PACKING INDEX OF SUBSETS IN POLISH GROUPS

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Abstract. For a subset $A$ of a Polish group $G$, we study the (almost) packing index $\text{ind}_P(A)$ (resp. $\text{Ind}_P(A)$) of $A$, equal to the supremum of cardinalities $|S|$ of subsets $S \subseteq G$ such that the family of shifts $\{xA\}_{x \in S}$ is (almost) disjoint (in the sense that $|xA \cap yA| < |A|$ for any distinct points $x, y \in S$). Subsets $A \subseteq G$ with small (almost) packing index are small in a geometric sense. We show that $\text{ind}_P(A) \in \mathbb{N} \cup \{\aleph_0, c\}$ for any $\sigma$-compact subset $A$ of a Polish group. If $A \subseteq G$ is Borel, then the packing indices $\text{ind}_P(A)$ and $\text{Ind}_P(A)$ cannot take values in the half-interval $[\text{sq}(\Pi^1_1), c)$ where $\text{sq}(\Pi^1_1)$ is a certain uncountable cardinal that is smaller than $c$ in some models of ZFC. In each non-discrete Polish Abelian group $G$ we construct two closed subsets $A, B \subseteq G$ with $\text{ind}_P(A) = \text{ind}_P(B) = c$ and $\text{Ind}_P(A \cup B) = 1$ and then apply this result to show that $G$ contains a nowhere dense Haar null subset $C \subseteq G$ with $\text{ind}_P(C) = \text{Ind}_P(C) = \kappa$ for any given cardinal number $\kappa \in [4, c]$.

1. Introduction

Given a Polish group $G$ and a non-empty subset $A \subseteq G$ with nice descriptive properties, we study all possible values of the packing index

$$\text{ind}_P(A) = \sup \{|S| : S \subseteq G \text{ and } \{xA\}_{x \in S} \text{ is disjoint}\}$$

of $A$, which indicates the smallness of the subset $A$ in a geometric sense. The papers [BL1], [BL2], [L1] and [L2] are devoted to constructing subsets with a given packing index. In particular, for every non-zero cardinal number $\kappa \leq c$ one can easily construct a subset $A \subseteq \mathbb{R}$ with $\text{ind}_P(A) = \kappa$. After discussing those results on the topological seminar at Wroclaw University the second author was asked by Krysztof Omiljanowski about possible restrictions on the packing index $\text{ind}_P(A)$ of subsets $A \subseteq \mathbb{R}$ having good descriptive properties (like being compact $\sigma$-compact, Borel, measurable or meager). This question was probably motivated by the well-known fact that the Continuum Hypothesis (though irresolvable in ZFC) has positive solution in the realm of Borel subsets of the real line: each uncountable Borel subset $A \subseteq \mathbb{R}$ has cardinality $c$ of continuum.

In this paper we shall give several partial answers to Omiljanowski’s question. On the one hand, we show that $\sigma$-compact subsets $A$ in Polish groups cannot have an intermediate packing index $\aleph_0 < \text{ind}_P(A) < c$. For a Borel subset $A$ of a Polish group we have a weaker result: $\text{ind}_P(A)$ cannot take the value in the interval $\text{sq}(\Pi^1_1) \leq \text{ind}_P(A) < c$ where $\text{sq}(\Pi^1_1)$ stands for the smallest cardinality $\kappa$ such that each coanalytic subset $X \subseteq 2^\omega \times 2^\omega$ contains a square $S \times S$ of size $|S \times S| = \kappa$ provided $X$ contains a square of size $\geq \kappa$. The value of the small uncountable

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cardinal $sq(\Pi^1_1)$ is not completely determined by ZFC Axioms: both the equality $sq(\Pi^1_1) = \mathfrak{c}$ and the strict inequality $sq(\Pi^1_1) < \mathfrak{c}$ are consistent with Martin Axiom, see [Sh].

On the other hand, for every infinite cardinal number $\kappa \leq \mathfrak{c}$ in each non-discrete Polish Abelian group $G$ we shall construct a nowhere dense Haar null subset $A \subset G$ with $\text{Ind}_P(A) = \text{Ind}_P(A) = \kappa$. Here $$\text{Ind}_P(A) = \sup\{|S| : S \subset G \text{ and } \{xA\}_{x \in S} \text{ is almost disjoint}\}$$ is the almost packing index of $A$. In the above definition, a family of shifts $\{xA\}_{x \in S}$ is defined to be almost disjoint if $|xA \cap yA| < |A|$ for all distinct $x, y \in S$.

To construct the nowhere dense Haar null subset $A \subset G$ with a given (almost) packing index, in each non-discrete Polish Abelian group $G$ we first construct a closed nowhere dense Haar null subset $C \subset G$ with $\text{Ind}_P(C) = 1$. The set $C$, being nowhere dense and Haar null, is small in the sense of category and measure, but is large in the geometric sense because for any two distinct points $x, y \in G$ the shifts $xC$ and $yC$ have intersection of cardinality $|xC \cap yC| = |G|$. In particular, $CC^{-1} = G$ and thus $C$ is a closed nowhere dense Haar null subset that algebraically generates the group $G$. This result can be seen as an extension of a result of S. Solecki [So] who proved that each non-locally compact Polish Abelian group $G$ is algebraically generated by a nowhere dense subset. Also it extends some results of [BP] §13. In fact, the closed Haar null subset $C \subset G$ with $\text{Ind}_P(C) = 1$ is constructed as the union $C = A \cup B$ of two closed subsets $A, B \subset G$ with $\text{Ind}_P(A) = \text{Ind}_P(B) = \mathfrak{c}$. This shows that the packing index is highly non-additive.

**Notation.** By $\omega$ we denote the first infinite ordinal, $\mathbb{N} = \omega \setminus \{0\}$ stands for the set of positive integers. Cardinals are identified with the initial ordinals of given cardinality; $\mathfrak{c}$ stands for the cardinality of continuum. All topological groups $G$ considered in this paper will be supplied with an invariant metric $\rho$ generating the topology of $G$. By $e$ we denote the identity element of $G$. For a point $x \in G$ and a real number $r$ by $B(x, r) = \{g \in G : \rho(g, x) < r\}$ we denote the open $r$-ball centered at $x$. Also for $x \in G$ we put $\|x\| = \rho(x, e)$. The invariantness of $\rho$ implies $\|x^{-1}\| = \|x\|$ and $\|xy\| \leq \|x\| + \|y\|$ for all $x, y \in G$.

