INTRODUCTION

Suppose $K \times M \to M$ is a Hamiltonian action of a compact Lie group $K$ on a symplectic manifold $M$, and that $\mu : M \to \mathfrak{t}^*$ is a proper equivariant moment map for this action. To avoid unnecessary symmetry we assume throughout this paper that there is at least one point in $\mu^{-1}(0)$ whose stabilizer is zero dimensional. The symplectic quotient, or reduction at zero, is given by the construction $M_0 := Z/K$ where $Z = \mu^{-1}(0)$. The infinitesimal orbit type stratification of $Z$ descends to a stratification of $M_0$ by symplectic orbifolds. When $0$ is a regular value of the moment map there is a single stratum. In this case the cohomology of $M_0$ can be studied via the surjective restriction:

$$H^*_K(M) \twoheadrightarrow H^*_K(Z) \cong H^*(M_0).$$

When $0$ is no longer a regular value then, in general, $H^*_K(Z)$ will have infinite dimension, and the pull-back $H^*(M_0) \to H^*_K(Z)$ will be neither surjective nor injective.

In this paper we show how, under certain conditions on the action, we can still extract topological information about $M_0$ from $H^*_K(Z)$ when the reduction is no longer regular. The information we obtain is a complete description of the middle perversity intersection cohomology $IH^*(M_0)$ as a graded vector space with non-degenerate pairing between classes of complementary degree. In many cases when dealing with singular spaces this appears to be more natural data than the ordinary cohomology.

Intersection cohomology has its most elegant description in terms of the derived category of constructible sheaves. We are interested in the relationship between two objects, the derived push-down $C'_K(Z)$ of the constant sheaf on $EK \times_K Z$, and the middle perversity intersection cohomology complex $IC'(M_0)$. The hypercohomology functor takes these respectively to $H^*_K(Z)$ and $IH^*(M_0)$. In §3 we show how the partial resolution constructed in [MS99] allows us to define (non-canonically) a morphism

$$\kappa : C'_K(Z) \to IC'(M_0).$$

This generalises a construction first used by Kirwan in [Kir86].

The intersection cohomology complex $IC'(M_0)$ is characterised as an object of the derived category by three axioms which we call normalisation, support and cosupport. The normalisation axiom, that a complex is quasi-isomorphic to the
constant sheaf on the unique open stratum, is always satisfied by \( C_K(Z) \). A simple check shows that the support axiom holds for \( C_K(Z) \) precisely when \( K \) has only finite stabilizers on \( Z \), or, equivalently, the reduction is regular. (Of course cosupport also holds for \( C_K(Z) \) in this situation because it is quasi-isomorphic to \( \mathcal{IC}'(M_0) \) under the regularity assumption.) In this paper we demand rather less, we allow support to fail, so the reduction need not be regular, but demand that cosupport holds for \( C_K(Z) \). We say an action for which this occurs is *almost-balanced*. The reason for the terminology is that the almost-balanced condition is equivalent to asking that the weights of the action on a normal fibre to an infinitesimal orbit type stratum be evenly distributed about the origin. This is explained more precisely in [Kiea]. As well as their theoretical interest there are interesting examples of almost-balanced actions. For instance the moduli space of bundles over a Riemann surface arises as a quotient by an almost-balanced action. The rank 2 case is discussed in some detail in [Kieb]. The ‘standard’ example of \( SU_2 \) acting on \( (\mathbb{P}^1)^n \) is also almost-balanced and we discuss this in §4.4. Very loosely, one expects actions arising in highly symmetric contexts to be almost-balanced.

We show in §4 that when the action is almost-balanced there is a canonical morphism

\[
\lambda : \mathcal{IC}^*(M_0) \longrightarrow C_K(Z).
\]

which splits \( \kappa \) i.e. \( \kappa \lambda = \text{id} \). If we make the slightly stronger assumption that the action is weakly balanced we can identify the subspace of \( H_K^*(Z) \) which corresponds to \( IH^*(M_0) \). The naturality of \( \lambda \) ensures that the intersection pairing corresponds to the normal product on equivariant cohomology.

Finally, in §5 we show that the same set of ideas can be applied to *any* circle action. The key point is that we can define a non-middle perversity \( n \) for which \( IH_n^*(M_0) \cong IH^*(M_0) \). The statements of the previous section then hold for arbitrary circle actions but with \( n \) replacing the middle perversity.
1. Intersection cohomology

Suppose \( X \) is a connected pseudomanifold of dimension \( 2n \) with a fixed even dimensional topological stratification \( \{ S_\alpha \} \). Let \( D(X) \) be the bounded below derived category of constructible sheaves on \( X \). This is a triangulated category with shift functor \([n]\) and truncation functors \( \tau_{<n} \) and \( \tau_{>n} \). Slightly more general than \( \tau_{<n} \) is the functor given by truncation on a closed subset, which was introduced by Deligne and defined in [GM83, §1.14]. Let \( Y \) be a closed subset of \( X \). Then there is a functor \( \tau^Y_{<n} \) and a morphism \( \tau^Y_{<n} A' \rightarrow A' \) which induces

\[
H^i_x(\tau^Y_{<n} A') \cong \begin{cases} 
0 & x \in Y \text{ and } i \geq n; \\
H^i_x(A') & \text{otherwise}
\end{cases}
\]

for any sheaf complex \( A' \). Of course \( \tau^X_{<n} = \tau_{<n} \).

We denote the cone on a morphism \( \alpha \) in \( D(X) \) by Con\( ' \langle \alpha \rangle \). The category \( D(X) \) also has a tensor product and Verdier duality functor. Stratified maps between spaces induce the standard derived functors \( f^*, Rf_*, f^! \) and \( Rf! \) of sheaf theory. A reasonably complete description of this category is given in [GM83, §1].

**Definition 1.1.** A perversity \( p \) is a function \( \{ S_\alpha \} \rightarrow \mathbb{Z} \). We say \( p \) is monotone if \( p(S_\alpha) \leq p(S_\beta) \) whenever \( S_\alpha \supset S_\beta \).

Given a perversity \( p \) we define full subcategories of \( D(X) \) by

\[
D^p_{\leq 0}(X) = \{ A' \mid H^i(j^!_\alpha A') = 0 \text{ for } i > p(S_\alpha) \}
\]
and

\[
D^0_{\leq 0}(X) = \{ A' \mid H^i(j^!_\alpha A') = 0 \text{ for } i < p(S_\alpha) \}.
\]

**Theorem 1.2** ([KS94, Ch. 7, 1.2.1]). The pair \( (D^p_{\leq 0}(X), D^0_{\leq 0}(X)) \) of subcategories defines a \( t \)-structure on \( D(X) \). There are left and right adjoints \( \tau^p_{\leq 0} \) and \( \tau^p_{\geq 0} \) respectively to the inclusions

\[
D^p_{\leq 0}(X) \hookrightarrow D(X) \quad \text{and} \quad D^p_{\geq 0}(X) \hookrightarrow D(X).
\]

Now suppose \( p \) is a monotone perversity with \( p(U) = 0 \) where \( U \) is the unique open stratum. We define two further perversities \( p^+ \) and \( p^- \) by

\[
p^\pm(S_\alpha) = \begin{cases} 
0 & S_\alpha = U \\
p(S_\alpha) \pm 1 & \text{otherwise}
\end{cases}
\]

**Theorem 1.3** ([GM83, §3]). There is a unique object

\[
IC^p_p(X) \in D^p_{\leq 0}(X) \cap D^0_{\geq 0}(X)
\]
with the property that \( IC^p_p(X)_{|U} \) is quasi-isomorphic to the constant sheaf \( \mathbb{Q} \) on \( U \).

