Formal Test Synthesis for Safety-Critical Autonomous Systems based on Control Barrier Functions

Prithvi Akella, Mohamadreza Ahmadi, Richard M. Murray, and Aaron D. Ames

Abstract—The prolific rise in autonomous systems has led to questions regarding their safe instantiation in real-world scenarios. Failures in safety-critical contexts such as human-robot interactions or even autonomous driving can ultimately lead to loss of life. In this context, this paper aims to provide a method by which one can algorithmically test and evaluate an autonomous system. Given a black-box autonomous system with some operational specifications, we construct a minimax problem based on control barrier functions to generate a family of test parameters designed to optimally evaluate whether the system can satisfy the specifications. To illustrate our results, we utilize the Robotarium as a case study for an autonomous system that claims to satisfy waypoint navigation and obstacle avoidance simultaneously. We demonstrate that the proposed test synthesis framework systematically finds those sequences of events (tests) that identify points of system failure.

I. INTRODUCTION

Autonomous systems have become increasingly pervasive in our everyday life, whether that be through the rise in interest for autonomous vehicles [2], intelligent defense systems [3], or even human/robot interaction [4]. This rise in prevalence has motivated a similar increase in questions regarding the efficacy of these systems in safety critical contexts. These questions are not entirely unfounded, however, as even in those cases when attempting to verify system efficacy, horrific accidents still occur e.g. recent autonomous car crashes. Nonetheless, the field is still pushing forward rapidly, and in the future, these autonomous systems will have to deal with even more complex, dynamic, and relatively unstructured environments. Coupled with the cost of failure, this increase in system complexity makes systematic test and evaluation of these systems all the more necessary.

Significant work on this issue has been carried out by the test and evaluation (T&E) community. Reachability analysis has been used to shape critical test cases from existing data [5]. At the discrete level, RRT has been used to efficiently search a feasible space to find critical sequences that identify failure of the underlying controller [6]. Tests based on a graph-search framework over clustered, critical situations, have been developed via exhaustive mission simulation of the underlying system [7]. Each of the aforementioned methods are model-based and not easily adaptable to other systems/testing environments as they are exhaustive. To address the issue of adaptivity, one approach adaptively samples the feasible space to generate successively harder tests [8]. That being said, the aforementioned frameworks require an accurate system model to function well, and except for the latter contribution, none are easily adaptable. However, as noted in a memo by the Department of Defense [9], a testing framework that is both adaptive/adversarial and formally guarantees safety is still highly sought after.

Prior work in the T&E community reference formal methods as a means by which one can formally guarantee safety/the lack thereof (see [10]). Formal methods, specifically linear and signal temporal logic (LTL & STL), have garnered significant interest in the controls community (see [11]–[15]). In each of these cases, the logical specification encodes a control objective whose satisfaction is formally guaranteed via the barrier-based controller. In this respect, control barrier functions are very useful in formally guaranteeing these logical specifications insofar as satisfying a specification and remaining within a safe set are both set-based arguments [16]. However, these formal guarantees require specific knowledge of the onboard controller and system dynamics - for the test engineer, this is oftentimes not the case.

Our Contribution: In this paper, the overarching goal is to start to bridge the work done in the controls and
the T&E community. Specifically, we address the issue of designing an adaptable/adversarial testing framework. Given an autonomous system with some operational specifications, we construct a minimax problem whose solution defines testing scenarios intended to optimally frustrate satisfaction of the given specifications without specific knowledge of the onboard control architecture. To this end, we begin by collecting data of the autonomous system satisfying the specifications. Then, we use the collected demonstration data to frame Linear Programs that develop approximate control barrier functions to develop a minimax game to solve for optimal testing parameters designed to frustrate satisfaction of the specifications. The proposed method is illustrated in Figure 1.

Outline: In Section II we review some preliminary definitions and formally define the problem under study. In Section III we detail the main result of the paper, i.e., a minimax game for test generation. In Section IV we couple the result with a linear program to systematically generate difficult tests. Finally, in Section V we illustrate our proposed methodology with a case study.

II. PROBLEM FORMULATION

In this section, we present some notions used in the sequel and formally define the problem under study.

A. Preliminaries

We consider a class of systems (to-be-tested) that can be modeled as a dynamical system with affine inputs:

\[ \dot{x} = f(x) + g(x)u, \quad x \in \mathcal{X} \subset \mathbb{R}^n, \quad u \in \mathcal{U} \subset \mathbb{R}^m. \] (1)

Furthermore, we will assume that both \( f(x) \) and \( g(x) \) are locally Lipschitz. For any function \( h(x) \),

\[
L_f h(x) \triangleq \nabla_x h(x)f(x), \\
L_g h(x) \triangleq \nabla_x h(x)g(x),
\]

are its Lie derivatives.

Formal Methods: We will define \( \mathcal{A} \) to be the set of atomic propositions from which the provided control objective, i.e., a temporal logic specification, has been synthesized. We use the following notation to represent the truth/lack thereof for an atomic proposition

\[ \forall \phi \in \mathcal{A}, \quad [\phi] \triangleq \{ x \in \mathcal{X} | \phi(x) = \text{TRUE} \}, \]

where \( \phi(x) \) denotes the atomic proposition evaluated at the state \( x \). In addition, we will define the symbols \( \neg, \land, \lor \) to correspond to negation, conjunction, and disjunction respectively. That is, \( \neg \phi = \text{TRUE} \) when \( \phi = \text{FALSE} \). Likewise \( \phi \land \omega = \text{TRUE} \) when \( \phi = \text{TRUE} \) and \( \omega = \text{TRUE} \), and \( \phi \lor \omega = \text{TRUE} \) when either \( \phi = \text{TRUE} \) or \( \omega = \text{TRUE} \).

