Cohomology of $\mathfrak{osp}(2|2)$ acting on the spaces of linear differential operators on the superspace $\mathbb{R}^{1|2}$

Nizar Ben Fraj Maha Boujelben*

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Abstract

We compute the first differential cohomology of the orthosymplectic Lie superalgebra $\mathfrak{osp}(2|2)$ with coefficients in the superspace of linear differential operators acting on the space of weighted densities on the (1, 2)-dimensional real superspace. We also compute the same, but $\mathfrak{osp}(1|2)$-relative, cohomology. We explicitly give 1-cocycles spanning these cohomology. This work is a simplest generalization of a result by Basdouri and Ben Ammar [Cohomology of $\mathfrak{osp}(1|2)$ with coefficients in $\mathcal{D}_{\lambda,\mu}$. Lett. Math. Phys.81, 239–251 (2007)].

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1 Introduction

The space of weighted densities with weight $\lambda$ (or $\lambda$-densities) on $\mathbb{R}$, denoted by:

$$ \mathcal{F}_\lambda = \left\{ f(dx)^\lambda \mid f \in C^\infty(\mathbb{R}) \right\}, \quad \lambda \in \mathbb{R}, $$

is the space of sections of the line bundle $(T^*\mathbb{R})^\lambda$ for positive integer $\lambda$. Let $\text{Vect}(\mathbb{R})$ be the Lie algebra of all vector fields $X_F = F \frac{d}{dx}$ on $\mathbb{R}$, where $F \in C^\infty(\mathbb{R})$. The Lie derivative $L_D$ along the vector field $D$ makes $\mathcal{F}_\lambda$ a $\text{Vect}(\mathbb{R})$-module for any $\lambda \in \mathbb{R}$:

$$ L_{X_F}(f(dx)^\lambda) = L_{X_F}^\lambda(f(dx)^\lambda) \quad \text{with} \quad L_{X_F}^\lambda(f) = F f' + \lambda f F', \quad (1.1) $$

where $f'$, $F'$ are $\frac{df}{dx}$, $\frac{dF}{dx}$. On the space $\mathcal{D}_{\lambda,\mu}$ of differential operators $\mathcal{F}_\lambda \to \mathcal{F}_\mu$ a $\text{Vect}(\mathbb{R})$-module structure is given by the formula:

$$ X_F \cdot A = L_{X_F}^\mu \circ A - A \circ L_{X_F}^\lambda, \quad (1.2) $$

for any differential operator $A : f(dx)^\lambda \mapsto (Af)(dx)^\mu$.

Lecomte, in [11], found the cohomology $H^1_{\text{diff}}(\mathfrak{sl}(2), \mathcal{D}_{\lambda,\mu})$ and $H^2_{\text{diff}}(\mathfrak{sl}(2), \mathcal{D}_{\lambda,\mu})$, where $\mathfrak{sl}(2)$ is realized as the Lie subalgebra of $\text{Vect}(\mathbb{R})$ spanned by $\{X_1, X_x, X_{x^2}\}$ and where $H^*_{\text{diff}}$

*Institut Supérieur de Sciences Appliquées et Technologie, Sousse, and Département de Mathématiques, Faculté des Sciences de Sfax, BP 802, 3038 Sfax, Tunisie. E-mails: ben-fraj_nizar@yahoo.fr, Maha.Boujelben@fss.rnu.tn
denotes the differential cohomology; that is, only cochains given by differential operators are considered. These spaces appear naturally in the problem of describing the deformations of the sl(2)-module $S_{\mu-\lambda} = \bigoplus_{k=0}^{\infty} F_{\mu-\lambda-k}$, the space of symbols of differential operators of $D_{\lambda,\mu}$. More precisely, the elements of $H^1(\mathfrak{sl}(2), V)$ classify the infinitesimal deformations of a sl(2)-module $V$ and the obstructions to integrability of a given infinitesimal deformation of $V$ are elements of $H^2(\mathfrak{sl}(2), V)$ (for examples, see [1, 2, 5, 12]).

Now, we can study the corresponding super structures. More precisely, we consider the superspace $\mathbb{R}^{1|n}$ equipped with a contact 1-form $\alpha_n$, and introduce the superspace $\mathfrak{S}_{\lambda}^n$ of $\lambda$-densities on the superspace $\mathbb{R}^{1|n}$. The spaces $\mathfrak{S}_{\lambda}^n$ are modules over $\mathcal{K}(n)$, the Lie superalgebra of contact vector fields on $\mathbb{R}^{1|n}$; the space $\mathfrak{D}_{\lambda,\mu}^n$ of differential operators $\mathfrak{S}_{\lambda}^n \rightarrow \mathfrak{S}_{\lambda}^n$ is, naturally, a $\mathcal{K}(n)$-module. The spaces $H^i_{\text{diff}}(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu}^n)$ for $i = 1$ and 2 need to be computed in order to describe deformations of the $\mathfrak{osp}(1|2)$-module $\mathfrak{S}_{\mu-\lambda}^{\lambda} = \bigoplus_{k\geq 0} \mathfrak{S}_{\mu-\lambda-k}^n$, a super analogue of $S_{\mu-\lambda}$, see [9].

In [3], Basdouri and Ben Ammar studied this question for $n = 1$. In this case, $\mathfrak{sl}(2)$ is replaced by the Lie superalgebra $\mathfrak{osp}(1|2)$ naturally realized as a subalgebra of $\mathcal{K}(1)$.

Since there seems to be no conceptual difference in the setting or results obtained in the study of the cohomology of $\mathfrak{osp}(n|2)$ acting on the spaces of linear differential operators on the superspace $\mathbb{R}^{1|n}$ for $n$ considered so far (0, 1 and 2 in this paper), the point is not to treat in further articles the cases $n = 3$ and so on. The point is that the behavior and certain properties of the Lie superalgebras $\mathfrak{osp}(n|2)$ and $\mathcal{K}(n)$ are similar for $n < 4$ (K); the cases $n = 0$ and $n = 1$ are particularly close. However, in several questions, the case $n = 2$ is exceptional due to an occasional isomorphism $\mathcal{K}(n) \simeq \text{Vect}(\mathbb{R}^{1|1})$ ([10]), and one never knows a priori which type of questions will make a given particular $n$ exceptional. We can expect that the properties of $\mathfrak{osp}(n|2)$ and $\mathcal{K}(n)$ become uniform only for $n > 6$. So somebody has to perform all the calculations in the hope to find an interesting result (such, for example, as mentioned in Subsection 4.3).

