THE BERNSTEIN CENTER IN NATURAL CHARACTERISTIC

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Abstract. Let $G$ be a locally profinite group and let $k$ be a field of positive characteristic $p$. Let $Z(G)$ denote the center of $G$ and let $\mathfrak{Z}(G)$ denote the Bernstein center of $G$, that is, the $k$-algebra of natural endomorphisms of the identity functor on the category of smooth $k$-linear representations of $G$. We show that if $G$ contains an open pro-$p$ subgroup but no proper open centralisers, then there is a natural isomorphism of $k$-algebras $\mathfrak{Z}(Z(G)) \cong \mathfrak{Z}(G)$. We also describe $\mathfrak{Z}(Z(G))$ explicitly as a particular completion of the abstract group ring $k[Z(G)]$. Both conditions on $G$ are satisfied whenever $G$ is the group of points of any connected smooth algebraic group defined over a local field of residue characteristic $p$. In particular, when the algebraic group is semisimple, we show that $\mathfrak{Z}(G) = k[Z(G)]$.

1. Introduction

Let $G$ be a locally profinite group and let $k$ be any field. Recall that a $k$-linear representation $V$ of $G$ is said to be smooth if every vector in $V$ is fixed by an open subgroup of $G$. The smooth $k$-linear representations of $G$ form an abelian category $\text{Mod}(G)$ and the Bernstein center $\mathfrak{Z}(G)$ of $\text{Mod}(G)$ is by definition the ring of natural endomorphisms of the identity functor on $\text{Mod}(G)$: it is naturally a commutative $k$-algebra which acts on every $V$ in $\text{Mod}(G)$ by $k$-linear endomorphisms commuting with the action of $G$ of $V$.

When $G$ is a reductive group over a local non-archimedean field of mixed characteristic $(0,p)$ and $k$ is the complex numbers, the Bernstein center $\mathfrak{Z}(G)$ was studied in detail by Bernstein in [Ber]. He found that $\mathfrak{Z}(G)$ decomposes as a direct product of smaller algebras that are now called the Bernstein components, and he found a parametrisation of these components in terms of $G$-conjugacy classes of cuspidal pairs on the Levi subgroups of $G$. Consequently, in this classical case, the Bernstein center is rather large, and it plays a fundamental role in the classical local Langlands correspondence [Hen].

Recently, there has been a lot of interest in a possible mod-$p$ local Langlands correspondence, where instead one works with smooth representations defined over a field $k$ of characteristic $p$; see, for example, [Vig04], [BP], [Sch] and [BH+]. It is therefore natural to enquire about the structure of $\mathfrak{Z}(G)$. There has been an expectation that in this case $\mathfrak{Z}(G)$ is small, which we will confirm in this paper by a precise explicit calculation.

We begin the description of our results by introducing the following completion of the group algebra $k[Z(G)]$ of the center $Z(G)$ of $G$:

$$k[Z(G)] := \varprojlim k[Z(G)/Z']$$

where $Z'$ runs over all compact open subgroups of $Z(G)$. By considering the natural action of $k[Z(G)]$ on the subspace $V^U$ of $U$-fixed points of a given object $V$ in $\text{Mod}(G)$ as $U$ ranges over all compact open subgroups of $G$, it is not difficult to see that the algebra $k[Z(G)]$ acts by $k[G]$-linear endomorphisms on every $V$ in $\text{Mod}(G)$ in a natural way. This defines a $k$-algebra...
homomorphism $\Phi_G : k[Z(G)] \to \mathcal{Z}(G)$ which is easily seen to be injective. Using this map we can now state our main result.

**Theorem 1.1.** Let $G$ be a locally profinite group that contains an open pro-$p$ subgroup and let $k$ be a field of characteristic $p$. Suppose that $G$ contains no proper open centralisers. Then the natural map $\Phi_G : k[Z(G)] \to \mathcal{Z}(G)$ is an isomorphism.

We show in Thm. 6.13 below that the assumptions on $G$ are satisfied whenever $G = G(\mathfrak{F})$ for some connected smooth algebraic group $G$ defined over a local nonarchimedean field $\mathfrak{F}$ of residue characteristic $p$. By applying Theorem 1.1 to $G$ and to $Z(G)$ in turn, we obtain the following

**Corollary 1.2.** Under the assumptions of Theorem 1.1, the natural map

$$\Phi_G \circ \Phi_Z^{-1} : \mathcal{Z}(Z(G)) \to \mathcal{Z}(G)$$

is an isomorphism of $k$-algebras.

Here is a brief summary of the contents of this paper. In §3 we use a result of Positselski $\text{Pos}_2$ to show that $\mathcal{Z}(G)$ is naturally isomorphic to the $k$-algebra of bi-equivariant $k$-linear endomorphisms $\text{End}_{\text{Mod}(G \times G)}(C_c^\infty(G, k))$ of the space $C_c^\infty(G, k)$ of compactly supported locally constant $k$-valued functions on $G$. In §4, we use the fact that $C_c^\infty(G, k)$ is isomorphic in $\text{Mod}(G)$ to the compact induction from $U$ to $G$ of the corresponding space of locally constant $k$-valued functions on some fixed open pro-$p$ subgroup $U$ of $G$, together with a version of the Mackey decomposition, to reduce the calculation of the bi-equivariant endomorphism algebra to the calculation of the fixed points in some profinite permutation modules $k[|U|]$ under a twisted conjugation action of various open subgroups $U_w$ of $U$. In §5 we crucially use the hypothesis that $k$ has characteristic $p$ to show that in this situation, all fixed points in such profinite permutation modules arise from the finite orbits. In §6 we use the hypothesis on open centralisers in $G$ to show that there are no such finite $U_w$-orbits in $G$ — unless $w$ happens to be central in $G$ — and deduce Theorem 1.1 which reappears as Theorem 6.10.

We also explain how to verify the hypothesis on open centralisers in the case of connected algebraic groups. Finally, in §7 we give a self-contained proof of Positselski’s Theorem, for the convenience of the reader unfamiliar with the categorical machinery in $\text{Pos}_2$.

We thank F. Pop and M.-F. Vignéras for pointing out that Prop. 6.12 is known to the experts and for indicating an argument. After this paper was finished, A. Dotto has informed us that he had proved (using different methods) that for certain $p$-adic reductive groups, the Bernstein center is a local ring when one fixes a central character.

The second author acknowledges support from Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy EXC 2044 –390685587, Mathematics Münster: Dynamics–Geometry–Structure.

2. **Notation**

Throughout $G$ is a locally profinite group, and $k$ is any field. If $G$ contains an open pro-$p$ subgroup then we call $G$ locally pro-$p$. By $\text{Op}_e(G)$ we denote the set of open compact subgroups of $G$. Furthermore, $\text{Mod}(G)$, resp. $\text{Vect}$, denotes the abelian category of all smooth $G$-representations in $k$-vector spaces, resp. of all $k$-vector spaces. As usual, $Z(G)$ is the center of $G$. Similarly, $Z(R)$, for a ring $R$, is the center of $R$. For any topological space $X$ we let $C_c^\infty(X, k)$ denote the $k$-vector space of $k$-valued locally constant functions with compact
support on \(X\). If \(X\) is compact, resp. discrete, we simply write \(C^\infty(X, k)\), resp. \(\mathcal{C}_c(X, k)\). For a compact open subset \(C \subseteq X\) let \(\text{char}_C \in \mathcal{C}_c^\infty(X, k)\) denote the characteristic function of \(C\). For any group \(\Gamma\) acting on a set \(Y\) and any subset \(P\) of \(Y\), we write \(\Gamma_P\) to denote the stabiliser of \(P\) in \(\Gamma\).

3. The Bernstein center of \(\text{Mod}(G)\)

The Bernstein center \(\mathfrak{Z}(G)\) of the abelian category \(\text{Mod}(G)\) by definition is the ring of natural endomorphisms of the identity functor on \(\text{Mod}(G)\). It obviously is a commutative \(k\)-algebra, which acts functorially on any object in \(\text{Mod}(G)\). In order to describe it in more down to earth terms we observe that the group \(G \times G\) acts smoothly on \(\mathcal{C}_c^\infty(G, k)\) by \((g_1, g_2)F(-) := F(g_1^{-1} \cdot g_2)\). In the following we write \(G_t\), resp. \(G_r\), and correspondingly \(g_t\) and \(g_r\), if we refer to the action of the left, resp. right, factor \(G\), and correspondingly \(g \in G\).

We first make the larger \(k\)-algebra \(\mathfrak{A}(G)\) of all natural endomorphisms of the forgetful functor \(\text{Mod}(G) \to \text{Vect}\) more explicit. An element of \(\mathfrak{A}(G)\), of course, is a family of \(k\)-linear endomorphisms \(T_V\) for any \(V\) in \(\text{Mod}(G)\) which commute, in an obvious sense, with any morphism in \(\text{Mod}(G)\). Any \(g \in G\) via its action on the \(V\) can be seen as an element \((g_V)_V \in \mathfrak{A}(G)\). Viewing \(\mathcal{C}_c^\infty(G, k)\) as an object in \(\text{Mod}(G)\) via the \(G_t\)-action we consider the \(k\)-algebra homomorphism

\[
\mathfrak{A}(G) \to \text{End}_k(\mathcal{C}_c^\infty(G, k))
\]

\[
(T_V)_V \mapsto T_{C_c^\infty(G, k)}.
\]

Observing that the \(G_t\)-action on \(\mathcal{C}_c^\infty(G, k)\) is by \(G_t\)-equivariant maps we see that any \(T_{\mathcal{C}_c^\infty(G, k)}\) has to commute with this \(G_r\)-action. The above map therefore is, in fact, a map

\[\Theta : \mathfrak{A}(G) \to \text{End}_{\text{Mod}(G_t)}(\mathcal{C}_c^\infty(G, k)).\]

We then have the following result due to Positselski ([Pos Prop. 3.6(a)]). See section 7 for an alternative proof.

**Proposition 3.1.** The map \(\Theta\) is an isomorphism.

In fact we will only need the injectivity of \(\Theta\) the proof of which is easy.