In this paper, we consider a subset of temporal logic (TL) operators, Future and Global, defined as follows (here \( \equiv \) denotes a logical equivalency):

\[ F \phi \equiv \exists t^* \geq 0 \text{ s.t. } x(t^*) \in [\phi], \]

\[ G \phi \equiv \forall t \geq 0, \quad x(t) \in [\phi]. \]

While this seems restrictive, these two operators can be composed to consider more complex TL specifications, such as \( \Box \phi = G(F \phi) \).

Control Barrier Functions (CBF): To provide a metric by which we measure satisfaction of the provided specification, we will establish a correspondence between these TL specifications and control barrier functions, \( h \). To start, we first define extended class-\( K \) functions, \( \alpha : (-b,a) \rightarrow (-\infty,\infty) \), to be those functions, \( \alpha \), that are strictly increasing and satisfy \( \alpha(0) = 0 \). Here, \( a,b > 0 \). Using these extended class-\( K \) functions, we can define Control Barrier Functions (CBF).

Definition 1 (Control Barrier Functions (CBF)): For a dynamical system of the form (1), a differentiable function, \( h : \mathbb{R}^n \rightarrow \mathbb{R} \) is considered a control barrier function if it satisfies the following criteria:

\[ \sup_{u \in \mathcal{U}} [L_fh(x) + L_g h(x)u + \alpha(h(x))] \geq 0, \quad \forall x \in \mathcal{X}, \]

where \( \alpha \) is an extended class-\( K \) function [16].

The usefulness of a CBF is in guaranteeing the forward invariance of its 0-superlevel set:

\[ C_h = \{ x \in \mathbb{R}^n \mid h(x) \geq 0 \}, \]

\[ \partial C_h = \{ x \in \mathbb{R}^n \mid h(x) = 0 \}. \]

Indeed, it was shown in Proposition 1 of [16] that a CBF, as in Definition 1, guarantees forward invariance of its 0-Superlevel set, \( C_h \). Here, we note that what we call control barrier functions are termed as zeroing control barrier functions in [17]. Finally, a finite time convergence control barrier function requires \( \alpha(x) = \gamma \text{sign}(x)|x|^\rho \) to ensure finite time convergence to the set, \( C_h \), by \( T = \frac{1}{\gamma(1-\rho)}|h(x_0)|^{1-\rho} \), provided \( h(x_0) \leq 0 \) [18].

B. Problem Statement

As mentioned earlier, the overarching test and evaluation goal is to validate an autonomous system’s capacity to satisfy a provided TL specification. However, as we have no knowledge of the controller on-board the system to-be-tested, not only do we have no metric of quantifying success for the TL specification, but we also do not have a systematic method of developing difficult tests by which to identify control system failures in satisfying the specification. We will show in the sequel that there exists a correspondence between CBFs and TL specifications. So, if we could determine these CBFs for the system at hand, we can use them to test the system against a given specification. This chain of reasoning is the basis for Figure 1. To that effect, we collect the following experimental data of the system satisfying the control objective:

Definition 2 (Data-Set): Define \( \mathcal{D}_i = \{(x_k^i, u_k^i) \in \mathbb{R}^n \times \mathbb{R}^m \mid k = 0,1,\ldots,T_i\} \) as the data-set of state, action
pairs for demonstration, $i$. Here, $k$ indexes time until $T_i$, which is the max time for the specific demonstration at hand. Then, define $\mathcal{D} = \{\mathcal{D}_1, \ldots, \mathcal{D}_r\}$ as the set of all provided demonstrations.

Assumption 1: For the provided data-set, $\mathcal{D}$, and associated specification, the data-set for each demonstration, $\mathcal{D}_i$, terminates when the specification is satisfied. e.g. for a specification defined as $F \phi \land G \omega$, where $\phi, \omega \in \mathcal{A}$, then for each $\mathcal{D}_i$,

- $x^i_0 \in [\phi]$ and $x^i_k \notin [\phi]$ for all $k = 0, 1, \ldots, T_i - 1$, and
- $x^i_k \in [\omega]$ for all $k = 0, 1, \ldots, T_i$.

We use the generated data-set, $\mathcal{D}$, to determine composite CBFs that mimic system behavior. We compose these CBFs from a candidate set of barrier functions defined as follows:

**Definition 3 (Candidate Barrier Set):** We call

$$\mathbf{B} \triangleq \{h_1, h_2, \ldots, h_q\},$$

a candidate barrier set for some provided, continuously differentiable functions, $\{h_i\}_{i=1}^q$.

Note that in the above definition, each component of the candidate barrier set may not be a valid CBF, i.e. $\mathbf{B}$ could be the set of all polynomials of degree, $n \leq q - 1$. Finally, we need to formalize how we specifically identify these testing scenarios.

**Definition 4 (Testing Parameters):** We define the vector, $d \in \mathbb{R}^p$, to be a collection of testing parameters used to generate tests e.g. the location of obstacles, time when a phenomena starts, etc.

