A New Method to Derive Low-Lying N-dimensional Quantum Wave Functions by Quadratures Along a Single Trajectory

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Abstract

We present a new method to derive low-lying N-dimensional quantum wave functions by quadrature along a single trajectory. The N-dimensional Schroedinger equation is cast into a series of readily integrable first order ordinary differential equations. Our approach resembles the familiar W.K.B. approximation in one dimension, but is designed to explore the classically forbidden region and has a much wider applicability than W.K.B.. The method also provides a perturbation series expansion and the Green’s functions of the wave equation in N-dimension, all by quadratures along a single trajectory. A number of examples are given for illustration, including a simple algorithm to evaluate the Stark effect in closed form to any finite order of the electric field.

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1. INTRODUCTION

In this paper, we begin with the generalization to arbitrary dimensions of a new method \[1\] in which the low-lying quantum wave function of a particle in a potential \(+V\) can be obtained by quadratures along the trajectory of a corresponding classical mechanical problem with \(-V\) as its potential. Let \(q = (q_1, q_2, \ldots, q_N)\) describes the \(N\)-dimensional coordinates of the particle, and \(\Phi(q)\) is the ground-state (or low-lying) quantum wave function that satisfies

\[ H\Phi(q) = E\Phi(q), \]  
\[
(1.1)
\]

where

\[ H = -\frac{1}{2} \nabla^2 + V(q) \]
\[
(1.2)
\]

is the Hamiltonian of a non-relativistic particle of unit mass, and

\[ \nabla^2 = \sum_{i=1}^{N} \frac{\partial^2}{\partial q_i^2}. \]
\[
(1.3)
\]

We assume \(V(q)\) to be bounded from below; therefore we can choose

\[ V(q) \geq 0. \]
\[
(1.4)
\]

Its minimum \(V(q) = 0\) may occur at more than one point, each with a non-vanishing curvature.

As in \[1\], we introduce a scale factor \(g^2\) by writing

\[ V(q) = g^2 v(q), \]
\[
(1.5)
\]

and consider the case of large \(g\). We express

\[ \Phi(q) = e^{-gS(q)} \]
\[
(1.6)
\]

in terms of a formal expansion:

\[ gS(q) = gS_0(q) + S_1(q) + g^{-1}S_2(q) + g^{-2}S_3(q) + \cdots \]
\[
(1.7)
\]
with a similar expansion for $E$,

$$E = gE_0 + E_1 + g^{-1}E_2 + g^{-2}E_3 + \cdots. \quad (1.8)$$

Substituting (1.6) - (1.8) into the Schrödinger equation (1.1) and equating the coefficients of $g^{-n}$ on both sides, we derive

$$(\nabla S_0)^2 = 2v,$$

$$\nabla S_0 \cdot \nabla S_1 = \frac{1}{2} \nabla^2 S_0 - E_0,$$

$$\nabla S_0 \cdot \nabla S_2 = \frac{1}{2} \left[ \nabla^2 S_1 - (\nabla S_1)^2 \right] - E_1,$$

$$\nabla S_0 \cdot \nabla S_3 = \frac{1}{2} \left[ \nabla^2 S_2 - 2(\nabla S_1) \cdot (\nabla S_2) \right] - E_2,$$

etc., where $\nabla$ is the $N$-dimensional gradient vector whose components are $\nabla_i = \partial / \partial q_i$ and

$$i = 1, 2, \cdots, N. \quad (1.10)$$

The underlying reason for considering the expressions (1.7) and (1.8) is that for large $g$ the ground-state quantum wave function has its amplitude centered around the minimum of $V(q)$; near the minimum, $V(q)$ may be approximated by a quadratic function of $q$, with the curvature of the corresponding potential surface proportional to $g^2$. Thus, for large $g$, the ground-state energy is $O(g)$ and so is the exponent of the ground-state wave function, as suggested by the ground-state energy and the corresponding wave function of a harmonic oscillator. The parameter $g^{-1}$ serves as a measure of anharmonicity.

It is useful to write the first line of (1.9) as

$$\frac{1}{2}(\nabla S_0)^2 - v(q) = e,$$  \quad (1.11)

where

$$e = 0 + . \quad (1.12)$$
Equation (1.11) is equivalent to the corresponding Hamilton-Jacobi equation in classical mechanics, with $-V(q)$ as the potential and $e$ the total energy. To solve (1.11), we pick a minimum of $V(q)$, say $q = 0$ with

$$V(0) = g^2 v(0) = 0. \quad (1.13)$$

Consider a trajectory $q(t)$ which begins at $q = 0$ when $t = 0$ and ends at $q_T$ when $t = T$; i.e.,

$$q(0) = 0 \quad \text{and} \quad q(T) = q_T. \quad (1.14)$$

Hamilton's action-integral is given by the minimum of the path integral

$$S_0(q_T, e) = \int_0^T \left[ \frac{1}{2} \sum_{i=1}^N \dot{q}_i(t)^2 - (-v(q(t))) \right] dt + T e, \quad (1.15)$$

where

$$T = \left( \frac{\partial S_0}{\partial e} \right)_{q_T} \quad (1.16)$$

is the total time of the trajectory from the initial point $0$ to the final point $q_T$. Since the potential in the classical problem $= -v(q) \leq 0$, the integrand in (1.15) is positive everywhere. Hence, given any final point $q_T$ and any total energy $e > 0$, the minimum of (1.15) exists, so is therefore the classical trajectory. (For clarity, assume the minimum to be unique, a restriction that will be removed later.) As $e \to 0^+$, the initial velocity $\dot{q}$ is proportional to $e^{1/2}$ which also approaches zero; the total time $T$ of the classical trajectory is $\propto e^{-1/2} \to \infty$, yielding

$$\lim_{e=0^+} T e = 0. \quad (1.17)$$

The same limit $e = 0^+$ of (1.15) gives $S_0$ of (1.11) - (1.12).

To go back to the first line of (1.9), we designate the end-point $q_T$ of the classical trajectory simply as $q$, the point of interest for the quantum solution $\Phi(q)$. Setting
\[ q_T = q. \] \hfill (1.18)

the solution \( S_0(q) \) of the first line in (1.9) can now be written as

\[ S_0(q) = \int_0^T \left[ \frac{1}{2} \sum_{i=1}^N q_i^2 - (-v(q)) \right] dt, \] \hfill (1.19)

where the integral is along the trajectory \( q_i(t) \) that satisfies the classical equations of motion

\[ \ddot{q}_i = \partial v/\partial q_i, \] \hfill (1.20)

the energy conservation

\[ \frac{1}{2} \sum_{i=1}^N \dot{q}_i(t)^2 - v(q(t)) = 0 + \] \hfill (1.21)

and the initial and final configurations (1.14) and (1.18).

The solution \( S_0(q) \) thus obtained is positive and continuous everywhere; it is an increasing function along the direction of the classical trajectory. In the case that the classical potential \( -v(q) \) has other absolute maxima besides 0, say

\[ -v(a_1) = -v(a_2) = \cdots = -v(a_n) = 0, \] \hfill (1.22)

\( S_0(q) \) is not analytic at these isolated points

\[ q = a_1, a_2, \cdots, a_n. \] \hfill (1.23)

For a total energy \( e > 0 \), along the classical trajectory the velocity must continue in its direction when the trajectory passes through any of the other maxima of \( -v(q) \), say \( q = a_n \). Thus, as \( e \to 0^+ \), \( \nabla S_0 \) becomes zero at \( q = a_n \), but its sign along the trajectory-direction remains the same before or after \( a_n \); this forces \( \nabla S_0 \) to develop a kink at \( q = a_n \). At the initial point \( q = 0 \), while \( -v(0) = 0 \) and therefore \( \nabla S_0 = 0 \) as \( e = 0^+ \), \( S_0(q) \) is analytic, since different trajectories emanating from \( q = 0 \) have to go along different directions. At infinity, it is easy to see that \( S_0(q) = \infty \), and therefore \( \Phi_0 \equiv e^{-gS_0} \) is zero.
To solve the second equation in (1.9), we require $S_1(q)$ to be also analytic at $q = 0$; this yields

$$E_0 = \frac{1}{2}(\nabla^2 S_0)_{\text{at } q=0}.$$ (1.24)

It is convenient to consider the surface

$$S_0(q) = \text{constant};$$ (1.25)

its normal is along the corresponding classical trajectory passing through $q$. Characterize each classical trajectory by the $S_0$-value along the trajectory and a set of $N - 1$ angular variables

$$\alpha = (\alpha_1(q), \alpha_2(q), \cdots, \alpha_{N-1}(q)),$$ (1.26)

so that each $\alpha$ determines one classical trajectory with

$$\nabla \alpha_j \cdot \nabla S_0 = 0,$$ (1.27)

where

$$j = 1, 2, \cdots, N - 1.$$ (1.28)

(As an example, we note that as $q \to 0$, $v(q) \to \frac{1}{2} \sum \omega_i^2 q_i^2$ and therefore $S_0 \to \frac{1}{2} \sum \omega_i q_i^2$. Consider the ellipsoidal surface $S_0 = \text{constant} \equiv \frac{1}{2} l^2$. For $l$ sufficiently small, each classical trajectory is normal to this ellipsoidal surface. A convenient choice of $\alpha$ could be simply any $N - 1$ orthogonal parametric coordinates on the surface.) Each $\alpha$ designates one classical trajectory, and vice versa. Every $(S_0, \alpha)$ is mapped into a unique set $(q_1, q_2, \cdots, q_N)$ with $S_0 \geq 0$ by construction. Depending on the problem, the converse may or may not be true; i.e. $(q_1, q_2, \cdots, q_N) \to (S_0, \alpha)$ could be either one to one, or one to many. In the latter case, we regard the points in the $q$-space as specified by the coordinates $(S_0, \alpha)$, rather than the original coordinates $(q_1, q_2, \cdots, q_N)$. 

7
Write
\[ S_1(q) = S_1(S_0, \alpha), \] (1.29)
the second line of (1.9) becomes
\[ (\nabla S_0)^2 \frac{\partial S_1}{\partial S_0} = \frac{1}{2} \nabla^2 S_0 - E_0, \] (1.30)
and therefore
\[ S_1(q) = S_1(S_0, \alpha) = \int_0^{S_0} \frac{dS_0}{(\nabla S_0)^2} \left\{ \frac{1}{2} \nabla^2 S_0 - E_0 \right\}, \] (1.31)
where the integration is taken along the classical trajectory of constant \( \alpha \). Likewise, the third, fourth and other lines of (1.9) lead to
\[ E_1 = \frac{1}{2} \left[ \nabla^2 S_1 - (\nabla S_1)^2 \right]_{q=0}, \] (1.32)
\[ S_2(q) = S_2(S_0, \alpha) = \int_0^{S_0} \frac{dS_0}{(\nabla S_0)^2} \left\{ \frac{1}{2} \nabla^2 S_1 - (\nabla S_1)^2 - E_1 \right\}, \] (1.33)
\[ E_2 = \frac{1}{2} \left[ \nabla^2 S_2 - 2(\nabla S_1) \cdot (\nabla S_2) \right]_{q=0}, \] (1.34)
\[ S_3(q) = S_3(S_0, \alpha) = \int_0^{S_0} \frac{dS_0}{(\nabla S_0)^2} \left\{ \frac{1}{2} \nabla^2 S_2 - 2(\nabla S_1) \cdot (\nabla S_2) - E_2 \right\}, \] (1.35)
etc. These solutions give the convenient normalization convention at \( q = 0 \),
\[ S(0) = 0 \] (1.36)
and\[ \Phi(0) = 1. \]
The above expressions for \( E_0, E_1, E_2, \cdots \) can be derived more directly by introducing
\[ \sigma(q) \equiv gS(q) - g S_0(q) \\
= S_1(q) + g^{-1} S_2(q) + g^{-2} S_3(q) + \cdots. \] (1.37)
Thus, the ground-state wave function $\Phi(q)$ is

$$\Phi = e^{-gS_0 - \sigma},$$  \hfill (1.38)

and the corresponding Schroedinger equation (1.1) gives

$$g\nabla S_0 \cdot \nabla \sigma = \frac{1}{2} [g\nabla^2 S_0 + \nabla^2 \sigma - (\nabla \sigma)^2] - E.$$  \hfill (1.39)

At $q = 0$, $v(q) = 0$ and $\nabla S_0 = 0$, on account of $(\nabla S_0)^2 = 2v$. Assuming $\nabla \sigma$ to be regular, we have

$$E = \frac{1}{2} [g\nabla^2 S_0 + \nabla^2 \sigma - (\nabla \sigma)^2] \text{ at } q = 0$$  \hfill (1.40)

which leads to (1.24), (1.32), (1.34), · · · for $E_0, E_1, E_2, \cdots$.

Example.

As an example, consider the special case

$$V(q) = \frac{g^2}{2} (q_1^2 + q_2^2 + \cdots + q_N^2).$$  \hfill (1.41)

From (1.9), one can readily show that the first line gives

$$S_0(q) = \frac{g}{2} (q_1^2 + q_2^2 + \cdots + q_N^2)$$  \hfill (1.42)

and (1.24) leads to

$$E_0 = N/2;$$  \hfill (1.43)

the other equations (1.31) - (1.35), etc., yield $E_1 = E_2 = \cdots = 0$ and $S_1 = S_2 = \cdots = 0$.

The result is the well known exact answer

$$\Phi(q) = exp\left[ -\frac{g}{2} (q_1^2 + q_2^2 + \cdots + q_N^2) \right]$$  \hfill (1.44)

and
\[ E = \frac{g}{2} N. \] (1.45)

The organization of the paper is as follows: In Section 2, we discuss the case of a separable potential, and show that the solution of the wave function is a product of one-dimensional functions, as expected. The particular one-dimensional example of

\[ V(x) = \frac{g^2}{2} (x^2 - a^2)^2 \] (1.46)

has been analyzed in ref. [1] using our method. In this problem, besides the anharmonicity parameter \( g^{-1} \), there is also a much smaller barrier penetration parameter

\[ e^{-\frac{4}{3} ga^3}. \] (1.47)

(The same problem has also been extensively studied in the literature[2-9], using mostly Feynman’s path integration method. Quite often, the two parameters \( g \) and \( a \) in (1.46) are related by \( 2ag = 1 \), which makes \( V = \frac{1}{2} y^2 (1 - gy)^2 \) with \( y = x + a \). The dimensionless anharmonicity parameter \( 1/\text{ga}^3 \) then becomes \( 8g^2 \) and the barrier penetration parameter (1.47) becomes \( e^{-1/6g^2} \). In our \( g^{-1} \)-expansion, we keep the parameter \( a \) fixed.)

The ground-state wave function \( \psi_{\text{even}}(x) \) is even in \( x \), and the first excited state is an odd function \( \psi_{\text{odd}}(x) \). Let \( E_{\text{even}} \) and \( E_{\text{odd}} \) be their corresponding eigenvalues. We can expand both their sum and difference as the following double series:

\[ \frac{1}{2}(E_{\text{odd}} + E_{\text{even}}) = ga \sum_{m,n} C_{mn}(ga^3)^{-m} e^{-\frac{2}{3} nga^3} \] (1.48)

and

\[ \frac{1}{2}(E_{\text{odd}} - E_{\text{even}}) = 4 \left( \frac{2}{\pi} g^3 a^5 \right)^{\frac{1}{2}} e^{-\frac{4}{3} ga^3} \sum_{m,n} c_{mn}(ga^3)^{-m} e^{-\frac{4}{3} (n-1)ga^3} \] (1.49)

where \( C_{mn} \) and \( c_{mn} \) are numerical coefficients, which can be explicitly expressed as definite integrals [1], with the results
\[ c_{m0} = 0 \quad \text{for all} \quad m, \]
\[ C_{00} = c_{01} = 1, \quad C_{10} = -\frac{1}{4}, \]
\[ C_{20} = -\frac{9}{2^6}, \quad C_{30} = -\frac{89}{2^9}, \quad \text{etc.} \]

The terms with \( n = 1 \) represent the one-instanton contributions\[2-9\], \( n = 2 \) the two-instanton contributions, etc. As one can see from the derivation, an important tool, in addition to the \( g^{-1} \) expansion (1.7) - (1.9) discussed above, is the use of the Sturm-Liouville type of Green’s functions. In this paper, we will show how to generalize such Green’s function technique to an \( N \)-dimensional problem. In Section 3, we introduce a compact notation which will pave the way towards the generalization.

