APPLICATION OF MULTIPLE FOURIER–LEGENDRE SERIES TO IMPLEMENTATION OF STRONG EXPONENTIAL MILSTEIN AND WAGNER–PLATEN METHODS FOR NON-COMMUTATIVE SEMILINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. The article is devoted to the application of multiple Fourier–Legendre series to implementation of strong exponential Milstein and Wagner–Platen methods for non-commutative semilinear stochastic partial differential equations with multiplicative trace class noise. These methods have strong orders of convergence $1.0 - \varepsilon$ and $1.5 - \varepsilon$ correspondingly (here $\varepsilon$ is an arbitrary small positive real number) with respect to the temporal discretization. The theorem on mean-square convergence of approximations of iterated stochastic integrals of multiplicities 1 to 3 with respect to the infinite-dimensional $Q$-Wiener process is formulated and proved. The results of the article can be applied to implementation of exponential Milstein and Wagner–Platen methods for non-commutative semilinear stochastic partial differential equations with multiplicative trace class noise.

1. Introduction

It is well-known that one of the effective approaches to the construction of high-order strong numerical methods (with respect to the temporal discretization) for stochastic partial differential equations (SPDEs) is based on the Taylor formula in Banach spaces and exponential formula for the mild solution of SPDEs [1]-[5]. A significant step in this direction was made in [2], [3], where the exponential Milstein and Wagner–Platen methods for semilinear SPDEs were constructed. Under the appropriate conditions [2], [3] these methods have strong orders of convergence $1.0 - \varepsilon$ and $1.5 - \varepsilon$ correspondingly (where $\varepsilon$ is an arbitrary small positive real number) with respect to the temporal discretization.

An important feature of the mentioned numerical methods is the presence in them of the so-called iterated stochastic integrals with respect to the infinite-dimensional $Q$-Wiener process [7]. The problem of numerical modeling of these stochastic integrals was solved in [2], [3] for the case when special commutativity conditions are fulfilled.

If the mentioned commutativity conditions are not fulfilled, which often corresponds to SPDEs in numerous applications, the numerical simulation of iterated stochastic integrals with respect to the infinite-dimensional $Q$-Wiener process becomes much more difficult. Note that the exponential Milstein scheme [2] contains the iterated stochastic integrals of multiplicities 1 and 2 with respect to the infinite-dimensional $Q$-Wiener process and the exponential Wagner–Platen scheme [3] contains the mentioned stochastic integrals of multiplicities 1 to 3. In [8] two methods of the mean-square approximation of iterated stochastic integrals from the Milstein scheme [2] have been considered. Note that the mean-square error of approximation of these stochastic integrals consists of two components [8].

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The first component is related with the finite-dimensional approximation of the infinite-dimensional $Q$-Wiener process while the second one is connected with the approximation of iterated Itô stochastic integrals with respect to the scalar standard Brownian motions. In the author’s publications [9], [10], [23], [24] the problem of the mean-square approximation of iterated stochastic integrals with respect to the scalar standard Brownian motions was adapted for iterated stochastic integrals with respect to the infinite-dimensional $Q$-Wiener process in the sense of second component of approximation error (see above) has been solved for an arbitrary multiplicity $k$ ($k \in \mathbb{N}$) of stochastic integrals. More precisely, in [9], [10] (also see [23], [24]) the method of generalized multiple Fourier series [11]-[54] for the approximation of iterated Itô stochastic integrals with respect to the scalar standard Brownian motions was adapted for iterated stochastic integrals with respect to the infinite-dimensional $Q$-Wiener process (in the sense of second component of the approximation error).

In this article, we extend the method for estimating the first component of approximation error from [3] for iterated stochastic integrals of multiplicities 1 to 3 with respect to the infinite-dimensional $Q$-Wiener process. In addition, we combine the obtained results with the results from [9], [10]. Thus, the results of the article can be applied to the implementation of exponential Milstein and Wagner–Platen schemes for semilinear SPDEs with multiplicative trace class noise and without the conditions of commutativity for SPDEs.

2. Exponential Milstein and Wagner–Platen Numerical Schemes for Non-Commutative Semilinear SPDEs

Let $U, H$ be separable $\mathbb{R}$-Hilbert spaces and $L_{HS}(U, H)$ be a space of Hilbert–Schmidt operators mapping from $U$ to $H$. Let $(\Omega, F, \mathbb{P})$ be a probability space with a normal filtration $\{F_t, t \in [0, \bar{T}]\}$, let $W_t$ be an $U$-valued $Q$-Wiener process with respect to $\{F_t, t \in [0, \bar{T}]\}$, which has a covariance trace class operator $Q \in L(U)$. Here $L(U)$ denotes all bounded linear operators mapping from $U$ to $U$. Consider an $\mathbb{R}$-Hilbert space $U_0 = Q^{1/2}(U)$ with a scalar product

$$\langle u, w \rangle_{U_0} = \left\langle Q^{-1/2}u, Q^{-1/2}w \right\rangle_U$$

for all $u, w \in U_0$.

Consider the semilinear parabolic SPDE

$$dX_t = (AX_t + F(X_t))dt + B(X_t)dW_t, \quad X_0 = \xi, \quad t \in [0, \bar{T}],$$

where nonlinear operators $F, B$ ($F : H \to H, B : H \to L_{HS}(U_0, H)$), linear operator $A : D(A) \subset H \to H$ as well as the initial value $\xi$ are assumed to satisfy the conditions of existence and uniqueness of the SPDE ([1]) mild solution (see [3], Assumptions A1–A4). It is well-known [6] that Assumptions A1–A4 [3] guarantee the existence and uniqueness (up to modifications) of the mild solution $X_t : [0, \bar{T}] \times \Omega \to H$ of the SPDE ([1])

$$X_t = \exp(At)\xi + \int_0^t \exp(A(t-\tau))F(X_\tau)d\tau + \int_0^t \exp(A(t-\tau))B(X_\tau)dW_\tau$$

with probability 1 (further w. p. 1) for all $t \in [0, \bar{T}]$, where $\exp(At)$, $t \geq 0$ is the semigroup generated by the operator $A$.

Consider eigenvalues $\lambda_i$ and eigenfunctions $e_i(x)$ of the covariance operator $Q$, where $i = (i_1, \ldots, i_d) \in J$, $x = (x_1, \ldots, x_d)$, and $J = \{i : i \in \mathbb{N}^d, \text{ and } \lambda_i > 0\}$.  

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The series representation of the \( Q \)-Wiener process has the following form \cite{7}

\[
W_t = \sum_{i \in J} e_i \sqrt{\lambda_i} w_i^{(i)} \quad \text{or} \quad W_t = \sum_{i \in J_M} e_i \langle e_i, W_t \rangle_U,
\]

where \( t \in [0, \bar{T}] \), \( w_i^{(i)} \) (\( i \in J \)) are independent standard Wiener processes and \( \langle \cdot, \cdot \rangle_U \) is a scalar product in \( U \).

Note that eigenfunctions \( e_i \), \( i \in J \) form an orthonormal basis of \( U \) \cite{7}. Consider the finite-dimensional approximation of \( W_t \) \cite{7}

\[
W_t^M = \sum_{i \in J_M} e_i \sqrt{\lambda_i} w_i^{(i)}, \quad t \in [0, \bar{T}],
\]

where

\[
J_M = \{ i : 1 \leq i_1, \ldots, i_d \leq M, \text{ and } \lambda_i > 0 \}.
\]

**Remark 1.** Obviously, without the loss of generality we can write \( J_M = \{ 1, 2, \ldots, M \} \).

Let \( \Delta > 0 \), \( \tau_p = p \Delta \) (\( p = 0, 1, \ldots, N \)), and \( N \Delta = \bar{T} \). Consider the following exponential Milstein and Wagner–Platen numerical schemes for the SPDE \cite{11} \cite{2}, \cite{3}

\[
Y_{p+1} = \exp \left( A \Delta \right) \left( Y_p + \Delta F(Y_p) + \int_{\tau_p}^{\tau_{p+1}} B(Y_p)dW_s + \right.
\]

\[
\left. + \int_{\tau_p}^{\tau_{p+1}} B'(Y_p) \left( \int_{\tau_p}^{s} B(Y_p)dW_s \right) dW_s \right),
\]

(an exponential Milstein scheme)

\[
Y_{p+1} = \exp \left( \frac{A \Delta}{2} \right) \times
\]

\[
\times \left( \exp \left( \frac{A \Delta}{2} \right) Y_p + \Delta F(Y_p) + \frac{\Delta^2}{2} F'(Y_p) \left( AY_p + F(Y_p) \right) + \int_{\tau_p}^{\tau_{p+1}} B(Y_p)dW_s + \right.
\]

\[
\left. + \frac{\Delta^2}{4} \sum_{i \in J} \lambda_i F''(Y_p) \left( B(Y_p)e_i, B(Y_p)e_i \right) + \int_{\tau_p}^{\tau_{p+1}} B'(Y_p) \left( \int_{\tau_p}^{s} B(Y_p)dW_s \right) dW_s + \right.
\]

\[
\left. + A \left( \int_{\tau_p}^{\tau_{p+1}} \int_{\tau_p}^{s} B(Y_p)dW_s ds - \frac{\Delta}{2} \int_{\tau_p}^{\tau_{p+1}} B(Y_p)dW_s \right) \right).
\]
\[
+ \Delta \int_{\tau_p}^{\tau_{p+1}} B'(Y_p) \left( AY_p + F(Y_p) \right) dW_s - \int_{\tau_p}^{\tau_{p+1}} \int B'(Y_p) \left( AY_p + F(Y_p) \right) dW_r ds + \\
+ \frac{1}{2} \int_{\tau_p}^{\tau_{p+1}} B''(Y_p) \left( \int B(Y_p) dW_\tau, \int B(Y_p) dW_\tau \right) dW_s + \\
+ \int_{\tau_p}^{\tau_{p+1}} F'(Y_p) \left( \int B(Y_p) dW_\tau \right) ds + \\
+ \int_{\tau_p}^{\tau_{p+1}} B'(Y_p) \left( \int B'(Y_p) \left( \int B(Y_p) dW_\theta \right) dW_\tau \right) dW_s \Bigg),
\]

(6)

(an exponential Wagner–Platen scheme)

where \( Y_p \) is an approximation of \( X_{\tau_p} \) (mild solution \( 2 \) at the time moment \( \tau_p \), \( p = 0, 1, \ldots, N \), and \( B', B'', F', F'' \) are Fréchet derivatives. In addition to the temporal discretization, the implementation of numerical schemes (5) and (6) also requires a finite-dimensional approximation of the spaces \( H, U \). Further, we will consider this approximation only for the space \( U \).

