Two-Way Quantum Number Distribution Based on Entanglement and Bell-State Measurements

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(Dated: January 29, 2019)

A scheme is proposed by which two parties, Alice and Bob, can securely exchange real numbers. The scheme requires Alice and Bob to share entanglement and both to perform Bell-state measurements. With a qubit system two real numbers can each be sent by Alice and Bob, resulting in four real numbers shared by them. The number of real numbers that can be shared increases if higher-dimensional systems are utilized. The number of significant figures of each shared real number depends upon the number of Bell-state measurements that Alice and Bob perform. The security of the scheme against individual eavesdropping attacks is analyzed and the effects of channel losses and errors discussed.

Since the introduction by Bennett and Brassard in 1984 of the first complete protocol (BB84) for quantum key distribution[1], many proposals for its variations, improvements and modifications have appeared and their experimental implementations have been developed[2]. The security of the BB84 protocol and its variations rely upon the quantum-mechanical principle that information gain in an attempt to distinguish between two non-orthogonal quantum states introduces a detectable disturbance in the state of the system. The sender, Alice, sends the signal to the receiver, Bob, in states chosen randomly from two conjugate bases. The eavesdropper, Eve, cannot guess the basis right every time, and her attempt to measure the signal in the wrong basis inevitably introduces an error to the key transmission.

An interesting alternative to the BB84 protocol is the scheme based on entangled pairs first proposed by Ekert in 1991 (E91)[3]. In its original version with spin (polarization)-entangled particles, Alice and Bob perform spin measurements along one of three directions. The measurement direction is chosen randomly and independently of each other. Measurement outcomes obtained when they measure along the same direction can be used for key generation, while those obtained when they measure along different directions are used to test Bell’s inequality. The security of the E91 protocol depends upon the fact that eavesdropping reduces the degree of correlation between the two members of the entangled pair and that this reduction manifests itself as a reduction in the degree of violation of Bell’s inequality.

In this work we propose a scheme which allows two parties, Alice and Bob, to simultaneously and securely exchange real numbers. As in the E91 protocol, the scheme requires Alice and Bob to share entanglement. Instead of performing measurements along randomly chosen directions, however, Alice and Bob are required to perform Bell-state measurements. The security of the scheme relies upon the fact that eavesdropping changes the outcome of the Bell-state measurements. This change in the outcome of the Bell-state measurements originates from the reduction in the degree of correlation between the two members of the entangled pair caused by eavesdropping. In this respect, the proposed scheme may be considered as a variation of the E91 protocol. The scheme, however, involves no random choice of bases or directions. With a qubit system, Alice and Bob each can send two real numbers to each other, resulting in four real numbers shared by them. The number of significant figures of each shared real number is determined by the number of Bell-state measurements that Alice and Bob perform. The protocol may thus be considered as a scheme to allow Alice and Bob to securely share integers (or collection of digits) whose length is determined by the number of Bell-state measurements they perform. The digits they share can be used for key generation for cryptographic purposes.

Let us suppose that Alice has an EPR (Einstein-Podolsky-Rosen) source that emits a large number N≫1 of entangled pairs one by one at a regular time interval, each pair in the same Bell state. The Bell state can be any of the four Bell states

\[ |\Phi_{00}\rangle_{AB} = |\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}} (|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B) \quad (1a) \]

\[ |\Phi_{01}\rangle_{AB} = |\Phi^-\rangle_{AB} = \frac{1}{\sqrt{2}} (|0\rangle_A |0\rangle_B - |1\rangle_A |1\rangle_B) \quad (1b) \]

\[ |\Phi_{10}\rangle_{AB} = |\Psi^+\rangle_{AB} = \frac{1}{\sqrt{2}} (|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B) \quad (1c) \]

\[ |\Phi_{11}\rangle_{AB} = |\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}} (|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B) \quad (1d) \]

but for the sake of the concreteness of argument, we take it as \[|\Phi_{00}\rangle_{AB}\]. Alice keeps the qubit A of each pair and sends the qubit B to Bob.

