ARITHMETIC PROGRESSIONS IN SETS OF SMALL DOUBLING

KEVIN HENRIOT

Abstract. We show that if a finite, large enough subset $A$ of an arbitrary abelian group satisfies the small doubling condition $|A + A| \leq (\log |A|)^{1-\epsilon} |A|$, then $A$ must contain a three-term arithmetic progression whose terms are not all equal, and $A + A$ must contain an arithmetic progression or a coset of a subgroup, either of which of size at least $\exp\left[ c(\log |A|)^{\delta} \right]$. This extends analogous results obtained by Sanders and, respectively, by Croot, Laba and Sisask in the case where the group is $\mathbb{Z}^s$ or $\mathbb{F}_q^n$.

1. Introduction

Our aim in this work is to generalize two types of results of additive combinatorics usually stated for dense subsets of the integers, namely Roth’s theorem [16] and Bourgain’s theorem on long arithmetic progressions in sumsets [2], to the case where the sets only have small doubling and live in an arbitrary abelian group. As in previous work of this nature [17,19,25,26], our motivation is to provide a link between two types of additive structure: small doubling on the one hand, and containment of arithmetic progressions in the set or its sumset on the other hand. Since the result we seek is known qualitatively by the modelling methods of Green and Ruzsa [7], we focus on the quantitative bounds that may be obtained for it.

Concerning the first topic of Roth’s theorem, we start by recalling the state-of-the-art bounds, which we state in the setting of a cyclic group. Here a $k$-term arithmetic progression in an abelian group is defined as a tuple $(x_1, \ldots, x_k)$, where $x_1, \ldots, x_k$ are group elements such that $x_2 - x_1 = \cdots = x_k - x_{k-1}$, and we say that it is trivial when $x_1, \ldots, x_k$ are all equal, and proper when they are all distinct; note that when the group has odd order every nontrivial three-term arithmetic progression is proper. The breakthrough work of Sanders [20] then, building on earlier work of Bourgain [3], has established that given a large enough, odd integer $N$, every subset of $\mathbb{Z}/NZ$ of density at least $(\log N)^{-1+o(1)}$ contains a proper three-term arithmetic progression. Under a density hypothesis, the generalization to finite abelian groups is not very challenging: indeed it can be essentially read out of [20] that any set of density at least $(\log |G|)^{-1+o(1)}$
in a finite abelian group $G$ of odd order contains a proper three-term arithmetic progression.

However, the situation is more complex when we only assume that the set in question, say $A$, has small doubling in the sense that $|A + A| \leq K|A|$. Since subsets of density $\alpha$ of a finite abelian group have doubling at most $K = \alpha^{-1}$, this includes the previous situation. We would then like to show that $K \leq (\log |A|)^{1-o(1)}$ forces $A$ to contain a proper three-term arithmetic progression, which would truly generalize the dense case, however this is not not obvious even in the case where $A$ is a set of integers. Indeed the direct approach, which proceeds by combining the standard Ruzsa modelling lemma [17] with the bounds for Roth’s theorem from [20], only yields an admissible range of $K \leq (\log |A|)^{1/4-o(1)}$. This is precisely what led Sanders [19] to design a more subtle approach which, for sets of integers, yields the range we seek.

**Theorem 1** (Sanders). There exists an absolute constant $c > 0$ such that the following holds. Suppose that $A$ is a finite set of integers such that

$$|A + A| \leq c(\log |A|)(\log \log |A|)^{-8} \cdot |A|.$$  

Then $A$ contains a proper three-term arithmetic progression.

This does not appear explicitly in the literature, but follows more or less directly from inserting Ruzsa’s modelling bound [17] into the argument of [19], taking also into account the latest bounds for Roth’s theorem [20]; we describe this in more detail at the end of the article. By this procedure, one can actually obtain a version of Theorem 1 for any group with good modelling in the sense of [7]. In the general abelian case, where available modelling arguments are by necessity much weaker [7], Sanders [19] also improves substantially on the bounds that would follow from a direct modelling approach.

**Theorem 2** (Sanders). There exists an absolute constant $c > 0$ such that the following holds. Suppose that $A$ is a finite subset of an abelian group such that

$$|A + A| \leq c(\log |A|)^{1/3}(\log \log |A|)^{-1} \cdot |A|.$$  

Then $A$ contains a nontrivial three-term arithmetic progression.

Note that the conclusion changed to yield a nontrivial arithmetic progression only; we say more on this later. The loss in the exponent of $\log |A|$ in comparison with the

\footnote{Throughout this introduction, we make the tacit assumption that all quantities appearing inside a double logarithm are at least $e^c$ in size.}
previous case is due to a limitation of the results on modelling; indeed via [7] it is only possible to Freiman-embed a set $A$ of doubling $K$ into a finite abelian group where its image has density $\exp[-CK^2 \log K]$. A construction by Green and Ruzsa [7] further shows that any modelling result of this type will feature an exponential loss in $\sqrt{K}$, at least if we insist on embedding the whole set. Fortunately, in a recent major advance on the polynomial Freiman-Ruzsa conjecture, Sanders [22] managed to sidestep this issue and obtained a correlation result which may be viewed as another form of modelling. This result may be applied to our situation to recover a range of doubling matching the current bounds for Roth’s theorem, for arbitrary abelian groups; this is the first observation of this paper.

**Theorem 3.** There exists an absolute constant $c > 0$ such that the following holds. Suppose that $A$ is a finite subset of an abelian group such that

$$|A + A| \leq c(\log |A|)(\log \log |A|)^{-7} \cdot |A|.$$ 

Then $A$ contains a nontrivial three-term arithmetic progression.

Here we say more on the issue of 2-torsion, which was already discussed by Sanders in [19]. In general, a set $A$ contains a nontrivial degenerate arithmetic progression $(x, y, x)$ if and only if $A - A$ contains an element of order 2; therefore in that case, Theorems 2 and 3 give only trivial information. Obtaining proper progressions in every case where it is possible (this excludes groups such as $\mathbb{F}_2^n$) is a thorny issue that has only been successfully addressed in work of Lev [13] and Sanders [18] in cases where the group rank is not too large; here we do not consider this issue.

The second topic we consider is that of long arithmetic progressions in sumsets, initiated by Bourgain [2] and further developed by Green [6]. Basing themselves on a fundamental new technique introduced by Croot and Sisask [5], these two last authors together with Laba [4] obtained a remarkable extension of Green’s result, which furthermore already works under a small doubling hypothesis.

**Theorem 4** (Croot, Laba, Sisask). There exists an absolute constant $c > 0$ such that the following holds. Let $K, L \geq 1$ be parameters, and suppose that $A, B$ are finite sets of integers such that $|A + B| \leq K|A|$ and $|A + B| \leq L|B|$. Then $A + B$ contains an arithmetic progression of length at least

$$\exp \left[ c \left( \frac{\log |A + B|}{K(\log L)^3} \right)^{1/2} \right]$$

provided $K \log^5(L \log |A|) \leq c \log |A + B|$. 
From the methods of [4], one can easily deduce that an analog result holds for subsets $A$ and $B$ of density $\alpha$ and $\beta$ of a finite abelian group, with $\alpha^{-1}$ and $\beta^{-1}$ in place of $K$ and $L$. Therefore we focus again on the case of small doubling in an arbitrary abelian group, to which the argument of [4] does not extend as it relies on a two-sets version of Ruzsa modelling [17]. The coveted generalization of Theorem 4 may however be recovered, again by using the Bogolyubov-Ruzsa lemma from [22], and establishing this is the second aim of this paper. Note that in the general abelian setting, we need to adapt the type of structure sought to allow for both cosets of subgroups and arithmetic progressions.

**Theorem 5.** There exists an absolute constant $c > 0$ such that the following holds. Let $K \geq 1$ be a parameter and suppose that $A$ is a finite subset of an abelian group such that $|A + A| \leq K|A|$. Then $A + A$ contains a set, which is either a proper arithmetic progression or a coset of a subgroup, of size at least

$$\exp\left[c\left(\frac{\log|A|}{K(\log K)^3}\right)^{1/2}\right] \quad \text{provided} \quad K \leq \frac{c\log|A|}{(\log \log|A|)^5}.$$ 

This recovers Theorem 4 in the symmetric case $A = B$, since in $\mathbb{Z}$ every nontrivial subgroup is infinite. We restrict to the symmetric case for simplicity; it seems feasible to obtain an asymmetric result of the shape of Theorem 4 from the methods of this paper, however we do not pursue this here.

Finally, we mention an application of results on arithmetic progressions in sets of small doubling, to the asymptotic size of restricted sumsets. This application was first observed independently by Schoen [24] and Hegyvári et al. [10] in the setting of integers, and later quantitatively strengthened by Sanders [19] in the more general setting of abelian groups. We write $A \hat{+} A$ for the set of sums of distinct elements of $A$ below.

**Corollary 1.** Suppose that $A$ is a finite nonempty subset of an abelian group. Then

$$|A \hat{+} A| \geq (1 - (\log|A|)^{-1+o(1)})|A + A|.$$ 

This improves upon the exponent $-\frac{1}{3}$ on the logarithm obtained by Sanders [19] via Theorem 2, since Theorem 3 is used instead. Note that by Behrend’s construction [14], the restricted sumset may have size as low as $(1 - e^{-c\sqrt{\log|A|}})|A + A|$ and therefore the bounds for this problem match those for Roth’s theorem closely.

Finally, we remark that by the finite modelling argument of Green and Ruzsa [7, Lemma 2.1], it suffices to prove all our results in the case where the group is finite abelian, and therefore we work under that hypothesis for the rest of the paper. This concludes our introduction and we discuss the structure of this paper in the next section.
Funding. This research was supported by a *contrat doctoral* from Université Paris 7 and by the ANR Caesar ANR-12-BS01-0011.

2. Overview

In this section we sketch the argument behind our results and outline the structure of this paper. We use the symbols $\approx$ and $\gtrsim$ to indicate statements that hold true up to certain negligible factors.

The first logical step in the proof of Theorem 3 consists in applying the correlation version of Sanders’ Bogolyubov-Ruzsa lemma [22] (Proposition 6) to deduce that a set $A$ of doubling $K$ has density $\gtrsim 1/K$ in (a translate of) a large Bourgain system $B$, a group-like object whose properties are recalled in Section 4. The second step is to obtain an efficient local version of Roth’s theorem (Proposition 2), which, roughly saying, asserts that a set $A$ of density $\alpha \gtrsim (\log |B|)^{-1}$ in a large Bourgain system $B$ contains many arithmetic progressions, and therefore a nontrivial one. This may be applied to the previous system $B$, for which $|B| \approx |A|$ and $\alpha \asymp 1/K$, under the condition $K \lesssim \log |A|$, thereby establishing Theorem 3. The local Roth theorem is developed in Section 6, drawing on analytic tools from Section 5, and it is combined in the preceding fashion with the correlation Bogolyubov-Ruzsa lemma in Section 7.

