1. Introduction

1.1. Algebraic Vafa-Witten invariants. In [17], Yuuji Tanaka and Richard Thomas proposed a definition of an SU(r) Vafa-Witten invariant [20]. Let \((S, H)\) be a polarized smooth complex surface with canonical bundle \(\omega_S\). A Higgs pair is a pair
\[
(E, \phi) \quad \text{with} \quad E \in \text{Coh}(S), \quad \phi : E \to E \otimes \omega_S.
\]
Choose a rank \(r\), Chern classes \(c_1, c_2\) on \(S\), and a line bundle \(M\) on \(S\) with \(c_1(M) = c_1\). Assume that \(r, c_1\) and \(c_2\) are chosen in such a way that stability and semistability
of Higgs pairs coincide (see Section 3). Let

\[ N_{r,M,c} \subseteq \{ (E, \phi) \mid \text{tr} \phi = 0, \text{rk}(E) = r, \det E \cong M, c_2(E) = c_2 \} \]

be the moduli space of Gieseker stable trace free Higgs pairs with fixed determinant. In [17] a symmetric perfect obstruction theory on \( N_{r,M,c} \) is constructed. Its dual complex is given by the cone

\[
\mathcal{R} \text{Hom}_\pi(E, E) \xrightarrow{\cdot \phi} \mathcal{R} \text{Hom}_\pi(E, E \otimes \omega_S) \to T,
\]

in which \((E, \phi)\) is a universal Higgs pair on \( N_{r,M,c} \times S \) and \( \pi : N_{r,M,c} \times S \to N_{r,M,c} \)

denotes the projection. The \( \mathbb{C}^* \)-action on \( N_{r,M,c} \), which is given by scaling the Higgs field, can be lifted to an equivariant structure on \( E \). It gives rise to a localized virtual class, which is used to define the Vafa-Witten invariant as

\[
\text{VW}_{r,c_1,c_2}(S) = \int_{(N_{r,M,c}^\perp)^{\mathbb{C}^*}} \frac{1}{e(N^{\text{vir}})},
\]

in which \( N^{\text{vir}} \) is the virtual normal bundle to \( (N_{r,M,c}^\perp)^{\mathbb{C}^*} \) in \( N_{r,M,c}^\perp \), and \( e(N^{\text{vir}}) \) denotes its equivariant Euler class.

A Higgs pair \((E, \phi)\) in the fixed locus \( N_{r,M,c}^\perp \) is equipped with a \( \mathbb{C}^* \)-action, and hence decomposes into weight spaces. As explained in [17], the Higgs field acts with weight 1. Thus, after twisting with some power of \( t \), we can write

\[
E = \bigoplus_{i=0}^{k} E_i \otimes t^{-i}
\]

\[
\phi = (\phi_1, \ldots, \phi_k) : E \to E \otimes \omega_S \otimes t,
\]

where the \( E_i \) are torsion free sheaves of rank \( r_i \) and \( \phi \) decomposes into maps

\[
\phi_i : E_{i-1} \to E_i \otimes \omega_S \otimes t \quad \text{for} \quad i = 1, \ldots, k.
\]

We will write

\[ \mathcal{M}_{(r_0, \ldots, r_k)} = \mathcal{M}_{(r_0, \ldots, r_k), c_1, c_2} \subseteq (N_{r,M,c}^\perp)^{\mathbb{C}^*} \]

for the open and closed locus of Higgs pairs with weight spaces of dimensions \( r_0, \ldots, r_k \). The locus

\[ \mathcal{M}_r = \left\{ (E, \phi) \in (N_{r,M,c}^\perp)^{\mathbb{C}^*} \mid \phi = 0 \right\} \]

is called the instanton branch [10]. It is isomorphic to the moduli space of torsion free rank \( r \) sheaves, and its contribution to the Vafa-Witten invariant is the (localized) virtual Euler characteristic (up to a sign). Its complement in the \( \mathbb{C}^* \)-fixed locus is called the monopole branch. In this paper, we will discuss the contribution of the locus \( \mathcal{M}_1 = \mathcal{M}_{(1 \ldots 1)} \) of Higgs pairs with 1-dimensional weight spaces to the monopole branch. As an application of [17], we will describe the structure of the generating series of the contributions of \( \mathcal{M}_1 \) to the Vafa-Witten invariant, and compute them in some cases.

In [14] (see also [19]), Maulik and Thomas define a refined version of the Vafa-Witten invariant, which we denote by

\[ \text{VW}_{r,c_1,c_2}(S, y). \]
It is a rational function in \( \sqrt{y} \), rather than a rational number. It specializes to the unrefined invariant at \( y = 1 \). The instanton contribution to the refined Vafa-Witten invariant is given, up to a sign and a power of \( y \), by the \( \chi_y \)-genus \([3]\) of the component \( \mathcal{M}_{(r)} \), which refines the virtual Euler characteristic \([9]\). We will discuss the contribution of \( \mathcal{M}_{1_r} \) to the refined invariant.

1.2. Nested Hilbert schemes. Fix a rank \( r \). For an \( r \)-tuple of non-negative integers \( n = (n_0, \ldots, n_{r-1}) \), and an \( (r - 1) \)-tuple \( \beta = (\beta_1, \ldots, \beta_{r-1}) \) of classes in \( H^2(S, \mathbb{Z}) \), let

\[
S^{[n_i]} := \text{Hilb}^{n_i}(S)
\]

denote the Hilbert schemes of \( n_i \) points on \( S \), and let \( \text{Hilb}_{\beta_i}(S) \) be the Hilbert schemes of curves on \( S \) with class \( \beta_i \). We will also write

\[
\text{Hilb}^{n_i}(S) := \text{Hilb}^{n_0} \times \cdots \times \text{Hilb}^{n_{r-1}} \times \text{Hilb}_{\beta_1} \times \cdots \times \text{Hilb}_{\beta_{r-1}}(S).
\]

The nested Hilbert scheme

\[
i : S^{[n]}_\beta \rightarrow \text{Hilb}^{n}_\beta(S)
\]

is defined as the incidence locus

\[
\{ I_0, \ldots, I_{r-1}, C_1, \ldots, C_{r-1} \mid I_{i-1}(-C_i) \subset I_i \}.
\]

The nested Hilbert schemes are studied in \([5]\), in which a perfect obstruction theory is constructed. Write \( I^{[n_i]} \) for the universal ideal sheaf on \( S^{[n_i]} \times S \), and let \( D_i \rightarrow \text{Hilb}_{\beta_i}(S) \times S \) be the universal curve with class \( \beta_i \). Finally, write \( \pi : S^{[n]}_\beta \times S \rightarrow S^{[n]}_\beta \) for the projection.

**Theorem 1.4** \([5]\). The nested Hilbert scheme \( S^{[n]}_\beta \) admits a perfect obstruction theory, the dual of which is given by a cone on

\[
\bigoplus_{i=0}^{r-1} R \mathcal{H}om_\pi \left( I^{[n_i]} \mid I^{[n_i]} \right) \rightarrow \bigoplus_{i=1}^{r-1} R \mathcal{H}om_\pi \left( I^{[n_{i-1}]} \mid I^{[n_i]}(D_i) \right),
\]

in which the LHS is the kernel of the trace map

\[
\bigoplus_{i=0}^{r-1} R \mathcal{H}om_\pi \left( I^{[n_i]} \mid I^{[n_i]} \right) \rightarrow R \pi_* \mathcal{O}_S.
\]

In \([7]\), Gholampour and Thomas give another construction of the perfect obstruction theory, using virtual resolutions of degeneracy loci of complexes. Moreover, they give a formula for the induced virtual class in the ambient space \([1.3]\). We will give the statement in the following restricted setting.

Let \( S \) be a surface satisfying

\[
H^1(\mathcal{O}_S) = 0 \quad \text{and} \quad p_g(S) > 0.
\]

For \( i = 0, \ldots, r - 1 \), let \( \mathcal{O}_S(\beta_i) \) be the line bundle with \( c_1(\mathcal{O}_S(\beta_i)) = \beta_i \), so we have

\[
\text{Hilb}_{\beta_i}(S) = |\mathcal{O}_S(\beta_i)| := \mathbb{P}(H^0(\mathcal{O}_S(\beta_i))].
\]

We will write

\[
\mathcal{F}(\beta_i) = \mathcal{F} \otimes \mathcal{O}_S(\beta_i)
\]
for any sheaf $\mathcal{F}$ on $S$.

**Theorem 1.5** (Comparison Theorem). After push-forward by $i$, the virtual class of $S^{[n]}_{\beta}$ is given by

$$i_*[S^{[n]}_{\beta}]^{\text{vir}} = \prod_{i=1}^{r-1} e \left( R\pi_* \mathcal{O}_S(\beta_i) - R\mathcal{H}om_{\pi} \left( \mathcal{I}^{[n-1]}, i_* \mathcal{I}^{[n]}(\beta_i) \right) \right)$$

$$\cap \left[ S^{[n_0]} \times \cdots \times S^{[n_{r-1}]} \right] \times \text{SW}(\beta_1) \times \cdots \times \text{SW}(\beta_{r-1})$$

$$\in A_{n_0+n_r} (\text{Hilb}^n_S(S))$$

in which

$$\text{SW}(\beta_i) \in A_0 (|\mathcal{O}_S(\beta_i)|) \cong \mathbb{Z}$$

is the Seiberg-Witten invariant of $\beta_i$, considered as a 0-cycle.

**Remark 1.6.** We write

$$e \left( R\pi_* \mathcal{O}_S(\beta_i) - R\mathcal{H}om_{\pi} \left( \mathcal{I}^{[n-1]}, i_* \mathcal{I}^{[n]}(\beta_i) \right) \right)$$

for $(n_{i-1} + n_i)$th Chern class of the K-theory class in the brackets, which has rank $n_{n-1} + n_i$. By the generalized Carlsson-Okounkov vanishing of [6], it is in fact the top Chern class, if non-zero.

**Remark 1.7.** For the definition and some basic properties of Seiberg-Witten classes of algebraic surfaces with $H^1(S) = 0$ and $p_g > 0$, we refer to [15, Section 6.3.1] or [14, Section 4].

The moduli space $\mathcal{M}_{1,r}$ is a union of nested Hilbert schemes $S^{[n]}_{\beta}$ [4, 17]. Moreover, the $\mathbb{C}^*$-localized virtual class from [17] agrees with the virtual class from Theorem 1.3. It follows [17] that the contribution of each component $\mathcal{M}_{1,r}$ to the Vafa-Witten invariant is topological. The observation that the generating series of these contributions is multiplicative cf. [5], leads to the following result.

**Notation 1.8.** We will write

$$\text{VW}_{1^r,c_1,c_2}(S,y)$$

for the contribution of $\mathcal{M}_{1,r} = \mathcal{M}_{1^r,c_1,c_2}$ to the refined Vafa-Witten invariant of [14].

**Theorem A.** Fix a rank $r \geq 1$. There are universal Laurent series, so independent of $S$, $A^{(r)}, B^{(r)}, C_{ij}^{(r)} \in \mathbb{Q}(\sqrt{y}|(q^\mathbb{Z}))$ \quad $1 \leq i \leq j \leq r-1$

such that for any surface $S$ with $H^1(S) = 0$ and $p_g > 0$, and any class $c_1 \in H^2(S,\mathbb{Z})$ such that semistability implies stability for all $c_2$, we have

$$\sum_{c_2 \in \mathbb{Z}} \text{VW}_{1^r,c_1,c_2}(S,y) \cdot q^{\frac{vd(r,c_1,c_2)}{2}} =$$

$$\left( A^{(r)} \right)^{\chi(S)} \left( B^{(r)} \right)^{K_3} \sum_\beta \delta_{c_1,\beta} \text{SW}(\beta_1) \cdots \text{SW}(\beta_{r-1}) \prod_{i \leq j} \left( C_{ij}^{(r)} \right)^{\beta_i \beta_j}$$

in which

- $vd(r,c_1,c_2) := 2rc_2 - (r-1)c_1^2 - (r^2-1)\chi(S);$  
- the second sum is over tuples $\beta = (\beta_1, \ldots, \beta_{r-1}) \in (H^2(S,\mathbb{Z}))^{r-1};$
- $\delta_{c_1,\beta} := \# \{ \gamma \in H^2(S,\mathbb{Z}) \mid c_1 - \sum_i i\beta_i = r \cdot \gamma \}.$
Remark 1.10. For fixed $r$, it is expected $[10]$, that (1.9) holds for all $c_1$. For general $c_1$, there might be values of $c_2$ for which there exist strictly semistable Higgs pairs. We don’t consider these cases. Vafa-Witten invariants in the case that there are strictly semistable Higgs pairs are discussed in $[18]$. However, the right hand side of (1.9) is defined for general $c_1$, and for the values of $c_2$ for which stability and semistability of Higgs pairs does coincide, the invariant

$$VW_{1^r,c_1,c_2}(S,y)$$

is given by the coefficient of $q^{\chi y (c_1,c_2)}$.

Remark 1.11. For odd rank $r$, the Laurent series have coefficients in $\mathbb{Q}(y)$, rather than in $\mathbb{Q}(\sqrt{y})$ (see Proposition 8.4).

The following corollary is implicitly in the statement of Theorem A.

Corollary 1.12. Let $S$ be a surface with $H^1(O_S) = 0$ and $p_g(S) > 0$, and let $(r,c_1,c_2)$ be Chern classes, for which stability=semistability. Then

$$VW_{1^r,c_1,c_2}(S,y)$$

is independent of the choice of a polarization of the surface $S$.

We will define the Laurent series in the theorem explicitly in terms of tautological integrals over products of Hilbert schemes of points on the surface $S$ (see Sections 5, 7 and 8). Although for surfaces with deg$(K_S) < 0$, the locus $M_{1^r}$ is empty by stability, the Hilbert schemes and the integrals are still defined. We will prove universality of these integrals for all surfaces (Proposition 7.5). As usual $[8]$, the coefficients of the power series can be determined by evaluating these integrals for $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, so we have access to toric methods, as we explain in Section 9.

