Statistical characteristics of formation and evolution of structure in the universe.

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ABSTRACT
An approximate statistical description of the formation and evolution of structure of the universe based on the Zel’dovich theory of gravitational instability is proposed. It is found that the evolution of DM structure shows features of self-similarity and the main structure characteristics can be expressed through the parameters of initial power spectrum and cosmological model. For the CDM-like power spectrum and suitable parameters of the cosmological model the effective matter compression reaches the observed scales \( R_{\text{wall}} \sim 20 – 25 h^{-1}\text{Mpc} \) with the typical mean separation of wall-like elements \( D_{\text{SLSS}} \sim 50 – 70 h^{-1}\text{Mpc} \). This description can be directly applied to the deep pencil beam galactic surveys and absorption spectra of quasars. For larger 3D catalogs and simulations it can be applied to results obtained with the core-sampling analysis.

It is shown that the interaction of large and small scale perturbations modulates the creation rate of early Zel’dovich pancakes and generates bias on the SLSS scale. For suitable parameters of the cosmological model and reheating process this bias can essentially improve the characteristics of simulated structure of the universe.

The models with \( 0.3 \leq \Omega_m \leq 0.5 \) give the best description of the observed structure parameters. The influence of low mass ”warm” dark matter particles, such as a massive neutrino, will extend the acceptable range of \( \Omega_m \) and \( h \).

Key words: cosmology: large-scale structure of the Universe — galaxies: clusters: general – theory.

1 INTRODUCTION

Over the past decade the large maps of the spatial galaxy distribution have been prepared and the unexpectedly complicated character of this distribution was established. The structure predicted by the Zel’dovich theory of gravitational instability (Zel’dovich 1970, 1978) was found already in the first wedge diagrams (Gregory & Thompson 1978) and now the Large Scale Structure (LSS) is seen in many observational catalogs, such as the CfA (de Lapparent, Geller & Huchra 1987; Ramella, Geller & Huchra 1992), the SRSS (da Costa et al. 1988), and the Las Campanas Redshift Survey (Shectman et al. 1996, hereafter LCRS). The observed high concentration of galaxies within the wall-like structure elements such as the Great Attractor (Dressler et al. 1987), and the Great Wall (de Lapparent, Geller & Huchra 1988) and the existence of extended under dense regions similar to the Great Void (Kirshner et al. 1983) put in the forefront the investigation of the Super Large Scale Structure (SLSS).

Now the SLSS is also found in many deep pencil beam redshift surveys (Broadhurst et al. 1990; Willmer et al. 1994; Buryak et al. 1994; Bellanger & de Lapparent 1995; Cohen et al. 1996) as a rich galaxy clumps with the typical separations in the range of \( (60 – 120) h^{-1}\text{Mpc} \). Here \( h=H_0/100\text{km/s/Mpc} \) is the dimensionless Hubble constant.

Further progress in the statistical description of the LSS & SLSS has been reached with the core-sampling method (Buryak et al. 1994) and the Minimal Spanning Tree technique (Barrow, Bhavsar & Sonoda 1985). Recent analysis of the LCRS performed by Doroshkevich et al. (1996 & 1997b, hereafter LCRS1 & LCRS2, Doroshkevich et al. 1998) revealed some statistical parameters of the wall-like SLSS component such as their typical separation, \( D_{\text{SLSS}} \approx 50 – 60 h^{-1}\text{Mpc} \), and the fraction of galaxies accumulated by the SLSS, which can reach \( \sim 50\% \). The same analysis indicates that formation of richer walls can be roughly described as an asymmetric 2D collapse of regions with a typical size \( R_{\text{wall}} \sim 20 – 25 h^{-1}\text{Mpc} \) that is about half of their typical
dispersion within the SLSS elements reaches 400 - 700km/s observed SLSS elements.

In this paper we present an approximate statistical description of the process of DM structure formation based on the nonlinear Zel’dovich theory. The potential of this approach is limited as the successive consideration of mutual interactions of the small and large scale perturbations becomes more and more cumbersome. In spite of this it allows us to obtain some interesting results. Thus, it is shown that formation of both LSS and SLSS is a joint process possessing some features of self-similarity. The main observed characteristics of LSS and SLSS are expressed through the structure functions of power spectrum and through the typical scales, set by the power spectrum, the time scale, set by the amplitude of perturbation, and the main parameters of of cosmological model. One of the most interesting such characteristics is the dynamical scale of the nonlinearity defined as the scale of essential DM concentration within high density walls. We show that for the CDM transfer function (Bardeen et al. 1986, hereafter BBKS) and Harrison – Zel’dovich primordial power spectrum and for cosmological models with lower matter density this scale of nonlinearity reaches 20 - 30h^{-1}Mpc that is comparable with typical scales of the observed SLSS elements.

Simulations (DMRT) show that even in cosmological models with a low matter density the simulated velocity dispersion within the SLSS elements reaches 400 - 700km/s, along each principal axis; this exceeds the observed value by a factor of ~1.5 - 2. Such a large and isotropic velocity dispersion is caused by the disruption of the walls into high density clouds. For smaller matter density of the universe this dispersion decreases but together with the fraction of matter accumulated by the walls. This means that other factors as, for example, the large scale bias in the spatial galaxy distribution relative to the more homogeneous distribution of DM and baryons could be essential for the successful reproduction of the observed SLSS. Such large scale bias caused by the interaction of small and large scale perturbations was discussed by Dekel & Silk (1986), and Dekel & Rees (1987), and estimated by Demiański & Doroshkevich (1999, hereafter Paper I).

The interaction of small and large scale perturbations is important during all evolutionary stages. Thus, even during early evolutionary periods the large scale perturbations modulate the rate of pancake formation. This modulation is seen as an acceleration of the pancake formation within deeper potential wells which later are transformed into the wall-like SLSS elements (Buryak et al. 1992; Demiański & Doroshkevich 1997; Paper I).

Evolution of structure was discussed and simulated many times (see, e.g., Sahni & Coles 1994; Doroshkevich et al. 1997a, hereafter DFGMM; for references, Sahni & Coles 1995). However, SLSS in the dark matter (DM) distribution similar to that seen in the LCRS was found only recently in a few simulations with the CDM-like power spectrum and Ω_mh^2 = 0.2 – 0.3, (Cole, Weinberg, Frenk & Ratra 1997; Doroshkevich, Müller, Retzäll & Turchaninov 1999, hereafter DMRT). Hence, for suitable cosmological models the evolution of small initial perturbations results in the SLSS formation.

Numerical simulations are now the best way to reproduce and to study the joint action of all the pertinent factors together and to obtain more representative description of the process of structure formation. Essential progress achieved recently both in the simulations and study of DM and ‘galaxy’ distributions (Governato et al. 1998; Jenkins et al. 1998; Doroshkevich, Fong & Makarova 1998; DMRT; Cole, Hatton, Weinberg & Frenk 1998) allows us to follow the structure evolution in a wide range of redshifts and to reveal differences between DM and galaxy distribution. Comparison of these results with observations and an approximate theoretical description stimulates further progress in our understanding of evolution of the universe.
able observational and simulated data. We conclude with Sec. 10 where a short discussion of the main results is presented. Some technical details are given in Appendixes I – IV.

2 STATISTICAL PARAMETERS OF PERTURBATIONS; VARIANCES AND TYPICAL SCALES

The simplest characteristics of perturbations are the variances of density and velocity perturbations. For a more detailed statistical description of the structure evolution it is necessary to use also the structure functions. They were introduced in Paper I and are briefly described in this Section and Appendix I. Here we consider only the CDM-like power spectrum but the same approach can be applied for other spectra as well.

Our analysis is based on the Zel’dovich theory which links the Eulerian, \( r_i \), and the Lagrangian, \( q_i \), coordinates of fluid elements (particles) by the expression

\[
  r_i = (1 + z)^{-1} q_i - B(z) S_i(q),
\]

where \( z \) denotes the redshift, \( B(z) \) describes growth of perturbations in the linear theory, and the potential vector \( S_i(q) = \partial \phi / \partial q_i \) characterizes the spatial distribution of perturbations. The Lagrangian coordinates of a particle, \( q_i \), are its unperturbed comoving coordinates.

For the flat universe with \( \Omega_m + \Omega_\Lambda = 1 \), \( \Omega_m \geq 0.1 \), the function \( B(z) \) can be approximated with a precision better then 10\% by the expression (Paper I)

\[
  B^{-3}(z) \approx \frac{1 - \Omega_m + 2\Omega_m(1 + z)^3}{1 + 1.2\Omega_m},
\]

and for an open universe with \( 0.1 \leq \Omega_m \leq 1 \), \( \Omega_\Lambda = 0 \) as

\[
  B^{-3}(z) \approx 1 + \frac{2.5\Omega_m}{1 + 1.5\Omega_m} z
\]

(Zel’dovich & Novikov 1983). For \( \Omega_m = 1 \), \( \Omega_\Lambda = 0 \) both expressions give \( B^{-3}(z) = 1 + z \).

The main characteristics of the perturbations are the variances of density \( \sigma_\rho^2 \), displacement \( \sigma_v^2 \), and components of the deformation tensor \( \sigma_D^2 \)

\[
  \sigma_v^2 = \frac{1}{2\pi^2} \int_0^\infty p(k)dk, \quad \sigma_D^2 = \frac{5}{2\pi^2} \int_0^\infty p(k)^2dk,
\]

where \( p(k) \) is the power spectrum, and \( k \) is the comoving wave number.

The power spectrum determines also two amplitude independent typical scales, which allow us to describe the process of structure formation and can be, possibly, estimated from the observed galaxy distribution. For the Harrison–Zel’dovich primordial power spectrum these scales, \( l_0 \) and \( l_c \), are defined as

\[
  l_0^{-2} = \int_0^\infty k^2 T^2(k/l_0)dk, \quad l_c^2 = \frac{5}{3} \sigma_D^2
\]

where \( T^2(x) \) is the transfer function and \( l_0 \equiv \Omega_m h^2 \text{Mpc}^{-1} \). For the CDM transfer function (BBKS) the scale \( l_0 \) and the typical masses of DM and baryonic components associated with the scale \( l_0 \) are

\[
  l_0 \approx 6.6 (\Omega_m h)^{-1} \sqrt{\frac{0.023}{m_{-2}}} \text{h}^{-1} \text{Mpc},
\]

\[
  M_0 = \frac{\pi}{6} < \rho > l_0^3 \approx \frac{2.5 \cdot 10^{13} M_\odot}{\Omega_m h^4}, \quad M_0^{(0)} = \frac{\Omega_\Lambda}{\Omega_m} M_0.
\]

For the CDM power spectrum the \( \sigma^2_\rho \) is divergent (logarithmically) at large \( k \) and the standard CDM transfer function should be truncated by introduction of an appropriate mass of DM particle, \( M_D \). This restriction allows, however, a significant interval for a possible mass of DM particles – up to \( 10^{16} - 10^{20} \text{eV} \). In this paper we will use the ratio \( l_c/l_0 \) as a measure of the mass of DM particles. For \( 0.5 \text{keV} \leq M_D \leq 10^{17} \text{keV} \) we have \( 0.2 \geq l_c/l_0 \geq 0.01 \) and for \( M_D \geq 5\text{keV} \) the following approximate expression can be used

\[
  l_c/l_0 \approx \frac{0.061}{\sqrt{\ln M_D}},
\]

where \( M_D \) is the mass of DM particle in keV. It turns out that the final results only weakly depend on the mass of DM particles. For the Harrison-Zel’dovich primordial power spectrum the dependence of the moments and typical scales on the mass of DM particles was discussed in Paper I. More cumbersome correlation and structure functions of perturbations were also discussed in Paper I and, partly, are presented in the Appendix I.

The amplitude of power spectrum can be taken from the measured anisotropy of the relic radiation (Stompor et al. 1995; Bunn & White 1997; Gorski et al. 1998). It can be fitted by

\[
  \sigma_\rho \approx 31.2 \sqrt{\frac{m_{-2}}{0.023}} \left( \frac{T_Q}{20 \mu K} \right) a(\Omega_m, \Omega_\Lambda) \text{Mpc},
\]

\[
  a(\Omega_m, \Omega_\Lambda = 1 - \Omega_m) = \Omega_m^{215-0.05ln\Omega_m},
\]

\[
  a(\Omega_m, \Omega_\Lambda = 0) = \Omega_m^{65-0.19ln\Omega_m},
\]

where \( T_Q \) is the amplitude of quadrupole component of anisotropy of the relic radiation. The time scale of the structure evolution is defined by the function

\[
  \tau(z) = \tau_B(z),
\]

\[
  \tau_0 = \frac{\sigma_\rho}{\sqrt{m_{-2}}}, \quad \tau_0 \approx 2.73 h^2 \Omega_m^{1.21} \left( \frac{m_{-2}}{0.023} \frac{T_Q}{20 \mu K} \right) a(\Omega_m, \Omega_\Lambda),
\]

\[
  \tau_0 \approx 2.73 h^2 \Omega_m^{1.65-0.19ln\Omega_m} \left( \frac{m_{-2}}{0.023} \frac{T_Q}{20 \mu K} \right), \quad \Omega_\Lambda = 0, \Omega_m \leq 1.
\]

Further on, as a rule, the dimensionless variables will be used. We will use \( l_0 \) as the unit of length, and \( < \rho > l_0 \) as the unit of surface density. This means also that such dimensionless characteristics of a pancake as the size of collapsed slab and resulting surface density of a pancake are identical. Below we will use both terms as well as the term ‘mass’, \( m \),
to characterize the surface density reached during formation of a pancake. The gravitational potential and displacement are measured in units of $\sigma_s l_0/\sqrt{3}$ and $\sigma_s/\sqrt{3}$, respectively.

