemma 6.
3 Blowup at boundary points

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^m$, $m \geq 7$, with boundary $S = \partial \Omega$. At a given point $a \in S$ we let $N(a)$ be the interior unit normal and choose an orthonormal basis $v_1(a), v_2(a)$ of $T_a S$. For $r_0 = r_0(\Omega) > 0$ we have a graph representation

$$f^a : D_{r_0} \to \mathbb{R}^3, \quad f^a(x) = a + x^1 v_1(a) + x^2 v_2(a) + \varphi^a(x) N(a),$$

such that

$$\|\varphi^a\|_{C^m(D_{r_0})} \leq C = C(\Omega).$$

Since $\varphi^a(0) = 0$ and $D\varphi^a(0) = 0$ we have

$$|\varphi^a(x)| \leq C|x|^2 \quad \text{and} \quad |D\varphi^a(x)| \leq C|x|.$$

We extend the graph parametrization to a diffeomorphism

$$F^a : Z_{r_0} = D_{r_0} \times (-r_0, r_0) \to \mathbb{R}^3, \quad F^a(x, z) = f^a(x) + z N(a).$$

Using indices $i, j, k = 1, 2$ we compute for $	ilde{g}^a = (F^a)^*(\cdot, \cdot)_{\mathbb{R}^3}$

$$\partial_i F^a(x, z) = v_i(a) + \partial_i \varphi^a(x) N(a),$$
$$\partial_3 F^a(x, z) = N(a),$$
$$\tilde{g}^a_{ij}(x, z) = \delta_{ij} + \partial_i \varphi^a(x) \partial_j \varphi^a(x)$$
$$\tilde{g}^a_{33}(x, z) = \partial_3 \varphi^a(x)$$
$$\tilde{g}^a_3(x, z) = 1.$$

Next consider the dilations

$$\sigma_\lambda : \mathbb{R}^3 \to \mathbb{R}^3, \quad \sigma_\lambda(x, z) = (\lambda x, \lambda z) \text{ where } \lambda > 0.$$

Clearly $\sigma_\lambda(Z_R) \subset Z_{r_0}$ for $R \leq \frac{r_0}{\lambda}$. We obtain the Riemannian isometry

$$F^{a, \lambda} : (Z_{\frac{r_0}{\lambda}}^1, \lambda^2 \tilde{g}^{a, \lambda}) \to F^a(Z_{r_0}) \subset \mathbb{R}^3, \quad F^{a, \lambda}(x, z) = F^a(\lambda x, \lambda z),$$

where the metric $\tilde{g}^{a, \lambda}$ is given by

$$\tilde{g}^{a, \lambda} : Z_{\frac{r_0}{\lambda}}^1 \to \mathbb{R}^{3 \times 3}, \quad \tilde{g}^{a, \lambda}(x, z) = \lambda^{-2}(\sigma_\lambda)^* \tilde{g}^a(x, z) = \tilde{g}^a(\lambda x, \lambda z).$$

The metric satisfies, as a function of $(\lambda, x, z)$ for $a \in S$ fixed,

$$\tilde{g}^{a, \lambda}(x, z) \in C^{m-1}([0, \frac{r_0}{2}] \times Z_2, \mathbb{R}^{3 \times 3}) \quad \text{where} \quad \tilde{g}^{a, 0}_{ij} = \delta_{ij}.$$
Moreover the above expansions yield bounds, for a constant $C = C(\Omega)$,

\[
\|\tilde{g}^{a,\lambda}_{ij} - \delta_{ij}\|_{C^{m-1}(Z_2)} \leq C\lambda^2,
\]

\[
\|\tilde{g}^{a,\lambda}_{ij}\|_{C^{m-1}(Z_2)} \leq C\lambda,
\]

\[
\tilde{g}^{a,\lambda}_{33} - 1 \equiv 0.
\]

We compute more precisely

\[
q_{ij}(x, z) := \left. \frac{\partial}{\partial \lambda} \tilde{g}^{a,\lambda}_{ij}(x, z) \right|_{\lambda=0} = \begin{cases} 
0 & \text{for } 1 \leq i, j \leq 2 \\
h^S_{ik}(a)x^k & \text{for } i = 1, 2 \text{ and } j = 3 \\
0 & \text{for } i = j = 3.
\end{cases}
\]

Taylor expansion yields for $C = C(\Omega)$

\[
\|\tilde{g}^{a,\lambda} - (\delta_{ij} + \lambda q_{ij})\|_{C^{m-3}(Z_2)} \leq C\lambda^2 \quad \text{where } 0 \leq \lambda \leq \lambda_0(\Omega).
\]

**Lemma 9** For $q_{ij}(x, z)$ as in (3.8), we have the following formulae:

\[
\begin{align*}
\int_{S^2_+} q(\nu, \nu) d\omega &= \frac{\pi}{2} H^S(a), \\
\int_{S^2_+} \text{tr}_{S^2} q d\omega &= -\frac{\pi}{2} H^S(a), \\
\int_{S^2_+} \text{tr}_{S^2} \nabla_\nu q d\omega &= \frac{\pi}{2} H^S(a), \\
\int_{S^2_+} \text{tr}_{S^2} \nabla_\cdot q d\omega &= -\frac{\pi}{2} H^S(a), \\
\int_{\partial S^2_+} q(\nu, e_3) ds &= -\pi H^S(a).
\end{align*}
\]

**Proof.** We compute writing $\omega = (\sin \theta \xi, \cos \theta)$ for $\xi \in S^1$, $0 \leq \theta \leq \frac{\pi}{2}$

\[
\int_{S^2_+} q(\nu, \nu) d\omega = 2 \int_0^{\frac{\pi}{2}} \int_{S^1} h^S(a)(\xi, \xi) \sin^3 \theta \cos \theta d\xi d\theta = \frac{\pi}{2} H^S(a).
\]

Since $\text{tr} q = 0$ we get

\[
\int_{S^2_+} \text{tr}_{S^2} q d\omega = -\int_{S^2_+} q(\nu, \nu) d\omega = -\frac{\pi}{2} H^S(a).
\]

Differentiating the equation $q(t\omega) = tq(\omega)$ at $t = 1$, we get

\[
\int_{S^2_+} \text{tr}_{S^2} \nabla_\nu q d\omega = -\int_{S^2_+} \text{tr}_{S^2} q d\omega = \frac{\pi}{2} H^S(a).
\]
Now we compute using the definition of $q$

$$\operatorname{tr}_{S^2} \nabla \cdot q(\cdot, \nu) = \operatorname{tr}_{S^3} \nabla \cdot q(\cdot, \nu) - \nabla_{\nu} q(\nu, \nu) = \langle \nu, e_3 \rangle \sum_{i=1}^{2} \partial_i q_{i3} + q(\nu, \nu).$$

Using $\sum_{i=1}^{2} \partial_i q_{i3} = h_{11}^S(a) + h_{22}^S(a) = H^S(a)$ we see

$$\int_{S^2_+} \operatorname{tr}_{S^2} q(\cdot, \nu) \, d\omega = -\int_0^{\frac{\pi}{2}} \int_{S^1} H^S(a) \cos \theta \sin \theta \, d\xi d\theta + \frac{\pi}{2} H^S(a) = -\frac{\pi}{2} H^S(a).$$

Finally we compute the boundary integral

$$\int_{\partial S^2_+} q(\nu, e_3) \, ds = -\int_{S^1} h(a)(\xi, \xi) \, d\xi = -\pi H(a).$$

