Directed Percolation and the Golden Ratio

Stephan M Dammer†, Silvio R Dahmen§ and Haye Hinrichsen∥
† Theoretische Physik, Fachbereich 10, Gerhard-Mercator-Universität Duisburg, 47048 Duisburg, Germany
§ Instituto de Física, Universidade Federal do Rio Grande do Sul (UFRGS), 91501-970 Porto Alegre RS Brazil
∥ Theoretische Physik, Fachbereich 8, Universität Wuppertal, 42097 Wuppertal, Germany

Abstract. Applying the theory of Yang-Lee zeros to nonequilibrium critical phenomena, we investigate the properties of a directed bond percolation process for a complex percolation parameter $p$. It is shown that for the Golden Ratio $p = (1 \pm \sqrt{5})/2$ and for $p = 2$ the survival probability of a cluster can be computed exactly.

PACS numbers: 02.50.-r, 64.60.Ak, 05.50.+q

Directed percolation (DP) is an anisotropic variant of ordinary percolation in which activity can only percolate along a given direction in space. Regarding this direction as a temporal degree of freedom, DP can be interpreted as a dynamical process. Directed percolation represents one of the most prominent universality classes of nonequilibrium phase transitions from a fluctuating active phase into a non-fluctuating absorbing state [1, 2, 3, 4].

A simple realization of DP is directed bond percolation. In this model the bonds of a tilted square lattice are conducting with probability $p$ and non-conducting with probability $1 - p$ (see Figure 1). The order parameter which characterizes the phase transition is the probability $P(\infty)$ that a randomly chosen site belongs to an infinite cluster. A cluster consists of all sites that are connected by a directed path of conducting bonds to the sites that generate the cluster at time $t = 0$. For $p > p_c$ this probability is finite whereas it vanishes for $p \leq p_c$. Close to the phase transition $P(\infty)$ is known to vanish algebraically as $P(\infty) \sim (p - p_c)^\beta$. Although DP can be defined and simulated easily, it is one of the very few systems for which – even in one spatial dimension – no analytical solution is known, suggesting that DP is a non-integrable process. In fact, the values of the percolation threshold and the critical exponents are not simple numerical fractions but seem to be irrational instead. Currently the best estimates for directed bond percolation in 1+1 dimensions are $p_c = 0.6447001(1)$ and $\beta = 0.27649(4)$ [5].

‡ To whom correspondence should be addressed (dammer@comphys.uni-duisburg.de)
Directed Percolation and the Golden Ratio

Figure 1. Lattice geometry of directed bond percolation in 1+1 dimensions. Sites can be either active \((s_{j,t} = 1)\) or inactive \((s_{j,t} = 0)\). Activity can percolate forward in time through bonds (solid lines) which are conducting with probability \(p\) and non-conducting with probability \(1 - p\).

In recent years a large variety of equilibrium phase transitions have been analyzed by studying the distribution of Yang-Lee zeros \([6, 7]\). Here the partition sum of a finite equilibrium system is expressed as a polynomial of the control parameter (usually a function of temperature). Typically the zeros of this polynomial lie on simpler geometric manifolds such as circles in the complex plane. As the system size increases these zeros approach the real axes at the phase transition point. This explains the crossover to a non-analytic behaviour at the transition in the limit of an infinite system. Even more recently, the concept of Yang and Lee has been generalized to integrable nonequilibrium systems \([8]\). The present work is motivated by an ongoing effort to apply similar techniques to non-integrable systems such as directed percolation \([9]\). Our aim is to understand the nature of nonequilibrium phase transitions in more detail and to search for a signature of integrability and non-integrability in the distribution of Yang-Lee zeros.

To apply the concept of Yang-Lee zeros to DP, we consider the order parameter in a finite system as a function of the percolation probability \(p\) in the complex plane. This can be done by studying the survival probability \(P(t)\), which is defined as the probability that a cluster generated in a single site at time \(t = 0\) survives up to time \(t\) (or even longer). Note that in the limit of infinite \(t\) the order parameter \(P(\infty)\) and \(P(t)\) coincide. The partition sum of an equilibrium system and the survival probability of DP show a similar behavior in many respects, e.g. they both are positive in the physically accessible regime and can be expressed as polynomials in finite systems. If the system size tends to infinity, both functions exhibit a non-analytic behavior at the phase transition as the Yang-Lee zeros in the complex plane approach the real line.

