Infinitesimal Characterizations for Strong Invariance and Monotonicity for Non-Lipschitz Control Systems

Michael Malisoff
Department of Mathematics
Louisiana State University
Baton Rouge, LA 70803-4918
malisoff@lsu.edu

August 6, 2018

Abstract
We provide new infinitesimal characterizations for strong invariance of multifunctions in terms of Hamiltonian inequalities and tangent cones. In lieu of the standard local Lipschitzness assumption on the multifunction, we assume a new feedback realizability condition that can in particular be satisfied by control systems that are discontinuous in the state variable. Our realization condition is based on H. Sussmann’s unique limiting property, and allows a more general class of feedback realizations than is allowed by the recent strong invariance characterizations [16]. We also give new nonsmooth monotonicity characterizations for control systems that may be discontinuous in the state.

Key Words: strong invariance, monotone control systems, nonsmooth analysis

1 Introduction
The theory of flow invariance plays an important role in much of modern control theory and optimization (see [1, 9, 10, 15, 22, 23]). For a given set valued dynamics $F$ evolving on $\mathbb{R}^n$ and a subset $S \subseteq \mathbb{R}^n$, the theory provides necessary and sufficient conditions under which the pair $(F, S)$ is strongly invariant, meaning, for each $T > 0$ and each trajectory $y : [0, T] \rightarrow \mathbb{R}^n$ of $F$ starting at a point in $S$ we have $y(t) \in S$ for all $t \in [0, T]$. For the special case where $F$ is locally Lipschitz and $S \subseteq \mathbb{R}^n$ is closed, infinitesimal characterizations for strong invariance are well known. For example, if $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is locally Lipschitz and nonempty, compact, and convex valued with linear growth and $S \subseteq \mathbb{R}^n$ is closed, then it is well known (cf. [9, Chapter 4]) that $(F, S)$ is strongly invariant if and only if $F(x) \subseteq T_S^C(x)$ for all $x \in S$, where $T_S^C$ denotes the Clarke tangent cone (cf. Section 2 and Appendix A for the relevant definitions). However, this cone characterization can fail if $F$ is non-Lipschitz as illustrated in the following simple example: Take $n = 1$, $S = \{0\}$, $F(0) = [-1, +1]$, and $F(x) = \{-\text{sign}(x)\}$ for $x \neq 0$. Then $T_S^C(0) = \{0\}$, so $F(0) \not\subseteq T_S^C(0)$. However, $(F, S)$ is strongly invariant. This example is covered by the main sufficient conditions for strong invariance in [16, 18].

On the other hand, consider the controlled differential inclusion

$$\dot{x} \in G(x, \alpha) := \prod_{i=1}^n g_i(x, \alpha_i) D_i(x)$$

where each factor $g_i : \mathbb{R}^n \times A \rightarrow \mathbb{R} : (x, a) \mapsto g_i(x, a)$ is locally Lipschitz, $A \subseteq \mathbb{R}^m$ is compact, $\alpha_i \in A := \{\text{measurable } [0, \infty) \rightarrow A\}$ for each $i$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, and each of the multifunctions

---

*The author thanks Eduardo Sontag for suggesting the problems addressed in this work and Peter Wolenski for commenting on an earlier draft. Supported by Louisiana Board of Regents Support Fund Grant LEQSF(2003-06)-RD-A-12.
\[ D_i : \mathbb{R}^n \rightarrow \mathbb{R} \] is Borel measurable. (Throughout this note, \( \prod_{i=1}^n S_i = S_1 \times \cdots \times S_n \) for subsets \( S_i \subseteq \mathbb{R} \) and \( \prod_{i=1}^n s_i = (s_1, \ldots, s_n) \) for points \( s_i \in \mathbb{R} \).) The dynamics (1) can be viewed as an uncontrolled differential inclusion \( \dot{x} \in F(x) \) by taking \( F(x) = \cup \{ G(x, a) : a \in A^n \} \). By definition, the trajectories of (1) are those absolutely continuous functions \( y : [0, T] \rightarrow \mathbb{R}^n \) defined for some \( T > 0 \) and inputs \( a(t) \in A \) that satisfy \( y(t) \in G(y(t), a(t)) \) for (Lebesgue) almost all (a.a.) \( t \in [0, T] \). The \( D_i \)'s can be interpreted as set valued state dependent disturbance perturbations acting on the individual components of the locally Lipschitz dynamics \( g = (g_1, g_2, \ldots, g_n) \). With this interpretation, the values \( t \mapsto \beta_i(t) \in D_i(y(t)) \) assumed by the disturbances are unknown to the controller; one only knows that each \( \beta_i(t) \) takes some value in \( D_i(y(t)) \) for each \( t \). However, the mappings \( g_i \) and \( D_i \), the inputs \( t \mapsto \alpha_i(t) \in A \), and the current state \( t \mapsto y(t) \) can be measured. The dynamics (1) include the example from the previous paragraph by taking \( n = 1 \) and \( g_1 \equiv 1 \). The objective is to find sufficient conditions in terms of the \( g_i \)'s and \( D_i \)'s under which all trajectories of (1) starting in a given closed set \( S \subseteq \mathbb{R}^n \) remain in \( S \), i.e., such that \( (G, S) \) is strongly invariant. Since the \( D_i \)'s are not necessarily Lipschitz (or even continuous), the dynamics \( G \) may be discontinuous in the state, so the usual strong invariance criteria for locally Lipschitz systems (cf. \cite{1, 9, 22}) do not apply. Moreover, the dynamics (1) are not in general tractable by the strong invariance results from \cite{16}, even if the \( D_i \)'s are singleton valued. For example, take \( n = 2 \), \( g_1 \equiv g_2 \equiv 1 \), \( D_1(x) \equiv \{ 1 \} \), \( D_2(x) = \{ 1 \} \) if \( x_2 \geq 0 \), and \( D_2(x) = \{ 10 \} \) if \( x_2 < 0 \). In this case, if \( f \) were a feedback realization for the \( G \)-trajectory \( y(t) = (t, t) \) satisfying the requirements of \cite{16} (cf. Section 3.3 for the relevant definitions), then \( (1, 1) = f(t, (t, t)) \) for all \( t \geq 0 \). Since \( f \) is continuous in the state variable, we can find \( \gamma > 0 \) such that \( f(t, x) \in [1/2, 3/2]^2 \) for all \( t \geq 0 \) and \( \| x \| \leq \gamma \). In particular if \( t \geq 0 \), \( \| x \| \leq \gamma \), and \( x_2 < 0 \), then \( f(t, x) \not\in \text{cone}(D_1(x) \times D_2(x)) \) is \( \{ \lambda(1, 10) : \lambda \geq 0 \} \), so the cone requirement on \( f \) from \cite{16} cannot be satisfied. We prove strong invariance results for (1) in Section 4.

Strong invariance theorems are of great independent interest because they have been applied in many areas of nonlinear analysis and dynamical systems theory. For some applications, it suffices to have characterizations of weak invariance, which is the less restrictive requirement that for each point \( \bar{x} \) in the constraint set \( S \), there exists at least one trajectory \( y \) of the dynamics starting at \( \bar{x} \) such that \( y(t) \in S \) for all \( t \geq 0 \). While weak invariance characterizations have been shown under very general assumptions on the dynamics (e.g., locally bounded convex values and closed graph), the standard results on strong invariance generally require locally Lipschitz multifunctions. Strong invariance theorems have important applications in uniqueness and regularity theory for solutions of Hamilton-Jacobi-Bellman equations, stability theory, differential games, monotone systems in biology, and elsewhere (cf. \cite{1, 2, 3, 9, 10, 22}). On the other hand, it is well appreciated that many important dynamics such as (1) may be non-Lipschitz or even discontinuous in the state and so are beyond the scope of the usual strong invariance methods. Therefore, the development of conditions guaranteeing strong invariance under less restrictive assumptions on the dynamics is a problem that is of considerable ongoing research interest.

This motivates the search for new infinitesimal characterizations for strong invariance for non-Lipschitz differential inclusions which is the focus of this note. (Here and in the sequel, “non-Lipschitz” means “not necessarily locally Lipschitz in the state variable”.) Donchev, Rios, and Wolenski \cite{12, 18} recently proved strong invariance characterizations under the somewhat less restrictive structural assumption of one sided Lipschitzness. A completely different approach was pursued by Krastanov, Malisoff, and Wolenski \cite{16} who gave a new Hamiltonian sufficient condition for strong invariance for a class of feedback realizable differential inclusions (see Section 3.3 for the relevant definitions). The results in \cite{16} do not require any of the usual structural assumptions on the dynamics that are generally needed in strong invariant systems theory, and therefore can be applied to a more general class of systems. However, \cite{16} requires a cone condition on the feedback realizations that is not in general satisfied for the dynamics we consider below (see Section 3.3 for more discussions on \cite{16}).

In this note, we provide a nontrivial extension of \cite{16} by proving strong invariance under an alternative feedback realizability condition that can in particular be satisfied by the general non-Lipschitz system (1) and other examples that are not tractable by known strong invariance results. Our condition is based on Malisoff’s coercive upper envelope approach from \cite{17} and Sussmann’s unique limiting condition from \cite{21}. We express our invariance results in terms of tangent cones and Hamiltonian inequalities. Our approach has the additional advantage over \cite{16} that it is preserved under “stacking” in the following
sense: If two dynamics \( \dot{y}_1 \in G_1(y_1, \alpha_1) \) and \( \dot{y}_2 \in G_2(y_2, \alpha_2) \) satisfy our feedback condition, then so does the “stacked” dynamic \( (\dot{y}_1, \dot{y}_2) \in G_1(y_1, \alpha_1) \times G_2(y_2, \alpha_2) \). The realizability condition in [16] is not preserved under “stacking”. Starting from this “stacking” property, our results lead to new infinitesimal monotonicity characterizations (see Section 5 below). Moreover, our results can still be applied even when the constraint set \( S \) is not necessarily a closed subset of \( \mathbb{R}^n \) (cf. Section 3.2 below). While our main theorem can be shown using Zorn’s Lemma, the proof we give below is constructive, and in particular leads to a new approach to building viable trajectories for Carathéodory dynamics; see Remark B.1.

This note is organized as follows. In Section 2, we state our feedback realizability property precisely and illustrate its applicability to dynamics such as (1) that are not tractable by the known strong invariance theory. In Section 3, we state our new necessary and sufficient conditions for strong invariance and we explain in detail how our conditions improve on the known results. We prove our main results in Section 4. Section 5 shows how our invariance characterizations lead to new nonsmooth monotonicity characterizations for systems with set valued state dependent non-Lipschitz disturbances. We review the relevant arguments from [16] and background from nonsmooth analysis in the appendices.

2 Feedback Realizability Hypothesis and Examples

Our main object of study is an autonomous differential inclusion \( \dot{x} \in F(x) \). This includes measurable controlled differential inclusions \( \dot{x} \in G(x, a) \) with control constraints \( a \in A \) by taking \( F(x) := G(x, A) \).

By a trajectory of \( \dot{x} \in F(x) \) on an interval \([0, T]\) starting at a point \( \bar{x} \in \mathbb{R}^n \), we mean an absolutely continuous function \( y : [0, T] \to \mathbb{R}^n \) for which \( y(0) = \bar{x} \) and \( \dot{y}(t) \in F(y(t)) \) for (Lebesgue) almost all \((a.a.) \) \( t \in [0, T] \).

We let \( \text{Traj}(F, \bar{x}) \) denote the set of all trajectories \( y : [0, T] \to \mathbb{R}^n \) for \( F \) starting at \( \bar{x} \) for each \( T > 0 \), and \( \text{Traj}(F, \bar{x}) := \bigcup_{T \geq 0} \text{Traj}(F, \bar{x}) \) and \( \text{Traj}(F) := \bigcup_{\bar{x} \in \mathbb{R}^n} \text{Traj}(F, \bar{x}) \). For a control system \( \dot{x} \in G(x, A) \), we let \( \text{Traj}(G, \alpha, \bar{x}) \) denote the set of all trajectories \( y : [0, T] \to \mathbb{R}^n \) for \( G(\cdot, \alpha(t)) \) starting at \( \bar{x} \) for each \( T > 0 \) and \( \alpha \in A \) := \{ measurable \( \alpha : [0, T] \to A \} \). In that case, we also set \( \text{Traj}(G, \alpha, \bar{x}) = \bigcup_{T \geq 0} \text{Traj}(G, \alpha, \bar{x}) \) and \( \text{Traj}(G, \bar{x}) = \bigcup_{\alpha \in A} \text{Traj}(G, \alpha, \bar{x}) \). For \( n > 1 \) (respectively, \( n = 1 \)), a mapping \( G \) defined on a Borel subset of \( \mathbb{R}^n \) is said to be measurable provided \( G \) is Borel (respectively, Lebesgue) measurable.

