Fast calibration of two-factor models for energy option pricing

Emanuele Fabbiani\textsuperscript{1}, Andrea Marziali\textsuperscript{2}, and Giuseppe De Nicolao\textsuperscript{1}

\textsuperscript{1}Department of Industrial and Computer Engineering, University of Pavia
\textsuperscript{2}Modeling and Pricing, A2A

\textsuperscript{2}22nd October 2018

Abstract

Deregulation of energy markets in the 90s boosted the interest in energy derivatives. Over the last two decades, more and more complex financial instruments were developed. Pricing exotic derivatives often involves Monte Carlo simulations, which rely on stochastic processes to model the underlyings: it is thus critical to choose appropriate models and precisely calibrate them, so that they reflect the market scenario.

Several models have been proposed in the literature, from the simple geometric Brownian motion to more complex mean-reverting, multi-factor models. To enable their calibration against listed vanilla options, it is required to compute the variance of their states. This paper presents a simple and general method to compute the covariance matrix of the state through a matrix Lyapunov differential equation, and discusses its numerical and analytical solutions.

The availability of an analytical solution paves the way to an efficient market calibration of model parameters. As case studies, EEX German electricity and TTF Dutch gas markets were considered. Two different single-factor models and a two-factor one were calibrated against market prices: out-of-sample validation showed that a two-factor model outperforms the other two approaches.

\textit{Index terms}—Pricing, Lyapunov equation, energy derivatives, volatility, market calibration
1 Introduction

Liberalization of energy markets spurred the adoption of a variety of energy derivatives. Accurate pricing models are thus required by any energy company. Since the 70s closed-form formulae exist to derive the no-arbitrage price of European vanilla options and spread options [3, 2, 18]. However, if more complex derivatives, involving a larger amount of underlyings, are considered, the price is usually obtained by numerical methods, most commonly Monte Carlo (MC) simulations [10].

The basic elements of MC simulations are the stochastic models of the underlyings, which should be carefully chosen and precisely calibrated, in order to tail the market scenario. Since the early 2000s, the geometric Brownian motion (GBM) model appeared inadequate for energy derivatives, where mean-reversion is usually observed. Since then, a number of diffusion models specific to energy commodities have been presented [17, 22, 24, 20]. Schwartz and Smith [21] proposed a two-factor mean-reverting model for oil prices. Barlow and coworkers [19] presented three different mean-reverting models and a calibration procedure against prices history, leveraging the Kalman filter. A wide literature covers also spike modelling through jump-diffusion processes [7, 12, 6, 13, 14].

An equally important branch of research in the energy sector focuses on modelling the forward curve [5, 8, 23], following the typical interest rate models first proposed by Heat and coworkers [11]. More recently, Kiesel et al. [15] proposed a two-factor model tailored to electricity futures, as well as a calibration procedure based on market prices.

Herein, we devote our attention to spot prices rather than the entire forward curve, discussing efficient approaches for the computation of the variance for a wide class of models. In particular, we exploit the fact that, when the underlying is represented by a linear stochastic system, its state covariance obeys a Lyapunov matrix differential equation. The variance of the underlying as a function of the maturity, obtained from the solution of this equation, can then be plugged into the Black formula to obtain the no-arbitrage price of vanilla options. A model calibration procedure can thus be implemented by minimising the difference between model-predicted and actual market prices. Market calibration guarantees an arbitrage-free price, so that calibrated models can be fed into MC simulations in order to price complex derivatives.

The Lyapunov equation approach applies to all models that can be written as linear stochastic systems, in spite of their order. Both numerical and analytical solutions of the Lyapunov equation are discussed. In either case, the key point is the computation of a Gramian integral, which can be performed analytically or through the numerical calculation of a matrix exponential. A comparison between the two approaches is carried out in terms of computational speed. Computational efficiency is a crucial factor, as market calibration calls for the repeated evaluation of pricing formulae.

We tested the proposed calibration procedure by pricing listed vanilla options collected during several trading days on EEX German electricity market and TTF gas market. We compared geometric Brownian motion with two mean-
reverting models: the Ornstein-Uhlenbeck process and a two-factor model with log-spot price mean reverting to a generalised Wiener process. Jump-diffusion models were not considered because either spikes were absent or their frequency and intensity were negligible in the considered datasets. The results show that the two-factor mean-reverting model outperforms the other contenders.