In this paper we consider the case $n = 2$. That is, we consider the orthosymplectic Lie superalgebra $\mathfrak{osp}(2|2)$ naturally realized as a subalgebra of $\mathcal{K}(2)$. We compute here $H^i_{\text{diff}}(\mathfrak{osp}(2|2), \mathfrak{D}_{\lambda,\mu}^2)$ and $H^i_{\text{diff}}(\mathfrak{osp}(2|2), \mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu}^2)$. Moreover, we give explicit formulae for all the nontrivial 1-cocycles. These spaces arise in the classification of infinitesimal deformations of the $\mathfrak{osp}(2|2)$-module $\mathfrak{S}_{\mu-\lambda}^{\lambda} = \bigoplus_{k\geq 0} \mathfrak{S}_{\mu-\lambda-k}^2$. We hope to be able to describe in the future all the deformations of this module.

## 2 Definitions and Notation

Recall that $C^\infty(\mathbb{R}^{1|2})$ consists of elements of the form:

$$F(x, \theta_1, \theta_2) = f_0(x) + f_1(x)\theta_1 + f_2(x)\theta_2 + f_{12}(x)\theta_1\theta_2,$$

where $f_0, f_1, f_2, f_{12} \in C^\infty(\mathbb{R})$, and where $x$ is the even indeterminate, $\theta_1$ and $\theta_2$ are odd indeterminates, i.e., $\theta_i\theta_j = -\theta_j\theta_i$. Let $|F|$ be the parity of a homogeneous function $F$. Let

$$\text{Vect}(\mathbb{R}^{1|2}) = \left\{ F_0\partial_x + F_1\partial_1 + F_2\partial_2 \mid F_i \in C^\infty(\mathbb{R}^{1|2}) \right\},$$

where $\partial_i = \frac{\partial}{\partial \theta_i}$. Let $\mathcal{K}(2)$ be the Lie superalgebra of contact vector fields on $\mathbb{R}^{1|2}$:

$$\mathcal{K}(2) = \left\{ X \in \text{Vect}(\mathbb{R}^{1|2}) \mid \text{there exists } F \in C^\infty(\mathbb{R}^{1|2}) \text{ such that } \mathfrak{L}_X(\alpha_2) = F\alpha_2 \right\},$$

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where \( \mathcal{L}_X \) is the Lie derivative along the vector field \( X \) and

\[
\alpha_2 = dx + \theta_1d\theta_1 + \theta_2d\theta_2.
\]

Any contact vector field on \( \mathbb{R}^{1|2} \) can be expressed as

\[
X_F = F \partial_x - \frac{1}{2}(-1)^{|F|}\sum_{i=1}^{2} \eta_i(F)\eta_i, \quad \text{where} \quad F \in C^\infty(\mathbb{R}^{1|2})
\]

and \( \eta_i = \partial_i - \theta_i\partial_x \). The contact bracket is defined by \([X_F, X_G] = X_{\{F, G\}}\):

\[
\{F, G\} = FG' - F'G - \frac{1}{2}(-1)^{|F|}\sum_{i=1}^{2} \eta_i(F) \cdot \eta_i(G).
\] (2.3)

The orthosymplectic Lie superalgebra \( \mathfrak{osp}(2|2) \) can be realized as a subalgebra of \( \mathcal{K}(2) \):

\[
\mathfrak{osp}(2|2) = \text{Span}(X_1, X_x, X_2, X_{\theta_1}, X_{\theta_2}, X_\theta_1, X_\theta_2, X_{\theta_1\theta_2}).
\]

We easily see that \( \mathfrak{osp}(1|2) \) is subalgebra of \( \mathfrak{osp}(2|2) \):

\[
\mathfrak{osp}(1|2) = \text{Span}(X_1, X_x, X_2, X_{\theta_1}, X_{\theta_2}) \simeq \text{Span}(X_1, X_x, X_2, X_{\theta_2}, X_{\theta_1}).
\]

We define the space of \( \lambda \)-densities as

\[
\mathfrak{F}_2^\lambda = \left\{ F(x_1, x_2, \theta_1, \theta_2) \alpha_2^\lambda \mid F(x_1, x_2, \theta_1, \theta_2) \in C^\infty(\mathbb{R}^{1|2}) \right\}.
\] (2.4)

As a vector space, \( \mathfrak{F}_2^\lambda \) is isomorphic to \( C^\infty(\mathbb{R}^{1|2}) \), but the Lie derivative of the density \( G\alpha_2^\lambda \) along the vector field \( X_F \) in \( \mathcal{K}(2) \) is now:

\[
\mathcal{L}_{X_F}(G\alpha_2^\lambda) = \mathcal{L}_{X_F}^\lambda(G\alpha_2^\lambda), \quad \text{with} \quad \mathcal{L}_{X_F}^\lambda(G) = \mathcal{L}_{X_F}(G) + \lambda F'G.
\] (2.5)

Here, we restrict ourselves to the subalgebra \( \mathfrak{osp}(2|2) \), thus we obtain a one-parameter family of \( \mathfrak{osp}(2|2) \)-modules on \( C^\infty(\mathbb{R}^{1|2}) \) still denoted by \( \mathfrak{F}_2^\lambda \). As an \( \mathfrak{osp}(1|2) \)-module, we have

\[
\mathfrak{F}_2^\lambda \simeq \mathfrak{F}_2^\lambda \oplus \Pi(\mathfrak{F}_2^{\lambda+1})
\] (2.6)

where \( \Pi \) is the change of parity operator.

Since \( -\eta_2^2 = \partial_x \), and \( \partial_i = \eta_i - \theta_i\eta_2 \), every differential operator \( A \in \mathfrak{D}_2^\lambda \) can be expressed in the form

\[
A(F\alpha_2^\lambda) = \sum_{\ell,m} a_{\ell,m}(x, \theta) \eta_1^\ell \eta_2^m(F)\alpha_2^\theta,
\] (2.7)

where the coefficients \( a_{\ell,m}(x, \theta) \) are arbitrary functions.

