ON THE KNOT QUANDLE OF THE TWIST-SPUN TREFOIL

AYUMU INOUE

ABSTRACT. We show that the knot quandle of the 3-, 4-, or 5-twist-spun trefoil is isomorphic to a quandle related to the 16-, 24-, or 600-cell respectively. We further show that the cardinality of the knot quandle of the m-twist-spun trefoil is finite if and only if $1 \leq m \leq 5$. This phenomenon is attributable to the fact that the regular tessellation $\{3, m\}$, in the sense of the Schl"afli symbol, consists of infinite triangles if $m$ is greater than or equal to 6.

1. INTRODUCTION

A quandle is an algebraic system, which has high compatibility with knot theory. Associated with a knot, which is not necessary to be classical, we have its knot quandle in a similar manner to the knot group. It is known, by Joyce [6] and Matveev [8], that every classical knots are completely distinguished by their knot quandles up to equivalence. Further we are able to obtain various invariants, e.g., quandle cocycle invariants [1], from knot quandles.

For each positive integer $m$, we have a typical 2-knot called the $m$-twist-spun trefoil. Here, a 2-knot means the image of a smooth embedding of the oriented 2-sphere $S^2$ into $\mathbb{R}^4$. The aim of this paper is to investigate the knot quandle of the $m$-twist-spun trefoil. Since the 1-twist-spun trefoil is equivalent to the trivial 2-knot, it is obvious that the knot quandle of the 1-twist-spun trefoil is isomorphic to the trivial quandle of order 1. Rourke and Sanderson [9] pointed out that the knot quandle of the 2-twist-spun trefoil is isomorphic to the dihedral quandle of order 3. On the other hand, as far as the author is aware, no one knows about the knot quandle of the $m$-twist-spun trefoil so far, if $m$ is greater than or equal to 3.

Recently the author [5] showed that the cardinality of the knot quandle of the 3-, 4-, or 5-twist-spun trefoil is equal to 8, 24, or 120 respectively. We note that these numbers are equal to the numbers of vertices of the 16-, 24-, and 600-cell respectively. This is more than coincidence. In this paper, we show that the knot quandle of the 3-, 4-, or 5-twist-spun trefoil is isomorphic to a quandle derived from rotational symmetries of the 16-, 24-, or 600-cell respectively (Theorem 5.1).

Behavior of the knot quandle of the $m$-twist-spun trefoil changes drastically whether $m$ is less than 6 or not. We show that the cardinality of the knot quandle of the $m$-twist-spun trefoil is infinite if $m$ is greater than or equal to 6, while it is finite if $1 \leq m \leq 5$ (Theorem 5.2). This phenomenon is attributable to the fact that the regular tessellation $\{3, m\}$, in the sense of the Schl"afli symbol, consists of infinite triangles if $m$ is greater than or equal to 6.

2010 Mathematics Subject Classification. 57Q45, 52B15.

Key words and phrases. quandle, knot quandle, twist-spun trefoil, symmetry, regular polytope, tessellation.
2. Quandle

In this section, we review some notions about quandles briefly and introduce several concrete quandles. We refer the reader to [2, 6, 7] for more details.

A quandle is a non-empty set $X$ equipped with a binary operation $*: X \times X \to X$ satisfying the following three axioms:

1. For each $x \in X$, $x * x = x$ (Q1)
2. For each $x \in X$, the map $* : X \to X$ ($w \mapsto w * x$) is bijective (Q2)
3. For each $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$ (Q3)

The notions of homomorphism, epimorphism and isomorphism are appropriately defined for quandles.

Suppose $(X, \ast)$ is a quandle. For each integer $n$, we have the binary operation $\ast^n : X \times X \to X$ sending $(x, y)$ to $(\ast^n y)(x)$. It is routine to check that this binary operation $\ast^n$ also satisfies the axioms of a quandle. A subset $S$ of $X$ is said to generate $X$ if any element of $X$ is obtained from elements of $S$ by taking operations $\ast$ and $\ast^{-1}$ iteratively.

We will refer to the following quandles in subsequent sections.

Example 2.1 (trivial quandle). Let $X$ be a finite set and $\ast$ the binary operation on $X$ given by $x * y = x$. Then it is easy to see that $\ast$ satisfies the axioms of a quandle. We refer to the quandle $(X, \ast)$ as the trivial quandle of order $|X|$. Here, $|X|$ denotes the cardinality of $X$ as usual.