3 STATISTICAL CHARACTERISTICS OF PANCAKES

In this section the mass function of Zel’dovich pancakes and its time evolution is given. This can be done using the main equation of Zel’dovich theory (2.1). The mass of compressed matter is measured by the Lagrangian size of compressed slab, $q$, or by the surface density of pancake, $<\rho>$. As it was noted above, both measures are identical in dimensionless notation. So, we will use both terms ‘size’, $q$, and ‘mass’, $m$, of a pancake to characterize its surface density.

Here we do not consider the transversal characteristics of structure elements and cannot discriminate, for example, the central part of a poorer pancake and periphery of richer pancake, if they have the same surface density or mass, $m$. In this sense our approach gives characteristics similar to that obtained with the core-sampling analysis, pencil beam observations or the distribution of absorption lines in spectra of QSOs.

3.1 The pancake formation

According to the relation (2.1) when two particles with different Lagrangian coordinates $q_1$ and $q_2$ meet at the same Eulerian point $r$ a pancake with the surface mass density $<\rho> = |q_1 - q_2|$ forms. Here we assume that all particles situated between these two boundary particles are also incorporated into the same pancake. This assumption is also made in the adhesion approach (see, e.g., Shandarin & Zel’dovich 1989). Formally, this condition can be written as

\[ q_{12} = q_1 - q_2 = \tau \cdot [S(q_1) - S(q_2)]. \]  

(3.1)

This means that the pancake formation process can be characterized by the scalar random function

\[ Q(q_{12}) = \frac{q_{12}}{q_{12}} \cdot [S(q_1) - S(q_2)] \]  

(3.2a)

under the condition

\[ q_{12} \times [S(q_1) - S(q_2)] = 0. \]

In a coordinate system with the first axis oriented along the vector $q_{12}$ the condition (3.1) can be rewritten more explicitly as

\[ Q(q_{12}) = \Delta S_1 = q_{12}/\tau, \quad \Delta S_2(q_{12}) = \Delta S_3(q_{12}) = 0, \]  

(3.2b)

\[ \Delta S_i = S_i(q_1) - S_i(q_2), \]

and formation of a pancake is described by the 1D matter flow between points $q_1 = q_{12}/q_{12}$ and $q_2 = q_{12}/q_{12}$.

As is shown in Appendix II, under the assumption of Gaussian distribution of perturbations and neglecting the (weak) correlations between matter motion in orthogonal directions, the probability of a pancake formation, for a given $q$ prior to the ‘time’ $\tau$ or for a given $\tau$ with a size larger then $q$, is identical to the probability to have $\Delta S_1 \geq q/\tau$ ($Q(q_{12})/q_{12} \geq 1/\tau$), so

\[ W_{\sigma}(q, \tau) = 1 - \frac{1}{8} \left[ 1 + \text{erf} \left( \frac{\mu(q)}{\sqrt{2\tau}} \right) \right]^3, \]  

(3.3)

where (see Appendix I)

\[ \mu(q) = \frac{q}{\sqrt{2[1 - G_{12}(q)]}}. \]

As is seen from (3.1), and (3.3) the pancake characteristics at the moment $\tau$ are expressed through the function $\mu(q)$ related to the initial power spectrum. This is an essential feature of the Zel’dovich theory allowing one to obtain approximate analytical description of pancake properties.

Equation (3.3) demonstrates that $\sim \frac{q}{\tau}$ of matter with $Q(q_{12}) \geq 0$ is compressed at least in one direction and for $\sim \frac{1}{\tau}$ of matter 3D expansion takes place what reflects the symmetry of initial distribution of the displacements, $\Delta S_i$. At first glance it seems that this conclusion is in contradiction with known results of Zel’dovich theory. Actually, the fraction of mass, $f_{DM}$, accumulated by pancakes can be estimated using the PDF for the largest principal value of deformation tensor, $dW(A_{ij})$ (Zel’dovich 1970; Doroshkevich 1970; Doroshkevich & Shandarin 1979; Shandarin & Zel’dovich 1989; Paper I). This PDF can be approximated (with a precision $\sim 10\%$) by the Gaussian function with $q_{12} > 3\sigma_D/\sqrt{2\pi}$, $\sigma_D^2 = (13/6 - 4.5/\pi)\sigma_D^2$. As in the Zel’dovich theory pancake formation is described by the relation $B(z)A_{ij} = 1$ (which follows directly from (2.1) ), so for the fraction of compressed matter with $\lambda_1 \geq 1/\tau$ we have

\[ f_{DM} \approx \frac{1}{2} \text{erf} \left( \frac{\sigma_D}{2\sigma_D} \left( \frac{l_c}{l_0} - \frac{3}{\sqrt{2\pi}} \right) \right) \]  

(3.5)

and for $\tau \gg l_c/l_0$, $f_{DM} \rightarrow 1$ (in the Zel’dovich theory $0.92 \leq f_{DM} \leq 1$). This shows that already during the early period of nonlinear evolution, at $\tau \approx l_c/l_0 \ll 1$, large fraction of matter $f_{DM} \geq 0.9$ is compressed into low mass pancakes. But for the CDM-like power spectrum the description of matter compression through the deformation tensor is appropriate only at small scales $q \leq q_0$, whereas for $q \gg q_0$, the correlations between matter flow in orthogonal directions rapidly decrease what can be seen directly from the expressions for the structure functions given in Appendix I.

At larger scales we have to use the more cumbersome description discussed above and the estimate (3.3) shows that at such scales the efficiency of matter integration into structure elements is only $\sim 0.875$ (more accurate estimates taking into account the correlation of displacements lowers this value to 0.79). This limit is reached already at small $\tau$, for $q \ll 1$, that means strong matter concentration within small structure elements. Further evolution does not change this limit and only redistributes – due to sequential merging – the compressed matter to more and more massive structure elements. Thus, the approximate estimates show that $\sim 21\%$ of matter is subjected to 3D compression, $\sim 21\%$ to 3D expansion, $\sim 29\%$ of matter is subjected to 2D compression and can be accumulated by filaments and $\sim 29\%$ is subjected to 1D compression and remains in pancakes. The difference between estimates (3.3) and (3.5) shows that $\sim 15 - 20\%$ of matter incorporated in small clouds, with $M \leq M_\theta$, is
not accumulated by larger pancakes, with \( M \geq M_d \), and remains distributed between those pancakes. These estimates can be changed because the Zel’’dovich approximation becomes invalid when strong matter compression is reached during pancakes formation.

### 3.2 The characteristics of pancake formation

The probability distribution function (PDF) for pancakes formed at the moment \( \tau \) can be found from (3.3) as

\[
N_{cr}(q, \tau) = -\frac{8}{7} \frac{dW_{cr}}{dq} = \frac{6}{7} \frac{d\mu}{dq} \Phi \left( \frac{\mu}{\sqrt{2\tau}} \right), \quad 0 < q < \infty,
\]

\[
\Phi(x) = e^{-x^2} [1 + erf(x)]^2.
\]

Simple analysis based on relations (3.3), (3.6) shows that the maximum of the PDF (3.6) is reached at \( q \approx 1.5q_0 \). The mean mass (surface density) of formed pancakes is described by the expression

\[
< m(\tau) > = \int_0^\infty qN_{cr} dq = \frac{8}{7} \int_0^\infty W_{cr} dq.
\]

The rate of formation of pancakes with mass \( q \) is

\[
N_r = \frac{8}{7} \frac{dW_{cr}(q, \tau)}{d\tau} = \frac{6}{7} \frac{\mu}{\sqrt{2\tau}} \Phi \left( \frac{\mu}{\sqrt{2\tau}} \right),
\]

and the maximal rate is reached for \( \mu(q) \approx \tau \).

These relations take into account the growth of pancakes size due to accretion of matter and form a reliable basis for further theoretical considerations of various characteristics of structure elements. However they cannot be applied directly to the observed and simulated matter distribution as they do not consider the possibility of merging of pancakes. In the process of merging, earlier formed pancakes are accumulated by larger ones. A more refined technique taking into account the pancake interaction should be used in order to describe the evolution of structure elements that are the pancakes that survived the merging process. The same problem appears in the Press-Schechter theory and it can be solved by introduction of the survival probability (Peacock & Heavens 1990; Bond et al. 1991). This problem will be discussed below.

### 3.3 Spatial characteristics of pancake distribution

Using the standard technique (see, e.g., BBKS) we can also find the mean cumulative comoving linear number density of pancakes, \( n(>q) \), which characterizes the pancake distribution along a random straight line

\[
n(>q)dr = \frac{3}{4\pi} \frac{\mu(q)}{q} < \nabla Q \cdot d\vec{r} > \Phi \left( \frac{\mu}{\sqrt{2\tau}} \right).
\]

When \( l_c \ll l_0 \) it is described by the following (approximate) expression:

\[
n(>q) \approx \frac{3}{4\pi\sqrt{q_0}} < \sqrt{6 + \ln(\sqrt{3} + \sqrt{2})} > \frac{\mu(q)}{q} \Phi \left( \frac{\mu}{\sqrt{2\tau}} \right),
\]

\[
l_c n(>q) \approx \frac{1.35}{\sqrt{6\pi}} \frac{\mu(q)}{q} \Phi \left( \frac{\mu}{\sqrt{2\tau}} \right),
\]

where \( < \mu_c \approx 0.5 \) describes the random orientation of a pancake and the chosen straight line (see also Appendix II).

To characterize the 3D distribution of higher peaks of the function \( Q(q) \) and the percolation process, in the course of which separate pancakes are integrated into the joint structure, the standard technique using the Euler characteristics can be applied. It is described in details by BBKS; Gott et al. (1989); Coles et al. (1996); Seto et al. (1997).

In our case the anisotropy of the pancake compression manifests itself as an anisotropy of correlation functions of displacements and their derivatives what results in the appearance of different Euler characteristics, \( n_{31}(q, \tau), n_{32}(q, \tau), n_{33}(q, \tau) \):

\[
n_{31} = \frac{3}{32\pi^2} < \frac{Q_{22}Q_{33} - Q_{23}^2}{\sigma_{11}\sigma_{22}\sigma_{33}} > < |Q_1| > \Phi \left( \frac{\mu}{\sqrt{2\tau}} \right),
\]

\[
Q_{ij} = \frac{\partial Q}{\partial q_i}, \quad Q_{ij} = \frac{\partial^2 Q}{\partial q_i \partial q_j}.
\]

where variances \( \sigma_{11}, \sigma_{22}, \sigma_{33} \) are defined by (II.4) and (II.6) and expressions for \( n_{32} = n_{33} \) can be found by cyclic permutation of indices of derivatives \( Q_i, Q_{ij} \).

Functions \( n_{32} \& n_{33} \) characterize the percolation process in the direction \( q_{12} \), and \( n_{31} \) characterizes this process in transversal directions. For the most interesting case \( q_0 \ll 1, q_0 \ll q \) we have

\[
l_0^3 n_{32} = l_0^3 n_{33} = n_0 \left( \frac{\mu^2}{\tau^2} - \frac{1 + 2q}{3(1 + q)} \right),
\]

\[
l_0^3 n_{31} = n_0 \left( \frac{\mu^2}{\tau^2} - 1 \right),
\]

\[
n_0 = \frac{3}{4\pi} \sqrt{\frac{\mu}{q_0}} \Phi \left( \frac{\mu}{\sqrt{2\tau}} \right).\]

The function \( n_{31} \) is similar to the standard expression for an isotropic Gaussian field (BBKS).

The first zero of Euler characteristic describes approximately the percolation when separate higher peaks are incorporated into a larger (in the limiting case - infinite) structure element (Tomita 1990; Mecke and Wagner 1991). The expressions (3.12) show that in the directions orthogonal to \( q_{12} \) the percolation takes place at the 'moment' \( \tau = \mu(q) \) whereas along \( q_{12} \) it occurs later, at \( \tau = \mu(q)\sqrt{3(1 + q)}/(1 + 2q) \). This fact favors two step percolation and percolation along the pancake surface takes place first. For a given \( q \) and \( \tau \) the ratio \( n(>q)/n_{31} \) and \( n(>q)/n_{32} \) measure the mean surface of formed pancakes.

