Abstract

We prove that in the Euclidean representation of the three dimensional massless Nelson model the $t = 0$ projection of the interacting measure is absolutely continuous with respect to a Gaussian measure with suitably adjusted mean. We also determine the Hamiltonian in the Fock space over this Gaussian measure space.

KEYWORDS: Nelson’s scalar field model, infrared regular representation, ground state, Gibbs measure on path space, cluster expansion
1 Introduction

The Nelson model \[6\] of a spinless electron coupled to a scalar massless Bose field is infrared divergent in 3 space dimensions. In its algebraic version the ground state representation of the Nelson Hamiltonian is not unitarily equivalent to the Fock representation \[3, 2\]. From the point of view of functional integrals, which is the one taken here, the \(t = 0\) projection of the interacting measure is singular with respect to the \(t = 0\) projection of the free measure \[5\]. Such a result leaves open the possibility of an explicit representation of the \(t = 0\) functional measure. In this letter we will settle the issue. In the last section we will explain the connection to the non-Fock representation.

The infrared divergence in Nelson’s model can be seen already on the level of the classical action. The Euclidean action for the electron is

\[
S_p(\{q_t\}) = \int_{-\infty}^{\infty} \left( \frac{1}{2} q_t^2 + V(q_t) \right) dt. \tag{1.1}
\]

Here \(t \mapsto q_t \in \mathbb{R}^3\) is a path of the electron. The electron’s mass is set equal to one only for convenience. \(V : \mathbb{R}^3 \to \mathbb{R}\) is a confining potential, which for technical reasons we require to be bounded from below, continuous and have the asymptotics \(V(x) = C|x|^{2s} + o(|x|^{2s})\) for large \(|x|\) with some constant \(C > 0\) and exponent \(s > 1\). The massless field \(\xi_t : \mathbb{R}^3 \to \mathbb{R}\) has the action

\[
S_f(\{\xi_t\}) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \frac{1}{2} \left( \partial_t \xi_t(x)^2 + \nabla \xi_t(x)^2 \right) dx dt. \tag{1.2}
\]

Finally, the electron is coupled locally and translation invariant as given by

\[
S_{\text{int}}(\{q_t, \xi_t\}) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} g(x - q_t) \xi_t(x) dx dt. \tag{1.3}
\]

Here \(g\) is a form factor, assumed to be a sufficiently smooth function and of rapid decrease, normalized such that \(\int_{\mathbb{R}^3} g(x) dx = e\), where \(e\) is the charge of the particle. The case of interest is \(e \neq 0\), since \(e = 0\) implies \(\hat{g}(0) = 0\) (\(\hat{g}\) the Fourier transform of \(g\)) and therefore regularizes the interaction in the infrared region. At a later stage we will need that \(|e| \leq e^*\) with some sufficiently small \(e^*\).

The infrared divergence can be deduced from the solution of the classical field equation which minimizes energy. An easy computation leads to the minimizer \(q_t = q_{\text{min}}\), where \(q_{\text{min}}\) corresponds to an absolute minimum of the potential, i.e. \(V(x) \geq V(q_{\text{min}})\), for all \(x \in \mathbb{R}^3\), and \(\xi_t = \xi_{\text{min}}\) given by

\[
\xi_{\text{min}}(x) = \Delta^{-1} g(x - q_{\text{min}}). \tag{1.4}
\]

In three dimensions \(\xi_{\text{min}}(x) \approx -e/(4\pi|x|)\) for large \(|x|\). On the other hand, for the free measure \(g = 0\) and \(\xi_{\text{min}} = 0\). If the total action \(S = S_p + S_f + S_{\text{int}}\) is used for constructing the Euclidean path measure (formally like \(\exp(-S)\)), then basically one has to compare two Gaussian fields of the same covariance, one with mean \(\Delta^{-1} g(x)\) and the other with zero mean. Because of slow decay of the Coulomb potential in 3 dimensions, these two measures cannot have a density with respect to each other.
Our arguments also offer a hint of how to improve on this situation. We reschedule the field action by adding a term dependent on a function $u$ to be chosen below:

$$S^u_f(\{\xi_t\}) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \left( \frac{1}{2} \left( \partial_t \xi_t(x)^2 + (\nabla \xi_t(x))^2 \right) + \xi_t(x) u(x) \right) dx dt.$$

