TRANSVERSAL INTERSECTION AND SUM OF POLYNOMIAL IDEALS

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ABSTRACT. In this paper we derive some conditions for transversal intersection of polynomial ideals. We exhibit some examples. Finally, as an application of the results proved, we compute the Betti numbers for ideals of the form $I_1(XY) + J$, where $X$ and $Y$ are matrices and $J$ is the ideal generated by the $2 \times 2$ minors of the matrix consisting of any two rows of $X$.

1. INTRODUCTION

Ideals $I$ and $J$ are said to intersect transversally if $I \cap J = IJ$. We have observed the interesting fact that for transversally intersecting ideals $I$ and $J$ in the polynomial ring $R$, the tensor product of minimal free resolutions of $R/I$ and $R/J$ is a minimal free resolution of $R/(I + J)$; see 2.2. As a part of a bigger study of understanding the syzygies of ideals of the form $I + J$, where $I$ and $J$ are both determinantal, we were motivated to look for criterion for transversal intersection of polynomial ideals; see 2.5. We have come across some natural classes of ideals in the polynomial ring which intersect transversally with the rational normal curves and with the determinantal ideals of the form $I_1(XY)$; see 3.6, 3.7. Let us briefly introduce ideals of the form $I_1(XY)$ and their sum with other determinantal ideals, which are extremely relevant in the field of algebra and geometry and therefore forms the central theme of our study.

Let $K$ be a field and $\{x_{ij}; 1 \leq i, j \leq n\}, \{y_j; 1 \leq j \leq n\}$ be indeterminates over $K$; $n \geq 2$. Let $R := K[x_{ij}, y_j]$ denote the polynomial algebra over $K$. Let $X$ denote an $n \times n$ matrix such that its entries are the variables...
and it is either generic or symmetric generic. Let \( Y = (y_j)_{n \times 1} \) be the \( n \times 1 \) column matrix. Let \( I_1(XY) \) denote the ideal generated by the polynomials \( g_j \), which are the \( 1 \times 1 \) minors or entries of the \( n \times n \) matrix \( XY \). The primality, primary decomposition and Betti numbers of ideals of the form \( I_1(XY) \) have been studied in [13] and [14], with the help of Gröbner bases for \( I_1(XY) \).

Ideals of the form \( I_1(XY) + J \) are particularly interesting when \( J \) is also determinantal. They occur in several geometric considerations like linkage and generic residual intersection of polynomial ideals, especially in the context of syzygies. Bruns-Kustin-Miller [11] resolved the ideal \( I_1(XY) + I_{\min(m,n)}(X) \), where \( X \) is a generic \( m \times n \) matrix and \( Y \) is a generic \( n \times 1 \) matrix. Johnson-McLoud [6] proved certain properties for the ideals of the form \( I_1(XY) + I_2(X) \), where \( X \) is a generic symmetric matrix and \( Y \) is either generic or generic alternating. These ideals We have considered the ideal \( I_1(XY) + I_2(\tilde{X}_{ij}) \) (see section 4), where \( \tilde{X}_{ij} \) is the matrix consisting of the \( i \)-th and the \( j \)-th rows of \( X \). In an attempt to prove the Cohen-Macaulay property of the ring \( R/(I_1(XY) + I_2(\tilde{X}_{ij})) \), \( 1 \leq i < j \leq n \), we ended up with an explicit construction of the minimal free resolution. In the process of doing so, we have encountered several examples of transversal intersection of ideals, vide Lemmas [6, 2] and [7, 4] linear quotients, vide Lemma [6, 3] and use the technique of iterated mapping cone along with [2, 2] as an effective tool.

2. Criterion for Transversal Intersection of Polynomial Ideals

**Definition 1.** Two ideals \( I \) and \( J \) in the polynomial ring are said to intersect transversally if \( I \cap J = IJ \).

**Definition 2.** Let \( T \subset R \) be a set of monomials. We define

\[
\text{supp}(T) = \{(i, j, 0) \mid x_{ij} \text{ divides } m \text{ for some } m \in T\} \cup \\
\{(0, 0, k) \mid y_k \text{ divides } m \text{ for some } m \in T\}.
\]

If \( T = \{m\} \), then we write \( \text{supp}(m) \) instead of \( \text{supp}(\{m\}) \).

**Lemma 2.1.** Let \( > \) be a monomial ordering on \( R \). Let \( I \) and \( J \) be ideals in \( R \) and let \( m(I) \) and \( m(J) \) denote unique minimal generating sets for their leading ideals \( \text{Lt}(I) \) and \( \text{Lt}(J) \) respectively. Then, \( I \cap J = IJ \) if \( \text{supp}(m(I)) \cap \text{supp}(m(J)) = \emptyset \). In other words, the ideals \( I \) and \( J \) intersect transversally if the set of variables occurring in the set \( m(I) \) is disjointed from the set of variables occurring in the set \( m(J) \).
Proof. Let \( f \in (I \cap J) \setminus IJ \). Let \( r \) denote the remainder term after division of \( f \) by a Gröbner basis of \( IJ \) with respect to the monomial order \( < \). Now \( r \in I \cap J \) implies that \( r \in I \) and therefore \( \text{Lt}(r) \in \text{Lt}(I) \). Hence, there exists monomial \( m_i \in m(I) \) such that \( m_i \mid \text{Lt}(r) \). Similarly, there exists monomial \( m_j \in m(J) \) such that \( m_j \mid \text{Lt}(r) \). Given that \( m_i \) and \( m_j \) are of disjoint support, we have \( m_im_j \mid \text{Lt}(r) \) and this proves that \( \text{Lt}(r) \in \text{Lt}(IJ) \), which is a contradiction. \( \square \)

The notion of transversal intersection of ideals \( I \) and \( J \) become particularly useful while resolving ideals of the form \( I + J \). We say that \( I \) and \( J \) intersect transversally if \( I \cap J = IJ \). Suppose that \( F \) resolves \( R/I \) and \( G \) resolvest\( R/J \) minimally. It is interesting to note that if \( I \) and \( J \) intersect transversally, then the tensor product complex \( F \otimes_R G \) resolves \( R/(I+J) \) minimally; see Lemma 2.2. Therefore, it is useful to know if two ideals intersect transversally, especially when one is trying to compute minimal free resolutions and Betti numbers for ideals of the form \( I + J \), through iterated techniques; see [5].

**Lemma 2.2.** Let \( I \) and \( J \) be graded ideals in the standard graded polynomial ring \( R = k[x_1, \ldots, x_n] \) over a field \( k \). Let us assume that \( I \cap J = I \cdot J \). Suppose that \( F \) and \( G \) are minimal graded free resolutions of \( I \) and \( J \) respectively. Then \( F \otimes_R G \) is a minimal graded free resolution for the graded ideal \( I + J \).

**Proof.** Suppose that \( \hat{R} = k[[x_1, \ldots, x_n]] \). We have \( I \otimes_R \hat{R} \cong I\hat{R} \) and \( (I \cap J) \otimes_R \hat{R} \cong I\hat{R} \cap J\hat{R} \), since \( \hat{R} \) is a flat \( R \) algebra (see Theorem 7.4 in [8]). Hence, \( I\hat{R} \cap J\hat{R} = (I\hat{R})(J\hat{R}) \). Let \( F_I \) and \( F_J \) denote minimal graded free \( R \)-resolutions of the ideals \( I \) and \( J \) respectively. Since \( \hat{R} \) is a flat \( R \) algebra and entries of each matrix that occurs in \( F_J \) are homogeneous, we have \( E_I = F_I \otimes_R \hat{R} \) is a minimal free resolution of \( I\hat{R} \). Similarly, \( E_J = F_J \otimes_R \hat{R} \) is a minimal free resolution of \( J\hat{R} \).

We first prove that \( E_I \otimes_R E_J \) is a minimal free resolution for \( I\hat{R} + J\hat{R} \). Consider the short exact sequence \( 0 \to I\hat{R} \to \hat{R} \to \hat{R}/I\hat{R} \to 0 \) and tensor it with \( \hat{R}/J\hat{R} \) over \( \hat{R} \). We get the exact sequence
\[
0 \to \text{Tor}_1^\hat{R}(\hat{R}/I\hat{R}, \hat{R}/J\hat{R}) \to \hat{R}/(I\hat{R} \cdot J\hat{R}) \to \hat{R}/J\hat{R} \to \hat{R}/(I\hat{R} + J\hat{R}) \to 0.
\]

The terms on the left are 0 since \( \hat{R} \) is a flat \( \hat{R} \) module. Moreover, the kernel of the map from \( I\hat{R}/I\hat{R} \cdot J\hat{R} \to \hat{R}/J\hat{R} \) is \( \hat{R} \cap J\hat{R}/I\hat{R} \cdot J\hat{R} \). Therefore \( \text{Tor}_1^\hat{R}(\hat{R}/I\hat{R}, \hat{R}/J\hat{R}) = 0 \) if and only if \( I\hat{R} \cap J\hat{R} = I\hat{R} \cdot J\hat{R} \). By Corollary 1 of Theorem 3 proved in [7], \( \text{Tor}_1^\hat{R}(\hat{R}/I\hat{R}, \hat{R}/J\hat{R}) = 0 \) implies
that $\text{Tor}^i_R\left(\hat{R}/I\hat{R},\hat{R}/J\hat{R}\right) = 0$ for all $i \geq 1$. Therefore, $H_i(\mathbb{E}_f \otimes_R \mathbb{E}_j) \simeq \text{Tor}^i_R\left(\hat{R}/I\hat{R},\hat{R}/J\hat{R}\right) = 0$ for all $i \geq 1$ and $H_0(\mathbb{E}_f \otimes_R \mathbb{E}_j) \simeq \hat{R}/(I\hat{R} + J\hat{R})$. This proves that $\mathbb{E}_f \otimes_R \mathbb{E}_j$ resolves $I\hat{R} + J\hat{R}$. The resolution is minimal since both $\mathbb{E}_f$ and $\mathbb{E}_j$ are minimal.

