Dimers and the Ising model

Jerzy Cisło Institute of Theoretical Physics
University of Wrocław
pl. M. Borna 9, 50-205 Wrocław, Poland.

February 2, 2008

Abstract
We present a connection between the ground state of the Ising Model and the dimer problem. We find the generating function for dimers as the appropriate limit of the free energy per spin for the Ising Model.

1. Dimers. Dimers are objects connecting two neighboring sites of the lattice. We consider a lattice which can be covered with dimers in the way that each site of the lattice belongs to exactly one dimer. How to calculate the number of such coverings for a given lattice was shown in 1961 by Kasteleyn, Fisher and Temperley in 1961 [1-3]. The connection between the Ising model on the plane lattice and the dimer problem is well known. In 1961, Fisher [4] gave a general rule connecting the partition function for the Ising model with a generating function for configurations of dimers on an appropriately constructed lattice. For the Ising model on the honeycomb, for instance, the dimers on 3-12 lattice are considered. Moreover, the Ising model—as well as the model of free fermions—are connected with the dimer problem on a nonplane lattice [5,6].

2. From the Ising model to the dimer problem. In this paper, we shall show that the doubled number of dimer coverings for a given lattice is equal to the degeneration of the ground state of the Ising model on a lattice dual to the one covered with dimers. Next, we will use this fact to find the generating function for the dimers on the chessboard lattice.

Let us begin with the description of our construction. Let $L$ be any planar lattice for which the dual lattice $L_d$ can be covered with dimers. Let us cover the lattice $L_d$ with dimers in one of the possible ways.
To every edge crossed by a dimer we assign the positive constant of interaction $E$, and to the remaining edges — the negative constant of interaction $-E$.

The energy of the interaction of two neighboring spins is defined by the product of these spins and the constant of interaction. The degeneration of the ground state equals the number of spin configurations with the lowest energy.

The degeneration of the ground state of our Ising model is equal to half of the number of the dimer coverings of the lattice $L_d$.

To prove the above statement, let us note that we get the smallest energy when exactly one edge of each elementary polygon of the lattice $L$ has the positive energy $E$.

Each configuration of this type corresponds to one dimer covering of the lattice $L_d$. Conversely, each dimer covering corresponds to two spin configurations with the lowest energy. To see that, we fix one spin and systematically define the neighboring ones: if the edge is positive—or negative but crossed—the next spin has the same sign; otherwise the next spin has the opposite sign.

Obviously, we can fix the first spin in two ways. Hence, the number of dimer coverings reads

$$Z_d = \frac{1}{2} \lim_{T \to 0^+} Z(T) e^{ME/kT},$$

where $Z$ is the statistical sum of the Ising model on the lattice $L$, $ME$ is a minimal energy of the system, and $M$ is the number of edges of the lattice $L$ minus number of dimers.

3. Example. Finally, we present an application of the procedure described above. Let us consider the chessboard Ising model with constants of interaction $E_1, E_2, E_3 < 0, E_4 > 0$ (see Fig.1).

We change the signs of $E_2$ and $E_4$ in every other column so that the edges crossed by dimers of the standard configuration have positive interaction constant (Fig.1). This transformation does not change the statistical sum.

Let $E_1 = -E - L_1 kT, E_2 = -E - L_2 kT, E_3 = -E - L_3 kT, E_4 = E + L_4 kT$.

Then

$$\lim_{T \to 0^+} Z(E_1, E_2, E_3, E_4) e^{ME/kT} = Z_d(x_1, x_2, x_3, x_4),$$

3
where $Z(x_1, x_2, x_3, x_4)$ is the generating function for the dimer problem on the chessboard lattice in the variables $x_i = \exp(2L_i - L1 - L2 - L3 - L4)$, $M = 3N/4$, and $N$ is the number of spins.

In 1951 Utiama obtained the formula for the free energy per spin for the Ising model on the chessboard lattice [6]:

$$ f(T) = -kT \lim_{N \to \infty} \frac{\ln Z(T)}{N} = $$

$$ = -kT \ln 2 - \frac{kT}{16\pi^2} \int_0^{2\pi} \int_0^{2\pi} \ln \frac{1}{2} (1 + C_1C_2C_3C_4 + S_1S_2S_3S_4 + (S_1S_2 + S_3S_4) \cos \phi - (S_1S_4 + S_2S_3) \cos \theta + S_2S_4 \cos(\phi + \theta) + S_1S_2 \cos(\phi - \theta)) d\theta d\phi, $$

where $C_i = \cosh(-2E_i/kT)$, and $S_i = \sinh(-2E_i/kT)$.

While calculating the thermodynamic limit, we suppose that the lattice grows in the same way in two directions.

Using this result we find

$$ \Psi(x_1, x_2, x_3, x_4) = \lim_{N \to \infty} \frac{\ln Z_d(x_1, x_2, x_3, x_4)}{N} = \frac{3}{4} - \lim_{T \to 0^+} \frac{f(T)}{kT} = $$

$$ = \frac{1}{16\pi^2} \int_0^{2\pi} \int_0^{2\pi} \ln(x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2(x_3x_4 - x_1x_2) \cos \phi + 2(x_2x_3 - x_1x_4) \cos \theta - x_1x_3 \cos(\phi + \theta) + x_2x_4 \cos(\phi - \theta)) d\theta d\phi. $$

In this way it was possible to find the generating function for the dimer problem without any reference to the general theory. Similar considerations may be repeated for the triangular and honeycomb lattices.
Fig.1. Lattices: $L$ (continues lines) and $L_d$ (dotted lines).

**References**

[1] Kasteleyn P.W. Physica **27** (1961) 1209-1225

[2] Fisher M.E. Phys.Rev. **124** (1961) 1664-1672

[3] Temporley H.N.V. Fisher M.E. Phil.Mag. **6** (1961) 1061-1063

[4] Fisher M.E. J.Math.Phys. **7** (1961) 1776

[5] Kasteleyn P.W. J.Math.Phys. **4** (1963) 287-293

[6] Utiana T. Prog.Theor.Phys. **6** (1951) 907