Tropicalization method in cluster algebras

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Abstract. This is a brief survey on the recently developing tropicalization method in cluster algebras and its applications to the periodicities of Y-systems and the associated dilogarithm identities.

1. Introduction

The nature of tropicalization is built into cluster algebras from the beginning by the series of fundamental works by Fomin, Zelevinsky, and Berenstein [FZ02, FZ03a, FZ03b, BFZ05, BZ05, FZ07].

Recall that there are two kinds of variables in cluster algebras, namely, cluster variables and coefficients. The cluster variables are generators of a cluster algebra itself, and they are the main characters in many applications of cluster algebras. Meanwhile, the coefficients are a little bit in a subsidiary position at first sight, but they turn out to be equally important. The coefficients take values in any choice of semifield. In particular, one can choose it to be a tropical semifield. Such a cluster algebra is said to be of geometric type in [FZ02], and this is where the tropicalization gets into business.

These ‘tropical’ coefficients were introduced and studied by several reasons:

i) Tropical coefficients are also regarded as frozen cluster variables (together with extended exchange matrices), and they naturally arise in various contexts in applications [FZ02].

ii) They appeared in the study of the periodicity conjecture of Y-systems, and the dynamics of these tropical coefficients are governed by the underlying Weyl group and root system [FZ03b]. This result is further used in the classification of the cluster algebras of finite type [FZ03a].

iii) A special case of tropical coefficients, called the principal coefficients play the central role of the structure theory of cluster variables and coefficients [FZ07].

Tropicalization also appeared (a little implicitly) in the formulation of cluster algebras based on algebraic tori by Fock and Goncharov [FG09a, FG09c, FG09b]. The importance of this approach becomes more manifest in the analogous formulation of quantum cluster algebras [FG09a, BZ05, FG09c, FG09b, Tra11].
More recently, the tropicalization method became more powerful with the help of the categorification of cluster algebras by generalized cluster categories developed by [CC06, BMR+06, DK08, FK10, Pal08, Ami09, Kel10a, KY11, Pla11b, Pla11a], etc. In particular, we have two remarkable applications of the tropicalization method, namely, the periodicity of seeds [Kel10a, Kel10b, IIK+10a, IIK+10b, IIK+10c, NT10] and the associated dilogarithm identities [Cha05, Nak11a, IIK+10b, IIK+10c, NT10, Nak11b]. As corollaries, they solved two long standing conjectures which arose in the study of integrable models in 1990’s, namely, the periodicities of Y-systems and the dilogarithm identities in conformal field theory. Furthermore, these dilogarithm identities have their quantum counterparts [Nag10, Kel11, Nag11a, Nag11b, KN11], and the tropicalization plays a crucial role also in this context.

In this note we give a brief overview of these developments, focusing on the role of tropicalization. Essentially all the results are collected from the literature; however, some part of the presentation (e.g., Section 3.2) is new.

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2. Cluster algebra with coefficients

We recall some basic facts on cluster algebras with coefficients in [FZ02, FZ03a, FZ07] with a little change of notation and terminology therein. Throughout this section we fix a finite index set $I$, say, $I = \{1, \ldots, n\}$.

2.1. Semifield.

**Definition 2.1.** We say that $\mathbb{P}$ is a semifield if $\mathbb{P}$ is a multiplicative abelian group endowed with the addition $\oplus$, which is commutative, associative, and distributive, i.e., $(a \oplus b)c = ac \oplus bc$.

The following two examples of semifields are important in this note.

**Example 2.2.** (a) **Universal semifield** $\mathbb{P}_{univ}(y)$. For an $I$-tuple of variables $y = (y_i)_{i \in I}$, it is the semifield consisting of all the rational functions of $y$ over $\mathbb{Q}$ with subtraction-free expressions.

(b) **Tropical semifield** $\mathbb{P}_{trop}(y)$. For an $I$-tuple of variables $y = (y_i)_{i \in I}$, it is the multiplicative free abelian group (i.e., the group of all the Laurent monomials of $y$ with coefficient 1) endowed with the following addition $\oplus$:

$$\bigoplus_{i \in I} y_i^{a_i} \oplus \bigoplus_{i \in I} y_i^{b_i} := \bigoplus_{i \in I} y_i^{\min(a_i, b_i)}.$$  

Later we use the following natural homomorphism of semifields

$$\pi_{\text{trop}} : \mathbb{P}_{\text{univ}}(y) \to \mathbb{P}_{\text{trop}}(y),$$

which sends $y_i \mapsto y_i$ and $c \mapsto 1$ ($c \in \mathbb{Q}_{>0}$).

2.2. Cluster algebra with coefficients. Let $(B, x, y)$ be a triplet with the following data:

- a *skew-symmetrizable* integer matrix $B = (b_{ij})_{i,j \in I}$; namely there is a positive integer diagonal matrix $D = \text{diag}(d_i)_{i \in I}$ such that $DB$ is skew-symmetric, i.e., $(DB)^T = -DB$, where $T$ is the transposition,

- an $I$-tuple $x = (x_i)_{i \in I}$ of formal variables,
• an $I$-tuple $y = (y_i)_{i \in I}$ of formal variables.

We fix $D$ such that $d_1, \ldots, d_n$ are coprime throughout. Thus, $D = I$ when $B$ itself is skew-symmetric. Below we consider the cluster algebra with coefficients $A(B, x, y)$, whose initial seed is $(B, x, y)$ and coefficients semifield is $\mathbb{P}_{\text{univ}}(y)$. Let us recall its definition to fix the convention.

Let $(B', x', y')$ be a triplet with the following data:

- a skew-symmetrizable integer matrix $B' = (b'_{ij})_{i, j \in I}$ such that $DB$ is skew-symmetric for the above $D$,
- an $I$-tuple $x' = (x'_i)_{i \in I}$ with $x'_i \in \mathbb{Q}(x)$, where $\mathbb{Z}[y] := \mathbb{Z}[y_{univ}(y)]$ is the group ring of $\mathbb{P}_{\text{univ}}(y)$, and $\mathbb{Q} := \mathbb{Q}[\mathbb{P}_{\text{univ}}(y)]$ is the fractional field of $\mathbb{Z}$.
- an $I$-tuple $y' = (y'_i)_{i \in I}$ with $y'_i \in \mathbb{P}_{\text{univ}}(y)$.

Then, for $(B', x', y')$ and $k \in I$, another triplet of the same kind $(B'', x'', y'')$, called the mutation of $(B', x', y')$ at $k$ and denoted by $\mu_k(B', x', y')$, is defined by the following rule:

\begin{align}
(2.3) \quad b''_{ij} &= \begin{cases} 
-b'_{ij} & i = k \text{ or } j = k \\
 b'_{ij} + [-b'_{ik}]_+ b'_{kj} + b'_{ik} [b'_{kj}]_+ & i, j \neq k,
\end{cases} \\
(2.4) \quad y''_i &= \begin{cases} 
 y_k^{-1} & i = k \\
 \frac{y_i (1 + y'_i)^{-b'_i}_+}{(1 + y'_k)^{-b'_k}_+} & i \neq k.
\end{cases} \\
(2.5) \quad x''_i &= \begin{cases} 
x^{-1}_i \left( \frac{1}{1 + y'_k} \prod_{j \in I} x'_j [b'_{ij}]_+ + \frac{1}{1 + y'_i} \prod_{j \in I} x'_j [-b'_{ji}]_+ \right) & i = k \\
x'_i & i \neq k.
\end{cases}
\end{align}

Here, for any integer $x$, we set $[x]_+ := \max(x, 0)$. The relations (2.4) and (2.5) are called the exchange relations. The involution property $\mu_k^2 = \text{id}$ holds.

Repeat mutations from the initial seed $(B, x, y)$ and collect all the obtained triplets $(B', x', y')$, which are called seeds. For each seed $(B', x', y')$, $B'$ is called an exchange matrix, $x'$ and $y'_i$ ($i \in I$) are called a cluster and a cluster variable, and $y'$ and $y'_i$ ($i \in I$) are called a coefficient tuple and a coefficient. The cluster algebra $A(B, x, y)$ with coefficients in $\mathbb{P}_{\text{univ}}(y)$ is the $\mathbb{Z}$-subalgebra of $\mathbb{Q}(x)$ generated by all the cluster variables.