(1.5)

For convenience, we represent this auxiliary function as the convolution

$$u(x) = (\varrho * h)(x) \equiv \int_{\mathbb{R}^3} e^{ik \cdot x} \hat{\varrho}(k) \hat{h}(k) dk$$

(1.6)

where $\hat{h}$ denotes the Fourier transform of $h$, of which we require that

1. $\hat{h}$ is a real-valued, even, bounded, and sufficiently smooth function;
2. $\hat{h}(0) = 1$.

The first condition is more for simplicity than necessity, the second is essential. The density $\exp(-S^u_f(\{\xi_t\}))$ provides the free Gaussian measure. Keeping the electron’s action and the total action unchanged, we obtain a rescheduled interaction of the form

$$S^u_{\text{int}}(\{q_t, \xi_t\}) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^3} (\varrho(x-q_t) - (\varrho * h)(x)) \xi_t(x) dx dt.$$

(1.7)

The function $u$ is chosen in such a way that the free Gaussian’s mean asymptotically agrees with the solution of minimal energy of the classical field equation. This then makes up for the possibility of the $t = 0$ interacting measure to be absolutely continuous with respect to the $t = 0$ projection of the above Gaussian measure. In the remainder of the letter this argument will be made sound.

## 2 Infrared regular representation

First we translate the actions to well defined measures on path space. $S_p$ is associated with the stationary $P(\phi)_1$-process $q_t$ on $C(\mathbb{R}, \mathbb{R}^3)$ defined by the stochastic differential equation

$$dq_t = (\nabla \log \psi_0) dt + dB_t$$

(2.1)

($dB_t$ denotes Brownian motion), with invariant measure $dN^0 = \psi_0^2(x) dx$. Here $\psi_0$ is the ground state of the Schrödinger operator $(-1/2)\Delta + V(q)$ generating the $P(\phi)_1$-process. The path measure of this process will be denoted by $\mathcal{N}^0$. The paths $Q = \{q_t : t \in \mathbb{R}\}$ are $\mathcal{N}^0$-almost surely continuous and $q_t$ is a time-reversible Markov process. Moreover, for our class of potentials $V$ the semigroup of this process is intrinsically ultracontractive [3].

The field $\{\xi_t(f) : f \in \mathcal{S}(\mathbb{R}^3), t \in \mathbb{R}\}$ with $f \in \mathcal{S}(\mathbb{R}^3)$, i.e. the Schwartz space over $\mathbb{R}^3$, is in its turn described by the Gaussian measure $\mathcal{G}$ on $C(\mathbb{R}, \mathcal{S}'(\mathbb{R}^3))$ resulting from the (modified) Euclidean action $S^u_f$. This Gaussian measure has mean

$$\mathbb{E}_\mathcal{G}[\xi_t(f)] = -\int_{\mathbb{R}^3} \frac{\hat{\varrho}(k) \hat{f}(k) \hat{h}(k)}{|k|^2} dk \equiv \int_{\mathbb{R}^3} f(x) \gamma(x) dx$$

(2.2)
(i.e., represented in position space in terms of the function $\gamma$), and covariance

$$\text{cov}_G(\xi_s(f_1), \xi_t(f_2)) = \int_{\mathbb{R}^3} \frac{\hat{f}_1(k)\hat{f}_2(k)}{2|k|} e^{-|k||t-s|} dk.$$  (2.3)

