Variational approach to second species periodic solutions of Poincaré of the 3 body problem

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Dedicated to Ernesto Lacomba on the occasion of his 65th birthday

Abstract
We consider the plane 3 body problem with 2 of the masses small. Periodic solutions with near collisions of small bodies were named by Poincaré second species periodic solutions. Such solutions shadow chains of collision orbits of 2 uncoupled Kepler problems. Poincaré only sketched the proof of the existence of second species solutions. Rigorous proofs appeared much later and only for the restricted 3 body problem. We develop a variational approach to the existence of second species periodic solutions for the nonrestricted 3 body problem. As an application, we give a rigorous proof of the existence of a class of second species solutions.

1 Introduction
Consider the plane 3-body problem with masses $m_1, m_2, m_3$. Suppose that $m_3$ is much larger than $m_1, m_2$, i.e. $\mu = (m_1 + m_2)/m_3$ is a small parameter. Then

$$m_1/m_3 = \mu \alpha_1, \quad m_2/m_3 = \mu \alpha_2, \quad \alpha_1 + \alpha_2 = 1.$$ 

Let $q_1, q_2 \in \mathbb{R}^2$ be positions of $m_1, m_2$ with respect to $m_3$ (Poincaré’s heliocentric coordinates) and $p_1, p_2 \in \mathbb{R}^2$ their scaled momenta. The motion of $m_1, m_2$ with

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respect to $m_3$ is described by a Hamiltonian system $(H_\mu)$ with Hamiltonian

$$H_\mu(q,p) = H_0(q,p) + \mu \left( \frac{|p_1 + p_2|^2}{2} - \frac{\alpha_1 \alpha_2}{|q_1 - q_2|} \right), \quad (1.1)$$

where

$$H_0(q,p) = \frac{|p_1|^2}{2\alpha_1} + \frac{|p_2|^2}{2\alpha_2} - \frac{\alpha_1}{|q_1|} - \frac{\alpha_2}{|q_2|}.$$

The Hamiltonian $H_\mu$ is a small perturbation of the Hamiltonian $H_0 = H_1 + H_2$ describing 2 uncoupled Kepler problems. The configuration space of system $(H_0)$ is $U^2 = U \times U$, where $U = \mathbb{R}^2 \setminus \{0\}$. The configuration space of the perturbed system $(H_\mu)$ is $U^2 \setminus \Delta$, where

$$\Delta = \{q = (q_1, q_2) \in U^2 : q_1 = q_2\}$$

represents collisions of $m_1, m_2$.

System $(H_\mu)$ has energy integral and the integral of angular momentum

$$G(q,p) = G_1 + G_2 = iq_1 \cdot p_1 + iq_2 \cdot p_2$$

corresponding to the rotational symmetry $e^{i\theta} : \mathbb{R}^2 \to \mathbb{R}^2$.

System $(H_0)$ has additional first integrals $H_1, H_2$ and $G_1, G_2$ – energies and angular momenta of $m_1, m_2$. In the domain

$$P = \{(q,p) \in U^2 \times \mathbb{R}^4 : H_1, H_2 < 0, G_1, G_2 \neq 0\},$$

orbits of $m_1, m_2$ are Kepler ellipses, and solutions of system $(H_0)$ are quasiperiodic with 2 frequencies.

Let $R \subset P$ be the regular domain – the set of points in $P$ such that the corresponding Kepler ellipses do not cross. For small $\mu > 0$ solutions of system $(H_\mu)$ in $R$ are $O(\mu)$-approximated by solutions of system $(H_0)$ on finite time intervals independent of $\mu$. By the classical perturbation theory, away from resonances the same is true on longer time intervals. Moreover, as proved by Arnold [3], for small $\mu > 0$, system $(H_\mu)$ has a large measure of invariant 2-dimensional KAM tori on which solutions are quasiperiodic, and thus well approximated (modulo rotation) by solutions of system $(H_0)$ on infinite time intervals.

In the singular domain $S \subset P$, where the corresponding Kepler ellipses cross, the classical perturbation theory does not work. Indeed, for almost any initial condition in $S$, solution of system $(H_0)$ is quasiperiodic with incommensurable frequencies, and so eventually $m_1, m_2$ simultaneously approach an intersection point of Kepler ellipses. Then the perturbation in (1.1) becomes large, and so it can not be ignored even in the first approximation in $\mu$.

For small $\mu > 0$ solutions of the 3 body problem $(H_\mu)$ in $S$ can be described as follows. The bodies $m_1, m_2$ move along nearly Kepler ellipses and after many revolutions they almost collide. Then they start moving along a new pair of

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1We identify $\mathbb{R}^2$ with $\mathbb{C}$, so multiplication by $i$ is rotation by $\pi/2$. 
nearly Kepler orbits. If the new energies $H_1, H_2$ are both negative, so the
new Kepler orbits are ellipses, then $m_1, m_2$ will again nearly collide after many
revolutions, and the process repeats itself. Thus almost collision solutions of
the 3-body problem ($H_\mu$) shadow chains of collision orbits of system ($H_0$).

Almost collision periodic solutions of system ($H_\mu$) were first studied by
Poincaré in New Methods of Celestial Mechanics. Poincaré named them second
species periodic solutions. However, he did not provide a rigorous existence
proof. Rigorous proofs appeared much later (see e.g. [10, 10, 17]) and only for the
restricted 3 body problem, circular and elliptic. In [10, 8] also chaotic second
species solutions of the circular and elliptic restricted problem were studied.

The goal of this paper is to develop a variational approach to almost collision
periodic orbits of the nonrestricted 3 body problem. As an application, we will
give a rigorous proof of the existence of a class of almost collision periodic orbits.
Chaotic almost collision orbits will be studied in another paper.

Remark 1.1. It is possible to fix the value of angular momentum $G$ and reduce
rotational symmetry. Then we obtain a Hamiltonian system with 3 degrees
of freedom. However, since reduction of the rotational symmetry considerably
complicates the Hamiltonian, it is simpler to work with the original Hamiltonian
system ($H_\mu$) with 4 degrees of freedom.

Remark 1.2. We consider only near collisions of small masses $m_1$ and $m_2$
and exclude near collisions of $m_1, m_2$ with $m_3$. In particular, triple collisions
are excluded. It is well known that double collisions can be regularized, but
the Levi-Civita regularization becomes singular as $\mu \to 0$. Understanding this
singularity is the base for our methods. Levi-Civita regularization was previously
used to study second species solutions for the restricted 3 body problem, see e.g.
[16, 10, 14].

Remark 1.3. Main results of this paper hold for more general Hamiltonians
with singularity, for example

$$H_\mu(q,p) = \frac{|p_1|^2}{2a_1(q)} + \frac{|p_2|^2}{2a_2(q)} + g(q,\mu) - \frac{\mu f(q,\mu)}{|q_1 - q_2|}, \quad a_1, a_2, f > 0,$$

where all functions are smooth without singularity at $q_1 = q_2$.

2 Main results

A solution of the Hamiltonian system ($H_\mu$) is determined by its projection to
the configuration space $U^2 \setminus \Delta$ which will be called a trajectory. Let $L_\mu$ be the
Lagrangian corresponding to $H_\mu$. A $T$-periodic trajectory $\gamma: \mathbb{R} \to U^2 \setminus \Delta$ is a
critical point of the Hamilton action functional

$$A_\mu(T, \gamma) = \int_0^T L_\mu(\gamma(t), \dot{\gamma}(t)) \, dt$$

(2.1)
on the space of $T$-periodic $W_{loc}^{1,2}$ curves in $U^2 \setminus \Delta$. We write $T$ explicitly in $A_u(T, \gamma)$ because later it will become a variable.

Any trajectory of system $(H_0)$ has the form $\gamma = (\gamma_1, \gamma_2)$, where $\gamma_j$ is a trajectory of the Kepler problem. For $\mu = 0$ the action functional $A = A_0$ splits:

$$A(T, \gamma) = \alpha_1 B(T, \gamma_1) + \alpha_2 B(T, \gamma_2), \quad \gamma = (\gamma_1, \gamma_2),$$  \hspace{1cm} (2.2)

where

$$B(T, \sigma) = \int_{0}^{T} \left( \frac{|\dot{\sigma}(t)|^2}{2} + \frac{1}{|\sigma(t)|} \right) dt$$  \hspace{1cm} (2.3)

is the action functional of the Kepler problem on the space of $T$-periodic $W_{loc}^{1,2}$ curves $\sigma : \mathbb{R} \to U$.

We have to consider also trajectories of system $(H_0)$ with collisions. A $T$-periodic curve $\gamma = (\gamma_1, \gamma_2) : \mathbb{R} \to U^2$ is called a periodic $n$-collision chain if there exist time moments

$$t = (t_1, \ldots, t_n), \quad t_1 < \ldots < t_n < t_{n+1} = t_1 + T, \hspace{1cm} (2.4)$$

such that:

- $\gamma$ has collisions at $t = t_j$, so that $\gamma(t_j) = (x_j, x_j) \in \Delta$, $\gamma_1(t_j) = \gamma_2(t_j) = x_j$.

- $\gamma|_{[t_j, t_{j+1}]}$ is a trajectory of system $(H_0)$ which will be called a collision orbit.

- Momentum $p(t) = (p_1(t), p_2(t)) = (\alpha_1 \dot{\gamma}_1(t), \alpha_2 \dot{\gamma}_2(t))$ changes at collisions so that the total momentum $y = p_1 + p_2$ is continuous:

$$y(t_j + 0) = y(t_j - 0) = y_j.$$  \hspace{1cm} (2.5)

Equivalently, the jump of momentum $p(t_j + 0) - p(t_j - 0)$ is orthogonal to $\Delta$ at $\gamma(t_j)$.

- The total energy $H_0$ does not change at collision:

$$H_0(\gamma(t_j), p(t_j + 0)) = H_0(\gamma(t_j), p(t_j - 0)).$$  \hspace{1cm} (2.6)

By (2.3), the total angular momentum $G = G_1 + G_2$ is preserved at collisions:

$$G(\gamma(t_j), p(t_j \pm 0)) = ix_j \cdot y_j.$$  

Hence $G$ is constant along $\gamma$. By (2.6), the total energy $H_0 = H_1 + H_2 = E$ is also constant along $\gamma$, but not the energies $H_1, H_2$ of $m_1, m_2$, or their angular momenta $G_1, G_2$.

A collision chain $\gamma$ is a broken trajectory of system $(H_0)$ -- a concatenation of collision orbits with reflections from $\Delta$. However, unlike for ordinary billiard
systems, $\Delta$ has codimension 2 in the configuration space $U^2$, so the change of the normal component of the momentum at collision is not uniquely determined. Thus there is no direct interpretation of collision chains as trajectories of a dynamical system. Such an interpretation is given later on.

Collision chains are limits of trajectories of system $(H_{\mu})$ which approach collisions as $\mu \to 0$. Indeed, we have:

**Proposition 2.1.** Let $\gamma_{\mu}$ be a $T_{\mu}$-periodic trajectory of system $(H_{\mu})$ which uniformly converges, as $\mu \to 0$, to a $T$-periodic curve $\gamma$. If $\gamma([0,T]) \cap \Delta$ is a finite set, then $\gamma$ is a periodic collision chain.

We say that $\gamma_{\mu}$ is an almost collision orbit shadowing the collision chain $\gamma$. A similar statement holds for nonperiodic collision chains.

