Numerical Simulations of the MRI and Real Disks

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Abstract. Numerical MHD codes have become extraordinarily powerful tools with which to study accretion turbulence. They have been used primarily to extract values for the classical $\alpha$ parameter, and to follow complex evolutionary development. Energy transport, which is at the heart of classical disk theory, has yet to be explored in any detail. Further topics that should be explored by simulation include nonideal MHD, radiation physics, and outburst behavior related to the temperature sensitivity of the resistivity.

1. Introduction: The Need for MHD Simulations

Magnetic fields, particularly weak magnetic fields, strongly destabilize differential rotational in astrophysical accretion disks, allowing them to live up to their name. The magnetorotational instability (MRI) is but one manifestation of a very general behavior of weak magnetic fields in plasmas: by forcing angular momentum and entropy transport to follow field lines, they allow free energy gradients (temperature and angular velocity) to become sources of instability. Despite the fact that they have only been recently understood in detail, the instabilities are not difficult to simulate and to study numerically. (A single processor workstation will do.) For this reason, numerical investigations of spontaneously arising, undriven accretion disk turbulence are based on the MRI.

Nevertheless, the notion that simple differential rotation is intrinsically unstable via nonlinear hydrodynamic instability (Dubrulle 1993) continues to attract adherents (e.g., Richard & Zahn 1999; Richard 2003), in part because there is as yet no mathematical proof of nonlinear stability. It is argued that shear layers are known to be linearly stable, but extremely sensitive to non-linear disruption, thus Keplerian differential should also be nonlinearly unstable, provided that the Reynolds number is sufficiently high.

In fact, one does not even have to resort to nonlinearity to find an example of shear induced instability. Linear global hydro instabilities can be present under some circumstances, provided that proper boundary conditions are specified. These “Papaloizou-Pringle” (Papaloizou & Pringle 1984; Goldreich, Goodman, & Narayan 1986) instabilities, which involve communication by trapped waves on either side of a corotation radius, can disrupt systems that would ordinarily be stable by the classical Rayleigh criterion of increasing outward specific angular momentum. Numerical study of the PP instability poses no unresolvable difficulties, and it may be followed well into its nonlinear regime (Hawley 1991). Indeed, the emergent characteristic behavior of this instability is an instructive
example of how a controlled simulation can be used in pursuing an answer to a well-posed problem. The PP instability creates spiral structure in the disk, rather than turbulence, since this turn out to be a more efficient angular momentum transport mechanism in a hydrodynamical flow. But both because the instability is very sensitive to the proper boundary conditions, and because the instability weakens dramatically when the pressure gradients vanish, the relevance PP instabilities to Keplerian disks is far from clear.

More to the point, numerical simulations offer no support for the notion that local differential rotation is itself nonlinearly unstable. Proponents of nonlinear instability argue that this is simply because existing codes do not have a high enough effective Reynolds number, and are thus too diffusive. I must confess to being somewhat baffled by this argument, since the nonlinear instability of a shear layer is readily demonstrated, and it certainly does not require outlandishly high Reynolds numbers. Moreover, tight convergence has been achieved between codes with very different diffusive properties, and these show unambiguous nonlinear stability of local Keplerian flow (Hawley, Balbus, & Winters 1999).

It has been recently noted that the local “shearing box” equations of motion (e.g. Balbus & Hawley 1998), which are fully nonlinear but do not include radial curvature terms, exhibit an exact scale invariance (Balbus 2004b). Specifically, if \( w(\mathbf{r}, t), P(\mathbf{r}, t), \) and \( \rho(\mathbf{r}, t) \) represent exact solutions for the velocity, pressure and density respectively, then

\[
\frac{1}{\beta}w(\beta\mathbf{r}, t), \quad \rho(\beta\mathbf{r}, t), \quad \frac{1}{\beta^2}P(\beta\mathbf{r}, t)
\]