Let us consider the following iterated Itô stochastic integrals

\[
I^{(r_1)}_{(1)T,t} = \int_t^T dW^{(r_1)}_{t_1},
\]

\[
I^{(r_0)}_{(10)T,t} = \int_t^T \int_t^{t_2} dW^{(r_1)}_{t_1} dt_2, \quad I^{(0r_2)}_{(01)T,t} = \int_t^T \int_t^{t_2} dt_1 dW^{(r_2)}_{t_2},
\]

\[
I^{(r_1r_2)}_{(11)T,t} = \int_t^T \int_t^{t_2} dW^{(r_1)}_{t_1} dW^{(r_2)}_{t_2},
\]

\[
I^{(r_1r_2r_3)}_{(111)T,t} = \int_t^T \int_t^{t_2} \int_t^{t_3} dW^{(r_1)}_{t_1} dW^{(r_2)}_{t_2} dW^{(r_3)}_{t_3},
\]

where \( r_1, r_2, r_3 \in J_M, \ 0 \leq t < T \leq T, \) and \( J_M \) is defined by (4).

Let us replace the infinite-dimensional \( Q \)-Wiener process in the iterated stochastic integrals from (5), (6) by its finite-dimensional approximation (3). Then we have w. p. 1

\[
\int_{\tau_p}^{\tau_{p+1}} B(Y_p) dW^M_s = \sum_{r_1 \in J_M} B(Y_p) e_{r_1} \sqrt{\lambda_{r_1}} I^{(r_1)}_{(1)(1)T,\tau_p,\tau_{p+1}},
\]

(7)
\begin{align}
A \left( \int_{\tau_p}^{\tau_p+1} \int_{\tau_p}^s B(Y_p) d\mathbf{W}_s^M ds \right) - \frac{\Delta}{2} \int_{\tau_p}^{\tau_p+1} B(Y_p) d\mathbf{W}_s^M 
&= A \int_{\tau_p}^{\tau_p+1} B(Y_p) \left( \frac{\tau_{p+1}}{2} - s + \frac{\tau_p}{2} \right) d\mathbf{W}_s^M = \\
&= \sum_{r_1 \in J_M} AB(Y_p) c_{r_1} \sqrt{\lambda_{r_1}} \left( \frac{\Delta}{2} I^{(r_1)}_{(1)\tau_{p+1},\tau_p} - I^{(0r_1)}_{(01)\tau_{p+1},\tau_p} \right),
\end{align}

\begin{align}
\Delta \int_{\tau_p}^{\tau_p+1} B'(Y_p) \left( AY_p + F(Y_p) \right) d\mathbf{W}_s^M - \int_{\tau_p}^{\tau_p+1} \int_{\tau_p}^s B'(Y_p) \left( AY_p + F(Y_p) \right) d\mathbf{W}_s^M ds 
&= \int_{\tau_p}^{\tau_p+1} B'(Y_p) \left( AY_p + F(Y_p) \right) e_{r_1} \sqrt{\lambda_{r_1}} I^{(0r_1)}_{(01)\tau_{p+1},\tau_p} \phantom{d\mathbf{W}_s^M}

&= \sum_{r_1 \in J_M} B'(Y_p) \left( AY_p + F(Y_p) \right) e_{r_1} \sqrt{\lambda_{r_1}} I^{(0r_1)}_{(01)\tau_{p+1},\tau_p}
\end{align}

\begin{align}
\int_{\tau_p}^{\tau_p+1} B'(Y_p) \left( \int_{\tau_p}^s B(Y_p) d\mathbf{W}_s^M \right) ds 
&= \sum_{r_1 \in J_M} F'(Y_p) B(Y_p) e_{r_1} \sqrt{\lambda_{r_1}} \left( \Delta I^{(r_1)}_{(1)\tau_{p+1},\tau_p} - I^{(0r_1)}_{(01)\tau_{p+1},\tau_p} \right)
\end{align}

\begin{align}
\int_{\tau_p}^{\tau_p+1} B'(Y_p) \left( \int_{\tau_p}^s B'(Y_p) \left( \int_{\tau_p}^{\tau_{p+1}} B(Y_p) d\mathbf{W}_s^M \right) d\mathbf{W}_s^M \right) d\mathbf{W}_s^M 
&= \sum_{r_1, r_2 \in J_M} B'(Y_p) \left( B'(Y_p) \left( B(Y_p) e_{r_1} \right) e_{r_2} \right) \sqrt{\lambda_{r_1}} \lambda_{r_2} I^{(r_1r_2)}_{(11)\tau_{p+1},\tau_p}
\end{align}

\begin{align}
\int_{\tau_p}^{\tau_p+1} B'(Y_p) \left( \int_{\tau_p}^s B'(Y_p) \left( \int_{\tau_p}^{\tau_{p+1}} B(Y_p) d\mathbf{W}_s^M \right) d\mathbf{W}_s^M \right) d\mathbf{W}_s^M 
&= \sum_{r_1, r_2, r_3 \in J_M} B'(Y_p) \left( B'(Y_p) \left( B(Y_p) e_{r_1} \right) e_{r_2} \right) e_{r_3} \sqrt{\lambda_{r_1}} \lambda_{r_2} \lambda_{r_3} I^{(r_1r_2r_3)}_{(111)\tau_{p+1},\tau_p}
\end{align}
\[ \int_{\tau_p}^{\tau_{p+1}} B''(Y_p) \left( \int_{\tau_p}^{s} B(Y_p) \, dW^M_{\tau}, \int_{\tau_p}^{s} B(Y_p) \, dW^M_{T} \right) \, dW^M_{s} = \]

\[ = \sum_{r_1, r_2, r_3 \in J_M} B''(Y_p) (B(Y_p) e_{r_1}, B(Y_p) e_{r_2}) e_{r_3} \sqrt{\lambda_{r_1} \lambda_{r_2} \lambda_{r_3}} \times \]

\[ \times \int_{\tau_p}^{\tau_{p+1}} \left( \int_{\tau_p}^{s} dW^{(r_1)}_{\tau}, \int_{\tau_p}^{s} dW^{(r_2)}_{\tau} \right) \, dW^{(r_3)}_{s}. \] (13)

Note that in (8)–(10) we used the Itô formula. Moreover, using the Itô formula we obtain

\[ \int_{\tau_p}^{s} dW^{(r_1)}_{\tau}, \int_{\tau_p}^{s} dW^{(r_2)}_{\tau} = I^{(r_1 r_2)}_{(11),s,\tau_p} + I^{(r_2 r_1)}_{(11),s,\tau_p} + 1_{\{r_1 = r_2\}} (s - \tau_p) \quad \text{w. p. 1.} \] (14)

From (14) we have

\[ \int_{\tau_p}^{\tau_{p+1}} \left( \int_{\tau_p}^{s} dW^{(r_1)}_{\tau}, \int_{\tau_p}^{s} dW^{(r_2)}_{\tau} \right) \, dW^{(r_3)}_{s} = I^{(r_1 r_2 r_3)}_{(111),\tau_{p+1},\tau_p} + I^{(r_2 r_1 r_3)}_{(111),\tau_{p+1},\tau_p} + 1_{\{r_1 = r_2\}} I^{(0 r_3)}_{(01),\tau_{p+1},\tau_p} \quad \text{w. p. 1.} \] (15)

After substituting (15) into (13), we obtain

\[ \int_{\tau_p}^{\tau_{p+1}} B''(Y_p) \left( \int_{\tau_p}^{s} B(Y_p) \, dW^M_{\tau}, \int_{\tau_p}^{s} B(Y_p) \, dW^M_{T} \right) \, dW^M_{s} = \]

\[ = \sum_{r_1, r_2, r_3 \in J_M} B''(Z) (B(Z) e_{r_1}, B(Z) e_{r_2}) e_{r_3} \sqrt{\lambda_{r_1} \lambda_{r_2} \lambda_{r_3}} \times \]

\[ \times \left( I^{(r_1 r_2 r_3)}_{(111),\tau_{p+1},\tau_p} + I^{(r_2 r_1 r_3)}_{(111),\tau_{p+1},\tau_p} + 1_{\{r_1 = r_2\}} I^{(0 r_3)}_{(01),\tau_{p+1},\tau_p} \right) \quad \text{w. p. 1.} \] (16)

Thus, for the implementation of numerical schemes (5) and (6) we need to approximate the following collection of iterated Itô stochastic integrals

\[ I^{(r_1)}_{(11),T,\tau_1}, I^{(0 r_1)}_{(01),T,\tau_1}, I^{(r_1 r_2)}_{(111),T,\tau_1}, I^{(0 r_1 r_2)}_{(011),T,\tau_1}, \]

where \( r_1, r_2, r_3 \in J_M, \ 0 \leq t < T \leq T. \)

The monographs \[22\] (Chapters 5, 6) and \[24\] or \[24\] (Chapters 1, 2, and 5) (also see \[11\]–[21], \[26\]–[74]) are devoted to the constructing of efficient methods (based on generalized multiple Fourier series) of the mean-square approximation of iterated Itô stochastic integrals with respect to components of the finite-dimensional Wiener process. These results are also adapted for iterated Stratonovich
stochastic integrals [10, 21, 28, 33, 35–42, 46, 49, 50, 52, 54]. In Sect. 3, we consider a very short review of the results from monographs [22] (Chapters 5, 6) and [23] or [24] (Chapters 1, 2, and 5).

3. Method of Approximation of Iterated Itô Stochastic Integrals Based on Generalized Multiple Fourier Series. The Case of Multiple Fourier–Legendre Series

Consider the following iterated Itô stochastic integrals

\[
J[\psi^{(k)}]_{T,t}^{(t_1, \ldots, t_k)} = \int_t^T \psi_k(t_k) \cdots \int_t^{t_2} \psi_1(t_1) \, dw_{t_1}^{(i_1)} \cdots dw_{t_k}^{(i_k)},
\]

where \(0 \leq t < T \leq \bar{T}\), every \(\psi_l(\tau) (l = 1, \ldots, k)\) is a continuous non-random function on \([t, T]\), \(w^{(i)}_x (i = 1, \ldots, m)\) are independent standard Wiener processes, \(w^{(0)}_x = \tau, i_1, \ldots, i_k = 0, 1, \ldots, m\). The case \(\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T])\) will be considered in Theorem 2 (see below).

Suppose that \(\{\phi_j(x)\}_{j=0}^{\infty}\) is a complete orthonormal system of functions in the space \(L_2([t, T])\).

Define the following function on the hypercube \([t, T]^k\)

\[
K(t_1, \ldots, t_k) = \begin{cases} 
\psi_1(t_1) \cdots \psi_k(t_k), & t_1 < \ldots < t_k \\
0, & \text{otherwise}
\end{cases} = \prod_{l=1}^{k} \psi_l(t_l) \prod_{l=1}^{k-1} 1_{\{t_l < t_{l+1}\}},
\]

where \(t_1, \ldots, t_k \in [t, T]\) for \(k \geq 2\) and \(K(t_1) \equiv \psi_1(t_1)\) for \(t_1 \in [t, T]\). Here \(1_A\) is the indicator of the set \(A\).