We call \( IC^p_p(X) \) the perversity \( p \) intersection cohomology complex. Its hypercohomology \( H^p_\ast(X) \) is the perversity \( p \) intersection cohomology. The uniqueness part of the proof rests on the following lemma which is proved by induction over the stratification:

**Lemma 1.4.** Suppose \( A'|_U \) is quasi-isomorphic to the constant sheaf \( \mathbb{Q} \). Then we have the following implications:

\[
\begin{align*}
A' &\in D^p_{\leq 0}(X) \quad \implies \quad \text{Hom}(A', IC^p_p(X)) \cong \mathbb{Q} \\
A' &\in D^+_{\leq 0}(X) \quad \implies \quad \text{Hom}(IC^p_p(X), A') \cong \mathbb{Q}.
\end{align*}
\]
We define the middle perversity to be the function \( m(S_\alpha) = \frac{1}{2} \text{codim} S_\alpha \). We will denote the middle perversity intersection cohomology complex simply by \( IC^\cdot(X) \).

The top perversity is given by \( t(S_\alpha) = 2m(S_\alpha) \). A simple check shows that \( IC^\cdot(X) \otimes 2 \in D^\cdot_{\leq 0}(X) \) and so lemma 1.4 guarantees there is a natural morphism to \( IC^\cdot_t(X) \).

**Theorem 1.5** \((\text{GM83, §5.2})\). This morphism gives rise to pairings

\[
IH^{n-i}(X) \otimes IH^{n+i}(X) \xrightarrow{f_X} IH^{2n}_t(X) \xrightarrow{\int_X} \mathbb{Q}
\]

which are non-degenerate when \( X \) is compact.

To simplify the notation we say that \( A^\cdot \) satisfies support \( \iff A^\cdot \in D^{\cdot-\leq 0}(X) \)

\( A^\cdot \) satisfies cosupport \( \iff A^\cdot \in D^{\cdot+\geq 0}(X) \).

Thus theorem 1.4 says that \( IC^\cdot(X) \) is characterised by the conditions that it satisfies support, cosupport and is the constant sheaf on \( U \).

**2. Singular symplectic quotients**

Suppose \((M,\omega)\) is a Hamiltonian \( K \)-space for some compact connected Lie group \( K \). By this we mean that \( K \) acts on \( M \) preserving the symplectic form \( \omega \) in such a way that there is an equivariant moment map \( \mu : M \rightarrow \mathfrak{k}^* \) satisfying

\[
\langle d\mu, a \rangle = \int_M a_M \omega
\]

where \( a \in \mathfrak{k} \) and \( a_M \) is the vector field on \( M \) arising from the infinitesimal action of \( a \). Throughout this paper we will assume that \( \mu \) is proper, and in this case call \( M \) a proper Hamiltonian \( K \)-space.

It is well known that if 0 is a regular value of \( \mu \) then the topological space \( M_0 = \mu^{-1}(0)/K \) can naturally be given the structure of a compact symplectic orbifold. When 0 is not a regular value Lerman and Sjamaar show in \([SL9\] \) that \( M_0 \) is a stratified symplectic space. We briefly describe the stratification but ignore the Poisson structure on the functions since this is unnecessary for our topological applications.

Let us put \( Z = \mu^{-1}(0) \). Suppose \( H \) is a compact subgroup of \( K \) with Lie algebra \( \mathfrak{h} \). Let \( Z_H \) be the subset of points of \( Z \) whose stabilizer is precisely \( H \). We also define subsets of \( Z \)

\[
Z_\mathfrak{h} = \{ z \in Z : \text{Lie Stab } z = \mathfrak{h} \} \quad \text{and} \quad Z_{(\mathfrak{h})} = \{ z \in Z : \text{Lie Stab } z \in (\mathfrak{h}) \}
\]

where \((\mathfrak{h})\) is the set of subalgebras of \( \mathfrak{k} \) conjugate to \( \mathfrak{h} \).

For each nonempty subset \( Z_{(\mathfrak{h})} \) of \( Z \) fix once and for all a representative \( \mathfrak{h} \) of the conjugacy class \((\mathfrak{h})\) and let \( H \) be the generic stabiliser of a point in \( Z_\mathfrak{h} \) (so that \( \text{Lie } (H) = \mathfrak{h} \)). The set \( I_Z \) of these Lie subalgebras forms an indexing set for a topological stratification \( S_\mathfrak{f} \) of \( M_0 \) by infinitesimal orbit types. The strata are the sets consisting of those points representing orbits in the subsets \( Z_{(\mathfrak{h})} \) for \( \mathfrak{h} \in I_Z \). These strata are symplectic orbifolds, with a symplectic structure induced from \( \omega \). This stratification makes \( M_0 \) into a compact topological pseudomanifold — see \([MS99, §3.1.1] \) for details.
2.1. Relation with algebraic quotients. There is a well known principle that symplectic and algebraic quotients are ‘the same’. Suppose $V$ is a smooth complex projective variety with a given embedding $V \hookrightarrow \mathbb{P}^n$. Let $G$ be a reductive algebraic group which acts on $V$ via a homomorphism $\rho : G \to GL_{n+1}$. Geometric invariant theory constructs a categorical quotient $V^{ss}/G$ of the Zariski open subset $V^{ss}$ of semistable points.

Let $K$ be a maximal compact subgroup of $G$. We may assume, by conjugating if necessary, that $K$ maps into $U_{n+1}$ under the homomorphism $\rho$. The Fubini-Study form on $\mathbb{P}^n$ restricts to a $K$-Kähler form on $V$ which is preserved by the action of $K$. Further the action of $K$ is Hamiltonian and there is a moment map $\mu : V \hookrightarrow \mathbb{P}^n \xrightarrow{\phi} \mathfrak{u}_{n+1}^* \xrightarrow{(dp)^*} \mathfrak{k}^*$ where $\phi$ is the moment map for the standard action of $U_{n+1}$ on $\mathbb{P}^n$.

**Theorem 2.1.** ([Kir84, Nes84]) The zero set $\mu^{-1}(0)$ of the moment map is contained within the semistable points $V^{ss}$. This inclusion induces a homeomorphism $\mu^{-1}(0)/K \to V^{ss}/G$.

More generally let $M$ be a proper Hamiltonian $K$-space. We fix a choice of compatible $K$-invariant almost complex structure $J$ and metric $g$ on $M$. Let $p \in Z_H \subset Z(h)$. Define

$$V_p = (T_p(Kp) \oplus JT_p(Kp))^\perp \leq T_pM$$

(2) $W_p = (V_p)_h^\perp \leq V_p$

where $\perp$ denotes the orthogonal complement with respect to $g$, and set

where $(V_p)_h$ is the subspace invariant under the infinitesimal $\mathfrak{h}$ action. $W_p$ has a Hermitian structure, induced from the triple $(J, g, \omega)$, with respect to which it becomes a Hamiltonian $H$-space. It follows from the local normal form for the moment map (see [Mar85, GS84]) that a neighbourhood of the point $q$ in $M_0$ representing the orbit $Kp$ is homeomorphic to the product of the reduction at zero of $W_p$ with $\mathbb{C}^r$ for some $r$. By a small extension of theorem 2.1 to the quasi-projective variety $W_p/H^\mathbb{C}$ with $\mathbb{C}^r$. Thus all symplectic reductions are locally homeomorphic to algebraic varieties.

