The space of nodal curves of type $p, q$
with given Weierstraße semigroup

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Abstract

We continue the investigation of curves of type $p, q$ started in [KKW]. We study the space of such curves and the space of nodal curves with prescribed Weierstraße semigroup. A necessary and sufficient criterion for a numerical semigroup to be a Weierstraße semigroup is given. We find a class of Weierstraße semigroups which apparently has not yet been described in the literature.

Introduction

Let $K$ be an algebraically closed field of characteristic 0. For relatively prime numbers $p, q \in \mathbb{N}$ with $1 < p < q$ a plane curve $C$ of type $p, q$ is the zero-set of a Weierstraß polynomial of type $p, q$

$$F(X, Y) := Y^p + bX^q + \sum_{\nu p + \mu q < pq} b_{\nu \mu} X^\nu Y^\mu \ (b_{\nu \mu} \in K, b \in K \setminus \{0\})$$

in $\mathbb{A}^2(K)$. Such curves are irreducible and have only one place $P$ at infinity, i.e. $P$ is the only point at infinity of the normalization of the projective closure $R$ of $C$. The Weierstraß semigroup $H(P)$ of $R$ at $P$ is also called the Weierstraß semigroup of $C$. It contains the semigroup $H_{pq}$ generated by $p$ and $q$ as a subsemigroup. Hence $H(P)$ is obtained from $H_{pq}$ by closing some of its $d := 1/2(p - 1)(q - 1)$ gaps. Remember that $H_{pq}$ is a symmetric semigroup with conductor $c := (p - 1)(q - 1)$. It is shown in [KKW] that any Weierstraß semigroup is the Weierstraß semigroup of a plane curve of type $p, q$ having only nodes as singularities if $p$ and $q$ are properly chosen.

By the substitution $X \mapsto 1/\sqrt{b} \cdot X, Y \mapsto Y$ the polynomial $F$ goes over into a normed Weierstraß polynomial of type $p, q$

$$Y^p - X^q + \sum_{\nu p + \mu q < pq} a_{\nu \mu} X^\nu Y^\mu$$

whose zero-set it isomorphic to $C$ and has the same place at infinity and the same Weierstraß semigroup. We call it the associated normed curve of $C$. In this paper we understand by curves of type $p, q$ the plane curves defined by normed Weierstraß polynomials of type $p, q$.

These curves can be identified with the points $\left(\{a_{\nu \mu}\}_{\nu p + \mu q < pq}\right) \in \mathbb{A}^n(K)$ associated with their equation where $n := 1/2(p + 1)(q + 1) - 1$. In Section 1 we describe the (locally closed)

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subsets of $\mathbb{A}^n(K)$ consisting of the various kinds of curves of type $p, q$. In particular we are interested in the set of nodal curves of type $p, q$. Such curves have at most $d$ nodes, and their Weierstraß semigroup has genus $g = d - l$ if $l$ is the number of the nodes. The singular nodal curves form a dense open set of an irreducible hypersurface $H \subset \mathbb{A}^n(K)$ whose properties are the main object of study in Section 1. It turns out that for any $l \in \{0, \ldots, d\}$ a nodal curve of type $p, q$ exists having exactly $l$ nodes (Theorem 1.6).

Given a numerical semigroup $H$ with $p \in H$ greater than the elements of a minimal system of generators of $H$ we construct in Section 2 a locally closed subset $V_{pq}(H)$ in some affine space, such that $H$ is a Weierstraß semigroup if and only if $V_{pq}(H) \neq \emptyset$ (Corollary 2.4). The set $V_{pq}(H)$ is explicitly described by polynomial vanishing and non-vanishing conditions, where "explicit" means that the polynomials are given by a formula or there is an algorithm to compute them. In principle the membership test for polynomial ideals allows then to decide whether $H$ is a Weierstraß semigroup or not. However for any $H$ of interest (i.e. where the result is not known) the number of conditions is huge so that the criterion seems only to be of theoretical interest and not feasible for a computer program.

By the simplification of nodal curves introduced in Section 3 the criterion allows to show without computations that every $H$ of the following kind is a Weierstraß semigroup: $H$ is obtained from the semigroup $H_{pq}$ generated by $p$ and $q$ ($1 < p < q$, with $p, q$ relatively prime) by closing all gaps of $H_{pq}$ which are greater than or equal to a given gap of $H_{pq}$ (Theorem 3.2). The hyperordinary semigroups defined by Rim and Vitulli [RV] belong to this class of semigroups. These authors have shown with a different method that hyperordinary semigroups are Weierstraß semigroups.

1 The space of plane curves of type $p, q$

Let $R := K[\{A_{\nu \mu}\}_{\nu p + \mu q < pq}]$ be the polynomial ring in the $n = \frac{1}{2}(p + 1)(q + 1) - 1$ indeterminates $A_{\nu \mu}$ ($\nu p + \mu q < pq$) over $K$. The generic (normed) Weierstraß polynomial

$$F = Y^p - X^q + \sum_{\nu p + \mu q < pq} A_{\nu \mu} X^\nu Y^\mu = A_{00} + \ldots$$

of type $p, q$ has the partial derivatives

$$F_X = -qX^{q-1} + \sum_{\nu p + \mu q < pq} \nu A_{\nu \mu} X^{\nu-1} Y^\mu = A_{10} + \ldots$$

$$F_Y = pY^{p-1} + \sum_{\nu p + \mu q < pq} \mu A_{\nu \mu} X^\nu Y^{\mu-1} = A_{01} + \ldots$$

where the dots represent terms containing $X$ or $Y$. We are interested in the ring

$$A = R[X, Y]/(F, F_X, F_Y)$$

as an $R$-Algebra. As a $K$-algebra it is isomorphic to the polynomial ring

$$K[\{A_{\nu \mu}\}_{(\nu, \mu) \neq (0, 0), (1, 0), (0, 1), X, Y}]$$
hence the image $R'$ of $R$ in $A$ is a domain. Moreover $\{F, F_X, F_Y\}$ is a regular sequence in $R[X, Y]$.

We endow $R[X, Y]$ with the grading given by $\deg(X) = p$, $\deg(Y) = q$ and $\deg(r) = 0$ for $r \in R$ and let $\mathcal{F}$ denote the corresponding degree filtration. Let $\mathcal{N} := R[X, Y]/(F_X, F_Y)$. The polynomial $F$ has degree form $Y^p - X^q$, and since the partial derivatives are homogeneous maps the degree form of $F_X$ is $-qX^{q-1}$ and that of $F_Y$ is $pY^{p-1}$. Since they form a regular sequence in $R[X, Y]$ we have $\text{gr}_\mathcal{F} \mathcal{N} = R[X, Y]/(X^{q-1}, Y^{p-1})$ (see [Ku2], B.12), hence $\mathcal{N}$ is a free $R$-module with the basis

$$B := \{\xi^\nu \eta^\mu\}_{\nu < q - 1, \mu < p - 1}$$

where $\xi, \eta$ are the residue classes of $X, Y$ in $\mathcal{N}$ (see [Ku2], B.6). In particular $\text{rank}(\mathcal{N}) = (p - 1)(q - 1) =: c$, and different basis elements have different degrees with respect to the residue grading of $\mathcal{F}$.

Since $A$ is finite over $R'$ we have $R' = R/q$ where the prime ideal $q$ is generated by an irreducible polynomial in $R$, hence $\mathcal{H} := \text{Spec}(R')$ is an irreducible hypersurface in $A^n(K) = \text{Spec}(R)$.