Intuitively, Proposition 2.1 is almost evident: a near collision of $m_1, m_2$ lasts a short time during which the influence of the non-colliding body $m_3$ is negligible. Then $m_1, m_2$ form a 2 body problem, so their total momentum $y = p_1 + p_2$ and total energy $H_0 = H_1 + H_2$ are almost preserved. This yields (2.5)–(2.6). This can be made into a rigorous proof, see e.g. [1]. A better way to prove Proposition 2.1 is by using the Levi-Civita regularization, see section 7.

Collision chains can be characterized as extremals of Hamilton’s action functional (2.2). Let $\Omega^T_n$ be the set of $\omega = (t, T, \gamma)$, where $t = (t_1, \ldots, t_n)$ satisfies (2.4) and $\gamma : \mathbb{R} \to U^2$ is a $T$-periodic $W_{loc}^{1,2}$ curve such that $\gamma(t_j) = (x_j, x_j) \in \Delta$. The collision times $t_j$ and collision points $x_j$ are not fixed. Then $\Omega^T_n$ can be identified with an open set in a Hilbert space (see (2.8) and collision times $t_j$ and collision points $x_j$ are smooth functions on $\Omega^T_n$.

**Remark 2.1.** If $\gamma(t) \notin \Delta$ for $t \neq t_j$, then time moments $t_j$ are determined by the curve $\gamma$, i.e. the projection $\Omega^T_n \to W^{1,2}(\mathbb{R}/TZ, U^2)$, $(t, T, \gamma) \to \gamma$, is injective. Then $\omega = (t, T, \gamma)$ is determined by $\gamma$. But $t_j$ are not continuous functions of $\gamma$, so we have to include the variables $t$ in the definition of $\Omega^T_n$.

**Remark 2.2.** In the one-dimensional calculus of variations Hilbert spaces are unnecessary: at least locally function spaces can be replaced by finite dimensional subspaces of broken extremals. We will use this approximation later on. However, in this section we use conventional $W^{1,2}$ setting.

The action functional $A(\omega) = A(T, \gamma)$ is a smooth function on $\Omega^T_n$.

**Proposition 2.2.** $\gamma$ is a $T$-periodic $n$-collision chain iff $\omega = (t, T, \gamma)$ is a critical point of the Hamilton action $A$ on $\Omega^T_n$.

**Proof.** If $\omega = (t, T, \gamma)$ is a critical point of $A$ on $\Omega^T_n$, then each segment $\gamma|_{[t_{i-1}, t_i]}$ is a collision orbit of system $(H_0)$. Then by the first variation formula [2],

$$dA(\omega) = \sum_{j=1}^{n} (\Delta h_j \, dt_j - \Delta y_j \cdot dx_j),$$
Proposition 2.3. is as follows: is the classical Maupertuis action. The Maupertuis principle for collision chains is a collision chain with energy $H$ is determined by $\hat{\gamma}_j \in \Omega_n \times W_n$. This defines a group action of $\mathbb{R}$ on $\Omega_n$. The map map $\Omega_n \to \mathbb{R}_+^n \times W_n$, $(t, T, \gamma) \to (s, \sigma)$, makes it possible to identify $\hat{\Omega}_n$ with $\mathbb{R}_+^n \times W_n$. We can represent $\sigma_j$ by $\sigma_j = \sigma_j(\tau) = \sigma_j(\tau) - (1 - \tau)x_j - \tau x_{j+1}$. Then $\sigma$ is determined by $(x, \hat{\sigma})$, where $x = (x_1, \ldots, x_n) \in U^n$ and $\hat{\sigma} = (\hat{\sigma}_1, \ldots, \hat{\sigma}_n)$. Hence $W_n \cong U^n \times W_0^{1,2}([0, 1], U^{2n})$. Finally, $\hat{\Omega}_n \cong \mathbb{R}_+^n \times W_n \equiv \mathbb{R}_+^n \times U^n \times W_0^{1,2}([0, 1], U^{2n})$.

\[ \hat{\Omega}_n \cong \mathbb{R}_+^n \times W_n \equiv \mathbb{R}_+^n \times U^n \times W_0^{1,2}([0, 1], U^{2n}), \quad \Omega_n \cong \hat{\Omega}_n \times \mathbb{R}. \]
The action functional gives a smooth function $A(s, \sigma)$ on $\tilde{\Omega}_n$, invariant under rotations: $A(s, e^{i\theta} \sigma) = A(s, \sigma)$. We deal with the rotation degeneracy later on.

Often it is convenient to use parametrization independent Jacobi’s form of the Maupertuis action functional – the length of $\gamma$ in the Jacobi’s metric $ds_E$:

$$J^E(\gamma) = \int_\gamma ds_E, \quad ds_E = \max_p \{ p \cdot dq : H_0(q,p) = E \}$$

$$= \sqrt{2(E + \alpha_1|q_1|^{-1} + \alpha_2|q_2|^{-1})(\alpha_1|dq_1|^2 + \alpha_2|dq_2|^2)}.$$

Then $J^E(\gamma) \leq A^E(T, \gamma)$ and if $\gamma$ is parametrized so that $H_0 \equiv E$, then $A^E(T, \gamma) = J^E(\gamma)$. Thus up to parametrization, extremals of $J^E$ and $A^E$ are the same. For a collision chain $\gamma$ corresponding to $(s, \sigma) \in \tilde{\Omega}_n$,

$$J^E(\gamma) = J^E(\sigma) = \sum_{j=1}^n J^E(\sigma)$$

is a function on $W_n$. We obtain

**Proposition 2.4.** If $\gamma$ is a periodic $n$-collision chain of energy $E$ then the corresponding $\sigma \in W_n$ is an extremal of the Jacobi action $J^E$. If $\sigma$ is an extremal of $J^E$, then if each $\sigma_j$ is reparametrized so that $H_0 \equiv E$, the corresponding $\gamma$ is a periodic $n$-collision chain.

We will not use Jacobi’s variational principle since $J^E$ is not a smooth function. A discrete version of Jacobi’s action is smooth and we will use it later on.

Similar variational principles hold for collision chains periodic in a rotating coordinate frame: $\gamma(t + T) = e^{i\Phi} \gamma(t)$ for some quasiperiod $T$ and phase $\Phi$. We call $\gamma$ periodic modulo rotation. If $\Phi \not\in 2\pi\mathbb{Q}$, then $\gamma$ is quasiperiodic in a fixed coordinate frame.

The corresponding function space is defined as follows. Let $\tilde{\Omega}_n$ be the set of all $(t, T, \gamma, \Phi)$, where $t, T$ are as before, $\Phi \in \mathbb{R}$ and the $W^{1,2}_{loc}$ curve $\gamma : \mathbb{R} \to U^2$ satisfies $\gamma(t + T) = e^{i\Phi} \gamma(t)$ and $\gamma(t_j) \in \Delta$. Then $\tilde{\Omega}_n$ can be identified with an open set in a Hilbert space and $t_j, x_j, T, \Phi$ are smooth functions on $\tilde{\Omega}_n$. In fact $\tilde{\Omega}_n \cong \Omega_n \times \mathbb{R}$. Indeed, $\hat{\gamma}(t) = e^{-iv\Phi/T}\gamma(t)$ is a $T$-periodic curve, so $(t, T, \hat{\gamma}) \in \tilde{\Omega}_n$.

Define the Maupertuis–Routh action functional on $\tilde{\Omega}_n$ by

$$A^{EG}(t, T, \gamma, \Phi) = A(T, \gamma) + ET - G\Phi. \quad (2.9)$$

This is a smooth function on $\tilde{\Omega}_n$ and we have:

**Proposition 2.5.** $\gamma$ is a periodic modulo rotation collision chain with energy $E$ and angular momentum $G$ iff $(t, T, \gamma, \Phi)$ is a critical point of the functional $A^{EG}$ on $\tilde{\Omega}_n$.

**Remark 2.3.** It seems natural to take $\Phi \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ since $\Phi + 2\pi$ gives the same collision chain. But then the functional $A^{EG}$ will be multivalued: defined modulo $2\pi G$. 

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Remark 2.4. The name of the functional $A^{EG}$ is motivated as follows. One can perform Routh’s reduction [2] of the rotational symmetry for fixed $G$ replacing the configuration space $U^2$ by $\tilde{U}^2 = U^2/T \cong \mathbb{R}_+^2 \times T$ and the Lagrangian by the so called Routh function. Then the functional $A^{EG}$ becomes the Maupertuis functional for the reduced Routh system. Probably this observation is due to Birkhoff [5]. However, Routh’s reduction makes the Lagrangian more complicated, so we do not use it in this paper.

There are several other possible variational principles for collision chains (for example, we may fix the phase $\Phi$), but in the present paper we will use only the ones given above.

Sufficient condition for the existence of a periodic orbit of system $(H_\mu)$ shadowing a given collision chain $\gamma$ requires that the chain is nontrivial in the following sense. Let $u(t) = \gamma_2(t) - \gamma_1(t)$ and let

$$v(t) = \alpha u(t) = \alpha_1 p_2(t) - \alpha_2 p_1(t), \quad \alpha = \alpha_1 \alpha_2,$$

be the scaled relative velocity of $m_1, m_2$. Let $v^\pm_j = v(t_j \pm 0)$ be relative collision velocities. Since

$$h^\pm_j = |y_j|^2/2 + |v^\pm_j|^2/2\alpha - |x_j|^{-1} = E,$$

equations (2.5)–(2.6) imply that $|v^+_j| = |v^-_j|$: relative speed is preserved at collision. We impose two essentially equivalent conditions:

**Direction change condition.** Relative collision velocity changes direction at collision: $\Delta v_j = v^+_j - v^-_j \neq 0, j = 1, \ldots, n$. In particular, $v^\pm_j \neq 0$.

**No early collisions condition.** $\gamma(t) \not\in \Delta$ for $t \neq t_j$.

If the direction change condition is not satisfied at some $t_j$, then $\gamma(t_j - 0) = \gamma(t_j + 0)$, and so $\gamma|_{[t_{j-1}, t_{j+1}]}$ is a smooth trajectory of system $(H_0)$. Deleting the collision time moment $t_j$ we obtain a $(n - 1)$-collision chain violating no early collisions condition.

Conversely, if $\gamma$ is a $n$-collision chain violating no early collisions condition, then adding an extra collision time moment, we obtain a $(n + 1)$-collision chain violating the changing direction condition. From now on we add these two equivalent conditions to the definition of a collision chain.

**Remark 2.5.** The changing direction condition implies that almost collision orbits $\gamma_\mu$ shadowing the collision chain $\gamma$ come $O(\mu)$-close to collision. Often almost collision orbits discussed in Astronomy come close to collision, but not too close, for example $O(\mu^\nu)$-close with $\nu \in (0, 1)$, see e.g. [15, 18, 14]. Such orbits change direction at near collision, but this change is small as $\mu \to 0$. Then the corresponding collision chains do not satisfy the changing direction condition: $\Delta v_j = v^+_j - v^-_j = 0$. Our methods do not work for such almost collision orbits.
The changing direction condition makes it possible to construct a shadowing orbit \( \gamma_{\mu} \) of system \((H_{\mu})\), but it does not prevent \( \gamma_{\mu} \) from having regularizable double collisions of \( m_1, m_2 \). To exclude such collisions we need to impose an extra condition:

**No return condition.** \( v_j^+ + v_j^- \neq 0, \quad j = 1, \ldots, n \).

But this condition is not as crucial as the changing direction condition, so we do not include it in the definition of a collision chain.