2.2. Partial desingularisation. In [Kir85] Kirwan showed how a singular geometric invariant theory quotient may be canonically desingularised by a particular sequence of blowups. This approach was extended to singular reductions of symplectic manifolds with proper moment map in [MS99]. Lack of canonicity in the definition of symplectic blowup means the resulting desingularisation is certainly not unique up to symplectomorphism. However it is unique up to homeomorphism and as we are interested in purely topological questions in this paper this is quite sufficient for our purposes.

Suppose that $M$ is a symplectic manifold with a Hamiltonian action of a compact connected Lie group $K$ for which the moment map $\mu : M \to \mathfrak{k}^*$ is proper. Let $\mathfrak{h} \in I_Z$ be a Lie subalgebra of $\mathfrak{k}$ indexing a stratum $S \in \mathcal{S}_\mathfrak{k}$ of maximal depth.

**Proposition 2.2** (partial desingularisation). We can find a continuous surjection $\tilde{M}_0 \to M_0$ where $\tilde{M}_0$ is the reduction of a proper Hamiltonian $K$-space $\tilde{M}$ such that
1. the restriction \( \widetilde{M}_0 \setminus \pi^{-1}S \rightarrow M_0 \setminus S \) is a homeomorphism;
2. if \( p \in \mathbb{Z}_H \subset \mathbb{Z}(b) \) and \( q \) its image in \( S \) then \( \pi^{-1}(q) \) is homeomorphic to \( \mathbb{P}W_p//H^C \) where \( W_p \leq T_pM \) is the Hermitian subspace defined in \( (3) \);
3. \( \widetilde{M}_0 \) has a stratification indexed by \( \tilde{I}_Z = I_Z \setminus \{ b \} \).

Applying this proposition inductively we can find a symplectic orbifold, arising as the regular reduction of a proper Hamiltonian \( K \)-space, which we call the partial desingularisation of \( M_0 \). The reader is referred to \([\text{MS99}]\) for a precise uniqueness statement. As remarked above, the only fact we will use is that it is unique up to homeomorphism.

### 3. The Kirwan Map

For a regular symplectic reduction \( M_0 \) of a proper Hamiltonian \( K \)-space \( M \) the Kirwan map is defined to be the composition

\[
H_K^*(M) \rightarrow H_K^*(Z) \cong H^*(M_0)
\]

of restriction to the zero set \( Z \) of the moment map and the natural isomorphism of the equivariant cohomology of \( Z \) with the cohomology of \( M_0 \). One of its most important properties is that it is surjective, see \([\text{Kir85}, 3.10]\). In this section we generalise this and define a map

\[
H_K^*(M) \rightarrow IH^*(M_0)
\]

to the intersection cohomology of the reduction even when the reduction is not regular — this idea was first discussed (for geometric invariant theory quotients) in \([\text{Kir86}]\).

We introduce the notation \( C_K(Z) \) for the derived push-forward of the constant sheaf with stalk \( \mathbb{Q} \) via the map \( EK \times_K Z \rightarrow M_0 \). Our aim is to construct a morphism \( \kappa : C_K(Z) \rightarrow IC^*(M_0) \) in \( \mathcal{D}(M_0) \). Inductively we may assume there is a morphism

\[
\bar{\kappa} : C_K(\bar{Z}) \rightarrow IC^*(\bar{M}_0)
\]

where \( \bar{Z} \) is the zero set of the moment map on \( \bar{M}_0 \). (In the base case of the induction when the reduction is regular there is a quasi-isomorphism.) Suppose that we have a ‘suitable’ morphism \( R\pi_*IC^*(\bar{M}_0) \rightarrow IC^*(M_0) \). We may then define \( \kappa \) by the composition

\[
C_K(Z) \rightarrow R\pi_*C_K(\bar{Z}) \rightarrow R\pi_*IC^*(\bar{M}_0) \rightarrow IC^*(M_0)
\]

of the equivariant pull-back \( \pi^* \), the morphism \( R\pi_*\bar{\kappa} \) and this suitable morphism. We then define the Kirwan map to be the composite

\[
H_K^*(M) \rightarrow H_K^*(Z) \rightarrow IH^*(M_0)
\]

where the second map is induced from \( \kappa \). In an abuse of notation we will often denote this second map by \( \kappa \) rather than the correct but clumsy \( H^*(M_0; \kappa) \).

How do we conjure up this ‘suitable’ morphism? Let us consider the inclusion \( \pi^{-1}S \rightarrow \bar{M}_0 \) where \( S \) was the maximal depth stratum of \( M_0 \) which we blew up in the first stage of the partial desingularisation. Here \( \pi^{-1}S \) plays the role of exceptional divisor in \( \bar{M}_0 \). In particular this inclusion is normally nonsingular of codimension 2. Thus, using the results of \([\text{GM83}, \S 5.4]\), we obtain a Gysin morphism

\[
R\pi_*IC^*(\bar{M}_0) \xrightarrow{\gamma} R\pi_*IC^*(\bar{M}_0)[2].
\]
Intuitively we think of $H^*(M_0; \gamma)$ as analogous to multiplication by the first Chern class of the normal bundle.

Recall from proposition 2.2 that if $q \in S$ is the image of $p \in Z_H$ then $\pi^{-1}(q)$ is homeomorphic to the geometric invariant theory quotient $\mathbb{P}W_p//H^C$. Further the embedding of $\pi^{-1}(q)$ into the restriction of the normal bundle of $\pi^{-1}S \to \tilde{M}_0$ to that fibre is homeomorphic to the embedding of the zero section into an anti-ample line bundle $L$ on $\mathbb{P}W_p//H^C$. There is an isomorphism

$$H^*_q(R\pi_!\mathcal{L}^*(\tilde{M}_0)) \cong H^*(\mathbb{P}W_p//H^C).$$

Let $r = \frac{1}{2}\text{codim } S = \frac{1}{2} \dim W_p//H^C$. By [BBDS84, §6] there is a hard Lefschetz theorem for the above intersection cohomology groups arising from the ample line bundle $L$.

