Infinite spin fields in $d = 3$ and beyond

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Abstract

In this paper we consider the frame-like formulation for the so called infinite (continuous) spin representations of the Poincare algebra. In the tree dimensional case we give explicit Lagrangian formulation for bosonic and fermionic infinite spin fields (including the complete sets of the gauge invariant objects and all the necessary extra fields). Moreover we find the supertransformations for the supermultiplet containing one bosonic and one fermionic fields leaving the sum of their Lagrangians invariant. Properties of such fields and supermultiplets in four and higher dimensions are also briefly discussed.

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Introduction

Besides the very well known finite components massless and massive representations of the Poincare algebra there are rather exotic so called infinite (or continuous) spin ones (see e.g. [1, 2]). In dimensions $d \geq 4$ they have infinite number of physical degrees of freedom and so may be of some interest for the higher spins theory. Indeed they attracted some attention last times [3, 4, 5, 6, 7, 8]. It has been noted several times that such infinite spin representations may be considered as a limit of massive higher spin ones where spin goes to infinity, mass goes to zero while their product being fixed. Moreover, recently Metsaev has shown the the metric-like Lagrangian formulation for the bosonic [9] and fermionic [10] fields in $AdS_d$ spaces with $d \geq 4$ can be constructed using exactly the same technique as was previously used for the gauge invariant formulation of massive higher spin bosonic [11] and fermionic [12] fields.

The current paper is devoted to the frame-like formulation for such infinite spin fields. In the first (and main) section we construct gauge invariant Lagrangian formulation for bosonic and fermionic cases. We also elaborate on the whole set of the gauge invariant objects (introducing all necessary extra fields) and rewrite our Lagrangians in the explicitly gauge invariant form. Moreover we managed to find supertransformations for the supermultiplet containing one bosonic and one fermionic infinite spin fields that leaves the sum of their Lagrangians invariant. For this we heavily use our previous results on the gauge invariant formulation for massive bosonic and fermionic higher spin fields in $d = 3$ [13, 14] (see also [15, 16, 17]) as well as on the massive higher spin supermultiplets [18, 19, 20]. In the last two sections we briefly discuss the properties of such fields and supermultiplets in $d = 4$ and $d \geq 5$ dimensions leaving explicit details to the forthcoming publication.

Notations and conventions We will work in the frame-like multispinor formalism where all objects are two, one or zero forms with a set of completely symmetric local spinor indices. We will follow mostly the conventions of [20] but we restrict ourselves with the flat Minkowski space.

1 Infinite spin fields in $d = 3$

In this section we develop the frame like formalism for the massless infinite spin bosonic and fermionic fields as well as for the supermultiplet containing such fields.

1.1 Infinite spin boson

As we have already noted there is a tight connection between the gauge invariant description for the massive finite spin fields and the one for the massless infinite spin ones. Thus we will follow the same approach as in [13] but this time without restriction on the number of components. So we introduce an infinite set of physical and auxiliary one-forms $\Omega^{(2k)}, \Phi^{(2k)}, 1 \leq k \leq \infty$ as well as one-form $A$ and zero-forms $B^{(2)}, \pi^{(2)}$ and $\varphi$. We begin with

\footnote{Note that in three dimensions such infinite spin bosonic field (as any massive higher spin boson) has just two physical degrees of freedom, while infinite spin fermionic field (as any massive higher spin fermion) has just one. However it is impossible to realize such representations using a finite number of components (see...}
the sum of kinetic terms for all these fields:

\[ L_0 = \sum_{k=1}^{\infty} (-1)^{k+1} \left[ k \Omega_{\alpha(2k-1)\beta} e^\gamma \Omega^{\alpha(2k-1)\gamma} + \Omega_{\alpha(2k)} d\Phi^\alpha(2k) \right] + E B_{\alpha(2)} B^\alpha(2) - B_{\alpha(2)} e^{\alpha(2)} dA - E \pi_{\alpha(2)} \pi^\alpha(2) + \pi_{\alpha(2)} E^\alpha(2) d\varphi \]  

(1)

as well as their initial gauge transformations:

\[ \delta_0 \Omega^{\alpha(2k)} = d\eta^{\alpha(2k)}, \quad \delta_0 \Phi^\alpha(2k) = d\xi^\alpha(2k) + e^\alpha_\beta \eta^{\alpha(2k-1)\beta}, \quad \delta_0 A = d\xi \]  

(2)