Alice has in her possession another set of N qubits, which we denote by the subscript \(\alpha\), each of which she prepared in the state \(|\psi\rangle_{\alpha} = a|0\rangle + b|1\rangle \) (\(|a|^2 + |b|^2 = 1\)). Since Alice prepared the qubits \(\alpha\) in this state, she and only she knows what \(a\) and \(b\) are, and she keeps them to herself. Alice performs a series of N Bell-state measurements on each pair of qubits \(\alpha\) and A. On the other side, Bob has in his possession yet another set of N qubits, which we denote by the subscript \(\beta\), each of which he prepared in the state \(|\psi\rangle_{\beta} = x|0\rangle + y|1\rangle \) (\(|x|^2 + |y|^2 = 1\)). Since Bob prepared the qubits \(\beta\) in this state, he and only he knows what \(x\) and \(y\) are, and he keeps
them to himself. Bob also performs a series of N Bell-state measurements on each pair of qubits β and B. The experimental scheme is depicted schematically in Fig. 1.

In order to find the probability $P_{ijkl}$ that Alice’s Bell-state measurement yields $|\Phi_{ij}\rangle_{\alpha A}$ and Bob’s Bell-state measurement yields $|\Phi_{kl}\rangle_{\beta B}$, we expand the total wave function $|\psi\rangle_{\alpha B} = |\psi\rangle_{\alpha A}|\psi\rangle_{\Phi_{00}}_{AB}$ in terms of $|\Phi_{ij}\rangle_{\alpha A}|\Phi_{kl}\rangle_{\beta B}$ as

$$|\psi\rangle_{\alpha B} = \sum_{i,j,k,l=0}^{1} |\Phi_{ij}\rangle_{\alpha A}|\Phi_{kl}\rangle_{\beta B} V_{ijkl}. \quad (2)$$

A straightforward algebra yields

$$V_{0000} = V_{0101} = V_{1010} = V_{1111} = \frac{1}{2\sqrt{2}} (xa + yb) \quad (3a)$$

$$V_{0010} = V_{1001} = V_{1101} = \frac{1}{2\sqrt{2}} (xa - yb) \quad (3b)$$

$$V_{0110} = V_{1011} = V_{1100} = \frac{1}{2\sqrt{2}} (xb - ya) \quad (3c)$$

$$V_{0100} = V_{1010} = V_{1001} = -V_{1110} = \frac{1}{2\sqrt{2}} (xb + ya) \quad (3d)$$

The probabilities $P_{ijkl}$’s are given by $P_{ijkl} = |V_{ijkl}|^2$. Alice and Bob can determine these probabilities experimentally from the result of their Bell-state measurements. They only need to count the number $N_{ijkl}$ of occurrences for the joint outcome $|\Phi_{ij}\rangle_{\alpha A}|\Phi_{kl}\rangle_{\beta B}$. The experimentally determined probabilities are then given by

$$P_{ijkl}^{exp} = \frac{N_{ijkl}}{N}. \quad (4)$$

Suppose now, however, that Alice and Bob each announce publicly her or his measurement result only when the outcome is $\Phi_{10}$ or $\Phi_{11}$. This is consistent with the realistic situation, because only these two Bell states can be unambiguously distinguished with linear optical means [4]. The probabilities that can be determined experimentally are then only $P_{1010}^{exp}$, $P_{1011}^{exp}$, $P_{1110}^{exp}$, and $P_{1111}^{exp}$. These probabilities are given theoretically as

$$P_{1010} = P_{1111} = \frac{1}{8} |xa + yb|^2 = \frac{1}{8} [\cos^2 a \cos^2 b + \sin^2 a \sin^2 b + 2 \cos \theta_a \cos \theta_b \sin \theta_a \sin \theta_b \cos (\phi_a + \phi_b)] \quad (5a)$$

$$P_{1011} = P_{1110} = \frac{1}{8} |xa - yb|^2 = \frac{1}{8} [\cos^2 a \cos^2 b + \sin^2 a \sin^2 b - 2 \cos \theta_a \cos \theta_b \sin \theta_a \sin \theta_b \cos (\phi_a + \phi_b)] \quad (5b)$$