2. The packing indices of $\sigma$-compact sets in Polish groups

In this section we show that the packing index of a $\sigma$-compact subset in a Polish groups cannot take an intermediate value between $\omega$ and $\mathfrak{c}$.

First we prove a useful

**Lemma 1.** Let $A$ be a subset of a Polish group $G$. If $\text{Ind}_P(A) < \mathfrak{c}$, then the closure of $AA^{-1}$ contains a neighborhood of the neutral element $e$ of $G$.

**Proof.** Fix any complete metric $\rho$ generating the topology of the Polish group $G$. Assuming that $AA^{-1}$ is not a neighborhood of $e$, we shall construct a perfect subset $K \subset G$ such that $(xA)_{x \in K}$ is disjoint, which will imply that $\text{Ind}_P(A) = |K| = \mathfrak{c}$.

Taking into account that the closed subset $AA^{-1}$ is not a neighborhood of $e$ in $G$, for any open neighborhood $U$ of $e$ we can find a point $b \in U \setminus AA^{-1}$ and an open neighborhood $V$ of $e$ such that $V^{-1}bV \subset U \setminus AA^{-1}$.

Using this fact, by induction construct a sequence $(b_n)_{n \in \omega}$ of points in $G$ and sequences $(U_n)_{n \in \omega}$ and $(V_n)_{n \in \omega}$ of open neighborhoods of $e$ in $G$ such that
Proof. The set \((N \cup \{\omega\})\) and thus by the condition (4). Therefore, the sequence \((\varepsilon_n)\) implies that the map \(\text{id} = (\varepsilon_i)_{i \in \omega}\) the infinite product

\[ f(\varepsilon) = \prod_{i=0}^{\infty} b_i^\varepsilon_i = \lim_{n \to \infty} f_n(\varepsilon) \]

where \(f_n(\varepsilon) = \prod_{i=0}^{n} b_i^\varepsilon_i\). Let us show that the latter limit exists. It suffices to check that \((f_n(\varepsilon))_{n \in \omega}\) is a Cauchy sequence in \((G, \rho)\).

The condition (5) implies that \(U_{i+1}^2 \subset U_i^2\) for all \(i\). This can be used as the inductive step in the proof of the inclusion \(U_n \cdots U_m \subset U_n^2\) for all \(m \geq n\). Then for every \(m \geq n\)

\[ f_m(\varepsilon) = f_n(\varepsilon) \cdot U_{n+1} \cdots U_m \subset f_n(\varepsilon) \cdot U_{n+1}^2 \subset f_n(\varepsilon)V_{n+1} \]

and thus

\[ \rho(f_m(\varepsilon), f_n(\varepsilon)) \leq \text{diam}_\rho(f_n(\varepsilon) \cdot V_{n+1}) < 2^{-n} \]

by the condition (4). Therefore, the sequence \((f_n(\varepsilon))_{n \in \omega}\) is Cauchy and the limit \(f(\varepsilon) = \lim_{n \to \infty} f_n(\varepsilon)\) exists. Moreover, the upper bound \(\rho(f_m(\varepsilon), f_n(\varepsilon)) \leq 2^{-n}\) implies that the map \(f : \{0, 1\}^\omega \to G\) is continuous. On the other hand, the inclusions \(f_m(\varepsilon) \subset f_n(\varepsilon) \cdot U_n^2, m \geq n\), imply that

\[ f(\varepsilon) \subset f_n(\varepsilon) \cdot U_n^2 \subset f_n(\varepsilon) \cdot U_n^3 \subset f_n(\varepsilon) \cdot V_{n+1} \]

This inclusion will be used for the proof of the injectivity of \(f\). We shall prove a little bit more: for any distinct vectors \(\varepsilon\) and \(\delta\) in \(\{0, 1\}^\omega\) we get \(f(\varepsilon)A \cap f(\delta)A = \emptyset\).

Find the smallest number \(n \in \omega\) such that \(\varepsilon_n \neq \delta_n\). We lose no generality assuming that \(\delta_n = 0\) and \(\varepsilon_n = 1\). It follows that \(f(\varepsilon) = f_n(\varepsilon)V_{n+1} = f_n(\varepsilon)\cdot b_nV_{n+1}\) while \(f(\delta) = f_n(\delta)V_{n+1} = f_n(\delta)\cdot V_{n+1} = f_{n-1}(\varepsilon)V_{n+1}\). Then

\[ (f(\varepsilon))^{-1}f(\delta) \subset V_{n+1}^{-1}b_nV_{n+1} \subset G \setminus AA^{-1} \]

by the condition (2) and hence \(f(\varepsilon)A \cap f(\delta)A = \emptyset\).

Thus the family \((xA)_{x \in K}\) is disjoint where \(K = f(\{0, 1\}^\omega)\). The injectivity of \(f\) implies that \(\text{ind}_P(A) \geq |K| = \omega\). \(\square\)

Now we can prove the main result of this section.

**Theorem 1.** If \(A\) is a \(\sigma\)-compact subset \(A\) in a Polish group \(G\), then \(\text{ind}_P(A) \in \mathbb{N} \cup \{\omega, \chi\}\). Moreover, if the set \(A\) is compact, then

1. \(\text{ind}_P(A) = \omega\) if \(G\) is not locally compact;
2. \(\text{ind}_P(A) \in \{\omega, \chi\}\) if \(G\) is locally compact but not compact;
3. \(\text{ind}_P(A) \in \mathbb{N} \cup \{\chi\}\) if \(G\) is compact.

**Proof.** If \(A\) is \(\sigma\)-compact, then so is the set \(AA^{-1} = \{xy^{-1} : x, y \in A\}\) and then the set \((G \setminus AA^{-1}) \cup \{e\}\) is a \(G_\delta\)-set in \(G\). In its turn, the subset

\[ X = \{(x, y) \in G \times G : y^{-1}x \in (G \setminus AA^{-1}) \cup \{e\}\} \]
is of type $G_δ$ in $G \times G$, being the preimage of the $G_δ$-subset $(G \setminus AA^{-1}) \cup \{e\}$ under the continuous map $g : G \times G \to G$, $g : (x, y) \mapsto y^{-1}x$.