Expressions (3.7) \& (3.12) show that at small \( \tau \) percolation takes place for pancakes with \( m \approx \langle m \rangle \approx 4\tau^2 \). This means that only the highest peaks with \( m \geq \langle m \rangle \) can be considered as discrete objects. The number
density of such peaks can be obtained from (3.12), for \( \mu/\tau > 1 \), as

\[
\ell_{q, q_{pk}}(> q, \tau) = m_0 \frac{\mu^2}{\tau^2}. \tag{3.13}
\]

Later, when objects with \( m \sim 1 \) are formed and an essential fraction of matter is concentrated in larger pancakes with \( m \sim < m > \geq \tau \), the percolation takes place through relatively low mass pancakes with \( m \leq < m > \). This ideal picture will be however strongly distorted by the matter compression or expansion in transversal directions and by pancake disruption due to the gravitational instability.

### 3.4 Pancake interaction

An important characteristic of pancake interaction is the two points PDF which gives the conditional probability to form a pancake with the size \( q \) at the moment \( \tau \) at the distance \( D_{sep} \) from a pancake with the size \( D_1 \) formed at the moment \( \tau_1 \). Here we will restrict our consideration to the simplest case when all three vectors, \( q \), \( D_1 \), \& \( D_{sep} \) are aligned along the same straight line that is the most interesting case for the small separations. Much more cumbersome case of arbitrary arrangement and/or orientation of these vectors will be considered separately.

In this simplest case the conditional probability to have \( \Delta S > q/\tau \) is

\[
W_c(q, \tau; D_1, D_{sep}) = 0.5 \cdot \text{erfc}(q_2/\sqrt{2}), \tag{3.14}
\]

\[
g(x, y, r_s) = \frac{x - r_s y}{\sqrt{1 - r_s^2}}, \quad g_2 = g[\mu(q)/\tau, \mu(D_1)/\tau_1, r_s].
\]

The function \( \mu(q) \) is given by (3.3) and the coefficient of correlation, \( r_s(q, D_1, D_{sep}) \), describing the interaction of two pancakes, can be found in Appendix III (III.1). The conditional distribution function of such pancakes can be written as

\[
N_c(q, \tau; D_1, D_{sep}) = \left[ \frac{2}{\pi} \frac{dq_2}{d^q} \right]^{-0.5 q_2^2}. \tag{3.15}
\]

These relations characterize the parameters of pancakes formed at the moment \( \tau \) under the condition of a pancake formation at the moment \( \tau_1 \) with the size \( D_1 \) and the separation \( D_{sep} \).

The basic relation (3.1) implies that two pancakes with sizes \( D_1 \) and \( D_2 \) and a separation \( |D_{sep}| \leq 0.5(D_1 + D_2) \) merge together and form a single pancake. For larger separations merging of pancakes can also be considered in the same manner as before, but using the Euler position of formed pancake

\[
r_{pan} = \int_{q_1}^{q_2} r(q, q_{12}) dq_{12} = \frac{1}{1 + z} \left[ q_{cent} - B(z) \Delta \phi_{12}/q_{12} \right], \tag{3.16}
\]

where \( q_{cent} = 0.5(q_1 + q_2) \) is the Lagrangian position of central point of the pancake and \( \Delta \phi_{12} = \phi(q_1) - \phi(q_2) \). As it is apparent from (3.16), \( r_{pan} \) depends on the potential difference, \( \phi(q_1) - \phi(q_2) \), and therefore the pancake evolution depends on the potential distribution.

As discussed in Paper I around each pancake there is a potential well with a typical size exceeding the mass \( m \) of collapsed pancake by a factor of \( 1.5 - 2 \). The matter infall into these potential wells stimulates merging of pancakes formed with a small separation. For larger separation the influence of this factor decreases progressively and the process of merging becomes (almost) random. These results are consistent with predictions of the adhesion model (Shandarin & Zel’dovich 1989).

The conditional probability that two pancakes with sizes \( D_1 \) and \( D_2 \) whose separation is \( |D_{sep}| \geq 0.5(D_1 + D_2) \) merge at the moment \( \tau \) is defined by the condition

\[
r_{pan}(D_1, \tau, q_{cent}) = n_{pan}(D_2, \tau, q_{cent} + D_{sep}),
\]

and instead of eq. (3.3) we have for the probability of merging

\[
W_{merg}(D_1, D_2, D_{sep}) = \frac{1}{2} \text{erfc} \left( \frac{\chi(D_1, D_2, D_{sep})}{\tau \sqrt{2}} \right). \tag{3.17}
\]

The function \( \chi(D_1, D_2, D_{sep}) \) is given by (III.6). We can extend applicability of the formula (3.17) for small separations by requiring that

\[
W_{merg}(D_1, D_2, D_{sep}) = 1, \quad |D_{sep}| \leq 0.5(D_1 + D_2). \tag{3.18}
\]

### 4 PANCAKE EVOLUTION AND FORMATION OF FILAMENTS

Similar technique can also be used to estimate the transversal size, the pancakes compression and/or expansion in transversal directions, and other properties of pancakes. As we are interested in the formation of structure elements with typical sizes \( q \gg q_0 \) the local description through the deformation tensor cannot be used and the imposed conditions make even approximate description of pancake evolution quite cumbersome (see, e.g., Governato et al. 1998). The general tendencies and rough characteristics of this evolution can only be outlined. Thus, for example, we can estimate the matter fraction compressed within filamentary-like elements and high density clumps as of about 50% whereas only ~29% of matter is subjected to 1D compression. This estimate implies that larger pancakes could also incorporate an essential fraction of filaments and clumps.

The formation of filaments as well as pancakes disruption are stimulated by the growth of density in the course of pancakes compression, and therefore, probably, during early evolutorial stages, filaments represent the most conspicuous elements of the structure. As was discussed in Sec. 3.3, filaments merge to form a joint network. Later, when larger pancakes are formed, the evolution of pancakes becomes slower and disruption of pancakes dominates. These expectations are consistent with the observed and simulated matter distribution. Thus, the conspicuous filaments are seen even at \( z = 3 \) (see, e.g., Governato et al. 1998; Jenkins et al. 1998) whereas disrupted walls dominate at small redshifts (LCRS1, LCRS2, DMRT).

#### 4.1 The characteristics of filament formation

Some approximate characteristics of filament distribution can be obtained by considering the formation of filaments as a sequential matter compression along two principal directions. Such two step compression results in formation of high
density "ridge" surrounded by a lower density anisotropic halo. In a coordinate system with the first and second axes oriented along the directions of maximal and intermediate compressions this process can be approximated described by two equations similar to (3.2):

\[ Q(q_{12}) = q_{12}/\tau, \quad Q(y_{12}) = y_{12}/\tau. \quad (4.1) \]

Here vectors \( \mathbf{q}_{12} \) and \( \mathbf{y}_{12} \) and functions \( Q(q_{12}) \) & \( Q(y_{12}) \) describe the deformation along the first and second coordinate axes respectively and additional conditions introduced in (3.2) are assumed to be fulfilled. As was discussed above the matter compression along the second axis is accelerated by the pancake formation and the function \( \tau_{f}(z) \) differs from that given by (2.9).

Bearing in mind these restrictions we will approximately characterize the probability of filament formation, \( W_{f}^{0} \), for given \( q \) & \( y \) prior to the ‘time’ \( \tau \) & \( \tau_{f} \) or for given \( \tau \) & \( \tau_{f} \) with sizes larger then \( q \) & \( y \), respectively, as the probability to have \( Q(q_{12})/q_{12} \geq Q(y_{12})/y_{12} \geq 1/\tau_{f}, Q(q_{12})/q_{12} \geq 1/\tau \):

\[ W_{f}^{0}(q, \tau; y, \tau_{f}) = 1 - \frac{3}{8}\exp \left( \frac{\mu(q)}{\sqrt{2\pi}} \right)^{2}. \quad (4.2) \]

The PDF for the filaments, \( N_{f}^{0} \), can be found from (4.2) as

\[ N_{f}^{0}(q, \tau; y, \tau_{f}) = \frac{8}{3\pi\tau\tau_{f}} \frac{d\mu(q)}{dq} \frac{d\mu(y)}{dy} \exp \left[ \frac{\mu(q)}{2\tau^{2}} - \frac{\mu(y)}{2\tau_{f}^{2}} \right] \times \left[ 1 + \text{erf} \left( \frac{\mu(y)}{2\tau_{f}} \right) \right]. \quad (4.3) \]

More adequate characteristic of filaments is the linear mass density, \( m_{f} = \pi\sigma g/4 \), defined as a mass per unit length of filament. Both in observed and simulated catalogues filaments usually form a less massive part of structure elements with \( m_{f} < 1 \). For such filaments \( q < 1, \mu(q) \approx \sqrt[4]{2}, y < 1, \mu(y) \approx \sqrt[4]{2} \) and the PDF \( N_{f}^{0}(m_{f}) \) can be found from (4.3) as

\[ N_{f}^{0}(m_{f}, \tau, \tau_{f}) \approx \frac{1}{2\pi < m_{f} > \sqrt{\xi}} \int_{0}^{\infty} \frac{dx}{x} e^{-\frac{x}{\tau} \left( 1+z \right)} \quad (4.4) \]

\[ \xi = \frac{m_{f}}{< m_{f} >}, \quad < m_{f} > = 4\pi^{2}\tau^{2}\tau_{f}. \]

The surface density of spatial distribution of filaments can be obtained using the same technique as in Sec. 3.3.

4.2 The pancake evolution

Separation of filaments into a special class of structure elements requires also a redefinition of population of ‘true’ pancakes. Now the probability of ‘true’ pancake formation defined by conditions \( Q(q_{12})/q_{12} \geq 1/\tau, \quad Q(y_{12})/y_{12} \leq 1/\tau \) is:

\[ W_{f}^{(p)}(q, \tau; y, \tau_{f}) = \frac{3}{8}\exp \left( \frac{\mu(q)}{\sqrt{2\tau}} \right)^{2} \left[ 1 + \text{erf} \left( \frac{\mu(y)}{\sqrt{2\tau_{f}}} \right) \right]^{2}. \]

instead of expressions (3.2) and (3.3). The PDF for such pancakes is

\[ N_{cr}^{0}(q, \tau) = \sqrt{\frac{2}{\pi\tau_{f}} \exp \left( -\frac{\mu^{2}}{2\tau_{f}^{2}} \right)}, \quad 0 \leq q \leq \infty. \quad (4.5) \]

This PDF differs from (3.6) by the form of the function \( \Phi(x) \) what decreases the PDF for larger \( \mu/\tau \).

As before, expression (4.5) characterizes pancakes by the surface mass density of collapsed matter, \( q \). However the pancake surface mass density varies with time after pancake formation due to transversal compression or expansion that results, in particular, in formation of filaments. Even if these transversal motions do not lead to such dramatic results they can change drastically the observed surface density of pancakes. So, the current surface mass density \( m_{p} = q/s_{p} \), where \( s_{p} \) describes the variation of pancake surface caused by transversal motions, is a more adequate characteristic of pancakes.

As was discussed above this period of pancake evolution is not adequately described by the Zel’dovich theory and our results become unreliable. To obtain qualitative characteristics of influence of these factors we can, for example, describe the variation of pancake’s surface as

\[ s_{p} \propto (1 - \tau^{*}Q(y_{12})/y_{12})(1 - \tau^{*}Q(z_{12})/z_{12}). \]

Here vectors \( \mathbf{y}_{12} \) and \( \mathbf{z}_{12} \), and functions \( Q(y_{12}) \) & \( Q(z_{12}) \), describe the deformation along the second and third coordinate axes respectively. The function \( \tau^{*} \) differs from that given by (2.9) and it can depend on transversal motions. Even so rough consideration shows that the exponential term in (4.5) is eroded, and the resulting PDF becomes power-like:

\[ N(m_{p} \ll 1) \propto m_{p}^{-1/2}, \quad N(m_{p} \gg 1) \propto m_{p}^{-2}. \quad (4.6) \]

The pancake disruption accelerates this erosion as well and makes the PDF more complicated.

This discussion shows that slowly evolving pancakes with slower transversal motions can be separated into a special subpopulation for which the surface density changes slowly, \( m_{p} \approx q \), and the PDF (4.5) correctly describes the pancake distribution during the essential period of evolution. This subpopulation is singled out by conditions

\[ |Q(z_{12})/z_{12}| \leq |Q(y_{12})/y_{12}| \leq \epsilon/\tau, \quad \epsilon < 1 \]

and the probability of existence of such pancakes is proportional to \( \tau^{-2} \). This factor describes the disappearance of such pancakes in the course of evolution. This subpopulation can be however quite rich (see., discussion in Sec. 9).

