VARIATIONAL PRINCIPLES IN THE FRAME OF CERTAIN GENERALIZED FRACTIONAL DERIVATIVES

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Abstract. In this article, we study generalized fractional derivatives that contain kernels depending on a function on the space of absolute continuous functions. We generalize the Laplace transform in order to be applicable for the generalized fractional integrals and derivatives and apply this transform to solve some ordinary differential equations in the frame of the fractional derivatives under discussion.

1. Introduction. Fractional calculus which is the extension of integer-order differentiation and integration to any order is one of the fastest growing fields. Although this calculus is as older as the usual calculus, only in the last few decades it started to attract scientists working in different areas because it was discovered that good results had emerged when fractional derivatives and integrals were used to model many real world phenomena [28, 29, 22, 27, 26, 13, 25]. One of the good features of this filed is that there is a variety of derivatives and integrals which allow a researcher to use the appropriate non-local operator which may be better used to describe several complex phenomena in real world problems. While scientists were focusing on the application of fractional calculus, others were interested in conceiving new types of non-local new fractional operators. Hadamard and generalized fractional operators were ones of them [23, 14, 12, 20, 21, 15, 4]. Even though there are many and different non-local fractional operators, because of the difficulty faced in describing some phenomena, researches started to be in need of new fractional operators which were believed to better understand real world problems. Among these new non-singular fractional operators we mention Caputo-Fabrizio, Atangana-Baleanu and Yang-Srivastava-Machado fractional derivatives [11, 8].

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The calculus of variation is an old and excellently established field with many applications in diversified areas of science and engineering [30]. The variational problems in the frame of fractional operators attracted many researchers working on the field of optimal control and Hamiltonian systems [6, 5, 10, 17, 18, 19, 1, 2, 9, 3].

In this work, we consider variational problems involving generalized fractional operators that contain two parameters and are reduced to the classical fractional operators when one of the parameters tends to a specific value. We present integration by parts formulas for operators with any order which enable to find the Euler-Lagrange equations emerging from variational problems involving a certain type of generalized fractional operators. We would like to mention that the same variational problem problem was considered in [7] for generalized fractional derivatives between 0 and 1. We believe that the integration by parts formula in Theorem 2.1 in [7] does not contain generalized fractional derivatives, but rather contains other types of fractional derivatives and hence the Euler-Lagrange equation found in [7] is not suitable for the variational problems we are tackling.

Our paper is organized as follows: In section 2, we state some definitions, lemmas and theorems needed in this work. In section 3, we present different integration-by-parts formulas. In section 4, we derive Euler-Lagrange equations for certain operators when one of the parameters tends to a specific value. We present integrals of operators that contain two parameters and are reduced to the classical fractional operators [28, 29, 22].

2. Preliminaries. First, let us recall some formulas from the classical fractional calculus [28, 29, 22].

The left Riemann-Liouville fractional of order \( \alpha, \Re(\alpha) > 0 \) is defined by

\[
(\_aI^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) \, ds. \tag{1}
\]

The right Riemann-Liouville fractional of order \( \alpha, \Re(\alpha) > 0 \) ending at \( b > a \) is defined by

\[
(I_b^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) \, ds. \tag{2}
\]

The left Riemann-Liouville fractional derivative of order \( \alpha, \Re(\alpha) \geq 0 \) is given as

\[
(\_aD^\alpha f)(t) = \frac{d^n}{dt^n} (\_aI^{n-\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s) \, ds, \tag{3}
\]

where \( n = \lceil \Re(\alpha) \rceil + 1 \). The right Riemann-Liouville fractional derivative of order \( \alpha, \Re(\alpha) \geq 0 \) reads

\[
(D_b^\alpha f)(t) = (-1)^n \frac{d^n}{dt^n} (I_b^{n-\alpha} f)(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b (s-t)^{n-\alpha-1} f(s) \, ds. \tag{4}
\]

The left Caputo fractional of order \( \alpha, \Re(\alpha) \geq 0 \) has the following form

\[
(\_cD^\alpha f)(t) = (\_aI^{n-\alpha}(f^{(n)}))(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) \, ds, \tag{5}
\]

where \( n = \lceil \Re(\alpha) \rceil + 1 \). While, the right Caputo fractional derivative of order \( \alpha, \Re(\alpha) \geq 0 \) reads

\[
(C_D^\alpha f)(t) = (I_b^{n-\alpha} (-1)^n f^{(n)})(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1} f^{(n)}(s) \, ds. \tag{6}
\]
The Hadamard type fractional integrals and derivatives were introduced in [23] as the following:

The left Hadamard fractional integral of order \( \alpha, \Re(\alpha) > 0 \) has the following form

\[
\left( {a}I_{\alpha} f \right)(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} \left( \ln \frac{s}{t} \right)^{\alpha-1} f(s) \frac{ds}{s}.
\]

(7)

The right Hadamard fractional integral of order \( \alpha, \Re(\alpha) > 0 \) ending at \( b > a \) is defined by

\[
\left( {b}I_{\alpha} f \right)(t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{b} \left( \ln \frac{s}{t} \right)^{\alpha-1} f(s) \frac{ds}{s}.
\]

(8)

