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Output feedback stable stochastic predictive control with hard control constraints

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Abstract—We present a stochastic predictive controller for discrete time linear time invariant systems under incomplete state information. Our approach is based on a suitable choice of control policies, stability constraints, and employment of a Kalman filter to estimate the states of the system from incomplete and corrupt observations. We demonstrate that this approach yields a computationally tractable problem that should be solved online periodically, and that the resulting closed loop system is mean-square bounded for any positive bound on the control actions. Our results allow one to tackle the largest class of linear time invariant systems known to be amenable to stochastic stabilization under bounded control actions via output feedback stochastic predictive control.

Index Terms—Stochastic predictive control, Kalman filter, constrained control.

I. Introduction

Optimization based control techniques have received tremendous attention because of their wide applicability, tractability, and capability to handle a variety of constraints at the synthesis stage while minimizing some performance objective. Stochastic predictive control (SPC) is an optimization based control technique where the actions are obtained by solving a finite horizon (of length say \( N_r \)) optimal control problem at each sampling instant involving an expected cost given the current state of the plant, where the system dynamics is affected by stochastic uncertainties. The underlying optimization problem yields a control policy [1, §3.4], [2]. Receding horizon implementation of SPC then consists of solving the optimization problem every recalculation interval of length \( N_r \), \( (N_r \leq N) \); the first \( N_r \) controls are applied to the plant, the rest are discarded, and the procedure repeated.

This article addresses SPC under output feedback and hard constraints on the control inputs. While there is a growing body of work on SPC with hard control constraints under perfect state information, the output feedback (partial measurements) case is more difficult, but perhaps more important. In most practical applications the entire set of system states cannot be measured, and several predictive control schemes for systems with incomplete state information have been proposed in [3]–[7]. In [3], [4], all the uncertainties are assumed to be bounded, and robust control techniques were adopted. In [5], [6], the process noise is considered to be unbounded but stability and recursive feasibility have not been addressed under hard bounds on the control inputs. Since optimization over feedback policies gives, in general, better control performance than that over open loop input sequences [8], performing SPC with policies is recommended. Adopting this approach, when perfect state information is available, saturated disturbance feedback policies were utilized in [9] for SPC. A full solution to the unconstrained SPC problem under output feedback was first provided in [10] by proposing innovation feedback, but this work did not involve hard control bounds.\(^1\) By generalizing the saturated disturbance feedback approach of [9], a Kalman filter was utilized in [7] under output feedback and bounded controls. Stability of linear systems under Gaussian noise is impossible to ensure with bounded controls, if the spectral radius of the system matrix is greater than unity. The bordering case of the system with a Lyapunov stable (but not asymptotically stable) system matrix is, in general, difficult to analyse. In [9], the property of stability of the closed-loop system was proved in the presence of large enough control authority and a Lyapunov stable system matrix. In other words, the optimal control problem in [7] turns out to be feasible only when the hard bound on the control is larger than a threshold that is a function of various statistical objects and the system data. This requirement severely restricts practical applicability of the controller in [7], and to overcome this restriction, here we formulate an alternative controller.

Since stochastic optimal control problems are generally not tractable, we follow the affine saturated innovation feedback policy approach of [7] to get a tractable deterministic surrogate of the underlying stochastic optimal control problem, and employ globally feasible drift conditions to ensure stochastic stability for any positive bound on control.\(^2\) Beyond enlarging the applicability of the main result of [7], in the current work we present a recursively feasible convex quadratic program (QP) to be solved periodically online as part of the SPC algorithm. From an algorithmic perspective, the second order cone program of [7] is replaced by a QP, which has significant numerical advantages [13].

The remainder of this article is organized as follows: In §II we establish the definitions of the plant and its properties. Important ingredients of SPC under output feedback are described in §III. We discuss stability issues in §IV. In §V we provide our main result that is validated via numerical experiments in §VI. We conclude in §VII and present our proofs in the Appendix.