To construct shadowing orbits we also need some nondegeneracy assumptions. We say that a \( T \)-periodic \( n \)-collision chain \( \gamma \) with energy \( E \) is **nondegenerate** if \( \omega = (t, T, \gamma) \in \Omega_n \) is a nondegenerate modulo symmetry critical point of the Maupertuis action \( A^E \) on \( \Omega_n \).

Due to time translation and rotation invariance, critical points of \( A^E \) are all degenerate: the group action \( \gamma(t) \rightarrow e^{i\theta}\gamma(t - \tau) \) of \( \mathbb{R} \times T \) preserves \( A^E \). We say that \( \omega = (t, T, \gamma) \in \Omega_n \) is nondegenerate modulo symmetry if the nullity of the quadric form \( d^2 A^E(\omega) \) on \( T_\omega \Omega_n \) is 2 — the lowest possible. Equivalently, the manifold \( M \subset \Omega_n \) obtained from \( \omega \) by the action of the group \( \mathbb{R} \times T \) is a nondegenerate critical manifold. Nondegeneracy modulo symmetry is equivalent to nondegeneracy of the corresponding critical point on \( \Omega_n/(\mathbb{R} \times T) \cong \tilde{\Omega}_n/T \).

As usual in the classical calculus of variations, the Hessian operator corresponding to \( d^2 A^E(\omega) \) is a sum of invertible and compact operators on the Hilbert space \( T_\omega \Omega_n \), so nondegeneracy modulo symmetry implies that the Hessian has bounded inverse on \( T_\omega \Omega_n/T_\omega M \). In fact, at least locally, \( A^E \) can be reduced to a finite dimensional discrete action functional (see section 5), so all Hilbert spaces involved are essentially finite dimensional.

Now two main results will be formulated.

**Theorem 2.1.** Let \( \gamma \) be a nondegenerate \( T \)-periodic collision chain with energy \( E \). Then for small \( \mu > 0 \) there exists a \( T_{\mu} \)-periodic orbit \( \gamma_{\mu} \) of system \((H_{\mu})\) with energy \( E \) which \( O(\mu) \) shadows \( \gamma \):

\[
T_{\mu} = T + O(\mu), \quad \gamma_{\mu}(t) = \gamma(t) + O(\mu), \quad t \in [0, T].
\]

If \( \gamma \) satisfies the no return condition, then \( \gamma_{\mu} \) has no collisions and there exist \( 0 < a < b \) independent of \( \mu \) such that

\[
\mu a \leq d(\gamma_{\mu}(t_j), \Delta) \leq \mu b. \tag{2.12}
\]

Due to time translation and rotation symmetry, 4 multipliers of \( \gamma_{\mu} \) (eigenvalues of the linear symplectic Poincaré map of \( \mathbb{R}^8 \)) are equal to 1. Nontrivial multipliers are \( \lambda_1, \lambda_1^{-1}, \lambda_2, \lambda_2^{-1} \), where \( \lambda_1(\mu) \) is real and large of order \( |\ln \mu| \), and \( \lambda_2(\mu) \) has a limit as \( \mu \rightarrow 0 \), where \( \lambda_2(0) \neq 1 \) is complex with \( |\lambda_2(0)| = 1 \) or real.

Next we consider collision chains with fixed energy \( E \) and angular momentum \( G \). Again we say that a periodic modulo rotation collision chain \( \gamma \) is nondegenerate if the corresponding \( (t, T, \gamma, \Phi) \in \tilde{\Omega}_n \) is a nondegenerate modulo symmetry critical point of the functional \( A^{E,G} \) on \( \tilde{\Omega}_n \). Thus it has only degeneracy coming from rotation and time translation invariance.

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Theorem 2.2. Let $\gamma$ be a $T$-periodic modulo rotation nondegenerate collision chain with energy $E$ and angular momentum $G$. Then for small $\mu > 0$ there exists a periodic modulo rotation orbit $\gamma_\mu$ of system $(H_\mu)$ with energy $E$ and angular momentum $G$ which $O(\mu)$-shadows $\gamma$:

$$\gamma_\mu(t + T_\mu) = e^{i\Phi_\mu} \gamma_\mu(t), \quad \gamma_\mu(t) = \gamma(t) + O(\mu), \quad t \in [0, T],$$

where

$$T_\mu = T + O(\mu), \quad \Phi_\mu = \Phi + O(\mu).$$

An estimate (2.12) holds also here. Even if $\gamma$ is periodic ($\Phi \in 2\pi\mathbb{Q}$), in general the shadowing orbit $\gamma_\mu$ will be periodic only modulo rotation, and thus quasiperiodic in a fixed coordinate frame.

To use Theorems 2.1–2.2, we need to find nondegenerate modulo symmetry collision chains. In general this is not easy. A simple application of Theorem 2.2, based on a perturbative approach, is given in section 3. More complex applications will be given in a future publication.

In section 4 a description of nondegenerate collision orbits is given. Using this description, in section 5 we reduce the action functionals to their discrete versions. Then in section 6 we formulate a local connection result – Theorem 6.1 – and use it to prove Theorem 2.1. The proof of Theorem 2.2 is similar. In section 7 we use Levi-Civita regularization to reduce Theorem 6.1 to Theorem 7.2 which is a generalization of the Shilnikov Lemma [21] to Hamiltonian systems with a normally hyperbolic critical manifold.

Remark 2.6. In this paper we do not attempt to use global variational methods. The reason is that although one can use global methods to find critical points of the action functionals, in general it is hard to check that the critical points satisfy the changing direction condition.

3 Restricted elliptic limit

Suppose that one of the small masses $m_1, m_2$ is much smaller than the other: $\alpha_1 \ll \alpha_2$. In the formal limit $\alpha_1 \to 0$ we obtain the restricted elliptic 3 body problem for which many second species periodic solutions were obtained in [7]. These results do not immediately extend to the case of small $\alpha_1 > 0$. However, we will show that they can be used to obtain many second species periodic solutions for the nonrestricted 3 body problem.

Let us fix energy $E$ and angular momentum $G$. For $\alpha_1 \ll \alpha_2$, the Maupertuis–Routh action functional

$$A_{EG}(t, T, \gamma, \Phi) = \alpha_1 B(T, \gamma_1) + \alpha_2 B(T, \gamma_2) + ET - G\Phi, \quad \gamma = (\gamma_1, \gamma_2), \quad (3.1)$$

on $\hat{\Omega}_n$ is a small perturbation of the Maupertuis–Routh action functional for the Kepler problem. Indeed, the functional $A_{0EG} = A_{EG}|_{\alpha_1=0}$ does not depend on $\gamma_1$:

$$A_{0EG}(t, T, \gamma, \Phi) = R_{EG}(T, \gamma_2, \Phi) = B(T, \gamma_2) + TE - G\Phi. \quad (3.2)$$
The condition $\gamma(t_j) \in \Delta$ imposes no restrictions on $\gamma_2$. Thus the functional $B^{EG}$ is defined on the set $\Pi$ of $(T, \sigma, \Phi)$, where $\sigma : \mathbb{R} \to U$ is a $W^{1,2}_{loc}$ curve such that $\sigma(t + T) = e^{i\Phi} \sigma(t)$. We have a submersion $\pi : \hat{\Omega}_n \to \Pi$, $(t, T, \gamma, \Phi) \to (T, \gamma_2, \Phi)$, and $A^0_{EG} = B^{EG} \circ \pi$.

The functional $B^{EG}$ is very degenerate, because all orbits of the Kepler problem with energy $E < 0$ are periodic with the same period $\tau = 2\pi(-2E)^{-3/2}$. Suppose $E, G$ are such that there exists an elliptic orbit $\Gamma$ of Kepler's problem with energy $E$ and angular momentum $G$. For definiteness let $G > 0$. Then $0 < (-2E)G < 1$. The major semiaxis and eccentricity of $\Gamma$ are

$$a = (-2E)^{-1}, \quad e = \sqrt{1 + 2EG}.$$

The Maupertuis action is

$$J_E(\Gamma) = \int_\Gamma y \cdot dx = 2\pi(-2E)^{-1/2}.$$

The counterclockwise elliptic orbit $\Gamma : \mathbb{R} \to U$ is defined uniquely modulo rotation and time translation.

**Proposition 3.1.** Let $E < 0$, $G > 0$ and $(-2E)G < 1$. Then all critical points $\omega = (T, \sigma, \Phi)$ of the functional $B^{EG}$ on $\Pi$ belong to one of the nondegenerate critical manifolds $M_m \subset \Pi$, $m \in \mathbb{N}$, obtained from $(m\tau, \Gamma, 0)$ by rotation and time translation of $\Gamma$. We have

$$B^{EG}|_{M_m} = B(m\tau, \Gamma) + m\tau E = mJ_E(\Gamma) = 2\pi m(-2E)^{-1/2}.$$

**Proof.** Let $(T, \sigma, \Phi) \in \Pi$ be a critical point of $B^{EG}$. Then $\sigma$ is a solution of the Kepler problem with energy $E$ and angular momentum $G$ and hence $\sigma$ is a time translation and rotation of $\Gamma$. Since $\Gamma$ is a non-circular orbit, quasiperiodicity condition $\sigma(t + T) = e^{i\Phi} \sigma(t)$ implies that $\Phi = 0 \mod 2\pi\mathbb{Z}$ and $T = m\tau$ for some $m \in \mathbb{N}$.

Next we need to check that $M_m$ is a nondegenerate critical manifold of $B^{EG}$. Essentially this is the same statement, but now we need to consider the linearized Kepler problem.

The second variation $d^2B^{EG}(\omega)$ at $\omega = (m\tau, \Gamma, 0)$ is a bilinear form on the tangent space $T_\omega \Pi$ which is the set of $\eta = (\theta, \xi, \phi)$, where $\theta, \phi \in \mathbb{R}$ and $\xi : \mathbb{R} \to \mathbb{R}^2$ is a vector field such that

$$\xi(t + m\tau) = \xi(t) + \dot{\Gamma}(t)\theta + i\Gamma(t)\phi. \quad (3.3)$$

The standard calculus of variations implies that if $\eta \in T_\omega \Pi$ belongs to the kernel of $d^2B^{EG}(\omega)$, then $\xi$ is a solution of the variational equation for $\Gamma$ which lie on the zero levels of the linear first integrals corresponding to the integrals of angular momentum and energy. The linear approximations at $\Gamma$ to the integrals of energy and angular momentum are

$$\dot{\Gamma}(t) \cdot \xi(t) - \dot{\xi}(t) \cdot \phi = 0, \quad i\Gamma(t) \cdot \xi(t) - i\dot{\Gamma}(t) \cdot \xi(t) \equiv 0.$$
Condition (3.3) gives
\[ i\Gamma(t) \cdot \dot{\Gamma}(t) \theta \equiv 0, \quad i\Gamma(t) \cdot \dot{\Gamma}(t) \phi \equiv 0. \]
Since \( \Gamma \) is noncircular, \( \theta = \phi = 0 \) and so \( \xi(t + m\tau) = \xi(t) \). It follows that \( \eta = (0, \xi, 0) \) is tangent to \( M_m \), i.e. the variation \( \xi(t) \) is obtained by time translation and rotation of \( \Gamma(t) \).

