Rapid Feature Evolution Accelerates Learning in Neural Networks

Haozhe Shan ∗
Center for Brain Science
Harvard University
Cambridge, MA 02138
hshan@g.harvard.edu

Blake Bordelon ∗
School of Engineering and Applied Science
Harvard University
Cambridge, MA 02138
blake_bordelon@g.harvard.edu

Abstract

Neural network (NN) training and generalization in the infinite-width limit are well-characterized by kernel methods with a neural tangent kernel (NTK) that is stationary in time. However, finite-width NNs consistently outperform corresponding kernel methods, suggesting the importance of feature learning, which manifests as the time evolution of NTKs. Here, we analyze the phenomenon of kernel alignment of the NTK with the target functions during gradient descent. We first provide a mechanistic explanation for why alignment between task and kernel occurs in deep linear networks. We then show that this behavior occurs more generally if one optimizes the feature map over time to accelerate learning while constraining how quickly the features evolve. Empirically, gradient descent undergoes a feature learning phase, during which top eigenfunctions of the NTK quickly align with the target function and the loss decreases faster than power law in time; it then enters a kernel gradient descent (KGD) phase where the alignment does not improve significantly and the training loss decreases in power law. We show that feature evolution is faster and more dramatic in deeper networks. We also found that networks with multiple output nodes develop separate, specialized kernels for each output channel, a phenomenon we termed kernel specialization. We show that this class-specific alignment does not occur in linear networks.

1 Introduction

Deep learning provides a flexible framework to solve difficult statistical inference problems across a variety of application areas [1]. During optimization of the statistical objective, useful features are often extracted by the neural network as the weights in the network evolve. Though feature learning appears to be crucial to the success of neural networks on large-scale problems (as well as enabling transfer learning on new, related problems), the precise way that neural network features evolve to accelerate learning is not well understood theoretically. In the limiting case of large widths and small learning rates, neural networks behave like linear models in their parameters and their predictions are governed by a static neural tangent kernel (NTK) [2, 3]. In this “lazy learning” limit, the neural network extracts no useful features and the loss evolves according to the alignment between the target values and the initial NTK [4]. In this limit, a precise characterization of training and generalization dynamics can be obtained [5, 6, 7], allowing the derivation of scaling laws of large width models [8, 9, 10]. At finite width and small initial weights, the NTK is no longer a static object, but evolves during training [11, 12]. Recent experiments performed by Baratin et al [13] on multiple network architectures showed that the NTK becomes aligned with the target function during training, providing a connection between NTK theory and feature evolution in finite width networks.

∗Equal Contribution; We flipped a coin to determine author order.

Preprint. Under review.
In this work, we aim to characterize, through experiments and exactly solvable toy models, how evolution of the NTK interacts with the loss dynamics to accelerate learning during gradient descent. We first provide empirical evidence that the loss dynamics for networks in a feature learning regime are qualitatively different from the frozen kernel limit and cannot be reproduced through a simple rescaling of the learning rate. We reproduce empirical findings of rapid evolution of the kernel early in training followed by a period of where the predictions evolve with the final kernel \[14, 13, 16\]. We provide a normative explanation of this kernel alignment phenomenon by considering optimal feature evolution. In this toy model, the kernel’s features evolve in the direction of the gradient of the loss one step ahead in time. In the continuous time limit, this generates differential equations which couple the loss dynamics to the dynamics of the kernel. This model reproduces the accelerated learning curve early in training and a saturating asymptotic alignment value. It further predicts a sharp transition between a kernel regime and a feature learning regime reminiscent of the spiked matrix problem \[17\]. We also provide a mechanistic model of how kernel alignment naturally occurs in neural networks during training by considering a two-layer linear network. For multi-class problems, the time varying kernel is a rank-4 tensor with entries which provides a scalar value to each pair of outputs, and pair of inputs. We show empirically a phenomenon of specialization, where the kernel for each class aligns to its own target function. We demonstrate that such specialization is not possible in deep linear neural networks but is possible in two layer non-linear neural networks, demonstrating the importance of nonlinearity in class-specific feature learning.

2 Neural Tangent Kernel Dynamics

We will now proceed to describe the setup for our study; namely the evolution of the NTK during optimization. For simplicity, we will first discuss scalar output functions and will extend our discussion to multiple class outputs in a later section. Let \( f(x, \theta) \) represent the output of a neural network with parameters \( \theta \) and input vector \( x \). We optimize the parameters \( \theta \) with gradient flow on a the MSE loss for a dataset with \( P \) examples \( D = \{(x^\mu, y^\mu)\}_{\mu=1}^P \). Such dynamics give

\[
\frac{d\theta}{dt} = -\eta \sum_{\mu=1}^P \frac{\partial f(x, \theta)}{\partial \theta} (f(x^\mu, \theta) - y^\mu)
\]

where \( \eta \) is the learning rate. Rather than studying the dynamics of the parameters \( \theta \), we follow Jacot et al and consider the dynamics of the network’s predictions \[2\]

\[
\frac{df(x^\mu, \theta)}{dt} = \frac{\partial f(x^\mu, \theta)}{\partial \theta} \cdot \frac{d\theta}{dt} = -\eta \sum_{\nu} K(x^\mu, x^\nu; \theta) (f(x^\nu) - y^\nu)
\]

where the \( K \) represents the NTK \( K(x, x'; \theta) = \frac{\partial f(x, \theta)}{\partial \theta} \cdot \frac{\partial f(x', \theta)}{\partial \theta} \). Equation \[2\] exactly describes gradient flow in arbitrary neural network models if we allow \( K \) to vary over the trajectory of parameters \( \theta(t) \) \[18\]. We will use \( K(\theta) \in \mathbb{R}^{P\times P} \) to denote the gram matrix over the \( P \) training examples.

