ON THE SUCCINCTNESS OF ATOMS OF DEPENDENCY

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Abstract. Propositional team logic is the propositional analog to first-order team logic. Non-classical atoms of dependence, independence, inclusion, exclusion and anonymity can be expressed in it, but for all atoms except dependence only exponential translation are known. In this paper, we systematically classify their succinctness in the positive fragment, where negation is only allowed at the level of literals, and in full propositional team logic with unrestricted negation. By introducing a variant of the Ehrenfeucht–Fraïssé game called formula size game into team logic, we obtain exponential lower bounds in the positive fragment for all atoms. In the full fragment, we present polynomial upper bounds again for all atoms.

1. Introduction

As a novel extension of classical logic, team semantics provides a framework for reasoning about whole collections of entities at once, as well as their relation with each other. Such a collection of entities is called a team. Originally, team semantics was introduced by Hodges \cite{Hod97} to provide a compositional approach to logic of incomplete information, such as Hintikka’s and Sandu’s independence-friendly logic IF \cite{HS89}.

In his seminal work, Väänänen \cite{Vä07} introduced dependence logic which extends first-order logic by so-called dependence atoms, atomic formulas \(=(x_1, \ldots, x_n; y)\) that intuitively express that the value of \(y\) depends only on the values of \(x_1, \ldots, x_n\). While in IF dependencies between variables are expressed with annotated quantifiers such as \(\exists y/\{x_1, \ldots, x_n\}\), in team semantics these can be expressed as genuine formulas. Accordingly, dependence logic formulas are evaluated on sets (teams) of first-order assignments. Besides the dependence atom, a multitude of other notions of interdependencies between variables were studied, such as the independence of variables \cite{GV13}, written \(x_1 \cdots x_n \perp y_1 \cdots y_m\), the inclusion \(x_1 \cdots x_n \subseteq y_1 \cdots y_n\) \cite{Gal12}, exclusion \(x_1 \cdots x_n \mid y_1 \cdots y_n\), and anonymity \(x_1 \cdots x_n \Upsilon y_1 \cdots y_n\) \cite{Vä19}, also known as non-dependence \cite{Rö18}. We generally refer to these expressions as atoms of dependency. In its original formulation, dependence logic is not closed under Boolean negation, and in fact basic laws such as the law of the excluded middle—that either \(\alpha\) or \(\neg\alpha\) holds in any given interpretation—fail. By adding a Boolean negation operator, often

\textsuperscript{2012 ACM CCS: Theory of computation} \rightarrow \text{Complexity theory and logic;}

Key words and phrases: team semantics, succinctness, dependence atom.
written \(\sim\), Väänänen [Vä07] introduced team logic as a strictly more powerful extension of dependence logic.

In the last decade, research on logics with team semantics outside of the first-order setting has thrived as well. A plethora of related systems has been introduced, most prominently for modal logic [Väa08], propositional logic [Yan14, YV16], and temporal logic [KMV15, KMVZ18]. Analogously to first-order team logics, variants with a Boolean negation were studied extensively [YV17, Mül14, KMSV15]. The atoms of dependency in these logics feature a fundamental difference to their first-order counterparts: First-order dependencies range over individuals of the universe, whereas the arguments of propositional dependency atoms are truth values. Due to this fact, propositional and modal team logics are powerful enough to express all the above dependency notions in terms of weaker syntactic ingredients.

Gogic et al. [GKPS95] argue that in addition to the computational complexity of a logic and which properties it can express, it is also important to consider how succinctly the logic can express those properties. The succinctness of especially modal and temporal logics has been an active area of research for the last couple of decades; see e.g. [Wil99, LSW01, EVW02, AI03, Mar03] for earlier work on the topic and [FvdHIK11, FvdHIK13, vDFvdHI14, vdHI14] for recent work. A typical result states that a logic \(\mathcal{L}_1\) is exponentially more succinct than another logic \(\mathcal{L}_2\). This means that there is a sequence of properties \((P_n)_{n \in \mathbb{N}}\) such that \(P_n\) is definable by \(\mathcal{L}_1\)-formulas \((\varphi_n)_{n \in \mathbb{N}}\), but every family \((\psi_n)_{n \in \mathbb{N}}\) of \(\mathcal{L}_2\)-formulas that defines \((P_n)_{n \in \mathbb{N}}\) is exponentially larger than \((\varphi_n)_{n \in \mathbb{N}}\).

In team semantics, the question of succinctness has received only little attention so far. In their paper, Hella et al. [HLSV14] are primarily concerned with the expressive power of modal dependence logic, but they also show that defining the dependence atom in modal logic with Boolean disjunction requires a formula of exponential size. Similarly, Kontinen et al. [KMSV17] investigate many aspects of modal independence logic and among them show that modal independence logic is exponentially more succinct than basic modal logic. Our paper is, to our knowledge, the first systematic look at succinctness for team semantics.

The most commonly used systematic methods for proving succinctness results are formula size games and extended syntax trees. Formula size games are a variant of Ehrenfeucht-Fraïssé games made to correspond to the size of formulas instead of the usual depth of some operator. They were first introduced by Adler and Immerman [AI03] for branching-time temporal logic CTL. The method of extended syntax trees was originally formulated by Grohe and Schweikardt [GS05] for first-order logic. The notion of extended syntax tree was actually inspired by the Adler-Immerman game, and in a certain sense these two methods are equivalent: an extended syntax tree can be interpreted as a winning strategy for one of the players of the corresponding formula size game. Both of these methods have been adapted to many languages, especially in the modal setting, see e.g. [FvdHIK11, vdHIK12, vDFvdHI14].

The formula size game we define in this paper is an adaptation of the games defined by Hella and Väänänen for propositional and first-order logic [HV15] and later by Hella and Vilander for basic modal logic [HV16]. The new games of Hella and Väänänen are variations of the original Adler-Immerman game with a key difference. In the original game, the syntax tree of the formula in question is constructed in its entirety and consequently the second player has an easy optimal strategy. Thus the original game is in some sense a single player game. The new variant uses a predefined resource that bounds the size of the constructed formula and only one branch of the syntax tree is constructed in one play. The second players decisions now truly matter as she gets to decide which branch that is.
Table 1:

| Property          | Connectives in $\Sigma$ | Result |
|-------------------|--------------------------|--------|
| Dependence $\sim=$  | $\land, \varnothing, *$  | poly   |
| $\equiv=$         | $\land, \varnothing, *$  | exp    |
|                  | $\land, \sim, *$         | poly   |
| Independence $\sim\bot_c$ | $\land, \varnothing, \lor$ | poly   |
| $\bot$            | $\land, \varnothing, *$  | exp    |
| $\bot_c$          | $\land, \sim, *$         | poly   |
| Inclusion $\sim\subseteq$ | $\land, \varnothing, \lor$ | poly   |
| $\subseteq$       | $\land, \varnothing, *$  | exp    |
|                  | $\land, \sim, *$         | poly   |
| Exclusion $\sim|$ | $\land, \varnothing, *$  | poly   |
|                  | $\land, \sim, *$         | poly   |
| Anonymity $\sim\Upsilon$ | $\land, \varnothing, \lor$ | poly   |
| $\Upsilon$        | $\land, \varnothing, *$  | exp    |
|                  | $\land, \sim, *$         | poly   |
| Parity $\sim\oplus$ | $\land, \varnothing, *$  | exp    |
| $\oplus$          | $\land, \varnothing, *$  | exp    |
|                  | $\land, \sim, \lor$      | poly   |

Table 1: The succinctness of team properties in propositional team logic. “$*$” means that the entry holds if $\lor$, $\land$, or both are available. The bounds are sharp in the following sense: “poly” means that there is a polynomial translation to $\text{PL}(\Sigma)$. “exp” means that there is an exponential translation to $\text{PL}(\Sigma)$, but no sub-exponential translation.

Contribution. In this paper we consider the succinctness of atoms of dependency. So far, it is known that these atoms can be expressed by exponentially large formulas (see Table 2), with only the dependence atom having a known polynomial size formula [HKVV18].

In Section 2 we define propositional team logic and the fragments we consider, and recall some useful known results.

In Section 3 we obtain exponential lower bounds in the positive fragment of propositional team logic, where Boolean negation only occurs in literals. Our lower bounds imply succinctness results between logics of the form $\text{PL}(\Sigma)$ and $\text{PL}(\Sigma \cup \{\subseteq\})$, and show that the known translations to the positive fragment (see Table 2) are asymptotically optimal.

Most of the lower bounds are obtained via the new formula size game for propositional team logic, including a lower bound for the parity of the cardinality of teams. The lower bounds for dependence and exclusion atoms are obtained via the notion of upper dimension, adapted from [HLSV14].

In Section 4 we polynomially define the negations of the considered atoms of dependency in the positive fragment. From this, as a corollary we obtain polynomial upper bounds for full propositional team logic. Moreover, we define parity polynomially in the full logic, even though both even and odd parities have exponential lower bounds in the positive fragment. See Table 1 for an overview of all results.
Finally, we consider algorithmic applications of our results and show that the complexities of satisfiability, validity and model checking for propositional and modal team logic remain the same after extension by some atoms of dependency.

2. Preliminaries

Definition 2.1 (Teams). A domain $\Phi$ is a finite set of atomic propositions. A $\Phi$-assignment is a function $s:\Phi\rightarrow\{0,1\}$. A $\Phi$-team $T$ is a (possibly empty) set of $\Phi$-functions, $T\subseteq\Phi\rightarrow\{0,1\}$. The set of all $\Phi$-teams is denoted by $\text{Tms}(\Phi)$.

Definition 2.2 (Splits). Let $T$ be a team. We say that an ordered pair $(T_1, T_2)$ of teams is a split of $T$, if $T_1, T_2 \subseteq T$ and $T_1 \cup T_2 = T$. We say that a split $(T_1, T_2)$ is strict if $T_1 \cap T_2 = \emptyset$. Otherwise it is lax. We denote the set of splits of $T$ by $\text{Sp}(T)$, and its subset of strict splits by $\text{SSp}(T)$.

Definition 2.3 (PL($\Sigma, \Phi$)-formulas). Let $\Sigma$ be a set of connectives $\circ$ each with a designated arity $\text{ar}(\circ) \geq 0$. A $\Phi$-literal is of the form $\top$, $\bot$, $\sim\top$, $\sim\bot$, $p$, $\sim p$, $\neg p$, or $\sim\neg p$, where $p \in \Phi$. The set of PL($\Sigma, \Phi$)-formulas is then the smallest set containing all $\Phi$-literals and closed under connectives in $\Sigma$, i.e., if $\varphi_1, \ldots, \varphi_n \in \text{PL}(\Sigma, \Phi)$ and $\text{ar}(\circ) = n$, then $\circ(\varphi_1, \ldots, \varphi_n) \in \text{PL}(\Sigma, \Phi)$.

Let $\text{Prop}(\varphi) \subseteq \Phi$ denote the set of propositional variables that occur in the formula $\varphi$. We will omit the domain $\Phi$ if it is not relevant in the context and write only $\text{PL}(\Sigma)$. We consider the following connectives:

$$
\begin{align*}
T \models & \top \quad \text{always}, \\
T \models & \bot \quad \text{never}, \\
T \models & p \quad \iff \forall s \in T: s(p) = 1, \\
T \models & \neg p \quad \iff \forall s \in T: s(p) = 0, \\
T \models & \sim \psi \quad \iff T \not\models \psi, \\
T \models & \psi \land \theta \quad \iff T \models \psi \land T \models \theta, \\
T \models & \psi \lor \theta \quad \iff T \models \psi \lor T \models \theta, \\
T \models & \exists(S,U) \in \text{Sp}(T): S \models \psi \land U \models \theta, \\
T \models & \forall(S,U) \in \text{Sp}(T): S \models \psi \lor U \models \theta, \\
T \models & \forall(S,U) \in \text{SSp}(T): S \models \psi \lor U \models \theta,
\end{align*}
$$

Note that, as usually in the context of team logic, one distinguishes a dual negation $\lnot$ and a contradictory negation $\sim$. For example, we have the equivalences $\lnot(p \lor q) \equiv \neg p \land \neg q$ and $\sim(p \lor q) \equiv \sim p \land \sim q$, but $\lnot(p \lor q) \not\equiv \sim(p \lor q)$.

We say $\varphi$ entails $\psi$, in symbols $\varphi \models \psi$, if $T \models \varphi$ implies $T \models \psi$ for all domains $\Phi \supseteq \text{Prop}(\varphi) \cup \text{Prop}(\psi)$ and $\Phi$-teams $T$. If $\varphi \models \psi$ and $\psi \models \varphi$, then we write $\varphi \equiv \psi$ and say that $\varphi$ and $\psi$ are equivalent.

Usually, for propositional team logic, $\land$, $\lor$ and $\sim$ are omitted since they are definable as $\varphi \land \psi \equiv \sim(\sim \varphi \lor \sim \psi)$, $\varphi \lor \psi \equiv \sim(\sim \varphi \land \sim \psi)$, and $\varphi \sim \psi \equiv \sim(\sim \varphi \lor \sim \psi)$. If they are removed entirely, the resulting fragment is the following.
Definition 2.4. Formulas in \( \text{PL}(\{\wedge, \alpha, \vee, \checkmark\}) \) are in positive normal form. \( \text{PL}(\{\wedge, \alpha, \vee, \checkmark\}) \) is the positive fragment of propositional team logic. A PL(\{\wedge\})-formula that contains no \( \sim \) is a purely propositional formula.

Yang and Väänänen [YV17] showed that already the positive fragment is expressively complete:

Proposition 2.5 ([YV17]). For every set \( P \) of \( \Phi \)-teams there is a formula \( \varphi \) in positive normal form such that \( T \in P \iff T \models \varphi \) for all \( \Phi \)-teams \( T \). Equivalently, for every \( \Sigma \) and formula \( \psi \in \text{PL}(\Sigma, \Phi) \) there is a formula \( \varphi \) in positive normal form such that \( \psi \equiv \varphi \).

We will consistently use the letters \( \alpha, \beta, \gamma, \ldots \) for purely propositional formulas, whereas \( \varphi, \psi, \theta, \ldots \) will denote arbitrary formulas.

Many fragments of team logic enjoy useful closure properties:

Definition 2.6. Let \( \varphi \) be a PL(\( \Sigma, \Phi \))-formula.

- \( \varphi \) is union closed if, for any set of \( \Phi \)-teams \( T \) such that \( \forall T \in T : T \models \varphi \) we have \( \bigcup T \models \varphi \).
- \( \varphi \) is downward closed if, for any \( \Phi \)-teams \( T_1, T_2 \), if \( T_2 \models \varphi \) and \( T_1 \subseteq T_2 \), we have \( T_1 \models \varphi \).
- \( \varphi \) is upward closed if, for any \( \Phi \)-teams \( T_1, T_2 \), if \( T_2 \models \varphi \) and \( T_1 \supseteq T_2 \), we have \( T_1 \models \varphi \).
- \( \varphi \) has the empty team property if \( \emptyset \not\models \varphi \).
- \( \varphi \) is flat if, for any \( \Phi \)-team \( T \), \( T \models \varphi \) if and only if \( \{ s \} \models \varphi \) for all \( s \in T \).

Clearly, a formula is flat if and only if it is union closed, downward closed, and has the empty team property.

Proposition 2.7. Let \( \varphi, \psi \in \text{PL}(\Sigma) \) such that at least one of \( \varphi \) and \( \psi \) is downward closed. Then \( \varphi \lor \psi \equiv \varphi \checkmark \psi \).

Proof. Obviously, \( \varphi \checkmark \psi \) entails \( \varphi \lor \psi \). Conversely, if \( T \models \varphi \lor \psi \) via some split \( (T_1, T_2) \) of \( T \), then either \( T_1 \setminus T_2 \) will still satisfy \( \varphi \) or \( T_2 \setminus T_1 \) will satisfy \( \psi \). So either \( (T_1 \setminus T_2, T_2) \) or \( (T_1, T_2 \setminus T_1) \) is a strict split of \( T \) witnessing \( \varphi \checkmark \psi \).\hfill\Box

Proposition 2.8. Every \( \sim \)-free \( \text{PL}(\{\wedge, \vee, \checkmark\}) \)-formula is flat.

Proof. Follows from Yang and Väänänen [YV17] for \( \text{PL}(\{\wedge, \vee\}) \), and together with Proposition 2.7 for \( \text{PL}(\{\wedge, \vee, \checkmark\}) \).\hfill\Box

An important property of propositional (and other) logics is locality, which means that formulas depend only on the assignment to variables that actually occur in the formula. This property can be generalized to team semantics.

Definition 2.9. If \( T \) is a \( \Psi \)-team and \( \Phi \subseteq \Psi \), the projection of \( T \) onto \( \Phi \), denoted \( T|\Phi \), is defined as the \( \Phi \)-team \{ \( s|\Phi | s \in T \} \), where \( s|\Phi \) is the the restriction of the function \( s \) to the domain \( \Phi \).

Definition 2.10. A formula \( \varphi \in \text{PL}(\Sigma, \Phi) \) has locality if, for any domain \( \Psi \supseteq \Phi \) and \( \Psi \)-team \( T \), it holds \( T \models \varphi \) if and only if \( T|\Phi \models \varphi \).