We denote the $t = 0$ stationary distribution of this Markov process by $G$; it is itself a Gaussian measure with mean (2.2) and covariance (2.3) taken with $s = t$. For easing the argument below, we introduce the centred field

$$\eta_t(f) = \xi_t(f) - \mathbb{E}_G[\xi_t(f)], \quad f \in \mathcal{S}(\mathbb{R}^3);$$  (2.4)

as an auxiliary variable; this obviously has then a Gaussian distribution with zero mean and the same covariance as (2.3) above. Note that $u$ introduces a shift in the mean of the unmodified Gaussian measure $\mathcal{G}_0$ on the same space (which with $h \equiv 0$ would be 0). We denote by $\mathcal{G}_0$ the $t = 0$ distribution of $\mathcal{G}_0$. This shift induces the unitary map

$$\mathcal{U} : L^2(\mathcal{S}'(\mathbb{R}^3), dG) \rightarrow L^2(\mathcal{S}'(\mathbb{R}^3), dG_0), \quad (\mathcal{U}F)(f) = F(f - \gamma),$$  (2.5)

with $F : \mathcal{S}'(\mathbb{R}^3) \rightarrow \mathbb{R}, f \in \mathcal{S}'(\mathbb{R}^3)$ and $\gamma$ given by (2.2), which we introduce here for later use.

The non-interacting joint particle-field process is set up on $C(\mathbb{R}, \mathbb{R}^3 \times \mathcal{S}'(\mathbb{R}^3))$ and described by the path measure $\mathcal{P}^0 = \mathcal{N}^0 \times G$ with $t = 0$ distribution $\mathcal{P}^0 = \mathcal{N}^0 \times G$. The interacting system can be described by taking $\mathcal{P}^0$ as reference process and modifying it with the density given by

$$\frac{d\mathcal{P}_T}{d\mathcal{P}^0}(q, \xi) = \frac{1}{Z_T} \exp \left( - \int_{-T}^T S_{\text{int}}^{\mu}(\{q_t, \xi_t\}) dt \right)$$  (2.6)

Here

$$Z_T = \int \exp \left( - \int_{-T}^T S_{\text{int}}^{\mu}(\{q_t, \xi_t\}) dt \right) d\mathcal{P}^0$$  (2.7)

is the normalizing partition function. To have a more explicit formula instead of (2.7) we first make the Gaussian part of the integration to obtain

$$Z_T^Q = \mathbb{E}_{\mathcal{N}^0_T} \mathbb{E}_G \left[ \exp \left( - \int_{-T}^T \int_{\mathbb{R}^3} \hat{\eta}_t(k)\hat{\theta}(k)(e^{ik \cdot q_t} - \hat{h}(k)) dk dt \right) \right]$$

$$= \mathbb{E}_{\mathcal{N}^0_T} [Z_T^Q]$$

where

$$Z_T^Q = \int \exp \left( - \int_{-T}^T \int_{\mathbb{R}^3} \hat{\eta}_t(k)\hat{\theta}(k)(e^{ik \cdot q_t} - \hat{h}(k)) dk dt \right) \times$$

$$\times \exp \left( \int_{-T}^T \int_{\mathbb{R}^3} \hat{\gamma}(k)\hat{\theta}(k)(e^{ik \cdot q_t} - \hat{h}(k)) dk dt \right) d\mathcal{G}_0$$

$$= \exp \left( -\frac{1}{2} \int_{\mathbb{R}^3} \frac{|\hat{\theta}(k)|^2}{|k|} \hat{h}(k) dk \int_{-\infty}^\infty ds \int_{-T}^T dt e^{-|k||t-s|}(e^{ik \cdot q_t} - \hat{h}(k)) \right) \times$$

$$\times \exp \left( \frac{1}{4} \int_{-T}^T \int_{-T}^T \frac{|\hat{\theta}(k)|^2}{|k|} dk (e^{ik \cdot q_t} - \hat{h}(k))(e^{-ik \cdot q_t} - \hat{h}(k)) e^{-|k||t-s|} dt ds \right).$$

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where arguments extend directly.

\[ Z^Q_T = C_T \ Z^Q_T \]

with \( C_T > 0 \) independent of \( Q \) and

\[ Z^Q_T = \exp \left( - \int_{-T}^T \int_{-T}^T W(q_s - q_t, s - t) dt ds \right) \times \exp \left( - \int_{-T}^T \int_{|s| > T} dt \int_{ \mathbb{R}^3 } ds \int_{ \mathbb{R}^3 } W(q_t - q_s, t - s) h(q) dq \right) \times \exp \left( - \int_{-T}^T ds \int_{|t| > T} dt \int_{ \mathbb{R}^3 } W(q_s - q_t, t - s) h(q) dq \right). \]

