Even and odd geometries on supermanifolds

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We analyze from a general perspective all possible supersymmetric generalizations of symplectic and metric structures on smooth manifolds. There are two different types of structures according to the even/odd character of the corresponding quadratic tensors. In general we can have even/odd symplectic supermanifolds, Fedosov supermanifolds and Riemannian supermanifolds. The geometry of even Fedosov supermanifolds is strongly constrained and has to be flat. In the odd case, the scalar curvature is only constrained by Bianchi identities. However, we show that odd Riemannian supermanifolds can only have constant scalar curvature. We also point out that the supersymmetric generalizations of AdS space do not exist in the odd case.

1 Introduction

The two main quadratic geometrical structures of smooth manifolds which play a significant role in classical and quantum physics are Riemannian metrics and symplectic forms. Riemannian geometry is not only basic for the formulation of general relativity but also for the very formulation of gauge field theories. The symplectic structure provides the geometrical framework for classical mechanics (see, e.g. \cite{1}) and field theories \cite{2}. The Fedosov method of quantization by deformation \cite{3} is also formulated in terms of symplectic structures and symplectic connections (the so-called Fedosov manifolds \cite{4}). The introduction of the concept of supermanifold by Berezin \cite{5} (see also \cite{6, 7}) opened new perspectives for geometrical approaches of supergravity and quantization of gauge theories \cite{8, 9, 10}. In summary, the geometry of manifolds and supermanifolds percolates all fundamental physical theories.

In this note we address the classification of possible extensions of symplectic and metric structures to supermanifolds in terms of graded symmetric and antisymmetric second-order tensor fields. The cases of even and odd symplectic and Riemannian supermanifolds are analyzed in some detail. Graded non-degenerate Poisson supermanifolds are described by symplectic supermanifolds that if equipped with a symmetric symplectic connection become graded Fedosov supermanifolds. The even case corresponds to a straightforward generalization of Fedosov manifold \cite{4} where the scalar curvature vanishes as for standard Fedosov manifolds. Graded metric supermanifolds equipped with the unique compatible symmetric connection also correspond to graded Riemannian supermanifold. The scalar
curvature is non trivial, in general, for odd Riemannian and Fedosov supermanifolds, but in the first case it must always be constant. There is a supersymmetric generalization of AdS space but it is trivial in the odd case.

The paper is organized as follows. In Sect. 2, we consider scalar structures which can be used for the construction of symplectic and metric supermanifolds. The properties of symmetric affine connections on supermanifolds and their curvature tensors are analyzed in Sect. 3. In Sect. 4, we introduce the concepts of even and odd Fedosov supermanifolds and even and odd Riemannian supermanifolds are analyzed in Sect. 5. Finally, we convey the main results in Sect. 6. We use the condensed notation suggested by DeWitt [11] and definitions and notations adopted in [12].

2 Scalar Fields

Let $\mathcal{M}$ be a supermanifold with a dimension $\dim \mathcal{M} = N$ and $\{x^i\}, \epsilon(x^i) = \epsilon_i$ a local system of coordinates on in the vicinity of a point $p \in \mathcal{M}$. Let us consider now the most general scalar structures on supermanifolds which can be defined in terms of graded second-rank symmetric and antisymmetric tensor fields.

In general, there exist eight types of second rank tensor fields with the required symmetry properties

\begin{align*}
\omega^{ij} &= -(1)^{\epsilon_i \epsilon_j} \omega^{ji}, \\
\Omega^{ij} &= -(1)^{\epsilon_i \epsilon_i} \Omega^{ji}, \\
E_{ij} &= -(1)^{\epsilon_i \epsilon_j} E_{ji}, \\
g_{ij} &= -(1)^{\epsilon_i \epsilon_j} g_{ji},
\end{align*}

Using these tensor fields (1)–(4) it is not difficult to built eight scalar structures on a supermanifold:

\begin{align*}
\{A, B\} &= \frac{\partial_r A}{\partial x^i} (1)^{\epsilon_i} \epsilon(\omega) \frac{\partial B}{\partial x^i}, \\
(A, B) &= \frac{\partial_r A}{\partial x^i} (1)^{\epsilon_i} \epsilon(\Omega) \frac{\partial B}{\partial x^i}, \\
E &= E_{ij} dx^j \wedge dx^i, \\
g &= g_{ij} dx^j dx^i,
\end{align*}

where $A$ and $B$ are arbitrary superfunctions.