Assuming that $\text{ind}_P(A) > \aleph_0$, we could find an uncountable subset $S \subset G$ with disjoint family $\{xA\}_{x \in S}$, which implies that $S \times S \subset X$. Since the Polish space $X$ contains the uncountable square $S \times S$, we can apply the Shelah’s result [Sh, 1.14] to conclude that $X$ contains the square $P \times P$ of a perfect subset $P \subset G$ (the latter means that $P$ is closed in $G$ and has no isolated point). It follows from $P \times P \subset X$ that the family $\{xA\}_{x \in P}$ is disjoint and thus $\mathfrak{c} = |P| \leq \text{ind}_P(A) \leq |G| = \mathfrak{c}$.

Now assuming that $A \subset G$ is compact we shall prove the items (1)–(3) of the theorem. The compactness of $A$ implies the compactness of $AA^{-1}$. If $AA^{-1}$ is not a neighborhood of $e$, then we can apply Lemma 1 to conclude that $\text{ind}_P(A) = \mathfrak{c}$. This is so if the group $G$ is not locally compact. So, next we assume that $AA^{-1}$ is a neighborhood of $e$. In this case the group is locally compact and we can take a neighborhood $U \subset G$ of $e$ with $UU^{-1} \subset AA^{-1}$.

Then for every $B \subset G$ with $B^{-1}B \cap AA^{-1} = \{e\}$ we get $B^{-1}B \cap UU^{-1} = \{e\}$, which implies that the family $(xB)_{x \in B}$ is disjoint and the set $B$ is at most countable, being discrete in the Polish space $G$. This proves the upper bounds $\text{ind}_P(A) \leq \aleph_0$.

If the group $G$ is not compact, then using the Zorn Lemma, we can find a maximal set $B \subset G$ with $B^{-1}B \cap AA^{-1} = \{e\}$. We claim that $BAA^{-1} = G$. Assuming the converse, we could find a point $b \in G \setminus BAA^{-1}$. Then the set $bA$ is disjoint from the set $BA$ and hence we can enlarge the set $B$ to the set $\tilde{B} = B \cup \{b\}$ such that $(xA)_{x \in \tilde{B}}$ is disjoint. The latter is equivalent to $\tilde{B}^{-1}\tilde{B} \cap AA^{-1} = \{e\}$ and this contradicts the maximality of $B$. The compactness of $AA^{-1}$ and non-compactness of $G = BAA^{-1}$ implies that $B$ is infinite and thus $\text{ind}_P(A) \geq |B| \geq \aleph_0$. This completes the proof of the second item of the theorem.

To prove the third item, assume that $G$ is compact. In this case $G$ carries a Haar measure $\mu$ which is a unique probability invariant $\sigma$-additive Borel measure on $G$. If $AA^{-1}$ is not a neighborhood of $e$, then $\text{ind}_P(A) = \mathfrak{c}$ by a preceding case. So we assume that $AA^{-1}$ is a neighborhood of $e$ and take another neighborhood $U$ of $e$ with $UU^{-1} \subset AA^{-1}$. Since finitely many shifts of $U$ cover the group $G$, we get $\mu(U) > 0$. Now given any subset $B \subset G$ with $B^{-1}B \cap AA^{-1} = \{e\}$, we get $B^{-1}B \cap UU^{-1} = \{e\}$. The latter equality implies that the family $(xB)_{x \in B}$ is disjoint and then $1 = \mu(G) \geq \mu(BU) = |B|\mu(U)$ implies that $|B| \leq 1/\mu(U)$. Consequently, the packing index $\text{ind}_P(A) \leq 1/\mu(U)$ is finite. \[\square\]

In light of this theorem two open questions arise naturally.

**Question 1.** Is there a compact group $G$ and a $\sigma$-compact subset $A$ with $\text{ind}_P(A) = \aleph_0$?

**Question 2.** Is there a Polish group $G$ and a Borel subset $A \subset G$ with $\aleph_0 < \text{ind}_P(A) < \mathfrak{c}$?

The latter question does not reduce to the $\sigma$-compact case because of the following example (in which $T = \mathbb{R}/\mathbb{Z}$ stands for the circle).

**Proposition 1.** The countable product $G = T^\omega$ contains a $G_δ$-subset $A \subset G$ such that $\text{ind}_P(A) = \mathfrak{c}$ but each $\sigma$-compact subset $B \subset T^\omega$ containing $A$ has $\text{ind}_P(B) < \aleph_0$. 

Proof. Let \( q : \mathbb{Z} \to T \) denote the quotient map, \( J = q([0, \frac{1}{2}]) \) be the half-circle, \( I = \overline{J} = q([0, \frac{1}{2}]) \) be its closure, and \( D = q([0, \frac{1}{2}]) \) be two opposite points on \( T \). It is clear that \( D^{-1}D \cap JJ^{-1} = \{ e \} \) while \( I \cdot I^{-1} = T \).

It follows that \( A = J^\omega \) is a \( G_\delta \)-subset of \( T^\omega \) with \( \text{ind}_P(A) = |D^\omega| = \aleph_1 \) because \( (D^\omega)^{-1}D^\omega \cap A = \{ e \} \).

Now given any \( \sigma \)-compact subset \( B \supset A \) in \( T^\omega \), we should check that \( \text{ind}_P(B) < \aleph_0 \). Replacing \( B \) by \( B \cap I^\omega \), if necessary, we can assume that \( B \subset I^\omega \). Since \( B \subset I^\omega \) contains the dense \( G_\delta \)-subset \( J^\omega \) of \( I^\omega \), the standard application of the Baire Theorem yields an non-empty open subset \( U \subset I^\omega \) with \( U \subset B \). We lose no generality assuming that \( U \) is of basic form \( U = V \times I^\omega \) for some \( n \in \omega \) and some open set \( V \subset I^n \). Observe that

\[
U^{-1}U = VV^{-1} \times I^\omega \setminus (I^\omega \setminus n)^{-1} = VV^{-1} \times I^\omega \setminus n
\]

is an open neighborhood of \( e \) in \( T^\omega \). Consequently, \( B B^{-1} \supset UU^{-1} \) is also an open neighborhood of \( e \) in \( T^\omega \). Proceeding as in the proof of the first item of Theorem 1 we can see that

\[
\text{ind}_P(B) \leq 1/\mu(V \times T^{\omega-n}) < \aleph_0.
\]

\[ \square \]

3. The Packing Indices of Analytic Sets in Polish Groups

In this section we shall give a partial answer to Question 2 related to the packing indices of Borel subsets in Polish groups. It is well-known that each Borel subset of a Polish space is analytic. We recall that a metrizable space \( X \) is analytic if \( X \) is a continuous image of a Polish space. A space \( X \) is coanalytic if for some Polish space \( Y \) containing \( X \) the complement \( Y \setminus X \) is analytic. The classes of analytic and coanalytic spaces are denoted by \( \Sigma_1^1 \) and \( \Pi_1^1 \), respectively. It is well-known that the intersection \( \Delta_0^1 = \Sigma_1^1 \cap \Pi_1^1 \) coincides with the class of all absolute Borel (metrizable separable) spaces. By \( \Sigma_0^2 \) and \( \Pi_0^2 \) we denote the classes of \( \sigma \)-compact and Polish spaces, respectively.

