Examples of manifolds of positive Ricci curvature with quadratically nonnegatively curved infinity and infinite topological type

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Abstract

In this paper, we construct a complete \(n\)-dim \((n \geq 6)\) Riemannian manifold with positive Ricci curvature, quadratically nonnegatively curved infinity and infinite topological type. This gives a negative answer to a conjecture by Jiping Sha and Zhongmin Shen [17] in the case of \(n \geq 6\).

1 Introduction

A complete noncompact Riemannian manifold is said to be of finite topological type if it is homeomorphic to the interior of a compact manifold with boundary. According to Cheeger-Gromoll’s Soul Theorem [5], the manifolds with nonnegative sectional curvature is of finite topological type. But it is not always true for complete manifolds with nonnegative Ricci curvature. For lower dimensional manifolds, we have a positive answer: in the case of \(n = 2\), all notions of curvature coincide; in the case of \(n = 3\), such manifolds were first studied by Schoen and Yau by using stable minimal surfaces [20] and finally classified by G. Liu [11]. For the higher dimensional case, counterexamples with positive Ricci curvature were constructed by Sha-Yang [18, 19] and Menguy [13, 14] (also see [3] for examples of Kähler manifolds). Thus, one thinks in general that, in order to get finiteness results under nonnegative Ricci curvature, some additional assumptions must be required for the higher dimensional case. Some well-known conditions are volume growth, diameter growth, and sectional curvature bounded below or the so-called quadratically nonnegatively curved infinity (see the precise definition below).

In order to put the paper into a proper perspective, we here recall some known results in this direction. Let \(M^n\) be an \(n\)-dimensional complete noncompact manifold, \(p_0 \in M\) a fixed point. By \(B(p_0, t)\) (resp. \(S(p_0, t)\)) denote

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the ball (resp. sphere) with the center at $p_0$ and the radius $t$. Set
\[
\text{diam}(p_0; t) = \sup_i \text{diam}(\Sigma_i, M \setminus B(p_0, \frac{1}{2}t)),
\]
where $\Sigma_i$ is a connected component of $S(p_0, t)$, $\text{diam}(\Sigma_i, M \setminus B(p_0, \frac{1}{2}t))$ denotes the interior diameter of $\Sigma_i$ in $M \setminus B(p_0, \frac{1}{2}t)$ when $M \setminus B(p_0, \frac{1}{2}t)$ is considered as a metric space with the induced metric and $\Sigma_i$ as its subset.

**Definition.** Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotonic function. A complete Riemannian manifold $M$ with the base point $p_0$ is said to have a diameter growth of order $o(f)$ (resp. $O(f)$) if and only if $f^{-1}(t)\text{diam}(p_0; t)$ converges to 0 (resp. remains bounded) as $t \to \infty$.

Abresch and Gromoll [2] showed the following

**Abresch-Gromoll Theorem:** If an $n$-dim complete noncompact manifold with nonnegative Ricci curvature is of diameter growth of order $o(t^{\frac{1}{n}})$ for some base point $p_0$, then it is of finite topological type provided the sectional curvature of $M$ is bounded below.

The proof depends on their estimate for the excess function and Grove-Shiohama critical points theory for the distance function [9] (cf. also [4]). Along the same line, later Jiping Sha and Zhongmin Shen [17] also proved that a complete manifold with nonnegative Ricci curvature and quadratically nonnegatively curved infinity is of finite topological type if it has either maximal volume growth or minimal volume growth.

Let $M^n$ be an $n$-dim complete noncompact Riemannian manifold with the base point $p_0 \in M^n$. Set
\[
K_{p_0}(t) = \inf_{M^n \setminus B(p_0, t)} K
\]
where $B(p_0, t)$ denotes the geodesic ball with center at $p_0$ and radius $t$, $K$ denotes the sectional curvature of $M$, and the infimum is taken over all the sections at all the points contained in $M \setminus B(p_0, t)$. Then, J. P. Sha and Z. M. Shen showed the following [17]

**Sha-Shen Theorem:** Let $M^n$ be an $n$-dim complete noncompact Riemannian manifold with the base point $p_0$ satisfying $\text{Ric} \geq 0$ and
\[
K_{p_0}(t) \geq -\left(\frac{K_0}{1 + t^\alpha}\right)^2
\]
for some $K_0 > 0$ and $\alpha$ satisfying $0 \leq \alpha \leq 1$. If $M$ furthermore satisfies
\[
\frac{\text{vol}(B(p_0, t))}{t^n} = \mu + o\left(\frac{1}{t^{(n-1)(1-\alpha)}}\right)
\]
where $\mu = \lim_{t \to \infty} \frac{\text{vol}(B(p_0, t))}{t^n} > 0$, then $M$ is of finite topological type.
When \( \alpha = 1 \), we call the above sectional curvature condition \textit{quadratically nonnegatively curved infinity}. For \( \alpha > 1 \), U. Abresch have actually showed in \cite{1} that if a complete \( n \)-dim manifold satisfies that the integral \( \int_0^\infty tK_{p_0}(t)dt \) converges, here \( K_{p_0}(t) = \max\{-K_{p_0}(t), 0\} \), then \( M^n \) is of finite topological type. Similar results also refer to \cite{4}.

Based on the above result and some related observation, Sha and Shen conjectured \cite{17} (also see \cite{21}) that a complete Riemannian manifold with nonnegative Ricci curvature and quadratically nonnegatively curved infinity is of finite topological type, without assuming the condition of volume growth.

Instead of the volume growth, similar to Abresch-Gromoll theorem one can also consider the condition of diameter growth. Applying the method of Abresch and Gromoll, one can get the following slight generalization of Abresch-Gromoll Theorem.

\textbf{Theorem 1.1.} (\cite{10}) Let \( M^n \) be an \( n \)-dim complete noncompact Riemannian manifold with the base point \( p_0 \) satisfying \( \text{Ric} \geq 0 \) and

\[ K_{p_0}(t) \geq -\left(\frac{K_0}{1 + t^\alpha}\right)^2 \]

for some \( K_0 > 0 \) and \( \alpha \) satisfying \( 0 \leq \alpha \leq 1 \). If \( M \) furthermore has diameter growth of order \( o(t^{\frac{(n-1)\alpha+1}{n}}) \). Then \( M^n \) is of finite topological type.

