The Turán number for the edge blow-up of trees: the missing case

Cheng Chi∗ Long-Tu Yuan†

Abstract

The edge blow-up of a graph is the graph obtained from replacing each edge of it by a clique of the same size where the new vertices of the cliques are all different. Wang, Hou, Liu and Ma determined the Turán number of the edge blow-up of trees except one particular case. Answering an problem posed by them, we determined the Turán number of this particular case.

1 Introduction

Given a family of graphs \( H \), a graph \( G \) is said to be \( H \)-free (\( H \)-free if \( H = \{ H \} \)) if \( G \) does not contain any copy of \( H \in H \) as a subgraph. A typical problem in extremal combinatorics is the following Turán-type problem: what is the maximum number of edges in an \( H \)-free graph on \( n \) vertices? The aforementioned number is called the extremal number for \( H \) and denoted by \( \text{ex}(n, H) \). Denote by \( \text{EX}(n, H) \) the set of \( H \)-free graphs on \( n \) vertices with \( \text{ex}(n, H) \) edges and call a graph in \( \text{EX}(n, H) \) an extremal graph for \( H \). We use \( \text{ex}(n, H) \) and \( \text{EX}(n, H) \) instead of \( \text{ex}(n, H) \) and \( \text{EX}(n, H) \) respectively when \( H = \{ H \} \).

Much interests has been attracted to this problem during the last few decades. In 1907, Mantel [7] determined the extremal number for triangle for all \( n \geq 3 \). Turán [11] extended Mantel’s result to complete graph with any given order in 1941.

Our notations are standard, see [1]. Given a graph \( H \) and a set of vertices \( A \subseteq V(H) \), we denote \( \min\{\deg_H(x) ; x \in A\} \) by \( \delta_H(A) \). Given a graph \( H \) and a positive integer \( p \geq 2 \), the edge blow-up of \( H \), denoted by \( H^{p+1} \), is the graph obtained from \( H \) by replacing each edge of \( H \) by a clique of size \( p + 1 \) where the new vertices of the cliques are all distinct. In [6, 8] and [13], \( \text{ex}(n, H^{p+1}) \) has been investigated for a large family of graphs \( H \). In [12], Wang, Hou, Liu and Ma determined the extremal number when \( H \) is tree satisfies some conditions and \( p \geq 3 \). Furthermore, the authors of [12] posed the following question.

∗School of Mathematical Sciences, East China Normal University, 500 Dongchuan Road, Shanghai 200240, China. Email: 52215500038@stu.ecnu.edu.cn.
†School of Mathematical Sciences and Shanghai Key Laboratory of PMMP, East China Normal University, 500 Dongchuan Road, Shanghai 200240, P.R. China. Email: ltyuan@math.ecnu.edu.cn. Supported in part by National Natural Science Foundation of China grant 11901554 and Science and Technology Commission of Shanghai Municipality (No. 18dz2271000).
Question 1.1. Give \( p \geq 3 \) and a tree \( T \) such that its two coloring classes \( A \) and \( B \) satisfying \(|A| \leq |B|\), determine \( \text{ex}(n,T^{p+1}) \) when \( \delta_T(A) = 1 \) and \( \alpha(T) > |B| \).

We solve this question. First we introduce some notations. Given two disjoint graphs \( G \) and \( H \), the disjoint union of \( G \) and \( H \), denoted by \( G \cup H \), is the graph with vertex set \( V(G) \cup V(H) \) and edge set \( E(G) \cup E(H) \). We use \( kG \) to denote the disjoint union of \( k \) copies of \( G \). The join of \( G \) and \( H \), denoted by \( G + H \), is the graph obtained from \( G \cup H \) by adding all edges of the form \( gh \), where \( g \in V(G) \) and \( h \in V(H) \). Denoted by \( P_n \), a path on \( n \) vertices, \( S_n \), a star on \( n \) vertices, \( C_n \), a cycle on \( n \) vertices, \( M_n \) a matching on \( n \) vertices and \( K_{n_1,...,n_p} \), the complete \( p \)-partite graph with the size of \( i \)-partite class \( n_i \). A \( p \)-partite Turán graph on \( n \) vertices, denoted by \( T(n,p) \), is a \( K_{n_1,...,n_p} \) with \( \sum_{i=1}^{p} n_i = n \) and \( |n_i - n_j| \leq 1 \) for \( 1 \leq i,j \leq p \). Let \( H'(n,p,q) = \overline{K}_{q-1} + T(n-q+1,p) \) and \( h'(n,p,q) = e(H'(n,p,q)) \). Let \( e(T(n,p)) = t(n,p) \).