We identify the curves of type $p, q$ with the closed points $\alpha := (\{a_{\nu \mu}\}) \in A^n(K)$ or with the maximal ideals $m = (\{A_{\nu \mu} - a_{\nu \mu}\}_{\nu + \mu < pq}) (a_{\nu \mu} \in K)$ of $R$. For $\alpha \in A^n$ we denote the curve with the equation $F(\alpha, X, Y) = 0$ by $C_\alpha$. The set $\text{Max}(A)$ can be identified with the set of singularities of the curves of type $p, q$. If a maximal ideal $\mathfrak{m}$ of $A$ with preimage $m = (\{A_{\nu \mu} - a_{\nu \mu}\})$ in $R$ is given, then $\mathfrak{m}$ corresponds to a singularity of the curve $C_\alpha$. Moreover

$$A_{\mathfrak{m}}/m_{\mathfrak{m}} = (K[C_\alpha]/J)_{\mathfrak{m}}$$

with the Jacobian ideal $J$ of $K[C_\alpha]$ and the image $\mathfrak{m}$ of $\mathfrak{m}$ in $K[C_\alpha]/J$.

**Proposition 1.1.** The singular curves of type $p, q$ are the closed points of the irreducible hypersurface $\mathcal{H} \subset A^n(K)$. The closed points of $A^n(K)$ outside of $\mathcal{H}$ are in one-to-one correspondence with the smooth curves of type $p, q$. Their Weierstraß semigroup is $H_{pq}$.

The last statement of the proposition follows from the fact that the Weierstraß semigroup of a smooth curve of type $p, q$ has genus $g = d$, hence no gaps of $H_{pq}$ have to be closed in it.

An example of a smooth curve of type $p, q$ is given by the equation $Y^p - X^q + a_{00} = 0 (a_{00} \neq 0)$.

As an $R$-module $A$ can be written

$$A = \mathcal{N}/ \sum_{\alpha < q - 1, \beta < p - 1} R \cdot \xi^\alpha \eta^\beta F(\xi, \eta)$$

and the relations

$$\xi^\nu = \frac{1}{q} \sum \nu A_{\nu \mu} \xi^{\nu - 1} \eta^\mu, \eta^{\nu - 1} = -\frac{1}{p} \sum \mu A_{\nu \mu} \xi^\nu \eta^{\mu - 1}$$

allow with the usual reduction process to write

$$\xi^\alpha \eta^\beta F(\xi, \eta) = \sum_{\nu < q - 1, \mu < p - 1} r_{\nu \mu}^{\alpha \beta} \cdot \xi^\nu \eta^\mu (r_{\nu \mu}^{\alpha \beta} \in R).$$
The \( c \times c \)-matrix \( M := (r_{i,j}) \) represents the multiplication by \( F(\xi, \eta) \) in \( N \), and since \( F(\xi, \eta) \) is not a zero-divisor in \( N \) we have an exact sequence of \( R \)-modules

\[
0 \to R^c \xrightarrow{M} R^c \to A \to 0.
\]

(2)

\( M \) is a relation matrix of the \( R \)-module \( A \) with respect to the basis \( B \) of \( N \). For \( 0 \leq l \leq c \) the \((c-l)\)-minors of \( M \) generate the \( l \)-th Fitting ideal \( F_l(A/R) \) of the \( R \)-module \( A \). In particular \( F_0(A/R) = (\Delta) \) with \( \Delta := \det(M) \), the norm of the multiplication map by \( F(\xi, \eta) \). Here \( \Delta \neq 0 \), the map given by \( M \) being injective. We have

\[
(0) \not\subset F_0(A/R) \subset F_1(A/R) \subset \cdots \subset F_c(A/R)
\]

By [Ku1], D.14

(3)

\[
\text{Ann}_R(A)^c \subset F_0(A/R) = (\Delta) \subset \text{Ann}_R(A)
\]

and therefore \( \text{Rad}(\text{Ann}_R(A)) = \text{Rad}(\Delta) \). As \( A \) is an \( R \)-algebra \( \text{Ann}_R(A) \) is the kernel \( q \) of the structure homomorphism \( R \to A \), hence \( \text{Rad}(\Delta) \) is also a prime ideal. It follows that \( \Delta = \alpha \Delta_0^c \) with an irreducible polynomial \( \Delta_0 \) of \( R \) which generates \( q \), an \( \alpha \in K \setminus \{0\} \) and an \( r \in \mathbb{N} \), hence \( R' = R/q = R/\langle \Delta_0 \rangle \) and the hypersurface \( \mathcal{H} \) is given by the equation \( \Delta_0 = 0 \).

For \( p \in \text{Spec}(R) \) and \( l \in \{0, \ldots, c\} \) we have the following formula for the minimal number of generators of the \( R_p \)-module \( A_p \)

\[
\mu_p(A) = \min\{l \mid F_l(A_p/R_p) = R_p\}
\]

(4)

([Ku1], D.8).

Let \( m = \{(A_{n,m} - a_{n,m})\} \) be a maximal ideal of \( R \) corresponding to the polynomial \( \bar{\Phi} := F(\alpha, X, Y) \in K[X, Y] \) and \( l \in \{0, \ldots, c\} \). Then by (4) \( F_l(A/R)_m = F_l(A_m/R_m) = R_m \) if and only if the \( R_m \)-module \( A_m \) has a minimal number of generators \( \leq l \), that is, if and only if

\[
\dim_K(K[X, Y]/(\bar{\Phi}, \bar{F}_X, \bar{F}_Y)) \leq l.
\]

If \( \mathfrak{m} \in \text{Max}(K[X, Y]) \) corresponds to a node of \( C_\alpha \), then

\[
\dim_K(K[X, Y]/(\bar{\Phi}, \bar{F}_X, \bar{F}_Y))_{\mathfrak{m}} = 1.
\]

If \( C_\alpha \) is a nodal curve, then \( \dim_K(K[X, Y]/(\bar{\Phi}, \bar{F}_X, \bar{F}_Y)) \) is the number of its nodes and (4) implies

**Lemma 1.2.** If \( C_\alpha \) has at most \( l \) nodes and no other singularities, then \( m \) is contained in the open set \( \text{Max}(R) \setminus V(F_l(A/R)) \) of \( \text{Max}(R) \). Conversely, if \( C_\alpha \) has \( l \) distinct nodes and \( m \in \text{Max}(R) \setminus V(F_l(A/R)) \), then \( C_\alpha \) is a nodal curve with exactly \( l \) nodes.

For the module of differentials of \( A/R \) we have

\[
\Omega^1_{A/R} = \text{Ad}X \oplus \text{Ad}Y/ (F_X(x, y)dx + F_{XY}(x, y)dy, F_{YX}(x, y)dx + F_{YY}(x, y)dy)
\]

with the residue classes \( x, y \) of \( X, Y \) in \( A \). Since the variables \( A_{00}, A_{01}, A_{10} \) have disappeared in the second derivatives the Hesse determinant \( \text{Hess}_F(X, Y) \) of \( F \) is a non-zero polynomial in \( A \). Take \( \mathfrak{m} \in \text{Max}(A) \) with preimage \( m \) in \( R \) corresponding to a point in \( \mathcal{H} \). Nodes are the singularities with non-vanishing Hesse determinant, hence \( \mathfrak{m} \) corresponds to a node of the curve given by \( m \) if and only if \( \mathfrak{m} \in \text{Max}(A) \setminus V(\text{Hess}_F) \). This is equivalent to each of the following conditions.
(i) \( \text{Hess}_F(X,Y) \) is a unit in \( A_{2\mathfrak{m}} \).

(ii) \( \Omega^1_{A_{\mathfrak{m}}/R} = 0 \).

(iii) \( \mathfrak{M} \) is unramified over \( R \) ([Ku1], 6.10).

From (ii) and (iii) we conclude

**Proposition 1.3.** The nodal curves of type \( p,q \) having at least one node correspond bijectively to the maximal ideals \( \mathfrak{m} \in V(\Delta_0) = \mathcal{H} \) with \( \mathfrak{m} \notin V(\text{Ann}_R(\Omega^1_{A/R})) \) or equivalently with \( A/R \) being unramified at \( \mathfrak{m} \).