1.3. Rank 2 and 3 conjectures. In $[10]$, Lothar Göttsche and Martijn Kool conjecture a formula for the generating series of the $\chi_y$-genus of the instanton branch for rank 2 and 3. Moreover they conjecture, motivated by $S$-duality $[20]$, that the generating series of refined Vafa-Witten invariants satisfies modular properties that relates the contributions of the instanton branch to those of the monopole branch. Using this, they give a conjectural formula for the contribution of the monopole branch to the refined Vafa-Witten invariants of rank 2 and 3. For rank 2, their conjectures refine the predictions in the physics literature $[20]$.

The formulas of $[10]$ that predict the monopole contributions to the Vafa-Witten invariants in rank 2 and 3, have precisely the structure of the generating series (1.9) of the $M_{1^r}$ contributions. This suggests that $M_{1^r}$ accounts for the entire monopole contribution.

Conjecture 1.13. For $S$ and $c_1$ as in Theorem 1.1 and $r$ prime, we have

$$VW_{1^r,c_1,c_2}(S,y) = VW_{1^r,c_1,c_2}(S,y)$$

for all $c_2 \in \mathbb{Z}$.

The conjecture has now been proved by Thomas in $[19]$.

Theorem 1.14 (Thomas). Conjecture 1.13 holds.
It follows that Theorem \[A]\ and Theorem \[1.13\] prove the structure of \[10\] Conjecture 1.5], generalized to arbitrary rank. The rank 2 and 3 conjectures of \[10\] give the universal series appearing in the formula explicitly in terms of functions

\[
\phi_{-2,1}(x, y), \Delta(x), \Theta_{A_2,(1,0)}(x, y), \eta(x), \theta_2(x, y), \theta_3(x, y), \text{ and } W_{\pm}(x, y),
\]

which we give in Appendix \[A\]. The following conjectures, which we formulate using the notation of Theorem \[A\], imply \[10, Remark 1.7 and Conjecture 1.5\].

**Conjecture 1.15.** For rank 2, the universal series appearing in Theorem \[A\] and defined in Section \[A\] are given by

\[
A^{(2)}(y) = q^{\frac{1}{2}} \frac{y^{\frac{1}{2}} - y^{-\frac{1}{2}}}{\phi_{-2,1}(q^2, y^2)^{-\frac{1}{2}}},
\]

\[
B^{(2)}(y) = q^{-\frac{1}{2}} \frac{\eta(q^2)}{\Theta_{A_2,(1,0)}(q^2, y)},
\]

\[
C_{11}^{(2)}(y) = -\frac{\theta_2(q, y)}{\theta_2(q, y)}.
\]

**Conjecture 1.16.** For rank 3, we have

\[
A^{(3)}(y) = q^{\frac{1}{2}} \frac{y^{\frac{1}{2}} - y^{-\frac{1}{2}}}{\phi_{-2,1}(q^3, y^3)^{-\frac{1}{2}}},
\]

\[
B^{(3)}(y) = q^{-\frac{1}{2}} \frac{\eta(q^3)W_-(q^{\frac{3}{2}}, y)}{\Theta_{A_2,(1,0)}(q^{\frac{3}{2}}, y)},
\]

\[
C_{12}^{(3)}(y) = W_+(q^{\frac{3}{2}}, y)W_-(q^{\frac{3}{2}}, y),
\]

\[
C_{11}^{(3)}(y) = C_{22}^{(3)}(y) = \frac{1}{W_-(q^{\frac{3}{2}}, y)}.
\]

As remarked before, the universality allows us to determine the first few terms of the power series of Theorem \[A\] by toric computations. We implemented the Atiyah-Bott localization formula for the surfaces $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ in Sage \[16\] and found agreement with Conjectures \[1.15\] and \[1.16\].

Define multiplicative subgroups

\[
U^{(r)}_N := 1 + q^N \mathbb{Q}(y^{\frac{1}{2}})[[q]] \subset \mathbb{Q}(y^{\frac{1}{2}})((q^{\frac{1}{2}}))^*.
\]

for all $r, N \geq 1$, and consider series

\[
P = cq^{\frac{1}{2}}(1 + p_1 q + p_2 q^2 + \ldots)
\]

and

\[
P' = c'q^{\frac{1}{2}}(1 + p'_1 q + p'_2 q^2 + \ldots)
\]

with $c, c' \in \mathbb{Q}(\sqrt{y})^*$, $p_i, p'_i \in \mathbb{Q}(\sqrt{y})$ and $z, z' \in \mathbb{Z}$. The Laurent series appearing in Theorem \[A\] and Conjectures \[1.15\] and \[1.16\] are all of this form. Then we have

\[
P \equiv P' \mod U^{(r)}_{N+1} \quad \text{(i.e. } P'P^{-1} \in U^{(r)}_{N+1})
\]

if and only if

\[
c = c', \quad z = z', \quad \text{and} \quad p_i = p'_i, \ldots, p_N = p'_N.
\]

**Theorem B.** Let $S$ be a surface with $H^1(O_S) = 0$ and $p_2(S) > 0$. The rank 2 conjectures of \[10\] correctly predict the first 7 terms of the universal series of Theorem \[A\]. The rank 3 conjectures correctly predict the first 6 terms. In other
words, the equations of Conjecture 1.15 hold modulo $U_7^{(2)}$, and the equations of Conjecture 1.16 hold modulo $U_6^{(3)}$.

Let $S$ be a surface as in Theorem A. Assume the Picard group
\[ \text{Pic}(S) = \mathbb{Z} \cdot [C] \]
of $S$ is generated by a smooth very ample canonical curve $C \in |K_S|$. Let $c_1 = K_S$. We have by Lemma 10.21 that for rank 3, the only $\beta$ that contributes to the RHS of (1.9), is $\beta = (K_S, 0)$. In rank 2, and in a slightly more general setting, the only contribution is given by $\beta = (K_S)$. It follows by Theorem A that we have
\[ (1.17) \sum_{c_2} VW_{1^r, K_S, c_2}(S, y) \cdot q^{\text{vol}(r, K_S, c_2)} = \left( -A^{(r)}(y) \right)^{\chi(O_S)} \left( B^{(r)}(y) C^{(r)}_{11}(y) \right) K_S^2 \]
for $r = 2, 3$, where we have used $SW(K_S) = (-1)^{\chi(O_S)}$ by e.g. [15 Proposition 6.3.4]. Now are toric computations are slightly faster, so we obtain the following result.

**Theorem B’.** Let $S$ be a surface with $H^1(O_S) = 0$ and $p_g(S) > 0$, and assume that the Picard group of $S$ is generated by a smooth very ample canonical curve. Then we have
\[ \sum_{c_2} VW_{1^2, K_S, c_2}(S, y) \cdot q^{\text{vol}(2, K_S, c_2)} \equiv \left( -q^{\phi_{-2,1}(q^2, y^2)^2 \Delta(q^3)^2} \right)^{\chi(O_S)} \left( \frac{-q^{\phi_{-2,1}(q^2, y^2)^2 \Delta(q^3)^2}}{\theta_2(q, y)} \right) K_S^2 \mod U_1^{(2)} , \]
\[ \sum_{c_2} VW_{1^3, K_S, c_2}(S, y) \cdot q^{\text{vol}(3, K_S, c_2)} \equiv \left( -q^{\phi_{-2,1}(q^3, y^3)^2 \Delta(q^3)^2} \right)^{\chi(O_S)} \left( \frac{-q^{\phi_{-2,1}(q^3, y^3)^2 \Delta(q^3)^2}}{\Theta_{2,1}(q^3, y^3)} \right) K_S^2 \mod U_8^{(3)} . \]

For $S$ a surface as in Theorem B’ and rank $r = 2$, the moduli space $M_{1^2}$ is smooth for $c_2 \leq 3$. In [17] and [19], this is used to compute the Vafa-Witten invariant by direct intersection-theoretic calculations. The rank 2 equation of Theorem B’ is proven modulo $U_3^{(2)}$ in [19]. In [17], it is proven in the unrefined case, obtained by setting $y = 1$, and $U_1^{(2)}$.

For rank 3, the moduli space $M_{1^3}$ is smooth if and only if $c_2 \leq 2$ (Proposition 10.22). This allows us to compute the Vafa-Witten invariants by the methods of [17] [19]. As a result, we obtain an alternative proof, by direct calculations, for the rank 3 equation of Theorem B’ modulo $U_3^{(3)}$.

**Remark 1.18.** In the restricted setting above, we also did the rank 2 computations in the unrefined case. The Göttsche-Kool conjecture reduces to the original formula from [20], which we were able to check modulo $U_1^{(2)}$.

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2. The moduli space

Let $S$ be a smooth projective surface with $p_g(S) > 0$ and $H^1(O_S) = 0$. Fix a rank $r$. As mentioned in the introduction, the locus $M_1^r$ of Higgs pairs with 1-dimensional weight spaces is a union of nested Hilbert schemes. In this section, we will introduce some notation and describe universal Higgs pairs over the connected components.

Write $s := r - 1$ and let $L = (L_0, \ldots, L_s)$ be an $r$-tuple of line bundles on $S$.

Notation 2.1. Define classes

$$\beta_i = c_1(L_i - L_{i-1} + \omega_S) \in H^2(S, \mathbb{Z})$$

for $i = 1, \ldots, s$, and write

$$\beta = \beta(L) = (\beta_1, \ldots, \beta_s).$$

We will also write

$$\beta^i = K_S - \beta_i$$

for $i = 1, \ldots, s$ and an $s$-tuple $\beta = (\beta_1, \ldots, \beta_s) \in (H^2(S, \mathbb{Z}))^s$. In particular, when $\beta = \beta(L)$, we have

$$\beta^i = c_1(L_{i-1} - L_i)$$

for $i = 1, \ldots, s$.

Remark 2.2. We will use the convention

$$s := r - 1$$

throughout the paper. Furthermore, $L$ will always denote an $s + 1$-tuple of line bundles on $S$, and $\beta$ an $s$-tuple of classes in $H^2(S, \mathbb{Z})$.

Consider the product of complete linear systems

$$\text{Hilb}_\beta(S) = \text{Hilb}_{\beta_1}(S) \times \cdots \times \text{Hilb}_{\beta_s}(S)$$

and write

$$|O_S(\beta_i)| \times S \xleftarrow{\text{pr}_i} \text{Hilb}_\beta(S) \times S \xrightarrow{q} S$$

for the projections, where $i = 1, \ldots, s$. We will write $O_{\beta_i}(1)$ for the canonical line bundle on $|O_S(\beta_i)|$. Let $t$ be an equivariant parameter for the trivial $\mathbb{C}^*$-action on a point.

For $i = 0, \ldots, s$, let $L_i$ be a $\mathbb{C}^*$-equivariant line bundle on $\text{Hilb}_\beta(S) \times S$, with fibres $L_i$ over $\text{Hilb}_\beta(S)$, such that the tautological sections

$$O_{|O_S(\beta_i)| \times S} \to O_{\beta_i}(1) \boxtimes O_S(\beta_i)$$

induce $\mathbb{C}^*$-equivariant maps

$$\phi_{L,i} : L_{i-1} \to L_i \otimes q^* \omega_S$$
for \( i = 1, \ldots, s \). In other words, we require
\[
L_i^* \otimes (q^* L_i) \in \pi^* \text{Pic} S_n \otimes t^2
\]
\[
L_i \otimes L_i^* \otimes q^* \omega_S = \text{pr}_1^* (O_{\beta_1}(1) \boxtimes O_S(\beta_i))
\]
for \( i = 1, \ldots, s \). Of course we could just define
\[
L_0 = \mathcal{O}_1
\]
\[
L_1 = \text{pr}_1^* O_{\beta_1}(1)
\]
\[
\vdots
\]
\[
L_s = L_s \otimes \text{pr}_1^* O_{\beta_s}(1) \otimes \cdots \otimes \text{pr}_s^* O_{\beta_s}(1)
\]
but we prefer the ambiguity. Define the locally free sheaf
\[
\text{E}_\mathcal{L} := (L_0 \otimes t^0) \oplus \cdots \oplus (L_s \otimes t^{-s}).
\]
The maps \( \phi_{\mathcal{L}, i} \) define a \( \mathbb{C}^* \)-equivariant Higgs field
\[
\phi_{\mathcal{L}} = (\phi_{\mathcal{L}, 1}, \ldots, \phi_{\mathcal{L}, s}) : E_\mathcal{L} \rightarrow E_\mathcal{L} \otimes \omega_S \otimes t.
\]

Now choose non-negative integers \( n = (n_0, \ldots, n_s) \) and write
\[
\text{Hilb}_\beta^n(S) = S^{[n_0]} \times \cdots \times S^{[n_s]} \times \text{Hilb}_\beta(S)
\]
as in the introduction. Let \( \mathcal{I}^{[n]} \) denote the universal ideal sheaf on \( S^{[n]} \times S \). Define the following sheaf on \( \text{Hilb}_\beta^n(S) \times S \), suppressing obvious pull-backs:
\[
\text{E}^{[n]}_\mathcal{L} := (L_0 \otimes \mathcal{I}^{[n_0]} \otimes t^0) \oplus \cdots \oplus (L_s \otimes \mathcal{I}^{[n_s]} \otimes t^{-s}).
\]
The nested Hilbert scheme is by definition the maximal subscheme
\[
i : S^{[n]} \rightarrow \text{Hilb}_\beta^n(S)
\]
such that \( \phi_{\mathcal{L}} \) restricts to a Higgs field
\[
\phi^{[n]}_{\mathcal{L}} : E^{[n]}_\mathcal{L} \rightarrow E^{[n]}_\mathcal{L} \otimes \omega_S \otimes t.
\]

Remark 2.3. Throughout the paper, the letter \( n \) is reserved for \( s+1 \)-tuples of non-negative integers. Also \( \text{Hilb}_\beta^n(S) \) will always denote a product of Hilbert schemes as above.

Proposition 2.4 ([5], [17]). Every connected component of \( \mathcal{M}_{1^r} \) is represented by a triple
\[
(S^{[n]}_\beta, E^{[n]}_\mathcal{L}, \phi^{[n]}_{\mathcal{L}}),
\]
as constructed above, which is unique modulo \( \text{Pic}(S^{[n]}_\beta) \times t^2 \), i.e. modulo the choices in the construction.

