CONTINUATION METHOD FOR PDE-CONSTRAINED GLOBAL OPTIMIZATION: ANALYSIS AND APPLICATION TO THE SHALLOW WATER EQUATIONS

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Abstract. This paper shows how a class of nonconvex optimization problems constrained by discretized nonlinear partial differential equations may be solved to global optimality using an interior point continuation method. The solution procedure rests on a nested homotopy. The inner homotopy solves a barrier problem by driving the barrier parameter to zero. The outer homotopy makes use of a linear PDE that approximates the nonlinear PDE. This outer homotopy deforms the approximating linear PDE to the nonlinear PDE in a manner that ensures that the discretized constraint gradients remain linearly independent. Provided that the objective is convex and the search space remains path-connected, it is shown how a continuation method applied to the nested homotopy yields globally optimal solutions. As a case study, an appropriate discretization and homotopy for the shallow water equations is presented, together with a numerical experiment that solves a nonconvex numerical optimal control problem to global optimality. The approach is suitable for closed-loop nonconvex model predictive control of large-scale cyber-physical systems.

Key words. optimal control, partial differential equations, homotopy continuation, bifurcation analysis, nonconvex programming, interior point methods, global optimization

AMS subject classifications. 37G10, 49J20, 49K40, 49N60, 65H20, 90C25, 90C26, 90C51

1. Introduction. Optimization problems constrained by discretized nonlinear partial differential equations arise in the context of numerical optimal control of cyber-physical systems, such as river systems including man-made structures such as adjustable weirs [21]. In general, these nonconvex problems cannot be solved to global optimality by a naive application of an interior point method. They can, however, be solved to global optimality using polynomial hierarchies [15], or using a homotopy method that tracks all zeroes of a deforming system of polynomials [3]. Both of these methods suffer from high computational complexity and cannot be applied to large problems in a closed-loop setting with tight limits on computation time.

In this paper, we look at the homotopy method from a different angle. Instead of tracking zeroes of a polynomial as in [3], we set up a homotopy between a convex relaxation and the nonconvex problem. In this way, the number of variables of the optimization problem does not increase (as they would with a Lasserre hierarchy), and we may restrict our attention to the tracking of a single solution. The resulting method is therefore readily applied to problems with a large number of variables.

We will now give a brief overview of this method. Let \( \theta \in [0, 1] \) be the deformation parameter, where \( \theta \) equal to zero corresponds to the convex approximation of the nonconvex optimization problem and \( \theta \) equal to one to the original nonconvex problem. By construction, the approximated convex problem only admits global optima. Let \( x_{cp} \) denote such a global optimum, and let \( S \) denote the space of all possible solutions for all \( \theta \in [0, 1] \). We describe a procedure to find an optimal solution of each stage of the deformation, starting from \( x_{cp} \). That is, we construct a well-behaved “problem-to-solution” function \( f : [0, 1] \to S \) where any \( f(\theta) \) is an optimal solution to the corresponding optimization problem deformed by \( \theta \) from \( f(0) = x_{cp} \). Here well-behaved is taken to mean that the function is continuous and does not contain any
singularities. Singularities would produce bifurcations and other undesired behavior [18]. These basic properties allow us to derive a method to find a solution for the nonconvex problem, \( f(1) \), starting from an optimal solution of the convex approximation, \( f(0) \), by tracing a uniquely defined path of solutions as \( \theta \) is taken from zero to one. These properties also allow us to prove that the solution at the end of the path, at \( \theta = 1 \), is a global optimum. In Section 2, we formally describe this approach and provide sufficient conditions to ensure that the path exists, is unique, and that the problem does not admit any other solutions. The results hinge on two newly defined notions: zero-convexity and path-stability.

In Section 4, we consider an application of the homotopy method to the shallow water equations. These equations occur when setting up decision support systems for river and canal systems, such as those managed by Rijnland water authority in the area around the city of Leiden and Amsterdam Schiphol Airport in the Netherlands (the area covers approximately 1175 \text{km}^2). At Rijnland, the method is in day-to-day use for closed-loop model predictive control of 4 primary pumping stations to control water levels and water quality in the primary canal system with a total length of approximately 370 km [25, 23].

2. Background. A general continuous optimization problem can be formulated in the following standard manner:

\[
\begin{align*}
\min_x f(x) \quad \text{subject to} \\
&g_i(x) \leq 0, \quad i \in \{1, \ldots, m\} \\
&h_j(x) = 0, \quad j \in \{1, \ldots, p\} \\
&x \in \mathbb{R}^n.
\end{align*}
\]

We assume throughout that all the functions are \textit{three times} continuously differentiable (cf. Proposition 3.6). We will refer to such functions as being \textit{smooth}. Let \( f \) denote the \textit{objective function} of problem \( (P) \).

One can reformulate \( (P) \) as an optimization problem of the type

\[
\begin{align*}
\min_x f(x) \quad \text{subject to} \\
&c(x) = 0, \\
&x_i \geq 0 \quad i \in \{1, \ldots, m\}, \\
&x \in \mathbb{R}^n.
\end{align*}
\]

Note that we make a distinction between bounded, nonnegative variables and unbounded variables.

2.1. Interior point methods. Interior point methods are used to find local minima of general optimization problems [27, 19, 12, 26]. We will briefly mention some notions that we will need for our purposes. The general idea is to find a solution by computing (approximate) solutions for a sequence of barrier problems. A barrier problem, for a parameter \( \mu > 0 \), is defined as:

\[
\begin{align*}
\min_x f(x) - \mu \sum_{i=1}^{m} \ln x_i \quad \text{subject to} \\
&c(x) = 0, \\
&x \in \mathbb{R}^n.
\end{align*}
\]
This reformulation allows us to remove the non-negativity constraints on the variables. As long as the algorithm starts with strictly positive $x_i$, $i \in \{1, \ldots, m\}$, the logarithmic barrier terms in the objective function will ensure that the solution coordinates remain strictly positive. Furthermore, if $f$ is a convex function and $c$ linear, then the reformulation also turns the convex optimization problem ($P'$) into a strictly convex optimization problem ($P_\mu$) with a unique solution. Generally speaking, for any sequence of barrier parameters $\mu$ converging to zero, the sequence of the corresponding solutions to the problems ($P_\mu$) converges to a solution of ($P$).\footnote{The exact conditions for such convergence are discussed in [12].}

The objective function in the barrier problem ($P_\mu$) is only defined for interior points:

**Definition 2.1.** Consider the barrier problem ($P_\mu$). A point $x$ is called an interior point if $x_i > 0$ for all $i \in \{1, \ldots, m\}$.

**Definition 2.2.** Consider the barrier problem ($P_\mu$). A point $x$ is called a feasible interior point if it is an interior point and if it satisfies the constraints $c(x) = 0$.

For a generic optimization problem, the standard strategy to find a local minimum is to use the method of the Lagrange multipliers. The Lagrangian of the problem ($P$) is \[ L(x, \lambda) := f(x) + \lambda^T c(x) \]
where $\lambda$ is the vector of Lagrangian multipliers. Any local minimum of ($P$) is a solution to the system of equations
\begin{equation}
\nabla_x L(x, \lambda) = 0, \quad c(x) = 0. \tag{2.1}
\end{equation}

**Remark 1.** Note that any inequality constrained optimization problem as ($P$) can be reformulated as ($P'$). Moreover, the interior point method can be used also for optimization problems where the variables are bounded.

Inequality constraints: constraints of the form $g(x) \leq 0$ can be handled by introducing nonnegative slack variables $s \geq 0$ such that $g(x) + s = 0$.