2.1 Static Neural Tangent Kernel Limit

The key insight from previous works which makes such dynamics analytically tractable is the fact that, under a certain initialization scheme and in the infinite width limit, the kernel matrix is constant during the timecourse of gradient descent \[2, 3, 19, 20\]. This means that the predictions of the model evolve linearly against a static matrix \( K(\theta(t)) \sim K_0 \). Let \( \Delta \in \mathbb{R}^P \) be a vector with entries \( \Delta^\mu = f(x^\mu, \theta) - y^\mu \), then we have

\[
\Delta = -\eta K_0 \Delta \implies \Delta(t) = -\exp(-\eta t K_0) y
\]

where \( K_0 \) is the kernel matrix at initialization. The loss \( L_t = ||\Delta(t)||^2 \) can be decomposed explicitly through an eigendecomposition of this initial kernel \( K_0 = \sum_k \lambda_k u_k u_k^\top \in \mathbb{R}^{P\times P} \) giving \( L_t = \sum_k \exp(-2\eta \lambda_k t) (u_k^\top y)^2 \). Generically, the initial NTK tends to exhibit power law spectra \( \lambda_k \sim k^{-b} \) and \( (u_k^\top y)^2 \sim k^{-a} \), which generates power-law loss scalings \( L_t \sim t^{-(a-1)/b} \) \[21\]. We compare neural network training dynamics against dynamics of kernel gradient descent with a fixed kernel (Fig.1) for a 2-layer ReLU NN and NTK on MNIST using the Neural Tangents API \[22\].

We found that the evolving kernel in neural networks significantly accelerates learning which cannot be accounted for by the increasing norm of the network’s kernel (Fig.1A).
Figure 1: Feature evolution alters the structure of the NTK and accelerates learning. A The loss for a two-layer $N = 500$ MLP when trained on a subset of MNIST (NN) is compared to kernel gradient descent with the initial kernel (KGD) and the initial kernel using a rescaled learning rate to account for the difference in norm of the NN’s NTK and the initial NTK (aKGD). We see that even the optimistically rescaling of the learning rate by the final NN’s NTK norm does not account for the gap in the loss. B The alignment between the NTK and the task kernel throughout training increases to an asymptote for the neural network but remains constant for the static kernel dynamics (KGD, aKGD). Average and standard deviation over five different initializations are plotted. C The norm of the kernel increases non-monotonically throughout training. D The eigenvalue spectrum for the NTK falls in a power law $\lambda_k \sim k^{-b}$ both before and after training, but with an decreased exponent $b$ after training. E The task power spectra also obey power laws $v_k^2 = (u_k \cdot y)^2 \sim k^{-a}$, again with a larger exponent $a$ after training indicating better alignment. F The alignment between the NTK for output channel 3 increases its alignment with the output labels for task 3 but not for others. This is the phenomenon of specialization, where the kernels for each output channel become aligned only to their respective target function.

2.2 Dynamics of the NTK

Since $K(\theta)$ depends explicitly on the parameters $\theta$, it generally will vary with time during optimization. Explicit differentiation in time $\frac{d}{dt} K(\theta) = \sum_i \frac{\partial K}{\partial \theta_i} \frac{d\theta_i}{dt}$ generates a differential equation which depends on higher derivatives of the neural network output. The quantities $\sum_i \frac{\partial K}{\partial \theta_i} \frac{df^\mu}{d\theta_i}$ are also time varying, and their time derivatives can be computed in terms of even higher derivatives. Repeating this procedure iteratively generates an infinite hierarchy of dynamical systems known as the neural tangent hierarchy [23]. Truncation of this hierarchy after two terms generates the lazy-learning infinite width limit dynamics. Truncation after the first four terms captures all corrections of order $1/N$ (where $N$ is the network width) in the large $N$ limit as was shown by Dyer et al [11]. Such corrections are complicated expressions involving third and fourth derivatives in the parameters $\theta$ and show impressive agreements with finite width network evolution. However, they are difficult to interpret. Our aim is to provide a more interpretable and analytically tractable model of the evolution of features during learning which is agnostic about network width.

3 Kernel alignment

Empirical studies of the NTK during training have revealed that early in training the kernel evolves to become more aligned with the task, giving a lower loss throughout time [12] [14]. The key quantities of interest we aim to track to understand feature evolution in neural networks is the loss $L_t = \sum_\mu (f(x^\mu, \theta) - y^\mu)^2$ and the kernel alignment metric [24]

$$A(t) = \frac{\langle yy^\top, K(\theta) \rangle_F}{\|K(\theta)\|_F \|yy^\top\|_F} = \frac{y^\top K(\theta)y}{\|K(\theta)\|_F \|y\|^2}$$

(4)
We see that, in general, alignment saturates at large time since the final kernel is a balance between
\( \Delta \) when trained on MNIST and CIFAR-10 \[13\]. In Figure 1, we show that alignment between the
kernel and target function increases during training for the neural network (NN) but not for the static
kernel (KGD). In the next sections we examine what increasing alignment demands of the dynamics
and why such dynamics are reasonable from the perspective of optimization. Figure 1C-E show that
the kernel evolves both in norm and in each separate eigenspace. The alignment also exhibits task
specialization as shown in F, which we describe in Sec 6.

4 Optimal Feature Evolution Exhibits Task Alignment

To provide a normative explanation of the kernel alignment phenomenon and explain why it is
reasonable from an optimization perspective, we analyze a learning problem where features are free
to change a small amount between timesteps during gradient descent. At each time, the predictions
on the training set are updated and the features \( \Psi_{t+1} = \frac{\partial L_{t+1}}{\partial \Psi} \) which define the kernel \( K = \Psi^\top \Psi \), are
also updated in the direction of steepest descent of the loss at the next time step

\[
\Psi_{t+1} = \Psi_t - \gamma \frac{\partial L_{t+1}}{\partial \Psi} \; \left| \Psi_t \right|, \quad L_{t+1}(\Psi) = ||(I - \eta \Psi^\top \Psi) \Delta_t||^2
\]

where \( \Delta_t = f_t - y_t \in \mathbb{R}^P \). We analyze a continuous time limit of the induced dynamics in the small
\( \eta \) limit (Appendix A.1). In this continuous time limit we model the dynamics of the residual \( \Delta(t) \) and the feature matrix \( \Psi(t) \in \mathbb{R}^{N \times P} \), giving the following coupled dynamical system

\[
\Delta(t) = -\eta \Psi(t)^\top \Delta(t) \quad \Psi(t) = \gamma \eta \Psi(t) \Delta(t) (\Delta(t))^\top
\]

