Quantum field theory with classical sources—linearized quantum gravity

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Abstract
In a previous work (Skagerstam 2018 arXiv:1801.09947v1 [quant-ph]) and in terms of an exact quantum-mechanical framework, \(\hbar\)-independent causal and retarded expectation values of the second-quantized electro-magnetic fields in the Coulomb gauge were derived in the presence of a conserved classical electric current. The classical \(\hbar\)-independent Maxwell’s equations then naturally emerged. In the present work, we extend these considerations to linear gravitational quantum deviations around a flat Minkowski space-time in a Coulomb-like gauge. The emergence of the classical, causal, and properly retarded linearized classical theory of general relativity with a conserved classical energy–momentum tensor is then outlined. The quantum-mechanical framework also provides for a simple approach to classical quadrupole gravitational radiation of Einstein and microscopic spontaneous graviton emission and/or absorption processes.

Keywords: quantum gravity, classical sources, causality, radiation processes

1. Introduction
In electro-dynamics it is natural to introduce gauge-dependent scalar and vector potentials. These potentials do not have to be local in space and time. It can then be a rather delicate issue to verify that gauge-independent observables obey the physical constraint of causality and that they also are properly retarded. Attention to this issue is often discussed in a classical...
framework. For instance one then shows in what manner various choices of gauge give rise to the same electro-magnetic field strengths (for recent discussions see, e.g. [2–10]). In [5] and, in particular [11], related issues are discussed in the context of gravitational interactions.

In a previous publication [1] we have shown that the time-evolution, as dictated by quantum mechanics, for massless and spin-one second-quantized electro-magnetic fields in the presence of a classical conserved current, automatically solves these issues. In the Coulomb gauge made use of, the corresponding non-propagating constraint degree of freedom was explicitly eliminated in terms of the physical charge density. The classical theory of Maxwell then naturally emerges in terms of expectation values of the second-quantized electro-magnetic fields by imposing conservation of the classical electric current. This is in line with more general \( S \)-matrix arguments due to Weinberg [12]. We also showed that various exact \( \hbar \)-independent radiative processes like the classical Vavilov–Čerenkov radiation [13, 14] can be obtained in a straightforward manner [15–17].

In the present paper we extend the work of [1] to linear gravitational quantum deviations around a flat Minkowski space-time in a Coulomb-like gauge in the presence of a classical and conserved energy–momentum tensor. The time-evolution as dictated by quantum mechanics, for a suitably defined massless and spin-two second-quantized gravitational field of two physical and propagating degrees of freedom, then automatically solves the issues of causality and retardation in a similar manner as in the case of electro-dynamics. It will, however, be argued that this is only true for certain propagating physical degrees of freedom and not for all degrees of freedom of the gravitational field as is sometimes claimed in the literature (see, e.g. [5], and [18] for various historical remarks). The corresponding Coulomb-like degrees of freedom, like the Newton’s gravitational potential, are then neither retarded nor causal. In fact, they are shown to be instantaneous. As constraints, these degrees of freedom are explicitly solved for in terms of the components of the energy–momentum tensor. The dynamical equations can, as in the case of electro-dynamics [1], be reduced to a system of decoupled harmonic oscillators with space-time dependent external forces. No pre-defined global causal order is assumed other than the deterministic time-evolution as prescribed by the Schrödinger equation. The classical weak-field gravitational theory of Einstein then naturally emerges in terms of expectation values of the second-quantized gravitational field for any initial quantum state again in line with more general \( S \)-matrix arguments due to Weinberg [12] and by Boulware and Deser [19] by imposing conservation of the classical energy–momentum tensor.

The paper is organized as follows. In section 2 we recall, for reasons of completeness, the classical version of Einstein’s theory of weak gravitational fields in vacuum in the presence of a space-time dependent conserved energy–momentum tensor, and a proper set of propagating degrees of freedom is constructed. The quantized degrees of freedom around a flat Minkowski space-time in the presence of a conserved space-time dependent energy–momentum tensor is outlined in section 3. In this section we also make some comments on the construction of a conserved energy–momentum tensor for these quantized gravitational degrees of freedom and the Weinberg–Witten theorem [20]. In section 4 we consider the issues of causality and retardation in terms of quantum-mechanical averages of the second-quantized gravitational degrees of freedom and geodesic deviation. The emergence of the classical weak-field limit of Einstein’s general theory of relativity is then outlined. In section 5 we also discuss, in a quantum-mechanical framework, Einstein’s classical gravitational quadrupole radiation as well as the microscopic spontaneous graviton emission process from a quantum source in terms of an excited hydrogen-like atom. In section 6 finally, we present some conclusions and remarks.
In an appendix, we present some calculational techniques as made use of in obtaining decay rates for the emission of gravitons.

2. Weak classical gravitational fields

We consider classical weak deviations from the flat Minkowski space-time in terms of the metric tensor, i.e. we write

\[ g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu}, \]  

(2.1)

where \(|h_{\mu \nu}| \ll 1\) for all \(\mu\) and \(\nu\). We will make use of the conventions that Greek indices run from 1 to 4, with the space-time x-coordinate components \(x^\mu = (x^1, x^2, x^3, x^4) \equiv (x, x^4 = ct)\) as well as \(\partial_\mu = \partial / \partial x^\mu\), and the diagonal metric \(\eta_{\mu \nu}\) has the signature \((1, 1, 1, -1)\). The Minkowski metric \(\eta_{\mu \nu}\) is used to raise or lower space-time Greek indices. When convenient, we will make use of the notation \(f(x) \equiv f(x, t)\) for space-time dependent fields. Apart from using the index 4 instead of 0 for time components, we follow the conventions of [21].

In general, the fundamental classical Einstein field equation takes the form

\[ G_{\mu \nu} \equiv R_{\mu \nu} - \frac{1}{2} \eta_{\mu \nu} R = -\kappa T_{\mu \nu}, \]  

(2.2)

with

\[ \kappa = 8 \pi G / c^4. \]  

(2.3)

In the weak-field limit, the Riemann–Christoffel curvature tensor \(R^{(1)}_{\lambda \mu \nu \kappa}\) is approximated by

\[ R^{(1)}_{\lambda \mu \nu \kappa} = \frac{1}{2} \left( \frac{\partial^2 h_{\lambda \nu}}{\partial x^\alpha \partial x^\mu} - \frac{\partial^2 h_{\mu \nu}}{\partial x^\alpha \partial x^\lambda} + \frac{\partial^2 h_{\mu \kappa}}{\partial x^\alpha \partial x^\lambda} \right). \]  

(2.4)

In general, \(R_{\mu \nu} \equiv R^{(1)}_{\lambda \mu \nu \lambda}\) as well as \(R \equiv R^{(1)}_{\lambda \lambda}\). The Einstein tensor \(G_{\mu \nu}\) is therefore approximated by \(G_{\mu \nu}^{(1)}\) as defined by

\[ G_{\mu \nu}^{(1)} = \frac{1}{2} \left( \delta_\mu^\alpha \delta_\nu^\beta \nabla^2 + \eta^{\alpha \beta} \partial_\mu \partial_\nu - \delta_\mu^\alpha \partial_\beta \partial_\nu - \delta_\nu^\beta \partial_\alpha \partial_\mu + \eta_{\mu \nu} \left( \partial^\alpha \partial^\beta - \eta^{\alpha \beta} \Box \right) \right) h_{\alpha \beta}, \]  

(2.5)

with \(\Box \equiv \partial^\alpha \partial_\alpha = \nabla^2 - \partial^2 / \partial (ct)^2\). \(G_{\mu \nu}^{(1)}\) is then such that \(\partial_\mu G_{\mu \nu}^{(1)} = 0\) and therefore we must impose conservation of the energy–momentum tensor, i.e. \(\partial_\mu T_{\mu \nu} = 0\).