With these definitions in place, the problem statement is as follows:

**Problem 1:** Given an autonomous system whose controller is unknown, $\mathcal{D}$, $\mathbf{B}$, and a TL specification the system intends to satisfy, devise an adaptive/adversarial strategy to identify a set of testing parameters $d$.

We show in the next section that these test parameters $d$ characterize a test scenario designed to validate that the autonomous system reliably satisfies a given TL specification.

### III. MAIN RESULT

This section will detail the main result of this paper - the minimax game formulated to generate optimal test parameters, $d^*$, designed to frustrate satisfaction of a TL specification expressed through CBFs.

A. Main Result

To preface the main result, we will make the following remark to simplify notation:

**Remark 1:** We denote $h^G_i, i \in \mathcal{I}$ to be a set of CBFs for a finite number of specifications of the type $F \phi_i$. Likewise, $h^G_j, j \in \mathcal{J}$ denote CBFs for specifications of the type $G \omega_j$. That is, $C_{h^G_i} \equiv [\phi_i], \forall i \in \mathcal{I}$, and $C_{h^G_j} \equiv [\omega_j], \forall j \in \mathcal{J}$.

In addition, we will make the following assumption to simplify the formulations in the sequel.

**Assumption 2:** We will assume that the CBFs $h^G_i, j \in \mathcal{J}$ depend on a set of test parameters $d$. That is, $h^G_i : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ and $h^G_j : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}$ whereas, $h^F_i : \mathbb{R}^n \to \mathbb{R}$.

We will define the following set of feasible inputs:

$$\mathcal{U}(x,d) = \{u \in \mathcal{U} | h^G_i(x,u,d) \geq -\alpha_j(h^G_j(x,d)), \forall j \in \mathcal{J}\},$$

where each $\alpha_j$ is the corresponding extended class-$\mathcal{K}$ function with respect to which $h^G_j$ is a CBF. Likewise, we will define:

$$x(t)|_{u(t)} = x(0) + \int_0^t (f(x(s)) + g(x(s))u(s)) \, ds,$$

to be the solution to equation (1) provided the input signal, $u(t)$.

Likewise, we will make the following assumption to frame the type of specifications accounted for by the testing framework to be detailed:

**Assumption 3:** We assume that the provided TL specification can be recast into the following form:

$$[\forall i \in \mathcal{I} (F \phi_i)] \land [\forall j \in \mathcal{J} (G \omega_j)], \phi_i, \omega_j \in \mathcal{A} \forall i, j, \quad (2)$$

with the following initial conditions:

$$\forall i \in \mathcal{I} (\phi_i(x(0))) = \text{FALSE}, \quad (3a)$$
$$\land \forall j \in \mathcal{J} (\omega_j(x(0))) = \text{TRUE}. \quad (3b)$$

Intuitively, specifications of type (2) denote control objectives wherein the system must ensure continued satisfaction of multiple control objectives while accomplishing at least one of a subset of tasks e.g. navigating to one of multiple waypoints while avoiding all obstacles. Equations (3a) and (3b) indicate that the system does not start in trivial states, wherein the specification (2) has already been satisfied. Finally, to account for an adversarial testing framework, we specify that the test parameters are a function of the current state, i.e. $d(x)$, where the specific functional form is expressed in Theorem [1].

Intuitively, the idea is that for tests to be adversarial to system action, they must, necessarily, depend on the system state.

Under the notation specified in Remark [1] the main result is as follows:

**Theorem 1 (Algorithmic Test Generation):** Given an autonomous system and a TL specification of the form in equation (2), the solution, $d^*(x)$, to the minimax game:

$$d^*(x) = \arg\max_{d \in \mathbb{R}^p} \max_{u \in \mathcal{U}(x,d)} \sum_{i \in \mathcal{I}} h^F_i (x,u), \quad \text{(Minimax)}$$

defines an optimal test parameter sequence, $d^*(x(t))$, predicted on a state trajectory, $x(t)|_{u(t)}$, and the control signal, $u(t)$, i.e., $d^*(x(t))$ identifies a sequence of test scenarios designed to ensure system satisfaction of the following specification:

$$[\forall i \in \mathcal{I} (G \neg \phi_i)] \lor [\forall j \in \mathcal{J} (F \neg \omega_j)], \phi_i, \omega_j \in \mathcal{A} \forall i, j. \quad (4)$$
B. Proof of Main Result

This section contains the lemmas necessary to prove the main result, Theorem 1. For all maximization/minimization problems contained within, we specify that infeasibility of the associated optimization problem corresponds to a value of $-\infty$, $\infty$ respectively.