The parameter \( \gamma \) controls the timescale of feature evolution so we refer to it as the feature learning rate;
in particular the \( \gamma \to 0 \) limit recovers the static kernel limit. Though this is a coupled nonlinear
set of ordinary differential equations, we can exploit a symmetry of the dynamics by noting that

\[
C = \gamma \Delta(t) (\Delta(t))^\top + \Psi(t)^\top \Psi(t) \quad \dot{C} = 0
\]

is a conserved matrix, revealing the elliptical geometry of this feature evolution model (see Appendix
A.1 for scalar quantities \( \Delta, \psi \), the equation \( \psi^2 + \gamma \Delta^2 = C \) defines an ellipse). At initialization the
matrix takes the value \( C = \gamma y y^\top + K_0 \), where \( K_0 \) is the initial kernel matrix. At the end of training
\( t \to \infty \), the loss is zero \( \Delta \sim 0 \). Exploiting this conservation law and the boundary conditions,
we therefore find the relation between the initial and final kernels \( C = K_\infty = \gamma y y^\top + K_0 \). This
allows comparison of the alignment \( A_t \) at initialization and at the end of training

\[
A_0 = \frac{y^\top K_0 y}{||K_0||_F ||y||^2}, \quad A_\infty = \frac{\gamma (y^\top y)^2 + y^\top K_0 y}{||K_0 + \gamma y y^\top||_F ||y||^2}
\]

We see that, in general, alignment saturates at large time since the final kernel is a balance between
the initial kernel \( K_0 \) and the update from the labels \( \gamma y y^\top \). The asymptotic alignment depends on
\( K_0, y \) and the feature learning rate \( \gamma \). As \( \gamma \to \infty \), perfect alignment is obtained \( A_\infty \to 1 \). In reality,
the feature learning rate \( \gamma \) likely depends on details of the network architecture, such as depth and
initial weight variance which facilitate feature learning. Further, we can verify that this dynamics
accelerates learning over the \( \gamma \to 0 \) gradient descent limit

\[
\dot{\Delta} = -\eta (C - \gamma \Delta \Delta^\top) \Delta = -\eta (K_0 + \gamma y y^\top - \gamma \Delta \Delta^\top) \Delta
\]

\[
\dot{\Psi} = \eta \gamma \Psi \left[ K_0 + \gamma y y^\top - K \right]
\]

While \( ||y^\top \Delta|| > ||\Delta^\top \Delta|| \) the loss \( L(t) = ||\Delta(t)||^2 \) decreases more rapidly than it would in the
\( \gamma \to 0 \) kernel limit, by recognizing the kernel as \( K = \Psi^\top \Psi \). This demonstrates a strict advantage
that feature evolution provides during learning. At large time \( t \), the \( \Delta \Delta^\top \) term becomes negligible
compared to the final kernel \( K_\infty = K_0 + \gamma y y^\top \) and the loss essentially evolves linearly with matrix
\( K_\infty \). Thus, the eigenvalues and eigenvectors of \( K_\infty \) become relevant to understanding the large \( t \)
loss dynamics. In the large feature learning limit, \( \gamma \to \infty \), the loss converges exponentially fast since
\( L(t) = ||\Delta(t)||^2 \sim \exp(-\eta \gamma t) \). On the other hand, if \( \gamma \) is small compared to the initial kernel, then
it essentially acts like a rank-1 perturbation to the initial kernel matrix. We see that this perturbation
essentially leaves the power law spectrum \( \lambda_k \sim k^{-b} \) unchanged for large \( k \), but achieves a different
Figure 2: Spectra and eigenvector alignment for final kernel $K_\infty = K_0 + \gamma y y^\top$ for MNIST even-vs-odd task under our optimal feature evolution model. $K_0$ is an infinite width, depth 3 ReLU NTK.

A. As $\gamma$ increases the top few eigenvalues of $K_\infty$ are altered. B. Increasing $\gamma$ also influences the task alignment spectrum $v_k^2 = (u_k^\top y)^2$. C. In particular, the alignment between $y$ and the top eigenvector $u_1$ of $K_\infty$ increases with $\gamma$, approaching 1 as $\gamma \to \infty$. A certain threshold $\gamma^* \approx 0.3$ separates the kernel regime from the strong feature learning regime. D. The final alignment also monotonically increases with feature learning strength $\gamma$. E. During learning, the alignment metric $A(t)$ starts at $A_0$ and monotonically approaches its maximum $A_\infty$. Larger $\gamma$ gives a faster rise and a higher asymptote. F. Increasing the speed of feature evolution accelerates learning. At large $\gamma$, the loss dynamics consist of two stages, first an exponential phase where the loss decays like $\sim (u_1 \cdot y)^2 e^{-2\eta \lambda_1 t}$. This is followed by a power law phase where all smaller eigenvalue modes are learned (Appendix A.1).

$v_k^2 \sim k^{-a'}$ scaling with a new exponent $a' > a$ after feature learning. As a consequence, the large $t$ loss scales like $L_t \sim t^{-(a'-1)/b}$ which possesses a better exponent than the lazy learning limit $\gamma = 0$.

This model allows us to rationalize analytically the two phases of learning during feature evolution as we show in the Appendix A.1, as it predicts exponential decrease in loss at small times followed by power laws at large time. In Figure 2 we show the effect of the feature learning rate $\gamma$ on the dynamics of learning and alignment for a depth-3 ReLU NTK on MNIST [25]. In A-B we show the spectra of the final kernel matrix $C = K_\infty = K_0 + \gamma y y^\top = \sum_k \lambda_k u_k u_k^\top$ and the projection of its eigenvectors $u_k$ onto the labels $y$, giving $v_k^2 = (u_k^\top y)^2$. We observe that only the top few eigenvalues change, but the task spectrum $v_k^2$ is strongly affected by feature evolution: the task spectrum decays much more rapidly with index $k$ for the feature learning models $\gamma > 0$ compared to the frozen NTK $\gamma = 0$ model. In C, we plot the alignment of the target function with the top-principal component $u_1$ as a function of $\gamma$. A sharp transition, reminiscent of the spiked-matrix model, appears which separates a regime with low alignment from a regime with almost perfect overlap with the top eigenfunction $u_1$ with the task $y$ [17, 26]. The final alignment $A_\infty$ also monotonically increases with $\gamma$. In E we demonstrate that the time to alignment also depends on the feature learning rate $\gamma$. Models with larger $\gamma$ converge to their asymptotic alignment more rapidly than models with small $\gamma$. This alignment phase early in training alters the loss dynamics as shown in Figure 2F. A transition from exponential behavior to power law behavior occurs once the asymptotic alignment is achieved.