We shall say that a subset \( X \subset 2^\omega \times 2^\omega \) contains a square of size \( \kappa \) if there is a subset \( A \subset 2^\omega \) with \( A \times A \subset X \) and \( |A \times A| = \kappa \). Given a class \( C \) of spaces denote by \( \text{sq}(C) \) the smallest cardinal \( \kappa \) such that each subspace \( X \in C \) of \( 2^\omega \times 2^\omega \) contains a square of size \( \kappa \) contains a square of size \( \kappa \). The Shelah’s result \([\text{Sh}]\) (applied in the proof of Theorem 1) guarantees that \( \text{sq}(\Pi_0^2) = \aleph_1 \). For other descriptive classes \( C \) the value \( \text{sq}(C) \) is not so definite and depends on Set-Theoretic Axioms. In particular, the Continuum Hypothesis implies that \( \text{sq}(\Sigma_0^2) = \text{sq}(\Sigma_1^1) = \text{sq}(\Pi_1^1) = \aleph_1 \).

On one hand, the strict inequality \( \text{sq}(\Pi_1^1) < \aleph_1 \) is consistent with \( \text{ZFC} + \text{MA} \), see \([\text{Sh}]\) 1.9, 1.10. However, there is a substantial difference between the classes \( \Pi_0^2 \) and \( \Sigma_0^2 \) of Polish and \( \sigma \)-compact spaces. By \([\text{Sh}]\) each Polish space \( X \subset 2^\omega \times 2^\omega \) containing an uncountable square contains a Perfect square. On the other hand, there is a \( \text{ZFC} \)-example of a \( \sigma \)-compact subspace \( X \subset 2^\omega \times 2^\omega \) that contains a square of size \( \aleph_1 \) but not the perfect one, see \([\text{K}]\).

Proposition 2. Let \( A \) be an analytic subset of a Polish group \( G \). If \( \text{ind}_P(A) \geq \text{sq}(\Pi_1^1) \), then \( \text{ind}_P(A) = \aleph_1 \).

Proof. Using the fact that each Polish space is a continuous one-to-one image of a zero-dimensional Polish space, we can show that \( \text{sq}(\Pi_1^1) \) coincides with the smallest
cardinal $\kappa$ such that for any Polish space $X$ a coanalytic subset $C \subset X \times X$ contains a square of size $c$ provided $C$ contains a square of size $\geq \kappa$.

Given an analytic subset $A$ of a Polish group $G$ we can see that both the sets $AA^{-1}$ and $AA^{-1} \setminus \{e\}$ are analytic and thus the set $C = \{(x, y) \in G \times G : y^{-1}x \notin AA^{-1} \setminus \{e\}\}$ is coanalytic.

Assuming that $\text{ind}_P(A) \geq \text{sq}(\Pi^1_1)$, we could find a subset $S \subset G$ of size $|S| \geq \text{sq}(\Pi^1_1)$ such that the family $\{xA\}_{x \in S}$ is disjoint. The latter is equivalent to $S^{-1} \cdot S \subset G \setminus (AA^{-1} \setminus \{e\})$ and to $S \times S \subset C$. By the equivalent definition of $\text{sq}(\Pi^1_1)$ (with $2^\omega$ replaced by any Polish space), the coanalytic subset $C \subset G \times G$ contains a square $K \times K$ of size $c$ (because it contains the square $S \times S$ of cardinality $|S \times S| \geq \text{sq}(\Pi^1_1)$). It follows from $K \times K \subset C$ that the family $\{xA\}_{x \in K}$ is disjoint and thus $\text{ind}_P(A) \geq |K| = c$.

A similar result holds for the almost packing index

\[
\text{Ind}_P(A) = \sup\{|S| : S \subset G \text{ and } \{xA\}_{x \in S} \text{ is almost disjoint}\}
\]

We recall that $\{xA\}_{x \in S}$ is \textit{almost disjoint} if $|xA \cap yA| < |A|$ for any distinct points $x, y \in S$.

In the proof of the following theorem we shall use a known fact of the Descriptive Set Theory saying that for a Borel subset $A \subset X \times Y$ in the product of two Polish spaces the set $\{y \in Y : |A \cap (X \times \{y\})| \leq \aleph_0\}$ is coanalytic in $Y$, see [Ke] 18.9.

**Proposition 3.** Let $A$ be a Borel subset of a Polish group $G$. If $\text{Ind}_P(A) \geq \text{sq}(\Pi^1_1)$, then $\text{ind}_P(A) = c$.

**Proof.** It follows from $\text{Ind}_P(A) \geq \text{sq}(\Pi^1_1) > \aleph_0$ that the space $G$ is uncountable. If $A$ is countable, then trivially, $\text{Ind}_P(A) = \text{ind}_P(A) = c$.

So we assume that $A$ is uncountable. First we show that the subset $C = \{x \in G : |A \cap xA| \leq \aleph_0\}$ is coanalytic. Consider the homeomorphism $h : G \times G \to G \times G$, $h : (x, y) \mapsto (x, y^{-1}x)$, and the Borel subset $B = h(A \times A) \subset G \times G$. Since $C = \{z \in G : |B \cap (G \times \{z\})| \leq \aleph_0\}$, we may apply the mentioned result [Ke] 18.9 to conclude that the set $C$ is coanalytic. Then the set $D = \{(x, y) \in G \times G : y^{-1}x \in C\}$ is coanalytic as the preimage of the coanalytic subset under a continuous map between Polish spaces.

Assuming that $\text{Ind}_P(A) \geq \text{sq}(\Pi^1_1)$, we could find a subset $S \subset X$ such that $|S| \geq \text{sq}(\Pi^1_1)$ such that for any distinct $x, y \in S$ the intersection $xA \cap yA$, being a Borel subset of cardinality $|xA \cap yA| < |A| \leq c$, is at most countable. Consequently, $y^{-1}x \in C$ and thus $S \times S \subset D$. By the equivalent definition of $\text{sq}(\Pi^1_1)$, the coanalytic set $D$ contains a square $K \times K$ of size $c$. It follows from $K^{-1}K \subset C$ that the family $\{xA\}_{x \in K}$ is almost disjoint. Consequently, $c = |K| \leq \text{ind}_P(A) \leq |G| = c$.