A multifunction \( F : \mathbb{R}^n \Rightarrow \mathbb{R}^n \) is said to have linear growth provided there exist positive constants \( c_1 \) and \( c_2 \) such that \( ||v|| \leq c_1 + c_2 ||x|| \) for all \( v \in F(x) \) and \( x \in \mathbb{R}^n \), where \( || \cdot || \) denotes the Euclidean supremum norm. For any interval \( I \), a function \( f : I \times \mathbb{R}^n \to \mathbb{R}^n \) is said to have linear growth (on \( I \)) provided \( x \mapsto F(x) := \{ f(t, x) \mid t \in I \} \) has linear growth. For any subsets \( C, P \subseteq \mathbb{R}^n \) and any constant \( \eta \in \mathbb{R} \), we set \( C + \eta P := \{ c + \eta p \mid c \in C, p \in P \} \). Also, \( B_n := \{ x \in \mathbb{R}^n \mid ||x|| \leq 1 \} \). A mapping \( F : \mathbb{R}^n \Rightarrow \mathbb{R}^n \) is said to be lower semicontinuous provided for each \( x \in \mathbb{R}^n \) and \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( F(x') + \varepsilon B_n \supseteq F(x) \) for all \( x' \in x + \delta B_n \); it is said to be closed (respectively, compact, convex, nonempty) valued provided \( F(x) \) is closed (respectively, compact, convex, nonempty) for each \( x \in \mathbb{R}^n \). We say that \( F \) is locally bounded provided \( F(\eta B_n) \) is bounded for each \( \eta > 0 \). Throughout this paper, we assume that all our mappings from \( \mathbb{R}^n \) are nonempty valued. Also, \( \text{int}(C) \) (respectively, \( \text{bd}(C) \)) denotes the interior (respectively, boundary) of any subset \( C \) of a Euclidean space. The \( i \)th component of a mapping \( F \) into \( \mathbb{R}^n \) is denoted by \( F_i \) for \( i = 1, 2, \ldots, n \).

A continuous function \( \omega : [0, \infty) \to [0, \infty) \) is called a modulus provided it is nondecreasing with \( \omega(0) = 0 \). For each \( T \geq 0 \), we let \( C(0, T] \) denote the set of all functions \( f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n \) that satisfy

\[
(C_1) \quad \text{For each } x \in \mathbb{R}^n, \text{ the map } t \mapsto f(t, x) \text{ is measurable};
\]

\[
(C_2) \quad \text{For each compact set } K \subseteq \mathbb{R}^n, \text{ there exists a modulus } \omega_{f,K} \text{ such that for all } t \in [0, T] \text{ and } x_1, x_2 \in K, \| f(t, x_1) - f(t, x_2) \| \leq \omega_{f,K}(\| x_1 - x_2 \|); \text{ and}
\]

\[
(C_3) \quad f \text{ has linear growth on } [0, T].
\]

This agrees with the definition of \( C(0, T] \) in [16]. Our main hypothesis is that each \( y \in \text{Traj}(F) \) is also the unique (generalized) solution of an appropriate initial value problem \( \dot{x} = f(t, x), \ x(0) = y(0) \) for a
feedback realization \( f \in C[0,T] \). However, we allow a more general class of feedback realizations than is allowed in [16]. We present our feedback realization hypothesis next.

We need the following additional definitions. We let \( \text{co} \) (resp., \( \text{cc} \)) denote the convex hull (resp., closed convex hull). For each subset \( P \) in Euclidean space, we set \( \text{cone} \{ P \} = \cup \{ \eta P : \eta \geq 0 \} \) (written \( \text{cone} \{ p \} \) when \( P = \{ p \} \) is singleton) and we define the projections \( \text{pr}_i \{ P \} = \{ p_i : \exists p = (p_1, p_2, \ldots, p_n) \in P \} \), so \( F_i(x) = \text{pr}_i \{ F(x) \} \) for each \( i \) and mapping \( F \) into \( \mathbb{R}^n \). We define the component cone (ccone) by \( \text{ccone} \{ P \} = \{ q \in \mathbb{R}^n : \exists p \in P \text{ s.t. } q_i \in \text{cone} \{ p_i \} \forall i \} \). When \( P = \{ p \} \) is singleton, we write this as \( \text{cone} \{ p \} \).

Notice that \( v \in \text{ccone} \{ P \} \) is a less restrictive condition than \( v \in \text{cone} \{ P \} \). For example, \( (1, -1) \in \text{cone} \{(2, -1) \} \setminus \text{cone} \{(2, 1) \} \). Note that if \( P_1, P_2 \subseteq \mathbb{R}^n \) and \( v_i \in \text{cone} \{ P_i \} \) for \( i = 1 \) and \( 2 \), then \( (v_1, v_2) \in \text{cone} \{ P_1 \times P_2 \} \subseteq \mathbb{R}^{2n} \). Given \( F : \mathbb{R}^n \Rightarrow \mathbb{R}^n, \bar{x} \in \mathbb{R}^n, \) and \( T > 0 \), we set

\[
C_F([0,T],\bar{x}) := \{ f \in C[0,T] : \exists \gamma > 0 \text{ s.t. } f(t,x) \in \text{cone} \{ F(x) \} \text{ for a.a. } t \in [0,\gamma) \text{ and all } x \in \bar{x} + \gamma \mathcal{B}_n \}.
\]

Given \( W \subseteq \mathbb{R}^n \), we call a mapping \( F : \mathbb{R}^n \Rightarrow \mathbb{R}^n \) \( W \)-weakly zeroing provided: \( \{ x \in W \text{ and } p \in F(x) \} \Rightarrow \{ (p_1, 0, 0, \ldots, 0), (0, p_2, 0, \ldots, 0), \ldots, (0, 0, 0, \ldots, p_n) \} \subseteq F(x) \). We simply say that \( F \) is weakly zeroing provided it is \( \mathbb{R}^n \)-weakly zeroing. Any \( F \) that is the product of one dimensional multi-functions \( F_i \) satisfying \( 0 \in F_i(x) \) for all \( i \) and \( x \) is weakly zeroing, but weak zeroing is more general since it allows images \( F(x) \) such as \( \{ p \in \mathbb{R}^2 : p_1 \geq 0, p_2 \geq 0, p_1 \leq 1 - p_1 \} \). We assume the following condition:

\( (U') \quad \text{For each } \bar{x}, T \geq 0, \text{ and } y \in \text{Traj}_T(F,\bar{x}), \text{ there exists } f \in C_F([0,T],\bar{x}) \text{ for which } y \text{ is the unique solution of } \dot{y}(t) = f(t,y(t)) \text{ on } [0,T] \text{ starting at } \bar{x}. \text{ Also, } F \text{ is weakly zeroing.} \)

When the conditions in \( (U') \) hold, we call \( f \) a feedback realization for the trajectory \( y \). The prime notation signifies that our condition is a variant of the realizability condition \( (U) \) from [16]; see Section 3.3 for a comparison of conditions \( (U) \) and \( (U') \). Since \( 0 \in F(x) \) for all \( x \in \mathbb{R}^n \), our dynamics are weakly (but not necessarily strongly) invariant for any constraint set \( S \). In our tangential characterizations of strong invariance, our weak zeroing requirement can be relaxed to the requirement that \( F \) be bd(S)-weakly zeroing for the closed constraint set \( S \) (see Section 3.1). Notice that feedback realizations \( f \) can depend on the initial value \( \bar{x} \) and the trajectory \( y \). For example, if \( F \) is compact-convex valued and Lipschitz and weakly zeroing, then (see [16]) we can satisfy \( (U') \) using \( f(t,x) := \text{proj}_{F(x)} \bar{y}(t) \) for any \( y \in \text{Traj}(F) \), where \( \text{proj}_{F(x)} \) denotes the (unique) closest point in \( F(x) \) to \( q \). The following non-Lipschitz examples show how Condition \( (U') \) can be satisfied without necessarily having \( f(t,x) \in F(x) \) for all \( x \) and a.a. \( t \).

**Example 2.1.** Choose \( n = 1, F(0) = [-1,+1], \text{ and } F(x) = \{-\text{sign}(x), 0\} \text{ for } x \neq 0. \) To verify Condition \( (U') \), let \( T > 0, \bar{x} \in \mathbb{R}, \text{ and } y \in \text{Traj}_T(F,\bar{x}) \) be given. Note that \( (F, \{0\}) \) is strongly invariant. Therefore, either (i) \( y(t) \) starts at some \( \bar{x} \neq 0 \) and then moves toward 0 at unit or zero speed or (ii) \( y(t) \equiv 0 \). If \( \bar{x} \neq 0, \text{ then } (U') \text{ is satisfied using } f(t,x) = \text{sign}(\bar{x})1_{D_0}(t), \text{ where } 1_{D_0}(t) \text{ is defined to be 0 when } y(t) = 0 \text{ or } y(t) = 0, \text{ and 1 otherwise. } \text{ Then } f(t,x) \in \text{cone} \{ F(x) \} \text{ for all } t \in [0,T] \text{ and } x \in \bar{x} + (|\bar{x}|/2)\mathcal{B}_1. \text{ If instead } \bar{x} = 0, \text{ then we choose } f(t,x) = 0 \text{ for all } x \in \text{cone} \{ F(x) \} \text{ for all } t \in [0,T] \text{ and } x \in \mathbb{R}. \)

**Example 2.2.** Consider the following dynamics evolving on \( \mathbb{R}^n \):

\[
\dot{x}(t) = \prod_{i=1}^n L_i(x) D_i(x),
\]

where we assume the following for each \( i \):

\( (H_1) \quad x \mapsto L_i(x) \subseteq \mathbb{R} \text{ is locally Lipschitz and compact valued.} \)
\( (H_2) \quad x \mapsto D_i(x) \subseteq \mathbb{R} \text{ is measurable, locally bounded, and closed valued.} \)
\( (H_3) \quad 0 \in D_i(x) \text{ for all } x \in \mathbb{R}^n, \text{ and } x \mapsto \text{cone} \{ D_i(x) \} \text{ is constant.} \)

where \( L_i(x)D_i(x) := \{ \lambda \delta : \lambda \in L_i(x), \delta \in D_i(x) \} \). Note that under \( (H_3) \), the only possible constant values for \( \text{cone} \{ D_i(x) \} \) are \( \{0\}, \mathbb{R}, [0,\infty), \text{ or } (-\infty,0] \). In particular, for each \( x, y \in \mathbb{R}^n \text{ and } \beta \in D_i(x), \) we have \( \beta \in \text{cone} \{ D_i(y) \} \). These hypotheses allow the systems (1) from the introduction by taking
there are measurable functions $g_i(x, A)$ for all $i$. To check that (2) satisfies ($U'$), let $\bar{x} \in \mathbb{R}^n$, $T > 0$, $y \in \text{Traj}_T(F, \bar{x})$, and $\phi : \mathbb{R}^n \to [0, 1]$ be any smooth (i.e., $C^\infty$) function that is identically one on $y$ and compactly supported. Then $y$ is also a trajectory of

$$G(x) := \prod_{i=1}^n \phi(x) \co \{L_i(x)\} D_i(x). \quad (3)$$

Since each component $x \mapsto \phi(x) \co \{L_i(x)\}$ is Lipschitz and compact and convex valued, standard parametrization results (cf. [4, Section 9.7]) allow us to rewrite (3) as $G(x) = \prod_{i=1}^n g_i(x, B_1) D_i(x)$ for some Lipschitz function $g = (g_1, g_2, \ldots, g_n)$ where $g_i : \mathbb{R}^n \times B_1 \to \mathbb{R}$ for all $i$. Since $y \in \text{Traj}_T(G, \bar{x})$, there are measurable functions $\alpha_i : [0, T] \to B_1$ and $\beta_i : [0, T] \to \mathbb{R}$, with $\beta_i(t) \in D_i(y(t))$ for a.a. $t \in [0, T]$, such that $y$ is the unique solution of

$$f(t, x) := \prod_{i=1}^n g_i(x, \alpha_i(t)) \beta_i(t) \quad (4)$$

on $[0, T]$ starting at $\bar{x}$. This follows from the (Generalized) Filippov Selection Theorem (see [22, p.72]). The local boundedness requirement in ($H_3$) guarantees that the $\beta_i$’s are (essentially) bounded. Using the fact that $\text{cone}\{\co D\} = \text{cone}\{D\}$ for all subsets $D \subseteq \mathbb{R}$, one can verify ($U'$) using the choice (4) of $f$.

**Example 2.3.** We next illustrate how dynamics satisfying ($U'$) can be “stacked” to build new dynamics that again satisfy ($U'$). Assume two dynamics $F_1, F_2 : \mathbb{R}^n \to \mathbb{R}^n$ satisfy condition ($U'$). Consider the “stacked” dynamics $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ defined by $F(x) = F_1(x_1) \times F_2(x_2)$. We claim that $F$ satisfies ($U'$) in dimension $2n$. To see why, let $T > 0$, $\bar{x} = (\bar{x}_1, \bar{x}_2) \in \mathbb{R}^n \times \mathbb{R}^n$, and $y \in \text{Traj}_T(F, \bar{x})$ be given. Write $y = (y_1, y_2) : [0, T] \to \mathbb{R}^n \times \mathbb{R}^n$. Since $y_i \in \text{Traj}_T(F_i, \bar{x}_i)$ for $1 \leq i \leq 2$, we can find $f_i \in C_F([0, T], \bar{x}_i)$ such that $y_i$ is the unique solution of the initial value problem $\dot{x}_i = f_i(t, x_i), x_i(0) = \bar{x}_i$ on $[0, T]$ for $1 \leq i \leq 2$. The requirement ($U'$) is then satisfied for $y$ using the feedback realization $f(t, x) := (f_1(t, x_1), f_2(t, x_2))$. On the other hand, since we do not in general have $f(t, x) \in \text{cone}\{F(x)\}$, the stacked dynamics may not satisfy the more restrictive requirement ($U$) from [16] (see Section 3.3).