For each $i < r$ whose embedding of $\pi_0$ is homeomorphic to the geometric invariant theory quotient $\text{Con}^\gamma$ and the local Lefschetz decomposition above are sufficient to induce a global decomposition

$$R\pi_!\mathcal{L}^*(\tilde{M}_0) \cong \mathcal{L}^*(M_0) \oplus \mathcal{B}^r$$

where $\mathcal{B}^r$ is supported on $S$. We will then take the projection to be the ‘suitable’ morphism. For ease of reading we put $\mathcal{A}^r = R\pi_!\mathcal{L}^*(\tilde{M}_0)$ and set

$$\mathcal{A}^i = (\tau_{<i} R\mu_! \mathcal{A}^r) \{2(i-r)\} \quad i = 1, \ldots, r - 1$$

where $\mu$ is the inclusion of $S$ in $M_0$. Consider the morphism $\mathcal{A}^i \oplus \ldots \oplus \mathcal{A}^{r-1} \overset{\alpha}{\longrightarrow} \mathcal{A}^r$ whose $i^{th}$ factor is

$$\mathcal{A}^i \to R\mu_! \mathcal{A}^r \{2(i-r)\} \overset{\gamma^{-i}}{\longrightarrow} \mathcal{A}^r.$$

For each $i < r - 2$ we also have a morphism $(1, -\gamma) : \mathcal{A}^i[2] \to \mathcal{A}^i+1 \oplus \mathcal{A}^i+2$ and hence can define

$$\mathcal{A}^i[2] \oplus \ldots \oplus \mathcal{A}^{r-2}[2] \overset{\beta}{\longrightarrow} \mathcal{A}^i \oplus \ldots \oplus \mathcal{A}^{r-1}$$

as the sum of these. It is clear that $\alpha \beta = 0$ so we can choose a factorisation

$$\mathcal{A}^i \oplus \ldots \oplus \mathcal{A}^{r-1} \overset{\alpha}{\longrightarrow} \mathcal{A}^r \quad \text{Con}^\gamma.$$

Since $\text{Con}^\gamma(\beta)$ is supported on $S$, we get a morphism $\text{Con}^\gamma(\varphi) \to i_* \iota^* \mathcal{A}^r = i_* \iota^* \mathcal{L}^*(M_0)$ where $i$ is the inclusion of the complement of $S$. Computing cohomology we find that $\text{Con}^\gamma(\varphi) \cong \tau_{<c} \text{Con}^\gamma(\varphi)$ and that the induced morphism $\text{Con}^\gamma(\varphi) \cong \tau_{<c} \text{Con}^\gamma(\varphi) \to \tau_{<c} \iota^* \mathcal{L}^*(M_0) \cong \mathcal{L}^*(M_0)$ is a quasi-isomorphism. Since $\pi$ is proper, Verdier duality applied to

$$R\pi_!\mathcal{L}^*(\tilde{M}_0) \to \text{Con}^\gamma(\varphi) \cong \mathcal{L}^*(M_0)$$

yields a morphism $\mathcal{L}^*(M_0) \to R\pi_!\mathcal{L}^*(\tilde{M}_0)$. Both are quasi-isomorphisms except on $S$ and it then follows from lemma [2.3] that they induce the desired splitting. Note that this decomposition is not canonical since we had to choose the lift $\varphi$. 
Pictorially we think of this decomposition as follows (with r=4):

\[
\begin{align*}
\text{initial} & = \text{mid 1} + \text{mid 2} + \text{viewport} - \text{final} \\
& = \text{mid 1} + \text{mid 2} + \text{viewport} + \text{viewport} - \text{final}
\end{align*}
\]

" \( R\pi_*\mathcal{IC}^*(\widetilde{M_0})|_S \cong \mathcal{IC}^*(M_0)|_S + \gamma \mathcal{A}_3^3 + \gamma^2 \mathcal{A}_2^2 + \gamma^3 \mathcal{A}_1^1 - \gamma^2 \mathcal{A}_1^1 \) "

**Conjecture 3.1.** The Kirwan map \( H^*_K(M) \to IH^*(M_0) \) is surjective.

There is various evidence suggesting that this conjecture should be true. First note that it follows from the equivariant Morse theory of the moment map that \( H^*_K(M) \to H^*_K(Z) \) is surjective (see [Kir84, 3.10]) so that what is in question is the surjectivity of \( H^*_K(Z) \to IH^*(M_0) \). In various special cases, for instance when the action of \( K \) is almost-balanced or \( K \) is a circle, we show below that this is surjective. In the GIT setting, the second author proved the surjectivity [Woo] by using the decomposition theorem. Recent work of Tolman suggests that this is so for arbitrary actions by higher dimensional tori.

4. The cosupport condition

When \( K \) acts quasi-freely on \( Z \) there is a quasi-isomorphism \( \mathcal{IC}^*(M_0) \cong \mathcal{C}_K^*(Z) \). In other words \( \mathcal{C}_K^*(Z) \) satisfies both the support and cosupport conditions. Since the equivariant cohomology of a point with respect to a non-trivial connected group is infinite dimensional we can easily check that

\( \mathcal{C}_K^*(Z) \) satisfies support \( \iff \) \( K \) acts quasi-freely on \( Z \).

It is very natural to ask: when does \( \mathcal{C}_K^*(Z) \) satisfy cosupport? Again there is a pleasant geometric interpretation of this condition.

**Definition 4.1.** For each connected component of a stratum of \( M_0 \) fix a point with generic stabiliser in the preimage of the component in \( Z \). We say that the action satisfies \( C(j) \) if, for each such point \( p \),

\[
\dim(W_p/\mathcal{H}^C) + j < 2 \min\{\operatorname{codim}_{W_p} S \mid S \in \mathcal{U}_p\},
\]

where \( \mathcal{H} = \text{Stab} p \), the subspace \( W_p \) of \( T_pM \) is as in [3] and \( \mathcal{U}_p \) is the set of unstable strata in the Morse stratification of \( \mathbb{P}W_p \) by the gradient flow of the norm square of the moment map associated to the \( \mathcal{H} \) action. Equivalently, using the connection with geometric invariant theory outlined in §2.1, we can express this as

\[
\dim(W_p/\mathcal{H}^C) + j < 2 \operatorname{codim}_{W_p}\{x \in W_p \mid 0 \in \overline{\mathcal{H}^C x}\}.
\]

Note that \( \{x \in W_p \mid 0 \in \overline{\mathcal{H}^C x}\} \) is \( \varphi_p^{-1}(0) \) where \( \varphi_p : W_p \to W_p/\mathcal{H}^C \) is the (algebraic) quotient map. This definition is independent of the chosen points in the preimages of the connected components of the strata.

**Lemma 4.2.** \( \mathcal{C}_K^*(Z) \in D^m_{\geq j} (M_0) \iff \) the action satisfies \( C(j) \).
Proof. Suppose \( p \) is in the preimage in \( Z \) of a point \( q \) in a connected component \( j_S : S \rightarrow M_0 \) of a stratum, and that \( H = \text{Stab} \, p \). There is a long exact sequence

\[
\ldots \rightarrow H^i_W(j_{S!} C_K(Z)) \rightarrow H^i_H(W_p) \rightarrow H^i_H(W_p \setminus \varphi_p^{-1}(0)) \rightarrow \ldots
\]

We see that \( \text{codim} \, \varphi_p^{-1}(0) = \min\{i | H^i_W(j_{S!} C_K(Z)) \neq 0 \} \). By the definition of \( D_{\geq j}^{n^+}(M_0) \) we see that \( C_K(Z) \) lies in this subcategory if, and only if,

\[
\text{codim} \, \varphi_p^{-1}(0) > \frac{1}{2} \text{codim} \, S + j = \frac{1}{2} \dim(W_p//H^C) + j
\]

which is precisely \( C(j) \).