Then following general scheme we add to the Lagrangian a set of cross terms gluing all these components together:

\[ L_1 = \sum_{k=1}^{\infty} (-1)^{k+1} \left[ \tilde{a}_k \Omega_{\alpha(2k)\beta(2)} e^{\beta(2)} \Phi^\alpha(2k) + a_k \Omega_{\alpha(2k)} e_{\beta(2)} \Phi^{\alpha(2k)\beta(2)} \right] + \tilde{a}_0 \Omega_{\alpha(2)} e^\alpha(2) A - a_0 \Phi_{\alpha(2)} E^\beta \gamma B^\alpha + \tilde{a}_0 \pi_{\alpha(2)} E^\alpha(2) A \]  

(3)

and introduce appropriate corrections for the gauge transformations:

\[ \delta_1 \Omega^{\alpha(2k)} = \frac{(k+2)}{k} a_k e^{\beta(2)} \eta^{\alpha(2k)\beta(2)} + \frac{a_{k-1}}{k(2k-1)} e^{\alpha(2)} \eta^{\alpha(2k-2)} \]

\[ \delta_1 \Phi^\alpha(2k) = a_k e^{\beta(2)} \xi^{\alpha(2k)\beta(2)} + \frac{(k+1)a_{k-1}}{k(k-1)(2k-1)} e^{\alpha(2)} \xi^{\alpha(2k-2)} \]

\[ \delta_1 \Omega^{\alpha(2)} = 3 a_1 e^{\beta(2)} \eta^{\alpha(2)\beta(2)}, \quad \delta_1 \Phi^\alpha(2) = a_1 e^{\beta(2)} \xi^{\alpha(2)\beta(2)} + 2 a_0 e^{\alpha(2)} \xi \]

\[ \delta_1 B^\alpha(2) = 2 a_0 e^{\alpha(2)}, \quad \delta_1 A = \frac{a_0}{2} e^{\alpha(2)} \xi^{\alpha(2)}, \quad \delta_1 \varphi = -\tilde{a}_0 \xi \]  

(4)

Here consistency of the gauge transformations with the Lagrangian requires:

\[ \tilde{a}_k = -\frac{(k+2)}{k} a_k, \quad \tilde{a}_0 = 2a_0 \]

At last we introduce mass-like terms for all components and appropriate corrections to the gauge transformations:

\[ L_2 = \sum_{k=1}^{\infty} (-1)^{k+1} b_k \Phi_{\alpha(2k-1)\beta} e^\gamma \Phi^{\alpha(2k-1)\gamma} + b_0 \Phi_{\alpha(2)} E^\alpha(2) \varphi + \tilde{b}_0 E \varphi^2 \]  

(5)

\[ \delta_2 \Omega^{\alpha(2k)} = \frac{b_k}{k} e^\alpha_\beta \xi^{\alpha(2k-1)\beta}, \quad \delta_2 \pi^\alpha(2) = b_0 \xi^{\alpha(2)} \]

(6)

Now we require that the whole Lagrangian \( L = L_0 + L_1 + L_2 \) will be invariant under the gauge transformations \( \delta = \delta_0 + \delta_1 + \delta_2 \). This produce the following general relations on the parameters:

\[ (k+2)^2 b_{k+1} = k(k+1) b_k \]

\[ \frac{2(k+2)(2k+3)}{(k+1)(2k+1)} a_k^2 = \frac{2(k+1)}{(k-1)} a_{k-1}^2 + 4b_k = 0 \]  

(7)

(8)

e.g. \[ \text{[6].} \]
as well as some relations for the lower components:

\[ 5a_1^2 - a_0^2 + 4b_1 = 0 \]

\[ a_0^2 = 64b_1, \quad b_0 = \frac{\hat{a}_0a_0}{4}, \quad b_0 = \frac{3a_0^2}{2} \]

The general solution of all these relations has two free parameters. In the massive finite spin case its just the mass and spin but in our case we choose \( a_0 \) and \( b_1 \) as the main ones. Then all other parameters can be expressed as follows:

\[ b_k = \frac{4b_1}{k(k + 1)^2} \tag{9} \]

\[ a_k^2 = \frac{k}{2(k + 3)} \left[ 3(k + 1) \right] \frac{2(2k + 2)}{a_0^2} - \frac{8k}{(k + 1)b_1} \tag{10} \]

Now we are ready to analyze the solution obtained. Let us begin with the case \( a_0^2 < 16b_1 \). In general it means that starting from some value of \( k \) all \( a_k^2 \) become negative so that we obtain non unitary theory. The only exceptions happen then one adjust the values of \( a^2 \) and \( b_1 \) so that at some \( k_0 \) we obtain \( c_{k_0} = 0 \). In this case we obtain unitary theory with the finite number of components and this case corresponds to the gauge invariant description for the massive bosonic field with the spin \( k_0 + 1 \). Let us turn to the case \( a_0^2 = 16b_1 \) (this corresponds to the case \( \mu_0 = 0 \) in [2]). In this case we obtain:

\[ a_k^2 = \frac{3k}{2(k + 1)(k + 2)(k + 3)}a_0^2 \tag{11} \]

so we get an unitary theory with infinite number of components. Note that for the case \( a_0^2 > 16b_1 \) we also obtain unitary theory but as it was shown by Metsaev [3] it corresponds to the tachionic infinite spin field. Thus in what follows we will restrict ourselves with the case \( a_0^2 = 16b_1 \) only.

