Planar and Nonplanar Konishi Anomalies and Effective Superpotential for Noncommutative $\mathcal{N} = 1$ Supersymmetric $U(1)$

Farhad Ardalan and Néda Sadooghi

Department of Physics, Sharif University of Technology
P.O. Box 11365-9161, Tehran-Iran

and

Institute for Studies in Theoretical Physics and Mathematics (IPM)
School of Physics, P.O. Box 19395-5531, Tehran-Iran
E-mails: ardalan, sadooghi@theory.ipm.ac.ir

Abstract

The Konishi anomalies for noncommutative $\mathcal{N} = 1$ supersymmetric $U(1)$ gauge theory arising from planar and nonplanar diagrams are calculated. Whereas planar Konishi anomaly is the expected $\star$-deformation of the commutative anomaly, nonplanar anomaly reflects the important features of nonplanar diagrams of noncommutative gauge theories, such as UV/IR mixing and the appearance of nonlocal open Wilson lines. We use the planar and nonplanar Konishi anomalies to calculate the effective superpotential of the theory. In the limit of vanishing $|\Theta p|$, with $\Theta$ the noncommutativity parameter, the noncommutative effective superpotential depends on a gauge invariant superfield, which includes supersymmetric Wilson lines, and has nontrivial dependence on the gauge field supermultiplet.

PACS No.: 11.15.Bt, 11.10.Gh, 11.25.Mj

Keywords: Noncommutative $\mathcal{N} = 1$ Supersymmetry, effective Superpotential.
1 Introduction

Quantum Field Theories with $\mathcal{N} = 1$ supersymmetry have been the subject of intense studies in the past 20 years. In particular it has been shown that in many cases the exact form of the effective superpotential can be determined using kinematical constraints such as holomorphy and various symmetries, and also approximate dynamical information about the asymptotic behavior of the superpotential [1]-[4].

Recently a new technique for evaluating the effective superpotential has been developed by Dijkgraaf and Vafa [5]. They conjectured that the exact effective superpotential for $\mathcal{N} = 1$ supersymmetric gauge theories can be constructed from an associated matrix model, and, that the diagrams relevant to the effective theory are indeed planar. Motivated by Dijkgraaf-Vafa’s conjecture, the authors of [6] derived the effective superpotential using a certain generalized Konishi anomaly. After this work many other authors have analyzed supersymmetric gauge theories along the line of Konishi anomaly method.

In a separate development [7], Ooguri and Vafa considered a novel deformation of $\mathcal{N} = 1$ supersymmetric gauge theories in four dimensions, which involves a non-vanishing anticommutation relation of fermionic coordinates of the superspace $\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta}$, (see also [8]). The consequences of this deformation have been explored further by Seiberg [9]. In particular, it is shown, that in the resulting $\mathcal{N} = 1/2$ theory the ordinary space-time coordinates $x$ do not commute, when the supercoordinate $\theta$ satisfy the anticommutation relation $\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta}$. In [7] Ooguri and Vafa have shown that the exact superpotential of this so called $C$–deformed gauge theory involves nonplanar diagrams. This theory has been intensely studied recently [10, 11]. In [11], the one-loop effective superpotential of an $\mathcal{N} = 1/2$ holomorphic Wess-Zumino model is calculated, where both fermionic and bosonic coordinates are made non(anti)commutative, and it is shown that planar and nonplanar contributions exhibit different behavior. Whereas planar diagrams yield an effective superpotential proportional to $\Phi \star \log \Phi$, nonplanar diagrams are UV divergent when bosonic noncommutativity is turned off. Here $\Phi$ is the Wess-Zumino hypermultiplet. Resumming the nonplanar diagrams, they are expressed as a $\star$-product including open Wilson lines in superspace. These supersymmetric open Wilson lines are indeed responsible for noncommutative UV/IR mixing [12].

In this paper, we consider noncommutativity of space coordinates only [13] and determine the exact effective superpotential for the $\mathcal{N} = 1$ supersymmetric $U(1)$ gauge theory with particular
emphasis on the role of nonplanar diagrams. To find the exact effective superpotential, we follow the method of Konishi anomaly [2, 3] without recourse to the matrix model. A matrix model formulation of noncommutative supersymmetric theories with bosonic noncommutativity has been presented in [14], where a $\star$-deformed Konishi anomaly is used to study Dijkgraaf and Vafa’s conjecture. We will show, using the previous results on anomaly in noncommutative gauge theories [15, 16, 17], that such a $\star$-deformed Konishi anomaly arises only from planar diagrams [18]. The theory, however, has both planar and nonplanar diagrams. Nonplanar diagrams do not contribute to the anomaly for large noncommutativity parameter $\Theta$. For small $\Theta$, though, they yield a finite contribution to the anomaly and have to be taken into account. Indeed, nonplanar anomaly seems to be the playing ground for many important features of noncommutative gauge theories discovered in the last few years.

This paper is organized as follows: As most of the intricacies of the effective action is in the form of the anomaly, we briefly review the results of [16] on planar and nonplanar anomalies of noncommutative field theories in Section 2. After deriving the global Noether current of the theory, we compute, using Fujikawa’s path integral method [19], the gauge covariant (planar) and invariant (nonplanar) anomalies corresponding to the covariant and invariant currents of $U(1)$ gauge theory. We will show that due to the UV/IR mixing [12], the nonplanar anomaly arises as a singularity at the limit of zero $|\Theta p|$ where $\Theta$ is the noncommutativity parameter and $p$ the momentum. In Section 3, we will then calculate the covariant (planar) and invariant (nonplanar) Konishi anomalies using a supersymmetric version of Fujikawa’s path integral method, originally introduced by Konishi [20]. To obtain $\star$-gauge invariant expressions for the invariant (nonplanar) Konishi anomaly, it is attached to a noncommutative supersymmetric open Wilson line, defined by the noncommutative generalization of supersymmetric Wilson lines [21].

There are three ingredients which are necessary for evaluating the exact effective superpotential. The first is the Konishi anomaly, whose nontrivial structure will be presented in Section 3. The second is the one-loop $\beta$-function of the theory. In section 4, the background field method will be used to determine the one-loop $\beta$-function of noncommutative $\mathcal{N} = 1$ supersymmetric $U(1)$ gauge theory with $N_f$ flavor pairs in the fundamental and antifundamental representation. This method has been previously used in [22] and [23], where the $\beta$-function of nonsupersymmetric field theory was determined and was shown that it exhibits a UV/IR mixing. In the
supersymmetric generalization, we will show that in the $|p| \gg \frac{1}{\sqrt{\Theta}}$ limit only planar diagrams contribute to the one-loop $\beta$-function, and that the theory is asymptotically free for $N_f < 3$. In the limit of $|p| \ll \frac{1}{\sqrt{\Theta}}$, however, where the nonplanar diagrams also contribute, the theory is IR free for any number of flavors.

Finally, the third ingredient is the anomaly corresponding to the $U_R(1)$ symmetry of the theory. As in the ordinary supersymmetric QCD, it receives contributions from the fermionic fields of the $N_f$ matter supermultiplets and also from the gaugino field in the gauge supermultiplet in the adjoint representation. We will calculate the anomaly corresponding to the R-symmetry separately using the Fujikawa Method and will show, that, due to UV/IR mixing, the anomaly arising from $N_f$ flavors in the matter supermultiplet in the fundamental representation receives contribution only in the limit of vanishing $|\Theta p|$, whereas the contribution of the gaugino in the adjoint representation to this R-anomaly appears for any $|\Theta p|$.

Having these special ingredients of noncommutative supersymmetric gauge theories on hand, in Section 4 we follow the conventional methods of ordinary SQCD, outlined in Refs. [1, 2, 24] (for a recent review see [25]), to find the effective superpotential of the gaugino condensate for two different limits of vanishing and arbitrary but finite $|\Theta p|$, separately. We also determine the Affleck-Dine-Seiberg meson field superpotential [26] for the noncommutative theory in these two limits. Section 5 is devoted to discussions.

2 Anomalies in Noncommutative $U(1)$ Gauge Theory

To begin we will briefly review the results of Refs. [15, 16]. The important observations there were that a noncommutative gauge theory with matter fields in fundamental representation consists of two different vector currents and axial vector currents and that the anomalies corresponding to these axial vector currents arise from planar and nonplanar diagrams of the noncommutative gauge theory. Whereas the planar anomaly [15], arising from planar diagrams, is the $\star$-deformation of the commutative Adler-Bell-Jackiw (ABJ) anomaly [27], the nonplanar anomaly reflects the unconventional and important behavior of nonplanar diagrams of noncommutative gauge theories. There are at least two important features related to the nonplanar anomaly of noncommutative gauge theories:

The first property is the UV/IR mixing indicating a certain singularity at $|\Theta p| \to 0$ limit. The second important point related to the nonplanar anomaly is the appearance of a general-
ized $\star$-product between $F_{\mu\nu}$ and its dual. In [16], we calculated perturbatively the contribution of nonplanar diagrams to the axial anomaly using two different regularization methods; dimensional and Pauli-Villars regularizations, and showed that the result for the nonplanar anomaly is not $\star$-gauge invariant if we consider only the triangle, square and pentagon diagrams as in the ordinary commutative non-Abelian gauge theories. To restore the $\star$-gauge invariance, it is necessary to attach $F_{\mu\nu}$ and its dual to an open Wilson line with a length proportional to $\sqrt{\Theta}$. The expansion of the Wilson line in the external gauge fields produces infinitely many diagrams whose contributions guarantee the gauge invariance of the final result [28, 29]. The invariant anomaly in noncommutative gauge theories has been also studied in a number of other works [30, 31, 32], and there are certain disagreements in the literature; here we will reproduce our previous results [16] by the Fujikawa method. But, when using the result of the anomaly for determination of the effective action of the gauginos in the supersymmetric theory we will not commit ourselves to a particular form of the anomaly, $S'$ in this paper. Any of the forms found for $S'$ in the literature may be substituted in the action.\(^1\)

In this section we first derive the global Noether currents of noncommutative nonsupersymmetric $U(1)$ gauge theory and resolve the ambiguity which arises in determining these currents. We then calculate the covariant (planar) and invariant (nonplanar) anomalies in noncommutative $U(1)$ gauge theory using Fujikawa’s path integral method [19], which can be easily generalized to the Konishi anomaly of supersymmetric gauge theories (see section 3).

Let us begin by fixing our notations and by recalling that noncommutative gauge theory is characterized by replacing the familiar product of functions with the $\star$-product

$$f(x) \star g(x) \equiv f(x + \xi) \exp \left( \frac{i\Theta_{\mu\nu}}{2} \frac{\partial}{\partial \xi^\mu} \frac{\partial}{\partial \zeta^\nu} \right) g(x + \zeta) \bigg|_{\xi = \zeta = 0},$$

(2.1)

where $\Theta_{\mu\nu}$ is a real antisymmetric matrix, and reflects the noncommutativity of the coordinates

$$[x^\mu, x^\nu] = i\Theta_{\mu\nu}. \quad (2.2)$$

The action of the noncommutative $U(1)$ gauge theory with matter fields in the fundamental representation is

$$S[A_\mu, \bar{\psi}, \psi] = -\frac{1}{4} \int d^4x \ F_{\mu\nu} \star F^{\mu\nu} + \int d^4x \ \bar{\psi}(x) \star (i\slashed{D} - m) \psi(x),$$

(2.3)

\(^1\)The vanishing anomaly of [30] is in fact excluded by the arguments of [31]: The non-vanishing of the covariant anomaly forces a nonzero value for the invariant anomaly (see the discussion in section 2.3).
with the field strength tensor

\[ F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu], \]  

(2.4)

and the covariant derivative

\[ D_\mu \psi(x) \equiv \partial_\mu \psi(x) + igA_\mu(x) \star \psi(x). \]  

(2.5)

The above action is invariant under global transformation of the matter fields

\[ \delta \psi(x) = i\alpha \psi(x), \quad \text{and} \quad \delta \bar{\psi}(x) = -i\alpha \bar{\psi}(x). \]  

(2.6)

The corresponding Noether current is given unambiguously by varying the Lagrangian density giving

\[ j_{\text{inv.}}^\mu(x) \equiv \bar{\psi}(x)\gamma^\mu \star \psi(x), \quad \text{with} \quad \partial_\mu j_{\text{inv.}}^\mu(x) = 0. \]  

(2.7)

We should emphasize that the covariant current, \( J_{\text{cov.}}^\mu \equiv \bar{\psi}_\alpha(\gamma^\mu)^{\alpha\beta} \psi_\beta \), is not conserved in the Noether sense, i.e. \( \partial_\mu J_{\text{cov.}}^\mu \neq 0 \). It satisfies the equation

\[ D_\mu J_{\text{cov.}}^\mu = 0. \]  

(2.8)

Similar variational procedure leads to the anomalous global axial vector current

\[ j_{\text{inv.}}^{\mu,5}(x) = -\bar{\psi}_\alpha(x) \star \psi_\beta(x) (\gamma_\mu \gamma_5)^{\alpha\beta}, \]  

(2.9)

resulting from the invariance of the action (2.3) under the global axial transformation \( \delta \psi = i\alpha \gamma_5 \psi \). A second axial vector current can also be defined

\[ J_{\text{cov.}}^{\mu,5}(x) = \psi_\beta(x) \star \bar{\psi}_\alpha(x) (\gamma_\mu \gamma_5)^{\alpha\beta}. \]  

(2.10)

These two currents satisfy \( \partial_\mu j_{\text{inv.}}^{\mu,5}(x) = 0 \) and \( D_\mu J_{\text{cov.}}^{\mu,5}(x) = 0 \) in the chiral limit.\(^2\)

In [15, 16] we have shown, using diagrammatic methods, that the anomaly corresponding to the covariant axial vector current, \( J_{\text{cov.}}^{\mu,5}(x) \), arises only from planar diagrams, whereas the anomaly corresponding to the invariant current, \( j_{\text{inv.}}^{\mu,5}(x) \), receives contributions only from the nonplanar diagrams. In the following section, we derive anew our previous results using Fujikawa’s path integral method [19].