Bounds on optimization variables: the analysis is readily extended to cases where $x_L \leq x \leq x_U$, by adjusting the barrier function to tend to infinity as $x \to x_L$ and $x \to x_U$ \cite{27, 19, 12, 26}:
\[ f(x, \theta) - \mu \sum_{i \in I_L} \ln((x - x_L) \cdot e_i) - \mu \sum_{i \in I_U} \ln((x_U - x) \cdot e_i), \]
where $e_i$ is the $i$th unit vector, $I_L$ is the set of indices such that $x_L \cdot e_i \neq -\infty$ and $I_U$ the set of indices such that $x_U \cdot e_i \neq \infty$.

**2.2. Parametric programming.** A parametric optimization problem is a particular type of optimization problem where the objective and constraint functions
depend on a parameter \( \theta \):

\[
\min_x f(x, \theta) \quad \text{subject to}
\]

\[
\begin{align*}
(P^{\theta}) & \quad c(x, \theta) = 0, \\
x_i & \ge 0 \quad i \in \{1, \ldots, m\}, \\
x & \in \mathbb{R}^n,
\end{align*}
\]

where \( x \) is the optimization variable, \( f(x, \theta) \) is the objective function and \( c(x, \theta) \) denotes the constraints.

The main idea of this paper is to continuously deform an optimization problem, such that \((P^{\theta=0})\) is a convex problem and \((P^{\theta=1})\) is the original nonconvex problem, and track the corresponding solution \( x^*(\theta) \). From the previous discussion, we know that any solution of \((P^{\theta})\) is the solution of a system of equations. Thus, we can equivalently (cf. Proposition (3.6)) consider the continuous deformation of a system of equations and track its solution.

### 2.3. Continuation methods.

Here we provide a brief overview of the classical continuation method [1]. Let \( F : \mathbb{R}^n \to \mathbb{R}^n \) denote the residual function for a system of nonlinear equations of the form

\[
F(x) = 0.
\]

In general, finding a solution \( x^* \) such that \( F(x^*) = 0 \) is a hard problem. If an initial guess \( x_0 \) is sufficiently close to a solution and the function \( F \) satisfies certain regularity properties, the Newton-Raphson method will converge to \( x^* \). But if \( x_0 \) is too far away, the Newton-Raphson method may diverge and a different approach is needed.

The continuation method is one such approach and we will now sketch the idea behind it. One approximates the residual function \( F \) with a suitable function \( \tilde{F} \), for which a solution \( \tilde{x}^* \) is known:

\[
\tilde{F}(\tilde{x}^*) = 0.
\]

A homotopy parameter \( \theta \) is then introduced to deform \( \tilde{F} \) into \( F \):

\[
G(x, \theta) := (1 - \theta) \tilde{F} + \theta F.
\]

With \( \tilde{x}^* \) given such that \( \tilde{F}(\tilde{x}^*, 0) = 0 \), we can increase \( \theta \) and solve \( G(x, \theta) = 0 \) for \( x \) starting from \( \tilde{x}^* \), which, if the increase in \( \theta \) was sufficiently small, will lie sufficiently close to the solution of \( G(x, \theta) = 0 \) for the Newton-Raphson method to converge. Continuing in this way, under suitable conditions, we arrive at a solution \( x^* \) such that \( F(x^*) = G(x^*, 1) = 0 \).

In the process, we have traced a path \( \theta \to x(\theta) \). By the implicit function theorem [20, Theorem 9.28], this path exists locally and uniquely, and is continuously differentiable, as long as \( \partial G / \partial x \) is nonsingular. This motivates the following definition:

**Definition 2.3.** Consider the homotopy \((2.2)\). A point \( x \) is called singular if the Jacobian matrix \( \partial G / \partial x \) is singular at \( x \).

At singular points, the path may (1) turn back on itself, (2) end, or (3) bifurcate into multiple paths. Figure 1 illustrates a path with a bifurcation, and highlights the point where \( \partial G / \partial x \) is singular. Clearly, this situation is undesirable and in the following we will look for conditions under which all points are nonsingular.
Fig. 1: A path with a bifurcation. The singular point is highlighted.

3. Continuation method for global optimization. Our objective is to construct, for any \( \mu > 0 \) and any \( \theta \in [0, 1] \), an optimization problem \((\mathcal{P}^\theta)\) such that we can track its solution using a continuation method. Using interior point methods, a problem \((\mathcal{P}^\theta)\) is solved by equivalently finding the solution of a system of equations (2.1). Continuation theory guarantees that the path of the solution of the system of equations does not bifurcate (i.e., it can be traced) if the Jacobian of the residual function is not singular.

More formally, we want to track the solution of the system of equations

\[
\begin{align*}
\nabla_x L_\mu(x, \lambda, \theta) &= 0, \\
c(x, \theta) &= 0.
\end{align*}
\]

where \( L_\mu(x, \lambda, \theta) := f(x, \theta) - \mu \sum_{i=1}^{m} \ln x_i + \lambda^T c(x, \theta) \) is the Lagrangian of the following parametric barrier problem:

\[
\begin{align*}
\min_x f(x, \theta) - \mu \sum_{i=1}^{m} \ln x_i & \quad \text{subject to} \\
c(x, \theta) &= 0, \\
x & \in \mathbb{R}^n.
\end{align*}
\]

Let \( F_\mu(x, \lambda, \theta) \) denote the residual of the system of equations (3.1). Then the system

\[
F_\mu(x, \lambda, \theta) = 0
\]

admits a unique solution path in a neighborhood of \( x^*, \lambda^*, \) and \( \theta^* \) if the Jacobian matrix \( \partial F_\mu/\partial (x, \lambda) \) is nonsingular at the point \( (x^*, \lambda^*, \theta^*) \).

In the following, we will need the notion of the tangent space of the constraint manifold:

**Definition 3.1** (e.g., [18]). Fix a \( \theta \in [0, 1] \) and a feasible interior point \( x \). We call the linear space

\[
T(x, \theta) := \{ y : \nabla_x c(x, \theta)y = 0 \}
\]

the tangent space of the constraints \( c(x, \theta) = 0 \) at \( x \).
3.1. Sufficient conditions for convergence to a global optimum. We are
now ready to discuss sufficient conditions for the path tracing procedure to converge to
a global optimum. To do so, we will need to introduce two new notions: zero-convexity
and path-stability.

**Definition 3.2.** We say that the parametric optimization problem \((P^\theta)\) is zero-
convex if the objective function \(x \mapsto f(x, 0)\) is a convex function, and the constraints
\(x \mapsto c(x, 0)\) are linear.

The notion of zero-convexity captures the idea that there should be a unique
solution at \(\theta = 0\), and that it should be possible to find this solution using standard
methods.

**Definition 3.3.** We say that the parametric optimization problem \((P^\theta)\) is path-
stable with respect to the interior point method if its barrier formulation \((P^\mu)\) does
not admit singular feasible points for any \(\mu > 0\) and any \(\theta \in [0, 1]\).

The concept of path-stability captures the idea that we seek a way to consistently
arrive at a uniquely related local minimum of the fully nonlinear problem at \(\theta = 1,
\) i.e., without path bifurcations along the way. The following Proposition provides a
useful characterization of path-stability:

**Proposition 3.4.** Consider the parametric optimization problem \((P^\mu)\). Fix a
\(\mu > 0\) and a \(\theta \in [0, 1]\). Let \(x\) denote a feasible point. The point \(x\) is nonsingular if
and only if the following two conditions hold:

1. The Hessian matrix \(\nabla^2_{xx} L_\mu(x, \lambda, \theta)\) is nonsingular on the tangent space
   \(T(x, \theta)\);
2. The Jacobian matrix of the constraints \(\nabla_x c(x, \theta)\) has full rank.

