The Physics of the Ultraviolet Renormalon

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Abstract

We review the physics of the ultraviolet renormalon. This mini-review is intended to be a sequel to the review by the same authors at the "QCD '96" conference last year.

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THE ULTRAVIOLET RENORMALON.

Last-year's review [1] did not cover this topic at all, mostly because the physics of the ultraviolet renormalon is quite different and much less transparent than that of the infrared renormalons. Very recently, however, there has appeared some preliminary evidence [2], obtained with the lattice simulations, that UV-renormalon type effects may be significant. In another development, these effects were reconsidered within a general dispersion relations framework [3].

**Basic facts.**

The best-known fact about UV renormalons in QCD [4] is that they dominate the perturbation expansions in $\alpha_s(Q^2)$ at large orders $n$:

$$f_{\text{pert}} = \sum_n a_n \alpha_s^n = c_{\text{renorm}} \sum_n (-1)^n n! b_0^n \alpha_s^n$$  \hspace{1cm} (1)

where $f_{\text{pert}}$ is a generic perturbative series for an observable $f$ and $c_{\text{renorm}}$ is a constant. The $n!$ behaviour of the expansion coefficients $a_n$ can be inferred from the simplest renormalon chain [4]. Evaluation of $\beta$ requires evaluation of loop corrections and will be neglected in this review. Moreover, Feynman integrals associated with the renormalon chain are dominated by very large $k_{\text{eff}}^2$:

$$k_{\text{eff}}^2 \sim e^n Q^2,$$  \hspace{1cm} (2)

where $Q^2$ is assumed to be large by itself. Because of the sign oscillations in (1) the series is Borel summable. Putting for simplicity $\beta = 0$,

$$\sum_n a_n \alpha_s^n \sim \int_0^\infty \frac{\exp(-t/b_0 \alpha_s)}{1+t} dt.$$  \hspace{1cm} (3)

There exist also other ways to circumvent the divergence of perturbative expansions associated with the UV-renormalon (1). The asymptotic nature of the expansion (1) can be characterized by the magnitude $\Delta$ where

$$\Delta = |a_n \alpha_s^n|_{\text{min}}$$  \hspace{1cm} (4)

and the minimization is understood with respect to $n$, with $\alpha_s$ fixed. It is easy to see that if an expansion in $\alpha_s(Q^2)$ is used, then

$$\Delta(\alpha_s(Q^2)) \sim \frac{\Lambda_{\text{QCD}}}{Q^2}.$$  \hspace{1cm} (5)

On the other hand, if the coupling is normalized at $\mu$ and an expansion in $\alpha_s(\mu^2)$ is considered then [3]:

$$\Delta(\alpha_s(\mu^2)) \sim \frac{\Lambda_{\text{QCD}}^2}{\mu^2} \frac{Q^4}{Q^2}.$$  \hspace{1cm} (6)

which implies that by choosing the normalization point $\mu^2$ high enough one can make the uncertainty of the perturbative expansion arbitrarily small.

The last observation which we would like to mention among the facts concerning UV renormalons is that the evaluation of the coefficients $a_n$ is most straightforward if one uses an operator product expansion utilizing the fact that $k_{\text{eff}}^2 \gg Q^2$ [5, 7, 8]. Moreover, operators of dimension $d = 6$ are relevant to the leading UV renormalon [4].
We shall give an example of such an operator later and now notice that in this way one can prove [4] that actually multirenormalon chains produce the same asymptotics as a single chain [4]). Thus, by $c_{\text{renorm}}$ in Eq. (1) one should actually understand a sum over contributions of graphs with various numbers of renormalon chains:

$$c_{\text{renorm}} \to \sum_n c_{n \text{ renorm}}.$$  

(7)

Thus a new sum is introduced and the convergence properties of this sum are not known. Moreover, “towers” of renormalon chains can be important as well and the corresponding $k^2_{\text{eff}}$ can be even much larger than indicated by (3).

Puzzles.