\(^2\)Relations between different currents and their connection to the commutative currents are discussed in [33, 34].
2.1 Covariant (Planar) $U_A(1)$ Anomaly

Consider the partition function of noncommutative $U(1)$ gauge theory with matter fields in the fundamental representation

$$Z = \int D\psi \ D\bar{\psi} \ e^{-iS_F[\psi,\bar{\psi}]},$$

(2.11)

where

$$S_F = i \int d^4x \left[ \bar{\psi}_\alpha(x) \star (\gamma_\mu)^{\alpha\beta} \partial_\mu \psi(x) + ig \bar{\psi}_\alpha(x) \star A_\mu(x)(\gamma^\mu)^{\alpha\beta} \psi_\beta(x) \right],$$

(2.12)

is the fermionic part of the action (2.3) in the massless limit. Requiring that $Z$ remains invariant under the following fundamental axial change of variables,

$$\delta_5 \psi(x) = i\alpha(x)\gamma_5 \star \psi(x),$$

(2.13)

the covariant (planar) axial anomaly can be determined. Under these local change of variables the fermionic part of the action transforms as

$$S_F \longrightarrow S'_F = S_F - \int d^4x \ D_\mu J_{\mu,5}^{\text{cov.}}(x) \star \alpha(x),$$

(2.14)

with the covariant current $J_{\mu,5}^{\text{cov.}}$ given in (2.10). The covariant derivative $D_\mu$ is defined by

$$D_\mu J_{\mu,5}^{\text{cov.}}(x) \equiv \partial_\mu J_{\mu,5}^{\text{cov.}} + ig [A_\mu, J_{\mu,5}^{\text{cov.}}].$$

(2.15)

To calculate the Jacobian of the transformations (2.13), we expand $\psi$ and $\bar{\psi}$ as a linear combination of the eigenfunctions $\varphi_n$ and $\varphi_n^\dagger$ of the Dirac operator

$$\psi(x) = \sum_n a_n \varphi_n(x), \quad \text{and} \quad \bar{\psi}(x) = \sum_n b_n \varphi_n^\dagger(x),$$

(2.16)

where

$$a_m = \int d^4x \varphi_m^\dagger(x) \star \psi(x),$$

(2.17)

transforms as

$$a'_m = \sum_n (\delta_{mn} + C_{mn}) a_n, \quad \text{with} \quad C_{mn} \equiv i \int d^4x \alpha(x) \star \varphi_{n,\beta}(x) \star \varphi_{m,\alpha}^\dagger(x)(\gamma_5)^{\alpha\beta}. \quad (2.18)$$

The Jacobian $\mathcal{J}$ therefore is

$$\mathcal{J} = \exp \left( \sum_n C_{nn} \right) = \exp \left( i \int d^4x \alpha(x) \star \sum_n \varphi_{n,\beta}(x) \star \varphi_{n,\alpha}^\dagger(x)(\gamma_5)^{\alpha\beta} \right),$$

(2.19)
and the measure $D\psi D\bar{\psi}$ transforms as

$$D\psi D\bar{\psi} \rightarrow D\psi' D\bar{\psi}' = \exp \left( -2i \int d^4x \, \alpha(x) \sum_n \varphi_{n,\beta}(x) \varphi_{n,\alpha}^\dagger(x) (\gamma_5)^{\alpha\beta} \right) D\psi D\bar{\psi}. \tag{2.20}$$

Combining the relations (2.14) and (2.20) we find that the partition function (2.11) remains invariant if and only if

$$D_{\mu} J_{\mu,5}^{\text{cov.}}(x) = 2 \lim_{M \to \infty} \left( \sum_n \left[ e^{-\frac{D^2}{M^2}} \varphi_n(x) \right] \varphi_{n,\alpha}^\dagger(x) (\gamma_5)^{\alpha\beta} \right). \tag{2.21}$$

As the $\varphi_n$’s transform in the fundamental representation, it can be easily checked that both sides of the above equation are covariant under $\star$-gauge transformation. Here, as in the commutative case, the r.h.s. of (2.21) must be regulated, using a gauge covariant Gaussian damping factor $\exp(-\frac{D^2}{2M^2})$ with $-D^2 = -D^2 - g^2 \sigma_{\mu\nu} F_{\mu\nu}$, $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_{\mu}, \gamma_{\nu}]$ and $M$ the regulator mass

$$D_{\mu} J_{\mu,5}^{\text{cov.}}(x) = \lim_{M \to \infty} \left( \sum_n \left[ e^{-\frac{D^2}{M^2}} \varphi_n(x) \right] \varphi_{n,\alpha}^\dagger(x) (\gamma_5)^{\alpha\beta} \right). \tag{2.22}$$

Transforming to the Fourier space and after some standard manipulations, we arrive at

$$D_{\mu} J_{\mu,5}^{\text{cov.}}(x) = 2 \lim_{M \to \infty} \text{tr} \left( \gamma_5 \frac{1}{2!} \left( -\frac{g}{2M^2} \sigma_{\mu\nu} F_{\mu\nu}(x) \right) \right) \int \frac{d^4k}{(2\pi)^4} e^{\frac{k^2}{M^2}} + O(\frac{1}{M^2}). \tag{2.23}$$

Taking the limit $M \to \infty$ and using $\text{tr}(\gamma_5 \sigma_{\mu\nu} \sigma_{\lambda\rho}) = 4i \varepsilon_{\mu\nu\lambda\rho}$, the only remaining finite term, the planar $U_A(1)$ anomaly is given by

$$D_{\mu} J_{\mu,5}^{\text{cov.}}(x) = -\frac{g^2}{16\pi^2} F_{\mu\nu}(x) \star \tilde{F}_{\mu\nu}(x), \tag{2.24}$$

and $\tilde{F}_{\mu\nu} \equiv \varepsilon_{\mu\nu\lambda\rho} F^{\lambda\rho}$. The result is indeed the expected $\star$-generalization of the Adler-Bell-Jackiw anomaly (ABJ) [27], in agreement with previous calculation [15, 17].

### 2.2 Noncommutative Open Wilson Line and Generalized $\star$-Product

In the next section we will calculate the $U(1)$ anomaly corresponding to the invariant current of the theory. We will see that in order to receive a gauge invariant expression for the so called invariant or nonplanar anomaly, we have to attach the anomaly to an open Wilson line with the length proportional to $\sqrt{\Theta}$, where $\Theta$ is the noncommutativity parameter. In this section we will review some general aspects of noncommutative open Wilson line in the nonsupersymmetric case.
The noncommutative open Wilson lines [28] are noncommutative generalization of the commutative Schwinger’s line integrals, and are defined by

\[ W(x, \ell) = P_\star \exp \left( i \int_0^1 d\sigma \frac{d\xi^\mu(\sigma)}{d\sigma} A_\mu(x + \xi(\sigma)) \right) \]  

where 0 ≤ \sigma ≤ 1, \( x \) is the basis of the Wilson line and \( \ell \) is its length. It transforms under the local gauge transformation as

\[ W(x, \ell) \to U(x) \star W(x, \ell) \star U^\dagger(x + \ell), \]  

with \( U(x) \equiv e^{i\alpha(x)} \) and \( \alpha(x) \) an arbitrary function. Generally for a \( \star \)-gauge covariant operator \( \mathcal{O}_{\text{cov.}}(x) \) transforming as

\[ \mathcal{O}_{\text{cov.}}(x) \to U(x) \star \mathcal{O}_{\text{cov.}}(x) \star U^\dagger(x), \]

one defines a modified Fourier transformation

\[ \tilde{\mathcal{O}}_{\text{inv.}}(k) = \int d^4x \mathcal{O}_{\text{cov.}}(x) \star W(x, \ell) \star e^{ikx} \equiv \int d^4x P_\star \left[ \mathcal{O}_{\text{cov.}}(x)W(x, \ell) \right] \star e^{ikx}, \]

which is manifestly \( \star \)-gauge invariant, if the length of the Wilson line is given by \( \ell_\mu = \Theta_{\mu
u}k^\nu \). The gauge invariant version of \( \mathcal{O}_{\text{cov.}} \) is given therefore by going back to the coordinate space

\[ \mathcal{O}_{\text{inv.}}(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \int d^4y P_\star \left[ \mathcal{O}_{\text{cov.}}(y)W(y, \ell = \Theta_{\mu\nu}k^\nu) \right] \star e^{igy}. \]

It is clear that the first term in the expansion of the Wilson line in terms of the small gauge field is the same gauge covariant operator \( \mathcal{O}_{\text{cov.}} \)

\[ \mathcal{O}_{\text{inv.}}(x) = \mathcal{O}_{\text{cov.}}(x) + \text{higher order terms}. \]

The situation changes if the gauge covariant operator is a product of \( n \) other gauge covariant operators. Suppose the operator \( \mathcal{Q}_{\text{cov.}} \) is a product of \( n \) operators \( \mathcal{O}_i \)

\[ \mathcal{Q}_{\text{cov.}}(x) = \mathcal{O}_1(x) \star \mathcal{O}_2(x) \star \cdots \star \mathcal{O}_n(x), \]

with

\[ \mathcal{O}_i(x) \to U(x) \star \mathcal{O}_i(x) \star U^\dagger(x). \]
In this case there are two possibilities to attach the operators to the Wilson line in order to build a gauge invariant operator. The first possibility is to attach $Q_{\text{cov.}}(x)$ itself at the end of the Wilson line. Its gauge invariant version is again given by (2.29)

$$Q_{\text{inv.}}(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \int d^4y \, P_\star \left[ Q_{\text{cov.}}(y) W(y, \ell = \Theta^{\mu
u} k_\nu) \right] \star e^{iky}. \quad (2.33)$$

Expanding the Wilson line in terms of the external gauge field leads to

$$Q_{\text{inv.}}(x) = O_1(x) \star O_2(x) \star \cdots \star O_n(x) + \text{higher order terms.} \quad (2.34)$$

The second possibility is to smear the gauge covariant operators $Q_{\text{cov.}}(x)$ along the Wilson line by attaching $O_i(x), i = 1, \cdots, n$ at different insertion points and eventually integrating over all these insertion points [29]

$$\bar{Q}_{\text{inv.}}(k) = \int d^4x \left( \prod_{i=1}^n \int_0^1 d\tau_i \right) P_\star \left[ W(x, C) \prod_{i=1}^n O_i \left( x + \xi(\tau_i) \right) \right] \star e^{ikx} \equiv \int d^4x \, L_\star \left[ W(x, C) \prod_{i=1}^n O_i(x) \right] \star e^{ikx}. \quad (2.35)$$

The new path ordering $L_\star$, defined in the above equation, expresses the smearing of $n$ covariant operators $O_i$ along the Wilson line using new parameters $\tau_i, i = 1, \cdots, n$. The gauge invariant version of $Q_{\text{cov.}}(x)$ in the $x$ space can therefore be given by an inverse Fourier transformation

$$Q_{\text{inv.}}(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \int d^4y \, L_\star \left[ W(x, C) \prod_{i=1}^n O_i(x) \right] \star e^{iky}. \quad (2.36)$$

The Wilson line appearing in the above equation can be expanded again in powers of the external gauge fields $A_\mu$. The crux of this expansion is that here, in contrast to the previous case (2.30), the first term of the expansion will include a generalized $\star$-product [28], $\star_n$, between $n$ gauge covariant operators

$$Q_{\text{inv.}}(x) = [O_1(x), \cdots, O_n(x)]_{\star_n} + \text{higher order terms.} \quad (2.37)$$

For $n = 2$, (2.37) reads

$$Q_{\text{inv.}}(x) = O_1(x) \star' O_2(x) + \text{higher order terms,} \quad (2.38)$$

with $\star'$-product given by

$$f(x) \star' g(x) \equiv f(x + \xi) \sin \left( \frac{\Theta^{\mu\nu}}{2} \frac{\partial}{\partial \xi} \frac{\partial}{\partial \zeta} \right) \left. g(x + \zeta) \right|_{\xi = \zeta = 0}. \quad (2.39)$$

We will now calculate the anomaly corresponding to gauge invariant current, where the open Wilson lines are introduced to provide the gauge invariance of the result.
2.3 Invariant (Nonplanar) $U_A(1)$ Anomaly

We will now use the invariance of the partition function (2.11) under the antifundamental local axial change of variables

$$\delta_5 \psi(x) = i \psi(x) \star \alpha(x) \gamma_5,$$  \hspace{1cm} (2.40)

to obtain the invariant (nonplanar) anomaly. Under this change of variable the fermionic part of the action (2.12) transforms as

$$S_F \rightarrow S'_F = S_F - \int d^4 x \partial_\mu j_{\text{inv.}}^{\mu,5}(x) \star \alpha(x),$$  \hspace{1cm} (2.41)

with $j_{\text{inv.}}^{\mu,5} = -\bar{\psi} \star \psi (\gamma_\mu \gamma_5)^{\alpha \beta}$ given in (2.9), and we get

$$\mathcal{D} \psi \mathcal{D} \bar{\psi} \rightarrow \mathcal{D} \psi' \mathcal{D} \bar{\psi}' = \exp \left( -2i \int d^4 x \alpha(x) \star \sum_n \varphi_{n,\alpha}^\dagger(x) \star (\gamma_5)^{\alpha \beta} \varphi_{n,\beta}(x) \bigg) \mathcal{D} \psi \mathcal{D} \bar{\psi},$$  \hspace{1cm} (2.42)

for the Jacobian of the transformation. The condition for the invariance of the partition function now reads

$$\partial_\mu j_{\text{inv.}}^{\mu,5}(x) = 2 \left( \sum_n \varphi_{n,\alpha}^\dagger(x) \star (\gamma_5)^{\alpha \beta} \varphi_{n,\beta}(x) \right).$$  \hspace{1cm} (2.43)

Note that since the l.h.s. of this equation is gauge invariant a gauge invariant version of Gaussian damping factor $\exp(-\psi^2/M^2) = \exp(-D^2/M^2 - g \sigma_{\mu \nu} F^{\mu \nu}/2)$ has to be used to regularize the r.h.s. To do this the field tensor $F^{\mu \nu}$ is to be smeared along an open Wilson line $W(y, C)$ with the length $C$.

3 We obtain

$$\partial_\mu j_{\text{inv.}}^{\mu,5}(x) = \lim_{M \to \infty} 2 \int \frac{d^4 p}{(2\pi)^4} e^{-ipx} \int d^4 y \sum_n \left[ e^{-\frac{p^2}{2M^2}} \varphi_{n,\alpha}^\dagger(y) \right] \times (\gamma_5)^{\alpha \beta} \star P_\star \left( W(y, C) \exp \left( -\frac{g}{2M^2} \int_0^1 d\tau F_{\mu \nu}(y + \tilde{p}\tau) \sigma^{\mu \nu} \bigg) \delta_{\alpha \beta} \varphi_{n,\delta}(y) \right) \star e^{ipy},$$  \hspace{1cm} (2.44)

Expanding now the exponential including the field strength tensor and using the completeness of the basis functions $\tilde{\varphi}_n(k)$

$$\sum_n \tilde{\varphi}_{n,\alpha}(k) \tilde{\varphi}_{n,\beta}(k') = (2\pi)^4 \delta(k - k') \delta_{\alpha \beta},$$

we find that the only remaining term in the limit $M \to \infty$ is given by

$$\partial_\mu j_{\text{inv.}}^{\mu,5}(x) = A_{\text{inv.}}^{\text{nonplanar}}(x),$$  \hspace{1cm} (2.45a)

3 The calculation of the unintegrated form of the invariant anomaly using the Seiberg-Witten map [32] suggests the smearing procedure used here.
where

\[
A_{\text{nonplanar}}^{\text{inv.}}(x) = \lim_{M \to \infty} \left. \frac{\text{2 tr} \left( \gamma^5 \sigma^{\mu
u} \sigma^{\rho\lambda} \right) \left( -\frac{g}{2M^2} \right)^2 \frac{1}{2!} \int \frac{d^4 p}{(2\pi)^4} \ e^{-ipx} \int \frac{d^4 k}{(2\pi)^4} e^{+\frac{k^2}{M^2}} \right] \times \int d^4 y \ e^{-iky} \ * P_* \left( W(y) * \int_0^1 d\tau_1 \int_0^1 d\tau_2 \ F_{\mu\nu}(y + p\tau_1) \ F_{\lambda\tau}(y + p\tau_2) \right) * e^{iky} * e^{ipy}
\]

\[=
\lim_{M \to \infty} \frac{ig^2}{M^4} \int \frac{d^4 p}{(2\pi)^4} \ e^{-ipx} \int \frac{d^4 k}{(2\pi)^4} e^{+\frac{k^2}{M^2}} \times \int d^4 y \ P_* \left( W(y - \tilde{k}) * \int_0^1 d\tau_1 \int_0^1 d\tau_2 \ F_{\mu\nu}(y - \tilde{k} + p\tau_1) \ F_{\lambda\tau}(y - \tilde{k} + p\tau_2) \right) * e^{ipy}.
\]

(2.45b)

Here the relation

\[f(y) * e^{iky} = e^{iky} * f(y - \tilde{k}),\]

with \(\tilde{k} = \Theta_{\mu\nu} k^\nu\) is used.