In this note we casually call $x'_i$ and $y'_i$ the $x$-variables and the $y$-variables. It is a little unfortunate and inconvenient that this conflicts with the notation and terminology (the $a$-variables and the $x$-variables) used by another standard and important references by Fock and Goncharov [FG09a, FG09b, FG09c].

For each seed $(B', x', y')$, define $\hat{y}'_i$ ($i \in I$) by

\begin{align}
(2.6) \quad \hat{y}'_i := y'_i \prod_{j \in I} x'_j [b'_{ij}]_+.
\end{align}

Then, $\hat{y}'_i$ satisfies the same exchange relation (2.4) as $y'_i$ [FZ07 Prop.3.9]; namely,

\begin{align}
(2.7) \quad \hat{y}''_i &= \begin{cases} 
 \hat{y}'_i^{-1} & i = k \\
 \frac{y'_i (1 + y'_k)^{-b'_i}_+}{(1 + y'_k)^{-b'_k}_+} & i \neq k.
\end{cases}
\end{align}
2.3. \(\varepsilon\)-expressions for mutations. The matrix mutation (2.3) is also written as

\[
b''_{ij} = \begin{cases} 
- b'_{ij} & \text{if } i = k \text{ or } j = k \\
 b'_{ij} + [-\varepsilon b'_{ik}] + b'_{kj} + b'_{ik}[\varepsilon b'_{kj}] & \text{if } i, j \neq k,
\end{cases}
\]

where \(\varepsilon \in \{1, -1\}\) and it is independent of the choice of \(\varepsilon\) \([BFZ05]\) Eq.(3.1)]. If \(\varepsilon = 1\), then it is the same as (2.3). In the same spirit, the exchange relations (2.4) and (2.5) have the expression (independent of the choice of \(\varepsilon = \pm 1\)) as

\[
y''_i = \begin{cases} 
 y'_{ik}^{-1} & \text{if } i = k \\
y'_i y_k^{[\varepsilon b'_{ki}]} + (1 \oplus y'_k)^{-b'_{ki}} & \text{if } i \neq k,
\end{cases}
\]

\[
x''_i = \begin{cases} 
x'_{ik}^{-1} \left( \prod_{j \in I} x'_{jk}^{[-\varepsilon b'_{jk}]} \right) \left( 1 + \hat{y}'_k \varepsilon \right) \right) & \text{if } i = k \\
x'_i \left( 1 \oplus y'_k \varepsilon \right) & \text{if } i \neq k,
\end{cases}
\]

where \(\hat{y}'_i\) is defined in (2.6). Let us call (2.8)–(2.10) the \(\varepsilon\)-expressions for their counterparts.

At first sight it seems useless to have such a redundant choice of the sign \(\varepsilon\). However, it turns out that there is actually a canonical choice of \(\varepsilon\) at each mutation in view of tropicalization. We will see more about it below.

2.4. Example. It is standard to identify a skew-symmetric matrix \(B\) with a quiver \(Q\) (without loop and 2-cycle) such that \(I\) is the set of the vertices of \(Q\), and \(t\) arrows are drawn from the vertex \(i\) to \(j\) if \(b_{ij} = t > 0\).

Example 2.3. Consider the cluster algebra \(\mathcal{A}(B, x, y)\) whose initial exchange matrix \(B\) and the corresponding quiver \(Q\) are given by

\[
B = \begin{pmatrix} 
0 & -1 \\
1 & 0
\end{pmatrix}, \quad Q = \begin{array}{c}
\circ \\
1 \\
2
\end{array} \xrightarrow{1 \rightarrow 2}.
\]

This is the cluster algebra of type \(A_2\) in the classification of \([FZ02]\). In particular, it is of finite type, namely, there are only finitely many seeds. Set \(\Sigma(0) = (Q(0), x(0), y(0))\) to be the initial seed \((Q, x, y)\), and consider the seeds \(\Sigma(t) = (Q(t), x(t), y(t))\) \((t = 1, \ldots, 5)\) by the following sequence of the alternative mutations of \(\mu_1\) and \(\mu_2\):

\[
\Sigma(0) \xrightarrow{\mu_1} \Sigma(1) \xrightarrow{\mu_2} \Sigma(2) \xrightarrow{\mu_1} \Sigma(3) \xrightarrow{\mu_2} \Sigma(4) \xrightarrow{\mu_1} \Sigma(5).
\]

According to (2.10), we set

\[
\hat{y}_1 = y_1 x_2, \quad \hat{y}_2 = y_2 x_1^{-1}.
\]
Then, using the exchange relations (2.8)–(2.10), we obtain the following explicit form of seeds:

$$
\begin{align*}
Q(0) & : \begin{cases} x_1(0) = x_1 \\ x_2(0) = x_2, \end{cases} \quad \begin{cases} y_1(0) = y_1 \\ y_2(0) = y_2, \end{cases} \\
Q(1) & : \begin{cases} x_1(1) = x_1^{-1} \frac{1 + \hat{y}_1}{1 \oplus y_1} \\ x_2(1) = x_2, \end{cases} \quad \begin{cases} y_1(1) = y_1^{-1} \\ y_2(1) = y_2(1 \oplus y_1), \end{cases} \\
Q(2) & : \begin{cases} x_1(2) = x_1^{-1} \frac{1 + \hat{y}_1}{1 \oplus y_1} \\ x_2(2) = x_2^{-1} \frac{1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2}{1 \oplus y_2 \oplus y_1 y_2}, \end{cases} \quad \begin{cases} y_1(2) = y_1^{-1}(1 \oplus y_2 + y_1 y_2) \\ y_2(2) = y_2^{-1}(1 \oplus y_1)^{-1}, \end{cases} \\
Q(3) & : \begin{cases} x_1(3) = x_1 x_2^{-1} \frac{1 + \hat{y}_2}{1 \oplus y_2} \\ x_2(3) = x_2^{-1} \frac{1 + \hat{y}_2 + \hat{y}_1 \hat{y}_2}{1 \oplus y_2 \oplus y_1 y_2}, \end{cases} \quad \begin{cases} y_1(3) = y_1(1 \oplus y_2 + y_1 y_2)^{-1} \\ y_2(3) = y_1 y_2(1 \oplus y_2)^{-1}(1 \oplus y_2)^{-1}. \end{cases} \\
Q(4) & : \begin{cases} x_1(4) = x_1 x_2^{-1} \frac{1 + \hat{y}_2}{1 \oplus y_2} \\ x_2(4) = x_1, \end{cases} \quad \begin{cases} y_1(4) = y_2^{-1} \\ y_2(4) = y_1 y_2(1 \oplus y_2)^{-1}, \end{cases} \\
Q(5) & : \begin{cases} x_1(5) = x_2 \\ x_2(5) = x_1, \end{cases} \quad \begin{cases} y_1(5) = y_2 \\ y_2(5) = y_1. \end{cases}
\end{align*}
$$

Here, the encircled vertices in quivers are the forward mutation points in the sequence (2.12). We observe the (half) periodicity of mutations of seeds.

2.5. C-matrices, G-matrices, and F-polynomials. Let us present the ‘separation formulas’ due to [FZ07] clarifying the fundamental structure of x- and y-variables.

**Theorem 2.4 (Separation formulas [FZ07] Prop.3.13 & Cor.6.3).** For each seed \((B', x', y')\) of any cluster algebra \(\mathcal{A}(B, x, y)\), there exist polynomials \(F'_i(y)\) \((i \in I)\) of \(y = (y_i)_{i \in I}\), an integer matrix \(C' = (c'_{ij})_{i,j \in I}\), and an integer matrix \(G' = (g'_{ij})_{i,j \in I}\) such that the following formulas hold:

\begin{align}
\begin{align*}
y'_i &= \prod_{j \in I} y'_{ij} \prod_{j \in I} F'_j(y)_{\oplus}, \\
x'_i &= \prod_{j \in I} x'_{ij} F'_j(y)_{\oplus}.
\end{align*}
\end{align}

Here, \(F'_i(y)_{\oplus}\) is the sum replacing the sum + in any subtraction-free expression of the polynomial \(F'_i(y)\) with the sum + in \(\mathbb{P}_{\text{univ}}(y)\).