We denote this open set of the hypersurface \( \mathcal{H} \) by \( \mathcal{H}_{pq} \). The additional assumption that \( \mathfrak{m} \notin V(F_l(A/R)) \) defines for each \( l = 1, \ldots, d \) an open subset \( U_l \) of \( \mathcal{H}_{pq} \) whose closed points correspond to the nodal curves of type \( p,q \) having at most \( l \) nodes. Set \( U_0 := \emptyset \). We have

\[
\mathcal{H}_{pq} = \bigcup_{l=1}^d \mathcal{H}_{pq}^l
\]

with the locally closed subset \( \mathcal{H}_{pq}^l := U_l \setminus U_{l-1} \) whose closed points correspond to the curves having exactly \( l \) nodes. The Weierstraß semigroups of the curves in \( \mathcal{H}_{pq}^l \) are certain semigroups \( H \) with \( p,q \in H \) having genus \( g = d - l \). By [KKW], Theorem 6.4 every Weierstraß semigroup \( H \) of genus \( g \) is the Weierstraß semigroup of an element of \( \mathcal{H}_{pq}^{d-g} \) for suitably chosen \( p,q \).

The curve \( C : (Y - b)^p - (X - a)^q + c(X - a)(Y - b) = 0 \ (a, b, c \in K, c \neq 0) \) has only one singularity at \( (a, b) \), and it is a node. Therefore \( \mathcal{H}_{pq}^1 \neq \emptyset \). The associated normed curve of the Lissajous curve of type \( p,q \) ([KKW], Example 2.4) has the maximal possible number \( d \) of nodes, hence \( \mathcal{H}_{pq}^d \neq \emptyset \).

For a domain \( B \) let \( Q(B) \) denote its quotient field.

**Proposition 1.4.** We have \( Q(R') = Q(A) \). Hence \( R' \to A \) induces a finite birational morphism \( K^{n-1}(K) \to \mathcal{H} \), and the hypersurface \( \mathcal{H} \) is rational.

**Proof.** Let \( \mathfrak{p} \) be the kernel of \( R \to A \). The inclusion \( R' \to A \) induces an injection \( Q(R') \to A_{\mathfrak{p}} \). Since \( A \) is a domain and integral over \( R' \) we have \( A_{\mathfrak{p}} = Q(A) \). Moreover \( F_1(A_{\mathfrak{p}}/R_{\mathfrak{p}}) = R_{\mathfrak{p}} \) since \( F_1(A_{\mathfrak{m}}/R_{\mathfrak{m}}) = R_{\mathfrak{m}} \) with the \( \mathfrak{m} \) belonging to the curve \( C \) above, as Fitting ideals are compatible with localization. Hence by (4) \( A_{\mathfrak{p}} \) is generated over \( R_{\mathfrak{p}} \) by one element, i.e. \( Q(A) = Q(R') \).

For the maximal ideals \( \mathfrak{m} \in \mathcal{H}_{pq}^l \) all \( \mathfrak{M} \in \text{Max}(A) \) lying over \( \mathfrak{m} \) are unramified over \( R \). Let \( \mathfrak{m} = \{ (a_{\nu\mu} - a_{\mu\nu}) \}_{\nu\mu+pq<\alpha} \in \mathcal{H}_{pq}^l \) with \( \alpha := \{ (a_{\nu\mu}) \} \in K^n \) be given, and let \( \mathfrak{M} \in \text{Max}(A) \) correspond to a node of the curve \( C_\alpha \). Set \( T_{\nu\mu} := a_{\nu\mu} - a_{\mu\nu} \) for short. The canonical homomorphism \( R_{\mathfrak{m}} \to A_{\mathfrak{M}} \) induces a local homomorphism \( \varphi : \widetilde{R}_{\mathfrak{m}} \to \widehat{A}_{\mathfrak{M}} \) of the completions which is surjective since \( A_{\mathfrak{M}} \) is unramified over \( R_{\mathfrak{m}} \). Here \( \widetilde{R}_{\mathfrak{m}} = K[[T_{\nu\mu}]] \) and \( \widehat{A}_{\mathfrak{M}} \) are regular local rings of dimension \( n \) resp. \( n - 1 \). Therefore \( \ker(\varphi) \) is generated by a power series \( \Delta_{\mathfrak{M}} \) of order 1, an irreducible factor of \( \Delta_0 \) considered as a power series in the \( T_{\nu\mu} \).
Thus \( \mathfrak{M} \) defines a smooth analytic branch \( \text{Spec}(\hat{R}_m/(\Delta_{\mathfrak{M}})) \) of \( \mathcal{H} \) near the point \( \alpha \). Different nodes of \( C_\alpha \) define different branches as the power series \( \Delta_0 \) cannot have multiple factors, \( \Delta_0 \) being an irreducible polynomial. The local ring \( R_m/(\Delta_0) \) is regular if and only if \( C_\alpha \) has only one node. Thus \( \mathcal{H}_1 \) is the set of regular points of \( \mathcal{H}_{pq} \).

Let \( \hat{A}_m \) be the completion of \( A_m := R_m \otimes A \) as an \( R_m \)-module. Then

\[
\hat{A}_m = \hat{A}_{\mathfrak{M}_1} \times \cdots \times \hat{A}_{\mathfrak{M}_l} = \hat{R}_m/(\Delta_{\mathfrak{M}_1}) \times \cdots \times \hat{R}_m/(\Delta_{\mathfrak{M}_l})
\]

by the Chinese Remainder Theorem. Since the Fitting ideals are compatible with localization and completion we obtain that

\[
F_0(\hat{A}_m/\hat{R}_m) = \hat{R}_m \cdot F_0(A/R) = \hat{R}_m \cdot \Delta = \hat{R}_m \cdot (\prod_{i=1}^l \Delta_{\mathfrak{M}_i}) = \hat{R}_m \cdot \Delta_0.
\]

Remember that \( \Delta = a\Delta_0^r \) with \( a \in K \setminus \{0\} \) and \( r \geq 1 \). Since we know that nodal curves of type \( p, q \) with at least one node exist for every \( p, q \) the above consideration implies that \( r = 1 \) and we have proved the irreducibility of \( \Delta \), the polynomial generating \( F_0(A/R) \). Thus the hypersurface \( \mathcal{H} \) is defined by \( F_0(A/R) = (\Delta) \).

We determine the leading form of \( \Delta_{\mathfrak{M}_i} \) which defines the tangent hyperplane of the branch \( \Delta_{\mathfrak{M}_i} = 0 \). As \( \Omega^1_{A_m/R} = 0 \) we see that \( \Omega^1_{A_m/K} \) is generated by the differentials \( dT_{\nu\mu} \). Moreover we have the relation

\[
\sum_{(\nu,\mu) \neq (0,0)} x^{\nu\gamma} y^{\mu} dT_{\nu\mu} = 0
\]

coming from \( dF = 0 \). Therefore \( \{dT_{\nu\mu} \mid (\nu,\mu) \neq (0,0)\} \) is a basis of \( \Omega^1_{A_m/K} \), and we obtain

\[
\Omega^1_{A_m/K} = \bigoplus_{(\nu,\mu) \neq (0,0)} \hat{A}_{\mathfrak{M}_i} \cdot dT_{\nu\mu}
\]