The preceding examples play an important role when we apply our results to monotone control systems in Section 5. In the next two sections, we state and prove our main strong invariance results. For the relevant background from nonsmooth analysis, see Appendix A.

**3 Statement and Discussion of Main Results**

**3.1 Sufficient Conditions for Strong Invariance**

Let $H_F : \mathbb{R}^n \times \mathbb{R}^n \to [-\infty, +\infty]$ denote the (upper) Hamiltonian for our dynamics $F$ defined by $H_F(x, d) := \sup \{\langle v, d \rangle : v \in F(x)\}$. For any subset $D \subseteq \mathbb{R}^n$, we write $H_F(x, D) \leq 0$ to mean that $H_F(x, d) \leq 0$ for all $d \in D$. By definition, this inequality holds vacuously if $D = \emptyset$.

**Theorem 1.** Let $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfy ($U'$) and $\Psi : \mathbb{R}^n \to \mathbb{R}$ be lower semicontinuous. If there exists an open set $U \subseteq \mathbb{R}^n$ containing $S := \{x \in \mathbb{R}^n : \Psi(x) \leq 0\}$ for which $H_F(x, \partial_P \Psi(x)) \leq 0$ for all $x \in U$, then $(F, S)$ is strongly invariant.

The statement of Theorem 1 is the same as the main theorem in [16] except Theorem 1 replaces the realization assumption ($U$) from [16] with ($U'$) (see Section 3.3 for a comparison of these two assumptions). We prove Theorem 1 below by constructing appropriate Euler polygonal arcs; see also Remark B.1 for an alternative nonconstructive proof based on Zorn’s Lemma. Condition ($U'$) allows us to apply our results to a broad class of examples that are not tractable by [16] or other known strong invariance results (see Section 2). Note that we require the Hamiltonian inequality in a neighborhood $U$ of $S$. The result is not true in general if the Hamiltonian condition is placed only on $S$. For example, take $n = 1$, $\Psi(x) = x^2$, and $F(x) \equiv \{0, 1\}$. Then $S = \{0\}$ and $H_F(0, \partial_P \Psi(0)) = 0$, but in this case $(F, S)$ is not
strongly invariant. The non-Lipschitz system in Example 2.1 is covered by Theorem 1 if we choose the verification function \( \Psi(x) = x^2 \). In that case, the Hamiltonian condition reads \( H_F(x, \Psi'(x)) = 0 \) for all \( x \in \mathbb{R} \), so our sufficient condition for strong invariance is satisfied.

Theorem 1 contains the usual sufficient condition for strong invariance for an arbitrary closed set \( S \subseteq \mathbb{R}^n \) by letting \( \Psi \) be the characteristic function \( \chi_S \) of \( S \); that is, \( \chi_S(x) = 0 \) if \( x \in S \) and is 1 otherwise. Then \( \partial_F \Psi(x) = \{0\} \) for all \( x \notin \text{bd}(S) \), and \( \partial_F \Psi(x) = N^F_S(x) \) for all \( x \in \text{bd}(S) \). This implies:

**Corollary 3.1.** Let \( F : \mathbb{R}^n \rightharpoonup \mathbb{R}^n \) satisfy \((U')\) and \( S \subseteq \mathbb{R}^n \) be closed. If \( H_F(x, N^F_S(x)) \leq 0 \) for all \( x \in \text{bd}(S) \), then \((F, S)\) is strongly invariant.

In fact, Corollary 3.1 is the special case of Theorem 1 using \( \mathcal{U} = \mathbb{R}^n \), \( \Psi = \chi_S \), and \( S = S \). For the proof of Theorem 1, see Section 4. The proof shows that Theorem 1 remains true even if the weakly zeroing requirement on \( F \) is relaxed to: There exists an open set \( \mathcal{G} \) containing \( \text{bd}(S) \) such that \( F \) is \( \mathcal{G} \)-weakly zeroing. Moreover, Corollary 3.1 remains true even if this requirement is relaxed to requiring that \( F \) be \( \text{bd}(S) \)-weakly zeroing.

### 3.2 Strong Invariance Characterizations

The converse of Corollary 3.1 does not hold, as illustrated by Example 2.1. There \((F, \{0\})\) is strongly invariant but \( N^F_{\{0\}}(0) = \mathbb{R} \), so the Hamiltonian condition in the corollary is not satisfied. This means that the converse of Theorem 1 does not hold. On the other hand, it is desirable to have a sufficient and necessary condition for strong invariance, that is, a strong invariance characterization. One would also hope to have such a characterization in the more general situation where a non-Lipschitz dynamic \( F \) evolves on an arbitrary open subset \( \mathcal{O} \subseteq \mathbb{R}^n \) and the constraint set \( S \subseteq \mathbb{R}^n \) is a (relatively) closed subset of \( \mathcal{O} \) but is not necessarily a closed subset of \( \mathbb{R}^n \); i.e., \( S = F \cap \mathcal{O} \) for some closed subset \( F \subseteq \mathbb{R}^n \).

Characterizations of this kind play an important role in the analysis of monotone control systems (see Section 5 below or [1]).

In this subsection, we provide such a characterization in terms of the Bouligand tangent cone \( T^n_F \) (see Appendix A for the relevant definitions). Strong invariance of \((F, S)\) under relatively closed state constraints was treated in [1]. There it was assumed that \( F \) is locally Lipschitz. Here we consider a more general dynamic \( F : \mathcal{O} \times A \rightharpoonup \mathbb{R}^n \) of the form

\[
\dot{x} = F(x, a) = \prod_{i=1}^n g_i(x, a)D_i(x), \quad x \in \mathcal{O}, \ a \in A
\]

where \( A \) is a compact subset of a Euclidean space, each \( g_i : \mathcal{O} \times A \to \mathbb{R} \) is locally Lipschitz, and

\( (H_4) \quad \text{Each } D_i \text{ is convex valued, lower semicontinuous, and satisfies } (H_2)-(H_3). \)

Our assumptions imply that \( \mathcal{O} \ni x \mapsto F(x, A) \) satisfies \((U')\) (see Example 2.2). We prove:

**Theorem 2.** Let \( \mathcal{O} \subseteq \mathbb{R}^n \) be open and \( S \) be a (relatively) closed subset of \( \mathcal{O} \). Let \( F : \mathcal{O} \times A \rightharpoonup \mathbb{R}^n \) be as above. Then \((F, S)\) is strongly invariant if and only if \( F(x, A) \subseteq T^n_F(x) \) for all \( x \in S \).

We defer the proof to Section 4 which also shows that the requirement that \( 0 \in D_i(x) \) for all \( i \) and \( x \) can be relaxed to: There exist compact sets \( N_1 \subseteq N_2 \subseteq \ldots \subseteq N_k \subseteq \ldots \) such that

(i) \( \mathcal{O} = \bigcup_k N_k \) and

(ii) \( 0 \in D_i(x) \) for all \( i \), all \( x \in \text{bd}(S \cap N_k) \), and all \( k \in \mathbb{N} \),

which imply that the pairs \((F, S \cap N_k)\) are weakly invariant.

**Remark 3.2.** The tangential condition from Theorem 2 remains sufficient for strong invariance of \((F, S)\) if \((H_4)\) is relaxed to requiring \( x \mapsto F(x, A) \) to satisfy \((U')\), by the same proof. This gives a sufficient condition for strong invariance for the dynamics (5) with relatively closed state constraints if we merely
assume \((H_2)-(H_4)\) without necessarily having convex valued or lower semicontinuous dynamics. On the other hand, Example 2.1 illustrates that the tangential condition is not necessary for strong invariance without assuming \((H_4)\) even if \(\mathcal{O} = \mathbb{R}^n\). In that example, there are no controls \(a_1\) and \((F, \{0\})\) is strongly invariant but \([-1, +1] = F(0) \not\subseteq T_{(0)}^{\mathcal{O}} = \{0\}\). This does not contradict Theorem 2 since \(F\) is not lower semicontinuous, so \((H_4)\) is not satisfied.

### 3.3 Relationship to Known Strong Invariance Results

Theorem 1 applies to a more general class of multifunctions than the known strong invariance results because it does not require the usual Lipschitz, one-sided Lipschitz, or other structural assumptions on \(F\), nor does it require the more restrictive feedback realizability condition from [16]. The papers [6, 8] provide strong invariance results for locally Lipschitz dynamics (see also [9, Chapter 4]). For locally Lipschitz \(F\), Clarke [6] showed that strong invariance of \((F, S)\) is equivalent to \(F(x) \subseteq T_S^x(x)\) for all \(x \in S\), where \(T_S^x\) denotes the Clarke tangent cone (cf. [9] or Appendix A for the relevant definitions). See [8] for Hilbert space versions and [22] for other strong invariance results for Lipschitz dynamics and nonautonomous versions. For strong invariance characterizations under somewhat more general structural conditions on \(F\) (e.g., dissipativity and one-sided Lipschitzness), see [11, 12, 18].

On the other hand, Theorem 1 does not make any such structural assumptions on \(F\) and allows non-Lipschitz dynamics such as (2) that are intractable by the standard strong invariance results. In [16], strong invariance was shown assuming the following on the multifunction \(F\):

\[
(U) \quad \text{For each } \bar{x} \in \mathbb{R}^n, T \geq 0, \text{ and } y \in \text{Traj}_T(F, \bar{x}), \text{ there exists } f \in C_F([0, T], \bar{x}) \text{ for which } y \text{ is the unique solution of the initial value problem } \dot{y}(t) = f(t, y(t)), \quad y(0) = \bar{x} \text{ on } [0, T].
\]

where \(C_F([0, T], \bar{x}) = \{ f \in C[0, T] : \exists \gamma > 0 \text{ s.t. } f(t, x) \in \text{cone}\{F(x)\} \forall x \in \bar{x} + \gamma B_n \text{ & a.a. } t \in [0, T]\} \text{ and } C[0, T] \text{ as defined in Section 2 above. An important difference between (U) and (U') is that (U) requires } f(t, x) \text{ to be locally in the cone of } F(x) \text{ rather than the larger cone so Condition } \text{(U)} \text{ from [16] is a more restrictive requirement than our Condition } \text{(U'). In other words, our condition } f(t, x) \in \text{cone}\{F(x)\} \text{ from (U')} \text{ means there are weights } \omega_1, \omega_2, \ldots, \omega_n \in [0, \infty) \text{ (possibly depending on } t \text{ and } x) \text{ such that } f_i(t, x) \in \omega_i F_i(x) \text{ for all } i \text{ while (U) makes the further restriction that } \omega_1 = \omega_2 = \ldots = \omega_n \text{ (see Section 1 for an example where (U') holds but these weights cannot be chosen to be equal).} \text{ The main results of [16] are the same as our Theorem 1 and Corollary 3.1 except with (U') replaced by (U).}

Condition (U) has an advantage because it does not require zeroing, but the argument in the next section shows that Corollary 3.1 remains true even if this requirement is relaxed to requiring that \(F\) be bd(S)-weakly zeroing. Notice however that if \(F_1, F_2 : \mathbb{R}^n \rightarrow \mathbb{R}^n\) are two dynamics that satisfy (U), then it is not in general the case that the “stacked” dynamic \(F(x) := F_1(x) \times F_2(x)\) satisfies (U). In particular, if \(\bar{x} = (\bar{x}_1, \bar{x}_2) \in \mathbb{R}^n \times \mathbb{R}^n\), and if also \(f_1, f_2 \in C[0, T]\) and \(\gamma > 0\) are such that \(f_1(t, x_j) \in \text{cone}\{F_1(x_j)\}\) for all \(x_j \in \bar{x}_j + \gamma B_n\), a.a. \(t \in [0, T]\), and \(j = 1, 2\), then it is not generally the case that \((f_1(t, x_1), f_2(t, x_2)) \in \text{cone}\{F_1(x_1) \times F_2(x_2)\}\), although we do have

\[
(f_1(t, x_1), f_2(t, x_2)) \in \text{cone}\{F_1(x_1) \times F_2(x_2)\} \text{ for all } x = (x_1, x_2) \in \bar{x} + \gamma B_{2n} \text{ and a.a. } t \in [0, T],
\]

so \(F\) satisfies \((U')\) in dimension \(2n\). Therefore, condition \((U')\) has the advantage that it is preserved under “stacking”. Using this “stacking” property, we prove nonsmooth monotonicity characterizations that extend the corresponding results from [1] (see Section 5). Theorem 2 extends the strong invariance characterization for locally Lipschitz controlled dynamics and relatively closed state constraints from [1, Theorem 4] by allowing non-Lipschitz disturbances \(D_1\). In particular, Theorem 2 gives a tangential characterization for strong invariance for non-Lipschitz dynamics defined on all of \(\mathcal{O} = \mathbb{R}^n\); our results are new even for this particular case.
4 Proof of Main Results