The paper is organized as follows: Section 2 recalls the fundamentals of the Black framework, and Section 3 concisely presents the considered models. The variance derivation through the Lyapunov equation is provided in Section 4. Section 5 describes the calibration procedure and its results on the test cases. Finally, in Section 6 some concluding remarks end the paper.

2 Black framework and different models

Plain vanilla options in the energy markets exhibit some peculiar features. Their underlyings are most often represented by averages of future prices on a given period: month, quarter or year (from now on, we call these underlyings simply "futures"). Black formulae are a widely accepted framework to price vanilla options on futures.

Assume that a future $F$ behaves like a geometric Brownian motion with zero mean and standard deviation $\sigma$:

$$dF(t) = \sigma F(t) dw(t)$$

where $w(t)$ is a Wiener process. Black showed that the no-arbitrage prices $c$ of a European call option and $p$ of a European put option on the future $F$ are:

$$c = e^{-rT} (F_0 N (d_1) - K N (d_2))$$
$$p = e^{-rT} (K N (-d_2) - F_0 N (-d_1))$$

where:

$$d_1 = \frac{\ln \left( \frac{F_0}{K} \right) + \frac{(\sigma^2/2)T}{\sigma \sqrt{T}}}{\sigma \sqrt{T}}$$
$$d_2 = \frac{\ln \left( \frac{F_0}{K} \right) - \frac{(\sigma^2/2)T}{\sigma \sqrt{T}}}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T}$$

$F_0$ being the price of the underlying future at time $t = 0$, that is when the option is traded, and $N(\cdot)$ the cumulative probability distribution of a standard Gaussian variable. The listed price $F_0$ takes into account the seasonal behaviour embedded in the corresponding maturity.

The only parameter in the Black formulae which is not established by the contract or by the market is the volatility $\sigma$, that only appears in the term $\sigma \sqrt{T}$. Recalling that $\sigma \sqrt{T}$ is the standard deviation of a GBM process at time $t$, this term can be interpreted as the uncertainty on the log-return of the underlying at maturity. In view of this, one could devise alternatives models for the evolution of the underlying, and then plug their standard deviation into the Black formulae.
to obtain $c$ and $p$. A rigorous proof of this general approach is provided in [21, 15].

Suppose that the variance of log-return of the underlying at time $t$ is given by a positive function of time $p(\cdot)$. The terms $d_1$ and $d_2$ can be written as:

\begin{align*}
    d_1 &= \ln \left( \frac{F_0}{K} \right) + \frac{1}{2} p(T) \sqrt{p(T)} \\
    d_2 &= \ln \left( \frac{F_0}{K} \right) - \frac{1}{2} p(T) \sqrt{p(T)} = d_1 - \sqrt{p(T)}
\end{align*}

The function $p(\cdot)$ is evaluated at a single time instant, so that pricing does not take into account the evolution of the underlying future after the maturity. This simplification is justified in most energy markets, including EEX and TTF, because the option maturity coincides (or almost coincides) with the beginning of the delivery period of the future.

The pricing formulae can be slightly simplified by noting that, nowadays, proxies for the risk-free interest rate — e.g. the rate charged by the European Central Bank or the yields of highly reliable national bonds — are close to zero, or even negative. In the following, therefore, it is assumed $r = 0$.

3 Mean-reverting models

Despite being simple and widely accepted, GBM does not take into account the phenomenon of mean-reversion. Prices of energy commodities and related futures tend to follow a long-term trend: if, for whatever reason, they get away from it, they tend to be pushed back to the trend within a short time span [21].

3.1 Single-factor models

The simplest mean-reverting model is the Ornstein-Uhlenbeck process:

\begin{align*}
    \begin{cases}
        x(t) = \ln s(t) \\
        dx(t) = \lambda (\mu - x(t)) dt + \sigma dw(t)
    \end{cases}
\end{align*}

where $\lambda$ accounts for the strength of the mean-reversion of the log-price $x(\cdot)$ to the long-term trend $\mu$. In order to set the model in the risk-neutral measure, a common practice is to centre the distribution of model prices around the listed futures. The value of $\mu$ thus disappears and its calibration becomes useless.

