PÓLYA CONJECTURE FOR THE NEUMANN EIGENVALUES

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Abstract. For a given bounded domain $\Omega \subset \mathbb{R}^n$ with $C^1$-smooth boundary, we prove the Pólya conjecture for the Neumann eigenvalues. In other words, we prove that

$$\mu_{k+1} \leq \frac{(2\pi)^2 k^{2/n}}{(\omega_n \cdot \text{vol}(\Omega))^{2/n}}$$

for all $k = 0, 1, 2, 3, \ldots$, where $\mu_k$ is the $k$-th Neumann eigenvalue of the Laplacian for $\Omega$.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with smooth boundary $\partial \Omega$. We consider the following Neumann eigenvalue problem:

\[
\begin{cases}
- \Delta v = \mu v & \text{in } \Omega, \\
\frac{\partial v}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where $\nu$ denotes the outward unit normal vector to $\partial \Omega$. As is well-known, the spectrum of the Neumann problem is discrete and consists of a sequence $\{\mu_k\}_{k=1}^{\infty}$ of eigenvalues (with finite multiplicity) written in increasing order according to their multiplicity:

$$0 = \mu_1 < \mu_2 \leq \cdots \leq \mu_k \leq \cdots \nearrow +\infty.$$ (1.2)

Weyl’s asymptotic formula (see [19] or [20]) says that

$$\mu_{k+1} \sim \frac{(2\pi)^2 k^{2/n}}{(\omega_n \cdot \text{vol}(\Omega))^{2/n}}$$ as $k \to \infty$, (1.3)

where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$ and vol$(\Omega)$ is the volume of $\Omega$. In 1961, G. Pólya [13] showed that for any regularly plane-covering domain (a plane-covering domain is one which can be used to tile the plane without gaps or overlaps but allowing rotations, and reflections of the fundamental domain)

$$\mu_{k+1} \leq \frac{(2\pi)^2 k^{2/n}}{(\omega_n \cdot \text{vol}(\Omega))^{2/n}},$$ for $k = 0, 1, 2, 3, \ldots$, (1.4)

(see also Kellner [6] for the complete proof of (1.4) for plane-covering domain). In [13], Pólya went on to conjecture that these inequalities hold for all domains.

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Despite a great deal of attention (see Payne’s review articles [11], [12], Yau’s problem list [21] and other works [22], [23], [24], Protter’s review article [15], [16] and even Pólya popular book [14]), this conjecture remain unsolved (see [1]). The only case of (1.4) that is known for arbitrary domains is \( k = 1 \) (using P. Szegő-Weinberger inequality (see [17] and [18]) for \( (\text{vol}(\Omega))^{2/n} \mu_2 \)).

More information on this problem can be found in the literatures [8], [1], [24], [11] and [12].

In this paper, by considering an equivalent eigenvalue problem of multi-Laplacian \((-\Delta)^m\) and by applying Kröger’s technique (see [7]), and finally by letting \( m \to +\infty \), we prove the Pólya conjecture for the Neumann eigenvalues. Our main result is the following:

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^n \ (n \geq 2) \) be a bounded domain with \( C^1 \)-smooth boundary \( \partial \Omega \), and let \( 0 = \mu_1 < \mu_2 \leq \mu_3 \leq \cdots \leq \mu_k \leq \cdots \) be the Neumann eigenvalues of the Laplacian on \( \Omega \). Then

\[
\mu_{k+1} \leq \frac{(2\pi)^{k/2}}{(\omega_n \cdot \text{vol}(\Omega))^{2/n}} \quad \text{for all } k = 0, 1, 2, 3, \cdots .
\]

2. Proof of main theorem

**Proof of theorem 1.1.** (i) We first assume that \( \Omega \) is a bounded domain with smooth boundary. In this case, the eigenvalue problem (1.1) is equivalent to the following eigenvalue problem of the multi-Laplacian:

\[
\begin{align*}
(-\Delta)^m \phi &= \mu^m \phi & \text{in } \Omega, \\
\frac{\partial ((-\Delta)^{m-1} \phi)}{\partial \nu} &= 0 & \text{on } \partial \Omega, \ l = 1, 2, \cdots, m.
\end{align*}
\]