5 A Simple Model of Task Alignment in Neural Networks

To better understand how kernel-task alignment arises during neural network training dynamics, we consider a simple toy model of a two-layer linear network. For input vector $x_\mu$, its output is given by $r_\mu = N^{-1} V^T W x_\mu$, where $V \in \mathbb{R}^{N \times 1}$, $W \in \mathbb{R}^{N \times N}$. We use gradient descent to solve the
The above analysis gives three insights about kernel alignment. (1) one origin of task alignment in
of dimensions term. 1 eigenfunction of spiked matrix problem (REF) where a rank-one matrix is added to a high-rank bulk. Whether the top
NTK evolution is the gradient for the last-layer readout $V$. Assuming that case where an L2 regularization is applied to all weights, either explicitly or implicitly through SGD $K$ allows us to characterize the NTK at infinite time, $\infty$ is the input covariance matrix and $V\infty$ $\mu,\mu$ $0 \approx \infty \mu \cdot \nabla T$ $x\mu$. This structure allows us to characterize the NTK at infinite time, $K\infty$, given by
\begin{equation}
(K\infty)_{\mu,\mu'} = \nabla V\infty r(x\mu) \cdot \nabla V\infty r(x\mu') + \nabla W\infty r(x\mu) \cdot \nabla W\infty r(x\mu')
= N^{-2} x\mu W\infty W\infty^T x\mu' + N^{-2} \|V\infty\|^2 x\mu^T x\mu'.
\end{equation}
Writing $W\infty = N \|V\infty\|^2 V\infty s^T + W\perp$, where $W\perp V\infty = 0$,
\begin{equation}
(K\infty)_{\mu,\mu'} = \|V\infty\|^2 y\mu y\mu' + N^{-2} x\mu W\perp W\perp x\mu' + N^{-2} \|V\infty\|^2 x\mu^T x\mu'.
\end{equation}
Here, the first term contributes a rank-1 component to $K\infty$ corresponding to the task alignment; the second and third term contributes a high-rank matrix, responsible for "bulk" eigenvalues. In the case where an L2 regularization is applied to all weights, either explicitly or implicitly through SGD dynamics, $W\perp \approx 0$. We thus drop the term quadratic in $W\perp$ and obtain the approximate expression (which we validate numerically in Fig[7]
\begin{equation}
K\infty \approx \|V\infty\|^2 yy^T + \|V\infty\|^2 \Sigma = \|V\infty\|^2 yy^T + \alpha_{K}\|V\infty\|^2 K_0,
\end{equation}
where $\Sigma \in \mathbb{R}^{P \times P}$ is the input covariance matrix and $K_0 \approx \alpha_K \Sigma$ which holds for large $N$. This expression reveals that the final NTK in this toy model resembles the optimal final kernel computed in Sec[4] with $\|V\infty\|^{-\gamma} \alpha_{K}'$ acting like the $\gamma$ parameter. A larger $\|V\infty\|^{-\gamma} \alpha_{K}'$ leads to stronger feature learning. Assuming that $\|V\infty\| \propto \alpha_K$, this conclusion is consistent with the suggestion that smaller weight initialization leads to stronger feature learning [27][28][29].

The above analysis gives three insights about kernel alignment. (1) one origin of task alignment in NTK evolution is the gradient for the last-layer readout $V$. (2) the expression for $K\infty$ resembles a spiked matrix problem (REF) where a rank-one matrix is added to a high-rank bulk. Whether the top eigenfunction of $K\infty$ aligns with $y$ depends on the relative strength of the rank-1 term and the bulk terms. (3) Even at infinite time, kernel alignment does not approach 1, due to presence of the bulk term.

6 Kernel Specialization in Networks with Multiple Outputs

For networks with $C$ output nodes (e.g. $C$-class classification networks), the NTK $K$ is a 4D tensor of dimensions $P \times P \times C \times C$. We define a matrix-valued "subkernel" as
\begin{equation}
K_{cc'} \in \mathbb{R}^{P \times P}: (K_{cc'})_{\mu\mu'} = K_{\mu\mu'cc'}.
\end{equation}
What is the mechanism for kernel specialization? We first asked whether this phenomenon can be demonstrated kernel specialization (Fig.4A). Does kernel specialization occur in neural network learning? Previous empirical studies of multi-output networks studies the "traced kernel", $K_{tr} = C^{-1} \sum_{c=1}^{C} K^{c,c}$, and found that it became aligned with all $C$ target functions \cite{13}. While this observation is consistent with kernel specialization, it can also arise from indiscriminate alignment where $K^{c,c'}$ becomes aligned with all $\{y^{c'}\}_{c'=1,...,C}$. To see whether kernel specialization occurs, we computed a kernel specialization matrix (KSM), defined as

$$KSM(c, c') \equiv \frac{A(K^{c,c}, y^{c'}, y^{c',T})}{C-1 \sum_{c''=1}^{C} A(K^{c'',c'}, y^{c'}, y^{c',T})},$$

If kernel specialization occurs, KSM should be higher when $c = c'$ (i.e. diagonal elements of the KSM). We computed the KSM for a two-layer $N = 500$ MLP trained on 10-class classification of MNIST digits and found that diagonal elements are indeed higher than off-diagonal ones, demonstrating kernel specialization (Fig.4B).

What is the mechanism for kernel specialization? We first asked whether this phenomenon can be reproduced by deep linear models with $C$ output nodes. We assume a typical architecture used for multiclass classification, where the network has $L$ shared hidden layers and $C$ linear readouts from the last hidden layer. We then train the network to minimize $C^{-1} P^{-1} \sum_{c=1}^{C} \sum_{\mu=1}^{P} (r^{c}_\mu - y^{c}_\mu)^2$, where $y^{c}_\mu = s^{cT} x_\mu$. We show that under a weak assumption about symmetry of target functions with respect to $c$, linear networks of arbitrary depth develop specialized kernels(see \ref{A.3}). To test this prediction, we trained a four-layer linear MLP with 10 output nodes on 10 linear target functions and computed its KSM. As predicted, the KSM does not show specialization (Fig.4B).

The lack of kernel specialization in linear networks suggests a crucial role for nonlinearity in feature learning. To understand how nonlinearity begets specialization, we analyzed a two-layer MLP with $C$ output nodes and elementwise nonlinearity $\phi(\cdot)$ in the hidden layer (and no bias for simplicity), output of the $c$th node is $r^{c}_\mu = N^{-1/2} V_c^T \phi(N^{-1/2} W x_\mu)$. The corresponding NTK can be decomposed into two terms:

$$K^{c,c}_{\mu,\mu'} = (K^{c,c}_{V})_{\mu,\mu'} + (K^{c,c}_{W})_{\mu,\mu'}$$

$$K^{c,c}_{\mu,\mu'} \equiv N^{-1} \|V_c\|^2 \phi((N^{-1/2} W x_\mu)^T \phi((N^{-1/2} W x_{\mu'}))$$

$$K^{c,c}_{\mu,\mu'} \equiv N^{-2} x_\mu^T x_{\mu'} [D_1(x_\mu)V_c]^T [D_1(x_{\mu'})V_c]$$
where \( D_1(x) = \nabla_{Wx} \phi(N^{-1/2}Wx) \). Here, while we cannot express \( K_{cc}^{V,V} \) in terms of target functions, the analogy with linear networks suggests that it aligns the kernel with all target functions. That it has no dependence on \( c \) other than an input-independent prefactor precludes it from contributing to kernel specialization. On the other hand, \( K_{cc}^{W,W} \) differs from its linear counterpart in that it has a scalar \( [D_1(x)\hat{V}_c]^T[D_1(x')\hat{V}_c] \), which introduces interaction between \( c \) and \( x, x' \). By exclusion, kernel specialization must arise from \( K_{cc}^{V,V} \).