The left Hadamard fractional derivative of order \( \alpha, \Re(\alpha) \geq 0 \) and \( n = [\Re(\alpha)] + 1 \) is given by

\[
\left( {a}D_{\alpha} f \right)(t) = \frac{t^{\delta}}{\Gamma(\alpha-n)} \left( \frac{d}{dt} \right)^{n} \left( \ln \frac{s}{a} \right)^{\delta-1} f(s) \frac{ds}{s},
\]

(9)

and in the space \( AC^\alpha[a, b] = \{ g : [a, b] \rightarrow \mathbb{C} : \delta^{\alpha-1}g(t) \in AC[a, b] \} \) equivalently by

\[
\left( {a}D_{\alpha} f \right)(t) = \frac{t^{\delta}}{\Gamma(\alpha-n)} \left( \frac{d}{dt} \right)^{n} \left( \ln \frac{s}{a} \right)^{\delta-1} f(s) \frac{ds}{s}.
\]

(10)

The right fractional derivative of order \( \alpha, \Re(\alpha) \geq 0 \) is defined by

\[
\left( {b}D_{\alpha} f \right)(t) = \frac{t^{\delta}}{\Gamma(\alpha-n)} \left( -\frac{d}{dt} \right)^{n} \left( \ln \frac{s}{b} \right)^{\delta-1} f(s) \frac{ds}{s},
\]

(11)

and in the space \( AC^\alpha[a, b] \) equivalently by

\[
\left( {b}D_{\alpha} f \right)(t) = \left( {b}I_{\alpha}^{\delta} \left( -\frac{d}{dt} \right)^{n} f \right)(t).
\]

(12)

For \( a < b, c \in \mathbb{R} \) and \( 1 \leq p < \infty \), define the function space

\[
X^{p}_{c}(a, b) = \{ f : [a, b] \rightarrow \mathbb{R} : \| f \|_{X^{p}_{c}} = \left( \int_{a}^{b} |t^{c} f(t)|^{p} dt \right)^{1/p} < \infty \}.
\]

For \( p = \infty, \| f \|_{X^{p}_{c}} = \text{ess sup}_{a \leq t \leq b} |t^{c} f(t)| \). The generalized left and right fractional integrals of order \( \alpha, \Re(\alpha) > 0 \) and \( \rho > 0 \) are defined in [20] as

\[
\left( {a}I_{\alpha}^{\rho} f \right)(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} \left( \frac{t^{\rho} - s^{\rho}}{\rho} \right)^{\alpha-1} f(s) \frac{ds}{s^{1-\rho}}
\]

(15)

and

\[
\left( {b}I_{\alpha}^{\rho} f \right)(t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{b} \left( \frac{s^{\rho} - t^{\rho}}{\rho} \right)^{\alpha-1} f(s) \frac{ds}{s^{1-\rho}}.
\]

(16)

respectively. It can be easily noticed that when \( \rho = 1 \), the integrals in (15) and (16) reduces to the integrals in (1) and (2), respectively. In addition, when the limits of
the integrals in (15) and (16) as \( \rho \to 0 \) are taken, one gets the Hadamard fractional derivatives in (7) and (8).

The left and right generalized fractional derivatives of order \( \alpha, \Re (\alpha) \geq 0, \) and \( \rho > 0 \) are defined by (see [21])

\[
(a D^{\alpha,\rho} f)(t) = \gamma^n (a I^{n-\alpha,\rho} f)(t) = \frac{1}{\Gamma(n-\alpha)} \gamma^n \int_{a}^{t} \left( \frac{t^\rho - s^\rho}{\rho} \right)^{n-\alpha-1} f(s) \frac{ds}{s^{1-\rho}},
\]

and

\[
(D_b^{\alpha,\rho} f)(x) = (-\gamma)^n (b I^{n-\alpha,\rho} f)(t) = \frac{1}{\Gamma(n-\alpha)} (-\gamma)^n \int_{t}^{b} \left( \frac{s^\rho - t^\rho}{\rho} \right)^{n-\alpha-1} f(s) \frac{ds}{s^{1-\rho}},
\]

respectively, where \( \gamma = t^{1-\rho} \frac{d}{dt} \). Putting \( \rho = 1 \) in (17) and (18) one gets the Riemann-Liouville fractional derivatives (3) and (4) and letting \( \rho \) tend to 0, one gets the Hadamard fractional derivatives (9) and (10).

For the functions in \( AC^n_{\gamma}[a, b] = \{ f : [a, b] \to \mathbb{C} \text{ and } \gamma^{-1} f \in AC[a, b], \gamma = t^{1-\rho} \frac{d}{dt} \} \) and \( C^n_{\gamma}[a, b] = \{ f : [a, b] \to \mathbb{C} \text{ and } \gamma^{-1} f \in C[a, b], \gamma = t^{1-\rho} \frac{d}{dt} \} \), the left and right generalized Caputo fractional derivatives of order \( \alpha, \Re (\alpha) > 0 \) are given respectively as in [15] by

\[
C_a D^{\alpha,\rho} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} \left( \frac{t^\rho - s^\rho}{\rho} \right)^{n-\alpha-1} (\gamma^n f)(s) \frac{ds}{s^{1-\rho}} = a I^{n-\alpha,\rho}(\gamma^n f)(t)
\]

and

\[
C D_b^{\alpha,\rho} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t}^{b} \left( \frac{s^\rho - t^\rho}{\rho} \right)^{n-\alpha-1} (\gamma^n f)(s) \frac{ds}{s^{1-\rho}} = b I^{n-\alpha,\rho}((-\gamma)^n f)(t).
\]

It should be noticed that the derivative in (19) becomes the left Caputo derivative (5) once one replaces \( \rho \) by 1 and the left Caputo-Hadamard derivative (12) if one takes the limit as \( \rho \) approaches 0. The same relation holds for (20) and (6) and (20) and (14).