Definition 1.2 (Simonovits [10]). Given a family of graphs \( \mathcal{L} \) with \( p(\mathcal{L}) = p \geq 3 \), let \( \mathcal{M} := \mathcal{M}(\mathcal{L}) \) be a family of minimal graphs \( M \) up to subgraph senses such that there exist a large constant \( t = t(\mathcal{L}) \) depending on \( \mathcal{L} \) such that there exists a graph \( L \in \mathcal{L} \) such that \( L \) is a subgraph of \( M \cup I_v + T \), where \( T = T(t,p-2) \) and \( I_v \) is an independent set on \( v \) vertices. We call \( \mathcal{M}(\mathcal{L}) \) the decomposition family of \( \mathcal{L} \).

A covering of a graph is a set of vertices \( U \) such that every edge of this graph meets at least one vertices of \( U \). An independent covering of a bipartite graph is a covering \( U \) such that no two vertices of \( U \) are adjacent. The covering number \( \beta(T) \) of a graph \( T \) is the minimum order of a covering of \( T \). The independent covering number \( q(T) \) of a bipartite graph \( T \) is the minimum order of an independent covering of \( T \). The independent number \( \alpha(T) \) of a graph \( T \) is the maximum order of a set of vertices such that no two of which are adjacent. For a family of graphs \( \mathcal{F} \) which contains at least one bipartite graph, the independent covering number of \( \mathcal{F} \) is defined by

\[
q(\mathcal{F}) = \min\{q(F) : F \in \mathcal{F} \text{ and } F \text{ is bipartite.}\}
\]

Theorem 1.3 (Liu [6]). Let \( p \geq 3 \) be an integer and \( T \) be a tree. Let coloring classes of \( T \) be \( A \) and \( B \), where \(|A| \leq |B|\). When \( n \) is sufficiently large, we have that

- if \( \delta_T(A) = 1 \) and \( \alpha(T) = |B| \), then \( \text{ex}(n,T^{p+1}) = h(n,p,|A|) \);
- if \( \delta_T(A) \geq 2 \), then \( \text{ex}(n,T^{p+1}) = h(n,p,|A|) + 1 \).

Furthermore, extremal graphs are characterized.

Wang, Hou, Liu and Ma [12] extended Liu’s result to a larger family of trees very recently. Before stating their results, we need follow definitions.

Define

\[
g_1(k) = \begin{cases} 
  k^2 - \frac{3}{2}k & \text{if } k \text{ is even;} \\
  k^2 - \frac{3k-1}{2} & \text{if } k \text{ is odd,}
\end{cases}
\]

and

\[
g_2(k) = \begin{cases} 
  k^2 - \frac{3}{2}k & \text{if } k \text{ is even;} \\
  k^2 - k & \text{if } k \text{ is odd.}
\end{cases}
\]
**Theorem 1.4** (Wang, Hou, Liu and Ma [12]). Let \( p \geq 3 \) be an integer and \( T \) be a tree. Let coloring classes of \( T \) be \( A \) and \( B \), where \(|A| \leq |B|\). Let \( A_0 = \{x \in A : \deg_T(x) = \delta_T(A)\} \) and \( B_0 = \{y \in B : |N(y) \cap A_0| \geq 2\} \). Denote by \( q = |A| \), \( k = \delta_T(A) \) and \( b + 2 = \delta(B_0) \). If \( k \geq 2 \), then for sufficiently large \( n \), we have \( \text{ex}(n, T^{p+1}) = \)

