Abstract  The cocenter of an affine Hecke algebra plays an important role in the study of representations of the affine Hecke algebra and the geometry of affine Deligne–Lusztig varieties (see for example, He and Nie in Compos Math 150(11):1903–1927, 2014; He in Ann Math 179:367–404, 2014; Ciubotaru and He in Cocenter and representations of affine Hecke algebras, 2014). In this paper, we give a Bernstein–Lusztig type presentation of the cocenter. We also obtain a comparison theorem between the class polynomials of the affine Hecke algebra and those of its parabolic subalgebras, which is an algebraic analog of the Hodge–Newton decomposition theorem for affine Deligne–Lusztig varieties. As a consequence, we present a new proof of the emptiness pattern of affine Deligne–Lusztig varieties (Görtz et al. in Compos Math 146(5):1339–1382, 2010; Görtz et al. in Ann Sci École Norm Sup, 2012).

Keywords  Affine Hecke algebras · Cocenters · Parabolic subalgebras
Mathematics Subject Classification  20C08 · 20F55 · 20E45

1 Introduction

1.1 Cocenter of affine Hecke algebras

The purpose of this paper is twofold. We use some ideas arising from affine Deligne–Lusztig varieties to study affine Hecke algebras (with arbitrary nonzero parameters), and we apply the results on affine Hecke algebras to affine Deligne–Lusztig varieties.

For simplicity, we only discuss the equal parameter case in the introduction. The case of unequal parameters and the twisted cocenters will also be presented in this paper.

Let $\mathcal{R} = (X, R, Y, R^\vee, F_0, S_0)$ be a based root datum and let $\tilde{W}$ be the associated extended affine Weyl group (see Sect. 2.1). An affine Hecke algebra $\mathcal{H}$ is a deformation of the group algebra of $\tilde{W}$. It is a free $\mathbb{Z}[v, v^{-1}]$-algebra with basis $\{T_w\}$, where $w \in \tilde{W}$. The relations among the $T_w$ are given in Sect. 2.3. This is the Iwahori–Matsumoto presentation of $\mathcal{H}$.

The cocenter $\bar{\mathcal{H}} = \mathcal{H}/[\mathcal{H}, \mathcal{H}]$ of $\mathcal{H}$ is a useful tool in the study of the representation theory of $\mathcal{H}$ and the structures of $p$-adic groups (e.g., affine Deligne–Lusztig varieties). We will discuss some applications of the cocenter as they serve as the motivation for this paper.

Let $R(\mathcal{H})$ be the Grothendieck group of representations of $\mathcal{H}$. Then, the trace map $Tr : \tilde{\mathcal{H}} \to R(\mathcal{H})^\ast$ relates the cocenter $\tilde{\mathcal{H}}$ to the representations of $\mathcal{H}$. This map was studied in [1, 11] for Hecke algebras of $p$-adic groups.

In [6], we provide a standard basis of the cocenter $\bar{\mathcal{H}}$, which is constructed as follows. For each conjugacy class $\mathcal{O}$ of $\tilde{W}$, we choose a minimal length representative $w_{\mathcal{O}}$. Then, the image of $T_{w_{\mathcal{O}}}$ in $\tilde{\mathcal{H}}$ is independent of the choice of $w_{\mathcal{O}}$ and the set $\{T_{w_{\mathcal{O}}}\}$, where $\mathcal{O}$ ranges over all the conjugacy classes of $\tilde{W}$, is a basis of $\bar{\mathcal{H}}$. This is the (normalized) Iwahori–Matsumoto presentation of $\bar{\mathcal{H}}$.

Moreover, for any $w \in \tilde{W}$,

$$T_w \equiv \sum_{\mathcal{O}} f_{w, \mathcal{O}} T_{w_{\mathcal{O}}} \mod [\mathcal{H}, \mathcal{H}]$$

for some $f_{w, \mathcal{O}} \in \mathbb{N}[v - v^{-1}]$. The coefficients $f_{w, \mathcal{O}}$ are called the class polynomials.

In [9], the first-named author proved the “dimension=degree” theorem, which relates the degrees of the class polynomials of $\mathcal{H}$ to the dimensions of the affine Deligne–Lusztig varieties of the corresponding $p$-adic group $G$.

1.2 Parametrization of conjugacy classes

However, the Iwahori–Matsumoto presentation of the cocenter is not convenient to use in the study of representations of $\mathcal{H}$. The reason is that a large number of representations of $\mathcal{H}$ are built on the induced representations $\text{Ind}_{\mathcal{H}_J}^{\mathcal{H}} (-)$, where $J \subset F_0$ and $\mathcal{H}_J$ is
the corresponding parabolic subalgebra. On the other hand, a conjugacy class \( \mathcal{O} \) of \( \tilde{W} \), which intersects with the parabolic subgroup \( \tilde{W}_J \), does not always have a minimal length elements in \( \tilde{W}_J \). Thus, \( T_{w_\mathcal{O}} \) are not contained in \( \mathcal{H}_J \) in general.

In order to relate the cocenter and representations of \( \mathcal{H} \), we need another presentation, namely the Bernstein–Lusztig presentation for the cocenter. To do this, we first establish a parametrization of conjugacy classes of \( \tilde{W} \).

**Proposition A** (Proposition 6.8) Let \( A \) be the set of pairs \((J, C)\), where \( J \subset F_0 \), \( C \) is a dominant elliptic conjugacy class of \( \tilde{W}_J \). The inclusion map gives a bijection

\[
\pi : A/\sim \to \text{conjugacy classes of } \tilde{W}.
\]

The definition of dominant elliptic conjugacy classes is given in Sect. 6.6. It is motivated by the Newton map of \( p \)-adic groups.

1.3 Bernstein–Lusztig basis of the cocenter

Based on the above parametrization, we give the Bernstein–Lusztig presentation of the cocenter.

**Theorem B** (Theorems 6.1 and 6.9) Let \((J, C) \in A \) and \( \mathcal{O} = \pi(J, C) \). Then

\[
T_{w_\mathcal{O}} \equiv T^{J}_{w_C} \mod [\mathcal{H}, \mathcal{H}].
\]

Here, \( w_\mathcal{O} \) is a minimal length representative of \( \mathcal{O} \) with respect to the length function \( \ell \) of \( \tilde{W} \), and \( w_C \) is a minimal length representative of \( C \) with respect to the length function \( \ell_J \) of \( \tilde{W}_J \). The description of the element \( T^{J}_{w_C} \) in \( \mathcal{H} \) uses Bernstein–Lusztig presentation. Thus, Theorem B gives a Bernstein–Lusztig presentation of the cocenter \( \mathcal{H} \).

Notice that the length function on \( \tilde{W} \) and the length function on \( \tilde{W}_J \) are different. Thus, the relationship between the element \( w_\mathcal{O} \) and \( w_C \) is hard to understand. On the other hand, the relationship between Iwahori–Matsumoto basis and Bernstein–Lusztig basis of \( \mathcal{H} \) is also complicated. It is amazing that the Iwahori–Matsumoto element \( T_{w_\mathcal{O}} \) and the Bernstein-Lusztig element \( T^{J}_{w_C} \) differ by an element in \([\mathcal{H}, \mathcal{H}]\).

Another interesting thing is that in the Bernstein–Lusztig basis of \( \mathcal{H} \), there are exactly \( N \) elements that are not represented by elements in a proper parabolic subalgebra of \( \mathcal{H} \), where \( N \) is the number of elliptic conjugacy classes of \( \tilde{W} \). On the other hand, Opdam and Solleveld showed in [16, Proposition 3.9] and [17, Theorem 7.1] that for affine Hecke algebras with positive real parameters, the dimension of the space of “elliptic trace functions” on \( \mathcal{H} \) also equals \( N \).

1.4 Application to affine Deligne–Lusztig varieties

Now, we discuss some applications to class polynomials in affine Hecke algebras and to affine Deligne–Lusztig varieties of a loop group. For simplicity, we only discuss split groups in the introduction.
Let $G$ be a split group over the field of Laurent series. We fix a Borel subgroup $B$ and a maximal torus $A \subset B$. The pair $(B, A)$ gives rise to a based root datum and the Iwahori–Weyl group $\tilde{W}$ of $G$. We call a parabolic subgroup of $G$ standard if it contains $B$ and semistandard if it contains $A$.

Let $P$ be a semistandard parabolic subgroup of $G$. The notion of $P$-alcove elements (in $\tilde{W}$) was introduced by Görtz, Haines, Kottwitz, and Reuman in [5] and generalized in [4] for non-split groups. Roughly speaking, $w$ is a $P$-alcove element if the finite part of $w$ lies in the finite Weyl group of $P$, and $w$ sends the fundamental alcove to a certain region of the apartment. See [5, Section 3] for a visualization. The $P$-alcove elements play an important role in the study of affine Deligne–Lusztig varieties.

**Theorem C** (Theorems 7.2 and 7.3) Let $P$ be a semistandard parabolic subgroup of $G$ and let $J \subset F_0$ and $z \in W_0$ such that $z$ is minimal in its right coset of $W_J$ and $P = z^{-1}P_Jz$. Here, $P_J$ is the standard parabolic subgroup corresponding to $J$ (and $F_0$). If $\tilde{w}$ is a $P$-alcove element, then

(1) $T_{\tilde{w}} \in \mathcal{H}_J + [\mathcal{H}, \mathcal{H}]$.

(2) Suppose that

$$T_{\tilde{w}} \equiv \sum_{\mathcal{O}} f_{\tilde{w}, \mathcal{O}} T_{w_{\mathcal{O}}} \mod [\mathcal{H}, \mathcal{H}],$$

$$T_{z\tilde{w}z^{-1}}^J \equiv \sum_{\mathcal{O}'} f_{z\tilde{w}z^{-1}, \mathcal{O}'} T_{w'_{\mathcal{O}'}}^J \mod [\mathcal{H}_J, \mathcal{H}_J].$$

Then $f_{\tilde{w}, \mathcal{O}} = \sum_{\mathcal{O}' \subset \mathcal{O}} f_{z\tilde{w}z^{-1}, \mathcal{O}'}$.

The Hodge–Newton decomposition theorem, which is proved in [5, Theorem 1.1.4], says that if $P$ is a semistandard parabolic subgroup of $G$ and $\tilde{w}$ is a $P$-alcove element, then the corresponding affine Deligne–Lusztig varieties for the group $G$ and for the semistandard Levi subgroup $M$ of $P$ are locally isomorphic.

Recall that there is a close relationship between the class polynomials and the affine Deligne–Lusztig varieties. Thus, Theorem C above can be regarded as an algebraic analog of the Hodge–Newton decomposition theorem in [5].

Combining Theorem C with the “degree=dimension” Theorem, we can derive an algebraic proof of [5, Theorem 1.1.2] and [4, Corollary 3.6.1] on the emptiness pattern of affine Deligne–Lusztig varieties.

2 Affine Hecke algebras

2.1 Based root datum

Let $\mathfrak{A} = (X, R, Y, R^\vee, F_0, S_0)$ be a based root datum, where $R \subset X$ is the set of roots, $R^\vee \subset Y$ is the set of coroots, and $F_0 \subset R$ is the set of simple roots. By definition, there exist a bijection $\alpha \mapsto \alpha^\vee$ from $R$ to $R^\vee$ and a perfect pairing $\langle, \rangle : X \times Y \to \mathbb{Z}$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ and the corresponding reflections $s_\alpha : X \to X$ stabilize $R$ and $s_\alpha^\vee : Y \to Y$ stabilizes $R^\vee$. Then, the set of simple reflections $S_0 = \{s_\alpha; \alpha \in F_0\}$
generates the Weyl group \( W_0 = W(R) \) of \( R \). We denote by \( R^+ \subset R \) the set of positive roots determined by \( F_0 \). Let \( X^+ = \{ \lambda \in X; \langle \lambda, \alpha'^\vee \rangle \geq 0, \forall \alpha \in R^+ \} \).

An automorphism of \( \mathfrak{g} \) is an automorphism \( \delta \) of \( X \) such that \( \delta(F_0) = F_0 \). Let \( \Gamma \) be a subgroup of automorphisms of \( \mathfrak{g} \).

### 2.2 Affine Weyl groups

Let \( V = X \otimes_\mathbb{Z} \mathbb{R} \). For \( \alpha \in R \) and \( k \in \mathbb{Z} \), set

\[
H_{\alpha,k} = \{ v \in V; \langle v, \alpha'^\vee \rangle = k \}.
\]

Let \( \mathfrak{S} = \{ H_{\alpha,k}; \alpha \in R, k \in \mathbb{Z} \} \). Connected components of \( V - \bigcup_{H \in \mathfrak{S}} H \) are called alcoves. Let

\[
C_0 = \{ v \in V; 0 < \langle v, \alpha'^\vee \rangle < 1, \forall \alpha \in R^+ \}
\]

be the fundamental alcove.