Example 2.2 (mosaic quandle). Suppose $n$ is an integer greater than or equal to 2. Let $B$ be the unit sphere in $\mathbb{R}^3$ if $2 \leq n \leq 5$, the Euclidean plane if $n = 6$, otherwise the hyperbolic plane. Consider a tessellation of $B$ by congruent regular triangles whose interior angles are $2\pi/n$. We note that this tessellation is known to be the regular tessellation $\{3, n\}$ in the sense of the Schlaffi symbol. Figure 1 depicts such tessellations for the case that $n$ is equal to 3, 6, or 8. Associated with the tessellation, let us consider the set

$$X = \left\{ (v, r_v) \mid v : \text{a vertex of the tessellation}, \quad r_v : B \to B \text{ the rotation about } v \text{ by } 2\pi/n \right\},$$

where, if $2 \leq n \leq 5$, a rotation about $v$ means a rotation about the line in which $v$ and the origin of $\mathbb{R}^3$ lie. We note that each $r_v$ is not only an isometry of $B$ but the one of the tessellation. Further these isometries are conjugate to each other. Let us define the binary operation $\ast$ on $X$ by

$$(v, r_v) \ast (w, r_w) = (r_w(v), r_w \circ r_v \circ r_w^{-1}) = (r_w(v), r_{r_w(v)}).$$

Then it is routine to check that $\ast$ satisfies the axioms of a quandle. We refer to the quandle $(X, \ast)$ as the mosaic quandle of type $\{3, n\}$, in this paper. The mosaic quandle of type $\{3, 2\}$, $\{3, 3\}$, $\{3, 4\}$, or $\{3, 5\}$ is known to be the dihedral quandle of order 3, the tetrahedral quandle, the octahedral quandle, or the icosahedral quandle respectively. We note that cardinalities of these mosaic quandles are equal to 3, 4, 6, and 12 respectively. On the other hand, the cardinality of the mosaic quandle of type $\{3, n\}$ is infinite if $n$ is greater than or equal to 6.

Example 2.3 (16-cell quandle). Let us consider the set

$$V = \{ \pm e_1, \pm e_2, \pm e_3, \pm e_4 \}$$
Figure 1. Regular tessellations \{3,3\}, \{3,6\}, and \{3,8\} consisting of 8 points in $\mathbb{R}^4$. Here, $e_i \in \mathbb{R}^4$ denotes the column vector whose $j$-th entry is $\delta_{ij}$ as usual. We note that the convex hull of $V$ is known as the 16-cell. Associated with $v \in V$, we define the $4 \times 4$ matrix $R_v$ as follows:

$R_{\pm e_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, \quad $R_{\pm e_2} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, \quad $R_{\pm e_3} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$, \quad $R_{\pm e_4} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

We note that each linear transformation $r_v : \mathbb{R}^4 \to \mathbb{R}^4$ sending $x$ to $R_v x$ is an isometry of the 16-cell, a $(2\pi/3)$-rotation about a plain in which $v$ and the origin of $\mathbb{R}^4$ lie, and these isometries are conjugate to each other. Let us consider the set

$$X = \{(v, r_v) \mid v \in V\}$$

and define its binary operation $*$ in the same manner as the mosaic quandle:

$$(v, r_v) * (w, r_w) = (r_w(v), r_w \circ r_v \circ r_w^{-1}) = (r_w(v), r_{r_w(v)}) .$$

Then the binary operation $*$ also satisfies the axioms of a quandle. We refer to the quandle $(X, *)$ as the 16-cell quandle, in this paper. We note that the cardinality of the 16-cell quandle is obviously equal to 8.

Example 2.4 (24-cell quandle). Let us consider the set

$$V = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq 4\}$$

consisting of 24 points in $\mathbb{R}^4$. We note that the convex hull of $V$ is known as the 24-cell. Associated with $v \in V$, we define the $4 \times 4$ matrix $R_v$ as follows:

$$R_{\pm(e_1+e_2)} = R_{\pm(e_1-e_2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} ,$$
We note that each linear transformation $r_v: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ sending $x$ to $r_v x$ is an isometry of the 24-cell, a $(\pi/2)$-rotation about a plane in which $v$ and the origin of $\mathbb{R}^4$ lie, and these isometries are conjugate to each other. Let us consider the set

$$X = \{(v, r_v) \mid v \in V\}$$

and define its binary operation $\ast$ in the same manner as the mosaic quandle:

$$(v, r_v) \ast (w, r_w) = (r_w(v), r_w \circ r_v \circ r_w^{-1}) = (r_w(v), r_{r_w(v)}(v)).$$

Then the binary operation $\ast$ also satisfies the axioms of a quandle. We refer to the quandle $(X, \ast)$ as the 24-cell quandle, in this paper. We note that the cardinality of the 24-cell quandle is obviously equal to 24.