The Borel summation brings in some puzzles as well. Indeed, the procedure implies that the sum over the rising branch of the perturbative expansion is equal to (one half of) the minimal product $a_n a_s^n(Q^2)$ which is of order $\Lambda^2_{\text{QCD}}/Q^2$. On the other hand, applying general dispersion relations to the quantity $f$ (whose perturbative expansion we are analyzing) one would conclude that the $1/Q^2$ piece is associated with resonances. This kind of logic is behind the QCD sum rules [3] but in that case resonances are dual to the IR sensitive part of perturbative graphs parametrized in terms of vacuum condensates. Now, UV renormalons bring a duality between terms $\Lambda^2_{\text{QCD}}/Q^2$ in the dispersive representation and Feynman graphs at momenta of order $Q^4/\Lambda^2_{\text{QCD}}$ which looks puzzling [10].

Turning next to Eqs. (1), (3) we note that for $\mu^2 \gg Q^2$ the perturbative expansion in $\alpha_s(\mu^2)$ is uniquely defined to an accuracy much better than $\Lambda^2_{\text{QCD}}/Q^2$. Moreover, if the whole procedure is self consistent the expansion in $\alpha_s(\mu^2)$ appears to converge to the Borel sum of the perturbative expansion in $\alpha_s(Q^2)$. In other words, it appears as if one can prove the validity of the Borel summation, which seems to be too strong a result to emerge without extra hypotheses.

Borel summation apparently implies a kind of generalization of the renormalization group to the power-like corrections [6, 11]. Indeed let us consider a large but finite ultraviolet cut off $\Lambda_{\text{UV}}$. Then we can write the bare Lagrangian as

$$L_{\text{bare}} = L_4(\alpha_{s,\text{bare}}) + \sum_i c_i \frac{\Lambda^2_{\text{UV}}}{\Lambda^2_{\text{UV}}} O_i(6)$$

(8)

where $L_4$ is the standard Lagrangian containing operators of dimension $d = 4$ while $O_i^6$ are all possible operators of dimension $d = 6$. Moreover, the standard renormalization group argument produces a relation between $\alpha_s(Q^2), \alpha_s(\Lambda^2_{\text{UV}}), \Lambda_{\text{UV}}/Q$. Similarly, the ultraviolet renormalon [11] is related to the insertions of the $d=6$ operators (8). Indeed let us estimate, by means of Eq. (2), the order of perturbative expansion which is affected by the UV cut off,

$$N \sim \ln \frac{\Lambda^2_{\text{UV}}}{Q^2}.$$  

(9)

Then the corresponding change in the Borel sum (3) is of order:

$$\int_{(\ln \Lambda^2_{\text{UV}}/Q^2)/\ln Q^2/\Lambda^2_{\text{QCD}}}^{\infty} \frac{\exp(-t/b_0 \alpha_s(Q^2))}{1 + t} \frac{dt}{t} \sim \frac{Q^2}{\Lambda^2_{\text{UV}}}$$

(10)
which corresponds to the matrix elements of $d = 6$ operators over states characterized by large momentum $Q$ (unlike the OPE used in the infrared region the matrix elements of the operators used to evaluate the UV renormalon are calculable perturbatively, see, e.g., [7]).

This argument leads us to believe that the "irrelevant" operators of $d = 6$ in Eq. (8) can be introduced only in conjunction with the UV renormalons. On the other hand, there are no theoretical means in fact to fix the constant in front of the coefficients $a_n$ due to UV renormalons. The reason was already mentioned above: multi-renormalon chains produce the same asymptotics as in Eq. (1). Thus, the coefficients $c_i$ could be fixed only in terms of all $c_n\text{renorm}$. However, these multi-renormalon chains are associated with momentum $k_{eff}^2$ even higher than (2) [7]. Thus Eq. (9) does not hold generally speaking beyond one-renormalon chain and Eq. (10) does not extend much beyond one renormalon chain and the assumption on the Borel summability appears to be a not-well-understood constraint.

All these questions can well be resolved upon further analysis. However, the time might be ripe for speculations as well.

Speculations.

