H-infinity Sliding Mode Controller Design for a Human Swing Leg System

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Abstract
In this paper, the H-infinity Sliding Mode Control (HSMC) is designed to produce a new dynamic output feedback controller for trajectory tracking of the nonlinear human swing leg system. The human swing leg system represents the support of human leg or the humanoid robot leg which is usually modeled as a double pendulum. The thigh and shank of a human leg is represented by two pendulum links and the hip joint will connect the upper body to the thigh and the knee joint will connect the thigh to the shank. The external torques (servo motors) are applied at the hip and knee joints to move the muscles of thigh and shank. The results show that the HSMC can robustly stabilize the system and achieve a desirable time response specification better than if only H-infinity or SMC is used. This controller achieves the following specifications: \( t_s = 0.85 \text{ sec} \), \( e_{ss} = 0.01\text{ deg} \) for hip joint and \( t_s = 0.75 \text{ sec} \), \( e_{ss} = 0.02\text{ deg} \) for knee joint.

Keywords: Human Swing Leg System, Full State Feedback Controller, H-infinity, Sliding Mode Control (SMC), Uncertain System.

1. Introduction
The ability of human to move from one place to another is called Human movement. Walking, jogging, and running gaits are considered the parts of Human movement. Walking is one of the main gaits of the human movement and running gaits are considered the parts of Human movement. Walking, jogging, and running movements are often called Human movement. Due to neurological injuries, such as spinal cord injury and stroke, which result in incomplete gait; there are a vast number of people who lose their walking ability [2].

In the last three decades, many researches have been done to control this system using different control approaches. Ono et al. [3] and Huang et al. [4], designed P controller for human swing leg system and the lost energy in joint self-impact stopper was obtained. A torque has been applied to the hip joint to restore the lost energy. Dallali et al. [5], presented a comparison between PID and Linear Quadratic Regulator (LQR) controllers. A better robustness was obtained from the LQR controller. The test was done on the robot leg from \(-11.5^\circ\) to \(11.5^\circ\) and obtained control action about 30 N.m. Gregg et al. [6], implemented virtual constraints that unify the stance period, coordinate ankle and knee control on a powered prosthetic leg. To simulate the torque limit of the experimental prosthesis, the saturate prosthesis, and obtained a torque has been applied to the hip joint to restore the lost energy. Dallali et al. [5], presented a comparison between PID and Linear Quadratic Regulator (LQR) controllers. A better robustness was obtained from the LQR controller. The test was done on the robot leg from \(-11.5^\circ\) to \(11.5^\circ\) and obtained control action about 30 N.m. Gregg et al. [6], implemented virtual constraints that unify the stance period, coordinate ankle and knee control on a powered prosthetic leg. To simulate the torque limit of the experimental prosthesis, the saturate prosthesis.
with maximum error of about 0.15% for the hip joint and 0.35% for the knee joint.
The H-infinity and Sliding Mode Control (SMC) techniques are used in the control theory to achieve stabilization and robust performance. The advantages of the H-infinity method over classical control methods which are applicable to problems with cross-coupling between channels in multivariable systems. Disadvantages of this method need a high level of mathematical understanding to apply it successfully and a reasonably good model of the system to be controlled [8]. In the SMC, the sliding surface is chosen to force the system state trajectory to move along it. The sliding surface is always chosen in such a way that led a control law guarantees to be stable [9]. SMC has been applied for uncertain systems, stochastic systems, time delay systems, and switched hybrid systems [10].

In this work, H-infinity, Sliding Mode Control and H-infinity Sliding Mode Control are the proposed controllers for the system. The main goal is to design the H-infinity Sliding Mode controller to stabilize the human swing leg system and achieve a desirable tracking.

2. System Mathematical Model

In human and humanoid, the motion of the hip and knee is achieved in the swing leg system while the ankle contribution can be neglected. The system is modeled as double pendulum with the thigh and shank represented as two links. The unconstrained double pendulum is shown in Fig.1. In this figure, the masses of thigh and shank are $m_1, m_2$ respectively. The lengths of thigh and shank are $l_1, l_2$ respectively. The hip and knee joint angles are $\theta_1$ and $\theta_2$ respectively. The applied external torques are $\tau_1$ and $\tau_2$ to move the thigh and shank links [7, 11, 12].