We call the above \(F'_i(y), C',\) and \(G'\) the F-polynomials, the C-matrix, and G-matrix of \((B', x', y')\), respectively. Columns of \(C'\) and \(G'\) are called c-vectors and g-vectors, respectively, in [FZ07]. It is conjectured in [FZ02] Sec.3 that the polynomial \(F'_i(y)\) itself is subtraction-free (positivity conjecture). However, we do not rely on this conjecture in this note.
The above formulas, together with categorification and tropicalization explained later, make the theory of cluster algebras very rich and powerful.

2.6. Sign-coherence and constant term conjectures. Let us recall two fundamental conjectures on cluster algebras, described as ‘tantalizing conjecture’ in \cite{FZ07}.

Conjecture 2.5 (\cite{FZ07} Conj.5.4). (i) (Sign-coherence conjecture) Every column of a C-matrix (i.e., every c-vector) is a nonzero vector and its nonzero components are either all positive or all negative.

(ii) (Constant term conjecture) Every F-polynomial has constant term 1.

It is known that the conjectures (i) and (ii) are equivalent \cite[Prop.5.6]{FZ07}. Note that the conjecture (i) is analogous to the positivity/negativity property of roots in root systems. So far, the following partial result is known.

Theorem 2.6 (\cite{DWZ10, Pla11b, Pla11a, Nag10}). The conjecture is true for any cluster algebra $A(B, x, y)$ with skew-symmetric matrix $B$.

All the known proofs of this theorem require some ‘extra’ machinery, namely, representation of quiver with potential $\cite{DWZ10}$, categorification by 2-Calabi-Yau category $\cite{Pla11b, Pla11a}$ or by 3-Calabi-Yau category $\cite{Nag10}$.

2.7. Tropical sign. Assuming the sign-coherence in Conjecture 2.5, to any seed $(B', x', y')$ and $k \in I$, one can uniquely assign the sign $\varepsilon_k \in \{1, -1\}$ defined by the sign of the $k$th column of the $C$-matrix of $(B', x', y')$. We call the sign $\varepsilon_k$ the tropical sign of $(B', x', y')$ at $k \in I$.

In the next section, we see that the tropical sign $\varepsilon_k$ is the canonical choice for $\varepsilon$ in the $\varepsilon$-expressions (2.8)–(2.10). The use of $\varepsilon$-expressions, together with the tropical sign, is important but overlooked until recently $\cite{Nag10, Qin10, NZ12, Kel11, Nag11a, Nag11b, KN11}$.

3. Tropicalization in cluster algebras

Here, we collect some basic properties of the tropicalization in cluster algebras.

3.1. Tropical $y$-variables. Let $(B', x', y')$ be any seed of a cluster algebra $A(B, x, y)$. Let $\pi_{trop}$ be the one in (2.2). For the notational simplicity, let us write $\pi_{trop}(\cdot)$ as $[\cdot]$. For the $y$-variables $y'_i \in P_{univ}(y)$, we call their images $[y'_i] \in P_{trop}(y)$ the tropical $y$-variables. They are identified with the principal coefficients in $\cite{FZ07}$.

Theorem 3.1 (\cite{FZ07} Prop.3.13 & Prop.5.2)).

\begin{align}
[y'_i] = \prod_{j \in I} y_j^{c_{ji}} \\
[F'_i(y)_B] = 1.
\end{align}

In other words, the formula (2.14) expresses the separation of the tropical part $[y'_i]$ and the nontropical part given by the $F$-polynomials. The sign-coherence conjecture in Conjecture 2.5 means that $[y'_i]$ is not 1 and its nonzero exponents in $y$ are either all positive or all negative. The constant term conjecture further implies that $y'_i$ has a Laurent expansion in $y$, and the tropical $y$-variable $[y'_i]$ is its leading term. Note that (3.2) is weaker than the constant term conjecture. For example, $[y_1 \oplus y_2] = 1$. 
Let us give the exchange relation for tropical \( y \)-variables. There are actually two versions, depending on whether one assumes the sign-coherence or not.

(i) Without assuming the sign-coherence. First, note that the exchange relation of tropical \( y \)-variables is formally the same with (2.9) by replacing \( y'_i \) with \([y'_i]\) and \( \oplus \) with the tropical sum \( \oplus \) in (2.1). Thus, we obtain the well-known exchange relation for \( C \)-matrices from (2.9) [FZ02 Prop.5.8]:

\[
(c''_{ij}) = \begin{cases} 
-c'_{ik} & j = k \\
 c'_{ij} + c'_{ik} [\varepsilon b'_{kj}] + [-\varepsilon c'_{ik}] b'_{kj} & j \neq k.
\end{cases}
\]  

(ii) Assuming the sign-coherence. Next, assume the sign-coherence, and let us take the tropical sign \( \varepsilon'_{ik} \) in Section 2.6 and set \( \varepsilon = \varepsilon'_{ik} \) in (2.9). Then, we have

\[
[1 \oplus y'_i \varepsilon'_{ik}] = 1,
\]

and we obtain a simplified form of the exchange relation for tropical \( y \)-variables

\[
(c''_{ij}) = \begin{cases} 
-c'_{ik} & j = k \\
 c'_{ij} + c'_{ik} \varepsilon b'_{kj} & j \neq k.
\end{cases}
\]  

In other words, by setting \( \varepsilon = \varepsilon'_{ik} \), the second line of the exchange relation (2.9) separates into the tropical part \( y'_i y'_k [\varepsilon b'_{ik}] \) and the nontropical part \( (1 \oplus y'_i \varepsilon'_{ik}) b'_{ik} \).

Note that (3.5) is equivalent to the exchange relation of \( C \)-matrices

\[
(c''_{ij}) = \begin{cases} 
-c'_{ik} & j = k \\
 c'_{ij} + c'_{ik} \varepsilon b'_{kj} & j \neq k,
\end{cases}
\]

which is also obtained directly from (3.3) by setting \( \varepsilon = \varepsilon'_{ik} \) [NZ12 Prop.1.3].

3.2. \( y \)-tropical \( x \)-variables. In the same spirit, let us interpret the \( G \)-matrices as \( x \)-variables in tropicalization. However, it is not the ‘tropical \( x \)-variables’ in the direct sense, but the one obtained through the tropicalization of \( y \)-variables (and also \( \tilde{y} \)-variables!). Hence, we call them the \( y \)-tropical \( x \)-variables here. Again, we consider two versions.

(i) Without assuming the sign-coherence. Let us recall the result of [FZ07]. We introduce the \( \mathbb{Z}^n \)-grading for the initial variables as follows:

\[
\deg x_i = \tilde{e}_i, \quad \deg y_i = -\tilde{b}_i,
\]

where \( \tilde{e}_i \) is the \( i \)th unit vector in \( \mathbb{Z}^n \), and \( \tilde{b}_i \) is the \( i \)th column of \( B \). It follows that

\[
\deg \tilde{y}_i = 0.
\]

Now we tropicalize the coefficients \( y'_i \mapsto [y'_i] \). This also induces the evaluations \( x'_i \mapsto [x'_i] \) and \( \tilde{y}'_i \mapsto [\tilde{y}'_i] \), where the coefficient of each monomial in \( x \) is tropicalized. More explicitly, from (2.15) and (3.7), we have

\[
[x'_i] = \left( \prod_{j \in I} x'_j \right) F'_i(\tilde{y}).
\]

In particular, by (3.7) and (3.8), \( [x'_i] \) is homogeneous and

\[
\deg [x'_i] = \tilde{g}'_i.
\]
where \( g'_k \) is the \( i \)th column of \( G' \). On the other hand, by the tropicalization of (2.10), we have

\[
[x'_{ii}'] = \begin{cases} 
[x'_{ik}]^{-1} \left( \prod_{j \in I} [x'_{jj}]^{-[\varepsilon'_{b_{jk}}]} \right) \frac{1 + [\hat{y}'_k]^{\varepsilon}}{[1 + \hat{y}'_k]^{\varepsilon}} & i = k \\
[x'_{ii}] & i \neq k.
\end{cases}
\]  

