Ruppeiner Geometry, Reentrant Phase transition and Microstructure of Born-Infeld AdS Black Hole

Naveena Kumara A.,* Ahmed Rizwan C.L.,† Kartheek Hegde,‡ and Ajith K.M.§

Department of Physics, National Institute of Technology Karnataka, Surathkal 575 025, India

Md Sabir Ali¶

Department of Physics, Indian Institute of Technology, Ropar, Rupnagar, Punjab 140 001, India
Abstract

Born-Infeld AdS black holes exhibits a reentrant phase transition for certain values of the Born-Infeld parameter $b$. This behaviour is an additional feature compared to the van der Waals like phase transition observed in charged AdS black holes. Therefore it is worth observing the underlying microscopic origin for this re-entrant phase transition. Depending on the value of the parameter $b$, the black hole system has four different cases: no phase transition, a reentrant phase transition with two scenarios, or a van der Waals-like phase transition. In this article, by employing the Ruppeiner geometry method in the parameter space of temperature and volume, we investigate the microstructure of Born-Infeld black hole via the phase transition study, which includes van der Waals like and re-entrant phase transition. We find that the microstructures of the black hole that lead to usual and re-entrant phase transitions are distinct in nature. The usual phase transition is characterised by the typical RN-AdS microstructure. The small black hole phase has a dominant repulsive interaction for the low temperature case. Interestingly, for the range of pressure in which the system displays a reentrant phase transition, the phase transition is from the large black hole phase to the intermediate black hole phase which possesses only dominant attractive interaction. We show that in the reentrant phase transition case, the intermediate black holes behave like a bosonic gas, and in the usual phase transition case the small black holes behave like a quantum anyon gas. In both cases the large black hole phase displays an interaction similar to the bosonic gas.

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*Electronic address: naviphysics@gmail.com
†Electronic address: ahmedrizwancl@gmail.com
‡Electronic address: hegde.kartheek@gmail.com
§Electronic address: ajith@nitk.ac.in
¶Electronic address: alimd.sabir3@gmail.com
I. INTRODUCTION

In physics, thermodynamics is a general and powerful tool to understand the physical properties of a system and a wide range of phenomena. The simplicity of thermodynamics lies in its inherent property that, the microscopic details are not necessary to explore the macroscopic physics. This property is particularly useful in dealing with the systems for which the microscopic details are not well understood, such as quantum gravity. Therefore the lack of full knowledge of quantum aspects of black hole offers a perfect scenario for the application of thermodynamics to probe it’s microscopic details.

Black holes being thermodynamic systems, exhibit a rich class of phase transitions, which are the key tools in probing their properties in black hole chemistry. Quite interesting facet of black hole thermodynamics is the phase transition and related phenomenon in AdS spaces. In recent times the interest on AdS black hole thermodynamics aroused among the researchers, basically from the identifying the cosmological constant with the thermodynamic pressure and the modification in the first law by including the corresponding variations [1, 2]. With this association, it was demonstrated that the phase transition features of AdS black holes can be seen as van der Waals like and/or reentrant phase transitions (RPT) [3–5].

A reentrant phase transition (RPT) occurs when the system undergoes more than one phase transition when there is a monotonic change in the thermodynamic variable such that, the initial and final macro states of the system are the same. In conventional thermodynamic systems, this phenomenon was first observed in the nicotine/water mixture, in a process with fixed percentage of nicotine an increase in the temperature. The system exhibited a reentrant phase transition during this process from initial homogeneous mixed state to an intermediate distinct nicotine/water phases and finally to a the homogeneous state [6]. This kind of reentrant phase transition is observed in variety of physical systems, more commonly in multicomponent fluid systems, gels, ferroelectrics, liquid crystals, binary gases etc. [7]. This phenomenon is not limited to condensed matter physics, for example, in a (2 + 1) dimensional Dirac oscillator, in a magnetic field, in non-commutative spacetime a similar phase transition is observed [8].