where we set

$$a = \cos \theta_a, \quad b = \sin \theta_a e^{i\phi_a}$$

$$x = \cos \theta_b, \quad y = \sin \theta_b e^{i\phi_b} \quad (6)$$

When the experimentally determined probabilities are substituted for the corresponding theoretical probabilities, Eqs. (5) constitute two equations that relate the four constants $\theta_a, \phi_a, \theta_b$, and $\phi_b$. Since Alice knows $\theta_a$ and $\phi_a$, she can use the two equations to solve for $\theta_b$ and $\phi_b$. Similarly, Bob knows $\theta_b$ and $\phi_b$, and therefore he can use the two equations to solve for $\theta_a$ and $\phi_a$. A third person, an eavesdropper, however, knows none of the four constants, and there is no way for her to determine the four unknown constants from the two equations. Thus, the method described above, with Alice and Bob announcing either their measurement result only when the outcome is $\Phi_{10}$ or $\Phi_{11}$, provides a means for Alice and Bob to securely share four real numbers. Without loss of generality we take them as $\cos \theta_a, \cos \phi_a, \cos \theta_b$, and $\cos \phi_b$, four real numbers less than 1.

The number of real numbers that can be shared increases if Alice and Bob use higher-dimensional systems. The generalized Bell states for a $d$-dimensional system qudit) can be defined as

$$|\Phi_{ij}\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_{q=0}^{d-1} \omega^{|q|}|q+i\rangle|q+j\rangle \quad (7)$$

where $\omega = e^{i \frac{2\pi}{d}}$. As before, we assume that each of the entangled pairs AB produced by the source is in $|\Phi_{00}\rangle_{AB}$. Alice performs a series of Bell-state measurements on each pair of the qudit A and another qudit $\alpha$ she prepared in the state $\sum_{i=0}^{d-1} a_i |i\rangle_{\alpha}$, while Bob performs a series

![FIG. 1: Experimental Scheme. The EPR(Einstein-Podolsky-Rosen) source emits entangled pairs in state $|\Phi_{00}\rangle_{AB}$. Alice performs Bell-state measurements on the qubit pairs $\alpha$ and $A$, and Bob on the qubit pairs $\beta$ and $B$. BSM stands for Bell-state measurement.](image-url)
of Bell-state measurements on each pair of the qudit B and another qudit $\beta$ he prepared in the state $\sum_{i=0}^{d-1} x_i |i\rangle _\beta$.

As in the qubit case, the total wave function $|\psi\rangle _{\alpha\beta AB}$ can be expanded in terms of the Bell states $|\Phi_{ijkl}\rangle _{\alpha A}$, and the probability amplitudes $V_{ijkl}$'s can be expressed in terms of $a_i$'s and $x_i$'s. We obtain

$$V_{ijkl} = \frac{1}{d^4} \omega^{i+j+k} \sum_{m=0}^{d-1} \omega^{-(j+i)m} a_m x_m$$

where all indices are evaluated modulo $d$. The probabilities $P_{ijkl}$'s are then determined by $P_{ijkl} = |V_{ijkl}|^2$.

The constants $a_i$'s and $x_i$'s constitute (4d-4) unknowns determined from experimentally determined probabilities $P_{ijkl}$'s. To Alice and Bob, however, there are only (2d-2) unknowns. By agreeing to publicly announce the measurement result only when the measurement outcome is among judiciously chosen Bell states, Alice and Bob can limit the number of probabilities that can be determined experimentally in such a way that the number of equations that relate the experimentally determined probabilities with the parameters $a_i$'s and $x_i$'s is greater than or equal to (2d-2) but less than (4d-4). This way, (4d-4) real numbers can be secretly shared between Alice and Bob.

We now discuss the security of the scheme described above against eavesdropping attacks. Perhaps the simplest attack that Eve can attempt is the intercept-resend attack depicted in Figure 2. In this attack Eve intercepts each qubit B being transmitted from Alice to Bob and keeps it, while she generates her own entangled pairs EF each qubit B being transmitted from Alice to Bob and attack depicted in Figure 2. In this attack, Eve intercepts and Alice and Bob would have performed their Bell-state measurements on the pairs $x_i$ and $x_j$, respectively. Note, however, that the qudits A and F are not entangled, and thus Alice's Bell-state measurement is completely independent of Bob's Bell-state measurement. As a result, all the probabilities $P_{ijkl}$'s should be the same, i.e.,