where \( \beta \) is an arbitrary scaling factor, are also exact solutions. This means that Keplerian flow is no more unstable at very small scales then at larger, computationally accessible scales. Any putative local disk instability in Keplerian flow would have to exhibit behavior very different from that seen in the classical breakdown of planar shear flow. Clearly, Coriolis forces are strongly stabilizing, rigorously so in the linear regime, and it is not surprising that the stabilization extends into the nonlinear regime as well. As we have already noted, this does not mean that any rotationally-based instability is proscribed, it simply means that local shear is insufficient to disrupt Keplerian flow. Nonlinear instability buffs are forced to argue that conditions beyond the reach of modern supercomputers are required to trigger instability, a position that may become increasingly untenable as time goes on. As a very practical matter, the MRI remains the only instability on the market that manages to sustain long term, unforced turbulence in numerical simulations of Keplerian disks. In the end, numericists really don’t have a choice.

2. Tools of the Trade: Correlated Fluctuations

Accretion disk turbulence cannot yet be directly observed in nature, and in the laboratory the MRI has only very recently been identified (Sisan et al. 2004). Three-dimensional local MHD simulations are almost ten years old at the time of this writing (Hawley, Gammie, & Balbus 1995). Three-dimensional global simulations followed shortly thereafter (Armitage 1998; Hawley 2000, 2001). In fact, two-dimensional global simulations date back two decades (Uchida &
Shibata 1985), well before MRI calculations appeared in local guise, and before the MRI had been identified in accretion disks. By now, global MHD simulations are a small industry, with several groups around the world investigating many different specialized problems. What can be learned from these efforts?

We must first understand what it is that we don’t understand, and for that we should begin with the tenets of classical disk theory (CDT). CDT is more widely known as viscous disk theory, a moniker that surely needs to be laid to rest. CDT helped to orient our intuitions during the early stages of disk theory, but now that this phase has past, it is time for disk viscosity to be jettisoned. The analogy has become more of a hindrance than anything else — in some papers enhanced viscosity and turbulence are viewed as competing processes! It is time for disk theorists to move out of the 1970s, and relegate “anomalous viscosity” to the cultural dustbin along with afros, leisure suits, and disco.

2.1. Stress Tensor

The fundamental transport quantity that is at the center of any theory of turbulent disk transport is the stress tensor. Unlike a viscous stress, the true stress tensor is entirely a property of the flow, not of the constituent fluid. It is defined in terms of correlated velocity fluctuations and correlated magnetic fields. Let \( u \) be the difference between the local disk velocity and pure Keplerian rotation on cylinders. Let \( u_A \) be the Alfvén velocity associated with the magnetic field \( B \), and mass density \( \rho \),

\[
\rho = \frac{B}{\sqrt{4\pi \rho}},
\]

and is not necessarily a fluctuating quantity. We assume that the magnitudes of \( u_A \) and \( u \) are comparable, and that both are much less than the unperturbed rotation velocity, \( R\Omega \), where \( R \) is the cylindrical radius. The density and pressure fluctuations, \( \delta \rho \) and \( \delta P \), are assumed to be small compared with their mean values, \( \bar{\rho} \) and \( \bar{P} \), respectively. The stress tensor has, in principle, six independent components, but in practice only the azimuthal components play an essential role in disk theory,

\[
T = \bar{\rho} \langle (uu_\phi - u_A u_{A\phi}) \rangle
\]

and we shall henceforth refer to these components as the stress tensor. Extracting an accurate mean value of \( T \) from numerical simulations requires patience, as very long time averaging may be required (Steinacker & Papaloizou 2002).