\(^1\)Recall that the innovation sequence is a quantity in the measurement update equation of Kalman filter, found by the difference between the measured output and the estimated output obtained from the estimated states [11, page no. 130].

\(^2\)Recall that drift conditions [12] relate the values of Lyapunov like functions with their conditional expectations, given the current state, at the next stage of the underlying process.
Let \( R, N, \) and \( N^* \) denote the set of real numbers, the non-negative integers, and the positive integers, respectively. \( I_d \) is the \( d \times d \) identity matrix and \( 0 \) is the matrix of appropriate dimension with 0 entries. For given \( \zeta \) and \( r \), we define the component-wise saturation function \( \text{sat}_{r, \zeta}^{(i)}(z) = \begin{cases} \frac{z(i)}{\zeta} & \text{if } |z(i)| \leq r, \\ \zeta & \text{if } |z(i)| > r, \text{ and} \\ -\zeta & \text{otherwise,} \end{cases} \) for each \( i = 1, \ldots, v \). For any sequence \( (s_n)_{n \in \mathbb{N}} \) taking values in some Euclidean space, we denote by \( s_{n, k} \) the vector \( \left[ s_n^T \ s_{n+1}^T \ \cdots \ s_{n+k-1}^T \right]^T \), \( k \in \mathbb{N} \). The notation \( \mathbb{E}_z[\cdot] \) stands for the conditional expectation with given \( z \). For a given vector \( C \), its \( i^{th} \) component is denoted by \( C^{(i)} \). Similarly, \( C^{(j : i)} \) denote the \( j^{th} \) row of a given matrix \( C \). \( \sigma_1(M) \) denotes the largest singular value of \( M \), and \( M^{*} \) is the Moore-Penrose pseudo-inverse of \( M \) [14, §6.1]. For \( \xi \in \mathbb{R} \) we let \( \xi_+ := \max(0, \xi) \), and \( \xi_- := \max(0, -\xi) \). Let \( \mathbb{R}_v(A, B) \) denote the matrix \( \begin{bmatrix} A^{N+1}B & A^{N+2}B & \cdots & B \end{bmatrix} \). We let \( |a, b| := \{ z \in \mathbb{R} \mid a < z < b \} \).

II. DYNAMICS AND OBJECTIVE FUNCTION

Consider a discrete time dynamical system
\[
\begin{align*}
    x_{t+1} &= Ax_t + Bu_t + w_t, \quad (1a) \\
    y_t &= Cx_t + z_t, \quad (1b)
\end{align*}
\]
where \( t \in \mathbb{N} \), and \( x_t \in \mathbb{R}^d \), \( u_t \in \mathbb{R}^m \), \( y_t \in \mathbb{R}^q \) are the states, the control inputs, and the outputs, respectively, at time \( t \). The process noise \( w_t \in \mathbb{R}^d \) and the measurement noise \( z_t \in \mathbb{R}^q \) are stochastic processes, the system matrices \( A, B \) and \( C \) are known and are of appropriate dimensions. The control \( u_t \) is constrained as per:
\[
u_t \in U := \{ v \in \mathbb{R}^m \mid \| v \|_\infty \leq u_{\text{max}} \} \text{ for all } t. \quad (2)
\]
We have the following assumptions:

(A1) The pair \((A, B)\) is stabilizable and the pair \((A, C)\) is observable.

(A2) The initial condition, the process and the measurement noise vectors are mutually independent and normally distributed with \( x_0 \sim \mathcal{N}(0, \Sigma_{x_0}) \), \( w_t \sim \mathcal{N}(0, \Sigma_w), z_t \sim \mathcal{N}(0, \Sigma_z) \), with \( \Sigma_{x_0} \geq 0, \Sigma_w > 0 \) and \( \Sigma_z > 0 \).

(A3) The system matrix \( A \) is Lyapunov stable.

(A4) \((A, \Sigma_{x_0}^{1/2})\) is controllable.