To start, we need to show that TL specification (4) and TL specification (2) are mutually exclusive. To that end, we have the following Lemma regarding relations between TL operators:

**Lemma 1:** The following relations are true:

\[ \neg G \phi \equiv F(\neg \phi), \]
\[ F \neg \phi \equiv G(\neg \phi). \]

**Proof:** For equation (5),

\[ \neg G \phi \equiv \exists \neg \phi \geq 0 \mid x(t^*) \in \neg \phi \equiv F(\neg \phi). \]

Likewise, for equation (6),

\[ F \neg \phi \equiv \forall t \geq 0 x(t) \in \neg \phi \equiv G(\neg \phi). \]

Using Lemma 1 and De Morgan’s Law, we can prove that the two TL specifications, (4) and (2), are mutually exclusive:

**Lemma 2:** TL specifications (4) and (2) are mutually exclusive.

**Proof:**

\[ \neg \left[ \bigvee_i (F \phi_i) \right] \cap \left[ \bigwedge_j (G \omega_j) \right] \]
\[ \equiv \neg \left[ \bigvee_i (F \phi_i) \right] \vee \neg \left[ \bigwedge_j (G \omega_j) \right] \]
\[ \equiv \left[ \bigwedge_i (\neg F \phi_i) \right] \vee \left[ \bigvee_j (\neg G \omega_j) \right] \]
\[ \equiv \left[ \bigwedge_i (G \neg \phi_i) \right] \vee \left[ \bigvee_j (F \neg \omega_j) \right] \]

Effectively, Lemma 2 proves that if $d^*(x(t))$ ensures system satisfaction of TL specification (4), then the sequence of test parameters did indeed identify a system failure insofar as the system failed to satisfy the specification (2). It remains, however, to show that minimax game (Minimax) defines a sequence, $d^*(x(t))$, that forces the system to satisfy (4). To that end, we have the following Lemma that draws a correspondence between CBFs and TL specifications:

**Lemma 3:** For an atomic proposition, $\phi \in A$, if there exists a function, $h_\phi(x)$, such that $C_{h_\phi} = \{ \phi \}$, then:

\[ G \phi \equiv h_\phi(x(t)) \geq 0, \forall t \geq 0, \]

and:

\[ F \phi \equiv \exists t^* < \infty \text{ s.t. } h_\phi(x(t^*)) \geq 0. \]

Furthermore, if $h_\phi(x)$ is a CBF, then $\exists u(t)$ such that $G \phi = \text{TRUE}$. Likewise, if $h_\phi(x)$ is an FTCBF, then $\exists u(t)$ such that $F \phi = \text{TRUE}$. 

**Proof:** For $F \phi$:

\[ F \phi \equiv \exists 0 \leq t^* < \infty \text{ s.t. } x(t^*) \in \phi, \]
\[ \equiv 0 \leq t^* < \infty \text{ s.t. } x(t^*) \in C_{h_\phi}, \]
\[ \equiv 0 \leq t^* < \infty \text{ s.t. } h_\phi(x(t^*)) \geq 0. \]

Hence, if $h_\phi(x)$ is an FTCBF wherein $h(x(0)) \leq 0$, then an input sequence, $u(t)$, that satisfies:

\[ L_f h(x(t)) + L_g h(x(t)) u(t) + \gamma \text{sign}(h(x(t))) |h(x(t))|^p \geq 0, \forall t \leq T = \frac{1}{\gamma(1-\rho)} |h(x_0)|^{1-p} \]

ensures $h(x(T)) \geq 0 \implies F \phi = \text{TRUE}$. $G \phi$ follows similarly.

**Lemma 3** provides a metric by which to verify that $d^*(x(t))$ ensures system satisfaction of specification (4). Specifically, Lemma 3 requires that $d^*(x(t))$ either ensure $h_i^F(x(t)) < 0 \forall i \in I$ and $\forall t \geq 0$, or $h_i^F(x(t)) < 0$ for at least one $j \in J$ and $t \geq 0$. To show this, we require the following definitions for the optimal cost, $s$, optimal input, $u^*$, and optimal test parameter, $d^*$:

\[ s(x(t), d) = \max_{u \in U(x(t), d)} \sum_{i \in I} h_i^F(x(t), u), \]
\[ u^*(x(t), d) = \arg\max_{u \in U(x(t), d)} \sum_{i \in I} h_i^F(x(t), u), \]
\[ d^*(x(t)) = \arg\min_{d \in \mathbb{R}^p} \sum_{i \in I} h_i^F(x(t), u^*(x(t), d)). \]

Here, we note that equation (9) is a re-casting of equation (Minimax) accounting for the optimal input, $u^*(x(t), d)$. In addition, we will define the following set of invalidating test parameters:

\[ \mathcal{D}(x) = \{ d \in \mathbb{R}^p \mid U(x, d) = \emptyset \}. \]

With the above definitions, we have the following Lemma:

**Lemma 4:** If, for some $x(t)$, $\mathcal{D}(x(t)) \neq \emptyset$, then the optimal solution, $d^*$, to equation (Minimax) is such that $d^* \in \mathcal{D}(x(t))$.

**Proof:** First, we note that,

\[ \forall d \in \mathcal{D}(x(t)), s(x(t), d) = -\infty. \]

The equation above comes from the infeasibility of maximization problem (7), which results in a value of $s = -\infty$. Furthermore, equation (Minimax) is equivalent to:

\[ d^*(x(t)) = \arg\min_{d \in \mathbb{R}^p} s(x(t), d). \]

Based on the Locally Lipschitz assumptions made for $f(x)$ and $g(x)$ in equation (1) and the requirement that a CBF, $h(x)$, is differentiable at least once, it is true that $L_f h_i^F(x), L_g h_i^F(x)$ are bounded $\forall i \in I$.

In addition, $\forall u \in U(x(t), d), u$ is bounded.

Therefore, $\hat{h}_i^F(x(t), u) = L_f h(x(t)) + L_g h(x(t)) u$ is bounded $\forall i \in I$.