Example 2.5 (600-cell quandle). Let us consider the set

$$V = \{\pm e_1, \pm e_2, \pm e_3, \pm e_4\} \cup \left\{ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \right\} \cup \left\{ \frac{1}{2}(\pm \phi e_{e(1)} \pm e_{e(2)} \pm \phi^{-1} e_{e(3)}) \mid e \in A_4 \right\}$$

consisting of 120 points in $\mathbb{R}^4$. Here, $\phi$ denotes the golden ratio $(1 + \sqrt{5})/2$ and $A_4$ the alternating group on $\{1, 2, 3, 4\}$. We note that the convex hull of $V$ is known as the 600-cell. Associated with $v \in V$, we define the $4 \times 4$ matrix $R_v$ as follows:

$$R_{\pm e_1} = R_{\pm \frac{1}{2}(\phi e_1 + e_2 - \phi^{-1} e_3)} = R_{\pm \frac{1}{2}(\phi e_1 + e_2 - \phi^{-1} e_3)} = R_{\pm \frac{1}{2}(\phi^{-1} e_1 + \phi e_2 + e_3)} = R_{\pm \frac{1}{2}(\phi^{-1} e_1 + \phi e_2 + e_3)} = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & \phi & -\phi^{-1} \\ 0 & \phi & -\phi^{-1} & 1 \\ 0 & \phi^{-1} & -1 & -\phi \end{pmatrix}.$$
\[ R_{\pm e_4} = R_{\pm \frac{1}{2}(e_2 - \phi e_3 + \phi^{-1} e_4)} = R_{\pm \frac{1}{2}(e_2 - \phi e_3 - \phi^{-1} e_4)} \]

\[ = R_{\pm \frac{1}{2}(\phi^{-1} e_2 - e_3 + e_4)} = R_{\pm \frac{1}{2}(\phi^{-1} e_2 - e_3 - e_4)} = \frac{1}{2} \begin{pmatrix} -\phi & 1 & \phi^{-1} & 0 \\ -1 & -\phi^{-1} & -\phi & 0 \\ -\phi^{-1} & -\phi & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \]

\[ R_{\pm e_3} = R_{\pm \frac{1}{2}(e_1 + \phi^{-1} e_3 - \phi e_4)} = R_{\pm \frac{1}{2}(e_1 - \phi^{-1} e_3 - \phi e_4)} \]

\[ = R_{\pm \frac{1}{2}(\phi^{-1} e_1 + e_3 - e_4)} = R_{\pm \frac{1}{2}(\phi^{-1} e_1 - e_3 - e_4)} = \frac{1}{2} \begin{pmatrix} -\phi^{-1} & 1 & 0 & -\phi \\ -1 & -\phi & 0 & -\phi^{-1} \\ -\phi^{-1} & 0 & 2 & 0 \\ -\phi & 0 & -1 & \phi^{-1} \end{pmatrix}, \]

\[ R_{\pm e_2} = R_{\pm \frac{1}{2}(e_1 + \phi e_2 - \phi^{-1} e_4)} = R_{\pm \frac{1}{2}(e_1 - \phi e_2 + \phi^{-1} e_4)} \]

\[ = R_{\pm \frac{1}{2}(\phi e_1 + \phi^{-1} e_2 + e_4)} = R_{\pm \frac{1}{2}(\phi e_1 - \phi^{-1} e_2 + e_4)} = \frac{1}{2} \begin{pmatrix} 1 & 0 & \phi^{-1} & \phi \\ 0 & 2 & 0 & 0 \\ -\phi^{-1} & 0 & -\phi & 1 \\ \phi & 0 & -1 & \phi^{-1} \end{pmatrix}, \]

\[ R_{\pm \frac{1}{2}(e_1 + e_2 - e_3 - e_4)} = R_{\pm \frac{1}{2}(e_1 - \phi^{-1} e_3 + \phi e_4)} = R_{\pm \frac{1}{2}(\phi e_2 - \phi^{-1} e_3 - e_4)} \]

\[ = R_{\pm \frac{1}{2}(\phi^{-1} e_1 + e_3 - e_4)} = R_{\pm \frac{1}{2}(\phi^{-1} e_1 - e_3 - e_4)} = \frac{1}{2} \begin{pmatrix} -\phi^{-1} & 1 & 0 & \phi \\ 0 & 1 & -\phi & -\phi^{-1} \\ -\phi & -1 & -\phi^{-1} & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \]

