OMEGA RESULTS FOR THE DIVISOR AND CIRCLE PROBLEMS

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1. Introduction

Let \( d(n) \) denote the number of divisors of \( n \) and \( r(n) \) the number of ways of writing \( n \) as the sum of two integer squares. Let \( \Delta(x) \) and \( P(x) \) denote the remainder terms in the asymptotic formulae

\[
\sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + \Delta(x)
\]

and

\[
\sum_{n \leq x} r(n) = \pi x + P(x).
\]

In 1916 G.H. Hardy [4] showed that

\[
\Delta(x) = \begin{cases} 
\Omega_+(x \log x)^{\frac{1}{4}} \log_2 x \\
\Omega_-(x^{\frac{1}{4}})
\end{cases}
\]

and that

\[
P(x) = \begin{cases} 
\Omega_-(x \log x)^{\frac{1}{4}} \\
\Omega_+(x^{\frac{1}{4}})
\end{cases}.
\]

Here and throughout \( \log_j \) denotes the \( j \)-th iterated logarithm, so that \( \log_2 = \log \log \), \( \log_3 = \log \log \log \) and so on. Recall that for a real valued function \( f \) and a positive function \( g \) the symbol \( f = \Omega(g) \) means that \( \limsup_{x \to \infty} |f(x)|/g(x) > 0 \). We write \( f = \Omega_+(g) \) if \( \limsup_{x \to \infty} f(x)/g(x) > 0 \), and \( f = \Omega_-(g) \) if \( \liminf_{x \to \infty} f(x)/g(x) < 0 \). Lastly \( f = \Omega_{\pm}(g) \) means that \( f = \Omega_+(g) \) and also \( f = \Omega_-(g) \).

Since Hardy, gradual progress had been made on the \( \Omega_- \) result for \( \Delta \) and the \( \Omega_+ \) result for \( P \) culminating in the work of K. Corrádi and I. Kátaï [1] who showed that for a positive constant \( c \)

\[
\Delta(x) = \Omega_-(x^{\frac{1}{4}} \exp(c(\log_2 x)^{\frac{1}{4}}(\log_3 x)^{-\frac{1}{4}}))
\]

and a similar \( \Omega_+ \) result for \( P(x) \). In 1981 J.L. Hafner [2] obtained the first improvements on the \( \Omega_+ \) result for \( \Delta \) and the \( \Omega_- \) result for \( P \). He showed that for some positive constants \( A \) and \( B \),

\[
\Delta(x) = \Omega_+(x \log x)^{\frac{1}{4}}(\log_2 x)^{3+2 \log 2/4} \exp(-A \sqrt{\log_3 x}) \quad \text{and} \quad P(x) = \Omega_-(x \log x)^{\frac{1}{4}}(\log_2 x)^{(\log 2)/4} \exp(-B \sqrt{\log_3 x}).
\]

Hafner observed that these results represented the limit of his method and that A. Selberg (unpublished) had obtained similar bounds. In this note we refine Hafner's results and show that the magnitudes of \( \Delta(x) \) and \( P(x) \) can be larger than the values given above. However, unlike Hafner's result, we cannot determine the sign of the large values we exhibit.

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Theorem 1. We have
\[
\Delta(x) = \Omega\left( (x \log x)^{\frac{1}{4}} (\log_2 x)^{\frac{3}{2}(2^{1/3} - 1)} (\log_3 x)^{-\frac{5}{8}} \right),
\]
and
\[
P(x) = \Omega\left( (x \log x)^{\frac{1}{4}} (\log_2 x)^{\frac{3}{2}(2^{1/3} - 1)} (\log_3 x)^{-\frac{5}{8}} \right).
\]

Note that \(\frac{3}{4}(2^{1/3} - 1) = 1.1398\ldots\) while \((3 + 2 \log 2)/4 = 1.0965\ldots\); also \(\frac{3}{4}(2^{1/3} - 1) = 0.1949\ldots\) while \((\log 2)/4 = 0.1732\ldots\).