The function \(K(t_1, \ldots, t_k)\) is piecewise continuous on the hypercube \([t, T]^k\). At this situation it is well known that the generalized multiple Fourier series of \(K(t_1, \ldots, t_k) \in L_2([t, T]^k)\) converges to \(K(t_1, \ldots, t_k)\) on the hypercube \([t, T]^k\) in the mean-square sense, i.e.

\[
\lim_{p_1, \ldots, p_k \to \infty} \left\| K(t_1, \ldots, t_k) - \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_1 \ldots j_k} \prod_{l=1}^{k} \phi_{j_l}(t_l) \right\|_{L_2([t, T]^k)} = 0,
\]

where

\[
C_{j_1 \ldots j_k} = \int_{[t, T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \phi_{j_l}(t_l) \, dt_1 \cdots dt_k
\]

is the Fourier coefficient and

\[
\|f\|_{L_2([t, T]^k)} = \left( \int_{[t, T]^k} f^2(t_1, \ldots, t_k) \, dt_1 \cdots dt_k \right)^{1/2}.
\]
Consider the discretization \( \{\tau_j\}_{j=0}^N \) of \([t,T]\) such that
\[
(21) \quad t = \tau_0 < \ldots < \tau_N = T, \quad \Delta_N = \max_{0 \leq j \leq N-1} \Delta\tau_j \rightarrow 0 \text{ if } N \rightarrow \infty, \quad \Delta\tau_j = \tau_{j+1} - \tau_j.
\]

**Theorem 1** [11] (2006) (also see [9], [10], [12]-[54]). Suppose that every \( \psi_j(\tau) \) \((l = 1, \ldots, k)\) is a continuous non-random function on \([t,T]\) and \(\{\phi_j(x)\}_{j=0}^\infty\) is a complete orthonormal system of continuous functions in \(L_2([t,T])\). Then
\[
J[\psi(k)]_{T,t}^{(i_1,\ldots,i_k)} = \lim_{p_1,\ldots,p_k \rightarrow \infty} \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_k \cdots j_1} \left( \prod_{l=1}^k \zeta_j^{(i_l)} - \prod_{l=1}^k \phi_j(\tau_l) \Delta w_{\tau_l}^{(i_l)} \right),
\]
where
\[
G_k = H_k \setminus L_k, \quad H_k = \left\{ (l_1, \ldots, l_k) : l_1, \ldots, l_k = 0, 1, \ldots, N-1 \right\},
\]
\[
L_k = \left\{ (l_1, \ldots, l_k) : l_1, \ldots, l_k = 0, 1, \ldots, N-1; l_g \neq l_r \ (g \neq r); g, r = 1, \ldots, k \right\}.
\]
l.i.m. is a limit in the mean-square sense, \(i_1, \ldots, i_k = 0, 1, \ldots, m\),
\[
(23) \quad \zeta_j^{(i)} = \int_t^T \phi_j(s) dw_s^{(i)}
\]
are independent standard Gaussian random variables for various \(i\) or \(j\) \((i \neq 0)\), \(C_{j_k \cdots j_1}\) is the Fourier coefficient \(20\), \(\Delta w_{\tau_l}^{(i)} = w_{\tau_l}^{(i)} - w_{\tau_l}^{(i)} (i = 0, 1, \ldots, m)\), \(\{\tau_j\}_{j=0}^N\) is the discretization of \([t,T]\), which satisfies the condition \(21\).

Note that in [11]-[24], [32] the version of Theorem 1 for systems of Haar and Rademacher–Walsh functions has been considered. Another version of Theorem 1 related to the application of complete orthonormal with weight \(r(t_1) \ldots r(t_k) \geq 0\) systems of functions in \(L_2([t,T]^k)\) has been considered in [22]-[24], [44]. A generalization of Theorem 1 to the case of an arbitrary complete orthonormal system of functions \(\{\phi_j(x)\}_{j=0}^\infty\) in the space \(L_2([t,T])\) as well as \(\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T])\) will be considered below (see Theorem 2).

In order to evaluate the significance of Theorem 1 for practice we will demonstrate its transformed particular cases for \(k = 1, \ldots, 5\) [11]-[54] \((cases \ k = 6, 7 \ and \ k > 7 \ can \ be \ found \ in \ [12]-[54])\)
\[
(24) \quad J[\psi(1)]_{T,t}^{(i_1)} = \lim_{p_1 \rightarrow \infty} \sum_{j_1=0}^{p_1} C_{j_1} \zeta_j^{(i_1)},
\]
\[
(25) \quad J[\psi(2)]_{T,t}^{(i_1,i_2)} = \lim_{p_1,p_2 \rightarrow \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} C_{j_2 j_1} \left( \zeta_j^{(i_1)} \zeta_j^{(i_2)} - \mathbb{1}_{\{i_1=i_2 \neq 0\}} \mathbb{1}_{\{j_1=j_2\}} \right),
\]
\[
J[\psi](t, T, t) = \text{l.i.m.}_{p_1, p_2, p_3 \to \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{j_3=0}^{p_3} C_{j_1 j_2 j_3} \left( \zeta^{(i_1)}_{j_1} \zeta^{(i_2)}_{j_2} \zeta^{(i_3)}_{j_3} - \right. \\
\left. - 1_{\{i_1=i_2 \neq 0\}} 1_{\{j_1=j_2\}} \zeta^{(i_3)}_{j_3} - 1_{\{i_1=i_3 \neq 0\}} 1_{\{j_2=j_3\}} \zeta^{(i_1)}_{j_1} - 1_{\{i_2=i_3 \neq 0\}} 1_{\{j_1=j_3\}} \zeta^{(i_2)}_{j_2} \right),
\]

(26)

\[
J[\psi](t, T, t) = \text{l.i.m.}_{p_1, \ldots, p_4 \to \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{j_3=0}^{p_3} \sum_{j_4=0}^{p_4} C_{j_1 j_2 j_3 j_4} \left( \prod_{i=1}^{4} \zeta^{(i)}_{j_i} - ight. \\
\left. - 1_{\{i_1=i_2 \neq 0\}} 1_{\{j_1=j_2\}} \zeta^{(i_3)}_{j_3} \zeta^{(i_4)}_{j_4} - 1_{\{i_1=i_3 \neq 0\}} 1_{\{j_1=j_3\}} \zeta^{(i_2)}_{j_2} \zeta^{(i_4)}_{j_4} - ight. \\
\left. - 1_{\{i_2=i_3 \neq 0\}} 1_{\{j_2=j_3\}} \zeta^{(i_1)}_{j_1} \zeta^{(i_4)}_{j_4} - 1_{\{i_2=i_4 \neq 0\}} 1_{\{j_2=j_4\}} \zeta^{(i_1)}_{j_1} \zeta^{(i_3)}_{j_3} + ight. \\
\left. + 1_{\{i_1=i_2 \neq 0\}} 1_{\{j_1=j_2\}} 1_{\{i_1=i_4 \neq 0\}} 1_{\{j_3=j_4\}} \zeta^{(i_2)}_{j_2} \zeta^{(i_3)}_{j_3} + ight. \\
\left. + 1_{\{i_1=i_4 \neq 0\}} 1_{\{j_1=j_4\}} 1_{\{i_1=i_3 \neq 0\}} 1_{\{j_3=j_2\}} \zeta^{(i_2)}_{j_2} \zeta^{(i_3)}_{j_3} \right),
\]

(27)

\[
J[\psi](t, T, t) = \text{l.i.m.}_{p_1, \ldots, p_5 \to \infty} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \sum_{j_3=0}^{p_3} \sum_{j_4=0}^{p_4} \sum_{j_5=0}^{p_5} C_{j_1 j_2 j_3 j_4 j_5} \left( \prod_{i=1}^{5} \zeta^{(i)}_{j_i} - ight. \\
\left. - 1_{\{i_1=i_2 \neq 0\}} 1_{\{j_1=j_2\}} \zeta^{(i_3)}_{j_3} \zeta^{(i_4)}_{j_4} \zeta^{(i_5)}_{j_5} - 1_{\{i_1=i_3 \neq 0\}} 1_{\{j_1=j_3\}} \zeta^{(i_2)}_{j_2} \zeta^{(i_4)}_{j_4} \zeta^{(i_5)}_{j_5} - ight. \\
\left. - 1_{\{i_2=i_3 \neq 0\}} 1_{\{j_2=j_3\}} \zeta^{(i_1)}_{j_1} \zeta^{(i_4)}_{j_4} \zeta^{(i_5)}_{j_5} - 1_{\{i_2=i_4 \neq 0\}} 1_{\{j_2=j_4\}} \zeta^{(i_1)}_{j_1} \zeta^{(i_3)}_{j_3} \zeta^{(i_5)}_{j_5} + ight. \\
\left. + 1_{\{i_1=i_2 \neq 0\}} 1_{\{j_1=j_2\}} 1_{\{i_1=i_4 \neq 0\}} 1_{\{j_3=j_5\}} \zeta^{(i_2)}_{j_2} \zeta^{(i_3)}_{j_3} \zeta^{(i_5)}_{j_5} + ight. \\
\left. + 1_{\{i_1=i_2 \neq 0\}} 1_{\{j_1=j_2\}} 1_{\{i_1=i_5 \neq 0\}} 1_{\{j_3=j_5\}} \zeta^{(i_2)}_{j_2} \zeta^{(i_3)}_{j_3} \zeta^{(i_4)}_{j_4} + ight. \\
\left. + 1_{\{i_1=i_3 \neq 0\}} 1_{\{j_1=j_3\}} 1_{\{i_2=i_5 \neq 0\}} 1_{\{j_3=j_5\}} \zeta^{(i_2)}_{j_2} \zeta^{(i_4)}_{j_4} \zeta^{(i_5)}_{j_5} + ight. \\
\left. + 1_{\{i_2=i_3 \neq 0\}} 1_{\{j_2=j_3\}} 1_{\{i_1=i_5 \neq 0\}} 1_{\{j_3=j_5\}} \zeta^{(i_1)}_{j_1} \zeta^{(i_4)}_{j_4} \zeta^{(i_5)}_{j_5} + ight. \\
\left. + 1_{\{i_2=i_4 \neq 0\}} 1_{\{j_2=j_4\}} 1_{\{i_1=i_5 \neq 0\}} 1_{\{j_3=j_5\}} \zeta^{(i_1)}_{j_1} \zeta^{(i_3)}_{j_3} \zeta^{(i_5)}_{j_5} + ight. \\
\left. + 1_{\{i_1=i_4 \neq 0\}} 1_{\{j_1=j_4\}} 1_{\{i_2=i_5 \neq 0\}} 1_{\{j_3=j_5\}} \zeta^{(i_2)}_{j_2} \zeta^{(i_3)}_{j_3} \zeta^{(i_4)}_{j_4} + ight. \\
\left. + 1_{\{i_1=i_5 \neq 0\}} 1_{\{j_1=j_5\}} 1_{\{i_2=i_4 \neq 0\}} 1_{\{j_3=j_4\}} \zeta^{(i_1)}_{j_1} \zeta^{(i_2)}_{j_2} \zeta^{(i_3)}_{j_3} + ight. \\
\left. + 1_{\{i_1=i_5 \neq 0\}} 1_{\{j_1=j_5\}} 1_{\{i_2=i_5 \neq 0\}} 1_{\{j_3=j_5\}} \zeta^{(i_1)}_{j_1} \zeta^{(i_2)}_{j_2} \zeta^{(i_4)}_{j_4} + ight. \\
\left. + 1_{\{i_2=i_5 \neq 0\}} 1_{\{j_2=j_5\}} \zeta^{(i_1)}_{j_1} \zeta^{(i_2)}_{j_2} \zeta^{(i_3)}_{j_3} \zeta^{(i_4)}_{j_4} \zeta^{(i_5)}_{j_5} \right),
\]

(28)

where \(1_A\) is the indicator of the set \(A\).
Consider the generalization of [24]–[28] for the case of an arbitrary $k$ ($k \in \mathbb{N}$) as well as for the case of an arbitrary complete orthonormal system of functions $\{\phi_j(x)\}_{j=0}^\infty$ in the space $L_2([t, T])$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T])$.