Let \( W = \mathbb{Z}R \rtimes W_0 \) be the affine Weyl group and \( S \supset S_0 \) be the set of simple reflections in \( W \). Then, \( (W, S) \) is a Coxeter group. Set \( \tilde{W} = (X \times W_0) \rtimes \Gamma = X \rtimes (W_0 \rtimes \Gamma) \). Then, \( W \) is a subgroup of \( \tilde{W} \). Both \( W \) and \( \tilde{W} \) can be regarded as groups of affine transformations of \( V \), which send alcoves to alcoves. For \( \lambda \in X \), we denote by \( t^\lambda \in W \) the corresponding translation. For any hyperplane \( H = H_{\alpha,k} \in \mathfrak{S} \) with \( \alpha \in R \) and \( k \in \mathbb{Z} \), we denote by \( s_H = t^{k\alpha} s_{\alpha} \in W \) the reflection of \( V \) along \( H \).

For any \( \tilde{w} \in \tilde{W} \), we denote by \( \ell(\tilde{w}) \) the number of hyperplanes in \( \mathfrak{S} \) separating \( C_0 \) from \( \tilde{w}C_0 \). By [10], the length function is given by the following formula

\[
\ell(t^k w \tau) = \sum_{\alpha, w^{-1}(\alpha) \in R^+} |\langle \chi, \alpha'^\vee \rangle| + \sum_{\alpha \in R^+, w^{-1}(\alpha) \in R^-} |\langle \chi, \alpha'^\vee \rangle - 1|.
\]

Here, \( \chi \in X, w \in W_0 \) and \( \tau \in \Gamma \).

If \( \tilde{w} \in \tilde{W} \), then \( \ell(\tilde{w}) \) is just the word length in the Coxeter system \( (W, S) \). Let \( \Omega = \{ \tilde{w} \in \tilde{W}; \ell(\tilde{w}) = 0 \} \). Then \( \tilde{W} = W \rtimes \Omega \).

### 2.3 Iwahori–Matsumoto presentation of affine Hecke algebras

Let \( q, s, t \in S \) be indeterminates. We assume that \( q^{\frac{1}{2}} = q^t \) if \( s, t \) are conjugate in \( \tilde{W} \).

Let \( \mathcal{A} = \mathbb{Z}[q_s^{\frac{1}{2}}, q_s^{-\frac{1}{2}}]_{s \in S} \) be the ring of Laurent polynomials in \( q_s^{\frac{1}{2}}, s \in S \) with integer coefficients.

The (generic) Hecke algebra \( \mathcal{H} \) associated with the extended affine Weyl group \( \tilde{W} \) is an associative \( \mathcal{A} \)-algebra with basis \( \{ T_{\tilde{w}}; \tilde{w} \in \tilde{W} \} \) subject to the following relations

\[
T_{\tilde{x}} T_{\tilde{y}} = T_{\tilde{x} \tilde{y}}, \quad \text{if} \quad \ell(\tilde{x}) + \ell(\tilde{y}) = \ell(\tilde{x} \tilde{y});
\]

\[
(T_s - q^{\frac{1}{2}})(T_s + q^{-\frac{1}{2}}) = 0, \quad \text{for} \quad s \in S.
\]
If $q^\frac{1}{s}$ is $q^\frac{1}{t}$ for all $s, t \in S$, then we call $\mathcal{H}$ the (generic) Hecke algebra with equal parameter.

This is the (normalized) Iwahori–Matsumoto presentation of $\mathcal{H}$. It reflects the structure of (quasi) Coxeter group $\tilde{W}$.

2.4 Bernstein–Lusztig presentation of affine Hecke algebras

In this section, we recall the Bernstein–Lusztig presentation of $\mathcal{H}$. It is used to construct a basis of the center of $\mathcal{H}$ and is useful in the study of representations of $\mathcal{H}$.

For any $\lambda \in X$, we may write $\lambda$ as $\lambda = \chi - \chi'$ for $\chi, \chi' \in X^+$. Now set $\theta_{\lambda} = T_{\chi} T_{\chi'}^{-1}$.

It is easy to see that $\theta_{\lambda}$ is independent of the choice of $\chi, \chi'$. The following results can be found in [14].

1. $\theta_{\lambda} \theta_{\lambda'} = \theta_{\lambda + \lambda'}$ for $\lambda, \lambda' \in X$.
2. The set $\{\theta_{\lambda} T_w; \lambda \in X, w \in W_0\}$ and $\{T_w \theta_{\lambda}; \lambda \in X, w \in W_0\}$ are $A$-bases of $\mathcal{H}$.
3. For $\lambda \in X^+$, set $\varepsilon_{\lambda} = \sum_{\lambda' \in W \cdot \lambda} \theta_{\lambda'}$. Then $\varepsilon_{\lambda}, \lambda \in X^+$ is an $A$-basis of the center of $\mathcal{H}$.
4. $\theta_{\chi} T_{s_{\alpha}} - T_{s_{\alpha}} \theta_{s_{\alpha} \chi} = \frac{1}{2} (q_{s_{\alpha}} - q_{-s_{\alpha}}) \frac{\theta_{\chi} - \theta_{s_{\alpha} \chi}}{1 - \theta_{s_{\alpha}}}$ for $\alpha \in F_0$ such that $\alpha^\vee \notin 2Y$ and $\chi \in X$.

The following special cases will be used a lot in this paper.

5. Let $\alpha \in F_0$ and $\chi \in X$. If $\langle \chi, \alpha^\vee \rangle = 0$, then $\theta_{\chi} T_{s_{\alpha}} = T_{s_{\alpha}} \theta_{\chi}$.
6. Let $\alpha \in F_0$ and $\chi \in X$. If $\langle \chi, \alpha^\vee \rangle = 1$, then $\theta_{s_{\alpha} \chi} = T_{s_{\alpha}} \theta_{\chi} T_{s_{\alpha}}^{-1}$.

2.5 Parabolic subalgebras

For any $J \subset F_0$, let $R_J$ be the set of roots spanned by $\alpha$ for $\alpha \in J$ and let $R_J^\vee$ be the set of coroots spanned by $\alpha^\vee$ for $\alpha \in J$. Let $\mathfrak{R}_J = (X, R_J, Y, R_J^\vee, J)$ be the based root datum corresponding to $J$. Let $W_J \subset W_0$ be the subgroup generated by $s_{\alpha}$ for $\alpha \in J$ and set $\tilde{W}_J = (X \times W_J) \rtimes \Gamma_J$. Here, $\Gamma_J = \{\delta \in \Gamma; \delta (R_J) = R_J\}$. As in Sect. 2.2, we set $\delta_J = \{H_{\alpha, k} \in \delta_J; \alpha \in R_J, k \in \mathbb{Z}\}$ and $C_J = \{v \in V; 0 < \langle v, \alpha^\vee \rangle < 1, \alpha \in R_J^\vee\}$. For any $\tilde{w} \in \tilde{W}_J$, we denote by $\ell_J(\tilde{w})$ the number of hyperplanes in $\delta_J$ separating $C_J$ from $\tilde{w} C_J$.

We denote by $\tilde{W}^J$ (resp. $\tilde{J}$) the set of minimal coset representatives in $\tilde{W}/W_J$ (resp. $W_J \setminus \tilde{W}$). For $J, K \subset F_0$, we simply write $\tilde{W}^J \cap \tilde{K}^J \tilde{W}$ as $\tilde{K} \tilde{W}^J$. We define $\tilde{W}^J$, $\tilde{W}^J_0$, $\tilde{W}^J_0$, and $\tilde{J} \tilde{W}^K$ in a similar way.

Let $\mathcal{H}_J \subset \mathcal{H}$ be the subalgebra generated by $\theta_{\lambda}$ for $\lambda \in X$ and $T_w$ for $w \in W_J \rtimes \Gamma_J$. We call $\mathcal{H}_J$ a parabolic subalgebra of $\mathcal{H}$.

It is known that $\mathcal{H}_J$ is the Hecke algebra associated with the extended affine Weyl group $\tilde{W}_J$ and the parameter function $p^\frac{1}{t}$, where $t$ ranges over simple reflections in $\tilde{W}_J$. The parameter function $p^\frac{1}{t}$ is determined by $q^\frac{1}{s}$ (see [16, 1.2]). We denote by $\{T_{\tilde{w}}^J\}_{\tilde{w} \in \tilde{W}_J}$ the Iwahori–Matsumoto basis of $\mathcal{H}_J$.

1 There is a more complicated formula for $\alpha^\vee \notin 2Y$ as well. However, we do not need it in this paper.
3 The Iwahori–Matsumoto presentation of \( \widetilde{\mathcal{H}} \)

3.1 Minimal length elements

We follow [6].

For \( \tilde{w}, \tilde{w}' \in \tilde{W} \) and \( s \in S \), we write \( \tilde{w} \overset{s}{\rightarrow} \tilde{w}' \) if \( \tilde{w}' = s \tilde{w}s \) and \( \ell(\tilde{w}') \leq \ell(\tilde{w}) \). We write \( \tilde{w} \rightarrow \tilde{w}' \) if there is a sequence \( \tilde{w} = \tilde{w}_0, \tilde{w}_1, \ldots, \tilde{w}_n = \tilde{w}' \) of elements in \( \tilde{W} \) such that for any \( k, \tilde{w}_{k-1} \overset{s}{\rightarrow} \tilde{w}_k \) for some \( s \in S \).

We write \( \tilde{w} \approx \tilde{w}' \) if \( \tilde{w} \rightarrow \tilde{w}' \) and \( \tilde{w}' \rightarrow \tilde{w} \). It is easy to see that \( \tilde{w} \approx \tilde{w}' \) if \( \tilde{w} \rightarrow \tilde{w}' \) and \( \ell(\tilde{w}) = \ell(\tilde{w}') \).

We call \( \tilde{w}, \tilde{w}' \in \tilde{W} \) elementarily strongly conjugate if \( \ell(\tilde{w}) = \ell(\tilde{w}') \) and there exists \( x \in W \) such that \( \tilde{w}' = x \tilde{w} x^{-1} \) and \( \ell(x \tilde{w}) = \ell(x) + \ell(\tilde{w}) \) or \( \ell(\tilde{w} x^{-1}) = \ell(x) + \ell(\tilde{w}) \). We call \( \tilde{w}, \tilde{w}' \) strongly conjugate if there is a sequence \( \tilde{w} = \tilde{w}_0, \tilde{w}_1, \ldots, \tilde{w}_n = \tilde{w}' \) such that for each \( i, \tilde{w}_{i-1} \) is elementarily strongly conjugate to \( \tilde{w}_i \). We write \( \tilde{w} \sim \tilde{w}' \) if \( \tilde{w} \) and \( \tilde{w}' \) are strongly conjugate. We write \( \tilde{w} \overset{t}{\sim} \tilde{w}' \) if \( \tilde{w} \sim \delta \tilde{w}' \delta^{-1} \) for some \( \delta \in \Omega \).

Now, we recall one of the main results in [6].

**Theorem 3.1** Let \( \mathcal{O} \) be a conjugacy class of \( \tilde{W} \) and \( \mathcal{O}_{\text{min}} \) be the set of minimal length elements in \( \mathcal{O} \). Then

1. For any \( \tilde{w}' \in \mathcal{O} \), there exists \( \tilde{w}'' \in \mathcal{O}_{\text{min}} \) such that \( \tilde{w}' \rightarrow \tilde{w}'' \).
2. Let \( \tilde{w}', \tilde{w}'' \in \mathcal{O}_{\text{min}} \), then \( \tilde{w}' \overset{\mathcal{O}}{\sim} \tilde{w}'' \).

3.2 Iwahori–Matsumoto presentation of the cocenter

Let \( h, h' \in \mathcal{H} \), we call \( [h, h'] = hh' - h'h \) the commutator of \( h \) and \( h' \). Let \( [\mathcal{H}, \mathcal{H}] \) be the \( \mathcal{A} \)-submodule of \( \mathcal{H} \) generated by all commutators. We call the quotient \( \mathcal{H}/[\mathcal{H}, \mathcal{H}] \) the cocenter of \( \mathcal{H} \) and denote it by \( \widetilde{\mathcal{H}} \).

It follows easily from definition that \( T_{\tilde{w}} \equiv T_{\tilde{w}'} \mod [\mathcal{H}, \mathcal{H}] \) if \( \tilde{w} \overset{\mathcal{O}}{\sim} \tilde{w}' \). Hence by Theorem 3.1 (2), for any conjugacy class \( \mathcal{O} \) of \( \tilde{W} \) and \( \tilde{w}, \tilde{w}' \in \mathcal{O}_{\text{min}}, T_{\tilde{w}} \equiv T_{\tilde{w}'} \mod [\mathcal{H}, \mathcal{H}] \). We denote by \( T_{\mathcal{O}} \) the image of \( T_{\tilde{w}} \) in \( \widetilde{\mathcal{H}} \) for any \( \tilde{w} \in \mathcal{O}_{\text{min}} \).