We now show that $F_I \otimes_R F_J$ is a minimal free resolution of $I\hat{R} + J\hat{R}$. Let $H_i$ be the $i$-th homology of the complex $F_I \otimes_R F_J$, then $H_i$ is a graded finitely generated $R$-module. Since $\hat{R}$ is a flat $R$ algebra and $\mathbb{E}_f \otimes_R \mathbb{E}_j$ is a minimal free resolution of $I\hat{R} + J\hat{R}$, we have $H_i \otimes_R \hat{R} = 0$. Let $m = \langle x_1, \ldots, x_n \rangle$ be the maximal relevant ideal in the standard graded polynomial ring $R$. Now $(H_i \otimes_R \hat{R}) \otimes_R R/m \cong (H_i/mH_i \otimes_R \hat{R}/m\hat{R}) \cong (H_i/mH_i \otimes_{R/mR} \hat{R}/m\hat{R}) = 0$. Therefore $H_i/mH_i = 0$ and using graded Nakayama $H_i = 0$. Since all entries of matrices that occur in $F_I \otimes_R F_J$ are homogeneous we have $F_I \otimes_R F_J$ is a minimal graded free resolution of $I + J$.

**Theorem 2.3 (Rees).** Let $N$ be an $R$ module, $h_1, \ldots, h_k$ be an $N$-regular sequence in $R$ and $J = \langle h_1, \ldots, h_k \rangle$. Let $Y = y_1, \ldots, y_k$ be indeterminates over $R$. If $F(y_1, \ldots, y_k) \in N[Y]$ is homogeneous of degree $r$ and $F(h_1, \ldots, h_k) \in J^{r+1}N$ then the coefficients of $F$ are in $JN$.

**Proof.** See Theorem 1.1.7 in [1].

**Lemma 2.4.** Let $h_1, \ldots, h_k$ be a regular sequence in $R$. Let $J$ denote the ideal $\langle h_1, \ldots, h_{k-1} \rangle$. Then $h_k$ is not a zero divisor in $R/J^r$, for every $r \geq 1$.

**Proof.** We use induction on $r$. For $r = 1$, the result follows from the fact that $h_1, \ldots, h_{k-1}$ is a regular sequence in $R$. Let us assume that $h_k$ is not a zero divisor in $R/J^{r-1}$. Let $h_kp \in J^r$, hence $h_kp \in J^{r-1}$. By the induction hypothesis we have $p \in J^{r-1}$. We can write

$$p = \sum_{\lambda_1 + \cdots + \lambda_{k-1} = r-1} \beta(\lambda_1, \ldots, \lambda_{k-1}) h_1^{\lambda_1} \cdots h_{k-1}^{\lambda_{k-1}}.$$

Let us consider the homogeneous polynomial $F(y_1, \ldots, y_{k-1})$ of degree $r - 1$ in $\hat{R}[y_1, \ldots, y_{k-1}]$, given by

$$F(y_1, \ldots, y_{k-1}) = \sum_{\lambda_1 + \cdots + \lambda_{k-1} = r-1} h_k \beta(\lambda_1, \ldots, \lambda_{k-1}) y_1^{\lambda_1} \cdots y_{k-1}^{\lambda_{k-1}}.$$

Then, $F(h_1, \ldots, h_{k-1}) = h_kp \in J^r$. By Theorem 2.3, $\{h_k \beta(\lambda_1, \ldots, \lambda_{k-1}) | \lambda_1 + \cdots + \lambda_{k-1} = r-1\} \subseteq J$. Given that $h_1, \ldots, h_k$ be a regular sequence in $R$, we have $\{\beta(\lambda_1, \ldots, \lambda_{k-1}) | \lambda_1 + \cdots + \lambda_{k-1} = r-1\} \subseteq J$. Hence, $p \in J^r$. □
Theorem 2.5. Let $I$ and $J$ be ideals in $R$, such that $J$ is generated by an $R/I$ regular sequence $g_1, g_2, \ldots, g_k$. Then,

(i) $I \cap J = IJ$.

(ii) $I \cap J^r = IJ^r$, for all positive integers $r$.

Proof. (i) We use induction on the length $k$ of the $R/I$-regular sequence generating the ideal $J$. For $k = 1$, let $\alpha \in I \cap J$. Then since $\alpha \in J$, we can write $\alpha = g_1 r_1$ for some $r_1 \in R$. Therefore, $r_1 g_1 \in I$ and $r_1 g_1 = 0 \in R/I$. The element $\overline{g_1}$ is not a zero divisor in $R/I$, by hypothesis. Therefore, $\overline{r_1} = \overline{0} \in R/I$ and hence $r_1 \in I$. This shows that $\alpha = r_1 g_1 \in IJ$. We assume that the statement is true for $k - 1$. Let $\alpha \in I \cap J$. Since $\alpha \in J$, we can write $\alpha = r_1 g_1 + r_2 g_2 + \cdots + r_k g_k$, for some $\alpha_1, \ldots, \alpha_k \in R$. Now $\alpha \in I$, therefore $\overline{r_1 g_1} \in R/(I + \langle g_1, g_2, \ldots, g_k-1 \rangle)$. The elements $\overline{g_1}, \overline{g_2}, \ldots, \overline{g_k}$ being a regular sequence in $R/I$, we have $\overline{r_k} = 0$, that is, $r_k \in I + \langle g_1, g_2, \ldots, g_k-1 \rangle$. Let $r_k = r_1 g_1 + r_2 g_2 + \cdots + r_{k-1} g_{k-1} + h$, where $h \in I$ and $r_1, r_2, \ldots, r_{k-1} \in R$. Therefore, $\alpha = (r_1 + t_1 g_1) g_1 + (r_2 + t_2 g_2) g_2 + \cdots + (r_{k-1} + t_{k-1} g_{k-1}) g_{k-1} + hg_k$. We have $hg_k \in I \cap J$. Hence it is enough to show that $\alpha' = \alpha - hg_k \in IJ$. Then $J' = \langle g_1, g_2, \ldots, g_k-1 \rangle$. Then $\{\overline{g_1}, \overline{g_2}, \ldots, \overline{g_k-1} \}$ being a part of a regular sequence, is a regular sequence in $R/I$. By the induction hypothesis, we have $I \cap J' = IJ'$. Now $\alpha \in I$ implies that $\alpha - ig_k \in I$ and therefore $\alpha' \in I$. Also, $\alpha' \in \langle g_1, g_2, \ldots, g_k-1 \rangle = J'$. Therefore $\alpha' \in I \cap J' = IJ'$.

(ii) We use induction on $r$. For $k = 1$ the result trivially holds for all $r \geq 1$ by (i). We assume by induction that for $I \cap J^s = IJ^s$ for all $1 \leq s < r$. Now we prove that the result holds good for $s = r$. Let $x_k \in I \cap J^r$. Every element of $J^r$ can be written in the form

$$\sum_{i_1+i_2+\cdots+i_k=r} \alpha_{i_1,i_2,\ldots,i_k} g_1^{i_1} \cdots g_k^{i_k}.$$ 

Therefore, we can write $x_k = g_k \gamma_k + \beta_k$, where $\beta_k \in \langle g_1, \ldots, g_{k-1} \rangle^r$ and $\gamma_k \in J^{r-1}$. We know that $g_k$ is not a zero divisor in $R/(I + \langle g_1, \ldots, g_k-1 \rangle)$. It follows by [3,1] that $g_k$ is also a non-zero divisor in $R/(I + \langle g_1, \ldots, g_k-1 \rangle^r)$. We know that $x_k \in I \subseteq I + \langle g_1, \ldots, g_k-1 \rangle^r$. Therefore, $\overline{x_k} = \overline{0}$ in $R/I + \langle g_1, \ldots, g_k-1 \rangle^r$ and hence $\overline{g_k} \overline{\gamma_k} = \overline{0}$. This proves that $\overline{\gamma_k} = \overline{0}$ in $R/I + \langle g_1, \ldots, g_k-1 \rangle^r$. Let $\gamma_k = i_k + \alpha_k$, where $i_k \in I$ and $\alpha_k \in \langle g_1, \ldots, g_{k-1} \rangle^r$. Hence, $i_k = \gamma_k - \alpha_k \in I \cap J^{r-1}$ and therefore $i_k \in I \cap J^{r-1} = IJ^{r-1}$, by the induction hypothesis on $r$. It follows that $g_k i_k \in IJ^r$. In order to show that $x_k = g_k (i_k + \alpha_k) + \beta_k \in IJ^r$, it is therefore enough to prove that $x_{k-1} := g_k i_k + \beta_k \in IJ^r$. We have $x_{k-1} \in I \cap \langle g_1, \ldots, g_{k-1} \rangle^r$. We continue this process to produce $x_{k-2}, x_{k-3}, \ldots, x_1$, such that $x_i \in I \cap \langle g_1, \ldots, g_{k-i+1} \rangle^r$, for every $i = 1, \ldots, k$. 

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In particular, \( x_1 \in I \cap (g_1)^r = I \cdot (g_1)^r \). Then, it follows that \( x_2 \in I \cdot (g_1, g_2)^r \). We can successively go back and prove that \( x_k \in IJ^r \). \( \Box \)

Recently, Professor G. Valla pointed out the resemblance of Lemma 2.5 in this paper and Lemma 1.1 in [12].

