Abstract. We show the existence of semiorthogonal decompositions of Donaldson-Thomas categories for $(-1)$-shifted cotangent derived stacks associated with $\Theta$-stratifications on them. Our main result gives an analogue of window theorem for categorical DT theory, which has applications to d-critical analogue of D/K equivalence conjecture, i.e., existence of fully-faithful functors (equivalences) under d-critical flips (flops). As an example of applications, we show the existence of fully-faithful functors of DT categories for Pandharipande-Thomas stable pair moduli spaces under wall-crossing at super-rigid rational curves for any relative reduced curve classes.

1. Introduction

1.1. Background. In [Todb], the author introduced $\mathbb{C}^*$-equivariant Donaldson-Thomas (DT) categories associated with $(-1)$-shifted cotangents $\Omega_{\mathfrak{M}}[-1]$ over quasi-smooth derived stacks $\mathfrak{M}$ with stability conditions. They are defined to be certain singular support quotients of derived categories of coherent sheaves on $\mathfrak{M}$, which via Koszul duality are regarded as gluing of dg-categories of $\mathbb{C}^*$-equivariant factorizations of super-potentials giving local d-critical charts of $\Omega_{\mathfrak{M}}[-1]$. The DT category depends on a choice of a stability condition, and it is an interesting problem to investigate wall-crossing phenomena of DT categories under a change of stability conditions. The above problem is motivated by a categorification of wall-crossing formula of DT invariants [JS12, KS], and also a d-critical analogue of D/K equivalence conjecture in birational geometry [HO, Kaw02]. A sequence of conjectures on wall-crossing of DT categories is proposed in [Todb], in the context of wall-crossing of one dimensional stable sheaves and Pandharipande-Thomas moduli spaces of stable pairs [PT09]. Some fundamental results on DT categories toward the above conjectures are also established in [Todb], including window theorem for DT categories which generalizes window theorem originally developed in GIT quotient stacks [HL15, BFK19].

In [Todb], we established two kinds of window theorem for DT categories, by constructing window subcategories in two different ways:

(i) Gluing of magic window subcategories by Halpern-Leistner-Sam [HLS20] (see [Todb, Section 5]).

(ii) Semiorthogonal summands of DT categories with complements described by Porta-Sala categorified Hall products [PS] (see [Todb, Section 6]).
The first window subcategory is independent of a stability condition, but only available for symmetric derived stacks (or more generally derived stacks with symmetric structures). The second window subcategory depends on a stability condition, and is more close to the original window subcategories for GIT quotient stacks \cite{HL15, BFK19}. A drawback of the second one is that it is available only in the situation that the categorified Hall products are a priori given, so in \cite{Todb}, Section 6 we only proved the window theorem for some special cases of PT moduli spaces.

The purpose of this paper is to generalize the second approach in \cite{Todb}, Section 6, using Halpern-Leistner’s theory of $\Theta$-stratifications \cite{HLb} instead of categorified Hall products \cite{PS}. Namely we show the existence of semiorthogonal decompositions of DT categories associated with $\Theta$-stratifications of $(-1)$-shifted cotangents derived stacks, whose semiorthogonal summands give window subcategories of the second type as above. The categorified Hall products are naturally interpreted in the context of $\Theta$-stratifications, and it turns out that many of the proofs of our main theorem follow from direct generalizations of the arguments in \cite{Todb}, Section 6. However we also need to give several modifications and prove some fundamental properties of $\Theta$-stacks, e.g. descriptions of stacks of filtered objects of $(-1)$-shifted cotangents in terms of $(-2)$-shifted conormals (see Section 4.1). Furthermore this generalization has more potentials for applications. For example, we show the existence of fully-faithful functors of DT categories for PT stable pair moduli spaces (see Section 4.1). Furthermore this generalization has more potentials for applications. For example, we show the existence of fully-faithful functors of DT categories for PT stable pair moduli spaces under wall-crossing at super-rigid rational curves for any relative reduced curve classes, which gives an evidence of the conjectures in \cite{Todb} that is not covered in \cite{loc. cit.}.

1.2. Semiorthogonal decompositions of DT categories. Let $\mathcal{M}$ be a quasi-smooth and QCA derived stack over $\mathbb{C}$ with classical truncation $\mathcal{M} = t_0(\mathcal{M})$, and

$$\Omega_{\mathcal{M}}[-1] := \text{Spec}_{\mathcal{M}} S_{\mathcal{O}_m}(T_{\mathcal{M}}[1])$$

the $(-1)$-shifted cotangent derived stack of $\mathcal{M}$. Given a $\mathbb{C}^*$-invariant open substack

$$\mathcal{N}^{ss} \subset \mathcal{N} := t_0(\Omega_{\mathcal{M}}[-1])$$

and its complement $\mathcal{Z} \subset \mathcal{N}$, we have the triangulated subcategory $\mathcal{C}_Z \subset D^b_{\text{coh}}(\mathcal{M})$ consisting of objects whose singular supports \cite{AG15} are contained in $\mathcal{Z}$. The $\mathbb{C}^*$-equivariant DT category in \cite{Todb} is modeled by the Verdier quotient (see Definition 3.5)

$$\mathcal{D}T^{\mathbb{C}^*}(\mathcal{N}^{ss}) := D^b_{\text{coh}}(\mathcal{M})/\mathcal{C}_Z.$$ (1.2)

The $\mathbb{Z}/2$-periodic version is also introduced in \cite{Todb}, whose basic model is (see Definition 3.9)

$$\mathcal{D}T^{Z/2}(\mathcal{N}^{ss}) := D^b_{\text{coh}}(\mathcal{M}_z)/\mathcal{C}_Z,$$ (1.3)

where $\mathcal{M}_z := \mathcal{M} \times \text{Spec} \mathbb{C}[e]$ for $\deg(e) = -1$ and $Z_z := (\mathcal{Z} \times A^1) \cup (\mathcal{N} \times \{0\})$. The above definitions are based on Koszul duality equivalences which, in the case that $\mathcal{M}$ is a quasi-smooth affine derived scheme, give equivalences of these categories with $\mathbb{C}^*$-equivariant or $\mathbb{Z}/2$-periodic triangulated categories of factorizations. The latter one (1.3) is recovered from the former one (1.2) under the compact generation of $\text{Ind}\mathcal{C}_Z$ up to idempotent completion (see Theorem 3.10), so in this paper we only focus on the $\mathbb{C}^*$-equivariant DT category (1.2).

We are interested in the case that the open substack (1.1) is the semistable locus with respect to a certain stability condition. For a choice of $\mathcal{L} \in \text{Pic}(\mathcal{M})_R$ with $l = c_1(\mathcal{L})$, the Hilbert-Mumford criterion with respect to maps from the $\Theta$-stack

$$\Theta := [A^1/\mathbb{C}^*] \to \mathcal{N}$$

defines the $l$-semistable locus $\mathcal{N}^{ss} \subset \mathcal{N}$, which generalizes Mumford’s GIT semistable locus for GIT quotient stacks depending on linearizations. The above viewpoint of defining $l$-semistable locus with respect to maps from the $\Theta$-stack is due to Halpern-Leistner \cite{HLb}, who further generalizes Kempf-Ness stratifications for GIT quotient stacks to $\Theta$-stratifications for more general stacks $\mathcal{X}$. In several cases, $\Theta$-stratifications are constructed from numerical data $(l, b)$ for $l \in H^2(\mathcal{X}, \mathbb{R})$ and a
positive definite $b \in H^4(\mathcal{X}, \mathbb{R})$. In our situation, by taking $l \in H^2(\mathcal{M}, \mathbb{R})$ as above, positive definite $b \in H^4(\mathcal{M}, \mathbb{R})$, and pulling them back to $\mathcal{N}$, we have the $\Theta$-stratification
\begin{equation}
\mathcal{N} = S_0^1 \sqcup \cdots \sqcup S_k^1 \sqcup \mathcal{N}^{l-ss}
\end{equation}
with center $Z_i^1 \subset S_0^1$. We show that each $Z_i^1$ is an open substack of $(-1)$-shifted cotangent over the stack of graded objects of $\mathcal{M}$ (see Subsection 6.2). So we have the DT category $\mathcal{D}T^{C^*}(Z_i^1)$, together with the decomposition into $C^*$-weight part with respect to the canonical $B C^*$-action on $Z_i^1$
$$\mathcal{D}T^{C^*}(Z_i^1) = \bigoplus_{j \in \mathbb{Z}} \mathcal{D}T^{C^*}(Z_i^1)_{wt=j}.$$ 

The following is the main result in this paper:

**Theorem 1.1.** (Corollary 6.13) Suppose that $\mathcal{M}$ admits a good moduli space $\mathcal{M} \to M$ satisfying the formal neighborhood theorem (see Definition 5.5). Then for each choice of $m_i \in \mathbb{R}$ for $1 \leq i \leq N$, there exists a semiorthogonal decomposition
$$\mathcal{D}T^{C^*}(\mathcal{N}) = \langle D_{1, < m_1}, \ldots, D_{N, < m_N}, \mathcal{W}_{m_+}, D_{N, \geq m_N}, \ldots, D_{1, \geq m_1} \rangle$$

satisfying the following:

(i) There exist semiorthogonal decompositions of the form
$$D_{i, < m_i} = \langle \cdots, D_{i, \Delta}(Z_i^1)_{wt=\lceil m_i \rceil-3}, D_{i, \Delta}(Z_i^1)_{wt=\lceil m_i \rceil-2}, D_{i, \Delta}(Z_i^1)_{wt=\lceil m_i \rceil-1}, \cdots \rangle,$$
$$D_{i, \geq m_i} = \langle D_{i, \Delta}(Z_i^1)_{wt=\lceil m_i \rceil}, D_{i, \Delta}(Z_i^1)_{wt=\lceil m_i \rceil+1}, D_{i, \Delta}(Z_i^1)_{wt=\lceil m_i \rceil+2}, \cdots \rangle.$$ 

(ii) The composition functor
$$\mathcal{W}_{m_+} \to \mathcal{D}T^{C^*}(\mathcal{N}) \to \mathcal{D}T^{C^*}(\mathcal{N}^{l-ss})$$

is an equivalence.

The subcategory $\mathcal{W}_{m_+}$ gives a desired window subcategory mentioned in the previous subsection. Halpern-Leistner [HLa] proved the existence of semiorthogonal decomposition of $D_{coh}(\mathcal{M}) (= \mathcal{D}T^{C^*}(\mathcal{N}))$ associated with $\Theta$-stratifications of $\mathcal{M}$ (not of $\mathcal{N}$), under some assumption on weights of the obstruction spaces of $\mathcal{M}$ at each center of $\Theta$-strata (see [HLa, Theorem 2.3.1]). The $\Theta$-stratification [HL] does not necessary a pull-back of the $\Theta$-stratification of $\mathcal{M}$, e.g. the semistable locus $\mathcal{N}^{l-ss}$ may be strictly bigger than the pull-back of $\mathcal{M}^{l-ss}$. So the result of [HLa, Theorem 2.3.1] does not imply Theorem 1.1. On the other hand if the above weight condition is satisfied, then the $\Theta$-stratification on $\mathcal{M}$ pulls back to a $\Theta$-stratification on $\mathcal{N}$, and the semiorthogonal decomposition in Theorem 1.1 recovers the semiorthogonal decomposition in [HLa, Theorem 2.3.1] (see Remark 6.14). In the case that $\mathcal{M}$ is the derived moduli stack of one dimensional sheaves on surfaces, Păduraru [Pâda] recently constructs semi-orthogonal decomposition of $D_{coh}(\mathcal{M})$ which is closely related to the one in Theorem 1.1 with the aim of constructing K-theoretic BPS Lie algebra of a surface whose quiver with super-potential version is obtained in [Pâdih].

The DT category $\mathcal{D}T^{C^*}(\mathcal{N}^{l-ss})$ depends on $l$, and we are interested in its behavior under a change of $l$. We use the result of Theorem 1.1 to compare $\mathcal{D}T^{C^*}(\mathcal{N}^{l-ss})$ under wall-crossing of $l$. We take another $\mathcal{L}' \in \text{Pic}(\mathcal{M})_R$ with $l' = c_1(\mathcal{L'})$ and set $l_\pm = l \pm \varepsilon l'$ for $0 < \varepsilon \ll 1$. Then we have the wall-crossing diagram for the good moduli space $\mathcal{N}^{l-ss} \to \mathcal{N}^{l-ss}$.

In [Toda], we introduced the notion of d-critical flips (flops) which are analogue of usual flips (flops) in birational geometry for Joyce’s d-critical loci [Joy15a]. If the above diagram is a d-critical flip.
(flop), then as an analogy of D/K equivalence conjecture [BO, Kaw02] we expect the existence of a fully-faithful functor (an equivalence)

$$\mathcal{D}T^*(\mathcal{N}^\ast_{d-ss}) \hookrightarrow (\sim)\mathcal{D}T^*(\mathcal{N}^\ast_{d+ss}).$$

We impose some assumption (see Assumption 6.17) on tangent complexes at closed points in $\mathcal{N}^\ast_{d-ss}$, which is typically satisfied for d-critical flips, and prove the following:

**Theorem 1.2.** (Theorem 6.18) Under Assumption 6.17, we have the inclusion $W_{m^*}^{-m} \subset W_{m^*}^{+m}$. In particular, we have the fully-faithful functor

$$\mathcal{D}T^*(\mathcal{N}^\ast_{d-ss}) \hookrightarrow \mathcal{D}T^*(\mathcal{N}^\ast_{d+ss}).$$

1.3. **Application: wall-crossing at (-1, -1)-curve.** In [Todb] we proposed several conjectures on wall-crossing of DT categories for Pandharipande-Thomas stable pair moduli spaces [PT09] on local surfaces, and proved them in several cases, e.g. MNOP/PT correspondence for reduced curve classes. As an application of Theorem 1.2, we give a further evidence to the above conjectures which is not covered in [Todb].

Let $S$ be a smooth projective surface over $\mathbb{C}$, and

$$\pi: X := \text{Tot}_S(\omega_S) \to S$$

the total space of canonical line bundle of $S$, which is a non-compact Calabi-Yau 3-fold called local surface. A PT stable pair is a pair $(F, s)$, where $F$ is a compactly supported one dimensional coherent sheaf on $X$ and $s: \mathcal{O}_X \to F$ is surjective in dimension one. For $(\beta, n) \in \text{NS}(S) \oplus \mathbb{Z}$, we have the moduli space of PT stable pairs

$$P_n(X, \beta) = \{(F, s) : \text{PT stable pair with } (\pi_*[F], \chi(F)) = (\beta, n)\},$$

which is a quasi-projective scheme. The moduli space of PT stable pairs is regarded as a moduli space of some stable objects in the derived category. Indeed for a fixed ample divisor $H$ on $S$, there is a one parameter family of stability conditions on the abelian category of D0-D2-D6 bound states, denoted by $\mu^t$-stability for $t \in \mathbb{R}$, whose $t \to \infty$ limit recovers the PT theory (see Subsection 7.2).

The numerical PT invariants and their wall-crossing formula play a key role in the study of curve counting invariants on Calabi-Yau 3-folds (see [Tod12]).

We denote by $P_n^t(X, \beta)$ the moduli space of $\mu^t$-semistable objects with numerical class $(\beta, n)$. It was observed in [Toda] that the wall-crossing diagram at $t > 0$

$$\begin{align*}
P_n^t(X, \beta) \quad &\quad P_n^{t-}(X, \beta) \\
\downarrow & \quad \downarrow \\
P_n^t(X, \beta) &
\end{align*}$$

is a d-critical flip (see [Toda Theorem 9.13]). Here we refer to [Todb Subsection 1.3.2] for an importance of the above wall-crossing in the context of numerical PT invariants. In [Todb], we introduced the categorical DT theory for D0-D2-D6 bound states

$$\mathcal{D}T^*(P_n^t(X, \beta)), \ t \notin W$$

following the basic model [1.2], where $W \subset \mathbb{R}$ is the set of walls. Based on the of D/K equivalence conjecture [BO, Kaw02], the following conjecture was proposed in [Todb]:

**Conjecture 1.3.** ([Todb Conjecture 4.24]) In the diagram (1.2), there exists a fully-faithful functor

$$\mathcal{D}T^*(P_n^{t-}(X, \beta)) \hookrightarrow \mathcal{D}T^*(P_n^{t+}(X, \beta)).$$
Let $C \subset S$ be a $(-1)$-curve, and $f: S \to T$ a birational contraction which contracts $C$ to a smooth surface $T$. Note that $C$ is a $(-1,-1)$-curve inside $X$, which is contracted to a conifold singularity by a flopping contraction $X \to Y$. We take an ample divisor $H$ on $S$ of the following form

$$H = af^*h - C, \ a \gg 0$$

where $h$ is an ample divisor on $T$. As an application of Theorem 1.2, we prove the following:

**Theorem 1.4.** (Theorem \[28\]) Conjecture \[3\] is true if $t \geq 1$ and $\beta$ is $f$-reduced, i.e., $f_*\beta$ is a reduced curve class on $T$.

In the situation of Theorem 1.4, a strictly polystable object at the wall is a direct sum of stable objects with line bundles on $C$. The wall-crossing in Theorem 1.4 is the one which appeared in \[13\], where the flop transformation of PT invariants was proved. The sequence of wall-crossing diagrams in this case is a d-critical minimal model program, which connects PT stable pair moduli space $P_n(X, \beta)$ with non-commutative stable pair moduli space $P_n^{nc}(X, \beta)$ (see Subsection 7.3).

$$P_n(X, \beta) = P^{(k)}(X, \beta) \dashrightarrow P^{(k-1)}(X, \beta) \dashrightarrow \cdots \dashrightarrow P^{(n)}(X, \beta) = P^{nc}(X, \beta).$$

Here $k \gg 0$ and $P_n^{nc}(X, \beta)$ is isomorphic to the moduli space of PT stable pairs for the non-commutative scheme $(Y, A\chi_{X})$, which is a non-commutative crepant resolution of $Y$ given in \[4\]. So Theorem 1.4 implies a sequence of fully-faithful functors

$$\mathcal{D}\mathcal{T}^c(P^{nc}(X, \beta)) \dashrightarrow \cdots \dashrightarrow \mathcal{D}\mathcal{T}^c(P^{(n-1)}(X, \beta)) \dashrightarrow \mathcal{D}\mathcal{T}^c(P_n(X, \beta)).$$

We also remark that Theorem 1.4 in the case that $f_*\beta = 0$ is proved in \[4\], Theorem 4.3.5], where PT moduli spaces are global critical loci so the original window theorem for GIT quotient stacks is enough to prove the result. In the situation that $f_*\beta$ is only a reduced class, we no longer have a global critical locus description of PT moduli spaces, and we need the result of Theorem 1.4.

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**1.5. Notation and convention.** In this paper, all the schemes or (derived) stacks are locally of finite presentation over $\mathbb{C}$, except formal fibers along with good moduli space morphisms. For a scheme or derived stack $Y$ and a quasi-coherent sheaf $\mathcal{F}$ on it, we denote by $\mathcal{S}_{\mathcal{O}_Y}(\mathcal{F})$ its symmetric product $\oplus_{i \geq 0} \text{Sym}^i_{\mathcal{O}_Y}(\mathcal{F})$. We omit the subscript $\mathcal{O}_Y$ if it is clear from the context. For a derived stack $\mathfrak{M}$, we always denote by $t_0(\mathfrak{M})$ the underlying stack given by the truncation. For an algebraic group $G$ which acts on $Y$, we denote by $[Y/G]$ the associated quotient stack.

All the dg-categories or triangulated categories are defined over $\mathbb{C}$. For a triangulated category $\mathcal{D}$ and a set of objects $\mathcal{S} \subset \mathcal{D}$, we denote by $\langle \mathcal{S} \rangle_{\text{ex}}$ the extension closure, i.e., the smallest extension closed subcategory of $\mathcal{D}$ which contains $\mathcal{S}$. For a triangulated subcategory $\mathcal{D}' \subset \mathcal{D}$, we denote by $\mathcal{D}/\mathcal{D}'$ its Verdier quotient. In the case that $\mathcal{D}$ is a dg-category and $\mathcal{D}' \subset \mathcal{D}$ is a dg-subcategory, its Drinfeld dg-quotient \[13\] is also denoted by $\mathcal{D}/\mathcal{D}'$. The category of dg-categories dgCat consists of dg-categories over $\mathbb{C}$ with morphisms given by dg-functors. By Tabuada \[5\], there is a cofibrantly generated model category structure on dgCat, whose localization by weak equivalences is denoted by Ho(dgCat). An equivalence between dg-categories is defined to be an isomorphism in Ho(dgCat).

For a dg-category $\mathcal{D}$, we denote by Ind $\mathcal{D}$ its dg-categorical ind-completion of $\mathcal{D}$ (denoted as $\widetilde{\mathcal{D}}$ in \[7\], also see \[9\] Section 5.3.3 for a dg-category in the context of $\infty$-category). When we discuss limits or ind-completions, we (implicitly or explicitly) take these functors on dg-enhancements or $\infty$-categorical enhancements (which will be obviously given in the context) and then take their homotopy categories.
Let $G$ be an algebraic group, $Y$ be a representation of $G$ and $\lambda: \mathbb{C}^* \to G$ a one parameter subgroup. We denote by $\langle \lambda, Y \rangle$ the sum of $\lambda$-weights of $Y$, which equals to the $\lambda$-weight of $\det(Y)$. By regarding $Y$ as a $\mathbb{C}^*$-representation via $\lambda$, we denote by $Y^\lambda \geq 0 \subset Y$ the direct summand consisting of non-negative $\mathbb{C}^*$-weights, and $\langle \lambda, Y^\lambda \geq 0 \rangle$ is also defined as above.

For an abelian category $\mathcal{A}$ and a collection of objects $(E_1, \ldots, E_k)$ in $\mathcal{A}$, its Ext-quiver is the quiver whose vertex set is $\{1, \ldots, k\}$, and the number of arrows from $i$ to $j$ is $\dim \text{Ext}^1(E_i, E_j)$.

2. Semiorthogonal decompositions of factorization categories: review

In this section, we review the dg or triangulated categories of factorizations of super-potentials, and their semiorthogonal decompositions associated with Kempf-Ness stratifications proved in [HL15, BFK19]. The above semiorthogonal decompositions are local models of semiorthogonal decompositions for DT categories.

2.1. Review of factorization categories. Here we review the theory of factorizations associated with super-potentials. The basic references are [Gr12, EP15, PV11].

Let $\mathcal{X}$ be a noetherian smooth algebraic stack over $\mathbb{C}$, $\mathcal{L} \to \mathcal{X}$ a line bundle and $w \in \Gamma(\mathcal{X}, \mathcal{L}^\otimes 2)$ a global section. A (quasi) coherent category $\mathcal{A}$ is called absolutely acyclic. The triangulated category of factorizations of $w$ is defined by the Verdier quotient

$$\text{MF}_*(\mathcal{X}, w) := \text{HMF}_*(\mathcal{X}, w)/\text{Acy}_*^{\text{abs}}, \quad * \in \{\text{qcoh, coh}\}.$$  

It admits a natural dg-enhancement by taking the Drinfeld quotient

$$(2.1) \quad \text{MF}_*(\mathcal{X}, w)_{\text{dg}} := \text{MF}_*(\mathcal{X}, w)_{\text{dg}}/\text{Acy}_*^{\text{abs, dg}}, \quad * \in \{\text{qcoh, coh}\}$$

where $\text{Acy}_*^{\text{abs, dg}}$ is the full dg-subcategory of $\text{MF}_*(\mathcal{X}, w)_{\text{dg}}$ consisting of absolutely acyclic objects in $\text{MF}_*(\mathcal{X}, w)_{\text{dg}}$. 

Let $\mathcal{Z} \subset \mathcal{X}$ be a closed substack. We define

$$(2.2) \quad \text{MF}_*(\mathcal{X}, w)_{\mathcal{Z}} := \ker(\text{MF}_*(\mathcal{X}, w) \to \text{MF}_*(\mathcal{X} \setminus \mathcal{Z}, w|_{\mathcal{X}\setminus \mathcal{Z}})).$$

Here $j: \mathcal{X} \setminus \mathcal{Z} \hookrightarrow \mathcal{X}$ is the open immersion. Then we have the equivalence (cf. [EP15 Theorem 1.10])

$$(2.3) \quad \text{MF}_*(\mathcal{X}, w)/\text{MF}_*(\mathcal{X}, w)_{\mathcal{Z}} \simeq \text{MF}_*(\mathcal{X} \setminus \mathcal{Z}, w|_{\mathcal{X}\setminus \mathcal{Z}}).$$

Let $\text{Crit}(w) \subset \mathcal{X}$ be the critical locus. We also have the equivalence (cf. [PV11 Corollary 5.3])

$$\text{MF}_*(\mathcal{X}, w)_{\text{Crit}(w)} \simeq \text{MF}_*(\mathcal{X}, w).$$

In particular by setting $\mathcal{Z}' = \mathcal{Z} \cap \text{Crit}(w)$, we have the equivalence

$$(2.4) \quad \text{MF}_*^\text{abs,coh}(\mathcal{X} \setminus \mathcal{Z}', w|_{\mathcal{X}\setminus \mathcal{Z}'}) \simeq \text{MF}_*^\text{abs,coh}(\mathcal{X} \setminus \mathcal{Z}, w|_{\mathcal{X}\setminus \mathcal{Z}}).$$

We will use the following two special versions of factorization categories. Let $\mathcal{Y}$ be a smooth stack and set $\mathcal{X} = \mathcal{Y} \times B\mu_2$, and $\mathcal{L}$ to be the line bundle on $\mathcal{X}$ induced by the weight one $\mu_2$-character. Then $\mathcal{L}^\otimes 2 \cong \mathcal{O}_\mathcal{X}$, so any regular function $w: \mathcal{Y} \to \mathbb{C}$ determines a global section $w \in \Gamma(\mathcal{X}, \mathcal{L}^\otimes 2)$. We set

$$\text{MF}_*^{\otimes 2}(\mathcal{Y}, w) := \text{MF}_*(\mathcal{Y} \times B\mu_2, w), \quad * \in \{\text{qcoh, coh}\}.$$
The above triangulated categories are \( \mathbb{Z}/2 \)-periodic, i.e. \( [2] \cong \text{id} \). When \( \mathcal{Y} \) is an affine scheme, the above triangulated category is equivalent to Orlov’s triangulated category of matrix factorizations \( \text{Orl09} \).

Let \( \mathbb{C}^* \) acts on a smooth stack \( \mathcal{Y} \) and set \( \mathcal{X} = [\mathcal{Y}/\mathbb{C}^*] \), and \( \mathcal{L} \) to be the line bundle on \( \mathcal{X} \) induced by the weight one \( \mathbb{C}^* \)-character. For a global section \( w \in \Gamma(\mathcal{X}, \mathcal{L}^\otimes 2) \), we set
\[
\text{MF}_{\mathbb{C}^*}^\ast(\mathcal{Y}, w) := \text{MF}_{\ast}([\mathcal{Y}/\mathbb{C}^*], w), \text{ } \ast \in \{\text{coh}, \text{qcoh}\}.
\]
For example, we will use the above construction for \( \mathcal{Y} = [Y/G] \) for a noetherian scheme \( Y \) with an action of an algebraic group \( G \) and an action of \( \mathbb{C}^* \) which commutes with the \( G \)-action, and \( w: Y \to \mathbb{C} \) is a \( G \)-invariant function with \( \mathbb{C}^* \)-weight two. Let \( \lambda: \mathbb{C}^* \to G \) be a one parameter subgroup contained in the center of \( G \) and the induced \( \mathbb{C}^* \)-action on \( Y \) trivial. Then we have the decomposition for \( \ast \in \{\text{coh}, \text{qcoh}\} \)
\[
\text{MF}_{\mathbb{C}^*}^\ast([Y/G], w) = \bigoplus_{j \in \mathbb{Z}} \text{MF}_{\mathbb{C}^*}^\ast([Y/G], w)_{\lambda \cdot \text{wt}=j}
\]
where \( \text{MF}_{\mathbb{C}^*}^\ast([Y/G], w)_{\lambda \cdot \text{wt}=j} \) is the weight \( j \)-part with respect to \( \lambda \). We define the subcategory
\[
\text{MF}_{\text{coh}}^\ast([Y/G], w)_{\lambda \cdot \text{above}} \subset \text{MF}_{\text{qcoh}}^\ast([Y/G], w)
\]
to be consisting of objects whose \( \lambda \)-weights are bounded above, and each \( \lambda \)-weight \( j \) part is an object in \( \text{MF}_{\text{coh}}^\ast([Y/G], w)_{\lambda \cdot \text{wt}=j} \).

For a closed substack \( Z \subset Y \), we define the subcategories
\[
\text{MF}_{\ast}^{Z/2}(\mathcal{Y}, w) \subset \text{MF}_{\ast}^{Z/2}(\mathcal{Y}, w) \subset \text{MF}_{\ast}^\ast(\mathcal{Y}, w)\]
in the similar way as \( \text{[2.2]} \). Here we assume that \( Z \) is \( \mathbb{C}^* \)-invariant in the latter case. The dg-categories \( \text{MF}_{\ast}^\ast(\mathcal{Y}, w)_{\text{dg}}, \text{MF}_{\ast}^\ast(\mathcal{Y}, w)_{\text{dg}} \) are also defined in the similar way as \( \text{[2.1]} \).

2.2. Kempf-Ness stratification. Here review Kempf-Ness stratifications associated with GIT quotients of reductive algebraic groups, following the convention of \( \text{[HL15, Section 2.1]} \). Let \( G \) be a reductive algebraic group with maximal torus \( \mathcal{T} \), which acts on a smooth affine scheme \( Y \). We denote by \( \mathcal{T} \) the character lattice of \( G \) and \( \mathcal{C} \) the cocharacter lattice of \( G \). For a one parameter subgroup \( \lambda: \mathbb{C}^* \to G \), let \( Y^{\lambda \geq 0}, Y^{\lambda = 0} \) be defined by
\[
Y^{\lambda \geq 0} := \{ y \in Y : \lim_{t \to 0} \lambda(t)(y) \text{ exists} \}, \quad Y^{\lambda = 0} := \{ y \in Y : \lambda(t)(y) = y \text{ for all } t \in \mathbb{C}^* \}.
\]
The Levi subgroup and the parabolic subgroup
\[
G^{\lambda = 0} \subset G^{\lambda \geq 0} \subset G
\]
are also similarly defined by the conjugate \( G \)-action on \( G \), i.e. \( g \cdot (-) = g(-)g^{-1} \). The \( G \)-action on \( Y \) restricts to the \( G^{\lambda \geq 0} \)-action on \( Y^{\lambda \geq 0} \), and the \( G^{\lambda = 0} \)-action on \( Y^{\lambda = 0} \). We note that \( \lambda \) factors through \( \lambda: \mathbb{C}^* \to G^{\lambda = 0} \), and it acts on \( Y^{\lambda = 0} \) trivially.

For an element \( l \in \text{Pic}([Y/G]_R) \), we have the open subset of \( l \)-semistable points
\[
Y^{l \text{-ss}} \subset Y
\]
characterized by the set of points \( y \in Y \) such that for any one parameter subgroup \( \lambda: \mathbb{C}^* \to G \) such that the limit \( z = \lim_{t \to 0} \lambda(t)(y) \) exists in \( Y \), we have \( \text{wt}(l_z) \geq 0 \). Let \( |\ast| \) is the Weyl-invariant norm on \( N_R \). The above subset of \( l \)-semistable points fits into the Kempf-Ness (KN) stratification
\[
(2.5) \quad Y = S_1 \sqcup S_2 \sqcup \cdots \sqcup S_N \sqcup Y^{l \text{-ss}}.
\]
Here for each \( 1 \leq i \leq N \) there exists a one parameter subgroup \( \lambda_i: \mathbb{C}^* \to T \subset G \), an open and closed subset \( S_i \) of \( (Y \setminus \bigcup_{i' < i} S_{i'})^{\lambda_i = 0} \) such that
\[
S_i = G \cdot Y_i, \quad Y_i := \{ y \in Y^{\lambda_i \geq 0} : \lim_{t \to 0} \lambda_i(t)(y) \in Z_i \}.
\]
Moreover by setting the slope to be

\[ \mu_i := -\frac{\text{wt}(l|z_i)}{[\lambda_i]} \in \mathbb{R} \]

we have the inequalities \( \mu_1 > \mu_2 > \cdots > 0 \). The above stratification gives an example of a \( \Theta \)-stratification in Definition 1.5 (see Example 1.10).

2.3. Semiorthogonal decomposition via KN stratification. In the setting of Subsection 2.2 let us consider a KN-stratification (2.5) with one parameter subgroup \( \lambda_i : \mathbb{C}^* \to G \) for each \( i \). Suppose that there is an auxiliary \( \mathbb{C}^* \)-action on \( Y \) which commutes with the \( G \)-action, and \( w : Y \to \mathbb{C}^* \) is a \( G \)-invariant function with auxiliary \( \mathbb{C}^* \)-weight two. We have the following diagram (see \[HL15\] Definition 2.2)

\[
\begin{array}{ccc}
[Y_i/G^{|\lambda_i|\geq 0}] & \xrightarrow{q_i} & [(Y \setminus \cup_{i' \prec i} S_{i'})/G] \\
\downarrow p_i & & \downarrow w \\
[Z_i/G^{|\lambda_i|=0}] & \xrightarrow{\tau_i} & \mathbb{C}.
\end{array}
\]

Here the left vertical arrow is given by taking the \( t \to 0 \) limit of the action of \( \lambda_i(t) \) for \( t \in \mathbb{C}^* \), and \( \tau_i, q_i \) are induced by the embedding \( Z_i \hookrightarrow Y, S_i \hookrightarrow Y \) respectively. Let \( \eta_i \in \mathbb{Z} \) be defined by

\[ \eta_i := \text{wt}_{\lambda_i}(\det(N^V_{S_i/Y}|z_i)). \]

Note that the auxiliary \( \mathbb{C}^* \)-action preserves \( S_i \) and \( Z_i \) by its commutativity with the \( G \)-action. We will use the following version of window theorem.