To derive Theorem 5, we need to obtain instead a local version of an almost-periodicity lemma of Croot et al. [4] (Proposition 9), drawing again on the tools of Section 5. This process, carried out in Section 8, requires a somewhat simpler version of Sanders’ Bogolyubov-Ruzsa lemma (Proposition 8) which deduces containment of a large Bourgain system in the sumset $2A - 2A$ from the hypothesis that $A$ has small doubling, and the rest of the argument follows the strategy of [4].

Finally, to illustrate some of the above ideas, we showcase the proof of Theorem 3 in the model setting of $\mathbb{F}_3^n$, where the proof of Sanders’ Bogolyubov-Ruzsa lemma [22] simplifies substantially. As an added benefit, the formidable bounds of Bateman and Katz [1] for caps in $\mathbb{F}_3^n$ yield a larger admissible range of doubling in this setting. The notation used in the proof is introduced in Section 3.

**Theorem 6.** There exist positive absolute constants $c$ and $\varepsilon$ such that the following holds. Suppose that $A$ is a subset of $\mathbb{F}_3^n$ such that

$$|A + A| \leq c (\log |A|)^{1+\varepsilon} \cdot |A|.$$ 

Then $A$ contains a proper three-term arithmetic progression.
Proof. Write $K = |A + A|/|A|$, so that we are assuming that $K \leq c(\log |A|)^{1+\varepsilon}$. The proof of [7, Proposition 6.1] readily adapts to $\mathbb{F}_3^n$, and shows that $A$ is Freiman-isomorphic to a subset of doubling $K$ and density at least $K^{-4}$ of another finite field $\mathbb{F}_3^n$, which we identify with $A$ from now on. By examining the proof of [22, Theorem A.1], which works equally well in $\mathbb{F}_m^3$, one may deduce that there exist a measure $\mu$ and a subspace $V$ of $\mathbb{F}_m^3$ of codimension at most $C(\log K)^4$ such that
\[
\langle 1_A * \mu_V * \mu_{A+A} * \mu, \mu\rangle_{L^2} \geq \frac{1}{2} \mu_G(A)/\mu_G(A + A).
\]
By the definition of $K$, and upon applying Hölder’s and Young’s inequalities, we obtain
\[
\frac{1}{2K} \leq \langle 1_A * \mu_V * \mu_{A+A} * \mu, \mu\rangle_{L^2} \\
\leq \|1_A * \mu_V * \mu_{A+A} * \mu\|_{\infty} \|\mu_A\|_{L^1} \\
\leq \|1_A * \mu_V\|_{\infty}.
\]
Therefore we may find $x$ such that $A' = (A - x) \cap V$ has density at least $\frac{1}{2K}$ in $V$. Since $V$ has codimension at most $C(\log K)^4$, it has size at least $|G|^{1/2}$ in our range of $K$. Applying [1, Theorem 1.1] to $A'$, we are then ensured to find a proper three-term arithmetic progression in $A'$ provided
\[
\frac{1}{2K} \geq C(\log |V|)^{-1+\varepsilon}
\]
and this concludes the proof since $\log |V| \approx \log |A|$.

3. Notation

In this section we introduce the notation used throughout the article.

Ambient group. We let $G$ denote a fixed, finite abelian group. The arguments of later sections all take place in this group unless otherwise stated.

$\mathbb{Z}$-actions. The group $G$ is naturally equipped with a structure of $\mathbb{Z}$-module, and we let $k \cdot x$ denote the action of a scalar $k \in \mathbb{Z}$ on an element $x \in G$. For a subset $X$ of $G$ and a subset $I$ of $\mathbb{Z}$, we further write
\[
k \cdot X = \{k \cdot x : x \in X\} \quad \text{and} \quad I \cdot x = \{k \cdot x : k \in I\}.
\]
Note that $\cdot$ is also used in other places for the regular multiplication of complex numbers, however it should be clear from the context which one is meant.

Functions. We define the averaging operator over a subset $X$ of $G$, which acts on the space of functions $f : G \to \mathbb{C}$, by $E_X f = |X|^{-1} \sum_{x \in X} f(x)$, and we write $E_{x \in X} f(x)$ when we want to keep the variable explicit. It is also convenient to introduce the operator
of translation on a function $f$ defined by $\tau_x f(u) = f(x + u)$ for all $x, u \in G$. We furthermore define the support of $f$ as $\text{Supp}(f) = \{x \in G : f(x) \neq 0\}$. On the physical space, we use the normalized counting measure so that for functions $f, g : G \to \mathbb{C}$, we let

$$
\|f\|_{L^p} = (\mathbb{E}_G |f|^p)^{1/p},
$$

(Scalar product) $$
\langle f, g \rangle_{L^2} = \mathbb{E}_G f \overline{g},
$$

(Convolutions) 
\[ f \ast g(x) = \mathbb{E}_{y \in G} f(y)g(x - y) \quad \forall x \in G. \]

We occasionally write $\|f\|_p$ for $\|f\|_{L^p}$, and we let $f^{(\ell)}$ denote the convolution of $f$ with itself $\ell$ times.

**Measures.** We identify measures $\mu$ on $G$ with functions $\mu : G \to \mathbb{R}_+$ via the identity $\mu(\{x\}) = |G|^{-1} \mu(x)$, so that $\mu(E) = (1_E, \mu)_{L^2}$ for every subset $E$ of $G$. We only consider probability measures; in other words, we always assume that $\|\mu\|_{L^1} = 1$. We write $\mu_A$ for the measure defined by $\mu_A(E) = |E \cap A|/|A|$ for every set $E$, which under our identification corresponds to the function $\mu_A = \mu_G(A)^{-1}1_A$.

**Fourier transform.** The Fourier transform over finite abelian groups is now a standard tool of additive combinatorics. It is very well explained for example in [9], and here we only recall its main properties.

Write $\mathbb{U}$ for the unit circle, then the dual group $\hat{G}$ is defined as the set of morphisms from $G$ to $\mathbb{U}$, called characters, and the Fourier transform of a function $f : G \to \mathbb{C}$ is defined by $\hat{f}(\gamma) = \langle f, \gamma \rangle_{L^2}$ at every character $\gamma$. We write $(f)^\wedge$ for the Fourier transform of $f$ when $f$ has a complicated expression.

We define the summation operator over a subset $\Delta$ of $\hat{G}$, which acts on the space of functions $F : \hat{G} \to \mathbb{C}$, by $\sum_{\Delta} F = \sum_{\gamma \in \Delta} F(\gamma)$. On the Fourier space, we use the counting measure so that for functions $F, G : \hat{G} \to \mathbb{C}$, we let

$$
\|F\|_{\ell^p} = \left(\sum_{\hat{G}} |F|^p\right)^{1/p},
$$

(Scalar product) $$
\langle F, G \rangle_{\ell^2} = \sum_{\hat{G}} F \overline{G}.
$$

The three classic formulæ of harmonic analysis then read as follows:

(Fourier inversion) $f = \sum_{\hat{G}} \hat{f}(\gamma) \gamma$,

(Parseval formula) $\langle f, g \rangle_{L^2} = \langle \hat{f}, \hat{g} \rangle_{\ell^2}$,

(Convolution identity) $(f \ast g)^\wedge = \hat{f} \cdot \hat{g}$. 


Other. We let $c$ and $C$ denote absolute positive constants, which may take different values at each occurrence. Given nonnegative functions $f$ and $g$, we let $f = O(g)$ or $f \ll g$ indicate the fact that there exists a constant $C$ such that $f \leq Cg$, and we let $f = \Theta(g)$ or $f \asymp g$ indicate that $f \ll g$ and $g \ll f$ hold simultaneously. We also write $\ell(x) = \log(e/x)$ for $x \geq 1$, since this quantity arises often in our computations. Note finally that in many occurrences of logarithms throughout the paper, one should replace $\log x$ by $\log e x$ for the results to be formally correct in all ranges of parameters; we leave this as a mental task to the reader to alleviate the notation. Other notation in this paper is introduced in the relevant section as needed.

4. Bourgain systems

In this section we recall the theory of Bourgain systems, which was introduced by Green and Sanders [8] as a generalization of the Bohr set technology of Bourgain [3]. In a sense these systems are the most general class of sets for which the strategy of density increment on Bohr sets, pioneered by Bourgain [3], may be carried out. What is needed for such an undertaking is for the set to behave approximately like a $d$-dimensional ball with respect to dilation, as axiomatized in the following definition.

Definition 1 (Bourgain system). A Bourgain system of dimension $d$ is a family of sets $\mathcal{B} = (B_{\rho})_{\rho > 0}$, where $B_{\rho}$ are subsets of $G$ such that, for all positive $\rho$ and $\rho'$,

- (containment of 0) $0 \in B_{\rho}$
- (symmetry) $-B_{\rho} = B_{\rho}$
- (nesting) $B_{\rho} \subset B_{\rho'}$ if $\rho \leq \rho'$
- (additive closure) $B_{\rho} + B_{\rho'} \subset B_{\rho + \rho'}$
- ($2^d$-covering) $\exists X_{\rho'}: B_{2\rho} \subset X_{\rho'} + B_{\rho}$ and $|X_{\rho'}| \leq 2^d$.

We write $B = B_1$, and we define the density of $\mathcal{B}$ as $b = |B|/|G|$.

We let the sets $B_{\rho}$, and sometimes also the dimension $d$ and the density $b$, be defined implicitly whenever we introduce a Bourgain system $\mathcal{B}$. We now describe two important classes of Bourgain systems: Bohr sets and coset progressions. To define the former, we consider the multiplicative analog $\| \cdot \|_{U}$ on the unit circle of the usual pseudo-norm $\| \cdot \|_{T} = d(\cdot, \mathbb{Z})$ on the torus, defined by $\|e(\theta)\|_{U} = \|\theta\|_{T}$ for every $\theta \in \mathbb{T}$. 


Definition 2 (Bohr set). Suppose that $\Gamma \subset \hat{G}$ and $\delta > 0$. The Bohr set of frequency set $\Gamma$ and radius $\delta$ is

$$B = B(\Gamma, \delta) = \{ x \in G : \| \gamma(x) \|_U \leq \delta \}.$$ 

The dimension of $B$ is $d = |\Gamma|$. We define the dilate of $B$ by $\rho > 0$ as the set $B_\rho = B(\Gamma, \rho \delta)$, and the Bohr system induced by $B$ as the system $\mathcal{B} = (B_\rho)_{\rho > 0}$.

The usual bounds for the size and growth of a Bohr set allow us to quickly estimate the dimension and density of the Bourgain system it induces.

Lemma 1. The system $\mathcal{B}$ induced by a Bohr set of dimension $d$ and radius $\delta \leq 1$ is a Bourgain system $\mathcal{B}$ of dimension at most $6d$ and density at least $\delta^d$.