Then, comparing the degrees of the both hand sides of (3.11), we have the exchange relation of \( G \)-matrices \cite[Prop.6.6 & Eq.(6.13)]:

\[
g'_{ij} = \begin{cases} 
-g'_{ik} + \sum_{\ell \in I} g'_{\ell k}[-\varepsilon'_{b_{\ell k}}] + \sum_{\ell \in I} b_{i \ell}[-\varepsilon'_{c_{i \ell}}] & j = k \\
g'_{ij} & j \neq k.
\end{cases}
\]  

(ii) Assuming the sign-coherence. Next, assume the sign-coherence, and let us take the tropical sign \( \varepsilon' \) in Section 2.6 and set \( \varepsilon = \varepsilon' \) in (3.11). Then, the factor \( [1 + \hat{y}'_k^{\varepsilon}] \) disappears. One can now think that \([x'_i] = x_i\) and \([\hat{y}_i] = \hat{y}_i\) are independent variables and that the exchange relations for \([x'_i]\) and \([\hat{y}_i]\) are given by (2.7) (with \( \hat{g}'_i \) replaced by \( \hat{\hat{y}}'_i \)) and (3.11) (with \([1 + \hat{y}'_k^{\varepsilon}] \) omitted).

We then make the ‘second tropicalization’ \([\cdot] : \mathbb{P}_{\text{univ}}(\hat{y}) \to \mathbb{T}_{\text{trop}}(\hat{y})\), which also induces the evaluation \([x'_i] \mapsto [[x'_i]]\). Let us call the images \([[x'_i]]\) the \( y \)-tropical \( x \)-variables. Then, from (3.9) and (3.11), we obtain

\[
[[x'_i]] = \prod_{j \in I} \hat{g}'_{ji},
\]  

and

\[
[[x''_i]] = \begin{cases} 
[[x'_i]]^{-1} \prod_{j \in I} [[x'_j]]^{-[\varepsilon'_1_{b_{jk}}]} & i = k \\
[[x'_i]] & i \neq k,
\end{cases}
\]  

which are parallel to (3.1) and (3.5). Thus, by setting \( \varepsilon = \varepsilon'_k \), the second line of the exchange relation (2.9) separates into the \( y \)-tropical part and the \( y \)-nontropical part.

Note that (3.14) is equivalent to the exchange relation of \( G \)-matrices

\[
g'_{ij} = \begin{cases} 
-g'_{ik} + \sum_{\ell \in I} g'_{\ell k}[-\varepsilon'_{b_{\ell k}}] & j = k \\
g'_{ij} & j \neq k.
\end{cases}
\]  

which is also obtained directly from (3.12) by setting \( \varepsilon = \varepsilon'_k \) \cite[Prop.1.3]{NZ12}.

### 3.3. Matrix form of mutations and duality.

In this subsection we assume the sign-coherence.

Following \cite{BFZ05}, let \( J_k \) be the \( n \times n \) diagonal matrix whose diagonal entries are all 1 except for the \( k \)th one which is \(-1\). For any \( n \times n \) matrix, say, \( A \), let \( [A]_k^{\varepsilon\cdot} \) denote the matrix whose entries are zero except for the \( k \)th row and the \((k,i)\)th entry is \([a_{ki}]_+\). Similarly, let \( [A]_k^{\cdot\varepsilon} \) denote the matrix whose entries are zero except for the \( k \)th column and the \((i,k)\)th entry is \([a_{ik}]_+\). For an \( n \times n \) matrix \( A \), \( k \in I \), and \( \varepsilon \in \{1,-1\} \), define

\[
P_{A,k,\varepsilon} = J_k + [\varepsilon A]_k^{\cdot\varepsilon}, \quad Q_{A,k,\varepsilon} = J_k + [\varepsilon A]_k^{\varepsilon\cdot}.
\]  

}\]
Then, the exchange relation with \( \varepsilon = \varepsilon' \), \( \varepsilon' \), and \( \varepsilon'' \) are written in the following matrix form \([BFZ05 \text{ Eq.}(3.1)], \ [NZ12 \text{ Prop.1.3}]):\n
\begin{align}
B'' &= Q'B', k, -\varepsilon' B' P'B', k, \varepsilon', \\
C'' &= C' P'B', k, \varepsilon'.
\end{align}

Thus, we have \( (3.19) \), which agrees with the involution property of the mutation \((3.20)\), and the fact \( C \).

For \((B'', x'', y'') = \mu_k(B', x', y')\), we have \( \varepsilon''_k = -\varepsilon'_k, \ v''_{ik} = -v'_{ik}, \) and \( v''_{ki} = -v'_{ki}, \) thus, we have

\begin{align}
P_{B'', k, \varepsilon''_k} &= P_{B', k, \varepsilon'_k}, \\
Q_{B'', k, -\varepsilon''_k} &= Q_{B', k, -\varepsilon'_k}, \\
(P_{B', k, \varepsilon'_k})^2 &= (Q_{B', k, -\varepsilon'_k})^2 = I,
\end{align}

which agrees with the involution property of the mutation \( \mu_k^2 = \text{id} \).

Also, using \(-B''^T = DB'D^{-1}, \) we obtain

\[Q_{B', k, -\varepsilon'_k} = D^{-1}(P_{B', k, \varepsilon'_k})^T D.\]

Thus, \((3.17)\) and \((3.18)\) are rephrased in a more uniform way as

\begin{align}
DB'' &= (P_{B', k, \varepsilon'_k})^T DB' P_{B', k, \varepsilon'_k}, \\
DC'' &= DC' P_{B', k, \varepsilon'_k}, \\
G'' D^{-1} &= G' D^{-1}(P_{B', k, \varepsilon'_k})^T.
\end{align}

The following duality of \( C \)- and \( G \)-matrices immediately follows from \((3.20)\), \((3.23)\), and the fact \( C = G = I \) for the initial seed \((B, x, y)\).

**Proposition 3.2.** \([NZ12 \text{ Eq.}(3.11)]\) Assuming the sign-coherence, we have

\[ (G' D^{-1})^T (DC') = I. \]

In other words, the column vectors of \( DC' \) and \( G' D^{-1} \) are dual basis of \( \mathbb{Q}^n \) to each other. (When \( B \) is skew-symmetric, they are dual basis of \( \mathbb{Z}^n \) to each other.)

**3.4. Basis changes on lattice.** Let \( L \simeq \mathbb{Z}^n \) be a lattice of rank \( n \). Following Fock and Goncharov \([FG09a, FG09c, FG09b]\), we assign a basis \( \vec{v}_1', \ldots, \vec{v}_n' \) of \( L \) to each seed \( \Sigma' = (B', x', y') \) such that under the mutation \( \Sigma'' = \mu_k(\Sigma') \) two bases are related by

\[ \varepsilon''_i = \begin{cases} 
-\vec{v}'_k, & i = k \\
\vec{v}'_i + [\varepsilon'_k \vec{v}'_{ki} + \vec{v}'_k], & i \neq k.
\end{cases} \]

This induces the coordinate transformation \( \tau_{\Sigma', \Sigma''} : \mathbb{Z}^n \to \mathbb{Z}^n \) whose matrix representation is given by \( P_{B', k, \varepsilon'_k} \) in \((3.10)\). To each seed \( \Sigma' \) we assign a skew-symmetric bilinear form \( \langle \cdot, \cdot \rangle_{\Sigma'} \) on \( L \) by \( \langle \vec{v}'_i, \vec{v}'_j \rangle_{\Sigma'} := d_i b_{ij}' \). The following is an immediate consequence of \((3.22)\).