**Proposition 2.1.** As a \( \mathfrak{osp}(1|2) \)-module, we have

\[
\mathfrak{D}_2^\lambda \simeq \mathfrak{D}_2^\lambda \oplus \mathfrak{D}_2^{\lambda+1} \oplus \Pi\left(\mathfrak{D}_2^\lambda \oplus \mathfrak{D}_2^{\lambda+1} \oplus \mathfrak{D}_2^{\lambda+2}\right).
\] (2.8)
Proof. Any element $F \in C^\infty(\mathbb{R}^2)$ can be uniquely written as follows: $F = F_1 + F_2 \theta_2$, where $\partial_2 F_1 = \partial_2 F_2 = 0$. Therefore, for any $X_H \in \mathfrak{osp}(1|2)$, we easily check that

$$\mathfrak{L}^\lambda_{X_H}(F) = \mathfrak{L}^\lambda_{X_H}(F_1) + \mathfrak{L}^{\lambda + \frac{1}{2}}_{X_H}(F_2) \theta_2.$$ 

Thus, the following map is an $\mathfrak{osp}(1|2)$-isomorphism:

$$\Phi_\lambda : \mathfrak{g}^2 \lambda \rightarrow \mathfrak{g}^{1}(\mathfrak{osp}(1|2)) \oplus \Pi(\mathfrak{g}^{1}(\mathfrak{osp}(1|2)))$$

$$F \mathfrak{a}_\lambda \mapsto \left( F_1 \mathfrak{a}_1^\lambda, \Pi(F_2 \mathfrak{a}_1^{\lambda + \frac{1}{2}}) \right) \quad (2.9)$$

So, we deduce an $\mathfrak{osp}(1|2)$-isomorphism:

$$\Psi_{\lambda, \mu} : \mathfrak{D}^{1}_{\lambda, \mu} \oplus \mathfrak{D}^{1}_{\lambda + \frac{1}{2}, \mu + \frac{1}{2}} \oplus \Pi \left( \mathfrak{D}^{1}_{\lambda, \mu + \frac{1}{2}} \oplus \mathfrak{D}^{1}_{\lambda + \frac{1}{2}, \mu} \right) \rightarrow \mathfrak{D}^{2}_{\lambda, \mu}$$

$$A \mapsto \Phi_{\mu}^{-1} \circ A \circ \Phi_{\lambda}. \quad (2.10)$$

Here, we identify the $\mathfrak{osp}(1|2)$-modules via the following isomorphisms:

$$\Pi \left( \mathfrak{D}^{1}_{\lambda, \mu + \frac{1}{2}} \right) \rightarrow \text{Hom}_{\text{diff}} \left( \mathfrak{g}^{1}, \Pi(\mathfrak{g}^{1}(\mathfrak{osp}(1|2))) \right) \quad \Pi(A) \mapsto \Pi \circ A,$$

$$\Pi \left( \mathfrak{D}^{1}_{\lambda + \frac{1}{2}, \mu} \right) \rightarrow \text{Hom}_{\text{diff}} \left( \mathfrak{g}^{1}(\mathfrak{osp}(1|2)), \Pi(\mathfrak{g}^{1}(\mathfrak{osp}(1|2))) \right) \quad \Pi(A) \mapsto A \circ \Pi,$$

$$\mathfrak{D}^{1}_{\lambda + \frac{1}{2}, \mu + \frac{1}{2}} \rightarrow \text{Hom}_{\text{diff}} \left( \Pi(\mathfrak{g}^{1}(\mathfrak{osp}(1|2)), \Pi(\mathfrak{g}^{1}(\mathfrak{osp}(1|2))) \right) \quad \Pi(A) \mapsto \Pi \circ A \circ \Pi.$$

$\square$

3 The space $H^1(\mathfrak{osp}(2|2), \mathfrak{D}^2_{\lambda, \mu})$

3.1 Lie superalgebra cohomology, see [7]

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra acting on a superspace $V = V_0 \oplus V_1$ and let $\mathfrak{t}$ be a subalgebra of $\mathfrak{g}$. (If $\mathfrak{t}$ is omitted it assumed to be $\{0\}$.) The space of $\mathfrak{t}$-relative $n$-cochains of $\mathfrak{g}$ with values in $V$ is the $\mathfrak{g}$-module

$$C^n(\mathfrak{g}, \mathfrak{t}; V) := \text{Hom}_{\mathfrak{g}}(\Lambda^n(\mathfrak{g}; \mathfrak{t}); V).$$

The coboundary operator $\delta_n : C^n(\mathfrak{g}, \mathfrak{t}; V) \rightarrow C^{n+1}(\mathfrak{g}, \mathfrak{t}; V)$ is a $\mathfrak{g}$-map satisfying $\delta_n \circ \delta_{n-1} = 0$. The kernel of $\delta_n$, denoted $Z^n(\mathfrak{g}, \mathfrak{t}; V)$, is the space of $\mathfrak{t}$-relative $n$-cocycles, among them, the elements in the range of $\delta_n$ are called $\mathfrak{t}$-relative $n$-coboundaries. We denote $B^n(\mathfrak{g}, \mathfrak{t}; V)$ the space of $n$-coboundaries.

By definition, the $n^{th}$ $\mathfrak{t}$-relative cohomology space is the quotient space

$$H^n(\mathfrak{g}, \mathfrak{t}; V) = Z^n(\mathfrak{g}, \mathfrak{t}; V)/B^n(\mathfrak{g}, \mathfrak{t}; V).$$

We will only need the formula of $\delta_n$ (which will be simply denoted $\delta$) in degrees 0 and 1: for $v \in C^0(\mathfrak{g}, \mathfrak{t}; V) = V^\mathfrak{t}$, $\delta v(g) := (-1)^{|g||v|} g \cdot v$, where

$$V^\mathfrak{t} = \{ v \in V \mid h \cdot v = 0 \quad \text{for all} \ h \in \mathfrak{t} \},$$

and for $\Upsilon \in C^1(\mathfrak{g}, \mathfrak{t}; V)$,

$$\delta(\Upsilon)(g, h) := (-1)^{|g||\Upsilon|} g \cdot \Upsilon(h) - (-1)^{|h|(|g|+|\Upsilon|)} h \cdot \Upsilon(g) - \Upsilon([g, h]) \quad \text{for any} \ g, h \in \mathfrak{g}.$$
3.2 The main theorem

The main result in this paper is the following:

**Theorem 3.1.** The space $H^1_{\text{diff}}(\mathfrak{osp}(2|2), \mathcal{D}_{\lambda,\mu})$ is purely even. It has the following structure:

$$H^1_{\text{diff}}(\mathcal{K}(2), \mathcal{D}_{\lambda,\mu}) \simeq \begin{cases} \mathbb{R}^2 & \text{if } \mu - \lambda = 0, \\ \mathbb{R}^3 & \text{if } (\lambda, \mu) = (-\frac{k}{2}, \frac{k}{2}) \text{ and } k \in \mathbb{N}\setminus\{0\}, \\ 0 & \text{otherwise}. \end{cases} \quad (3.11)$$