Given a deterministic initial price, the variance of $x$ at time $t$ is

\begin{align*}
    \text{Var} [x(t)] = \frac{\sigma^2}{2\lambda} \left( 1 - e^{-2\lambda t} \right)
\end{align*}

The variance is asymptotically constant, which is not a good model for the variability of the returns. A more realistic model should indeed combine features of both geometric Brownian motion and mean-reverting models: an initial
fast-growing volatility, due to diverting phenomena, and an asymptotic ever-increasing uncertainty.

3.2 Two-factor models

The literature proposes several two-factor models. Among them, of particular interest is the Log-spot price mean reverting to generalised Wiener process model (LMR-GW) [19]. Its first equation is an Ornstein-Uhlenbeck process that accounts for the short-term variations and reverts to the long-term drift, driven by a geometric Brownian motion, represented by the second equation. The main features of the two processes are thus combined in a single model. The full set of equations is:

\[
\begin{align*}
    s(t) &= e^{x_1(t)} \\
    dx_1(t) &= \lambda (x_2(t) - x_1(t)) dt + \sigma_1 dw_1(t) \\
    dx_2(t) &= \mu dt + \sigma_2 dw_2(t)
\end{align*}
\]  

where \( w_1(\cdot) \) and \( w_2(\cdot) \) are independent Wiener processes.

To exploit the LMR-GW model for pricing, an expression of its variance is required. The problem is not new in the literature: in particular, it is worth mentioning the analytical solution worked out by Schwartz and Smith for a differently formulated second-order model [21]. Their derivation was targeted to that specific model, so that its extension to other two-factor or higher-order models is not straightforward.

In the next section, leveraging the theory of linear stochastic systems, we show that the variance of a wide class of models can be computed in a systematic way.

4 Variance derivation by Lyapunov equation

4.1 Continuous-time stochastic linear systems

Consider the state-space description of a continuous-time stochastic linear system:

\[
\begin{align*}
    dx(t) &= Ax(t) dt + Bdw(t) \\
    y(t) &= Cx(t)
\end{align*}
\]  

where \( A, B \) and \( C \) are matrices of suitable dimensions, \( x(\cdot) \) is the \( n \)-dimensional state and \( w(\cdot) \) an \( m \)-dimensional vector of independent Wiener processes. In the second equation \( y(\cdot) \) is the output of the system. To complete the description of the system, initial values for both the expected value and the variance of the state are required:

\[
\begin{align*}
    \bar{x}_0 &:= E[x(0)] \in \mathbb{R}^n \\
    P_0 &:= \text{Var}[x(0)] \in \mathbb{R}^{n \times n}
\end{align*}
\]
4.2 The Lyapunov equation

Let \( P(t) = \text{Var} \left[ x(t) \right] \) denote the covariance matrix of the system state. From (11), it follows that \( P(\cdot) \) satisfies the Lyapunov matrix differential equation:

\[
\frac{dP(t)}{dt} = AP(t) + P(t)A^T + BB^T
\]

under the initial condition \( P(0) = P_0 \) [9]. A solution to this equation is given by the matrix version of the Lagrange formula [16, 9]:

\[
P(t) = e^{At}P_0e^{A^Tt} + \int_0^t e^{A(t-z)}BB^Te^{A^T(t-z)}dz
\]

where the notation \( e^M \) denotes the matrix exponential of \( M \).

4.3 Ornstein-Uhlenbeck model

In order to illustrate the usage of the Lyapunov equation, we apply it to the Ornstein-Uhlenbeck process, whose variance is known. The Ornstein-Uhlenbeck process is a particular case of (11) when:

\[
A = -\lambda, \quad B = \sigma, \quad C = 1, \quad P_0 = p_0 = \text{Var} (x_0)
\]

The \( \mu \) parameter does not impact the variance, so it can be neglected. By applying (15), one obtains:

\[
P(t) = e^{At}P_0e^{A^Tt} + \int_0^t e^{A(t-z)}BB^Te^{A^T(t-z)}dz
\]

\[
= p_0e^{-2\lambda t} + \frac{\sigma^2}{2\lambda} (1 - e^{-2\lambda t})
\]

If the initial state of the system is deterministic, its variance is zero. The expression then simplifies to equation (9).