**Proof.** Consider the Jacobian matrix

\[
\frac{\partial F_\mu}{\partial (x, \lambda)} = \nabla^2_{xx} L_\mu(x, \lambda, \theta).
\]

Define a local basis \(e_1, \ldots, e_\ell\) that spans the tangent space \(T(x, \theta)\), and a basis
\(e_{\ell+1}, \ldots, e_n\) that spans its orthogonal complement. With respect to these bases, the
Jacobian matrix has the form

\[
\begin{pmatrix}
\nabla^2_{e_1, \ldots, e_{\ell+1}, \ldots, e_n} L_\mu & \nabla^2_{e_{\ell+1}, \ldots, e_n} L_\mu & \cdots & \nabla^2_{e_n} L_\mu \\
0 & 0 & \cdots & 0 \\
\nabla^2_{e_1, \ldots, e_{\ell+1}, \ldots, e_n} c & \nabla^2_{e_{\ell+1}, \ldots, e_n} c & \cdots & \nabla^2_{e_n} c
\end{pmatrix},
\]

since \(dc(x, \theta)/de_i = \nabla_x c(x, \theta)e_i = 0\) if \(e_i \in T(x, \theta)\). Clearly, this matrix is nonsingular
if and only if conditions (1) and (2) hold.

**Remark 2.** Condition (2) in Proposition (3.4) is also known as the linear
independence constraint qualification (LICQ) [11, 13].

**Remark 3.** Proposition (3.4) relates to Theorem 2.1 from Poore and Tiahrt [18]
in two ways:

1. The result from Poore and Tiahrt is more general in the sense that it considers
   inequality-constrained systems directly;
2. Proposition (3.4) is more general in the sense that it applies to all feasible
   points, as opposed to solutions only.

**Remark 4.** For an example of a problem where the Hessian of the Lagrangian is
singular on the tangent space, consider the two-dimensional toy problem: \(\min x_1^2 + x_2^2\)
subject to the constraint \(x_1^2 + x_2^2 = 1\). Note how every feasible point of the toy problem
is a non-strict local minimum.
Lemma 3.5. Fix a $\mu > 0$ and a $\theta \in [0, 1]$. If the parametric optimization problem $(P^\theta_\mu)$ is zero-convex and path-stable with respect to the interior point method, then the Hessian matrix

$$\nabla^2_{xx} L_\mu(x, \lambda, \theta)$$

is positive definite on the tangent space $T(x, \theta)$ for all feasible interior points $x$, and all $\lambda$.

Proof. Fix a $\mu > 0$. For $\theta = 0$, path-stability implies that $\nabla^2_{xx} L_\mu(x, \lambda, 0)$ is non-singular on the tangent space $T(x, \theta)$ (Proposition (3.4)). Therefore, the eigenvalues of $\nabla^2_{xx} L_\mu(x, \lambda, 0)$ on the tangent space cannot be zero. Zero-convexity, together with the convexity of the barrier terms and the linearity of the constraints, implies that the eigenvalues of $\nabla^2_{xx} L_\mu(x, \lambda, 0)$ cannot be negative. Therefore, the matrix $\nabla^2_{xx} L_\mu(x, \lambda, 0)$ must be positive definite on the tangent space $T(x, \theta)$.

For $\theta > 0$, note that the eigenvalues – being the roots of the characteristic polynomial – vary continuously with $\mu$, $\theta$, $x$, and $\lambda$ (14) since $L_\mu$ is twice continuously differentiable. As such, a negative eigenvalue can only arise if there would exist a $\theta > 0$ and $(x, \lambda)$ such that $\nabla^2_{xx} L_\mu(x, \lambda, \theta)$ would have a zero eigenvalue on the tangent space $T(x, \theta)$, and hence would be singular there. But this would contradict the assumption of path-stability due to Proposition 3.4.

The following Proposition shows that finding a solution for the system of Equations (3.1) is indeed equivalent to solving the optimization problem $(P^\theta_\mu)$, provided that zero-convexity and path-stability hold (cf. Lemma 3.5):

Proposition 3.6. [Edwards, [10], Theorem 8.9] For any $\mu > 0$ and $\theta \in [0, 1]$, a solution $(x, \lambda)$ of the system of Equations (3.1) is a strict local minimum of the parametric optimization problem $(P^\theta_\mu)$ if

1. the Lagrangian $L_\mu(x, \lambda, \theta)$ is three times continuously differentiable in a neighborhood of $(x, \lambda)$; and
2. the Hessian matrix $\nabla^2_{xx} L_\mu(x, \lambda, \theta)$ is positive definite on the tangent space $T(x, \theta)$.

In order to describe the sufficient conditions that ensure the existence of no more than one solution, we will need to recall the notion of path-connected set:

Definition 3.7 (e.g., [16]). A set $X$ is path-connected if for any $x_1, x_2 \in X$, there exists a continuous function $f : [0, 1] \rightarrow X$ such that $f(0) = x_1$ and $f(1) = x_2$.

The following Theorem describes sufficient conditions for our parametric optimization problem to have at most one solution, so that any solution must be the global optimum:

Theorem 3.8. Consider the parametric barrier problem $(P^\theta_\mu)$. Assume that

1. the problem is zero-convex;
2. the problem is path-stable;
3. the set of feasible interior points is path-connected.

Then for any $\mu > 0$ and any $\theta \in [0, 1]$, the barrier problem $(P^\theta_\mu)$ has at most one unique solution. This unique solution is its global optimum.

Proof. Fix $\mu > 0$ and $\theta \in [0, 1]$. Suppose that the system of equations $F_\mu(x, \lambda, \theta) = 0$ admits two different solutions. From Lemma (3.5) and Proposition (3.6), it follows that these are strict local minima. Connect the two strict local minima with a continuous path of feasible points. The objective function is continuous, and the image of the path forms a compact set. Therefore, by the extreme value theorem [20], the
objective function must attain a maximum somewhere on the path. Since the endpoints of the path are strict local minima, the local maximum must lie in the interior of the path, and it must yield an objective value exceeding the objective values for the two local minima. But the existence of such a local maximum on the path would contradict the positive definiteness of the Hessian matrix implied by Lemma (3.5).

For problems that satisfy the sufficient conditions of Theorem (3.8), the \textit{central path} \cite{19} of the interior point method is uniquely defined.

To solve an optimization problem $(P)$ that has an associated parametric optimization problem $(P^\mu_\theta)$ that satisfies the conditions of Theorem (3.8), we provide a feasible starting point, and then let the interior point method implementation drive $\mu \to 0$, while ensuring that at every iteration $\mu > 0$ \cite{27, 14, 26}.

Note that the requirement to start with a feasible starting point is not restrictive. This starting point can be obtained either from a simulation computation prior to the optimization run, or alternatively, it can be obtained using a continuation algorithm. The continuation algorithm in turn may be seeded using the solution of the problem at $\theta = 0$, which is a convex problem that may be solved using an interior point method without a starting point \cite{19}.

\textbf{Remark 5.} Even if the search space is not path-connected, zero-convex and path stable optimization problems can be solved to local optimality using the continuation method. Due to path-stability, the homotopy path cannot bifurcate, and hence the local optimum of the nonconvex optimization problem is uniquely defined by the global optimum of the convex problem at $\theta = 0$.

\section{Application to the shallow water equations.} In the present section we prove that the sufficient conditions for global optimality hold when considering the one-dimensional shallow water equations. The analysis is lengthy and illustrates that while our homotopy method is powerful, practical application requires a good amount of preparatory analysis.

In Subsection 4.1, we describe the one-dimensional shallow water equations, linear approximations to the equations, and explain their discretization. In Subsection 4.4, we prove that our discretization of the shallow water equations satisfies the notions of zero-convexity and path-stability, and can be included in an optimization problem in such a way that the search space remains path-connected. The proofs use well-known results from real analysis, linear algebra, and general topology. In Subsection 4.5, we consider a numerical example where we solve an optimization problem subject to the discretized shallow water equations.

\subsection*{4.1. The shallow water equations.} In the present section, we summarize the one-dimensional shallow water equations. These are also known as the \textit{Saint-Venant} equations \cite{9}.

