Exact Solution of the $N$-dimensional Radial Schrödinger Equation with Pseudoharmonic Potential via Laplace Transform Approach

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Abstract

The second order $N$-dimensional Schrödinger equation with pseudoharmonic potential is reduced to a first order differential equation by using the Laplace transform approach and exact bound state solutions are obtained. Our results generalize earlier work done for various potential combinations in the case of lower dimensions.

Keywords: Laplace Transformation Approach, Exact solution, Pseudoharmonic potential, Schrödinger Equation

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I. INTRODUCTION

Schrödinger equation has long been recognised as an essential tool for the study of atoms, nuclei, molecules and their spectral behaviors. Much effort has been put in to find the exact bound state solution of this non-relativistic equation for various potentials describing the nature of bonding or the nature of vibration of quantum systems. A large number of research workers all around the world continue to study the ever fascinating Schrödinger equation, which has wide application over vast areas of theoretical physics. The Schrödinger equation is traditionally solved by with operator algebraic method [1], power series method [2-3] or path integral method [4]. Recently, the Fourier transform method has been used by several authors [5-7] to solve the Schrödinger equation for one dimensional harmonic oscillator. In addition the Morse and Mie-type potentials, pseudoharmonic potential and Delta function potential have also generated a lot of interest in the scientific community.

There are various alternative methods in the literature to solve Schrödinger equation, but one of the most efficient and economic way is the Laplace transformation approach (LTA). The success of this approach lies in reduction of the second order differential equations into first order equations, which can then be solved by standard methods. LTA was first used by Schrödinger to derive the radial eigenfunctions of the Hydrogen atom [8]. Englefield [9] used this technique to solve the Coulomb problem. Using the same methodology, the Schrödinger equation has also been solved for various other potentials, namely, pseudoharmonic, Mie-type [10], Dirac delta [11], Morse-type [12] and quantum harmonic oscillator [13].

Recently, N-dimensional Schrödinger equations have received focal attention in literature. The hydrogen atom in five dimensions and isotropic oscillator in eight dimensions have been discussed by Davtyan and co-workers [14]. Chatterjee has reviewed [15] several methods commonly adopted for the study of N dimensional Schrödinger equations in the large N limit, where a relevant 1/N expansion can be used. Later Yanez and co-workers have investigated the position and momentum information entropies of N-dimensional system [16]. The quantization of angular momentum in N-dimension has been described by Al-Jaber [17]. Other recent studies of Schrödinger equation in higher dimension include isotropic harmonic oscillator plus inverse quadratic potential [18], N-dimensional radial Schrödinger equation with the Coulomb potential [19]. In the present article we discuss the exact solutions of the N-dimensional radial Schrödinger equation with pseudoharmonic potential using Laplace
II. BOUND STATE SPECTRUM

The time independent Schrödinger equation for a particle of mass $M$ in $N$-dimensional space has the form [15, 20]

$$-\frac{\hbar^2}{2M} \nabla^2_N \psi + V\psi = E\psi,$$  \hspace{1cm} (1)

where $\nabla^2_N$ is the Laplacian operator in the polar coordinates $(r, \theta_1, \theta_2, \ldots, \theta_{N-2}, \varphi)$ of $R^N$. Here $r$ is the hyperradius and $\theta_1, \theta_2, \ldots, \theta_{N-2}, \varphi$ are the hyperangles. The form of $\nabla^2_N$ is given by

$$\nabla^2_N = r^{1-N} \frac{\partial}{\partial r} \left( r^{N-1} \frac{\partial}{\partial r} \right) + \Lambda^2_N(\Omega) \frac{1}{r^2},$$ \hspace{1cm} (2)

where $\Lambda^2_N(\Omega)$ is the hyperangular momentum operator [15, 21], given by,

$$\Lambda^2_N = -\sum_{i,j=1 \atop i \neq j}^N \Lambda_{ij}, \quad \Lambda_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i},$$

for all Cartesian components $x_i$ of the $N$-dimensional vector $(x_1, x_2, \ldots, x_N)$.