In \( \Omega^1_{A_m/K} \) there is the relation \( d\Delta_{\mathfrak{M}_i} = \sum_{\nu\mu + \mu q < pq} \partial \Delta_{\mathfrak{M}_i}/\partial T_{\nu\mu} \cdot dT_{\nu\mu} \), and by (7)

\[
\sum_{(\nu,\mu) \neq (0,0)} (\partial \Delta_{\mathfrak{M}_i}/\partial T_{\nu\mu} - x^{\nu\gamma} y^{\mu} \cdot (\partial \Delta_{\mathfrak{M}_i}/\partial T_{00}) dT_{\nu\mu} = 0
\]

which implies in \( \hat{A}_{\mathfrak{M}_i} \) the relations

\[
\partial \Delta_{\mathfrak{M}_i}/\partial T_{\nu\mu} = x^{\nu\gamma} y^{\mu} \cdot (\partial \Delta_{\mathfrak{M}_i}/\partial T_{00})
\]

for all \( \nu, \mu \). Since \( \Delta_{\mathfrak{M}_i} \) has order 1, at least one of the partial derivatives must be a unit in \( \hat{A}_{\mathfrak{M}_i} \), hence so must be the partial with respect to \( T_{00} \). Let \( (\xi, \eta) \in K^2 \) be the node corresponding to \( \mathfrak{M} \). Considering the above relations modulo \( \mathfrak{M}\hat{A}_{\mathfrak{M}_i} \) we find that

\[
\partial \Delta_{\mathfrak{M}_i}/\partial T_{\nu\mu} \mid 0 = \xi^{\nu\gamma} \eta^{\mu} \cdot \partial \Delta_{\mathfrak{M}_i}/\partial T_{00} \mid 0
\]
where the last partial does not vanish, hence the leading form of $\Delta_M$ is

\[(8) \quad L_{2R} \Delta_{2R} = \partial \Delta_{2R} / \partial T_{00} |_{0} \cdot \sum_{\nu p + \mu q < pq} \xi^\nu \eta^\mu T_{\nu \mu}.
\]

Collecting everything we obtain

**Proposition 1.5.** At the closed points of $H_{pq}^l$ the hypersurface $H$ has $l$ regular branches with tangent hyperplanes given by (8).

Let $(\xi_1, \eta_1), \ldots, (\xi_i, \eta_i)$ be the nodes of $C_\alpha$. We shall see in Lemma 2.1 that the matrix $(\xi_1^\nu \eta_1^\mu)_{\nu p + \mu q < pq, i = 1, \ldots, l}$ has rank $l$. Thus the $L_{2R} \Delta_{2R}, (i = 1, \ldots, l)$ are linearly independent over $K$ and the $\Delta_{2R}$ form part of a regular system of parameters of $\hat{R}_m$.

For the defining polynomial $\Delta$ of $H$ this means the following: If we expand $\Delta$ as a polynomial in the $T_{\nu \mu} = A_{\nu \mu} - a_{\nu \mu}$ its form of lowest degree is up to a constant factor the product of the $l$ homogenous linear polynomials $L_{2R} \Delta_{2R}$ which moreover are linearly independent over $K$.

Formula (6) implies that

\[(9) \quad \hat{R}_m \cdot F_k(A/R) = F_k(\hat{A}_m/\hat{R}_m) = (\{\Delta_{2R_{i_1}}, \ldots, \Delta_{2R_{i_{k+1}}}\}_{i_1 < \cdots < i_{k+1}}).
\]

One can prove (9) by first showing it when the $\Delta_{2R}$ are variables in a polynomial ring and by passing then to the completion. Thus the ideals $\hat{R}_m \cdot F_k(A/R)$ ($k = 0, \ldots, l - 1$) are radical ideals of height $k + 1$ in $\hat{R}_m$, and so are the $F_k(A_\alpha/R_\alpha)$ in $R_\alpha$.

**Theorem 1.6.** For any $l$ with $1 \leq l \leq d$ there is a nodal curve of type $p, q$ with exactly $l$ nodes, i.e. $H_{pq}^l \neq \emptyset$.

**Proof.** Let $\mathfrak{m} \in \text{Max}(R)$ correspond to the curve $C$ associated to the Lissajous curve of type $p, q$. There is a $g \in R$ such that $\mathfrak{m} \in D(g)$ and that the closed points in $D(g) \cap H$ correspond to nodal curves. Then by the above the $F_k(A_g/R_g)$ ($k = 0, \ldots, d$) form a strictly increasing sequence of radical ideals in $R_g$. Choose a maximal ideal $\mathfrak{n} \in D(g)$ such that

$\mathfrak{F}_{i-1}(A_g/R_g) \subset \mathfrak{n} R_g$, $\mathfrak{F}_i(A_g/R_g) \not\subset \mathfrak{n} R_g$.

Then the curve corresponding to $\mathfrak{n}$ has exactly $l$ nodes. \[\square\]
Examples 1.7. The Weierstraß semigroups of the curves in $H_{pq}^l$ are numerical semigroups $H$ with $p, q \in H$ and genus $d - l$. If $l \leq p/2$ all possible $H$ of this kind do occur, see [KKW], Example 5.4. For $l = 1$ the semigroup $H$ is obtained from $H_{pq}$ by closing one gap $c - 1 - (ap + bq), (a, b \in \mathbb{N})$. We must have $a = b = 0$, otherwise more than one gap would be closed. Therefore $H = \langle p, q, c - 1 \rangle$ and any curve in $H_{pq}^1$ has this Weierstraß semigroup. In $H_{pq}^2$ we have the Weierstraß semigroups $\langle p, q, c - 1 - p \rangle$ and $\langle p, q, c - 1 - q \rangle$. The curves in $H_{pq}^d$ are the nodal curves of type $p, q$ for which the normalization of its projective closure has genus 0. Their Weierstraß semigroup is $\mathbb{N}$.

The hypersurface $H$ contains many lines.

Proposition 1.8. Let $\alpha \neq \beta$ be closed points of $H$ such that $\text{Sing}(C_\alpha) \cap \text{Sing}(C_\beta) \neq \emptyset$, and let $L$ be the line through $\alpha$ and $\beta$. Then $L \subset H$, and for almost all closed $\gamma \in L$ the curve $C_\gamma$ has the singular set $\text{Sing}(C_\alpha) \cap \text{Sing}(C_\beta)$.

Proof. Set $H := F(\beta, X, Y) - F(\alpha, X, Y)$ and $D := V(H)$. Then $H$ and $F(\alpha, X, Y)$ are relatively prime and $\text{Sing}(C_\alpha) \cap \text{Sing}(D) = \text{Sing}(C_\alpha) \cap \text{Sing}(C_\beta)$. By [KKW], Proposition 3.1 the curve $F(\alpha, X, Y) + d \cdot H = F(\alpha + d(\beta - \alpha), X, Y) = 0$ has for almost all $d \in K \setminus \{0\}$ the singular set $\text{Sing}(C_\alpha) \cap \text{Sing}(C_\beta) \neq \emptyset$. It follows that $L \subset H$. \qed

Corollary 1.9. For any closed point $\alpha \in H$ there is at least one line $L$ with $\alpha \in L \subset H$.

Proof. Let $(a, b)$ be a singularity of $C_\alpha$, and let $C_\beta$ be a nodal curve with $(a, b)$ as its only node. It can be chosen such that $\alpha \neq \beta$. Then $H$ contains by 1.8 the line through $\alpha$ and $\beta$. \qed

Corollary 1.10. Let $L \subset H$ be a line through a closed point $\alpha$ where $C_\alpha$ is a nodal curve. Then for almost all closed points $\gamma \in L$ the curves $C_\gamma$ have the same Weierstraß semigroup.

Proof. Since $\alpha \in H_{pq}$ and this set is open in $H$ almost all $C_\gamma$ with $\gamma \in L$ are nodal curves having by 1.8 the same set of nodes. By [KKW], Corollary 4.3 they also have the same Weierstraß semigroup. \qed

2 Which numerical semigroups are Weierstraßsemigroups?

Let $H$ be a numerical semigroup of genus $g$ and let $p < q$ be relatively prime numbers from $H$. The semigroup $H_{pq}$ has $d$ gaps $\gamma_1 < \cdots < \gamma_d$ which can be written

$$\gamma_i = (p - 1)(q - 1) - 1 - (a_ip + b_iq)$$

with a unique $(a_i, b_i) \in \mathbb{N}^2$. Of these gaps $l := d - g$ are closed in $H$. We want to decide whether a nodal curve $C$ of type $p, q$ with $l$ nodes exists such that $H$ is the Weierstraß semigroup of $C$. 