Below we present some formulas that will be used later in this work.

**Theorem 2.1.** [21] Let \( \alpha > 0, 1 \leq p \leq \infty \) and \( c \in \mathbb{R} \). Then for \( f \in X^p_c(a, b) \) where \( a > 0, \rho > 0, \) we have

\[
(a D^{\alpha,\rho} f) (a) = f \text{ and } D_b^{\alpha,\rho} I_{b}^{\alpha,\rho} f = f
\]

**Lemma 2.2.** Let \( \Re (\alpha) \geq 0, \rho > 0 \) and \( n = \lceil \Re (\alpha) \rceil \). If \( f \in AC^n_{\gamma}[a, b], \) where \( 0 < a < b < \infty \). Then,

\[
\left( C_a D^{\alpha,\rho} f \right)(t) = \left( a D^{\alpha,\rho} f \right)(t) - \sum_{k=0}^{n-1} \frac{\gamma^k f(a)}{\Gamma(k-\alpha+1)} \frac{(t^\rho - a^\rho)}{\rho}^{k-\alpha},
\]

\[
\left( C D_b^{\alpha,\rho} f \right)(t) = \left( D_b^{\alpha,\rho} f \right)(t) - \sum_{k=0}^{n-1} \frac{(-1)^k \gamma^k f(b)}{\Gamma(k-\alpha+1)} \frac{(b^\rho - t^\rho)}{\rho}^{k-\alpha}.
\]

**Proof.** The proof can be executed using Definition 3.1 and Lemma 2.8 in [15].
3. Integration by parts. In this section, we present some integration by parts formulas involving the generalized fractional integrals and derivatives presented in the previous section.

In order to prove an integration by part formula for the generalized type fractional derivatives and integrals we introduce the following function spaces: For $p \geq 1$ and $\alpha > 0$, we define

$$aI_{\alpha, \rho}^\varphi(X_p) = \{ f : f = aI_{\alpha, \rho}^\varphi, \varphi \in X_p(a, b) \}. \quad (24)$$

and

$$I_{\alpha, \rho}^\psi(X_p) = \{ f : f = I_{\alpha, \rho}^\psi, \psi \in X_p(a, b) \}. \quad (25)$$

**Theorem 3.1.** Let $\alpha > 0$, $p \geq 1$, $q \geq 1$, and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$). Then

- If $\varphi(t) \in X_p^c(a, b)$ and $\psi(t) \in X_q^c(a, b)$, then

$$\int_a^b \psi(t)(aI_{\alpha, \rho}^\varphi)(t)\frac{dt}{t^{1-\rho}} = \int_a^b \varphi(t)(I_{\alpha, \rho}^\psi)(t)\frac{dt}{t^{1-\rho}}.$$  

- If $f(t) \in I_{\alpha, \rho}^\psi(X_p)$ and $g(t) \in aI_{\alpha, \rho}^\varphi(X_q)$, then

$$\int_a^b f(t)(aD_{\alpha, \rho}^\varphi)(t)\frac{dt}{t^{1-\rho}} = \int_a^b (D_{\alpha, \rho}^\psi)(t)g(t)\frac{dt}{t^{1-\rho}}.$$

**Proof.**

- From (15), we have

$$\int_a^b \psi(t)(aI_{\alpha, \rho}^\varphi)(t)dt = \int_a^b \psi(t) \left[ \frac{1}{\Gamma(\alpha)} \int_a^t (t^\rho - s^\rho)^{\alpha-1} \frac{ds}{s^{1-\rho}} \right] \frac{dt}{t^{1-\rho}} = \int_a^b \varphi(s) \left[ \frac{1}{\Gamma(\alpha)} \int_s^b (t^\rho - s^\rho)^{\alpha-1} \frac{ds}{s^{1-\rho}} \right] \frac{dt}{t^{1-\rho}} = \int_a^b \varphi(s)(I_{\alpha, \rho}^\psi)(s)\frac{ds}{s^{1-\rho}}.$$  

It should be noted that Dirichlet formula was used in the second step of the proof.

- From definition and the first part, we have

$$\int_a^b f(t)(aD_{\alpha, \rho}^\varphi)(t)\frac{dt}{t^{1-\rho}} = \int_a^b (I_{\alpha, \rho}^\psi)(t).(aD_{\alpha, \rho}^\varphi)(t)\frac{dt}{t^{1-\rho}} = \int_a^b \psi(t).(aI_{\alpha, \rho}^\varphi)(t)\frac{dt}{t^{1-\rho}} = \int_a^b (D_{\alpha, \rho}^\psi)(t)g(t)\frac{dt}{t^{1-\rho}}.$$  

Below we present integration by parts formula for functions in the space $AC_n^\alpha[a, b]$ or $C_n^\alpha(a, b)$. 