\[ R_{\pm \frac{1}{2}(e_1 - e_2 - e_3 - e_4)} = R_{\pm \frac{1}{2}(e_1 + \phi^{-1} e_2 + \phi e_4)} = R_{\pm \frac{1}{2}(\phi e_1 - \phi^{-1} e_2 - e_4)} \]

\[ = R_{\pm \frac{1}{2}(e_2 + \phi e_3 + \phi^{-1} e_4)} = R_{\pm \frac{1}{2}(\phi e_1 + e_3 - \phi^{-1} e_4)} = \frac{1}{2} \begin{pmatrix} 1 & 0 & \phi^{-1} & -\phi \\ -1 & -\phi^{-1} & \phi & 0 \\ 1 & 1 & 1 & 1 \\ -1 & \phi & 0 & -\phi^{-1} \end{pmatrix}, \]

\[ R_{\pm \frac{1}{2}(e_1 - e_2 + e_3 - e_4)} = R_{\pm \frac{1}{2}(e_1 - \phi^{-1} e_2 + \phi e_4)} = R_{\pm \frac{1}{2}(\phi e_2 - \phi^{-1} e_3 + e_4)} \]

\[ = R_{\pm \frac{1}{2}(\phi^{-1} e_1 - e_3 + e_4)} = R_{\pm \frac{1}{2}(\phi^{-1} e_1 - e_3 - e_4)} = \frac{1}{2} \begin{pmatrix} -\phi^{-1} & 0 & \phi & -1 \\ -\phi & -\phi^{-1} & 0 & 1 \\ 1 & -1 & 1 & 1 \\ 0 & \phi & \phi^{-1} & 1 \end{pmatrix}, \]

\[ R_{\pm \frac{1}{2}(e_1 - e_2 - e_3 + e_4)} = R_{\pm \frac{1}{2}(e_1 + \phi e_2 - \phi^{-1} e_4)} = R_{\pm \frac{1}{2}(\phi e_1 + e_3 - \phi^{-1} e_4)} \]

\[ = R_{\pm \frac{1}{2}(\phi e_1 + \phi^{-1} e_2 + e_4)} = R_{\pm \frac{1}{2}(\phi e_2 - \phi^{-1} e_3 - e_4)} = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 & 1 \\ \phi^{-1} & 1 & 0 & -\phi \\ -\phi & 1 & -\phi^{-1} & 0 \\ 0 & -1 & -\phi & -\phi^{-1} \end{pmatrix}, \]

\[ R_{\pm \frac{1}{2}(e_1 + e_2 - e_3 + e_4)} = R_{\pm \frac{1}{2}(e_1 + \phi^{-1} e_2 - \phi e_4)} = R_{\pm \frac{1}{2}(\phi e_3 - \phi^{-1} e_3 - e_4)} \]

\[ = R_{\pm \frac{1}{2}(\phi^{-1} e_1 + e_2 - e_4)} = R_{\pm \frac{1}{2}(\phi^{-1} e_1 - e_2 - e_4)} = \frac{1}{2} \begin{pmatrix} -\phi^{-1} & \phi & -1 & 0 \\ 0 & -\phi^{-1} & -1 & -\phi \\ -\phi & 0 & 1 & -\phi^{-1} \\ -1 & 1 & 1 & 1 \end{pmatrix}, \]

\[ R_{\pm \frac{1}{2}(e_1 + e_2 + e_3 + e_4)} = R_{\pm \frac{1}{2}(e_1 - \phi e_2 - \phi^{-1} e_4)} = R_{\pm \frac{1}{2}(\phi e_1 + e_3 + \phi^{-1} e_4)} \]

\[ = R_{\pm \frac{1}{2}(\phi e_2 + \phi^{-1} e_3 + e_4)} = R_{\pm \frac{1}{2}(\phi e_1 - e_2 + \phi^{-1} e_3)} = \frac{1}{2} \begin{pmatrix} 1 & -\phi^{-1} & \phi & 0 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & -\phi^{-1} & \phi \\ 1 & \phi & 0 & -\phi^{-1} \end{pmatrix}, \]
We note that each linear transformation \( r_v : \mathbb{R}^4 \to \mathbb{R}^4 \) sending \( x \) to \( R_v x \) is an isometry of the 600-cell, a \((2\pi/5)\)-rotation about a plain in which \( v \) and the origin of \( \mathbb{R}^4 \) lie, and these isometries are conjugate to each other. Let us consider the set

\[
X = \{ (v, r_v) \mid v \in V \}
\]

and define its binary operation \(*\) in the same manner as the mosaic quandle:

\[
(v, r_v) * (w, r_w) = (r_w(v), r_w \circ r_v \circ r_w^{-1}) = (r_w(v), r_{r_w(v)}).
\]

Then the binary operation \(*\) also satisfies the axioms of a quandle. We refer to the quandle \((X, *)\) as the 600-cell quandle in this paper. We note that the cardinality of the 600-cell quandle is obviously equal to 120.