The stress tensor plays two mathematically related, but physically distinct roles in disk theory: (1) it is directly proportional to the outward angular momentum flux through the fluid, and hence to the classical \( \alpha \) parameter; and (2) it is directly proportional to the rate at which energy is extracted from the background differential rotation of the disk. The latter is the free energy source that powers the turbulent fluctuations themselves. The central tenet of CDT is that this energy goes nowhere. Instead, it is all locally dissipated and radiated away on the spot. The reader will recall the well-known factor of 3 discrepancy between the radiated energy and the local orbital energy that is released. This is usually explained as a nonlocal consequence of the presence of an energy flux, but I prefer to think of it as the result of the local injection of the free energy of differential rotation. In any case, this assumption of local energy dissipation is very powerful, and needs to be carefully checked.
2.2. Fluctuation Equations

**Hydrostatic equilibrium**  The steady state disk equations have been worked out under assumptions of section 2.1. The fundamental equation of hydrostatic equilibrium is

$$ R\Omega^2 e_R = \frac{1}{\rho} \nabla P + \nabla \Phi $$

(3)

where $e_R$ is a unit vector in the radial direction, and $\Phi$ is the central potential. Notice that hydrostatic equilibrium may be expressed in terms of mean values, and is not directly affected by the turbulent fluctuations. Equation (3), and those to follow, are two-dimensional: the process of extracting the mean averages out the azimuthal structure. Hydrostatic equilibrium breaks down, of course, in regions of thermal or transonic flow.

**Mass conservation**  By way of contrast, transport processes are intimately linked to correlations in fluctuating quantities, and to the magnetic field. (This is true even if the field is weak.) The mass flux is

$$ \langle \rho v \rangle = \bar{\rho} \bar{v} + \langle \delta \rho \delta v \rangle, $$

(4)

and in steady state

$$ \nabla \cdot \langle \rho v \rangle = 0 $$

(5)

The final fluctuating term in equation (4) is generally comparable to the mean flow term, though it is dropped in viscosity analogues.

**Angular momentum**  Angular momentum conservation is expressed by the steady state equation

$$ \nabla \cdot \langle R^2 \Omega \langle \rho v \rangle + \bar{R} T \rangle = 0 $$

(6)

The first term represents that angular momentum transported directly by the mass flux, whereas the second is flux passing through the disk gas. In CDT, only radial transport is considered, and $T$ is replaced by a viscous stress.

**Energy conservation**  Energy is transported through the body of the disk via the thermal energy flux,

$$ F_{\text{th}} = \frac{\gamma \bar{P}}{\gamma - 1} \langle \delta v \delta \tau \rangle $$

(7)

where $\gamma$ is the adiabatic index, and $\tau$ is the normalized temperature $kT/\mu$ ($k$ is the Boltzmann constant and $\mu$ the mean mass per particle). This form of $F_{\text{th}}$ is very general. The velocity-temperature correlation is responsible for convective transport, as well as the energy transported by ordinary sound waves. The energy equation for a turbulent disk is (Balbus 2004a):

$$ -T_{\rho \phi} \frac{d\Omega}{d\ln R} = \nabla \cdot F_{\text{th}} + \bar{\tau} \langle \rho v \rangle \cdot \nabla S + Q_{\text{rad}} $$

(8)

where $T_{\rho \phi}$ is the radial component of $T$, $Q_{\text{rad}}$ is the volumetric radiative loss rate, and $S$ is in essence the mean entropy,

$$ S \equiv \frac{1}{\gamma - 1} \ln \frac{\bar{P}}{\bar{\rho}^\gamma} $$
Equation (8) states that the rate at which energy is locally extracted from the differential rotation is equal to the rate at which it is carried away by waves, dissipated as heat, and radiated away. In CDT, all terms on the right side of the equation are ignored except for radiative losses, $Q_{\text{rad}}$. This is justified in the following sense. It is customary to express the stress tensor in terms of a simple dimensionless scaling,

$$T_{R\phi} = \alpha P$$

(9)

where $\alpha$ is a dimensionless parameter that embodies both the magnitudes of the velocity fluctuations as well as their relative degree of correlation. This will of course be immediately recognized as the relationship that gives “$\alpha$-disk” theory its name. Since there is more than one correlation tensor of interest here, let us relabel the above alpha as $\alpha_L$, since $T_{R\phi}$ is associated with angular momentum transport. We are also free to invoke a scaling law of the form

$$F_E = \alpha_E P \bar{\tau},$$

(10)

where $F_E$ is a radial energy flux. The neglect of this thermal energy flux will then be justified if

$$\frac{\bar{\tau}}{R\Omega \alpha_L} \ll 1$$

(11)