Theorem 3.2. Let $\alpha > 0, n = [\alpha] + 1$ and $f \in X^\rho_c(a,b)$, $c \leq \rho$, $g \in AC^n_\gamma[a,b]$ or $C^n_\gamma[a,b]$. Then

$$
\int_a^b f(t) \left( c \frac{D^\alpha g(t)}{t^{1-\rho}} \right) \frac{dt}{t^{1-\rho}} = \int_a^b g(t) \left( D^\alpha_b f(t) \right) \frac{dt}{t^{1-\rho}} + \sum_{k=0}^{n-1} (\gamma^k g(t)(I^k_b - 1, \rho)f(t)) \bigg|_a^b.
$$

(26)

In particular, if $0 < \alpha < 1$ then

$$
\int_a^b f(t) \left( c \frac{D^\alpha g(t)}{t^{1-\rho}} \right) \frac{dt}{t^{1-\rho}} = \int_a^b g(t) \left( D^\alpha_b f(t) \right) \frac{dt}{t^{1-\rho}} + g(t)(I^1_b - 1, \rho)f(t) \bigg|_a^b.
$$

(27)

Proof.

$$
\int_a^b f(t) \left( c \frac{D^\alpha g(t)}{t^{1-\rho}} \right) \frac{dt}{t^{1-\rho}} = \int_a^b f(t)(I^{n-\alpha, \rho}g^n)(t) \frac{dt}{t^{1-\rho}}
$$

$$
= \int_a^b (\gamma^n g(t)(I^{n-\alpha, \rho}f(t) \frac{dt}{t^{1-\rho}} \quad \text{from Theorem 3.1}
$$

$$
= \int_a^b (\gamma^{n-1} g(t)(I^{n-\alpha, \rho}f(t))'(t) dt
$$

$$
= (\gamma^{n-1} g(t)(I^{n-\alpha, \rho}f(t)) \bigg|_a^b
$$

$$
= \int_a^b (\gamma^{n-1} g(t)(I^{n-\alpha, \rho}f(t))'(t) dt
$$

$$
= (\gamma^{n-1} g(t)(I^{n-\alpha, \rho}f(t)) \bigg|_a^b
$$

$$
+ \int_a^b (\gamma^{n-1} g(t)(I^{n-1-\alpha, \rho}f(t)) \frac{dt}{t^{1-\rho}},
$$

where, the last step is obtained by using the second part of Theorem 2.1. Repeating the same procedure $n - 1$ times, one gets

$$
\int_a^b f(t) \left( c \frac{D^\alpha g(t)}{t^{1-\rho}} \right) \frac{dt}{t^{1-\rho}} = \sum_{k=0}^{n-1} (\gamma^{n-1-k} g(t)(I^{n-k-\alpha, \rho}f(t)) \bigg|_a^b
$$

$$
+ \int_a^b g(t)(I^{n-\alpha, \rho}f(t) \frac{dt}{t^{1-\rho}}
$$

$$
= \sum_{k=0}^{n-1} (\gamma^{n-1-k} g(t)(I^{n-k-\alpha, \rho}f(t) \bigg|_a^b
$$

$$
+ \int_a^b g(t)(D^\alpha_b f(t) \frac{dt}{t^{1-\rho}}
$$
Corollary 1. Let $\alpha > 0$, $n = [\alpha] + 1$ and $f \in X^p_c(a,b)$, $c \leq \rho$, $g \in AC^n_\gamma[a,b]$ or $C^n_\gamma[a,b]$. Then,

\[
\int_a^b f(t) (a D^{\alpha,\rho} g)(t) \frac{dt}{t^{1-\rho}} = \int_a^b g(t) (D^{\alpha,\rho}_b f)(t) \frac{dt}{t^{1-\rho}} + \sum_{k=0}^{n-1} (\gamma^k g)(b)(I^{k-\alpha,\rho}_{a} f)(b^-).
\]

In particular, if $0 < \alpha < 1$ then

\[
\int_a^b f(t) (a D^{\alpha,\rho} g)(t) \frac{dt}{t^{1-\rho}} = \int_a^b g(t) (D^{\alpha,\rho}_b f)(t) \frac{dt}{t^{1-\rho}} + g(b)(I^{k-\alpha,\rho}_b f)(b^-). \tag{29}
\]

Proof. Using Lemma 2.2, one obtains the following:

\[
\int_a^b f(t) (a D^{\alpha,\rho} g)(t) \frac{dt}{t^{1-\rho}} = \int_a^b f(t) (C^{\alpha,\rho}_a D^{\alpha,\rho} g)(t) \frac{dt}{t^{1-\rho}} + \int_a^b \sum_{k=0}^{n-1} (\gamma^k g)(a^+) (I^{k-\alpha,\rho}_{a} f)(a^+)\frac{dt}{t^{1-\rho}}.
\]