3. Transversal Intersection with the Rational Normal Curve

Let \( S = K[x_1, \ldots, x_{n+1}] \), where \( x_i \)'s are indeterminates over the field \( K \). Let \( N = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_2 & x_3 & x_4 & \cdots & x_{n+1} \end{pmatrix} \) and let \( I_2(N) \) denote the ideal generated by the \( 2 \times 2 \) minors of the matrix \( N \) in \( S \). The ideal \( I_2(N) \) is the defining ideal of the rational normal curve in the projective space under the standard parametrization. Our aim in this section is to show that some natural classes of ideals \( J \) in the polynomial ring intersect transversally with the ideal \( I_2(N) \). This information helps us write the minimal free resolution of the sum ideal \( I_2(N) + J \), since the tensor product complex of the minimal free resolutions of \( I_2(N) \) and \( J \) resolve \( I + J \) minimally; see Lemma 3.7 in [13]. The main theorems in this section are Theorems 3.6 and 3.7. We first prove the following Lemmas.

**Lemma 3.1.** Let \( I \) and \( J \) be ideals in \( S \), such that \( J = \langle x \rangle \), where \( x \) is a non-zero polynomial in \( S \). Then \( I \cap J = IJ \) if and only if \( \bar{x} \) is not a zero divisor in \( R/I \).

**Proof.** Let \( \bar{xg} = \bar{0} \) in \( R/I \). Then, \( xg \in I \) and also \( xg \in J \). Therefore, \( xg \in I \cap J = IJ \). We can write \( xg = xg', \) for some \( g' \in I \). This shows that \( x(g - g') = 0 \) an hence \( g = g' \in I \). The converse follows from Lemma 2.5. \( \Box \)

**Lemma 3.2.** The ideals \( I_2(N) \) and \( \langle x_1^a + x_{n+1}^b \rangle \), where \( a, b \in \mathbb{N} \), intersects transversally.

**Proof.** The ideal \( I_2(N) \) is kernel of the homomorphism \( \zeta : S \to k[s, t] \) defined as \( \zeta(x_i) = s^{n-i+1}t^{i-1} \), \( 1 \leq i \leq n+1 \). Therefore, the ideal \( I_2(N) \) is a prime ideal. Again, \( x_1^a + x_{n+1}^b \notin I_2(N) \), since \( \zeta(x_1^a + x_{n+1}^b) = s^{na} + t^{nb} \neq 0 \). Therefore, \( x_1^a + x_{n+1}^b \) is not a zero divisor in \( R/I_2(N) \). Hence, by lemma 3.1, the ideals \( I_2(N) \) and \( \langle x_1^a + x_{n+1}^b \rangle \) intersect transversally. \( \Box \)

**Lemma 3.3.** Let \( a, b \) be natural numbers. Then,

(i) The projective dimension of \( R/(I_2(N) + \langle x_1^a + x_{n+1}^b \rangle) \) is \( n \).

(ii) \( R/(I_2(N) + \langle x_1^a + x_{n+1}^b \rangle) \) is Cohen-Macaulay.
Proof. (i) The minimal free resolution of $R/I_2(N)$ is given by the Eagon-Northcott complex of length $n - 1$. The ideals $I$ and $\langle x_1^a + x_{n+1}^b \rangle$ intersect transversally by Lemma 3.2. Therefore, a minimal free resolution of $R/(I_2(N) + \langle x_1^a + x_{n+1}^b \rangle)$ is given by the tensor product complex of the minimal free resolutions of $R/I_2(N)$ and $R/(x_1^a + x_{n+1}^b)$ by Lemma 3.2. Therefore, it follows that the projective dimension of $R/(I_2(N) + \langle x_1^a + x_{n+1}^b \rangle)$ is $n (= n - 1 + 1)$.

(ii) Projective dimension of $R/(I_2(N) + \langle x_1^a + x_{n+1}^b \rangle)$ is $n$, therefore, by the Auslander Buchsbaum theorem $\text{depth}(R/(I_2(N) + \langle x_1^a + x_{n+1}^b \rangle)) = 1$. Again $I_2(N)$ is a prime ideal of height $n - 1$ and $x_1^a + x_{n+1}^b$ is a non-zero divisor of $R/I_2(N)$, therefore, by the Krull’s principal ideal theorem height of $I_2(N) + \langle x_1^a + x_{n+1}^b \rangle$ is $n$. Hence, $\dim(R/(I_2(N) + \langle x_1^a + x_{n+1}^b \rangle)) = 1$ and therefore $R/(I_2(N) + \langle x_1^a + x_{n+1}^b \rangle)$ is Cohen-Macaulay.

Lemma 3.4. Let $a, b, c$ be natural numbers. Let $p$ be a prime ideal such that $I_2(N) + \langle x_1^a + x_{n+1}^b, x_n^c \rangle \subset p$. Then, $p = \langle x_1, \ldots, x_{n+1} \rangle$ and hence the height of the ideal $I_2(N) + \langle x_1^a + x_{n+1}^b, x_n^c \rangle$ is $n + 1$.

Proof. We use induction to prove $x_{n-i} \in p$, for all $0 \leq i \leq n - 2$. Since $p$ is a prime ideal and $x_n^c \in I_2(N) + \langle x_1^a + x_{n+1}^b, x_n^c \rangle \subset p$, therefore $x_n \in p$. Let us assume that $x_{n-i} \in p$ for some $i$. Again $x_{n-i}x_{n-i-2} - x_{n-i-1}^2 \in I_2(N) + \langle x_1^a + x_{n+1}^b, x_n^c \rangle \subset p$. By the induction hypothesis $x_{n-i} \in p$, therefore we have $x_{n-i-1} \in p$. Therefore $x_{n-i} \in p$ for all $0 \leq i \leq n - 2$.

As $x_1x_{n+1} - x_2x_n \in p$ and $x_n \in p$ we have $x_1x_{n+1} \in p$ hence $x_1$ or $x_{n+1} \in p$. If $x_1 \in p$, then $x_1^a \in p$ which implies that $x_{n+1}^b \in p$. Hence $x_{n+1} \in p$. If $x_{n+1} \in p$, then similarly we can show that $x_1 \in p$ and hence $p = \langle x_1, \ldots, x_{n+1} \rangle$.

Lemma 3.5. $x_n^d$ is not a zero divisor in $R/(I_2(N) + \langle x_1^a + x_{n+1}^b \rangle)$.

Proof. Suppose that it is a zero divisor, then it is contained in an associated prime ideal of $R/(I_2(N) + \langle x_1^a + x_{n+1}^b \rangle)$. But, $R/(I_2(N) + \langle x_1^a + x_{n+1}^b \rangle)$ being Cohen-Macaulay, the prime ideal has to be minimal. We know that the height of the ideal $I_2(N) + \langle x_1^a + x_{n+1}^b \rangle$ is $n$. Hence, any minimal prime ideal of $R/(I_2(N) + \langle x_1^a + x_{n+1}^b \rangle)$ has height $0$. In other words, any minimal prime ideal containing $I_2(N) + \langle x_1^a + x_{n+1}^b \rangle$ has height $n$. But any prime ideal containing both $(I_2(N) + \langle x_1^a + x_{n+1}^b \rangle)$ and $x_n$ has height $n + 1$ from the previous lemma. Hence it cannot be minimal.

Theorem 3.6. Let $J$ denote the ideal $\langle x_1^a + x_{n+1}^b, x_n^c \rangle$ in the polynomial ring $S$, such that $a, b, c$ are in $\mathbb{N}$. Then, $I_2(N) \cap J = IJ$. 
Proof. We have seen in Lemmas 3.2 and 3.5 that $x_1^a + x_2^b \cdots x_n^c$ is a regular sequence in $R/I_2(N)$. Hence, by Theorem 2.5, the ideals $I_2(N)$ and $J$ intersect transversally.

Theorem 3.7. Let $T = k[x_{ij}, y_i]$, where $i, j \in \{1, \ldots, n\}$. Let

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

be generic matrices of indeterminates $x_{rs}$ and $y_k$. Let $f_r = \sum_{s=1}^n x_{rs}y_s$, for $1 \leq r \leq n$. Then, $I_1(XY) = \mathcal{I} = \langle f_1, \ldots, f_n \rangle$.

Let $H = \begin{pmatrix} x_{i1} & x_{i2} & \cdots & x_{i(i-1)} & x_{i(i+1)} & \cdots & x_{i(n-1)} & x_{in} \\ x_{i2} & x_{i3} & \cdots & x_{i(i+1)} & x_{i(i+2)} & \cdots & x_{in} & x_{pq} \end{pmatrix}$, $1 \leq p, q, i \leq n$ and $p \neq i, p \neq q$.

Let $J = I_2(H)$ be the rational normal curve. Then $\mathcal{I} \cap J = \mathcal{I}J$.

Proof. Let us consider the monomial order

- $x_{11} > x_{22} > \cdots > x_{nn}$;
- $x_{rs}, y_s < x_{nn}$ for every $1 \leq r \neq s \leq n$.

Then $\{f_1, \ldots, f_n\}$ forms a Gröbner basis for the ideal $\mathcal{I}$ and $\text{Lt}(f_i) = x_{ri}y_i$. Therefore, $\text{supp}(\text{Lt}(\mathcal{I})) \cap \text{supp}(\text{Lt}(J)) = \emptyset$. Hence by Theorem 2.5 $\mathcal{I} \cap J = \mathcal{I}J$.

4. Resolution of Sums of Ideals

- If $X$ is generic and $i < j$; let $\tilde{X}_{ij} = \begin{pmatrix} x_{i1} & x_{i2} & \cdots & x_{in} \\ x_{j1} & x_{j2} & \cdots & x_{jn} \end{pmatrix}$.
- If $X$ is generic symmetric and $i < j$; let

  $$\tilde{X}_{ij} = \begin{pmatrix} x_{1i} & \cdots & x_{ii} & \cdots & x_{ij} & \cdots & x_{in} \\ x_{1j} & \cdots & x_{ij} & \cdots & x_{jj} & \cdots & x_{jn} \end{pmatrix}.$$  

- Let $G_{ij}$ denote the set of all $2 \times 2$ minors of $\tilde{X}_{ij}$.
- Let $I_2(\tilde{X}_{ij})$ denote the ideal generated $G_{ij}$.