Obviously \(\mathfrak{Z}(G) = \{(T_V)_V \in \mathfrak{A}(G) : (T_V \circ g_V)_V = (g_V \circ T_V)_V\}\) for any \(g \in G\). But as a consequence of Prop. 3.1 the condition on the right hand side only needs to be verified on \(\mathcal{C}_c^\infty(G, k)\). This implies the following description.

**Corollary 3.2.** \(\mathfrak{Z}(G) = \text{End}_{\text{Mod}(G \times G)}(\mathcal{C}_c^\infty(G, k)).\)

We will now write down some obvious elements in \(\mathfrak{Z}(G)\).

**Definition 3.3.** Let \(Z\) be a locally profinite abelian group, and let \(\widehat{k[Z]}\) denote the following completion of \(k[Z]\):

\[
\widehat{k[Z]} := \lim_{\substack{\longrightarrow \cr Z' \in \text{Op}_Z(Z)}} k[Z/Z'].
\]

To get some feeling for this completion, consider the following special case.

**Remark 3.4.** Let \(Z\) be a locally profinite abelian group. Suppose there is a compact open subgroup \(Z_0\) of \(Z\) and a discrete subgroup \(A\) of \(Z\) such that \(Z = A \times Z_0\). Then \(\widehat{k[Z]}\) is
isomorphic to the usual completed group ring of the profinite group $Z_0$ but with coefficients taken in the discrete $k$-algebra $k[A]$: 

\[
\hat{k}[Z] \cong k[A][[Z_0]] := \lim_{Z_1 \in \text{Op}_c(Z_0)} k[A][Z_0/Z_1]
\]

If $A$ is finite this simplifies to $\hat{k}[Z] = k[A] \otimes_k k[[Z_0]]$.

**Example 3.5.** Suppose that $Z = \mathfrak{f}^\times$ for some finite extension $\mathfrak{f}$ of $\mathbb{Q}_p$. Fix a uniformiser $\pi \in \mathfrak{o}_\mathfrak{f}$, let $A$ be the subgroup of $Z$ generated by $\pi$ and let $Z_0 := \mathfrak{o}_\mathfrak{f}^\times$. Then $Z, A$ and $Z_0$ satisfy the conditions of Remark 3.4. Therefore in this case, $\hat{k[\mathfrak{f}^\times]}$ is the completed group ring of the compact abelian $p$-adic Lie group $\mathfrak{o}_\mathfrak{f}^\times$ with coefficients in the Laurent polynomial ring $k[A] \cong k[\pi, \pi^{-1}]$ in the variable $\pi$.

**Lemma 3.6.** $\hat{k[Z(G)]]}$ acts naturally on every $V \in \text{Mod}(G)$ by $k[G]$-linear endomorphisms.

**Proof.** For every compact open subgroup $H$ of $G$, restrict the $Z := Z(G)$-action on $V$ to $V^H = \{ v \in V : h \cdot v = v \text{ for all } h \in H \}$. Since the action of $Z$ on $V$ commutes with the action of $H$, $V^H$ is a $Z$-submodule of $V$. The normal subgroup $Z \cap H$ of $Z$ acts trivially on $V^H$ and therefore the $Z$-action on $V^H$ factors descends to a well-defined action of $Z/(Z \cap H)$ on $V^H$. By the definition of the completion $\hat{k[Z]}$, there is a canonical map from this completion to $\hat{k[Z/(Z \cap H)]}$, because $Z \cap H$ is a compact open subgroup of $Z$. In this way, every $V^H$ is naturally a $\hat{k[Z]}$-module, via this homomorphism. Let $J$ be another compact open subgroup of $G$, contained in $H$. Then $V^H$ is contained in $V^J$, and both are $\hat{k[Z]}$-modules as explained above. We observe that the inclusion map $V^H \to V^J$ is in fact $\hat{k[Z]}$-linear.

We have now defined a $\hat{k[Z]}$-action on $V^H$ for every compact open subgroup $H$ of $G$ and we have checked that these actions are compatible with the inclusions $V^H \to V^J$ whenever $J \subseteq H$ is a smaller compact open subgroup. Therefore there is a well-defined action of $\hat{k[Z]}$ on $V = \bigcup_H V^H$.

It remains to check that the $\hat{k[Z]}$-action we have constructed commutes with the given $G$-action on $V$. Fix $x \in \hat{k[Z]}$, $g \in G$ and $v \in V$. Choose a compact open subgroup $H$ of $G$ such that $v \in V^H$; then $v$ and $g \cdot v$ both lie in $V^J$ where $J := H \cap gHg^{-1} \in \text{Op}_c(G)$. The action of $x$ on $V^J$ is equal to the action of its image $\overline{x}$ in $k[Z(J)]$. Now $zJ \cdot (g \cdot v) = g \cdot (zJ \cdot v)$ for every $zJ \in Z(J)$ so $x \cdot (g \cdot v) = \overline{x} \cdot (g \cdot v) = g \cdot (\overline{x} \cdot v) = g \cdot (x \cdot v)$ as required. \(\square\)

Let $x \mapsto \Phi_V(x) \in \text{End}_{\text{Mod}(G)}(V)$ denote the action of $x \in \hat{k[Z(G)]]}$ on $V \in \text{Mod}(G)$ that was constructed in Lemma 3.6.

**Lemma 3.7.** The map $x \mapsto \Phi(x) := (\Phi_V(x))_V$ is a $k$-algebra homomorphism 

\[
\Phi : \hat{k[Z(G)]]} \to \mathfrak{g}(G).
\]

**Proof.** The action of $k[Z(G)]$ commutes with every morphism $\varphi : V \to W$ in $\text{Mod}(G)$. This implies that $\Phi_V(x) \circ \varphi = \varphi \circ \Phi_V(x)$ for every $x \in \hat{k[Z(G)]]}$. So $(\Phi_V(x))_V$ does define an element $\Phi(x)$ in $\mathfrak{g}(G)$. The verification that $\Phi$ is a $k$-algebra homomorphism is straightforward. \(\square\)

**Proposition 3.8.** Let $C := C^\infty_c(G, k)$, regarded as an object in $\text{Mod}(G)$. Then

1. $\Phi_C : \hat{k[Z(G)]]} \to \text{End}_{\text{Mod}(G)}(C)$ is injective, and
Proof. (1) Write $Z := Z(G)$. It is enough to show that $k[Z/(Z \cap U)]$ acts faithfully on $C^U = C^\infty_c(G,k)^{U_x} = C^c_c(U\backslash G,k)$ for every compact open subgroup $U$ of $G$. For any $x \in G$, consider the characteristic function $\text{char}_{Ux} \in C_c(U\backslash G,k)$ of the coset $Ux \in U\backslash G$; if $z \in Z$ then $z_x \text{char}_U = \text{char}_{Uz}$ shows that $\text{char}_U$ generates a $k[Z/(Z \cap U)]$-submodule of $C^U$ which is free of rank one. Hence $k[Z/(Z \cap U)]$ acts faithfully on $C^U$.

(2) This follows from the construction of $\Phi_C : k[Z] \to \text{End}_{\text{Mod}(G)}(C)$: it suffices to see that $\Phi_C(k[Z])$ commutes with the action of $Gr_r$, but $\Phi_C(z)(f) = z f$ for any $f \in C$ and any $z \in Z$, and $z_f$ commutes with $Gr_r$. □

We now can easily compute the Bernstein center for commutative groups.

**Proposition 3.9.** If $G = Z$ is commutative then the maps

$$k[Z] \xrightarrow{\Phi} \mathcal{Z}(Z) \xrightarrow{\Theta} \text{End}_{\text{Mod}(Z)}(C^\infty_c(Z,k))$$

are isomorphisms.

**Proof.** We already know from Cor. 3.2 that the right hand map is an isomorphism. But for this proof we only use its injectivity which is very easy to see (cf. section 7). It therefore suffices to establish that the composed map $\Theta \circ \Phi = \Phi_{C^\infty_c(Z,k)}$ is bijective. We observe that

$$\text{End}_{\text{Mod}(Z)}(C^\infty_c(Z,k)) = \text{Hom}_{\text{Mod}(Z)} \left( \bigcup_{Z' \in \text{Op}_c(Z)} C_c(Z/Z',k), C^\infty_c(Z,k) \right)$$

$$= \lim_{Z' \in \text{Op}_c(Z)} \text{Hom}_{\text{Mod}(Z)}(C_c(Z/Z',k), C^\infty_c(Z,k))$$

$$= \lim_{Z' \in \text{Op}_c(Z)} \text{End}_{\text{Mod}(Z)}(C_c(Z/Z',k)) .$$

This shows that $\Phi_{C^\infty_c(Z,k)} = \lim_{Z' \in \text{Op}_c(Z)} \Phi_{C_c(Z/Z',k)}$ and reduces us to the case of a discrete commutative group $Z$. But in this case $C_c(Z,k)$ is a free module of rank one over $k[Z]$. □

**Corollary 3.10.** The composed map

$$\mathcal{Z}(Z(G)) \xrightarrow{\Phi^{-1}} k[Z(G)] \xrightarrow{\Phi} \mathcal{Z}(G)$$

between Bernstein centers is an injective homomorphism of $k$-algebras.

4. THE UNIVERSAL ENDOMORPHISM RING

One of our goals is the computation of the endomorphism ring $\text{End}_{\text{Mod}(G \times G)}(C^\infty_c(G,k))$. We fix a compact open subgroup $U \subseteq G$. We also choose a set $W \subseteq G$ of representatives of the double cosets of $U$ in $G$. For the background for the subsequent points A) to D) see [Vig96] Chap. I.5.