\[
\begin{cases}
  h(n, p, q) + g_1(k) & \text{if } k \text{ is even}; \\
  h(n, p, q) + g_2(k) & \text{if } k \text{ is odd and } B_0 = \emptyset; \\
  h(n, p, q) + g_1(k) & \text{if } k \text{ is odd and } 0 \leq b \leq q - 1 - \left\lceil \frac{k-1}{q-1} \right\rceil; \\
  h'(n, p, q) + g_2(k) + \lfloor (q-1)(b-1)/2 \rfloor & \text{if } k \text{ is odd and } b \geq \max \left\{ 1, q - 1 - \left\lceil \frac{k-1}{q-1} \right\rceil \right\}.
\end{cases}
\]

Furthermore, all extremal graphs are characterized.

Now we set \( \mathcal{M} = \mathcal{M}(T^{p+1}) \) and \( q = q(\mathcal{M}) \). If there exists a graph \( T' \in \mathcal{M} \) such that \( \beta(T') \leq q - 1 \), then we let \( \mathcal{B} := \mathcal{B}(T) \) be the family of graph \( T'[A_{T'}] \), where \( T' \in \mathcal{M} \) and \( A_{T'} \) is a covering set with size at most \( q - 1 \) of \( T' \). If \( \beta(T') \geq q \) for every \( T' \in \mathcal{M} \), then we set \( \mathcal{B} = \{K_q\} \).

**Theorem 1.5.** Let \( p \geq 3 \) be an integer and \( T \) be a tree. Let coloring classes of \( T \) be \( A \) and \( B \), where \(|A| \leq |B|\). If \( \delta_T(A) = 1 \) and \( \alpha(T) > |B| \), then for sufficiently large \( n \), we have

\[
\text{ex}(n, T^{p+1}) = h'(n, p, q) + \text{ex}(q - 1, \mathcal{B})
\]

where \( q \) and \( \mathcal{B} \) are defined as above. Furthermore, all extremal graphs are characterized.

**Remark.** Combining with the results in [6] and [8], the extremal number for \( T^{p+1} \) is determined, where \( T \) is an arbitrary tree and \( p \geq 3 \).

A double broom \( B(\ell, s, t) \) is a tree obtained from a path \( P_\ell \) by attaching \( s \) pendant edges to one end vertex of \( P_\ell \) and \( t \) pendant edges to the other end vertex of \( P_\ell \), where \( \ell, s, t \geq 2 \). The double broom \( B(7, 5, 3) \) is as in Figure 1.

![Double broom B(7, 5, 3)](image)

**Corollary 1.6.** Let \( p \geq 3 \) be an integer and \( T = B(2k, s, t) \) be a double broom satisfying \( k, s, t \geq 2 \). Then for sufficiently large \( n \), we have

\[
\text{ex}(n, T^{p+1}) = h(n, p, k + 1)
\]

Furthermore, \( H(n, p, k + 1) \) is the unique extremal graph.

**Proof.** It can be easily checked that \( q(\mathcal{M}(T^{p+1})) = k + 1 \) and \( \beta(T') \geq k + 1 \) holds for every \( T' \in \mathcal{M}(T^{p+1}) \). Furthermore, we have \( \alpha(T) = k - 1 + s + t > k + \min\{s, t\} = |B| \) and \( \delta_T(A) = 1 \). Therefore, the result holds by applying Theorem 1.5 with \( q = k + 1 \) and \( \mathcal{B} = \{K_q\} \). 

2 Preliminaries

2.1 Technical lemmas

Given a graph $T$, a vertex split on some vertex $v \in V(T)$ is defined by replacing $v$ by an independent set of size $\deg_T(v)$ in which each vertex is adjacent to exactly one distinct vertex in $N_T(v)$. The family of graphs that can be obtained by applying vertex split on some $U \subseteq V(T)$ is denoted by $\mathcal{H}(T)$. The following lemma can help us to determine the graphs in $\mathcal{M}(T^{p+1})$.