Lemma 4.1. Suppose that $X$ is either generic or generic symmetric. The set $G_{ij}$ is a Gröbner basis for the ideal $I_2(\tilde{X}_{ij})$, with respect a suitable monomial order.
Proof. We choose the lexicographic monomial order given by the following ordering among the variables: 
\( x_{st} > x_{s't'} \) if \((s', t') >_{\text{lex}} (s, t)\) and 
\( y_{n-1} > \cdots > y_1 > x_{st} \) for all \( s, t \). We now apply Lemma 4.2 in [13] for the matrix \( X^t \) and for \( k = 2 \).

Our aim in this paper is to prove the following theorem:

**Theorem 4.2.** Let \( X = (x_{ij}) \) be either the generic or the generic symmetric matrix of order \( n \). Let \( 1 \leq i < j \leq n \).

1. The total Betti numbers for the ideal \( I_2(\tilde{X}_{ij}) + \langle g_i, g_j \rangle \) are given by
   \[ b_0 = 1, \quad b_1 = \binom{n}{2} + 2, \quad b_2 = 2\binom{n}{3} + n, \quad b_{i+1} = i\binom{n}{i} + (i-2)\binom{n}{i+1}, \quad \text{for} \quad 2 \leq i \leq n - 2 \quad \text{and} \quad b_n = n - 2. \]

2. Let \( 1 \leq k \leq n - 2 \). Let \( \beta_{k,p} \) denote the \( p \)-th total Betti number for the ideal \( I_2(\tilde{X}_{ij}) + \langle g_i, g_j, g_{l_1}, \ldots, g_{l_k} \rangle \), such that \( 1 \leq l_1 < \ldots < l_k \leq n \) and \( l_t \) is the smallest in the set \( \{1, 2, \ldots, n\} \setminus \{i, j, l_1, \ldots, l_{t-1}\} \), for every \( 1 \leq t \leq k \). They are given by \( \beta_{k,0} = 1, \quad \beta_{k,p} = \beta_{k-1,p-1} + \beta_{k-1,p} \) for \( 1 \leq p \leq n + k - 1 \) and \( \beta_{k,n+k} = n - 2 \).

In particular, the total Betti numbers for the ideal \( I_1(XY) + I_2(\tilde{X}_{ij}) \) are \( \beta_{n-2,0}, \beta_{n-2,1}, \ldots, \beta_{n-2,2n-2} \).

5. Preliminaries and some Homological Lemmas

We first recall some useful results on determinantal ideals pertaining to our work. We refer to [3], [4], [10] for detailed discussions on these.

**Lemma 5.1.** Let \( h_1, h_2, \ldots, h_n \in R \) be such that with respect to a suitable monomial order on \( R \), the leading terms of them are mutually coprime. Then, \( h_1, h_2, \ldots, h_n \) is a regular sequence in \( R \).

**Proof.** See Lemma 2.1 in [14].

**Theorem 5.2.** Let \( K \) be a field and let \( x_{ij} : 1 \leq i \leq m, 1 \leq j \leq n \) be indeterminates over \( K \). Let \( A = (x_{ij}) \) be the \( m \times n \) matrix of indeterminates and \( I_m(A) \) denotes the ideal generated by the maximal minors of \( A \). The set of maximal minors of \( A \) is a universal Gröbner basis for the ideal \( I_m(A) \).

**Proof.** See [3].

**The Eagon-Northcott Complex.** We present the relevant portion from the book [4] here. Let \( F = R^f \) and \( G = R^g \) be free modules of finite rank over
the polynomial ring $R$. The *Eagon-Northcott complex* of a map $\alpha : F \rightarrow G$ (or that of a matrix $A$ representing $\alpha$) is a complex

$$
EN(\alpha) : 0 \rightarrow (\text{Sym}_{f-g}G)^* \otimes \wedge^f F \xrightarrow{d_{f-g+1}} (\text{Sym}_{f-g-1}G)^* \otimes \wedge^{f-1} F \xrightarrow{d_{f-g}} \cdots \rightarrow (\text{Sym}_2 G)^* \otimes \wedge^{g+2} F \xrightarrow{d_3} G^* \otimes \wedge^{g+1} F \xrightarrow{d_{g+1}} F \xrightarrow{\wedge^g} \wedge^g G.$$

Here $\text{Sym}_k G$ is the $k$-th symmetric power of $G$ and $M^* = \text{Hom}_R(M, R)$. The map $d_j$ are defined as follows. First we define a diagonal map 

$$(\text{Sym}_k G)^* \rightarrow G^* \otimes (\text{Sym}_{k-1} G)^*$$

$u \mapsto \sum_i u'_i \otimes u''_i$

as the dual of the multiplication map $G \otimes \text{Sym}_{k-1} G \rightarrow \text{Sym}_k G$ in the symmetric algebra of $G$. Next we define an analogous diagonal map

$$\wedge^k F \rightarrow F \otimes \wedge^{k-1} F$$

$v \mapsto \sum_i v'_i \otimes v''_i$

as the dual of the multiplication in the exterior algebra of $F^*$.

**Theorem 5.3** (Eagon-Northcott). The Eagon-Northcott complex is a free resolution of $R/I_g(\alpha)$ iff $\text{grade}(I_g(\alpha)) = f - g + 1$ where $I_g(\alpha)$ denotes the $g \times g$ minors of the matrix $A$ representing $\alpha$.

**Proof.** See [4]. \(\square\)

**Mapping Cone.** We present the relevant portion from the book [10] here. Let $R$ be the polynomial ring. Let $\phi : (U, d) \rightarrow (U', d')$ be a map of complexes of finitely generated $R$-modules. The mapping cone of $\phi$ is the complex $W$ with differential $\delta$ defined as follows. Let $W_i = U_{i-1} \oplus U'_i$, with $\delta|_{U_{i-1}} = -d + \phi : U_{i-1} \rightarrow U_{i-2} \oplus U'_{i-1}$ and $\delta|_{U'_i} = d' : U'_i \rightarrow U'_{i-1}$ for each $i$.

**Theorem 5.4.** Let $M$ be an ideal minimally generated by the polynomials $f_1, \ldots, f_r$. Set $M_i = \langle f_1, \ldots, f_i \rangle$, for $1 \leq i \leq r$. Thus, $M = M_r$. For each $i \geq 1$, we have the short exact sequence

$$0 \rightarrow S/(M_i : f_{i+1}) \xrightarrow{f_{i+1}} S/M_i \rightarrow S/M_{i+1} \rightarrow 0.$$ 

If resolutions of $S/M_i$ and $S/(M_i : f_{i+1})$ are known then we can construct a resolution of $S/M_{i+1}$ by the mapping cone construction.

**Proof.** See Construction 27.3 in [10]. \(\square\)
Lemma 5.5. Let
\[ R^{a_1} \xrightarrow{A_1} R^{a_2} \xrightarrow{A_2} R^{a_3} \]
be an exact sequence of free modules. Let \( Q_1, Q_2, Q_3 \) be invertible matrices of sizes \( a_1, a_2, a_3 \) respectively. Then,
\[ R^{a_1} Q_2^{-1} A_1 Q_1 \xrightarrow{} R^{a_2} Q_3^{-1} A_2 Q_2 \xrightarrow{} R^{a_3} \]
is also an exact sequence of free modules.

Proof. The following diagram is a commutative diagram of free modules and the vertical maps are isomorphisms:
\[
\begin{array}{c}
R^{a_1} \xrightarrow{A_1} R^{a_2} \xrightarrow{A_2} R^{a_3} \\
Q_1 \uparrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quito
row from $A_{n+1}$, $A'_{n-1}$ the matrix obtained by deleting the l-th column from $A_{n-1}$ and $A'_n$ the matrix obtained by deleting the l-th row and m-th column from $A_n$. Then, the sequence

$$\ldots \rightarrow R^β_{n+1} A'_{n+1} \rightarrow R^β_{n} A'_{n} \rightarrow R^β_{n-1} A'_{n-1} \rightarrow R^β_{n-2} \rightarrow \ldots$$

is exact.

**Proof.** The fact that the latter sequence is a complex is self evident. We need to prove its exactness. By the previous lemma we may assume that $l = m = 1$, for we choose elementary matrices to permute rows and columns and these matrices are always invertible. Now, due to exactness of the first complex we have $A_{n-1}A_n = 0$. This implies that the first column of $A_{n-1}$ is 0, which implies that $\text{Im}(A_{n-1}) = \text{Im}(A'_{n-1})$. Therefore, the right exactness of $A_{n+1}$ is preserved. By a similar argument we can prove that the left exactness of $A'_n$ is preserved.

Let $(x)$ denote a tuple with entries from $R$. If $(x) \in \ker(A'_n)$, then $(0, x) \in \ker(A_n)$. There exists $(y) \in R^β_{n+1}$ such that $A_{n-1}(y) = (0, x)$. It follows that $A'_{n-1}(y) = (x)$, proving the left exactness of $A'_n$. By a similar argument we can prove the right exactness of $A'_n$.

**Lemma 5.8.** Let $A$ be a $q \times p$ matrix over $R$ with $a_{ij} = \pm 1$, for some $i$ and $j$. Let $C$ be a $p \times s$ matrix and $B$ a $r \times q$ matrix over $R$. There exist an invertible $q \times q$ matrix $X$ and an invertible $p \times p$ matrix $Y$, such that

(i) $(XAY)_{kj} = δ_{ki}$ and $(XAY)_{ik} = δ_{jk}$, that is

$$XAY = \begin{pmatrix}
\cdots & 0 & \cdots \\
\cdots & 0 & \cdots \\
\vdots & \vdots & \vdots \\
0 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots \\
0 & 0 & \cdots \\
\cdots & 0 & \cdots \\
0 & \cdots & \cdots \\
\end{pmatrix} ; \quad 1 \text{ at the } (i,j)-\text{th spot.}
$$

(ii) $(Y^{-1}C)_{kl} = C_{kl}$ for $k \neq j$ and $(Y^{-1}C)_{jl} = C_{jl} + \sum_{t \neq i}(a_{lt})C_{tl}$

(iii) $(BX^{-1})_{kl} = B_{kl}$ for $l \neq i$ and $(BX^{-1})_{ki} = B_{ki} + \sum_{t \neq i}(a_{ti})B_{kt}$.