The bilinear operation $\{A, B\}$ (5) obeys the following symmetry property

$$\{A, B\} = -(1)^{\epsilon(\omega) + \epsilon(A) + \epsilon(B) + \epsilon(\Omega)} \{B, A\}$$

which in the even case ($\epsilon(\omega) = 0$) reduces to

$$\{A, B\} = -(1)^{\epsilon(A) + \epsilon(B)} \{B, A\}$$

and in the odd case ($\epsilon(\omega) = 1$) to

$$\{A, B\} = -(1)^{\epsilon(A) + 1 + \epsilon(B) + 1} \{B, A\}.$$

On the other hand, the bilinear operation $(A, B)$ (6) has the symmetry property

$$(A, B) = -(1)^{\epsilon(\omega) + \epsilon(A) + \epsilon(B) + \epsilon(\Omega)} (B, A)$$

which in the even case ($\epsilon(\omega) = 0$) reduces to

$$(A, B) = -(1)^{\epsilon(A) + \epsilon(B)} (B, A)$$
and in the odd case \((\epsilon(\omega) = 1)\) to

\[
(A, B) = -(-1)^{(\epsilon(A)+1)(\epsilon(B)+1)}(B, A).
\] (14)

One can easily check that in the even case \((\epsilon(\omega) = 0)\) the bilinear operation \(\{A, B\}\) satisfies the Jacobi identity

\[
\{A, \{B, C\}\} - \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0
\] (15)

if and only if \(\omega\) satisfies

\[
\omega^{ij} \partial \omega_{kl} (1)^{\epsilon_1 \epsilon_i l} + \omega^{ij} \partial \omega_{ik} (1)^{\epsilon_1 \epsilon_k i} + \omega^{ij} \partial \omega_{li} (1)^{\epsilon_1 \epsilon_k i} \equiv 0.
\] (16)

In the odd case there is no possibility of satisfying the Jacobi identity for the operation \(\{A, B\}\).

On the contrary, the Jacobi’s identity for \((A, B)\) can be satisfied

\[
(A, (B, C)) - (C, (A, B)) + (B, (C, A)) = 0
\] if and only if \(\Omega\) is odd, \(\epsilon(\Omega) = 1\), and satisfies

\[
\Omega^{ij} \partial \Omega_{kl} (1)^{\epsilon_1 \epsilon_i l} + \Omega^{ij} \partial \Omega_{ik} (1)^{\epsilon_1 \epsilon_k i} + \Omega^{ij} \partial \Omega_{li} (1)^{\epsilon_1 \epsilon_k i} \equiv 0.
\] (17)

Therefore, because of the identities \([15, 16]\) and \([17]\), one can identify \(A, B\) \((\epsilon(\{A, B\}) = \epsilon(A) + \epsilon(B)\) and \((A, B)\) \((\epsilon((A, B)) = \epsilon(A) + \epsilon(B) + 1)\) with the Poisson bracket and the antibracket respectively.

It is also possible to combine the Poisson bracket associated to \(\omega\) and the antibracket into the so-called graded Poisson bracket (see, for example, [13, 14, 15, 16]) in the following bilinear operation

\[
\{A, B\}_g = \frac{\partial A}{\partial x^i} (1)^{\epsilon_1 \epsilon_i} \omega_{ij} \frac{\partial B}{\partial x^j}, \quad \omega_{ij} = -(-1)^{\epsilon_1 \epsilon_2 \epsilon_3} \omega_{ij},
\] (18)

\[
\epsilon(\{A, B\}_g) = \epsilon(\omega) + \epsilon(A) + \epsilon(B).
\] (19)

From \([19]\) it follows the symmetry property

\[
\{A, B\}_g = -(-1)^{\epsilon(A) + \epsilon(\omega_1) + \epsilon(\omega_2) + \epsilon(\omega_3)} \{B, A\}_g.
\] (20)