The Riemann–Christoffel curvature tensor equation (2.4) has a local gauge invariance, i.e. it is invariant under infinitesimal coordinate transformations

\[ x_\mu \rightarrow x_\mu' = x_\mu - \xi_\mu, \]  

(2.6)

where we only consider \(\xi_\mu\) and terms like \(\partial_\mu \xi_\mu\) to first order. This is so since equation (2.6) and the tensor properties of \(g_{\mu \nu}\) imply

\[ h_{\mu \nu} \rightarrow h'_{\mu \nu} = h_{\mu \nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \]  

(2.7)

It then follows that \(G_{\mu \nu}^{(1)}\) has the same invariance. For our purposes, we impose Lorentz-covariant equations of motion and regard (2.7) as a conventional local gauge-transformation of the gauge-field degrees of freedom \(h_{\mu \nu}\).

\[ \text{A more recent account by Weinberg (2008).} \]
It is now well-known that the components $G^{(1)}_{\mu\nu} = -\kappa T_{\mu\nu}$ of Einstein’s equation (2.2) in the weak-field limit, required to have conservation of the energy–momentum tensor, i.e. $\partial^\mu T_{\mu\nu} = 0$, contain at most time-derivatives of first order and can therefore be used as initial conditions for second-order equations of motion (also see, e.g. section 7 in [21] and for more recent discussions [22, 23]). We therefore consider the components $G^{(1)}_{\mu\nu} = -\kappa T_{\mu\nu}$ to be constraint equations similar to the constraint equation $\nabla \cdot E = -\nabla^2 \phi = \rho/\epsilon_0$ in electrodynamics in the Coulomb gauge (see, e.g. [1]). The components $G^{(1)}_{\mu\nu} = -\kappa T_{\mu\nu}$ can therefore be used to eliminate degrees of freedom from the ten degrees of freedom of $h_{\mu\nu}$ to be clarified in more detail below.

Instead of making use of an extension of the Helmholtz decomposition of vector fields (see, e.g. [1] for an elementary discussion and further references) to tensor fields, we find it convenient and more transparent to analyze the corresponding constraints by considering the Fourier transform of Einstein’s equation (2.2) in the weak-field limit. We therefore define

$$T_{\mu\nu}(k) = \int d^4x e^{-ik\cdot x} T_{\mu\nu}(x),$$

(2.8)

and

$$h_{\mu\nu}(k) = \int d^4x e^{-ik\cdot x} h_{\mu\nu}(x).$$

(2.9)

Equations (2.2) and (2.5) therefore lead to

$$\left(\delta_\mu^\alpha \delta_\nu^\beta k^2 + \eta^{\alpha\beta} k_\mu k_\nu - \delta_\mu^\alpha k_\nu k^\beta + \eta_{\mu\nu} (k^\alpha k^\beta - \eta^{\alpha\beta} k^2)\right) h_{\alpha\beta}(k) = 2\kappa T_{\mu\nu}(k),$$

(2.10)

where $k^2 \equiv k^2 - k_4^2$. The gauge-invariance of $G^{(1)}_{\mu\nu}$ under the transformation (2.7) now allows us to introduce four Coulomb-like or radiation gauge conditions $\partial_\mu h_{\mu\nu}(x) = 0$, i.e.

$$k_\mu h_{\mu\nu}(k) = 0,$$

(2.11)

where Latin indices run from 1 to 3. With the gauge choice equation (2.11), we find from equation (2.10) the constraint equations

$$k^2 h_{4\alpha}(k) + k_4 k_\mu h_{\mu\nu}(k) = 2\kappa T_{4\alpha}(k),$$

(2.12)

as well as

$$k^2 h_{4\alpha}(k) - k^2 h_0(k) = \kappa T_{\mu\nu}^{\mu\nu}(k).$$

(2.13)

The transversality gauge-condition (2.11), conservation of the energy–momentum tensor, i.e. $k_\mu T^{\mu\nu}(k) = 0$, as well as equation (2.12), then imply that $k^2 k_\mu h_{\mu\nu}(k) = 2\kappa k_\nu T_{\mu\nu}(k) = 2\kappa k_\nu T_{4\alpha}(k)$.

We therefore obtain

$$k^2 h_0(k) = 2\kappa T_{4\alpha}(k).$$

(2.14)

By making use of equation (2.14), we can therefore write equation (2.12) in the following form where the transversality of $h_{4\alpha}(k)$ is explicit:

$$k^2 h_{4\alpha}(k) = 2\kappa P_{\alpha\beta} T_{4\beta}(k).$$

(2.15)

Here $P_{\alpha\beta}$ is given by

$^8$ See footnote 7.
\begin{equation}
\mathcal{P}_{ij} \equiv \delta_{ij} - \hat{k}_i \hat{k}_j = \sum_{\lambda} \epsilon^*_{ij}(\mathbf{k}; \lambda) \epsilon_{ij}(\mathbf{k}; \lambda),
\end{equation}

expressed, for later purposes, in terms of polarization vectors \( \epsilon(\mathbf{k}; \lambda) \), with \( \lambda = \pm \) for complex-valued circular polarization and with \( \lambda = 1, 2 \) for real-valued linear polarization, obeying the transversality condition \( \mathbf{k} \cdot \epsilon(\mathbf{k}; \lambda) = 0 \), and where we have defined the unit vector \( \hat{k} \equiv \mathbf{k}/|\mathbf{k}| \). The circular polarization degrees of freedom obey the rule \( \epsilon(-\mathbf{k}; \pm) = \epsilon^*(\mathbf{k}; \pm) \). In the case of linear polarization the unit vectors \( \epsilon(\mathbf{k}; \lambda) \) are such that \( \epsilon(-\mathbf{k}; \lambda) = (-1)^{\lambda+1} \epsilon(\mathbf{k}; \lambda) \) and

\begin{equation}
\epsilon(\mathbf{k}; \pm) = \frac{1}{\sqrt{2}} \left( \epsilon(\mathbf{k}; 1) \pm i \epsilon(\mathbf{k}; 2) \right).
\end{equation}

Conservation of the energy–momentum tensor allows us, furthermore, to rewrite the constraint equation (2.13) in the form

\begin{equation}
\mathbf{k}^2 h_{44}(\mathbf{k}) = 2\kappa \left( P_{\mu \nu} T^{\mu \nu}(k) - \frac{1}{2} T^{\mu \mu}(k) \right).
\end{equation}

In the Newtonian limit, equation (2.18) reduces to \( \mathbf{k}^2 h_{44}(\mathbf{k}) = \kappa T_{44}(k) = 8\pi G\rho(k)/c^2 \) in terms of the mass density \( \rho(k) \), i.e. \( h_{44}(x) = -2\Phi(x)/c^2 \), as it should, where \( \Phi(x) \) is the classical Newtonian gravitational potential such that \( \nabla^2 \Phi(x) = 4\pi G \rho(x) \).