By Proposition 2.4, the connected components of \( \mathcal{M}_{1^r} \) are naturally indexed by tuples \( L = (L_0, \ldots, L_s) \) and \( n = (n_0, \ldots, n_s) \).

Definition 2.5. We denote a connected component of \( \mathcal{M}_{1^r} \) represented by a triple \((S^{[n]}_\beta, E^{[n]}_\mathcal{L}, \phi^{[n]}_{\mathcal{L}})\) by \( \mathcal{M}^{[n]}_L \).

Obviously, not every pair \((L, n)\) corresponds to a component of \( \mathcal{M}_{1^r} \). The nested Hilbert scheme might be empty, or the Higgs pairs in the family \((E^{[n]}_\mathcal{L}, \phi^{[n]}_{\mathcal{L}})\) might be unstable. The other restriction is the Chern data of the Higgs pairs. We will address stability in Section 3. We finish this section with a lemma regarding the second issue.
Lemma 2.6. The total Chern class of any fibre $E$ of $E^\beta_\lambda$ over $S^\beta_n$ is given by
\[
c(E) = 1 + \sum_{i=0}^s c_1(L_i) + |n| \cdot pt + \sum_{0 \leq i < j \leq s} c_1(L_i)c_1(L_j)
\]
\[
= 1 + (s + 1)c_1(L_0) - \sum_{i=1}^s (s + 1 - i)\beta^i
\]
\[
+ |n| \cdot pt + \frac{s}{2s + 2}c_1^2 - \sum_{1 \leq i < j \leq s} \frac{i(s + 1 - j)}{s + 1} \beta^i \beta^j - \sum_{1 \leq i \leq s} \frac{i(s + 1 - i)}{2s + 2}(\beta^i)^2,
\]
where
\[
c_1 = c_1(L_0) + \ldots + c_1(L_s)
\]
\[
|n| = n_0 + \ldots + n_s,
\]
and $pt$ denotes the Poincaré dual of a point.

Proof. This is a straightforward computation. For the second equation, note that we have
\[
(s + 1)c_1(L_i) = \sum_{k=1}^i -k\beta^k + \sum_{k=i+1}^s (s + 1 - k)\beta^k + c_1
\]
for $i = 0, \ldots, s$. Substituting this into
\[
\sum_{0 \leq i < j \leq s} c_1(L_i)c_1(L_j)
\]
and interchanging sums gives the result. $\Box$

3. Stability

By Proposition 2.4 the connected components of $M_{1r}$ are isomorphic to nested Hilbert schemes $S^\beta_n$, with
\[
\beta = (\beta_1, \ldots, \beta_s) \quad \text{and} \quad n = (n_0, \ldots, n_s)
\]
tuples of divisor classes on $S$ and integers respectively. The Hilbert scheme is empty if and only if one of the $\beta_i$'s is not effective, or $\beta_i = 0$ and $n_{i-1} < n_i$ for some $i$. Obviously, the virtual class of the nested Hilbert scheme vanishes in this case.

We will give dual conditions on $\beta$ and $n$, which hold whenever the Higgs pairs parametrized by $S^\beta_n$ are Gieseker unstable, and which in turn imply the vanishing of the virtual class. We recall the definition of stability of Higgs pairs.

Definition 3.1. Let $H$ be a polarization of the surface $S$. A Higgs pair $(E, \phi)$ is called slope stable (resp. slope semistable) if
\[
\frac{\deg(F)}{\rk(F)} < \frac{\deg(E)}{\rk(E)} \quad \text{(resp.)} \quad \frac{\deg(F)}{\rk(F)} \leq \frac{\deg(E)}{\rk(E)}.
\]
for every $\phi$-invariant subsheaf $0 \neq F \subseteq E$ with $\rk(F) < \rk(E)$. It is called Gieseker stable (resp. Gieseker semistable) if we have inequalities of polynomials in $m$
\[
\frac{\chi(F(mH))}{\rk(F)} < \frac{\chi(E(mH))}{\rk(E)} \quad \text{(resp.)} \quad \frac{\chi(F(mH))}{\rk(F)} \leq \frac{\chi(E(mH))}{\rk(E)}.
\]
for every proper $\phi$-invariant subsheaf $0 \neq F \subseteq E$. By "(semi)stable", we will always mean Gieseker (semi)stable.
Let $E = E_0 \oplus \ldots \oplus E_s$ be a sum of torsion free rank 1 sheaves, equipped with a Higgs field

$$\phi = (\phi_1, \ldots, \phi_s) : E \to E \otimes \omega_S$$
given by homomorphisms

$$\phi_i : E_{i-1} \to E_i \otimes \omega_S \quad \text{for} \quad i = 1, \ldots, s.$$ 

Note that all Higgs pairs in $\mathcal{M}_{1r}$ are of this form.

**Lemma 3.2.** Assume that $(E, \phi)$ is indecomposable, i.e. $\phi_i \neq 0$ for $i = 1, \ldots, s$ and assume that

$$\deg(E_{i-1}) \geq \deg(E_i) \quad \text{for} \quad i = 1, \ldots, s.$$ 

Then the pair $(E, \phi)$ is slope semistable. It is moreover slope stable unless

$$\deg(E_0) = \ldots = \deg(E_s).$$

**Proof.** Let $F \subset E$ be a $\phi$-invariant Higgs field. Let $j$ maximal, such that

$$(3.3) \quad F \subset E_j \oplus \ldots \oplus E_s.$$ 

I claim that $F$ has rank $s + 1 - j$. It follows that if $F$ is a destabilizing subsheaf, so is $E_j \oplus \ldots \oplus E_s$.

In order to prove the claim, consider the filtration

$$F = F^0 \supset \ldots \supset F^{s-j} \supset F^{s+1-j} = 0$$

of $F$, given by

$$F^i = K_{S}^{-i} \otimes \phi^i(F),$$

so we have

$$F^i \subset E_{j+i} \oplus \ldots \oplus E_s$$

for $i = 0, \ldots, s+1-j$. Note that for $i = 0, \ldots, s-j$, by injectivity of $\phi_{j+i} \cdots \phi_{j+1}$, and by the choice of $j$, the composition

$$F^i \subset E_{j+i} \oplus \ldots \oplus E_s \to E_{j+i}$$

is non-zero, and hence its image has rank 1, since $E_{j+i}$ is torsion-free. On the other hand, its kernel contains $F^{i+1}$. It follows that we have

$$\text{rk} F > \text{rk} F^1 > \ldots > \text{rk} F^{s+1-j} = 0$$

and hence, $\text{rk}(F) = s + 1 - j$ by $(3.3)$, proving the claim.

It follows that $(E, \phi)$ is slope semistable if and only if

$$\frac{\sum_{i=0}^{j} \deg(E_i)}{s + 1 - j} \leq \frac{\sum_{i=0}^{s} \deg(E_i)}{s + 1} = \frac{\deg(E)}{\text{rk}(E)}$$

for $j = 0, \ldots, s$. This clearly holds when $\deg(E_i) \leq \deg(E_{i-1})$ for all $i$. Finally note that $(E, \phi)$ is slope stable if one of the inequalities is strict. \hfill $\square$

The hypothesis of Lemma 3.2 certainly holds when $c_1(E_{i-1}) - c_1(E_i)$ is effective for each $i$. In this case, the condition

$$\deg(E_0) = \ldots = \deg(E_s)$$

implies that

$$c_1(E_0) = \ldots = c_1(E_s).$$

Although the Higgs pair $(E, \phi)$ is not slope stable, it might still be Gieseker (semi)stable.
Lemma 3.4. Assume that \((E, \phi)\) is indecomposable and that
\[c_1(E_0) = \ldots = c_1(E_s) \quad \text{and} \quad c_2(E_0) \leq \ldots \leq c_2(E_s)\).
Then the pair \((E, \phi)\) is Gieseker semistable. It is Gieseker stable unless
\[c_2(E_0) = \ldots = c_2(E_s)\).

Proof. The proof is similar to the proof of Lemma 3.2. Simply note that by
Grothendieck-Riemann-Roch the hypothesis implies
\[\chi(E_i(m)) \geq \chi(E_i(m))\]
for \(i = 1, \ldots, s\), with equality whenever \(n_{i-1} = n_i\).

Let \(L_0, \ldots, L_s\) be line bundles on \(S\) and let \(n = (n_0, \ldots, n_s)\) be non-negative integers. Let \(\beta = \beta(L)\) (and \(\beta_i\) and \(\beta^i\) for \(i = 1, \ldots, s\)) be given as in Notation 2.1
and consider the flat family of Higgs pairs \((E[n], \phi[n])\) over the base \(S[n]\), as defined in Section 2.

In terms of \(\beta\) and \(n\), Lemma 3.2 and Lemma 3.4 tell us that whenever the
family \((E[n], \phi[n])\) is not Gieseker semistable, there is an \(i \in \{1, \ldots, s\}\) such that
the divisor class \(\beta^i\) is not effective, or such that \(\beta^i = 0\) and \(n_{i-1} > n_i\) (compare
to the introduction of this section!). As we will see in the following proposition,
this suffices to show that we have \(i_*[S[n]]_{\text{vir}} = 0\) in this case (recall that we write
\(i : S[n] \to \text{Hilb}_n(S)\)).

Proposition 3.5. Assume that
\[i_*[S[n]]_{\text{vir}} \neq 0\].
Then the family \((E[n], \phi[n])\) of Higgs pairs is (Gieseker) semistable for any polarization
of \(S\). It is stable unless \(L_0 = \ldots = L_s\) and \(n_0 = \ldots = n_s\).

Proof. By [15] Proposition 6.3.4, Theorem 1.5 and the hypothesis, we have
\[\text{SW}(\beta_1) \cdots \text{SW}(\beta_s) = (-1)^s \chi(O_S) \text{SW}(\beta_1) \cdots \text{SW}(\beta_s)
\neq 0\]
and hence \(\beta^i \geq 0\) for \(i = 0, \ldots, s\), by definition of the Seiberg-Witten class. By
Lemma 3.2 the fibres of \((E[n], \phi[n])\) are slope-stable, and hence Gieseker stable,
unless \(L_0 = \ldots = L_s\). Assume the latter. We need to show that \(n_{i-1} \leq n_i\) for all
\(i\). Assume that \(n_{i-1} > n_i\) for some \(i\). Then the nested Hilbert scheme
\[i : S[n_{i-1}, n_{i-1}] \to S[n] \times S[n_{i-1}]
\]
is empty, and we have by Serre duality and Theorem 1.5
\[e(R\pi_*\omega_S - R\mathcal{H}om_{\pi}(\mathcal{I}[n-1], \mathcal{I}[n]) \otimes \omega_S))\]
\[= (-1)^{n_{i-1}+n_i} e(R\pi_*\mathcal{O}_S - R\mathcal{H}om_{\pi}(\mathcal{I}[n], \mathcal{I}[n]))\]
\[= (-1)^{n_{i-1}+n_i} i_*[S[n_{i-1}]_{\text{vir}}\]
\[= 0\].

By the assumption \(L_0 = \ldots = L_s\), we have in particular \(\beta_i = K_S\). Using Theorem 1.5 again, we find \(i_*[S[n]]_{\text{vir}} = 0\), contradicting
the hypothesis. \(\square\)
Recall (Proposition 2.4), that $\mathcal{M}_{1^r,c_1,c_2}$ is a union of Hilbert schemes $S_{\beta}^{[n]}$. Write will also write $i$ for the morphism

$$i: \mathcal{M}_{1^r,c_1,c_2} \to \bigcup_{\beta,n} \text{Hilb}_{\beta}^{[n]}(S)$$

given on each connected component of $\mathcal{M}_{1^r,c_1,c_2}$ by the inclusion

$$i: S_{\beta}^{[n]} \to \text{Hilb}_{\beta}^{[n]}(S).$$

By the vanishing of Proposition 3.5, we can sum in the following proposition over all pairs $(L, n)$ or $(\beta, n)$, rather than the ones that correspond to connected components of stable Higgs pairs. In particular, the push-forward by $i$ of the virtual class does not depend on the polarization of the surface $S$.

**Proposition 3.6.** Fix $r, c_1$ and $c_2$ such that Gieseker semistability of Higgs pairs implies Gieseker stability. Then have

$$i_* [\mathcal{M}_{1^r,c_1,c_2}]^{\text{vir}} = \sum_{L,n} \delta_{L,c_1} i_* [S_{\beta(L)}^{[n]}]^{\text{vir}}$$

$$= \sum_{\beta,n} \delta_{\beta,c_1} i_* [S_{\beta}^{[n]}]^{\text{vir}}$$

in which

$$\delta_{L,c_1} = \begin{cases} 1 & \text{if } c_1(L_0) + \ldots + c_1(L_s) = c_1 \\ 0 & \text{else,} \end{cases}$$

$$\delta_{\beta,c_1} = \# \left\{ \gamma \in H^2(S, \mathbb{Z}) \mid c_1 - \sum_{i=1}^s i \beta^i = (s+1) \cdot \gamma \right\},$$

and the sums are taken over

$$L, n \quad \text{with} \quad c_2 = |n| + \sum_{0 \leq i < j \leq s} c_1(L_i)c_1(L_j),$$

$$\beta, n \quad \text{with} \quad c_2 = |n| + \frac{r-1}{2r} c_1^2 + \sum_{1 \leq i < j \leq s} \frac{i(s+1-j)}{r} \beta^i \beta^j$$

$$+ \sum_{1 \leq i \leq s} \frac{i(s+1-i)}{2r} (\beta^i)^2.$$
Example 3.8. If $H^2(S, \mathbb{Z})$ is $r$-torsion free, the multiplicity $\delta_{\beta, c_1}$ reduces to the Kronecker $\delta$ for an identity in the group $H^2(S, \mathbb{Z})/rH^2(S, \mathbb{Z})$. For rank 2 and 3 we have:

$$
\delta_{(\beta_1), c_1} = \begin{cases} 1 & \text{if } c_1 \equiv \beta^1 \mod 2H^2(S, \mathbb{Z}) \\ 0 & \text{else} \end{cases}
$$

$$
\delta_{(\beta_1, \beta_2), c_1} = \begin{cases} 1 & \text{if } c_1 + \beta^2 \equiv \beta^1 \mod 3H^2(S, \mathbb{Z}) \\ 0 & \text{else} \end{cases}
$$

4. Vafa-Witten integrals

Choose line bundles $L = (L_0, \ldots, L_\ell)$ on $S$ and let $\beta = \beta(L) = (\beta_1, \ldots, \beta_\ell)$ and $\mathcal{L} = (L_0, \ldots, L_\ell)$ be defined as in Section 2. Let $n = (n_0, \ldots, n_\ell)$ be non-negative integers. Recall that we write

$$
E^{[n]}_\mathcal{L} = L_0 \otimes t_{[n_0]} \otimes \cdots \otimes L_\ell \otimes t_{[n_\ell]} \otimes t^{-s}
$$

for the sheaf on

$$
\text{Hilb}^n_\beta(S) \times S = S^{[n_0]} \times \cdots \times S^{[n_\ell]} \times |\mathcal{O}_S(\beta_1)| \times \cdots \times |\mathcal{O}_S(\beta_\ell)| \times S
$$

and for its restriction to the nested Hilbert scheme

$$
i : S^{[n]}_{\beta} \to \text{Hilb}^n_\beta(S),
$$

over which we have a canonically defined Higgs field $\phi_{\mathcal{L}} : E^{[n]}_\mathcal{L} \to E^{[n]}_\mathcal{L} \otimes \omega_S \otimes t$.