If the tensor fields \(\omega_{ij}^{kl}\) satisfy the identities

\[
\omega_{ij}^{kl} \partial \omega_{ij} (1)^{\epsilon_1 \epsilon_i \epsilon_j} + \omega_{ij}^{kl} \partial \omega_{ij} (1)^{\epsilon_1 \epsilon_i \epsilon_j} + \omega_{ij}^{kl} \partial \omega_{ij} (1)^{\epsilon_1 \epsilon_i \epsilon_j} \equiv 0,
\] (20)

then \(\{A, B\}_g\) satisfies the Jacobi identity

\[
\{A, \{B, C\}_g\} - \{B, \{C, A\}_g\} + \{C, \{A, B\}_g\} = 0
\] (21)

with \(\epsilon_1 \epsilon_2 \epsilon_3 = (\epsilon(A) + \epsilon(\omega_1) + \epsilon(\omega_2) + \epsilon(\omega_3))\) and plays the role of a graded Poisson bracket.

A supermanifold \(\mathcal{M}\) equipped with a Poisson bracket is called a Poisson supermanifold, \((\mathcal{M}, \{\})).\) Usually a manifold \(\mathcal{M}\) equipped with an non-degenerate antibracket is called an antisymplectic supermanifold \((\mathcal{M}, \{\})\) or, sometimes, an odd Poisson supermanifold (see, for example, [14, 16]).

In Eq. \(\{3\}\) \(\omega\) denotes a generic graded differential 2-form. If \(\omega\) is closed

\[
d\omega = E_{ijk} dx^k \wedge dx^j \wedge dx^i = 0
\] (22)
and non-degenerate, then it defines a graded (even or odd) symplectic supermanifold \((M, E)\) \([6]\). In terms of tensor fields \(E_{ij}\) the condition \((22)\) can be expressed as

\[
E_{ij,k}(-1)^{\epsilon_k e_k} + E_{jk,i}(-1)^{\epsilon_j e_j} + E_{ki,j}(-1)^{\epsilon_k e_j} = 0, \quad E_{ij} = (-1)^{\epsilon_i e_j} E_{ji}
\]  

(23)

and in terms of inverse tensor fields \(E^{ij}\) Eqs. \((23)\) can be rewritten in the form

\[
E^{jl} \frac{\partial E^{jk}}{\partial x^l}(-1)^{\epsilon_l (\epsilon_k + \epsilon(E))} + E^{kl} \frac{\partial E^{ij}}{\partial x^l}(-1)^{\epsilon_k (\epsilon_j + \epsilon(E))} + E^{ij} \frac{\partial E^{kl}}{\partial x^l}(-1)^{\epsilon_j (\epsilon_i + \epsilon(E))} = 0,
\]

(24)

where \(E^{ij} = (-1)^{\epsilon(E) + \epsilon_i \epsilon_j} E^{ji}\). Identifying \(E^{ij}\) with the tensor field \(\omega^{ij}\) in \((5)\), one gets in the even case \((\epsilon(E) = 0)\) the Poisson bracket for which the Jacobi identity \((15)\) follows from \((21)\). Therefore, in the even case there is one-to-one correspondence between non-degenerate Poisson supermanifolds and an even symplectic supermanifolds. In the odd case \((\epsilon(E) = 1)\), if we assume \(E^{ij} = \Omega^{ij}\) in \((6)\) then \(E^{ij}\) defines an antibracket for which the Jacobi identity \((17)\) follows from \((24)\). Therefore antisymplectic supermanifolds can be identified with odd symplectic manifolds.

If the tensor field \(g_{ij}\) in \((5)\) is non-degenerate, one has a graded metric that can provide a supermanifold \(M\) with a graded (even or odd) metric structure, giving rise to a Riemannian supermanifold \((M, g)\). On the other hand, the inverse tensor field \(g^{ij}\) also defines a bilinear operation with symmetry properties \((11)\) or \((13)\) but it does not satisfy the Jacobi identity.