4.4 LMR-GW two-factor model

The state-space formulation of (10) is:

\[
\begin{align*}
  dx_1 (t) &= \lambda (x_2 (t) - x_1 (t)) dt + \sigma_1 dw_1 (t) \\
  dx_2 (t) &= \mu + \sigma_2 dw_2 (t) \\
  y(t) &= x_1 (t)
\end{align*}
\]

The drift parameter \( \mu \) does not affect the variance and is therefore neglected. The system can be rearranged in the standard form by letting:

\[
A = \begin{bmatrix} -\lambda & \lambda \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}
\]

6
Let $P(\cdot)$ and $P_0$ denote the state covariance and its initial state:

$$
P(t) = \begin{bmatrix} P_{11}(t) & P_{12}(t) \\ P_{21}(t) & P_{22}(t) \end{bmatrix}, \quad P_0 = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \quad (20)
$$

Observe that:

$$
e^{At} = \begin{bmatrix} e^{-\lambda t} & 1 - e^{-\lambda t} \\ 0 & 1 \end{bmatrix} \quad (21)
$$

By applying the Lagrange formula (15), after some algebraic manipulation, the following analytical solution is found:

$$
P_{11}(t) = \left( p_1 - 2p_{12} + p_2 - \frac{\sigma_1^2 + \sigma_2^2}{2\lambda} \right) e^{-2\lambda t} + 
2 \left( p_{12} - p_2 + \frac{\sigma_2^2}{\lambda} \right) e^{-\lambda t} + \sigma_2^2 t + \frac{\sigma_2^2 - 3\sigma_1^2}{2\lambda} + p_2
$$

$$
P_{12}(t) = P_{21}(t) = \left( p_{12} - p_2 + \frac{\sigma_2^2}{\lambda} \right) e^{-\lambda t} + \sigma_2^2 t + \frac{\sigma_2^2}{\lambda}
$$

$$
P_{22}(t) = \sigma_2^2 t + p_2 \quad (22)
$$

As in the case of the Ornstein-Uhlenbeck model, if the initial state of the system is deterministic, its variance $P_0$ is null:

$$
P_{11}(t) = -\frac{\sigma_1^2 + \sigma_2^2}{2\lambda} e^{-2\lambda t} + \frac{2\sigma_2^2}{\lambda} e^{-\lambda t} + \sigma_2^2 t + \frac{\sigma_2^2 - 3\sigma_1^2}{2\lambda}
$$

$$
P_{12}(t) = P_{21}(t) = \frac{\sigma_2^2}{\lambda} e^{-\lambda t} + \sigma_2^2 t - \frac{\sigma_2^2}{\lambda}
$$

$$
P_{22}(t) = \sigma_2^2 t \quad (23)
$$

As the log-price of the underlying is represented by $x_1$, for pricing purposes the only relevant term is $P_{11}$.

### 4.5 Numerical solution of the Lyapunov equation

As seen in the previous section, the matrix exponential $e^{At}$ is key to solve the Lyapunov equation. If an analytical solution is not available, one could choose a numerical approximation. An efficient numerical procedure relies on the following lemma, which is stated and proved in [4].

**Theorem 1** (Exponential of triangular matrix). Let $A_{11}$, $A_{12}$ and $A_{22}$ be matrices of suitable dimensions. Let

$$
F = \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix} = \exp \left( \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} h \right) \quad (24)
$$

7
Then the following equations hold:

\[
F_{11} = e^{A_{11}h} \quad (25)
\]
\[
F_{22} = e^{A_{22}h} \quad (26)
\]
\[
F_{12} = \int_0^h e^{A_{11}(h-s)} A_{12} e^{A_{22}s} ds \quad (27)
\]

Starting from the previous result, one can define \( F(t) \) as:

\[
F(t) = \begin{bmatrix} F_{11}(t) & F_{12}(t) \\ 0 & F_{22}(t) \end{bmatrix} = \exp \left( \begin{bmatrix} A & BSB^T \\ 0 & -A^T \end{bmatrix} t \right) \quad (28)
\]

to get:

\[
F_{11}(t) = e^{At} \quad (29)
\]
\[
F_{22}(t) = e^{-A^T t} \quad (30)
\]
\[
F_{12}(t) = \int_0^t e^{A(t-z)} BB^T e^{-A^T t} dz \quad (31)
\]

With a few more manipulations:

\[
F_{12}(t) = \int_0^t e^{A(t-z)} BB^T e^{A^T (t-z)} dz \cdot F_{22}(t) \quad (32)
\]

Which implies:

\[
\int_0^t e^{A(t-z)} BB^T e^{A^T (t-z)} dz = F_{12}(t) F_{22}^{-1}(t) \quad (33)
\]

The whole solution of the differential Lyapunov matrix equation can thus be expressed as follows - time dependency is omitted for readability:

\[
P(t) = F_{11} P_0 F_{22}^{-1} + F_{12} F_{22}^{-1} = (F_{11} P_0 + F_{12}) F_{22}^{-1} \quad (34)
\]

Note that \( F_{22} \) is always invertible because it is a matrix exponential. The above procedure is general-purpose: as long as the model for the underlying can be transformed into a linear stochastic system, Algorithm 1 can be applied.