The components \( h_{44}(k) \) and \( h_{44}(k) \) are therefore fixed by the constraint equations (2.15) and (2.18). The spatial trace \( h_{44}(k) \) is, in addition, determined by these constraints according to equation (2.14). When transformed back to space-time coordinates these constraints will only involve spatial derivatives evaluated at equal time. The constraint equation \( G^{(1)}_{\mu \nu} = -\kappa T_{\mu \nu} \) has therefore been solved for. Here we remark that the Coulomb-like degrees of freedom \( h_{44}(x) \), \( h_{44}(x) \), and \( h_{44}(x) \) are not retarded and, in fact, they are instantaneous in time.

The dynamical Fourier transformed wave-equation for the components \( h_{ij}(k) \) that follows from equation (2.10) can conveniently now be written in the following form:

\begin{equation}
k^2 \left( h_{ij}(k) - \frac{1}{2} P_{ij}(k) \right) = 2\kappa P_{ij,lm} T^{lm}(k).
\end{equation}

Here we have defined the tensor projection operator \( P_{ij,lm} \) by

\begin{equation}
P_{ij,lm} \equiv \frac{1}{2} \left( P_{ij} P_{lm} + P_{ij} P_{lm} - P_{ij} P_{lm} \right),
\end{equation}

such that \( P_{ij,lm} P^{lm,tk} = P_{ij,tk} \). Both sides of equation (2.19) are now by construction transverse, symmetric, and traceless in the indices \( ij \). The gauge-invariant combination

\begin{equation}
\chi_{ij}(k) \equiv h_{ij}(k) - \frac{1}{2} P_{ij} h_{00}(k),
\end{equation}

then contains two-degrees of freedom corresponding to the degrees of freedom of an on-shell spin-two massless field, i.e. a propagating gravitational field. In terms of a formal inverse Fourier transform \( \delta^T_{ij} \equiv (\delta_{ij} - \partial_i \partial_j/\nabla^2) \) of the projection operator \( P_{ij} \), as defined by equation (2.16), the transverse, traceless, and gauge-invariant spatial components \( h^{TT}_{ij}(x, t) \) of the tensor \( h_{\mu \nu}(x, t) \) can now be written in the well known form (see, e.g. [24])

\begin{equation}
h^{TT}_{ij}(x, t) = \delta^{TT}_{ij} h_{00}(x, t).
\end{equation}

Here we have defined the projection operator.
\[ \delta_{TT}^{\ell \ell_1} \equiv \delta_{\ell \ell_1}^{T} - \frac{1}{2} \delta_{\ell \ell_1}^{TT} , \]  

(2.23)

which in equation (2.22), of course, can be expressed in terms of explicit non-local space-integrals (see, e.g. [1]). It then follows that \( h_{TT}^{\ell}(x, t) = \chi_{\ell}^{\ell}(x, t) \). Similarly, apart from a factor \( 2\kappa \), the inverse Fourier transform of the right-hand side of equation (2.19) corresponds to the transverse and traceless part \( T_{\ell_1}^{TT}(x, t) = \delta_{\ell \ell_1}^{TT} T_{\ell_1} \) of the tensor \( T_{\ell_1}(x, t) \). At this point one remarks that equation (2.19) does not mathematically prescribe the space-time causal and/or retardation properties of \( \chi_{\ell}(x, t) \) as, e.g. discussed by Rohrlich [5]. In section 4 we will, however, verify that quantum mechanics leads to the correct causal and retarded behaviour of suitable expectation values of a second-quantized version of the field \( \chi_{\ell}(x, t) \).

### 3. The second-quantized gravitational field

A second-quantized interaction picture version of the components \( \chi_{\ell}(x) \) in equation (2.21) takes a conventional form, i.e.

\[ \chi_{\ell}(x) = \sum_{k, \lambda} \sqrt{\frac{2 \hbar \kappa}{\sqrt{\omega_k}}} \left( \epsilon_{\ell}(k; \lambda) a_{k\lambda} e^{i k \cdot x} + \epsilon_{\ell}^*(k; \lambda) a_{k\lambda}^* e^{-i k \cdot x} \right) , \]

(3.1)

with \( \lambda = 1, 2 \) or \( \lambda = \pm \) for gravitons with \( k \cdot x = k \cdot x - \omega_k t \) using \( k_4 = \omega_k / c = |k| \), analogous to the second-quantized transverse electro-magnetic potential \( A_T(x) \) for photons in the Coulomb gauge (see, e.g. [1]). The presence of the factor \( 2\kappa \) is similar to the appearance of the factor \( 1/2\epsilon_0 \) in the second-quantized electro-magnetic field potential \( A_T(x) \) and also makes \( \chi_{\ell}(x) \) dimensionless. An overall numerical factor is then determined from, e.g. a suitable canonical commutation relation (see, e.g. [12, 25, 26]) or, as will be verified below, from consistency with the classical Einstein field equation in terms of expectation values. It is assumed that the free-field Hamiltonian \( H_0 \) used in the definition of the interaction picture takes the form

\[ H_0 = \sum_{k, \lambda} \hbar \omega_k (a_{k\lambda}^* a_{k\lambda} + 1/2) . \]

(3.2)

The components of the two, in general complex-valued, polarization tensors \( \epsilon_{\ell}(k; \lambda) \) in equation (3.1) are normalized in such a way that

\[ \text{Tr} \left[ \epsilon^*(k; \lambda) \epsilon(k; \lambda') \right] \equiv \sum_{k, \ell} \epsilon_{\ell}(k; \lambda) \epsilon_{\ell}(k; \lambda') = \delta_{\lambda \lambda'} . \]

(3.3)

An explicit representation in terms of the real-valued, symmetric and transverse photon linear polarization unit vectors \( \epsilon(k; \lambda) \), with \( \lambda = 1, 2 \), in section 2 is

\[ \epsilon_{\ell}(k; 1) = \frac{1}{\sqrt{2}} \left( \epsilon_i(k; 1) \epsilon_j(k; 1) - \epsilon_i(k; 2) \epsilon_j(k; 2) \right) , \]

(3.4)

and

\[ \epsilon_{\ell}(k; 2) = \frac{1}{\sqrt{2}} \left( \epsilon_i(k; 1) \epsilon_j(k; 2) + \epsilon_i(k; 2) \epsilon_j(k; 1) \right) . \]

(3.5)