There are recent speculations [3] which turn the puzzles discussed above into positive statements. Namely a $1/Q^2$ non-perturbative correction is postulated to exist in the running coupling $\alpha_s(Q^2)$ itself and then these corrections make the question on the Borel summability of the UV renormalon somewhat irrelevant because there is an extra source of $1/Q^2$ terms. The argument is based on a dispersion representation for the running coupling $\alpha_s(Q^2)$ and goes back to ideas expressed in the fifties [12]. Namely the running coupling $\alpha_s(Q^2)$ in the leading log approximation

$$\alpha_s(Q^2) = \frac{1}{b_0 \ln(Q^2/\Lambda_{QCD}^2)}$$

contains a pole at $Q^2 = \Lambda_{QCD}^2$ which is not present at any finite order of perturbation theory. If one removes this pole from the imaginary part and invokes analyticity then one arrives at (for a recent discussion see [13]) a modified expression for the running coupling:

$$\bar{\alpha}_s(Q^2) = \frac{1}{b_0 \ln(Q^2/\Lambda_{QCD}^2)} + \frac{\Lambda_{QCD}^2}{b_0(\Lambda_{QCD}^2 - Q^2)}$$

(11)

which at large $Q^2$ differs from the standard coupling by a $1/Q^2$ term. Although such terms are not detectable formally via perturbative expansions, common wisdom tells us that the nonperturbative corrections start with $Q^{-4}$ corrections since the lowest dimension of a gauge invariant operator, $(G^a_{\mu\nu})^2$ is four. Eq. (11) does introduce therefore a new kind of non-perturbative correction at large $Q^2$ based on the idea of the duality of the $1/Q^2$ corrections at large $Q^2$ to the region of small $Q^2$, $Q^2 \sim \Lambda_{QCD}^2$ in the dispersion representation. Let us note that the reasoning for the introduction of the $1/Q^2$ correction by itself is not compelling in fact. One can always impose relations on the imaginary part in such a way as to get rid of the $1/Q^2$ term at large $Q^2$ (see, e.g., [14]).

Note also that according to the logic outlined above the $1/Q^2$ term in $\alpha_s^2(Q^2)$ does not reduce directly to that in $\alpha_s(Q^2)$ and therefore the $1/Q^2$ correction cannot be
removed by a mere redefinition of the coupling. Indeed originally we have an expansion in $\alpha_s(Q^2)$ and are removing now the corresponding single pole in the perturbative expression for each term. For example:

$$\frac{1}{\ln^2 Q^2 / \Lambda^2_{QCD}} \rightarrow \frac{1}{\ln^2 Q^2 / \Lambda^2_{QCD}} - \frac{\Lambda^2_{QCD}}{Q^2 - \Lambda^2_{QCD}},$$

and there is no small factor $\alpha_s(Q^2)$ in front of the $1/Q^2$ correction despite the fact that we started with $\alpha_s^2(Q^2)$. Therefore, all terms in the $\alpha_s(Q^2)$ expansion give comparable contribution to the $1/Q^2$ correction. This collapse of the whole perturbative expansion as far as power like corrections are concerned is similar to collapse of contributions of all the UV-renormalon chains mentioned above (see Eq. (7)).

Physical picture.

Although the speculations above did allow us to settle the puzzles outlined in subsection 2, it is an absolutely an open question whether these speculations are correct. The point which we emphasize here is that if the non-perturbative $1/Q^2$ corrections in $\alpha_s(Q^2)$ do exist, this would signify new physics, not a simple definition or redefinition of the coupling. To substantiate the point let us consider [15] for a moment, a very different problem at first sight, that is the static potential between a heavy quark and antiquark at short distances.

At short distances the potential is dominated of course by a Coulomb-like piece so that the actual problem is power-like corrections to it. These corrections have been considered many times (see, e.g., [16]) and various approaches so far have given similar results:

$$\lim_{r \to 0} V(r) = -\frac{C_F \alpha_s(r)}{r} + c_2 r^2 \Lambda^2_{QCD}$$

where $C_F = 4/3$ and we consider the potential in the color-singlet channel. Note that short distances, that is $r \ll \Lambda^{-1}_{QCD}$ are considered. At large distances $V(r)$ is a confining, linear in $r$ potential.