Proof. The first four properties of a Bourgain system are easy to check. Further, by three applications of [27, Lemma 4.20] we obtain $|B_{4\rho}| \leq 2^{6d}|B_{\rho/2}|$, and therefore by Ruzsa’s covering lemma we may find a set $X_\rho$ such that

$$B_{2\rho} \subset X_\rho + B_{\rho/2} - B_{\rho/2} \subset X_\rho + B_\rho$$

and $|X_\rho| \leq |B_{2\rho} + B_{\rho/2}|/|B_{\rho/2}| \leq 2^{6d}$. Working through the argument in that reference, one could extract a better bound $2^{2d}$, but this would not affect our end results much.

The bound on the density may be read directly from [27, Lemma 4.20]. An alternate reference for these estimates is [11, Section 5].

In our definition of a coset progression, we write $[x, y]_\mathbb{Z} = \{ n \in \mathbb{Z} : x \leq n \leq y \}$ for reals $x \leq y$.

Definition 3 (Coset progression). Let $L \in \mathbb{R}^d_+$ and $\omega \in G^d$ where $d \geq 1$, and let $H$ be a subgroup of $G$. The coset progression of dimension $d$ determined by $L, \omega, H$ is

$$M = M(L, \omega, H) = [-L_1, L_1]_\mathbb{Z} \cdot \omega_1 + \cdots + [-L_d, L_d]_\mathbb{Z} \cdot \omega_d + H.$$ 

We define the dilate of $M$ by $\rho > 0$ as $M_\rho = M(\rho L, \omega, H)$, and the coset progression system induced by $M$ as the system $\mathcal{M} = (M_\rho)_{\rho > 0}$.

The dimension of the Bourgain system induced by a coset progression may be estimated by a simple covering argument.

Lemma 2. The system $\mathcal{M}$ induced by a $d$-dimensional coset progression $M$ is a Bourgain system of dimension at most $3d$. 


Proof. It is again rather simple to derive the first four properties of a Bourgain system for $\mathcal{M}$, and we now concern ourselves with the fifth. The dilate of $M$ by $\rho > 0$ is

$$M_\rho = [-\rho L_1, \rho L_1]_Z \cdot \omega_1 + \cdots + [-\rho L_d, \rho L_d]_Z \cdot \omega_d + H.$$ 

To obtain the covering property, first observe that for any $k \in \mathbb{N} \geq 0$, one may cover the interval $[-k, k]_Z$ by three translates of $[-\frac{k}{2}, \frac{k}{2}]_Z$ (this is sharp for $k$ odd), and that this still holds for any real $k \geq 0$. Therefore, for every $1 \leq i \leq d$, we may find a set $T_i$ with $|T_i| \leq 3$ such that $[-2\rho L_i, 2\rho L_i]_Z \subset T_i + [-\rho L_i, \rho L_i]_Z$. Consequently, for any $\rho > 0$ we have a covering

$$M_{2\rho} \subset \bigcup_{t \in T_1 \times \cdots \times T_d} (t_1 \cdot \omega_1 + \cdots + t_d \cdot \omega_d + M_\rho) = X_\rho + M_\rho$$

for a certain set $X_\rho$ of size at most $|T_1| \cdots |T_d| \leq 3^d$. □

With these examples covered, we now work exclusively within the framework of Bourgain systems. We start by defining a few basic operations on these systems.

Lemma 3 (Dilation). Suppose that $\lambda \in (0, 1]$ and that $\mathcal{B}$ is a Bourgain system of dimension $d$ and density $b$. Then the dilated system $\mathcal{B}_\lambda = (\mathcal{B}_\lambda)_\rho > 0$ is a Bourgain system of dimension at most $d$ and density at least $(\lambda/2)^d \cdot b$.

Proof. Let $\lambda \in (0, 1]$, and choose $k \geq 0$ such that $2^{-(k+1)} < \lambda \leq 2^{-k}$. By the covering property of Definition 1, we have $|B_\rho| \leq 2^d |B_{\rho/2}|$ for every $\rho > 0$, from which it follows by iteration that $|B| \leq 2^{(k+1)d} |B_{1/2^{k+1}}| \leq (2/\lambda)^d |B_\lambda|$. That $\mathcal{B}_\lambda$ is a $d$-dimensional Bourgain system is obvious, and the bound on the density follows from the previous computation. □

Definition 4 (Sub-Bourgain systems). Suppose that $\mathcal{B}$ and $\mathcal{B}'$ are two Bourgain systems. We say that $\mathcal{B}$ is a sub-Bourgain system of $\mathcal{B}'$, and we write $\mathcal{B} \leq \mathcal{B}'$, when $B_\rho \subset B'_\rho$ for all $\rho > 0$. For $\lambda \in (0, 1]$, we also write $\mathcal{B} \leq_{\lambda} \mathcal{B}'$ when $\mathcal{B} \leq \mathcal{B}'_\lambda$.

The properties of an intersection of Bourgain systems were derived in [19, Lemma 3.4], whose proof we reproduce here for completeness.

Lemma 4 (Intersection). Suppose that $\mathcal{B}^{(1)}, \ldots, \mathcal{B}^{(k)}$ are Bourgain systems of dimensions $d_1, \ldots, d_k$ and densities $b_1, \ldots, b_k$. Then the intersection system

$$\mathcal{B}_1 \wedge \cdots \wedge \mathcal{B}_k = (B^{(1)}_\rho \cap \cdots \cap B^{(k)}_\rho)_{\rho > 0}$$

is a Bourgain system of dimension at most $2(d_1 + \cdots + d_k)$ and of density at least $4^{-(d_1 + \cdots + d_k)} b_1 \cdots b_k$. 

Proof. The first four properties of a Bourgain system are again easy to check, and we now consider the covering property. Let \( \rho > 0 \). For each \( 1 \leq i \leq k \), apply the covering property of \( B^{(i)} \) twice to obtain a set \( T_i \) of size at most \( 4^d \) such that \( B^{(i)}_{2\rho} \subset T_i + B^{(i)}_{\rho/2} \). Distributing intersection over union, we have then

\[
\bigcap_{1 \leq i \leq d} B^{(i)}_{2\rho} = \bigcup_{(t_1, \ldots, t_k) \in T_1 \times \cdots \times T_k} \bigcap_{1 \leq i \leq k} \left( t_i + B^{(i)}_{\rho/2} \right).
\]

Now pick an element \( x(t) \) in each nonempty intersection \( \bigcap_i (t_i + B^{(i)}_{\rho/2}) \). Then for each element \( x \) of \( \bigcap_i B^{(i)}_{2\rho} \), we may find an element \( t \in \prod_i T_i \) such that

\[
x - x(t) \in \bigcap_i \left( B^{(i)}_{\rho/2} - B^{(i)}_{\rho/2} \right) \subset \bigcap_i B^{(i)}_\rho.
\]

This yields the desired covering with \( X_\rho \) defined as the set of all \( x(t) \).

To estimate the density of the intersection, first apply Ruzsa’s covering lemma for each \( 1 \leq i \leq k \) to obtain a covering of the form

\[
G \subset T_i + B^{(i)}_{1/4} - B^{(i)}_{1/4} \subset T_i + B^{(i)}_{1/2}
\]

where \( T_i \) is a set of size \( |T_i| \leq 4^d b_i^{-1} \). From \( G \subset \bigcap_i (T_i + B^{(i)}_{1/2}) \), it follows that

\[
G = \bigcup_{(t_1, \ldots, t_k) \in T_1 \times \cdots \times T_k} \bigcap_{1 \leq i \leq k} \left( t_i + B^{(i)}_{1/2} \right) = \bigcup_{t \in T_1 \times \cdots \times T_k} A(t)
\]

where \( A(t) \) are sets satisfying \( A(t) - A(t) \subset \bigcap_i B^{(i)} \). By the pigeonhole principle, we may also find a point \( t \) such that

\[
|A(t)| \geq \frac{|G|}{|T_1| \cdots |T_k|} \geq 4^{-(d_1 + \cdots + d_k)} b_1 \cdots b_k |G|,
\]

which yields the desired density estimate since \( |A(t) - A(t)| \geq |A(t)| \). \( \square \)

We consider one last operation on Bourgain systems; since it is so simple we leave it as an exercise to the reader.

Lemma 5 (Homomorphic image). Suppose that \( \mathcal{B} \) is a Bourgain system of dimension \( d \), and \( \phi \) is an endomorphism of \( G \). Then the image system \( \phi(\mathcal{B}) = (\phi(B_\rho))_{\rho > 0} \) is a Bourgain system of dimension at most \( d \).

Finally, we recall the essential notion of regularity introduced by Bourgain [3] for Bohr sets, and which has a natural analogue for Bourgain systems. We let\(^2\) \( C_0 = 2^5 \) and \( C_1 = 2^6 \) in what follows for definiteness, although the exact values are unimportant.

\(^2\)These precise constants, featured in subsequent lemmas, are derived in [11, Section 6].
Definition 5 (Regular Bourgain system). We say that a Bourgain system $\mathcal{B}$ of dimension $d$ is regular when, for every $|\rho| \leq \frac{1}{C_0d}$,

$$1 - C_0|\rho|d \leq \frac{|B_{1+\rho}|}{|B|} \leq 1 + C_0|\rho|d.$$ 

In practice one can always afford to work with regular Bourgain systems, as is the case with Bohr sets, via [19, Proposition 3.5] which we now quote.

Lemma 6. Suppose that $\mathcal{B}$ is a Bourgain system. Then there exists $\lambda \in \left[\frac{1}{2}, 1\right]$ such that $\mathcal{B}_\lambda$ is regular.

The regularity computations in subsequent sections rely on the following $L^1$ estimate.

Lemma 7. Suppose that $\mathcal{B}$ is a regular Bourgain system of dimension $d$ and $\mu$ is a measure on $G$ with support in $B_\rho$, where $0 < \rho \leq \frac{1}{C_1d}$. Then

$$\|\mu_B * \mu - \mu_B\|_{L^1} \leq C_1\rho d.$$ 

Proof. For every $y \in B_\rho$, the function $\mu_{y+B} - \mu_B$ has support in $B_{1+\rho} \setminus B_{1-\rho}$, so that

$$\|\mu_{y+B} - \mu_B\|_{L^1} \leq \frac{|B_{1+\rho}| - |B_{1-\rho}|}{|B|} \leq 2C_0\rho d.$$ 

Averaging over $y \in G$ with weights $\mu(y)$, and using the triangle inequality, we recover the desired estimate. \qed

5. Spectral analysis on Bourgain systems

This section is concerned with collecting all the analytic information we need about the large spectrum of the indicator functions of certain sets. The main task is to obtain a large structured set on which all characters of the large spectrum take values close to 1, since such a set may be later used for purposes of a density-increment-based iteration, or to locate long arithmetic progressions.

When considering indicator functions of subsets of Bohr sets, the information we seek is provided by the spectral analysis developed by Sanders [21], and the aim of this section is therefore to obtain a similar analysis for Bourgain systems. Note that such a process was already carried out in the earlier article [19], however we benefit here from the more efficient analysis of the local spectrum from [21]. To be specific, there is now a local analog of Chang’s bound [21, Lemma 4.6] which supersedes the earlier local analog of Bessel’s inequality [19, Proposition 4.4]. We now give the precise statements, and in that regard it is useful to recall the following definitions.
Definition 6 (Annihilation). Let $\nu \in (0, 2]$ be a parameter, and suppose that $T$ is a subset of $G$ and $\Delta$ is a subset of $\hat{G}$. We say that $\Delta$ is $\nu$-annihilated by $T$ when
\[ |1 - \gamma(t)| \leq \nu \quad \text{for all } t \in T \text{ and } \gamma \in \Delta. \]
When $\mathcal{B}$ is a Bourgain system, we say that it $\nu$-annihilates $\Delta$ when $B$ does.