3 Connections in Supermanifolds

Let us consider a covariant derivative \(\nabla\) (or an affine connection \(\Gamma\)) on a supermanifold \(M\). In each local coordinate system \(\{x\}\) the covariant derivative \(\nabla\) is described by its components \(\nabla_i (\epsilon(\nabla_i) = \epsilon_i)\), which are related to the components the affine connection \(\Gamma^{ij}_{jk}\) \((\epsilon(\Gamma^{ij}_{jk}) = \epsilon_i + \epsilon_j + \epsilon_k)\) by

\[
e^i \nabla_j = e^k \Gamma^i_{k j}(-1)^{\epsilon_k (\epsilon_i + 1)}, \quad \epsilon_i \nabla_j = -e_k \Gamma^k_{ij}
\]

(25)

where \(\{\epsilon_i\}\) and \(\{e^i\}\) are the associated bases of the tangent \(TM\) and cotangent \(T^*M\) spaces respectively. The action of the covariant derivative on a tensor field of any rank and type is given in terms of \(\nabla\) or \(\epsilon\) or \(\Gamma\).

From here on, we shall consider only symmetric connections

\[
\Gamma^i_{jk} = (-1)^{\epsilon_j \epsilon_k} \Gamma^i_{kj}.
\]

(26)

The curvature tensor field \(R^i_{mjk}\) is defined in terms of the commutator of covariant derivatives, \([\nabla_i, \nabla_j] = \nabla_i \nabla_j - (-1)^{\epsilon_i \epsilon_j} \nabla_j \nabla_i\), whose action on a vector field \(T^i\) is

\[
T^i [\nabla_j, \nabla_k] = (-1)^{\epsilon_m (\epsilon_i + 1)} T^m R^i_{mjk}.
\]

(27)

The choice of factor in r.h.s \((27)\) is dictated by the requirement that the contraction of tensor fields of types \((1, 0)\) and \((1, 3)\) yield a tensor field of type \((1, 2)\). A straightforward calculation yields

\[
R^i_{mjk} = -\Gamma^i_{mj,k} + \Gamma^i_{mk,j}(-1)^{\epsilon_j \epsilon_k} + \Gamma^i_{jn} \Gamma^m_{mk}(-1)^{\epsilon_j \epsilon_m} - \Gamma^i_{kn} \Gamma^m_{mj}(-1)^{\epsilon_k (\epsilon_m + \epsilon_j)}.
\]

(28)

The curvature tensor field has a generalized antisymmetry,

\[
R^i_{mjk} = (-1)^{\epsilon_j \epsilon_k} R^i_{mkj};
\]

(29)

and satisfies the Jacobi identity,

\[
(-1)^{\epsilon_m \epsilon_k} R^i_{mjk} + (-1)^{\epsilon_j \epsilon_m} R^i_{jkm} + (-1)^{\epsilon_k \epsilon_j} R^i_{kmj} = 0.
\]

(30)
Using the Jacobi identity for the covariant derivatives,
\[ [\nabla_i, [\nabla_j, \nabla_k]](1)\epsilon^i\epsilon_k + [\nabla_k, [\nabla_i, \nabla_j]](1)\epsilon^k\epsilon_j + [\nabla_j, [\nabla_k, \nabla_i]](1)\epsilon^j\epsilon_i \equiv 0, \]
(31)
one obtains the Bianchi identity,
\[ (-1)^{\epsilon_i\epsilon_j} R^n_{mjk;i} + (-1)^{\epsilon_j\epsilon_k} R^n_{mij;k} + (-1)^{\epsilon_k\epsilon_j} R^n_{mkij} \equiv 0, \]
(32)
with the notation \( R^n_{mjk;i} := R^n_{mjk} \nabla_i \).

4 Symplectic supermanifolds

Let us consider a symplectic supermanifold \((\mathcal{M}, \omega)\), i.e. a supermanifold \(\mathcal{M}\) with a closed non-degenerate graded differential 2-form \(\omega\)
\[ \omega = \omega_{ij} dx^i \wedge dx^j, \quad \omega_{ij} = -(-1)^{\epsilon_i\epsilon_j} \omega_{ji}. \]
(33)
The closure condition of \(\omega\), \(d\omega = 0\), can be rewritten as
\[ \omega_{ij,k}(-1)^{\epsilon_i\epsilon_k} + \omega_{jk,i}(-1)^{\epsilon_j\epsilon_k} + \omega_{ki,j}(-1)^{\epsilon_j\epsilon_k} = 0 \]
in terms of the inverse tensor field \(\omega^{ij}\)
\[ \omega^{ij} = -(-1)^{\epsilon(\omega)+\epsilon_i\epsilon_j} \omega^{ji}. \]
(35)
and do coincide with identities [20]. It means that in the even case \((\epsilon(\omega) = 0)\) \(\omega^{ij}\) defines a non-degenerate Poisson bracket while in the odd case \((\epsilon(\omega) = 1)\) it defines an antibracket. Therefore in the even case there is a one-to-one correspondence between even symplectic supermanifolds and non-degenerate Poisson supermanifold. In the odd case any antisymplectic supermanifold is nothing but an odd symplectic supermanifold.