Applying the separation variable method [18] in Eq. (1) and taking the separation constant as $\ell(\ell + N - 2)$ [22] with $\ell = 0, 1, 2, \ldots$, the $N$-dimensional hyperradial or in short the "radial" Schrödinger equation becomes

$$\left[ \frac{d^2}{dr^2} + \frac{N-1}{r} \frac{d}{dr} - \frac{\ell(\ell + N - 2)}{r^2} - \frac{2M}{\hbar^2} [V(r) - E] \right] R(r) = 0,$$ \hspace{1cm} (3)

where $E$ is the energy eigenvalue and $\ell$ is the orbital angular momentum quantum number.

Inserting the pseudoharmonic potential having the form [10]

$$V(r) = a_1 r^2 + \frac{a_2}{r^2} + a_3,$$ \hspace{1cm} (4)

into Eq. (3) and using the following abbreviations

$$\nu(\nu + 1) = \ell(\ell + N - 2) + \frac{2M}{\hbar^2} a_2; \quad \mu^2 = \frac{2M}{\hbar^2} a_1; \quad \epsilon^2 = \frac{2M}{\hbar^2} (E - a_3),$$ \hspace{1cm} (5)

we obtain

$$\left[ \frac{d^2}{dr^2} + \frac{N-1}{r} \frac{d}{dr} - \frac{\nu(\nu + 1)}{r^2} - \mu^2 r^2 + \epsilon^2 \right] R(r) = 0.$$ \hspace{1cm} (6)
Assuming a solution of type \( R(r) = r^{-k}f(r) \) with \( k > 0 \) and changing the variable as \( y = r^2 \) and taking \( f(r) = \chi(y) \), we obtain

\[
4y \frac{d^2 \chi}{dy^2} + 2(N - 2k) \frac{d \chi}{dy} + \left[ \frac{k(k + 1) - k(N - 1) - \nu(\nu + 1)}{y} - \mu^2 y + \epsilon^2 \right] \chi = 0, \tag{7}
\]

In order to get the Laplace transform of the above differential equation in usual form we require a restriction on the parameters as

\[
k(k + 1) - k(N - 1) - \nu(\nu + 1) = 0, \tag{8}
\]

Hence we have

\[
y \frac{d^2 \chi}{dy^2} - \left( k - \frac{N}{2} \right) \frac{d \chi}{dy} - \frac{1}{4}(\mu^2 y - \epsilon^2) \chi = 0, \tag{9}
\]

Introducing the Laplace transform \( F(t) = \mathcal{L}\{\chi(y)\} \) [23, 27, 28] Eq. (9) can read

\[
\left( t^2 - \frac{\mu^2}{4} \right) \frac{dF}{dt} + \left[ \left( k - \frac{N}{2} + 2 \right) t - \frac{\epsilon^2}{4} \right] F = 0, \tag{10}
\]

The solution of the last equation can be written easily as

\[
F(t) = C \left( t + \frac{\mu}{2} \right)^{-\frac{\epsilon^2}{4\mu} - \frac{1}{2}(k - \frac{N}{2} + 2)} \left( t - \frac{\mu}{2} \right)^{\frac{\epsilon^2}{4\mu} - \frac{1}{2}(k - \frac{N}{2} + 2)}, \tag{11}
\]

In order to get a finite wave function it should be [10, 12, 19]

\[
\frac{\epsilon^2}{4\mu} - \frac{1}{2}(k - \frac{N}{2} + 2) = n, n = 0, 1, 2, \ldots \tag{12}
\]

In this manner

\[
F(t) = C' \sum_{j=0}^{n} \frac{(-1)^j n!}{(n - j)! j!} \left( t + \frac{\mu}{2} \right)^{-(k - \frac{N}{2} + 2 + j)} \tag{13}
\]

So using the inverse Laplace transformation such as \( \chi(y) = \mathcal{L}^{-1}\{F(t)\} \) [23, 27, 28]

\[
\chi(y) = C'' \sum_{j=0}^{n} \frac{(-1)^j n!}{(n - k)! k!} \Gamma(k - \frac{N}{2} + 2) y^{k - \frac{N}{2} + 1 + j} e^{-\frac{\mu}{2} y}, \tag{14}
\]