The following 1-cocycles span the corresponding cohomology spaces:

$$\begin{align*}
\mathcal{Y}_{\lambda,\lambda}(X_G) &= G' \\
\mathcal{T}_{\lambda,\lambda}(X_G) &= \begin{cases} \eta_1 \eta_2(G) & \text{if } \lambda = 0 \\
2 \lambda \eta_1 (\partial_2(G)) - (-1)^{|\cal G|} (\partial_2(G) \eta_1 + \theta_2 \eta_2 \eta_1(G) \eta_2) & \text{if } \lambda \neq 0 \end{cases} \\
\mathcal{T}_{-\frac{k}{2},\frac{k}{2}}(X_G) &= G' \eta_1 \eta_2^{k-1} \\
\mathcal{T}_{-\frac{k}{2},\frac{k}{2}}(X_G) &= k \eta_1(\partial_2(G)) \eta_1 \eta_2^{2k-1} - (-1)^{|\cal G|} \left( \partial_2(G) \eta_1^{2k+1} - \eta_1(\theta_2 \partial_2(G)) \eta_1^{2k+1} \right) \\
\mathcal{T}_{-\frac{k}{2},\frac{k}{2}}(X_G) &= (k - 1) G'' \eta_1 \eta_2^{2k-3} + (-1)^{|\cal G|} \left( \eta_2(G') \eta_1^{2k-1} - \eta_1(G') \eta_2^{2k-1} \right) \\
\end{align*} \quad (3.12)$$

The proof of Theorem 3.1 will be the subject of Section 5. In fact, we need first the description of $H^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathcal{D}_{\lambda,\mu})$ and the $\mathfrak{osp}(1|2)$-relative cohomology $H^1_{\text{diff}}(\mathfrak{osp}(2|2), \mathfrak{osp}(1|2); \mathcal{D}_{\lambda,\mu})$. To describe the latter one, we need also the description of some bilinear $\mathfrak{osp}(1|2)$-invariant mappings.

4 Invariant Operators and Cohomology of $\mathfrak{osp}(1|2)$

4.1 Invariant bilinear differential operators

Observe that, as a $\mathfrak{osp}(1|2)$-module, we have

$$\mathfrak{osp}(2|2) \simeq \mathfrak{osp}(1|2) \oplus \Pi(\mathfrak{h}),$$

where $\mathfrak{h}$ is the subspace of $\mathfrak{g}_{-\frac{1}{2}}$ spanned by $\{\theta_1 \alpha_1^{-\frac{1}{2}}, x \alpha_1^{-\frac{1}{2}}, \alpha_1^{-\frac{1}{2}}\}$. In fact, it is easy to see that, for the adjoint action, the Lie superalgebra $\mathcal{K}(2)$ is isomorphic to $\mathfrak{g}_{-1}$ which is isomorphic, as $\mathfrak{osp}(1|2)$-module, to $\mathfrak{g}_{-1} \oplus \Pi(\mathfrak{g}_{-1})$. So, the space $\mathfrak{osp}(2|2)$ is isomorphic, as a $\mathfrak{osp}(1|2)$-module, to $\Phi(\mathfrak{osp}(2|2))$, where $\Phi(\mathfrak{osp}(2|2))$ is given by (2.49). More precisely, any element $X_F$ is decomposed into $X_F = X_{F_1} + X_{F_2 \theta_2}$ where $\partial_2 F_1 = 0, \partial_2 F_2 = 0$, and then $X_{F_1} \in \mathfrak{osp}(1|2)$ and $X_{F_2 \theta_2}$ is identified to $\Pi(F_2 \alpha_1^{-\frac{1}{2}}) \in \Pi(\mathfrak{h})$ and it will be denoted $X_{\mathfrak{f}_{\mathfrak{f}_2}}$.

To compute the $\mathfrak{osp}(1|2)$-relative cohomology of $\mathfrak{osp}(2|2)$, we need the description of $\mathfrak{osp}(1|2)$-invariant mappings form $\mathfrak{h} \otimes \mathfrak{g}_{\lambda} \rightarrow \mathfrak{g}_{\mu}$. To do that, we first, describe the $\mathfrak{sl}(2)$-invariant mappings form $\mathfrak{h} \otimes \mathfrak{f}_{\mathfrak{f}_2} \rightarrow \mathfrak{f}_{\mathfrak{f}_2}$. Obviously, as a $\mathfrak{sl}(2)$-module, we have $\mathfrak{h} \simeq \mathfrak{h}_0 \oplus \Pi(\mathfrak{h}_1)$, where $\mathfrak{h}_0$ is the subspace of $\mathfrak{f}_{-\frac{1}{2}}$ spanned by $\{x(dx)^{-\frac{1}{2}}, (dx)^{-\frac{1}{2}}\}$ and $\mathfrak{h}_1$ is the subspace of $\mathfrak{f}_0$ spanned by 1.
Lemma 4.1. (see [3]) Let \( A : \mathfrak{h}_0 \otimes \mathcal{F}_\lambda \to \mathcal{F}_\mu, (h dx^{-\frac{k}{2}}, f dx^{\lambda}) \mapsto A(h, f) dx^\mu \) be an \( \mathfrak{sl}(2) \)-invariant nontrivial bilinear differential operator. Then \( \mu = \lambda + k - \frac{1}{2} \) where \( k \) is a non-negative integer satisfying
\[
k(k-1)(2\lambda + k - 1)(2\lambda + k - 2) = 0,
\]
and, up to a scalar factor, the map \( A \) is given by:
\[
A(h, f) = h f^{(k)} + k(2\lambda + k - 1) h' f^{(k-1)}.
\]

By a straightforward computation, we can also check the following lemma.

Lemma 4.2. Let \( B : \mathfrak{h}_1 \otimes \mathcal{F}_\lambda \to \mathcal{F}_\mu, (h, f dx^{\lambda}) \mapsto B(h, f) dx^\mu \) be a nontrivial \( \mathfrak{sl}(2) \)-invariant bilinear differential operator, then
\[
\mu = \lambda \quad \text{or} \quad (\lambda, \mu) = (\frac{1-k}{2}, \frac{1+k}{2}) \quad \text{and} \quad B(h, f) = ah f^{(\mu - \lambda)},
\]
where \( k \in \mathbb{N} \) and \( a \in \mathbb{R} \).

