Bound State Boundary S-matrix of the sine-Gordon Model

Subir Ghoshal
Department of Physics and Astronomy
Rutgers University
P.O.Box 849, Piscataway, NJ 08855-0849

Abstract
We study the boundary S-matrix for the reflection of bound states of the two-dimensional sine-Gordon integrable field theory in the presence of a boundary.

1. Introduction
In a recent paper [1], two-dimensional integrable field theory with a boundary has been studied. An essential ingredient of an integrable field theory in infinite space is the existence of an infinite number of mutually commuting integrals of motion. In the presence of the boundary, for arbitrary boundary conditions, these “charges” no longer remain conserved. However, sometimes, it is possible to modify these charges with special “integrable” boundary conditions so that the modified charges are indeed conserved. Then such a theory may be called a “two-dimensional integrable field theory with a boundary”.

An integrable “bulk” field theory enjoys the property that its multi-particle S-matrix amplitude factorizes into a product of an appropriate number of two-particle S-matrix amplitudes. The latter satisfy several constraints, namely, Yang-Baxter equation, unitarity and crossing symmetry [2]. These constraints enable one to compute the exact S-matrix up to the so-called “CDD”-factors. It has been known for quite some time how to generalise

---

1 E-mail: GHOSHAL@ruhets.rutgers.edu
this factorizable structure of the S-matrix in the presence of a reflecting boundary\cite{3}. In addition to the “bulk” two-particle S-matrices one needs to introduce specific “boundary reflection amplitudes” for reflections of various particles in the theory off the boundary. The latter have to satisfy appropriate generalisations of the constraints of the bulk theory - the boundary Yang-Baxter equation, the boundary unitarity condition and the boundary cross-unitarity condition. The last of these was introduced in\cite{1}. Thus one can in a way similar to the bulk case, pin down the factorizable boundary S-matrix, again, upto the “CDD”-factors.

In\cite{1}, the boundary S-matrices of the soliton scattering in the sine-Gordon model with a particular integrable boundary condition (which we call the “boundary sine-Gordon model”), were obtained. In the present work we compute the boundary S-matrix for reflections of the soliton-antisoliton bound states (the breathers) off the boundary. We employ the ideas of “boundary bootstrap” as discussed in\cite{1,4}.

2. Boundary S-matrix of the sine-Gordon Model

In this section we review the study of the S-matrices of the sine-Gordon model in the presence of the boundary\cite{1}. The bulk SG model is described by the action\cite{2},

\[
\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx a(\phi, \partial_{\mu} \phi)
\]

(2.1)

where,

\[
a(\phi, \partial_{\mu} \phi) = \frac{1}{2}(\partial_{\mu} \phi)^2 - \frac{m^2}{\beta^2} \cos \beta \phi
\]

(2.2)

where \(\phi(x, y)\) is a scalar field and \(\beta\) is a dimensionless coupling constant. The model is integrable both classically and quantum mechanically\cite{3,6}. In the quantum theory, the discrete symmetry \(\phi \to \phi + \frac{2\pi}{\beta} N, N \in \mathbb{Z}\) is spontaneously broken at \(\beta^2 < 8\pi\)\cite{7}; in this domain the theory is massive and its particle spectrum consists of a soliton-antisoliton pair \((A, \bar{A})\) (with equal masses) and a number (which depends on \(\beta\)) of neutral particles (“quantum breathers”) \(B_n, n = 1, 2, ..., \lambda\), where

\[
\lambda = \frac{8\pi}{\beta^2} - 1
\]

(2.3)

The soliton (antisoliton) carries a positive (negative) unit of “topological charge”

\[
q = \frac{\beta}{2\pi} \int_{-\infty}^{\infty} dx \frac{\partial}{\partial x} \phi(x, y) = \frac{\beta}{2\pi} [\phi(+\infty, y) - \phi(-\infty, y)]
\]

(2.4)
The charge conjugation $C : A \leftrightarrow \bar{A}$ is related to $\phi \leftrightarrow -\phi$ symmetry of (2.2). The particles $B_n$ are neutral (they are interpreted as the soliton-antisoliton bound states), $B_n$ with even (odd) $n$ being $C$-even ($C$-odd). The masses of $B_n$ are

$$m_n = 2M_s \sin\left(\frac{n\pi}{2\lambda}\right); \quad n = 1, 2, \ldots < \lambda,$$

where $M_s$ is the soliton mass.