Our method also applies to the remainder term in the \(k\)-divisor problem (also called the Piltz divisor problem). Let \(k \geq 2\) be an integer and let \(d_k(n)\) denote the number of ways of expressing \(n\) as a product of \(k\) factors. Let \(\Delta_k(x)\) denote the remainder term in the asymptotic formula for \(\sum_{n \leq x} d_k(n)\); that is,
\[
\sum_{n \leq x} d_k(n) = \text{Res}_{s=1} \zeta(s) \frac{x^s}{s} + \Delta_k(x).
\]

G. Szegö and A. Walfisz [7, 8] showed that \(\Delta_k(x) = \Omega^*(\left( (x \log x)^{(k-1)/(2k)} (\log_2 x)^{k-1} \right)\)

where \(\Omega^* = \Omega_+\) if \(k = 2, 3\) and \(\Omega^* = \Omega_-\) if \(k \geq 4\). Hafner [3] improved this to
\[
\Delta_k(x) = \Omega^*(\left( (x \log x)^{\frac{k-1}{2k}} (\log_2 x)^{\frac{k-1}{2k} + k-1} \exp(-A_k \sqrt{\log_3 x}) \right)
\]

for some positive constant \(A_k\). We exhibit larger values of \(|\Delta_k(x)|\) but as in Theorem 1 we cannot control the sign of these values (except when \(k \equiv 3 \pmod{4}\)).

Theorem 2. With notations as above
\[
\Delta_k(x) = \Omega\left( (x \log x)^{\frac{k-1}{2k}} (\log_2 x)^{\frac{k-1}{2k} (k^{2k/(k+1)} - 1) (\log_3 x)^{-\frac{1}{4} - \frac{k-1}{8k}} \right).
\]

The above estimate holds with \(\Omega_+\) in place of \(\Omega\) if \(k \equiv 3 \pmod{8}\), and with \(\Omega_-\) in place of \(\Omega\) if \(k \equiv 7 \pmod{8}\).

For large \(k\) the exponent of \(\log_2 x\) in our result is \(\sim k^2/2\) while that in Hafner’s is \(\sim (k \log k)/2\).

We now describe our method, using \(\Delta(x)\) for illustration. One knows that \(\Delta(x^2)\) is given by the conditionally convergent series
\[
\frac{x^{7/2}}{\pi \sqrt{2}} \sum_{n=1}^{\infty} \frac{d(n)}{n^{7/2}} \cos(4\pi \sqrt{nx} - \pi/4).
\]

By smoothing a little, one may restrict the sum above to the terms \(n \leq N\) weighted appropriately, and it suffices (roughly speaking) to give omega results for the truncated series \(\sum_{n \leq N} d(n) n^{-\frac{7}{2}} \cos(4\pi \sqrt{nx} - \pi/4)\). Let \(M\) denote a set of \(M\) positive integers. By Dirichlet’s Theorem on diophantine approximation we may find \(x \in [X, 6^M X]\) such
that $\| 2\sqrt{mx} \| \leq 1/6$ for each $m \in \mathcal{M}$.

If we select $\mathcal{M}$ to be the first $M$ integers, and take $M = \lfloor \log X \rfloor = N$ then we obtain Hardy’s omega result. Hafner exploits the uneven distribution of $d(n)$ by selecting $\mathcal{M}$ such that $d(m)$ is large for $m \in \mathcal{M}$. To ensure that the terms $n \leq N$, $n \notin \mathcal{M}$ do not cancel the contribution of the terms $m \leq N$, $m \in \mathcal{M}$, Hafner imposes the restriction $\sum_{n \leq N, n \notin \mathcal{M}} d(n)n^{-\frac{3}{4}} = o(N^{\frac{1}{4}} \log N)$. Optimizing this argument leads to his $\Omega_+$ result. We argue instead as follows: For an integer parameter $L$, we first find $x \in [X, (6L)^M X]$ such that $\| 2\sqrt{mx} \| \leq 1/(6L)$ for each $m \in \mathcal{M}$. Then for each of the $L$ points $\ell x$ ($1 \leq \ell \leq L$) we see that the terms $m \leq M$, $m \in \mathcal{M}$ pull in the same direction.

We then show that for one of these points the contribution of the terms $n \leq N$, $n \notin \mathcal{M}$ is not too destructive. The effect is essentially to eliminate Hafner’s restriction, and this accounts for our improvement. Our argument really works for $\sum_{n \leq N} d(n)n^{-\frac{3}{4}} \cos(4\pi \sqrt{nx})$, so that it is first necessary to remove the phase $-\pi/4$. It is in this step that we lose knowledge of the sign of the large values we exhibit.