Although each time [16] there is a particular model behind Eq. (13) the physics can be explained in a very simple way [15]. Namely, consider an Abelian case and represent the potential as an integral over electric fields of two charged particles:

$$V(r) = \frac{1}{4\pi} \int d^3r' \mathbf{E}_1(r') \cdot \mathbf{E}_2(r + r').$$

In particular, the Coulomb potential can be obtained of course by integrating over the fields $\mathbf{E}_{1,2}$ of two point charges. On the other hand, if the electric fields are modified at large distances, then there arises a correction to the Coulomb energy at small distances as well.

Consider as an example two charges of opposite signs in a cavity of size $R$, $R \gg r$. Then the electric field of the charges, which is that of a dipole at large distances in empty space, changes at $r' \sim R$. The corresponding change in $V(r)$ is of order

$$\delta V(r) \sim \alpha r^2 \int_R^\infty d^3r' \frac{\alpha r^2}{(r')^6} \sim \frac{\alpha r^2}{R^3},$$

which is in agreement with the correction to the static potential above. Indeed, in QCD one just assumes that the perturbative fields are changed at distances $R_{cr} \sim \Lambda^{-1}_{QCD}$ where the running coupling $\alpha_s(R_{cr})$ becomes of order unit.
Let us introduce now an alternative model according to which the (color) electrostatic field of quarks is a correct zeroth-order approximation only as far as it exceeds some critical value of order $\Lambda_{QCD}^2$:

$$\left(\mathbf{E}^a\right)_{cr}^2 \sim \Lambda_{QCD}^4$$

(16)

while weaker fields do not penetrate the vacuum because of its specific, confining properties. From this condition we get an estimate of distances $R_{cr}$ where the electrostatic field of quarks is strongly modified:

$$\frac{\alpha_s r^2}{R^6_{cr}} \leq \Lambda_{QCD}^4$$

(17)

where for simplicity we have neglected the effect of the running of $\alpha_s(r')$. The change in the potential is then of order:

$$\lim_{r \to 0} \delta V \sim \frac{\alpha_s r^2}{R^6_{cr}} \sim \alpha^{1/2} r \Lambda_{QCD}^2,$$

(18)

i.e., we get a linear in $r$ leading correction to the potential at short distances.

Now, we can turn back to the $1/Q^2$ correction to $\alpha_s(Q^2)$. The point is that such a correction would also bring a linear correction to the potential at short distances and it is natural to speculate that the mechanism behind this $1/Q^2$ correction is the same as discussed above in the case of the potential.

Although condition (16) might look very natural it is worth emphasizing that to realize such a condition we need small size non-perturbative fluctuations in the physical QCD vacuum. Indeed, according to (17) $R_{cr} \to 0$ if $r \to 0$. If one tries to speculate, what kind of fluctuations these could be, it is natural to turn to the dual superconductor picture of confinement [17] (for a recent review see [18]). Magnetic monopoles are a crucial field configuration in this case. The magnetic monopoles of QCD were introduced [19] in the Abelian projection of QCD where they appear as singular objects. Although this could be an artifact of the gauge fixing [19], convincing evidence for existence of monopoles as physical objects was obtained in just this gauge (see [20] and references therein). If the physical size of monopoles is indeed vanishing the linear potential at large distances, could well continue to $r \to 0$. This can be inferred, for example, from the results of ref.(21). One may even say that measurements on the lattice of the potential at short distances could give meaning to the notion of a monopole size.