**Theorem 2.1.** (\[HL15\] BFK19) For each \( i \), we take \( m_i \in \mathbb{R} \).

(i) For each \( j \in \mathbb{Z} \), the composition

\[ q_i \circ p_i^* : \text{MF}_{\text{coh}}^\times([Z_i/G^{|\lambda_i|=0}], w|_{Z_i}, \lambda_i, \text{wt} = j) \to \text{MF}_{\text{coh}}^\times([S_i/G], w|_{S_i}) \to \text{MF}_{\text{coh}}^\times([(Y \setminus \cup_{i' \prec i} S_{i'})/G], w) \]

is fully-faithful, whose essential image is denoted by \( D_{i,j} \).

(ii) There exist semiorthogonal decomposition

\[ \text{MF}_{\text{coh}}^\times([(Y \setminus \cup_{i' \prec i} S_{i'})/G], w) = \left( \ldots, D_{i,[m_i]-2} \circ D_{i,[m_i]-1} \circ W_{i,m_i}, D_{i,[m_i]}, D_{i,[m_i]+1}, \ldots \right). \]

Here \( W_{i,m_i} \) consists of factorizations \((P, d_P)\) satisfying that

\[
\tau_i^*(P, d_P) \in \bigoplus_{j \in [m_i, m_i + \eta_i]} \text{MF}_{\text{coh}}^\times([Z_i/G^{|\lambda_i|=0}], w|_{Z_i}, \lambda_i, \text{wt} = j).
\]

(iii) The composition functor

\[
W_{i,m_i} \hookrightarrow \text{MF}_{\text{coh}}^\times([(Y \setminus \cup_{i' \prec i} S_{i'})/G], w) \to \text{MF}_{\text{coh}}^\times([(Y \setminus \cup_{i' \prec i} S_{i'})/G], w)
\]

is an equivalence.

As a consequence of Theorem 2.1 we have the following window theorem. Let

\[ W^d_{m_i}([Y/G], w) \subset \text{MF}_{\text{coh}}^\times([Y/G], w) \]

be the subcategory of objects \((P, d_P)\) satisfying the condition (2.7) for all \( i \). Then the composition functor

\[
W^d_{m_i}([Y/G], w) \hookrightarrow \text{MF}_{\text{coh}}^\times([Y/G], w) \to \text{MF}_{\text{coh}}^\times([Y^{l-ss}/G], w)
\]

is an equivalence. Note that the above window subcategory depends on a choice of \( m_i \).
2.4. Application for KN stratification of critical locus. We will apply Theorem 2.1 for a KN stratification of \( \text{Crit}(w) \)
\[
\text{Crit}(w) = S'_1 \sqcup S'_2 \sqcup \cdots \sqcup S'_N \sqcup \text{Crit}(w)^{\text{loc}}
\]
in the following way. After discarding KN strata \( S_i \subset Y \) with \( \text{Crit}(w) \cap S_i = \emptyset \), the above stratification is obtained by restricting a KN stratification \( 2.4 \) for \( Y \) to \( \text{Crit}(w) \). Let \( \lambda_i : C^* \to G \) be a one parameter subgroup for \( S'_i \) with center \( Z'_i \subset S'_i \). We define \( Z_i \subset Y \) to be the union of connected components of the \( \lambda_i \)-fixed part of \( Y \) which contains \( Z'_i \), and \( \overline{Z}_i \subset Y \) is the set of points \( y \in Y \) with \( \lim_{t \to 0} \lambda_i(t)y \in Z_i \). Similarly to (2.6), we have the diagram
\[
\begin{align*}
& \overline{Z}_i / G^\lambda_i = 0 \quad \overline{Y}_i / G^\lambda_i = 0 \quad Y / G \\
& \quad \quad \downarrow \quad \quad \downarrow \\
& [\overline{Z}_i / G^\lambda_i = 0] \quad [\overline{Y}_i / G^\lambda_i = 0].
\end{align*}
\]
Here the left horizontal arrows are open and closed immersions. By noting the equivalence \( 2.7 \), together with \( \text{Crit}(w) \cap Z'_i = \text{Crit}(w|\overline{Z}_i) \) and \( Z_i = Z_i \setminus \cup_{\nu < i} S'_\nu \), the result of Theorem 2.1 implies the semiorthogonal decomposition
\[
\text{MF}^C_{\text{coh}}(([Y \setminus \cup_{\nu < i} S'_\nu]) / G], w) = (\cdots, D_{i, [m_i] - 2}, D_{i, [m_i] - 1}, W_{i, m_i}, D_{i, m_i}, D_{i, [m_i] + 1}, \cdots)
\]
with equivalences
(2.8)
\[
\overline{\varphi}_{i, j, \pi} : \text{MF}^C_{\text{coh}}(([\overline{Z}_i \setminus \cup_{\nu < i} S'_\nu]) / G^\lambda_i = 0], w|\overline{Z}_i)_{\lambda_i, \text{wt} = j} \sim \sim D_{i, j}.
\]
The subcategory \( W_{i, m_i} \) consists of factorizations \( (\pi, d\pi) \) such that

\[
(P, d\pi)|_{([\overline{Z}_i \setminus \cup_{\nu < i} S'_\nu]) / G^\lambda_i = 0]} \in \bigoplus_{j \in [m_i, m_i, +1]} \text{MF}^C_{\text{coh}}(([\overline{Z}_i \setminus \cup_{\nu < i} S'_\nu]) / G^\lambda_i = 0], w|\overline{Z}_i)_{\lambda_i, \text{wt} = j}
\]
where \( \pi_i = \text{wt}_{\lambda_i} \det(I_{\pi_i})^\nu |\overline{Z}_i \), and the composition functor

\[
W_{i, m_i} \hookrightarrow \text{MF}^C_{\text{coh}}(([Y \setminus \cup_{\nu < i} S'_\nu]) / G], w) \to \text{MF}^C_{\text{coh}}(([Y \setminus \cup_{\nu < i} S'_\nu]) / G], w)
\]
is an equivalence.

3. Categorical DT theory for \((-1)\)-shifted cotangent derived stacks: review

The \( C^* \)-equivariant DT category for \((-1)\)-shifted cotangent derived stacks is defined in [Todb] based on Koszul duality equivalence between derived categories of coherent sheaves on quasi-smooth affine derived schemes and triangulated categories of \( C^* \)-equivariant factorizations. The \( Z/2 \)-periodic version of Koszul duality equivalence is also proved in [Tode], which is used to give \( Z/2 \)-periodic version of DT categories. In this section, we review Koszul duality, its functorial properties, and the constructions of DT categories.

3.1. Koszul duality equivalence. Let \( Y \) be a smooth affine scheme with a section \( s \) of a vector bundle \( V \to Y \). We denote by \( \Omega \) the derived zero locus of \( s \)
\[
\Omega = \text{Spec} R(V \to Y, s),
\]
where \( R(V \to Y, s) \) is the Koszul complex
\[
R(V \to Y, s) := \left( \cdots \to \bigwedge^2 V^\vee \xrightarrow{\partial} V^\vee \xrightarrow{\partial} \mathcal{O}_Y \to 0 \right).
\]
Let \( V^\vee \to Y \) be the total space of the dual vector bundle of \( V \). There is an associated function on \( V^\vee \), given by
\[
w : V^\vee \to \mathbb{C}, \quad w(x, v) = \langle s(x), v \rangle, \quad x \in Y, \quad v \in V^\vee |_x.
\]
It is well-known that the critical locus of the above function is the classical truncation of the \((-1)\)-shifted cotangent over \(\mathfrak{U}\) (see [JT17]),

\[(3.3) \quad t_0(\Omega_\mathfrak{U}[-1]) = \text{Crit}(w) \subset V^\vee.\]

Let \(C^*\) acts on the fibers of \(V^\vee \to Y\) by weight two, so that \(w\) is of weight two. Let \(G\) be an affine algebraic group which acts on \(Y\) such that \((V, s)\) is \(G\)-equivariant. We have the following Koszul duality equivalence which relates derived category of coherent sheaves on \([\mathfrak{U}/G]\) with the triangulated category of \(C^*\)-equivariant factorizations of \(w\) on \([V^\vee/G]\) (here we refer to Subsection 3.3 for \(D^b_{\text{coh}}(-)\) for derived stacks):

**Theorem 3.1.** (cf. [Isi13, Shi12, Hir17, Todb] Theorem 2.3.3, Lemma 2.3.10) There is an equivalence of triangulated categories

\[(3.4) \quad \Phi: D^b_{\text{coh}}([\mathfrak{U}/G]) \sim \text{MF}_{\text{coh}}^C([V^\vee/G], w),\]

which extends to the equivalence

\[(\Phi^{\text{ind}}): \text{Ind} D^b_{\text{coh}}([\mathfrak{U}/G]) \sim \text{MF}_{\text{coh}}^C([V^\vee/G], w).\]

The equivalence (3.4) is constructed in the following way. Let \(K_s\) be the following \(C^*\)-equivariant factorization of \(w\)

\[K_s := (\mathcal{O}_{V^\vee} \otimes_{\mathcal{O}} \mathcal{O}_\mathfrak{U}, d_{K_s}).\]

Here the \(C^*\)-action is given by the grading

\[\mathcal{O}_{V^\vee} \otimes_{\mathcal{O}} \mathcal{O}_\mathfrak{U} = S_{\mathcal{O}} V (-2) \otimes_{\mathcal{O}} S_{\mathcal{O}} V (V^\vee (1)),\]

and the weight one map \(d_{K_s}\) is given by

\[d_{K_s} = 1 \otimes d_{\mathcal{O}_\mathfrak{U}} + \eta: \mathcal{O}_{V^\vee} \otimes_{\mathcal{O}} \mathcal{O}_\mathfrak{U} \to \mathcal{O}_{V^\vee} \otimes_{\mathcal{O}} \mathcal{O}_\mathfrak{U} (1),\]

where \(\eta \in V \otimes_{\mathcal{O}} V^\vee \subset \mathcal{O}_{V^\vee} \otimes_{\mathcal{O}} \mathcal{O}_\mathfrak{U}\) corresponds to \(id \in \text{Hom}(V, V)\), and (1) indicates the shift of \(C^*\)-weight by one. We also equip the diagonal \(G\)-equivariant structure on \(K_s\). Then \(\Phi\) is given by

\[\Phi(-) = K_s \otimes_{\mathcal{O}_\mathfrak{U}} (-), \quad D^b_{\text{coh}}([\mathfrak{U}/G]) \to \text{MF}_{\text{coh}}^C([V^\vee/G], w).\]

### 3.2. Singular supports of (ind) coherent sheaves.

The theory of singular supports of coherent sheaves on \(\mathfrak{U}\) is developed in [AG15] following the earlier work [BIK08]. Here we recall its definition. Let \(\text{HH}^* (\mathfrak{U})\) be the Hochschild cohomology

\[\text{HH}^* (\mathfrak{U}) := \text{Hom}_{\mathfrak{U} \times \mathfrak{U}} (\mathcal{O}_\mathfrak{U}, \mathcal{O}_\mathfrak{U}).\]

Here \(\Delta: \mathfrak{U} \to \mathfrak{U} \times \mathfrak{U}\) is the diagonal. Then it is shown in [AG15] Section 4] that there exists a canonical map \(\mathcal{H}^1 (\mathfrak{T}_\mathfrak{U}) \to \text{HH}^2 (\mathfrak{U})\), so the map of graded rings

\[\mathcal{O}_{\text{Crit}(w)} = S_{\mathcal{O}_V} (\mathcal{H}^1 (\mathfrak{T}_\mathfrak{U})) \to \text{HH}^2 (\mathfrak{U}) \to \text{Nat}_{D^b_{\text{coh}} (\mathfrak{U})} (id, id[2*]).\]

Here \(\text{Nat}_{D^b_{\text{coh}} (\mathfrak{U})} (id, id[2*])\) is the group of natural transformations from \(id\) to \(id[2*]\) on \(D^b_{\text{coh}} (\mathfrak{U})\), and the right arrow is defined by taking Fourier-Mukai transforms associated with morphisms \(\Delta, \mathcal{O}_\mathfrak{U} \to \Delta_s \mathcal{O}_\mathfrak{U}[2*]\). The above maps induce the map for each \(F \in D^b_{\text{coh}} (\mathfrak{U})\),

\[\mathcal{O}_{\text{Crit}(w)} \to \text{Hom}^{2*} (F, F).\]

The above map defines the \(C^*\)-equivariant \(\mathcal{O}_{\text{Crit}(w)}\)-module structure on \(\text{Hom}^{2*} (F, F)\), which is finitely generated by [AG15] Theorem 4.1.8]. Below a closed subset \(Z \subset \text{Crit}(w)\) is called conical if it is invariant under the fiberwise \(C^*\)-action on \(\text{Crit}(w)\). For \(F \in D^b_{\text{coh}} (\mathfrak{U})\), its singular support is a conical closed subset

\[\text{Supp}^s (F) \subset \text{Crit}(w)\]

defined to be the support of \(\text{Hom}^{2*} (F, F)\) as \(\mathcal{O}_{\text{Crit}(w)}\)-module.
For a conical closed subset $Z \subset \text{Crit}(w)$ invariant under the $G$-action, let $\mathcal{Z} = [Z/G]$ and set

$$C_Z \subset D^b_{\text{coh}}([\mathcal{U}/G]), \; \text{Ind} \ C_Z \subset \text{Ind} D^b_{\text{coh}}([\mathcal{U}/G])$$

be the triangulated subcategories consisting of objects whose singular supports are contained in $\mathcal{Z}$, and its ind-completion respectively (see (8.11) for singular supports for derived stacks).

**Proposition 3.2.** ([Todb] Proposition 2.3.9) The equivalences in Theorem 3.1 restrict to the equivalences

$$\Phi : C_Z \sim \text{MF}_{\text{coh}}^C([V^\vee/G], w)_\mathcal{Z}, \; \Phi^{\text{ind}} : \text{Ind} C_Z \sim \text{MF}_{\text{coh}}^C([V^\vee/G], w)_\mathcal{Z}.$$ 

In particular by (2.23), the equivalences in Theorem 3.1 descend to the equivalences

$$\Phi : D^b_{\text{coh}}([\mathcal{U}/G])/C_Z \sim \text{MF}_{\text{coh}}^C([V^\vee/G] \setminus \mathcal{Z}, w),$$

$$\Phi^{\text{ind}} : \text{Ind} D^b_{\text{coh}}([\mathcal{U})/\text{Ind} C_Z \sim \text{MF}_{\text{coh}}^C([V^\vee/G] \setminus \mathcal{Z}, w).$$

### 3.3. Some functorial properties of Koszul duality equivalence.

We review some functorial properties of Koszul duality equivalence in Theorem 3.1 based on [Todb] Section 2.4. For $i = 1, 2$, let $Y_i$ be smooth affine schemes, $V_i \to Y_i$ be vector bundles with sections $s_i$. Suppose that affine algebraic groups $G_i$ act on $Y_i$ such that $(V_i, s_i)$ are $G_i$-equivariant. Let us consider a commutative diagram

$$\begin{array}{ccc}
V_1 & \xrightarrow{g} & V_2 \\
\downarrow{s_1} & & \downarrow{s_2} \\
Y_1 & \xrightarrow{f} & Y_2,
\end{array}$$

which is $G_i$-equivariant with respect to a group homomorphism $\phi : G_1 \to G_2$. The top morphism $g$ is a composition $V_1 \xrightarrow{g'} f^* V_2 \xrightarrow{f} V_2$, where $g'$ is a morphism of $G_1$-equivariant vector bundles on $Y_1$. The diagram (3.6) induces the diagram for smooth stacks $\mathcal{Y}_i = [Y_i/G_i]$, $\mathcal{V}_i = [V_i/G_i]$,

$$\begin{array}{ccc}
\mathcal{V}_1 & \xrightarrow{g} & \mathcal{V}_2 \\
\downarrow{s_1} & & \downarrow{s_2} \\
\mathcal{Y}_1 & \xrightarrow{f} & \mathcal{Y}_2,
\end{array}$$

which also induces the morphism of derived stacks

$$f : [\mathcal{U}_1/G_1] \to [\mathcal{U}_2/G_2], \; \mathcal{U}_i := \text{Spec} \mathcal{R}(V_i \to Y_i, s_i).$$

Then we have the push-forward functor (see [DG13] Section 3.6)

$$f^{\text{ind}} : \text{Ind} D^b_{\text{coh}}([\mathcal{U}_1/G_1]) \to \text{Ind} D^b_{\text{coh}}([\mathcal{U}_2/G_2]).$$

Let $w_i : V_i^\vee \to \mathbb{C}$ be given as in (3.2) defined from $(Y_i, V_i, s_i)$. The diagram (3.6) induces the following diagram

$$\begin{array}{ccc}
\mathcal{V}_1^\vee & \xrightarrow{g} & \mathcal{V}_2^\vee \\
\downarrow{p_1} & & \downarrow{p_2} \\
Y_1 & \xrightarrow{f} & Y_2
\end{array}$$

Here $w$ is determined by

$$w = f^* s_2 \in \Gamma(Y_1, f^* V_2) \subset \Gamma(Y_1, S(f^* V_2)).$$
The commutativity of (3.6) implies that the diagram (3.8) is also commutative. One can check the following identity in $f^*V_2^\vee$ (see [Todb, Lemma 2.4.2])

$$g^{-1}(\text{Crit}(w_1)) \cap f^{-1}(\text{Crit}(w_2)) = t_0(\Omega_{\mathcal{U}_2}[-1] \times_{\mathcal{U}_2} \mathcal{U}_1).$$

We define the following functor

$$f_* \circ g^*: MF^C_{qcoh}([V_1^\vee/G_1], w_1) \xrightarrow{g^*} MF^C_{qcoh}([f^*V_2^\vee/G_1], \overline{\pi}) \xrightarrow{f_*} MF^C_{qcoh}([V_2^\vee/G_2], w_2).$$

Then the following diagram commutes (see [Todb, Lemma 2.2.4]):

$$\begin{array}{ccc}
\text{Ind} D^b_{coh}([\mathcal{U}_1/G_1]) & \xrightarrow{\Phi^\text{ind}_1} & MF^C_{coh}([V_1^\vee/G_1], w_1) \\
\downarrow f^\text{ind} & & \downarrow f_* \circ g^* \\
\text{Ind} D^b_{coh}([\mathcal{U}_2/G_2]) & \xrightarrow{\Phi^\text{ind}_2} & MF^C_{coh}([V_2^\vee/G_2], w_2).
\end{array}$$

Here the horizontal arrows are equivalences in Theorem 3.1.

Suppose that the morphism $f: \mathcal{Y}_1 \to \mathcal{Y}_2$ in the diagram (3.7) is a proper morphism of smooth stacks, so in particular $f: [\mathcal{U}_1/G_1] \to [\mathcal{U}_2/G_2]$ is proper. Then the diagram (3.11) restricts to the commutative diagram

$$\begin{array}{ccc}
D^b_{coh}([\mathcal{U}_1/G_1]) & \xrightarrow{\Phi_1} & MF^C_{coh}([V_1^\vee/G_1], w_1) \\
\downarrow f_1 & & \downarrow f_* \circ g^* \\
D^b_{coh}([\mathcal{U}_2/G_2]) & \xrightarrow{\Phi_2} & MF^C_{coh}([V_2^\vee/G_2], w_2).
\end{array}$$

Moreover we have the continuous right adjoint of $f^\text{ind}_1$ (see [Ga13, Section 10.1])

$$f^! : \text{Ind} D^b_{coh}([\mathcal{U}_2/G_2]) \to \text{Ind} D^b_{coh}([\mathcal{U}_1/G_1]).$$

On the other hand, we also have the functor

$$g_* \circ f^! : MF^C_{qcoh}([V_2^\vee/G_2], w_2) \xrightarrow{f^!} MF^C_{qcoh}([f^*V_2^\vee/G_1], \overline{\pi}) \xrightarrow{g_*} MF^C_{qcoh}([V_1^\vee/G_1], w_1)$$

which is a right adjoint of the functor (3.10). Then the following diagram commutes (see [Todb, Lemma 2.4.6]):

$$\begin{array}{ccc}
\text{Ind} D^b_{coh}([\mathcal{U}_2/G_2]) & \xrightarrow{\Phi^\text{ind}_2} & MF^C_{coh}([V_2^\vee/G_2], w_2) \\
\downarrow f^! & & \downarrow g_* \circ f^! \\
\text{Ind} D^b_{coh}([\mathcal{U}_1/G_1]) & \xrightarrow{\Phi^\text{ind}_1} & MF^C_{coh}([V_1^\vee/G_1], w_1).
\end{array}$$

Suppose that $f: [\mathcal{U}_1/G_1] \to [\mathcal{U}_2/G_2]$ is not necessarily proper, but it is quasi-smooth. Then we have the functor (see [GR17, Section 3.1])

$$f^*: D^b_{coh}([\mathcal{U}_2/G_2]) \to D^b_{coh}([\mathcal{U}_1/G_1]).$$

If furthermore $f^*V_1^\vee \to V_1^\vee$ in the diagram (3.8) is proper, then we have the functor

$$g_1 \circ f^*: MF^C_{coh}([V_2^\vee/G_2], w_2) \xrightarrow{f^*} MF^C_{coh}([f^*V_2^\vee/G_1], \overline{\pi}) \xrightarrow{g_1} MF^C_{coh}([V_1^\vee/G_1], w_1)$$

$$f^*$$. 

If furthermore $g_1 \circ f^*$ is proper, then we have the functor

$$g_1 \circ f^*: MF^C_{coh}([V_2^\vee/G_2], w_2) \xrightarrow{f^*} MF^C_{coh}([f^*V_2^\vee/G_1], \overline{\pi}) \xrightarrow{g_1} MF^C_{coh}([V_1^\vee/G_1], w_1)$$

$$f^*$$.
which is a left adjoint of the functor (3.10). Then the following diagram commutes (see Todb Lemma 2.4.7):

\[
\begin{array}{ccc}
D^b_{\text{coh}}([\mathcal{U}_2/G_2]) & \xrightarrow{\phi_2} & \text{MF}^{\text{c}}_{\text{coh}}([V_2^\vee/G_2], w_2) \\
\downarrow f & & \downarrow g \circ f^* \\
D^b_{\text{coh}}([\mathcal{U}_1/G_1]) & \xrightarrow{\phi_1} & \text{MF}^{\text{c}}_{\text{coh}}([V_1^\vee/G_1], w_1).
\end{array}
\]

3.4. Quasi-smooth derived stacks. Below, we denote by $\mathcal{M}$ a derived Artin stack over $\mathbb{C}$. This means that $\mathcal{M}$ is a contravariant $\infty$-functor from the $\infty$-category of affine derived schemes over $\mathbb{C}$ to the $\infty$-category of simplicial sets

\[\mathcal{M} : d\text{Aff}^{\text{op}} \to \text{SSets}\]

satisfying some conditions (see Toe14 Section 3.2 for details). Here $d\text{Aff}^{\text{op}}$ is defined to be the $\infty$-category of commutative simplicial $\mathbb{C}$-algebras, which is equivalent to the $\infty$-category of commutative differential graded $\mathbb{C}$-algebras with non-positive degrees. The classical truncation of $\mathcal{M}$ is denoted by

\[\mathcal{M} := t_0(\mathcal{M}) : \text{Aff}^{\text{op}} \hookrightarrow d\text{Aff}^{\text{op}} \to \text{SSets}\]

where the first arrow is a natural functor from the category of affine schemes to affine derived schemes.

For an affine derived scheme $\mathcal{U} = \text{Spec} R$ for a cdga $R$, we set $D_{\text{qcoh}}(\mathcal{U})_{\text{dg}}$ to be the dg-category of $R$-modules localized by quasi-isomorphisms. The dg-category of quasi-coherent sheaves on $\mathcal{M}$ is defined to be the limit in the $\infty$-category of dg-categories (see Toe14 Section 4.1)

\[D_{\text{qcoh}}(\mathcal{M})_{\text{dg}} := \lim_{\mathcal{U} \to \mathcal{M}} D_{\text{qcoh}}(\mathcal{U})_{\text{dg}}.\]

Here the limit is taken for the $\infty$-category of smooth morphisms $\alpha : \mathcal{U} \to \mathcal{M}$ for affine derived schemes $\mathcal{U}$ with 1-morphisms given by smooth morphisms $f : \mathcal{U} \to \mathcal{U}'$ commuting with maps to $\mathcal{M}$, and pull-back $f^* : D_{\text{qcoh}}(\mathcal{U}')_{\text{dg}} \to D_{\text{qcoh}}(\mathcal{U})_{\text{dg}}$ is assigned for each $f$. The homotopy category of $D_{\text{qcoh}}(\mathcal{M})_{\text{dg}}$ is denoted by $D_{\text{qcoh}}(\mathcal{M})$, which is a triangulated category. We have the dg and triangulated subcategories

\[D^b_{\text{coh}}(\mathcal{M})_{\text{dg}} \subset D_{\text{qcoh}}(\mathcal{M})_{\text{dg}}, \quad D^b_{\text{coh}}(\mathcal{M}) \subset D_{\text{qcoh}}(\mathcal{M})\]

consisting of objects which have bounded coherent cohomologies.

A morphism of derived stacks $f : \mathcal{N} \to \mathcal{M}$ is called quasi-smooth if $L_f$ is perfect such that for any point $x \to \mathcal{M}$ the restriction $L_f|_x$ is of cohomological amplitude $[-1, 1]$. Here $L_f$ is the $f$-relative cotangent complex. A derived stack $\mathcal{M}$ over $\mathbb{C}$ is called quasi-smooth if $\mathcal{M} \to \text{Spec} \mathbb{C}$ is quasi-smooth. By [BBBJ15 Theorem 2.8], the quasi-smoothness of $\mathcal{M}$ is equivalent to that $\mathcal{M}$ is a 1-stack, and any point of $\mathcal{M}$ lies in the image of a 0-representable smooth morphism $\alpha : \mathcal{U} \to \mathcal{M}$, where $\mathcal{U}$ is an affine derived scheme obtained as a derived zero locus as in [31]. In this case, we have

\[D^b_{\text{coh}}(\mathcal{M})_{\text{dg}} = \lim_{\mathcal{U} \to \mathcal{M}} D^b_{\text{coh}}(\mathcal{U})_{\text{dg}}\]

where the limit is taken for the $\infty$-category $\mathcal{I}$ of smooth morphisms $\alpha : \mathcal{U} \to \mathcal{M}$ where $\mathcal{U}$ is equivalent to an affine derived scheme of the form $[\mathcal{U}]$. In this paper when we write $\lim_{\mathcal{U} \to \mathcal{M}}$ for a quasi-smooth $\mathcal{M}$, the limit is always taken for the $\infty$-category $\mathcal{I}$ as above. For a quasi-smooth derived stack $\mathcal{M}$, we denote by $\text{Ind} D^b_{\text{coh}}(\mathcal{M})_{\text{dg}}$ the dg-category of its ind-coherent sheaves (see Gai13 Section 10), and also Gai13 Section 11.3, Proposition 11.4.3 for its equivalence with $*$-pull back version)

\[\text{Ind} D^b_{\text{coh}}(\mathcal{M})_{\text{dg}} := \lim_{\mathcal{U} \to \mathcal{M}} \text{Ind} D^b_{\text{coh}}(\mathcal{U})_{\text{dg}},\]

where the limit is taken for the $\infty$-category $\mathcal{I}$ as above. Its homotopy category is denoted by $\text{Ind} D^b_{\text{coh}}(\mathcal{M})$. 

Following [DG13, Definition 1.1.8], a derived stack $\mathcal{M}$ is called QCA (quasi-compact and with affine automorphism groups) if the following conditions hold:

(i) $\mathcal{M}$ is quasi-compact;
(ii) The automorphism groups of its geometric points are affine;
(iii) The classical inertia stack $I_{\mathcal{M}} := \Delta \times_{\mathcal{M} \times \mathcal{M}} \Delta$ is of finite presentation over $\mathcal{M}$.

The QCA condition will be useful since in this case $\text{Ind} \, D^b_{\text{coh}}(\mathcal{M})_{\text{dg}}$ is compactly generated with compact objects $D^b_{\text{coh}}(\mathcal{M})_{\text{dg}}$ (see [DG13, Theorem 3.3.5]).

3.5. Good moduli spaces for Artin stacks. In general for a classical Artin stack $\mathcal{M}$, its good moduli space is an algebraic space $M$ together with a quasi-compact morphism,

$$\pi_{\mathcal{M}} : M \to \mathcal{M}$$

satisfying the following conditions (cf. [Alp13, Section 1.2]):

(i) The push-forward $\pi_{\mathcal{M}*} : \text{QCoh}(\mathcal{M}) \to \text{QCoh}(M)$ is exact.
(ii) The induced morphism $\mathcal{O}_M \to \pi_{\mathcal{M}*}\mathcal{O}_M$ is an isomorphism.

The good moduli space morphism $\pi_{\mathcal{M}}$ is universally closed. Moreover for each closed point $y \in M$, there exists a unique closed point $x \in \pi_{\mathcal{M}}^{-1}(y)$, and its automorphism group $\text{Aut}(x)$ is reductive (see [Alp13, Theorem 4.16, Proposition 12.14]).

Let $\mathfrak{M}$ be a quasi-smooth derived stack such that $\mathcal{M} = t_0(\mathfrak{M})$ admits a good moduli space $\pi_{\mathcal{M}} : \mathcal{M} \to M$ as above. For a closed point $y \in M$, let $\widehat{\mathcal{M}}_y$ be the formal fiber defined by

$$\widehat{\mathcal{M}}_y := \mathcal{M} \times_M \text{Spec} \widehat{\mathcal{O}}_{M,y}.$$

By a standard deformation theory argument, there exists a derived stack $\widehat{\mathfrak{M}}_y$ (unique up to equivalence) together with the following Cartesian diagrams (see [Todb, Lemma 5.2.5])

\begin{equation}
\begin{array}{ccc}
\widehat{\mathcal{M}}_y & \xrightarrow{\pi_y} & \text{Spec} \widehat{\mathcal{O}}_{M,y} \\
\downarrow & & \downarrow \\
\mathfrak{M} & \xrightarrow{\pi_{\mathcal{M}}} & M.
\end{array}
\end{equation}

Below we use the same symbol $y \in \mathcal{M}$ to denote the unique closed orbit in the fiber of $\pi_{\mathcal{M}} : \mathcal{M} \to M$ at $y$. Let $G_y := \text{Aut}(y)$, which is a reductive algebraic group. We denote by $\widehat{\mathcal{H}}^0(\mathfrak{T}_\mathcal{M}|_y)$ the formal fiber along $\mathcal{H}^0(\mathfrak{T}_\mathcal{M}|_y) \to \mathcal{H}^0(\mathfrak{T}_\mathcal{M}|_y)/G_y$ at the origin, i.e.

$$\widehat{\mathcal{H}}^0(\mathfrak{T}_\mathcal{M}|_y) := \mathcal{H}^0(\mathfrak{T}_\mathcal{M}|_y) \times_{\mathcal{H}^0(\mathfrak{T}_\mathcal{M}|_y)/G_y} \text{Spec} \mathcal{O}_{\mathcal{H}^0(\mathfrak{T}_\mathcal{M}|_y)/G_y,0}.$$

Definition 3.3. ([Todb, Definition 5.2.3, Lemma 5.2.5]) The derived stack $\mathfrak{M}$ with a good moduli space $\pi_{\mathcal{M}} : \mathcal{M} \to M$ satisfies the formal neighborhood theorem if for any closed point $y \in M$, there is a $G_y$-equivariant formal morphism (called Kuranishi map)

$$\kappa_y : \widehat{\mathcal{H}}^0(\mathfrak{T}_\mathcal{M}|_y) \to \mathcal{H}^1(\mathfrak{T}_\mathcal{M}|_y)$$

such that $\kappa_y(0) = 0$, and by setting $\widehat{\mathfrak{U}}_y$ to be the derived zero locus of $\kappa_y$, we have an equivalence

$$[\widehat{\mathfrak{U}}_y/G_y] \sim \widehat{\mathfrak{M}}_y$$

which sends $0$ to $y$ and identities on stabilizer groups.

Remark 3.4. It is proved in [Todb, Section 7.4] that if $\mathfrak{M}$ is a quasi-smooth derived stack obtained as moduli spaces of stable sheaves or stable pairs, then it satisfies formal neighborhood theorem. In this case, a Kuranishi map is obtained from an $A_{\infty}$-structure of the derived category of coherent sheaves on surfaces.
3.6. (-1)-shifted cotangent derived stacks. Let \( \mathcal{M} \) be a quasi-smooth derived stack. We denote by \( \Omega_{\mathcal{M}}[-1] \) the (-1)-shifted cotangent derived stack of \( \mathcal{M} \)

\[
p : \Omega_{\mathcal{M}}[-1] := \text{Spec}_{\mathcal{M}}(\mathbb{T}_{\mathcal{M}}[1]) \rightarrow \mathcal{M}.
\]

Here \( \mathbb{T}_{\mathcal{M}} \in D_{\text{coh}}^b(\mathcal{M}) \) is the tangent complex of \( \mathcal{M} \), which is dual to the cotangent complex \( \mathbb{L}_{\mathcal{M}} \) of \( \mathcal{M} \). The derived stack \( \Omega_{\mathcal{M}}[-1] \) admits a natural (-1)-shifted symplectic structure \([PTV13, Cal19]\), which induces a d-critical structure \([Joy15b]\) on its classical truncation \( \mathcal{N} \)

\[
p_0 : \mathcal{N} := t_0(\Omega_{\mathcal{M}}[-1]) \rightarrow \mathcal{M}.
\]

Let \( \mathcal{M}_1, \mathcal{M}_2 \) be quasi-smooth derived stacks with truncations \( \mathcal{M}_i = t_0(\mathcal{M}_i) \). Let \( f : \mathcal{M}_1 \rightarrow \mathcal{M}_2 \) be a morphism. Then the morphism \( f_{\text{st}} \) induces a d-critical structure \([Joy15b]\) on its classical truncation \( \mathcal{N} \)

\[
p_0 : \mathcal{N} := t_0(\Omega_{\mathcal{M}}[-1]) \rightarrow \mathcal{M}.
\]

Let us take a conical closed substack \( Z \subset \mathcal{N} = t_0(\Omega_{\mathcal{M}}[-1]) \).

