Probabilistic methods for physics

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Abstract. We present an asymptotic method giving a probability of presence of the iterated spots of $\mathbb{R}^d$ by a polynomial function $f$. We use the well-known Perron Frobenius operator (PF) that lets certain sets and measure invariant by $f$. Probabilistic solutions can exist for the deterministic iteration. If the theoretical result is already known, here we quantify these probabilities. This approach seems interesting to use for computing situations when the deterministic methods don’t run. Among the examined applications, are asymptotic solutions of Lorenz, Navier-Stokes or Hamilton’s equations. In this approach, linearity induces many difficult problems, all of whom we have not yet resolved.

1. Presentation of the problem

The Perron-Frobenius’s equation (PF) is a very difficult functional equation. Ulam and Von Neumann gave the first and only known asymptotic solution for the logistic $f(a) = 4(a - a^2)$ in $\mathbb{R}$.

Here, we study an iteration $f$, which is a polynomial function of $\mathbb{R}^d$ in $\mathbb{R}^d$. The main new result for physics is the random behaviour of solutions of special differential equations. It is impossible to expose all our results and demonstrations in this 4-pages paper. The detailed references and proofs can be found in the preprint hal-00691097, v3[1] after some little corrections.

Here, we insist only on the leading points to understand the process we used to obtain these results.

1.1 The main hypothesis H0

The polynomial iteration $f$ applies a compact set $C$ of $\mathbb{R}^d$ in $C$. $f(0) = 0$. Let $\lambda$ be the real eigenvalues of the linear part of $f$ at $0$. There is no resonance if we have $1 \neq \lambda^n$ for all $n \in \mathbb{N}^d$.

We iterate indefinitely this application $f^{(k)} = f \circ f^{(k-1)}$. Then, we obtain infinity of spots in $C$. When $k \to \infty$, these spots can converge toward some points:

• Fixed points: $f(\alpha) = \alpha$;
• Cycles such as $f^{(p)}(\alpha_p) = \alpha_p$ but $f \circ f^{(p)}(\alpha_p) \neq \alpha_p$;
• Sets $B$ where exists an invariant probability measure $P$ called Perron Frobenius (PF) (see[3]):

$$P(B) = P \circ f^{-1}(B)$$
In this paper, we want to find the measure $P$ solution of this functional equation on $C$. We note the generic collection of $f^{(n)}$ as $f \in C(f)$; $\text{Fix}(f)$ is the set of the fixed points of $f$.

2 - The resolving equation $L$ and the resolving deviation $e^n(y)$

First, we transform all measure $P$ by Stieltjes-Laplace's $L$ in order to capture the fixed point 0:

$$\phi(y) = L(P) = \lim_{\epsilon \to 0^+} \epsilon^x dP(x)$$

As in [3], for all $f$ and $P$, we have: $P_f = P \circ f^{-1}$ and

$$\phi_f(y) = \int e^{xy} dP_f(x) = \int e^{yx} dP(x).$$

Second, we compute $\phi_f(y)$ in 2 steps:

- As $C$ is compact, we can expand in convergent series: $\phi(y) = \sum_n b_n y^n$.
- We translate the origin with $a$: $x = a + u$; then, for all measure $P$ on $C$, we have:

$$\phi(y) = E(e^{y}) \text{ becomes } \phi_a(y) = \phi(y)e^{ya} = \sum_n b_n \partial^n(e^{ya}) / \partial a^n,$$

$$\phi_f(y) = E(e^{yf}) \text{ becomes } \phi_f(y,a) = \sum_n b_n \partial^n(e^{yk}) / \partial a^n.$$

Thence, if $P$ is the PF measure for $f$, then $\theta_f(\phi_a) = \phi_f(y,a) - \phi(y,a) = 0$ is the resolving equation $L$. If $a = 0$, we have $\theta_f(\phi) = \sum_n b_n (y^n - H_n(y)) = 0$ with $H_n(y) = \partial^n e^{yk} / \partial a^n|_{a=0}$.

2.1. The resolving deviation

We call resolving deviation $e^n(y)$, the polynomial defined by:

$$e^n(y) = y^n - H_n(y)$$

Then: $\theta_f(\phi) = \sum_n b_n e^n(y)$

2.2. The ideal $E_n$

If $Z(e^n)$ is the set of zeros of $e^n(y)$, let $Z(E_n) = Z(e^n) \cap \bigcap_{\ell=1}^{d+1} Z(e^{n+1})$ \ell = 1,...d be the set of these common zeros. $E_n$ is the ideal generated by these $d + 1$ polynomials $e^n$, $e^{n+1}$, $\ell = 1,2,...,d$.

If $\varphi(y)$ is a solution of $L$ and $\phi(y)$ the solution of (PF), then, the relation between $\varphi(y)$ and $\phi(y)$ is: $\phi(y,a) = \partial^2 \varphi(y,a) / \partial y \partial a = \partial(y\varphi(y,a))/ \partial a$ at $a = 0$. ($\theta(\phi) = 0$ is an identity for all $y$ and $a$).

