Abstract

We introduce two remarkable identities written in terms of single commutators and anticommutators for any three elements of arbitrary associative algebra. One is a consequence of other (fundamental identity). From the fundamental identity, we derive a set of four identities (one of which is the Jacobi identity) represented in terms of double commutators and anticommutators. We establish that two of the four identities are independent and show that if the fundamental identity holds for an algebra, then the multiplication operation in that algebra is associative. We find a generalization of the obtained results to the super case and give a generalization of the fundamental identity in the case of arbitrary elements. For nondegenerate even symplectic (super)manifolds, we discuss analogues of the fundamental identity.

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1 Introduction

Algebras endowed with a bracket satisfying the Jacobi identity are currently used extensively in the formulations of classical and quantum theories. For example, the formulations of Classical Mechanics and Classical Field Theory are based on symplectic manifolds endowed with the Poisson bracket satisfying the Jacobi identity \[1, 2\]. One of the main ingredients in consistent formulation of quantum mechanics is the commutator of operators, which satisfies the Jacobi identity. The main object of General Relativity is the Riemann tensor, which in particular satisfies the Bianchi identity following from the Jacobi identity for covariant derivatives. The quantization of dynamical systems with constraints in the Hamiltonian formalism \[3, 4, 5, 6\] assumes the use of even symplectic supermanifolds \[7, 8\] endowed with superextension of the commutator and a Poisson bracket satisfying the generalized Jacobi identity. The quantization of dynamical systems with constraints in the Hamiltonian formalism \[7, 8\] assumes the use of even symplectic supermanifolds endowed with an antibracket (a super Poisson bracket \[9\]) that satisfies the generalized Jacobi identity. The list of such descriptions can be continued.

Here, we consider the Jacobi identities naturally existing for any associative algebra from a new standpoint (in our understanding). For this, we use a remarkable identity for any three elements of a given associative algebra presented in terms of only single commutators. The Jacobi identity written, as is known, in terms of double commutators and anticommutators follows from this identity. Using the anticommutator, we introduce a second (fundamental) identity for an arbitrary associative algebra written for three elements of the algebra in terms of single commutators and anticommutators. We show that the first identity is a consequence of the second. This allows speaking of this identity as a fundamental (basic) identity for any associative algebra. For identities (one of which is the Jacobi identity) in terms of double commutators and anticommutators can be derived from the fundamental identity. Among these identities, two are independent. For any algebra, we prove that if the fundamental identity is satisfied, then the multiplication operation is associative. We generalize the basic relations and statements to the case of superalgebras.

This paper is organized as follows. In Section 2, we introduce the fundamental identity written in terms of single commutators and anticommutators for arbitrary associative algebras and derive a set of four (reducible) identities in terms of double commutators and anticommutators. We show that an algebra endowed with the fundamental identity is an associative algebra. In Section 3, we extend the results obtained in the preceding sections to the super case. In Section 4, we study a new interesting identity that exists for nondegenerate symplectic (super)manifolds. In Section 5, we generalize the basic identities discussed in Section 2 to the case of an arbitrary number of elements. Finally, we make some concluding concluding remarks in Section 6.
2 Remarkable identities in associative algebras

We consider an arbitrary associative algebra \( A \) with elements \( X \in A \). Let \( T_i, i = 1, 2, ..., n \) be a basis in \( A \). Then there exists the decomposition \( X = x^i T_i \) for any element \( X \) in \( A \). Because \( T_i T_j \in A \), we have

\[
T_i T_j = F_{ij}^k T_k, \quad (2.1)
\]

where \( F_{ij}^k \) are structure constants of the algebra. In terms of \( F_{ij}^k \), the associativity conditions \( (XY)Z = X(YZ) \) is

\[
F_{ij}^n F_{mk}^l = F_{ij}^m F_{nk}^l. \quad (2.2)
\]

These constants can be uniquely represented as sum of symmetric and antisymmetric terms

\[
F_{ij}^k = \frac{1}{2} c_{ij}^k + \frac{1}{2} f_{ij}^k, \quad (2.3)
\]

where \( c_{ij}^k \) and \( f_{ij}^k \) have the symmetries

\[
c_{ij}^k = c_{ji}^k, \quad f_{ij}^k = -f_{ji}^k. \quad (2.4)
\]

The commutator \([\cdot, \cdot]\) and anticommutator \(\{\cdot, \cdot\}\) in this algebra are defined for any two elements \( X, Y \in A \) by the relations

\[
[X, Y] = XY - YX, \quad \{X, Y\} = XY + YX, \quad (2.5)
\]

which are elements of \( A \).