In order to do this, let us consider the unordered set $\{1, 2, \ldots, k\}$ and separate it into two parts: the first part consists of $r$ unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k - 2r$ numbers. So, we have

\begin{equation}
(29) \quad ((\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}), \{q_1, \ldots, q_{k-2r}\}),
\end{equation}

where $\{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}\} = \{1, 2, \ldots, k\}$, braces mean an unordered set and parentheses mean an ordered set.

We will say that (29) is a partition and consider the sum with respect to all possible partitions

\begin{equation}
(30) \quad \sum_{((\{g_1, g_2\}, \ldots, \{g_{2r-1}, g_{2r}\}),(q_1, \ldots, q_{k-2r})): (g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}) = \{1, 2, \ldots, k\}} a_{g_1, g_2, \ldots, g_{2r-1}, g_{2r}, q_1, \ldots, q_{k-2r}}.
\end{equation}

**Theorem 2** [23] (Sect. 1.11), [24] (Sect. 15). Suppose that $\{\phi_j(x)\}_{j=0}^\infty$ is an arbitrary complete orthonormal system of functions in the space $L_2([t, T])$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T])$. Then the following expansion

\begin{equation}
(31) \quad J_{[T, t]}^{\psi^{(k)}(i_1, \ldots, i_k)} = \lim_{p_1, \ldots, p_k \to \infty} \prod_{j_1=0}^{p_1} \ldots \prod_{j_k=0}^{p_k} C_{j_k, \ldots, j_1} \left( \prod_{l=1}^{k} C_{i_l}^{(i)} \right) + \sum_{r=1}^{[k/2]} (-1)^r \times \prod_{s=1}^{r} 1_{i_{2s-1} = i_{2s} \neq 0} 1_{j_{2s-1} = j_{2s}} \prod_{l=1}^{k-2r} C_{j_{2l}}^{(i_l)}
\end{equation}

converging in the mean-square sense is valid, where $[x]$ is an integer part of a real number $x$; another notations are the same as in Theorem 1.

In particular, from (31) for $k = 5$ we obtain

\begin{equation}
\begin{aligned}
J_{[T, t]}^{\psi^{(5)}(i_1, \ldots, i_5)} &= \lim_{p_1, \ldots, p_5 \to \infty} \prod_{j_1=0}^{p_1} \ldots \prod_{j_5=0}^{p_5} C_{j_5, \ldots, j_1} \left( \prod_{l=1}^{5} C_{i_l}^{(i)} \right) - \\
&= \sum_{((g_1, g_2), (q_1, q_2, q_3)): (g_1, g_2, q_1, q_2, q_3) = (1, 2, 3, 4, 5)} 1_{i_{q_1} = i_{q_2} \neq 0} 1_{j_{q_1} = j_{q_2}} \prod_{l=1}^{3} C_{j_{q_l}}^{(i_{q_l})} + \\
&\quad + \sum_{((g_1, g_2), (q_3, q_4, q_5)): (g_1, g_2, q_3, q_4, q_5) = (1, 2, 3, 4, 5)} 1_{i_{q_1} = i_{q_2} \neq 0} 1_{j_{q_1} = j_{q_2}} 1_{i_{q_3} = i_{q_4} \neq 0} 1_{j_{q_3} = j_{q_4}} C_{j_{q_5}}^{(i_{q_5})}.
\end{aligned}
\end{equation}
The last equality obviously agrees with (28). Note that the correctness of formulas (24)–(28) can be verified by the fact that if
\[ i_1 = \ldots = i_5 = \ldots = i_j = 1, \ldots, m \]
and \( \psi_1(s), \ldots, \psi_6(s) \equiv \psi(s) \), then we can derive from (24)–(28) the well known equalities (the cases \( k = 2, 3 \) were discussed in details in [12]–[24], [32]–[41]).

\[
J[\psi^{(1)}]_{T,t} = \frac{1}{1!} \delta^{(i)}_{T,t},
\]
\[
J[\psi^{(2)}]_{T,t} = \frac{1}{2!} \left( \delta^{(i)}_{T,t} \right)^2 - \Delta_{T,t},
\]
\[
J[\psi^{(3)}]_{T,t} = \frac{1}{3!} \left( \delta^{(i)}_{T,t} \right)^3 - 3 \delta^{(i)}_{T,t} \Delta_{T,t},
\]
\[
J[\psi^{(4)}]_{T,t} = \frac{1}{4!} \left( \delta^{(i)}_{T,t} \right)^4 - 6 \delta^{(i)}_{T,t} \Delta_{T,t} + 3 \Delta^2_{T,t},
\]
\[
J[\psi^{(5)}]_{T,t} = \frac{1}{5!} \left( \delta^{(i)}_{T,t} \right)^5 - 10 \delta^{(i)}_{T,t} \Delta_{T,t} + 15 \delta^{(i)}_{T,t} \Delta^2_{T,t}
\]
w. p. 1, where
\[
\delta^{(i)}_{T,t} = \int_t^T \psi(s) dW^{(i)}_s, \quad \Delta_{T,t} = \int_t^T \psi^2(s) ds.
\]

The above equalities can be independently obtained using the Itô formula and Hermite polynomials [61].

Assume that \( J[\psi^{(k)}]_{T,t} \) is an approximation of the stochastic integral (17), which is the expression on the right-hand side of (31) before passing to the limit \( \lim_{p_1, \ldots, p_k \to \infty} \). Let us denote

\[
E^{(i_1 \ldots i_k)_{p_1 \ldots p_k}} = M \left\{ \left( J[\psi^{(k)}]_{T,t} - J[\psi^{(k)}]_{T,t} \right)^{p_1 \ldots p_k} \right\},
\]

(32) \[ I_k = \| K \|_{L^2([t,T]^k)}^2 = \int_{[t,T]^k} K^2(t_1, \ldots, t_k) dt_1 \ldots dt_k, \quad E^{(i_1 \ldots i_k)_{p_1 \ldots p_k}} \mid_{p_1 = \ldots = p_k = p} \overset{\text{def}}{=} E^{(i_1 \ldots i_k)_{p}}. \]

In [21], [22], [24], [32], [34] it was shown that

(33) \[ E^{(i_1 \ldots i_k)_{p_1 \ldots p_k}} \leq k! \left( I_k - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k \ldots j_1}^2 \right), \]

where \( i_1, \ldots, i_k = 1, \ldots, m \) for \( 0 < T - t < \infty \) and \( i_1, \ldots, i_k = 0, 1, \ldots, m \) for \( 0 < T - t < 1 \). Note that the estimate (33) is valid under the conditions of Theorem 2.
Let us consider some approximations of iterated Itô stochastic integrals using Theorems 1, 2 and multiple Fourier–Legendre series.

The complete orthonormal system of Legendre polynomials in the space $L_2([t, T])$ looks as follows

\begin{equation}
\phi_j(x) = \sqrt{\frac{2j+1}{T-t}} P_j\left(\left(x - \frac{T+t}{2}\right) \frac{2}{T-t}\right), \quad j = 0, 1, 2, \ldots,
\end{equation}

where $P_j(x)$ is the Legendre polynomial.

Using the system of functions (34) and Theorems 1, 2 we obtain the following approximations of iterated Itô stochastic integrals (9)-(18) (also see early publications (22) (1997), (23) (1998), (24) (2000))

\begin{equation}
I_{(i)T,t}^{(i_1)} = \sqrt{T-t} \zeta_{0}^{(i_1)},
\end{equation}

\begin{equation}
I_{(0)T,t}^{(i_1)} = \frac{(T-t)^{3/2}}{2} \left( \zeta_{0}^{(i_1)} + \frac{1}{\sqrt{3}} \zeta_{1}^{(i_1)} \right),
\end{equation}

\begin{equation}
I_{(10)T,t}^{(i_2)} = \frac{(T-t)^{3/2}}{2} \left( \zeta_{0}^{(i_1)} - \frac{1}{\sqrt{3}} \zeta_{1}^{(i_1)} \right),
\end{equation}

\begin{equation}
I_{(11)T,t}^{(i_1)} = \frac{T-t}{2} \left( \zeta_{0}^{(i_1)} \zeta_{0}^{(i_2)} + \sum_{i=1}^{q} \frac{1}{\sqrt{4i^2-1}} \left( \zeta_{i-1}^{(i_1)} \zeta_{i}^{(i_2)} - \zeta_{i}^{(i_1)} \zeta_{i-1}^{(i_2)} \right) - \mathbf{1}_{\{i_1=i_2\}} \right),
\end{equation}

\begin{equation}
I_{(111)T,t}^{(i_2j_2j_3)} = \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_3j_2j_1} \left( \zeta_{j_1}^{(i_2)} \zeta_{j_2}^{(i_3)} - \mathbf{1}_{\{i_1=i_2\}} \right) \mathbf{1}_{\{j_1=j_2\}} \zeta_{j_3}^{(i_3)} - \mathbf{1}_{\{i_2=i_3\}} \mathbf{1}_{\{j_2=j_3\}} \zeta_{j_1}^{(i_1)}
\end{equation}

\begin{equation}
I_{(111)T,t}^{(i_1i_2i_3)} = \frac{1}{6}(T-t)^{3/2} \left( \zeta_{0}^{(i_1)} \right) - \frac{3}{8} \zeta_{0}^{(i_1)},
\end{equation}

\begin{equation}
C_{j_3j_2j_1} = \sqrt{(2j_1+1)(2j_2+1)(2j_3+1)} \frac{(T-t)^{3/2}}{8} C_{j_3j_2j_1},
\end{equation}

\begin{equation}
\tilde{C}_{j_3j_2j_1} = \int_{-1}^{1} P_{j_3}(z) \int_{-1}^{z} P_{j_2}(y) \int_{-1}^{y} P_{j_1}(x) dx dy dz,
\end{equation}
APPLICATION OF MULTIPLE FOURIER–LEGENDRE SERIES

Table 1. Minimal numbers $q$, $q_1$ such that $E^{(i_1 i_2)q} E^{(i_1 i_2 i_3)q_1} \leq (T - t)^4$, $q_1 \ll q$.

| $T - t$ | 0.08222 | 0.05020 | 0.02310 | 0.01956 |
|---------|---------|---------|---------|---------|
| $q$     | 19      | 51      | 235     | 328     |
| $q_1$   | 1       | 2       | 5       | 6       |

Table 2. Coefficients $C_{3jk}$.