Section 4 contains the main body of this paper. We depart from the initial approach based on the Hamilton-Jacobi equation; instead, we consider two \( N \)-dimensional Hamiltonians:

\[ H = -\frac{1}{2} \nabla^2 + V(q) \]  

(1.51)

with, as before,

\[ V(q) = g^2 v(q) \geq 0, \]

and

\[ \mathcal{H} = H + \epsilon U(q). \]  

(1.52)

Assuming that the ground-state wave function \( \Phi(q) = e^{-gS(q)} \) of the unperturbed Hamiltonian \( H \) is known, we set out to derive the perturbation series expansion for the corresponding ground-state wave function \( \Psi(q) \) of \( \mathcal{H} \) with \( \epsilon U \) as the perturbation. Again, as we shall see, to each order in \( \epsilon \), the perturbed \( \Psi(q) \) can be obtained by quadrature along a single trajectory connecting 0, the point that the unperturbed \( \Phi(q) \) is maximum (i.e. \( S \) is minimum), to the point \( q \) of interest; the trajectory is determined by being the normal to the \( S = \) constant surfaces along the entire course of the trajectory. (It may be called the quantum trajectory, in
contrast to the classical trajectory determined by the Hamilton-Jacobi equation.) The form of the new perturbation series differs from the usual expression; the result is of course the same, as will be illustrated by a few examples. The new perturbation series formula will also be shown to cover the previous analysis in Sections 1-3 based on the classical trajectory derived from the Hamilton-Jacobi equation as special cases.

In the same Section 4, the Green’s function $G$ of the N-dimensional Hamiltonian $H$ will be derived, with $G$ satisfying

$$ (H - E)G = I \quad (1.53) $$

and $E$ being the ground-state energy of $H$. Usually, the Green’s function and the perturbation series expansion to each order $\epsilon^n$, require either an infinite sum over all excited levels of $H$, or equivalently, a sum over all possible paths through Feynman’s path integration method. The Green’s function and the perturbation series formula derived here are different; they depend only on quadratures along a single trajectory.

In Section 5, we give examples to illustrate the applications of the new perturbation series, as well as to clarify the properties of the Green’s function. In Section 6, we show how the trajectory-quadrature formulation can be extended to derive excited states, and in Section 7 we discuss how generalization can be made to allow for singular potentials, such as an attractive Coulomb potential with perturbations. As we shall see, with our new approach we can evaluate, e.g., the Stark effect to any finite order of the perturbation by quadratures along the radial trajectory.

Although our methods are elementary, they become surprisingly powerful for the following reason. For example, in (1.38) by writing the wave function in the form of $e^{-gS_0 - \sigma}$, where $g$ is large, we cause the cross-term $g \nabla S_0 \cdot \nabla \sigma$ to be much larger than $\frac{1}{2} \nabla^2 \sigma - (\nabla \sigma)^2$. Therefore the differential equation for $\sigma$ can be written order by order in such a way that $\frac{1}{2} \nabla^2 \sigma - (\nabla \sigma)^2$ is known from previous orders, and $\nabla S_0 \cdot \nabla \sigma = (\nabla S_0)^2 \frac{\partial \sigma}{\partial S_0}$ is inferred from the equation. In
effect, to each order, \( \sigma \) now satisfies a first order ordinary differential equation (along the trajectory of constant \( \alpha \)), not a second order partial differential equation. This makes possible a direct solution through quadratures. With variations, this theme runs through all the sections to be presented in this paper (sometimes \( e^{-\sigma} \) is written \( e^{-\tau} \), or \( \chi \), and sometimes the known exponent is called \( gS \), rather than \( gS_0 \)).

2. SEPARABLE POTENTIAL

For clarity, we consider the special case of dimension \( N = 2 \). The discussions below for a separable potential can be trivially extended to any \( N > 2 \). Let the two dimensional coordinate \( q \) in the previous section be designated

\[
q = (x, y)
\]  

(2.1)

and the quantum mechanical potential \( V(q) \) and \( \nabla^2 \) in (1.2) be

\[
V(q) = g^2v(q) = g^2[v_x(x) + v_y(y)]
\]

(2.2)

and

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.
\]

(2.3)

As in (1.4) and (1.13), we assume

\[
v_x(x) \geq 0, \quad v_y(y) \geq 0
\]

(2.4)

and at the point \( x = y = 0 \),

\[
v_x(0) = v_y(0) = 0.
\]

(2.5)

The classical Lagrangian density for the corresponding potential \(-v(q)\) is
\[ \frac{1}{2} [\ddot{x}(t)^2 + \ddot{y}(t)^2] + v_x(x) + v_y(y), \]  

(2.6)

where the dots denote time-derivatives, as before. The equations of motion are

\[ \ddot{x}(t) = \frac{d}{dx} v_x(x) \]

and

\[ \ddot{y}(t) = \frac{d}{dy} v_y(y). \]

(2.7)

As in (1.14), we consider a trajectory that begins at \( x(0) = y(0) = 0 \) (the maximum, or one of the maxima, of \(-v\)) at \( t = 0 \) and ends at

\[ x(T) = x_T \quad \text{and} \quad y(T) = y_T \]

(2.8)

when \( t = T > 0 \), in which, \( x_T \) and \( y_T \) are variables independent of \( T \). The action integral along the trajectory is given by the sum of

\[ A_x(x_T, T) = \int_{0}^{T} \left[ \frac{1}{2} \dot{x}^2 - (-v_x(x)) \right] dt \]

and

\[ A_y(y_T, T) = \int_{0}^{T} \left[ \frac{1}{2} \dot{y}^2 - (-v_y(y)) \right] dt \]

(2.9)

The minimum of \( A_x(x_T, T) \) and \( A_y(y_T, T) \), keeping \( x_T, y_T \) and \( T \) fixed, gives the Lagrangian equations of motion (2.7), and the derivatives of \( A_x \) and \( A_y \) with respect to the final variables \( x_T, y_T \) and \( T \) are

\[ \frac{\partial}{\partial x_T} A_x(x_T, T) = \ddot{x}(T) \]

\[ \frac{\partial}{\partial y_T} A_y(y_T, T) = \ddot{y}(T) \]

\[ \frac{\partial}{\partial T} A_x(x_T, T) = -e_x \]

(2.10)

and

\[ \frac{\partial}{\partial T} A_y(y_T, T) = -e_y. \]
where $e_x$ and $e_y$ are the energies in the $x$ and $y$ directions, given by

$$e_x = \frac{1}{2} x^2 - v_x(x)$$

and

$$e_y = \frac{1}{2} y^2 - v_y(y).$$

For a given final configuration characterized by $x_T$, $y_T$ and $T$, both $e_x$, $e_y$, and therefore also their sum

$$e = e_x + e_y$$

are determined. Following (1.15), we transform the independent variables from $x_T$, $y_T$ and $T$ to $x_T$, $y_T$ and $e$. We write

$$S_0(x_T, y_T, e) = A_x(x_T, T) + A_y(y_T, T) + T e,$$

where $T$ is a dependent variable, determined by

$$T = T(x_T, y_T, e) = \left( \frac{\partial S_0}{\partial e} \right)_{x_T, y_T}.$$  

Given $x_T$, $y_T$ and $e$, the classical trajectory is given by the minimum of (2.13), from which $T$, $e_x$, $e_y$ are also determined. Thus, we can regard

$$e_x = e_x(x_T, y_T, e),$$

and

$$e_y = e_y(x_T, y_T, e).$$

Hamilton’s action integral (1.15) becomes

$$S_0(x_T, y_T, e) = X_0(x_T, e_x) + Y_0(y_T, e_y),$$

where
\[ X_0(x_T, e_x) = A_x(x_T, T) + T \, e_x \]  

and  

\[ Y_0(y_T, e_y) = A_y(y_T, T) + T \, e_y, \]  

in which \( T \) is given by (2.14). Combining (2.10) and (2.17), we can readily show that  

\[ \left( \frac{\partial S_0}{\partial x_T} \right)_{y_T,e} = \left( \frac{\partial X_0}{\partial x_T} \right)_{x_T} = \dot{x} \, (T) \]  

\[ \left( \frac{\partial S_0}{\partial y_T} \right)_{x_T,e} = \left( \frac{\partial Y_0}{\partial y_T} \right)_{y_T} = \dot{y} \, (T) \]  

(2.18)  

and  

\[ \left( \frac{\partial X_0}{\partial e_x} \right)_{x_T} = \left( \frac{\partial Y_0}{\partial e_y} \right)_{y_T} = \left( \frac{\partial S_0}{\partial e} \right)_{x_T,y_T} = T. \]  

Along each classical trajectory, when the final time variable changes from \( T \) to \( T + dT \), the end point moves from \( (x_T, y_T) \) to \( (x_T + \dot{x} \, (T) dT, y_T + \dot{y} \, (T) dT) \); correspondingly,  

\[ dS_0 = \left( \frac{\partial X_0}{\partial x_T} \right)_{e_x} \dot{x} \, (T) dT + \left( \frac{\partial Y_0}{\partial y_T} \right)_{e_y} \dot{y} \, (T) dT + e \, dT \]  

(2.19)  

Combining the first two lines of (2.18) with (2.19) and taking the limit \( e = 0^+ \), we find  

\[ dS_0 = \left( \left( \frac{\partial X_0}{\partial x_T} \right)^2 + \left( \frac{\partial Y_0}{\partial y_T} \right)^2 \right) dT = (\nabla S_0)^2 dT. \]  

(2.20)  

Along the trajectory we also have  

\[ dT = dx_T/ \dot{x} \, (T) = dy_T/ \dot{y} \, (T), \]  

(2.21)  

which, together with (2.18), leads to  

\[ dT = dx_T/ \left( \frac{\partial X_0}{\partial x_T} \right)_{e_x} = dy_T/ \left( \frac{\partial Y_0}{\partial y_T} \right)_{e_y}. \]  

(2.22)  

Substituting (2.20) and (2.22) into (1.31) and labeling \( x_T, y_T \) simply as \( x, y \), we obtain  

\[ S_1(x, y) = X_1(x) + Y_1(y), \]  

(2.23)
where

\[
X_1(x) = \int_0^x dx \left( \frac{\partial X_0}{\partial x} \right)^{-1} \left\{ \frac{1}{2} \frac{\partial^2 X_0}{\partial x^2} - E_{0,x} \right\},
\]

\[
Y_1(x) = \int_0^y dy \left( \frac{\partial Y_0}{\partial y} \right)^{-1} \left\{ \frac{1}{2} \frac{\partial^2 Y_0}{\partial y^2} - E_{0,y} \right\}
\]

(2.24)

with

\[
E_{0,x} = \frac{1}{2} \left( \frac{\partial^2 X_0}{\partial x^2} \right)_{x=0}
\]

and

\[
E_{0,y} = \frac{1}{2} \left( \frac{\partial^2 Y_0}{\partial y^2} \right)_{y=0}
\]

(2.25)

and

\[
E_0 = E_{0,x} + E_{0,y},
\]

(2.26)

consistent with (1.24).

Likewise, \( S_2, S_3, \cdots \) can be written as sums of the form \( X_2(x) + Y_2(y) \), \( X_3(x) + Y_3(y) \), \( \cdots \), with

\[
X_2(x) = \int_0^x dx \left( \frac{\partial X_0}{\partial x} \right)^{-1} \left\{ \frac{1}{2} \frac{\partial^2 X_1}{\partial x^2} - \left( \frac{\partial X_1}{\partial x} \right)^2 \right\} - E_{1,x},
\]

\[
Y_2(y) = \int_0^y dy \left( \frac{\partial Y_0}{\partial y} \right)^{-1} \left\{ \frac{1}{2} \frac{\partial^2 Y_1}{\partial y^2} - \left( \frac{\partial Y_1}{\partial y} \right)^2 \right\} - E_{1,y}
\]

(2.27)

\[
X_3(x) = \int_0^x dx \left( \frac{\partial X_0}{\partial x} \right)^{-1} \left\{ \frac{1}{2} \frac{\partial^2 X_2}{\partial x^2} - 2 \frac{\partial X_1}{\partial x} \frac{\partial X_2}{\partial x} \right\} - E_{2,x},
\]

\[
Y_3(y) = \int_0^y dy \left( \frac{\partial Y_0}{\partial y} \right)^{-1} \left\{ \frac{1}{2} \frac{\partial^2 Y_2}{\partial y^2} - 2 \frac{\partial Y_1}{\partial y} \frac{\partial Y_2}{\partial y} \right\} - E_{2,y}
\]

where

\[
E_{1,x} = \frac{1}{2} \left( \frac{\partial^2 X_1}{\partial x^2} - \left( \frac{\partial X_1}{\partial x} \right)^2 \right)_{x=0},
\]
\[ E_{1,y} = \frac{1}{2} \left( \frac{\partial^2 Y_1}{\partial y^2} - \left( \frac{\partial Y_1}{\partial y} \right)^2 \right) \text{at } y = 0, \quad (2.28) \]

\[ E_{2,x} = \frac{1}{2} \left( \frac{\partial^2 X_2}{\partial x^2} - \frac{\partial X_1}{\partial x} \frac{\partial X_2}{\partial x} \right) \text{at } x = 0, \]

\[ E_{2,y} = \frac{1}{2} \left( \frac{\partial^2 Y_2}{\partial y^2} - \frac{\partial Y_1}{\partial y} \frac{\partial Y_2}{\partial y} \right) \text{at } y = 0, \]

etc.. The corresponding wave function factors \( e^{-gS_0}, e^{-S_1}, e^{-g^{-1}S_2}, \ldots \) are all products \( e^{-g(X_0+Y_0)}, e^{-(X_1+Y_1)}, e^{-g^{-1}(X_2+Y_2)}, \ldots \), etc..

3. A COMPACT NOTATION

In this section we shall go over completely to the variables \( S_0, \alpha \) and express the \( g^{-1} \) expansion ((1.7) and (1.31)-(1.35)) described in Section 1 as iteration of an integral operator along a classical trajectory, that is for variable \( S_0 \) and fixed \( \alpha \). As we shall see, our results can be written in a compact form by introducing an implicit matrix notation in which the indices are two values of \( S \).

In terms of the function \( \sigma(q) \), introduced by (1.37), the wave function \( \Phi(q) \) is

\[ \Phi(q) = e^{-gS_0(q)-\sigma(q)}. \quad (3.1) \]

Because

\[ (\nabla S_0)^2 = 2v, \quad (3.2) \]

the Schrödinger equation (1.1) is equivalent to the following equation for \( e^{-\sigma} \):

\[ [g(\nabla S_0) \cdot \nabla + T] e^{-\sigma(q)} = [-\frac{g}{2} \nabla^2 S_0 + E] e^{-\sigma(q)}, \quad (3.3) \]

where

\[ T = -\frac{1}{2} \nabla^2. \quad (3.4) \]
Its solution is given by (1.37) and (1.40), which can be written in a more compact form, as we shall see.