One of the nice and general features of the frame like formalism is that for each field (physical or auxiliary) one can construct corresponding gauge invariant object. For the case at hands we will follow massive case in [17, 20]. For almost all fields corresponding gauge invariant objects can be directly constructed from the known form for the gauge transformations given above (here for the later convenience we chaged normalization for the zero-forms \( B^{\alpha(2)} \Rightarrow 2a_0B^{\alpha(2)}, \pi^{\alpha(2)} \Rightarrow b_0\pi^{\alpha(2)} \)):

\[
\begin{align*}
\mathcal{R}^{\alpha(2k)} & = d\Omega^{\alpha(2k)} + \frac{b_k}{k}e_\beta^\alpha \Phi^{\alpha(2k-1)\beta} + \frac{(k + 2)}{k}a_0e_\beta^{(2)\alpha}\Omega^{\alpha(2k-1)\beta} + \frac{a_{k-1}}{k(2k - 1)}e^{\alpha(2)}\Omega^{\alpha(2k-2)} \\
\mathcal{T}^{\alpha(2k)} & = d\Phi^{\alpha(2k)} + e_\beta^\alpha \Phi^{\alpha(2k-1)\beta} + a_k e_\beta^{(2)\alpha} \Phi^{\alpha(2k-1)\beta} + \frac{(k + 1)a_k}{k(2k - 1)}e^{\alpha(2)}\Phi^{\alpha(2k-2)} \\
\mathcal{R}^{\alpha(2)} & = d\Omega^{\alpha(2)} + b_1 e_\beta^\alpha \Phi^{\alpha(2)\beta} + 3a_1 e_\beta^{(2)\alpha} \Omega^{\alpha(2)\beta} - a_0^2 E_\beta B^{\alpha\beta} + b_0 E^{\alpha(2)}\Phi \\
\mathcal{T}^{\alpha(2)} & = d\Phi^{\alpha(2)} + e_\beta^\alpha \Omega^{\alpha\beta} + a_1 e_\beta^{(2)\alpha} \Phi^{\alpha(2)}\beta + 2a_0 e^{\alpha(2)} A \\
\mathcal{A} & = dA - 2a_0 E_{\alpha(2)} B^{\alpha(2)} + \frac{a_0}{4} e_\alpha^{(2)} \Phi^{\alpha(2)} \\
\Phi & = d\varphi - \frac{\sqrt{3}}{2} a_0^2 e_\alpha^{(2)} \pi^{\alpha(2)} + 2\sqrt{3}a_0 A
\end{align*}
\]
But to construct gauge invariant objects for $B^{\alpha(2)}$ and $\pi^{\alpha(2)}$ one must introduce a first pair of the so called extra fields $^{2}B^{(4)}_{\alpha}$ and $^{2}\pi^{\alpha(4)}$:

\[
B^{\alpha(2)} = dB^{\alpha(2)} - \Omega^{\alpha(2)} + b_{1}e_{\beta}^{\alpha}\pi^{\alpha\beta} + 3a_{1}e_{\beta(2)}B^{\alpha(2)}\beta(2) \\
\Pi^{\alpha(2)} = d\pi^{\alpha(2)} + e^{\alpha}_{\beta}B^{\beta} - \Phi^{\alpha(2)} - \frac{1}{\sqrt{3}}e^{\alpha(2)}\varphi + a_{1}e_{\beta(2)}\pi^{\alpha(2)}\beta(2)
\]

which transform as follows:

\[
\delta B^{\alpha(4)} = \eta^{\alpha(4)}, \quad \delta \pi^{\alpha(4)} = \xi^{\alpha(4)}
\]

But to construct gauge invariant objects for these new fields one must introduce the next pair of extra fields and so on. This results in the infinite chain of zero forms $B^{\alpha(2k)}$ and $\pi^{\alpha(2k)}$, $1 \leq k \leq \infty$ with the following set of gauge invariant objects:

\[
B^{\alpha(2k)} = dB^{\alpha(2k)} - \Omega^{\alpha(2k)} + \frac{a_{k-1}}{k(2k-1)}e^{\alpha(2k-2)}B^{(2k-2)} \\
\Pi^{\alpha(2k)} = d\pi^{\alpha(2k)} - \Phi^{\alpha(2k)} + e^{\alpha}_{\beta}B^{\alpha(2k-1)}\beta + a_{k}e_{\beta(2k)}\pi^{\alpha(2k)}\beta(2) \\
+ \frac{(k+1)a_{k-1}}{(k-1)(2k-1)}e^{\alpha(2)}\pi^{\alpha(2k-2)}
\]