The quantity we seek to annihilate is then the following.

Definition 7 (Large spectrum). Suppose that $\eta \in (0, 1]$ be a parameter and $f : G \to \mathbb{C}$ is a function. The $\eta$-large spectrum of $f$ is the level set of $\hat{G}$ defined by
\[ \text{Spec}_\eta(f) = \{ |\hat{f}| \geq \eta \|f\|_{L^1} \}. \]

We also need to recall one piece of terminology from [21, Section 4], which is only used in this section. Write $\mathbb{D}$ for the unit disk, and let $\mu$ be any measure on $G$. Given a parameter $\theta \in (0, 1]$, we say that a set $\Lambda$ of characters is $(\theta, \mu)$-dissociated when, for every function $\omega : \Lambda \to \mathbb{D}$, we have
\[ \int \prod_{\lambda \in \Lambda} \left(1 + \text{Re}[\omega(\lambda)\lambda]\right) \, d\mu \leq e^\theta, \]
and when $\theta = 1$ we simply say that $\Lambda$ is $\mu$-dissociated. We may now quote two lemmas of local spectral analysis from [21], with minor tweaks in both cases.

Lemma 8 (Local Chang bound). Let $\eta \in (0, 1]$ be a parameter, and suppose that $B$ is a subset of $G$ and $X$ is a subset of $B$ of density $\tau$. Then every $\mu_B$-dissociated subset of $\text{Spec}_\eta(\mu_X)$ has size at most $C\eta^{-2} \log \tau^{-1}$.

Proof. This is [21, Lemma 4.6], specialized to the case where $f = \mu_X$ and $\mu = \mu_B$, so that with the notation from there $L_{\mu_X, \mu_B} = \tau^{-1/2}$. □

Lemma 9 (Annihilating locally dissociated sets). Let $\nu \in (0, 1]$ be a parameter. Suppose that $\mathcal{B}$ is a regular Bourgain system, $\Delta$ is a set of characters, and $m$ is the size of the largest $\mu_B$-dissociated subset of $\Delta$, or 1 if there is no such subset. Then there exists a Bohr set $\tilde{B}$ of dimension at most $m$ and radius equal to $c/m$ such that $\Delta$ is $\nu$-annihilated by $B_{2^m/d^2} \cap \tilde{B}_\nu$.

Proof. This is [21, Lemma 6.3] with $\eta = 1$ and $m = \max(k, 1)$, and two minor tweaks: $\mathcal{B}$ is a Bourgain system instead of a Bohr set and a few changes of variables have been effected. Since the proof requires only a regularity estimate of the type of Lemma 7, the generalization to Bourgain systems is immediate. □
As usual these two ingredients combine to show that the large spectrum of a dense subset of a Bourgain system may be efficiently annihilated. Before carrying this out, we introduce a last definition which serves to simplify our technical statements.

**Definition 8.** Let \( m \geq 1 \) be a parameter and suppose that \( \mathcal{B} \) is a Bourgain system. We say that \( \mathcal{B} \) is \( m \)-controlled when it has dimension at most \( m \) and density at least \( \exp[-Cm \log m] \).

We are now ready to introduce the main technical tool of this paper. Recall that \( \ell(x) \) stands for \( \log(e/x) \) here and throughout the article.

**Proposition 1** (Local spectrum annihilation). Let \( \eta, \nu \in (0,1] \) be parameters. Suppose that \( \mathcal{B} \) is a regular Bourgain system and \( X \) is a subset of \( \mathcal{B} \) of relative density \( \tau \). Then \( \text{Spec}_\eta(\mu_X) \) is \( \nu \)-annihilated by a regular Bourgain system of the form

\[
\mathcal{B}_{c\nu/d^2 m} \cap \tilde{\mathcal{B}}_{\nu}, \quad \text{where} \quad m \leq C\eta^{-2}\ell(\tau)
\]

and \( \tilde{\mathcal{B}} \) is an \( m \)-controlled Bourgain system.

**Proof.** Let \( m \) denote the size of the largest \( \mu_\mathcal{B} \)-dissociated subset of \( \text{Spec}_\eta(\mu_X) \), or 1 when there is no such set. By Lemma 8, we have \( m \leq C\eta^{-2}\ell(\tau) \). By Lemma 9, we also know that \( \text{Spec}_\eta(\mu_X) \) is \( \nu \)-annihilated by a regular Bourgain system \( \overline{\mathcal{B}} := \mathcal{B}_{c\nu/d^2 m} \cap \tilde{\mathcal{B}}_{\nu} \), where \( \tilde{\mathcal{B}} \) is the Bourgain system induced by a Bohr set of dimension \( d \leq m \) and radius \( \delta = c/m \). By Lemma 6, we may further ensure that \( \overline{\mathcal{B}} \) is regular up to dilating it by a factor \( \asymp 1 \), which does not affect the shape of the above intersection except in the value of the constants. By Lemma 1, we also see that \( \tilde{\mathcal{B}} \) has dimension at most \( 6m \) and density at least \( \exp[-Cm \log m] \), so that the result follows by replacing \( 6m \) with \( m \) and adapting the constants. \( \square \)

### 6. Roth’s theorem for Bourgain systems

This section is concerned with a local version of Roth’s theorem [16], first considered by Sanders [19], which applies to dense subsets of a Bourgain system. Since the pioneering work of Bourgain [3], modern proofs of Roth’s theorem [20, 21] all share the same global structure and proceed by an iteration on subsets of Bohr sets. An important observation made in [19] is that this iteration may be initialized inside a certain Bohr set instead of the whole group, and further that one may perform the same iteration on Bourgain systems in place of Bohr sets.

However the quantitative estimates obtained in [19] correspond roughly in strength to a range of \( \alpha \gtrsim (\log N)^{-1/3} \) in Roth’s theorem, while the best-known range, also by
Sanders [20], is now \( \alpha \gtrsim (\log N)^{-1} \). Conceptually, there is no obstacle in obtaining this better quantitative dependency with Bourgain systems, and for the same local initialization, however on a technical level it is not entirely straightforward as most density-increment statements then take a different shape. We carry out this process in this section; since it is not the right place here to present the whole argument of [20], we only include the main structural results we need from it and indicate the changes that need to be done to other. Unfortunately, this means that the reader needs either to be conversant with [20], or to read this section conditionally on Proposition 4 below. What we obtain eventually is the following quantitative improvement of [19, Theorem 5.1].

**Proposition 2** (Local Sanders-Roth theorem). Suppose that \( \mathcal{B} \) is a regular Bourgain system and \( A \) is a subset of \( \mathcal{B} \) of relative density \( \alpha \) such that \( A - A \) contains no element of order 2. Then

\[
\langle 1_A \ast 1_A, 1_{2A} \rangle_{L^2} \geq \exp \left[ -C(\alpha^{-1} + d)\ell(\alpha)^{6}\ell(\alpha/d) \right] \cdot b^2.
\]

We make a brief comment here on the shape of the above proposition. The three-term arithmetic progressions contained in a set \( A \) are precisely the triples \((x, y, z)\) of \( A^3 \) such that \( x + z = 2 \cdot y \). The assumption on \( A \) shows that the change of variables \( y \mapsto 2 \cdot y \) is injective on \( A \), from which we see that the total number of such progressions is equal to \( \langle 1_A \ast 1_A, 1_{2A} \rangle_{L^2} \cdot |G|^2 \). We invite the reader to keep this observation in mind, as it is used implicitly in later arguments.

We now present our modified version of the argument of [20]. To begin with, we reconstitute the \( L^2 \) density-increment strategy entirely as it takes a different form for Bourgain systems, which determines the shape of iterative statements. The following lemma is the usual argument that allows one to pass from large energy of the Fourier transform over a character set, to a density increment on any set annihilating those characters.

**Lemma 10.** Let \( \rho, \kappa \in (0, 1] \) be parameters. Suppose that \( \mathcal{B} \) is a regular Bourgain system, \( A \) is a subset of \( \mathcal{B} \) of relative density \( \alpha \), \( T \) is a subset of \( \mathcal{B}_\rho \) and \( \Delta \) is a set of characters. Assume also that \( \rho \leq c\kappa \alpha/d \) and write \( f_A = 1_A - \alpha 1_B \). Then if

\[
\sum_{\Delta} |\hat{f}_A|^2 \geq \kappa \alpha^2 b \quad \text{and} \quad \Delta \text{ is } \frac{1}{2}\text{-annihilated by } T,
\]

we have \( \|1_A \ast \mu_T\|_\infty \geq (1 + 2^{-3}\kappa)\alpha \).

**Proof.** For every character \( \gamma \in \Delta \) we know that \( |1 - \gamma| \leq 1/2 \) on \( T \), and therefore \( |\hat{\mu}_T(\gamma) - 1| \leq \mathbb{E}_T|1 - \gamma| \leq \frac{1}{2} \) and \( |\hat{\mu}_T(\gamma)| \geq \frac{1}{2} \). Inserting this into the energy lower
bound, we have, via Parseval,
\[ \frac{1}{4} \kappa \alpha^2 b \leq \sum_{\hat{G}} |\hat{f}_A|^2 |\hat{\mu}_T|^2 = \langle f_A * \mu_T, f_A * \mu_T \rangle_{L^2}. \]
Expanding this scalar product, and with the help of Lemma 7, we obtain
\[
\frac{1}{4} \kappa \alpha^2 b \leq \|1_A * \mu_T\|_2^2 - 2\alpha \langle 1_A * \mu_T, 1_B * \mu_T \rangle_{L^2} + \alpha^2 \langle 1_B * \mu_T, 1_B * \mu_T \rangle_{L^2}
\]
\[
= \|1_A * \mu_T\|_2^2 - 2\alpha b \langle 1_A, \mu_B * \mu_T * \mu_{-T} \rangle_{L^2} + \alpha^2 b \langle 1_B, \mu_B * \mu_T * \mu_{-T} \rangle_{L^2}
\]
\[
= \|1_A * \mu_T\|_2^2 - (1 + O(\frac{\rho d}{\alpha})) \alpha^2 b.
\]
Choosing \( \rho \leq c \kappa \alpha / d \), we have then
\[
(1 + 2^{-3} \kappa) \alpha^2 b \leq \|1_A * \mu_T\|_2^2
\]
\[
\leq \|1_A * \mu_T\|_\infty \|1_A * \mu_T\|_1
\]
\[
= \|1_A * \mu_T\|_\infty \cdot \alpha b.
\]
Dividing both sides by \( \alpha b \) concludes the proof. \( \square \)

As usual this may be combined with a statement on the local annihilation of the large spectrum, such as Proposition 1, to recover an \( L^2 \)-density increment lemma.