Here

\[ W(q, t) = -\frac{1}{4} \int_{\mathbb{R}^3} \frac{\hat{\delta}(k)^2}{|k|} e^{-|k||q|} \cos(k \cdot q) \ dk \]

This implies that the marginal distribution \( N_T \) of \( P_T \) for the particle paths satisfies

\[ \frac{dN_T}{dN^0} = \frac{Z^Q_T}{Z_T} \]

with \( Z_T = \int Z^Q_T dN^0 \). The \( t = 0 \) single time distribution of \( N_T \) will be denoted by \( N_T \).

**Lemma 2.1** There is some \( e^0 > 0 \) such that for all \( |e| \leq e^0 \) the weak local limit \( \lim_{T \to \infty} N_T = \mathcal{N} \) exists and is a Gibbs probability measure on \( C(\mathbb{R}, \mathbb{R}^3) \). Moreover, \( \mathcal{N} \) does not depend on \( h \), i.e. it coincides with the limiting measure constructed for \( h \equiv 0 \).

**Proof:** By using cluster expansion, the Gibbs measure for \( h \equiv 0 \) can be shown to exist for \( W \) at sufficiently weak couplings \( |e| \leq e^* \). Here we have an interaction which differs from the cases studied in [4] by the last two factors in (2.10). However, they appear as extra energies coming from interactions between pieces of paths lying inside \([-T, T]\) with constant boundary conditions in \( \mathbb{R} \setminus [-T, T] \) weighted by \( h \), to which our previous arguments extend directly. \( \Box \)

For any fixed path \( Q = \{ q_t : t \in \mathbb{R} \} \in C(\mathbb{R}, \mathbb{R}^3) \) consider now the conditional distribution \( P^Q_T = P_T(\cdot | \{ q_t \} = Q) \) as a probability measure on \( C(\mathbb{R}, \mathbb{R}^3 \times S'(\mathbb{R}^3)) \). Since the interaction is linear in \( \xi_t \), \( P^Q_T \) is itself a Gaussian measure with covariance (2.3) and mean

\[ E_{P^Q_T}[\xi_t(f)] = \int_{\mathbb{R}^3} \hat{f}(k) \hat{g}^{Q,t}_T(k) dk \]

where

\[ \hat{g}^{Q,t}_T(k) = \hat{g}_T(k) \frac{\hat{\delta}(k)}{2|k|} \int_{-T}^T e^{-|k||t-\tau|} \left( e^{ikq_\tau} - \hat{h}(k) \right) d\tau \]

\[ = \frac{\hat{\delta}(k)}{2|k|} \int_{-T}^T e^{-|k||t-\tau|} e^{ikq_\tau} d\tau - \frac{\hat{\delta}(k)}{2|k|} \int_{|\tau| > T} e^{-|k||t-\tau|} d\tau. \]
Lemma 2.2 There is some $e^* > 0$ such that for all $|e| \leq e^*$ the weak local limit $\lim_{T \to \infty} \mathcal{P}_T = \mathcal{P}$ exists and is a probability measure on $C(\mathbb{R}, \mathbb{R}^3 \times \mathcal{S}'(\mathbb{R}^3))$. Moreover, $\mathcal{P}$ coincides with the path measure for $h \equiv 0$.