| $k$ | 0     | 1     | 2     | 3     | 4     | 5     | 6     |
|-----|-------|-------|-------|-------|-------|-------|-------|
| 0   | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| 1   | $\frac{1}{128}$ | 0     | $-\frac{1}{32}$ | 0     | $\frac{1}{64}$ | 0     | 0     |
| 2   | $\frac{1}{128}$ | 0     | $-\frac{1}{32}$ | 0     | $\frac{1}{64}$ | 0     | 0     |
| 3   | $\frac{1}{128}$ | 0     | $-\frac{1}{32}$ | 0     | $\frac{1}{64}$ | 0     | 0     |
| 4   | $-\frac{1}{128}$ | 0     | $\frac{1}{32}$ | 0     | $-\frac{1}{64}$ | 0     | 0     |
| 5   | $-\frac{1}{128}$ | 0     | $\frac{1}{32}$ | 0     | $-\frac{1}{64}$ | 0     | 0     |
| 6   | 0     | $-\frac{1}{32}$ | 0     | $\frac{1}{64}$ | 0     | 0     | 0     |

Standard Gaussian random variables $\zeta_{ij}^{(i)}$ ($i \neq 0$) are defined by (23), and

\[
I_{(11)}^{(i_1 i_2)} = \lim_{q \to \infty} I_{(11)}^{(i_1 i_2)q},
\]

\[
I_{(111)}^{(i_1 i_2 i_3)} = \lim_{q_1 \to \infty} I_{(111)}^{(i_1 i_2 i_3)q_1}.
\]

Note that $T - t \ll 1$ ($T - t$ is an integration step with respect to the temporal variable). Thus $q_1 \ll q$ (see Table 1 [9]-[24], [35]). Moreover, the values $C_{3jk}$ do not depend on $T - t$. This feature is important because we can use a variable integration step $T - t$. Coefficients $C_{3jk}$ are calculated once and before the start of the numerical scheme. Some examples of exact calculation of coefficients $C_{3jk}$ via Python programming language can be found in Table 2 (the database with 270,000 exactly calculated Fourier–Legendre coefficients was described in [55], [56]).

According to the notations introduced above, we have

\[
E^{(i_1 i_2)q} = M \left\{ \left( I_{(11)}^{(i_1 i_2)} - I_{(11)}^{(i_1 i_2)q} \right)^2 \right\},
\]

\[
E^{(i_1 i_2 i_3)q_1} = M \left\{ \left( I_{(111)}^{(i_1 i_2 i_3)} - I_{(111)}^{(i_1 i_2 i_3)q_1} \right)^2 \right\}.
\]

Then for pairwise different $i_1, i_2, i_3 = 1, \ldots, m$ we obtain [9]-[24], [34]

\[
E^{(i_1 i_2)q} = \frac{(T - t)^2}{2} \left( \frac{1}{2} - \sum_{i=1}^{q} \frac{1}{4i^2 - 1} \right),
\]

(40)
(41) \[ E^{(i_1i_2i_3)} q_1 = \frac{(T-t)^3}{6} - \sum_{j_1,j_2,j_3=0}^{q_1} C_{j_1,j_2,j_3}^2. \]

On the basis of the presented approximations of iterated Itô stochastic integrals we can see that increasing of multiplicities of these integrals leads to increasing of orders of smallness with respect to \( T-t \) \( (T-t \ll 1) \) in the mean-square sense for iterated Itô stochastic integrals. This leads to sharp decrease of member quantities in the approximations of iterated Itô stochastic integrals, which are required for achieving the acceptable accuracy of approximation \( (q_1 \ll q) \).

From (41) we obtain [9]-[24], [35], [54] (for more details see [55]-[58])

(42) \[ E^{(i_1i_2i_3)} q_1 \bigg|_{q_1=6} \approx 0.01956000(T-t)^3. \]

4. APPROXIMATION OF ITERATED STOCHASTIC INTEGRALS OF MULTIPLICITY \( k \) WITH RESPECT TO THE \( Q \)-WIENER PROCESS

Consider the iterated Itô stochastic integral with respect to the \( Q \)-Wiener process in the following form

\[
I[\Phi^{(k)}(Z),\psi^{(k)}]_{T,t} = \int_t^T \Phi_k(Z) \left( \int_t^{t_3} \Phi_2(Z) \left( \int_t^{t_2} \Phi_1(Z) \psi_1(t_1) dW_{t_1} \right) \psi_2(t_2) dW_{t_2} \right) \cdots \psi_k(t_k) dW_{t_k}, \tag{43}
\]

where \( Z : \Omega \to H \) is an \( \mathbf{F}_t/B(H) \)-measurable mapping, \( \Phi_k(v) \left( \cdots (\Phi_2(v) (\Phi_1(v)) \cdots) \right) \) is a \( k \)-linear Hilbert–Schmidt operator mapping from \( \underbrace{U_0 \times \cdots \times U_0}_k \) to \( H \) for all \( v \in H \), and \( \psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t,T]) \).

Let \( I[\Phi^{(k)}(Z),\psi^{(k)}]_{t,T}^M \) be an approximation of the stochastic integral \( I[\Phi^{(k)}(Z),\psi^{(k)}]_{t,T} \)

\[
I[\Phi^{(k)}(Z),\psi^{(k)}]_{t,T} = \sum_{r_1,r_2,\ldots,r_k \in J_M} \Phi_k(Z) (\cdots (\Phi_2(Z) (\Phi_1(Z) e_{r_1}) e_{r_2}) \cdots) e_{r_k} \times
\]

\[
\sqrt{\lambda_{r_1} \lambda_{r_2} \ldots \lambda_{r_k}} J[\psi^{(k)}]_{I_{t,T}^M}^{(r_1r_2\cdots r_k)}, \tag{44}
\]
where $0 \leq t < T \leq \bar{T}$ and

$$J[\psi(k)]_{T,t}^{(r_1 \ldots r_k)} = \int_t^T \psi_k(t) \cdots \int_t^{t_1} \psi_2(t_2) \cdots \int_t^{t_1} \psi_1(t_1) d\mathbf{w}_{t_1}^{(r_1)} d\mathbf{w}_{t_2}^{(r_2)} \cdots d\mathbf{w}_{t_k}^{(r_k)}$$

is the iterated Itô stochastic integral $^\dagger_r$, $r_1, r_2, \ldots, r_k \in J_M$.

Let $I[\Phi(k)(Z), \psi(k)]_{T,t}^{M \times p_1 \ldots p_k}$ be an approximation of the stochastic integral $^\dagger_2$

$$I[\Phi(k)(Z), \psi(k)]_{T,t}^{M \times p_1 \ldots p_k} = \sum_{r_1, r_2, \ldots, r_k \in J_M} \Phi_k(Z) (\ldots (\Phi_2(Z) (\Phi_1(Z) e_{r_1}) e_{r_2}) \ldots) e_{r_k} \times$$

$$\times \sqrt{\lambda_{r_1} \lambda_{r_2} \cdots \lambda_{r_k}} J[\psi(k)]_{T,t}^{(r_1 \ldots r_k) \times p_1 \ldots p_k},$$

(45)

where $J[\psi(k)]_{T,t}^{(r_1 \ldots r_k) \times p_1 \ldots p_k}$ is defined as a prelimit expression on the right-hand side of $^\dagger_3$

$$J[\psi(k)]_{T,t}^{(r_1 \ldots r_k) \times p_1 \ldots p_k} = \sum_{j_1=0}^{p_1} \cdots \sum_{j_k=0}^{p_k} C_{j_1 \ldots j_k} (\prod_{l=1}^{k} \zeta_{j_l}^{(r_l)}) + \sum_{m=1}^{[k/2]} (-1)^m \times$$

$$\times \sum_{((\{s_1, s_2\}, \ldots, \{s_{2m-1}, s_{2m}\}), \{q_1, \ldots, q_{2m}\}) \in \mathcal{T}_{2 \times 2m} \times \mathcal{T}_{2 \times 2m} \times \{1, 2, \ldots, k\}} \prod_{s=1}^{m} 1_{r_{2s-1} = r_{2s} 
eq 0} 1_{j_{2s-1} = j_{2s}} \prod_{l=1}^{k-2m} \zeta_{j_{q_l}}^{(r_l)}.$$

Let $U, H$ be separable $\mathbb{R}$-Hilbert spaces, $U_0 = Q^{1/2}(U)$, and $L(U, H)$ be the space of linear and bounded operators mapping from $U$ to $H$. Let $L(U, H)_0 = \{T|_{U_0} : T \in L(U, H)\}$ (here $T|_{U_0}$ is the restriction of operator $T$ to the space $U_0$). It is known $^\dagger_0$ that $L(U, H)_0$ is a dense subset of the space of Hilbert–Schmidt operators $L_{HS}(U_0, H)$.

**Theorem 3** $^\dagger_9, ^\dagger_{10}, ^\dagger_{23} - ^\dagger_{25}$. Suppose that $\{\phi_j(x)\}_{j=0}^\infty$ is an arbitrary complete orthonormal system of functions in the space $L_2([t, T])$ and $\psi_1(\tau), \ldots, \psi_k(\tau) \in L_2([t, T])$. Furthermore, let the following conditions be satisfied:

1. $Q \in L(U)$ is a nonnegative and symmetric trace class operator ($\lambda_i$ and $\varepsilon_i$ ($i \in J$) are its eigenvalues and eigenfunctions (which form an orthonormal basis of $U$) correspondingly), and $W_\tau$, $\tau \in [0, T]$ is an $U$-valued $Q$-Wiener process.
2. $Z : \Omega \rightarrow H$ is an $\mathcal{F}_t/B(H)$-measurable mapping.
3. $\Phi_1 \in L(U, H)_0$, $\Phi_2 \in L(H, L(U, H)_0)$, and $\Phi_k(v) (\ldots (\Phi_2(v)(\Phi_1(v)) \ldots)$ is a $k$-linear Hilbert–Schmidt operator mapping from $U_0 \times \ldots \times U_0$ to $H$ for all $v \in H$ such that

$$||\Phi_k(Z) (\ldots (\Phi_2(Z) (\Phi_1(Z)e_{r_1}) e_{r_2}) \ldots) e_{r_k} ||_H^2 \leq L_k < \infty$$

w. p. 1 for all $r_1, r_2, \ldots, r_k \in J_M, M \in \mathbb{N}$. Then
\[ M \left\{ \left\| I[\Phi^{(k)}(Z), \psi^{(k)}]_{T,t}^M - I[\Phi^{(k)}(Z), \psi^{(k)}]_{T,t}^{M,p_1...p_k} \right\|_H^2 \right\} \leq \]

\[ \leq L_k(k!)^2 (\text{tr } Q)^k \left( I_k - \sum_{j_1=0}^{p_1} \ldots \sum_{j_k=0}^{p_k} C_{j_k...j_1}^2 \right), \]

where \( I_k \) is defined by (32), \( \text{tr } Q = \sum_{i \in J} \lambda_i \), and

\[ C_{j_k...j_1} = \int_{[t,T]^k} K(t_1, \ldots, t_k) \prod_{l=1}^{k} \phi_{j_l}(t_l) dt_1 \ldots dt_k, \]

\[ K(t_1, \ldots, t_k) = \begin{cases} 
\psi_1(t_1) \ldots \psi_k(t_k), & t_1 < \ldots < t_k \\
0, & \text{otherwise} 
\end{cases} \]

**Remark 2.** It should be noted that the right-hand side of the inequality (47) is independent of \( M \) and tends to zero if \( p_1, \ldots, p_k \to \infty \) due to the Parseval equality.

