Equivalence of ensembles in Curie–Weiss models using coupling techniques

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Joel L. Lebowitz has been one of the driving forces and main supporters of mathematical statistical physics for over half a century. It is a particular honour and a pleasure to dedicate this, in relation humble, update on foundations of statistical mechanics to him.

Abstract

We consider equivalence of ensembles for two mean field models: the discrete, standard Curie–Weiss model and its continuum version, also called the mean-field spherical model. These systems have two thermodynamically relevant quantities and we consider the three associated standard probability measures: the microcanonical, canonical, and grand canonical ensembles. We prove that there are ranges of parameters where at least two of the ensembles are equivalent. The equivalence is not restricted to proving that the ensembles have the same thermodynamic limit of the specific free energy but we also give classes of observables whose ensemble averages agree in the limit. Moreover, we obtain explicit error estimates for the difference in the ensemble averages. The proof is based on a construction of suitable couplings between the relevant ensemble measures, proving that their Wasserstein fluctuation distance is small enough for the error in the ensemble averages to vanish in the thermodynamic limit. A crucial property for these estimates is permutation invariance of the ensemble measures.

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1 Introduction

In this paper, we consider the equivalence of ensembles for two mean field models related to the Curie–Weiss model. Our aim is to cover not only equivalence of specific free energies in the thermodynamic limit but also to obtain explicit error estimates for a class of observables, such as local moments, between the expectations in the relevant ensembles.

Both of the models have two thermodynamically relevant quantities. We consider the related standard ensembles, i.e., microcanonical, canonical, and grand canonical ensembles, each with two parameters associated with the two thermodynamically relevant quantities. For the sake of completeness, we will give an overview of the standard ensemble theory in Sec. 1.1. There we also introduce notations and terminology which will be used later for defining the ensemble measures of the two Curie–Weiss models.

The first of the models is the regular Curie–Weiss model. For a recent overview of its main properties and motivation, we refer to [6]. We will define, in detail, the related microcanonical and canonical ensembles in Sec. 4. One can find many relevant details and tricks associated with the computation of the canonical partition function in [6]. For the purposes of this paper, the microcanonical ensemble and associated calculations are self-contained and can already be followed with the tools provided here.

The second model is a continuum modification of the Curie–Weiss model called the mean-field spherical model. The model is studied in [5] and it is a simplification of the Berlin-Kac model introduced in [1]. In [5], the authors consider the thermodynamic properties of the microcanonical and canonical ensembles. In Sec. 5, we explore the mean-field spherical model in a slightly generalized set-up, namely, by also considering the density of the system to be a free parameter. This allows to study also the properties of the grand canonical ensemble which is not explored in [5].

For both of these models, we will give detailed proofs of explicit rates of convergence of finite marginal distributions and/or finite moments of all order between the ensembles of the models. The main result here is the development of novel methods which employ rigorous and well-understood analysis of the thermodynamic properties of the ensembles in order to prove a form of weak convergence of the probability measures corresponding to the different ensembles. The main mathematical tools for the rigorous control of expectations in the various ensemble measures are couplings of the ensemble measures and the related Wasserstein distance between them, with suitably chosen “cost functions.” We give a brief review of couplings and Wasserstein metric in Sec. 2.

A crucial property of the ensemble measures and of the couplings constructed here is their invariance under permutations of the particle labels. The permutation invariance improves the control of differences of expectations under the ensemble measures, allowing to bound the error by the above Wasserstein distance. The method is similar to how translation invariance has been used in [7] for the supercritical Berlin–Kac spherical model, and it is described in detail in Sec. 3.
Another tool for such an estimation is the Laplace method of asymptotic analysis for such integrals. The method and how it applies to the above error estimation is also discussed in Sec. 4.

We postpone more detailed discussion about further related previous works, and how the present estimates connect to these, at the end of Introduction, to Sec. 1.2.

1.1 Heuristics of two constraint equilibrium measures

To fix notations and terminology, we give here a brief and informal introduction to the basic ideas of equilibrium statistical mechanics of systems with two constraints. In the following, $\mathcal{S}$ is some arbitrary state space with some fixed positive reference measure $d\phi$. The two fixed quantities will be called the energy $H : \mathcal{S} \to \mathbb{R}$ and the particle number $N : \mathcal{S} \to \mathbb{R}$. We use $V > 0$ to represent the number of degrees of freedom of the system and we focus on the properties of the system for large $V$. It is typically related to the “volume” of the state space $\mathcal{S}$ in some way.

We represent the constraints using, at the moment somewhat formal, delta function notations; the rigorous meaning of the notations will be discussed later. The microcanonical ensemble with energy density $\varepsilon \in \mathbb{R}$ and particle density $\rho \in \mathbb{R}$ is then given by

$$
\mu^{\varepsilon,\rho;V}(d\phi) := \frac{1}{Z_{MC}(\varepsilon,\rho;V)} \delta(H[\phi] - \varepsilon V) \delta(N[\phi] - \rho V) \, d\phi.
$$

The canonical ensemble with inverse temperature $\beta \in \mathbb{R}$ and particle density $\rho \in \mathbb{R}$ is given by

$$
\mu^{\beta,\rho;V}(d\phi) := \frac{1}{Z_{C}(\beta,\rho;V)} e^{-\beta H[\phi]} \delta(N[\phi] - \rho V) \, d\phi.
$$

Finally, the analogously defined grand canonical ensemble with inverse temperature $\beta \in \mathbb{R}$ and chemical potential $\mu \in \mathbb{R}$ is

$$
\mu^{\beta,\mu;V}(d\phi) := \frac{1}{Z_{GC}(\beta,\mu;V)} e^{-\beta (H[\phi] - \mu N[\phi])} \, d\phi.
$$

Most often not all of the parameter values listed above can be allowed, for example, when the computation of the associated normalization constant $Z$ would yield zero or infinity. We will state the appropriate parameter ranges for each of the models and ensembles considered here later, as part of their definition.

These ensembles can be represented in an alternative way by considering them as mixtures of the more constrained ensembles. Indeed, we have

$$
\mu^{\beta,\rho;V}(d\phi) = \int d\varepsilon e^{-V \beta \varepsilon} Z_{MC}(\varepsilon,\rho;V) e^{-V \beta \varepsilon} \int d\varepsilon \mu^{\varepsilon,\rho;V}(d\phi) \tag{1.1}
$$

and

$$
\mu^{\beta,\mu;V}(d\phi) = \int d\varepsilon d\rho e^{-V(\beta \varepsilon - \beta \rho \mu)} Z_{MC}(\varepsilon,\rho;V) e^{-V(\beta \varepsilon - \beta \rho \mu)} \int d\varepsilon d\rho e^{-V(\beta \varepsilon - \beta \rho \mu)} Z_{MC}(\varepsilon,\rho;V) \mu^{\varepsilon,\rho;V}(d\phi).
$$

Next, we define the specific microcanonical entropy or microcanonical entropy per degrees of freedom by

$$
s(\varepsilon,\rho;V) := \frac{1}{V} \ln Z_{MC}(\varepsilon,\rho;V).
$$

We define the specific canonical free energy or canonical free energy per degrees of freedom by

$$
f_{C}(\beta,\rho;V) := -\frac{1}{V} \ln Z_{C}(\beta,\rho;V).
$$

Note that we do not divide here by $\beta$, as would be common for definition of a free energy: this would not be convenient for our models since also zero and negative values of $\beta$ may occur here.
Similarly, the specific grand canonical free energy or grand canonical free energy per degrees of freedom is defined here by

\[ f_{GC}(\beta, \mu; V) := -\frac{1}{V} \ln Z_{GC}(\beta, \mu; V). \]

Now, note that

\[ e^{-V\beta \varepsilon} Z_{MC}(\varepsilon, \rho; V) = e^{-V(\beta \varepsilon - s(\varepsilon, \rho; V))}, \]

and

\[ e^{-V(\beta \varepsilon - \beta \mu \rho)} Z_{MC}(\varepsilon, \rho; V) = e^{-V(\beta \varepsilon - \beta \mu \rho - s(\varepsilon, \rho; V))}. \]

Assuming that the limits exist, we define

\[ s(\varepsilon, \rho) := \lim_{V \to \infty} s(\varepsilon, \rho; V), \quad f_C(\beta, \rho) = \lim_{V \to \infty} f_C(\beta, \rho; V), \quad f_{GC}(\beta, \mu) := \lim_{V \to \infty} f_{GC}(\beta, \mu; V). \]

Heuristically applying the theory of Laplace-type integrals, we then should have

\[
\begin{align*}
 f_C(\beta, \rho) &= \lim_{V \to \infty} f_C(\beta, \rho; V) = -\lim_{V \to \infty}\frac{1}{V} \ln \int d\varepsilon e^{-V(\beta \varepsilon - s(\varepsilon, \rho; V))} \\
 &= \inf_{\varepsilon} \{ \beta \varepsilon - \lim_{V \to \infty} s(\varepsilon, \rho; V) \} \\
 &= \inf_{\varepsilon} \{ \beta \varepsilon - s(\varepsilon, \rho) \},
\end{align*}
\]

and, similarly, for the grand canonical ensemble, one finds that

\[
 f_{GC}(\beta, \mu) = \inf_{\varepsilon, \rho} \{ \beta \varepsilon - \beta \mu \rho - s(\varepsilon, \rho) \}.
\]

The above formal computation shows the basic connection between the specific microcanonical entropy and the specific free energies of the other ensembles. In particular, these objects are connected by the Legendre transform. Typically, this results in a one-to-one correspondence between the parameters \( \varepsilon \) and \( \rho \) in the microcanonical ensemble with the associated free parameters \( \beta \) and \( \mu \). Assuming that the above thermodynamic limits exist and agree with each other using this correspondence, we say that the ensembles are thermally equivalent.

This terminology is not completely standard but we wish to make a distinction to another notion of equivalence of ensembles occurring later, namely, equivalence of the random fields \( \phi \) generated by each of the three ensembles as probability measures. It is indeed often taken for granted that thermal equivalence implies some form of equivalence of the random fields. However, proving this is often not obvious and obtaining useful error estimates is even less so. One goal of the present contribution is to show by way of examples how coupling techniques may be employed to control such errors.

To be explicit, consider the relationship between the specific microcanonical entropy and specific canonical free energy. Fix a particle density \( \rho \) such that \( s(\varepsilon, \rho) \) is strictly convex in the energy density variable \( \varepsilon \). Using the convexity of \( s \), one can compute that the mapping \( \varepsilon \mapsto \beta \varepsilon - s(\varepsilon, \rho) \) attains its unique smallest value for \( \varepsilon^* \) which satisfies \( \beta = \partial_\varepsilon s(\varepsilon^*, \rho) \). In particular, the monotonicity of \( \partial_\varepsilon s(\varepsilon, \rho) \) in the variable \( \varepsilon \) ensures that there always exists a unique \( \varepsilon^* \) such that \( \beta = \partial_\varepsilon s(\varepsilon^*, \rho) \), and, conversely, if one is given an energy density \( \varepsilon^* \), then there will exist a \( \beta \) such that the mapping \( \varepsilon \mapsto \beta \varepsilon - s(\varepsilon, \rho) \) has a unique minimum at this specific \( \varepsilon^* \). This relationship, which can be read off of the given equations as the involutive nature of the Legendre transform for strictly convex functions, is the correspondence alluded to above. The analysis of the specific free energy for the grand canonical ensemble and the relationship between the specific microcanonical entropy follows the same pattern, but with additional detail due to having two parameters to minimize over.
The theory of Laplace-type integrals is well-developed and allows one to compute explicit asymptotics of such integrals. In particular, one is typically interested in second-order fluctuations. Indeed, from the specific free energies, we obtain

\[ \langle H \rangle_{\beta,\rho; V} = \partial_{\beta} f_C(\beta, \rho; V), \]
\[ \langle H^2 \rangle_{\beta,\rho; V} - \langle (H)_{\beta,\rho; V} \rangle^2 = -\partial_{\beta}^2 f_C(\beta, \rho; V). \]

Using the theory of Laplace-type integrals, we typically have

\[ \lim_{V \to \infty} \langle H \rangle_{\beta,\rho; V, MC} = \partial_{\beta} f_C(\beta, \rho), \]

and

\[ \lim_{V \to \infty} \frac{\langle H^2 \rangle_{\beta,\rho; V, MC} - \langle (H)_{\beta,\rho; V, MC} \rangle^2}{V} = -\partial_{\beta}^2 f_C(\beta, \rho). \]

The first limit implies that the energy density of the canonical system converges to a constant, which, in turn, implies that the energy density behaves like \( O(1) \) for large \( V \). The contents of the second limit imply that the standard deviation of the energy density of the canonical system behaves like \( O(V^{-\frac{1}{2}}) \). This behaviour is typical for systems with a convex specific microcanonical entropy.

However, in addition to analysing the thermodynamic properties of the system, the Laplace-type analysis offers us something more. Indeed, if we return to the alternative representation of the canonical ensemble and we denote the minimizing \( \varepsilon \) of \( f_C(\beta, \rho) \) by \( \varepsilon^* \), then, for some suitable class of observables \( g(\phi) \), one might expect that

\[ \lim_{V \to \infty} \left| \langle g \rangle_{\varepsilon^*,\rho; V, MC} - \langle g \rangle_{\beta,\rho; V, C} \right| = 0. \]

We then say that the two ensembles are equivalent in this observable class. For instance, if the above result would hold for every function \( g : S \to \mathbb{C} \) which is Lipschitz continuous, we could say that the microcanonical and canonical ensembles are Lipschitz observable equivalent. Analogously, if the result holds for all polynomials \( g \) of the field whose degree is not allowed to grow with \( V \), we say that the ensembles are equivalent in their local moments.

In this paper, we will introduce a class of such functions \( g \) for some statistical mechanical models which have precisely this property. This serves to motivate further study of such function classes and development of the methods used here.

### 1.2 Related works and further motivation

There has always been considerable interest in trying to classify the “correct” notions of convergence of the equilibrium ensembles. For a particularly illuminating and modern account on some of the various notions which have been considered, we refer to [9] and its references. Thermodynamic equivalence from the point of view of large deviations and convexity properties of entropy is considered in great generality in [9]. Here, we approach the problem more from the point of view of equivalence of generic local expectation values, and the additional facilitating ingredient is label permutation invariance of the studied equilibrium ensembles. For rigorous applications of the ensembles as approximations in non-equilibrium phenomena, such as for local thermal equilibrium, it would be important to be able to estimate the error in the approximation. This is the second goal for the example cases in the present contribution.

In fact, such rigorous proofs are already available in the literature, albeit for different systems from the ones studied here. A very detailed mathematical account of such a convergence has been given in [2] starting from uniform distributions on the intersection of a simplex and a sphere. By appropriately parametrizing the radius of the sphere, and considering the behaviour of finite
dimensional marginals and moments of this uniform distribution as the dimension of the space grows, the author was able to rigorously prove that a phase transition occurs for this specific system. In particular, the author was able to prove that in the high dimensional limit the finite marginal distributions of the given uniform distributions are tensor products of the measures \( \mu(dx) = dx \, Ae^{-rx^2-sx} \) where \( dx \) is the Lebesgue measure on \([0, \infty)\), \( r \geq 0, \ s > 0 \), and \( A \) is a normalizing constant. The author showed that for certain radii of the sphere, we would have \( r > 0 \), and, beyond a certain critical radius, we would have \( r = 0 \), which signals a phase transition.

Another work in this direction, which cites the previous article, is given in [4]. In this work the authors consider the convergence of the microcanonical and grand canonical measures related to the Bose–Hubbard model. The commonality between both [2] and [4] is that the models they are considering are defined on state spaces with strictly positive unbounded elements. Such a feature seems to be a key property of these models since both of these works observe a phase transition into a state which can be characterized as containing a condensate.

In fact, a fairly satisfying account of ensembles with unbounded strictly positive phase spaces has been given in [8]. In this work the author proves a form of the equivalence of ensembles for systems with multiple constraints satisfying certain conditions, and the results are quite general as to their applicability. However, the main theorems presented there hold for phase spaces which are defined on \([0, \infty)^N\) rather than \(\mathbb{R}^N\), and, furthermore, the assumptions of the main theorem do not hold for the ensembles we are considering here.

Finally, let us mention the origin of the continuum model we are considering. In the work [5], the authors consider a further simplification to the Berlin–Kac model introduced in [1]. In particular, the nearest neighbour Ising model is replaced by a mean-field Hamiltonian, and, as can be seen from the contents of the article, the thermodynamic properties of the microcanonical and canonical ensembles become exactly computable. However, the authors do not consider the properties of local observables in their analysis.

Our approach differs significantly from those of the above previous works and their associated models. In particular, we will employ various coupling methods to prove convergence of finite dimensional marginal distributions and finite moments of all orders. In addition, our arguments do not hinge on definitions of the microcanonical ensembles with thin-set approximations. Instead, we define the microcanonical ensembles directly as constrained measures and explore their properties via analytic rather than probabilistic methods. Undoubtedly, the proof of the local convergence of observables and moments of the Curie–Weiss models is physics folklore. In particular, we refer to [3] for the standard reference on models of this specific type. However, for the second model introduced in [5] there does not seem to be proofs pertaining to the convergence of finite dimensional marginals or finite moments. There is a considerable amount of fine structure which much be considered to give a full account of the equivalence of ensembles at this level.

Lastly, we will mention that the main purpose of this paper is to display the specific methods of coupling and their relationship with the local convergence properties of the equilibrium ensembles. The thermodynamic properties of these systems are already well-known and have been studied extensively, but we wish to give an alternate, simpler and more accurate, account of the two models present in this paper, with the hope that the ideas used here generalize to less well studied models.

2 Couplings and Wasserstein distances

In this section we collect some of the basic notions of couplings needed here. More thorough introduction is available for instance in [10].

2.1 Couplings and transport maps

In this paper, we will frequently make use of the notion of coupling between probability measures. Let \( X \) be a sample space and let \( \Sigma \) be a \( \sigma \)-algebra on \( X \). Let \( \mu_1 \) and \( \mu_2 \) be two probability measures on \( X \). Define the coordinate projections \( P_1 : X \times X \to X \) and \( P_2 : X \times X \to X \) by \( P_1(x, y) := x \)
and \( P_2(x, y) := y \). A probability measure \( \gamma \) on a sample space \( X \times X \) with a \( \sigma \)-algebra \( \Sigma \otimes \Sigma \) is called a coupling if \( \gamma \circ P_1^{-1} = \mu_1 \) and \( \gamma \circ P_2^{-1} = \mu_2 \). Here, and in the following, \( P^{-1} \) will be used not only to denote the inverse of a mapping \( P \), but also for the associated map which takes a set to its preimage under \( P \).

In this paper, we will often give the definitions of probability measures with the explicit assumption that they can be constructed by simply giving suitable values of the expectations of measurable functions. For example, if \( X \) is a locally compact Hausdorff space and we are able to construct a bounded positive linear functional \( L \) on \( C_c(X) \), the space of continuous functions with compact support equipped with the supremum norm, such that \( \|L\| = 1 \), then by the Riesz–Markov–Kakutani representation theorem, there exists a unique Radon probability measure \( \mu \) on \( X \) such that \( L(f) = \langle f \rangle_\mu \) for all \( f \in C_c(X) \).

For the contents of this paper, we will use the following equivalent notion of coupling. Let \( f : X \to \mathbb{R} \) be a measurable function. A probability measure \( \gamma \), as defined in the previous paragraph, is a coupling if

\[
\langle f \circ P_1 \rangle_\gamma = \langle f \rangle_{\mu_1}, \quad \langle f \circ P_2 \rangle_\gamma = \langle f \rangle_{\mu_2}
\]

holds for all such functions \( f \). One typically says that the marginal distributions of \( \gamma \) are given by \( \mu_1 \) and \( \mu_2 \).

In this paper, we will sometimes refer to specific types of couplings as transport maps. Let \( \mu_1 \) be a probability measure as before, and let \( T : X \to X \) be a measurable map. Define the probability measure \( \mu_2 \) by setting \( \mu_2(A) := \mu_1(T^{-1}(A)) \) for all \( A \in \Sigma \). Such a probability measure \( \mu_2 \) is called the pushforward measure of \( \mu_1 \) by the map \( T \). We then denote \( \mu_2 = T_\ast \mu_1 \). This notion is also sometimes called the abstract change of variables due to an equivalent definition of the pushforward measure. If \( f : X \to \mathbb{R}_+ \) is a characteristic function of a measurable set, we may set

\[
\langle f \rangle_{\mu_3} = \int_X \mu_1(dx) \ f(T(x)) ,
\]

(2.1)

and this defines a positive measure \( \mu_3 \) on \( \Sigma \). Then, it is straightforward to check that \( \mu_3 \) indeed is a probability measure for which (2.1) holds for every non-negative measurable function \( f \). In addition, \( \mu_3 = \mu_2 \), and thus (2.1) provides an alternative definition of \( T_\ast \mu_1 \).

When \( \mu_2 \) and \( \mu_1 \) are measures such that there is a measurable map \( T \) for which \( \mu_2 = T_\ast \mu_1 \), we call \( T \) a transport map from the measure \( \mu_1 \) to \( \mu_2 \). A transport map \( T \) can always be used to construct a coupling between \( \mu_1 \) and \( \mu_2 \) as follows: If \( g : X \times X \to \mathbb{R}_+ \) is a measurable function, we define a probability measure \( \gamma \) by setting

\[
\langle g \rangle_\gamma = \int_X \mu_1(dx) \ g(x, T(x)).
\]

One can go through analogous steps as above and show that \( \gamma \) is then indeed a coupling of \( \mu_1 \) and \( \mu_2 = T_\ast \mu_1 \).