Proposition 4.3. Let \( A : \mathfrak{h} \times \mathfrak{f}_\lambda \to \mathfrak{g}_1 \), \((H \alpha_1^{-\frac{1}{2}}, F \alpha_1^\lambda) \mapsto A(H, F) \alpha_1^\mu \) be a non-zero \( \mathfrak{osp}(1|2) \)-invariant bilinear differential operator. Then one of the following holds:

i) If \( \mu = \lambda + k - \frac{1}{2} \) where \( k \) is a non-negative integer satisfying \( k(k-1)(2\lambda + k - 1) = 0 \), then, up to a scalar factor, the map \( A \) is given by:
\[
A(H, F) = H F^{(k)} + k(2\lambda + k - 1) H' F^{(k-1)} - (-1)^{|H'|} k H' \eta_1(H) \eta_1(F^{(k-1)}). \tag{4.13}
\]

ii) If \( \mu = \lambda + k \), where \( k \) is a non-negative integer satisfying \( k(2\lambda + k)(2\lambda + k - 1) = 0 \), then, up to a scalar factor, the map \( A \) is given by:
\[
A(H, F) = (-1)^{|H|} H \eta_1(F^{(k)}) + (2\lambda + k) \left( \eta_1(H) F^{(k)} + k H' \eta_1(F^{(k-1)}) \right). \tag{4.14}
\]

Remark 4.4. For \( k = 0, 1 \), the operators (4.13) and (4.14) are not only \( \mathfrak{osp}(1|2) \)-invariant, but also \( \mathcal{K}(1) \)-invariant.

Proof. Let \( A = A_0 + A_1 \) be the decomposition of \( A \) into even and odd parts. As \( \mathfrak{sl}(2) \)-module, we have
\[
\mathfrak{h} \times \mathfrak{g}_1 \simeq \mathfrak{h}_0 \otimes \mathcal{F}_\lambda \oplus \mathfrak{h}_0 \otimes \Pi(\mathcal{F}_{\lambda + \frac{1}{2}}) \oplus \Pi(\mathfrak{h}_1) \otimes \mathcal{F}_\lambda \oplus \Pi(\mathfrak{h}_1) \otimes \Pi(\mathcal{F}_{\lambda + \frac{1}{2}}). \tag{4.15}
\]
So, the map \( A_0 \) is decomposed into four maps:
\[
\begin{align*}
\mathfrak{h}_0 \otimes \mathcal{F}_\lambda & \to \mathcal{F}_\mu, & \mathfrak{h}_0 \otimes \Pi(\mathcal{F}_{\lambda + \frac{1}{2}}) & \to \Pi(\mathcal{F}_{\mu + \frac{1}{2}}), \\
\Pi(\mathfrak{h}_1) \otimes \mathcal{F}_\lambda & \to \Pi(\mathcal{F}_{\mu + \frac{1}{2}}), & \Pi(\mathfrak{h}_1) \otimes \Pi(\mathcal{F}_{\lambda + \frac{1}{2}}) & \to \mathcal{F}_\mu
\end{align*}
\tag{4.16}
\]
and \( A_1 \) is also decomposed into four maps:
\[
\begin{align*}
\mathfrak{h}_0 \otimes \mathcal{F}_\lambda & \to \Pi(\mathcal{F}_{\mu + \frac{1}{2}}), & \mathfrak{h}_0 \otimes \Pi(\mathcal{F}_{\lambda + \frac{1}{2}}) & \to \mathcal{F}_\mu, \\
\Pi(\mathfrak{h}_1) \otimes \mathcal{F}_\lambda & \to \mathcal{F}_\mu, & \Pi(\mathfrak{h}_1) \otimes \Pi(\mathcal{F}_{\lambda + \frac{1}{2}}) & \to \Pi(\mathcal{F}_{\mu + \frac{1}{2}}).
\end{align*}
\tag{4.17}
\]
Observe that the change of parity \( \Pi \) commutes with the \( \mathfrak{sl}(2) \)-action, therefore, according to Lemma 4.1 and Lemma 4.2, we can deduce the expressions of the operators (4.16) and (4.17). We conclude by using the invariance property with respect to \( X_{\theta_1} \) and \( X_{x \theta_1} \). \qed
4.2 The space $H^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathcal{D}^2_{\lambda,\mu})$

Let $\mathfrak{g} = \mathfrak{f} \oplus \mathfrak{p}$ be a Lie superalgebra, where $\mathfrak{f}$ is a subalgebra and $\mathfrak{p}$ is a $\mathfrak{f}$-module such that $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{f}$. Consider a 1-cocycle $\Upsilon \in Z^1(\mathfrak{g}, V)$, where $V$ is a $\mathfrak{g}$-module. The cocycle relation reads

$$
\Upsilon([g,h]) = (-1)^{|g||\Upsilon|}g \cdot \Upsilon(h) + (-1)^{|h||g|+|\Upsilon|}h \cdot \Upsilon(g) = 0, \quad g, h \in \mathfrak{g}.
$$

Denote $\Upsilon_\mathfrak{f} = \Upsilon|_{\mathfrak{f}}$ and $\Upsilon_\mathfrak{p} = \Upsilon|_{\mathfrak{p}}$. Obviously, $\Upsilon_\mathfrak{f}$ is a 1-cocycle over $\mathfrak{f}$ and if $\Upsilon_\mathfrak{f} = 0$ then $\Upsilon_\mathfrak{p}$ is $\mathfrak{f}$-invariant. Thus, the space $H^1(\mathfrak{g}, V)$ is closely related to the space $H^1(\mathfrak{f}, V)$. Furthermore, $\Upsilon_\mathfrak{f}$ and $\Upsilon_\mathfrak{p}$ subject to the following equations:

$$
\Upsilon_\mathfrak{p}([h, p]) = (-1)^{|h||\Upsilon|}h \cdot \Upsilon_\mathfrak{p}(p) + (-1)^{|p|(|h|+|\Upsilon|)}p \cdot \Upsilon_\mathfrak{f}(h) = 0, \quad h, p \in \mathfrak{f}, \quad p \in \mathfrak{p}, \quad (4.18)
$$

$$
\Upsilon_\mathfrak{f}([p, p']) = (-1)^{|p||\Upsilon|}p \cdot \Upsilon_\mathfrak{f}(p') + (-1)^{|p'||(|p|+|\Upsilon|)}p' \cdot \Upsilon_\mathfrak{p}(p) = 0, \quad p, p' \in p. \quad (4.19)
$$