Proposition 2.11 ([YV17]). Every \( \text{PL}(\{\wedge, \sim, \checkmark\}) \)-formula has locality.

Note that locality quickly fails if we admit strict splitting \( \checkmark \) (cf. Yang and Väänänen [YV17]). The formula \( \psi := \sim p \checkmark \sim p \checkmark \sim p \) is an easy counter-example to the locality of \( \text{PL}(\{\checkmark\}) \). No team with domain \{ \( p \) \} does satisfy \( \psi \), since it needs at least three assignments in the team, but for example the maximal \{ \( p, q \} \)-team satisfies \( \psi \).
\[
\begin{align*}
= (\vec{\alpha}, \vec{\beta}) & \equiv \bigvee_{\vec{s} \in \{\top, \bot\}^n} (\vec{\alpha} = \vec{s} \land \bigwedge_{i=1}^{m} (\beta_i \lor \neg \beta_i)) \\
\vec{\alpha} \perp \vec{\beta} & \equiv \bigvee_{\vec{s} \in \{\top, \bot\}^k} (\vec{\gamma} = \vec{s} \land \vec{\alpha} \perp \vec{\beta}) \\
\vec{\alpha} \perp \vec{\beta} & \equiv \bigwedge_{\vec{s} \in \{\top, \bot\}^n} \bigwedge_{\vec{s}' \in \{\top, \bot\}^m} ((\vec{\alpha} \neq \vec{s}) \lor (\vec{\beta} \neq \vec{s}') \lor E(\vec{\alpha} = \vec{s} \land \vec{\beta} = \vec{s}')) \\
\vec{\alpha} \subseteq \vec{\beta} & \equiv \bigwedge_{\vec{s} \in \{\top, \bot\}^n} ((\vec{\alpha} \neq \vec{s}) \lor E(\vec{\beta} = \vec{s})) \\
\vec{\alpha} | \vec{\beta} & \equiv \bigwedge_{\vec{s} \in \{\top, \bot\}^n} ((\vec{\alpha} \neq \vec{s}) \lor (\vec{\beta} \neq \vec{s})) \\
\vec{\alpha} \Upsilon \vec{\beta} & \equiv \bigvee_{\vec{s} \in \{\top, \bot\}^n} (\vec{\alpha} = \vec{s} \land \bigvee_{i=1}^{m} (E\beta_i \land E\neg\beta_i))
\end{align*}
\]

Table 2: Exponential translations of atoms in the positive fragment, where \(\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)\), \(\vec{\beta} = (\beta_1, \ldots, \beta_m)\) \((n = m \text{ for } \subseteq \text{ and } |)\) and \(\vec{\gamma} = (\gamma_1, \ldots, \gamma_k)\).

**Definition 2.12** (Satisfiability). A formula \(\varphi\) is \(\Phi\)-satisfiable if \(T \models \varphi\) for at least one \(\Phi\)-team \(T\).

The domain is crucial here: The previous example formula \(\psi\) is \(\{p, q\}\)-satisfiable, but not \(\{p\}\)-satisfiable.

We proceed with the definition of the size of a formula. The literature contains many different accounts of what should be considered formula size. We take as our basic concept the length of the formula as a string. Since in team semantics the domain is often fixed and finite, we consider each proposition symbol to be only one symbol in the string. In Section 3 we will define a different measure of formula size for the purposes of the formula size game, but we formulate our results for length.

**Definition 2.13.** The length of a formula \(\varphi \in \text{PL}(\Sigma)\), denoted by \(|\varphi|\), is the length of \(\varphi\) as a string, counting proposition symbols as one symbol.

If \(\alpha\) is a purely propositional formula and not a proposition, then technically \(\neg\alpha\) is not a formula; then by \(\neg\alpha\) we refer to the formula that is obtained from \(\alpha\) by pushing \(\neg\) inwards using classical laws, i.e., \(\neg(\beta \land \gamma) := (\neg\beta \lor \neg\gamma)\) and \(\neg(\beta \lor \gamma) := (\neg\beta \land \neg\gamma)\). In the positive fragment, we define the shorthands \(\neg\top := \bot\) and \(\neg\bot := \top\), which defines exactly the non-empty teams, and \(E\alpha := \top \lor (NE \land \alpha)\), which expresses that at least one assignment in the team satisfies the purely propositional formula \(\alpha\). For tuples \(\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)\) and \(\vec{\beta} = (\beta_1, \ldots, \beta_n)\) of purely propositional formulas, we write \(\vec{\alpha} = \vec{\beta}\) for the formula \(\bigwedge_{i=1}^{n} ((\alpha_i \land \beta_i) \lor (\neg\alpha_i \land \neg\beta_i))\) and \(\vec{\alpha} \neq \vec{\beta}\) for \(\neg(\vec{\alpha} = \vec{\beta})\).
By slight abuse of notation, we will write \( s(\varphi) \) even if \( \varphi \) is not a proposition, and mean
\[
s(\varphi) = \begin{cases} 
1 & \text{if } \{s\} \models \varphi, \\
0 & \text{else.}
\end{cases}
\]
If \( \vec{\varphi} = (\varphi_1, \ldots, \varphi_n) \) is a tuple of formulas, then \( s(\vec{\varphi}) \) is short for the vector \( (s(\varphi_1), \ldots, s(\varphi_n)) \in \{0,1\}^n \).

We consider the following atoms of dependency, where \( \vec{\varphi}, \vec{\psi}, \vec{\theta} \) are (possibly empty) tuples of formulas:

**Dependence:** \( = (\vec{\varphi}; \vec{\psi}) \):
\[
T \models = (\vec{\varphi}; \vec{\psi}) \iff \forall s, s' \in T: s(\vec{\varphi}) = s'(\vec{\varphi}) \Rightarrow s(\vec{\psi}) = s'(\vec{\psi})
\]

**Independence:** \( \vec{\varphi} \perp \vec{\psi} \):
\[
T \models \vec{\varphi} \perp \vec{\psi} \iff \forall s, s' \in T: \exists s'' \in T: s(\vec{\varphi}) = s''(\vec{\varphi}) \text{ and } s'(\vec{\psi}) = s''(\vec{\psi})
\]

**Conditional independence:** \( \vec{\varphi} \perp \vec{\psi} \vec{\theta} \):
\[
T \models \vec{\varphi} \perp \vec{\psi} \vec{\theta} \iff \forall s, s' \in T: \text{ if } s(\vec{\psi}) = s'(\vec{\psi}) \text{ then } \\
\exists s'' \in T: s(\vec{\varphi} \vec{\psi}) = s''(\vec{\varphi} \vec{\psi}) \text{ and } s'(\vec{\theta}) = s''(\vec{\theta})
\]

**Inclusion:** \( \vec{\varphi} \subseteq \vec{\psi} \), where \( \vec{\varphi} \) and \( \vec{\psi} \) have equal length:
\[
T \models \vec{\varphi} \subseteq \vec{\psi} \iff \forall s \in T \exists s' \in T: s(\vec{\varphi}) = s'(\vec{\psi})
\]

**Exclusion:** \( \vec{\varphi} \nmid \vec{\psi} \), where \( \vec{\varphi} \) and \( \vec{\psi} \) have equal length:
\[
T \models \vec{\varphi} \nmid \vec{\psi} \iff \forall s \in T \forall s' \in T: s(\vec{\varphi}) \neq s'(\vec{\psi})
\]

**Anonymity:** \( \vec{\varphi} \Upsilon \vec{\psi} \):
\[
T \models \vec{\varphi} \Upsilon \vec{\psi} \iff \forall s \in T \exists s' \in T: s(\vec{\varphi}) = s'(\vec{\varphi}) \text{ and } s(\vec{\psi}) \neq s'(\vec{\psi})
\]

Originally, the dependence and independence atoms were introduced by Väänänen [Vä07] and Grädel and Väänänen [GV13]. Inclusion and exclusion were considered by Galliani [Gal12]. The anonymity atom is due to Väänänen [Vä19]. The propositional counterparts of all the above atoms, except for the anonymity atom, were first studied by Yang [Yan14].

**Proposition 2.14.** Let \( \Sigma = \{\land, \lor, \lor'\} \) or \( \Sigma = \{\land, \lor, \lor''\} \). The atoms of dependence, conditional independence, inclusion, exclusion and anonymity are expressible by \( \text{PL}(\Sigma) \)-formulas of size \( 2^{O(n)} \).

**Proof.** See Table 2 for \( \text{PL}(\{\land, \lor, \lor\}) \)-formulas defining each atom. For \( \text{PL}(\{\land, \lor, \lor''\}) \), it is easy to check that replacing each occurrence of \( \lor \) with \( \lor' \) leads to an equivalent formula. \( \square \)
3. Exponential lower bounds for team properties

Though the length of a formula is the most immediate measure of formula size, it is not the most practical one in terms of defining a formula size game. For a measure better suited to the game we have chosen the number of literals in a formula, which we call width.

**Definition 3.1.** The *width* of a formula \( \varphi \in PL(\Sigma) \), denoted by \( \text{wd}(\varphi) \), is defined recursively as follows:

- \( \text{wd}(l) = 1 \) for a literal \( l \),
- \( \text{wd}(\psi \chi \theta) = \text{wd}(\psi) + \text{wd}(\theta) \), where \( \chi \in \Sigma \) is binary,
- \( \text{wd}(\neg \psi) = \text{wd}(\psi) \), where \( \neg \in \Sigma \) is unary.

For the actual upper and lower bounds we prove, the difference between length and width is inconsequential. The number of binary connectives, and therefore parentheses, depends on the number of literals and the number of negations of either kind for a minimal formula is bounded by the number of literals. Note that for the game we also assume formulas to be in negation normal form, but this doesn’t affect the width of formulas.

3.1. A formula size game for team semantics. Let \( \mathcal{A}_0 \) and \( \mathcal{B}_0 \) be sets of \( \Phi \)-teams and let \( k_0 \) be a natural number. Let \( \Sigma \subseteq \{ \forall, \land, \lor, \forall, \land \} \) be a set of connectives. Note that if the strong negation \( \sim \) is freely available in the fragment under consideration, then either neither or both of a pair of dual operators must be included in \( \Sigma \).

The formula size game \( FS_{k_0}^\Sigma(\mathcal{A}_0, \mathcal{B}_0) \) for \( PL(\Sigma) \) has two players, \( S \) (Samson) and \( D \) (Delilah). Positions of the game are of the form \((k, \mathcal{A}, \mathcal{B})\), where \( \mathcal{A} \) and \( \mathcal{B} \) are sets of teams and \( k \) is a natural number.

The goal of \( S \) is to construct a formula \( \varphi \) that separates \( \mathcal{A} \) from \( \mathcal{B} \), which means that \( T \vDash \varphi \) for every team \( T \in \mathcal{A} \) and \( T \not\vDash \varphi \) for every team \( T \in \mathcal{B} \). In other words, \( \varphi \) separates \( \mathcal{A} \) from \( \mathcal{B} \) if \( \mathcal{A} \vDash \varphi \) and \( \mathcal{B} \vDash \neg \varphi \).

The starting position is \((k_0, \mathcal{A}_0, \mathcal{B}_0)\). If \( k_0 = 0 \), \( D \) wins the game. In a position \((k, \mathcal{A}, \mathcal{B})\) with \( k \geq 1 \), \( S \) must make one of \( |\Sigma| + 1 \) moves to continue the game. The available moves are the ones given by \( \Sigma \) and the literal move. The moves work as follows:

- \( \forall \)-move: \( S \) chooses subsets \( \mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A} \) such that \( \mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A} \) and natural numbers \( k_1, k_2 > 0 \) such that \( k_1 + k_2 = k \). Then \( D \) chooses \( i \in \{1, 2\} \). The game continues from the position \((k_1, \mathcal{A}_i, \mathcal{B})\).
- \( \wedge \)-move: Same as the \( \forall \)-move with the roles of \( \mathcal{A} \) and \( \mathcal{B} \) switched.
- \( \lor \)-move: For every team \( A \in \mathcal{A} \), \( S \) chooses a split \( (A_1, A_2) \). Let \( \mathcal{A}_i = \{ A_i \mid A \in \mathcal{A}_i \} \) for \( i \in \{1, 2\} \). For every team \( B \in \mathcal{B} \), \( S \) chooses a function \( f_B : \text{Sp}(B) \rightarrow \{1, 2\} \). Let \( \mathcal{B}_i = \{ f_B^{-1}(i) \mid B \in \mathcal{B} \} \) for \( i \in \{1, 2\} \). Finally, \( S \) chooses natural numbers \( k_1, k_2 > 0 \) such that \( k_1 + k_2 = k \). Then \( D \) chooses a number \( i \in \{1, 2\} \). The game continues from the position \((k_1, \mathcal{A}_i, \mathcal{B}_i)\).
- \( \otimes \)-move: Same as the \( \lor \)-move with the roles of \( \mathcal{A} \) and \( \mathcal{B} \) switched.
- \( \forall \)-move: Same as the \( \lor \)-move except all splits \( (A_1, A_2) \) and \( (B_1, B_2) \) considered are strict.
- \( \otimes \)-move: Same as the \( \forall \)-move with the roles of \( \mathcal{A} \) and \( \mathcal{B} \) switched.
- Literal move: \( S \) chooses a \( \Phi \)-literal \( l \). If \( l \) separates \( \mathcal{A} \) from \( \mathcal{B} \), \( S \) wins. Otherwise, \( D \) wins.

Note that since for all the connective moves \( k_1, k_2 > 0 \), the resource \( k \) decreases in each move, and if \( k = 1 \), only the literal move is available. Thus, in a finite number of moves, \( S \) must make a literal move which will end the game and one of the players will win.
We first prove that the game is indeed connected to the width of PL($\Sigma$)-formulas. As per usual for these types of games, the winning strategies of the players are key.

**Theorem 3.2.** Let $A_0$ and $B_0$ be sets of teams and let $k_0 \in \mathbb{N}$. Then the following conditions are equivalent:

1. $S$ has a winning strategy for the game $FS_{k_0}^\Sigma(A_0, B_0)$.
2. There is a formula $\phi \in PL(\Sigma)$ with $wd(\phi) \leq k_0$ which separates $A_0$ from $B_0$.

**Proof.** We prove the equivalence of (1)$_{k_0}$ and (2)$_{k_0}$ by induction on $k_0$.

If $k_0 = 0$, then $D$ wins the game immediately so $S$ does not have a winning strategy. Correspondingly there are no $PL(\Sigma)$-formulas with width 0.

Let $k_0 > 1$ and assume that the equivalence of (1)$_k$ and (2)$_k$ holds for all natural numbers $k < k_0$ and all sets of teams $A$ and $B$.

(1)$_{k_0} \Rightarrow$ (2)$_{k_0}$: Let $\delta$ be a winning strategy of $S$ for the game $FS_{k_0}^\Sigma(A_0, B_0)$. We divide the proof into cases according to the first move of $\delta$. We handle all operators possibly in $\Sigma$ except for dual cases.

- **Literal move:** Since $S$ is playing according to the winning strategy $\delta$, the literal $l$ chosen by $S$ separates $A_0$ from $B_0$. In addition, $wd(l) = 1 \leq k_0$.

- **$\ominus$-move:** Let $(k_1, A_1, B_0)$ and $(k_2, A_2, B_0)$ be the successor positions chosen by $S$ according to $\delta$. Since $\delta$ is a winning strategy, $S$ has a winning strategy for both games $FS_{k_1}^\Sigma(A_1, B_0)$ and $FS_{k_2}^\Sigma(A_2, B_0)$. By induction hypothesis, there are formulas $\psi_1$ with $wd(\psi_1) \leq k_1$ that separate $A_1$ from $B_0$. Let $\phi = \psi_1 \ominus \psi_2$. We have $A_0 = A_1 \cup A_2$ so $A_0 \vDash \phi$. On the other side we have $B_0 \vDash \psi_2$ so $B \vDash \phi$. Finally $wd(\phi) = wd(\psi_1) + wd(\psi_2) \leq k_1 + k_2 = k_0$.

- **$\lor$-move:** Let $(k_1, A_1, B_1)$ and $(k_2, A_2, B_1)$ be the successor positions chosen by $S$ according to $\delta$. Again by induction hypothesis there are formulas $\psi_i$ with $wd(\psi_i) \leq k_i$ which separate $A_i$ from $B_i$. Let $\varphi = \psi_1 \lor \psi_2$. For each $A \in A_0$ S chose a split $(A_1, A_2)$. Now $A_1 \vDash \psi_1$ and $A_2 \vDash \psi_2$ so $A \vDash \varphi$. On the other side, for each $B \in B_0$, S chose a function $f_B : Sp(B) \rightarrow \{1, 2\}$. For each $(B_1, B_2) \in Sp(B)$, if $f_B(B_1, B_2) = i$, then $B_i \vDash \psi_i$. Thus $B \vDash \varphi$. The width of $\varphi$ is as in the previous case.

- **$\forall$-move:** Same as the $\lor$-move except all splits considered are strict.

(2)$_{k_0} \Rightarrow$ (1)$_{k_0}$: Let $\psi \in PL(\Sigma)$ with $wd(\psi) \leq k_0$ which separates $A_0$ from $B_0$. We give the first move of the winning strategy of $S$ and obtain the rest of it by the induction hypothesis. We divide the proof into cases according to the outermost connective of $\psi$. We again handle only one of each pair of dual cases.