**Question 3.** Let $A$ be a Borel subset of a Polish group $G$. Is there an at most countable subset $C \subset A$ such that $\text{ind}_P(A \setminus C) = \text{Ind}_P(A)$?

The other problem concerns the cardinals $\text{sq}(C)$ for various descriptive classes $C$. If such a class $C$ contains the square of a countable metrizable space, then $\aleph_1 \leq \text{sq}(C) \leq c$ and thus $\text{sq}(C)$ falls into the category of the so-called small uncountable cardinals, see [V]. However unlike to other typical small uncountable cardinals, $\text{sq}(C)$ does no collapse to $c$ under the Martin Axiom, see [Sh].
Problem 1. Explore possible values and inequalities between classical small uncountable cardinals and the cardinals $sq(C)$ for various descriptive classes $C$.

4. Relation of the packing index to other notions of smallness

Taking into account that a subset $A$ with large packing index $\text{ind}_P(A)$ is geometrically small, it is natural to consider the relation of the packing index to some other known concepts of smallness, in particular, the smallness in the sense of Baire category and the measure.

We recall that a subset $A$ of a topological space $X$ is meager if $A$ can be written as the countable union of nowhere dense subsets. We shall need the following classical fact.

Proposition 4 (Banach-Kuratowski-Pettis). For any analytic non-meager subset $A$ of a Polish group $G$ the set $AA^{-1}$ contains a neighborhood of the neutral element of $G$.

A similar result holds for analytic subsets that are not Haar null. We recall that a subset $A$ of a topological group $G$ is called Haar null if there is a Borel probability measure $\mu$ on $G$ such that $\mu(xAy) = 0$ for all $x, y \in G$. This notion was introduced by J.Christensen [C] and thoroughly studied in [THJ]. In particular, a subset $A$ of a locally compact group $G$ is Haar null if and only if $A$ has zero Haar measure. Yet, Haar null sets exists in non-locally compact groups (admitting no invariant measure).

Proposition 5 (Chistensen). If an analytic subset $A$ of a Polish group $G$ is not Haar null, then $AA^{-1}$ contains a neighborhood of the neutral element of $G$.

We shall use those propositions to prove

Theorem 2. Let $A$ be an analytic subset of a Polish group $G$. If $\text{ind}_P(A) > \aleph_0$, then $A$ is meager and Haar null.

Proof. Otherwise, we can apply Propositions 4 or 5 to conclude that $AA^{-1}$ contains a neighborhood $U$ of the neutral element $e$ of $G$. Find another neighborhood $V \subset G$ of $e$ with $VV^{-1} \subset U \subset AA^{-1}$.

Since $\text{ind}_P(A) > \aleph_0$ there is an uncountable subset $S \subset X$ such that the family $\{xA\}_{x \in S}$ is disjoint, which is equivalent to $S^{-1}S \cap AA^{-1} = \{e\}$. It follows from the choice of $V$ that $S^{-1}S \cap VV^{-1} \subset S^{-1}S \cap AA^{-1} = \{e\}$ and thus the family $\{xV\}_{x \in S}$ is disjoint. Since $V$ is an open neighborhood of $e$, the set $S$, being discrete in $G$, is at most countable. This contradiction completes the proof. \hfill \Box

5. The packing index and unions

It is known that the countable union of meager (resp. Haar null) subsets of a Polish group is meager (resp. Haar null). In contrast, the union of two subsets $A, B \subset G$ with large packing index need not have large packing index. A simplest example is given by the sets $A = \mathbb{R} \times \{0\}$ and $B = \{0\} \times \mathbb{R}$ on the plane $\mathbb{R}^2$. They have packing indices $\text{ind}_P(A) = \text{ind}_P(B) = c$ but $\text{ind}_P(A \cup B) = 1$. In fact, this situation is typical. According to [L2], each infinite group $G$ contains two sets $A, B \subset G$ such that $\text{ind}_P(A) = \text{ind}_P(B) = |G|$ but $\text{Ind}_P(A \cup B) = 1$. The following theorem is a topological version of this result.
Theorem 3. Each non-discrete Polish Abelian group $G$ contains two closed subsets $A, B \subset G$ such that $\text{ind}_P(A) = \text{ind}_P(B) = \aleph$ and $\text{Ind}_P(A \cup B) = 1$.

Proof. Fix an invariant metric $\rho$ generating the topology of $G$. This metric is complete because $G$ is Polish. Since $G$ is Abelian, we use the additive notation for the group operation on $G$. The neutral element of $G$ will be denoted by $0$.

We define a subset $D$ of $G$ to be $\varepsilon$-separated if $\rho(x, y) \geq \varepsilon$ for any distinct points $x, y \in D$. By the Zorn lemma, each $\varepsilon$-separated subset can be enlarged to a maximal $\varepsilon$-separated subset of $G$.

Put $\varepsilon_{-1} = \varepsilon_0 = 1$ and choose a maximal $2\varepsilon_0$-separated subset $H_0 \subset G$ containing zero. Proceeding by induction we shall define a sequence $(h_n)_{n \in \mathbb{N}} \subset G$ of points, a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ of positive real numbers and a sequence $(H_n)_{n \in \mathbb{N}}$ of subsets of $G$ such that for every $n > 0$

(i) $B(0, \varepsilon_{n-1}) \setminus B(0, 33\varepsilon_n)$ is not empty and $\varepsilon_n < 2^{-6}\varepsilon_{n-1}$;

(ii) $\|h_n\| = 5\varepsilon_n$,

(iii) $H_n \supset \{0, h_n\}$ is a maximal $2\varepsilon_n$-separated subset in $B(0, 8\varepsilon_{n-1})$.