$$P_{ijkl} = \frac{1}{16}, \quad i, j, k, l = 0 \text{ or } 1$$

Alice and Bob can check if $P_{1010}$ or $P_{1111}$ is the same as (or close to) $P_{0110}$ or $P_{1010}$. If they feel that the two probabilities are too close to trust, they discard the data and restart from the beginning. It is possible that the two probabilities are the same (or close) not because of Eve's attack because Alice and Bob happen to choose $\phi_a$ and $\phi_b$ such that $\cos(\phi_a + \phi_b) \approx 0$. This case will also have to be discarded. If Eve attacks not all but only a part of the qudits B, the two probabilities may not be sufficiently close to be detected. In this case Alice and Bob must resort to digit comparison to detect Eve's attack. Due to the attack, the real numbers $\cos \theta_a$ and $\cos \phi_a$ (cos $\theta_b$ and $\cos \phi_b$) computed by Bob(Alice) from the experimentally determined probabilities will deviate from the correct values that Alice (Bob) initially assigned. By comparing a few digits (e.g., a digit at the third decimal point of $\cos \theta_a$) and checking if they agree, Alice and Bob can check against Eve's attacks.

Another possible mode of attack is the "entangle-measure" attack depicted in Fig. 3. In this attack, Eve prepares a set of ancilla qubits E each in state $|0\rangle _E$, entangle each of them with the qubit B by performing a CNOT operation with the qubit B as the control bit and the qubit E as the target bit, and performs a Bell-state measurement upon each pair of the qubit E and another qubit $\eta$ a set of which she prepares in state, say, $|u\rangle _\eta + |v\rangle _\eta$. In this mode of attack, Eve's role is indistinguishable from Bob's role, and she can obtain as much information as Bob can. In order to find the effect of the entangle-measure attack upon the probabilities $P_{ijkl}$'s, we expand the six-qubit wave function $|\psi\rangle _{\alpha\beta\eta ABE} = (a|0\rangle_\alpha + b|1\rangle_\alpha)(x|0\rangle_\beta + y|1\rangle_\beta)(u|0\rangle_\eta + v|1\rangle_\eta)\sqrt{2}([0\rangle_A |0\rangle_B |0\rangle_E + |1\rangle_A |1\rangle_B |1\rangle_E]$ in terms of the product of the Bell states as

$$|\psi\rangle _{\alpha\beta\eta ABE} = \sum_{i,j,k,l,m,n=0} |\Phi_{ijkl}\rangle _{\alpha A} |\Phi_{kl}\rangle _{\beta B} |\Phi_{mn}\rangle _{\eta E} V_{ijklmn}$$

and calculate the probabilities according to $P_{ijkl} = |V_{ijklmn}|^2$. 

\[\text{FIG. 2: Eve's intercept-resend attack}\]

\[\text{FIG. 3: Eve's entangle-measure attack}\]
\[ \sum_{m,n=0}^{1} |V_{ijklmn}|^2. \] A straightforward algebra yields

\[ P_{0000} = P_{0101} = P_{1010} = P_{1111} = \]
\[ P_{0001} = P_{0100} = P_{1011} = P_{1110} = \frac{1}{8} \left( |xa|^2 + |yb|^2 \right) \] (10a)
\[ P_{0010} = P_{0111} = P_{1000} = P_{1101} = \frac{1}{8} \left( |xb|^2 + |ya|^2 \right) \] (10b)

In particular, the probabilities \( P_{0101}, P_{1011}, P_{1101}, \) and \( P_{1110} \) are all the same in this case and given by

\[ P_{1010} = P_{1111} = P_{1011} = P_{1101} = \frac{1}{8} \left( \cos^2 \theta_a \cos^2 \theta_b + \sin^2 \theta_a \sin^2 \theta_b \right) \] (11)

The entangle-measure attack can thus be detected using the same method employed to detect the intercept-resend attack. If Eve entangles every qubit B with her ancilla qubit E, it can be detected by checking if \( P_{0101} \) or \( P_{1111} \) is the same as (or close to) \( P_{0111} \) or \( P_{1101} \). In general, however, Alice and Bob should perform digit comparison to detect the attack, because Eve can attack only a part of the qubits B.