From our remarks above the ideal omega result for $\Delta(x)$ seems the following. Arrange the sequence $d(n)n^{-\frac{3}{4}}$ in descending order, and let $S(M)$ denote the sum of the first $M$ largest values. Then $\Delta(x) = \Omega(x^{\frac{1}{4}} S(\log x))$. One can show that $S(M) = M^{\frac{1}{2}} (\log M)^{\frac{3}{4}(2^{\alpha/3}-1)+o(1)}$; thus Theorem 1 essentially obtains this ideal omega result.

We may model $\sqrt{2}\Delta(x)x^{-\frac{1}{4}}$ by a random trigonometric series $\sum_{n=1}^{\infty} d(n)n^{-\frac{3}{4}} \cos(X_n)$ where the $X_n$ are independent random variables uniformly distributed on $[0, 2\pi)$. The work of H.L. Montgomery and A.M. Odlyzko [6] provides estimates for the probability of large values attained by this trigonometric series. This suggests that the omega result obtained in Theorem 1 represents the true maximal order of $\Delta(x)$ up to $(\log_2 x)^{o(1)}$.

2. The key Lemma

Let $f(1), f(2), \ldots$ be a sequence of non-negative real numbers and $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$ be a non-decreasing sequence of non-negative real numbers. We suppose that $\sum_{n=1}^{\infty} f(n) < \infty$ and consider the trigonometric series

$$F(x) := \sum_{n=1}^{\infty} f(n) \cos(2\pi \lambda_n x + \beta)$$

where $\beta \in \mathbb{R}$.

Lemma 3. Let $L \geq 2$ and $N \geq 1$ be integers. Let $\mathcal{M}$ be a set of integers such that $\lambda_m \in \left[\frac{\lambda_N}{2}, \frac{3\lambda_N}{2}\right]$ for each $m \in \mathcal{M}$. For any $X \geq 2$ there exists a point $x \in [X/2, (6L)^{M+1} X]$ such that

$$|F(x)| \geq \frac{1}{8} \sum_{m \in \mathcal{M}} f(m) - \frac{1}{L-1} \sum_{\lambda_n \leq 2\lambda_N} f(n) - \frac{4}{\pi^2 X \lambda_N} \sum_{n} f(n).$$

\footnote{Here $\| \cdot \|$ denotes the distance from the nearest integer.}

\footnote{This is not entirely accurate since the terms at $n$ and $nm^2$ are obviously correlated. With this caveat the model is plausible, see D.R. Heath-Brown [5].}
If $\beta \equiv 0 \pmod{2\pi}$ then there is a point $x \in [X/2, (6L)^{M+1}X]$ such that

$$F(x) \geq \frac{1}{8} \sum_{m \in \mathcal{M}} f(m) - \frac{1}{L-1} \sum_{n \leq 2\lambda} f(n) = \frac{2}{\pi^2 X \lambda N} \sum_{n} f(n). \tag{2}$$

If $\beta \equiv \pi \pmod{2\pi}$ then the conclusion (2) holds with $-F(x)$ in place of $F(x)$.

**Proof.** Let $K(u) = \left(\frac{\sin(\pi u)}{\pi u}\right)^2$ be Fejér’s kernel and recall that $\int_{-\infty}^{\infty} K(u) e(-uy) du = \max(0, 1 - |y|) =: k(y)$, say. Consider

$$\int_{-\infty}^{\infty} \lambda N K(\lambda N u) e(-\lambda N u) F(x + u) du$$

$$= \frac{1}{2} \sum_{n} f(n) \int_{-\infty}^{\infty} \lambda N K(\lambda N u) e(-\lambda N u) \left(e^{i\beta} e(\lambda_n (x + u)) + e^{-i\beta} e(-\lambda_n (x + u))\right) du$$

$$= \frac{e^{i\beta}}{2} \sum_{n} f(n) e(\lambda_n x) k\left(\frac{\lambda N - \lambda_n}{\lambda N}\right),$$

since $k((\lambda N + \lambda_n)/\lambda N) = 0$. Setting

$$F_1(x) = \frac{1}{2} \sum_{n} f(n) \cos(2\pi \lambda_n x) k\left(\frac{\lambda N - \lambda_n}{\lambda N}\right),$$

we deduce that

$$F_1(x) \leq \int_{-\infty}^{\infty} \lambda N K(\lambda N u) |F(x + u)| du$$

$$\leq \int_{-X/2}^{X/2} \lambda N K(\lambda N u) |F(x + u)| du + \int_{|u| > X/2} \frac{1}{\pi^2 \lambda N u^2} \sum_{n} f(n) du$$

$$\leq \max_{u \in [-X/2, X/2]} |F(x + u)| + \frac{4}{\pi^2 X \lambda N} \sum_{n} f(n). \tag{3}$$