Consider the coordinate transformation

$$q_1, q_2, q_3, \cdots, q_N \rightarrow S_0, \alpha_1, \alpha_2, \cdots, \alpha_{N-1}$$

(3.5)

with $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{N-1})$ denoting the set of $N - 1$ orthogonal angular coordinates introduced in (1.26) - (1.28). Define the $\theta_0$-function:

$$\theta_0(S_0 - \mathcal{S}_0) = \begin{cases} 
1 & \text{if } 0 \leq S_0 < \mathcal{S}_0 \\
0 & \text{if } 0 \leq \mathcal{S}_0 < S_0
\end{cases}$$

(3.6)

where $S_0$ and $\mathcal{S}_0$ vary from 0 to $\infty$. For $S_0 > 0$, its derivative gives Dirac’s $\delta$-function:

$$\frac{\partial}{\partial S_0} \theta_0(S_0 - \mathcal{S}_0) = \delta(S_0 - \mathcal{S}_0).$$

(3.7)

Let us define $C_0(\alpha)$ as a square matrix in the $S_0$-space, with its matrix element

$$(S_0|C_0(\alpha)|\mathcal{S}_0) = g^{-1}\theta_0(S_0 - \mathcal{S}_0)/(|\nabla S_0|^2)^{1/2}.$$  

(3.8)

An important feature is that each matrix element of $C_0(\alpha)$ connects two points $q(S_0, \alpha)$ and $\mathcal{q}(\mathcal{S}_0, \alpha)$ along the same classical trajectory that satisfies (1.20) - (1.21), and therefore having the same angular variables $\alpha_1, \alpha_2, \cdots, \alpha_{N-1}$. Throughout the paper, the variables $S_0$ and $\mathcal{S}_0$ vary between 0 and $\infty$, with the end points 0 and $\infty$ treated as boundaries. All gradients refer to differentiations with respect to the Cartesian coordinates; i.e.,

$$\nabla_j = \frac{\partial}{\partial q_j} \quad \text{and} \quad \nabla^2 = \sum_{j=1}^{N} \frac{\partial^2}{\partial q_j^2},$$

(3.9)

as before. On account of (1.27) and (3.6), for $S_0 > 0$,

$$\nabla S_0 \cdot \nabla \theta_0(S_0 - \mathcal{S}_0) = (\nabla S_0)^2 \delta(S_0 - \mathcal{S}_0),$$

(3.10)
where the \( j \)th component of \( \nabla \theta_0(S_0 - \overline{S}_0) \) is
\[
\nabla_j \theta_0(S_0 - \overline{S}_0) = \frac{\partial}{\partial q_j} \theta_0(S_0 - \overline{S}_0)
\]
with \( q_j = q_j(S_0, \alpha) \). From (3.8), we also have
\[
g \nabla S_0 \cdot \nabla (S_0|C_0(\alpha)|\overline{S}_0) = \delta(S_0 - \overline{S}_0). \tag{3.11}
\]

Like \( C_0(\alpha) \), \( \theta_0 \) also denotes a square matrix whose matrix elements are
\[
(S_0|\theta_0|\overline{S}_0) = \theta_0(S_0 - \overline{S}_0). \tag{3.12}
\]

Matrix multiplications follow the usual rule; e.g.,
\[
(S_0|C_0\theta_0|\overline{S}_0) = \int_0^\infty d\overline{S}_0(S_0|C_0|\overline{S}_0)(\overline{S}_0|\theta_0|\overline{S}_0). \tag{3.13}
\]

Introduce \( h_0^2(\alpha) \) to be a diagonal matrix whose elements are
\[
(S_0|h_0^2(\alpha)|\overline{S}_0) = \delta(S_0 - \overline{S}_0)/(\nabla S_0)^2. \tag{3.14}
\]

Thus,
\[
(S_0|\theta_0 h_0^2(\alpha)|\overline{S}_0) = \theta_0(S_0 - \overline{S}_0)/(\nabla S_0)^2, \tag{3.15}
\]
but
\[
(S_0|h_0^2(\alpha)\theta_0|\overline{S}_0) = \theta_0(S_0 - \overline{S}_0)/(\nabla S_0)^2. \tag{3.16}
\]

Equations (3.8) and (3.11) can be written in their matrix forms
\[
C_0 = g^{-1} \theta_0 h_0^2 \tag{3.17}
\]
and
\[
g \nabla S_0 \cdot \nabla C_0 = 1, \tag{3.18}
\]
in which the $\alpha$-dependences of $C_0$ and $h_0^2$ are suppressed.

**Theorem 1.** The solution of the wave equation (3.3) - (3.4) for $e^{-\sigma}$ satisfies

$$
e^{-\sigma(S_0, \alpha)} = 1 + \int_0^\infty dS_0(S_0)[1 + C_0(\alpha)T]^{-1}C_0(\alpha)|\overline{S}_0)\bigl[-\frac{g}{2}\nabla^2S_0 + E\bigr]e^{-\sigma(S_0, \alpha)}.$$ 

(3.19)

**Proof (i)** We first establish the existence of the inverse matrix $[1 + C_0(\alpha)T]^{-1}$. Assuming the opposite, there would be a column matrix $f(\alpha)$ which satisfies

$$[1 + C_0(\alpha)T]f(\alpha) = 0.$$ 

(3.20)

Applying $\nabla S_0 \cdot \nabla$ from the left, using (3.18) and on account of $T = -\frac{1}{2}\nabla^2$, we have for the matrix element $f(S_0, \alpha)$ of $f(\alpha)$:

$$g\nabla S_0 \cdot \nabla f(S_0, \alpha) - \frac{1}{2}\nabla^2 f(S_0, \alpha) = 0.$$ 

(3.21)

Hence,

$$\nabla \cdot \bigl[e^{-2gS_0}\nabla f(S_0, \alpha)\bigr] = 0.$$ 

(3.22)

Multiplying this equation by $f(S_0, \alpha)$ and integrating over all space, we derive, through partial integration,

$$\int d^N q \ e^{-2gS_0}[\nabla f(S_0, \alpha)]^2 = 0,$$ 

(3.23)

provided

$$e^{-2gS_0}f\nabla f = 0 \quad \text{at} \quad S_0 = \infty.$$ 

(3.24)

It is clear that (3.23) implies $\nabla f(S_0, \alpha) = 0$; i.e., $f(S_0, \alpha) = \text{constant}$, which makes $-\frac{1}{2}\nabla^2 f = 0$. It follows then from (3.20),

$$f(S_0, \alpha) = 0.$$ 

21
(ii) From (3.3),
\[ g(\nabla S_0) \cdot \nabla + T)(e^{-\sigma} - 1) = \left[ -\frac{g}{2} \nabla^2 S_0 + E \right] e^{-\sigma}. \tag{3.25} \]

Next, applying \( g(\nabla S_0) \cdot \nabla \) onto \( 1 + C_0T \) and using (3.18), we find
\[ (g(\nabla S_0) \cdot \nabla)[1 + C_0T] = g(\nabla S_0) \cdot \nabla + T; \tag{3.26} \]
therefore, (3.25) becomes
\[ [g(\nabla S_0) \cdot \nabla][1 + C_0T](e^{-\sigma} - 1) = \left[ -\frac{g}{2} \nabla^2 S_0 + E \right] e^{-\sigma}. \tag{3.27} \]

On the other hand, assuming (3.19) we have
\[ e^{-\sigma} - 1 = (1 + C_0T)^{-1}C_0 \left[ -\frac{g}{2} \nabla^2 S_0 + E \right] e^{-\sigma} \tag{3.28} \]
which satisfies (3.27). In deriving the above, we see that for \( f = e^{-\sigma} - 1 \), (3.24) is satisfied, which ensures the applicability of \((1 + C_0T)^{-1}\). The theorem is then proved.

According to (3.17), \( C_0 = O(g^{-1}) \), and therefore, for \( g^2 \gg 1 \), we may expand
\[ (1 + C_0T)^{-1}C_0 = C_0 + (-C_0T)C_0 + (-C_0T)^2C_0 + \cdots. \tag{3.29} \]

Neglecting \( O(g^{-1}) \), \( \sigma \approx S_1 \) and approximating \((1 + C_0T)^{-1}C_0\) by \( C_0 \), (3.28) reduces to
\[ e^{-S_1} - 1 = C_0 \left[ -\frac{g}{2} \nabla^2 S_0 + E \right] e^{-S_1}. \tag{3.30} \]

Taking its derivative through \( g\nabla S_0 \cdot \nabla \), we have, as expected,
\[ g[\nabla S_0 \cdot \nabla S_1]e^{-S_1} = \left[ \frac{g}{2} \nabla^2 S_0 \right] e^{-S_1} \tag{3.31} \]
which, upon the approximation \( E \approx gE_0 \), gives the second line of (1.9). Likewise, we can derive (1.31) - (1.35) for \( S_1, E_1, S_2, E_2, \cdots \) from (3.28).
4. GREEN'S FUNCTIONS AND PERTURBATION SERIES

As we have seen, the $g^{-1}$ expansion enables us to probe the ground-state wave function $\Phi(q)$ along a classical trajectory; to each order $g^{-n}$, the $N$-dimensional quantum wave function can be calculated through quadratures, by performing a finite number of definite integrals along the same trajectory. To the same order of accuracy, the corresponding energy is also determined by a finite number of differentiations at a single point where the potential has an absolute minimum, say at $q = 0$. This result may seem unfamiliar, since the Schroedinger wave function $\Phi(q)$ and its eigenvalue $E$ depend sensitively on the boundary conditions at $\infty$. But in our approach, the determination of $\Phi(q)$ is based only on a single trajectory connecting 0 and $q$. The boundary condition at infinity appears almost automatically, because the classical action (i.e., Hamilton’s action integral) satisfies the property:

$$S_0(q) \to \infty \quad \text{as} \quad q \to \infty. \quad \text{(4.1)}$$

Since the classical action is calculated for a potential $-V(q)$, which has its maximum at 0, this condition (4.1) imposes only a general restriction on the potential; in any case, it is considered to be an input in the new method. To explore further the essential features of the underlying mechanism, we shall in this section generalize our formalism without any explicit use of the Hamilton-Jacobi equation.

In this section we shall derive several new forms of the Green’s function for the $N$-dimensional Schroedinger equation; these are all based on quadratures along a single specified trajectory. With the Green’s function we will be able to arrive at a new perturbation series expansion, different from the usual familiar ones. As we shall see (near the end of the section), the material discussed in previous sections 1-3 will appear as special cases. As mentioned in the Introduction, the new Green’s function will also enable us to extend the multi-instanton calculations developed for the one-dimensional problem in ref. [1] to arbitrary dimensions. In what follows, we shall first give a number of definitions, then for clarity, arrange the main
results in the form of three theorems (Theorem 2, 3 and 4 below). Examples of the new perturbation series and the Green’s function will be given in Section 5.

Consider two $N$-dimensional Hamiltonians:

$$H = -\frac{1}{2} \nabla^2 + V(q)$$  \hspace{1cm} (4.2)

with $V(q) \geq 0$, as before, and

$$\mathcal{H} = H + \epsilon U(q).$$  \hspace{1cm} (4.3)

Their ground-state wave functions are $\Phi(q)$ and $\Psi(q)$ respectively:

$$H\Phi(q) = E\Phi(q)$$  \hspace{1cm} (4.4)

and

$$\mathcal{H}\Psi(q) = E\Psi(q).$$  \hspace{1cm} (4.5)

Let

$$\epsilon \Delta \equiv E - E.$$  \hspace{1cm} (4.6)

Write, as before,

$$V(q) = g^2 v(q),$$  \hspace{1cm} (4.7)

$$\Phi(q) = e^{-gS(q)}$$  \hspace{1cm} (4.8)

and

$$\Psi(q) = e^{-gS(q) - \tau(q)}.$$  \hspace{1cm} (4.9)

We have, from (4.4) - (4.9),
\[
\frac{g^2}{2}(\nabla S)^2 - \frac{g^2}{2} \nabla^2 S - g^2 v + E = 0 \quad (4.10)
\]

and
\[
g \nabla S \cdot \nabla \tau + \frac{1}{2} \left[(\nabla \tau)^2 - \nabla^2 \tau \right] = \epsilon (U - \Delta) \quad (4.11)
\]

In the subsequent discussions in this section, we assume that \( S(q) \) is known, and our objective is to compute \( \tau(q) \), assuming that
\[
|\epsilon| << 1. \quad (4.12)
\]

We assume further that \( S(q) \) has an overall behavior similar to the Hamilton-Jacobi action \( S_0(q) \) given by (1.19); in particular, \( S(q) \) has an absolute minimum at \( q = 0 \). (This convention will be adopted here on.) Therefore
\[
\nabla S = 0 \quad \text{at} \quad q = 0. \quad (4.13)
\]

(Note that from here on the absolute minimum of \( V(q) \) may or may not be at the same point.) Throughout this section, the role of the Hamilton-Jacobi action integral \( S_0 \) in previous sections will be replaces by \( S \). As in (1.26) - (1.28), we introduce a set of \( N - 1 \) angular variables
\[
\beta = (\beta_1(q), \beta_2(q), \cdots, \beta_{N-1}(q)), \quad (4.14)
\]

which satisfy
\[
\nabla \beta_j \cdot \nabla S = 0 \quad (4.15)
\]

with
\[
j = 1, 2, \cdots, N - 1. \quad (4.16)
\]

Each point \( q \) in the \( N \)-dimensional space will now be designated by
(4.17)

\[(S, \beta_1, \beta_2, \cdots, \beta_{N-1}),\]

instead of \((q_1, q_2, q_3, \cdots, q_N)\). At \(\mathbf{q}\), let \(\hat{S}, \hat{\beta}_1, \hat{\beta}_2, \cdots, \hat{\beta}_{N-1}\) form a set of \(N\) orthonormal unit vectors, which are the normals of the \(S = \text{constant}, \beta_1 = \text{constant}, \cdots, \beta_{N-1} = \text{constant}\) surfaces at that point. Correspondingly, a line element \(d\mathbf{q}\) can be written as

\[d\mathbf{q} = \hat{S} h_S dS + \sum_{j=1}^{N-1} \hat{\beta}_j h_j d\beta_j; \tag{4.18}\]

the gradient is given by

\[\nabla = \hat{S} \frac{1}{h_S} \frac{\partial}{\partial S} + \sum_{j=1}^{N-1} \hat{\beta}_j \frac{1}{h_j} \frac{\partial}{\partial \beta_j}, \tag{4.19}\]

and

\[T = -\frac{1}{2} \nabla^2 \tag{4.20}\]

can be decomposed into two parts:

\[T = T_S + T_\beta, \tag{4.21}\]

with

\[T_S = -\frac{1}{2h_S h_\beta} \frac{\partial}{\partial S} (h_\beta \frac{\partial}{\partial S}), \tag{4.22}\]

in which

\[h_\beta = \prod_{j=1}^{N-1} h_j, \tag{4.23}\]

and

\[T_\beta = -\frac{1}{2h_S h_\beta} \sum_{j=1}^{N-1} \frac{\partial}{\partial \beta_j} (h_S h_\beta \frac{\partial}{\partial \beta_j}). \tag{4.24}\]

It follows from (4.18),

\[h_S^2 = [(\nabla S)^2]^{-1}, h_1^2 = [(\nabla \beta_1)^2]^{-1}, \cdots, h_j^2 = [(\nabla \beta_j)^2]^{-1}, \cdots. \tag{4.25}\]
The volume element in the $\mathbf{q}$-space is

$$d^N \mathbf{q} = h_S h_\beta dS d\beta$$  \hspace{1cm} (4.26)

where

$$d\beta = \prod_{j=1}^{N-1} d\beta_j.$$  \hspace{1cm} (4.27)