$\square$

For $l = m - 1 \geq 6$ and $0 \leq \lambda \leq \lambda_0(\Omega)$ the metric $\tilde{g}^{a,\lambda}$ belongs to the neighborhood $G$ of the standard metric as in Lemma 6. We put $w^{a,\lambda} = w[\tilde{g}^{a,\lambda}]$ and $q^{a,\lambda} = \tilde{g}^{a,\lambda} - \delta$. The Taylor expansion from Lemma 8 then yields

$$\left| W\left(w^{a,\lambda}, \tilde{g}^{a,\lambda}\right) - \frac{2\pi}{\lambda} - \int_{S^2_+} \left(-\frac{1}{2} \operatorname{tr}_{S^2} q^{a,\lambda} + q^{a,\lambda}(\nu, \nu) + \operatorname{tr}_{S^2} \nabla \cdot q(\cdot, \nu) - \frac{1}{2} \operatorname{tr}_{S^2} \nabla_{\nu} q \right) \, d\omega \right| + \int_{\partial S^2_+} q^{a,\lambda}(\nu, e_3) \, ds \leq C \left| q^{a,\lambda} \right| _{C^0(Z_2)}^2.$$

Now $\left| q^{a,\lambda} - \lambda q \right| _{C^{m-2,\alpha}(Z_2)} \leq C \lambda^2$ by (3.9), hence evaluating the integrals shows

$$\left| W\left(w^{a,\lambda}, \tilde{g}^{a,\lambda}\right) - \frac{2\pi}{\lambda} - \pi H(a) \right| \leq C \lambda \quad \text{where } C = C(\Omega). \quad (3.10)$$

Transforming back yields the following result, where by $\mathcal{M}^{k,\alpha}(S)$ we denote the set of $C^{k,\alpha}$ immersions of $S^2_+$ meeting $S$ orthogonally from inside along the boundary.

**Proposition 1** Let $\Omega$ be of class $C^m$ for $m \geq 7$, and $k := m - 2$. Then for $a \in S$ and $0 < \lambda \leq \lambda_0$ the $\phi^{a,\lambda}(\omega) = F^{a,\lambda}((1 + w^{a,\lambda}(\omega))\omega)$ belong to $\mathcal{M}^{k,\alpha}(S)$, have area $\mathcal{A}(\phi^{a,\lambda}) = 2\pi \lambda^2$, are centered at $a \in S$ and satisfy

$$\left| W(\phi^{a,\lambda}) - \frac{2\pi}{\lambda} + \pi H(a) \right| \leq C \lambda \quad \text{where } C = C(\Omega).$$

In particular we see that $\inf_{f \in \mathcal{M}(S)} W(f) < 2\pi$. 
Remark. Suppose that a sequence of immersions $f_k \in \mathcal{M}(S)$ satisfies

$$
(3.11) \quad \text{diam } f_k(D) \to 0, \quad A(f_k) \leq C, \quad L(f_k|_{\partial \Sigma}) \leq C.
$$

It is not difficult to show that then $\lim \inf_{k \to \infty} \mathcal{W}(f_k) \geq 2\pi$. Thus for a $\mathcal{W}$-minimizing sequence $f_k$ in $\mathcal{M}(S)$ one of the bounds in $(3.11)$ must be violated in view of Proposition 1. For $\Omega$ convex the length bound could in fact be dropped using the Gauß Bonnet theorem. Global bounds for the Willmore energy of surfaces with free boundary are proved in recent work by Volkmann [21].

In the following lemma we check how the constrained solutions transform when changing the orthonormal basis $v_1(a), v_2(a)$ used to identify $T_aS$ with $\mathbb{R}^2$.

**Lemma 10** Let $w^{a,\lambda}$ be the solution with respect to the basis $v_{1,2} = v_{1,2}(a)$ of $T_aS$, and let $T \in SO(2)$. Then the corresponding solution $w^{T,a,\lambda}$ with respect to the basis $v^T_j = T_{ij}v_i$ is given by

$$
w^{T,a,\lambda} = w^{a,\lambda} \circ T, \quad \text{where we identify } T \doteq \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}.
$$

In particular we have $\phi^{T,a,\lambda} = \phi^{a,\lambda} \circ T$.

**Proof.** We compute

$$
f^a(Tx) = a + (Tx)^1v_1 + (Tx)^2v_2 + \varphi(Tx)N(a) \\
= a + (T_{11}x^1 + T_{12}x^2)v_1 + (T_{21}x^1 + T_{22}x^2)v_2 + \varphi(Tx)N(a) \\
= a + x^1(T_{11}v_1 + T_{21}v_2) + x^2(T_{12}v_1 + T_{22}v_2) + \varphi(Tx)N(a).
$$

This shows $\varphi^{T,a}(x) = \varphi^a(Tx)$ and $f^{T,a}(x) = f^a(Tx)$ on $D_{r_0}$. It follows that $F^{T,a,\lambda}(x, z) = F^{a,\lambda}(Tx, z)$ and hence

$$
g^{T,a,\lambda} = \lambda^{-2}(F^{T,a,\lambda})^*(\cdot, \cdot) = \lambda^{-2}(F^{a,\lambda} \circ T)^*(\cdot, \cdot) = T^*g^{a,\lambda}.
$$

The boundary value problem $(2.11)$ is Riemannian invariant, that is

$$
B[w^{a,\lambda} \circ T, g^{T,a,\lambda}] = B[w^{a,\lambda}, g^{a,\lambda}] \circ T = 0, \\
Q^i[w^{a,\lambda} \circ T, g^{T,a,\lambda}] = Q^i[w^{a,\lambda}, g^{a,\lambda}] \circ T = 0, \quad \text{for } i = 1, 2, \\
Q^3[w^{a,\lambda} \circ T, g^{T,a,\lambda}] = Q^3[w^{a,\lambda}, g^{a,\lambda}] = 0, \\
Q^4[w^{a,\lambda} \circ T, g^{T,a,\lambda}] = T^{-1}Q^4[w^{a,\lambda}, g^{a,\lambda}] = 0.
$$

By uniqueness in Lemma 6 we conclude that $w^{T,a,\lambda} = w^{a,\lambda} \circ T$. \hfill \square
We now study the reduced energy function

\begin{equation}
\overline{W} : S \times [0, \lambda_0) \to \mathbb{R}, \quad \overline{W}(a, \lambda) = W(a, \lambda) = W(\phi^a, \lambda) = W(\omega^a, \tilde{g}^a).
\end{equation}

We already know that

\[ \overline{W}(a, 0) \equiv 2\pi, \quad \nabla_a \overline{W}(a, 0) \equiv 0 \quad \text{and} \quad \frac{\partial \overline{W}}{\partial \lambda}(a, 0) = -\pi H(a). \]

For further computations we assume w.l.o.g. that 0 \in S, \ N(0) = e_3, and chose an orthonormal frame \( v_1, v_2 \in C^{m-1}(U, \mathbb{R}^3) \) on a neighborhood \( U \subset S \) such that

\begin{equation}
Dv_i v_j (0) = h^S(0)(v_i, v_j) N^S(0).
\end{equation}

The \( v_i(a) \) can be obtained for instance by Gram-Schmidt applied to the coordinate vectors of the local graph representation. In order to have \( \overline{W} \) of class \( C^r \) for \( r \geq 1 \), we assume in the following that \( m = 6 + 2r \). Taking \( k = 4 \), one then checks that the map

\[ S \times [0, \lambda_0] \to C^{4+r}(Z_2, \mathbb{R}^{3 \times 3}), \quad (a, \lambda) \mapsto \tilde{g}^a, \]

is of class \( C^r \), which implies also \( \overline{W} \in C^r(S \times [0, \lambda_0]) \).