### 2.2 Wasserstein distance and coupling optimisation

For the moment, we will specialize to considering probability measure on \( \mathbb{R}^n \). Let \( \mu_1 \) and \( \mu_2 \) be probability measures on \( \mathbb{R}^n \) and let \( f : \mathbb{R}^n \to \mathbb{R} \) be a bounded 1-Lipschitz function with respect to the \( \| \cdot \|_p \)-norm for some \( p \geq 1 \). Suppose there exists a coupling \( \gamma \) of \( \mu_1 \) and \( \mu_2 \). Using the properties of probability measures, we have

\[
\left| \langle f \rangle_{\mu_1} - \langle f \rangle_{\mu_2} \right| = \left| \langle f \circ P_1 - f \circ P_2 \rangle_\gamma \right| \leq \langle |f \circ P_1 - f \circ P_2| \rangle_\gamma \leq \langle \|x_1 - x_2\|_p \rangle_\gamma .
\]

(2.2)

On the last line, we have used the short hand notation \( x_i = P_i(x), \ i = 1, 2 \), for clarity. One should note that the coupling does not appear on the left hand side of this inequality, and, we are thus free to minimize this inequality with respect to all couplings \( \gamma \). Since there always exists at
least one coupling, given by the the product coupling \( \gamma = \mu_1 \otimes \mu_2 \), and since the functions \( f \) are bounded, then for any coupling the middle expression has a uniform upper-bound. Therefore,

\[
|\langle f \rangle_{\mu_1} - \langle f \rangle_{\mu_2}| \leq \inf_{\gamma} \langle \|x_1 - x_2\|_\gamma \rangle.
\]

Naturally, we can swap the norm \( \| \cdot \|_p \) for any cost function \( c(x,y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \) with enough regularity as long as we can relate the difference of the expectations somehow to the given cost function.

For \( p \geq 1 \), define \( \mathcal{P}_p(\mathbb{R}^n) \) to be the space of Radon probability measures with finite \( p \)th moments, i.e., assuming that \( \langle \|x\|_p^p \rangle < \infty \). Consider \( \mu_1, \mu_2 \in \mathcal{P}_p(\mathbb{R}^n) \). Given also some \( q \geq 1 \), we denote the \( p \)-Wasserstein distance between \( \mu_1 \) and \( \mu_2 \) with respect to the \( q \)-norm by \( W_{p,q}(\mu_1, \mu_2) \). Explicitly,

\[
W_{p,q}(\mu_1, \mu_2) = \left( \inf_{\gamma} \int_{\mathbb{R}^n \times \mathbb{R}^n} \gamma(dx,dy) \|x - y\|_q^p \right)^{1/p},
\]

and, since \( \|x\|_q \leq N^{1/q} \max_j |x_j| \), is straightforward to check that then \( W_{p,q}(\mu_1, \mu_2) < \infty \).

The \( p \)-Wasserstein distance has been studied comprehensively and applied in a great variety of circumstances. However, for the purposes of this paper, we will be more interested in slightly modified cost functions which are similar in nature to the \( p \)-Wasserstein distances. The main drawback of many of the methods and papers associated with the Wasserstein distances is that the focus has been on the case where the dimension of the space \( n \) is fixed. In the context of statistical mechanics, we are typically interested in asymptotic properties for arbitrarily large \( n \).

### 3 Two methods of coupling and main lemmas

For the purposes of this section and for the definition of the lattice model later, let us fix some shorthand notations first. Given \( N \in \mathbb{N} \), we denote the collection of first \( N \) integers as follows

\[
[N] := \{1, 2, \ldots, N\},
\]

and we denote the group of permutations of its elements by \( S_N \). Given a subset \( I \subset [N] \), of a length \( n := |I| \), there is a unique bijection \( \pi_I : I \to [n] \) which retains the order of the elements in the subsequence. We let \( \pi_I \in S_N \) denote the extension of \( \pi_I \) which is obtained by permuting the elements in \( [N] \setminus I \) in an order preserving manner into the set \( [N] \setminus [n] \). In addition, every bijection \( \pi_I \) as above defines a projection \( P_I : \mathbb{R}^N \to \mathbb{R}^n \) via the formula \( (P_I x)_j := x_{\pi_I^{-1}(j)}, \ j \in [n] \). Analogously, given a permutation \( \pi \in S_N \), the corresponding coordinate permutation will be denoted \( Q_\pi : \mathbb{R}^N \to \mathbb{R}^N \). Explicitly, we set \( (Q_\pi x)_j := x_{\pi^{-1}(j)}, \ j \in [N] \) (note that using the inverse permutation in the formula will result in a map which will send coordinate \( i \) into coordinate \( \pi(i) \)).

Given \( y \in \mathbb{R} \), there is a unique integer \( k \in \mathbb{Z} \) for which \( k \leq y < k + 1 \), and we denote this by using the “floor” notation, \( k := \lfloor y \rfloor \). In particular, given \( n, N \in \mathbb{N} \) such that \( n \leq N \) and setting \( k = \lfloor N/n \rfloor \) we have \( k \in \mathbb{N} \) and \( k \) satisfies \( kn \leq N < (k + 1)n \).

**Definition 3.1** (Permutation invariance of measures on \( \mathbb{R}^N \)). Given \( N \in \mathbb{N} \), a probability measure \( \mu \) on \( \mathbb{R}^N \), we say that \( \mu \) is permutation invariant, if for every integrable function \( f : \mathbb{R}^N \to \mathbb{R} \) and a permutation \( \pi \in S_N \), we have \( f \circ Q_\pi \in L^1(\mu) \) and

\[
(f \circ Q_\pi)_\mu = (f)_\mu.
\]

Finally, instead of using a standard \( p \)-norm to measure distances in \( \mathbb{R}^N \), we scale it suitably with \( N \) so that the Wasserstein cost function becomes an average over particle labels. The benefits of this definition will become apparent in Sec. [3].
Definition 3.2 (Specific p-norm fluctuation distance). Suppose \( p \geq 1 \) and \( N \in \mathbb{N} \). Let \( \mu_1 \) and \( \mu_2 \) be two Radon probability measures on \( \mathbb{R}^N \) such that the \( p \)-th moments under both measures are finite. Their specific \( p \)-norm fluctuation distance \( w_p \) is then defined as

\[
   w_p(\mu_1, \mu_2; N) := \left( \inf_{\gamma} \int_{\mathbb{R}^N \times \mathbb{R}^N} \gamma(dx, dy) \left( \frac{1}{N} \sum_{i=1}^{N} |x_i - y_i|^p \right)^{\frac{1}{p}} \right)^{\frac{1}{p}},
\]

where the infimum is taken over all couplings of \( \mu_1 \) and \( \mu_2 \).

Clearly, this definition relates to the standard \( p \)-norm Wasserstein distance mentioned earlier via a scaling: \( w_p = N^{-\frac{1}{p}} W_{p,p} \).

3.1 The direct coupling method

To highlight the benefits of the above definitions, we offer the following fundamental Lemma which will be used to prove the main theorems of this paper. It should be stressed that the key assumption is to specialize to permutation invariant measures. We aim to consider local expectations, i.e., \( \langle \gamma \rangle \) for functions \( \gamma : \mathbb{R}^N \rightarrow \mathbb{R} \) which depend only on components \( x_i, i \in I \), where \( I \subset [N] \) can be otherwise arbitrary but it has a bounded size, i.e., \(|I|\) remains bounded when \( N \rightarrow \infty \). In particular, note that then there is some \( f : \mathbb{R}^{|I|} \rightarrow \mathbb{R} \) such that \( F = f \circ P_I \).

Lemma 3.3. Suppose \( p \geq 1 \) and \( N \in \mathbb{N} \). Let \( \mu_1 \) and \( \mu_2 \) be two permutation invariant Radon probability measures on \( \mathbb{R}^N \) such that the \( p \)-th moments under both measures are finite. Consider a subset \( I \subset [N] \). If \( f : \mathbb{R}^{|I|} \rightarrow \mathbb{R} \) is a bounded 1-Lipschitz function with respect to the \( \| \cdot \|_p \) norm, then we have

\[
   \left| \langle f \circ P_I \rangle_{\mu_1} - \langle f \circ P_I \rangle_{\mu_2} \right| \leq \left( \frac{1}{1 - \frac{1}{p}} \right)^{\frac{1}{p}} w_p(\mu_1, \mu_2; N). \]

Proof. For the proof, set \( n := |I| \) and \( k := \lfloor N/n \rfloor \) when \( k \in \mathbb{N} \) and \( k \) satisfies \( kn \leq N < (k + 1)n \). We define the sets \( I_i \subset [N] \), \( i \in [k] \), by setting \( I_i := I \) and, for \( i > 1 \), we proceed inductively by selecting \( |I| \) elements from the set \([N] \setminus \left( \bigcup_{j=1}^{i-1} I_j \right) \) to be the set \( I_i \). The collection of sets \( I_i \) are disjoint and \( \bigcup_{i=1}^{k} I_i \subset [N] \). For any \( i \), there is a permutation in \( S_N \) which is bijection between \( I_i \) and \( I \). Thus by the assumed permutation invariance of the measures, we have \( \langle f \circ P_{I_i} \rangle = \langle f \circ P_I \rangle \) for either measure and all \( i \). Therefore,

\[
   \langle f \circ P_I \rangle_{\mu_1} - \langle f \circ P_I \rangle_{\mu_2} = \frac{1}{k} \sum_{i=1}^{k} \left( \langle f \circ P_{I_i} \rangle_{\mu_1} - \langle f \circ P_{I_i} \rangle_{\mu_2} \right). 
\]

Suppose then that \( \gamma \) is a coupling between \( \mu_1 \) and \( \mu_2 \). Then \( \langle f \circ P_{I_i} \rangle_{\mu_j} = \langle f \circ P_I \circ P_{I_i} \rangle_{\gamma} \) for both \( j = 1, 2 \). Again resorting to the shorthand notations \( x_j := P_j x \), we can rewrite

\[
   \langle f \circ P_I \rangle_{\mu_1} - \langle f \circ P_I \rangle_{\mu_2} = \langle f(P_{I_1}x_1) - f(P_{I_2}x_2) \rangle_{\gamma}. 
\]

The absolute value of this expression can now be estimated using the assumed 1-Lipschitz property of \( f \). Combining the results and using the triangle inequality we thus obtain

\[
   \left| \langle f \circ P_I \rangle_{\mu_1} - \langle f \circ P_I \rangle_{\mu_2} \right| \leq \frac{1}{k} \sum_{i=1}^{k} \left( \|P_{I_1}x_1 - P_{I_2}x_2\|_{\gamma} \right) \leq \left( \frac{1}{k} \sum_{i=1}^{k} \left( \|P_{I_1}x_1 - P_{I_2}x_2\|_p \right)^{\frac{p}{p}} \right)^{\frac{1}{p}},
\]

where in the last step we have used Hölder’s inequality. Since the sets \( I_i \) are disjoint, here \( \sum_{i=1}^{k} \|P_{I_1}x_1 - P_{I_2}x_2\|_p \leq \sum_{i=1}^{N} \|x_i - (x_2)_i\|_p \). Therefore,

\[
   \left| \langle f \circ P_I \rangle_{\mu_1} - \langle f \circ P_I \rangle_{\mu_2} \right| \leq \left( \frac{1}{k} \sum_{i=1}^{k} \left( \|x_1 - x_2\|_p \right)^{\frac{p}{p}} \right)^{\frac{1}{p}}.
\]
Because the left hand side of the above estimate does not depend on the coupling $\gamma$, we can take the infimum over all possible couplings. Then using the relation between $k$ and $n$ stated in the beginning of the proof, we obtain

$$\left| \langle f \circ P_1 \rangle_{\mu_1} - \langle f \circ P_1 \rangle_{\mu_2} \right| \leq \left( \frac{n}{1 - \frac{p_0}{p}} \right)^{\frac{1}{p}} w_p(\mu_1, \mu_2; N),$$

as desired. \hfill \Box

The first theorem concerned bounds on local observables which were bounded 1-Lipschitz functions. This next variant of the previous lemma concerns estimation of arbitrary finite moments.

**Theorem 3.4.** Suppose $p > 1$ and $N \in \mathbb{N}$. Let $\mu_1$ and $\mu_2$ be two permutation invariant Radon probability measures on $\mathbb{R}^N$ such that the $p_0$:th moments of both measures are finite for some $p_0 \geq p$.

Let $J$ be a finite sequence of elements in $[N]$ where elements may be repeated. Let $n_J := |J|$ denote the length of the sequence and $I \subset [N]$ the collection of elements occurring in the sequence, i.e., set $I := \{ J|, \ell \in [n_J] \}$. For any $x \in \mathbb{R}^N$, we then let $x^J$ denote the power

$$x^J := \prod_{\ell=1}^{n_J} x_{J\ell}.$$

Assuming also $n_J \leq p_0 + 1 - \frac{p_0}{p}$, it follows that

$$\left| \langle x^J \rangle_{\mu_1} - \langle x^J \rangle_{\mu_2} \right| \leq n_J M(J, p)^{n_J - 1} \left( \frac{|I|}{1 - \frac{p_0}{p}} \right)^{\frac{1}{p}} w_p(\mu_1, \mu_2; N),$$

where $M(J, p) = 1$ if $n_J = 1$, and otherwise

$$M(J, p) := \max_{i \in I} \left( \left| \langle x_i | q(n_J - 1) \rangle_{\mu_1} \right|^{\frac{1}{n_J}}, \left| \langle x_i | q(n_J - 1) \rangle_{\mu_2} \right|^{\frac{1}{n_J}} \right).$$

with $q = \frac{p_0}{p - 1}$.

**Proof.** Generalized Hölder’s inequality implies that

$$\langle |x^J| \rangle \leq \prod_{\ell=1}^{n_J} \langle |x_{J\ell}|^{n_J} \rangle^{\frac{1}{n_J}}.$$

Since $1 \leq n_J \leq p_0$, the assumptions guarantee that $x^J$ is integrable with respect to both $\mu_1$ and $\mu_2$. On the other hand, $n_J \leq p_0 + 1 - \frac{p_0}{p}$ implies $q(n_J - 1) \leq p_0$, so that also $M(J, p) < \infty$.

First, note that for $x, y \in \mathbb{R}^N$, we have

$$x^J - y^J = \sum_{i=1}^{n_J} (x_{Ji} - y_{Ji}) \prod_{j < i} x_{Jj} \prod_{k > i} y_{Jk}.$$

There are $n_J$ factors in each of the products under the sum. Thus for any coupling $\gamma$ between $\mu_1$ and $\mu_2$ and, for simplicity, replacing $x_1, x_2$ by $x, y$, we find using the generalized Hölder’s inequality

$$\left| \langle x^J - y^J \rangle_{\gamma} \right| \leq \sum_{i=1}^{n_J} \left| (x_{Ji} - y_{Ji}) \prod_{j < i} |x_{Jj}| \prod_{k > i} |y_{Jk}| \right| \gamma

\leq \sum_{i=1}^{n_J} \langle |x_{Ji} - y_{Ji}|^{n_J} \rangle_{\gamma} \prod_{j < i} \left( \langle |x_{Jj}|^{n_J} \rangle_{\gamma} \right)^{\frac{1}{n_J}} \prod_{k > i} \left( \langle |y_{Jk}|^{n_J} \rangle_{\gamma} \right)^{\frac{1}{n_J}},$$

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where \( q' := q(n_J - 1) \), so that indeed \( \frac{1}{p} + (n_J - 1)\frac{1}{q'} = 1 \), as required by the Hölder’s inequality. Apart from the first term, the remaining \( n_J - 1 \) terms are all bounded by \( M(J, p) \). Therefore,

\[
\left| \langle x^J - y^J \rangle \right| \leq M(J, p)^{n_J - 1} \sum_{i=1}^{n_J} \langle |x_i - y_i|^p \rangle \leq M(J, p)^{n_J - 1} n_J \left( \frac{1}{n_J} \sum_{i=1}^{n_J} \langle |x_i - y_i|^p \rangle \right) ^\frac{1}{p},
\]

where Hölder’s inequality has been used in the second step. Here, even if there are repetitions in the sequence \( J \), we have \( \frac{1}{n_J} \sum_{i=1}^{n_J} |x_i - y_i|^p \leq \|x - y\|_{p, f}^p \). Therefore,

\[
\left| \langle x^J - y^J \rangle \right| \leq n_J M(J, p)^{n_J - 1} \left( \|x - y\|_{p, f}^p \right) ^\frac{1}{p}.
\]

To finish the proof, one should notice that the label subset \( I \) which appears in this theorem can be regarded in the same way as in the proof of Lemma 3.3.3 Using the assumed permutation invariance to clone the labels yields collections of subsequences \( J(\ell) \) and subsets \( I_k \) for \( \ell \in [k] \), where \( k := [N/n] \), \( n := |I| \). Since \( \langle x^J \rangle_{\mu_1} = \langle x^{J(\ell)} \rangle_{\mu_i} \), by construction, we find using permutation invariance that

\[
\left| \langle x^J \rangle_{\mu_1} - \langle x^J \rangle_{\mu_2} \right| \leq \frac{1}{k} \sum_{\ell=1}^{k} \left| \langle x^{J(\ell)} - y^{J(\ell)} \rangle \right| \leq n_J M(J, p)^{n_J - 1} \left( \frac{|I|}{n} \right) \left( \delta_{\mu_1, \mu_2} \right) ,
\]

as desired.

\[ \square \]

### 3.2 Free energy method

By applying the previous lemma, we are now able to produce two distinct types of coupling proofs which concern the ensembles discussed in the introduction.

**Theorem 3.5.** Let \( \mu_{MC}^{\varepsilon, \rho; N} \) be a permutation invariant probability measure corresponding to a microcanonical ensemble with energy density \( \varepsilon \) and particle density \( \rho \). In addition, assume that if we fix a possible energy density \( \varepsilon' \), then for any other possible energy density \( \varepsilon \) there exists a constant \( C(\varepsilon, \rho) > 0 \) independent of \( \varepsilon \) and \( N \), but possibly dependent on \( \varepsilon \) and \( \rho \), such that

\[
w_p(\mu_{MC}^{\varepsilon, \rho; N}, \mu_{MC}^{\varepsilon', \rho; N}; N) \leq C(\varepsilon, \rho) |\varepsilon - \varepsilon'|
\]

for some \( p \geq 1 \). Suppose also that the microcanonical and canonical measures, for some parameter \( \beta \), have finite \( p \)-th moments.

Fix \( n < \infty \) and consider any \( I \subset \mathbb{N} \) of length \( n \). Let \( f : \mathbb{R}^{|I|} \to \mathbb{R} \) be a bounded 1-Lipschitz function with respect to the \( \| \cdot \|_p \) norm. Then

\[
\left| \langle f \circ P_I \rangle_{MC}^{\varepsilon, \rho; N} - \langle f \circ P_I \rangle_{C}^{\beta, \rho; N} \right| \leq C(\varepsilon, \rho) \left( \frac{|I|}{1 - |I|/N} \right) ^\frac{1}{p} \left( \sigma_{\beta, \rho; N} \left( \frac{H}{N} \right) + |\varepsilon - \langle H \rangle_{C}^{\beta, \rho; N} \right) ,
\]

where the canonical standard deviation of energy density reads explicitly

\[
\sigma_{\beta, \rho; N} \left( \frac{H}{N} \right) = \sqrt{\frac{(H^2)_{\beta, \rho; N} - \langle H \rangle_{C}^{\beta, \rho; N}^2}{N^2}}.
\]

Using the notation of the specific free energies, the same result can be rewritten as

\[
\left| \langle f \circ P_I \rangle_{MC}^{\varepsilon, \rho; N} - \langle f \circ P_I \rangle_{C}^{\beta, \rho; N} \right| \leq C(\varepsilon, \rho) \left( \frac{|I|}{1 - |I|/N} \right) ^\frac{1}{p} \left( \frac{1}{\sqrt{N}} \sqrt{-\partial_{\beta} f_c(\beta, \rho; N)} + |\varepsilon - \partial_{\beta} f_c(\beta, \rho; N)| \right).
\]
Proof. By the relation (1.1), we have
\[
\langle f \circ P_l \rangle_{\text{MC}}^\varepsilon,\rho - \langle f \circ P_l \rangle_\rho^\varepsilon N
= \frac{1}{\int d\varepsilon' e^{-N\beta\varepsilon'} Z_{MC}(\varepsilon',\rho; N)} \int d\varepsilon' e^{-N\beta\varepsilon'} Z_{MC}(\varepsilon',\rho; N) \left( \langle f \circ P_l \rangle_{\text{MC}}^\varepsilon,\rho - \langle f \circ P_l \rangle_\rho^\varepsilon N \right).
\]
Applying Lemma 3.3 together with the assumptions of this theorem, we thus obtain
\[
\langle f \circ P_l \rangle_{\text{MC}}^\varepsilon,\rho - \langle f \circ P_l \rangle_\rho^\varepsilon N
\leq \frac{1}{\int d\varepsilon' e^{-N\beta\varepsilon'} Z_{MC}(\varepsilon',\rho; N)} \int d\varepsilon' e^{-N\beta\varepsilon'} Z_{MC}(\varepsilon',\rho; N) \left( \frac{|I|}{1 - \frac{p}{q}} \right)^{\frac{1}{2}} C(\varepsilon,\rho) |\varepsilon - \varepsilon'|.
\]
Since
\[
|\varepsilon - \varepsilon'| \leq \left| \varepsilon - \frac{\langle H \rangle_{\text{C}}^{\beta,\rho,N}}{N} \right| + \left| \frac{\langle H \rangle_{\text{C}}^{\beta,\rho,N}}{N} - \varepsilon' \right|,
\]
where the first term on the right hand side does not depend on \(\varepsilon'\), we obtain by Hölder’s inequality an estimate
\[
\langle f \circ P_l \rangle_{\text{MC}}^\varepsilon,\rho - \langle f \circ P_l \rangle_\rho^\varepsilon N
\leq C(\varepsilon,\rho) \left( \frac{|I|}{1 - \frac{p}{q}} \right)^{\frac{1}{2}} \left( \left| \varepsilon - \frac{\langle H \rangle_{\text{C}}^{\beta,\rho,N}}{N} \right| + \left| \frac{\langle H \rangle_{\text{C}}^{\beta,\rho,N}}{N} - \varepsilon' \right|^2 \right)^{\frac{1}{2}}
\]
\[
= C(\varepsilon,\rho) \left( \frac{|I|}{1 - \frac{p}{q}} \right)^{\frac{1}{2}} \left( \sigma_{\text{C}}^{\beta,\rho,N} \frac{H}{N} + \left| \varepsilon - \frac{\langle H \rangle_{\text{C}}^{\beta,\rho,N}}{N} \right| \right),
\]
as desired. Then we use the generic properties listed in Sec. 1.1 to express the result in terms of the canonical free energy.