Factorizable scattering of solitons is described by the commutation relations

$$A(\theta)\bar{A}(\theta') = a(\theta - \theta')A(\theta')A(\theta), \quad \bar{A}(\theta)\bar{A}(\theta') = a(\theta - \theta')\bar{A}(\theta')\bar{A}(\theta),$$

$$A(\theta)\bar{A}(\theta') = b(\theta - \theta')A(\theta')A(\theta) + c(\theta - \theta')A(\theta')\bar{A}(\theta),$$

where $A(\theta)$ and $\bar{A}(\theta)$ are soliton and antisoliton creation operators and the two-particle scattering amplitudes $a, b, c$ are given by

$$a(\theta) = \sin(\lambda(\pi - u))\rho(u),$$

$$b(\theta) = \sin(\lambda u)\rho(u),$$

$$c(\theta) = \sin(\lambda\pi)\rho(u),$$

where $u = -i\theta$ and

$$\rho(u) = -\frac{1}{\pi} \Gamma(\lambda)\Gamma(1 - \lambda u/\pi)\Gamma(1 - \lambda + \lambda u/\pi) \prod_{l=1}^{\infty} \frac{F_l(u)F_l(\pi - u)}{F_l(0)F_l(\pi)};$$

$$F_l(u) = \frac{\Gamma(2l\lambda - \lambda u/\pi)\Gamma(1 + 2l\lambda - \lambda u/\pi)}{\Gamma((2l + 1)\lambda - \lambda u/\pi)\Gamma(1 + (2l - 1)\lambda - \lambda u/\pi)}.$$  (2.8)

The amplitudes of $AB_n$ and $B_nB_m$ scatterings can be found in [2]. The amplitudes $b(\theta)$ and $c(\theta)$ have simple poles in the “physical strip”, $Re\theta = 0$, $0 < Im\theta < \pi$, i.e. $0 < u < \pi$ for

$$u_n = \pi - \frac{n\pi}{\lambda}, \quad n = 1, 2, \ldots < \lambda$$  (2.9)

These are interpreted as the neutral bound states $B_n$. From the pole terms

$$b(\theta) \simeq \frac{f^+ f^-}{\theta - iu_n}$$

$$c(\theta) \simeq \frac{f^+ f^-}{\theta - iu_n}$$  (2.10)
the vertices $f_{ij}^n$ can be extracted and they are

$$f_{n}^{++} = f_{n}^{+-}(-1)^n = f_{n}^{++}$$

$$f_{n}^{--} = f_{n}^{+-}$$  \hspace{1cm} (2.11)

The field theory in the presence of the boundary can be defined by the following action

$$\int_{-\infty}^{\infty} dy \int_{-\infty}^{0} dx \, a(\phi, \partial_\mu \phi) + \int_{-\infty}^{\infty} dy \, b(\phi_B, \frac{d}{dy} \phi_B)$$  \hspace{1cm} (2.12)

where $\phi_B = \phi(0, y)$. Based on an explicit computation of the first non-trivial integral of motion, it was argued in [1], that the SG model is integrable in the semi-infinite space with

$$b(\phi) = -M \cos\left(\frac{\beta}{2} (\phi - \phi_0)\right)$$  \hspace{1cm} (2.13)

where $M$ and $\phi_0$ are free parameters. The factorizable boundary scattering theory associated with (2.13) was developed in [1]. The boundary S-matrix of the solitons and antisolitons can be conveniently described by the following “commutation relations” between soliton and antisoliton creation operators $A(\theta)$ and $\bar{A}(\theta)$ and the formal “boundary creation operator” $B$ (see [1]),

$$A(\theta)B = P_+(\theta)A(\theta)B + Q_+(\theta)\bar{A}(\theta)B;$$

$$\bar{A}(\theta)B = P_-(\theta)\bar{A}(\theta)B + Q_-(\theta)A(\theta)B$$  \hspace{1cm} (2.14)

Here $P_+, Q_+, P_-, Q_-$ are the amplitudes of soliton (antisoliton) one-particle boundary scattering processes shown in Fig.1. Except for the case $M = \infty$, the boundary value $\phi(x = 0, y)$ is not fixed in the boundary field theory (2.13) and hence the topological charge

$$q = \frac{\beta}{2\pi} \int_{-\infty}^{0} dx \frac{\partial}{\partial x} \phi(x, y)$$  \hspace{1cm} (2.15)

is not conserved. That is why we allow processes described by $Q_\pm$. The above amplitudes have to satisfy constraints resulting from the boundary Yang-Baxter equation (BYB), the

---

2 we disregard here, the possibility of the presence of additional “boundary degrees of freedom” other than the boundary value of the “bulk” field $\phi(x, y)$. 