In directed bond percolation the survival probability is given by the sum over the weights of all possible configurations of bonds. Each conducting bond contributes to the weight with a factor \(p\), while each non-conducting bond contributes with a factor \(1 - p\). However, the states of those bonds which do not touch the actual cluster are irrelevant as they do not contribute to the survival of the cluster. Therefore, it is sufficient to consider...
Directed Percolation and the Golden Ratio

Figure 2. Example of a cluster which survives until time $t = 3$ (solid lines). The bonds belonging to its hull are shown as dashed lines. Left: The corresponding weight to the survival probability $P(3)$ in (1) is $p^4(1-p)^2$. Right: For the probability $R(3)$ the following row of bonds has to be taken into account as well, contributing an additional factor $(1-p)^4$ thus leading to a different weight in (4) which is given by $p^4(1-p)^6$.

the sum over all possible clusters $C$ of bonds connected to the origin. Each cluster is weighted by the contributions of the conducting bonds belonging to the cluster and the non-conducting bonds belonging to its hull. More precisely, the survival probability can be expressed as

$$P(t) = \sum_C p^n(1-p)^m,$$  \hfill (1)

where the sum runs over all clusters reaching the horizontal row at time $t$. For each cluster $n$ denotes the number of its bonds, while $m$ is the number of bonds belonging to its hull (see left part of Figure 2). Note that $m$ does not include bonds connecting sites at time $t$ and $t+1$ since the cluster may survive even longer. Summing up all weights in Equation (1), one obtains a polynomial of degree $t^2 + t$. As can be verified, the first few polynomials are given by

$$P(0) = 1$$
$$P(1) = 2p - p^2$$
$$P(2) = 4p^2 - 2p^3 - 4p^4 + 4p^5 - p^6$$
$$P(3) = 8p^3 - 4p^4 - 10p^5 - 3p^6 + 18p^7 + 5p^8 - 30p^9 + 24p^{10} - 8p^{11} + p^{12}$$
$$P(4) = 16p^4 - 8p^5 - 24p^6 - 8p^7 + 6p^8 + 84p^9 - 29p^{10} - 62p^{11} - 120p^{12} + 244p^{13} + 75p^{14} - 470p^{15} + 495p^{16} - 268p^{17} + 83p^{18} - 14p^{19} + p^{20}.$$

As $t$ increases, the number of cluster configurations grows rapidly, leading to complicated polynomials with very large coefficients. The distribution of zeros in the complex plane for the polynomial $P(t = 15)$ is shown in Figure 3. As can be seen, the distribution reminds of a fractal, being perhaps a signature of the non-integrable nature of DP. As expected, the zeros approach the phase transition point from above and below.

While a statistical analysis of the distribution of zeros will be presented elsewhere, we will focus in the present work on a particularly surprising observation. This is the existence of certain points on the real line where the polynomials can be solved exactly
for all values of $t$. Beside the trivial points $p = 0$ (where $P(t) = \delta_{t,0}$) and $p = 1$ (where $P(t) = 1$), we find a $t$-independent zero at $p = 2$ and, even more surprisingly, a very simple solution if $p$ is equal to one of the Golden Ratios $(1 \pm \sqrt{5})/2$. The Golden Ratios are the roots of the quadratic equation $p^2 = p + 1$ and play an important role not only in number theory [10] but also in other fields ranging from chaotic systems [11] to arts [12]. Although these special points are located outside the physically accessible regime $0 \leq p \leq 1$, their existence may help to understand the structure of Yang-Lee zeros in the complex plane.

**Time-independent zero at $p = 2$:**
For $p = 2$ and $t \geq 1$ all polynomials $P(t)$ vanish identically. This can be shown as follows. Let us consider the probability $R(t)$ that a cluster dies out at time $t$, i.e., the row at time $t$ is the last row reached by a cluster. Obviously $R(t)$ is related to the survival probability by

$$R(t) = P(t) - P(t + 1).$$

(3)

Clearly, $R(t)$ can be expressed as a weighted sum over the same set of clusters as in (1). However, in the present case the weights differ from those in Equation (1) by the number of non-conducting bonds in the clusters hull between $t$ and $t+1$ since it is now required that all sites at time $t+1$ are inactive (see right part of Figure 2). This means that $R(t)$ can be expressed as

$$R(t) = \sum_C p^n (1-p)^m (1-p)^{2k},$$

(4)

where $n$, $m$ and $C$ have the same meaning as in (1) and $k$ is the number of active sites in the horizontal row at time $t$. Obviously, for $p = 2$ the additional factor $(1-p)^{2k}$

---

**Diagram:** Distribution of zeros of the polynomial $P(15)$ in the complex plane.
Figure 4. Decomposition of the configurational sum of non-surviving clusters into three subsets. Open and closed bonds are denoted by solid (dashed) lines. Bonds which are not shown may be either open or closed. The box includes all bonds contributing to the factor $\hat{Q}_1$ while all other bonds contribute to the factor $\hat{Q}_2$. The proof shows that the configurations in (a) and (b) cancel so that only the configurations of (c) contribute to $Q$. Iterating the procedure by shifting the box to the right, it can be shown that all sites at $t-2$ and $t-1$ have to be zero.

drops out so that $P(t) = R(t)$ for all values of $t$. Moreover, $R(0) = P(0) = 1$ (for $p = 2$). Combining these results with Equation (3) we arrive at $P(t) = 0$ for $t > 0$, which completes the proof.