In black hole physics, reentrant phase transition was first observed in four-dimensional Born Infeld AdS black holes [4], where the initial and final phases are large black holes and the intermediate phase is an intermediate black hole. For this case, LBH/IBH/LBH
reentrant phase transition takes place when the temperature is decreased monotonically in a certain range of pressure. However higher dimensional Born Infeld black holes do not show reentrant phase transitions [9]. Interestingly, rotating black holes in dimensions $d \geq 6$ show reentrant phase transitions [10]. In subsequent studies, the RPT in higher dimensional single spinning and multi spinning Kerr black holes in anti-de Sitter and de Sitter spacetime background were also investigated [11–13]. Reentrant phase transitions are also observed in gravity theories consisting of higher-curvature corrections [14–18]. The reentrant phase transition of Born Infeld black hole was also analysed in a different perspective, wherein the charge of the system was varied, and the cosmological constant (pressure) was kept fixed [19]. Furthermore, the the relationship between the RPT and the photon sphere of BI spacetime studied [20]. The question that is still unanswered for a reentrant phase transition is, “what is the underlying microstructure that leads to it?”

It is a well established notion that the geometric methods can serve as a tool to understand microscopic interaction of the black hole [21–42]. Originally, the geometric methods were developed from Gaussian thermodynamic fluctuation theory [43], in which a metric is constructed by choosing a suitable thermodynamic potential in a phase space which constitutes other thermodynamic variables, the corresponding curvature scalar encodes details about phase transitions and critical points. The sign of the curvature scalar is an indicator of the nature of interaction, this convention is adopted from the Ruppeiner geometry applications to conventional thermodynamic systems [44–47].

It is worth noting that the investigation of the microstructure of black hole system has been one of the major challenges in black hole physics for the past few decades. Even though string theory and loop quantum gravity provide tools to understand the quantum gravity, there is no complete theoretical description for that. Therefore the microscopic physics of black holes is confined to some speculative assumptions. The success of black hole thermodynamics in understanding various aspects of black hole, prompts us to speculate some relation between micro-dynamics and thermodynamics. However the process is in the reverse sense compared to statistical physics, where the microscopic knowledge is sought from macroscopic details. However, when we seek the nature of microstructure, we do not have a clear picture about the black hole constituents, we take an abstract concept of black hole molecule.

This article is organised as follows. In the next section, we discuss the thermodynamics
and the phase structure of the black hole. We will consider the van der Waals case (which will be called SPT throughout the article which stands for standard phase transition) and RPT case separately. Then the microstructure study is carried out by constructing the Ruppeiner geometry using the fluctuation coordinates as temperature and volume (section III). In the last section (IV) we present our results.

II. THERMODYNAMICS AND PHASE STRUCTURE OF THE BLACK HOLE

In this section we present thermodynamics and phase structure of the Born-Infeld AdS black hole. The action for Einstein gravity in the presence of Born-Infeld field has the following form \[ S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left( \mathcal{R} - 2\Lambda + 4b^2 \left( 1 - \sqrt{1 + \frac{F_{\mu\nu}F^{\mu\nu}}{2b^2}} \right) \right). \] (1)

Here \( \mathcal{R} \) and \( \Lambda \) are the Ricci scalar and cosmological constant, respectively. The Born Infeld parameter \( b \) with dimension of mass represents the maximal electromagnetic field strength, and is related to the string tension (the identification is motivated from string theory [49]), and \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the electromagnetic field tensor, where \( A_\mu \) is the vector potential. We study the extended thermodynamics of the black hole, where the cosmological constant serves the role of thermodynamic pressure. Their relation to the AdS radius \( l \) are given by,

\[ \Lambda = -\frac{3}{l^2}, \quad \text{and} \quad P = \frac{3}{8\pi l^2}. \] (2)

The solution to the Einstein field equations in the static and spherically symmetric spacetime background yields [50–52],

\[ ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \] (3)

\[ F = \frac{Q}{\sqrt{r^4 + Q^2/b^2}} dt \wedge dr, \] (4)

with the metric function of the form,

\[ f(r) = 1 + \frac{r^2}{l^2} - \frac{2M}{r} + \frac{2b^2}{r} \int_r^\infty \left( \sqrt{r^4 + \frac{Q^2}{b^2}} - r^2 \right) dr \] (5)