Using Theorem 3.2, one finds out that

\[
\int_a^b f(t) (a D^{\alpha,\rho} g)(t) \frac{dt}{t^{1-\rho}} = \int_a^b g(t) (D^{\alpha,\rho}_b f)(t) \frac{dt}{t^{1-\rho}} + \sum_{k=0}^{n-1} (\gamma^k g)(a^+) (I^{k-\alpha,\rho}_{a} f)(a^+)\frac{dt}{t^{1-\rho}} + \sum_{k=0}^{n-1} (\gamma^k g)(b^-)(I^{k-\alpha,\rho}_{a} f)(b^-).
\]

Hence, the identity (28) holds. \qed
Theorem 3.3. Let $\alpha > 0, n = [\alpha] + 1$ and $f \in X_c^\alpha(a, b)$, $c \leq \rho$, $g \in AC^{\alpha}_{\gamma}[a, b]$ or $C^n[a, b]$. Then
\[
\int_a^b f(t) C^\alpha_{\gamma} g(t) dt \bigg/ \frac{t - a}{t^{1 - \rho}} = \int_a^b g(t) (a D^\alpha f)(t) dt \bigg/ \frac{t - a}{t^{1 - \rho}}
- \sum_{k=0}^{n-1} (-1)^k (\gamma^k g(t)(a I^{k-\alpha+1, \rho} f)(t) |_{a}^{b}.
\]

In particular, if $0 < \alpha < 1$ then
\[
\int_a^b f(t) C^\alpha_{\gamma} g(t) dt \bigg/ \frac{t - a}{t^{1 - \rho}} = \int_a^b g(t) (a D^\alpha f)(t) dt \bigg/ \frac{t - a}{t^{1 - \rho}}
- g(t)(a I^{1-\alpha, \rho} f)(t) \bigg/ \frac{t - a}{t^{1 - \rho}}.
\]

Proof.
\[
\int_a^b f(t) C^\alpha_{\gamma} g(t) dt \bigg/ \frac{t - a}{t^{1 - \rho}} = \int_a^b f(t) (a I^{n-\alpha, \rho} ((-\gamma)^n g)) dt \bigg/ \frac{t - a}{t^{1 - \rho}}
= \int_a^b ((-\gamma)^n g)(t)(a I^{n-\alpha, \rho} f)(t) dt \bigg/ \frac{t - a}{t^{1 - \rho}} \text{ by Theorem 3.1}
= (-1)^n \int_a^b (\gamma^{n-1} g)(t) (a I^{n-\alpha, \rho} f)(t) dt
= (-1)^n \{ \left( (\gamma^{n-1} g)(t)(a I^{n-\alpha, \rho} f)(t) \bigg/ \frac{t - a}{t^{1 - \rho}} \right) \}_{a}^{b}
- \int_a^b (\gamma^{n-1} g)(t)(a I^{n-\alpha, \rho} f)'(t) dt
= (-1)^n \{ \left( (\gamma^{n-1} g)(t)(a I^{n-\alpha, \rho} f)(t) \bigg/ \frac{t - a}{t^{1 - \rho}} \right) \}_{a}^{b}
- \int_a^b (\gamma^{n-1} g)(t)(a I^{n-\alpha, \rho} f)(t) dt \bigg/ \frac{t - a}{t^{1 - \rho}} \},
\]

Repeating the same procedure $n - 1$ times, one gets
\[
\int_a^b f(t) C^\alpha_{\gamma} g(t) dt \bigg/ \frac{t - a}{t^{1 - \rho}} = (-1)^n \left\{ \sum_{k=0}^{n-1} (-1)^k (\gamma^{n-1-k} g)(t)(\gamma^k (a I^{n-\alpha, \rho} f))(t) \bigg/ \frac{t - a}{t^{1 - \rho}} \right\}
+ (-1)^n \int_a^b g(t)(\gamma^n (a I^{n-\alpha, \rho} f)(t) dt \bigg/ \frac{t - a}{t^{1 - \rho}}
= \sum_{k=0}^{n-1} (-1)^{n+k} (\gamma^{n-1-k} g)(t)(a I^{n-k-\alpha, \rho} f)(t) \bigg/ \frac{t - a}{t^{1 - \rho}}
+ \int_a^b g(t)(a D^{\alpha, \rho} f)(t) dt \bigg/ \frac{t - a}{t^{1 - \rho}}
= \sum_{k=0}^{n-1} (-1)^k (\gamma^k g)(t)(a I^{k-\alpha+1, \rho} f)(t) \bigg/ \frac{t - a}{t^{1 - \rho}}
+ \int_a^b g(t)(a D^{\alpha, \rho} f)(t) dt \bigg/ \frac{t - a}{t^{1 - \rho}}.
\]

\qed
Corollary 2. Let $\alpha > 0, n = [\alpha] + 1$, $\rho > c$ and $f \in X^p_c(a, b)$, $c \leq \rho$, $g \in AC^\gamma[a, b]$ or $C^n_\gamma[a, b]$. Then,
\[
\int_a^b f(t) (D_b^{\alpha, \rho} g)(t) \frac{dt}{t^{1-\rho}} = \int_a^b g(t)(a D^{\alpha, \rho} f)(t) \frac{dt}{t^{1-\rho}} + \sum_{k=0}^{n-1} ((-\gamma)^k g)(a^+) (a^{k-\alpha+1, \rho} f)(a^+). \tag{32}
\]