In particular, since \( C_K(Z) \) satisfies cosupport if, and only if, it lies in the subcategory \( D_{\geq 0}^{n^+}(M_0) \), we have

\[
C_K(Z) \text{ satisfies cosupport } \iff \text{ the action satisfies } C(0).
\]

**Lemma 4.3.** \( C(0) \Rightarrow \tau_{\leq 0}^{n^+} C_K(Z) \cong \mathcal{I}C^*(M_0) \Rightarrow C(-1) \).

Proof: This follows from the triangle associated to \( \tau_{\leq 0}^{n^+} C_K(Z) \rightarrow C_K(Z) \) and the above lemma.

**Remark 4.4.** If the condition \( C(0) \) is satisfied, we say the \( K \) action on \( M \) is almost-balanced. The terminology came from the fact that \( C(0) \) can be viewed as a condition on the distribution of the weights of the maximal torus action on the normal space to each stratum [Kiea]. For example, if the weights of the maximal torus action of \( H \) on \( W_p \) is symmetric with respect to the origin, for each \( \mathfrak{h} \in I_Z \) and \( p \in Z_h \), then the \( K \) action is almost balanced. This is the case for the quotient construction yielding the moduli space of bundles on a Riemann surface ([Kiea] Proposition 7.3). We compute a simpler example in §4.4.

When the action is almost-balanced the above lemma shows that there is a natural morphism \( \lambda : \mathcal{I}C^*(M_0) \cong \tau_{\leq 0}^{n^+} C_K(Z) \rightarrow C_K(Z) \) which we interpret as a generalisation of the pull-back which exists when the action of \( K \) on \( Z \) is quasi-free.

**Proposition 4.5.** Suppose the action of \( K \) on \( M \) is almost balanced. Then \( \kappa \lambda = \text{id} \) and so the intersection cohomology of \( M_0 \) is naturally identified with a subspace of the equivariant cohomology of \( Z \).

Proof: It is easy to check that the composition

\[
\mathcal{I}C^*(M_0) \xrightarrow{\lambda} C_K(Z) \xrightarrow{\kappa} \mathcal{I}C^*(M_0)
\]

is non-zero, but then it must then be a quasi-isomorphism by 1.4.

4.1. **Finding the right subspace.** In order to make practical use of the observation that \( IH^*(M_0) \) embeds into \( H_K^*(Z) \) when the action is almost-balanced we need to be able to give a concrete description of the image. To this end we introduce an auxiliary complex \( \mathcal{V}^*(Z) \).

Suppose \( \mathfrak{h} \) is in the indexing set \( I_Z \) for the stratification of \( M_0 \). Let \( Y_{(b)} \) be the closure in \( Z \) of the corresponding subset \( Z_{(b)} \), and similarly let \( Y_{\mathfrak{h}} \) be the closure of \( Z_{\mathfrak{h}} \). Let \( N^{H_0} \) be the normaliser in \( K \) of \( H_0 = \exp \mathfrak{h} \), the identity component of the generic stabiliser of points in \( Z_{\mathfrak{h}} \). Let \( N^{H_0} \) be the identity component of \( N^{H_0} \).
Notice that $Y_{(b)}$ is homeomorphic to $KY_b$ and consider the resolution
$$K \times_{N^H_0} Y_b \to KY_b = Y_{(b)}$$
which is an isomorphism over $KZ_b$. Since the normal subgroup $H_0$ of $N^H_0$ acts trivially upon $Y_b$, we get morphisms
\[(5)\] $C_K(Y_{(b)}) \to C^*_N H_0(Y_b) \cong R\psi_b^*(C^*_N H_0(Y_b) \otimes H^{H_0})$
where $H^{H_0}$ is the constant sheaf on $Y_b/N^H_0$ with stalk $H^{*}_{H_0}$ and $\psi_b$ is the quotient map $Y_b/N^H_0 \to Y_b/N^{H_0}$. Define the complex $\mathcal{L}_b(Z) \in D(M_0)$ to be the extension by zero to $M_0$ of
$$R\psi_b^*(C^*_N H_0(Y_b) \otimes \tau_{\geq n_b} H^{H_0})$$
where $n_b$ is half the real codimension of the stratum $Z_{(b)}/K$ of $M_0$.

**Definition 4.6.** Define $V'(Z)$ (up to quasi-isomorphism) by the distinguished triangle
$$V'(Z) \to C_K(Z) \to \bigoplus_{b \in I_Z} \mathcal{L}_b(Z)$$
where the morphism $C'_K(Z) \to \mathcal{L}_b(Z)$ is given by restricting to $Y_{(b)}$, applying the morphisms in (5) and truncating in the second factor.

Let $V^*(Z)$ be the image of the hypercohomology of $V'(Z)$ in $H^*_K(Z)$. Restriction to $Y_{(b)}$ composed with the morphisms (5) induces a map
$$H^*_K(Z) \to [H^*_N H_0/Y_b (Y_b) \otimes H^*_H]^{\pi_0 N^H_0}$$
where the square brackets denote the invariant part under the action of the finite group $\pi_0 N^H_0 \cong N^H_0/N_0^{H_0}$. We can check that $V^*(Z)$ consists of those classes in $H^*_K(Z)$ whose image under this map lies in the direct summand
$$[H^*_N H_0/Y_b (Y_b) \otimes H^*_H]^{\pi_0 N^H_0}$$
for each $Z \in I_Z$.

**Assumption 4.7.** For the remainder of this section we will make the technical assumption that the action is not only almost-balanced but in fact weakly-balanced i.e. not only does $C'_K(Z)$ satisfy cosupport (which means $C'_K(Z) \in D^{m_+}_{\geq 0} (M_0)$) but so does each of the $C^*_N H_0/Y_b (Y_b)$ (by which we mean $C^*_N H_0(Y_b) \in D^{m_+}_{\geq 0} (Y_b/N^{H_0})$, or equivalently $\mathcal{L}_b(Z) \in D^{m_+}_{\geq 1} (M_0)$). These latter conditions can be interpreted geometrically in a similar fashion to the interpretation of the almost-balanced condition — see [Kies]. The moduli space of bundles on a Riemann surface arises as a quotient by a weakly-balanced action and we compute another example in §4.4.

**Lemma 4.8.** Suppose the action of $K$ is weakly-balanced. Then there is a direct sum decomposition $V'(Z) \cong \mathcal{IC}'(M_0) \oplus \mathcal{F}'(Z)$.

**Proof.** It follows immediately from the weakly-balanced assumption that the morphism $\mathcal{IC}'(M_0) \cong \tau_{<0} C'_K(Z) \to C'_K(Z)$ can be factored through $V'(Z)$ since $\tau_{<0} \mathcal{L}_b(Z) = 0$. It is easy to check that the resulting composition
$$\mathcal{IC}'(M_0) \to V'(Z) \to C'_K(Z) \xrightarrow{\kappa} \mathcal{IC}'(M_0)$$
is not zero and so must then be a quasi-isomorphism by lemma 4.4. \qed
Thus the intersection cohomology $IH^*(M_0)$ embeds as a subspace of $V^*(Z) \subset H^*_K(Z)$. We now show that the image of the embedding is precisely this subspace.

**Theorem 4.9.** Suppose the action of $K$ is weakly-balanced. Then the restriction of the Kirwan map to $V^*(Z)$ is an isomorphism onto $IH^*(M_0)$.