### 4.6 Numerical and analytical solution: comparison

The most common market calibration procedures require the pricing formulae - and thus the variance function - to be evaluated several times. Moreover, in real use cases, they are run over multiple securities. Hence, even small differences in execution time can be greatly amplified, making practically useless even theoretically valid methods. For this reason tests were carried out to compare the computational efficiency of the two methods.
Algorithm 1 Variance of a linear stochastic system.

1. Consider a linear stochastic system

\[
\begin{align*}
\begin{cases}
    dx(t) &= Ax(t) \, dt + Bdw(t) \\
y(t) &= Cx(t)
\end{cases}
\end{align*}
\]  

(35)

where \( \text{Var} \left[ x(0) \right] = P_0 \).

2. Compute

\[
F = \begin{bmatrix}
    F_{11}(t) & F_{12}(t) \\
    0 & F_{22}(t)
\end{bmatrix}
= \exp \left( \begin{bmatrix}
    A & BB^T \\
    0 & -Ar
\end{bmatrix} t \right)
\]  

(36)

3. Get the variance \( P(t) \) of the state \( x \) as:

\[
P(t) = (F_{11}(t) P_0 + F_{12}(t)) F_{22}^{-1}(t)
\]  

(37)

Both analytical and numerical approaches were implemented using Python and its SciPy package. Matrix exponential was calculated by the \texttt{expm} function of the \texttt{scipy.linalg} module, which employs the Padé approximant, improved with scaling and squaring methods [1].

The hardware was a commercial off-the-shelf personal computer, running an Intel i5 3340M two-core CPU and a 16 GB RAM.

As a test case, an LMR-GW model with market calibrated parameters was considered. A time window of 30 days was set, within which the variance was computed at \( M \) time instants, with \( M \) ranging from 1 to 30,000. Each run was repeated 10 times and the median of the CPU times was computed - see Table 1. Speedups were derived as the ratio of the CPU time of the numerical solution to that of the analytical one. Speedups are consistently over one order of magnitude and are almost constant for large enough \( M \) - see Fig. 1.

| Evaluations | Analytical | Numerical | Speedup |
|-------------|------------|-----------|---------|
| 1           | 0.000047   | 0.001238  | 26.34   |
| 10          | 0.000256   | 0.012116  | 47.33   |
| 100         | 0.0026     | 0.0985    | 37.50   |
| 1,000       | 0.0259     | 0.9956    | 39.41   |
| 3,000       | 0.0738     | 3.0751    | 41.77   |
| 5,000       | 0.1239     | 5.0849    | 41.45   |
| 10,000      | 0.2885     | 10.5574   | 37.44   |
| 20,000      | 0.4958     | 20.8032   | 42.17   |
| 30,000      | 0.7394     | 31.4745   | 42.83   |

Table 1: CPU time and speedup. All times are in seconds.
Figure 1: CPU time against number of evaluations - semilog scale.

5 Market calibration

The models described in the previous sections can be calibrated either against the history of the underlying or the current market prices of listed options. Historical calibration is adequate for forecasting and risk management tasks, but not for pricing, in which case market calibration is preferred. In particular, the optimal model parameters are those which better explain the real market price of listed vanilla options.

Let $P_i$, $i = 1, \ldots, n$, be the prices of $n$ European vanilla options traded on a certain market. Let also $\hat{P}_i$ be the price of the $i$-th option given by the Black formula, where the variance computed according to the chosen model has been plugged into. The loss function can be defined as follows:

$$L = \sum_{i=1}^{n} (P_i - \hat{P}_i)^2$$

so that the vector $\theta^*$ of optimal parameters can be found as

$$\theta^* = \arg \min_{\theta} L (\theta)$$

which has to be solved by numerical methods.