**A) Transitivity of induction:** We have the $G_c$-equivariant isomorphism

(1) $$\text{ind}^{G_c}_{U^c}(C^\infty_c(U,k)) \xrightarrow{\sim} C^\infty_c(G,k)$$

$$\phi \mapsto F_\phi(g) := \phi(g)(1) .$$
Note that $\phi(g)(u) = F_{\phi}(gu)$ for $u \in U$. One checks that the $G_r$-action on $C^\infty_c(G, k)$ corresponds under this isomorphism to the $G_r$-action on $\text{ind}^{G_t}_{U_t}(C^\infty(U, k))$ given by

$$ (h, \phi) \mapsto h_r(\phi)(g)(u) := \phi(guh)(1). \tag{2} $$

In particular, for $h = v \in U$ we have

$$ v_r(\phi)(g)(u) = \phi(guv)(1) = (uv)^{-1}_t(\phi(g))(1) = \phi(g)(uv) = v_r(\phi(g))(u) $$

and hence

$$ v_r(\phi)(g) = v_r(\phi(g)). \tag{3} $$

**B) Frobenius reciprocity:** We have the $U_t$-equivariant embedding

$$ C^\infty(U, k) \rightarrow \text{ind}^{G_t}_{U_t}(C^\infty(U, k)) $$

$$ f \mapsto \iota(f)(g) := \begin{cases} g^{-1}_t f & \text{if } g \in U, \\ 0 & \text{otherwise.} \end{cases} $$

It gives rise to the isomorphism

$$ \text{End}_{\text{Mod}(G_t)}(\text{ind}^{G_t}_{U_t}(C^\infty(U, k))) \cong \text{Hom}_{\text{Mod}(U_t)}(C^\infty(U, k), \text{ind}^{G_t}_{U_t}(C^\infty(U, k))) $$

$$ \alpha \mapsto \alpha^\iota := \alpha \circ \iota. \tag{4} $$

Later on we will be interested only in $G_r$-equivariant $\alpha$’s. The main point being, as it will turn out, the $U_r$-equivariance we note here only the following. For $v \in U$ we compute, using (3) in the last equality,

$$ \iota(v_r f)(g) = \begin{cases} g^{-1}_r (v_r f) & = \begin{cases} v_r(g^{-1}_t f) & = v_r(\iota(f))(g) = v_r(\iota(f))(g), \\ 0 & \end{cases} \end{cases} $$

which means that the map $\iota$ is $U_r$-equivariant. Hence with $\alpha$ also $\alpha^\iota$ is $U_r$-equivariant.

**C) Mackey decomposition:** The disjoint decomposition $G = \bigcup_{w \in W} U_w U$ gives rise to the $U_r$-invariant decomposition

$$ \text{ind}^{G_t}_{U_t}(C^\infty(U, k)) = \bigoplus_{w \in W} \text{ind}^{(U_w U)_t}_{U_t}(C^\infty(U, k)). $$

We put $U_w := U \cap wUw^{-1}$. For any $w \in W$ the map

$$ \text{ind}^{(U_w U)_t}_{U_t}(C^\infty(U, k)) \rightarrow \text{ind}^{U_w}_{U_t}(w_\ast \text{res}_{(U_w U)^{-1}}^{U}(U_w U, k)) $$

$$ \phi \mapsto \tilde{\phi}(u) := \phi(uw) $$

is a $U_t$-equivariant isomorphism. Here the functor $w_\ast$ sends a $U_w^{-1}$-representation $Y$ to the $U_w$-representation $w_\ast Y := Y$ but on which $U_w$ acts through the homomorphism $u \mapsto w^{-1}uw$. We see that

$$ \text{ind}^{G_t}_{U_t}(C^\infty(U, k)) \rightarrow \bigoplus_{w \in W} \text{ind}^{U_w}_{U_t}(w_\ast C^\infty(U, k)) $$

$$ \phi \mapsto (\phi_w(u) := \phi(uw))_w $$

is a $U_t$-equivariant isomorphism. Using (3) one checks that this map also is $U_r$-equivariant where $U_r$ acts on the target in a way which is induced in each summand by the $U_r$-action on $C^\infty(U, k)$.  

\footnote{This map does depend on the choice of the set $W$.}
D) Second Frobenius reciprocity: Let $Y$ be any object in $\operatorname{Mod}(U_w)$. Since $U_w$ is open and hence of finite index in $U$ the $U_w$-equivariant map

$$\text{ind}_{U_w}^U(Y) \longrightarrow Y$$

induces the isomorphism

$$\operatorname{Hom}_{\operatorname{Mod}(U)}(C^\infty(U, k), \text{ind}_{U_w}^U(Y)) \xrightarrow{\cong} \operatorname{Hom}_{\operatorname{Mod}(U_w)}(C^\infty(U, k), Y)$$

$$\beta \mapsto [f \mapsto \beta(f)(1)].$$

For $Y = w_*C^\infty(U, k)$ this becomes the isomorphism

$$\operatorname{Hom}_{\operatorname{Mod}(U)}(C^\infty(U, k), \text{ind}_{U_w}^U(w_*C^\infty(U, k))) \xrightarrow{\cong} \operatorname{Hom}_{\operatorname{Mod}(U_w)}(C^\infty(U, k), w_*C^\infty(U, k))$$

$$\beta \mapsto [f \mapsto \beta(f)(1)].$$

In this case the map (6) is visibly $U_r$-equivariant. Hence (7) respects $U_r$-equivariance. The combination of (4), (5), and (7) leads to the following chain of maps:

$$\operatorname{End}_{\operatorname{Mod}(G)}(\text{ind}_{U_w}^G(C^\infty(U, k)))$$

$$\cong$$

$$\operatorname{Hom}_{\operatorname{Mod}(U_w)}(C^\infty(U, k), \text{ind}_{U_w}^G(C^\infty(U, k)))$$

$$\cong$$

$$\operatorname{Hom}_{\operatorname{Mod}(U_w)}(C^\infty(U, k), \bigoplus_{w \in W} \text{ind}_{U_w}^U(w_*C^\infty(U, k)))$$

$$=\quad$$

$$\operatorname{Hom}_{\operatorname{Mod}(U_w)}(C^\infty(U, k), \prod_{w \in W} \text{ind}_{U_w}^U(w_*C^\infty(U, k)))$$

$$=\quad$$

$$\prod_{w \in W} \operatorname{Hom}_{\operatorname{Mod}(U_w)}(C^\infty(U, k), w_*C^\infty(U, k))$$

We denote the composite map by $\alpha \mapsto \Omega(\alpha) = (\Omega_w(\alpha))_{w \in W}$. Its explicit description is

$$(9)\quad \Omega_w(\alpha)(f) = (\alpha \circ \iota(f))(w).$$

We have seen already that $\Omega$ respects $U_r$-equivariance. It is clear that the image of $\Omega$ is

$$(10)\quad \operatorname{im}(\Omega) = \{\omega_w : \text{for any } f \in C^\infty(U, k) \text{ we have } \omega_w(f) = 0 \text{ for all but finitely many } w \in W\}.$$
To reformulate the target of \( \Omega \) under the presence of the \( U_r \)-equivariance condition we first observe that

\[
\text{(11)} \quad \text{Hom}_{\text{Mod}(U_w \times U)}(C^\infty(U, k), w \cdot C^\infty(U, k)) = \{ \beta \in \text{End}_{\text{Mod}(U_r)}(C^\infty(U, k)) : \beta((u \cdot (w^{-1}uw)(\beta(-))) \text{ for any } u \in U_w \}. 
\]

The \( U_r \)-action on \( C^\infty(U, k) \) extends uniquely to an action of the completed group ring \( k[[U_r]] \). The \( U_r \)-action is equivariant for this \( k[[U_r]] \)-action. This gives rise to a ring homomorphism

\[
k[[U_r]] \xrightarrow{\cong} \text{End}_{\text{Mod}(U_r)}(C^\infty(U, k))
\]

which is easily seen to be an isomorphism. Using this identification the equality (11) becomes the equality

\[
\text{(12)} \quad \text{Hom}_{\text{Mod}(U_w \times U)}(C^\infty(U, k), w \cdot C^\infty(U, k)) = \{ \lambda \in k[[U]] : \lambda u = (w^{-1}uw)\lambda \text{ for any } u \in U_w \} =: k[[U]]_w.
\]

We conclude that \( \Omega \) restricts to an injective map

\[
\Omega_U : \text{End}_{\text{Mod}(G \times U)}(\text{ind}^G_U(C^\infty(U, k))) \longrightarrow \prod_{w \in W} k[[U]]_w
\]

with the additional property, by (11), that any element \( (\lambda_w)_w \) in the image of \( \Omega_U \) satisfies:

\[
\text{(13)} \quad \Omega_U : \text{End}_{\text{Mod}(G \times U)}(\text{ind}^G_U(C^\infty(U, k))) \longrightarrow \prod_{w \in W} k[[U]]_w
\]

For any \( f \in C^\infty(U, k) \) we have \( \lambda_w(f) = 0 \) for all but finitely many \( w \in W \).

In section 6 we will study the vector spaces \( k[[U]]_w \) based upon a formalism developed in the next section.

5. Fixed points in profinite permutation modules

In this section, we extend [Ard] Prop. 2.1 to a more general setting.

Let \( \Gamma \) be a profinite group and let \( Y \) be a profinite set equipped with a continuous action \( \text{act} : \Gamma \times Y \to Y \) of \( \Gamma \). We begin with some topological preliminaries.

**Lemma 5.1.** Let \( P \) be a closed and open subset of \( Y \). Then the stabiliser \( \Gamma_P \) is open in \( \Gamma \).

**Proof.** Note that \( \text{Op}_c(\Gamma) \) is a directed set under reverse inclusion. First, we show that

\[
\bigcap_{H \in \text{Op}_c(\Gamma)} HP = P.
\]

Let \( x \) lie in the intersection; then for each \( H \in \text{Op}_c(\Gamma) \) we can find \( y_H \in P \) and \( h_H \in H \) such that \( x = h_H y_H \). Regarding \( H \mapsto h_H \) as a net in \( \Gamma \), we observe that \( h_H \to 1 \) because \( \bigcap_{H \in \text{Op}_c(\Gamma)} H = \{1\} \). Now \( P \) is compact because \( P \) is closed in \( Y \) and \( Y \) is compact. So, by [Kel] Thm. 5.2, we can find a subnet \( \varphi : B \to \text{Op}_c(\Gamma) \) such that \( h_{\varphi(b)}(y) \to y \) for any \( y \in P \).

Note that \( h_{\varphi(b)} \to 1 \) still, so \( (h_{\varphi(b)}, y_{\varphi(b)}) \to (1, y) \) in \( \Gamma \times Y \). Since \( \text{act} : \Gamma \times Y \to Y \) is continuous, we conclude that \( x = h_{\varphi(b)} y_{\varphi(b)} \to 1 \cdot y = y \). Therefore \( x = y \in P \) because \( Y \) is Hausdorff.