- **$\psi$ is a literal:** We know that $\varphi$ separates $A_0$ from $B_0$ so $S$ wins by making a literal move.

- **$\varphi = \psi_1 \ominus \psi_2$:** $S$ chooses $A_i = \{A \in A_0 \mid A \vDash \psi_i\}$ for $i \in \{1, 2\}$, $k_1 = wd(\psi_1)$ and $k_2 = k - k_1$. Since $\varphi$ separates $A_0$ from $B_0$, we have $A_0 \vDash \varphi$ so $A_1 \cup A_2 = A$. On the other side, $B_0 \vDash \varphi$ so $B \vDash \varphi$ and $B \vDash \varphi$ for every $B \in B_0$. No, now matter which number $i \in \{1, 2\}$ D chooses, in the following position $(k_1, A_i, B_0)$, the formula $\psi_i$ will separate $A_i$ from $B_0$. In addition, $k_1 \leq wd(\psi_1)$ and $k_2 = k_0 - k_1 \leq wd(\psi_2) - wd(\psi_1) = wd(\psi_2)$. By induction hypothesis S has a winning strategy for both games $FS_{k_i}^\Sigma(A_i, B_0)$.

- **$\varphi = \psi_1 \lor \psi_2$:** Again we have $A_0 \vDash \varphi$ so for every $A \in A$, there is a split $(A_1, A_2)$ such that $A_1 \vDash \psi_1$ and $A_2 \vDash \psi_2$. $S$ choose such a split for every $A \in A_0$. On the other side, $B_0 \vDash \varphi$ so for every $B \in B$ and every split $(B_1, B_2)$ we have $B_1 \vDash \psi_1$ or $B_2 \vDash \psi_2$. For each $B \in B_0$, $S$ chooses $f_B$ so that if $f_B(B_1, B_2) = i$, then $B_i \vDash \psi_i$. Now, no matter which number $i \in \{1, 2\}$ D chooses, in the following position $(k_i, A_i, B_i)$, the formula $\psi_i$...
will separate $A_i$ from $B_i$. S deals with the resource $k_0$ just like in the previous case. By induction hypothesis $S$ has a winning strategy for both games $F_{k_0}^{\Sigma}(A_i, B_i)$.

- $\varphi = \psi_1 \lor \psi_2$: Same as the $\lor$-case except all splits considered are strict.

Before we move on to the lower bounds, we prove a very standard lemma for formula size games stating that if at any time the same team ends up on both sides of the game, $D$ wins.

**Lemma 3.3.** If in a position $P = (k, A, B)$ there is a team $T \in A \cap B$, $D$ has a winning strategy from position $P$.

**Proof.** As long as there is $T \in A \cap B$, if $S$ makes a literal move, $D$ wins. We show that $D$ can maintain this condition. We again skip the cases of dual operators.

- $\ominus$-move: $S$ chooses sets $A_1, A_2 \subseteq A$. Since $A_1 \cup A_2 = A$, we have $T \in A_i$ for some $i \in \{1, 2\}$.
  Then $D$ chooses the following position $(k_i, A_i, B)$ and we have $T \in A_i \cap B$.

- $\lor$-move: Let $(T_1, T_2)$ be the split $S$ chooses for $T$ on the left side. On the right side $S$ must choose $i = f_T(T_1, T_2) \in \{1, 2\}$. Then $D$ chooses the following position $(k_i, A_i, B_i)$ and we have $T_i \in A_i \cap B_i$.

- $\lor$-move: Same as the $\lor$-move except the split must be strict.

Since $S$ must eventually make a literal move, $D$ wins the game.

3.2. Lower bounds via the game. In this section we use the formula size game to show lower bounds for the lengths of formulas defining atoms of dependency in the positive fragment of propositional team logic. We first state all of the bounds as a theorem and prove them in the rest of the section.

For natural numbers $k$ and $m$, $[k]_m$ is the remainder of $k$ modulo $m$.

**Theorem 3.4.** Let $\Sigma = \{\ominus, \land, \lor, \forall\}$, $n, m \geq 1$, and $\Phi_n = \{p_1, \ldots, p_n\}$.

- If $m \leq 2^n$ and $k < m$, then a PL($\Sigma$)-formula, that defines the property $|T| \equiv k \pmod{m}$ of $\Phi_n$-teams $T$, has length at least $2^n - [2^n - k]_m$. In particular, a formula that defines even parity has length at least $2^n$.

- A PL($\Sigma$)-formula that defines cardinality $k \leq 2^n$ of $\Phi_n$-teams has length at least $k$.

- A PL($\Sigma$)-formula that defines $p_1 \cdots p_n \leq q_1 \cdots q_m$ has length at least $2^n$.

- A PL($\Sigma$)-formula that defines $p_1 \cdots p_n \perp q_1 \cdots q_m$ has length at least $2^{n+m}$.

- A PL($\Sigma$)-formula that defines $p_1 \cdots p_n(q_1 \cdots q_m)$ has length at least $2^{n+1}$.

Our approach to proving these bounds is similar to that of Hella and Väänänen in [HV15]. They used a formula size game for propositional logic to show that defining the parity of the number of ones in a propositional assignment of length $n$ requires a formula of length $n^2$. We focus on teams that differ only by one assignment and define a measure named *density* as in [HV15], although our definition is slightly different.

**Definition 3.5.** Let $T$ be a team. A team $T'$ is a neighbour of $T$, if $T' = T \setminus \{s\}$ for some assignment $s \in T$.

Let $A$ be a set of teams. We denote the number of neighbours of $T$ in the set $A$ by

$$N(T, A) = |\{A \mid A \in A, A \text{ is a neighbour of } T\}|.$$  

The *density* of the pair $(A, B)$ is

$$D(A, B) = \max\{N(A, B) \mid A \in A\}.$$
We shall use density as an invariant for the formula size game. Essentially we will show that a certain number of the resource \( k \) must be expended before a literal move can be made. First we show that literal moves cannot be made when density is too high.

**Lemma 3.6.** If \( D(\mathcal{A}, \mathcal{B}) > 1 \), then no literal separates \( \mathcal{A} \) from \( \mathcal{B} \).

**Proof.** If \( D(\mathcal{A}, \mathcal{B}) > 1 \), at least one team \( A \in \mathcal{A} \) has two neighbours \( B_1, B_2 \in \mathcal{B} \). Now any positive literal \( l \) (with respect to \( \sim \)) true in \( A \) is also true in \( B_1 \) and \( B_2 \) since they are subteams of \( A \). On the other hand, since \( B_1 \) and \( B_2 \) are different neighbours of \( A \), we have \( B_1 \cup B_2 = A \). For a negative literal \( \sim l \), assume that \( B_1 \not\sim l \) and \( B_2 \not\sim l \). This means that \( B_1 \models l \) and \( B_2 \models l \) so by union closure, \( A \models l \). Thus \( A \not\models l \). \( \square \)

We proceed to show that density behaves well with respect to the moves of the game.

**Lemma 3.7.** In a position \((k, \mathcal{A}, \mathcal{B})\), if \( S \) makes a \( \sqcap \)-move, and the possible following positions are \((k_1, \mathcal{A}_1, \mathcal{B})\) and \((k_2, \mathcal{A}_2, \mathcal{B})\), then
\[
D(\mathcal{A}_1, \mathcal{B}) + D(\mathcal{A}_2, \mathcal{B}) \geq D(\mathcal{A}, \mathcal{B}).
\]

**Proof.** Let \( A \) be one of the teams in \( \mathcal{A} \) with most neighbours in \( \mathcal{B} \). Since \( \mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A} \), we may assume by symmetry that \( A \in \mathcal{A}_1 \). Since all the same neighbours of \( A \) are still in \( \mathcal{B} \), we get \( D(\mathcal{A}_1, \mathcal{B}) + D(\mathcal{A}_2, \mathcal{B}) \geq D(\mathcal{A}_1, \mathcal{B}) \geq D(\mathcal{A}, \mathcal{B}) \). \( \square \)

**Lemma 3.8.** In a position \((k, \mathcal{A}, \mathcal{B})\), if \( S \) makes a \( \land \)-move, and the possible following positions are \((k_1, \mathcal{A}, \mathcal{B}_1)\) and \((k_2, \mathcal{A}, \mathcal{B}_2)\), then
\[
D(\mathcal{A}, \mathcal{B}_1) + D(\mathcal{A}, \mathcal{B}_2) \geq D(\mathcal{A}, \mathcal{B}).
\]

**Proof.** Let \( A \) be one of the teams in \( \mathcal{A} \) with most neighbours in \( \mathcal{B} \). Since \( \mathcal{B}_1 \cup \mathcal{B}_2 = \mathcal{B} \), the neighbours of \( A \) are split between \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) so \( D(\mathcal{A}, \mathcal{B}_1) + D(\mathcal{A}, \mathcal{B}_2) \geq N(\mathcal{A}, \mathcal{B}_1) + N(\mathcal{A}, \mathcal{B}_2) \geq N(\mathcal{A}, \mathcal{B}) = D(\mathcal{A}, \mathcal{B}) \). \( \square \)

**Lemma 3.9.** In a position \((k, \mathcal{A}, \mathcal{B})\), if \( S \) makes a \( \lor \)-move or \( \lor \)-move, and the possible following positions are \((k_1, \mathcal{A}, \mathcal{B}_1)\) and \((k_2, \mathcal{A}, \mathcal{B}_2)\), then
\[
D(\mathcal{A}, \mathcal{B}_1) + D(\mathcal{A}, \mathcal{B}_2) \geq D(\mathcal{A}, \mathcal{B})
\]

or \( D \) has a winning strategy from one of the following positions.

**Proof.** Let \( A \) be one of the teams in \( \mathcal{A} \) with the most neighbours in \( \mathcal{B} \) and let \((A_1, A_2)\) be the (strict) split of \( A \) chosen by \( S \). Suppose \( B = A \setminus \{a\} \) is a neighbour of \( A \) in \( \mathcal{B} \). Then \((B_1, B_2) := (A_1 \setminus \{a\}, A_2 \setminus \{a\})\) is a split of \( B \), and is strict if \((A_1, A_2)\) is strict. Let \( f_B : (S)Sp(B) \rightarrow \{1, 2\} \) be the function chosen by \( S \) for the team \( B \) and \( i := f_B(B_1, B_2) \). If \( a \notin A_i \), then \( A_i = B_i \in \mathcal{A_i} \cap \mathcal{B}_i \) and by Lemma 3.3, \( D \) has a winning strategy from the position \((k_i, \mathcal{A}_i, \mathcal{B}_i)\). Consequently, we proceed with the case where \( a \in A_i \) for all \( A, B \) as above. Then \( B_i = A_i \setminus \{a\} \) is a neighbour of \( A_i \) in \( \mathcal{B}_i \), i.e., on the opposite side in the position \((k_i, \mathcal{A}_i, \mathcal{B}_i)\).

We see that for each neighbour \( B \) of \( A \), we obtain a neighbour of \( A_1 \) in \( \mathcal{B}_1 \), or one of \( A_2 \) in \( \mathcal{B}_2 \). Furthermore, if \( B = A \setminus \{a\} \) and \( B' = A \setminus \{a'\} \) are distinct neighbours of \( A \), then \( A_1 \setminus \{a\} \) and \( A_1 \setminus \{a'\} \) are distinct neighbours of \( A_1 \). For this reason, \( D(\mathcal{A}_1, \mathcal{B}_1) + D(\mathcal{A}_2, \mathcal{B}_2) \geq D(\mathcal{A}, \mathcal{B}) \). \( \square \)

For the rest of this section, we look at a fragment \( PL(\Sigma) \) with operators from \( \Sigma = \{\sqcup, \land, \lor, \lor\} \). All results are lower bounds for this fragment and are naturally inherited by any fragment \( PL(\Sigma') \), where \( \Sigma' \subseteq \Sigma \).

We gather the above lemmas as the following theorem stating the usefulness of density.
Theorem 3.10. If \( k_0 < D(\mathbb{A}_0, \mathbb{B}_0) \), then \( D \) has a winning strategy in the game \( FS_{k_0}^\Sigma (\mathbb{A}_0, \mathbb{B}_0) \).

Proof. We define a strategy \( \delta \) for \( D \) and show that if \( D \) plays according to \( \delta \), the condition \( k < D(\mathbb{A}, \mathbb{B}) \) is maintained in all positions \((k, \mathbb{A}, \mathbb{B}) \).

Let \((k, \mathbb{A}, \mathbb{B}) \) be a position of the game \( FS_{k_0}^\Sigma (\mathbb{A}_0, \mathbb{B}_0) \). By induction hypothesis, \( k < D(\mathbb{A}, \mathbb{B}) \).

- If \( S \) makes a \( \emptyset \)-move, then by Lemma 3.7, \( D(\mathbb{A}_1, \mathbb{B}) + D(\mathbb{A}_2, \mathbb{B}) \geq D(\mathbb{A}, \mathbb{B}) \). Assume for contradiction that \( k_1 \geq D(\mathbb{A}_i, \mathbb{B}) \) for \( i \in \{1, 2\} \). Then
  \[
  k = k_1 + k_2 \geq D(\mathbb{A}_1, \mathbb{B}) + D(\mathbb{A}_2, \mathbb{B}) \geq D(\mathbb{A}, \mathbb{B}) > k,
  \]
which is a contradiction. Therefore \( k_i < D(\mathbb{A}_i, \mathbb{B}) \) for some \( i \in \{1, 2\} \) and \( D \) chooses that \( i \) to continue the game.

- The case of a \( \land \)-move is similar, utilizing Lemma 3.8.

- If \( S \) makes a \( \lor \)-move, then by Lemma 3.9, \( D \) has a winning strategy from a following position \((k_i, \mathbb{A}_i, \mathbb{B}_i) \) or \( D(\mathbb{A}_1, \mathbb{B}_1) + D(\mathbb{A}_2, \mathbb{B}_2) \geq D(\mathbb{A}, \mathbb{B}) \). In the first case \( D \) chooses the position \((k_i, \mathbb{A}_i, \mathbb{B}_i) \) and uses the winning strategy to win the game. In the second case \( D \) chooses a following position that maintains the condition \( k < D(\mathbb{A}, \mathbb{B}) \) just like in the \( \emptyset \)-case above.

- If \( S \) makes a literal move, since \( D(\mathbb{A}, \mathbb{B}) > k \geq 1 \), by Lemma 3.6, \( D \) wins the game. \( \square \)

Lemma 3.11. No set \( \mathbb{A} \) of \( \Phi \)-teams can be defined with a \( PL(\Sigma) \)-formula of width less than \( D(\mathbb{A}, \text{Tms}(\Phi) \setminus \mathbb{A}) \).

Proof. Let \( \mathbb{B} := \text{Tms}(\Phi) \setminus \mathbb{A} \). Now defining \( \mathbb{A} \) amounts to separating \( \mathbb{A} \) from \( \mathbb{B} \). If \( k \leq D(\mathbb{A}, \mathbb{B}) \), then by Theorem 3.10, \( D \) has a winning strategy in the game \( FS_{k_0}^\Sigma (\mathbb{A}, \mathbb{B}) \) and by Theorem 3.2, \( \mathbb{A} \) and \( \mathbb{B} \) cannot be separated by a formula with width \( k \). \( \square \)

With the above lemma, we are now in the position to prove the main theorem of this section.

Proof of Theorem 3.4. We find in each case a team which satisfies the desired property \( \mathbb{A} \) and has the desired number of neighbours which do not. Then \( D(\mathbb{A}, \text{Tms}(\Phi) \setminus \mathbb{A}) \) is greater than or equal to the desired number and the claim follows from Lemma 3.11 along with the fact that length is always greater than width.

- First is the cardinality \( k \) \((\text{mod} \ m) \) of \( \Phi_\nu \)-teams. Let \( k' = 2^n - \lfloor 2^n - k \rfloor_m \). We first note that \( k' \leq 2^n \) so there is a \( \Phi_\nu \)-team \( T_1 \) with cardinality \( k' \). Furthermore,
  \[
  k' \equiv \lfloor 2^n - 2^n + k \rfloor_m \equiv k \mod m.
  \]
Now \( |T_1| \equiv k \mod m \) and \( T_1 \) has \( k' \) neighbours with smaller cardinality.

- For a specific cardinality \( k \leq 2^n \), if \( T_2 \) is any team with cardinality \( k \), then \( T_2 \) clearly has \( k \) neighbours with a smaller cardinality.

- Next is the inclusion atom \( p_1 \cdots p_n \subseteq q_1 \cdots q_n \). If \( s(p_1) \cdots s(p_n) \) is a binary representation of the number \( i \), we denote this by \( s(\overline{p}) = i \). For \( i \in \{0, \ldots, 2^n - 1\} \), let \( s_i \) be the assignment with \( s_i(\overline{p}) = i \) and \( s_i(\overline{q}) = [i + 1]_{2^n} \). Let \( T_3 := \{ s_i \mid i \in \{0, \ldots, 2^n - 1\} \} \). Now \( \overline{p} \) and \( \overline{q} \) both get all possible values so \( T_3 \models p_1 \cdots p_n \subseteq q_1 \cdots q_n \). Furthermore, for any \( s_i \in T_3 \), we have \( T_3 \setminus \{s_i\} \not\models p_1 \cdots p_n \subseteq q_1 \cdots q_n \) since \( \overline{p} \) gets the value \([i + 1]_{2^n} \) but \( \overline{q} \) does not. Thus there are \(|T_3| = 2^n \) neighbours of \( T_3 \) which do not satisfy the inclusion atom.