The critical manifold \( N_m = \pi^{-1}(M_m) \subset \hat{\Omega}_n \) of \( A^{0\text{EG}} \) corresponding to \( M_m \) is (up to time translation and rotation)
\[ N_m = \{(t, m\tau, \sigma, \Gamma, 0) \in \hat{\Omega}_n : \sigma(t + m\tau) = \sigma(t), \ \sigma(t_j) = \Gamma(t_j)\}. \]
This is an infinite dimensional nondegenerate critical manifold of \( A^{0\text{EG}} \). For nonzero \( \alpha_1 \), by (3.1),
\[ A^{\text{EG}}|_{N_m} = \alpha_1 \left( B(T, \sigma) + Em\tau - 2\pi m(-2E)^{-1/2} \right) + 2\pi m(-2E)^{-1/2}. \]
By a standard property of nondegenerate critical manifolds [17], any nondegenerate modulo symmetry critical point \( \omega \in N_m \) of \( A^{\text{EG}}|_{N_m} \) for small \( \alpha_1 > 0 \) gives a nondegenerate modulo symmetry critical point of \( A^{\text{EG}} \), and hence a nondegenerate modulo symmetry collision chain with energy \( E \) and angular momentum \( G \).

Up to an additive constant and a constant multiple, \( A^{\text{EG}}|_{N_m} \) is Hamilton’s action \( B(m\tau, \sigma) \) for the Kepler problem. It is defined on the set \( \Pi_{\Gamma,m} \) of \( (t, \sigma) \), where \( \sigma : \mathbb{R} \to U \) is an \( m\tau \)-periodic curve such that \( \sigma(t_j) = \Gamma(t_j) \). Thus \( B(m\tau, \sigma) = B_{\Gamma,m}(t, \sigma) \) is precisely the action functional whose critical points are collision chains of the elliptic restricted 3 body problem. This functional was studied in [7], and many of its nondegenerate critical points were found for small eccentricity (almost circular \( \Gamma \)), i.e. \( (-2E)G \) close to 1. Also the changing direction and no early collisions condition was verified in [7], and this carries out for small \( \alpha_1 > 0 \). We obtain

**Theorem 3.1.** Let \( 0 < (-2E)|G| < 1 \) be close to 1. Then for sufficiently small \( \alpha_1 > 0 \) there exist many collision chains \( \gamma \) such that for sufficiently small \( \mu > 0 \), \( \gamma \) is \( O(\mu) \)-shadowed by a second species periodic modulo rotation solution \( \gamma_\mu \) of the nonrestricted 3 body problem with given \( E, G \).

This result can be improved by using a more quantitative statement from [7]. The obtained second species solutions are periodic in a rotating coordinate frame and quasiperiodic in a fixed coordinate frame. Proper periodic orbits will be obtained in a future publication; for them reduction to the restricted elliptic problem is impossible.

### 4 Collision action function

Collision chains can be represented as critical points of a function of a finite number of variables – discrete action functional. This is needed for the proof
of Theorems 2.1–2.2 and in subsequent publications. Since collision chains are concatenations of collision orbits, we need to describe collision orbits first.

A collision orbit \( \gamma = (\gamma_1, \gamma_2) \) of system \((H_0)\) is a pair of Kepler orbits joining the points \( x_-, x_+ \in U \). Thus description of collision orbits is reduced to the classical Lambert’s problem \([22]\) of joining the points \( x_-, x_+ \) by a Kepler orbit.

First we join the points \( x_-, x_+ \) by a Kepler orbit \( \Gamma : [0, \tau] \to U \) with fixed energy \( E < 0 \), or, equivalently, fixed major semi axis \( a = (-2E)^{-1} \). Due to scaling invariance of Kepler’s problem without loss of generality set \( a = 1 \).

Then a Kepler ellipse passing through \( x_-, x_+ \) is determined by the second focus \( F \) such that

\[
\left| x_- \right| + \left| x_- - F \right| = 2, \quad \left| x_+ \right| + \left| x_+ - F \right| = 2.
\]

The solution \( F = F(x_-, x_+) \) of these equations exists and smoothly depends on \( x_\pm \) if the corresponding circles intersect transversely, i.e. \( (x_-, x_+) \) lie in the set

\[
X = \{(x_-, x_+) \in U^2 : \left| x_+ \right| - \left| x_- \right| < \left| x_+ - x_- \right| < 4 - \left| x_- \right| - \left| x_+ \right| \}.
\]

For \( (x_-, x_+) \in X \) there exist two solutions \( F \) of equations (4.1), and we take one of them, for definiteness the one on the left side of the segment \( x_- x_+ \).

Let \( \Gamma(x_-, x_+) \) be the counter clock wise simple arc of the constructed Kepler ellipse joining the points \( x_- \) and \( x_+ \). Let

\[
f(x_-, x_+) = \int_{\Gamma} y \cdot dx = \int_{\Gamma} (2|x|^{-1} - 1)^{1/2}\,dx
\]

be the Maupertuis action of \( \Gamma \). This is a smooth rotation invariant function on \( X \):

\[
f(e^{i\theta}x_-, e^{i\theta}x_+) = f(x_-, x_+).
\]

Remark 4.1. By Lambert’s Theorem \([22]\), \( f \) is a function of \( s_\pm = |x_-| + |x_+| \pm \left| x_- - x_+ \right| \) only. An explicit formula is

\[
f(x_-, x_+) = W(s_+) \pm W(s_-),
\]

where

\[
W(s) = \frac{1}{2} \sqrt{(4 - s)s} + 2 \arctan \sqrt{\frac{s}{4 - s}}.
\]

Plus is taken if \( x_+ = e^{i\theta}x_- \) with \( \theta \in [\pi, 2\pi] \) and minus if \( \theta \in (0, \pi] \). One can check that \( f \) is smooth at \( \theta = \pi \).

Due to scaling invariance of the Kepler problem, for arbitrary negative energy \( E < 0 \), the Maupertuis action of a simple counter clock wise arc \( \Gamma = \Gamma(E, x_-, x_+) \) connecting the points \( (x_-, x_+) \in X_E = (-2E)^{-1}X \) is

\[
f(E, x_-, x_+) = \int_{\Gamma} y \cdot dx = \int_{\Gamma} (2|\left| x \right|^{-1} + E))^{1/2}\,dx
\]

\[
= (-2E)^{-1/2}f((-2E)x_-, (-2E)x_+).
\]

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Next we describe Kepler orbits $\Gamma : [0, \tau] \to U$ connecting $x_-, x_+$ while making $n = [\Gamma]$ full revolutions around 0. To define the number $n \in \mathbb{Z}$ of revolutions, set

$$n = [(\theta(\tau) - \theta(0))/2\pi], \quad \Gamma(t) = r(t)e^{i\theta(t)}.$$ 

**Proposition 4.1.** For any $(x_-, x_+) \in X_E$ and any $n \in \mathbb{Z}$:

- There exists a Kepler orbit $\Gamma = \Gamma_n(E, x_-, x_+) : [0, \tau] \to U$ of energy $E$ joining the points $x_-, x_+$ and making $[\Gamma] = n$ revolutions.
- $\Gamma$ smoothly depends on $E, x_-, x_+$.
- The Maupertuis action of $\Gamma$ is
  
  $$J_n(E, x_-, x_+) = \int_{\Gamma} y \cdot dx = (-2E)^{-1/2}(2\pi|n| + (\text{sgn } n)f((-2E)x_-, (-2E)x_+)). \quad (4.3)$$

- $\tau = \tau_n(E, x_-, x_+) = \frac{\partial}{\partial E} J_n(E, x_-, x_+). \quad (4.4)$

- $\frac{\partial^2}{\partial E^2} J_n(E, x_-, x_+) > 0, \quad (x_-, x_+) \in X_E. \quad (4.5)$

The orbits $\Gamma_n(E, x_-, x_+)$ are nondegenerate (have non-conjugate end points) and any nondegenerate connecting orbit with $E < 0$ is obtained in this way.

For the classical Lambert’s problem [22], when $\Gamma$ is a simple elliptic arc, $n = 0$ or $n = -1$ depending on if $\Gamma$ is a counterclockwise or clockwise. We set $\text{sgn } 0 = 1$.

The first term in (4.3) is the Maupertuis action for $n$ complete revolutions around the Kepler ellipse, and the second is the action of a simple elliptic arc. Equation (4.4) follows from the first variation formula; it is essentially Kepler’s time equation. So only inequality (4.5) is non-evident. It is enough to check it for the classical Lambert’s problem with $n = 0, 1$. Then (4.5) can be deduced from the explicit formula (4.6), although the computation is not trivial. An equivalent statement was proved in [19].

Next we consider Lambert’s problem for fixed time $\tau > 0$. This problem involves solving the transcendental Kepler’s equation so there is no explicit formula for the solution. Let $D_n \subset \mathbb{R}_+ \times U^2$ be the open set which is the image of the diffeomorphism

$$(E, x_-, x_+) \to (\tau_n(E, x_-, x_+), x_-, x_+), \quad E < 0, \quad (x_-, x_+) \in X_E.$$ 

**Proposition 4.2.** For any $n \in \mathbb{Z}$ and any $(\tau, x_-, x_+) \in D_n$:

- There exists a nondegenerate Kepler orbit $\Gamma = \Gamma_n(\tau, x_-, x_+) : [0, \tau] \to U$ with $[\Gamma] = n$ full revolutions joining the points $x_-$ and $x_+$. 

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• \(\Gamma\) smoothly depends on \(\tau, x_-, x_+\).

• Hamilton’s action

\[
B(\tau, \Gamma) = F_n(\tau, x_-, x_+)
\]

is a smooth function on \(D_n\) and

\[
\frac{\partial^2}{\partial \tau^2} F_n(\tau, x_-, x_+) < 0, \quad (\tau, x_-, x_+) \in D_n.
\]

• All nondegenerate connecting orbits with \(E < 0\) are \(\Gamma_n(\tau, x_-, x_+)\) for some \(n \in \mathbb{Z}\) and \((\tau, x_-, x_+) \in D_n\).

Proof. We need to find the energy \(E < 0\) such that the connecting orbit \(\Gamma = \Gamma_n(E, x_- x_+)\) in Proposition 4.1 has given time \(\tau = \tau_n(E, x_- x_+).\) Then

\[
(F_n(\tau, x_-, x_+) + \tau E) \Big|_{\tau = \tau_n(E, x_- x_+)} = J_n(E, x_- x_+).
\]

Hence \(J_n\) and \(-F_n\) are Legendre transforms of each other:

\[
J_n(E, x_- x_+) = \max_{\tau} (F_n(\tau, x_-, x_+) + \tau E), \quad F_n(\tau, x_-, x_+) = \min_E (J_n(E, x_- x_+) - \tau E).
\]

Since \(J_n(E, x_- x_+)\) is convex with respect to \(E\), its Legendre transform \(-F_n(\tau, x_-, x_+)\) is convex in \(\tau\) and smooth. \(\square\)

The initial and final total momenta of \(\gamma\) are given by the first variation formula

\[
y_+ = \frac{\partial}{\partial x_+} F_n(\tau, x_-, x_+), \quad y_- = -\frac{\partial}{\partial x_-} F_n(\tau, x_-, x_+).
\]

(4.7)

Now it is easy to describe nondegenerate collision orbits \(\gamma = (\gamma_1, \gamma_2)\) of system \((H_0)\). Denote by \(k = [\gamma] = (k_1, k_2) \in \mathbb{Z}^2, k_j = [\gamma_j]\), the rotation vector of \(\gamma\). We obtain

**Proposition 4.3.** For any \(k = (k_1, k_2) \in \mathbb{Z}^2\) and any \((\tau, x_-, x_+) \in V_k = D_{k_1} \cap D_{k_2}:

• There exists a nondegenerate collision orbit \(\gamma : [0, \tau] \to U^2, \gamma = \gamma(k, \tau, x_-, x_+), \) with collision points \(\gamma(0) = (x_-, x_-), \) \(\gamma(\tau) = (x_+, x_+).\)

• \(\gamma\) smoothly depends on \((\tau, x_-, x_+) \in V_k\).