Following the theme of the direct coupling method, the approach can also then be applied to the case of finite moments.

Theorem 3.6. Let \(\mu_{\text{MC}}^{\varepsilon,\rho,N}\) be a permutation invariant probability measure corresponding to a microcanonical ensemble with energy density \(\varepsilon\) and particle density \(\rho\). In addition, assume that if we fix a possible energy density \(\varepsilon'\) then for any other possible energy density \(\varepsilon\) there exists a constant \(C(\varepsilon,\rho) > 0\) independent of \(\varepsilon'\) and \(N\), but possibly dependent on \(\varepsilon\) and \(\rho\) such that
\[
w_p(\mu_{\text{MC}}^{\varepsilon,\rho,N}, \mu_{\text{MC}}^{\varepsilon',\rho,N}; N) \leq C(\varepsilon,\rho) |\varepsilon - \varepsilon'|^p
\]
for some \(p > 1\). Suppose also that the microcanonical and canonical measures, for some parameter \(\beta\), have finite \(p_0\)th moments for some \(p_0 \geq p\).

Let \(J\) be a finite sequence of elements in \([N]\) where elements may be repeated, let \(n_J := |J|\), and suppose that \(n_J \leq p_0 + 1 - \frac{p_0}{p}\). Collect into \(I \subset [N]\) the elements occurring in the sequence. It follows that
\[
\left| \langle \phi^J \rangle_{\text{MC}}^{\varepsilon,\rho,N} - \langle \phi^J \rangle_\rho^\varepsilon N \right| \leq C(\varepsilon,\rho)n_J M(J, p)^{n_J - 1} \left( \frac{|I|}{1 - \frac{p}{q}} \right)^{\frac{1}{2}} \left( \sigma_{\text{C}}^{\beta,\rho,N} \frac{H}{N} + \left| \varepsilon - \frac{\langle H \rangle_{\text{C}}^{\beta,\rho,N}}{N} \right| \right),
\]
where, using the dual exponent \(q = \frac{p}{p-1}\),
\[
M(J, p) := \max_{i \in I} \left( \left| x_i \right|^q (n_J - 1) / \mu_1, \left| x_i \right|^q (n_J - 1) / \mu_2 \right) < \infty.
\]
Proof. The proof is almost identical to the proof of the previous theorem. In order to isolate the moments of the canonical ensemble, one needs an additional application of Hölder’s inequality.

For suitable ensembles, these theorems together imply that with bounded moments, one can achieve an explicit rate of convergence of the finite dimensional moments and marginals of the ensembles.

3.3 Rigorous asymptotic analysis of Laplace-type integrals

First, we will fix some notation and definitions.

Definition 3.7 (Asymptotic equivalence). Let \( f, g : \mathbb{R} \to \mathbb{R} \) be suitable functions so that the following limits and quotients exist. We say that \( f \) and \( g \) are asymptotically equivalent at \( a \in \mathbb{R} := [-\infty, \infty] \) if

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = 1.
\]

Asymptotic equivalence will be denoted \( f \sim g \) without reference to the limiting point \( a \) if it is clear from context.

Furthermore, we say that a function \( f \) admits an asymptotic power series representation at a point \( a \in \mathbb{R} \) if there is a sequence of constants \( (a_k)_{k \in \mathbb{N}_0} \) and some \( \mu \geq 0 \) such that

\[
\lim_{x \to a} (x - a)^{-N-\mu} \left( f(x) - \sum_{k=0}^{N} a_k (x - a)^{k+\mu} \right) = 0
\]

for all \( N \in \mathbb{N}_0 \). Since this implies that \( f(x) - \sum_{k=0}^{N-1} a_k (x - a)^{k+\mu} \sim a_N (x - a)^{N+\mu} \) whenever \( a_N \neq 0 \), we will use the notation

\[
f(x) \sim \sum_{k=0}^{\infty} a_k (x - a)^{k+\mu}
\]

to denote the above without reference to \( N \), even if the power series on the right does not converge.

Analogously, we define asymptotic power series representation as \( x \to \infty \) by requiring that \( x \mapsto f(1/x), x > 0 \), has an asymptotic power series at 0. Explicitly, we then require

\[
\lim_{x \to \infty} x^{N+\mu} \left( f(x) - \sum_{k=0}^{N} a_k x^{-k-\mu} \right) = 0,
\]

for all \( N \in \mathbb{N}_0 \), and denote this by

\[
f(x) \sim \sum_{k=0}^{\infty} a_k x^{-k-\mu}
\]

The asymptotic analysis of Laplace-type integrals has been studied extensively. For completeness, we will present below a general form of the asymptotics of Laplace-type integrals.

Theorem 3.8. Let \( h : [a, b] \to \mathbb{R} \) be a function satisfying the following conditions:

- \( h \) attains a global minimum at the left end-point \( a \), \( h(x) > h(a) \) for all \( x \in (a, b) \), and for every \( \delta > 0 \) we have \( \inf_{x \in [a+\delta, b]} \{ h(x) - h(a) \} > 0 \).

- \( h \) admits a power series representation at the left end-point \( a \) of the form

\[
h(x) \sim h(a) + \sum_{s=0}^{\infty} a_s (x - a)^{s+\mu}
\]

for some \( \mu > 0 \) and with \( a_0 \neq 0 \).
• $h$ is differentiable in a neighbourhood of $a$ and the previous power series representation can be term-wise differentiated to give

$$h'(x) \sim \sum_{s=0}^{\infty} a_s (s + \mu)(x - a)^{s+\mu-1}.$$  

• $h'$ is continuous in a neighbourhood of $a$ except possibly at $a$.

Suppose also that $\varphi : [a, b] \to \mathbb{R}$ is a function satisfying all of the following:

• $\varphi$ is continuous in a neighbourhood of $a$ except possibly at $a$.

• $\varphi$ admits a power series representation at the left end-point $a$ of the form

$$\varphi(x) \sim \sum_{s=0}^{\infty} b_s (x - a)^{s+\alpha-1}$$

for some $\alpha \in \mathbb{C}$ such that $\text{Re} \alpha > 0$, and with $b_0 \neq 0$.

Furthermore, suppose that there exists $M > 0$ such that for all $\lambda \geq M$, the integral $I(\lambda)$ defined by

$$I(\lambda) := \int_{a}^{b} dx \, \varphi(x) e^{-\lambda h(x)},$$

converges absolutely.

Then, as $\lambda \to \infty$, 

$$I(\lambda) \sim e^{-\lambda h(a)} \sum_{s=0}^{\infty} \Gamma \left( \frac{s + \alpha}{\mu} \right) \frac{c_s}{\lambda^{s+\alpha}},$$

where the coefficients $c_s$ are expressible in terms of $a_s$ and $b_s$, and, in particular, we have

$$c_0 = \frac{b_0}{\mu a_0^\mu}.$$

**Proof.** The proof is given [11], chapter 2 “Classical Procedures”, section 1 “Laplace’s method”. □

The previous theorem can be applied to all the Laplace-type integrals that will be used in this paper. To be explicit, the most typical usage of this theorem will be for the case where $h : [a, c] \to \mathbb{R}$ is a twice continuously differentiable strictly convex function, which implies that $h''(x) > 0$ for all $x \in [a, c]$. If there exists $b \in (a, c)$ such that $h''(b) = 0$, then this point $b$ is the global minimum of $h$ and one can consider the function $h$ on the intervals $[a, b]$ and $[b, c]$. Note that the previous theorem holds precisely for $h$ on the interval $[b, c]$ since $h$ attains its global minimum at the left-end point $b$. For the interval $[a, b]$, one instead considers the mapping $\tilde{h}(x) := h(-x)$ defined on the interval $[-b, -a]$. One finds that the mapping $\tilde{h}$ attains a global minimum at $-b$ and again the contents of the previous theorem hold.

The role of the mapping $\varphi : [a, c] \to \mathbb{R}$ does not change. In particular, if $\varphi$ admits a power series representation at any point on this interval, then it necessarily also admits power series representations when using one sided limits.

In the case of a strictly convex $h$, at the global minimum $b$, we have $\mu = 2$ and $a_0 = \frac{1}{2} h''(b) \neq 0$. The mapping $\varphi$ is of more importance. In particular, suppose that $\varphi$ is a smooth function such that for some finite $i \in \mathbb{N}$ and for all $k < i$, we have $\varphi^{(k)}(b) = 0$ and $\varphi^{(i)}(b) \neq 0$. In the notation of the previous theorem, this would correspond to the situation where $\alpha = i + 1$ and $b_0 = \frac{1}{i!} \varphi^{(i)}(b)$. Applying the previous theorem, we then would have

$$\int_{a}^{c} dx \, \varphi(x) e^{-\lambda h(x)} \sim e^{-\lambda h(b)} \Gamma \left( \frac{i+1}{2} \right) \frac{1}{i!} \varphi^{(i)}(b) \frac{1}{\lambda^{i+1}} \frac{1}{2} \left( \frac{1}{2} h''(b) \right)^{\frac{i+1}{2}} \lambda^{-\frac{i+1}{2}}.$$
and
\[
\frac{\int_a^c dx \varphi(x)e^{-\lambda h(x)}}{\int_a^c dx e^{-\lambda h(x)}} \sim \frac{\Gamma \left( \frac{d+1}{2} \right)}{\Gamma \left( \frac{d}{2} \right)} \frac{1}{n! \varphi^{(i)}(b)} \frac{1}{(\frac{1}{2} h''(b))^\frac{1}{2}} \frac{1}{\lambda^\frac{d}{2}}.
\]

The primary message from this is that the order of the first non-zero derivative of \( \varphi \) determines the rate of vanishing of such Laplace-type integrals, in particular, in this case we would have
\[
\frac{\int_a^c dx \varphi(x)e^{-\lambda h(x)}}{\int_a^c dx e^{-\lambda h(x)}} = O(\lambda^{-\frac{d}{2}}), \quad \lambda \to \infty.
\]
Furthermore, there will be some cases in which \( h'(x) \neq 0 \) for any \( x \in (a,c) \). In such a situation, it will also typically be so that \( h'(x) < 0 \) for all \( x \in (a,c) \), this implies that \( h \) is minimized at the right end point \( c \), and, by considering the mapping \( h(x) = h(-x) \) again, we see that the mapping \( h \) is now minimized at its left end point \( -c \) and thus the theorems hold again. In such a situation, we have \( \mu = 1 \) instead of \( \mu = 2 \).

4 Discrete Curie–Weiss model

We begin by presenting the Curie–Weiss Hamiltonian.

**Definition 4.1** (Curie–Weiss Hamiltonian). Let \( \Lambda \subset \mathbb{Z}^d \) be a \( d \)-dimensional finite square lattice with size \( N := |\Lambda| \). We define the space of spin configurations \( \mathcal{S} := \{-1, 1\}^\Lambda \). Let \( J > 0 \) and \( h \in \mathbb{R} \). Define the Hamiltonian \( H : \mathcal{S} \to \mathbb{R} \) and the particle number \( N : \mathcal{S} \to \mathbb{R} \) by
\[
H[\phi] := -\frac{J}{2N} \sum_{x,y \in \Lambda} \phi(x)\phi(y) - h \sum_{x \in \Lambda} \phi(x), \quad N[\phi] := \sum_{x \in \Lambda} \phi(x)^2.
\]
We also define the magnetisation \( M : \mathcal{S} \to \mathbb{R} \) by
\[
M[\phi] := \sum_{x \in \Lambda} \phi(x),
\]
and we note that the Hamiltonian can be written in terms of the magnetisation by
\[
H[\phi] = -\frac{J}{2N} M[\phi]^2 - hM[\phi] = -\frac{J}{2N} \left( M[\phi] + \frac{hN}{J} \right)^2 + \frac{h^2 N}{2J}.
\]
Furthermore, we define the energy density \( \varepsilon_N : \mathcal{S} \to \mathbb{R} \) and magnetisation density \( m_N : \mathcal{S} \to \mathbb{R} \) by
\[
\varepsilon_N[\phi] := \frac{H[\phi]}{N} \quad \text{and} \quad m_N[\phi] := \frac{M[\phi]}{N}.
\]
We remark that the particle number function is superfluous in this model, as it is obvious that \( N[\phi] := N \) for all \( \phi \in \mathcal{S} \), but, it is included due to its relevance to the continuum model we will consider in Sec. 5.

4.1 Microcanonical ensembles

We will now present two closely related microcanonical ensembles, along with a number of lemmas and coupling theorems needed for the main theorems.

**Definition 4.2** (Fixed energy density/Microcanonical ensemble). Let \( \varepsilon \in \mathbb{R} \) be such that \( \varepsilon \in \text{Ran}[\varepsilon_N] \). Define the set \( \mathcal{S}_\varepsilon := \{ \phi \in \mathcal{S} : \varepsilon_N[\phi] = \varepsilon \} \). The microcanonical ensemble with energy density \( \varepsilon \) is defined via its action on functions \( f : \mathcal{S} \to \mathbb{R} \) by
\[
\langle f \rangle_{MC}^{\varepsilon} := \frac{1}{|\mathcal{S}_\varepsilon|} \sum_{\phi \in \mathcal{S}_\varepsilon} f(\phi).
\]
In some sense, the fixed energy ensemble for some values of $J$ and $h$ is not necessarily fundamental as it can be represented as a convex combination of fixed magnetisations. To this end, we define the fixed magnetisation ensemble analogously.

**Definition 4.3** (Fixed magnetisation ensemble/Microcanonical ensemble). Let $m \in \mathbb{R}$ be such that $m \in \text{Ran}[m_N]$. Define the set $S_m := \{ \phi \in \mathcal{S} : m_N[\phi] = m \}$. The microcanonical ensemble with magnetisation density $m$ is defined via its action on bounded 1-Lipschitz functions $f : \mathcal{S} \to \mathbb{R}$ by

$$\langle f \rangle_{MC}^{m} := \frac{1}{|S_m|} \sum_{\phi \in S_m} f(\phi).$$

We will always use the lower case letter $m$ to specify magnetisation densities and $\epsilon$ for energy densities so that there is no ambiguity.

The energy density can be written in terms of the magnetisation density as

$$\epsilon_n[\phi] = -\frac{J}{2} \left( m_N[\phi] + \frac{h}{J} \right)^2 + \frac{h^2}{2J} \iff m_{N,\pm}[\phi] = \frac{h}{J} \pm \sqrt{\frac{h^2}{2J} - \frac{2\epsilon_n[\phi]}{J}},$$

from which it is clear that for some values of $h$ and $J$ there are multiple magnetisation densities which give the same energy density. The following lemma makes the previous statements more quantitative.

**Lemma 4.4.** Let $\epsilon \in \mathbb{R}$ be such that $\epsilon \in \text{Ran}[\epsilon_N]$ and define $m_{\pm} = -\frac{h}{J} \pm \frac{h^2}{2J} - \frac{2\epsilon_N}{J}$. We have

$$\langle f \rangle_{MC}^{\epsilon} = \frac{|S_{m_{+}}|}{|S_{m_{+}}| + |S_{m_{-}}|} \langle f \rangle_{MC}^{m_{+},\epsilon} + \frac{|S_{m_{-}}|}{|S_{m_{+}}| + |S_{m_{-}}|} \langle f \rangle_{MC}^{m_{-},\epsilon},$$

with the convention that $\langle f \rangle_{MC}^{m_{-},\epsilon} = 0$ and $|S_m| = 0 \text{ if } m \notin \text{Ran}[m_{\pm}]$.

**Proof.** Since $\epsilon \in \text{Ran}[\epsilon_N]$, it follows that $\epsilon \leq \frac{h^2}{2J}$. If $\epsilon = \frac{h^2}{2J}$, we have $m_{+} = m_{-}$, $S_m = S_{\epsilon}$ for $m = m_{\pm}$, and thus $\langle f \rangle_{MC}^{\epsilon} = \frac{1}{2} \langle f \rangle_{MC}^{m_{+},\epsilon} + \frac{1}{2} \langle f \rangle_{MC}^{m_{-},\epsilon}$. Hence, (4.1) holds in this case.

We may thus assume that $\epsilon < \frac{h^2}{2J}$, when $m_{-} < m_{+}$. We have $S_{\epsilon} = S_{m_{+}} \cup S_{m_{-}}$ and $S_{m_{+}} \cap S_{m_{-}} = \emptyset$. Thus, $|S_{\epsilon}| = |S_{m_{+}}| + |S_{m_{-}}|$ and

$$\frac{1}{|S_{\epsilon}|} \sum_{\phi \in S_{\epsilon}} f(\phi) = \frac{1}{|S_{m_{+}}| + |S_{m_{-}}|} \left( \frac{1}{|S_{m_{+}}|} \sum_{\phi \in S_{m_{+}}} f(\phi) + \frac{1}{|S_{m_{-}}|} \sum_{\phi \in S_{m_{-}}} f(\phi) \right),$$

from which the statement follows. \qed

In fact, for most values of the external field $h$, one of the above fixed magnetisation density ensembles in Lemma 4.3 dominates the other. To study this, for any field configuration $\phi \in \mathcal{S}$ we denote the collection of positive spins by $\Lambda_{+}[\phi] := \phi^{-1}(1)$ and the collection of negative spins by $\Lambda_{-}[\phi] := \phi^{-1}(-1)$. The total positive spin $M_{+}$ and total negative spin $M_{-}$ are then defined as the number of elements in these sets, i.e., $M_{\pm} = |\Lambda_{\pm}|$. Since $\phi$ can only take these two values, we then clearly have

$$M_{+} + M_{-} = N, \quad M_{+} - M_{-} = N,$$

where $N := |\Lambda|$ and $M := M[\phi]$ denotes the total magnetisation. For later use, let us point out that by these relations we have $M_{+} = \frac{N+M}{2}$ and $M_{-} = \frac{N-M}{2}$.

We now can prove the following lemma.

**Lemma 4.5.** Let $m, m' \in \text{Ran}[m_N] \setminus \{-1, 1\}$. Define $f : (-1, 1) \to \mathbb{R}$ by $f(m) := -\frac{1+m}{2} \ln \frac{1+m}{2} - \frac{1-m}{2} \ln \frac{1-m}{2}$. It follows that

$$\frac{|S_m|}{|S_{m'}|} \sim e^{-N(f(m') - f(m))} \left( \frac{1 - (m')^2}{1 - m^2} \right)^{\frac{1}{2}},$$

and, if $|m'| < |m|$, then $f(m') - f(m) > 0$. 

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Proof. Denote $M = mN$ and recall that, if $\phi \in S_m$, then $M[\phi] = M$. Since $M_+ = \frac{M+N}{2}$, we thus have

$$|S_m| = \left(\frac{N}{M+N}\right)^{\frac{N}{2}} = \frac{N!}{\left(\frac{N}{2}\right)! \left(\frac{N}{2}\right)!} = (N+1)^{-1} \left(\frac{\Gamma\left(\frac{N+M}{2}+1\right)}{\Gamma\left(\frac{N-M}{2}+1\right)}\right)^{-1},$$

where $\Gamma$ denotes the standard Gamma function. Using the Beta function, we have

$$\frac{\Gamma\left(\frac{N+M}{2}+1\right)}{\Gamma\left(\frac{N-M}{2}+1\right)} = B\left(N+1, \frac{N}{2}+1, \frac{N}{2}+1\right) = \int_0^1 dt \, t^{\frac{N+M}{2}} (1-t)^{\frac{N-M}{2}}.$$

Define $h : (0, 1) \to \mathbb{R}$ by $h(t) := -\frac{1+m}{2} \ln t - \frac{1-m}{2} \ln(1-t)$. We have $h'(t) = \frac{1-m}{2} \frac{1}{t} - \frac{1+m}{2} \frac{1}{1-t}$ and $h''(t) = \frac{1+m}{2} \frac{1}{t^2} + \frac{1-m}{2} \frac{1}{(1-t)^2}$. It is thus evident that $h$ is strictly convex, $h'(0) = 0 \iff t = \frac{1+m}{1-m}$ and $h''(\frac{1+m}{1-m}) = \frac{1}{1-m^2}$. Using Theorem 3.8, we thus have

$$\frac{|S_m|}{|S_m|} = \frac{1}{\int_0^1 dt \, e^{-N\left(-\frac{1+m}{2} \ln t - \frac{1-m}{2} \ln(1-t)\right)}} \sim e^{-N\left(h\left(\frac{1+m}{1-m}\right) - h\left(\frac{1-m}{1+m}\right)\right) \left(1 - (m')^2\right) \frac{1}{1-m^2}} = e^{-N(f(m')-f(m)) \left(1 - (m')^2\right) \frac{1}{1-m^2}}.$$

The function $f$ satisfies $f'(m) = \frac{1}{2} \ln \frac{1+m}{1-m}$ and $f''(m) = -\frac{1}{1-m^2}$. It follows that $f$ is strictly concave and obtains a maximum at $m = 0$. Furthermore, the function $f$ is even, and, as a result, $f(m') > f(m) \iff |m'| < |m|$. □

By Lemma 4.3, we see that of the two magnetisation densities appearing in Lemma 4.4 the one with a smaller absolute value dominated the other one. This statement is formalized in the next proposition.