The Leibniz rules for the commutator and anticommutator follow from (2.5),

\[
[X, Y Z] = [X, Y] Z + Y [X, Z], \quad \{X, Y Z\} = \{X, Y\} Z - Y \{X, Z\}. \quad (2.6)
\]

It is clear that

\[
[T_i, T_j] = f_{ij}^k T_k, \quad \{T_i, T_j\} = c_{ij}^k T_k. \quad (2.7)
\]

The following remarkable identities written in terms of single commutators and anticommutators exist for any associative algebra \( A \) and for any \( X, Y, Z \in A \) (without any reference to a basis):

\[
[X, Y Z] + [Z, XY] + [Y, ZX] \equiv 0, \quad (2.8)
\]

\[
[X, Y Z] + \{Y, Z X\} - \{Z, XY\} \equiv 0. \quad (2.9)
\]

From these identities, we can derive a set of identities in terms of double commutators and anticommutators. In particular, the Jacobi identity

\[
[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] \equiv 0 \quad (2.10)
\]
follows from (2.8). Moreover, we can deduce the identity containing the anticommutator from (2.8),

\[ [X, \{Y, Z\}] + [Z, \{X, Y\}] + [Y, \{Z, X\}] \equiv 0 \]  

(2.11)

Similarly, we can derive the following identities from (2.9):

\[ [X, \{Y, Z\}] - \{Z, [X, Y]\} + \{Y, [Z, X]\} \equiv 0, \]

(2.12)

\[ [X, [Y, Z]] + \{Y, \{Z, X\}\} - \{Z, \{X, Y\}\} \equiv 0. \]

(2.13)

We note that identities (2.8) and (2.9) are not independent, because summing (2.9) over cyclic permutations gives identity (2.8). Therefore, we have reason to regard identity (2.9) as the fundamental identity in associative algebras because the identities (2.8), (2.10), (2.11), (2.12) and (2.13) can be derived from it. In turn, the set of identities (2.10) - (2.13) is not independent. Indeed, summing of (2.12) over cyclic permutations gives identity (2.11). The same operation applied to identity (2.13) leads to Jacobi identity (2.10) [10]. We note that identities (2.12), (2.13) were also discussed for associative algebras in [11]. It is clear that having identities (2.12) - (2.13) and explicit realization of commutator and anticommutator (2.5), we can reproduce fundamental identity (2.9).

We note that from associativity condition (2.2), we can derive analogues of identities (2.8)-(2.13) in terms of structure constants. In particular, the Jacobi identities for the antisymmetric parts \( f_{ij}^k \) of the structure constants have the form

\[ f_{ij}^m f_{mk}^n + f_{ki}^m f_{mj}^n + f_{jk}^m f_{ni}^n \equiv 0. \]

(2.14)

The identities containing symmetric and antisymmetric parts of structure constants can be written as

\[
\begin{align*}
&c_{ij}^m f_{mk}^n + c_{ki}^m f_{mj}^n + c_{jk}^m f_{ni}^n \equiv 0, \\
&f_{ij}^m c_{mk}^n - f_{ki}^m c_{mj}^n + c_{jk}^m f_{mi}^n \equiv 0, \\
&c_{ij}^m c_{mk}^n - c_{ki}^m c_{mj}^n + f_{jk}^m f_{ni}^n \equiv 0.
\end{align*}
\]

(2.15)