Define a class

$$
T^{[n]}_\mathcal{L} := R\mathcal{H}om_{\mathcal{L}}(E^{[n]}_\mathcal{L}, E^{[n]}_\mathcal{L} \otimes \omega_S \otimes t)_0 - R\mathcal{H}om_{\mathcal{L}}(E^{[n]}_\mathcal{L}, E^{[n]}_\mathcal{L})_0 \in K_0(\text{Hilb}^n_\beta(S)),
$$

and denote its pull-back to $S^{[n]}_{\beta}$ by the same symbol. Note that $T^{[n]}_\mathcal{L}$ depends only on $\beta$, rather than on $L$, or the choice of $\mathcal{L}$. We will write

$$
N^{[n]}_\mathcal{L} := T^{[n]}_\mathcal{L} - \left(T^{[n]}_\mathcal{L}\right)^{\mathbb{C}^*}
$$

for its moving part. Let $e$ denote the $\mathbb{C}^*$-equivariant Euler class, and define the rational number

$$(4.1) \quad \text{VW}^{[n]}_\beta := \int_{[S^{[n]}_{\beta}]} e \cdot \left(N^{[n]}_\mathcal{L}\right).$$

In the case that $(E^{[n]}_\mathcal{L}, \phi_{\mathcal{L}})$ represents a connected component

$$
\mathcal{M}^{[n]}_\mathcal{L} = (S^{[n]}_{\beta}, E^{[n]}_\mathcal{L}, \phi^{[n]}_{\mathcal{L}}) \subset \mathcal{M}_{1, c_1, c_2},
$$

$T^{[n]}_\mathcal{L}$ is the class in $K$-theory of the cone in the introduction, and hence equals the $\mathbb{C}^*$-localized obstruction theory of [17] on $\mathcal{M}^{[n]}_\mathcal{L}$. Over $\mathcal{M}^{[n]}_\mathcal{L}$, the class $N^{[n]}_\mathcal{L}$ is the virtual normal bundle to the $\mathbb{C}^*$ fixed locus $(N^{[n]}_{1, c_1, c_2})^{\mathbb{C}^*}$ in $N^{[n]}_{1, c_1, c_2}$. By definition of the Vafa-Witten invariant [1.2], the contribution of the connected component $\mathcal{M}^{[n]}_\mathcal{L}$ is given by $\text{VW}^{[n]}_\beta$.

If the Higgs pair $(E^{[n]}_\mathcal{L}, \phi_{\mathcal{L}})$ contains fibres that are not Gieseker semistable, it does not represent a connected component of any $\mathcal{M}_{1, c_1, c_2}$, and hence does not contribute to the Vafa-Witten invariant. On the other hand, by Proposition 5.4, we
have $V_{\beta}^{[n]} = 0$ in this case. It follows that, using the notation from Proposition 3.6 we have:

$$V_{W_{\beta}}^{[n]} = \sum_{L,n} \delta_{L,c_1} V_{W_{\beta}^{[L]}}^{[n]}$$

$$= \sum_{\beta,n} \delta_{\beta,c_1} V_{W_{\beta}}^{[n]}.$$

Now define a line bundle

$$K_{L}^{[n]} := \det \left( T_{L}^{[n]} \right)$$
on $\text{Hilb}_{\beta}^{[n]}(S)$. Note that $T_{\beta}^{[n]}$ is defined as the difference between a complex and its dual, up to a factor $t$. Hence its determinant is by construction a square, up to a factor $t$. Hence, after choosing once and for all a square root of $t$, the line bundle $K_{L}^{[n]}$ has a canonical square root, denoted by $(K_{L}^{[n]})^{ \frac{1}{2}}$. Over $S_{\beta}^{[n]}$, the bundle $K_{L}^{[n]}$ restricts to the virtual canonical bundle [19], and its square root restricts to the canonical square root of [19, Proposition 2.6].

By [19], the contribution to the refined invariant can be computed by

$$V_{W_{\beta}}^{[n]}(y) := \left[ \int_{[S_{\beta}^{[n]}]_{\text{vir}}} \frac{\text{ch} \left( (K_{L}^{[n]})^{ \frac{1}{2}} \right)}{\text{ch} \left( \Lambda_{\cdot} (N_{L}^{[n]})^{\vee} \right)} \text{Td} \left( (T_{L}^{[n]})^{\vee} \right) \right]_{\text{ch}(t) = y},$$

where ch and Td denote the $\mathbb{C}^{*}$-equivariant Chern character and Todd class respectively. Again, in the language of Proposition 3.6 we have

$$V_{W_{\beta}}^{[n]}(y) = \sum_{L,n} \delta_{L,c_1} V_{W_{\beta}^{[L]}}^{[n]}(y)$$

$$= \sum_{\beta,n} \delta_{\beta,c_1} V_{W_{\beta}}^{[n]}(y).$$

By Theorem 1.5 we have

$$i_* [S_{\beta}^{[n]}]_{\text{vir}} = \prod_{i=1}^{s} e \left( R\pi_* (O_{S}(\beta_i) - R\mathcal{H}om_{\pi} \left( T^{[n-1]}_{\beta_i}, T^{[n]}_{\beta_i} \right) ) \right)$$

$$\cap \left[ S^{[n_0]} \times \cdots \times S^{[n_s]} \right] \times \text{SW}(\beta_1) \times \cdots \times \text{SW}(\beta_s).$$

The factor

$$\text{SW}(\beta) := \text{SW}(\beta_1) \times \cdots \times \text{SW}(\beta_s)$$

$$\in A_0 \left( |O_{S}(\beta_1)| \times \cdots \times |O_{S}(\beta_s)| \right)$$

annihilates all Chern classes in the integrands of (4.1) and (4.2) that are pulled back from

$$|O_{S}(\beta_1)| \times \cdots \times |O_{S}(\beta_s)|.$$

It follows that we can rewrite (4.1) and (4.2) as integrals over

$$\text{Hilb}^{n}(S) = S^{[n_0]} \times \cdots \times S^{[n_s]}.$$
and classes

\[ T_L^{[n]} := \mathcal{H}om_{\pi}(E_L^{[n]}, E_L^{[n]} \otimes \omega_S \otimes t)_0 - \mathcal{H}om_{\pi}(E_L^{[n]}, E_L^{[n]})_0, \]

\[ N_L^{[n]} := T_L^{[n]} - \left( T_L^{[n]} \right)^{C^*}, \quad \text{and} \]

\[ K_L^{[n]} := \det \left( T_L^{[n]} \right) \]

in \( K_0(\text{Hilb}^n(S)) \). Again, note that since \( H^1(O_S) = 0 \), the classes \( T_L^{[n]}, N_L^{[n]}, \) and \( K_L^{[n]} \) depend on \( \beta = \beta(L) \), rather than on \( L \). We have, now considering \( \text{SW}(\beta) \) as an integer,

\[ VW_{\beta}^{[n]} = \text{SW}(\beta) \int_{[\text{Hilb}^n(S)]} \frac{1}{e(N_L^{[n]})} \times \prod_{i=1}^s e \left( R\pi_* \mathcal{O}(\beta_i) - R\mathcal{H}om_{\pi}(I^{[n_i-1]}, I^{[n_i]}(\beta_i)) \right) \]

and

\[ VW_{\beta}^{[n]}(y) = \text{SW}(\beta) \left[ \int_{[\text{Hilb}^n(S)]} \frac{\text{ch} \left( (K_L^{[n]})^{C^*} \right)}{\text{ch}(\Lambda^*(N_L^{[n]}))} \text{Td} \left( (T_L^{[n]})^{C^*} \right) \times \prod_{i=1}^s e \left( R\pi_* \mathcal{O}(\beta_i) - R\mathcal{H}om_{\pi}(I^{[n_i-1]}, I^{[n_i]}(\beta_i)) \right) \right]_{\text{ch}(t)=y}. \]

5. REMOVING TRACE

We can normalize the generating series

\[ \sum_n VW_{\beta}^{[n]} q^n \]

by dividing through the leading term. In terms of the integrals of (4.4), this comes down to considering ‘traceless’ integrants. By this we mean the following. Note that \( N_L \) can be written as a linear combination of terms of the form

\[ R\mathcal{H}om_{\pi}(E, F) \]

with \( E \) and \( F \) torsion free rank 1 sheaves. We will replace each such term by

\[ R\mathcal{H}om_{\pi}(E, F) - R\mathcal{H}om_{\pi}(\det E, \det F). \]

In Section 6, we will deal with the leading term of the generating series separately.

We keep the notation from the previous section. Moreover, we will write

\[ E_L := E_L^{[0]} = L_0 \otimes t^0 \oplus \ldots \oplus L_s \otimes t^{-s} \]

for the vector bundle on \( S \), and furthermore

\[ T_L := T_L^{[0]} = R\text{Hom}(E_L, E_L \otimes \omega_S \otimes t)_0 - R\text{Hom}(E_L, E_L)_0, \]

\[ N_L := N_L^{[0]} = T_L - (T_L)^{C^*}, \]

\[ K_L := K_L^{[0]} = \det T_L^l. \]
for the classes in the equivariant $K$-group of a point. Finally, we will also use the notation

$$T_{L,0}^{[n]} = T_L^{[n]} - T_L, \quad N_{L,0}^{[n]} = N_L^{[n]} - N_L, \quad \text{and} \quad K_{L,0}^{[n]} = K_L^{[n]} \otimes K_L^*,$$

for the classes in $K_0(\text{Hilb}^n(S))$, where we suppress pull-backs from the point. Define

$$F_n(S, \beta) := \frac{1}{e(N_L)}$$

and

$$Q_n(S, \beta) := \int_{\text{Hilb}^n(S)} \frac{1}{e(N_L^{[n]}_{L,0})} \prod_{i=1}^s e \left( R\pi_* \mathcal{O}(\beta_i) - R\mathcal{H}om_{\pi}(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\beta_i)) \right),$$

so we have

$$\text{VW}_{\beta}^{[n]} = \text{SW}(\beta)F_n(S, \beta)Q_n(S, \beta).$$

In the refined case, define

$$F(S, \beta, y) := \left[ \frac{\text{ch}(K_L^*)}{\text{ch}(\Lambda^*(N_L^*))} \text{Td}(\mathcal{T}^*) \right]_{\text{ch}(t) = y}$$

and

$$Q_n(S, \beta, y) := \left[ \int_{\text{Hilb}^n(S)} \frac{\text{ch}(K_{L,0}^{[n]} \mathcal{T}^*)}{\text{ch}(\Lambda^*(N_{L,0}^{[n]} \mathcal{T}^*))} \text{Td}(\mathcal{T}_{L,0}^{[n]} \mathcal{T}^*) \times \prod_{i=1}^s e \left( R\pi_* \mathcal{O}(\beta_i) - R\mathcal{H}om_{\pi}(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\beta_i)) \right) \right]_{\text{ch}(t) = y},$$

so that

$$\text{VW}_{\beta}^{[n]}(y) = \text{SW}(\beta) \ F(S, \beta, y) \ Q_n(S, \beta, y).$$

**Remark 5.1.** A priori, $Q_n(S, \beta, y)$ is a rational function in $\sqrt{y}$, due to the fractional exponent of the virtual canonical bundle. However, an easy computation shows that the equivariant parameter $t$ appears in $K_{L,0}^{[n]}$, with even exponent, and hence, $Q_n(S, \beta, y)$ is in fact a rational function in $y$.

In the Section 5, we will determine $F(S, \beta)$ under the assumption $\text{SW}(\beta) \neq 0$. In Section 6, we will show that the numbers $Q_n(S, \beta)$ are given by universal polynomials $P_n(S, \beta)$ in the Chern numbers of $S$ and $\beta = (\beta_1, \ldots, \beta_s)$. We will deal with the refined version at the same time.