**Proposition 4.6.** Let $h \neq 0$, $\varepsilon \in \mathbb{R}$ be such that $\varepsilon \in \text{Ran}[\varepsilon_N]$, and $m_\pm$ be as in Lemma 4.4 such that $|m_\pm| \neq 1$. If $|m_+| > |m_-|$, set $m = m_-$ and $m' = m_+$, and, if $|m_-| > |m_+|$, set $m = m_+$ and $m' = m_-$. For all functions $g : S \to \mathbb{R}$ which are bounded by some $K > 0$, we have

$$\left|\langle g \rangle_{MC}^{\varepsilon;N} - \langle g \rangle_{MC}^{m;N}\right| \leq \frac{2K}{|S_m|},$$

in the notation of Lemma 4.3. It follows that

$$\langle g \rangle_{MC}^{\varepsilon;N} = \langle g \rangle_{MC}^{m;N} + O(e^{-cN}),$$

for a positive global constant $c > 0$.

**Proof.** We have

$$\langle g \rangle_{MC}^{\varepsilon;N} = \frac{|S_m|}{|S_m|} \langle g \rangle_{MC}^{m;N} + \frac{|S_m'|}{|S_m|} \langle g \rangle_{MC}^{m';N},$$

and

$$\left|\langle g \rangle_{MC}^{\varepsilon;N} - \langle g \rangle_{MC}^{m;N}\right| = \frac{|S_m'|}{|S_m|} \left|\langle g \rangle_{MC}^{m';N} - \langle g \rangle_{MC}^{m;N}\right| \leq \frac{2K}{|S_m|},$$

as desired. □
By the above proposition, the case \( h \neq 0 \) requires an analysis of the fixed magnetisation density ensembles instead of the fixed energy density ensemble. However, the case for \( h = 0 \) is different. For future reference, we present the \( h = 0 \) case as a proposition, although it is a direct corollary of Lemma 4.4.

**Proposition 4.7.** Let \( h = 0, \varepsilon \in \text{Ran}[\varepsilon_n] \), and \( m \pm \) as in Lemma 4.4. For functions \( f : S \to \mathbb{R} \) integrable with respect to both \( \mu_{MC}^{m_+;N} \) and \( \mu_{MC}^{m_-;N} \), we have

\[
(f)_{\varepsilon;MC} = \frac{1}{2} (f)_{MC}^{m_+;N} + \frac{1}{2} (f)_{MC}^{m_-;N}.
\]

**Proof.** For \( h = 0 \), the allowed magnetisation densities have the same absolute value but different signs. As a result, the amount of configurations is the same for both magnetisations.

Due to the simple combinatorial nature of the fixed magnetisation density ensembles, we are able to compute the \( w_1 \) fluctuation distance between two different fixed magnetisation ensembles explicitly. We have the following theorem.

**Theorem 4.8.** Let \( m \in \text{Ran}[m_N] \) and \( m' \in \text{Ran}[m_N] \) and denote \( M = mN \) and \( M' = m'N \). Then,

\[
w_1(\mu_{MC}^{mN};\mu_{MC}^{m',N};N) = |m' - m|.
\]

**Proof.** By symmetry, it suffices to prove the result in the case \( M' > M \); note that if \( M' = M \), then \( \mu_{MC}^{mN} = \mu_{MC}^{m'N} \). First, consider any field configuration \( \phi \in S \) such that \( M[\phi] = M \). Note that \( \Lambda_+[\phi] \) corresponds to the sites on the lattice for which \( \phi \) has a positive spin. Now, consider another field configuration \( \phi' \in S \) such that \( \phi'[\Lambda_+|\phi] = \phi \) and \( M[\phi'] = M' \). Such a configuration \( \phi' \) can be constructed by taking the configuration \( \phi \) and flipping negative spins to positive spins or vice versa until we obtain the magnetisation \( M' \).

As before, define \( M_+ := |\Lambda_+[\phi]| \) and \( M_- := N - M_+ \), and set also \( M'_+ := |\Lambda_+|\phi'|\). Then \( M_+ = \frac{N + M}{2} \), \( M_- = \frac{N - M}{2} \), and \( M_+ < M'_+ \). Denote \( \Delta := M'_+ - M_+ = \frac{M' - M}{2} > 0 \). The number of field configurations with magnetisation \( M \) is given by \( \binom{N}{M_+} \) or, equivalently, by \( \binom{N}{M_-} \). In order to go from magnetisation \( M \) to \( M' \), we must flip \( \Delta \) negative sites to positive sites. The number of ways to do this is \( \binom{M}{\Delta} \). Define \( \gamma : S \times S \to \mathbb{R} \) by

\[
\gamma(\phi, \phi') := \mathbb{1}(\phi \in S_m, \phi' \in S_{m'}, \phi'|_{\Lambda_+|\phi} = \phi|_{\Lambda_+|\phi}) \frac{1}{\binom{N}{M_+}} \frac{1}{\binom{M_+}{\Delta}}.
\]

By construction, we have

\[
\sum_{\phi' \in S} \gamma(\phi, \phi') = \frac{\mathbb{1}(\phi \in S_m)}{\binom{N}{M_-}} \frac{1}{\binom{M_-}{\Delta}} \sum_{\phi' \in S_{m'}} \mathbb{1}(\phi'|_{\Lambda_+|\phi} = \phi|_{\Lambda_+|\phi}) = \frac{\mathbb{1}(\phi \in S_m)}{\binom{N}{M_-}} \frac{1}{\binom{M_-}{\Delta}} \binom{M_-}{\Delta}.
\]

In the other direction, if we fix \( \phi' \) with magnetisation \( M' \) and consider the number of ways to go to a field configuration \( \phi \) which agrees on the positive lattice sites of \( \phi \), then clearly we must take \( \Delta = \frac{M' - M}{2} \) positive sites of \( \phi' \) and flip them negative. The number of ways to do this is given by \( \binom{M_+}{\Delta} \) and we thus have

\[
\sum_{\phi \in S} \gamma(\phi, \phi') = \frac{\mathbb{1}(\phi' \in S_{m'})}{\binom{N}{M_-}} \frac{1}{\binom{M_-}{\Delta}} \sum_{\phi \in S_m} \mathbb{1}(\phi'|_{\Lambda_+|\phi} = \phi|_{\Lambda_+|\phi}) = \frac{\mathbb{1}(\phi' \in S_{m'})}{\binom{N}{M_-}} \frac{1}{\binom{M_-}{\Delta}} \binom{M_+}{\Delta}.
\]
Now, we have the following simple binomial coefficient manipulations
\[
\binom{M'}{2}\binom{N}{M'} = \binom{M'}{2}\binom{N}{M'} = \frac{(N-M_-)! (M_+ - \Delta)!}{(N-M_+)! (M_- - \Delta)!} = \frac{(N-M_-)! (M_+ - \Delta)!}{(M_+)! (M_- + M_+ - M_-)!} = 1.
\]

It follows that
\[
\sum_{\phi \in \mathcal{S}} \gamma(\phi, \phi') = \frac{\mathbb{1}(\phi' \in \mathcal{S}_{m'})}{(M')_l} = \frac{\mathbb{1}(\phi' \in \mathcal{S}_{m'})}{|\mathcal{S}_{m'}|}.
\]

This verifies that \( \gamma \) is indeed a coupling between the fixed magnetisation density ensembles with different magnetisations \( m \) and \( m' \). For such a coupling, by construction, we have
\[
\gamma(\phi, \phi') \| \phi - \phi' \|_1 = 2\Delta \gamma(\phi, \phi') = (M' - M) \gamma(\phi, \phi'),
\]
from which it follows that
\[
w_1(\mu_{MC}^{m;N}, \mu_{MC}^{m';N} : N) \leq \frac{M' - M}{N} = m' - m.
\]

On the other hand, if \( \eta \) is any other coupling of \( \mu_{MC}^{m;N} \) and \( \mu_{MC}^{m';N} \), we also have
\[
\frac{1}{N} \int_{\mathcal{S} \times \mathcal{S}} \eta(d\phi, d\phi') \sum_{x \in \Lambda} |\phi(x) - \phi'(x)| \geq \frac{1}{N} \left| \sum_{x \in \Lambda} \phi(x) - \sum_{x \in \Lambda} \phi'(x) \right| = m' - m.
\]

This implies that the coupling \( \gamma \) is an optimal coupling, and, we have
\[
w_1(\mu_{MC}^{m;N}, \mu_{MC}^{m';N} : N) = m' - m.
\]

This completes the proof assuming \( M' > M \), and hence by symmetry, also the proof of the Theorem.

We have an optimal control of the \( w_1 \) fluctuation distance between two different fixed magnetisation densities in Theorem 4.9. By Proposition 4.6, this suffices to control the dependence of expectations if \( h \neq 0 \). For \( h = 0 \), we need a bound for two different ensembles with different energy densities.

**Theorem 4.9.** Let \( h = 0 \) and \( \varepsilon, \varepsilon' \in \text{Ran}[\varepsilon_N] \). If \( \varepsilon \neq 0 \), we have
\[
w_1(\mu_{MC}^{0;N}, \mu_{MC}^{\varepsilon';N} : N) = \left| \sqrt{-\frac{2\varepsilon}{J}} - \sqrt{-\frac{2\varepsilon'}{J}} \right| \leq \frac{2}{J} \frac{1}{\sqrt{-\frac{2\varepsilon}{J}}} |\varepsilon - \varepsilon'|,
\]
and, if \( \varepsilon = 0 \),
\[
w_1(\mu_{MC}^{0;N}, \mu_{MC}^{\varepsilon';N} : N) = \sqrt{-\frac{2\varepsilon'}{J}}.
\]

**Proof.** For both \( \varepsilon \) and \( \varepsilon' \) let \( m_\pm \) and \( m'_\pm \) be the corresponding magnetisation densities in Lemma 4.4.

Let \( \gamma_+ : \mathcal{S} \times \mathcal{S} \to \mathbb{R} \) be the coupling constructed in Theorem 4.8 between the constant magnetisation ensembles \( m_+ \) and \( m'_+ \), and let \( \gamma_- : \mathcal{S} \times \mathcal{S} \to \mathbb{R} \) be the coupling between the \( m_- \) and \( m'_- \) ensembles. Define \( \gamma : \mathcal{S} \times \mathcal{S} \to \mathbb{R} \) by
\[
\gamma(\phi, \phi') := \frac{1}{2} \gamma_+(\phi, \phi') + \frac{1}{2} \gamma_-(\phi, \phi').
\]
Then for an arbitrary observable \( f : S \to \mathbb{R} \), we have \( \langle f(\phi) \rangle_\gamma = \frac{1}{2} \langle f \rangle_{\mu_{MC}^{\varepsilon,N}} + \frac{1}{2} \langle f \rangle_{\mu_{MC}^{\varepsilon,N}} = \langle f \rangle_{\mu_{MC}^{\varepsilon,N}} \) by Proposition 4.7. Similarly, one checks that the right marginal is given by \( \mu_{MC}^{\varepsilon,N} \), and thus \( \gamma \) is indeed a coupling of \( \mu_{MC}^{\varepsilon,N} \) and \( \mu_{MC}^{\varepsilon,N} \).

Furthermore, by direct calculation, we have

\[
\gamma(\phi, \phi') |\phi - \phi'|_1 = \gamma(\phi, \phi') \frac{|m_+ N - m' N| + |m_0 N - m'_0 N|}{2} = \gamma(\phi, \phi') N \sqrt{\frac{-2\varepsilon}{J} - \frac{2\varepsilon'}{J}}.
\]

Following the same procedure as before, we have an optimal coupling for which

\[
w_1(\mu_{MC}^{\varepsilon,N}, \mu_{MC}^{\varepsilon',N} ; N) = \left| \sqrt{-\frac{2\varepsilon}{J}} - \sqrt{-\frac{2\varepsilon'}{J}} \right| \leq \frac{2}{J} \frac{1}{\sqrt{-\frac{2\varepsilon}{J}}} |\varepsilon - \varepsilon'|.
\]

Note that the last inequality only holds for \( \varepsilon \neq 0 \).

### 4.2 Canonical ensembles

Next, we will focus on the properties of the canonical ensembles.

**Definition 4.10** (Fluctuating energy/canonical ensemble). Let \( \beta > 0 \). The canonical ensemble with inverse temperature \( \beta \) is defined via its action on bounded 1-Lipschitz functions \( f : S \to \mathbb{R} \) by

\[
\langle f \rangle_{\beta,N}^C := \frac{1}{\sum_{\phi \in S} e^{-\beta H[\phi]}} \sum_{\phi \in S} e^{-\beta H[\phi]} f(\phi) = \frac{1}{\sum_{\varepsilon \in \text{Ran}[\varepsilon]} e^{-\beta \varepsilon N} |\varepsilon|} \sum_{\varepsilon \in \text{Ran}[\varepsilon]} e^{-\beta \varepsilon N} |\varepsilon| \langle f \rangle_{\mu_{MC}^{\varepsilon,N}}.
\]

The representation on the right-most side of the above equality is called the energy representation of the canonical measure.

**Definition 4.11.** (Fluctuating magnetisation/canonical ensemble) Let \( \mu \in \mathbb{R} \). The canonical ensemble with magnetic potential \( \mu \) is defined via its action on bounded 1-Lipschitz functions \( f : S \to \mathbb{R} \) by

\[
\langle f \rangle_{\mu,N}^C := \frac{1}{\sum_{\phi \in S} e^{-\mu M[\phi]}} \sum_{\phi \in S} e^{-\mu M[\phi]} f(\phi) = \frac{1}{\sum_{m \in \text{Ran}[m]} e^{-\mu m N} |m|} \sum_{m \in \text{Ran}[m]} e^{-\mu m N} |m| \langle f \rangle_{\mu_{MC}^{m,N}}.
\]

The representation on the far right is called the magnetisation representation of this ensemble.

In the following lemmas, we will characterize some of the relevant asymptotic properties of the previously defined ensembles. For the canonical ensemble, we will need to rely on the previously given theorems and lemmas concerning Laplace-type integrals. We will start with the fluctuating magnetisation ensemble where these Laplace-type methods are not needed since the sums may be evaluated explicitly.

**Lemma 4.12.** The partition function of the fluctuating magnetisation ensemble is given by

\[
Z_C(\mu; N) := \sum_{\phi \in S} e^{-\mu M[\phi]} = (2 \cosh(\mu))^N,
\]

and the specific free energy is given by

\[
f_C(\mu; N) := -\frac{1}{N} \ln Z_C(\mu; N) = -\ln(2 \cosh(\mu)).
\]
The average and standard deviation of the magnetisation density are given by

\[
\langle M \rangle_G^{\mu;N} \frac{1}{N} = -\tanh(\mu),
\]

and

\[
\sigma_G^{\mu;N} \left( \frac{M}{N} \right) = \frac{1}{\sqrt{N}} \sqrt{1 - \tanh^2(\mu)}.
\]

Proof. We have

\[
Z_G(\mu; N) = \sum_{\phi \in S} e^{-\mu M[\phi]} = \sum_{\phi \in S} \prod_{x \in \Lambda} e^{-\mu \phi(x)} = \prod_{x \in \Lambda} (e^\mu + e^{-\mu}) = (2 \cosh(\mu))^N.
\]

The rest of the results following by differentiating the free energy with respect to \( \mu \) and dividing appropriately by the degrees of freedom \( N \).

For the fixed average energy canonical ensemble, we will first analyse the properties of the function which will eventually determine the asymptotics of the ensemble.

Lemma 4.13. Let \( \psi_\beta : [0, \infty) \to \mathbb{R} \) be defined by

\[
\psi_\beta(z) := \frac{1}{2} z^2 - \ln \cosh(\sqrt{\beta} J z).
\]

For all \( \beta \leq \frac{1}{2} \), \( \psi_\beta \) has a unique global minimum at \( z = 0 \).

For every \( \beta > \frac{1}{2} \), there exists \( m \in (0, 1) \) such that \( \psi_\beta \) has a unique global minimum at \( z = z(\beta) > 0 \) at which \( \psi_\beta'(z(\beta)) > 0 \), and \( z(\beta) \) satisfies \( \tanh^2(\sqrt{\beta} J z(\beta)) = m \). We also have the converse, for every \( m \in (0, 1) \) there exists \( \beta > \frac{1}{2} \) and \( z(\beta) > 0 \) such that \( \psi_\beta \) has a unique global minimum at \( z = z(\beta) \) and \( z(\beta) \) satisfies \( \tanh^2(\sqrt{\beta} J z(\beta)) = m \).

Proof. Suppose \( 0 < \beta \leq \frac{1}{2} \). We have for all \( z > 0 \),

\[
\psi_\beta'(z) = z - \sqrt{\beta J} \tanh(\sqrt{\beta J} z)
\]

and

\[
\psi_\beta''(z) = 1 - \beta J (1 - \tanh^2(\sqrt{\beta J} z)) > 0.
\]

If \( z \geq 1 \), there is \( c > 0 \) such that \( c \leq \tanh(\sqrt{\beta J} z) < 1 \), and thus \( \psi_\beta''(z) \geq 1 - \beta J (1 - c^2) > 0 \). It follows that \( \lim_{z \to \infty} \psi_\beta'(z) = \infty \), and in the opposite limit we have \( \lim_{z \to 0} \psi_\beta'(z) = 0 \). Because \( \psi_\beta'' > 0 \), we know that \( \psi_\beta \) is a continuous strictly increasing function, and by the above limit values, we know that \( \psi_\beta \) obtains every value on the positive real line and it is one-to-one. In particular, \( \psi_\beta'(z) > 0 \) for all \( z > 0 \), and thus \( \psi_\beta \) is a continuous, strictly increasing function. Since \( \psi_\beta(z) = 0 \) at \( z = 0 \), we have \( \psi_\beta(z) > 0 \) for all \( z > 0 \).

Consider then \( \beta > \frac{1}{2} \) and let \( \varepsilon := 1 - \frac{1}{\beta J} \). Then \( 0 < \varepsilon < 1 \) and, since \( \beta J = 1 + \beta J \varepsilon \), we have

\[
\psi_\beta''(z) = -\beta J (\varepsilon - \tanh^2(\sqrt{\beta J} z)).
\]

The mapping \( z \mapsto \tanh^2(\sqrt{\beta J} z) \) is strictly increasing on the interval \([0, \infty)\), continuous, is zero at \( z = 0 \), and obtains the limiting value \( 1 \) when \( z = \infty \). It follows that there exists a unique \( z(\beta) > 0 \) such that \( \tanh^2(\sqrt{\beta J} z(\beta)) = \varepsilon \). From the previous observations, we see that \( \psi_\beta''(z) > 0 \) if \( z > z(\beta) \), \( \psi_\beta''(z) < 0 \) if \( 0 < z < z(\beta) \), and \( \psi_\beta''(z) = 0 \) if \( z = z(\beta) \). Returning to the function \( \psi_\beta \), we see that \( \psi_\beta \) is strictly decreasing on the interval \((0, z(\beta))\) and it is strictly increasing on the interval \((z(\beta), \infty)\). Because \( \psi_\beta'(z) \to 0 \) as \( z \to 0^+ \), it follows that \( \psi_\beta'(z) < 0 \) for all \( 0 < z \leq z(\beta) \), \( \psi_\beta'(z) \) has a global minimum at \( z = z(\beta) \), it subsequently increases. As in the first case, we
also find that $\psi_\beta'(z) \to \infty$ as $z \to \infty$, and thus there exists a unique $\delta(\beta) > z(\beta)$ such that $\psi_\beta'(\delta(\beta)) = 0$. Therefore, we can conclude that $\psi_\beta$ attains a global minimum at $z = \delta(\beta)$, it is decreasing on the interval $[0, \delta(\beta)]$, and increasing on the interval $[\delta(\beta), \infty)$. Since $\delta(\beta) > z(\beta)$, we have $\psi_\beta''(\delta(\beta)) > 0$, as claimed above.

The previous analysis shows that if $\beta > \frac{1}{m}$, then there exists a unique $\delta(\beta) > 0$ such that $\psi_\beta$ is minimized at $z = \delta(\beta)$ and $\psi_\beta'(\delta(\beta)) = 0$. We also have the converse result. Let $m \in (0, 1)$. There exists $t \in (0, \infty)$ such that $\tanh^2(t) = m$, and, because the hyperbolic tangent is an increasing function, we have $t = \tanh^{-1}(\sqrt{m})$. We note that

$$
\frac{t}{\beta J} - \tanh(t) = 0 \iff \beta = \frac{t}{J \tanh(t)} > \frac{1}{J}.
$$

The inequality on the right side occurs due to $t > \tanh(t)$ for $t > 0$. Fix $\beta$ to be

$$
\beta = \frac{t}{J \tanh(t)} = \frac{1}{\sqrt{m}}.
$$

For such a $\beta$, define $\delta(\beta) := \frac{t}{\sqrt{m}} = \sqrt{\tanh(t)}t$. We have

$$
\psi_\beta'(\delta(\beta)) = \delta(\beta) - \sqrt{\beta J} \tanh(\sqrt{\beta J} \delta(\beta)) = \sqrt{\tanh(t)}t - \sqrt{\frac{t}{\tanh(t)}} \tanh(t) = 0.
$$

This confirms that for every $m \in (0, 1)$ there exists an inverse temperature $\beta > \frac{1}{m}$ and a minimizing $z = \delta(\beta)$ such that $\psi_\beta(z)$ is globally minimized for $z = \delta(\beta)$ and $\tanh^2(\sqrt{\beta J} \delta(\beta)) = m$.  