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Let \( \gamma_j < \cdots < \gamma_{j_i} \) be the gaps of \( H_{pq} \) which are closed in \( H \). Further let \( A_H(X_i, Y_1, \ldots, X_l, Y_l) \) be the matrix
\[
\begin{pmatrix}
X_1^{a_{i_1}} Y_1^{b_{i_1}} & \cdots & X_1^{a_{i_l}} Y_1^{b_{i_l}} \\
\vdots & & \vdots \\
X_l^{a_{i_1}} Y_l^{b_{i_1}} & \cdots & X_l^{a_{i_l}} Y_l^{b_{i_l}}
\end{pmatrix}
\]
and \( D_H(X_1, Y_1, \ldots, X_l, Y_l) := \text{det}(A_H(X_1, Y_1, \ldots, X_l, Y_l)) \) its determinant.

**Lemma 2.1.** If \( H \) is the Weierstraß semigroup of a nodal curve \( C : F = 0 \) of type \( p, q \) with the nodes \((\xi_1, \eta_1), \ldots, (\xi_l, \eta_l)\), then \( D_H(\xi_1, \eta_1, \ldots, \xi_l, \eta_l) \neq 0 \).

**Proof.** If \( D_H(\xi_1, \eta_1, \ldots, \xi_l, \eta_l) = 0 \), then there exists a non-zero \( \lambda = (\lambda_1, \ldots, \lambda_l) \in K^l \) such that \( A_H \cdot \lambda^l = 0 \). Assume that \( \lambda_1 = \cdots = \lambda_{l-1} = 0 \), \( \lambda_l \neq 0 \). Let \( x, y \) denote the images of \( X, Y \) in the function field \( K(C) \) of \( C \) and \( P \) the place at infinity of \( C \). The function
\[
\Phi(x, y) := \lambda_i x^{a_{i_j}} y^{b_{i_j}} + \cdots + \lambda_l x^{a_{i_l}} y^{b_{i_l}} \in K[C]
\]
satisfies \( \Phi(\xi_i, \eta_i) = 0 \) \( (i = 1, \ldots, l) \). If follows from [KKW], Proposition 4.2 that \( \text{ord}_P(\Phi(x, y) dx) + 1 = \gamma_{j_i} \) is a gap of \( H \), contradicting the fact that \( \gamma_{j_i} \) was a gap of \( H_{pq} \) closed in \( H \). \( \square \)

With the generic Weierstraß polynomial \( F(\{A_{\nu \mu}\}, X, Y) \in R[X, Y] \) of type \( p, q \) and \( l \) with \( 1 \leq l \leq d \) set
\[
T := R[X_1, Y_1, \ldots, X_l, Y_l]/\{(\nu F(X_i, Y_i), F_X(X_i, Y_i), F_Y(X_i, Y_i))\}_{i=1, \ldots, l}.
\]

Let \( C_L : F(\{a_{\nu \mu}\}, X, Y) = 0 \) be the normed curve associated to the Lissajous curve of type \( p, q \), and let \( (\xi_i, \eta_i) (i = 1, \ldots, d) \) be its nodes. \( C_L \) has the Weierstraß semigroup \( \mathbb{N} \). By Lemma 2.1 we have \( D_N(\xi_1, \eta_1, \ldots, \xi_l, \eta_l) \neq 0 \). Therefore the columns of this determinant corresponding to the gaps \( \gamma_{j_1}, \ldots, \gamma_{j_l} \) are linearly independent over \( K \), and there are nodes \( (\xi_1, \eta_1), \ldots, (\xi_l, \eta_l) \) (say) such that \( D_H(\xi_1, \eta_1, \ldots, \xi_l, \eta_l) \neq 0 \) too.

Let \( \delta \) be the image of \( D_H(X_1, Y_1, \ldots, X_l, Y_l) \) and \( t \) that of \( \prod_{i=1}^l \text{Hess}_F(X_i, Y_i) \) in \( T \). Then \( t \cdot \delta \) is not contained in the maximal ideal corresponding to the point \((\{a_{\nu \mu}^l\}, \xi_1, \eta_1, \ldots, \xi_l, \eta_l)\) and hence \( t \cdot \delta \) is not nilpotent. Therefore \( S_H := T_{t, \delta} \) is not the zero-ring. Now the elements of \( \text{Max}(S_H) \) correspond bijectively to the \( (\beta, \xi_1, \eta_1, \ldots, \xi_l, \eta_l) \) where the \((\xi_i, \eta_i)\) are nodes of the curve \( C_\delta \) and have the additional property that \( D_H(\xi_1, \eta_1, \ldots, \xi_l, \eta_l) \neq 0 \). In particular the nodes are distinct.

Let \( h \) be the set of the \((a_{j_i}, b_{j_i}) \in \mathbb{N}^2 (i = 1, \ldots, l)\) corresponding to the gaps of \( H_{pq} \) which are closed in \( H \). Let \( x_i, y_j \) be the images of the \( X_i, Y_i \) in \( S_H \) and denote the images of the \( A_{\nu \mu} \) also by \( A_{\nu \mu} \) \((\nu p + \mu q < pq)\).

**Lemma 2.2.** We have \( \Omega_{S_H/K}^1 = \bigoplus_{(\nu, \mu) \in h} S_H dA_{\nu \mu} \). In particular \( S_H \) is a regular \( K \)-algebra, equidimensional of dimension \( n - 1 \). Further \( S_H \) is unramified over \( K[\{A_{\nu \mu}\}_{(\nu, \mu) \in h}] \).
Proof. The module of differentials has the presentation
\[
\Omega^1_{S_H/K} = \bigoplus_{\nu p + \mu q < pq} S_H dA_{\nu\mu} \oplus \bigoplus_{i=1}^l S_H dX_i \oplus S_H dY_i / U
\]
where \(U\) is generated by
\[
\sum_{\nu p + \nu q < pq} x_i^\nu y_i^\mu dA_{\nu\mu},
\]
\[
\sum_{\nu p + \nu q < pq} \nu x_i^\nu y_i^\mu dA_{\nu\mu} + F_{X_i}(x_i, y_i) dX_i + F_{XY}(x_i, y_i) dY_i
\]
and
\[
\sum_{\nu p + \nu q < pq} \mu x_i^\nu y_i^\mu dA_{\nu\mu} + F_{Y_i}(x_i, y_i) dX_i + F_{Y_0}(x_i, y_i) dY_i
\]
\((i = 1, \ldots, l)\). Since \(\text{Hess}_C(x_i, y_i)\) \((i = 1, \ldots, l)\) and \(D_H(x_1, y_1, \ldots, x_l, y_l)\) are units in \(S_H\), the statement about \(\Omega^1_{S_H/K}\) follows, and the remaining assertions are clear by the differential criterion of regularity ([Ku1], 7.2).

Now let \(U_{pq}^l(H) := \text{Spec}(S_H) \setminus V(F_i(A/R)S_H)\). By Lemma 1.2 the closed points \((\alpha, \xi_1, \eta_1, \ldots, \xi_l, \eta_l)\) of the scheme \(U_{pq}^l(H)\) are those for which the curve \(C_\alpha\) has no singularities but the nodes \((\xi_i, \eta_i)\) which satisfy \(D_H(\xi_1, \eta_1, \ldots, \xi_l, \eta_l) \neq 0\). These \(C_\alpha\) have a Weierstraß semigroup which is obtained from \(H_{pq}\) by closing \(l\) of its gaps, but may be different from \(H\).