**Theorem 3.2** (1) The elements \( \{ T_{\mathcal{O}} \} \), where \( \mathcal{O} \) ranges over all the conjugacy classes of \( \tilde{W} \), span \( \widetilde{\mathcal{H}} \) as an \( \mathcal{A} \)-module.

(2) If \( q_{it}^s = q_{it}^t \) for all \( s, t \in S \), then \( \{ T_{\mathcal{O}} \} \) is a basis of \( \widetilde{\mathcal{H}} \).

We call \( \{ T_{\mathcal{O}} \} \) the Iwahori–Matsumoto presentation of the cocenter \( \widetilde{\mathcal{H}} \) of affine Hecke algebra \( \mathcal{H} \).

The equal parameter case was proved in [6, Theorems 5.3 and 6.7]. Part (1) for the unequal parameter case can be proved in the same way as in loc. cit. We expect that Part (2) remains valid for unequal parameter case. One possible approach is to use the classification of irreducible representations and a generalization of density theorem. We do not go into details in this paper.
4 Some length formulas

4.1 The strategy to prove Theorem B

The strategy to prove Theorem B in this paper is as follows. For a given conjugacy class $O$, we

- construct a minimal length element in $O$, which is used for the Iwahori–Matsumoto presentation of $\overline{H}$;
- construct a suitable $J$, and an element in $O \cap \tilde{W}_J$, of minimal length in its $\tilde{W}_J$-conjugacy class, which is used for the Bernstein-Lusztig presentation of $\overline{H}$;
- find the explicit relationship between the two different elements.

To do this, we need to relate the length function on $\tilde{W}$ with the length function on $\tilde{W}_J$ for some $J \subset F_0$. This is what we will do in this section. Another important technique is the “partial conjugation” method introduced in [7], which will be discussed in the next section.

4.2 Comparison between two length functions

Let $n = \sharp(W_0 \times \Gamma)$. For any $\tilde{w} \in \tilde{W}$, $\tilde{w}^\rho = t^\lambda$ for some $\lambda \in X$. We set $v_{\tilde{w}} = \lambda/n \in V$ and call it the Newton point of $\tilde{w}$. Let $\overline{\nu}_{\tilde{w}}$ be the unique dominant element in the $W_0$-orbit of $v_{\tilde{w}}$. Then, the map $\tilde{W} \to V, \tilde{w} \mapsto \overline{\nu}_{\tilde{w}}$ is constant on the conjugacy class of $\tilde{W}$. For any conjugacy class $O$, we set $v_O = \overline{\nu}_{\tilde{w}}$ for any $\tilde{w} \in O$ and call it the Newton point of $O$.

For $\tilde{w} \in \tilde{W}$, set

$$V_{\tilde{w}} = \{ v \in V; \tilde{w}(v) = v + v_{\tilde{w}} \}.$$ 

By [6, Lemma 2.2], $V_{\tilde{w}} \subset V$ is a non-empty affine subspace and $\tilde{w}V_{\tilde{w}} = V_{\tilde{w}} + v_{\tilde{w}} = V_{\tilde{w}}$. Let $p : \tilde{W} = X \times (W_0 \times \Gamma) \to W_0 \times \Gamma$ be the projection map. Let $u$ be an element in $V_{\tilde{w}}$. By the definition of $V_{\tilde{w}}$, $V_p(\tilde{w}) = \{ v - u; v \in V_{\tilde{w}} \}$. In particular, $v_{\tilde{w}} \in V_p(\tilde{w})$.

Let $E \subset V$ be a convex subset. Set $\mathfrak{H}_E = \{ H \in \mathfrak{H}; E \subset H \}$ and $W_E \subset W$ to be the subgroup generated by $s_H$ with $H \in \mathfrak{H}_E$. We say a point $p \in E$ is regular in $E$ if for any $H \in \mathfrak{H}_E$, $p \in H$ implies that $E \subset H$. Then, regular points of $E$ form an open dense subset of $E$. This construction will be applied to the case, where $E$ is an affine subspace of $V_{\tilde{w}}$ stabilized by $\tilde{w}$.

For any $\lambda \in V$, set $J_\lambda = \{ \alpha \in F_0; \langle \lambda, \alpha^\vee \rangle = 0 \}$.

The following results are proved in [6, Propositions 2.4 and 2.7] and will be used many times in this paper.

1. For any $\tilde{w} \in \tilde{W}$, there exists $\tilde{w}' \in \tilde{W}$ with $\tilde{w} \to \tilde{w}'$ and $\tilde{C}_0$ contains a regular point of $V_{\tilde{w}'}$.
2. If $\tilde{w} \in \tilde{W}$ and $\tilde{C}_0$ contains a regular point of some affine subspace $E \subset V_{\tilde{w}'}$ such that $\tilde{w}E = E + v_{\tilde{w}} = E$, then

$$\ell(\tilde{w}) = \langle \overline{\nu}_{\tilde{w}}, 2\rho^\vee \rangle + \sharp \mathfrak{H}_E(C, \tilde{w}C),$$
where $C$ is the connected component of $V - \cup_{H \in \mathcal{H}'} H$ containing $C_0$.

**Proposition 4.1** Let $\tilde{w} \in \tilde{W}$ such that $\tilde{C}_0$ contains a regular point $e$ of $V_{\tilde{w}}$. Then, $\tilde{w}$ is of minimal length in its conjugacy class if and only if it is of minimal length in its $W_{V_{\tilde{w}}}$-conjugacy class.

**Proof** Note that for any $x \in W_{V_{\tilde{w}}}$, $\tilde{C}_0$ contains a regular point of $V_{\tilde{w}} = x^{-1}V_{\tilde{w}} = V_{x^{-1}\tilde{w}x}$. By Sect. 4.2 (1) and (2), the minimal length of elements in the conjugacy class of $\tilde{w}$ equals

$$\langle \tilde{v}_{\tilde{w}}, 2\rho^\vee \rangle + \min_C \mathcal{H}_{V_{\tilde{w}}}(C, \tilde{w}C) = \langle \tilde{v}_{\tilde{w}}, 2\rho^\vee \rangle + \min_{x \in W_{V_{\tilde{w}}}} \mathcal{H}_{V_{\tilde{w}}}(xC_0, \tilde{w}xC_0)$$

$$= \langle \tilde{v}_{\tilde{w}}, 2\rho^\vee \rangle + \mathcal{H}_{V_{\tilde{w}}}(C_0, x^{-1}\tilde{w}xC_0)$$

$$= \min_{x \in W_{V_{\tilde{w}}}} \mathcal{H}(x^{-1}\tilde{w}x),$$

where $C$ ranges over all connected components of $V - \cup_{H \in \mathcal{H}'} H$ and $\rho^\vee = \frac{1}{2} \sum_{\alpha \in R^+} \alpha^\vee$.

**Proposition 4.2** Let $\tilde{w} \in \tilde{W}$ such that $\tilde{C}_0$ contains a regular point of some affine subspace $E \subset V_{\tilde{w}}$ stabilized by $\tilde{w}$. Let $J \subset F_0$. Assume there exists $z \in J W_0$ such that $z\tilde{w}z^{-1} \in \tilde{W}_J$. If $\mathcal{H}_{zE} \subset \mathcal{H}_J$, then

$$\mathcal{H}(\tilde{w}) = \mathcal{H}(z\tilde{w}z^{-1}) + \langle \tilde{v}_{\tilde{w}}, 2\rho^\vee \rangle - \langle \tilde{v}_{z\tilde{w}z^{-1}}, 2\rho^\vee \rangle,$$

where $\tilde{v}_{z\tilde{w}z^{-1}}$ denotes the unique $J$-dominant element in the $W_J$-orbit of $z\tilde{w}z^{-1}$ and $\rho^\vee_J = \frac{1}{2} \sum_{\alpha \in R^+} \alpha^\vee$. In particular, if $J \subset J_{z\tilde{w}z^{-1}}$, then

$$\mathcal{H}(\tilde{w}) = \mathcal{H}(z\tilde{w}z^{-1}) + \langle \tilde{v}_{\tilde{w}}, 2\rho^\vee \rangle.$$

**Remark** When $E = V_{\tilde{w}}$, we have $\mathcal{H}_{V_{z\tilde{w}z^{-1}}} \subset \mathcal{H}_J$. So the formula $(*)$ holds in this case.

**Proof** By Sect. 4.2 (2), we have

$$\mathcal{H}(\tilde{w}) = \langle \tilde{v}_{\tilde{w}}, 2\rho^\vee \rangle + \mathcal{H}(C, \tilde{w}C),$$

where $C$ is the connected component of $V - \cup_{H \in \mathcal{H}_E} H$ containing $C_0$. Since $z \in J W_0$, $zC_0 \subset C_J$ and hence $C_J$ contains a regular point of $zE \subset zV_{\tilde{w}} = V_{z\tilde{w}z^{-1}}$, which is stabilized by $z\tilde{w}z^{-1}$. Applying Sect. 4.2 (2) to $\mathcal{H}_{zE}$ instead of $\mathcal{H}$, we obtain

$$\mathcal{H}_{zE}(z\tilde{w}z^{-1}) = \langle \tilde{v}_{z\tilde{w}z^{-1}}, 2\rho^\vee \rangle + \mathcal{H}_{zE}(C', z\tilde{w}z^{-1} C'),$$

where $C'$ is the connected component of $V - \cup_{H \in \mathcal{H}_{zE}} H$ containing $C_J$. Since $zC = C'$, the map $H \mapsto zH$ induces a bijection between $\mathcal{H}_{zE}(C, \tilde{w}C)$ and $\mathcal{H}_{zE}(C', z\tilde{w}z^{-1} C')$. \(\square\)
Lemma 4.3 Let $J \subset F_0$ and $z \in J W_0$. Let $s \in S$ and $t = z s z^{-1}$.

1. If $t \in \tilde{W}_J$, then $\ell_J(t) = 1$.
2. If $t \notin \tilde{W}_J$, then $z s = x z'$ for some $x \in \tilde{W}_J$ with $\ell_J(x) = 0$ and $z' \in J W_0$.

Proof Assume $s = s_H$ is the reflection along some hyperplane $H \in S$. Since $s \in S$, $\tilde{C}_0$ contains some regular point of $H$. Since $z \in J W_0$, $z \tilde{C}_0 \subset C_J$. If $t \notin \tilde{W}_J$, then $\tilde{C}_J$ contains some regular point of $H' = z H$ and hence $t = s H'$ is of length one with respect to $\ell_J$.

If $t \in W_0 - W_J$, then $s \in S_0$ and $z s \in J W_0$. In this case, $z' = z s$ and $x = 1$. If $t \notin W_J \cup W_0$, then $s = t^0 s_0$ for some maximal coroot $\theta^\vee$ with $z(\theta) \notin R_J$. Then, $z s = t^{z(\theta)} u z'$ for some $u \in W_J$ and $z' \in J W_0$. Let $s \in R_j^+$. Since $z', z \in J W_0$ and that $\theta^\vee$ is a maximal coroot, we have
\[
\langle z(\theta), \theta^\vee \rangle = \begin{cases} 1, & \text{if } u^{-1}(\theta) < 0; \\ 0, & \text{Otherwise.} \end{cases}
\]

In other words, $\ell_J(t^{z(\theta)} u) = 0$.

Corollary 4.4 Let $\tilde{w}' \in \tilde{W}$ and $z' \in J W_0$ such that $z' \tilde{w}' z'^{-1} \in \tilde{W}_J$. Let $s \in S$ such that $\tilde{w}'$ and $\tilde{w} = s \tilde{w}'$'s are of the same length. Let $z$ be the unique minimal element of the coset $W J z p(s)$. Then, $z \tilde{w} z^{-1}$ and $z' \tilde{w}' z'^{-1}$ belong to the same $\tilde{W}_J$-conjugacy class and
\[
\ell_J(z \tilde{w} z^{-1}) = \ell_J(z' \tilde{w}' z'^{-1}).
\]

Proof The first statement follows form the construction of $z$.

Without loss of generality, we may assume that $\tilde{w}'$'s $> \tilde{w}' > s \tilde{w}'$. Let $t = z' s z'^{-1}$.

If $t \in \tilde{W}_J$, then $\ell_J(t) = 1$ by Lemma 4.3. Since $\tilde{w}' > s \tilde{w}'$, the reflection hyperplane $H \in S$ of $s$ separates $C_0$ from $\tilde{w}' C_0$. Hence, $z' H$ separates $C_J$ from $z' \tilde{w}' z'^{-1} C_J$ since $z' C_0 \subset C_J$, which means that $z' \tilde{w}' z'^{-1} > t z' \tilde{w}' z'^{-1}$. Similarly, $z' \tilde{w}' z'^{-1} > z' \tilde{w}' z'^{-1}$. Therefore, $\ell_J(z \tilde{w} z^{-1}) = \ell_J(t z' \tilde{w}' z'^{-1} t) = \ell_J(z' \tilde{w}' z'^{-1})$.