The first ratio is an inverse Mach number for the rotation velocity, and will be small (by definition) for a thin Keplerian disk. This presumably includes CV disks. The energy flux term will then be negligible if $\alpha_E$ does not much exceed $\alpha_L$. While this is not a priori unreasonable, neither is it very well tested. It is the kind of question that should be asked of numerical simulations.

### 2.3. Global Simulations

Most of the global disk simulations to date follow the following prototype. The gas is a polytropic torus, in the gravitational field of a “black hole.” This means that the potential field is given by (Paczyński & Wiita 1980):

$$\Phi = -\frac{GM}{|\mathbf{r} - \mathbf{r}_g|},$$

(12)

where $r_g$ is the usual Schwarzschild (or gravitational) radius, $2GM/c^2$. (Symbols have their usual meanings.) This initial state is chosen to be a torus because it allows a simple hydrostatic equilibrium to be constructed, but it is also physically reasonable: a reservoir of angular momentum bearing material located at a large distance from the origin. The equilibrium is of course nonmagnetic; as soon as a small field is added, the MRI quickly dominates the evolution.

The original motivation for introducing the Paczyński-Wiita potential was that it captured some key features of true black hole dynamics in a user-friendly Newtonian format. For the numericist, this is a particularly convenient potential function, since the inner boundary is pure outflow, or, from the point of view of the hole, an inflow at $r = r_g$. The use of this potential when the central object is not a black hole is obviously a cheat, but it does finesse the problem of how to handle the numerically intractable disk-star boundary layer. The price
to be paid is that all information of this observationally critical region is lost. But if the scope of the simulation is restricted to the study of the fundamental turbulent transport properties in the body of the disk, this approach makes some sense, particularly at this early stage. One must be willing to accommodate the possibility, however, that the existence of the inner boundary layer could cause global changes in ways that are difficult to foresee.

If we take at face value the notion that numerical codes calculate the evolution of a strictly polytropic disk, equation (8) leaves only one choice for its dominant balance:

\[-T_R \phi \frac{d\Omega}{d\ln R} = \nabla \cdot F_{th}\]  

(13)

This is a balance between the thermal energy flux divergence and the rate at which energy is exchanged with differential rotation, a purely adiabatic process. In the case of WKB waves, it leads to the conservation of wave action, rather than wave energy. More generally, this balance would involve extraction of the free energy of differential rotation followed by its active transport through the disk. This would involve an adiabatic coupling between evanescent and wavelike modes. Does this really happen in simulations, let alone real disks?

It hasn’t been checked. ZEUS-like codes, which base their energetics on an internal, as opposed to a total energy equation, tend to be lossy when computing small scale flow structure. This is not necessarily bad, and in fact it mimics the behavior of real disks. In numerical tests of the stability of hydrodynamical flow, these energy losses were tracked and their rapid growth used as a hallmark for the presence of turbulence (Balbus, Hawley, & Stone 1996). These losses introduce an effective $Q_{rad}$ term in the calculation, even though it is not explicitly included in the code. The interesting question is how much free energy is locally “radiated” and how much is transported elsewhere, since this will determine whether local disk models make sense. The fact that simulations do not lead to very thin disks suggests that local grid losses are not overwhelming, but a detailed accounting should be part of the numerical diagnostics. The existence of thin disks as well as the local $R^{-3/4}$ scaling law seen in some eclipsing binaries like Z Cha (Frank, King, & Raine 2002) are certainly consistent with local dissipation, but it is not obvious that they require it.