**Remark 2.6.** Mosaic quandles are able to be defined, in the same manner, for every regular tessellations \( \{p, q, r, \ldots \} \). The 16-, 24-, and 600-cell quandles are nothing less than mosaic quandles of type \( \{3, 3, 4\} \), \( \{3, 4, 3\} \), and \( \{3, 3, 5\} \) respectively.

**Remark 2.7.** The 16- and 24-cell quandles are referred as SmallQuandle(8, 1) and SmallQuandle(24, 2) in the GAP package Rig [11] respectively.

3. **Presentation of a Quandle**

As well as groups, we have a presentation of a quandle. We will define the knot quandle of the \( m \)-twist-spun trefoil by giving its presentation. Thus, in this section, we review the definition of a presentation of a quandle briefly. We further do some facts on presentations. We refer the reader to [7] for more details.

Suppose \( S \) is a non-empty set. Let \( FQ(S) \) be the subset of the free group \( F(S) \) on \( S \) consisting of the elements of the form \( g^{-1}ag \) with some \( a \in S \) and \( g \in F(S) \).

In other words, \( FQ(S) \) is the union of conjugacy classes of \( F(S) \) containing the elements of \( S \). We will denote the conjugation \( g^{-1}ag \) by \( a^g \) in the remaining. It is routine to check that the binary operation \(*\) on \( FQ(S) \) given by

\[
a^g * b^h = a^{gh} = a^{gh^{-1}bh}
\]

satisfies the axioms of a quandle. We call the quandle \((FQ(S), *)\) the free quandle on \( S \).

For a given subset \( R \) of \( FQ(S) \times FQ(S) \), we consider to enlarge \( R \) by repeating the following moves:

(a) For each \( x \in FQ(S) \), add \((x, x)\) in \( R \)
(b) If \((x, y)\) is an element of \( R \), then add \((y, x)\) in \( R \)
(c) If \((x, y)\) and \((y, z)\) are elements of \( R \), then add \((x, z)\) in \( R \)
A consequence of $R$ is an element of an expanded $R$ by a finite sequence of the above moves. Let $x \sim y$ denote that $(x, y)$ is a consequence of $R$. Then it is obvious, by definition, that $\sim$ is an equivalence relation on $FQ(S)$. Further it is routine to check that the binary operation $\ast$ on $FQ(S)$ is passed to the quotient $FQ(S)/\sim$ and still satisfies the axioms of a quandle.

A quandle $(X, \ast_X)$ is said to have a presentation $(S \mid R)$ if $(X, \ast_X)$ is isomorphic to the quandle $(FQ(S)/\sim, \ast)$. We refer to an element of $S$ or $R$ as a generator or relation of $(S \mid R)$ respectively. We will write a relation $(x, y) \in R$ as $x = y$ in the remaining. We further abbreviate a presentation $\langle \{s_1, s_2, \ldots, s_n\} \mid \{r_1, r_2, \ldots, r_m\} \rangle$ to $\langle s_1, s_2, \ldots, s_n \mid r_1, r_2, \ldots, r_m \rangle$.

For each quandle $(X, \ast_X)$, a map $f : S \to X$ induces a quandle homomorphism $f_\ast : FQ(S) \to X$ in a natural way. Further $f$ induces an well-defined quandle homomorphism $f_* : FQ(S)/\sim \to X$ if and only if the equation $f_\ast(x) = f_\ast(y)$ holds for each relation $x = y$ in $R$. We note that $f_*$ is surjective if and only if the image of $f$ generates $X$.

Fenn and Rourke [4] essentially showed the following theorem which is similar to the Tietze’s theorem for group presentations.