- For the independence atom \( p_1 \cdots p_n \perp q_1 \cdots q_m \), let \( T_4 \) be the full team with domain \( \{p_1, \ldots, p_n, q_1, \ldots, q_m\} \). Clearly \( T_4 \models p_1 \cdots p_n \perp q_1 \cdots q_m \) and \( T_4 \setminus \{s\} \not\models p_1 \cdots p_n \perp
\( q_1 \ldots q_m \) for any assignment \( s \in T_4 \). Thus there are \( |T_4| = 2^{n+m} \) neighbours of \( T_4 \) which do not satisfy the independence atom.

- Finally, for the anonymity atom \( p_1 \ldots p_n \forall q \), let \( T_5 \) be the full team with domain \( \{p_1, \ldots, p_n, q\} \). Clearly \( T_5 \models p_1 \ldots p_n \forall q \) and \( T_5 \setminus \{s\} \not\models p_1 \ldots p_n \forall q \) for any \( s \in T_5 \).

We now have \( |T_5| = 2^{n+1} \) neighbours of \( T_5 \) which do not satisfy the anonymity atom. \( \square \)

In the above, we did not prove lower bounds for the atoms of dependence and exclusion. The reason for this is that the invariant we use for the formula size game is density, which is defined via the neighbourship relation. The remaining two atoms are downward closed, so a team in \( \mathcal{A} \) cannot have any neighbours in \( \mathcal{B} \) at all. Thus the above strategy fails for these two atoms. We present a different approach in the next section.

3.3. Lower bounds via upper dimension. For the lower bounds of dependence and exclusion atoms we employ the notion of upper dimension, which was successfully used to prove lower bounds by Hella et al. [HLSV14]. Their paper mainly concerns the expressive power of modal dependence logic, but at the end it is shown that defining the dependence atom in modal logic with Boolean disjunction \( \otimes \) requires a formula with length at least \( 2^n \). However, the logic they consider again has downward closure. As the positive fragment is expressively complete, it is obviously not downward closed. Hence, we adapt the technique of Hella et al. accordingly. We first state the lower bounds as a theorem and then prove it in this section.

**Theorem 3.12.** Let \( \Sigma = \{\otimes, \land, \lor, \forall, \exists\} \) and \( n \geq 1 \).

- A \( \text{PL}(\Sigma) \)-formula that defines \( = (p_1 \ldots p_n; q) \) has length at least \( 2^n \).
- A \( \text{PL}(\Sigma) \)-formula that defines \( p_1 \ldots p_n | q_1 \ldots q_n \) has length at least \( 2^n \).

When proving this theorem, we will assume that \( \Sigma = \{\otimes, \land, \lor\} \). We will show in the next subsection that this imposes no restriction on the results, as for every \( \text{PL}(\{\otimes, \land, \lor, \forall, \exists\}) \)-formula that has downward closure, the empty team property and locality, there is an equivalent \( \text{PL}(\{\otimes, \land, \lor\}) \)-formula of at most the same size.

**Definition 3.13.** Let \( \varphi \in \text{PL}(\Sigma, \Phi) \). We say that a set of pairs of \( \Phi \)-teams \( \mathcal{G}(\varphi) \) is a generator of \( \varphi \), if for any \( \Phi \)-team \( T \) it holds that \( T \models \varphi \) precisely if there is \( (S, U) \in \mathcal{G}(\varphi) \) such that \( S \subseteq T \subseteq U \). The upper dimension \( \text{Dim}(\mathcal{G}) \) of \( \mathcal{G} \) is the number of distinct upper bounds in \( \mathcal{G} \):

\[
\text{Dim}(|\mathcal{G}|) := |\{U : (S, U) \in \mathcal{G}\}|.
\]

The upper dimension of \( \varphi \), denoted \( \text{Dim}(\varphi) \), is the minimal upper dimension of a generator of \( \varphi \):

\[
\text{Dim}(\varphi) := \min\{\text{Dim}(\mathcal{G}) \mid \mathcal{G} \text{ is a generator of } \varphi\}.
\]

**Lemma 3.14. Let \( \varphi, \psi \in \text{PL}(\Sigma, \Phi) \) and \( \Phi = \{p_1, \ldots, p_n\} \). We have the following estimates:**

- \( \text{Dim}(l) = 1 \) for any \( \Phi \)-literal \( l \),
- \( \text{Dim}(\varphi \land \psi) \leq \text{Dim}(\varphi) \cdot \text{Dim}(\psi) \),
- \( \text{Dim}(\varphi \lor \psi) \leq \text{Dim}(\varphi) \cdot \text{Dim}(\psi) \),
- \( \text{Dim}(\varphi \otimes \psi) \leq \text{Dim}(\varphi) + \text{Dim}(\psi) \),

**Proof.** For the binary connectives, let \( \mathcal{G}(\varphi) \) and \( \mathcal{G}(\psi) \) be minimal generators of \( \varphi \) and \( \psi \), respectively.
Let $T$ be the full $\Phi$-team. Any positive literal $l \in \{p, \neg p, \top, \bot \mid p \in \Phi\}$ has flatness, so 
\[
\{(\emptyset, \{s \in T \mid \{s\} \vdash l\})\}
\]
generates $l$. Any negative literal $l \in \{\neg p, \neg \neg p, \neg \top, \neg \bot \mid p \in \Phi\}$ is upward closed, so 
\[
\{\{(\{s\}, T) \mid s \in T : \{s\} \vdash l\}\}
\]
generates $l$.

For the conjunction, we show that the set $G(\cap) := \{(S_1 \cup S_2, U_1 \cap U_2) \mid (S_1, U_1) \in G(\varphi), (S_2, U_2) \in G(\psi)\}$ is a generator of $\varphi \land \psi$. Let $T \models \varphi \land \psi$. Now $T \models \varphi$, so there are $(S_1, U_1) \in G(\varphi)$ such that $S_1 \subseteq T \subseteq U_1$. The same goes for $\psi$ and $(S_2, U_2)$. Together, $S_1 \cup S_2 \subseteq T \subseteq U_1 \cap U_2$. Conversely, suppose $S_1 \cup S_2 \subseteq T \subseteq U_1 \cap U_2$. Then in particular $S_1 \subseteq T \subseteq U_1$ and $S_2 \subseteq T \subseteq U_2$, so $T \models \varphi, \psi$, hence $T \models \varphi \land \psi$.

Thus $G(\cap)$ is a generator of $\varphi \land \psi$ and
\[
\dim(\varphi \land \psi) \leq \dim(G(\cap)) \leq \dim(G(\varphi)) \cdot \dim(G(\psi)) = \dim(\varphi) \cdot \dim(\psi).
\]

For the lax disjunction, let $G(\cup) := \{(S_1 \cup S_2, U_1 \cup U_2) \mid (S_1, U_1) \in G(\varphi), (S_2, U_2) \in G(\psi)\}$. Let $T \models \varphi \lor \psi$. Then there is a split $(T_1, T_2)$ of $T$ such that $T_1 \models \varphi$ and $T_2 \models \psi$. Now there are $(S_1, U_1) \in G(\varphi)$ and $(S_2, U_2) \in G(\psi)$ such that $S_1 \subseteq T_i \subseteq U_i$ for $i \in \{1, 2\}$. It is easy to see that then $S_1 \cup S_2 \subseteq T \subseteq U_1 \cup U_2$.

Conversely, assume $(S_1, U_1) \in G(\varphi)$ and $(S_2, U_2) \in G(\psi)$ such that $S_1 \cup S_2 \subseteq T \subseteq U_1 \cup U_2$. Define $T_i := (T \cap U_i) \cup S_i$. Then $(T_1, T_2)$ is a split of $T$, and $S_i \subseteq T_i \subseteq U_i$ (w.l.o.g. $S_i \subseteq U_i$). Consequently, $T_1 \models \varphi$ and $T_2 \models \psi$, so $T \models \varphi \lor \psi$. Thus $G(\cup)$ is a generator of $\varphi \lor \psi$ and
\[
\dim(\varphi \lor \psi) \leq \dim(G(\cup)) \leq \dim(G(\varphi)) \cdot \dim(G(\psi)) = \dim(\varphi) \cdot \dim(\psi).
\]

For the Boolean disjunction, we show that $G(\varphi \cup \psi)$ is a generator of $\varphi \oplus \psi$. Let $T \models \varphi \oplus \psi$. Then $T \models \varphi$ or $T \models \psi$, and there is $(S, U) \in G(\varphi) \cup G(\psi)$ such that $S \subseteq T \subseteq U$. Likewise, if $(S, U) \in G(\varphi) \cup G(\psi)$ exists such that $S \subseteq T \subseteq U$, then either $(S, U) \in G(\varphi)$ or $(S, U) \in G(\psi)$. Hence $T \models \varphi$ or $T \models \psi$. Thus $G(\varphi) \cup G(\psi)$ is a generator of $\varphi \oplus \psi$ and
\[
\dim(\varphi \oplus \psi) \leq \dim(G(\varphi) \cup G(\psi))
\leq \dim(G(\varphi)) + \dim(G(\psi)) = \dim(\varphi) + \dim(\psi).
\]

We include the next lemma for the sake of completeness. It is identical to its counterpart for (downward closed) modal dependence logic.

Let $\text{occ}_{\oplus}(\varphi)$ denote the number of occurrences of $\oplus$ inside $\varphi$, formally:

- $\text{occ}_{\oplus}(l) := 0$ for a literal $l$,
- $\text{occ}_{\oplus}(\varphi \lor \theta) := \text{occ}_{\oplus}(\varphi) + \text{occ}_{\oplus}(\theta)$, where $\lor \in \{\land, \lor\}$,
- $\text{occ}_{\oplus}(\varphi \land \theta) := \text{occ}_{\oplus}(\varphi) + \text{occ}_{\oplus}(\theta) + 1$.

**Lemma 3.15** ([HLSV14, Proposition 5.9]). Let $\varphi \in \text{PL}(\Sigma)$. Then $\dim(\varphi) \leq 2^{\text{occ}_{\oplus}(\varphi)}$.

**Proof.** By induction on $\varphi$, using the previous lemma. For literals $\varphi = l$, $\dim(l) = 1 = 2^0 = 2^{\text{occ}_{\oplus}(l)}$. For $\lor \in \{\land, \lor\}$, it holds that
\[
\dim(\psi \lor \theta) \leq \dim(\psi) \cdot \dim(\theta) \leq 2^{\text{occ}_{\oplus}(\psi)} \cdot 2^{\text{occ}_{\oplus}(\theta)} = 2^{\text{occ}_{\oplus}(\psi) + \text{occ}_{\oplus}(\theta)} = 2^{\text{occ}_{\oplus}(\varphi)}
\]
and for the Boolean disjunction,
\[
\dim(\psi \land \theta) \leq \dim(\psi) + \dim(\theta) \leq \dim(\psi) \cdot \dim(\theta) + 1 \leq 2^{\text{occ}_{\oplus}(\psi)} \cdot 2^{\text{occ}_{\oplus}(\theta)} + 1 \leq 2^{\text{occ}_{\oplus}(\psi) + \text{occ}_{\oplus}(\theta) + 1} = 2^{\text{occ}_{\oplus}(\varphi)}.
\]

Next, we show that the upper dimension of the dependence atom and the exclusion atom is at least doubly exponential.
Lemma 3.16. Let \( n \geq 1 \), let \( p_1, \ldots, p_n, q_1, \ldots, q_n \in \Phi \) be pairwise distinct propositions, \( \vec{p} = (p_1, \ldots, p_n) \), and \( \vec{q} = (q_1, \ldots, q_n) \). Then \( \Dim(=(\vec{p}; q_1)) \geq 2^n \) and \( \Dim(\vec{p} \mid \vec{q}) \geq 2^n - 2 \).

Proof. Let \( T \) be the full \( \Phi \)-team. Suppose that \( G \) is a generator of \( =(\vec{p}; q_1) \). For each \( f : \{0,1\}^n \to \{1,0\} \), pick the subteam

\[
X(f) := \{ s \in T \mid f(s(\vec{p})) = s(q_1) \}
\]

of \( T \). Then \( f_1 \neq f_2 \) implies \( X(f_1) \neq X(f_2) \). As a consequence, there are \( 2^n \) such teams. Moreover, every \( X(f) \) satisfies \( =(\vec{p}; q_1) \) and is maximal in the sense that \( X(f) \subseteq Y \subseteq T \) implies \( Y \models =(\vec{p}; q_1) \).

For the sake of contradiction, now assume \( |G| < 2^n \). Then there are two pairs \( (S_1, U), (S_2, U) \in G \) such that \( X(f_1), X(f_2) \subseteq U \) for distinct \( f_1, f_2 \). By maximality, \( U = X(f_1) = X(f_2) \). But then \( f_1 = f_2 \), contradiction. Hence \( |G| \geq 2^n \).

For the exclusion atom, we proceed similarly. A function \( f : \{0,1\}^n \to \{1,0\} \) is non-constant if \( f(\vec{b}) \neq f(\vec{b'}) \) for some \( \vec{b}, \vec{b'} \in \{0,1\}^n \). Now, for all non-constant \( f : \{0,1\}^n \to \{1,0\} \), define the subteam

\[
X(f) := \{ s \in T \mid f(s(\vec{p})) = 1 \text{ and } f(s(\vec{q})) = 0 \}
\]

of \( T \). Then \( X(f) \models \vec{p} \mid \vec{q} \), since for every \( \vec{b} \in \{0,1\}^n \) either \( s(\vec{p}) \neq \vec{b} \) for all \( s \in X(f) \) or \( s(\vec{q}) \neq \vec{b} \) for all \( s \in X(f) \). Also, \( f_1 \neq f_2 \) implies \( X(f_1) \neq X(f_2) \): If, say, \( 1 = f_1(\vec{b}) \neq f_2(\vec{b}) = 0 \), then there is an assignment \( s \in T \) such that \( s \in X(f_1) \setminus X(f_2) \), namely where \( s(\vec{p}) = \vec{b} \) and \( s(\vec{q}) \) is arbitrary such that \( f_1(s(\vec{q})) = 0 \) (as \( f_1 \) is non-constant). Consequently, there are \( 2^n - 2 \) such teams (omitting the constant functions, where \( X_f \) would just be empty).

We show that these teams are again maximal, i.e., \( X(f) \subseteq Y \subseteq T \) implies \( Y \nvdash \vec{p} \mid \vec{q} \). The remaining proof is then just as for the dependence atom. Hence assume that \( Y \) contains an assignment \( s \notin X(f) \). Then \( f(s(\vec{p})) = 0 \) or \( f(s(\vec{q})) = 1 \). By symmetry, we consider only the first case. As \( f \) is non-constant, there exists \( \vec{b} \in \{0,1\}^n \) with \( f(\vec{b}) = 1 \). Now, pick \( s' \in T \) such that \( s'(\vec{p}) = \vec{b} \) and \( s'(\vec{q}) = s(\vec{p}) \). Then \( f(s'(\vec{p})) = 1 \) and \( f(s'(\vec{q})) = 0 \), so \( s' \in X(f) \subseteq Y \). Hence \( s, s' \in Y \), but \( s'(\vec{q}) = s(\vec{p}) \). As a result, \( Y \nvdash \vec{p} \mid \vec{q} \).

We conclude the section with the following proof of the exponential lower bounds for the dependence and exclusion atoms.

Proof of Theorem 3.12. We consider the exclusion atom, the dependence atom works analogously. Suppose that \( \varphi \in \text{PL}(\Sigma) \) is equivalent to \( p_1 \cdots p_n \mid q_1 \cdots q_n \). Then by Lemma 3.16, \( \Dim(\varphi) \geq 2^{2^n} - 2 \), as the upper dimension is a purely semantic property. However, by Lemma 3.15, \( \Dim(\varphi) \leq 2^{\text{occ}_{\varphi}(\varphi)} \leq 2^{2^n} - 2 \). With \( n \geq 1 \), the resulting inequality \( 2^{2^n} - 2 \leq 2^{2^n} - 2 \) implies \( |\varphi| \geq 2^n \).

3.4. From lax to strict lower bounds. We show that it suffices to prove the lower bounds for the dependence and exclusion atoms for the restricted operator set \( \Sigma = \{ \emptyset, \land, \lor \} \).

Definition 3.17. A formula \( \varphi \) is interval closed if, for all domains \( \Phi \) and \( \Phi \)-teams \( T_1, T_2, T_3 \), we have that \( T_1 \subseteq T_2 \subseteq T_3 \) and \( T_1, T_3 \models \varphi \) implies \( T_2 \models \varphi \).

Proposition 3.18. Every formula \( \varphi \in \text{PL}(\{\land, \lor, \top\}) \) is union closed and interval closed.
Lemma 3.20. Let \( T \cup T' \models \varphi \) whenever \( T, T' \models \varphi \), since there is only a finite number of \( \Phi \)-teams. By induction on \( \varphi \), we prove the following statement that entails both properties: If \( \Phi \) is a domain \( T, T_1, T_2, T_3 \) are \( \Phi \)-teams such that \( T_1, T_3 \models \varphi \) and \( T_1 \subseteq T_2 \subseteq T_1 \cup T_3 \), then \( T_2 \not\models \varphi \). This contains the special cases \( T_2 = T_1 \cup T_3 \) for union and \( T_1 \subseteq T_3 \) for interval closure.