Here \( Z \) is called conical if it is invariant under the fiberwise \( \mathbb{C}^* \)-action on \( \mathcal{N} \rightarrow \mathcal{M} \). Let \( \alpha : \mathcal{U} \rightarrow \mathcal{M} \) be a smooth morphism such that \( \mathcal{U} \) is of the form \((3.1)\). We have the associated conical closed subscheme

\[
\alpha^* Z := \alpha^*(\alpha^*)^{-1}(Z) \subset t_0(\Omega_{\mathcal{U}}[-1]) = \text{Crit}(\alpha).
\]

Here \( w \) is given as in \((3.2)\). As in \([AG15]\), we define

\[
\mathcal{C}_Z, dg := \lim_{\mathcal{U} \twoheadrightarrow \mathcal{M}} C_{\mathcal{U}^* Z, dg} \subset D_{\text{coh}}^b(\mathcal{M})_{dg}, \quad \text{Ind}\mathcal{C}_Z, dg := \lim_{\mathcal{U} \twoheadrightarrow \mathcal{M}} \text{Ind} C_{\mathcal{U}^* Z, dg} \subset \text{Ind} D_{\text{coh}}^b(\mathcal{M})_{dg},
\]

whose homotopy categories are denoted by \( \mathcal{C}_Z \), \( \text{Ind}\mathcal{C}_Z \), respectively.

3.7. DT categories for (-1)-shifted cotangents. For a quasi-smooth and QCA derived stack \( \mathcal{M} \), let us take an open substack \( \mathcal{N}^{ss} \) and its complement \( \mathcal{Z} \),

\[
\mathcal{N}^{ss} \subset \mathcal{N}, \quad \mathcal{Z} := \mathcal{N} \setminus \mathcal{N}^{ss}.
\]

In the case that \( \mathcal{N}^{ss} \) is \( \mathbb{C}^* \)-invariant so that \( \mathcal{Z} \) is a conical closed substack, the \( \mathbb{C}^* \)-equivariant dg or triangulated DT categories were defined in \([Todh]\) as Drinfeld or Verdier quotient:

**Definition 3.5.** \([Todh]\) Definition 3.2.2) We define \( \mathbb{C}^* \)-equivariant DT categories for \( \mathcal{N}^{ss} \) as

\[
D\mathcal{T}_{\mathbb{C}^*}(\mathcal{N}^{ss})_{dg} := D_{\text{coh}}^b(\mathcal{M})_{dg}/C_{\mathcal{Z}, dg}, \quad D\mathcal{T}_{\mathbb{C}^*}(\mathcal{N}^{ss}) := D_{\text{coh}}^b(\mathcal{M})/C_{\mathcal{Z}}.
\]

The above definition was based on the Koszul duality equivalence in Theorem 3.1, which gives an interpretation of the above categories as gluing of dg-categories of \( \mathbb{C}^* \)-equivariant factorizations.

Let \( \mathcal{W} \subset \mathcal{M} \) be a closed substack, and take the open derived substack \( \mathcal{M}_0 \subset \mathcal{M} \) whose truncation is \( \mathcal{M} \setminus \mathcal{W} \). We have the following conical closed substack

\[
\mathcal{Z}_0 := \mathcal{Z} \setminus p_0^{-1}(W) \subset \mathcal{N}_0 := t_0(\Omega_{\mathcal{M}}[-1]).
\]

Note that \( \mathcal{N}_0 = \mathcal{N} \setminus p_0^{-1}(W) \), and we have the following open immersion

\[
\mathcal{N}_0^{ss} : = \mathcal{N}_0 \setminus \mathcal{Z}_0 \rightarrow \mathcal{N} \setminus \mathcal{Z} = \mathcal{N}^{ss}.
\]

The following lemma is useful to replace the quotient categories to give equivalent DT categories:
Lemma 3.6. ([Todb, Lemma 3.2.9]) Suppose that \( p_0^{-1}(W) \subseteq Z \), or equivalently the open immersion \( _W Z \subseteq Z \) is an isomorphism. Then the restriction functor gives an equivalence

\[
\mathcal{D}\mathcal{T}^C(\mathcal{N}^*_{\text{ss}}) \sim \mathcal{D}\mathcal{T}^C(\mathcal{N}^*_{\text{ss}}).
\]

The ind-completions of DT categories are described as follows:

Proposition 3.7. ([Todb, Proposition 3.2.7, Theorem 7.2.2]) Suppose that \( \text{Ind}
\mathcal{C}_{Z,\text{dg}} \) is compactly generated. Then we have an equivalence

\[
\text{Ind} \, D^b_{\text{coh}}(\mathfrak{M})_{\text{dg}}/\text{Ind}
\mathcal{C}_{Z,\text{dg}} \sim \text{Ind} \, \mathcal{D}\mathcal{T}^C_{\text{ss}}(\mathcal{N}^*_{\text{ss}})._{\text{dg}}.
\]

If \( \mathcal{M} \) admits a good moduli space, we have the following compact generation of the subcategory of fixed singular supports:

Proposition 3.8. ([Todc, Theorem 7.2.2]) For a quasi-smooth and QCA derived stack \( \mathfrak{M} \), suppose that \( \mathcal{M} = t_0(\mathfrak{M}) \) admits a good moduli space. Then \( \text{Ind} \, \mathcal{C}_{Z,\text{dg}} \) is compactly generated. In particular, the equivalence (3.19) holds.

Let \( \mathfrak{M}_w := \mathfrak{M} \times \text{Spec} \, \mathbb{C}[\varepsilon] \) for \( \text{deg}(\varepsilon) = -1 \) and set

\[
Z_{\varepsilon} := \mathbb{C}^* (Z \times \{1\}) \cup (N \times \{1\}) \subset N \times \mathbb{A}^1 = t_0(\Omega_{\mathfrak{M}_w}[-1]).
\]

Here \( \mathbb{C}^* \) acts on fibers of \( N \times \mathbb{A}^1 \rightarrow \mathcal{M} \) by weight two. Then \( Z_{\varepsilon} \) is a conical closed substack. The \( \mathbb{Z}/2 \)-periodic DT categories in Definition 3.5 as follows:

Definition 3.9. ([Todc, Definition 4.2]) We define \( \mathbb{Z}/2 \)-periodic DT categories for \( \mathcal{N}^*_{\text{ss}} \) as

\[
\mathcal{D}\mathcal{T}^{\mathbb{Z}/2}_{\text{ss}}(\mathcal{N}^*_{\text{ss}})_{\text{dg}} := D^b_{\text{coh}}(\mathfrak{M}_w)_{\text{dg}}/\mathcal{C}_{Z_{\varepsilon},\text{dg}}, \quad \mathcal{D}\mathcal{T}^{\mathbb{Z}/2}(\mathcal{N}^*_{\text{ss}}) := D^b_{\text{coh}}(\mathfrak{M}_w)/\mathcal{C}_{Z_{\varepsilon}}.
\]

The above definition is also based on the Koszul duality equivalence. If \( \mathfrak{M} = U \) as in (3.1), then there is an equivalence (see Todc, Proposition 3.13)

\[
D^b_{\text{coh}}(U)/\mathcal{C}_{Z_{\varepsilon}} \sim \text{MF}_{\text{coh}}^Z(V^\vee \setminus Z, w).
\]

Moreover it is proved in Todc that, up to idempotent completion, the \( \mathbb{Z}/2 \)-periodic DT category is recovered from the \( \mathbb{C}^* \)-equivariant DT category \( \mathcal{D}\mathcal{T}^C_{\text{ss}}(\mathcal{N}^*_{\text{ss}}) \) defined in Todb.

Theorem 3.10. ([Todc, Theorem 4.9]) Suppose that \( Z \subset N \) is a conical closed substack such that \( \text{Ind} \, \mathcal{C}_Z \subset \text{Ind} \, D^b_{\text{coh}}(\mathfrak{M}) \) is compactly generated. Then there is an equivalence

\[
\mathcal{D}\mathcal{T}^{\mathbb{Z}/2}_{\text{ss}}(\mathcal{N}^*_{\text{ss}})_{\text{dg}} \simeq \text{RHom}(\mathbb{C}[u^\pm 1], \text{Ind} \, \mathcal{D}\mathcal{T}^C_{\text{ss}}(\mathcal{N}^*_{\text{ss}}))^{\text{fp}}.
\]

Here \( \text{deg}(u) = 2, \mathbb{C}[u^\pm 1] \) is regarded as a dg-category with one object, and \( \text{RHom}(-, -) \) is an inner Hom of dg-categories (see Toë07, Corollary 6.4).

By the above theorem, a fully-faithful functor (equivalence) of \( \mathbb{C}^* \)-equivariant DT categories induce a fully-faithful functor (equivalence) of idempotent completions of \( \mathbb{Z}/2 \)-periodic DT categories (see the argument of Todc, Theorem 5.4). Therefore in what follows, we focus on the \( \mathbb{C}^* \)-equivariant DT categories.

4. The stacks of filtered and graded objects: review

Halpern-Leistner [HLP] developed the moduli theory of maps from \( \Theta = [\mathbb{A}^1/\mathbb{C}^*] \) to a fixed Artin stack, whose moduli stack is called the stack of filtered objects. It is used to define the notion of \( \Theta \)-stratifications, which generalizes Harder-Narasimhan stratifications for moduli stacks of coherent sheaves and Kempf-Ness stratifications for GIT quotient stacks. The above moduli theory is extended to the case of maps to derived Artin stacks by Halpern-Leistner-Preygel [HLP]. In this section, we review the theory of stacks of filtered objects, \( \Theta \)-stratifications, and some of their properties.
4.1. Quasi-coherent sheaves on theta stacks. Let Θ be the stack defined by

Θ := [A¹/ℂ*].

Here ℂ* acts on A¹ by weight one. Here we review some basic properties of quasi-coherent sheaves on the stacks Θ × T and BC* × T for a derived stack T. By the Rees construction, we have the following lemma:

Lemma 4.1. ([HLa, Section 1.1]) (i) The ∞-category \( D_{\text{qcoh}}(\Theta \times T) \) is equivalent to the ∞-category of diagrams

\[
(4.1) \quad \mathcal{E}_* = \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{E}_{-1} \to \cdots
\]

where \( \mathcal{E}_i \in D_{\text{qcoh}}(T) \).

(ii) The ∞-category \( D_{\text{qcoh}}(BC^* \times T) \) decomposes into

\[
D_{\text{qcoh}}(BC^* \times T) = \bigoplus_{i \in \mathbb{Z}} D_{\text{qcoh}}(T)_{\text{wt}=i}
\]

where \( D_{\text{qcoh}}(T)_{\text{wt}=i} \) is a copy of \( D_{\text{qcoh}}(T) \) corresponding to the ℂ*-weight \( i \) part.

An object \( \mathcal{E} \in D_{\text{qcoh}}(\Theta \times T) \) determines a diagram (4.1) by

\[
\cdots \to \pi_T^*(\mathcal{O}(i+1) \boxtimes \mathcal{E}) \to \pi_T^*(\mathcal{O}(i) \boxtimes \mathcal{E}) \to \pi_T^*(\mathcal{O}(i-1) \boxtimes \mathcal{E}) \to \cdots,
\]

where \( \pi_T : \Theta \times T \to T \) is the projection, \( \mathcal{O}(i) \) is a line bundle on \( \Theta \) determined by the ℂ*-character of weight \( i \). Conversely a diagram (4.1) determines an object of \( D_{\text{qcoh}}(\Theta \times T) \) by

\[
\mathcal{E}_* \mapsto \bigoplus_{i \in \mathbb{Z}} \mathcal{E}_i,
\]

where the action of \( t \in \mathcal{O}_{A^1} = \mathbb{C}[t] \) is given by \( \mathcal{E}_{i+1} \to \mathcal{E}_i \).

We denote by

\[
(4.2) \quad i_1 : \text{Spec } \mathbb{C} \to \Theta, \quad i_0 : BC^* \to \Theta
\]

the morphisms corresponding to \( 1 \in A^1, 0 \in A^1 \) respectively. Under the above Rees construction, the pull-back functors

\[
i_1^* : D_{\text{qcoh}}(\Theta \times T) \to D_{\text{qcoh}}(T), \quad i_0^* : D_{\text{qcoh}}(\Theta \times T) \to D_{\text{qcoh}}(BC^* \times T)
\]

are described as

\[
(4.3) \quad i_1^*(\mathcal{E}_*) = \colim_{i \to -\infty} \mathcal{E}_i, \quad i_0^*(\mathcal{E}_*) = \bigoplus_{i \in \mathbb{Z}} \text{Cone}(\mathcal{E}_{i+1} \to \mathcal{E}_i).
\]

Here \( \text{Cone}(\mathcal{E}_{i+1} \to \mathcal{E}_i) \) lies in \( D_{\text{qcoh}}(T)_{\text{wt}=i} \).

Let \( \pi_T : \Theta \times T \to T \) and \( \pi_T^* : BC^* \times T \to T \) be the projections. We have the pull-back functors

\[
\pi_T^* : D_{\text{qcoh}}(T) \to D_{\text{qcoh}}(\Theta \times T), \quad \pi_T^* : D_{\text{qcoh}}(T) \to D_{\text{qcoh}}(BC^* \times T).
\]

Their left adjoint functors

\[
\pi_T^- : D_{\text{qcoh}}(\Theta \times T) \to D_{\text{qcoh}}(T), \quad \pi_T^+ : D_{\text{qcoh}}(BC^* \times T) \to D_{\text{qcoh}}(T)
\]

are given by (see [HLa, Lemma 1.3.1])

\[
(4.4) \quad \pi_T^-(\mathcal{E}_*) = \text{Cone} \left( \mathcal{E}_1 \to \colim_{i \to -\infty} \mathcal{E}_i \right), \quad \pi_T^+(\mathcal{E}_* \boxplus \bigoplus_{i \in \mathbb{Z}} \mathcal{E}_i) = \mathcal{E}_0.
\]
4.2. **The stacks of filtered objects.** Let \( \mathcal{M} \) be a derived Artin stack with affine stabilizers. The \( \infty \)-functor

\[
dAf f^\text{op} \to \text{Sset}, \quad T \mapsto \text{Map}(\Theta \times T, \mathcal{M})
\]

is a derived Artin stack \([\text{HLP}]\), denoted by \( \text{Filt}(\mathcal{M}) \) and called the *stack of filtered objects*. Similarly the \( \infty \)-functor

\[
dAf f^\text{op} \to \text{Sset}, \quad T \mapsto \text{Map}(\text{BC}^* \times T, \mathcal{M})
\]

is a derived Artin stack, denoted by \( \text{Grad}(\mathcal{M}) \) and called the *stack of graded objects*. There exist natural maps

\[
\begin{align*}
\text{Filt}(\mathcal{M}) & \xrightarrow{\sigma} \mathcal{M} \\
\sigma & \downarrow \text{ev}_0 \downarrow \quad \text{ev}_1 \\
\text{Grad}(\mathcal{M}) & &
\end{align*}
\]

Here \( \text{ev}_0, \text{ev}_1, \sigma, \tau \) are induced by morphisms \( t_0, i_1 \) in \([\text{HLP}]\), the projection \( \Theta \to \text{BC}^* \) and the natural morphism \( \text{Spec} \mathbb{C} \to \text{BC}^* \) respectively. Note that we have \( \tau = \text{ev}_1 \circ \sigma \). Moreover we have

\[
t_0(\text{Filt}(\mathcal{M})) = \text{Filt}(t_0(\mathcal{M})), \quad t_0(\text{Grad}(\mathcal{M})) = \text{Grad}(t_0(\mathcal{M})).
\]

We note that there is a natural action of \( \text{BC}^* \) on \( \text{Grad}(\mathcal{M}) \) acting on the source of the maps \( \text{BC}^* \times T \to \mathcal{M} \). Consequently there is a natural morphism into the inertia stack

\[
(\text{C}^*)_{\text{Grad}(\mathcal{M})} \to I_{\text{Grad}(\mathcal{M})}.
\]

We have the decomposition of the derived category \( D^b_{\text{coh}}(\text{Grad}(\mathcal{M})) \) into weight space categories with respect to the above \( \text{BC}^* \)-action

\[
D^b_{\text{coh}}(\text{Grad}(\mathcal{M})) = \bigoplus_{j \in \mathbb{Z}} D^b_{\text{coh}}(\text{Grad}(\mathcal{M}))_{\text{wt} = j}.
\]

**Example 4.2.** Let \( \Phi \) be an affine derived scheme of the form \([\text{P}3]\). Suppose that a reductive algebraic group \( G \) acts on \( Y \) such that \( (V, s) \) is \( G \)-equivariant. For a one parameter subgroup \( \lambda: \mathbb{C}^* \to G \), we have the following commutative diagram (see the notation in Subsection \([\text{P}3]\))

\[
\begin{array}{c}
V^\lambda = 0 \quad V^\lambda \geq 0 \\
\downarrow s^\lambda = 0 \quad \downarrow s^\lambda \geq 0 \\
Y^\lambda = 0 \quad Y^\lambda \geq 0
\end{array}
\]

Here the horizontal arrows from left to right are natural inclusions, and the horizontal arrows from left to right are given by taking the limits for the \( \lambda \)-actions. By setting \( \Phi^\lambda \geq 0 \), \( \Phi^\lambda = 0 \) to be the derived zero loci of \( s^\lambda \geq 0 \), \( s^\lambda = 0 \) respectively, we have the following diagram

\[
\begin{array}{c}
[\Phi^\lambda \geq 0 / G^\lambda \geq 0] \quad \xrightarrow{\text{ev}_1} \quad [\Phi / G] \\
\downarrow \sigma \quad \downarrow \text{ev}_0 \\
[\Phi^\lambda = 0 / G^\lambda = 0].
\end{array}
\]

It is proved in \([\text{HLa}]\) Lemma 1.6.1 that \( [\Phi^\lambda \geq 0 / G^\lambda \geq 0] \) is an open and closed substack of \( \text{Filt}(\Phi / G) \), \( [\Phi^\lambda = 0 / G^\lambda = 0] \) is its center, and any connected component of \( \text{Filt}(\Phi / G) \) is a component of \( [\Phi^\lambda \geq 0 / G^\lambda \geq 0] \) for a unique conjugacy class of \( \lambda \). The decomposition \((4.7)\) at the components \( [\Phi^\lambda = 0 / G^\lambda = 0] \) is

\[
D^b_{\text{coh}}([\Phi^\lambda = 0 / G^\lambda = 0]) = \bigoplus_{j \in \mathbb{Z}} D^b_{\text{coh}}([\Phi^\lambda = 0 / G^\lambda = 0]_{\text{wt} = j})
\]

where each component is \( \lambda \)-weight \( j \)-part with respect to \( \lambda: \mathbb{C}^* \to G^\lambda = 0 \).
The cotangent complexes of \( \mathcal{L}_{\text{Filt}(\mathcal{M})} \), \( \mathcal{L}_{\text{Grad}(\mathcal{M})} \) are described in the following way. Let us consider the following diagrams

\[
\begin{array}{ccc}
\Theta \times \text{Filt}(\mathcal{M}) & \xrightarrow{\text{ev}_F} & \mathcal{M} \\
\pi_F \downarrow & & \downarrow \pi_G \\
\text{Filt}(\mathcal{M}) & & \text{Grad}(\mathcal{M})
\end{array}
\]

where horizontal arrows are universal maps and the vertical arrows are projections. Then we have \( \text{(4.8)} \)

\[
\mathcal{L}_{\text{Filt}(\mathcal{M})} = \pi_{F+}(\text{ev}_F^* \mathcal{L}_{\text{Filt}(\mathcal{M})}), \quad \mathcal{L}_{\text{Grad}(\mathcal{M})} = \pi_{G+}(\text{ev}_G^* \mathcal{L}_{\text{Grad}(\mathcal{M})}).
\]

Here \( \pi_{F+}, \pi_{G+} \) are given by \( \text{(4.4)} \). In the case that \( \mathcal{M} \) is quasi-smooth, we have the following:

**Lemma 4.3.** \((\text{HLa} \text{ Lemma 2.2.4})\) If \( \mathcal{M} \) is quasi-smooth, then \( \text{Filt}(\mathcal{M}) \) and \( \text{Grad}(\mathcal{M}) \) are also quasi-smooth and the morphism \( \text{ev}_0 \) in the diagram \( \text{(4.9)} \) is a quasi-smooth morphism.

As for the morphism \( \text{ev}_1 \), the following \( \Theta \)-reductive condition is introduced in \( \text{[HLb]} \):

**Definition 4.4.** \((\text{HLb} \text{ Definition 4.16})\) A derived stack \( \mathcal{M} \) is called \( \Theta \)-reductive if the morphism \( \text{ev}_1 \) satisfies the valuative criterion of properness.

For example if \( \mathcal{M} = t_0(\mathcal{M}) \) admits a good moduli space, then \( \mathcal{M} \) is \( \Theta \)-reductive (see \( \text{[AHLH]} \text{ Theorem A} \)). If \( \mathcal{M} \) is quasi-smooth and \( \Theta \)-reductive, then the diagram \( \text{(4.5)} \) is a diagram of quasi-smooth derived stacks such that \( \text{ev}_0 \) is quasi-smooth and \( \text{ev}_1 \) is proper.

### 4.3. Theta stratifications.

Let \( \mathcal{S} \subset \text{Filt}(\mathcal{M}) \) be an open and closed substack. The center of \( \mathcal{S} \) is defined to be the open and closed substack \( \mathcal{S} := \sigma^{-1}(\mathcal{S}) \subset \text{Grad}(\mathcal{M}) \). The diagram \( \text{(4.5)} \) restricts to the diagrams

\[
\begin{array}{ccc}
\mathcal{S} & \xrightarrow{\text{ev}_1} & \mathcal{M} \\
\sigma \downarrow & & \downarrow \tau \\
\mathcal{S}_0 & \xrightarrow{\text{ev}_0} & \mathcal{Z}
\end{array}
\]

Here the right diagram is obtained from the left by taking classical truncations. From \( \text{(4.6)} \), note that \( \mathcal{S} \subset \text{Filt}(\mathcal{M}) \) is open and closed, and \( \mathcal{Z} \) is the center of \( \mathcal{S} \). Following \( \text{[HLb]} \text{ [HLa]} \), the substack \( \mathcal{S} \subset \text{Filt}(\mathcal{M}) \) is called a \( \Theta \)-stratum if the map

\[
\text{ev}_1 : \mathcal{S} \hookrightarrow \text{Filt}(\mathcal{M}) \to \mathcal{M}
\]

in the diagram \( \text{(4.10)} \) is a closed immersion. The last condition is also equivalent to that \( \text{ev}_1 : \mathcal{S} \to \mathcal{M} \) is a closed immersion of classical stacks.

**Definition 4.5.** \((\text{HLb} \text{ Definition 2.2})\) A \( \Theta \)-stratification of \( \mathcal{M} \) indexed by a totally ordered set \( I \) with minimal element \( 0 \in I \) consists of

(i) a collection of open substacks \( \mathcal{M}_{c} \subset \mathcal{M} \) for \( c \in I \) such that \( \mathcal{M}_{c} \subset \mathcal{M}_{c'} \) when \( c < c' \),

(ii) a \( \Theta \)-stratum \( \mathcal{S}_c \subset \text{Filt}(\mathcal{M}_{c}) \) for all \( c \in I \) with \( \mathcal{M}_{c} \setminus \text{ev}_1(\mathcal{S}_c) = \mathcal{M}_{<c} \),

(iii) for every \( x \in \mathcal{M} \), there is a minimal \( c \in I \) such that \( x \in \mathcal{M}_{<c} \).

The semistable locus is defined to be \( \mathcal{M}^{ss} := \mathcal{M}_{\leq 0} \subset \mathcal{M} \).

We will often regard \( \mathcal{S}_c \) as a locally closed substack of \( \mathcal{M} \) by \( \mathcal{S}_c \xrightarrow{\text{ev}_1} \mathcal{M}_{\leq c} \subset \mathcal{M} \), and write the \( \Theta \)-stratification as

\[
\mathcal{M} = \mathcal{M}^{ss} \bigcup_{c>0} \mathcal{S}_c.
\]

Note that

\[
H^*(\Theta, \mathbb{R}) \cong \mathbb{R}[q], \quad q = c_1(\mathcal{O}(1))
\]
where $\mathcal{O}(1)$ is the line bundle on $\Theta$ induced by the weight one $\mathbb{C}^*$-character. Then for any $\mathbb{R}$-line bundle $\mathcal{L}$ on $\mathfrak{M}$ with $l = c_1(\mathcal{L})$, we have $q^{-1}f^*l \in \mathbb{R}$.

**Definition 4.6.** A point $p \in \mathfrak{M}(\mathbb{C})$ is called $l$-semistable if for any $f : \Theta \to \mathfrak{M}$ with $f(1) \cong p$, we have $q^{-1}f^*l \geq 0$.

A map $f : BC^* \to \mathfrak{M}$ is called non-degenerate if the induced morphism $\mathbb{C}^* \to \text{Aut}(f(0))$ is non-trivial, and $f : \Theta \to \mathfrak{M}$ is called non-degenerate if $f(0) : BC^* \to \mathfrak{M}$ is non-degenerate. For an element $b \in H^4(\mathfrak{M}, \mathbb{R})$, we have the following positivity condition:

**Definition 4.7.** ([HLa, Definition 3.85]) An element $b \in H^4(\mathfrak{M}, \mathbb{R})$ is called positive definite if for any non-degenerate $f : BC^* \to \mathfrak{M}$, we have $q^{-2}f^*b > 0$.

For $(l, b)$ such that $b$ is positive definite and a non-degenerate map $f : \Theta \to \mathfrak{M}$, we set

$$\mu_{l,b}(f) := -\frac{q^{-1}f^*l}{\sqrt{q^{-2}f^*b}} \in \mathbb{R}. \quad (4.11)$$

**Definition 4.8.** ([HLa, Definition 3.1.2]) The numerical invariant $\mu_{l,b}$ is said to define the $\Theta$-stratification if there is a $\Theta$-stratification of $\mathfrak{M}$ whose strata $\mathcal{S} \subset \text{Filt}(\mathfrak{M})$ are ordered by the values of $\mu$, and each $f \in \mathcal{S}$ with $f(1) \cong p$ is the unique maximizer of $\mu$ (up to composition with a ramified covering $(-)^n : \Theta \to \Theta$) satisfying the condition $f(1) \cong p$. The $l$-semistable locus is denoted by $\mathfrak{M}^{l\text{-ss}} \subset \mathfrak{M}$.

In the case that $\mathcal{M}$ admits a good moduli space, the following result is proved in [HLa].

**Theorem 4.9.** ([HLa, Theorem 3.1.3]) If $\mathcal{M}$ admits a good moduli space, then $\mu_{l,b}$ given by (4.11) defines the $\Theta$-stratification

$$\mathcal{M} = \mathcal{S}_1 \sqcup \cdots \sqcup \mathcal{S}_N \sqcup \mathcal{M}^{l\text{-ss}},$$

such that $\mathcal{M}^{l\text{-ss}}$ also admits a good moduli space.

**Example 4.10.** In the situation of Subsection 2.2, let us take

$$l \in H^2(BG, \mathbb{R}) = M_\mathbb{R}, \ b \in H^4(BG, \mathbb{R})$$

such that $b$ is positive definite. We regard them as elements of $H^2([Y/G], \mathbb{R}), \ H^4([Y/G], \mathbb{R})$ by the pull-back of the projection $[Y/G] \to BG$. A choice of $b$ corresponds to a Weyl-invariant norm $|\cdot|$ on $N_\mathbb{R}$. By taking the quotient stacks of the stratification (2.3), we have the $\Theta$-stratification of the quotient stack $\mathcal{Y} = [Y/G]$ with respect to $\mu_{l,b}$

$$\mathcal{Y} = \mathcal{S}_1 \sqcup \mathcal{S}_2 \sqcup \cdots \sqcup \mathcal{S}_N \sqcup \mathcal{Y}^{l\text{-ss}}.$$

Later we will use the following lemma:

**Lemma 4.11.** Let $\pi_\mathcal{M} : \mathcal{M} \to \mathcal{M}$ be a good moduli space.

(i) We have the commutative diagram

$$(4.12) \quad \text{Filt}(\mathcal{M}) \overset{ev_1}{\longrightarrow} \mathcal{M} \quad \text{ev}_0 \downarrow \quad \pi_\mathcal{M} \downarrow \quad \mathcal{M}. \quad \text{Grad}(\mathcal{M}) \overset{\pi_\text{\mathcal{M}\text{-ss}}}{\longrightarrow} \mathcal{M}.$$

(ii) For a morphism $M' \to M$ between algebraic spaces, we set $\mathcal{M}' = \mathcal{M} \times_\mathcal{M} M'$ so that $\pi_{\mathcal{M}'} : \mathcal{M}' \to \mathcal{M}'$ is a good moduli space morphism. Then the pull-back of the diagram (4.12) by $M' \to M$ is isomorphic to the diagram

$$\text{Filt}(\mathcal{M}') \overset{ev_1}{\longrightarrow} \mathcal{M}' \quad \text{ev}_0 \downarrow \quad \pi_{\mathcal{M}'} \downarrow \quad \mathcal{M}' \quad \text{Grad}(\mathcal{M}') \overset{\pi_{\mathcal{M}'\text{-ss}}}{\longrightarrow} \mathcal{M}'.$$
Proof. (i) Let $g$ be the map $g : \text{Spec } \mathbb{C} \to BC^* \to \Theta$ where the first map is the natural one and the second one is $i_0$. Then for any classical scheme $T$ and a map $f : \Theta \times T \to \mathcal{M}$, we have the commutative diagram

$$
\begin{array}{ccc}
T \xrightarrow{(g, \text{id})} \Theta \times T & \xrightarrow{f} & \mathcal{M} \\
\downarrow (g, \text{id}) & & \downarrow \pi_{\mathcal{M}} \\
T & \xrightarrow{\pi_{\mathcal{M}}} & \mathcal{M}
\end{array}
$$

Here the middle vertical arrow is the projection which is nothing but the good moduli space morphism for $\Theta \times T$, and the bottom arrows are induced morphisms on good moduli spaces. The commutative diagram (4.12) follows from the above commutative diagram.

(ii) The claim follows from the following Cartesian diagrams (see [HLb, Corollary 1.30.1])

$$
\begin{array}{ccc}
\text{Filt}(\mathcal{M}') & \xrightarrow{\pi_{\mathcal{M}} \circ \tau_{\mathcal{M}}} & \text{Grad}(\mathcal{M}') \\
\downarrow \pi_{\mathcal{M}'} \circ \sigma & & \downarrow \pi_{\mathcal{M}'} \circ \sigma \\
\mathcal{M}' & \xrightarrow{\mathcal{M}'} & \mathcal{M}
\end{array}
$$

Here top horizontal arrows are induced by $\mathcal{M}' \to \mathcal{M}$.

\[\square\]

5. The stacks of filtered and graded objects for $(-1)$-shifted cotangents

In this section, we describe the stack of filtered objects for the $(-1)$-shifted cotangent $\Omega_{\mathfrak{M}}[-1]$ in terms of $(-2)$-shifted conormal stacks over $\text{Filt}(\mathfrak{M})$. This is a generalization of the result proved in [Tod20, Proposition 3.1] that the moduli stacks of exact sequences of local surfaces are isomorphic to $(-2)$-shifted conormal stacks over the moduli stacks of exact sequences of surfaces.

5.1. Description via $(-2)$-shifted conormals. For a derived stack $\mathfrak{M}$, let $\Omega_{\mathfrak{M}}[-1]$ be the $(-2)$-shifted cotangent stack of $\mathfrak{M}$.

\[
\begin{array}{ccc}
\text{Filt}(\Omega_{\mathfrak{M}}[-1]) & \xrightarrow{\mathcal{M}} & \Omega_{\mathfrak{M}}[-1] \\
\downarrow \sigma & & \downarrow \tau \\
\text{Grad}(\Omega_{\mathfrak{M}}[-1])
\end{array}
\]

On the other hand, we have the following morphism from the diagram (4.5)

\[
\begin{array}{ccc}
\text{Filt}(\Omega_{\mathfrak{M}}[-1]) & \xrightarrow{(\text{ev}_0, \text{ev}_1)} & \Omega_{\mathfrak{M}}[-1] \\
\downarrow \sigma & & \downarrow \tau \\
\text{Grad}(\Omega_{\mathfrak{M}}[-1])
\end{array}
\]

(5.1)

We denote by $\Omega_{(\text{ev}_0, \text{ev}_1)}[-2]$ the $(-2)$-shifted conormal stack for the above morphism

\[
\begin{array}{ccc}
\text{Filt}(\Omega_{\mathfrak{M}}[-1]) & \xrightarrow{\mathcal{M}} & \Omega_{\mathfrak{M}}[-1] \\
\downarrow \sigma & & \downarrow \tau \\
\text{Grad}(\Omega_{\mathfrak{M}}[-1])
\end{array}
\]

(5.2)

Note that in the diagram (4.5), we have the distinguished triangles on $\text{Filt}(\mathfrak{M})$

\[
\begin{array}{ccc}
\text{ev}_1^* \mathbb{L}_{\mathfrak{M}} & \to & \mathbb{L}_{\mathfrak{M}} \to \mathbb{L}_{(\text{ev}_0, \text{ev}_1)} \\
\downarrow \mathbb{L}_{\text{Grad}(\mathfrak{M})} & & \downarrow \mathbb{L}_{(\text{ev}_0, \text{ev}_1)} \\
\text{ev}_0^* \mathbb{L}_{\text{Grad}(\mathfrak{M})} & \to & \mathbb{L}_{\mathfrak{M}} \to \mathbb{L}_{\mathfrak{M}}[1]
\end{array}
\]

(5.3)

The last arrows induce the morphisms

\[
\begin{array}{ccc}
\Omega_{(\text{ev}_0, \text{ev}_1)}[-2] & \xrightarrow{h_1} & \Omega_{\mathfrak{M}}[-1] \\
\downarrow h_0 & & \\
\Omega_{\text{Grad}(\mathfrak{M})}[-1]
\end{array}
\]

(5.4)

We have the following relationship between the stack of filtered objects for $(-1)$-shifted cotangent stack and $(-2)$-shifted conormal stack of the morphism (5.1).
**Proposition 5.1.** There exists an equivalence of derived stacks

\[
\text{Filt}(\Omega_{\mathcal{M}}[-1]) \sim \Omega_{(ev_0,ev_1)}[-2]
\]

such that the following diagram is commutative

\[
\begin{array}{ccc}
\text{Filt}(\mathcal{M}) & \leftarrow & \text{Filt}(\Omega_{\mathcal{M}}[-1]) \\
\downarrow \sim & & \downarrow \sim \\
\text{Filt}(\mathcal{M}) & \leftarrow & \Omega_{(ev_0,ev_1)[-2]} \Omega_{\mathcal{M}}[-1].
\end{array}
\]

Here the top left arrow is induced by the projection \(\Omega_{\mathcal{M}}[-1] \to \mathcal{M}\), the bottom left arrow is the natural morphism \(5.2\).