As defined in equations (7) and (8), it is also true that

\[ s(x(t), d) = \sum_{i \in I} \hat{h}_i^F(x(t), u^*(x(t), d)). \]
As each component, $\hat{h}^F$, is finite and $|\mathcal{I}| < \infty$, the following is true:

$$\exists M < \infty \text{ s.t. } |s(x(t),d)| < M, \quad \forall d \notin D(x(t)). \quad (12)$$

By definition of argmin and using equations $[10]$, $[11]$, and $[12]$, we have that $d^*(x(t)) \in D(x(t))$.

With Lemma $[4]$ we can show that the sequence, $d^*(x(t))$, attempts to force the system to satisfy, $\forall j(\mathcal{F} \neg \omega_j)$. We will show this first for a single $G:\omega$:

**Lemma 5:** If, for a given state trajectory, $x(t)$, $\omega(x(0)) = TRUE$, and $D(x(t)) \neq \emptyset \forall t \geq 0$ with $|\mathcal{J}| = 1$, then:

$$\forall \delta > 0, \exists t^*_\delta \in (0, \infty) \text{ s.t. } h_\omega(x(t^*)), d^*(t^*) < \delta,$$

where $h_\omega$ is the CBF corresponding to $G:\omega$.

**Proof:** Via Lemma $[4]$ we know that $\forall t \geq 0$, $d^*(x(t)) \in D(x(t))$. As $|\mathcal{J}| = 1$, this implies that $\forall t \geq 0$, $h_\omega(x(t), u, d^*(x(t))) < -\alpha(h_\omega(x(t), d^*(x(t))))$, $\forall u \in \mathcal{U}$.

As $\alpha(\cdot) \in \mathcal{K}$ (abbreviating $d^*(x(t))$ to $d^*(t)$):

$$h_\omega(x(t), d^*(t)) < \beta(h_\omega(x(0), d^*(0)), t),$$

where $\beta(\cdot)$ is a class-$\mathcal{K}$ function. As a result:

$$\exists t^* \in (0, \infty) \text{ s.t. } h_\omega(x(t), d^*(0)), t^* \leq \delta,$$

choosing $t^*_\delta = t^*$ completes the proof.

**Lemma 6:** If $\phi(x(0)) = FALSE$ and $|\mathcal{I}| = 1$, then the test parameter sequence, $d^*(x(t))$, is guaranteed to find a system trajectory, $x(t)|_{\omega^*(x(t),d^*(x(t)))}$, that satisfies $G \neg \phi$ provided a trajectory exists wherein:

$$\hat{h}_\phi(x(t), u^*(x(t), d(t))) \leq 0, \quad \forall t \geq 0, \quad (13)$$

$$D(x(t)) = \emptyset, \quad \forall t \geq 0, \quad (14)$$

for some $d(t)$.

**Proof:** First, we denote $\hat{h}_\phi(x)$ to be the CBF corresponding to $F:\phi$. It follows from Lemma $[5]$ then:

$$\phi(x(0)) = FALSE \equiv \hat{h}_\phi(x(0)) < 0. \quad (15)$$

From equation $[15]$, to prove $G \neg \phi$, it is sufficient to prove:

$$\hat{h}_\phi(x(t), u(t)) \leq 0, \quad \forall t \geq 0, \quad (16)$$

as if true:

$$\hat{h}_\phi(x(t)) = \hat{h}_\phi(x(0)) + \int_0^t \hat{h}_\phi(x(s), u(s))ds,$$

$$< \int_0^t \hat{h}_\phi(x(s), u(s))ds \leq 0,$$

$$\implies x(t)|_{u(t)} \in [-\phi], \quad \forall t \geq 0 \equiv G \neg \phi.$$

As a result, all that remains is to show that equation $[13]$ is satisfied by $d^*(x(t))$. Here, equation $[14]$ ensures that the results of Lemma $[4]$ do not apply, as otherwise $d^*(x(t)) \in D(x(t))$ and we cannot make a statement regarding $s(x(t), d^*)$. Then, by definition of argmin and equation $[9]$ (abbreviating $d^*(x(t))$ to $d^*(t)$):

$$\hat{h}_\phi(x(t), u^*(x(t), d^*(t))) \leq \hat{h}_\phi(x(t), u^*(x(t), d(t))),$$

which results in:

$$\hat{h}_\phi(x(t), u^*(x(t), d^*(t))) \leq 0. \quad (17)$$

From equation $[17]$ and the sufficiency proof predicated on equation $[16]$, we have:

$$x(t)|_{u^*(x(t), d^*(t))} \in [-\phi], \quad \forall t \geq 0 \equiv G \neg \phi.$$

With the aforementioned lemmas, we are now ready to prove Theorem $[1]$.  

**Proof:** [Theorem $[1]$] If both $|\mathcal{I}| = 1$ and $|\mathcal{J}| = 1$, then the result stems directly from Lemmas $[5]$ and $[6]$. First we note the following is true:

$$D(x(t)) = \emptyset \lor (D(x(t)) \neq \emptyset) = TRUE, \quad \forall t \geq 0. \quad (18)$$

As a result, it is true that $\forall t \geq 0$, the optimal test parameter sequence, $d^*(x(t))$, attempts to ensure that the following is true:

$$\left(\hat{h}_\phi(x(t), u^*(x(t), d^*(x(t)))) \leq 0 \lor \right)\left(\hat{h}_\omega(x(t), u, d^*(x(t))) < -\alpha h_\omega(x(t), d^*(x(t))) \forall u \in \mathcal{U} \right)$$

$$\forall t \geq 0.$$

Hence, if either $D(x(t)) = \emptyset$ or $D(x(t)) \neq \emptyset$ persist $\forall t \geq 0$, then the results of Lemmas $[5]$ and $[6]$ ensure $d^*(x(t))$ attempts to force the system to satisfy, $G \neg \phi \lor \mathcal{F} \omega$. We lose the guarantee on $G \neg \phi$ that we had in Lemma $[6]$ as we can no longer ensure $D(x(t)) = \emptyset, \forall t \geq 0$. However, whenever $D(x(t)) = \emptyset$, $d^*(x(t))$ will steer the system away from achieving $F \phi$, if feasible.