**Proposition 3** (\( L^2 \) density-increment). Let \( \kappa, \eta \in (0, 1] \) be parameters. Suppose that \( B, \hat{B} \) are Bourgain systems and \( B \) is regular, \( A \) is a subset of \( B \) of relative density \( \alpha \) and \( X \) is a subset of \( \hat{B} \) of relative density \( \tau \). Assume also that \( \hat{B} \leq_\rho B \) with \( \rho \leq c \kappa \alpha / d \) and write \( f_A = 1_A - \alpha 1_B \). Then if
\[
\sum_{\text{Spec}_{\eta}(\mu_X)} |\hat{f}_A|^2 \geq \kappa \alpha^2 b,
\]
there exists an \( m \)-controlled Bourgain system \( \tilde{B} \) such that
\[ \mathcal{B} = B_{c/d^2} m \wedge \tilde{B} \quad \text{is regular,} \]
\[ m \leq C \eta^{-2} \ell(\tau), \]
\[ \|1_A * \mu_{\mathcal{B}}\|_\infty \geq (1 + 2^{-3} \kappa) \alpha. \]

**Proof.** By Proposition 1, \( \text{Spec}_{\eta}(\mu_X) \) is \( \frac{1}{4} \)-annihilated by a regular Bourgain system of the form \( \mathcal{B} = \hat{B}_{c d^2} m \wedge \tilde{B} \), where \( \tilde{B} = \tilde{B}_{1/2} \) and \( \tilde{B} \) is an \( m' \)-controlled Bourgain system with \( m' \leq C \eta^{-2} \ell(\tau) \). Note that by Lemma 3, \( \tilde{B} \) is \( O(m') \)-controlled. Applying then Lemma 10 with \( \Delta = \text{Spec}_{\eta}(\mu_X) \) and \( T = \mathcal{B} \leq \tilde{B} \) concludes the proof. \( \square \)
We now take a big step forward and claim that the following analog of [20, Lemma 6.2] holds. This involves a careful examination of the argument of [20], and we regret imposing the double-checking process below on the reader, however past this point our argument is again self-contained.

**Proposition 4** (Iterative lemma on two scales). *Suppose that $B, B'$ are regular Bourgain systems, $A$ is a subset of $B$ of relative density $\alpha$ and $A'$ is a subset of $B'$ of relative density $\alpha'$. Assume also that $B' \leq \rho B$ with $\rho \leq c \alpha / d$. Then either*

(i) *(Many three-term arithmetic progressions)*

$$\langle 1_A \ast 1_{A'}, 1_{-A} \rangle_{L^2} \geq \exp \left[ -C \alpha^{-1} \ell(\alpha') - C' \ell(\alpha'/d') \right] b b',$$

(ii) *(Density increment)*

*There exists an $m$-controlled Bourgain system $\mathcal{B}$ with\*[20, Proposition 4.1] which is just an iteration of the previous lemma.*

The first type of $L^2$ density-increment appears in the proof of [20, Lemma 4.2] on p. 626 with parameters $\kappa \approx 1, \eta \approx \alpha^{1/2}, \tau \gg \alpha'$, so that $m \leq C \alpha^{-1} \ell(\alpha')$ upon applying Proposition 3. The same density-increment is featured in [20, Proposition 4.1] which is just an iteration of the previous lemma.

A second type of density-increment arises in the proof of [20, Corollary 5.2] on pp. 630–632 which involves certain densities $\sigma$ and $\lambda$, and which features parameters $\kappa \approx \lambda, \eta \approx 1$,

$$\tau \geq \exp [-C \lambda^{-2} \ell(\sigma) \ell(\lambda \alpha)^2 \ell(\alpha)]$$

so that

$$m \leq C \lambda^{-2} \ell(\sigma) \ell(\lambda \alpha)^2 \ell(\alpha)$$

upon applying Proposition 3. This is finally combined with [20, Proposition 4.1] on p. 633 to obtain [20, Lemma 6.2], to the effect that we either have an $L^2$ density-increment of the first type, or of the second type with $\lambda \approx 1$ and $\sigma \geq \exp [-C \alpha^{-1} \ell(\alpha')]$,
and therefore such that \( \kappa \asymp 1 \) and \( m \leq C\alpha^{-1}\ell(\alpha)^3\ell(\alpha') \) in the application of Proposition 3. Choosing \( B'' = B'_{c\alpha'/d'} \) in (the Bourgain system version of) [20, Lemma 6.2] and using Lemma 3, we obtain an alternative case (i) of the desired shape.

Since, by Lemma 7, Bourgain systems satisfy the same regularity estimates as Bohr sets, we may replace the latter by the former and apply Proposition 3 everywhere as claimed, thereby obtaining the desired iterative lemma. Finally, the constant \( 2^{-13} \) may be extracted from [20] although its precise value is unimportant; it is just convenient to write down an explicit value for later computations. \( \Box \)

At this point we recall a simple technique, originating in Bourgain’s proof of Roth’s theorem [3, (5.13)–(5.18)], which allows one to pass from two scales to one in iterative statements.

**Lemma 11.** Let \( \theta \in (0,1] \) be a parameter. Suppose that \( \mathcal{B}, \mathcal{B}', \mathcal{B}'' \) are Bourgain systems, \( \mathcal{B} \) is regular and \( A \) is a subset of \( B \) of relative density \( \alpha \). Assume also that \( \mathcal{B}' \leq \rho \mathcal{B} \) and \( \mathcal{B}'' \leq \rho \mathcal{B} \) with \( \rho \leq c\theta\alpha/d \). Then either

\[
\max (\|1_A \ast \mu_{\mathcal{B}'}\|_\infty, \|1_A \ast \mu_{\mathcal{B}''}\|_\infty) \geq (1 + \frac{\theta}{2})\alpha
\]

or there exists \( x \) such that \( 1_A \ast \mu_{\mathcal{B}'}(x) \geq (1 - \theta)\alpha \) and \( 1_A \ast \mu_{\mathcal{B}''}(x) \geq (1 - \theta)\alpha \).

**Proof.** A quick regularity computation via Lemma 7 yields

\[
\mathbb{E}_B (1_A \ast \mu_{\mathcal{B}'} + 1_A \ast \mu_{\mathcal{B}''}) = \langle 1_A, \mu_{\mathcal{B}} \ast \mu_{\mathcal{B}'} \rangle + \langle 1_A, \mu_{\mathcal{B}} \ast \mu_{\mathcal{B}''} \rangle
= 2\alpha + O(\rho d)
\geq (2 - \frac{\theta}{2})\alpha
\]

provided that \( \rho \leq c\theta\alpha/d \). By the pigeonhole principle, there exists \( x \in G \) such that

\[
1_A \ast \mu_{\mathcal{B}'}(x) + 1_A \ast \mu_{\mathcal{B}''}(x) \geq (2 - \frac{\theta}{2})\alpha.
\]

Assuming that we are not in the first case of the lemma, we have

\[
1_A \ast \mu_{\mathcal{B}'}(x) \geq (2 - \frac{\theta}{2})\alpha - (1 + \frac{\theta}{2})\alpha = (1 - \theta)\alpha
\]

and similarly for \( 1_A \ast \mu_{\mathcal{B}''}(x) \). \( \Box \)

With this technique in hand, we may modify Proposition 4 so as to make the iteration easier to perform. Once this is done, Proposition 2 is derived by a standard, yet computationally intensive iterative process. For this argument to work however, we need to make the assumption that the set \( A \) contains no degenerate arithmetic progressions at each step of the iteration.
Proposition 5 (Final iterative lemma). Suppose that $G$ has odd order, $\mathcal{B}$ is a regular Bourgain system, and $A$ is a subset of $\mathcal{B}$ of relative density $\alpha$ such that $A - A$ contains no element of order 2. Then either

(i) (Many three-term arithmetic progressions)
\[ \langle 1_A * 1_A, 1_{2,A} \rangle_{L^2} \geq \exp \left[ -C\alpha^{-1}\ell(\alpha) - Cd\ell(\alpha/d) \right] \cdot b^2, \]

(ii) (Density increment)
there exist Bourgain systems $\widehat{\mathcal{B}}, \widehat{\mathcal{B}}$ and an element $u \in \{1, -2\}$ such that
\[ \widehat{\mathcal{B}} = \widehat{\mathcal{B}} \land \widehat{\mathcal{B}} \text{ is regular}, \]
\[ \widehat{\mathcal{B}} = u \cdot \mathcal{B}_{(\alpha/2d)c}, \quad \widehat{b} \geq \exp \left[ -Cd\ell(\alpha/d) \right] \cdot b, \]
\[ \widehat{d} \leq C\alpha^{-1}\ell(\alpha)^4, \quad \widehat{b} \geq \exp[-C\alpha^{-1}\ell(\alpha)^5], \]
\[ \|1_A * \mu_{\widehat{\mathcal{B}}}\|_\infty \geq (1 + 2^{-14})\alpha. \]

Proof. Let $\theta = 2^{-15}$ and define regular Bourgain systems $\mathcal{B}' = \mathcal{B}_{\alpha/d}$ and $\mathcal{B}'' = \mathcal{B}'_{\alpha/d}$ with the help of Lemma 6. Now apply Lemma 11 to $A$ and $\mathcal{B}, \mathcal{B}', \mathcal{B}''$: in the first case of that lemma, we are in the second case of the proposition, while in the second case we may find an element $x$ such that $A' := (A - x) \cap \mathcal{B}'$ has relative density $\alpha' \geq (1 - 2^{-15})\alpha$ in $B'$, and $A'' := (A - x) \cap \mathcal{B}''$ has relative density at least $\frac{1}{2}\alpha$ in $B''$; the latter weak bound suffices for our purposes.