**Proof.** We first describe the relative cotangent complex \(L_{(ev_0,ev_1)}\) for the morphism \(5.1\). Let \(ev_F: \Theta \times \text{Filt}(\mathcal{M}) \to \mathcal{M}\) be the universal morphism as in the diagram \(5.3\). By the Rees construction in Lemma \(4.1\), the object \(ev^*_F \mathbb{L}_{\mathcal{M}} \in D_{qcoh}(\Theta \times \text{Filt}(\mathcal{M}))\) is described by a diagram

\[
\begin{array}{ccc}
\text{ev}^*_F \mathbb{L}_{\mathcal{M}} = & \cdots & (ev^*_F \mathbb{L}_{\mathcal{M}})_1 \to (ev^*_F \mathbb{L}_{\mathcal{M}})_0 \to (ev^*_F \mathbb{L}_{\mathcal{M}})_{-1} \to \cdots
\end{array}
\]

for \((ev^*_F \mathbb{L}_{\mathcal{M}})_i \in D_{qcoh}(\text{Filt}(\mathcal{M}))\). By \(5.2\) and \(4.3\), we have

\[
(5.8) \quad \mathbb{L}_{\text{Filt}(\mathcal{M})} = \text{Cone} \left( (ev^*_F \mathbb{L}_{\mathcal{M}})_1 \to \text{colim} (ev^*_F \mathbb{L}_{\mathcal{M}})_i \right).
\]

From the commutative diagram

\[
\begin{array}{ccc}
\Theta \times \text{Filt}(\mathcal{M}) & \xrightarrow{ev_F} & \mathcal{M} \\
\downarrow \sim & & \downarrow \sim \\
\Theta \times \text{Filt}(\mathcal{M}) & \xrightarrow{(i_0,\text{id})} & \mathcal{M}
\end{array}
\]

and \(5.3\), we have

\[
(5.9) \quad ev^*_F \mathbb{L}_{\mathcal{M}} = i^*_1 ev^*_F \mathbb{L}_{\mathcal{M}} = \text{colim} (ev^*_F \mathbb{L}_{\mathcal{M}})_i.
\]

Also from the commutative diagram

\[
\begin{array}{ccc}
BC^* \times \text{Filt}(\mathcal{M}) & \xrightarrow{(i_0,\text{id})} & BC^* \times \text{Grad}(\mathcal{M}) \\
\downarrow (i_0,\text{id}) & & \downarrow \text{ev}_G \\
\Theta \times \text{Filt}(\mathcal{M}) & \xrightarrow{ev_F} & \mathcal{M}
\end{array}
\]

and \(5.3\), \(5.4\), we have

\[
(5.10) \quad ev_0^* \mathbb{L}_{\text{Grad}(\mathcal{M})} \cong ((i_0,\text{id})^* ev^*_G \mathbb{L}_{\mathcal{M}})_{\text{wt}=0} \\
\cong ((i_0,\text{id})^* ev^*_F \mathbb{L}_{\mathcal{M}})_{\text{wt}=0} \\
\cong \text{Cone} ((ev^*_F \mathbb{L}_{\mathcal{M}})_1 \to (ev^*_F \mathbb{L}_{\mathcal{M}})_0).
\]

We have the commutative diagram from \(5.8\), \(5.9\), \(5.10\), where horizontal arrows are distinguished triangles

\[
\begin{array}{ccc}
\mathbb{L}_{\text{Grad}(\mathcal{M})} & \xrightarrow{ev_0^*} & \mathbb{L}_{\text{Filt}(\mathcal{M})} \\
\sim & & \sim \\
\text{Cone} ((ev^*_F \mathbb{L}_{\mathcal{M}})_1 \to (ev^*_F \mathbb{L}_{\mathcal{M}})_0) & \xrightarrow{\sim} & \text{Cone} ((ev^*_F \mathbb{L}_{\mathcal{M}})_1 \to ev^*_1 \mathbb{L}_{\mathcal{M}}) & \text{Cone} ((ev^*_F \mathbb{L}_{\mathcal{M}})_0 \to ev^*_1 \mathbb{L}_{\mathcal{M}}).
\end{array}
\]
Therefore there is an isomorphism

\[ L_{ev_0} \cong \text{Cone} \left( (ev_1^* F_{L, M})_0 \to ev_1^* L_{M} \right) \]

which makes the above diagram commutative. We also have the commutative diagram, where horizontal arrows are distinguished triangles

\[
\begin{align*}
\text{ev}_1^* L_{M} & \longrightarrow \text{ev}_0^* L_{M} \rightarrow \text{ev}_0^* \text{ev}_1^* L_{M} \\
\text{ev}_1^* L_{M} \rightarrow \text{Cone} \left( (ev_1^* F_{L, M})_0 \to ev_1^* L_{M} \right) & \rightarrow (ev_1^* F_{L, M})_0[1] \rightarrow ev_1^* L_{M}[1]
\end{align*}
\]

Therefore there is an isomorphism

\[
L_{(ev_0, ev_1)} \cong (ev_1^* F_{L, M})_0[1]
\]

which makes the above diagram commutative.

Let \( T \) be a derived stack, and consider a diagram

\[
\begin{array}{ccc}
\text{Filt}(\Omega_{M}[-1]) & \longrightarrow & \text{Filt}(\mathcal{M}) \\
\downarrow & & \downarrow \\
\Theta \times T & \longrightarrow & \mathcal{M} \\
\text{id}, f & & \\
\Theta \times \text{Filt}(\mathcal{M}) & \longrightarrow & \mathcal{M},
\end{array}
\]

By the definition of the stack of filtered objects, the above diagram corresponds to a diagram

\[
\begin{array}{ccc}
\Omega_{\mathcal{M}}[-1] & \longrightarrow & \Omega_{\mathcal{M}}[-1] \\
\downarrow & & \downarrow \\
\Theta \times T & \longrightarrow & \mathcal{M} \\
g & & \\
\Theta \times \text{Filt}(\mathcal{M}) & \longrightarrow & \mathcal{M}.
\end{array}
\]

Here \( f, g \) fit into the commutative diagram

\[
\begin{array}{ccc}
\Theta \times T & \longrightarrow & \text{Filt}(\mathcal{M}) \\
\downarrow (\text{id}, f) & & \downarrow g \\
\Theta \times \text{Filt}(\mathcal{M}) & \longrightarrow & \mathcal{M}.
\end{array}
\]

By the definition of \( \Omega_{\mathcal{M}}[-1] \) and the above commutative diagram, giving a dotted arrow in (5.14) is equivalent to giving a morphism

\[
\mathcal{O}_{\Theta \times T} \to g^* \mathbb{L}_{\mathcal{M}}[-1] = (\text{id}, f)^* F_{L, M} \mathbb{L}_{\mathcal{M}}[-1].
\]

Under the Rees construction, the above morphism corresponds to a commutative diagram

\[
\begin{array}{cccc}
\cdots & \longrightarrow & 0 & \longrightarrow \mathcal{O}_T \longrightarrow \mathcal{O}_T \longrightarrow \cdots \\
\downarrow & & \downarrow \text{id} & & \downarrow \text{id} & & \downarrow \\
\cdots & \longrightarrow & f^* (ev_1^* F_{L, M})_1[-1] & \longrightarrow f^* (ev_1^* F_{L, M})_0[-1] & \longrightarrow f^* (ev_1^* F_{L, M})_{0}[-1] & \longrightarrow \cdots
\end{array}
\]

Together with the isomorphism (5.12), we see that giving a morphism (5.15) is equivalent to giving a morphism

\[
\mathcal{O}_T \to f^* (ev_1^* F_{L, M})_0[-1] \cong f^* \mathbb{L}_{(ev_0, ev_1)}[-2].
\]
It follows that giving a dotted arrow in (5.13) is equivalent to giving a dotted arrow in the diagram

\[
\begin{array}{c}
\Omega_{(ev_0, ev_1)}[-2] \\
\downarrow f \\
\text{Filt}({\mathcal{M}}).
\end{array}
\]

Therefore we obtain an equivalence (5.2) together with the left commutative diagram in (5.6). The right commutative diagram in (5.6) follows from the right square in the diagram (5.11). □

In the proof of the above proposition, we described the cotangent complex of \( \text{Filt}({\mathcal{M}}) \) as in (5.8). Now suppose that \( {\mathcal{M}} \) is quasi-smooth and QCA, so in particular \( \text{Filt}({\mathcal{M}}) \) is also quasi-smooth by Lemma 4.3. Then \( \text{L}_{\text{Filt}({\mathcal{M}})} \) is perfect, so we have its determinant line bundle \( \text{det} \text{L}_{\text{Filt}({\mathcal{M}})} \). Later we will use its description. We use the following commutative diagram

\[
\begin{array}{c}
\text{Filt}({\mathcal{M}}) \\
\downarrow (\pi, id) \\
\Theta \times \text{Grad}({\mathcal{M}}) \\
\downarrow (i_0, id) \\
\Theta \times \text{Grad}({\mathcal{M}}) \to \Theta \times \text{Filt}({\mathcal{M}}) \\
\downarrow ev_0 \\
\text{Grad}({\mathcal{M}}) \to \text{ev}_0 \text{Filt}({\mathcal{M}}).
\end{array}
\]

(5.17)

\[
\text{Lemma 5.2. There is an isomorphism of line bundles}
\]

\[
\text{det} \text{L}_{\text{Filt}({\mathcal{M}})} \cong ev_0^* \text{det} (\pi_G + (id, \sigma)^* \text{ev}_F^* \text{L}_{{\mathcal{M}}}).
\]

Proof. Let \( (\text{ev}_F^* \text{L}_{{\mathcal{M}}})_i \) be as in (5.7), and set

\[
\text{gr}_i (\text{ev}_F^* \text{L}_{{\mathcal{M}}}) := \text{Cone} ((\text{ev}_F^* \text{L}_{{\mathcal{M}}})_{i+1} \to (\text{ev}_F^* \text{L}_{{\mathcal{M}}})_i).
\]

We also set \( \text{gr}_{\geq j} (\text{ev}_F^* \text{L}_{{\mathcal{M}}}) \) to be the direct sum of (5.18) for all \( i \geq j \), and \( \text{gr}_* (\text{ev}_F^* \text{L}_{{\mathcal{M}}}) \) to be the direct sum of (5.18) for all \( i \). Then we have

\[
(i_0, id)^* \text{ev}_F^* \text{L}_{{\mathcal{M}}} \cong \text{gr}_* (\text{ev}_F^* \text{L}_{{\mathcal{M}}})
\]

and they are perfect, so we have \( \text{gr}_i (\text{ev}_F^* \text{L}_{{\mathcal{M}}}) = 0 \) for \( |i| \gg 0 \) and each \( \text{gr}_* (\text{ev}_F^* \text{L}_{{\mathcal{M}}}) \) is also perfect. It follows that each \( (\text{ev}_F^* \text{L}_{{\mathcal{M}}})_i \) is also perfect. From the diagram (5.17), we have the isomorphisms

\[
\text{ev}_0^* \text{det} (\pi_G + (id, \sigma)^* \text{ev}_F^* \text{L}_{{\mathcal{M}}}) \cong \text{det} (\pi_F + (id, ev_0)^* (id, \sigma)^* \text{ev}_F^* \text{L}_{{\mathcal{M}}})
\]

\[
\cong \text{det} (\pi_F + (pr, id)^* (i_0, id)^* \text{ev}_F^* \text{L}_{{\mathcal{M}}})
\]

\[
\cong \text{det} \text{Cone} (\text{gr}_{\geq 1} (\text{ev}_F^* \text{L}_{{\mathcal{M}}}) \to \text{gr}_* (\text{ev}_F^* \text{L}_{{\mathcal{M}}}))
\]

\[
\cong \text{det} \text{Cone} (\text{ev}_F^* \text{L}_{{\mathcal{M}}})_{i+1} \to \text{colim} (\text{ev}_F^* \text{L}_{{\mathcal{M}}})_i
\]

\[
\cong \text{det} \text{L}_{\text{Filt}({\mathcal{M}})}.
\]

□

5.2. The stacks of graded objects on \((-1)\)-shifted cotangents. In a way similar to Proposition 5.1 we have the following lemma:

\[
\text{Lemma 5.3. There exists an equivalence of derived stacks}
\]

\[
\text{Grad}({\mathcal{M}})[-1] \sim \Omega_{\text{Grad}({\mathcal{M}})}[-1]
\]

such that the following diagram is commutative

\[
\begin{array}{c}
\text{Grad}({\mathcal{M}}) \\
\downarrow \sim \\
\Omega_{\text{Grad}({\mathcal{M}})}[-1] \\
\downarrow \sim \\
\text{Filt}({\mathcal{M}}) \\
\downarrow \sim \\
\Omega_{(ev_0, ev_1)}[-2].
\end{array}
\]

(5.20)
Here the top left arrow is induced by the projection $\Omega_{\mathcal{M}}[-1] \to \mathcal{M}$, the bottom left arrow is the natural projection, and the right vertical arrow is given in Proposition 5.1.

Proof. Let $T$ be a derived stack, and consider a diagram

\[
\begin{array}{ccc}
\text{Grad}(\Omega_{\mathcal{M}}[-1]) & \xrightarrow{i} & \text{Grad}(\mathcal{M}) \\
T \xrightarrow{f} & & \xrightarrow{g} \\
\end{array}
\]

By the definition of the stack of graded objects, the above diagram corresponds to a diagram

\[
\begin{array}{ccc}
\Omega_{\mathcal{M}}[-1] & \xrightarrow{g} & \mathcal{M} \\
\text{BC}^* \times T \xrightarrow{\text{id}, f} & & \xrightarrow{g} \\
\end{array}
\]

(5.21)

Here $f, g$ fit into the commutative diagram

\[
\begin{array}{ccc}
\text{BC}^* \times \mathcal{M} & \xrightarrow{\text{id}, f} & \text{BC}^* \times \text{Grad}(\mathcal{M}) \\
\xrightarrow{g} & & \xrightarrow{\text{ev}_G} \mathcal{M}. \\
\end{array}
\]

By the definition of $\Omega_{\mathcal{M}}[-1]$ and the above commutative diagram, giving a dotted arrow in (5.21) is equivalent to giving a morphism

\[ \mathcal{O}_{\text{BC}^* \times T} \to g^* L_{\mathcal{M}}[-1] = (\text{id}, f)^* \text{ev}_G^* L_{\mathcal{M}}[-1]. \]

By (4.4) and (4.9), giving the above morphism is equivalent to giving a morphism

\[ \mathcal{O}_T \to f^* (\text{ev}_G^* L_{\mathcal{M}})_{wT=0}[-1] = f^* L_{\text{Grad}(\mathcal{M})}[-1]. \]

Therefore giving a dotted arrow in (5.21) is equivalent to giving a dotted arrow in the diagram

\[
\begin{array}{ccc}
\Omega_{\text{Grad}(\mathcal{M})}[-1] & \xrightarrow{i} & \text{Grad}(\mathcal{M}) \\
T \xrightarrow{f} & & \xrightarrow{g} \\
\end{array}
\]

which implies the equivalence (5.19) together with the left commutative diagram in (5.20).

Finally we check that the right diagram in (5.20) is commutative. Let us consider a diagram (5.13) and its composition with $\text{ev}_0^\Omega$

\[
\begin{array}{ccc}
\text{Filt}(\Omega_{\mathcal{M}}[-1]) & \xrightarrow{\text{ev}_0^\Omega} & \text{Grad}(\Omega_{\mathcal{M}}[-1]) \\
\xrightarrow{i} & & \xrightarrow{f} \\
T \xrightarrow{g} & & \xrightarrow{\text{Filt}(\mathcal{M})} \\
\end{array}
\]

The composition of the above dotted arrow with $\text{ev}_0^\Omega$ corresponds to the composition of the dotted arrow in the diagram (5.14) with $i_1 \times \text{id}_T: \text{BC}^* \times T \to \Theta \times T$

\[
\begin{array}{ccc}
\Omega_{\mathcal{M}}[-1] & \xrightarrow{i} & \Theta \times \mathcal{M} \\
\text{BC}^* \times T \xrightarrow{\text{id}_0, \text{id}_T} & & \xrightarrow{g} \\
\end{array}
\]
It corresponds to the morphism given by the pull-back of \((5.13)\) by \((i_0 \times \text{id}_T)\)
\[
\mathcal{O}_T \to ((i_0, \text{id}_T)^* g^* \mathcal{L}_\mathfrak{M})_{\text{wt}=0}[-1] = \text{Cone}(f^*(\text{ev}_F^* \mathcal{L}_\mathfrak{M})_1 \to f^*(\text{ev}_F^* \mathcal{L}_\mathfrak{M})_0)[-1],
\]
which is the composition of \((5.13)\) with the natural morphism
\[
f^*(\text{ev}_F^* \mathcal{L}_\mathfrak{M})_0[-1] \to \text{Cone}(f^*(\text{ev}_F^* \mathcal{L}_\mathfrak{M})_1 \to f^*(\text{ev}_F^* \mathcal{L}_\mathfrak{M})_0)[-1].
\]
Then the right commutative diagram in \((5.20)\) follows from the commutative diagram
\[
\begin{array}{c}
\text{Cone}(f^*(\text{ev}_F^* \mathcal{L}_\mathfrak{M})_1 \to f^*(\text{ev}_F^* \mathcal{L}_\mathfrak{M})_0)[-1] \\
\sim \\
L_{(\text{ev}_0, \text{ev}_1)}[-2] \rightarrow \text{ev}_0^* L_{\text{Grad}(\mathfrak{M})}[-1].
\end{array}
\]
Here the left vertical arrow is \((5.12)\) and the the right vertical arrow is \((5.10)\), and the horizontal arrows are natural morphisms. \(\square\)

5.3. Compatibility with singular supports. We now show that, using Proposition \(5.1\) and Lemma \(5.3\) some natural functors induced by the stacks of filtered objects preserve singular supports in some sense. We first consider the following commutative diagram
\[
\begin{array}{ccc}
\text{Filt}(\Omega^\mathfrak{M}[-1]) & \longrightarrow & \text{Filt}(\mathfrak{M}) \\
\text{ev}_0^\Omega & & \text{ev}_0 \\
\text{Grad}(\Omega^\mathfrak{M}[-1]) & \longrightarrow & \text{Grad}(\mathfrak{M}).
\end{array}
\]
Here the horizontal arrows are induced by the projection \(\Omega^\mathfrak{M}[-1] \to \mathfrak{M}\).

Lemma 5.4. The diagram \((5.22)\) induces the isomorphisms of connected components,
\[
\pi_0(\text{Filt}(\Omega^\mathfrak{M}[-1])) \xrightarrow{\cong} \pi_0(\text{Filt}(\mathfrak{M}))
\]
\[
\pi_0(\text{Grad}(\Omega^\mathfrak{M}[-1])) \xrightarrow{\cong} \pi_0(\text{Grad}(\mathfrak{M})).
\]

Proof. By Proposition \(5.1\) and Lemma \(5.3\) there exist \(\mathbb{C}^*\)-actions on \(\text{Filt}(\Omega^\mathfrak{M}[-1]), \text{Grad}(\Omega^\mathfrak{M}[-1])\)
which specialize to \(\text{Filt}(\mathfrak{M}), \text{Grad}(\mathfrak{M})\) respectively. Therefore the horizontal arrows in \((5.23)\) are isomorphisms. The vertical isomorphisms are proved in \((HL)\) Lemma 1.24. \(\square\)

Given a quasi-compact open and closed substack \(\mathcal{E} \subset \text{Filt}(\mathfrak{M})\) with the associated diagram \((4.10)\), by Lemma \(5.4\) we have the associated diagrams for \(\Omega^\mathfrak{M}[-1]\) and its classical truncation
\[
\begin{array}{ccc}
\mathcal{E}^\Omega & \xrightarrow{\text{ev}_1^\Omega} & \Omega^\mathfrak{M}[-1], \\
\sigma^\Omega & & \tau^\Omega \\
3^\Omega & \xrightarrow{\text{ev}_0^\Omega} & \mathcal{E}^\Omega
\end{array}
\quad
\begin{array}{ccc}
\mathcal{S}^\Omega & \xrightarrow{\text{ev}_1^\Omega} & t_0(\Omega^\mathfrak{M}[-1]). \\
\sigma^\Omega & & \\
\mathcal{Z}^\Omega & \xrightarrow{\text{ev}_0^\Omega} & \mathcal{S}^\Omega
\end{array}
\]
Here \(\mathcal{E}^\Omega\) is the open and closed substack of \(\text{Filt}(\Omega^\mathfrak{M}[-1])\) corresponding to \(\mathcal{E}\) by the top isomorphism in \((5.23)\), \(3^\Omega\) is the center of \(\mathcal{E}^\Omega\), and the right diagram is induced by the left one by taking the classical truncation. By Proposition \(5.1\) and Lemma \(5.3\) we have the commutative isomorphisms
\[
\begin{array}{ccc}
\mathcal{Z}^\Omega & \xrightarrow{\text{ev}_0^\Omega} & \mathcal{S}^\Omega \\
\cong & & \cong \\
t_0(\Omega^\mathfrak{M}[-1]) & \xrightarrow{\text{ev}_0} & t_0(\Omega^\mathfrak{M}[-1])
\end{array}
\]
\[
\begin{array}{ccc}
\mathcal{S}^\Omega & \xrightarrow{\text{ev}_1^\Omega} & t_0(\Omega^\mathfrak{M}[-1]). \\
\cong & & \cong \\
\mathcal{Z}^\Omega & \xrightarrow{\text{ev}_0^\Omega} & \mathcal{S}^\Omega
\end{array}
\]
Suppose that 𝜓 is quasi-smooth and Θ-reductive, so that in the diagram (4.10) the morphism \( e_v^0 \) is quasi-smooth and \( e_v^1 \) is proper. Then we have the functor
\[
(5.25) \quad \Upsilon: D^b_{\text{coh}}(3) \xrightarrow{e_v^0} D^b_{\text{coh}}(\mathcal{S}) \xrightarrow{e_v^1} D^b_{\text{coh}}(\mathcal{M}).
\]
Let \( T_0 \subset Z^\Omega \) and \( T_1 \subset t_0(\Omega_\mathcal{M}[−1]) \) be conical closed substacks such that the following holds:
\[
(5.26) \quad (e_v^0)^{-1}(T_0) \subset (e_v^1)^{-1}(T_1) \subset S^\Omega.
\]
Below we identify \( T_0 \) as a closed substack in \( t_0(\Omega_\mathcal{M}[−1]) \) by the left isomorphism in (5.24). We have the following lemma:

**Lemma 5.5.** \( (\text{Tod}20 \; \text{Proposition 2.4}) \) Under the condition (5.26), the functor \( \Upsilon \) in (5.25) preserves singular supports, i.e., it restricts to the functor
\[
\Upsilon: C_{T_0} \to C_{T_1}.
\]

**Proof.** Using the notation of the diagram (3.16), the diagram (4.10) induces the following diagram (5.27)

Here the middle square is Cartesian by \( \text{Tod}20 \) (2.18). As \( e_v^0 \) is quasi-smooth and \( e_v^1 \) is proper, the morphism \( e_v^0 \) is a closed immersion and \( e_v^1 \) is proper, so \( h_1 \) is also proper. The morphisms \( h_0, h_1 \) are restrictions of the classical truncations of the morphisms in (5.4). By \( \text{AG15} \) Lemma 8.4.2, the functor \( e_v^0 \) in (5.27) restricts to the functor
\[
e_v^0: C_{T_0} \to C_{e_v^0(e_v^1)^{-1}(T_0)}.
\]
By \( \text{AG15} \) Lemma 8.4.5, the functor \( e_v^1 \) in (5.25) restricts to the functor
\[
e_v^1: C_{e_v^0(e_v^1)^{-1}(T_0)} \to C_{e_v^1(e_v^0)^{-1}e_v^0(e_v^1)^{-1}(T_0)}.
\]

The lemma follows from
\[
e_v^1(e_v^1)^{-1}e_v^0(e_v^1)^{-1}(T_0) = h_1(h_0)^{-1}(T_0) \subset T_1.
\]

Here the last inclusion follows from (5.26) and the diagram (5.24). \( \square \)

### 5.4. Comparison of Θ-strata.
Finally in this section, we observe a relationship between Θ-strata for \( \mathcal{M} \) and \( t_0(\Omega_\mathcal{M}[−1]) \). Let \( \mathcal{S} \subset \text{Filt}(\mathcal{M}) \) be as in the previous subsection, and set \( S = t_0(\mathcal{S}) \) as in the diagram (4.10). We have the commutative diagram
\[
(5.28) \quad \mathcal{S}^\Omega \xrightarrow{e_v^0} t_0(\Omega_\mathcal{M}[−1]) \xrightarrow{e_v^1} \mathcal{M}.
\]

Here the left vertical arrow is induced by the projection \( \Omega_\mathcal{M}[−1] \to \mathcal{M} \).

**Lemma 5.6.** Suppose that \( \mathcal{M} \) is quasi-smooth and the morphism \( e_v^1: \mathcal{S} \to \mathcal{M} \) is a closed immersion. The \( e_v^1: \mathcal{S}^\Omega \to t_0(\Omega_\mathcal{M}[−1]) \) is a closed immersion.
Proof. From the distinguished triangle (5.33), we have the exact sequence of coherent sheaves on $S$
$$
H^{-2}(L_{e_0}) \to H^{-2}(L_{(e_0,e_1)}) \to H^{-1}(ev^*_1 L_{\mathfrak{M}}).
$$
By Lemma 4.3 we have $H^{-2}(L_{e_0}) = 0$. Together with the middle vertical isomorphism in (5.24),
we have the closed immersion
(5.29)
$$
\mathfrak{S}^\Omega \hookrightarrow S \times_M t_0(\Omega_{\mathfrak{M}}[-1]).
$$
Therefore the lemma holds.

By the above lemma, a $\Theta$-strata $S \hookrightarrow M$ gives rise to a $\Theta$-strata $\mathfrak{S}^\Omega \hookrightarrow t_0(\Omega_{\mathfrak{M}}[-1])$. On the other hand, the diagram (5.28) is not necessary Cartesian (see Example 5.7 below), so a $\Theta$-stratification
for $M$ does not necessary pull-back to a $\Theta$-stratification for $t_0(\Omega_{\mathfrak{M}}[-1])$.

**Example 5.7.** Let $V_0, V_1$ be finite dimensional $\mathbb{C}^*$-representations, and set
$$
\mathfrak{U} = \text{Spec} \mathbb{C}[V_1'][1] \oplus V_0', \quad \mathfrak{M} = [\mathfrak{U}/\mathbb{C}^*].
$$
Here the differential on $\mathbb{C}[V_1'[1] \oplus V_0']$ is zero. Then we have
$$
t_0(\Omega_{\mathfrak{M}}[-1]) = [(V_0 \oplus V_1')/\mathbb{C}^*]
$$
and a diagram (5.28) is of the form
$$
\begin{array}{ccc}
[(V_0 \oplus V_1')^{\lambda \geq 0}/\mathbb{C}^*] & \longrightarrow & [(V_0 \oplus V_1')/\mathbb{C}^*] \\
/ \downarrow & & / \downarrow \\
[V_0^{\lambda \geq 0}/\mathbb{C}^*] & \longrightarrow & [V_0/\mathbb{C}^*]
\end{array}
$$
for some one parameter subgroup $\lambda: \mathbb{C}^* \to \mathbb{C}^*$. Each horizontal arrow is a closed immersion, but
the above diagram is Cartesian if and only if $V_1^{\lambda > 0} = 0$, or equivalently $H^{-1}(\mathbb{L}_{\mathfrak{M}}|_0) = V_1'$ has only
non-negative weights.

We have the following sufficient condition for the diagram (5.28) to be a Cartesian:

**Lemma 5.8.** Suppose that $H^{-1}(\mathbb{L}_{\mathfrak{M}}|_z)$ has only non-negative weights, where $(-)|_z$ is the pull-back
via $Z \hookrightarrow \mathfrak{S} \hookrightarrow \mathfrak{M}$. Then the diagram (5.28) is a Cartesian.

Proof. From the proof of Lemma 5.6 we have the exact sequence
(5.30)
$$
0 \to H^{-2}(L_{(e_0,e_1)}|_z) \to H^{-1}(\mathbb{L}_{\mathfrak{M}}|_z) \to H^{-1}(L_{e_0}|_z).
$$
Here we have pulled back objects on $\mathfrak{G}$ via $Z \hookrightarrow \mathfrak{S} \hookrightarrow \mathfrak{G}$. We also have the distinguished triangle
$$
\mathbb{L}_\lambda|_z \to \mathbb{L}_0|_z \to \mathbb{L}_{e_0}|_z.
$$
By [HL3] Lemma 1.5.5, the object $\mathbb{L}_{e_0}|_z$ has only non-positive weights such that $\mathbb{L}_\lambda|_z$ is the direct
summand of the weight zero part. Therefore $\mathbb{L}_{e_0}|_z$ has only negative weights. By the assumption
that $H^{-1}(\mathbb{L}_{\mathfrak{M}}|_z)$ has only non-negative weights, it follows that the last map in (5.30) is a zero
map. Therefore the second map in (5.30) is an isomorphism, so the closed immersion (5.29) is an
isomorphism on $Z \hookrightarrow \mathfrak{S}$. Since any point in $\mathfrak{S}$ is specialized to a point in $Z$, and the locus in $\mathfrak{S}$ where
(5.29) is an isomorphism is open, it follows that the closed immersion (5.29) is an isomorphism. □

6. Semiorthogonal decompositions of DT categories

In this section, we give a proof of Theorem 1.1. In [Tod1] Section 6], we proved a similar result
for moduli spaces of PT stable pairs using categorified Hall products [PS]. The argument in this
section is almost a repetition of [Tod1] Section 6] with some modifications.
6.1. **Theta stratifications of \((-1)\)-shifted cotangents.** Let \(\mathfrak{M}\) be a quasi-smooth and QCA derived stack, such that its classical truncation \(\mathcal{M} = t_0(\mathfrak{M})\) admits a good moduli space

\[ \pi_\mathcal{M} : \mathcal{M} \to M. \]

Let \(\mathcal{N}\) be the classical truncation of the \((-1)\)-shifted cotangent

(6.1)

\[ p_0 : \mathcal{N} = t_0(\Omega_\mathfrak{M}[-1]) \to \mathcal{M}. \]

Here we discuss \(\Theta\)-stratification on \(\mathcal{N}\). First we have the following lemma.

**Lemma 6.1.** The stack \(\mathcal{N}\) also admits a good moduli space \(\mathcal{N} \to N\).

**Proof.** Let \(f\) be the composition

\[ f : \mathcal{N} \xrightarrow{p_0} \mathcal{M} \xrightarrow{\pi_\mathcal{M}} M. \]

We set \(N := \text{Spec}_M f_*\mathcal{O}_N\). Since \(p_0\) is an affine morphism, the natural morphism \(\mathcal{N} \to N\) is checked to be a good moduli space morphism. \(\square\)

Let us take \(L \in \text{Pic}(\mathcal{M})\) with \(l = c_1(L) \in H^2(\mathcal{M}, \mathbb{R})\) and \(b \in H^4(\mathcal{M}, \mathbb{R})\) such that \(b\) is positive definite. We also regard them as elements in \(\text{Pic}(\mathcal{N})\), \(H^2(\mathcal{N}, \mathbb{R})\) and \(H^4(\mathcal{N}, \mathbb{R})\) by the pull-back via (6.1). By Theorem 4.9 and Lemma 6.1, there is a \(\Theta\)-stratification with respect to \(l\) and \(b\)

(6.2)

\[ N = S_1^\Omega \sqcup \cdots \sqcup S_n^\Omega \sqcup N^\text{ss}. \]

For each \(i\), we denote by \(S_i^\Omega\) the union of \(S_j^\Omega\) for \(j \leq i\) which is a closed substack of \(N\). We have the diagram

(6.3)

\[
\begin{array}{ccc}
S_i^\Omega & \xrightarrow{ev_i^\Omega} & N \\
\downarrow \sigma_i^\Omega & & \downarrow \tau_i^\Omega \\
Z_i^\Omega & \xrightarrow{ev_0^\Omega} & N_{\leq i-1}
\end{array}
\]

where \(Z_i^\Omega\) is the center of \(S_i^\Omega\). Note that we have open immersions

\[ S_i^\Omega \subset \text{Filt}(N \setminus S_{\leq i-1}^\Omega) \subset \text{Filt}(N), \quad Z_i^\Omega \subset \text{Grad}(N \setminus S_{\leq i-1}^\Omega) \subset \text{Grad}(N) \]

where the first inclusions are open and closed, and the second inclusions are induced by the open immersion \(N \setminus S_{\leq i-1}^\Omega \subset N\).