The existence of fundamental identity (2.9) allows discussing the associativity of multiplication in the algebra from a new standpoint. Indeed, let \( \mathcal{A} \) be an algebra, then we can introduce the commutator and anticommutator by rules (2.5) as elements of \( \mathcal{A} \). We suppose that fundamental identity (2.9) is satisfied. We know that identities (2.11)-(2.13) follow from (2.9). We can then show that the multiplication in \( \mathcal{A} \) is associative. For this, we introduce the multiplication for any two elements \( X, Y \in \mathcal{A} \) as

\[ XY = \frac{1}{2} \left( [X, Y] + \{X, Y\} \right) \]

(2.16)
and we verify that equality

\[(XY)Z = X(YZ)\]  \hspace{1cm} (2.17)

holds. Indeed, from (2.16) that

\[(XY)Z = [[X,Y], Z] + \{X, \{Y, Z\}\} + \{X, Y\}, Z\}, \hspace{1cm} (2.18)
\]

\[X(YZ) = [X, [Y, Z]] + [X, \{Y, Z\}] + \{X, [Y, Z]\} + \{X, Y, Z\}\}. \hspace{1cm} (2.19)

For the difference between (2.18) and (2.19), we obtain

\[(XY)Z - X(YZ) = -[Z, [X, Y]] - [X, [Y, Z]] -
\]

\[-[Z, \{X, Y\}] - [X, \{Y, Z\}] + \{Z, [X, Y]\} - \{X, [Y, Z]\} +
\]

\[+\{Z, \{X, Y\}\} - \{X, \{Y, Z\}\}. \hspace{1cm} (2.20)\]

With identity (2.13) and Jacobi identity (2.10), which follows from it, taking into account, relation (2.20) becomes

\[(XY)Z - X(YZ) = -[Z, \{X, Y\}] - [X, \{Y, Z\}] + \{Z, [X, Y]\} - \{X, [Y, Z]\}. \hspace{1cm} (2.21)\]

Expressing \([Z, \{X, Y\}]\) and \([X, \{Y, Z\}]\) using identity (2.12), we now obtain associativity condition (2.17). Therefore, for any algebra \(\mathcal{A}\), fundamental identity (2.9) is equivalent to the associativity condition. We formulate this result as a theorem:

**Theorem 1.** For the associativity of multiplication in an algebra \(\mathcal{A}\), it is necessary and sufficient that identity (2.9), where the commutators and anticommutators are defined by rule (2.5), is satisfied.

Any associative algebra has the natural structure of a Lie algebra if the Lie bracket is defined in terms of associative multiplication. The converse does not hold in general: the Lie bracket in the general case does not allow introducing an associative multiplication. But the following theorem holds.

**Theorem 2.** For any algebra \(\mathcal{A}\) equipped with two bilinear operations \([\cdot, \cdot]\) and \(\{\cdot, \cdot\}\) with the symmetry properties:

\[[X, Y] = -[Y, X], \quad \{X, Y\} = \{Y, X\}, \hspace{1cm} (2.22)\]

and satisfying the identities (2.12) and (2.13), associative multiplication can be introduced by rule (2.16).

The proof of Theorem 2 is in fact contained in relations (2.17) - (2.21).

We note that the above results do not use assumptions about the finiteness or infiniteness of a system of basis vectors \(\{T_i\}\) of the algebra \(\mathcal{A}\) or even the existence of a basis at all.
3 Superextension

The results obtained in the preceding section can be extended to any associative superalgebras $\mathcal{A}_s$ with elements $X \in \mathcal{A}_s$ having the Grassmann parity $\varepsilon(X)$. Let $T_i, i = 1, 2, ..., n, \varepsilon(T_i) = \varepsilon_i$ be a basis in $\mathcal{A}_s$ such that there exist the decomposition $X = x^i T_i, \varepsilon(x^i) = \varepsilon(X) + \varepsilon_i$, for any element $X \in \mathcal{A}_s$. Because $T_i T_j \in \mathcal{A}_s$, we have

$$T_i T_j = F_{ij}^k T_k, \quad (3.1)$$

where $F_{ij}^k, (\varepsilon(F_{ij}^k) = \varepsilon_i + \varepsilon_j + \varepsilon_k)$, are structure constants. In terms of $F_{ij}^k$ the associativity condition $(XY)Z = X(YZ)$ is

$$F_{ij}^n F_{nk}^m = F_{jk}^n F_{in}^m (-1)^{\varepsilon_i + \varepsilon_j + \varepsilon_k}. \quad (3.2)$$