**Exact solution for $p$ at the Golden Ratio:**
For $p = (1 \pm \sqrt{5})/2$ we find that the survival probability ‘oscillates’ between two different values, namely

$$P(t) = \begin{cases} 1 & \text{if } t \text{ is even} \\ \frac{\pm \sqrt{5} - 1}{2} & \text{if } t \text{ is odd.} \end{cases} \quad (5)$$

To prove this result, we first verify that (5) is indeed satisfied for $t = 0$ and $t = 1$. Then we show that

$$P(t) = P(t-2) \quad \text{for } t \geq 2 \quad \text{and} \quad p = (1 \pm \sqrt{5})/2. \quad (6)$$

However, instead of analyzing the survival probability directly, it turns out to be more convenient to consider the complementary probability $Q(t) = 1 - P(t)$ that a cluster does not survive until time $t$. Obviously, $Q(t)$ is the sum over the weights of all clusters which do not reach the horizontal row at time $t$, i.e., we impose the boundary condition $s_{0,t} = s_{1,t} = \ldots = s_{t,t} = 0$. Depending on the states of the two sites $s_{0,t-1}$ and $s_{0,t-2}$ at the left edge of the clusters, this set of clusters may be separated into three different subsets, namely,

(a) a subset where $s_{0,t-1} = s_{0,t-2} = 1$,
(b) a subset where $s_{0,t-1} = 0$ and $s_{0,t-2} = 1$, and
(c) a subset where $s_{0,t-1} = s_{0,t-2} = 0$.

Next we show that the weights of the clusters in the subsets (a) and (b) cancel each other. To this end we note that the weighted sum $\hat{Q}(t)$ over all clusters in both subsets may be decomposed into two independent factors $\hat{Q}(t) = \hat{Q}_1\hat{Q}_2$, where $\hat{Q}_1$ depends only
on the state of the three bonds between the sites \(s_{0,t-2}, s_{0,t-1}, s_{0,t}, \) and \(s_{1,t}\) (inside the box in Figure 4), while \(\hat{Q}_2\) accounts for all other relevant bonds. Obviously, the first factor is given by

\[
\hat{Q}_1^{(a)} = p(1-p)^2, \quad \hat{Q}_1^{(b)} = 1-p, \tag{7}
\]

while \(\hat{Q}_2\) takes the same value in both subsets. Thus, if \(p\) is given by the Golden Ratio, we obtain \(\hat{Q}_1^{(a)} + \hat{Q}_1^{(b)} = \hat{Q}_1(t) = 0\) and therefore the weights of subsets (a) and (b) cancel. Consequently, all remaining contributions to \(Q(t)\) come from the clusters in subset (c) where the sites \(s_{0,t-1}\) and \(s_{0,t-2}\) are inactive.

Now we can iterate this procedure by successively considering the sites \(s_{j,t-1}\) and \(s_{j,t-2}\) from the left to the right, where \(j = 1 \ldots t-2\). In this way it can be shown that all these sites have to be inactive as well. Therefore, the only surviving contributions are those in which the entire row of sites at \(t-2\) is inactive, implying that \(Q(t) = Q(t-2)\). The proof of Equation (5) then follows by induction.

To summarize, we have shown that the survival probability \(P(t)\) of a (1+1)-dimensional directed bond percolation process can be computed exactly at certain points on the physically non-accessible part of the real axes. We hope that this observation may help to understand the distribution of Yang-Lee zeros in the complex plane. Moreover, we expect similar special points to exist in other realizations of DP and related models.

Acknowledgments

This work was supported by the DAAD/CAPES within the German-Brazilian cooperation project PROBRAL – “Rigorous Results in Nonequilibrium Statistical Mechanics and Nonlinear Physics”.

References

[1] Kinzel W 1983 Annals of the Israel Physical Society vol 5, ed. by Deutscher G, Zallen R, and Adler J (Bristol: Adam Hilger)
[2] Marro J and Dickman R 1999 Nonequilibrium phase transitions in lattice models (Cambridge: Cambridge University Press)
[3] Hinrichsen H 2000 Adv. Phys. 49 815
[4] Hinrichsen H 2000 Braz. J. Phys. 30 69
[5] Jensen I 1996 Phys. Rev. Lett. 77 4988
[6] Yang C N and Lee T D (1952) Phys. Rev. 78 404; Lee T D and Yang C N (1952) Phys. Rev. 87 410
[7] Derrida B, De Seze L, and Itzykson C (1983) J. Stat. Phys. 33 559
[8] Arndt P F (2000) Phys.Rev.Lett. 84 814
[9] Arndt P F, Dahmen S R, and Hinrichsen H (2001) Physica A 295 128
[10] see e.g.: Schroeder M 1999 Number Theory in Science and Communication (Springer Series in Information Science, vol. 7, Springer Verlag); Vajda S 1989 Fibonacci and Lucas numbers, and the Golden Section (Chichester: Horwood)
[11] Ozorio de Almeida A M 1988 Hamiltonian Systems: Chaos and Quantization (Cambridge: Cambridge University Press)
[12] Ghyka M 1977 The Geometry of Art and Life (New York: Dover Publications)