Lemma 2.1 (Liu [6]). Given $p \geq 3$ and any graph with $\chi(H) \leq p - 1$, we have $\mathcal{M}(H^{p+1}) = \mathcal{H}(H)$. In particular, a matching of size $e(H)$ is in $\mathcal{M}(H^{p+1})$.

Theorem 2.2 (Erdős and Stone [3]). For all integers $p \geq 1$, $N \geq 1$, and every $\varepsilon > 0$, there exists an integer $n_0(\varepsilon, N, p + 1)$ such that every graph with $n \geq n_0$ vertices and at least $t(n, p) + \varepsilon n^2$ edges contains $T(N, p + 1)$ as a subgraph.

3 Proof of Theorem 1.5

Given a tree $T$ with coloring classes $A$ and $B$ satisfying $|A| \leq |B|$, $\delta_T(A) = 1$ and $\alpha(T) > |B|$. Now we set $\mathcal{M}$ be the decomposition family of $T^{p+1}$. It follows from Lemma 2.1 that $\mathcal{M} = \mathcal{H}(T)$. Furthermore, $\mathcal{M}$ contains a matching of size $t$, where $t = e(T)$.

Let $\mathcal{U}_n$ be the family of graphs obtained from $H'(n,p,q)$ by embedding a copy of $Q \in \text{EX}(q-1,B)$ in $K_{q-1}$ in $H'(n,p,q)$. The definition of $\mathcal{B}$ implies every $H_n \in \mathcal{U}_n$ is $T^{p+1}$-free, and hence we have

$$\text{ex}(n,T^{p+1}) \geq h'(n,p,q) + \text{ex}(q-1,B) \quad (2)$$

Let $\phi(n) := \text{ex}(n,T^{p+1}) - h'(n,p,q) - \text{ex}(q-1,B)$ and $K = \max\{\phi(n) : n \leq n_0\}$, where $n_0$ is a large constant depending on $p$ and $T$. Clearly, $\phi(n)$ is a non-negative integer. For the upper bound, we will show that if $n > n_0$ and $\phi(n) > 0$, then there exists an $n_4$ depending on $p$ and $T$ such that $\phi(n) < \phi(n - n_4p)$. This would imply that if $n = n_0 + mn_4p$, then $\phi(n) < K - m$, and hence the theorem holds for $n \geq n_0 + Kn_4p$.

Let $n_1$ be a sufficiently large constant. Let $n_0 = n_0(\varepsilon, n_1p,p)$ be the constant from Theorem 2.2, where $\varepsilon = 1/(2p(p-1))$. Let $L_n$ be a $T^{p+1}$-free graph with $\text{ex}(n,T^{p+1})$ edges, where $n \geq n_0$. Equation 2 and Theorem 2.2 imply that $L_n$ contains a $T = T(n_1p,p)$ with partite class $\tilde{B}^0_1, \cdots, \tilde{B}^0_p$ as a subgraph. Note that $M_{2t} \in \mathcal{M}$. It follows from the definition of decomposition family and the fact that $L_n$ is $T^{p+1}$-free that $\nu(L_n[\tilde{B}^0_i]) \leq t$ for $i \in [p]$. Let the maximum matching in $L_n[\tilde{B}^0_0]$ be $\{x_1y_1, \cdots, x_iy_i\}$ with $t_i \leq t$. Let $B^0_i = \tilde{B}^0_i \setminus \{x_1y_1, \cdots, x_iy_i\}$. By the definition of $B^0_i$, there is no edge in $L_n[B^0_i]$. Hence there is an induced subgraph $T_0 = T(n_2p,p)$ of $L_n$ with partite class $B^0_1, \cdots, B^0_p$ obtained by deleting $2t$ vertices from each $\tilde{B}_i$, where $n_2 = n_1 - 2t$. 