Here:

\[
\delta B^{\alpha(2k)} = \eta^{\alpha(2k)}, \quad \delta \pi^{\alpha(2k)} = \xi^{\alpha(2k)}
\]

Now we have an infinite set of gauge one-forms as well as an infinite set of Stueckelberg zero-forms. As in the massive finite spin case \[17, 20\] this allows us to rewrite the Lagrangian in the explicitly gauge invariant form:

\[
L = -\frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k+1}[R^{\alpha(2k)}\Pi^{\alpha(2k)} + T^{\alpha(2k)}B^{\alpha(2k)}] + \frac{1}{2}e^{\alpha(2)}B^{\alpha(2)}\Phi
\]

By construction each term here is separately gauge invariant and the explicit values for all coefficients are determined by the so called extra field decoupling conditions:

\[
\frac{\delta L}{\delta B^{\alpha(2k)}} = 0, \quad \frac{\delta L}{\delta \pi^{\alpha(2k)}} = 0, \quad 2 \leq k \leq \infty
\]

### 1.2 Fermionic case

In this case we will also follow the construction for the massive finite spin field \[14\] but this time for the infinite set of components. So we introduce a set of one-forms $\Psi^{\alpha(2k+1)}$, $0 \leq k \leq \infty$ and a zero-form $\psi^{\alpha}$. Once again we begin with the sum of kinetic terms for all fields:

\[
\frac{1}{k}L_{0} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2}\Psi^{\alpha(2k+1)}d\Psi^{\alpha(2k+1)} + \frac{1}{2}\psi^{\alpha}E^{\alpha}_{\beta}d\psi^{\beta}
\]

\(^2\)Recall that extra fields are the fields that do not enter the free Lagrangian but are necessary for the construction of the whole set of gauge invariant objects. Moreover such fields play important role in the construction of the interactions.
as well as with their initial gauge transformations:
\[ \delta_0 \Psi^{\alpha(2k+1)} = d\zeta^{\alpha(2k+1)} \] (17)

Now we add a set of cross terms gluing them together
\[ \frac{1}{i} L_1 = \sum_{k=1}^{\infty} (-1)^{k+1} c_k \Psi_{\alpha(2k-1)} \beta(2) e^{\beta(2)} \Psi^{\alpha(2k-1)} + c_0 \Psi_{\alpha} E^{\alpha} \beta \psi^{\beta} \] (18)

and corresponding corrections to the gauge transformations:
\[ \delta_1 \Psi^{\alpha(2k+1)} = c_{k+1} e^{\beta(2)} \zeta^{\alpha(2k+1)} \beta(2) + c_k \frac{k}{2(k+1)} e^{\alpha(2)} \zeta^{\alpha(2k-1)} , \]
\[ \delta_1 \psi^{\alpha} = c_0 \zeta^{\alpha} \] (19)

At last we add the mass-like terms for all fields and appropriate corrections to the gauge transformations:
\[ \frac{1}{i} L_2 = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{d_k}{2} \Psi_{\alpha(2k)} \beta \gamma e^{\beta(2)} \Psi^{\alpha(2k)} \gamma - \frac{m_0}{2} E_{\alpha} \psi^{\alpha} \] (20)

\[ \delta_2 \Psi^{\alpha(2k+1)} = \frac{d_k}{(2k+1)} e^{\alpha} \beta \zeta^{\alpha(2k+1)} \beta \] (21)

Now we require that the whole Lagrangian \( L = L_0 + L_1 + L_2 \) will be invariant under the gauge transformations \( \delta = \delta_0 + \delta_1 + \delta_2 \). This produce a number of general relations on the parameters
\[ (2k+5)d_{k+1} = (2k+3)d_k \] (22)
\[ \frac{(k+2)(2k+1)}{(k+1)(2k+3)} c_{k+1}^2 - c_k^2 + \frac{d_k^2}{(2k+1)} = 0 \] (23)

as well as
\[ \frac{8}{3} c_1^2 - c_0^2 + 4d_0^2 = 0, \quad d_0 = \frac{m_0}{3} \]