The trajectory of a given $\beta$ defines a continuous curve normal to the $S = \text{constant}$ surfaces; along any such trajectory we will introduce the following matrices, $\theta$, $C(\beta)$, $D(\beta)$ and $G(\beta)$, each of which has its matrix element connecting two points $(S, \beta)$ and $(\overline{S}, \beta)$ along the same trajectory. The matrix element of $\theta$ is given by

$$(S|\theta|\overline{S}) = \theta(S - \overline{S}) = \begin{cases} 1 & \text{if } 0 \leq \overline{S} < S \\ 0 & \text{if } 0 \leq S < \overline{S}, \end{cases}$$  \hspace{1cm} (4.28)

We now define $C$, $D$ and $G$ in matrix notations by (cf. (3.13)-(3.17))

$$C \equiv g^{-1}\theta h_S^2,$$  \hspace{1cm} (4.29)

$$D \equiv -2e^{-gS}\theta e^{2gS} h_S h_\beta h_\beta h_S$$  \hspace{1cm} (4.30)

and

$$G \equiv D(1 + T_\beta D)^{-1} = (1 + DT_\beta)^{-1} D,$$  \hspace{1cm} (4.31)

in which the $\beta$-dependence of the matrices is not exhibited explicitly and $e^{\pm gS}$, $h_S$, $h_\beta$ are all diagonal matrices. (The conditions for the existence of inverse matrices, like $(1 + T_\beta D)^{-1}$ will be given later, after (4.62).) For example, the matrix element of $C$ is

$$(S|C|\overline{S}) = g^{-1}\theta(S - \overline{S})/(\nabla S)^2.$$  \hspace{1cm} (4.32)

Similar to (3.13) and (3.15)-(3.16), the product of any two matrices, say $A$ and $B$, defined along the same trajectory (i.e., sharing the same $\beta$) has its matrix element given by the usual rule
\[(S|AB|\overline{S}) = \int_0^\infty d\overline{S}(S|A|\overline{S})(\overline{S}|B|\overline{S}). \]  \hspace{1cm} (4.33)

The matrix element of its derivative \(\frac{\partial}{\partial S} AB\) is given by
\[(S|\frac{\partial}{\partial S} AB|\overline{S}) = \int_0^\infty d\overline{S} \frac{\partial}{\partial S}(S|A|\overline{S})(\overline{S}|B|\overline{S}), \]  \hspace{1cm} (4.34)

with \(\frac{\partial}{\partial S}\) operating only on the left index \(S\), whereas that of \(\frac{\partial}{\partial \beta_j} AB\) is
\[(S|\frac{\partial}{\partial \beta_j} AB|\overline{S}) = (S|\frac{\partial A}{\partial \beta_j} B + A(\frac{\partial B}{\partial \beta_j})|\overline{S}). \]  \hspace{1cm} (4.35)

**Theorem 2.**

1. \(g\nabla S \cdot \nabla C = I\), \hspace{1cm} (4.36)

2. \(D\) and \(G\) satisfy
\[D = e^{-gS}(1 + CT_S)^{-1}C e^{gS} = e^{-gS}C(1 + T_S C)^{-1}e^{gS} \]  \hspace{1cm} (4.37)

and
\[G = e^{-gS}(1 + CT)^{-1}C e^{gS} = e^{-gS}C(1 + TC)^{-1}e^{gS}. \]  \hspace{1cm} (4.38)

3. Furthermore, \(D\) and \(G\) are the Green's functions of \(T_S + V - E\) and \(T + V - E\):
\[(T_S + V - E)D = I, \]  \hspace{1cm} (4.39)
\[(H - E)G = (T + V - E)G = I, \]  \hspace{1cm} (4.40)

where \(I\) denotes the unit matrix whose matrix element is \(\delta(S - \overline{S})\).

**Proof**

1. For \(S > 0\),
\[\frac{\partial}{\partial S} \theta(S - \overline{S}) = \delta(S - \overline{S}). \]  \hspace{1cm} (4.41)

On account of the orthogonality \(\nabla S \cdot \nabla \beta_j = 0\), we have
\[\nabla S \cdot \nabla = (\nabla S)^2 \frac{\partial}{\partial S}. \]  \hspace{1cm} (4.42)
Eq. (4.36) follows.

2. Define
\[ \overline{D} \equiv e^{gS} De^{-gS} \quad \text{and} \quad \overline{G} \equiv e^{gS} Ge^{-gS}. \] (4.43)

Then, from (4.31) we see that
\[ \overline{D} = -2\theta e^{2gS} \frac{h_S}{h_\beta} e^{-2gS} h_S h_\beta, \] (4.44)
\[ \frac{\partial \overline{D}}{\partial S} = -2e^{2gS} \frac{h_S}{h_\beta} e^{-2gS} h_S h_\beta, \] (4.45)
\[ \frac{\partial}{\partial S} \left( h_\beta \frac{\partial \overline{D}}{\partial S} \right) = -4g e^{2gS} \theta e^{-2gS} h_S h_\beta - 2h_S h_\beta. \] (4.46)

and, because of (4.22),
\[ T_S \overline{D} = \frac{2g}{h_S h_\beta} e^{2gS} \theta e^{-2gS} h_S h_\beta + 1. \] (4.47)

Multiplying by \( C \) on the left, we find
\[ CT_S \overline{D} = 2\theta \frac{h_S}{h_\beta} e^{2gS} \theta e^{-2gS} h_S h_\beta + C = -\overline{D} + C, \]
and therefore
\[ (1 + CT_S) \overline{D} = C, \] (4.48)

or
\[ \overline{D} = (1 + CT_S)^{-1} C = C(1 + T_S C)^{-1}. \] (4.49)

Since according to (4.43)
\[ D = e^{-gS} \overline{D} e^{gS}. \] (4.50)

(4.37) is proved.
By using (4.31), we derive

\[
\frac{\partial D}{\partial S} = -g D - 2e^S \frac{h_S}{h_\beta} \theta e^{-gS} h_S h_\beta,
\] (4.51)

\[
\frac{h_\beta}{h_S} \frac{\partial D}{\partial S} = -g \frac{h_\beta}{h_S} D - 2e^S \theta e^{-gS} h_S h_\beta,
\] (4.52)

and

\[
\frac{\partial}{\partial S} \left( \frac{h_\beta}{h_S} \frac{\partial D}{\partial S} \right) = -g \left[ \frac{\partial}{\partial S} \left( \frac{h_\beta}{h_S} \right) \right] D - 2h_S h_\beta.
\] (4.53)

On the other hand, because \( \Phi = e^{-gS} \) satisfies \((T_S + V - E)\Phi = 0\) and since

\[
T_S \Phi = \frac{1}{2h_S h_\beta} \frac{\partial}{\partial S} \left( \frac{h_\beta}{h_S} \frac{\partial}{\partial S} \right) e^{-gS} = \frac{1}{2h_S h_\beta} \left\{ -g \left[ \frac{\partial}{\partial S} \left( \frac{h_\beta}{h_S} \right) \right] + g^2 \frac{h_\beta}{h_S} \right\} e^{-gS},
\] (4.54)

we have

\[
\frac{g}{2h_S h_\beta} \left\{ \frac{\partial}{\partial S} \left( \frac{h_\beta}{h_S} \right) \right\} = -V + E.
\] (4.55)

Therefore (4.53) leads to

\[-\frac{1}{2h_S h_\beta} \frac{\partial}{\partial S} \left( \frac{h_\beta}{h_S} \frac{\partial D}{\partial S} \right) + (V - E) D = 1; \] (4.56)

i.e., \( D \) satisfies (4.39) and is the Green’s function of \( T_S + V - E \).

From (4.31) and the identity

\[
1 = \left[ 1 - T_\beta D(1 + T_\beta D)^{-1} \right] (1 + T_\beta D)
\] (4.57)

we have

\[
(T_S + V - E) D = \left[ 1 - T_\beta D(1 + T_\beta D)^{-1} \right] \left( 1 + T_\beta D \right), \] (4.58)

i.e.,

\[
(T_S + V - E) D(1 + T_\beta D)^{-1} = 1 - T_\beta D(1 + T_\beta D)^{-1}.
\] (4.59)
From (4.31), \( D(1 + T_\beta D)^{-1} \) is \( G \); we derive

\[
(T + V - E)G = 1,
\]

which establishes (4.40).

From \( \overline{G} = e^{gS}Ge^{-gS} \) and since \( e^{gS} \) commutes with \( T_\beta \), it follows that

\[
\overline{G} = \overline{D}(1 + T_\beta \overline{D})^{-1}.
\]

By using \( \overline{D} = C(1 + T_S C)^{-1} \), we have

\[
1 + T_\beta \overline{D} = 1 + T_\beta C(1 + T_S C)^{-1}
\]

\[
= (1 + T_S C + T_\beta C)(1 + T_S C)^{-1}.
\]

Its inverse is

\[
(1 + T_\beta \overline{D})^{-1} = (1 + T_S C)(1 + TC)^{-1}
\]

which leads to

\[
\overline{G} = \overline{D}(1 + T_S C)(1 + TC)^{-1} = C(1 + TC)^{-1},
\]

and that gives (4.38). From the above expression, we see that

\[
\overline{G}^{-1} = C^{-1} + T.
\]

Likewise, from (4.49) and (4.61), we can write

\[
\overline{D}^{-1} = C^{-1} + T_S
\]

and

\[
\overline{G}^{-1} = \overline{D}^{-1} + T_\beta.
\]
These relations make transparent the connections between the original equations (4.38), (4.37) and (4.31).

To complete the proof, we have to examine the conditions, under which $1 + CT$, $1 + CT_S$ and $1 + DT_\beta$ have inverses.

As in (3.20), we assume $f_1 = f_1(S, \beta)$ to satisfy

$$(1 + CT)f_1 = 0. \quad (4.63)$$

Operating on its left by $-2g\nabla S \cdot \nabla$, we derive

$$-2g\nabla S \cdot \nabla f_1 + \nabla^2 f_1 = 0.$$ 

Hence

$$\nabla \cdot (e^{-2gS}\nabla f_1) = 0. \quad (4.64)$$

Multiplying by $f_1 d^N q$ and integrating over all space, we have

$$\int e^{-2gS}(\nabla f_1)^2 d^N q = 0, \quad (4.65)$$

provided

$$e^{-2gS}f_1 \nabla f_1 = 0 \quad \text{at} \quad \infty. \quad (4.66)$$

From (4.65), we see that the only solution is $\nabla f_1 = 0$; i.e., $f_1 = \text{constant}$. However, from the original equation (4.63), it follows then $f_1 = 0$. Consequently, $(1 + CT)^{-1}$ exists provided (4.66) holds.

Next, we examine the equation

$$(1 + CT_S)f_2 = 0. \quad (4.67)$$

Operating $g\nabla S \cdot \nabla$ on its left, we find
\[ g \nabla S \cdot \nabla f_2 + T_S f_2 = 0. \quad (4.68) \]

Multiplying (4.68) by \( f_2 h_S h_\beta dS \) and integrating from \( S = 0 \) to \( \infty \), we have

\[ \int_0^\infty f_2 \frac{\partial}{\partial S} \left( \frac{h_\beta}{h_S} e^{-2gS} \frac{\partial f_2}{\partial S} \right) dS = 0. \quad (4.69) \]

Assuming

\[ (f_2 \frac{1}{h_S} \frac{\partial f_2}{\partial S}) e^{-2gS} h_\beta = 0 \text{ at } S = 0 \text{ and } S = \infty, \quad (4.70) \]

(which is the \( S \)-component equivalence of (4.66)), we derive

\[ \int_0^\infty e^{-2gS} \left( \frac{1}{h_S} \frac{\partial f_2}{\partial S} \right)^2 h_\beta h_s dS = 0. \quad (4.71) \]

Recognizing \( d^N q = h_S h_\beta dS d\beta \) is positive, it follows then \( \frac{\partial f_2}{\partial S} = 0 \). Substituting this back to (4.67), we find

\[ f_2 = 0. \quad (4.72) \]

Hence, \( (1 + CT_S)^{-1} \) exists, provided (4.70) holds.

Lastly, we consider

\[ (1 + DT_\beta) f_3 = 0. \quad (4.73) \]

Applying \( T_S + V - E \) from the left and using (4.39), we obtain

\[ (T_S + T_\beta + V - E) f_3 = 0. \quad (4.74) \]

Thus, the inverse of \( (1 + DT_\beta)^{-1} \) exists, provided we restrict ourselves to a Hilbert space which excludes the ground-state of \( H \). Theorem 2 is then proved.

It is important to note that the existence of \( (1 + CT)^{-1} \), and therefore \( G \), as defined by (4.38), is free from the last restriction.
We return now to (4.3)-(4.6). The following theorem will enable us to derive the ground-state wave function \( \Psi = e^{-gS-\tau} \) of \( H + \epsilon U \) in terms of \( e^{-gS} \), the ground-state wave function of \( H \):

**Theorem 3**

\[
e^{-\tau} = 1 + (1 + CT)^{-1}C\epsilon(-U + \Delta)e^{-\tau}, \tag{4.75}
\]

or equivalently,

\[
\Psi = e^{-gS} + (1 + DT\beta)^{-1}D\epsilon(-U + \Delta)\Psi
= e^{-gS} + G\epsilon(-U + \Delta)\Psi. \tag{4.76}
\]

**Proof.** Eq. (4.11) can also be written as

\[
(g\nabla S \cdot \nabla + T)e^{-\tau} = \epsilon(-U + \Delta)e^{-\tau},
\]

where \( T = -\frac{1}{2}\nabla^2 \), as before. Since from (4.36) \( g\nabla S \cdot \nabla C = 1 \), and since \( \nabla(e^{-\tau} - 1) = \nabla e^{-\tau} \), the above expression gives

\[
g\nabla S \cdot \nabla(1 + CT)(e^{-\tau} - 1) = \epsilon(-U + \Delta)e^{-\tau}. \tag{4.77}
\]

Hence,

\[
e^{-\tau} - 1 = (1 + CT)^{-1}C\epsilon(-U + \Delta)e^{-\tau}, \tag{4.78}
\]

as can be derived by substituting (4.78) into (4.77). Thus (4.75) follows. Multiplying (4.78) by \( e^{-gS} \) on the left gives

\[
\Psi = e^{-gS} + e^{-gS}(1 + CT)^{-1}C\epsilon(-U + \Delta)e^{-\tau} \tag{4.79}
\]

which leads to (4.76), on account of (4.31) and (4.38). Theorem 3 is thereby established.