It is clear that if \( \alpha = 0 \), the result is just Abresch-Gromoll Theorem. The theorem can actually be stated as Furthermore if \( M \) has diameter \( \text{diam}(p_0; t) < \delta(n, K_0)t \frac{(n-1)\alpha+1}{n} \) for \( t \) large enough, then \( M^n \) is of finite topological type, here \( \delta(n, K_0) = \frac{2K_0-\cosh^{-1}(\cosh^{2}K_0)/(1/K_0)^{\frac{n-1}{n}}}{4+\alpha} \) is smaller than 1.

So, from the above results, it seems that under the conditions of nonnegative Ricci curvature and certain sectional curvature bound below, in order to guarantee the manifolds being of finite topological type, some additional conditions, like diameter growth or volume growth, must be assumed. The purpose of this paper then is to construct examples of complete Riemannian manifolds of positive Ricci curvature with quadratically nonnegatively curved infinity and infinite topological type, which gives a negative answer to Sha-Shen’s conjecture.

As mentioned before, earlier examples with positive (or nonnegative) Ricci curvature and infinite topological type have been constructed by Sha and Yang \cite{18}, which have sectional curvature bounded below but diameter growth of order \( O(t^\frac{1}{2}) \); moreover, they have infinite second Betti number. Later, Perelman \cite{16} gave a more general construction of 4-dimensional compact examples with positive Ricci curvature, big volume and arbitrary large Betti numbers. Then, Menguy applied Perelman’s construction to obtain the following two noncompact examples of infinite topological type.

The first one \cite{14} tells us that the condition of sectional curvature bounded below in Abresch-Gromoll Theorem is necessary:
Menguy’s Example with Bounded Diameter: There exists a complete 4-dimensional manifold with

\[ \text{positive Ricci curvature,} \]
\[ \text{infinite topological type, and} \]
\[ \text{bounded diameter growth.} \]

The second one [13] corresponds to Sha-Shen’s Theorem for \( \alpha = 1 \):

Menguy’s Example with Euclidean Volume Growth: There exists a complete 4-dimensional manifold \( M \) with

\[ \text{Ric}_{M^4} > 0, \]
\[ \text{vol}(B(p_0, t)) = O(t^4), \]
\[ K_{p_0}(t) \geq -\left(\frac{h(t)}{1+t}\right)^2, \]

and infinite topological type, where \( h(t) \) is an arbitrary positive function going to infinity when \( t \) is going to infinity.

According to the constructions of Perelman and Menguy, it is not difficult to give the following examples.

A Direct Generalization of Menguy’s Examples: For any \( 0 \leq \alpha \leq 1 \), there exists a complete 4-dimensional manifold with

\[ \text{Ric}_{M^4} > 0, \]
\[ \text{diam}(p_0; t) = O(t^\alpha), \]
\[ \text{vol}(B(p_0, t)) = O(t^{(n-1)\alpha+1}), \]
\[ K_{p_0}(t) \geq -\left(\frac{h(t)}{1+t^{2\alpha-1}}\right)^2, \]

and infinite topological type, where \( h(t) \) is an arbitrary positive function going to infinity when \( t \) is going to infinity.

All the above examples, however, do not give a negative answer to Sha-Shen’s conjecture. Here, we’ll try to construct a 6-dimensional example, which gives a negative answer to Sha-Shen’s conjecture, as follows.

Theorem 1.2. There exists a 6-dimensional complete Riemannian manifold \( M \) with the base point \( p_0 \) satisfying

\[ \text{Ric} > 0, \]
\[ K_{p_0}(t) \geq -\left(\frac{K_0}{1+t}\right)^2, \]
\[ \lim_{t \to \infty} \frac{\text{vol}(B_t(p_0))}{t^6} = 0, \]
\[ \text{diam}(p_0; t) = O(t), \]
which is of infinite topological type.

It is easy to get the higher dimensional examples with the same properties by taking a metric product of the above 6-dimensional example with the standard spheres. Here, we want to point out that the examples with quadratically nonnegatively curved infinity and infinite topological type has been constructed by Abresch \[1\]; also, Gromov \[8\] (also see \[12\]) showed that any (smooth paracompact) noncompact manifold \(M\) admits a complete Riemannian metric with quadratically nonnegatively curved infinity and quadratically nonpositively curved infinity (the definition is similar to that of quadratically nonnegatively curved infinity). But the Ricci curvatures of these examples are not nonnegative in general.

To construct such examples, we need some basic tools. For the first one and second one, the readers can refer to \[16\]; for the last two, one can refer to \[13\].

T1. Perelman’s Neck: Let \((S^n, g = dt^2 + B^2(t) d\sigma^2)\) be a rotationally symmetric metric satisfying

(i) sectional curvature > 1;

(ii) \(0 \leq t \leq \pi R, \max t\{B(t)\} = r, \) there exists \(\rho < \frac{\sinh t_0}{100} < \frac{1}{100} \) s.t. \(\sin(2t_0) = \frac{1}{100} \cosh \frac{t_0}{100} > 0\) such that \(0 < r < \rho < R\) and \(r^{n-1} < \rho^n\).

Then there exists a metric of \(S^n \times [0, 1]\) such that

1) \(\text{Ric} > 0;\)

2) the boundary component \(S^n \times \{0\}\) is concave, with normal curvatures equal to \(-\lambda,\) and is isometric to \(S^n(\rho\lambda^{-1}),\) for some \(\lambda > 0;\)

3) the boundary component \(S^n \times \{1\}\) is strictly convex, with all its normal curvatures bigger than 1, and is isometric to \((S^n, g).\)

T2. \(C^0\) Smoothing: Let \(M_1, M_2\) be compact smooth manifolds of positive Ricci curvature, with isometric boundaries \(\partial M_1 \cong \partial M_2 = X.\) Suppose that the normal curvatures of \(\partial M_1\) are bigger than the negatives of the corresponding normal curvatures of \(\partial M_2.\) Then the result \(M_1 \cup_X M_2\) of gluing \(M_1\) and \(M_2\) can be smoothed near \(X\) to produce a manifold of positive Ricci curvature.