As in the bosonic case the general solution for all these relations has two free parameters and we choose \( c_0 \) and \( m_0 \) this time. Then all other coefficients can be expressed as follows:
\[ d_k = \frac{m_0}{(2k+3)} \] (24)
\[ c_k^2 = \frac{(2k+1)^2}{4(k+1)} c_0^2 - \frac{k}{2(2k+1)} m_0^2 \] (25)

The properties of this solution appears to be the same as in the bosonic case. Namely, for the case \( m_0^2 > 2c_0^2 \) in general we obtain non unitary theory. The only exceptions appear if one adjust this parameters so that at some \( k_0 \) we get \( c_{k_0} = 0 \). In this case we obtain unitary theory with finite number of components which corresponds to the gauge invariant description for massive fermionic field with spin \( k_0 + 3/2 \). For the \( m_0^2 = 2c_0^2 \) (this corresponds to \( \mu_0 = 0 \) in [10]) we obtain
\[ c_k^2 = \frac{c_0^2}{4(k+1)(2k+1)} \] (26)
that corresponds to the unitary massless infinite spin field while for the \( m_0^2 < c_0^2 \) we again obtain tachionic infinite spin case. As in the bosonic case in what follows we will restrict ourselves with the case \( m_0^2 = 2c_0^2 \) only.

Now we proceed with the construction of the full set of gauge invariant objects. For all one-forms the construction is pretty straightforward (again for the later convenience we changed normalization for the zero-form \( \psi^\alpha \Rightarrow c_0\psi^\alpha \):

\[
F^{\alpha(2k+1)} = d\Psi^{\alpha(2k+1)} + \frac{d_k}{(2k+1)} e^\alpha_\beta \Psi^{(2k)\beta} + c_{k+1} e^\beta_\gamma (2k+1) \Psi^{(2k+1)\beta(2)} + \frac{c_k}{k(2k+1)} e^{(2)}_\alpha \Psi^{(2k-1)}
\]

\[
\mathcal{F}^\alpha = D\Psi^\alpha + d_0 e^\alpha_\beta \Psi^\beta + c_1 e^\beta_\gamma (2k+1) \Psi^{(2\beta(2)} - c_0^2 E^\alpha_\beta \Psi^\beta
\]

But to construct gauge invariant object for the zero-form one must introduce a first extra field:

\[
C^\alpha = d\psi^\alpha - \Psi^\alpha + d_0 e^\alpha_\beta \psi^\beta + c_1 e^\beta_\gamma (2k+1) \psi^{(2\beta)} - c_0^2 E^\alpha_\beta \psi^\beta
\]

Then to construct gauge invariant object for this field one must introduce the second one and so on. This results in the infinite set of zero-forms with the corresponding gauge invariant objects:

\[
C^{\alpha(2k+1)} = d\psi^{\alpha(2k+1)} - \Psi^{\alpha(2k+1)} + \frac{d_k}{(2k+1)} e^\alpha_\beta \psi^{(2k)\beta} + c_{k+1} e^\beta_\gamma (2k+1) \psi^{(2k+1)\beta(2)} + \frac{c_k}{k(2k+1)} e^{(2)}_\alpha \psi^{(2k-1)}
\]

where

\[
\delta\psi^{(2k+1)} = \zeta^{(2k+1)}
\]

Now we have an infinite set of one-form and zero-form fields and their gauge invariant two and one forms. This allows us to rewrite the Lagrangian in the explicitly gauge invariant form:

\[
\mathcal{L} = -\frac{i}{2} \sum_{k=0}^{\infty} (-1)^{k+1} \mathcal{F}^{\alpha(2k+1)} C^{\alpha(2k+1)}
\]

As in the bosonic case each term is separately gauge invariant while the specific values of all coefficients are determined by the extra field decoupling condition:

\[
\frac{\delta \mathcal{L}}{\delta\psi^{\alpha(2k+1)}} = 0, \quad 1 \leq k \leq \infty
\]

### 1.3 Infinite spin supermultiplet

It is interesting (see e.g. [1]) that similarly to the usual massless and massive fields such massless infinite spin fields also can form supermultiplets. In \( d = 3 \) the minimal supermultiplets contains just one bosonic and one fermionic fields. Due to tight relation with gauge invariant formulation for the massive higher spin fields and supermultiplets here we will heavily use the results of our recent paper [20]. The main difference (besides the infinite set of components) is the essentially different expressions for the coefficients \( a_k \) and \( c_k \).
The general strategy will be to find explicit form of the supertransformations for all fields such that all gauge invariant two and one forms transform covariantly and to check the invariance of the Lagrangian. Let us begin with the bosonic fields. For the general case \( k \geq 2 \) we will use the following ansatz:

\[
\begin{align*}
\delta \Omega^{(2k)} & = i \rho_k \Psi^{(2k-1)} \zeta + i \sigma_k \Psi^{(2k)} \zeta \\
\delta \Phi^{(2k)} & = i \alpha_k \Psi^{(2k-1)} \zeta + i \beta_k \Psi^{(2k)} \zeta
\end{align*}
\]

and require that the corresponding two-form transform covariantly:

\[
\begin{align*}
\delta R^{(2k)} & = i \rho_k F^{(2k-1)} \zeta + i \sigma_k F^{(2k)} \zeta \\
\delta T^{(2k)} & = i \alpha_k F^{(2k-1)} \zeta + i \beta_k F^{(2k)} \zeta
\end{align*}
\]