From (4.75), we may expand \( e^{-\tau} \) as a power series of \( \epsilon \):
\[ e^{-\tau} = 1 + \epsilon(1 + CT)^{-1}C(-U + \Delta) + \epsilon^2[(1 + CT)^{-1}C(-U + \Delta)]^2 \\
+ \cdots + \epsilon^n[(1 + CT)^{-1}C(-U + \Delta)]^n + \cdots. \] (4.80)

Likewise, from (4.76), we derive the perturbation series

\[ \Psi = \Psi_0 + \epsilon \Psi_1 + \epsilon^2 \Psi_2 + \cdots + \epsilon^n \Psi_n + \cdots, \] (4.81)

where

\[ \Psi_0 = e^{-gS}, \] (4.82)

and for \( n \geq 1, \)

\[ \Psi_n = [G(-U + \Delta)]^n e^{-gS}, \] (4.83)

with \( G = (1 + DT_\beta)^{-1}D, \) or \( e^{-gS}(1 + CT)^{-1}Ce^{gS}. \)

From (4.83), we have

\[ \Psi_n = G(-U + \Delta)\Psi_{n-1}. \] (4.84)

Eqs. (4.76) and (4.83) can be directly established by using \((H - E)G = 1; \) i.e., \( G \) being the Green’s function of \((H - E). \) Likewise, since

\[ (T_S + V - E)e^{-gS} = 0 \]

and

\[ (T_S + V - E)\Psi = [-T_\beta + \epsilon(-U + \Delta)]\Psi, \]

\( \Psi \) also satisfies

\[ \Psi = e^{-gS} + D(-T_\beta + \epsilon(-U + \Delta)]\Psi, \] (4.85)

on account of (4.39), \( D \) being the Green’s function of \((T_S + V - E). \)
The new perturbation series (4.81)-(4.83) is most effective, when it is complemented by an additional expansion in powers of \( g^{-1} \), the anharmonicity parameter. From \( C = g^{-1} \theta h^2_S \) and (4.37)-(4.38), we see that the three Green's functions \( C, D \) and \( G \) are all \( O(g^{-1}) \). The \( n^{th} \) order perturbative wave function \( e^n \Psi_n \) is given by (4.83). By using (4.32) for \( G \), \( \Psi_n \) can be written as a power series in \( D \), and therefore also in \( g^{-1} \):

\[
\Psi_n = \left\{ [D - DT_\beta D + (DT_\beta)^2 D - \cdots + (-)^m (DT_\beta)^m D + \cdots ](-U + \Delta) \right\}^n e^{-gS}, \quad (4.86)
\]
in which each \( D \) assumes the form given by (4.31); i.e.,

\[
D = -2e^{-gS} \theta e^{2gS} \frac{hS}{h_\beta} \theta e^{-gS} \theta h_\beta,
\]
a double definite integral along the trajectory. Through (4.38), \( \Psi_n \) can also be expressed as a different power series in \( g^{-1} \) in terms of \( C \):

\[
\Psi_n = e^{-gS}\{ [C - CT C + \cdots + (-)^m (CT)^m C + \cdots ](-U + \Delta) \}^n, \quad (4.87)
\]
in which each \( C \) is a single definite integral along the same trajectory. The equivalence between these two expressions rests on (4.37), which gives

\[
D = e^{-gS}\{ C - CT_\theta C + \cdots + (-)^m (CT_\theta)^m C + \cdots \} e^{gS}.
\]

The identity between this power series expression of \( D \) and its double integral form given above has its origin in the Sturm-Liouville construction of the Green’s function for a one-dimensional Schroedinger equation. Recalling that \( e^{-gS} \) is a solution of the second order ordinary differential equation in \( S \),

\[
(T_\theta V + E)e^{-gS} = 0, \quad (4.88)
\]
we can form an irregular solution \( F \) which satisfies the same equation

\[
(T_\theta + V - E)F = 0 \quad (4.89)
\]
with
\[
F(S) = e^{-gS} \int_0^S e^{2g\bar{S}} \left( h_S \right)_{S_{d\bar{S}}},
\]
(4.90)
in which the angular variable \( \beta \) is kept fixed and not exhibited explicitly in the argument. The Green’s function \( D \) is related to the Wronskian-type expression by
\[
(S|D|\bar{S}) = 2[e^{-g\bar{S}} F(S) - F(S)e^{-g\bar{S}}] \left( h_S h_\beta \right)_{\bar{S}_{\beta}} \theta(S - \bar{S}).
\]
(4.91)

In ref. [1], we used a one-dimensional example to demonstrate how this form of the Green’s function can enable us to evaluate the multi-instanton expansion. Here we have extended the application of a one-dimensional Green’s function to an \( N \)-dimensional problem.

**Theorem 4.** The energy shift \( \epsilon \Delta \) is given by
\[
\epsilon \Delta = \frac{\int e^{-g\bar{S}} eU \Psi dNq}{\int e^{-g\bar{S}} \Psi dNq}
\]
(4.92)
where \( dNq = h_S h_\beta dSd\beta \), as in (4.26) and the integration is extended over all space.

**Proof.** From (4.31) or (4.91), the matrix element of \( D(\beta) \) is
\[
(S|D(\beta)|\bar{S}) = -2e^{-g\bar{S}} \int_0^S d\bar{S} \left( h_S \right)_{\bar{S}_{\beta}} \theta(\bar{S} - S) \cdot e^{2g\bar{S}} e^{-g\bar{S}} \left( h_S h_\beta \right)_{\bar{S}_{\beta}},
\]
(4.93)
where the subscripts \( \bar{S}, \beta \) and \( \bar{S}, \beta \) indicate the arguments of the functions inside the parenthesis.

Let \( \chi \) denotes the column matrix
\[
\chi(S, \beta) \equiv [-T_\beta + \epsilon(-U + \Delta)] \Psi(S, \beta).
\]
Eq. (4.85) becomes
\[
\Psi = e^{-gS} + D\chi.
\]
As \( S \to \infty \), the left side should be zero, but the right side is, on account of (4.93), controlled by the \( \bar{S} \)-integration in the upper region when \( \bar{S} \) is near \( S \); i.e., writing \( D\chi \) as the product of \(-2e^{-gS}\) times.
\[ \int_0^\infty d\bar{S}e^{2g\bar{S}}(h_{S})_{\bar{S},\beta} \int_0^\infty d\bar{S}e^{-g\bar{S}}(h_{S}h_{\beta})_{\bar{S},\beta}\chi(\bar{S},\beta), \]

we see that the factor to the right of (and including) the \( \bar{S} \)-integration sign approaches \( e^{2gS} \) times

\[ \int_0^\infty d\bar{S}e^{-g\bar{S}}(h_{S}h_{\beta})_{\bar{S},\beta}\chi(\bar{S},\beta), \]  

which would make \( D\chi \to \infty \) unless \((4.94)\) is zero. Thus, the convergence of \( \Psi \) at \( \infty \) requires

\[ \int_0^\infty d\bar{S}e^{-g\bar{S}}h_{S}h_{\beta}[-T_{\beta} + \epsilon(-U + \Delta)]\Psi = 0 \]  

(4.95)

at all \( \beta \). Integrating over the angular variables \( d\beta = \prod_{j=1}^{N-1} d\beta_j \), and because of

\[ -h_{S}h_{\beta}T_{\beta} = \frac{1}{2} \sum_{j=1}^{N-1} \frac{\partial}{\partial \beta_j} h_{S}h_{\beta} \frac{\partial}{\partial \beta_j} \]

and

\[ \int d\beta h_{S}h_{\beta}T_{\beta}\Psi = 0 \]

we derive \((4.92)\). This result is well known. (Since \((H + \epsilon U)\Psi = (E + \epsilon \Delta)\Psi\), the multiplication of \( e^{-gS} \) on the left and integrating over all space gives immediately \((4.92)\).) We have demonstrated that the same result can also be derived, though somewhat awkwardly, by the Green’s function method developed in this paper.

The usual perturbation series requires to each order \( n \), either a product of infinite sums over all excited levels of the unperturbed Hamiltonian \( H \), or a sum over all possible paths through Feynman’s path integration method. The perturbation series formula derived here is different; it depends only on quadratures along a single trajectory, as will be illustrated by examples in the next section.
Remarks. Before leaving this section, we wish to address the connection between the perturbation series developed in this section and the previous series expansion given by (3.28)-(3.29). In sections 1-3, we discuss the relation between the Schroedinger wave function

\[ e^{-gS} = e^{-gS_0 - \sigma} \]

and the classical Hamilton’s action integral \( S_0 \), which satisfies

\[ g^2 (\nabla S_0)^2 = 2V = 2g^2 v. \]

Thus formally we may regard \( e^{-gS_0} \) as the solution of a Schroedinger-like wave equation whose eigenvalue happens to be 0:

\[ \left( -\frac{1}{2} \nabla^2 + V_c \right) e^{-gS_0} = 0 \quad (4.97) \]

where

\[ V_c = g^2 v - \frac{1}{2} g \nabla^2 S_0. \quad (4.98) \]

Since \( e^{-gS} \) satisfies

\[ \left( -\frac{1}{2} \nabla^2 + V_c + \frac{1}{2} g \nabla^2 S_0 \right) e^{-gS} = E e^{-gS}, \quad (4.99) \]

we may treat (4.97) as the unperturbed Schroedinger equation, and (4.99) as the perturbed one, with \( \frac{1}{2} g \nabla^2 S_0 \) as the perturbative Hamiltonian and \( E \) as the energy shift. The corresponding small \( \epsilon \)-parameter that characterize the perturbation becomes \( g^{-1} \), since \( \frac{1}{2} g \nabla^2 S_0 \) is \( O(g^{-1}) \) times \( V_c \). This explains the similarity between (3.28) for \( e^{-\sigma} \) and (4.75) for \( e^{-\tau} \); it also identifies the \( g^{-1} \)-expansion with the perturbative \( \epsilon \)-expansion.
5. EXAMPLES

From (4.31) and (4.38), the Green’s function $G$ can be either expressed in terms of the single integral form $C$, or equivalently, the double integral form $D$:

$$G = e^{-gS}(1 + CT)^{-1}Ce^{gS} \quad (5.1)$$

or

$$G = (1 + DT_\beta)^{-1}D. \quad (5.2)$$

These two complement each other in their applications. As in (4.8) - (4.9), let $e^{-gS}$ be the ground-state wave function of $H$. The corresponding ground-state wave function

$$\Psi = e^{-gS-\tau} \quad (5.3)$$

of $H + \epsilon U$ can also be expressed either in terms of $C$ (eq.(4.75)):

$$e^{-\tau} = 1 + (1 + CT)^{-1}C\epsilon(-U + \Delta)e^{-\tau}, \quad (5.4)$$

or in terms of $D$ (eq.(4.76)):

$$\Psi = e^{-gS} + (1 + DT_\beta)^{-1}D\epsilon(-U + \Delta)\Psi. \quad (5.5)$$

As shown in (4.80) and (4.86), both can be used for the perturbation series expansion. In this section, we shall examine several examples which illustrate the different merits of these two formulations.

For clarity, consider the one-dimensional case, in which (5.2) and (5.5) reduce to $G = D$ and

$$\Psi = e^{-gS} + D\epsilon(-U + \Delta)\Psi. \quad (5.6)$$

As in (4.43), introduce
\[ \overline{D} \equiv e^{gS} De^{-gS}, \]

which, in accordance with (4.48) and since \( T_\delta = T \) in the one-dimensional case, is related to \( C \) by

\[ (1 + CT)\overline{D} = C. \]  \hfill (5.7)

In terms of \( \overline{D} \), (5.6) becomes

\[ e^{-\tau} = 1 + \overline{D} \epsilon (-U + \Delta) e^{-\tau}. \]  \hfill (5.8)

In order that the above expression is equivalent to the alternative form (5.4) in terms of \( C \), we need to convert (5.7) into

\[ \overline{D} = (1 + CT)^{-1} C. \]  \hfill (5.9)

According to (4.66), this requires

\[ e^{-2gS} f_1 \nabla f_1 = 0 \]  \hfill (5.10)

at the boundary, where, for (5.8), the function \( f_1 \) is

\[ f_1 = \overline{D} (-U + \Delta) e^{-\tau}, \]  \hfill (5.11)

and in one-dimension, the boundary consists of

\[ x = \pm \infty. \]  \hfill (5.12)

We recall further that \( C \) satisfies (4.36) which is now an ordinary first order differential equation

\[ g \frac{dS}{dx} \frac{dC}{dx} = 1; \]  \hfill (5.13)

this determines \( C \) up to an integration constant, which can be set by requiring

\[ C = 0 \quad \text{at} \quad S = 0. \]  \hfill (5.14)
Now, $D$ is the solution of the second order differential equation

\[-\frac{1}{2} \frac{d^2}{dx^2} + V - E)D = 1, \tag{5.15}\]

and therefore contains two independent integration constants. In view of (5.9) and (5.14), we require

\[D = 0 \quad \text{at} \quad S = 0 \tag{5.16}\]

and write for the right hand side of (5.11)

\[\overline{D}(-U + \Delta)e^{-\tau} = -2 \int_0^x e^{2gS(y)} dy \int_{-\infty}^y e^{-2gS(z)} (-U + \Delta)e^{-\tau(z)} dz \tag{5.17}\]

so that (5.10) holds at the boundary $x = -\infty$. (For convenience, we choose $S = 0$ at $x = 0$.)

In order to satisfy (5.10) at the other boundary $x = \infty$, we set

\[\int_{-\infty}^{\infty} e^{-2gS(x) - \tau(x)} (-U + \Delta)dx = 0; \tag{5.18}\]

otherwise, (5.17) would $\sim e^{2gS}$ at $x = \infty$, violating (5.10) - (5.11). Thus, the energy shift $\Delta$ is also determined, and (5.18) is a special case of (4.92). On the other hand, if we use (5.11), expressing $G$ in terms of the single integral form $C$, then $\Delta$ would be determined by differentiations at $S = 0$, similar to (1.32) and (1.34). In the following, we shall solve two examples by using $G$, first in terms of $C$, then in terms of $D$.

**Example 1.** Consider a one-dimensional harmonic oscillator with

\[H = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{g^2}{2} x^2 \tag{5.19}\]

and a perturbative potential

\[\epsilon U = \epsilon x^{2p}, \tag{5.20}\]

where $p$ is a positive integer. The unperturbed ground-state wave function is
\[ \Phi = e^{-gS} = e^{-\frac{g}{2}x^2}. \quad (5.21) \]

Since \( S = \frac{1}{2}x^2 \), we have
\[ h_S^2 = (dS/dx)^{-2} = x^{-2} = (2S)^{-1}. \quad (5.22) \]

For \( n > 0 \) and in accordance with (5.14),
\[ Cx^{2n} = \int_0^S \frac{dS}{g2S}x^{2n} = \frac{1}{g(2n)}x^{2n}. \quad (5.23) \]

**Lemma 1**
\[ (1 + CT)^{-1}C[x^{2n} - \Gamma_{1n}] = \sum_{m=1}^{n} \Gamma_{mn}x^{2m}. \quad (5.24) \]

where
\[
\begin{align*}
\Gamma_{nn} &= \frac{1}{g2n}, & \Gamma_{n-1n} &= \frac{2n - 1}{2g^2(2n - 2)}, \\
\Gamma_{mn} &= \frac{(2n - 1)(2n - 3) \cdots (2m + 1)}{m(2g)^{n-m+1}} \quad \text{for} \quad 1 \leq m \leq n - 1.
\end{align*}
\]

and in particular
\[ \Gamma_{1n} = \frac{(2n - 1)!!}{(2g)^n}. \]

**Proof** Since \( T = -\frac{1}{2}d^2/dx^2 \),
\[ (1 + CT)^{-1}C = C - CTC + (-CT)^2C + \cdots + (-CT)^mC + \cdots \]
and
\[ -TCx^{2n} = \frac{1}{2g} (2n - 1)x^{2n-2}, \]
we find
\[
\begin{align*}
-CTCx^{2n} &= \frac{2n - 1}{2g^2(2n - 2)}x^{2n-2} = \Gamma_{n-1n}x^{2n-2}, \\
(-CT)^2Cx^{2n} &= \frac{(2n - 1)(2n - 3)}{2^2g^3(2n - 4)}x^{2n-4} = \Gamma_{n-2n}x^{2n-4},
\end{align*}
\]
\[ (-CT)^n C x^{2n} = \frac{(2n-1)!!}{(2g)^n} x^2 = \Gamma_n x^2. \]

But, \((-CT)^n C x^{2n} = CT \Gamma_n\), which by itself would be \(\infty\). On the other hand,

\[ (-CT)^n C x^{2n} - CT \Gamma_n = 0; \quad (5.26) \]

Lemma 1 is then proved.