The shallow water equations describe situations in fluid dynamics where the horizontal length scale is large compared to the water depth. The Saint-Venant equations are given by the momentum equation

\begin{equation}
\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{Q^2}{A} \right) + gA \frac{\partial H}{\partial x} + g \frac{Q|Q|}{AR} = 0,
\end{equation}

with longitudinal coordinate $x$, time $t$, discharge $Q$, water level $H$, cross section $A$, hydraulic radius $R := A/P$, wetted perimeter $P$, Chézy friction coefficient $C$, gravitational constant $g$, and by the mass balance (or continuity) equation

\begin{equation}
\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0.
\end{equation}
The cross section $A : H \mapsto A(H)$ and wetted perimeter $P : H \mapsto P(H)$ are three times continuously differentiable functions such that for all $H$, it holds that $A > 0$, $dA/dH > 0$, $d^2A/dH^2 \geq 0$, $P > 0$ and $dP/dH > 0$. The conditions $dA/dH > 0$ and $dP/dH > 0$ state that the cross sectional area and the wetted perimeter are strictly increasing functions of the water level, and the condition $d^2A/dH^2 \geq 0$ states that the channel width $dA/dH$ is a non-decreasing function of the water level.

For our purposes, we require that these functions must be defined for all $H$, i.e., including $H < H_b$. We do this in order to be able to produce “imaginary” solutions where $H < H_b$. This construction improves the topology of the search space (cf. Corollary (4.3)), eventually leading to the global optimality result. Note that in Subsection 4.4, we will discuss ways to impose “soft” constraints on water levels.

A simple approach to building such functions is to let $A$ and $P$ approach their natural $H = H_b$ values asymptotically as $H \to -\infty$, and to extrapolate smoothly as $H \to \infty$. Such functions can be set up by fitting a quartic B-Spline [8] to bathymetry data over the range of physically feasible water levels (perturbing $A(H_b)$ away from zero if necessary), and flanking the B-Spline fits with the appropriate smooth extrapolations.

In practice, however, it is typically not required to set up such extrapolations. In Subsection 4.4, we will show that if a solution is found to the optimization problem, this solution must be a globally optimal solution. It is easy to check, a posteriori, whether a solution satisfies $H > H_b$ everywhere. If a soft lower bound is set on $H$ (cf. Subsection 4.4), an optimum with $H \leq H_b$ can only arise if channel reaches fall dry due to a lack of water.

In the remainder of this paper, we will restrict our attention to smooth, subcritical solutions of the Saint-Venant equations. Correct handling of supercritical phenomena requires additional attention as discussed in, e.g., [22].

4.2. A linear approximation to the shallow water equations. The mass balance equation (4.2) and the momentum equation (4.1) are both, in general, nonlinear. The inclusion of these equations as equality constraints in an optimization problem results in a problem that is nonconvex. In the present section, we develop linear approximations of these equations.

Starting from a globally optimal solution of the convex optimization problem subject to the linear approximation of the Saint-Venant equations, we may use the continuation method to find a solution of the nonlinear problem. In Subsection 4.4 we will show that this solution is the only solution, and hence the global optimum of the barrier formulation of the nonlinear problem.

We start by defining a nominal water level $\bar{H}$. Typically, this level would correspond to a mean water level or a level setpoint. We obtain a linear approximation to the mass balance equation by considering a rectangular cross section with nominal width $\bar{w} := (dA/dH)(\bar{H})$:

\begin{equation}
\frac{\partial Q}{\partial x} + \bar{w} \frac{\partial H}{\partial t} = 0.
\end{equation}

We now turn our attention towards the momentum equation. The water level gradient $\partial H/\partial x$ is a primary driver of the flow and the direction thereof. In order to maintain directional variability in the linear model we, therefore, need to retain the water level gradient as-is. Hence, we linearize the pressure term around $\partial H/\partial x = 0$ and $A = \bar{A}$ with $\bar{A} := A(\bar{H})$, and obtain the linearized pressure term $\tilde{g}\bar{A}\partial H/\partial x$.

The quadratic nature of the friction term cannot be maintained in a linear model. We apply the nominal cross section $\bar{A}$, which results in a nominal hydraulic radius
Fig. 2: Staggered grid with an upstream level boundary and a downstream flow boundary. Here, \( H_1 \) and \( Q_5 \) are boundary variables, whereas all other variables are internal.

\[
\mathcal{R} = \frac{A}{P} \quad \text{with} \quad P := P(H), \quad \text{and linearize around} \quad Q = Q_0. \quad \text{The choice of} \quad Q_0 \quad \text{does not express a preferred flow direction due to the presence of the absolute value function.}
\]

The convective acceleration term \( \partial (Q^2/A) / \partial x \) is of limited significance in subcritical river wave propagation scenarios [17], and it turns out that we can show path-stability if we leave it out of the linear approximation. This is the same approximation that is used to derive the so-called inertial wave equations. We obtain the following linear approximation to the momentum equation:

\[
\frac{\partial Q}{\partial t} + gA \frac{\partial H}{\partial x} + g \frac{Q}{\overline{Q}} A R C^2 = 0.
\]

4.3. Semi-implicit discretization on a staggered grid. We discretize our hydraulic equations on a staggered grid and semi-implicitly in time, analogous to the approaches set out in, e.g., [4, 7, 22]. The pressure term is discretized semi-implicitly in the sense that the levels are evaluated at time \( t_j \), whereas the cross section is evaluated at time \( t_{j-1} \). The friction term is discretized semi-implicitly in the sense that the discharge \( Q \) is evaluated at time \( t_j \), whereas the cross section and hydraulic radius are evaluated at time \( t_{j-1} \). The convective acceleration term is discretized explicitly in time. The von Neumann stability of such semi-implicit discretizations is analyzed in, e.g., [6].

In the following, we will refer to those variables which lie between two other hydraulic variables as interior variables. All other hydraulic variables are referred to as boundary variables. The staggered grid, and the distinction between interior and boundary variables, is illustrated in Figure 2.

Throughout the paper we assume, without loss of generality, that the grid nodes are numbered as in Figure 2. That is, every interior variable \( H_i \) has the variables \( Q_{i-1} \) and \( Q_i \), respectively, to its left and to its right. Such variables exist by construction. Similarly, any interior variable \( Q_i \) has the variable \( H_i \) to its left and \( H_{i+1} \) to its right.

We now introduce the homotopy parameter \( \theta \) interpolating between the linear and nonlinear equations. Following interpolation of the linear and nonlinear mass balance equations, (4.3) and (4.2), respectively, and discretization on our staggered grid, we obtain the discretized homotopic mass balance equation

\[
c_{i,j} := \frac{Q_i(t_j) - Q_{i-1}(t_j)}{\Delta x} + \theta \frac{A_i(H_i(t_j)) - A_i(H_i(t_{j-1}))}{\Delta t} + (1 - \theta)w \frac{H_i(t_j) - H_i(t_{j-1})}{\Delta t} = 0
\]

\forall i \in I_H \quad \forall j \in \{1, \ldots, T\}
with the index set $I_H$ such that every $H_i$, $i \in I_H$, is an interior variable. This is a mildly nonlinear, mass-conservative formulation as in [7].