T3. \(C^1\) Smoothing: Let \(M\) be a \(C^2\) Riemannian manifold of positive Ricci curvature with some discrete points \(C^1\) continuous, then we can smooth it at these points to be a \(C^2\) manifold of positive Ricci curvature.

T4. Perelman’s Property: Let \(M^n\) be a compact hypersurface in \(\bar{M}^{n+1}\). We denote \(N\) the outward normal unit vectors, \(II_N\) its second fundamental form, \(K_{\text{int}}\) its intrinsic curvatures and \(SM \subset TM\) its unit tangent bundle. We say that \(M^n\) satisfies Perelman’s property if for all \(x \in M\) and all linearly independent \(X, Y\) in \(S_x M,\)

\[K_{\text{int}}(X, Y) > \max_{Z \in S_x M} II_N(Z, Z).\]
The idea for construction benefit from the constructions of Perelman [16] and Menguy [13, 14, 15]. We begin with a doubly warped product metric $ds^2 = dt^2 + u^2(t)(dx^2 + f^2(t, x)d\sigma^2) + g^2(t)d\theta^2$, where $d\sigma^2, d\theta^2$ are the standard metric of the 2-dim sphere $S^2$. For the first part $dt^2 + u^2(t)(dx^2 + f^2(t, x)d\sigma^2)$ of the metric, we assume that it is the spherical metric for $t_i < t < t_i + 2r_i$, here $t_i + 2r_i < t_{i+1}$ (for the choice of $t_i$ and $r_i$, cf. §2), and $u(t) = O(t)$. Thus the metric on this part is similar to the compact example of Perelman [16], which is a double suspension over a small 2-sphere so that one gets a singular 4-sphere with the set of its singular points being a circle. Then it allows us to remove the geodesic ball $B_{\frac{r_i}{2}}(o_i)(o_i = (t_i + r_i, 0))$ and glue in a $\mathbb{C}P^2$ via a neck of Perelman, provided with $f(t, x)$ almost like $R_0 \sin x$ ($R_0 \ll 1$) and smoothed near $x = 0$. We can continue this surgery when $i$ is going to infinity. Then, we get a manifold with infinite second Betti number, which is of course infinite topological type.

Actually, this is exactly the constructing process of Menguy’s Example with Euclidean Volume Growth [13]. However, the radius $r_i$ in Menguy’s example must be of order $o(t)$ (more precisely, $r_i = \frac{t_i}{h(t_i)}$, here $h$ is a certain function going to infinity as $t \to \infty$), this forces the sectional curvature to be a little weaker than quadratically nonnegatively curved infinity. But the little gap $h(t)$ is essential for $-\frac{\gamma \alpha}{\alpha} \geq 0$ so that Ricci curvature can be controlled to be positive. In our construction, we set $r_i = rt_i = O(t)$ to get exact quadratically nonnegatively curved infinity (here $r$ is a positive constant). To ensure that Ricci curvature is positive, we add the second product part $\times g(t)S^2$ to control the curvatures through the new directions. In order to make the surgery of gluing $\mathbb{C}P^2$ work, we set $g(t)$ to be constant for $t_i + \frac{r_i}{6} < t < t_i + \frac{11r_i}{6}$. Moreover, in order to make Ricci curvature to be positive, the order $\gamma$ of $g(t)(= O(t^\gamma))$ must be sufficiently small. Clearly, by means of Sha-Shen’s theorem, the manifolds constructed is not Euclidean volume growth.

2 The Main Block

We begin with the metric

$$ds^2 = dt^2 + u^2(t)(dx^2 + f^2(t, x)d\sigma^2) + g^2(t)d\theta^2,$$

where both $d\sigma^2$ and $d\theta^2$ are the standard metric of the sphere $S^2$. $T, X, \Sigma_1, \Sigma_2, \Theta_1, \Theta_2$ are an orthonormal basis corresponding to the forms $dt^2, dx^2, d\sigma^2, d\theta^2$ respectively.

Given some constants $c, \gamma, \alpha$ and $t_1$ satisfying

$$0 < c < \frac{1}{3}, \quad 0 < \gamma < \frac{1}{4}, \quad \alpha > 1, \quad t_1 > 1,$$
Set
\[ K = K(c) = \frac{1 - c^2}{c^2}, \]
and define \( \psi = \psi(c) \) by
\[ \sin(\sqrt{K}\psi) = \sqrt{1 - c^2}. \]
Take \( r > 0 \) satisfying
\[ r \leq r(c) = \frac{\pi}{4\sqrt{K}} - \frac{\psi}{2}, \]
Note that \( c \) and \( r \) are fixed constants while \( \gamma, \alpha \) and \( t_1 \) are to be determined.
We define
\[
\begin{align*}
    t_i &= t_1 \alpha^i, \\
    r_i &= rt_i, \\
    \psi_i &= \psi t_i, \\
    K_i &= \frac{K}{t_i^2},
\end{align*}
\]
and
\[ \Delta = \sqrt{K_i}(2r_i + \psi) = \sqrt{K}(2r + \psi). \]