4
unitarity equation and the cross-unitarity equation [1], described in Fig.2, Fig.3 and Fig.4 respectively. The BYB alone gives the solution [8], [1],

\[ P_+ (\theta) = \cos(\xi + \lambda u)R(u); \]
\[ P_- (\theta) = \cos(\xi - \lambda u)R(u); \]
\[ Q_+ (\theta) = \frac{k_+}{2} \sin(2\lambda u)R(u); \]
\[ Q_- (\theta) = \frac{k_-}{2} \sin(2\lambda u)R(u), \]

where again \( u = -i\theta; \xi, k_\pm \) are free parameters and \( R(u) \) is an arbitrary function. One can set \( k_+ = k_- = k \) by using a gauge transformation

\[ A(\theta) \rightarrow e^{i\alpha} A(\theta), \bar{A}(\theta) \rightarrow e^{-i\alpha} \bar{A}(\theta) \]

The function \( R(u) \) can be determined using the “boundary unitarity” and the “boundary cross-unitarity” equations [1]

\[ R(u) = R_0(u)R_1(u) \]

where

\[ R_0(u) = \frac{F_0(u)}{F_0(-u)}; \]
\[ F_0(u) = \frac{\Gamma(1 - 2\lambda u/\pi)}{\Gamma(\lambda - 2\lambda u/\pi)} \times \prod_{k=1}^{\infty} \frac{\Gamma(4\lambda k - 2\lambda u/\pi)\Gamma(1 + 4\lambda k - 2\lambda u/\pi)\Gamma(\lambda(4k + 1))\Gamma(1 + \lambda(4k - 1))}{\Gamma(\lambda(4k + 1) - 2\lambda u/\pi)\Gamma(1 + \lambda(4k - 1) - 2\lambda u/\pi)\Gamma(1 + 4\lambda k)\Gamma(4\lambda k)} \]

and

\[ R_1(u) = \frac{1}{\cos \xi} \sigma(\eta,u)\sigma(i\vartheta,u) \]

where

\[ \sigma(x, u) = \frac{\Pi(x, \pi/2 - u)\Pi(-x, \pi/2 - u)\Pi(x, -\pi/2 + u)\Pi(-x, -\pi/2 + u)}{\Pi^2(x, \pi/2)\Pi^2(-x, \pi/2)}; \]
\[ \Pi(x, u) = \prod_{l=0}^{\infty} \frac{\Gamma(1/2 + (2l + 1/2)\lambda + x/\pi - \lambda u/\pi)\Gamma(1/2 + (2l + 3/2)\lambda + x/\pi)}{\Gamma(1/2 + (2l + 3/2)\lambda + x/\pi - \lambda u/\pi)\Gamma(1/2 + (2l + 1/2)\lambda + x/\pi)} \]

This is the general solution for generic \( \lambda \). For integer \( \lambda \) there are additional solutions.
solves

\[ \sigma(x, u)\sigma(x, -u) = [\cos(x + \lambda u) \cos(x - \lambda u)]^{-1}; \quad \sigma(x, \pi/2 - u) = \sigma(x, \pi/2 + u), \]

and the parameters \( \eta \) and \( \vartheta \) are determined through the equations

\[ \cos(\eta) \cosh(\vartheta) = -\frac{1}{k} \cos \xi; \quad \cos^2(\eta) + \cosh^2(\vartheta) = 1 + \frac{1}{k^2}. \] (2.20)

3. Bound State Boundary S-matrix

In this section we discuss the scattering of bound states \( B_n \) of the sine-Gordon model off the boundary. One starts with the commutation relation (Fig.5)

\[ B_n(\theta)B = R_B^{(n)}(\theta)B_n(-\theta)B \] (3.1)

where \( B_n(\theta) \) creates the bound state \( B_n \) with rapidity \( \theta \). The bound state boundary scattering amplitude \( R_B^{(n)}(\theta) \) can be derived from the “boundary bootstrap equation” [1,4] \[ f_{i_1i_2}(u_n)R^{i_1}_{j_1}(u + \frac{u_n}{2})S_{j_2j_1}^{i_2j_1}(2u)R^{i_2}_{j_2}(u - \frac{u_n}{2}) = f_{j_1j_2}^{i_1i_2}(u_n)R_B^{(n)}(u) \] (3.2)

where \( R_j^i \) stand for the amplitudes (2.16) and \( S_{ij}^{kl}(u) \) represent the two-particle bulk S-matrices (2.7) with \( u = -i\theta \). This equation is illustrated in Fig.6. Furthermore the solution must satisfy the following:

1. Boundary Unitarity Equation:

\[ R_B^{(n)}(u)R_B^{(n)}(-u) = 1 \] (3.3)

This equation can be obtained as a consistency condition by applying (3.1) twice.