In particular, if $0 < \alpha < 1$ then
\[
\int_a^b f(t) (D_b^{\alpha, \rho} g)(t) \frac{dt}{t^{1-\rho}} = \int_a^b g(t)(a D^{\alpha, \rho} f)(t) \frac{dt}{t^{1-\rho}} + g(a^+) (a^{1-\alpha, \rho} f)(a^+). \tag{33}
\]

Proof. In the light of Lemma 2.2, we have
\[
\int_a^b f(t) (D_b^{\alpha, \rho} g)(t) \frac{dt}{t^{1-\rho}} = \int_a^b (c D_b^{\alpha, \rho} g)(t) \frac{dt}{t^{1-\rho}} + \sum_{k=0}^{n-1} \frac{((-\gamma)^k g)(b)}{\Gamma(k - \alpha + 1)} \int_a^b f(t) \left(\frac{b^\rho - t^\rho}{\rho}\right)^{k-\alpha} \frac{dt}{t^{1-\rho}} = \int_a^b (c D_b^{\alpha, \rho} g)(t) \frac{dt}{t^{1-\rho}} + \sum_{k=0}^{n-1} ((-\gamma)^k g)(b^-) (a^{k-\alpha+1, \rho} f)(b^-).
\]

Once one utilizes Theorem 3.3, one obtains
\[
\int_a^b f(t) (D_b^{\alpha, \rho} g)(t) \frac{dt}{t^{1-\rho}} = \int_a^b g(t)(a D^{\alpha, \rho} f)(t) \frac{dt}{t^{1-\rho}} \bigg|_{a}^{b} - \sum_{k=0}^{n-1} ((-\gamma)^k g)(a^{k-\alpha+1, \rho} f)(t) \bigg|_{a}^{b} + \sum_{k=0}^{n-1} ((-\gamma)^k g)(b^-) (a^{k-\alpha+1, \rho} f)(b^-) = \int_a^b g(t)(a D^{\alpha, \rho} f)(t) \frac{dt}{t^{1-\rho}} + \sum_{k=0}^{n-1} ((-\gamma)^k g)(a^+) (a^{k-\alpha+1, \rho} f)(a^+).
\]

Hence, identity (32) is proved. \qed

4. Euler-Lagrange equations. In this section, we consider Euler-Lagrange equations that are derived from the necessary conditions that certain functionals to have extremum for a given function.

Consider the functional $J : C^n_\gamma[a, b] \to \mathbb{R}$ of the form
\[
J(y) = \int_a^b L\left(t, y(t), (C_{\alpha} D^{\alpha, \rho} y)(t)\right) \frac{dt}{t^{1-\rho}}, \tag{34}
\]
where \( 0 \leq a < b, \rho > 0, \alpha > 0, n = [\alpha] + 1, y^{(i)}(a) = a_i \) (constant), \( y^{(i)} = b_i \) (constant) \( i = 0, 1, 2, \ldots, n - 1 \) and \( L : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) with \( L_2 = \frac{\partial L}{\partial a} \) is in \( \mathcal{C}_n^\alpha[a, b] \).

**Theorem 4.1.** For the functional \( J \) in (34) to have a local extremum in \( S = \big\{ z \in C_n^\alpha[a, b] : z^{(i)}(a) = a_i \) (constant), \( z^{(i)} = b_i \) (constant) \big\} \) at some \( y \in S \), it is necessary that \( y \) satisfies the Euler-Lagrange equation

\[
L_2(t) + (D_1^\alpha g(t)) = 0, \quad \forall t \in [a, b], \quad (35)
\]

where \( L_2 = \frac{\partial L}{\partial y} \) and \( (D_1^\alpha g(t)) \) are continuous on \([a, b]\).

**Proof.** Let \( J \) have a local extremum at \( \hat{y} \in S \). Let \( \epsilon \in \mathbb{R} \) and define a set of curves \( y(t) = \hat{y} + \epsilon g(t), \) where \( g(t) \) is an arbitrary function such that \( g^{(i)}(a) = g^{(i)}(b) = 0, i = 0, 1, 2, \ldots, n - 1 \). Now

\[
J(\epsilon) = J(\hat{y} + \epsilon g(t)) = \int_a^b L(t, \hat{y}(t) + \epsilon g(t), (C_aD_1^\alpha \hat{y})(t) + \epsilon (C_aD_1^\alpha g)(t)) \frac{dt}{t^{1-\rho}}
\]

is a function of \( \epsilon \) that has an extremum when \( \epsilon = 0 \). That is \( J'(0) = 0 \). Thus we have

\[
\int_a^b \left[ L_2(t) + (D_1^\alpha g(t)) \right] \frac{dt}{t^{1-\rho}} = 0.
\]

Now using the integration by parts formula (26) in Theorem 3.2, one gets

\[
\int_a^b \left[ L_2(t) + (D_1^\alpha g(t)) \right] \frac{dt}{t^{1-\rho}} = \sum_{k=0}^{n-1} (\gamma^k g)(t) (I_1^{1-k+\alpha}(L_3))(t) \bigg|_a^b = 0.
\]

Now because \( g^{(i)}(a) = g^{(i)}(b) = 0, i = 0, 1, 2, \ldots, n - 1 \), we have \( (\gamma^{n-k}) g)(t) = 0, \) \( k = 0, 1, 2, \ldots, n - 1 \). Hence, we have

\[
\int_a^b \left[ L_2(t) + (D_1^\alpha g(t)) \right] g(t) \frac{dt}{t^{1-\rho}} = 0.
\]

Since \( g(t) \) is arbitrary, it follows that

\[
L_2(t) + (D_1^\alpha g(t)) = 0, \quad \forall t \in [a, b].
\]

\[\square\]

When \( \alpha = \rho = 1 \), Theorem 4.1 coincides with the classical case [30].

