Non-symmetry constraints of the AKNS system yielding integrable Hamiltonian systems

Wen-Xiu Ma∗
Department of Mathematics, City University of Hong Kong, Kowloon, Hong Kong, P. R. China

Si-Ming Zhu†
Department of Mathematics, Zhongshan University, Guangzhou 510275, P. R. China

Abstract

This paper aims to show that there exist non-symmetry constraints which yield integrable Hamiltonian systems through nonlinearization of spectral problems of soliton systems, like symmetry constraints. Taking the AKNS spectral problem as an illustrative example, a class of such non-symmetry constraints is introduced for the AKNS system, along with two-dimensional integrable Hamiltonian systems generated from the AKNS spectral problem.

1 Introduction

The nonlinearization process yields integrable Hamiltonian systems from spectral problems of soliton systems [1]-[4]. Much excitement in the study of nonlinearization comes from a kind of specific symmetry constraints [5, 6]. It is due to symmetry constraints that the nonlinearization technique is so powerful in generating integrable Hamiltonian systems [7, 8]. Usually, taking non-symmetry constraints leads to spectral problems to non-Hamiltonian systems, which are difficult to be handled. However, there appears a natural question of whether there exist any non-symmetry constraints which can still force spectral problems of soliton systems to be integrable Hamiltonian systems. The answer is yes. This paper aims to show that it is possible to generate integrable Hamiltonian systems from spectral problems of soliton systems by employing non-symmetry constraints of soliton systems.

In the following section, we take the AKNS spectral problem as an illustrative example and recall some known results related to symmetry constraints of the AKNS system for reference. Then in Section 3, we move on to discuss non-symmetry constraints of the AKNS system. A class of non-symmetry constraints is introduced for the AKNS system,
which generates two-dimensional integrable Hamiltonian systems from the AKNS spectral problem. Finally in Section 4 a comparison between our integrable systems and other integrable systems and some concluding remarks are given.

2 AKNS system and symmetry constraints

Let us take the AKNS spectral problem [9]

\[ \phi_x = U \phi, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad U = U(u, \lambda) = \begin{pmatrix} -\lambda & q \\ r & \lambda \end{pmatrix}, \quad u = \begin{pmatrix} q \\ r \end{pmatrix}, \quad (2.1) \]

with \( \lambda \) being a spectral parameter, as an illustrative example. Its adjoint spectral problem reads as

\[ \psi_x = -U^T \psi, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \text{i.e.} \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} \lambda & -r \\ -q & -\lambda \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (2.2) \]

where \( T \) denotes the transpose operation of matrices. The spectral problem (2.1) or its adjoint spectral problem (2.2) yields the AKNS hierarchy

\[ u_t = \begin{pmatrix} q \\ r \end{pmatrix}_t = K_n = \Phi^n \begin{pmatrix} -2q \\ 2r \end{pmatrix} = JG_n = J\frac{\delta \tilde{H}_n}{\delta u}, \quad n \geq 0, \quad (2.3) \]

where the Hamiltonian operator \( J \), the recursion operator \( \Phi \), and the Hamiltonian functionals \( \tilde{H}_n \) are defined by

\[ J = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \quad \Phi = \begin{pmatrix} -\frac{1}{2} \partial + q \partial^{-1}r & q \partial^{-1}q \\ -r \partial^{-1}r & \frac{1}{2} \partial - r \partial^{-1}q \end{pmatrix}, \quad (2.4) \]

\[ \tilde{H}_n = \int H_n \, dx, \quad H_n = \int_0^1 <G_n(\lambda u), u> \, d\lambda, \quad n \geq 0, \quad (2.5) \]

where \( <\cdot, \cdot> \) denotes the standard inner product of \( \mathbb{R}^2 \). The first nonlinear integrable system in this soliton hierarchy is the AKNS system of nonlinear Schrödinger equations

\[ u_t = \begin{pmatrix} q \\ r \end{pmatrix}_t = K_2 = \begin{pmatrix} -\frac{1}{2}q_{xx} + q^2r \\ \frac{1}{2}r_{xx} - qr^2 \end{pmatrix}. \quad (2.6) \]