5 STATISTICAL CHARACTERISTICS OF DARK MATTER STRUCTURE ELEMENTS

Results obtained above allow us to find the approximate PDF and other characteristics of structure elements. The structure element with a size (mass) \( m \) is defined as a pancake with the size \( m \) formed at a moment \( \tau \) that not merged with any other pancake.
5.1 Merging of dark matter structure elements

As before the characteristics of structure element at a moment $\tau$ are expressed through characteristics of initial perturbations. The approximate expression for the PDF of structure element can be written in a form similar to the known equation of coagulation:

$$N_{str}(m, \tau) =$$

$$\int_0^\infty dy \int_0^m dx N_c(x, \tau) N_c(m-x, x, y, \tau) \frac{dW_{merg}(x, m-x, y)}{dy}$$

$$- N_c(m, \tau) \int_0^\infty dx \int_0^\infty dy N_c(x, m, y, \tau) \frac{dW_{merg}(x, m, y)}{dy}.$$

The functions $N_c, N_e$ & $W_{merg}$ are given by (3.6), (3.15), (3.17) & (3.18). Here the first term describes the formation of two pancakes with sizes $x$ & $m - x$ and a separation $y$ and their merging to a pancake with the size $m$ while the second term describes merging of the pancake of size $m$ with another pancake. If the mass exchanged during merging is incorporated into the forming pancake then the first term in (5.1) has to be appropriately changed.

Here, as the first step of investigation, we will use the simpler approximate approach based on the survival probability of a pancake with size $m$ to avoid merging with larger pancakes with sizes $x \geq m$. For the more interesting case of smaller pancakes with $q_0 \ll m \leq x < 1$, the most probable process is the formation of two pancakes with sizes $m$ & $x \geq m$ at a small separation $|D_{sep}| \leq 0.5(x + m)$ that means, as follows from (3.1), formation of one structure element with a size $x$. This process is described by the second term in (5.1). Because in this case the probability of merging quickly decreases for larger separations $|D_{sep}| \geq 0.5(m + x)$ and the function $W_c(m, x, D_{sep})$ weakly depends on $D_{sep}$, we will use the approximate expression for the probability of merging, $P_{merg}$,

$$P_{merg}(m, \tau) \approx 2W_c(m, \tau; \frac{m}{\tau}, \frac{m}{\tau}).$$

and for the survival probability, $P_{surv}$,

$$P_{surv}(m, \tau) \approx 1 - P_{merg}(m, \tau).$$

In spite of the approximate character of this approach, it allows us to obtain reasonable estimates of the expected efficiency of merging and of the large scale bias. As it is directly seen from (5.4), for $\mu(m) \leq \tau$,

$$P_{surv}(m, \tau) \propto \mu(m)\tau^{-1},$$

what characterizes the impact of pancake merging.

![Figure 1](https://example.com/fig1.png)

Figure 1. The PDF of structure elements, $N_{str}(m, \tau)$, the mass distribution function, $mN_{str}(m, \tau)$, and the fraction of compressed matter, $f(> m)$, vs. the masses (sizes) of structure elements, $m/m_m$, are plotted for five moments of time: $\tau = 0.1$ (solid line), $\tau = 0.3$ (dashed line), $\tau = 0.5$ (dot-dashed line), $\tau = 0.7$ (dot-dot-dot-dashed line), $\tau = 2$ (long dashed line).

5.2 Statistical characteristics of structure elements

With this survival probability the approximate PDF for the structure elements, $N_{str}(q, \tau)$, can be written as follows:

$$N_{str}(m, \tau) \propto P_{surv}(m, \tau) N_c(m, \tau).$$

These relations allow us to obtain also the approximate mass distribution function, $N_{str}(m, \tau)$, characterizing the distribution of compressed matter over the structure elements. For the more interesting case $m \ll 1$ we have

$$N_{str}(m, \tau) \approx \frac{24}{17\sqrt{2\pi} \tau} \frac{d\Phi}{dm} \left(\frac{\mu}{\sqrt{2\tau}}\right) \cdot erf\left(\frac{\mu}{\sqrt{2\tau}}\right),$$

where $\mu(m)$ is the median mass of structure elements $m_n, \mu(m) \tau$. In the general case this function can be obtained numerically. The mean cumulative comoving linear number density of structure elements, $n_{str}(> m, \tau)$, is

$$n_{str}(> m, \tau) \approx \int_m^\infty dm \frac{dn(> m, \tau)}{dm} P_{surv}(m, \tau)$$

$$= n(> m, \tau) P_{surv}(m, \tau) + \int_m^\infty dm n(> m, \tau) \frac{dP_{surv}(m, \tau)}{dm}.$$
distribution function is rapidly changing due to the merging of smaller pancakes. Later the evolution is slower and it is sustained by growth of median mass due to the progressive matter concentration within richer structure elements.

The same effect is clearly seen in Fig. 2 where the mean linear number density of structure elements, \( n_{\text{str}}(> m, \tau) \), is plotted for the same time moments. Rapid merging of low mass pancakes and formation of more massive pancakes changes the shape of this function at \( m \ll m_m \) while at \( m \sim m_m \) evolution is very slow. The function \( D_{\text{str}}(f) \) plotted in Fig. 2 (bottom panel) is similar to that found in simulations (DFGMM).

Time dependence of the mean, \( < m > \), and median, \( m_m \), masses of structure elements is plotted in Fig. 3 (top panel) together with fits

\[
< m(\tau) > \approx 9 \tau^2 / (1 + 4 \tau), \quad \tau \leq 1.5, \quad (5.8a)
\]

\[
m_m(\tau) \approx 0.03 + 2.2 \tau, \quad \tau \leq 1.5. \quad (5.8b)
\]

Faster growth of \( < m(\tau) > \) in comparison to (3.7) is caused by merging of pancakes as it is described by (5.5). In Fig. 3 (top panel) the \( D_{\text{str}}(m_m, \tau) \) is plotted together with the fit

\[
D_{\text{str}}(m_m, \tau) \approx 1.5 \cdot l_0 \frac{\tau}{\tau + 0.07} \quad (5.8c)
\]

Figure 2. Top panel: the mean comoving linear density of a pancake, \( l_0 n_{\text{str}}(> m, \tau) \), vs. masses \( m/m_m \) are plotted for five moments of time: \( \tau = 0.1 \) (solid line), \( \tau = 0.3 \) (dashed line), \( \tau = 0.5 \) (dot-dashed line), \( \tau = 0.7 \) (dot-dot-dot-dashed line), \( \tau = 2 \) (long dashed line). Bottom panel: the mean separations of pancakes, \( D_{\text{sep}}(> m, \tau) / l_0 \), vs. the fraction of compressed matter \( f(> m) \), for the same time moments.

Figure 3. Top panel: Time dependence of the mean mass, \( < m > \), (solid line), median mass, \( m_m \), (dashed line) and the mean separation \( D_{\text{sep}}(m_m)/l_0 \). The fits (5.8) are plotted by dotted lines. Bottom panel: Evolution of the mean comoving linear number density of structure elements, \( n_{\text{str}}l_0 \), vs. \( \tau \) for four threshold masses \( m/l_0 \geq 0.05 \) (solid line), \( m/l_0 \geq 0.1 \) (dashed line), \( m/l_0 \geq 0.3 \) (dot-dashed line), \( m/l_0 \geq 0.5 \) (dot-dot-dot-dashed line). A power law \( \tau^{-0.4} \) is plotted by a dotted line.

Bottom panel in Fig. 3 shows evolution of the linear number density of structure elements, \( l_0 n \), caused by their exponentially fast formation at smaller \( \tau \), and later, by successive merging which is well fitted by the expression \( D_{\text{str}}(> q, \tau) \propto \tau^{-0.4} \) what is slower then that found in DFGMM.

5.3 Parameters of structure in observed catalogues

In previous Sections the formation and evolution of structure was described in a comoving space. But in observed catalogues the redshift position of galaxies along the line of sight is used. This difference distorts the parameters of observed structure with respect to the theoretical expectations (see, e.g., Melott et al. 1998; Hui, Kofman & Shandarin 1999). To take into account this distortions the theoretical relations given above need to be modified.

In observational catalogues the distance to a galaxy is defined by its observed velocity which can be found from
\[ \tau(z) \to \tau_e(z) = \tau(z) \cos \varphi + \sin \varphi, \]

(5.10)

where \( \varphi \) is a random angle between the line of sight and the direction of pancake compression. It is evident that \( 1 \leq \tau_e / \tau \leq \sqrt{\beta^2 + 1} \), and for \( \varphi = \pi / 2 \) \( \tau_e = \tau \). This means that the observed structure parameters will be randomly increased with respect to those found in the comoving space.

The value of this distortion depends on the cosmological parameters and redshift and is superposed with distortions generated by the structure disruption, selection effects, and other random factors. The well known example of such distortion is the effect of so called “finger of God”. The direct comparison of simulated structure parameters in comoving and redshift spaces performed in DMRT demonstrates a moderate dependence of the main structure properties on these factors. But some properties of the structure such as characteristics of galaxy distribution within walls are different in comoving and redshift spaces. More details can be found in Sec. 9 and DMRT.

6 FORMATION OF VOIDS

The same method can be used to solve the complementary but more complicated problem of void formation. According to the Zel’dovich theory the probability to find a ‘void’ with a size \( r > r_v \) is identical to the probability that the inequality

\[ (1 + z)r = q_1 - q_2 - \tau \Delta S_{12} \geq (1 + z)r_v \]

holds under the conditions that pancakes have been formed near the boundary points \( q_1 \) and \( q_2 \) and in the absence of any pancakes between points \( q_1 \) and \( q_2 \). This is equivalent to the probability that the inequality

\[ \Delta S_{12} \geq (q_2 - (1 + z)r_v) / \tau \]

holds under the same conditions.

More accurately we can consider the under dense regions bounded by pancakes with masses \( m_1 \) and \( m_2 \) (or with masses exceeding the threshold mass, \( m_{th} \)) in the absence of any pancakes with masses \( m \geq m_{th} \) between points \( q_1 \) and \( q_2 \). The methods discussed above and supplemented by the void probability function technique could be used, in principle, to obtain such probability but the derivation becomes very cumbersome due to many additional conditions. Numerical methods similar to that used by Sahni et al. (1994) could be more suitable for such an investigation.

Let us remind however that the parameters of under dense regions are closely linked to the spatial distribution of initial gravitational potential what allows to specify conveniently the large scale perturbations. This link is clearly demonstrated by the adhesion approach (see, e.g., Shandarin & Zel’dovich 1989) and was discussed in details in Paper I. It can also be described with the technique considered above. Thus, for example, the large scale modulation of the pancake distribution by the spatial distribution of this potential can be described as a modulation of the effective ‘time’ moment \( \tau_{eff} \).

To illustrate this statement we can consider the pancake formation for a given potential difference, \( \Delta \phi \), between two points with comoving coordinates \( x_1 \) and \( x_1 + D_{cell} \). In this case the distribution function of the pancake with a size \( m \ll D_{cell} \) at the point with a comoving coordinate \( x_1 \leq y \leq x_1 + D_{cell} \) is given by (3.6) and (5.5) with

\[ \frac{1}{\tau_{eff}(x)} \approx \frac{1}{\tau} \left( 1 - \kappa \phi(D_{cell}, y) \frac{\Delta \phi}{\sigma_\phi} \right), \quad 0 \leq y \leq D_{cell}, \]

(6.3)

\[ \sigma_\phi^2 = 2G_0(D_{cell}), \quad \kappa \phi = G_{12}(y) - G_{12}(D_{cell} - y). \]

Here \( \sigma_\phi \) is the dispersion of gravitational potential difference at the points with the separation \( D_{cell} \). Functions \( G_0 \) and \( G_{12} \) are introduced in Appendix I.

As is seen from (6.3) the efficiency of pancake formation depends directly on the coordinate \( y \) and the potential difference, \( \Delta \phi(D_{cell}) \). It is amplified in regions with \( \kappa \phi \Delta \phi < 0 \), and depressed in regions with \( \kappa \phi \Delta \phi > 0 \).

The same large scale modulation modifies the Euler position of pancake and instead of Eq. (3.16) we obtain after averaging of \( \Delta \phi(m)/m \) (under the condition \( m \ll D_{cell} \))

\[ < (1 + z)\tau_{pan} > = y + \tau \bar{G}_e(y, D_{cell}) \frac{\Delta \phi}{\sigma_\phi}, \]

(6.4)

\[ \bar{G}_e(y, D) = yG_{12}(y) + (D - y)G_{12}(D - y), \]

and for the mean Eulerian linear number density of pancakes

\[ n_c(y) = \left( \frac{\partial r_{pan}}{\partial y} \right)^{-1} = \frac{1 + z}{1 + \tau \bar{G}_n(y, D_{cell}) \Delta \phi}. \]

(6.5)

\[ \bar{G}_n(y, D) = G_{12}(y) - G_{12}(D - y) + y^2G_{23}(y) - (D - y)^2G_{23}(D - y). \]

These expressions describe the matter infall into wells of initial gravitational potential and clearly demonstrate the close connection of under dense regions with the spatial distribution of initial gravitational potential. More detailed discussion of this effect can be found in Paper I.