4.2. **Proof of theorem 4.9.** Let $F^*(Z)$ be the image of the hypercohomology of $F^*(Z)$ in $H^*_K(Z)$. It follows from lemma 4.8 that it is sufficient to show that $F^*(Z)$ is zero. Inductively we will assume that $F^*(\tilde{Z})$ is zero, where

$$ F^*(\tilde{Z}) \cong IC^*(\tilde{M}_0) \oplus F^*(\tilde{Z}) $$

and $F^*(\tilde{Z})$ is the image of the hypercohomology of $F^*(\tilde{Z})$ in $H^*_K(\tilde{Z})$.

Let $S$ be the deepest stratum of $M_0$ (which is the centre of the first blowup in the partial desingularisation procedure) and set $r = \frac{1}{2}\text{codim } S$. We also introduce closed subsets $C \subset Z$ (for centre) and $E \subset \tilde{Z}$ (for exceptional divisor) which are respectively the preimages of $S$ under $Z \to M_0$ and $\tilde{Z} \to M_0 \to Z$.

**Lemma 4.10.** We can choose a factorisation

$$ V^*(Z) \xrightarrow{\alpha} \tau^S_{<r} C^*_K(Z) $$

$$ \xrightarrow{\beta} C^*_K(Z) $$

where $\tau^S_{<r}$ is the truncation over a closed subset functor (see \[3\]).

**Proof.** This follows immediately from definition 4.6 of $V^*(Z)$. ☐

**Lemma 4.11.** We can choose a morphism $\beta$ so that

$$ V^*(Z) \xrightarrow{\beta} R\pi_* V^*(\tilde{Z}) $$

$$ \xrightarrow{\beta} C^*_K(Z) \xrightarrow{\beta} R\pi_* C^*_K(\tilde{Z}) $$

commutes.

**Proof.** For each $h \in I_\tilde{Z}$, let $\tilde{Y}_h$ denote the $H_0 = \exp h$ fixed point set in $\tilde{Z}$ just like $Y_h$ in $Z$. Then since the map $\tilde{Z} \to Z$ is equivariant, we have the commutative diagram

$$ K \times_{N_{h_0}} \tilde{Y}_h \to K\tilde{Y}_h \to \tilde{Z} $$

$$ K \times_{N_{h_0}} Y_h \to K\tilde{Y}_h \to Z. $$

This induces the following commutative diagram

$$ C^*_K(Z) \xrightarrow{\beta} R\pi_* C^*_K(\tilde{Z}) $$

$$ \bigoplus_{h \in I_\tilde{Z}} L_h(Z) \xrightarrow{\beta} \bigoplus_{h \in I_\tilde{Z}} R\pi_* L_h(\tilde{Z}) $$
by assigning zero for $\eta \in I_2 \setminus I_2$. The claim now follows from the definitions. □

**Lemma 4.12.** If $\eta \in F^*(Z)$ then $0 = \pi^* \eta|_E \in H^*_K(E)$ where $E$ is defined above.

**Proof.** We can decompose $R\pi_* V'(\tilde{Z})$ into a direct sum $R\pi_* IC'(\tilde{M}_0) \oplus R\pi_* F'(\tilde{Z})$ using (8). It follows from lemma 4.8 that the composition

$$R\pi_* IC'(\tilde{M}_0) \oplus R\pi_* F'(\tilde{Z}) \cong R\pi_* V'(\tilde{Z}) \rightarrow R\pi_* C'_K(\tilde{Z}) \rightarrow R\pi_* IC'(\tilde{M}_0)$$

is a quasi-isomorphism on the first term. Now let us consider the composition

$$F'(Z) \rightarrow V'(Z) \xrightarrow{\beta} R\pi_* V'(\tilde{Z}) \cong R\pi_* IC'(\tilde{M}_0) \oplus R\pi_* F'(\tilde{Z}).$$

By the above and lemma 4.11 the projection onto the first term of the RHS is, up to a quasi-isomorphism, the same as

$$F'(Z) \rightarrow V'(Z) \rightarrow C'_K(Z) \rightarrow R\pi_* C'_K(\tilde{Z}) \rightarrow R\pi_* IC'(\tilde{M}_0).$$

In particular if we decompose $R\pi_* IC'(\tilde{M}_0)$ as in (9) into $IC'(\tilde{M}_0) \oplus B'$ then by definition $F'(Z) \rightarrow IC'(\tilde{M}_0)$ is zero and the choice of $\alpha$ in lemma 4.10 induces a unique lift of $F'(Z) \rightarrow B'$ to $\tau_{<r} B'$.

The inductive assumption that $F^*(\tilde{Z}) = 0$ now tells us that the two morphisms

$$\tau_{<r} C'_K(Z) \xrightarrow{\alpha'} \tau_{<r} C'_K(\tilde{Z}) \rightarrow R\pi_* C'_K(\tilde{Z}) \xrightarrow{\beta'} \tau_{<r} B'$$

induced by $\alpha$ and $\beta$ respectively must give the same map on hypercohomology.

The almost-balanced assumption guarantees that there is a quasi-isomorphism $\tau_{<r} C'_K(Z) \cong \tau_{<r} R\pi_* C'_K(Z)$, where $i$ is the inclusion of the open complement to $S$. So $\tau_{<r} R\pi_* C'_K(\tilde{Z})$ decomposes as $\tau_{<r} C'_K(Z) \oplus C'$ where $C'$ is defined (up to quasi-isomorphism) by the distinguished triangle

$$C' \rightarrow \tau_{<r} R\pi_* C'_K(\tilde{Z}) \rightarrow \tau_{<r} R\pi_* C'_K(Z).$$

Furthermore because $B'$ is supported on $S$ it includes into $\tau_{<r} R\pi_* C'_K(\tilde{Z})$ as a subobject of $C'$.

The final observation we require is that the spectral sequence computing the hypercohomology of $R\pi_* C'_K(\tilde{Z})|_S$ degenerates at the $E_2$-term. This follows from Deligne’s criterion and the surjectivity of the restriction $H_{0p}(P\,W_p) \rightarrow H_{0p}(P\,W_p^{ss})$. Hence the spectral sequence for $\tau_{<r} R\pi_* C'_K(\tilde{Z})|_S$ must also degenerate and there is an injection

$$(7) \quad H^*(S; \tau_{<r} R\pi_* C'_K(\tilde{Z})|_S) \rightarrow H^*(S; R\pi_* C'_K(\tilde{Z})|_S) \cong H^*_K(E)$$

where $E$ is the ‘exceptional divisor’ in $\tilde{Z}$. But now we are done for we have shown that the image of the class $\eta \in F^*(\tilde{Z})$ pulled back to $H^*_K(E)$ lies in two subspaces, namely $H^*(S; \tau_{<r} C'_K(Z)|_S)$ and $H^*(S; C')$, whose intersection is zero. □
We can now complete the proof of theorem 4.13. Let \( \eta \in F^*(Z) \). By lemmas 4.11 and 4.12 we see that \( \pi^*\eta \in H^*_K(\tilde{Z}) \) lies in the subspace \( V^*(\tilde{Z}) \cong H^*(\tilde{M}_0) \) and restricts to zero on \( E \). Furthermore by the definition of \( F^*(Z) \) we know that it projects to zero in the summand \( IH^*(M_0) \) of \( IH^*(\tilde{M}_0) \). It follows from 4.13 and the fact that \( B' \) is supported on \( S \) that we must have \( \pi^*\eta = 0 \).