![Figure (1): Schematics human swing leg](image)

The dynamic equations of the system using Lagrange’s method can be represented by [7]:

$$\ddot{\theta}_1 = K_1 (r_2 + l_2 \sin \theta_2 - r_3 \sin \theta_1) + K_2 (r_2 \cos \theta_2 - K_2 \cos \theta_1 - \theta_1) - K_2 \cos \theta_2 - K_2 \cos \theta_1 - K_2 \cos \theta_1 + K_2 \cos \theta_2)$$

where $K_1 = (m_1 + m_2)l_1^2$, $K_2 = m_l^2 l_2^2$, $K_3 = (m_1 + 2m_2)g$. Assume the state variables [11]:

$x_1 = \theta_1$: angular position of the upper link.
$x_2 = \theta_2$: angular position of the lower link.
$x_3 = \dot{\theta}_1$: angular velocity of the upper link.
$x_4 = \dot{\theta}_2$: angular velocity of the lower link.

so that

$$\dot{x}_1 = x_3$$

$$\dot{x}_2 = x_4$$

By linearizing eq. (5) to eq. (8) using Jacobians’ method with the following initial condition: $(\theta_1, \theta_2) = (30^\circ, 10^\circ)$, $(\dot{\theta}_1, \dot{\theta}_2) = (0.1, 0.25) \text{ rad/sec}$, with $(\tau_1, \tau_2) = (0.5, 0.3) \text{ N.m}$, and the resulting state space representation for the system is:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where $A, B, C$ and $D$ are obtained as:

$$A =$$

$$0 \quad 0 \quad 1 \quad 0$$

$$0 \quad 0 \quad 0 \quad 1$$

$$B =$$

$$0 \quad 0 \quad 0 \quad 0$$

$$0 \quad 0 \quad 0 \quad 0$$

$$C = [1 \quad 0 \quad 0 \quad 0]$$

$$D = [0 \quad 0 \quad 0 \quad 0]$$

The parameters of human swing leg system are given in Table 1.

![Table 1: The parameters of human swing leg model](image)

Consider the system with uncertainty and external disturbance with matching condition is described as:

$$\dot{x} = (A + \Delta A)x + (B + \Delta B)u + d$$

where $x \in R^n$, $A$ and $\Delta A \in R^{n \times n}$, $B$ and $\Delta B \in R^{n \times m}$ and $u \in R^m$. The matrices $\Delta A$ and $\Delta B$ represent the uncertainties in the matrices $A$ and $B$ respectively and $d$ refers to external disturbance. Assume the perturbations are about $\pm 10\%$ of system parameters i.e. $m_1, m_2 = 0.1 (1 \pm 10\%)$ and $l_1, l_2 = 0.55 (1 \pm 10\%)$.
substituting eq. (9) in the resulting equation, yields [11, 17]:
\[ \dot{S} = F(Ax + Bu) \] (18)
The sliding mode control law is:
\[ u = u_o + u_s \] (19)
where \( u_o \) is the nominal control and \( u_s \) is discontinuous control input if the proposed controller is Sliding Mode Control only, but \( u_o \) will be designed based on H-infinity Control if the proposed controller is H-infinity Sliding Mode Control.
The vector \( S(t) = 0 \) represents the intersection of all \( m \) sliding hyperplanes passing through the origin of the state space. The \( F \) matrix is computed by minimizing a quadratic objective function involving the state vector and the effective input, that is, by solving a linear quadratic (LQ) problem. This approach is called the Optimal Sliding Mode controller where the objective function to be minimized is [16]:
\[ J = \int_0^\infty (x^T Q x) \, dt \] (20)
For the LQ control, it is necessary to have the input term in the quadratic objective function and the input term is not presented in the objective function in eq. (20), and the constraints are that the system is on the intersection of \( m \) sliding hyperplanes. The \( F \) matrix is not specified a priori and will come out as a solution to the problem. Using the similarity transformation [11, 16]:
\[ q = H x \] (21)
where \( q \in \mathbb{R}^n \) represents the transformed state. The \( H \in \mathbb{R}^n \) represents the transformation matrix is:
\[ H = \begin{bmatrix} n & B \end{bmatrix}^T \] (22)
where the matrix \( n \in \mathbb{R}^{n \times (n-m)} \) are the columns of this matrix are composed of basis vectors of the null space of \( B^T \). Differentiating eq. (21) and substituting eq. (15) and eq. (21) in the resulting equation, yields:
\[ \dot{q} = A q + B u + \Delta A q + \Delta B u + \epsilon \] (23)
where
\[ A = H A H^{-1}, \Delta A = H B A \delta H^{-1} = \begin{bmatrix} 0 & 0 \\ \Delta A_{21} & \Delta A_{22} \end{bmatrix} \]
\[ B = H B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \Delta B = H B B \delta = \begin{bmatrix} 0 \\ \Delta B_2 \end{bmatrix} \] and \( \epsilon = H B \delta \) (24)
The matrix \( H \) has a special structure, the first \((n-m)\) rows of \( B \) turn out to be zeroes and the vector \( q \) is decomposed as follows:
\[ q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \] (25)
where \( q_1 \) and \( q_2 \) are \((n-m)\) and \( m \)-dimensional vectors, respectively. Partitioning eq. (23), yields:
\[ \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} A_{21} & A_{22} \\ \Delta A_{21} & \Delta A_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u + \begin{bmatrix} 0 \\ \Delta B_2 \end{bmatrix} \] (26)
Substituting eq. (21) in eq. (20), yields:
\[ J = \int_0^\infty (q^T Q q) \, dt \] (27)
where,
\[ Q_q = (H^{-1})^T Q H^{-1} \] (28)
If \( Q = Q^T \geq 0 \), then \( Q_q = Q_q^T \geq 0 \); because the signs of eigenvalues are preserved under congruence transformation. Partitioning \( Q_q \) to form the partition of \( q \) in eq. (28) as:
\[ Q_q = \begin{bmatrix} Q_r & N \\ N^T & R \end{bmatrix} \]  
(29)

Substituting eq. (25) and eq. (29) in eq. (27), yields:
\[ J = \int_0^\infty (q_1^T Q_r q_1 + 2q_1^T N q_2 + q_2^T R q_2) \, dt \]  
(30)

The matrix Q is selected by trial and error to be:
\[ Q = 10^5 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.0018 & 0.0007 \\ 0 & 0 & 0.0007 & 0.0003 \end{bmatrix} \]  
(31)

From the selected Q matrix and according to eq. (28) and eq. (29) the matrices \( Q_q, Q_r, R \) and \( N \) are obtained as follows:
\[ Q_q = 10^5 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.0018 & 0.0007 \\ 0 & 0 & 0.0007 & 0.0003 \end{bmatrix} \]  
(31)

\[ Q_r = 10^5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \]

\[ R = \begin{bmatrix} 182.9 & 71.7 \\ 71.7 & 30.4 \end{bmatrix} \]

\[ N = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \]  
(32)

for \( S = 0 \), the \((n-m)\) dimensional dynamics is represented by [16]:
\[ \dot{q}_1 = \mathcal{A}_{11} q_1 + \mathcal{A}_{12} q_1 \]  
(33)

Then sliding hyperplanes can be described as:
\[ S = K q_1 + q_2 \]  
(34)

where the gain matrix \( K \) is obtained for the minimum value of \( J \):
\[ K = R^{-1} (\mathcal{A}_{12}^T P_2 + N^T) \]  
(35)

where \( P_2 \) matrix is found by the solution of the Riccati equation:
\[ P_2 (\mathcal{A}_{11} - \mathcal{A}_{12} R^{-1} N^T) + (\mathcal{A}_{11} - \mathcal{A}_{12} R^{-1} N^T) P_2 - P_2 \mathcal{A}_{12} R^{-1} \mathcal{A}_{12}^T P_2 + Q_q - NR^{-1} N^T = 0 \]  
(36)

where
\[ P_2 = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \]  
(37)