**Proposition 3.3.** \([FG09a \text{ Lemma 1.7}]\) For any seeds \( \Sigma' \) and \( \Sigma'' \), we have \( \langle \tilde{a}, \tilde{b} \rangle_{\Sigma'} = (\tilde{a}, \tilde{b})_{\Sigma''} (\tilde{a}, \tilde{b} \in L) \). Thus, the form \( \langle \cdot, \cdot \rangle_{\Sigma'} \) does not depend on \( \Sigma' \).

Fock and Goncharov formulated cluster algebras (and quantum cluster algebras) starting from the basis change \([3.20] \ [FG09a, FG09c, FG09b]\) without the tropical sign \( \varepsilon'_k \). Modifying the formulation to include \( \varepsilon'_k \) is straightforward, though we need to establish the sign-coherence in advance. See \([Nag10 \text{ Eq.}(2.3)]\) for an interpretation of \((3.25)\) in view of Ginzburg’s differential graded algebra.
3.5. Categorification by generalized cluster category. In this subsection we assume that $B$ is skew-symmetric.

The categorification of cluster algebras by triangulated categories (2-Calabi-Yau realization) was started by [CC06, BMR] with cluster categories when the initial quiver $Q$ is type $ADE$. Later it has been gradually extended to general quivers [DK08, FK10, Pal08, Ami09, Kel10a, KY11, Pal11b, Pal11a]. Here we summarize the most general result of [Pal11b, Pal11a] without explaining detail.

For the quiver $Q$ corresponding to $B$, using the principal extension $\tilde{Q}$ of $Q$ and a potential $W$ on it, a triangulated category $\mathcal{C} = C_{(Q,W)}$, called the generalized cluster category of $Q$, is defined. Then, to each seed $(B', x', y')$ of the cluster algebra $\mathcal{A}(B, x, y)$, one can canonically assign a rigid object $T' = \bigoplus_{i \in I} T'_i$ in $\mathcal{C}$.

**Theorem 3.4** ([Pal11b, Pal11a]). Let $T = \bigoplus_{i \in I} T_i$ be the rigid object assigned to the initial seed $(B, x, y)$. Then, for any seed $(B', x', y')$, the following formulas hold.

\begin{align}
(3.26) & \quad \tilde{Q}' = \text{quiver of } \text{End}_\mathcal{C}(T'), \\
(3.27) & \quad c'_{ij} = -\text{ind}_{T'}(T_i[1])_j = \text{ind}_{T_i}^\mathcal{C}(T_i)_j, \\
(3.28) & \quad d'_{ij} = \text{ind}_{T'}(T'_i)_j, \\
(3.29) & \quad F'_i(y) = \sum_{e \in \mathbb{Z}^L_{\geq 0}} \chi(\text{Gr}_e(\text{Hom}_\mathcal{C}(T, T'_i[1])) \prod_{j \in I} y_j^{e_j}).
\end{align}

Here, $\text{Gr}_e(\cdot)$ the quiver Grassmannian with dimension vector $e$, and $\chi(\cdot)$ is the Euler characteristic.

In brief, the rigid object $T'$ provides all the information of the seed $(B', x', y')$. Let us present some consequence of this remarkable theorem to tropicalization.

Recall that the matrices $G'$ and $G'$ determine each other (Proposition 3.2). By (3.25), the matrix $G'$ carries the information of the index of $T'$. Moreover, the index of $T'$ uniquely determines $T'$ itself [Pal11a Prop.3.1]. Thus, by Theorem 3.4 we recover the seed $(B', x', y')$. Therefore, we obtain the following corollary.

**Corollary 3.5.** The tropical $y$-variables $[y'_i]$ $(i \in I)$ determines the seed $(B', x', y')$.

In our application, it is useful to formulate this corollary in terms of periodicity.

**Definition 3.6.** Let $\nu : I \rightarrow I$ be a bijection, and let $(B', x', y')$ be a seed. For an $I$-sequence $(k_1, \ldots, k_L)$, let $\langle B''', x''', y''' \rangle = \mu_{k_L} \cdots \mu_{k_1}(B', x', y')$. We say $(k_1, \ldots, k_L)$ is a $\nu$-period of $(B', x', y')$ if

\begin{align}
(3.30) & \quad b''_{\nu(i)\nu(j)} = b'_{ij}, \quad x''_{\nu(i)} = x'_i, \quad y''_{\nu(i)} = y'_i, \quad (i, j \in I).
\end{align}

**Example 3.7.** In Example 2.3 $(1, 2, 1, 2, 1)$ is a $(12)$-period of $(Q, x, y)$, where $(12)$ is the transposition of 1 and 2.

**Corollary 3.8** ([IK+10a Th.5.1]). An $I$-sequence $(k_1, \ldots, k_L)$ is a $\nu$-period of $(B', x', y')$ if and only if

\[ [y''_{\nu(i)}] = [y'_i] \quad (i \in I). \]

In other words, the periodicity of seeds follows from the periodicity of tropical $y$-variables. (We conjecture that it is true also for skew-symmetrizable $B$.) It turns out that this criterion is very powerful.
4. Application I: Periodicities of Y-systems

We apply the tropicalization method to prove certain periodicities of seeds corresponding to the conjectured periodicities of Y-systems.

4.1. Background: Periodicity conjecture of Y-systems. For a cluster algebra \( \mathcal{A}(B, x, y) \) of finite type, starting from \((B, x, y)\), repeat any finite sequence of mutations sufficiently many times. Since there are only finitely many numbers of seeds by definition, eventually it returns to the initial quiver. Thus, we have a periodicity of seeds. Aside from these examples, however, periodicity of seeds is a rare event and so far there is no systematic way to find it. Fortunately, we had some source predicting infinitely many periodicities of seeds, which appeared prior to cluster algebras; that is, the periodicity conjecture of Y-systems [Zam91, RTV93, KNS94].

The Y-systems are systems of functional algebraic equations, and they were introduced and studied in 90’s (see [KNS10] and references therein). They are associated with pairs \((X, \ell)\) of a Dynkin diagram \(X\) of finite type and an integer \(\ell \geq 2\) (called level). For simplicity, let us concentrate on the case when \(X\) is simply laced. Let \(I\) be the set of vertices of \(X\), and consider a family of variables \(\{Y_m^a(u) \mid a \in I; m = 1, \ldots, \ell - 1; u \in \mathbb{Z}\}\). The Y-system is the following system of equations:

\[
Y_m^a(u - 1)Y_m^a(u + 1) = \frac{\prod_{b \in I; b \sim a} (1 + Y_m^b(u))}{(1 + Y_m^{a-1}(u^{-1}))(1 + Y_m^{a+1}(u^{-1}))}.
\]

Here, \(a \sim b\) means \(a\) and \(b\) are adjacent in \(X\), and \(Y_0^a(u)^{-1} = Y_\ell^a(u)^{-1} = 0\). The periodicity conjecture of the Y-system claimed that

\[
Y_m^a(u + 2(h + \ell)) = Y_m^a(u),
\]

where \(h\) is the Coxeter number of \(X\). When \(X\) is nonsimply laced, the Y-system looks more complicated and we omit to present it, but the periodicity (4.2) still holds by replacing \(h\) with the dual Coxeter number \(h^\vee\). (For simply laced case \(h = h^\vee\)). The periodicity (4.2) was proved for \(X = A_1\) by [FS95, GT96] and for \(X = A_n\) by [Sze09, Vol07], by using explicit solutions of (4.1).

It turned out that these Y-systems are a part of relations among \(y\)-variables in certain cluster algebras, and the periodicity conjecture (4.2) is translated into the periodicity of seeds in the corresponding cluster algebras. Fomin and Zelevinsky first recognized this fact in the simplest case of simply laced \(X\) and \(\ell = 2\), and proved the periodicity [FZ03b]. Then, the case of simply laced \(X\) and general \(\ell\) was proved using the categorification and the Auslander-Reiten theory of quiver representations [Kel10a, Kel10b]. Finally, the most general case including the nonsimply laced \(X\) was proved using the tropicalization method [IIK+10a, IIK+10b]. Below we explain the essence of this method.