For each \( r \) let \( \mathcal{Y}_r := \{ y_0, \ldots, y_r \} \) denote the set of observations up to time \( t \). Let us fix an optimization horizon \( N \in \mathbb{N}^* \) and recalculation interval \( N_r \in \mathbb{N}^* \) such that \( N_r \leq N \). We define the cost \( V_r \) as
\[
V_r := \mathbb{E}_{y_0} \left[ \sum_{k=0}^{N-1} \left( \| x_{t+k} \|^2_{Q_k} + \| u_{t+k} \|^2_{R_k} \right) + \| x_{t+N} \|^2_{Q_N} \right], \quad (3)
\]
for \( t \in \mathbb{N} \) and \( s \geq t \). Let us define \( \hat{x}_{t|s} := \mathbb{E}_{y_0} \left[ x_t \right] \) and \( P_{t|s} := \mathbb{E}_{y_0} \left[ (x_t - \hat{x}_{t|s})(x_t - \hat{x}_{t|s})^\top \right] \). For simplicity of notation we denote \( \hat{x}_{t|t} \) by \( \hat{x} \) and \( P_{t|t} \) by \( P_t \). We need a fundamental related to Kalman filtering, which is stated in [8, p.102]. Let us define the estimation error vector \( e_t := x_t - \hat{x}_t \). Let
\[
K_t := (AP_t A^\top + \Sigma_w) C^\top \left( C(AP_t A^\top + \Sigma_w) C^\top + \Sigma_c \right)^{-1},
\]
\[
\Gamma_t := I_d - K_t C, \text{ and } \phi_t := \Gamma_t A.
\]

III. OUTPUT FEEDBACK POLICIES

In this section we describe two important ingredients of SPC in the presence of incomplete state information. First of all, we need to estimate the states of the system given its output, and secondly, choose a suitable policy in terms of saturated innovations [10].

A. Estimator

For \( t, s \in \mathbb{N} \) and \( t \geq s \) we define \( \hat{x}_{t|s} := \mathbb{E}_{y_0} \left[ x_t \right] \) and \( P_{t|s} := \mathbb{E}_{y_0} \left[ (x_t - \hat{x}_{t|s})(x_t - \hat{x}_{t|s})^\top \right] \). For simplicity of notation we denote \( \hat{x}_{t|t} \) by \( \hat{x} \) and \( P_{t|t} \) by \( P_t \). We need a fundamental related to Kalman filtering, which is stated in [8, p.102]. Let us define the estimation error vector \( e_t := x_t - \hat{x}_t \). Let
\[
K_t := (AP_t A^\top + \Sigma_w) C^\top \left( C(AP_t A^\top + \Sigma_w) C^\top + \Sigma_c \right)^{-1},
\]
\[
\Gamma_t := I_d - K_t C, \text{ and } \phi_t := \Gamma_t A.
\]
The filter equations in [8, p.102] are used to find the evolution of the estimator error over one optimization horizon as follows:

\[ e_{t:N+1} = F_t e_t + G_t w_{t:N} - H_t s_{t:N+1}, \] (7)

where \( F_t := \begin{bmatrix} I_t & \phi_t & \phi_{t+1} & \cdots & \phi_{t+N-1} \\ \phi_t & \phi_{t+1} & \cdots & \phi_{t+N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{t+N-2} & \phi_{t+N-3} & \cdots & \phi_{t+N-1} \\ \phi_{t+N-1} & \phi_{t+N-2} & \cdots & \phi_{t+N-1} \end{bmatrix}, \]

\[ G_t := \begin{bmatrix} \Gamma_t & 0 & \cdots & 0 \\ 0 & G_{t+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G_{t+N-1} \end{bmatrix}, \]

\[ H_t := \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \phi_{t+1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{t+N-2} & \cdots & \phi_{t+N-1} & 0 \end{bmatrix}. \]