\(^9\text{Due to unfortunate printing errors, some references are missing. In particular 28 in this publication is our [25]. Also the reference to Weinberg after their equation (4.18) should be to our [12].}\)
With, e.g. \( \mathbf{k} = (0, 0, k) \) then the matrix elements of \( \epsilon_{ij}(\mathbf{k}; \lambda) \) are given by

\[
e_{11}(\mathbf{k}; 1) = -e_{22}(\mathbf{k}; 1) = 1/\sqrt{2}, \quad e_{12}(\mathbf{k}; 2) = e_{21}(\mathbf{k}; 2) = 1/\sqrt{2}, \quad \text{and} \quad e_{ij}(\mathbf{k}; \lambda) = 0 \text{ for } i = 1, 2, 3 \quad [12].
\]

By construction \( \epsilon_{ij}(\mathbf{k}; \lambda) \) obey the rule \( \epsilon_{ij}(-\mathbf{k}; \lambda) = (-1)^{\lambda+1} \epsilon_{ij}(\mathbf{k}; \lambda) \). In passing we notice that the tensor nature of a classical metric deviation \( \chi_\mu(\mathbf{x}) \) has been reported in a recent and remarkable precise test in terms of observed classical gravitational waves from the binary black hole coalescence event GW170814 [27]. In terms of the normalization equation (3.3), we remark that the tensor projection operator \( P_{ij,lm} \) as defined by equation (2.20) can be obtained as follows

\[
P_{ij,lm} = P_{ij,lm}(\mathbf{k}) \equiv \sum_{\lambda=1,2} \epsilon_{ij}(\mathbf{k}; \lambda) \epsilon_{lm}(\mathbf{k}; \lambda).
\]

(3.6)

If we express the quantum field \( \chi_\mu(x) \) in terms of complex circular polarization we can make use of real-valued transverse photon linear polarization unit vectors \( \epsilon(\mathbf{k}; \lambda) \) and equation (2.17). We can then write

\[
\epsilon_{ij}(\mathbf{k}; \pm) \equiv \epsilon_{ij}(\mathbf{r}; \pm) = \frac{1}{\sqrt{2}} (\epsilon_{ij}(\mathbf{k}; 1) \pm i \epsilon_{ij}(\mathbf{k}; 2)),
\]

(3.7)

such that \( \epsilon_{ij}^*(\mathbf{k}; \pm) = \epsilon_{ij}(-\mathbf{k}; \pm) \). The corresponding creation operators \( a_{ij,\lambda}^* \) are given by

\[
a_{ij,\lambda}^* = \frac{1}{\sqrt{2}} (a_{ij,\lambda} \pm i a_{ij,\lambda}^*)
\]

(3.8)

We then observe that, by construction,

\[
\sum_{\lambda=1,2} \epsilon_{ij}(\mathbf{k}; \lambda) a_{ij,\lambda}^* = \sum_{\lambda=\pm} \epsilon_{ij}^*(\mathbf{k}; \lambda) a_{ij,\lambda}^*,
\]

(3.9)

which simply states that the quantum field \( \chi_\mu(x) \) in equation (3.1) does not depend on the actual realization of the choice of polarization degrees of freedom. Under a rotation around the \( \mathbf{k} \)-axis with an angle \( \theta \) one now readily finds that \( a_{ij,\lambda} \to \exp(\pm i 2 \theta) a_{ij,\lambda} \) due to the rotation properties of the polarization tensors \( \epsilon_{ij}(\mathbf{k}; \lambda) \), for \( \lambda = 1, 2 \). From this we conclude that single graviton states \( |\mathbf{k}; \pm \rangle \equiv a_{ij,\lambda}^* |0\rangle \) carry intrinsic helicities \( \pm 2 \hbar \).

In addition to the free field Hamiltonian \( H_0 \) in equation (3.2), and analogous to the second-quantization of the electro-magnetic field in the Coulomb gauge, we can now construct a momentum operator \( P \) and a helicity operator \( \Sigma \) according to

\[
P \equiv \sum_{\mathbf{k} \lambda} \hbar k a_{ij,\lambda}^* a_{ij,\lambda},
\]

(3.10)

and

\[
\Sigma \equiv 2\hbar \sum_{\mathbf{k}} \hat{k} (a_{ij,\lambda}^* a_{ij,\lambda} - a_{ij,\lambda} a_{ij,\lambda}^*).
\]

(3.11)

The complete set of commuting operators \( H_0, P, \) and \( \Sigma \) are now such that diagonal one-particle states \( |\mathbf{k}; \pm \rangle \) carry all the appropriate quantum numbers of a spin-two particle, i.e. a graviton. A complete set of physical Fock-states can then be generated in a conventional manner. In addition to the intrinsic spin angular momentum \( \Sigma \), photon as well as graviton states can also carry conventional orbital angular momentum \( \mathbf{L} \), which for photons plays an important role in many current contexts (see, e.g. [28] and references cited therein), but will not be of concern in the present paper.

For our purposes, the equations (3.2), (3.10) and (3.11) are given by construction. Nevertheless, a gauge-invariant energy–momentum tensor for the gravitational field \( \chi_\mu(x) \) can
be found, as first discussed in various forms and in the context of classical gravitational waves by Einstein [29]\(^\text{10}\), and is defined by (also see, e.g. [24, 26, 30–34]\(^\text{11}\) for other considerations)

\[ t_{\mu\nu}(x) \equiv \frac{1}{4\kappa} \left( \partial_\mu \chi_{ij}(x) \partial_\nu \chi_{ij}(x) - \frac{1}{2} \eta_{\mu\nu} \partial_\alpha \chi_{ij}(x) \partial_\alpha \chi_{ij}(x) \right). \quad (3.12) \]

At the classical level, one procedure to motivate equation (3.12) is, e.g. to consider the Einstein tensor \( G_{\mu\nu} \) to second-order in the deviation \( h_{\mu\nu} \), denoted by \( G^{(2)}_{\mu\nu} \). An effective energy–momentum tensor for the gravitational degrees of freedom can then be defined by \(-G^{(2)}_{\mu\nu}/\kappa\) (see, e.g. [32], section 7.6 and references cited therein), a procedure also used in various studies of low energy effective quantum gravity [35]. By a straightforward calculation in the gauge equation (2.11), and by considering the source-free limit \( T_{\mu\nu} \to 0 \), we then obtain a gauge-invariant energy–momentum tensor \( t_{\mu\nu}^{(\text{grav})} \) for the free gravitational field \( \chi_{ij}(x) \), i.e.

\[ t_{\mu\nu}^{(\text{grav})}(x) \equiv -\frac{1}{2\kappa} \left( \chi_{ij}(x) \partial_\mu \partial_\nu \chi_{ij}(x) - \frac{1}{2} \eta_{\mu\nu} \chi_{ij}(x) \partial_\alpha \partial_\alpha \chi_{ij}(x) \right). \quad (3.13) \]

With the normalization of second-quantized graviton field \( \chi_{ij}(x) \) with two polarization degrees of freedom in equation (3.1), an additional factor 1/2 must then be included in equation (3.13). When performing space-integration over the quantization volume \( V \) one verifies that equations (3.12) and (3.13) then leads to the same result. Conventional conservation of energy–momentum can now be expressed in terms of expectation values of the effective and conserved energy–momentum tensor \( T_{\mu\nu} + \langle t_{\mu\nu} \rangle \).