Now each \( HP \) is the continuous image in \( Y \) of the compact space \( H \times P \) under \( \text{act} \), and is therefore compact. Since \( Y \) is Hausdorff, each \( HP \) is closed in \( Y \). By the above,

\[
Y \setminus P = \bigcup_{H \in \text{Op}_c(\Gamma)} Y \setminus HP
\]
is an open covering of $Y \setminus P$. Since $P$ is also open, its complement $Y \setminus P$ is closed and hence compact. This implies that $P = H_1 P \cap \cdots \cap H_n P$ for some $H_1, \cdots, H_n \in \text{Op}_c(\Gamma)$. But then the open subgroup $H_1 \cap \cdots \cap H_n$ of $\Gamma$ stabilises $P$. 

Now we can record a $\Gamma$-equivariant version of the well-known topological characterisation of profinite sets. Let $\mathcal{P}$ be the set of open partitions $\mathcal{P}$ of $Y$ so that every $P \in \mathcal{P}$ is a finite discrete quotient space of $Y$. These form an inverse system if we order $\mathcal{P}$ by refinement, so that $\mathcal{P}_2 \geq \mathcal{P}_1$ if and only if every $P_2 \in \mathcal{P}_2$ is contained in some $P_1 \in \mathcal{P}_1$. By [Wil] Prop. 1.1.7, the natural map $Y \xrightarrow{\cong} \varprojlim_{\mathcal{P} \in \mathcal{P}} \mathcal{P}$ is an isomorphism of profinite sets.

We say that $Q \in \mathcal{P}$ is $\Gamma$-stable if $gQ \in Q$ for all $Q \in Q$ and all $g \in \Gamma$. Like $\mathcal{P}$, the set $\mathcal{Q} \subset \mathcal{P}$ of $\Gamma$-stable open partitions of $Y$ forms a directed set under refinement.

**Lemma 5.2.** Every $\mathcal{P} \in \mathcal{P}$ admits a $\Gamma$-stable refinement $Q \in \mathcal{Q}$.

**Proof.** Because $Y$ is compact, $\mathcal{P}$ is a finite set, so using Lemma [5.1] we can find an open subgroup $H$ of $\Gamma$ which stabilises each member of $\mathcal{P}$. Fix a complete set $\{g_1, \cdots, g_k\}$ of left coset representatives of $H$ in $\Gamma$, and define

$$Q := \{g_1 P_1 \cap \cdots \cap g_k P_k : P_1, \cdots, P_k \in \mathcal{P}\} \setminus \{\emptyset\}.$$ 

Then every member of $Q$ is a non-empty closed and open subset of $\Gamma$, and since $Y = \bigcup_{P \in \mathcal{P}} g_i P$ for each $i = 1, \cdots, k$, we see that

$$Y = \bigcap_{i=1}^k \bigcup_{P \in \mathcal{P}} g_i P = \bigcup_{P_1, \cdots, P_k \in \mathcal{P}} g_1 P_1 \cap \cdots \cap g_k P_k = \bigcup_{Q \in Q} Q.$$

So, $Q$ forms a covering of $Y$ by non-empty closed and open subsets. Let $Q = g_1 P_1 \cap \cdots \cap g_k P_k$ and $Q' = g_1 P'_1 \cap \cdots \cap g_k P'_k$ be two members of $Q$ such that $Q \cap Q' \neq \emptyset$. Then

$$\emptyset \neq Q \cap Q' = (g_1 P_1 \cap \cdots \cap g_k P_k) \cap (g_1 P'_1 \cap \cdots \cap g_k P'_k) = g_1 (P_1 \cap P'_1) \cap \cdots \cap g_k (P_k \cap P'_k)$$

shows that $P_i \cap P'_i \neq \emptyset$ for each $i = 1, \cdots, k$. Since $\mathcal{P}$ is a partition, $P_i = P'_i$ for each $i = 1, \cdots, k$ and hence $Q = Q'$. So, $Q$ is a partition of $Y$.

Finally, to see that $Q$ is $\Gamma$-stable, consider the permutation action of $\Gamma$ on $\Gamma/H$. Fix $x \in \Gamma$ and write $xg_i H = x_i g_i H$ for some permutation $i \mapsto x \cdot i$ of $\{1, \cdots, k\}$. Since $H$ stabilises each $P \in \mathcal{P}$, we have $xg_i P = g_{x_i} P$ for all $i = 1, \cdots, k$ and all $P \in \mathcal{P}$. Then for any $Q = g_1 P_1 \cap \cdots \cap g_k P_k \in \mathcal{Q}$, we see that

$$xQ = x(g_1 P_1 \cap \cdots \cap g_k P_k) = g_{x_1} P_1 \cap \cdots \cap g_{x_k} P_k = g_1 P_{x^{-1}1} \cap \cdots \cap g_k P_{x^{-1}k} \in \mathcal{Q}.$$ 

Thus $Q$ is the required $\Gamma$-stable refinement of $\mathcal{P}$. 

We regard each $Q \in \mathcal{Q}$ as a finite discrete space; it admits a natural $\Gamma$-action. Lemma [5.2] implies that this action is continuous. If $Q \in \mathcal{Q}$ is a refinement of $\mathcal{R} \in \mathcal{Q}$, then the natural map $\pi_{\mathcal{Q}\mathcal{R}} : \mathcal{Q} \to \mathcal{R}$ which sends $Q \in \mathcal{Q}$ to the unique element $R \in \mathcal{R}$ containing $Q$ is $\Gamma$-equivariant, so the profinite set $\varprojlim_{Q \in \mathcal{Q}} Q$ carries a natural continuous $\Gamma$-action. From Lemma [5.2] we easily deduce the following

**Corollary 5.3.** The natural $\Gamma$-equivariant map $Y \xrightarrow{\cong} \varprojlim_{Q \in \mathcal{Q}} Q$ is an isomorphism.

Let $R$ be any possibly noncommutative ring.
**Definition 5.4.** The permutation module $R[[Y]]$ of $Y$ is the projective limit

$$R[[Y]] := \lim_{\substack{\longrightarrow \ Q \in \mathfrak{Q}}} R[Q]$$

of the usual permutation $R[\Gamma]$-modules $R[Q]$ as $Q$ ranges over the set $\mathfrak{Q}$ of $\Gamma$-stable open partitions of $Y$. It is equipped with the projective limit topology of the discrete topologies.

By functoriality, there is a natural $\Gamma$-action on $R[[Y]]$ and we would like to describe the $R$-submodule of $\Gamma$-invariants $R[[Y]]^\Gamma$ in $R[[Y]]$.

**Definition 5.5.**

1. Let $\Gamma \backslash Y$ denote the set of $\Gamma$-orbits in $Y$.
2. Let $\Delta_\Gamma(Y) := \{ y \in Y : |\Gamma \cdot y| < \infty \}$ be the set of points whose $\Gamma$-orbit is finite.
3. For each $O \in \Delta_\Gamma(Y)$, let $\nu_Y(O) := \sum_{y \in O} y \in R[Y] \subset R[[Y]]$ denote the orbit sum of $O$.

Any such finite orbit sum is $\Gamma$-invariant, so we obtain a natural $R$-linear map

$$\nu_Y : R[\Gamma \backslash \Delta_\Gamma(Y)] \to R[[Y]]^\Gamma$$

which sends the basis vector $O \in R[\Gamma \backslash \Delta_\Gamma(Y)]$ to its orbit sum $\nu_Y(O)$. Evidently this map is injective.

**Remark 5.6.** Since the action of $\Gamma$ on $Y$ is continuous, the stabiliser $\Gamma_y$ of any point $y \in Y$ is closed. Since a closed subgroup of finite index in $\Gamma$ is open, we see that $y \in Y$ lies in $\Delta_\Gamma(Y)$ if and only if $y$ is fixed by some open subgroup $H$ of $\Gamma$, which we may as well take to be normal:

$$\Delta_\Gamma(Y) = \bigcup_{H \in \text{Op}_c(\Gamma)} Y^H = \bigcup_{N \in \text{Op}_c(\Gamma) \text{ normal}} Y^N.$$

We begin with the well-known, but instructive case where $Y$ is finite.

**Lemma 5.7.** Suppose that $Y$ is finite. Then $\nu_Y$ is a bijection.

For any two $Q \supseteq R$ in $\mathfrak{Q}$ we introduce the $R$-linear map

$$\sigma_{QR} : R[\Gamma \backslash Q] \to R[\Gamma \backslash R]$$

$$C \mapsto \frac{|C|}{|\pi_{QR}(C)|} \pi_{QR}(C).$$

Note that the fraction actually is an integer. It is easy to check that the diagram

$$\begin{array}{ccc}
R[\Gamma \backslash Q] & \xrightarrow{\sigma_{QR}} & R[\Gamma \backslash R] \\
\nu_Q & \cong & \cong \\
& R[Q]^\Gamma & \xrightarrow{R[\pi_{QR}]} R[R]^\Gamma
\end{array}$$

is commutative. By passing to the projective limit we obtain an $R$-linear topological isomorphism

$$\lim_{\substack{\longrightarrow \ Q \in \mathfrak{Q}}} R[\Gamma \backslash Q] \xrightarrow{\cong} R[[Y]]^\Gamma$$

w.r.t. the projective limit topologies. Note that on the target this topology coincides with the subspace topology from $R[[Y]]$. 
Proof. Let $x = (x_Q)_{Q \in \mathcal{Q}} \in \prod_{Q \in \mathcal{Q}} R[\Gamma \setminus Q]$ and write $x_Q = \sum_{C \in \Gamma \setminus Q} x(C) \mathcal{C}$ for all $Q \in \mathcal{Q}$. Then $x \in \varprojlim_{Q \in \mathcal{Q}} R[\Gamma \setminus Q]$ if and only if for all $P \geq Q$ in $\mathcal{Q}$ and all $\mathcal{C} \in \Gamma \setminus Q$ we have

$$x(C) = \sum_{B} x(B)$$

where the sum runs over the finite set $\{B \in \Gamma \setminus P : \pi_{PQ}(B) = \mathcal{C} \text{ and } |B| = |\mathcal{C}|\}$.