It follows from (i) that the series $\sum_{n \in \mathbb{N}} \varepsilon_n$ is convergent and thus for any sequence $x_n \in H_n$, $n \in \mathbb{N}$, the series $\sum_{n \in \mathbb{N}} x_n$ is convergent (because $\|x_n\| < 8\varepsilon_{n-1}$ for all $n \in \mathbb{N}$ according to (iii)). So it is legal to consider the sets of sums

$$\Sigma_0 = \{ \sum_{n \in \omega} x_{2n} : (x_{2n})_{n \in \omega} \in \prod_{n \in \omega} H_{2n} \},$$

$$\Sigma_1 = \{ -\sum_{n \in \omega} x_{2n+1} : (x_{2n+1})_{n \in \omega} \in \prod_{n \in \omega} H_{2n+1} \}.$$

Let $A$ and $B$ be the closures of the sets $\Sigma_0$ and $\Sigma_1$ in $G$. It remains to prove that the sets $A, B$ have the desired properties: $\text{ind}_P(A) = \text{ind}_P(B) = \aleph$ and $\text{Ind}_P(A \cup B) = 1$. This will be done in three steps.

1. First we prove that $\text{ind}_P(A) = \aleph$. By Lemma this equality will follow as soon as we check that $\overline{A - A}$ is not a neighborhood of the neutral element $0$ in $G$. It suffices for every $k \in \mathbb{N}$ to find a point $g \in B(0, \varepsilon_{2k}) \setminus \overline{A - A}$. By condition (ii), there is a point $g \in G$ with $33\varepsilon_{2k+1} \leq \|g\| < \varepsilon_{2k}$. We claim that $g \notin \overline{A - A} = \Sigma_0 - \Sigma_0$. More precisely,

$$\text{dist}(g, \overline{A - A}) = \text{dist}(g, \Sigma_0 - \Sigma_0) \geq \min\{\varepsilon_{2k+1}, \varepsilon_{2k}/2\}.$$

Take any two distinct points $x, y \in \Sigma_0$ and find infinite sequences $(x_{2n})_{n \in \mathbb{N}}, (y_{2n})_{n \in \mathbb{N}} \in \prod_{n \in \omega} H_{2n}$ with $x = \sum_{n \in \omega} x_{2n}$ and $y = \sum_{n \in \omega} y_{2n}$.

Let $m = \min\{n \in \omega : x_{2n} \neq y_{2n}\}$. If $m \geq k + 1$, then

$$\|x - y\| = \|\sum_{n \geq m} x_{2n} - y_{2n}\| \leq \sum_{n \geq m} \|x_{2n}\| + \|y_{2n}\| \leq$$

$$\leq 2 \sum_{n \geq m} \varepsilon_{2n-1} \leq 32 \varepsilon_{2n-1} \leq 32 \varepsilon_{2k+1} < \|g\| - \varepsilon_{2k+1}$$

and hence $\rho(x - y, g) \geq \varepsilon_{2k+1}$.
If $m \leq k$, then
\[
\|x - y\| = \|(x_{2m} - y_{2m}) + \sum_{n > m} (x_{2n} - y_{2n})\| \geq \\
\geq \|x_{2m} - y_{2m}\| - \sum_{n > m} (\|x_{2n}\| + \|y_{2n}\|) \geq \\
\geq 2\varepsilon_{2m} - 32\varepsilon_{2m+1} \geq \frac{3}{2}\varepsilon_{2m} > \|g\| + \frac{1}{2}\varepsilon_{2m}
\]
and again $\rho(x - y, g) \geq \frac{1}{4}\varepsilon_{2m} \geq \frac{1}{5}\varepsilon_{2k}$. 

2. In the same manner we can prove that $\text{ind}_p(B) = \kappa$.

3. It remains to prove that $\text{Ind}_p(A \cup B) = 1$. First we recall some standard notation. Denote by $2^{<\omega} = \bigcup_{n \in \omega} 2^n$ the set of finite binary sequences. For any sequence $s = (s_0, \ldots, s_{n-1}) \in 2^{<\omega}$ and $i \in 2 = \{0, 1\}$ by $|s| = n$ we denote the length of $s$ and by $s^*i = (s_0, \ldots, s_{n-1}, i)$ the concatenation of $s$ and $i$. For a finite or infinite binary sequence $s = (s_i)_{i \leq \omega}$ and $l \leq n$ let $s|l = (s_i)_{i < l}$. The set $2^\omega$ is a tree with respect to the partial order: $s \leq t$ iff $s|t = t|l$ where $l = |s| \leq |t|$. 

The equality $\text{ind}_p(A \cup B) = 1$ will follow as soon as we prove that $|A \cap (g + B)| \geq \kappa$ for all $g \in G$. We shall construct a sequence of points $\{x_s\}_{s \in 2^{<\omega}}$ such that for every sequence $s \in 2^{<\omega}$ the following conditions hold:

(1) $x_s \in H_{|s|} \subset B(0, 8\varepsilon_{|s| - 1})$;
(2) $\|x_s - x_{s'}\| > \varepsilon_n$;
(3) $\|g - \sum_{t \leq s} x_t\| < 7\varepsilon_{|s|}$.

We start choosing a point $x_0 \in H_0$ with $\rho(x_0, g) < 2\varepsilon_{-1} = 2\varepsilon_0$. Such a point $x_0$ exists because $H_0$ is a maximal $(0, 2\varepsilon_0)$-separated set in $G$. Next we proceed by induction. Suppose that for some $n$ the points $x_s, s \in 2^{<n}$, have been constructed. Given a sequence $s \in 2^{n-1}$ we need to define the points $x_s0$ and $x_s1 \in H_n$. Let $g_s = g - \sum_{t \leq s} x_t$. Since $\|g_s + h_n\| \leq \|g_s\| + \|h_n\| < 7\varepsilon_{n-1} + 5\varepsilon_n < 8\varepsilon_{n-1}$ and $H_n$ is a maximal $2\varepsilon_n$-separated subset in $B(0, 8\varepsilon_{n-1})$, there are two points $x_s0, x_s1 \in H_n$ with $\rho(g_s, x_s0) < 2\varepsilon_n$ and $\rho(g_s + h_n, x_s1) < 2\varepsilon_n$. The condition (2) follows from
\[
\|x_s0 - x_s1\| \geq \|g_s - (g_s + h_n)\| = \|g_s - x_s0\| + \|g_s + h_n - x_s1\| > \\
5\varepsilon_n - 2\varepsilon_n = \varepsilon_n.
\]

The condition (3) follows from the estimates
\[
\|g - \sum_{t \leq s} x_t\| = \|g - x_s0\| < 2\varepsilon_n = 2\varepsilon_{|s0|}
\]
and
\[
\|g - \sum_{t \leq s} x_t\| = \|g - x_{s'1} - \sum_{t \leq s} x_t\| = \|g_s + h_n - x_{s'1} - h_n\| \leq \\
\leq \|g_s + h_n - x_{s'1}\| + \|h_n\| < 2\varepsilon_n + 5\varepsilon_n = 7\varepsilon_{|s'1|}.
\]