4.2. Sign-arrow coordination. Fortunately, for the cluster algebra corresponding to any Y-system, the initial exchange matrix \(B\) is skew-symmetric, even when \(X\) is nonsimply laced. Thus, one can apply Corollary 3.8 and the sign-coherence.

To study periodicity, it is efficient to work with the quiver \(Q\) corresponding to \(B\). Recall the exchange relation (3.5) of tropical \(y\)-variables. The only nontrivial term therein is \([y_{k'}][e_k b_{k'i}]^+\) and this term appears only when \(e_k b_{k'i} > 0\). In particular,
when there is no multiple arrows between \textit{k} and \textit{i}, which is the case for the Y-systems, the tropical exchange relation (3.5) is rephrased in the following simple rule:

i) (Sign-arrow coordination) For \( \textit{i} \neq \textit{k} \), multiply \( y'_\textit{k} \) to \( y'_\textit{i} \) only when one of the following situations occurs:

\[
\varepsilon'_\textit{k} > 0 \quad \text{or} \quad \varepsilon'_\textit{k} < 0 \tag{4.3}
\]

ii) Invert \( y'_\textit{k} \).

4.3. Examples. Let us demonstrate, by examples, how the tropicalization method works to prove periodicities of seeds corresponding to (4.2). Our goal here is to understand the period \( 2(h + \ell) \) in (4.2) for the case \( (X, \ell) = (A_3, 3) \) in Example 4.3. Recall that the Coxeter number of \( A_n \) is \( n + 1 \).

Example 4.1. The case \( (X, \ell) = (A_2, 2) \), where \( h = 3 \) and \( \ell = 2 = 5 \).

The corresponding cluster algebra is of type \( A_2 \), and the (half) periodicity of seeds is just the one presented in Example 2.3. The length 5 of the (half) period \((1, 2, 1, 2, 1)\) coincides with \( h + \ell = 3 + 2 \).

Using Corollary 3.8 and the rules i) and ii) in Section 4.2, one can prove this periodicity much more efficiently by the following diagrammatic calculation of tropical \( y \)-variables:

![Diagram](image)

Here, the vectors \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} -1 \\ -1 \end{pmatrix} \), for example, represent \( y_1 \) and \( y_1^{-1}y_2^{-1} \), respectively. Thus, they are c-vectors (leading from the bottom). Observe that, at the forward mutation points (which are framed in the diagram), the exponents of the variables \([y_u (u)]\) for \( u = 0, 1 \) are positive and identified with the negative simple roots \(-\alpha_1, -\alpha_2 \) of \( A_2 \), while the ones for \( u = 2, 3, 4 \) are negative and identified with the positive roots \( \alpha_1, \alpha_1 + \alpha_2, \alpha_2 \) of \( A_2 \). This separation of the region of \( u \) naturally explains the formula of the period \( 5 = 3 + 2 \). Fomin and Zelevinsky explained and generalized this phenomenon to any simply laced \( X \) using the piecewise-linear analogue of the Coxeter element \([FZ03b]\).

For later use, let us present a similar diagram for the opposite quiver of \( Q \).

Example 4.2. The case \( (X, \ell) = (A_3, 2) \), where \( h = 4 \) and \( \ell = 2 = 6 \).

Let us present one more example of level 2. The corresponding cluster algebra is the cluster algebra of type \( A_3 \). We choose the following initial quiver \( Q \) with
Let \( \mu_+ = \mu_1 \mu_3 \) and \( \mu_- = \mu_2 \). Set \( \Sigma(0) = (Q(0), x(0), y(0)) = (Q, x, y) \), and consider the following sequence of mutations in the backward direction:

\[
\begin{align*}
\Sigma(-6) & \xrightarrow{\mu_+} \Sigma(-5) & \xrightarrow{\mu_-} \Sigma(-4) & \xrightarrow{\mu_+} \Sigma(-3) & \xrightarrow{\mu_-} \Sigma(-2) & \xrightarrow{\mu_+} \Sigma(-1) & \xrightarrow{\mu_-} \Sigma(0)
\end{align*}
\]

By doing the backward mutations (at the points which are not framed in the diagram below) for tropical \( y \)-variables, one can show the (half) periodicity.

Here, the vectors \((1, 0, 0)\) and \((0, -1, -1)\), for example, represent \( y_1 \) and \( y_2^{-2}y_3^{-3} \), respectively: Again, at the forward mutation points, (which are framed in the diagram), \( [y_i(u)] \) for \( u = -1, -2, -3, -4 \) are identified with the positive roots \( \alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 \), of \( A_3 \), while the ones for \( u = -5, -6 \) are identified with the negative simple roots \(-\alpha_1, -\alpha_2, -\alpha_3\) of \( A_3 \). This explains the formula for the (half) period \( 6 = 4 + 2 \).

For later use, let us present a similar diagram for the opposite quiver with opposite parity:

\[
\begin{align*}
\mu_- & \xleftrightarrow{\mu_+} 0 0 1 & \xrightarrow{\mu_-} 0 0 1 & \xrightarrow{\mu_+} 0 0 1 & \xrightarrow{\mu_-} 0 0 1 & \xrightarrow{\mu_+} 0 0 1 & \xrightarrow{\mu_-} 0 0 1 \\
y(-6) & \xrightarrow{\mu_+} 0 1 0 & \xrightarrow{\mu_-} 0 1 0 & \xrightarrow{\mu_+} 0 1 0 & \xrightarrow{\mu_-} 0 1 0 & \xrightarrow{\mu_+} 0 1 0 & \xrightarrow{\mu_-} 0 1 0 \\
\end{align*}
\]

Example 4.3. The case \((X, \ell) = (A_3, 3)\) with \( h + \ell = 4 + 3 = 7 \).

This is our main example. The corresponding cluster algebra has the following initial quiver \( Q \) with parity assignment to each vertex.

\[
Q =
\]

Let \( \mu_+ \) and \( \mu_- \) be the composite mutations at the vertices with + and −, respectively. Then, we consider the sequence of mutations in the both forward and
backward directions:

\[ \cdots \leftrightarrow \Sigma(-2) \leftrightarrow \Sigma(-1) \leftrightarrow \Sigma(0) \leftrightarrow \Sigma(1) \leftrightarrow \Sigma(2) \leftrightarrow \cdots. \]  

We can prove the desired (half) period of 7 by the following diagrammatic calculation.

A closer look at the diagram reveals the following factorization property [Nak11a].

(i) In the region \( u = 0, 1, 2 \), the variables \([y_i(u)]\) at the forward mutation points (framed one) transform like the positive roots of \(A_2\) in each column. Compare it with the second diagram in Example 4.1. This is because during this region they remain positive and then, by the sign-arrow coordination (4.3), the horizontal arrows can be ignored in the mutation.

(ii) In the region \( u = -1, -2, -3, -4 \), the variables \([y_i(u)]\) at the forward mutation points transform like the positive roots of \(A_3\) in each column. Compare it with two diagrams in Example 4.2. Again, this is because during this region they remain negative and then, by the sign-arrow coordination (4.3), the vertical arrows can be ignored in the mutation.

This explains the formula of periodicity \( 7 = 4 + 3 \). One can easily generalize the argument to any simply laced \( X \) and any \( \ell \), using the result of [FZ03b] for \( \ell = 2 \).
The periodicities of seeds corresponding to the Y-systems of nonsimply laced type can be proved in the same spirit, but they are necessarily more complicated (thus, perhaps more intriguing). See [IK+10a, IK+10b].