By Dirichlet’s Theorem (see for example §8.2 of [10]), for any $X \geq 2$ there exists a point $x_0$ in $[X, (6L)^M X]$ such that $\|\lambda_m x_0\| \leq 1/(6L)$ for each $m \in \mathcal{M}$. Consider

$$\sum_{\ell = -L}^{L} k\left(\frac{\ell}{L}\right) F_1(\ell x_0) = \frac{1}{2} \sum_{n} f(n) k\left(\frac{\lambda N - \lambda_n}{\lambda N}\right) \sum_{\ell = -L}^{L} k\left(\frac{\ell}{L}\right) \cos(2\pi \lambda_n \ell x_0).$$

The sum over $\ell$ is $\frac{1}{2} \left(\frac{\sin(\pi L \lambda_n x_0)}{\sin(\pi \lambda_n x_0)}\right)^2$ which is always non-negative. Further if $n \in \mathcal{M}$ then each term in the sum is at least $\cos(2\pi/6) = \frac{1}{2}$ and so the sum here is at least $L/2$. Thus we see that

$$\sum_{\ell = -L}^{L} k\left(\frac{\ell}{L}\right) F_1(\ell x_0) \geq \frac{L}{4} \sum_{m \in \mathcal{M}} f(m) k\left(\frac{\lambda N - \lambda_m}{\lambda N}\right) \geq \frac{L}{8} \sum_{m \in \mathcal{M}} f(m),$$
since \(\lambda_m \in [\lambda_N/2, 3\lambda_N/2]\) for all \(m \in \mathcal{M}\). Since \(F_1(\ell x_0) = F_1(-\ell x_0)\) we deduce that for some \(1 \leq \ell_0 \leq L\)

\[
F_1(\ell_0 x_0) \geq \frac{1}{8} \sum_{m \in \mathcal{M}} f(m) - \frac{F_1(0)}{L - 1} \geq \frac{1}{8} \sum_{m \in \mathcal{M}} f(m) - \frac{1}{L - 1} \sum_{\lambda_n \leq 2\lambda_N} f(n).
\]

Using this in (3) we obtain the first assertion of the Lemma.

Suppose now that \(\beta \equiv 0 \pmod{2\pi}\). We start with

\[
\int_{-\infty}^{\infty} 2\lambda_N K(2\lambda_N u)F(x + u)du = \sum_n f(n) \cos(2\pi \lambda_n x)k\left(\frac{\lambda_n}{2\lambda_N}\right)
\]

and letting \(F_2(x)\) denote the RHS above, we deduce that

\[
F_2(x) \leq \max_{u \in [-X/2, X/2]} F(x + u) + \frac{2}{\pi^2X\lambda_N} \sum_n f(n).
\]

We then argue as in the preceding paragraph and obtain the estimate (2). The case \(\beta \equiv \pi \pmod{2\pi}\) follows since \(\cos(t + \pi) = -\cos(t)\).

3. Proof of Theorem 1

Let \(X\) be large. Uniformly in \(X \leq x \leq X^3\) we have (see (12.4.4) of [10])

\[
\Delta(x) = \frac{x^{\frac{3}{2}}}{\pi\sqrt{2}} \sum_{n \leq X^3} \frac{d(n)}{n^{\frac{3}{2}}} \cos \left(4\pi\sqrt{n x} - \frac{\pi}{4}\right) + O(X^\epsilon).
\]

We will apply the result of §2 taking \(f(n) = d(n)n^{-\frac{3}{2}}\) if \(n \leq X^3\) and \(f(n) = 0\) for larger \(n\), \(\lambda_n = 2\sqrt{n}\), and \(\beta = -\frac{\pi}{4}\). Then for \(\sqrt{X} \leq x \leq X^\frac{3}{2}\) we have \(\Delta(x^2) = \frac{\pi}{4}\sqrt{n}\Delta(x) + O(X^\epsilon)\), so that it suffices to establish an \(\Omega\) result for \(F\).