We let $\widehat{A}'' = -2 \cdot A''$ and $\widehat{\mathcal{B}}'' = -2 \cdot \mathcal{B}''$, so that from the injectivity of $y \mapsto 2 \cdot y$ on $A''$ and the bound $|\widehat{\mathcal{B}}''| \leq |\mathcal{B}''|$, we deduce that $\widehat{A}''$ has density at least $\frac{1}{2}\alpha$ in $\widehat{\mathcal{B}}''$. Furthermore, by Lemma 5, we see that $\widehat{\mathcal{B}}''$ is a Bourgain system of dimension at most $d''$ and, since $\widehat{\mathcal{B}}''$ contains $\widehat{A}''$, of density at least $\frac{1}{2}ab''$. Observe finally that with these choices of $A'$ and $A''$, we have
\[ (1_A * 1_{A}, 1_{2-A})_{L^2} = (1_{A-x} * 1_{2-A}, 1_{x-A})_{L^2} \geq (1_{A'} * 1_{A''}, 1_{A'})_{L^2}. \]

We now apply Proposition 4 to the sets $A'$ and $\widehat{A}'$, located respectively in $\mathcal{B}'$ and $\widehat{\mathcal{B}}'$. In the first case of that proposition, it follows from (6.1) and Lemma 3 that we are in the first case of the proposition we seek to prove. In the second case of Proposition 4, we obtain a regular Bourgain system $\mathcal{B} = \mathcal{B} \land \mathcal{B}$ where
\[ \mathcal{B} = ( -2 \cdot \mathcal{B}'')_{(\alpha/2d)c} = -2 \cdot (\mathcal{B}'')(\alpha/2d)c = -2 \cdot \mathcal{B}_{(\alpha/2d)c}, \]
and $\mathcal{B}$ is $C\alpha^{-1}\ell(\alpha)^4$-controlled, and such that
\[ \|1_A * \mu_{\mathcal{B}}\|_\infty \geq \|1_A * \mu_{\mathcal{B}}\|_\infty \geq (1 + 2^{-13})\alpha' \geq (1 + 2^{-14})\alpha. \]
Applying Lemma 3 to \( \hat{B} = \hat{B}''_{(\alpha/2d)c} \), recalling that \( \hat{b}'' \geq \frac{1}{2} \alpha b'' \), and via Definition 8, we conclude that we are in the second case of the proposition that we intend to prove.

\[ \square \]

**Proof of Proposition 2.** We construct iteratively sequences of subsets \( A_i \) of regular Bourgain systems \( \mathcal{B}^{(i)} \) of density \( \alpha_i \), such that \( A_i \) is contained in a translate of \( A \). Since \( A_i - A_i \) is a subset \( A - A \), it does not contain any element of order 2 either. We initiate the iteration with \( A_1 = A \) and \( \mathcal{B}^{(1)} = \mathcal{B} \).

At each step we apply Proposition 5 to the set \( A_i \), and in the first case of that proposition we stop the iteration, while in the second case we let \( \mathcal{B}^{(i+1)} = \mathcal{B}^{(i)} \) with the notation from there, and we pick \( x_i \) and \( A_{i+1} = (A_i - x_i) \cap \mathcal{B}^{(i)} \) so that \( A_{i+1} \) has relative density \( \alpha_{i+1} = \| 1_{A_i} \ast \mu_{\mathcal{B}^{(i)}} \|_{\infty} \) in \( \mathcal{B}^{(i)} \).

Since \( \alpha_{i+1} \geq (1 + c) \alpha_i \) whenever \( A_{i+1} \) is defined, the iteration proceeds for a number of steps bounded by \( \mathcal{C}(\ell(\alpha)) \). At each step, we obtain Bourgain systems \( \hat{B}^{(i)} \) and \( \hat{B}^{(i)} \) and an element \( u_i \in \{1, -2\} \) such that

\[
\mathcal{B}^{(i+1)} = \hat{B}^{(i)} \wedge \hat{B}^{(i)} \text{ is regular,}
\]

and, since \( \alpha_i \geq \alpha \), such that

\[
\hat{B}^{(i)} = u_i \cdot \mathcal{B}^{(i)}_{(\alpha_i/2d)c}, \quad \hat{b}_i \geq \exp \left[ - C d_i \ell(\alpha/d_i) \right] \cdot b_i,
\]

\[
\tilde{d}_i \leq C \alpha^{-1} \ell(\alpha)^4, \quad \tilde{b}_i \geq \exp \left[ - C \alpha^{-1} \ell(\alpha)^5 \right].
\]

Iterating \( i - 1 \) times (6.2) and (6.3), we obtain a Bourgain system of the form

\[
\mathcal{B}^{(i)} = \hat{B}^{(i-1)} \wedge u_{i-1} \cdot \left( \ldots u_2 \cdot \left( \hat{B}_2^{(i)} \wedge u_1 \cdot \hat{B}_1 \ldots \right) \ldots \right)
\]

where the stars stand for certain dilations. This is not exactly an intersection of Bourgain systems, however the argument used in the proof of Lemma 4 is easily adapted to show that \( \mathcal{B}^{(i)} \) has dimension at most

\[
d_i \leq 2(d + \tilde{d}_1 + \cdots + \tilde{d}_{i-1}).
\]

By (6.4) and since \( i \leq \mathcal{C}(\ell(\alpha)) \), this yields \( d_i \leq 2d + C \alpha^{-1} \ell(\alpha)^5 \).

Applying Lemma 4 to the intersection (6.2), and with (6.3) and (6.4), we also obtain

\[
b_{i+1} \geq 4^{-\left(\tilde{d}_i + \tilde{d}_i\right)} \cdot \hat{b}_i \cdot \tilde{b}_i \geq \exp \left[ - C(\alpha^{-1} + d) \ell(\alpha)^5 \ell(\alpha/d) \right] \cdot b_i.
\]

Iterating this at most \( \mathcal{C}(\ell(\alpha)) \) times, we obtain

\[
b_i \geq \exp \left[ - C(\alpha^{-1} + d) \ell(\alpha)^6 \ell(\alpha/d) \right] \cdot b.
\]
When the algorithm stops, we have therefore

$$\langle 1_{A_i} \ast 1_{A_i}, 1_{2-A_i} \rangle_{L^2} \geq \exp \left[ -C\alpha^{-1}\ell(\alpha) - C d_i \ell(\alpha/d_i) \right] \cdot b_i^2.$$  

Inserting the bounds on $d_i$ and $b_i$ in the above, and recalling that $A_i$ is contained in a translate of $A$, this concludes the proof.

7. From small doubling to three-term arithmetic progressions

This section is concerned with the proof of Theorem 3 and the related Corollary 1. As mentioned before, an extremely important tool for us is the recent correlation-based Bogolyubov-Ruzsa lemma of Sanders [22]. In our situation, it serves to pass from a set of small doubling to one with high density in a coset progression, which is a particular type of Bourgain system. The local Sanders-Roth theorem of the previous section may then be applied to this new set, to show that it contains a nontrivial three-term arithmetic progression; this is the main observation of this paper. We now quote the main result of [22], with a minor tweak to ensure regularity.

**Proposition 6** (Correlation Bogolyubov-Ruzsa lemma [22]). Let $K \geq 1$ be a parameter, and suppose that $A$ is a subset of $G$ such that $|A + A| \leq K|A|$. Then there exists a $d$-dimensional coset progression $M$ inducing a regular Bourgain system and such that

$$\|1_A \ast \mu_M\|_{\infty} \geq \frac{1}{2K},$$

$$d \leq C (\log K)^6,$$

$$|M| \geq \exp \left[ -C (\log K)^6 (\log \log K) \right] \cdot |A|.$$  

**Proof.** Without the regularity condition, this is [22, Theorem 10.1] with $A = S$ and $\varepsilon = \frac{1}{2}$. To obtain regularity, one may simply follow the proof in [22], stopping just before the application of [22, Lemma 10.2], and dilating by a certain constant factor the coset progression $M$ obtained at this point. By Lemmas 3 and 6, one may choose this constant so that the dilated induced Bourgain system is regular, while losing at most a factor $e^{-C (\log K)^6}$ in size, and the rest of the proof goes unchanged. \qed

It is crucial for our argument that this statement makes no assumption of density on the set $A$, whereas the earlier Bogolyubov-Chang-type lemma [19, Proposition 6.1] used by Sanders does. In terms of bounds, we could also allow for $d \leq K^{1+o(1)}$ and $|M| \geq e^{-C K^{1+o(1)} |A|}$ in Proposition 6, without affecting the quality of bounds in Theorem 3; however we do not know of any argument significantly simpler than that of [22] to obtain such estimates.
We now present the proof of Theorem 3, following the usual approach of estimating the total number of three-term arithmetic progressions, only to compare it later to the number of trivial ones. Corollary 1 then follows by inserting the bound of Theorem 3 into the argument of [19].

**Proposition 7.** Let $K \geq 1$ be a parameter. Suppose that $A$ is a subset of $G$ such that $|A + A| \leq K|A|$ and $A - A$ contains no element of order 2. Then

$$\langle 1_A * 1_A, 1_{2\cdot A} \rangle_{L^2} \geq \exp\left[-CK(\log K)^{7}\right] \cdot \mu_G(A)^2.$$  

**Proof.** Let $M$ be the coset progression given by Proposition 6, and write $\mathcal{M}$ for its induced regular Bourgain system. By the correlation conclusion, we may pick an element $x$ such that $A' = (A-x) \cap M$ has relative density $\frac{1}{2K}$ in $M$. Applying then Proposition 2 to $A'$ and $\mathcal{M}$, we obtain

$$\langle 1_A * 1_A, 1_{2\cdot A} \rangle_{L^2} \geq \langle 1_{A'} * 1_{A'}, 1_{2\cdot A'} \rangle_{L^2} \geq \exp[-C(K + d)(\log K)^6(\log Kd)] \cdot \mu_G(M)^2.$$  

This yields the desired estimate upon inserting the bounds from Proposition 6. \(\square\)

**Proof of Theorem 3.** Write $K = |A + A|/|A|$. If $A - A$ contains an element $x - y$ of order 2, we readily find a nontrivial, degenerate arithmetic progression $(x, y, x)$ in $A$. Otherwise, Proposition 7 tells us that $A$ possesses at least $e^{-CK(\log K)^{7}}|A|^2$ three-term arithmetic progressions, while the number of trivial ones is at most $|A|$. By the assumption on $K$, we are then ensured to find at least one nontrivial arithmetic progression in $A$. \(\square\)

**Proof of Corollary 1.** It suffices to insert the bounds of Theorem 3 in the proof of [19, Theorem 1.5] on pp. 230–231. \(\square\)

8. FROM SMALL DOUBLING TO LONG ARITHMETIC PROGRESSIONS

In this section we derive Theorem 5, basing ourselves on the approach of Croot et al. [4], which divides roughly into three steps. In the first step, one produces a large, structured set of almost periods of the convolution of the set $A$ under consideration with itself. The second step is to show, by a packing argument, that the set $A + A$ necessarily contains a translated copy of subset of this set of almost-periods of a certain size. The third step is to pick such a subset with basic additive structure, such as an arithmetic progression.

The original argument of [4] is based on Ruzsa’s modelling lemma [17], which has no efficient equivalent for general abelian groups, and therefore we need to use again a modelling approach based on the Bogolyubov-Ruzsa lemma of Sanders. In contrast
with the previous section however, we now need a version of this lemma that provides us with a containment conclusion, and for this we quote [22, Theorem 1.1].

**Proposition 8** (Containment Bogolyubov-Ruzsa lemma [22]). Let $K \geq 1$ be a parameter, and suppose that $A$ is a subset of $G$ such that $|A + A| \leq K|A|$. Then there exists a $d$-dimensional coset progression $M$ contained in $2A - 2A$ and such that

$$d \leq C(\log K)^6 \quad\text{and}\quad |M| \geq \exp\left[-C(\log K)(\log \log K)\right] \cdot |A|.$$

As noted in [22, Section 3], this version can be deduced from Proposition 6. The containment conclusion is sufficient in our situation, because the Croot-Sisask lemma works under a doubling hypothesis, whereas the iterative argument used in the proof of Roth’s theorem requires an assumption of density instead. Our reason for emphasizing this point is that the containment version above is easier to obtain than the correlation one, and is explained in depth in a survey by Sanders [23]. Although the type of structure obtained there is different, consisting of a convex coset progression instead, this would not affect our argument much since this object is also a Bourgain system, as can be seen from [23, Section 4].

