Planarity of a unit graph part -III $|\text{Max}(R)| \geq 3$ case

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Abstract
The rings considered in this article are commutative with identity $1 \neq 0$. Recall that the unit graph of a ring $R$ is a simple undirected graph whose vertex set is the set of all elements of the ring $R$ and two distinct vertices $x, y$ are adjacent in this graph if and only if $x + y \in U(R)$ where $U(R)$ is the set of all unit elements of ring $R$. We denote this graph by $UG(R)$. In this article we classified rings $R$ with $|\text{Max}(R)| \geq 3$ such that $UG(R)$ is planar.

Keywords
Planar graph, $(Ku_1^3)$ and $(Ku_2^3)$.

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Contents

1 Introduction .......................................................... 1413
2 On the classification of rings $R$ with $|\text{Max}(R)| \geq 3$ in order that $UG(R)$ is planar ......................... 1414

References ............................................................. 1416

1. Introduction

We first recall the following definitions and results from graph theory. A graph $G=(V,E)$ is said to be complete if every pair of distinct vertices of $G$ are adjacent in $G$. A complete graph on $n$ vertices is denoted by $K_n$ [4, Definition 1.1.11]. A graph $G=(V,E)$ is said to be bipartite if the vertex set can be partitioned into two nonempty subsets $X$ and $Y$ such that each edge of $G$ has one end in $X$ and other in $Y$. The pair $(X,Y)$ is called a bipartition of $G$. A bipartite graph $G$ with bipartition $(X,Y)$ is denoted by $G(X,Y)$. A bipartite graph $G(X,Y)$ is said to be complete if each vertex of $X$ is adjacent to all the vertices of $Y$. If $G(X,Y)$ is a complete bipartite graph with $|X|=m$ and $|Y|=n$, then it is denoted by $K_{m,n}$ [4, Definition 1.1.12]. Let $G=(V,E)$ be a graph. By a clique of $G$, we mean a complete subgraph of $G$ [4, Definition 1.2.2]. We say that the clique number of $G$ equals $n$ if $n$ is the largest positive integer such that $K_n$ is a subgraph of $G$ [4, p.185]. The clique number of a graph $G$ is denoted by the notation $\omega(G)$. If $G$ contains $K_n$ as a subgraph for all $n \geq 1$, then we set $\omega(G) = \infty$. A graph $G$ is said to be planar if it can be drawn in a plane in such a way that no two edges of $G$ intersect in a point other than a vertex of $G$ [4, Definition 8.1.1]. Two adjacent edges of a graph $G$ are said to be in series if their common vertex is of degree two [5, p.9]. Two graphs are said to be homeomorphic if one graph can be obtained from the other graph by the creation of edges in series (i.e by insertion of vertices of degree two) or by the merger of edges in series [5, p.100]. Recall from [5, p.93] that $K_3$ is referred to as Kuratowski’s first graph and $K_{3,3}$ is referred to as Kuratowski’s second graph. A celebrated theorem of Kuratowski says that a necessary and sufficient condition for a graph $G$ to be planar is that $G$ does not contain either of Kuratowski’s two graphs or any graph homeomorphic to either of them [5, Theorem 5.9].

In view of Kuratowski’s Theorem, [5, Theorem 5.9] we introduce the following definitions. We say that a graph $G=(V,E)$ satisfies $Ku_1$ if $G$ does not contain $K_3$ as a subgraph and we say that graph $G=(V,E)$ satisfies $Ku_2$ if $G$ does not contain $K_{3,3}$ as a subgraph. We say that a graph $G=(V,E)$ satisfies $Ku_1^3$ if $G$ satisfies $Ku_1$ and moreover, $G$ does not contain any subgraph homeomorphic to $K_3$. We say that a graph $G=(V,E)$ satisfies $Ku_2^3$ if $G$ satisfies $Ku_2$ and moreover, $G$ does not contain any subgraph homeomorphic to $K_{3,3}$.

If a graph $G$ is planar, then it follows from Kuratowski’s theorem [5, Theorem 5.9] that $G$ satisfies both $Ku_1^3$ and $Ku_2^3$. Hence $G$ satisfies both $Ku_1$ and $Ku_2$. It is interesting to note that a graph $G$ may be nonplanar even if it satisfies both
We do not know an example of a graph $G$ such that $G$ satisfies $K_{u_1}$ but $G$ does not satisfy $K_u^*$. Therefore, there exists at most one $i \in \{1, 2, 3, \ldots, n\}$ such that $|F_i| \geq 4$. 