2. Boundary Cross-unitarity Equation:

\[ R_B^{(n)}(\frac{\pi}{2} - u) = R_B^{(n)}(\frac{\pi}{2} + u)S^{(n,n)}(2u) \] (3.4)

where \( S^{(n,n)}(2u) \) is the scattering amplitude for \( B_n + B_n \to B_n + B_n \). These two conditions have been discussed in detail in [1]. Fig.3 and Fig.4 illustrate (3.3) and (3.4) respectively if particles \( a \) and \( b \) are both taken to be \( B_n \). The equation (3.2) yields \( R_B^{(n)}(u) \) when the
expressions (2.11) for $f_{ij}^n$, (2.7) for $S_{ij}^{kl}(u)$ and (2.16), (2.18)-(2.20) for $R_j^n$ are used. It can be written as

$$R_B^{(n)}(u) = R_0^{(n)}(u) R_1^{(n)}(u)$$

(3.5)

where

$$R_0^{(n)}(u) = (-1)^{n+1} \frac{\cos \left( \frac{u}{2} + \frac{n\pi}{4} \right) \cos \left( \frac{u}{2} - \frac{n\pi}{4} \right) \sin \left( \frac{u}{2} + \frac{n\pi}{4} \right)}{\cos \left( \frac{u}{2} - \frac{n\pi}{4} \right) \cos \left( \frac{u}{2} + \frac{n\pi}{4} \right) \sin \left( \frac{u}{2} - \frac{n\pi}{4} \right)}$$

$$\times \prod_{l=1}^{n-1} \frac{\sin(u + \frac{l\pi}{2\lambda}) \cos^2 \left( \frac{u}{2} - \frac{n\pi}{4} - \frac{l\pi}{4\lambda} \right)}{\sin(u - \frac{l\pi}{2\lambda}) \cos^2 \left( \frac{u}{2} + \frac{n\pi}{4} + \frac{l\pi}{4\lambda} \right)}$$

(3.6)

We find that $R_1^{(n)}(u)$, which contains the boundary parameters $\eta$ and $\vartheta$ is different depending on whether $n$ is even or odd

$$R_1^{(2n)}(u) = S^{(2n)}(\eta, u) S^{(2n)}(i\vartheta, u)$$

(3.7)

where

$$S^{(2n)}(x, u) = \prod_{l=1}^{n} \frac{\sin(u) - \cos \left( \frac{x}{\lambda} - (l - \frac{1}{2}) \frac{\pi}{2\lambda} \right) \sin(u) - \cos \left( \frac{x}{\lambda} + (l - \frac{1}{2}) \frac{\pi}{2\lambda} \right)}{\sin(u) + \cos \left( \frac{x}{\lambda} - (l - \frac{1}{2}) \frac{\pi}{2\lambda} \right) \sin(u) + \cos \left( \frac{x}{\lambda} + (l - \frac{1}{2}) \frac{\pi}{2\lambda} \right)}$$

(3.8)

$$n = 1, 2, ..., < \frac{\lambda}{2}$$

and

$$R_1^{(2n-1)}(u) = S^{(2n-1)}(\eta, u) S^{(2n-1)}(i\vartheta, u)$$

(3.9)

with

$$S^{(2n-1)}(x, u) = \frac{\cos \left( \frac{x}{\lambda} \right) - \sin(u) \prod_{l=1}^{n-1} \frac{\sin(u) - \cos \left( \frac{x}{\lambda} - \frac{l\pi}{2\lambda} \right) \sin(u) - \cos \left( \frac{x}{\lambda} + \frac{l\pi}{2\lambda} \right)}{\sin(u) + \cos \left( \frac{x}{\lambda} - \frac{l\pi}{2\lambda} \right) \sin(u) + \cos \left( \frac{x}{\lambda} + \frac{l\pi}{2\lambda} \right)}}{\cos \left( \frac{x}{\lambda} \right) + \sin(u)}$$

(3.10)

$$n = 1, 2, ..., < \frac{\lambda + 1}{2}$$

One can check that the solution so obtained satisfies (3.3) and (3.4).