The energy–momentum tensor \( t_{\mu\nu} \) in equation (3.12) is, by construction, now such that the quantized gravitational field \( \chi_{ij} \) in equation (3.1) leads to the correct diagonal free-field Hamiltonian, i.e.

\[ H_0 = \int_V d^3x t_{44}(x,t), \quad (3.14) \]

analogous to the case of the free second-quantized transverse electro-magnetic field \( A_T \). A momentum operator \( P \) can also be obtain from equation (3.12) in terms of \( t = (t^{41}, t^{42}, t^{43}) \) according to

\[ P = \frac{1}{c} \int_V d^3x t(x,t). \quad (3.15) \]

The orbital angular momentum is then given by \( L = \int d^3x x \times t(x,t)/c \). The helicity operator \( \Sigma \) can also be expressed in terms of the of quantum field \( \chi_{ij} \) by taking the properties of \( \chi_{ij} \) under rotations into account in a standard manner, but will not be of importance in the present context. The total angular momentum is then given \( J = L + \Sigma \) analogous to the case of the second-quantized radiation field \( A_T(x) \).

We observe that in many classical expositions one neglect the second term in the energy–momentum equation (3.12) and performs an averaging procedure over several characteristic wave-lengths of gravitational radiation (see, e.g. [23, 24, 31–33]). We, however, follow a conventional second-quantized field theory procedure making use of equation (3.12) in order to verify, e.g. equation (3.14). Furthermore, and according to the Weinberg–Witten theorem [20], massless particles with helicity larger than one cannot carry a Lorentz-covariant energy–momentum tensor. As a theorem this appears to contradict the construction above.

\(^{10}\) A numerical factor of two in front of the quadrupole formula is, however, missing.

\(^{11}\) See footnote 9.
The theorem follows from a consideration of the rotational properties of matrix elements like \( \langle -\mathbf{k} \pm | t_{\mu \nu} | \mathbf{k} \pm \rangle \) around the \( \mathbf{k} \)-axis. Due to the properties of the polarization tensor \( \epsilon_{ij}(k; \lambda) \) in equation (3.1) it, however, follows from an explicit calculation of such on-shell matrix elements, making use of the definition of \( t_{\mu \nu} \) according to equations (3.12) and (3.1) for the quantum field \( \chi_{ij}(x) \), that such matrix elements actually vanish identically and in this sense the Weinberg–Witten theorem is therefore avoided.

4. The causality issue

The well-known interaction picture Hamiltonian for a classical current \( \mathbf{j}(x,t) \) interaction with the second-quantized and transverse electro-magnetic field \( \mathbf{A}_i(x,t) \) takes an analogous form for the gravitational field \( \chi_{ij}(x,t) \) (see, e.g. [12, 26])

\[
H_i(t) = -\frac{1}{2} \int \mathcal{d}^3x T_{ij}(x,t) \chi_{ij}(x,t)
\]

\[
= -\sum_{k,\lambda} \sqrt{\frac{\hbar e^2}{2V\omega_k}} \left( \epsilon_{ij}(k; \lambda) a_{k\lambda} e^{-i\omega_{ij}T} + \epsilon_{ij}(k; \lambda) a_{k\lambda}^* e^{i\omega_{ij}T} \right),
\]

where we make use of the spatial Fourier transform

\[
T_{ij}(k, t) \equiv \int \mathcal{d}^3x e^{i\mathbf{k} \cdot \mathbf{x}} T_{ij}(x,t).
\]

The total Hamiltonian of the system will, in general, also involve instantaneous act-at-a-distance c-number valued Newtonian-like terms (see, e.g. [12, 26]) which, however, will not be of any concern in the present paper.

Due to the linear dependence of \( a_{k\lambda} \) and \( a_{k\lambda}^* \) in equation (4.1), it is now straightforward to exactly solve for the unitary quantum dynamics (see, e.g. [36–39] for early discussions). Indeed, with \( |\psi(t)\rangle \equiv \exp(\text{i}tH_0/\hbar)|\psi(t)\rangle \), where we for convenience make the choice \( t_0 = 0 \) for the initial time, the state \( |\psi(t)\rangle \) is such that

\[
i\hbar \frac{d|\psi(t)\rangle}{dt} = H_i(t)|\psi(t)\rangle,
\]

where for observables

\[
O_i(t) \equiv \exp(\text{i}tH_0/\hbar)|O\rangle \exp(-\text{i}tH_0/\hbar),
\]

and where, in our case, \( H_0 \) is given by equation (3.14). The time-evolution for \( |\psi(t)\rangle \) is then given by

\[
|\psi(t)\rangle = \exp \left( \frac{\text{i}}{\hbar} \phi(t) \right) \exp \left( -\frac{\text{i}}{\hbar} \int_0^t dt' H_i(t') \right) |\psi(0)\rangle,
\]

for any initial pure state \( |\psi(0)\rangle \). The c-number phase \( \phi(t) \), which will be of no importance in our considerations, can be computed according to

\[
i\phi(t) = \frac{1}{2\hbar} \int_0^t dt' \left[ N(t'), H_i(t') \right] = \frac{1}{2\hbar} \int_0^t dt' \int_0^{t'} dt'' \left[ H_i(t''), H_i(t') \right],
\]

\[12 \text{ See footnote 9.}
\]

\[13 \text{ See footnote 9.}\]
with

\[ N(t) \equiv \int_0^t dt' H_i(t'), \tag{4.7} \]

since \([N(t'), H_i(t')]\) now is a \(c\)-number.

Apart from the phase-factor \(\phi(t)\), the time-evolution in the interaction picture can therefore be expressed in terms of a conventional multi-mode displacement operator \(U(\alpha)\) as (see, e.g. [40–42]) given by

\[ U(\alpha) \equiv \exp \left( \frac{i}{\hbar} \int_0^t dt' H_i(t') \right) = \prod_{k\lambda} \exp \left( i \alpha_{k\lambda}(t) a_{k\lambda}^\dagger - \alpha_{k\lambda}(t) a_{k\lambda} \right). \tag{4.8} \]

where, in our case,

\[ \alpha_{k\lambda}(t) \equiv \frac{i}{\hbar} \sqrt{\frac{\hbar c^2}{2V\omega_k}} \int_0^t dt' \epsilon^{\omega_k t'} \epsilon^*_k(k;\lambda) T^{\alpha}_{ij}(k,t'). \tag{4.9} \]

Since expectation values are independent of the picture used, i.e.