First of all this gives us an important relation

\[
c_0^2 = 6a_0^2
\]

Recall that the parameters \( a_0 \) and \( c_0 \) are the main dimension-full parameters that determine the whole construction for the bosonic and fermionic fields. So this relation plays the same role as the requirement that masses of bosonic and fermionic fields in the supermultiplet must be equal. Further, we obtain explicit expressions for all parameters

\[
\begin{align*}
\alpha_k^2 & = k \hat{\alpha}^2, & \beta_k^2 & = \frac{(k + 1)}{2k(2k + 1)} \hat{\alpha}^2 \\
\sigma_k^2 & = \frac{3a_0^2}{4k(k + 1)^2} \hat{\alpha}^2, & \rho_k^2 & = \frac{3a_0^2}{8k^3(k + 1)(2k + 1)} \hat{\alpha}^2
\end{align*}
\]

where \( \hat{\alpha} \) is an arbitrary parameter that can be fixed by the normalization of the superalgebra.

For the three bosonic components that require separate consideration we obtain:

\[
\begin{align*}
\delta \Omega^{(2)} & = i \rho_1 \Psi^{(2)} \zeta + i \sigma_1 \Psi^{(2)} \zeta - \frac{i \sqrt{3} a_0^2}{4} \hat{\alpha} e^{\alpha(2)} \psi^\beta \zeta \\
\delta A & = \frac{i \hat{\alpha}}{2} \Psi^{(2)} \zeta + \frac{i \sqrt{3} a_0}{2} \hat{\alpha} \psi e^{\alpha(2)} \psi, & \delta \varphi & = -i \sqrt{3} a_0 \hat{\alpha} \psi^\alpha \zeta
\end{align*}
\]

At last the supertransformations for the zero-forms look like:

\[
\begin{align*}
\delta B^{(2k)} & = i \sigma_k \psi^{(2k)} \zeta + i \rho_k \psi^{(2k-1)} \zeta \\
\delta \pi^{(2k)} & = i \beta_k \psi^{(2k)} \zeta + i \alpha_k \psi^{(2k-1)} \zeta
\end{align*}
\]

where all coefficients \( \alpha_k, \beta_k, \rho_k, \sigma_k \) are the same as above.

Now let us turn to the fermionic components. For the general case \( k \geq 1 \) we will consider the following ansatz:

\[
\begin{align*}
\delta \Psi^{(2k+1)} & = \frac{\alpha_k}{2(k + 1)} \Omega^{(2k)} \zeta + 2(k + 1) \beta_k + \Omega^{(2k+1)} \zeta \\
& + \gamma_k \Phi^{(2k)} \zeta + \delta_k \Phi^{(2k+1)} \zeta
\end{align*}
\]
Then the requirement that the corresponding two-forms transform covariantly:

\[ \delta \mathcal{F}^{(2k+1)} = \frac{\alpha_k}{(2k+1)} \mathcal{R}^{(2k)} \zeta^\alpha + 2(k+1)\beta_{k+1} \mathcal{R}^{(2k+1)} \zeta^\beta \]

\[ + \gamma_k \mathcal{T}^{(2k)} \zeta^\alpha + \delta_k \mathcal{T}^{(2k+1)} \zeta^\beta \]  

(37)

gives us the same relation on the parameters \(a_0\) and \(c_0\) as before and also gives:

\[ \gamma_k^2 = \frac{3a_0^2}{4k(k+1)^2(2k+1)^2} \hat{\alpha}^2 \]

\[ \delta_k^2 = \frac{3a_0^2}{2(k+1)(k+2)(2k+3)} \hat{\alpha}^2 \]

Again there is a couple of components that need to be considered separately:

\[ \delta \Psi^\alpha = 2\beta_1 \Omega^{\alpha\beta} \zeta_\beta + \delta_0 \Psi^{\alpha\beta} \zeta_\beta + a_0 \hat{\alpha} e_{\beta}(2) B^{(2)} \zeta^\alpha + \sqrt{3}a_0 \hat{\alpha} A^{(2)} \zeta^\alpha - \frac{\sqrt{3}a_0}{2} \hat{\alpha} \varphi \zeta^\alpha \zeta^\beta \]

\[ \delta \psi^\alpha = \frac{2\sqrt{3}}{3} \hat{\alpha} B^{\alpha\beta} \zeta_\beta + \frac{a_0}{2} \hat{\alpha} \pi^{\alpha\beta} \zeta_\beta + \frac{\hat{\alpha}}{2} \varphi \zeta^\alpha \]  