Since the optimization is done every \( N_r \) time steps, we find \( \hat{x}_{t+N_r} \) by using (1) and the filter equations [8, p.102] as:

\[ \hat{x}_{t+N_r} = A^{N_r} \hat{x}_t + R_{N_r} (A,B) u_{t:N_r} + \Xi_{t+N_r-1}, \] (8)

where

\[ \Xi_{t+N_r-1} = \begin{bmatrix} A^{N_r-1} K_r C & \cdots & K_{t+N_r-1} C \end{bmatrix} e_{t:N_r} + \begin{bmatrix} A^{N_r-1} K_r C & A^{N_r-2} K_{t+1} C & \cdots & K_{t+N_r-1} C \\ A^{N_r-1} K_r C & A^{N_r-2} K_{t+1} C & \cdots & K_{t+N_r-1} C \end{bmatrix} w_{t:N_r} + \begin{bmatrix} A^{N_r-1} K_r C & A^{N_r-2} K_{t+1} C & \cdots & K_{t+N_r-1} C \end{bmatrix} s_{t:N_r+1}. \]

The quantity \( \Xi_{t+N_r-1} \) in (8) admits a bound:

**Proposition 1.** ([7, Proposition 3]) There exists some \( T', \beta > 0 \) such that \( E_{\theta_0} ||\Xi_{t+N_r-1}|| \leq \beta \) for all \( t \geq T' - N_r + 1 \).

**B. Control policy class**

We select the affine parametrization of control policies [16-18] under output feedback. It is demonstrated in [10] that, in the absence of control bounds, optimization over innovation feedback \( (y_t - \hat{y}_t) \) leads to convex problems. The innovation sequence for one optimization horizon is then given by:

\[ y_{t:N+1} - \hat{y}_{t:N+1} = C F_t e_t + C G_t w_{t:N} + (I - C H_t) s_{t:N+1}, \] (9)

where \( \hat{y}_{t:N+1} = C \hat{x}_{t:N+1}, F_t G_t \) and \( H_t \) are as in (7), and \( C \) is defined in (4). However, this policy is inadmissible because controls are bounded. To satisfy hard bounds on the controls while retaining computational tractability, we consider affine parametrization in terms of the saturated values of innovation feedback: We periodically minimize the cost (3) over the following causal feedback policy for \( \ell = 0, \ldots, N-1 \),

\[ u_{t+\ell} = \eta_{t+\ell} + \sum_{i=0}^{\ell} \theta_{\ell,i+1} \psi_i (y_{t+i} - \hat{y}_{t+i}), \] (10)

**IV. Stability**

It is well known that the construction of a robust positively invariant set is not possible in the presence of Gaussian noise [1, §3.4]. Therefore, standard deterministic Lyapunov based arguments for proving stability are not applicable. Moreover, a linear stochastic system cannot be globally stabilized with the help of bounded controls [15, Theorem 1.7, Open problem 1.3] if the nominal plant has spectral radius larger than unity. Under this fundamental limitation, we restrict our attention to Lyapunov stable plants for ensuring stability. We establish the property of mean-square boundedness of the closed-loop plant under (A3). This class of systems is indeed, till date, the largest class of linear time invariant systems that can be globally stabilized with bounded control actions. We recall the following definition:

**Definition 1.** ([12, §III.A]) An \( \mathbb{R}^d \)-valued random process \( \{x_t\}_{t \in \mathbb{N}_0} \) with given initial measurement of output \( \tilde{y}_0 \) is said to be mean-square bounded if

\[ \sup_{t \in \mathbb{N}} E_{\theta_0} [\|x_t\|^2] < +\infty. \]
Note that a Lyapunov stable system matrix $A$ can be decomposed \cite{19} into a Schur stable part $A_s$ and an orthogonal part $A_o$, as:

$$
\begin{align*}
[\hat{x}^o_{t+1}, \hat{x}^o_t] &= \left[ A_o \hat{x}^o_t + B_o u_t + \Xi^o_t, A_s \hat{x}^o_t + B_s u_t + \Xi^o_t \right],
\end{align*}
$$