We now proceed to the proof, starting with the following lemma which serves to collect together certain computations from [4] on $L^p$ and $L^{p/2}$ norms of convolutions.

**Lemma 12.** Let $p \geq 2$ and $K \geq 1$ be parameters. Suppose that $A$ is a subset of $G$ such that $|A + A| \leq K|A|$. Then

$$\mu_G(A + A)^{1/p} \leq K^{1/2}\|1_A * \mu_A\|_{p/2}^{1/2} \quad\text{and}\quad \|1_A * \mu_A\|_{p/2}^{1/2} \leq K^{1/2}\|1_A * \mu_A\|_p.$$

**Proof.** By Hölder’s inequality we have

$$\mu_G(A) = \mathbb{E}_G 1_A * \mu_A \leq \mu_G(A + A)^{1-2/p}\|1_A * \mu_A\|_{p/2},$$

from which the first estimate follows upon rearranging and taking square roots. To obtain the second, apply Cauchy-Schwarz and the first estimate in

$$\left[\mathbb{E}_G (1_A * \mu_A)^{p/2}\right]^2 \leq \mu_G(A + A)\|1_A * \mu_A\|_p^p \leq K^{p/2}\|1_A * \mu_A\|_{p/2}^{p/2}\|1_A * \mu_A\|_p^p.$$

The result follows upon taking $p$-th roots, then dividing both sides by $\|1_A * \mu_A\|_{p/2}^{1/2}$.

An important tool from [4] is a version of the Croot-Sisask lemma [5] that serves to smooth the convolution of two sets by an iterated convolution factor. The precise statement we need is a standard consequence of [4, Theorem 6.1]; an exposition of it by the author may be found in [12, Section 7].
Lemma 13 (Croot-Sisask $L^p$-smoothing). Let $K, L \geq 1$, $\theta \in (0, K^{-1/2}]$, $p \in 2\mathbb{N}$, $\ell \in \mathbb{N}$ be parameters. Suppose that $A, S, T$ are subsets of $G$ such that $|A + S| \leq K|A|$ and $|S + T| \leq L|S|$. Then there exists a subset $X$ of $T$ of size $|X| \geq (2L)^{-Cp^{\ell^2}/\theta^2}|T|$ such that

$$
\|1_A \ast \mu_S - 1_A \ast \mu_S \ast \lambda_X^{(\ell)}\|_p \leq \theta \|1_A \ast \mu_S\|_{p/2}^{1/2}
$$

where $\lambda_X = \mu_X \ast \mu_{-X}$.

As anticipated, our first step is to produce a set of almost-periods of the convolution of a small doubling set with itself. Following [4], this is done by first smoothing this convolution by the iterated convolution of a certain set $X$, with the difference that this set is now localized to a Bourgain system, which is taken to be a coset progression later on. Via the Fourier transform, any set annihilating the large spectrum of $X$ induces a set of almost-periods of the smoothed convolution, and via the results of Section 5, we may choose this annihilator to be a large Bourgain system. Here we make a small parenthesis on notation: throughout this section, $a \sim b$ stands for $b/2 \leq a \leq 2b$.

Proposition 9. Let $K \geq 1$ and $p \in 2\mathbb{N}$ be parameters. Suppose that $B$ is a regular Bourgain system and $A$ is a subset of $G$ such that $|A + A| \leq K|A|$ and $B \subset 2A - 2A$. Then there exist $m \geq 1$ and Bourgain systems $\overline{B}, \widetilde{B}$ such that $\widetilde{B}$ is $m$-controlled and

$$
\overline{B} = B_{c/(Kd^2m)} \wedge \tilde{B}_{c/K},
$$

$$
m \leq CpK(\log K)^3,
$$

and for every $x \in \overline{B}$,

$$
\|1_A \ast \mu_A - \tau_x 1_A \ast \mu_A\|_p \leq \frac{1}{2} \|1_A \ast \mu_A\|_p.
$$

Proof. First observe that, by the Plünnecke-Ruzsa-Petridis inequality [15],

$$
|A + B| \leq |3A - 2A| \leq K^5|A|,
$$

and therefore we may apply Lemma 13 with $(S, T) = (A, B)$ and $L = K^5$, for parameters $\theta$ and $\ell$ to be determined later. This yields a subset $X$ of $B$ of relative density $\tau$ such that

$$
\tau \geq \exp\left[-Cp^{\ell^2}/\theta^2 \log K\right],
$$

$$
\|1_A \ast \mu_A - 1_A \ast \mu_A \ast \lambda_X^{(\ell)}\|_p \leq \theta \|1_A \ast \mu_A\|_{p/2}^{1/2}.
$$

We write $I$ for the identity operator on functions, and given $x \in G$ we define the function $\hat{x} : \hat{G} \to G$ which maps $\gamma$ to $\gamma(x)$. Consider now an arbitrary element $x$ of $G$,
then by the triangle inequality and (8.2), we have
\[
\|(I - \tau_x)(A \ast \mu_A)\|_p \leq \|(I - \tau_x)(1_A \ast \mu_A - 1_A \ast \mu_A \ast \lambda_X)\|_p + \|1_{(A + A) \cup (A + A - x)} \cdot (I - \tau_x)1_A \ast \mu_A \ast \lambda_X\|_p
\]
\[
\leq 2\theta\|1_A \ast \mu_A\|_{\frac{1}{p/2}}^2 + 2\mu_G(A + A)^{1/p}\|(I - \tau_x)1_A \ast \mu_A \ast \lambda_X]\infty.
\]
By Parseval, we have further
\[
(8.3) \|(I - \tau_x)1_A \ast \mu_A\|_p \leq 2\theta\|1_A \ast \mu_A\|_{\frac{1}{p/2}}^2 + 2\mu_G(A + A)^{1/p}\sum_{\tilde{G}}|\tilde{I}_A||\hat{\mu}_A|\hat{\mu}_X|^{2}\|1 - \tilde{x}|.
\]
Invoking now Proposition 1 with a parameter \(\nu \in (0, 1]\), and recalling (8.1), we infer that \(\text{Spec}_{1/2}(\mu_X)\) is \(\nu\)-annihilated by \(\tilde{\mathcal{B}} = \mathcal{B}_{ca/\delta^m} \cap \tilde{\mathcal{B}}_\nu\), where \(\tilde{\mathcal{B}}\) is an \(m\)-controlled Bourgain system with \(m \leq C\ell^2\theta^2 - \log K\). From now on we restrict to \(x \in \mathcal{B}\), so that, by considering separately the summation over \(\text{Spec}_{1/2}(\mu_X)\) in (8.3), we obtain
\[
\|(I - \tau_x)1_A \ast \mu_A\|_p \leq 2\theta\|1_A \ast \mu_A\|_{\frac{1}{p/2}}^2 + 2(\nu + 2^{1-2\ell})\mu_G(A + A)^{1/p}\sum_{\tilde{G}}|\tilde{I}_A||\hat{\mu}_A|.
\]
By Parseval we know that \(\sum_{\tilde{G}}|\tilde{I}_A||\hat{\mu}_A| = 1\). Applying finally Lemma 12, we obtain
\[
\|(I - \tau_x)1_A \ast \mu_A\|_p \leq (2\theta + 2\nu K^{1/2} + 2^{2-2\ell}K^{1/2})\|1_A \ast \mu_A\|_{1/p}^{1/2}
\]
\[
\leq (2\theta + 2\nu K^{1/2} + 2^{2-2\ell}K^{1/2})K^{1/2}\|1_A \ast \mu_A\|_p.
\]
Choosing \(\theta = K^{-1/2}/8, \nu = K^{-1}/16\) and \(\ell \sim C\log K\), we obtain the desired \(L^p\)-estimate, and the bound on \(m\) follows by inserting the value of these parameters.

Secondly, we need the following packing argument which may be extracted from the computations of [4], but whose proof we include for completeness. In practice we specialize \(f\) below to \(1_A \ast \mu_A\) which has \(A + A\) as support.

**Lemma 14.** Let \(p \geq 2\) be a parameter. Suppose that \(f : G \rightarrow \mathbb{C}\) and \(R \subset G\) are such that, for all \(t \in R\),
\[
\|(I - \tau_t)f\|_p \leq \frac{1}{2}\|f\|_p.
\]
Then for every subset \(T\) of \(R\) of size \(|T| < 2^p\), there exists a translate \(x \in G\) such that \(x + T \subset \text{Supp}(f)\).

**Proof.** Given a subset \(T\) of \(R\), consider the quantity
\[
I := \sum_{t \in T}\|f - \tau_t f\|_p^p,
\]
so that by the assumptions of the lemma, we have at once \(I \leq |T| \cdot 2^{-p}\|f\|_p^p\).
Now assume for contradiction that for every $x \in G$, the translate $x + T$ is not contained in $\text{Supp}(f)$; then for every $x \in G$ we may find an element $t \in T$ such that $f(x + t) = 0$. Exchanging summations, this yields the lower bound

$$I = \mathbb{E}_G \sum_{t \in T} |f - \tau_t f|^p \geq \mathbb{E}_G |f|^p.$$

Combining both bounds on $I$, we obtain

$$\|f\|_p^p \leq |T| 2^{-p} |f|_p^p.$$

We obtain a contradiction if $|T| < 2^p$, and therefore we find a translated copy of $Y$ in the support of $f$ in that case. \hfill \Box

Last, we need an analog for Bourgain systems in abelian groups of the well-known fact, used in [4], that Bohr sets of $\mathbb{Z}_N$ of radius $\delta$ and dimension $d$ contain arithmetic progressions of length $\delta N^d$.

**Lemma 15.** Suppose that $B$ is a Bourgain system of dimension $d$ and $h \geq d$, and assume that $|B| \geq 2^h$. Then there exists a subset $T$ of $B$, which is either a proper arithmetic progression or a subgroup, of size $\frac{1}{4} |B|^{1/2h} \leq |T| \leq |B|^{1/2h}$.

**Proof.** Let $\eta = 2 |B|^{-1/2h} \in (0, 2^{-2}]$ so that, by Lemma 3, we have

$$|B_\eta| \geq \exp \left[ \log |B| - d \log \frac{2}{\eta} \right] \geq |B|^{1/2}.$$

Let $N = \lfloor \eta^{-1/2} \rfloor$, so that we have a sumset containment

$$N^2 B_\eta \subset B_{N^2 \eta} \subset B. \quad (8.4)$$

Since $\eta^{-1/2} \geq 2$, we have also $\frac{1}{2} \eta^{-1/2} \leq N \leq \eta^{-1/2}$.

We are now in one of two cases. In the first, there exists an element $x$ in $B_\eta$ of order $N$, thus the arithmetic progression $T = [0, N-1]_Z \cdot x$ is proper and, by (8.4), contained in $B$. Since $|T| = N$, we have also $\frac{1}{4} |B|^{1/2h} \leq |T| \leq |B|^{1/2h}$.