Proof. Let $P \geq Q$ and write $x_P = \sum_{B \in \Gamma \setminus P} x(B)B$. Then by (15), we have

$$\sigma_{PQ}(x_P) = \sum_{\mathcal{C} \in \Gamma \setminus Q} \left( \sum_{\pi_{PQ}(B) = \mathcal{C}} x(B) \frac{|B|}{|\mathcal{C}|} \right) \mathcal{C}.$$

Let $B \in \Gamma \setminus P$ and let $\mathcal{C} = \pi_{PQ}(B)$. Then $|B|/|\mathcal{C}| = [\Gamma \setminus \pi_{PQ}(P) : \Gamma_P]$ for any $P \in \mathcal{B}$. Since $\Gamma$ is a pro-$p$ group, the index $[\Gamma \setminus \pi_{PQ}(P) : \Gamma_P]$ is a power of $p$. Since $R$ is a ring of characteristic $p$, $|B|/|\mathcal{C}|$ is zero in $R$ if $|B| > |\mathcal{C}|$. Hence

$$\sigma_{PQ}(x_P) = \sum_{\mathcal{C} \in \Gamma \setminus Q} \left( \sum_{\pi_{PQ}(B) = \mathcal{C}} x(B) \right) \mathcal{C}.$$

The result follows, because $x \in \varprojlim_{Q \in \mathcal{Q}} R[\Gamma \setminus Q]$ if and only if $\sigma_{PQ}(x_P) = x_Q$ for all $P \geq Q$. \qed
the indexing set \( \{ Q \in \Omega : Q \geq \mathcal{R} \} \) of this projective system is directed, we deduce from [Bou] Chap. I, §9.6, Prop. 8(b) that the projective limit \( \varprojlim_{Q \geq \mathcal{R}} \mathcal{S}(Q) \) is non-empty. Choose a family \((\mathcal{C}(Q))_{Q \geq \mathcal{R}}\) in this limit. Then since each member of this family is a finite set of size \(|D|\), it follows from the definition of \( \mathcal{S}(Q) \) that the transition maps \( \pi_{PQ} : \mathcal{P} \to \mathcal{Q} \) restrict to bijections \( \pi_{PQ} : \mathcal{C}(P) \to \mathcal{C}(Q) \) for every \( P \geq Q \geq \mathcal{R} \). Therefore the projective limit \( \varprojlim_{Q \geq \mathcal{R}} \mathcal{C}(Q) \) is non-empty. So we may choose an element \((P(Q))_{Q \geq \mathcal{R}}\) in this projective limit. Although we think of \( P(Q) \) as a point in the finite set \( Q \) we may view \( P(Q) \) correspondingly as a non-empty open and closed subset of \( Y \). By definition of \( \pi_{Q\mathcal{R}} \), we see that for each \( P \geq Q \geq \mathcal{R} \), \( P(P) \) is contained in \( P(Q) \). Since \( \Omega \) is directed and forms a cofinal family of finite open partitions of \( Y \) by Lemma [5.2] and since \( Y \) is a profinite set, we see that \( \bigcap_{Q \geq \mathcal{R}} P(Q) \) is a singleton point, \( \{ y_D \} \) say. Now, \( \Gamma_{P(P)} \subseteq \Gamma_{P(Q)} \) for all \( P \geq Q \geq \mathcal{R} \) since \( \pi_{PQ} : \mathcal{P} \to \mathcal{Q} \) is \( \Gamma \)-equivariant and since its restriction \( \pi_{PQ} : \mathcal{C}(P) \to \mathcal{C}(Q) \) is a bijection \( \mathcal{P}(Q) \to \mathcal{Q}(P) \). On the other hand, \( | \Gamma : \Gamma_{P(Q)} | = | \Gamma : \Gamma_{P(Q)} | = | \mathcal{C}(Q) | = | D | \) shows that in fact \( \Gamma_{P(Q)} = \Gamma_{P(\mathcal{R})} \) for all \( Q \geq \mathcal{R} \). Since \( \bigcap_{Q \geq \mathcal{R}} \Gamma_{P(Q)} \) evidently stabilises \( \bigcap_{Q \geq \mathcal{R}} \mathcal{P}(Q) \), we conclude that the finite index subgroup \( \Gamma_{P(\mathcal{R})} \) of \( \Gamma \) fixes \( y_D \) and hence \( y_D \in \Delta_\Gamma(Y) \). Hence \( \sigma_{\mathcal{R}}(\eta(\Gamma \cdot y_D)) = \Gamma \cdot P(\mathcal{R}) = D \).

Finally, since the inclusion \( \sigma_{\mathcal{R}}(\text{im} \eta) \subseteq \text{im} \sigma_{\mathcal{R}} \) in (17) is clear, take \( x = (x_D)_{Q \in \Omega} \in \varprojlim_{Q \in \Omega} R[\Gamma \setminus \mathcal{Q}] \) and consider its image \( \sigma_{\mathcal{R}}(x) = x_{\mathcal{R}} = \sum_{x(D) \neq 0} x(D)D \in R[\Gamma \setminus \mathcal{R}] \). Then

\[
x_{\mathcal{R}} = \sigma_{\mathcal{R}} \eta \left( \sum_{x(D) \neq 0} x(D) \Gamma \cdot y_D \right) \in \sigma_{\mathcal{R}}(\text{im} \eta).
\]

This shows that \( \text{im} \sigma_{\mathcal{R}} \subseteq \sigma_{\mathcal{R}}(\text{im} \eta) \) and completes the proof. \( \square \)

**Corollary 5.10.** Let \( \Gamma \) be a pro-p group which acts continuously on the profinite set \( Y \) and let \( R \) be a ring of characteristic \( p \). Suppose that \( \Delta_\Gamma(Y) \) is closed in \( Y \). Then \( \Delta_\Gamma(Y) \) is also a profinite set, and the natural maps

\[
j : R[[\Gamma \setminus \Delta_\Gamma(Y)]]^\Gamma \to R[[\Delta_\Gamma(Y)]]^\Gamma \quad \text{and} \quad i : R[[\Delta_\Gamma(Y)]]^\Gamma \to R[[Y]]^\Gamma
\]

are both isomorphisms.

**Proof.** The first assertion is clear since any closed subspace of a profinite set is again profinite. The two maps in question appear in the following commutative diagram

\[
\begin{array}{ccc}
R[[\Gamma \setminus \Delta_\Gamma(Y)]] & \xrightarrow{\nu_Y} & R[[\Delta_\Gamma(Y)]] \\
\downarrow & & \downarrow \quad j \\
R[[Y]]^\Gamma & \xrightarrow{i} & R[[\Delta_\Gamma(Y)]]^\Gamma
\end{array}
\]

and are both injective. The maps \( \nu_Y \) and \( \nu_{\Delta_\Gamma(Y)} \) both have dense image by Prop. [5.9] since \( \Delta_\Gamma(Y) \) is profinite and since \( \Delta_\Gamma(\Delta_\Gamma(Y)) = \Delta_\Gamma(Y) \). The maps \( i \) and \( j \) have closed image, by the functoriality of the \( Y \to R[[Y]] \) construction. Hence they are both isomorphisms. \( \square \)

In general, \( \Delta_\Gamma(Y) \) need not be closed in \( Y \) as the following example shows.

**Example 5.11.** Let \( \Gamma = \mathbb{Z}_p \). For each \( n \geq 0 \), let \( Y_n \) be the disjoint union of \( n + 1 \) \( \Gamma \)-orbits \( Y_n(0), Y_n(1), \ldots, Y_n(n) \), where \( |Y_n(m)| = p^m \) for all \( n \geq m \geq 0 \). Choose base points
\( y_n(m) \in Y_n(m) \) for each \( n \geq m \geq 0 \). For each \( n \geq 0 \), define \( \pi_{n+1} : Y_{n+1} \rightarrow Y_n \) to be the unique \( \Gamma \)-equivariant map such that

\[
\pi_{n+1}(y_{n+1}(m)) = \begin{cases} 
y_n(m) & \text{if } n \geq m \geq 0 
y_n(n) & \text{if } m = n + 1.
\end{cases}
\]

The restriction of \( \pi_{n+1} \) to \( Y_{n+1}(m) \) is an isomorphism onto \( Y_n(m) \) whenever \( n \geq m \geq 0 \). The projective limit \( Y = \lim_{\leftarrow} Y_n \) is naturally a profinite set equipped with a continuous \( \Gamma \)-action.

For each \( m \geq 0 \), let \( y_\infty(m) := (y_n(m))_{n=0}^\infty \in Y \). Then the \( \Gamma \)-orbit of \( y_\infty(m) \) has size exactly \( p^m \) for all \( m \geq 0 \), so that \( y_\infty(m) \in \Delta_\Gamma(Y) \) for all \( m \geq 0 \).

Suppose now that \( z \in Y \) has infinite \( \Gamma \)-orbit. Write \( z = (z_n)_{n=0}^\infty \) where \( z_n \in Y_n \) and suppose for a contradiction that for some \( n \geq 0 \), \( z_n \notin Y_n(n+1) \). Then \( z_n \in Y_n(m) \) for some \( m \leq n \). Then for all \( k \geq n \), \( z_k \) must lie in the preimage of \( Y_n(m) \) in \( Y_k \), which is \( Y_k(m) \) by construction. But then \( p^n \mathbb{Z}_p \) fixes \( z_k \) for all \( k \geq n \) and hence \( p^n \mathbb{Z}_p \) fixes \( z \) as well. This contradicts the hypothesis that the \( \Gamma \)-orbit of \( z \) was infinite. Therefore \( z_n \) must lie in \( Y_n(n+1) \) for each \( n \geq 0 \), and we conclude that \( Y \setminus \Delta_\Gamma(Y) \) consists of a single infinite \( \Gamma \)-orbit. Since this orbit is necessarily closed, it follows that \( \Delta_\Gamma(Y) = \bigcup_{m=0}^{\infty} \Gamma \cdot y_\infty(m) \) is open.

Finally, for any \( n \geq 0 \), the image of both \( z \) and \( y_\infty(n+1) \) in \( Y_n \) is contained in \( Y_n(n+1) \). Hence the image of \( z \) in \( Y_n \) is contained in the image of \( \Delta_\Gamma(Y) \). Hence \( z \) lies in the closure of \( \Delta_\Gamma(Y) \), but \( z \notin \Delta_\Gamma(Y) \). So, \( \Delta_\Gamma(Y) \) is not closed.