**Theorem 3.1.** Assume that a quandle has two distinct presentations. Then the presentations are related to each other by a finite sequence of the following moves or their inverse:

(T1) Choose a consequence of the set of relations, and then add it to the set of relations

(T2) Choose an element $x$ of the free quandle on the set of generators, and then introduce a new generator $s$ in the set of generators and the new relation $s = x$ in the set of relations

We refer to the above moves and their inverse as Tietze moves.

4. Twist-spin construction of a 2-knot and the knot quandle

The twist-spin construction, introduced by Zeeman [12], is a typical method to obtain a 2-knot. The $m$-twist-spin trefoil will be defined through this construction. Thus, in this section, we review the twist-spin construction briefly. We further do the definition of the knot quandle of a 2-knot which is obtained through the construction. We refer the reader to [2, 7, 10] for more details.

Let $k$ be an oriented knotted arc which is properly embedded into the upper half space $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0\}$. Choose and fix a 3-ball $B$ in the interior of $\mathbb{R}^3_+$ so that $B$ wholly contains the knotted part of $k$. We assume that $k$ intersects with $\partial B$ only at the north and south poles of $B$. The left-hand side of Figure 2 illustrates the situation. Suppose $m$ is a positive integer. Spin $\mathbb{R}^3_+$ 360 degrees in $\mathbb{R}^4$ along $\partial \mathbb{R}^3_+$. Further, at the same time, rotate $B$ $360m$ degrees in $\mathbb{R}^3_+$ along the axis of $B$ connecting between the north and south poles. Then the locus of $k$ yields a 2-knot. We refer to this 2-knot as the $m$-twist-spin of $k$.

Associated with the $m$-twist-spin of $k$, we have its knot quandle as follows. Suppose $\pi : \mathbb{R}^3_+ \to \mathbb{R}^2_+ = \{(s, t) \in \mathbb{R}^2 \mid t \geq 0\}$ is the orthogonal projection mapping $(x, y, z)$ to $(x, z)$. Deforming $k$ slightly if necessary, we may assume that each singularity of the plain curve $\pi(k)$ is a transversal double point. The diagram of
$k$ is the plain curve $\pi(k)$ with over/under information on each double point. We always indicate over/under information by breaks in the under-passing segments. The right-hand side of Figure 2 is the diagram of the knotted arc depicted in the left-hand side. We note that the diagram consists of disjoint oriented arcs embedded into $\mathbb{R}^2_+$. Let $d$ be the diagram of $k$ and $S$ the set consists of the arcs of $d$. For each double point of $d$ to which arcs $x$, $y$ and $z$ are related as depicted in Figure 3, we consider the relation $x \ast y = z$. We let $R_1$ be the set consisting of these relations. Suppose $s$ and $t$ are the arcs of $d$ which contain the image of the start and end points of $k$ respectively. For each arc $u$ of $d$ other than $s$ or $t$, we consider the relation $u \ast^m s = u$. We let $R_2$ be the set consisting of these relations. The knot quandle of the $m$-twist-spin of $k$ is the quandle which has the presentation $\langle S \mid R_1 \cup R_2 \rangle$. For example, the knot quandle of the $m$-twist-spin of the knotted arc depicted in the left-hand side of Figure 2 has the following presentation:

$$\langle a, b, c, d, e \mid a \ast d = b, c \ast e = b, e \ast c = d, c \ast b = d, b \ast^m a = b, c \ast^m a = c, d \ast^m a = d \rangle.$$
5. The knot quandle of the twist-spun trefoil

The \( m \)-twist-spun trefoil is defined to be the \( m \)-twist-spin of the oriented knotted arc depicted in the left-hand side of Figure 4, for each positive integer \( m \). Since the knotted arc has the diagram depicted in the right-hand side of Figure 4, the knot quandle of the \( m \)-twist-spun trefoil has the following presentation:

\[
\langle a, b, c, d \mid a \ast c = b, b \ast d = c, c \ast b = d, b \ast^m a = b, c \ast^m a = c \rangle
\]

\[
= \langle a, b, c \mid a \ast c = b, (b \ast c) \ast b = c, b \ast^m a = b, c \ast^m a = c \rangle
\]

\[
= \langle a, c \mid (a \ast c) \ast a = c, c \ast^m a = c \rangle.
\]

Here, equality means being related to each other by Tietze moves. In this section, we investigate the knot quandle of the \( m \)-twist-spun trefoil.

As mentioned in Section 1, the knot quandle of the 1- or 2-twist-spun trefoil is respectively isomorphic to the trivial quandle of order 1 or the dihedral quandle of order 3. We are able to check the fact straightforwardly applying Tietze moves on presentations of the knot quandles. Further, utilizing presentations, we have the following theorem.