3. Disk Morphology

Disk morphology predictions may be more robust than is often recognized. The angular velocity $\Omega$ tends to be constant on cylinders, since $z$-dependent rotation profiles are unstable (Goldreich & Schubert 1967). If so, the right hand side of equation (3) is a pure gradient, and whatever the actual constitutive relationship between $P$ and $\rho$ may be, the equilibrium functional relationship between these two quantities will be that isobaric and isochoric surfaces coincide. Moreover, these surfaces are also equipotentials, provided that the effective centrifugal potential is included (e.g., Frank et al. 2002).

In principle, the rotation profile $\Omega(R)$ is a free functional parameter, but in practice almost all MRI simulations show that a Keplerian power law rapidly
emerges:
\[ \Omega^2 = \frac{GM \cos \beta}{R^3} \]  
where \( \cos \beta \) is a proportionality constant not very different from unity. The reason for this seems to be connected with the vigorous outward angular momentum transport that always accompanies the MRI in disks, which spreads the initial torus radially, diluting the dynamical effects of pressure gradients in the process. The disk then seems to follow a simple a quasi-Keplerian profile, linear in \( GM \), with residual pressure support allowing rotation at slightly below its pressure-free value. For a polytropic equation of state, \( P = K \rho^\gamma \), the equilibrium density profile satisfies
\[ \frac{K \gamma \rho^{\gamma-1}}{\gamma - 1} = \mathcal{H}_\infty + \frac{GM}{r} \left(1 - \frac{\cos \beta}{\cos \lambda}\right) \]  
where \( \mathcal{H}_\infty \) is the enthalpy at infinity, and \( \lambda \) is the colatitude, \( \pi/2 - \theta \), where \( \theta \) is the usual spherical angle. An isothermal equation of state yields:
\[ \rho = \rho_\infty \exp \left[\frac{GM}{r} \left(1 - \frac{\cos \beta}{\cos \lambda}\right)\right] \]  
There is a qualitative difference between an adiabatic and an isothermal disk. An adiabatic disk has a well-defined sharp edge at the value of \( \lambda \) corresponding to \( \rho = 0 \), whereas an isothermal disk trails off exponentially above the disk boundary \( \lambda = \beta \). The equipotential surfaces themselves satisfy the equation
\[ \frac{1}{r} \left(1 - \frac{\cos \beta}{\cos \lambda}\right) = C = \text{constant}. \]  

Contours for a fiducial example are shown in Figure 1. When \( C > 0 \), the contours are open, extending to infinity. When \( C < 0 \), they curl back through the midplane and reconnect at the origin. The critical \( C = 0 \) contour is the straight line \( \lambda = \beta \), dividing the two classes. Notice that all contours converge to this critical contour as \( r \to 0 \).

The zone of “contour convergence” seems to be the site of the launching of a jet-like outflow in numerical simulations, giving some credence to this analytic model. (The jet appears to be thermal, rather than magneto-centrifugal, in origin.) The tight grouping of the equipotential surfaces in the convergence zone allows dissipative processes to move fluid elements from one equipotential surface to another fairly easily in this region. Incoming disk material cannot be swallowed as fast as it arrives, and is forced onto upward onto open equipotential surfaces. The inner most allowable surface will always coincide more closely to a constant angular momentum cylinder then will surface closer to the disk. This will be the easiest contour for the fluid to flow along as it leaves the disk, requiring a minimum of angular momentum change. And indeed, the simulations show a pile-up of material close to the last open equipotential surface, confined against this surface from the inside by a low density magnetic corona. Significantly, the isothermal runs do not show the formation of a jet. This can be understood both because the softer equation of state allows material to be more easily swallowed.
Figure 1. Equipotential adiabatic disk contours for the case when $\Omega$ is 0.84 of its Keplerian value, corresponding to an opening wedge angle of 45°, marked by the separatrix line marked OUTFLOW. Open contours become very closely packed near the $\rho = 0$ boundary.

at the center of flow, and because the density is exponentially curtailed above the cone $\lambda = \beta$.