To test these predictions, we measured alignment between \( K_{3,3}^{V} \) and \( K_{3,3}^{W} \) with \( y_3 \) and other target functions in a two-layer ReLU network trained on MNIST with SGD. As predicted, \( K_{3,3}^{V} \) becomes aligned with all target functions to a similar extent (Fig.5A) while \( K_{3,3}^{W} \) preferentially becomes aligned with \( y_3 \) (Fig.5A). These results show that kernel alignment in multi-output neural networks is driven by two distinctive mechanisms: part of the kernel becomes aligned with all target functions while part of the kernel specializes to fit individual target functions.

Figure 5: Kernel alignment is driven by both indiscriminate alignment and kernel specialization. A. Gradient w.r.t. \( V \) contributes indiscriminate alignment to all target functions. B. Gradient w.r.t. \( W \) causes specialized alignment of individual subkernels. Average of 5 runs; standard error shown.

7 Rate of Feature Learning Increases with Network Depth

In Sec.4 the rate of feature learning is controlled by a parameter \( \gamma \). Larger \( \gamma \) is associated with faster kernel alignment and better final alignment. We next asked how network architecture, specifically depth, impacts the effective \( \gamma \) in neural network training. We trained ReLU MLPs of different depth on the 10-class MNIST classification and tracked kernel alignment over time (Fig.6A). Deeper networks is associated with both faster alignment during training and better alignment after training, consistent with a higher feature learning rate in .

To quantify how quickly each architecture learns features, we estimated the effective \( \gamma \) in each of the architecture. To do that, we first rewrite the final kernel alignment after optimal feature evolution (discussed in Sec. as

\[
A(K_\infty, yy^T) = \frac{(yy^T, K_\infty)}{||K_\infty||^2} = \frac{K_\infty \cdot yy^T + \gamma ||y||^4}{||y||^2 \sqrt{||K_0||^2 + 2 \gamma (yy^T, K_0)^F + \gamma^2 ||y||^4}}.
\]

All variables other than \( \gamma \) on the R.H.S. can be computed using \( K_0 \). We then solve this equation for \( \gamma \) (which can be shown to always have a real, positive root if \( A(K_\infty, yy^T) > A(K_0, yy^T) \)) to get an estimate of \( \gamma \) in neural networks. To ensure fair comparison between architectures, we divide \( K_0, K_\infty \) of each architecture by \( ||K_0|| \). The results for MLPs of different depth trained on 10-class MNIST are shown in Fig.6D. Consistent with the observations in Fig.6A, deeper networks have larger estimated \( \gamma \), suggesting that their features evolve more rapidly during training and achieve higher alignment.

8 Conclusion

This work demonstrated the role of kernel alignment in learning dynamics, both through experiment and theory. We showed empirically that learning is accelerated through rapid feature alignment early in training and that this acceleration cannot be accounted for by a simple increase in the scale of the
Figure 6: Depth Increases the Feature Learning Rate and Specialization. A The alignment curves for ReLU networks of varying depth. Deeper models reach a higher alignment value and begin increasing alignment earlier. This is consistent with our optimal evolution model, where larger feature learning rate $\gamma$ causes faster alignment dynamics and also a higher asymptote. B The cross-class alignment matrix between kernel for class $c$ and task $c'$ for a 3-layer network. C The same is plotted for a depth 5-layer network. The deeper network achieves a greater degree of specialization. (Compare B and C with Fig.4A) D Estimated feature learning rate ($\gamma$) in MLPs of different depth. Deeper networks have higher $\gamma$, suggesting that their features evolve faster.

kernel over time, but is a consequence of the top eigensystem of the kernel evolving to match the task. Varying the depth allowed us to demonstrate that architectural features in the neural network control the rate of feature learning and consequently alter the kernel alignment curve. We identified the new phenomenon of kernel specialization, where the kernel for each output channel aligns preferentially to its own target function. We show that such alignment does not occur in linear networks but happens rapidly in non-linear networks. A normative model of feature evolution, where features are updated to to reduce the predicted error one step in advance reproduces many of the empirical phenomena we documented above. First, we find that the feature learning rate controls the asymptotic value of the alignment metric as well as the dynamics as well as the timescale of feature learning: faster feature learning implies higher asymptotic alignment. We also theoretically study the final kernel alignment of linear networks and show that their feature learning rate depend on depth and initial weight scale. The final kernel for two layer linear networks indicates. Lastly, in a theoretical analysis of the specialization phenomenon we demonstrate why nonlinearity is necessary to account for the class-specific alignment of sub-blocks of the rank-4 kernel tensor with the target functions corresponding to each class.

Limitations

A primary limitation of the present study is the focus on the mean-square error loss. Though this loss is especially tractable to analyze theoretically, an ideal theory of kernel evolution should extend to more commonly used losses for supervised learning (eg cross-entropy). We expect that richer phenomena, such as cross-class interference in the rank-4 kernel tensor could emerge from the cross-entropy loss during the feature evolution phase. In addition, our optimal feature evolution model does not explain mechanically how a neural network’s gradient features could evolve to have high alignment with the target function throughout training. Though our asymptotic alignment matches the value predicted for linear networks, studying how the dynamics of a NTK differs from this optimal evolution dynamics is an interesting future direction.
References

[1] Yann LeCun, Yoshua Bengio, and Geoffrey Hinton. Deep learning. nature, 521(7553):436–444, 2015.

[2] Arthur Jacot, Franck Gabriel, and Clément Hongler. Neural tangent kernel: Convergence and generalization in neural networks, 2020.

[3] Jaehoon Lee, Lechao Xiao, Samuel S Schoenholz, Yasaman Bahri, Roman Novak, Jascha Sohl-Dickstein, and Jeffrey Pennington. Wide neural networks of any depth evolve as linear models under gradient descent. Journal of Statistical Mechanics: Theory and Experiment, 2020(12):124002, Dec 2020.