\begin{equation}
(12)
\end{equation}

where $\hat{x}^o_t \in \mathbb{R}^{d_o}$, $\hat{x}^o_{t+1} \in \mathbb{R}^{d_o}$, and $d = d_o + d_s$. Linear systems with Schur stable matrices are automatically stable (in suitable sense) under bounded controls, but stability of orthogonal systems in presence of stochastic noise is not obvious. The following stability constraint for orthogonal subsystem was presented in \cite{7}:

$$
\begin{align*}
\|A_s \hat{x}^o_t + R_s(A_o, B_o)u_t\| \leq \|\hat{x}^o_t\| - (\beta + \epsilon')/2
\end{align*}
$$

\begin{equation}
(13)
\end{equation}

where $\epsilon' > 0$, $\beta$ is as defined in Proposition 1, and $\kappa$ is the reachability index of the matrix pair $(A_o, B_o)$. Mean square boundedness of the closed-loop system with the above stability constraint was proved in \cite{7} under large enough control authority. In particular, it was assumed in \cite{7} that

$$
\sum_{\max} \geq \sigma_1 \left( R_N(A_o, B_o) \right)^2 (\beta + \epsilon').
$$

Let us consider the following example.

**Example 1.** Let us consider the dynamics (12) when $A_o = 1, d_s = 0$:

$$
\begin{align*}
\hat{x}^o_{t+1} &= \hat{x}^o_t + u_t + \Xi^o_t,
\hat{y}^o_t &= \hat{x}^o_t, \quad \hat{y}^o_{t+1} = \hat{x}^o_t + \zeta^o_t.
\end{align*}
$$

\begin{equation}
(14)
\end{equation}

Then the drift condition (13) given in \cite{7} requires a scalar control such that $|\hat{x}^o_t + u_t| - |\hat{x}^o_t| \leq (\beta + \epsilon')/2$ for all $\hat{x}^o_t$. Here equality holds when $u_t$ has magnitude $\beta + \epsilon'$ and sign opposite to that of $\hat{x}^o_t$. Since $\beta$ depends on the bound on the uncertainty (see Proposition 1), stability does not follow from \cite{7, Theorem 1} when $\sum_{\max} < \beta + \epsilon'$.

However, this lower bound on the control authority is artificial and we show that it can be removed if different drift conditions are employed, resulting in different stability constraints. We have the following Lemma.

**Lemma 1.** Consider the orthogonal part of the system (12). If there exists a $\kappa$-history dependent policy such that for any given $r > 0$, and $0 < \zeta < \sum_{\max} u_{\sum_{\max}},$ and for any $\kappa t = 0, 1, 2, \ldots$, the control $u_t \in \mathbb{U}$ is chosen such that for $j = 1, 2, \ldots, \kappa$ the following conditions hold:

$$
\begin{align*}
E_x \left[ (A_o^{(t+j)})^\top R_s(A_o, B_o)u_t x_t \right]^{(j)} &\leq -\zeta
\end{align*}
\begin{equation}
(15a)
\end{equation}
$$

if $(A_o^{(t+j)})^\top \hat{x}^o_t \geq 0, r,$

$$
\begin{align*}
E_x \left[ (A_o^{(t+j)})^\top R_s(A_o, B_o)u_t x_t \right]^{(j)} &\geq 0
\end{align*}
\begin{equation}
(15b)
\end{equation}
$$

if $(A_o^{(t+j)})^\top \hat{x}^o_t \leq 0, r$.

Successive application of this policy renders the orthogonal part of the closed-loop system (12) mean-square bounded; i.e. there exists $\gamma_o > 0$ such that $E_{\theta_0} \left[ |\hat{x}^o_t|^2 \right] \leq \gamma_o$, for all $t \in \mathbb{N}$.