In the super case, the commutator $[\cdot, \cdot]$ and anticommutator $\{\cdot, \cdot\}$ are introduced for any two elements $X, Y \in \mathcal{A}_s$ by the relations

$$[X, Y] = XY - (-1)^{\varepsilon(X)\varepsilon(Y)} YX, \quad \{X, Y\} = XY + (-1)^{\varepsilon(X)\varepsilon(Y)} YX \quad (3.3)$$

with obvious symmetries properties

$$[X, Y] = -[Y, X](-1)^{\varepsilon(X)\varepsilon(Y)}, \quad \{X, Y\} = \{Y, X\}(-1)^{\varepsilon(X)\varepsilon(Y)}. \quad (3.4)$$

The Leibniz rules follow from (3.3)

$$[X, Y Z] = [X, Y] Z + Y[X, Z](-1)^{\varepsilon(X)\varepsilon(Y)},$$

$$\{X, Y Z\} = \{X, Y\} Z - Y\{X, Z\}(-1)^{\varepsilon(X)\varepsilon(Y)}. \quad (3.5)$$

For basis elements, we have

$$[T_i, T_j] = f_{ij}^k T_k, \quad \{T_i, T_j\} = c_{ij}^k T_k, \quad (3.6)$$

where structure constants $f_{ij}^k$ and $c_{ij}^k$ have the symmetry properties

$$c_{ij}^k = c_{ji}^k (-1)^{\varepsilon_i \varepsilon_j}, \quad f_{ij}^k = -f_{ji}^k (-1)^{\varepsilon_i \varepsilon_j}. \quad (3.7)$$

They can be identified with symmetric and antisymmetric parts of the structure coefficients $F_{ij}^k$

$$c_{ij}^k = F_{ij}^k + F_{ji}^k (-1)^{\varepsilon_i \varepsilon_j}, \quad f_{ij}^k = F_{ij}^k - F_{ji}^k (-1)^{\varepsilon_i \varepsilon_j}. \quad (3.8)$$

For any associative superalgebra $\mathcal{A}_s$ and for any $X, Y, Z \in \mathcal{A}_s$, we have the identities

$$[X, Y Z](-1)^{\varepsilon(X)\varepsilon(Z)} + [Z, XY](-1)^{\varepsilon(Z)\varepsilon(Y)} + [Y, ZX](-1)^{\varepsilon(Y)\varepsilon(X)} \equiv 0 \quad (3.9)$$

$$[X, Y Z](-1)^{\varepsilon(X)\varepsilon(Z)} + \{Y, ZX\}(-1)^{\varepsilon(Y)\varepsilon(X)} - \{Z, XY\}(-1)^{\varepsilon(Z)\varepsilon(Y)} \equiv 0, \quad (3.10)$$

For any two elements $X, Y \in \mathcal{A}_s$, we have the identity

$$\{X, Y\} = [X, Y] - (-1)^{\varepsilon(X)\varepsilon(Y)} [Y, X] \quad (3.11)$$

For any three elements $X, Y, Z \in \mathcal{A}_s$, we have the identity

$$\{X, Y Z\} = \{X, Y\} Z + (-1)^{\varepsilon(X)\varepsilon(Y)} Y\{X, Z\} \quad (3.12)$$
which generalize relations (2.8) and (2.9). Identity (3.9) can be derived from (3.10) and can be regarded as the fundamental identity for associative superalgebras. A set of identities in terms of double commutators and anticommutators is

\[
[X, [Y, Z]](-1)^{\delta(X)\delta(Z)} + [Z, [X, Y]](-1)^{\delta(Z)\delta(Y)} + [Y, [Z, X]](-1)^{\delta(Y)\delta(X)} = 0, \tag{3.11}
\]