As in (4.9), the eigenstate of

\[ \mathcal{H} = \frac{1}{2} \frac{d^2}{dx^2} + \frac{g^2}{2} x^2 + \epsilon x^{2p} \]

is \(e^{-\frac{1}{2}g x^2 - \tau}\). Write

\[ e^{-\tau} = 1 + \sum_{l=1}^{\infty} a_l x^{2l}. \quad (5.27) \]

From (4.75)

\[ \sum_{l=1}^{\infty} a_l x^{2l} = (1 + CT)^{-1} C \epsilon (-x^{2p} + \Delta)(1 + \sum_{l=1}^{\infty} a_l x^{2l}). \quad (5.28) \]

By using (5.24) and (5.27), we derive

\[ \Delta = \Gamma_{1p} + \sum_{l=1}^{\infty} a_l (\Gamma_{1l+p} - \Delta \Gamma_{1l}) \quad (5.29) \]

and for \(n \geq 1\)

\[ a_n = -\epsilon \Gamma_{np} - \epsilon \sum_{l=1}^{\infty} a_l (\Gamma_{nl+p} - \Delta \Gamma_{nl}). \quad (5.30) \]

where \(\Gamma_{mn} = 0\) for \(m > n\); otherwise it is given by (5.25). From (5.25), we see that

\[ \epsilon \Delta = -a_1. \quad (5.31) \]

Expand
\[ \epsilon \Delta = \epsilon \Delta(1) + \epsilon^2 \Delta(2) + \epsilon^3 \Delta(3) + \cdots \]

and

\[ a_n = \epsilon a_n(1) + \epsilon^2 a_n(2) + \epsilon^3 a_n(3) + \cdots ; \]

it follows then

\[ \Delta(1) = \Gamma_{1 \ p}, \quad a_n(1) = -\Gamma_{n \ p}, \]

\[ \Delta(2) = \sum_{m=1}^{p} \Gamma_{m \ p}(-\Gamma_{1 \ m+p} + \Gamma_{1 \ m} \Gamma_{1 \ p}) \quad (5.32) \]

\[ a_n(2) = \sum_{m=1}^{p} \Gamma_{m \ p}(\Gamma_{n \ m+p} - \Gamma_{n \ m} \Gamma_{1 \ p}), \]

etc. (Because of (5.30), \( a_n(2) = 0 \) for \( n > 2p \).)

As a special case, for

\[ \epsilon U = \epsilon x^4 \quad (5.33) \]

and therefore \( p = 2 \), we find

\[ \epsilon \Delta(1) = \frac{3\epsilon}{4g^2}, \quad \epsilon^2 \Delta(2) = -\frac{21\epsilon^2}{8g^5}, \quad (5.34) \]

the same as the usual perturbation results.

**Example 2** Using the same one-dimensional harmonic oscillator Hamiltonian (5.19) as the unperturbed \( H \), we consider now an odd power perturbation potential

\[ \epsilon U = \epsilon x^{2p+1} \quad (5.35) \]

The analysis given below is parallel to that in the first example, but with some changes, as will be indicated.

For \( n \geq 0 \),

\[ C x^{2n+1} = \int_{0}^{S} \frac{dS}{g2S} x^{2n+1} = \frac{1}{g(2n+1)} x^{2n+1}. \quad (5.36) \]
It is straightforward to establish the following:

**Lemma 2**

\[(1 + CT)^{-1} C x^{2n+1} = \sum_{m=0}^{n} \gamma_{m \ n} x^{2m+1} \]  
\[(5.37)\]

where

\[
\begin{align*}
\gamma_{n \ n} &= \frac{1}{g(2n + 1)}, & \gamma_{n-1 \ n} &= \frac{n}{g^2(2n - 1)}, \\
\gamma_{m \ n} &= \frac{n(n-1) \cdots (m + 1)}{g^{n-m+1}(2m + 1)} \quad \text{for} \quad 0 \leq m < n
\end{align*}
\]  
\[(5.38)\]

and in particular

\[
\gamma_{0 \ n} = \frac{n!}{g^{n+1}}.
\]

Instead of (5.27), we now set

\[
e^{-\tau} = 1 + \sum_{n=1}^{\infty} b_n x^n.
\]  
\[(5.39)\]

From (4.75),

\[
\sum_{n=1}^{\infty} b_n x^n = (1 + CT)^{-1} C \varepsilon (-x^{2p+1} + \Delta)(1 + \sum_{n=1}^{\infty} b_n x^n).
\]  
\[(5.40)\]

It is convenient to extend the definitions of \(\gamma_{m \ n}\) and \(\Gamma_{m \ n}\) (as in (5.25)), by defining

\[
\gamma_{m \ n} = \Gamma_{m \ n} = 0 \quad \text{for} \quad m > n
\]  
\[(5.41)\]

and

\[
\Gamma_{0 \ n} = 0.
\]

Using the two Lemmas, we derive

\[
\Delta = \sum_{l} (b_{2l+1} \Gamma_{1 \ l+p+1} - \Delta b_{2l} \Gamma_{1 \ l}),
\]

\[
b_{2n} = -\varepsilon \sum_{l} (b_{2l+1} \Gamma_{n \ l+p+1} - \Delta b_{2l} \Gamma_{n \ l})
\]  
\[(5.42)\]
\[ b_{2n+1} = -\epsilon \gamma_n \cdot \sigma - \epsilon \sum_l (b_{2l} \gamma_n \cdot \sigma + \Delta b_{2l+1} \gamma_n \cdot \sigma). \]

Here and in the following, whenever the range of the summation index is not exhibited, it reads

\[ l \ (or \ m, \ n) = 0, 1, 2, \cdots, \infty. \] (5.43)

By comparing the first two equations in (5.42), we have

\[ \epsilon \Delta = b_2. \] (5.44)

It can be readily verified that the following power series expansions hold:

\[ \epsilon \Delta = \epsilon^2 \Delta(2) + \epsilon^4 \Delta(4) + \epsilon^6 \Delta(6) + \cdots \]

\[ b_{2n} = \epsilon^2 b_{2n}(2) + \epsilon^4 b_{2n}(4) + \epsilon^6 b_{2n}(6) + \cdots, \] (5.45)

\[ b_{2n+1} = \epsilon b_{2n+1}(1) + \epsilon^3 b_{2n+1}(3) + \epsilon^5 b_{2n+1}(5) + \cdots, \]

in which

\[ b_{2n+1}(1) = -\gamma_n \cdot \sigma, \quad \Delta(2) = -\sum_l \gamma_l \cdot \sigma \Gamma_1 l + \sigma + 1, \]

\[ b_{2n}(2) = \sum_l \gamma_l \cdot \sigma \Gamma_n l + \sigma + 1, \]

\[ b_{2n+1}(3) = -\sum_{l,m} \gamma_l \cdot \sigma \Gamma_m l + \sigma + 1 \gamma_m m + \sigma - \Delta(2) \sum m \gamma_m m \gamma_m \cdot \sigma, \] (5.46)

\[ \Delta(4) = -\Delta(2) \sum_{l,m} (\gamma_l \cdot \sigma \Gamma_m l + \sigma + 1 + \Gamma l m + \sigma + 1 \gamma_m m + \sigma + 1) \]

\[ -\sum_{l,m,n} \gamma_l \cdot \sigma \Gamma_m l + \sigma + 1 \gamma_m m + \sigma + 1 \gamma_1 n + \sigma + 1, \]

etc.

In the special case of \( p = 0 \), we have

\[ \epsilon U = \epsilon x, \] (5.47)
the above expressions lead to
\[ \epsilon^2 \Delta(2) = -\frac{\epsilon^2}{2g^2} \]  \hspace{1cm} (5.48)

and all other \( \Delta(2n) = 0 \). This confirms the exact result, since

\[ V + \epsilon U = \frac{g^2}{2} x^2 + \epsilon x = \frac{g^2}{2} (x + \frac{\epsilon}{g^2})^2 - \frac{\epsilon^2}{2g^2}. \]  \hspace{1cm} (5.49)

The lowest eigenvalue of \( V \) is \( \frac{g^2}{2} \), and that of \( V + \epsilon U \) is \( \frac{g^2}{2} - \frac{\epsilon^2}{2g^2} \). Likewise, \( (5.45)-(5.46) \) yield

\[ b_1 = \epsilon b_1(1) = -\epsilon \gamma_0 \Gamma_0 = -\frac{\epsilon}{g}, \]
\[ b_2 = \epsilon^2 b_2(2) = \epsilon^2 \gamma_0 \Gamma_1 \Gamma_1 = -\frac{\epsilon^2}{2g^2}, \]  \hspace{1cm} (5.50)
\[ b_3 = \epsilon^3 b_3(3) = -\epsilon^3 \gamma_0 \Gamma_1 \Gamma_1 = -\frac{\epsilon^3}{6g^3}, \]

etc., so that

\[ e^{-\tau} = 1 + \sum_{n=1}^{\infty} b_n x^n = e^{-\epsilon/g} \]

which leads to the expected expression

\[ e^{-gS-\tau} \propto e^{-\frac{\epsilon}{g}(x + \frac{\epsilon}{g})^2}. \]  \hspace{1cm} (5.51)

So far we have only used the Green’s function \( G \) in terms of the single integral form \( C \).

Next, we will repeat the same examples, but using the double integral form \( D \).

Example 1 (repeat) We return to the same simple harmonic oscillator problem \( (5.19) - (5.20) \), with

\[ H = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{g^2}{2} x^2 \]

and the perturbation

\[ \epsilon U = \epsilon x^{2p}. \]
In order to derive the perturbation series (5.27) - (5.32) for $e^{-\tau}$ and $\Delta$ using $D$, we have to deal with integrals of the type

$$Dx^{2n} = -2 \int_0^x e^{gy^2} dy \int_{-\infty}^y e^{-gz^2} z^{2n} dz.$$  \hspace{1cm} (5.52)

Introduce

$$\xi = \sqrt{g}x, \quad \eta = \sqrt{g}y, \quad \zeta = \sqrt{g}z.$$

(5.52) becomes

$$Dx^{2n} = -\frac{2}{g^{n+1}} \int_0^\xi e^{\eta^2} d\eta \int_{-\infty}^\eta e^{-\zeta^2} \zeta^{2n} d\zeta.$$ \hspace{1cm} (5.53)

Let $H_l(\zeta)$ be the usual Hermite polynomials, obtained by the generating function

$$e^{-t^2 + 2t\zeta} = \sum_{l=0}^{\infty} \frac{t^l}{l!} H_l(\zeta).$$ \hspace{1cm} (5.54)

The integrand in (5.53) can be expressed in terms of $H_l(\zeta)$ through

$$\zeta^{2n} = \sum_{m=0}^{n} \frac{(2n)!}{2^{2n} m! (2n - 2m)!} H_{2n-2m}(\zeta).$$ \hspace{1cm} (5.55)

For $l \geq 1$,

$$e^{-\xi^2} H_l(\xi) = \frac{d}{d\xi} [e^{-\xi^2} H_{l-1}(\xi)];$$

therefore

$$\int_0^\xi e^{\eta^2} d\eta \int_{-\infty}^\eta e^{-\zeta^2} H_l(\zeta) d\zeta = -\int_0^\xi H_{l-1}(\eta) d\eta$$

$$= -\frac{1}{2l} [H_l(\xi) - H_l(0)]$$ \hspace{1cm} (5.56)

and

$$DH_l(\sqrt{g}x) = \frac{2}{g^2} \int_0^\xi e^{\eta^2} d\eta \int_{-\infty}^\eta e^{-\zeta^2} H_l(\zeta) d\zeta$$

$$= \frac{1}{lg} [H_l(\xi) - H_l(0)].$$ \hspace{1cm} (5.57)
Since $H_0(\zeta) = 1$, by separating out in (5.55) the term $m = n$ in the sum, we have

\[ \zeta^{2n} = g^n \Gamma_{1\,n} + \sum_{m=0}^{n-1} \frac{(2n)!}{2^{2m}m!(2n-2m)!} H_{2n-2m}(\zeta), \]

(5.58)

where

\[ \Gamma_{1\,n} = \frac{(2n-1)!!}{(2g)^n}, \]

the same as (5.25). From (5.57) and (5.58), it follows that

\[ \overline{D}(x^{2n} - \Gamma_{1\,n}) = \frac{(2n)!}{(4g)^n} \sum_{l=1}^{n} \frac{1}{(n-l)!(2l)!2l[H_{2l}(\sqrt{g}x) - H_{2l}(0)]}. \]

(5.59)

The identity between the above expression and $(1 + CT)^{-1}C(x^{2n} - \Gamma_{1\,n})$ given by (5.24) can be established by first applying $T$ onto (5.57). Through the well known formula

\[ \frac{d^2}{d\xi^2} H_l(\xi) = 2\xi \frac{d}{d\xi} H_l(\xi) - 2l H_l(\xi), \]

(5.60)

we find

\[ T \overline{D}H_l(\sqrt{g}x) = \frac{1}{l}[-x \frac{d}{dx} H_l(\sqrt{g}x) + lH_l(\sqrt{g}x)]. \]

(5.61)

Next, apply $C$ onto (5.61):

\[ C T \overline{D}H_l(\sqrt{g}x) = \frac{1}{g} \int_0^x \frac{dx}{x} T \overline{D}H_l(\sqrt{g}x) \]

\[ = -\frac{1}{lg}[H_l(\sqrt{g}x) - H_l(0)] + CH_l(\sqrt{g}x). \]

(5.62)

Combining (5.57) and (5.62), we obtain

\[ (1 + CT) \overline{D}H_l(\sqrt{g}x) = CH_l(\sqrt{g}x). \]

(5.63)

Hence, the multiplication of $(1 + CT)$ and (5.59) leads to

\[ (1 + CT) \overline{D}(x^{2n} - \Gamma_{1\,n}) = C(x^{2n} - \Gamma_{1\,n}), \]

(5.64)
which gives the equality between (5.59) and (5.24). Thus, both forms of \( G \), (5.1) and (5.2), yield the same perturbation series expansion (5.31) - (5.32).

It is instructive to note that in (5.24),

\[
(1 + CT)^{-1}C[x^{2n} - \Gamma_1 n] = \sum_{m=1}^{n} \Gamma_m n x^{2m},
\]

each term, \((1 + CT)^{-1}Cx^{2n}\) or \((1 + CT)^{-1}C\Gamma_1 n\), is by itself \(\infty\), which necessitates the combination that appears on the left-hand side. Using the single-integral form \( C \) for \( G \), the energy shift \( \Delta \) is required to make the integral finite at \( x = 0 \). On the other hand, \( \mathcal{D} x^{2n} \), or

\[
\mathcal{D} \Gamma_1 n = -2 \int_{0}^{x} e^{gy^2} dy \int_{-\infty}^{y} e^{-gz^2} \Gamma_1 n dz,
\]

is well-defined at any finite \( x \). As \( x \to \infty \), either \( \mathcal{D} x^{2n} \) or

\[
\mathcal{D} \Gamma_1 n \sim -2 \Gamma_1 n \int_{-\infty}^{\infty} e^{-gz^2} dz \cdot e^{g x^2}
\]

approaches \( \infty \), which, when multiplied by \( e^{-\frac{1}{2} g x^2} \), would lead to a perturbed wave function \( e^{-gS-\tau} \) divergent at \( x = \infty \). As shown in (5.17), this requires the energy shift \( \Delta \) to be determined by (5.18); in this example, it is equivalent to have the subtraction \( -\mathcal{D} \Gamma_1 n \) in (5.59).