2.3. Theorem 1

When $n \to \infty$ with $\not= \lambda^n$ (no resonance):

- 1. There exists an integer $m$ such as $(\varphi_n(y) - 1)^m \in E_n$. With a probability 1, the distribution defined by $\varphi_n(y)$ is constant on the algebraic affine real manifolds defined by $E_n$.
- 2. These results are valid for all $f, \in C(f)$.

The demonstration needs the following steps: First, in order to vanish $\theta_f(\varphi_n)$ for a solution $\varphi_n(y)$, $e^n(y)$ must be null when $n \to \infty$. Second, if we have $e^n(y) = 0$, by translation of the origin, we must have $e^{n+1}(y) = 0 \ell = 1,...d$; then, immersing the problem in the
algebraically closed field of the complex numbers, the Hilbert’s theorem of zeros (nullstellensatz) says \((q_n - 1)^m \in \mathbb{E}_n\).

3. Approximation of the zeros by the Riemann-Debbie’s steepest descent method
The main idea is the following:

- The leading term of \(e^n(y)\) is \(y^n(1 - \lambda^n)\). If \(|\lambda^n| > 1\), \(y^n\) is negligible with respect to \(H_n(y)\). So, the asymptotic distribution of the real zeros of \(H_n(y)\) gives us the solution when \(y \to \infty\) with \(n\) and \(y = n_j s_j\). But we can search this distribution for all \(\lambda^n\). As Plancherel and Rotach\([4]\) did for the Hermitian polynomials, we represent \(H_n(y)\) in the field of the complexes with the Cauchy’s integral:

\[
H_{n,1}(y) = \frac{\partial^{n-1} e^{y f(a)}/\partial a^{n-1}}{y^{n-1}} \bigg|_{a=0} = c \int_{\Gamma} \frac{e^{yf(a)} da}{a^{n-1}}
\]

We call the Plancherel-Rotach’s function PRF: \(\gamma(a) = yf(a) - n \log a\). \(\Gamma\) is a closed polydisk around the fixed point 0 of \(f\), \(a \in \mathbb{C}^d\). \(n-1 = (n_1-1, \ldots, n_d-1)\). \(c\) is a finite non-null constant.

- The steepest descent’s method says: if \(a\) is critical point of \(\gamma(a)\), \(ce^{\gamma(a)}\) gives an approximation of \(H_{n,1}(y)\) when \(y \to \infty\). As \(yf(a)\) is polynomial in \(a\), we can refer to Delabaere \([2]\) for a formulation of the conditions (general position). The definite negative Hessian of \(\gamma(a)\) at \(a\) is sufficient condition.

- As only the real zeros of \(H_{n,1}(y)\) interest us, we can neglect all factors \(\neq 0\) in the representation of \(H_{n,1}(y)\). Then, we can write this equivalence \(H_{n,1}(y) = e^{yf(a)}\) at \(a\). But, \(H_{n,1}(y) = 0\) iff the critical point \(a = a(y)\) has complex coordinates. Then, \(H_{n,1}(y)^{-1} e^{\Re(\gamma(a)) - \int_0 \sin(\Im(\gamma(a))) - \sin(\Im(\gamma(a)))} \) and \(\Re(\gamma(a))\) must be maximum.

3.3. Theorem 2
If the critical point \(a \in Z(J_y)\) is in general position, then, when \(n \to \infty\):

- 1. \(\gamma(a)\) have one common index of derivation \(n\) for all coordinates of \(y\) and \(y = ns\).

- 2. If \(a\) has \(p\) complex coordinates, the \(d-p\) dimensional manifolds solutions are indexed by a code \(k \in [1, n-1]^p\) and verify \(s_j \Im f_i(a) - \theta_i = k_i \pi / n\), with the code \(k_i \in (1, \ldots, n-1)\). Between two zeros at time \(n-1\) there is one zero at time \(n\): then we have a Rolle’s foliation.

- 3. If \(k_\ell = \lim k_\ell / n\) for \(\ell = 1, \ldots, p\), then \(k\) is asymptotically a random uniform vector on the unit cube \([0,1]^p\). Then, the \(d-p\) dimensional manifolds are indexed by the random vector \(k \in [0,1]^p\) and verify the random equation \(s_j \Im f_i(a) - \theta_i = k_\ell \pi / n\), \(\ell = 1, 2, \ldots, p\).