5. Application II: Dilogarithm identities

5.1. Dilogarithm functions. Define the Euler dilogarithm \( \text{Li}_2(x) \) and Rogers dilogarithm \( L(x) \) by the following integrals [Lew81, Kir95, Zag07].

\[
\text{Li}_2(x) = -\int_0^x \left\{ \frac{\log(1-y)}{y} \right\} dy \quad (x \leq 1),
\]

\[
L(x) = -\frac{1}{2} \int_0^x \left\{ \frac{\log(1-y)}{y} + \frac{\log y}{1-y} \right\} dy \quad (0 \leq x \leq 1).
\]

We restrict the argument \( x \) in the above to avoid the branches coming from the logarithm. Two functions are related by

\[
L(x) = \text{Li}_2(x) + \frac{1}{2} \log x \log(1-x) \quad (0 \leq x \leq 1),
\]

\[
-L\left(\frac{x}{1+x}\right) = \text{Li}_2(-x) + \frac{1}{2} \log x \log(1+x) \quad (0 \leq x).
\]

The Rogers dilogarithm satisfies the following properties:

\[
L(0) = 0, \quad L(1) = \zeta(2) = \frac{\pi^2}{6},
\]

\[
L(x) + L(1-x) = \frac{\pi^2}{6} \quad (0 \leq x \leq 1),
\]

\[
L(x) + L(y) + L\left(\frac{1-x}{1-xy}\right) + L(1-xy) + L\left(\frac{1-y}{1-xy}\right) = \frac{\pi^2}{2} \quad (0 \leq x, y \leq 1).
\]

The relations (5.6) and (5.7) are called Euler’s identity and Abel’s identity (or the pentagon identity). Using (5.6), the pentagon identity is also written as follows:

\[
L(x) + L(y) = L\left(\frac{x(1-y)}{1-xy}\right) + L(xy) + L\left(\frac{y(1-x)}{1-xy}\right).
\]

5.2. Background: Dilogarithm identities in CFT. In the late 80’s there appeared a remarkable conjecture on the dilogarithm identities of central charges in conformal field theory (CFT) [Kir89, KR90, BR90, Kun93]. They are associated with pairs \((X, \ell)\) of a Dynkin diagram \(X\) of finite type and an integer \(\ell \geq 2\) (level), which are the same data for the Y-systems in Section 4.1.

For simplicity, let us concentrate on the case when \(X\) is simply laced case as in Section 4.1. For a family of variables \(\{Y_m^{(a)}\} | a \in \mathcal{I}; m = 1, \ldots, \ell - 1\)\, we introduce the following system of algebraic equations, which is called the constant Y-system:

\[
(Y_m^{(a)})^2 = \frac{\prod_{b \in \ell, b \sim a} (1 + Y_m^{(b)})}{(1 + Y_m^{(a)} - 1)(1 + Y_m^{(a)} - 1)},
\]

where \(Y_0^{(a)} = Y_0^{(a)} - 1 = 0\). Namely, it is the Y-system (4.1) with the condition that the variables \(Y_m^{(a)}(u)\) are constant with respect to \(u\).

**Theorem 5.1** ([NK09]). There exists a unique real positive solution of (5.9).
The dilogarithm conjecture claimed that for the above real positive solution of (5.9), the following equality holds.

\[ \frac{6}{\pi^2} \sum_{a \in I} \sum_{m=1}^{\ell-1} L \left( \frac{Y_m(a)}{1 + Y_m(a)} \right) = \frac{\ell \dim g}{h + \ell} - n, \]

where \( h \) is the Coxeter number of \( X \), \( n = |I| \) = rank \( X \), and \( g \) is the simple Lie algebra of type \( X \). The equality (5.10) was proved for \( X = A_n \) by \[Kir89\] by using an explicit solution of (5.9). The first term of the right hand side of (5.10) is the central charge of the Wess-Zumino-Witten model of type \( X \) and level \( \ell \). The right hand side itself is the central charge of the parafermion CFT of type \( X \) and level \( \ell \). See \[KNS10\] and references therein for further background of the equality (5.10).

Gliozzi and Tateo proposed the functional generalization of (5.10) in accordance with the periodicity conjecture (4.2) of the Y-systems \[GT95\]. Namely, for any real positive solution of the Y-system (4.1), the following identity holds.

\[ \frac{6}{\pi^2} \sum_{a \in I} \sum_{m=1}^{\ell-1} \sum_{u=0}^{2(h+\ell)-1} L \left( \frac{Y_m(a)}{1 + Y_m(a)} \right) = 2hn(\ell - 1). \]

Indeed, the equality (5.10) follows from (5.11) by specializing it to the constant solution with respect to \( u \). The identity (5.11) was proved for \( X = A_1 \) by \[GT96\] and \[PS95\] by using explicit solutions of the Y-systems. It was proved for simply laced \( X \) and \( \ell = 2 \) by \[Cha05\] using the cluster algebra method based on the result of \[PZ03a\]. Then, it was proved in full generality by \[Nak11a\], \[IIK+10a\], \[IIK+10b\] using the tropicalization method. It was further generalized to the dilogarithm identities associated with periods of seeds in cluster algebras \[Nak11b\].

### 5.3. Classical dilogarithm identities

In this subsection we assume that \( B \) is skew-symmetric.

Let us present so far the most general dilogarithm identities associated with periods of seeds in cluster algebras \[Nak11b\]. Since \( x \)-variables are irrelevant, let us concentrate on \( y \)-seeds’ \((B',y')\). Let \((k_0, \ldots, k_{N-1})\) be a \( \ell \)-period of the initial seed \((B, y)\). Set \((B(0), y(0)) = (B, y)\), and consider the sequence of mutations,

\[ (B(0), y(0)) \overset{\mu_{k_0}}{\leftrightarrow} (B(1), y(1)) \overset{\mu_{k_1}}{\leftrightarrow} \cdots \overset{\mu_{k_{N-1}}}{\leftrightarrow} (B(N), y(N)) \]

Let \( \varepsilon_t \) be the tropical sign of \((B(t), y(t))\) at \( k_t \). We call the sequence \((\varepsilon_0, \ldots, \varepsilon_{N-1})\) the tropical sign-sequence of (5.12).

**Theorem 5.2** (Classical dilogarithm identities \[Nak11b\]). The following equalities hold for any evaluation of the initial \( y \)-variables \( y_i \) (\( i \in I \)) in \( \mathbb{R}_{>0} \).

\[ \sum_{t=0}^{N-1} \varepsilon_t L \left( \frac{y_{k_t}(t)^{\varepsilon_t}}{1 + y_{k_t}(t)^{\varepsilon_t}} \right) = 0, \]

\[ \frac{6}{\pi^2} \sum_{t=0}^{N-1} L \left( \frac{y_{k_t}(t)}{1 + y_{k_t}(t)} \right) = N_-, \]

\[ \frac{6}{\pi^2} \sum_{t=0}^{N-1} L \left( \frac{1}{1 + y_{k_t}(t)} \right) = N_+, \]

where \( N_+ \) and \( N_- \) (\( N_+ + N_- = N \)) are the total numbers of 1 and \(-1\) among \( \varepsilon_0, \ldots, \varepsilon_{N-1} \), respectively.
Three identities are equivalent to each other due to (5.6). Applying (5.14) for the periods corresponding to the Y-systems and counting the number \( N_{-} \), we obtain (5.11), thus proving the dilogarithm identities (5.10) as well.

**Example 5.3 (Pentagon identity).** Apply Theorem 5.2 to the period in Example 2.3. We have \((k_0, k_1, k_2, k_3, k_4) = (1, 2, 1, 2, 1)\) and

\[
y_1(0) = y_1, \quad y_2(1) = y_2(1 + y_1), \quad y_1(2) = y_1^{-1}(1 + y_2 + y_1y_2),
\]

(5.16)

\[
y_2(3) = y_1^{-1}y_2^{-1}(1 + y_2), \quad y_1(4) = y_2^{-1},
\]

\[\varepsilon_0 = \varepsilon_1 = 1, \quad \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = -1.\]

Inserting these data into (5.13), we obtain

\[
L \left( \frac{y_1}{1+y_1} \right) + L \left( \frac{y_2(1+y_1)}{1+y_2+y_1y_2} \right) - L \left( \frac{y_1}{1+y_1} \right) - L \left( \frac{y_1y_2}{1+y_2+y_1y_2} \right) - L \left( \frac{y_2}{1+y_2} \right) = 0.
\]

(5.17)

By setting \( x = y_1/(1+y_1), y = y_2(1+y_2)/(1+y_2+y_1y_2) \), it coincides with the pentagon identity (5.8).