2.1 Construction of \( u(t) \)

The construction of \( u(t) \) is similar with the one in [15].
For \( t_i < t < t_i + 2r_i \), we set
\[ u(t) = \frac{1}{\sqrt{K_i}} \sin(\sqrt{K_i}(t - t_i + \psi_i)), \]
For \( t_i + 2r_i < t < t_{i+1} \), we set
\[ u(t) = t_i w\left(\frac{t}{t_i}\right), \]
where \( w(t) \) is a \( C^2 \) function defined on \([1 + 2r, \alpha]\), which is independent of \( i \). Actually we can define it as follows. First set
\[ w(t) = \min\{ \frac{\sin \Delta}{\sqrt{K}} + (t - 1 - 2r) \cos \Delta + c + 1 \frac{\alpha}{2 \log 1 + 2r} (t \log \frac{t}{1 + 2r} - t + 1 + 2r), ct \} \]
and then smoothen \( w(t) \) to be a \( C^2 \) function. Since at \( t = 1 + 2r \)
\[ \frac{\sin \Delta}{\sqrt{K}} < c(1 + 2r), \]
and at $t = \alpha$

$$\sin \frac{\Delta}{\sqrt{K}} + (\alpha - 1 - 2r) \cos \Delta + \frac{c + 1}{2 \log \frac{\alpha}{1 + 2r}} (\alpha \log \frac{\alpha}{1 + 2r} - \alpha + 1 + 2r) > \alpha,$$

provided with $\cos \Delta \geq \frac{c + 1}{\log \frac{\alpha}{1 + 2r}}$, which is possible for $\alpha \geq \alpha_0(c, r)$. Thus $u(t)$ is $C^1$ at the endpoints $t_i + 2r_i = (1 + 2r)t_i$ and $t_i + 1 = \alpha t_i$.

On the other hand, if $w(t) = ct$,

$$\begin{cases}
w_t \equiv c \\
w_{tt} \equiv 0,
\end{cases}$$

if $w(t) = \frac{\sin \Delta}{\sqrt{K}} + (t - 1 - 2r) \cos \Delta + \frac{c + 1}{2 \log \frac{\alpha}{1 + 2r}} (t \log \frac{t}{1 + 2r} - t + 1 + 2r),
\begin{cases}
\cos \Delta \leq w_t \leq \frac{1 + 3c}{2} < 1 \\
-3w_{tt} + \frac{1}{2} \frac{\alpha (1 - 2\gamma)}{t^2} > 0,
\end{cases}$$

when $\cos(\Delta) \geq \frac{c + 1}{\log \frac{\alpha}{1 + 2r}}$ and $\alpha \geq \alpha_1(c, r, \gamma)$.

**Conclusion:** When $\alpha \geq \alpha_1(c, r, \gamma)$, we have $u(t)$ satisfying, for $t > t_1$,

$$\begin{cases}
\cos \Delta \leq u_t \leq c, \quad -\frac{u_t}{u} = \frac{K}{c^2} & t < t_i + 2r_i \\
\cos \Delta \leq u_t \leq \frac{1 + 3c}{2}, \quad -3u_{tt} + \frac{1}{2} \frac{\alpha (1 - 2\gamma)}{t^2} > 0 & t_i + 2r_i < t < t_{i+1}.
\end{cases}$$

### 2.2 Construction of $g(t)$

Now we need to construct a function $g(t)$ to control the curvature component $-\frac{u_t}{u}$ for $t_i + 2r_i < t < t_{i+1}$. More precisely, we let $\gamma < \frac{1}{4}$ and $-\frac{u_t}{u} \geq \frac{\alpha (1 - 2\gamma)}{2t^2}$ on these intervals. On the other hand, to keep the surgery unaffected, we set $g(t)$ to be constants on $t_i + \frac{r_i}{6} < t < t_i + \frac{11r_i}{6}$. Moreover, to make the Ricci curvature $Ric(\Theta_k, \Theta_k)$ positive, we need the component $K(\Theta_1, \Theta_2) = \frac{1}{g^2} - \frac{\gamma^2}{g^2}$. This is why the dimensions of our examples can not be smaller than 6.

For $t_1 < t < t_1 + \frac{r_1}{6}$, we set

$$g(t) \equiv g(t_1 + \frac{r_1}{6}) = (t_1 + \frac{r_1}{6})^{2} = (1 + \frac{r}{6})^{2}t_1^{2}.$$  

For $t_i + \frac{r_i}{6} < t < t_{i+1} + \frac{r_{i+1}}{6} = \alpha(t_i + \frac{r_i}{6})$, we set

$$g(t) = \begin{cases}
0 & t_i + \frac{r_i}{6} < t < t_i + \frac{11r_i}{6} \\
\frac{6\gamma^2}{(1 + 2r_i)t_i^2} \left( t - (t_i + \frac{11r_i}{6}) \right) & t_i + \frac{11r_i}{6} < t < t_i + 2r_i \\
\frac{6\gamma^2}{(1 + 2r_{i+1})t_{i+1}^2} \left( t_{i+1} + \frac{r_{i+1}}{6} - t \right) & t_{i+1} < t < t_{i+1} + \frac{r_{i+1}}{6}.
\end{cases}$$
where \( \beta = \frac{\log \alpha}{\log \alpha - \log(1+2r) + \frac{r(1+r)}{6(1+2r)}} = \beta(r, \alpha) \) ensures that

\[
g(t_i + \frac{r_i}{6}) = (t_i + \frac{r_i}{6})^\gamma, \quad \forall i \geq 1.
\]

Thus for \( t_i + \frac{r_i}{6} < t < t_{i+1} + \frac{r_{i+1}}{6} = \alpha(t_i + \frac{r_i}{6}) \),

\[
g(t) \leq g(t_{i+1} + \frac{r_{i+1}}{6}) = \alpha^\gamma(t_i + \frac{r_i}{6})^\gamma \leq \alpha^\gamma t^\gamma,
\]

and

\[
g(t) \geq g(t_i + \frac{r_i}{6}) = (t_i + \frac{r_i}{6})^\gamma \geq \alpha^{-\gamma t^\gamma}.
\]

Note that for \( \alpha \geq \alpha_3(r) \), we can have \( \frac{1}{2} \leq \beta \leq 2 \). Then

\[
|g_t| \leq \frac{2(1 + \frac{r}{6})\gamma}{t},
\]

and

\[
\begin{cases}
|g_{tt}| \leq \frac{12\gamma(1 + \frac{rt}{6})}{rt^2} & t_i < t < t_{i} + 2r_i \\
|g_{tt}| \geq \frac{\gamma(1 - \frac{2\gamma}{2})}{2r^2} & t_{i} + 2r_i < t < t_{i+1}.
\end{cases}
\]