The factor $R_0^{(n)}(u)$ contains poles in the “physical strip” $0 < u < \frac{\alpha}{2}$ located at $u = \frac{\alpha}{2} - \frac{n\pi}{2\lambda}$ for $\lambda > 1$. These poles can be explained as follows. In the “direct channel” of the scattering $B_n + B_n \rightarrow B_n + B_n$ the corresponding amplitude $S^{(n,n)}(u)$ shows a pole at $u = \frac{n\pi}{\lambda}$ [2]. This pole corresponds to the propagation of a real bound state $B_{2n}$. As discussed in [1], the boundary state $|B >$ associated with the boundary condition (2.13) is expected to contain the contributions of the zero-momentum particles $B_{2n}$. Therefore
the amplitude $R_{B}^{(n)}(u)$ must show a pole at $u = \frac{\pi}{2} - \frac{n\pi}{2\lambda}$ as illustrated in Fig.7. The pole at $u = \frac{\pi}{2}$ is necessary to satisfy (3.4) at $u = 0$ since $S^{(n,n)}(0) = -1$.

It is difficult to discuss the behavior of $R_{1}^{(n)}(u)$ for general values of the boundary parameters $\eta$ and $\vartheta$. As in [1], we will consider only two special cases:

a) Fixed Boundary Condition: In this case (2.19) becomes [1],

$$R_{1}(u) = \frac{1}{\cos \xi} \sigma(\xi, u)$$

and consequently we get

$$R_{1}^{(n)}(u) = S^{(n)}(\xi, u)$$

There exist poles in the “physical strip” $0 < u < \frac{\pi}{2}$ in $R_{1}^{(n)}(u)$ and these represent some “boundary bound states”.

b) Free Boundary Condition: In this case $\eta = \frac{\pi}{2}(\lambda + 1)$ and $\vartheta = 0$ [1]. There exists a pole in $R_{1}^{(n)}(u)$ for both even and odd values of $n$ at $u = \frac{n\pi}{2\lambda}$. This pole corresponds to a “boundary bound state” propagating along the boundary as illustrated in Fig.8. As discussed in [1], the energy of this “boundary bound state” should be $e_{n} = e_{0} + m_{n}\cos\left(\frac{n\pi}{2\lambda}\right)$, where $e_{0}$ is the ground state energy. Using (2.5), we get $e_{n} = e_{0} + M_{s}\sin\left(\frac{n\pi}{2\lambda}\right)$. The existence of this set of bound states with energies $e_{n}$ can be seen also from the soliton boundary S-matrices (2.16). For the “free” boundary conditions, $P_{+}(\theta) = P_{-}(\theta) = P_{\text{free}}(\theta)$ and $Q_{+}(\theta) = Q_{-}(\theta) = Q_{\text{free}}(\theta)$ both show poles at $\theta = i\frac{\pi}{2} - i\frac{n\pi}{\lambda}$. Energies of the corresponding “boundary bound states”, when computed, agree with $e_{n}$ found above.

Acknowledgement

I would like to thank A.B.Zamolodchikov for advice, inspiration and numerous illuminating discussions.
References

[1] S.Ghoshal and A.B.Zamolodchikov, ”Boundary S-matrix and Boundary State in Two-dimensional Integrable Quantum Field Theory”, Rutgers Preprint, RU-93-20, hep-th/9306002.

[2] A.B.Zamolodchikov, Al.B.Zamolodchikov. Ann. Phys. 120 (1979), 253.

[3] I. Cherednik. Theor. Math. Phys. 61, 35 (1984), p.977.

[4] A. Fring, R. Koberle. “Factorized Scattering in the Presence of Reflecting Boundaries”. Preprint USP-IFQSC/TH/93-06, 1993.

[5] L. Takhtadjian, L. Faddeev. Theor. Math. Phys. 21, (1974) p.160.

[6] V. Korepin, L. Faddeev. Theor. Math. Phys. 25 (1975) p.147.

[7] S. Coleman. Phys. Rev. D11, (1975) p.2088.

[8] H. J. De Vega, A. Gonzalez Ruiz. Preprint LPTHE 92-45.
\[ a = P_+ \]

\[ \bar{a} = P_- \]

\[ A = Q_+ \]

\[ \bar{A} = Q_- \]

**FIG 1**

\[ a \]

\[ \bar{a} \]

\[ c \]

\[ d \]

\[ \theta_1 \]

\[ \theta_2 \]

\[ b \]

**FIG 2**

\[ a \]

\[ \bar{a} \]

\[ c \]

\[ d \]

\[ \theta_1 \]

\[ \theta_2 \]

**FIG 3**

\[ a \]

\[ \bar{a} \]

\[ b \]

**FIG 4**
\[ \alpha = u + u_n / 2 \]
\[ \sigma = u + u_n / 2 \]

\[ \pi/\lambda = R^{(n)}(\theta) \]