It will be shown in Proposition 3.1 that the scheme \(U_{pq}^l(H)\) is not empty. In order to decide whether \(H\) is the Weierstraß semigroup of a nodal curve of type \(p, q\) we need a further consideration which is inspired by [Ha], IV.4.

Let \(\gamma_{i_1} < \cdots < \gamma_{i_g}\) be the gaps of \(H\), \(\gamma_{i_k} = c - 1 - (a_{i_k} p + b_{i_k} q)\). Then
\[
\{\gamma_{i_1}, \ldots, \gamma_{i_g}\} \cup \{\gamma_{j_1}, \ldots, \gamma_{j_m}\}
\]
is the set of all gaps of \(H_{pq}\). In \(H_{pq}\) there are \(d - i_k\) gaps > \(\gamma_{i_k}\), and \(H\) has \(g - k\) gaps > \(\gamma_{i_k}\). Hence there are \((d - i_k) - (g - k) = l - (i_k - k)\) gaps of \(H_{pq}\) which are > \(\gamma_{i_k}\) and are closed in \(H\). Therefore \(\gamma_{j_m} > \gamma_{i_k}\) if and only if \(m > i_k - k\).

Let \(s_k\) be the column
\[
\begin{pmatrix}
X_1^{a_{i_k}} Y_1^{b_{i_k}} \\
\vdots \\
X_l^{a_{i_k}} Y_l^{b_{i_k}}
\end{pmatrix}
\]
and \(D_k^n(X_1, Y_1, \ldots, X_l, Y_l)\) for \(m \in \{1, \ldots, i_k - k\}\) the determinant of the matrix which is obtained from \(A_H\) by replacing its \(m\)-th column by \(s_k\). These are \(\sum_{k=1}^g (i_k - k) = \sum_{k=1}^g (g + 1)\) determinants. Let \(J\) be the ideal generated by their images in \(S_H\). If the semigroup \(H\) is obtained from \(H_{pq}\) by closing its \(l\) greatest gaps, then no \(D^n_k\) are present, and we set \(J = (0)\). Let \(V_{pq}(H) := U_{pq}^l(H) \cap V(J)\).
Theorem 2.3. The closed points of $V_{pq}(H)$ correspond to the nodal curves of type $p, q$ having the Weierstraß semigroup $H$, i.e. $H$ is the Weierstraß semigroup of such a curve if and only if $V_{pq}(H) \neq \emptyset$.

Proof. a) Let $Q := (\alpha, \xi_1, \eta_1, \ldots, \xi_l, \eta_l) \in V_{pq}(H)$. Since $Q \in U_{pq}^l(H)$ the curve $C_\alpha$ is a nodal curve with the nodes $(\xi_1, \eta_1), \ldots, (\xi_l, \eta_l)$ and $D_H(\xi_1, \eta_1, \ldots, \xi_l, \eta_l) \neq 0$. Moreover

\begin{equation}
D^n_k(\xi_1, \eta_1, \ldots, \xi_l, \eta_l) = 0 \quad \text{for} \quad k = 1, \ldots, g \quad \text{and} \quad m = 1, \ldots, \lambda_k - k.
\end{equation}

Further for any $k \in \{1, \ldots, g\}$ the linear system of equations

\[ A_H(\xi_1, \eta_1, \ldots, \xi_l, \eta_l) \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_l \end{pmatrix} = -s_k(\xi_1, \eta_1, \ldots, \xi_l, \eta_l) \]

has a unique solution. By Cramer’s rule (1) implies that $\lambda_1 = \cdots = \lambda_k - k = 0$. Let $x, y$ denote the images of $X, Y$ in the function field of $C_\alpha$. The polynomial $\Phi_k(X, Y) := X^{a_{ik}} Y^{b_{ik}} + \sum_{m > \lambda_k - k}^{\lambda_m X^{a_{jm}} Y^{b_{jm}}}$ vanishes at the nodes $(\xi_i, \eta_i)$ $(i = 1, \ldots, l)$, and since $\gamma_{jm} > \gamma_{ik}$ for $m > \lambda_k - k$ the differential $\omega_k := \frac{\Phi_k(x,y)}{F_p(x,y)} dx$ has order $\text{ord}_P(\omega_k) = \gamma_{ik} - 1$ at the place at infinity of $C_\alpha$. By [KKW], Proposition 4.2 $\gamma_{i_1}, \ldots, \gamma_{i_g}$ are gaps of the Weierstraß semigroup of $C_\alpha$, i.e. $H$ is this semigroup.

b) Let $H$ be the Weierstraß semigroup of a nodal curve $C_\alpha : F(\alpha, X, Y) = 0$ of type $p, q$ with $l$ distinct nodes $(\xi_1, \eta_1), \ldots, (\xi_l, \eta_l)$. We show that $Q := (\alpha, \xi_1, \eta_1, \ldots, \xi_l, \eta_l) \in V_{pq}(H)$. By the discussion above we know already that $Q \in U_{pq}^l(H)$.

Let $\Omega_{\infty}$ be the vector space of differentials with non-negative order at the place $P$ at infinity of $C_\alpha$. According to [KKW], Lemma 4.1 we can choose a basis $\{\omega_1, \ldots, \omega_l\}$ of the vector space $\Omega$ of holomorphic differentials on $R$ such that $\omega_k := \frac{\Phi_k(x,y)}{F_p(x,y)} dx$ with

\[ \Phi_k(x,y) = x^{a_{ik}} y^{b_{ik}} + \lambda_{ik+1} x^{a_{ik+1}} y^{b_{ik+1}} + \cdots + \lambda_{k,l} x^{a_{kl}} y^{b_{kl}} \quad (k = 1, \ldots, g) \]

and $\text{ord}_P(\omega_k) + 1 = \gamma_{ik}$. By elementary transformations we attain that

\[ \Phi_k(x,y) = x^{a_{ik}} y^{b_{ik}} + \lambda_{r,k} x^{a_{r,i}} y^{b_{r,i}} + \cdots + \lambda_{l,k} x^{a_{l,i}} y^{b_{l,i}} \quad (k = 1, \ldots, g), \]

with certain $\lambda_{i,k} \in K$ where $r = i_k - k + 1$. Since $\Phi_k(\xi_i, \eta_i) = 0$ $(i = 1, \ldots, l)$ and $\lambda_{i,m} = 0$ $(m = 1, \ldots, \lambda_{i} - k)$ Cramer’s rule implies that $D^n_k(\xi_1, \eta_1, \ldots, \xi_l, \eta_l) = 0$ for $k = 1, \ldots, g$ and $m = 1, \ldots, \lambda_k - k$. Hence $Q \in V(J) \cap U_{pq}^l(H) = V_{pq}(H)$. 

\[ \square \]

Theorem 2.3 and [KKW], Theorem 6.4 imply
Corollary 2.4. Let \( p \) be greater than the elements of the minimal system of generators of \( H \). Then \( H \) is a Weierstraß semigroup if and only if \( V_{pq}(H) \neq \emptyset \).

