A Note on a result due to Ankeny and Rivlin

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Abstract. Let \( p(z) = a_0 + a_1z + a_2z^2 + a_3z^3 + \cdots + a_nz^n \) be a polynomial of degree \( n \) having no zeros in the unit disk. Then it is well known that for \( R \geq 1 \),
\[
\max_{|z|=R} |p(z)| \leq \left( \frac{R^n + 1}{2} \right) \max_{|z|=1} |p(z)|.
\]
In this paper, we consider polynomials with gaps, having all its zeros on the circle \( S(0,K) := \{ z : |z| = K \} \), \( 0 < K \leq 1 \), and estimate the value of \( \left( \max_{|z|=R} |p(z)| \right)^s \) for any positive integer \( s \).

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1. Introduction

Let \( p(z) = \sum_{j=0}^{n} a_jz^j \) be a polynomial of degree \( n \). We will denote
\[
M(p, r) := \max_{|z|=r} |p(z)|, \quad r > 0,
\]
\[
||p|| := \max_{|z|=1} |p(z)|,
\]
and
\[
D(0,K) := \{ z : |z| < K \}, \quad K > 0.
\]
Bernstein observed the following result, which in fact is a simple consequence of the maximum modulus principle (see [8, p. 137]). This inequality is also known as the Bernstein’s inequality.

Theorem 1.1. Let \( p(z) = \sum_{j=0}^{n} a_jz^j \) be a polynomial of degree \( n \). Then for \( R \geq 1 \),
\[
M(p, R) \leq R^n ||p||. \tag{1.1}
\]
Equality holds for \( p(z) = \alpha z^n \), \( \alpha \) being a complex number.

\[1\]This is a preprint of a paper whose final and definite form is published open access in Applied Mathematics E-Notes. See http://www.math.nthu.edu.tw/amen/ for the final version.
For polynomial of degree $n$ not vanishing in the interior of the unit circle, Ankeny and Rivlin [1] proved the following result.

**Theorem 1.2 (Ankeny and Rivlin [1]).** Let $p(z) = \sum_{j=0}^{n} a_j z^j \neq 0$ in $D(0, 1)$. Then for $R \geq 1$,

$$M(p, R) \leq \left( \frac{R^n + 1}{2} \right) ||p||.$$  

(1.2)

Here equality holds for $p(z) = \alpha + \beta z^n / 2$, where $|\alpha| = |\beta| = 1$.

In 2005, Gardner, Govil and Musukula [3] proved the following generalization and sharpening of Theorem 1.2.

**Theorem 1.3.** Let $p(z) = a_0 + \sum_{j=t}^{n} a_j z^j, 1 \leq t \leq n$, be a polynomial of degree $n$ and $p(z) \neq 0$ in $D(0, K), K \geq 1$. Then for $R \geq 1$,

$$M(p, R) \leq \left( \frac{R^n + s_0}{1 + s_0} \right) ||p|| - \left( \frac{R^n - 1}{1 + s_0} \right) m - \frac{n}{1 + s_0} \left[ \frac{(||p|| - m)^2 - (1 + s_0)^2 |a_n|^2}{(||p|| - m)} \right].$$

(1.3)

where $m = \min_{|z|=K} |p(z)|$, and

$$s_0 = K^{t+1} \cdot \frac{\frac{1}{n} \cdot \frac{|a_t|}{|a_0| - m} K^{t-1}}{\frac{1}{n} \cdot \frac{|a_t|}{|a_0| - m} K^{t+1} + 1}.$$  

Several research monographs have been written on this subject of inequalities (see for example Govil and Mohapatra [3], Milovanović, Mitrović and Rassias [7], Rahman and Schmeisser [9], and recent article of Govil and Nwaeze [5]).

While trying to obtain an inequality analogous to (1.2) for polynomials not vanishing in $D(0, K), K \leq 1$, Dewan and Ahuja [2] were able to prove this only for polynomials having all the zeros on the circle $S(0, K) := \{z : |z| = K\}, 0 < K \leq 1$.

**Theorem 1.4.** Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n$ having all its zeros on $S(0, K), K \leq 1$. Then for $R \geq 1$ and for every positive integer $s$,
\[ \{M(p, R)\}^s \leq \left[ \frac{K^{n-1}(1 + K) + (R^n s - 1)}{K^{n-1} + K^n} \right] \{M(p, 1)\}^s. \] (1.4)

For \( s = 1 \), the Theorem 1.1 reduces to

**Corollary 1.5.** Let \( p(z) = \sum_{j=0}^{n} a_j z^j \) be a polynomial of degree \( n \) having all its zeros on \( S(0, K), K \leq 1 \). Then for \( R \geq 1 \),

\[ M(p, R) \leq \left[ \frac{K^{n-1}(1 + K) + (R^n - 1)}{K^{n-1} + K^n} \right] M(p, 1). \] (1.5)

In same spirit, we prove the following results

**2. Main Results**

**Theorem 2.1.** Let \( p(z) = z^m \left[ a_{n-m} z^{n-m} + \sum_{j=\mu}^{n-m} a_{n-m-j} z^{n-m-j} \right] \), \( 1 \leq \mu \leq n-m, 0 \leq m \leq n-1 \), be a polynomial of degree \( n \), having \( m \)-fold zeros at origin and remaining \( n-m \) zeros on \( S(0, K), K \leq 1 \). Then for \( R \geq 1 \) and every positive integer \( s \),

\[ [M(p, R)]^s \leq L(\mu; K, m, n, s)[M(p, 1)]^s, \] (2.1)

where

\[ L(\mu; K, m, n, s) = \frac{n(K^{n-m-2\mu+1} + K^{n-m-\mu+1}) + (R^n s - 1)[n + mK^{n-m-2\mu+1} + mK^{n-m-\mu+1} - m]}{n(K^{n-m-2\mu+1} + K^{n-m-\mu+1})}. \]