(38)

At last for the Stueckelberg zero-forms we obtain:

\[ \delta \psi^{(2k+1)} = \frac{\alpha_k}{(2k+1)} B^{(2k)} \zeta^\alpha + 2(k+1)\beta_{k+1} B^{(2k+1)} \zeta^\beta \]

\[ + \gamma_k \pi^{(2k)} \zeta^\alpha + \delta_k \pi^{(2k+1)} \zeta^\beta \]  

(39)

where all parameters \(\alpha_k, \beta_k, \gamma_k\) and \(\delta_k\) are the same as before.

We have explicitly checked that the sum of the bosonic and fermionic Lagrangians is invariant under these supertransformations up to the terms proportional to the auxiliary fields \(B^{(2)}\) and \(\pi^{(2)}\) equations in the same way as in the case of massive higher spin supermultiplets [20].

2 Infinite spin fields in \(d = 4\)

Similarly to the three dimensional case in \(d = 4\) there exist just one bosonic and one fermionic infinite spin representations corresponding to the completely symmetric (spin-)tensors. Metric-like gauge invariant Lagrangian formulation (valid also in \(d > 4\)) has been constructed recently [9, 10]. Frame-like Lagrangian formulation can be straightforwardly obtained from the frame-like gauge invariant formalism for the massive completely symmetric (spin-)tensors developed in [21]. These results will be presented elsewhere.

The complete set of the gauge invariant objects for the massive bosonic higher spin fields in \(d \geq 4\) has been constructed in [22]. It requires the following three sets of fields:

\[ \Phi_{\mu}^{a(k),b(l)}, \quad S^{a(k),b(l)} \quad 0 \leq k \leq s - 1, \quad 0 \leq l \leq k \]

\[ W^{a(k),b(l)} \quad k \geq s, \quad 0 \leq l \leq s - 1 \]
where notation $\Phi_{\mu}^{a(k),b(l)}$ means that local indices correspond to the Young tableau with two rows. Thus we have two finite sets of gauge one-forms and Stueckelberg zero-forms as well as infinite number of gauge invariant zero-forms. As in the three dimensional case one can try to consider the limit where spin goes to infinity and mass goes to zero, but in $d > 3$ it appears to be rather involved task. As for the analogous formulation for the massive fermionic higher spin fields to the best of our knowledge it still remain to be elaborated.

As it quite well known in $d = 4$ there exist two type of massive higher spin $N = 1$ supermultiplets corresponding to the integer or half-integer superspins:

$$\begin{pmatrix}
s + 1 \frac{1}{2} \\
\frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
A_s \\
\Psi_{s+\frac{1}{2}}
\end{pmatrix}
\Rightarrow
\sum_{k=1}^{s}
\begin{pmatrix}
\Phi_{k+\frac{1}{2}} \\
\Psi_{k-\frac{1}{2}}
\end{pmatrix}
\oplus
\begin{pmatrix}
\Phi_{\frac{1}{2}} \\
\frac{1}{2}
\end{pmatrix}
\oplus
\begin{pmatrix}
A_{s+1} \\
\Psi_{s+\frac{1}{2}}
\end{pmatrix}
\oplus
\sum_{k=1}^{s}
\begin{pmatrix}
\Phi_{k+\frac{1}{2}} \\
\Psi_{k-\frac{1}{2}}
\end{pmatrix}
\oplus
\begin{pmatrix}
\Phi_{\frac{1}{2}} \\
\frac{1}{2}
\end{pmatrix}
$$

Their explicit Lagrangian description was constructed in [23] using gauge invariant description for massive bosonic and fermionic higher spin fields. The main idea was that massive supermultiplet can be constructed out of the appropriately chosen set of the massless ones. The decomposition of these two massive supermultiplets into the massless one looks as follows:

It was crucial for the whole construction that each pair of bosonic fields with equal spins must have opposite parities and one has to consider a kind of duality mixing between these fields. Moreover such mixing arises already at the massless supermultiplets level so that even in the massless infinite spin limit these pairs do not decouple and we still have two infinite spin bosonic and two infinite spin fermionic components. It is still possible that by abandoning parity one can construct the supermultiplet containing just one bosonic and one fermionic fields but it remains to be checked.