**Conclusion:** When \( \alpha \geq \alpha_3(r) \), we have a \( C^1 \) function \( g(t) \) satisfying, for \( t > t_1 \),

\[
\alpha^{-\gamma t^\gamma} \leq g(t) \leq \alpha^\gamma t^\gamma,
\]

\[
|g_t| \leq \frac{2(1 + \frac{r}{6})\gamma}{t},
\]

and

\[
\begin{cases}
|g_{tt}| \leq \frac{12\gamma(1 + \frac{rt}{6})}{rt^2} & t_i < t < t_{i} + 2r_i \\
|g_{tt}| \geq \frac{\gamma(1 - \frac{2\gamma}{2})}{2r^2} & t_{i} + 2r_i < t < t_{i+1}.
\end{cases}
\]

### 2.3 Construction of \( f(t, x) \)

The function \( f(t, x) \) here is actually \( f_\epsilon(t)(x) \), which depends on the parameter \( \epsilon < 1 \). Then we will set \( f_\epsilon(x) \) to be almost \( R_0 \sin x \) (\( R_0 \) is sufficiently small and determined when gluing in \( \mathbb{CP}^2 \)‘s) and smoothed near \( x = 0 \). The original idea comes from \[16\] and the details are given in \[13\] and \[14\].

Let \( \phi \in C^2(\mathbb{R}) \),

\[
\phi(x) = \begin{cases} 
1 & x \leq 0 \\
0 & x \geq 1.
\end{cases}
\]

Set

\[
\phi_\epsilon(x) = \phi(\frac{x - \epsilon}{\epsilon^\frac{1}{4}} - \epsilon) = \begin{cases} 
1 & x \leq \epsilon \\
0 & x \geq \epsilon^\frac{1}{4}.
\end{cases}
\]

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Let $b = b(\varepsilon), l = l(\varepsilon)$ and $\delta = \delta(\varepsilon)$ to be determined. Then, set

$$f_\varepsilon(x) = \begin{cases} 
\sin(tx) / t & 0 \leq x \leq b \\
\sin(lb) e^{(1-\varepsilon)\log \frac{b}{\varepsilon}} & b \leq x \leq \varepsilon \\
R_0 \sin(x + \delta \phi_\varepsilon(x)) & \varepsilon \leq x \leq \frac{\pi}{2}
\end{cases}$$

and $f_\varepsilon(x) = f_\varepsilon(\pi - x)$ for $\frac{\pi}{2} \leq x \leq \pi$. Through this symmetry extension, all the things we will do next around $o_i = (t_i + r_i, 0)$ (including the surgery of gluing in $\mathbb{C}P^2$'s) can and must be done around $o'_i = (t_i + r_i, \pi)$ (we have to remove $B_{\frac{\pi}{2} r_i}(o'_i)$ since Ricci curvatures on these geodesic balls are not necessarily positive). Thus we just consider the situation of $0 \leq x \leq \frac{\pi}{2}$ from now on. The following two propositions and their proofs can be found in [14].

**Proposition 2.1.** \(\forall \eta > 0, \text{ there exists } \varepsilon_0 > 0 \text{ such that } \forall \varepsilon < \varepsilon_0, \forall x \in [0, \pi], \text{ one has} \)

\[
-f_{\varepsilon}(xx) < 1 - \eta, \\
1 - \frac{(f_{\varepsilon})^2}{f_{\varepsilon}^2} \geq 1 - \eta.
\]

**Proposition 2.2.** \(\forall \varepsilon < \varepsilon_0, \forall x \in [0, \frac{\pi}{2}], \text{ one has} \)

\[
A(x) = \tan(x) \left( \frac{(f_{\varepsilon})_{xx}}{f_{\varepsilon}} - \cot(x) \right) < 2\varepsilon.
\]

Now let $\varepsilon = \varepsilon(t) < \varepsilon_0$ be a smooth function, which satisfies

$$\varepsilon(t) = \begin{cases} 
\varepsilon_i & t_{i-1} + \frac{3}{2}r_i < t < t_i + \frac{1}{2}r_i \\
\varepsilon_{i+1} & t_i + \frac{3}{2}r_i < t < t_{i+1} + \frac{1}{2}r_{i+1}
\end{cases}$$

where \(\{\varepsilon_i\}\) is a decreasing sequence of positive constants to be determined.

Set $f(t, x) = f_{\varepsilon(t)}(x)$. Note that $f(t, x) = R_0 \sin x$ for $\varepsilon_i \leq x \leq \frac{\pi}{2}$, and $f(t, x) = f_{\varepsilon(i)}(x)$ for $t_{i-1} + \frac{3}{2}r_i < t < t_i + \frac{1}{2}r_i$, $f(t, x) = f_{\varepsilon(i+1)}(x)$ for $t_i + \frac{3}{2}r_i < t < t_{i+1} + \frac{1}{2}r_{i+1}$. Equivalently, this is to say that $f_i \neq 0$ only arises in $\{(t, x) \mid t_i + \frac{1}{2}r_i \leq t \leq t_i + \frac{3}{2}r_i, 0 \leq x \leq \varepsilon_i\}$. Thus, if $\{(t, x) \mid t_i + \frac{1}{2}r_i \leq t \leq t_i + \frac{3}{2}r_i, 0 \leq x \leq \varepsilon_i\} \subset B_{\frac{\pi}{2} r_i}(o_i)$, then $f_i \equiv 0$ outside $B_{\frac{\pi}{2} r_i}(o_i)$. To prove this, we only need to show

$$d((t_i + r_i \pm \frac{1}{2}r_i, \varepsilon_i), (t_i + r_i, 0)) < \frac{4}{5} r_i.$$
In fact,
\[
d((t_i + r_i \pm \frac{1}{2} r_i, \epsilon^\frac{t}{2}), (t_i + r_i, 0)) \leq d((t_i + r_i \pm \frac{1}{2} r_i, \epsilon^\frac{t}{2}), (t_i + r_i \pm \frac{1}{2} r_i, 0)) + d((t_i + r_i \pm \frac{1}{2} r_i, 0), (t_i + r_i, 0))
\]
\[
\leq \int_0^{\epsilon^\frac{t}{2}} u(t_i + \frac{3}{4} r_i) dx + \frac{1}{2} r_i
\]
\[
\leq \epsilon^\frac{t}{2} t_i + \frac{1}{2} r_i
\]
Thus we can just choose \( \epsilon < (\frac{r \sqrt{K}}{5})^4 \).