Proof: By using (2.14) it is immediate that $\lim_{T \to \infty} \hat{g}^Q_t = \hat{g}^Q_t$ exists and coincides with the mean for $\mathcal{P}^Q$ with $h \equiv 0$. Thus we have also that $\mathcal{P}_T^Q \to \mathcal{P}^Q$ exists. Moreover, since $\mathcal{P}_T(E \times F) = \int_F \mathcal{P}_T^Q(E) dN_T$, for all $T > 0$, where $E$ and $F$ are appropriate measurable sets, we also have the claim of the lemma. Hence the joint particle-field path measure does not depend on $h$, i.e. on how the reshuffling of the free field and interaction energies is performed. \hfill \square

$\mathcal{P}$ and $\mathcal{P}^0$ are Markov processes whose $t = 0$ distributions we denote below by $\mathcal{P}$ resp. $\mathcal{P}^0$. In the modified function space representation Nelson’s Hamiltonian is $H$ on the Hilbert space $\mathcal{H}^0 = L^2(\mathbb{R}^3 \times \mathcal{S}'(\mathbb{R}^3), d\mathcal{P}^0)$ acting like

\begin{equation}
(F, e^{-tH}G)_{\mathcal{H}^0} = \mathbb{E}_{\mathcal{P}^0} \left[ F(q_0, \xi_0)G(q_t, \xi_t) \exp(-\int_0^t \mathcal{L}_{\text{int}}'(\{q_s, \xi_s\})ds) \right], \quad t > 0
\end{equation}

with suitable $F$ and $G$, and where $\mathcal{L}_{\text{int}}'(\{q_t, \xi_t\}) = \int_{\mathbb{R}^3} (\varphi(x - q_t) - (\varphi \ast h)(x))\xi_t(x)dx$ is the Lagrangian corresponding to (1.7). The bottom of the spectrum of $H$ will be denoted by $E_0$. We define the Euclidean Hamiltonian $H_{\text{euc}}$ acting on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3 \times \mathcal{S}'(\mathbb{R}^3), d\mathcal{P})$ as the self-adjoint operator generating the symmetric contracting semigroup $T_t$ given by

\begin{equation}
(F, T_t G)_{\mathcal{H}} = \mathbb{E}_{\mathcal{P}} [F(q_0, \xi_0)G(\xi_t, q_t)], \quad t > 0.
\end{equation}

Theorem 2.3 Suppose $|e| \leq e^*$ with some $e^* > 0$. Then $H$ has a unique strictly positive ground state $\Psi_0$ corresponding to the lowest eigenvalue $E_0$, and $H - E_0$ is unitarily equivalent with $H_{\text{euc}}$.

Proof: From [3] we know that the existence of a strictly positive ground state of $H$ implies that $H - E_0$ and $H_{\text{euc}}$ are unitarily equivalent. Moreover, by Theorem 4.1 of the same reference we also know that for proving existence of the ground state it suffices to show that

\begin{equation}
\lim_{T \to \infty} \inf \left( \frac{d\mathcal{P}_T}{d\mathcal{P}^0} \right)^{1/2} d\mathcal{P}^0 > 0.
\end{equation}

We have

\[
\frac{d\mathcal{P}_T}{d\mathcal{G}}(\xi) = \exp \left( \int_{\mathbb{R}^3} (\xi(k) - \hat{\gamma}(k))\hat{\gamma}(k)dk \int_{-T}^T (e^{ik\cdot q_t} - \hat{h}(k))e^{-|k||t|}dt \right) \times \\
\exp \left( -\frac{1}{4} \int_{\mathbb{R}^3} dk \int_{-T}^T \int_{-T}^T \frac{|\hat{\gamma}(k)|^2}{|k|} (e^{ik\cdot q_s} - \hat{h}(k))(e^{-ik\cdot q_s} - \hat{h}(k))e^{-|k|(|t|+|s|)}dtds \right).
\]