[4] Lenaic Chizat, Edouard Oyallon, and Francis Bach. On lazy training in differentiable programming, 2020.

[5] Alberto Bietti and Julien Mairal. On the inductive bias of neural tangent kernels, 2019.

[6] Greg Yang and Hadi Salman. A fine-grained spectral perspective on neural networks, 2020.

[7] Blake Bordelon, Abdulkadir Canatar, and Cengiz Pehlevan. Spectrum dependent learning curves in kernel regression and wide neural networks. In International Conference on Machine Learning, pages 1024–1034. PMLR, 2020.

[8] Yasaman Bahri, Ethan Dyer, Jared Kaplan, Jaehoon Lee, and Utkarsh Sharma. Explaining neural scaling laws. arXiv preprint arXiv:2102.06701, 2021.

[9] Stefano Spigler, Mario Geiger, and Matthieu Wyart. Asymptotic learning curves of kernel methods: empirical data versus teacher–student paradigm. Journal of Statistical Mechanics: Theory and Experiment, 2020(12):124001, Dec 2020.

[10] Joel Hestness, Sharan Narang, Newsha Ardalani, Gregory Diamos, Heewoo Jun, Hassan Kianinejad, Md. Mostofa Ali Patwary, Yang Yang, and Yanqi Zhou. Deep learning scaling is predictable, empirically, 2017.

[11] Ethan Dyer and Guy Gur-Ari. Asymptotics of wide networks from feynman diagrams. arXiv preprint arXiv:1909.11304, 2019.

[12] Mario Geiger, Stefano Spigler, Arthur Jacot, and Matthieu Wyart. Disentangling feature and lazy training in deep neural networks. Journal of Statistical Mechanics: Theory and Experiment, 2020(11):113301, nov 2020.

[13] Aristide Baratin, Thomas George, César Laurent, R Devon Hjelm, Guillaume Lajoie, Pascal Vincent, and Simon Lacoste-Julien. Implicit regularization via neural feature alignment. In Arindam Banerjee and Kenji Fukumizu, editors, Proceedings of The 24th International Conference on Artificial Intelligence and Statistics, volume 130 of Proceedings of Machine Learning Research, pages 2269–2277. PMLR, 13–15 Apr 2021.

[14] Stanislav Fort, Gintare Karolina Dziugaite, Mansheej Paul, Sepideh Kharaghani, Daniel M. Roy, and Surya Ganguli. Deep learning versus kernel learning: an empirical study of loss landscape geometry and the time evolution of the neural tangent kernel, 2020.

[15] Guillaume Leclerc and Aleksander Madry. The two regimes of deep network training, 2020.

[16] Stanislaw Jastrzębski, Maciej Szymczak, Stanislaw Fort, Devansh Arpit, Jacek Tabor, Kyunghyun Cho*, and Krzysztof Geras*. The break-even point on optimization trajectories of deep neural networks. In International Conference on Learning Representations, 2020.

[17] Amelia Perry, Alexander S. Wein, Afonso S. Bandeira, and Ankur Moitra. Optimality and sub-optimality of PCA I: Spiked random matrix models. The Annals of Statistics, 46(5):2416 – 2451, 2018.

[18] Pedro Domingos. Every model learned by gradient descent is approximately a kernel machine, 2020.

[19] Sanjeev Arora, Simon S Du, Wei Hu, Zhiyuan Li, Ruslan Salakhutdinov, and Ruosong Wang. On exact computation with an infinitely wide neural net. arXiv preprint arXiv:1904.11955, 2019.

[20] Chaoyue Liu, Libin Zhu, and Misha Belkin. On the linearity of large non-linear models: when and why the tangent kernel is constant. In H. Larochelle, M. Ranzato, R. Hadsell, M. F. Balcan, and H. Lin, editors, Advances in Neural Information Processing Systems, volume 33, pages 15954–15964. Curran Associates, Inc., 2020.
[21] Maksim Velikanov and Dmitry Yarotsky. Universal scaling laws in the gradient descent training of neural networks, 2021.

[22] Roman Novak, Lechao Xiao, Jiri Hron, Jaehoon Lee, Alexander A. Alemi, Jascha Sohl-Dickstein, and Samuel S. Schoenholz. Neural tangents: Fast and easy infinite neural networks in python. In International Conference on Learning Representations, 2020.

[23] Jiaoyang Huang and Horng-Tzer Yau. Dynamics of deep neural networks and neural tangent hierarchy. In Hal Daumé III and Aarti Singh, editors, Proceedings of the 37th International Conference on Machine Learning, volume 119 of Proceedings of Machine Learning Research, pages 4542–4551. PMLR, 13–18 Jul 2020.

[24] Corinna Cortes, Mehryar Mohri, and Afshin Rostamizadeh. Algorithms for learning kernels based on centered alignment. J. Mach. Learn. Res., 13(1):795–828, March 2012.

[25] Yann LeCun and Corinna Cortes. MNIST handwritten digit database. 2010.

[26] Gérard Ben Arous, Song Mei, Andrea Montanari, and Mihai Nica. The landscape of the spiked tensor model. Communications on Pure and Applied Mathematics, 72(11):2282–2330, 2019.

[27] Lénaïc Chizat, Edouard Oyallon, and Francis Bach. On lazy training in differentiable programming. In H. Wallach, H. Larochelle, A. Beygelzimer, F. d’Alché-Buc, E. Fox, and R. Garnett, editors, Advances in Neural Information Processing Systems, volume 32. Curran Associates, Inc., 2019.

[28] Song Mei, Theodor Misiakiewicz, and A. Montanari. Mean-field theory of two-layers neural networks: dimension-free bounds and kernel limit. In COLT, 2019.

[29] Greg Yang and Edward J. Hu. Feature learning in infinite-width neural networks, 2021.