**Corollary 3.** Consider a functional of the form

\[
\int_a^b L(t, y_1(t), y_2(t), \ldots, y_m(t), (C_aD_1^\alpha y_1)(t), (C_aD_1^\alpha y_2)(t), \ldots, (C_aD_1^\alpha y_m)(t)) \frac{dt}{t^{1-\rho}},
\]

where \( 0 \leq a < b, \rho > 0, \alpha > 0, n = [\alpha] + 1, y_j^{(i)}(a) = a_{ij}, y_j^{(i)} = b_{ij} \) \( i = 0, 1, 2, \ldots, n - 1, j = 1, 2, \ldots, m \) and \( L : [a, b] \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} \) with \( \frac{\partial L}{\partial a} \in \mathcal{C}_n^\alpha[a, b], \) \( \forall j = 1, 2, \ldots, m \). For the functional in (36) to have a local extremum, it is necessary that \( y_j, j = 1, 2, \ldots, m \) satisfy the Euler-Lagrange equations

\[
\frac{\partial L}{\partial y_j} + \left[ (D_1^\alpha \frac{\partial L}{\partial a}) \right] = 0, \quad \forall t \in [a, b], \quad j = 1, 2, \ldots, m.
\]
Now we consider a functional $J : C^m_\gamma[a, b] \to \mathbb{R}$ involving left and right generalized fractional derivatives with of the form

$$J(y) = \int_a^b L \left( t, y(t), \frac{\partial^\alpha y(t)}{\partial t^\alpha}, \frac{\partial^\beta y(t)}{\partial t^\beta} \right) \frac{dt}{t^{1-\rho}},$$

where $0 \leq \alpha < \beta$, $\rho > 0$, $0 < \alpha \leq \beta$, $n = [\alpha] + 1$, $m = [\beta] + 1$, $y^{(i)}(a) = a_i$ (constant), $y^{(i)} = b_i$ (constant) $i = 0, 1, 2, ..., m - 1$ and $L : [a, b] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $L_3 = \frac{\partial L}{\partial y}$ and $L_4 = \frac{\partial L}{\partial \frac{\partial^\beta y}{\partial t^\beta}}$ are in $C^m_\gamma[a, b]$.

**Theorem 4.2.** For the functional $J$ in (38) to have a local extremum in $S = \left\{ z \in C^m_\gamma[a, b] : z^{(i)}(a) = a_i \ (\text{constant}), z^{(i)} = b_i \ (\text{constant}) \right\}$ at some $y \in S$, it is necessary that $y$ satisfies the Euler-Lagrange equation

$$L_2(t) + (D_b^{\alpha-\rho} L_3)(t) + ( a D^{\alpha-\rho} L_4)(t) = 0, \ \forall t \in [a, b],$$

where $L_2 = \frac{\partial L}{\partial y}$, $(D_b^{\alpha-\rho} L_3)(t)$ and $( a D^{\alpha-\rho} L_4)(t)$ are continuous on $[a, b]$.

**Proof.** Let $J$ have a local extremum at $\hat{y} \in S$. Let $\epsilon \in \mathbb{R}$ and define a set of curves $y(t) = \hat{y} + \epsilon g(t)$, where $g(t)$ is an arbitrary function such that $g^{(i)}(a) = g^{(i)}(b) = 0, i = 0, 1, 2, ..., m - 1$. Now

$$J'(\epsilon) = J(\hat{y} + \epsilon g(t)) = \int_a^b L \left( t, \hat{y}(t) + \epsilon g(t), \frac{\partial^\alpha (\hat{y} + \epsilon g)(t)}{\partial t^\alpha}, \frac{\partial^\beta \hat{y}(t) + \epsilon \frac{\partial^\beta g(t)}{\partial t^\beta}}{\partial t^\beta} \right) \frac{dt}{t^{1-\rho}}$$

is a function of $\epsilon$ that has an extremum when $\epsilon = 0$. That is

$$J'(0) = \int_a^b \left[ L_2(t)g(t) + L_3(t) \frac{\partial^\alpha g(t)}{\partial t^\alpha} + L_4(t) \frac{\partial^\beta g(t)}{\partial t^\beta} \right] \frac{dt}{t^{1-\rho}} = 0.$$

Now using the integration by parts formulas (26) in Theorem 3.2 and (30) in Theorem 3.3 one gets

$$\int_a^b \left[ L_2(t)g(t) + (D_b^{\alpha-\rho} L_3)(t)g(t) + ( a D^{\alpha-\rho} L_4)(t) \right] g(t) \frac{dt}{t^{1-\rho}}$$

$$+ \sum_{k=0}^{n-1} (\gamma_k g)(t) \int_a^b (a I^{k-\alpha+1-\rho} L_3)(t) \frac{dt}{t^{1-\rho}} - \sum_{k=0}^{m-1} (-\gamma_k)^m g(t) \int_a^b (a I^{k-\beta+1-\rho} L_4)(t) \frac{dt}{t^{1-\rho}} = 0.$$