**Lemma 2.3.** Let $n \geq 3$ and let $R = F_1 \times F_2 \times F_3 \times \cdots \times F_n$, where $F_i$ is a field for each $i \in \{1, 2, 3, \ldots, n\}$. If $UG(R)$ satisfies $(K_{u_2}^*)$, then $F_i \in \{Z_2, Z_3, F_4, \mathbb{Z}_5\}$ for each $i \in \{1, 2, 3, \ldots, n\}$. 

**Proof.** Assume that $UG(R)$ satisfies $(K_{u_2}^*)$. Then $UG(R)$ satisfies $(K_{u_2})$. Hence, we obtain from Remark 2.2 (i) that $F_i \in \{Z_2, Z_3, F_4, \mathbb{Z}_5\}$ for each $i \in \{1, 2, 3, \ldots, n\}$. We want to show that $F_i \not\cong \mathbb{Z}_5$ for each $i \in \{1, 2, 3, \ldots, n\}$. Suppose that $F_i = \mathbb{Z}_5$ for some $i \in \{1, 2, 3, \ldots, n\}$. Without loss of generality, we can assume that $F_1 = \mathbb{Z}_5$. In such a case, we know from Remark 2.2 (ii) that $F_i \in \{Z_2, Z_3\}$ for each $i \in \{2, \ldots, n\}$. Since $|U(\mathbb{Z}_5)| = 4$, it follows from [10, Lemma 3.2] that $|U(F_2 \times F_3 \times \cdots \times F_n)| \leq 2$. Hence, there exists at most one $i \in \{2, 3, \ldots, n\}$ such that $F_i = Z_3$. We consider the following cases.

**Case (1)** $F_1 = Z_2$ for each $i \in \{2, 3, \ldots, n\}$

In this case, $R \cong Z_2 \times T$ as rings, where $T = F_2 \times F_3 \times \cdots \times F_n$ is such that char$(T) = 2$. We know from [10, Proposition 3.12] that $UG(R)$ does not satisfy $(K_{u_2}^*)$. This is in contradiction to the assumption that $UG(R)$ satisfies $(K_{u_2}^*)$.

**Case (2)** $F_1 = Z_3$ for a unique $i \in \{2, 3, \ldots, n\}$

Without loss of generality, we can assume that $F_2 = Z_3$. Note that $F_1 = Z_2$ for each $i \in \{3, \ldots, n\}$. Let us denote the ring $F_1 \times \cdots \times F_i$ by $T_i$. Then $R \cong Z_3 \times Z_3 \times T_i$ as rings. Observe that char$(T_i) = 2$. Consider the mapping $f : V(U(\mathbb{Z}_5 \times T_1)) = \mathbb{Z}_5 \times T_1 \to V(U(\mathbb{Z}_5 \times Z_3 \times T_1)) = \mathbb{Z}_3 \times \mathbb{Z}_3 \times T_1$ by $f(x, y) = (x, 1, y)$ for any $(x, y) \in \mathbb{Z}_3 \times T_1$. It is clear that $f$ is one-one and for any $(x_1, y_1), (x_2, y_2) \in \mathbb{Z}_3 \times T_1$ are adjacent in $V(\mathbb{Z}_5 \times T_1)$ if and only if $f(x_1, y_1)$ and $f(x_2, y_2)$ are adjacent in $U(\mathbb{Z}_5 \times Z_3 \times T_1)$. Therefore, $UG(\mathbb{Z}_3 \times T_1)$ is isomorphic to a subgraph of $UG(\mathbb{Z}_3 \times Z_3 \times T_1)$. We know from [10, Proposition 3.12] that $UG(\mathbb{Z}_3 \times T_1)$ does not satisfy $(K_{u_2}^*)$. Hence, $UG(\mathbb{Z}_3 \times Z_3 \times T_1)$ does not satisfy $(K_{u_2}^*)$. This is in contradiction to the assumption that $UG(R)$ satisfies $(K_{u_2}^*)$.

Thus if $UG(R)$ satisfies $(K_{u_2}^*)$, then $F_i \in \{Z_2, Z_3, F_4, \mathbb{Z}_5\}$ for each $i \in \{1, 2, 3, \ldots, n\}$. 