If $t \notin \tilde{W}_J$, then $z s = x^{-1} z'$ for some $x \in \tilde{W}_J$ with $\ell_J(x) = 0$. Hence, $z \tilde{w} z^{-1} = x^{-1} z' \tilde{w}' z'^{-1} x$ and $\ell_J(z \tilde{w} z^{-1}) = \ell_J(z' \tilde{w}' z'^{-1})$.

Proposition 4.5 Let $O$ be a conjugacy class of $\tilde{W}$ and $J \subset F_0$ such that $O \cap \tilde{W}_J \neq \emptyset$. Let $\tilde{w} \in O_{\min}$ and $z \in J W_0$, such that $z \tilde{w} z^{-1} \in \tilde{W}_J$. Then, $z \tilde{w} z^{-1}$ is of minimal length (with respect to $\ell_J$) in its $\tilde{W}_J$-conjugacy class.

Proof By Sect. 4.2 (1), there exists $\tilde{w} \rightarrow \tilde{w}' \in O_{\min}$ such that $\tilde{C}_0$ contains a regular point of $V_{\tilde{w}}$. By Corollary 4.4, it suffices to consider the case that $\tilde{C}_0$ contains a regular point of $V_{\tilde{w}}$. By Propositions 4.1 and 4.2,
\[
\ell_J(z \tilde{w} z^{-1}) = \min_{x \in W_{V_{\tilde{w}}}} \ell_J(z x \tilde{w} x^{-1} z^{-1}) = \min_{y \in W_{\tilde{V} z \tilde{w} z^{-1}}} \ell_J(y z \tilde{w} z^{-1} y^{-1}).
\]

Note that $\tilde{C}_J$ contains a regular point of $V_{\tilde{w} z^{-1}}$. Applying Proposition 4.1 to $\ell_J$ and $z \tilde{w} z^{-1}$ that we obtain the desired result.
5 A family of partial conjugacy classes

5.1 The map \( U \) on Coxeter groups

In this section, we consider an arbitrary Coxeter group \((W, S)\).

We first recall a criterion on whether the left multiplication by a given reflection decreases the length. This is obtained using the map \( U \) defined below and is used to prove the main result Theorem 5.5 in this section.

Let \( T = \bigcup_{w \in W} wSw^{-1} \subset W \) be the set of reflections in \( W \). Let \( \Gamma = \{ \pm 1 \} \times T \). For \( s \in S \), define \( U_s : R \to R \) by \( U_s(\epsilon, t) = (\epsilon(-1)^{\delta_{s,t}}, sts) \), where \( \delta \) is the Kronecker symbol.

Let \( \tilde{W} = W \rtimes \Gamma \). For any \( \delta \in \Gamma \), define \( U_\delta : R \to R \) by \( U_\delta(\epsilon, t) = (\epsilon, \delta(t)) \). Then \( U_\delta U_s U_\delta^{-1} = U_{\delta(s)} \) for \( s \in S \) and \( \delta \in \Gamma \).

We have the following result.

Proposition 5.1
(1) There is a unique homomorphism \( U \) of \( \tilde{W} \) into the group of permutations of \( R \) such that \( U(s) = U_s \) for all \( s \in S \) and \( U(\delta) = U_\delta \) for all \( \delta \in \Gamma \).
(2) For any \( w \in \tilde{W} \) and \( t \in T \), \( tw < w \) if and only if for \( \epsilon = \pm 1 \), \( U(w^{-1})(\epsilon, t) = (-\epsilon, w^{-1}tw) \).

The case \( \Gamma = \{ 1 \} \) is in [15, Proposition 1.5 and Lemma 2.2]. The general case can be reduced to that case easily.

5.2 Partial conjugacy classes

Let \( J \subset S \). We consider the action of \( W_J \) on \( \tilde{W} \) by \( w \cdot w' = w w' w^{-1} \) for \( w \in W_J \) and \( w \in \tilde{W} \). Each orbit is called a \( W_J \)-conjugacy class or a partial conjugacy class of \( \tilde{W} \) (with respect to \( W_J \)). We set \( \Gamma_J = \{ \delta \in \Gamma; \delta(J) = J \} \).

Lemma 5.2 Let \( I \subset S \) and \( w \in W_I \rtimes \Gamma_I \). Then, \( w \) is of minimal length in its \( W_I \)-conjugacy class if and only if \( w \) is of minimal in its \( W \)-conjugacy class.

Proof The “if” part is trivial.

Now, we show the “only if” part. Suppose that \( w \) is a minimal length element in its \( W_I \)-conjugacy class. An element in the \( W \)-conjugacy class of \( w \) is of the form \( xwx^{-1} \) for some \( x \in W \). Write \( x = x_1 y \), where \( x_1 \in W_I \) and \( y \in W_I \). Then, \( x_1 y y^{-1} \in W_I \) is in the \( W_I \)-conjugacy class of \( w \). Hence, \( \ell(y y^{-1}) \geq \ell(y) \). Now
\[
\ell(xwx^{-1}) \geq \ell(x_1 (yy^{-1})) - \ell(x_1) = \ell(x_1) + \ell(yy^{-1}) - \ell(x_1)
\]
\[
= \ell(yy^{-1}) \geq \ell(w).
\]

Thus, \( w \) is a minimal length element in its \( W \)-conjugacy class. \( \square \)
5.3 The subset $I(J, w)$

In general, a $W_J$-conjugacy class in $\tilde{W}$ may not contain any element in $W_J \rtimes \Gamma_J$. To study the minimal length elements in this partial conjugacy class, we introduce the notation $I(J, w)$, a subset of $J$.

For any $w \in J\tilde{W}$, set

$$I(J, w) = \max\{K \subset J; wKw^{-1} = K\}.$$ 

Since $w(K_1 \cup K_2)w^{-1} = wK_1w^{-1} \cup wK_2w^{-1}$, $I(J, w)$ is well-defined. We have that

(a) $I(J, w) = \bigcap_{i \geq 0} w^{-i}Jw^i$.

Set $K = \bigcap_{i \geq 0} w^{-i}Jw^i$. Let $s \in I(J, w)$. Then $w^i sw^{-i} \in I(J, w) \subset J$ for all $i$. Thus $s \in K$. On the other hand, $wKw^{-1} \subset K$. Since $K$ is a finite set, $wKw^{-1} = K$. Thus $K \subset I(J, w)$.

(a) is proved.

**Lemma 5.3** Let $w \in J\tilde{W}$ and $x \in W_J$. Then $x \in W_1(J, w)$ if and only if $w^{-i}xw^i \in W_J$ for all $i \in \mathbb{Z}$.

**Proof** If $x \in W_1(J, w)$, then $w^{-1}xw \in W_{w^{-1}(J, w)} = W_1(J, w)$. So $w^{-i}xw^i \in W_J$ for all $i \in \mathbb{Z}$.

Suppose that $w^{-i}xw^i \in W_J$ for all $i \in \mathbb{Z}$. We write $x$ as $x = ux_1$, where $u \in W_{J \cap w^{-1}Jw}$ and $x_1 \in J \cap w^{-1}Jw$. By [13, 2.1(a)], $wx_1 \in Jw$ and $wx \in W_J(wx_1)$. Since $wxw^{-1} \in W_J$, $wx \in W_Jw$. Therefore $W_J(wx_1) \cap W_Jw \neq \emptyset$ and $wx_1 = w$.

So $x_1 = 1$ and $x \in W_{J \cap w^{-1}Jw}$.

Applying the same argument, $x \in W_{\bigcap_{i \geq 0} w^{-i}Jw^i} = W_1(J, w)$. $\square$

5.4 A family of partial conjugacy classes

Similar to Sect. 3.1, for $w, w' \in \tilde{W}$, we write $w \rightarrow_J w'$ if there is a sequence $w = w_0, w_1, \ldots, w_n = w'$ of elements in $\tilde{W}$ such that for any $k$, $w_{k-1} \rightarrow w_k$ for some $s \in J$. The notations $\sim_J$ and $\approx_J$ are defined in a similar way.

The following result is proved in [7, Corollary 3.8] (see also [6, Theorem 2.5]).

**Theorem 5.4** Let $\mathcal{O}$ be a $W_J$-conjugacy class of $\tilde{W}$. Then, there exists a unique element $\tilde{w} \in J\tilde{W}$ and a $W_1(J, \tilde{w})$-conjugacy class $C$ of $W_1(J, \tilde{w})\tilde{w}$ such that $\mathcal{O} \cap W_1(J, \tilde{w})\tilde{w} = C$. In this case,

(1) for any $\tilde{w}' \in \mathcal{O}$, there exists $x \in W_1(J, \tilde{w})$ such that $\tilde{w}' \rightarrow_J x\tilde{w}$.

(2) for any two minimal length elements $x\tilde{w}, x'\tilde{w}$ of $C$, $x\tilde{w} \sim_{J, \tilde{w}} x'\tilde{w}$.

Now, we prove the following result.

**Theorem 5.5** Let $I \subset J \subset S$ and let $w, u \in \tilde{W}$ be of minimal length in the same $W_J$-conjugacy class such that $u \in J\tilde{W}$, $w \in J\tilde{W}$ and $wIw^{-1} = I$. Then, there exists $h \in I(J, w)W_J I$ such that $hIh^{-1} \subset I(J, u)$ and $hwh^{-1} = u$. 
Remark This result can be interpreted as a conjugation on a family of partial conjugacy classes in the following sense. Let \( I \) be the set of \( W_I \)-conjugacy classes that intersects \( W_I w \) and \( I \) be the set of \( W_J \)-conjugacy classes that intersects \( W_I(J,u)u \). Then

1. There is an injective map \( I \to I \), which sends a \( W_I \)-conjugacy class \( O_1 \) in \( I \) to the unique \( W_J \)-conjugacy class \( O_2 \) in \( I \) that contains \( O_1 \).
2. Conjugating by \( h \) sends \( O_1 \cap W_I w \) into \( O_2 \cap W_I(J,u)u \).

Proof Let \( b = f e \in W_J \) with \( f \in W_J \) and \( e \in W_I \) such that \( bwb^{-1} = u \). We show that

(a) \( f w f^{-1} = u \).

Set \( x = e w e^{-1} w^{-1} \). Then \( x \in W_I \) and \( u = f x w f^{-1} \). Suppose that \( x \neq 1 \). Then \( s x < x \) for some \( s \in I \). Set \( t = f s f^{-1} \). Since \( f \in W_I \), \( t f = f s > f \). Thus \( U(f^{-1})(e, t) = (e, f^{-1} t f) = (e, s) \). Since \( s x < x \), \( U(x^{-1})(e, s) = (-e, x^{-1} s x) \).

Notice that \( w \in I W \) with \( w I w^{-1} = I \) and \( f^{-1} \in I W \). Thus \( w f^{-1} \in I W \). Hence \( U(f w^{-1})(-e, x^{-1} s x) = (-e, f w^{-1} x^{-1} s x w f^{-1}) \).

Therefore, \( U(f w^{-1} x^{-1} f^{-1})(e, t) = (-e, f w^{-1} x^{-1} s x w f^{-1}) \) and

\[
 tfx w f^{-1} = f (sx) w f^{-1} < f x w f^{-1} = u.
\]

Applying this argument successively, we have \( f w f^{-1} < u \). This contradicts our assumption that \( u \) is of minimal length in the \( W_J \)-conjugacy class containing \( f w f^{-1} \).

Hence \( x = 1 \) and \( f w f^{-1} = u \).

(a) is proved.

Now, write \( f \) as \( f = \pi h \) with \( \pi \in W_{I(J,u)} \) and \( h \in I(J,u)W_J \). Then \( w = h^{-1} (\pi^{-1} u \pi^{-1} u^{-1}) u h \). Similar to the proof of (a), we have that \( w \geq h^{-1} u h \). By our assumption, \( w \) is a minimal length element in the \( W_J \)-conjugacy class of \( h^{-1} u h \).

Thus \( w = h^{-1} u h \).

For any \( x \in W_I \) and \( i \in \mathbb{Z} \), \( h w^i = u^i h \) and

\[
 u^i (hxh^{-1}) u^{-i} = (hw^i)x(hw^i)^{-1} = h(w^i x w^{-i}) h^{-1} \in h W_I h^{-1} \subset W_J.
\]

By Lemma 5.3, \( h x h^{-1} \in W_I(J,u) \). Thus \( h W_I h^{-1} \subset W_I(J,u) \). Since \( h \in I(J,u)W_I \), we have \( h I^{-1} h^{-1} \subset I(J,u) \).

6 Bernstein–Lusztig presentation of the cocenter of \( \mathcal{H} \)

6.1 Main result

We fix a conjugacy class \( O \) of \( \tilde{W} \) and will construct a subset \( J \) of \( F_0 \), as small as possible, such that \( T_{W_0} \in \mathcal{H}_J + [\mathcal{H}, \mathcal{H}] \).