This same rationale extends to the case wherein $|\mathcal{I}| \neq 1$ and/or $|\mathcal{J}| \neq 1$. Since $[13]$ holds, for the multi-specification case, $d^*(x(t))$ attempts to ensure:

$$\left(\sum_{i \in \mathcal{I}} \hat{h}^i (x(t), u^*(x(t), d^*(x(t)))) \leq 0 \lor \right)\left(\hat{h}^i (x(t), u, d^*(x(t))) < -\alpha (\hat{h}^i (x(t), d^*(x(t))) \forall u \in \mathcal{U} \right)$$

$$\forall t \geq 0,$$

and for at least one $j$.

For the first inequality in equation $[19]$, the following implication is true:

$$\sum_{i \in \mathcal{I}} \hat{h}^i (x(t), u^*(x(t), d^*(x(t)))) \leq 0 \implies \forall i \in \mathcal{I} \left(\hat{h}^i (x(t), u^*(x(t), d^*(x(t)))) \leq 0 \right) = TRUE.$$
Implication (20) can be deduced from a contradiction. If, for
the same implication, we were to assume the LHS of (20)
to be true and the RHS to be false, then:
\[
\forall i \in I \left( \hat{h}_i^F(x(t), u^*(x(t), d^*(x(t)))) \leq 0 \right) \implies \text{FALSE} \implies
\forall i \in I \left( \hat{h}_i^F(x(t), u^*(x(t), d^*(x(t)))) > 0 \right) \implies \text{TRUE} \implies
\sum_{i \in I} \hat{h}_i^F(x(t), u^*(x(t), d^*(x(t)))) > 0,
\]
which is a contradiction.

As a result, \(d^*(x(t))\) attempting to ensure equation (19)
is equivalent to saying \(d^*(x(t))\) attempts to ensure:
\[
\forall i \in I \left( \hat{h}_i^F(x(t), u^*(x(t), d^*(x(t)))) \leq 0 \right) \lor
\forall j \in J \left( \hat{h}_j^G(x(t), u, d^*(x(t))) < -\alpha_j \left( \hat{h}_j^G(x(t), d^*(x(t))) \right) \right),
\]
\(\forall u \in \mathcal{U}\), \(\forall t \geq 0\). (21)

Coupled with the initial conditions (3a) and (3b), \(d^*(x(t))\) attempting to ensure equation (21) is equivalent to saying \(d^*(x(t))\) attempts to ensure system satisfaction of equation (4), which is the desired result.

IV. TEST SYNTHESIS

This section provides some additions to the main result that make it extensible to the problem at hand. Specifically, we formulate a linear program to extend the results of Theorem 1 to generate test cases wherein we have no prior knowledge of the controller on-board the system. Likewise, we have a corollary that permits a predictive form of equation Minimax such as the one used to generate the tests in Section V.

To start, we want to use the results of Theorem 1 to see if the provided autonomous system satisfies the associated TL specification. However, as we do not know the controller onboard the system, we do not have any CBFs with which to define the minimax game in Theorem 1. That being said, Lemma 3 in the Appendix provides us a method by which to determine these CBFs from the system demonstration data, \(\mathcal{D}\). First, we define an estimated CBF (e-CBF) to be a convex combination of component functions in \(\mathcal{B}\), where \(p_j\) below denote the weights for said combination:
\[
h^*(x) = \sum_{j=1}^{\vert \mathcal{B} \vert} p_j h_j(x), \quad \forall h_j \in \mathcal{B}.
\]

By default, Lemma 3 indicates that specification satisfaction requires the associated CBF to be positive. As a result, we will choose a cost function that is minimized when \(\text{e-CBF}\) is most positive over all demonstrations:
\[
J(\mathcal{B}, x, p) \triangleq -\sum_{i=1}^{\vert \mathcal{D} \vert} \sum_{k=0}^{\vert \mathcal{T}_i \vert} \sum_{j=1}^{\vert \mathcal{B} \vert} p_j h_j(x_k^i).
\]

Likewise, Assumption 1 dictates that demonstrations end upon satisfaction of the control objective. Therefore, for the estimated CBF to correspond to Future type constraints, \(\text{e-CBF}\) should be positive at the end of each demonstration. Similarly, for Global type constraints, \(\text{e-CBF}\) should be positive over the length of all demonstrations. This results in the following Corollary:

**Corollary 1:** For a given data-set, \(\mathcal{D}\), and a candidate set of functions, \(\mathcal{B}\), the solution, \(p^*\), to the following linear program:
\[
p^* = \arg \min_{p \in [0,1]} J(\mathcal{B}, x, p), \quad \text{(CBF-LP)}
\]
subject to (22a) or (22b),
\[
\begin{align*}
& p_j \geq 0, \quad \forall j = 1, 2, \ldots, \vert \mathcal{B} \vert, \\
& \sum_{j=1}^{\vert \mathcal{B} \vert} p_j = 1,
\end{align*}
\]
determines an estimated CBF, \(\text{e-CBF}\), for specifications of type \(F \phi\) (constraint (22a)) or type \(G \phi\) (constraint (22b)). Furthermore, for \(F \phi\):
\[
C_h \cap \{\phi\} \subseteq \{x_T^i\} \quad \forall i = 1, 2, \ldots, r,
\]
and for \(G \phi\):
\[
C_h \cap \{\phi\} \subseteq \{x_k^i\} \quad \forall k = 0, 1, \ldots, T_i \text{ and } i = 1, 2, \ldots , r.
\]

**Proof:** Assumption 1 requires that \(\phi = \text{TRUE}\) at each \(T_i\) for \(\mathcal{D}_i\). If a solution to equation (CBF-LP) exists with constraint (22a), then we have:
\[
x_T^i_k \in \{\phi\},
\]
\[
h^*(x_T^i_k) = \sum_{j=1}^{\vert \mathcal{B} \vert} p_j h_j(x_T^i_k) \geq 0,
\]
for any solution, \(p^*\). As a result, \(C_h \cap \{\phi\} \subseteq \{x_T^i_k\} \quad \forall i = 1, 2, \ldots, r\), which only implies set equivalence up to the provided data. As a result, Lemma 3 only applies over the provided data-set where the equivalence holds. To show the same for Global type specifications, replace constraint (22a) with (22b) and the proof follows similarly.

As the CBFs generated via Corollary 1 are not exact, the results of Theorem 1 cannot be guaranteed. However, they are very useful in generating tests as will be shown in Section V.

Secondly, the minimax game in Theorem 1 may be non-convex and/or calculation of the solution may be computationally difficult. However, as the minimax game depends only on the current state, we can calculate the optimal test parameters for some subset of points and define the actual test to be an interpolation of the parameters defined
Integrator systems and developed the following game from testing parameter we control. We estimated that the robots here, as the estimated CBF’s for $F$ identified: Equation (CBF-LP) 

$\text{minimize } \sum_{i=1}^{N} h^*_i(x_i), \quad \text{subject to } \quad \hat{h}^*_i(x_{i-1}, u_i, d) \geq -\beta h^*_o(x_{i-1}, d), \quad x_i = x_{i-1} + u_i \Delta t, \quad \forall i = 1, 2, \ldots, N, \quad \|d - x_0\| \geq r_o.$

In equation (Simulation Game) above, $N = 2$, $\beta = 100$, and $r_o = 0.175$. For large values of $\beta$, there is less of an implied assumption about system behavior as it decays to the boundary of the estimated safe region. As a result, large $\beta$ values permit equation (Simulation Game) to account for a wider range of system behavior when solving for $d^*$. In addition, $r_o$ constrains against trivial solutions wherein $d = x_0$, which makes the inner maximization problem infeasible.

To quantify how “hard” a test/demonstration is, we define:

- $H_g^i \triangleq \frac{1}{T_i+1} \sum_{k=0}^{T_i} \left| \hat{h}_g(x_k) \right|$ to be the average time the system spent outside the goal. Here, $\hat{h}_g$ denotes a normalized version of our estimated CBF, $h^*_g$, such that $-1 \leq \hat{h}_g(x_k) \leq 0, \forall k = 0, 1, \ldots, T_i$, and $T_i$ is the max time for our Demonstrations/tests as defined in Definition 2.

- $H_o^i \triangleq 1 - \frac{1}{T_i+1} \sum_{k=0}^{T_i} \hat{h}_o(x_k)$ to be the average time spent collision free. Here, $\hat{h}_o$ denotes a normalized version of our estimated CBF, $h^*_o$ such that $0 \leq \hat{h}_o(x_k) \leq 1, \forall k = 0, 1, \ldots, T_i$.

To note, tests drive $H_o^i \rightarrow 1$ in an effort to drive $H_g^i \rightarrow 1$ which denotes system safety failure and inability to reach the goal, respectively.

Figures 2 and 3 show the results of simulations based on the obstacle locations outputted by minimax.
The method detailed involves estimation of approximate goal in doing so, is to provide a mathematical framework and evaluation for verification and validation of autonomous setup here mimics the same single-agent case shown in the forced the system to satisfy specification (4). An example of these cases, notice how the estimated CBF, $d^*_i(x(t))$, at least with respect to the estimated CBF, $h^*_{ij}$. Data for all 20 single-agent simulations are compared against the provided data-set, $D$, in Figure 3. Under normal operation, the demonstration data is relatively consistent $i.e.$ $H_{ij}^i$ hovers just below 0.4 and $H_{ij}^i$ hovers just around 0.7 for all demonstrations, $i = 1, 2, \ldots, 20$. However, for all test simulations, $H_{ij}^i > 0.4$ and $H_{ij}^i > 0.8$ further corroborating that the test parameter sequence generates difficult tests, and in 7/20 cases wherein $H_{ij}^i = 1$, also forced the system to satisfy specification [4]. An example of an experimental demonstration of the test framework can be seen in an accompanying video (linked here: [1]). The setup here mimics the same single-agent case shown in the examples in Figure 2.