**Lemma 4.14.** Let $\psi_\beta : [0, \infty) \to \mathbb{R}$ be defined as in Lemma 4.13. The partition function of the canonical ensemble for $h = 0$ is given by

$$
Z_C(\beta; N) := \sum_{\phi \in \mathcal{S}} e^{-\beta H[\phi]} = \frac{\sqrt{N} 2^{N+1}}{\sqrt{2\pi}} \int_0^\infty dz e^{-N \psi_\beta(z)},
$$

and the specific free energy is given by

$$
f_C(\beta; N) := -\frac{1}{N} \ln Z_C(\beta; N) = -\frac{N + 1}{N} \ln 2 - \frac{\ln N}{2} - \frac{1}{N} \ln \int_0^\infty dz e^{-N \psi_\beta(z)}.
$$

The average and standard deviation of the energy density are given by

$$
\frac{\langle H \rangle_C^{\beta, N}}{N} = \frac{\int_0^\infty dz (\partial_\beta \psi_\beta(z)) e^{-N \psi_\beta(z)}}{\int_0^\infty dz e^{-N \psi_\beta(z)}},
$$

and

$$
\sigma_C^{\beta, N} \left( \frac{H}{N} \right) = \left( \frac{\int_0^\infty dz (\partial_\beta \psi_\beta(z))^2 e^{-N \psi_\beta(z)}}{\int_0^\infty dz e^{-N \psi_\beta(z)}} + \left( \frac{\int_0^\infty dz (\partial_\beta \psi_\beta(z)) e^{-N \psi_\beta(z)}}{\int_0^\infty dz e^{-N \psi_\beta(z)}} \right)^2 - \frac{1}{N} \int_0^\infty dz \left( \frac{\partial_\beta^2 \psi_\beta(z) e^{-N \psi_\beta(z)}}{\int_0^\infty dz e^{-N \psi_\beta(z)}} \right) \right)^{1/2},
$$

where

$$
\partial_\beta \psi_\beta(z) = -\frac{\sqrt{J} z}{2 \sqrt{\beta}} \tanh(\sqrt{J} z),
$$

and

$$
\partial_\beta^2 \psi_\beta(z) = \frac{\sqrt{J} z}{4 \beta^2} \tanh(\sqrt{J} z) + \frac{J z^2}{4 \beta} (\tanh^2(\sqrt{J} z) - 1).
$$
Proof. For $\phi \in \mathcal{S}$, using Gaussian linearisation, we have

$$e^{-\beta H[\phi]} = e^{\left(\frac{\beta J}{\sqrt{2\pi}} M[\phi]\right)^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz \ e^{-\frac{1}{2}\beta z^2 + \left(\frac{\beta J}{\sqrt{2\pi}}\right) M[\phi]}.$$ 

Using the same calculation as in Lemma 4.12, we have

$$\sum_{\phi \in S} e^{-\beta H[\phi]} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz \ e^{-\frac{1}{2}\beta z^2} \sum_{\phi \in S} e^{\left(\frac{\beta J}{\sqrt{2\pi}}\right) M[\phi]}$$

$$= \frac{2^N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dz \ e^{-\frac{1}{2}\beta z^2} \left(\sqrt{\frac{\beta J}{N}}\right)^N$$

$$= \frac{\sqrt{N} 2^N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dw \ e^{-\frac{1}{2}(\frac{\beta}{N} w^2 - \ln \cosh(\sqrt{\beta J} w))}$$

$$= \frac{\sqrt{N} 2^{N+1}}{2\pi} \int_0^{\infty} dz \ e^{-N\psi_\beta(z)},$$

from which it follows that

$$-\frac{1}{N} \ln Z_N(\beta) = -\frac{N+1}{N} \ln 2 - \frac{1}{2} \frac{\ln N}{N} - \frac{1}{N} \ln \int_0^{\infty} dz \ e^{-N\psi_\beta(z)}.$$

The rest of the computation follows by taking derivatives of the specific free energy with respect to $\beta$ and dividing by the degrees of freedom $N$. All derivatives with respect to $\beta$ will involve the corresponding derivatives of $\psi_\beta$ whose computation is straightforward.

Finally, we will present the asymptotics of the average and standard deviation of the energy density for the canonical ensemble as a lemma.

**Proposition 4.15.** For $\beta < \frac{1}{2}$, we have

$$\frac{\langle H \rangle_{\beta,N}}{N} = O(N^{-1}), \quad \sigma_{\beta,N}^2 \left(\frac{H}{N}\right) = O(N^{-1}).$$

and, if $\beta = \frac{1}{2}$,

$$\frac{\langle H \rangle_{\beta,N}}{N} = O(N^{-\frac{1}{2}}), \quad \sigma_{\beta,N}^2 \left(\frac{H}{N}\right) = O(N^{-\frac{1}{2}}).$$

For $\beta > \frac{1}{2}$, there always exists $m \in (0,1)$, and, for every $m \in (0,1)$, there always exists $\beta > \frac{1}{2}$ such that

$$\frac{\langle H \rangle_{\beta,N}}{N} = -\frac{J}{2} m + O(N^{-\frac{1}{2}}), \quad \sigma_{\beta,N}^2 \left(\frac{H}{N}\right) = O(N^{-\frac{1}{2}}).$$

Proof. For $\beta \leq \frac{1}{2}$, the mapping $\psi_\beta(z)$ is minimized for $z = 0$. By direct computation, we then have

$$\partial_\beta \psi_\beta(0) = \partial_\beta \psi_\beta'(0) = 0, \quad \partial_\beta \psi_\beta''(0) \neq 0,$$

$$(\partial_\beta \psi_\beta)^2(0) = \left(\left(\partial_\beta \psi_\beta\right)^2\right)'(0) = \left(\left(\partial_\beta \psi_\beta\right)^2\right)''(0) = \left(\left(\partial_\beta \psi_\beta\right)^2\right)'''(0) = 0, \quad \left(\left(\partial_\beta \psi_\beta\right)^2\right)'''(0) \neq 0,$$

$$\partial_\beta^2 \psi_\beta(0) = \partial_\beta^2 \psi_\beta'(0) = \partial_\beta^2 \psi_\beta''(0) = \partial_\beta^2 \psi_\beta'''(0) = 0, \quad \partial_\beta^3 \psi_\beta''(0) \neq 0.$$

In addition, if $\beta < \frac{1}{2}$,

$$\psi_\beta(0) = \psi_\beta'(0) = 0, \quad \psi_\beta''(0) > 0,$$
and if \( \beta = \frac{1}{J} \),

\[
\psi(0) = \psi_0(0) = \psi_0''(0) = \psi_0'''(0) = 0, \quad \psi_0''''(0) > 0.
\]

Using the asymptotics of the Laplace-type integrals from Sec. 3.3 we have for \( \beta < \frac{1}{J} \),

\[
\int_0^\infty dz \frac{\partial_\beta \psi_\beta(z)e^{-N\psi_\beta(z)}}{\int_0^\infty dz e^{-N\psi_\beta(z)}} = O(N^{-1}),
\]

(4.2)

\[
\int_0^\infty dz \frac{(\partial_\beta \psi_\beta(z))^2e^{-N\psi_\beta(z)}}{\int_0^\infty dz e^{-N\psi_\beta(z)}} = O(N^{-2}),
\]

(4.3)

\[
\int_0^\infty dz \frac{\partial_\beta^2 \psi_\beta(z)e^{-N\psi_\beta(z)}}{\int_0^\infty dz e^{-N\psi_\beta(z)}} = O(N^{-2}).
\]

(4.4)

Therefore, by Lemma 4.14 we find

\[
\frac{\langle H \rangle^{\beta,N}}{N} = O(N^{-1}), \quad \sigma^{\beta,N}_C \left( \frac{H}{N} \right) = O(N^{-1}),
\]

as desired. If \( \beta = \frac{1}{J} \), repeating the analysis implies that all three bounds in (4.2)–(4.4) hold if on the right hand side the power of \( N \) is divided by 2. Hence, the same holds for the conclusion and yields the stated bounds.

For \( \beta > \frac{1}{J} \), we need to be more explicit with the leading term. Recall that by Lemma 4.13 for every \( \beta > \frac{1}{J} \) there exists an \( m \in (0,1) \) such that \( \beta = \frac{\tanh^{-1}(\sqrt{m})}{\sqrt{J}} \) and \( z_0 = \frac{\tanh^{-1}(\sqrt{m})}{\sqrt{J}} \) is the unique minimizer of \( \psi_\beta \). For this minimizing \( z_0 \), we have

\[
\partial_\beta \psi_\beta(z_0) = \frac{\sqrt{J} \tanh^{-1}(\sqrt{m})}{2\sqrt{J}} \tanh \left( \frac{\sqrt{J} \tanh^{-1}(\sqrt{m})}{\sqrt{J}} \right) = \frac{J}{2}. \tag{4.11}
\]

Whenever possible, we will use \( m \) instead of \( \beta \) if it is pertinent to the computation at hand. By Lemma 4.13 we have

\[
\psi_\beta(z_0) < 0, \quad \psi_\beta'(z_0) = 0, \quad \psi_\beta''(z_0) > 0,
\]

\[
\partial_\beta \psi_\beta(z_0) - \left( \frac{J}{2} \right) = 0, \quad \partial_\beta \psi_\beta'(z_0) \neq 0,
\]

\[
\left( \partial_\beta \psi_\beta(z_0) - \left( \frac{J}{2} \right) \right)^2 = 0, \quad 2 \partial_\beta \psi_\beta'(z_0) \left( \partial_\beta \psi_\beta(z_0) - \left( \frac{J}{2} \right) \right) = 0, \quad 2(\partial_\beta \psi_\beta'(z_0))^2 \neq 0.
\]

Note that

\[
\int_0^\infty dz \left( \partial_\beta \psi_\beta(z) \right)^2 e^{-N\psi_\beta(z)} - \left( \int_0^\infty dz \partial_\beta \psi_\beta(z)e^{-N\psi_\beta(z)} \right)^2 = \int_0^\infty dz \left( \partial_\beta \psi_\beta(z) - \left( \frac{J}{2} \right) \right)^2 e^{-N\psi_\beta(z)} - \left( \int_0^\infty dz \left( \partial_\beta \psi_\beta(z) - \left( \frac{J}{2} \right) \right) e^{-N\psi_\beta(z)} \right)^2.
\]

It follows from Lemma 4.13 that

\[
\int_0^\infty dz \frac{(\partial_\beta \psi_\beta(z))^2e^{-N\psi_\beta(z)}}{\int_0^\infty dz e^{-N\psi_\beta(z)}} - \left( \int_0^\infty dz \frac{\partial_\beta \psi_\beta(z)e^{-N\psi_\beta(z)}}{\int_0^\infty dz e^{-N\psi_\beta(z)}} \right)^2 = O(N^{-1}),
\]

\[
\int_0^\infty dz \frac{\partial_\beta^2 \psi_\beta(z)e^{-N\psi_\beta(z)}}{\int_0^\infty dz e^{-N\psi_\beta(z)}} = O(1).
\]

Collecting the terms, we see that

\[
\frac{\langle H \rangle^{\beta,N}}{N} = -\frac{J}{2}m + O(N^{-\frac{1}{2}}), \quad \sigma^{\beta,N}_C \left( \frac{H}{N} \right) = O(N^{-\frac{1}{2}}),
\]

as desired. \( \square \)
4.3 Convergence of finite marginal distributions and finite moments

The main theorems formulated in the coupling section concern 1-Lipschitz functions with respect to some norm \( ||\cdot||_p \). Since the domain set is finite, all functions \( f : \{-1, 1\}^I \to \mathbb{R} \) are automatically Lipschitz functions with respect to all of these norms. However, being a 1-Lipschitz function, i.e., being a function for which its optimal Lipschitz constant \( K \), defined by

\[
K := \max_{\phi, \psi \in \{-1, 1\}^I, \phi \neq \psi} \frac{|f(\phi) - f(\psi)|}{\|\phi - \psi\|_p},
\]
satisfies \( K \leq 1 \), is a property which depends on the choice of norm, and restricts the class of allowed functions. Naturally, if \( f \) is a function with \( K > 1 \), then we can apply the results below to the 1-Lipschitz function \( \frac{1}{K} f \), and the conclusions for the original function \( f \) will be the same, as long as the constant \( K \) remains bounded in \( N \). In particular, the asymptotic bounds are valid for all fields \( f \) defined on a fixed set \( I \). The choice of using \( p = 1 \) norm below is partially a matter of convenience, due to equivalence of the finite set \( p \)-n norms, but one should be careful in the application of the result if the size of the set \( I \) is allowed to become unbounded as \( N \to \infty \).

We can now state the full convergence theorems.

**Theorem 4.16.** Let \( m \in \text{Ran}[\varepsilon] \) and \( \mu = \tanh^{-1}(-m) \). Let \( I \subset \Lambda \) be a finite index set, and let \( f : \{-1, 1\}^I \to \mathbb{R} \) be a bounded 1-Lipschitz function with respect to the \( ||\cdot||_1 \) norm. It follows that

\[
\langle f \circ P_I \rangle_{MC}^m = \langle f \circ P_I \rangle_{C}^m + O(N^{-\frac{\beta}{2}}).
\]

*Proof.* The result follows by applying the free energy method presented in Theorem 3.10 along with the \( w \)-fluctuation distance bound presented in Theorem 4.13 and the equations in Lemma 4.12. \( \square \)

**Theorem 4.17.** Let \( h \neq 0 \) and let \( \varepsilon \in \text{Ran}[\varepsilon] \). Let \( m \) be as in Proposition 4.6 and let \( \mu \) be as in Theorem 4.10. Let \( I \subset \Lambda \) be a finite index set, and let \( f : \{-1, 1\}^I \to \mathbb{R} \) be a bounded 1-Lipschitz function with respect to the \( ||\cdot||_1 \) norm. Then,

\[
\langle f \circ P_I \rangle_{MC}^\varepsilon = \langle f \circ P_I \rangle_{C}^{\varepsilon} + O(N^{-\frac{\beta}{2}}).
\]

*Proof.* By Proposition 4.6 it follows that

\[
\langle f \circ P_I \rangle_{MC}^\varepsilon = \langle f \circ P_I \rangle_{C}^{m} + O(e^{-cN}),
\]

and, by Theorem 4.10 it follows that

\[
\langle f \circ P_I \rangle_{MC}^m = \langle f \circ P_I \rangle_{C}^{m} + O(N^{-\frac{\beta}{2}}).
\]

Combining these and considering the slower rate of convergence, the result follows. \( \square \)

**Theorem 4.18.** Let \( h = 0 \) and let \( \varepsilon \in \text{Ran}[\varepsilon] \). Let \( I \subset \Lambda \) be a finite index set, and let \( f : \{-1, 1\}^I \to \mathbb{R} \) be a bounded 1-Lipschitz function with respect to the \( ||\cdot||_1 \) norm.

If \( \varepsilon \neq 0 \), let \( \beta = \frac{\tanh^{-1}(\sqrt{-\frac{\mu}{\varepsilon}})}{\sqrt{-\frac{\mu}{\varepsilon}}} > \frac{1}{2} \). It follows that

\[
\langle f \circ P_I \rangle_{MC}^\varepsilon = \langle f \circ P_I \rangle_{C}^{\beta} + O(N^{-\frac{\beta}{2}}).
\]

If \( \varepsilon = 0 \), it follows that for any \( \beta < \frac{1}{2} \), we have

\[
\langle f \circ P_I \rangle_{MC}^{0} = \langle f \circ P_I \rangle_{C}^{\beta} + O(N^{-\frac{\beta}{2}}),
\]

and, for \( \beta = \frac{1}{2} \), we have

\[
\langle f \circ P_I \rangle_{MC}^{0} = \langle f \circ P_I \rangle_{C}^{\beta} + O(N^{-\frac{\beta}{2}}).
\]

\[\text{25}\]
Proof. If $\varepsilon \neq 0$, then the result follows by applying the free energy method presented in Theorem 3.5 along with the $w_1$ fluctuation distance bound presented in Theorem 4.9 and the asymptotics presented in Lemma 4.14.

If $\varepsilon = 0$, then observe that the $w_1$ bound in Theorem 4.9 is not Lipschitz in the appropriate sense to directly apply Theorem 3.5. However, following the proof of Theorem 3.5, we can apply the following inequality

$$\langle w_1 \left( \mu_{MC}^0; \mu_{MC}^\varepsilon : N \right) \rangle_C \leq \sqrt{\frac{2}{J}} \left( \sigma \left( \frac{H}{N} \right)_C \right)^{1/2},$$

where the upper-index $\frac{H}{N}$ is random variable with respect to the canonical ensemble. It follows that

$$\left| \langle f \circ P_I \rangle_{MC}^{0,N} - \langle f \circ P_I \rangle_{C}^{\varepsilon,N} \right| \leq C \left( \sigma \left( \frac{H}{N} \right)_C \right)^{1/2},$$

for a global constant $C > 0$, and, by considering the asymptotics presented in Lemma 4.14, for $\beta < \frac{1}{J}$, it follows that

$$\langle f \circ P_I \rangle_{MC}^{0,N} = \langle f \circ P_I \rangle_{C}^{\varepsilon,N} + O(N^{-\frac{1}{2}}),$$

and, for $\beta = \frac{1}{J}$, we find

$$\langle f \circ P_I \rangle_{MC}^{0,N} = \langle f \circ P_I \rangle_{C}^{\varepsilon,N} + O(N^{-\frac{1}{4}}).$$

4.4 Remark on choice of cost function

For the microcanonical ensemble with fixed magnetisation density, the $w_1$ choice of cost function is natural since the $|| \cdot ||_1$-norm satisfies

$$|M[\phi] - M[\phi']| \leq ||\phi - \phi'||_1.$$

However, for the fixed energy density case with $h = 0$, one might also consider the following specific fluctuation distance

$$\inf_{\gamma} \frac{1}{N^2} \int_S (d\phi, d\phi') \frac{J}{2} \sum_{x,y \in \Lambda} |\phi(x)\phi(y) - \phi'(x)\phi'(y)|.$$

Indeed, if we denote the above by $w_H(\cdot, \cdot; N)$, then, we have

$$w_H(\mu_{MC}^{\varepsilon,N}, \mu_{MC}^{\varepsilon',N}; N) \geq |\varepsilon - \varepsilon'|.$$

However, we also have

$$\frac{J}{2} \sum_{x,y \in \Lambda} |\phi(x)\phi(y) - \phi'(x)\phi'(y)| \leq \frac{J}{2} \sum_{x,y \in \Lambda} (|\phi(x) - \phi'(x)||\phi(y)| + |\phi(y) - \phi'(y)||\phi(x)|)$$

$$= N J \sum_{x \in \Lambda} |\phi(x) - \phi'(x)|.$$

This implies that

$$w_H(\mu_{MC}^{\varepsilon,N}, \mu_{MC}^{\varepsilon',N}; N) \leq J w_1(\mu_{MC}^{\varepsilon,N}, \mu_{MC}^{\varepsilon',N}; N).$$

Although this cost function might be more natural to consider, the computations regarding it are more difficult, and, as such we opted for the $w_1$ distance instead. There is perhaps a type of optimisation one can do with the help of the cost functions, but we have not explored such optimisations in this paper.
5 Continuum Curie–Weiss model

In the second model considered here, the “spin-field” $\phi$ is allowed to take all real values otherwise being similar to the earlier discrete Curie–Weiss model.

**Definition 5.1** (Continuum Curie–Weiss Hamiltonian). Let $\Lambda \subset \mathbb{Z}^d$ be a $d$-dimensional finite square lattice with size $N := |\Lambda|$. We define the space of field configurations $\mathcal{S} := \mathbb{R}^\Lambda$. Let $J > 0$ and $h \in \mathbb{R}$. Define the Hamiltonian $H : \mathcal{S} \to \mathbb{R}$ and the particle number $N : \mathcal{S} \to \mathbb{R}$ by

$$H[\phi] := -\frac{J}{2N} \sum_{x,y \in \Lambda} \phi(x)\phi(y) - h \sum_{x \in \Lambda} \phi(x), \quad N[\phi] := \sum_{x \in \Lambda} \phi(x)^2.$$ 

We also define the magnetization $M : \mathcal{S} \to \mathbb{R}$ by

$$M[\phi] := \sum_{x \in \Lambda} \phi(x),$$

and we note that the Hamiltonian can be written in terms of the magnetization,

$$H[\phi] = -\frac{J}{2N} M[\phi]^2 - hM[\phi] = -\frac{J}{2N} \left( M[\phi] + \frac{hN}{J} \right)^2 + \frac{h^2N^2}{2J}.$$ 

Furthermore, we define the energy density $\varepsilon : \mathcal{S} \to \mathbb{R}$ and magnetization density $m : \mathcal{S} \to \mathbb{R}$ by $\varepsilon[\phi] := \frac{H[\phi]}{N}$ and $m[\phi] := \frac{M[\phi]}{N}$. 

In this model, the particle number function $N$ is much more relevant than in the discrete case. For this Hamiltonian, we will need to consider probability measures described by products of delta functions. To properly resolve them, we begin with an observation concerning a matrix relevant to the definitions of the ensembles. In the following, we employ the notation $M_N(\mathbb{R})$ for the collection of real $N \times N$ matrices.

**Lemma 5.2.** Define $M \in M_N(\mathbb{R})$ by $M_{ij} := 1$ for all $i \in [N]$ and $j \in [N]$. There exists an orthogonal matrix $U \in M_N(\mathbb{R})$ which diagonalizes $M$ such that for $x \in \mathbb{R}^N$, we have

$$(Ux)_1 = \frac{1}{\sqrt{N}} \sum_{i=1}^N x_i.$$ 

**Proof.** A simple analysis shows that $M$ has an eigenvalue $N$ with no degeneracy, and an eigenvalue 0 with $N - 1$ fold degeneracy. Collecting the eigenvalues into a diagonal matrix results in $D \in M_N(\mathbb{R})$ for which $D_{11} := N$ and $D_{ij} := 0$ for all other $i \in [N]$ and $j \in [N]$. Since $M$ is a real symmetric matrix, then there exists an orthogonal matrix $Q \in M_N(\mathbb{R})$ such that $Q^T D Q = M$.