In our situation, $\mathfrak{g} = \mathfrak{osp}(2|2)$, $\mathfrak{f} = \mathfrak{osp}(1|2)$, $\mathfrak{p} = \Pi(\mathfrak{h})$ and $V = \mathcal{D}^2_{\lambda,\mu}$. Thus, as a first step towards the proof of Theorem 4.11, we shall need to compute $H^1(\mathfrak{osp}(1|2), \mathcal{D}^2_{\lambda,\mu})$. According to isomorphism (2.8), we can see that the knowledge of $H^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathcal{D}^1_{\lambda,\mu})$ allows us to compute $H^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathcal{D}^2_{\lambda,\mu})$:

$$
H^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathcal{D}^2_{\lambda,\mu}) \simeq H^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathcal{D}^1_{\lambda,\mu}) \oplus H^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathcal{D}^1_{\lambda+\frac{1}{2},\mu}) \oplus H^1_{\text{diff}}(\mathfrak{osp}(1|2), \Pi(\mathcal{D}^1_{\lambda,\mu+\frac{1}{2}})) \oplus H^1_{\text{diff}}(\mathfrak{osp}(1|2), \Pi(\mathcal{D}^1_{\lambda+\frac{1}{2},\mu})).
$$

(4.20)

Of course, we can deduce the structure of $H^1_{\text{diff}}(\mathfrak{osp}(1|2), \Pi(\mathcal{D}^1_{\lambda,\mu}))$ from $H^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathcal{D}^1_{\lambda,\mu})$. Indeed, to any $\Upsilon \in Z^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathcal{D}^1_{\lambda,\mu})$ corresponds $\Upsilon^\prime \in Z^1_{\text{diff}}(\mathfrak{osp}(1|2), \Pi(\mathcal{D}^1_{\lambda,\mu}))$ where $\Upsilon^\prime(X_G) = \Pi(\sigma \circ \Upsilon(X_G))$ with $\sigma(F) = (-1)^{|F|}F$. Obviously, $\Upsilon$ is a coboundary if and only if $\Upsilon^\prime$ is a coboundary. Thus, we recall the space $H^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathcal{D}^1_{\lambda,\mu})$ which was computed in [3]:

$$
H^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathcal{D}^1_{\lambda,\mu}) \simeq \begin{cases} 
\mathbb{R} & \text{if } \lambda = \mu, \\
\mathbb{R}^2 & \text{if } \lambda = \frac{1-k}{2}, \mu = \frac{k}{2}, \quad k \in \mathbb{N} \setminus \{0\}, \\
0 & \text{otherwise}.
\end{cases}
$$

A basis for the space $H^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathcal{D}^1_{\lambda,\mu})$ is given by the cohomology classes of the 1-cocycles $\Gamma_{\lambda,\mu}$ and $\Gamma^\prime_{\lambda,\mu}$ defined by:

$$
\Gamma_{\lambda,\mu}(X_G) = G',
$$

$$
\Gamma^\prime_{\frac{1-k}{2},\frac{k}{2}}(X_G) = (-1)^{|G|} \pi_1(G) \eta_{1}^{2k-1},
$$

$$
\Gamma^\prime_{\frac{1-k}{2},\frac{k}{2}}(X_G) = (-1)^{|G|} (k-1) \pi_1(G) \eta_{1}^{2k-3} + \pi_1(G) \eta_{1}^{2k-2}.
$$

(4.21)

4.3 The space $H^1_{\text{diff}}(\mathfrak{osp}(2|2), \mathfrak{osp}(1|2); \mathcal{D}^2_{\lambda,\mu})$

In this subsection we compute the space $H^1_{\text{diff}}(\mathfrak{osp}(2|2), \mathfrak{osp}(1|2); \mathcal{D}^2_{\lambda,\mu})$ and we prove that it is nontrivial which is not the case for $n = 1$: $H^1_{\text{diff}}(\mathfrak{osp}(1|2), \mathfrak{sl}(2); \mathcal{D}^1_{\lambda,\mu}) = 0$, see [3]. Moreover, the first author, in [9], proved that the space $H^1_{\text{diff}}(\mathcal{K}(2), \mathcal{K}(1); \mathcal{D}^2_{\lambda,\mu})$ is nontrivial while the space $H^1_{\text{diff}}(\mathcal{K}(1), \text{Vect}(\mathbb{R}); \mathcal{D}^1_{\lambda,\mu})$ is trivial, see [4]. Hence, the case $n = 2$ appears as a special case.
Theorem 4.5. \( \dim H^1_{\text{diff}}(\mathfrak{osp}(2|2), \mathfrak{osp}(1|2); \mathcal{D}^2_{\lambda, \mu}) \leq 1. \) It is 1 only if \( \lambda = \mu \neq 0 \) or \( (\lambda, \mu) = (-\frac{k}{2}, \frac{k}{2}) \), where \( k \in \mathbb{N} \setminus \{0\} \). The cohomology classes of the following 1-cocycles generate the corresponding spaces:

\[
\begin{align*}
\bar{\Upsilon}_{\lambda, \lambda}(X_G) &= 2\lambda \pi_1(\partial_2(G)) - (-1)^{|G|} \left( \partial_2(G)\pi_1 + \theta_2 \pi_2 \pi_1(G)\pi_2 \right), \\
\bar{\Upsilon}_{-\frac{k}{2}, -\frac{k}{2}}(X_G) &= k \pi_1(\partial_2(G))\pi_1\pi_2^{2k-1} - (-1)^{|G|} \left( \partial_2(G)\pi_2^{2k+1} - \pi_1(\theta_2 \partial_2(G))\pi_1^{2k+1} \right).
\end{align*}
\]

(4.22)

To prove Theorem 4.5, we need the following classical fact:

Lemma 4.6. Let \( \mathfrak{g} \) be a Lie superalgebra and \( A : U \otimes V \to W \) a bilinear map, where \( U, V \) and \( W \) are \( \mathfrak{g} \)-modules. We consider the following associated maps

\[
\begin{align*}
A_1 : \Pi(U) \otimes V &\to W, \quad A_2 : \Pi(U) \otimes \Pi(V) \to \Pi(W), \\
A_3 : \Pi(U) \otimes V &\to \Pi(W), \quad A_4 : \Pi(U) \otimes \Pi(V) \to W
\end{align*}
\]

defined by

\[
\begin{align*}
A_1(\Pi(u) \otimes v) &= (-1)^{|u|}A(\Pi(u) \otimes v), \quad A_2(\Pi(u) \otimes \Pi(v)) = (-1)^{|u|}\Pi(A(\Pi(u) \otimes v)), \\
A_3(\Pi(u) \otimes v) &= (-1)^{|u|}\Pi(A(\Pi(u) \otimes v)), \quad A_4(\Pi(u) \otimes \Pi(v)) = (-1)^{|u|}A(\Pi(u) \otimes v).
\end{align*}
\]

The maps \( A_1, A_2, A_3 \) and \( A_4 \) are \( \mathfrak{g} \)-invariant if and only if \( A \) is \( \mathfrak{g} \)-invariant.