**Proof.** (i) We prove for $a_{ij} = 1$. The other case is similar. We take $Y = \Pi_{k \neq j}E_{jk}(-a_{ik})$ and $X = \Pi_{k \neq i}E_{ki}(-a_{kj})$, where $E_{kl}(α)$ denotes the matrix $E$ with $E_{kl} = α$, $E_{tt} = 1$ and $E_{ut} = 0$ for $u \neq t$ and $(u,t) \neq (k,l)$.

(ii) and (iii) are easy to verify. 

□
Lemma 5.9. Let $A$ be a $q \times p$ matrix, $C$ be a $p \times s$ matrix and $B$ a $r \times q$ matrix over $R$. The matrices $A$, $B$ and $C$ satisfy property $P_{ij}$ if they satisfy the following conditions:

- $A_{ij} = 1$, $A_{ik} \in m$ for $k \neq j$ and $A_{kj} \in m$ for $k \neq i$;
- $B_{ki} \in m$, for $1 \leq k \leq r$;
- $C_{jl} \in m$, for $1 \leq l \leq s$.

The matrices $XAY$, $BX^{-1}$ and $Y^{-1}C$ satisfy property $P_{ij}$, if $A$, $B$, $C$ satisfy property $P_{ij}$.

Proof. This follows from the above lemma since $a_{ik}$ and $a_{kj}$ belong to $m$. □

6. Betti Numbers of $I_1(XY) + I_2(\tilde{X}_{ij})$

Lemma 6.1. Let $X$ be generic or generic symmetric matrix. Let $i < j$.

(i) $\text{ht}(I_2(\tilde{X}_{ij})) = n - 1$.

(ii) The Eagon-Northcott complex minimally resolves the ideal $I_2(\tilde{X}_{ij})$.

Proof. (i) We show that $f_1, \ldots, f_{n-1}$, given by $f_k = x_{ik}x_{j,k+1} - x_{jk}x_{i,k+1}$, $1 \leq k \leq n - 1$ form a regular sequence.

Let us first assume that $X$ is generic. We take the lexicographic monomial order induced by the following ordering among the variables: $x_{i1} > x_{i2} > \cdots > x_{in} > x_{j2} > x_{j3} > \cdots > x_{jn} > x_{j1} > x_{kl}$, such that $x_{kl}$ are those variables which do not appear in $\tilde{X}_{ij}$ and the variables $y_p$ are smaller than $x_{j1}$. Then, $\text{Lt}(f_k) = x_{ik}x_{j,k+1}$ and hence $\text{gcd} (\text{Lt}(f_k), \text{Lt}(f_l)) = 1$ for every $k \neq l$. Therefore, $f_1, \ldots, f_{n-1}$ is a regular sequence by Lemma 5.1 and hence $\text{ht}(I_2(\tilde{X})) \leq n - 1$. On the other hand, $\text{ht}(I_2(\tilde{X})) \leq n - 1$, by Theorem [13.10] in [8]. Hence, $\text{ht}(I_2(\tilde{X})) \leq n - 1$.

If $X$ is generic symmetric, then we have to choose the lexicographic monomial order induced by $x_{ii} > x_{ij} > x_{1i} > x_{2i} > \cdots > x_{i-1,i} > x_{i,i+1} > \cdots > x_{in} > x_{jj} > \cdots > x_{iji} > \cdots > x_{j-1,j} > x_{jj+1} > \cdots > x_{jn}$ and variables $x_{kl}$ not appearing in $\tilde{X}_{ij}$ and the variables $y_p$ are smaller than $x_{jn}$.

(ii) The height of $I_2(\tilde{X}_{ij})$ is $n - 1$, which is the maximum. Hence, the Eagon-Northcott complex minimally resolves the ideal $I_2(\tilde{X}_{ij})$. □

Lemma 6.2. Let $X$ be generic or generic symmetric. Let $i < j$. Then $I_2(\tilde{X}_{ij}) \cap \langle g_i \rangle = I_2(\tilde{X}_{ij}) \cdot \langle g_i \rangle$, that is, the ideals $I_2(\tilde{X}_{ij})$ and $\langle g_i \rangle$ intersect transversally.
Proof. Let $X$ be generic. We choose the lexicographic monomial order given by the following ordering among the variables: $x_{st} > x_{s't'}$ if $(s', t') > (s, t)$ and $y_n > y_{n-1} > \cdots > y_1 > x_{st}$ for all $s, t$. Then, by Lemma 4.1 the set of all $2 \times 2$ minors forms a Gröbner basis for the ideal $I_2(\widetilde{X}_{ij})$. Clearly, the minimal generating set $m(\text{Lt}(I_2(\widetilde{X}_{ij})))$ doesn’t involve the indeterminates $x_i$ and $y_n$, whereas $\text{Lt}(g_i) = x_i y_n$. Hence, the supports of $m(\text{Lt}(I_2(\widetilde{X}_{ij})))$ and $m(\text{Lt}(g_i))$ are disjoint. Therefore, by Lemma 2.1 we are done.

Let $X$ be generic symmetric. Once we choose the correct monomial order, the rest of the proof is similar to the generic case. Suppose that $(i, j) = (n-1, n)$. We choose the lexicographic monomial order given by the following ordering among the variables:

$y_1 > y_n > y_{n-1} > \cdots > y_2 > x_{n-1,n-1} > x_{n-1,n} > x_{1,n-1} > x_{2,n-1} > \cdots > x_{n-2,n-1}$

$> x_{nn}$

$> x_{1n} > \cdots > x_{n-2,n}$

$> x_{st}$ for all other $s, t$. □

Suppose that $(i, j) \neq (n-1, n)$. We choose the lexicographic monomial order given by the following ordering among the variables:

$y_n > y_{n-1} > \cdots > y_1 > x_{ii} > x_{ij} > x_{1i} > x_{2i} > \cdots > x_{i-1,i} > x_{i,i+1} > \cdots > x_{in}$

$> x_{jj} > \cdots > x_{j-1,j} > x_{j,j+1} > \cdots > x_{jn}$

$> x_{st}$ for all other $s, t$. □

Lemma 6.3. Let $X$ be generic and $i < j$. Then, $(I_2(\widetilde{X}_{ij}) + \langle g_i \rangle : g_j) = \langle x_{i1}, \ldots, x_{in} \rangle$. If $X$ is generic symmetric and $i < j$, then $(I_2(\widetilde{X}_{ij}) + \langle g_i \rangle : g_j) = \langle x_{i1}, \ldots, x_{i-1,i}, x_{ii}, \ldots, x_{in} \rangle$.

Proof. Let $X$ be generic. We have $x_{it} g_j = x_{jt} g_i + \sum_{k=1}^{n} (x_{it} x_{jk} - x_{ik} x_{jt}) y_k$. Hence, $\langle x_{i1}, \ldots, x_{in} \rangle \subseteq (I_2(\widetilde{X}_{ij}) + \langle g_i \rangle : g_j)$. Moreover, $I_2(\widetilde{X}_{ij}) + \langle g_i \rangle \subseteq \langle x_{i1}, \ldots, x_{in} \rangle$ and $g_j \notin \langle x_{i1}, \ldots, x_{in} \rangle$. The ideal $\langle x_{i1}, \ldots, x_{in} \rangle$ being a prime ideal, it follows that $\langle x_{i1}, \ldots, x_{in} \rangle \supseteq (I_2(\widetilde{X}_{ij}) + \langle g_i \rangle : g_j)$. The proof for the generic symmetric case is similar. □

7. Minimal Free Resolution of $I_2(\widetilde{X}_{ij}) + \langle g_i, g_j \rangle$

Our aim is to construct a minimal free resolution for the ideal $I_2(\widetilde{X}_{ij}) + \langle g_1, \ldots, g_n \rangle$. We have proved that the ideals $I_2(\widetilde{X}_{ij})$ and $\langle g_i \rangle$ intersect transversally; see §6.2. The ideal $I_2(\widetilde{X}_{ij}) + \langle g_i \rangle$ can therefore be resolved.
minimally by Theorem 2.2. We have also proved that the ideal \( I_2(\tilde{X}_{ij}) + \langle g_i \rangle \) and the ideal \( \langle g_j \rangle \) have linear quotient; see 6.3. Therefore, the ideal \( I_2(\tilde{X}_{ij}) + \langle g_i, g_j \rangle \) can be resolved by the mapping cone construction. A minimal free resolution can then be extracted from this resolution by applying Lemma 5.9. Next, we will show that the ideal \( I_2(\tilde{X}_{ij}) + \langle g_i, g_j \rangle \) intersects transversally with the ideal \( \langle g_i \rangle \), if \( l_1 \) is the minimum in the set \( \{1, 2, \ldots, n\} \setminus \{i, j\} \); see Lemma 7.4. Therefore, the ideal \( I_2(\tilde{X}_{ij}) + \langle g_i, g_j, g_{l_1} \rangle \) can be resolved minimally by Theorem 2.2. Proceeding in this manner, we will be able to show that the ideals \( I_2(\tilde{X}_{ij}) + \langle g_i, g_j, g_{l_1}, \ldots, g_{l_k} \rangle \) and \( \langle g_{l_{k+1}} \rangle \) intersect transversally, if \( 1 \leq l_1 < \ldots < l_k < l_{k+1} \leq n \) and \( l_{k+1} \) is the smallest in the set \( \{1, 2, \ldots, n\} \setminus \{i, j, l_1, \ldots, l_k\} \); see Lemma 7.4. This finally gives us a minimal free resolution for the ideal \( I_2(\tilde{X}_{ij}) + \langle g_i, g_j, g_{l_1}, \ldots, g_{l_{n-2}} \rangle \), with \( 1 \leq l_1 < \ldots < l_{n-2} \leq n \) and \( l_t \notin \{i, j\} \) for every \( t \).