• Hamilton’s action of \(\gamma\) is

\[
S_k(\tau, x_-, x_+) = A(\tau, \gamma) = \alpha_1 F_{k_1}(\tau, x_-, x_+) + \alpha_2 F_{k_2}(\tau, x_-, x_+). \quad (4.8)
\]

\[
\frac{\partial^2}{\partial \tau^2} S_k(\tau, x_-, x_+) < 0, \quad (\tau, x_-, x_+) \in V_k. \quad (4.9)
\]
By the first variation formula [2],

\[ dS_k(\tau, x_-, x_+) = y_+ \cdot dx_+ - y_- \cdot dx_- - E \, d\tau, \]

where \( E \) is the total energy of the collision orbit \( \gamma \), and \( y_\pm = y(t_\pm) \) are total momenta at collisions. Thus

\[
y_+ = \frac{\partial S_k}{\partial x_+}(\tau, x_-, x_+), \quad y_- = -\frac{\partial S_k}{\partial x_-}(\tau, x_-, x_+), \quad E = -\frac{\partial S_k}{\partial \tau}(\tau, x_-, x_+). \tag{4.10}
\]

Remark 4.2. We will not need this in the present paper, but for almost all \((\tau, x_-, x_+) \in V_k\) the collision action satisfies the twist condition

\[
\det \frac{\partial^2 S_k}{\partial \tau \partial x_+}(\tau, x_-, x_+) \neq 0. \tag{4.11}
\]

Thus \( S_k \) is the generating function of a symplectic collision map \((\tau, x_-, y_-) \to (\tau, x_+, y_+)\). This will be important for the study of chaotic collision chains.

Remark 4.3. It can happen that the collision orbit \( \gamma \) has early collisions: \( \gamma(t) \in \Delta \) for some \( t \in (0, \tau) \). To avoid this, we may need to delete from \( V_k \) a zero measure set, see [7].

Let us now fix energy \( E < 0 \) and look for collision orbits of system \((H_0)\) with energy \( E \). The map

\[
(\tau, x_-, x_+) \to \left( \frac{\partial S_k}{\partial \tau}(\tau, x_-, x_+), x_-, x_+ \right)
\]

is a diffeomorphism of \( V_k \) onto an open set \( W_k \subset \mathbb{R} \times U^2 \). Let \( L^E_k \) be the Legendre transform of \(-S_k\) with respect to \( \tau \):

\[
L^E_k(x_-, x_+) = \max_\tau (S_k(\tau, x_-, x_+) + \tau E) = \left( S_k(\tau, x_-, x_+) + \tau E \right)|_{\tau=\tau_0(E,x_-,x_+)},
\]

where \( \tau_0 \) is obtained by solving the last equation (4.10). Then \( L^E_k \) is a smooth function on

\[
W^E_k = \{(x_-, x_+) : (E, x_-, x_+) \in W_k \}. \]

We obtain

**Proposition 4.4.** Let \( E < 0 \). For any \((x_-, x_+) \in W^E_k\) there exists a unique collision orbit \( \gamma : [0, \tau] \to U^2 \) of energy \( E \) such that \( \gamma(0) = (x_-, x_-), \gamma(\tau) = (x_+, x_+) \). Its Maupertuis action

\[
L^E_k(x_-, x_+) = \int_\gamma p \cdot dq = \int_\gamma ds_E
\]

is a smooth function on \( W^E_k \). The total momenta at collision are

\[
y_+ = \frac{\partial L^E_k}{\partial x_+}(x_-, x_+), \quad y_- = -\frac{\partial L^E_k}{\partial x_-}(x_-, x_+). \tag{4.12}
\]
In terms of actions functions (4.3) for the Kepler problem,

\[
L_k^E(x_-, x_+) = \min_{E_1+E_2=E} (\alpha_1 J_k_1(E_1, x_-, x_+) + \alpha_2 J_k_2(E_2, x_-, x_+)).
\]

Remark 4.4. Due to homogeneity of the Kepler problem,

\[
L_k^E(x_-, x_+) = (-2E)^{-1/2}L_k((-2E)x_-, (-2E)x_+),
\]

where \(L_k\) corresponds to energy \(E = -1/2\).

Remark 4.5. The action functions \(S_k\) and \(L_k^E\) can not be expressed in elementary functions. However they admit simple asymptotic representation for large \(k\). This will be done in a subsequent publication.

In the next section we use the action functions \(S_k\) and \(L_k^E\) to represent collision chains as critical points of discrete action functionals.

5 Discrete variational principles

For a given sequence \(k = (k^1, \ldots, k^n) \in \mathbb{Z}^{2n}, k^j \in \mathbb{Z}^2\), define a discrete Hamilton’s action by

\[
A_k(s, x) = \sum_{j=1}^{n} S_k(s_j, x_j, x_{j+1}),
\]

where \(s = (s_1, \ldots, s_n) \in \mathbb{R}^+_n, x = (x_1, \ldots, x_n) \in U^n\) and \(S_k\) is the action function on \(V_k\) defined by (4.3). The domain of \(A_k\) is

\[
V_k = \{(s, x) \in \mathbb{R}^+_n \times U^n : (s_j, x_j, x_{j+1}) \in V_k, x_{n+1} = x_1\}.
\]

Any \((s, x) \in V_k\) defines \((t, T, \gamma) \in \Omega_n\) as follows. Take \(t = (t_1, \ldots, t_n)\) so that \(s_j = t_{j+1} - t_j\) and set

\[
\gamma(t) = \gamma(k^j, s_j, x_j, x_{j+1})(t - t_j), \quad t_j \leq t \leq t_{j+1},
\]

where \(\gamma(k^j, s_j, x_j, x_{j+1}) : [0, s_j] \to U^2\) is the collision orbit in Proposition 4.3. Then \(\gamma = \gamma_k(s, x)\) is a broken trajectory of system \((H_0)\) with period \(T = \sum_{j=1}^{n} s_j\) and Hamilton’s action \(A_k(s, x) = A(T, \gamma)\). Of course \((t, \gamma)\) is defined modulo time translation, so we identify curves which differ by time translation. Thus we defined an embedding \(\iota : V_k \to \Omega_n = \Omega_n/\mathbb{R}\) and \(A_k = A \circ \iota\).

For collision chains with fixed period \(T\), we restrict \(A_k\) to

\[
V_k^T = \{(s, x) \in V_k : \sum_{j=1}^{n} s_j = T\}.
\]

Proposition 5.1. Any critical point \((s, x)\) of \(A_k\) on \(V_k^T\) defines a \(T\)-periodic collision chain \(\gamma = \gamma_k(s, x)\).
Indeed, critical points of $A_k$ satisfy
\[
\frac{\partial S_{k,j}}{\partial x_j}(s_j, x_j, x_{j+1}) + \frac{\partial S_{k,j-1}}{\partial x_j}(s_{j-1}, x_{j-1}, x_j) = 0, 
\]
(5.1)
\[
\frac{\partial S_{k,j}}{\partial s_j}(s_j, x_j, x_{j+1}) = -E, 
\]
(5.2)
where $-E$ is the Lagrange multiplier. By (4.10), $E$ is the energy of the corresponding collision chain $\gamma$. The total momentum at collision is
\[
y_j = -\frac{\partial S_{k,j}}{\partial x_j}(s_j, x_j, x_{j+1}) = \frac{\partial S_{k,j-1}}{\partial x_j}(s_{j-1}, x_{j-1}, x_j). 
\]
Thus $\gamma$ satisfies (2.5)–(2.6).

Proposition 5.1 follows also from Proposition 2.2. Indeed, the functional $A_k$ is the restriction of Hamilton’s action $A$ to the set $\iota(V_k) \subset \Omega_n$ of broken extremals. This set is obtained by equating to zero the differential of $A$ for fixed $t, T, x$.

In Proposition 5.1 the period $T$ is fixed. For collision chains with fixed energy $E < 0$ we consider a discrete Maupertuis action functional on $V_k$:
\[
A_k^E(s, x) = A_k(s, x) + ET, 
\]
where $T$ is a function on $V_k$, and $A_k^E(s, x)$ is the Maupertuis action (2.7) of the broken trajectory $\gamma = \gamma_k(s, x)$. We obtain

**Proposition 5.2.** To any critical point $(s, x)$ of $A_k^E$ on $V_k$ there corresponds a periodic collision chain $\gamma$ with energy $E$. All nondegenerate collision chains with energy $E$ are obtained in this way from nondegenerate modulo rotation critical points of some $A_k^E$.

**Remark 5.1.** Hamilton’s action is invariant under rotations: $A_k(s, e^{i\theta}x) = A_k(s, x)$. Thus every critical point of the functional $A_k^E$ is degenerate. To obtain nondegenerate critical points we should consider the quotient functional $\tilde{A}_k^E$ on the quotient space
\[
\tilde{V}_k = V_k/T \subset \mathbb{R}^n_+ \times \mathbb{U}^n, 
\]
where $\mathbb{U}^n = U^n/T \cong \mathbb{R}^n_+ \times T^{n-1}$.

Let us now fix energy $E < 0$ and angular momentum $G$ and consider periodic modulo rotation collision chains $\gamma$ with given $E, G$. We obtain the discrete Maupertuis–Routh action functional
\[
A_k^{EG}(s, x, \Phi) = \sum_{j=1}^{N} S_{k,j}(s_j, x_j, x_{j+1}) + ET - G\Phi, 
\]
\[
x_{n+1} = e^{i\Phi}x_1, 
\]
\[
T = \sum_{j=1}^{N} s_j. 
\]
The independent variables are $s = (s_1, \ldots, s_n)$, $x = (x_1, \ldots, x_n)$ and $\Phi$, so the domain of $A_k^{EG}$ is
\[
\tilde{V}_k = \{ (s, x, \Phi) \in \mathbb{R}^n_+ \times U^n \times \mathbb{R} : (s_j, x_j, x_{j+1}) \in V_k, \ x_{n+1} = e^{i\Phi}x_1 \}.
\]
Proposition 5.3. To any critical point \((s, x, \Phi)\) of \(A_{k}^{E}\) there corresponds a periodic modulo rotation collision chain \(\gamma = \gamma_{k}(s, x, \Phi)\) with energy \(E\) and angular momentum \(G\). Any nondegenerate periodic modulo rotation collision chain with energy \(E\) and angular momentum \(G\) is obtained from a nondegenerate modulo rotation critical point of some \(A_{k}^{E}\).