A proof of Lemma 1 is given in the appendix. In order to implement the drift conditions (15) with the help of a tractable optimization program, we consider the first $k$ blocks of (11)

$$
\begin{align*}
u_{t,x} = (\eta_t)^{(j)} x_t x_t^\top + (\Theta_t)^{(j)} x_t x_t^\top \psi_y(y_{t,N+1} - \hat{y}_{t,N+1})
\end{align*}
\begin{equation}
(16)
\end{equation}
$$

and substitute them in (15). Now by setting $\kappa t = 0$, we get the following stability constraints for $\epsilon > 0$:

$$
\begin{align*}
\left( (A_o^{(j)})^\top R_s(A_o, B_o) (\eta_t)^{(j)} x_t x_t^\top \right)_{(j)} \leq -\zeta
\end{align*}
\begin{equation}
(17a)
\end{equation}
$$

if $(A_o^{(j)})^\top \hat{x}^o_t \geq 0 + \epsilon$, and

$$
\begin{align*}
\left( (A_o^{(j)})^\top R_s(A_o, B_o) (\eta_t)^{(j)} x_t x_t^\top \right)_{(j)} \geq 0
\end{align*}
\begin{equation}
(17b)
\end{equation}
$$

if $(A_o^{(j)})^\top \hat{x}^o_t \leq -\epsilon$. The amount of constant negative drift $\zeta$ can be selected from the interval $[0, \sum_{\max}]$ to respect the hard bound on control.

**V. MAIN RESULTS**

In this section we recast the constrained optimal control problem (6) as a convex quadratic optimization program that is recursively feasible, and its receding horizon implementation ensures mean-square boundedness of the system states. We can select the recalculation interval $N_r$ such that $k \leq N_r \leq N$. For simplicity, we choose $N_r = k \leq N$. Let us define $\Sigma_{\psi w} := E_{\theta_0} \left[ w_{t,N} \psi(y_r y_r - \hat{y}_{r,N+1}) \right], \Sigma_{\psi} := E_{\theta_0} \left[ \psi(y_r y_r - \hat{y}_{r,N+1}) \right], \Sigma_{\psi} = E_{\theta_0} \left[ \psi(y_r y_r - \hat{y}_{r,N+1}) \right], \alpha := B^\top Q \Sigma_{\psi}$.

We have the following theorems:

**Theorem 1.** For every time $t = 0, 1, \ldots, 2N_r - 1$, the optimal control problem (6) can be written as the following convex quadratic (globally) feasible program:

$$
\begin{align*}
\min_{\eta_t, \Theta_t} & \quad \eta^\top \alpha \eta_t + \eta^\top \Theta_t \alpha \psi \Sigma_{\psi w} + 2 \eta^\top \psi B^\top Q A \hat{x}_t
\end{align*}
\begin{equation}
(18)
\end{equation}
$$

s. t. $|\eta^\top (i)| + \|\Theta_t (i)\|^2 \psi_y \leq u_{\sum_{\max}} \forall i = 1, \ldots, N_r$, (19) stability constraints (17).

The matrices $\Sigma_{\psi w}, \Sigma_{\psi}$ and $\Sigma_{\psi}$ required in the above optimization program depend on $y_{r,N+1} - \hat{y}_{r,N+1}$, which depend on the time variant quantity $P_t$. Since $P_t$ converges asymptotically to a stationary value \cite[\S 4.4, \Sigma_{\psi w}, \Sigma_{\psi} and $\Sigma_{\psi}$ can be easily computed offline empirically via classical Monte Carlo methods \cite{20}. Computations for determining our policy were carried out in the MATLAB-based software package YALMIP \cite{21}, and were solved using SDPT3-4.0 \cite{22}.

**Theorem 2.** The successive applications of the controls given by the optimization program in Theorem 1 above renders the closed-loop system mean-square bounded for any positive bound on control.

Proofs of Theorem 1 and Theorem 2 are provided in the appendix.
The controller of [7] is infeasible.