In the following we will calculate the first term in the expansion of the open Wilson line in the powers of external gauge field and we will show that as from the perturbative calculation performed in [16], a generalized \(*\)-product will emerge between the two field strength tensors. Taking (2.45b) and expanding the Wilson line in the orders of small external gauge field, we arrive at

\[
A_{\text{nonplanar}}^{\text{inv.}}(x) \bigg|_{\text{first term}} = \lim_{M \to \infty} \frac{ig^2}{M^4} \int \frac{d^4 k}{(2\pi)^4} e^{+\frac{k^2}{M^2}} \int \frac{d^4 p}{(2\pi)^4} e^{-ipx} \times \int d^4 y \ P_* \left( \int_0^1 d\tau_1 \int_0^1 d\tau_2 \ F_{\mu\nu}(y - \tilde{k} + p\tau_1) \ * \tilde{F}^{\mu\nu}(y - \tilde{k} + p\tau_2) \right) * e^{ipy},
\]

(2.47)

with \[
\int_{\tau_0}^{\tau} d\tau_1 \int_{\tau_0}^{\tau_1} d\tau_2 \ O_1(x + \xi(\tau_1)) \ O_2(x + \xi(\tau_2)) \equiv \\
\int_{\tau_0}^{\tau} d\tau_2 \int_{\tau_0}^{\tau_2} d\tau_1 \ O_1(x + \xi(\tau_1)) \ * \ O_2(x + \xi(\tau_2)) + \int_{\tau_0}^{\tau} d\tau_1 \int_{\tau_0}^{\tau_1} d\tau_2 \ O_2(x + \xi(\tau_2)) \ * \ O_1(x + \xi(\tau_1)),
\]

with \(\xi(\tau) \equiv \Theta^{\mu\nu} p_{\nu} \tau\). In the momentum space, after integrating over \(y\), we get

\[
A_{\text{nonplanar}}^{\text{inv.}}(x) \bigg|_{\text{first term}} = \lim_{M \to \infty} \frac{ig^2}{M^4} \int \frac{d^4 k_1}{(2\pi)^4} \ e^{+\frac{k_1^2}{M^2}} \int \frac{d^4 k_2}{(2\pi)^4} e^{-i(k_1 + k_2) \cdot x} e^{2i(k_1 + k_2) \cdot \tilde{k}} \times \left\{ \int_0^{\tau_2} \int_0^{\tau_1} e^{-ik_1 \cdot x_1} e^{(1 + 2\tau_1) - 2\tau_2} + \int_0^{\tau_1} \int_0^{\tau_2} e^{ik_1 \cdot x_1} e^{(1 - 2\tau_1) + 2\tau_2} \right\} F_{\mu\nu}(k_1) \ * \tilde{F}_{\mu\nu}(k_2),
\]

(2.48)
with \( k_1 \times k_2 \equiv \frac{\Theta_{\mu
u}}{2} k_1 \mu k_2 \nu \). Performing the integration over \( \tau_i \), \( i = 1, 2 \) and using

\[
\int_0^1 d\tau_2 \int_0^{\tau_2} d\tau_1 \, e^{-i k_1 \times k_2 (1 + 2\tau_1 - 2\tau_2)} + \int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 \, e^{+i k_1 \times k_2 (1 - 2\tau_1+2\tau_2)} = \frac{\sin(k_1 \times k_2)}{k_1 \times k_2},
\]

we obtain

\[
\mathcal{A}_{\text{nonplanar}}^{\text{inv.}}(x) \bigg|_{\text{first term}} = \lim_{M \to \infty} + \frac{ig^2}{M^4} \int \frac{d^4 k}{(2\pi)^4} \left[ e^{+i \frac{k^2}{M^2}} \left( 1 + i k + \frac{1}{2} M^2 (k_1 + k_2) \Theta_{\mu\nu} \right)^2 \right] e^{-i (k_1 + k_2) x} \times e^{2i (k_1 + k_2) \times k} F_{\mu\nu}(k_1) \frac{\sin(k_1 \times k_2)}{k_1 \times k_2} \tilde{F}_{\mu\nu}(k_2).
\]

Next the integration over \( k \) in the Euclidean space can be performed using

\[
e^{+ \frac{k^2}{M^2}} e^{2i (k_1 + k_2) \times k} = e^{+ \frac{1}{M^2} (k_1 + k_2) \Theta_{\mu\nu} \Theta_{\rho\sigma} q^\rho q^\sigma},
\]

with \( q \circ q \equiv q_\mu \Theta^{\mu\nu} \Theta_{\nu\sigma} q^\sigma \). The first term of \( \mathcal{A}_{\text{inv.}}^{\text{nonplanar}} \) is therefore

\[
\mathcal{A}_{\text{inv.}}^{\text{nonplanar}}(x) \bigg|_{\text{first term}} = \lim_{M \to \infty} - \frac{g^2}{16\pi^2} \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} e^{- \frac{M^2}{4} (k_1 + k_2) \circ (k_1 + k_2)} \times e^{-i k_1 x} F_{\mu\nu}(k_1) \frac{\sin(k_1 \times k_2)}{k_1 \times k_2} \tilde{F}_{\mu\nu}(k_2) e^{-i k_2 x}.
\]

Now we are in the position to discuss the celebrated UV/IR mixing introduced in [12]. As it is explained there, in noncommutative gauge theories, involving both the UV cutoff \( M \) and the IR cutoff \(|\Theta q|\), the two limits \( M \to \infty \) and \(|\Theta q| \to 0\) do not commute. To show this in the case of anomalies, let us define \( q \equiv k_1 + k_2 \) in (2.51), and consider the limit \( M^2 \frac{q \circ q}{4} \to 1 \) or \( \frac{q \circ q}{4} \to \frac{1}{M^2} \).

This limit is equivalent with taking first the limit \( M \to \infty \) and then \(|\Theta q| \to 0\). In this case, even before taking \(|\Theta q| \to 0\) the exponent \( \exp(- \frac{M^2 q \circ q}{4}) \) vanishes. Thus in limit \( M \to \infty \) and for any value of \(|\Theta q|\), the first term in the expansion of \( \mathcal{A}_{\text{inv.}}^{\text{nonplanar}} \) vanishes. In the opposite case, i.e. when we take first \(|\Theta q| \to 0\) and then \( M \to \infty \), a finite anomaly arises. This limit can be understood as the limit \( \frac{M^2 q \circ q}{4} \ll 1 \) or \( \frac{q \circ q}{4} \ll \frac{1}{M^2} \), too [12]. In this case the exponent \( \exp(- \frac{M^2 q \circ q}{4}) \to 1 \) and we are left with a finite nonplanar anomaly.

\[
\mathcal{A}_{\text{inv.}}^{\text{nonplanar}}(x) \bigg|_{\text{first term}} = - \frac{g^2}{16\pi^2} F_{\mu\nu}(x) \star F_{\mu\nu}(x),
\]

where the field strength tensors \( F_{\mu\nu} \)'s are as before defined in the noncommutative space by (2.4). Taking the limit \( M \to \infty \), which is performed in both cases, remove the unphysical cutoff dependence in the final expression for the anomaly.
Summarizing the above results from UV/IR mixing, the invariant (nonplanar) anomaly vanishes by taking the limit $M \to \infty$ keeping $|\Theta q|$ arbitrary but finite. But, when we take the $|\Theta q| \to 0$ limit before $M \to \infty$, we find

$$A_{\text{inv.}}^{\text{nonplanar}}(x) = -\frac{g^2}{16\pi^2} F_{\mu\nu}(x) \ast \tilde{F}^{\mu\nu}(x) + \cdots,$$

where the ellipses denote the higher order terms arising from the expansion of the Wilson line in the orders of the external gauge field. Note that the contributions from the Wilson line guarantees the $\ast$-gauge invariance of the unintegrated form of the anomaly. This is shown in [32], where the unintegrated invariant anomaly is determined using a Seiberg-Witten map. One of the conditions which must be fullfilled to find this map is that the integrated form of the invariant anomaly should be identical with the integrated form of the $\ast$-gauge covariant anomaly (2.24).

Indeed, integrating the expression (2.53) over the noncommutative space, leads to a gauge invariant result identical with the integrated form of the covariant anomaly (2.24). If the divergence of $j_{\text{inv.}}^{\mu.5}$ did vanish for all values of $|\Theta q|$ [30], a contradiction would arise when we looked at the integrated version of the covariant and invariant anomaly. This contradiction and its resolution, using the nonplanar anomaly [16], was first discussed in [31], where the non-conservation of the axial charge corresponding to two currents $J_{\text{cov.}}^{\mu.5}$ and $j_{\text{inv.}}^{\mu.5}$ is studied. Using the cyclic symmetry of the $\ast$-product under integration, both currents $J_{\text{cov.}}^{\mu.5}$ and $j_{\text{inv.}}^{\mu.5}$ lead to the same gauge invariant axial charge defined by $Q_5 \equiv \int d\hat{x} J_{\text{cov.}}^{0.5} = \int d\hat{x} j_{\text{inv.}}^{0.5}$. The covariant anomaly (2.24) implies that $Q_5$ is not conserved. Hence a vanishing of $j_{\text{inv.}}^{\mu.5}$ for all value of $|\Theta q|$ would lead to an inconsistency. However, this obvious contradiction is resolved, if we note that a finite (nonplanar) anomaly arises from a singularity at $|\Theta q| \to 0$ as is given in (2.53). The integrated version of the nonplanar anomaly receives therefore a finite contribution from $|\Theta q| \to 0$ limit.

As we have explained above, the Wilson line in the final expression for the invariant anomaly is necessary to preserve the $\ast$-gauge invariance of the unintegrated form of the anomaly. Expanding the Wilson line in the order of the external gauge field leads to infinitely many terms, which can be understood as the contribution from infinitely many diagrams, where more and more gauge fields are inserted to the ordinary triangle diagrams. As it is stated in [16], this result can be regarded as the noncommutative generalization of Adler-Bardeen’s nonrenormalization theorem [35]. Note that all additional terms due to the Wilson line attachment, and
higher order in the external gauge field can be written as a total derivative. Thus, once we integrate over the full expression of the invariant (nonplanar) anomaly including the Wilson line, we are left with the same integrated form of the covariant (planar) anomaly. This question is also addressed in [32].

We shall further note that the cancellation of the nonplanar anomaly in the case when we take first $M \to \infty$ and for arbitrary but finite $|\Theta q|$, can be understood in the framework of a noncommutative Green-Schwarz mechanism of anomaly cancellation [30, 31]. In fact, in the this limit the gauge theory is still coupled to the string theory, so that a nonvanishing tree level contribution from Ramond-Ramond charges is responsible for the cancellation of the nonplanar anomaly in a mechanism similar to the Green-Schwarz mechanism of anomaly cancellation in the commutative case [36].

3 Noncommutative Konishi Anomaly

The action of noncommutative $\mathcal{N} = 1$ supersymmetric $U(1)$ gauge theory is given by

$$S = S_{\text{matter}} + S_{\text{gauge}},$$

(3.1a)

with

$$S_{\text{matter}} = \int d^4 x \ d^2 \theta \ d^2 \bar{\theta} \ \Phi(x, \bar{\theta}) \star e^{V(x, \theta, \bar{\theta})} \star \Phi(x, \theta),$$

(3.1b)

where $\Phi$ and $V$ are the standard chiral matter field and gauge field supermultiplets, and

$$S_{\text{gauge}} = -\frac{1}{16\pi i} \int d^4 x \ d^2 \theta \ \tau W_\alpha(x, \theta) \star W^\alpha(x, \theta) + \text{h.c.},$$

(3.1c)

with holomorphic coupling $\tau \equiv \frac{4\pi i}{g^2} + \frac{\vartheta}{2\pi}$ including the gauge coupling and the $\vartheta$-angle. Using the definition $W_\alpha = \bar{D}^2(e^{-V})_\star D_\alpha(e^V)_\star$, it can be shown that the gauge action involves both $F_{\mu\nu} \star F_{\mu\nu}$ and $F_{\mu\nu} \star \tilde{F}^{\mu\nu} = \varepsilon_{\mu\nu\lambda\rho} F^{\mu\nu} \star F^{\lambda\rho}$ term that are proportional to $\frac{1}{g^2}$ and the $\vartheta$-angle, respectively. The above action (3.1a-c) is invariant under local $\star$-gauge transformation of matter fields in the fundamental representation

$$\Phi(x, \theta) \to \Phi'(x, \theta) = e^{i\Lambda(x, \theta)} \star \Phi(x, \theta),$$

(3.2)

and $\star$-gauge transformation of the gauge supermultiplet

$$e^{V} \to e^{i\tilde{\Lambda}} \star e^{V} \star e^{-i\Lambda},$$

(3.3)
where Λ and ¯Λ are arbitrary chiral and antichiral superfields.

Consider now the matter part of the action (3.1b). It is easy to check the invariance of this action under global transformation of the chiral matter field Φ

$$\Phi \rightarrow \Phi' = e^{iA}\Phi,$$

(3.4)

where $A$ is a constant parameter. To find the global currents of this theory, we proceed as in the nonsupersymmetric theory and define two independent local change of variables corresponding to the same global transformation (3.4)

$$\delta_A \Phi(z) = iA(z) \ast \Phi(z),$$
$$\delta_A \Phi(z) = i\Phi(z) \ast A(z).$$

(3.5)

Here $z$ stands for the collective supercoordinates $x$ and $\theta$. The corresponding currents to this change of variables are

$$J_{\text{cov.}}(z) \equiv \Phi(z) \ast \bar{\Phi}(z) \ast e^{V(z)},$$
$$J_{\text{inv.}}(z) \equiv \bar{\Phi}(z) \ast e^{V(z)} \ast \Phi(z).$$

(3.6)

The first and second currents are covariant and invariant under the $\ast$-gauge transformation, respectively. In the following we will determine the anomaly in the above symmetry corresponding to the currents (3.6) separately.