Proof. (Theorem 4.5): Consider a 1-cocycle \( \Upsilon \) over \( \mathfrak{osp}(2|2) \) vanishing on \( \mathfrak{osp}(1|2) \). Thus, the equations (4.18) and (4.19) become

\[
\begin{align*}
X_G \cdot \Upsilon(X_H) - (-1)^{|G||\Upsilon|} \Upsilon([X_G, X_H]) &= 0, \tag{4.23} \\
(-1)^{|H_1||\Upsilon|} X_{H_1} \cdot \Upsilon(X_{H_2}) - (-1)^{|H_1||\Upsilon|}|H_2| X_{H_2} \cdot \Upsilon(X_{H_1}) &= 0, \tag{4.24}
\end{align*}
\]

for all \( X_H, X_{H_1}, X_{H_2} \in \Pi(\mathfrak{h}) \) and \( X_G \in \mathfrak{osp}(1|2) \). According to the isomorphism (2.9), the map \( \Upsilon \) is decomposed into four components:

\[
\begin{align*}
\Pi(\mathfrak{h}) \times \mathfrak{g}^1_\lambda &\to \mathfrak{g}^1_\mu, \quad \Pi(\mathfrak{h}) \times \Pi(\mathfrak{g}^1_{\lambda+\frac{1}{2}}) &\to \Pi(\mathfrak{g}^1_{\mu+\frac{1}{2}}), \\
\Pi(\mathfrak{h}) \times \mathfrak{g}^1_\lambda &\to \Pi(\mathfrak{g}^1_{\mu+\frac{1}{2}}), \quad \Pi(\mathfrak{h}) \times \Pi(\mathfrak{g}^1_{\lambda+\frac{1}{2}}) &\to \mathfrak{g}^1_\mu. \tag{4.25}
\end{align*}
\]

The equation (4.23) expresses the \( \mathfrak{osp}(1|2) \)-invariance of each of these bilinear maps. Thus, using Proposition 4.3 Lemma 4.6 and equation (4.24), we prove that, if \( \Upsilon \) is an odd 1-cocycle then, up to a scalar factor, \( \Upsilon \) is given by (with \( a, b \in \mathbb{R} \) and \( k \in \mathbb{N} \setminus \{0\} \)):

\[
\Upsilon = \begin{cases} 
\delta (a \partial_2^k + b(\bar{\eta}_1 + \theta_2 \bar{\eta}_1 \bar{\eta}_2) \partial_2^{k-1}) & \text{if } (\lambda, \mu) = (\frac{k}{2}, \frac{k}{2}), \\
\delta (a \partial_2^k + b \theta_2 \bar{\eta}_1 \bar{\eta}_2 \partial_2^{k-1}) & \text{if } (\lambda, \mu) = (-\frac{k}{2}, \frac{k-1}{2}), \\
\delta (\partial_2) & \text{if } \mu = \lambda + \frac{1}{2} \text{ and } \lambda \neq 0, -\frac{1}{2}, \\
\delta (\theta_2) & \text{if } \mu = \lambda - \frac{1}{2}, \\
0 & \text{otherwise.}
\end{cases}
\]
Now, if $\Upsilon$ is an even 1-cocycle, by the same arguments as above, we get:

$$\Upsilon = \begin{cases} 
  a \tilde{\Upsilon}_{\frac{k}{2}, \frac{k}{2}} + b \delta (\tilde{\eta}_1 \partial_2 \partial_{k-1}^1) & \text{if } (\lambda, \mu) = (-\frac{k}{2}, \frac{k}{2}), \\
  a \tilde{\Upsilon}_{\lambda, \lambda} + b \delta (\theta_2 \tilde{\eta}_2) & \text{if } \lambda = \mu \neq 0, \\
  \delta (\theta_2 (a \tilde{\eta}_1 + b \tilde{\eta}_2)) & \text{if } \lambda = \mu = 0, \\
  0 & \text{otherwise}.
\end{cases}$$

where $\tilde{\Upsilon}_{\lambda, \lambda}$ and $\tilde{\Upsilon}_{\frac{k}{2}, \frac{k}{2}}$ are those given in (4.22). Therefore, in order to complete the proof of Theorem 4.5, we have to study the cohomology classes of the 1-cocycles $\tilde{\Upsilon}_{\lambda, \lambda}$ and $\tilde{\Upsilon}_{\frac{k}{2}, \frac{k}{2}}$.

**Lemma 4.7.** The maps $\tilde{\Upsilon}_{\lambda, \lambda}$ and $\tilde{\Upsilon}_{\frac{k}{2}, \frac{k}{2}}$ are nontrivial $osp(1|2)$-relative 1-cocycles.

**Proof.** First, we can easily see that, for any even element $F \in C^\infty(\mathbb{R}^{1|2})$,

$$\tilde{\Upsilon}_{\frac{k}{2}, \frac{k}{2}} \left( X_{\theta_1 \theta_2} \right) (F \alpha_{2, \frac{k}{2}}) = -k \bar{\eta}_1 \bar{\eta}_2 \alpha_{2, \frac{k}{2}} (F \alpha_{2, \frac{k}{2}})$$

(4.26)

Next, assume that there exists an even operator $A \in \mathcal{D}^{2, \frac{k}{2}, \frac{k}{2}}$ such that $\tilde{\Upsilon}_{\frac{k}{2}, \frac{k}{2}}$ is equal to $\delta A$, that is

$$\tilde{\Upsilon}_{\frac{k}{2}, \frac{k}{2}}(X_G) = \Sigma_{X_G}^\frac{k}{2} \circ A - A \circ \Sigma_{X_G}^\frac{k}{2}.$$ 

(4.27)

The operator $A$ is of the form (2.7); the condition (4.27) implies that its coefficients are constants (which is equivalent to the fact that $X_1 \cdot A = 0$). Then, it is now easy to check that the condition (4.27) has no solution: using formula (2.3), we can see that the expression (4.26) never appear in the right hand side of (4.27). This is a contradiction with our assumption. Similarly, we prove that the cocycle $\tilde{\Upsilon}_{\lambda, \lambda}$ is nontrivial. Lemma 4.7 is proved. Thus we have completed the proof of Theorem 4.5. \hfill $\square$

**Corollary 4.8.** Up to a coboundary, any 1-cocycle $\Upsilon \in Z^1_{\text{diff}}(osp(2|2), \mathcal{D}^{2, \lambda, \mu})$ is invariant with respect to the vector field $X_1 = \partial_x$. That is, the map $\Upsilon$ can be expressed with constant coefficients.