- Let \( \varphi \) be a positive resp. negative literal. Then the proposition holds by flatness resp. upward closure.
- Let \( \varphi = \psi \land \theta \). Then clearly \( T_2 \models \psi \land \theta \) by induction hypothesis.
- Let \( \varphi = \psi \lor \theta \). Then there are splits \((S_1, U_1)\) of \( T_1 \) and \((S_3, U_3)\) of \( T_3 \) such that \( S_1, S_3 \models \psi \) and \( U_1, U_3 \not\models \theta \). Define teams \( S_2 := T_2 \cap (S_1 \cup S_3) \) and \( U_2 := T_2 \setminus (U_1 \cup U_3) \). Then clearly \( S_2 \subseteq S_1 \cup S_3 \), and since \( S_1 \subseteq T_1 \subseteq T_2 \), we obtain \( S_1 \subseteq S_2 \). Hence \( S_1 \subseteq S_2 \subseteq S_1 \cup S_3 \). Similarly, \( U_1 \subseteq U_2 \subseteq U_1 \cup U_3 \). As \( T_2 = S_2 \cup U_2 \), by induction hypothesis \( T_2 \not\models \psi \lor \theta \).
- Let \( \varphi = \psi \land \theta \). There are strict splits \((S_1, U_1)\) of \( T_1 \) and \((S_3, U_3)\) of \( T_3 \) such that \( S_1, S_3 \models \psi \) and \( U_1, U_3 \not\models \theta \). We proceed as before, but now \((S_2, U_2)\) must be a strict split. Let \( S_2 := (T_2 \cap (S_1 \cup S_3)) \setminus U_1 \) and \( U_2 := T_2 \setminus S_2 \). Clearly \( S_2 \subseteq S_1 \cup S_3 \), and since \( S_1 \) and \( U_1 \) are disjoint, from \( S_1 \subseteq T_1 \subseteq T_2 \) it follows that \( S_1 \subseteq S_2 \). Similarly, \( U_1 \subseteq U_2 \) follows from the fact that \( U_1 \subseteq T_1 \subseteq T_2 = S_2 \cup U_2 \) and \( U_1 \cap S_2 = \emptyset \). Finally, we show \( U_2 \subseteq U_1 \cup U_3 \). Suppose that some \( t \in U_2 \setminus (U_1 \cup U_3) \) exists. As \( U_2 \subseteq T_2 \subseteq T_1 \cup T_3 = S_1 \cup S_3 \cup U_1 \cup U_3 \), then \( t \in S_1 \cup S_3 \). But by definition of \( S_2 \) then \( t \in S_2 \). Contradiction to \( S_2 \cap U_2 = \emptyset \).

Corollary 3.19. For every domain \( \Phi \) and \( \Phi \)-satisfiable formula \( \varphi \in \text{PL}(\{\land, \lor, \exists\}) \) there is a \( \subseteq \)-maximal \( \Phi \)-team \( M^{\Phi}(\varphi) \) that satisfies \( \varphi \).

The prototypical counter-example showing that already \( \text{PL}(\{\exists\}) \) is not union closed is the constancy atom \( \models (p) = p \land \neg p \) (cf. Galliani [Gal12, Corollary 4.10]). Neither is it interval closed, as the formula \( \models (p) = \exists \psi \land \neg \psi \) shows: \( \varphi \) is true in the empty team and in the full \( \{p\}\)-team, but not in \( \{s \mid s(p) = 1\} \).

For what follows, we define the relaxation \( \varphi^* \) of a formula \( \varphi \) as the formula where every occurrence of \( \lor \) is replaced by \( \land \).

Lemma 3.20. Let \( \varphi \in \text{PL}(\{\land, \lor, \exists\}) \). If \( \varphi^* \) is \( \Phi \)-satisfiable for some domain \( \Phi \), then \( \varphi \) is \( \Psi \)-satisfiable for some domain \( \Psi \supseteq \Phi \).

Proof. The idea is that any lax splitting can be simulated by a strict one, by duplicating assignments in the team such that no assignment needs to be used in both halves of the splitting. If \( T \) is a \( \Phi \)-team and \( \Psi \supseteq \Phi \), the team

\[
T[\Psi] := \{ s : \Psi \rightarrow \{0, 1\} \mid s|\Phi \in T \}
\]

is intuitively obtained from \( T \) by padding all assignments in \( T \) with all possible values for propositions \( p \in \Psi \setminus \Phi \). Observe that \( T[\Psi]|\Phi = T \).

To now prove the lemma, we show the following stronger statement by induction on \( \varphi \): If \( \varphi^* \) is satisfied by a \( \Phi \)-team \( T \), then there is a domain \( \Psi \supseteq \Phi \) such that, for all teams \( X \) (of any domain \( \Psi' \supseteq \Psi \), but we will silently assume this below), \( X|\Psi = T[\Psi] \) implies \( X \not\models \varphi \).

- If \( \varphi \) is a literal, then \( \varphi^* = \varphi \). Since \( \varphi \) has locality by Proposition 2.11 we can set \( \Psi := \Phi \).
- Suppose \( \varphi = \psi_1 \land \psi_2 \) and \( T \models \varphi^* = \psi_1^* \land \psi_2^* \). By induction hypothesis there are \( \Psi_1, \Psi_2 \supseteq \Phi \) such that \( X|\Psi_i = T[\Psi_i] \) implies \( X \not\models \psi_i \). We set \( \Psi := \Psi_1 \cup \Psi_2 \). Then \( X|\Psi = T[\Psi] \) implies \( X|\Psi_i = T[\Psi_i] \) and hence \( X \not\models \varphi \).
Suppose \( \varphi = \varphi_1 \lor \varphi_2 \) or \( \varphi = \varphi_1 \uparrow \varphi_2 \). In both cases, \( \varphi^* = \psi_1^* \lor \psi_2^* \). If \( T \models \varphi^* \), then there is a (possibly lax) split of \( T \) into \( \Phi \)-teams \( (S_1, S_2) \) such that \( S_i \models \psi_i^* \), for \( i \in \{1, 2\} \). Again there are domains \( \Psi_1, \Psi_2 \supseteq \Phi \) such that \( X_i \upharpoonright \Psi_i = S_i[\Psi_i] \) implies \( X_i \models \psi_i \).

We pick a proposition \( p \) such that \( p \not\in \Psi_1 \cup \Psi_2 \), and let \( \Psi := \Psi_1 \cup \Psi_2 \cup \{p\} \). Now assume \( X \upharpoonright \Psi = T[\Psi] \) for a team \( X \). We have to show that \( X \models \psi_1 \lor \psi_2 \). Divide \( X \) into the following subteams:

\[
\begin{align*}
Y_1 &:= \{ s \in X \mid s|\Phi \in S_1 \cap S_2 \text{ and } s(p) = 1 \} \\
Y_2 &:= \{ s \in X \mid s|\Phi \in S_1 \cap S_2 \text{ and } s(p) = 0 \} \\
Z_1 &:= \{ s \in X \mid s|\Phi \in S_1 \setminus S_2 \} \\
Z_2 &:= \{ s \in X \mid s|\Phi \in S_2 \setminus S_1 \}
\end{align*}
\]

Clearly, \( (Y_1 \cup Z_1, Y_2 \cup Z_2) \) is a strict split of \( X \). To show \( Y_i \cup Z_i \models \psi_i \) for \( i \in \{1, 2\} \), we demonstrate \( (Y_i \cup Z_i)[\Psi_i] = S_i[\Psi_i] \).

\( \subseteq \) := Let \( \hat{s} \in (Y_i \cup Z_i)[\Psi_i] \). Then \( \hat{s} = s|\Phi \) for some \( s \in Y_i \) or \( s \in Z_i \). In both cases, \( s|\Phi \in S_i \), so \( \hat{s} \in S_i[\Psi_i] \) since \( \hat{s}|\Phi = s|\Phi \).

\( \supseteq \) := Let \( s \in S_i[\Psi_i] \). Then \( s|\Phi \in S_i \subseteq T \), so there is some \( \hat{s} \in T[\Psi] \) that extends \( s|\Phi \), that is, \( \hat{s}|\Psi_i = s \), and with \( \hat{s}(p) \) chosen such that \( \hat{s}(p) = 1 \) if \( i = 1 \). Moreover, since \( X \upharpoonright \Psi = T[\Psi] \), there is now also some \( s' \in X \) that extends \( \hat{s} \), i.e., \( s'|\Psi = \hat{s} \). Ultimately, \( s'|\Phi \in S_i \), which implies \( s' \in Y_i \cup Z_i \) and hence \( s = s'|\Psi_i \in (Y_i \cup Z_i)[\Psi_i] \).

\( \square \)

**Theorem 3.21.** If \( \varphi \in \text{PL}(\{\ominus, \land, \lor, \uparrow, \downarrow\}) \) is downward closed and has locality, then \( \varphi \) is equivalent to its relaxation \( \varphi^* \).

**Proof.** The direction \( \varphi \models \varphi^* \) is straightforward to prove by induction (this requires neither downward closure nor locality).

For the other direction, \( \varphi^* \models \varphi \), we first bring \( \varphi \) and \( \varphi^* \) into a normal form consisting only of Boolean disjunctions of \( \text{PL}(\Sigma') \)-formulas, where \( \Sigma' := \{\land, \lor, \uparrow, \downarrow\} \), using the distributive laws

\[
\begin{align*}
\theta_1 \circ (\theta_2 \otimes \theta_3) &\equiv (\theta_1 \circ \theta_2) \otimes (\theta_1 \circ \theta_3) \\
(\theta_1 \circ \theta_2) \circ \theta_3 &\equiv (\theta_1 \circ \theta_3) \otimes (\theta_2 \circ \theta_3)
\end{align*}
\]

for \( \circ \in \Sigma' \) (cf. Galliani [Gal16, Proposition 5] and Virtema [Vir17, Proposition 6.2]). We obtain \( \varphi \equiv \bigotimes_{i=1}^n \psi_i \) and \( \varphi^* \equiv \bigotimes_{i=1}^n \psi_i^* \) for suitable \( \psi_1, \ldots, \psi_n \in \text{PL}(\Sigma') \).

We require the following claim: If \( \Phi \) is a domain and \( \psi \in \text{PL}(\Sigma') \) is \( \Phi \)-satisfiable, then \( M^\Phi(\psi) = M^\Phi(\psi^*) \) (these teams exist due to Corollary 3.19). Before we proceed to prove the claim, we show how it implies \( \varphi^* \models \varphi \).

Let \( T \) be an arbitrary \( \Phi \)-team, where \( \Phi \supseteq \text{Prop}(\varphi) \), such that \( T \models \varphi^* \). Then \( T \models \psi_i^* \) for some \( i \). By Lemma 3.20, there is a domain \( \Psi \supseteq \Phi \) such that \( \psi_i \in \Psi \)-satisfiable. Pick an arbitrary \( \Psi \)-team \( T' \) such that \( T'[\Phi] = T \); then still \( T' \models \psi_i^* \) by locality of \( \psi_i^* \). Consequently, \( M' := M^\Phi(\psi_i^*) \) exists and \( T' \subseteq M' \). By the claim, \( M^\Phi(\psi_i) = M' \). In particular, \( M' \) satisfies \( \psi_i \) and hence \( \varphi \). By assumption of the lemma, \( \varphi \) has locality, so \( M'[\Phi] \models \varphi \). Moreover, \( \varphi \) is downward closed, and \( T' \subseteq M' \) implies \( T'[\Phi] \subseteq M'[\Phi] \), so we conclude that \( T = T'[\Phi] \) satisfies \( \varphi \). Since \( T \) was arbitrary, this proves \( \varphi^* \models \varphi \).

Finally, we prove the claim: If \( \psi \in \text{PL}(\Sigma') \) is \( \Phi \)-satisfiable for some domain \( \Phi \), then \( M^\Phi(\psi) = M^\Phi(\psi^*) \). The proof is by induction on \( \psi \). In the following, we omit \( \Phi \) and write simply \( M(\psi) \) and \( M(\psi^*) \).

- If \( \psi \) is a literal, then \( \psi = \psi^* \) and we are done.
• If \( \psi = \theta_1 \land \theta_2 \), then \( \theta_1 \) and \( \theta_2 \) are satisfiable, so \( M(\theta_1) \) and \( M(\theta_2) \) exist. We show that \( M(\psi) = M(\theta_1) \cap M(\theta_2) \). Since \( M(\psi) \vDash \psi \), we have \( M(\psi) \vDash \theta_1, \theta_2 \), and so \( M(\psi) \subseteq M(\theta_1) \cap M(\theta_2) \) by definition of \( M(\theta_1) \) and \( M(\theta_2) \). Conversely, \( M(\theta_1) \cap M(\theta_2) \vDash \theta_1, \theta_2 \) due to interval closure, since \( M(\psi) \subseteq M(\theta_1) \cap M(\theta_2) \subseteq M(\theta_1), M(\theta_2) \). As a consequence, \( M(\theta_1) \cap M(\theta_2) \subseteq M(\psi) \). We use a similar argument for \( \psi^* = \theta_1^* \land \theta_2^* \) and by induction hypothesis obtain \( M(\psi) = M(\theta_1) \cap M(\theta_2) = M(\theta_1^*) \cap M(\theta_2^*) = M(\psi^*) \).

• If \( \psi = \theta_1 \lor \theta_2 \), then \( M(\theta_1) \) and \( M(\theta_2) \) again exist. Clearly, \( M(\theta_1) \cup M(\theta_2) \vDash \psi \), so \( M(\theta_1) \cup M(\theta_2) \subseteq M(\psi) \). Likewise, since \( M(\psi) \vDash \psi \), \( M(\psi) \) has a split \( (S_1, S_2) \) such that \( S_i \vdash \theta_i \) and hence \( S_i \subseteq M(\theta_i), \) for \( i \in \{1, 2\} \). Consequently, \( M(\psi) \subseteq M(\theta_1) \cup M(\theta_2) \). As before, we obtain \( M(\psi) = M(\theta_1) \cup M(\theta_2) = M(\theta_1^*) \cup M(\theta_2^*) = M(\psi^*) \).

• Finally, let \( \psi = \theta_1 \lor \theta_2 \). As in the lax case, \( M(\psi^*) = M(\theta_1^* \lor \theta_2^*) = M(\theta_1^*) \cup M(\theta_2^*) = M(\theta_1) \cup M(\theta_2) \subseteq M(\psi) \). However, the direction \( M(\theta_1) \cup M(\theta_2) \subseteq M(\psi) \) is more involved. Unlike before, the split \( (M(\theta_1), M(\theta_2)) \) does not witness \( M(\theta_1) \cup M(\theta_2) \vDash \psi \), since it is not necessarily strict.

Instead, we construct disjoint teams \( S_1, S_2 \) such that \( M(\theta_2) \subseteq S_1 \cup S_2 \vDash \psi \), as then \( M(\theta_2) \subseteq M(\psi) \), and by symmetry \( M(\theta_1) \cup M(\theta_2) \subseteq M(\psi) \).

First, some strict split \( (T_1, T_2) \) of \( M(\psi) \) must exist such that \( T_1 \vdash \theta_1 \). In particular, \( T_i \subseteq M(\theta_i) \). From this, we define \( S_1 := (M(\theta_1) \setminus M(\theta_2)) \cup T_1 \) and \( S_2 := M(\theta_2) \setminus S_1 \).

Now \( T_1 \subseteq S_1 \subseteq M(\theta_1) \) and \( T_1 = M(\theta_1) \), so by interval closure, \( S_1 \vdash \theta_1 \). Likewise, as \( T_2 \cap T_1 = \emptyset \) and \( T_2 \subseteq M(\theta_2) \), we have \( T_2 \cap S_1 = \emptyset \), so \( T_2 \subseteq S_2 \). Since clearly \( S_2 \subseteq M(\theta_2) \), again by interval closure we obtain \( S_2 \vdash \theta_2 \).

\( \square \)

4. Polynomial upper bounds for team properties

In this section, we complement the exponential lower bounds presented in Theorem 3.4 by polynomial upper bounds in the fragment \( \text{PL}(\{\emptyset, \land, \lor\}) \). Notably, among these polynomially definable properties are the negations of all atoms of dependency considered previously. This exhibits a striking asymmetry of succinctness between the standard atoms of dependency and their negations. The formulas we use feature heavy use of the shorthand defined in Section 2.

As with the lower bounds in the previous section, we will first state the theorem and prove it with a series of lemmas. The length of a tuple \( \vec{\gamma} = (\gamma_1, \ldots, \gamma_n) \) of formulas is \( |\vec{\gamma}| := \sum_{i=1}^n |\gamma_i| \). The negation of a formula \( \varphi \) is \( \sim \varphi \).

**Theorem 4.1.** Let \( \Sigma \supseteq \{\emptyset, \land, \lor\} \). Let \( \vec{\alpha} = (\alpha_1, \ldots, \alpha_n), \, \vec{\beta} = (\beta_1, \ldots, \beta_m), \) and \( \vec{\gamma} = (\gamma_1, \ldots, \gamma_k) \) be tuples of purely propositional formulas.