We define

\[ \overline{S}_i^\Omega \subset \text{Filt}(N), \quad \overline{Z}_i^\Omega \subset \text{Grad}(N) \]

to be the smallest open and closed substacks which contain \(S_i^\Omega, Z_i^\Omega\) respectively. Note that \(\overline{Z}_i^\Omega\) is the center of \(\overline{S}_i^\Omega\), and the diagram (6.3) extends to the diagram

(6.4)

\[
\begin{array}{ccc}
\overline{S}_i^\Omega & \xrightarrow{ev_i^\Omega} & N \\
\downarrow \sigma_i^\Omega & & \downarrow \tau_i^\Omega \\
\overline{Z}_i^\Omega & \xrightarrow{ev_0^\Omega} & \overline{N}_{\leq i-1}
\end{array}
\]

We have the following lemma:

**Lemma 6.2.** The following diagrams are Cartesian

(6.5)

\[
\begin{array}{ccc}
S_i^\Omega & \xrightarrow{ev_i^\Omega} & S_i^\Omega \\
\downarrow \square & & \downarrow \square \\
N \setminus S_{\leq i-1}^\Omega & \xrightarrow{ev_i^0} & Z_i^\Omega \\
\end{array}
\]

\[
\begin{array}{ccc}
S_i^\Omega & \xrightarrow{ev_i^\Omega} & \overline{S}_i^\Omega \\
\downarrow \square & & \downarrow \square \\
N \setminus S_{\leq i-1}^\Omega & \xrightarrow{ev_i^0} & \overline{Z}_i^\Omega \\
\end{array}
\]
Proof. The left diagram is Cartesian by the unique maximizer condition in Definition 4.8. As for the right diagram, let us take a map \( f: \Theta \to \mathcal{N} \) in \( S^i_\Omega \) and suppose that \( f(0): BC^* \to \mathcal{N} \) lies \( Z^\Omega_i \). Then \( f(0) \) factors as

\[
f(0): BC^* \to \mathcal{N} \setminus S^\Omega_{\leq i-1} \subset \mathcal{N}.
\]

Therefore \( f(1) \in \mathcal{N} \setminus S^\Omega_{\leq i-1} \) as \( \mathcal{N} \setminus S^\Omega_{\leq i-1} \subset \mathcal{N} \) is open. By the left Cartesian diagram in (6.5), we conclude that \( f \) lies in \( S^\Omega_i \).

We define open and closed substacks

\[
S_i \subset \text{Filt}(\mathcal{M}), \ Z_i \subset \text{Grad}(\mathcal{M})
\]

to be corresponding to \( S^\Omega_i, Z^\Omega_i \) under the isomorphisms (5.23). Then \( Z_i \) is the center of \( S_i \) and we have the following diagram as in (4.10)

(6.6)

Note that, although \( S^\Omega_i \) forms a \( \Theta \)-stratification for \( \mathcal{N} \), the morphism \( e\nu_1 \) in (6.6) is not necessary a closed immersion so that \( S_i \) may not form a \( \Theta \)-stratification of \( \mathcal{M} \). We set \( T^\Omega_i \) to be

\[
T^\Omega_i := Z^\Omega_i \setminus Z_i \subset Z^\Omega_i \to t_0(\Omega_3[-1]).
\]

Here the last isomorphism is given in Lemma 5.3. We regard \( T^\Omega_i \) as a closed substack of \( t_0(\Omega_3[-1]) \) by the above isomorphism, which is conical. Following Definition 3.5, the DT categories for \( Z^\Omega_i \) and \( \mathcal{N} \setminus S^\Omega_{\leq i-1} \) are defined by

\[
DT^C(\mathcal{N}) := D^b_{\text{coh}}(\mathcal{M})/C_{T^\Omega_i}, \ DT^C(\mathcal{N} \setminus S^\Omega_{\leq i-1}) := D^b_{\text{coh}}(\mathcal{M})/C_{S^\Omega_{\leq i-1}}.
\]

Since \( \mathcal{M} \) is quasi-smooth and \( \Theta \)-reductive, in the diagram (6.6) the morphism \( e\nu_0 \) is quasi-smooth and \( e\nu_1 \) is proper. Therefore for each \( i \), we have the functor

(6.7)

\[
\Upsilon_i: D^b_{\text{coh}}(Z^\Omega_i) \xrightarrow{e\nu_0} D^b_{\text{coh}}(S_i) \xrightarrow{e\nu_1} D^b_{\text{coh}}(\mathcal{M}).
\]

Lemma 6.3. The functor (6.7) descends to the functor

(6.8)

\[
\overline{\Upsilon}_i: DT^C(S^\Omega_i) \to DT^C(\mathcal{N} \setminus S^\Omega_{\leq i-1}).
\]

Proof. By the Cartesian squares (6.5) and Lemma 6.2 we have the identity in the diagram (6.3)

(6.9)

\[
(e\nu_0^\Omega)^{-1}(T^\Omega_i) = (e\nu_0^\Omega)^{-1}(S^\Omega_{\leq i-1}) = S^\Omega_i \setminus S^\Omega_i \subset S^\Omega_i.
\]

Therefore by Lemma 5.5, the functor (6.7) sends \( C_{T^\Omega_i} \) to \( C_{S^\Omega_{\leq i-1}} \). By taking the quotients, we obtain the functor (6.8) as a descendant of (6.7).

The weight decomposition (4.7) respects the singular supports, so we have the decomposition of the DT category for \( Z^\Omega_i \) into weight space categories (see [Toda Subsection 3.2.3])

\[
DT^C(\mathcal{N}) = \bigoplus_{j \in \mathbb{Z}} DT^C(\mathcal{N})_{\text{wt}=j}.
\]

We denote by

\[
DT^C(\mathcal{N}) \xrightarrow{p_j} DT^C(\mathcal{N})_{\text{wt}=j}
\]
the projection onto the weight space category, inclusion from the weight space category, respectively.

We define $\Upsilon_{i,j}$ to be the composition

$$\Upsilon_{i,j} : DT^C(\Omega_{i \leq j}^\infty) \to DT^C(\Omega_{i}^\infty) \to DT^C(\mathcal{N} \setminus S_{\leq i-1}^\infty).$$

### 6.2. Descriptions along formal fibers.

Here we give a description of the diagram (6.6), formally locally on the good moduli space $M$. Below we assume that the good moduli space morphism $\pi : M \to M$ satisfies the formal neighborhood theorem in Definition 3.3. We take a closed point $y \in M$ and use the same symbol $y$ to denote the unique closed point in the fiber of $\pi_M : M \to M$ at $y$, and $G_y := \text{Aut}(y)$. By the formal neighborhood theorem, there exists a $G_y$-equivariant Kuranishi map

$$\kappa_y : \hat{\mathcal{M}}_y \to \mathcal{Z}_i, \quad (6.11)$$

as in Definition 3.3 and an equivalence

$$\mathcal{M}_y \sim [\hat{\mathcal{M}}_y/G_y]$$

(see the notation in Subsection 3.5).

From Lemma 4.11, we have the commutative diagram

Here $S_i, Z_i$ are classical truncations of $\mathfrak{S}_i, \mathfrak{Z}_i$ in the diagram (6.6). By taking the fiber products with Spec $\hat{O}_{M,y} \to M$, we obtain the diagram

$$\text{(6.12)}$$

Here $\hat{S}_{i,y} := S_i \times_M \text{Spec } \hat{O}_{M,y}$ is an open and closed substack of $\text{Filt}(\hat{M}_y)$ by Lemma 4.11 (ii), $\hat{Z}_{i,y}$ is its center, and

$$\hat{\mathcal{M}}_{i,y} \subset \text{Filt}(\hat{M}_y), \quad \hat{\mathcal{S}}_{i,y} \subset \text{Grad}(\hat{M}_y)$$

are the corresponding components under the isomorphisms (4.9). We have the following lemma:

**Lemma 6.4.** The following diagrams are Cartesians:

$$\text{(6.13)}$$

**Proof.** The diagrams (6.13) are Cartesians on classical truncations by the constructions. It is enough to show that

$$\tilde{j}^* \mathcal{L}_\mathfrak{S} \cong \mathcal{L}_{\hat{\mathcal{S}}_{i,y}}, \quad \tilde{j}^* \mathcal{L}_{\mathfrak{Z}} \cong \mathcal{L}_{\hat{\mathcal{Z}}_{i,y}}.$$
Since we have the isomorphism \( \hat{\iota}^* y L \cong \hat{\iota}^* M \), the above isomorphisms easily follow from the descriptions of cotangent complexes of stacks filtered objects and graded objects in terms of \( L \) as in (4.9).

For a one parameter subgroup \( \lambda : C^* \to G_y \), from Example 4.2 and Lemma 4.11 (ii) we have the following diagram

\[
\begin{array}{ccc}
\mathcal{H}^0(T\mathcal{M}|_y)^{\lambda \geq 0}/G_y^{\lambda \geq 0} & \xrightarrow{\hat{\iota}^* y} & \mathcal{H}^0(T\mathcal{M}|_y)/G_y \\
\downarrow & & \downarrow \\
\mathcal{H}^0(T\mathcal{M}|_y)^{\lambda = 0}/G_y^{\lambda = 0} & \xrightarrow{\hat{\iota}^* y} & \mathcal{H}^0(T\mathcal{M}|_y)/G_y.
\end{array}
\]

By taking the formal fibers at \( 0 \in \mathcal{H}^0(T\mathcal{M}|_y)/G_y \), we obtain the commutative diagram

\[
\begin{array}{ccc}
\hat{\mathcal{H}}^0(T\mathcal{M}|_y)^{\lambda \geq 0}/G_y^{\lambda \geq 0} & \xrightarrow{\hat{\iota}^* y} & \hat{\mathcal{H}}^0(T\mathcal{M}|_y)/G_y \\
\downarrow & & \downarrow \\
\hat{\mathcal{H}}^0(T\mathcal{M}|_y)^{\lambda = 0}/G_y^{\lambda = 0} & \xrightarrow{\hat{\iota}^* y} & \hat{\mathcal{H}}^0(T\mathcal{M}|_y)/G_y.
\end{array}
\]

Since \( \kappa_y \) in (6.11) is \( G_y \)-equivariant it restricts to the following maps

\[
\begin{align*}
\kappa_y^{\lambda \geq 0} : \hat{\mathcal{H}}^0(T\mathcal{M}|_y)^{\lambda \geq 0} & \to \hat{\mathcal{H}}^1(T\mathcal{M}|_y)^{\lambda \geq 0}, \\
\kappa_y^{\lambda = 0} : \hat{\mathcal{H}}^0(T\mathcal{M}|_y)^{\lambda = 0} & \to \hat{\mathcal{H}}^1(T\mathcal{M}|_y)^{\lambda = 0},
\end{align*}
\]

which are \( G_y^{\lambda \geq 0} \)-equivariant, \( G_y^{\lambda = 0} \)-equivariant, respectively. We define

\[
\hat{\mathcal{U}}^{\lambda \geq 0} \hookrightarrow \hat{\mathcal{H}}^0(T\mathcal{M}|_y)^{\lambda \geq 0}, \quad \hat{\mathcal{U}}^{\lambda = 0} \hookrightarrow \hat{\mathcal{H}}^0(T\mathcal{M}|_y)^{\lambda = 0}
\]

to be the derived zero loci of \( \kappa_y^{\lambda \geq 0}, \kappa_y^{\lambda = 0} \) respectively.

By Example 4.2, for each \( 1 \leq i \leq N \) there is a one parameter subgroup \( \lambda_i : C^* \to G_y \) such that each component of the outer diagram in (6.12)

\[
\tilde{S}_{t,y} \xrightarrow{ev_1} \tilde{M}_{t,y} \xrightarrow{ev_0} \tilde{Z}_{t,y}
\]

is equivalent to the diagram of the following form for \( \lambda = \lambda_i \)

\[
\hat{\mathcal{U}}^{\lambda \geq 0} \hookrightarrow \hat{\mathcal{H}}^0(T\mathcal{M}|_y)^{\lambda \geq 0}, \quad \hat{\mathcal{U}}^{\lambda = 0} \hookrightarrow \hat{\mathcal{H}}^0(T\mathcal{M}|_y)^{\lambda = 0}
\]

6.3. **Descriptions of \((-1)\)-shifted cotangents along formal fibers.** Here we give a formal local description of the stack \( \mathcal{N} \) and its \( \Theta \)-stratification (6.2). Let us consider the composition

\[
\mathcal{N} \xrightarrow{p_y} \mathcal{M} \xrightarrow{\pi_M} M.
\]

We denote by \( \tilde{\mathcal{N}}_y \) the formal fiber of the above morphism at \( y \in M \), i.e.

\[
\tilde{\mathcal{N}}_y := \mathcal{N} \times_M \text{Spec} \tilde{O}_{M,y}.
\]
Let $\kappa_y$ be the Kuranishi map (6.11). We define the following function, determined by $\kappa_y$ as in 3.2

$$\hat{w}_y: \left[ \left( \mathcal{H}^0(\mathcal{T}_{\mathcal{M}}|_y) \oplus \mathcal{H}^1(\mathcal{T}_{\mathcal{M}}|_y)^\vee \right) / G_y \right] \to \mathbb{C}, \quad \hat{w}_y(-, -) = (\kappa_y(-), -).$$

Then from the identity (3.3), the stack $\hat{\mathcal{N}}_y$ is isomorphic to the critical locus of $\hat{w}_y$

$$\hat{\mathcal{N}}_y \cong \text{Crit}(\hat{w}_y) \subset \left[ \left( \mathcal{H}^0(\mathcal{T}_{\mathcal{M}}|_y) \oplus \mathcal{H}^1(\mathcal{T}_{\mathcal{M}}|_y)^\vee \right) / G_y \right].$$

We next give a formal local description of the $\Theta$-stratification (6.2) for $\mathcal{N}$. We have the following commutative diagram

\[
\begin{array}{cccccc}
S^0 & \to & S^1 & \to & \mathcal{N} & \to & M \\
\downarrow \text{ev}^0 & & \downarrow \text{ev}^1 & & \downarrow \pi_N & & \downarrow \pi_M \\
\mathcal{S}^0 \times \mathbb{C} & \to & \mathcal{S}^1 \times \mathbb{C} & \to & \mathcal{N} & \to & \text{Spec } \mathcal{O}_{M,y}.
\end{array}
\]

Here the left Cartesian diagram follows from Lemma 6.2, the middle square follows from Lemma 4.11, and the right bottom arrow is the induced morphism on good moduli spaces. By taking the fiber products with $\text{Spec } \mathcal{O}_{M,y} \to M$, we obtain the commutative diagram

\[ (6.16) \]

\[
\begin{array}{cccccc}
\hat{S}^0_{i,y} & \to & \hat{S}^1_{i,y} & \to & \hat{\mathcal{N}}_y & \to & \hat{M}_y \\
\downarrow \text{ev}^0 & & \downarrow \text{ev}^1 & & \downarrow \pi_N & & \downarrow \pi_M \\
\mathcal{S}^0_{i,y} \times \mathbb{C} & \to & \mathcal{S}^1_{i,y} \times \mathbb{C} & \to & \hat{\mathcal{N}}_y & \to & \text{Spec } \mathcal{O}_{M,y}.
\end{array}
\]

By [HLb Corollary 1.30.1], the stacks $\hat{S}^0_{i,y}, \mathcal{S}^0_{i,y}$ are open and closed substacks

$$\hat{S}^0_{i,y} \subset \text{Filt}(\hat{\mathcal{N}}_y \setminus \mathcal{S}^0_{i-1,y}), \quad \mathcal{S}^0_{i,y} \subset \text{Filt}(\mathcal{N}_y),$$

and the $\Theta$-stratification (6.2) induces the $\Theta$-stratification on $\hat{\mathcal{N}}_y$

$$\hat{\mathcal{N}}_y = \hat{S}^0_{1,y} \sqcup \cdots \sqcup \mathcal{S}^0_{N,y} \sqcup \hat{\mathcal{N}}^{ss}_y.$$
We have the following commutative diagram

$$
\begin{array}{c}
\left[ \left( \hat{H}^0(T_{3|y})^{\lambda \geq 0} \oplus \hat{H}^1(T_{3|y})^{\lambda = 0} \right) / G_0 \right] \xrightarrow{q_2} \left[ \left( \hat{H}^0(T_{3|y}) \oplus \hat{H}^1(T_{3|y})^{\lambda \geq 0} \right) / G_y \right] \\
\left[ \left( \hat{H}^0(T_{3|y})^{\lambda \geq 0} \oplus \hat{H}^1(T_{3|y})^{\lambda = 0} \right) / G_y \right] \xrightarrow{r_2} \left[ \left( \hat{H}^0(T_{3|y})^{\lambda \geq 0} \oplus \hat{H}^1(T_{3|y})^{\lambda \geq 0} \right) / G_y \right] \\
\left[ \left( \hat{H}^0(T_{3|y})^{\lambda = 0} \oplus \hat{H}^1(T_{3|y})^{\lambda = 0} \right) / G_y \right] \xrightarrow{r_1} \left[ \left( \hat{H}^0(T_{3|y})^{\lambda = 0} \oplus \hat{H}^1(T_{3|y})^{\lambda = 0} \right) / G_y \right] \\
\left[ \left( \hat{H}^0(T_{3|y})^{\lambda = 0} \oplus \hat{H}^1(T_{3|y})^{\lambda = 0} \right) / G_y \right] \xrightarrow{f_2} \left[ \left( \hat{H}^0(T_{3|y})^{\lambda = 0} \oplus \hat{H}^1(T_{3|y})^{\lambda = 0} \right) / G_y \right] \\
\end{array}
$$

Here $f_1, r_2, g_2$ are given by projections onto the corresponding weight spaces, and $g_1, r_1, f_2, g_2$ are given by inclusions from the corresponding weight spaces. In order to simplify the notation, we write the above diagram as

(6.17)

where each $(X_i, w_i)$ is

$$
\begin{align*}
& w_1 : X_1 = \left[ \left( \hat{H}^0(T_{3|y})^{\lambda \geq 0} \oplus \hat{H}^1(T_{3|y})^{\lambda = 0} \right) / G_y \right] \xrightarrow{\tilde{w}_1} \mathbb{C}, \\
& w_2 : X_2 = \left[ \left( \hat{H}^0(T_{3|y})^{\lambda \geq 0} \oplus \hat{H}^1(T_{3|y})^{\lambda \geq 0} \right) / G_y \right] \xrightarrow{\tilde{w}_2} \mathbb{C}, \\
& w_3 : X_3 = \left[ \left( \hat{H}^0(T_{3|y})^{\lambda = 0} \oplus \hat{H}^1(T_{3|y})^{\lambda = 0} \right) / G_y \right] \xrightarrow{\tilde{w}_3} \mathbb{C}, \\
& w_4 : X_4 = \left[ \left( \hat{H}^0(T_{3|y})^{\lambda = 0} \oplus \hat{H}^1(T_{3|y})^{\lambda = 0} \right) / G_y \right] \xrightarrow{\tilde{w}_4} \mathbb{C}, \\
& w_5 : X_5 = \left[ \left( \hat{H}^0(T_{3|y})^{\lambda = 0} \oplus \hat{H}^1(T_{3|y})^{\lambda = 0} \right) / G_y \right] \xrightarrow{\tilde{w}_5} \mathbb{C}, \\
& w_6 : X_6 = \left[ \left( \hat{H}^0(T_{3|y})^{\lambda = 0} \oplus \hat{H}^1(T_{3|y})^{\lambda = 0} \right) / G_y \right] \xrightarrow{\tilde{w}_6} \mathbb{C}.
\end{align*}
$$

We note that, by the Koszul duality equivalence in Theorem 3.1, we have equivalences

(6.18)

$$
\begin{align*}
& \tilde{\Phi}_y : D^b_{\text{coh}}(\hat{H}/G_y) \xrightarrow{\sim} \text{MF}^c_{\text{coh}}(X_1, w_1), \\
& \tilde{\Phi}^{\lambda \geq 0}_y : D^b_{\text{coh}}(\hat{H}^{\lambda \geq 0}/G_y) \xrightarrow{\sim} \text{MF}^c_{\text{coh}}(X_3, w_3), \\
& \tilde{\Phi}^{\lambda = 0}_y : D^b_{\text{coh}}(\hat{H}^{\lambda = 0}/G_y) \xrightarrow{\sim} \text{MF}^c_{\text{coh}}(X_5, w_5).
\end{align*}
$$
The diagram (6.17) induces the diagram

\[(6.19)\]

\[q_1^{-1}(\text{Crit}(w_5)) \cap (q_2 r_1)^{-1}(\text{Crit}(w_3)) \cap q_2^{-1}(\text{Crit}(w_1)) \xrightarrow{q_3} \text{Crit}(w_5).\]

Here \(q_3\) is induced by the zero section of \(q_1: \mathcal{X}_0 \to \mathcal{X}_5\), and \(q_4 := q_2 \circ q_3\).

Let \(\lambda = \lambda_i\) be the one parameter subgroup of \(G_y\) corresponding to the \(i\)-th \(\Theta\)-strata as in (6.14).

Then from the diagrams (5.24), (5.27) together with (3.9), one can show that (see the proof of [Tod, Proposition 3.2.4]) the diagram (6.19) is isomorphic to the following diagram in (6.16)

\[(6.20)\]

\[\begin{array}{ccc}
\mathcal{S}_{i,y}^\Omega & \xrightarrow{ev_i^\Omega} & \mathcal{N}_y^\Omega \\
\sigma_i^\Omega \downarrow & & \downarrow \tau_i^\Omega \\
\mathcal{N}_{i,y}^\Omega & & \mathcal{P}_{i,y}^\Omega.
\end{array}\]

6.4. Adjoint functors of \(\Upsilon_i\) and \(\Upsilon_{i,j}\). Let us take the ind-completion of the functor (6.7)

\[(6.21)\]

\[\Upsilon_i^{\text{ind}}: \text{Ind} DT^C(\mathcal{Z}_i^\Omega) \to \text{Ind} DT^C(\mathcal{N} \setminus \mathcal{S}_{\leq i-1}^\Omega).\]

We see that the above functor admits a right adjoint, explicitly described in terms of the diagram (6.6) as follows:

**Lemma 6.5.** The functor (6.21) admits a right adjoint functor of the form

\[(6.22)\]

\[\Upsilon_i^R = ev_0^{\text{ind}} ev_1^! : \text{Ind} DT^C(\mathcal{N} \setminus \mathcal{S}_{\leq i-1}^\Omega) \to \text{Ind} DT^C(\mathcal{Z}_i^\Omega).\]

Here \(ev_0, ev_1\) are given in the diagram (6.4).

**Proof.** Let us take the ind-completion of the functor (6.7)

\[(6.23)\]

\[\Upsilon_i^{\text{ind}}: \text{Ind} D^b_{\text{coh}}(\mathcal{Z}_i) \xrightarrow{ev_i^\Omega} \text{Ind} D^b_{\text{coh}}(\mathcal{S}_i) \xrightarrow{ev_i^{\text{ind}}} \text{Ind} D^b_{\text{coh}}(\mathcal{N}).\]

By Proposition 3.7, the functor (6.21) is a descendant of the functor (6.23). It admits a right adjoint

\[(6.24)\]

\[ev_0^{\text{ind}} ev_1^! : \text{Ind} D^b_{\text{coh}}(\mathcal{N}) \xrightarrow{ev_1^!} \text{Ind} D^b_{\text{coh}}(\mathcal{S}_i) \xrightarrow{ev_0^{\text{ind}}} \text{Ind} D^b_{\text{coh}}(\mathcal{Z}_i).\]

Here see [DG13] Section 3.6, 3.7.7, [GR17] Proposition 3.16] for the above adjoint functors for ind-coherent sheaves. We show that the above functor (6.21) restricts to the functor

\[(6.25)\]

\[ev_{0*}^{\text{ind}} ev_1^!: \text{Ind} \mathcal{C}_{\mathcal{S}_{\leq i-1}^\Omega} \to \text{Ind} \mathcal{C}_{\mathcal{Z}_i^\Omega}.\]

Since singular supports have point-wise characterizations (see [AG13 Proposition 6.2.2]), it is enough to prove the above claim formally locally on the good moduli space \(M\), i.e. for any closed point \(y \in M\) and the diagram (6.15), it is enough to show that the functor

\[(6.26)\]

\[ev_{0*}^{\text{ind}} ev_1^!: \text{Ind} D^b_{\text{coh}}(\mathcal{U}_y^\lambda / G_y) \xrightarrow{ev_1^!} \text{Ind} D^b_{\text{coh}}(\mathcal{U}_y^\lambda / G_y^\lambda) \xrightarrow{ev_0^{\text{ind}}} \text{Ind} D^b_{\text{coh}}(\mathcal{U}_y^\lambda / G_y^\lambda).\]

restricts to the functor

\[(6.27)\]

\[ev_{0*}^{\text{ind}} ev_1^!: \text{Ind} \mathcal{C}_{\mathcal{S}_{\leq i-1,y}^\Omega} \to \text{Ind} \mathcal{C}_{\mathcal{T}_{i,y}^\Omega}.\]

Here \(\mathcal{T}_{i,y}^\Omega := \mathcal{Z}_{i,y}^\Omega \setminus \mathcal{N}_{i,y}^\Omega\) in the diagram (6.10). By the identity (6.9), we also have the following identity in the diagram (6.20)

\[(6.28)\]

\[(ev_1^\Omega)^{-1}(\mathcal{T}_{i,y}^\Omega) = (ev_1^\Omega)^{-1}(\mathcal{S}_{\leq i-1,y}^\Omega) \subset \mathcal{S}_{i,y}^\Omega.\]
By the above identity together with the fact that the Cartesian square in (6.17) is a derived Cartesian and $f_2$ is proper, we can apply Proposition 3.2.14] to conclude that the functor (6.26) restricts to the functor (6.27).

By Proposition 3.7 and Lemma 6.6 below, the desired right adjoint (6.22) is obtained by taking the quotients of both sides in (6.24) by the subcategories in (6.25).

We have used the following lemma, whose proof is straightforward.

**Lemma 6.6.** ([Orl09] Lemma 1.1) Let $\mathcal{D}, \mathcal{D}'$ be triangulated categories and $\mathcal{C} \subset \mathcal{D}, \mathcal{C}' \subset \mathcal{D}'$ be full triangulated subcategories. Let $F: \mathcal{D} \to \mathcal{D}', G: \mathcal{D}' \to \mathcal{D}$ be an adjoint pair of exact functors such that $F(\mathcal{C}) \subset \mathcal{C}'$, $G(\mathcal{C}') \subset \mathcal{C}$. Then they induce functors

$$F: \mathcal{D}/\mathcal{C} \to \mathcal{D}'/\mathcal{C}', \quad G: \mathcal{D}'/\mathcal{C}' \to \mathcal{D}/\mathcal{C}$$

which are adjoints as well.

Let $\Upsilon_{i,j}^R$ be the functor obtained by composing $\Upsilon_i^R$ in Lemma 6.5 with the projection onto the weight $j$ part

$$\Upsilon_{i,j}^R: \text{Ind} \mathcal{DT}^{C^*}(\mathcal{N} \setminus S_{\leq i}^{\Omega}) \xrightarrow{\Upsilon_i^R} \text{Ind} \mathcal{DT}^{C^*}(Z_i^{\Omega}) \xrightarrow{\text{pr}_j} \text{Ind} \mathcal{DT}^{C^*}(Z_i^{\Omega})_{\text{wt}=j}. \tag{6.29}$$

**Proposition 6.7.** The functor (6.29) restricts to the functor

$$\Upsilon_{i,j}^R: \mathcal{DT}^{C^*}(\mathcal{N} \setminus S_{\leq i}^{\Omega}) \to \mathcal{DT}^{C^*}(Z_i^{\Omega})_{\text{wt}=j}, \tag{6.10}$$

giving a right adjoint of the functor (6.10).

**Proof.** It is enough to show that the composition

$$\text{Ind} D^b_{\text{coh}}(\mathcal{M}) \xrightarrow{\text{ev}_1^\text{ind}} \text{Ind} D^b_{\text{coh}}(\mathcal{G}_1) \xrightarrow{\text{ev}_1} \text{Ind} D^b_{\text{coh}}(\mathcal{G}_1) \xrightarrow{\text{pr}_j} \text{Ind} D^b_{\text{coh}}(\mathcal{G}_1)_{\text{wt}=j}$$

in the diagram (6.6) sends $D^b_{\text{coh}}(\mathcal{M})$ to $D^b_{\text{coh}}(\mathcal{G}_1)_{\text{wt}=j}$. For $\mathcal{E} \in D^b_{\text{coh}}(\mathcal{M})$, we have

$$\text{pr}_j \circ \text{ev}^\text{ind}_0 \circ \text{ev}_1(\mathcal{E}) \in \text{Ind} D^b_{\text{coh}}(\mathcal{G}_1)_{\text{wt}=j} \subset \text{Ind} D^b_{\text{coh}}(\mathcal{G}_1). \tag{6.30}$$

Here the subscript + indicates bounded below subcategory with respect to the natural t-structure on ind-coherent sheaves, and the fact that $\text{ev}^\text{ind}_0$, $\text{ev}_1$ preserve these subcategories is proved in [Gai13 Lemma 3.4.4]. By [Gai13 Proposition 1.2.4], we have the equivalence

$$\text{Ind} D^b_{\text{coh}}(\mathcal{G}_1)_{\text{wt}=j} \simeq D^{b_{\text{coh}}}(\mathcal{G}_1)_{\text{wt}=j}. \tag{6.11}$$

Therefore it is enough to show that the object (6.11) is cohomological bounded above and has coherent cohomologies. Since this is a local property, it is enough to check this formally locally on $M$, i.e. it is enough to show that for each closed $y \in M$ we have

$$\hat{t}_y \circ \text{pr}_j \circ \text{ev}^\text{ind}_0 \circ \text{ev}_1(\mathcal{E}) \in D^b_{\text{coh}}(\mathcal{G}_y)_{\text{wt}=j}. \tag{6.12}$$

Here $\hat{t}_y$ is given in the diagram (6.13). By the base change properties for functors of ind-coherent sheaves (see [Gai13 Lemma 3.6.9, Proposition 7.1.6]), it is enough to show that the composition functor from the diagram (6.16)

$$\text{Ind} D^b_{\text{coh}}(\mathcal{G}_y \cap G_y) \xrightarrow{\text{ev}_1} \text{Ind} D^b_{\text{coh}}(\mathcal{G}_y \cap G_y) \xrightarrow{\text{ev}^\text{ind}} \text{Ind} D^b_{\text{coh}}(\mathcal{G}_y \cap G_y) \xrightarrow{\text{pr}_j} \text{Ind} D^b_{\text{coh}}(\mathcal{G}_y \cap G_y)_{\text{wt}=j}$$

restricts to the functor

$$\text{pr}_j \circ \text{ev}^\text{ind}_0 \circ \text{ev}_1: D^b_{\text{coh}}(\mathcal{G}_y \cap G_y) \to D^b_{\text{coh}}(\mathcal{G}_y \cap G_y)_{\text{wt}=j}. \tag{6.17}$$
Below we use the notation of the diagram \( \ref{6.17} \). By the equivalences \( \ref{6.18} \) and using \( \ref{5.11} \) and \( \ref{6.13} \), we are reduced to showing that the composition of functors

\[
\begin{align*}
\text{MF}_{\text{coh}}^c(X_1, w_1) & \xrightarrow{f_2^*} \text{MF}_{\text{coh}}^c(X_2, w_2) \xrightarrow{g_2^*} \text{MF}_{\text{coh}}^c(X_3, w_3) \xrightarrow{g_1^*} \text{MF}_{\text{coh}}^c(X_4, w_4) \\
\xrightarrow{f_1^*} & \text{MF}_{\text{coh}}^c(X_5, w_5) \xrightarrow{pr} \text{MF}_{\text{coh}}^c(X_5, w_5)_{\lambda \cdot \omega t = j}
\end{align*}
\]

restricts to the functor

\[
\text{MF}_{\text{coh}}^c(X_1, w_1) \to \text{MF}_{\text{coh}}^c(X_5, w_5)_{\lambda \cdot \omega t = j}.
\]

By the derived base change, the above composition functor is equivalent to the following composition

\[
\text{MF}_{\text{coh}}^c(X_1, w_1) \xrightarrow{f_2^*} \text{MF}_{\text{coh}}^c(X_2, w_2) \xrightarrow{\tau_x} \text{MF}_{\text{coh}}^c(X_6, w_6) \xrightarrow{\varphi} \text{MF}_{\text{coh}}^c(X_5, w_5)_{\lambda \cdot \omega t = j}.
\]

Since \( f_2 : X_2 \to X_1 \) is a representable morphism of smooth stacks, it is quasi-smooth and \( f_2^* \) is given by \( f_2^*(-) = \omega f_2 \). Therefore \( f_1^* f_2^* \) gives the functor

\[
pr_1 q_1^* : \text{MF}_{\text{coh}}^c(X_6, w_6) \to \text{MF}_{\text{coh}}^c(X_5, w_5)_{\lambda \cdot \omega t = j}.
\]

It is enough to show that the functor \( pr_1 q_1^* \) gives the functor

\[
(6.31)
pr_1 q_1^* : \text{MF}_{\text{coh}}^c(X_6, w_6) \to \text{MF}_{\text{coh}}^c(X_5, w_5)_{\lambda \cdot \omega t = j}.
\]

The morphism \( q_1 \) factors as

\[
q_1 : X_6 \xrightarrow{q_1^*} X_7 := [A/G_y^{\lambda > 0}] \xrightarrow{q_1''} X_5,
\]

where \( A := (\mathcal{H}^0(T_{\mathfrak{M}^1}|_y)^{\lambda = 0} \oplus (\mathcal{H}^1(T_{\mathfrak{M}^1}|_y)^{\lambda = 0})^\vee \) and \( G_y^{\lambda > 0} \) acts on it through the projection \( G_y^{\lambda \geq 0} \to G_y^{\lambda = 0} \). Since \( \mathcal{H}^0(T_{\mathfrak{M}^1}|_y)^{\lambda > 0} \oplus (\mathcal{H}^1(T_{\mathfrak{M}^1}|_y)^{\lambda > 0}) \) has positive \( \lambda \)-weights, by [Tohö Lemma 2.2.3] the push-forward \( q_1'' \) restricts to the functor

\[
q_1'' : \text{MF}_{\text{coh}}^c(X_6, w_6) \to \text{MF}_{\text{coh}}^c(X_7, w_7)_{\lambda \cdot \omega t = j},
\]

where \( w_7 = q_1'' w_5 \). Let \( G_y^{\lambda > 0} \) be the kernel of the projection \( G_y^{\lambda \geq 0} \to G_y^{\lambda = 0} \). Then \( G_y^{\lambda > 0} \) is unipotent, so it admits a filtration of normal subgroups

\[
0 = G_0 \subset G_1 \subset \cdots \subset G_k = G_y^{\lambda > 0}
\]

such that each subquotient \( G_i/G_{i-1} \) is isomorphic to the additive group \( \mathbb{A}^{m_i} \), for some \( m_i \). By setting \( Q_i = G_y^{\lambda \geq 0}/G_i \), we have the factorizations of \( q_1'' \)

\[
q_1'' : X_7 = [A/Q_0] \to [A/Q_1] \to \cdots \to [A/Q_k] = X_5.
\]

Here each \( Q_i \) acts on \( A \) through the projection \( Q_i \to G_y^{\lambda = 0} \). Since \( [A/Q_{i-1}] \to [A/Q_i] \) is a \( \mathbb{A}^{m_i} \)-gerbe, the push-forward of a coherent sheaf along with the above morphism is quasi-isomorphic to a bounded complex of coherent sheaves. Therefore the functor \( q_1'' \) gives

\[
q_1'' : \text{MF}_{\text{coh}}^c(X_7, w_7) \to \text{MF}_{\text{coh}}^c(X_5, w_5).
\]

Therefore \( q_1 = q_1'' \circ q_1^* \) restricts to the functor

\[
q_1 : \text{MF}_{\text{coh}}^c(X_6, w_6) \to \text{MF}_{\text{coh}}^c(X_5, w_5)_{\lambda \cdot \omega t = j},
\]

which concludes that \( pr_1 q_1^* \) gives the functor \( \ref{6.31} \).