The closed points of \( V_{pq}(H) \) are the \((\{a_{\nu \mu}\}_{\nu p + \mu q < pq}, \xi_1, \eta_1, \ldots, \xi_l, \eta_l) \in K^{n+2l} \) which are zeros of the polynomials

\[
F(X_i, Y_i), \quad F_X(X_i, Y_i), \quad F_Y(X_i, Y_i) \quad (i = 1, \ldots, l)
\]

and of

\[
D^m_k(X_1, Y_1, \ldots, X_i, Y_i) \quad (k = 1, \ldots, g, m = 1, \ldots, i_k - k)
\]

and not zeros of the polynomials \( D_H(X_1, Y_1, \ldots, X_i, Y_i), \text{Hess}_F(X_i, Y_i) \quad (i = 1, \ldots, l) \) and of at least one of the \( N := \binom{c}{2} (c - l) \)-minors \( h_t \quad (t = 1, \ldots, N) \) of the matrix \( M = (r_{\nu \mu}^\beta) \) defined in Section 1. Let \( a \) be the ideal in \( \mathbb{R}[X_1, Y_1, \ldots, X_i, Y_i] \) generated by the polynomials (2) and (3). By Theorem 2.3 and Hilbert’s Nullstellensatz \( H \) is the Weierstraß semigroup of a nodal curve of type \( p, q \) if and only if there exists \( t \in \{1, \ldots, N\} \) such that

\[
h_t \cdot \prod_{i=1}^l \text{Hess}_F(X_i, Y_i) \cdot D_H(X_1, Y_1, \ldots, X_i, Y_i) \not\in \text{Rad}(a).
\]

One can try to decide this by the radical membership test (see e.g. [Kr-R], page 219). However the number \( N \) of necessary tests increases rapidly with \( p \) and \( q \), and so do the degrees of the involved polynomials. A sufficient condition is that (4) holds for a \((c - 1)\)-minor \( h_t \) of the matrix \( M \) which requires \( c^2 \) tests in the worst case, but with no guarantee of a success.

The polynomials in (2),(3) and (4) all belong to \( \mathbb{Q}[[A_{\nu \mu}], X_1, Y_1, \ldots, X_i, Y_i] \). Therefore (4) holds true if and only if it holds true for \( K = \mathbb{Q} \), the field of algebraic numbers. In other words, the property of \( H \) to be a Weierstraß semigroup is independent of the choice of the base field. For example we can test it for \( K = \mathbb{C} \).

The projection \( \mathbb{A}^{n+2l}(K) \to \mathbb{A}^n(K) \) \((\alpha, \xi_1, \eta_1, \ldots, \xi_l, \eta_l) \mapsto \alpha\) maps the locally closed set \( V_{pq}(H) \) onto a constructible set \( V_{pq}^H \subset \mathcal{H}^H_{pq} \) whose closed points correspond bijectively to the nodal curves of type \( p, q \) with the Weierstraß semigroup \( H \). We have

\[
\mathcal{H}^H_{pq} = \bigcup_H V_{pq}^H
\]

where \( H \) runs over the numerical semigroups containing \( p \) and \( q \) with \( d - l \) gaps.

3 Simplification of nodal curves and a class of Weierstraß semigroups

Let \( 1 < p < q \) be relatively prime integers and \( d = \frac{1}{2}(p - 1)(q - 1) \). In Theorem 1.6 we have seen that for any \( l \in \{1, \ldots, d\} \) there is a nodal curve of type \( p, q \) with exactly \( l \) nodes. The following proposition gives a more precise statement and a different proof.

**Proposition 3.1.** Let \( H \) be a numerical semigroup which is obtained from \( H_{pq} \) by closing \( l \) of its gaps. Then \( V^l_{pq}(H) \neq \emptyset \).
As an immediate consequence we get

**Theorem 3.2.** Let $H$ be the numerical semigroup which is obtained from $H_{pq}$ by closing its $l$ greatest gaps. Then $H$ is a Weierstraß semigroup.

In fact, for $H$ as in 3.2 no determinants $D^m_k$ occur. Therefore $V_{pq}(H) = U^l_{pq}(H)$ which is not empty by 3.1, and Theorem 2.3 implies that $H$ is the Weierstraß semigroup of a nodal curve of type $p,q$. \hfill □

In order to prove 3.1 we need some preparations. Since $U^l_{pq}(H)$ is defined over $\mathbb{Q}$ we may assume that $K = \mathbb{C}$. Let $R := \mathbb{C}[\{A_{pq}\}]$ and $F \in R[X,Y]$ the generic Weierstraß polynomial of type $p,q$. We have $\text{Spec}(R) = \mathbb{A}^n(\mathbb{C})$ with $n = 1/2(p + 1)(q + 1) - 1$. In $\text{Spec}(R[X,Y]) = \mathbb{A}^n(\mathbb{C}) \times \mathbb{A}^2(\mathbb{C})$ we consider the smooth subschemes $V(F,F_X,F_Y) \cong \mathbb{A}^{n-1}(\mathbb{C})$ and $V(F_X,F_Y) \cong \mathbb{A}^n(\mathbb{C})$. Let $R' = R/(\Delta)$ be the image of $R$ in $R[X,Y]/(F,F_X,F_Y)$ and

$$\mathcal{H}^l_{pq} \subset \mathcal{H}_{pq} \subset \mathcal{H} = \text{Spec}(R') \subset \text{Spec}(R) = \mathbb{A}^n(\mathbb{C})$$

as in Section 1. Further let $\pi : \mathbb{A}^n(\mathbb{C}) \times \mathbb{A}^2(\mathbb{C}) \to \mathbb{A}^n(\mathbb{C})$ be the projection onto the first factor. Its restriction $\pi_0 : V(F,F_X,F_Y) \to \mathbb{A}^n(\mathbb{C})$ to $V(F,F_X,F_Y)$ is finite and has image $\mathcal{H}$. For a closed point $\alpha \in \mathcal{H}^l_{pq}$ the corresponding curve $C_\alpha$ has $l$ nodes $(x_1,y_1),\ldots,(x_l,y_l)$ and no other singularities.

We endow $\mathbb{C}^m (m > 0)$ with its standard norm $||\ ||$ and standard topology. For $P \in \mathbb{C}^m$ and $\epsilon > 0$ let $U_\epsilon(P) := \{Q \in \mathbb{C}^m | ||Q - P|| < \epsilon\}$ denote the $\epsilon$-neighborhood of $P$.

The proof of the following proposition is inspired by arguments of Benedetti-Risler [BR], Lemma 5.5.9 and Pecker [P] in real algebraic geometry.

**Proposition 3.3** (Simplification of nodal curves). Let $P_{i_1},\ldots,P_{i_k}$ be distinct nodes of $C_\alpha$ ($1 \leq \lambda \leq l$). Given $\epsilon > 0$ and $\delta > 0$ there exists $\beta \in U_\epsilon(\alpha)$ such that the curve $C_\beta : F(\beta,X,Y) = 0$ has $\lambda$ distinct nodes $Q_1,\ldots,Q_\lambda$ and no other singularities where $Q_k \in U_\delta(P_{i_k})$ for $k = 1,\ldots,\lambda$.

We obtain Proposition 3.1 by applying 3.3 to the normed curve $C_\alpha$ associated to the Lissajous curve of type $p,q$. Let $(x_i,y_i)$ ($i = 1,\ldots,d$) be the nodes of $C_\alpha$ and $\gamma_i = (p - 1)(q - 1) - 1 - (a_ip + b_iq)$ ($i = 1,\ldots,d$) the gaps of $H_{pq}$. Then the determinant

$$D_H(x_1,y_1,\ldots,x_d,y_d) = \det\left(\begin{array}{cc} x_i & y_i \\ a_j & b_j \end{array}\right)_{i,j=1,\ldots,d}$$

does not vanish by Lemma 2.1. Let $\gamma_{jk}$ ($k = 1,\ldots,l$) be the gaps of $H_{pq}$ which are closed in $H$. Consider the columns of $(x_i y_i)$ corresponding to the $(a_{jk},b_{jk})$ ($k = 1,\ldots,l$). Since they are linearly independent there exist nodes $P_{i_k} := (x_{i_k},y_{i_k})$ of the curve $C_\alpha$ such that

$$D_H(x_{i_1},y_{i_1},\ldots,x_{i_l},y_{i_l}) \neq 0.$$

By Proposition 3.3 there is a nodal curve $C_\beta : F(\beta,X,Y) = 0$ with exactly $l$ nodes $Q_k = (\xi_k,\eta_k)$ ($k = 1,\ldots,l$) which are arbitrarily close to the $P_{i_k}$. Then for a suitable $\beta$ also $D_H(\xi_1,\eta_1,\ldots,\xi_l,\eta_l) \neq 0$, and it follows that $(\beta,\xi_1,\eta_1,\ldots,\xi_l,\eta_l) \in U^l_{pq}(H)$. \hfill □