This AKNS system has its associated spectral problem

\[ \phi_t = V^{(2)} \phi, \quad V^{(2)}(u, \lambda) = \begin{pmatrix} -\lambda^2 + \frac{1}{2} qr & \lambda q - \frac{1}{2} q_x \\ \lambda r + \frac{1}{2} r_x & \lambda^2 - \frac{1}{2} qr \end{pmatrix}, \quad (2.7) \]

which implies that (2.6) is equivalent to a zero curvature equation \( U_t - V_x^{(2)} + [U, V^{(2)}] = 0 \).

The AKNS system (2.6) also has a tri-Hamiltonian structure

\[ u_t = K_2 = J_0 \frac{\delta \tilde{H}_2}{\delta u} = J_1 \frac{\delta \tilde{H}_1}{\delta u} = J_2 \frac{\delta \tilde{H}_0}{\delta u}, \quad (2.8) \]

2
where the Hamiltonian operators $J_i$, $0 \leq i \leq 2$, and the Hamiltonian functionals $\tilde{H}_i$, $0 \leq i \leq 2$, are given by

$$J_0 = J = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \quad J_1 = \Phi J = \begin{pmatrix} 2q\partial^{-1}q & \partial - 2q\partial^{-1}r \\ \partial - 2r\partial^{-1}q & 2r\partial^{-1}r \end{pmatrix}, \quad (2.9)$$

$$J_2 = \Phi J_1 = \begin{pmatrix} q\partial^{-1}q - \partial q\partial^{-1}q & -\frac{1}{2}\partial^2 + q\partial^{-1}r\partial + \partial q\partial^{-1}r \\ \frac{1}{2}\partial^2 - r\partial^{-1}q - r\partial^{-1}q\partial & \partial r\partial^{-1}r - r\partial^{-1}r\partial \end{pmatrix}, \quad (2.10)$$

$$\tilde{H}_0 = \int qr \, dx, \quad \tilde{H}_1 = \frac{1}{4} \int (qr_x - q_x r) \, dx, \quad \tilde{H}_2 = \frac{1}{8} \int (qr_{xx} + q_{xx} r - 2q^2 r^2) \, dx. \quad (2.11)$$

A proof that $J_0 + \alpha J_1 + \beta J_2$ is Hamiltonian for all $\alpha$ and $\beta$ can be found in [10].

It is known [5] that the spectral problem (2.1) and the adjoint spectral problem (2.2) become a finite-dimensional integrable Hamiltonian system

$$\phi_{ix} = -\frac{\partial H}{\partial \psi_i}, \quad \psi_{ix} = \frac{\partial H}{\partial \phi_i}, \quad H = \lambda \phi_2 \psi_2 - \lambda \psi_1 \psi_1 + \phi_1 \phi_2 \psi_1 \psi_2, \quad i = 1, 2, \quad (2.12)$$

when we employ a symmetry constraint

$$K_0 = E J \delta \lambda / \delta u = J \psi^T \frac{\partial U}{\partial u} \psi = J \begin{pmatrix} \phi_1 \psi_2 \\ \phi_2 \psi_1 \end{pmatrix}, \quad (2.13)$$

with $E$ being a normalized constant, which leads to two constraints on the potentials

$$q = \phi_1 \psi_2, \quad r = \phi_2 \psi_1. \quad (2.14)$$

Note $J \delta \lambda / \delta u$ is a symmetry of the AKNS system (2.6) due to $\lambda_t = 0$. Actually it can directly be shown that $J(\phi_1 \psi_2, \phi_2 \psi_1)^T$ is a symmetry of the AKNS system (2.10), that is to say,

$$\sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} -2\phi_1 \psi_2 \\ 2\phi_2 \psi_1 \end{pmatrix}$$