### 6. The Leading Term

In this section we compute the factor $F(S, \beta, y)$. Let $L = (L_0, \ldots, L_s)$ be an $(s + 1)$-tuple of line bundles on $S$, and let $\beta = \beta(L)$ be given as in Notation 2.1. Also recall that we write

$$E_L = L_0 \otimes t^0 \oplus \cdots \oplus L_s \otimes t^{-s}.$$

Assume that

$$\text{SW}(\beta) = \text{SW}(\beta_1) \cdots \text{SW}(\beta_s) \neq 0.$$

Then, by [15, Proposition 6.29], we have $(\beta^i)^2 = (\beta^i K_S)$, or equivalently $\chi(\beta^i) = \chi(\mathcal{O}_S)$ for $i = 1, \ldots, s$. For $0 \leq i \leq j \leq s$, we have by Serre duality

$$R\text{Hom}(L_i, L_j) = \chi(L_i^* \otimes L_j) = \chi(L_j^* \otimes L_i \otimes \omega_S) = R\text{Hom}(L_j, L_i \otimes \omega_S).$$
It follows that we have
\[ T_L = R\text{Hom}(E_L, E_L \otimes \omega_s \otimes t)_0 - R\text{Hom}(E_L, E_L)_0 \]
\[ = \sum_{i=0}^{s} (t^{i+1} - t^{-i}) \left( \sum_{j=0}^{s-i} \chi(L^*_{i+j} \otimes L_j \otimes \omega_s) - \sum_{j=1}^{s-i} \chi(L^*_{i+j} \otimes L_{j-1}) \right) 
- (t - 1) \cdot \chi(O_s) \]
\[ = \sum_{i=1}^{s} (t^{i+1} - t^{-i}) \left( \sum_{j=0}^{s-i} \chi(\beta^{i+j} + \ldots + \beta^{i+j} + K_S) - \sum_{j=1}^{s-i} \chi(\beta^j + \ldots + \beta^{i+j}) \right), \]

where the second sum starts with \( i = 1 \), since the coefficient of \((t - 1)\) is

\[ \left( \sum_{j=0}^{s} \chi(K_S) - \sum_{j=1}^{s} \chi(\beta^j) \right) - \chi(O_s) = 0 \]

by the assumption \( \text{SW}(\beta) \neq 0 \). Note that in particular, we have

(6.1) \[ T_L = N_L. \]

Moreover, note that
\[ \chi(\beta^{i+j} + \ldots + \beta^{i+j} + K_S) = \frac{(\beta^{1+j} + \ldots + \beta^{i+j} + K_S) \cdot (\beta^{1+j} + \ldots + \beta^{i+j})}{2} + \chi(O_S) \]
\[ = \sum_{j<k<l \leq i+j} \beta^k \beta^l + \chi(O_S). \]

and similarly
\[ \chi(\beta^j + \ldots + \beta^{i+j}) = \frac{(\beta^j + \ldots + \beta^{i+j}) \cdot (\beta^j + \ldots + \beta^{i+j} - K_S)}{2} + \chi(O_S) \]
\[ = \sum_{j \leq k < l \leq i+j} \beta^k \beta^l + \chi(O_S). \]

It follows that for \( k \leq l \), the multiplicity with which the term
\[ (t^{i+1} - t^{-i}) \cdot \beta^k \beta^l \]
appears in \( T_L \) is given by
\[ \mu(i, k, l) := \# \{ j \mid 0 \leq j < k \leq l \leq i + j \leq s \} - \# \{ j \mid 0 < j < k \leq l \leq i + j \leq s \} \]
\[ = \# \{ j \mid 0, l - i \leq j \leq k - 1, s - i \} - \# \{ j \mid 1, l - i \leq j \leq k, s - i; k < l \} \]
\[ = \begin{cases} 1 & \text{if } l \leq i; k > s - i \\ -1 & \text{if } l > i; k \leq s - i; l - k \leq i \\ 0 & \text{else} \end{cases} \]
\[ \min \{ i, s - l + 1, k, s - i + 1 \} \text{ if } k = l. \]
As a matrix, $\mu(i,-,-)$ looks, at least for $i \geq s+1-i$ like,

$$[\mu(i,k,l)]_{1 \leq k,l \leq s} = \begin{pmatrix}
1 & & & & & -1 \\
2 & & & & & \\
& \ddots & & & & \\
& & \ddots & & & \\
& & & m & 1 & \ldots & 1 \\
& & & & \ddots & \ddots & \\
& & & & & \ddots & 1 \\
& & & & & & m \\
& & & & & & \\
& & & & & & 2 \\
& & & & & & 1
\end{pmatrix}$$

where $m = s-i+1$, and where the triangles of 1’s and $-1$’s are respectively $2i-s-1$ and $s-i$ entries wide. So we have

$$N_L = T_L = \sum_{i=1}^{s} \left( \chi(O_S) + \sum_{k \leq l} \mu(i,k,l) \cdot \beta^k \beta^l \right) \cdot (t^{i+1} - t^{-i}) \cdot \left( t^{i+1} - t^{-i} \right).$$

We define the following rational numbers:

- $F_0^{(s+1)} := \frac{(-1)^{s+1}}{s+1}$;
- $F_{kk}^{(s+1)} := \frac{(-1)^{s+k}}{s+k}$ for $1 \leq k \leq s$;
- $F_{kl}^{(s+1)} := \frac{l(s+1-k)}{(s+1)(l-k)}$ for $1 \leq k < l \leq s$.

**Proposition 6.2.** Assume that $SW(\beta) \neq 0$. Then we have

$$F(S, \beta) = (F_0^{(s+1)})^{\chi(O_S)} \prod_{k \leq l} \left( F_{kl}^{(s+1)} \right)^{\beta^k \beta^l}.$$

**Proof.** For $k \leq l$, the contribution of the term $\mu(i,k,l) \cdot \beta^k \beta^l \cdot (t^{i+1} - t^{-i})$ to

$$F(S, \beta) = \frac{1}{e(N_L)}$$

is given by

$$\left( \frac{1}{e(t^{i+1} - t^{-i})} \right)^{\mu(i,k,l) \cdot \beta^k \beta^l} = \left( \frac{-i}{t^{i+1}} \right)^{\mu(i,k,l) \cdot \beta^k \beta^l},$$

and similarly for $\chi(O_S)(t^{i+1} - t^{-i})$. From here, the result is obtained by an easy computation. \(\square\)

Similarly, define the following rational functions in $y^{1/2}$:
\[ F_0^{(s+1)}(y) := \frac{(-1)^s y^{s/2}}{1 + y + \ldots + y^s}; \]
\[ F_{kk}^{(s+1)}(y) := (-1)^k y^{\binom{s+1-k}{2}} \min\{k, s+1-k\} \prod_{i=1}^{\min\{k, s+1-k\}} \frac{y^i - 1}{y^{s+2-i} - 1} \text{ for } 1 \leq k \leq s; \]
\[ F_{kl}^{(s+1)}(y) := \frac{(y^l - 1)(y^{s+1-k} - 1)}{(y^{s+1} - 1)(y^l-k - 1)} \text{ for } 1 \leq k < l \leq s. \]

**Proposition 6.3.** Assume that \( \text{SW}(\beta) \neq 0 \). Then we have
\[
F(S, \beta, y) = \left( F_0^{(s+1)}(y) \right)^{\chi(O_S)} \prod_{k \leq l} \left( F_{kl}^{(s+1)}(y) \right)^{\beta_k^k \beta_l^l}. 
\]

**Proof.** Recall (6.1) that \( T_L \) has no fixed part, so we have
\[
F(S, \beta, y) = \left( \frac{\text{ch}(K_L^\ast)}{\text{ch}(\Lambda^\ast(N_L^\ast))} \right) \left( \text{Td}((T_L)^C) \right)_{\text{ch}(t) = y} 
= \left[ \frac{\text{ch}(\text{det}(N_L^\ast)^\ast)}{\text{ch}(\Lambda^\ast(N_L^\ast))} \right]_{\text{ch}(t) = y}.
\]
The contribution of the term \( \mu(i, k, l) \cdot \beta_k^k \beta_l^l \cdot (t^{i+1} - t^{-i}) \) to \( F(S, \beta, y) \) is given by
\[
\left( \frac{\text{ch}(\text{det}(t^{i+1} - t^{-i})^\ast)}{\text{ch}(\Lambda^\ast(t^{i+1} - t^{-i})^\ast))} \right)_{\text{ch}(t) = y} = \left( -y^{1/2} \frac{y^i - 1}{y^{i+1} - 1} \right)^{\mu(i, k, l) \cdot \beta_k^k \beta_l^l},
\]
and similarly for \( \chi(O_S)(t^{i+1} - t^{-i}) \). Again, from this point the result follows by a straight-forward computation. \( \square \)

**Remark 6.4.** If \( r = s+1 \) is odd, note that \( F(S, \beta, y) \) is a function in \( y \), rather than in \( \sqrt{y} \), for any \( \beta = (\beta_0, \ldots, \beta_s) \) with \( \text{SW}(\beta) \neq 0 \).

**Example 6.5.** For rank 2 we have
\[
F(S, \beta, y) = \left( -\frac{y^{1/2}}{1 + y} \right)^{\chi(O_S) + \beta_1^1 \beta_1^1},
\]
and for rank 3
\[
F(S, \beta, y) = \left( \frac{y}{1 + y + y^2} \right)^{\chi(O_S) + \beta_1^1 \beta_1^1 + \beta_2^2 \beta_2^2} \left( \frac{(y + 1)^2}{1 + y + y^2} \right)^{\beta_1^1 \beta_2^2}.
\]

7. **Universality**

Let \( S \) be a smooth projective surface, not necessarily with \( H^1(O_S) = 0 \) or \( p_g > 0 \). For non-negative integers \( n = (n_0, \ldots, n_s) \) and classes \( \beta = (\beta_1, \ldots, \beta_s) \), consider the number \( Q_n(S, \beta) \) defined in Section 5 as an integral over
\[
\text{Hilb}^n(S) = S^{[n_0]} \times \cdots \times S^{[n_s]}.
\]

Using the notation
\[
q^n = q_0^n \cdots q_s^n,
\]
we form the generating series
\[ \sum_n Q_n(S, \beta) q^n. \]

The following universality result (Proposition 7.2, or rather its refined version (Proposition 7.5), is the main ingredient for the proof of Theorem A.

**Remark 7.1.** In Section 5, the integrals \( Q_n(S, \beta) \) were defined in terms of a lift of \( \beta \) to a vector of line bundles \( L \), such that \( \beta = \beta(L) \) (see Notation 2.1). Since we do not assume \( H^1(O_S) = 0 \) in this section, this lift involves a lift of the \( \beta_i \) to divisor classes of \( S \). We assume we have made such choice, and we consider \( \beta \) as a vector of classes in \( A^1(S) \). By Proposition 7.2, \( Q_n(S, \beta) \) does not depend on this choice.

**Proposition 7.2.** For each symbol
\[ \mathfrak{m} \in \left\{ \chi(O_S), K_S^2, K_S \beta_1^i, \beta_1^i \beta_1^j \right\}_{1 \leq i \leq j \leq s}, \]
there is a power series \( A_{\mathfrak{m}}^{(s+1)}(q) \in \mathbb{Q}[[q_0, \ldots, q_s]] \), starting with 1, and depending only on \( s \), such that
\[ \sum_n Q_n(S, \beta) q^n = \prod_{\mathfrak{m}} (A_{\mathfrak{m}}^{(s+1)}(q))^{n_\mathfrak{m}} \]
for any smooth projective surface \( S \) and classes \( \beta_1, \ldots, \beta_s \in A^1(S) \).

**Proof.** By the techniques of [2] (see also [11]), the integral \( Q_n(S, \beta) \) can be universally expressed as a polynomial \( P_n(S, \beta) \) in the Chern numbers of \( S \) and the classes \( \beta^1, \ldots, \beta^s \). Following [8, Proposition 2.3], it suffices to show that the generating series is multiplicative, i.e., that we have
\[ \sum_n Q_n(S \sqcup S', \beta + \beta') q^n = \sum_n Q_n(S, \beta) q^n \cdot \sum_n Q_n(S', \beta') q^n \]
for surfaces \( S \) and \( S' \) and \( s \)-tuples \( \beta \) and \( \beta' \) of classes in \( A^1(S) \) and \( A^1(S') \) respectively.