\[ = 1 + \frac{r^2}{l^2} - \frac{2M}{r} + \frac{2b^2r^2}{3} \left( 1 - \sqrt{1 + \frac{Q^2}{b^2r^4}} \right) + \frac{4Q^2}{3r^2} 2F_1 \left( \frac{1}{4}, \frac{1}{2}, \frac{5}{4}; \frac{Q^2}{b^2r^4} \right). \] (6)
Where, \( _2F_1 \) is the hypergeometric function, the parameters \( M \) and \( Q \) are the ADM mass and the asymptotic charge of the solution, respectively. We obtain mass of the black hole by setting \( f(r_+) = 0 \),

\[
M = \frac{r_+}{2} + \frac{r_+^3}{2l^2} + \frac{b^2r_+^3}{3} \left( 1 - \sqrt{1 + \frac{Q^2}{b^2r_+^4}} \right) + \frac{2Q^2}{3r_+^2} \binom{1}{1} \binom{1}{2} \binom{5}{4} \left( \frac{r_+}{2} \cdot \frac{5}{4} - \frac{Q^2}{b^2r_+^4} \right). \tag{7}
\]

The Hawking temperature and the corresponding entropy are given by,

\[
T = \frac{1}{4\pi r_+} \left( 1 + \frac{3r_+}{l^2} + 2b^2r_+^2 \left( 1 - \sqrt{1 + \frac{Q^2}{b^2r_+^4}} \right) \right), \quad S = \frac{A}{4} = \pi r_+^2 \tag{8}
\]

The electric potential \( \Phi \) and the electric polarization \( B \), which is conjugate to \( b \) and is referred to as BI vacuum polarization, measured at infinity with respect to the event horizon they are calculated to be,

\[
\Phi = \frac{Q}{r_+} \binom{1}{1} \binom{1}{2} \binom{5}{4} \left( \frac{r_+}{2} \cdot \frac{5}{4} - \frac{Q^2}{b^2r_+^4} \right) \tag{9}
\]

\[
B = \frac{2}{3} br_+^3 \left( 1 - \sqrt{1 + \frac{Q^2}{b^2r_+^4}} + \frac{Q^2}{3b r_+^4} \right) \binom{1}{1} \binom{1}{2} \binom{5}{4} \left( \frac{r_+}{2} \cdot \frac{5}{4} - \frac{Q^2}{b^2r_+^4} \right) \tag{10}
\]

Interpreting the mass \( M \), as enthalpy rather than the internal energy of the black hole, we obtain the first law of thermodynamics as,

\[
dM = TdS + VdP + \Phi dq + Bdb, \tag{11}
\]

where \( V = \frac{4\pi r_+^3}{3} \) is the thermodynamic volume of the system. The thermodynamic quantities of BI-AdS black hole satisfy the Smarr formula, in addition to the first law of thermodynamics, which is obtained by scaling argument as,

\[
M = 2 (TS - VP) + \Phi Q - Bb. \tag{12}
\]

The equation of state of the black hole system is,

\[
P = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2} - \frac{b^2}{4\pi} \left( 1 - \sqrt{1 + \frac{Q^2}{b^2r_+^4}} \right). \tag{13}
\]

The black hole shows a van der Waals like phase transition, which depends on the value of the Born Infeld coupling coefficient \( b \). The critical values corresponding to this phase transition is obtained by employing the conditions,

\[
(\partial_{r_+} P)_T = (\partial_{r_+} P)_T = 0. \tag{14}
\]
The critical values are \([4, 20]\),

\[
P_c = \frac{1 - 16x Q^2}{8\pi r_{+c}^2} - \frac{b^2}{4\pi} \left( 1 - \frac{1}{4\pi r_{+c}^2} \right), \quad (15)
\]

\[
T_c = \frac{1 - 8x Q^2}{2\pi r_{+c}}, \quad (16)
\]

\[
r_{c+} = \frac{1}{2} \left( \frac{1}{x_k^2} - \frac{16Q^2}{b^2} \right)^{\frac{1}{2}}, \quad (17)
\]

where

\[
x_k = 2\sqrt{\frac{p}{3}} \cos \left( \frac{1}{3} \arccos \left( \frac{3q}{2p} \sqrt{\frac{-3}{p}} \right) - \frac{2\pi k}{3} \right), \quad k = 0, 1, 3, \quad (18)
\]

\[
p = -\frac{3b^2}{32Q^2}, \quad q = \frac{b^2}{256Q^4}. \quad (19)
\]

Since, the critical point corresponding to \(x_2\) is always a complex number, effectively we have only two critical points. Based on these two values we can classify the system into four different cases which depend on the value of \(b\).