In other words, if \( \phi_k \) is the eigenfunction corresponding to the \( k \)-th Neumann eigenvalue \( \mu_k \), then \( \mu_k \) (respectively, eigenfunction \( \phi_k \)) must be the \( k \)-th eigenvalue (respectively, eigenfunction \( \phi_k \)) of problem (2.1). The converse is still true.

Applying Green’s formula, we see that the eigenvalues \( \mu_k \) is given by the following variational formulas:

\[
\mu_1 = 0 \quad \text{and } \phi_1 \equiv 1,
\]

\[
\mu_k = \frac{\int_{\Omega} |(-\Delta)^{m/2} \phi_k(x)|^2 \, dx}{\int_{\Omega} |\phi_k(x)|^2 \, dx} = \inf_{\phi \in C^\infty(\Omega), \frac{\partial \phi}{\partial \nu}|_{\partial \Omega} = \cdots = \frac{\partial((-\Delta)^{m-1} \phi)}{\partial \nu}|_{\partial \Omega} = 0} \frac{\int_{\Omega} |(-\Delta)^{m/2} \phi(x)|^2 \, dx}{\int_{\Omega} |\phi(x)|^2 \, dx}, \quad k = 2, 3, \cdots,
\]

where

\[
|(-\Delta)^{m/2} \phi|^2 = \begin{cases} |\Delta^{m/2} \phi|^2 & \text{if } m \text{ is even,} \\ \frac{1}{2} |\nabla ((-\Delta)^{(m-1)/2} \phi)|^2 & \text{if } m \text{ is odd.}
\end{cases}
\]

Let \( \{\phi_j\}_{j=1}^k \) be the set of orthonormal eigenfunctions corresponding to the Neumann eigenvalues \( \{\mu_j\}_{j=1}^k \). By the regularity of elliptic equations, we known that \( \phi_j \in C^\infty(\Omega) \) for every \( j \geq 1 \). Following from Li-Yau’s method (see [9] or [7]), we consider the function defined by

\[
\Phi(x, y) = \sum_{j=1}^k \phi_j(x) \phi_j(y) \quad x, y \in \Omega.
\]
The projection of $h_z(y) \equiv e^{i(y,z)}$ onto the subspace of $L^2(\Omega)$ spanned by $\phi_1, \ldots, \phi_k$ can be written in terms of the Fourier transform $\hat{\Phi}$ of $\Phi$ with respect to the $x$-variable:

$$\int_\Omega \Phi(x,y) e^{i(x,z)} \, dx = (2\pi)^{-n/2} \hat{\Phi}(z,y).$$

Since $\phi_k \in C^\infty(\Omega)$ is the $k$-th eigenfunction corresponding to $\mu_k^m$ for (2.1), we obtain an upper bound for $\mu_{k+1}^m$ is given by

$$\inf_r \frac{\int_{B_r} \int_\Omega |\Delta_y^{m/2}(h_z(y) - (2\pi)^{n/2} \hat{\Phi}(z,y))|^2 \, dy \, dz}{\int_{B_r} \int_\Omega |h_z(y) - (2\pi)^{n/2} \hat{\Phi}(z,y)|^2 \, dy \, dz},$$

where $B_r$ denotes the ball with radius $r$ and center 0 in $\mathbb{R}^n$ for an arbitrary $r$ with

$$r > 2\pi \left( \frac{k}{\omega_n \cdot \text{vol}(\Omega)} \right)^{1/n}.$$

Thus, we have

$$\int_{B_r} \int_\Omega |\Delta_y^{m/2}(h_z(y) - (2\pi)^{n/2} \hat{\Phi}(z,y))|^2 \, dy \, dz$$