We also have the left adjoints as follows:

**Lemma 6.8.** The functor \( \ref{6.18} \) admits a left adjoint

\[
(6.32)
\Upsilon_{d,j} : DT^{\text{coh}}(\mathcal{N} \setminus S_{2j-1}^{d}) \to DT^{\text{coh}}(Z_{2j}^{d})_{\omega t = j}.
\]
Proof. In order to simplify the notation, we write \( \mathcal{M}_1 = \mathcal{M}, \mathcal{M}_2 = \mathcal{S}_1 \) and \( \mathcal{M}_3 = \mathcal{S}_i \) in the diagram (6.10). We denote by \( \mathcal{D}_i \) the Serre duality equivalence for \( D^b_{\text{coh}}(\mathcal{M}_i) \), given by

\[ \mathcal{D}_i := \text{Hom}(-, \omega_{\mathcal{M}_i}), \quad \omega_{\mathcal{M}_i} := \det \mathcal{L}_{\mathcal{M}_i}[\text{rank } \mathcal{L}_{\mathcal{M}_i}]. \]

We have the following adjoint pairs for the functors between \( \text{Ind} D^b_{\text{coh}}(\mathcal{M}_1) \) and \( \text{Ind} D^b_{\text{coh}}(\mathcal{M}_3)_{\text{wt} = j} \),

\[ \text{pr}_j \mathcal{D}_3 \circ \text{ev}_0 \circ \text{ev}_1 \mathcal{D}_1 \cong \mathcal{D}_i \text{ev}_1 \circ \text{ev}_0 \mathcal{D}_3 \mathcal{D}_i j. \]

Here the above functors are given by

\[ \text{Ind} D^b_{\text{coh}}(\mathcal{M}_1) \xrightarrow{\mathcal{D}_2} \text{Ind} D^b_{\text{coh}}(\mathcal{M}_1)^{\text{op}} \xrightarrow{\text{ev}_1^*} \text{Ind} D^b_{\text{coh}}(\mathcal{M}_2)^{\text{op}} \xrightarrow{\text{ev}_1} \text{Ind} D^b_{\text{coh}}(\mathcal{M}_3)^{\text{op}} \xrightarrow{\mathcal{D}_3} \text{Ind} D^b_{\text{coh}}(\mathcal{M}_3)_{\text{wt} = j}. \]

Since \( \text{ev}_1 \) is proper, using [Ca13, Corollary 9.5.9] we have

\[ \mathcal{D}_1 \circ \text{ev}_1 \circ \text{ev}_0 \circ \mathcal{D}_3 \circ \mathcal{D}_i j \cong \text{ev}_1 \circ \mathcal{D}_2 \circ \text{ev}_0 \circ \mathcal{D}_3 \circ \mathcal{D}_i j \cong \text{ev}_1 \circ \text{ev}_0 \circ \mathcal{D}_3 \circ \mathcal{D}_i j. \]

By Lemma 5.3, \( \text{ev}_0 \) is quasi-smooth. Also by Lemma 5.2, the relative dualizing complex \( \omega_{\mathcal{E}_{\mathcal{M}_0}} \) is of the form \( \omega_{\mathcal{E}_{\mathcal{M}_0}} \mathcal{L}[k] \) for a line bundle \( \mathcal{L} \) on \( \mathcal{M}_3 \) and \( k \in \mathbb{Z} \). Therefore we conclude that

\[ \mathcal{D}_1 \circ \text{ev}_1 \circ \text{ev}_0 \circ \mathcal{D}_3 \circ \mathcal{D}_i j (-) \cong \text{ev}_1 \circ \text{ev}_0 \circ \mathcal{D}_3 \circ \mathcal{D}_i j (-) \otimes \mathcal{L}[k]. \]

The composition \( \mathcal{L} \circ \mathcal{D}_i j \) is isomorphic to \( \mathcal{D}_3 \circ \mathcal{D}_i j (-) \otimes \mathcal{L} \), where \( \mathcal{L} \) is the equivalence
diagram (6.33)

\[ \mathcal{L} \circ \mathcal{D}_i j = \text{Ind} D^b_{\text{coh}}(\mathcal{M}_3)_{\text{wt} = j} \cong \text{Ind} D^b_{\text{coh}}(\mathcal{M}_3)_{\text{wt} = j + \text{wt} \mathcal{L}}. \]

Therefore we have the adjoint pair for the functors between \( \text{Ind} D^b_{\text{coh}}(\mathcal{M}_1) \) and \( \text{Ind} D^b_{\text{coh}}(\mathcal{M}_3)_{\text{wt} = j} \)

(6.33)

\[ \mathcal{L} \circ \mathcal{D}_i j : \text{Ind} D^b_{\text{coh}}(\mathcal{M}_3)_{\text{wt} = j} \circ \text{Ind} D^b_{\text{coh}}(\mathcal{M}_3)_{\text{wt} = j} \]

Since the dualizing functors and tensor products with line bundle preserve singular supports and compact objects, as in the proof of Lemma 6.5 the LHS of (6.33) induces the functor (6.32) giving a left adjoint of the functor (6.10).

6.5. Semiorthogonal decompositions of DT categories. Here we give a proof of Theorem 1.1. We first show that the functors \( \Upsilon_{i,j} \) are fully-faithful.

**Proposition 6.9.** The functor \( \Upsilon_{i,j} \) in (6.10) is fully-faithful.

Proof. By Proposition 6.7, it is enough to show that the natural transform

(6.34)

\[ \text{id} \to \Upsilon_{i,j}^* \circ \Upsilon_{i,j} \]

is an isomorphism. This is a local property, so it is enough to check the isomorphism (6.34) after pulling back via \( \mathcal{L}_y : \mathcal{M}_y \to \mathcal{M} \) at each closed point \( y \in M \). Let \( \lambda_i : C^* \to \mathcal{G}_y \) be the one parameter subgroup corresponding to the \( i \)-th \( \Theta \)-strata as in (6.14), and set \( \lambda = \lambda_i \). By the base change, we are reduced to showing that the functor

\[ \text{ev}_1 \circ \text{ev}_0^*: D^b_{\text{coh}}(\mathcal{M}_{\mathcal{L} = 0}/\mathcal{G}_{\lambda = 0})_{\text{wt} = j} \to D^b_{\text{coh}}(\mathcal{M}_{\mathcal{L} = 0}/\mathcal{G}_{\lambda = 0}) \]

from the diagram (6.11) is fully-faithful. We show that the above functor is fully-faithful through the Koszul duality equivalences in (6.18).

Below we use the notation in the diagram (6.17). Let us consider the following composition functor

\[ \text{MF}^C_{\text{coh}}(\mathcal{X}_5, w_5)_{\lambda, \text{wt} = j} \xrightarrow{f^1_z} \text{MF}^C_{\text{coh}}(\mathcal{X}_4, w_4) \xrightarrow{g^1_u} \text{MF}^C_{\text{coh}}(\mathcal{X}_3, w_3) \]

\[ \xrightarrow{g^2} \text{MF}^C_{\text{coh}}(\mathcal{X}_2, w_2) \xrightarrow{f^2_z} \text{MF}^C_{\text{coh}}(\mathcal{X}_1, w_1). \]
By (3.12) and (3.14), it is enough to show that the descendant of the above composition functor
\[(6.35)\quad f_2 \circ g_2 \circ g_1 \circ f_1 : \text{MF}_{\text{coh}}(\mathcal{X}_5 \setminus \widehat{T}_i, w_5)_{\lambda\text{-wt}=j} \to \text{MF}_{\text{coh}}(\mathcal{X}_1 \setminus \mathcal{S}_{i-1}^\Omega, w_1)\]
is fully-faithful. Note that we have
\[g_1(-) = g_1(- \otimes \det(H^1(T_{\mathfrak{M}|y})^{\lambda>0})^{\vee}[-\dim H^1(T_{\mathfrak{M}|y})^{\lambda>0}]).\]
Here \(\det(H^1(T_{\mathfrak{M}|y})^{\lambda>0})^{\vee}\) is a \(G_y^{\lambda>0}\)-character, which descends to a \(G_y^{\lambda=0}\)-character by the projection \(G_y^{\lambda>0} \to G_y^{\lambda=0}\) as \(\mathbb{C}^*\) is commutative. Therefore it is written as \(f_1^* \det(H^1(T_{\mathfrak{M}|y})^{\lambda>0})^{\vee}\) where we regard \(\det(H^1(T_{\mathfrak{M}|y})^{\lambda>0})^{\vee}\) as a \(G_y^{\lambda=0}\)-character. It follows that we have
\[f_2 \circ g_2 \circ g_1 \circ f_1(-) \cong q_2 \circ g_1(- \otimes \det(H^1(T_{\mathfrak{M}|y})^{\lambda>0})^{\vee}[-\dim H^1(T_{\mathfrak{M}|y})^{\lambda>0}]).\]
Here the first functor is an equivalence
\[(6.36)\quad \otimes \det(H^1(T_{\mathfrak{M}|y})^{\lambda>0})^{\vee} : \text{MF}_{\text{coh}}(\mathcal{X}_5 \setminus \widehat{T}_i, w_5)_{\lambda\text{-wt}=j} \xrightarrow{\sim} \text{MF}_{\text{coh}}(\mathcal{X}_5 \setminus \widehat{T}_i, w_5)_{\lambda\text{-wt}=j-\langle \lambda, H^1(T_{\mathfrak{M}|y})^{\lambda>0} \rangle}.\]
By the identity (6.28), we have \(\widehat{T}_i = (T^{\mathfrak{M}})^{-1}(S^\Omega_{\leq i-1,y})\) in the diagram (6.20). Therefore from the equivalence (2.3), we conclude that the functor
\[(6.37)\quad \text{MF}_{\text{coh}}(\mathcal{X}_5 \setminus \widehat{T}_i, w_5)_{\lambda\text{-wt}=j-\langle \lambda, H^1(T_{\mathfrak{M}|y})^{\lambda>0} \rangle} \to \text{MF}_{\text{coh}}(\mathcal{X}_1 \setminus \mathcal{S}_{i-1}^\Omega, w_1)\]
is fully-faithful. Therefore the functor (6.35) is fully-faithful.

We denote by
\[\mathcal{D}_{i,j} \subset \mathcal{D}T^C^* (\mathcal{N} \setminus \mathcal{S}_{i-1}^\Omega)\]
the essential images of the functor \(\Phi_{i,j}\) in (6.10), which is equivalent to \(\mathcal{D}T^C^* (Z_i^\Omega)_{\text{wt}=j}\) by Proposition (6.9). We have the following semiorthogonality of these subcategories:

**Lemma 6.10.** For \(j > j'\), we have
\[\text{Hom}(\mathcal{D}_{i,j}, \mathcal{D}_{i,j'}) = 0.\]

**Proof.** It is enough to show the vanishing
\[\Phi_{i,j}^R \circ \Phi_{i,j}^L \equiv 0, \quad j > j'.\]
As in the proof of Proposition (6.9), it is enough to prove this formally locally on the good moduli space \(M\). Note that the LHS of (6.37) have \(\lambda\)-weight \(j - \langle \lambda, H^1(T_{\mathfrak{M}|y})^{\lambda>0} \rangle\). Therefore by Theorem (2.1), the essential images of the functors (6.37) are semiorthogonal for \(j > j'\), so the lemma follows.

For each \(1 \leq i \leq N\) and \(m \in \mathbb{Z}\), we define
\[\mathcal{W}_{i,m} := \bigcap_{j \geq m} \ker(\Phi_{i,j}^R) \cap \bigcap_{j < m} \ker(\Phi_{i,j}^L) \subset \mathcal{D}T^C^* (\mathcal{N} \setminus \mathcal{S}_{i-1}^\Omega).\]
The following is the main result in this section, which gives a window theorem for DT categories associated with \(\Theta\)-stratification on \((-1)\)-shifted cotangents.

**Theorem 6.11.** There exists a semiorthogonal decomposition
\[(6.38)\quad \mathcal{D}T^C^* (\mathcal{N} \setminus \mathcal{S}_{i-1}^\Omega) = (\ldots, \mathcal{D}_{i,m-2}, \mathcal{D}_{i,m-1}, \mathcal{W}_{i,m}, \mathcal{D}_{i,m}, \mathcal{D}_{i,m+1}, \ldots),\]
such that the composition functor
\[(6.39)\quad \mathcal{W}_{i,m} \hookrightarrow \mathcal{D}T^C^* (\mathcal{N} \setminus \mathcal{S}_{i-1}^\Omega) \twoheadrightarrow \mathcal{D}T^C^* (\mathcal{N} \setminus \mathcal{S}_{i}^\Omega)\]
is an equivalence.
Proof. We first show the semiorthogonal decomposition \((6.38)\). From Lemma \([6.10]\) and the definition of \(W_{i,m}\), the RHS of \((6.38)\) is semiorthogonal. It is enough to show that the RHS generates the LHS. For an object \(E\) in the LHS, the proof of Proposition \([6.4]\) shows that \(\Upsilon^R_{i,j}(E) = 0\) for \(j \gg 0\) formally locally at the good moduli space \(p \in M\). Since \(M \to M\) is universally closed, there exists an open neighborhood of \(p \in M\) on which \(\Upsilon^R_{i,j}(E) = 0\) for \(j \gg 0\). As \(M\) is quasi-compact, we conclude that \(\Upsilon^R_{i,j}(E) = 0\) for \(j \gg 0\). A similar argument shows that \(\Upsilon^L_{i,j}(E) = 0\) for \(j \ll 0\).

Suppose that \(E\) is not an object in \(W_{i,m}\). Then there is \(j_1 \geq m\) such that \(\Upsilon^R_{i,j_1}(E) = 0\) for \(j > j_1\) and \(\Upsilon^L_{i,j_1}(E) \neq 0\), or there is \(j_1' < m\) such that \(\Upsilon^R_{i,j_1'}(E) = 0\) for \(j < j_1'\) and \(\Upsilon^L_{i,j_1'}(E) \neq 0\). Below we assume the former case. The latter case is similarly discussed. We have the distinguished triangle \(\Upsilon_{i,j_1} \Upsilon^R_{i,j_1}(E) \to E \to E_1\), where \(\Upsilon^R_{i,j_1}(E_1) = 0\) for \(j \geq j_1\). Repeating the above constructions for \(E_1\), we obtain the desired semiorthogonal decomposition \((6.38)\).

If \(\Upsilon^R_{i,j}(E_3) = 0\) for all \(j < m\), then \(E_3 \in W_{i,m}\). Otherwise there is \(j_1' < m\) such that \(\Upsilon^R_{i,j_1'}(E_3) \neq 0\) for \(j < j_1'\) and \(\Upsilon^L_{i,j_1'}(E_3) \neq 0\). Similarly to above, we have the distinguished triangle \(E_4 \to E_3 \to \Upsilon_{i,j_1'} \Upsilon^R_{i,j_1'}(E_4)\) such that \(\Upsilon^R_{i,j_1'}(E_4) = 0\) for \(j \leq j_1'\). We also have \(\Upsilon^R_{i,j_1}(E_4) = 0\) for \(j \geq m\) by applying \(\Upsilon^R_{i,j_1}\) to the above triangle and using Lemma \([6.10]\). By repeating the above construction for \(E_4\), we obtain the distinguished triangle \(E_5 \to E_4 \to E_6\), \(E_6 \in \langle D_{i,j_1}, \ldots, D_{i,m-1} \rangle\), \(E_5 \in W_{i,m}\).

Therefore we obtain the desired semiorthogonal decomposition \((6.38)\).

Below we show that the composition functor \((6.39)\) is an equivalence. By Lemma \([6.5]\), any object in \(D_{i,j}\) has singular supports contained in \(S^{\Omega}_{\leq i}\). Therefore the composition \(D_{i,j} \to DT^C(\mathcal{N} \setminus S^{\Omega}_{\leq i-1}) \to DT^C(\mathcal{N} \setminus S^{\Omega}_{\leq i})\) is zero. It follows that by the semiorthogonal decomposition \((6.38)\) the functor \((6.39)\) is essentially surjective. It remains to show that \((6.39)\) is fully-faithful. Let \(\iota : U \to M\) be an étale morphism for an affine scheme \(U\), and take the following Cartesian diagrams

\[
\begin{array}{ccc}
\mathfrak{M}_U & \xrightarrow{\iota_{\mathfrak{M}}} & \mathcal{M} U \\
\downarrow & \square & \downarrow i \\
\mathfrak{M} & \xrightarrow{\iota} & M
\end{array}
\]

such that \(\iota_{\mathfrak{M}}\) is étale. Here the above diagram exists since the \(\infty\)-category of étale morphisms with target \(\mathcal{M}\) is equivalent to that with target \(\mathfrak{M}\). We have the fully-faithful functor (see [Tod5] Lemma 7.2.3)

\[
DT^C(\mathcal{N} \setminus S^{\Omega}_{\leq i}) \hookrightarrow \lim_{U \to M} \left( \text{D}^b_{\text{coh}}(\mathfrak{M}_U)/C_{\text{qcoh}}(\mathfrak{M}_U) \right).
\]

Let \(E_1, E_2\) be an object in \(W_{i,m}\). By the above fully-faithful functor, it is enough to show that the natural morphism

\[
\text{Hom}_{D^b_{\text{coh}}}(\mathfrak{M}_U)/C_{\text{qcoh}}(\mathfrak{M}_U) (i^*_\mathfrak{M} E_1, i^*_\mathfrak{M} E_2) \to \text{Hom}_{D^b_{\text{coh}}}(\mathfrak{M}_U)/C_{\text{qcoh}}(\mathfrak{M}_U) (i^*_\mathfrak{M} E_1, i^*_\mathfrak{M} E_2)
\]

is an isomorphism. The above morphism is regarded as a morphism in \(D^b_{\text{qcoh}}(U)\), so it is enough to show the above isomorphism formally locally at any point in \(U\). Similarly to Proposition \([6.7]\) we prove the corresponding claim for derived categories of factorizations via Koszul duality.

For each \(y \in M\), we use the notation of the diagram \((6.17)\). Let \(D_{i,j}\) be the essential image of the functor \((6.35)\). Since the functor \((6.35)\) is the composition of \((6.36)\) and \((6.37)\), by Theorem \([2.1]\) we have the semiorthogonal decomposition

\[
\text{MF}^C_{\text{coh}}(X_1 \setminus S^{\Omega}_{\leq i-1,y}, w_1) = \langle \ldots, \hat{D}_{i,m-2}, \hat{D}_{i,m-1}, \hat{W}_{i,m}, \hat{D}_{i,m}, \hat{D}_{i,m+1}, \ldots \rangle.
\]
such that the composition functor
\[ \tilde{\mathcal{W}}_{i,m} \to \text{MF}_{\text{coh}}^C(\mathcal{X}_1 \setminus \mathcal{S}^{\mathbb{O}_{<1}}_{i-1,y}, w_1) \to \text{MF}_{\text{coh}}^C(\mathcal{X}_1 \setminus \mathcal{S}^{\mathbb{O}_{<1}}_{i,y}, w_1) \]
is an equivalence. Since \( \Upsilon_{i,j}^L \) and \( \Upsilon_{i,j}^R \) commute with base change over \( M \), for any object \( \mathcal{E} \in \mathcal{W}_{i,m} \) we have \( \tilde{\Phi}_y(\gamma_y^* \mathcal{E}) \in \tilde{\mathcal{W}}_{i,m} \) where \( \gamma_y : \tilde{\mathcal{M}}_y \to \mathcal{M} \) is the morphism in (3.13) and \( \tilde{\Phi}_y \) is the equivalence
\[ \tilde{\Phi}_y : D_{\text{coh}}^b((\tilde{\mathcal{M}}_y / G_y))/C_{\mathcal{S}^{\mathbb{O}_{<1}}_{i-1,y}} \cong \text{MF}_{\text{coh}}^C(\mathcal{X}_1 \setminus \mathcal{S}^{\mathbb{O}_{<1}}_{i-1,y}, w_1) \]
given by (3.5). It follows that the natural morphism
\[ \text{Hom}_{D_{\text{coh}}^b(\tilde{\mathcal{M}}_y)/C_{\mathcal{S}^{\mathbb{O}_{<1}}_{i-1,y}}}(\tilde{\gamma}_y^* \mathcal{E}_1, \tilde{\gamma}_y^* \mathcal{E}_2) \to \text{Hom}_{D_{\text{coh}}^b(\tilde{\mathcal{M}}_y)/C_{\mathcal{S}^{\mathbb{O}_{<1}}_{i-1,y}}}(\tilde{\gamma}_y^* \mathcal{E}_1, \tilde{\gamma}_y^* \mathcal{E}_2) \]
is an isomorphism. Using Lemma 6.12 below, we conclude the formal local isomorphism of (6.40). □

**Lemma 6.12.** In the setting of the diagram (3.7), suppose that \( M \) is affine. Let \( Z \subset t_0(\Omega_{\mathcal{M}})[-1] \) be a conical closed substack. Then for any \( \mathcal{E}_1, \mathcal{E}_2 \in D_{\text{coh}}^b(\mathcal{M}) \), we have the isomorphism
\[ \text{Hom}_{D_{\text{coh}}^b(\mathcal{M})/C_{\mathcal{Z}}}^{\text{coh}}(\mathcal{E}_1, \mathcal{E}_2) \otimes_{\mathcal{O}_M} \hat{\mathcal{O}}_{M,y} \cong \text{Hom}_{D_{\text{coh}}^b(\mathcal{M})/C_{\mathcal{Z}}}^{\text{coh}}(\tilde{\gamma}_y^* \mathcal{E}_1, \tilde{\gamma}_y^* \mathcal{E}_2). \]

**Proof.** Noting Proposition 3.7 and Lemma 6.6 we have the isomorphisms
\[ \text{Hom}_{D_{\text{coh}}^b(\mathcal{M})/C_{\mathcal{Z}}}^{\text{coh}}(\tilde{\gamma}_y^* \mathcal{E}_1, \tilde{\gamma}_y^* \mathcal{E}_2) \cong \text{Hom}_{\text{Ind}(D_{\text{coh}}^b(\mathcal{M})/C_{\mathcal{Z}})}^{\text{coh}}(\tilde{\mathcal{E}}_1, \tilde{\mathcal{E}}_2) \]
\[ \cong \text{Hom}_{\text{Ind}(D_{\text{coh}}^b(\mathcal{M})/C_{\mathcal{Z}})}^{\text{coh}}(\mathcal{E}_1, \mathcal{E}_2) \otimes_{\mathcal{O}_M} \hat{\mathcal{O}}_{M,y}. \]
Here the second isomorphism is the projection formula for ind-coherent shaves (see [GRTW] Proposition 3.3.7), where \( \hat{\mathcal{O}}_{M,y} \in D_{\text{coh}}^b(M) \) acts on \( \text{Ind}(D_{\text{coh}}^b(\mathcal{M})/C_{\mathcal{Z}}) \) by the tensor product. Since \( \mathcal{E}_2 \) is a compact object in \( \text{Ind}(D_{\text{coh}}^b(\mathcal{M})/C_{\mathcal{Z}}) \) (see the proof of [Todh] Proposition 3.2.7), we have the isomorphism
\[ \text{Hom}_{D_{\text{coh}}^b(\mathcal{M})/C_{\mathcal{Z}}}^{\text{coh}}(\mathcal{E}_1, \mathcal{E}_2) \otimes_{\mathcal{O}_M} \hat{\mathcal{O}}_{M,y} \cong \text{Hom}_{\text{Ind}(D_{\text{coh}}^b(\mathcal{M})/C_{\mathcal{Z}})}^{\text{coh}}(\mathcal{E}_1, \mathcal{E}_2) \otimes_{\mathcal{O}_M} \hat{\mathcal{O}}_{M,y}. \]
Therefore the lemma holds. □

For an interval \( I \subset \mathbb{R} \cup \{-\infty, \infty\} \), we set
\[ D_{i,j} := (D_{i,j} : j \in I) \subset DT^C(\mathcal{N} \setminus \mathcal{S}^{\mathbb{O}_{<1}}_{i-1}). \]
Note that for each \( m \in \mathbb{R} \) the semiorthogonal decomposition in Theorem 6.11 implies that
\[ DT^C(\mathcal{N} \setminus \mathcal{S}^{\mathbb{O}_{<1}}_{i-1}) = \langle D_{1, < m}, W_{i,m}, D_{i, > m} \rangle, \]
where \( W_{i,m} := W_{i,[m]} \). As a corollary of Theorem 6.11 we have the following:

**Corollary 6.13.** For each choice of \( m_i \in \mathbb{R} \) for \( 1 \leq i \leq N \), there exists a semiorthogonal decomposition
\[ DT^C(\mathcal{N}) = \langle D_{1, < m_1}, \ldots, D_{N, < m_N}, W_{m_1}, D_{N, \geq m_N}, \ldots, D_{1, \geq m_1} \rangle \]
such that the composition functor
\[ W_{m_1} \to DT^C(\mathcal{N}) \to DT^C(\mathcal{N}_{\text{ss}}) \]
is an equivalence.

**Remark 6.14.** In [HL3] Theorem 2.3.1, Halpern-Leistner proves the existence of semiorthogonal decomposition of \( D_{\text{coh}}^b(\mathcal{M}) \), associated with the \( \Theta \)-stratification for \( \mathcal{M} \)
\[ \mathcal{M} = \mathcal{G}_1 \sqcup \cdots \sqcup \mathcal{G}_N \sqcup \mathcal{M}_{\text{ss}} \]
under the additional assumption that each \( H^{-1}(\mathcal{X}_i | \mathcal{Z}_i) \) has only non-negative weights. Here \( Z_i \)
is the center of \( \mathcal{S}_i = t_0(\mathcal{G}_i) \). By Lemma 6.8, the above weight assumption implies that the \( \Theta \)-stratification (6.22) for \( \mathcal{M} \) pulls back to the \( \Theta \)-stratification (6.2) for \( \mathcal{N} \). In this case we have
\[ \mathcal{DT}^C(\mathcal{N}_{\text{ss}}) = D_{\text{coh}}^b(\mathfrak{M}_{\text{ss}}), \mathcal{DT}^C(\mathcal{Z}) = D_{\text{coh}}^b(\mathfrak{Z}) \] and the semiorthogonal decomposition in Corollary [6.15] coincides with the one proved in [HLa, Theorem 2.3.1]. On the other hand, the proof of Corollary [6.13] is applied without the above weight assumption. Indeed the semistable locus \( \mathcal{N}_{\text{ss}} \) may be strictly bigger than the pull-back of \( \mathcal{M}_{\text{ss}} \), and we need categorical DT theory to formulate the semiorthogonal decomposition in Corollary [6.13] without the above weight assumption.

From the proof of Theorem [6.11] one can characterize the subcategory \( \mathcal{W}_m \) in [6.41] in terms of formal fibers along with the good moduli space morphism \( \pi_M : M \to L \). Let us take a closed point \( y \in M \) and use the notation in the diagram [6.17]. For each \( 1 \leq i \leq N \), by taking the one parameter subgroup \( \lambda_i \) as in [6.14], we set

\[ (6.43) \quad \eta_i := \langle \lambda_i, \det L^\vee \rangle = \langle \lambda_i, -H^0(\mathbb{T}_{\mathcal{M}|y})^{\lambda_i<0} + H^1(\mathbb{T}_{\mathcal{M}|y})^{\lambda_i>0} + H^{-1}(\mathbb{T}_{\mathcal{M}|y})^{\lambda_i<0} \rangle. \]

Here \( q_2 \) is a morphism in [6.17]. We also fix \( \mathcal{Z} \in \text{Pic}(\mathfrak{M}) \) with \( \mathcal{T} = c_1(\mathcal{Z}) \) satisfying

\[ (6.44) \quad \text{wt}(\tau^*\mathcal{Z}) \notin \mathbb{Q} \]

for the diagram [6.6] for all \( 1 \leq i \leq N \), e.g. \( \mathcal{T} = \varepsilon \cdot \mathcal{L} \) for \( \varepsilon \notin \mathbb{Q} \). We define the subcategory

\[ \mathcal{W}_y \subset \text{MF}_{\text{coh}}^C(\mathcal{X}_1, w_1) \]

to be consisting of factorizations \((\mathcal{P}, d_{\mathcal{P}})\) such that, for the inclusion

\[ (6.45) \quad \tau : \mathcal{X}_5 \setminus \tilde{T}_{i,y} \hookrightarrow \mathcal{X}_1 \setminus \tilde{S}_{\leq i-1,y} \]

from the \( \lambda_i \)-fixed part, we have

\[ (6.46) \quad \tau^*(\mathcal{P}, d_{\mathcal{P}}) \in \bigoplus_{j \in [\frac{1}{\eta_i + (\lambda_i, \delta)} + \frac{1}{\eta_i + (\lambda_i, \delta)}]} \text{MF}_{\text{coh}}^C(\mathcal{X}_5 \setminus \tilde{T}_{i,y}, w_5)_{\lambda_i, \text{wt}=j} \]

for all \( 1 \leq i \leq N \). Here \( \delta \) is given by

\[ (6.47) \quad \delta := \frac{1}{2} \text{det}(H^1(\mathbb{T}_{\mathcal{M}|y}))^\vee + \mathcal{Z} \in K(BG_{\mathfrak{m}}), \]

which is independent of \( i \). We also set

\[ (6.48) \quad m_i := \frac{1}{2} \text{wt}(\sigma^* \det L^\vee_{\text{ev}_1}) + \text{wt}(\tau^*\mathcal{Z}) \]

in the notation of the diagram [6.6].

**Proposition 6.15.** For a choice of \( m_i \) in [6.48], an object \( \mathcal{E} \in \mathcal{DT}^C(N) \) lies in \( \mathcal{W}_m \) if and only if for any closed point \( y \in M \), for the morphism \( \tilde{\gamma}_y : \tilde{\mathcal{M}}_y \to \mathfrak{M} \) in [3.16] and an equivalence \( \tilde{\Phi}_y \) in [6.18], we have \( \tilde{\Phi}_y(\tilde{\gamma}_y^*\mathcal{E}) \in \mathcal{W}_y \).

**Proof.** By Theorem [2.1] and the argument of Theorem [6.11] an object \( \mathcal{E} \in \mathcal{DT}^C(N) \) lies in \( \mathcal{W}_m \) if and only if for any closed point \( y \in M \) and \( 1 \leq i \leq N \), for the morphism [6.45] and one parameter subgroup \( \lambda_i \) as in [6.14], we have

\[ (6.49) \quad \tau^*\tilde{\Phi}_y(\tilde{\gamma}_y^*\mathcal{E}) \in \bigoplus_{j \in [m_i - (\lambda_i, H^1(\mathbb{T}_{\mathcal{M}|y})^{\lambda_i>0}) + \eta_i, m_i - (\lambda_i, H^0(\mathbb{T}_{\mathcal{M}|y})^{\lambda_i<0}) + \eta_i]} \text{MF}_{\text{coh}}^C(\mathcal{X}_5 \setminus \tilde{T}_{i,y}, w_5)_{\lambda_i, \text{wt}=j}. \]

Here the weight shift by \( (\lambda_i, H^1(\mathbb{T}_{\mathcal{M}|y})^{\lambda_i>0}) \) is due to the equivalence [6.30]. From the diagram [6.15], we have the distinguished triangle

\[ \text{ev}_1^* L_{[\tilde{\mathcal{M}}_y/G_y]} \to L_{[\tilde{\mathcal{M}}_y/G_y]^{\lambda_i>0}} \to L_{\text{ev}_1} \]

which gives the identity

\[ \text{wt}(\sigma^* \det L^\vee_{\text{ev}_1}) = \langle \lambda_i, H^0(\mathbb{T}_{\mathcal{M}|y})^{\lambda_i<0} - H^1(\mathbb{T}_{\mathcal{M}|y})^{\lambda_i<0} - H^{-1}(\mathbb{T}_{\mathcal{M}|y})^{\lambda_i<0} \rangle. \]
It follows that, by \((6.33)\) we have the identities
\[
\frac{1}{2} \operatorname{wt}(\sigma^* \det L^{\psi}_{\text{ev}}) + \operatorname{wt}(\tau^* \Theta) - \langle \lambda_i, \mathcal{H}^1(T_{\mathcal{M}}|_y)^{\lambda_i > 0} \rangle = \frac{1}{2} \langle \lambda_i, \mathcal{H}^0(T_{\mathcal{M}}|_y)^{\lambda_i < 0} - \mathcal{H}^1(T_{\mathcal{M}}|_y)^{\lambda_i > 0} - \mathcal{H}^{-1}(T_{\mathcal{M}}|_y)^{\lambda_i < 0} \rangle + \langle \lambda_i, \Theta \rangle - \frac{1}{2} \langle \lambda_i, \mathcal{H}^1(T_{\mathcal{M}}|_y) \rangle = -\frac{1}{2} \eta_i + \left( \lambda_i, \frac{1}{2} \det(\mathcal{H}^1(T_{\mathcal{M}}|_y))^\vee \right).
\]

Together with the condition \((6.44)\), we see that the condition \((6.49)\) is equivalent to the condition \((6.46)\) for \(\mathcal{P} = \hat{\Phi}_y(t^* \mathcal{E})\). Therefore the proposition holds. \(\square\)

6.6. Inclusions of window subcategories. Let us take another \(\mathcal{L} \in \text{Pic}(\mathcal{M})\) with \(l' = c_1(\mathcal{L}')\), and set
\[
l_{\pm} := l \pm l', \quad 0 < \varepsilon \ll 1.
\]

Then we have \(\mathcal{N}^{l \pm \text{-ss}} \subset \mathcal{N}^{l \text{-ss}}\), and the \(\Theta\)-stratification for \((l_{\pm}, b)\) is a refinement of the \(\Theta\)-stratification \((6.2)\) for \((l, b)\)
\[
(6.50) \quad \mathcal{N} = S_1^0 \sqcup \cdots \sqcup S_N^0 \sqcup S_{N+1}^{0 \pm} \sqcup \cdots \sqcup S_{N+k_\pm}^{0 \pm} \sqcup \mathcal{N}^{l \pm \text{-ss}},
\]
where
\[
(6.51) \quad \mathcal{N}^{l \text{-ss}} = S_{N+1}^{0 \pm} \sqcup \cdots \sqcup S_{N+k_\pm}^{0 \pm} \sqcup \mathcal{N}^{l \pm \text{-ss}}
\]
is the \(\Theta\)-stratification of \(\mathcal{N}^{l \text{-ss}}\) for \((l_{\pm}, b)\) restricted to \(\mathcal{N}^{l \text{-ss}}\).