The gain matrix \( K \) becomes:
\[ K = \begin{bmatrix} 0.2813 & 0.8035 \\ -1.2053 & -1.7065 \end{bmatrix} \]  
(38)

From eq. (17) and eq. (34), \( F \) matrix is:
\[ F = \begin{bmatrix} K & I_n \end{bmatrix} H = \begin{bmatrix} 0.4926 & -0.6944 & 0.4926 & -0.6944 \\ -0.6944 & 1.9705 & -0.6944 & 1.9705 \end{bmatrix} \]  
(39)

The selection of matrix \( F \) or matrix \( K \) is to ensure the system stability on the intersection of all hyperplanes. The standard LQ problem provided \( R > 0 \) and if \( Q \) is selected to be positive definite, \( R \) will be positive definite [16].

In SMC, \( u_o \) is conducted to satisfy the desired specification for nominal system corresponds to \( \dot{S} = 0 \), from eq. (18), eq. (19) becomes:
\[ u_o = -(FB)^{-1} FA x \]  
(40)

The uncertain parameters in human swing leg system are represented as a structured parameter uncertainty. This problem can be defined by the configuration in Fig.3. The \( G \) is the plant and \( d(t) \) is the disturbance, whose effect on the output signal to be minimize. The input \( u_o(t) \) is the control signal which used to achieve this goal. The output \( e(t) \) represents the signal to be minimized and the output \( y(t) \) represents the system states which are available for feedback [17-19].

\[ d \]
\[ u_o \]
\[ G \]
\[ e \]
\[ K_\infty \]
\[ y = x \]

**Figure (3):** A full state feedback H-infinity control structure.

Therefore, the full state feedback H-infinity controller will be expressed as [8, 16]:
\[ \dot{x} = A x + B u_o + B d \]  
(41)

\[ e = C_x x + D_1 u_o \]  
(42)

\[ y = x \]  
(43)

where \( d = (A_1 x + B_1 u_o + \delta) \) is the disturbance matrix, and \( C_1, D_1 \) is the controller design matrix. The solution of the corresponding H-infinity problem based on Riccati equation requires the following assumptions to be satisfied [8, 16]:
- \((A, B)\) is stabilizable, \((C, A)\) is detectable.
- \( C_1^T D_1 = 0 \) and \( D_1^T D_1 = I \).

The H-infinity norm of the closed-loop transfer function between error and disturbance \( T_{ed} \) should be less than a given value of \( \gamma \):
\[ \| T_{ed}(s) \|_\infty < \gamma \]  
(44)

\[ d \] tries to maximize the cost function \( J \) while the control signal \( u_o \) tries to minimize it. The physical meaning of the relation given above competes to each other within infimum (inf) and supremum (sup). This problem of inf – sup optimization can be written as [16]:
\[ \inf_d \sup_{u_o} d J(u_o, d) < \infty \]  
(45)

and
\[ J(e, d) = \int_0^\infty (e^T e - \gamma^2 d^T d) \, dt \]  
(46)

assume that \( d \) and \( u_o \) have the following structures:
\[ d = K_d x \]  
(47)

and
\[ u_o = -K_\infty x \]  
(48)

where \( K_d \) and \( K_\infty \) represent the feedback gains for disturbance and controller. Substituting eq. (47) and eq. (48) in eq. (42), yields:
\[ e = (C_1 - D_1 K_\infty) x \]  
(49)

therefore
\[ e^T e = x^T (C_1^T C_1 + K_\infty^T K_\infty) x \]  
(50)

and
\[ d^T d = (K_d x)^T (K_d x) = x^T (K_d^T K_d) x \]  
(51)

Substituting eq. (50) and eq. (51) in eq. (45), gives:
\[ J(u_o, d) = \int_0^\infty x^T (C_1^T C_1 + K_\infty^T K_\infty - \gamma^2 K_d^T K_d) x \, dt \]  
(52)

and substituting eq. (47) and eq. (48) in eq. (41), yields:
\[ \dot{x} = (A - B K_\infty + B K_d) x \]  
(53)

where
\[ K_0 = B^T P_1, \quad K_d = \frac{1}{r^2} B^T P_1 \] (54)