**5.4. Quantum pentagon identity.** Following [FV93, FK94], define the quantum dilogarithm \( \Psi_q(x) \), for \( |q| < 1 \) and \( x \in \mathbb{C} \), by

\[
\Psi_q(x) = \sum_{n=0}^{\infty} \frac{(-qx)^n}{(q^2;q^2)_n} = \frac{1}{(q^{-x};q^2)_{\infty}}, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k).
\]

(5.18)

It satisfies the following recursion relations.

\[
\Psi_q(q^{\pm 2}x) = (1 + q^{\pm 1}x)^{\pm 1}\Psi_q(x).
\]

(5.19)

The following properties explain why it is considered as a quantum analogue of the dilogarithm [FV93, FK94].

(a). Asymptotic behavior: In the limit \( q \to 1^- \),

\[
\Psi_q(x) \sim \exp \left( -\frac{\text{Li}_2(-x)}{2 \log q} \right).
\]

(5.20)

(b). Quantum pentagon identity: If \( UV = q^2 VU \), then

\[
\Psi_q(U)V \Psi_q(V) = \Psi_q(V)\Psi_q(q^{-1}UV)\Psi_q(U).
\]

(5.21)

Moreover, in the limit \( q \to 1^- \), the relation (5.21) reduces to the pentagon identity (5.8).

**5.5. Quantum dilogarithm identities.** In this subsection we assume that \( B \) is skew-symmetric.

We present a quantum counterpart of the dilogarithm identities (5.13), which is a generalization of (5.21). To do that, we use the quantum cluster algebras by [FG09a, FG09c]. In short, it replaces classical \( y \)-seeds \((B', y')\) with the quantum one \((B', Y')\) in the following way: First, the initial quantum \( y \)-variables are noncommutative and satisfy the relation

\[
Y_i Y_j = q^{2b_{ij}} Y_j Y_i.
\]

(5.22)
Second, for the mutation \((B'', Y'') = \mu_k(B', Y')\), the \(\varepsilon\)-expression of the exchange relation \((2.22)\) is replaced with

\[
Y_i'' = \begin{cases} 
Y_i'^{-1} & i = k \\
q^{b_{ik}[\varepsilon b_{ik}]} Y_i' Y_j' [\varepsilon b_{ik}]+ \prod_{m=1}^{[b_{ik}]} (1 + q^{-\text{sgn}(b_{ik})(2m-1)}) Y_k'^{-\varepsilon - \text{sgn}(b_{ik})} & i \neq k.
\end{cases}
\]

Formally setting \(q = 1\), quantum \(y\)-seeds reduce to the classical one.

Let \(\varepsilon_k'\) be the tropical sign of \((B', y')\) at \(k\). In analogy with the classical case, we introduce the tropical quantum \(y\)-variables \([Y_i']\) by the initial condition \([Y_i'] = Y_i\) and the exchange relation

\[
[Y_i'] = \begin{cases} 
[Y_i'^{-1}] & i = k \\
q^{b_{ik}[\varepsilon b_{ik}]} [Y_i'] [\varepsilon b_{ik}] & i \neq k.
\end{cases}
\]

This gives the tropical part of \((5.23)\) with \(\varepsilon = \varepsilon_k'\) therein.

On the other hand, the nontropical part of \((5.23)\) is given by the adjoint action of the quantum dilogarithm; More precisely,

\[
\text{Ad}(\Psi_q(\mu_{\varepsilon_k^i}(\varepsilon_k^k)))\Psi_q(\mu_{\varepsilon_k^k}(\varepsilon_k^k)) = Y_i' \Psi_q(q^{-2b_{ik}} Y_k'^{-\varepsilon_k^k} \varepsilon_k^k \Psi_q(Y_k'^{-\varepsilon_k^k})) = Y_i' \prod_{m=1}^{[b_{ik}]} (1 + q^{-\varepsilon_k^k \text{sgn}(b_{ik})(2m-1)}) Y_k'^{-\varepsilon_k^k - \text{sgn}(b_{ik})},
\]

where we use \((5.19)\) in the last equality. This is where the quantum dilogarithm is involved in quantum cluster algebras. Actually, in \cite{FG09a}, this important property (without \(\varepsilon_k'\)) was employed as the definition of the exchange relation of the quantum \(y\)-variables. The factor \(\varepsilon_k'\) was introduced by \cite{Kel11} based on the work \cite{Nag11a}.

A \(\nu\)-period of a quantum \(y\)-seed is defined in the same way as the classical case. One can show that an \(I\)-sequence is a \(\nu\)-period of a quantum \(y\)-seed \((B', y')\) if and only if it is a \(\nu\)-period of the corresponding \(\varepsilon\)-sequence \((\varepsilon_{N-1})\). Using the result of \cite[Theorem 6.1]{BZ05} and \cite[Lemma 2.22]{FG09c},

**Theorem 5.4 (Quantum dilogarithm identities \cite{Kel11, Nag11a}).** Suppose that \((k_0, \ldots, k_{N-1})\) is a \(\nu\)-period of \((B, Y)\), and let \((\varepsilon_1, \ldots, \varepsilon_{N-1})\) be the tropical sign-sequence of \((5.12)\). Then, the following identity holds.

\[
\Psi_q([Y_{k_0}(0)]^{\varepsilon_0} \cdots [Y_{k_{N-1}}(N-1)]^{\varepsilon_{N-1}} = 1.
\]

**Example 5.5 (Quantum pentagon identity).** We continue to use the data in Examples 2.3 and 5.3. For the initial quantum \(y\)-seed \((B, Y)\), we have

\[
Y_1 Y_2 = q^3 Y_2 Y_1.
\]

For the quantum \(y\)-seeds corresponding to \((2.12)\), we have

\[
\begin{align*}
Y_1(0) &= Y_1, & Y_1(1) &= Y_1^{-1}, & Y_1(2) &= Y_1^{-1}(1 + qY_2 + Y_1Y_2), \\
Y_2(0) &= Y_2, & Y_2(1) &= Y_2(1 + qY_1), & Y_2(2) &= Y_2^{-1}(1 + q^{-1}Y_1)^{-1}, \\
Y_1(3) &= Y_1(1 + qY_2 + Y_1Y_2)^{-1}, & Y_1(4) &= Y_2^{-1}, & Y_1(5) &= Y_2, \\
Y_2(3) &= q^{-1}Y_1^{-1}Y_2^{-1}(1 + qY_2), & Y_2(4) &= q^{-1}Y_1Y_2(1 + q^{-1}Y_2), & Y_2(5) &= Y_1,
\end{align*}
\]

Second, for the mutation \((B'', Y'') = \mu_k(B', Y')\), the \(\varepsilon\)-expression of the exchange relation \((2.22)\) is replaced with

\[
Y_i'' = \begin{cases} 
Y_i'^{-1} & i = k \\
q^{b_{ik}[\varepsilon b_{ik}]} Y_i' Y_j' [\varepsilon b_{ik}]+ \prod_{m=1}^{[b_{ik}]} (1 + q^{-\text{sgn}(b_{ik})(2m-1)}) Y_k'^{-\varepsilon - \text{sgn}(b_{ik})} & i \neq k.
\end{cases}
\]
Thus, the quantum dilogarithm identity (5.26) is

\[
\Psi_q(Y_1) \Psi_q(Y_2) \Psi_q(Y_1)^{-1} \Psi_q(q^{-1}Y_1Y_2)^{-1} \Psi_q(Y_2)^{-1} = 1.
\]

It coincides with the quantum pentagon relation (5.21).

In [KN11] several forms of quantum dilogarithm identities are given, where the identities (5.26) are called the **tropical form**. Rewriting them in the **local form** therein, then taking the semiclassical limit \( q \to 1 \) with the saddle point method, the classical dilogarithm identities (5.13) can be recovered [KN11].

In summary, we obtain the following scheme, which vastly generalizes the classical and quantum pentagon identities:

\[
\begin{array}{c}
\text{periods of quantum cluster algebras} \\
\text{quantum dilogarithm identities} \\
\text{periods of classical cluster algebras} \\
\text{classical dilogarithm identities}
\end{array}
\]

\[
\begin{array}{c}
\text{semiclassical limit}
\end{array}
\]

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