- **Case 1 (SPT),** \(b < b_0\): For this condition, the system behaves like a Schwarzschild AdS black hole. The large black hole phase is stable and the small black hole phase is unstable. Therefore there is no van der Waals like phase transition in this case.

- **Case 2 (RPT),** \(b_0 < b < b_1\): In this condition, the system is characterised by two critical points \(c_0\) and \(c_1\). However \(c_0\) is an unstable point as it has a higher Gibbs free energy. In this scenario, the system exhibits a zeroth order phase transition and a van der Waals like first order phase transition, between a large black hole phase and an intermediate black hole phase. These successive transitions is together termed as a reentrant phase transition.

- **Case 3 (RPT),** \(b_1 < b < b_2\): This is another case of reentrant phase transition displayed by the black hole. However, here the critical point \(c_0\) lies in the negative pressure region.

- **Case 4 (no PT),** \(b_2 < b\): Here the black hole exhibits the typical van der Waals like phase transition with one critical point.
Figure 1: The coexistence curve and the metastable curves for the SPT of the black hole. The transition curve separates SBH and LBH phases. The region between the coexistence curve and metastable curves are the metastable SBH and LBH phases in respective region. We have taken $Q = 1$ and $b = 1$.

The values of the parameter $b$ where the phase transition behaviour changes, are given by,

$$b_0 = \frac{1}{\sqrt{8Q}} \approx \frac{0.3536}{Q}, \quad b_1 = \frac{\sqrt{3 + 2\sqrt{3}}}{6Q} \approx \frac{0.4237}{Q}, \quad b_2 = \frac{1}{2Q} = \frac{0.5}{Q}. \quad (20)$$

For SPT situation, we introduce the reduced parameters, which are defined as,

$$P_r = \frac{P}{P_c}, \quad T_r = \frac{T}{T_c}, \quad V_r = \frac{V}{V_c} \quad (21)$$

Since there are two critical points in the RPT case, there is no unique way of defining the reduced parameters. In the following, we study the phase structure of the black hole corresponding to SPT and RPT, by considering case 1 and case 3, respectively. There exists no analytic expression for the coexistence curve of the system, therefore we obtain it via a numerical method by employing the swallow tail behaviour of Gibbs free energy. The expression for the Gibbs free energy is given by,

$$G(T, P) = \frac{1}{4} \left[ r_+ - \frac{8\pi}{3} P r_+^3 - \frac{2b^2 r_+^2}{3} \left( 1 - \sqrt{1 + \frac{Q^2}{b^2 r_+^4}} \right) + \frac{8Q^2}{3r_+^2} F_1 \left( \frac{1}{4}, \frac{1}{2}, \frac{5}{4}; \frac{Q^2}{b^2 r_+^4} \right) \right]. \quad (22)$$

We find the coexistence curve for the cases $b_2 < b$ and $b_1 < b < b_2$ separately. The coexistence curves so obtained in the $P - T$ plane are inverted for $T - V$ plane for both cases. To study
Figure 2: The coexistence curve and the metastable curves for the RPT of the black hole. In the temperature region $T_z$ and $T_c$ the black hole shows only the usual LBH-SBH van der Waals phase like transition. In the temperature range $T_t$ to $T_z$ there is a LBH-IBH-LBH reentrant phase transition. This region is enlarged in the inlet. For the temperatures $T < T_t$, there is no phase transition. The pink line represents a zeroth-order phase transition between IBH-LBH and is not a coexistence curve. We have taken $Q = 1$ and $b = 0.45$.

the phase structure of the black hole, we also find the metastable curves, which are defined by

$$ (\partial V P)_T = 0 \quad \text{and} \quad (\partial V T)_P = 0. $$

(23)

III. RUPPEINER GEOMETRY AND MICROSTRUCTURE OF THE BLACK HOLE

In this section, we investigate the microstructure of four-dimensional Born-Infeld AdS black hole using Ruppeiner geometry, constructed in the parameter space with fluctuation coordinates as temperature and volume. Particularly, we are interested in the microstructure that leads to the reentrant phase transition. As we will see, the underlying microstructure for the van der Waals phase transition has similarity with the RN-AdS black hole phase transition, whereas the reentrant phase transition emerges from a different nature of the black hole microstruture.