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**Proposition 3.3.** In the following we consider $S := V(F_X, F_Y) \cap \mathbb{C}^n \times \mathbb{C}^2$ and $T := V(F_X, F_Y) \cap \mathbb{C}^n \times \mathbb{C}^2$ as submanifolds of $\mathbb{C}^n \times \mathbb{C}^2$. Then $S \cong \mathbb{C}^{n-1}$ is a hypersurface in $T \cong \mathbb{C}^n$. We shall study the holomorphic maps $\pi : \mathbb{C}^n \times \mathbb{C}^2 \to \mathbb{C}^n$ and $\pi_0 : S \to \mathbb{C}^n$ corresponding to the morphisms $\pi$ and $\pi_0$ from above in the neighborhood of $\alpha \in \mathbb{C}^n$. We have

$$\pi_0^{-1}(\alpha) = \{\alpha\} \times \text{Sing}(C_\alpha) = \{(\alpha, x_i, y_i) | i = 1, \ldots, l\}.$$ 

**Lemma 3.4.** Given $\delta > 0$ there are for small $\epsilon > 0$ open neighborhoods $U_i$ of $(\alpha, x_i, y_i)$ on $S (i = 1, \ldots, l)$ with the following properties:

(i) The $U_i$ are pairwise disjoint and

$$\pi_0^{-1}(U_\epsilon(\alpha)) = \bigcup_{i=1}^l U_i, \ U_i \subset U_\epsilon(\alpha) \times U_\delta(x_i, y_i) \text{ for } i = 1, \ldots, l.$$ 

(ii) $\pi(U_i) \subset U_\epsilon(\alpha)$ is a submanifold of codimension 1 $(i = 1, \ldots, l)$ and the map $\pi_0 : U_i \to \pi(U_i)$ is biholomorphic.

(iii) For any subset $\{j_1, \ldots, j_l\} \subset \{1, \ldots, l\}$ with $\lambda$ distinct elements $\pi(U_{j_1}) \cap \cdots \cap \pi(U_{j_\lambda})$ is a submanifold of $U_\epsilon(\alpha)$ of codimension $\lambda$.

Using the lemma we can finish the proof of Proposition 3.3 as follows: Since $\mathcal{H}_{pq}$ is open in $\mathcal{H}$ we can choose in $3.4$ an $\epsilon > 0$ such that $U_\epsilon(\alpha) \cap \mathcal{H} \subset \mathcal{H}_{pq}$. Then for all $\beta \in U_\epsilon(\alpha) \cap \mathcal{H}$ it follows that $C_\beta$ is a nodal curve of type $p, q$. By dimension reasons the set

$$B := \pi(U_{i_1}) \cap \cdots \cap \pi(U_{i_\lambda}) \setminus \bigcup_{i \notin \{i_1, \ldots, i_\lambda\}} \pi(U_i)$$

is not empty. Moreover since the $U_i \subset U_\epsilon(\alpha) \times U_\delta(x_i, y_i)$ are pairwise disjoint and $\pi_0 : U_i \to \pi(U_i)$ is bijective, for any $\beta \in B$ the fiber $\pi_0^{-1}(\beta)$ consists of exactly $\lambda$ points $(\beta, Q_k) \in U_{i_k}$ and $Q_k \in U_\delta(P_k)$ for $k = 1, \ldots, \lambda$. \hfill $\square$

**Lemma 3.4.** (i) We shall apply the Implicit Function Theorem to the map $(F_X, F_Y) : \mathbb{C}^n \times \mathbb{C}^2 \to \mathbb{C}^2$ given by $F_X$ and $F_Y$. Remember that $S \cong \mathbb{C}^{n-1}$ is a hypersurface of $T = \{(\beta, x, y) | F_X(\beta, x, y) = F_Y(\beta, x, y) = 0\}$. The Jacobian of the map $(F_X, F_Y)$ has rank 2 at the points $(\alpha, x_i, y_i)$ since the Hessian $\text{Hess}_{\alpha}$ is one of its 2-minors and $\text{Hess}_{\alpha}(\alpha, x_i, y_i) \neq 0$ for $i = 1, \ldots, l$.

The Implicit Function Theorem states that there exist $\epsilon_0 > 0$ and $\delta_0 > 0$ and holomorphic maps $\varphi_i : U_{\epsilon_0}(\alpha) \to U_{\delta_0}(x_i, y_i)$ with $\varphi_i(\alpha) = (x_i, y_i)$ such that $T \cap U_{\epsilon_0}(\alpha) \times U_{\delta_0}(x_i, y_i)$ is the graph $\Gamma_{\varphi_i} = \{(\beta, \varphi_i(\beta)) | \beta \in U_{\epsilon_0}(\alpha)\}$ of $\varphi_i$ $i = 1, \ldots, l$. The morphism $\pi_0$ of $\mathbb{C}$-schemes is finite. Then the underlying continuous map $\pi_0$ is closed with respect to the standard topology, as is well-known. Further $U := \bigcup_{i=1}^l S \cap \Gamma_{\varphi_i}$ is an open neighborhood of $\pi_0^{-1}(\alpha)$ on $S$. Hence $W := \mathbb{C}^n \setminus \pi_0(U)$ is an open neighborhood of $\alpha$ in $\mathbb{C}^n$ such that $\pi_0^{-1}(W) \subset U$. For small $\epsilon \leq \epsilon_0$ we have $\pi_0^{-1}(U_\epsilon(\alpha)) \subset U$ and so

$$\pi_0^{-1}(U_\epsilon(\alpha)) = \bigcup_{i=1}^l U_i$$

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where $U_i := S \cap (\Gamma_{\varphi_i} \cap \pi^{-1}(U_\varepsilon(\alpha))) = S \cap \Gamma_{\varphi_i} | U_\varepsilon(\alpha)$ is an open neighborhood of $(\alpha, x_i, y_i)$ on $S$ $(i = 1, \ldots, l)$. For small $\varepsilon > 0$, as $\varphi_1, \ldots, \varphi_l$ are continuous functions, the $U_1, \ldots, U_l$ are pairwise disjoint and $U_i \subset U_\varepsilon(\alpha) \times U_\delta(x_i, y_i)$ for $i = 1, \ldots, l$.

(ii) Since $U_i \subset \Gamma_{\varphi_i} | U_\varepsilon(\alpha)$ is a submanifold of codimension 1 and $\pi : \Gamma_{\varphi_i} | U_\varepsilon(\alpha) \rightarrow U_\varepsilon(\alpha)$ is biholomorphic $\pi(U_i) \subset U_\varepsilon(\alpha)$ is likewise a submanifold of codimension 1 and $\pi : U_i \rightarrow \pi(U_i)$ is biholomorphic.

(iii) The gradient of $F$ at $(\alpha, x_i, y_i)$ has the form $(v_i, 0, 0)$ with $v_i := (\{x_i^\nu y_i^\mu\}_{\nu p + \mu q < pq})$ for $i = 1, \ldots, l$. By 2.1 the vectors $v_i$ are linearly independent, and $v_i$ is normal to the hypersurface $\pi(U_i)$ at $\alpha$. It follows that $\pi(U_{i_1}) \cap \cdots \cap \pi(U_{i_\lambda})$ is for small $\varepsilon > 0$ a submanifold of $U_\varepsilon(\alpha)$ of codimension $\lambda$.

In connection with Theorem 3.2 we have a question: Given a Weierstraß semigroup close its greatest gap. Do we get again a Weierstraß semigroup?

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