### 5. Approximation of Iterated Stochastic Integrals From the Exponential Milstein and Wagner–Platen Schemes for SPDEs

This section is devoted to the approximation of iterated stochastic integrals from the Milstein scheme (5) and Wagner–Platen scheme (6) for SPDEs. These integrals have the following form

\[ J_1[B(Z)]_{T,t} = \int_t^T B(Z) dW_s, \]

\[ J_2[B(Z)]_{T,t} = A \left( \int_t^s \int_t^s B(Z) dW_{r,s} - \frac{(T-t)}{2} \int_t^T B(Z) dW_s \right), \]

\[ J_3[B(Z), F(Z)]_{T,t} = \int_t^T B'(Z) \left( \int_t^{t_2} \left( AZ + F(Z) \right) dt_1 \right) dW_{t_2}, \]

\[ J_4[B(Z), F(Z)]_{T,t} = \int_t^T F'(Z) \left( \int_t^{t_2} B(Z) dW_{t_1} \right) dt_2, \]
\begin{align}
I_1[B(Z)]_{T,t} &= \int_t^T B'(Z) \left( \int_t^s B(Z) dW_s \right) dW_t, \\
I_2[B(Z)]_{T,t} &= \int_t^T B'(Z) \left[ \int_t^{t_2} B'(Z) \left( \int_t^s B(Z) dW_t \right) dW_t \right] dW_{t_2}, \\
I_3[B(Z)]_{T,t} &= \int_t^T B''(Z) \left( \int_t^{t_2} B(Z) dW_{t_1}, \int_t^{t_2} B(Z) dW_{t_2} \right) dW_{t_2},
\end{align}

where \( Z : \Omega \to H \) is an \( \mathcal{F}_t / \mathcal{B}(H) \)-measurable mapping, \( 0 \leq t < T \leq T \).

Note that according to (7)–(10), (35), and (36) we can write w. p. 1

\begin{align}
J_1[B(Z)]_{T,t}^M &= \int_t^T B(Z) dW_s^M = (T-t)^{1/2} \sum_{r_1 \in J_M} B(Z) e_{r_1} \sqrt{\lambda_{r_1} \zeta_0^{(r_1)}}, \\
J_2[B(Z)]_{T,t}^M &= A \left( \int_t^T \int_t^s B(Z) dW_{t_\tau} dW_t - \frac{(T-t)}{2} \int_t^T B(Z) dW_t \right) = \\
&= -\frac{(T-t)^{3/2}}{2 \sqrt{3}} \sum_{r_1 \in J_M} AB(Z) e_{r_1} \sqrt{\lambda_{r_1} \zeta_1^{(r_1)}}, \\
J_3[B(Z), F(Z)]_{T,t}^M &= \int_t^T B'(Z) \left( \int_t^{t_2} (AZ + F(Z)) dt_1 \right) dW_{t_2} = \\
&= \frac{(T-t)^{3/2}}{2} \sum_{r_1 \in J_M} B'(Z) \left( AZ + F(Z) \right) e_{r_1} \sqrt{\lambda_{r_1} \left( \zeta_0^{(r_1)} + \frac{1}{\sqrt{3}} \zeta_1^{(r_1)} \right)}, \\
J_4[B(Z), F(Z)]_{T,t}^M &= \int_t^T F'(Z) \left( \int_t^{t_2} B(Z) dW_{t_1} \right) dt_2 = \\
&= \frac{(T-t)^{3/2}}{2} \sum_{r_1 \in J_M} F'(Z) B(Z) e_{r_1} \sqrt{\lambda_{r_1} \left( \zeta_0^{(r_1)} - \frac{1}{\sqrt{3}} \zeta_1^{(r_1)} \right)},
\end{align}

where \( \zeta_0^{(r_1)}, \zeta_1^{(r_1)} \) (\( r_1 \in J_M \)) are independent standard Gaussian random variables.
Let $I_1[B(Z)]_{T,t}^{M}$, $I_2[B(Z)]_{T,t}^{M}$, $I_3[B(Z)]_{T,t}^{M}$ be approximations of stochastic integrals (52)–(54), which have the following form (see (11), (12), and (16))

$$I_1[B(Z)]_{T,t}^{M} = \int_{t}^{T} B'(Z) \left( \int_{t}^{s} B(Z) dW_{s}^{M} \right) dW_{t}^{M} = \sum_{r_1, r_2 \in J_M} B'(Z) (B(Z)e_{r_1}) e_{r_2} \sqrt{\lambda_{r_1} \lambda_{r_2}} f_{(11)T,t}^{(r_1r_2)} \times$$

$$I_2[B(Z)]_{T,t}^{M} = \int_{t}^{T} B'(Z) \left( \int_{t}^{t_2} B(Z) dW_{t_1}^{M} \right) \left( \int_{t_2}^{t_3} B(Z) dW_{t_2}^{M} \right) dW_{t_3}^{M} = \sum_{r_1, r_2, r_3 \in J_M} B'(Z) (B(Z)e_{r_1})(B(Z)e_{r_2}) e_{r_3} \sqrt{\lambda_{r_1} \lambda_{r_2} \lambda_{r_3}} \times$$

$$I_3[B(Z)]_{T,t}^{M} = \int_{t}^{T} B''(Z) \left( \int_{t}^{t_2} B(Z) dW_{t_1}^{M} \right) \left( \int_{t_2}^{t_3} B(Z) dW_{t_2}^{M} \right) \left( \int_{t_3}^{t_4} B(Z) dW_{t_3}^{M} \right) dW_{t_4}^{M} = \sum_{r_1, r_2, r_3 \in J_M} B''(Z) (B(Z)e_{r_1})(B(Z)e_{r_2}) e_{r_3} \sqrt{\lambda_{r_1} \lambda_{r_2} \lambda_{r_3}} \times$$

$$\times \left( f_{(111)T,t}^{(r_1r_2r_3)} + f_{(111)T,t}^{(r_2r_1r_3)} + f_{(01)T,t}^{(r_1r_2)} + f_{(11)T,t}^{(r_1r_2r_3)} \right).$$

Let $I_1[B(Z)]_{T,t}^{M,q}$, $I_2[B(Z)]_{T,t}^{M,q}$, $I_3[B(Z)]_{T,t}^{M,q}$ be approximations of stochastic integrals (58)–(60), which are represented as follows

$$I_1[B(Z)]_{T,t}^{M,q} = \sum_{r_1, r_2 \in J_M} B'(Z) (B(Z)e_{r_1}) e_{r_2} \sqrt{\lambda_{r_1} \lambda_{r_2}} f_{(11)T,t}^{(r_1r_2)} \times$$

$$I_2[B(Z)]_{T,t}^{M,q} = \sum_{r_1, r_2, r_3 \in J_M} B'(Z) (B'(Z)e_{r_1})(B(Z)e_{r_2}) e_{r_3} \times$$

$$\times \sqrt{\lambda_{r_1} \lambda_{r_2} \lambda_{r_3}} f_{(111)T,t}^{(r_1r_2r_3)} \times$$

$$I_3[B(Z)]_{T,t}^{M,q} = \sum_{r_1, r_2, r_3 \in J_M} B''(Z) (B(Z)e_{r_1})(B(Z)e_{r_2}) e_{r_3} \sqrt{\lambda_{r_1} \lambda_{r_2} \lambda_{r_3}} \times$$
where $q \geq 1$ and the approximations $I^{(111)T, t}_{(111)T, t}$, $I^{(r_1 r_2 r_3)q}_{(111)T, t}$, $I^{(r_2 r_1 r_3)q}_{(111)T, t}$, $I^{(r_1 r_3)q}_{(111)T, t}$ are defined by (38), (39).

Recall that $L^{1, 2}_{HS}(U_0, H)$ is a space of Hilbert–Schmidt operators mapping from $U_0$ to $H$. Let $L^{(2)}_{HS}(U_0, H)$ and $L^{(3)}_{HS}(U_0, H)$ be spaces of bilinear and 3-linear Hilbert–Schmidt operators mapping from $U_0 \times U_0$ to $H$ and from $U_0 \times U_0 \times U_0$ to $H$ correspondingly. Furthermore, let $\|\cdot\|_{L^{(2)}_{HS}(U_0, H)}$, $\|\cdot\|_{L^{(3)}_{HS}(U_0, H)}$ be operator norms in these spaces.

**Theorem 4** ([59] also see [23, 25, 60]). Let the conditions 1, 2 of Theorem 3 be fulfilled. Let $B(v)$ be a Hilbert–Schmidt operator mapping from $U_0$ to $H$ for all $v \in H$, $B'(v)$ be a nonlinear Hilbert–Schmidt operator mapping from $U_0 \times U_0$ to $H$ for all $v \in H$, and $B''(v)(B'(v))(B(v))$, $B''(v)(B(v), B(v))$ be 3-linear Hilbert–Schmidt operators mapping from $U_0 \times U_0 \times U_0$ to $H$ for all $v \in H$ (we suppose that Fréchet derivatives $B'$, $B''$ exist (see Sect. 2)). Moreover, let there exists a constant $C$ such that w. p. 1

\[
\left\|B(Z)Q^{-\alpha}\right\|_{L^{1, 2}_{HS}(U_0, H)} < C,
\left\|B'(Z)(B(Z))Q^{-\alpha}\right\|_{L^{(2)}_{HS}(U_0, H)} < C,
\left\|B''(Z)(B(Z), B(Z))Q^{-\alpha}\right\|_{L^{(3)}_{HS}(U_0, H)} < C,
\]

for some $\alpha > 0$. Then

\[
\begin{align*}
\left\|I_1[B(Z)]_{T, t} - I_1[B(Z)]_{T, t}\right\|_{M, q}^2 & \leq \left(\frac{1}{2} - \sum_{j=1}^{q} \frac{1}{4 j^2 - 1}\right) + K_Q \left(\sup_{i \in J \setminus J_M} \lambda_i\right)^{2\alpha} \\
\left\|I_2[B(Z)]_{T, t} - I_2[B(Z)]_{T, t}\right\|_{M, q}^2 & \leq \left(\frac{1}{6} - \sum_{j_1, j_2, j_3 = 0}^{q} C_{j_1 j_2 j_3}^2\right) + L_Q \left(\sup_{i \in J \setminus J_M} \lambda_i\right)^{2\alpha} \\
\end{align*}
\]
\[ M \left\{ \left\| I_3[B(Z)]_{T,t} - I_3[B(Z)]_{T,t}^{M,q} \right\|_H^2 \right\} \leq \]

\[ (T - t)^3 \left( C_2 (\text{tr } Q)^3 \left( \frac{1}{6} - \sum_{j_1,j_2,j_3=0}^q \hat{C}^2_{j_3,j_2,j_1} \right) + M_Q \left( \sup_{i \in J \setminus J_M} \lambda_i \right)^{2a} \right), \]

where \( q \in \mathbb{N}, C_0, C_1, C_2, K_Q, L_Q, M_Q < \infty, \) and

\[ \hat{C}_{j_3,j_2,j_1} = \frac{\sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)}}{8} C_{j_3,j_2,j_1}, \]

\[ \tilde{C}_{j_3,j_2,j_1} = \frac{1}{\sqrt{-1}} \int_{-1}^{z} P_{j_3}(z) \int_{-1}^{y} P_{j_2}(y) \int_{-1}^{x} P_{j_1}(x) dx dy dz, \]

where \( P_j(x) (j = 0, 1, 2, \ldots) \) is the Legendre polynomial.