5.1 EEX and TTF energy options

The proposed formulas are tested by solving a real-world market calibration problem. The ideal field of application would be that of complex derivatives,
which, however, are typically traded over the counter, so that there is no benchmark for their pricing. For this reason, the test is rather performed on vanilla options from TTF gas and EEX electricity markets, chosen because of their high liquidity.

Options on electricity and gas futures can be exercised, thus converted into actual futures, on the last trading day, typically two and five days before the start of the delivery month, for EEX and TTF respectively. These times coincide with the option maturity.

The dataset consisted in options on monthly futures listed in 57 consecutive days, between 01/11/2017 and 25/01/2018. The total number of different delivery periods was 25 for TTF and 7 for EEX. The number of listed options changed day by day, ranging between 430 and 596 for EEX and between 342 and 720 for TTF.

On each trading day, 70% of the available options were randomly assigned in the training set, while the remaining 30% were included in the test set. Three models, namely GBM, Ornstein-Uhlenbeck, and LMR-GW, were calibrated on the training set, and then asked to price the options of the test set. The Mean Absolute Error (MAE) between prediction and actual market price was then computed. Letting $n$ represent the number of options in the test set,

$$ MAE = \frac{1}{n} \sum_{i=1}^{n} \left| P_i - \hat{P}_i \right| $$

(40)

The MAE index is preferred over relative measures - as Mean absolute Percentage Error (MAPE) - because option prices are often close to zero, thus distorting the value of relative metrics.

5.2 Market calibration results

Figure 2 presents the Mean Absolute Error on the test set as a function of the day (left) as well as its statistical distribution (right) on both TTF (top) and EEX (bottom).

It is apparent that the two mean-reverting models outperform GBM on both EEX and TTF. Moreover, the two-factor model LMR-GW performs better than Ornstein-Uhlenbeck, especially on TTF options, while performances are more comparable on EEX. Note that MAE can take very different values from day to day: this is possibly explained by the different sets of options listed on different days.

The reason of different performances is hard to assess from aggregated data. To gain some insight, an in-depth analysis can be performed on sample days. In particular, we present results relative to the TTF market on 28th November 2017 as a case study.

To compare the models it may be useful to compare their implied volatilities. Given the price $v$ of an option, the implied volatility is the value of $\sigma$ such that the proper Black formula - either (2) or (3) - returns $v$. When GBM is the model of the underlying, the implied volatility is obviously constant and equal
Figure 2: Mean absolute errors on test sets.
to the tuned value of $\sigma$. However, when a different model is employed, implied volatility may become a function of time (i.e. maturity), strike and spot price. As far as single- and two-factor mean-reverting models are considered, neither strike or spot price appear in the variance, which, however, depends on the maturity. Thanks to the presence of both exponential and constants terms in the variance, these models account for the Samuelson effect, according to which volatility increases as the maturity decreases.

Again with reference to TTF options listed on 28th November 2018, charts in Figures 3 display the implied volatility surfaces derived from the models against the actual implied volatilities of listed options. Differently from others, these charts include the whole dataset.

![Implied Volatility Surfaces](image)

(a) GBM implied volatility.  
(b) Ornstein-Uhlenbeck implied volatility  
(c) LMR-GW implied volatility  
(d) Implied volatilities against maturity

Figure 3: Implied volatility against maturity and moneyness.

From the figure, the inadequacy of GBM to follow implied volatility is apparent. A much better fitting is achieved by the two mean-reverting models, with a definitely superior performance obtained by LMR-GW. Remarkably, even if Ornstein-Uhlenbeck and LMR-GW implied volatilities do not depend on moneyness, they explain fairly well the actual implied volatility.
5.3 Remarks on Calibration Procedures

The availability of fast methods for variance computation is crucial for the possibility of routinely performing market calibrations as the one reported in this section.

With respect to the test here described, performed on the same hardware described in Section 4.6, the calibration of LMR-GW models using analytical formulae took about 2 minutes in net CPU time, while slightly more than 28 minutes were required using numerical procedures based on matrix exponential.

6 Conclusion

In this paper, the problem of market calibration of stochastic models for energy commodities was tackled. Tuned models are the key building blocks to price complex derivatives. To enable the usage of Black formulae, the variance of the model must be derived. Despite results already existed for several models, it appeared of interest the description of a general and systematic approach. The key observation is that, whenever the model can be written as a stochastic linear system, its variance satisfies a matrix Lyapunov differential equation.