4.1 Proof of Theorem 1

Fix $T > 0$, $\bar{x} \in \text{bd}(S)$, $\varepsilon > 0$, and $f \in C^\prime_F([0,T], \bar{x})$. By definition, we can then find $\gamma \in (0,1)$ such that $f_i(t, x) \in \text{cone}(F_i(x))$ for all $i$, a.a. $t \in [0,\gamma)$, and all $x \in \bar{x} + \gamma B_n$. We define the mollification $f_\varepsilon$ by (30) in Appendix A below and let $f_{\varepsilon,i}$ denote its $i$th component for $i = 1,2,\ldots,n$. By reducing $\gamma$ as necessary without relabelling, we can assume that $\bar{x} + \gamma B_n \subseteq U$. By also reducing $T > 0$, we also assume $T \in (0,\gamma)$. For each $i \in \{1,2,\ldots,n\}$, $t \in [0,T]$, $x \in \mathbb{R}^n$, and $r > 0$, we then set

$$G_{f,i}^\varepsilon[t,x,r] = \prod_{j=1}^n R_{ij}^\varepsilon, \quad G_{f_i}^\varepsilon[t,x,r] = \prod_{j=1}^n G_{f_j}^\varepsilon[t,x,r], \quad g_f[t,x,r] := 1 + \sup \{ \|p\| : p \in G_f^\varepsilon[t,x,r] \},$$

where $R_j^\varepsilon \equiv \{0\}$ for all $j \neq i$ and

$$R_j^\varepsilon = G_{f_j}^\varepsilon[t,x,r] := \overline{\text{co}} \{ f_{\varepsilon,i}(t,y) : \|y - x\| \leq 1/r \}.$$

It follows that $G_{f,i}^\varepsilon[t,x,r] \subseteq g_f[t,x,r] B_n$ everywhere for each $i$. By our linear growth assumption (C$_3$) from Section 2, the sets $G_{f,i}^\varepsilon$ are compact. The following consequence of the Clarke-Ledyaev Mean Value Inequality (Theorem 5 from Appendix A) extends Claim 4.1 from [16] to our more general feedback realizations $f$:

**Claim 4.1.** If $x \in \mathbb{R}^n$, $t \geq 0$, $k \in \mathbb{N}$, and $h > 0$ are such that

$$0 < h \leq \frac{1}{32k g_f[t,x,k/n]} \quad \text{and} \quad x + h g_f[t,x,k/n] B_n \subseteq \bar{x} + \frac{2\gamma}{3} B_n,$$

then

$$\Psi(x + hv) \leq \Psi(x) + \frac{h}{k}$$

holds for some $v \in G_f^\varepsilon[t,x,k/n]$.

**Proof.** Set $Y_1 = x + hG_{f,1}^\varepsilon[t,x,16k]$ which is compact and convex. We first find $v_1 \in G_{f,1}^\varepsilon[t,x,16k]$ such that

$$\Psi(x + (v_1,0,\ldots,0)) \leq \Psi(x) + \frac{h}{nk}.$$

Suppose no such $v_1$ exists. Since $\Psi$ is lower semicontinuous and $Y_1$ is compact, we would then have

$$\delta := \frac{h}{nk} < \min_{y \in Y_1} \Psi(y) - \Psi(x).$$

Notice that $g_f[t,x,16k] \leq g_f[t,x,k/n]$. By (6), we can therefore find $\lambda \in (0,\frac{1}{16k})$ satisfying

$$x + h g_f[t,x,16k] B_n + \lambda B_n \subseteq x + h g_f[t,x,k/n] B_n + \lambda B_n \subseteq \bar{x} + \gamma B_n \subseteq U.$$

Next we apply Theorem 5 from Appendix A with the choices $Y = Y_1$ and $\delta$ defined by (9). It follows that there exist $z \in [x,Y_1] + \lambda B_n$ and $\zeta \in \partial \Psi(z)$ for which

$$\delta < \min_{y \in Y_1} \langle \zeta, y - x \rangle = \min_{v \in G_{f,1}^\varepsilon[t,x,16k]} \langle \zeta, hv \rangle,$$

where $[x,Y_1]$ denotes the closed convex hull of $x$ and $Y_1$. By (10), $z \in \bar{x} + \gamma B_n \subseteq U$. Also, (6) combined with the choice of $\lambda$ gives $\|z - x\| \leq h g_f[t,x,16k] + \lambda \leq h g_f[t,x,k/n] + \lambda \leq \frac{1}{16k}$. Therefore

$$f_{\varepsilon,1}^\varepsilon(t,z) := (f_{\varepsilon,1}(t,z),0,0,\ldots,0) \in G_{f,1}^\varepsilon[t,x,16k].$$
Since \( z \in \bar{x} + \gamma B_n \), we know that \( f_1(s,z) \in \text{cone}\{F(z)\} \) for a.a. \( s \in [0,T] \). Since \( F \) is weakly zeroing, this gives \( (f_1(s,z),0,0,\ldots,0) \in \text{cone}\{F(z)\} \) for a.a. \( s \in [0,T] \). Since \( z \in \mathcal{U} \), our Hamiltonian hypothesis then gives

\[
(\zeta, (f_1(s,z),0,0,\ldots,0)) \leq 0 \quad \text{for a.a. } s \in [0,T].
\] (13)

Therefore, (11)-(13) give the contradiction

\[
0 < \delta \leq h(\zeta, f_{x_1}(t,z)) = h \int_{\mathbb{R}} \eta(t-s)(\zeta, (f_1(s,z),0,0,\ldots,0)) ds \leq 0,
\] (14)

so there must exist \( v_1 \in G_{f_1}^e[t,x,16k] \subseteq G_{f_1}^e[t,x,k/n] \subseteq g_f[t,x,k/n]B_1 \) satisfying (8). Assume \( n > 1 \).

Set \( v_1^o = (v_1,0,0,\ldots,0) \in \mathbb{R}^n \). Then \( (x + hv_1^o) + hg_f[t,x,k/n]((0) \times B_{n-1}) \subseteq x + hg_f[t,x,k/n]B_n \), so (10) gives \( (x + hv_1^o) + hg_f[t,x,k/n]((0) \times B_{n-1}) + \lambda B_n \subseteq \bar{x} + \gamma B_n \). Moreover,

\[
G_{f_2}^e[t,x + hv_1^o,16k] \subseteq G_{f_2}^e[t,x,k/n],
\] (15)

because if \( ||y - (x + hv_1^o)|| \leq \frac{1}{16k} \), then (6) gives \( ||y - x|| \leq \frac{1}{16k} + h|v_1| \leq \frac{1}{16k} + \frac{1}{16k} \leq \frac{2}{16k} \leq \frac{2}{k} \), which gives (15). We can therefore apply the preceding argument with \( x \) replaced by \( x + hv_1^o \), and with \( Y_1 \) replaced by the new compact convex set

\[
Y_2 := (x + hv_1^o) + hG_{f_2}^e[t,x + hv_1^o,16k] \subseteq (x + hv_1^o) + hg_f[t,x,k/n]((0) \times B_{n-1})
\]

to find \( z \in [x + hv_1^o, Y_2] + \lambda B_n \) such that \( f_{x_2,2}(t,z) := (0,f_{x,2}(t,z),0,0,\ldots,0) \in G_{f_2}^e[t,x + hv_1^o,16k] \) and \( v_2 \in G_{f_2}^e[t,x + hv_1^o,16k] \subseteq G_{f_2}^e[t,x,k/n] \) such that

\[
\Psi(x + hv_1^o + h(0,v_2,0,0,\ldots,0)) \leq \Psi(x + hv_1^o) + \frac{h}{nk}.
\] (16)

Next we argue by induction. Proceeding inductively, we apply the same argument but with \( x \) replaced by \( \bar{x}_{i-1} := x + h(v_1,v_2,\ldots,v_{i-1},0,0,\ldots,0) \) and with the set \( Y_1 \) replaced by the new compact convex set \( Y_{i} := \bar{x}_{i-1} + hG_{f_i}^e[t,\bar{x}_{i-1},16k] \). Since \( v_i \in G_{f_r}^e[t,x,k/n] \) for all \( r < i \), (6) implies that \( |v_i| \leq \frac{1}{16k} \) for all \( r < i \), so the proof of (15) shows \( G_{f,i}^e[t,\bar{x}_{i-1},16k] \subseteq G_{f_i}^e[t,x,k/n] \) and

\[
[\bar{x}_{i-1}, Y_{i}] + \lambda B_n \subseteq \bar{x}_{i-1} + hg_f[t,x,k/n]((0) \times B_{n-i-1}) + \lambda B_n \subseteq x + (hg_f[t,x,k/n] + \lambda)B_n \subseteq \bar{x} + \gamma B_n \subseteq \mathcal{U} \quad \text{(by (10)).}
\]

This allows us to find \( v_i \in G_{f_i}^e[t,\bar{x}_{i-1},16k] \subseteq G_{f_i}^e[t,x,k/n] \) such that

\[
\Psi(x + h(v_1,v_2,\ldots,v_i,0,0,\ldots,0)) \leq \Psi(\bar{x}_{i-1}) + \frac{h}{nk}.
\] (17)

for \( i = 2,3,\ldots,n \). Choosing \( v = (v_1,v_2,\ldots,v_n) \in G_{f_n}^e[t,x,k/n] \), we obtain (7) by summing the inequalities in (8) and (17) over \( i = 2,\ldots,n \).

Now set \( D := \bar{x} + \frac{n}{2}B_n \subseteq \mathcal{U} \). Let \( \omega_{f,K} \) be a modulus of continuity for \( x \mapsto f(t,x) \) on \( K := D + nB_n \) for all \( t \in [0,T] \) (see Condition (C2)). Then \( \omega_{f,K} \) is also a modulus of continuity of \( K \ni x \mapsto f_{x,t}(x) \) for all \( t \in [0,T] \) and \( \varepsilon > 0 \). The following claim parallels Claim 4.2 from [16]:

**Claim 4.2.** Let \((t,x,k) \in [0,T] \times D \times \mathbb{N} \) and \( v \in G_{f}^e[t,x,k/n] \). Then \( \| v - f_{x,t}(x) \| \leq \omega_{f,K}(n/k) + 1/k \).

**Proof.** Using the Carathéodory Theorem and the definition of \( G_{f}^e[t,x,k/n] \), we can write

\[
v = \Delta + \prod_{i=1}^{n} \left[ \sum_{j=1}^{2} \alpha_{i,j}f_{x,ij}(t,y_{i,j}) \right],
\]

where...
where
\[ \|y_{i,j} - x\| \leq n/k \forall i,j; \quad \alpha_{i,j} \in [0,1] \forall i,j; \quad \sum_{j=1}^{2} \alpha_{i,j} = 1 \forall i; \quad \text{and} \quad \|\Delta\| \leq 1/k. \]

In particular, \( x \in K \) and \( y_{i,j} \in K \) for all \( i \) and \( j \). This gives
\[ |v_i - f_{\epsilon,i}(t,x)| \leq |v_i - \sum_{j=1}^{2} \alpha_{i,j} f_{\epsilon,i}(t,y_{i,j})| + \left| \sum_{j=1}^{2} \alpha_{i,j} \{ f_{\epsilon,i}(t,y_{i,j}) - f_{\epsilon,i}(t,x) \} \right| \leq \frac{1}{k} + \omega_{f,K} \left( \frac{n}{k} \right) \]
for \( i = 1,2,\ldots,n \). The claim follows by applying the supremum norm.

Next define \( \delta(D) := 1 + c_1 + nc_2 + c_2 \max\{\|v\| : v \in D\} \). It follows that
\[ G^f_0[t,x,k/n] \subseteq \{c_1 + c_2(\|x\| + n/k)\}E_n \subseteq \delta(D)E_n \forall t \in [0,T], \ x \in D, \ k \in \mathbb{N}. \tag{18} \]

Theorem 1 now follows from a slight variant of the argument in [16] which we include as Appendix B.

4.2 Proof of Theorem 2

We first make some general observations that relate our tangential and Hamiltonian conditions for strong invariance for an arbitrary multifunction \( G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \). We let \( H_G \) denote the (upper) Hamiltonian for \( G \) (see Section 3.1). We also set \( D^o := \{p \in \mathbb{R}^n : \langle p,d \rangle \leq 0 \forall d \in D\} \) for each subset \( D \subseteq \mathbb{R}^n \); i.e., \( D^o \) is the polar set for \( D \). Note that \( H_G(x,D) \leq 0 \) holds if and only if \( G(x) \subseteq D^o \).

**Lemma 4.1.** Let \( G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \) and \( R \) be a closed subset of \( \mathbb{R}^n \). (i) If \( G(x) \subseteq T^B_R(x) \) for all \( x \in R \), then \( H_G(x,N^p_R(x)) \leq 0 \) for all \( x \in R \). (ii) If \( G \) is closed, convex, and nonempty valued and lower semicontinuous, and if \( H_G(x,N^p_R(x)) \leq 0 \) for all \( x \in R \), then \( G(x) \subseteq T^B_R(x) \) for all \( x \in R \).

**Proof.** Since \( T^B_R(x) \subseteq (N^p_R(x))^o \) for all \( x \in R \) (cf. [9, Exercise II.7.1(d)] or Appendix A), the assumptions of (i) imply \( G(x) \subseteq (N^p_R(x))^o \) for all \( x \in R \). This establishes part (i). To prove part (ii), first note that if its hypotheses imply \( G(x) \subseteq (N^p_R(x))^o \) for all \( x \in R \). Let \( N^C_R \) and \( T^C_R \) denote the Clarke normal and tangent cones, respectively (cf. Appendix A for the relevant definitions). We claim that \( G(x) \subseteq (N^C_R(x))^o \) for all \( x \in R \). To verify this claim, fix \( x \in R \) and \( v \in G(x) \). By Theorem 6 in Appendix A, we can find a continuous selection \( s : \mathbb{R}^n \rightarrow \mathbb{R}^n \) of \( G \) for which \( s(x) = v \). Therefore, if \( R \ni x_1 \rightarrow x \) and \( N^p_R(x) \ni \zeta \rightarrow \zeta \in \mathbb{R}^n \), then \( s(x_t) \in G(x_t) \) implies \( \langle s(x_t), \zeta_t \rangle \leq 0 \) for all \( t \). Passing to the limit as \( t \rightarrow \infty \) gives \( \langle v, \zeta \rangle \leq 0 \). Therefore, the characterization (27) for \( N^C_R \) in Appendix A gives \( v \in (N^C_R(x))^o \), which proves the claim. It follows that \( G(x) \subseteq (N^C_R(x))^o = T^C_R(x) \subseteq T^B_R(x) \) for all \( x \in R \), which proves (ii).