7 LARGE SCALE BIAS AND CHARACTERISTICS OF BIASED COLD MATTER DISTRIBUTION

7.1 Large scale bias

In observations we have to deal with the galaxy distribution which is, probably, biased relative to the spatial distribution of dark matter in the considered range of scales. The qualitative physical ground of such large scale bias is transparent and was discussed already ten years ago by Dekel & Silk (1986), and Dekel & Rees (1987). It can be caused by the joint action of large scale perturbations and reheating of the baryonic component of matter. The interaction of large and small scale perturbations shows up as an excess of low mass pancakes and acceleration of their growth.
within richer wall-like elements and is seen in simulations as a strong concentration of high density peaks within filaments and richer walls. It results in an excess of compressed cold gas inside "proto" walls before reheating. But after reheating further compression and cooling of gas are inhibited due to the growth of entropy of uncompressed gas. This excess of cold, low entropy baryonic matter, reached at the reheating redshift, \( z = z_h \), can be considered as the excess of galaxies within richer walls that is the large scale bias in spatial galaxy distribution.

In Paper I the modulation of the rate of pancake formation by the large scale perturbations of initial gravitational potential was discussed. Here we consider the direct interaction of earlier and later pancakes, that allows us to obtain more reliable estimates of the large scale bias. With the technique developed above we can consider only the direct interaction of two population of pancakes, namely, those created at \( z = z_h \), and at \( z = 0 \), and we have to neglect the mutual influence of intermediate population of pancakes accumulated by the walls. Such an influence can in principle be considered but the description becomes very cumbersome. So, our estimate presented below is actually the lower limit of the possible large scale bias. Perhaps, this problem can be studied in simulations which take into account all important factors together (see, e.g., Sahni et al. 1994; Cole et al. 1998).

For the warm dark matter model discussed in Paper I the mass of DM particles restricts the minimal size of created pancakes to the correlation scale of initial density perturbations, \( r_c \). For the CDM power spectrum considered here there are no natural limits and, in principle, pancakes with arbitrarily low mass can form. Formally, it is due to the divergence of higher moments of power spectrum and, because of this, the typical scale \( r_c \) is zero. In this case, however, the typical scale \( q_o \) introduced by (3.4) plays similar role and discriminates the process of earlier low mass pancake formation, described through the deformation tensor, and later formation of structure elements discussed above. The typical mass of such low mass pancakes given by the expression (3.4) exceeds the estimates of minimal baryonic mass formed before the reheating, \( M_{\text{min}} \sim 10^6M_\odot \) (Tegmark et al. 1997), and the value \( q_o \) can be accepted as a natural limit of pancake size. Further evolutionary history of low mass pancakes is uncertain, and this problem should be considered separately.

Thus, for the CDM power spectrum we assume that:

(i) All baryons accumulated by structure elements with a size \( m \geq q_o \), at the redshift \( z = z_h \), \( \tau = \tau_h \), are incorporated into observed 'galaxies'.

(ii) The formation of such clouds is interrupted at the redshift \( z = z_h \), \( \tau = \tau_h \) by the instantaneous reheating of the uncompressed gas.

Both assumptions are very restrictive but they can be used for rough estimates of the efficiency of this mechanism of generation of the large scale bias. The moment of reheating is, probably, restricted to \( 7 \leq z \leq 15 \). The lower limit is imposed by the observational constrains. The upper limit is imposed by the very fast growth of the inverse Compton cooling for larger redshifts.

This means that for the models with \( \Omega_m \sim 0.3 - 1 \) the reheating probably occurred at \( \tau_h = \tau(z_h) \approx (0.15 - 0.3)\tau_0 \).

7.2 Statistical characteristics of biased cold matter distribution

The biasing factor on the LSS and SLSS scales can be estimated in the same manner as the probability of merging (5.2). Now we are interested in the mass of cold matter collapsed at \( \tau = \tau_h \) which was accumulated at the moment \( \tau \) by a pancake with a size \( D \gg m_{\text{cld}} \). The approximate PDF for this mass can be written as

\[
N_{\text{cld}}(m_{\text{cld}}, \tau_h) \approx 2N_c \left( m_{\text{cld}}, \tau_h; D, \tau, \frac{m_{\text{cld}} + D}{2} \right)
\]

\[
\approx N_{\text{cr}}(m_{\text{cld}}, \tau_1),
\]

\[
\frac{1}{\tau_1} = \frac{1}{\tau_h} - \frac{\kappa_1(D)}{\tau}, \quad \kappa_1(D) \approx \frac{1}{(2 + D)(1 + D)}.
\]

These relations show that, as it was discussed in Paper I, the formation of earlier pancakes within larger structure elements is accelerated, what is seen in (7.1) as a shift of the effective evolutionary 'time', \( \tau_1 \), relatively to the real reheating 'time' moment, \( \tau_h \). The corresponding mean 'mass' of such pancakes can be found as

\[
<m_{\text{cld}}(\tau_h) > \approx 4\tau_1^2 \approx 4\tau_h^2(1 + 1.5\tau_h/\tau), \quad D < 1.
\]

The dimensionless biasing factor

\[
b_{\text{bias}} = \frac{<m_{\text{cld}}>}{<m>} - 1 \approx 1.5\tau_h/\tau,
\]

is about of 1/5 – 2/5 and weakly depends on the size \( D \) of the considered structure element. The relation (7.1) describes also the excess of larger pancakes induced by the influence of larger structure elements

\[
\frac{N_{\text{cld}}(m_{\text{cld}}, \tau_h)}{N_{\text{cr}}(m, \tau_h)} \approx \frac{\tau_h}{\tau_1} \exp \left( \frac{m_{\text{cld}}}{2\tau_1^2} \right).
\]

As is seen from (7.1) the bias decreases, for \( \tau \gg \tau_h \). This means that the indirect multi step interaction becomes specially important and the intermediate population of pancakes formed at \( \tau_h \leq \tau^* \leq \tau \) with a size \( x^* \leq D \) and accumulated by a structure element with a size \( D \), at the moment \( \tau \), will enhance the bias and increase the concentration of cold matter within larger pancakes.

These results are consistent with the conclusions of Paper I, Cole et al. (1998) and Sahni et al. (1994) and demonstrate the sensitivity of the considered bias to the detailed characteristics of the process of galaxy formation. They show that the over and under dense regions found now in the observed galaxy distribution can, probably, be related even at high redshifts to a spatial modulation of sizes of the earlier pancakes. The larger pancakes are more closely associated with over dense regions whereas the under dense regions are occupied preferentially by low mass pancakes.

Summarizing, we can infer that the influence of large scale bias generated by a combined action of reheating and large scale perturbations can be important and can distort characteristics of the observed galaxy distribution relative to the same characteristics of DM distribution. Under suitable conditions the fraction of baryons accumulated by galaxies can be suppressed in low massive pancakes and under dense regions and increased in richer structure elements.
8 DYNAMICAL CHARACTERISTICS OF THE PANCAKES

In this section we will find two dynamical characteristics of pancakes, namely, the velocity of pancakes as a whole and the dispersion of matter velocity within pancakes caused by compression of pancakes.

The velocity of an infalling particle, \( v_i \), can be found from (2.1) as

\[
v_i(q, z) = \frac{d}{dt} \left[ \frac{H(z)}{1 + z} |q_i - \beta(z) B(z) S_i(q) | \right],
\]

where \( H(z) \) is the Hubble constant.

The distribution function of peculiar velocity is Gaussian as

\[
\beta = 1 - \frac{1 + z}{z} \frac{d B(z)}{d z},
\]

where \( \phi(z) \) is the Hubble constant.

The functions \( \beta(z) \) can be found from (2.2) and (2.3) as

\[
\beta \approx 2 - B^2(z) \left[ \frac{1 - \Omega_m}{1 + \Omega_m \Omega_m} - 1 \right], \quad \Omega_m + \Omega_\Lambda = 1,
\]

where \( \Omega_m \) is the matter density parameter.

For \( \Omega_m = 1 \) both expressions give \( \beta = 2 \).

The general relation for the velocity of a particle (8.1) and the velocity of a pancake as a whole through the momentum conservation law, we have

\[
v_{pan} = \frac{1}{q_{12}} \int_{q_{27}} v(q) dq = H(z) \frac{1 + z}{1} \left[ q_{cent} - \beta B(z) \frac{\Delta \phi_{12}}{q_{12}} \right].
\]

The velocity dispersion, \( \sigma_v \), vs. the size of "proto walls". \( q = R_{wall} \), for the redshift \( z = 0 \), for five cosmological models with \( h = 0.5, \Omega_m = 1, \Omega_\Lambda = 0 \) (solid line), and \( h = 0.7, \Omega_m = 0.5, \Omega_\Lambda = 0.5 \) (dashed line), \( \Omega_m = 0.5, \Omega_\Lambda = 0 \) (dotted line), \( \Omega_m = 0.3, \Omega_\Lambda = 0 \) (long-dashed line). Dotted lines show observational estimates of the size of "proto walls" \( R_{wall} \sim 30 h^{-1} \text{Mpc} \), and the velocity dispersion \( \sigma_v \sim 400 \text{km/s} \).

The velocity dispersion within a pancake with a given mass \( q = q_1 - q_2 \), caused by the matter infall into pancake, is defined by the energy conservation law. The velocity of a particle with the coordinate \( q_3 \) relative to the pancake is

\[
v_{12} = \frac{1}{q_{12}} H(z) \left[ q_{12} - \beta B(z) \frac{\Delta \phi_{12}}{q_{12}} \right].
\]

The mean peculiar velocity, \( < v > \), and its dispersion, \( \sigma_v \), is

\[
\sigma_v^2 = \frac{H^2(z)}{1 + z} \left[ \beta(z) - 1 \right]^2 B^2(z) \frac{2G(m_{12})}{q_{12}^2}.
\]

The distribution function of peculiar velocity is Gaussian as before. In dimensional variables, \( z = 0 \), and for \( q_{12} \ll 1 \), we have:

\[
\sigma_v(0) = H_0 |\beta(0) - 1| \sigma_z / \sqrt{3} = H_0 |\beta(0) - 1| H_0 / 3.
\]

These relations show that for smaller \( q_{12} \) the pancakes velocity is close to the mean velocity dispersion and decrease \( \propto q_{12}^{-1/2} \), for \( q_{12} > 1 \). The vortexless character of the initial velocity perturbations is clearly seen in eq. (8.4), (8.5).

Variations of the pancake velocity with the transversal coordinates can distort the shape of pancake and, in principle, can destroy the pancake. The impact of this factor can be characterized through the expression (8.4) as a correlation of pancake velocity at different transversal coordinates. This correlation is described by the normalized function \( r_v \) (see Appendix IV), and for \( q_{12}, q_{34} \ll 1 \), we have

\[
r_v = q_{12} \cdot q_{34} / q_{12} q_{34},
\]

what demonstrates the regular character of velocity variations. For larger \( q_{12} \) and \( q_{34} \) the velocity dispersion (8.6) decreases and therefore this factor becomes less important.

![Figure 4. The velocity dispersion, \( \sigma_v \), vs. the size of "proto walls".](image-url)

The velocity dispersion within a pancake with a given mass \( q = q_1 - q_2 \), caused by the matter infall into pancake, is defined by the energy conservation law. The velocity of a particle with the coordinate \( q_3 \) relative to the pancake is

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\sigma_v^2 = \frac{1 + z}{H^2(z)} \left[ \beta(z) - 1 \right]^2 B^2(z) \frac{2G(m_{12})}{q_{12}^2}.
\]

The distribution function of peculiar velocity is Gaussian as before. In dimensional variables, \( z = 0 \), and for \( q_{12} \ll 1 \), we have:

\[
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r_v = q_{12} \cdot q_{34} / q_{12} q_{34},
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what demonstrates the regular character of velocity variations. For larger \( q_{12} \) and \( q_{34} \) the velocity dispersion (8.6) decreases and therefore this factor becomes less important.
9 COMPARISON WITH SIMULATED AND OBSERVED STRUCTURE PARAMETERS

Some characteristics of large scale galaxy distribution can be extracted from available catalogues of galaxies and clusters of galaxies and catalogues of absorbers in spectra of quasars. Some characteristics of large scale DM distribution can be derived from available simulations. All these characteristics are distorted by selection effects, bias between spatial distribution of DM and galaxies, and other factors what complicates the direct comparison of characteristics obtained with different methods and data bases.

Nonetheless, the comparison of even so distorted characteristics with the expected characteristics discussed above can be interesting as it allows one to compare various estimates and illustrates their sensitivity to basic cosmological parameters and other factors. More detailed discussion can be found in DMRT & LCRS2.