[30] Andrew M. Saxe, James L. McClelland, and Surya Ganguli. Exact solutions to the nonlinear dynamics of learning in deep linear neural networks. In Yoshua Bengio and Yann LeCun, editors, 2nd International Conference on Learning Representations, ICLR 2014, Banff, AB, Canada, April 14-16, 2014, Conference Track Proceedings, 2014.
A Appendix

A.1 Optimal Feature Evolution Induces Kernel Alignment

Let $\Delta = f - y$ and let $\Psi \in \mathbb{R}^{N \times P}$ represent the feature matrix whose inner product gives the kernel $K = \Psi^{\top} \Psi$. We will first discuss a discrete time dynamical system before taking a gradient flow limit. Thus, we index $\Delta_t$ as the error at time $t$ and $\Psi_t$ as the features at time $t$. We consider the following simulataneous updates to $\Psi_t$ and $\Delta_t$

$$
\Delta_{t+1} (\Psi_t) = \Delta_t - \eta \Psi_t^\top \Psi_t \Delta
$$

$$
\Psi_{t+1} = \Psi_t - \gamma \eta \frac{\partial}{\partial \Psi_t} ||\Delta_{t+1}(\Psi_t)||^2 = \Psi_t - \frac{1}{2} \eta \gamma \frac{\partial}{\partial \Psi_t} ||\Delta_t - \eta \Psi_t^\top \Psi_t \Delta_t||^2
$$

(21)

Expanding the last term and computing the derivative gives

$$
\frac{1}{2} \frac{\partial}{\partial \Psi_t} ||\Delta_t - \eta \Psi_t^\top \Psi_t \Delta_t||^2 = \frac{1}{2} \frac{\partial}{\partial \Psi_t} \left[ ||\Delta_t||^2 - 2 \Delta_t^\top \Psi_t \Delta_t + \Delta_t^\top \Psi_t^\top \Psi_t \Delta_t \right] = -\eta \Psi_t \Delta_t^\top + \eta^2 \Psi_t \Psi_t^\top \Delta_t + \eta \Psi_t \Delta_t^\top + O(\eta^2)
$$

(22)

Taking the $\eta \to 0$ limit while taking the distance in time between adjacent steps to zero, we find the following gradient flow dynamics

$$
\dot{\Delta}(t) = -\eta \Psi(t) \Psi(t) \Delta(t) \quad \dot{\Psi}(t) = \gamma \eta \Psi(t) \Delta(t) \Delta(t)^\top
$$

(23)

This is a collection of coupled nonlinear ODEs. The $\gamma \to 0$ limit recovers lazy learning where the features do not evolve. Increasing $\gamma$ increases the rate at which features evolve, thus we name it the feature learning rate. Despite the nonlinearity, we will attempt to solve these equations to gain insight into such optimal feature updates. The key trick is that the equations can be decoupled through the use of a conservation law. To motivate the conservation law, consider the scalar version of these differential equations

$$
\dot{\psi} = -\psi^2 \Delta \quad \dot{\psi} = \gamma \psi \Delta^2
$$

(24)

Note that $\frac{1}{2} \frac{d}{dt} \Delta^2 = -\Delta^2 \psi^2$ while $\frac{1}{2} \frac{d}{dt} \psi^2 = \gamma \psi^2 \Delta^2$. A particular linear combination of these terms reveals a conservation law

$$
\frac{\gamma}{2} \frac{d}{dt} \Delta^2 + \frac{1}{2} \frac{d}{dt} \psi^2 = \frac{1}{2} \frac{d}{dt} \left[ \gamma \Delta^2 + \psi^2 \right] = -\gamma \Delta^2 \psi^2 + \gamma \Delta^2 \psi^2 = 0
$$

(25)

Thus $\gamma \Delta^2 + \psi^2$ is a conserved quantity throughout the dynamics. This indicates that, in $(\Delta, \psi)$ space, the trajectory can only move along an ellipse, with the ratio of axis lengths determined by $\sqrt{\gamma}$. Using the conservation law, we can introduce a constant $C = \gamma \Delta^2 + \psi^2 = \gamma \Delta_0^2 + \psi_0^2$, where $\Delta_0$ and $\psi_0$ are the initial values. We note the similarity between these elliptical differential equations and the hyperbolic geometry of gradient descent in deep linear neural networks [30], where the conservation laws have the form $a^2 - b^2 = C$. Using our discovered elliptical conservation law, the differential equations can now be decoupled

$$
\dot{\Delta} = -(C - \gamma \Delta^2) \Delta \quad \dot{\psi} = \psi(C - \psi^2)
$$

(26)

Letting $u = \frac{1}{2} \Delta^2$ and $v = \frac{1}{2} \gamma^2$, we find $\dot{u} = -u(C - \gamma u)$ and $\dot{v} = v(C - v)$, which give solutions of the form

$$
u = \frac{CA}{A + e^{2Ct}} \quad v = \frac{CB}{B + e^{-2Ct}}
$$

(27)

for constants $A$ and $B$ determined by the initial conditions. This indicates that the loss and the kernel power increases as logistic functions. Since $u$ represents the loss, this indicates that at large times, a scaling of $u \sim \exp(-2(\psi_0^2 + \gamma \Delta_0^2)v)$ is obtained, which improves with increasing $\gamma$. 

12
The scalar case was illuminating since it allowed us to identify a conservation law and solve the differential equation. We now aim to extend this argument to arbitrary dimensional matrices $\Psi(t) \in \mathbb{R}^{N \times d}$ and vectors $\Delta(t) \in \mathbb{R}^d$. Inspired by the elliptical geometry in the scalar case, we make the following ansatz that $C = \gamma \Delta \Delta^\top + \Psi^\top \Psi$ is conserved. Indeed, explicit differentiation reveals this to be the case.

\[
\frac{d}{dt} \left[ \gamma \Delta \Delta^\top + \Psi^\top \Psi \right] = -\eta \gamma \Psi^\top \Psi \Delta \Delta^\top - \eta \gamma \Delta \Delta^\top \Psi^\top \Psi + \eta \gamma \Delta \Delta^\top \Psi^\top \Psi + \eta \gamma \Psi^\top \Psi \Delta \Delta^\top = 0
\]  

(28)

Thus $C = \gamma \Delta \Delta^\top + \Psi^\top \Psi$ is a conserved matrix. We can use this fact to again decouple the dynamics

\[
\dot{\Delta} = -\eta (C - \gamma \Delta \Delta^\top) \Delta \quad \dot{\Psi} = \eta \Psi(C - \Psi^\top \Psi)
\]  

(29)

Positive $\gamma$ has the effect of accelerating convergence of $\Delta$ to zero, while the initial condition and final conditions can be explicitly related $C = K_{\infty} = \gamma y y^\top + \Psi_0^\top \Psi_0$, demonstrating that the kernel will align more with the labels after training. The dynamics of the loss and the kernel can be examined in the eigenbasis of $C$. Let $C = K_{\infty} = \sum_k c_k v_k v_k^\top$ and let $\Delta = \sum_k \delta_k v_k$ and $K = \Psi^\top \Psi = \sum_{k,\ell} A_{k,\ell} v_k v_\ell^\top$ for symmetric matrix $A$. This generates the following differential equations

\[
\frac{d}{dt} \ln \delta_k = -\eta c_k + \eta \gamma \sum_\ell \delta_\ell^2 \\
\dot{A}_{k,\ell} = \eta A_{k,\ell} (c_k + c_\ell) - 2\eta \sum_j A_{k,j} A_{j,\ell}
\]  