\[
[X, \{Y, Z\]}(-1)^{\delta(X)\delta(Z)} + [Z, \{X, Y\}](-1)^{\delta(Z)\delta(Y)} + [Y, \{Z, X\}](-1)^{\delta(Y)\delta(X)} = 0, \tag{3.12}
\]

\[
[X, \{Y, Z\}](1)^{\delta(X)\delta(Z)} - \{Z, [X, Y]\}(-1)^{\delta(Z)\delta(Y)} + \{Y, [Z, X]\}(-1)^{\delta(Y)\delta(X)} = 0, \tag{3.13}
\]

\[
[X, [Y, Z]](-1)^{\delta(X)\delta(Z)} + \{Y, \{Z, X\}\}(-1)^{\delta(X)\delta(Y)} - \{Z, \{X, Y\}\}(-1)^{\delta(Z)\delta(Y)} = 0. \tag{3.14}
\]

Identities (3.11) and (3.12) respectively follow from (3.13) and (3.14) by summing over cyclic permutations.

The identities in terms of symmetric and antisymmetric parts of structure constants follow from associativity condition (3.2),

\[
f_{ij} n f_{nk} m (-1)^{\delta_i\delta_k} + f_{ki} n f_{nj} m (-1)^{\delta_k\delta_j} + f_{jk} n f_{mi} m (-1)^{\delta_j\delta_i} = 0, \tag{3.15}
\]

We can again consider the associativity of multiplication operation in superalgebras from a new standpoint. For this, we consider a superalgebra \(A_s\). We introduce the commutator and anticommutator by rule (3.3) as elements of \(A_s\). We suppose that fundamental identity (3.10) is satisfied. Using the representation of the multiplication for any two elements \(X, Y \in A_s\) in the form

\[
XY = \frac{1}{2} \left( [X, Y] + \{X, Y\} \right) \tag{3.16}
\]

and repeating the proof given in Section 2, we obtain the associativity of multiplication

\[
(XY)Z = X(YZ). \tag{3.17}
\]

Consequently, we have the following theorem.

**Theorem 3.** For associativity of multiplication in a superalgebra \(A_s\), it is necessary and sufficient that identity (3.10), where the commutators and anticommutators are defined by rule (3.3), is satisfied.
Any associative superalgebra has a natural structure of a Lie superalgebra if the Lie superbracket is defined in terms of associative multiplication by the formula 
\[ [X, Y] = XY - YX(-1)^{\varepsilon(X)\varepsilon(Y)}. \]
The converse does not hold, generally speaking: the Lie superbracket in the general case does not allow introducing an associative multiplication. But the following theorem holds.

**Theorem 4.** Let \( \mathcal{A}_s \) be a superalgebra equipped with two bilinear operations \([\cdot, \cdot]\) and \(\{\cdot, \cdot\}\) with the symmetry properties:

\[ [X, Y] = -(\varepsilon(X)\varepsilon(Y))[Y, X], \quad \{X, Y\} = (-1)^{\varepsilon(X)\varepsilon(Y)}\{Y, X\}. \tag{3.18} \]

If these operations satisfy identities (3.13) and (3.14), then an associative multiplication can be introduce in \( \mathcal{A}_s \).

Indeed, we define the multiplication \( X \circ Y \) by rule (3.16),

\[ X \circ Y = \frac{1}{2}([X, Y] + \{X, Y\}), \]

and apply the proof given in Section 2. We then obtain the associativity of this multiplication. In terms of this multiplication, the binary operations introduced above have the usual representation

\[ [X, Y] = X \circ Y - Y \circ X(-1)^{\varepsilon(X)\varepsilon(Y)}, \quad \{X, Y\} = X \circ Y + Y \circ X(-1)^{\varepsilon(X)\varepsilon(Y)}. \tag{3.19} \]

We note that the above consideration does not contain any assumptions concerning a basis in the superalgebra.