Let us turn our attention to practical issues concerning the proposed scheme. Suppose Alice and Bob want to securely share 4 real numbers less than 1 (\( \cos \theta_a, \cos \phi_a, \cos \theta_b, \cos \phi_b \)) each accurate to D decimal points, or equivalently 4 integers each of length D, or equivalently 4D digits. How many times do Alice and Bob each need to perform Bell-state measurements? When a sufficiently large number \( N \gg 1 \) of Bell-state measurements are made, the number \( N_{ijkl}^{\text{exp}} \) of times the joint outcome \( |\Phi_{ijkl}\rangle_{A,B} \) is counted lies within the range defined as

\[ NP_{ijkl} - \sqrt{2NP_{ijkl}(1-P_{ijkl})} \leq N_{ijkl}^{\text{exp}} \leq NP_{ijkl} + \sqrt{2NP_{ijkl}(1-P_{ijkl})} \] (12)

where \( P_{ijkl} \) is the exact theoretical probability given, for example, for a qubit system by the absolute square of \( V_{ijkl} \) given by Eqs. (5). Thus, the experimentally determined probabilities \( P_{ijkl}^{\text{exp}} = N_{ijkl}^{\text{exp}}/N \) are accurate to \( \sim \sqrt{2NP_{ijkl}(1-P_{ijkl})}/N \). Taking \( N = 10^n \), \( P_{ijkl}^{\text{exp}} \)’s are accurate to \( \frac{\sqrt{2}}{2} \) decimal points. The real values \( \cos \theta_a, \cos \phi_a, \cos \theta_b, \) and \( \cos \phi_b \) that are determined from these experimentally determined probabilities should also be accurate to \( \frac{\sqrt{2}}{2} \) decimal points. We conclude therefore that, for Alice and Bob to securely share 4D digits, they should perform \( \sim 10^{2D} \) Bell-state measurements each. The proposed scheme has therefore a rather low efficiency of \( \sim 4D10^{-2D} \).

The efficiency of the scheme can be enhanced by noting that the efficiency decreases exponentially with D. Instead of trying to obtain 4D digits in a single experiment consisting of \( \sim 10^{2D} \) measurements, Alice and Bob can opt to divide it into many independent experiments each with different values of parameters \( a, b, x \) and \( y \). For example, consider the situation where Alice and Bob want to share 400 digits. They can achieve it by performing \( \sim 10^{200} \) Bell-state measurements in a single experiment and obtaining 4 real numbers accurate to 100 decimal points, i.e., 400 digits. Alternatively, they can choose to shoot for only four digits in a single experiment by performing \( \sim 10^2 \) Bell-state measurements and obtaining 4 real numbers accurate only to one decimal point. They can then repeat the experiment 100 times, each time with different values of \( a, b, x \) and \( y \) to obtain 400 real numbers each accurate to one decimal point, i.e., 400 digits. Using this “divide-repeat” strategy, the number of Bell-state measurements performed by Alice and Bob is reduced to \( \sim 10^4 \) and the efficiency is enhanced to \( \sim 400/10^4 = 4 \times 10^{-2} \). Even if some (two or three) of these four digits obtained from each single experiment need to be used for checking against eavesdropping attacks, the efficiency still remains to be \( \sim 10^{-2} \). If Alice and Bob feel that they need more digits than two or three from each single experiment to be used for the security check, they can make \( \sim 10^4 \) Bell-state measurements and obtain eight digits in a single experiment. They can then repeat the experiment 50 times to obtain 400 digits altogether. The efficiency in this case is \( \sim 1.6 \times 10^{-5} \). Another way of increasing the efficiency is to use high-dimensional systems. Since the number of real numbers that can be shared increases with increased dimension, the efficiency also increases by going to high-dimensional systems.