(30)

To zero-th order in $\gamma$, the loss scales like $L_t = \sum_k (v_k^\top y)^2 \exp(-2\eta c_k t)$, which in general will decay more quickly than the loss for the frozen kernel, since $K_{\infty}$ is more aligned with $y$ than $K_0$. When $\gamma$ is small but non-negligible, we expect $(v_k^\top y)^2 \gg (v_\ell^\top y)^2$ for $k > 1$. We thus get a loss that looks like $L_t = D e^{-2\eta c_1 t} + \sum_{k > 1} (v_k^\top y)^2 \exp(-2\eta c_k t)$. The first term dominates at small times since it has a large prefactor constant $D = (v_1^\top y)^2$, however once $t \approx 1/c_1$, the tail sum dominates and the loss falls as a power law, with a possibly improved exponent due to better alignment.

Now, let’s consider the kernel’s dynamics. First, at small times $A_{k,\ell}$ is non-diagonal since $K_0$ and $K_{\infty}$ do not necessarily commute. These off diagonal terms will eventually decay due to the $-\sum_j A_{k,j} A_{j,\ell}$ term. Once $A$ is approximately diagonal, the dynamics for the diagonal terms are

\[
\dot{A}_{k,k} = 2\eta A_{k,k} c_k - 2\eta A_{k,k}^2
\]

(31)

This is identical to the scalar equations studied above which we can solve exactly

\[
A_{k,k}(t) = \frac{B_k c_k}{B_k + e^{-2\eta c_k t}}
\]

(32)

for some constants $B_k$ determined by the initial values $A_{k,k}(0)$. Thus, $A_{k,k}(t)$ increase as logistic functions with a time constant given by $c_k$. Thus, the kernel is approximately

\[
K(t) \sim \sum_k A_{k,k}(t) v_k v_k^\top
\]

(33)

which gives an alignment of

\[
\langle y y^\top, K(t) \rangle = \frac{1}{\gamma} \langle K_{\infty} - K_0, K(t) \rangle_F = \frac{1}{\gamma} \sum_k (c_k - A_{k,k}(0)) A_{k,k}(t)
\]

(34)

which increases as a weighted sum of logistic functions. The norm of the kernel grows as $||K(t)||_F^2 = \sum_k A_{k,k}(t)^2$ so the alignment curve has the form

\[
A(t) = \frac{\sum_k (c_k - A_{k,k}(0)) A_{k,k}(t)}{\sqrt{\sum_k A_{k,k}(t)^2} \sqrt{||K_0||_F^2 - 2 \sum_k c_k A_{k,k}(t) + \sum_k A_{k,k}(t)^2}}
\]

(35)
We derived a general expression for \( K^{c,c'}(x, x') \), defined in Eq\[15\] for networks of any depth and show that it cannot show kernel specialization under an assumption of symmetry between target functions. Define \( f_l(x) = W_l^T W_{l-1}^T \ldots W_1^T x \). Then

\[
N^{L+1} K^{c,c'}(x, x') = \nabla_{\Theta} r(c)(x)^T \nabla_{\Theta} r(c')(x')
\]

\[
= \delta(c - c') f_L(x)^T f_L(x')
\]

\[
+ V^{cT} W_L^T W_{L-1}^T V^{c'} f_{L-1}(x)^T f_{L-1}(x')
\]

\[
+ V^{cT} W_L^T W_{L-1}^T W_{L-2}^T W_{L-2}^T V^{c'} f_{L-2}(x)^T f_{L-2}(x')
\]

\[
+ V^{cT} W_L^T W_{L-1}^T W_{L-2}^T W_{L-2}^T W_{L-3}^T W_{L-3}^T V^{c'} f_{L-3}(x)^T f_{L-3}(x')
\]

\[
\ldots
\]

Defining scalar functions for \( l < L \)

\[
\alpha_l(c, c') \equiv V^{cT} W_L^T W_{L-1}^T \ldots W_{l+1}^T W_{l+1}^T \ldots W_{L-1}^T W_L^T V^{c'},
\]

one has

\[
N^{L+1} K^{c,c'}(x, x') = \delta(c - c') f_L(x)^T f_L(x') + \sum_{l=0}^{L-1} \alpha_l(c, c') f_l(x)^T f_l(x').
\]

It is thus a weighted sum of covariance of activations in all layers and the input. We make a class-symmetry ansatz that

\[
\forall l, c, c': \alpha_l(c, c) = \alpha_l(c', c') = \tilde{\alpha}_l.
\]

To see why this ansatz is reasonable, define \( \tilde{V}_l^c \equiv W^{l+1T} \ldots W_{L-1}^T W_{L-1}^T \ldots W_{l+1}^T W_{l+1}^T \ldots W_l^T W_l^T V^c \). \( \alpha_l(c, c) = ||\tilde{V}_l^c||^2 \); after learning, \( y(c)(x) = r(c)(x) = N^{-(L+1)/2} (W^{l+1T} \ldots W_{L-1}^T W_{L-1}^T \ldots W_{l+1}^T W_{l+1}^T \ldots W_l^T W_l^T V^c)^T f_l(x) \). We then assume the covariance of \( f_l(x) \) projected along the direction of \( V^c \) to be approximately the same across \( c \) and that \( r(c)(x) \) to have approximately the same variance. This would suggest that \( \tilde{V}_l^c \) should have the same norm across \( c \).

Under the class-symmetry ansatz,

\[
N^{L+1} K^{c,c'}(x, x') = f_L(x)^T f_L(x') + \sum_{l=0}^{L-1} \tilde{\alpha}_l f_l(x)^T f_l(x')
\]

does not have \( c \) dependence and thus cannot specialize.
A.4 Experimental Details

We train our models on a Google Colab GPU and include code to reproduce all experimental results in the supplementary materials. To match our theory, we use fixed learning rate SGD. Both evaluation of the infinite width kernels and training were performed with the Neural Tangents API [22]. In Figure 2 we use the infinite width kernel on 1000 MNIST even-odd digits, encoding the output as a single binary variable. In all other experiments, we used one-hot labels for the 10 different classes.