To set up the Ruppeiner geometry, we consider two subsystems of an isolated thermodynamic system with a total entropy $S$. The smaller subsystem is assigned with an entropy $S_B$
and the larger subsystem with entropy $S_E$. Viewing the larger subsystem as a thermal bath,
we can take $S_B \ll S_E \approx S$. Therefore, if the system is described by a set of independent
variables $x^0$ and $x^1$,

$$S(x^0, x^1) = S_B(x^0, x^1) + S_E(x^0, x^1).$$  \hspace{1cm} (24)$$

When the system is in thermal equilibrium, the entropy $S$ is in its local maximum $S_0$. Taylor
expanding the entropy in the neighbourhood of the local maximum $x^\mu = x^\mu_0$, we obtain

$$S = S_0 + \frac{\partial S_B}{\partial x^\mu} \bigg|_{x^\mu_0} \Delta x^\mu_B + \frac{\partial S_E}{\partial x^\mu} \bigg|_{x^\mu_0} \Delta x^\mu_E$$

$$+ \frac{1}{2} \frac{\partial^2 S_B}{\partial x^\mu \partial x^\nu} \bigg|_{x^\mu_0} \Delta x^\mu_B \Delta x^\nu_B + \frac{1}{2} \frac{\partial^2 S_E}{\partial x^\mu \partial x^\nu} \bigg|_{x^\mu_0} \Delta x^\mu_E \Delta x^\nu_E + \ldots$$ \hspace{1cm} (25)

Since the first derivatives vanish for the equilibrium condition, we have

$$\Delta S = S - S_0 = \frac{1}{2} \frac{\partial^2 S_B}{\partial x^\mu \partial x^\nu} \bigg|_{x^\mu_0} \Delta x^\mu_B \Delta x^\nu_B + \frac{1}{2} \frac{\partial^2 S_E}{\partial x^\mu \partial x^\nu} \bigg|_{x^\mu_0} \Delta x^\mu_E \Delta x^\nu_E + \ldots$$ \hspace{1cm} (26)

$$\approx \frac{1}{2} \frac{\partial^2 S_B}{\partial x^\mu \partial x^\nu} \bigg|_{x^\mu_0} \Delta x^\mu_B \Delta x^\nu_B,$$ \hspace{1cm} (27)

where we have truncated the higher order terms and the second term. The second term is
negligible compared to first term, as the entropy $S_E$ of the thermal bath is close to that of
the whole system, and its second derivative with respect to the intensive variables $x^\mu$ are
smaller than those of $S_B$.

In Ruppeiner geometry method, the entropy $S$ is taken as the thermodynamical potential
and its fluctuation $\Delta S$ is related to to the line element $\Delta l^2$, which is the measure of the
distance between two neighbouring fluctuation states of the thermodynamic system [43].

The Ruppeiner line element is given by,

$$\Delta l^2 = \frac{1}{k_B} g_{\mu \nu}^R \Delta x^\mu \Delta x^\nu$$ \hspace{1cm} (29)$$

where $k_B$ is the Boltzmann constant and the metric $g_{\mu \nu}^R$ is,

$$g_{\mu \nu}^R = - \frac{\partial^2 S_B}{\partial x^\mu \partial x^\nu}$$ \hspace{1cm} (30)$$

Since the line element is related to the distance between the neighbouring fluctuation states,
the metric $g_{\mu \nu}^R$ must be encoded with the microscopic details of the system. Taking the
fluctuation coordinates $x^\mu$ as temperature and volume, and Helmholtz free energy of the
system as the thermodynamic potential, the line element $\Delta l^2$ can be shown to have the following form [22],

$$\Delta l^2 = \frac{C_V}{T^2} \Delta T^2 - \frac{(\partial V P)_T}{T} \Delta V^2$$

(31)

where $C_V$ is the heat capacity at constant volume,

$$C_V = T \left( \frac{\partial S}{\partial T} \right)_V .$$

(32)