### 6. Twisted actions on completed group rings

From here on we assume that the field \( k \) has characteristic \( p \). We keep our fixed compact open subgroup \( U \) of \( G \) but assume that it is pro-\( p \), and we let \( w \in G \) be any fixed element. Recall that \( U_w = U \cap wUw^{-1} \) is an open subgroup of \( U \). We introduce the twisted conjugation action of \( U_w \) on \( U \), given by

\[
U_w \times U \rightarrow U
(u, x) \mapsto (w^{-1}uw)xu^{-1}.
\]

Let \( \Delta U_w(U) \) be the set of finite \( U_w \)-orbits in \( U \) under this action and for any such orbit \( O \) recall that \( \nu_G(O) \in k[U] \) denotes the sum of all its elements. This action extends to a twisted
conjugation action of $U_w$ on $k[[U]]$, and $k[[U]]_w = k[[U]]^{U_w,*}$ simply is the subspace of fixed points under this action. Since $U$ is pro-$p$, Prop. 5.9 gives the following result.

**Proposition 6.1.** The $k$-vector subspace of $k[[U]]$ spanned by the orbit sums $v_O(U)$ for $O \in \Delta_{U_w}(U)$ is dense in $k[[U]]^{U_w,*} = k[[U]]_w$.

Next we make the following simple observation.

**Lemma 6.2.** The $U_w$-orbit of an element $x \in U$ under the twisted action is finite if and only if the centraliser $C_G(wx)$ of the element $wx \in G$ is open in $G$.

**Proof.** Obviously the orbit of $x$ is finite if and only if the stabiliser $S_x := \{ u \in U_w : xu = (w^{-1}uw)x \}$ is open in $U_w$. But for a general $g \in G$ (with $w,x$ fixed) we have $xg = (w^{-1}gw)x$ if and only if $(wx)g = g(wx)$ if and only if $g \in C_G(wx)$. So, $S_x = C_G(wx) \cap U_w$. Thus, the orbit of $x$ is finite if and only if $S_x$ is open in $U_w$ if and only if $C_G(wx)$ is open in $G$. □

**Lemma 6.3.** Suppose that $G$ contains no proper open centralisers. Then $Z(U) = Z(G) \cap U$.

**Proof.** Obviously $Z(G) \cap U \subseteq Z(U)$. Let therefore $x \in Z(U)$. Then $U \subseteq C_G(x)$ so that $C_G(x)$ is open in $G$. Since $G$ contains no proper open centralisers, we conclude that $x \in Z(G)$. □

We assume from now on that $G$ contains no proper open centralisers.

**Proposition 6.4.** For any $w \in W$, we have $\Delta_{U_w}(U) = U \cap w^{-1}Z(G)$.

**Proof.** The $U_w$-orbit of $x \in U$ is finite if and only if the centraliser $C_G(wx)$ is open in $G$ by Lemma 6.2. But this is equivalent to $wx \in Z(G)$ because $G$ contains no proper open centralisers. □

**Corollary 6.5.** Let $w \in W$. Then

$$\Delta_{U_w}(U) = \begin{cases} U \cap Z(G) & \text{if } w \in Z(G), \\ \emptyset & \text{if } w \notin Z(G). \end{cases}$$

**Proof.** This follows immediately from Prop. 6.4. □

For convenience we choose a set of representatives $Z \subseteq Z(G)$ for the double cosets of $U$ contained in $Z(G)U$. We further assume that $Z \subseteq W$.

**Proposition 6.6.** For any $w \in W$, we have

$$k[[U]]_w = \begin{cases} k[[U]] & \text{if } w \in Z, \\ 0 & \text{if } w \notin Z. \end{cases}$$

**Proof.** Prop. 6.4 implies that $\Delta_{U_w}(U)$ is closed in $U$, so we may apply Cor. 5.10 to obtain

$$k[[U]]_w = k[[U]]^{U_w,*} = k[[\Delta_{U_w}(U)]^{U_w,*}.$$

Since $\Delta_{U_w}(U) = \emptyset$ when $w \notin Z$ by Cor. 6.3, we may assume that $w \in Z$. But then since $w \in Z(G)$, we have $U_w = U$ and the twisted action is the usual conjugation action of $U$ on itself. The result follows, because $\Delta_{U_w}(U) = Z(U)$ by Cor. 6.3 and Lemma 6.3. □

Write $Z := Z(G)$. The following is straightforward.

**Lemma 6.7.** The restriction map $\text{Res}_G^Z : C_c^\infty(G,k) \to C_c^\infty(Z,k)$ is surjective, and $k[Z]$-equivariant.
Let \( C := \text{ind}_{U}^{G}(C^{\infty}(U, k)) \) and let \( C_{Z} := \text{ind}_{Z(U)}^{Z}(C^{\infty}(Z(U), k)) \). From (13) together with Prop. 6.6 we have the injective map
\[
\Omega_{U} : \text{End}_{\text{Mod}(G \times U)}(C) \rightarrow \prod_{z \in Z} \text{End}_{\text{Mod}(Z(U))}(C^{\infty}(Z(U), k)) = \prod_{z \in Z} k[[Z(U)]].
\]
On the other hand, since \( Z \) is a transversal for \( Z(U) \) in \( Z \), replacing \( G \) by \( Z \) gives us the map \( \Omega_{Z(U)} \) out of \( \text{End}_{Z(U)}(C_{Z}) \) with the same target. These two maps appear in the following diagram:

\[
\begin{array}{ccc}
\widehat{k[Z]} & \xrightarrow{\Phi_{C}} & \text{End}_{\text{Mod}(G \times G)}(C) \xrightarrow{\subseteq} \text{End}_{\text{Mod}(G \times U)}(C) \\
\Phi_{C_{Z}} & & \Omega_{U} \\
\text{End}_{Z(U)}(C_{Z}) & \xrightarrow{\Omega_{Z(U)}} & \prod_{z \in Z} k[[Z(U)]]
\end{array}
\]

**Lemma 6.8.** The diagram (18) is commutative.

**Proof.** First, consider the following diagram:

\[
\begin{array}{ccc}
C^{\infty}(U, k) & \xrightarrow{\iota^{G}} & \text{ind}_{U}^{G}(C^{\infty}(U, k)) \xrightarrow{F_{G}} C_{c}^{\infty}(G, k) \\
\text{Res}_{Z(U)}^{U} & & \text{Res}_{Z(U)}^{G} \\
C^{\infty}(Z(U), k) & \xrightarrow{\iota^{Z}} & \text{ind}_{Z(U)}^{Z}(C^{\infty}(Z(U), k)) \xrightarrow{\psi} C_{c}^{\infty}(Z, k)
\end{array}
\]

where \( F_{G} \) and \( F_{Z} \) are the maps of Transitivity of Induction from (11), and \( \iota^{G} \) and \( \iota^{Z} \) are the Frobenius reciprocity embeddings appearing in (11). From the definitions of \( F \) and \( \iota \) it is straightforward to verify that the composed horizontal maps are simply the extension by zero maps. As \( Z(U) = U \cap Z \) by Lemma 6.3 the outer rectangle is commutative. Since \( F_{G} \) and \( F_{Z} \) are isomorphisms in \( \text{Mod}(G_{\ell}) \) and since \( \text{Res}_{Z}^{G} : C_{c}^{\infty}(G, k) \rightarrow C_{c}^{\infty}(Z, k) \) is \( k[Z]- \)linear by Lemma 6.7 there is a unique \( k[Z]- \)linear map \( \psi \) making the entire diagram commutative. Note that the inverse of \( F_{G} \) is given as follows: \( F_{G}^{-1}(f)(g)(u) = f(gu) \) for all \( f \in C_{c}^{\infty}(G, k), g \in G \) and \( u \in U \). Now, given \( q \in \text{ind}_{U}^{G_{\ell}} C^{\infty}(U, k), z \in Z \) and \( y \in Z(U), \) we calculate
\[
\psi(q)(z)(y) = (F_{Z}^{-1} \circ \text{Res}_{Z}^{G} \circ F_{G})(q)(z)(y) = (\text{Res}_{Z}^{G} F_{G}(q))(z)(y)
\]
\[
= F_{G}(q)(z)(y) = q(z)(y)(1) = (q^{-1}(q))(z)(y) = q(z)(y).
\]

Therefore the map \( \psi \) is given on \( q \in \text{ind}_{U}^{G_{\ell}} C^{\infty}(U, k) \) by the explicit formula
\[
(20) \quad \psi(q) = \text{Res}_{Z(U)}^{U} \circ \text{Res}_{Z}^{G} q.
\]

Fix \( x \in \widehat{k[Z]} \). Let \( \alpha := \Phi_{C}(x) \) denote the action of \( x \) on \( C \), and let \( \beta := \Phi_{C_{Z}}(x) \) denote the action of \( x \) on \( C_{Z} \). Using formula (19), we see that to prove the Lemma, it will be enough to show that \( \Omega_{z}(\alpha) = \Omega_{z}(\beta) \) for every \( z \in Z \). To this end, fix \( h \in C^{\infty}(Z(U), k) \) and choose \( f \in C^{\infty}(U, k) \) such that \( h = \text{Res}_{Z(U)}^{U} f \) using Lemma 6.7 applied to \( U \) in place of \( G \). Then we
can calculate as follows:
\[
\begin{align*}
\Omega_z(\beta)(h) &= \beta((\hat{Z}(\text{Res}^U_{Z(U)} f))(z)) \quad \text{by (19)} \\
&= \beta(\psi(\ell^G f))(z) \quad \text{by (19)} \\
&= \psi(\alpha(\ell^G f))(z) \\
&= \text{Res}^U_{Z(U)}(\alpha(\ell^G f))(z) \quad \text{by (20)} \\
&= \text{Res}^U_{Z(U)} \Omega_z(\alpha)(f) \quad \text{by (1)} \\
&= \Omega_z(\alpha)(\text{Res}^U_{Z(U)} f) \\
&= \Omega_z(\alpha)(h)
\end{align*}
\]
where on the third line we used the $k[Z]$-linearity of $\psi$, and on the sixth line we used the $k[[Z(U)]]$-linearity of $\text{Res}^U_{Z(U)} : C^\infty(U, k) \to C^\infty(Z(U), k)$. The result follows, since $k[[Z(U)]]$ acts faithfully on $C^\infty(Z(U), k)$.