Note that
\[
\begin{align*}
\text{Hilb}^n(S \sqcup S') & = (S \sqcup S')^{[n_0]} \times \cdots \times (S \sqcup S')^{[n_s]} \\
& = \bigsqcup_{i_0+j_0=n_0} (S^{[i_0]} \times S^{[j_0]}) \times \cdots \times \bigsqcup_{i+s+j_s=n_s} (S^{[i_s]} \times S^{[j_s]}) \\
& = \bigsqcup_{i_0+j_0=n_0, i_s+j_s=n_s} S^{[i_0]} \times S^{[j_0]} \times \cdots \times S^{[i_s]} \times S^{[j_s]} \\
& = \bigsqcup_{i+j=n} \text{Hilb}^i(S) \times \text{Hilb}^j(S'),
\end{align*}
\]
(7.3)
in which the last sum is taken over \( s + 1 \)-tuples \( i = (i_0, \ldots, i_s) \) and \( j = (j_0, \ldots, j_s) \) of non-negative integers with \( n = i + j \). Consider the universal ideal sheaves
\[ \mathcal{I}^{[n_k]}_{S \sqcup S'}, \quad \text{for} \quad k = 0, \ldots, s \]
on
\[ \text{Hilb}^n(S \sqcup S') \times (S \sqcup S'). \]
For fixed $i$ and $j$ with $i + j = n$ and for $k = 0, \ldots, s$, we will write
\[
\begin{align*}
p_k & : \text{Hilb}^i(S) \times \text{Hilb}^j(S') \to S^{[i]} \times S \\
q_k & : \text{Hilb}^i(S) \times \text{Hilb}^j(S') \to S^{[jk]} \times S'
\end{align*}
\]
for the projections. Over the components in the decomposition (7.3), the universal sheaves are given by
\[
\mathcal{T}_{S \sqcup S'}^{[i]} \mid_{\text{Hilb}^i(S) \times \text{Hilb}^j(S') \times (S \sqcup S')} = p_k^* \mathcal{T}_S^{[i]} \oplus q_k^* \mathcal{T}_{S'}^{[j]}. 
\]
Write
\[
\pi: S \to *, \quad \pi': S' \to * \quad \text{and} \quad \pi \sqcup \pi': S \sqcup S' \to *
\]
for the projections. Let $M$ and $M'$ be a line bundles on $S$ and $S'$ respectively. It follows that for $0 \leq k, l \leq s$ we have
\[
(7.4) \quad R\mathcal{H}om_{\pi \sqcup \pi'}(\mathcal{T}^{[n]})_i \mathcal{T}^{[n]}_i \oplus (M \oplus M') = \sum_{i+j=n} R\mathcal{H}om_{\pi}(p_k^* \mathcal{T}_S^{[i]}, p_l^* \mathcal{T}_S^{[j]} \otimes M) \oplus R\mathcal{H}om_{\pi'}(q_k^* \mathcal{T}_{S'}^{[j]}, q_l^* \mathcal{T}_{S'}^{[j]} \otimes M')
\]
in the ring
\[
K_0(\text{Hilb}^n(S \sqcup S')) = \bigoplus_{i+j=n} K_0(\text{Hilb}^i(S) \times \text{Hilb}^j(S')).
\]
For any pair $i$ and $j$ of $(s+1)$-tuples of non-negative integers, write
\[
\begin{align*}
p & : \text{Hilb}^i(S) \times \text{Hilb}^j(S') \to \text{Hilb}^i(S) \\
q & : \text{Hilb}^i(S) \times \text{Hilb}^j(S') \to \text{Hilb}^j(S')
\end{align*}
\]
Let $L$ and $L'$ be $s+1$-tuples of line bundles on $S$ and $S'$ respectively, such that $\beta = \beta(L)$ and $\beta' = \beta(L')$ (see notation (2.1)). Consider the $K$-theory classes
\[
N_{L+L',0}^{[i]}, \quad N_{L,0}^{[i]} \quad \text{and} \quad N_{L',0}^{[j]},
\]
as defined in Section 4. Note that it is immediate from the definition, that these classes do not depend on the choice of $L$ and $L'$ (see also Remark 7.1). By definition, $N_{L+L',0}^{[i]}$ is linear combination of classes of the form (7.4), and we find
\[
N_{L+L',0}^{[i]} = \sum_{i+j=n} p^* N_{L,0}^{[i]} + q^* N_{L',0}^{[j]}.
\]
It follows that
\[
\frac{1}{e(N_{L+L',0}^{[i]})} = \sum_{i+j=n} \frac{1}{e(N_{L,0}^{[i]})} \cdot \frac{1}{e(N_{L',0}^{[j]})}.
\]
Finally, the corresponding multiplicative property of the factor
\[
\prod_{i=1}^s \left( R\pi_* \mathcal{O}(\beta_i) - R\mathcal{H}om_{\pi}(\mathcal{T}^{[n-1]}, \mathcal{T}^{[n]}(\beta_i)) \right)
\]
in the integrant of $Q_n(S, \beta)$ follows from the generalized Carlsson-Okounkov vanishing of [6] (see also [11]). Integrating gives the result.

The proof of Proposition 7.2 also gives the following refined result.
Proposition 7.5. For each symbol

\[ \mathfrak{m} \in \left\{ \chi(O_S), K_S^2, K_S \beta^i, \beta^i \beta^j \right\} \]

there is a power series \( A_{\mathfrak{m}}^{(s+1)}(y) \in \mathbb{Q}(y)[[q_0, \ldots, q_s]] \), starting with 1, such that

\[ \sum_n Q_n(S, \beta, y) q^n = \prod_{\mathfrak{m}} (A_{\mathfrak{m}}^{(s+1)}(y))^{q_n} \]

for any smooth projective surface \( S \) and classes \( \beta_1, \ldots, \beta_s \in A^1(S) \).

**Proof.** A similar proof holds, using the multiplicative properties of ch, \( \Lambda^* \), det, and Td. Note that by Remark 5.1, the universal series take coefficients in \( \mathbb{Q}(y) \), rather than in \( \mathbb{Q}(\sqrt{y}) \). \( \square \)

8. **Proof of Theorem A**

In this section, we will identify

\[ q := q_0 = \ldots = q_s, \]

so the equation in Proposition 7.5 becomes

\[ \sum_n Q_n(S, \beta, y) q^n = \prod_{\mathfrak{m}} (A_{\mathfrak{m}}^{(s+1)}(y))^{q_n} \]

in the ring \( \mathbb{Q}(y)[[q]] \), where we use the notation \( |n| = n_0 + \ldots + n_s \).

Let \( L = (L_0, \ldots, L_s) \) be line bundles on a surface \( S \), and let \( \beta = \beta(L) \) be given as in Notation 2.1. For non-negative integers \( n = (n_0, \ldots, n_s) \) and ideal sheaves \( I_i \in S^{[n_i]} \), consider the sheaf

\[ E = L_0 \otimes I_0 \oplus \ldots \oplus L_s \otimes I_s. \]

In the language of Section 2, \( E \) is a fibre of the family \( E_S^{[n]} \) of sheaves on \( S \) over \( \text{Hilb}_r^\nu(S) \). Recall that we use the convention \( r = s + 1 \). By Lemma 2.1, we have

\[ \text{vd}(r, \beta, n) := \text{vd}(r, c_1(E), c_2(E)) \]

\[ = 2rc_2(E) - (r - 1)c_1(E)^2 - (r^2 - 1)\chi(O_S) \]

\[ = 2r|n| - 2 \sum_{i<j} i(r - j)\beta^i \beta^j - \sum_i i(r - i)\beta^i - (r^2 - 1)\chi(O_S). \]

In other words, we have

\[ q^{\frac{|n|}{2(r+1)}} = q_{\text{vd}(r, \beta, n)} = q_{\text{vd}(r, \beta, n)} \]

\[ = q^n \left( q^{-\frac{(r+1)}{2}} \right)^{\chi(O_S)} \prod_{i<j} \left( q^{-\frac{i(r+1)}{2}} \right)^{\beta^i \beta^j} \prod_i q^{-\frac{i(r+1)}{2}} \left( q^{-\frac{i(r+1)}{2}} \right)^{\beta^i \beta^j}. \]

Finally, recall that for any surface \( S \) with \( H^1(O_S) = 0 \) and \( p_g(S) > 0 \), and for \( \beta \) with

\[ \text{SW}(\beta) = \text{SW}(\beta_1) \cdots \text{SW}(\beta_s) \neq 0 \]

we have, by Proposition 0.3

\[ F(S, \beta, y) = \left( F_0^{(r)}(y) \right)^{\chi(O_S)} \prod_{k \leq l} \left( F_{kl}^{(r)}(y) \right)^{\beta^k \beta^l}, \]

where \( F(S, \beta, y) \) is defined as in Section 5.
Define the following Laurent series in $q^{\frac{1}{r}}$ with coefficients in $\mathbb{Q}(\sqrt{y})$:

$$A^{(r)} := q^{\frac{-\chi^2(y)}{2r}} F_0^{(r)}(y) \frac{A^{(r)}_{X(O_S)}(y)}{\chi(O_S)} (-1)^{(r-1)}$$

$$B^{(r)} := A^{(r)}_{K^2_S}(y)$$

$$C_{ij}^{(r)} := q^{\frac{-\chi^2(y)}{2r}} F_{ij}^{(r)}(y) A^{(r)}_{\beta_i \beta_j}(y) \text{ for } 1 \leq i < j \leq r - 1;$$

$$C_{ii}^{(r)} := q^{\frac{-\chi^2(y)}{2r}} F_{ii}^{(r)}(y) A^{(r)}_{\beta_i \beta_i}(y) A^{(r)}_{\beta_i S} S, \beta, y q^n.$$  

**Proof of Theorem 3** First note that, by definition, the Laurent series are universal in the sense that they only depend on $r$. Now let $S$ be a surface with $H^1(O_S) = 0$ and $p_g(S) > 0$. For $\beta \in H^2(S, \mathbb{Z})$ with $\text{SW}(\beta) \neq 0$, we have by (8.1), (8.2) and (8.3)

$$(A^{(r)})^i(O_S) (B^{(r)})^i K^2_S \prod_{i \leq j} (C_{ij}^{(r)})^{\beta_i \beta_j} = (-1)^{(r-1)} \chi(O_S) q^{\frac{\chi(r, \beta, 0)}{2r}} F(S, \beta, y) \sum_n Q_n(S, \beta, y) q^n.$$  

By Proposition 6.3.4, we have

$$\text{SW}(\beta) = (-1)^{(r-1)} \chi(O_S) \text{SW}(\beta^1) \cdots \text{SW}(\beta^{r-1}).$$

It follows that

$$(A^{(r)})^i(O_S) (B^{(r)})^i K^2_S \sum_{\beta} \delta_{c_1, \beta} \text{SW}(\beta^1) \cdots \text{SW}(\beta^{r-1}) \prod_{i \leq j} (C_{ij}^{(r)})^{\beta_i \beta_j} = \sum_{\beta} \delta_{c_1, \beta} \text{SW}(\beta) F(S, \beta, y) \sum_n Q_n(S, \beta, y) q^{\frac{\chi(r, \beta, n)}{2r}}$$

$$= \sum_{\beta, n} \delta_{c_1, \beta} \text{VW}_n^{[\beta]}(y) q^{\frac{\chi(r, \beta, n)}{2r}}.$$  

Fix $r, c_1$ and $c_2$ and write

$\text{vd} = \text{vd}(r, c_1, c_2) = 2rc_2 - (r - 1)c_1^2 - (r^2 - 1)\chi(O_S).$  

Then coefficient of $q^{\frac{1}{r}}$ is given by

$$\sum_{\beta, n} \delta_{c_1, \beta} \text{VW}_n^{[\beta]}(y),$$

in which the sum is taken over $\beta = (\beta_1, \ldots, \beta_s)$ and $n = (n_0, \ldots, n_s)$ with $\text{vd} = \text{vd}(r, \beta, n)$. By equation (8.3), this is exactly the contribution of $\mathcal{M}_{r, c_1, c_2}$ to the Vafa-Witten invariant. \hfill \qed

**Proposition 8.4.** Let $S$ be a surface with $H^1(O_S) = 0$ and $p_g(S) > 0$, and let Chern $r, c_1$ and $c_2$ be Chern classes such that semistability implies stability. If $r$ is odd, we have

$$\text{VW}_{1, c_1, c_2}(S, y) \in \mathbb{Q}(y) \subset \mathbb{Q}(\sqrt{y}).$$

**Proof.** By Proposition 7.5 and Remark 6.4, the Laurent series $A^{(r)}, B^{(r)}$ and $C_{ij}^{(r)}$ have coefficients in $\mathbb{Q}(y)$. \hfill \qed
9. Computations

In order to determine the coefficients of the series
\[ A^{(s+1)}_{\mathcal{M}} \quad \text{for} \quad \mathcal{M} \in \left\{ \chi(O_S), K^2_S, K_S^2, \beta^i \beta^j \right\}_{1 \leq i \leq j \leq s}, \]
up to some degree \( N \), it suffices to evaluate the integrals
\[ Q_n(S, \beta) = \int_{[\text{Hilb}^n(S)]} \prod_{i=1}^s \left( R\pi_* O(\beta_i) - R\pi_\mathcal{M} \circ \pi_\mathcal{M} \left( \mathcal{I}^{[a]-1}, \mathcal{I}^{[a]}(\beta_i) \right) \right) \]
for \(|n| \leq N \) on \( \mathbb{P}^2 \) and \( \mathbb{P}^1 \times \mathbb{P}^1 \) and sufficiently many different \( \beta \), as in [8], and as we will also explain in this section (see Section 5 for notation and definitions). Write for fixed \( s \)
\[ \log A^{(s+1)}_{\mathcal{M}} = \sum_{n \geq 0} a^n q^n \]
in which the sum is taken over non-zero \( s+1 \)-tuples of non-negative integers and we use the notation
\[ q^n = q_0^{n_0} \cdots q_s^{n_s}, \]
as usual. Also write
\[ a_n = \left( a^n_{\chi(O_S)} a^n_{K^2_S} a^n_{K^2_S} \cdots a^n_{K^2_S} a^n_{\beta^i \beta^j} \cdots a^n_{\beta^i \beta^j} \right) \]
for each \( n \). Then, by Proposition 7.2, we have for any surface \( S \), and any \( s \)-tuple of algebraic classes \( \beta \in (H^2(S, \mathbb{Z}))^s \)
\[ \log \sum_n Q_n(S, \beta) q^n = \sum_{\mathcal{M}} \mathcal{M} \log A^{(s+1)}_{\mathcal{M}} = \sum_n a_n \cdot \left( \begin{array}{c} \chi(O_S) \\ K^2_S \\ \beta^i \beta^j \\ \vdots \\ \beta^s \beta^s \end{array} \right) q^n. \]

For \( s+1 \)-tuples of non-negative integers \( m \) and \( n \), write
\[ m \leq n \quad \text{iff} \quad m_0 \leq n_0, \ldots, m_s \leq n_s. \]
We can determine the \( n \)-th coefficient of each power series \( A^{(r)}_{\mathcal{M}} \) by evaluating the integrals \( Q_m(S, \beta) \) for each \( m \leq n \), and for sufficiently many surfaces \( S \) and \( s \)-tuples divisor classes \( \beta \) such that the vectors
\[ \left( \begin{array}{c} \chi(O_S) \\ K^2_S \\ \beta^i \beta^j \\ \vdots \\ \beta^s \beta^s \end{array} \right) \]
span a \( \mathbb{Q} \)-linear space of full rank. For example, \( a^n_{\chi(O_S)} \) and \( a^n_{K^2_S} \) can be found by evaluating \( Q_m(\mathbb{P}^2, (0 \cdots 0)) \) and \( Q_m(\mathbb{P}^1 \times \mathbb{P}^1, (0 \cdots 0)) \) for \( m \leq n \), and inverting the matrix
\[ \begin{pmatrix} \chi(O_{\mathbb{P}^2}) & \chi(O_{\mathbb{P}^1 \times \mathbb{P}^1}) \\ K^2_{\mathbb{P}^2} & K^2_{\mathbb{P}^1 \times \mathbb{P}^1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 9 & 8 \end{pmatrix}. \]
Similarly, we can determine the other coefficients, by choosing sufficiently many and sufficiently general \( \beta \). The following trick is just a reformulation of this.

