SUPPLEMENTARY INFORMATION

Enhanced diffusion by reversible binding to active polymers

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Mathematical model - Derivations

The detailed derivations of analytical equations in the theoretical model are presented here. Firstly, based on the equations introduced in Eqs. 1, 2 and 3 of the manuscript, we now show the detailed steps behind the derivation of (a) the mean position of the plate \( \langle x \rangle \), and (b) the
mean squared position \( \langle x^2 \rangle \) and the corresponding diffusivity \( D_c \) from reversible binding of chains as a function of the number of attached tethers \( N_a \). Secondly, we derive the combined diffusivity \( D(N_a) \) presented as Eq. 5 in the manuscript using the fluctuation-dissipation theorem. Lastly, we present the limiting case when the active temperature \( T_c \) of the chains are much greater than the ambient temperature \( T \) of the medium.

**Mean plate position**

The mean of the plate position \( x(t) \) over time is defined by:

\[
\langle x \rangle = \int_{-\infty}^{\infty} xp(x, t) dx \quad (S1)
\]

Using equation 1 of the main manuscript that gives the time derivative \( dp(x, t)/dt \) we have:

\[
\frac{d\langle x \rangle}{dt} = \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} w(\delta) \left[ p(x - \delta, t) - p(x, t) \right] d\delta \, dx
= \int_{-\infty}^{\infty} w(\delta) \int_{-\infty}^{\infty} xp(x - \delta, t) d\delta dx - \langle x \rangle \int_{-\infty}^{\infty} w(\delta) d\delta \quad (S2)
\]

Let us consider the first term on the right hand side of the above equation. The integration over \( x \) is performed first followed by the integration in \( \delta \). Since the frequency of movement to the left and right is the same, we have \( w(\delta > 0) = w(\delta < 0) = w(\delta) \). The first term can therefore be written in as

\[
\int_{-\infty}^{\infty} w(\delta) \int_{-\infty}^{\infty} xp(x - \delta, t) d\delta dx = \int_{0}^{\infty} w(\delta)(S_1(\delta) + S_1(-\delta)) d\delta \quad (S3)
\]

where

\[
S_1(\delta) = \int_{-\infty}^{\infty} xp(x - \delta, t) dx \quad (S4)
\]
Now, the term $S_1(\delta) + S_1(-\delta)$ can be further simplified by collecting terms in the integral of $x$ which have the same probability $p$. This is done by first identifying that the variable $x$ in above integral in Eq. S4 is the integration variable and can be substituted by any variable ranging from $\infty$ to $\infty$ to yield the same result. Choosing $x' = x - 2\delta$ we have for a given $\delta$

$$S_1(-\delta) = \int_{-\infty}^{\infty} x' p(x' + \delta, t) dx'$$

$$= \int_{-\infty}^{\infty} (x - 2\delta)p(x - 2\delta + \delta, t) dx$$

$$= \int_{-\infty}^{\infty} (x - 2\delta)p(x - \delta, t) dx$$  \hspace{1cm} (S5)$$

We now combine the terms in Eq. S3 to give

$$\frac{d\langle x \rangle}{dt} = \int_{0}^{\infty} w(\delta) \int_{-\infty}^{\infty} (x + x - 2\delta)p(x - \delta, t) dx \ d\delta - \int_{-\infty}^{\infty} w(\delta) \langle x \rangle d\delta$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} w(\delta) \int_{-\infty}^{\infty} 2(x - \delta)p(x - \delta, t) d\delta - \int_{-\infty}^{\infty} w(\delta) \langle x \rangle d\delta$$

$$= \int_{-\infty}^{\infty} w(\delta) (\langle x \rangle - \langle x \rangle) d\delta$$

$$= 0$$  \hspace{1cm} (S6)$$

where the property $\int_{-\infty}^{\infty} w(\delta) (\cdot) d\delta = 2 \int_{0}^{\infty} w(\delta) (\cdot) d\delta$ is used in line 2 above since $w(\delta)$ is an even function. Thus we have shown that the mean position of the plate is unchanging in time since there is no bias in movement to the right or left due to the kicks from polymer chains.
Mean square displacement

The mean square displacement of the plate is:

\[ \langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 p(x,t) dx \]  \hspace{1cm} (S7)

Following a similar procedure as the mean, using equation 1 of the main manuscript, the time derivative of the MSD is given in terms of the time derivative \( dp(x,t)/dt \) as :

\[
\frac{d\langle x^2 \rangle}{dt} = \int_{-\infty}^{\infty} w(\delta) \left( \int_{-\infty}^{\infty} x^2 (p(x-\delta,t) - p(x,t)) dx \right) d\delta
\]

\[
= \int_{-\infty}^{\infty} w(\delta) \int_{-\infty}^{\infty} x^2 p(x-\delta,t) dx d\delta - \langle x^2 \rangle \int_{-\infty}^{\infty} w(\delta) d\delta \]  \hspace{1cm} (S8)