Let \(L, M\) and \(N\) be parameters to be chosen shortly and suppose that \((6L)^M + 1 \leq \sqrt{X}\). Let \(\mathcal{M}\) be a set of \(M\) integers in \([N/4, 9N/4]\). Then (1) of Lemma 3 shows that there exists a point \(x \in [X/2, X^{\frac{3}{2}}]\) such that

\[
|F(x)| \geq \frac{1}{8} \sum_{m \in \mathcal{M}} d(m) n^{\frac{3}{2}} - \frac{1}{L - 1} \sum_{n \leq 4N} d(n)n^{\frac{3}{2}} - \frac{2}{\pi^2 X \sqrt{N}} \sum_{n \leq X^3} d(n)n^{\frac{3}{2}}
\]

\[
\geq \frac{1}{18N^{\frac{3}{2}}} \sum_{m \in \mathcal{M}} d(m) + O\left(\frac{N^{\frac{3}{2}} \log N}{L} + \frac{\log X}{X^{\frac{3}{2}}}\right).
\]

(4)

Choose \(L = (\log_2 X)^{10}\) and let \(\lambda\) be a positive real number (we shall see that \(\lambda = 2^{\frac{3}{2}}\) optimally). We take \(\mathcal{M}\) to be the set of integers in \([N/4, 9N/4]\) having exactly \([\lambda \log_2 N]\) distinct prime factors. The cardinality of \(\mathcal{M}\) is

\[
M \asymp \frac{N}{\log N} \frac{(\log_2 N)^{[\lambda \log_2 N] - 1}}{([\lambda \log_2 N] - 1)!} \asymp \frac{N}{\sqrt{\log_2 N}} (\log N)^{\lambda - 1 - \lambda \log \lambda},
\]
upon using Stirling’s formula and Theorem 4 of II.6.1 of G. Tenenbaum [9] for example. If we take $N = c \log X (\log_2 X)^{1 - \lambda + \lambda \log (\log_3 X)^{-\frac{1}{4}}}$ for a suitably small positive constant $c$ then the condition $(6L)^{M+1} \leq \sqrt{X}$ is satisfied. Upon noting that each $m \in \mathcal{M}$ satisfies $d(n) \geq 2^{[\lambda \log_2 N]} \gg (\log N)^{\lambda \log 2}$ we deduce from (4) that for some $x \in [X/2, X^{\frac{2}{3}}]$ 

$$|F(x)| \gg \frac{M (\log N)^{\lambda \log 2}}{N^{\frac{1}{4}}} + O\left(\frac{N^{\frac{1}{4}}}{(\log_2 X)^9} + 1\right)$$

$$\gg \frac{(\log X)^{\frac{1}{4}}}{(\log_3 X)^{\frac{1}{8}}} (\log_2 X)^{\lambda \log 2 + \frac{1}{4}(\lambda - 1 - \lambda \log \lambda)} + O((\log X)^{\frac{1}{4}}(\log_2 X)^{(1 - \lambda + \lambda \log \lambda)/4 - 9}).$$

The optimal choice of $\lambda$ is $\lambda = 2^{\frac{1}{4}}$ which gives the omega result for $\Delta(x)$ claimed in Theorem 1.

The proof for $P(x)$ is similar. By modifying the argument in Titchmarsh [10; §12.4] we obtain that uniformly in $X \leq x \leq X^3$

$$P(x) = -\frac{x^{\frac{1}{4}}}{\pi} \sum_{n \leq X^3} r(n) \frac{r(n)}{n^{\frac{1}{4}}} \cos \left(2\pi \sqrt{nx} + \frac{\pi}{4}\right) + O(X^\epsilon).$$

We now apply the result of §2 taking $f(n) = r(n)n^{-\frac{1}{4}}$ for $n \leq X^3$ and $f(n) = 0$ for larger $n$, $\lambda_n = \sqrt{n}$ and $\beta = \frac{\pi}{4}$. Then for $\sqrt{X} \leq x \leq X^{\frac{2}{3}}$ we have $P(x^2) = -\frac{\sqrt{X}}{\pi} F(x) + O(X^\epsilon)$ so that it suffices to establish an $\Omega$ result for $F$. Let $L$, $M$ and $N$ be parameters to be chosen and suppose $(6L)^{M+1} \leq \sqrt{X}$. Let $\mathcal{M}$ be a set of $M$ integers in $[N/4, 9N/4]$. Then (1) of Lemma 3 shows that there is a point $x \in [X/2, X^{\frac{2}{3}}]$ with 

$$|F(x)| \geq \frac{1}{18N^{\frac{1}{4}}} \sum_{m \in \mathcal{M}} r(m) + O\left(\frac{N^{\frac{1}{4}}}{L} + \frac{1}{X^{\frac{1}{4}}}\right).$$