Demonstrations use the facts that \(a\) is critical, and, not only \(H_{n,1}(y) = 0\), but also \(H_{n,1+i}(y) = 0\), \(i = 1, 2, \ldots, d\). Unfortunately, in many problems, the Hessian is not definite (as in the Henon’s case\([1]\)).
4. Applications to ODE and PDE

Many physical problems can be written with a differential equation as $\partial a / \partial t = F(a)$, where $F(a)$ is an application of $\mathbb{R}^d$ in $\mathbb{R}^d$. In this equation, the unknown variables are a vector $a$ of $\mathbb{R}^d$ and the known variables are a vector $t$ of $\mathbb{R}^k$. Here, the number of equations equals the number of unknown variables $a$. We write the differential equation in terms of iteration:

4.1. A differential iteration is the application $f(a,\delta)$ of $\mathbb{R}^d$ in $\mathbb{R}^d$, defined by $f(a,\delta) = a + \delta F(a)$ for all fixed path $\delta_0 > \delta > 0$ of $\mathbb{R}^k$.

In order to use the previous results, we assume the hypothesis H:

The polynomial application $F$ iterates a compact set $C \subset \mathbb{R}^d$ in $C$ and we can apply the steepest descent’s method to $f(a,\delta)$. But here, the steepest descent’s method will be managed, especially if the Hessian is degenerate.

4.2. The iteration $f(a,\delta)$ is said partly linear in $a = (a,b)$ if $F$ linear in $b$.

We denote: $F(a) = (A(a)b + B(a),C(a)b + D(a))$ where $a = (a,b)$ with $a \in \mathbb{R}^p$ and $b \in \mathbb{R}^{dp}$.

For all each zero $\alpha = (\alpha,\beta)$ of $F$ we note $a = u + \alpha$ and $u = (u,v)$ in $\mathbb{R}^p$ and $y = (x,y) \in \mathbb{R}^d$. For every zero $\alpha$ of $F$, we define a partly linear iteration: $G_\alpha(u) = (u + \tau A(u + \alpha)v,v)$ at $\alpha$.

4.3. Theorem 3

If the differential iteration $f(a,\delta) = a + \delta F(a)$, applying $C \subset \mathbb{R}^d$ in $C$ for all $\delta_0 > \delta > 0$ with $\delta = t_0 / n \rightarrow 0 \in \mathbb{R}^k$, verify the hypotheses H of (4.1) for every fixed point $\alpha \in C$:

* 1. The invariant distribution dominates the fixed point according to $\lambda \tau$ will be positive or negative. In the EDO case, the condition $\Sigma \lambda < 0$ implies generally the convergence toward the fixed point. We have resonance with $\Sigma \lambda = 0$. For the PDE, the planes $\lambda \tau = 0$ get big gaps.

* 2. If the Hessian of $yF(a)$ is definite, there exists, besides the fixed points of $F$, only cyclical orbits with uniform density and period $T$ such as $\int_0^T F(a(t + t_0))dt = 0$ for all $t_0$.

* 3. If $F$ is partly linear, then asymptotically:

  the critical point $a = (a,b)$ de $\gamma_\alpha(a)$ can tends to the critical point of either $\gamma_\alpha(a)$ at $\theta$ or $\gamma_\alpha(a)$ at $\alpha$. Thence, we find the asymptotic invariant solution among the solutions of each iteration $G_\alpha(a)$. As $G_\alpha$ is partly linear, we have families of random manifolds by iteration of the critical point.

* 4. If we have two fixed points $\theta$ or $\alpha$, we shall take the distribution at $\theta$ or at $\alpha$ according to $\text{Re}(x\tau A(a)\beta)$ is positive or negative at the critical point. The dominating distribution must get the greatest contribution of the steepest descent’s method. The critical points realising $\text{Re}(x\tau A(a)\beta) = 0$ induce a route of communication between the two sets.

But, owing to the degeneracy of the Hessian, it is clear that the present description doesn’t give the complete solution for the points * 3 and * 4, as for the Lorenz or Navier Stokes attractor where the main solutions are random ovals[1]. We don’t have studied cycles and their zones of domination. We observe that many terms of $F$ vanish asymptotically.

5. Conclusion
Some asymptotic solutions of a differential equation can be randomized. The probabilistic theory brings information whereas the deterministic approach cannot explain everything.

- Many ideas on chaos must be corrected: the definition of chaos, as a situation between determinism and probability, seems devoid of interest. The multiple errors of approximation making unpredictable the behaviour does not give a good mathematical argument. Notions of sensibility to the initial conditions have to be seen differently. We have only zones of domination or attraction.

- On the other hand, the degeneracy of the Hessian remains a difficult problem.

References

[1] G. Cirier 2012 Probabilistic and asymptotic methods with the Perron Frobenius’s operator.

Preprint, hal-00691097, v. 2.

[2] E. Delabaere and C. J. Howls 2002 Global asymptotic for multiple integrals with boundaries,

[3] A. Lasota, M.C. Mackey1991 Chaos, Fractals and Noise, Sec. Ed., Berlin, Springer.

[4] M. Plancherel et W. Rotach 1929 Sur les valeurs asymptotiques des polynômes d’Hermite

\[ H_n(x) = (-1)^n e^{x^2/2} d^n e^{-x^2/2} / dx^n \], Com. Math. Helvetici 1, EMS, p 227 - 254.