Fix a surface \( S \) with \( K_S^2 \neq 0 \) (e.g. \( S = \mathbb{P}^2 \)), and let \( b_1, \ldots, b_s \) be formal parameters. Let \( \beta \) be given by

\[
\beta_1^b = b_1 K_S, \ldots, \beta_s^b = b_s K_S.
\]

Then we have

\[
\log \sum_n Q_n(S, \beta) q^n = \sum_n a_n \cdot \left( \begin{array}{c} \chi(O_S) \\ K_S^2 \\ b_1 K_S^2 \\ \vdots \\ b_s K_S^2 \\ b_1 b_2 K_S^2 \\ \vdots \\ b_s^2 K_S^2 \end{array} \right) q^n.
\]

After determining the coefficient of \( q^n \) by evaluating \( Q_m(S, \beta) \) for \( m \leq n \), we find back the coefficient \( a_n^{\beta_1 K_S} \) as \( \frac{1}{K_S^2} \) times the coefficient of \( b_1 \). Similarly, the coefficient of \( b_j b_1 \) is \( a_n^{\beta_2 K_S} \).

By Proposition 7.3, the discussion above also applies to the refined invariant. Simply replace \( Q_n(S, \beta) \) by its refined counterpart \( Q_n(S, \beta, y) \), and note that the coefficients \( a_n^b \) lie in \( \mathbb{Q}(y) \).

Let \( S \) be any toric surface with a torus \( T \), and assume that we have equipped all line bundles appearing in the integral with an equivariant structure. Then, by applying the Atiyah-Bott localization formula, we obtain

\[
Q_n(S, \beta) = \sum_{F \in (\mathrm{Hilb}^n(S))^T} \int_{(\mathrm{Hilb}^n(S))^T} e \left( \left( R \pi_* \mathcal{O}(\beta_i) - R \mathcal{H}om_{\pi}(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\beta_i)) \right) \middle|_{F} \right) e \left( N_{L,0}^{[n]} \right) e(\mathcal{T}_{\mathrm{Hilb}^n(S), F})
\]

(9.1) \[= \sum_{F \in (\mathrm{Hilb}^n(S))^T} \int e \left( -T_{L,0}^{[n]} \right) e(\mathcal{T}_{\mathrm{Hilb}^n(S), F})\]

in which \( e() \) denotes the equivariant Euler class for the torus \( T \times \mathbb{C}^* \).

Remark 9.2. In the factor

\[
e \left( \left( R \pi_* \mathcal{O}(\beta_i) - R \mathcal{H}om_{\pi}(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\beta_i)) \right) \middle|_{F} \right)
\]

in the formula above, the Euler class \( e(\cdot) \) should a priori be the \( T \)-equivariant Chern class \( c_{n_{i-1}+n_i}^T(\cdot) \). But by \[1, \text{ Lemma 6}], and Lemma \[9.3] below, the class

\[
\left( R \pi_* \mathcal{O}(\beta_i) - R \mathcal{H}om_{\pi}(\mathcal{I}^{[n_{i-1}]}, \mathcal{I}^{[n_i]}(\beta_i)) \right) \middle|_{F} \in K^T(F)
\]

can be represented by an \( n_{i-1}+n_i \)-dimensional representation of the torus \( T \) (rather than by a formal difference of \( T \)-representations). It follows that the \( T \)-equivariant top Chern class agrees with \( T \)-equivariant Euler class.

Remark 9.3. The compact form of the expression (9.1) is due to the fact that it is obtained by applying the Atiyah-Bott localization formula twice. The virtual version of the formula, due to Graber and Pandharipande \[12], expresses by definition the
contributions \( [21] \) of nested Hilbert schemes \( S^{[n]}_S \) to the monopole branch of the Vafa-Witten invariant for a surface \( S \) with \( p_g(S) > 0 \). The second time, however, we applied the formula to Hilbert schemes of points on a toric surface.

Similarly, we have

\[
Q_n(S, \beta, y) = \sum_{F \in (\text{Hilb}^n(S))^T} \int \sum_{i} \frac{e \left( \left( \left( R\pi_* \mathcal{O}(\beta_i) - R\mathcal{H}om_{\pi}(\mathcal{T}^{[n-1]}_F, \mathcal{T}^{[n]}_F(\beta_i)) \right) \right) | F \right)}{e(T_{\text{Hilb}^n(S)} F)} \times \frac{\text{ch} \left( (K^n_{L,0})^* | F \right)}{\text{ch}(\Lambda^* L_0 \mid F)} Td \left( (\mathcal{T}^{[n]}_F \mid C^* | F \right) = \sum_{F \in (\text{Hilb}^n(S))^T} \int \frac{\text{ch} \left( (K^n_{L,0})^* | F \right)}{\text{ch}(\Lambda^* L_0 \mid F)}
\]

where \( \text{ch} \) and \( \text{Td} \) denote the \( T \times \mathbb{C}^* \)-equivariant Chern character and Todd class respectively. We have used the equation

\[
\text{ch}(\Lambda^* (L^*)) = 1 - \exp(-\alpha) = \frac{e(L)}{\text{Td}(L)}
\]

for any 1-dimensional \( T \)-representation \( L \) with \( c_1(L) = \alpha \).

Let \( F \in \text{Hilb}^n(S) \) be a \( T \)-fixed point. Let \( 0 \leq i, j \leq s \), and write

\[
I = \mathcal{T}^{[n]}_F \quad \text{and} \quad J = \mathcal{T}^{[n]}_F.
\]

The class \( \mathcal{T}^{[n]}_F \) is a linear combination of classes of the form

\[
\left( R\pi_* M - R\mathcal{H}om_{\pi}(\mathcal{T}^{[n]}_F, \mathcal{T}^{[n]}_F \otimes M) \right) | F = \chi(M) - R\text{Hom}_S(I, J \otimes M),
\]

where \( M \) is a \( T \times \mathbb{C}^* \)-equivariant line bundle on \( S \).

**Lemma 9.4.** Let \( \{ U_\sigma \}_{\sigma = 1, \ldots, e(S)} \) be the maximal open cover of \( S \) by affine \( T \)-fixed subsets, cf. \([9, \text{Section 4}]\). Then we have

\[
\chi(M) - R\text{Hom}_S(I, J \otimes M) = \sum_{\sigma = 1}^{e(S)} \Gamma(U_\sigma, M) - R\text{Hom}_{U_\sigma}(I|_{U_\sigma}, J|_{U_\sigma} \otimes M|_{U_\sigma}).
\]

**Proof.** Write \( U_{\sigma \tau} = U_\sigma \cap U_\tau \) for \( \sigma < \tau \). Since \( I \) and \( J \) are ideal sheaves of \( \mathbb{C}^* \)-fixed 0-dimensional subschemes of \( S \), and \( U_{\sigma \tau} \) does not contain any fixed points, we have

\[
\Gamma(U_{\sigma \tau}, \mathcal{E}xt^i(I, J \otimes M)) = \Gamma(U_\sigma \cap U_\tau, \mathcal{E}xt^i(I|_{U_\sigma \cap U_\tau}, (J \otimes M)|_{U_\sigma \cap U_\tau}) = \Gamma(U_{\sigma \tau}, \mathcal{E}xt^i(O, M))
\]

for any \( i \), and a similarly for intersections \( U_\sigma \cap U_\tau \cap U_\nu \). Now use the local-to-global spectral sequence and the Cech complex for the covering \( \{ U_\sigma \} \) (cf. \([13, \text{Section 4.6}]\)), to order to compare the classes \( \chi(M) \) and \( R\text{Hom}_S(I, J \otimes M) \).

Now \([1, \text{Lemma 6}]\), and also the proof of \([9, \text{Proposition 4.1}]\), give an explicit expressions for the RHS of \((9.5)\). This allows us to compute \( Q_n(S, \beta) \). We have implemented the computation in Sage \([16] \) for \( S = \mathbb{P}^1 \times \mathbb{P}^1 \) and \( S = \mathbb{P}^2 \) and arbitrary rank. Part of the results for rank 3 are listed in Appendix \([13] \).
10. Smooth moduli spaces

In the case that the monopole branch of the moduli space of \( \mathbb{C}^* \)-fixed Higgs pairs is smooth, there is a direct method to compute the Vafa-Witten invariants. Let \( S \) be a surface with \( H^1(\mathcal{O}_S) = 0, p_g > 0 \), and assume that \( \text{Pic}(S) \) is generated by a smooth very ample canonical curve \( C \). In this case, the only Seiberg-Witten basic classes of \( S \) are 0 and \( K_S \). For rank 2, the monopole branch

\[
\mathcal{M}_{12} = \mathcal{M}_{12, K_S, c_2} \subset (\mathcal{N}_{2, \omega_S, c_2})^{\mathbb{C}^*}
\]

is smooth precisely when \( c_2 = 0, 1, 2, 3 \) \cite{17}. In particular, the virtual class is given by the Euler class of the obstruction bundle and the Vafa-Witten invariants can be computed using the intersection theory of (smooth) nested Hilbert schemes of points on the surface and the smooth canonical curve. This method, which is carried out in \cite{17} (unrefined) and \cite{19} (refined), can be generalized to rank 3, but only for \( c_2 = 0, 1, 2 \). We have done the computation in this setting, and have found that they confirm our results (see the discussion after Theorem B). 

Let \( (E, \phi) \in \mathcal{M}_{13, K_S, c_2} \) be a Higgs pair, so \( E \) can be written as

\[
E = I_0 \otimes \omega_S^a \oplus I_1 \otimes \omega_S^b \oplus I_2 \otimes \omega_S^c
\]

in which \( I_i \in S^{[n_i]} \) for \( i = 0, 1, 2 \) and \( a, b, c \in \mathbb{Z} \) such that \( a + b + c = 1 \), and we have \( \phi = (\phi_1, \phi_2) \), where \( \phi_1 \) and \( \phi_2 \) are non-zero homomorphisms

\[
\phi_1: I_0 \otimes \omega_S^a \to I_1 \otimes \omega_S^{b+1}
\]

\[
\phi_2: I_1 \otimes \omega_S^b \to I_2 \otimes \omega_S^{c+1}.
\]

Lemma 10.1. We have \( (a, b, c) = (1, 0, 0) \).

Proof. Slope semistability of \( E \) implies that

\[
e \leq \frac{1}{3}, \quad \frac{b + c}{2} \leq \frac{1}{3}.
\]

On the other hand, by the existence of the maps \( \phi_1 \) and \( \phi_2 \) we have

\[
a \leq b + 1, \quad b \leq c + 1.
\]

It is easy to see that the only integral solution to these inequalities together with \( a + b + c = 1 \) is \( (a, b, c) = (1, 0, 0) \).

\[\square\]

Proposition 10.2. Let \( S \) be given as above. Then \( \mathcal{M}_{13, K_S, c_2} \) is smooth if and only if \( c_2 \leq 2 \). In particular, we have

\[
\mathcal{M}_{13, K_S, 1} \cong (S^{[1]} \times |K_S|) \sqcup \mathcal{C};
\]

\[
\mathcal{M}_{13, K_S, 2} \cong (S^{[2]} \times |K_S|) \sqcup (S^{[1]} \times |K_S|) \sqcup (S^{[1]} \times \mathcal{C}) \sqcup \mathcal{C}^{[2]}_{|K_S|}.
\]

in which

\[
\mathcal{C} \longrightarrow S
\]

\[
|K_S|
\]

is the universal canonical curve, and \( \mathcal{C}^{[2]}_{|K_S|} \to |K_S| \) the relative Hilbert scheme of points.
Proof. Note that for \( I_i \in S^{[n_i]} \), \( i = 0, 1, 2 \), we have
\[
c_2(I_0 \otimes \omega_S \oplus I_1 \oplus I_2) = n_0 + n_1 + n_2.
\]
By Lemma 10.1 we find
\[
\mathcal{M}_{1^3, K_S, c_2} \cong \bigsqcup_{|n| = c_2, n_1 \leq n_0} S_{(0, K_S)}^{[n_0, n_1, n_2]},
\]
and in particular
\[
\mathcal{M}_{1^3, K_S, 1} = S_{(0, K_S)}^{[1,0,0]} \cup S_{(0, K_S)}^{[0,0,1]} \cong S^{[1]} \times |K_S| \cup C;
\]
\[
\mathcal{M}_{1^3, K_S, 2} = S_{(0, K_S)}^{[2,0,0]} \cup S_{(0, K_S)}^{[1,1,0]} \cup S_{(0, K_S)}^{[1,0,1]} \cup S_{(0, K_S)}^{[0,0,2]} \cong (S^{[2]} \times |K_S|) \cup (S^{[1]} \times |K_S|) \cup (S^{[1]} \times C) \cup C_{[K_S]}^{[2]}.
\]
The total spaces of the universal canonical curve \( C \), and of relative Hilbert scheme of points \( C_{[K_S]}^{[2]} \) are smooth by the the assumption that \( K_S \) is very ample.