Let \(\Gamma\) be a symmetric connection of a symplectic supermanifold \((\mathcal{M}, \omega)\). The corresponding covariant derivative \(\nabla\) has to verify the compatibility condition \(\omega \nabla = 0\) with the symplectic structure \(\omega\). In each local coordinate system \(\{x^i\}\) the compatibility condition can be expressed as
\[ \omega_{ij} \nabla_k = \omega_{ij,k} - \Gamma_{ijk} + \Gamma_{jik}(-1)^{\epsilon_i\epsilon_j} = 0, \quad \omega_{ij} = -(-1)^{\epsilon_i\epsilon_j} \omega_{ji}. \]
(36)
in terms of the components \(\Gamma^i_{jk} (\nabla_i)\) of the symplectic connection \(\nabla\), where we use the notation
\[ \Gamma_{ijk} = \omega_{im} \Gamma^n_{jk} = \epsilon(\Gamma_{ijk}) = \epsilon(\omega) + \epsilon_i + \epsilon_j + \epsilon_k. \]
(37)
A symplectic supermanifold \((\mathcal{M}, \omega)\) equipped with a symmetric symplectic connection \(\Gamma\) is called a Fedosov supermanifold \((\mathcal{M}, \omega, \Gamma)\).

Let us consider now curvature tensor \(R_{ijkl}\) of a symplectic connection
\[ R_{ijkl} = \omega_{im} R^n_{jk} = \epsilon(R_{ijkl}) = \epsilon(\omega) + \epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l, \]
(38)
where \(R^n_{jk}\) is defined in [23]. This tensor has the following symmetry properties
\[ R_{ijkl} = -(-1)^{\epsilon^k\epsilon^l} R_{ijlk}, \quad R_{ijkl} = (-1)^{\epsilon^i\epsilon^j} R_{jikl} \]
(39)
and satisfies the identity
\[ R_{ijkl} + (-1)^{\epsilon_i(\epsilon_k+\epsilon_j)} R_{iljk} + (-1)^{\epsilon_k(\epsilon_i+\epsilon_j)} R_{klij} + (-1)^{\epsilon_i(\epsilon_j+\epsilon_k)} R_{jkl} = 0. \]
(40)
The last statement can be derived from the Jacobi identity
\[(−1)^{ε_1}R_{ijkl} + (−1)^{ε_2}R_{iljk} + (−1)^{ε_3}R_{iklj} = 0,\]  
(41)
together with a cyclic change of indices [17]. The identity (40) involves different components of the curvature tensor with cyclic permutation of all indices, but the sign factors depend on the Grassmann parities of the indices and do not follow a cyclic permutation rule, similar to that of Jacobi identity, but are defined by the permutation of the indices that maps a given set into the original one.

From the curvature tensor, \(R_{ijkl}\), and the inverse tensor field \(ω^{ij}\) of the symplectic structure \(ω_{ij}\), one can construct the only canonical tensor field of type (0, 2),
\[K_{ij} = ω_{kn}R_{nikj}(−1)^{ε_i+ε_j}= R_{ikj}^k (−1)^{ε_k(ε_i+1)}, \quad ε(K_{ij}) = ε_i + ε_j.\]  
(42)
This tensor \(K_{ij}\) is the Ricci tensor and satisfies the relations [18]
\[1 + (−1)^{ε(ω)}[K_{ij} − (−1)^{ε_i}K_{ji}] = 0.\]  
(43)
In the even case \(K_{ij}\) is symmetric whereas in the odd case there are not restrictions on its (generalized) symmetry properties.