Moreover,

\begin{equation}
\frac{d\mathcal{P}_T}{d\mathcal{P}^0}(q, \xi) = \mathbb{E}_{\mathcal{N}_T} \left[ \frac{d\mathcal{P}_T}{d\mathcal{G}}(\xi)|q_0 = q \right] d\mathcal{N}_T(q).
\end{equation}
By using that for some $c > 0$
\[
\frac{1}{c} \leq \frac{dN_T}{dN^0} \leq c
\]  
(2.19)
obtained similarly as in [4], in combination with Jensen’s inequality we obtain
\[
\int \left( \frac{dP_T}{d\nu} (q, \xi) \right)^{1/2} dP^0(q)
\]
(2.20)
\[
= \int E_{N_T} \left[ \left( \frac{dP}{d\nu} \right)^{1/2} |q = q_0 \right] \left( \frac{dN_T}{dN^0} (q) \right)^{1/2} dP^0
\]
\[
\geq \frac{1}{\sqrt{c}} \int \exp \left( E_{N_T} \left[ \frac{1}{2} \int_{t^{3}} (\xi(k) - \gamma(k)) \hat{\vartheta}(k) \int_{-T}^{T} e^{-|k| |t|}(e^{ikqt} - \hat{h}(k)) dt dk |q = q_0 \right] \right)
\times \exp \left( -\frac{1}{8} \int_{t^{3}} dk \int_{-T}^{T} \frac{1}{|k|} \hat{\vartheta}(k)^2 m_T(k; t) e^{-|k| (|t| + |s|)} ds dt \right) dP^0
\]
\[
\geq \frac{1}{\sqrt{c}} \exp \left( -\frac{c}{8} \int_{t^{3}} dk \int_{-T}^{T} \frac{1}{|k|} \hat{\vartheta}(k)^2 m_T(k; t) e^{-|k| (|t| + |s|)} ds dt \right)
\]
where
\[
m_T(k; s, t) = E_{N_T} \left[ (e^{ikqt} - \hat{h}(k))(e^{-ikqs} - \hat{h}(k)) \right].
\]  
(2.21)
We will use two different estimates of this function. One is the uniform bound
\[
|m_T(k; s, t)| \leq c_1,
\]  
(2.22)
and the other is
\[
|m_T(k; s, t)| \leq |k|^2 \int_{t^{3}} \int_{t^{3}} p^T_{s,t}(q_1, q_2) (|q_1| + b)(|q_2| + b) dN^0(q_1) dN^0(q_2)
\]  
(2.23)
for $|t - s| \geq 1$, where $p^T_{s,t}$ is the probability density with respect to $dN^0(q_1) dN^0(q_2)$ of the measure $N_T$ with $q_1 = q_1$, $q_2 = q_2$, and $b, c_1 > 0$ are some constants. Using the intrinsic hypercontractivity of the reference measure and the cluster expansion of [4] we have in this case that
\[
|p^T_{s,t}(q_1, q_2)| \leq c_2
\]  
(2.24)
with some $c_2 > 0$, moreover by the exponential fall-off of the stationary measure of the reference process
\[
\int_{t^{3}} (|q| + b) dN^0(q) = c_3 < \infty,
\]  
(2.25)
with some $c_3 > 0$. Thus we estimate the exponent in (2.20) by writing
\[
\int_{t^{3}} dk \int_{-T}^{T} \int_{-T}^{T} \frac{1}{|k|} \hat{\vartheta}(k)^2 m_T(k; s, t) e^{-|k| (|t| + |s|)} ds dt
\]
\[
\leq \int_{t^{3}} \frac{1}{|k|} \left( c_1 \int_{-T \leq s, t \leq T} e^{-|k| (|t| + |s|)} ds dt + c_2 c_3^2 |k|^2 \int_{|t|, |s| \leq T} e^{-|k| (|t| + |s|)} ds dt \right) dk
\]
\[
\leq \int_{t^{3}} \frac{1}{|k|} \left( A_1 + A_2 \right) dk < \infty
\]
with some constants $A_1, A_2 > 0$. This then yields a non-zero uniform lower bound on \((2.17)\), and completes the proof of the theorem.

\[ \square \]

### 3 Hamiltonian in the new representation

The Hamiltonian $H$ of the infrared regular representation is defined through \((2.15)\). For having a more concrete expression the natural strategy is to build the Fock space $\mathcal{F}$ over $L^2(S'(\mathbb{R}^3), dG)$ and to transform $H$ unitarily to an operator acting on $L^2(\mathbb{R}^3, dq) \otimes \mathcal{F}$. We do this in three steps.