**Phenomenology.**

The phenomenology of the UV renormalons is in its infancy. There exists only a single experimental indication [2] obtained on a lattice that $\alpha_s(Q^2)$ does receive a $1/Q^2$ correction at large $Q^2$:

$$\alpha_s(Q^2) = \frac{1}{b_0 \ln Q^2 / \Lambda_{QCD}^2} + c_{lat} \frac{\Lambda_{QCD}^2}{Q^2}$$

(19)

with $c_{lat} > 0$. Note that this positiveness of $c_{lat}$ rules out the most naive identification with the $1/Q^2$ correction in Eq. (11). Beyond this naive level, however, even this observation should be taken with caution. Indeed as is discussed above (see Eq. (12)) the $1/Q^2$ corrections due to all orders in $\alpha_s(Q^2)$ are of the same order and Eq. (13) should
be taken rather as a mnemonic for the results of the measurements of \( \langle \alpha_s(G_{\mu\nu})^2 \rangle \) than a real determination of \( \alpha_s(Q^2) \) with inclusion of powerlike corrections. It would be of great importance of course to further check that nonperturbative corrections at large \( Q^2 \) start with \( 1/Q^2 \).

As for direct measurements of the \( V(r) \) on the lattice (see [22] and references therein) they do not indicate any change of linear in \( r \) behaviour at large \( r \) to the \( r^2 \) behaviour at short distances predicted by the "standard" QCD (see Eq. (13)) and in this sense one might say that expectations of a linear correction at short distances are confirmed (see Eq. (18)). However existing measurements do not target power corrections to the Coulomb potential at short distances specifically and no statement on these corrections has been made.

Direct evaluation of ultraviolet renormalon contribution to various observables was tried in a few papers, see e.g. [23]. The result is scheme dependent and usually does not represent a leading power correction. It is more in the spirit of the present review to consider rather UV-renormalon inspired phenomenology. Namely one starts with UV renormalon contributions to various quantities and assumes, explicitly or tacitly, that they are enhanced in some way to a phenomenologically significant level. The approach is similar to that one widely practiced for IR renormalons (for a review see [4]). And although such a phenomenology can be truly successful only if non-perturbative \( 1/Q^2 \) corrections exist and are relatively large, it can be developed without discussing the mechanism of the enhancement. There are some encouraging qualitative results obtained. First, the UV renormalons are universal in the sense that their contributions are determined in terms of anomalous dimensions of a few operators of dimension six [7, 8]. These operators can be split furthermore into operators which can be added directly to the bare QCD Lagrangian (see Eq. (8)) and operators which are process dependent. The latter set depends, for example, on whether one considers correlator of vector or pseudoscalar currents. It turns out that the process-independent operators dominate. It is not trivial since these operators appear first on the two-renormalon-chain level while process-dependent operators can be associated already with a single chain. Moreover, the quark operators are then the same that are postulated within the Nambu-Jona-Lasinio model indicating for the first time a possible connection between this phenomenological model and fundamental QCD [4, 24, 25].

Moreover, one may try to explain [24] by inclusion of \( 1/Q^2 \) terms the irregularities [26] in the QCD sum rules in various channels. Essentially, in the same spirit as described in subsections 2.3 one speculates now that UV renormalons are dual to the pion (for other possible explanations of the failure of the standard QCD sum rules in the pion channel see [28]). On the other hand, there can be no such duality for the \( \rho \) [27]. For this to be true, the leading quark operator of dimension six should be

\[
O_{\text{lead}}^{(6)} = (\bar{q} \tau^i \gamma_5 q)(\bar{q} \tau^i \gamma_5 q) + (\bar{q} \tau^i q)(\bar{q} \tau^i q)
\]

where \( \tau^i \) are the Pauli matrices in the flavour space of \( u \) and \( d \) quarks. This difference between the \( \rho \) - and \( \pi \) - channels is a nontrivial check of the whole scheme. It is amusing therefore that a direct calculation of two renormalon chains [25] does indicate this difference:

\[
(\text{UV renormalon})_{PS} = 18(\text{UV renormalon})_V
\]
giving hopes for the relevance of renormalons.

**Conclusions.**

One might say that there is a recent trend to view the UV-renormalon physics in a way similar to the IR physics. Namely one assumes that nonperturbative fluctuations produce $1/Q^2$ corrections which enhance the UV renormalon contributions. This analogy between UV and IR physics is far from trivial since it assumes the existence of small-size nonperturbative fluctuations in QCD. This strategy may well be wrong. On the other hand, if it is confirmed by measurements it would provide new insight into the physics of confinement.

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