For \( m = 0 \), we have

**Corollary 2.2.** Let \( p(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j} \), \( 1 \leq \mu \leq n \), be a polynomial of degree \( n \), having all zeros on \( |z| = K, K \leq 1 \). Then for \( R \geq 1 \) and every positive integer \( s \),

\[ [M(p, R)]^s \leq L(\mu; K, n, s)[M(p, 1)]^s, \] (2.2)

where

\[ L(\mu; K, n, s) = \frac{K^{n-\mu}(K^{1-\mu} + K) + (R^n s - 1)}{K^{n-2\mu+1} + K^{n-\mu+1}}. \]

If we set \( \mu = 1 \) into Corollary 2.2, we get the following result of Dewan and Ahuja \cite{2}.
Corollary 2.3. Let \( p(z) = \sum_{j=0}^{n} a_j z^j \), be a polynomial of degree \( n \), having all zeros on \( |z| = K, K \leq 1 \). Then for \( R \geq 1 \) and every positive integer \( s \),

\[
[M(p, R)]^s \leq L(1; K, n, s)[M(p, 1)]^s,
\]

where

\[
L(1; K, n, s) = \frac{K^{n-1}(1 + K) + (R^ns - 1)}{K^{n-1} + K^n}.
\]

3. Lemmas

For the proof Theorem 2.1 we need the following lemmas. The first lemma is due to Kumar and Lal [6].

Lemma 3.1. Let \( p(z) = \sum_{j=0}^{n} a_j z^j \), be a polynomial of degree \( n \), having \( m - \text{fold} \) zeros at origin and remaining \( n - m \) zeros on \( |z| = K, K \leq 1 \).

\[
\max_{|z|=1} |p'(z)| \leq \frac{n + m(K^{n-m-2\mu+1} + K^{n-m-\mu+1} - 1)}{K^{n-m-2\mu+1} + K^{n-m-\mu+1}} \max_{|z|=1} |p(z)|.
\]

The next lemma is the Bernstein inequality given in Theorem 1.1.

Lemma 3.2. Let \( p(z) \) be a polynomial of degree \( n \). Then for \( R \geq 1 \),

\[
M(p, R) \leq R^n M(p, 1).
\]

4. Proof

Proof of Theorem 2.1. By Lemma 3.1, we have

\[
\max_{|z|=1} |p'(z)| \leq \frac{n + m(K^{n-m-2\mu+1} + K^{n-m-\mu+1} - 1)}{K^{n-m-2\mu+1} + K^{n-m-\mu+1}} \max_{|z|=1} |p(z)|.
\]

Applying Lemma 3.2 to the polynomial \( p'(z) \) which is of degree \( n - 1 \), it follows that for all \( R \geq 1 \) and \( \theta \in [0, 2\pi) \),

\[
|p'(Re^{i\theta})| \leq \max_{|z|=R} |p'(z)|
\leq R^{n-1} \max_{|z|=1} |p'(z)|
\leq R^{n-1} \left[ \frac{n + m(K^{n-m-2\mu+1} + K^{n-m-\mu+1} - 1)}{K^{n-m-2\mu+1} + K^{n-m-\mu+1}} \right] \max_{|z|=1} |p(z)|.
\]

So for each \( \theta \in [0, 2\pi) \) and \( R \geq 1 \), we obtain
\[ [p(Re^{i\theta})]^s - [p(e^{i\theta})]^s = \int_1^R \frac{dp(te^{i\theta})}{dt} \, dt \]
\[ = \int_1^R s[p(te^{i\theta})]^{s-1}p'(te^{i\theta})e^{i\theta} \, dt. \]

This implies that
\[ |p(Re^{i\theta})|^s \leq |p(e^{i\theta})|^s + s \int_1^R |p(te^{i\theta})|^{s-1}|p'(te^{i\theta})| \, dt. \]

So,
\[ [M(p, R)]^s \leq [M(p, 1)]^s + s \int_1^R \left[ t^n M(p, 1) \right]^{s-1} |p'(te^{i\theta})| \, dt \]
\[ \leq [M(p, 1)]^s + s \int_1^R t^{ns-n} [M(p, 1)]^{s-1} t^{n-1} n + m(\frac{K^{n-m-2\mu+1} + K^{n-m-\mu+1}}{K^{n-m-2\mu+1} + K^{n-m-\mu+1}}) M(p, 1) dt \]
\[ = [M(p, 1)]^s + s \left[ n + m(\frac{K^{n-m-2\mu+1} + K^{n-m-\mu+1}}{K^{n-m-2\mu+1} + K^{n-m-\mu+1}}) \right] [M(p, 1)]^s \int_1^R t^{ns-n} \, dt \]
\[ = [M(p, 1)]^s + [M(p, 1)]^s \left[ n + m(\frac{K^{n-m-2\mu+1} + K^{n-m-\mu+1}}{K^{n-m-2\mu+1} + K^{n-m-\mu+1}}) \right] \frac{R^{ns} - 1}{ns} \]
\[ = [M(p, 1)]^s \left[ 1 + \frac{R^{ns} - 1}{n(K^{n-m-2\mu+1} + K^{n-m-\mu+1})} \right]. \]

This yields
\[ [M(p, R)]^s \leq [M(p, 1)]^s \left[ n(K^{n-m-2\mu+1} + K^{n-m-\mu+1}) + \left[ n + m(\frac{K^{n-m-2\mu+1} + K^{n-m-\mu+1}}{K^{n-m-2\mu+1} + K^{n-m-\mu+1}} - 1) \right] (R^{ns} - 1) \right]. \]

This completes the proof.

\[ \square \]

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