**Theorem 5.1.** The knot quandle of the 3-, 4-, or 5-twist-spun trefoil is isomorphic to the 16-, 24-, or 600-cell quandle respectively.

**Proof.** Choose adjacent vertices \( v \) and \( w \) of the 16-, 24-, or 600-cell \( P \) as follows:

\[
\begin{cases}
  v = e_1, \ w = e_2 & \text{if } P \text{ is the 16-cell,} \\
  v = e_1 + e_2, \ w = e_2 + e_4 & \text{if } P \text{ is the 24-cell,} \\
  v = e_1, \ w = -\frac{1}{2}(\phi^{-1}e_1 + \phi e_3 - e_4) & \text{if } P \text{ is the 600-cell.}
\end{cases}
\]

Then it is routine to check that the set \( \{(v, r_v), (w, r_w)\} \) generates the 16-, 24-, or 600-cell quandle respectively. Further \( (v, r_v) \) and \( (w, r_w) \) satisfy the following
equations:
\[(v, r_v) * (w, r_w) * (v, r_v) = (w, r_w),\]
\[(w, r_w) * 3 (v, r_v) = (w, r_w) \text{ if } P \text{ is a 16-cell},\]
\[(w, r_w) * 4 (v, r_v) = (w, r_w) \text{ if } P \text{ is a 24-cell},\]
\[(w, r_w) * 5 (v, r_v) = (w, r_w) \text{ if } P \text{ is a 600-cell}.\]

We thus have the epimorphism \(\varphi\) from the knot quandle of the 3-, 4-, or 5-twist-spun trefoil to the 16-, 24-, or 600-cell quandle, respectively, sending \(a\) to \((v, r_v)\) and \(c\) to \((w, r_w)\). On the other hand, the author [5] showed that the cardinality of the knot quandle of the 3-, 4-, or 5-twist-spun trefoil is equal to 8, 24, or 120 respectively. Since these numbers coincide with cardinalities of the 16-, 24-, and 600-cell quandles respectively, \(\varphi\) is not only an epimorphism but an isomorphism. 

A similar argument to the above works for the other mosaic quandles. Let \((X, \ast)\) be the mosaic quandle of type \(\{3, m\}\), and \(v\) and \(w\) adjacent vertices of the regular tessellation \(\{3, m\}\) related to \(X\). Then it is routine to check that the set \(\{(v, r_v), (w, r_w)\}\) generates \(X\). Further \((v, r_v)\) and \((w, r_w)\) satisfy the following equations:

\[\{(v, r_v) * (w, r_w) * (v, r_v) = (w, r_w), \quad (w, r_w) * m (v, r_v) = (w, r_w).\]

We thus have the epimorphism from the knot quandle of the \(m\)-twist-spun trefoil to \((X, \ast)\) sending \(a\) to \((v, r_v)\) and \(c\) to \((w, r_w)\). Since the cardinality of \(X\) is infinite if \(m\) is greater than or equal to 6, we immediately have the following theorem.

**Theorem 5.2.** The cardinality of the knot quandle of the \(m\)-twist-spun trefoil is infinite if \(m\) is greater than or equal to 6, while it is finite if \(1 \leq m \leq 5\).

### 6. Discussion

We wrap up our study with the following discussion. Suppose \(X\) is a subset of a group which is closed under conjugation by the elements of \(X\).\(^1\) Then \(X\) is a quandle with the binary operation \(\ast\) given by \(x \ast y = yxy^{-1}\). We refer to this quandle \((X, \ast)\) as the **conjugation quandle** on \(X\).

Associated with the 16-, 24-, and 600-cell quandles, let us consider the following sets consisting of the isometries related to the quandles respectively:

\[X_{16} = \{r_{e_1}, r_{e_2}, r_{e_3}, r_{e_4}\},\]

\[X_{24} = \{r_{e_1+e_2}, r_{e_3+e_4}, r_{e_1+e_3}, r_{e_1-e_3}, r_{e_1+e_4}, r_{e_2+e_3}\},\]

\[X_{600} = \left\{ r_{e_1}, r_{e_2}, r_{e_3}, r_{e_4}, \right. \]

\[
\left. r_{\frac{1}{3}(e_1+e_2+e_3+e_4)}; \frac{1}{2}(e_1-e_2-e_3-e_4); \frac{1}{2}(e_1-e_2+e_3+e_4); \frac{1}{2}(e_1-e_2+e_3+e_4); \frac{1}{2}(e_1+e_2+e_3+e_4); \frac{1}{2}(e_1+e_2+e_3+e_4); \frac{1}{2}(e_1+e_2+e_3+e_4); \frac{1}{2}(e_1+e_2+e_3+e_4); \frac{1}{2}(e_1+e_2+e_3+e_4); \right. \]

\[
\left. \frac{1}{2}(e_1+e_2+e_3+e_4); \frac{1}{2}(e_1+e_2+e_3+e_4); \frac{1}{2}(e_1+e_2+e_3+e_4); \frac{1}{2}(e_1+e_2+e_3+e_4); \right. \]

\[\left\}.\right.\]