Writing out the above matrix multiplication componentwise explicitly, we find for all $i, j$

$$NQ_{i1}Q_{1j} = 1.$$ 

In particular, then $|Q_{i1}| = \frac{1}{\sqrt{N}}$ for all $i \in [N]$, and thus for each $i$ there is $\sigma_i \in \{\pm 1\}$ such that $Q_{i1} = \sigma_i \frac{1}{\sqrt{N}}$. Using a proof by contradiction, one can see that, in fact, the elements $Q_{i1}$ must either all be negative or all be positive. Now, define $U \in M_N(\mathbb{R})$ by $U := -Q$ if the elements $Q_{i1}$ are all negative, and $U := Q$ if the elements $Q_{i1}$ are all positive. It follows that $U$ is an orthogonal matrix and, by definition, we have

$$(Ux)_1 := \sum_{j=1}^N U_{1j}x_j = \frac{1}{\sqrt{N}} \sum_{i=1}^N x_i.$$ 

This completes the proof of the Lemma. 

[]
Next, we will give two examples of how to apply δ-function calculation rules to resolve the ones relevant to the Curie–Weiss system, for both microcanonical and canonical ensembles. It is possible to prove the validity of these manipulations under the assumptions made in the Examples, for instance, following the discussion in Appendix A of [7].

**Example 5.3.** Let \( U : S \to \mathbb{R} \times \mathbb{R}^{N-1} \) be an orthogonal matrix satisfying Lemma 5.2. Suppose \( \rho > 0 \) and \( m \in (-\sqrt{\rho}, \sqrt{\rho}) \). For bounded 1-Lipschitz functions \( f : S \to \mathbb{R} \), we have

\[
\int_S d\phi \, \delta \left( \sum_{x \in \Lambda} \phi(x) - mN \right) \delta \left( \sum_{x \in \Lambda} \phi(x)^2 - \rho N \right) f(\phi) = \int_\mathbb{R} dz \int_{\mathbb{R}^{N-1}} d\psi \, \delta(\sqrt{N} z - mN) \delta(z^2 + ||\psi||^2 - \rho N) (f \circ U^{-1})(z, \psi)
\]

\[
= \frac{(N(\rho - m^2))^{\frac{N-3}{2}}}{2\sqrt{N}} \int_{\mathbb{R}^{N-2}} d\Omega \, (f \circ U^{-1})(m\sqrt{N}, \sqrt{N}(\rho - m^2)\Omega).
\]

Here we have first made a change of variables to \((z, \psi) = U\phi\) and then used spherical coordinates system to integrate out the resulting δ-functions. Since the left hand side does not depend on the choice of the matrix \( U \), all choices must result in the same value for the integral on the right hand side.

**Example 5.4.** Let \( U : S \to \mathbb{R} \times \mathbb{R}^{N-1} \) be an orthogonal matrix satisfying Lemma 5.2. Fix \( h \in \mathbb{R} \) and \( \rho > 0 \), and suppose \( \varepsilon \in \mathbb{R} \) satisfies \( \varepsilon < \frac{h^2}{\rho} \). Define then

\[
m_- := -h + \sqrt{\frac{h^2}{\rho} - \frac{2\varepsilon}{\rho}}, \quad m_+ := -h - \sqrt{\frac{h^2}{\rho} - \frac{2\varepsilon}{\rho}}.
\]

Then, \( m_-, m_+ \) are distinct real numbers, and we assume furthermore that \( m_+^2 < \rho \) and \( m_-^2 < \rho \).

Then for bounded 1-Lipschitz functions \( f : S \to \mathbb{R} \), we may proceed as in the previous Example to conclude that

\[
\int_S d\phi \, \delta \left( \sum_{x,y \in \Lambda} \phi(x)\phi(y) - h \sum_{x \in \Lambda} \phi(x) - \varepsilon N \right) \delta \left( \sum_{x \in \Lambda} \phi(x)^2 - \rho N \right) f(\phi)
\]

\[
= \int_\mathbb{R} dz \int_{\mathbb{R}^{N-1}} d\psi \, \delta \left( -\frac{J}{2} z^2 - h\sqrt{N} z - \varepsilon N \right) \delta(z^2 + ||\psi||^2 - \rho N) (f \circ U^{-1})(z, \psi)
\]

\[
= \frac{1}{J \sqrt{\frac{h^2}{\rho} - \frac{2\varepsilon}{\rho}}} \int_\mathbb{R} dz \int_{\mathbb{R}^{N-1}} d\psi \, \delta \left( z - m_+\sqrt{N} \right) \delta(z^2 + ||\psi||^2 - \rho N) (f \circ U^{-1})(z, \psi)
\]

\[
+ \frac{1}{J \sqrt{\frac{h^2}{\rho} - \frac{2\varepsilon}{\rho}}} \int_\mathbb{R} dz \int_{\mathbb{R}^{N-1}} d\psi \, \delta \left( z - m_-\sqrt{N} \right) \delta(z^2 + ||\psi||^2 - \rho N) (f \circ U^{-1})(z, \psi)
\]

\[
= \frac{1}{J \sqrt{\frac{h^2}{\rho} - \frac{2\varepsilon}{\rho}}} \frac{(N(\rho - m_+^2))^{\frac{N-3}{2}}}{2\sqrt{N}} \int_{\mathbb{R}^{N-2}} d\Omega \, (f \circ U^{-1})(m_+\sqrt{N}, \sqrt{N}(\rho - m_+^2)\Omega)
\]

\[
+ \frac{1}{J \sqrt{\frac{h^2}{\rho} - \frac{2\varepsilon}{\rho}}} \frac{(N(\rho - m_-^2))^{\frac{N-3}{2}}}{2\sqrt{N}} \int_{\mathbb{R}^{N-2}} d\Omega \, (f \circ U^{-1})(m_-\sqrt{N}, \sqrt{N}(\rho - m_-^2)\Omega).
\]

We will utilize these forms for more explicit definitions of the ensembles and in the proof concerning the boundedness of moments of the microcanonical ensembles.

**5.1 Microcanonical ensembles**

From here on, whenever the mapping \( U \) is present, we are always referring to the mapping \( U \) defined by a matrix satisfying Lemma 5.2. We fix the choice of this matrix in the following. Analogously
with the discrete case, we begin with definitions of the two microcanonical ensembles, related to fixed magnetization and to fixed energy.

**Definition 5.5** (Fixed magnetization density and particle density/microcanonical ensemble). Let \( \rho > 0 \) and \( m \in (-\sqrt{\rho}, \sqrt{\rho}) \). The microcanonical ensemble with particle density \( \rho \) and magnetization density \( m \) is defined via its action on bounded 1-Lipschitz functions \( f : S \to \mathbb{R} \) by

\[
(f)_{MC}^{m, \rho; N} := \frac{1}{|S^{N-1}|} \int_{S^{N-2}} d\Omega \left( f \circ U^{-1} \right)(m\sqrt{N}, \sqrt{N}(\rho - m^2))\Omega).
\]

Furthermore, we define the specific microcanonical partition function by

\[
Z_{MC}(m, \rho; N):= \rho^m \frac{Z_{MC}(m_+, \rho; N)}{Z_{MC}(m_-, \rho; N)} \frac{(f)_{MC}^{m_+, \rho; N}}{(f)_{MC}^{m_-, \rho; N}}.
\]

and the specific microcanonical entropy by

\[
s_{MC}(m, \rho) := \frac{1}{2} \ln(\rho - m^2).
\]

**Definition 5.6** (Fixed energy density and particle density/microcanonical ensemble). Let \( \rho > 0 \), and \( \varepsilon, m_+, \) and \( m_- \) be as in Example 5.4, in particular, assume \( \varepsilon < \frac{\mu}{2} \) and \( m_-, m_+ < \rho \). The microcanonical ensemble with energy density \( \varepsilon \) and particle density \( \rho > 0 \) is then defined via its action on bounded 1-Lipschitz functions \( f : S \to \mathbb{R} \) by

\[
(f)_{MC}^{\varepsilon, \rho; N} := \frac{Z_{MC}(m_+, \rho; N)}{Z_{MC}(m_-, \rho; N)} \frac{Z_{MC}(m_+, \rho; N)}{Z_{MC}(m_-, \rho; N)} \frac{(f)_{MC}^{m_+, \rho; N}}{(f)_{MC}^{m_-, \rho; N}}.
\]

If \( \varepsilon < \frac{\mu}{2} \) but \( \min(m_-, m_+) < \rho \leq \max(m_-, m_+) \), we set \( (f)_{MC}^{\varepsilon, \rho; N} := (f)_{MC}^{m, \rho; N} \) with \( m := m_+ \)

if \( |m_+| < |m_-| \), and \( m := m_- \), otherwise.

If \( \varepsilon = \frac{\mu}{2} \) and \( \rho > \frac{\mu}{2} \), we set \( (f)_{MC}^{\varepsilon, \rho; N} := (f)_{MC}^{m_-, \rho; N} \) with \( m := -\frac{\mu}{2} \).

One can indeed verify by using the calculations in Examples 5.3 and 5.4 that these measures correspond to \( \delta \)-function definitions, resolved in the manner used in the Examples. The second definition serves as an explanation of the choice of multiplicative constants in the definition of the microcanonical partition function and specific entropy. In the degenerate case \( \varepsilon = \frac{\mu}{2} \), we would have above \( m_+ = m_- = -\frac{\mu}{2} = m \), and in this case the \( \delta \)-function definition in Example 5.4 does not really make sense since it would contain a singular term \( \delta \left((z - m\sqrt{N})^2\right) \). Instead, in analogy with the discrete case, we define in this case the fixed energy microcanonical ensemble via the corresponding fixed magnetization ensemble.

Note that in addition to values of \( (\varepsilon, \rho) \) for which there are no solutions to the constraints, we have also left undefined the degenerate energy ensembles for which \( \varepsilon \leq \frac{\mu}{2} \) but \( \min(m_-, m_+) = \rho \), as well as the degenerate magnetization ensembles with \( m^2 = \rho \). In these cases, the dimensionality of the solution manifold does not increase with \( N \), and all solutions have \( \psi = 0 \). As such, the resulting degenerate ensemble does not have standard thermodynamic behaviour.

We begin by estimating the fluctuation distance of two fixed magnetization ensembles by constructing a suitable transport map between them.

**Theorem 5.7.** Let \( \rho > 0 \) and let \( m, m' \in (-\sqrt{\rho}, \sqrt{\rho}) \). We have

\[
w_{2}(\mu_{MC}^{m, \rho; N}, \mu_{MC}^{m', \rho; N}; N) \leq \left( 1 + \frac{2}{\sqrt{1 - m^2 / \rho}} \right) |m - m'|.
\]
In order for this mapping to act on the correct coordinate space, we define
\[ T(z, \psi) := \left( m' \sqrt{N}, \sqrt{N(\rho - m'^2)} \right) \].

Note that then for any \( \Omega \in \mathbb{S}^{N-2} \) we have
\[ T(m \sqrt{N}, \sqrt{N(\rho - m^2)} \Omega) = (m' \sqrt{N}, \sqrt{N(\rho - (m')^2)} \Omega). \]
In order for this mapping to act on the correct coordinate space, we define \( T' : \mathbb{R}^N \rightarrow \mathbb{R}^N \) by
\[ T' := U^{-1} \circ T \circ U. \]
Then for any observable \( f \) we obtain directly from the definitions a relation
\[ \langle f \circ T' \rangle_{\mu_{MC}'} = \langle f \rangle_{\mu_{MC}'.} \]

Therefore, \( T' \) is a transport map from the measure \( \mu_{MC}' \) to \( \mu_{MC}'. \) Let \( \gamma \) denote the associated coupling as defined in Sect. 2.1. This yields an estimate
\[ w_2(\mu_{MC}', \mu_{MC}' ; N)^2 \leq \int \gamma(d\phi, d\psi) \frac{1}{N} \| \phi - \psi \|_2^2 = \int \mu_{MC}'(d\phi) \frac{1}{N} \| \phi - T'\phi \|_2^2 \]
\[ = \frac{1}{N} \int_{\mathbb{S}^{N-2}} d\Omega \left\| U^{-1}(m \sqrt{N}, \sqrt{N(\rho - m^2)} \Omega) - U^{-1}(m' \sqrt{N}, \sqrt{N(\rho - m'^2)} \Omega) \right\|^2 \]
\[ = \frac{1}{N} \int_{\mathbb{S}^{N-2}} d\Omega \left\| (m \sqrt{N}, \sqrt{N(\rho - m^2)} \Omega) - (m' \sqrt{N}, \sqrt{N(\rho - m'^2)} \Omega) \right\|^2 \]
\[ = (m - m')^2 + \left( \sqrt{\rho - m^2} - \sqrt{\rho - m'^2} \right)^2 \leq \left( 1 + \frac{4\rho}{\rho - m^2} \right) \frac{1}{m^2} (m - m')^2. \]
Since \( 1 + x^2 \leq (1 + x)^2 \) for \( x \geq 0 \), we obtain the stated bound after taking a square root. \( \Box \)

To study the fixed energy ensembles, we begin with a Lemma which implies that, as in the discrete case, for \( h \neq 0 \), one of the fixed magnetization measures dominates in the fixed energy ensemble.

Lemma 5.8. If \( h \neq 0 \), then
\[ \max\{Z_{MC}(m_+, \rho; N), Z_{MC}(m_-, \rho; N)\} = \rho \left( \frac{|h|}{J} - \sqrt{\frac{h^2}{J^2} - \frac{2\varepsilon}{J}} \right)^{\frac{2}{2 - \delta}}. \]

Proof. If we consider the mapping \( m \mapsto Z_{MC}(m, \rho; N) \), then it is clear that \( Z_{MC}(m, \rho; N) \geq Z_{MC}(m', \rho; N) \) for all \( |m| \leq |m'| \) \( < \sqrt{\rho} \). Now, note that
\[ m^2_\pm = \frac{h^2}{J^2} + \frac{2\varepsilon}{J} \pm \frac{h}{J} \sqrt{\frac{h^2}{J^2} - \frac{2\varepsilon}{J}}. \]
If \( h > 0 \), then \( m^2_+ > m^2_+ \implies |m_-| > |m_+| \), and, if \( h < 0 \), then \( m^2_- < m^2_- \implies |m_+| > |m_-| \).

We remark that for all of the above \( h, \rho \) the set \( \mathcal{E}_{h, \rho} \) contains \( \varepsilon = 0 \) and an interval of negative values of \( \varepsilon \). In particular, \( \mathcal{E}_{h, \rho} \) is non-empty. Also, in case \( h = 0 \), we have \( \mathcal{E}_{0, \rho} = \left( -\frac{h^2}{2\varepsilon}, 0 \right] \).

Definition 5.9. For \( h \in \mathbb{R} \) and \( \rho > 0 \), we define the set of possible energy densities \( \mathcal{E}_{h, \rho} \) by
\[ \mathcal{E}_{h, \rho} := \left\{ \varepsilon \in \mathbb{R} : \varepsilon \leq \frac{h^2}{2\varepsilon}, \left| \frac{|h|}{J} - \sqrt{\frac{h^2}{J^2} - \frac{2\varepsilon}{J}} \right| < \sqrt{\rho} \right\}. \]
Theorem 5.10. Let $h \neq 0$, $\rho > 0$, and suppose $\varepsilon \in \mathcal{E}_{h,\rho}$. For integrable functions $f : S \to \mathbb{R}$ satisfying $|\langle f \rangle_{\text{MC}}|^{\max} \leq K$ for some $K \geq 0$ and whenever the ensemble is defined, we have

$$\left| \langle f \rangle_{\text{MC}}^{\varepsilon,\rho,N} - \langle f \rangle_{\text{MC}}^{m,\rho,N} \right| \leq 2K \frac{\rho - \left( |h| + \sqrt{|h|^2 - 2\varepsilon} \right)^2}{\rho - \left( |h| - \sqrt{|h|^2 - 2\varepsilon} \right)^2}, \quad (5.1)$$

where

$$m = -\frac{h}{J} + \text{sgn}(h)\sqrt{\frac{h^2}{J^2} - \frac{2\varepsilon}{J}}.$$

In addition,

$$\langle f \rangle_{\text{MC}}^{\varepsilon,\rho,N} = \langle f \rangle_{\text{MC}}^{m,\rho,N} + O(e^{-cN})$$

for some positive constant $c > 0$.

Proof. If $\varepsilon = \frac{h^2}{2J}$, we have here $m = -\frac{h}{J}$, and the Theorem is trivially true since then $\langle f \rangle_{\text{MC}}^{\varepsilon,\rho,N} = \langle f \rangle_{\text{MC}}^{m,\rho,N}$. The same holds for those values where $\varepsilon < \frac{h^2}{2J}$ and $\rho \leq \max(m_-^2, m_+^2)$.

In the remaining cases, we necessarily have $\varepsilon < \frac{h^2}{2J}$ and $m_-^2, m_+^2 < \rho$. Thus we may repeat the computation made in the discrete case, yielding an estimate

$$\left| \langle f \rangle_{\text{MC}}^{\varepsilon,\rho,N} - \langle f \rangle_{\text{MC}}^{m,\rho,N} \right| \leq \left( \langle f \rangle_{\text{MC}}^{m_-,\rho,N} + \langle f \rangle_{\text{MC}}^{m_+,\rho,N} \right) Z_{\text{MC}}(m_-\text{sgn}(h), \rho; N) Z_{\text{MC}}(m_+\text{sgn}(h), \rho; N).$$

The results follow since the term inside the absolute values on the right hand side in (5.1) is strictly less than one.

If $h = 0$, just like the discrete Curie–Weiss model, there exists a suitable coupling which can be constructed from the couplings used for the fixed magnetization ensembles.

Theorem 5.11. Let $h = 0$ and $\varepsilon', \varepsilon \in \left(-\frac{\rho J}{2}, 0\right]$. For $\varepsilon \neq 0$, we have

$$w_2(\mu_{\text{MC}}^{\varepsilon,\rho,N}, \mu_{\text{MC}}^{\varepsilon',\rho,N}; N) \leq \frac{2}{J} \frac{1}{\sqrt{2\varepsilon}} \sqrt{1 - \left( -\frac{2\varepsilon}{\rho} \right)},$$

and, for $\varepsilon = 0$, we have

$$w_2(\mu_{\text{MC}}^{0,\rho,N}, \mu_{\text{MC}}^{\varepsilon',\rho,N}; N) \leq \frac{2}{\sqrt{J}} |\varepsilon'|^{\frac{1}{2}}.$$

Proof. Now $m_{\pm} = \pm \sqrt{\frac{-2\varepsilon}{J}}$, and thus if $\varepsilon \neq 0$, we have $0 < |m_{\pm}| < \rho$. Therefore, these ensembles are defined as in the first case in Definition 5.8.

Suppose first that $\varepsilon, \varepsilon' < 0$ and let $m_{\pm}$ and $m'_{\pm}$ be corresponding positive and negative magnetization densities to $\varepsilon$ and $\varepsilon'$, respectively. In analogy with the discrete case, we define $T : \mathbb{R} \times \mathbb{R}^{N-1} \to \mathbb{R} \times \mathbb{R}^{N-1}$ by

$$T(z, \psi) := \mathbb{1}(z \geq 0) \left( m'_+\sqrt{N}, \sqrt{(\rho - m'_+^2)} N \frac{\psi}{||\psi||_2} \right) + \mathbb{1}(z < 0) \left( m'_-\sqrt{N}, \sqrt{(\rho - m'_-^2)} N \frac{\psi}{||\psi||_2} \right).$$
Now, let $U \in M_N(\mathbb{R})$ be the same unitary mapping as before. By setting $T' := U^{-1} \circ T \circ U$ and going through the same calculations as earlier, one can confirm that $(f \circ T')_{\MC}^{N, \rho} = (f')_{\MC}^{N, \rho}$ for all observables $f$. Thus $T'$ is a transport map and the associated coupling yields a bound

$$w_2(\mu_{MC}^{\varepsilon, \rho; N}, \mu_{MC}^{\varepsilon', \rho; N}; N)^2 \leq \frac{1}{2} \sum_{\sigma = \pm 1} \left[ (m_\sigma - m'_\sigma)^2 + \left( \sqrt{\rho} - \sqrt{\rho} - (m'_\sigma)^2 \right)^2 \right]$$

$$= \frac{2}{J} (\sqrt{\varepsilon} - \sqrt{\varepsilon'})^2 + \left( \sqrt{\rho} - \frac{2}{J} \sqrt{\rho} - \frac{2}{J} \varepsilon' \right)^2 \leq \frac{4}{J^2} (\rho - \frac{2}{J} \rho)^2 (\varepsilon - \varepsilon')^2.$$

If $\varepsilon = 0 > \varepsilon'$, we have $m = 0$ but it still holds that $(f)_{\MC}^{\varepsilon, \rho; N} = \frac{1}{2} (f')_{\MC}^{m, \rho; N} + \frac{1}{2} (f')_{\MC}^{m, \rho; N}$. Proceeding as above then yields an estimate

$$w_2(\mu_{MC}^{0, \rho; N}, \mu_{MC}^{\varepsilon, \rho; N}; N)^2 \leq \frac{2}{J} (-\varepsilon') + \left( \sqrt{\rho} - \frac{2}{J} \sqrt{\rho} - \frac{2}{J} \varepsilon' \right)^2 \leq \frac{2}{J} |\varepsilon'| + \frac{4}{J^2} \varepsilon'^2 \leq \frac{4}{J} |\varepsilon'|.$$

The bound is also trivially true if $\varepsilon = \varepsilon' = 0$. Combining the above estimates proves the statement in the Theorem. \qed

### 5.2 Canonical ensembles

We will define the canonical ensembles this time with the help of the microcanonical ensembles.