VI. NUMERICAL EXPERIMENT

We present numerical experiments to illustrate our results and compare our approach against [7]. Let us consider the four dimensional linear stochastic system (1) with matrices lifted from [7]:

\[
A = \begin{bmatrix} 0.9 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = I_4.
\]

The simulation data was chosen to be \( \bar{x} \sim N(0,I_4), \quad w_t \sim N(0,I_4), \quad \xi_t \sim N(0,I_4). \)

We solved a constrained finite-horizon optimal control problem corresponding to states and control weights \( Q = I_4, Q_f = I_4, R = 1. \) We selected an optimization horizon, \( N = 5, \) recalculation interval \( N_r = k = 3 \) and simulated the system responses. We selected the nonlinear bounded term \( \psi(\cdot) \) in our policy to be a vector of scalar sigmoidal functions

\[
\varphi(\xi) = \frac{1 - e^{-\xi}}{1 + e^{-\xi}}
\]

applied to each coordinate of innovation sequence. We plot empirical mean square bound with respect to \( u_{\text{max}} \) picked from the set \( \{1, 2, 3, 4, 5, 10, 20\}. \) All the averages are taken over 100 sample paths for 90 time steps. The optimization program of [7] becomes infeasible when \( u_{\text{max}} \leq 3, \) because the stability constraints used in [7] require a larger bound on the controls. Therefore, we modified the optimization algorithm of [7]. For our purposes we have forced the control values to 0 whenever the termination code of the solver is not equal to 0. Our proposed controller performs better than this modified controller for \( u_{\text{max}} = 1, 2, \) and yields similar performance for \( u_{\text{max}} = 3. \) For \( u_{\text{max}} \geq 4, \) all three controllers perform similarly in terms of the mean square bound of the closed-loop states.

VII. EPILOGUE

We presented a tractable method for predictive control of linear stochastic control systems in the presence of incomplete and corrupt state information. We proved that given any fixed control bound, our receding horizon strategy yields a closed-loop system with mean-square bounded states whenever the system matrix \( A \) is Lyapunov stable. Our results are valid when the process and measurement uncertainties are independent and Gaussian. In future work, we aim to incorporate control channel noise into our framework.
Now [23, Theorem 2.1] guarantees the existence of constants \( C_1^{(j)}, C_2^{(j)} > 0, j = 1, \ldots, d_o \), such that
\[
\sup_{t \in \mathbb{N}} \mathbb{E}_t \left[ \left( \langle z_t^{(j)} \rangle_+ \right)^2 \right] \leq C_1^{(j)} \quad \text{and} \quad \sup_{t \in \mathbb{N}} \mathbb{E}_t \left[ \left( \langle z_t^{(j)} \rangle_- \right)^2 \right] \leq C_2^{(j)}.
\]
Since \( |\delta| = \delta_+ + \delta_- = \delta_+ + (-\delta)_+ \) for any \( \delta \in \mathbb{R}^d \), and for any \( \delta \in \mathbb{R}^{d_r} \), \( |\delta|^2 = \sum_{j=1}^{d_r} |\delta^{(j)}_+|^2 \leq 2 \sum_{j=1}^{d_r} (\langle \delta^{(j)}_+ \rangle^2 + \langle \delta^{(j)}_- \rangle^2) \), we see at once that the preceding bounds imply \( \mathbb{E}_t \mathbb{E}_t \mathbb{E}_t \| z_t \|^2 < C \) for some constant \( C > 0 \). Since \( \tilde{x}_t \) is derived from \( x_t \) by an orthogonal transformation, it immediately follows that \( \mathbb{E}_t \mathbb{E}_t \mathbb{E}_t \| \tilde{x}_t \|^2 \leq C \). A standard argument (e.g., as in [15]) now suffices to conclude from mean-square boundedness of the \( k \)-subsampled process \( (\tilde{x}_t)^k \) the same property of the original process \( (x_t)^k \).

\section*{Proof of Theorem 1.} Consider the objective function (3). We substitute the stacked state vector \((4a)\) and stacked control vector \((11)\) in the objective function. By observing that \( \mathbb{E}_t \left[ \psi(\tilde{y}_t; z_t) \right] = 0 \) and \( \mathbb{E}_t \left[ x_t \right] = \tilde{x}_t \), and removing terms independent of decision variables, we get the desired objective function (18). Therefore, the objective function in (3) is equivalent to (18) under the constraints \((4a)\) and \((11)\).