In the second case, every element of $B_\eta$ has order at most $N$. Since $|B_\eta| \geq |B|^{1/2} \geq N$, we may pick $N - 1$ distinct nonzero elements $x_1, \ldots, x_{N-1} \in B_\eta$ and consider the subgroup $T$ they generate, viz.

$$T = \langle x_1, \ldots, x_{N-1} \rangle_Z = [0, N-1]_Z \cdot x_1 + \cdots + [0, N-1]_Z \cdot x_{N-1}.$$

By (8.4) it follows again that $T$ is contained in $B$, and the size of $T$ satisfies

$$\frac{1}{4} |B|^{1/2h} \leq N \leq |T| \leq N^2 \leq |B|^{1/2h}. \hfill \Box$$
We are now ready to combine the previous propositions into a proof of Theorem 5.

Proof of Theorem 5. By Proposition 8, we may find a $d$-dimensional coset progression $M \subset 2A - 2A$ such that
\begin{equation}
    d \leq (\log K)^O(1) \quad \text{and} \quad |M| \geq \exp \left[ - (\log K)^O(1) \right] \cdot |A|.
\end{equation}

Up to dilating $M$ by a constant factor, which preserves the above bounds by Lemma 3, we may assume via Lemma 6 that $M$ induces a regular Bourgain system $\mathcal{M}$. By Lemma 2, that system also satisfies the dimension bound (8.5).

Applying now Proposition 9 with $B = M$ and a parameter $p \in 2\mathbb{N}$ to be determined later, we obtain Bourgain systems $\overline{B}, \tilde{B}$ such that
\begin{align}
    \overline{B} &= \mathcal{M}_{(1/2dpK)^O(1)} \wedge \tilde{B}_{c/K}, \\
    \tilde{d} &\leq CpK(\log K)^3, \\
    \tilde{b} &\geq \exp \left[ - CpK(\log pK)(\log K)^3 \right],
\end{align}

where we have unfolded Definition 8, and such that
\begin{equation}
    \| (I - \tau_x) 1_A * \mu_A \| \leq \frac{1}{2} \| 1_A * \mu_A \|_p \quad \text{for all} \quad x \in \overline{B}.
\end{equation}

Applying Lemma 4 to the intersection (8.6), and considering (8.5) and (8.7), we obtain
\begin{equation}
    \tilde{d} \ll (\log K)^O(1) + pK(\log K)^3 \ll pK(\log K)^3
\end{equation}

and we let $h = CpK(\log K)^3 \geq \tilde{d}$. By Lemmas 3 and 4, we also obtain
\begin{equation}
    \mu_G(\overline{B}) \geq \exp \left[ - C\tilde{d}(\log dpK) \right] \mu_G(M) \cdot \exp \left[ - C\tilde{d}\log K \right] \tilde{b}
\end{equation}

and therefore, by (8.5), (8.7) and (8.8),
\begin{equation}
    |\overline{B}| \geq \exp \left[ - CpK(\log pK)(\log K)^3 \right] \cdot |A|.
\end{equation}

Both the conditions $|\overline{B}| \geq |A|^{1/2}$ and $|\overline{B}| \geq 2^h$ are satisfied provided
\begin{equation}
    pK(\log pK)(\log K)^3 \leq c \log |A|.
\end{equation}

Considering that $\overline{B} \subset M \subset 2A - 2A$, we thus have a rough estimate $|A|^{1/2} \leq |\overline{B}| \leq |A|^4$. By Lemma 15, we may therefore find a subset $T$ of $\overline{B}$, which is either a proper arithmetic progression or a subgroup, of size bounded by
\begin{equation}
    \frac{1}{h} |A|^{1/8h} \leq \frac{1}{h} |\overline{B}|^{1/4h} \leq |T| \leq |\overline{B}|^{1/2h} \leq |A|^{2/h}.
\end{equation}
Recalling our choice $h = C p K (\log K)^3$ and (8.10), this shows that

$$|T| = \exp \left[ \Theta \left( \frac{\log |A|}{pK(\log K)^3} \right)^{1/2} \right].$$

The condition $|T| < 2^p$ is therefore satisfied if we choose

$$p \sim C \left( \frac{\log |A|}{K(\log K)^3} \right)^{1/2}.$$

It remains to check the conditions $p \geq 2$ and (8.10); these are seen to be satisfied for

$$K \leq \frac{c \log |A|}{(\log \log |A|)^5}$$

after a tedious, yet elementary computation. This yields the final size estimate

$$|T| = \exp \left[ \Theta \left( \frac{\log |A|}{K(\log K)^3} \right)^{1/2} \right]$$

and since we verified the conditions $|T| < 2^p$ and (8.9), an application of Lemma 14 with $f = 1_A * \mu_A$ and $R = \overline{B}$ concludes the proof. \hfill \Box

9. Remarks

In this section we collect together certain remarks of expository or exploratory nature which have not found their way into the main text.

We first wish to explain in more detail how Theorem 1 follows from the results of the literature. Consider a set of integers $A$ of doubling $K$, then for the purpose of finding arithmetic progressions in $A$, we may instead assume that $A$ is a subset of a cyclic group of odd order of density $\gg K^{-4}$ and doubling $K$, via a partial Freiman isomorphy [17]. Applying [19, Proposition 6.1] to $A$, one obtains a regular Bohr set of dimension $d \ll K \log K$ and density $b \geq \exp[-CK(\log K)^2]$, on which a certain translate of $A$ has density $\gg K^{-1}$. In that setting, Proposition 2 of this article is just [20, Theorem 1.1], initializing the iterative argument from there on a Bohr set instead of the whole group; there is no need to consider Bourgain systems or 2-torsion. Proposition 2 thus specialized shows that $A$ contains at least $\exp[-CK(\log K)^8] \cdot |A|^2$ three-term arithmetic progressions, and therefore at least one nontrivial progression for $K = |A + A|/|A|$ in the range specified by Theorem 1.

Secondly, we remark that the modelling argument used in Sections 7 and 8 could likely be adapted to other problems on dense sets, such as solving translation-invariant equations or finding long arithmetic progressions in $A + A + A$, to obtain a generalization of these results to the case of sets of small doubling in an arbitrary abelian group.
However, it is not clear to the author whether it is worth pursuing such generalizations, given the current lack of combinatorial applications of the kind of Corollary 1 for results of this type.

REFERENCES

1. M. Bateman and N. H. Katz, *New bounds on cap sets*, J. Amer. Math. Soc. 25 (2012), no. 2, 585–613.
2. J. Bourgain, *On arithmetic progressions in sums of sets of integers*, A tribute to Paul Erdős, Cambridge Univ. Press, Cambridge, 1990, pp. 105–109.
3. __________, *On triples in arithmetic progression*, Geom. Funct. Anal. 9 (1999), no. 5, 968–984.
4. E. Croot, I. Laba, and O. Sisask, *Arithmetic progressions in sumsets and $L^p$-almost-periodicity*, Combin. Probab. Comput. 22 (2013), no. 3, 351–365.
5. E. Croot and O. Sisask, *A probabilistic technique for finding almost-periods of convolutions*, Geom. Funct. Anal. 20 (2010), no. 6, 1367–1396.
6. B. Green, *Arithmetic progressions in sumsets*, Geom. Funct. Anal. 12 (2002), no. 3, 584–597.
7. B. Green and I. Z. Ruzsa, *Freiman’s theorem in an arbitrary abelian group*, J. Lond. Math. Soc. (2) 75 (2007), no. 1, 163–175.
8. B. Green and T. Sanders, *A quantitative version of the idempotent theorem in harmonic analysis*, Ann. of Math. (2) 168 (2008), no. 3, 1025–1054.
9. H. Hatami, *Fourier analysis of finite abelian groups*, lecture notes (2011), [http://www.cs.mcgill.ca/%7Ehatami/comp760/lectures2-3.pdf](http://www.cs.mcgill.ca/%7Ehatami/comp760/lectures2-3.pdf).
10. N. Hegyvári, F. Hennecart, and A. Plagne, *A proof of two Erdős’ conjectures on restricted addition and further results*, J. Reine Angew. Math. 560 (2003), 199–220.
11. K. Henriot, *Bourgain’s bounds for Roth’s theorem*, expository note (2013), [http://dms.umontreal.ca/~henriot/bourgainroth.pdf](http://dms.umontreal.ca/~henriot/bourgainroth.pdf).
12. __________, *Notes on the Croot-Sisask lemma*, expository note (2013), [http://dms.umontreal.ca/~henriot/almostp.pdf](http://dms.umontreal.ca/~henriot/almostp.pdf).
13. V. F. Lev, *Progression-free sets in finite abelian groups*, J. Number Theory 104 (2004), no. 1, 162–169.
14. N. Lyall, *Behrend’s example*, expository note (2005), [http://www.math.uga.edu/%7Elyall/REU/Behrend.pdf](http://www.math.uga.edu/%7Elyall/REU/Behrend.pdf).
15. G. Petridis, *New proofs of Plünnecke-type estimates for product sets in groups*, Combinatorica 32 (2012), no. 6, 721–733.
16. K. F. Roth, *On certain sets of integers*, J. London Math. Soc. 28 (1953), no. 1, 104–109.
17. I. Z. Ruzsa, *Arithmetical progressions and the number of sums*, Period. Math. Hungar. 25 (1992), no. 1, 105–111.
18. T. Sanders, *Roth’s theorem in $\mathbb{Z}_n^2$*, Anal. PDE 2 (2009), no. 2, 211–234.
19. __________, *Three-term arithmetic progressions and sumsets*, Proc. Edinb. Math. Soc. (2) 52 (2009), no. 1, 211–233.
20. __________, *On Roth’s theorem on progressions*, Ann. of Math. (2) 174 (2011), no. 1, 619–636.
21. __________, *On certain other sets of integers*, J. Anal. Math. 116 (2012), no. 1, 53–82.
22. ______., *On the Bogolyubov-Ruzsa lemma*, Anal. PDE 5 (2012), no. 3, 627–655.
23. ______., *The structure theory of set addition revisited*, Bull. Amer. Math. Soc. (N.S.) 50 (2013), no. 1, 93–127.
24. T. Schoen, *The cardinality of restricted sumsets*, J. Number Theory 96 (2002), no. 1, 48–54.
25. J. Solymosi, *Arithmetic progressions in sets with small sumsets*, Combin. Probab. Comput. 15 (2006), no. 4, 597–603.
26. Y. V. Stanchescu, *Planar sets containing no three collinear points and non-averaging sets of integers*, Discrete Math. 256 (2002), no. 1-2, 387–395.
27. T. Tao and V. H. Vu, *Additive combinatorics*, Cambridge Studies in Advanced Mathematics, vol. 105.

Département de mathématiques et de statistique, Université de Montréal, CP 6128 succ. Centre-Ville, Montréal QC H3C 3J7, Canada

*Email address: henriot@dms.umontreal.ca*