**Remark 3.** Note that the estimate like \( (64) \) has been derived in \([8]\) (also see \([2]\)) with the difference connected with the first term on the right-hand side of \( (63) \). In \([8]\) the authors used the Karhunen–Loève expansion of the Brownian bridge process for the approximation of iterated Itô stochastic integrals with respect to the scalar standard Wiener processes. In this article we apply Theorem 1 and the system of Legendre polynomials for obtaining the first term on the right-hand side of \( (64) \).

**Proof.** The estimate \( (65) \) directly follows from Theorem 3 of this article (the first term on the right-hand side of \( (63) \)) and Theorem 1 from \([8]\) (the second term on the right-hand side of \( (63) \)). Further \( C_3, C_4, \ldots \) denote various constants.

Let us prove the estimates \( (64), (65) \). Using the elementary inequality \( (a + b)^2 \leq 2(a^2 + b^2) \) and Theorem 3, we obtain

\[ M \left\{ \left\| I_3[B(Z)]_{T,t} - I_3[B(Z)]_{T,t}^{M,q} \right\|_H^2 \right\} \leq \]

\[ \leq 2 \left( M \left\{ \left\| I_2[B(Z)]_{T,t} - I_2[B(Z)]_{T,t}^{M} \right\|_H^2 \right\} + M \left\{ \left\| I_2[B(Z)]_{T,t} - I_2[B(Z)]_{T,t}^{M,q} \right\|_H^2 \right\} \right) \leq \]

\[ \leq 2M \left\{ \left\| I_2[B(Z)]_{T,t} - I_2[B(Z)]_{T,t}^{M} \right\|_H^2 \right\} + C_5(T - t)^3 (\text{tr } Q)^3 \left( \frac{1}{6} - \sum_{j_1,j_2,j_3=0}^q \hat{C}^2_{j_3,j_2,j_1} \right), \]

\[ M \left\{ \left\| I_3[B(Z)]_{T,t} - I_3[B(Z)]_{T,t}^{M,q} \right\|_H^2 \right\} \leq \]
Repeating with an insignificant modification the proof of Theorem 3 for the case \( k = 3 \) (see for details [9] (pp. 39–44) or [10], [23]–[25]), we have

\[
M \left\{ \left\| I_3[B(Z)]_{T,t} - I_3[B(Z)]_{T,t}^{M,q} \right\|_H^2 \right\} \leq 4C(3!)^2 (\text{tr } Q)^3 (T-t)^3 \left( \frac{1}{6} - \sum_{j_1, j_2, j_3 = 0}^q \hat{C}_{j_3}^2 \right),
\]

where constant \( C \) has the same meaning as constant \( L_k \) in Theorem 3 (\( k \) is the multiplicity of the iterated stochastic integral).

Combining (67) and (68), we obtain

\[
M \left\{ \left\| I_3[B(Z)]_{T,t} - I_3[B(Z)]_{T,t}^{M,q} \right\|_H^2 \right\} \leq 2M \left\{ \left\| I_3[B(Z)]_{T,t} - I_3[B(Z)]_{T,t}^{M} \right\|_H^2 \right\} + C_4 (T-t)^3 (\text{tr } Q)^3 \left( \frac{1}{6} - \sum_{j_1, j_2, j_3 = 0}^q \hat{C}_{j_3}^2 \right).
\]

Let us evaluate the values

\[
M \left\{ \left\| I_2[B(Z)]_{T,t} - I_2[B(Z)]_{T,t}^{M} \right\|_H^2 \right\}, \quad M \left\{ \left\| I_3[B(Z)]_{T,t} - I_3[B(Z)]_{T,t}^{M} \right\|_H^2 \right\}.
\]

Using the elementary inequality \((a + b + c)^2 \leq 3(a^2 + b^2 + c^2)\), we have

\[
M \left\{ \left\| I_2[B(Z)]_{T,t} - I_2[B(Z)]_{T,t}^{M} \right\|_H^2 \right\} \leq 3 \left( E_{T,t}^{1,M} + E_{T,t}^{2,M} + E_{T,t}^{3,M} \right),
\]

\[
M \left\{ \left\| I_3[B(Z)]_{T,t} - I_3[B(Z)]_{T,t}^{M} \right\|_H^2 \right\} \leq 3 \left( G_{T,t}^{1,M} + G_{T,t}^{2,M} + G_{T,t}^{3,M} \right),
\]

where
We have

\[
E_{T,t}^{1,M} = M \left\{ \int_t^T \left[ \int_t^{t_2} B'(Z) \left( \int_t^{t_2} B(Z) d\left( W_{t_1} - W_{t_1}^M \right) \right) dW_{t_2} \right] dW_{t_3} \right\}_H^2,
\]

\[
E_{T,t}^{2,M} = M \left\{ \int_t^T \left[ \int_t^{t_2} B'(Z) \left( \int_t^{t_2} B(Z) dW_{t_1}^M \right) d\left( W_{t_2} - W_{t_2}^M \right) \right] dW_{t_3} \right\}_H^2,
\]

\[
E_{T,t}^{3,M} = M \left\{ \int_t^T \left[ \int_t^{t_2} B'(Z) \left( \int_t^{t_2} B(Z) dW_{t_1}^M \right) d\left( W_{t_2} - W_{t_2}^M \right) \right] dW_{t_3} \right\}_H^2,
\]

\[
G_{T,t}^{1,M} = M \left\{ \int_t^T \left[ \int_t^{t_2} B''(Z) \left( \int_t^{t_2} B(Z) dW_{t_1}, \int_t^{t_2} B(Z) d\left( W_{t_1} - W_{t_1}^M \right) \right) dW_{t_2} \right] \right\}_H^2,
\]

\[
G_{T,t}^{2,M} = M \left\{ \int_t^T \left[ \int_t^{t_2} B''(Z) \left( \int_t^{t_2} B(Z) d\left( W_{t_1} - W_{t_1}^M \right), \int_t^{t_2} B(Z) dW_{t_1}^M \right) dW_{t_2} \right] \right\}_H^2,
\]

\[
G_{T,t}^{3,M} = M \left\{ \int_t^T \left[ \int_t^{t_2} B''(Z) \left( \int_t^{t_2} B(Z) dW_{t_1}^M, \int_t^{t_2} B(Z) dW_{t_1}^M \right) d\left( W_{t_2} - W_{t_2}^M \right) \right] \right\}_H^2.
\]
\[(72) \quad \leq C_6 \int_{t_1}^{T} \int_{t_1}^{t_3} M \left\{ \left\| \int_{t_1}^{t_2} B(Z) d \left( W_{t_1} - W_{t_1}^M \right) \right\|_H^2 \right\} dt_2 dt_3 \leq \]

\[(73) \quad \leq C_6 \left( \sup_{i \in \mathcal{J} \setminus \mathcal{J}_M} \lambda_i \right)^{2\alpha} \int_{t_1}^{T} \int_{t_1}^{t_2} M \left\{ \left\| B(Z) Q^{-\alpha} \right\|_{L_{H^\alpha}(U_0,H)}^2 \right\} dt_1 dt_2 dt_3 \leq \]

\[(74) \quad \leq C_7 \left( \sup_{i \in \mathcal{J} \setminus \mathcal{J}_M} \lambda_i \right)^{2\alpha} (T - t)^3. \]

Note that the transition from (72) to (73) was made by analogy with the proof of Theorem 1 in [8] (also see [2]). More precisely, taking into account the relation \(Q^{\alpha} e_i = \lambda_i^{\alpha} e_i\), we have (see [8, Sect. 3.1])

\[M \left\{ \left\| \int_{t_1}^{t_2} B(Z) d \left( W_{t_1} - W_{t_1}^M \right) \right\|_H^2 \right\} = \]

\[= M \left\{ \left\| \sum_{i \in \mathcal{J} \setminus \mathcal{J}_M} \sqrt{\lambda_i} \int_{t_1}^{t_2} B(Z) e_i d w_{t_1}^{(i)} \right\|_H^2 \right\} = \]

\[= \sum_{i \in \mathcal{J} \setminus \mathcal{J}_M} \lambda_i \int_{t_1}^{t_2} M \left\{ \left\| B(Z) Q^{-\alpha} e_i \right\|_H^2 \right\} dt_1 = \]

\[= \sum_{i \in \mathcal{J} \setminus \mathcal{J}_M} \lambda_i^{1+2\alpha} \int_{t_1}^{t_2} M \left\{ \left\| B(Z) Q^{-\alpha} e_i \right\|_H^2 \right\} dt_1 = \]

\[= \left( \sup_{i \in \mathcal{J} \setminus \mathcal{J}_M} \lambda_i \right)^{2\alpha} \int_{t_1}^{t_2} M \left\{ \sum_{i \in \mathcal{J} \setminus \mathcal{J}_M} \lambda_i \left\| B(Z) Q^{-\alpha} e_i \right\|_H^2 \right\} dt_1 \leq \]

\[\leq \left( \sup_{i \in \mathcal{J} \setminus \mathcal{J}_M} \lambda_i \right)^{2\alpha} \int_{t_1}^{t_2} M \left\{ \sum_{i \in \mathcal{J}} \left\| B(Z) Q^{-\alpha} e_i \right\|_H^2 \right\} dt_1 = \]

\[(75) \quad = \left( \sup_{i \in \mathcal{J} \setminus \mathcal{J}_M} \lambda_i \right)^{2\alpha} \int_{t_1}^{t_2} M \left\{ \left\| B(Z) Q^{-\alpha} \right\|_{L_{H^\alpha}(U_0,H)}^2 \right\} dt_1. \]
Further we also will use the estimate like \((75)\). We have

\[ E^{2,M}_{T,t} = \]

\[ \leq C_8 \int_t^T M \left\{ \left\| B'(Z) \left( \int_t^{t_3} B(Z) d\mathbf{W}_t^M \right) d\left( \mathbf{W}_{t_2} - \mathbf{W}_t^M \right) \right\|_{L_{H^S(U_0, H)}}^2 \right\} dt_3 \leq \]

\[ \leq C_8 \left( \sup_{i \in J \setminus J_M} \lambda_i \right)^{2\alpha} \int_t^T \int_t^{t_3} M \left\{ \left\| B'(Z) \left( \int_t^{t_2} B(Z) d\mathbf{W}_t^M \right) Q^{-\alpha} \right\|_{L_{H^S(U_0, H)}}^2 \right\} dt_2 dt_3 \leq \]

\[ \leq C_9 \left( \sup_{i \in J \setminus J_M} \lambda_i \right)^{2\alpha} (T-t)^3, \]

\[ E^{3,M}_{T,t} \leq \left( \sup_{i \in J \setminus J_M} \lambda_i \right)^{2\alpha} \times \]

\[ \int_t^T M \left\{ \left\| B'(Z) \left( \int_t^{t_3} B'(Z) \left( \int_t^{t_2} B(Z) d\mathbf{W}_t^M \right) d\mathbf{W}_t^M \right) Q^{-\alpha} \right\|_{L_{H^S(U_0, H)}}^2 \right\} dt_3 \leq \]

\[ \leq C_{10} \left( \sup_{i \in J \setminus J_M} \lambda_i \right)^{2\alpha} \int_t^T \int_t^{t_3} M \left\{ \left\| B'(Z) \left( B'(Z) (B(Z)) Q^{-\alpha} \right\|_{L_{H^S(U_0, H)}}^2 \right\} \left( \frac{(t_3-t)^2}{2} \right) dt_3 \leq \]

\[ \leq C_{11} \left( \sup_{i \in J \setminus J_M} \lambda_i \right)^{2\alpha} (T-t)^3. \]