Let \( c < 1/(1 + t) \) be a sufficiently small constant. If there exists a vertex \( x_1 \in L_n - T_0 \) such that \( x_1 \) is adjacent to at least \( c^2n_2 \) vertices of each partite class of \( T_0 \), then \( T_0 \) contains a \( T_1 = T(c^2n_2p, p) \) such that each vertex of which is joint to \( x_1 \). Generally, if there exists a vertex \( x_i \in L_n - T_{i-1} - \{x_1, \cdots, x_{i-1}\} \) such that \( u \) is adjacent to at least \( c^2n_2 \) vertices of each partite class of \( T_{i-1} \), then \( T_{i-1} \) contains \( T_i = T(c^2n_2p, p) \) such that each vertex of which is joint to \( x_1, \cdots, x_i \). Thus we can define a sequence of graphs recursively. However, it follows from the definition of \( L_n \) and \( q \) that the above process stops at last after \( T_{q-1} \).

Suppose to the contrary, let \( V(T_q) = B_1^q \cup \cdots \cup B_p^q \). Note that the graph induced by \( B_1^q \cup \{x_1, \cdots, x_q\} \) contains some element of \( \mathcal{M} \) by the definition of \( q \). Then \( L_n \) contains a copy of \( T^{p+1} \) by the definition of decomposition family, a contradiction.

Now suppose that the above process ends with \( T_s \) with \( s \leq q - 1 \). Let \( E = \{x_1, \cdots, x_s\} \) and the partite class of \( T_s \) be \( B_1^s, \cdots, B_p^s \). Denote \( |B_i^s| \) by \( n_3 \) for convenience. We can partition the remaining vertices into following set: Let \( x \in V(L_n) \setminus (T_s \cup E) \). If there exists an \( i \in [p] \) such that \( x \) is adjacent to less than \( c^2n_3 \) vertices of \( B_i^s \) and is adjacent to at least \( (1 - c)n_3 \) vertices of \( B_j^s \) for all \( j \neq i \), then let \( x \in C_i \). If there exists an \( i \in [p] \) such that \( x \) is adjacent to less than \( c^2n_3 \) vertices of \( B_i^s \) and is adjacent to less than \( (1 - c)n_3 \) vertices of \( B_j^s \) for some \( j \neq i \), then let \( x \in D \). It follows from the definition of \( T_s \) that \( C_1 \cup \cdots \cup C_p \cup D \) is a partition of \( V(L_n) \setminus (T_s \cup E) \). Note that for a \( S \subset B_i^s \cup C_i \) with \( |S| \leq 2t \), the common neighbourhoods of \( S \) in \( B_j^s \) is at least \( (1 - 2tc)n_3 \geq n_3/2 \), where \( c \neq i \). It follows from the definition of decomposition family and Lemma 2.1 that \( \nu(L_n[B_i^s \cup C_i]) \leq t \). Now consider the edges joining \( B_i^s \) and \( C_i \) and select a maximum matching, say \( y_1z_1, \cdots, y_tz_t \) with \( y_i, z_i \in B_i^s \), \( z_i \in C_i \) and \( 1 \leq z_i \leq t \leq t_i \). Let \( X_i = \bigcup_{z_i=1}^{t_i} (N_{L_n}(z_i) \cap B_i^s) \). Then \( |X_i| \leq tc^2n_3 \) by the definition of \( C_i \). Let \( C_i = C_i \cup X_i \) and \( B_i^s \cap X_i \); then \( L_n[B_i^s \cup C_i] \) contains no edge by the maximality of \( y_1z_1, \cdots, y_tz_t \). Hence it is possible to move \( tc^2n_3 \) vertices from \( B_i^s \) to \( C_i \) to obtain \( B_i^s \) and \( C_i' \) such that \( B_i^s \subset B_i^s \) and \( C_i \subset C_i' \). Let \( n_4 = (1 - tc^2)n_3 = \left|B_i^s\right|, T_s^* = T(n_4p, p) \) and \( \hat{L} = L_n - T_s^* \). Then \( T_s^* \) is an induced subgraph of \( L_n \) and the vertices of \( \hat{L} \) can be partitioned into \( p + 2 \) sets \( C_1', \cdots, C_p', D \) and \( E \) such that

- every \( x \in E \) is adjacent to each vertex of \( T_s^* \) and \( |E| = s \),
- every \( x \in C_i' \) is adjacent to no vertex of \( B_i^s \) and is adjacent to at least \( (1 - c - tc^2)n_3 \) vertices of \( B_j^s \) for all \( j \neq i \).
- every \( x \in D \) is adjacent to at most \( c^2n_3 \) vertices of \( B_i^s \) and is adjacent to at most \( (1 - c)n_3 \) vertices of \( B_j^s \) for some \( i, j \in [p] \) with \( i \neq j \).