By Sect. 4.2 (1), there exists \( \tilde{w}' \in O_{\text{min}} \) such that \( \tilde{C}_0 \) contains a regular point \( e' \) of \( V_{\tilde{w}'} \). We choose \( v \in V \) such that \( V_{\tilde{w}'} = V_{\tilde{w}'} + v \) and \( v, v_{\tilde{w}'} \in \tilde{C} \) for some Weyl Chamber \( C \). Let \( \tilde{v} \) be the unique dominant element in \( W_0 \cdot v \). We write \( J \) for \( J_{v_{\tilde{w}}} \cap J_{\tilde{v}} \).
Let $z \in J W_0$ with $z(v_{\tilde{w}'}) = v_\mathcal{O}$ and $z(v) = \tilde{v}$. Set $\tilde{w}_0 = z \tilde{w}' z^{-1}$. By Proposition 4.5, $\tilde{w}_0$ is of minimal length (with respect to $\ell_J$) in its $\tilde{W}_J$-conjugacy class.

Unless otherwise stated, we keep the notations in the rest of this section. The main result of this section is

**Theorem 6.1** We keep the notations in Sect. 6.1. Then

$$T_{w_\mathcal{O}} \equiv T_{\tilde{w}_0}^J \mod [\mathcal{H}, \mathcal{H}].$$

6.2 Idea of the proof

The idea of the proof is as follows.

Suppose $\tilde{w}_0 = t^{\lambda_0} w_0$ and $\tilde{w}' = t^{\lambda'} w'$. Then, we need to compare $T_{\tilde{w}_0}^J$ and $T_{\tilde{w}'}^J$.

Although $\lambda_0$ and $\lambda$ are in the same $W_0$-orbit, the relationship between $T_{\tilde{w}_0}^J$ and $T_{\tilde{w}'}^J$ is complicated. Roughly speaking, we write $\lambda_0$ as $\lambda_0 = \mu_1 - \mu_2$ for $J$-dominant coweights, i.e., $\langle \mu_1, \alpha_i \rangle, \langle \mu_2, \alpha_i \rangle > 0$ for $i \in J$. Then

$$T_{\tilde{w}}^J = T_{\tilde{w}_0}^J (T_{\tilde{w}_0}^J)^{-1} = \theta_{\mu_1} \theta_{\mu_2}^{-1}.$$

The right hand side is not easy to compute.

To overcome the difficulty, we replace $\tilde{w}_0$ by another minimal length element $\tilde{w}'$ in its $\tilde{W}_J$-conjugacy class whose translation part is $J$-dominant and replace $\tilde{w}'$ by another minimal length element $\tilde{w}_2$ in $\mathcal{O}$ and study the relationship between $\tilde{w}_1$ and $\tilde{w}_2$ instead. The construction of $\tilde{w}_1$ and $\tilde{w}_2$ uses “partial conjugation action”.

6.3 Several technical Lemmas

Recall that $e' \in \tilde{C}_0$ is a regular element of $V_{\tilde{w}'_0}$. Set $e = z(e')$.

Since $e \in z(\tilde{C}_0)$ and $e$ is a regular point of $V_{\tilde{w}_0}$, we have

1. $0 \leq |\langle e, \alpha^\vee \rangle| \leq 1$ for any $\alpha \in R$.
2. $\langle e, \alpha^\vee \rangle \geq 0$ for any $\alpha \in R_f^+$. 
3. If $e \in H_{\alpha, k}$ for some $\alpha \in R$ and $k \in \mathbb{Z}$, then $V_{\tilde{w}_0} \subset H_{\alpha, k}$ and $\alpha \in R_J$. In particular, $J_e \subset J$ and by (2), $R_{J_e} = \{ \alpha \in R; \langle e, \alpha^\vee \rangle = 0 \}$.

By Theorem 5.4, the $W_{J_e}$-conjugacy class of $\tilde{w}_0$ contains a minimal length element $\tilde{w}_1$ of the form $\tilde{w}_1 = t^\lambda w_1 x_1$ with $\lambda \in X$, $w_1 \in W_j \rtimes \Gamma_J$ and $x_1 \in W_{1(J_e,t^\lambda w_1)}$ such that $t^\lambda w_1 \in J_e \tilde{W}$ and $x_1$ is of minimal length in its $\text{Ad}(w_1)$-twisted conjugacy class of $W_{1(J_e,t^\lambda w_1)}$.

By Theorem 5.4, there exists a minimal length element $\tilde{w}_2$ in the $W_0$-conjugacy class of $\tilde{w}_0$ with $\tilde{w}'$, which has the form $\tilde{w}_2 = t^\lambda w_2 x_2$ such that $t^\lambda w_2 \in S_0 \tilde{W}$ and $x_2 \in W_{1(S_0,t^\lambda w_2)}$. Since $\tilde{w}_2 \in \mathcal{O}_{\text{min}}$, $\tilde{w}_2 \in \mathcal{O}_{\text{min}}$.

The outline of the proof is as follows.

First, we show that $\tilde{w}_1$ is of minimal length in its $\tilde{W}_J$-conjugacy class and write down $T_{\tilde{w}_1}^J$ in terms of Bernstein-Lusztig presentation. This is done in Lemma 6.3 and uses properties on $\lambda$ and $e$ established in Lemma 6.2.
Then, we use the element $y$ in Lemma 6.5 to conjugacy $\lambda$ to $\tilde{\lambda}$ and $T_{\tilde{w}_1}^J$ to an element in $\theta_J^* H_0$, where $H_0$ is the finite Hecke algebra.

In the end, we modify the finite part using conjugation by the element $h$ in Proposition 6.6 so that the final element we obtain is in $\mathcal{O}_{\min}$.

**Lemma 6.2** Keep notations in Sect. 6.3. Then, we have

1. For any $\alpha \in R^+$, $(\tilde{w}_1(e), \alpha^\vee) > -1$.
2. $(\lambda, \alpha^\vee)^* \leq -1$ for any $\alpha \in R^+$.
3. $(\lambda, \alpha^\vee) > 0$ for any $\alpha \in R^+_J$.
4. For any $\alpha \in R^+$, if $(\lambda, \alpha^\vee) = -1$, then $(e, \alpha^\vee) < 0$.

**Proof** For any $\alpha \in R^+$,

$$(\lambda, \alpha^\vee) + (w_1(e), \alpha^\vee) = (\lambda + w_1 x_1(e), \alpha^\vee) = (\tilde{w}_1(e), \alpha^\vee) = (e + v_O, \alpha^\vee).$$

(1) By Sect. 6.3 (1), $(e, \alpha^\vee)^* \leq -1$. So $(e + v_O, \alpha^\vee)^* \leq -1$. If $(e + v_O, \alpha^\vee)^* = -1$, then $(e, \alpha^\vee)^* = -1$. Therefore $\alpha \in R^+_J$ by Sect. 6.3 (3), which contradicts Sect. 6.3 (2).

(2) By Sect. 6.3 (1), $(w_1(e), \alpha^\vee) \leq 1$. By (1), $(\lambda, \alpha^\vee) = (\tilde{w}_1(e), \alpha^\vee) - (w_1(e), \alpha^\vee) > -2$. Since $(\lambda, \alpha^\vee) \in \mathbb{Z}$, $(\alpha^\vee)^* \geq -1$.

(3) By Sect. 6.3 (2), $0 \leq (e, \alpha^\vee) = (v_O + e, \alpha^\vee) = (\lambda, \alpha^\vee) + (w_1(e), \alpha^\vee)$. If $(\lambda, \alpha^\vee) < 0$, then by Sect. 6.3 (1), $(e, \alpha^\vee) = 0$ and hence $\alpha \in R^+_J$ by Sect. 6.3 (4). Since $t^h w_1 \in J^* \tilde{W}$, $(\lambda, \alpha^\vee)^* \geq 0$, which is a contradiction. Therefore, $(\lambda, \alpha^\vee)^* \geq 0$ for all $\alpha \in R^+_J$.

(4) Suppose that $(e, \alpha^\vee) \geq 0$. Since $(\lambda, \alpha^\vee) = -1$ and $(w_1(e), \alpha^\vee) \leq 1$, $(e + v_O, \alpha^\vee) \leq 0$. Thus $(e, \alpha^\vee) = (v_O, \alpha^\vee) = 0$. Therefore $\alpha \in R^+_J$ by Sect. 6.3 (3), which contradicts (3).

**Lemma 6.3** We keep the notations in Sect. 6.3. Then

1. $\tilde{w}_1 \in \tilde{W}_J$ is of minimal length (with respect to $\ell_J$) in its $\tilde{W}_J$-conjugacy class.
2. $t^h w_1 \in J^* \tilde{W}$.

**Remark** By (2), $T_{\tilde{w}_1}^J = \theta_J T_{\tilde{w}_1}^{-1} T_{x_1}$.

**Proof** Since $W_{J_e} \subset W_J$ fixes $V_{\tilde{w}_0}$, we have $\tilde{w}_1 \in \tilde{W}_J$ and $V_{\tilde{w}_0} = V_{\tilde{w}_1}$.

(1) Since $\ell(\tilde{w}_1) \leq \ell(\tilde{w}_0)$, we have $\ell_J(\tilde{w}_1) \leq \ell_J(\tilde{w}_0)$ by Proposition 4.2. By Proposition 4.5, $\tilde{w}_0$ is a minimal length (with respect to $\ell_J$) in its conjugacy class of $\tilde{W}_J$. So is $\tilde{w}_1$.

(2) It suffices to show that $(\lambda, \alpha^\vee)^* \geq 1$ for any $\alpha \in R^+_J$ with $w_1^{-1}(\alpha) < 0$. Suppose that $(\lambda, \alpha^\vee)^* < 1$. By Lemma 6.2 (3), $(\lambda, \alpha^\vee) = 0$. Hence by Sect. 6.3 (2),

$$0 \geq (e, w_1^{-1}(\alpha^\vee)) = (w_1(e), \alpha^\vee) = (\tilde{w}_1(e), \alpha^\vee) = (e + v_O, \alpha^\vee) \geq 0.$$

Thus by Sect. 6.3 (2) again, $(e, \alpha^\vee) = 0$ and $\alpha \in R^+_J$. However, $t^h w_1 \in J_e \tilde{W}$ by our construction. Hence $(\lambda, \alpha^\vee)^* \geq 1$, which is a contradiction. □
Combining Lemma 6.3 with Proposition 4.2 (where we take \( E = z^{-1} V_{w_1} = V_{w'} \)), we obtain

**Corollary 6.4** We have

\[
\ell(z^{-1} t^2 w_1 z) = \langle v_0, 2\rho \vee \rangle + \langle \lambda, 2\rho \vee \rangle - \ell(w_1),
\]

\[
\ell(z^{-1} t^2 w_1 x_1 z) = \langle v_0, 2\rho \vee \rangle + \langle \lambda, 2\rho \vee \rangle - \ell(w_1) + \ell(x_1).
\]

**Lemma 6.5** Let \( y \in J_\lambda W_0 \) be the unique element such that \( y(\lambda) = \tilde{\lambda} \). Then

1. \( \ell(y w_1 y^{-1}) = 2\ell(y) + \ell(w_1) \).
2. \( \langle \tilde{\lambda}, 2\rho \vee \rangle = \langle v_0, 2\rho \vee \rangle + \langle \lambda, 2\rho \vee \rangle + 2\ell(y) \).
3. \( y J_\lambda y^{-1} \subset J_\lambda \).

**Proof** By definition, for any \( \alpha \in R^+ \), \( y(\alpha) \in R^+ \) if and only if \( \langle \lambda, \alpha \vee \rangle \geq 0 \). By Lemma 6.2 (2), \( \ell(y) = 2|\alpha \in R^+; \langle \lambda, \alpha \vee \rangle = -1 | \).

(1) Let \( \alpha \in R^+ \) such that \( w_1^{-1}(\alpha) < 0 \). Then \( \alpha \in R^+ \) since \( w_1 \in W_J \times \Gamma_J \). Hence \( \langle \lambda, \alpha \vee \rangle \geq 0 \) by Lemma 6.2 (3). Therefore, \( y(\alpha) > 0 \). Hence \( \ell(y w_1) = \ell(y) + \ell(w_1) \).

To show \( \ell(y w_1 y^{-1}) = \ell(y w_1) + \ell(y^{-1}) \), we have to prove that for any \( \beta \in R^+ \) with \( y(\beta) < 0 \), we have \( y w_1(\beta) \in R^+ \).

Assume \( y(\beta) < 0 \). Then \( \langle \lambda, \beta \vee \rangle < 0 \). Thus \( \langle \lambda, \beta \vee \rangle = -1 \) by Lemma 6.2 (2).