Combining \((66), (70), (74) - (77)\), we obtain \((64)\). We have
Let us estimate the right-hand side of (78). Let \( s > t \). For fixed \( M \in \mathbb{N} \) and for some \( N > M \) \((N \in \mathbb{N})\) we have

\[
G_{T,t}^{1,M} = \int_{t}^{T} M \left\{ \left\| B''(Z) \left( \int_{t}^{t_2} B(Z) dW_{t_1}, \int_{t}^{t_2} \|B(Z) d(\bar{W}_{t_1} - W_{t_1}^M)\| \right) \right\|_{L_H s(U_0,H)}^2 \right\} dt_3 \leq \]

\[
\leq C_{12} \int_{t}^{T} M \left\{ \left\| \int_{t}^{t_2} B(Z) dW_{t_1} \right\|_{H}^4 \right\}^{1/2} \left( M \left\{ \left\| \int_{t}^{t_2} B(Z) d(\bar{W}_{t_1} - W_{t_1}^M) \right\|_{H}^4 \right\} \right)^{1/2} \right\} dt_3 \leq \]

\[
\leq C_{13} \int_{t}^{T} \left( M \left\{ \left\| B(Z) \right\|_{L_H s(U_0,H)}^4 \right\} \right)^{1/2} \left( M \left\{ \left\| B(Z) \right\|_{H}^4 \right\} \right)^{1/2} \right\} dt_3 \leq \]

\[
\leq C_{14} \int_{t}^{T} \left( t_2 - t \right) \left( M \left\{ \left\| \int_{t}^{t_2} B(Z) d(\bar{W}_{t_1} - W_{t_1}^M) \right\|_{H}^4 \right\} \right)^{1/2} \right\} dt_3. \]

(78)

Let us estimate the right-hand side of (78). Let \( s > t \). For fixed \( M \in \mathbb{N} \) and for some \( N > M \) \((N \in \mathbb{N})\) we have

\[
M \left\{ \left\| \int_{t}^{s} B(Z) d(\bar{W}_{t_1}^N - W_{t_1}^M) \right\|_{H}^4 \right\} = \]

\[
= M \left\{ \left\| \sum_{j \in J_N \setminus J_M} \sqrt{\lambda_j} B(Z) e_j (w_s^{(j)} - w_t^{(j)}) + \sum_{j' \in J_N \setminus J_M} \sqrt{\lambda_{j'}} B(Z) e_{j'} (w_s^{(j')} - w_t^{(j')}) \right\|_{H}^2 \right\} \]

\[
= \sum_{j,j',t' \in J_N \setminus J_M} \sqrt{\lambda_j \lambda_{j'} \lambda_t \lambda_{t'}} M \left\{ \left\langle B(Z) e_j, B(Z) e_{j'} \right\rangle_H \left\langle B(Z) e_t, B(Z) e_{t'} \right\rangle_H \right\} \times \]

\[
\times M \left\{ (w_s^{(j)} - w_t^{(j)}) (w_s^{(j')} - w_t^{(j')}) (w_s^{(t)} - w_t^{(t)}) (w_s^{(t')} - w_t^{(t')}) \right\} F_t \right\} = \]
\begin{align*}
& = 3(s - t)^2 \sum_{j \in J_N \setminus J_M} \lambda_j^2 M \left\{ \left\| B(Z)e_j \right\|_H^4 \right\} + \\
& + (s - t)^2 \sum_{j, j' \in J_N \setminus J_M (j \neq j')} \lambda_j \lambda_{j'} \left( M \left\{ \left\| B(Z)e_j \right\|_H^2 \left\| B(Z)e_{j'} \right\|_H^2 \right\} + 2 \left\langle B(Z)e_j, B(Z)e_{j'} \right\rangle \right) \leq \\
& \leq 3(s - t)^2 \left( \sum_{j \in J_N \setminus J_M} \lambda_j^2 M \left\{ \left\| B(Z)e_j \right\|_H^4 \right\} + \\
& + \sum_{j, j' \in J_N \setminus J_M (j \neq j')} \lambda_j \lambda_{j'} M \left\{ \left\| B(Z)e_j \right\|_H^2 \left\| B(Z)e_{j'} \right\|_H^2 \right\} \right) = \\
& = 3(s - t)^2 M \left\{ \left( \sum_{j \in J_N \setminus J_M} \lambda_j \left\| B(Z)e_j \right\|_H^2 \right)^2 \right\} \leq \\
& \leq 3(s - t)^2 \left( \sup_{i \in J_N \setminus J_M} \lambda_i \right)^{4\alpha} M \left\{ \left( \sum_{j \in J_N \setminus J_M} \lambda_j \left\| B(Z)Q^{-\alpha}e_j \right\|_H^2 \right)^2 \right\} \leq \\
& \leq C_{15} (s - t)^2 \left( \sup_{i \in J_N \setminus J_M} \lambda_i \right)^{4\alpha} M \left\{ \left\| B(Z)Q^{-\alpha} \right\|_{L_H \cap \{U_0, H\}}^4 \right\}.
\end{align*}

Performing the passage to the limit \( \lim_{N \to \infty} \) in (79) and using (78), we have

\begin{align*}
(80) \quad & G_{T,t}^{1,M} \leq C_{16} \left( \sup_{i \in J \setminus J_M} \lambda_i \right)^{2\alpha} (T - t)^3.
\end{align*}

Absolutely analogously we obtain

\begin{align*}
(81) \quad & G_{T,t}^{2,M} \leq C_{17} \left( \sup_{i \in J \setminus J_M} \lambda_i \right)^{2\alpha} (T - t)^3.
\end{align*}

Let us estimate \( G_{T,t}^{3,M} \). We have
\[ G_{T,t}^{3,M} \leq \left( \sup_{i \in J \setminus J_M} \lambda_i \right)^{2\alpha} \times \]

\[ \times \int_t^T M \left\{ \left\| B''(Z) \left( \int_t^{t_2} B(Z) dW_{t_1} - \int_t^{t_1} B(Z) dW_{t_1} \right) \right\|_{L^{\infty}(U_0,H)}^2 \right\} dt_2 \leq \]

\[ \leq \left( \sup_{i \in J \setminus J_M} \lambda_i \right)^{2\alpha} \sum_{i \in J} \sum_{j,l \in J_M} \lambda_i \lambda_j \lambda_l \int_t^T (t_2 - t)^2 \times \]

\[ \times \left( M \left\{ \left\| B''(Z)(B(Z)e_j, B(Z)e_l) Q^{-\alpha} e_i \right\|_H \right\} + \right. \]

\[ + M \left\{ \left\| B''(Z)(B(Z)e_j, B(Z)e_l) Q^{-\alpha} e_i \right\|_H \times \right. \]

\[ \times \left\| B''(Z)(B(Z)e_l, B(Z)e_l) Q^{-\alpha} e_i \right\|_H \left. \right\} \right\} dt_2 \leq \]

\[ \leq C_{18} \left( \sup_{i \in J \setminus J_M} \lambda_i \right)^{2\alpha} (T - t)^3. \]

Combining (69), (71), and (80)–(82), we get (65). Theorem 4 is proved.

Let us consider the convergence analysis for the stochastic integrals (49)–(51) (convergence of the stochastic integral (48) follows from (75) (see Theorem 1 in [8] or [2])).

Using the Itô formula, we obtain w. p. 1 [3]

\[ J_2[B(Z)]_{T,t} = \int_t^T \left( \frac{T}{2} - s + \frac{t}{2} \right) AB(Z) dW_s, \]

\[ J_3[B(Z), F(Z)]_{T,t} = \int_t^T (s - t) B'(Z) \left( AZ + F(Z) \right) dW_s. \]
Suppose that

$$M \left\{ \left\| B'(Z) \left( AZ + F(Z) \right) Q^{-\alpha} \right\|^2_{LHS(U_0, H)} \right\} < \infty,$$

$$M \left\{ \left\| A B(Z) Q^{-\alpha} \right\|^2_{LHS(U_0, H)} \right\} < \infty$$

for some $\alpha > 0$.

Then by analogy with (75) we get

$$M \left\{ \left\| J_2[B(Z)]_{T, t} - J_2[B(Z)]_{T, t}^M \right\|^2_H \right\} \leq$$

$$\leq C_{19}(T - t)^3 \left( \sup_{i \in J \setminus J_M} \lambda_i \right)^{2\alpha},$$

$$M \left\{ \left\| J_3[B(Z), F(Z)]_{T, t} - J_3[B(Z), F(Z)]_{T, t}^M \right\|^2_H \right\} \leq$$

$$\leq C_{20}(T - t)^3 \left( \sup_{i \in J \setminus J_M} \lambda_i \right)^{2\alpha},$$

where $J_2[B(Z)]_{T, t}^M$, $J_3[B(Z), F(Z)]_{T, t}^M$ are defined by (55), (56).

Moreover, in conditions of Theorem 4 we obtain

$$M \left\{ \left\| J_4[B(Z), F(Z)]_{T, t} - J_4[B(Z), F(Z)]_{T, t}^M \right\|^2_H \right\} =$$

$$= M \left\{ \left\| \int_t^T F'(Z) \left( \int_t^{t_2} B(Z) d \left( W_{t_1} - W_{t_1}^M \right) \right) dt_2 \right\|^2_H \right\} \leq$$

$$\leq (T - t) \int_t^T M \left\{ \left\| F'(Z) \left( \int_t^{t_2} B(Z) d \left( W_{t_1} - W_{t_1}^M \right) \right) \right\|^2_H \right\} dt_2 \leq$$
\[
\leq C_{21}(T-t) \int_t^T M \left\{ \left\| B(Z) d (W_{t_1} - W_{t_1}^M) \right\|_H^2 \right\} dt_2 \leq
\]

\[
\leq C_{21}(T-t) \left( \sup_{i \in J \setminus J_M} \lambda_i \right)^{2\alpha} T \int_t^{t_2} M \left\{ \left\| B(Z) Q^{-\alpha} \right\|_{L_{HS}(U_0,H)}^2 \right\} dt_1 dt_2 \leq
\]

\[
\leq C_{22}(T-t)^3 \left( \sup_{i \in J \setminus J_M} \lambda_i \right)^{2\alpha},
\]

where \( J_4(B(Z), F(Z))_{t_1,t} \) is defined by \([57]\).

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