We discussed both numerical and analytical approaches to the solution of the Lyapunov equation and tested their computational efficiency, which is a crucial factor when the variance calculation enters calibration procedures relying on numerical optimization. We found the analytical solution to be 30 to 40 times faster than the numerical one.

In order to give a practical demonstration of the usefulness of efficient market calibration, we put three models - geometric Brownian motion, Ornstein-Uhlenbeck, LMR-GW - to the test on the pricing of vanilla options. With negligible computational effort, we showed that the LMR-GW model outperforms other approaches, thanks to its superior ability to interpolate the implied volatility surface.

References

[1] Awad H. Al-Mohy and Nicholas J. Higham. A new scaling and squaring algorithm for the matrix exponential. SIAM Journal on Matrix Analysis and Applications, 31(3):970–989, 2009.

[2] Fischer Black. The pricing of commodity contracts. Journal of Financial Economics, 3(1):167 – 179, 1976.

[3] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. The Journal of Political Economy, 81(3):637–654, May 1973.

[4] Tongwen Chen and Bruce Francis. Optimal Sampled-Data Control Systems. Springer-Verlag New York Inc, 1995.
[5] Les Clewlow and Chris Strickland. *Energy derivatives: pricing and risk management*. Lacima Publ., 2000.

[6] Cyriel De Jong. The nature of power spikes: A regime-switch approach. *Studies in Nonlinear Dynamics & Econometrics*, 10(3), 2006.

[7] Shijie Deng. *Stochastic models of energy commodity prices and their applications: Mean-reversion with jumps and spikes*. University of California Energy Institute Berkeley, 2000.

[8] Alexander Eydeland and Krzysztof Wolyniec. *Energy and power risk management: New developments in modeling, pricing, and hedging*, volume 206. John Wiley & Sons, 2003.

[9] Zoran Gajic and Muhammad Tahir Javed Qureshi. *Lyapunov Matrix Equations in System Stability and Control*. Academic Press, 1995.

[10] Paul Glasserman. *Monte Carlo methods in financial engineering*, volume 53. Springer Science & Business Media, 2013.

[11] David Heath, Robert Jarrow, and Andrew Morton. Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation. *Econometrica: Journal of the Econometric Society*, pages 77–105, 1992.

[12] Ronald Huisman and Ronald Mahieu. Regime jumps in electricity prices. *Energy economics*, 25(5):425–434, 2003.

[13] Joanna Janczura and Rafał Weron. An empirical comparison of alternate regime-switching models for electricity spot prices. *Energy economics*, 32(5):1059–1073, 2010.

[14] Joanna Janczura and Rafał Weron. Efficient estimation of Markov regime-switching models: An application to electricity spot prices. *ASTA Advances in Statistical Analysis*, 96(3):385–407, 2012.

[15] Rudiger Kiesel, Gero Schindlmaier, and Reik H. Borger. A two-factor model for the electricity forward market. *Quantitative Finance*, 9(3):279–287, 2009.

[16] Huibert Kwakernaak and Raphael Silvan. *Linear Optimal Control Systems*. Wiley, 1972.

[17] Julio J Lucia and Eduardo S Schwartz. Electricity prices and power derivatives: Evidence from the nordic power exchange. *Review of derivatives research*, 5(1):5–50, 2002.

[18] William Margrabe. The value of an option to exchange one asset for another. *The Journal of Finance*, 33(1):177–186, 1978.
[19] Yuri Gusev Martin Barlow and Manpo Lai. Calibration of multifactor models in electricity market. *International Jurnal of Theoretical and applied Finance*, 7(2):101–120, 2004.

[20] Dragana Pilipovic. *Energy risk: Valuing and managing energy derivatives*. McGraw Hill Professional, 2007.

[21] Eduardo Schwartz and James E. Smith. Short-term variations ans long-term dynamics in commodity pricing. *Management Science*, 46(7):893–911, July 2000.

[22] Eduardo S Schwartz. The stochastic behavior of commodity prices: Implications for valuation and hedging. *The journal of finance*, 52(3):923–973, 1997.

[23] Fanelli Viviana and S Musti. Modelling electricity forward curve dynamics in the italian market. *Quaderni del dipartimento di Scienze Economiche, Matematiche e Statistiche dell’Universit di Foggia*, 20, 2008.

[24] Rafal Weron. *Modeling and forecasting electricity loads and prices: A statistical approach*, volume 403. John Wiley & Sons, 2007.