Returning to the proof of Theorem 2, assume \( F(x,A) \subseteq T^B_S(x) \) for all \( x \in S \). We show that \( (F,S) \) is strongly invariant by extending an argument from the appendix of [1] to our more general situation where the dynamics \( F \) may be discontinuous. Let \( M \) and \( N \) be any compact subsets of \( \mathbb{R}^n \) contained in \( O \) such that \( M \subseteq \text{int}(N) \). Then \( E := S \cap N \) is a closed subset of \( \mathbb{R}^n \). Let \( \phi : \mathbb{R}^n \rightarrow \mathbb{R} \) be any \( C^\infty \) function that is identically 1 on \( M \), strictly positive on \( \text{int}(N) \), and zero elsewhere. Extend \( F \) to \( \mathbb{R}^n \) by defining \( F(x,A) = \{0\} \) outside \( O \) and define the multifunction \( F^F : \mathbb{R}^n \times A \rightrightarrows \mathbb{R}^n \) by \( F^f(x,a) = \phi(x) F(x,a) \). As observed in [1, Lemma A.4, p. 1696], we have:

**Lemma 4.2.** For each \( x \in E \), either \( F^f(x,A) = \{0\} \) or \( T^B_E(x) = T^B_S(x) \).

Therefore, since \( T^B_E(x) \) is a cone for all \( x \in E \), we get \( F^F(x,A) \subseteq T^B_S(x) \) for all \( x \in E \). Applying Lemma 4.1(i) with \( G(x) = F^F(x,A) \) and \( R = E \) gives \( H^F_E(x,N^p_E(x)) \leq 0 \) for all \( x \in E \). Since the multifunction \( x \mapsto F^F(x,A) \) also satisfies \((U')\), Corollary 3.1 implies \((F^F,E)\) is strongly invariant. Therefore, if \( T > 0 \) and \( y : [0,T] \rightarrow O \) is any trajectory of \( F \) such that \( y(0) \in S \), and if we specialize the preceding argument to the compact set \( M = \{y(t) : 0 \leq t \leq T\} \), then \( y \in \text{Traj}(F^f) \) (because \( \phi \equiv 1 \) on \( M \)) and \( y(0) \in E \), so \( y \) remains in \( E \subseteq S \). This shows \((F,S)\) is strongly invariant.
Conversely, assume that \((F, S)\) is strongly invariant. We need to show that \(F(x, A) \subseteq T^B_S(x)\) for all \(x \in S\). To this end, fix \(\bar{x} \in S\) and \(\bar{a} \in A\). Since \(S \subseteq \mathcal{O}\) and \(\mathcal{O} \subseteq \mathbb{R}^n\) is open, we can find \(\mu > 0\) such that \(M := \bar{x} + \mu B_n \subseteq \mathcal{O}\). Choose any compact set \(N \subseteq \mathcal{O}\) such that \(M \subseteq \text{int}(N)\), and let \(\phi : \mathbb{R}^n \to \mathbb{R}\) be any \(C^\infty\) function satisfying the requirements above; i.e., \(\phi \equiv 1\) on \(M\), \(\phi > 0\) on \(\text{int}(N)\), and \(\phi \equiv 0\) outside \(\text{int}(N)\). Define \(F^\phi\) as before. Then \(x \mapsto F^\phi(x, \bar{a})\) is closed and convex valued and lower semicontinuous.

Let \(E = S \cap N\) as before, \(\zeta \in N_E^F(\bar{x})\), and \(v \in F^\phi(\bar{x}, \bar{a})\) be such that
\[
H_{F^\phi(\cdot, \bar{a})}(\bar{x}, \zeta) = \langle \zeta, v \rangle.
\]
Such a \(v\) exists because the sets \(D_t(\bar{x})\) are compact. Reapplying Michael’s Selection Theorem (Theorem 6 in Appendix A) provides a continuous selection \(s : \mathbb{R}^n \to \mathbb{R}^n\) of \(x \mapsto F^\phi(x, \bar{a})\) such that \(s(\bar{x}) = v\).

The characterization (26) of the proximal normal cone in Appendix A gives a constant \(\sigma \geq 0\) such that \(\langle \zeta, x' - \bar{x} \rangle \leq \sigma ||x' - \bar{x}||^2\) for all \(x' \in E\). Let \(z\) be a \(C^1\) local solution of the initial value problem \(\dot{z} = s(z), z(0) = \bar{x}\), which we can assume remains in \(M\). Since \(\phi \equiv 1\) on \(M\), \(z\) is also a trajectory of \(F\). Since we are assuming \((F, S)\) is strongly invariant, \(z\) also stays in \(S\). Since \(S \cap M \subseteq E\), we get \(\langle \zeta, z(t) - \bar{x} \rangle \leq \sigma ||z(t) - \bar{x}||^2\) for small \(t > 0\). Since \(\dot{z}(0) = v\), dividing by \(t > 0\) and letting \(t \to 0^+\) gives
\[
\langle \zeta, v \rangle \leq \sigma \lim_{t \to 0} \frac{||z(t) - \bar{x}||^2}{t} = 0.
\]

Since \(\phi(\bar{x}) = 1\), and since \(\zeta \in N_E^F(\bar{x})\) and \(\bar{a} \in A\) were arbitrary, (19)-(20) give \(H_{F^\phi(\cdot, \bar{a})}(\bar{x}, N_E^F(\bar{x})) \leq 0\).

The preceding argument applies to any \(\bar{x} \in \text{int}(N) \cap S\), by choosing \(\phi\) to be 1 near \(\bar{x}\) and compactly supported on \(N\), so \(H_{F^\phi(x, \bar{a})}(\bar{x}, N_E^F(\bar{x})) \leq 0\) for all \(x \in \text{int}(N) \cap S\). If we now fix such a function \(\phi\), then we have \(\phi(x)F(x, A) = F^\phi(x, A) \equiv \{0\}\) for all \(x \notin \text{int}(N)\), so \(H_{F^\phi(x, \bar{a})}(\bar{x}, N_E^F(\bar{x})) \leq 0\) for all \(x \in N \cap S = \mathcal{E}\).

Applying Lemma 4.1(ii) with \(G(x) = F^\phi(x, \bar{a}), \mathcal{R} = \mathcal{E}\), and any \(\bar{a} \in A\) gives \(\phi(x)F(x, A) \subseteq T^B_S(x)\) for all \(x \in E\). If \(x \in E \cap \text{int}(N)\) and \(F^\phi(x, A) = \{0\}\), then \(F(x, A) = \{0\}\), because \(\phi(x) > 0\); while if \(x \in E \cap \text{int}(N)\) and \(F^\phi(x, A) \neq \{0\}\), then Lemma 4.2 and the fact that \(\phi(x) > 0\) give \(F(x, A) \subseteq T^B_S(x)\). Therefore, \(F(x, A) \subseteq T^B_S(x)\) for all \(x \in E \cap \text{int}(N) = S \cap \text{int}(N)\). Now we fix \(\bar{x} \in S\) and \(\mu > 0\) such that \(M := \bar{x} + \mu B_n \subseteq \mathcal{O}\) and apply the preceding argument to all possible compact sets \(N \subseteq \mathcal{O}\) containing \(M\) in their interiors. Since \(\mathcal{O}\) is the union of the interiors of such sets \(N\), the preceding argument gives \(F(x, A) \subseteq T^B_S(x)\) for all \(x \in S \cap \mathcal{O} = S\), as desired. This proves Theorem 2.

5 Infinitesimal Characterizations of Monotonicity

5.1 Background and Statement of Results

In this section, we use Theorem 2 to prove new characterizations of monotonicity for control systems that may be non-Lipschitz in the state. Monotone control systems were introduced by Angeli and Sontag in [1] and have since been applied extensively in systems biology (cf. [1, 2, 3]). To simplify our exposition, we only consider monotone control systems evolving on Euclidean space with input values in \(\mathcal{U} := \mathcal{B}_m\) but our results can be adapted to systems whose inputs are valued in any ordered Banach space. The relevant definitions are as follows. We are given two closed cones \(K \subseteq \mathbb{R}^n\) and \(K_u \subseteq \mathbb{R}^m\) (called positivity cones) which we assume are convex, nonempty, and pointed (i.e., \(K \cap (-K) = \{0\}\) and \(K_u \cap (-K_u) = \{0\}\)). We define orders on \(\mathbb{R}^n\) and \(\mathbb{R}^m\) as follows: \(x_1 \geq x_2 \in \mathbb{R}^n\) if and only if \(x_1 - x_2 \in K\), and \(u_1 \geq u_2 \in \mathbb{R}^m\) if and only if \(u_1 - u_2 \in K_u\). The ordering on \(\mathcal{U}\) induces an order on the set of controls \(\mathcal{U}_\infty := \{\text{measurable } \alpha : [0, \infty) \to \mathcal{U}\} \text{ as follows: } \alpha_1 \geq \alpha_2 \in \mathcal{U}_\infty \text{ if and only if } \alpha_1(t) - \alpha_2(t) \in K_u \text{ for a.a. } t \geq 0\). We set \(\mathcal{U}^{[1]} = \{(u_1, u_2) \in \mathcal{U} \times \mathcal{U} : u_1 \geq u_2 \}\) and \(\mathcal{U}^{[\infty]} = \{\alpha_1, \alpha_2) \in \mathcal{U}_\infty \times \mathcal{U}_\infty : \alpha_1 \geq \alpha_2 \} \text{.}

Our main object of study in this section is the controlled dynamic
\[
\dot{x} \in G(x, u) := \sum_{i=1}^n g_i(x, u)D_i(x), \quad u \in \mathcal{U}, \quad x \in \bar{X}
\]
evolving on an open set \(\bar{X} \subseteq \mathbb{R}^n\) in which each \(g_i : \bar{X} \times \mathcal{U} \to \mathbb{R}\) is locally Lipschitz, \((H_4)\) is satisfied, and the \(D_i's\) take all their values in some compact set \(D \subseteq \mathbb{R}\). The dynamic can be viewed as a locally
Lipschitz dynamic with non-Lipschitz disturbances \( D_i \) acting on its individual components. Since we are assuming \( 0 \in D_i(x) \) for all \( i \) and \( x \), the dynamic (21) is clearly weakly invariant for any state constraint set \( S \subseteq X \), since it allows all constant trajectories. However, since (21) is not necessarily Lipschitz in the state variable, it may not be strongly invariant for some state constraints. Following [1], we also assume there is a closed set \( X \subseteq \hat{X} \) that is the closure of its interior such that all trajectories of

\[
G_D(x, u) = \prod_{i=1}^{n} g_i(x, u) D, \quad u \in U, \ x \in \hat{X}
\]

starting in \( X \) remain in \( X \). We set \( V = \text{int}(X) \). Since \( G_D \) is locally Lipschitz in \( x \) and \( D \subseteq \mathbb{R} \) is compact, the strong invariance of \( (G_D, X) \) can be checked using standard tangent cone conditions (cf. [1, Appendix A]). On the other hand, because we allow non-Lipschitz \( D_i \)'s, (21) is not in general tractable by the standard strong invariance characterizations. We also define \( G^{[2]} : \mathbb{R}^{2n} \times U^{[2]} \Rightarrow \mathbb{R}^{2n} \) by

\[
G^{[2]}(x, u) = G(x_1, u_1) \times G(x_2, u_2); \ i.e., \ two \ \text{“stacked” copies of } G \text{ with ordered inputs.}
\]

Again following [1], we also consider more general orders given by arbitrary closed sets \( \Gamma \subseteq X \times X \) as follows: We say that \( x_1 \geq x_2 \in X \) provided \( (x_1, x_2) \in \Gamma \). This includes the special case of state spaces ordered by positivity cones \( K \) by choosing \( \Gamma = \{(x_1, x_2) \in X \times X : x_1 - x_2 \in K \} \). With the order on \( \mathbb{R}^n \) expressed this way, we set \( \Gamma_o = \Gamma \cap (V \times V) \). We always assume the following approximation property, which parallels the approximation requirement in [1]:

\[
(A) \quad \text{For each } \xi \in \text{bd}(\Gamma), T > 0, \text{ and } y \in \text{Traj}_T(G^{[2]}, \xi), \text{ there exist } T' \in (0, T] \text{ and two sequences } \Gamma_o \ni \xi_k \to \xi \text{ and } \text{Traj}_{T'}(G^{[2]}, \xi_k) \ni y_k \to y \text{ uniformly on } [0, T'] \text{ as } k \to \infty.
\]