9.1 Observed and simulated parameters of the structure

Analysis of galaxy distribution in the Las Campanas Redshift Survey (Shectman et al. 1996) shows that the richer and poorer structure elements can be assigned to wall-like and filamentary populations, but the accurate demarcation of these elements is problematic. Both subpopulations accumulate about 40 – 60% of galaxies. Filaments fill the gaps between the walls and form a random network with the mean cell size \( D_{\text{cell}} \sim 10 – 12h^{-1}\) Mpc at \( z = 0 \). The richer walls can be formed due to an anisotropic compression of matter within slices with a typical size of \( D_{\text{wall}} \sim 50 – 60h^{-1}\) Mpc and with a typical separation of \( D_{\text{wall}} \sim 50 – 60h^{-1}\) Mpc (LCRS1 & LCRS2). The walls are significantly disrupted by the small scale clustering of galaxies (see, e.g., Fig. 5 in Ramella et al. 1992). The velocity dispersion within such wall-like elements was estimated as \( \sigma_{\text{obs}} \sim 350 – 400\) km/s (Oort 1983a). The bulk velocities of galaxies are now estimated as \( \sigma_{\text{obs}} \sim 400\) km/s (see, e.g., Dekel 1997).

The observed surface density of structure elements is heavily distorted by the selection effects and small scale clustering and is equally well fitted to a power law, exponential distribution, and functions discussed in Secs. 4 & 5.

The structure in the DM distribution was simulated and analyzed by DMRT for the SCDM model with \( \Omega_m = 1 \), \( h = 0.5 \). ACDM model with \( \Omega_m = 0.35 \), \( \Omega_L = 0.65 \), \( h = 0.7 \) and OCDM model with \( \Omega_m = 0.5 \), \( h = 0.6 \). For all these models the normalization to the two year COBE data were used. These simulations reproduce the structure with the main parameters similar to those found in observations. The analysis shows that walls accumulate about 40% of DM with a mean size of "proto walls" \( D_{\text{wall}} \sim 15 – 20h^{-1}\) Mpc and with a mean separation of walls and filaments \( D_{\text{wall}} \sim 50 – 70h^{-1}\) Mpc, \( D_{\text{fil}} \sim 9 – 14h^{-1}\) Mpc, respectively.

In all the models the simulated velocity dispersion within walls, \( \sigma_{\text{sim}} \), is the same along all three principal directions of walls and it is generated by the wall curvature and disruption into a system of high density clumps. The walls are seen usually as ordered sets of irregular high density clumps connected by lower density bridges. The degree of wall disruption depends on the code used and reached resolution, and it is more conspicuous in the models with larger \( \Omega_m \). In the redshift space this inner structure is partly eroded and characteristics of DM walls are similar to that found for the observed galaxy distribution.

The isotropy of simulated velocity dispersion \( \sigma_u \) confirms the essential influence of small scale clustering on the properties of both observed and simulated structure elements. The instability of a thin compressed layer was analyzed in the linear approximation by Doroshkevich (1980), and Vishniac (1983), and was recently simulated by Valinia et al. (1997). The clustering rate and parameters of formed clusters depend on the density of compressed matter and properties of transversal velocity field.

9.2 Comparison of simulated and expected structure properties

Theoretical parameters of DM wall-like structure elements are listed in Table I for the median mass \( m = m_\text{med} \) and for \( l_c/l_0 = 0.056 \). The variations of \( D_{\text{wall}} \) listed in Table I are moderate and, in the range of precision of our approximate consideration, these values are similar to those observed and simulated. The differences become more significant for the OCDM models. The value of \( D_{\text{wtr}} \) is sensitive to the assumed mass threshold (see Figs. 2 & 6) and can be adjusted.

The simulated structure elements are also distorted by the small scale clustering but owing to the rich statistics of such elements a more detailed quantitative comparison of simulated and expected properties of the structure can be performed. To do this we will consider the PDFs (4.4) and (4.5) for the surface density of filaments and pancakes, the PDFs for velocity \( u \) and velocity dispersion \( \sigma_u \) and direct estimates of important parameters

\[
\tau_u = \frac{\sqrt{\langle \sigma_u^2 \rangle}}{H_0 l_0 (\beta - 1)} \quad m_\text{w}/l_0 = 2\sqrt{3/\tau_u} \quad (9.1)
\]

following from relations (8.7) and (8.11).

To characterize the filamentary and wall-like subpopulations of structure separately the analysis was performed both in the comoving and redshift spaces for three subsamples of structure elements. The first contained all particles, the second and third incorporated the richer and poorer structure elements with \( N_{\text{mem}} > 200 \) and \( N_{\text{mem}} < 200 \), respectively. Here \( N_{\text{mem}} \) is the number of DM particles within a structure element bounded by the threshold density \( n_{\text{thr}} / n > = 1.7, 1.1 \) & 1.1 for the SCDM, OCDM and ACDM models. In all the cases the second subsamples accumulates about 40 – 45% of particles.

To obtain the required PDF a set of rectangular sampling cores was prepared and the number of DM particles in the intersection of separate structure elements with these cores was taken as a characteristic of surface density of structure elements. For each of the cluster its velocity \( u \) and velocity dispersion \( \sigma_u \) along the core were also found. To depress the impact of small scale clustering the analysis was performed for the core size \( l_{\text{core}} = 10h^{-1}\) Mpc, for the second subsample, and for \( l_{\text{core}} = 4h^{-1}\) Mpc, for the first and third subsamples when the core size is restricted by the separation of filaments. The random intersection of cores and structure elements generates significant excess of poorer clusters. To depress this effect poorer clusters were rejected and the truncated PDFs were considered. Even so, the number of clusters used was \( \sim 3 \) 500, for the second, and \( \sim 10 \) 000, for the first...
Table 1. Main expected parameters of the SLSS.

| $\Omega_m$ | $\Omega_\Lambda$ | $h$ | $l_0$ | $\tau_0$ | $R_{\text{wall}}$ | $D_{\text{str}}$ | $\tau_u$ | $m_u/l_0$ | $\tau_v$ | $m_v/l_0$ | $\tau_p$ |
|------------|-----------------|-----|------|---------|-----------------|-----------------|--------|-----------|---------|-----------|--------|
| 1          | 0               | 0.5 | 13.2 | 0.7     | $\sim 20.7$   | $\sim 19.8$    | 0.52   | 1.2       | 0.42    | 1.0       | 0.41   |
| 0.5        | 0.5             | 0.7 | 18.9 | 0.6     | $\sim 25.5$   | $\sim 26.8$    | 0.28   | 0.5       | 0.27    | 0.8       | 0.35   |
| 0.5        | 0               | 0.6 | 22.0 | 0.3     | $\sim 14.5$   | $\sim 26.9$    | 0.40   | 0.9       | 0.36    | 1.3       | 0.44   |
| 0.35       | 0.65            | 0.7 | 26.9 | 0.4     | $\sim 23.5$   | $\sim 37.1$    | 0.40   | 0.9       | 0.36    | 1.3       | 0.44   |
| 0.35       | 0               | 0.7 | 26.9 | 0.2     | $\sim 12.0$   | $\sim 34.5$    | 0.28   | 0.5       | 0.27    | 0.8       | 0.35   |

Figure 5. The simulated PDFs of surface density of structure elements, $N_{\text{str}}(m)$, vs. $m/\langle m \rangle$ for the full sample (top panel) and subsamples of richer (middle panel) and poorer (bottom panel) structure elements for the ΛCDM model. The fits (9.3) and (9.4) are plotted by solid lines, the power fits are plotted by dashed lines.

and third subsamples. The random orientation of structure elements and cores can increase the measured surface density (up to two times for homogeneous matter distribution, Kendall & Moran 1963), and reduces the measured velocity of structure elements along the core, $u_c$, by a factor of $\sqrt{3}$ and we have

$$u_c = u \cos \varphi, \quad \tau_u = \sqrt{3} \tau_{uc}. \quad (9.2)$$

To compare the simulated and theoretical PDFs the two parameters fits were used. For the second subsample the distribution of surface density was fitted to the expression

$$N_w = \frac{a_w}{\sqrt{x_w}} \exp(-x_w) \exp(f(\sqrt{x_w})) \quad x_w = \frac{b_w m_p}{<m_p>}, \quad (9.3)$$

which reproduces approximately the theoretical relations (4.4) and gives better fits. Here parameters $b_w$ & $b_f$ describe the truncated character of fitted PDF whereas parameters $a_w$ & $a_f$ provide their normalization. For velocity of clusters, $u$, the PDF is found to be well fitted to the expected Gaussian distribution, the expression (9.3) with parameters $a_w$ & $b_w$ fits the velocity dispersion $\sigma_v$.

For ΛCDM model these PDFs are plotted in Figs. 5 and 6 together with the best fits. It turns out that different PDFs verify the accepted discrimination of two populations of structure elements. For the second subsample the theoretical estimate of mean wall size

$$<m_p>/l_0 \approx 8(0.5 + 1/\pi) \tau_0^2 \approx 6.55 \tau_0^2, \quad (9.5)$$

links the parameters $\tau_0$ and $<m_p>$. For the ΛCDM and OCDM models the parameters $\tau_u$ (for all samples) and $<m_v/l_0>$, $\tau_v = \sqrt{<m_v/l_0>/6.55}$, $<m_p/l_0>$ & $\tau_p = \sqrt{<m_p/l_0>/6.55}$ (for second subsample) listed in Table I are consistent with the simulated amplitude. For the first reproduce approximately the theoretical relations (4.4) and gives better fits. Here parameters $b_w$ & $b_f$ describe the truncated character of fitted PDF whereas parameters $a_w$ & $a_f$ provide their normalization. For velocity of clusters, $u$, the PDF is found to be well fitted to the expected Gaussian distribution, the expression (9.3) with parameters $a_w$ & $b_w$ fits the velocity dispersion $\sigma_v$.

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and third subsamples the distributions plotted in Fig. 6 and corresponding parameter \( m_{l}/l_{0} \) characterize the typical mass of dominant numerous poorer elements and it is about half of that for the second subsample.

The formal precision of these estimates can be taken as \( \sim 7 - 10\% \) (precision of fits) but a real precision can be estimated by the comparison with the value of the input parameter \( \tau_{0} \) also listed in Table I. Differences between results obtained in comoving and redshift spaces do not exceed the reached precision. For the SCDM model the expected and reconstructed parameters differ by a factor of \( \sim 1.5 \) what can be partly caused by the strong wall disruption. Moreover, this simple description is correct for \( m_{l}/l_{0} \leq 1 \) whereas more cumbersome general relations describe the properties of richer walls which dominate the SCDM model.

In all these cases the simulated mass distribution can be equally well fitted to a power law with the exponent \( \kappa \sim 1.5 - 2 \). Such distribution is similar to that described analytically for the matter concentration within a set of clusters with a surface density \( \sigma_{cl}\sim r^{-\gamma} \) (see also discussion in Miralda-Escude et al. 1996). In this case the measured PDF and mass function are also expressed by the power law

\[
dW_{cl}\propto \sigma_{cl}^{r} dr\propto \sigma_{cl}^{2/\gamma} d\sigma_{cl}. \tag{9.6}
\]

Of course, profiles of separate irregular clusters can vary over a wide range and such description and interpretation illustrate only the important role of small scale clustering.

### 9.3 Evolution of DM structure

Results obtained in Secs. 3, 4 & 5 show that during the most interesting period \( \tau \gg l_{c}/l_{0} \) evolution of DM structure shows features of self-similarity and important characteristics of structure can be expressed as functions of the parameter \( \mu(q)/\tau \). This approximate self-similarity is caused by the Zel’dovich approximation and occurs for any distribution function of initial perturbations and any power spectrum. It becomes more transparent for simple structure functions, at \( q_{0} \ll q \leq 1 \) (Appendix I), typical for the CDM like initial power spectrum, but becomes more cumbersome for \( \tau \sim l_{c}/l_{0} \ll 1 \), and \( \tau \gg 1 \), when the influence of the scales \( q_{0} L_{0} \), introduced by (1.6), (1.7) becomes important. This self-similarity allows us to characterize the structure evolution by the time dependence of several typical parameters such as the mean and median masses, \( <m> \& m_{m} \).

This approximate self-similarity is partly violated due to the evolution of pancakes, discussed in Sec. 4, and small scale pancake disruption as they are not described by the Zel’dovich theory. Nonetheless, the observations and simulations show that in a universe dominated by cold DM particles general properties of matter distribution are quite similar at high and small redshifts (see more detailed discussion in Governato et al. 1998; Jenkins et al. 1998). Self-similarity of structure evolution was previously discussed for the scale-free power spectra (see, e.g., Efstathiou et al. 1988).

The evolution of observed structure formed by galaxies can however be far from the self-similar evolution of DM structure due to the small and large scale bias. Nonetheless, as it was discussed above, the main parameters of DM structure can be compared with observed galaxy distribution. Thus, in particular, the observed separation of filaments, at \( z = 0 \), is about 4 - 6 times smaller than the typical separation of walls (see, e.g., Efstathiou et al. 1988). Such result can be attributed to pancakes formed by compression of a slice with thickness \( m_{f} \sim 0.1 - 0.05 m_{m} \approx 1 - 2h^{-1}\text{Mpc} \) at redshift \( z \sim 3 - 5 \). In smaller pancakes the formation of galaxies can be suppressed and they can be associated with weak Ly-\( \alpha \) absorbers observed far from galaxies (Morris et al. 1993; Stocke et al. 1995; Shull et al. 1996).