### 3.1 Covariant (Planar) Konishi Anomaly

Using the method presented in [20], we will calculate the anomaly corresponding to the covariant current $J_{\text{cov.}}$ (3.6). We consider the invariance of the partition function of noncommutative $\mathcal{N} = 1$ supersymmetric $U(1)$

$$\mathcal{Z} = \int \mathcal{D}\Phi \mathcal{D}\bar{\Phi} \ e^{-iS_{\text{matter}}},$$

(3.7)

under the local fundamental change of variable $\delta_A \Phi(z) = iA(z) \ast \Phi(z)$. The variation of the matter field action $S_{\text{matter}}$ corresponding to this change of variable is given by

$$\delta_A S_{\text{matter}} = \int d^8z \ A(z) \ast J_{\text{cov.}}(z).$$

(3.8)

The Jacobian $\mathcal{J}$ of this transformation can be easily calculated using the same method from nonsupersymmetric field theory presented in the previous section, and reads

$$\mathcal{J} = \exp \left( i \sum_n \int d^8z \ A(z) \ast \Phi_n(z) \ast \bar{\Phi}_n(z) \right),$$

(3.9)
where $\phi_n(z)$’s are a complete orthonormal basis of the Hilbert space. Combining the two results (3.8) and (3.9), and using the invariance of the partition function under the above *fundamental* change of variables, we arrive first at

$$-\frac{\tilde{D}^2}{4} J_{\text{cov.}}(z) = -i \sum_n \frac{\tilde{D}^2}{4} \left( \Phi_n(z) \star \bar{\Phi}_n(z) \right).$$

(3.10)

The r.h.s. of the above equation is then to be regulated using a Gaussian damping factor $e^{L/M^2}$, with the operator

$$L \equiv \frac{1}{16} \tilde{D}^2(e^{-V})_\star D^2(e^V)_\star,$$

(3.11)

and $M$ the Pauli-Villars mass. The chiral operator $L$ is manifestly supersymmetric invariant, gauge covariant and contains $\partial$ as the lowest component. Since the damping factor transforms covariantly under $\star$-gauge transformation, we have to insert it on the r.h.s. of (3.10) so that the resulting expression remains gauge covariant. We therefore obtain

$$-\frac{\tilde{D}^2}{4} J_{\text{cov.}}(z) = \lim_{M \to \infty} -i \sum_n e^{\frac{k_1^2}{M^2}} \frac{\tilde{D}^2}{4} \left( \Phi_n(z) \star \bar{\Phi}_n(z) \right).$$

(3.12)

Going now to the Fourier space we get

$$-\frac{\tilde{D}^2}{4} J_{\text{cov.}}(z) = \lim_{M \to \infty} -i \int \frac{d^4k_1}{(2\pi)^4} e^{\frac{k_1^2}{M^2}} \frac{\tilde{D}^2}{4} e^{ik_1z} \star e^{-ik_2z} \sum_n \Phi_n(k_1) \bar{\Phi}_n(k_2),$$

$$= \lim_{M \to \infty} -\frac{i}{2M^4} \int \frac{d^4k}{(2\pi)^4} e^{+\frac{k^2}{M^2}} W_\alpha(x, \theta) \star W^\alpha(x, \theta),$$

(3.13)

integrating over $k$ we arrive at the planar (covariant) Konishi anomaly corresponding to $J_{\text{cov.}}$

$$-\frac{\tilde{D}^2}{4} J_{\text{cov.}}(z) = S_{\text{planar, cov.}}(z), \quad \text{with} \quad S_{\text{planar, cov.}}(z) \equiv -\frac{1}{32\pi^2} W_\alpha(z) \star W^\alpha(z).$$

(3.14)

This is the only finite term surviving the limit $M \to \infty$ (see also [20]). As $W_\alpha$’s transform covariantly under $\star$-gauge transformation

$$W_\alpha(z) \to e^{i\Lambda(z)} \star W_\alpha(z) \star e^{-i\Lambda(z)},$$

(3.15)

clearly $S_{\text{planar, cov.}}(z)$ transforms also covariantly. The full Konishi equation for nonvanishing tree level superpotential reads

$$-\frac{\tilde{D}^2}{4} J_{\text{cov.}}(z) = \Phi(z) \star \frac{\partial W_{\text{tree}}(z)}{\partial \Phi(z)} + S_{\text{planar, cov.}}(z).$$

(3.16)
3.2 Invariant (Nonplanar) Konishi Anomaly

In this section we will derive the Konishi anomaly corresponding to the invariant current

\[ J_{\text{inv.}}(z) = \Phi(z) \star e^V \star \Phi(z), \]

of (3.6). The antifundamental change of variable \( \delta_A \Phi(z) = i \Phi(z) \star A(z) \) leads to

\[ \delta_A S_{\text{matter}} = \int d^8z \ A(z) \star J_{\text{inv.}}(z), \tag{3.17} \]

and the Jacobian reads

\[ J = \exp \left( i \sum_n \int d^8z \ A(z) \star \Phi_n(z) \star \Phi_n(z) \right), \tag{3.18} \]

with \( \Phi_n(z) \) the same orthonormal and complete basis as was introduced previously. Combining now the results from (3.17) and (3.18), and using the invariance of the partition function under global transformation \( \delta_A \Phi = i A \Phi \) we arrive first at

\[ - \frac{\bar{D}^2}{4} J_{\text{inv.}} = - \int d^8z \ A(z) \star \Phi_n(z) \star \Phi_n(z). \tag{3.19} \]

To regulate the r.h.s. we have to insert a \( \star \)-gauge invariant version of gauge covariant damping factor \( e^{L/M^2} \). As in the nonsupersymmetric case we smear the operator \( L \) (3.11) along an open Wilson line \( W(y, \theta; C) \)

\[ - \frac{\bar{D}^2}{4} J_{\text{inv.}}(x, \theta) = \lim_{M \to \infty} - i \int d^4p (2\pi)^4 \epsilon^{-ipx} \int d^4y \sum_n \frac{\bar{D}^2}{4} P_{\star} \left[ \Phi_n(y, \theta) \star W(y, \theta; C) \right. \]

\[ \star \exp \left( \frac{1}{M^2} \int \frac{1}{0} d\tau \ L(x + \bar{p}\tau, \theta) \right) \star \Phi_n(y, \theta) \] \( \star e^{ipy} \). \tag{3.20} \]

Here we have introduced the supersymmetric version of an open Wilson line \( W(y, \theta; C) \), as was used for the nonsupersymmetric case (2.44). The explicit form of the \( \star \)-generalization of the supersymmetric Wilson line for the commutative case introduced in [21] reads

\[ W(C_{z_1, z_2}) = \exp \left( \int_{C_{z_1, z_2}} ds \ z^A A_A \right), \quad \dot{z}^A = \frac{dz^M}{ds} e^A_M, \tag{3.21} \]

with \( e^A_M \) the vierbein matrix [37]

\[ e^A_M = \begin{pmatrix} \delta^m_{\alpha} & 0 & 0 \\ i\sigma^{m\alpha}_{\alpha\beta} \bar{\beta}^\beta & \delta^\mu_\alpha & 0 \\ i (\theta\sigma^\epsilon)^{\dot{\alpha}}_\dot{\epsilon} & 0 & \delta^{\dot{\alpha}}_\mu \end{pmatrix}. \tag{3.22} \]
We have used the following notations
\[ z^M = (x^m, \theta^\mu, \bar{\theta}_\dot{\mu}), \quad e^M_A \frac{\partial}{\partial z^M} \equiv D_A \equiv \left( \partial_\alpha, D_\alpha, D^\dot{\alpha} \right), \]
\[ D_\alpha = \frac{\partial}{\partial \theta^\alpha} + i \left( \sigma^m \bar{\theta}_\dot{\mu} \right)_\alpha \partial_m, \quad \bar{D}^\dot{\alpha} = \frac{\partial}{\partial \bar{\theta}^\dot{\alpha}} + i (\theta \sigma^m e^\dot{\alpha}) \partial_m, \] (3.23)
and introduced the superfield \( A_A \) for the gauge field supermultiplets \( U \) and \( V \) (matrices in the Lie algebra)
\[ A_\alpha \equiv e^{-V} D_\alpha e^V, \quad A^\dot{\alpha} \equiv e^{-U} \bar{D}^\dot{\alpha} e^U, \]
\[ A_A \equiv \frac{1}{4} i e^{\dot{\alpha}} \bar{e}_A \left( D_\alpha A^\dot{\beta} + \bar{D}^\dot{\beta} A_\alpha + \{ A_\alpha, A^\dot{\beta} \} \right). \] (3.24)
As in the nonsupersymmetric case the supersymmetric Wilson line (3.21) transforms under \( \star \)-gauge transformation as
\[ \mathcal{W}(x, \theta; \ell) \to e^{i \Lambda(x, \theta)} \star \mathcal{W}(x, \theta; \ell) \star e^{-i \Lambda(x+\ell, \theta)}. \] (3.25)
Consider again the expression on the r.h.s. of (3.20). Going to the Fourier space and performing the same manipulations as in equation (2.45a-b) we arrive at
\[ -\frac{\bar{D}^2}{4} J_{\text{inv.}} = S_{\text{inv.}}^{\text{nonplanar}}, \] (3.26a)
where
\[ S_{\text{inv.}}^{\text{nonplanar}}(x, \theta) \equiv \lim_{M \to \infty} -\frac{i}{M^4} \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \int \frac{d^4k}{(2\pi)^4} e^{+\frac{k^2}{M^2}} \int d^4y \mathcal{P} \left[ \mathcal{W}(y - \tilde{k}, \theta; \ell = \tilde{\ell}) \right. \]
\[ \left. \times \int_0^1 d\tau_1 \mathcal{W}_\alpha(y - \tilde{k} + \tau_1 \tilde{p}, \theta) \star \int_0^1 d\tau_2 \mathcal{W}^\alpha(y - \tilde{k} + \tau_2 \tilde{p}, \theta) \right] \star e^{ipy}. \] (3.26b)
Using now the expansion of the supersymmetric Wilson line (3.21) and going through the same algebraic manipulations following from (2.45a-b) to (2.51) in the nonsupersymmetric case, we obtain the lowest term of the nonplanar Konishi anomaly in the expansion of the supersymmetric Wilson line in powers of the superfield \( A_A \)
\[ S_{\text{inv.}}^{\text{nonplanar}}(x, \theta) \bigg|_{\text{first term}} = -\frac{1}{32\pi^2} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} e^{-\frac{k_1^2}{\lambda^2}} qqq \]
\[ \times e^{-ik_1 x} \mathcal{W}_\alpha(k_1, \theta) \frac{\sin (k_1 \times k_2)}{(k_1 \times k_2)} \mathcal{W}_\alpha(k_2, \theta) e^{-ik_2 x} + \text{higher order terms}, \] (3.27)
with \( q \equiv k_1 + k_2 \). Here similar UV/IR mixing phenomena as in the nonsupersymmetric case occur, \( i.e. \) the invariant (nonplanar) Konishi anomaly appears only when we take first \( |\Theta q| \to 0 \) and then \( M \to \infty \) [see the discussion leading to (2.52) in the previous section]

\[
S_{\text{inv.}}^{\text{nonplanar}}(x, \theta) \bigg|_{\text{first term}} = -\frac{1}{32\pi^2} W_\alpha(x, \theta) \ast' W^\alpha(x, \theta).
\] (3.28)

In summary the invariant (nonplanar) Konishi anomaly vanishes if we take first \( M \to \infty \) keeping \( |\Theta q| \) arbitrary but finite and appears as a singularity at \( |\Theta q| \to 0 \) limit

\[
S_{\text{inv.}}^{\text{nonplanar}}(x, \theta) = -\frac{1}{32\pi^2} W_\alpha(x, \theta) \ast' W^\alpha(x, \theta) + \cdots,
\] (3.29)

where the ellipses denote the contribution of the higher order terms in the external gauge multiplet arising from the expansion of the attached Wilson line.

The full Konishi equation, including the contributions of the tree level superpotential, is given by

\[
-\frac{\bar{D}^2}{4} J_{\text{inv.}}(z) = \frac{\partial W_{\text{tree}}(z)}{\partial \Phi(z)} \ast \Phi(z) + S_{\text{inv.}}^{\text{nonplanar}}(z).
\] (3.30)

### 4 Effective Superpotential for Noncommutative \( \mathcal{N} = 1 \) Supersymmetric \( U(1) \)

In this section, we will determine the effective superpotential of noncommutative \( \mathcal{N} = 1 \) supersymmetric \( U(1) \) gauge theory. Assuming that the original theory consists of one gauge supermultiplet and \( 2N_f \) fundamental and antifundamental chiral matter fields \( Q_i \) and \( \tilde{Q}_i \) with \( i = 1, \cdots, N_f \), the effective field theory will depend on the noncommutative generalization of the meson and gaugino bilinears. The fundamental matter multiplet \( Q_i \) and antifundamental matter multiplet \( \tilde{Q}_i \) transform as

\[
Q_i \to e^{iA} \ast Q_i, \quad \text{and} \quad \tilde{Q}_i \to \tilde{Q}_i \ast e^{-iA},
\] (4.1)

and the noncommutative \( \ast \)-gauge invariant meson is defined by

\[
T_{ij} \equiv \tilde{Q}_i \ast Q_j, \quad \text{with} \quad i, j = 1, \cdots, N_f.
\] (4.2)

Further, \( S \propto W_\alpha \ast W^\alpha \) and \( S' \propto W_\alpha \ast' W^\alpha + \cdots \), with the extra terms denoting the contributions from the expansion of the attached Wilson line, will be used as the relevant gaugino superfields.

As in the ordinary supersymmetric commutative theories [3], it can be shown using holomorphy and other symmetry arguments that the effective superpotential consists of a tree level
superpotential $W_{\text{tree}}$ and a nonperturbative term, the Veneziano-Yankielowicz (VY) dynamical superpotential $W_{\text{dyn}}$ [1, 2]

$$W_{\text{eff}}(T, S; \Lambda_{N_f}) = W_{\text{tree}}(T; m, \lambda) + W_{\text{dyn}}(T, S; \Lambda_{N_f}),$$

where $m$ and $\lambda$ are bare parameters of the superpotential\(^4\) and $\Lambda_{N_f}$ is the holomorphic intrinsic scale for a theory with $N_f$ massless flavors. It is defined as in ordinary commutative supersymmetric theories by the $\vartheta$-angle

$$\Lambda_{N_f} = |\Lambda_{N_f}| e^{i\vartheta/\beta_{N_f}}.$$  

(4.4)

Here $|\Lambda_{N_f}|$ is the intrinsic scale of the noncommutative U(1) gauge theory that enters through dimensional transmutation. It is given by the coefficient $\beta_{N_f}$ of the one-loop $\beta$-function $\beta(g) = -\frac{g^3}{16\pi^2} \beta_{N_f}$ of the theory

$$|\Lambda_{N_f}| \equiv \mu \exp \left( - \frac{8\pi^2}{\beta_{N_f} g^2(\mu)} \right).$$  

(4.5)

The subscript $N_f$ denotes the number of massless flavors.