To construct orbits of system \((H_{\mu})\) shadowing the collision chain \(\gamma\) corresponding to a critical point \((s, x)\), we need to verify the changing direction condition. For \(k = (k_{1}, k_{2}) \in \mathbb{Z}^{2}\) denote

\[ R_{k}(\tau, x_{-}, x_{+}) = F_{k_{1}}(\tau, x_{-}, x_{+}) - F_{k_{2}}(\tau, x_{-}, x_{+}). \]

By (4.7) the relative collision velocities (2.10) of a collision orbit \(\gamma = \gamma_{k}(\tau, x_{-}, x_{+}) : [0, \tau] \rightarrow U^{2}\) are given by

\[ \dot{u}(0) = -\frac{\partial R_{k}}{\partial x_{-}}(\tau, x_{-}, x_{+}), \quad \dot{u}(\tau) = \frac{\partial R_{k}}{\partial x_{+}}(\tau, x_{-}, x_{+}). \]

Thus the changing direction condition for the collision chain corresponding to \((s, x)\) can be expressed as follows:

\[ \frac{\partial R_{k_{j}}}{\partial x_{j}}(s_{j}, x_{j}, x_{j+1}) + \frac{\partial R_{k_{j-1}}}{\partial x_{j}}(s_{j-1}, x_{j-1}, x_{j}) \neq 0. \] (5.3)

We have

\[ A_{k} = \alpha_{1}B_{k_{1}} + \alpha_{2}B_{k_{2}}, \] (5.4)

where \(k = (k_{1}, k_{2})\) with \(k_{j} = (k_{1}^{j}, \ldots, k_{n}^{j}) \in \mathbb{Z}^{n}\) and

\[ B_{k_{j}}(s, x) = \sum_{i=1}^{n} F_{k_{j}}(s_{i}, x_{i}, x_{i+1}) \] (5.5)

is the discrete action functional for the Kepler problem. If \((s, x)\) is a critical point of \(A_{k}\) with respect to \(x\), then by (5.1), the changing direction condition (5.3) is equivalent to

\[ \frac{\partial}{\partial x_{j}}B_{k_{j}}(s, x) \neq 0, \quad j = 1, \ldots, n. \] (5.6)

Next we reformulate the shadowing Theorems 2.1–2.2.

Theorem 5.1. Let \((s, x) \in V_{k}\) be a nondegenerate modulo rotation critical point of \(A_{k}^{E}\) satisfying the changing direction condition (5.6). Then for sufficiently small \(\mu > 0\) the corresponding \(T\)-periodic collision chain \(\gamma\) is \(O(\mu)\)-shadowed modulo time translation by an almost collision \(T_{\mu}\)-periodic orbit \(\gamma_{\mu}\) of the 3 body problem with period \(T_{\mu} = T + O(\mu)\).
Theorem 5.2. Let \((s, x, \Phi) \in \hat{V}_k\) be a nondegenerate modulo rotation critical point of \(A^{EG}_k\) satisfying the changing direction condition (7.6). Then for sufficiently small \(\mu > 0\), the corresponding collision chain \(\gamma\) is \(O(\mu)\)-shadowed modulo rotation and time translation by an almost collision periodic modulo rotation orbit \(\gamma_\mu\) of the 3 body problem with energy \(E\) and angular momentum \(G\).

These discrete versions of Theorems 2.1–2.2 are most suitable for applications. In a future publication we will use them in [9] to find many nontrivial second species solutions.

For a dynamical systems reformulation, it is convenient to introduce Jacobi’s discrete action functional

\[
J_k^E(x) = \sum_{j=1}^{n} L_k^E(x_j, x_{j+1}).
\]

It is defined on

\[
W_k^E = \{x = (x_1, \ldots, x_n) : (x_j, x_{j+1}) \in W_k^j, x_{n+1} = x_1\}.
\]

Equating to 0 the derivatives of \(A_k^{EG}(s, x)\) with respect to \(s\), we obtain:

Proposition 5.4. Any nondegenerate modulo symmetry periodic collision chain with energy \(E\) corresponds to a nondegenerate modulo rotation critical point \(x\) of some \(J_k^E\).

A critical point \(x = (x_1, \ldots, x_n)\) of \(J_k^E\) is a \(n\)-periodic trajectory of a discrete Lagrangian system \((L^E)\) with a multivalued discrete Lagrangian \(L^E = \{L_k^E\}_{k \in \mathbb{Z}^2}\):

\[
\frac{\partial}{\partial x_j}(L_k^E(x_{j-1}, x_j) + L_k^E(x_j, x_{j+1})) = 0, \quad j = 1, \ldots, n.
\]

Thus description of second species solutions is reduced to the dynamics of a discrete Lagrangian system \((L^E)\). Under a twist condition, a periodic trajectory of system \((L^E)\) corresponds to a periodic trajectory \((x_j, y_j), y_j = -\frac{\partial}{\partial x_j}L_k^E(x_j, x_{j+1})\), of a sequence of symplectic twist maps \((x_j, y_j) \rightarrow (x_{j+1}, y_{j+1})\) with generating functions \(L_k^E\). We postpone this reformulation to a future paper, where we deal with chaotic almost collision orbits.

The minimal degeneracy of a critical point of \(J_k^E\) is at least 1 due to rotational symmetry \(J_k^E(e^{i\theta}x) = J_k^E(x)\). This implies that the discrete Lagrangian system (or the corresponding symplectic map) has an integral of angular momentum \(G = ix_j \cdot y_j\). One can perform Routh’s reduction in this discrete Lagrangian system reducing it to one degree of freedom [12], but this complicates the discrete Lagrangian.

For periodic modulo rotation collision chains with fixed \(E, G\) we have:
Proposition 5.5. Any nondegenerate periodic modulo rotation collision chain with energy \( E \) corresponds to a nondegenerate modulo rotation critical point \((x, \Phi)\) of the discrete Jacobi–Routh action functional

\[
J^E_k(x, \Phi) = J^E_\mu(x) - G\Phi, \quad x_{n+1} = e^{i\Phi}x_1.
\]

The proofs of Theorems 5.1 and 5.2 are modifications of the proof of Theorem 2.1 in [6]. They are based on the Levi-Civita regularization and shadowing. The proof of Theorem 5.1 will be given in the next section. The proof of Theorem 5.2 is similar and will be omitted.

6 Proof of Theorem 5.1

For \( \mu \neq 0 \), the action functional \((2.1)\) of system \((H_\mu)\) is singular when \( \gamma \) approaches \( \Delta \). We will formulate a variational problem for almost collision orbits of system \((H_\mu)\) with given energy \( E \) which has no singularity at \( \Delta \).

Let us fix energy \( E < 0 \). Trajectories of system \((H_\mu)\) with energy \( E \) are extremals of the Jacobi action functional

\[
J^E_\mu(\gamma) = \int_\gamma ds^E_\mu, \quad ds^E_\mu = \max_p \{ p \cdot dq : H_\mu(q, p) = E \}.
\]

Away from \( \Delta \), the functional \( J^E_\mu \) is a regular perturbation of the Jacobi functional \( J^E \) for system \((H_0)\). Regularizing \( J^E_\mu \) near \( \Delta \) requires some preparation.

First we describe local behavior of trajectories of system \((H_0)\) colliding with \( \Delta \). We will use the variables (this is a version of Jacobi’s variables)

\[
x = \alpha_1 q_1 + \alpha_2 q_2, \quad y = p_1 + p_2, \quad u = q_2 - q_1, \quad v = \alpha_1 p_2 - \alpha_2 p_1.
\]

Thus \( x \) is the center of mass of \( m_1, m_2 \), \( y \) is their total momentum, \( u \) is their relative position, and \( v \) is the scaled relative velocity. The change is symplectic:

\[
p \cdot dq = p_1 \cdot dq_1 + p_2 \cdot dq_2 = y \cdot dx + v \cdot du.
\]

The inverse change is

\[
q_1 = x - \alpha_2 u, \quad q_1 = x + \alpha_1 u, \quad p_1 = \alpha_1 y - v, \quad p_2 = \alpha_2 y + v.
\]

For solutions of system \((H_0)\), \( y = \dot{x} \) and \( v = \alpha \dot{u} \), where \( \alpha = \alpha_1 \alpha_2 \).

Let \( \gamma \) be a trajectory of system \((H_0)\) with energy \( E \). We denote by \((x(t), y(t), u(t), v(t))\) its representation in Jacobi’s variables. If \( \gamma \) has a collision at \( t = 0 \), i.e. \( u(0) = 0 \), \( x(0) = x_0 \), then

\[
H_0 = |y_0|^2/2 + |v_0|^2/2\alpha - |x_0|^{-1} = E.
\]

We assume that collisions occurs with nonzero relative speed \( v_0 \neq 0 \). Then there exists \( \delta > 0 \) such that \((x_0, y_0)\) lies in a compact set

\[
M = M^E_\delta = \{(x_0, y_0) : \lambda(x_0, y_0) = E - |y_0|^2/2 + |x_0|^{-1} \geq \delta, \ |x_0| \geq \delta \}.
\]

We fix \( \delta > 0 \). Eventually it will taken sufficiently small. Denote \( B_\rho = \{ u \in \mathbb{R}^2 : |u| \leq \rho \} \) and \( S_\rho = \partial B_\rho \). We have

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Lemma 6.1. Take any \( \delta > 0 \) and let \( \rho > 0 \) be sufficiently small. Then for any \((x_0, y_0) \in M\) and any \( u_+ \in S_\rho \), there exists a trajectory \( \gamma_+ : [0, \tau_+] \to U^2 \) of system \((H_0)\) with energy \( E \) such that:

- \( u(t) \in B_\rho \) for \( 0 \leq t \leq \tau_+ \) and \( x(0) = x_0, \ y(0) = y_0, \ u(0) = 0, \ u(\tau_+) = u_+ \).
- \( \gamma_+ \) smoothly depends on \((x_0, y_0, u_+) \in M \times S_\rho\) and

\[
\tau_+ = \tau_+(x_0, y_0, u_+) = \rho \sqrt{\alpha/2\lambda(x_0, y_0)} + O(\rho^2),
\]

\[
x(\tau_+) = \xi_+(x_0, y_0, u_+) = x_0 + \rho \sqrt{\alpha/2\lambda(x_0, y_0)}y_0 + O(\rho^2). \tag{6.3}
\]

- The Maupertuis action of \( \gamma_+ \) has the form

\[
J^E(\gamma_+) = \int_{\gamma_+} p \cdot dq = a_+(x_0, y_0, u_+) \tag{6.4}
\]

\[
= \rho \sqrt{2\alpha/\lambda(x_0, y_0)}(E + |x_0|^{-1}) + O(\rho^2).
\]

Remark 6.1. On \( S_\rho \) we use the polar coordinate \( \theta \), where \( u = re^{i\theta} \). Thus \( O(\rho^2) \) means a function of \( x_0, y_0, \theta \) whose \( C^2 \) norm is bounded by \( cp^2 \) with \( c \) independent of \( \rho \).