**Example 2 (repeat).** To complete the one-dimensional analysis, we continue with the same simple harmonic oscillator \( H \) as the unperturbed Hamiltonian, but with the perturbation given by

\[
eU = \varepsilon x^{2n+1}.
\]

Since by repeated partial integrations,

\[
\int_{-\infty}^{y} e^{-gz^2} z^{2n+1} dz = -\frac{1}{2g} [y^{2n} + \frac{n}{g} y^{2(n-1)} + \frac{n(n-1)}{g^2} y^{2(n-2)} + \ldots + \frac{n!}{g^n} e^{-gy^2}],
\]

51
we find

\[ \mathcal{D}_x^{2n+1} = -2 \int_0^x e^{gy^2} dy \int_{-\infty}^y e^{-gz^2} z^{2n+1} dz \]

\[ = \sum_{m=0}^n \gamma_m n x^{2m+1}, \]

the same as (5.37) for \((1 + CT)^{-1} C x^{2n+1}\), with \(\gamma_m n\) given by (5.38). It is straightforward to see that using either \(\mathcal{D}\), or \((1 + CT)^{-1} C\), we can arrive at the same perturbation series (5.45) and (5.46).
6. EXCITED STATES

In this section, we apply the new approach to excited states. Let $e^{-gS}$ be the ground-state of (1.1)

$$He^{-gS} = Ee^{-gS}$$

where

$$H = -\frac{1}{2}\nabla^2 + g^2 v$$

and, as in (1.4)–(1.5),

$$v(q) \geq 0. \quad (6.1)$$

In this section, we assume $e^{-gS}$ to be already known; our purpose is to derive the excited states $\Psi_{ex}$, which satisfies

$$H\Psi_{ex} = (E + \mathcal{E})\Psi_{ex} \quad (6.2)$$

Write

$$\Psi_{ex} = \chi(q)e^{-gS}. \quad (6.3)$$

Since

$$\nabla^2\Psi_{ex} = (\nabla^2\chi - 2g\nabla\chi \cdot \nabla S)e^{-gS} + \chi(\nabla^2e^{-gS}),$$

(6.1)-(6.3) lead to

$$g\nabla S \cdot \nabla \chi - \frac{1}{2}\nabla^2 \chi = \mathcal{E} \chi. \quad (6.4)$$

The expansion (1.7),

$$gS = gS_0 + S_1 + g^{-1}S_2 + g^{-2}S_3 + \cdots, \quad (6.5)$$
will now be accompanied by similar expansions for $\chi$ and $E$:

$$\chi = \chi_0 + g^{-1}\chi_1 + g^{-2}\chi_2 + \cdots$$  \hspace{1cm} (6.6)$$

and

$$E = gE_0 + E_1 + g^{-1}E_2 + \cdots.$$  \hspace{1cm} (6.7)$$

Substituting (6.5) - (6.7) into (6.4) and equating the coefficients of $g^{-n}$ on both sides, we obtain the following first order partial differential equations for $\chi_0$, $\chi_1$, $\chi_2$, $\cdots$:

$$(\nabla S_0 \cdot \nabla - E_0)\chi_0 = 0,$$  \hspace{1cm} (6.8)$$

$$(\nabla S_0 \cdot \nabla - E_0)\chi_1 = (-\nabla S_1 \cdot \nabla + \frac{1}{2}\nabla^2 + E_1)\chi_0,$$  \hspace{1cm} (6.9)$$

$$(\nabla S_0 \cdot \nabla - E_0)\chi_2 = (-\nabla S_1 \cdot \nabla + \frac{1}{2}\nabla^2 + E_1)\chi_1 + (-\nabla S_2 \cdot \nabla + E_2)\chi_0,$$  \hspace{1cm} (6.10)$$

etc. To see how these equations can be solved we first give a simple example and then address the general solution.

**Example** Take the example

$$v(q) = \frac{1}{2}(\nu_1^2 q_1^2 + \nu_2^2 q_2^2 + \cdots + \nu_N^2 q_N^2),$$  \hspace{1cm} (6.11)$$

and correspondingly,

$$S_0(q) = \frac{1}{2}(\nu_1^2 q_1^2 + \nu_2^2 q_2^2 + \cdots + \nu_N^2 q_N^2)$$

$$S_1(q) = S_2(q) = \cdots = 0,$$  \hspace{1cm} (6.12)$$

$$E = \frac{g}{2}(\nu_1 + \nu_2 + \cdots + \nu_N).$$  \hspace{1cm} (6.13)$$

Introduce, as in (1.26)-(1.28), the variables

$$(S_0, \alpha)$$
with \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{N-1}) \) and
\[
\nabla S_0 \cdot \nabla \alpha_j = 0. \tag{6.14}
\]

In terms of these new coordinates \( S_0 \) and \( \alpha \), (6.8) becomes
\[
(\nabla S_0)^2 \left( \frac{\partial \ln \chi_0}{\partial S_0} \right)_{\alpha} = E_0, \tag{6.15}
\]
which will be integrated along the constant-\( \alpha \) trajectory. The constant-\( \alpha \) trajectory is determined by the solution of the Hamilton-Jacobi equation (1.11); it describes a classical trajectory with \(-v\) as the potential and a positive infinitesimal energy, starting from \( q = 0 \) at time near \(-\infty\), and \( q = (q_1, q_2, \ldots, q_N) = (S_0, \alpha) \) at time \( t \). As \( t \) increases to \( t + dt \), the end of the trajectory moves from \( q = (S_0, \alpha) \) to
\[
(q_1 + dq_1, q_2 + dq_2, \ldots, q_N + dq_N) = (S_0 + dS_0, \alpha) \tag{6.16}
\]
keeping \( \alpha \) constant. In accordance with (2.20) - (2.22), we have
\[
dt = \frac{dS_0}{(\nabla S_0)^2} = \frac{dq_1}{\nu_1 q_1} = \frac{dq_2}{\nu_2 q_2} = \cdots = \frac{dq_N}{\nu_N q_N}. \tag{6.17}
\]
When \( S_0 \to 0 \), each \( q_i \) must also \( \to 0 \) along the trajectory. Since \( \chi_0 \) is a single-valued function of \( q \), we can classify \( \chi_0 \) by its power dependence on \( q_i \), as \( q_i \to 0 \). Take this to be
\[
\chi_0 \to q_1^{n_1} q_2^{n_2} \cdots q_N^{n_N}. \tag{6.18}
\]

Since
\[
(\frac{\partial \ln \chi_0}{\partial S_0})_{\alpha} = \sum_{i=1}^{N} (\frac{\partial \ln q_i}{\partial S_0})_{\alpha} \frac{\partial \ln \chi_0}{\partial \ln q_i}. \tag{6.19}
\]

From (6.17), we see that along the classical (constant \( \alpha \) ) trajectory
\[
(\frac{\partial \ln q_i}{\partial S_0})_{\alpha} = \frac{\nu_i}{(\nabla S_0)^2}, \tag{6.20}
\]
and consequently
\[
(\frac{\partial \ln \chi_0}{\partial S_0})_\alpha = \frac{1}{(\nabla S_0)^2} \sum_{i=1}^N \nu_i \frac{\partial \ln \chi_0}{\partial q_i}. \tag{6.21}
\]

As \( S_0 \to 0 \), under the assumption (6.18), we have
\[
(\nabla S_0)^2(\frac{\partial \ln \chi_0}{\partial S_0})_\alpha \to \sum_{i=1}^N n_i \nu_i; \tag{6.22}
\]
therefore (6.15) yields
\[
\mathcal{E}_0 = \sum_{i=1}^N n_i \nu_i \tag{6.23}
\]
and
\[
(\nabla S_0)^2(\frac{\partial \ln \chi_0}{\partial S_0})_\alpha = \sum_{i=1}^N n_i \nu_i \tag{6.24}
\]
along the entire trajectory for all \( S_0 > 0 \). (It is reassuring that (6.23) is independent of \( \alpha \).) It follows then
\[
\chi_0 = q_1^{n_1} q_2^{n_2} \cdots q_N^{n_N} \tag{6.25}
\]
at all \( \mathbf{q} \).

In this example, because \( S_1 = 0 \), (6.9) becomes
\[
(\nabla S_0)^2(\frac{\partial \chi_1}{\partial S_0})_\alpha - \mathcal{E}_0 \chi_1 = \mathcal{E}_1 \chi_0 + \frac{1}{2} \nabla^2 \chi_0,
\]
which, on account of (6.15) and (6.25), can also be written as
\[
\chi_0(\nabla S_0)^2[\frac{\partial}{\partial S_0}(\frac{\chi_1}{\chi_0})]_\alpha = \mathcal{E}_1 \chi_0 + \frac{1}{2} \left[ \frac{n_1(n_1 - 1)}{q_1^2} + \frac{n_2(n_2 - 1)}{q_2^2} + \cdots + \frac{n_N(n_N - 1)}{q_N^2} \right] \chi_0. \tag{6.26}
\]
Thus, keeping \( \alpha \) fixed,
\[
\chi_1 = \chi_0 \int \frac{dS_0}{(\nabla S_0)^2} \left( \mathcal{E}_1 + \frac{1}{2} \left[ \frac{n_1(n_1 - 1)}{q_1^2} + \frac{n_2(n_2 - 1)}{q_2^2} + \cdots + \frac{n_N(n_N - 1)}{q_N^2} \right] \right), \tag{6.27}
\]

56
where the integration is along the (classical) trajectory normal to the $S_0 = \text{constant}$ surfaces. We leave (6.27) in the indefinite integral form, since the integration constant can be eliminated by the transformation

$$\chi_1 \rightarrow \chi_1 + \text{constant} \cdot \chi_0$$

which affects only the overall normalization factor of $\chi$. (This convention will be adopted below for the general case as well.)

For any $\mathcal{E} \neq 0$, and any partition

$$\mathcal{E}_1 = m_1 \nu_1 + m_2 \nu_2 + \cdots + m_N \nu_N$$

(where $m_i$ can be an arbitrary number) gives

$$\int \frac{dS_0}{(\nabla S_0)^2} \mathcal{E}_1 = m_1 \ln q_1 + m_2 \ln q_2 + \cdots + m_N \ln q_N.$$

This leads to an inadmissible wave function, and consequently

$$\mathcal{E}_1 = 0.$$  \hfill (6.29)

Hence, by using (6.17), (6.27) and (6.29), we find for this example

$$\chi_1 = -\frac{\chi_0}{4} \left[ \frac{n_1(n_1 - 1)}{\nu_1 q_1^2} + \frac{n_2(n_2 - 1)}{\nu_2 q_2^2} + \cdots + \frac{n_N(n_N - 1)}{\nu_N q_N^2} \right].$$  \hfill (6.30)

Recalling that the Hermite polynomial $H_n(x)$ is

$$H_n(z) = (2z)^n[1 - \frac{n(n - 1)}{4z^2} + \cdots],$$

the sum

$$\chi = \chi_0 + g^{-1} \chi_1 + \cdots$$

can be shown to be, apart from an overall normalization factor,
\[ H_{n_1}(\sqrt{g\nu_1 q_1})H_{n_2}(\sqrt{g\nu_2 q_2}) \cdots H_{n_N}(\sqrt{g\nu_N q_N}). \quad (6.31) \]

Returning to the general case, because \( q = 0 \) is the minimum of \( v(q) \), as \( q \to 0 \), \( v(q) \) depends quadratically on \( q \). Write

\[ v(q) \to \frac{1}{2}(\nu_1^2 q_1^2 + \nu_2^2 q_2^2 + \cdots + \nu_N^2 q_N^2); \quad (6.32) \]

correspondingly,

\[ S_0(q) \to \frac{1}{2}(\nu_1 q_1^2 + \nu_2 q_2^2 + \cdots + \nu_N q_N^2). \quad (6.33) \]

Again, we shall classify \( \chi_0(q) \) according to its behavior as \( q \to 0 \), by using (6.18). As in this example, this leads to

\[ \mathcal{E}_0 = n_1 \nu_1 + n_2 \nu_2 + \cdots + n_N \nu_N; \quad (6.34) \]

where, as before, \( n_1, n_2, \cdots, n_N \) are positive integers. From (6.8), it follows that

\[ \ln \chi_0 = \mathcal{E}_0 \int \frac{dS_0}{(\nabla S_0)^2} \quad (6.35) \]

with the integration taken along the trajectory of constant \( \alpha \), and the integration constant is determined by the normalization condition (6.18), as \( S_0 \to 0 \).

It is convenient to characterize the limit \( S_0 \to 0 \), by introducing along the trajectory an overall scale factor \( \lambda \) (of the dimension \([q_i]\)). Because of (6.17), for sufficiently small \( S_0 \), we set

\[ S_0 \propto \lambda^2, \quad q_i \propto \lambda, \quad (6.36) \]

and therefore, on account of (6.17),

\[ \frac{dS_0}{(\nabla S_0)^2} = \frac{1}{\nu_1} d\ln q_1 = \frac{1}{\nu_2} d\ln q_2 = \cdots = \frac{1}{\nu_N} d\ln q_N \propto d\ln \lambda. \quad (6.37) \]

We now turn to (6.9); as in (6.26)-(6.27), it can be written as
\[ \chi_0(\nabla S_0)^2\left[ \frac{\partial}{\partial S_0} \frac{\chi_1}{\chi_0} \right] = \left( \frac{1}{2} \nabla^2 - \nabla S_1 \cdot \nabla \right) \chi_0 + \mathcal{E}_1 \chi_0, \]

and therefore

\[ \chi_1 = \chi_0 \int \frac{dS_0}{(\nabla S_0)^2} \left[ \frac{1}{\chi_0} \left( \frac{1}{2} \nabla^2 - \nabla S_1 \cdot \nabla \right) \chi_0 + \mathcal{E}_1 \right]. \quad (6.38) \]

Near \( S_0 = 0 \), we may use (6.36) to expand the first term inside the square bracket as a power series in \( \lambda \):

\[ \frac{1}{\chi_0} \left( \frac{1}{2} \nabla^2 - \nabla S_1 \cdot \nabla \right) \chi_0 = b_{-2} \lambda^{-2} + b_{-1} \lambda^{-1} + b_0 + b_1 \lambda + b_2 \lambda^2 + \cdots, \quad (6.39) \]

where \( b_{-2}, b_{-1}, b_0, b_1, \cdots \) are constants. Because of (6.37), in order that \( \chi_1 \) be analytic at \( q = 0 \), we must require

\[ \mathcal{E}_1 = -b_0; \quad (6.40) \]

otherwise, \( \chi_1 \) would have a term proportional to \( \chi_0 \ln \lambda \), which is not admissible. The function \( \chi_1 \) is given by the integral (6.38) along the \( \alpha \)-constant trajectory. In a similar way, we can determine \( \mathcal{E}_2, \mathcal{E}_3, \cdots \) and obtain the solutions \( \chi_2, \chi_3, \cdots \), in terms of quadrature along the classical (constant \( \alpha \)) trajectory.

### 7. PERTURBATION AROUND AN ATTRACTIVE COULOMB POTENTIAL

Let \( H_c \) be the Hamiltonian for a Coulomb potential:

\[ H_c = -\frac{1}{2} \nabla^2 - \frac{g^2}{r}, \quad (7.1) \]

where

\[ g^2 = Z e^2 \quad (7.2) \]
with \( \nabla^2 \) denoting the three-dimensional Laplacian, \( r \) the radius, \( Ze \) the nuclear charge and \( -e \) the electronic charge. Consider a problem in which there is an additional perturbation \( \epsilon U \); the corresponding Hamiltonian is

\[
H = H_c + \epsilon U,
\]

where \( U \) is not singular at the origin. Let \( \psi_c \) and \( \psi \) be the ground-states of \( H_c \) and \( H \); i.e.,

\[
H_c \psi_c = E_c \psi_c \quad \text{(7.4)}
\]

and

\[
H \psi = E \psi. \quad \text{(7.5)}
\]

### 7.1 Isotropic Case

We first discuss the case that \( U(r) \) depends only on \( r \). As we shall see, with modification our method can be adapted to derive \( \psi \) by quadratures along the radial trajectory. The solution to the Coulomb problem is well known:

\[
\psi_c = e^{-g^2 r} \quad \text{and} \quad E_c = -\frac{1}{2} g^4. \quad \text{(7.6)}
\]

This suggests\(^{[10]} \) that instead of \((1.7)-(1.8)\), a different \( g \)-power expansion is needed, one that should conform to the form \((7.6)\) of the Coulomb wave function.