Now because $g^{(i)}(a) = g^{(i)}(b) = 0, i = 0, 1, 2, ..., m - 1$, we have

$$\int_a^b \left[ L_2(t) + (D_b^{\alpha-\rho} L_3)(t) + ( a D^{\alpha-\rho} L_4)(t) \right] g(t) \frac{dt}{t^{1-\rho}} = 0$$

Because $g(t)$ is arbitrary, one gets

$$L_2(t) + (D_b^{\alpha-\rho} L_3)(t) + ( a D^{\alpha-\rho} L_4)(t) = 0, \ \forall t \in [a, b].$$

**Remark 1.** When $\rho = 1$, the Euler-Lagrange equation (39) coincides with the Euler-Lagrange equation (19) in [5]. When $\alpha = 1$ and $\rho \in (0, 1]$, the results in [24] are reobtained.
To support the obtained results we study an example of physical interest under Theorem 4.1. This example was studied in [16] and [17] when $\rho = 1$. It was also considered in [2] in the frame of Atangana-Baleanu fractional derivative.

**Example.** We study the following action,

$$J(y) = \int_0^b \left[ \frac{1}{2} \left( \frac{C_0^\alpha y(t)}{D^\rho y(t)} \right)^2 - V(y(t)) \right] t^{\rho - 1} dt,$$

where $0 < \alpha < 1$ and with $y(0)$, $y(b)$ are assigned or with the natural boundary condition $\left( I_b^{1-\alpha,\rho} \frac{C_0^\alpha y(t)}{D^\rho y(t)} \right)(t) = 0$. Then, by applying Theorem 4.1 we obtain

$$(D_b^{\alpha,\rho} \left( \frac{C_0^\alpha y(t)}{D^\rho y(t)} \right))(t) - \frac{dV}{dy}(t) = 0 \text{ for all } t \in [0, b].$$

Next, we solve the above fractional Euler-Lagrange equations for certain potential functions.

- We consider the free particle case $V \equiv 0$: The Euler-Lagrange equations will be reduced to

$$(D_b^{\alpha,\rho} \frac{C_0^\alpha y(t)}{D^\rho y(t)})(t) = 0.$$

By applying $I_b^{\alpha,\rho}$ to both sides and making use of (2.10) in Theorem 2.7 of [15] we reach at

$$\left( \frac{C_0^\alpha y(t)}{D^\rho y(t)} \right)(t) = \frac{I_b^{1-\alpha,\rho} \frac{C_0^\alpha y(b^-)}{D^\rho y(b^-)} \left( \frac{b^\rho - t^\rho}{\rho} \right)^{\alpha - 1}}{\rho^{\alpha - 1} \Gamma(\alpha)} = 0.$$

In particular, if $\left( \frac{C_0^\alpha y(t)}{D^\rho y(t)} \right)(t)$ is continuous at $b$, we have $\left( \frac{C_0^\alpha y(t)}{D^\rho y(t)} \right)(t) = 0$ and hence

$$y(t) = c_1.$$  \hspace{1cm} (40)

Using the condition $y(0) = A$, the solution reads

$$y(t) = A.$$  \hspace{1cm} (41)

We remark here that as $\alpha \to 1$ and $\rho \to 1$, we get the classical case.

- Let $V(y) = cy^2/2$. Then, the associated fractional Euler-Lagrange equation under the assumption that $\left( \frac{C_0^\alpha y(t)}{D^\rho y(t)} \right)(t)$ is continuous at $b$ becomes

$$(D_b^{\alpha,\rho} \frac{C_0^\alpha y(t)}{D^\rho y(t)})(t) = cy(t).$$

Applying $I_b^{\alpha,\rho}$ and $I_0^{\alpha,\rho}$, successively together with the usage of (2.10) in Theorem 2.7 and Theorem 3.6 in [15] one after the other, we reach at the integral equation

$$y(t) = y(0) + c \left( \frac{D^\alpha y(t)}{0 I_b^{\alpha,\rho} \frac{C_0^\alpha y(t)}{D^\rho y(t)}} \right)(t).$$  \hspace{1cm} (42)

Notice that, when $\alpha \to 1$ and $\rho \to 1$ we get the classical result.

5. **Conclusions.** In this paper, we presented some integration by parts formulas associated with a specific type of generalized fractional operators involving two parameters and coming down to the classical operators when one of these parameters is fixed. Thus, the integration by parts found in the paper are the generalization of the existing ones in the literature. Moreover, we considered some variational problems that embody some generalized fractional operators and found the Euler-Lagrange equations corresponding to these problems. It was remarked that under some conditions the Euler-Lagrange equations found match with the ones found previously and hence they can be considered as generalization of the former results.
We have assumed that the parameter $\rho$ must be positive. In fact, the case $\rho \to 0$ will be reduced to the Hadamard-type variational problems and this will be part of our investigations in future works.

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