4 Nondegenerate symplectic (super)manifolds

We consider a nondegenerate symplectic supermanifold \((\mathcal{M}, \omega)\), where \(\mathcal{M}\) is a supermanifold and \(\omega\) is a nondegenerate closed 2-form with the Grassmann parity \(\varepsilon(\omega(X,Y)) = \varepsilon(X) + \varepsilon(Y) + \varepsilon(\omega)\), where \(X\) and \(Y\) are elements of the cotangent space of \(\mathcal{M}\). We speak of an even symplectic supermanifold if \(\varepsilon(\omega) = 0\) and of an odd symplectic supermanifold if \(\varepsilon(\omega) = 1\).

It is well known (see, e. g., [1]) that an even symplectic supermanifold is the foundation for describing classical dynamical systems in the Hamiltonian formalism. In turn, the covariant quantization of gauge theories (Batalin-Vilkovisky method [9]) is based on a nondegenerate odd symplectic supermanifolds. Any nondegenerate closed symplectic structure defines the Poisson superbracket \(\{F, G\}\) \((\varepsilon(\{F, G\}) = \varepsilon(F) + \varepsilon(G) + \varepsilon(\omega))\), which for any two scalar functions \(F, G\) on \(\mathcal{M}\) is a scalar under general changes of coordinates on \(\mathcal{M}\). The Poisson superbracket has the properties of antisymmetry

\[ \{F, G\} = -\{G, F\}(-1)^{(\varepsilon(F)+\varepsilon(\omega))(\varepsilon(G)+\varepsilon(\omega))} \tag{4.1} \]
the linearity
\[ \{F + G, H\} = \{F, H\} + \{G, H\} \tag{4.2} \]
and satisfied the Leibniz rule
\[ \{F, GH\} = \{F, G\}H + \{F, H\}G(-1)^{\varepsilon(H)\varepsilon(G)} \tag{4.3} \]
and the Jacobi identity
\[
\begin{align*}
\{F, \{G, H\}\} & - \{F, G\}H - \{F, H\}G + \{G, \{F, H\}\} \\
& \equiv 0,
\end{align*} \tag{4.4}
\]
which is consequence of the closedness of symplectic structure.

In even case \((\varepsilon(\omega) = 0)\), the Poisson superbracket coincides with the superextension of the Poisson bracket. In odd case \((\varepsilon(\omega) = 1)\), the Poisson superbracket is the antibracket, which is one of the fundamental operations in the BV quantization method \([9, 12]\) and it is known in mathematics as the Buttin bracket \([13]\).

We note that in even case \((\varepsilon(\omega) = 0)\), the identity
\[
\{F, \{G, H\}\}(-1)^{(\varepsilon(F)\varepsilon(H))} + \{H, \{F, G\}\}(-1)^{(\varepsilon(H)\varepsilon(G))} + \{G, \{H, F\}\}(-1)^{(\varepsilon(G)\varepsilon(F))} \equiv 0.
\tag{4.5}
\]
follows from (4.1) and (4.3). Unfortunately, in contrast to associative algebras, we cannot regard this identity as fundamental for nondegenerate closed even symplectic supermanifolds, because we cannot deduce Jacobi identity (4.4) from (4.5). Nevertheless if the canonical quantization is applied to a dynamical system for which the phase space is described by a nondegenerate even closed symplectic supermanifold, then the Poisson bracket is transformed into the commutator \(\{F, G\} \rightarrow (i\hbar)^{-1}[\hat{F}, \hat{G}]\), and the identity (4.5) reduces to (3.9) for the operators \(\hat{F}, \hat{G}, \hat{H}\).