Moreover, we have \( \beta \not\in R_J^+ \) and \( \langle e, \beta \vee \rangle < 0 \) by Lemma 6.2 (3) and (4). Since \( w_1 \in W_J \times \Gamma_J \), \( w_1(\beta) > 0 \). By Lemma 6.2 (1),

\[
-1 < \langle w_1(e), w(\beta \vee) \rangle = \langle \lambda, w_1(\beta \vee) \rangle + \langle w_1(e), w_1(\beta \vee) \rangle = \langle \lambda, w_1(\beta \vee) \rangle + \langle e, \beta \vee \rangle
\]

\[
< \langle \lambda, w_1(\beta \vee) \rangle.
\]

Therefore \( \langle \lambda, w_1(\beta \vee) \rangle \geq 0 \) and \( y w_1(\beta) \in R^+ \).

(2) By Theorem 3.1 and Lemma 6.2 (2), we have that

\[
\langle \tilde{\lambda}, 2\rho \vee \rangle = \sum_{\alpha \in R^+} |\langle \lambda, \alpha \vee \rangle| = \sum_{\alpha \in R^+} \langle \lambda, \alpha \vee \rangle + 2|\alpha \in R^+; \langle \lambda, \alpha \vee \rangle = -1 |
\]

\[
= \langle \lambda, 2\rho \vee \rangle + 2\ell(y).
\]

Since \( w_1 \in W_J \times \Gamma_J \), \( w_1^k(\rho \vee - \rho \vee) = \rho \vee - \rho \vee \) for all \( i \in \mathbb{Z} \). Let \( m = |W_J \times \Gamma_J| \). Then

\[
\sum_{k=1}^m w_1^k(\tilde{\lambda}) = m v_0 \quad \text{and}
\]

\[
\langle \tilde{\lambda}, 2\rho \vee \rangle - \langle \lambda, 2\rho \vee \rangle = 2\ell(y) + \langle \lambda, 2(\rho \vee - \rho \vee) \rangle
\]

\[
= 2\ell(y) + \frac{1}{m} \sum_{k=1}^m \langle \lambda, 2w_1^{-k}(\rho \vee - \rho \vee) \rangle = 2\ell(y) + \frac{1}{m} \sum_{k=1}^m \langle w_1^{-k}(\lambda), 2(\rho \vee - \rho \vee) \rangle
\]

\[
= 2\ell(y) + \langle v_0, 2(\rho \vee - \rho \vee) \rangle = 2\ell(y) + \langle v_0, 2\rho \vee \rangle.
\]

(3) Notice that \( W_{J_\lambda} y W_{J_\lambda} = W_{J_\lambda} (y W_{J_\lambda} y^{-1}) y = W_{J_\lambda} y \) and \( y \in J_\lambda W_0 \), we see that \( y \) is the unique minimal element of the double coset \( W_{J_\lambda} y W_{J_\lambda} \), that is, \( y \in J_\lambda W_0 J_\lambda \).

Moreover, \( y W_{J_\lambda} y^{-1} \subset W_{J_\lambda} \). Thus, \( y \) sends simple roots of \( J_\lambda \) to simple roots of \( J_\lambda \). \( \square \)
Proposition 6.6  Set \( I = yI(J_e, t^k w_1)y^{-1} \). Then, there exists \( h \in I(J_{\tilde{\kappa}}, w_2) \) such that

1. \( hIh^{-1} \subset I(J_{\tilde{\kappa}}, w_2) \).
2. \( w_2 = hyw_1y^{-1}h^{-1} \).
3. Both \( w_2 \) and \( yw_1y^{-1} \) are of minimal lengths in their common \( W_{J_{\tilde{\kappa}}} \)-conjugacy class.
4. \( hy\tilde{w}_1y^{-1}h^{-1} \in O_{\text{min}} \).

Remark  By Lemma 6.5 (3), \( I \subset y(J_{\tilde{\kappa}}) \subset J_{\tilde{\kappa}} \). Moreover, we have \( yw_1y^{-1} \in I W_0 \) and \( yw_1y^{-1}Iyw_1^{-1}y^{-1} = I \) by the construction of \( w_1 \).

Proof  By Theorem 5.4, there exists a minimal length element in the \( W_0 \)-conjugacy class of \( t^k w_1 \) of the form \( t^k uc \), where \( u \in J_{\tilde{\kappa}}(W_0 \rtimes \Gamma) \) and \( c \in W_I(J_{\tilde{\kappa}}, u) \). Again by Theorem 5.4, there exists \( c' \in W_I(J_{\tilde{\kappa}}, u) \) such that \( uc' \) is of minimal length in the \( W_{J_{\tilde{\kappa}}} \)-conjugacy class of \( uc \). Note that \( t^k uc \) and \( t^k uc' \) are in the same \( W_{J_{\tilde{\kappa}}} \)-conjugacy class. So by the choice of \( t^k uc \), we have

\[
\ell(t^k) - \ell(u) + \ell(c) = \ell(t^k uc) \leq \ell(t^k uc') = \ell(t^k) - \ell(u) + \ell(c')
\]

that is, \( \ell(c) \leq \ell(c') \). Hence \( \ell(uc) = \ell(u) + \ell(c) \leq \ell(u) + \ell(c') = \ell(uc') \). Therefore, (a) \( uc \) is of minimal length in its \( W_{J_{\tilde{\kappa}}} \)-conjugacy class.

By Corollary 6.4, \( \ell(z^{-1}t^k w_1z) = \langle v_\Omega, 2\rho^\vee \rangle + \langle \lambda, 2\rho_\Omega^\vee \rangle - \ell(w_1) \). Applying Lemma 6.5, we have

\[
\ell(yw_1y^{-1}) = 2\ell(y) + \langle v_\Omega, 2\rho^\vee \rangle + \langle \lambda, 2\rho_\Omega^\vee \rangle - \ell(z^{-1}t^k w_1z).
\]

On the other hand, \( \ell(t^k uc) = \langle \tilde{\lambda}, 2\rho^\vee \rangle - \ell(u) + \ell(c) \). Hence

\[
\ell(uc) = \langle \tilde{\lambda}, 2\rho^\vee \rangle + 2\ell(c) - \ell(t^k uc).
\]

Since \( t^k uc \) and \( t^k yw_1y^{-1} \) are in the same \( W_0 \)-conjugacy class, then \( uc \) and \( yw_1y^{-1} \) are in the same \( W_{J_{\tilde{\kappa}}} \)-conjugacy class. By (a) and Lemma 6.5, we see that

\[
0 \leq \ell(yw_1y^{-1}) - \ell(uc) = \ell(t^k uc) - \ell(z^{-1}t^k w_1z) - 2\ell(c).
\]

Notice that by our construction, \( t^k uc \) is of minimal length in the \( W_0 \)-conjugacy class of \( z^{-1}t^k w_1z \). Hence \( c = 1 \) and \( \ell(yw_1y^{-1}) = \ell(u) \). By (a), both \( u \) and \( yw_1y^{-1} \) are of minimal lengths in their common \( W_{J_{\tilde{\kappa}}} \)-conjugacy class.

By Proposition 5.5, there exists \( \tilde{h} \in I(J_{\tilde{\kappa}}, u) \) such that \( u = hyw_1y^{-1}h^{-1} \) and \( hIh^{-1} \subset I(J_{\tilde{\kappa}}, u) \). Thus, \( hy\tilde{w}_1y^{-1}h^{-1} \in t^k uW_1(J_{\tilde{\kappa}}, u) = t^k uW_{I(S_0, t^k u)} \). The \( W_0 \)-conjugacy class of \( \tilde{w}_1 \) intersects both \( t^k uW_{I(S_0, t^k u)} \) and \( t^k w_2W_{I(S_0, t^k w_2)} \). By Theorem 5.4, \( w_2 = u \).
By definition, \( x_1 \) is a minimal length element in the \( \text{Ad}(w_1)\)-twisted conjugacy class by \( W_J(J_\lambda, w_1) \). Thus, \( h_x y_1^{-1}h^{-1} \) is of minimal length in its \( \text{Ad}(w_2)\)-twisted conjugacy class by \( W_h h_1 h^{-1} \). By Lemma 5.2, \( h_x y_1^{-1}h^{-1} \) is of minimal length in its \( \text{Ad}(w_2)\)-twisted conjugacy class by \( W_I(J_\lambda, w_2) \). Thus by Theorem 5.4, \( h_x \tilde{w}_1 y_1^{-1}h^{-1} = \tilde{x}_2 w_2 (h_x y_1^{-1}h^{-1}) \) is of minimal length in the \( W_0\)-conjugacy class of \( \tilde{w} \). So \( h_x \tilde{w}_1 y_1^{-1}h^{-1} \in \mathcal{O}_{\min} \). □

6.4 Proof of Theorem 6.1

Now we prove Theorem 6.1.

By Lemma 6.3, \( \tilde{w}_0 \) and \( \tilde{w}_1 \) are of minimal lengths (with respect to \( \ell_J \)) in their \( \tilde{W}_J\)-conjugacy class. Hence by Sect. 3.2,

\[ T_J^{\tilde{w}_0} = T_J^{\tilde{w}_1} = \theta_{\lambda} T_{w_1}^{-1} T_x \mod [\mathcal{H}_J, \mathcal{H}_J]. \]  

(a)

Let \( x' = y x_1 y^{-1} \in W_I \subset W_{J_\lambda} \) and \( x'' = h x' h^{-1} \in W_h h_1 h^{-1} \). We show that

\[ \theta_{\lambda} T_{w_1}^{-1} T_x \equiv \theta_{\lambda} T_{w_1}^{-1} T_x' \mod [\mathcal{H}, \mathcal{H}]. \]  

(b)

Let \( y = s_{r'} \cdots s_{s_1} \) be a reduced expression. For each \( k \), let \( \alpha_k \) be the positive simple root corresponding to \( s_k \) and let \( \lambda_k = s_k \cdots s_{s_1}(\lambda) \). Since \( y s_1 \cdots s_{s_1}(\alpha_k) < 0 \), then

\[ \langle \lambda_{k-1}, \alpha_k \rangle = \langle \lambda, s_k \cdots s_{s_1}(\alpha_k) \rangle = \langle y \lambda, y s_k \cdots s_{s_1}(\alpha_k) \rangle < 0. \]

By Lemma 6.2 (2), \( \langle \lambda_{k-1}, \alpha_k \rangle = -1 \). By Sect. 2.4 (6), \( T_{s_k} \theta_{\lambda_{k-1}} = \theta_{\lambda_k} T_{s_k}^{-1} \). Applying it successively, we have that

\[ T_y \theta_{\lambda} = T_{s_r} \cdots T_{s_1} \theta_{\lambda} = \theta_{y(\lambda)} T_{s_r}^{-1} \cdots T_{s_1}^{-1} = \theta_{\tilde{x}} T_{y^{-1}}^{-1}. \]

Since \( y \in J_{\lambda} W_0 \), we have \( \ell(x'y) = \ell(y x_1) = \ell(x') + \ell(y) = \ell(y) + \ell(x_1) \).

By Lemma 6.5, \( \ell(y w_1^{-1} y^{-1}) = 2 \ell(y) + \ell(w_1) \). Hence \( T_y T_{x_1} T_{y^{-1}} = T_{x'} \) and \( T_y T_{w_1}^{-1} T_{y^{-1}} = T_{y w_1^{-1} y^{-1}} \). Therefore,

\[ T_y \theta_{\lambda} T_{w_1}^{-1} T_{x_1} T_{y^{-1}} = \theta_{\lambda} T_{y^{-1} w_1^{-1} y^{-1}} T_{x_1} T_{y^{-1}} = \theta_{\lambda} (T_{y^{-1} w_1^{-1} y^{-1}} (T_y T_{x_1} T_{y^{-1}})) \]

\[ = \theta_{\lambda} T_{y w_1^{-1} y^{-1}} T_{x'}. \]

(b) is proved.