Let the number of edges joining \( T_s^* \) and \( \hat{L} \) in graph \( L_n \) denoted by \( e_L \). Then we have

\[
e(L_n) = e(\hat{L}) + e_L + e(T_s^*)
\]

Let \( H_n \in \mathcal{H}_n \) and \( T_s'' \) be an induced copy of \( T(n_4p, p) \) in \( H_n \). Let \( H_{n-n_4p} = H_n - T_s'' \) and \( e_H \) be the number of edges joining \( T_s'' \) and \( H_{n-n_4p} \) in graph \( H_n \). Then

\[
e(H_n) = e(H_{n-n_4p}) + e_H + e(T_s'')
\]
Since \( \hat{L} \) contains no copy of \( T^{p+1} \), we have \( e(\hat{L}) \leq e(L_{n-n_4p}) \), where \( L_{n-n_4p} \in EX(n-n_4p, T^{p+1}) \). Obviously we have \( e(T'_q) = e(T''_q) \). Simple calculation show that

\[
e_H = (q-1)n_4p + (n-n_4p-q+1)n_4(p-1) \\
= (q-1)n_4 + (n-n_4p)n_4(p-1)
\]  

(3)

It follows from the definition of \( C'_i \), \( D \) and \( E \) that

\[
e_L \leq sn_4 + (n-n_4p-s-|D|)n_4(p-1) + |D|((p-2)n_4 + (1-c+c^2)n_3) \\
= sn_4 + (n-n_4p)n_4(p-1) - |D|(n_4 - (1-c+c^2)n_3) \\
\leq (q-1)n_4 + (n-n_4p)n_4(p-1) - |D|n_3(c - (t+1)c^2) \\
= e_H - |D|n_3(c - (t+1)c^2)
\]  

(4)

Hence we have

\[\phi(n) = e(L_n) - e(H_n)\]
\[\leq e(L_{n-n_4p}) - e(H_{n-n_4p}) + e_L - e_H\]
\[= \phi(n-n_4p) + e_L - e_H\]

If \( e_L - e_H < 0 \), then we have \( \phi(n) < \phi(n-n_4p) \), where \( n_4 \leq n_2 \). Hence we suppose that \( e_L - e_H \geq 0 \). Combined with Equation 3 and 4 we conclude that \( e_L = e_H \). (Note that \( c < 1/(1+t) \) is sufficiently small.) Note that \( e_L = e_H \) holds if and only if \( |D| = 0, s = q-1 \) and \( C'_i \) is complete to \( B'_i \) for \( i \in [p] \) and \( j \neq i \).

If \( e(L_n[E]) \) contains some copy of \( B' \in \mathcal{B} \), then \( L_n \) contains a copy of \( T^{p+1} \) by the definition of \( \mathcal{B} \). Hence we conclude that \( L_n[E] \) is \( \mathcal{B} \)-free and \( e(L_n[E]) \leq \text{ex}(q-1, \mathcal{B}) \). The rest of the proof will be divided into two cases.

**Case 1.** \( q = q(T) \).

In this case, note that \( T \) is a tree. Clearly, \( T \) admits a unique proper 2-coloring and hence \( q(T) = \lfloor A \rfloor \) holds. Note that \( \delta_T(A) = \min \{ \deg_T(x) : x \in A \} = 1 \) by assumptions. Hence there exists a vertex \( u \in A \) such that \( N_T(u) = \{ v \} \). Since \( \lfloor A \rfloor = q(T) = q \), we can find a copy of \( (T - \{ u \})^{p+1} \) using vertices in \( E \cup B'_1 \cup \cdots \cup B'_p \) in \( L_n \). Let \( \phi \) be an embedding from \( (T - \{ u \})^{p+1} \) to \( L_n \) such that \( \text{Im} \psi \subseteq E \cup B'_1 \cup \cdots \cup B'_p \) and \( \psi(A \setminus \{ u \}) = E \).