Figure 2 shows a detailed simulation by De Villiers, Hawley, & Krolik (2003). The plot displays azimuthally averaged density contours after a period of time corresponding to 10 orbital periods at the pressure maximum of the initial torus used in the simulation. This is a general relativistic calculation of an adiabatic gas, but except very near the “plunging region”, the gravitational potential is nearly Newtonian. The tear drop cross section predicted by analytic theory and inner jets are both present.

4. Beyond Ideal MHD

4.1. Hall and Ohmic Processes

At temperatures below 2000 K, the level of ionization in a CV disks can fall below the minimum needed to keep the gas in the ideal MHD regime. Assuming
that dust grains are unimportant, the relative mutual drift velocities of ions, electron, and neutrals gives rise to ambipolar diffusion, Hall electromotive forces (HEMFs), and Ohmic dissipation. Ambipolar diffusion is important in interstellar clouds, but for the much denser disks of interest here, HEMFs and Ohmic losses are of primary interest. Numerical MHD has recently begun to study these nonideal effects, which can have a decisive influence over disk behavior.

Ohmic losses were studied in local disk calculations by Fleming & Stone (2003), who found that MHD turbulence could be maintained only if the magnetic Reynolds number

\[ Re_M \equiv \frac{c_s^2}{\eta \Omega} \]

where \( \eta \) is the resistivity, exceeded a critical value, \( Re_M(\text{crit}) \). The value of \( Re_M(\text{crit}) \) depends upon the field geometry. For runs with a mean vertical field present, Fleming & Stone found critical values of about 100; these increased to \( 10^4 \) for runs without a mean vertical field. Once \( Re_M < Re_M(\text{crit}) \), turbulence is quickly suppressed. The addition of HEMFs (Sano & Stone 2002) seems not to alter very much the values of \( Re_M(\text{crit}) \), but the saturations levels are more sensitive. These can be significantly raised by the presence of the Hall effect.

Numerical non-ideal MHD offers yet another venue of opportunity. The ohmic resistivity is extraordinarily temperature sensitive in the transition regime,
because it depends upon the electron ionization fraction. If thermal in origin, this fraction depends upon Boltmann factors of the form \( \exp(-I/kT) \) where the ionization energy \( I \) is well above \( kT \), which accounts for the temperature sensitivity. Small temperature fluctuations cause large changes in the ionization fraction. An increase in \( T \) leads to a decrease in \( \eta \), and, it may be expected, to higher levels of turbulence. This is a prescription for yet higher temperatures, and a possible runaway. A similar instability is associated with \( T \) decreases and suppression of turbulence. How this resolves itself is not yet known, but it seems to me that the problem is within the computational realm.

Applications of this behavior to DN eruptions have been explored (Gammie & Menou 1998; Menou 2000; Sano & Stone 2003), but interpreting the quiescent stage in this context remains problematic. Fleming & Stone (2003) make the interesting and provocative suggestion that magnetically “dead” zones may be affected by surrounding lower density magnetically active regions. In this view, the dead zones may be only morbid, and host an active stress tensor maintained by outwardly transporting trailing density waves. If this is in fact the way that such low ionization regions behave, it is a bit of a coup for numerical studies of accretion disks. It is difficult to imagine a compelling \textit{a priori} analytic argument, for example, that would give life to the dead zone in this way.