**Proof.** The 1-cocycle condition reads:

$$X_1 \cdot \Upsilon(X_F) - (-1)^{|F||\Upsilon|} X_F \cdot \Upsilon(X_1) - \Upsilon([X_1, X_F]) = 0.$$ 

(4.28)

But, from (4.21) and Theorem 4.5, it follows that, up to a coboundary, we have $\Upsilon(X_1) = 0$, and therefore the equation (4.28) becomes

$$X_1 \cdot \Upsilon(X_F) - \Upsilon([X_1, X_F]) = 0$$

which is nothing but the invariance property of $\Upsilon$ with respect to $X_1$. \hfill $\square$

## 5 Proof of Theorem 3.1

Consider a 1-cocycle $\Upsilon \in Z^1_{\text{diff}}(osp(2|2), \mathcal{D}^{2, \lambda, \mu})$. If $\Upsilon|_{osp(1|2)}$ is trivial then the 1-cocycle $\Upsilon$ is completely described by Theorem 4.5. Thus, assume that $\Upsilon|_{osp(1|2)}$ is nontrivial. Of course, up to coboundary, the general form of $\Upsilon|_{osp(1|2)}$ is given by (4.21) together with the isomorphism...
while $\Upsilon_{\Pi(b)}$ can be essentially described by equation (4.18) and Corollary 4.8. More precisely, according to (4.21) and the isomorphism (4.20), the 1-cocycle $\Upsilon$ can be nontrivial only, a priori, if $\lambda = \mu$, or $\lambda = \mu \pm \frac{1}{2}$, or $(\lambda, \mu) = (\lambda, \mu) = (\frac{1}{2}, \frac{k}{2})$, $(\frac{1}{2}, \frac{k-1}{2})$ where $k \in \mathbb{N}$. Thus, we have to distinguish all these cases. Hereafter all $\varepsilon$'s are constants.

**The case where $\lambda = \mu$**

Considering (4.21) and the isomorphism (4.20), we see that there are two subcases:

i) $\lambda = \mu \neq 0$. In this case, the map $\Upsilon_{\text{osp}(1|2)}$ is, a priori, given by

$$
\Upsilon_{\text{osp}(1|2)}(X_{G_1})(F\alpha_2^\lambda) = (\varepsilon_1 G_1' F_1 + \varepsilon_2 G_1' F_2 \theta_2) \alpha_2^\lambda,
$$

where $F = F_1 + F_2 \theta_2$, with $\partial_2 F_1 = \partial_2 F_2 = 0$. By direct computation, using equations (4.18)–(4.19) and Corollary 4.8 we deduce that

$$
\varepsilon_1 = \varepsilon_2 \quad \text{and} \quad \Upsilon_{\text{osp}(1|2)}(X_{G_2})(F\alpha_2^\lambda) = \varepsilon_1 (-1)^{|F|} G_2' F \theta_2 \alpha_2^\lambda.
$$

Hence $\Upsilon$ is a multiple of $\Upsilon_{\lambda, \lambda}$, see (3.12).

ii) $\lambda = \mu = 0$. Here the map $\Upsilon_{\text{osp}(1|2)}$ is, a priori, given by

$$
\Upsilon_{\text{osp}(1|2)}(X_{G_1})(F) = \left( \varepsilon_1 G_1' F_1 + \left( \varepsilon_2 G_1' F_2 + (-1)^{|F_1|} (\varepsilon_3 G_1' \overline{\eta_1}(F_1) + \varepsilon_4 \eta_1(G_1' F_1)) \right) \theta_2 \right).
$$

The same arguments, as above, show that $\varepsilon_1 = \varepsilon_2$, $\varepsilon_3 = 0$ and

$$
\Upsilon_{\text{osp}(1|2)}(X_{G_2})(F) = \left( \varepsilon_1 G_2' \theta_2 + \varepsilon_4 (-1)^{|G_2|} \eta_1(G_2) \right) F.
$$

Hence $\Upsilon$ is linear combination of $\Upsilon_{0,0}$ and $\tilde{\Upsilon}_{0,0}$, see (3.12).

**The case where $\mu - \lambda = k$ and $2\lambda = -k \neq 0$**

In this case, the map $\Upsilon_{\text{osp}(1|2)}$ is, a priori, given by

$$
\Upsilon_{\text{osp}(1|2)}(X_{G_1})(F\alpha_2^{-\frac{k}{2}}) = \\
\left((-1)^{|F_1|} (\varepsilon_1 G_1' \overline{\eta_1}(F_1) + \varepsilon_2 \left( k G_1' \overline{\eta_1}^{k-1}(F_1) + \eta_1(G_1') \overline{\eta_1}^{2k}(F_1) \right) \theta_2 \\
+ (-1)^{|F_2|} (\varepsilon_3 (k-1) G_1' \overline{\eta_1}^{2k-3}(F_2) + \eta_1(G_1') \overline{\eta_1}^{2k-2}(F_2))) \right) \alpha_2^{-\frac{k}{2}}.
$$

Again by the same arguments, we prove that we have $\varepsilon_4 = -\varepsilon_1$, $\varepsilon_3 = \varepsilon_2$ and

$$
\Upsilon_{\text{osp}(1|2)}(X_{G_2})(F\alpha_2^{-\frac{k}{2}}) = \\
\left( \varepsilon_1 G_2' \overline{\eta_1}(F^{(k-1)}) \theta_2 - \varepsilon_2 G_2' \overline{\eta_1}^{2k-1}(F) \right) \alpha_2^{-\frac{k}{2}}.
$$

Hence $\Upsilon$ is linear combination of $\Upsilon_{-\frac{k}{2}, -\frac{k}{2}}$ and $\tilde{\Upsilon}_{-\frac{k}{2}, -\frac{k}{2}}$, see (3.12).

For the cases where $\lambda = \mu \pm \frac{1}{2}$, or $(\lambda, \mu) = (\frac{1}{2}, \frac{k}{2})$, $(\frac{1}{2}, \frac{k-1}{2})$, the same arguments as before, show that $\Upsilon$ is trivial. This completes the proof.

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