• The dependence atom \( = (\vec{\alpha}; \vec{\beta}) \) is equivalent to the negation of a \( \text{PL}(\Sigma) \)-formula of length \( O(|\vec{\alpha}| \cdot |\vec{\beta}|) \).

• The exclusion atom \( \vec{\alpha} \mid \vec{\beta} \) is equivalent to the negation of a \( \text{PL}(\Sigma) \)-formula of length \( O(n|\vec{\alpha}| |\vec{\beta}|) \).

• The inclusion atom \( \vec{\alpha} \subseteq \vec{\beta} \) is equivalent to the negation of a \( \text{PL}(\Sigma) \)-formula of length \( O(n|\vec{\alpha}| |\vec{\beta}|) \).

• The conditional independence atom \( \vec{\alpha} \perp_{\vec{\gamma}} \vec{\beta} \) is equivalent to the negation of a \( \text{PL}(\Sigma) \)-formula of length \( O(n(n + m + k)|\vec{\alpha}| \cdot |\vec{\beta}| |\vec{\gamma}|) \).

• The anonymity atom \( \vec{\alpha} \forall \vec{\beta} \) is equivalent to the negation of a \( \text{PL}(\Sigma) \)-formula of length \( O(n|\beta| + |\vec{\alpha}|) \).
Additionally, for the dependence and exclusion atoms, \( \Sigma \supseteq \{ \emptyset, \land, \lor \} \) yields the same result. Furthermore, all these formulas are logspace-computable.

**Proof.** The results follow from Lemmas 4.2 to 4.7. For the formulas that are equivalent to the negations of the dependence and exclusion atom, note that every occurrence of \( \lor \) in them is of the form \( \alpha \lor \varphi \) for purely propositional \( \alpha \). But then \( \alpha \lor \varphi \equiv \alpha \lor \varphi \) by Proposition 2.7. For this reason, these results hold for \( \Sigma \supseteq \{ \emptyset, \land, \lor \} \) as well. \( \square \)

Note that for the parity of teams there is no such asymmetry and we have exponential lower bounds for both even and odd cardinality. Nevertheless, we will present a polynomial upper bound in a stronger logic than PL(\( \{ \emptyset, \land, \lor, \lor \} \)) for parity in the subsequent subsection.

Next, we proceed to prove each of the cases.

**Dependence atom.** It is well-known that the dependence atom can be efficiently rewritten by means of other connectives in most flavors of team logic that have unrestricted negation (see, e.g., [Vä07, KMSV15, HKVV18]). For the sake of completeness, we will also state such a formula here.

The following formula expresses the negation of the dependence atom \( = (\vec{a}; \vec{b}) \) and has length \( \mathcal{O}(|\vec{a}\vec{b}|) \). Recall the defined abbreviations \( E_{\alpha} := \top \lor (\neg \bot \land \alpha) \) and \( (\vec{a} = \vec{b}) := \land_{i=1}^{n}(\alpha_i \land \beta_i) \lor (\neg \alpha_i \land \neg \beta_i) \), which we will extensively use in this section.

\[
\varphi(\vec{a}; \vec{b}) := \top \lor \left( \land_{i=1}^{n}(\alpha_i \lor \neg \alpha_i) \land \bigvee_{i=1}^{m}(E_{\beta_i} \land E_{\neg \beta_i}) \right)
\]

**Lemma 4.2.** \( \sim = (\vec{a}; \vec{b}) \equiv \varphi(\vec{a}; \vec{b}) \).

**Proof.** Analogously to [HKVV18, Proposition 2.5]. \( \square \)

In what follows, we require the abbreviation \( \alpha \leftrightarrow \varphi := \neg \alpha \lor (\alpha \land \varphi) \), or equivalently, with strict splitting, \( \alpha \leftrightarrow \varphi := \neg \alpha \lor (\alpha \land \varphi) \). It was introduced by Galliani [Gal15]. Its semantics is the following:

\[
T \models \alpha \leftrightarrow \varphi \iff T_\alpha \models \varphi,
\]

where \( T_\alpha := \{ s \in T \mid s \models \alpha \} \).

For the remaining atoms, we introduce two auxiliary formulas \( \theta(\vec{a}; \vec{b}; \gamma) \) and \( \theta^\neq(\vec{a}; \vec{b}; \gamma) \) that are used several times. Here, \( \vec{a} = (\alpha_1, \ldots, \alpha_n) \) and \( \vec{b} = (\beta_1, \ldots, \beta_n) \) are tuples of propositional formulas, \( \gamma \) is a single propositional formula, and \( n \geq 1 \). Roughly speaking, these formulas compare the values of \( \vec{a} \) in assignments of \( T_\gamma \) with those of \( \vec{b} \) in assignments of \( T_{\neg \gamma} \). In the implementation below, they furthermore impose that \( T_\gamma \) is a singleton w.r.t. \( \vec{a} \), i.e., \( T_\gamma \) is non-empty and \( s(\vec{a}) = s'(\vec{a}) \) for all \( s, s' \in T_\gamma \).

It turns out that the common atoms of dependency can be reduced to such a pattern of assignment-wise comparison between distinct subteams.
Below, we give \( PL(\{\emptyset, \land, \lor\}) \)-formulas that define \( \theta^= (\vec{\alpha}; \vec{\beta}; \gamma) \) and \( \theta^\neq (\vec{\alpha}; \vec{\beta}; \gamma) \) with length \( O(n|\gamma| + |\vec{\alpha}| + |\vec{\beta}|) \):

\[
\theta^= (\vec{\alpha}; \vec{\beta}; \gamma) := (\gamma \iff (\text{NE} \land = (\vec{\alpha}))) \land \bigwedge_{i=1}^{n} \left( (\gamma \land (\alpha_i = l)) \lor (\neg \gamma \land (\beta_i = l)) \right)
\]

\[
\theta^\neq (\vec{\alpha}; \vec{\beta}; \gamma) := (\gamma \iff (\text{NE} \land = (\vec{\alpha}))) \lor \bigvee_{i=1}^{n} \left( E_{\gamma} \land \bigvee_{l \in \{\top, \bot\}} ((\gamma \land (\alpha_i = l)) \lor (\neg \gamma \land (\beta_i \neq l))) \right)
\]

We proceed with proving their correctness.

**Lemma 4.3.** Let \( \approx \in \{=, \neq\} \). Let \( T \) be a team. Then \( T \models \theta^\approx (\vec{\alpha}; \vec{\beta}; \gamma) \) if and only if \( T_\gamma \neq \emptyset \), \( s(\vec{\alpha}) = s'(\vec{\alpha}) \) for all \( s, s' \in T_\gamma \), and \( s(\vec{\beta}) \approx s'(\vec{\beta}) \) for all \( s \in T_{\neg \gamma}, s' \in T_\gamma \).

**Proof.** Clearly, \( T \models (\gamma \iff (\text{NE} \land = (\vec{\alpha}))) \) if and only if \( T_\gamma \neq \emptyset \) and there is \( b \in \{0, 1\}^n \) such that \( s(\vec{\alpha}) = b \) for all \( s \in T_\gamma \). For the second part, as the proof for \( \theta^= \) is straightforward; let us consider \( \theta^\neq \).

For the first direction, suppose that \( T \models \theta^\neq (\vec{\alpha}; \vec{\beta}; \gamma) \). Then \( T \) can be divided into teams \( Y_1 \cup \cdots \cup Y_n \) such that \( Y_i \cap T_\gamma \neq \emptyset \) and each \( Y_i \) satisfies the respective Boolean disjunction. Now let \( s \in T_{\neg \gamma} \) be arbitrary and, say, \( s \in Y_i \). Let \( l \in \{\top, \bot\} \) according to the true disjunct, i.e., \( Y_i \models (\gamma \land (\alpha_i = l)) \lor (\neg \gamma \land (\beta_i \neq l)) \). Then some \( s^* \in Y_i \cap T_\gamma \) exists, and moreover \( s^*(\alpha_i) \neq s(\beta_i) \) since \( s^* \in T_\gamma \). Consequently, \( s(\vec{\beta}) \neq b \) as desired.

Conversely, assume that there is a unique \( b \) such that \( s(\vec{\alpha}) = b \) for \( s \in T_\gamma \), and that \( s(\vec{\beta}) \neq b \) holds for all \( s \in T_{\neg \gamma} \). We divide \( T \) into teams \( Y_1 \cup \cdots \cup Y_n \) such that

\[
Y_i := T_\gamma \cup \{s \in T_{\neg \gamma} \mid s(\beta_i) \neq a_i\}.
\]

It is easy to see that this is a split of \( T \); as otherwise some \( s \in T_{\neg \gamma} \) is left over with \( s(\beta_i) = b_i \) for all \( i \in \{1, \ldots, n\} \), contradicting our assumption that \( s(\vec{\beta}) \neq b \). Clearly \( Y_i \models E_{\gamma} \), as \( T_i \neq \emptyset \) and \( T_\gamma \subseteq Y_i \). It remains to check that setting \( l := \top \) if \( a_i = 1 \) (resp. \( l := \bot \) if \( a_i = 0 \)) renders \( (\gamma \land (\alpha_i = l)) \lor (\neg \gamma \land (\beta_i \neq l)) \) true in \( Y_i \).

**Exclusion atom.** The following formula expresses the negation of the exclusion atom \( \vec{\alpha} \mid \vec{\beta} \) and has length \( O(n|\alpha \vec{\beta}|) \), where \( \vec{\alpha} = (\alpha_1, \ldots, \alpha_n) \) and \( \vec{\beta} = (\beta_1, \ldots, \beta_n) \).

\[
\varphi(\vec{\alpha}; \vec{\beta}) := E(\vec{\alpha} = \vec{\beta}) \land \bigvee_{i=1}^{n} \left( \top \lor (E_{\neg \gamma} \land \theta^= (\vec{\alpha}; \vec{\beta}; \gamma)) \right)
\]

First, we explain our approach intuitively. A team \( T \) fails to satisfy the exclusion atom precisely if some assignment \( s \) agrees on \( \vec{\alpha} \) and \( \vec{\beta} \) (this is easy to check), or when no assignment does so, then \( s(\vec{\alpha}) = s'(\vec{\beta}) \) for distinct \( s, s' \). If we exclude the first case, however, \( s \) and \( s' \) must disagree on some \( \alpha_i \). We exploit this fact by using the bit \( \alpha_i \) as a “handle” and compare \( s(\vec{\alpha}) \) and \( s'(\vec{\beta}) \) via \( \theta^= \).

**Lemma 4.4.** \( \sim \vec{\alpha} \mid \vec{\beta} \equiv \varphi(\vec{\alpha}; \vec{\beta}) \).
Proof. Suppose $T \not\models \bar{\alpha} \mid \bar{\beta}$. By definition, there are $s, s' \in T$ such that $s(\bar{\alpha}) = s'(\bar{\beta})$. First, if $s(\bar{\alpha}) = s'(\bar{\alpha})$, then $T \models E(\bar{\alpha} = \bar{\beta})$ via $s'$. Otherwise, $s$ and $s'$ disagree on some $\alpha_i$, i.e., there is $\gamma \in L := \{\alpha_i, \neg \alpha_i \mid 1 \leq i \leq n\}$ such that $s(\gamma) = 1$ and $s'(\gamma) = 0$. Then the split $(T \setminus \{s, s'\}, \{s, s'\})$ satisfies the Boolean disjunct with index $\gamma$, as clearly $\{s, s'\} \models E_{\neg \gamma}$ and $\{s, s'\} \models \theta(\bar{\alpha}; \bar{\beta}; \gamma)$.

For the other direction, assume that $T \models \varphi(\bar{\alpha}; \bar{\beta})$. Then either $T \models E(\bar{\alpha} = \bar{\beta})$ and we are done, or there exist $\gamma \in L$ and some split $(S, U)$ of $T$ such that $U \models E_{\neg \gamma} \land \theta(\bar{\alpha}; \bar{\beta}; \gamma)$. As $U \models \theta(\bar{\alpha}; \bar{\beta}; \gamma)$, we have $U_{\gamma} \neq \emptyset$ and $s_1(\bar{\alpha}) = s_2(\bar{\beta})$ for all $(s_1, s_2) \in U_{\gamma} \times U_{\neg \gamma}$. Since both $U_{\gamma}$ (by $\theta(=)$) and $U_{\neg \gamma}$ (by $E(=)$) are non-empty, we conclude that $T \not\models \bar{\alpha} \mid \bar{\beta}$. \hfill \qed

**Inclusion atom.** The following formula expresses the negation of the inclusion atom $\bar{\alpha} \subseteq \bar{\beta}$ and has length $\mathcal{O}(n|\bar{\alpha}|\bar{\beta}|)$, where $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\bar{\beta} = (\beta_1, \ldots, \beta_n)$.

$$
\varphi(\bar{\alpha}; \bar{\beta}) := \bigwedge_{\gamma \in \{\beta_1, \ldots, \beta_n\}} \left( \gamma \lor \left( \left( (\alpha_i \neq \beta_i) \land \theta(\bar{\alpha}; \bar{\beta}; \gamma) \right) \right) \right)
$$

Again, we first explain our approach. A team $T$ falsifies the inclusion atom precisely if there is a "pivot" assignment $s^* \in T$ such that $s^*(\bar{\alpha}) \neq s(\bar{\beta})$ for all $s \in T$. In particular, then $s^*(\alpha_i) \neq s^*(\beta_i)$ for some $i$. As for the exclusion atom, we can exploit this fact by using the bit $\beta_i$ as a "handle": We erase all $s$ from the team that agree with $s^*$ on $\beta_i$ (they cannot satisfy $s(\bar{\beta}) = s^*(\bar{\alpha})$ anyway) and have only assignments remaining from which $s^*$ can be easily distinguished via $\beta_i$, again permitting successive comparisons of all bits.

**Lemma 4.5.** $\sim \bar{\alpha} \subseteq \bar{\beta} \models \varphi(\bar{\alpha}; \bar{\beta})$.

Proof. Let $T \not\models \bar{\alpha} \subseteq \bar{\beta}$. We show that $T \models \varphi(\bar{\alpha}; \bar{\beta})$. By definition, there is $s^* \in T$ such that $s^*(\bar{\alpha}) \neq s(\bar{\beta})$ for all $s \in T$. In particular, $s^*(\alpha_i) \neq s^*(\beta_i)$ for some $i \in \{1, \ldots, n\}$. Let $\gamma \in \{\beta_i, \neg \beta_i\}$ such that $s^*(\gamma) = 1$, and consider the subteam $S := \{s^*\} \cup T_{\neg \gamma}$ of $T$. We show that the Boolean disjunct with index $\gamma$ is satisfied by the split $(T \setminus S, S)$. Clearly, $T \setminus S \models \gamma$. Moreover, $S_{\gamma} = \{s^*\} \models \alpha_i \neq \beta_i$. Finally, $S \models \theta(\bar{\alpha}; \bar{\beta}; \gamma)$ holds since $s^*(\bar{\alpha}) \neq s(\bar{\beta})$ for all $s \in T_{\neg \gamma}$ by assumption.

Conversely, assume $T \models \varphi(\bar{\alpha}; \bar{\beta})$ with $1 \leq i \leq n$ and $\gamma \in \{\beta_i, \neg \beta_i\}$ chosen according to a satisfying disjunct of $\varphi(\bar{\alpha}; \bar{\beta})$. By the formula, $T$ can be divided into $X \cup S$ with $X \models \gamma$, $S_{\gamma} \models (\alpha_i \neq \beta_i)$, and $S \models \theta(\bar{\alpha}; \bar{\beta}; \gamma)$. By the latter formula, in particular $S_{\gamma}$ must be non-empty, i.e., contain some assignment $s^*$. It satisfies $s^*(\alpha_i) \neq s^*(\beta_i)$. We conclude by showing that for all $s \in T$ it holds $s^*(\bar{\alpha}) \neq s(\bar{\beta})$, and hence $T \not\models \bar{\alpha} \subseteq \bar{\beta}$. For all $s \in S_{\neg \gamma} = T_{\neg \gamma}$, this once more follows since $S \models \theta(\bar{\alpha}; \bar{\beta}; \gamma)$. For all $s \in T_{\gamma}$, as $s \models \gamma$, we have $s^*(\alpha_i) \neq s^*(\beta_i) = s(\beta_i)$, since $s^*(\gamma) = s(\gamma) = 1$. \hfill \qed
Independence atom. The following formula expresses the negation of the conditional independence atom $\bar{\alpha} \perp_{\vec{\gamma}} \vec{\beta}$ and has length $O(n(n + m + k)|\bar{\alpha}|\vec{\beta}|\vec{\gamma}|)$, where $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\vec{\beta} = (\beta_1, \ldots, \beta_m)$, and $\vec{\gamma} = (\gamma_1, \ldots, \gamma_k)$.

$$\varphi(\bar{\alpha}; \vec{\beta}; \vec{\gamma}) := \bigvee_{\delta \in \{\alpha_i - \alpha_i | 1 \leq i \leq n\}} \bigvee_{\varepsilon \in \{\beta_j - \beta_j | 1 \leq j \leq m\}} \big( [ (\neg \delta \land \theta^e(\bar{\alpha}\vec{\gamma}; \vec{\beta}; \vec{\gamma}; \varepsilon)) \lor (\neg \varepsilon \land \theta^e(\vec{\beta}; \vec{\beta}; \vec{\gamma}; \delta)] \big) \land ((\delta \lor \varepsilon) \leftrightarrow (\vec{\gamma}))$$

Lemma 4.6. $\sim\bar{\alpha} \perp_{\vec{\gamma}} \vec{\beta} \equiv \varphi(\bar{\alpha}; \vec{\beta}; \vec{\gamma})$.