For example, for \( m = 10 \) we can take \( r = 2 \) and deduce

\begin{equation}
\frac{\partial}{\partial \lambda} \nabla_a \overline{W}(a, 0) = \nabla_a \frac{\partial \overline{W}}{\partial \lambda}(a, 0) = -\pi \nabla H(a).
\end{equation}

**Theorem 2** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain of class \( C^8 \). Put \( S = \partial \Omega \) and

\[ \mathcal{M}_{4, \lambda}(S) = \{ f \in \mathcal{M}_{4, \lambda}(S) : \mathcal{A}(f) = 2\pi \lambda^2 \}. \]

For any \( \lambda \in (0, \lambda_0] \) and \( a \in S \) the following are equivalent:

1. \( a \) is a critical point of \( \overline{W}(\cdot, \lambda) \).
2. \( \phi^a, \lambda \) is a critical point of the Willmore functional in \( \mathcal{M}_{4, \lambda}(S) \).
3. \( \phi^a, \lambda \) solves the boundary value problem
   \[ \Delta H + |A^o|^2 H = \alpha H \quad \text{for some} \ \alpha \in \mathbb{R}, \]
   \[ \frac{\partial f}{\partial \eta} = N^S \circ f \quad \text{along} \ \partial S^2_+ \]
   \[ \frac{\partial H}{\partial \eta} + h^S(\nu, \nu) H = 0 \quad \text{along} \ \partial S^2_+. \]

**Corollary 1** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain of class \( C^8 \). Then for any \( \lambda \in (0, \lambda_0] \) there exist two different critical points of the Willmore functional in \( \mathcal{M}_{4, \lambda}(S) \), corresponding to the extrema of the function \( \overline{W}(\cdot, \lambda) \).
Proof. From Proposition 1 we have for \(a_1, a_2 \in S\)
\[
\pi |H^S(a_1) - H^S(a_2)| \leq \frac{|\overline{W}(a_1, \lambda) - \overline{W}(a_2, \lambda)|}{\lambda} + C\lambda.
\]
If there is a sequence \(\lambda_k \searrow 0\) such that each function \(\overline{W}(\cdot, \lambda_k)\) is constant, then \(H^S\) must be constant and hence \(\Omega\) is a round ball by Alexandroff’s theorem. By symmetry we then have infinitely many critical points. On the other hand, if \(\overline{W}(\cdot, \lambda)\) is not constant, then it attains its extrema at different points \(a_1(\lambda), a_2(\lambda) \in S\). The surfaces \(\phi^{a_i(\lambda), \lambda}\) are then geometrically different, since the \(a_i(\lambda)\) are their barycenters.

As noted in [19, 23] the number of critical points is in fact bounded below by the Ljusternik-Shnirelman category of \(S\), which equals three if \(S\) is a surface of higher genus. We have also the following fact about the concentration points for \(\lambda \searrow 0\).

Corollary 2 Let \(\Omega \subset \mathbb{R}^3\) be a bounded domain of class \(C^{10}\), and assume that the \(\phi^{a_k, \lambda_k}\) are critical points of the Willmore functional in \(M_{\lambda_k}(S)\), where \(\lambda_k \to 0\) and \(a_k \to a \in S\). Then \(\nabla H^S(a) = 0\).

Proof. We have \(\nabla_a \overline{W}(a_k, \lambda_k) = 0\) by assumption. Using \(\nabla_a \overline{W}(a, 0) \equiv 0\) which follows from \(\overline{W}(a, 0) \equiv 2\pi\), we get
\[
0 = \frac{\nabla_a \overline{W}(a_k, \lambda_k) - \nabla_a \overline{W}(a_k, 0)}{\lambda_k} = \int_0^{\lambda_k} \frac{\partial}{\partial \lambda} \nabla_a \overline{W}(a_k, \lambda) d\lambda \to \frac{\partial}{\partial \lambda} \nabla_a \overline{W}(a, 0).
\]

Claim (1) follows from (3.14).

Proof of Theorem 2. For \(\lambda_0 > 0\) sufficiently small, we show that critical points of \(\overline{W}(\cdot, \lambda), \lambda \in (0, \lambda_0]\), correspond to critical points of the Willmore functional in \(M^{4, \alpha}(S)\). Consider the constrained solutions
\[
\phi^{a, \lambda}(\omega) = F^{a, \lambda}(\omega + w^{a, \lambda}(\omega)\omega), \quad \omega \in S_+^2.
\]

For fixed \(\lambda\) the family \(\phi^{a, \lambda}\) is a variation in \(M^{4, \alpha}(S)\). Now
\[
H[\phi^{a, \lambda}] = H[w^{a, \lambda}, (F^{a, \lambda})^*\langle \cdot, \cdot \rangle] = \lambda^{-1} H[w^{a, \lambda}, \tilde{g}^{a, \lambda}],
\]
\[
W[\phi^{a, \lambda}] = W[w^{a, \lambda}, (F^{a, \lambda})^*\langle \cdot, \cdot \rangle] = \lambda^{-3} W[w^{a, \lambda}, \tilde{g}^{a, \lambda}].
\]

Thus for \(\psi_i = \psi_i[w^{a, \lambda}, \tilde{g}^{a, \lambda}], i = 0, 1, 2\), as in (2.6) we have
\[
W[\phi^{a, \lambda}] \in \text{Span} \{\psi_0, \psi_1, \psi_2\}.
\]

We further have along \(\partial S_+^2\)
\[
\langle \nu[\phi^{a, \lambda}], N^S \circ \phi^{a, \lambda} \rangle = 0 \quad \text{and} \quad \left(\frac{\partial H}{\partial \eta} + h^S(\nu, \nu)H\right)[\phi^{a, \lambda}] = 0.
\]

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For the two-dimensional barycenter defined in Lemma 12 we see
\[ \pi_S \left( \int_{S^2} \phi^{a,\lambda} \, d\mu_{\phi^{a,\lambda}} \right) = F^{a,\lambda}(C[w^{a,\lambda}, \tilde{g}^{a,\lambda}]) = F^{a,\lambda}(0) = a. \]

This summarizes the conditions for constrained solutions. Next we study variations corresponding to the parameter \( a \).

Assume that \( 0 \in S, \ N^S(0) = e_3 \), is a critical point for the function \( \overline{W}^\lambda = \overline{W}(\cdot, \lambda) \). Choose an orthonormal frame \( v_{1,2}(a) \in C^7(U, \mathbb{R}^3) \) nearby, such that \( \nabla_S v_j(0) = (D_{v_j})^T(0) = 0 \). The map \( F^{a,\lambda} \) is given explicitly by
\[ F^{a,\lambda}(x, z) = a + \lambda x^a + (\varphi^0(\lambda x) + \lambda z) N^S(a), \quad \text{where} \quad x^a = x^1 v_1(a) + x^2 v_2(a). \]

Taking the derivative \( \frac{\partial}{\partial a} \) at \( a = 0 \) gives
\[ \frac{\partial F^{a,\lambda}}{\partial a}(x, z)|_{a=0} = e_i + (\varphi^0(\lambda x) + \lambda z) W^S(0) e_i + \lambda \frac{\partial x^a}{\partial a_i}(x)|_{a=0} + \frac{\partial \varphi^0}{\partial a_i}(\lambda x)|_{a=0} e_3. \]