Let us consider the composition
\[
(6.52) \quad \mathcal{N}^{l \text{-ss}} \hookrightarrow \mathcal{N} \xrightarrow{\mathcal{P}_0} \mathcal{M} \xrightarrow{\hat{\Phi}} M.
\]

We take a closed point \(x \in \mathcal{N}^{l \text{-ss}}\), and denote by \(y \in M\) its image under the above composition. As before, we use the same symbol \(y \in \mathcal{M}\) to denote the unique closed point in the fiber of \(\pi_{\mathcal{M}}: \mathcal{M} \to M\) at \(y\). Note that \(p_0(x) \in \mathcal{M}\) may not be a closed point so that it may not be isomorphic to \(y \in \mathcal{M}\). In the notation of the diagram \((6.17)\), the closed point \(x \in \mathcal{N}^{l \text{-ss}}\) corresponds to a closed point \(x \in \mathcal{X}^{l \text{-ss}}\), where \(\mathcal{X}^{l \text{-ss}} \subset \mathcal{X}_1\) is the semistable locus with respect to the \(G_y\)-character \(\mathcal{L}_y\). Let \(G_x \subset G_y\) be the stabilizer subgroup of \(x\). We define \(W_x\) to be the tangent space of the stack \(\mathcal{X}^{l \text{-ss}}\) at \(x\), i.e.
\[
W_x := \mathcal{H}^0(T_{\mathcal{X}_1^{l \text{-ss}}}|_x)
\]
which is a \(G_x\)-representation. Here \(T_{\mathcal{X}_1^{l \text{-ss}}}\) is the tangent complex of \(\mathcal{X}_1^{l \text{-ss}}\).

Remark 6.16. The tangent complex of \(\mathcal{X}^{l \text{-ss}}\) is given by
\[
T_{\mathcal{X}_1^{l \text{-ss}}} = \left( \mathcal{H}^{-1}(T_{\mathcal{M}}|_y) \otimes \mathcal{O}_{\mathcal{X}_1^{l \text{-ss}}} \xrightarrow{\mathcal{P}_0} \left( \mathcal{H}^0(T_{\mathcal{M}}|_y) \oplus \mathcal{H}^1(T_{\mathcal{M}}|_y)^\vee \right) \otimes \mathcal{O}_{\mathcal{X}_1^{l \text{-ss}}} \right).
\]
The restriction of \(T_{\mathcal{X}_1^{l \text{-ss}}}\) to \(x\) is the two term complex of \(G_x\)-representations
\[
T_{\mathcal{X}_1^{l \text{-ss}}}|_x = \left( \mathcal{H}^{-1}(T_{\mathcal{M}}|_y) \xrightarrow{\mathcal{P}_0} \mathcal{H}^0(T_{\mathcal{M}}|_y) \oplus \mathcal{H}^1(T_{\mathcal{M}}|_y)^\vee \right).
\]
The kernel of \(\mu_x\) is the Lie algebra of \(G_x\). The \(G_x\)-representation \((6.52)\) is given by the cokernel of \(\mu_x\).

We impose the following assumption:

Assumption 6.17. For each closed point \(x \in \mathcal{N}^{l \text{-ss}}\), there is a decomposition of \(G_x\)-representations
\[
W_x = S_x \oplus U_x\quad \text{such that } S_x \text{ is a symmetric } G_x\text{-representation, i.e. } S_x \cong S_x^*\text{ as } G_x\text{-representations, and}
\]
\[
(6.53) \quad W_x^{l \text{-ss}} = S_x^{l \text{-ss}} \oplus U_x.
\]
Here the $l_-$-semistable loci on $G_x$-representations are defined with respect to the $G_x$-character $(\mathcal{L} \otimes (\mathcal{L}')^{-\varepsilon})|_x$.

For $1 \leq i \leq N$ and $N < i \leq N + k_+$, we set $m_i$ and $m_i^\pm$ as in (6.48), with a choice of $\mathcal{I}$ to be (6.54)
\[
\mathcal{I} = \varepsilon_+ \cdot l_+ + \varepsilon_- \cdot l_-
\]
for generic $(\varepsilon_+, \varepsilon_-) \in \mathbb{R}^2$. We denote the above choice by $m_i^\pm$, i.e.
\[
m_i^\pm = \{m_1, \ldots, m_N, m_{N+1}^\pm, \ldots, m_{N+k_+}^\pm\}.
\]
By Corollary 6.13, we have subcategories
\[
\mathcal{W}_i^\pm \subset \mathcal{W}_i \subset \mathbf{DT}^{C^*}(\mathcal{N})
\]
such that the compositions
\[
\mathcal{W}_i^\pm \hookrightarrow \mathbf{DT}^{C^*}(\mathcal{N}) \twoheadrightarrow \mathbf{DT}^{C^*}(\mathcal{N}^{l^\pm-ss})
\]
are equivalences.

**Theorem 6.18.** Under Assumption 6.17 we have the inclusion $\mathcal{W}_i^\pm \subset \mathcal{W}_i^\pm$. In particular, we have the fully-faithful functor
\[
\mathbf{DT}^{C^*}(\mathcal{N}^{l^\pm-ss}) \hookrightarrow \mathbf{DT}^{C^*}(\mathcal{N}^{l^\pm-ss}).
\]

**Proof.** For $i > N$, we use the subscript $\pm$ to denote corresponding objects associated with the strata $S_l^\pm$ in (6.50), e.g. $Z_l^\pm$ for the center of $S_l^\pm$, $\lambda^\pm_l$ for one parameter subgroups in (6.13) for the $i$-th $\Theta$-strata with $N < i \leq N + k_+$, denote $\eta^\pm_l$ for (6.43), etc. By Proposition 6.15, it is enough to show the inclusion $\mathcal{W}_y^\pm \subset \mathcal{W}_y^\pm$ for any closed point $y \in M$. We define
\[
\mathcal{W}_y^\pm \subset \mathbf{MF}^{Z/2}_c(X_1, w_1)
\]
to be the subcategories defined by the same condition (6.40) with respect to $l_+$ and $l$ for the $Z/2$-periodic triangulated categories of factorizations. Since the forgetful functor $\mathbf{MF}^{Z/2}_c(X_1, w_1) \to \mathbf{MF}^{Z/2}_c(X_1, w_1)$ is conservative, it is enough to show the inclusion
\[
\mathcal{W}_y^\pm \subset \mathbf{MF}^{Z/2}_c(X_1, w_1)
\]
(6.55)
Let $X_1^{lss} \to X_1^{lss}$ be the good moduli space for $X_1^{lss}$. For a closed point $x \in X_1^{lss}$, we also denote by $x \in X_1^{lss}$ the unique closed point contained in the fiber of $X_1^{lss} \to X_1^{lss}$ at $x$. Let $W_x$ be the formal fiber of $W_x$ at zero. By Luna’s étale slice theorem, we have the Cartesian square
\[
\begin{array}{ccc}
\mathbf{DT}^{C^*}(\mathcal{N}^{l^\pm-ss}) & \longrightarrow & \mathbf{DT}^{C^*}(\mathcal{N}^{l^\pm-ss}) \\
\mathbf{DT}^{C^*}(\mathcal{N}^{l^\pm-ss}) & \longrightarrow & \mathbf{DT}^{C^*}(\mathcal{N}^{l^\pm-ss})
\end{array}
\]
(6.56)
which identifies the left vertical arrow with the formal fiber of the right vertical arrow at $x$. Let
\[
i^*w_1: \mathbf{DT}^{C^*}(\mathcal{N}^{l^\pm-ss}) \to \mathbf{DT}^{C^*}(\mathcal{N}^{l^\pm-ss})
\]
be the pull-back of $w_1: X_1 \to X_1$ by the top morphism in (6.54). Then its critical locus is isomorphic to a formal fiber along with the good moduli space morphism $X^{lss} \to N^{lss}$. The $\Theta$-stratification (6.51) is local on the good moduli space, so it induces the $\Theta$-stratification on $\text{Crit}(\eta^*w_1)$

\[
\text{Crit}(\eta^*w_1) = \mathcal{S}_{N+1, x} \sqcup \cdots \sqcup \mathcal{S}_{N+k_+, x} \sqcup \text{Crit}((\eta^*w_1)_{l^\pm-ss})
\]
(6.57)
\[
\text{Crit}(\eta^*w_1) = \mathcal{S}_{N+1, x} \sqcup \cdots \sqcup \mathcal{S}_{N+k_+, x} \sqcup \text{Crit}((\eta^*w_1)_{l^\pm-ss})
\]
We denote by
\[
\mathcal{W}_x^{\pm/2} \subset \mathbf{MF}^{Z/2}_c(W_x/G_x, \eta^*w_1)
\]
the window subcategory in Subsection 2.3 associated with the \( \Theta \)-stratification \((6.57)\), i.e. the subcategory of objects \( P \in \mathcal{P} \) of the window subcategory in Subsection 2.3 associated with the \( \Theta \)-stratification \((6.57)\), i.e. the subset

\[
\tau_{\pm} : \left[ \frac{[\hat{W}_x^{\lambda_i^\pm = 0}/G_x^{\lambda_i^\pm = 0}}{\mathcal{S}_{\leq 1-1,x}} \right] \to \left[ \frac{\hat{W}_x^\pm /G_x}{\mathcal{S}_{\leq 1-1,x}} \right]
\]

of the inclusion from the \( \lambda_i^\pm \)-fixed part, we have

\[
\tau_{\pm}^* P \in \bigoplus_{j \in \left[-\frac{1}{2} \eta_i^+ + (\lambda_i^+, \delta), \frac{1}{2} \eta_i^+ + (\lambda_i^+, \delta)\right]} \mathrm{MF}_{\text{coh}}^w\left( \frac{[\hat{W}_x^{\lambda_i^\pm = 0}/G_x^{\lambda_i^\pm = 0}}{\mathcal{S}_{\leq 1-1,x}} \right) \circ \mathcal{S}_{\leq 1-1,x} = \mathcal{S}_{\leq 1-1,x}
\]

for all \( N < i \leq N + k_\pm \), where \( \delta \in \text{Pic}(BG_x) \) is the \( G_y \)-character \((6.47)\) restricted to \( G_x \subset G_y \) with the choice of \( \check{I} \) given by \((6.54)\). From the equivalence of the composition

\[
\frac{\hat{W}_x^{Z/2,l}}{\mathcal{W}_y^{Z/2,l}} \subset \frac{\hat{W}_x^{Z/2,l}}{\mathcal{W}_y^{Z/2,l}} \subset \frac{\hat{W}_x^{Z/2,l}}{\mathcal{W}_y^{Z/2,l}}
\]

we can identify the subcategories \( \frac{\hat{W}_x^{Z/2,l}}{\mathcal{W}_y^{Z/2,l}} \subset \frac{\hat{W}_x^{Z/2,l}}{\mathcal{W}_y^{Z/2,l}} \subset \frac{\hat{W}_x^{Z/2,l}}{\mathcal{W}_y^{Z/2,l}} \subset \frac{\hat{W}_x^{Z/2,l}}{\mathcal{W}_y^{Z/2,l}} \) with the subcategories in \( \mathrm{MF}_{\text{coh}}^X(\mathcal{X}_1^{\text{ss}}, w_1) \) consisting of objects satisfying the condition \((6.40)\) for all \( N < i \leq N + k_\pm \) with respect to \( k_\pm \). Since the latter condition is local on the good moduli space \( X_1^{\text{ss}} \to X_1^{\text{ss}} \), an object \( \mathcal{E} \in \mathrm{MF}_{\text{coh}}^X(\mathcal{X}_1^{\text{ss}}, w_1) \) is an object in \( \frac{\hat{W}_x^{Z/2,l}}{\mathcal{W}_y^{Z/2,l}} \) if and only if for any closed point \( x \in X_1^{\text{ss}} \) we have \( \mathcal{E} \in \frac{\hat{W}_x^{Z/2,l}}{\mathcal{W}_y^{Z/2,l}} \). On the other hand, it is proved in \( \text{[Todb]} \) Proposition 5.1.7 (also see \( \text{[KT]} \) Proposition 2.6) that, under Assumption \((6.17)\), we have the inclusion

\[
\frac{\hat{W}_x^{Z/2,l}}{\mathcal{W}_y^{Z/2,l}} \subset \frac{\hat{W}_x^{Z/2,l}}{\mathcal{W}_y^{Z/2,l}}.
\]

Therefore the inclusion \((6.35)\) holds. \( \square \)

**Remark 6.19.** The results of Theorem \((6.11)\) and Theorem \((6.12)\) rely on the existence of good moduli space of \( \mathcal{M} \). In a situation we are interested in, in many cases \( \mathcal{M} \) does not admit a good moduli space. However it is sometimes possible to find an open embedding \( \mathcal{M} \subset \mathcal{M}' \) for a quasi-smooth derived stack \( \mathcal{M}' \) such that \( \mathcal{M}' = t_0(\mathcal{M})' \) admits a good moduli space. Then using Lemma \((3.6)\) we can work with the singular support quotients of \( D^b_{\text{coh}}(\mathcal{M})' \) and apply the above results. We will apply this idea in Section 7 using the moduli stacks of perverse coherent systems.

### 7. Categorical wall-crossing at \((-1,-1)\)-curve

In \( \text{[Todb]} \) Section 6, we proved Conjecture \((1.3)\) for reduced curve classes using categorified Hall products. In this section, we use Theorem \((6.18)\) to prove Conjecture \((1.3)\) under wall-crossing at \((-1,-1)\)-curves for reduced curve classes.

#### 7.1. Categorical PT theory for local surfaces

Let \( S \) be a smooth projective surface over \( \mathbb{C} \), and \( X \) the associated local surface

\[
\pi : X := \text{Tot}_S(\omega_S) \to S.
\]

Let \( \text{Coh}_{\leq 2}(X) \subset \text{Coh}(X) \) be the abelian subcategory of compactly supported coherent sheaves \( F \) on \( X \) with \( \dim \text{Supp}(F) \leq 1 \). For each \( (\beta, n) \in \text{NS}(S) \oplus \mathbb{Z} \), we denote by

\[
P_n(X, \beta)
\]

the moduli space of Pandharipande-Thomas stable pairs \(\text{PT}09\) parameterizing pairs \((F, s)\) where \( F \in \text{Coh}_{\leq 1}(X) \) is a pure one dimensional sheaf and \( s : \mathcal{O}_X \to F \) is surjective in dimension one, satisfying \( (\pi_*[F], \chi(F)) = (\beta, n) \).

For \( (\beta, n) \in \text{NS}(S) \oplus \mathbb{Z} \), let

\[
\mathcal{M}_n^1(S, \beta)
\]

be the derived moduli stack of pairs \((F, s)\), where \( F \in \text{Coh}_{\leq 1}(S) \) and \( s : \mathcal{O}_S \to F \) satisfying \( ([F], \chi(F)) = (\beta, n) \) (see \( \text{[Todb]} \) Subsection 4.1.1 for its precise formulation). It is proved in \( \text{[Todb]} \)
Theorem 4.1.3] that \( \mathfrak{M}_n^1(S, \beta) \) is quasi-smooth, and the classical truncation of its \((-1)\)-shifted cotangent is isomorphic to the component of the moduli stack of \textit{D0-D2-D6 bound states}, i.e. objects in the following subcategory

\[
\mathcal{A}_X := (\mathcal{O}_{\overline{X}}, \text{Coh}_{\leq 1}(X)[{-1}])_{\text{ex}} \subset D^b_{\text{coh}}(\overline{X}).
\]

Here \( X \subset \overline{X} \) is a projective compactification of \( X \). In particular, we have the open immersion

\[
P_n(X, \beta) \subset t_0(\Omega_{\mathfrak{M}_n^1(S, \beta)}[-1]),
\]
sending a stable pair \((F, s)\) to the two term complex \((\mathcal{O}_{\overline{X}} \to F)\).

Let \( \mathfrak{M}_n^1(S, \beta)_{qc} \subset \mathfrak{M}_n^1(S, \beta) \) be a quasi-compact derived open substack such that

\[
P_n(X, \beta) \subset t_0(\Omega_{\mathfrak{M}_n^1(S, \beta)_{qc}}[-1]) \subset t_0(\Omega_{\mathfrak{M}_n^1(S, \beta)}[-1]).
\]

We have the following conical closed substack

\[
Z^{P\text{-us}} := t_0(\Omega_{\mathfrak{M}_n^1(S, \beta)_{qc}}[-1]) \setminus P_n(X, \beta) \subset t_0(\Omega_{\mathfrak{M}_n^1(S, \beta)}[-1]).
\]

Following Definition 3.5, the DT category for PT moduli spaces on the local surface is defined in [Todb] as follows:

**Definition 7.1.** ([Todb] Definition 4.2.1) The \( \mathbb{C}^* \)-equivariant DT category for \( P_n(X, \beta) \) is defined by

\[
\text{DT}^{\mathbb{C}^*}(P_n(X, \beta)) := D^b_{\text{coh}}(\mathfrak{M}_n^1(S, \beta))/\mathcal{C}_{Z^{P\text{-us}}}. \]

### 7.2. Wall-crossing of D0-D2-D6 bound states.

There are some variants of stable pair moduli spaces, depending on choices of stability conditions. Let

\[
\text{cl}: K(\mathcal{A}_X) \to \mathbb{Z} \oplus \text{NS}(S) \oplus \mathbb{Z}
\]

be a group homomorphism defined by \( \text{cl}(\mathcal{O}_X) = (1, 0, 0) \) and \( \text{cl}(F[-1]) = (\pi_*[F], \chi(F)) \) for \( F \in \text{Coh}_{\leq 1}(X) \). We also fix an ample divisor \( H \) on \( S \). For \( E \in \mathcal{A}_X \) with \( \text{cl}(E) = (r, \beta, n) \) and \( t \in \mathbb{R} \), we set \( \mu_t^1(E) \in \mathbb{Q} \cup \{\infty\} \) to be

\[
\mu_t^1(E) := \begin{cases} 
  t, & \text{if } r > 0, \\
  n/H \cdot \beta, & \text{if } r = 0.
\end{cases}
\]

**Definition 7.2.** ([Todb] Definition 4.2.5) An object \( E \in \mathcal{A}_X \) is \( \mu_t^1 \)-stable for any exact sequence \( 0 \to E_1 \to E \to E_2 \to 0 \) in \( \mathcal{A}_X \) with \( E_i \neq 0 \), we have \( \mu_t^1(E_1) < (\leq) \mu_t^1(E_2) \).

By [AHL11] Theorem 7.25, it admits a good moduli space (also see Remark 7.15)

\[
P_n^+(X, \beta) \to P_n^+(X, \beta).
\]

There is a finite set of walls \( W \subset \mathbb{R} \) such that the above morphism is an isomorphism if \( t \notin W \). If \( t \in W \cap \mathbb{R}_{>0} \), then we have the following wall-crossing diagram

\[
\begin{array}{c}
P_n^+(X, \beta) \\
\downarrow \quad \downarrow \\
P_n^+(X, \beta) \\
\end{array}
\begin{array}{c}
P_n^-(X, \beta) \\
P_n^-(X, \beta) \\
P_n^-(X, \beta) \\
\end{array}
\]

(7.3)

which is a \textit{d-critical flip} (see [Toda, Theorem 9.13]), i.e. a \textit{d-critical} analogue of usual flip in birational geometry. Moreover we have (see [Tod10] Theorem 3.21)

\[
P_n^+(X, \beta) = P_n(X, \beta), \quad t \gg 0
\]

By taking \( t_1 > t_2 > \cdots > t_k > 0 \) which do not lie on walls, we have the sequence of \textit{d-critical flips}

\[
P_n(X, \beta) \dashrightarrow P_n^1(X, \beta) \dashrightarrow \cdots \dashrightarrow P_n^k(X, \beta).
\]

(7.4)
In particular in the diagram (7.4), we have the chain of fully-faithful functors

\[ \mathcal{F} \otimes \mathcal{O}_Z : Z \subseteq X \text{ is a compactly supported closed subscheme with } \pi_*(Z) \leq \beta \}

Then \( n(\beta) > -\infty \) by [Tod09, Lemma 3.10]. We will use the following lemma:

**Lemma 7.3.** (Todb Lemma 4.2.7) If \( P^t_n(X, \beta') \neq \emptyset \) for \( 0 < \beta' \leq \beta \) and \( t > 0 \), we have \( n' \geq n(\beta) \).

Similarly to the moduli spaces of PT stable pairs, there exists a quasi-compact derived open substack \( \mathcal{M}_{t}^n(S, \beta)_{qc} \subset \mathcal{M}_{t}^n(S, \beta) \) such that

\[ P^t_n(X, \beta) \subset t_0(\Omega_{\mathcal{M}_{t}^n(S, \beta)_{qc}}[-1]) \subset t_0(\Omega_{\mathcal{M}_{t}^n(S, \beta)})[-1]). \]

Similarly to (7.1), we have the conical closed substack

\[ \mathcal{Z}_{t-us} := t_0(\Omega_{\mathcal{M}_{t}^n(S, \beta)_{qc}}[-1]) \setminus P^t_n(X, \beta) \subset t_0(\Omega_{\mathcal{M}_{t}^n(S, \beta)})[-1]). \]

**Definition 7.4.** (Todb, Definition 4.2.9) The \( \mathbb{C}^* \)-equivariant DT category for \( P^t_n(X, \beta) \) is defined by

\[ \mathcal{D}T^\mathbb{C}^*(P^t_n(X, \beta)) := D^b_{coh}(\mathcal{M}_{t}^n(S, \beta))/\mathcal{Z}_{t-us}. \]

**Remark 7.5.** By Lemma 7.6, the DT categories in Definition 7.4 and Definition 7.4 are independent of a choice of \( \mathcal{M}_{t}^n(S, \beta)_{qc} \) up to equivalence.

As an analogy of D/K equivalence conjecture in birational geometry [BO, Kaw02], the following conjecture is proposed in [Todb].

**Conjecture 7.6.** (Todb Conjecture 4.24) In the diagram (7.3), there exists a fully-faithful functor

\[ \mathcal{D}T^\mathbb{C}^*(P^t_n(X, \beta)) \rightarrow \mathcal{D}T^\mathbb{C}^*(P^t_n(X, \beta)). \]

In particular in the diagram (7.4), we have the chain of fully-faithful functors

\[ \mathcal{D}T^\mathbb{C}^*(P^t_n(X, \beta)) \rightarrow \cdots \rightarrow \mathcal{D}T^\mathbb{C}^*(P^t_n(X, \beta)) \rightarrow \mathcal{D}T^\mathbb{C}^*(P_n(X, \beta)). \]

**7.3. Wall-crossing at \((-1, -1)\)-curve.** Suppose that the surface \( S \) contains a \((-1)\)-curve \( C \subset S \). Let

\[ f: S \rightarrow T \]

be a birational contraction which contracts \( C \) to a smooth point in \( T \). We have the closed embeddings

\[ C \hookrightarrow S \rightarrow X, \quad N_{C/X} = \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1), \]

where \( i \) is the zero section. Let \( h \) be an ample divisor on \( S \). For \( X = \text{Tot}_S(\omega_S) \), we have the commutative diagram

\[ \begin{array}{ccc} X & \xrightarrow{g} & Y := \text{Spec}_T \left( \bigoplus_{k \geq 0} m_p^k \otimes \omega_T^{-k} \right) \\ \pi \downarrow & & \downarrow \pi \\ S & \xrightarrow{f} & T \end{array} \]

where \( p = f(C) \in T, m_p \subset \mathcal{O}_T \) is the ideal sheaf of \( p \). The morphism \( g \) is a flopping contraction which contracts a \((-1, -1)\)-curve \( C \) to a conifold point in \( Y \). We take an ample class \( H \in \text{NS}(S) \) of the form

\[ H = a \cdot f^* h - C, \quad a \gg 0. \]

Let \( W \subset \mathbb{R} \) be the set of walls with respect to the \( \mu^1 \)-stability and the above choice of \( H \).
Lemma 7.7. For a fixed $(\beta, n) \in \text{NS}(S) \oplus \mathbb{Z}$ such that $\beta$ is effective, we take $a \gg 0$ so that
\begin{equation}
(7.6) \quad a > \max \left\{ \frac{2n - 2n(\beta) + C \cdot \beta'}{h \cdot f_s \beta'} : 0 < \beta' \leq \beta, f_s \beta' \neq 0 \right\}.
\end{equation}
Then we have $W \cap \mathbb{R}_{\geq 1/2} \subset \mathbb{Z}_{\geq 1}$. For $k \in \mathbb{Z}_{\geq 1}$ and $t = k$, any strictly $\mu^1_\ast$-polystable object in $\mathcal{A}_X$ with $\text{cl}(-) = (1, \beta, n)$ is of the form
\begin{equation}
(7.7) \quad I = I_1 \oplus \mathcal{O}_C(k - 1)^{\oplus m} [-1],
\end{equation}
where $m \in \mathbb{Z}_{\geq 1}$ and $I_1 \in \mathcal{A}_X$ is $\mu^1_\ast$-stable.

Proof. For $t \geq 1/2$, let $I \in \mathcal{A}_X$ be a strictly $\mu^1_\ast$-polystable object with $\text{cl}(I) = (1, \beta, n)$. It is of the form
\begin{equation}
I = I_1 \oplus F_2 [-1]
\end{equation}
where $I_1 \in \mathcal{A}_X$ is $\mu^1_\ast$-stable, $F_2 [-1] \in \text{Coh}_{\leq 1}(\mathcal{X}) [-1]$ is $\mu^1_\ast$-semistable, such that
\begin{equation}
\text{cl}(I_1) = (1, \beta_1, n_1), \quad \text{cl}(F_2 [-1]) = (0, \beta_2, n_2), \quad \frac{n_2}{H \cdot \beta_2} = t \geq \frac{1}{2}.
\end{equation}
Since we have $(\beta, n) = (\beta_1, n_1) + (\beta_2, n_2)$, we have
\begin{equation}
(7.8) \quad n_1 = n - n_2 \leq n - \frac{1}{2} (ah \cdot f_s \beta_2 - C \cdot \beta_2).
\end{equation}
Since $\beta_1 \leq \beta$, we have $n_1 \geq n(\beta)$ by Lemma 7.3. Therefore for $a \gg 0$ satisfying (7.6), we have $f_s \beta_2 = 0$, so $\beta_2$ is of the form $\beta_2 = m[C]$. Then $F_2$ is a semistable sheaf supported on $C$, so it is of the form $F_2 = \mathcal{O}_C(k - 1)^{\oplus m}$ for some $m \in \mathbb{Z}_{\geq 1}$ and $k \in \mathbb{Z}$. As $\mu^1_\ast(F_2) = t = k$, we have $t \in \mathbb{Z}_{\geq 1}$, and obtain the desired form (7.7) for $\mu^1_\ast$-polystable objects.

Lemma 7.8. For $a \gg 0$ satisfying (7.6) and $t \geq 1/2$, an object $I \in \mathcal{A}_X$ with $\text{cl}(I) = (1, \beta, n)$ is $\mu^1_\ast$-semistable if and only if the following conditions are satisfied:

(i) $\text{g}_s \mathcal{H}^1(I)$ is zero dimensional,
(ii) $\text{Hom}(\mathcal{O}_C(m)[-1], I) = 0$ for $m + 1 > t$,
(iii) $\text{Hom}(I, \mathcal{O}_C(m)[-1]) = 0$ for $m > t + 1$,
(iv) $\text{Hom}(\mathcal{O}_X[-1], I) = 0$ for $x \in X \setminus C$.

Proof. For a $\mu^1_\ast$-semistable object $I \in \mathcal{A}_X$, the conditions (ii), (iii) and (iv) are obviously satisfied. As for (i), suppose that $\text{g}_s \mathcal{H}^1(I)$ is not zero dimensional. By the $\mu^1_\ast$-stability, we have $\mu^1_\ast(\mathcal{H}^1(I)) \geq t \geq 1/2$. By setting $\text{cl}(\mathcal{H}^0(I)) = (1, \beta''', n''')$ and $\text{cl}(\mathcal{H}^1(I)[-1]) = (0, \beta', n')$, we have $n'' = n - n' \geq n(\beta)$ since $\mathcal{H}^0(I) = I_Z$ for a compactly supported closed subscheme $Z \subset X$ with $\pi_s[Z] \leq \beta$. Therefore we obtain
\begin{equation}
\frac{n - n(\beta)}{(af^+ h - C) \cdot \beta'} \geq \frac{n'}{(af^+ h - C) \cdot \beta'} \geq \frac{1}{2},
\end{equation}
which contradicts to (7.6).

Conversely, suppose that $I \in \mathcal{A}_X$ satisfies (i) to (iv). Let $I \to F[-1]$ be a surjection in $\mathcal{A}_X$ such that $F \in \text{Coh}_{\leq 1}(\mathcal{X})$ with $\mu^1_\ast(F) < t$. As $\text{g}_s \mathcal{H}^1(I)$ is zero dimensional, $\text{g}_s F$ is also zero dimensional. If $F \neq 0$, then there is a surjection $F \to \mathcal{O}_C(m)$ for $t > m + 1$, which contradicts to (iii). Therefore $F = 0$. Then by taking the Harder-Narasimhan filtration of $I$ for $\mu^1_\ast$-stability, we have the exact sequence $0 \to F_1[-1] \to I \to I_2 \to 0$ in $\mathcal{A}_X$ such that $F_1 \in \text{Coh}_{\leq 1}(\mathcal{X})$ with $\mu^1_\ast(F_1) > t$ and $I_2$ is $\mu^1_\ast$-semistable of rank one. We set $\text{cl}(F_1[-1]) = (0, \beta_1, n_1)$ and $\text{cl}(I_2) = (1, \beta_2, n_2)$. Then $n_2 = n - n_1 \geq n(\beta)$ by Lemma 7.3. It follows that
\begin{equation}
\frac{n - n(\beta)}{(af^+ h - C) \cdot \beta_1} \geq \frac{n_1}{(af^+ h - C) \cdot \beta_1} = \mu^1_\ast(F_1[-1]) \geq t \geq \frac{1}{2}.
\end{equation}
which contradicts to \( (7.6) \) if \( f_*, \beta_1 \neq 0 \). Therefore \( f_*, \beta_1 = 0 \) so \( g_* F_1 \) is zero dimensional. If \( F_1 \neq 0 \), then there is an injection \( O_C(m) \hookrightarrow F_1 \) for \( m + 1 > t \) or \( O_x \hookrightarrow F_1 \) for \( x \in X \setminus C \), which contradicts to (ii) or (iv). Therefore we conclude that \( F_1 = 0 \), i.e. \( I \) is \( \mu^1_i \)-semistable. \( \square \)

Below we take \( a \gg 0 \) as in Lemma \( 7.7 \) and set
\[
P_n^{(k)}(X, \beta) := P_n^{k+1-0}(X, \beta)
\]
for \( k \in \mathbb{Z}_{\geq 0} \), which equals to \( P_n^{k+0}(X, \beta) \) if \( k \geq 1 \) by Lemma \( 7.7 \). The wall-crossing diagram \( (7.3) \) in this case at \( t = k \) is
\[
P_n^{(k)}(X, \beta) \quad \xrightarrow{P_n^{k=0}(X, \beta)} \quad P_n^{(k-1)}(X, \beta)
\]
The assertion in Conjecture \( 7.6 \) is specialized as follows:

**Conjecture 7.9.** For each \( k \in \mathbb{Z}_{\geq 1} \), there exists a fully-faithful functor
\[
\mathcal{D}T^C_+ (P_n^{(k-1)}(X, \beta)) \hookrightarrow \mathcal{D}T^C_+ (P_n^{(k)}(X, \beta)).
\]

### 7.4. Perverse coherent sheaves

We will describe the moduli spaces \( P_n^{(k)}(X, \beta) \) in terms of perverse coherent sheaves introduced in \([\text{Bri02, VdB04}]\), defined as follows:

**Definition 7.10.** \([\text{Bri02, VdB04}]\) The subcategory of perverse coherent sheaves
\[
\text{Per}(S/T) \subset D_{\coho}^b(S)
\]
is defined to be the subcategory of objects \( E \in D_{\coho}^b(S) \) satisfying the following conditions:

(i) we have \( Rf_* E \in \text{Coh}(T) \),

(ii) we have \( \text{Hom}^<0(O_C(-1)[1], E) = \text{Hom}^<0(E, O_C(-1)[1]) = 0. \)

The subcategory \( \text{Per}(S/T) \) is the heart of a bounded t-structure on \( D_{\coho}^b(S) \) which contains \( O_S \). We will denote by \( \mathcal{P}H^i(\cdot) \) the \( i \)-th cohomology with respect to the above perverse t-structure. By setting
\[
\mathcal{E} := O_S \oplus O_S(-C), \quad A_S := f_* \text{End}(\mathcal{E}),
\]
we have the equivalence (see \([\text{VdB04}]\))
\[
(7.9) \quad Rf_* R\mathcal{H}om(\mathcal{E}, \cdot) : \text{Per}(S/T) \xrightarrow{\sim} \text{Coh}(A_S).
\]
We also have the following abelian subcategories for \( i = 0, 1 \)
\[
\text{Per}_{\leq i}(S/T) := \{ E \in \text{Per}(S/T) : \dim \text{Supp}(Rf_* E) \leq i \}.
\]

The subcategories
\[
\text{Per}_{\leq i}(X/Y) \subset \text{Per}(X/Y) \subset D_{\coho}^b(X)
\]
are also defined in a similar way, using the flopping contraction \( g : X \to Y \) instead of \( f : S \to T \), imposing an additional condition that \( Rg_* E \) is compactly supported for \( i = 1 \). Similarly to \( (7.9) \), we have the equivalence
\[
(7.10) \quad Rg_* R\mathcal{H}om(\pi^* \mathcal{E}, \cdot) : \text{Per}(X/Y) \xrightarrow{\sim} \text{Coh}(A_X)
\]
where \( A_X = g_* \text{End}(\pi^* \mathcal{E}) \). In particular, the push-forward gives a functor
\[
\pi_* : \text{Per}_{\leq i}(X/Y) \to \text{Per}_{\leq i}(S/T).
\]
For \( i = 0 \), the abelian categories \( \text{Per}_{\leq 0}(S/T) \), \( \text{Per}_{\leq 0}(X/Y) \) are generated by their simple objects, and described as the extension closures
\[
(7.11) \quad \text{Per}_{\leq 0}(S/T) = \langle \mathcal{S}_0, \mathcal{S}_1, O_x : x \in S \setminus C \rangle_{\text{ex}}, \quad \text{Per}_{\leq 0}(X/Y) = \langle \mathcal{S}_0, \mathcal{S}_1, O_x : x \in X \setminus C \rangle_{\text{ex}}
\]
where $S_0 := 0_C$ and $S_1 := 0_C(-1)[1]$.