We now turn our attention to the momentum equation. Interpolating between the linear and nonlinear momentum equations (4.4) and (4.1), respectively, and discretizing on our staggered grid, we obtain the discretized homotopic momentum equation

\begin{equation}
\frac{d_i}{dt} := \left. \frac{Q_i(t_j) - Q_i(t_{j-1})}{\Delta t} \right|_{t=t_j} + \theta e_{i,j}
+ g \left( \theta A_{i+\frac{1}{2}}(t_{j-1}) + (1 - \theta) \bar{A} \right) \frac{H_{i+1}(t_j) - H_i(t_j)}{\Delta x}
+ g \left( \frac{P_{i+\frac{1}{2}}(t_{j-1}) \text{sabs } Q_i(t_{j-1})}{A_{i+\frac{1}{2}}(t_{j-1})^2} + (1 - \theta) \frac{\bar{P} \text{sabs } \bar{Q}}{A^2} \right) \frac{Q_i(t_j)}{C_i^2} = 0
\end{equation}

with

\begin{align*}
A_{i+\frac{1}{2}}(t_j) &:= \frac{1}{2} \left( A_i(H_i(t_j)) + A_{i+1}(H_{i+1}(t_j)) \right); \\
P_{i+\frac{1}{2}}(t_j) &:= \frac{1}{2} \left( P_i(H_i(t_j)) + P_{i+1}(H_{i+1}(t_j)) \right),
\end{align*}

convective acceleration $e_{i,j}$, and the index set $I_Q$ such that every $Q_i$, $i \in I_Q$, is an interior variable. The parameter $C_i$ indicates the local friction coefficient, and $H_i^b$ indicates the local bottom level. In order to avoid singular derivatives, we use

\[ \text{sabs } x := \sqrt{x^2 + \varepsilon}, \]

where $\varepsilon$ is a small constant, as a smooth approximation for $|x|$.

Note that we have used a single set of constant nominal values $\bar{\pi}$, $\bar{A}$, $\bar{P}$, and $\bar{Q}$ for the entire reach. This is sufficient for the development of the theory.

The convective acceleration term $e_{i,j}$ must be discretized explicitly in time in order to be able to prove path-stability. In the following, we consider a finite difference approximation to the convective acceleration term, in which the finite differences are taken in the upstream direction (a so-called upwind scheme). In order to ensure a smooth formulation regardless of the flow direction, we replace the Heaviside function with a logistic function, finally obtaining

\[ e_{i,j} := \text{sH}(Q_i(t_{j-1})) \frac{2Q_i(t_{j-1}) - Q_{i-1}(t_{j-1})}{A_{i+\frac{1}{2}}(t_{j-1})} \frac{Q_i(t_{j-1}) - Q_{i-1}(t_{j-1})}{\Delta x} \]
\[ + (1 - \text{sH}(Q_i(t_{j-1}))) \frac{2Q_i(t_{j-1}) - Q_{i+1}(t_{j-1}) - Q_i(t_{j-1})}{A_{i+\frac{1}{2}}(t_{j-1})} \frac{Q_i(t_{j-1})^2}{\Delta x} \]
\[ - \frac{Q_i(t_{j-1})^2}{A_{i+\frac{1}{2}}(t_{j-1})^2} \frac{A_{i+1}(H_{i+1}(t_{j-1})) - A_i(H_i(t_{j-1}))}{\Delta x}, \]

with the logistic function

\[ \text{sH}(x) := \frac{1}{1 + e^{-Kx}}, \]

and steepness factor $K > 0$. 

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4.4. Homotopy convergence analysis. We consider a numerical optimal control problem subject to the dynamics (4.5) - (4.6) imposed as equality constraints between flow variables $Q$ and water level variables $H$. Let $F_\mu(x, \lambda, \theta) = 0$, where

$$
F_\mu(x, \lambda, \theta) = 0,
$$

and the vector $x$ contains the variables $Q$, $H$ denote the primal equation system (2.1) corresponding to this optimization problem. In particular, we denote with $x_{hyd}$ the vector of the interior hydraulic variables.

In the following, we will show zero-convexity, path-stability, and path-connectedness of the search space for this type of problem, provided that the following assumptions hold:

**BND** None of the interior hydraulic variables are bounded. All free boundary variables have both a lower bound as well as an upper bound such that the lower bound is strictly less than the upper bound.

**ICO** Initial values $Q_i(t_0)$ and $H_i(t_0)$ are provided and replaced into the model so that the variables at $t_0$ are no longer included in the optimization problem.

**HBC** Any water level boundary conditions are fixed, i.e., if $H_i$ is a water level boundary, then $H_i(t_j) = v_j$ for some time series $\{v_j\}_{j \in \{0, \ldots, T\}}$. Furthermore, the values $v_j$ are replaced into the model so that the variables $H_i(t_j)$ are no longer included in the optimization problem.

**QBC** There is at least one free flow boundary condition. Any two free flow boundary conditions must have at least one interior flow variable situated in between.

**OBJ** The objective function is three times continuously differentiable and convex.

Condition BND is trivially satisfied. If needed, bounds and constraints on the interior hydraulic variables can be mimicked by including suitable penalty terms in the objective function. For example, the three times continuously differentiable and convex objective term

$$
sg(x) := w_g \begin{cases} 
0 & \text{if } g(x) \leq 0, \\
g(x)^4 & \text{otherwise}, 
\end{cases}
$$

with weighting factor $w_g > 0$ can be seen as a “soft” encoding of the convex inequality constraint $g(x) \leq 0$. The soft constraint may be violated, but it is expensive to do so. We will now briefly mention the physical intuition behind soft constraints. Sometimes there is too little or too much water in a system to satisfy a minimum or maximum water level. We need our decision support system to produce control strategies in such situations as well, however, and therefore bounds on interior hydraulic variables must not be rigid.

The condition ICO may be satisfied by providing a complete initial state, possibly computed using a state estimation algorithm prior to the optimization run.

The condition HBC, requiring the water level variables at the boundaries to have fixed values, is hardly restrictive. A downstream water level variable only occurs in the momentum equation for the adjacent flow variable. A free downstream level therefore translates to a “free” downstream discharge. The downstream level variable may therefore be omitted, resulting in the adjacent flow variable becoming the new free boundary variable. There is no requirement for $Q$ boundaries to be fixed.

The condition QBC is trivially satisfied. If two free flow boundaries would not have an interior flow variable situated in between, then we would only be imposing

\[\nabla^2_{xx} s_g(x) = 12 g(x)^2 \nabla_x g(x) \nabla^2_x g(x) + 4 g(x)^3 \nabla^2_{xx} g(x)\]

is positive semidefinite, whence $s_g$ is convex.

In order to ensure that penalty violations dominate other objective terms as $\mu \to 0$, it is sometimes convenient to set $w_g = 1/\mu$. 

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\textsuperscript{2}The Hessian matrix $\nabla^2_{xx} s_g(x) = 12 g(x)^2 \nabla_x g(x) \nabla^2_x g(x) + 4 g(x)^3 \nabla^2_{xx} g(x)$ is positive semidefinite, whence $s_g$ is convex.

\textsuperscript{3}In order to ensure that penalty violations dominate other objective terms as $\mu \to 0$, it is sometimes convenient to set $w_g = 1/\mu$. 

---
the continuity equation, but not the momentum equation.

The condition OBJ states that the objective of optimization problem must be convex. This is not restrictive in the sense that all standard convex objectives, such as minimization in the 1, 2, or \(\infty\) norms, are allowed.

The remainder of this section is dedicated to proving that, under the assumptions mentioned above, the problem is zero-convex (Proposition 4.1), path-stable (Proposition 4.5) and its feasible solutions are path-connected (corollary 4.3). These three properties combined will allow us to deduce that the nonconvex optimization problem has a unique global optimum (Theorem 4.6). Moreover, as is discussed in Remark (6), this methodology is able to find all solutions of interest.

**Remark 6.** Suppose that the original nonconvex optimization problem, prior to its transformation to a barrier formulation, has a solution. Then this solution is of one of the following two types:

1. an interior point; or
2. a point with any number of flow boundary bounds active.

Points of type (1) can be reached by a sequence of interior points such that the objective function values of the original and the barrier problems converge as \(\mu \to 0\). Points of type (1) therefore cannot obtain lower objective values than those reached using the interior point method.