First we map $L^2(\mathbb{R}^3, d\mathcal{N}_0)$ to $L^2(\mathbb{R}^3, dq)$ through the similarity transformation $R: \phi(q) \mapsto \psi_0(q)\phi(q)$. Thereby the particle Hamiltonian becomes

\[ H_p = -\frac{1}{2}\Delta + V(q). \tag{3.1} \]

Next we deal with the Hamiltonian for the free field. First $\mathcal{U}$ defined by \((2.5)\) is used to map $L^2(S'(\mathbb{R}^3), dG)$ into $L^2(S'(\mathbb{R}^3), dG_0)$. This map induces a transformation between the shifted, $G$, and non-shifted, $G_0$, Gaussian processes, and keeps their generator the same. Then by the Wiener-Itô transform $W: L^2(S'(\mathbb{R}^3), dG_0) \rightarrow \mathcal{F}$ this is further mapped into Fock space (for details see e.g. \[7\]). Finally we transform the interaction. By \((2.15)\) in $L^2(\mathbb{R}^3 \times S'(\mathbb{R}^3), dq \times dG)$ the interaction is multiplication by

\[ \int_{\mathbb{R}^3} \hat{\varphi}(k)\hat{\xi}(k) \left( e^{ik\cdot q} - \hat{h}(k) \right) e^{-ik\cdot x} \, dk \tag{3.2} \]

which under $\mathcal{U}$ goes over to the multiplication operator

\[ \int_{\mathbb{R}^3} \hat{\varphi}(k)\hat{\xi}(k) - \hat{\gamma}(k) \left( e^{ik\cdot q} - \hat{h}(k) \right) e^{-ik\cdot x} \, dk \tag{3.3} \]

on $L^2(\mathbb{R}^3 \times S'(\mathbb{R}^3), dq \times dG_0)$. Consider now

\[ \mathcal{V} = (1 \otimes W)(1 \otimes \mathcal{U})(R \otimes 1) \tag{3.4} \]

as an isometry from $L^2(\mathbb{R}^3 \times S'(\mathbb{R}^3), d\mathcal{P}_0)$ to $L^2(\mathbb{R}^3, dq) \otimes \mathcal{F}$. Then

\[ \mathcal{V}H\mathcal{V}^{-1} = \left( -\frac{1}{2}\Delta + V(q) \right) \otimes 1 + 1 \otimes \int_{\mathbb{R}^3} |k|a^*(k)a(k) \, dk \]

\[ + \int_{\mathbb{R}^3} \frac{\hat{\varphi}(k)}{\sqrt{|k|}} \left( e^{ik\cdot q} - \hat{h}(k) \right) \otimes a(k) + \left( e^{-ik\cdot q} - \hat{h}(k) \right) \otimes a^*(k) \, dk \]

\[ - \int_{\mathbb{R}^3} \frac{|\hat{\varphi}(k)|^2}{k^2} \hat{h}(k) \left( e^{ik\cdot q} - \hat{h}(k) \right) \otimes 1 \, dk \equiv H^\text{ren}_N. \tag{3.5} \]

The standard Nelson Hamiltonian $H_N$ corresponds to setting $\hat{h} \equiv 0$ in \((3.3)\). We see that the subtraction of $\hat{h}$ regularizes the interaction at small $k$ and makes $H^\text{ren}_N$ a well
defined Hamiltonian having a ground state in \( L^2(\mathbb{R}^3, dq) \otimes \mathcal{F} \). In an operator theoretic context, Arai proposed \( H_{\text{ren}}^N \) of the form 3.3 for the special case \( \hbar \equiv 1 \) and proved that under similar assumptions on the potential as ours \( H_{\text{ren}}^N \) has a ground state [1].

Obviously, for obtaining the ground state expectation of an operator \( A \) of physical interest, one can use \( H_{\text{ren}}^N \) only in conjunction with the transformation \( A \mapsto \mathcal{V}A\mathcal{V}^{-1} \). Thus expectations of \( A \) obtained in the standard way by first introducing the cut-off Hamiltonian, taking ground state expectations with it, and subsequently removing the cut-off are identical with computing averages of \( \mathcal{V}A\mathcal{V}^{-1} \) by using \( H_{\text{ren}}^N \). As proved above, the infrared limit is thus correctly taken care of.

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