### 10 SUMMARY AND DISCUSSION

In this paper we continue investigations and statistical description of the process of structure formation and evolution initiated in our previous papers (Buryak et al. 1992; Demiański & Doroshkevich 1992, 1997; DFGMM; Paper I) and based on the nonlinear theory of gravitational instability (Zel’dovich 1970). The new elements discussed above are the approximate statistical description of the nonlinear structure evolution manifesting itself as a successive matter concentration into more and more massive structure elements and the estimates of the large scale modulation of the spatial distribution of luminous matter relative to DM and baryonic components. We show that, as has been discussed in Sec. 9.4, the evolution of DM structure demonstrates some features of self-similarity and the main characteristics of the DM structure can be expressed through the structure functions of initial perturbations and through parameters of the initial power spectrum and cosmological models. It is shown – in accordance with simulations – that in a low density models the nonlinear evolution occurs on the scale \( \sim 20 - 25h^{-1}\text{Mpc} \) and results in formation of wall-like component of the structure of the universe.

Results discussed in Section 3 show that, for the CDM-like power spectra, the high nonlinear matter compression into low mass pancakes and the percolation take place already at \( \tau \approx l_{c}/l_{0} < 1 \). Later evolution leads to a rapid growth of typical mass of DM pancakes. The theoretical description discussed above can be directly applied to the interpretation of deep pencil beam galactic surveys and absorption spectra of quasars, what allows us to consider together rich observational data accumulated by these methods for both small and large redshifts. For larger 3D catalogs and simulations this description can be applied to results obtained with the core-sampling analysis as it was demonstrated in Sec. 9.2.

Comparison with simulations shows that the velocity dispersion of matter compressed within the wall-like elements is more sensitive to the cosmological parameters, large scale modulation of spatial galaxy distribution and properties of DM component. The small scale clustering generates the essential difference between the expected and simulated velocity dispersion and allows us to discriminate the cosmological models with respect to the matter density, \( \Omega_{m} \), and the composition of DM. This clustering is smaller for the models with \( \Omega_{m} \leq 0.5 \), but for models with \( \Omega_{m} \leq 0.3 \) it is more difficult to reproduce the sizes, separations of observed walls, and the high matter concentration within the SLSS. The observed structure parameters are best reproduced for the models with \( \Omega_{m}h \approx 0.2 - 0.3 \). These values are consistent with estimates obtained both from the simulations of cluster of galaxies (Bahcall & Fan 1998; Cole et al. 1997, ...
Two factors can extend the set of acceptable cosmological parameters. The first is a more complicated composition of DM component, what means that an essential fraction of DM can be associated with relatively hot and/or low mass particles (see, e.g., Colombi et al. 1996; Brustein & Hadad 1998). The simple estimates based on the Trimain - Gunn relation (Trimain & Gunn 1979) show that if the recently formed wall-like SLSS is sensitive to the influence of low mass relic particles, with \( M_p \lesssim 3 - 5 \, \text{eV} \), then the structure properties at higher redshifts are sensitive to more massive particles as well. This means that for the MDM models with more complicated DM composition the agreement between observed and simulated properties of the LSS and SLSS can be achieved in a broader range of \( \Omega_m \) and \( h \) and, moreover, the analysis of observed structure evolution can specify the DM composition. The models including some fraction of unstable dark matter particles (Turner et al. 1984; Doroshkevich et al. 1989) seem also to be perspective. Unstable particles can produce reheating by photo decay whereas the hot products of decay will delay both formation and disruption of the walls.

The second factor is the large scale modulation of spatial galaxy distribution. The formation of LSS and SLSS is always accompanied by bias on the same scales, because both processes are caused by the same large scale perturbations and, so, are strongly correlated in space. The efficiency of such bias depends on many factors, such as the redshift of reheating period, and the formation and evolution of cold clouds. The rough estimates of the bias presented in Sec. 7 show that the bias factor could be quite high (see also Demiański and Doroshkevich 1997; Paper I) and together with other factors it can essentially distort the spatial distribution of galaxies with respect to the DM distribution. These estimates could be enhanced by taking into account the mutual interaction of pancakes with various sizes and moments of formation. More representative estimates of the bias can, probably, be obtained from simulations such as, for example, Sahni et al. (1994) and Cole et al. (1998).

This mechanism of bias generation implies early, for \( z \gtrsim 5 \), formation of the main fraction of low entropy gaseous clouds, that can be identified with 'proto galaxies'. The reheating does not prevent further formation of DM pancakes, which can be identified with a population of gas clouds responsible for weaker absorption lines observed at high redshifts (Miralda-Escude et al. 1996; Hernquist et al. 1996) but it inhibits the formation of high density, low entropy gaseous clouds and delays the formation of galaxies in extended under dense regions. Further transformation of formed earlier 'proto galaxies' into observed galaxies is a slow and complicated process continuing up to now and earlier formation of such clouds is not in contradiction with the observed peak of galaxy formation at redshifts \( z \approx 2 - 3 \) (Steidel et al. 1996).

The observed strong variations of galaxy distribution with respect to the DM and intergalactic gas distribution, \( \rho_{gal}/\rho_{gas} \), can be, partly, associated with this bias. Indeed, if in clusters of galaxies this ratio is found to be \( \rho_{gal}/\rho_{gas} \gtrsim 0.2 \) (see, e.g., White et al. 1993) then, for example, within Bootes Void \( \rho_{gal}/\rho_{gas} \to 0 \) (Weistrop et al. 1992). The existence of 'invisible' structure elements, which are now seen as gas clouds responsible for weak Ly-\( \alpha \) absorption lines situated far from galaxies (\( \approx 5 - 6 h^{-1}\text{Mpc} \), Morris et al. 1993; Stocke et al. 1995; Shull et al. 1996) can also be considered as an evidence in favor of large scale bias.

Simulations show also existence of bias in the distribution of DM and 'galaxies' identified with the highest peaks in the initial density distribution (see, e.g., Eke et al. 1996). Such bias is similar, in some important respects, to that generated by reheating. It operates on really large scales (Bower et al. 1993) and can be essential for the SLSS formation as well. It results in structure composed of filaments and sheets in the distribution of 'galaxies' (Doroshkevich, Fong & Makarova 1998). Other mechanisms of bias formation discussed recently (Coles 1993; Sahni & Coles 1995; Tegmark & Peebles 1998), operate on significantly smaller scales.

Simulations allow us to test the joint action of various factors on the structure parameters and, therefore, they now seem to offer more perspective way for detailed investigations of the structure formation and evolution. They need, however, to be essentially improved in order to discriminate the spatial distribution and other parameters of 'galaxies' and dark matter. The methods used for the measurement and description of simulated and observed structure should be also improved as now they provide us with limited information about structure properties. These restrictions become especially important at higher redshifts, where the simulated density contrast is small. Large dispersions of measured values make also any comparison of simulated and theoretical structure parameters more difficult. Nonetheless, such approach seems to be perspective because further progress in these directions can be reached.

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Figure 7. Structure functions $G_0$ (top panel) and $G_0, G_1, G_{12}$ & $-G_{23}$ vs. Lagrangian coordinate $q/l_0$ for the SCDM power spectrum.

$q_{12} = |\bar{q}_1 - \bar{q}_2|$ and for the perturbations of gravitational potential we have

$$3 < \Delta \phi_{12} \Delta \phi_{34} > = G_0(x_{14}) - G_0(x_{13}) + G_0(x_{23}) - G_0(x_{24}),$$

$$\Delta \phi_{12} = \phi(\mathbf{q}_1) - \phi(\mathbf{q}_2), \quad \Delta \phi_{34} = \phi(\mathbf{q}_3) - \phi(\mathbf{q}_4).$$

(I.1)

Differentiation of this structure function gives other structure functions:

$$3 < \Delta \phi(x_{13}) \Delta \phi(x_{23}) > = (x_{13} - x_{23})G_1(x_{13}) - (x_{13} - x_{23})G_1(x_{23}),$$

$$3 < S_i(x_1)S_j(x_2) > = \Delta_{ij}G_1(x_{12}) + \frac{x_i x_j}{x^2} G_2(x_{12})|_{x=x_{12}},$$

$$3 | \mathbf{q}_k \Delta_i \phi + x_i \Delta_k \phi + x_j \Delta_i \phi | G_2(x) + \frac{x_i x_j}{x^2} G_3(x)|_{x=x_{12}},$$

(I.2)

where the projection operator $\Delta_{ij} = \delta_{ij} - x_i x_j / x^2$ is used and $x = q/l_0$. The scale $l_0$ was introduced by (2.5).

In the general case these functions can be expressed through the power spectrum, $p(k)$, and spherical Bessel functions and we obtain

$$G_0(q) = \frac{3}{2(2\pi)^3/2} \int_0^\infty \frac{p(k)}{k^2} \left( 1 - \sin \frac{k q}{y} \right) dk,$$

$$G_1(q) = \frac{3}{(2\pi)^3/2} \int_0^\infty p(k) y^{-3/2} J_{3/2}(y) dy = G_0'/x,$$

$$G_2(q) = -\frac{3 l_0^3}{(2\pi)^3/2} \int_0^\infty k^2 p(k) y^{-5/2} J_{5/2}(y) dk = G_1'/x,$$

$$G_3(q) = \frac{3 l_0^4}{(2\pi)^3/2} \int_0^\infty k^4 p(k) y^{-7/2} J_{7/2}(y) dy = G_2'/x,$$

where $y = k q / x \approx q / l_0$.

$$G_{12}(x) = G_1(x) + x^2 G_2(x) = G_0''(x),$$

$$G_{23}(x) = 3G_2(x) + x^2 G_3(x) = G_1'/x, \quad (I.4)$$

$$G_{234}(x) = 3G_2(x) + 6x^2 G_3(x) + x^4 G_4(x) = G_1''.$$  

$$G_1(0) = G_{12}(0) = 1, \quad G_{23}(0) = G_{234}(0) = \frac{l_0^2}{l_0^2}, \quad (I.5)$$

and the scale $l_0$ was introduced by (2.5).

For realistic power spectra these functions cannot be expressed analytically and they have to be calculated numerically. For the Harrison - Zel’dovich primordial power spectrum and for the CDM transfer function (BBKS), for $x = q/l_0 \geq 6l_0^2/l_0^2$, $6l_0^2/l_0^2 \ll 1$ these functions can be approximated by

$$G_0(x) \approx \frac{1}{a_0} \ln (1 + x + a_0 x^2)$$

$$-\frac{1}{2a_0 \sqrt{a_0 - 1}} \left( \frac{x \sqrt{a_0 - 1}}{2 + x} \right),$$

$$G_1(x) \approx |1 + x + a_0 x^2|^{-1} x^2,$$

$$G_{12}(x) \approx G_{2}^2(x)|1 - a_0 x^2|^{-1}, \quad (I.6)$$

$$G_{23}(x) \approx -2G_{2}^2(x) (1 + 3a_0 x^2 - a_0^2 x^3)/x,$$

$$G_{234}(x) \approx 6G_{2}^2(x) (1 + 3a_0 x^2 + 4a_0^2 x^3 - a_0^3 x^4),$$

$$a_0 = 5 \left( l_0 / L_0 \right)^2 \approx 0.3, \quad L_0 \approx 4.1 l_0,$$

where the typical scale $L_0$ is defined as

$$L_0 = 3\pi \int_0^\infty p(k) \left( 1 - \sin \frac{k L_0}{k L_0} \right) k^{-2} dk / \int_0^\infty p(k) dk.$$

The functions (I.3) together with fits (I.6) are plotted in Fig. 7. The more cumbersome expression

$$G_1(x) \approx (1 + q_0) \left[ 1 + \sqrt{q_0^2 + x^2} + a_0 x^2 \right]^{-1},$$

$$G_{12} \approx G_{2}^2(1 + q_0)^{-1} \left[ 1 - a_0 x^2 + q_0^2 / \sqrt{q_0^2 + x^2} \right]^{-1}, \quad (I.7)$$

$$q_0 = 6l_0^2/l_0^2 \left[ 1 + \sqrt{1 + 12l_0^2/l_0^2} \right]^{-1} \ll 1$$

can be used for all $x$. In this case the functions $G_2(x)$, $G_{23}(x)$ and $G_{234}(x)$ can be found through the relations (I.4).

Appendix II

Distribution function of pancakes.

We begin with the distribution function for the difference of displacements at points with coordinates $\mathbf{q}_1$ and $\mathbf{q}_2$.
The normalization and notation introduced in Appendix I are used. The dimensionless ‘time’ $\tau$ is introduced by (2.9).