Up to now we have assumed an ideal situation where there are no losses and no errors. In general, however, losses and errors are unavoidable and their effects must be taken into account. Due to losses, only \( N \eta \) qubits out of \( N \) qubits sent from Alice will be detected by Bob, where \( \eta \) is the probability that a single photon sent from Alice is detected at Bob’s detectors. If one considers only the channel losses, it is given by \( \eta = 10^{-\alpha l c}/10 \), where \( \alpha \) is the absorption coefficient, \( l \) is the length of the channel(fiber) and \( c \) accounts for a distance-independent loss in the channel. The probabilities \( P_{ijkl}^{\text{exp}} \)’s should then be determined by comparing the number \( N_{ijkl}^{\text{exp}} \)’s not to \( N \) but to \( N \eta \). A more accurate determination of the probabilities can be obtained if one lets Bob announce his measurement result every time he receives a qubit B. He
should announce whether the outcome of his Bell-state measurement is $\Phi_{10}$ or $\Phi_{11}$ or inconclusive (corresponding to the case where the outcome is either $\Phi_{00}$ or $\Phi_{01}$ but he cannot distinguish between the two). The number $N_{ijkl}^{\text{exp}}$ can then simply be normalized to the number of times Bob has made his announcement. Errors can occur during generation, transmission and detection of qubits and can seriously limit the performance of our proposed scheme. Under ideal errorless conditions, the number of digits that Alice and Bob share can be increased simply by increasing the number of qubits they prepare and the number of measurements they perform. When errors are present, however, the error rate limits the number of meaningful digits that Alice and Bob share through a single experiment, and it may be meaningless to increase the number of measurements to be made in a single experiment beyond a certain level. For example, suppose the error rate is 5%. The accuracy of the probabilities $P_{ijkl}^{\text{exp}}$ is determined from the experiment cannot be better than 5%, which means that even the digit at the second decimal point of the real numbers $\cos \theta_a, \cos \phi_a, \cos \theta_b$ and $\cos \phi_b$ determined from these probabilities is not guaranteed to be accurate. It is then best for Alice and Bob to shoot for four digits, one digit for each real number, in a single experiment. The number of measurements that can guarantee the accuracy of the digit at the first decimal point is $\sim 10^2$ and it is in this case meaningless to increase the number of measurements well beyond $\sim 10^2$ in one experiment. When more digits are desired to be shared, Alice and Bob need to repeat the process of $\sim 10^2$ measurements with different sets of parameters $a, b, x$ and $y$. Thus, the “divide-repeat” strategy is not only desirable to enhance the efficiency but also required to make the scheme work in the presence of errors.

We are now in a position to propose a protocol for quantum number distribution which allows two parties, Alice and Bob, to share securely a certain number of digits. Alice and Bob should proceed as follows.

(1) Alice prepares $N$ entangled pairs AB, each in state $|\Phi_{00}\rangle_{AB}$, keeps the qubit A and sends the qubit B to Bob. Alice has another set of $N\eta$ qubits $\alpha$ and divides them into $N\eta/100$ groups with each group consisting of 100 qubits. (We assume for simplicity that $N\eta$ is an integral multiple of 100.) She prepares the qubits in the jth group (j=1, 2, ..., $N\eta/100$) in state $\left( a_j|0\rangle_\alpha + b_j|1\rangle_\alpha \right)$. Bob has a set of $N\eta$ qubits $\beta$ and divides them into $N\eta/100$ groups with each group consisting of 100 qubits. He prepares the qubits in the jth group in state $\left( x_j|0\rangle_\beta + y_j|1\rangle_\beta \right)$.

(2) Bob takes the first 100 qubits B he receives and the 100 qubits in the first group of the qubits $\beta$. He performs a Bell-state measurement on each qubit pair $\beta B$ and announces publicly where the outcome of each measurement is $\Phi_{10}$ or $\Phi_{11}$ or inconclusive. Alice takes the 100 qubits A, entangled partners of the first 100 qubits B that Bob received, and the 100 qubits in the first group of qubits $\alpha$. She performs a Bell-state measurement on each qubit pair $\alpha A$ and announces publicly whether the outcome of each measurement is $\Phi_{10}$ or $\Phi_{11}$.

(3) Alice and Bob count the numbers $N_{ijkl}^{\text{exp}}$, $N_{1010}^{\text{exp}}$, $N_{1011}^{\text{exp}}$, and $N_{1111}^{\text{exp}}$ of joint occurrences of $|\Phi_{10}\rangle_{\alpha A}|\Phi_{10}\rangle_{\beta B}$, $|\Phi_{11}\rangle_{\alpha A}|\Phi_{11}\rangle_{\beta B}$, $|\Phi_{10}\rangle_{\alpha A}|\Phi_{11}\rangle_{\beta B}$, and $|\Phi_{11}\rangle_{\alpha A}|\Phi_{10}\rangle_{\beta B}$, and determine the corresponding probabilities $P_{ijkl}^{\text{exp}} = N_{ijkl}^{\text{exp}}/100$. From the probabilities, they determine $\cos \theta_a, \cos \phi_a, \cos \theta_b$ and $\cos \phi_b$, each to the first decimal point. They now share 4 digits.