**Conclusion:** For \( \epsilon < \epsilon_0(c, r) \), Proposition 2.1, Proposition 2.2 are always true, and \( ft \equiv 0 \) outside \( B_{\frac{5}{5} r_i}(o_i) \).

**2.4 Ric > 0 outside \( B_{\frac{5}{5} r_i}(o_i) \)**

Since \( ft \equiv 0 \) outside \( B_{\frac{5}{5} r_i}(o_i) \), the nonzero Ricci curvatures are as follows.

\[
Ric(T, T) = -3 \frac{\partial u}{u} - 2 \frac{\partial g t}{g}
\]
\[
\geq \begin{cases} \[3u - \frac{\gamma(1 - 2\gamma)}{t} > 0, & t_i + 2r_i < t < t_{i+1}, \\
3K \frac{t}{t_i} - \frac{24\gamma(1 + \gamma)}{rt_i^2} > 0, & t_i < t < t_i + 2r_i, \end{cases}
\]
provided with \( \gamma < \gamma_0(c, r) \);

\[
Ric(X, X) = -2 \frac{\partial f}{f} \frac{1}{u^2} - 2 \left( \frac{\partial u}{u} \right)^2 - 2 \left( \frac{u_t}{u} \right)^2 - 2 \left( \frac{u_t}{u} \right) \frac{2(1 + \frac{r}{6})\gamma}{t} - \frac{1}{3} \frac{\gamma(1 - 2\gamma)}{t^2}
\]
\[
= \frac{2}{u^2} \left[ 1 - \frac{\eta - \left( \frac{1 + 3c}{2} \right)^2[1 + 2(1 + \frac{r}{6})\gamma]}{t} - \frac{1}{3} \frac{\gamma(1 - 2\gamma)}{t^2} \right]
\]
\[
\geq \frac{2}{(1 + 3c)^2 t^2} \left[ 1 - \frac{\eta - \left( \frac{1 + 3c}{2} \right)^2[1 + 2(1 + \frac{r}{6})\gamma]}{t} - \frac{1}{3} \frac{\gamma(1 - 2\gamma)}{t^2} \right]
\]
\[
= \frac{2}{(1 + 3c)^2 t^2} \left[ 1 - \frac{\eta - \left( \frac{1 + 3c}{2} \right)^2[1 + 2(1 + \frac{r}{6})\gamma + \frac{1}{6} \gamma(1 - 2\gamma)]}{t} \right]
\]
\[
> 0,
\]
provided with \( 1 - \eta > \left( \frac{1 + 3c}{2} \right)^2[1 + 2(1 + \frac{r}{6})\gamma + \frac{1}{6} \gamma(1 - 2\gamma)] \), which is possible when \( \eta < \eta_0(c) \) and \( \gamma < \gamma_0(c, r, \eta) \);
\[ \text{Ric}(\Sigma_j, \Sigma_j) = \left(1 - \frac{f_x^2}{f^2} - \frac{f_{xx}}{f} \right) \frac{1}{u^2} - 2\left(\frac{u_t}{u}\right)^2 - 2\frac{u_t}{u} \frac{gt}{g} - \frac{u_{tt}}{u}, \]

which is totally similar to \( \text{Ric}(X, X); \)

\[ \text{Ric}(\Theta_k, \Theta_k) = \frac{1}{g^2} - (\frac{gt}{g})^2 - \frac{g_{tt}}{g} - 3\frac{u_t}{u} \frac{gt}{g} \geq \frac{1}{\alpha^2 t^2} - \frac{4(1 + \frac{t}{6})^2 \gamma^2}{t^2} - \frac{12\gamma(1 + 2r)(1 + \frac{t}{6})}{rt^2} - \frac{3(1 + 3c)(1 + \frac{t}{6})\gamma}{t^2 \cos \Delta}, \]

provided with \( t_1 > t_1(c, r, \gamma). \)

3 Establishing Perelman’s Property for \( \partial B_{\frac{4}{5}, \xi}(o_i) \)

For \( t_i + \frac{r_i}{6} < t < t_i + \frac{11r_i}{6}, \) the metric becomes

\[ ds^2 = dt^2 + u^2(t) [dx^2 + f^2(t, x) d\sigma^2] + (t_i + \frac{r_i}{6})^2 \gamma d\theta^2. \]

In this section, we will show that if \( \epsilon \) is sufficiently small, “the intrinsic curvatures of \( \partial B_{\frac{4}{5}, \xi}(o_i) \)” are strictly bigger than “the square of the normal curvatures”, that is \( \partial B_{\frac{4}{5}, \xi}(o_i) \) satisfies Perelman’s property (see T4 in Section1). Thus after rescaling \( \partial B_{\frac{4}{5}, \xi}(o_i) \) by the maximal of the absolute value of the normal curvatures, we obtain a rotationally symmetric metric \((S^3, g)\) with intrinsic curvature bigger than 1 and normal curvatures less than or equal to 1. Then, by T1 and T2 in Section 1, we can remove \( B_{\frac{4}{5}, \xi}(o_i) \times (t_i + \frac{r_i}{6}) \gamma S^2 \) and glue with a \( \mathbb{CP}^2 \) via a Perelman’s neck (after rescaling by the inverse of the maximal of the absolute value of the normal curvatures) with a metric product \( \times (t_i + \frac{r_i}{6}) \gamma S^2 \). Note that the boundary component \( S^n \times \{1\} \) is glued with our main block along \( \partial B_{\frac{4}{5}, \xi}(o_i) \) and the boundary component \( S^n \times \{0\} \) is glued with \( \mathbb{CP}^2 \) in the same way of Perelman’s construction \[16\].