Choose $L = (\log_2 X)^{10}$ and let $\lambda$ be a positive real number (we shall see that the optimal choice of $\lambda$ is $2^{\frac{1}{4}}$). We take $\mathcal{M}$ to be the set of integers in $[N/4, 9N/4]$ having exactly $[\lambda \log_2 N]$ distinct prime factors all of which are $1$ (mod 4). Modifying the arguments in II.6 of Tenenbaum [9] we see that the cardinality of $\mathcal{M}$ is 

$$M \asymp \frac{N}{\log N} \frac{([\lambda \log_2 N] - 1)!}{(\pi \log_2 N)} \asymp \frac{N}{\sqrt{\log N}} (\log N)^{\lambda - 1 - \lambda \log \lambda - \lambda \log 2}.$$

If we let $N = c \log X (\log_2 X)^{1 - \lambda + \lambda \log \lambda + \lambda \log 2} (\log_3 X)^{-\frac{1}{4}}$ for a suitably small positive constant $c$ then the condition $(6L)^{M+1} \leq \sqrt{X}$ is met. Upon noting that $r(m) \geq 2^{[\lambda \log_2 N]} \gg (\log_2 X)^{\lambda \log 2}$ for all $m \in \mathcal{M}$ we obtain from (5) that for some $x \in [X/2, X^{\frac{2}{3}}]$

$$|F(x)| \gg \frac{(\log X)^{\frac{1}{4}}}{(\log_3 X)^{\frac{1}{8}}} (\log_2 X)^{\lambda \log 2 + \frac{1}{4}(\lambda - 1 - \lambda \log \lambda)} + O((\log X)^{\frac{1}{4}}(\log_2 X)^{(1 - \lambda + \lambda \log 2)/4 - 9}).$$

The optimal choice for $\lambda$ is $\lambda = 2^{\frac{1}{4}}$ which establishes this case of Theorem 1.
4. Proof of Theorem 2

Proposition 4. Let $x \geq 2$ and $N \geq 2$ be real numbers. Then for a fixed integer $k \geq 2$

$$\frac{N^\frac{k}{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \Delta_k(x^{k}e^{u/x})e^{-u^2N^\frac{k}{2}} du = O(x^\frac{k}{2}e^{cN^{\frac{k}{2}}})$$

$$+ \frac{x^\frac{k}{2}}{\pi\sqrt{k}} \sum_{n=1}^{\infty} \frac{d_k(n)}{n^\frac{k}{2+\epsilon}} \exp(-\pi^2(n/N)^\frac{k}{2}) \cos \left(2\pi kn^\frac{1}{2}x + \frac{k^2 - 3}{4\pi} \right).$$

Assuming Proposition 4 we now prove Theorem 2. We apply the result of §2 taking $f(n) = d_k(n)n^{-\frac{k+1}{2k}} \exp(-\pi^2(n/N)^\frac{k}{2})$, $\lambda_n = kn^\frac{1}{2}$ and $\beta = \frac{k^2 - 3}{4\pi}$. By Proposition 4 it suffices to establish $\Omega$ results for the corresponding $F(x)$ where we suppose that $X/2 \leq x \leq X^2$ say. (The error term in Proposition 4 is negligible for our choice of $N$ which will be $O(X^\epsilon)$.)

We choose $L = (\log_2 X)^{k+2}$ and select $\mathcal{M}$ to be the set of integers in $[2^{-k}N, (3/2)^kN]$ containing exactly $[\lambda \log_2 N]$ distinct prime factors; here $\lambda$ is a positive real number which will be optimally chosen as $k^{\frac{2k}{k+1}}$. As in §3, we see that the cardinality of $\mathcal{M}$ is $M \asymp N(\log N)^{-1-\lambda \log \lambda}(\log_2 N)^{-\frac{1}{2}}$. If we choose $N = c_k \log X(\log_2 X)^{1+\lambda \log \lambda - \lambda(\log_3 X)^{\frac{1}{2}}}$ for a suitably small positive constant $c_k$ then the condition $(6L)^{M+1} \leq X$ is met. Since $d_k(m) \geq k^{[\lambda \log_2 N]} \asymp (\log_2 X)^{\lambda \log k}$ for each $m \in \mathcal{M}$, Lemma 3 then establishes that for some $x \in [X/2, X^2]$ we have