satisfies the linearized system of the AKNS system (2.6):

$$\sigma_{1t} = -\frac{1}{2} \sigma_{1xx} + 2qr\sigma_1 + q^2 \sigma_2, \quad \sigma_{2t} = \frac{1}{2} \sigma_{2xx} - r^2 \sigma_1 - 2qr \sigma_2, \quad (2.15)$$

when $\phi$ and $\psi$ satisfy two systems of (2.1) and (2.2) and evolve according to

$$\phi_t = V^{(2)} \phi = V^{(2)} (u, \lambda) \phi, \quad \psi_t = -(V^{(2)})^T \psi = -(V^{(2)} (u, \lambda))^T \psi, \quad (2.16)$$

with $V^{(2)}$ being defined by (2.7). Therefore, (2.13) is a symmetry constraint indeed, because $K_0$ on the left side of (2.13) and $J \delta \lambda / \delta u$ on the right side of (2.13) are all symmetries of the AKNS system (2.6). Moreover, we can show that (2.13) is a symmetry constraint of each system in the AKNS hierarchy (2.3).
3 Non-symmetry constraints yielding integrable Hamiltonian systems

In what follows, we are going to present a class of non-symmetry constraints that still yield integrable Hamiltonian systems through nonlinearization of the spectral problem (2.1) and the adjoint spectral problem (2.2). Let us first assume that two constraints between the potentials and the eigenfunctions and adjoint eigenfunctions are defined by

\[ q = q(\phi_1, \phi_2, \psi_1, \psi_2), \quad r = r(\phi_1, \phi_2, \psi_1, \psi_2). \]  

(3.1)

Notice that a vector-valued function \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \), i.e.

\[ f = (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n, \]

is a gradient \( f = \text{grad} \ h \) of some function \( h : \mathbb{R}^n \rightarrow \mathbb{R} \) if and only if

\[ \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}, \quad 1 \leq i < j \leq n. \]

It follows that the spectral problem (2.1) and the adjoint spectral problem (2.2) is a Hamiltonian system under the constraints (3.1) and the symplectic form

\[ \omega^2 = d\phi_1 \wedge d\psi_1 + d\phi_2 \wedge d\psi_2, \]  

(3.2)

if and only if

\[
\begin{cases}
\frac{\partial q}{\partial \phi_1} \phi_2 = \frac{\partial r}{\partial \psi_1} \psi_2, & \frac{\partial q}{\partial \psi_1} \psi_1 = \frac{\partial r}{\partial \phi_2} \phi_1, \\
\frac{\partial q}{\partial \phi_1} \psi_1 = \frac{\partial r}{\partial \phi_2} \psi_2, & \frac{\partial q}{\partial \phi_2} \phi_2 = \frac{\partial r}{\partial \psi_1} \phi_1, \\
\frac{\partial q}{\partial \phi_2} \phi_1 = \frac{\partial r}{\partial \psi_1} \psi_1, & \frac{\partial r}{\partial \phi_2} \phi_1 = \frac{\partial r}{\partial \psi_2} \psi_2.
\end{cases}
\]  

(3.3)

Our case has \( n = 4 \), and the functions \( f_i \) and the variables \( x_i \) are chosen to be

\[ f_1 = \lambda \psi_1 - r \psi_2 \quad (= \psi_{1x}), \quad f_2 = -q \psi_1 - \lambda \psi_2 \quad (= \psi_{2x}), \]

\[ f_3 = \lambda \phi_1 - q \phi_2 \quad (= -\phi_{1x}), \quad f_4 = -r \phi_1 - \lambda \phi_2 \quad (= -\phi_{2x}), \]

and

\[ (x_1, x_2, x_3, x_4) = (\phi_1, \phi_2, \psi_1, \psi_2). \]