Since the expected value of a convex function is convex and (18) is obtained by substituting the affine functions of the decision variables into a quadratic function, the objective function (18) is convex quadratic [24]. The constraint (19) is an affine function of the decision variables \( q_t, \Theta_t \); hence it is convex. The constraint (19) is equivalent to the hard control constraint (2). This constraint is obtained by utilizing the fact that \( \psi : \mathbb{R}^{d_r(N+1)} \rightarrow \mathbb{R}^{d(N+1)} \) is component-wise symmetric with respect to the origin and the innovation sequence is mean zero Gaussian. The stability constraints (17) are obviously convex.

To see their feasibility, let us consider \( u_t, x_t \) (16), set \((q_t, \Theta_t)_{1:x_m} = -R_x(A_o, B_o)^o A_o^o \sigma_{x,t}^o (\tilde{x}_t^o), (\Theta_t)_{1:x_m} = 0 \), and observe that \( u_t, x_t \) satisfies (17) for all \( t \). Moreover, for \( t = 0, \ldots, \kappa - 1 \),
\[
\| u_t \|_2 + \| x_t \|_2 \leq \| u_t \|_2 + \| x_t \|_2 \leq \sigma_t (R_x(A_o, B_o)^o)^\# \| d_{t+\kappa} \|_2 \leq u_{\text{max}} \text{ whenever } 0 < \kappa < \frac{\| d_{t+\kappa} \|_2 \sigma_t (R_x(A_o, B_o)^o)}{u_{\text{max}}},
\]
which implies \( u_{t+\kappa} \in U \) [25, Theorem 4].

\section*{Proof of Theorem 2.} Choose \( T := N_F \left( \frac{T}{k} \right) \geq T \), while \( T' \) is according to the Lemma 2, then for every \( t \geq T \)
\[
\mathbb{E}_t \left[ \| x_t \|_2 \right] \leq 2 \mathbb{E}_t \left[ \| x_t - \hat{x}_t \|_2 \right] + 2 \mathbb{E}_t \left[ \| \hat{x}_t \|_2 \right] \leq 2 \rho + 2 \mathbb{E}_t \left[ \| \hat{x}_t \|_2 \right] \text{ from [7, Lemma 8]}
\]
\[
= 2 \rho + 2 \left( \mathbb{E}_t \left[ \| \hat{x}_t \|_2 \right] + \mathbb{E}_t \left[ \| \hat{x}_t \|_2 \right] \right) \leq 2 \left( \rho + \gamma_x + \mathbb{E}_t \left[ \| x_t \|_2 \right] \right) \text{ from [7, Lemma 10]}
\]
\[
\leq 2 \left( \rho + \gamma_x + \gamma_x \right) \text{ from Lemma 1}
\]
\[
= \gamma \quad \text{for all } t \geq T.
\]
By using the tower property of the conditional expectation, we get
\[
\mathbb{E}_t [\mathbb{E}_t \left[ \| x_t \|_2 \right]] = \mathbb{E}_t [\| x_t \|_2] \leq \gamma' \quad \text{for all } t \geq T.
\]
\[
\text{For } t = 0, \ldots, T - 1, \quad x_t = A_t^{-1} x_t + R_t(A, B_t) \left[ u_{t_0}^T \cdots u_{t-1}^T \right]^T + R_t(A, L_d) \left[ w_{t_0}^T \cdots w_{t-1}^T \right]^T.
\]
Because \( w_{t_j} \)'s are zero mean Gaussian and the control is bounded, we have
\[
\mathbb{E}_t [\| x_t \|_2] \leq \gamma_t \quad \text{for } t = 0, \ldots, T - 1.
\]

Define \( \gamma := \max \left\{ \gamma_t, \gamma_{T-1} \right\} \). We conclude that
\[
\sup_{t \in \mathbb{N}} \mathbb{E}_t \left[ \| x_t \|_2 \right] \leq \gamma.
\]