Before calculating the superpotential of noncommutative supersymmetric U(1) theory, we will compute the one-loop $\beta$-function of the theory and also the anomaly corresponding to $U_R(1)$ symmetry. These are necessary to determine the effective superpotentials, as in the ordinary commutative supersymmetric gauge theories, using the selection rules.

4.1 Prerequisites

One-Loop $\beta$-Function

As we will need the $\beta$-function of the noncommutative theory for our calculations, we will present a supersymmetric extension of the result in [22] and [23], where the effect of nonplanar diagrams on the one-loop $\beta$-function of the noncommutative nonsupersymmetric U(1) are studied.

Let us consider a noncommutative theory consisting of $n_f$ Weyl fermions in the fundamental representation and $n_s$ complex scalars. Using the background field method, the effective coupling of the theory is given by

$$\frac{1}{g^2(p)} = \frac{1}{g^2} + \Pi(p),$$

(4.6)

\(^4\)A complete proof of non-renormalization theorem of noncommutative supersymmetric theories will be presented elsewhere.
with $\Pi(p)$ is defined by the vacuum polarization tensor
\begin{equation}
\Pi_{\mu\nu} = (p^2 \delta_{\mu\nu} - p_\mu p_\nu)\Pi(p) + \frac{\bar{p}_\mu \bar{p}_\nu}{\bar{p}^2} \bar{\Pi}(p).
\end{equation}

It is given explicitly by
\begin{equation}
\Pi(p) = \frac{1}{8\pi^2} \left[ \int_0^1 dx \sum_{j_a} \alpha(j_a) \left( d(j_a)(1 - 2x)^2 - 4C(j_a) \right) \left( \ln \frac{\Delta}{4\pi \mu^2} + 2K_0(|\bar{p}|\sqrt{\Delta}) \right) 
+ \frac{1}{2} \int_0^1 dx \sum_{j_f} \alpha(j_f) \left( d(j_f)(1 - 2x)^2 - 4C(j_f) \right) \ln \frac{\Delta}{4\pi \mu^2} \right],
\end{equation}
with $\Delta = x(1 - x)p^2$ and the modified Bessel function
\[ K_\nu(\alpha z) \equiv \frac{\alpha^\nu}{2} \int_0^\infty \frac{dt}{t^{\nu+1}} e^{-\frac{\alpha t}{2}} (t + \alpha t^2), \]
which appears only in the contribution of the nonplanar diagrams. Adding over all $j_a$ and $j_f$ for the fields in adjoint (ghosts and gauge fields) and fundamental (Weyl fermions and complex scalars) representation, respectively, we arrive first at
\begin{equation}
\Pi(p) = \frac{1}{8\pi^2} \int_0^1 dx \left[ (4 - (1 - 2x)^2) \left( \ln \frac{\Delta}{4\pi \mu^2} + 2K_0(|\bar{p}|\sqrt{\Delta}) \right) 
- \left( \frac{n_f}{2}(1 - (1 - 2x)^2) + \frac{n_s}{2}(1 - 2x)^2 \right) \ln \frac{\Delta}{4\pi \mu^2} \right].
\end{equation}

This expression can be compared with the result in [23]. The values of $\alpha(j), d(j)$ and $C(j)$ are listed below

|           | ghosts | gauge fields | Weyl fermions | Complex scalars |
|-----------|---------|--------------|---------------|----------------|
| $d(j_a)$  | 1       | 4            | 2             | 1              |
| $C(j_a)$  | 0       | 2            | $\frac{1}{2}$ | 0              |
| $\alpha(j_a)$ | 1       | $-\frac{1}{2}$ | $\alpha(j_f)$ | $\frac{n_f}{2}$ | $-n_s$ |

Using
\begin{equation}
\beta(g; p, \Theta) \equiv p_\mu \frac{\partial}{\partial p_\mu} g(p) \equiv -\frac{g^3}{2} p_\mu \frac{\partial \Pi(p)}{\partial p_\mu},
\end{equation}
and the relations
\begin{equation}
p_\mu \frac{\partial}{\partial p_\mu} \ln \frac{\Delta}{4\pi \mu^2} = 2, \quad \text{and} \quad p_\mu \frac{\partial}{\partial p_\mu} 2K_0(|\bar{p}|\sqrt{\Delta}) = -4K_1(|\bar{p}|\sqrt{\Delta}) \frac{\bar{p}^2 \Delta}{|\bar{p}| \sqrt{\Delta}}.
\end{equation}
the one-loop $\beta$-function of the theory for an arbitrary noncommutativity parameter reads

$$\beta(g; p, \Theta) \equiv -\frac{g^3}{8\pi^2} \left[ -\frac{n_f}{3} - \frac{n_s}{6} + \int_0^1 dx \left( 4 - (1 - 2x)^2 \right) \left( 1 - 2K_1(|\tilde{p}|\sqrt{\Delta})\frac{\tilde{p}^2 \Delta}{|\tilde{p}| \sqrt{\Delta}} \right) \right]. \quad (4.12)$$

Note that here, the $\beta$-function depends in general on the momentum $p$. This is due to the breaking of Lorentz invariance. Momentum dependent $\beta$-functions were previously calculated explicitly in the so called noncommutative dipole theories [38].

In the limit $|p| \gg \frac{1}{\sqrt{\Theta}}$, only the planar diagrams contribute to the $\beta$-function. This can be shown using the relation $\lim_{z \to \infty} K_1(z) = 0$ in (4.12). However, in the limit $|p| \ll \frac{1}{\sqrt{\Theta}}$, both planar and nonplanar diagrams contribute to the one-loop $\beta$-function. This can be shown by taking the limit $\lim_{z \to 0} K_1(z) = \frac{1}{z}$ in (4.12). This reflects the UV/IR duality indicated in [22] and [23]. For a nonsupersymmetric theory with $n_f$ massless Weyl fermions in the fundamental representation and $n_s$ complex scalars, the $\beta$-function reads therefore

$$\beta(g; p, \Theta) = \begin{cases} -\frac{g^3}{16\pi^2} \beta_{\ell,n_f} & \text{with } \beta_{\ell,n_f} = +2 \left( \frac{11}{3} - \frac{n_f}{3} - \frac{n_s}{6} \right), \quad |p| \gg \frac{1}{\sqrt{\Theta}}, \\ -\frac{g^3}{16\pi^2} \beta_{s,n_f} & \text{with } \beta_{s,n_f} = -2 \left( \frac{11}{3} + \frac{n_f}{3} + \frac{n_s}{6} \right), \quad |p| \ll \frac{1}{\sqrt{\Theta}}. \end{cases} \quad (4.13)$$

The subscript $\ell$ and $s$ in $\beta_{\ell,n_f}$ and $\beta_{s,n_f}$ label the coefficient of the one-loop $\beta$-function for two cases of $|p| \gg \frac{1}{\sqrt{\Theta}}$ and $|p| \ll \frac{1}{\sqrt{\Theta}}$, respectively. Note that since the matter fields are in the fundamental representation, they do not receive any contribution from nonplanar diagrams [22, 23]. The sign of the terms proportional to $n_f$ and $n_s$ are therefore the same for $|p| \gg \frac{1}{\sqrt{\Theta}}$ and $|p| \ll \frac{1}{\sqrt{\Theta}}$ limits. As for the part arising from the fields in the adjoint representation, i.e. terms proportional to $11/3$ in (4.13), they have opposite signs on the first and second line of (4.13). This is due to the contribution from the nonplanar parts [see also [23], where the same phenomenon occurs]. According to this result, for $|p| \ll \frac{1}{\sqrt{\Theta}}$ limit, the coefficient of the one-loop $\beta$-function, including the minus sign in front of $\frac{g^3}{16\pi^2}$, is positive. Thus in this limit, the theory turns out to be IR free for all values of $n_f$ and $n_s$.

To arrive at the supersymmetric extension of one-loop $\beta$-function in our case, we set $n_f = n_s$ and use the values of $\alpha(j), d(j)$ and $C(j)$ from the table.

---

Note that in [23] since the matter fields are in the adjoint representation, in contrast to our calculation, the terms proportional to $n_f$ and $n_s$ have also the opposite signs in $|p| \gg \frac{1}{\sqrt{\Theta}}$ and $|p| \ll \frac{1}{\sqrt{\Theta}}$ limits.
Using the above values in the expression (4.8), the $\sum_{j_a} \alpha(j_a) d(j_a)$ for the fields in the adjoint representation as well as $\sum_{j_f} \alpha(j_f) d(j_f)$ for the fields in the fundamental representation vanish. This was also expressed in [22, 23]. The remaining part is

$$
\Pi(p) = \frac{1}{8\pi^2} \int_0^1 dx \sum_{j_a} \alpha(j_a) \left(-4C(j_a)\right) \left(2K_0 \left(|\tilde{p}|\sqrt{\Delta}\right)\right) + \frac{1}{8\pi^2} \int_0^1 dx \left[ \sum_{j_a} \alpha(j_a)(-4C(j_a)) + \frac{1}{2} \sum_{j_f} \alpha(j_f)(-4C(j_f)) \right] \ln \frac{\Delta}{4\pi\mu^2},
$$

(4.14)

leading to

$$
\beta(g; p, \Theta) \equiv -\frac{g^3}{8\pi^2} \left[ -\frac{n_f}{2} + 3 \int_0^1 dx \left( 1 - 2K_1(\tilde{p}|\sqrt{\Delta}|) \frac{\tilde{p}^2\Delta}{\tilde{p}|\sqrt{\Delta}|} \right) \right].
$$

(4.15)

Now choosing $n_f = 2N_f$ and performing the same analysis as above, we arrive at the coefficient of the one-loop $\beta$-function for $|p| \gg \frac{1}{\sqrt{\Theta}}$ and $|p| \ll \frac{1}{\sqrt{\Theta}}$.

$$
\beta(g; p, \Theta) = \begin{cases} 
-\frac{g^3}{16\pi^2}b_{e, N_f} & \text{with } b_{e, N_f} = +2 \left(3 - N_f\right), \quad |p| \gg \frac{1}{\sqrt{\Theta}}, \\
-\frac{g^3}{16\pi^2}b_{s, N_f} & \text{with } b_{s, N_f} = -2 \left(3 + N_f\right), \quad |p| \ll \frac{1}{\sqrt{\Theta}}.
\end{cases}
$$

(4.16)

The first expression on the r.h.s. of (4.14), is the contribution from nonplanar diagrams, appearing only in front of the fields in the adjoint representation. Further, since the matter fields are in the fundamental representation and therefore do not receive any nonplanar contributions, the sign of the part proportional to $N_f$ does not change in two limits of $|p| \gg \frac{1}{\sqrt{\Theta}}$ and $|p| \ll \frac{1}{\sqrt{\Theta}}$.

Our results on the one-loop $\beta$-function can also be compared with [43] and the references therein.

According to this result, in $|p| \gg \frac{1}{\sqrt{\Theta}}$ limit, the theory is asymptotically free only for $N_f < 3$, whereas the theory is IR-free in $|p| \ll \frac{1}{\sqrt{\Theta}}$ limit for any arbitrary value of $N_f$.

---

Note that the coefficient of the $\beta$-function of noncommutative supersymmetric $\mathcal{N} = 1$ U(1) gauge theory differs from the coefficient of the $\beta$-function of the commutative supersymmetric SQCD [4] by a factor of 2. This is the same discrepancy as in the nonsupersymmetric case [22, 39].
Noncommutative $U_R(1)$ Anomaly

Let us concentrate on the anomalies corresponding to the axial $U_A(1)$ and R-symmetry $U_R(1)$ transformations

\[
U_A(1) : \quad Q(x, \theta) \to e^{i \alpha} Q(x, \theta), \quad \text{and} \quad W_\alpha(x, \theta) \to W_\alpha(x, \theta),
\]

\[
U_R(1) : \quad Q(x, \theta) \to e^{i \alpha} Q(x, e^{-3i \alpha/2} \theta) \quad \text{and} \quad W_\alpha(x, \theta) \to e^{3i \alpha/2} W_\alpha(x, e^{-3i \alpha/2} \theta). \quad (4.17)
\]

The $U_R(1)$ anomaly receives contribution from the $N_f$ chiral fermion pair $\psi_i^L$ and $\tilde{\psi}_i^L$, with $i = 1, \cdots, N_f$ in the matter supermultiplet $Q$ and $\tilde{Q}$, and from the chiral gaugino $\lambda_L$ in the gauge supermultiplet.\(^7\) Whereas the chiral fermions $\psi_L$ and $\tilde{\psi}_L$ transform in the fundamental and antifundamental representations, respectively, the gaugino $\lambda_L$ transforms in the adjoint representation. In section 2, we calculated only the $U(1)$ anomaly of matter fields in the fundamental representation. In this section we will compute the anomaly arising from gauginos in the adjoint representation. Surprisingly, this result is also affected by the UV/IR mixing. So that for vanishing $|\Theta p|$ the R-anomaly corresponding to the adjoint gauginos vanishes. In the opposite case, however, i.e. if we consider the case of arbitrary but finite $|\Theta p|$, it appears again and therefore decouple from the R-anomaly arising from the contribution from fermions in the fundamental representation. This decoupling is in contrast to ordinary commutative $N_c$ color SQCD.

From section 2, it is simple to find the contribution to the $U_R(1)$ anomaly of the fermions in the fundamental and antifundamental representation. Here we shall focus only on the nonplanar (invariant) anomaly, because after the Noether procedure the only current arising from the global $U_R(1)$ transformation is the invariant current, whose corresponding anomaly vanishes for arbitrary but finite $|\Theta p|$. When we take the limit $|\Theta p| \to 0$, its value is given by

\[
\mathcal{A}_\psi = 2N_f \alpha R(\psi) \left( -\frac{1}{32\pi^2} F_{\mu\nu} \star F^{\mu\nu} + \cdots \right), \quad (4.18)
\]

where $R(\psi)$ is the R-charge of the chiral fermion $\psi_L$. The extra terms are the contributions of the open Wilson line.