\[ \langle \mathcal{O}(t) \rangle \equiv s \langle \psi(t)|\mathcal{O}(t)|\psi(t)\rangle_s = i \langle \psi(t)|\mathcal{O}_i(t)|\psi(t)\rangle_t, \tag{4.10} \]

and by upon acting with the displacement operator \(U(\alpha)\) in equation (4.8) on, e.g. a initial vacuum state, we can now evaluate the exact expectation value of second-quantized field \(\chi_\alpha(x,t)\) in a manner similar to the analysis for QED situation [1]. We then obtain the following \(\hbar\)-independent weak-field limit result

\[ \langle \chi_\alpha(x,t) \rangle = 2\kappa c^2 \int d^3x' \sum_k \frac{1}{V} P_{ijlm}(k) a^{ij}(x-x') \int_0^t dt' \frac{\sin(\omega_k(t-t'))}{\omega_k} T_{lm}(x',t'), \tag{4.11} \]

in the large volume \(V\) limit, with \(\sum_k = V \int d^3k/(2\pi)^3\), and making use of partial integrations. Here \(G_\alpha(x,t)\) is the retarded Green’s function

\[ G_\alpha(x,t) \equiv \int \frac{d^3k}{(2\pi)^3} e^{ikx} \sin(\omega t)/\omega \Theta(t) = \frac{\delta(t - |x|/c)}{4\pi c^2|x|}, \tag{4.12} \]

for \(t \geq 0\). Under time-reversal \(t \rightarrow -t\) we remark that we can keep the expressions above with \(t \rightarrow |t|\), i.e. we preserve time-reversal invariance provided that the source \(T_{ij}(x,t)\) is also time-reversed. In the derivation of equation (4.11) we have made use of equation (3.6) in the summation over the polarization degrees of freedom. A more explicitly form of equation (4.11) is the well-known causal and properly retarded expression

\[ \langle \chi_\alpha(x,t) \rangle = \frac{4G}{c^2} \int d^3x' T^{TT}_{ij}(x',t - |x-x'|/c)/|x-x'|. \tag{4.13} \]

The quantum mechanical expectation value \(\langle \chi_\alpha(x,t) \rangle\) in equation (4.13) obeys the classical equation of motion

\[ \square \langle \chi_\alpha(x,t) \rangle = -2\kappa T^{TT}_{ij}(x,t). \tag{4.14} \]

Equation (4.14) is, of course, the Fourier transform of Einstein’s classical dynamical equation (2.19). If the initial state is changed from the vacuum state to an arbitrary pure state an additional contribution to equation (4.13) appears corresponding to a homogeneous solution of the
wave equation equation (2.19) for $\chi_{ij}$. Together with the constraint equations $G_{\mu\nu}^{(1)} = -\kappa T_{\mu\nu}$, required to have the energy–momentum tensor $T_{\mu\nu}$ conserved, the complete set of the classical and linearized Einstein’s equations thus emerge from the quantum-mechanical framework outlined above analogous to the situation in QED [1].

The gauge-invariant Riemann–Christoffel tensor $R_{\lambda\mu\nu\kappa}(x,t)$ in the weak field limit according to equation (2.4) plays an important role in, e.g. the detection of gravitational waves (see, e.g. [21–24, 30–33, 43])14. Its causal and retardation properties in the far-field limit of a sufficiently localize energy–momentum tensor $T_{\mu\nu}(x,t)$ are therefore of importance. In the case of two test particles their spatial separation, i.e. the corresponding geodesic deviation, is then determined by, e.g. the components $R_{\lambda\mu\nu\kappa}(x,t)$ which we now express in terms of a space-time Fourier transform of equation (2.4), i.e.

$$R_{\lambda\mu\nu\kappa}^{(1)}(k) = \frac{1}{2} \left( k^2 h_{\lambda\mu}(k) - k_\lambda k_\mu h_{\nu\kappa}(k) - k_\mu k_\kappa h_{\lambda\nu}(k) + k_\nu k_\kappa h_{\lambda\mu}(k) \right).$$  \hspace{1cm} (4.15)

In terms of $h_{\lambda\mu}(k) = \chi_{\lambda\mu}(k) + P_{\lambda\mu}h_{\mu\nu}(k)/2$ we obtain

$$R_{\lambda\mu\nu\kappa}^{(1)}(k) = \frac{k^2}{2} \chi_{\lambda\mu}(k) + \frac{\kappa}{k^4} \left( \frac{k^2}{2} P_{\lambda\mu}T_{\nu\kappa}(k) - k_\lambda \left( k_\mu P_{\nu\kappa}T_{\lambda\mu}(k) + k_\nu k_\kappa T_{\lambda\mu}(k) - \frac{1}{2} T_{\lambda\mu}(k) \right) \right).$$ \hspace{1cm} (4.16)

where we have made use of the constraints equations (2.14), (2.15) and (2.18). When inverting the Fourier transform $R_{\lambda\mu\nu\kappa}^{(1)}(k)$ we make use of the fact that the integrals $\int d^3x T_{\lambda\mu\nu\kappa}(x,t)$ are time-independent for a sufficiently well localized energy–momentum tensor $T_{\mu\nu}(x,t)$. The far-field limit of $R_{\lambda\mu\nu\kappa}^{(1)}(x,t)$ is then obtained from equation (4.16) with the properly causal and retarded gauge-invariant result

$$\langle R_{\lambda\mu\nu\kappa}^{(1)}(x,t) \rangle = -\frac{1}{2c^2} \frac{\partial^2}{\partial t^2} (\chi_{\lambda\mu}(x,t)).$$ \hspace{1cm} (4.17)

For quantum fields in general, intrinsic quantum fluctuations are present [44, 45] and the related considerations for photons applies also for quantum states of gravitons. Since the Riemann–Christoffel tensor $R_{\lambda\mu\nu\kappa}^{(1)}(x,t)$ has now been promoted to an interaction-picture quantum-field, we remark that intrinsic quantum fluctuations will be associated with expectation values like in equation (4.17). As long as the equations of motion are linear averaging procedures over a space-time resolution can be implemented as in the case of electro-magnetic interactions as discussed in, e.g. [1]. Since gravity couples to an effective conserved energy–momentum tensor non-linearities naturally emerge [12, 19, 35] and the corresponding averaging procedures require clarification which, however, goes beyond the scope of the present paper (in this context also see [46]15).

5. Gravitational radiation processes

The emission of soft photons and/or gravitons [25, 26]16 can be investigated in terms of the formalism discussed above. The classical expression for gravitational quadrupole emission

14 See footnote 7.

15 During the final preparation of the present paper we became aware about the work by [46], where the fundamental role of unavoidable quantum fluctuations may play a fundamental role in cosmology.