**Proposition 6.9.** Let $G$ be a locally pro-$p$ group which contains no proper open centralisers and let $k$ be a field of characteristic $p$. For $C := \text{ind}^G_U(C^\infty(U, k))$ the natural map

\[ \Phi_C : k[\hat{Z}(G)] \xrightarrow{\cong} \text{End}_{\text{Mod}(G \times G)}(C) \]

is an isomorphism.

**Proof.** Suppose given $(\lambda_z)_{z \in Z} \in \prod_{z \in Z} k[[Z(U)]]$ such that for all $h \in C^\infty(U, k)$, $\lambda_z(h) = 0$ for all but finitely many $z \in Z$. Let $f \in C^\infty(Z(U), k)$ be given and choose $h \in C^\infty(U, k)$ extending $f$ using Lemma 6.7. Then $\lambda_z(f) = \lambda_z(\text{Res}^U_{Z(U)}(h)) = \text{Res}^U_{Z(U)}(\lambda_z(g)) = 0$ for all but finitely many $z \in Z$. In view of condition (14), this shows that in the diagram (15), the image of $\Omega_U$ is contained in the image of $\Omega_{Z(U)}$. Write $C_Z := \text{ind}_{Z(U)}^U(C^\infty_c(Z, k))$. Since $\Omega_U$ is injective by (13), this gives us an injective map $\theta$ which appears as the diagonal arrow in the diagram (15):

and which makes the bottom triangle commutative. Since the rectangle commutes by Lemma 6.8 and since $\Omega_{Z(U)}$ is injective, the top triangle commutes. Now, $C_Z$ is isomorphic to $C^\infty_c(Z, k)$ by (1), so Prop. 6.9 implies that $\Phi_{C_Z}$ is an isomorphism. Since $\theta$ is injective, we can now conclude that $\Phi_C$ is surjective. But since $C$ is isomorphic to $C^\infty_c(G, k)$ in $\text{Mod}(G)$ by (1), $\Phi_C$ is also injective by Prop. 3.8 (1), and this completes the proof.

**Theorem 6.10.** Let $G$ be a locally pro-$p$ group which contains no proper open centralisers and let $k$ be a field of characteristic $p$. Then the natural map

\[ \Phi : k[\hat{Z}(G)] \xrightarrow{\cong} \mathcal{Z}(G) \]

from Lemma 3.7 is an isomorphism.
Proof. By Prop. 3.8(2), we have the commutative diagram

\[ \begin{array}{ccc}
k[Z(G)] & \xrightarrow{\Phi} & \mathfrak{z}(G) \\
\Phi_C & \searrow & \downarrow \Theta \\
& \text{End}_{\text{Mod}(G \times G)}(C) &
\end{array} \]

Now $\Phi_C$ is an isomorphism by Proposition 6.9 and $\Theta$ is injective by the injectivity part of Prop. 3.11. Hence $\Phi$ is an isomorphism. \qed

Corollary 6.11. Let $G$ be a locally pro-$p$ group which contains no proper open centralisers and let $k$ be a field of characteristic $p$. Then the $k$-algebra homomorphism $\mathfrak{z}(Z(G)) \xrightarrow{k} \mathfrak{z}(G)$ from Cor. 3.11 is an isomorphism.

Finally, we specialise to the case where $G = G(\mathfrak{f})$ for some connected smooth algebraic group $G$ over a local nonarchimedean field $\mathfrak{f}$ of residue characteristic $p$. Certainly then $G$ is locally pro-$p$ (cf. Thm. in §II.IV.8 and Cor. 2 in §II.IV.9). We quickly remind the reader that the smoothness condition is automatic if $\mathfrak{f}$ has characteristic zero ([DC] II §6.1.1). We need the following fact which is known to the experts. We provide a proof since we could not find one in the literature.

Proposition 6.12. Let $X$ be a smooth scheme over the local field $\mathfrak{f}$, and let $Y \subseteq X$ be a Zariski open dense subset. Then $Y(\mathfrak{f})$ is dense in $X(\mathfrak{f})$ w.r.t. the valuation topology.

Proof. First we recall the elementary fact that the density of $Y$ in $X$ implies the density of $Y \cap X'$ in any open subset $X'$ of $X$ (check that any nonempty open subset of $X'$ must meet $Y$ non-trivially). Hence we may assume that $X$ is affine.

By passing to one of the finitely many connected components of $X$ we may further assume that $X$ is connected. If $X(\mathfrak{f}) = \emptyset$ then there is nothing to prove. We therefore assume that $X(\mathfrak{f}) \neq \emptyset$. Then $X$ is geometrically connected by [Sta] Lemma 33.7.14. The smoothness finally implies that $X$ is geometrically irreducible. Therefore we may use the implicit function theorem in the form of [GPR] Thm. 9.2. It says the following:

Let $x \in X(\mathfrak{f})$ be a point and let $t = (t_1, \ldots, t_d)$ be a system of local parameters at $x$. Choose a sufficiently small open neighbourhood $x \in U \subseteq X$ on which the $t_i$ are defined. This gives us a morphism $U \to \mathbb{A}^d$ to the affine space over $\mathfrak{f}$ which sends $x$ to 0. Then there exists an open neighbourhood $x \in B \subseteq U(\mathfrak{f})$ w.r.t. the valuation topology which is mapped by $t$ homeomorphically onto a polydisk $B_r(0) \subseteq \mathbb{A}^d(\mathfrak{f})$ around 0 of sufficiently small radius $r$.

We apply this now to a point $x \in X(\mathfrak{f}) \setminus Y(\mathfrak{f})$. The open neighbourhood $x \in U \subseteq X$ can be chosen to be affine. By the irreducibility of $X$ the Zariski closed subset $U \setminus Y$ of $U$ cannot be equal to $U$. Hence the restriction map $\mathcal{O}(U) \to \mathcal{O}(U \setminus Y)$ is not injective. We pick a function $f \neq 0$ in the kernel. By possibly making the polydisk $B$ smaller we may view $f = F(t_1, \ldots, t_d)$ as a nonzero formal power series convergent on $B$. The zero set of this power series is nowhere dense in $B$ by the comment after Cor. 5 on p. 198 in [BGR]. Hence we can find a sequence $(x_n)_{n \in \mathbb{N}}$ of points in $B \subseteq U(\mathfrak{f})$ such that

$$\lim_{n \to \infty} x_n = x \text{ w.r.t. the valuation topology and } f(x_n) \neq 0 \text{ for any } n \in \mathbb{N}.$$ 

The latter says that each $x_n \in Y(\mathfrak{f})$, because $f$ vanishes on $U \setminus Y$. The former then implies that $x$ lies in the closure of $Y(\mathfrak{f})$ w.r.t. the valuation topology. \qed
Theorem 6.13. Let $G$ be a connected smooth algebraic group over a local nonarchimedean field $\mathfrak{F}$ of residue characteristic $p$, and let $k$ be a field of characteristic $p$. Then $G = G(\mathfrak{F})$ contains no proper open centralisers, and we have $k[\mathcal{Z}(G)] \cong \mathfrak{F}(\mathcal{Z}(G)) \cong \mathfrak{F}(G)$.

Proof. Suppose that the centraliser $C_G(y)$ of some $y \in G$ is proper; then $y \notin \mathcal{Z}(G)$. Now, $C_G(y) = C_G(y)(\mathfrak{F})$, where the centraliser $C_G(y)$ in $G$ of the closed point $y$ is a closed algebraic $\mathfrak{F}$-subgroup of $G$ by [DG] II §1 Cor. 3.7. Since $y \notin Z(G)$, $G \setminus C_G(y)$ is a nonempty Zariski open subset of $G$. Since $G$ is irreducible by [DG] II §5 Thm. 1.1 it also is Zariski dense in $G$. So we may apply Prop. 6.12 with $X = G$ and $Y = G \setminus C_G(y)$ to obtain that $G \setminus C_G(y) = (G \setminus C_G(y))(\mathfrak{F})$ is dense in $G(\mathfrak{F}) = G$. Hence $C_G(y)$ cannot be open in $G$.

The second part of the assertion now follows from Thm. 6.10 and Cor. 6.11.

Corollary 6.14. Suppose that the connected smooth algebraic group $G$ is semisimple and that $k$ has characteristic $p$; then the Bernstein center of $\text{Mod}(G)$ is $\mathfrak{F}(G) = k[\mathcal{Z}(G)]$.

Proof. Under the present assumption $\mathcal{Z}(G)$ is finite.

7. Appendix: Positselski’s Theorem

We give a proof of Prop. 3.1 which is more direct than the one in [Pos].

For the injectivity of $\Theta$ let $(T_V)_{V} \in \mathfrak{A}(G)$ such that $T^{c}(\mathcal{O}_{G,k}) = 0$. Since $C_c(G/U,k)$, for any compact open subgroup $U \subseteq G$, is a $G_c$-subrepresentation of $C^\infty_c(G,k)$ it follows that $T^{c}(\mathcal{O}_{G,k}) = 0$. Now consider any object $V$ in $\text{Mod}(G)$. Any vector $v \in V$ is fixed by some compact open subgroup $U_v \subseteq G$. By Frobenius reciprocity we obtain a $G$-equivariant map $\rho_v : C_c(G/U_v,k) \to V$ sending the characteristic function of $U_v$ to $v$. We deduce that $T^{\text{im}(\rho_v)} = 0$. By varying the vector $v$ finally obtain that $T_V = 0$.

The argument for the surjectivity of $\Theta$ is slightly more involved.

Lemma 7.1. Inside the algebra $\text{End}_k(C^\infty_c(G,k))$ the subalgebras $\text{End}_{\text{Mod}(G)}(C^\infty_c(G,k))$ and $\text{End}_{\text{Mod}(G_{\mathfrak{F}})}(C^\infty_c(G,k))$ are mutually centralisers of each other.