Points of type (2) can also be reached by a sequence of interior points. By Lemma (4.2), and the implicit function theorem, there exists a neighborhood around the boundary variables of the solution for which the constraint manifold is defined (here we temporarily disregard the active bounds, which are arbitrary from the point of view of the dynamics). This means that we can perturb away from the bounds into the interior, and construct a sequence of interior points that converges to the solution. Therefore the same reasoning as for points of type (1) applies.

**Proposition 4.1.** Assume OBJ. Then the optimization problem \(F_\mu(x, \lambda, \theta) = 0\) is zero-convex.

**Proof.** The convexity of the optimization problem at \(\theta = 0\) is implied by OBJ, and the parametric definitions of the hydraulic constraints (4.5) - (4.6), which are linear for \(\theta = 0\). \(\Box\)

**Lemma 4.2.** Assume BND, ICO, and HBC. Then the gradients of the hydraulic constraints (4.5) - (4.6) form a basis of the space of the interior \(Q\) and \(H\) variables.

**Proof.** Since the number of interior hydraulic variables equals the number of hydraulic constraints, the statement of the lemma is equivalent to showing that the gradients of the hydraulic constraints are linearly independent over the space of the interior \(Q\) and \(H\) variables; i.e., that the equation

\[
\sum_{i \in I_H, j \in \{1, \ldots, T\}} \alpha_{i,j} \frac{\partial c_{i,j}}{\partial x_{hyd}} + \sum_{i \in I_Q, j \in \{1, \ldots, T\}} \beta_{i,j} \frac{\partial d_{i,j}}{\partial x_{hyd}} = 0
\]

is satisfied only if \(\alpha_{i,j} = \beta_{i,j} = 0\) for all \(i \in I_H\) and \(i \in I_Q\), respectively, and all \(j \in \{1, \ldots, T\}\).

Let \(x_{i,j}\) denote the subvector of the interior variables at interior discretization point \(i\) and time step \(t_j\), i.e.,

\[x_{i,j} := (H_i(t_j), H_{i+1}(t_j), Q_{i-1}(t_j), Q_i(t_j)).\]

For the sake of our proof, we only need to consider the partial derivatives of \(c_{i,j}\)
and \( d_{i,j} \) restricted to \( x_{i,j} \). The only non-zero terms are then equal to:

\[
\begin{align*}
\frac{\partial c_{i,j}}{\partial x_{hyd}} \bigg|_{x_{i,j}} &= \left( \phi_{i,j}, 0, \frac{-1}{\Delta x}, \frac{1}{\Delta x} \right); \\
\frac{\partial d_{i,j}}{\partial x_{hyd}} \bigg|_{x_{i,j}} &= \left( -\psi_{i,j}, \psi_{i,j}, 0, \tau_{i,j} \right),
\end{align*}
\]

where the vertical bar indicates restriction to a subvector, and where

\[
\begin{align*}
\phi_{i,j} &= \frac{1}{\Delta t} \left( \theta \frac{\partial A_i}{\partial H_i}(H_i(t_j)) + (1 - \theta)w \right); \\
\psi_{i,j} &= \frac{g}{\Delta x} \left( \theta \frac{A_i + \frac{1}{2}}{A_i + \frac{1}{2}}(t_{j-1}) + (1 - \theta)A \right); \\
\tau_{i,j} &= \frac{1}{\Delta t} + \theta \frac{g}{C_i^2} \frac{P_i + \frac{1}{2}(t_{j-1})sabs Q_i(t_{j-1})}{A_i + \frac{1}{2}(t_{j-1})^2} + (1 - \theta) \frac{g}{C_i^2} \frac{P_{sabs}Q_i}{A_i^2}.
\end{align*}
\]

We constructed the functions \( A \) and \( P \) such that \( A > 0, \frac{\partial A}{\partial H} > 0, \) and \( P > 0 \) for every \( H \). Furthermore, \( sabs Q > 0 \) for all \( Q \). Therefore the terms \( \phi_{i,j}, \psi_{i,j} \) and \( \tau_{i,j} \) must be nonzero for every time step \( j \in \{1, \ldots, T\} \) and for any \( i \in I_Q \).

We now proceed to prove that equation (4.7) admits a unique solution. We first illustrate the reasoning for the simpler case when \( T = 1 \). For this we want to show that equation (4.8)

\[
\sum_{i \in I_H} \alpha_i \frac{\partial c_{i,1}}{\partial x_{hyd}} \bigg|_{x_{i,j}} + \sum_{i \in I_Q} \beta_i \frac{\partial d_{i,1}}{\partial x_{hyd}} = 0,
\]

is satisfied only if \( \alpha_i = \beta_i = 0 \) for all \( i \in I_H \) and \( i \in I_Q \), respectively.

Consider the \(|I_H| + |I_Q|\)-square matrix \( M \) obtained by stacking on top of each other the gradients of the hydraulic constraints; i.e., the matrix whose rows are the gradients of the hydraulic constraints. The columns of \( M \) are indexed by the interior variables \( x_{hyd} \) and Equation (4.8) has a unique solution if and only if the rows of \( M \) are linearly independent. When permuting rows such that continuity and momentum equations alternate, and columns such that flow and level variables alternate, \( M \) is a tridiagonal matrix having the property that the \((m, n)\)-entry of this matrix is nonzero if and only if \(|m - n| \leq 1 \). Clearly, such a matrix has full rank and thus its rows are linearly independent. As stated previously, this is equivalent to showing that Equation (4.8) has a unique solution.

We will now prove the more general statement. Equation (4.7) is satisfied only if it holds even when we consider only part of its variables; i.e., equation

\[
\sum_{i \in I_H, j \in \{1, \ldots, T\}} \alpha_{i,j} \frac{\partial c_{i,j}}{\partial x_{hyd}} \bigg|_{x_{i,j}} + \sum_{i \in I_Q, j \in \{1, \ldots, T\}} \beta_{i,j} \frac{\partial d_{i,j}}{\partial x_{hyd}} \bigg|_{x_{i,j}} = 0
\]

holds for any subvector \( \bar{x} \) of \( x_{hyd} \). We will use this simple observation to argue about the \( \alpha \)'s and \( \beta \)'s in Equation (4.7).

Let \( x_T \) denote the subvector of \( x_{hyd} \) that contains all the internal hydraulic variables at time step \( T \), i.e., the variables \( H_i(t_T), Q_i(t_T) \) for \( i \in I_H \) and \( i \in I_Q \), respectively. As the variables of \( x_T \) appear only in the gradients of the constraints \( c_{i,T}, d_{i,T} \)
for \( i \in I_H \) and \( i \in I_Q \), respectively, the following holds:

\[
0 = \sum_{i \in I_H, j \in \{1, \ldots, T\}} \alpha_{i,j} \frac{\partial c_{i,j}}{\partial x_{\text{hyd}}} \bigg|_{x_T} + \sum_{i \in I_Q, j \in \{1, \ldots, T\}} \beta_{i,j} \frac{\partial d_{i,j}}{\partial x_{\text{hyd}}} \bigg|_{x_T} \\
= \sum_{i \in I_H} \alpha_{i,T} \frac{\partial c_{i,T}}{\partial x_{\text{hyd}}} \bigg|_{x_T} + \sum_{i \in I_Q} \beta_{i,T} \frac{\partial d_{i,T}}{\partial x_{\text{hyd}}} \bigg|_{x_T}.
\]

We claim that the above equation has a solution only when all the \( \alpha_{i,T} \) and the \( \beta_{i,T} \) are equal to zero. Let \( M_T \) be the matrix whose rows are the gradients of the hydraulic constraints \( \partial c_{i,T} \big|_{x_T} \), \( \partial d_{i,T} \big|_{x_T} \) for \( i \in I_H \) and \( i \in I_Q \), respectively, restricted to the variables \( x_T \). That is, \( M_T \) is a \(|I_H| + |I_Q|\)-square matrix. When permuting rows such that continuity and momentum equations alternate, and columns such that flow and level variables alternate, \( M_T \) is a tridiagonal matrix whose \((m, n)\)-entry is non-zero if and only if \(|m - n| \leq 1 \) and, hence, the rows of \( M_T \) are linearly independent. By construction, this is equivalent to saying that \( \alpha_{i,T} = \beta_{i,T} = 0 \) for all \( i \in I_H \) and \( i \in I_Q \), respectively.