After completing the inductive construction, we can use the condition (3) to see that for every infinite binary sequence $s \in 2^\omega$ we get
\[
g = \sum_{n \in \omega} x_{s|n} = \sum_{n \in \omega} x_{s|2n} + \sum_{n \in \omega} x_{s|2n+1}.
\]

We claim that the set
\[
D_0 = \{ \sum_{n \in \omega} x_{s|2n} : s \in 2^\omega \}
\]
lies in the intersection $A \cap (g + B)$. It is clear that $D_{0} \subset \Sigma_{0} \subset A$. To see that $D_{0} \subset g + B$, take any point $x \in D_{0}$ and find an infinite binary sequence $s \in 2^{\omega}$ with $x = \sum_{n} x_{s|2n}$. Then

$$x = \sum_{n} x_{s|2n} + \sum_{n} x_{s|2n+1} - \sum_{n} x_{s|2n+1} \in g + \Sigma_{1} \subset g + B.$$  

It remains to prove that $|D_{0}| \geq \mathfrak{c}$. Note that the set $D_{0}$, being a continuous image of the Cantor cube $2^{\omega}$, is compact. Now the equality $|D_{0}| = \mathfrak{c}$ will follow as soon as we check that $D_{0}$ has no isolated points. Given any sequence $s \in 2^{\omega}$ and $\delta > 0$ we should find a sequence $t \in 2^{\omega}$ such that

$$0 < \| \sum_{n \omega} x_{s|2n} - \sum_{n \omega} x_{t|2n} \| < \delta.$$  

Find even number $2m \in \omega$ such that $\sum_{n \geq m} 2 \omega 2n - 1 < \delta$ and take any sequence $t \in 2^{\omega}$ such that $t|2m - 1 = s|2m - 1$ but $t|2m \neq s|2m$. Then

$$\| \sum_{n \omega} x_{s|2n} - \sum_{n \omega} x_{t|2n} \| = \| \sum_{n \geq m} x_{s|2n} - \sum_{n \geq m} x_{t|2m} \| \leq \sum_{n \geq m} \| x_{s|2n} \| + \| x_{t|2m} \| \leq \sum_{n \geq m} 32 \omega 2n - 1 < \delta.$$  

On the other hand the lower bound $\| x_{s|2m} - x_{t|2m} \| > \varepsilon_{2m}$ supplied by (2) implies

$$\| \sum_{n \omega} x_{s|2n} - \sum_{n \omega} x_{t|2n} \| = \| \sum_{n \geq m} x_{s|2n} - \sum_{n \geq m} x_{t|2n} \| \geq \| x_{s|2m} - x_{t|2m} \| - \| \sum_{n \geq m} (x_{s|2n} - x_{t|2n}) \| > \varepsilon_{2m} - \sum_{n \geq m} 16 \omega 2n - 1 > 2 \varepsilon_{2m} - 32 \omega 2m + 1 > 0$$  

(the latter two inequalities follow from (i)). Now we see that $|D_{0}| = \mathfrak{c}$ and thus $|A \cap (g + B)| \geq |D_{0}| = \mathfrak{c}$, which implies that $\text{Ind}_{P}(A \cup B) = 1$. 

6. HAAR AND UNIVERSALLY NULL SETS WITH UNIT PACKING INDEX

In this section in each non-discrete Polish group $G$ we shall construct geometrically large subsets which are small in the sense of measure.

**Theorem 4.** Each non-discrete Polish group $G$ contains a closed nowhere dense Haar null subset $C$ with $\text{Ind}_{P}(C) = 1$ and thus $CC^{-1} = G$.

**Proof.** By Theorem 3 the group $G$ contains two closed subsets $A, B \subset G$ with $\text{ind}_{P}(A) = \text{ind}_{P}(B) = \mathfrak{c}$ and $\text{Ind}_{P}(A \cup B) = 1$. By Theorem 2 the sets $A, B$ are Haar null. Then the union $C = A \cup B$ is Haar null, and, being closed in $G$, is nowhere dense. 

Under the set-theoretic assumption $\text{cov}(\mathbb{K}) = \mathfrak{c}$ in each non-discrete Polish group $G$ we can construct an universally null subset $A \subset G$ with $\text{Ind}_{P}(A) = 1$. We recall that a subset $A$ of a topological space $X$ is called universally null if $\mu(A) = 0$ for every continuous probability Borel measure $\mu$ on $X$. A measure $\mu$ on $X$ is continuous if $\mu(\{x\}) = 0$ for all $x \in X$. It is clear that each universally null subset of a non-discrete Polish group in Haar null.
By \( \text{cov}(\mathbb{K}) \) we denote the smallest cardinality of a cover of the real line by meager subsets. It is known [11W19.4] that the equality \( \text{cov}(\mathbb{K}) = \mathfrak{c} \) is equivalent to MA_{countable}, the Martin Axiom for countable posets. In particular, \( \text{cov}(\mathbb{K}) = \mathfrak{c} \) holds under MA, the Martin Axiom.

**Theorem 5.** Under \( \text{cov}(\mathbb{K}) = \mathfrak{c} \) each non-discrete Polish group \( G \) contains a universally null subset \( A \) with \( \text{Ind}_P(A) = 1 \).

*Proof.* Fix a countable dense subset \( \{x_n\}_{n \in \omega} \subset G \) and a decreasing sequence \( (U_n)_{n \geq 0} \) of open neighborhoods of the unit \( e \) of \( G \) with \( e \in \bigcap_{n \geq 0} U_n \). To each function \( f : \omega \to \mathbb{N} \) we can assign a dense \( G_\delta \)-subset \( D_f = \bigcap_{n \in \omega} \bigcup_{k \geq n} x_k U_{f(k)} \) of \( G \). Let \( \mathbb{N}^\omega = \{f_\alpha : \alpha < \mathfrak{c} \} \) be any enumeration of the set \( \mathbb{N}^\omega \).

The group \( G \), being Polish and non-discrete, has size \( |G| = \mathfrak{c} \). Let \( G \times G = \{(g_\alpha, g'_\alpha) : \alpha < \mathfrak{c} \} \) be any enumeration of the product \( G \times G \). This enumeration induces an enumeration \( G = \{g_\alpha : \alpha < \mathfrak{c} \} \) of \( G \) such that \( |\{\alpha : g_\alpha = g\}| = \mathfrak{c} \) for each element \( g \in G \). Put \( G_\alpha = \bigcap_{\beta < \alpha} D_{f_\beta} \) for \( \alpha < \mathfrak{c} \).