Denote by \( N \) the outward normal unit vectors to \( T \partial B_{\frac{4}{5}, \xi}(o_i) \),

\[ N = T \cos \xi + X \sin \xi, \]

and

\[ Y = X \cos \xi - T \sin \xi. \]

Thus

\[ II_N(Y, Y) = \sqrt{K_i} \cot(\frac{A}{5} \sqrt{K_i r_i}), \]
since the $T - X$ plane is isometric to $S^2(\frac{1}{\sqrt{K_i}})$,

$$II_N(\Sigma_j, \Sigma_j) = \frac{u_t}{u} \cos \xi + \frac{f_x}{f_u} \sin \xi.$$ 

Note that if $f(t, x) = \sin x$,

$$\frac{u_t}{u} \cos \xi + \frac{f_x}{f_u} \sin \xi = \sqrt{K_i} \cot(\frac{4}{5} \sqrt{K_i r_i}).$$

Thus,

$$|II_N(\Sigma_j, \Sigma_j) - \sqrt{K_i} \cot(\frac{4}{5} \sqrt{K_i r_i})| \leq A(x) \frac{\cot x |\sin \xi|}{u}$$

$$= A(x) |\sqrt{K_i} \cot(\frac{4}{5} \sqrt{K_i r_i}) - \frac{u_t}{u} \cos \xi|$$

$$\leq A(x)(|\sqrt{K_i} \cot(\frac{4}{5} \sqrt{K_i r_i}) + |\frac{u_t}{u}|)$$

$$\leq 2\epsilon [\frac{\sqrt{K} \cot(\frac{4}{5} \sqrt{K r})}{t_i} + \frac{1 + 3c}{2ct_i}]$$

$$= 2[\sqrt{K} \cot(\frac{4}{5} \sqrt{K r}) + \frac{1 + 3c}{2c}] \frac{\epsilon}{t_i}$$

$$= D(c, r) \frac{\epsilon}{t_i} \leq D(c, r) \frac{\epsilon_i}{t_i}.$$ 

For the intrinsic curvatures, by Gauss equation, we have

$$K_{int}(\Sigma_1, \Sigma_2) = K(\Sigma_1, \Sigma_2) + (II_N(\Sigma_j, \Sigma_j))^2$$

$$\geq \frac{1 - f_x}{f_x^2} \frac{1}{u^2} - \frac{u_t^2}{u^2} + \left[\frac{\sqrt{K} \cot(\frac{4}{5} \sqrt{K r})}{t_i} - D(c, r) \frac{\epsilon}{t_i}\right]^2$$

$$\geq \frac{1 - \eta - (\frac{1+3c}{2})^2}{u^2} - \frac{u_t^2}{u^2} + \left[\frac{\sqrt{K} \cot(\frac{4}{5} \sqrt{K r})}{t_i} - D(c, r) \frac{\epsilon}{t_i}\right]^2$$

$$\geq \frac{1 - \eta - (\frac{1+3c}{2})^2}{c^2 t_i^2} - \frac{u_t^2}{u^2} + \left[\frac{\sqrt{K} \cot(\frac{4}{5} \sqrt{K r})}{t_i} - D(c, r) \frac{\epsilon}{t_i}\right]^2$$

$$\geq \left[\frac{\sqrt{K} \cot(\frac{4}{5} \sqrt{K r})}{t_i} + D(c, r) \frac{\epsilon_i}{t_i}\right]^2.$$
provided with \( \epsilon \) sufficiently small. Similarly,

\[
K_{\text{int}}(Y, \Sigma_j) = K(X, \Sigma_j) \cos^2 \xi + K(T, \Sigma_j) \sin^2 \xi + (II_{N}(Y, Y)(II_{N}(\Sigma_j, \Sigma_j))
\geq \left( -\frac{f_{xx}}{f} \frac{1}{u^2} - \frac{u_t^2}{u^2} \right) \cos^2 \xi + \left( -\frac{u_{tt}}{u} \right) \sin^2 \xi
+ \sqrt{K} \cot\left( \frac{4}{5} \sqrt{Kr} \right) \frac{\sqrt{K} \cot\left( \frac{4}{5} \sqrt{K} \right)}{t_i} - D(c, r) \frac{\epsilon}{t_i}\]
\geq 1 - \eta - \left( \frac{1+3\epsilon}{c^2t_i^2} \right)^2 \cos^2 \xi + \frac{K}{t_i^2} \sin^2 \xi
+ \sqrt{K} \cot\left( \frac{4}{5} \sqrt{Kr} \right) \frac{\sqrt{K} \cot\left( \frac{4}{5} \sqrt{K} \right)}{t_i} - D(c, r) \frac{\epsilon}{t_i}\]
\geq \left[ \sqrt{K} \cot\left( \frac{4}{5} \sqrt{Kr} \right) \right. 
\left. \frac{\sqrt{K} \cot\left( \frac{4}{5} \sqrt{K} \right)}{t_i} + D(c, r) \frac{\epsilon}{t_i} \right]^2,
\]
provided with \( \epsilon \) sufficiently small.

### 4 Gluing in \( \mathbb{CP}^2 \)'s

To glue in a \( \mathbb{CP}^2 \) along \( \partial B_{\frac{1}{\sqrt{Kr}}}(o_1) \) via a neck of Perelman, we need to check that \( \partial B_{\frac{1}{\sqrt{Kr}}}(o_1) \) rescaled by \( \frac{\sqrt{K} \cot\left( \frac{4}{5} \sqrt{K} \right)}{t_i} + D(c, r) \frac{\epsilon}{t_i} \) is exactly \((S^3, g)\) in T1 in Section 1. Even though we could not write out the precise expression of the metric, we can check it in the limit case (thanks to the Perelman’s neck, it only requires some rough characterization of the geometric shape).