**Definition 5.12.** (Fluctuating magnetization and fixed particle density/canonical ensemble) Let $\rho > 0$ and $\mu \in \mathbb{R}$. The canonical ensemble with magnetic potential $\mu$ and particle density $\rho$ is defined via its action on bounded 1-Lipschitz functions $f : S \to \mathbb{R}$ by

$$(f)_{C}^{\mu, \rho; N} := \frac{1}{\int_{-\sqrt{\sigma}}^{\sqrt{\sigma}} \rho e^{-\mu N \rho} \lim_{N \to \infty} Z_{MC}(m, \rho; N) (f)_{MC}^{m, \rho; N}} \int_{-\sqrt{\sigma}}^{\sqrt{\sigma}} \rho e^{-\mu N \rho} Z_{MC}(m, \rho; N) (f)_{MC}^{m, \rho; N} d\rho$$

$$= \frac{1}{\int_{-\sqrt{\sigma}}^{\sqrt{\sigma}} \rho e^{-N(\mu - s_{MC}(m, \rho))} (\rho - m^2)^{-\frac{1}{2}}} \int_{-\sqrt{\sigma}}^{\sqrt{\sigma}} \rho e^{-N(\mu - s_{MC}(m, \rho))} (\rho - m^2)^{-\frac{1}{2}} (f)_{MC}^{m, \rho; N}.$$ 

Furthermore, we define the canonical partition function by

$$Z_{C}(\mu, \rho; N) := \int_{-\sqrt{\sigma}}\sqrt{\sigma} \rho e^{-N(\mu - s_{MC}(m, \rho))} (\rho - m^2)^{-\frac{1}{2}},$$

and the specific canonical free energy by

$$f_{C}(\mu, \rho; N) := -\frac{1}{N} \ln Z_{C}(\mu, \rho; N).$$

The case $h \neq 0$ is taken care of by the previous definition. We will refer to the special case of $h = 0$ as the fluctuating energy density ensemble.

**Definition 5.13** (Fluctuating energy density and fixed particle density/canonical ensemble). Let $h = 0$. Suppose $\rho > 0$ and $\beta \in \mathbb{R}$. The canonical ensemble with inverse temperature $\beta$ and particle
Theorem 5.14. Fix $\rho > 0$ and define $\psi_\mu : (-\sqrt{\rho}, \sqrt{\rho}) \to \mathbb{R}$ by

$$\psi_\mu(m, \rho) := \mu m - \frac{1}{2} \ln(\rho - m^2).$$

We have

$$\langle \frac{M}{N} \rangle^\mu = \frac{\int_{-\sqrt{\rho}}^{\sqrt{\rho}} dm \, e^{-N\psi_\mu(m, \rho)}(\rho - m^2)^{-\frac{1}{2}} m}{\int_{-\sqrt{\rho}}^{\sqrt{\rho}} dm \, e^{-N\psi_\mu(m, \rho)}(\rho - m^2)^{-\frac{1}{2}}},$$

and

$$\sigma_G^{\mu, N}(\frac{M}{N}) = \sqrt{\frac{N}{\int_{-\sqrt{\rho}}^{\sqrt{\rho}} dm \, e^{-N\psi_\mu(m, \rho)}(\rho - m^2)^{-\frac{1}{2}} m^2}} \left( \frac{\int_{-\sqrt{\rho}}^{\sqrt{\rho}} dm \, e^{-N\psi_\mu(m, \rho)}(\rho - m^2)^{-\frac{1}{2}}}{\int_{-\sqrt{\rho}}^{\sqrt{\rho}} dm \, e^{-N\psi_\mu(m, \rho)}(\rho - m^2)^{-\frac{1}{2}}} \right)^2.$$
Proof. The first part of the theorem follows directly by differentiating the specific free energies with respect to $\mu$ and dividing by the degrees of freedom $N$ appropriately.

Next, for fixed $\rho > 0$, we compute

$$
\psi_\mu'(m,\rho) = \mu + \frac{m}{\rho - m^2}, \quad \psi_\mu''(m,\rho) = \frac{1}{\rho - m^2} + \frac{2m^2}{(\rho - m^2)^2} = \frac{\rho + m^2}{(\rho - m^2)^2} > 0.
$$

When considering $\psi_\mu$ as a map of the variable $z$, it follows that $\psi_\mu$ is strictly concave for all $\mu \in \mathbb{R}$ and there is thus a unique global minimum at $m \in (-\sqrt{\rho}, \sqrt{\rho})$ which satisfies $\psi_\mu'(m,\rho) = 0$. First, if $\mu = 0$, then clearly the minimizing $m = 0$. If $\mu \neq 0$, we have

$$
\psi_\mu'(m,\rho) = 0 \iff m = \frac{1}{2\mu} \pm \sqrt{\left(\frac{1}{2\mu}\right)^2 + \rho}.
$$

Next, if $\mu > 0$, then

$$
\frac{1}{2\mu} + \sqrt{\left(\frac{1}{2\mu}\right)^2 + \rho} > \sqrt{\rho},
$$

and thus the minimizing $m$ must be

$$
m = \frac{1}{2\mu} - \sqrt{\left(\frac{1}{2\mu}\right)^2 + \rho} \in (-\sqrt{\rho}, 0).
$$

If $\mu < 0$, then

$$
\frac{1}{2\mu} - \sqrt{\left(\frac{1}{2\mu}\right)^2 + \rho} = -\left(\sqrt{\left(\frac{1}{2\mu}\right)^2 + \rho} - \frac{1}{2\mu}\right) < -\sqrt{\rho},
$$

and thus the minimizing $m$ must satisfy

$$
m = \frac{1}{2\mu} + \sqrt{\left(\frac{1}{2\mu}\right)^2 + \rho} \in (0, \sqrt{\rho}).
$$

The conclusion is that, if $|m| < \sqrt{\rho}$, then

$$
\psi_\mu'(m,\rho) = 0 \iff m = \mathbb{1}(\mu \neq 0) \left(\frac{1}{2\mu} - \text{sgn}(\mu) \sqrt{\left(\frac{1}{2\mu}\right)^2 + \rho}\right).
$$

Furthermore, the above relation goes both ways. For every $(\mu,\rho)$ there exists a unique minimizing $m$ for the above equation, and, for every $m \in (-\sqrt{\rho}, \sqrt{\rho})$, there exists $\mu \in \mathbb{R}$ such that the given $m$ is the minimizing term. This can be seen by simply studying the given equation above and considering the limits $|\mu| \to 0$ and $|\mu| \to \infty$ and using the continuity on the open intervals $(-\infty, 0)$ and $(0, \infty)$.

The asymptotics of the average and standard deviation of magnetization density are given by the asymptotics of Laplace type integrals. We have

$$
\left\langle \frac{M}{N} \right\rangle_{C}^{\mu,\rho;N} = \mathbb{1}(\mu \neq 0) \left(\frac{1}{2\mu} - \text{sgn}(\mu) \sqrt{\left(\frac{1}{2\mu}\right)^2 + \rho}\right) + O(N^{-\frac{1}{2}}), \quad \sigma_{C}^{\mu,\rho;N} \left(\frac{M}{N}\right) = O(N^{-\frac{1}{2}}),
$$

as desired.

Next, we present the asymptotics of the $h = 0$ case.

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Theorem 5.15. Let $h = 0$. Fix $\rho > 0$ and define $\psi_\beta: \left(-\frac{\rho J}{2}, 0\right] \to \mathbb{R}$ by

$$\psi_\beta(\varepsilon, \rho) := \beta \varepsilon - \frac{1}{2} \ln \left(\rho + \frac{2\varepsilon}{J}\right).$$

We have

$$\langle H \rangle^{\beta,\rho,N}_{\mathbb{C}} = \frac{\int_0^\infty d\varepsilon e^{-N\psi_\beta(\varepsilon, \rho)} \left(-\frac{\rho}{J}\right)^{\frac{1}{2}} \left(\rho + \frac{2\varepsilon}{J}\right)^{-\frac{3}{2}} \varepsilon}{\int_0^\infty d\varepsilon e^{-N\psi_\beta(\varepsilon, \rho)} \left(-\frac{\rho}{J}\right)^{\frac{1}{2}} \left(\rho + \frac{2\varepsilon}{J}\right)^{-\frac{3}{2}}}.$$

and

$$\sigma^{\beta,\rho,N}_{\mathbb{C}} \left(\frac{H}{N}\right) = \left[\frac{\int_0^\infty d\varepsilon e^{-N\psi_\beta(\varepsilon, \rho)} \left(-\frac{\rho}{J}\right)^{\frac{1}{2}} \left(\rho + \frac{2\varepsilon}{J}\right)^{-\frac{3}{2}} \varepsilon^2}{\int_0^\infty d\varepsilon e^{-N\psi_\beta(\varepsilon, \rho)} \left(-\frac{\rho}{J}\right)^{\frac{1}{2}} \left(\rho + \frac{2\varepsilon}{J}\right)^{-\frac{3}{2}}} - \left(\frac{\int_0^\infty d\varepsilon e^{-N\psi_\beta(\varepsilon, \rho)} \left(-\frac{\rho}{J}\right)^{\frac{1}{2}} \left(\rho + \frac{2\varepsilon}{J}\right)^{-\frac{3}{2}} \varepsilon^2}{\int_0^\infty d\varepsilon e^{-N\psi_\beta(\varepsilon, \rho)} \left(-\frac{\rho}{J}\right)^{\frac{1}{2}} \left(\rho + \frac{2\varepsilon}{J}\right)^{-\frac{3}{2}}}\right)^{\frac{1}{2}}\right].$$

Furthermore, if we fix $\rho$, then for every $\beta \geq \frac{1}{\rho J}$, there exists $\varepsilon \in \left(-\frac{\rho J}{2}, 0\right]$ such that $\psi_\beta$ is minimized for $(\varepsilon, \rho)$, and, for every $\varepsilon \in \left(-\frac{\rho J}{2}, 0\right]$, there exists $\beta \geq \frac{1}{\rho J}$ such that $\psi_\beta$ is minimized at $(\varepsilon, \rho)$. For such $\beta$ and $\varepsilon$, the following asymptotics holds

$$\langle H \rangle^{\beta,\rho,N}_{\mathbb{C}} = -\frac{J\rho}{2} \left(1 - \frac{1}{\beta J}\right) + O(N^{-\frac{1}{2}}), \quad \sigma^{\beta,\rho,N}_{\mathbb{C}} \left(\frac{H}{N}\right) = O(N^{-\frac{1}{2}}). \quad (5.2)$$

If $\beta < \frac{1}{\rho J}$, the mapping $\psi_\beta$ is always minimized at $(0, \rho)$, and the following asymptotics hold

$$\langle H \rangle^{\beta,\rho,N}_{\mathbb{C}} = O(N^{-1}), \quad \sigma^{\beta,\rho,N}_{\mathbb{C}} \left(\frac{H}{N}\right) = O(N^{-1}).$$

Proof. First, we compute

$$\psi'_\beta(\varepsilon, \rho) = \beta - \frac{1}{\rho + \frac{2\varepsilon}{J}}, \quad \psi''_\beta(\varepsilon) = \frac{2}{J^2 \left(\rho + \frac{2\varepsilon}{J}\right)^2}.$$

It follows that $\psi_\beta$ is strictly convex and obtains a unique global minimum when $\psi'_\beta(\varepsilon, \rho) = 0$. Computing it from the above, we see that $\psi'_\beta(\varepsilon, \rho) = 0 \iff \varepsilon = -\frac{J}{2} \left(1 - \frac{1}{\beta J}\right)$. In particular, we see that for every $\varepsilon \in \left(-\frac{J}{2}, 0\right]$ there exists $\beta \geq \frac{1}{\rho J}$ such that the given $\varepsilon$ minimizes $\psi_\beta$, and, conversely, for every $\beta \geq \frac{1}{\rho J}$ there exists a minimizing value $\varepsilon \in \left(-\frac{\rho J}{2}, 0\right]$. Furthermore, if $\beta < \frac{1}{\rho J}$, then $\psi'_\beta$ is strictly negative on the entire interval, and, as a result $\psi_\beta$ is minimized for $\varepsilon = 0$.

For the asymptotics, if $\beta \geq \frac{1}{\rho J}$, then the asymptotics are standard and we have

$$\langle H \rangle^{\beta,\rho,N}_{\mathbb{C}} = -\frac{J\rho}{2} \left(1 - \frac{1}{\beta J}\right) + O(N^{-\frac{1}{2}}), \quad \sigma^{\beta,\rho,N}_{\mathbb{C}} \left(\frac{H}{N}\right) = O(N^{-\frac{1}{2}}).$$

If $\beta = \frac{1}{\rho J}$, then we need to choose half-integer values of “$\alpha$” in the Laplace method, but this will not alter the scaling of the asymptotics for the above ratios. However, if $\beta < \frac{1}{\rho J}$, then $\psi'_\beta(\varepsilon, \rho) < 0$ for all $\varepsilon$, and since then “$\mu = 1$” in the Laplace method, it follows that

$$\langle H \rangle^{\beta,\rho,N}_{\mathbb{C}} = O(N^{-1}), \quad \sigma^{\beta,\rho,N}_{\mathbb{C}} \left(\frac{H}{N}\right) = O(N^{-1}).$$

This completes the proof of the Theorem. ☐
5.3 Grand canonical ensembles

Finally, we will present the grand canonical ensembles and the direct coupling method. If one considers microcanonical to be the most fundamental ensemble, this will result in substantial simplification of computation of its expectation values in the thermodynamic limit since these can now be computed using the grand canonical ensemble which is a Gaussian measure.

Definition 5.16 (Fluctuating magnetization and particle density/grand canonical ensemble). Let \( \mu \in \mathbb{R} \) and \( \eta > 0 \). The grand canonical ensemble with magnetic potential \( \mu \) and chemical potential \( \eta \) is defined via its action on bounded \( 1 \)-Lipschitz functions \( f : \mathcal{S} \to \mathbb{R} \) by

\[
\langle f \rangle_{GC}^{\mu,\eta;N} := \frac{1}{Z_{GC}(\mu,\eta;N)} \int_{\mathbb{R}^N} d\phi \, e^{-\mu M[\phi] - \eta N[\phi]} f(\phi).
\]

The definition may be rewritten using the same parametrization of the integrals as for the microcanonical ensemble. The result is summarized in the following Lemma.

Lemma 5.17. Let \( \mu \in \mathbb{R} \) and \( \eta > 0 \). We have for all bounded \( 1 \)-Lipschitz functions \( f : \mathcal{S} \to \mathbb{R} \)

\[
\langle f \rangle_{GC}^{\mu,\eta;N} = \frac{1}{Z_{GC}(\mu,\eta;N)} \int_{-\infty}^{\infty} dz \, e^{-\eta (z + \frac{\mu \sqrt{\eta}}{N})^2} \int_{0}^{\infty} dr \, r^{N-2} e^{-\eta r^2}
\]

\[
\times \frac{1}{|S^{N-2}|} \int_{S^{N-2}} d\Omega \, (f \circ U^{-1})(z,r\Omega),
\]

where

\[
Z_{GC}(\mu,\eta;N) := \int_{-\infty}^{\infty} dz \, e^{-\eta (z + \frac{\mu \sqrt{\eta}}{N})^2} \int_{0}^{\infty} dr \, r^{N-2} e^{-\eta r^2}.
\]

We can now construct a direct coupling between the microcanonical ensemble and the grand canonical ensemble.

Theorem 5.18. Suppose \( \rho > 0 \), \( m \in (-\sqrt{\rho}, \sqrt{\rho}) \), \( \mu \in \mathbb{R} \) and \( \eta > 0 \) satisfy the relations

\[
m = -\frac{\mu}{2\eta}, \quad \rho = \frac{1}{2\eta} + \frac{\mu^2}{4\eta^2}.
\]

Then,

\[
w_2(\mu_{GC}^{\mu,\eta;N}, \mu_{MC}^{m,\rho;N};N) \leq \frac{1}{\sqrt{\rho - m^2}} \frac{1}{\sqrt{N}},
\]

which implies

\[
w_2(\mu_{GC}^{\mu,\eta;N}, \mu_{MC}^{m,\rho;N};N) = O(N^{-\frac{1}{2}}).
\]

Proof. Define \( T : \mathbb{R} \times \mathbb{R}^{N-1} \to \mathbb{R} \times \mathbb{R}^{N-1} \) by

\[
T(z,\psi) := \left( m\sqrt{N}, \sqrt{N(\rho - m^2)} \frac{\psi}{||\psi||_2} \right),
\]

and set \( T' := U^{-1} \circ T \circ U \). It follows that \( \langle f \circ T_{GC}^{\mu,\eta;N} \rangle = \langle f \rangle_{MC}^{m,\rho;N} \), and thus \( T' \) is a transport
map. Therefore, using the related coupling we find an estimate
\[
w_2(\mu_{GC}^{\mu,\eta,N}, \mu_{MC}^{m,\rho,N}; N)^2
\leq \frac{1}{N} \int_{-\infty}^{\infty} dz \, e^{-\eta(z^2 + \frac{z^2}{2\eta})} \int_{-\infty}^{\infty} dz' \, e^{-\eta(z' + \frac{z'^2}{2\eta})} \left( z - m\sqrt{N} \right)^2
\]
\[
+ \frac{1}{N} \int_{R^{N-1}} d\psi \, e^{-\eta|\psi|^2} \int_{R^{N-1}} d\psi' \, e^{-\eta|\psi'|^2} \left| ||\psi|| - \sqrt{N}(\rho - m^2) \right|^2
\]
\[
= \frac{1}{N} \int_{-\infty}^{\infty} dz \, e^{-\frac{z^2}{4\eta}} \int_{-\infty}^{\infty} dz' \, e^{-\frac{z'^2}{4\eta}} \left( z + \sqrt{N} \right)^2 \left( z' + \frac{z'^2}{2\eta} \right)^2 \left( \frac{z}{\sqrt{2\eta}} - \left( m\sqrt{N} + \frac{\mu\sqrt{N}}{2\eta} \right) \right)^2
\]
\[
+ \frac{1}{N} \int_{R^{N-1}} d\psi \, e^{-\frac{|\psi|^2}{2\eta}} \int_{R^{N-1}} d\psi' \, e^{-\frac{|\psi'|^2}{2\eta}} \left| ||\psi|| - \sqrt{N}(\rho - m^2) \right|^2.
\]

We compute
\[
\frac{\mu\sqrt{N}}{2\eta} = -m\sqrt{N}, \quad \frac{1}{\sqrt{2\eta}} = \sqrt{\rho - m^2} \iff \mu = -\frac{m}{\rho - m^2}, \quad \eta = \frac{1}{2(\rho - m^2)}.
\]
The converse result states that
\[
m = -\frac{\mu}{2\eta}, \quad \rho = \frac{1}{2\eta} + \frac{\mu^2}{4\eta^2}.
\]

It follows that for every pair \((m, \rho)\) for which the microcanonical ensemble exists, there exists a pair \((\mu, \eta)\) such that the grand canonical ensemble exists, and, the converse result holds as well.

For such a pair satisfying the equations given above, we have
\[
\frac{1}{N} \int_{-\infty}^{\infty} dz \, e^{-\frac{z^2}{4\eta}} \int_{-\infty}^{\infty} dz' \, e^{-\frac{z'^2}{4\eta}} \left( \frac{z}{\sqrt{2\eta}} - \left( m\sqrt{N} + \frac{\mu\sqrt{N}}{2\eta} \right) \right)^2 = \frac{1}{N} \frac{1}{2\eta} = \frac{1}{N} \frac{1}{4(\rho - m^2)},
\]
and
\[
\frac{1}{N} \int_{R^{N-1}} d\psi \, e^{-\frac{|\psi|^2}{2\eta}} \int_{R^{N-1}} d\psi' \, e^{-\frac{|\psi'|^2}{2\eta}} \left| ||\psi|| - \sqrt{N}(\rho - m^2) \right|^2
\]
\[
= \frac{1}{N} \frac{1}{2\eta} \int_{R^{N-1}} d\psi \, e^{-\frac{|\psi|^2}{2\eta}} \int_{R^{N-1}} d\psi' \, e^{-\frac{|\psi'|^2}{2\eta}} \left| ||\psi|| - \sqrt{N} \right|^2.
\]
We have \(||\psi||^2 - N = 1 + \sum_{i=1}^{N-1} (\psi_i^2 - 1)\), and thus
\[
(||\psi||^2 - N)^2 = \sum_{i=1}^{N-1} (\psi_i^2 - 1)^2 + \sum_{i\neq j} (\psi_i^2 - 1)(\psi_j^2 - 1) - 2 \sum_{i=1}^{N-1} (\psi_i^2 - 1) + 1.
\]

Therefore,
\[
\left| ||\psi|| - \sqrt{N} \right|^2 = \frac{(||\psi||^2 - N)^2}{(||\psi|| + \sqrt{N})^2}
\]
\[
\leq \frac{1}{N} \left( \sum_{i=1}^{N-1} (\psi_i^2 - 1)^2 + \sum_{i\neq j} (\psi_i^2 - 1)(\psi_j^2 - 1) - 2 \sum_{i=1}^{N-1} (\psi_i^2 - 1) + 1 \right).
\]
It follows that
\[ \frac{1}{\beta} \int_{\mathbb{R}^N} d\psi \ e^{-\beta |\psi|^2} \left| \left| \psi \right| \right|^2 \leq \frac{2N-1}{N}. \]

Combining all the terms, we find
\[ w_2(\mu_{\beta,\mu}^{\nu,\nu}; N)^2 \leq \frac{1}{N} \frac{1}{2\eta} \frac{3N-1}{N} = \frac{1}{4(\rho - m^2)} \frac{3N-1}{N^2} \leq \frac{1}{(\rho - m^2)N}, \]
which implies the bound stated in the Theorem.

If \( h \neq 0 \), the microcanonical energy ensemble is well-approximated by a microcanonical magnetization ensemble whose grand canonical theory we already covered above. For the case of \( h = 0 \), we consider the following grand canonical energy ensembles.