Now, one can define the scalar curvature tensor \(K\) by the formula
\[K = ω^{ji}K_{ij}(−1)^{ε_i+ε_j} = ω^{ji}ω_{kn}R_{nikj}(−1)^{ε_i+ε_j+ε_κ(ε_κ+1)}.\]  
(44)
From the symmetry properties of \(R_{ijkl}\), it follows that
\[1 + (−1)^{ε(ω)}]K = 0,\]  
(45)
which proves that as in the case of Fedosov manifolds [4] the scalar curvature \(K\) vanishes.

However, for odd Fedosov supermanifolds \(K\) is, in general, non-vanishing. This fact was quite recently used in Ref. [13] to generalize the BV formalism [8].

Let us consider the Bianchi identity (32) in the form
\[R^n_{mijk} = R^n_{mikj}(−1)^{ε_κ} + R^n_{mjki}(−1)^{ε_i(ε_j+1)} \equiv 0.\]  
(46)
Contracting indices \(i\) and \(n\) with the help of (42) we obtain
\[K_{mji} - K_{mki}^{j} = (−1)^{ε_κ} + R^n_{mjki}(−1)^{ε_i(ε_j+1)} \equiv 0.\]  
(47)
Now using the relations
\[K^i_j = \omega^{jk}K_{kj}(−1)^{ε_i}, K^i_j = \omega^{jk}K_{kj}^{i}(−1)^{ε_k}\]  
(48)
\[K^i_j = (−1)^{ε_κ} + ε_κ(ε_j+1),\]  
(49)
it follows that
\[K_{ij} = [1 − (−1)^{ε(ω)}]K^j_{ij}(−1)^{ε_j(ε_i+1)}.\]  
(50)
In the odd case this implies that
\[K_{ij} = 2K^j_{ij}(−1)^{ε_j(ε_i+1)}.\]  
(51)
In the even case \(K_{ij} = 0\) but in that case the relation (50) does not provide any new information because in this case \(K = 0\).
5 Riemannian supermanifolds

Let \( \mathcal{M} \) be a supermanifold equipped both with a metric structure \( g \)
\[ g = g_{ij} \, dx^i \wedge dx^j, \quad g_{ij} = (-1)^{\epsilon_i \epsilon_j} g_{ji}, \quad \epsilon(g_{ij}) = \epsilon(g) + \epsilon_i + \epsilon_j, \] (52)
and a symmetric connection \( \Delta \) with a covariant derivative \( \nabla \) compatible with the super-Riemannian metric \( g \)
\[ g_{ij} \nabla_k = g_{ij,k} - g_{im} \Delta^m_{jk} - g_{jm} \Delta^m_{ik} (-1)^{\epsilon_i \epsilon_j} = 0. \] (53)

It is easy to show that as in the case of Riemannian geometry there exists the unique symmetric connection \( \Delta^i_{jk} \) which is compatible with a given metric structure. Indeed, proceeding in the same way as in the usual Riemannian geometry one obtains the generalization of celebrated Christoffel formula for the connection in supersymmetric case \[ \Delta^i_{ki} = \frac{1}{2} \epsilon^j \left( g_{ij,k} (-1)^{\epsilon_k \epsilon_i} + g_{jk,i} (-1)^{\epsilon_i \epsilon_j} - g_{ki,j} (-1)^{\epsilon_k \epsilon_j} \right) (-1)^{\epsilon_j \epsilon_i + \epsilon_j + \epsilon_i} \epsilon(g)(\epsilon_i + \epsilon_i). \] (54)

It is straightforward to show that the symbols \( \Delta^i_{ki} \) in (54) are transformed according with transformation laws for connections. A metric supermanifold \( (\mathcal{M}, g) \) equipped with a (even or odd) symmetric connection \( \Delta \) compatible with a given metric structure \( g \) is called a (even or odd) Riemannian supermanifold \( (\mathcal{M}, g, \Delta) \).