$$|F(x)| \gg (\log X)^\frac{k-1}{2k}(\log_2 X)^{\frac{k+1}{2k}(\lambda-1-\lambda \log \lambda) + \lambda \log k}(\log_3 X)^{\frac{k-1}{2k} + \frac{k-1}{4\pi}} + O\left(\frac{N^\frac{k-1}{2k}(\log N)^{k-1}}{L}\right).$$

Choosing optimally $\lambda = k^{\frac{2k}{k+1}}$ we obtain the desired omega result for $F(x)$ and hence Theorem 2. When $k \equiv 3 \pmod{8}$ then $\beta \equiv 0 \pmod{2\pi}$ and when $k \equiv 7 \pmod{8}$ then $\beta \equiv \pi \pmod{2\pi}$, and so in these cases Lemma 3 leads to the one sided omega results claimed in Theorem 2.

It remains lastly to prove Proposition 4. The proof is based on a standard procedure using Perron’s formula, shifting contours, invoking the functional equation for $\zeta(s)$, and then applying the method of stationary phase. One can also extract Proposition 4 from the work of Hafner [3] (see (3.2.8)). For the sake of completeness we supply a proof.

Proof of Proposition 4. Write $D_k(x) = \sum_{n \leq x} d_k(n)$ and consider

$$\frac{N^\frac{k}{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} D_k(x^{k}e^{u/x})e^{-u^2N^\frac{k}{2}} du.$$ 

By Perron’s formula this is, for some $c > 1$,

$$= \frac{N^\frac{k}{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta(s) x^{ks} \exp \left(\frac{us}{x} - u^2N^\frac{s}{2s}\right) \frac{ds}{s} du$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta(s) x^{ks} \frac{N^\frac{k}{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left(-\left(uN^\frac{k}{2}x - \frac{s}{2N^\frac{k}{2}x}\right)^2 + \frac{s^2}{4N^\frac{k}{2}x^2}\right) du \frac{ds}{s}$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta(s) x^{ks} \exp \left(\frac{s^2}{4N^\frac{k}{2}x^2}\right) ds.$$
We move the line of integration above to the line $a-i\infty$ to $a+i\infty$ where we take $a = -\frac{1}{\log x}$. The pole at 0 gives an amount $O(1)$ while the pole at $s = 1$ contributes

$$\text{Res}_{s=1} \zeta(s) \frac{k^s}{s} \exp \left( \frac{s^2}{4N \gamma x^2} \right) = \frac{N^{\frac{1}{2}}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2 N \gamma} \left( \text{Res}_{s=1} \zeta(s) \frac{(x^k e^{u/x})^s}{s} \right) du.$$  

We conclude that

$$\frac{N^{\frac{1}{2}}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \Delta_k(x^k e^{u/x}) e^{-u^2 N \gamma} du = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) k^x \exp \left( \frac{s^2}{4N \gamma x^2} \right) ds + O(1).$$

We use the functional equation $\zeta(s) = \chi(s) \zeta(1-s)$ where $\chi(s) = 2^{s-1} \pi^s \sec(\pi s/2)/\Gamma(s)$ and expand $\zeta(1-s)^k = \sum_{n=1}^{\infty} d_k(n)n^{s-1}$. Then the above becomes

$$\sum_{n=1}^{\infty} d_k(n) \frac{1}{n} \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \chi(s) k^x n^s \exp \left( \frac{s^2}{4N \gamma x^2} \right) ds + O(1).$$

Call the integral in (6) above $I_n$. The integral over the line segment from $a-i$ to $a+i$ gives an amount $\ll n^a \log x$ and note that the integrand at $a-it$ is the complex conjugate of the integrand at $a+it$. Thus

$$I_n = \text{Re} \frac{1}{\pi i} \int_1^{\infty} \chi(a+it) (x^k n)^{a+it} \exp \left( \frac{a^2 + 2ait - t^2}{4N \gamma x^2} \right) dt + O(n^a \log x).$$