This agrees with the approximation condition posited in [1, p.1686] for locally Lipschitz dynamics \( G \), since in that case the convergence of \( y_k \) to \( y \) follows from continuous dependence on initial values. Condition \((A)\) is satisfied if for example (i) \( X \) is convex (or, even more generally, strictly star-shaped with respect to some interior point \( \xi \in V \)) and (ii) there exists a neighborhood \( N \) of \( \text{bd}(X \times X) \) such that \( x \to G^{[2]}(x, U^{[2]}) \) is Lipschitz on \( N \). In this case, the existence of a sequence \( \Gamma_o \ni \xi_k \to \xi \) in \((A)\) for each \( \xi \in \text{bd}(X \times X) \) follows from (i) and the argument from [1, p.1686]. Given \( y \in \text{Traj}(G^{[2]}, \xi) \), the existence of trajectories \( \text{Traj}(G^{[2]}, \xi_k) \ni y_k \to y \) uniformly in \((A)\) then follows because near \( \xi \), we can write \( G^{[2]}(x, \alpha(t)) = J(t, x, B_o) \) for some Lipschitz parametrization \( J \) and the input \( \alpha \in U^{[2]} \) for \( y \) (see [4, Chapter 9]), so \( y = J(t, y, \beta) \) for some input \( \beta \) and small times, and then we can apply continuous dependence on initial conditions to the dynamics \( J \). It suffices to check \((A)\) for \( \xi \in \text{bd}(X \times X) \), since \( \Gamma_o \cap \text{bd}(X \times X) \subseteq \Gamma_o \). On the other hand, we do not need to assume Lipschitzness of (21) in a neighborhood of \( \text{bd}(\Gamma_o) \) as was needed in [1]. We call (21) monotone provided:

\[
(M) \quad \text{If } \alpha_1 \geq \alpha_2 \in U_{\infty}, x_1 \geq x_2 \in X, T > 0, \phi_1 \in \text{Traj}_T(G, \alpha_1, x_1), \text{ and } \phi_2 \in \text{Traj}_T(G, \alpha_2, x_2), \text{ then } \phi_1(t) \geq \phi_2(t) \text{ for all } t \in [0, T].
\]

In other words, (21) is monotone provided its flow map preserves the orders on its inputs and initial states. Condition \((M)\) differs slightly from the definition of monotonicity in [1] because our non-Lipschitz dynamics generally admit multiple solution trajectories for some choices of inputs and initial states. Note that \( G \) is monotone if and only if \( (G^{[2]}, \Gamma) \) is strongly invariant. Moreover, \( G^{[2]} \) satisfies the hypotheses of Theorem 2 (see Example 2.2). For state spaces ordered by positivity cones \( K \), our main result is:

**Theorem 3.** The dynamics (21) is monotone if and only if the following condition is satisfied for all \( \xi_1, \xi_2 \in V \): \( (\xi_1 \geq \xi_2, u_1 \geq u_2) \Rightarrow G(\xi_1, u_1) - G(\xi_2, u_2) \subseteq T^B_K(\xi_1 - \xi_2) \).

It is easy to check that the tangent cone condition in Theorem 3 is equivalent to the following for each \( \xi_1, \xi_2 \in V \): \( (\xi_1 - \xi_2 \in \text{bd}(K), u_1 \geq u_2) \Rightarrow G(\xi_1, u_1) - G(\xi_2, u_2) \subseteq T^B_K(\xi_1 - \xi_2) \). This is because \( T^B_K(x) = \mathbb{R}^n \) for all \( x \in \text{int}(K) \). For orders induced by closed sets \( \Gamma \subseteq X \times X \), we also prove:

**Theorem 4.** The dynamics (21) is monotone if and only if the following condition is satisfied for all \( \xi_1, \xi_2 \in V \): \( (\xi_1 \geq \xi_2, u_1 \geq u_2) \Rightarrow G^{[2]}(\xi, u) \subseteq T^B_F(\xi) \).
Our theorems extend the monotonicity characterizations in [1] by allowing non-Lipschitz dynamics. The dynamics in [1] take the form \( G(x, u) = \{ g(x, u) : u \in U \} \) where \( g \) is continuous and locally Lipschitz in \( x \) locally uniformly in \( u \). Since we allow each factor \( x \mapsto D_i(x) \) in our dynamics (21) to be set valued and non-Lipschitz, our dynamics may be non-Lipschitz in the state. Therefore, the strong invariance theory from [9, Chapter 4] used in [1] no longer applies. Instead, we prove Theorems 3-4 using our new strong invariance theory from Section 4.

### 5.2 Proof of Theorems 3 and 4

The following lemmas parallel the corresponding lemmas in [1, Section III].

**Lemma 5.1.** \((G, \mathcal{V})\) is strongly invariant.

**Proof.** Let \( \tilde{x} \in \mathcal{V}, \alpha \in \mathcal{U}_\infty, T > 0, \) and \( y \in \text{Traj}_T(G, \alpha, \tilde{x}) \). We show \( y(t) \in \mathcal{V} \) for all \( t \in [0, T] \). Note that there exist measurable functions \( \beta_i : [0, T] \to \mathbb{R} \) with \( \beta_i(t) \in D_i(y(t)) \) for all \( i \) and a.a. \( t \in [0, T] \) such that \( y \) is the unique solution of

\[
\dot{x} = \prod_{i=1}^{n} g_i(x, \alpha(t))\beta_i(t)
\]

(23)
on \([0, T]\) starting at \( \tilde{x} \). This follows from the Filippov Selection Theorem as in Example 2.2. For each \( \tilde{x} \in \mathcal{V} \), let \([0, T] \ni t \mapsto z(t, \tilde{x})\) be the solution of (23) starting at \( \tilde{x} \). Notice that \( z(\cdot, \tilde{x}) \) is a trajectory for \( G_D(x, \alpha(t)) \) from (22) for each \( \tilde{x} \in \mathcal{V} \) and therefore remains in \( X \) by assumption. Fix \( t_0 \in [0, T] \).

Viewing (23) as a system with control inputs \( \alpha \) and \( \tilde{y} \), and arguing as in [1, Lemma III.6] (see [20, Lemma 4.3.8]) shows that the image \( \mathcal{F}(\mathcal{V}) \) of the final point map \( \mathcal{V} \ni \tilde{x} \mapsto \mathcal{F}(\tilde{x}) := z(t_0, \tilde{x}) \in X \) contains an open neighborhood \( W \) of \( \mathcal{F}(\tilde{x}) \). By assumption, \( W \subseteq \mathcal{F}(\mathcal{V}) \subseteq X \), so \( W \subseteq \text{int}(X) = \mathcal{V} \). Therefore, \( y(t_0) = \mathcal{F}(\tilde{x}) \in W \subseteq \mathcal{V} \). Since \( t_0 \in [0, T] \) was arbitrary, \( y \) stays in \( \mathcal{V} \). This proves the lemma.

Recall that \( G \) is monotone if and only if \((G^{[2]}, \Gamma)\) is strongly invariant. We next assume that the order on \( \mathbb{R}^n \) is expressed in terms of a closed set \( \Gamma \subseteq X \times X \) as above.

**Lemma 5.2.** The dynamics \( G \) is monotone if and only if \((G^{[2]}, \Gamma_o)\) is strongly invariant.

**Proof.** If \( G \) is monotone, then \((G^{[2]}, \Gamma)\) is strongly invariant. By the previous lemma, \((G^{[2]}, \mathcal{V} \times \mathcal{V})\) is also strongly invariant, so the sufficiency follows because \( \Gamma_o = \Gamma \cap (\mathcal{V} \times \mathcal{V}) \). Conversely, assume \((G^{[2]}, \Gamma_o)\) is strongly invariant. We show \((G^{[2]}, \Gamma)\) is strongly invariant using our approximation hypothesis (A). Let \( T > 0 \) and \( z : [0, T] \to \mathbb{R}^n \times \mathbb{R}^n \) be any trajectory of \( G^{[2]} \) starting at a point in \( \Gamma \), and define

\[
\bar{t} \overset{\text{def}}{=} \sup \{ t \in [0, T] : z(s) \in \Gamma \forall s \in [0, t] \}.
\]

(24)

We need to show that \( \bar{t} = T \). Suppose the contrary, so \( \bar{t} < T \). Notice that \( z(s) \in \Gamma \) for all \( s \in [0, \bar{t}] \). In particular, \( \xi := z(\bar{t}) \in \text{bd}(\Gamma) \). We apply (A) to the trajectory \([0, T - \bar{t}] \ni t \mapsto y(t) := z(t + \bar{t})\) of \( G^{[2]} \) starting at \( \xi \). This gives \( T' \in (0, T - \bar{t}] \) and sequences \( \Gamma_o \ni \xi_k \to \xi \) and \( \text{Traj}_{T'}(G^{[2]}, \xi_k) \ni y_k \to y \)

uniformly on \([0, T']\). By hypothesis, the \( y_k \)’s remain in \( \Gamma_o \subseteq \Gamma \), so their uniform limit also remains in the closed set \( \Gamma \) on \([0, T']\). Therefore, \( z \) remains in \( \Gamma \) on \([0, \bar{t} + T']\), which contradicts our definition of \( \bar{t} \) in (24). This establishes that \((G^{[2]}, \Gamma)\) is strongly invariant and completes the proof of the lemma.

We next relate the tangential conditions for \( G \) and \( G^{[2]} \) from Theorems 3-4.

**Lemma 5.3.** For any \( \xi \in (\xi_1, \xi_2) \in \Gamma_o \) and \( u = (u_1, u_2) \in U^{[2]} \), the following three conditions are equivalent: (a) \( G(\xi_1, u_1) - G(\xi_2, u_2) \subseteq T_K^b(\xi_1 - \xi_2) \), (b) \( G^{[2]}(\xi, u) \subseteq T_B^b(\xi) \), and (c) \( G^{[2]}(\xi, u) \subseteq T_B^b(\xi) \).

**Proof.** First fix \( \xi = (\xi_1, \xi_2) \in \Gamma_o \), \( u = (u_1, u_2) \in U^{[2]} \) and values \( v_i \in D_i(\xi_1) \) and \( w_i \in D_i(\xi_2) \). Set

\[
f(x, u) = (f_1(x_1, u_1), f_2(x_2, u_2)) = \left( \prod_{i=1}^{n} g_i(x_i, u_i)v_i, \prod_{i=1}^{n} g_i(x_i, u_i)w_i \right).
\]

(25)
Then $f$ is locally Lipschitz in $x$ although its trajectories are not necessarily trajectories of $G^{[2]}$. The lemma follows if the following are equivalent for all $\xi \in \Gamma_0$ and $u \in U^{[2]}$:

\[(a') f_1(\xi_1, u_1) - f_2(\xi_2, u_2) \in T^B_\Gamma(\xi_1 - \xi_2), \quad (b') f(\xi, u) \in T^B_{\Gamma_0}(\xi), \quad \text{and} \quad (c') f(\xi, u) \in T^B_\Gamma(\xi).
\]

Since (25) is locally Lipschitz in $x$, this equivalence follows from the proof of Lemma III.9 in [1].

We can now prove our two monotonicity theorems. Assume (21) is monotone, $(\xi_1, \xi_2) \in \Gamma_0$, and $u^o = (u^o_1, u^o_2) \in U^{[2]}$. By Lemma 5.2, $(G^{[2]}, \Gamma_0)$ is strongly invariant. Applying Theorem 2 from Section 4 to the dynamics $\xi \mapsto G^{[2]}(\xi, u^o)$ in dimension $2n$ with $O = V \times V$ and $S = \Gamma_0$ gives $G^{[2]}(\xi, u^o) \in T^B_{\Gamma_0}(\xi)$ for all $\xi \in \Gamma_0$. The tangential conditions from both theorems therefore follow from Lemma 5.3. Conversely, assume either of these tangential conditions. By Lemma 5.2, monotonicity of (21) follows once we show that $(G^{[2]}, \Gamma_0)$ is strongly invariant, but this follows from Theorem 2 applied to $G^{[2]}$ because (b) of Lemma 5.3 is satisfied for all $u \in A := U^{[2]}$ and $\xi \in \Gamma_0$. This concludes the proof of our theorems.

Remark 5.4. The tangential conditions from our monotonicity theorems remain sufficient for monotonicity if we drop the convex valuedness and lower semicontinuity assumptions on the $D_i$’s and keep all our other assumptions the same (by the same proof). Moreover, the requirement that $0 \in D_i(x)$ for all $i$ and $x$ can be relaxed to requiring an increasing sequence $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_k \subseteq \ldots$ of compact subsets of $V \times V$ such that (i) $V \times V = \bigcup_k N_k$ and (ii) $0 \in D_i(x_1) \times D_i(x_2)$ for all $i \in \{1, 2, \ldots, n\}$, all $x = (x_1, x_2) \in \text{bd}(\Gamma_0 \cap N_k)$, and all $k$ (see Section 3.2).