Observe that the equality \( \text{cov}(\mathbb{K}) = \mathfrak{c} \) implies that \( |G_\alpha \cap g_\alpha G_\alpha| = \mathfrak{c} \). Otherwise \( G \) can be presented as the union of \( < \mathfrak{c} \) many meager subsets:

\[
G = (G_\alpha \cap g_\alpha G_\alpha) \cup \bigcup_{\beta < \alpha} (G \setminus D_{f_\beta}) \cup \bigcup_{\beta < \alpha} (G \setminus g_\alpha D_{f_\beta}).
\]

The equality \( |G_\alpha \cap g_\alpha G_\alpha| = \mathfrak{c} \), \( \alpha < \mathfrak{c} \), allows us to construct inductively a transfinite sequence of points \( \{a_\alpha : \alpha < \mathfrak{c} \} \subset G \) such that \( a_\alpha \in G_\alpha \cap g_\alpha G_\alpha \setminus \{a_\beta, g_\beta a_\beta : \beta < \alpha\} \) for all \( \alpha < \mathfrak{c} \).

The choice of the enumeration \( G = \{g_\alpha : \alpha < \mathfrak{c} \} \) implies that the set \( A = \{a_\alpha, g_\alpha a_\alpha : \alpha < \mathfrak{c} \} \) has \( \text{Ind}_P(A) = 1 \).

It remains to check that \( A \) is universally null. Given a \( \sigma \)-additive Borel probability continuous measure \( \mu \) on \( G \), for every \( n \in \omega \) find a number \( f(n) \in \mathbb{N} \) such that \( \mu(a_\alpha U_{f(n)}) < 2^{-n} \). It follows that the dense \( G_\delta \)-subset \( D_f \) has measure \( \mu(D_f) = 0 \). Find an ordinal \( \alpha \) such that \( f_\alpha = f \) and observe that \( a_\beta, g_\beta a_\beta \in G_{f_\alpha} \subset D_f \) for all \( \beta > \alpha \). The inequality \( \text{cov}(\mathbb{K}) < \text{non}(\mathbb{L}) \) following from the Cichon’s diagram (see [W]) guarantees that \( \mu(\{a_\beta, g_\beta a_\beta : \beta \leq \alpha\}) = 0 \) and hence

\[
\mu(A) \leq \mu(\{a_\beta, g_\beta a_\beta : \beta \leq \alpha\}) + \mu(D_f).
\]

\[ \square \]

**Remark 1.** Theorem 5 cannot be proved in ZFC because in Laver’s model of ZFC each universally null subset \( A \) of a Polish group \( G \) has size \( |A| < \mathfrak{c} \), which implies that \( \text{ind}_P(A) = \text{Ind}_P(A) = \mathfrak{c} \).

7. **Constructing small subsets with a given (sharp) packing index**

In this section we shall show that Theorem 3 cannot be reversed: nowhere dense Haar null sets can have arbitrary packing index. In fact, we shall construct such sets \( A \) with an arbitrary (sharp) packing index

\[
\text{ind}_P^+(A) = \sup\{|S|^+ : S \subset G \text{ and } \{xA\}_{x \in S} \text{ is disjoint}\},
\]

\[
\text{Ind}_P^+(A) = \sup\{|S|^+ : S \subset G \text{ and } \{xA\}_{x \in S} \text{ is almost disjoint}\}.
\]
The formulas
\[
\text{ind}_P(A) = \sup \{ \kappa : \kappa < \text{ind}_P^\sharp(A) \} \quad \text{and} \quad \text{Ind}_P(A) = \sup \{ \kappa : \kappa < \text{Ind}_P^\sharp(A) \}
\]
show that the sharp packing indices carry more information about a set $A$ comparing to the usual packing indices.

All possible values of the sharp packing indices of subsets of a given Abelian group are determined by the following result proved in [12].

**Proposition 6.** Let $G$ be an infinite Abelian group and $L \subseteq G$ be a subset with $\text{Ind}_P(L) = 1$. For a cardinal $\kappa \in [2, |G|^+]$ the following conditions are equivalent:

1. there is a subset $A \subseteq G$ with $\text{ind}_P^\sharp(A) = \kappa$;
2. there is a subset $A \subseteq L$ with $\text{ind}_P^\sharp(A) = \text{Ind}_P^\sharp(A) = \kappa$;
3. if $|G/[G]|_2 \leq 2$, then $\kappa \neq 4$ and if $G = [G]_3$, then $\kappa \neq 3$.

Here $[G]_p = \{ x \in G : x^p = e \}$ for $p \in \{2, 3\}$.

Combining Proposition [6] with Theorems [3] and [5] we obtain the main result of this section.

**Theorem 6.** Let $G$ be a non-discrete Polish Abelian group and $\kappa$ be a cardinal such that (i) $2 \leq \kappa \leq |G|^+$, (ii) $k \neq 3$ if $G = [G]_3$, and (iii) $k \neq 4$ if $|G/[G]|_2 \leq 2$.

1. The group $G$ contains a nowhere dense Haar null subset $A$ such that $\text{ind}_P^\sharp(A) = \text{Ind}_P^\sharp(A) = \kappa$;
2. If $\text{cov}(\mathcal{K}) = \mathfrak{c}$, then $G$ contains a universally null subset $A$ with $\text{ind}_P^\sharp(A) = \text{Ind}_P^\sharp(A) = \kappa$.

Taking into account that $\text{ind}_P(A) = \sup \{ \kappa : \kappa < \text{ind}_P^\sharp(A) \}$, we can apply Theorem [6] to deduce the following corollary.

**Corollary 1.** Let $G$ be a non-discrete Polish Abelian group and $\kappa$ be a cardinal such that (i) $1 \leq \kappa \leq |G|$, (ii) $k \neq 2$ if $G = [G]_3$, and (iii) $k \neq 3$ if $|G/[G]|_2 \leq 2$.

1. The group $G$ contains a nowhere dense Haar null subset $A$ such that $\text{ind}_P(A) = \text{Ind}_P(A) = \kappa$;
2. If $\text{cov}(\mathcal{K}) = \mathfrak{c}$, then $G$ contains a universally null subset $A$ with $\text{ind}_P(A) = \text{Ind}_P(A) = \kappa$.

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