The Riccati equation required to solve this optimal problem is:

\[ P_1 A + A^T P_1 + Q_1 - P_1 B \left( I - \frac{1}{r^2} \right) B^T P_1 = 0 \] (55)

where \( P_1 \) is a positive semi-definite symmetric matrix, and \( Q_1 = C_1^T C_1 \). By trial and error \( Q_1 \) matrix is selected as:

\[ Q_1 = \begin{bmatrix} 400 & 0 & 0 & 0 \\ 0 & 300 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 20 & 0 & 0 & 0 \end{bmatrix} \] (56)

\[ C_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 20 & 0 & 0 & 0 \\ 0 & 17.3205 & 0 & 0 \\ 0 & 0 & 3.1633 & 0 \end{bmatrix} \]

The values of \( P_1, K_c \) and \( K_d \) of the system are:

\[ P_1 = \begin{bmatrix} 71.1377 & 2.3911 & 1.3238 & 0.4643 \\ 2.3911 & 56.3147 & 0.3972 & 0.2880 \\ 1.3238 & 0.3972 & 0.2491 & 0.0888 \\ 0.4643 & 0.2880 & 0.0888 & 0.0596 \end{bmatrix} \]

\[ K_c = \begin{bmatrix} -32.9699 & -0.4319 & 6.1050 & 0.2348 \\ -0.4319 & 29.1696 & 0.1993 & 5.5801 \\ 6.1050 & 0.1993 & 4.0353 & 0.1552 \\ 0.2348 & 5.5801 & 0.1552 & 3.6883 \end{bmatrix} \]

By trial and error \( \gamma \) is selected to be:

\[ \gamma = 1.23 \] (58)

To compute the discrete controller, eq. (48) is substituted in eq. (15) to give:

\[ \dot{x} = (A - B K_\alpha) x + B u_s + B \{(A_\delta - B_\delta K_\alpha) x + B_\delta u_s + \delta \} \] (59)

In the reaching phase, assuming that the closed-loop system has no finite-escape time, the switching surface is guaranteed reached in finite time by control law despite the disturbance or uncertainty. The sliding motion is completely independent of the uncertainty [20]. For human swing leg system, the suitable \( u_s \) is represented as a unit control signal. This leads to what is called "unity sliding mode control". For HSMC, \( u_s \) is

\[ u_s \to -\sigma(\|x\|) \frac{\nu T \nu}{\nu T \nu} \] (60)

for SMC, \( u_s \) is:

\[ u_s \to -(FB)^{-1} \sigma_{SMC}(\|x\|) \frac{\nu T \nu}{\nu T \nu} \] (61)

The Lyapunov function with a positive definite matrix \( P \) is:

\[ V = x^T P x = S^T S = x^T F^T P x \] (62)

where \( P = F^T F \), differentiating \( V \) with respect to time, such that \( V \leq 0, \forall x \neq 0 \):

\[ \dot{V} = V V^T \dot{x} = 2 S^T \dot{S} = 2 x^T F^T F \dot{x} \] (63)

where \( V V^T = 2 x^T F^T F = 2 x^T P \), then substituting eq. (54) in eq. (63), yields:

\[ \dot{V} = V V^T (A - B K_\alpha) x + V V^T B u_s + V V^T B ((A_\delta - B_\delta K_\alpha) x + B_\delta u_s + \delta) \] (64)

where

\[ V V^T (A - B K_\alpha) x = 2 x^T P (A - B K_\alpha) x = -x^T Q_{HSMC} x \leq 0 \] (65)

where

\[ P = \begin{bmatrix} 0.7248 & -1.7103 & 0.7248 & -1.7103 \\ -1.7103 & 4.3649 & -1.7103 & 4.3649 \\ 0.7248 & -1.7103 & 0.7248 & -1.7103 \\ -1.7103 & 4.3649 & -1.7103 & 4.3649 \end{bmatrix} \]

\[ Q_{HSMC} = \begin{bmatrix} 1.0608 & -2.2816 & 0.1733 & 0.4246 \\ -2.6606 & 5.7658 & -0.4342 & 1.0731 \\ 1.0608 & -2.2816 & 0.1733 & -1.4246 \\ -2.6606 & 5.7658 & -0.4342 & 1.0731 \end{bmatrix} \] (66)