The proof is obtained by a simple shooting argument, because \( H_0 \) has no singularity at \( \Delta \):

\[
x(t) = x_0 + ty_0 + O(t^2), \quad u(t) = tv_0/\alpha + O(t^2).
\]

It remains to solve the equation \( u(\tau_+) = u_+ \) for \( \tau_+ \) and \( v_0 \), where \( |v_0| = \sqrt{2\alpha/\lambda(x_0, y_0)} \). \( \square \)

Similarly, we have a trajectory \( \gamma_- : [\tau_- , 0] \to U^2 \) of system \((H_0)\) with energy \( E \) such that \( x(0) = x_0, \ y(0) = y_0, \ u(0) = 0, \ u(\tau_-) = u_- \). Then

\[
\tau_- = \tau_-(x_0, y_0, u_-), \quad x(\tau_-) = \xi_-(x_0, y_0, u_-), \quad J^E(\gamma_-) = a_-(x_0, y_0, u_-).
\]

If \((x_0, y_0) \in M\), then \( x_0 \) belongs to

\[
D = D^E_\delta = \{ x : \delta \leq |x| \leq (\delta - E)^{-1} \}.
\]

Let \( \Sigma_\rho \) be the boundary of the tubular neighborhood \( N_\rho \) of \( D^2 \subset \Delta \):

\[
\Sigma_\rho = \{ q : x \in D, \ u \in S_\rho \}, \quad N_\rho = \{ q : x \in D, \ u \in B_\rho \}.
\]

Fix arbitrary large \( C > 0 \) and let

\[
K_\rho = \{ (q_0, q_+) \in D^2 \times \Sigma_\rho : |q_0 - q_+| \leq C \rho \}.
\]

Lemma 6.2. If \( \rho > 0 \) is sufficiently small, then for any \((q_0, q_+) \in K_\rho \), there exists a trajectory \( \gamma : [0, \tau_+] \to N_\rho \) of system \((H_0)\) with energy \( E \) joining \( q_0 \) with \( q_+ \). Moreover \( \gamma \) smoothly depends on \((q_0, q_+)\) and its Maupertuis action has the form

\[
J^E(\gamma) = d_E(q_0, q_+) \tag{6.5}
\]

\[
= \sqrt{2\alpha^{-1}(E + |x_0|^{-1})}|x_+ - x_0|^2 + \alpha^2 \rho^2 + O(\rho^2).
\]
Here \( d_E(q_0, q_+) \) is the distance in the Jacobi metric \( ds_E \).

**Proof.** The condition \( \gamma(\tau_+) = q_+ = (x_+, u_+) \) gives \( u(\tau_+) = u_+ \), \( x(\tau_+) = x_+ \). Then (6.3) makes it possible to determine

\[
y_0 = \eta_+(q_0, q_+) = \sqrt{\frac{2(E + |x_0|^{-1})}{|x_+ - x_0|^2 + \alpha^2 \rho^2}}(x_+ - x_0) + O(\rho). \tag{6.6}
\]

Next we connect points \( q_-, q_+ \in \Sigma_\rho \) by a reflection trajectory of energy \( E \).

\begin{itemize}
  \item \( \rho \) sufficiently small. Then for any \( (q_-, q_+) \in P_\rho \):
  \begin{itemize}
    \item There exist \( \tau_- \) and a broken trajectory \( \gamma: [\tau_-, \tau_+] \to N_\rho \) with energy \( E \) such that \( \gamma(0) = q_0 = (x_0, x_0) \in D^2 \), \( \gamma|_{[0, \tau_+]} \), \( \gamma|_{[\tau_-, 0]} \) are trajectories of system \( (H_0) \), \( \gamma(\tau) = q_\pm \) and there is no jump of total momentum at collision: \( y(0) = y(-0) = y_0 \).
    \item \( \gamma \) smoothly depends on \( (q_-, q_+) \in P_\rho \).
    \item The Maupertuis action of \( \gamma \) has the form
  \[
  J_E(\gamma) = \int_{\gamma} ds_E = g_E(q_-, q_+) = d_E(q_0, q_+) \tag{6.7}
  \]
  \[
  = \sqrt{2(E + |x_0|^{-1})(|x_+ - x_-|^2 + 4\alpha^2 \rho^2)} + O(\rho^2). \tag{6.8}
  \]
\end{itemize}

\[
x_0 = \xi(q_-, q_+) = \frac{(x_+ + x_-)}{2} + O(\rho^2),
\]

\[
y_0 = \eta(q_-, q_+) = \sqrt{\frac{2(E + |x_0|^{-1})}{|x_+ - x_-|^2 + 4\alpha^2 \rho^2}}(x_+ - x_-) + O(\rho), \tag{6.9}
\]

\[
\tau_\pm = \tau_\pm(q_-, q_+) = \pm \sqrt{\frac{|x_+ - x_-|^2 + 4\alpha^2 \rho^2}{8(E + |x_0|^{-1})}} + O(\rho^2).
\]

**Proof.** We find \( x_0 \) from the equation

\[
\frac{\partial}{\partial x_0}(d_E(q_0, q_-) + d_E(q_0, q_+)) = 0 \quad \Leftrightarrow \quad \eta_+(q_0, q_+) = \eta_-(q_0, q_-),
\]

where \( \eta_\pm \) is defined in (6.3). Differentiating (6.3), we see that the Hessian matrix

\[
\sqrt{\frac{8(E + |x_0|^{-1})}{\alpha(|x_+ - x_-|^2 + 4\alpha^2 \rho^2)}} \left( I - \frac{(x_+ - x_-) \otimes (x_+ - x_-)}{|x_+ - x_-|^2 + 4\alpha^2 \rho^2} + O(\rho) \right)
\]

is nondegenerate. By the implicit function theorem, the solution \( x_0 = \xi(q_-, q_+) \) is smooth.

A similar result holds for the perturbed system \( (H_\rho) \), but it is no longer easy to prove. Fix an arbitrary small constant \( \delta > 0 \).
Lemma 6.3. Let \( \rho > 0 \) be sufficiently small. There exists \( \mu_0 > 0 \) such that for all \( \mu \in (0, \mu_0] \), any \( (x_0, y_0) \in M \) and any \( u_\pm \in S_\rho \) such that \( |u_+ + u_-| \geq \delta \rho \):

- There exist \( t_- < 0 < t_+ \) and a trajectory \( \gamma : [t_-, t_+] \to N_\rho \) of system \( (H_\mu) \) with energy \( E \) such that \( u(t_\pm) = u_\pm, x(0) = x_0, y(0) = y_0 \).
- \( \gamma \) smoothly depends on \( (x_0, y_0, u_-, u_+, \mu) \in M \times S_\rho^2 \times (0, \mu_0] \) and converges to a concatenation of trajectories \( \gamma_\pm \) in Lemma 6.1 as \( \mu \to 0 \).
- The Maupertuis action of \( \gamma \) has the form

\[
J^E_\mu(\gamma) = \int p \cdot dq = f^E_\mu(x_0, y_0, u_-, u_+) = a_+(x_0, y_0, u_+) + a_-(x_0, y_0, u_-) + \mu \hat{a}(x_0, y_0, u_-, u_+, \mu),
\]

where \( \hat{a} \) is \( C^2 \) bounded on \( M \times S_\rho^2 \times (0, \mu_0] \).

- \( t_\pm = \tau_\pm(x_0, y_0, u_+) + \mu \hat{\tau}_\pm(x_0, y_0, u_+, u_-), \)

\[
x(t_\pm) = x^\pm_\mu(x_0, y_0, u_-, u_+)
= \xi_\pm(x_0, y_0, u_+) + \mu \hat{\xi}_\pm(x_0, y_0, u_-, u_+),
\]

where \( \hat{\tau}_\pm \) and \( \hat{\xi}_\pm \) are uniformly \( C^1 \) bounded on \( M \times S_\rho^2 \times (0, \mu_0] \).

- If \( |u_+ - u_-| \geq \delta \rho \), then

\[
\mu a \leq \min_{t \in [t_-, t_+]} d(\gamma(t), \Delta) \leq \mu b, \quad 0 < a < b.
\]

The proof of Lemma 6.3 is given in section 7. It is based on Levi-Civita regularization and a generalization of Shilnikov’s Lemma [20], see also [21], to normally hyperbolic critical manifolds of a Hamiltonian system.

Next we deduce a local connection theorem. Fix arbitrary small \( \delta > 0 \), arbitrary large \( C > 0 \) and let

\[
Q_\rho = \{(q_-, q_+) \in \Sigma_\rho^2 : |q_- - q_+| \leq C \rho, |u_+ + u_-| \geq \delta \rho \}.
\]

Theorem 6.1. Let \( \rho > 0 \) be sufficiently small. There exists \( \mu_0 > 0 \) such that for all \( (q_-, q_+, \mu) \in Q_\rho \times (0, \mu_0] \):

- There exist \( t_- < 0 < t_+ \) and a trajectory \( \gamma : [t_-, t_+] \to N_\rho \) of system \( (H_\mu) \) with energy \( E \) such that \( \gamma(t_\pm) = q_\pm \) and the minimum of \( d(\gamma(t), \Delta) \) is attained at \( t = 0 \).

- \( \gamma \) smoothly depends on \( (q_-, q_+, \mu) \in Q_\rho \times (0, \mu_0] \) and converges to a reflection trajectory in Proposition 6.7 as \( \mu \to 0 \).
• The Maupertuis action of $\gamma$ has the form

$$J^E_\mu(\gamma) = \int_\gamma p \cdot dq = g^E_\mu(q_-, q_+ - t\gamma) + \mu\hat{g}(q_-, q_+, \mu),$$

where $\hat{g}$ is $C^2$ bounded on $Q_\rho \times (0, \mu_0]$.

• If $|u_+ - u_-| \geq \delta \rho$, then (6.13) holds.

Thus the action $J^E_\mu(\gamma) = g^E_\mu(q_-, q_+)$ has a limit $gE(q_-, q_+)$ as $\mu \to 0$ which is the action of the reflection orbit in Proposition 6.1. The condition that the distance to $\Delta$ is attained at $t = 0$ is needed only to exclude time translations, so that $t_\pm$ are uniquely defined.

**Proof.** We need to find $(x_0, y_0)$ such that $x_\pm^\pm(x_0, y_0, u_-, u_+) = x_\pm$. Since the implicit function theorem worked in the proof of Proposition 6.1 by (6.12), for small $\mu > 0$ it will work also here.

**Proof of Theorem 6.7.** Let $\gamma$ be a nondegenerate $n$-collision chain with energy $E$. Let $t = (t_1, \ldots, t_n)$ be collision times, $x = (x_1, \ldots, x_n)$, $\gamma(t_j) = (x_j, x_j)$, the corresponding collision points and $y_j$ the collision total momenta. Take $\delta > 0$ so small that the collision points and the collision speeds $v^\pm_j = v(t_j \pm 0)$ satisfy

$$|x_j| \geq \delta, \quad |v^+_j| = |v^-_j| \geq \delta \sqrt{2\alpha}.$$  

Then $(x_j, y_j) \in M$ and $x_j \in D$.

Take small $\rho > 0$ and let $t^\pm_j = t_j \pm s^\pm_j$ be the closest to $t_j$ times when $q^\pm_j = \gamma(t^\pm_j) \in \Sigma_\rho$. Since $\gamma$ satisfies the changing direction condition, $(q^-_j, q^+_j) \in Q_\rho$ if $C > 0$ is taken sufficiently large and $\rho > 0$ sufficiently small. Moreover for $\xi^\pm_j \in \Sigma_\rho$ close to $q^\pm_j$, we have $(\xi^-_j, \xi^+_j) \in Q_\rho$. Thus by Theorem 6.1 for small $\mu > 0$ the points $\xi^\pm_j$ can be joined in $N_\rho$ by a trajectory $\gamma^\rho_j$ of system $(H_\rho)$ with energy $E$ and the Maupertuis action $J^E_\mu(\gamma^\rho_j) = g^E_\mu(\xi^-_j, \xi^+_j)$.