To find the contribution to $U_R(1)$ anomaly corresponding to the chiral gaugino field $\lambda_L$, which in contrast to the matter fields are in the adjoint representation, we follow the Fujikawa

\(^7\)For convenience, we have taken only the left handed matter fields $\psi_L \equiv \frac{1+i\gamma_5}{2} \psi$, $\tilde{\psi}_L \equiv \frac{1+i\gamma_5}{2} \tilde{\psi}$ and gaugino field $\lambda_L \equiv \frac{1+i\gamma_5}{2} \lambda$.\]
method outlined in section 2. The partition function corresponding to the gauginos in the adjoint representation is given by

$$Z = \int \mathcal{D}\lambda_L \mathcal{D}\bar{\lambda}_L \ e^{-iS_\lambda[\lambda_L,\bar{\lambda}_L]},$$

(4.19)

with $S_\lambda = \int d^4x \mathcal{L}_\lambda$, and the Lagrangian density

$$\mathcal{L}_\lambda \equiv i\bar{\lambda}_L(x) \not\!D \lambda_L(x),$$

(4.20)

where the covariant derivative acts on the gaugino field in the adjoint representation

$$D_\mu \lambda_L(x) = \partial_\mu \lambda_L(x) + ig[A_\mu(x), \lambda_L(x)].$$

(4.21)

Under the antifundamental local change of variables

$$\delta \lambda_L(x) = iR(\lambda)\lambda_L(x) \star \alpha(x), \quad \text{and} \quad \delta \bar{\lambda}_L(x) = -iR(\lambda)\alpha(x) \star \bar{\lambda}_L(x),$$

(4.22)

the action transforms as

$$\delta S_\lambda = -\int d^4x \ D_\mu J^\mu_\lambda(x) \star \alpha(x), \quad \text{with} \quad J^\mu_\lambda = \bar{\lambda}_L \gamma^\mu \lambda_L.$$ 

(4.23)

Further using the Jacobian of the transformation we have

$$\mathcal{D}\lambda_L \mathcal{D}\bar{\lambda}_L \rightarrow \mathcal{D}\lambda'_L \mathcal{D}\bar{\lambda}'_L = \exp\left(-2iR(\lambda) \int d^4x \ \alpha(x) \star \sum_n \varphi^\dagger_{n,\alpha}(x) (\gamma_5)^{\alpha\beta} \varphi_{n,\beta}(x)\right) \mathcal{D}\lambda_L \mathcal{D}\bar{\lambda}_L.$$ 

(4.24)

The invariance of the partition function leads therefore to the anomaly corresponding to the gaugino field

$$\mathcal{A}_\lambda = \lim_{M \to \infty} \ 2 \ R(\lambda) \sum_n \varphi^\dagger_{n,\alpha}(x) (\gamma_5)^{\alpha\beta} \left(e^{-\not\!P^2/2M^2}\right)_\beta^\delta \varphi_{n,\delta}(x),$$

(4.25)

where we have introduced the regulator $M$ in the damping factor $\exp\left(-\not\!P^2/2M^2\right)$. Using the definition $\not\!P^2 = D_\mu D^\mu + \frac{i}{2} \sigma^{\mu\nu} [D_\mu, D_\nu]$, noting that the functions $\varphi_n$’s are in the adjoint representation and using the completeness of $\varphi_n$’s in the Fourier space, we arrive at

$$\mathcal{A}_\lambda = \lim_{M \to \infty} \ 2 \ R(\lambda) \int \frac{d^4k}{(2\pi)^4} \ e^{\frac{k^2}{2M^2}} e^{-i k x} \ * \ \gamma_5 \ \exp\left(\frac{i}{2M^2} \sigma^{\mu\nu} [D_\mu, D_\nu]\right) \ * \ e^{i k x}$$

$$= \lim_{M \to \infty} \frac{i g^2}{M^4} \ R(\lambda) \ v^{\mu\nu\rho\lambda} \int \frac{d^4k}{(2\pi)^4} \ e^{-\frac{k^2}{2M^2}} e^{-i k x} \ * \ [F_{\mu\nu}(x), [F_{\rho\lambda}(x), e^{i k x}]]_\star.$$ 

(4.26)
Using further the relation (2.46), we arrive at

$$\mathcal{A}_\lambda = \lim_{M \to \infty} \frac{ig^2}{M^4} R(\lambda) \varepsilon^{\mu\nu\rho\lambda} \int \frac{d^4k}{(2\pi)^4} e^{\frac{k^2}{M^2}}$$

$$\times \left( F_{\mu\nu}(x - \bar{k}) \star F_{\rho\lambda}(x - \bar{k}) - 2F_{\mu\nu}(x - \bar{k}) \star F_{\rho\lambda}(x) + F_{\mu\nu}(x) \star F_{\rho\lambda}(x) \right). \quad (4.27)$$

The three terms appearing on the r.h.s. of (4.27) are evaluated separately. After integrating over \(k\), the first term, for instance, leads to

$$\lim_{M \to \infty} + ig^2 \frac{M^4}{4\pi^2} R(\lambda) \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} e^{\frac{k_1^2}{M^2} - \frac{1}{2}M^2} F_{\mu\nu}(k_1) e^{ik_1x} e^{ik_2x} F^{\mu\nu}(k_2)$$

$$= \lim_{M \to \infty} - \frac{g^2}{16\pi^2} R(\lambda) \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} e^{\frac{-M^2}{(k_1+k_2)^2} + \frac{1}{2}M^2} F_{\mu\nu}(k_1) e^{ik_1x} e^{ik_2x} F^{\mu\nu}(k_2). \quad (4.28)$$

As for the invariant anomaly, a UV/IR mixing occurs here. Following the same arguments as in the previous section, and calculating the contributions of other two terms, it turns out the anomaly corresponding to the gauginos in the adjoint representation vanishes in the limit of vanishing \(|\Theta p|\), and for arbitrary but finite \(|\Theta p|\), it is given by

$$\mathcal{A}_\lambda = - \frac{g^2}{16\pi^2} R(\lambda) F_{\mu\nu} \star F^{\mu\nu}. \quad (4.29)$$

Let us now summarize our results about the anomalies corresponding to \(U_A(1)\) and \(U_R(1)\) symmetries. As we have seen in the limit of vanishing \(|\Theta p|\), \(\delta_A \mathcal{L}\) is given by

$$\delta_A \mathcal{L} = 2N_f \alpha \left( - \frac{1}{32\pi^2} F_{\mu\nu} \star F^{\mu\nu} + \cdots \right), \quad (4.30)$$

and \(\delta_R \mathcal{L}\) by

$$\delta_R \mathcal{L} = 2N_f \alpha R(\psi) \left( - \frac{1}{32\pi^2} F_{\mu\nu} \star F^{\mu\nu} + \cdots \right). \quad (4.31)$$

In (4.30) and (4.31) the ellipses denote the higher order terms in the expansion of the open Wilson line. For an arbitrary but finite \(|\Theta p|\), however, \(\delta_A \mathcal{L}\) vanishes and \(\delta_R \mathcal{L}\) is given by

$$\delta_R \mathcal{L} = 2\alpha R(\lambda) \left( - \frac{1}{32\pi^2} F_{\mu\nu} \star F^{\mu\nu} \right). \quad (4.32)$$

The above results will be used in the next section to calculate the effective superpotential of the noncommutative supersymmetric theory. Before proceeding let us describe our strategy:

As was mentioned before, the full effective superpotential of the theory consists of two parts, the dynamical Veneziano-Yanckielowicz (VY) superpotential [1, 2], and the tree level
superpotential. We will determine the noncommutative VY superpotential, as in the ordinary
commutative case [2], using the differential equations assuring that the symmetries of the
original theory are also preserved in the effective theory. To do this, we use the anomalies
corresponding to \( U_A(1) \) and \( U_R(1) \) symmetries and will first determine the relevant degrees of
freedom for the effective theory, and then, using the corresponding selection rules, we arrive at
the VY effective superpotential.

As for the anomalies, we have seen that, due to UV/IR mixing, a singularity appears in the
limit of vanishing \( |\Theta_p| \). To begin, in section 4.2, we will first consider the case of vanishing
\( |\Theta_p| \). In this case the relevant degree of freedom of the effective theory, apart from the meson
field \( T \equiv \bar{Q} \ast Q \), is the gauge invariant Konishi anomaly \( S' \propto W_\alpha \ast' W^\alpha + \cdots \), which contains the invariant anomaly \( F_{\mu\nu} \ast' \tilde{F}_{\mu\nu} + \cdots \) as a component. Here the effective superpotential will
be determined using the selection rules from the corresponding invariant anomalies, and from
the coefficient of the one-loop \( \beta \)-function in \( |p| \ll \frac{1}{\sqrt{\Theta}} \) limit i.e. \( b_{s,N_f} \) from (4.16). This program
ends up with an effective superpotential for the gauge invariant superfield \( S' \) and the meson
field \( T \). We will integrate out each of these superfields to find first the effective superpotential
for \( S' \) and then the Affleck-Dine-Seiberg (ADS) effective superpotential [26, 3, 24] for \( T \).

We then continue to determine the effective superpotential for arbitrary but finite \( |\Theta_p| \)
in section 4.3. According to the results for the anomalies corresponding to \( U_A(1) \) and \( U_R(1) \)
symmetries, the relevant degrees of freedom are the meson superfield \( T \), and the gauge covariant
Konishi anomaly \( S \propto W_\alpha \ast W^\alpha \), which contains the gauge covariant anomaly \( F_{\mu\nu} \ast \tilde{F}_{\mu\nu} \) as a
component. Using these anomalies and the coefficient of the one-loop \( \beta \)-function, \( b_{c,N_f} \) from
(4.16), in \( |p| \gg \frac{1}{\sqrt{\Theta}} \) limit, we will determine the effective superpotential for the gauge covariant
superfield \( S \) and the meson field \( T \). Again the effective superpotentials will be determined for
\( S \) and \( T \) separately.

4.2 Case 1: Effective superpotential for \( S' \)

Here we consider the limit of vanishing \( |\Theta_p| \). According to our results from the anomalies
corresponding to \( U_A(1) \) and \( U_R(1) \) symmetries in this limit, (4.30) and (4.31), the relevant
degrees of freedom in this case are the invariant (nonplanar) Konishi anomaly

\[
S' \equiv -S_{\text{nonplanar}}^{\text{inv.}} = +\frac{1}{32\pi^2} W_\alpha \ast' W^\alpha + \cdots ,
\]
with the extra terms denoting the corrections arising from the Wilson line attachment, and the meson field \( T_{ij} \equiv \bar{Q}_i \star Q_j \). The effective action is therefore given by

\[
I_{\text{eff}} = \int d^2 \theta \, d^4 \chi \, W_{\text{dyn}}(T, S').
\] (4.33)

As indicated in section 2, we will find \( W_{\text{dyn}} \) in terms of \( S' \) without necessarily committing ourselves to the particular form above. Varying this effective action with respect to the \( U_A(1) \) and \( U_R(1) \) transformations and using the results from (4.30) and (4.31) with \( R(\psi) = -1/2 \), we arrive at two differential equations determining \( W_{\text{dyn}} \) uniquely

\[
T \frac{\partial W_{\text{dyn}}}{\partial T} = -N_f S',
\] (4.34)

and

\[
-W_{\text{dyn}} + S' \frac{\partial W_{\text{dyn}}}{\partial S'} + \frac{2}{3} T \frac{\partial W_{\text{dyn}}}{\partial T} = + \frac{N_f}{3} S'.
\] (4.35)

Putting the first equation (4.34) in (4.35), it is given by

\[
-W_{\text{dyn}} + S' \frac{\partial W_{\text{dyn}}}{\partial S'} - N_f S' = 0.
\] (4.36)

The solution to the differential equations (4.34) and (4.36) reads therefore

\[
W_{\text{dyn}}(T, S'; m, \lambda; \Lambda_{N_f}, \Lambda_\Theta) = S' \left( \log \left( \frac{S^{\alpha + N_f}}{\Lambda_\Theta^{\alpha} \Lambda_{N_f}^{\kappa} \det T} \right) - N_f \right).
\] (4.37)

To determine the exponent \( \alpha \) and \( \kappa \) of the holomorphic scale \( \Lambda_{N_f} \) and the new mass scale \( \Lambda_\Theta \equiv \frac{1}{\sqrt{\Theta}} \), we use the selection rules as in the ordinary commutative SQCD [24, 25]. Requiring that the effective superpotential is invariant under \( U_R(1) \) and \( U_A(1) \) transformations, and using the results from (4.30) and (4.31), as well as the coefficient of the one-loop \( \beta \)-function in the \( |p| \ll \Lambda_\Theta \) limit, \( \text{i.e.} \ b_{s,N_f} \) from (4.16), the axial and \( R \)-charges and the mass dimension of the quantities appearing in the dynamical superpotential are determined. Table 1 summarizes these results.

The values \( \kappa = -b_{s,N_f} \) and \( \alpha = N_f + b_{s,N_f} \) with \( b_{s,N_f} = -2(3 + N_f) \) guarantee that the argument of logarithm in (4.37) is dimensionless and has vanishing \( R \)- and axial charges. The dynamical superpotential for the superfields \( S' \) and \( T \) reads therefore

\[
W_{\text{dyn}}(T, S'; \Lambda_{N_f}, \Lambda_\Theta) = +S' \left( \log \left( \frac{S^{N_f} \Lambda_\Theta^{N_f+6} \Lambda_{N_f}^{2(3+N_f)} \det T} {\det T} \right) - N_f \right).
\] (4.38)
Adding, as in the ordinary case, the tree level superpotential

$$W_{\text{tree}} = m \, \text{tr} \, T + \lambda \, \text{tr} \, T \star T,$$

(4.39)

with $\star$-products replacing the ordinary products, to the dynamical superpotential (4.38), the full effective superpotential reads

$$W_{\text{eff.}}(T, S'; m, \lambda; \Lambda_{N_f}, \Lambda_\Theta) =$$

$$= m \, \text{tr} \, T + \lambda \, \text{tr} \, T^2 + S' \left( \log \left( \frac{S'^{2(N_f) \Lambda_\Theta^{+(N_f+6)}}}{\Lambda_{N_f}^{+2(3+N_f) \det T}} \right) - N_f \right).$$

(4.40)

Integrating out $S'$ the Affleck-Dine-Seiberg (ADS) superpotential of the theory is given by

$$W_{\text{eff.}}^{\text{ADS}} = m \, \text{tr} \, T + \lambda \, \text{tr} \, T^2 - N_f \left( \frac{\Lambda_{N_f}^{-2(3+N_f) \Lambda_\Theta^{+(N_f+6)}}}{\det T} \right)^{-\frac{1}{N_f}}.$$ 

(4.41)

Further as in the ordinary case, integrating the meson field $T$ from (4.40) leads to the VY superpotential for the pure gauge theory as a function of $S'$. Doing this, we arrive first at the noncommutative Konishi equation

$$m \, < T_i > + 2\lambda \, < T_i^2 > = S',$$

(4.42)

where $T_i's, i = 1, \cdots N_f$ are the diagonal elements of the meson matrix. This equation can be solved using the factorization $< T_i^2 >= < T_i >^2$ as in the ordinary commutative SQCD theory [40], to yield the noncommutative meson field $< \text{tr} \, T >$ as a function of $S'$. Choosing $N_f^+$ eigenvalues to reduce to $T = 0$ and $N_f^-$ eigenvalues to $T = -m/2\lambda$, the classical vacua of the theory, we arrive at [40, 41]

$$< \text{tr} \, T > = N_f^+ \left( -\frac{m}{4\lambda} + \frac{m}{4\lambda} \sqrt{1 + \frac{8\lambda S'}{m^2}} \right) + N_f^- \left( -\frac{m}{4\lambda} - \frac{m}{4\lambda} \sqrt{1 + \frac{8\lambda S'}{m^2}} \right),$$

(4.43a)
and
\[<\text{tr } T^2> = N_f^+ \left( -\frac{m}{4\lambda} + \frac{m}{4\lambda} \sqrt{1 + \frac{8\lambda S'}{m^2}} \right)^2 + N_f^- \left( -\frac{m}{4\lambda} - \frac{m}{4\lambda} \sqrt{1 + \frac{8\lambda S'}{m^2}} \right)^2. \tag{4.43b}\]