the \( A_\delta, B_\delta \) and \( \delta \) are bounded, so

\[ \| A_\delta - B_\delta K_\alpha \| \leq \alpha \| B_\delta \| \leq \beta, \| \delta \| \leq \varepsilon \] (67)

and

\[ \| A_\delta - B_\delta K_\alpha \| x + B_\delta u_s + \delta \leq \alpha \| x \| + \beta \sigma(\|x\|) + \varepsilon \] (68)

where \( \alpha, \beta \) and \( \varepsilon \) are positive constant, substituting eq. (65), (67) and (68) in eq. (64), yields:

\[ \dot{V} \leq -x^T Q_{HSMC} x - \sigma(\|x\|) \| B^T \nu \nu \| + \| B^T \nu \nu \| (\alpha \| x \| + \beta \sigma(\|x\|) + \varepsilon) \leq 0 \] (69)

where \( \{ \sigma(\|x\|) (1 - \beta) - \alpha \sigma(\|x\|) - \varepsilon \geq 0 \) so that \( \sigma(\|x\|) (1 - \beta) - \alpha \sigma(\|x\|) - \varepsilon \geq 0 \)

where \( k > 0 \) is scalar gain to replace the inequality from control equation.

Let \( k = 0.1 \), so that \( \alpha = 8.3988, \beta = 0.2487, \varepsilon = 0.1414, \sigma(\|x\|) = 5.75 \) (71)

For SMC, to compute the discontinuous controller, eq. (65) becomes:

\[ \dot{V} = V V^T (A - B (FB)^{-1} FA) x + B u_s + B ((A_\delta - B_\delta (FB)^{-1} FA) x + B_\delta u_s + \delta) = V V^T (A - B (FB)^{-1} FA) x + V V^T B u_s + V V^T B ((A_\delta - B_\delta (FB)^{-1} FA) x + B_\delta u_s + \delta) \] (72)

Let

\[ V V^T (A - B (FB)^{-1} FA) x = 2 x^T P (A - B (FB)^{-1} FA) \] (73)

where \( Q_{SMC} = 2 P (A - B (FB)^{-1} FA) \) equals to:

\[ Q_{SMC} = \begin{bmatrix} 0.0931 & -0.0703 & 0.0007 & 0.0016 \\ -0.2267 & 0.1800 & -0.0026 & -0.0044 \\ 0.0931 & -0.0703 & 0.0009 & 0.0016 \\ -0.2267 & 0.1800 & -0.0026 & -0.0026 \end{bmatrix} \] (74)

\( A_\delta, B_\delta \) and \( \delta \) are bounded as:

\[ \| A_\delta - B_\delta (FB)^{-1} FA \| \leq \alpha_{SMC}, \| B_\delta \| \leq \beta, \| \delta \| \leq \varepsilon \] (75)

and

\[ \| A_\delta - B_\delta (FB)^{-1} FA \| x + B_\delta u_s + \delta \leq \alpha_{SMC} \| x \| + \beta \sigma_{SMC}(\|x\|) + \varepsilon \] (76)

where \( \alpha_{SMC}, \beta \) and \( \varepsilon \) are positive constants and \( \| u_s \| \leq \sigma_{SMC}(\|x\|) \).

Substituting eq. (73), eq. (75) and eq. (76) in eq. (72), yields:
\[
\dot{V} \leq -x^TQ_{SMC}x - \sigma_{SMC}(\|x\|)B^TPV + \|B^TPV\| \{\sigma_{SMC}(\|x\|) + \beta \sigma_{SMC}(\|x\|) + \varepsilon\} \leq -\|B^TPV\| \{\sigma_{SMC}(\|x\|)(1 - \beta) - \alpha_{SMC}\|x\| - \varepsilon\} \leq 0 \quad (77)
\]

The \{\sigma_{SMC}(\|x\|)(1 - \beta) - \alpha_{SMC}\|x\| - \varepsilon\} \geq 0, so that
\[
\sigma_{SMC}(\|x\|) = \frac{1}{1 - \beta} [\alpha_{SMC}\|x\| + k + \varepsilon] \quad (78)
\]