Let us consider the deformation of spherical cloud with a diameter $q$. For the case $q \gg q_0$ the general deformation of cloud can be characterized by the 2D random scalar function $\Theta(\theta, \phi) = [S(q/2) - S(-q/2)]$, $q^2$ instead of the deformation tensor $d_{ik}$. Expansion of this function into spherical harmonics allows one to obtain an approximate description of deformation of the cloud with a reasonable accuracy. Now we will take into account the spherical and quadrupole deformations only. These modes describe successfully the transformation of the cloud into filaments and sheets while higher order deformations relate mainly to the cloud disruption. In this case we can consider the deformation of the cloud in three principal directions, namely, $x, y, k\zeta$. The dispersion of displacement differences in these directions are

$$\sigma_x^2 = \langle [S_x(q/2) - S_x(-q/2)]^2 \rangle > 2[1 - G_{12}(q)], \quad (II.1)$$

and their correlations are described by the coefficient $r_{xy}$:

$$r_{xy} = \langle [S_x(q/2) - S_x(-q/2)][S_y(q/2) - S_y(-q/2)] \rangle >$$

$$\approx \frac{q^2 G_2(q/\sqrt{2})}{2[1 - G_{12}(q)]} \approx \frac{0.25\sqrt{2}}{1 + q/\sqrt{2}} \left( \frac{1 + q}{1 + q/\sqrt{2}} \right)^2, \quad (II.2)$$

for $q_0 < q$. Taking into account the approximate character of our consideration and because of $r_{xy} = r_{zx} = r_{zy} \ll 1$ we will neglect these correlations and will consider the deformation along three principal axes as independent. Under this assumption we have for the distribution of maximal displacement difference $\Delta S_x \geq \Delta S_y \geq \Delta S_z$ and for $\zeta = \Delta S_x/\sigma_x$

$$\frac{dW(\zeta)}{d\zeta} = \frac{3}{4\sqrt{2}\pi} e^{-\zeta^2/2} \left[ 1 + erf \left( \frac{\zeta}{\sqrt{2}} \right) \right]^2, \quad \zeta > 0, \quad (II.3)$$

$$\frac{dW(\zeta)}{d\zeta} = \frac{3}{4\sqrt{2}\pi} e^{-\zeta^2/2} \left[ 1 - erf \left( \frac{\zeta}{\sqrt{2}} \right) \right]^2, \quad \zeta < 0. \quad (II.3)$$

These relations show that under the above assumptions $1/8$ of matter is compressed in all directions and $1/8$ expanded in all directions, whereas $3/8$ of matter is compressed along $x$,$y$,$z$ axes and $3/8$ of matter is compressed along $x$ axis and expanded along $y$,$z$ axes. This means that $7/8$ of matter is compressed at least along one of the axes. The influence of correlation does not distort the symmetry between compression and expansion and, at $q \lesssim 1$, increases the probability to find $\Delta S_x \geq \Delta S_y \geq \Delta S_z \geq 0$ up to 0.21.

Let us now consider the matter compression along the direction of maximal compression $\mathbf{q} = \mathbf{q}_1 - \mathbf{q}_2$. In this case the dispersions of displacement differences $\Delta S_i = S_i(q_1) - S_i(q_2)$ are:

$$\sigma_{i1}^2 = \langle \Delta S_i \Delta S_i \rangle > 2[1 - G_{12}(q)], \quad (II.4)$$

$$\sigma_{i2}^2 = \langle \Delta S_i \Delta S_j \rangle = \sigma_{j2}^2 = \langle \Delta S_j \Delta S_j \rangle > 2[1 - G_{12}(q)], \quad (II.4)$$

$$\langle \Delta S_i \Delta S_j \rangle = \langle \Delta S_j \Delta S_i \rangle = 0. \quad (II.4)$$

For the distribution function (II.2) the probability to have $\Delta S_i \geq q/\tau$ is

$$W_{c_i}(q) = 1 - \frac{1}{8} \left[ 1 + erf \left( \frac{\mu(q)}{\sqrt{2}\tau} \right) \right]^3, \quad (II.5)$$

The function $\mu(q)$ characterizes the intersection of two particles before the time moment $\tau$ or for the pancake separation $\geq q$ and, in fact, $(8/7)W_{c_i}$ is the cumulative distribution function for pancakes with masses $> q$, for a given $\tau$.

The standard technique (see, e.g., BBKS) can be used in order to find the mean comoving linear density of pancakes with a given $\Delta S_1 = q/\tau$ and a pancake separation $\geq q$. The mean pancake is expressed through the characteristics of the derivations of function $Q$ introduced by (3.2), $Q_i = \partial Q/\partial q_i$, and for $q_0 \ll 1$ $Q \approx 6/q_0$.

$$Q_1 = \frac{1}{\mu(q)} < Q_i > Q_i \geq 0, \quad (III.6)$$

$$Q_2 = \frac{1}{\mu(q)} \sigma_{i2}^2 \geq Q_4 \geq 2/q_0 \quad (q_0 \text{ is introduced by (1.7)}). \quad (III.7)$$

where $\mu(q)$ characterizes the random angle of the intersection. For $l_c/l_0 \ll 1$, and $q \gg q_0$, we obtain (3.10).

**Appendix III**

**Interactions of pancakes.**

In order to investigate the correlations of the pancake properties we need to consider the intersection of the points $q_1$ and $q_2$, $q_{12} = q_1 - q_2$, at the moment $\tau_1$, and points $q_3$, $q_{34} = q_3 - q_4$, at the moments $\tau_2$, respectively. We consider the simplest case when the vectors $\mathbf{q}_{12}$ and $\mathbf{q}_{34}$ are situated along the same line.

It is convenient to characterize the pancakes by their sizes, $D_1$ and $D_2$, and by the separation of their central points, $D_{sep}$. In this case we have

$q_{12} = D_1$, \quad $q_{34} = D_2$,

$q_{31} = D_{sep} - 0.5(D_1 + D_2)$, \quad $q_{41} = D_{sep} - 0.5(D_1 + D_2)$,

$q_{32} = D_{sep} + 0.5(D_1 + D_2)$, \quad $q_{42} = D_{sep} + 0.5(D_1 - D_2)$.

The dispersions and the correlation of displacements $\Delta S(D_1)$ and $\Delta S(D_2)$ can be written as

$$\sigma_1^2(D_1) = \sigma_{11}(D_1), \quad \sigma_2^2(D_2) = \sigma_{11}(D_2), \quad (III.1)$$

$$\sigma_1 \sigma_2 \sigma_{rs} = G_{12}(q_{13}) - G_{12}(q_{63}) + G_{12}(q_{42}) - G_{12}(q_{44}), \quad (III.2)$$

and $\sigma_{rs}$ is a symmetric function of $D_{sep}$. For $D_{sep} \to \infty \quad \sigma_{rs} \to 0$, and for $D_{sep} = 0$

$$\sigma_1 \sigma_2 \sigma_{rs} = 2 \left[ G_{12} \left( \frac{D_1 + D_2}{2} \right) - G_{12} \left( \frac{D_1 + D_2}{2} \right) \right]. \quad (III.2)$$

$$r_{rs} = \frac{D_1, D_2, 0 \approx -\mu(D_1) \mu(D_2)G_{23}(D_1/2), \quad D_2 \ll 1. \quad (III.2)$$

For $D_{sep} = 0$, \quad $D_1 = D_2$, \quad $r_{rs} = 1$. 



\[ \mu(q) = \frac{q}{\sqrt{2[1 - G_{12}(q)]}}. \]
For a given $\Delta S(D_1) = D_1/\tau$, the conditional mean values $< \Delta S_c(D_2, D_{sep}) >$ and dispersions $\sigma_c(D_2, D_{sep})$ are described by the standard relations

\[ < \Delta S_c(D_2, D_{sep}) > = \sigma_c \mu_{cD}/\tau, \quad \sigma_c(D_2, D_{sep}) = \sigma_c \sqrt{1 - r_s^2}, \]

and the conditional probability of the pancake formation with $\Delta S(D_2, D_{sep}) \geq D_2/\tau_2$ under the condition $\Delta S(D_1) = D_1/\tau_1$ is

\[ W_c(D_2, \tau_2; D_1, \tau_1, D_{sep}) = 0.5 \text{erfc} \left( \frac{g_2}{\sqrt{2}} \right). \]  

(III.3)

These relations allow us to describe the formation of pancakes with $D_2 \leq D_1$ at the moment $\tau_2 \leq \tau_1$ and, in particular, to find the function $< m_{cD}(D_{sep}) >$ and $N_m(D_{sep})$. For $\tau_1 = \tau_2$ the same relations describe the accumulation of the smaller pancake ($D_2$) by the larger one ($D_1$).

More complicated relations describe the pancake coagulation. As it was noted in Sec. 3.4 this process is characterized by the function

\[ \psi(D_1, D_2, D_{sep}) = \frac{\phi(q_{cent} + D_1/2) - \phi(q_{cent} - D_1/2)}{D_1}, \]

and the condition of coagulation is

\[ \psi(D_1, D_2, D_{sep}) = D_{sep}/\tau. \]  

(III.4)

To obtain the statistical description of this process the conditional mean value $< \psi'(D_1, D_2, D_{sep}) >$ and dispersion $\sigma'_\psi$ need to be found. Using the expressions given in Appendix I and notation introduced above we obtain:

\[ \frac{1}{2} \sigma'_\psi = \frac{G_0(D_1)}{D_1^2} + \frac{G_0(D_2)}{D_2^2} + \frac{G_0(q_{31}) - G_0(q_{41}) - G_0(q_{32}) + G_0(q_{42})}{D_1 D_2}. \]  

(III.5)

The cross correlation of functions $\psi$ and $\Delta S(q_{31})$, $\Delta S(q_{34})$ is described as following:

\[ < \psi \Delta S(D_1) > = \mu(D_1) r_\psi, \quad < \psi \Delta S(D_2) > = \mu(D_2) r_\psi, \]

\[ r_\psi = \frac{q_{34} G_1(q_{31}) - q_{41} G_1(q_{41}) - q_{32} G_1(q_{32}) + q_{42} G_1(q_{42})}{D_1 D_2 \sigma_\psi}. \]

For the conditional mean value and dispersion of $\psi$ we have

\[ < \psi' > = r_\psi \left( \frac{\mu(D_1) - \mu(D_2)}{1 - r_s^2} + \frac{2 \mu(D_1) \mu(D_2)}{1 + r_s} \right), \]

\[ \left( \frac{\sigma'_\psi}{\sigma_\psi} \right)^2 = 1 - r_s^2 \left( \frac{\mu(D_1) - \mu(D_2)}{1 - r_s^2} + \frac{2 \mu(D_1) \mu(D_2)}{1 + r_s} \right). \]

Finally for the required function $\chi(D_1, D_2, D_{sep})$ we obtain

\[ \chi = [D_{sep} - < \psi' >] / \sigma_\psi. \]  

(III.6)

The dynamical characteristics of pancakes.

A correlation of pancake velocity described by (8.4) in points with different transversal coordinates is conventionally characterized by the correlation coefficient

\[ r_v = \frac{< [\phi(q_1) - \phi(q_2)][\phi(q_3) - \phi(q_4)] >}{\sqrt{< [\phi(q_1) - \phi(q_2)]^2 > < [\phi(q_3) - \phi(q_4)]^2 >}}, \]

\[ G_0(q_{14}) - G_0(q_{13}) + G_0(q_{23}) - G_0(q_{24}). \]  

(IV.1)

For small $q_{12}, q_{34}$ it transforms into (8.8).

The important characteristic of the pancake is the velocity profile around the points $q_1$ & $q_2$ under the condition $\Delta S(q_{12}) = q_{12}/\tau$. It is described by the mean conditional profile of displacement

\[ < S(q_1) > = \frac{q_1 - q_2 G_1(q_{13}) - G_1(q_{23})}{\tau} 2[1 - G_1(q_{12})], \]  

(IV.2)

and the conditional dispersion

\[ \sigma_s^2(q_1) = 1 - \frac{[G_1(q_{13}) - G_1(q_{23})]^2}{2[1 - G_1(q_{12})]^2}. \]  

(IV.3)

Using the expression (8.9) the velocity dispersion within pancakes can be found as follows:

\[ \frac{(1 + z)^2}{H^2(z)} \sigma_v^2 = \frac{q_{31}^2}{12} (1 - \beta(z))^2 + 2 \beta(z) f_1(q_{12}) - \beta^2(z) f_2(q_{12}) + \tau^2 \beta^2(z) f_3(q_{12})], \]  

(IV.4)

\[ f_1(q_{12}) = 1 - \frac{12 G_0(q_{12}) - 0.5 q_{2}^2 G_1(q_{12})}{q_{12}^2}, \]

\[ f_2(q_{12}) = 1 - \frac{3 J_{12}}{1 - G_1(q_{12})^2}, \]

\[ f_3(q_{12}) = 1 - \frac{J_{12}}{2[1 - G_1(q_{12})]} - \frac{2 G_0(q_{12})}{q_{12}^2}, \]

\[ J_{12} = q_{12}^{-1} \int_{q_{2}}^{q_{12}} dq_2 [G_1(q_{12} - q_{1}) - G_1(q_{2} - q_{1})]^2. \]  

Appendix IV