16 See footnote 9.
from a given, sufficiently well localized, source can, in addition, be obtained from the classical expression equation (4.13) in a standard manner (see, e.g. [21, 24, 30, 31, 33, 43])\textsuperscript{17}. We, however, notice the following. In view of diagonal form of the free-field Hamiltonian according to equations (3.2) or (3.14), and the coherent state generated by the interaction equation (4.1), it is now actually quite easy to find the time-dependence of the energy of the gravitons emitted. We first recall that conservation of the energy–momentum tensor \( T_{\mu\nu} \) leads to the well-known identity

\[
\int d^3 x T_{ij}(x, t) = \frac{1}{2c^2} \frac{\partial^2}{\partial t^2} \int d^3 x x_{x,y} T_{44}(x, t), \tag{5.1}
\]

for a sufficiently well-localized source \( T_{\mu\nu}(x, t) \). By enforcing the dipole approximation

\[
T_{ij}(k, t) \equiv \int d^3 x e^{ik \cdot x} T_{ij}(x, t) \approx \int d^3 x T_{ij}(x, t), \tag{5.2}
\]

and after a summation over polarizations using equation (3.6), one finds, disregarding the divergent vacuum contribution in \( H_0 \) in equation (3.2), that

\[
E(t) \equiv \langle \psi(t)|H_0|\psi(t)\rangle = \sum_k \frac{1}{\hbar} \left[ \sqrt{\frac{\hbar c^2}{2V_\omega k}} \right]^2 P_{ij,lm}(\hat{k}) \times \int_{t_0}^t dt' \int_{t_0}^t dt'' e^{i\omega_k(t' - t'')} T_{ij}(t') T_{lm}(t''), \tag{5.3}
\]

where used has been made of equation (4.9). Here we have defined

\[
T_{ij}(t) \equiv \int d^3 x T_{ij}(x, t), \tag{5.4}
\]

which now can be expressed in terms of quadrupole moments according to equation (5.1). We now extend the time interval to \((-\infty, \infty)\), and define the Fourier transform \( D_{ij}(\omega) \) of the moment \( \int d^3 x x_{x,y} T_{44}(x, t)/c^2 \) in terms of the energy density \( T_{44}(x, t) = c^2 \rho(x, t) \) according to

\[
D_{ij}(\omega) \equiv \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{i\omega t} D_{ij}(t), \tag{5.5}
\]

where

\[
D_{ij}(t) \equiv \int d^3 x x_{x,y} \rho(x, t). \tag{5.6}
\]

It then follows that

\[
E(t) = \frac{4\pi G}{5c^5} \int_0^{\infty} d\omega \omega^6 \langle Q_{ij}(\omega) Q_{ij}(\omega) \rangle. \tag{5.7}
\]

The angular average in equation (5.3) has than been obtained by making use of the result

\[
\int \frac{d\Omega}{4\pi} P_{ij,lm}(\hat{k}) D_{ij}(\omega) D_{lm}(\omega) = \frac{2}{5} \langle Q_{ij}(\omega) Q_{ij}(\omega) \rangle, \tag{5.8}
\]

expressed in terms of the gravitational quadrupole tensor \( Q_{ij}(\omega) \) as defined by

\textsuperscript{17} See footnote 7.
\[ Q_{kl}(\omega) \equiv D_{kl}(\omega) - \frac{1}{3} \delta_{kl} D_{mm}(\omega). \] (5.9)

The total gravitational energy emitted by the classical source according to equation (5.7) agrees with well-known results (see, e.g. [21]18 section 10).

According to equation (5.3) we now also observe that
\[ E(t) = \frac{G}{5\pi c^5} \int_0^\infty d\omega \int_0^t dt' \int_0^{t'} dt'' \hat{Q}_0(t') \hat{Q}_0(t'') \frac{\partial^2}{\partial t'\partial t''} \omega^{(t''-t)}, \] (5.10)
where the angular average has been obtained by making use of a similar expressions like equation (5.1), and we then obtain the famous Einstein expression [29]19 quadrupole gravitational radiated power \( P(t) \equiv \partial E(t)/\partial t \), i.e.
\[ P(t) = \frac{G}{5\pi c^5} \int_{-\infty}^\infty d\omega \int_0^t dt' \hat{Q}_0(t') \hat{Q}_0(t'') \omega^{(t''-t)} = \frac{G}{5\pi c^5} \hat{Q}_0(t) \hat{Q}_0(t). \] (5.12)

If this power will be observed at a large distance \( R \) from the classical source, the time-parameter \( t \) in equation (5.12) should, of course, be replaced by the retarded time \( t = R/c \).

Furthermore, we can now also make use of the interaction equation (4.1) in order to evaluate the total decay rate \( \Gamma \) for the microscopic transition \(|i\rangle \rightarrow |f\rangle \), with \(|i\rangle = |a_i\rangle \otimes |0\rangle \) and \(|f\rangle = |a_f\rangle \otimes |k\lambda\rangle \), of spontaneous emission of a graviton with the energy \( \hbar \omega_k \) from an excited hydrogen atom in the initial atomic state \(|a_i\rangle \) to the final atomic state \(|a_f\rangle \) in first-order in time-dependent perturbation theory in a standard manner. We then make use of \( T_{4k}(x,t) = m_e c^2 \delta^{(3)}(x-x(t)) \) in equation (5.1), where \( x(t) \) is the position of the electron at time \( t \) in the interaction picture. The relevant matrix element for the spontaneous one-graviton transition from the hydrogen atomic state \(|a_i\rangle = |nlm\rangle = |3dm\rangle \) to the final atomic ground state \(|a_f\rangle = |1s\rangle \) is then given by
\[ \langle f|H_I(0)|i\rangle = \frac{m_e c^2 \omega_f^2}{4c} \sqrt{\frac{16\pi G \hbar}{V \omega_k}} \epsilon_m^*(k\lambda) \alpha x^2 |a_f\rangle. \] (5.13)

Here \( \hbar \omega_k \equiv E_i - E_f \equiv \hbar \omega_k \) corresponds to energy conservation, and where, for the well localized quantum atomic states, we make use of the dipole approximation \( \mathbf{k} \cdot \mathbf{x} \approx 0 \). By summing over all polarization degrees of freedom and directions of the emitted graviton, making use of equation (3.6), the decay rate \( \Gamma \), for large time-scales \( t \), with finite \( \omega_k - \omega_f \), can then be obtained in a straightforward manner with the result (see appendix)
\[ \Gamma \equiv \frac{2\pi}{h^2} \int d^3 k \frac{V}{(2\pi)^3} \sum_\lambda \delta(\omega_k - \omega_f) |\langle i|H_I(0)|f\rangle|^2 = \frac{3^8}{2^{10}} \frac{G^{12} a_B^4 \omega_f^5}{5 \hbar c^5}, \] (5.14)
in the large volume \( V \) limit, independent of the energy-degeneracy quantum number \( m \) and where \( a_B = \hbar /\alpha m_e c \) is the Bohr radius for a reduced mass \( \approx m_e \) of the atomic hydrogen two-body system. This result agrees with [47] and with the semi-classical result as reported in [48].

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18 See footnote 7.
19 See footnote 10.
but not with results as quoted in \[21, 49\]^{20}. If \(n_{k\lambda}\) gravitons are present in the initial state, the rate \(\Gamma\) in equation (5.14) should be multiplied by the factor \((1 + n_{k\lambda})\) corresponding to stimulated emission. Absorption graviton processes can be studied in a similar manner.