Notice that \( h \in W_{J_\lambda} \). By Sect. 2.4 (5), \( T_h \theta_{\lambda} T_{h}^{-1} = \theta_{\tilde{x}} \). By Proposition 6.6,

\[ \ell(y w_1 y^{-1} h^{-1}) = \ell(h^{-1} w_2) = \ell(h^{-1}) + \ell(w_2) = \ell(y w_1 y^{-1}) + \ell(h^{-1}). \]
Thus $T_h T^{-1} y w_1^{-1} y^{-1} T_h = T^{-1} w_2^{-1}$. Since $h \in I(J_\lambda, w_2)$ and $h(I) \subset I(J_\lambda, w_2)$, we have that $\ell(x''h) = \ell(hx') = \ell(h) + \ell(x'') = \ell(x''') + \ell(h)$ and $T_h T_{x'} T_h^{-1} = T_{x''}$. So

$$T_h \theta_j T_{y w_1^{-1} y^{-1} T_h} T_{x'} T_h^{-1} = \theta_j (T_h T_{y w_1^{-1} y^{-1} T_h}) (T_h T_{x'} T_h^{-1}) = \theta_j T_{w_2^{-1} x''} T_{x''}.$$

By Proposition 6.6, $h y \tilde{w} y^{-1} h^{-1} = t_{x''} w x''$ and $\tilde{w}'$ are both of minimal lengths in $O$. By Theorem 3.1 and Sect. 3.2,

$$T_{\tilde{w}'} \equiv T_{t_{x''} w x''} = \theta_j T_{w_2^{-1} x''} T_{x''} \equiv \theta_j T_{y w_1^{-1} y^{-1} T_{x'}} \mod [\mathcal{H}, \mathcal{H}].$$

Combining (a), (b) and (c),

$$T_{w_O} \equiv T_j \mod [\mathcal{H}, \mathcal{H}].$$

**Example 6.7** Let’s consider the extended affine Weyl group $\tilde{W}$ associated with $GL_8$. Here $\tilde{W} \cong Z^8 \rtimes S_8$, where the permutation group $S_8$ of $\{1, 2, \ldots, 8\}$ acts on $Z^8$ in a natural way. Let $\tilde{u} = t^x \sigma$ with $\chi = [\chi_1, \ldots, \chi_8]$ and $\sigma \in S_8$. Then

$$\ell(\tilde{u}) = \sum_{i < j, \sigma(i) < \sigma(j)} |\lambda_i - \lambda_j| + \sum_{i < j, \sigma(i) > \sigma(j)} |\lambda_i - \lambda_j - 1|.$$

Take $\chi = [1, 1, 1, 1, 0, 0, 0] \in Z^8$ and $x = (6, 3, 1)(7, 4, 8, 5, 2) \in S_8$. Let $\tilde{w}' = t^x \chi \in S_8 \tilde{W}$. Then, $\ell(\tilde{w}') = 1$ and $\tilde{w}'$ is an minimal length element in its conjugacy class.

Let $J = \{(1, 2), (2, 3), (4, 5), (6, 7), (7, 8)\} \subset S_0$ and $\tilde{w} \in \tilde{W}_J = Z^8 \rtimes W_J$ with $\lambda = [1, 1, 0, 1, 1, 0, 0] \in Z_8$ and $w = (3, 2, 1)(7, 5, 8, 6, 4)$. Then $\ell_J(\tilde{w}) = 0$. In particular, $\tilde{w}$ is of minimal length (in the sense of $\ell_J$) in its conjugacy class of $\tilde{W}_J$.

By Theorem 6.1,

$$T_{\tilde{w}'} \equiv T_j \mod [\mathcal{H}, \mathcal{H}].$$

### 6.5 Elliptic conjugacy classes

Let $J \subset F_0$. We call an element $w \in W_J \rtimes \Gamma_J$ elliptic if $V^w \subset V^W_J$ and an element $\tilde{w} \in \tilde{W}_J$ elliptic if $p(\tilde{w})$ is elliptic in $W_J \rtimes \Gamma_J$. By definition, if $\tilde{w}$ is elliptic in $\tilde{W}_J$, then $v_{\tilde{w}} \in V^W_J$.

A conjugacy class $O$ in $W_J \rtimes \Gamma_J$ or $\tilde{W}_J$ is called elliptic if $\tilde{w}$ is elliptic for some (or, equivalently any) $\tilde{w} \in O$.

Now, we choose the choice of $v$ in Sect. 6.1. If we assume furthermore that $v$ is a regular point of $V^{p(\tilde{w})}$ (i.e., for any $\alpha \in R$, if $\langle v, \alpha^\vee \rangle = 0$, then $V^{p(\tilde{w})} \subset H_{\alpha, 0}$), then
\[ \tilde{v} = z(v) \] is a regular point of \( V^{p(\tilde{w}_0)} \). Thus \( V^{p(\tilde{w}_0)} \subset \cap_{\alpha \in R_J} H_{\alpha,0} = V^{W_J} \). Hence \( \tilde{w}_0 \) is elliptic in \( \tilde{W}_J \).

To guarantee the existence of \( z \in W_0 \) in Sect. 6.1, we can assume \( v \) lies in a sufficiently small neighborhood of \( v_{\tilde{w}'} \). Let \( z_0 \in W_0 \) such that \( z_0(v) = \tilde{v} \). Then \( z_0(v_{\tilde{w}'}) = \tilde{v}_{\tilde{w}'} \) by the choice of \( v \). Therefore, \( \tilde{z} \) is the minimal element of the coset \( W_J z_0 \).

### 6.6 Parametrization of conjugacy classes of affine Weyl groups

Let \( O \) be a conjugacy class and \( \tilde{w}, \tilde{w}' \in O \) with \( v_{\tilde{w}} = v_{\tilde{w}'} = v_O \). Let \( x \in \tilde{W} \) such that \( x \tilde{w} x^{-1} = \tilde{w}' \). Then \( x \in \tilde{W}_J O \). In particular, the set \( \{ \tilde{w} \in O \cap \tilde{W}_J O ; \tilde{v}_{\tilde{w}} = v_O \} \) is a single \( \tilde{W}_J O \)-conjugacy class.

Let \( J \subset F_0 \) and \( C \) be an elliptic conjugacy class of \( \tilde{W}_J \). We say that \( C \) is dominant if \( v_{\tilde{w}} \) is dominant for some (or, equivalently any) \( \tilde{w} \in C \).

Let \( A \) be the set of pairs \( (J, C) \), where \( J \subset F_0 \), \( C \) is a dominant elliptic conjugacy class of \( \tilde{W}_J \). For any \( (J, C) \), \( (J', C') \in A \), we write \( (J, C) \sim (J', C') \) if \( v_{\tilde{w}_1} = v_{\tilde{w}_1}' \) for \( \tilde{w} \in C \) and \( \tilde{w}' \in C' \) and there exists \( x \in J'(W_{J_{\tilde{w}_{\tilde{w}_1}}} \rtimes \Gamma_{J_{\tilde{w}_{\tilde{w}_1}}})^J \) such that \( x J x^{-1} = J' \) and \( x C x^{-1} = C' \).

#### Proposition 6.8

The map from \( A \) to the set of conjugacy classes of \( \tilde{W} \) sending \( (J, C) \) to the unique conjugacy class \( O \) of \( \tilde{W} \) with \( C \subset O \) gives a bijection from \( A / \sim \) to the set of conjugacy classes of \( \tilde{W} \).

**Proof.** If \( (J, C) \sim (J', C') \), then \( C \) and \( C' \) are in the same conjugacy class of \( \tilde{W} \). On the other hand, suppose that \( C \) and \( C' \) are in the same conjugacy class \( O \). Let \( \tilde{w} \in C \). Then \( v_{\tilde{w}} \in V^{W_J} \) and \( J \subset J_{v_{\tilde{w}}} \). Similarly, \( J' \subset J_{v_{\tilde{w}'}}, \) then \( C, C' \subset \{ \tilde{w}_1 \in O \cap \tilde{W}_J O ; v_{\tilde{w}_1} = v_O \} \) is in the same \( \tilde{W}_J O \)-conjugacy class. In particular, there exists \( x \in W_{J_{v_{\tilde{w}'}}, \Gamma_{J_{v_{\tilde{w}'}}}} \) such that \( x \tilde{w} x^{-1} \in C' \). Hence, \( p(x \tilde{w} x^{-1}) \) is an elliptic element in \( W_{J'} \rtimes \Gamma_{J'} \). By \[ 3 \), Proposition 5.2], \( x = x' x_1 \) for some \( x' \in W_{J'}, x_1 \in J'(W_{J_{v_{\tilde{w}'}}} \rtimes \Gamma_{J_{v_{\tilde{w}'}}})^J \) such that \( x_1 J x_1^{-1} = J' \). Hence \( x_1 \tilde{W}_J x_1^{-1} = \tilde{W}_{J'} \) and \( x_1 C x_1^{-1} = C' \). \( \square \)

Now combining Theorems 6.1 and 3.2, we have

**Theorem 6.9**

1. The elements \( \{ T^J_{O} \}_{(J,O) \in A / \sim} \) span \( \tilde{\mathcal{H}} \) as an \( A \)-module.
2. If \( q_\alpha^{1/2} = q_t^{1/2} \) for all \( s, t \in S \), then \( \{ T^J_{O} \}_{(J,O) \in A / \sim} \) is a basis of \( \tilde{\mathcal{H}} \).

This gives Bernstein-Lusztig presentation of the cocenter \( \tilde{\mathcal{H}} \).

### 7 P-alcove elements and the cocenter of \( \mathcal{H} \)

#### 7.1 P-alcove elements

For any \( \alpha \in R \) and an alcove \( C \), let \( k(\alpha, C) \) be the unique integer \( k \) such that \( C \) lies in the region between the hyperplanes \( H_{\alpha,k} \) and \( H_{\alpha,k-1} \). For any alcoves \( C \) and \( C' \), we say that \( C \geq_{\alpha} C' \) if \( k(\alpha, C) \geq k(\alpha, C') \).
The notion of $P$-alcove elements was introduced in [5] in the study of split $p$-adic groups. The notion of $(J, z)$-alcove elements, introduced in [4], is a generalization of the notion of $P$-alcove elements, which works for non-split groups and for affine Weyl groups together with diagram automorphism. We recall the definition in [4, §4.1].

Let $J \subset F_0$ and $z \in W_0$. We say an element $\tilde{w} \in \tilde{W}$ is a $(J, z)$-alcove element if

1. $z\tilde{w}z^{-1} \in \tilde{W}_J$ and
2. $\tilde{w}C_0 \gtrless_{\alpha} C_0$ for all $\alpha \in z^{-1}(R^+ - R^+_J)$.

Note that if $\tilde{w}$ is a $(J, z)$-alcove element, then it is also a $(J, uz)$-alcove element for any $u \in W_J$.

If $\tilde{w}$ is a $(J, z)$-alcove element, we may also call $\tilde{w}$ a $P$-alcove element, where $P = z^{-1}P_Jz$ is a semistandard parabolic subgroup of the connected reductive group $G$ associated with the root datum $\mathfrak{R}$.

**Lemma 7.1** Let $\tilde{w} \in \tilde{W}$ be a $(J, z)$-alcove and let $s \in S$.

1. If $\ell(\tilde{w}) = \ell(s\tilde{w}s)$, then $s\tilde{w}s$ is a $(J, zp(s))$-alcove element;
2. If $\tilde{w} > s\tilde{w}s$, then $zp(s)z^{-1} \in W_J$. Moreover, both $s\tilde{w} \text{ and } s\tilde{w}s$ are $(J, z)$-alcove elements.

**Remark** In part (2), $s\tilde{w}$ and $s\tilde{w}s$ are also $(J, zp(s))$-alcove elements.

**Proof** Part (1) is proved in [4, Lemma 4.4.3].

Assume $\tilde{w} > s\tilde{w}s$ and $s = s_H$ are the reflection along $H = H_{\alpha, k} \in \mathfrak{S}$ for some $\alpha \in R$ and $k \in \mathbb{Z}$. By replacing $\alpha$ by $-\alpha$ if necessary, we can assume that $z(\alpha) \in R^+$. If $z(\alpha) \notin R_J$, then $\alpha, p(\tilde{w})(\alpha) \in z^{-1}(R^+ - R^+_J)$. Note that $\tilde{w} > s\tilde{w}s$, so $H, \tilde{w}H \in \mathfrak{S}(C_0, \tilde{w}C_0)$. Hence $\tilde{w}C_0 >_{\alpha} C_0$ and $\tilde{w}C_0 >_{p(\tilde{w})(\alpha)} C_0$ since $\tilde{w}$ is a $(J, z)$-alcove. Applying $\tilde{w}$ to the first inequality, we have $\tilde{w}^2C_0 >_{p(\tilde{w})(\alpha)} \tilde{w}C_0$. Hence, both $C_0$ and $\tilde{w}^2C_0$ are separated from $\tilde{w}C_0$ by $\tilde{w}H$. In other words, $C_0$ and $\tilde{w}^2C_0$ are on the same side of $\tilde{w}H$. So $\tilde{w}C_0 >_{\alpha} C_0$ and $\tilde{w}^2C_0 >_{p(\tilde{w})(\alpha)} \tilde{w}C_0$, which cannot happen at the same time. That is a contradiction. The “moreover” part follows from [4, Lemma 4.4.2].

**Theorem 7.2** Let $\tilde{w} \in \tilde{W}$, $J \subset F_0$ and $z \in J W_0$ such that $\tilde{w}$ is a $(J, z)$-alcove. Then $T_{\tilde{w}} \in H_J + [\mathcal{H}, \mathcal{H}]$.

**Proof** We argue by induction on the length of $\tilde{w}$. Suppose that $\tilde{w}$ is of minimal length in its conjugacy class. By Sect. 4.2 (1) and Lemma 7.1, we may assume further that $\tilde{C}_0$ contains a regular point of $V_{\tilde{w}}$.