Let \(k = 0.1\), and then \(\sigma_{SMC} = 18.675, \beta = 0.2487, \varepsilon = 0.1414\),
\[
(\beta)_{\sigma_{SMC}} = \begin{bmatrix} 0.0928 & 0.0363 \\ \end{bmatrix}, \quad \sigma_{SMC}(\|x\|) = 50.72, \quad (\beta)_{\sigma_{SMC}} = \begin{bmatrix} 0.0928 & 0.0363 \\ \end{bmatrix}
\]

so that
\[
\dot{V} \leq -k\|B^TPV\|, \forall \|x\| \neq 0 \quad (79)
\]

Eq. (79) implies that in a finite time, the trajectory reaches the sliding surface and remains on the sliding surface for all future time. Note that the reaching time is related to the magnitude of the gain \(\sigma_{SMC}(\|x\|)\) for SMC or \(\sigma(\|x\|)\) for HSMC directly, and the unit vector term \(\|B^TPV\|\) of the control law is effective on the finite reaching time [15, 20]. The SMC law for the system dynamics in eq. (19) is:
\[
u = -[\beta]_{\sigma_{SMC}(\|x\|)}B^TPV \quad (80)
\]

and for HSMC is:
\[
u = -K_{\sigma_{SMC}(\|x\|)}B^TPV \quad (81)
\]

The SMC has a chattering problem, which is a very high frequency oscillation of the sliding variable around the sliding manifold. The chattering is undesirable for the real systems and actuator. Many approaches have been projected to overcome the chattering phenomenon. In this work, to eliminate the chattering phenomena a small positive value (\(\mu\)) is added to denominator of the unit vector to approximate the discontinuous controller \(u_0\) as [20]:
\[
u = B^TPV \quad (82)
\]

\[
\frac{\|B^TPV\|}{\|B^TPV\|+\mu}
\]

where \(\mu\) is a positive value. For \(S \neq 0\), it can be observed that point wise, when \(\mu\) approaches to 0 then:
\[
\lim_{\mu \rightarrow 0} \frac{\|B^TPV\|}{\|B^TPV\|+\mu} = \frac{\|B^TPV\|}{\|B^TPV\|} \quad (83)
\]

The control law for SMC becomes
\[
u = u_0 - (\beta)_{\sigma_{SMC}(\|x\|)}B^TPV \quad (84)
\]

and the control law for HSMC becomes
\[
u = u_0 - \sigma_{HSMC}(\|x\|)B^TPV \quad (85)
\]

In this work, the suitable value of \(\mu\) is selected by trial and error to be 0.2 for the full state feedback SMC and 0.1 for the full state feedback HSMC. These selected values for both controllers made the sliding surface and control action very smooth trajectories.

4. Results and Discussion

Fig.4 illustrate the behavior of the human swing leg system without controller and the system’s eigenvalues are \(\{0.0463 \pm 5.4358i, -0.0643 \pm 3.6555i\}\). It means that the system is unstable because it has roots in the right hand side of \(s\)-plane. That is, the design of a suitable controller to stabilize the system and achieve the desirable specifications is required.

Figure (4): Time response of human swing leg system for hip and knee joints.

A comparison among the three designed controllers to specify the best robust controller for the system depending on the time response specifications, the coupling effect the control action, and the robustness is presented.

Fig.5 shows the tracking case of the nonlinear human swing leg system with H-infinity controller by the Simulink MATLAB program.

Fig.6 represents the Simulink diagram in MATLAB for the nonlinear human swing leg system which can use it for full state feedback SMC and HSMC controller with necessary modification.

Figure (5): Simulink diagram of the nonlinear human swing leg system with H-infinity controller.
Figure (6): Simulink diagram of the nonlinear human swing leg system SMC or HSMC controller.

Fig. 7 represents the time response of angular position and control actions for hip and knee joints for the system using the three proposed controllers with initial conditions \(30^\circ, 10^\circ, (0.3, 0.25) \text{ rad/sec}\). The state trajectories obtained using HSMC approaches the equilibrium state faster than using H-inf or SMC approaches. Also, HSMC has given better performance rather than if only one of them is used with Low control effort.