6. Final remarks

A quantum-mechanical framework offers a platform to study causality and retardation issues in the classical theory of Maxwell [1] as well as in the weak-field limit of Einstein’s general theory of relativity. Second-quantization of physical degrees of freedom and current conservation for a classical source, leads to the well established classical theory for electro-magnetism in terms of expectation values of quantum fields. Inherent and unavoidable quantum fluctuations of these expectation values will always be present \([44–46]\)\^{21}. As we have now explicitly verified, the same reasoning applies for a second-quantized gravitational field around a flat metric in the presence of a sufficiently weak and conserved classical energy–momentum tensor. The weak-field limit of Einstein’s general theory of relativity than naturally emerges expressed also in terms of expectation values of quantum fields.

The overwhelming experimental support for Maxwell’s classical theory does not necessarily imply the existence of photons as quantum states and doubts on the existence of such quantum states are sometimes put forward (see, e.g. [50]). However, it is clear that the quantum-mechanical derivation of the classical theory necessarily implies the existence of single particle quantum states corresponding to a photon. The recent experimental spectacular discovery of gravitational radiation \([27, 51–55]\) is an additional verification of the correctness of Einstein’s general theory of relativity. Outstanding as such observations are, they do not necessarily prove the existence of gravitons as physical quantum states. In view of electromagnetism and photons as quantum states, one would, however, encounter a fundamental difficulty in understanding a possible physical non-existence of gravitons. This is so since the arguments we have presented concerning the emergence of the weak-field limit of Einstein’s classical theory of gravity from quantum mechanics, follow the same line of reasoning as in the case of electromagnetism [1]. Therefore, as for the existence of photons, they necessarily predict the existence of gravitons. Various arguments have, however, been put forward that, due to the weak coupling of gravitons to matter, it may, nevertheless, be impossible to detect single gravitons. In this context we observe that one may, in addition to the analysis of Dyson [56]\^{22}, also consider the possible detectability of quantum states of gravitons in the form of, e.g. a Fock state \(|n\rangle \equiv (a^*_{k\lambda})^n|0\rangle/\sqrt{n!}\) for a macroscopically large number \(n\). Such quantum states have, of course, a trivial expectation value of the quantized gravitational field equation (3.1) but non-zero intrinsic quantum fluctuations. Related discussions on the existence of gravitons can be found in, e.g. [47, 57, 58].

We have also observed that various classical \(\hbar\)-independent expressions for weak perturbations around a flat space-time easily and naturally follow from the quantum-mechanical framework provided the classical source has a conserved energy–momentum tensor. In the weak-field limit such expressions are supposed to be exact results. We now observe that the power spectrum of gravitons emitted from a classical source, with a conserved energy–momentum tensor, is exactly obtained from first-order time-dependent perturbation theory as can be seen by inspection of equation (5.3). It may come as a surprise that a first-order quantum-mechanical calculation can give an exact \(\hbar\)-independent answer. An explanation of this,
as it seems, remarkable fact can now be traced back to the factorization of the time-evolution operator in terms of a displacement operator for quantum states in the interaction picture according to equation (4.5) making use of equation (4.1). The phase $\phi(t)$ then contains the non-perturbative effects of all higher-order corrections to the first-order result. As a matter of fact, similar features are known to happen also in some other situations. As is well-known, the famous differential cross-section for Rutherford scattering can be obtained exactly by making use of the first-order Born approximation. All higher order corrections will then contribute with an overall phase for probability amplitudes which follows from the exact solution (see the excellent discussion in [59]). The classical Thomson cross-section for low-energy light scattering on a charged particle is also exactly obtained from a Born approximation due to the existence of an exact low-energy theorem in quantum electro-dynamics (see, e.g. the discussion in section 8 of [60]).

As alluded to above, the weak coupling of gravitons to matter makes it perhaps of a more academic issue to discuss various features of quantum states for single graviton. Recently it has, however, been observed that well-aligned pairs of photons, with the same helicity quantum numbers, can emulate the quantum-mechanical properties of spin-two gravitons [61]. In such a sense one may even consider entangled pairs of such emulated gravitons and, e.g. carry out EPR-correlation experiments in terms of the CHSH Bell-like inequalities [62, 63] in a similar manner as in the famous Aspect et al [64] realization of entangled pairs of photons. For photons the $\theta$-angle dependence under rotations around the $k$-direction according to $a_{k\pm} \rightarrow \exp(\pm i\theta) a_{k\pm}$, using the definition equation (3.8), will then be replaced by $\theta \rightarrow 2\theta$ for emulated gravitons. The corresponding quantum-mechanical CHSH Bell-like correlations to be measured for entangled pairs of emulated gravitons can then easily be worked out simply by the replacement $\theta \rightarrow 2\theta$ but will not be discussed in detail here [65]. Quantum optics for gravitons can therefore, at least, be contemplated in terms of emulated gravitons.

Finally, we speculate, that if the sources of the emulated gravitons [61] can be described by a conserved energy–momentum tensor, the theoretical arguments presented in this exposition should apply. The characteristic coupling constant $\kappa$ could then, of course, take a completely different value and thereby open up a possible window for experimental investigations of classical as well quantum-mechanical features of emulated gravitons in a completely different context.

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23 Also discussed in section 1.3.2 of Itzykson and Zuber (1980). All radiative corrections to Thomson scattering vanishes in the low-energy limit. This theorem goes back to Thirring (1950). More details can be found in section 19.3 Bjorken and Drell (1965).
Appendix. Decay processes

The decay rate $\Gamma$ is defined by

$$\Gamma \equiv \int d\Omega \sum_{\lambda=1,2} \frac{d\Gamma}{d\Omega} = G m_{\text{grav}}^2 \frac{5}{\hbar c^5} Q,$$  \hspace{1cm} (A.1)

where, for the one-graviton emission process of a hydrogen-like atom $|3dm\rangle \otimes |0\rangle \rightarrow |1s\rangle \otimes |k\lambda\rangle$, we have defined

$$Q \equiv \int \frac{d\Omega}{4\pi} D_{ij}^2 \sum_{\lambda=1,2} \epsilon_{ij}(k; \lambda) \epsilon^{*}_{kl}(k; \lambda).$$  \hspace{1cm} (A.2)

The matrix elements $D_{ij}$ are then obtained from the transition matrix elements equation (5.13) in the main text, i.e.

$$D_{ij} \equiv \int dx_1 x_1 \chi_{3dm}(x) x_i \chi_{1s}(x).$$  \hspace{1cm} (A.3)

In the case with $m = 0$ the only non-zero components are then $D_{33}/2 = -D_{11} = -D_{22} = a_B^2 \sqrt{3/2} \hbar^5$ in terms of the Bohr radius $a_B$ as in [47]. It can be verified that in this case the polarization sum in equation (A.2) is now analogous to equation (5.8) in the main text where we now have defined the quadrupole matrix elements $Q_{kl}$ by

$$Q_{kl} \equiv D_{kl} - \frac{1}{3} \delta_{kl} D_{nn},$$  \hspace{1cm} (A.4)

in terms of equation (A.3). By combining equations (A.1)–(A.4) the result equation (5.14) in the main text is obtained. Even though such a decay is most likely impossible to observe, this analysis shows that we have properly normalized the graviton quantum field $\chi_{ij}$ in equation (3.1) since the same analysis for a purely classical source leads to the well-known Einstein quadrupole formula as discussed in the main text.

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