We note that these sets are closed under conjugation by their elements. Further it is routine to check that the conjugation quandles of \(X_{16}, X_{24}\) and \(X_{600}\) are isomorphic to the tetrahedral quandle, the octahedral quandle, and the icosahedral quandle respectively.

\(^1\)We note that \(X\) should not always be a union of conjugacy classes of the group. For example, although it is not a conjugacy class of the symmetry group \(\mathcal{S}_n\) if \(n \geq 3\), we are able to choose a subset consisting of a single transposition of \(\mathcal{S}_n\) as \(X\).
Clark et al. [3] showed that the 16- and 24-cell quandles are ‘abelian’ extensions of the tetrahedral quandle and the octahedral quandle respectively. It seems that the relationship between the 16- or 24-cell quandle and the conjugation quandle of $X_{16}$ or $X_{24}$ leads the fact. Thus the author expects that the 600-cell quandle is an ‘abelian’ extension of the icosahedral quandle. Clark et al. [3] further showed that the icosahedral quandle has an ‘abelian’ extension whose cardinality is equal to 120. It gives us an evidence for the claim.

Recall that the tetrahedral quandle, the octahedral quandle, and the icosahedral quandle are the mosaic quandles of type $\{3, 3\}$, $\{3, 4\}$, and $\{3, 5\}$ respectively. Further, in light of Theorem 5.1, the 16-, 24-, or 600-cell quandle is isomorphic to the knot quandle of the 3-, 4-, or 5-twist-spun trefoil respectively. The author thus wonders that the knot quandle of the $m$-twist-spun trefoil is an ‘abelian’ extension of the mosaic quandle of type $\{3, m\}$ for each $m \geq 3$.

Acknowledgments

The author is partially supported by JSPS KAKENHI Grant Number JP16K17591.

References

1. J. S. Carter, M. Elhamdadi, M. Graña and M. Saito, Cocycle knot invariants from quandle modules and generalized quandle homology, Osaka J. Math. 42 (2005), no. 3, 499–541.
2. J. S. Carter, S. Kamada and M. Saito, Surfaces in 4-space, Encyclopaedia of Mathematical Sciences 142, Low-Dimensional Topology III, Springer-Verlag, Berlin, 2004.
3. W. E. Clark, M. Saito and L. Vendramin, Quandle coloring and cocycle invariants of composite knots and abelian extensions, J. Knot Theory Ramifications 25 (2016), no. 5, 1650024, 34 pp.
4. R. Fenn and C. Rourke, Racks and links in codimension two, J. Knot Theory Ramifications 1 (1992), no. 4, 343–406.
5. A. Inoue, On the knot quandle of a fibered knot, finiteness and equivalence of knot quandles, preprint, available at https://arxiv.org/abs/1807.08977.
6. D. Joyce, A classifying invariant of knots, the knot quandle, J. Pure Appl. Algebra 23 (1982), no. 1, 37–65.
7. S. Kamada, Surface-knots in 4-space, An introduction, Springer Monographs in Mathematics, Springer, Singapore, 2017.
8. S. V. Matveev, Distributive groupoids in knot theory, (in Russian), Mat. Sb. (N.S.) 119 (161) (1982), 78–88 (English translation: Math. USSR-Sb. 47 (1984), 73–83).
9. C. Rourke and B. Sanderson, A new classification of links and some calculations using it, preprint, available at https://arxiv.org/abs/math/0006062.
10. S. Satoh, Surface diagrams of twist-spun 2-knots, Knots 2000 Korea, Vol. 1 (Yongpyong), J. Knot Theory Ramifications 11 (2002), no. 3, 413–430.
11. L. Vendramin, Rig — A GAP package for racks and quandles, available at http://mate.dm.uba.ar/~lvendram/rig/.
12. E. C. Zeeman, Twisting spun knots, Trans. Amer. Math. Soc. 115 (1965), 471–495.