**Remark 7.11.** By [VdB04], the heart $\text{Per}_{\leq 1}(X/Y)$ is obtained as a tilting of $\text{Coh}_{\leq 1}(X)$ by the torsion pair $(T, F)$, i.e.

$$\text{Per}_{\leq 1}(X/Y) = (\mathcal{F}[1], T)_{\text{ex}} \subset D^b_{\text{coh}}(X)$$

where $\mathcal{F}$ consists of sheaves $F$ with $g_*F = 0$. By taking Harder-Narasimhan filtration, we see that $\mathcal{F}$ is the extension closure of objects $0_C(m)$ with $m \leq -1$.

### 7.5. Moduli stacks of perverse coherent systems

The notion of perverse coherent systems is defined as follows.

**Definition 7.12.** A perverse coherent system is a pair

$$(7.12) \quad (F, s), \quad F \in \text{Per}_{\leq 1}(S/T), \quad s : 0_S \to F.$$

A perverse coherent system $(F, s)$ is called a pure quotient if $s$ is surjective in $\text{Per}(S/T)$ and $F$ does not have a non-zero subobject in $\text{Per}_{\leq 0}(S/T)$.

We denote by

$$\mathcal{M}_{n}^{\text{per}}(S, \beta)$$

the derived moduli stack of perverse coherent systems $(F, s)$ where $F \in \text{Per}_{\leq 1}(S/T)$ and $s : 0_S \to F$, satisfying $([F], \chi(F)) = (\beta, n) \in \text{NS}(S) \oplus \mathbb{Z}$.

**Lemma 7.13.** The derived stack $\mathcal{M}_{n}^{\text{per}}(S, \beta)$ is quasi-smooth.

**Proof.** The tangent complex of $\mathcal{M}_{n}^{\text{per}}(S, \beta)$ at a perverse coherent system $(0_S \to F)$ is given by

$$\mathcal{R} \Gamma(0_S \to F, F).$$

Therefore it is enough to show that $\mathcal{R} \Gamma^{\leq i}(F) = 0$ for any $F \in \text{Per}_{\leq 1}(S/T)$. The first vanishing holds since $\mathcal{R}F, F \in \text{Coh}_{\leq 1}(T)$. As for the second vanishing, by the Serre duality we have

$$(7.13) \quad \text{Hom}(F, F[i]) = \text{Hom}(F, F \otimes 0_S[2 - i])^\vee.$$ 

Let $(T, F)$ be a torsion pair of $\text{Coh}_{\leq 1}(S)$ as in Remark 7.11. Then $F \otimes 0_S \subset F$ and $T \otimes 0_S \subset \text{Coh}_{\leq 1}(X)$, so we have

$$\text{Per}_{\leq 1}(S/T) \otimes 0_S \subset (\text{Per}_{\leq 1}(S/T), \text{Per}_{\leq 1}(S/T)[-1])_{\text{ex}}.$$

Therefore $(7.13)$ vanishes for $i \geq 3$. \qed

We have the derived open substack

$$\mathcal{P}_{n}^{\text{per}}(S, \beta) \subset \mathcal{M}_{n}^{\text{per}}(S, \beta)$$

corresponding to perverse coherent systems $(T, F)$ such that $p^\text{Cok}(s) \in \text{Per}_{\leq 0}(S/T)$, where $p^\text{Cok}(-)$ denotes the cokernel in the heart $\text{Per}(S/T)$. Its classical truncation is denoted by $\mathcal{P}_{n}^{\text{per}}(S, \beta)$.

**Lemma 7.14.** The moduli stack $\mathcal{P}_{n}^{\text{per}}(S, \beta)$ admits a good moduli space

$$(7.14) \quad \mathcal{P}_{n}^{\text{per}}(S, \beta) \to \mathcal{P}_{n}^{\text{per}}(S, \beta)$$

where each closed point of $\mathcal{P}_{n}^{\text{per}}(S, \beta)$ corresponds to a direct sum

$$(7.15) \quad I = I_0 \oplus (V_0 \otimes 0_S[-1]) \oplus (V_1 \otimes S_1[-1]) \oplus \bigoplus_{i=1}^{k} (W_i \otimes 0_{x_i}[-1])$$

where $I_0 = (0_S \xrightarrow{\alpha} F_0)$ for a pure quotient $(F_0, s_0), F[-1] := (0 \to F)$ for $F \in \text{Per}_{\leq 1}(S/T)$ and $V_0, V_1, W_i$ are finite dimensional vector spaces and $x_1, \ldots, x_i \in S \setminus C$ are distinct points. Moreover the good moduli space morphism $(7.14)$ satisfies the formal neighborhood theorem.
Proof. Let $\text{Per}_{\leq 1}^+(S/T)$ be the abelian category of pairs $(\mathcal{O}_S^\text{pr} \to F)$ for $r \in \mathbb{Z}_{\geq 0}$ and $F \in \text{Per}_{\leq 1}(S/T)$. Similarly to the moduli stack at MNOP/PT wall in [Toda Subsection 4.2.1], the moduli stack $\mathcal{P}_n^{\text{per}}(S, \beta)$ is identified with the moduli stack of semistable objects in $\text{Per}_{\leq 1}(S/T)$ with respect to the function

$$p_r: (\mathcal{O}_S^\text{pr} \to F) \mapsto \begin{cases} 0, & r > 0, \\ [I, F] \cdot h, & r = 0. \end{cases}$$

(7.16)

Namely a perverse coherent system $I = (\mathcal{O}_S \to F)$ is an object in $\mathcal{P}_n^{\text{per}}(S, \beta)$ if and only if for any exact sequence $0 \to I_1 \to I \to I_2 \to 0$ in $\text{Per}_{\leq 1}(S/T)$ we have $p_r(I_1) \leq 0 \leq p_r(I_2)$. Then the existence of good moduli space follows from [AHLH Theorem 7.25] (also see Remark 7.14).

For any perverse coherent system $(F, s)$ with $\mathcal{P}\text{Cok}(s) \in \text{Per}_{\leq 0}(S/T)$, the object $E = (\mathcal{O}_S \to F)$ admits a unique filtration

$$E_1 \subset E_2 \subset E, \ E_1 = (0 \to F'''), \ E_2/E_1 = (\mathcal{O}_S \to F'), \ E/E_2 = (0 \to F''')$$

where $F''$, $F'''$ are objects in $\text{Per}_{\leq 0}(S/T)$ and $(F', s')$ is a pure quotient. The objects $(0 \to F''')$, $(0 \to F'''')$ are semistable with respect to (7.16), $(\mathcal{O}_S \to F')$ is stable with respect to (7.16), and all of them satisfy $p_r(-) = 0$. Therefore any stable object with respect to $p_r$ is either a pure quotient or $F''''[-1]$ for a simple object in $\text{Per}_{\leq 0}(S/T)$. Then from the description of simple objects of $\text{Per}_{\leq 0}(S/T)$ in (7.11), we obtain the description of semistable objects (7.15). The formal neighborhood theorem holds by the argument of [Toda Lemma 7.4.3].

Remark 7.15. We need to take a little case in applying [AHLH Theorem 7.25], since the function (7.16) is not additive on $K(\text{Per}_{\leq 1}^+(S/T))$. However the argument in loc. cit. is applied as follows. For a map $f: \Theta \to \mathcal{M}_n^{\text{per}}(S, \beta)$ corresponding to a $\mathbb{Z}$-weighted filtration $I_\bullet = \cdots \to I_{w+1} \to I_w \to I_{w-1} \to \cdots$ in $\text{Per}_{\leq 1}^+(S/T)$, we set

$$l(f) := \sum_{w \in \mathbb{Z}} w \cdot p_r(I_w/I_{w+1}).$$

For $I = (\mathcal{O}_S^\text{pr} \to F) \in \text{Per}_{\leq 1}^+(S/T)$, we set $\text{rk}(I) := r$. Since there is no scaling automorphism if $r > 0$, we have $\text{rk}(I_0/I_1) = 1$ and $\text{rk}(I_w/I_{w+1}) = 0$ if $w \neq 0$. Then we immediately see that $I = (\mathcal{O}_S \to F)$ in $\mathcal{M}_n^{\text{per}}(S, \beta)$ satisfies $\mathcal{P}\text{Cok}(s) \in \text{Per}_{\leq 0}(S/T)$ if and only if for any map $f: \Theta \to \mathcal{M}_n^{\text{per}}(S, \beta)$ with $f(1) \sim I$, we have $l(f) \leq 0$. The function $l$ obviously satisfies the condition in [AHLH Remark 6.16], so we can apply [AHLH Theorem 7.25] as mentioned in loc. cit.. A similar argument also applies to the construction of the good moduli space (7.16).

The notion of perverse coherent systems and pure quotients are similarly defined for $\text{Per}(X/Y)$. We denote by $\mathcal{P}_n^{\text{per}}(X, \beta)$ the classical moduli stack of perverse coherent systems $(F, s)$ for $F \in \text{Per}(X/Y)$ and $s: \mathcal{O}_X \to F$ satisfies $\mathcal{P}\text{Cok}(s) \in \text{Per}_{\leq 0}(X/Y)$, where $\mathcal{P}\text{Cok}(-)$ is the cokernel in $\text{Per}(X/Y)$. Similarly to Lemma 7.14 we have the following:

Lemma 7.16. The moduli stack $\mathcal{P}_n^{\text{per}}(X, \beta)$ admits a good moduli space

$$\mathcal{P}_n^{\text{per}}(X, \beta) \to \mathcal{P}_n^{\text{per}}(X, \beta)$$

(7.17)

whose closed points correspond to direct sums of the form

$$I = I_0 \oplus (V_0 \otimes S_0[-1]) \oplus (V_1 \otimes S_1[-1]) \oplus \bigoplus_{i=1}^k (W_i \otimes \mathcal{O}_{x_i}[-1])$$

(7.18)

where $I_0 = (\mathcal{O}_X \to F_0)$ for a pure quotient $(F_0, s_0)$, $V_0, V_1, W_i$ are finite dimensional vector spaces and $x_1, \ldots, x_k \in X \setminus C$ are distinct points.
Lemma 7.17. The moduli stack \( \mathcal{P}^\text{per}_n(X, \beta) \) is isomorphic to the moduli stack of objects in the extension closure

\[(7.20) \quad I \in (\mathcal{O}_X, \text{Per}_{\leq 1}(X/Y)[-1])_{\text{ex}}\]

satisfying \( p\mathcal{H}^1(I) \in \text{Per}_{\leq 0}(X/Y) \) and \( \text{cl}(I) = (1, \beta, n) \).

Proof. From the identification of \( \mathcal{P}^\text{per}_n(X, \beta) \) with the moduli stack of pairs \((7.19)\), the lemma follows from the same arguments of [Odel3] Lemma 4.2.2 applied for the non-commutative scheme \((Y, A_X)\).

Lemma 7.18. For \( t \geq 1/2 \), we have the open immersion

\[(7.21) \quad \mathcal{P}^t_n(X, \beta) \subset \mathcal{P}^\text{per}_n(X, \beta)\]

Proof. Let \( I \in A_X \) be an object corresponding to a point in \( \mathcal{P}^t_n(X, \beta) \). We have the exact sequence

\[0 \to \mathcal{H}^0(I) \to I \to \mathcal{H}^1(I)[-1] \to 0\]

in \( A_X \) such that any Harder-Narasimhan factor \( F \) of \( \mathcal{H}^1(I) \) satisfies \( \chi(F)/H \cdot [F] \geq t \). Then as in the proof of Lemma 7.7, a choice \( a \gg 0 \) implies that \( F \) is supported on \( C \), hence \( F \) is a direct sum of \( \mathcal{O}_C(k-1) \) for \( k \geq 1 \). In particular we have \( \mathcal{H}^1(I) \in \text{Per}_{\leq 0}(X/Y) \).

We also have \( \mathcal{H}^0(I) \cong I_Z \) for a compactly supported closed subscheme \( Z \subset X \) with \( \dim Z \leq 1 \). Since \( R^1g_*\mathcal{O}_X = 0 \), we have \( R^1g_*\mathcal{O}_Z = 0 \). Moreover \( \text{Hom}(\mathcal{O}_Z, \mathcal{O}_C(-1)) = 0 \) since \( \mathcal{O}_X \to \mathcal{O}_Z \) is a surjection of coherent sheaves and \( \text{Hom}(\mathcal{O}_X, \mathcal{O}_C(-1)) = 0 \). It follows that \( \mathcal{O}_Z \in \text{Per}_{\leq 1}(X/Y) \). Therefore the morphism \( \mathcal{O}_X \to \mathcal{O}_Z \) is a morphism in \( \text{Per}(X/Y) \) which is generically surjective outside \( C \), hence its cokernel in \( \text{Per}(X/Y) \) is an object in \( \text{Per}_{\leq 0}(X/Y) \). Therefore to show that \( I \) is an object in \( \mathcal{P}^t_n(X, \beta) \) satisfying \( p\mathcal{H}^1(I) \in \text{Per}_{\leq 0}(X/Y) \), so we obtain the the desired open immersion from Lemma 7.17.

Lemma 7.19. In the setting of Lemma 7.18, an object \( I = (\mathcal{O}_X \xrightarrow{\delta} F) \) in \( \mathcal{P}^\text{per}_n(X, \beta) \) is an object in \( \mathcal{P}^t_n(X, \beta) \) if and only if the following conditions are satisfied:

(i) \( F \) is a coherent sheaf,
(ii) \( \text{Hom}(\mathcal{O}_C(m), F) = 0 \) for \( m + 1 > t \),
(iii) \( \text{Hom}(I, \mathcal{O}_C(m)[-1]) = 0 \) for \( t > m + 1 \geq 1 \),
(iv) \( \text{Hom}(\mathcal{O}_x, F) = 0 \) for \( x \in X \setminus C \).

Proof. The only if direction is obvious, so we only prove the if direction. Suppose that \( I \) satisfies (i) to (iv). Note that the condition (ii) is equivalent to \( \text{Hom}(\mathcal{O}_C(m)[-1], I) = 0 \) for \( m + 1 > t \) by applying \( \text{Hom}(\mathcal{O}_C(m), -) \) to the exact sequence

\[0 \to F[-1] \to I \to \mathcal{O}_X \to 0\]

in \( A_X \). Similarly (iii) is equivalent to \( \text{Hom}(\mathcal{O}_x[-1], I) = 0 \) for \( x \in X \setminus C \). Therefore by Lemma 7.8, it is enough to show that \( g_*\mathcal{H}^1(I) \) is zero dimensional and \( \text{Hom}(I, \mathcal{O}_C(m)[-1]) = 0 \) for \( m \leq -1 \). The second condition is obvious since for \( m \leq -1 \) we have \( \mathcal{O}_C(m)[-1] \in \text{Per}_{\leq 0}(X/Y)[-2] \). As for the first condition, let us take the exact sequence \( 0 \to \text{Im}(s) \to F \to \text{Cok}(s) \to 0 \) in \( \text{Coh}(X) \). As \( F \) is a sheaf, we have \( \text{Cok}(s) \in \text{Coh}(X) \cap \text{Per}(X/Y) \). By the argument of Lemma 7.18 we also have \( \text{Im}(s) \in \text{Coh}(X) \cap \text{Per}(X/Y) \). It follows that we have the exact sequence of perverse coherent systems

\[0 \to (\mathcal{O}_X \to \text{Im}(s)) \to (\mathcal{O}_X \xrightarrow{\delta} F) \to (0 \to \text{Cok}(s)) \to 0.
\]
Therefore we obtain the surjection \( p^* \mathcal{H}^1(I) \to \text{Cok}(s) \) in \( \text{Per}(X/Y) \). As \( p^* \mathcal{H}^1(I) \in \text{Per}_{\leq 0}(X/Y) \), we have \( \text{Cok}(s) \in \text{Per}_{\leq 0}(X/Y) \), hence \( \mathcal{H}^1(I) = \text{Cok}(s) \) satisfies that \( g_* \mathcal{H}^1(I) \) is zero dimensional. □

We also define the open substack
\[
(7.22) \quad P^{nc}_n(X, \beta) \subset P^{per}_n(X, \beta)
\]
to be consisting of pairs \((F, s)\) in \( P^{per}_n(X, \beta) \) satisfying \( \text{Hom}(\text{Per}_{\leq 0}(X/Y), F) = 0 \). Then under the equivalence \((7.10)\), the above stack is identified with the moduli stack of pairs \((7.19)\) such that \( F \in \text{Coh}_{\leq 1}(A_X) \) is pure one dimensional and \( s \) is surjective in dimension one. Therefore \( P^{nc}_n(X, \beta) \) is regraded as an analogue of PT stable pair moduli space for the non-commutative scheme \((Y, A_X)\).

**Lemma 7.20.** For \( t = 1 - 0 \), we have the isomorphism
\[
P^{t=1-0}_n(X, \beta) \cong P^{nc}_n(X, \beta).
\]

**Proof.** For \( F \in \text{Per}_{\leq 1}(X/Y) \), by Remark 7.11 there is an exact sequence
\[
0 \to \mathcal{H}^{-1}(F)[1] \to F \to \mathcal{H}^0(F) \to 0
\]
in \( \text{Per}_{\leq 1}(X/Y) \) such that \( \mathcal{H}^{-1}(F)[1] \in \text{Per}_{\leq 0}(X/Y) \). Therefore the condition \( \text{Hom}(\text{Per}_{\leq 0}(X/Y), F) = 0 \) implies that \( F \) is a sheaf. The above condition also implies that
\[
\text{Hom}(\mathcal{O}_C(m), F) = 0 \quad (m \geq 0), \quad \text{Hom}(\mathcal{O}_x, F) = 0 \quad (x \in X \setminus C).
\]
It follows that a pair \((F, s)\) in \( P^{nc}_n(X, \beta) \) is a \( \mu_1 \)-semistable object in \( A_X \) by Lemma 7.19.

Conversely for a pair \((F, s)\) in \( P^{t=1-0}_n(X, \beta) \), we have \( \text{Hom}(\text{Per}_{\leq 0}(X/Y), F) = 0 \) by Lemma 7.19 and the description of generators of \( \text{Per}_{\leq 0}(X/Y) \) in \((7.11)\). Therefore it is a pair in \( P^{nc}_n(X, \beta) \). □

By the above lemma, the sequence of d-critical flips in \((7.4)\) is in this case a sequence
\[
P_n(X, \beta) = P^{(k)}_n(X, \beta) \rightarrow P^{(k-1)}_n(X, \beta) \rightarrow \cdots \rightarrow P^{(0)}_n(X, \beta) = P^{nc}_n(X, \beta)
\]
where \( k \gg 0 \), which connects commutative stable pair moduli space \( P_n(X, \beta) \) with non-commutative stable pair moduli space \( P^{nc}_n(X, \beta) \) by d-critical minimal model program.

### 7.6 The case of relative reduced curve classes.

**Definition 7.21.** A class \( \beta \in \text{NS}(S) \) is called \( f \)-reduced if \( f_* \beta \) is a reduced class, i.e. any effective divisor on \( T \) with class \( f_* \beta \in \text{NS}(T) \) is a reduced divisor.

If an effective divisor \( D \) on \( S \) is of class \( \beta \) which is \( f \)-reduced, then \( D \) is of the form \( k[C] + [C'] \) where \( k \in \mathbb{Z}_{\geq 0} \) and \( C' \) is an effective divisor which does not contain \( C \). We have the following lemma.

**Lemma 7.22.** Suppose that \( \beta \) is \( f \)-reduced. Then we have the isomorphism over \( P^{per}_n(S, \beta) \)
\[
(7.23) \quad P^{per}_n(X, \beta) \xrightarrow{\cong} t_0(\Omega^{per}_n(S, \beta)[-1]).
\]

**Proof.** Similarly to Lemma 7.17, the left hand side is identified with the moduli stack of pairs \((7.19)\) such that \( F \) has a reduced one dimensional support on \( Y \) and \( \text{Cok}(s) \in \text{Coh}_{\leq 0}(A_X) \). Then the lemma follows from the argument of [Toda] Lemma 5.5.4 for the non-commutative scheme \((Y, A_X)\). □

For \( t \geq 1/2 \), we define the conical closed substack
\[
Z^{per, t-us}_n \subset t_0(\Omega^{per}_n(S, \beta)[-1])
\]
to be the complement of the open embedding \((7.21)\) via the isomorphism in Lemma 7.22. We have the following alternative description of DT categories in this case:

**Lemma 7.23.** Suppose that \( \beta \) is \( f \)-reduced. Then for \( t \geq 1/2 \), there is an equivalence
\[
\mathcal{D}^C(T^C(P^{per}_n(X, \beta))) \xrightarrow{\cong} D^b_{\text{coh}}(P^{per}_n(S, \beta))/\mathcal{C}_{\text{Z^{per, t-us}}}.
\]
Proposition 7.24. The open substack (7.21) is the $\mathbb{P}_{\text{per}}(S, \beta)$.

Proof. Let $\mathfrak{M}_{\text{per}}^\text{et}(S, \beta) \subset \mathbb{P}_{\text{per}}^\text{et}(S, \beta)$ be the derived open substack of $(\mathcal{O}_S \to F)$ such that $F \in \text{Coh}(S)$. Then by Lemma [7.18] and Lemma [7.22] we can take $\mathfrak{M}_1^\text{et}(S, \beta)_{\text{qs}}$ to be $\mathbb{P}_{\text{per}}^\text{et}(S, \beta)$ in Definition [7.4]. Then the lemma follows from Lemma [3.6].

Below we describe the open substack (7.21) in terms of semistable points with respect to some $\mathbb{R}$-line bundle on $\mathbb{P}_{\text{per}}^\text{et}(X, \beta)$. We denote by

$$(\mathcal{O}_S \times \mathcal{P}) \to \mathcal{F}_\mathbb{P} = D^\text{b}(S \times \mathbb{P}_{\text{per}}^\text{et}(S, \beta))$$

the universal perverse coherent systems. For $t \geq 1/2$, we define the following $\mathbb{R}$-line bundle on $\mathbb{P}_{\text{per}}^\text{et}(S, \beta)$

$$\mathcal{L}_t := \text{det} \, R\mathcal{P}_t(\mathcal{F}_\mathbb{P})^{-1} \otimes \text{det} \, R\mathcal{P}_t(\mathcal{F}_\mathbb{P} \boxtimes \mathcal{O}_S(C))^{-t}.$$  

Here $p \circ q : S \times \mathbb{P}_{\text{per}}^\text{et}(S, \beta) \to \mathbb{P}_{\text{per}}^\text{et}(X, \beta)$ is the projection. We also denote its pull-back to the $(-1)$-shifted cotangent $t_0(\mathfrak{M}_{\text{per}}^\text{et}(S, \beta) \mathcal{H})$ by $\mathcal{L}_t$, and also regard it as an $\mathbb{R}$-line bundle on $\mathbb{P}_{\text{per}}^\text{et}(X, \beta)$ via the isomorphism (7.24). By setting $I_t := c_1(\mathcal{L}_t)$, we have the following proposition:

**Proposition 7.24.** The open substack (7.21) is the $I_t$-semistable locus, i.e.

\[ \mathbb{P}_t^\text{et}(X, \beta) = \mathbb{P}_{\text{per}}^\text{et}(X, \beta)^{(I_t)}\text{-ss}. \]

Proof. Let $I = (\mathcal{O}_X \to F)$ be an object corresponding to a point in $\mathbb{P}_{\text{per}}^\text{et}(X, \beta)$ which is $I_t$-semistable. Suppose that it is not an object in $\mathbb{P}_t^\text{et}(X, \beta)$. Then by Lemma [7.19] it violates one of the conditions (i) to (iv). Suppose that it violates (i), i.e. $F$ is not a sheaf. Then by Remark [7.11] there is an exact sequence

$$0 \to \mathcal{O}_C(1) \to F \to F' \to 0$$

in $\text{Per}_{\leq 1}(X/Y)$ for some $m \leq -1$. So there is a map $f : \Theta \to \mathbb{P}_{\text{per}}^\text{et}(X, \beta)$ such that $f(1) \sim I$ and

$$f(0) \sim (0 \to \mathcal{O}_C(m)[1]) \oplus (\mathcal{O}_X \to F'),$$

where $(0 \to \mathcal{O}_C(m)[1])$ has $\mathbb{C}^*$-weight 1 and $(\mathcal{O}_X \to F')$ has $\mathbb{C}^*$-weight zero. Then we have

$$q_1 f^* I_t = (t - 1) \chi(\mathcal{O}_C(m)[1]) - t \chi(\mathcal{O}_C(m - 1)[1])$$

$$= -t + m + 1 < 0$$

which violates the $I_t$-semistability. Similarly one can show that if $I$ violates one of other conditions in Lemma [7.19] then it violates the $I_t$-semistability. Therefore $I$ is an object in $\mathbb{P}_t^\text{et}(X, \beta)$.

Conversely suppose that $I = (\mathcal{O}_X \to F)$ is an object in $\mathbb{P}_t^\text{et}(X, \beta)$, and take a map $f : \Theta \to \mathbb{P}_{\text{per}}^\text{et}(X, \beta)$ with $f(1) \sim I$. The map $f$ corresponds to a $\mathbb{Z}$-weighted filtration of perverse coherent systems

$$I_* = (\cdots \to I_{w+1} \to I_w \to I_{w-1} \to \cdots)$$

such that $I_0/I_1$ is of the form $(\mathcal{O}_X \to F_0)$ and $I_w/I_{w+1}$ for $w \neq 0$ is of the form $(0 \to F_w)$ for $F_w \in \text{Per}_{\leq 0}(X/Y)$. We have

$$q_1 f^* I_t = \sum_{w \geq 0} w \cdot \{(t - 1) \chi(I_w/I_{w+1}) - t \chi(I_w/I_{w+1}(C))\}$$

$$= \sum_{w < 0} -(t - 1) \chi(I_w) + t \chi(I_w(C)) + \sum_{w > 0} (t - 1) \chi(I_w) - t \chi(I_w(C)).$$

For $w > 0$, the object $I_w$ is of the form $(0 \to F_w)$ for $F_w \in \text{Per}_{\leq 0}(X/Y) \cap \text{Coh}(X)$ satisfying $\text{Hom}(\mathcal{O}_C(m), F_w) = 0$ for $m + t > t$ and $\text{Hom}(\mathcal{O}_x, F_w) = 0$ for $x \not\in X \setminus C$ by Lemma [7.19]. Therefore $F_w$ lies in the extension closure of $\mathcal{O}_C(m)$ for $m + 1 \leq t$. Since we have

$$t - m - 1 \geq 0,$$

the second sum in (7.25) is non-negative. Similarly $I/I_w$ for $w < 0$ is of the form $(0 \to F'_w)$ where $F'_w$ lies in the extension closure of $\mathcal{O}_C(m)$ for $m + 1 \geq t$, $\mathcal{O}_x$ for $x \in X \setminus C$ and $\mathcal{O}_C(m)[1]$ for $m \leq -1$. Therefore the first sum in (7.25) is also non-negative, and we conclude that $I$ is $I_t$-semistable. □
We also set \( b \in H^4(P_{per}^n(S, \beta), \mathbb{Q}) \) by
\[
(7.26) \quad b := \text{ch}_2(Rp_{PP}(\mathcal{F}_P)) + \text{ch}_2(Rp_{PP}(\mathcal{F}_P \boxtimes \mathcal{O}_S(C))).
\]

**Lemma 7.25.** The element (7.26) is positive definite in the sense of Definition 4.7.

**Proof.** Let \( f: BC^* \to P^r_n(S, \beta) \) be a non-degenerate morphism. The morphism \( f \) corresponds to a perverse coherent system \( I \) with \( C^* \)-automorphisms. By taking the decomposition into \( C^* \)-weight part, it decomposes into
\[
I = (\mathcal{O}_S \to F_0) \oplus \bigoplus_{i=1}^k F_i[-1].
\]
Here \( (\mathcal{O}_S \to F_0) \) has weight 0 and \( F_i \in \text{Per}_{\leq 0}(S/T) \) has non-zero weight \( w_i \). It follows that
\[
(7.27) \quad q^{-2} f^* b = \sum_{i=1}^k w_i^2 \cdot \chi(F_i \otimes (\mathcal{O}_S \oplus \mathcal{O}_S(C))).
\]
As \( f \) is non-degenerate, we have \( k \geq 1 \). We also have
\[
\chi(F_i \otimes (\mathcal{O}_S \oplus \mathcal{O}_S(C))) = \chi(Rf_*R\text{Hom}(\mathcal{E}, F_i)) > 0
\]
since \( Rf_*R\text{Hom}(\mathcal{E}, F_i) \) is a zero dimensional sheaf, which is non-zero by the equivalence (7.9). Therefore (7.27) is positive, and the lemma holds. \( \square \)

**Remark 7.26.** We have the \( \Theta \)-stratifications of \( P^r_n(X, \beta) \) and \( P^r_n(S, \beta) \) with respect to \((l_t, b)\)
\[
P^r_n(X, \beta) = S^1_1 \sqcup \cdots \sqcup S^1_N \sqcup P^r_n(X, \beta),
\]
\[
P^r_n(S, \beta) = S_1 \sqcup \cdots \sqcup S_M \sqcup P^r_n(S, \beta).
\]
The second \( \Theta \)-stratification does not necessarily satisfy the weight condition in Remark 6.14 so that it is not pulled-back to the first \( \Theta \)-stratification. Indeed for \( t \gg 0 \), a PT stable pair on \( X \) does not push-forward to a PT stable pair on \( S \), since there exist stable pairs on \( X \) thickened into the fiber direction of \( X \to S \) along \( C \). So \( P^r_n(X, \beta) \) is strictly bigger than the pull-back of \( P^r_n(S, \beta) \), and \( D^C(X, \beta) \) is not necessary equivalent to \( D^C_S(P^r_n(S, \beta)) \).

**Lemma 7.27.** Assumption 6.17 is satisfied for \( N = P^r_n(X, \beta) \), \( l = l_t \) for \( t = k \in \mathbb{Z}_{\geq 1} \) and \( l_\pm = l_{t+\pm} \) for \( 0 < \varepsilon \ll 1 \).

**Proof.** Let us take a closed point \( y \in P^r_n(S, \beta) \) corresponding to a polystable object \((7.15)\), and we use the same symbol \( y \in P^r_n(S, \beta) \) for the corresponding closed point. Then as in [Iod], Remark 5.4.2, the \( G_y \)-representation
\[
\mathcal{H}^0(T_{P^r_n(S, \beta)}|_y) \oplus \mathcal{H}^1(T_{P^r_n(S, \beta)}|_y)\nu
\]
is the space of representations of the Ext-quiver \( Q_y \) associated with the collection in \( D^b_{\text{coh}}(X) \)
\[
\{l^*_0 = (\mathcal{O}_X \to i_*F_0), S_0[-1], S_1[-1], i_*O_{Z_1}[-1], \ldots, i_*O_{Z_k}[-1]\}
\]
where \( i: S \hookrightarrow X \) is the zero section, with dimension vector \( \nu \) given by
\[
\nu = (1, \dim V_0, \dim V_1, \dim W_1, \ldots, \dim W_k).
\]
For \( t = k \in \mathbb{Z}_{\geq 1} \), let us take a closed point \( x \in P^r_n(X, \beta)^{t-ss} \) which maps to \( y \) by the composition
\[
P^r_n(X, \beta) \xrightarrow{\xi} P^r_n(S, \beta) \xrightarrow{\xi_y} P^r_n(S, \beta).
\]
By Lemma 7.27 and Proposition 7.24, the point \( x \) is represented by a \( \mu^+_t \)-polystable perverse coherent system of the form (7.17). The corresponding closed point
\[
x \in (\mathcal{H}^0(T_{P^r_n(S, \beta)}|_y) \oplus \mathcal{H}^1(T_{P^r_n(S, \beta)}|_y)\nu)^{t-ss}
\]
Conjecture 7.9 is true if Theorem 7.28.

character (7.28), it is easy to see that (6.53) holds (see [Todb, Lemma 5.1.12]). □

Theorem 6.18. Singular support of coherent sheaves and the geometric Langlands conjecture

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Kavli Institute for the Physics and Mathematics of the Universe (WPI), University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, 277-8583, Japan.
E-mail address: yukinobu.toda@ipmu.jp