We have \( \frac{\partial a_i}{\partial a}(0) = h_{ij}^S(0) e_3 \), thus
\[ \frac{\partial x^a}{\partial a_i}(x)|_{a=0} = (x^1 h_{1i}^S(0) + x^2 h_{2i}^S(0)) e_3. \]

Next we write \( \frac{\partial \varphi^a}{\partial a_i}|_{a=0} \) in terms of the graph function \( \varphi^{a=0} \), using the equation
\[ \langle F^{a,\lambda}(x, 0), e_3 \rangle = \varphi^0(\pi_{\mathbb{R}^2} F^{a,\lambda}(x, 0)). \]

The derivative \( \frac{\partial}{\partial a} \) yields at \( a = 0 \)
\[ \lambda(x^1 h_{1i}^S(0) + x^2 h_{2i}^S(0)) + \frac{\partial \varphi^a}{\partial a_i}|_{a=0} = \langle \nabla \varphi^0(\lambda x), e_i + \varphi^0(\lambda x) W^S(0) e_i \rangle. \]

Rearranging gives
\[ (3.15) \quad \frac{\partial \varphi^a}{\partial a_i}|_{a=0} = -\lambda(x^1 h_{1i}^S(0) + x^2 h_{2i}^S(0)) + (\delta_{ij} - \varphi^0(\lambda x) h_{ij}^S(0)) \partial_j \varphi^0(\lambda x). \]

Reinserting yields the formula
\[ (3.16) \quad \frac{\partial F^{a,\lambda}}{\partial a_i}(x, z)|_{a=0} = e_i - (\varphi^0(\lambda x) + \lambda z) h_{ij}^S(0) e_j + (\delta_{ij} - \varphi^0(\lambda x) h_{ij}^S(0)) \partial_j \varphi^0(\lambda x) e_3. \]

By the assumptions on \( \varphi^0 \) we have
\[ \frac{1}{\lambda} \| \varphi^0(\lambda x) \|_{C^8(B_2)} + \| D \varphi^0(\lambda x) \|_{C^7(B_2)} \leq C \lambda \quad \text{for} \quad \lambda \leq \frac{r_0}{2}. \]
This implies
\[ \left\| \frac{\partial F_{a_i}^\lambda}{\partial a^i} \right\|_{a=0} - e_i \right\|_{C^7(Z_2)} \leq C \lambda. \]

Now consider the \( \phi_{a_i}^\lambda = F_{a_i}^\lambda \circ f_{a_i}^\lambda \) where \( f_{a_i}^\lambda(\omega) = (1 + w_{a_i}^\lambda(\omega)) \omega \). We have
\[ \frac{\partial W}{\partial a_i}(0, \lambda) = \frac{\partial}{\partial a_i} W(\phi_{a_i}^\lambda)_{a=0} = DW(\phi_{0}^\lambda). \left( \frac{\partial F_{a_i}^\lambda}{\partial a^i} \right|_{a=0} + f_{a_i}^0 \circ D F_{0,\lambda} \frac{\partial f_{a_i}^\lambda}{\partial a^i} \right|_{a=0} \right). \]

We transform back to the reference chart, defining the vector field
\[ X_i : S^2_+ \to \mathbb{R}^3, \quad X_i(\omega) = \lambda D F_{0,\lambda}^{-1}(f_{0,\lambda}(\omega))^{-1} \cdot Y_i(\omega). \]

By Riemannian invariance, we then get
\[ D_w W(f_{0,\lambda}^0, g_{0,\lambda}^0) \cdot X_i = \lambda D W(\phi_{0}^\lambda) \cdot Y_i. \]

We want to show that \( X_i \approx e_i \) for sufficiently small \( \lambda > 0 \). From the definition \( F^0(x, z) = (x, z + \varphi^0(x)) \) we see \( (F^0)^{-1}(x, z) = (x, z - \varphi^0(x)) \), thus
\[ (DF^0)^{-1}(\lambda x, \lambda z) = D((F^0)^{-1})(F^0(\lambda x, \lambda z)) = \text{Id} - d \varphi^0(\lambda x) \otimes e_3. \]

Now \( \lambda D F_{0,\lambda}(x, z) = DF^0(\lambda x, \lambda z) \), which yields
\[ \lambda(D F_{0,\lambda})^{-1} \frac{\partial F_{a_i}^\lambda}{\partial a^i} \bigg|_{a=0} = e_i - (\varphi^0(\lambda x) + \lambda z) h_{ij}^S(0) e_j + (\delta_{ij} - \varphi^0(\lambda x) h_{ij}^S(0)) \partial_j \varphi^0(\lambda x) e_3 - (\partial_i \varphi^0(\lambda x) - (\varphi^0(\lambda x) + \lambda z) h_{ij}^S(0) \partial_j \varphi^0(\lambda x)) e_3 = e_i - (\varphi^0(\lambda x) + \lambda z) h_{ij}^S(0) e_j + \lambda z h_{ij}^S(0) \partial_j \varphi^0(\lambda x) e_3. \]

In particular
\[ (3.17) \quad \left\| \lambda(D F_{0,\lambda})^{-1} \frac{\partial F_{a_i}^\lambda}{\partial a^i} \bigg|_{a=0} - e_i \right\|_{C^7(Z_2)} \leq C \lambda. \]

The functions \( w_{a_i}^\lambda \) are defined as the solutions of the equation \( Q[w, g_{a_i}^\lambda] = 0 \), taking \( k = 4 \) in Lemma 6. From (2.12) we have the bound
\[ \|w^0,\lambda\|_{C^{4.0}(Z_2)} \leq C \|\tilde{g}^0,\lambda - \delta\|_{C^5(Z_2)} \leq C \lambda. \]

To estimate \( \frac{\partial w_{a_i}^\lambda}{\partial a^i} \bigg|_{a=0} \), we compute
\[ 0 = \frac{\partial}{\partial a^i} Q[w_{a_i}^\lambda, g_{a_i}^\lambda]_{a=0} = D_w Q[w_{a_i}^\lambda, g_{a_i}^\lambda] \cdot \frac{\partial w_{a_i}^\lambda}{\partial a^i} \bigg|_{a=0} + D_3 Q[w_{a_i}^\lambda, g_{a_i}^\lambda] \cdot \frac{\partial g_{a_i}^\lambda}{\partial a^i} \bigg|_{a=0}. \]
For $\lambda > 0$ sufficiently small we have $\|\tilde{g}^{0,\lambda} - \delta\|_{C^5(Z_2)}$ small and hence $\|w^{0,\lambda}\|_{C^4,\alpha(S^2_+)}$ small, so that $D_wQ[w^{0,\lambda}, \tilde{g}^{0,\lambda}]$ is close to the invertible Operator $L = D_wQ[0, \delta]$. Thus we can estimate
\[
\left\| \frac{\partial w^{a,\lambda}}{\partial a^i}\right\|_{a=0}\|_{C^4,\alpha(S^2_+)} \leq C \left\| D_{\tilde{g}}Q[w^{0,\lambda}, \tilde{g}^{0,\lambda}] \cdot \frac{\partial \tilde{g}^{a,\lambda}}{\partial a^i}\right\|_{a=0}\|_{C^0,\alpha(S^2_+) \times C^1,\alpha(S^2_+ \times \mathbb{R} \times \mathbb{R}^2)}
\leq C \left\| \frac{\partial \tilde{g}^{a,\lambda}}{\partial a^i}\right\|_{a=0}\|_{C^5(Z_2)}
\leq C \lambda.
\]