We point out that from (3.3) we can further obtain two conditions similar to the last two conditions in (3.3):

\[ \frac{\partial q}{\partial \phi_1} \phi_1 = \frac{\partial q}{\partial \psi_2} \psi_2, \quad \frac{\partial r}{\partial \phi_2} \phi_2 = \frac{\partial r}{\partial \psi_1} \psi_1. \]
Now the construction of the constraints \((3.1)\) yielding Hamiltonian systems becomes the problem of finding solutions to the system of differential equations \((3.3)\). Fortunately, by inspection, a solution to the system \((3.3)\) is found to be
\[
q = \alpha \phi_1 \psi_2 + g_1 (\phi_2 \psi_1), \quad r = \alpha \phi_2 \psi_1 + g_2 (\phi_1 \psi_2),
\]
where \(\alpha\) is an arbitrary constant, and \(g_1, g_2 : \mathbb{R} \to \mathbb{R}\) are two arbitrary functions. Changing the constraints \((3.4)\) into the following form
\[
K_0 = \begin{pmatrix} -2q \\ 2r \end{pmatrix} = \begin{pmatrix} -2(\alpha \phi_1 \psi_2 + g_1 (\phi_2 \psi_1)) \\ 2(\alpha \phi_2 \psi_1 + g_2 (\phi_1 \psi_2)) \end{pmatrix} = \alpha \begin{pmatrix} -2\phi_1 \psi_2 \\ 2\phi_2 \psi_1 \end{pmatrix} + \begin{pmatrix} -2g_1 (\phi_2 \psi_1) \\ 2g_2 (\phi_1 \psi_2) \end{pmatrix},
\]
\[
(3.5)
\]
it is not difficult to see that the constraints \((3.4)\) with \(g_1^2 + g_2^2 \neq 0\) are not generated from symmetry constraints of the AKNS system \((2.6)\), because the vector fields \(K_0\) and \((-2\phi_1 \psi_2, 2\phi_2 \psi_1)^T\) are symmetries but all nonzero vector fields
\[
\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} -2g_1 (\phi_2 \psi_1) \\ 2g_2 (\phi_1 \psi_2) \end{pmatrix}
\]
\[
(3.6)
\]
are not symmetries of the AKNS system \((2.4)\). This can be shown by observing the terms involving \(q^2\) and \(r^2\) in the linearized system \((2.13)\). Keeping \((2.1)\), \((2.2)\) and \((2.16)\) in mind, we find that \(\sigma_{1t}, -1/2\sigma_{1xx}\) and \(2qr\sigma_{1}\) do not contain any term involving \(q^2\) and thus the first equation of the linearized system \((2.13)\) requires \(g_2 = 0\). Similarly, we can find that the second equation of the linearized system \((2.15)\) requires \(g_1 = 0\). This will contradict our assumption of the nonzero condition on \((\sigma_1, \sigma_2)^T\): \(g_1^2 + g_2^2 \neq 0\). Therefore, \((\sigma_1, \sigma_2)^T = (-2g_1 (\phi_2 \psi_1), 2g_2 (\phi_1 \psi_2))^T\) with \(g_1^2 + g_2^2 \neq 0\) are not symmetries of the AKNS system \((2.6)\), and so the constraints \((3.5)\) with \(g_1^2 + g_2^2 \neq 0\) are not symmetry constraints of \((2.6)\), because \(K_0 - 2\alpha(-\phi_1 \psi_2, \phi_2 \psi_1)^T\) is a symmetry of \((2.6)\). Moreover, we believe that \((3.3)\) is not a symmetry constraint of the other systems in the AKNS hierarchy \((2.3)\), either.