Let $\mu \in V$ be a dominant vector such that $J = J_\mu$. Since $\tilde{w}$ is a $(J, z)$-alcove, then $zp(\tilde{w})z^{-1}(\mu) = \mu$, that is, $z^{-1}(\mu) + V_{\tilde{w}} = V_{\tilde{w}}$. Moreover

$$R^+ - R^+_J \subset \{ \alpha \in R^+; \langle z(\nu_{\tilde{w}}), \alpha \rangle \geq 0 \}. \tag{a}$$

2 In fact, for $\tilde{w} \in X \times W_0$ and $\delta \in \Gamma$, $\tilde{w}\delta$ is a $(J, z)$-alcove element if and only if $\tilde{w}C_0$ is a $(J, z^{-1}, \delta)$-alcove in [4, §4.1]. This is a generalization of the $P$-alcove introduced in [5].
Let \( v = v_w + \epsilon z^{-1}(\mu) \) with \( \epsilon \) a sufficiently small positive real number. We have \( V_w = V_{\tilde{w}} + v \). Let \( z_1 = uz \) with \( u \in W_J \) such that \( \langle z_1(v), \alpha^\vee \rangle \geq 0 \) for each \( \alpha \in R^+_J \). Let \( \beta \in R^+ - R^+_J \). By (a), \( \langle z_1(v_{\tilde{w}}), \beta^\vee \rangle = \langle z(v_{\tilde{w}}), u^{-1}(\beta^\vee) \rangle \geq 0 \). Moreover, \( \langle z_1 z^{-1}(\mu), \beta^\vee \rangle = \langle \mu, u^{-1}(\beta^\vee) \rangle > 0 \). Hence \( \langle z_1(v), \beta^\vee \rangle > 0 \). So \( z_1(v) \) is dominant. Since \( v \) lies in a sufficiently small neighborhood of \( v_{\tilde{w}}, z_1(v_{\tilde{w}}) \) is also dominant. Now applying Proposition 6.1 (2), \( T_{\tilde{w}} \in H_{J_{\tilde{w}w} \cap J_{\tilde{w}}} + [H, H] \).

Let \( \alpha \in R_{J_{\tilde{w}w} \cap J_{\tilde{w}}} \). Then \( \langle z_1(v), \alpha^\vee \rangle = \langle z_1(v_{\tilde{w}}), \alpha^\vee \rangle = 0 \). Hence \( \langle z_1 z^{-1}(\mu), \alpha^\vee \rangle = \langle \mu, \alpha^\vee \rangle = 0 \). Thus \( J_{\tilde{w}w} \cap J_{\tilde{w}} \subseteq J \). The statement holds for \( \tilde{w} \).

Now assume that \( \tilde{w} \) is not of minimal length in its conjugacy class and the statement holds for all \( \tilde{w}' \in \tilde{W} \) with \( \ell(\tilde{w}') < \ell(\tilde{w}) \).

By Theorem 3.1, there exist \( \tilde{w}_1 \equiv \tilde{w} \) and \( s \in S \) such that \( \ell(s\tilde{w}_1s) < \ell(\tilde{w}_1) = \ell(\tilde{w}) \). Then

\[
T_{\tilde{w}} \equiv T_{\tilde{w}_1} \equiv T_{s\tilde{w}_1s} + (q_s^z - q_{s^{-1}}^z)T_{s\tilde{w}_1} \mod [H, H].
\]

Here \( \ell(s\tilde{w}_1s), \ell(s\tilde{w}_1) < \ell(\tilde{w}) \). By Lemma 7.1, \( \tilde{w}_1, s\tilde{w}_1s, s\tilde{w}_1 \) are \( (J, z_1) \)-alcove elements for some \( z_1 \in J W_0 \). The statement follows from induction hypothesis.

### 7.2 Class polynomials

We introduce the class polynomials, following [6, Theorem 5.3]. Suppose that \( q_s^\frac{1}{t} = q_t^\frac{1}{s} \) for all \( s, t \in S \). We simply write \( v \) for \( q_s^\frac{1}{t} \). In this case, the parameter function \( p_t^\frac{1}{s} \) in Sect. 2.5 also equals to \( v \).

Let \( \tilde{w} \in \tilde{W} \). Then for any conjugacy class \( O \) of \( \tilde{W} \), there exists a polynomial \( f_{\tilde{w}, O} \in \mathbb{Z}[v - v^{-1}] \) with nonnegative coefficient such that \( f_{\tilde{w}, O} \neq 0 \) for only finitely many \( O \) and

\[
T_{\tilde{w}} \equiv \sum_O f_{\tilde{w}, O} T_O \mod [H, H]. \tag{a}
\]

The polynomials can be constructed explicitly as follows.

If \( \tilde{w} \) is a minimal element in a conjugacy class of \( \tilde{W} \), then we set \( f_{\tilde{w}, O} = \begin{cases} 1, & \text{if } \tilde{w} \in O \\ 0, & \text{if } \tilde{w} \notin O \end{cases} \). Suppose that \( \tilde{w} \) is not a minimal element in its conjugacy class and that for any \( \tilde{w}' \in \tilde{W} \) with \( \ell(\tilde{w}') < \ell(\tilde{w}) \), \( f_{\tilde{w}', O} \) is already defined. By Theorem 3.1, there exist \( \tilde{w}_1 \approx \tilde{w} \) and \( s \in S \) such that \( \ell(s\tilde{w}_1s) < \ell(\tilde{w}_1) = \ell(\tilde{w}) \). In this case, \( \ell(s\tilde{w}) < \ell(\tilde{w}) \) and we define \( f_{\tilde{w}, O} \) as

\[
f_{\tilde{w}, O} = (v_s - v_s^{-1}) f_{s\tilde{w}_1, O} + f_{s\tilde{w}_1s, O}.
\]

**Theorem 7.3** Let \( \tilde{w} \in \tilde{W}, J \subset F_0 \) and \( z \in J W_0 \) such that \( \tilde{w} \) is a \( (J, z) \)-alcove. Let

\[
T_{\tilde{w}} \equiv \sum_O f_{\tilde{w}, O} T_{w O} \mod [H, H];
\]
where $O$ and $O'$ run over all the conjugacy classes of $\tilde{W}$ and $\tilde{W}_J$, respectively, in the above summations. Then

$$f_{\tilde{w}, O} = \sum_{O' \subset O} f_{\tilde{w}z^{-1}, O'}.$$  

**Proof** We argue by induction on the length of $\tilde{w}$. If $\tilde{w}$ is of minimal length in its conjugacy class, then by Proposition 4.5, $\tilde{w}z^{-1}$ is also a minimal length element (with respect to $\ell_J$) in its $\tilde{W}_J$-conjugacy class. The statement holds in this case.

Now assume that $\tilde{w}$ is not of minimal length in its conjugacy class and the statement holds for all $\tilde{w}' \in \tilde{W}$ with $\ell(\tilde{w}') < \ell(\tilde{w})$.

By Theorem 3.1, there exist $\tilde{w}_1 \approx \tilde{w}$ and $s \in S$ such that $\ell(s \tilde{w}_1 s) < \ell(\tilde{w}_1) = \ell(\tilde{w})$. By Corollary 4.4 and Lemma 7.1, there exists $z_1 \in J W_0$ such that $\tilde{w}_1, s \tilde{w}_1 s, s \tilde{w}_1$ are $(J, z_1)$-alcove elements and $z \tilde{w}_1 z^{-1} \approx z_1 \tilde{w}_1 z_1^{-1}$ with respect to $\tilde{W}_J$.

Let $t = z s z^{-1}$. Then by Lemmas 4.3 and 7.1, $t \in \tilde{W}_J$ and $\ell(t) = 1$. By the proof of Corollary 4.4, $\ell_J(tz_1 \tilde{w}_1 z_1^{-1}t^{-1}) < \ell_J(z_1 \tilde{w}_1 z_1^{-1})$. So by the construction of class polynomials,

$$f_{\tilde{w}, O} = f_{\tilde{w}_1, O} = (v - v^{-1}) f_{s \tilde{w}_1, O} + f_{s \tilde{w}_1 s, O};$$

$$f_{\tilde{w}z^{-1}, O'} = f_{\tilde{w}_1 z_1^{-1}, O'} = (v - v^{-1}) f_{t z_1 \tilde{w}_1 z_1^{-1}, O'} + f_{t z_1 \tilde{w}_1 z_1^{-1}t^{-1}, O'}.$$  

The statement follows from induction hypothesis. $\Box$

### 7.3 Affine Deligne–Lusztig varieties

In the rest of this section, we discuss some application to affine Deligne–Lusztig varieties.

Let $\mathbb{F}_q$ be the finite field with $q$ elements. Let $k$ be an algebraic closure of $\mathbb{F}_q$. Let $F = \mathbb{F}_q((\epsilon))$, the field of Laurent series over $\mathbb{F}_q$, and $L = k((\epsilon))$, the field of Laurent series over $k$.

Let $G$ be a quasi-split connected reductive group over $F$, which splits over a tamely ramified extension of $F$. Let $\sigma$ be the Frobenius automorphism of $L/F$. We denote the induced automorphism on $G(L)$ also by $\sigma$.

Let $I$ be a $\sigma$-invariant Iwahori subgroup of $G(L)$. The $I$-double cosets in $G(L)$ are parameterized by the extended affine Weyl group $W_G$. The automorphism on $W_G$ induced by $\sigma$ is denoted by $\delta$. Set $\tilde{W} = W_G \rtimes \langle \delta \rangle$.

For $\tilde{w} \in W_G$ and $b \in G(L)$, set

$$X_{\tilde{w}}(b) = \{ gI \in G(L)/I; \; g^{-1}b \sigma(g) \in I \tilde{w} I \}.$$  

This is the affine Deligne–Lusztig variety attached to $\tilde{w}$ and $b$. It plays an important role in arithmetic geometry. We refer to [4,5] and [9] for further information.
The relationship between the affine Deligne–Lusztig varieties and the class polynomials of the associated affine Hecke algebra is found in [9, Theorem 6.1].

**Theorem 7.4** Let \( b \in G(L) \) and \( \tilde{w} \in \tilde{W} \). Then

\[
\dim(X_{\tilde{w}}(b)) = \max_{\mathcal{O}} \left\{ \frac{1}{2}(\ell(\tilde{w}) + \ell(w_\mathcal{O}) + \deg(f_{\tilde{w}\delta, \mathcal{O}})) - \langle \tilde{\nu}_b, 2\rho \rangle \right\},
\]

where \( \mathcal{O} \) ranges over the \( \tilde{W} \)-conjugacy class of \( W_G \delta \subset \tilde{W} \) such that \( \nu_\mathcal{O} \) equals the Newton point of \( b \) and \( \kappa_G(x) = \kappa_G(b) \) for some (or equivalently, any) \( x \in W_G \) with \( x\delta \in \mathcal{O} \). Here \( \kappa_G \) is the Kottwitz map [12].

### 7.4 Emptiness criterion for affine Deligne–Lusztig varieties

For \( J \subset F_0 \), let \( M_J \) be the corresponding Levi subgroup of \( G \) defined in [4, 3.2] and \( \kappa_J \) the Kottwitz map for \( M_J(L) \). As a consequence of Theorem 7.3, we have

**Theorem 7.5** Let \( \tilde{w} \in W_G \) and \( z \in W_0 \). Suppose \( \tilde{w}\delta \) is a \((J, z)\)-alcove element. Then for any \( b \in M_J(L) \), \( X_{\tilde{w}}(b) = \emptyset \) unless \( \kappa_J(z\tilde{w}\delta(z)^{-1}) = \kappa_J(b) \).

**Remark** This result was first proved in [5, Theorem 1.1.2] for split groups and then generalized to tamely ramified groups in [4, Corollary 3.6.1]. The approach there is geometric, using Moy-Prasad filtration. The approach here is more algebraic.

**Proof** Assume \( X_{\tilde{w}}(b) \neq \emptyset \). By Theorem 7.4, there exists a conjugacy class \( \mathcal{O} \) of \( W_G \delta \) such that \( f_{\tilde{w}\delta, \mathcal{O}} \neq 0 \), \( \nu_\mathcal{O} = \tilde{\nu}_b \) and \( \kappa_G(b) = \kappa_G(x) \) for some (or equivalently, any) \( x \in W_G \) with \( x\delta \in \mathcal{O} \). By Theorem 7.3, there exists a \( \tilde{W}_J \)-conjugacy class \( \mathcal{O}' \subset \mathcal{O} \) such that \( f_{\tilde{w}\delta, \mathcal{O}'} \neq 0 \). Choose \( b' \in M_J(L) \) such that \( \nu_{b'} = \nu_{\mathcal{O}'} \) and \( \kappa_J(b') = \kappa_J(x') \) for some (or equivalently, any) \( x' \in W_M \) with \( x'\delta \in \mathcal{O}' \). By [4, Proposition 3.5.1], \( b \) and \( b' \) belong to the same \( \sigma \)-conjugacy class of \( M_J(L) \). Since the affine Deligne-Lusztig variety \( X_{\tilde{w}}^{M_J}(z\tilde{w}\delta(z)^{-1})(b') \) for \( M_J \) is non-empty, we have \( \kappa_J(z\tilde{w}\delta(z)^{-1}) = \kappa_J(b') = \kappa_J(b) \).

\( \square \)

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