5 Generalization of basic identities

We note that for any associative algebra, we have the identities
\[
[X_1, X_2X_3 \cdots X_n] + \text{cycle}(X_1, X_2, ..., X_n) \equiv 0, \quad n = 3, 4, ...
\tag{5.1}
\]
which generalize identity (2.8). The Jacobi identity
\[
[[X_1, X_2], X_3] + [[X_3, X_1], X_2] + [[X_2, X_3], X_1] \equiv 0
\]
follows from (5.1) for \(n = 3\)
\[
[X_1, X_2X_3] - [X_2, X_1X_3] + \text{cycle}(X_1, X_2, X_3) \equiv 0.
\tag{5.2}
\]
The generalized Jacobi identity for \( n = 4 \) was discussed in [14] and had the form
\[
[[[X_1, X_2], X_3], X_4] + [[[X_2, X_1], X_4], X_3] + [[[X_3, X_4], X_1], X_2] + [[[X_4, X_3], X_2], X_1] \equiv 0. \quad (5.3)
\]
This identity can be obtained from (5.1). Indeed, a direct verification shows that it in fact coincides with
\[
[X_1, X_2 X_3 X_4] - [X_2, X_1 X_3 X_4] + [X_4, X_3 X_2 X_1] - [X_4, X_3 X_1 X_2] +
+ cycle(X_1, X_2, X_3, X_4) \equiv 0. \quad (5.4)
\]
The generalization of identity (5.3) for \( n = 5, 6, ... \) was given in [15], and these generalized Jacobi identities can again be derived from fundamental identities (5.1).

We can also suggest a generalization of identity (2.9). For four elements, we have
\[
[X_1, X_2 X_3 X_4] - \{X_4, X_1 X_2 X_3\} + \{X_3, X_4 X_1 X_2\} + [X_2, X_3 X_4 X_1] \equiv 0. \quad (5.5)
\]
In general, for any associative algebra, there exist the identities
\[
[X_1, X_2 X_3 \cdots X_n] - \{X_n, X_1 X_2 \cdots X_{n-1}\} + \{X_{n-1}, X_n X_1 \cdots X_{n-2}\} +
+ [X_{n-2}, X_{n-1} X_n X_1 \cdots X_{n-3}] + [X_{n-3}, X_{n-2} X_{n-1} X_n X_1 \cdots X_{n-4}] +
+ \cdots + [X_2, X_3 \cdots X_n X_1] \equiv 0, \quad n \geq 4. \quad (5.6)
\]
The proof of (5.6) is based on the obvious identity
\[
[X_n, X_1 X_2 \cdots X_{n-1}] + [X_{n-1}, X_n X_1 \cdots X_{n-2}] +
+ \{X_n, X_1 X_2 \cdots X_{n-1}\} - \{X_{n-1}, X_n X_1 \cdots X_{n-2}\} \equiv 0, \quad n \geq 2. \quad (5.7)
\]
Applying identity (5.1) to (5.7), we then obtain identity (5.6). As in the case of (2.8), identity (5.1) can be derive from (5.6). For this, we sum (5.6) over cyclic permutations. We have
\[
((n - 2)[X_1, X_2 X_3 \cdots X_n] - \{X_n, X_1 X_2 \cdots X_{n-1}\} + \{X_{n-1}, X_n X_1 \cdots X_{n-2}\}) +
+ cycle(X_1, X_2, ..., X_n) \equiv 0. \quad (5.8)
\]
But
\[
(\{X_n, X_1 X_2 \cdots X_{n-1}\} - \{X_{n-1}, X_n X_1 \cdots X_{n-2}\}) + cycle(X_1, X_2, ..., X_n) \equiv 0
\]
and identity (5.1) follows from (5.8). This allows speaking about identities (5.6) as fundamental for any associative algebra.
6 Conclusions

We have discussed identity (2.9) for arbitrary associative algebra and analog (3.10) of this identity for an arbitrary associative (super)algebra. We proposed regarding these identities as fundamental because they are described in terms of single commutators and anticommutators in contrast to the identities for algebras usually discussed (see [10, 11]), which are in fact consequences of these identities.

We proved (Theorem 2 (Theorem 4)) that any algebra or superalgebra endowed with two bilinear operations (commutator and anticommutator) satisfying identities (2.12) and (2.13) or (3.13) and (3.14) is an associative algebra or superalgebra. We can stress that there were no assumptions in the proof concerning the finiteness or even the existence of a basis for a given algebra or superalgebra.

We introduced identity (4.4) for any nondegenerate even symplectic supermanifold and discussed an application of this identity in the canonical quantization of dynamical systems.

Finally, we proposed a generalization of the basic identities to the case of an arbitrary number of elements involved in these relations (see (5.1) and (5.6)).

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1Note that for algebras this fact was proved in [10].
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