Write

\[
\psi = e^{-S}. \quad \text{(7.7)}
\]

Expand

\[
S = g^2 S_0 + S_1 + g^{-2} S_2 + g^{-4} S_3 + \cdots + g^{-(2n-2)} S_n + \cdots \quad \text{(7.8)}
\]

and
\[E = g^4 E_0 + g^2 E_1 + E_2 + g^{-2} E_3 + \cdots + g^{-(2n-4)} E_n + \cdots. \quad (7.9)\]

Since
\[
\nabla^2 \psi = [ (\nabla S)^2 - \nabla^2 S ] \psi ,
\]
(7.5) becomes
\[
- \frac{1}{2} (\nabla S)^2 + \frac{1}{2} \nabla^2 S - \frac{g^2}{r} + \epsilon U = E . \quad (7.10)
\]
Substituting (7.8)-(7.9) into (7.10) and equating the coefficients of \( g^4, g^2, g^0, \cdots, g^{-2m}, \cdots \) on both sides, we obtain
\[
(\nabla S_0)^2 = -2E_0 \quad (7.11)
\]
\[
\nabla S_0 \cdot \nabla S_1 = \frac{1}{2} \nabla^2 S_0 - \frac{1}{r} - E_1 \quad (7.12)
\]
\[
\nabla S_0 \cdot \nabla S_2 = -\frac{1}{2} (\nabla S_1)^2 + \frac{1}{2} \nabla^2 S_1 + \epsilon U - E_2 , \quad (7.13)
\]
\[
\nabla S_0 \cdot \nabla S_3 = -\nabla S_1 \cdot \nabla S_2 + \frac{1}{2} \nabla^2 S_2 - E_3 , \quad (7.14)
\]

etc.; for \( n > 1 \),
\[
\nabla S_0 \cdot \nabla S_{2n} = - \sum_{m=1}^{n-1} \nabla S_m \cdot \nabla S_{2n-m} - \frac{1}{2} (\nabla S_n)^2 + \frac{1}{2} \nabla^2 S_{2n-1} - E_{2n} \quad (7.15)
\]
and for \( n \geq 1 \)
\[
\nabla S_0 \cdot \nabla S_{2n+1} = - \sum_{m=1}^{n} \nabla S_m \cdot \nabla S_{2n+1-m} + \frac{1}{2} \nabla^2 S_{2n} - E_{2n+1} . \quad (7.16)
\]

For the ground-state, these \( S_m (r) \) are all radial functions. Since \( U(r) \) is regular at \( r = 0 \), we can set
\[
U(0) = 0 . \quad (7.17)
\]
We adopt the same normalization condition (1.36), with
\[
\psi(0) = 1 \quad \text{and} \quad S(0) = 0 . \quad (7.18)
\]
From (7.11), it follows that
\[ \frac{\partial S_0}{\partial r} = \sqrt{-2E_0} \]
and
\[ S_0 = \sqrt{-2E_0} r. \]  

Substituting these expressions into (7.12), we find
\[ \sqrt{-2E_0} \frac{\partial S_1}{\partial r} = \frac{1}{r}(\sqrt{-2E_0} - 1) - E_1. \]  

Because \( S_1 \) should be regular at \( r = 0 \), \( \sqrt{-2E_0} - 1 = 0 \) and therefore[10]
\[ E_0 = -\frac{1}{2} \quad \text{and} \quad S_0 = r \]  
confirming (7.6); (7.20) then leads to
\[ \frac{\partial S_1}{\partial r} = -E_1 \]
and
\[ S_1 = -E_1 r. \]

Since
\[ \nabla^2 r = \frac{2}{r}, \]  
we see that (7.13) becomes
\[ \frac{\partial S_2}{\partial r} = -\frac{1}{2}E_1^2 - \frac{E_1}{r} + \epsilon U(r) - E_2. \]

In order that \( S_2(r) \) be regular at \( r = 0 \),
\[ E_1 = 0, \]  
and consequently, on account of (7.22),
\[ S_1(r) = 0 \] \hspace{1cm} \text{(7.25)}

and

\[ \frac{\partial S_2}{\partial r} = \epsilon U(r) - E_2. \] \hspace{1cm} \text{(7.26)}

By using (7.23), we see that as \( r \to 0 \)

\[ \nabla^2 S_2 \to (\nabla^2 S_2)_0 = \frac{2}{r^2} (\frac{\partial S_2}{\partial r})_0 = \frac{2}{r} [\epsilon U(0) - E_2]. \] \hspace{1cm} \text{(7.27)}

From (7.14), we see that in order to have \( S_3 \) also regular at \( r = 0 \), \( \left( \frac{\partial S_2}{\partial r} \right)_0 \) must be 0; this yields on account of (7.17),

\[ E_2 = \epsilon U(0) = 0 \] \hspace{1cm} \text{(7.28)}

\[ \frac{\partial S_2}{\partial r} = \epsilon U(r) \] \hspace{1cm} \text{(7.29)}

and

\[ \frac{\partial S_3}{\partial r} = \frac{\epsilon}{2r} \frac{\partial}{\partial r} (r^2 U) - E_3. \] \hspace{1cm} \text{(7.30)}

As \( r \to 0 \), (7.14) implies

\[ \frac{\partial S_3}{\partial r} \to \left( \frac{\partial S_3}{\partial r} \right)_0 = \left[ \frac{\epsilon}{2r^2} \frac{\partial}{\partial r} (r^2 U) \right]_0 - E_3 \] \hspace{1cm} \text{(7.31)}

and, on account of (7.23),

\[ \nabla^2 S_3 \to \frac{2}{r} \left( \frac{\partial S_3}{\partial r} \right)_0. \] \hspace{1cm} \text{(7.32)}

Hence by using (7.15) for \( S_4 \) and in order that \( S_4 \) should be regular at \( r = 0 \),

\[ \left( \frac{\partial S_3}{\partial r} \right)_0 = 0, \] \hspace{1cm} \text{(7.33)}

and therefore
\[ E_3 = \frac{\epsilon}{2} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 U \right) \right]_0. \]  

(7.34)

The same reasoning yields for all \( m \geq 1 \), at \( r = 0 \),

\[ (\frac{\partial S_m}{\partial r})_0 = 0. \]  

(7.35)

From (7.34), we see that for \( \epsilon U(r) = \epsilon r^l \)

\[ E_3 = \begin{cases} 
\frac{3}{2} \epsilon & \text{if } \ l = 1 \\
0 & \text{if } \ l > 1.
\end{cases} \]  

(7.36)

**Example**

To carry the analysis further, let us consider the case \( l = 2 \); i.e.,

\[ \epsilon U(r) = \epsilon r^2. \]  

(7.37)

In this example, the above results give

\[ E_1 = E_2 = E_3 = 0 \]
\[ S_1 = 0, \quad S_2 = \frac{1}{3} \epsilon r^3 \]  

(7.38)

and

\[ S_3 = \epsilon r^2. \]

Set \( n = 2 \) in (7.15), we have

\[ \frac{\partial S_4}{\partial r} = -\frac{1}{2} \epsilon^2 r^4 + 3 \epsilon - E_4. \]

Because of (7.35), at \( r = 0 \), \( \frac{\partial S_4}{\partial r} = 0 \); hence,

\[ E_4 = 3 \epsilon \]  

(7.39)

and

64
In a similar way, we derive

\[ \begin{align*}
E_5 &= 0, \\
S_5 &= -\frac{7}{8} \epsilon^2 r^4, \\
E_6 &= 0, \\
S_6 &= -\frac{43}{12} \epsilon^2 r^3 + \frac{1}{14} \epsilon^3 r^7, \\
E_7 &= 0, \\
S_7 &= -\frac{43}{4} \epsilon^2 r^2 + \frac{13}{12} \epsilon^3 r^6, \\
E_8 &= -\frac{129}{4} \epsilon^2, \quad \text{etc.}
\end{align*} \]  
(7.40)

Putting together, for \( \epsilon U = \epsilon r^2 \), we find

\[ \begin{align*}
S &= g^2 r + \frac{\epsilon}{3g^2} r^3 + \frac{\epsilon}{g^4} r^2 - \frac{\epsilon^2}{10g^6} r^5 \\
&\quad - \frac{7\epsilon^2}{8g^8} r^4 + \frac{1}{g^{10}} \left( -\frac{43}{12} \epsilon^2 r^3 + \frac{1}{14} \epsilon^3 r^7 \right) \\
&\quad + \frac{1}{g^{12}} \left( -\frac{43}{4} \epsilon^2 r^2 + \frac{13}{12} \epsilon^3 r^6 \right) + O\left( \epsilon^3 g^{14} \right)
\end{align*} \]  
(7.41)

and

\[ \begin{align*}
E &= -\frac{g^4}{2} + \frac{3\epsilon}{g^4} - \frac{129}{4g^{12}} \epsilon^2 + O\left( \frac{\epsilon^3}{g^{20}} \right).
\end{align*} \]  
(7.42)

In the usual perturbation series, to each order of the perturbation, the derivation of the perturbed Coulomb wave function requires summations over an infinite number of excited bound-states, plus the continuum. Here, to each order \( \epsilon^m \), the perturbed wave function can be obtained in closed form by quadratures along the radial trajectory.

The perturbed energy can also be derived in an alternative way by using the integral form (4.92):

\[ \begin{align*}
E &= -\frac{1}{2} g^4 + \int_0^\infty e^{-g^2 r - S} \epsilon U r^2 dr \\
&\quad + \int_0^\infty e^{-g^2 r - S} r^2 dr.
\end{align*} \]  
(7.43)

To first order in \( \epsilon \), we can approximate the factor \( e^{-S} \) by \( e^{-g^2 r} \) in the integrals. For \( U = r^2 \),
\[
E = -\frac{1}{2}g^4 + \epsilon \int_0^\infty e^{-2g^2r^4} dr + O(\epsilon^2)
\]
\[
= -\frac{1}{2}g^4 + \frac{3\epsilon}{g^4} + O(\epsilon^2).
\] (7.45)

To calculate \(E\) to the accuracy of \(O(\epsilon^2)\), we need the wave function to \(O(\epsilon)\). From (7.42),

\[
e^{-S} = e^{-g^2r}[1 - \epsilon\left(\frac{r^3}{3g^2} + \frac{r^2}{g^4}\right) + O(\epsilon^2)].
\] (7.46)

Substituting this expression into (7.44), we find the second term on its right-hand side to be:

\[
\epsilon \int_0^\infty e^{-g^2r^2}r^4[1 - \epsilon\left(\frac{r^3}{3g^2} + \frac{r^2}{g^4}\right)]dr
\]
\[
\int_0^\infty e^{-g^2r^2}[1 - \epsilon\left(\frac{r^3}{3g^2} + \frac{r^2}{g^4}\right)]dr,
\] (7.47)

confirming (7.43).

7.2 Stark Effect

Consider the anisotropic example, in which the perturbation is

\[
\epsilon U = \epsilon r \cos a,
\] (7.48)

where \(a\) is the polar angle; i.e.,

\[
r^2 = x^2 + y^2 + z^2 \quad \text{and} \quad z = r \cos a.
\] (7.49)

Replacing \(\epsilon U(r)\) by \(\epsilon r \cos a\), we see that (7.1)-(7.28) remain intact with \(E_1 = E_2 = S_1 = 0\); (7.29) becomes

\[
\frac{\partial S_2}{\partial r} = \epsilon r \cos a,
\] (7.50)

and therefore

\[
S_2 = \frac{1}{2} \epsilon r^2 \cos a,
\] (7.51)

and
\[ \nabla^2 S_2 = 2 \epsilon \cos a. \tag{7.52} \]

Thus, (7.14) gives
\[ \frac{\partial S_3}{\partial r} = \epsilon \cos a - E_3, \tag{7.53} \]
which yields, on account of (7.49),
\[ S_3 = \epsilon z - E_3 r \tag{7.54} \]
and
\[ \nabla^2 S_3 = -\frac{2E_3}{r}. \tag{7.55} \]

Since \( S_4 \) satisfies (7.15) for \( n = 2 \),
\[ \frac{\partial S_4}{\partial r} = -\frac{1}{2}(\nabla S_2)^2 + \frac{1}{2} \nabla^2 S_3 - E_4. \]
In order that \( S_4 \) not have a \( \ln r \) singularity,
\[ E_3 = 0 \]
and therefore
\[ S_3 = \epsilon z. \tag{7.56} \]
In order that \( \nabla^2 S_4 \) not have a \( 1/r \) singularity,
\[ E_4 = 0. \tag{7.57} \]
Thus,
\[ S_4 = -\frac{1}{24} \epsilon^2 r^3(1 + 3 \cos^2 a). \tag{7.58} \]
Likewise, it is straightforward to derive
\[ E_5 = 0, \quad S_5 = -\frac{7}{16} \epsilon^2 r^2 (1 + \cos^2 a), \quad (7.59) \]
\[ E_6 = -\frac{9}{4} \epsilon^2, \quad S_6 = \frac{1}{16} \epsilon^3 r^4 \cos a (1 + \cos^2 a), \quad (7.60) \]
\[ E_7 = 0, \quad S_7 = -\frac{13}{48} \epsilon^3 r^3 \cos a (3 + \cos^2 a), \quad (7.61) \]
\[ E_8 = 0, \quad S_8 = \frac{53}{16} \epsilon^3 r^2 \cos a - \frac{1}{128} \epsilon^4 r^5 (1 + 10 \cos^2 a + 5 \cos^4 a), \quad (7.62) \]
\[ E_9 = 0, \quad S_9 = \frac{53}{8} \epsilon^3 r^3 \cos a - \frac{99}{512} \epsilon^4 r^4 (1 + 6 \cos^2 a + \cos^4 a), \quad (7.63) \]
\[ E_{10} = 0, \quad S_{10} = -\frac{761}{384} \epsilon^4 r^3 (1 + 3 \cos^2 a) + O(\epsilon^5), \quad (7.64) \]
\[ E_{11} = 0, \quad S_{11} = -\frac{3131}{256} \epsilon^4 r^2 (1 + \cos^2 a) + O(\epsilon^5), \quad (7.65) \]
\[ E_{12} = -\frac{3555}{64} \epsilon^4, \quad (7.66) \]

etc.

Combining these results together, for the potential \(-\frac{g^2}{r} + \epsilon \cos a\), we find that, in powers of \(\epsilon\), the wave function \(e^{-S}\) and the energy are given by

\[
S = g^2 r + \frac{\epsilon r}{g^4} \cos a (1 + \frac{1}{2} g^2 r) - \frac{\epsilon^2 r^2}{g^8} \left[ \frac{7}{16} (1 + \cos^2 a) + \frac{1}{24} g^2 r (1 + 3 \cos^2 a) \right] \\
+ \frac{\epsilon^3 r}{g^{16}} \cos a \left[ \frac{53}{8} (1 + \frac{1}{2} g^2 r) + \frac{13}{48} (g^2 r)^2 (3 + \cos^2 a) + \frac{1}{16} (g^2 r)^3 (1 + \cos^2 a) \right] \\
+ O(\epsilon^4) \quad (7.67)
\]

and

\[
E = -\frac{1}{2} g^4 - \frac{9 \epsilon^2}{4 g^8} - \frac{3555}{64} \epsilon^4 \frac{g^4}{g^{20}} + O(\epsilon^6). \quad (7.68)
\]

The \(-\frac{g^2}{r}\) is known\([11,12]\). Our method gives closed expressions for both the wave function and the energy to any finite order of \(\epsilon\).
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