In the last estimate we used the definition of $\tilde{g}^{a,\lambda}$, the formula (3.15) and the $C^8$ bound on $\varphi^0$. For $f^{a,\lambda}(\omega) = (1 + w^{a,\lambda}(\omega))\omega$ we obtain
\[
\|f^{0,\lambda}(\omega) - \omega\|_{C^4,\alpha(S^2_+)} + \left| \frac{\partial f^{a,\lambda}}{\partial a^i}\right|_{a=0}\|_{C^4,\alpha(S^2_+)} \leq C \lambda.
\]

Now we have
\[
X_i(\omega) = \left( \lambda(DF^{0,\lambda})^{-1} \frac{\partial F^{a,\lambda}}{\partial a^i}\right)_{a=0}\|_{f^{0,\lambda}(\omega)} + \lambda \frac{\partial f^{a,\lambda}}{\partial a^i}\right|_{a=0}.
\]

Combining (3.17) and (3.18) we conclude
\[
\|X_i - e_i\|_{C^4,\alpha(S^2_+)} \leq C \lambda.
\]

Now write $Y_i = \varphi_i\nu^{\varphi,\lambda} + D\varphi^{0,\lambda}\tau_i$, and compute
\[
\left. \frac{\partial}{\partial a^i}W(\varphi^{a,\lambda})\right|_{a=0} = \frac{1}{2} \int_{S^2_+} W[\varphi^{0,\lambda}] \varphi_i d\mu_g + \frac{1}{2} \int_{S^2_+} \left( \varphi_i \frac{\partial H}{\partial \eta} - \varphi^{a,\lambda} H - \frac{1}{2} g(\tau_i, \eta) \right) ds_g.
\]

As $\varphi^{a,\lambda}(\partial S^2_+) \subset S$ the vector $Y_i(\omega)$ is tangent to $S$ at $\varphi^{0,\lambda}(\omega)$, for any $\omega \in \partial S^2_+$. Since $\frac{\partial \varphi^{0,\lambda}}{\partial \eta} = N^S \circ \varphi^{0,\lambda}$ we get $g(\tau_i, \eta) \equiv 0$ along $\partial S^2_+$. Furthermore
\[
\frac{\partial H}{\partial \eta} = -h^S(\nu, \nu) H \quad (\text{boundary condition for } \varphi^{a,\lambda})
\]
\[
\frac{\partial \varphi_i}{\partial \eta} = -h^S(\nu, \nu) \varphi_i \quad (\text{admissibility as in (1.17)}).
\]

Thus all boundary terms cancel and we get putting $\xi_i = \tilde{g}^{0,\lambda}(X_i, \nu^{\varphi,\lambda})$
\[
\left. \frac{\partial}{\partial a^i}W(\varphi^{a,\lambda})\right|_{a=0} = \frac{1}{2} \int_{S^2_+} (\tilde{W}[\varphi^{0,\lambda}], Y_i) d\mu_{\varphi^{0,\lambda}}
= \frac{\lambda^3}{2} \int_{S^2_+} \tilde{g}^{0,\lambda}(\tilde{W}[f^{0,\lambda}, g^{0,\lambda}], X_i) d\mu_{f^{0,\lambda}}
= \frac{\lambda^3}{2} \int_{S^2_+} W[f^{0,\lambda}, \tilde{g}^{0,\lambda}] \xi_i d\mu_{f^{0,\lambda}}.
\]
The first variation formula for the area yields
\[
\frac{\partial}{\partial a_i} \mathcal{A}(\phi^{a,\lambda})|_{a=0} = \int_{S^2_+} H[\phi^{0,\lambda}] \varphi_i \, d\mu_{\phi^{0,\lambda}} + \int_{\partial S^2_+} g(\tau_i, \eta) \, ds_{\phi^{0,\lambda}}.
\]
Since \(g(\tau_i, \eta) \equiv 0\) along \(\partial S^2_+\), we get by transforming the integral
\[
\frac{\partial}{\partial a_i} \mathcal{A}(\phi^{a,\lambda})|_{a=0} = \lambda \int_{S^2_+} H[f^{0,\lambda}, \tilde{g}^{0,\lambda}] \xi_i \, d\mu_{f^{0,\lambda}}.
\]
But \(\mathcal{A}(\phi^{a,\lambda}) \equiv 2\pi \lambda^2\) for all \(a \in S\), therefore we have
\[
\int_{S^2_+} H[f^{0,\lambda}, \tilde{g}^{0,\lambda}] \xi_i \, d\mu_{f^{0,\lambda}} = 0 \quad \text{for } i = 1, 2.
\]
Now if \(0 \in S\) is a critical point for \(\bar{W}(\cdot, \lambda)\), then we also get
\[
\int_{S^2_+} W[f^{0,\lambda}, \tilde{g}^{0,\lambda}] \xi_i \, d\mu_{f^{0,\lambda}} = 0 \quad \text{for } i = 1, 2.
\]
By construction there exist \(\alpha, \beta_{1,2} \in \mathbb{R}\) such that for \(\psi_i = \psi_i[w^{0,\lambda}, \tilde{g}^{0,\lambda}]\) as in (2.6)
\[
W[f^{0,\lambda}, \tilde{g}^{0,\lambda}] = \alpha \psi_0 + \beta_i \psi_i.
\]
With respect to the metric \(g^{0,\lambda} = (f^{0,\lambda})^* \tilde{g}^{0,\lambda}\), the functions \(\xi_i\) are \(L^2\)-orthogonal to both \(W[f^{0,\lambda}, \tilde{g}^{0,\lambda}]\) and \(H[f^{0,\lambda}, \tilde{g}^{0,\lambda}]\). This yields
\[
0 = \langle \xi_i, W[f^{0,\lambda}, \tilde{g}^{0,\lambda}] \rangle_{L^2(S^2_+, g^{0,\lambda})} = \sum_{j=1}^2 \langle \xi_i, \psi_j \rangle_{L^2(S^2_+, g^{0,\lambda})} \beta_j \quad \text{for } i = 1, 2.
\]
From (3.18), (3.19) we have \(\|\xi_i - \langle \omega, e_i \rangle\|_{C^0(S^2)} \leq C \lambda\). Recalling that \(\psi_i = \sqrt{\frac{3}{2\pi}} \langle \omega, e_i \rangle\) for \(w = 0, \tilde{g} = 0\), we conclude
\[
\left| \langle \xi_i, \psi_j \rangle_{L^2(S^2_+, g^{0,\lambda})} - \sqrt{\frac{2\pi}{3}} \delta_{ij} \right| \leq C \lambda.
\]
This implies \(\beta_1 = \beta_2 = 0\) for \(\lambda \leq \lambda_0 = \lambda_0(\Omega)\), and we conclude \(W[\phi^{0,\lambda}] = \alpha H[\phi^{0,\lambda}]\) as claimed.

For the reverse implication assume that \(\phi^{0,\lambda}\) is critical for the Willmore functional in \(\mathcal{M}_{\lambda, \alpha}(S)\), i.e. \(W[\phi^{0,\lambda}] = \alpha H[\phi^{0,\lambda}]\) for some \(\alpha \in \mathbb{R}\). Then we compute
\[
\frac{\partial}{\partial a^i} W(\phi^{a,\lambda})|_{a=0} = \langle \tilde{W}[\phi^{0,\lambda}], \frac{\partial \phi^{a,\lambda}}{\partial a^i} \rangle_{a=0} L^2
\]
\[
= \alpha \langle \tilde{H}[\phi^{0,\lambda}], \frac{\partial \phi^{a,\lambda}}{\partial a^i} \rangle_{a=0} L^2
\]
\[
= \frac{\partial}{\partial a^i} \mathcal{A}(\phi^{a,\lambda})|_{a=0}
\]
\[
= 0.
\]
Hence $a = 0$ is a critical point of $W(\cdot, \lambda)$, which finishes the proof of the theorem. $\square$

We finally prove a purely local existence result.