$$= \int_{B_r} \int_\Omega |\Delta_y^{m/2}h_z(y)|^2 \, dy \, dz$$

$$- 2 \, \text{Re} \int_{B_r} \int_\Omega \langle (-\Delta_y)^{m/2}(h_z(y) - (2\pi)^{n/2} \hat{\Phi}(z,y)), (-\Delta_y)^{m/2}((2\pi)^{n/2} \hat{\Phi}(z,y)) \rangle \, dy \, dz$$

$$- (2\pi)^n \int_{B_r} \int_\Omega |\Delta_y^{m/2}\hat{\Phi}(z,y)|^2 \, dy \, dz.$$

Firstly, in view of the boundary conditions, we find by applying Green’s formula that

$$\int_{B_r} \int_\Omega \langle (-\Delta_y)^{m/2}(h_z(y) - (2\pi)^{n/2} \hat{\Phi}(z,y)), (-\Delta_y)^{m/2}(\hat{\phi}_j(z)\phi_j(y)) \rangle \, dy \, dz$$

$$= \int_{B_r} \int_\Omega (h_z(y) - (2\pi)^{n/2} \hat{\Phi}(z,y)) \mu_j^m \hat{\phi}_j(z)\phi_j(y) \, dy \, dz = 0,$$

so that

$$-2 \, \text{Re} \int_{B_r} \int_\Omega \langle (-\Delta_y)^{m/2}(h_z(y) - (2\pi)^{n/2} \hat{\Phi}(z,y)), (-\Delta_y)^{m/2}((2\pi)^{n/2} \hat{\Phi}(z,y)) \rangle \, dy \, dz = 0.$$

Next, from a direct calculation we obtain

$$\int_{B_r} \int_\Omega |\Delta_y^{m/2}h_z(y)|^2 \, dy \, dz = \int_{B_r} |z|^{2m} \text{vol}(\Omega) \, dz = \frac{r^{n+2m}}{n+2m} \left( n\omega_n \cdot \text{vol}(\Omega) \right),$$
where \(n\omega_n\) is the \((n-1)\)-dimensional volume of \(\partial B_1 = \{x \mid x \in \mathbb{R}^n, \sqrt{x_1^2 + \cdots + x_n^2} = 1\}\). Finally, we have

\[
(2.5) \quad (2\pi)^n \int_{B_r} \int_{\Omega} |\Delta^{m/2}_y \hat{\Phi}_k(z, y)|^2 \, dy \, dz
\]

\[
= (2\pi)^n \int_{B_r} \int_{\Omega} |(-\Delta_y)^{m/2} \hat{\Phi}_k(z, y)|^2 \, dy \, dz
\]

\[
= (2\pi)^n \int_{B_r} \int_{\Omega} \left( \sum_{j=1}^k \hat{\phi}_j(z) \phi_j(y) \right) \left( \sum_{j=1}^k (-\Delta_y)^m \phi_j(z) \phi_j(y) \right) \, dy \, dz
\]

\[
= (2\pi)^n \sum_{j=1}^k \mu_j^m \int_{B_r} |\hat{\phi}_j(z)|^2 \, dz.
\]

Combining (2.2) - (2.6), we obtain

\[
(2.7) \quad \mu_{k+1}^m \leq \frac{n+2}{n} \left( n\omega_n \cdot \text{vol}(\Omega) \right) - \frac{(2\pi)^n \sum_{j=1}^k \mu_j^m \int_{B_r} |\hat{\phi}_j(z)|^2 \, dz}{(2\pi)^n \sum_{j=1}^k \int_{B_r} |\hat{\phi}_j(z)|^2 \, dz}.
\]