We follow a similar procedure as the previous section to write the first term of the above equation as \( \int_{-\infty}^{\infty} w(\delta)(S_2(\delta) + S_2(-\delta))d\delta \) where

\[ S_2(\delta) = \int_{-\infty}^{\infty} x^2 p(x-\delta,t) \]  \hspace{1cm} (S9)
Combining the appropriate terms in $S_2(\delta)$ and $S_2(-\delta)$, we have

\[
\frac{d\langle x^2 \rangle}{dt} = \int_{-\infty}^{\infty} w(\delta) \int_{-\infty}^{\infty} (x^2 + (x - 2\delta)^2) p(x - \delta, t) dx \ d\delta - \langle x^2 \rangle \int_{-\infty}^{\infty} w(\delta) d\delta
\]

\[
= \int_{-\infty}^{\infty} w(\delta) \int_{-\infty}^{\infty} (2x^2 + 4\delta^2 - 4x\delta) p(x - \delta, t) dx \ d\delta - \langle x^2 \rangle \int_{-\infty}^{\infty} w(\delta) d\delta
\]

\[
= \int_{-\infty}^{\infty} \frac{1}{2} w(\delta) \int_{-\infty}^{\infty} 2(x^2 + 2\delta^2 - 2x\delta) p(x - \delta, t) dx \ d\delta - \langle x^2 \rangle \int_{-\infty}^{\infty} w(\delta) d\delta
\]

\[
= \int_{-\infty}^{\infty} w(\delta) \int_{-\infty}^{\infty} (\delta^2 + (x - \delta)^2) p(x - \delta, t) dx \ d\delta - \langle x^2 \rangle \int_{-\infty}^{\infty} w(\delta) d\delta
\]

\[
= \int_{-\infty}^{\infty} w(\delta) \int_{-\infty}^{\infty} \delta^2 p(x - \delta, t) dx \ d\delta + \int_{-\infty}^{\infty} w(\delta) \int_{-\infty}^{\infty} (x - \delta)^2 p(x - \delta, t) dx \ d\delta - \langle x^2 \rangle \int_{-\infty}^{\infty} w(\delta) d\delta
\]

\[
= \int_{-\infty}^{\infty} w(\delta) \int_{-\infty}^{\infty} (\delta^2 p(x - \delta, t) + \langle x^2 \rangle - \langle x^2 \rangle) dx \ d\delta
\]

\[
= \int_{-\infty}^{\infty} w(\delta) \delta^2 \int_{-\infty}^{\infty} p(x - \delta, t) dx \ d\delta
\]

where the property $\int_{-\infty}^{\infty} w(\delta) (\cdot) d\delta = 2 \int_{0}^{\infty} w(\delta) (\cdot) d\delta$ is used again in line 3.

As the summation of the probability function, $p(x, t)$ over all possible values of $x$ adds to 1, i.e. $\int_{-\infty}^{\infty} p(x) dx = 1$, we obtain:

\[
\frac{d\langle x^2 \rangle}{dt} = \int_{-\infty}^{\infty} \delta^2 w(\delta) d\delta = k_a (N - N_a) \int_{-\infty}^{\infty} \delta^2 P_a(\delta, N_a) d\delta + k_d N_a \int_{-\infty}^{\infty} \delta^2 P_d(\delta, N_a) d\delta \quad (S10)
\]

where we used equation 3 of the main manuscript to obtain the terms on the right end. Now
the diffusion constant becomes $D = 1/2 \ d\langle x^2 \rangle/dt$ and using Eq. 4 of the main manuscript
we have the variances of $P_a(\delta, N_a)$ and $P_d(\delta, N_a)$ to give

$$D_c(N_a) = \frac{1}{2} \sigma^2 \left[ \frac{k_a(N - N_a)}{(N_a + 1)^2} + \frac{k_d}{N_a} \right]$$  

(S11)

which is the derivation of Eq. 4 of the manuscript.

We note that the above expression for diffusivity may also be equivalently derived using a Laplace and Fourier transforms of the space and time scales in the Fokker-Planck equation.

**Effective Diffusivity**

The Langevin equation for the plate with two types of drag forces and corresponding force fluctuations is given by

$$(\zeta_m + \zeta_c)v_p(t) = F_m(t) + F_c(t)$$  

(S12)

The auto-correlation function of force fluctuations from the chains $F_c$ and the medium $F_m$ are given by the fluctuation-dissipation theorem as

$$\langle F_c(t)F_c(t') \rangle = 2\zeta_c k_B T_c \delta(t' - t)$$

$$\langle F_m(t)F_m(t') \rangle = 2\zeta_m k_B T \delta(t' - t)$$  

(S13)

where the delta function is an approximation for time differences $|t' - t| > \tau_R$, the average lifetime of the bonds. Assuming the the cross-correlation of the two forces vanishes, $\langle F_c(t')F_m(t) \rangle = 0$, we multiply the Langevin equation at times $t'$ and $t$ to get

$$(\zeta_m + \zeta_c)^2 \langle v_p(t')v_p(t) \rangle = \langle F_m(t')F_m(t) \rangle + \langle F_c(t')F_c(t) \rangle$$

$$\langle v_p(t')v_p(t) \rangle = \frac{1}{(\zeta_m + \zeta_c)^2} \left( 2\zeta_m k_B T \delta(t' - t) + 2\zeta_c k_B T_c \delta(t' - t) \right)$$

$$\langle v_p(t')v_p(t) \rangle = 2k_B \frac{\zeta_m T + \zeta_c T_c}{(\zeta_m + \zeta_c)^2} \delta(t' - t).$$  