Now the above functions could be written by using the confluent hypergeometric functions [18, 27]

\[
\chi(y) = C_n e^{-\frac{\mu}{2} y} y^{k - \frac{N}{2} + 1} F_1\left(-n, k - \frac{N}{2} + 2, y\right), \tag{15}
\]
So the radial wave function is given as
\[ R_{n\ell}(r) = C_{n\ell}e^{-\frac{a}{2}r^2}r^{k-N+2} F_1(-n, k - \frac{N}{2} + 2, r^2). \] (16)
where the normalization constant \( C_{n\ell} \) can be evaluated from the condition [18]
\[ \int_0^{\infty} [R_{n\ell}(r)]^2 r^{N-1}dr = 1, \] (17)
The formula \( F(-q, a + 1, b) = \frac{a!}{(q+a)!}L_q^a(b) \) where \( L_q^a(b) \) are the Laguerre polynomials [27] can be useful here and matching as \( q = n, \ a = k - \frac{N}{2} + 1, \ b = r^2 \) we have
\[ _1F_1(-n, k - \frac{N}{2} + 2, r^2) = \frac{n!(k - \frac{N}{2} + 1)!}{(k - \frac{N}{2} + 1 + n)!} L_n^{(k - \frac{N}{2} + 1)}(r^2), \] (18)
So using the following formula for the Laguerre polynomials
\[ \int_0^{\infty} x^A e^{-x^2} [L_B^A(x)]^2 dx = \frac{\Gamma(A + B + 1)}{B!}, \] (19)
we write the normalization constant
\[ C_{n\ell} = \sqrt{2} \mu^{\frac{1}{2}(k-\frac{N}{2}+2)} \sqrt{\frac{n!}{\Gamma(k - \frac{N}{2} + n + 2)}} \frac{(k - \frac{N}{2} + 1 + n)!}{n!(k - \frac{N}{2} + 1)!}. \] (20)
Finally, the energy eigenvalues are obtained from Eq. (12) as
\[ E_{n\ell} = \frac{\hbar^2}{2M} e^2 + a_3 = a_3 + \sqrt{8\hbar^2 a_1 M \left[ n + \frac{1}{2} + \frac{1}{4} \sqrt{(N + 2\ell - 2)^2 + \frac{8M a_2}{\hbar^2}} \right]}, \] (21)
and we write the corresponding normalized wave functions as
\[ R_{n\ell}(r) = \sqrt{2} \mu^{\frac{1}{2}(k-\frac{N}{2}+2)} \sqrt{\frac{n!}{\Gamma(k - \frac{N}{2} + n + 2)}} \frac{(k - \frac{N}{2} + 1 + n)!}{n!(k - \frac{N}{2} + 1)!} \times e^{-\frac{a}{2}r^2} r^{(k-N+2)} F_1(-n, k - \frac{N}{2} + 2, r^2), \] or
\[ R_{n\ell}(r) = \sqrt{2} \mu^{\frac{1}{2}(k-\frac{N}{2}+2)} \sqrt{\frac{n!}{\Gamma(k - \frac{N}{2} + n + 2)}} \times e^{-\frac{a}{2}r^2} r^{(k-N+2)} L_n^{(k - \frac{N}{2} + 1)}(r^2). \] (22)
The complete orthonormalized energy eigenfunctions of the \( N \)-dimensional Schrödinger equation with pseudoharmonic potential can be given by
\[ \psi(r, \theta_1, \theta_2, \ldots, \theta_{N-2}, \phi) = \sum_{n, \ell, m} C_{n\ell} R_{n\ell}(r) Y^m_\ell(\theta_1, \theta_2, \ldots, \theta_{N-2}, \phi), \] (23)
where \( Y^m_\ell(\theta_1, \theta_2, \ldots, \theta_{N-2}, \phi) \equiv Y^m_\ell(\Omega) \) are the hyperspherical harmonics of degree \( \ell \) on the \( S^{N-1} \) sphere. These harmonics are the root of the equation
\[ \Lambda^2_N(\Omega) Y^m_\ell(\Omega) + \ell(\ell + N - 2) Y^m_\ell(\Omega) = 0. \] (24)
which is the separated part of Eq. (1).
III. DEPENDENCY OF THE SOLUTIONS ON $N$ AND $a_1, a_2, a_3$

1. Isotropic Harmonic Oscillator

(a) Three dimensions ($N=3$).

For this case $a_1 = \frac{1}{2} M \omega^2$ and $a_2 = a_3 = 0$ which gives from Eq. (21)

$$E_{n\ell} = \hbar \omega \left( 2n + \ell + \frac{3}{2} \right),$$

(25)

From Eq. (5) and Eq. (8) we obtain $k = \ell + 1$, and get radial eigenfunctions from Eq. (22). The result agrees with those obtained in Ref. [18].