References

[1] D. Angeli and E. Sontag, Monotone control systems. IEEE Trans. Autom. Control 48 (2003), 1684–1698.
[2] D. Angeli and E. Sontag, Interconnections of monotone systems with steady-state characteristics. Optimal Control, Stabilization, and Nonsmooth Analysis (M. de Queiroz, M. Malisoff and P. Wolenski, Eds.), Lecture Notes in Control and Information Sciences Vol. 301, Springer-Verlag, Heidelberg, 2004, 343-349.
[3] D. Angeli and E. Sontag, Multistability in monotone input/output systems. Systems and Control Letters 51 (2004), 185–202.
[4] J.-P. Aubin and H. Frankowska, Set-Valued Analysis. Systems and Control: Foundations and Applications 2. Birkhäuser Boston, Inc., Boston, MA, 1990.
[5] J. Borwein and A. Lewis, Convex Analysis and Nonlinear Optimization. Theory and Examples. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC 3. Springer-Verlag, New York, 2000.
[6] F. Clarke, Generalized gradients and applications. Trans. Amer. Math. Soc. 205 (1975), 247–262.
[7] F. Clarke and Y. Ledyaev, Mean value inequalities in Hilbert space. Trans. Amer. Math. Soc. 344 (1994), 307–324.
[8] F. Clarke, Y. Ledyaev and M. Radulescu, Approximate invariance and differential inclusions in Hilbert spaces. J. Dynam. Control Syst. 3 (1997), 493–518.
[9] F. Clarke, Y. Ledyaev, R. Stern and P. Wolenski, Nonsmooth Analysis and Control Theory. Graduate Texts in Mathematics 178. Springer-Verlag, New York, 1998.
[10] F. Clarke and R. Stern, State constrained feedback stabilization. SIAM J. Control Optim. 42 (2003), 422–441.
[11] T. Donchev, Properties of the reachable set of control systems. Systems and Control Letters 46 (2002), 379–386.
[12] T. Donchev, V. Rios and P. Wolenski, A characterization of strong invariance for perturbed dissipative systems. Optimal Control, Stabilization, and Nonsmooth Analysis (M. de Queiroz, M. Malisoff and P. Wolenski, Eds.), Lecture Notes in Control and Information Sciences Vol. 301, Springer-Verlag, Heidelberg, 2004, 343-349.
A Background in Nonsmooth Analysis

In this appendix, we review the necessary background in nonsmooth analysis; see [9] for a more complete treatment and Section 2 above for the relevant notation. Let $S \subseteq \mathbb{R}^n$ be closed and $x \in S$. A vector $\zeta \in \mathbb{R}^n$ is called a proximal normal of $S$ at $x$ provided there exists a constant $\sigma = \sigma(\zeta, x) \geq 0$ such that

$$
\langle \zeta, x' - x \rangle \leq \sigma |x' - x|^2.
$$

(26)

for all $x' \in S$. The set of all proximal normal vectors of $S$ at $x$ is denoted by $N^P_S(x)$. The following local characterization of $N^P_S$ also holds: For any $\delta > 0$, $\zeta \in N^P_S(x)$ if and only if there exists $\sigma = \sigma(\zeta, x) \geq 0$ such that (26) holds for all $x' \in S \cap (x + \delta B_n)$. We also define the Clarke tangent cone $T^C_S$ and the Bouligand (a.k.a. contingent) tangent cone $T^B_S$ to subsets $S \subseteq \mathbb{R}^n$ as follows. We say that $v \in T^C_S(x)$ provided for each sequence $x_i \in S$ converging to $x$ and each sequence $t_i > 0$ decreasing to 0, there exists a sequence $v_i \rightarrow v$ such that $x_i + t_i v_i \in S$ for all $i$. In particular, if $S = \{0\}$, then $T^C_S(0) = \{0\}$. The Bouligand tangent cone to $S$ is defined by

$$
T^B_S(x) := \{q \in \mathbb{R}^n : \exists t_i \in (0, \infty) \text{ and } S \ni x_i \rightarrow x \text{ s.t. } t_i \downarrow 0 \text{ and } (1/t_i)(x_i - x) \rightarrow q \} \quad \forall x \in S.
$$

Then $T^B_S(x) \subseteq (N^P_S(x))^o$ for all $x \in S$ and all closed sets $S \subseteq \mathbb{R}^n$, where the superscript $o$ denotes the polar set; i.e., $D^o := \{p \in \mathbb{R}^n : \langle p, d \rangle \leq 0 \forall d \in D\}$ for each $D \subseteq \mathbb{R}^n$. The Clarke and Bouligand tangent cones can differ for some sets $S$ but are known to agree when $S$ is a closed convex subset of $\mathbb{R}^n$. We also use the Clarke normal cone $N^C_R$ which can be characterized for any closed subset $R \subseteq \mathbb{R}^n$ as follows:

$$
N^C_R(x) = \overline{\text{co}} \left\{ \lim_{i \to \infty} \zeta_i : \zeta_i \in N^P_R(x_i), R \ni x_i \rightarrow x \right\}
$$

(27)

Then $(N^C_R(x))^o = T^C_R(x) \subseteq T^B_R(x)$ for all $x \in R$. 

---

[13] L. Evans, **Partial Differential Equations.** Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998.
[14] H. Frankowska, *Letter to Michael Malisoff,* March 11, 2005.
[15] H. Frankowska, S. Plaskacz and T. Rzezuchowski, *Measurable viability theorems and the Hamilton-Jacobi-Bellman equation.* Journal of Differential Equations 116 (1995), 265-305.
[16] M. Krastanov, M. Malisoff and P. Wolenski, *On the strong invariance property for non-Lipschitz dynamics.* LSU Mathematics Electronic Preprint Series No. 2004-5 and ArXiv math.OC/0403180.
[17] M. Malisoff, *Viscosity solutions of the Bellman equation for exit time optimal control problems with non-Lipschitz dynamics.* ESAIM: Control, Optimisation and Calculus of Variations 6 (2001), 415–441.
[18] V. Rios and P. Wolenski, *A characterization of strongly invariant systems for a class of non-Lipschitz multifunctions.* Proceedings of the 42nd IEEE Conference on Decision and Control (Maui, HI, December 2003), 2593-2594.
[19] R. Rockafellar and R. Wets, *Variational Analysis.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] Vol. 317, Springer-Verlag, Berlin, 1998.
[20] E. Sontag, *Mathematical Control Theory. Deterministic Finite-Dimensional Systems. Second Edition.* Texts in Applied Mathematics 6. Springer-Verlag, New York, 1998.
[21] H. Sussmann, *Uniqueness results for the value function via direct trajectory-construction methods.* Proceedings of the 42nd IEEE Conference on Decision and Control (Maui, HI, December 2003), 3293-3298.
[22] R. Vinter, *Optimal Control. Systems and Control: Foundations and Applications.* Birkhäuser Boston, Inc., Boston, MA, 2000.
[23] P. Wolenski and Y. Zhuang, *Proximal analysis and the minimal time function.* SIAM J. Control Optim. 36 (1998), 1048–1072.
Next assume $f : \mathbb{R}^n \to (-\infty, \infty]$ is lower semicontinuous and $x \in \text{domain}(f) := \{x' : f(x') < \infty\}$. Then $\zeta \in \mathbb{R}^n$ is called a proximal subgradient for $f$ at $x$ provided there exist $\sigma > 0$ and $\eta > 0$ such that

$$f(x') \geq f(x) + \langle \zeta, x' - x \rangle - \sigma\|x' - x\|^2$$

(28)

for all $x' \in x + \eta \mathcal{B}_n$. The (possibly empty) set of all proximal subgradients for $f$ at $x$ is denoted by $\partial P f(x)$. When $f$ is the characteristic function of a closed set $S \subseteq \mathbb{R}^n$ (i.e., $f(x) = 0$ if $x \in S$ and $f(x) = 1$ otherwise), it follows from the local characterization of $N^P_S$ and (28) that $\partial P f(x) = N^P_S(x)$ for all $x \in \text{bd}(S)$ and $\partial P f(x) = \{0\}$ otherwise.

We next state a version of the Clarke-Ledyaev Mean Value Inequality (cf. [9, p. 117] for its proof). Let $[x, Y]$ denote the closed convex hull of $x \in \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^n$.

**Theorem 5.** Assume $x \in \mathbb{R}^n, Y \subseteq \mathbb{R}^n$ is compact and convex, and $\Psi : \mathbb{R}^n \to \mathbb{R}$ is lower semicontinuous. Then for any $\delta < \min_{y \in Y} \Psi(y) - \Psi(x)$ and $\lambda > 0$, there exist $z \in [x, Y] + \lambda \mathcal{B}_n$ and $\zeta \in \partial P \Psi(z)$ such that $\delta < \langle \zeta, y - x \rangle$ for all $y \in Y$.

A function $s : \mathbb{R}^n \to \mathbb{R}^n$ is called a selection of $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ provided $s(x) \in F(x)$ for all $x \in \mathbb{R}^n$. The following result is known as Michael’s Selection Theorem (see [5, Chapter 8] or [19, Corollary 5.59]).

**Theorem 6.** Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be lower semicontinuous and closed, convex, and nonempty valued. Let $x \in \mathbb{R}^n$ and $v \in F(x)$. Then there exists a continuous selection $s$ of $F$ for which $s(x) = v$.

The following is a variant of the well known “compactness of trajectories” lemma which we use in Appendix B. Its proof is a special case of the proof of [9, Theorem IV.1.11].

**Lemma A.1.** Let $\tilde{x} \in \mathbb{R}^n, T > 0, \tilde{f} \in C[0, T]$ be also continuous in $t$, and $y_k : [0, T] \to \mathbb{R}^n$ be a sequence of uniformly bounded absolutely continuous functions satisfying $y_k(0) = \tilde{x}$ for all $k$. Assume

$$\dot{y}_k(t) \in \tilde{f} (\tau_k(t), y_k(t) + r_k(t)) + \delta_k(t) \mathcal{B}_n$$

(29)

for a.a. $t \in [0, T]$ and all $k$, where $\delta_k : [0, T] \to [0, \infty)$ is a sequence of measurable functions that converges to $0$ in $L^2[0, T]$ as $k \to \infty$, $r_k : [0, T] \to \mathbb{R}$ is a sequence of measurable functions converging uniformly to $0$ as $k \to \infty$, and $\tau_k : [0, T] \to [0, \infty)$ is a sequence of measurable functions converging uniformly to $\tau(t) \equiv t$ as $k \to \infty$. Then there exists a trajectory $y$ of $\dot{y} = \tilde{f}(t, y), y(0) = \tilde{x}$ such that a subsequence of $y_k$ converges to $y$ uniformly on $[0, T]$.

We apply Lemma A.1 to continuous mollifications of our feedback realizations $f \in C[0, T]$ defined as follows. We first define the standard mollifier

$$\eta(t) = \begin{cases} C \exp \left(\frac{1}{|t|^2 - 1}\right), & |t| < 1 \\ 0, & |t| \geq 1 \end{cases}$$

where the constant $C > 0$ is chosen so that $\int_{\mathbb{R}} \eta(s) ds = 1$. For each $\varepsilon > 0$ and $t \in \mathbb{R}$, set $\eta_\varepsilon(t) := \eta(t/\varepsilon)/\varepsilon$. Notice that $\int_{\mathbb{R}} \eta_\varepsilon(t) dt = 1$ for all $\varepsilon > 0$. Define the following mollifications $f \in C[0, T]$:

$$f_\varepsilon(t, x) := \int_{\mathbb{R}} f(s, x) \eta_\varepsilon(t - s) ds$$

(30)

with the convention that $f(s, x) = 0$ for all $s \notin [0, T]$. Then $f_\varepsilon \in C[0, T]$ and $(t, x) \mapsto f_\varepsilon(t, x)$ is continuous for all $\varepsilon > 0$. (See [13, Appendix C] for the theory of convolutions and mollifiers.) We apply Lemma A.1 to a sequence $f_{\varepsilon(k)}$ of mollifications with $\varepsilon(k) > 0$ converging to zero using ideas from the usual proof that $f_{\varepsilon(k)}(\cdot, x) \to f(\cdot, x)$ in $L^1[0, T]$ for each $x$ and $f \in C[0, T]$ as $k \to \infty$.

**Remark A.2.** If $\tau_k(t) \equiv t$ for all $k$ in Lemma A.1, then the conclusions of the lemma remain true even if the $t$-continuity hypothesis on $f \in C[0, T]$ is omitted. This follows from the proof of the compactness of trajectories lemma in [9] and is applied to (46) below.
B Additional Proofs

In this appendix, we summarize the argument from [16] needed to complete our proof of Theorem 1. We continue to use the notation from Section 4.1 and we set

\[
T' := \min \left\{ T, \frac{\gamma}{32\delta(D)} \right\} \quad \text{and} \quad h_k := \frac{\gamma}{32k\delta(D)}
\]

(31)

for all \( k \in \mathbb{N} \). Choose \( N > 2 \) such that

\[
D + h_k\delta(D)B_n \subseteq \bar{x} + \frac{2\gamma}{3}B_n \quad \forall k \geq N.
\]

(32)

By the choices of \( \gamma \in (0,1) \) and \( \delta(D) \),

\[
0 < h_k \leq \frac{1}{32kgyf[t, x, k/n]} \forall t \in [0,T], x \in D, k \in \mathbb{N}.
\]

(33)

Next we define \( c(k) = \text{Ceiling}(T'/h_k) \); i.e., \( c(k) \) is the smallest integer \( \geq T'/h_k \). For each \( k \geq N \), we then define a partition \( \pi(k) : 0 = t_0,k < t_1,k < \cdots < t_{c(k),k} = T' \) by setting \( t_i,k = t_{i-1,k} + h_k \) for \( i = 1, 2, \ldots, c(k) - 1 \).