Since $|\gamma(t_j, t_{j+1})|$ is nondegenerate and, by no early collisions condition, does not come near $\Delta$, for $\xi^+_j$ close to $q^+_j$ and $\xi^-_{j+1}$ close to $q^-_{j+1}$ and small $\mu > 0$, the points $\xi^+_j$ and $\xi^-_{j+1}$ can be joined by a trajectory $\sigma^\rho_j$ of system $(H_\rho)$ with energy $E$ and the Maupertuis action $J^E_\mu(\sigma^\rho_j) = h^E_\mu(\xi^+_j, \xi^-_{j+1})$. This trajectory smoothly depends on $\mu$ also for $\mu = 0$, and $h^E(\xi^+_j, \xi^-_{j+1})$ is the Maupertuis action of a connecting trajectory of system $(H_0)$.

Combine the trajectories $\gamma^\rho_j, \sigma^\rho_j$ in a broken trajectory $\gamma_\mu$ with energy $E$ and Maupertuis action

$$f_\mu(\xi) = J^E_\mu(\gamma_\mu) = \sum_{j=1}^n (J^E_\mu(\gamma^\rho_j) + J^E_\mu(\sigma^\rho_j)) = \sum_{j=1}^n (g^E_\mu(\xi^+_j, \xi^-_{j+1}) + h^E_\mu(\xi^-_j, \xi^+_j)), \quad \xi = (\xi^-_1, \xi^+_1, \ldots, \xi^-_n, \xi^+_n) \in \Sigma_\rho^2.$$
The function $f_\mu$ has a limit $f_0$ as $\mu \to 0$ and

$$f_\mu(\xi) = f_0(\xi) + \mu \tilde{f}(\xi, \mu),$$

where

$$f_0(\xi) = \sum_{j=1}^{n} (d(\xi_j, \xi_j^+) + h_0^E(\xi_j^+, \xi_{j+1}^-) + d(\xi_{j+1}, \xi_{j+1}^-)),$$

and $\xi_j = \xi(\xi_j^-, \xi_j)$ is defined by (6.9). The remainder $\tilde{f}(\xi, \mu)$ is $C^2$ bounded on $Y \times (0, \mu_0]$, where the neighborhood $Y \subset \Sigma^2_{\rho}$ of $q = (q_{-1}, q_1, \ldots, q_{-n}, q_n)$ is independent of $\mu$.

Looking for critical points with respect to $\xi_j^\pm$ with fixed $\xi_j$ we obtain $f_0 = J^E_k(\xi_1, \ldots, \xi_n)$ for some $k \in \mathbb{Z}^2n$, and $J^E_k$ has a nondegenerate modulo rotation critical point $x$. Thus $f_0$ has a nondegenerate critical point $q \in \Sigma^2_{\rho}$. Then for small $\mu > 0$ the function $f_\mu(\xi)$ has a nondegenerate modulo rotation critical point $\xi_\mu$ close to $q$. The corresponding broken trajectory $\gamma_\mu$ has no break of velocity at intersection points $\xi_j^\pm$ with $\Sigma_{\rho}$ and hence $\gamma_\mu$ is a periodic trajectory of system $(H_\mu)$ with energy $E$.

\section{Levi-Civita regularization}

In this section we prove Lemma 6.3. In the Jacobi variables (6.1), the Hamiltonian $H_\mu$ takes the form

$$H_\mu = \frac{(1 + \mu)|y|^2}{2} + \frac{|v|^2}{2\alpha} - \frac{\alpha_1}{|\alpha_2 u - x|} - \frac{\alpha_2}{|\alpha_1 u + x|} - \frac{\mu \alpha}{|u|}.$$

Let us perform the Levi-Civita regularization on the fixed energy level $H_\mu = E$. We identify $u, v \in \mathbb{R}^2 = \mathbb{C}$ with complex numbers and make a change of variables

$$u = \xi^2, \quad v = \eta/2\xi.$$

Since

$$v \cdot du = \text{Re} (v \, d\bar{u}) = \text{Re} (\eta \, d\bar{\xi}) = \eta \cdot d\xi,$$

the change is symplectic:

$$p \cdot dq = y \cdot dx + \eta \cdot d\xi. \quad (7.1)$$

Finally, we obtain a transformation

$$q_1 = x - \alpha_2 \xi^2, \quad q_1 = x + \alpha_1 \xi^2, \quad p_1 = \alpha_1 y - \eta/2\xi, \quad p_2 = \alpha_2 y + \eta/2\xi.$$

The Levi-Civita map

$$g : \mathbb{R}^2 \times \mathbb{R}^2 \times U \times \mathbb{R}^2 \to (\mathbb{R}^4 \setminus \Delta) \times \mathbb{R}^4, \quad g(x, y, \xi, \eta) = (q_1, q_2, p_1, p_2),$$
is a symplectic double covering undefined at $\xi = 0$ which corresponds to the collision set $\Delta$.

Let

$$H^E_\mu(x, y, \xi, \eta) = |\xi|^2(H_\mu \circ g - E) + \mu \alpha$$

$$= \frac{|\eta|^2}{8\alpha} - |\xi|^2 \left(E + \frac{\alpha_1}{|\alpha_2\xi^2 - x|} + \frac{\alpha_2}{|\alpha_1\xi^2 + x|} - \frac{(1 + \mu)|y|^2}{2}\right).$$

Let $\Sigma^E_\mu = \{H_\mu = E\}$ and $\Gamma^E_\mu = \{H^E_\mu = \mu \alpha\}$. Since $g(\Gamma^E_\mu) = \Sigma^E_\mu$, the map $g$ takes orbits of system $(H^E_\mu)$ on $\Gamma^E_\mu$ to orbits of system $(H_\mu)$ on $\Sigma^E_\mu$. The time parametrization is changed: the new time is given by $d\tau = |\xi|^2dt$. In the following we will continue to denote the new time by $t$.

The singularity at $\Delta$ disappeared after regularization. The regularized Hamiltonian $H^E_\mu$ is smooth on

$$\mathcal{P} = \{(x, y, \xi, \eta) \in U \times \mathbb{R}^6 : x \neq \alpha_2\xi^2, x \neq -\alpha_1\xi^2\},$$

which means excluding collisions of $m_1$ and $m_2$ with $m_3$. The parameter $\mu \alpha$ may be regarded as new energy. The rotation group and the integral of angular momentum are now

$$(x, y, \xi, \eta) \rightarrow (e^{i\theta} x, e^{i\theta} y, e^{i\theta/2}\xi, e^{i\theta/2}\eta), \quad G = ix \cdot y + i\xi \cdot \eta/2.$$

The Hamiltonian $H^E_0$ has a critical manifold $\xi = \eta = 0$ which is contained in the level set $\Gamma^E_0$ of $H^E_0$. We have

$$H^E_0(x, y, \xi, \eta) = \frac{|\eta|^2}{8\alpha} - |\xi|^2\lambda(x, y) + O(|\xi|^4).$$

Collisions of $m_1, m_2$ with nonzero relative velocity correspond to the solutions asymptotic to

$$\mathcal{M} = M \times \{(0, 0)\},$$

where $M$ is as in [32]. This is a compact normally hyperbolic symplectic critical manifold for $H^E_0$. We obtain

**Theorem 7.1.** Collision orbits of system $(H_0)$ with energy $E$ correspond to orbits of system $(H^E_\mu)$ doubly asymptotic to $\mathcal{M}$. Orbits of system $(H_\mu)$ with energy $E$ passing $O(\mu)$-close to the singular set $\Delta$ correspond to orbits of system $(H^E_\mu)$ on the level $\Gamma^E_\mu$ passing $O(\sqrt{\mu})$-close to $\mathcal{M}$.

Next we translate Lemma 6.2 to the new variables.

Let $r > 0$ and let $\mathcal{N}_r = M \times B_r$ be a tubular neighborhood of $\mathcal{M}$ in $\mathcal{P}$. By the stable and unstable manifold theorems for normally hyperbolic invariant manifolds [13], if $r > 0$ is small enough, for any $(x_0, y_0) \in M$ and $\xi_- \in S_r$ there exists a unique solution $\zeta_- : [0, +\infty) \to \mathcal{N}_r$, $\zeta_-(t) = (x(t), y(t), \xi(t), \eta(t))$, of system $(H^E_\mu)$ such that $\xi(0) = \xi_-$ and $\zeta(\infty) = (x_0, y_0, 0, 0) \in \mathcal{M}$. We denote its action by

$$J(\zeta_-) = \int_{\zeta_-} y \cdot dx + \eta \cdot d\xi = J_-(x_0, y_0, \xi_-).$$
Since the stable and unstable manifolds are smooth, $J_-$ is a smooth function on $M \times S_r$. Similarly we define the function $J_+$ on $M \times S_r$ as the action of a solution $\zeta_+$ asymptotic to $M$ as $t \to -\infty$.

We have an analog of Shilnikov’s Lemma [20]. Fix small $\varepsilon > 0$ and denote

$$Q_r = \{(x_0, y_0, \xi_-, \xi_+) \in M \times S_r^2 : \xi_- \cdot \xi_+ \geq \varepsilon^2 r^2\}.$$

**Theorem 7.2.** There exists $r > 0$ and $\mu_0 > 0$ such that for any $(x_0, y_0, \xi_-, \xi_+, \mu) \in Q_r \times (0, \mu_0)$:

- There exists $T > 0$ and a solution
  $$\zeta(t) = (x(t), y(t), \xi(t), \eta(t)) \in \mathcal{N}_r, \quad t \in [-T, T],$$
  of system $(\mathcal{H}_\mu)$ on $\Gamma_\mu$ such that
  $$x(0) = x_0, \quad y(0) = y_0, \quad \xi(-T) = \xi_-, \quad \xi(T) = \xi_+.$$  \hspace{1cm} (7.2)
- $\zeta$ smoothly depends on $(x_0, y_0, \xi_-, \xi_+, \mu) \in Q_r \times (0, \mu_0]$.
- The Maupertuis action is a smooth function on $Q_r \times (0, \mu_0]$ and has the form
  $$J(\zeta) = \int_{\zeta} y \cdot dx + \eta \cdot d\xi = J_-(x_0, y_0, \xi_-) + J_+(x_0, y_0, \xi_+) + \mu \hat{J}(x_0, y_0, \xi_-, \xi_+, \mu),$$
where $\hat{J}$ is $C^2$ bounded on $Q_r \times (0, \mu_0]$.

A result very similar to Theorem 7.2 was proved in [6]. A complete proof of Theorem 7.2 will be published in [9].

**Proof of Lemma 6.3** We set $\rho = r^2$ and $u = \xi^2$. For given $u_+ \in S_r$ take $\xi_+ \in S_r$ such that $\xi_+ \cdot \xi_- \geq 0$. If $\xi_+ \cdot \xi_- > 0$, then $u_+ \neq -u_-$. Moreover for given $\delta > 0$ there exists $\varepsilon > 0$ such that $|u_- + u_+| \geq \delta \rho$ implies $\xi_- \cdot \xi_+ \geq \varepsilon^2 r^2$. If $\zeta$ is a trajectory in Theorem 7.2 then the corresponding trajectory $g(\zeta)$ of system $(H_\mu)$ satisfies the conditions of Lemma 6.3 In particular, by (7.1),

$$J_\pm(x_0, y_0, \xi_\pm) = a_\pm(x_0, y_0, u_\pm).$$

\[\square\]

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