**Definition 5.19** (Fluctuating energy and particle density/grand canonical ensemble). Suppose \( h = 0, \mu > 0 \) and \( \beta < \frac{2m}{\mu} \). The grand canonical ensemble with inverse temperature \( \beta \) and chemical potential \( \mu \) is defined via its action on bounded \( 1 \)-Lipschitz functions \( f : S \to \mathbb{R} \) by
\[
(f)^{\beta,\mu;N} := \frac{1}{Z_{\beta,\mu}(N)} \int_{\mathbb{R}^N} d\phi \ e^{-\beta H[\phi] - \mu N[\phi]} \int_{\mathbb{R}^N} d\phi \ e^{-\beta H[\phi]} f(\phi).
\]

The definition may be rewritten using the same parametrization of the integrals as for the microcanonical ensemble. The result is summarized in the following Lemma.

**Lemma 5.20.** Let \( h = 0 \). Let \( \mu > 0 \) and \( \beta < \frac{2m}{\mu} \). Then
\[
(f)^{\beta,\mu;N} = \frac{1}{Z_{\beta,\mu}(N)} \int_{-\infty}^{\infty} dz \ e^{-(\mu - \frac{\beta z^2}{2})z^2} \int_{0}^{\infty} dr \ r^{N-2} e^{r^2} \int_{[S^{N-2}]} d\Omega \ (f \circ U^{-1})(z, \Omega),
\]
where
\[
Z_{\beta,\mu}(N) := \int_{-\infty}^{\infty} dz \ e^{-(\mu - \frac{\beta z^2}{2})z^2} \int_{0}^{\infty} dr \ r^{N-2} e^{r^2}.
\]

For the fixed energy density ensemble, there is only a single value of energy density for which a direct coupling can be constructed.

**Theorem 5.21.** Suppose \( h = 0 \) and \( \mu > 0 \) are given. Define \( \rho = \frac{1}{2\mu} \). Then, for all \( \beta < \frac{2m}{\mu} \), we have
\[
w_2(\mu_{\beta,\mu}^{\nu,\nu}; N)^2 \leq \frac{1}{N} \frac{1}{2} \left( \mu - \frac{\beta}{2} \right) + \frac{1}{N} \frac{1}{\mu} \left( \frac{\rho}{1 - \rho \beta J} + 2\rho \right),
\]
implying
\[
w_2(\mu_{\beta,\mu}^{\nu,\nu}; N)^2 = O(N^{-\frac{1}{2}}).
\]

**Proof.** Let us begin by considering the more general case with \( h \in \mathbb{R} \) and \( \mu, \rho > 0 \) arbitrary. Let \( m_+ \) and \( m_- \) be the corresponding negative and positive magnetization densities to the given \( \epsilon \). Define \( T : \mathbb{R} \times \mathbb{R}^{N-1} \to \mathbb{R} \times \mathbb{R}^{N-1} \) by
\[
T(\psi, \psi') := \mathbb{1}(z \geq 0) \left( m_+ \sqrt{N} \frac{\psi}{||\psi||_2} + \mathbb{1}(z < 0) \left( m_- \sqrt{N} \frac{\psi}{||\psi||_2} \right) \right).
\]
Define $T' := U^{-1} \circ T \circ U$. It follows that $\langle f \circ T' \rangle_{\text{GC}}^{\beta,\mu,N} = \langle f \rangle_{\text{MC}}^{\varepsilon,\rho,N}$, and $T'$ is a transport map. Using the associated coupling, we find

$$w_2(\mu_{\text{GC}}^{\beta,\mu,N}, \mu_{\text{MC}}^{\varepsilon,\rho,N}; N)^2 \leq \frac{1}{N} \int_0^\infty \frac{1}{\beta} \int_0^\infty dz e^{-\frac{2z}{\beta \sqrt{N}}} \left( z - \sqrt{-\frac{2\varepsilon}{\beta \sqrt{N}}} \right)^2 + \frac{1}{N} \int_{\mathbb{R}^{N-1}} \frac{1}{\beta} \int_{\mathbb{R}^{N-1}} d\psi e^{-\frac{\beta \rho}{2} \|\psi\|^2} \left( \|\psi\| - \sqrt{N \left( \rho - \left( -\frac{2\varepsilon}{J} \right) \right)} \right)^2$$

where $\beta < 0$ and $\varepsilon > 0$. Note that the above holds for all $\beta < 2\mu$ if $\varepsilon = 0$. However, under the assumptions listed in the Theorem, i.e., if $h = 0 = \varepsilon$, $\mu > 0$, $\rho = \frac{1}{2\mu}$, we find via the same computation as above that

$$w_2(\mu_{\text{GC}}^{\beta,\mu,N}, \mu_{\text{MC}}^{\varepsilon,\rho,N}; N)^2 \leq \frac{1}{N} \int_0^\infty \frac{1}{\beta} \int_0^\infty dz e^{-\frac{2z}{\beta \sqrt{N}}} \left( z - \sqrt{-\frac{2\varepsilon}{\beta \sqrt{N}}} \right)^2 + \frac{1}{N} \frac{1}{\beta} \frac{1}{2} = \frac{1}{N} \frac{1}{\beta} \frac{1}{2} = \frac{1}{N} \frac{\rho}{\mu - \beta J + \rho}.$$ 

Note that for $\varepsilon \neq 0$, we have

$$w_2(\mu_{\text{GC}}^{\beta,\mu,N}, \mu_{\text{MC}}^{\varepsilon,\rho,N}; N)^2 \leq \frac{1}{N} \int_0^\infty \frac{1}{\beta} \int_0^\infty dz e^{-\frac{2z}{\beta \sqrt{N}}} \left( z - \sqrt{-\frac{2\varepsilon}{\beta \sqrt{N}}} \right)^2 + \frac{1}{N} \frac{1}{\beta} \frac{1}{2} = \frac{1}{N} \frac{\rho}{\mu - \beta J + \rho}.$$ 

which does not provide any additional convergence for the local expectation error estimates. However, under the assumptions listed in the Theorem, i.e., if $h = 0 = \varepsilon$, $\mu > 0$, $\rho = \frac{1}{2\mu}$, we find via the same computation as above that

$$w_2(\mu_{\text{GC}}^{\beta,\mu,N}, \mu_{\text{MC}}^{\varepsilon,\rho,N}; N)^2 \leq \frac{1}{N} \frac{1}{\beta} \frac{1}{2} = \frac{1}{N} \frac{\rho}{\mu - \beta J + \rho}.$$ 

Note that the above holds for all $\beta < \frac{2\mu}{\rho J}$. □

For the cases $\beta \geq \frac{1}{\rho J}$, we must introduce another class of target measures.

**Definition 5.22.** Let $\mu \geq 0$ and $\eta > 0$. We define an alternate grand canonical ensemble with parameters $\mu$ and $\eta$ via its action on bounded 1-Lipschitz functions $f : S \to \mathbb{R}$ by

$$\langle f \rangle_{\text{AGC}}^{\mu,\eta} := \frac{1}{Z_{\text{AGC}}(\mu, \eta, N)} \int_S d\phi e^{-\eta \|\phi\|^2} \cosh(\mid\mu M[\phi]\mid) f(\phi),$$

where

$$Z_{\text{AGC}}(\mu, \eta, N) = \int_S d\phi e^{-\eta \|\phi\|^2} \cosh(\mid\mu M[\phi]\mid).$$

By direct computation, we also have then

$$\langle f \rangle_{\text{AGC}}^{\mu,\eta} = \frac{1}{2} \langle f \rangle_{\text{GC}}^{\mu,\eta} + \frac{1}{2} \langle f \rangle_{\text{MC}}^{\varepsilon,\rho,N}.$$ 

Comparing this to the definition of the microcanonical ensemble, we find using Lemma 5.18 that if $h = 0$ and $-\frac{\beta J}{2} < \varepsilon < 0$, then, with $\mu = \sqrt{-\frac{2\varepsilon}{\rho J + \mu}}$ and $\eta = \frac{1}{2}\mu \sqrt{2\varepsilon}$,

$$w_2(\mu_{\text{AGC}}^{\mu,\eta}, \mu_{\text{MC}}^{\varepsilon,\rho,N}; N) = O(N^{-\frac{1}{2}}).$$

One should note that there is no direct coupling of this new alternate grand canonical ensemble to the microcanonical ensemble because the probability measures are not disjoint. However, the individual grand canonical ensembles do converge suitably to the fixed magnetization density case, and thus we still have the desired local convergence properties. We also remark that the case $\mu = 0$ corresponds to the regular grand canonical ensemble given by a Gaussian measure with $\beta = 0$. 

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5.4 Convergence of finite marginal distributions and finite moments

In this section we collect the implications of the earlier bounds for the equivalence of ensembles in the continuum model. We find that in the relevant parameter ranges all three ensembles, microcanonical, canonical, and grand canonical, are equivalent both for bounded 1-Lipschitz observables with the exception of the $h = 0$ microcanonical energy ensemble for which one needs to use a convex combination of two grand canonical ones, as given by the alternate grand canonical ensemble. The microcanonical and appropriate grand canonical ensembles are also equivalent in their local moments. The errors in the related expectations are shown to be bounded by $N^{-\frac{1}{2}}$.

The main theorems concerning the convergence of moments required the boundedness of single moments of all degrees. To this end, we will employ the following lemma.

Lemma 5.23. Let $f : \mathbb{R} \to \mathbb{R}$ be integrable with respect to all Gaussian measures. Let $x \in \Lambda$ and define $P_x : \mathcal{S} \to \mathbb{R}$ by $P_x(\phi) = \phi_x$. It follows that for all $\rho > 0$ and $m \in (-\sqrt{\rho}, \sqrt{\rho})$, we have

$$\langle f \circ P_x \rangle_{MC}^{m,\rho,N} = O(1).$$

Proof. A direct calculation using the delta function definition of the measures shows that

$$\langle f \circ P_x \rangle_{MC}^{m,\rho,N} = \frac{1}{C(m, \rho, N)} \int_{-\infty}^{\infty} d\phi_x f(\phi_x) \mathbb{1} \left( (\phi_x-m)^2 \leq (\rho-m^2)(N-1) \right) \left( 1 - \frac{(\phi_x-m)^2}{(\rho-m^2)(N-1)} \right)^{\frac{N-4}{2}},$$

where

$$C(m, \rho, N) := \int_{-\infty}^{\infty} d\phi_x \mathbb{1} \left( (\phi_x-m)^2 \leq (\rho-m^2)(N-1) \right) \left( 1 - \frac{(\phi_x-m)^2}{(\rho-m^2)(N-1)} \right)^{\frac{N-4}{2}}.$$

For $N \geq 5$, we have

$$\mathbb{1} \left( (\phi_x-m)^2 \leq (\rho-m^2)(N-1) \right) \left( 1 - \frac{(\phi_x-m)^2}{(\rho-m^2)(N-1)} \right)^{\frac{N-4}{2}} \leq \mathbb{1} \left( (\phi_x-m)^2 \leq (\rho-m^2)(N-1) \right) e^{-\frac{(\phi_x-m)^2}{2(\rho-m^2)}} \leq e^{-\frac{(\phi_x-m)^2}{2(\rho-m^2)}}.$$

By the dominated convergence theorem, using the assumed integrability of $f$, we have

$$\lim_{N \to \infty} \langle f \circ P_x \rangle_{MC}^{m,\rho,N} = \frac{1}{\int_{-\infty}^{\infty} d\phi_x e^{-\frac{(\phi_x-m)^2}{2(\rho-m^2)}}} \int_{-\infty}^{\infty} d\phi_x f(\phi_x) e^{-\frac{(\phi_x-m)^2}{2(\rho-m^2)}} < \infty,$$

which implies that

$$\langle f \circ P_x \rangle_{MC}^{m,\rho,N} = O(1).$$

Lemma 5.24. Let $f : \mathbb{R} \to \mathbb{R}$ be integrable with respect to all Gaussian measures. Let $x \in \Lambda$ and define $P_x : \mathcal{S} \to \mathbb{R}$ by $P_x(\phi) = \phi_x$. Then for all $\rho > 0$, $h \in \mathbb{R}$, and $\varepsilon \in E_{h,\rho}$, we have

$$\langle f \circ P_x \rangle_{MC}^{\varepsilon,\rho,N} = O(1).$$

Proof. By definition, $\langle f \circ P_x \rangle_{MC}^{\varepsilon,\rho,N}$ is either equal to one of the expectations studied in the previous Lemma, or it is a convex combination of $\langle f \circ P_x \rangle_{MC}^{m,\rho,N}$ and $\langle f \circ P_x \rangle_{MC}^{m,\rho,N}$. In both of these cases, the result remains bounded as $N \to \infty$. 

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First, we will state the convergence result for the microcanonical and canonical ensembles.

**Theorem 5.25.** Let \( \rho > 0 \) and \( m \in (-\sqrt{\rho}, \sqrt{\rho}) \) and define \( \mu := -\frac{m}{\rho - m^2} \). Let \( I \subset \Lambda \) be a finite index set, and let \( f : \mathbb{R}^{|I|} \to \mathbb{R} \) be a bounded 1-Lipschitz function with respect to the \( \| \cdot \|_2 \)-norm. It follows that
\[
\langle f \circ P_{I}^{m,\rho,N} \rangle_{MC} = \langle f \circ P_{I}^{m,\rho,N} \rangle_{C} + O(N^{-\frac{1}{2}}).
\]

**Proof.** The result follows by applying the free energy coupling presented in Theorem 5.15, along with the \( w_2 \) bound presented in Theorem 5.17 and with the asymptotics presented in Theorem 5.14. \( \square \)

**Theorem 5.26.** Suppose \( h \neq 0 \) and \( \rho > 0 \). Assume \( \varepsilon \in \mathcal{E}_{h,\rho} \). Define \( m := -\frac{h}{\rho} + \text{sgn}(h)\sqrt{\frac{h^2}{\rho^2} - \frac{2\varepsilon}{\rho}} \) and \( \mu := -\frac{m}{\rho - m^2} \). Let \( I \subset \Lambda \) be a finite index set, and let \( f : \mathbb{R}^{|I|} \to \mathbb{R} \) be a bounded 1-Lipschitz function with respect to the \( \| \cdot \|_2 \)-norm. It follows that
\[
\langle f \circ P_{I}^{\varepsilon,\rho,N} \rangle_{MC} = \langle f \circ P_{I}^{m,\rho,N} \rangle_{C} + O(N^{-\frac{1}{2}}).
\]

**Proof.** By Theorem 5.10, we have
\[
\langle f \circ P_{I}^{\varepsilon,\rho,N} \rangle_{MC} = \langle f \circ P_{I}^{m,\rho,N} \rangle_{C} + O(e^{-cN}),
\]
and, by Theorem 5.24, we have
\[
\langle f \circ P_{I}^{m,\rho,N} \rangle_{MC} = \langle f \circ P_{I}^{m,\rho,N} \rangle_{C} + O(N^{-\frac{1}{2}}),
\]
from which the result follows. \( \square \)

**Theorem 5.27.** Let \( h = 0 \). Suppose \( \rho > 0 \) and \( \varepsilon \in \left(-\frac{2\rho}{\rho}, 0\right) \). Let \( I \subset \Lambda \) be a finite index set, and let \( f : \mathbb{R}^{|I|} \to \mathbb{R} \) be a bounded 1-Lipschitz function with respect to the \( \| \cdot \|_2 \)-norm. For \( \varepsilon \in \left(-\frac{2\rho}{\rho}, 0\right) \), define \( \beta := \frac{1}{\rho + \frac{2\rho}{\rho}} \). It follows that
\[
\langle f \circ P_{I}^{\varepsilon,\rho,N} \rangle_{MC} = \langle f \circ P_{I}^{\beta,\rho,N} \rangle_{C} + O(N^{-\frac{1}{2}}).
\]

For \( \varepsilon = 0 \), let \( \beta < \frac{1}{\rho} \) be arbitrary. It follows that
\[
\langle f \circ P_{I}^{0,\rho,N} \rangle_{MC} = \langle f \circ P_{I}^{0,\rho,N} \rangle_{C} + O(N^{-\frac{1}{2}}),
\]

**Proof.** If \( \varepsilon \in \left(-\frac{2\rho}{\rho}, 0\right) \), the result follows by applying the free energy coupling presented in Theorem 5.15, along with the \( w_2 \) bound presented in Theorem 5.11 and the asymptotics presented in Theorem 5.14.

If \( \varepsilon = 0 \), then observe that the \( w_2 \) bound in Theorem 5.11 is not Lipschitz in the appropriate sense to directly apply Theorem 5.15. However, following the proof of Theorem 5.15, we can apply the following inequality
\[
\langle w_2 \left( \muMC, \muMC^{\rho,N} ; N \right) ^{\beta,\rho,N} \rangle_{C} \leq \frac{2}{\sqrt{\beta}} \left( \langle -H/N \rangle_{C}^{\beta,\rho,N} \right)^{\frac{1}{2}},
\]
where the upper-index \( \frac{H}{\beta} \) is a non-positive random variable of the canonical ensemble. It follows that
\[
\left| \langle f \circ P_{I}^{0,\rho,N} \rangle_{MC} - \langle f \circ P_{I}^{0,\rho,N} \rangle_{C} \right| \leq C \left( \langle -H/N \rangle_{C}^{\beta,\rho,N} \right)^{\frac{1}{2}},
\]
for a global constant \( C > 0 \). By considering the asymptotics presented in Theorem 5.15, we have
\[
\langle f \circ P_{I}^{0,\rho,N} \rangle_{MC} = \langle f \circ P_{I}^{0,\rho,N} \rangle_{C} + O(N^{-\frac{1}{2}}).
\]
Finally, we will characterize the microcanonical and grand canonical convergence.

**Theorem 5.28.** Let $\rho > 0$ and $m \in (-\sqrt{\rho}, \sqrt{\rho})$ and define $\mu := -\frac{m}{\rho-m^2}$ and $\eta := \frac{1}{2(\sqrt{\rho} - m)}$. Let $I \subset \Lambda$ be a finite index set, and let $f : \mathbb{R}^{|I|} \rightarrow \mathbb{R}$ be a bounded $1$-Lipschitz function with respect to the $|| \cdot ||_2$-norm or let $f$ be a finite product of finite order moments. It follows that

$$
\langle f \circ P_I \rangle_{MC}^{\rho, \eta} = \langle f \circ P_I \rangle_{GC}^{\mu, \eta} + O(N^{-\frac{1}{2}}).
$$

**Proof.** The result follows by applying the direct coupling method in Lemma 5.28 along with the $w_2$ bound given in Theorem 5.18. The convergence of the finite dimensional moments follows from Theorem 5.24 and the fact that the moments of both relevant ensembles are bounded by Lemma 5.23.

**Theorem 5.29.** Suppose $h \neq 0$ and $\rho > 0$. Assume $\varepsilon \in \mathcal{E}_{h, \rho}$. Define $m := -\frac{1}{h} + \text{sgn}(h)\sqrt{\frac{1}{\rho^2} - \frac{4}{\rho^4}}$ and set then $\mu := \frac{m}{\rho-m^2}$ and $\eta := \frac{1}{2(\sqrt{\rho} - m)}$. Let $I \subset \Lambda$ be a finite index set, and let $f : \mathbb{R}^{|I|} \rightarrow \mathbb{R}$ be a bounded $1$-Lipschitz function with respect to the $|| \cdot ||_2$-norm or let $f$ be a finite product of finite order moments. It follows that

$$
\langle f \circ P_I \rangle_{MC}^{\varepsilon, \rho} = \langle f \circ P_I \rangle_{GC}^{\mu, \eta} + O(N^{-\frac{1}{2}}).
$$

**Proof.** The proof follows from Theorem 5.28 via the same steps as in the proof of Theorem 5.26.

**Theorem 5.30.** Suppose $h = 0$ and $\rho > 0$. Assume $\varepsilon \in \left( -\frac{1}{2}, 0 \right]$ and define $\mu := \frac{1}{\rho} - \frac{2}{\rho} = \frac{1}{\rho} - \frac{2}{\rho}$ and $\eta := \frac{1}{2(\rho - \frac{1}{2})}$. Let $I \subset \Lambda$ be a finite index set, and let $f : \mathbb{R}^{|I|} \rightarrow \mathbb{R}$ be a bounded $1$-Lipschitz function with respect to the $|| \cdot ||_2$-norm or a finite product of finite order moments. It follows that

$$
\langle f \circ P_I \rangle_{MC}^{\varepsilon, \rho} = \langle f \circ P_I \rangle_{GC}^{\mu, \eta} + O(N^{-\frac{1}{2}}).
$$

**Proof.** The result follows by splitting the fixed energy density ensemble into its fixed magnetization entropy ensembles and applying Theorem 5.28.

5.5 RemarK on choice of cost function

For this model, it should be observed that the $w_2$ convergence is a natural choice of convergence from the perspective that it implies both $w_1$ and $w_2$ convergence simultaneously, which in turn implies that the magnetization density converges along with the energy density. Without this property, a coupling of suitable strength between the microcanonical and grand canonical measures seems unlikely. Observe that

$$
|w_1 \left( \mu_{MC}^{\rho, N}, \mu_{MC}^{\rho, N}, N \right)| \leq \left( \mu_{MC}^{m, \rho, N}, \mu_{MC}^{m, \rho, N}, N \right).
$$

Employing the lower bound in the triangle inequality, we have

$$
|w_1 \left( \mu_{MC}^{m, \rho, N}, \mu_{MC}^{m, \rho, N} \right)| \geq |m - m'|.
$$

Now, if we consider Theorem 5.7, then we have

$$
|w_1 \left( \mu_{MC}^{m, \rho, N}, \mu_{MC}^{m, \rho, N} \right)| \leq \left( 1 + \frac{2}{\sqrt{1 - m'^2}} \right) |m - m'|.
$$

Thus, even though optimality of the transport was not necessarily achieved as in the discrete case, the best possible scaling in the dependence on changes in the parameter $m$ was still obtained here.
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