Indeed, for \( i \) sufficiently large, as \( \epsilon \) goes to 0, the metric of \( B_{\frac{1}{\sqrt{Kr}}}(o_1) \) converges to

\[
dt^2 + \left( \frac{1}{\sqrt{K_i}} \sin(\sqrt{K_i}(t - t_i + \psi_i)) \right)^2 dx^2 + (R_0 \sin x)^2 d\sigma^2.
\]

Thus the metric of \( \partial B_{\frac{1}{\sqrt{Kr}}}(o_1) \) converges to

\[
\left( \frac{t_i \sin\left( \frac{4}{5} \sqrt{Kr} \right)}{\sqrt{K}} \right)^2 dy^2 + (R_0 \sin y)^2 d\sigma^2.
\]

Rescaled by \( \frac{\sqrt{K} \cot\left( \frac{4}{5} \sqrt{K} \right)}{t_i} \) and let \( t = y \cos\left( \frac{4}{5} \sqrt{K} \right) \), then the metric becomes

\[
dt^2 + (R_0 \cos\left( \frac{4}{5} \sqrt{K} \right) \sin \frac{t}{\cos\left( \frac{4}{5} \sqrt{K} \right)})^2 d\sigma^2.
\]

Choosing \( R_0 \) sufficiently small, the metric is exactly what we want.
5 Smooth near the origin

For \( t_1 < t < t_1 + \frac{r_1}{6} \), we already have

\[
\begin{cases}
    u(t) = \frac{1}{\sqrt{K_1}} \sin(\sqrt{K_1}(t - t_1 + \psi)) \\
    g(t) \equiv (t_1 + \frac{r_1}{6})^\gamma.
\end{cases}
\]

Then, for the hypersurface \( t = t_1 \), the intrinsic metric is

\[
ds^2 = (ct_1)^2 [dx^2 + f^2_\epsilon(x) d\sigma^2].
\]

Rescaled by \( t_1 \) and let \( t = ct \), when \( \epsilon \) goes to 0, it converges to

\[
dt^2 + (cR_0 \sin \frac{t}{c})^2 d\sigma^2.
\]

Similarly, choosing \( R_0 \) sufficiently small, the metric satisfies the conditions in Perelman’s neck.

On the other hand,

\[
II_{-T}(X, X) = II_{-T}(\Sigma_j, \Sigma_j) = -\frac{1}{t_1},
\]

\[
K_{int}(\Sigma_1, \Sigma_2) = K(\Sigma_1, \Sigma_2) + \frac{1}{t_1^2} = 1 - f^2_\epsilon \frac{1}{u^2} \geq 1 - \frac{1 - \eta t}{c^2 t_1^2}.
\]

\[
K_{int}(X, \Sigma_j) = K(X, \Sigma_j) + \frac{1}{t_1^2} = -f_{xx} \frac{1}{f} \frac{1}{u^2} \geq 1 - \frac{1 - \eta t}{c^2 t_1^2}.
\]

Setting \( 1 - \eta > c^2 \), we can continue the manifold for \( t \leq t_1 \) via a neck of Perelman and glue the end of the neck in a \( \mathbb{CP}^2 \) with a metric product \( \times (t_1 + \frac{r_1}{6})^\gamma S^2 \).

6 Sectional Curvatures

Now, we show that our example has quadratically nonnegatively curved infinity.

For necks and \( \mathbb{CP}^2 \)'s, after rescaling by \( \frac{\sqrt{K} \cot(\frac{\pi}{3} \sqrt{K} r)}{t_1} + D(c, r) \frac{\epsilon}{t_1} \),

\[
K_{p_0}(t) \geq -(K_0)^2 (\frac{\sqrt{K} \cot(\frac{\pi}{3} \sqrt{K} r)}{t_1} + D(c, r) \frac{\epsilon}{t_1})^2 \geq -\frac{K_0^2(c, r)}{t_1^2}.
\]

For the main block,

\[
K(T, X) = K(T, \Sigma_j) = -\frac{ut}{u} > -\frac{1}{3} \frac{t}{t_1^2} \gamma(1 - 2\gamma),
\]

\[
K(T, \Theta_k) = -\frac{gt}{g} \geq -\frac{12\gamma(1 + \frac{r_1}{3})(1 + 2r)}{rt^2}.
\]
\[
K(X, \Sigma_j) = -\frac{f_{xx}}{f} \frac{1}{u^2} - \frac{u_t}{u} \geq 1 - \eta - \left(\frac{1 + 3c}{2}\right)^2 \frac{1}{t^2} > 0,
\]
\[
K(X, \Theta_k) = K(\Sigma_j, \Theta_k) = -\frac{u_t}{u} \frac{g_t}{g} \geq -\frac{(1 + 3c)(1 + \frac{r}{6})\gamma}{\cos \Delta} \frac{1}{t^2},
\]
\[
K(\Sigma_1, \Sigma_2) = \frac{1 - f^2}{f^2} \frac{1}{u^2} - \frac{u_t^2}{u^2} \geq 1 - \eta - \left(\frac{1 + 3c}{2}\right)^2 \frac{1}{t^2} > 0,
\]
\[
K(\Theta_1, \Theta_2) = \frac{1}{g^2} - \frac{g_t^2}{g^2} \geq \frac{1}{t^{2\gamma}} - \frac{4(1 + \frac{r}{6})^2\gamma^2}{t^2} > 0.
\]

Thus we have
\[
K_{p_0}(t) \geq -\frac{K^2_0(c, r, \gamma)}{t^2}.
\]

7 Diameter growth and Volume growth

For the diameter growth, we have
\[
0 < \pi \cos \Delta = \liminf_{t \to \infty} \frac{\text{diam}(p_0; t)}{t} < \limsup_{t \to \infty} \frac{\text{diam}(p_0; t)}{t} = \pi c.
\]

For the volume growth, we have
\[
0 < V_1 = \liminf_{t \to \infty} \frac{\text{Vol}(B(p_0, t))}{t^{4+2\gamma}} < \limsup_{t \to \infty} \frac{\text{Vol}(B(p_0, t))}{t^{4+2\gamma}} = V_2 < +\infty.
\]

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