Stirling’s formula gives that $\chi(a+it) = (2\pi/t)^{a+it-\frac{1}{2}} e^{it(t+\frac{1}{2})} (1 + O(1/t))$. Hence

$$I_n = \text{Re} \frac{1}{\pi i} \int_1^{\infty} (x^k n)^{a+it} \frac{2\pi}{t} (k^{a+it-\frac{1}{2}}) e^{ik(t+\frac{1}{2})} \exp \left( - \frac{t^2}{4N \gamma x^2} \right) dt + O(n^a x^{\frac{1}{2}-1+\epsilon} \sqrt{N}).$$

We use the method of stationary phase (which occurs at $t = 2\pi n^\frac{1}{k} x$) to evaluate the above integral. We split the cases when $|t - 2\pi xn^\frac{1}{k}| \leq (xn^\frac{1}{k})^\frac{3}{2}$ and when $|t - 2\pi xn^\frac{1}{k}| > (xn^\frac{1}{k})^\frac{3}{2}$. In the first case (call $y = t - 2\pi xn^\frac{1}{k}$ so that $|y| \leq (xn^\frac{1}{k})^\frac{3}{2}$ here) we get by a Taylor expansion

$$\text{Re} \frac{(xn^\frac{1}{k})\frac{3}{2}-1}{2\pi^2 i} \int_{|y| \leq (xn^\frac{1}{k})^\frac{3}{2}} \exp \left( - \pi^2 \left( \frac{n}{N} \right)^\frac{2}{k} + i \left( \frac{k\pi}{4} + 2\pi k n^\frac{1}{k} x - \frac{k y^2}{4\pi x n^\frac{1}{k}} \right) \right) \times \left( 1 + O \left( \frac{1}{(xn^\frac{1}{k})\frac{3}{2}} \right) \right) dy.$$  

Using $\int_{|z| \leq T} e^{-iz^2} dz = \sqrt{\pi} e^{-i\pi/4} + O(T^{-1})$ we obtain that the above is

$$\frac{(xn^\frac{1}{k})\frac{3}{2}-1}{\pi \sqrt{k}} \exp \left( - \pi^2 \left( \frac{n}{N} \right)^\frac{2}{k} \right) \left( \cos \left( 2\pi k n^\frac{1}{k} x + \frac{k \pi}{4} + 3 \frac{\pi}{4} \right) + O \left( \frac{1}{(xn^\frac{1}{k})\frac{3}{2}} \right) \right).$$
To handle the second case we note that for any $1 \leq y \leq 2\pi x n^{\frac{k}{2}} - (xn^{\frac{1}{2}})^{\frac{3}{5}}$ we have (see Lemma 4.2 of [10])
\[
\int_{1}^{y} \exp \left( i \left( t \log(x^{k}n) + kt + \frac{k\pi}{4} - kt \log \frac{t}{2\pi} \right) \right) dt \ll \frac{1}{\log(2\pi x n^{\frac{k}{2}}/y)}.
\]

Using this and integration by parts we see that the integral in (7) over the range $1 \leq t \leq 2\pi x n^{\frac{k}{2}} - (xn^{\frac{1}{2}})^{\frac{3}{5}}$ is
\[
\ll (xn^{\frac{k}{2}})^{\frac{3}{5} - \frac{4}{3}n} \exp\left(\frac{1}{\log(2\pi x n^{\frac{k}{2}}/y)} \right) + n^a x^{\frac{k}{2} - 1} N^{\frac{k}{2}}.
\]

The same bound applies to the integral over the range $t \geq 2\pi x n^{\frac{k}{2}} + (xn^{\frac{1}{2}})^{\frac{3}{5}}$. Putting these estimates together we find that
\[
I_n = \frac{(xn^{\frac{k}{2}})^{\frac{k-1}{2}}}{\pi \sqrt{k}} \exp\left(-\pi^2 \left( \frac{n}{N} \right)^{\frac{3}{2}} \right) \cos \left( 2\pi kn^{\frac{1}{2}} x + \frac{k-3}{4} \pi \right)
\]
\[
+ O(n^a x^{\frac{k}{2} - 1 + \epsilon} \sqrt{N} + (xn^{\frac{1}{2}})^{\frac{k}{2} - \frac{3}{5}} \exp(-n/N^{\frac{3}{2}})).
\]

Using this in (6) we obtain the Proposition.

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