**Theorem 3** Let $\Omega$ be a bounded domain of class $C^{12}$. If $a \in S = \partial \Omega$ is a nondegenerate critical point of $H^S$, then there exists a $C^1$ curve $\gamma(\lambda) \in S$ for $\lambda \in [0, \lambda_0)$, such that $\gamma(0) = a$ and each $\phi^{\gamma(\lambda) \cdot \lambda}$, $\lambda > 0$, is a critical point of $W(f)$ in $\mathcal{M}^4_{\lambda}(S)$.

We need the following calculus lemma.

**Lemma 11** Let $u \in C^2(S \times (-\lambda_0, \lambda_0))$ be a given function satisfying $u(\cdot, 0) \equiv 0$, and let $v : S \times (-\lambda_0, \lambda_0) \to \mathbb{R}$ be defined by

$$v(a, \lambda) = \begin{cases} \lambda^{-1}u(a, \lambda) & \text{for } \lambda \neq 0, \\ \partial_\lambda u(a, 0) & \text{for } \lambda = 0, \end{cases}$$

Then $v$ is of class $C^1(S \times (-\lambda_0, \lambda_0))$, having the derivatives

$$\nabla v(a, \lambda) = \begin{cases} \lambda^{-1}\nabla u(a, \lambda) & \text{for } \lambda \neq 0, \\ \partial_\lambda \nabla u(a, 0) & \text{for } \lambda = 0, \end{cases}$$

$$\partial_\lambda v(a, \lambda) = \begin{cases} \lambda^{-2}(\lambda \partial_\lambda u(a, \lambda) - u(a, \lambda)) & \text{for } \lambda \neq 0, \\ \frac{1}{2} \partial_\lambda^2 u(a, 0) & \text{for } \lambda = 0. \end{cases}$$

**Proof.** We have using $u(x, 0) = 0$

$$|\lambda^{-1}u(x, \lambda) - \partial_\lambda u(a, 0)| = \left| \int_0^\lambda (\partial_\lambda u(x, s) - \partial_\lambda u(a, 0)) \, ds \right| \to 0 \quad \text{for } x \to a, \lambda \to 0.$$ 

This shows that $v$ is continuous. For the $C^1$ property it is sufficient to prove that the stated derivatives are also continuous. In the case of $\nabla v$ the argument above applies (noting that $\nabla u$ is $C^1$ by assumption). For $\partial_\lambda v$ we compute

$$\lambda^{-2}(\lambda \partial_\lambda u(x, \lambda) - u(x, \lambda)) - \frac{1}{2} \partial_\lambda^2 u(a, 0)$$

$$= \lambda^{-2} \int_0^\lambda (\partial_\lambda u(x, \lambda) - \partial_\lambda u(x, t)) \, dt - \frac{1}{2} \partial_\lambda^2 u(a, 0)$$

$$= \lambda^{-2} \int_0^\lambda \int_t^\lambda (\partial_\lambda^2 u(x, s) - \partial_\lambda^2 u(a, 0)) \, ds \, dt$$

$$\to 0 \quad \text{for } a \to x, \lambda \to 0.$$ 

The lemma is proved. $\square$
Proof of Theorem. We apply the lemma to the function \( u(a, \lambda) = \nabla \bar{W}(a, \lambda) \), where \( \bar{W}(\cdot, 0) = 2\pi \) and hence \( u(a, 0) = \nabla \bar{W}(a, 0) \equiv 0 \). This needs \( \bar{W} \in C^3(S \times (-\lambda_0, \lambda_0)) \), which is true for \( \Omega \in C^{12} \). We obtain from (3.14) (taking one more derivative \( \nabla_a \))

\[
\begin{align*}
v(a, 0) &= \partial_3 \nabla \bar{W}(a, 0) = -\pi \nabla S(a) \\
\nabla v(a, 0) &= \partial_3 \bar{W}(a, 0) = -\pi \nabla^2 S(a).
\end{align*}
\]

Now assume for \( 0 \in S \) that \( \nabla S(a) = 0 \) and \( \nabla^2 S(a) \) nondegenerate. Then the implicit function theorem, applied to \( v(a, \lambda) \), yields a neighborhood \( U \times (-\varepsilon, \varepsilon) \) and a \( C^1 \)-curve \( a = \gamma(\lambda) \), such that for \( (a, \lambda) \in U \times (-\varepsilon, \varepsilon) \) one has

\[
v(a, \lambda) = 0 \iff a = \gamma(\lambda).
\]

For \( \lambda \neq 0 \) we thus get

\[
\nabla \bar{W}(a, \lambda) = 0 \iff a = \gamma(\lambda).
\]

The theorem now follows from Theorem 2.

\[\Box\]

4 Appendix: Construction of the barycenter

The concept of Riemannian barycenter is due to Karcher [11]. For our purposes we only need a local version, which does not involve e.g. Riemannian comparison theory. Let \( U = D_3(0) \subset \mathbb{R}^2 \), \( V = B_{\frac{3}{2}}(0) \subset \mathbb{R}^3 \). For \( x \in U \), \( v \in V \) we put

\[
c_{x,v} : [0, 1] \to Z_2, \quad c_{x,v}(t) = x + tv.
\]

Further let \( X = \{ \phi \in C^2([0, 1], \mathbb{R}^3) : \phi(0) = \phi'(0) = 0 \} \) and

\[
X_\varepsilon = \{ \phi \in X : \| \phi \|_{C^0([0, 1])} < \varepsilon \}.
\]

We finally put \( G_\varepsilon = \{ \tilde{g} \in C^l(Z_2, \mathbb{R}^{3 \times 3}) : \| \tilde{g} - \delta \|_{C^l(Z_2)} < \varepsilon \} \) for \( l \geq 1 \), and consider

\[
F : U \times V \times X_\varepsilon \times G_\varepsilon \to C^0([0, 1], \mathbb{R}^3), \quad F[x, v, \phi, \tilde{g}] = \phi'' + \tilde{\Gamma} \circ c(c', c')|_{c = c_{x,v} + \phi}.
\]

We claim that \( F \) is of class \( C^{l-1} \). Write \( F = F_2 \circ F_1 \) where \( F_1 \) is the affine map

\[
F_1 : U \times V \times X_\varepsilon \to C^2([0, 1], \mathbb{R}^3), \quad F_1[x, v, \phi] = c_{x,v} + \phi.
\]

\( F_1 \) is continuous and hence smooth. The nonlinear map \( F_2 \) is given by

\[
F_2 : C^2([0, 1], Z_2) \times G_\varepsilon \to C^0([0, 1], \mathbb{R}^3), \quad F_2[c, \tilde{g}] = \phi'' + \tilde{\Gamma} \circ c(c', c').
\]