Similar to [7], by the induction assumption

\[
(2.8) \quad \mu_k^m \leq \frac{(r^{n+2m}/(n+2m)) \left( n\omega_n \cdot \text{vol}(\Omega) \right) - \sum_{j=1}^{k-1} \mu_j^m}{(r^n/n) \left( n\omega_n \cdot \text{vol}(\Omega) \right) - (k-1)},
\]

we ready to prove the following claim that

\[
(2.9) \quad \mu_{k+1}^m \leq \inf_{r > 2^n (k/(\omega_n \cdot \text{vol}(\Omega)))^{1/n}} \frac{(r^{n+2m}/(n+2m))(n\omega_n \cdot \text{vol}(\Omega)) - (2\pi)^n \sum_{j=1}^k \mu_j^m}{(r^n/n)(n\omega_n \cdot \text{vol}(\Omega)) - (2\pi)^n k}.
\]

From (2.8), we have

\[
(2.10) \quad \mu_k^m \leq \frac{\left( (r^{n+2m}/(n+2m)) \left( n\omega_n \cdot \text{vol}(\Omega) \right) - \sum_{j=1}^{k-1} \mu_j^m \right) - \mu_k^m}{\left( (r^n/n) \left( n\omega_n \cdot \text{vol}(\Omega) \right) - (k-1) \right) - 1}
\]

Noticing that \(0 \leq \int_{B_r} |\hat{\phi}_k(z)|^2 \, dz < 1\), we find by (2.10) that

\[
(2.11) \quad \frac{\sum_{j=1}^k \mu_j^m (1 - \int_{B_r} |\hat{\phi}_j(z)|^2 \, dz)}{\sum_{j=1}^k (1 - \int_{B_r} |\hat{\phi}_j(z)|^2 \, dz)} \leq \frac{(r^{n+2m}/(n+2m)) \left( n\omega_n \cdot \text{vol}(\Omega) \right) - \sum_{j=1}^k \mu_j^m}{(r^n/n) \left( n\omega_n \cdot \text{vol}(\Omega) \right) - k},
\]
which implies
\[
\frac{(r^{n+2m}/(n + 2m)) \frac{n \omega \cdot \text{vol}(\Omega)}{(2\pi)^m} - \sum_{j=1}^{k} \mu_j^m}{((r^n/n) \frac{n \omega \cdot \text{vol}(\Omega)}{(2\pi)^n} - k) + \sum_{j=1}^{k} (1 - \int_{B_r} |\hat{\phi}_j(z)|^2 dz)} 

\leq \frac{(r^{n+2m}/(n + 2m)) \frac{n \omega \cdot \text{vol}(\Omega)}{(2\pi)^m} - \sum_{j=1}^{k} \mu_j^m}{(r^n/n) \frac{n \omega \cdot \text{vol}(\Omega)}{(2\pi)^n} - k}.
\]

Combining this and (2.7), we obtain that the claim (2.9) is true.

In (2.9), if we estimate \( \sum_{j=1}^{k} \mu_j^m \) below by 0 and if we put
\[
r = (2\pi) \left( \frac{(1 + \frac{1}{m}) k}{\omega_n \cdot \text{vol}(\Omega)} \right)^{\frac{1}{n}},
\]
then
\[
\mu_{k+1} \leq \frac{n(m+1)}{n + 2m} (1 + \frac{1}{m})^2 (2\pi)^2 \frac{1}{(\omega_n \cdot \text{vol}(\Omega))} \left( \frac{2m}{k} \right)^{2/n}, \quad \text{for all } m \geq 1.
\]
That is,
\[
\mu_{k+1} \leq \left( \frac{n(m+1)}{n + 2m} \right)^{\frac{1}{n}} (1 + \frac{1}{m})^2 (2\pi)^2 \left( \frac{1}{\omega_n \cdot \text{vol}(\Omega)} \right)^{2/n} k^{2/n}, \quad \text{for all } m \geq 1.
\]
Note that
\[
\lim_{m \to +\infty} \left( \frac{n(m+1)}{n + 2m} \right)^{\frac{1}{n}} (1 + \frac{1}{m})^2 = 1.
\]
In inequality (2.12), by letting \( m \to +\infty \) we conclude that
\[
\mu_{k+1} \leq (2\pi)^2 \left( \frac{1}{\omega_n \cdot \text{vol}(\Omega)} \right)^{2/n} k^{2/n}.
\]