Studying the proof of lemma 4.12 more carefully we see that we can actually conclude that not only is \( \pi^*\eta|_E = 0 \) but that \( \eta|_C = 0 \) where \( C \subset Z \) is the ‘centre’ of the blowup defined above on page 11 since it lies in \( \mathbb{H}^*(S; \tau_{\mathcal{C}_K}^S\mathcal{C}_K(Z)|_S) \) which injects into \( H^*_K(E) \). Finally we note that

\[
\ker(H^*_K(Z) \to H^*_K(\tilde{Z})) \cap \ker(H^*_K(Z) \to H^*_K(C)) = 0
\]

so that \( \eta = 0 \) as desired. (8) follows from the observation that if \( \tilde{M} \to U \) is the blowup of a neighborhood of \( Z \) along a submanifold containing \( C \) such that \( \tilde{M}/K = M_0 \), then the nonminimal strata in \( \tilde{M} \) with respect to the norm square of the moment map retract onto those in the exceptional divisor \( \tilde{E} \) and hence \( \ker(H^*_K(M) \to H^*_K(\tilde{Z})) \cong \ker(H^*_K(\tilde{E}) \to H^*_K(E)) \).

4.3. The intersection pairing. Since \( M_0 \) is compact (by the properness assumption on the moment map) there is an intersection pairing

\[
IH^*(M_0) \otimes IH^*(M_0) \to \mathbb{Q}.
\]

This arises from a morphism \( \mathcal{IC}^*(M_0)^{\otimes 2} \to \mathcal{IC}_i(M_0) \) where the latter is the intersection cohomology complex with top perversity \( t(S) = \text{codim}(S) \). When the action is almost-balanced there is a second natural morphism

\[
\mathcal{IC}^*(M_0)^{\otimes 2} \to C^*_K(Z)^{\otimes 2} \to C^*_K(Z) \to \mathcal{IC}^*(M_0) \to \mathcal{IC}_i(M_0)
\]

induced from the product on equivariant cohomology.

**Lemma 4.13.** These two morphisms \( \mathcal{IC}^*(M_0)^{\otimes 2} \to \mathcal{IC}_i(M_0) \) are the same.

**Proof.** It is easy to check that \( \mathcal{IC}^*(M_0)^{\otimes 2} \in \mathcal{D}^-_{\leq 0}(M_0) \). Then by lemma 4.4 we have

\[
\text{Hom}(\mathcal{IC}^*(M_0)^{\otimes 2}; \mathcal{IC}_i(M_0)) \cong \mathbb{Q}.
\]

Since on the non-singular stratum the two morphisms are both just the product on the constant sheaf with stalk \( \mathbb{Q} \) we are done. \( \square \)

**Lemma 4.14.** Let \( \Sigma \) be the union of the singular strata of \( M_0 \). Then

\[
\mathbb{H}^\dim M_0(\Sigma; \mathcal{IC}^*(M_0)^{\otimes 2}) = 0.
\]

**Proof.** Note that the maximal degree in which the hypercohomology of a complex \( \mathcal{C} \) does not vanish is bounded by

\[
\max \{ i + \dim \text{supp}H^i(\mathcal{C}) \}.
\]

Now since \( \mathcal{IC}^*(M_0)^{\otimes 2} \in \mathcal{D}^-_{\leq 0}(M_0) \) we see that this quantity is strictly less than \( \max \{ \dim S_\alpha + t(S_\alpha) \} = \dim M_0 \). \( \square \)

**Remark 4.15.** Let \( d = \dim M_0 \). It follows from the above lemma that the product on equivariant cohomology gives us maps \( V^i(Z) \otimes V^{d-i}(Z) \to V^d(Z) \) and, by a similar argument, that \( V^d \) is the image of the pull-back \( H^d(M_0) \to H^*_K(Z) \).
**Definition 4.16.** We say a coadjoint orbit $O$ is close to 0 if $\mu^{-1}(O)$ is contained in the set $M^{ss}$ of points which flow to $Z$ under the gradient flow of the norm square of the moment map. (This depends upon the choices of compatible metric on $M$ and invariant metric on $t$.)

**Proposition 4.17.** Suppose the coadjoint orbit $O$ is close to 0 and contains a regular value of $\mu$ (and hence consists entirely of regular values). Let $\eta, \zeta \in H^{*}_{K}(M)$ be two classes whose restrictions to $H^{*}_{K}(Z)$ lie in $V^{*}(Z)$. Then the intersection pairing of $\kappa(\eta|_{Z})$ with $\kappa(\zeta|_{Z})$ in $IH^{*}(M_{0})$ is given by

$$\langle \eta|_{\mu^{-1}(O)}| \zeta|_{\mu^{-1}(O)} \rangle.$$

**Proof.** Since $\mu^{-1}(O) \subset M^{ss}$ the flow induces an equivariant map $\mu^{-1}(O) \to \mu^{-1}(0) = Z$ which is a homeomorphism between dense open sets and so we have

$$H^{*}_{K}(Z) \leftarrow H^{*}(M_{0}) \quad \text{and} \quad H^{*}_{K}(\mu^{-1}(O)) \leftarrow H^{*}(\mu^{-1}(O)/K).$$

Let $d = \dim M_{0}$. By lemma 4.13 and remark 4.12 we also have a commutative diagram

$$V^{i}(Z) \otimes V^{d-i}(Z) \leftarrow IH^{i}(M_{0}) \otimes IH^{d-i}(M_{0}) \quad \text{and} \quad V^{d}(Z) \leftarrow H^{d}(M_{0})$$

where the horizontal maps are isomorphisms. The result follows by composing these two diagrams and noting that $H^{d}(M_{0}) \to H^{d}(\mu^{-1}(O)/K)$ preserves evaluations against the fundamental class because $\mu^{-1}(O)/K \to M_{0}$ is ‘birational’. \qed

**4.4. Example.** (cf. [Kir86, §4]) Let $M = (\mathbb{P}^{1})^{n}$ acted on diagonally by $SU_{2}$ (in the obvious way). Identifying $\mathbb{P}^{1}$ with $S^{2} \subset \mathbb{R}^{3}$ and $su_{2}$ with $\mathbb{R}^{3}$ we have a moment map

$$\mu : M \to \mathbb{R}^{3}$$

$$(x_{1}, \ldots, x_{n}) \mapsto \sum x_{i}$$

for this action. We can check that this action is almost-balanced. The zero set $Z = \mu^{-1}(0)$ consists of $n$-tuples of points balanced about the origin. Let $T \subset SU_{2}$ be the maximal torus

$$\left( \begin{array}{cc} e^{it} & 0 \\ 0 & e^{-it} \end{array} \right).$$

The Weyl group of $SU_{2}$ is $\mathbb{Z}_{2}$ and $H^{*}_{SU_{2}}(M) = [H^{*}_{T}(M)]^{\mathbb{Z}_{2}}$ is the $\mathbb{Z}_{2}$-invariant part of $H^{*}_{T}(M)$. The $T$-equivariant cohomology is generated by classes $\alpha_{1}, \ldots, \alpha_{n}, \beta$ in degree 2 subject to the relations $\alpha_{i}^{2} = \beta^{2}$. The Weyl group acts trivially on the $\alpha_{i}$ and takes $\beta$ to $-\beta$. Hence

$$H^{*}_{SU_{2}}(M) \cong \mathbb{C}[\alpha_{1}, \ldots, \alpha_{n}, \beta^{2}]/(\alpha_{i}^{2} = \beta^{2} : i = 1, \ldots, n).$$