Proof. Since $k[G_{\mathfrak{F}}] \subseteq \text{End}_{\text{Mod}(G_{\mathfrak{F}})}(C^\infty_c(G,k))$ and $k[G_{\mathfrak{F}}] \subseteq \text{End}_{\text{Mod}(G_{\mathfrak{F}})}(C^\infty_c(G,k))$ it is immediate that the centraliser of one algebra is contained in the other algebra. For the opposite inclusions we have to show that $\beta \circ \alpha = \alpha \circ \beta$ for any $\alpha \in \text{End}_{\text{Mod}(G_{\mathfrak{F}})}(C^\infty_c(G,k))$ and $\beta \in \text{End}_{\text{Mod}(G_{\mathfrak{F}})}(C^\infty_c(G,k))$. Since

$$\bigcup_{U} C_c(U \setminus G,k) = C^\infty_c(G,k) = \bigcup_{U} C_c(G/U,k),$$

where $U$ runs over all compact open subgroups of $G$, both homomorphisms $\alpha$ and $\beta$ are completely determined by their values on the characteristic functions $\text{char}_U$. Moreover, $\alpha(\text{char}_U) \in C_c(U \setminus G,k)$ and $\beta(\text{char}_U) \in C_c(G/U,k)$. We have to check that $\beta \circ \alpha(\text{char}_U) = \alpha \circ \beta(\text{char}_U)$ for any $U$ and any $x \in G$. Let $\alpha(x) = \sum_{i} c_i \text{char}_{U_i g_i}$ and $\beta(\text{char}_{xU^{-1}}) = \sum_{j} d_j \text{char}_{h_j xU^{-1}}$ for finitely many constants $c_i, d_j \in k$ and elements $g_i, h_j \in G$ (depending on $x$ and $U$). We
compute
\[
β \circ α(\text{char}_{xU}) = β(\sum_i c_i \text{char}_{xU_i}) = \sum_i c_i β(\text{char}_{xU_i}) = \sum_i c_i β(\text{char}_{xU_i x^{-1} \cdot xU_i})
\]
\[
= \sum_i c_i β(\text{char}_{xU_i})^{-1} \text{char}_{xU_i} = \sum_i c_i (\text{char}_{xU_i})^{-1} β(\text{char}_{xU_i x^{-1}}) = \sum_i c_i d_j (\text{char}_{xU_i})^{-1} \text{char}_{hxU_i} = \sum_i c_i d_j \text{char}_{hxU_i}
\]

using that α is \(G_r\)-equivariant, resp. β is \(G_r\)-equivariant, in the first, resp. fifth, equality. Correspondingly
\[
α \circ β(\text{char}_{xU}) = α(β(\text{char}_{xU x^{-1}})) = α(\sum_j d_j \text{char}_{hxU_i})
\]
\[
= \sum_j d_j α(\text{char}_{hxU_i}) = \sum_i c_i d_j \text{char}_{hxU_i}.
\]

Lemma 7.2. Suppose that \(U_i\), for \(i = 1, 2\), are compact open subgroups of \(G\); any map \(ρ \in \text{Hom}_{\text{Mod}(G)}(C_c(G/U_1, k), C_c(G/U_2, k))\) is the restriction of a map \(\tilde{ρ} \in \text{End}_{\text{Mod}(G)}(C_c^∞(G, k))\).

Proof. By Frobenius reciprocity we have
\[
\text{Hom}_{\text{Mod}(G)}(C_c(G/U_1, k), C_c(G/U_2, k)) = C_c(U_1 \setminus G/U_2, k)
\]
with \(ρ\) corresponding to \(ρ(\text{char}_{U_1})\) and
\[
\text{End}_{\text{Mod}(G)}(C_c^∞(G, k)) = \text{Hom}_{\text{Mod}(G)}(\bigcup_{U ⊆ U_i} C_c(G/U, k), C_c^∞(G, k))
\]
\[
= \lim_{U \subseteq U_i} \text{Hom}_{\text{Mod}(G)}(C_c(G/U, k), C_c^∞(G, k))
\]
\[
= \lim_{U \subseteq U_i} C_c(U \setminus G, k)
\]
with \(\tilde{ρ}\) corresponding to \((\tilde{ρ}(\text{char}_{U}))_U\). The transition map \(C_c(U' \setminus G, k) \rightarrow C_c(U \setminus G, k)\), for \(U' \subseteq U\), in the projective system is given by \(\text{char}_{U'}g \mapsto \text{char}_{Ug}\). Suppose that \(ρ(\text{char}_{U_1}) = \sum_i c_i \text{char}_{xU_i}\). Then \((\sum_i c_i \text{char}_{xU_i})_U\) defines a \(\tilde{ρ}\) as in the assertion which restricts to \(ρ\).

We now fix a \(T \in \text{End}_{\text{Mod}(G)}(C_c^∞(G, k))\), and we have to construct a \((T_V)_V \in \mathfrak{A}(G)\) such that \(T_{C_c^∞(G,k)} = T\). Of course, we turn the latter condition into the definition \(T_{C_c^∞(G,k)} := T\) and extend it in several steps to all of \(\text{Mod}(G)\).

Step 1: Let \(\mathfrak{C}_0\) denote the full subcategory of \(\text{Mod}(G)\) with the single object \(C_c^∞(G, k)\). Then Lemma 7.1 tells us that \(T_{C_c^∞(G,k)} = T\) is a natural transformation of the forgetful functor \(\mathfrak{C}_0 \rightarrow \text{Vect}\).

Step 2: Let \(\mathfrak{C}_1\) denote the full subcategory of \(\text{Mod}(G)\) with objects representations of the form \(C_c(G/U, k)\) for \(U \subseteq G\) some compact open subgroup. We have \(C_c(G/U, k) = C_c^∞(G, k)\up_r \subseteq C_c^∞(G, k)\). The map \(T\) being \(G_r\)-equivariant it respects the subrepresentation \(C_c(G/U, k)\), and we may define \(T_{C_c(G/U, k)} := T|C_c(G/U, k)\). It follows from Lemma 7.2 and Step 1 that this defines a natural transformation of the forgetful functor \(\mathfrak{C}_1 \rightarrow \text{Vect}\).

Step 3: Let \(\mathfrak{C}\) denote the full subcategory of \(\text{Mod}(G)\) with objects arbitrary direct sums of objects in \(\mathfrak{C}_1\). If \(V = \oplus_{i \in I} V_i\) with \(V_i \in \mathfrak{C}_1\) then we simply define \(T_V := \oplus_{i \in I} T_{V_i}\). Let
$V' = \bigoplus_{j \in J} V'_j$ be another object in $\mathbb{C}$ with $V'_j$ in $\mathbb{C}_1$ and let $\gamma : V \to V'$ be a homomorphism in $\text{Mod}(G)$. There are homomorphisms $\gamma_{i,j} : V_i \to V'_j$ in $\text{Mod}(G)$ such that $\gamma((v_i)_i) = (\sum_i \gamma_{i,j}(v_i))_j$. Using Step 2 in the fourth equality we compute

$$T_{V'} \circ \gamma((v_i)_i) = T_{V'}((\sum_i \gamma_{i,j}(v_i))_j) = (\sum_i T_{V'}(\gamma_{i,j}(v_i)))_j = (\sum_i \gamma_{i,j}(T_V(v_i)))_j = \gamma((T_V(v_i))_i) = \gamma \circ T_V((v_i)_i).$$

This shows that our definition extends $T$ further to a natural transformation of the forgetful functor $\mathbb{C} \to \text{Vect}$.

**Step 4:** In the proof of injectivity we have seen already that for any $V$ in $\text{Mod}(G)$ there is a surjective homomorphism $C \to V$ in $\text{Mod}(G)$ with $C$ in $\mathbb{C}$.

**Step 5:** Consider a surjective homomorphism $\beta : C \to V$ as well as a surjective homomorphism $\gamma : C' \to \ker(\beta)$ as in Step 4. By Step 3 we then have the commutative diagram

$$\begin{array}{ccc}
C' & \xrightarrow{\gamma} & C' \\
\downarrow{\iota} & & \downarrow{\iota} \\
C & \xrightarrow{\gamma} & C.
\end{array}$$

This shows that $T_C$ respects $\ker(\beta)$ and induces therefore a map $T_V$ on $V$. We first check that $T_V$ is independent of the choices. For this let $\beta_i : C_i \to V$, for $i = 1, 2$, be two surjective homomorphisms as in Step 4. At first we consider the case that there is a homomorphism $\gamma : C_1 \to C_2$ in $\text{Mod}(G)$ such that $\beta_2 \circ \gamma = \beta_1$. Let $\tau_1$ be the linear endomorphism of $V$ induced by $T_{C_1}$. Given a $v \in V$ we choose a preimage $x \in C_1$ such that $\beta_1(x) = v$. Using Step 3 in the third equality we compute

$$\tau_1(v) = \beta_1(T_{C_i}(x)) = \beta_2 \circ \gamma \circ T_{C_1}(x) = \beta_2 \circ T_{C_2} \circ \gamma(x) = \tau_2(\beta_2(\gamma(x))) = \tau_2(\beta_1(x)) = \tau_2(v).$$

In general without the presence of a $\gamma$ we consider the commutative diagram

$$\begin{array}{ccc}
C_1 & \xrightarrow{\beta_1} & C_1 \oplus C_2 \xrightarrow{\beta_1 + \beta_2} C_2 \\
\downarrow{\iota} & & \downarrow{\iota} \\
V & \xrightarrow{\beta} & V.
\end{array}$$

with the horizontal maps being the canonical ones and apply the previous case to both halves.

**Step 6:** Finally we have to show that the thus defined $T_V$ constitute a natural transformation of the forgetful functor $\text{Mod}(G) \to \text{Vect}$, i.e., we have to check that, for any homomorphism $\gamma : V_1 \to V_2$ in $\text{Mod}(G)$ the diagram

$$\begin{array}{ccc}
V_1 & \xrightarrow{\gamma} & V_2 \\
\downarrow{\iota} & & \downarrow{\iota} \\
V_1 & \xrightarrow{\gamma} & V_2
\end{array}$$

is commutative.
is commutative. We choose two surjective homomorphisms $\beta_i : C_i \rightarrow V_i$, for $i = 1, 2$, as in Step 4 and obtain the commutative diagram

$$
\begin{array}{ccc}
C_1 & \xrightarrow{\gamma} & C_1 \oplus C_2 \\
\downarrow{\beta_1} & & \downarrow{\gamma \circ \beta_1 + \beta_2} \\
V_1 & \rightarrow & V_2.
\end{array}
$$

It then follows from Step 3 that $\gamma \circ T_{V_1} = T_{V_2} \circ \gamma$.

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