The Ruppeiner curvature scalar can be obtained directly from Eq. 31 by using the conventional definitions of Riemannian geometry. The curvature scalar so calculated is encoded with the interaction details of the microstructure of the system. The positive sign $R > 0$ and negative sign $R < 0$ are associated with repulsive and attractive interaction, respectively. If the Ruppeiner curvature vanishes, it implies that there is no effective interaction between the microscopic molecules. Moreover, the speculation is that the Ruppeiner scalar diverges at the critical point. However, the Ruppeiner scalar obtained from the line element (31) has pathologies as the heat capacity $C_V$ vanishes for the black hole system. To overcome this issue, a normalised Ruppeiner scalar is defined as [22],

$$R_N = C_V R$$

(33)

where $R$ is the curvature scalar calculated from Eq. 31. The normalised curvature scalar diverges at the critical point of phase transition.

We obtained the normalised curvature scalar for the four dimensional Born-Infeld AdS black hole using the above definitions. The SPT and RPT cases are analysed separately. First we will analyse the SPT case. It is found that the behaviour of $R_N$ for SPT case is similar to that of RN-AdS black hole. The functional behaviour of $R_N$ against thermodynamic volume $V$ for a fixed pressure is studied in fig. (3).

We analyse SPT in reduced parameter space (fig. 3). For $T_r < 1$, below critical temperature, $R_N$ has two negative divergences. These two divergences gradually come closer as temperature increases, and merge together at $T_r = 1$. The divergence at critical temperature occurs at $V_r = 1$. We also observe the behaviour of $R_N$ near the origin (shown in the inlets), as $V_r \to 0$, which is the small black hole phase. The value of $R_N$ goes to positive values for small values of $V_r$. This implies that the SBH phase has a dominant repulsive interaction.
Figure 3: The behaviour of the normalised Ruppeiner curvature scalar $R_N$ with the reduced volume $V_r$ at constant temperature for SPT. The inlets show the enlarged portion near the origin. $R_N$ has positive values in the small region as $V_r \to 0$.

On the other hand the LBH phase is always characterised by the dominant attractive interaction. For temperatures above the critical value both divergences disappear. To check whether the repulsive interaction regions are thermodynamically stable, we place the regions of $R_N$ with different sign in $T_r - V_r$ plane along with the coexistence and the metastable curves (fig. 4(a)). The sign changing curve is obtained by setting $R_N = 0$. In fig. 4(a), the shaded region corresponds to the negative sign of $R_N$. As we can see, the small black hole region overlaps with this, representing SBH with repulsive interactions. The nature of interaction can be better understood from behaviour of $R_N$ along the coexistence curve 4(b). From the diagram it is clear that LBH phase always has dominant attractive interaction, whereas, the SBH phase switches to dominant repulsive interaction for low temperature values. Both branches diverge to negative infinity at the critical point. This behaviour is exactly the same one which is seen in RN-AdS black holes [22, 23] and Hayward black holes.
Figure 4: 4(a): The sign changing curve of $R_N$ (black dotted line), metastable curve (blue dashed line) and the coexistence curve (red solid line) for the SPT. The shaded region (grey) corresponds to positive $R_N$, otherwise $R_N$ is negative. 4(b): The behaviour of normalised Ruppeiner curvature scalar $R_N$ along the coexistence curve. The red (solid) line and blue (dashed) line corresponds to large black hole and small black hole, respectively. Near origin region is shown in inlet, where $R_N$ takes negative values.