Now we have to match the RG-invariant scale \( \Lambda_{N_f} \) and \( \Lambda_\Theta \) defined for a theory with \( N_f \) massless flavors, to a new combination for a theory without massless flavors, i.e. a pure gauge theory. To do this we begin by adding the mass term \( m \text{ tr } T \) to the dynamical ADS superpotential of a theory with \( N_f \) massless flavors
\[W_{\text{eff}} = m \text{ tr } T - N_f \left( \frac{\Lambda_{N_f}^{-2(3+N_f)} \Lambda_\Theta^{+(N_f+6)}}{\det T} \right)^{-\frac{1}{N_f}}. \tag{4.44}\]

Integrating out only \( N_f - n_f \) massive flavors from \( N_f \) flavors, we arrive at the superpotential for a theory with \( n_f \) massless flavors
\[W_{\text{eff}} = -n_f \left( \frac{m^{(N_f-n_f)} \Lambda_{N_f}^{-2(3+N_f)} \Lambda_\Theta^{+(n_f+6)}}{\det \hat{T}} \right)^{-\frac{1}{n_f}}. \tag{4.45}\]

Here \( \hat{T} \) is the \( n_f \times n_f \) meson matrix built from \( n_f \) massless flavors. Comparing now this superpotential with the dynamical ADS superpotential with \( n_f \) massless flavors
\[W_{\text{eff}} = -n_f \left( \frac{\hat{\Lambda}_{n_f}^{-2(n_f+3)} \hat{\Lambda}_\Theta^{+(n_f+6)}}{\det \hat{T}} \right)^{-\frac{1}{n_f}}, \tag{4.46}\]
we arrive at the following consistent scale matching
\[\hat{\Lambda}_{n_f}^{-2(n_f+3)} \hat{\Lambda}_\Theta^{+(n_f+6)} = m^{N_f-n_f} \Lambda_{N_f}^{-2(N_f+3)} \Lambda_\Theta^{+(N_f+6)}. \tag{4.47}\]

This matching equation can be now used to define the scale for a pure gauge theory, indicated by \( n_f = 0 \)
\[\frac{\hat{\Lambda}_\Theta}{\Lambda_0} = \left( m^{N_f} \Lambda_{N_f}^{-2N_f} \right)^{\frac{1}{2}} \frac{\Lambda_\Theta}{\Lambda_{N_f}}. \tag{4.48}\]

Inserting (4.43a-b) and (4.48) in the effective superpotential (4.40), we arrive at the effective superpotential of noncommutative \( \mathcal{N} = 1 \) supersymmetric \( U(1) \) gauge theory with all \( N_f \) massive flavors integrated out
\[W_{\text{eff}}(S'; m, \lambda; \hat{\Lambda}_0, \hat{\Lambda}_\Theta) = +6S' \log \frac{\hat{\Lambda}_\Theta}{\Lambda_0} - \frac{N_f}{2} \frac{m^2}{8\lambda} - N_f \frac{m^2}{8\lambda} \sqrt{1 + \frac{28\lambda S'}{m^2}} + (N_f^+ - N_f^-) \frac{m^2}{8\lambda} \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{8\lambda S'}{m^2}} \right)^{N_f^+} \times \left( \frac{1}{2} - \frac{1}{2} \sqrt{1 + \frac{8\lambda S'}{m^2}} \right)^{N_f^-} + S' \log \left[ \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{8\lambda S'}{m^2}} \right)^{N_f^+} \left( \frac{1}{2} - \frac{1}{2} \sqrt{1 + \frac{8\lambda S'}{m^2}} \right)^{N_f^-} \right]. \tag{4.49}\]
This result can be compared with the effective superpotential of ordinary commutative $N_c$ colors SQCD which depends on the commutative gaugino bilinear $S \propto \text{tr}(W_\alpha W^\alpha)$ [40]

$$W_{\text{eff.}} = - N_c S \left( \log \frac{S}{\Lambda^3_0} - 1 \right) - \frac{N_f}{2} S - N_f \frac{m^2}{8\lambda} + (N_f^+ - N_f^-) \frac{m^2}{8\lambda} \sqrt{1 + \frac{8\lambda S}{m^2}}$$

$$+ S \log \left[ \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{8\lambda S}{m^2}} \right)^{N_f^+} \left( \frac{1}{2} - \frac{1}{2} \sqrt{1 + \frac{8\lambda S}{m^2}} \right)^{N_f^-} \right].$$

(4.50)

Although an apparent similarity between these two superpotentials exists, the noncommutative superpotential (4.49) is a nontrivial function of gauge field supermultiplet appearing in the Wilson line attachment of $S'$.

### 4.3 Case 2: Effective superpotential for $S$

Here we consider the case of arbitrary but finite $|\Theta p|$. The relevant degrees of freedom in this case are the covariant (planar) Konishi anomaly

$$S \equiv -S_{\text{planar}}^{\text{cov.}} = + \frac{1}{32\pi^2} W_\alpha \star W^\alpha,$$

and the meson field $T_{ij} \equiv \tilde{Q}_i \star Q_j$. The effective action can therefore be given by

$$I_{\text{eff.}} = \int d^2 \theta \ d^4 x \ W_{\text{dyn.}}(T, S).$$

(4.51)

Using the corresponding results $\delta A L = 0$ and $\delta R L$ from (4.32) with $R(\lambda) = 3/2$, we arrive at two differential equations determining $W_{\text{dyn.}}$ uniquely,

$$T \frac{\partial W_{\text{dyn.}}}{\partial T} = 0,$$

(4.52)

and

$$-W_{\text{dyn.}} + S \frac{\partial W_{\text{dyn.}}}{\partial S} + \frac{2}{3} T \frac{\partial W_{\text{dyn.}}}{\partial T} = -S.$$

(4.53)

Plugging the first equation (4.52) in (4.53), the second equation reads

$$-W_{\text{dyn.}} + S \frac{\partial W_{\text{dyn.}}}{\partial S} + S = 0,$$

(4.54)

which can be solved to yield

$$W_{\text{dyn.}}(T, S; \Lambda_{N_f}, \Lambda_{\Theta}) = S \left( \log \left( \frac{S^{-1}}{\Lambda_{\Theta}^N \Lambda_{N_f}} \right) + 1 \right).$$

(4.55)
To determine the two exponent $\alpha$ and $\kappa$, we require the invariance of the effective superpotential with respect to $U_A(1)$ and $U_R(1)$ transformations, and use the coefficient of the one-loop $\beta$-function in the $|p| \gg \Lambda_\Theta$ limit, \textit{i.e.} $b_{\ell,N_f}$ from (4.16). This is in fact admissible since in this case, $|\Theta p|$ is arbitrary but finite. The R-charge and the axial charge of the quantities appearing in the superpotential are therefore determined. Table 2 summarizes these results.

|                | $U_R(1)$-charge | $U_A(1)$-charge | $m$-dim |
|----------------|-----------------|-----------------|---------|
| $\det T$      | 3$N_f$          | 2$N_f$          | 2$N_f$  |
| $\Lambda_\Theta$ | 0               | 0               | +1      |
| $(\Lambda_{N_f})^{b_{\ell,N_f}}$ | 3               | 0               | $b_{\ell,N_f}$ |
| $S$            | 3               | 0               | 3       |
| $W_{dyn.}$     | 3               | 0               | 3       |

Table 2: $U_R(1)$, $U_A(1)$ and mass dimension for case 2.

The exponent $\alpha$ and $\kappa$ in the dynamical superpotential can be determined correspondingly. They are given by $\kappa = -b_{\ell,N_f} = -2(3 - N_f)$ from the coefficient of the one-loop $\beta$-function in the $|p| \gg \Lambda_\Theta$ limit, and $\alpha = b_{\ell,N_f} - 3 = 3 - 2N_f$. The dynamical superpotential as a function of $S$ reads therefore

$$W_{dyn.}(S; \Lambda_{N_f}, \Lambda_\Theta) = S \left( \log \left( \frac{S^{-1} \Lambda_{N_f}^{2N_f-3}}{\Lambda_{N_f}^{2(3-N_f)}} \right) + 1 \right). \quad (4.56)$$

The full effective superpotential in this case is given by taking (4.39) and adding it to the noncommutative dynamical superpotential (4.56)

$$W_{eff.}(T, S; m, \lambda; \Lambda_{N_f}, \Lambda_\Theta) = m \tr T + \lambda \tr T^2 - S \left( \log \left( \frac{S \Lambda_{N_f}^{3-2N_f}}{\Lambda_{N_f}^{2(3-N_f)}} \right) - 1 \right). \quad (4.57)$$

Integrating out the $S$ field, we arrive at the ADS effective superpotential

$$W_{eff.}^{ADS} = m \tr T + \lambda \tr T^2 + \Lambda_{N_f}^{6-2N_f} \Lambda_{\Theta}^{2N_f-3}. \quad (4.58)$$

Further integrating out the meson field $T$ from the full superpotential (4.57), we arrive first at

$$m + 2\lambda (T)_i = 0, \quad (4.59)$$
which can be solved to yield the noncommutative meson field \( < \text{tr} \, T > = -\frac{N_f m}{2\lambda} \). Replacing these results back in the full effective superpotential (4.57) and noting that \( < \text{tr} \, T^2 > = \frac{N_f m^2}{4\lambda^2} \), we have

\[
W_{\text{eff.}}(< \text{tr} \, T >) = -\frac{N_f m^2}{4\lambda} - S \left( \log \frac{S \Lambda_0^3}{\Lambda_0^6} - 1 \right),
\]

(4.60)

where the scale matching \( \frac{\Lambda_0^6}{\Lambda_0^6} = \frac{\Lambda_0^{6-2N_f}}{\Lambda_0^{1-2N_f}} \), is used.

5 Discussion

In the first part of this work, because of the significant role that subtleties of the nonplanar contributions to the anomaly play in the effective action, planar and nonplanar anomalies of noncommutative \( U(1) \) gauge theory with matter fields in the fundamental representation are calculated anew using Fujikawa’s path integral method [15, 16].

In the second part of this work, the Konishi anomalies of noncommutative \( \mathcal{N} = 1 \) supersymmetric \( U(1) \) gauge theory, with bosonic noncommutativity are calculated. As in the non-supersymmetric case, the invariant (nonplanar) Konishi anomaly involves a supersymmetric noncommutative open Wilson line with a length proportional to the noncommutative length scale \( \sqrt{\Theta} \) and exhibits therefore the same UV/IR mixing.

Finally, in the last part of this work, the exact effective superpotential for noncommutative \( \mathcal{N} = 1 \) supersymmetric \( U(1) \) gauge theory with \( N_f \) flavors in the fundamental and \( N_f \) in antifundamental representation is calculated in terms of the relevant degrees of freedom, the meson field \( T_{ij} \equiv \tilde{Q}_i \star Q_j \) and the superfields \( S \propto W_\alpha \star W^\alpha \) and \( S' \propto W_\alpha \star' W^\alpha + \cdots \), where the extra terms are the contributions from the expansion of Wilson line.

The VY superpotential for the superfields \( S \) and \( S' \) as well as the ADS superpotential for the meson fields \( T \) are affected by UV/IR mixing which has its origins both in their dependence on the planar and nonplanar anomalies and in their \( \beta \)-function dependence. We note that, in contrast to the ordinary SQCD, the contribution of the gauginos in the adjoint representation to the \( U_R(1) \) anomaly appears only for arbitrary but finite noncommutative momenta, whereas the chiral fermions in the fundamental representation contribute only in the limit of vanishing \( |\Theta p| \) to the \( U_R(1) \) anomaly.

A novel feature of the present work is the appearance of new scale \( \Lambda_\Theta \equiv \frac{1}{\sqrt{\Theta}} \) in the superpotential. In particular the noncommutative version of Affleck-Dine-Seiberg superpotential
of the meson fields depends on this noncommutative mass scale. The noncommutative scale appears also elsewhere in the literature [22, 42]. We must emphasize that although the dependence of the effective action on the superfield $S'$ and $S$ is familiar and, up to some numerical factors, similar to the ordinary $N_c$ colors SQCD [40], the dependence of $S'$ on the gauge field supermultiplet arising from the Wilson line is highly nontrivial.

6 Acknowledgment

We would like to thank M. Alishahiha for collaboration at the early stages of this work. We are also grateful to H. Arfaei and S. Parvizi for discussions. F. A. is indebted to R. Banerjee for discussions.

References

[1] G. Veneziano, and S. Yankielowicz, An Effective Lagrangian for the pure $\mathcal{N} = 1$ Supersymmetric Yang-Mills Theory, Phys. Lett. B113 (1982) 231.

[2] T. R. Taylor, G. Veneziano, and S. Yankielowicz, Supersymmetric QCD and its Massless Limit: An Effective Lagrangian Analysis, Nucl. Phys. B218 (1983) 493.

[3] K. A. Intriligator, R. G. Leigh, and N. Seiberg, Exact Superpotentials in Four-Dimensions, Phys. Rev. D50 (1994) 1092, arXiv: hep-th/9403198.

[4] M. E. Peskin, Duality in Supersymmetric Yang-Mills Theory, in *Boulder 1996, Fields, strings and duality*, 729, arXiv: hep-th/9702094.

[5] R. Dijkgraaf, and C. Vafa, A Perturbative Window into Nonperturbative Physics, arXiv: hep-th/0208048.

R. Dijkgraaf, and C. Vafa, On Geometry and Matrix Models, Nucl. Phys. B644 (2002) 21, arXiv: hep-th/0207106.

R. Dijkgraaf, and C. Vafa, Matrix Models, Topological Strings, and Supersymmetric Gauge Theories, Nucl. Phys. B644 (2002) 3, arXiv: hep-th/0206255.

[6] F. Cachazo, M. R. Douglas, N. Seiberg, and E. Witten, Chiral Rings and Anomalies in Supersymmetric Gauge Theory, JHEP 0212 (2002) 071, arXiv: hep-th/0211170.
[7] H. Ooguri, and C. Vafa, *The C Deformation of Gluino and Nonplanar Diagrams*, Adv. Theor. Math. Phys. 7 (2003) 53, arXiv: hep-th/0302109.

H. Ooguri, and C. Vafa, *Gravity Induced C Deformation*, Adv. Theor. Math. Phys. 7 (2004) 405, arXiv: hep-th/0303063.

[8] R. Casalbuoni, *On the Quantization of Systems with Anticommutating Variables*, Nuovo Cim. A 33 (1976) 115.

R. Casalbuoni, *The Classical Mechanics for Bose-Fermi Systems*, Nuovo Cim. A 33 (1976) 389.

S. Ferrara, and M. A. Lledo, *Some Aspects of Deformations of Supersymmetric Field Theories*, JHEP 0005 (2000) 008, arXiv: hep-th/0002084.

D. Klemm, S. Penati, and L. Tamassia, *Non(anti)commutative Superspace*, Class. Quant. Grav. 20 (2003) 2905, arXiv: hep-th/0104190.

R. Abbaspur, *Generalized Noncommutative Supersymmetry from a New Gauge Symmetry*, arXiv: hep-th/0206170.

J. de Boer, P. A. Grassi, and P. Nieuwenhuizen, *Non-Commutative Superspace from String Theory*, Phys. Lett. B574 (2003) 98, arXiv: hep-th/0302078.