The curvature tensor of the connection \( \Delta \) is \[ R_{ijkl} = g_{in} R^n_{jkl}, \quad \epsilon(R_{ijkl}) = \epsilon(g) + \epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l, \] (55)
where \( R^n_{jkl} \) is given by (28) by replacing \( \Gamma^i_{jk} \) for \( \Delta^i_{jk} \). The curvature tensor has the following symmetry properties \[ R_{ijkl} = -(-1)^{\epsilon_k \epsilon_l} R_{ijlk}, \quad R_{ijkl} = -(-1)^{\epsilon_i \epsilon_j} R_{jikl}, \quad R_{ijkl} = R_{kl ij} (-1)^{(\epsilon_i + \epsilon_j)(\epsilon_k + \epsilon_l)}. \] (56)

From the curvature tensor \( R_{ijkl} \) and the inverse tensor field \( g^{ij} \) of the metric \( g_{ij} \) defined by
\[ g^{ij} = (-1)^{\epsilon_i + \epsilon_j} g^{ji}, \quad \epsilon(g^{ij}) = \epsilon(g) + \epsilon_i + \epsilon_j, \] (57)
one can define the only independent tensor field of type \( (0,2) \):
\[ R_{ij} = R^k_{ikj} (-1)^{\epsilon_k(\epsilon_i+1)} = g^{kn} R_{nikj} (-1)^{(\epsilon_k + \epsilon_i)(\epsilon_k + \epsilon_i)} + \epsilon_i \epsilon_k, \] (58)
\[ \epsilon(R_{ij}) = \epsilon_i + \epsilon_j. \]

It is the generalized Ricci tensor which obeys the symmetry
\[ R_{ij} = -(-1)^{\epsilon(g) + \epsilon_i \epsilon_j} R_{ji}. \] (59)

A further contraction between the metric and Ricci tensors defines the scalar curvature
\[ \mathcal{R} = g^{ij} R_{ij} (-1)^{\epsilon_i + \epsilon_j}, \quad \epsilon(\mathcal{R}) = \epsilon(g) \] (60)
which, in general, is non vanishing. Notice that for an odd metric structure the scalar curvature tensor squared is identically equal to zero, \( \mathcal{R}^2 = 0. \)

Let us consider now relations which follow from the Bianchi identity (52). Repeating all arguments given in the end of previous Section one can derive the following relation between the scalar curvature and the Ricci tensor
\[ \mathcal{R}_{ij} = [1 + (-1)^{\epsilon(g)}] R^i_{ij} (-1)^{\epsilon_j(\epsilon_i+1)}. \] (61)
In the even case we have

\[ \mathcal{R}_{ij} = 2 \mathcal{R}^j_{\ ijl}(-1)^{\epsilon_j(\epsilon_i+1)}, \]  

(62)

which is a supersymmetric generalization of the well known relation of Riemannian geometry \[19\]. In the odd case \( \mathcal{R}_{ij} = 0 \) and the relation \[61\] implies that \( \mathcal{R} = \text{const.} \).

Therefore, odd Riemann supermanifolds can only have constant scalar curvature \( \mathcal{R} = \text{const.} \).

It is well known that special types of Riemannian manifolds play an important role in modern quantum field theory. In particular, a consistent formulation of higher spin field theories is possible on AdS space (see, for example \[20\]). In this case the curvature, Ricci and scalar curvature tensors have the form

\[ \mathcal{R}_{ijkl} = R(g_{ik}g_{jl} - g_{il}g_{jk}), \quad \mathcal{R}_{ij} = (N - 1)Rg_{ij}, \quad \mathcal{R} = N(N - 1)R, \]

where \( N \) is the dimension of the Riemannian manifold \( \mathcal{M} \) with a metric tensor \( g_{ij} \) and \( R \) is constant.

Let us analyze the structure of supersymmetric extensions of AdS spaces \[63\]. If \( g_{ij} \) is the graded metric tensor \[52\] of the AdS space one can define the following combination of metric tensors

\[ T_{ijkl} = g_{ik}g_{jl}(-1)^{(g)(\epsilon_i+\epsilon_k)+\epsilon_k\epsilon_j} \]

(64)

which transforms as a tensor field. Therefore a natural generalization of \[63\] satisfies that

\[ \mathcal{R}_{ijkl} = R(g_{ik}g_{jl}(-1)^{(g)(\epsilon_i+\epsilon_k)+\epsilon_k\epsilon_j} - g_{il}g_{jk}(-1)^{(g)(\epsilon_i+\epsilon_j)+\epsilon_i\epsilon_j}) = \]

\[ = (g_{ik} R g_{jl}(-1)^{\epsilon_k\epsilon_j} - g_{il} R g_{jk}(-1)^{\epsilon_i\epsilon_j})(-1)^{(g)}, \]