(4) As a check for the accuracy of the experiment, Alice and Bob check if $P_{1010}^{\text{exp}}$ and $P_{1111}^{\text{exp}}$ agree at least to the first decimal point. If not, they discard the data and restart. They do the same checking for $P_{1011}^{\text{exp}}$ and $P_{1110}^{\text{exp}}$. As a check against eavesdropping attacks, they check if $P_{1010}^{\text{exp}}$ (or $P_{1111}^{\text{exp}}$) is sufficiently different from $P_{1011}^{\text{exp}}$ (or $P_{1110}^{\text{exp}}$). If not, they discard the data and restart. As a further check against eavesdropping attacks, Alice and Bob each take two of the four digits they share (they could take one or three digits depending upon the level of confidence) and publicly compare and check if each of the two pairs agree. If the agreement is found, then they each keep the remaining two digits as the key. If not, they discard the data and restart.

(5) The steps (2)-(4) are repeated $\frac{N\eta}{100}$ times, each time with a different set of 100 qubits each of $A, B, \alpha$ and $\beta$. When all measurements are completed successfully, Alice and Bob have collected between them $2 \times \frac{N\eta}{100}$ real numbers each accurate to the first decimal point, i.e., $2 \times \frac{N\eta}{100}$ digits. The $2 \times \frac{N\eta}{100}$ digits constitute the final key.

We note that if the error rate is below 1%, Alice and Bob can shoot for 8 digits instead of 4 digits in a single experiment, by dividing the qubits into groups of $10^4$ qubits instead of 100 qubits, and performing $\sim 10^4$ Bell-state measurements instead of 100 measurements in a single experiment. Each single experiment will then produce 4 real numbers accurate to the second decimal point, i.e., eight digits. This way Alice and Bob have more qubits available for digit comparison, but the efficiency will be lower.

In conclusion we have proposed a protocol based on entanglement and Bell-state measurements that allows two parties to exchange real numbers securely. As compared with the standard quantum cryptographic protocols such as BB84, our proposed protocol suffers from the low efficiency. With the help of the “divide-repeat” strategy, however, its efficiency can be increased to $\sim 10^{-2}$. As the security of the proposed protocol relies upon the fact that an act of eavesdropping changes the outcome of the
Bell-state measurements, the protocol requires the process of digit comparison to protect against eavesdropping attacks. The protocol, however, does not require random choice between two conjugate bases as in BB84 nor the Bell’s inequality test as in E91. The proposed protocol appears to protect itself well against eavesdropping attacks. It is secure, in particular, against an individual attack in which Eve attacks every qubit transmitted from Alice to Bob, because such an all-out attack leaves its mark on the probabilities. If Eve attacks only a part of the qubits, then Alice and Bob have to perform information reconciliation, which consists of checking if some randomly selected digits they share agree. Ironically, the low efficiency of the protocol works to help this digit comparison process effective. Because of the low efficiency, the information on whether there were eavesdropping attacks is contained in a relatively small number of digits produced by the protocol. The agreement between just a small number of pairs of digits can thus be considered as a strong indication for the absence of eavesdropping attacks.

We note that our proposed protocol provides a way of two-way communication, allowing simultaneous mutual exchange of information between Alice and Bob. Alice and Bob are simultaneously both the sender and the receiver of information, while in standard cryptographic protocols information usually flows one way.

On the practical side, a successful operation of the proposed protocol requires generation, distribution, and detection of entanglement at a single-photon level, a difficult but not an impossible task. It requires, in particular, a large number of Bell-state measurements to be performed. We emphasize, however, that only two of the four Bell states are required to be distinguished. The distinction of the two Bell states is possible using only linear optical means and therefore can be accomplished without too much difficulty with the present technology.

Channel errors must be minimized for a successful operation of the protocol. If our protocol is to work at all, the error rate must be kept below $\sim 10\%$, because the error rate of over $\sim 10\%$ will not guarantee the accuracy of the digit even at the first decimal point.

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