(ii) For any bounded domain with \( C^1 \)-smooth boundary, we can choose a sequence \( \{\Omega_p\}_{p=1}^{\infty} \) of bounded domains with smooth boundaries such that \( \Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_p \subset \cdots \subset \Omega \) and \( \Omega_p \) converge to \( \Omega \) as \( p \to +\infty \). By this property and inequality (2.13), we have inequality (1.4) for arbitrary bounded domain with \( C^1 \)-smooth boundary. \( \square \)

**Remark 2.1.** (a) Let \( \Omega \) be bounded domain in \( \mathbb{R}^n \) \( (n \geq 2) \), and let \( 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \) be the eigenvalues of the Dirichlet boundary problem for the Laplace operator:
\[
\begin{cases}
-\Delta u = \lambda u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]
In 1912, Weyl [19] or [20] established the asymptotic formula
\[
\lambda_k \sim (2\pi)^2 \left( \frac{1}{\omega_n \cdot \text{vol}(\Omega)} \right)^{2/n} k^{2/n} \quad \text{as } k \to +\infty.
\]
Pólya [13] in 1961 conjectured that
\[
\lambda_k \geq (2\pi)^2 \left( \frac{1}{\omega_n \cdot \text{vol}(\Omega)} \right)^{2/n} k^{2/n} \quad \text{for all } k.
\]
Li and Yau [9] proved that the eigenvalues $\lambda_k$ satisfy the inequality

$$\lambda_k \geq \frac{n}{n+2} \frac{(2\pi)^2}{(\omega_n \text{vol}(\Omega))^{2/n}} k^{2/n} \quad \text{for all } k,$$

which gave a partial answer for the Pólya conjecture with a factor. In fact, Li and Yau in [9] established a sharp inequality concerning the sum of the first eigenvalues

$$\sum_{j=1}^{k} \lambda_j \geq \frac{n}{n+2} (2\pi)^2 \left( \frac{n}{\omega_n \text{vol}(\Omega)} \right)^{2/n} k^{1+\frac{2}{n}} \quad \text{for all } k.$$

If we add an assumption $\partial \Omega \in C^\infty$ for the bounded domain $\Omega$, and if we consider the equivalent multi-Laplacian eigenvalue problem:

$$\begin{cases}
(-\Delta)^m u = \lambda^m u & \text{in } \Omega, \\
(-\Delta)^{l-1} u = 0 & \text{on } \partial \Omega,
\end{cases}$$

for $l = 1, \ldots, m,$

(2.16)

then by completely similar to Li-Yau’s method (see [9]) for the above operator $(-\Delta)^m$ with the corresponding Dirichlet boundary conditions one can immediately get that

$$\lambda^m_k \geq \left( \frac{n}{n+2m} \right) \frac{(2\pi)^{2m} k^{2m}}{(\omega_n \cdot \text{vol}(\Omega))^{(2m)/n}}.$$

(2.17)

This inequality implies the Pólya conjecture for the Dirichlet eigenvalues by letting $m \to +\infty$ (see [5]). Finally, using a sequence $\{\Omega_p\}_{p=1}^\infty$ of bounded domains with smooth boundaries such that $\Omega_p$ converge to $\Omega$ as $p \to +\infty$, it follows that the Pólya conjecture for the Dirichlet eigenvalues is true for arbitrary bounded domain $\Omega$.

(b) From (1.5) and (2.15), we immediately get that for every bounded domain $\Omega \subset \mathbb{R}^n$ with $C^1$-smooth boundary, the following inequality holds

$$\mu_{k+1} \leq \lambda_k \quad \text{for } k = 1, 2, 3, \ldots$$

(2.18)

which is a famous inequality and has been proved by Friedlander in [4] (see also Mazzeo [10]).

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