The composition \( C^2 \times C^{l-1} \to C^0, (c, \tilde{\Gamma}) \mapsto \tilde{\Gamma} \circ c \), is of class \( C^{l-1} \). Namely differentiating \( l - 1 \) times with respect to \( c \) leaves exactly a \( C^0 \) function. Since we can build \( F_2 \) from
\( \tilde{\Gamma} \circ c \) by linear or bilinear operations, it is also of class \( C^{l-1} \). Assuming from now on \( l \geq 2 \), we have

\[
D_c F_2[c, \tilde{g}] f = \phi'' + 2 \tilde{\Gamma} \circ c(\phi', c') + (D \tilde{\Gamma}) \circ c(\phi, c', c')
\]

\[
D_{\tilde{g}} F_2[c, \tilde{g}] h = \frac{1}{2} \left( \tilde{g}^{kp}(\partial_i h_{jp} + \partial_j h_{ip} - \partial_p h_{ij}) \right) \circ c(c', c')' e_k
\]

\[
- \frac{1}{2} \left( \tilde{g}^{km} h_{mn} \tilde{g}^{np} (\partial_i \tilde{g}_{jp} + \partial_j \tilde{g}_{ip} - \partial_p \tilde{g}_{ij}) \right) \circ c(c', c')' e_k.
\]

In particular

\[
F[x, v, 0, \delta] = 0 \quad \text{and} \quad D_\phi F[x, v, 0, \delta] \psi = \psi''.
\]

The map \( D_\phi F[x, v, 0, \delta] : X \to C^0([0, 1], \mathbb{R}^3) \) is an isomorphism, in fact the equation \( \psi'' = f \) has the unique solution \( \psi \in X \) given by

\[
\psi(u) = \int_0^u \int_0^s f(t) dt \, ds.
\]

By the implicit function theorem, the set of solutions of \( F[x, v, \phi, \tilde{g}] = 0 \) near \([0, v_0, 0, \delta]\) is given as a \( C^{l-1} \) graph

\[
\phi = \phi[x, v, \tilde{g}],
\]

i.e. the corresponding curves \( c[x, v, \tilde{g}] = c_{x,v} + \phi[x, v, \tilde{g}] \) are geodesics with respect to \( \tilde{g} \) having initial data \( c(0) = x, c'(0) = v \). The exponential mapping is now given by

\[
\exp : U \times V \times G_\varepsilon \to Z_2, \quad \exp^\tilde{g}_x(v) = c[x, v, \tilde{g}](1).
\]

Now \( D \exp^\tilde{g}_x = \text{Id}_{\mathbb{R}^3} \), Thus for \( l \geq 2 \) and \( \varepsilon > 0 \) small we get

\[
\|D \exp^\tilde{g}_x - \text{Id}_{\mathbb{R}^3}\|_{C^0(V)} \leq \varepsilon_0 \quad \text{for} \quad \tilde{g} \in G_\varepsilon, \ x \in U = D_\theta(0).
\]

This gives for \( v, w \in V \)

\[
|\exp^\tilde{g}_x(v) - \exp^\tilde{g}_x(w)| = \left| \int_0^1 D \exp^\tilde{g}_x((1-t)v + tw) \cdot (v - w) \, dt \right|
\]

\[
\geq |v - w| - \left| \int_0^1 (D \exp^\tilde{g}_x((1-t)v + tw) - \text{Id}_{\mathbb{R}^3}) \cdot (v - w) \, dt \right|
\]

\[
\geq (1 - \varepsilon_0)|v - w|.
\]

This shows that \( \exp^\tilde{g}_x \) is injective on \( V = B^\frac{\varepsilon}{2}(0) \). We further estimate

\[
|\exp^\tilde{g}_x(v) - v| = \left| \int_0^1 (D \exp^\tilde{g}_x(tv) - \text{Id}_{\mathbb{R}^3}) \, dt \cdot v \right| \leq \varepsilon_0 |v|.
\]

We now show that \( \exp^\tilde{g}_x(V) \cap B^\frac{\varepsilon}{4}(0) \) is a closed subset of \( B^\frac{\varepsilon}{4}(0) \). Assume that \( \exp^\tilde{g}_x(v_k) \to p \in B^\frac{\varepsilon}{4}(0) \). From the above we then have

\[
|v_k| - \frac{5}{4} < |v_k| - |\exp^\tilde{g}_x(v_k)| \leq |v_k - \exp^\tilde{g}_x(v_k)| \leq \varepsilon_0 |v_k|,
\]

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which implies \(|v_k| \leq (1 - \varepsilon_0)^{-1} \frac{5}{7} < \frac{3}{7}\) for appropriate \(\varepsilon_0 > 0\). Up to a subsequence, we thus have \(v_k \to v \in V\) and \(\exp^{\tilde{g}}_x(v) = p\). Now \(\exp^{\tilde{g}}_x(V) \cap B_{\frac{4}{3}}(0)\) is also open by the inverse function theorem, hence we have \(B_{\frac{4}{3}}(0) \subset \exp^{\tilde{g}}_x(V)\), and we obtain the inverse

\[(\exp^{\tilde{g}}_x)^{-1} : B_{\frac{4}{3}}(0) \to V.\]

Of course we are not claiming that \(\exp^{\tilde{g}}_x\) maps all of \(V\) into \(B_{\frac{4}{3}}(0)\). The inverse is of class \(C^{l-1}\) in all variables \(x \in U, p \in B_{\frac{4}{3}}(0)\) and \(\tilde{g} \in C^l(Z_2)\). Namely let \(\exp^{\tilde{g}_0}_x(v_0) = p_0 \in B_{\frac{4}{3}}(0)\), where \(v_0 \in V\). Consider the equation

\[\exp^{\tilde{g}}_x(v) - p = 0.\]

By the implicit function theorem, the set of solutions has a local representation \(v = v[x, p, \tilde{g}]\) which is of class \(C^{l-1}\). But the local inverse equals the global inverse, and hence also the global inverse is of class \(C^{l-1}\) as claimed.

**Lemma 12 (two-dimensional barycenter)** Assume \(w : \mathbb{S}^2_+ \to \mathbb{R}, \tilde{g} : Z_2 \to \mathbb{R}^{3 \times 3}\) belong to the neighborhoods \(W_\varepsilon, G_\varepsilon\) given by

\[\|w\|_{C^1(\mathbb{S}^2_+)} < \varepsilon \quad \text{and} \quad \|\tilde{g} - \delta\|_{C^l(Z_2)} < \varepsilon \quad \text{where} \quad l \geq 2.\]

For \(\varepsilon > 0\) small we then have a welldefined function

\[X[w, \tilde{g}] : U \to \mathbb{R}^2, X[w, \tilde{g}](x) = -\pi_{\mathbb{R}^2} \left( \int_{\mathbb{S}^2_+} (\exp^{\tilde{g}}_x)^{-1}(f(\omega)) \, d\mu_{\tilde{g}}(\omega) \right),\]

and there is a unique point \(x \in U\) with \(X[w, \tilde{g}](x) = 0\). This point \(x = C[w, \tilde{g}]\) is called the two-dimensional barycenter of \((\text{the radial graph of})\) \(w\) with respect to \(\tilde{g}\). The map \(C[w, \tilde{g}]\) is of class \(C^{l-1}\).