We analyse underlying microstucture of the black hole that results in the RPT using the same method. The normalised Ruppeiner curvature scalar is plotted against the volume for constant temperatures in fig 5. Compared to the van der Waals case, here we observe an additional divergence for all allowed temperatures. For temperature $T < T_{c1}$ there exist three divergences. At $T = T_{c1}$ one divergence disappears. Interestingly, one divergence remains even for the temperatures $T > T_{c1}$. Contrary to the van der Waals case, $R_N$ always takes negative values implying only a dominant attractive interaction. We confirm this result by looking at the region covered by the sign changing curve (fig. 6(a)). The negative $R_N$ values does not overlap with any stable state of the system. Note that the transition line of zeroth-order phase transition (solid magenta line) also lies outside the shaded region. The behaviour along the transition curve is analysed in fig. 6(b). We consider the first-order phase transition coexistence curve and the zeroth-order transition curve. The zeroth-order transition line is not a coexistence line. The region of temperature where the phase transition takes place, $T_t < T < T_{c1}$, is only considered, as below the temperature $T_t$ there is no phase transition. Both intermediate and large black hole phases have dominant attractive interaction. That is, during a reentrant phase transition in Born-Infeld AdS black hole,
Figure 5: The behaviour of the normalised curvature scalar $R_N$ against the volume $V$ at constant temperature for the RPT. For $T < T_{c1}$ there are three divergences. There are two divergences near the origin which are shown in inlets. For temperature $T = T_{c1}$ one divergence disappears. Unlike van der Waals case, for the temperatures $T > T_{c1}$ we still have one divergence. We take $Q = 1$ and $b = 0.45$.

both the zeroth-order and first-order transition, preserve the nature of interaction between the microstructures. The existence of homogeneous dominant attractive interaction in the black hole for RPT resembles the microstructure of five dimensional neutral Gauss Bonnet AdS black holes [27]. This difference in the microstructure interaction for van der Waals and reentrant phase transition in Born-Infeld AdS black hole tells that the microstructure is determined by the coupling parameter $b$. During a RPT, the initial and final phases must be same from macroscopic point of view. It is clear from fig. 6(b) that the magnitude and sign of two LBH phases are same in the reentrant region.
Figure 6:  

(a): The sign changing curve (black dotted line) of $R_N$, metastable curve (blue dashed line) and the coexistence curve (red solid line) for the RPT. The shaded region (grey) corresponds to positive $R_N$, elsewhere $R_N$ is negative. The region near the origin is enlarged and shown in inlet. Note that, the small magenta line corresponding to the zeroth-order phase transition lies outside the shaded region.

(b): The behaviour of normalised curvature scalar $R_N$ along the transition line of first-order and zeroth-order phase transition. The red (solid) line and blue (dashed) line corresponds to large black hole and intermediate black hole, respectively. Both diverge at the critical point. The IBH branch suffers a discontinuity near the triple point $T_t$. The dot-dashed magenta line (shown in the inlet) corresponds to the LBH phase of zeroth-order phase transition, which is a miniature of the other LBH branch with a divergence at $T_z$, the corresponding termination point of transition. We take $Q = 1$ and $b = 0.45$.

IV. DISCUSSIONS

In this article, we have studied the microstructure of four-dimensional Born Infeld AdS black hole employing the Ruppeiner geometry method. We have constructed the Ruppeiner curvature scalar in the parametric space, where the temperature and volume are the fluctuation coordinates. The phase structure and the corresponding microscopic interactions for both the standard van der Waals (SPT) and the reentrant phase transitions (RPT) exhibited by the black hole were investigated. We found that the microstructure that leads to RPT is distinct from that of SPT. Our study shows that the Born Infeld coupling coefficient $b$ determines the microscopic interaction of the black hole. Since the analytic investigation is not possible due to the complexity of the spacetime we have carried out the study numerically.

The microstructure that corresponds to the SPT is analogous to that of the RN-AdS black
hole. The small black hole phase shows a dominant repulsive interaction in certain range of parameters. The large black hole phase is always characterised by the dominant attractive interaction. These are inferred from the sign of the Ruppeiner curvature scalar $R_N$. For both SBH and LBH branches it diverges near the critical point. Since there is only one critical point in the van der Waals case, we conveniently defined the reduced parameters and all analysis were carried out in terms of them. The black hole which has four different cases, depending on the value of $b$, shows the distinct RPT for certain temperature range. There are two RPT cases with two critical points in each. The RPT case has a different microstructure compared to SPT case, wherein no repulsive interaction present for both the intermediate and large black hole phases. The dominant attractive interaction does not change its nature during the zeroth-order and first-order phase transition in RPT. Both branches diverge near the physically meaningful critical point. The unchanging microstructure interaction during the RPT is analogous to the microstructure nature of five dimensional neutral Gauss Bonnet black hole. We believe that this study will help us to shed more light on black hole microstructure in general.

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