Let us now take the constraints \((3.4)\), and then the spectral problem \((2.1)\) and the adjoint spectral problem \((2.2)\) are nonlinearized into a Hamiltonian system
\[
\phi_{ix} = -\frac{\partial H(g_1, g_2)}{\partial \psi_i}, \quad \psi_{ix} = \frac{\partial H(g_1, g_2)}{\partial \phi_i}, \quad i = 1, 2,
\]
\[
(3.7)
\]
with the Hamiltonian function
\[
H(g_1, g_2) = \lambda (\phi_1 \psi_1 - \phi_2 \psi_2) - \alpha \phi_1 \phi_2 \psi_1 \psi_2 - h_1 (\phi_2 \psi_1) - h_2 (\phi_1 \psi_2),
\]
\[
(3.8)
\]
where \(h_1, h_2 : \mathbb{R} \to \mathbb{R}\) are two anti-derivative functions of \(g_1, g_2\), respectively. This Hamiltonian system has a second integral of motion
\[
F = \phi_1 \psi_1 + \phi_2 \psi_2.
\]
\[
(3.9)
\]
It can be generated as follows \[3, 12\]

\[
F = \text{tr} \bar{V}, \quad \bar{V} = \phi \psi^T = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} (\psi_1, \psi_2) = \begin{pmatrix} \phi_1 \psi_1 & \phi_1 \psi_2 \\ \phi_2 \phi_1 & \phi_2 \psi_2 \end{pmatrix}.
\]

Since we have \(\bar{V}_x = [U, \bar{V}]\) provided that (2.1) and (2.2) hold, we can compute that

\[
F_x = \text{tr}(\bar{V}_x) = \text{tr}[U, \bar{V}] = 0.
\]

This means that the function \(F\) is an integral of motion of the spectral problem (2.1) and the adjoint spectral problem (2.2) with any potentials \(q\) and \(r\), and hence \(F\) is also an integral of motion the Hamiltonian system defined by (3.7) and (3.8), where \(q\) and \(r\) are the special functions defined by (3.4).

Two integrals of motion \(H(g_1, g_2)\) and \(F\) are functionally independent and of course they commute with each other, i.e. the Poisson bracket of \(H(g_1, g_2)\) and \(F\) is equal to zero,

\[
\{H(g_1, g_2), F\} = -\left< \frac{\partial H(g_1, g_2)}{\partial \psi}, \frac{\partial F}{\partial \phi} \right> - \left< \frac{\partial H(g_1, g_2)}{\partial \phi}, \frac{\partial F}{\partial \psi} \right> = 0,
\]

since \(F\) is an integral of motion of the Hamiltonian system (3.7). These two properties guarantee that the Hamiltonian system defined by (3.7) and (3.8) is Liouville integrable [11]. Therefore the spectral problem (2.1) and the adjoint spectral problem (2.2) are nonlinearized into an integrable Hamiltonian system under the constraints (3.4), which are not of symmetry type when \(g_1^2 + g_2^2 \neq 0\).

The class of integrable Hamiltonian systems generated above has two degrees of freedom. They contain the Hamiltonian system (2.12) under a symmetry constraint if we take \(\alpha = 1\) and \(g_1 = g_2 = 0\). Another interesting integrable Hamiltonian system is associated with the case of \(\alpha = 0\) and \(g_1(y) = g_2(y) = y:\)

\[
\phi_{1x} = -\lambda \phi_1 + \phi_2^2 \psi_1, \quad \phi_{2x} = \phi_1^2 \psi_2 + \lambda \phi_2, \quad \psi_{1x} = \lambda \psi_1 - \phi_1 \psi_2^2, \quad \psi_{2x} = -\phi_2 \psi_1^2 - \lambda \psi_2. \tag{3.10}
\]