Now we will show that \( B'_1 \cup C'_i \) is an independent set of \( L_n \) for each \( i \). It suffices to show \( C'_i \) is an independent set for each \( i \) since there is no edge incident with \( B'_i \). We assume that \( \psi(v) \in B'_i \), where \( i \neq i. \) In fact, if there is an edge \( u'u'' \) in \( L_n[C'_i] \), then we can choose \( u_j \in B'_j \) such that \( u_j \notin \text{Im} \psi \) for \( j \in [p] \setminus \{ i, i' \} \). It can be seen immediately from the definition of \( B'_i, C'_i \) and \( E \) that \( u', u'' \) and \( \psi(v) \) together with all \( u_j \) forms a copy of \( K_{p+1} \). Furthermore, it can be verified that the mapping constructed above is an embedding from \( T^{p+1} \) to \( L_n \). This completes the proof for this case.

**Case 2.** \( q < q(T) \).

Let \( F \in \mathcal{M} \) such that \( q(F) = q \). Let \( A_F \) and \( B_F \) be coloring classes of \( F \) such that \( q(F) = |A_F| \). Now we show that \( \min \{ \deg_F(x) : x \in A_F \} = 1 \). It follows from the definition of decomposition family that \( \min \{ \deg_F(x) : x \in A_F \} \geq 1 \).
If $A_F$ contains a vertex $u$ which is obtained by splitting a vertex in $T$, then the result follows since $\deg_F(u) = 1$. Now we assume that every $u \in A_F$ is not a vertex obtained by splitting a vertex in $T$. Then we have $u \in V(T)$ for every $u \in A_F$. By lemma 2.1, we may assume $F$ is obtained by splitting $X \subseteq V(T)$. It is easy to see that $X \cap A_F = \emptyset$. Otherwise, we can find a vertex obtained by splitting a vertex in $T$, a contradiction. Let the vertices obtained by splitting $X$ in $T$ be $Y$ and $Z = B_F \setminus X$. It is clear that $V(T)$ is the disjoint union of $A_F$, $X$ and $Z$. Furthermore, $V(F)$ is the disjoint union of $A_F$, $Y$ and $Z$. Note that we have $E_T(A_F, Z) = E_F(A_F, Z)$ by the definition of $Z$. It follows from the definition that $Y \cup Z$ is an independent set of $F$. Note that $\delta(F) \geq 1$ since $F$ is obtained by splitting $X \subseteq V(T)$. Hence every $y \in Y$ is adjacent to some $v \in A_F$ in graph $F$. Therefore, $A_F$ is an independent covering of $T$. Then we have $q(T) \leq |A_F| = q < q(T)$, a contradiction.

Hence we have $\min\{\deg_F(x) : x \in A_F\} = 1$. Then similar arguments in Case 1 show that $B'_i \cup C'_i$ is an independent set of $L_n$.

Note that $B'_i \cup C'_i$ is an independent set of $L_n$ for each $i$ in both cases, then we have

$$
\begin{align*}
\epsilon(L_n) & \leq L_n[E] + \epsilon_{L_n}(E, V(L_n) \setminus E) + \sum_{1 \leq i < j \leq p} \epsilon_{L_n}(B'_i \cup C'_i, B'_j \cup C'_j) \\
& \leq \text{ex}(q - 1, B) + (q - 1)(n - q + 1) + \sum_{1 \leq i < j \leq p} |B'_i \cup C'_i||B'_j \cup C'_j| \\
& \leq \text{ex}(q - 1, B) + (q - 1)(n - q + 1) + t(n - q + 1, p) \\
& = \text{ex}(q - 1, B) + h'(n, p, q)
\end{align*}
$$

which contradicts the fact that $\phi(n) > 0$. The theorem follows.

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