For \( c_2 \geq 3 \), it is e.g. easy to see that
\[
S_{(0, K_S)}^{[1,1]} \cong (S \times C) \cup (\Delta_S \times |K_S|) \subset S \times S \times |K_S|
\]
has two irreducible components with non-empty intersection. For an ideal sheaf \( I \) on \( S \), let \( Z_I \) denote the corresponding subscheme. Then, for \( k \geq 1 \), the scheme \( S_{(0, K_S)}^{[1,1,k]} \cong S_{[K_S]}^{[1,k]} \)
\[
= \{ p \in S, I \in S^{[k]}, C \in |K_S| : Z_I \subset C \cup p \}
\]
has two components given by the conditions \( p \in Z_I \), and \( Z_I \subset C \) respectively. Hence, it is singular at points in the intersection given by \( p \in Z_I \subset C \). It follows that \( \mathcal{M}_{1^r, K_S, c_2} \) is singular for \( c_2 > 2 \). \( \square \)

In order to give the connected components of \( \mathcal{M}_{1^3, K_S, 1} \) and \( \mathcal{M}_{1^3, K_S, 2} \), rather than their isomorphism classes, we need if suffices to specify universal Higgs pairs. For \( c_2 = 1 \), the connected components of \( \mathcal{M}_{1^3, K_S, 1} \), together with the restrictions of the universal sheaf on \( \mathcal{M}_{1^3, K_S, 1} \times S \) are given as follows:
\[
S^{[1]} \times |K_S|, \quad \mathcal{I}^{[1]} \otimes \omega_S \oplus t^{-1} \oplus \mathcal{O}_{|K_S|}(1) \otimes t^{-2};
\]
\[
C, \quad \omega_S \oplus t^{-1} \oplus j^* (\mathcal{I}^{[1]} \otimes \mathcal{O}_{|K_S|}(1)) \otimes t^{-2}.
\]
in which \( j : C \times S \to S^{[1]} \times |K_S| \times S \) is the inclusion. We have suppressed pull-backs along the several projections. For \( c_2 = 2 \), we have
\[
S^{[2]} \times |K_S|, \quad \left( \mathcal{I}^{[2]} \otimes \omega_S \right) \oplus t^{-1} \oplus \mathcal{O}_{|K_S|}(1) \otimes t^{-2};
\]
\[
S^{[1]} \times |K_S|, \quad \left( \mathcal{I}^{[1]} \otimes \omega_S \right) \oplus \left( \mathcal{I}^{[1]} \otimes t^{-1} \right) \oplus \mathcal{O}_{|K_S|}(1) \otimes t^{-2};
\]
\[
S^{[1]} \times C, \quad \left( \mathcal{I}^{[1]} \otimes \omega_S \right) \oplus t^{-1} \oplus \left( j^* \left( \mathcal{I}^{[1]} \otimes \mathcal{O}_{|K_S|}(1) \right) \otimes t^{-2} \right);
\]
\[
C_{[K_S]}^{[2]}, \quad \omega_S \oplus t^{-1} \oplus \left( j_2^* \left( \mathcal{I}^{[2]} \otimes \mathcal{O}_{|K_S|}(1) \right) \otimes t^{-2} \right).
\]
in which we have written

\[ j_2 : C_{[K_S]}^{[2]} \times S \to S^{[2]} \times |K_S| \times S \]

for the inclusion. Again, we have suppressed pull-backs along projections. Now define Higgs fields \( \phi = (\phi_1, \phi_2) \) by the several natural inclusions of ideal sheaves.

As the moduli spaces are smooth, the obstruction sheaves have constant rank, and we can compute their K-theory class using Theorem 1.4. The virtual class of each component is now given by the Euler class of the obstruction bundle. We have

\[
[S^{[1]} \times |K_S|]^{\text{vir}} = e(K_S^* + \Omega_{|K_S|}(1)) \nonumber \\
= (-1)^{p_g} \cdot [C], 
onumber \\
[C]^{\text{vir}} = e(\Omega_{|K_S|}(1)) \nonumber \\
= (-1)^{p_g-1} \cdot [C], 
onumber \\
[S^{[2]} \times |K_S|]^{\text{vir}} = e(\omega_S^{[2]} + \Omega_{|K_S|}(1)) \nonumber \\
= [\omega_S^{[2]}]_K \cap (-H)^{p_g-1} \nonumber \\
= (-1)^{p_g-1} \cdot [C^{[2]}], 
onumber \\
[S^{[1]} \times |K_S|]^{\text{vir}} = e(H^0(\omega_S)) \nonumber \\
= 0, 
\]

\[
[S^{[1]} \times C]^{\text{vir}} = e(\omega_S^* + \Omega_{|K_S|}(1)) \nonumber \\
= (-1)^{p_g} \cdot [C \times C], 
\]

\[
[C^{[2]}]^{\text{vir}} = e(\Omega_{|K_S|}(1)) \nonumber \\
= (-1)^{p_g-1} \cdot [C^{[2]}], 
\]

in which \( H := c_1(\mathcal{O}_{|K_S|}(1)) \). It follows that the computation of the contribution of the Vafa-Witten invariant reduces to a computation in the intersection rings of \( C \) and \( C^{[2]} \). Using Grothendieck-Riemann-Roch to compute the Chern classes of the relative Hom complexes, this is a straight forward computation. The details are similar to the computations in [17] and [19].

11. Comparison to the Göttsche-Kool conjectures

The Laurent series that appear in Theorem A and are defined in Section 8 are given for rank 2 by

\[
A^{(2)} = q^{-\frac{2}{3}} \frac{1}{y^{-\frac{3}{2}} + y^\frac{3}{2}} A^{(2)}_{\mathcal{O}_S}(y), 
\]

\[
B^{(2)} = A^{(2)}_{K_S^2}(y), 
\]

\[
C^{(2)}_{11} = -q^{-\frac{1}{3}} \frac{1}{y^{-\frac{3}{2}} + y^\frac{3}{2}} A^{(2)}_{\mathcal{O}_{\beta_1 \beta_1}}(y) A^{(2)}_{\beta_1 K_S^2}(y), 
\]
and for rank 3 by

\[ A^{(3)} = q^{-\frac{1}{3}} \frac{1}{y^{-1} + 1 + y} A^{(3)}_{\mathcal{O}(\mathcal{O})}(y), \]
\[ B^{(3)} = A^{(3)}_{K^2}(y), \]
\[ C_{11}^{(3)} = q^{-\frac{1}{3}} \frac{1}{y^{-1} + 1 + y} A^{(3)}_{\beta^1 \beta^1 K^2}(y), \]
\[ C_{22}^{(3)} = q^{-\frac{1}{3}} \frac{1}{y^{-1} + 1 + y} A^{(3)}_{\beta^2 \beta^2 K^2}(y), \]
\[ C_{12}^{(3)} = q^{-\frac{1}{3}} \frac{(1 + y)^2}{1 + y + y^2} A^{(3)}_{\beta^1 \beta^2}(y). \]

In Section 9, we have computed the first 7 terms in rank 2, and the first 6 terms in rank 3 of the power series \( A^{(r)} \) appearing above. In Appendix B we have listed the first few terms of the rank 3 power series. This allows us to check the equations of Conjectures 1.15 and 1.16 term by term, leading to Theorem B. As an example, let us just check one term of \( C_{12}^{(3)} \). We have

\[ C_{12}^{(3)} = q^{-\frac{1}{3}} \frac{(1 + y)^2}{1 + y + y^2} A^{(3)}_{\beta^1 \beta^2}(y) \]
\[ = q^{-\frac{1}{3}} \frac{(1 + y)^2}{1 + y + y^2} \left( 1 + \frac{y^4 + 6y^3 + 6y^2 + 6y + 1}{(y + 1)^2 y} q + \ldots \right). \]

On the other hand we have

\[ W(q^{\frac{1}{3}}, y) = \frac{\Theta_{A_{2}(0,0)}(q^{\frac{1}{3}}, y)}{\Theta_{A_{2}(1,0)}(q^{\frac{1}{3}}, y)} \]
\[ = \frac{1 + (y^2 + 2y + 2y^{-1} + y^{-2}) q + \ldots}{(y + 1 + y^{-1})q^{\frac{1}{3}} + (y^2 + 1 + y^{-2}) q^{\frac{1}{3}} + \ldots} \]
\[ = q^{-\frac{1}{3}} \frac{1}{y + 1 + y^{-1}} \frac{1 + (y^2 + 2y + 2y^{-1} + y^{-2}) q + \ldots}{1 + (y - 1 + y^{-1}) q + \ldots} \]
\[ = q^{-\frac{1}{3}} \left( \frac{1}{y + 1 + y^{-1}} + \frac{y^2 + y + 1 + y^{-1} + y^{-2}}{y + 1 + y^{-1}} q + \ldots \right), \]

and hence

\[ W(q^{\frac{1}{3}}, 1) = q^{-\frac{1}{3}} \frac{1}{3}(1 + 5 q + \ldots). \]
It follows that

\[
W_+(q^\frac{1}{2}, y)W_-(q^\frac{1}{2}, y)
= W(q^\frac{1}{2}, y) + 3W(q^\frac{1}{2}, 1)
= q^{-\frac{1}{3}} \left( 1 + \frac{1}{y + 1 + y^{-1}} + \left( \frac{y^2 + y + 1 + y^{-1} + y^{-2}}{y + 1 + y^{-1}} + 5 \right) q + \ldots \right)
\]

\[
= q^{-\frac{1}{3}} \left( \frac{y + 2 + y^{-1}}{y + 1 + y^{-1}} + \left( \frac{y^2 + y + 1 + y^{-1} + y^{-2} + 5(y + 1 + y^{-1})}{y + 1 + y^{-1}} \right) q + \ldots \right)
\]

\[
= q^{-\frac{1}{3}} \left( \frac{(1 + y)^2}{1 + y + y^2} \left( 1 + \frac{y^4 + 6y^3 + 6y^2 + 6y + 1}{(y + 1)^2} q + \ldots \right) \right)
\]

\[
\equiv C_{12}^{(3)}(y) \mod U_2^{(3)},
\]

where have used the notation

\[
U_2^{(3)} = 1 + q^2 Q(y^\frac{1}{2})[q] \subset Q(y^\frac{1}{2})((q^\frac{1}{2}))^*.
\]

from the introduction.

Appendix A. Functions appearing in the Göttsche-Kool conjectures

The following definitions are taken from [10]:

\[
\phi_{-2,1}(x, y) := (y^\frac{1}{2} - y^{-\frac{1}{2}}) \prod_{n=1}^{\infty} \frac{(1 - x^n y^2)(1 - x^n y^{-1})^2}{(1 - x^n)^4}
\]

\[
\Delta(x) := x \prod_{n \in \mathbb{Z}_{>0}} (1 - x^n)^{24}
\]

\[
\eta(x) := x^{\frac{1}{24}} \prod_{n \in \mathbb{Z}_{>0}} (1 - x^n)
\]

\[
\theta_2(x, y) := \sum_{n \in \mathbb{Z} + \frac{1}{2}} x^n y^n
\]

\[
\theta_3(x, y) := \sum_{n \in \mathbb{Z}} x^n y^n
\]

\[
\Theta_{A_2, (0,0)}(x, y) := \sum_{(m,n) \in \mathbb{Z}^2} x^{2(m^2 - mn + n^2)} y^{m+n}
\]

\[
\Theta_{A_2, (1,0)}(x, y) := \sum_{(m,n) \in \mathbb{Z}^2} x^{2(m^2 - mn + n^2 + m - n + 4)} y^{m+n}
\]

\[
W(x, y) := \frac{\Theta_{A_2, (0,0)}(x, y)}{\Theta_{A_2, (1,0)}(x, y)}.
\]

Finally, the functions \(W_\pm(x, y)\) are defined as the roots of the following polynomial in \(\omega\):

\[
\omega^2 - (W(x, y)^2 + 3W(x, y)W(x, 1)) \omega + W(x, y) + 3W(x, 1) = 0.
\]

We will use the convention that \(W_-(x, y)\) is the one with leading term

\[
x^\frac{1}{2}(y^{-1} + 1 + y).
\]
Appendix B. Rank 3 results

We set $q := q_0 = q_1 = q_2$ and print the first few terms of $A_{\mathfrak{g}_0}^{(3)}(y)$ for

$$
\mathfrak{g} \in \left\{ \chi(O_S), (K_S^2), (\beta^1 K_S), (\beta^2 K_S), (\beta^1 \beta^1), (\beta^2 \beta^2), (\beta^1 \beta^2) \right\}.
$$

$$
A_{\chi(O_S)}^{(3)}(y) \equiv 1 + \frac{y^6 + 10y^3 + 1}{y^3} \mod q^4
$$

$$
A_{K_S^2}^{(3)}(y) \equiv 1 - \frac{(y^2 + y + 1)^2}{(y + 1)^2 y} q
- \frac{2y^4 + 7y^3 + 12y^2 + 7y + 2}{y^3 y^2 (y + 1)^4} q^2
+ \frac{1}{y^3(y + 1)^6} \left( 5y^{12} + 39y^{11} + 150y^{10} + 382y^9 + 705y^8 + 1002y^7
+ 1121y^6 + 1002y^5 + 705y^4 + 382y^3 + 150y^2 + 39y + 5 \right) q^3 \mod q^4
$$

$$
A_{\beta^1 K_S}^{(3)}(y) \equiv A_{\beta^2 K_S}^{(3)}(y)
\equiv 1 + \frac{1}{2} \frac{(y^2 + y + 1)(y - 1)^2}{(y + 1)^2 y} q
+ \frac{1}{8} \frac{(23y^4 + 68y^3 + 142y^2 + 68y + 23)(y^2 + y + 1)^2}{(y + 1)^4 y^2} q^2
- \frac{y^2 + y + 1}{16y^3(y + 1)^6} \left( 15y^{10} + 244y^9 + 1006y^8 + 2790y^7 + 4719y^6 + 5780y^5
+ 4719y^4 + 2790y^3 + 1006y^2 + 244y + 15 \right) q^3 \mod q^4
$$

$$
A_{\beta^1 \beta^1}^{(3)}(y) \equiv A_{\beta^2 \beta^2}^{(3)}(y)
\equiv 1 - \frac{1}{2} \frac{y^4 + 3y^3 + 6y^2 + 3y + 1}{(y + 1)^2 y} q
- \frac{1}{8} \frac{1}{(y + 1)^4 y^2} \left( 5y^8 + 30y^7 + 109y^6 + 218y^5 + 280y^4
+ 218y^3 + 109y^2 + 30y + 5 \right) q^2
+ \frac{1}{16y^3(y + 1)^6} \left( 11y^{12} + 115y^{11} + 571y^{10} + 1868y^9 + 4205y^8 + 6845y^7
+ 8026y^6 + 6845y^5 + 4205y^4 + 1868y^3 + 571y^2 + 115y + 11 \right) q^3 \mod q^4
$$

$$
A_{\beta^1 \beta^2}^{(3)}(y) \equiv 1 + \frac{y^4 + 6y^3 + 6y^2 + 6y + 1}{(y + 1)^2 y} q
- \frac{y^6 + y^5 + 8y^4 + 8y^3 + 8y^2 + y + 1}{(y + 1)^2 y^2} q^2
+ \frac{y^6 + 3y^4 + 4y^3 + 3y^2 + 1}{y^2(y + 1)^2} q^3 \mod q^4
$$
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