Let \( x_{T-1} \) denote the subvector of \( x_{\text{hyd}} \) that contains all the internal hydraulic variables at time step \( T-1 \), i.e., the variables \( H_i(t_{T-1}), Q_i(t_{T-1}) \) for \( i \in I_H \) and \( i \in I_Q \), respectively. Using the fact that the variables of \( x_{T-1} \) appear only in the gradient of the constraints \( \partial c_{i,T-1}, \partial d_{i,T-1}, \partial d_{i,T} \) for \( i \in I_H \) and \( i \in I_Q \), respectively, and that all the \( \alpha_{i,T} \) and the \( \beta_{i,T} \) are equal to zero, we have that:

\[
0 = \sum_{i \in I_H, j \in \{1, \ldots, T\}} \alpha_{i,j} \frac{\partial c_{i,j}}{\partial x_{\text{hyd}}} \bigg|_{x_{T-1}} + \sum_{i \in I_Q, j \in \{1, \ldots, T\}} \beta_{i,j} \frac{\partial d_{i,j}}{\partial x_{\text{hyd}}} \bigg|_{x_{T-1}} \\
= \sum_{i \in I_H} \alpha_{i,T-1} \frac{\partial c_{i,T-1}}{\partial x_{\text{hyd}}} \bigg|_{x_{T-1}} + \sum_{i \in I_Q} \beta_{i,T-1} \frac{\partial d_{i,T-1}}{\partial x_{\text{hyd}}} \bigg|_{x_{T-1}}.
\]

By looking at the square matrix \( M_{T-1} \) whose rows are the gradients of the constraints \( \partial c_{i,T-1}, \partial d_{i,T-1} \) restricted to the variables \( x_{T-1} \), we can use the same argument as before to show that all the \( \alpha_{i,T-1} \) and the \( \beta_{i,T-1} \) must be equal to zero.

Repeating the reasoning when considering Equation (4.9) for the internal hydraulic variables at time step \( T-2 \), then \( T-3 \) and so on, we can conclude that all the \( \alpha_{i,j} \) and \( \beta_{i,j} \) in equation (4.7) must be equal to zero. Here, we have used the fact that the initial condition is fully specified (ICO). This concludes the proof. \( \square \)

**Remark 7.** Lemma 4.2 crucially depends on the semi-implicit discretization of the pressure and friction terms. If these terms were discretized fully implicitly, then particular combinations of \( H \) and \( Q \) would lead to vanishing \( \psi_{i,j} \) and hence to singular points. In the same vein, vanishing \( \psi_{i,j} \) and \( \tau_{i,j} \) introduce singular points when considering a time-implicit discretization of the convective acceleration term.

**Corollary 4.3.** Assume \( \text{BND} \), \( \text{ICO} \), \( \text{HBC} \), and \( \text{QBC} \). Then, for any \( \theta \in [0, 1] \), the set of feasible interior points is non-empty and path-connected.

**Proof.** Fix \( \theta \in [0, 1] \). Partition the vector \( x \) into interior hydraulic and boundary components,

\[ x := (x_{\text{hyd}}, x_{\text{bdy}}). \]

For any \( x_{\text{bdy}} \), we may integrate the dynamics forwards in time, starting from the initial conditions (ICO). In [5], Theorem 1 and 2, Casulli shows that the solution of
the equations (4.5) and (4.6) exists and is unique. Note here that our construction is one-dimensional, so that for Casulli’s element volume $V$ we use

$$V = \Delta x (\theta A + (1 - \theta)\pi H).$$

We also do not treat wetting and drying in same way. By our construction $A > 0$ and $\partial A/\partial H > 0$ for all $H$, so that Casulli’s Theorem 1 always holds. Hence, we have a unique map $g : x_{bdy} \mapsto x_{hyd}$ that is defined for all $x_{bdy}$. By Lemma (4.2) and the implicit function theorem [20], this function $g$ is locally continuous, hence continuous everywhere$^4$.

Finally, note that the set of interior boundary variables is convex, hence path-connected. Therefore the image under $g$ is also path-connected. Since the interior hydraulic variables are unbounded (BND), the set of feasible interior points is the direct sum of the boundary variables and their image under $g$. Since both sets are path-connected, the set of feasible interior points is also path-connected.

\textbf{Lemma 4.4.} Assume BND, HBC, and OBJ. Partition the vector $x$ into interior hydraulic and boundary components,

$$x := (x_{hyd}, x_{bdy}).$$

Then the Hessian of the Lagrangian with respect to $x_{bdy}$, $\nabla^2_{x_{bdy}x_{bdy}} \mathcal{L}_\mu(x, \lambda, \theta)$, is positive definite.

\textbf{Proof.} Since we assume all boundary variables to be bounded (BND), the second derivatives of the logarithmic barrier functions with respect to $x_{bdy}$ form a positive definite diagonal matrix. To this we add the Hessian of the objective function $f$ with respect to $x_{bdy}$, which is positive semi-definite due to the convexity of $f$ (OBJ).

As the sign of the Lagrange multipliers is not known a-priori, the Hessians of the constraints may have an indefinite contribution to the Hessian of the Lagrangian, potentially resulting in a loss of positive semi-definiteness. This, however, can only happen for constraints that are nonlinear in $x_{bdy}$. As per (HBC), $H$ boundaries are fixed and hence do not occur as optimization variables. Free $Q$ boundaries do occur, but only the mass balance equation (4.5) and the convective acceleration term in the momentum equation (4.6) depend on boundary $Q$ variables. Both the mass balance equation and the convective acceleration term are linear in the boundary flow variables, and therefore do not contribute to $\nabla^2_{x_{bdy}x_{bdy}} \mathcal{L}_\mu$.

Hence, the Hessian matrix $\nabla^2_{x_{bdy}x_{bdy}} \mathcal{L}_\mu$ is positive definite.

\textbf{Proposition 4.5.} Assume BND, ICO, HBC, QBC, and OBJ. Then the optimization problem $F_\mu(x, \lambda, \theta) = 0$ is path-stable with respect to the interior point method.

\textbf{Proof.} Fix $\mu > 0$ and $\theta \in [0, 1]$. Partition the vector $x$ into interior hydraulic and boundary components,

$$x := (x_{hyd}, x_{bdy}).$$

(4.10)

With respect to the partitioning (4.10) the Jacobian matrix $\partial F_\mu(x, \lambda, \theta)/\partial(x, \lambda)$ has

$^4$If we would have used a non-mass-conservative discretization of the continuity equation, with $\partial A/\partial t$ discretized as $\partial A/\partial H(t_{j-1}): \Delta H(t_j)/\Delta x$, the equations (4.5) - (4.6) would both be linear in the variables at time step $t_j$. Then, existence, uniqueness, and continuity would follow directly from Lemma (4.2) and the continuity of the matrix inverse in the matrix coefficients.
The fact that every flow boundary variable appears in at least one and at most two hydraulic constraints, and that every hydraulic constraint depends on no more than one boundary variable (QBC), implies that the block 
\[ \nabla^2 x_{\text{bdy}} \nabla^2 x_{\text{hyd}} L \mu \nabla^T x_{\text{bdy}} C \]  
contains at least \( \dim x_{\text{bdy}} \times \dim x_{\text{hyd}} \) linearly independent columns with a single nonzero. This allows us to, using elementary column operations, eliminate the block 
\[ \nabla^2 x_{\text{bdy}} x_{\text{hyd}} L \mu \nabla^T x_{\text{bdy}} C \] .