**Lemma 2.4.** Let $n \geq 3$ and let $R = F_1 \times F_2 \times F_3 \times \cdots \times F_n$, where $F_i$ is a field for each $i \in \{1, 2, 3, \ldots, n\}$. Suppose that $F_1 = F_4$ for some $i \in \{1, 2, 3, \ldots, n\}$. If $UG(R)$ satisfies $(K_{u_2}^*)$, then $F_j = Z_2$ for all $j \in \{1, 2, 3, \ldots, n\}\{i\}$.

**Proof.** We are assuming that $F_1 = F_4$ for some $i \in \{1, 2, 3, \ldots, n\}$ and $UG(R)$ satisfies $(K_{u_2}^*)$. Without loss of generality, we can assume that $F_1 = F_4$. Since $|U(F_4)| = 3$ and $UG(R)$ satisfies $(K_{u_2})$, it follows from [10, Lemma 3.2] that $|U(F_2 \times F_3 \times \cdots \times F_n)| \leq 2$. We claim that $F_j = Z_2$ for each $j \in \{2, 3, \ldots, n\}$. Suppose that $F_j = Z_4$ for some $j \in \{2, 3, \ldots, n\}$. Without loss of generality, we can assume that $F_2 = Z_3$. Since $|U(Z_3 \times Z_3)| = 4$, it follows that $F_j = Z_2$ for each $j \in \{3, \ldots, n\}$. Let us denote the ring $F_2 \cdots \times F_n$ by $T_i$. Observe that char$(T_i) = 2$ and $R \cong F_4 \times Z_3 \times T_i$ as rings. We know from [10, Proposition 3.22] that $UG(R)$ does not satisfy $(K_{u_2})$. Therefore, there exists at most one $i \in \{1, 2, 3, \ldots, n\}$ such that $|F_i| \geq 4$.
3.18] that $UG(\mathbb{F}_4 \times \mathbb{Z}_3 \times T_1)$ does not satisfy $(Ku_2^r)$. This is in contradiction to the assumption that $UG(R)$ satisfies $(Ku_2^r)$.

Thus if $UG(R)$ satisfies $(Ku_2^r)$, then $F_j = \mathbb{Z}_2$ for each $j \in \{1, 2, 3, \ldots, n\}\backslash\{i\}$.

**Proposition 2.5.** Let $R = \mathbb{Z}_3 \times \mathbb{Z}_3 \times S$, where $S$ is a nonzero ring. Then $UG(R)$ does not satisfy $(Ku_2^r)$.

**Proof.** We consider the following cases.

**(Case 1)** $2 \notin U(S)$

Let $V_1 = \{(0, 1, 0), (0, 2, 1), (1, 2, 1)\}$ and let $V_2 = \{(1, 2, 0), (2, 2, 0), (1, 1, 1)\}$. Note that $V_1$ and $V_2$ are independent sets of $UG(R)$, $(0, 2, 1)$ is adjacent to both $(1, 2, 0)$ and $(2, 2, 0)$ in $UG(R)$, and $(1, 2, 1)$ is adjacent to $(1, 2, 0)$ in $UG(R)$. Let $H$ be the subgraph of $UG(R)$ induced on $V_1 \cup V_2 \cup \{(1, 0, 0), (2, 0, 0), (0, 0, 0)\}$. It is not hard to verify that $(0, 1, 0) - (1, 0, 1) - (1, 2, 0), (0, 1, 0) - (2, 0, 1) - (2, 2, 0), (0, 2, 1) - (1, 0, 0) - (1, 1, 1), (2, 1, 2) - (2, 0, 2) - (2, 2, 1) - (2, 2, 0), (2, 1, 2) - (0, 0, 0) - (1, 1, 1)$ are paths in $UG(R)$. Consider the subgraph $H_1$ of $H$ shown in Figure 1. Observe that $H_1$ is homeomorphic to $K_{3,3}$. Therefore, we obtain that $UG(R)$ does not satisfy $(Ku_2^r)$.

![Figure 1. H1](image1.png)

**(Case 2)** $2 \in U(S)$

Let $V_1 = \{(0, 1, 2), (0, 2, 0), (0, 2, 2)\}$ and let $V_2 = \{(1, 2, 2), (2, 0, 2), (1, 0, 2)\}$. Let $H$ be the subgraph of $UG(R)$ induced on $V_1 \cup V_2 \cup \{(1, 0, 0)\}$. Note that $(0, 2, 0)$ and $(0, 2, 2)$ are adjacent to each element of $V_2$ in $H$, $(0, 1, 2)$ is adjacent to both $(2, 0, 2)$ and $(1, 0, 2)$ in $H$, $(0, 1, 2) - (1, 0, 0) - (1, 2, 2)$ is a path in $H$. Let us denote the edges of $H$, $(1, 0, 0) - (0, 2, 2)$ and $(1, 2, 2) - (1, 0, 2)$ by $e_1$ and $e_2$. Let $H_2 = H \setminus \{e_1, e_2\}$. The subgraph $H_2$ of $UG(R)$ is shown in Figure 2.