We next define sequences \( x_{0,k}, x_{1,k}, x_{2,k}, \ldots, x_{c(k),k} \) for \( k \geq N \) as follows. We set \( x_{0,k} = \bar{x} \) and

\[
x_{1,k} = \bar{x} + (t_{1,k} - t_{0,k})v_{o,k},
\]

(34)

where \( v = v_{o,k} \in G_f^r[0, \bar{x}, k/n] \) satisfies the requirement from Claim 4.1 for the pair \( (t, x) = (0, \bar{x}) \) and \( h = h_k \). By (18) and (34), we get

\[
\|x_{1,k} - \bar{x}\| \leq h_k\delta(D) = \frac{\gamma}{32k},
\]

(35)

so \( x_{1,k} \in \bar{x} + \frac{\gamma}{32}B_n = D \). If \( c(k) \geq 2 \), then we set

\[
x_{2,k} = x_{1,k} + (t_{2,k} - t_{1,k})v_{1,k},
\]

(36)

where \( v_{1,k} \in G_f[t_{1,k}, x_{1,k}, k/n] \) satisfies the requirement from Claim 4.1 for the pair \( (t, x) = (t_{1,k}, x_{1,k}) \) and \( h = h_k \). Then (18) and (36) give \( \|x_{2,k} - x_{1,k}\| \leq h_k\delta(D) = \frac{\gamma}{32k} \), so

\[
\|x_{2,k} - \bar{x}\| \leq \frac{\gamma}{16k},
\]

(37)

by (35). By (37), \( x_{2,k} \in D \). We now repeat this process except with \( x_{2,k} \in D \) replacing \( x_{1,k} \). Proceeding inductively gives sequences \( v_{i,k} \in G_f^r[t_{i,k}, x_{i,k}, k/n] \) and \( x_{i,k} \) that satisfy

\[
x_{i+1,k} = x_{i,k} + (t_{i+1,k} - t_{i,k})v_{i,k}
\]

(38)

for each index \( i = 0, 1, \ldots, c(k) - 1 \). The \( v_{i,k} \)'s are chosen using Claim 4.1 with the choices \( h = h_k \) and \( (t, x) = (t_{i,k}, x_{i,k}) \) for all \( i \) and \( k \). The choices of \( T' \) and \( k \geq 2 \) and (38) give

\[
\|x_{i,k} - \bar{x}\| \leq \frac{c(k)\gamma}{32k} \leq \left( \frac{T'}{h_k} + 1 \right) \frac{\gamma}{4k} \leq \frac{\gamma}{2}
\]

for all \( i \) and \( k \). It follows that the sequences \( x_{i,k} \) lie in \( D \).

For each \( k \geq N \), we then choose \( x_{\pi(k)}(t) \) to be the unique polygonal arc satisfying \( x_{\pi(k)}(0) = \bar{x} \) and

\[
\hat{x}_{\pi(k)}(t) = f_c(\tau_k(t), x_{\pi(k)}(t) + r_k(t)) + z_k(\tau_k(t))
\]

(39)

for all \( t \in [0, T'] \setminus \pi(k) \), where \( \tau_k(t) \) is the partition point \( t_{i,k} \in \pi(k) \) immediately preceding \( t \) for each \( t \in [0, T'] \),

\[
z_k(t_{i,k}) := v_{i,k} - f_c(t_{i,k}, x_{\pi(k)}(t_{i,k})) \quad \forall i, k
\]

(40)
the \( v_{i,k} \in G^\gamma_{\varepsilon}[t_{i,k}, x_{\pi(k)}(t_{i,k}), k/n] \) satisfy the conclusions from Claim 4.1 for the pairs \((t, x) = (t_{i,k}, x_{i,k})\) and \( h = h_k \), and
\[
 r_k(t) := x_{\pi(k)}(r_k(t)) - x_{\pi(k)}(t) \quad \forall t \in [0, T'], \forall k.
\]
Then \( x_{\pi(k)} \) is the polygonal arc connecting the points \( x_{i,k} \) for \( i = 0, 1, 2, \ldots, c(k) \) so \( x_{n+1} = x_{\pi(k)}(t_{i,k}) \).

Since \((t, x) \mapsto f_\varepsilon(t, x)\) is continuous, we can use Claim 4.2 and the forms of \( z_k \) and \( r_k \) in (40)-(41) to check that (39) satisfies the requirements of our compactness of trajectories lemma with \( y_k = x_{\pi(k)} \) (see [16]), so we can find a subsequence of \( x_{\pi(k)} \) that converges uniformly to a trajectory \( y_\varepsilon \) of \( \dot{y} = f_\varepsilon(t, y) \), \( y(0) = \bar{x} \). By possibly passing to a subsequence without relabelling, we can assume that \( x_{\pi(k)} \to y_\varepsilon \) uniformly on \([0, T']\). Since \( t_{i+1,k} - t_{i,k} \in [0, h_k] \) and \( x_{i+1,k} = x_{i,k} + (t_{i+1,k} - t_{i,k})v_{i,k} \in D \) for all \( i = 0, 1, \ldots, c(k) - 1 \) and \( k \geq N \), conditions (32) and (33) and Claim 4.1 give
\[
\Psi(x_{i,k}) - \Psi(x_{i-1,k}) \leq \frac{h_k}{k}
\]
for \( i = 1, 2, \ldots, c(k) \). Summing the inequalities (42) over \( i \) and noting that \( h_k \leq \gamma \) gives
\[
\Psi(x_{i,k}) \leq \Psi(\bar{x}) + \frac{c(k)h_k}{k} \leq \Psi(\bar{x}) + \frac{1}{k}(T' + \gamma)
\]
for all \( i \) and \( k \). Since \( x_{\pi(k)}(r_k(t)) = x_{i,k} \) for all \( t \in [t_{i,k}, t_{i+1,k}] \) and all \( i \) and \( k \), (43) gives
\[
\Psi(x_{\pi(k)}(r_k(t))) \leq \Psi(\bar{x}) + \frac{1}{k}(T' + \gamma)
\]
for all \( t \in [0, T'] \). Since \(|r_k(t) - t| \leq h_k \to 0 \) as \( k \to +\infty \) for all \( t \in [0, T'] \), \( x_{\pi(k)}(r_k(t)) \to y_\varepsilon(t) \) for all \( t \in [0, T'] \) as \( k \to +\infty \). Moreover, \( \Psi \) is lower semicontinuous, so it follows from (44) that
\[
\Psi(y_\varepsilon(t)) \leq \Psi(\bar{x})
\]
for all \( t \in [0, T'] \).

Now let \( y_{1/i} : [0, T'] \to \mathbb{R}^n \) be the trajectory obtained by the preceding argument with the choice \( \varepsilon = 1/i \) for each \( i \in \mathbb{N} \). Note that \( y_{1/i}(t) \in D \) for all \( i \) and \( t \) because each of the polygonal arcs \( x_{\pi(k)} \) constructed above joins points in \( D \) and \( D \) is closed and convex. Moreover,
\[
\dot{y}_{1/i}(t) = f(t, y_{1/i}(t)) + [f_{1/i}(t, y_{1/i}(t)) - f(t, y_{1/i}(t))]
\]
for all \( i \) and almost all \( t \in [0, T'] \). One can check (see [16]) that the \( y_{1/i} \)'s are also equicontinuous, so we can assume (by passing to a subsequence) there is a continuous function \( y : [0, T'] \to D \) such that \( y_{1/i} \to y \) uniformly on \([0, T']\) (by the Ascoli-Arzelà lemma). One can show (cf. [16, Claim 4.3]) that
\[
f_{1/i}(t, y_{1/i}(t)) - f(t, y_{1/i}(t)) \to 0 \quad \text{in} \quad L^2[0, T']
\]
as \( i \to \infty \). The proof of (47) is based on the standard proof that \( f_{1/i}(\cdot, x) - f(\cdot, x) \to 0 \) in \( L^1[0, T'] \) for each \( x \) (see for example [13, pp. 630-1]) and property \( (C_2) \) from the definition of \( C[0, T] \). It therefore follows from Remark A.2 and the form of the dynamics (46) that a subsequence of \( y_{1/i} \) converges uniformly to some trajectory of \( f \) on \([0, T']\). This must be the aforementioned function \( y \), which is therefore a trajectory of \( f \). Again using the lower semicontinuity of \( \Psi \), (45) applied along the sequence \( \varepsilon = 1/i \) gives
\[
\Psi(y(t)) \leq \liminf_{i \to \infty} \Psi(y_{1/i}(t)) \leq \Psi(\bar{x}) \leq 0
\]
for all \( t \in [0, T'] \). In particular, \( y \in \text{Traj}_T(F, \bar{x}) \) and \( y : [0, T'] \to S \).

Finally, we prove the strong invariance assertion of the theorem. We assume \( x_0 \in S \), \( T \geq 0 \), and \( z \in \text{Traj}_T(F, x_0) \). Set
\[
\bar{t} := \sup \{ t \geq 0 : \Psi(z(s)) \leq 0 \forall s \in [0, t] \}.
\]
We next show that $\bar{t} = T$ (by contradiction), which would imply that $z$ remains in $S$ on $[0, T]$ and establish the theorem. Suppose $\bar{t} < T$. The lower semicontinuity of $\Psi$ gives $\Psi(z(\bar{t})) \leq 0$. The definition of $\bar{t}$ implies that $z(\cdot) \in S$ on $[0, \bar{t}]$ and in particular $\bar{x} := z(\bar{t}) \in \text{bd}(S) \subseteq U$. Next we let $f \in \mathcal{C}'([0, T-\bar{t}], \bar{x})$ satisfy the requirement $(U')$ for $F$ and the following trajectory for $F$:

$$[0, T-\bar{t}] \ni t \mapsto y(t) := z(t + \bar{t}).$$  \hspace{1cm} (49)$$

By definition, we can then find $\gamma \in (0, 1)$ satisfying $f(t, x) \in \text{cone}\{F(x)\}$ for a.a. $t \in [0, \gamma)$ and all $x \in \bar{x} + \gamma \mathcal{B}_n$. By reducing $\gamma > 0$ without relabelling, we can assume $\bar{x} + \gamma \mathcal{B}_n \subseteq U$.

By uniqueness of solutions of the initial value problem $\dot{y} = f(t, y)$, $y(0) = z(\bar{t})$ on $[0, T-\bar{t}]$, the above argument applied to $f$ and $\bar{x} = z(\bar{t})$ gives $\bar{t} \in (0, T-\bar{t})$ such that

$$\Psi(z(\bar{t} + t)) \leq 0 \ \forall t \in [0, \bar{t}].$$  \hspace{1cm} (50)$$

Here we use the fact that the trajectory for $f$ on $[0, T']$ starting at $\bar{x}$ (and valued in $S$) that we constructed above (where we can assume $T' < \gamma$) can be extended to a new trajectory for $f$ defined on all of $[0, T-\bar{t}]$ by the linear growth assumption $(C_3)$ on $f$ and so coincides with (49) by our uniqueness assumption in $(U')$. We can therefore set $t = T'$. Since $z$ remains in $S$ on $[0, \bar{t}]$, we conclude from (50) that $z$ remains in $S$ on $[0, t + \bar{t}]$. This contradicts the definition (48) of $\bar{t}$ and proves Theorem 1.

**Remark B.1.** The preceding proof provides a constructive approach to finding viable trajectories for our Carathéodory feedback realizations $f$ that remain in $S$. As suggested by [14], an alternative but highly nonconstructive proof of Theorem 1 would proceed as follows. Let $x(t)$ be any trajectory of $F$ starting in $S$. By Condition $(U')$, $x(t)$ admits a feedback realization $f \in \mathcal{C}[0, T]$, and our Hamiltonian assumption gives $\langle (f(t, x), 0), q \rangle \leq 0$ for all $q \in N^\circ_{\psi(x)}(x, \Psi(x)) = \{ (\xi, -1) : \xi \in \partial \Psi(x) \}$, almost all $t \geq 0$, and all $x \in U$, where $\text{epi} \ (\Psi) := \{ (x, r) : r \geq \Psi(x) \}$ is the epigraph of $\Psi$. It is not hard to deduce from this (cf. [9]) that $(f(t, x), 0) \in T^\circ_{\text{epi} \ (\Psi)}(x, r)$ for almost all $t \geq 0$, all $x \in U$, and all $(x, r) \in \text{epi} \ (\Psi)$, where $T^\circ_{\text{epi} \ (\Psi)}$ denotes the Bouligand tangent cone. Applying the measurable viability theorem (see [15, Section 4]) to the dynamics $F(t, x) = \{(f(t, x), 0)\}$ provides a trajectory $t \mapsto (\phi(t), \Psi(x(0)))$ for $\dot{x} = f(t, x)$, $\dot{y} = 0$ starting at $(x(0), \Psi(x(0)))$ that stays in $\text{epi} \ (\Psi)$. This requires $f$ to be modified outside $U$ in the usual way. Hence, $\Psi(\phi(t)) \leq \Psi(x(0)) \leq 0$ for all $t$. By the uniqueness part of Condition $(U')$, $\phi(t) \equiv x(t)$ so $x(t)$ stays in $S$. Unfortunately, the preceding alternative argument is highly nonconstructive, since the proof of the measurable viability theorem relies on Zorn’s Lemma to construct the trajectory $\phi(t)$. One natural question which should be considered is how our Euler constructions from our proof could be used to build numerical schemes for approximating viable trajectories for Carathéodory dynamics. This type of result could be useful in physical applications. This question will be addressed by the author in future research.