The overall Jacobian now has the form
\[ \begin{pmatrix} B & \nabla^2 x_{\text{bdy}} x_{\text{hyd}} L \mu & \nabla^T x_{\text{bdy}} C \\ 0 & \nabla^2 x_{\text{bdy}} x_{\text{hyd}} L \mu & \nabla^T x_{\text{bdy}} C \\ \nabla x_{\text{hyd}} C & \nabla x_{\text{hyd}} C & 0 \end{pmatrix}, \]

where \( B \) denotes the square block that results as a byproduct from the elementary column operations applied to eliminate the block \( \nabla^2 x_{\text{bdy}} x_{\text{hyd}} L \mu \).

Lemma 4.2 implies that the square block \( \nabla x_{\text{hyd}} C \) has full rank. This allows us to, using elementary row operations, transform the top-left block \( \nabla^2 x_{\text{bdy}} x_{\text{hyd}} L \mu \) into a nonsingular, upper-triangular matrix \( D \). The overall Jacobian now has the form
\[ \begin{pmatrix} D & B' & \nabla^2 x_{\text{bdy}} x_{\text{hyd}} L \mu & \nabla^T x_{\text{bdy}} C \\ 0 & \nabla^2 x_{\text{bdy}} x_{\text{hyd}} L \mu & \nabla^T x_{\text{bdy}} C \\ \nabla x_{\text{hyd}} C & \nabla x_{\text{hyd}} C & 0 \end{pmatrix}, \]

where \( B' \) denotes the block that results as a byproduct from the elementary row operations applied to obtain the nonsingular upper-triangular matrix \( D \).

Lemma 4.4 implies that \( \nabla^2 x_{\text{bdy}} x_{\text{hyd}} L \mu \) is a nonsingular matrix. Consequently, the submatrix
\[ A := \begin{pmatrix} D & B' \\ 0 & \nabla^2 x_{\text{bdy}} x_{\text{hyd}} L \mu \end{pmatrix} \]
is also nonsingular. Taking the Schur complement \( [24] \) with respect to the matrix \( A \), and using the fact that Lemma (4.2) implies that \( \nabla x C \) is full rank, we have
\[
\text{rank} \left( \frac{\partial F_\mu(x, \lambda, \theta)}{\partial (x, \lambda)} \right) = \text{rank} A + \text{rank}(\nabla x C A^{-1} \nabla^T x c) = \text{rank} A + \text{rank} \nabla x C = \dim x + \dim c.
\]

which is exactly the dimensionality of our Jacobian. Hence, the Jacobian matrix \( \partial F_\mu(x, \lambda, \theta)/\partial (x, \lambda) \) is nonsingular.

Combining the above results for zero-convexity, path-stability, and the path-connectedness of the search space, we obtain the following uniqueness theorem for optimization problems constrained by the shallow water equations:

**Theorem 4.6.** Assume BND, ICO, HBC, QBC, and OBJ. Then the nonconvex optimization problem \( F_\mu(x, \lambda, \theta) = 0 \) has a unique solution for every \( \mu > 0 \) and every \( \theta \in [0, 1] \).

Proof. Existence follows from Corollary (4.3). Uniqueness follows from Propositions (4.1) and (4.5), Corollary (4.3), and Theorem (3.8).}

Theorem 4.6 implies that the optimization problem \( F_\mu(x, \lambda, \theta) = 0 \) can be solved to global optimality using a continuation method.
Fig. 3: Staggered grid for the example problem.

Table 1: Parameters for the example problem.

| Parameter | Value                          | Description                                      |
|-----------|--------------------------------|--------------------------------------------------|
| $T$       | 72                             | Index of final time step                         |
| $\Delta t$ | 600 s                         | Time step size                                   |
| $H_i^{b}$ | ($-4.90, -4.92, \ldots, -5.10$) m | Bottom level                                    |
| $l$       | 10000 m                        | Total channel length                             |
| $A_i(H_i)$ | $50 \cdot (H - H_i^{b})$ m$^2$ | Channel cross section function                   |
| $P_i(H_i)$ | $50 + 2 \cdot (H - H_i^{b})$ m | Channel wetted perimeter function                |
| $C_i$     | $(40, 40, \ldots, 40)$ m$^{0.5}$/s | Chézy friction coefficient                      |
| $\mathcal{H}$ | 0.0 m                        | Nominal level in linear model for entire reach   |
| $\bar{Q}$ | 100 m$^3$/s                    | Nominal discharge in linear model for entire reach |
| $H_i(t_0)$ | (0.000, -0.025, $\ldots$, -0.222) m | Initial water levels at $H$ nodes |
| $Q_i(t_0)$ | (100, 100, $\ldots$, 100) m$^3$/s | Initial discharge at $Q$ nodes                  |
| $\varepsilon$ | $10^{-12}$                    | Absolute value approximation smoothness parameter |
| $K$       | 10                             | Convective acceleration steepness factor          |

4.5. Numerical example. We consider a single river reach with 10 uniformly spaced water level nodes and rectangular cross section, an upstream inflow boundary condition provided with a fixed time series, as well as a controllable downstream release boundary condition. The grid is illustrated in Figure 3, and the hydraulic parameters and initial conditions are summarized in Table 1. The model starts from steady state: the initial flow rate is uniform and the water level decreases linearly along the length of the channel.

To give a physical context for this problem, suppose this model represents a channel downstream of a reservoir and upstream of an adjustable weir with limited capacity. The weir is trying to dampen the sudden pulse of water shown in Figure 4(b) released by the reservoir.

Our optimization objective is to keep the water level at the $H$ nodes at 0 m above datum:

$$\min \sum_{i=1}^{10} \sum_{j=1}^{T} H_i(t_j)^2$$

subject to the adjustable weir flow constraint

$$100 \text{ m}^3/\text{s} \leq Q_{10} \leq 200 \text{ m}^3/\text{s}.$$ 

The solution to the optimization problem is plotted in Figure 4. By releasing water in anticipation of the inflow using the decision variable $Q_{10}$, the optimization
is able to reduce water level fluctuations and keep the water levels close to the target level.

This optimization problem was implemented in Python using the CasADi package [2] for algorithmic differentiation, and connected to the IPOPT optimization solver [26]. On a MacBook Pro with 2.9 GHz Intel Core i5 CPU, the example takes approximately 0.4 s to solve. The complete source code is available online at https://github.com/jbaayen/homotopy-example.

5. Conclusions. In the first part of this paper, we provided sufficient conditions for a (nonconvex) optimization problem to be solvable to global optimality with a continuation method.

The proof rests on a homotopy between a convex relaxation of the optimization problem and the original nonconvex optimization problem. This homotopy may be leveraged by a numerical continuation method to find a global optimum of the nonconvex problem. Unlike a Lasserre hierarchy, the resulting method does not suffer from an increasing number of optimization variables. The low computational complexity renders the method suitable for closed-loop model predictive control of large-scale cyber-physical systems.

In the second part of the paper, an application was presented to optimization problems subject to the discretized shallow water equations, a system of partial differential equations that describes flow in rivers and canals. The convergence of the homotopy procedure was analyzed, and a numerical example was presented. A software implementation of the method for the shallow water equations is in day-to-day use for closed-loop model predictive control of the primary waterways of the Rijnland water authority. The Rijnland system comprises approximately 370 km of primary waterways, controlled using 4 primary pumping stations, and covers the city of Leiden and Amsterdam Schiphol Airport in the Netherlands.
6. Acknowledgments. The authors would like to thank Jan van Schuppen, Pierre Archambeau, Krzysztof Postek, Jakub Mareček, and Dirk Schwanenberg for their comments.

TKI Delta Technology provided part of the funding under projects DEL021 and DEL029.

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