The mixing angles for the bosonic components take rather different values for the two type of the supermultiplets but as it can be seen from their explicit expressions in [23] in the infinite spin limit they all become equal. At the same time the main structural difference between them — the presence of the left most multiplet $(A_{s+1}, \Phi_{s+1/2})$ in the infinite spin limit disappears so both type of massive supermultiplets produce the same result (up to some field redefinitions).

3 Infinite spin fields in $d \geq 5$

Contrary to the three and four dimensional cases in $d \geq 5$ there exist an infinite number of such infinite spin representations. Let us briefly remind how their classification arises [1]. For the massless fields we have $p_\mu^2 = 0$ and by the Lorentz transformations one can always bring this vector to the canonical form $p_\mu = (1, 0, \ldots, 0, 1)$. This leads to the so called little group
(i.e. group of transformations leaving this vector intact) that besides the group $SO(d - 2)$ contains pseudo translations $T_i, i = 1, 2, \ldots, d - 2$ that are specific combinations of spatial rotations and Lorentz boosts. Usual finite helicities massless representations correspond to the case where all $T_i = 0$ while to construct infinite spin representations one can follow the same root as for the Poincare group itself. Namely one can consider eigenvectors for this pseudo translations $T_i |\xi_i \rangle = \xi_i |\xi_i \rangle$, $\xi_i^2$ being invariant. By using $SO(d - 2)$ transformations one can always bring such vector to the form $(1, 0, \ldots, 0)$ and this in turn leads to the so called short little group $SO(d - 3)$ leaving this vector intact. Thus infinite spin representations are determined by the corresponding representations of this short little group.

Now it is clear that for the $d = 3$ and $d = 4$ this short little group is trivial that is why we have just one bosonic and one fermionic representations while in $d \geq 5$ there exist infinitely many ones. For example in $d = 5$ and $d = 6$ such representations can be labeled by the parameter $l$ taking integer $l = 0, 1, 2, \ldots$ or half integer $l = \frac{1}{2}, \frac{3}{2}, \ldots$ values for the bosonic and fermionic cases correspondingly. Lagrangian formulation for such representations can be obtained form the frame-like gauge invariant formulation for the massive mixed symmetry bosonic and fermionic fields corresponding to the Young tableau $Y(k, l)$ with two rows developed in [24, 25, 26]. Namely one has to consider a limit where mass goes to zero, $k$ goes to infinity while $l$ being fixed. This construction will be presented in the forthcoming publication so here let us just illustrate how the spectrum of such representations looks like (by the spectrum we mean a collection of usual massless fields that we have to combine to obtain infinite spin one).

Completely symmetric case considered before corresponds to the $l = 0$ and has the following spectrum (dot stands for the scalar field):

\[
\ldots \begin{array}{cccccccc}
|\quad| \quad| \quad| \quad| \quad| \quad| \quad|
\end{array}
\ldots
\]

For the first non-trivial case $l = 1$ we will have two infinite chains of components:

\[
\begin{array}{cccccccc}
| & | & | & | & | & | & |
\end{array}
\ldots
\]

\[
\begin{array}{cccccccc}
| & | & | & | & | & | & |
\end{array}
\ldots
\]

The first line begins with the antisymmetric second rank tensor, the it contains hook and the whole set of long hooks, while in the second line we again have completely symmetric tensors starting with the vector field this time.

Let us give here one more concrete example for $l = 3$:

\[
\begin{array}{cccccccc}
| & | & | & | & | & | & |
\end{array}
\ldots
\]

\[
\begin{array}{cccccccc}
| & | & | & | & | & | & |
\end{array}
\ldots
\]

\[
\begin{array}{cccccccc}
| & | & | & | & | & | & |
\end{array}
\ldots
\]

\[
\begin{array}{cccccccc}
| & | & | & | & | & | & |
\end{array}
\ldots
\]

\[
\begin{array}{cccccccc}
| & | & | & | & | & | & |
\end{array}
\ldots
\]

\[
\begin{array}{cccccccc}
| & | & | & | & | & | & |
\end{array}
\ldots
\]

\[
\begin{array}{cccccccc}
| & | & | & | & | & | & |
\end{array}
\ldots
\]
Hopefully the general pattern is clear now. In general in the upper left corner we have a rectangular diagram with length $l$. Moving to the right we add one box to the first row, while moving down we cut one box from the second row until we end again with the completely symmetric tensors in the bottom line.

**Acknowledgments**

Author is grateful to the I. L. Buchbinder and T. V. Snegirev for collaboration. Also author is grateful to the organizers of the ”Workshop on higher spin gauge theories”, 26-28 April 2017, UMONS, Mons, Belgium for the kind hospitality during the workshop.

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