Figure (7): The states trajectories of the nonlinear human swing leg system using H-infinity controller, SMC and HSMC.

For tracking, assume the desired trajectory that the system will be follow is described as:

\[
\begin{align*}
\theta_{\text{hip desired}} &= 4 \sin 0.5t + 3 \sin t + \sin 1.5t + 20, \\
\theta_{\text{knee desired}} &= 0.5 \times \theta_{\text{hip desired}}
\end{align*}
\]  

Fig. 8 represents the time response of angular positions and control actions for hip and knee joints of the system using the three proposed controllers for specific trajectory. It can be noticed that HSMC is better and faster to follow the desired trajectory than H-infinity or SMC controller with Low control effort.

Figure (8): Tracking properties for the nonlinear human swing leg system using H-infinity controller, SMC and HSMC.

Fig. 9 represents states trajectories of nonlinear system when applying external disturbance for hip and knee joints to show the ability of the three proposed controllers.
controllers to overcome external disturbance. The disturbance signal which applied on the system is about 10% from the desired trajectory and it will be applied at specific time period (i.e. $t_{low}, t_{high}$). The disturbance equation describes as:

$$\theta_{\text{hip disturbance}} = \begin{cases} 
0 & 0 \leq t < t_{low} \\
0.4 \sin 0.5t + 0.3 \sin t + 0.1 \sin 1.5t + 2 & t_{low} \leq t \leq t_{high} \\
0 & t > t_{high}
\end{cases}$$

$$\theta_{\text{knee disturbance}} = 0.5 \times \theta_{\text{hip disturbance}} \quad (87)$$

It is noticed that SMC and HSMC can overcome the external disturbance better than H-infinity with Low control effort.

![Angular position for hip and knee respectively.](image)

![Control action for hip and knee respectively.](image)

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**Figure (9):** Disturbance rejection properties on state trajectories of the nonlinear human swing leg system using H-infinity controller, SMC and HSMC.

Fig.10 shows the system position trajectories of the system with a perturbation of $\pm 10\%$ in parameters of the system. It is apparent that the full state feedback H-infinity controller and SMC can effectively compensate the system parameters perturbation without eliminating the steady state error. The HSMC can effectively compensate the system parameters perturbation and achieve the required robustness for the system with a desirable time response better than H-infinity and SMC when used separately.

![Time response of the nonlinear system with $\pm 10\%$ perturbation in system parameters.](image)
Fig. 11 represents the states trajectories of nonlinear system when applying step inputs for hip and knee joints to show the ability of the three proposed controllers to overcome the cross coupling problem. It is noticed that H-infinity and HSMC can overcome this problem better than SMC. The trajectories of state using HSMC approaches to the steady state faster than the trajectories of state obtained using H-infinity or SMC approaches.

\[ t_s = 3.3 \text{ sec with } e_{ss} = 0.02^\circ \text{ for hip joint and } t_s = 3.1 \text{ sec with } e_{ss} = 0.04^\circ \text{ for knee joint. In HSMC, the } t_s = 0.85 \text{ sec with } e_{ss} = 0.01^\circ \text{ for hip joint and } t_s = 0.75 \text{ sec with } e_{ss} = 0.02^\circ \text{ for knee joint. So that, HSMC has given better performance than if only one of them is used.} \]

5. Conclusions

In this paper, H-infinity, SMC and HSMC were proposed to achieve trajectory tracking and stabilization of the uncertain and nonlinear human swing leg system. The results showed that the effectiveness and the applicability of the proposed controller. The design of the robust control using H-infinity and SMC for this nonlinear and uncertain systems was necessary. Although the H-infinity controller improves the transient response of the system and compensates the cross coupling effect, there was still error. The SMC can effectively attenuates the error but the resulting transient response was not desirable in addition to the high effect of the cross coupling. The design using the HSMC was very efficient and it has given a response better than if only one of them was used. The superiority of the HSMC was because it can achieve the design requirements that arise from both H-infinity and SMC simultaneously.

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