Proof. For the direction from left to right, assume $T \not\models \bar{\alpha} \perp_{\vec{\gamma}} \vec{\beta}$. Then there are $s^*, s^0 \in T$ such that $s^*(\vec{\gamma}) = s^0(\vec{\gamma})$, but for all $s \in T$ it holds either $s(\bar{\alpha}\vec{\gamma}) \neq s^*(\bar{\alpha}\vec{\gamma})$ or $s(\vec{\beta}) \neq s^0(\vec{\beta})$. In particular, there must be $i, j$ such that $s^*(\alpha_i) \neq s^0(\alpha_i)$ and $s^*(\beta_j) \neq s^0(\beta_j)$. Let $\delta \in \{\alpha_i - \alpha_i\}$ and $\varepsilon \in \{\beta_j - \beta_j\}$ such that $s^*(\varepsilon) = s^0(\delta) = 1$ and $s^*(\delta) = s^0(\varepsilon) = 0$. In order to now satisfy the Boolean disjunct with index $\delta, \varepsilon$, we define subteams

$$S := \{s^*\} \cup \{s \in T \mid s(\delta) = s(\varepsilon) = 0, s(\bar{\alpha}\vec{\gamma}) \neq s^*(\bar{\alpha}\vec{\gamma})\}$$

$$U := \{s^0\} \cup \{s \in T \mid s(\delta) = s(\varepsilon) = 0, s(\vec{\beta}) \neq s^0(\vec{\beta})\}$$

of $T$. We show that the (in fact strict) split $(T \setminus (S \cup U), S \cup U)$ satisfies the disjunction. First, $T \setminus (S \cup U) \models \delta \lor \varepsilon$ due to the fact that $T \setminus (S \cup U) \subseteq T_\delta \cup T_\varepsilon$. Furthermore, $S \cup U \models (\delta \lor \varepsilon) \leftrightarrow (\vec{\gamma})$, since $(S \cup U)_{\delta \lor \varepsilon} = \{s^*, s^0\}$. For the part in brackets, consider the (again strict) split $(S, U \setminus S)$ of $S \cup U$. Again, clearly $S \not\models \neg \delta$ and $U \not\models \neg \varepsilon$. Finally, both $S \models \theta^e(\bar{\alpha}\vec{\gamma}; \vec{\beta}; \vec{\gamma}; \varepsilon)$ and $U \models \theta^e(\vec{\beta}; \vec{\beta}; \vec{\gamma}; \delta)$ hold.

For the other direction, assume $T \models \varphi(\bar{\alpha}; \vec{\beta}; \vec{\gamma})$ with the Boolean disjunction satisfied with indices $\delta \in \{\alpha_i - \alpha_i | 1 \leq i \leq n\}$ and $\varepsilon \in \{\beta_j - \beta_j | 1 \leq j \leq m\}$. Then $T$ can be divided into $X \cup S \cup U$ where

- $X \models \delta \lor \varepsilon$,
- $S \models \neg \delta \land \theta^e(\bar{\alpha}\vec{\gamma}; \vec{\beta}; \vec{\gamma}; \varepsilon)$,
- $U \models \neg \varepsilon \land \theta^e(\vec{\beta}; \vec{\beta}; \vec{\gamma}; \delta)$,
- $S \cup U \models (\delta \lor \varepsilon) \leftrightarrow (\vec{\gamma})$.

Then, by $\theta^e$, assignments $s^* \in S_\varepsilon$ and $s^0 \in U_\delta$ exist. Now, for the sake of contradiction, suppose that $T \models \bar{\alpha} \perp_{\vec{\gamma}} \vec{\beta}$. As $S_\varepsilon \cup U_\delta \models (\vec{\gamma})$ and hence $s^*(\vec{\gamma}) = s^0(\vec{\gamma})$, due to independence, another assignment $s \in T$ must exist such that $s(\bar{\alpha}\vec{\gamma}) = s^*(\bar{\alpha}\vec{\gamma})$, and $s(\vec{\beta}) = s^0(\vec{\beta})$.

However, $s \not\models X$, since $s(\bar{\alpha}) = s^*(\bar{\alpha})$ implies $s \not\models \delta$ and $s(\vec{\beta}) = s^0(\vec{\beta})$ implies $s \not\models \varepsilon$. Consequently, $s \in S \cup U$. For this reason, either $s(\bar{\alpha}\vec{\gamma}) \neq s^*(\bar{\alpha}\vec{\gamma})$, or $s(\vec{\beta}) \neq s^0(\vec{\beta})$, contradiction to $s(\bar{\alpha}\vec{\gamma}) = s^*(\bar{\alpha}\vec{\gamma})$ and $s(\vec{\beta}) = s^0(\vec{\beta})$.

Anonymity atom. The following formula expresses the negation of the unary anonymity atom $\bar{\alpha}\gamma\beta$ and has length $O(n|\beta| + |\bar{\alpha}|)$, where $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$.

$$\varphi(\bar{\alpha}; \beta) := \bigvee_{\gamma \in \{\beta, \neg \beta\}} \delta \lor \theta^e(\bar{\alpha}; \vec{\alpha}; \gamma)$$

Lemma 4.7. $\sim\bar{\alpha}\gamma\beta \equiv \varphi(\bar{\alpha}; \beta)$.
Proof. Suppose \( T \not\models \vec{a} Y \vec{\beta} \). Then there is \( s^* \in T \) such that \( s(\vec{a}) = s^*(\vec{a}) \) implies \( s(\vec{\beta}) = s^*(\vec{\beta}) \) for all \( s \in T \). Let \( \gamma \in \{\vec{\beta}, \neg \vec{\beta}\} \) such that \( s^* \models \gamma \), and consider the split \((T \setminus \{S, \bar{S}\}) \) of \( T \) defined by \( S := \{s^*\} \cup T_\gamma \). Then \( T \setminus S \models \gamma \). Moreover, \( S \not\models \theta \vec{\alpha} \vec{\beta} (\vec{a}; \vec{\alpha}; \vec{\gamma}) \), since \( S_\gamma = \{s^*\} \) and \( s(\vec{a}) \neq s^*(\vec{a}) \) for all \( s \in S_\gamma \).

For the other direction, suppose there is \( \gamma \in \{\vec{\beta}, \neg \vec{\beta}\} \) such that \( S \models \gamma \) and \( U \models \theta \vec{\alpha} \vec{\beta} (\vec{a}; \vec{\alpha}; \vec{\gamma}) \) for some split \((S, U)\) of \( T \). Then there exists \( s^* \in U_\gamma \) such that \( s^*(\vec{a}) \neq s(\vec{a}) \) for all \( s \in U_\gamma \). Clearly, now \( s^*(\vec{\beta}) = s(\vec{\beta}) \) for all \( s \in S \cup U_\gamma \), so ultimately \( s^*(\vec{\beta}) = s(\vec{\beta}) \) for all \( s \in T \), hence \( T \not\models \vec{a} Y \vec{\beta} \).

In the first-order setting, Rönnholm [Rö18, Remark 2.31] demonstrated that the general anonymity atom can be expressed via the unary anonymity atom and the splitting disjunction. In the lemma below, we show that this can also be done via strict splitting. This yields the following formulas are equivalent:

**Lemma 4.8.** The following formulas are equivalent:

1. \( \vec{a} Y \vec{\beta} \),
2. \( \bigvee_{i=1}^m \vec{a} Y \vec{\beta}_i \),
3. \( \bigvee_{i=1}^m \vec{a} Y \vec{\beta}_i \).

**Proof.** For (2) \( \Rightarrow \) (1), we follow Rönnholm [Rö18]. Suppose \( T \models \bigvee_{i=1}^m \vec{a} Y \vec{\beta}_i \) via the split of \( T \) into \( Y_1 \cup \cdots \cup Y_m \), where \( Y_i \models \vec{a} Y \vec{\beta}_i \). To see that \( T \models \vec{a} Y \vec{\beta} \), let \( s \in T \) be arbitrary. For some \( i \), now \( s \in Y_i \). Consequently, there is \( s' \in Y_i \) such that \( s(\vec{a}) = s'(\vec{a}) \) but \( s(\vec{\beta}) \neq s'(\vec{\beta}) \). But as \( Y_i \subseteq T \) and \( s \) was arbitrary, (1) follows.

The step (3) \( \Rightarrow \) (2) is clear, since every strict split of a team is a split.

It remains to show (1) \( \Rightarrow \) (3). Here, we adapt the proof of Rönnholm [Rö18] for \( \vec{V} \).

Suppose that \( T \models \vec{a} Y \vec{\beta} \) holds. Define subteams \( Y_i \) of \( T \) by

\[
Y_i := \{ s \in T \mid \exists s' \in T : s'(\vec{a}) = s(\vec{a}) \text{ but } s(\vec{\beta}) \neq s'(\vec{\beta}) \},
\]

as in the proof of Rönnholm [Rö18], but additionally define teams \( Z_i := Y_i \setminus \bigcup_{j < i} Y_j \) for \( 1 \leq i \leq m \), where \( Y_0 := \emptyset \). We show that \( Z_1 \cup \cdots \cup Z_m \) forms a strict split of \( T \). The sets \( Z_1, \ldots, Z_m \) are pairwise disjoint, as \( Z_i \subseteq Y_i \) but \( Z_j \cap Y_i = \emptyset \) when \( i < j \). Next, let \( s \in T \) be arbitrary. Define

\[
I := \{ i \in \{1, \ldots, m\} \mid \exists s' \in T : s(\vec{a}) = s'(\vec{a}) \text{ but } s(\vec{\beta}) \neq s'(\vec{\beta}) \}.
\]

By assumption (1), \( I \) is non-empty and hence contains a minimal element \( i \). But then \( s \in Y_i \setminus \bigcup_{j < i} Y_j = Z_i \). Consequently, \( T = \bigcup_{i=1}^m Z_i \).

Finally, we need to show that \( Z_i \models \vec{a} Y \vec{\beta}_i \). For this, let now \( s \in Z_i \) be arbitrary. By definition of \( Z_i \), there exists \( s' \in T \) with \( s(\vec{a}) = s'(\vec{a}) \) and \( s(\vec{\beta}) \neq s'(\vec{\beta}) \). It suffices to show that \( s' \in Z_i = Y_i \setminus \bigcup_{j < i} Y_j \). As \( s' \in Y_i \) follows from the definition of \( Y_i \), assume \( s' \in Y_j \) for some \( j < i \). Then by symmetry also \( s \in Y_j \), contradiction to \( s \in Z_i \). Hence \( s' \notin Y_j \) for all \( j < i \), so \( s' \in Z_i \).
4.1. Defining team properties with the strong negation. With the negations of dependency atoms definable in PL(\{\land, \lor, \forall\}), it is an easy corollary that the atoms themselves are definable when additionally the strong negation \sim is available. In the next theorem, we prove this, generalize the part on the anonymity atom \Upsilon, and furthermore expand the results to also work with \check{\lor}, which we previously considered only for the downward closed atoms =\{\cdot, \cdot\} and | in Theorem 4.1.

Theorem 4.9. Let \Sigma = \{\sim, \land, \lor\} or \Sigma = \{\sim, \land, \check{\lor}\}. Let \vec{\alpha} = (\alpha_1, \ldots, \alpha_n), \vec{\beta} = (\beta_1, \ldots, \beta_m), and \vec{\gamma} = (\gamma_1, \ldots, \gamma_k) be tuples of purely propositional formulas.

- The dependence atom =(\vec{\alpha}; \vec{\beta}) is equivalent to a PL(\Sigma)-formula of length \mathcal{O}(|\vec{\alpha}|).
- The exclusion atom \vec{\alpha} \subseteq \vec{\beta} is equivalent to a PL(\Sigma)-formula of length \mathcal{O}(n|\vec{\alpha}|).
- The inclusion atom \vec{\alpha} \subseteq \vec{\beta} is equivalent to a PL(\Sigma)-formula of length \mathcal{O}(n|\vec{\beta}|).
- The conditional independence atom \vec{\alpha} \perp \check{\gamma} \vec{\beta} is equivalent to a PL(\Sigma)-formula of length \mathcal{O}(n(n + m + k)|\vec{\alpha}|).
- The anonymity atom \vec{\alpha}\Upsilon \vec{\beta} is equivalent to a PL(\Sigma)-formula of length \mathcal{O}(n|\vec{\beta}| + m|\vec{\alpha}|).

Furthermore, all these formulas are logspace-computable.

Proof. We extend Theorem 4.1. As all the formulas constructed there use \theta^= or \theta^\neq as subformulas, we first show that these can be expressed in \{\sim, \land, \lor\} and \{\sim, \land, \check{\lor}\}. For \{\sim, \land, \lor\} this is straightforward, since after rewriting \lor via \land and \sim, the resulting formulas are in PL(\{\sim, \land, \lor\}). Next to \{\sim, \land, \check{\lor}\}. Here, in \theta^=, the lax splitting \check{\lor} can equivalently be replaced by \check{\land} due to Proposition 2.7, as any occurrence of \check{\lor} has at least one purely propositional argument. However, this does not hold for \theta^\neq. Nevertheless, it is easy to see that \theta^\neq(\vec{\alpha}; \vec{\beta}; \gamma) is equivalent to

\[(\gamma \leftrightarrow (\text{NE} \land =(\vec{\alpha})) \land \sim(\check{\land} \check{\lor} \sim(=(\gamma) \land \theta^=(\vec{\alpha}; \vec{\beta}; \vec{\gamma})).\]

With \theta^= and \theta^\neq expressible in PL(\{\sim, \land, \lor\}) and PL(\{\sim, \land, \check{\lor}\}), the proof boils down to Lemma 4.7 and 4.8 for the anonymity atom, and to Theorem 4.1 due to the closure of the logic under Boolean negation.

Next, we again consider the parity of the cardinality of teams, i.e., is there a formula that is true precisely on teams with even cardinality? This differs from the other considered team properties in that both the property and its negation have exponential lower bounds in PL(\{\lor, \land, \check{\lor}, \check{\land}\}) (see Theorem 3.4). Nevertheless, we show that it is polynomially definable when linearly many negations are nested inside the formula, which was not necessary for the results of Theorem 4.9.

Theorem 4.10. Let |\Phi| = n. The class of \Phi-teams of odd cardinality is defined by a PL(\land, \sim, \check{\lor})-formula of length \mathcal{O}(n^2).

We use =(X), for a set X \subseteq \Phi of propositions, as abbreviation for =(\vec{p}) where \vec{p} = (p_1, \ldots, p_n) lists all propositions in X = \{p_1, \ldots, p_n\}. Then we can define \underline{1} := \text{NE} \land =(\Phi). Clearly, a \Phi-team T satisfies \underline{1} iff |T| = 1.

The formula expressing odd cardinality is now recursively defined as follows:

\[\varphi(\varepsilon) := \sim \bot\]
\[\varphi(p\vec{q}) := \underline{1} \check{\lor} \sim(\underline{1} \check{\land} (\sim(\check{\lor}(p) \land (1 \check{\lor}=(p)))) \check{\lor} (=(p) \land \sim\varphi(\vec{q})))\]

The proof of the theorem will follow from the next lemma.
Lemma 4.11. Let $T \in \text{Tms}(\Phi)$ and let $\vec{p}$ list all propositions in $\Phi$. Then $T \vDash \varphi(\vec{p})$ if and only if $|T|$ is odd.

Proof. Let $\vec{q} = (q_1, \ldots, q_m)$ be a tuple of variables. We show for any $\Phi$-team satisfying $=_{(\Phi \setminus \{q_1, \ldots, q_m\})}$ that $T \vDash \varphi(\vec{q})$ if and only if $|T|$ is odd. The proof is by induction on $|\vec{q}|$.

Clearly, if $T =_{(\Phi)}$, then $|T|$ is odd if $T \neq \emptyset$ iff $T \vDash \varphi(\varepsilon)$. For the induction step, we show that $|T|$ is odd iff $T \vDash \varphi(p \vec{q})$, assuming $T =_{(\Phi \setminus p \vec{q})}$.

The crucial subformula is $\psi := \big[1 \Updownarrow (\sim(\phi) \land (1 \Updownarrow \phi))\big] \Updownarrow (\phi \land \sim \varphi(\vec{q}))$. We prove, for any $\Phi$-team $S$ satisfying $=_{(\Phi \setminus \{q_1, \ldots, q_m\})}$ that $S \vDash \psi$ iff at least one of $|S_p|$ and $|S_{\neg p}|$ is odd. It is not hard to see that $|T|$ is odd iff there exists $s \in T$ such that both $|(T \setminus \{s\})_p|$ and $|(T \setminus \{s\})_{\neg p}|$ are even, which as a consequence is equivalent to the full formula $1 \Updownarrow \sim \psi$.