The only non-discrete subgroups which have fixed point are conjugates of $T$. The fixed points of $T$ correspond to sequences $(a_{1}, \ldots, a_{n})$ where $a_{i} = \pm 1$ respectively.
according to whether we take 0 or $\infty$ in the $i^{th}$ $\mathbb{P}^1$. If $F$ is the fixed point corresponding to $(a_1, \ldots, a_n)$ then $\mu_T(F) = \sum a_i$, the $T$-equivariant Euler class $e_T$ of the normal bundle to $F$ is $(\prod a_i)^{\beta_1}$ and $\alpha|_F = a_i \beta \in H^r_T(F)$.

Since $\mu(F) = 0$ implies that $\mu_T(F) = 0$ it is clear that the reduction at zero is seriously singular (i.e. has worse than orbifold singularities) if, and only if, $n$ is even. Put $n = 2m$. We note that the singularities of the reduction $M_0$ of $M$ are isolated. So the restriction of $\eta \in H^*_2(M)$ to $Z$ lies in $V^*$ if, and only if, either $\deg \eta < 2m - 3$ or $\eta|_{Z^{(1)}} = 0$. Furthermore it follows from the Poincaré polynomial calculations of [Kir84, §5] that the restriction

$$H^r_{SU_2}(M) \longrightarrow H^r_{SU_2}(Z)$$

is an isomorphism for $r < 2m$. We deduce that, for $r < 2m - 3$, there is an isomorphism $V^r \cong H^r_{SU_2}(M)$.

The reduction $M_0$ can, via geometric invariant theory, be given a description as a projective algebraic variety. In particular its intersection cohomology will satisfy the hard Lefschetz theorem i.e. there are isomorphisms

$$IH^{2m-3-i}(M_0) \cong IH^{2m-3+i}(M_0)$$

given by powers of a map $IH^r(M_0) \rightarrow IH^{r+2}(M_0)$. A moment’s thought shows that on $V^*$ this map must be given by multiplication by the Kähler class $\omega = \sum \alpha_j$. (Note that any multiple of $\omega$ restricts to an element of $V^*$ since

$$\omega|_{Z^{(1)}} = \sum \sum a_j \beta = 0.$$)

Finally since the equivariant cohomology of $M$ vanishes in odd degree we deduce that so does that of $M_0$ and, in particular, $IH^{2m-3}(M_0) = 0$. Hence

$$IH^r(M_0) \cong \begin{cases} H^r_{SU_2}(M) & r = 2m - 3 - i, \; i \geq 0 \\ \omega^i H^r_{SU_2}(M) & r = 2m - 3 + i, \; i \geq 0 \end{cases}$$

with the intersection pairing given by taking the product in $H^*_{SU_2}(M)$ and evaluating against the fundamental class of the reduction of $M$ at any regular coadjoint orbit close to 0. The full Lefschetz decomposition can be computed by considering the action of powers of $\omega$.

5. Circle actions

As we would expect the circle case is far simpler than the general one. Let $M$ be a proper Hamiltonian $S^1$-space, $Z$ be the zero set of an equivariant moment map and $M_0 = Z/S^1$ be the reduction. Let $\mathcal{F}_0$ be the set of fixed point components of $S^1$ which lie in $Z$. Each such component corresponds to an isomorphic singularity in $M_0$. For each $F \in \mathcal{F}_0$ define $d(F)$ (respectively $e(F)$) to be the minimum (respectively maximum) of the numbers of positive and negative weights of $S^1$ on normal fibre $W_F$ to $F$ in $M$.

The normal fibre to the singularity corresponding to $F \in \mathcal{F}_0$ in $M_0$ is given by the quotient $W_F/\mathbb{C}^*$. We can easily check that removing the vertex we have

$$(W_F \setminus \{0\})/\mathbb{C}^* \cong (W^+_F \setminus \{0\}) \times_{\mathbb{C}^*} (W^-_F \setminus \{0\})$$

where $W^\pm_F$ are the positive and negative weight spaces. This is the total space of some $\mathbb{C}^{d(F)} \setminus \{0\}$ bundle over a weighted projective space of dimension $d(F) - 1.$
Define a perversity \( n \) by
\[
 n(S) = \begin{cases} 
 0 & \text{codim } S = 0 \\
 2d(F) - 1 & S \cong F.
\end{cases}
\]

**Lemma 5.1.** There is a quasi-isomorphism \( \mathcal{IC}^*(M_0) \cong \mathcal{IC}_n(M_0) \) and \( C_K(Z) \) lies in the full subcategory \( D_{\geq 0}^+(M_0) \).

**Proof.** Both statements follow almost immediately from the above calculation of the normal fibre. \( \square \)

We now have the same situation as in the last section but with the perversity \( n \leq m \) replacing the middle perversity \( m \). Arguing analogously we deduce

**Theorem 5.2.** The intersection cohomology \( IH^*(M_0) \) is isomorphic (via the Kirwan map) to the subspace
\[
\{ \eta \in H^*_S(Z) \mid \eta|_F \in H^*(F) \otimes H^\leq 2d(F)-1_S \}.
\]

The intersection pairing is given by taking the product of classes and evaluating against the fundamental class of either of the shift resolutions \( M_{n+\epsilon} \) where \( 0 \neq \epsilon \in \mathbb{R} \).

**Corollary 5.3.** The intersection Betti numbers of \( M_0 \) are given by the Poincaré polynomial
\[
P^S_t(Z) - \frac{1}{1-t^2} \sum_{F \in R_0} t^{2d(F)} P_t(F).
\]

**Example 5.4.** We consider a linear circle action on the projective space \( \mathbb{P}^n \). Let \( p, q \) and \( r \) denote the number of positive, negative, and zero weights respectively, so that \( n = p + q + r - 1 \). Let us assume that \( p \leq q \) (the other case being entirely similar). Using equivariant Morse theory, we get
\[
P^S_t(Z) = (P_t(\mathbb{P}^n) - t^{2q+2r} P_t(\mathbb{P}^{n-1}) - t^{2p+2r} P_t(\mathbb{P}^{q-1}))/ (1-t^2)
= (1 + t^2 + \cdots + t^{2p+2r-2} - t^{2q+2r} - \cdots - t^{2n})/(1-t^2).
\]

Hence, by the above corollary, the intersection Poincaré polynomial is
\[
P^S_t(Z) - \frac{t^{2p} P_t(\mathbb{P}^{q-1})}{1-t^2} = (1 - t^{2p})(1 - t^{2q+2r})/(1-t^2)^2
\]
which is a palindromic polynomial of degree \( 2n - 2 \).

Intersection pairings can be also computed by using the splitting above — see \[Kiea\].

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