### VI. CONCLUSION

In this paper, we attempt to solve the problem of test and evaluation for verification and validation of autonomous systems, wherein the specific controllers are unknown. The goal in doing so, is to provide a mathematical framework designed to root out system inefficiencies in an effort to ensure confidence in those systems that pass the procedure. The method detailed involves estimation of approximate control barrier functions to frame a minimax game that is guaranteed to choose test parameters to frustrate system satisfaction of a provided temporal logic specification. In the future, we aim to extend this work to richer specification classes and formalize an iterative testing procedure based on our framework.

### REFERENCES

[1] “Video of experiment.” [https://vimeo.com/402779550](https://vimeo.com/402779550)
[2] P. Koopman and M. Wagner, “Autonomous vehicle safety: An interdisciplinary challenge,” *IEEE Intelligent Transportation Systems Magazine*, vol. 9, no. 1, pp. 90–96, 2017.
[3] J. V. Hook, W. Seto, V. Nguyen, Z. Hasnain, L. Gallagher, T. Halpin-Chan, V. Varamahaththy, and M. Angulo, “Autonomous swarms of high speed maneuvering surface vessels for the central test evaluation improvement program,” in *Unmanned Systems Technology XXI* (C. M. Shoemaker, H. G. Nguyen, and P. L. Muench, eds.), vol. 11021, pp. 140 – 149, International Society for Optics and Photonics, SPIE, 2019.
[4] M. Ahmadi, A. Singleteray, J. W. Burdick, and A. D. Ames, “Safe policy synthesis in multi-agent POMDPs via discrete-time barrier functions,” in *58th IEEE Conference on Decision and Control*, (Nice,France), 2019.
[5] M. Althoff and S. Lutz, “Automatic generation of safety-critical test scenarios for collision avoidance of road vehicles,” in *2018 IEEE Intelligent Vehicles Symposium (IV)*, pp. 1326–1333, IEEE, 2018.
[6] M. Koschi, C. Pek, S. Maierhofer, and M. Althoff, “Computationally efficient safety falsification of adaptive cruise control systems,” in *2019 IEEE Intelligent Transportation Systems Conference (ITSC)*, pp. 2879–2886, IEEE, 2019.
[7] T. A. Wheeler and M. J. Kochenderfer, “Critical factor graph situation clusters for accelerated automotive safety validation,” in *2019 IEEE Intelligent Vehicles Symposium (IV)*, pp. 2133–2139, IEEE, 2019.
[8] G. E. Mullins, P. G. Stankiewicz, and S. K. Gupta, “Automated generation of diverse and challenging scenarios for test and evaluation of autonomous vehicles,” in *2017 IEEE international conference on robotics and automation (ICRA)*, pp. 1443–1450, IEEE, 2017.
[9] B. A. Haugh, D. A. Sparrow, and D. M. Tate, “The status of test, evaluation, verification, and validation (tev&v) of autonomous systems,” 2018.
[10] H. Abbas, G. Fainekos, S. Sankaranarayanan, F. Ivančićundefined, and A. Gupta, “Probabilistic temporal logic falsification of cyber-physical systems,” *ACM Trans. Embed. Comput. Syst.*, vol. 12, May 2013.
[11] S. Srinivasan, S. Coogan, and M. Egerstedt, “Control of multi-agent systems with finite time control barrier certificates and temporal logic,” in *2018 IEEE Conference on Decision and Control (CDC)*, pp. 1991–1996, Dec 2018.
[12] L. Lindemann and D. V. Dimarogonas, “Control barrier functions for multi-agent systems under conflicting local signal temporal logic tasks,” *IEEE Control Systems Letters*, vol. 3, pp. 757–762, July 2019.
[13] L. Lindemann and D. V. Dimarogonas, “Decentralized control barrier functions for coupled multi-agent systems under signal temporal logic tasks,” in *2019 18th European Control Conference (ECC)*, pp. 89–94, June 2019.
[14] T. Wongpiromsarn, U. Topcu, and A. Lamperski, “Automata theory meets barrier certificates: Temporal logic verification of nonlinear systems,” *IEEE Transactions on Automatic Control*, vol. 61, pp. 3344–3355, Nov 2016.
[15] M. Ahmadi, A. Singleteray, J. W. Burdick, and A. D. Ames, “Barrier functions for multiagent-pomdp with dtl specifications,” *arXiv preprint arXiv:2003.09267*, 2020.
[16] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada, “Control barrier function based quadratic programs for safety critical systems,” *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 3861–3876, 2016.
[17] L. Wang, A. D. Ames, and M. Egerstedt, “Safety barrier certificates for collision-free multirobot systems,” *IEEE Transactions on Robotics*, vol. 33, no. 3, pp. 661–674, 2017.
[18] A. Li, L. Wang, P. Pierpaoli, and M. Egerstedt, “Formally Correct Composition of Coordinated Behaviors Using Control Barrier Certificates,” in *IEEE International Conference on Intelligent Robots and Systems*, pp. 3723–3729, Institute of Electrical and Electronics Engineers Inc., dec 2018.
[19] S. Wilson, P. Glotfelter, L. Wang, S. Mayya, G. Notomista, M. Mote, and M. Egerstedt, “The robotarium: Globally impact-ful opportunities, challengers, and lessons learned, in remote-access, distributed control of multi-robot systems..."