To prove the claim, suppose $S$ as above. For the first direction, w.l.o.g. $|S_p|$ is odd. Pick $s \in S_p$ arbitrarily and consider the split $(S_p \cup \{s\}, S_{\neg p} \setminus \{s\})$ of $S$. For the second component, $S_p \setminus \{s\} \vDash \neg \varphi(\vec{q})$ by induction hypothesis. For the first component, either $S_{\neg p}$ is empty and $S_p \cup \{s\} \vDash 1$, or $S_{\neg p}$ is non-empty and $S_{\neg p} \cup \{s\} \vDash \sim \neg \phi \land (1 \Updownarrow \neg \phi)$. In both cases, $S \vDash \psi$.

For the other direction, suppose $S \vDash \psi$ via the strict split $(U, V)$ such that $V \vDash \neg \phi \land (1 \Updownarrow \neg \phi)$, and either $U \vDash 1$ or $U \vDash \sim \phi \land (1 \Updownarrow \phi)$. We distinguish the latter two cases.

- $U \vDash 1$: Then $U, V \vDash \phi$ and $|U|$ is even by induction hypothesis. Both $U$ and $V$ odd, and one of them equals $S_p$ or $S_{\neg p}$, depending on whether $U$ and $V$ agree on $\phi$ or not.
- $U \vDash \sim \phi \land (1 \Updownarrow \phi)$: Due to symmetry, we can assume $V \subseteq S_p$ and $S_{\neg p} \subseteq U$. By the formula, $U$ has a strict split $(X, Y)$ such that $|X| = 1$ and $Y \vDash \phi$. Let $Z = S_p \setminus V$. Either $Z \subseteq X$ or $Z \subseteq Y$, as $X$ and $Y$ do not agree on $\phi$, but each is constant in $\phi$. If $Z \subseteq X$, then $Z = X$ and $|U \cup X| = |S_p|$ is odd and we are done. If $Z \subseteq Y$, then $S_{\neg p} \subseteq X$, hence $S_{\neg p} = X$ and $|S_{\neg p}|$ is odd.

Note that in the positive fragment, an exponential formula is possible. The following formulas depend on the domain $\Phi$ and recursively define even and odd parity of cardinality of teams:

\[
\varphi^{\text{even}}(\varepsilon) := \bot \\
\varphi^{\text{odd}}(\varepsilon) := \top \\
\varphi^{\text{even}}(\vec{q}) := (\phi \land \varepsilon^{\text{odd}}(\vec{q})) \lor (\neg \phi \land \varepsilon^{\text{odd}}(\vec{q})) \\
\varphi^{\text{odd}}(\vec{q}) := (\neg \phi \land \varepsilon^{\text{even}}(\vec{q})) \lor (\phi \land \varepsilon^{\text{even}}(\vec{q}))
\]

Theorem 4.12. Let $|\Phi| = n$. If $\Sigma = \{\lor, \land, \lor\}$ or $\Sigma = \{\land, \lor, \lor\}$, then the class of $\Phi$-teams of odd resp. even cardinality is definable by a PL$(\Sigma)$-formula of length $2^{\Theta(n)}$.

Proof. First of all, observe that the formula $(\phi \land \varphi \lor \neg \phi \land \varphi')$ is equivalent to $(\phi \land \varphi) \lor (\neg \phi \land \varphi')$ for all $\varphi, \varphi'$ and propositions $\phi$, since any split satisfying the former formula is necessarily strict, and hence satisfies the latter. As a consequence, it suffices to show that the formulas stated above, which use $\lor$, define parity.

Let $\Phi = \{p_1, \ldots, p_n\}$ be a domain, and $T$ a $\Phi$-team. Let $\vec{p} = (p_1, \ldots, p_n)$ list all variables in $\Phi$. We prove by induction on $n$ that $T \vDash \varphi^{\text{even}}(\vec{p})$ iff $|T|$ is even, and $T \vDash \varphi^{\text{odd}}(\vec{p})$ iff $|T|$ is odd.

First, if $\Phi = \emptyset$, then either $T = \emptyset$ and $T \vDash = \varphi^{\text{even}}(\varepsilon)$, or $T = \emptyset$ and $T \vDash = \varphi^{\text{odd}}(\varepsilon)$. For the inductive step, observe that $|T|$ is even iff $|T_p|$ and $|T_{\neg p}|$ have equal parity,
and is odd iff they have different parity, where \( p \in \Phi \) is an arbitrary proposition. Furthermore, \( T_p \) and \( T_p| (\Phi \setminus \{p\}) \) have the same cardinality (the same goes for \( T_{\neg p} \)). Additionally, \( T_p \) and \( T_p| (\Phi \setminus \{p\}) \) satisfy the same \( \text{PL}(\Phi \setminus \{p\}, \Sigma) \)-formulas by Proposition 2.11. Hence the equivalence immediately follows by induction hypothesis.

4.2. Algorithmic applications. In this final section, we consider the implications for the computational complexity of a variant of team logic called modal team logic. Usually, this logic is defined without any atoms of dependency whatsoever. Using the results of this paper, we show that the complexity does not change if any of the dependency atoms that we considered is added to the logic.

We define several complexity classes for which the problems introduced below are complete. Some of them are defined in terms of alternating Turing machines which were introduced by Chandra et al. [CKS81].

For each \( k \geq 0 \), we define the function \( \text{exp}_k \) as \( \text{exp}_0(n) := n \) and \( \text{exp}_{k+1}(n) := 2^{\text{exp}_k(n)} \).

**Definition 4.13.** For \( k \geq 0 \), \( \text{ATIME-ALT}(\text{exp}_k, \text{poly}) \) is the class of problems decided by an alternating Turing machine with at most \( p(n) \) alternations and runtime at most \( \text{exp}_k(p(n)) \), for a polynomial \( p \).

**Definition 4.14.** \( \text{TOWER}(\text{poly}) \) is the class of problems that are decided by a deterministic Turing machine in time \( \text{exp}_{p(n)}(1) \) for some polynomial \( p \).

**Modal team logic.** Modal team logic MTL, introduced by Müller [Müll14], extends both classical modal logic ML and propositional team logic \( \text{PL}(\{\land, \lor, \neg\}) \). As before, let \( \Phi \) be a set of propositions. The syntax of MTL is given by the following grammar, where \( p \in \Phi \):

\[
\varphi ::= \top | \bot | p | \neg \varphi | \varphi \land \varphi | \varphi \lor \varphi | \Box \varphi | \Diamond \varphi,
\]

Hence classical modal logic ML is simply the fragment of all \( \neg \)-free formulas of MTL.

A Kripke structure over \( \Phi \) is a tuple \( K = (W, R, V) \) where \( (W, R) \) is a directed graph and \( V : \Phi \rightarrow 2^W \). A team in \( K \) is a subset of \( W \). The set of splits of \( T \), \( \text{Sp}(T) \), is defined as for propositional teams.

The image of a team \( T \) is denoted by \( RT \) and is the set \( \{w' \in W \mid w \in T, (w, w') \in R\} \). A successor team \( T' \) of \( T \) is a subset of \( RT \) such that for all \( w \in T \) there exists \( w' \in T' \) such that \( (w, w') \in R \).

Let again \( \text{Prop}(\varphi) \) denote the set of propositional variables that occur in the formula \( \varphi \). MTL-formulas \( \varphi \) are evaluated on pairs \( (K, T) \), where \( K \) is a Kripke structure over some set \( \Phi' \supseteq \text{Prop}(\varphi) \) of propositions and \( T \) is a team in \( K \). The satisfaction relation \( (K, T) \models \varphi \) is
defined recursively:

\((K, T) \models \top\) always,

\((K, T) \models \bot\) never,

\((K, T) \models p \iff T \subseteq V(p),\)

\((K, T) \models \neg p \iff T \cap V(p) = \emptyset,\)

\((K, T) \models \sim \psi \iff (K, T) \nvdash \psi,\)

\((K, T) \models \psi \land \theta \iff (K, T) \models \psi \text{ and } (K, T) \models \theta,\)

\((K, T) \models \psi \lor \theta \iff \exists (S, U) \in \text{Sp}(T) : (K, S) \models \psi \text{ and } (K, U) \models \theta,\)

\((K, T) \models \Diamond \psi \iff (K, T') \models \psi \text{ for some successor team } T' \text{ of } T,\)

\((K, T) \models \Box \psi \iff (K, RT) \models \psi,\)

where \(p \in \Phi.\) An MTL-formula \(\varphi\) is called **satisfiable** if there exists a Kripke structure \(K\) over \(\text{Prop}(\varphi)\) and a team \(T\) in \(K\) such that \((K, T) \models \varphi.\) The formula \(\varphi\) is **valid** if \((K, T) \models \varphi\) holds for every Kripke structure \(K\) over \(\text{Prop}(\varphi)\) and team \(T\) in \(K.\)

The **satisfiability problem** of modal team logic is the set of its satisfiable formulas. Analogously, the **validity problem** is the set of valid formulas. Finally, the **model checking problem** is the set of tuples \((K, T, \varphi)\), where \(\varphi\) is a formula, \(K\) is a Kripke structure over \(\Phi' \supseteq \text{Prop}(\varphi)\), \(T\) is a team in \(K\), and \((K, T) \models \varphi.\)

Let \(\leq_m^{\log}\) denote logspace-computable reductions. A problem \(B\) is \(\leq_m^{\log}\)-hard for a complexity class \(C\) if \(A \in C\) implies \(A \leq_m^{\log} B\), and \(B\) is \(\leq_m^{\log}\)-complete if it is \(\leq_m^{\log}\)-hard and \(B \in C.\)

**Theorem 4.15** ([Lüc18, Mü14]). The satisfiability and validity problem of MTL are \(\leq_m^{\log}\)-complete for TOWER(poly). The model checking problem of MTL is \(\leq_m^{\log}\)-complete for PSPACE.

Let \(\text{md}(\varphi)\) denote the **modal depth** of \(\varphi\), recursively defined as

\[
\text{md}(\top) := \text{md}(\bot) := 0, \\
\text{md}(p) := \text{md}(\neg p) := 0, \\
\text{md}(\sim \varphi) := \text{md}(\varphi), \\
\text{md}(\varphi \land \psi) := \max\{\text{md}(\varphi \lor \psi) := \max\{\text{md}(\varphi), \text{md}(\psi)\}, \\
\text{md}(\Diamond \varphi) := \text{md}(\Box \varphi) := \text{md}(\varphi) + 1. \\
\]

Then MTL\(_k\) is the restriction of MTL to formulas with modal depth at most \(k.\) ML\(_k\) is the analogous fragment of ML. It is easy to see that MTL\(_0\) can be identified with the logic PL\(\{\land, \land, \lor, \lor\}).

**Theorem 4.16** ([Lüc18]). The satisfiability and validity problem of MTL\(_k\) are \(\leq_m^{\log}\)-complete for ATIME-ALT\(\text{exp}_k, \text{poly}\).

Next, we give the formal semantics of the modal atoms of dependency. They are analogous to the propositional case (cf. p. 7). Below, let \(K = (W, R, V)\) be a Kripke structure. For every \(w \in W\), we define

\[
w(\varphi) = \begin{cases} 
1 & \text{if } (K, \{w\}) \models \varphi, \\
0 & \text{else}.
\end{cases}
\]
Then the atoms are defined as follows:

**Dependence:** \(=(\vec{\varphi}; \vec{\psi})\):

\[(K, T) \vDash (\vec{\varphi}; \vec{\psi}) \iff \forall w, w' \in T : w(\vec{\varphi}) = w'(\vec{\varphi}) \Rightarrow w(\vec{\psi}) = w'(\vec{\psi})\]

**Independence:** \(\vec{\varphi} \perp \vec{\psi}\):

\[(K, T) \vDash \vec{\varphi} \perp \vec{\psi} \iff \forall w, w' \in T : \exists w'' \in T : w(\vec{\varphi}) = w''(\vec{\varphi}) \text{ and } w'(\vec{\psi}) = w''(\vec{\psi})\]

**Conditional independence:** \(\vec{\varphi} \perp_{\vec{\psi}} \vec{\theta}\):

\[(K, T) \vDash \vec{\varphi} \perp_{\vec{\psi}} \vec{\theta} \iff \forall w, w' \in T : \text{ if } w(\vec{\psi}) = w'(\vec{\psi}) \text{ then } \exists w'' \in T : w(\vec{\varphi}) = w''(\vec{\varphi}) \text{ and } w'(\vec{\theta}) = w''(\vec{\theta})\]

**Inclusion:** \(\vec{\varphi} \subseteq \vec{\psi}\), where \(\vec{\varphi}\) and \(\vec{\psi}\) have equal length:

\[(K, T) \vDash \vec{\varphi} \subseteq \vec{\psi} \iff \forall w \in T \exists w' \in T : w(\vec{\varphi}) = w'(\vec{\psi})\]

**Exclusion:** \(\vec{\varphi} \mid \vec{\psi}\), where \(\vec{\varphi}\) and \(\vec{\psi}\) have equal length:

\[(K, T) \vDash \vec{\varphi} \mid \vec{\psi} \iff \forall w \in T \forall w' \in T : w(\vec{\varphi}) \neq w'(\vec{\psi})\]

**Anonymity:** \(\vec{\varphi} \Upsilon \vec{\psi}\):

\[(K, T) \vDash \vec{\varphi} \Upsilon \vec{\psi} \iff \forall w \in T \exists w' \in T : w(\vec{\varphi}) = w'(\vec{\psi}) \text{ and } w(\vec{\psi}) \neq w'(\vec{\psi})\]

The modal atom of dependence is due to Väänänen [Vään08]. Ebbing et al. [EHM+13] generalized it to take arbitrary modal formulas as arguments. The modal independence atom is due to Kontinen et al. [KMSV17]. The modal inclusion atom is due to Hella and Stumpf [HS15]. Finally, the definitions of the exclusion and anonymity atoms are analogous to the propositional and first-order atoms. Using the polynomial translations presented in this paper, the complexity theoretic classification of fragments of MTL now carries over to their extensions by the above atoms of dependency.

**Theorem 4.17.** Let \(\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)\), \(\vec{\beta} = (\beta_1, \ldots, \beta_m)\), and \(\vec{\gamma} = (\gamma_1, \ldots, \gamma_k)\) be tuples of ML_{k}\-formulas.

- The dependence atom \((\vec{\alpha}; \vec{\beta})\) is equivalent to an MTL_{k}\-formula of length \(O(|\vec{\alpha}|\vec{\beta}|)\).
- The exclusion atom \(\vec{\alpha} | \vec{\beta}\) is equivalent to an MTL_{k}\-formula of length \(O(n|\vec{\alpha}|\vec{\beta}|)\).
- The inclusion atom \(\vec{\alpha} \subseteq \vec{\beta}\) is equivalent to an MTL_{k}\-formula of length \(O(n|\vec{\alpha}|\vec{\beta}|)\).
- The conditional independence atom \(\vec{\alpha} \perp_{\vec{\gamma}} \vec{\beta}\) is equivalent to an MTL_{k}\-formula of length \(O(n(n+m+k)|\vec{\alpha}|\vec{\beta}|\vec{\gamma}|)\).
- The anonymity atom \(\vec{\alpha} \Upsilon \vec{\beta}\) is equivalent to an MTL_{k}\-formula of length \(O(n|\vec{\beta}| + m|\vec{\alpha}|)\).

Furthermore, all these formulas are logspace-computable.

**Proof.** Proven exactly as the case of lax splitting in Theorem 4.9.

**Corollary 4.18.** Let \(\mathcal{A}\) be an arbitrary subset of the atoms \(\{=, \cdot, \perp, \subseteq, |, \Upsilon\}\). Let \(\mathcal{L}(\mathcal{A})\) resp. \(\mathcal{L}(\mathcal{A})\) be the extension of the logic \(\mathcal{L}\) by the atoms in \(\mathcal{A}\). Then the following holds:

- The satisfiability and validity problem of MTL\((\mathcal{A})\) are \(\leq_{\text{M}}^{\text{log}}\)-complete for TOWER\(\text{(poly)}\).
- The satisfiability and validity problem of MTL_{k}\(\mathcal{A}\) are \(\leq_{\text{M}}^{\text{log}}\)-complete for ATIME-ALT\(\text{(exp}_{k}\text{, poly)}\).
- The model checking problem of MTL\(\mathcal{A}\) is \(\leq_{\text{M}}^{\text{log}}\)-complete for PSPACE.
5. Conclusion

In this paper, we classified common atoms of dependency with respect to their succinctness in various fragments of propositional team logic. We showed that the negations of these atoms all can be polynomially expressed in the positive fragment of propositional team logic, while the atoms themselves can only be expressed in this fragment in formulas of exponential size. This implies polynomial upper bounds for the atoms in full propositional team logic with unrestricted contradictory negation. For the lower bounds, we adapted formula size games to the team semantics setting, and refined the approach with the notion of upper dimension.

In further research, comparing the atoms of dependency in terms of succinctness could be interesting. For example, do the lower bounds for the inclusion atoms still hold if we consider the positive fragment together with dependence atoms? Adding moves corresponding to atoms of dependency to the formula size game would enable looking into the relative succinctness.

Acknowledgment

The authors want to thank Lauri Hella and Raine Rönholm for pointing out references essential for this work, and Juha Kontinen for being the initiating force behind it. This work was supported in part by a joint grant of the DAAD (57348395) and the Academy of Finland (308099).

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