(65)

where \( R \) \((\epsilon(R) = \epsilon(g))\) is a constant. The Ricci tensor satisfies

\[ \mathcal{R}_{ij} = g^{kl} \mathcal{R}_{likj}(-1)^{(g)+1}(-1)^{(\epsilon_k+\epsilon_i)+\epsilon_i\epsilon_k} = R(N - 1)g_{ij}(-1)^{(g)} \]

(66)

and the scalar curvature tensor verifies that

\[ \mathcal{R} = RN(N - 1), \]

(67)

where we denote

\[ N = \delta_1^i(-1)^{\epsilon_i} \]

(68)

and \( N \) is nothing but the difference between the number of bosonic and fermionic dimensions of the supermanifold.

The above Riemannian tensors obey the following symmetry properties

\[ \mathcal{R}_{ijkl} = -(-1)^{\epsilon_k\epsilon_j}\mathcal{R}_{ijkl}, \quad \mathcal{R}_{ijkl} = -(-1)^{(g)+\epsilon_i\epsilon_j}\mathcal{R}_{ijkl}, \]

\[ \mathcal{R}_{ijkl} = (-1)^{(\epsilon_i+\epsilon_j)(\epsilon_k+\epsilon_l)}\mathcal{R}_{klji} + [1 - (-1)^{(g)}]g_{il} R g_{jk}(-1)^{(g)+\epsilon_i(\epsilon_j+\epsilon_k)}, \]

\[ \mathcal{R}_{ij} = (-1)^{\epsilon_i\epsilon_j}\mathcal{R}_{ji}. \]

It is easy to show that in the even case \((\epsilon(g) = 0)\) all required symmetry properties for \( \mathcal{R}_{ijkl} \) and \( \mathcal{R}_{ij} \) are satisfied. Therefore the supersymmetric generalization of \[63\] has the form

\[ \mathcal{R}_{ijkl} = R(g_{ik}g_{jl}(-1)^{\epsilon_k\epsilon_j} - g_{il}g_{jk}(-1)^{\epsilon_i\epsilon_j+\epsilon_i\epsilon_k}), \quad \mathcal{R}_{ij} = R(N - 1)g_{ij}, \quad \mathcal{R} = RN(N - 1). \]

(69)

In the odd case \((\epsilon(g) = 1)\) there exists only one possibility to satisfy the symmetry requirements: the vanishing of all curvature tensors

\[ R = 0 \rightarrow \mathcal{R}_{ijkl} = 0, \quad \mathcal{R}_{ij} = 0, \quad \mathcal{R} = 0. \]

(70)
6 Conclusions

There are two natural geometric structures of supermanifolds defined by symmetric and antisymmetric graded tensor fields of the second rank: the Poisson bracket defined by an antisymmetric even tensor field of type $(2, 0)$ and the antibracket given by an symmetrical odd tensor field of type $(2, 0)$. We have shown that the geometric structures of even and odd symplectic supermanifolds equipped with a symmetric connection compatible with a given symplectic structure are very similar, although only in the even case the scalar curvature has to vanish. In similar way, the structures of even and odd Riemannian supermanifolds equipped with the unique symmetric connection compatible with a given metric structure are also very similar. However, odd Riemannian supermanifolds are strongly constrained by the fact that their scalar curvature has to be constant whereas in the even case the curvature can have any value. It is quite remarkable that the strongest restrictions on the curvatures arise only for even symplectic and odd Riemannian manifolds. In the case of even Riemannian or odd symplectic manifolds, the curvature tensors can be non null and non-constant, respectively. There are several practical implications of the above formal results. The antisymplectic supermanifold underlying the Batalin-Vilkovisky quantization method is just an odd Fedosov supermanifold which as we have shown can have an arbitrary non-vanishing curvature. On the other hand, even Riemannian supermanifolds admit even AdS superspaces as special case, but there is no analogue for odd Riemannian supermanifolds, i.e. there are not odd supersymmetric AdS spaces.

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