**Proof.** Let \(f(\omega) = \omega + w(\omega)\). Fixing a coordinate system on \(\mathbb{S}^2_+\), we consider the map

\[(4.1) \quad U \times W_\varepsilon \times G_\varepsilon \to C^0(\mathbb{S}^2_+, \mathbb{R}^3), [x, w, \tilde{g}] \mapsto (\exp^{\tilde{g}}_x)^{-1} \circ f \sqrt{\det g}.\]

By standard rules for product and composition, the right hand side belongs to \(C^0(\mathbb{S}^2_+, \mathbb{R}^3)\); in particular \(X[w, \tilde{g}]\) is well-defined. We claim that the map \((4.1)\) is of class \(C^{l-1}\) in all three variables. For this we recall that \(\Psi[x, p, \tilde{g}] = (\exp^{\tilde{g}}_x)^{-1}(p)\) is of class \(C^{l-1}\). For \(\omega \in \mathbb{S}^2_+\) fixed we have the \(C^{l-1}\) composition

\[U \times W_\varepsilon \times G_\varepsilon \xrightarrow{C^\infty} U \times B_{\frac{4}{3}}(0) \times G_\varepsilon \xrightarrow{\Psi} V \xrightarrow{(x, \tilde{g}) \mapsto (x, f(\omega), \tilde{g})} \Psi[x, f(\omega), \tilde{g}].\]

Now all derivatives with respect to \(x, w, \tilde{g}\) up to order \(l - 1\) depend also continuously on \(\omega\), which yields the claim. For \(\tilde{g} = \delta\) we have \((\exp^{\delta}_x)^{-1}(p) = p - x\) which implies

\[X[w, \delta](x) = \mu_{\delta}(\mathbb{S}^2_+) \left( x - \pi_{\mathbb{R}^2} \int_{\mathbb{S}^2_+} f(\omega) \, d\mu_{\delta}(\omega) \right),\]
By definition of the barycenter we have \( X[0, \delta](0) = 0 \) and \( D_x X[0, \delta](x) = 2\pi \text{Id}_{\mathbb{R}^2} \). Thus by the implicit function theorem there is a unique point \( x \in U \) with \( X[w, \tilde{g}](x) = 0 \), and the resulting map \( x = C[w, \tilde{g}] \) is of class \( C^{l-1} \).

From the proof we note the explicit formula

\[
(4.2) \quad C[w, \delta] = \pi_{\mathbb{R}^2} \left( \int_{S^2_+} f(\omega) \, d\mu_g(\omega) \right).
\]

We consider the two coordinates \( C^i[f, \tilde{g}] \) of the barycenter as functionals depending on \( w \) resp. \( f \), and we now compute the corresponding \( L^2 \) gradient. Consider a compactly supported variation of \( f \) in direction \( \phi = \varphi \nu \). Then we have

\[
\frac{\partial}{\partial \varepsilon} (g_{ij})|_{\varepsilon=0} = -2\varphi h_{ij} \quad \text{and} \quad \frac{\partial}{\partial \varepsilon} d\mu_{g_\varepsilon}|_{\varepsilon=0} = -\varphi H \, d\mu_g.
\]

The first variation of \( X[f, \tilde{g}] \) is then

\[
\frac{\partial}{\partial \varepsilon} X[f_\varepsilon, \tilde{g}](x)|_{\varepsilon=0} = -\pi_{\mathbb{R}^2} \int_{S^2_+} D((\exp_{\varepsilon}^{\tilde{g}})^{-1})(f(\omega)) \cdot \phi(\omega) \, d\mu_g(\omega)
\]

\[
+ \pi_{\mathbb{R}^2} \int_{S^2_+} (\exp_{\varepsilon}^{\tilde{g}})^{-1}(f(\omega))H(\omega)\varphi(\omega) \, d\mu_g(\omega).
\]

By definition of the barycenter we have

\[
0 = \frac{\partial}{\partial \varepsilon} X[f_\varepsilon, \tilde{g}](C[f_\varepsilon, \tilde{g}])|_{\varepsilon=0}
\]

\[
= -\pi_{\mathbb{R}^2} \int_{S^2_+} D((\exp_{\varepsilon}^{\tilde{g}})^{-1})(f(\omega)) \phi(\omega) \, d\mu_g(\omega)|_{x=C[f, \tilde{g}]}
\]

\[
+ \pi_{\mathbb{R}^2} \int_{S^2_+} (\exp_{\varepsilon}^{\tilde{g}})^{-1}(f(\omega))H(\omega)\varphi(\omega) \, d\mu_g(\omega)|_{x=C[f, \tilde{g}]}
\]

\[
- \pi_{\mathbb{R}^2} \int_{S^2_+} D_x(\exp_{\varepsilon}^{\tilde{g}})^{-1}(f(\omega)) \, d\mu_g(\omega)|_{x=C[f, \tilde{g}]} \cdot \frac{\partial}{\partial \varepsilon} C[f_\varepsilon, \tilde{g}]|_{\varepsilon=0}.
\]

This implies the formula

\[
\frac{\partial}{\partial \varepsilon} C[f_\varepsilon, \tilde{g}]|_{\varepsilon=0} = \left( \pi_{\mathbb{R}^2} \int_{S^2_+} D_x(\exp_{\varepsilon}^{\tilde{g}})^{-1}(f(\omega)) \, d\mu_g(\omega) \right)^{-1}|_{x=C[f, \tilde{g}]}
\]

\[
\cdot \left( -\pi_{\mathbb{R}^2} \int_{S^2_+} D((\exp_{\varepsilon}^{\tilde{g}})^{-1})(f(\omega)) \phi(\omega) \, d\mu_g(\omega)|_{x=C[f, \tilde{g}]}
\]

\[
+ \pi_{\mathbb{R}^2} \int_{S^2_+} (\exp_{\varepsilon}^{\tilde{g}})^{-1}(f(\omega))H(\omega)\varphi(\omega) \, d\mu_g(\omega)|_{x=C[f, \tilde{g}]}. \]
Under reparametrizations of $f$ the barycenter remains the same, hence the $L^2$ gradient of $C^i[f, \tilde{g}]$ is normal along $f$. Taking the $\tilde{g}$ inner product with $\nu$ yields a scalar function, which we denote by $\text{grad}_{L^2} C^i[w, \tilde{g}]$ in slight abuse of notation. We now conclude

$$
\sum_{i=1}^2 \text{grad}_{L^2} C^i[f, \tilde{g}] e_i = \left( \pi_{\mathbb{R}^2} \int_{S^2_+} D_x (\exp_{\tilde{x}})^{-1}(f(\omega)) \, d\mu_{\tilde{g}}(\omega) \right) |_{x=C[f,\tilde{g}]}^{-1} \cdot \pi_{\mathbb{R}^2} \left( (\exp_{\tilde{x}})^{-1}(f) H - \left( (\exp_{\tilde{x}})^{-1}(f) \nu \right) \right) |_{x=C[f,\tilde{g}]}.
$$

(4.3)

In the Euclidean case $\tilde{g} = \delta$ we have $\exp_{x} v = x + v$, which yields for $i = 1, 2$

$$
\pi_{\mathbb{R}^2} \int_{S^2_+} D_x (\exp_x)^{-1}(f(\omega)) \, d\mu_{\tilde{g}}(\omega) = -\mu_{\tilde{g}}(S^2_+) \text{Id}_{\mathbb{R}^2},
$$

$$
\text{grad}_{L^2} C^i[f, \delta] = \frac{1}{\mu_{\tilde{g}}(S^2_+)} \langle \nu - (f - C[f, \delta])H, e_i \rangle.
$$

Specializing further to $f_0(\omega) = \omega$, we see

$$
\text{grad}_{L^2} C^i[f_0, \delta](\omega) = -\frac{3}{2\pi} \langle \omega, e_i \rangle.
$$

(4.4)

For $w \in C^{k,\alpha}(S^2_+)$ and $\tilde{g} \in C^l(\mathbb{Z}_2, \mathbb{R}^{3\times3})$ where $l \geq k + 1$, one deduces $\text{grad}_{L^2} C^i[w, \tilde{g}] \in C^{k-2,\alpha}(S^2_+)$. Moreover as a functional into $C^{k-4,\alpha}(S^2_+)$, it is of class $C^{l-k+1}$.

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