(b) Arbitrary $N$ dimensions.

Here $N$ is an arbitrary constant and $a_1 = \frac{1}{2} M \omega^2, a_2 = a_3 = 0$. We have the energy eigenvalues from Eq. (21)

$$E_{n\ell} = \hbar \omega \left( 2n + \ell + \frac{N}{2} \right),$$

(26)

Solving Eq. (5) and Eq. (8) with positive sense gives $k = \ell + N - 2$ and one can easily obtain the normalization constant as

$$C_{n\ell} = \left[ \frac{2(M\omega)^{(\ell+N/2)n!}}{\Gamma(\ell+N/2+n)} \right]^{1/2} \frac{(n + \ell + N - 2)!}{n!(\ell + N - 2)!}.$$

(27)

The radial wave functions are given in Eq. (22) with the normalization constant. The results obtained here agree with those found in some earlier work [18], [25] and [26].

2. Isotropic Harmonic Oscillator plus Inverse Quadratic Potential

(a) Two dimensions ($N=2$).

Here we have $a_1 = \frac{1}{2} M \omega^2, a_2 \neq 0$ and $a_3 = 0$ where $\omega$ is the circular frequency of the particle, so from Eq. (20) we obtain

$$C_{n\ell} = \left[ \frac{2(M\omega)^{(\ell+N/2)n!}}{\Gamma(k+n+1)} \right]^{1/2} \frac{(k+n)!}{n!k!},$$

(28)

where $k$ can be obtained from Eq. (5) and Eq. (8) giving $k = \sqrt{\nu(\nu + 1)} = \sqrt{\ell^2 + \frac{2M}{\hbar^2} a_2}$. Hence Eq. (21) gives the energy eigenvalues of the system

$$E_{n\ell} = \hbar \omega (2n + k + 1),$$

(29)

This result have already been obtained in Refs. [18] and [24].

6
(b) Three dimensions (N=3).

Here \( a_1 = \frac{1}{2} M \omega^2 \), \( a_2 \neq 0 \) and \( a_3 = 0 \) which gives the energy eigenvalues as

\[
E_{nl} = \frac{\hbar \omega}{2} \left[ 4n + 2 + \sqrt{(2\ell + 1)^2 + \frac{8Ma_2}{\hbar^2}} \right], \tag{30}
\]

Solving Eq. (8) we get \( k = \nu + 1 \) and the radial wave function can hence be obtained from Eq. (22) which also corresponds to the result obtained in Ref. [18].

3. 3-Dimensional Schrödinger Equation with Pseudoharmonic Potential

Here Eq. (8) gives \( k = \nu + 1 \) \((k > 0)\) with

\[
C_{nl} = \mu^{\frac{1}{2}(\nu+\frac{3}{2})} \sqrt{2\Gamma(\nu+n+\frac{3}{2})} \frac{1}{n!} \left[ \Gamma \left( \nu + \frac{3}{2} \right) \right]^{-1}, \tag{31}
\]

and the normalized wave functions

\[
R_{nl}(r) = \mu^{\frac{1}{2}(\nu+\frac{3}{2})} \sqrt{2\Gamma(\nu+n+\frac{3}{2})} \frac{1}{n!} \left[ \Gamma \left( \nu + \frac{3}{2} \right) \right]^{-1} e^{-\frac{\mu}{2} r^2} r^{\nu+1} _1F_1(-n,\nu+\frac{3}{2},r^2) \tag{32}
\]

with the energy eigenvalues

\[
E_{nl} = a_3 + \frac{\hbar^2}{2M} \epsilon^2 = a_3 + \sqrt{\frac{8\hbar^2 a_1}{M}} \left[ n + \frac{1}{2} + \frac{1}{4} \sqrt{(2\ell + 1)^2 + \frac{8Ma_2}{\hbar^2}} \right], \tag{33}
\]

This result corresponds exactly to the ones given in Ref. [10].

IV. CONCLUSIONS

We have investigated some aspects of N-dimensional hyperradial Schrödinger equation for pseudoharmonic potential by Laplace transformation approach. It is found that the energy eigenfunctions and the energy eigenvalues depend on the dimensionality of the problem. The present work generalises several earlier results. The general results obtained in this article have been verified with earlier reported results, which were obtained for certain special values of potential parameters and dimensionality.
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