The difficulty with the low state has been to avoid having the \( \alpha \) parameter drop all the way to zero. Gammie & Menou (1998) suggested global hydro-instabilities, Menou (2000) tidal interactions, and Sano & Stone (2003), in their numerical calculation, treated the low state “by hand”. The view put forth above offers another possibility: the quiescent disk state might be a hybrid wave-turbulent system, while the high state is a flow fully governed by the MRI. In its low state, the disk is obstructed by a sluggish but extended central zone, while accretion occurs more easily in a rarified atmosphere around the low ionization region. Even in the low hybrid state, the primary accretion process may still be largely MHD, but affecting only the tenuous gas.

4.2. Radiation

Radiation is essential to CDT, since this is the fate of all the free energy that is locally dissipated by turbulence. Very little work has been done, however, to include radiative losses in numerical simulations. What has been done, has focused on the case where the radiation energy rivals or dominates the thermal energy density, which can happen in the inner regions of black hole accretion. The linear stability of a magnetized, stratified, radiative gas has been investigated by Blaes & Socrates (2001). Despite the complexity of the full problem, the MRI emerges at the end of the day unscathed, its classical stability criterion \( d\Omega^2/dR > 0 \) remaining intact.

The proper interpretation of a linear analysis is slightly unclear, since the unperturbed state is already presumably fully turbulent due to the MRI; a rotationally stable disk lacks an internal energy source. The interplay between MHD turbulence and radiation is intrinsically nonlinear. Turner, Stone, & Sano (2002) have studied a local, radiative, axisymmetric, shearing box. The linear calculations of Blaes & Socrates were confirmed in detail, and the nonlinear flow fully developed. As in standard MRI simulations, the stress is dominated by the Maxwell component, which is a factor of a few larger than the Reynolds terms.
Clearly, much could be learned by returning to the case where radiation is an important loss mechanism, but does not dominate the local energy density—i.e., the Shakura & Sunyaev (1973) prototype. The technical expertise brought to bear on radiation dominated disks would serve this simpler problem well. Establishing that a radiative shearing box does, in fact, evolve toward something resembling the classical Shakura-Sunyaev paradigm would be an important milestone for accretion theory.

5. Summary

- The MRI remains the leading candidate for the origin of enhanced angular momentum and energy transport in accretion disks. A compelling case for sustained hydrodynamical turbulence has yet to be made.

- Despite very real limitations, the most serious of which is probably still the limited dynamical range, numerical simulations have become truly powerful. They have allowed us to follow the global evolution of polytropic disks, and to witness the build-up of a Keplerian disk, corona, and central jet—starting from nothing but a simple constant angular momentum torus. Many of these features can be understood analytically, at least in a crude sense.

- Energetics (including radiation and dissipative physics) remains in its infancy—we know less than we think. Numerical disk models have yet to be placed on the observational plane.

- We have not tested under what conditions the fundamental formula of phenomenological disk theory

\[ Q_\perp = -T_{R\phi} \frac{d\Omega}{d\ln R} \]

is valid. Even the eclipse mapping observations that show an \( R^{-3/4} \) power law dependence in the temperature profile of CV disks do not require this assumption to be valid, though they are certainly consistent with it. Investigating this question numerically should be a central goal for numericists.

- A closely related point: We are completely ignorant of the role of thermal energy transport, \( \delta v \delta T \). It need not be small relative to rotational transport, and can be extracted from simulations.

- The consequences for the MRI of departures from ideal MHD in the form of Hall EMFs and ohmic resistance are critical to understand. Only one team, Sano & Stone, has studied this numerically, and then only on fairly coarse grids. Resolution may be key here. The question that needs to be thoroughly explored is under what circumstances does the MRI turn off?

- The temperature dependence of the resistivity has not been touched numerically. Clearly, there is a real potential to explore eruptive behavior here.
• For all their foibles and limitations, numerical MRI simulations have taken their place at the helm of theoretical accretion disk studies, and they are likely to remain there for the foreseeable future.

Acknowledgements

I thank J.-P. Lasota for helpful comments on the manuscript. Support from NASA grants NAG5-13288 and NNG04GK77G is gratefully acknowledged.

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