Let us assume that \( X \) is generic and \( i = 1 \) and \( j = 2 \). The proofs for the general \( i \) and \( j \), with \( i < j \) would be similar according to the aforesaid scheme. The proofs in the case when \( X \) is generic symmetric would be similar as well. Comments for general \( i < j \) and the symmetric case have been made whenever necessary.

7.1. A minimal free resolution for \( I_2(\tilde{X}_{12}) + \langle g_1, g_2 \rangle \). The minimal free resolution of \( I_2(\tilde{X}_{12}) \) is given by the Eagon-Northcott complex, which is the following:

\[
\mathbb{E}_* : 0 \rightarrow E_{n-1} \rightarrow \cdots \rightarrow E_k \xrightarrow{\delta_k} E_{k-1} \rightarrow \cdots \rightarrow E_0 \rightarrow R/I_2(\tilde{X}) \rightarrow 0
\]

where \( E_0 \cong R^1 \), \( E_k = R^{k(\binom{n}{k+1})} \) and for each \( k = 0, \ldots, n-2 \), the map \( \delta_k : E_k \rightarrow E_{k-1} \) is defined as

\[
\delta_k \left( (e_{i_1} \land \cdots \land e_{i_{k+1}}) \otimes v_2^{k-1} \right) = \sum_{s=1}^{k+1} x_{2s}(e_{i_1} \land \cdots \land e_{i_s} \land \cdots \land e_{i_{k+1}}) \otimes v_2^{k-2}
\]

\[
\delta_k \left( (e_{i_1} \land \cdots \land e_{i_{k+1}}) \otimes v_1^{k-1} \right) = \sum_{s=1}^{k+1} (-1)^{s+1} x_{1s}(e_{i_1} \land \cdots \land e_{i_s} \land \cdots \land e_{i_{k+1}}) \otimes v_1^{k-2}
\]

\[
\delta_k \left( (e_{i_1} \land \cdots \land e_{i_{k+1}}) \otimes v_1^j v_2^{k-j-1} \right) = \sum_{s=1}^{k+1} (-1)^{s+1} x_{1s}(e_{i_1} \land \cdots \land e_{i_s} \land \cdots \land e_{i_{k+1}}) \otimes v_1^{j-1} v_2^{k-j-1}
\]

\[
+ \sum_{s=1}^{k+1} x_{2s}(e_{i_1} \land \cdots \land e_{i_s} \land \cdots \land e_{i_{k+1}}) \otimes v_1^j v_2^{k-j-2}
\]
for every ordered \( k + 1 \) tuple \((i_1, i_2, \ldots, i_{k+1})\), with \( 1 \leq i_1 < \cdots < i_{k+1} \leq n \) and for every \( j = 1, 2, \cdots, k - 2 \).

A minimal resolution of \( \langle g_1 \rangle \) is given by

\[
\mathbb{G}_*: 0 \rightarrow R \xrightarrow{g_1} R \rightarrow R/\langle g_1 \rangle \rightarrow 0.
\]

The ideals \( I_2(\tilde{X}_{12}) \) and \( \langle g_1 \rangle \) intersect transversally, by Lemma 6.2. Therefore, by Lemma 2.2 a minimal free resolution for \( I_2(\tilde{X}_{12}) + \langle g_1 \rangle \) is given by the tensor product complex

\[
\mathbb{E} \otimes \mathbb{G}_* : 0 \rightarrow E_{n-1} \rightarrow \cdots \rightarrow E_{k+1} \oplus E_k \xrightarrow{\psi_{k+1}} E_k \oplus E_{k-1} \rightarrow \cdots \rightarrow E_0 \rightarrow R/I_2(\tilde{X}_{12}) + \langle g_1 \rangle \rightarrow 0
\]

such that \( \psi_k : E_k \oplus E_{k-1} \rightarrow E_{k-1} \oplus E_{k-2} \) is the map defined as

\[
\psi_k (e_{i_1} \wedge \cdots \wedge e_{i_{k+1}}) \otimes v^j_1 v^k-j-1_2 = \delta_k ((e_{i_1} \wedge \cdots \wedge e_{i_{k+1}}) \otimes v^k-j-1_2)
\]

\[
\psi_k (e_{i_1} \wedge \cdots \wedge e_{i_k}) \otimes v^j_1 v^k-j-2_2 = (-1)^{k-1} g_1 ((e_{i_1} \wedge \cdots \wedge e_{i_k}) \otimes v^k-j-2_2) + \delta_{k-1} ((e_{i_1} \wedge \cdots \wedge e_{i_k}) \otimes v^j_1 v^k-j-2_2).
\]

Now we find a minimal free resolution for \( I_2(\tilde{X}_{12}) + \langle g_1, g_2 \rangle \) by mapping cone. Let \( C_k := (\mathbb{E} \otimes \mathbb{G}_*)_k \). We have proved in Lemma 6.3 that \( I_2(\tilde{X}_{12}) + \langle g_1, g_2 \rangle = (x_{11}, x_{12}, \cdots, x_{1n}) \); which is minimally resolved by the Koszul complex. Let us denote the Koszul Complex by \((\mathbb{F}; \sigma_k)\), where \( \sigma_k \) is the \( k \)-th differential. We first construct the connecting map \( \tau : \mathbb{F} \rightarrow \mathbb{E} \otimes \mathbb{G}_* \). Let us write \( F_k := R^{k(n)} \) and \( C_k := R^{k(k+1)} \oplus R^{(k-1)(n)} \). The map \( \tau_k : F_k \rightarrow C_k \) is defined as:

\[
\tau_k (e_{i_1} \wedge \cdots \wedge e_{i_k}) = \sum_j y_j (e_{i_1} \wedge \cdots \wedge e_{i_k} \wedge e_j) \otimes v^k-j-1_2 -(e_{i_1} \wedge \cdots \wedge e_{i_k}) \otimes v^k-j-2_2.
\]

Let us choose the lexicographic ordering among the \( k \) tuples \((i_1, \ldots, i_k)\), such that \( 1 \leq i_1 < \cdots < i_k \leq n \) in order to write an ordered basis for \( R^{(k)(n)} \). We define lexicographic ordering among the tuples \((i_1, \ldots, i_{k+1}, k - j, j)\), for \( j = 0, \ldots, k \) and \( k = 1, \ldots, n - 1 \) to order the basis elements for \( R^{k(k+1)} \). Moreover, in the free module \( C_k = R^{k(k+1)} \oplus R^{(k-1)(n)} \), we order the basis elements in such a way that those for \( R^{k(k+1)} \) appear first. The matrix representation of \( \tau_k \) with respect to the chosen ordered bases is the
Theorem 7.1. The following diagram commutes for every $k = 1, \ldots, n - 1$:

\[
\begin{array}{c}
F_k \xrightarrow{\tau_k} C_k \\
\downarrow \sigma_{k+1} & \downarrow \psi_{k+1} \\
F_{k+1} \xrightarrow{\tau_{k+1}} C_{k+1}
\end{array}
\]

Proof. It suffices to prove the statement for a basis element $(e_{i_1} \wedge \cdots \wedge e_{i_{k+1}})$ of $F_{k+1}$. Without loss of generality we consider $(e_1 \wedge \cdots \wedge e_{k+1})$. We first compute $(\tau_k \circ \sigma_{k+1})(e_1 \wedge \cdots \wedge e_{k+1})$.

\[
(e_1 \wedge \cdots \wedge e_{k+1}) \xrightarrow{\sigma_{k+1}} \sum_{j=1}^{k+1} (-1)^{j+1} x_{1j} (e_1 \wedge \cdots \wedge \hat{e}_j \wedge \cdots \wedge e_{k+1})
\]

\[
\xrightarrow{\tau_k} \sum_{j=1}^{k+1} (-1)^{j+1} x_{1j} \left[ \sum_{s=1}^{n} y_s (e_1 \wedge e_2 \wedge \cdots \wedge \hat{e}_j \wedge \cdots \wedge e_{k+1} \wedge e_s) \otimes v_1^{k-1} \right]
\]

\[
- \sum_{j=1}^{k+1} (-1)^{j+1} x_{1j} (e_1 \wedge e_2 \wedge \cdots \wedge \hat{e}_j \wedge \cdots \wedge e_{k+1}) \otimes v_1^{k-2}.
\]
We now compute \((ψ_{k+1} \circ τ_{k+1})(e_1 \wedge \ldots \wedge e_{k+1})\).