(S14)
Using the above velocity correlation function it is straightforward to obtain the diffusivity by integration, to obtain Eq. 5 in the manuscript as

\[
D(N_a) = \int_0^\infty dt \langle v_p(t)v_p(0) \rangle = k_B \frac{\zeta_m T + \zeta_c T_c}{(\zeta_m + \zeta_c)^2}.
\] (S15)

**Extreme limit of** \(T_c \gg T\)

In the limit of extremely active chains, we have a condition where effective temperature of the chains is much higher than the ambient temperature of the surrounding medium such that \(T/T_c \ll 1\). Consequently Eq. S15 can be re-written as

\[
D(N_a) = \frac{k_B T_c}{\zeta_c} \frac{\zeta_m (T/T_c)}{(\zeta_m + \zeta_c)^2} + \frac{\zeta_c^2}{(\zeta_m + \zeta_c)^2} \left( \frac{\zeta_m}{\zeta_c} \frac{T}{T_c} + 1 \right)
\]

\[
\approx \frac{k_B T_c}{\zeta_c} \frac{1}{(\zeta_m/\zeta_c + 1)^2}
\]

\[
= \frac{\sigma^2}{2(k_a + k_d)} \left( \frac{p(N - N_a)}{(N_a + 1)^2} + \frac{1 - p}{N} \right) \frac{1}{(\zeta_m/\zeta_c + 1)^2}
\] (S16)

The effective diffusion therefore is a function of the ratio of the friction coefficients \(\zeta_m/\zeta_c\).

In the case of the macroscopic realization of this system in the sequel of this paper, we have \(\zeta_m \ll \zeta_c\) as the surrounding medium is air and provide little resistance to motion compared to the friction from binding and unbinding.

**Ratio of timescales** \(\tau_R/\tau_b\)

The well known result of Rouse time \(\tau_R\) of the fundamental mode of chain fluctuations is given by

\[
\tau_R = \frac{\zeta_b N_k^2 b^2}{3\pi^2 k_B T_a} = \frac{N_k \zeta_b}{\pi^2} \left( \frac{N_k b^2}{3k_B T_a} \right) = \frac{1}{\pi^2} \frac{N_k \zeta_b}{K}
\] (S17)
where $\zeta_b$ is the friction coefficient of one bead in the polymer chain of size $b$, $l_c = N_k b$ is the contour length of the chain, $T_a$ is the active temperature of the chain, and $K = 3k_B T / N_k b^2$ is the effective Gaussian spring constant of the chain.

The bound time of a Brownian particle with friction coefficient $\zeta_p$ bound to a harmonic oscillator with spring constant $K$ is given by

$$\tau_b = \frac{\zeta_p}{K} \quad \text{(S18)}$$

Taking a spherical geometry each bead of the polymer and a rod configuration for the plate, the corresponding hydrodynamic friction coefficient in a medium of viscosity $\eta$ are given by

$$\begin{align*}
\zeta_b &= 6\pi \eta b \\
\zeta_p &= \frac{4}{3}\pi \eta L_p \\
\frac{\zeta_b}{\zeta_p} &= \frac{9}{2} \frac{b}{L_p} \quad \text{(S19)}
\end{align*}$$

Using Eqs. and , we finally get

$$\frac{\tau_R}{\tau_b} = \frac{9}{2\pi^2} \frac{N_k b}{L_p} = \frac{9}{2\pi^2} \left( \frac{l_c}{L_p} \right) \quad \text{(S20)}$$
Figures

Comparison of predictions from Monte-Carlo simulations and model

Figure S1: Plots of effective diffusivity normalized to the free diffusivity $D_f$ shown as a function of the number of attachable chains $N$ for different values of $\tau_R/\tau_b$ in (a), (b) and (c) at active temperature $T_a = 10T$. Plot (d) shows an diffusivity for an extremely low value of $\tau_R/\tau_b$ at $N = 10$ as a function of the fraction bound $p$. We note that both Monte-Carlo simulations0 and the model agree on the maximum at high value of $p$. The Monte-Carlo results are shown as circles and model prediction as lines.
Critical temperature for enhanced diffusion

Figure S2: Plot of normalized critical active temperature $T_a/T$ required to achieve enhanced diffusivity, $D > D_f$ for different values of $\tau_R/\tau_b$. The log plot corresponds to a linear relationship between the x and y axis parameters.