This system interchanges two constraints on the potentials \(q\) and \(r\) in the symmetry case (2.14) but it corresponds to a non-symmetry case. A more general example can be presented by choosing

\[
g_1(y) = h_1(y) = \beta_1 y^m + \gamma_1 e^y, \quad g_2(y) = h_2(y) = \beta_2 y^n + \gamma_2 e^y, \tag{3.11}
\]

where \(\beta_i\) and \(\gamma_i\) are arbitrary constants and \(m, n\) are non-negative integers. The resulting integrable system reads as

\[
\begin{align*}
\phi_{1x} &= -\lambda \phi_1 + \alpha \phi_1 \phi_2 \psi_2 + \beta_1 \phi_2^{m+1} \psi_1^m + \gamma_1 \phi_2 e^{\phi_2 \psi_1}, \\
\phi_{2x} &= \lambda \phi_2 + \alpha \phi_1 \phi_2 \psi_1 + \beta_2 \phi_1^{n+1} \psi_2^n + \gamma_2 \phi_1 e^{\phi_1 \psi_2}, \\
\psi_{1x} &= \lambda \phi_1 - \alpha \phi_2 \psi_1 \psi_2 - \beta_2 \phi_1^{n+1} \psi_2^n - \gamma_2 \psi_2 e^{\phi_1 \psi_2}, \\
\psi_{2x} &= -\lambda \psi_2 - \alpha \phi_1 \psi_1 \psi_2 - \beta_1 \phi_2^{m+1} \psi_1^m - \gamma_1 \psi_1 e^{\phi_2 \psi_1},
\end{align*} \tag{3.12}
\]
which can be put into the following Hamiltonian system

\[ \phi_{ix} = -\frac{\partial H_s(h_1, h_2)}{\partial \psi_i}, \quad \psi_{ix} = \frac{\partial H_s(h_1, h_2)}{\partial \phi_i}, \quad i = 1, 2 \]  

(3.13)

with the Hamiltonian function

\[
H_s(h_1, h_2) = \lambda(\phi_1 \psi_1 - \phi_2 \psi_2) - \alpha \phi_1 \phi_2 \psi_1 \psi_2 - \frac{\beta_1}{m+1}(\phi_2 \psi_1)^{m+1} - \frac{\beta_2}{n+1}(\phi_1 \psi_2)^{n+1} - \gamma_1 e^{\phi_2 \psi_1} - \gamma_2 e^{\phi_1 \psi_2}.
\]

(3.14)

4 Concluding remarks

Our integrable systems above are just a class of two-dimensional integrable Hamiltonian systems. Each term of the Hamiltonian functions defined by (3.8) mixes two kinds of the variables \( \phi_1, \phi_2 \) and the variables \( \psi_1, \psi_2 \), which shows a different feature from some well-known dynamical systems such as the Stäckel systems [13], the many-body systems of interacting particles [13, 14, 15], the Toda lattice [16]. Some specific interesting cases of two-dimensional integrable Hamiltonian with polynomial energy were also analyzed (see, for example, [17, 18, 19]). On the other hand, a general Hamiltonian function \( H = H(\phi_1, \phi_2, \psi_1, \psi_2) \) that commutes with \( F \) defined by (3.9) can be found by solving a specific differential equation

\[
\phi_1 \frac{\partial H}{\partial \phi_1} + \phi_2 \frac{\partial H}{\partial \phi_2} = \psi_1 \frac{\partial H}{\partial \psi_1} + \psi_2 \frac{\partial H}{\partial \psi_2}.
\]

But the resulting Hamiltonian systems may not be associated with spectral problems of soliton systems.

We emphasize that the paper aims to provide an example that non-symmetry constraints generate integrable Hamiltonian systems from spectral problems of soliton systems. Our result shows that the constraints between potentials and eigenfunctions and/or adjoint eigenfunctions yielding integrable Hamiltonian systems can be both of symmetry type and of non-symmetry type. However, non-symmetry constraints are not so powerful as symmetry constraints in generating integrable Hamiltonian systems. In the AKNS case, we don’t think that time parts of Lax pairs, for example, the system (2.16), can be transformed into integrable Hamiltonian systems under the non-symmetry constraint (3.5), although (3.4) provides a Bäcklund transformation of the integrable AKNS system (2.6). It is also interesting to solve the above two-dimensional integrable Hamiltonian systems, especially (3.10) and (3.12), and to extend them to many-body integrable Hamiltonian systems.

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