![Figure 2. H2](image2.png)

Observe that $H_2$ is homeomorphic to $K_{3,3}$. Hence, we obtain that $UG(R)$ does not satisfy $(Ku_2^r)$.

**Proposition 2.6.** Let $R_1 = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ ($n$ factors, $n \geq 1$) and let $R = R_1 \times \mathbb{Z}_6$. Then $UG(R)$ is planar.

**Proof.** Observe that $|R| = 2^n \times 6$, $|UG(R)| = |U(R_1)| \times |U(\mathbb{Z}_6)| = 2$, and $2 \notin U(R)$. Therefore, we obtain from [2, Proposition 3.4 (ii)] that $deg_{UG(R)} = 2$ for any $r \in R$. Note that any element of $R_1$ is of the form $(x_1, \ldots, x_n)$, where $x_i \in \{0, 1\}$ for each $i \in \{1, \ldots, n\}$. Let $(x_1, \ldots, x_n)$ be any element of $R_1$. Let $H$ be the component of $UG(R)$ containing $(x_1, \ldots, x_n)$ in $UG(R)$, and $(1, 0) - (1, 1) - (1, 2) - (0, 1) - (0, 2) - (0, 0)$ is a path in $UG(R)$. Consider the subgraph $H_1$ of $H$ shown in Figure 1. Observe that $H_1$ is homeomorphic to $K_{3,3}$. Therefore, we obtain that $UG(R)$ does not satisfy $(Ku_2^r)$.

**Theorem 2.7.** Let $n \geq 3$ and let $R = F_i \times F_2 \times \cdots \times F_n$, where $F_i$ is a field for each $i \in \{1, 2, 3, \ldots, n\}$. Then the following statements are equivalent:

(i) $UG(R)$ is planar.

(ii) $UG(R)$ satisfies both $(Ku_1^r)$ and $(Ku_2^r)$.

(iii) $UG(R)$ satisfies $(Ku_2^r)$.

(iv) There exists at most one $i \in \{1, 2, 3, \ldots, n\}$ such that $F_i \not\simeq \mathbb{Z}_2$. If $i \in \{1, 2, 3, \ldots, n\}$ is such that $F_i \not\simeq \mathbb{Z}_2$, then $F_i \in \{\mathbb{Z}_3, \mathbb{F}_4\}$.

**Proof.** (i) $\Rightarrow$ (ii) This follows from Kuratowski’s theorem [5, Theorem 5.9].

(ii) $\Rightarrow$ (iii) This is clear.

(iii) $\Rightarrow$ (iv) We are assuming that $UG(R)$ satisfies $(Ku_2^r)$. We know from Lemma 2.3 that $F_i \in \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{F}_4\}$ for each $i \in \{1, 2, 3, \ldots, n\}$. If $F_i = \mathbb{F}_4$ for some $i \in \{1, 2, 3, \ldots, n\}$, then we know from Lemma 2.4 that $F_i = \mathbb{Z}_2$ for all $j \in \{1, 2, 3, \ldots, n\}\backslash\{i\}$. If $F_i = \mathbb{Z}_3$ for some $i \in \{1, 2, 3, \ldots, n\}$, then it follows from Lemma 2.5 that $F_j = \mathbb{Z}_2$ for all $j \in \{1, 2, 3, \ldots, n\}\backslash\{i\}$.

(iv) $\Rightarrow$ (i) If $F_i = \mathbb{Z}_2$ for all $i \in \{1, 2, 3, \ldots, n\}$, then we know from [10, Proposition 3.5] that $UG(R)$ is planar. If $F_i = \mathbb{F}_4$ for some $i \in \{1, 2, 3, \ldots, n\}$, then by hypothesis, $F_j = \mathbb{Z}_2$ for each $j \in \{1, 2, 3, \ldots, n\}\backslash\{i\}$. In such a case, it follows from [10, Proposition 3.9] that $UG(R)$ is planar. If $F_i = \mathbb{Z}_3$ for some $i \in \{1, 2, 3, \ldots, n\}$, then by hypothesis, $F_j = \mathbb{Z}_2$ for each $j \in \{1, 2, 3, \ldots, n\}\backslash\{i\}$, in which case, it follows from Proposition 2.6 that $UG(R)$ is planar.