[9] N. Seiberg, *Noncommutative Superspace, \( \mathcal{N} = 1/2 \) Supersymmetry, Field Theory and String Theory*, JHEP 0306 (2003) 010, arXiv: hep-th/0305248.

[10] N. Berkovits, and N. Seiberg, *Superstrings in Graviphoton Background and \( \mathcal{N} = 1/2 + 3/2 \) Supersymmetry*, JHEP 0307 (2003) 010, arXiv: hep-th/0306226.

R. Britto, B. Feng, and S. J. Rey, *Deformed Superspace, \( \mathcal{N} = 1/2 \) Supersymmetry and (Non)Renormalization Theorems*, JHEP 0307 (2003) 067, arXiv: hep-th/0306215.

S. Terashima, and J. T. Yee, *Comments on Noncommutative Superspace*, JHEP 0312 (2003) 053, arXiv: hep-th/0306237.

T. Araki, K. Ito, A. Ohtsuka, *Supersymmetric Gauge Theories on Noncommutative Superspace*, Phys. Lett. B573 (2003) 209, arXiv: hep-th/0307076.
M. T. Grisaru, S. Penati, and A. Romagnoni, Two-loop Renormalization for Nonanti-commutative $\mathcal{N}=1/2$ Supersymmetric WZ Model, JHEP 0308 (2003) 003, arXiv: hep-th/0307099.

[11] R. Britto, B. Feng, and S.-J. Rey, Non(anti)commutative Superspace, UV/IR Mixing and Open Wilson Lines, JHEP 0308 (2003) 001, arXiv: hep-th/0307091.

[12] Sh. Minwalla, M. Van Raamsdonk, and N. Seiberg, Noncommutative Perturbative Dynamics, JHEP 0002 (2000) 020, arXiv: hep-th/9912072.

[13] N. Seiberg, and E. Witten, String Theory and Noncommutative Geometry, JHEP 9909 (1999) 032, arXiv: hep-th/9908142.

M. R. Douglas, and N. A. Nekrasov, Noncommutative Field Theory, Rev. Mod. Phys. 73 (2002) 977, arXiv: hep-th/0106048.

[14] H. Kawai, T. Kuroki, and T. Morita, Dijkgraaf-Vafa Theory as Large-N Reduction, Nucl. Phys. B664 (2003) 185, arXiv: hep-th/0303210.

[15] F. Ardalan, and N. Sadooghi, Axial Anomaly in Noncommutative $R^4$, Int. J. Mod. Phys. A16 (2001) 3151, arXiv: hep-th/0002143.

[16] F. Ardalan, and N. Sadooghi, Anomaly and Nonplanar Diagrams in Noncommutative Gauge Theories, Int. J. Mod. Phys. A17 (2002) 123, arXiv: hep-th/0009233.

[17] J. M. Garcia-Bondia, and C. P. Martin, Chiral Gauge Anomalies on Noncommutative $R^4$, Phys. Lett. B479 (2000) 321, arXiv: hep-th/0002171.

L. Bonora, M. Schnabl, and A. Tomasiello, A Note on Consistent Anomalies in Noncommutative YM Theories, Phys. Lett. B485 (2000) 311, arXiv: hep-th/0002210.

L. Bonora, and A. Sorin, Chiral Anomalies in Noncommutative YM Gauge Theories, Phys. Lett. B521 (2001) 421, arXiv: hep-th/0109204.

C. P. Martin, The Covariant Form of the Gauge Anomaly on Noncommutative $R^{2N}$, Nucl. Phys. B623 (2002) 150, arXiv: hep-th/0110046.

R. Banerjee, and S. Ghosh, Seiberg-Witten Map and the Axial Anomaly in Noncommutative Field theory, Phys. Lett. B533 (2002) 162, arXiv: hep-th/0110177.
[18] M. T. Grisaru, and S. Penati, *Noncommutative Supersymmetric Gauge Anomaly*, Phys. Lett. **B504** (2001) 89, arXiv: hep-th/0010177.

[19] K. Fujikawa, *Path Integral for Gauge Theories with Fermions*, Phys. Rev. **D21** (1980) 2848, Erratum-ibid. **D22** (1980) 1499.

[20] K.-i. Konishi, and K.-i. Shizuya, *Functional Integral Approach to Chiral Anomalies in Supersymmetric Gauge Theories*, Nuovo Cim. **A90** (1985) 111.

D. Amati, K. Konishi, Y. Meurice, G. C. Rossi, and G. Veneziano, *Nonperturbative Aspects in Supersymmetric Gauge Theories*, Phys. Rept. **162** (1988) 169.

K. Konishi, *Anomalous Supersymmetry Transformation of some Composite Operators in SQCD*, Phys. Lett. **B135** (1984) 439.

[21] S. Marculescu, and L. Mezincescu, *Phase-Factors and Point-Splitting in Supersymmetry*, Nucl. Phys. **B181** (1981) 127.

R. L. Mkrytchyan, *Dynamics of Loops in Superspace*, Nucl. Phys. **B198** (1982) 295.

M. Awada, and F. Mansouri, *Supersymmetric Wilson Loops, Superstring-like Observables, and the Natural Coupling of Superstring to Supersymmetric Gauge Theories*, Phys. Lett. **B384** (1996) 111, arXiv: hep-th/9512098.

[22] V. V. Khoze, and G. Travaglini, *Wilsonian Effective Actions and the IR/UV Mixing in Noncommutative Gauge Theories*, JHEP **0101** (2001) 026, arXiv: hep-th/0011218.

[23] L. Alvarez-Gaumé, and M. A. Vazquez-Mozo, *General Properties of Noncommutative Field Theories*, Nucl. Phys. **B668** (2003) 293, arXiv: hep-th/0305093.

[24] K. Intriligator, and N. Seiberg, *Lectures on Supersymmetric Gauge Theories and Electric-Magnetic Duality*, Nucl. Phys. Proc. Suppl. **45BC** (1996) 1, in Trieste HEP Cosmology (1995) 25, in Boulder TASI 95 (1995) 223, in *Erice 1996, Effective theories and fundamental interactions* 237, arXiv: hep-th/9509066.

[25] J. Terning, *TASI 2002 Lectures; Nonperturbative Supersymmetry*, arXiv: hep-th/0306119.

[26] I. Affleck, M. Dine, and N. Seiberg, *Dynamical Supersymmetry Breaking in Supersymmetric QCD*, Nucl. Phys. **B241** (1984) 493.
S. L. Adler, *Tests of the Conserved Vector Currents and Partially Conserved Axial-Vector Current Hypotheses in High-Energy Neutrino Reactions*, Phys. Rev. **135** (1964) B963.

S. L. Adler, *Consistency Conditions on the Strong Interactions Implied by a Partially Conserved Axial Vector Current*, Phys. Rev. **137** (1965) B1022.

J.S. Bell, and R. Jackiw, *A PCAC Puzzle π0 → γγ in the Sigma Model*, Nuovo Cim. A **60** (1969) 47.

S.-J. Rey, and R. von Unge, *S Duality, Noncritical Open String and Noncommutative Gauge Theory*, Phys. Lett. **B499** (2001) 215, arXiv: hep-th/0007089.

D. J. Gross, A. Hashimoto, and N. Itzhaki, *Observables of Noncommutative Gauge Theories*, Adv. Theor. Math. Phys. **4** (2000) 893, arXiv: hep-th/0008075.

S. R. Das, and S.-J. Rey, *Open Wilson Lines in Noncommutative Gauge Theory and Tomography of Holographic Dual Supergravity*, Nucl. Phys. **B590** (2000) 453, arXiv: hep-th/0008042.

M. R. Garousi, *Noncommutative World-Volume Interactions on D-Brane and Dirac-Born-Infeld Action*, Nucl. Phys. **B579** (2000) 279, arXiv: hep-th/9909214; *Transformation of the Dirac-Born-Infeld Action under the Seiberg-Witten Map*, Nucl. Phys. **B602** (2001) 527, arXiv: hep-th/0011147; *Dirac-Born-Infeld Action, Seiberg-Witten Map and Wilson Lines*, Nucl. Phys. **B611** (2001) 467, arXiv: hep-th/0105139.

H. Liu, J. Michelson, *-*Trek: The One-Loop N=4 Noncommutative SYM Action*, Nucl. Phys. **B614** (2001) 279, arXiv: hep-th/0008205; *Ramond-Ramond Couplings of Noncommutative D-Branes*, Phys. Lett. **B518** (2001) 143, arXiv: hep-th/0104139; *Supergravity Couplings of Noncommutative D-Branes*, Nucl. Phys. **B615** (2001) 169, arXiv: hep-th/0101016.

D. Zanon, *Noncommutative Perturbation in Superspace*, Phys. Lett. **B504** (2001) 101, arXiv: hep-th/0009196; *Noncommutative N = 1, N = 2 Super U(N) Yang-Mills: UV/IR Mixing and Effective Action Results at One Loop*, Phys. Lett. **B502** (2001) 265, arXiv: hep-th/0012009.

T. Mehen, and M. B. Wise, *Generalized *-Product, Wilson-Lines and the Solution of the Seiberg-Witten Equations*, JHEP **0012** (2000) 008, arXiv: hep-th/0010204.
A. Santambrogio, and D. Zanon, *One-Loop Four-Point Function in Noncommutative \( \mathcal{N} = 4 \) Yang-Mills Theory*, JHEP **0101** (2001) 024, arXiv: hep-th/0010275.

S. R. Das, and S. P. Trivedi, *To Noncommutative Branes, Open Wilson Lines and Generalized Star Products*, JHEP **0102** (2001) 046, arXiv: hep-th/0011131.

M. Percini, A. Santambrogio, and D. Zanon, *The One-Loop Effective Action of Noncommutative \( \mathcal{N} = 4 \) Super Yang-Mills is Gauge Invariant*, Phys. Lett. **B504** (2001) 131, arXiv: hep-th/0011140.

Y. Okawa, and H. Ooguri, *An Exact Solution to Seiberg-Witten Equation of Noncommutative Gauge Theory*, Phys. Rev. **D64** (2001) 046009, arXiv: hep-th/0104036.

S. Mukhi, and N. V. Suryanarayana, *Couplings of Noncommutative Branes to Ramond-Ramond Backgrounds*, JHEP **0105** (2001) 023, arXiv: hep-th/0104045.

S. R. Das, S. Mukhi, and N. V. Suryanarayana, *Derivative Corrections from Noncommutativity*, JHEP **0108** (2001) 039, arXiv: hep-th/0106024.

Y. Kiem, S.-J. Rey, H.-T. Sato, and J.-T. Yee, *Open Wilson Lines and Generalized Star Product in Noncommutative Scalar Field Theories*, Phys. Rev. **D65** (2002) 026002, arXiv: hep-th/0106121; *Anatomy of One-Loop Effective Action in Noncommutative Scalar Field Theories*, Eur. Phys. J. **C22** (2002) 757, arXiv: hep-th/0107106; Y. Kiem, S.-S. Kim, S.-J. Rey, and H.-T. Sato, *Anatomy of Two-Loop Effective Action in Noncommutative Field Theories*, Nucl. Phys. **B641** (2002) 256, arXiv: hep-th/0110066; *Interacting Open Wilson Lines in Noncommutative Field Theories*, Phys. Rev. **D65** (2002) 046003, arXiv: hep-th/0110215.

K. Kaminsky, Y. Okawa, and H. Ooguri, *Quantum Aspects of Seiberg-Witten Map in Noncommutative Chern-Simons Theory*, Nucl. Phys. **B663** (2003) 33, arXiv: hep-th/0301133.

S.-J. Rey, *Exact Answers to Approximate Questions: Noncommutative Dipole, Open Wilson Lines and UV-IR Duality*, in *Les Houches 2001, Gravity, gauge theories and strings* 587, arXiv: hep-th/0207108.

[29] H. Liu, *-*Trek II: \( *_n \) Operations, Open Wilson Lines and the Seiberg-Witten Map*, Nucl. Phys. **B614** (2001) 305, arXiv: hep-th/0011125.
[30] K. Intriligator, and J. Kumar, “*-Wars Episode 1: The Phantom Anomaly,” Nucl. Phys. B620 (2002) 315, arXiv: hep-th/0107199.

[31] A. Armoni, E. Lopez, and S. Theisen, Nonplanar Anomalies in Noncommutative Theories and the Green-Schwarz Mechanism, JHEP 0206 (2002) 050, arXiv: hep-th/0203165.

[32] R. Banerjee, Anomalies in Noncommutative Gauge Theories, Seiberg-Witten Transformation and Ramond-Ramond Couplings, Int. J. Mod. Phys. A19 (2004) 613, arXiv: hep-th/0301174.

[33] R. Banerjee, C.-K. Lee, and H. S. Yang, Seiberg-Witten Maps for Currents and Energy Momentum Tensors in Noncommutative Gauge Theories, Phys. Rev. D70 (2004) 065015, arXiv: hep-th/0312103.

[34] R. Banerjee, and K. Kumar, Maps for the Currents and Anomalies in Noncommutative Gauge Theories: Classical and Quantum Aspects, arXiv: hep-th/0404110.

[35] S. L. Adler, and W. A. Bardeen, Absence of Higher Order Corrections in the Anomalous Axial Vector Divergence Equation, Phys. Rev. 182 (1969) 1517.

[36] M. B. Green, and J. H. Schwarz, Anomaly Cancellation in Supersymmetric D=10 Gauge Theory and Superstring Theory, Phys. Lett. B149 (1984) 117.

[37] J. Wess, in Topics in quantum fields theories and gauge theories (Springer Verlag, Berlin, 1978), p. 81.

[38] N. Sadooghi, and M. Sorosh, Noncommutative Dipole QED, Int. J. Mod. Phys. A18 (2003) 97, arXiv: hep-th/0206009.

[39] T. Nakajima, Conformal Anomalies in Noncommutative Gauge Theories, Phys. Rev. D66 (2002) 085008, arXiv: hep-th/0108158.

[40] A. Brandhuber, H. Ita, H. Nieder, Y. Oz, and Ch. Römelsberger, Chiral Rings, Superpotentials and the Vacuum Structure of $\mathcal{N}=1$ Supersymmetric Gauge Theories, Adv. Theor. Math. Phys. 7 (2003) 269, arXiv: hep-th/0303001.

[41] B. M. Gripaios, and J. F. Wheater, Veneziano-Yankielowicz Superpotential in N=1 Susy Gauge Theories, Phys. Lett. B587 (2004) 150, arXiv: hep-th/0307176.
[42] C.-S. Chu, V. V. Khoze, and G. Travaglini, *Dynamical Breaking of Supersymmetry in Noncommutative Gauge Theories*, Phys. Lett. B513 (2001) 200, arXiv: hep-th/0105187.

[43] J. Levell, and G. Travaglini, *Effective Actions, Wilson Lines and the IR/UV Mixing in Noncommutative Supersymmetric Gauge Theories*, JHEP 0403 (2004) 021, arXiv: hep-th/0308008.