\[
(\psi_{k+1} \circ τ_{k+1}) \rightarrow \sum_{s=1}^{n} y_s (e_1 \wedge \ldots \wedge e_{k+1} \wedge e_s) \otimes v_1^{k+1} \\
- (e_1 \wedge \ldots \wedge e_{k+1}) \otimes v_1^{k+1} \\
\psi_{k+1} \rightarrow \sum_{s,j=1}^{n} \left(-(1)^{j+1} y_{s} x_{1j}(e_1 \wedge \ldots \wedge e_j \wedge \ldots \wedge e_{k+1} \wedge e_s) \otimes v_1^{k-1} \\
- (1)^k g_1(e_1 \wedge \ldots \wedge e_j \wedge \ldots \wedge e_{k+1} \wedge e_j) \otimes v_1^{k-1} \\
- \sum_{j=1}^{k+1} (1)^{j+1} x_{1j}(e_1 \wedge \ldots \wedge e_j \wedge \ldots \wedge e_{k+1} \wedge e_j) \otimes v_1^{k-2} \right) \\
\right)
\]

\[
= \sum_{j=1}^{k+1} \sum_{s=1}^{n} \left[x_{1j} y_{s}(-1)^{j+1}(e_1 \wedge \ldots \wedge e_j \wedge \ldots \wedge e_{k+1} \wedge e_s) \otimes v_1^{k-1} \right] \\
+ \sum_{s=1}^{n} (-1)^{s+1} y_{s} x_{1s}(e_1 \wedge \ldots \wedge e_s \wedge \ldots \wedge e_{k+1} \wedge e_s) \otimes v_1^{k-1} \\
- (1)^k g_1(e_1 \wedge \ldots \wedge e_s \wedge \ldots \wedge e_{k+1} \wedge e_s) \otimes v_1^{k-1} \\
- \sum_{j=1}^{k+1} (1)^{j+1} x_{1j}(e_1 \wedge \ldots \wedge e_s \wedge \ldots \wedge e_{k+1} \wedge e_j) \otimes v_1^{k-2} \right) \\
\right)
\]

\[
= \sum_{j=1}^{k+1} \sum_{s=1}^{n} \left[x_{1j} y_{s}(-1)^{j+1}(e_1 \wedge \ldots \wedge e_j \wedge \ldots \wedge e_{k+1} \wedge e_s) \otimes v_1^{k-1} \right] \\
+ \sum_{s=1}^{n} (-1)^{s+1}(-1)^{k+1-s} y_{s} x_{1s}(e_1 \wedge \ldots \wedge e_s \wedge \ldots \wedge e_{k+1} \wedge e_s) \otimes v_1^{k-1} \\
- (1)^k g_1(e_1 \wedge \ldots \wedge e_s \wedge \ldots \wedge e_{k+1} \wedge e_s) \otimes v_1^{k-1} \\
- \sum_{j=1}^{k+1} (1)^{j+1} x_{1j}(e_1 \wedge \ldots \wedge e_s \wedge \ldots \wedge e_{k+1} \wedge e_j) \otimes v_1^{k-2} \right) \\
\right)
\]

\[
= \sum_{j=1}^{k+1} \sum_{s=1}^{n} \left[x_{1j} y_{s}(-1)^{j+1}(e_1 \wedge \ldots \wedge e_j \wedge \ldots \wedge e_{k+1} \wedge e_s) \otimes v_1^{k-1} \right] \\
+ (-1)^k g_1(e_1 \wedge \ldots \wedge e_s \wedge \ldots \wedge e_{k+1} \wedge e_j) \otimes v_1^{k-1} \\
- (1)^k g_1(e_1 \wedge \ldots \wedge e_j \wedge \ldots \wedge e_{k+1} \wedge e_j) \otimes v_1^{k-1} \\
- \sum_{j=1}^{k+1} (1)^{j+1} x_{1j}(e_1 \wedge e_s \wedge \ldots \wedge e_{k+1} \wedge e_j) \otimes v_1^{k-2} \right) \\
\right)
\]

\[
= \sum_{j=1}^{k+1} \sum_{s=1}^{n} \left[x_{1j} y_{s}(-1)^{j+1}(e_1 \wedge \ldots \wedge e_j \wedge \ldots \wedge e_{k+1} \wedge e_s) \otimes v_1^{k-1} \right] \\
- \sum_{j=1}^{k+1} (1)^{j+1} x_{1j}(e_1 \wedge \ldots \wedge e_s \wedge \ldots \wedge e_{k+1} \wedge e_j) \otimes v_1^{k-2} \right). \]
Hence the mapping cone \( M(\mathbb{E} \otimes \mathbb{G}; \mathbb{F}) \) gives us the resolution for \( I_2(\tilde{X}_{12}) + \langle g_1, g_2 \rangle \) as described in [5]. However, this resolution is not minimal. We now construct a minimal free resolution from \( M(\mathbb{E} \otimes \mathbb{G}; \mathbb{F}) \).

A free resolution for the ideal \( I_2(\tilde{X}_{12}) + \langle g_1, g_2 \rangle \) has been constructed in 3.1, which is given by

\[
0 \rightarrow D_{n+2} \xrightarrow{d_{n+2}} D_{n+1} \cdots \xrightarrow{d_{k+1}} D_k \xrightarrow{d_k} D_{k-1} \cdots \xrightarrow{d_1} D_1 \xrightarrow{d_0} D_0 \rightarrow 0,
\]

such that \( D_k = F_{k-1} \oplus C_k = R^{(k-1)}(n) \oplus (R^{(k+1)}(n) \oplus R^{(k-1)}(n)) \) and \( d_k = (-\sigma_{k-1} + \tau_{k-1}, \psi_k) \). Let us recall that the map \( \psi \) is the differential in the free resolution for \( I_2(\tilde{X}_{12}) + \langle g_1 \rangle \), the map \( \sigma \) is the differential in the Koszul resolution for \( \langle x_{11}, x_{12}, \ldots, x_{1n} \rangle \) and \( \tau \) is the connecting homomorphism between the complexes defined in 3.1. Let us order bases for \( F_{k-1} \) and \( C_k \) with respect to the lexicographic ordering. Finally we order basis for \( D_k \) in such a way that the basis elements for \( F_{k-1} \) appear first, followed by the basis elements for \( C_k \). Therefore, the matrix representation for the differential map \( d_k \) is given by

\[
\begin{pmatrix}
-\sigma_{k-1} & 0 \\
\tau_{k-1} & \psi_k
\end{pmatrix} =
\begin{pmatrix}
A & 0 \\
0 & -1
\end{pmatrix}
\]

The entries in the matrices representing \( \sigma_{k-1} \) and \( \psi_k \) belong to the maximal ideal \( \langle x_{ij}, y_j \rangle \), since both are differentials of minimal free resolutions. The block matrix \( A \) has also elements in the maximal ideal \( \langle x_{ij}, y_j \rangle \). The only block which has elements outside the maximal ideal \( \langle x_{ij}, y_j \rangle \) is in the identity block appearing in \( \tau_{k-1} \). Therefore, it is clear from the matrix representation of the map \( d_k \) that we can apply Lemma 5.9 repeatedly to get rid of non-minimality. Hence, we get a minimal free resolution and the total
Betti numbers for the ideal $I_2(\tilde{X}_{12}) + \langle g_1, g_2 \rangle$ are

\[
\begin{align*}
    b_0 &= 1, \\
    b_1 &= \binom{n}{2} + 2, \\
    b_2 &= 2 \cdot \binom{n}{3} + n, \\
    b_{k+1} &= k \binom{n}{k+1} + (k-1) \binom{n}{k} + \binom{n}{k-1} - \binom{n}{k}, \quad \text{for } 2 \leq k \leq n - 1, \\
    b_n &= n - 2.
\end{align*}
\]

7.2. A minimal free resolution for $I_2(\tilde{X}_{ij}) + \langle g_1, \ldots, g_n \rangle$.

**Lemma 7.2.** Let $G_k = G_{12} \cup \{g_1, g_2, \ldots, g_k\}$, $1 \leq k \leq n$, where $G_{12}$ is the set of all $2 \times 2$ minors of $\tilde{X}_{12}$ defined in the list of notations in section 2. The set $G_k$ is a Gröbner basis for the ideal $I_2(\tilde{X}_{12}) + \langle g_1, \ldots, g_k \rangle$ with respect to a suitable monomial order.

**Proof.** We take the lexicographic monomial ordering in $R$ induced by the following ordering among the indeterminates:

\[
x_{nn} > \cdots > x_{tt} > \cdots > x_{33} > y_1 > \cdots > y_n > x_{11} > \cdots > x_{1n} > x_{21} > \cdots > x_{2n} > x_{st} \text{ for other } s, t.
\]

Then, we observe that for every $s \geq 3$, $\text{Lt}(g_s)$ is coprime with $\text{Lt}(g_t)$ for every $1 \leq t \leq k; t \neq s$ and also coprime with $\text{Lt}(h)$ for every $h \in \mathcal{G}$. Moreover, by Lemma 4.1 $G_{12}$ is a Gröbner basis for $I_2(\tilde{X}_{12})$. Therefore, we only have to test the $S$-polynomials $S(g_1, g_2), S(g_1, h)$ and $S(g_2, h)$, for $h \in \mathcal{G}$.

We can write $S(g_1, g_2) = \sum_{k=1}^{n} \binom{12}{1k} y_k$ and note that $\text{Lt}(\binom{12}{1k}) \leq \text{Lt}(S(g_1, g_2))$ for every $1 \leq k \leq n$. Hence, $S(g_1, g_2) \rightarrow_{G_k} 0$. We note that, if $i \neq 1$ then the leading terms of $g_1$ and $\binom{12}{st}$ are mutually coprime and therefore $S(g_1, \binom{12}{st}) \rightarrow_{G_k} 0$. Next, the expression $S(g_1, \binom{12}{1t}) = x_{1t} g_2 + \sum_{s 
eq t} \binom{12}{st} y_s$ shows that $S(g_1, \binom{12}{1t}) \rightarrow_{G_k} 0$. Similarly, if $s \neq 1$
then the leading terms of \( g_2 \) and \([12]st\) are mutually coprime and therefore \( S(g_2, [12]st) \to g_k 0 \). The proof for \( S(g_2, [12]1t) \) is similar to that of \( S(g_1, [12]st) \).