Let $R$ be a semiquasiloclal ring with $|Max(R)| = n \geq 3$. If $UG(R)$ satisfies $(Ku_2^r)$, then we know from [10, Proposition 2.3 and Remark 2.4] that there exist finite local rings $(R_i, m_i)$ for each $i \in \{1, 2, 3, \ldots, n\}$ such that $R \cong R_1 \times R_2 \times R_3 \times \cdots \times R_n$ as rings. Let us denote $R_1 \times R_2 \times R_3 \times \cdots \times R_n$ by $T$. If $UG(T)$ satisfies $(Ku_2^r)$, then we know from Lemma 2.1 that $R_i$ is not a field for at most one $i \in \{1, 2, 3, \ldots, n\}$. We assume that $R_i$ is not a field for some $i \in \{1, 2, 3, \ldots, n\}$. 

1415
and try to classify such rings \( T = R_1 \times R_2 \times R_3 \times \cdots \times R_n \) in order that \( UG(T) \) is planar.

**Remark 2.8.** Let \( t \geq 2 \) and let \( R = R_1 \times F_2 \times \cdots \times F_t \), where \( R_1 \) is a quasilocal ring which is not a field and \( F_i \) is a field for each \( i \in \{2, \ldots, t\} \). If \( |F_i| \geq 3 \) for some \( i \in \{2, \ldots, t\} \), then \( UG(R) \) does not satisfy \( (Ku_2)^t \).

**Proof.** Without loss of generality, we can assume that \( |F_2| \geq 3 \). Then either \( t = 2 \) or \( t \geq 3 \). If \( t = 2 \), then it follows from \([10, \text{Proposition 3.14}]\) that \( UG(R) \) does not satisfy \( (Ku_2)^2 \). Suppose that \( t \geq 3 \). Let \( T = F_3 \times \cdots \times F_n \). Note that \( R \cong R_1 \times F_2 \times T \) as rings. In this case, we obtain from \([10, \text{Proposition 3.15}]\) that \( UG(R) \) does not satisfy \( (Ku_2)^t \).

**Theorem 2.9.** Let \( n \geq 3 \) and let \( R = R_1 \times F_2 \times F_3 \times \cdots \times F_n \), where \( R_1 \) is a quasilocal ring which is not a field and \( F_i \) is a field for each \( i \in \{2,3,\ldots,n\} \). The following statements are equivalent:

1. \( UG(R) \) is planar.
2. \( UG(R) \) satisfies both \( (Ku_4^2) \) and \( (Ku_3^2) \).
3. \( UG(R) \) satisfies \( (Ku_3^2) \).
4. \( R_1 \) is isomorphic to one of the rings from the collection \( \{Z_4, \frac{Z_3[X]}{X^2Z_2[X]}\} \) and \( F_i \cong Z_2 \) for each \( i \in \{2,3,\ldots,n\} \).

**Proof.** (i) \( \Rightarrow \) (ii) This follows from Kuratowski’s theorem \([5, \text{Theorem 5.9}]\).

(ii) \( \Rightarrow \) (iii) This is clear.

(iii) \( \Rightarrow \) (iv) We are assuming that \( UG(R) \) satisfies \( (Ku_3^2) \) and so, \( UG(R) \) satisfies \( (Ku_2) \). It follows from \([10, \text{Lemma 2.2}]\) that \( UG(R_1) \) satisfies \( (Ku_2) \). Therefore, we obtain from (iii) \( \Rightarrow \) (iv) of \([9, \text{Lemma 2.4}]\) that \( R_1 \) is isomorphic to one of the rings from the collection \( \{Z_4, \frac{Z_3[X]}{X^2Z_2[X]}\} \). Moreover, we know from Remark 2.8 that \( F_i \cong Z_2 \) for each \( i \in \{2,3,\ldots,n\} \).

(iv) \( \Rightarrow \) (i) We are assuming that \( R_1 \) is isomorphic to one of the rings from the collection \( \{Z_4, \frac{Z_3[X]}{X^2Z_2[X]}\} \) and \( F_i \cong Z_2 \) for each \( i \in \{2,3,\ldots,n\} \). Now, it follows from \([10, \text{Proposition 3.7}]\) that \( UG(R) \) is planar. \( \square \)

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