**Remark.** The corresponding result for \( i < j \) in general would be the following:

**Lemma 7.3.** Let \( \mathcal{G}_{i,j,k} = \mathcal{G}_{ij} \cup \{g_i, g_j, g_{i_1}, \ldots, g_{i_{k-2}}\} \), \( 1 \leq k \leq n \), \( 1 \leq l_1 < \cdots < l_{k-2} \leq n \) and \( l_t \) is the smallest in the set \( \{1, 2, \ldots, n\} \setminus \{i, j, l_1, \ldots, l_{t-1}\} \); \( \mathcal{G}_{ij} \) denotes the set of all \( 2 \times 2 \) minors of \( \tilde{X}_{ij} \) defined in the list of notations in section 2. The set \( \mathcal{G}_{i,j,k} \) is a Gröbner basis for the ideal \( I_2(\tilde{X}_{ij}) + \langle g_1, \ldots, g_k \rangle \) with respect to a suitable monomial order.

**Proof.** While proving this statement with \( i < j \) arbitrary, we have to choose the following monomial orders. The rest of the proof remains similar.

Suppose that \( X \) is generic, we choose the lexicographic monomial ordering in \( R \) induced by the following ordering among the indeterminates:

\[
x_{nn} > \cdots > x_{jj} > \cdots > x_{ii} > \cdots > x_{11} > y_1 > \cdots > y_n
\]

\[
> x_{i1} > \cdots > x_{in}
\]

\[
> x_{j1} > \cdots > x_{jn}
\]

\[
x_{st} \quad \text{for all other } s, t.
\]

If \( X \) is generic symmetric, we choose the lexicographic monomial ordering in \( R \) induced by the following ordering among the indeterminates:

\[
x_{nn} > \cdots > x_{jj} > \cdots > x_{ii} > \cdots > x_{11} > y_1 > \cdots > y_n
\]

\[
> x_{ii} > x_{ij} > x_{i1} > x_{2i} > \cdots > x_{i-1,i} > x_{i,i+1} > \cdots > x_{in}
\]

\[
> x_{jj} > \cdots > x_{j-1,j} > x_{jj+1} > \cdots > x_{jn}
\]

\[
x_{st} \quad \text{for all other } s, t.
\]

**Lemma 7.4.** The ideals \( I_2(\tilde{X}_{12}) + \langle g_1, \ldots, g_k \rangle \) and \( \langle g_{k+1} \rangle \) intersect transversally, for every \( 2 \leq k \leq n - 1 \).

**Proof.** Suppose not, then, there exists \( h_{k+1} \not\in I_2(\tilde{X}_{12}) + \langle g_1, \ldots, g_k \rangle \) such that \( h_{k+1}g_{k+1} \in I_2(\tilde{X}_{12}) + \langle g_1, \ldots, g_k \rangle \). Let us choose the same monomial order on \( R \) as defined in Lemma 7.2. Upon division by elements of \( \mathcal{G}_k \), we may further assume that \( \text{Lt}(h) \nmid \text{Lt}(h_{k+1}) \) for every \( h \in \mathcal{G}_k \), since \( \mathcal{G}_k \) is a Gröbner basis for the ideal \( I_2(\tilde{X}_{12}) + \langle g_1, \ldots, g_k \rangle \) by Lemma 7.2. On the other hand \( h_{k+1}g_{k+1} \in I_2(\tilde{X}_{12}) + \langle g_1, \ldots, g_k \rangle \) and therefore \( \text{Lt}(h) \mid \text{Lt}(h_{k+1}) \), for some \( h \in \mathcal{G}_k \), since \( \text{Lt}(h) \) and \( \text{Lt}(g_{k+1}) \) are mutually coprime, - a contradiction. \( \square \)
Remark. The corresponding result for $i < j$ in general would be the following: The ideals $I_2(\tilde{X}_{ij}) + \langle g_i, g_j, g_{i_1}, \ldots, g_{k_l} \rangle$ and $\langle g_{k_{l+1}} \rangle$ intersect transversally, if $1 \leq i_1 < \ldots < i_k < k_{l+1} \leq n$ and $k_{l+1}$ is the smallest in the set $\{1, 2, \ldots, n\} \setminus \{i, j, l_1, \ldots, l_k\}$, for every $1 \leq k \leq n - 3$. The proof is essentially the same as above after we use the Lemma 7.3.

Proof of Theorem 4.2. Part (1) of the theorem has been proved in 5.1. We now prove part (2) under the assumption $i = 1$, $j = 2$. Let the minimal free resolution of $I_2(\tilde{X}_{12}) + \langle g_1, g_2, \ldots, g_k \rangle$ be $(L_\ast, \delta)$. By Lemma 7.3 and Lemma 2.2 the minimal free resolution of $I_2(\tilde{X}_{12}) + \langle g_1, \ldots, g_{k+1} \rangle$ is given by the tensor product of $(L_\ast, \delta)$ and $0 \rightarrow R \rightarrow \tilde{X}_{12} \rightarrow 0$, and that is precisely $(\mathbb{K}, \Delta)$, with $K_p = L_p \oplus L_{p-1}$ and $\Delta_p = (\lambda_p, (-1)^p g_{k+1} + \lambda_{p-1})$. Let $\beta_{k, p}, 0 \leq p \leq n + k$, denote the $p$-th total Betti number for the ideal $I_2(\tilde{X}_{12}) + \langle g_1, \ldots, g_k \rangle$. Then, the total Betti numbers $\beta_{k+1, p}, 0 \leq p \leq n + k + 1$ for the ideal $I_2(\tilde{X}_{12}) + \langle g_1, \ldots, g_{k+1} \rangle$ are given by $\beta_{k+1, 0} = 1$, $\beta_{k+1, p} = \beta_{k, p-1} + \beta_{k, p}$ for $1 \leq p \leq n + k$ and $\beta_{k+1, n+k+1} = n - 2$. The proof for general $i < j$ follows similarly according to the strategy discussed in the beginning of section 5.

In particular, the total Betti number $\beta_{n-2, p}$ for the ideal $I_2(\tilde{X}_{ij}) + \langle g_1, \ldots, g_n \rangle$ are given by $\beta_{n-2, 0} = 1$, $\beta_{n-2, p} = \beta_{n-3, p-1} + \beta_{n-3, p}$ for $1 \leq p \leq 2n - 3$ and $\beta_{n-2, 2n-2} = n - 2$. \hfill $\Box$

Example. We show the Betti numbers at each stage for $n = 4$ and $n = 5$.

\begin{align*}
n = 4: & \quad 1 \quad 6 \quad 8 \quad 3 \\
& \quad 1 \quad 7 \quad 14 \quad 11 \quad 3 \\
& \quad 1 \quad 8 \quad 12 \quad 7 \quad 2 \\
& \quad 1 \quad 9 \quad 20 \quad 19 \quad 9 \quad 2 \\
& \quad 1 \quad 10 \quad 29 \quad 39 \quad 28 \quad 11 \quad 2 \\
& \quad 1 \quad 10 \quad 20 \quad 5 \quad 4 \\
& \quad 1 \quad 11 \quad 30 \quad 25 \quad 9 \quad 4 \\
& \quad 1 \quad 12 \quad 25 \quad 25 \quad 14 \quad 3 \\
& \quad 1 \quad 13 \quad 37 \quad 50 \quad 39 \quad 17 \quad 3 \\
& \quad 1 \quad 14 \quad 50 \quad 87 \quad 89 \quad 56 \quad 20 \quad 3 \\
& \quad 1 \quad 15 \quad 64 \quad 137 \quad 176 \quad 145 \quad 76 \quad 23 \quad 3
\end{align*}

Theorem 7.5. The ring $R/I_1(XY) + I_2(\tilde{X}_{ij}), 1 \leq i < j \leq n$ is Cohen-Macaulay.

Proof. We know from 4.2 that the projective dimension of $R/I_1(XY) + I_2(\tilde{X}_{ij})$ is $2n - 2$. We claim that the elements of the set $P \cup Q$ forms a regular sequence, where $P = \{x_{ik}x_{j,k+1} - x_{jk}x_{i,k+1} \mid 1 \leq k \leq n - 1\}$ and
\[ \mathcal{Q} = \{ g_t \mid 1 \leq t \leq n, \ t \neq j \}. \] Suppose that \( X \) is generic and \( j < n \). We consider the matrices

\[
\mathbf{x}_{ij} = \begin{pmatrix}
x_{1i} & \cdots & \hat{x}_{ji} & \cdots & x_{ni} & x_{ji} \\
x_{ij} & \cdots & \hat{x}_{jj} & \cdots & x_{nj} & x_{jj} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_{1n} & \cdots & \hat{x}_{jn} & \cdots & x_{nn} & x_{jn}
\end{pmatrix}, \quad \mathbf{y}_{ij} = \begin{pmatrix}
y_1 \\
\vdots \\
y_j \\
\vdots \\
y_n \\
y_j
\end{pmatrix}
\]

Then we have, \( I_1(XY) = I_1(\mathbf{x}_{ij}\mathbf{y}_{ij}) \). We consider the lexicographic monomial order

\[
x_{nn} > \cdots > \hat{x}_{jj} > \cdots > \hat{x}_{ii} > \cdots x_{11} > y_j > y_n > \cdots > \hat{y}_j > \cdots y_1 > x_{1i} > \cdots \hat{x}_{ji} > \cdots > x_{ni} > x_{ji} > x_{1j} > \cdots \hat{x}_{jj} > \cdots > x_{nj} > x_{jj} > \text{other indeterminates}.
\]

Then, \( \text{Lt}(g_i) = x_{ji}y_j \) and \( \text{Lt}(g_k) = x_{tt}y_t \), for \( 1 \leq t \leq n \) and \( t \neq i, j \). Therefore, \( \text{Lt}(x_{ik}x_{jk,k+1} - x_{jk}x_{ik,k+1}) = x_{ik}x_{jk,k+1} \) for \( 1 \leq k \leq n - 1 \). The set \( \mathcal{P} \cup \mathcal{Q} \) forms a regular sequence by \([5]\) since the leading terms of the elements are mutually disjoint. The proof is similar in the case \( j = n \). Similarly one can prove in the case when \( X \) is generic symmetric. \( \square \)

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