ON QUANTUM STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. Existence and uniqueness theorems for quantum stochastic differential equations with nontrivial initial conditions are proved for coefficients with completely bounded columns. Applications are given for the case of finite-dimensional initial space or, more generally, for coefficients satisfying a finite localisability condition. Necessary and sufficient conditions are obtained for a conjugate pair of quantum stochastic cocycles on a finite-dimensional operator space to strongly satisfy such quantum stochastic differential equations. This gives an alternative approach to quantum stochastic convolution cocycles on a coalgebra.

Introduction

The investigation of quantum stochastic differential equations (QSDE) for processes acting on symmetric Fock spaces dates back to Hudson and Parthasarathy’s founding paper of quantum stochastic calculus ([HP1]). As usual in stochastic analysis, these equations are understood as integral equations. By a weak solution is meant a process, consisting of operators (or mappings), whose matrix elements satisfy certain ordinary integral equations. Quantum stochastic analysis also harbours a notion of strong solution. The first existence and uniqueness theorems ([HP1]) dealt with the constant-coefficient operator QSDE with finite-dimensional noise space; these were soon extended to the mapping QSDE by Evans and Hudson ([Ev]). Further extensions to the case of infinite-dimensional noise were obtained in [HP3], [MoS] and [Fag], and clarified in [Mey] and [LW1]. Solutions of such QSDE’s yield quantum stochastic, or Markovian, cocycles ([Acc]). The converse is also true under various hypotheses ([HuL], [Bra]); in [LW2] it was proved that any sufficiently regular cocycle on a $C^*$-algebra satisfies some QSDE weakly, and moreover if the cocycle is also completely positive and contractive, then it satisfies the equation strongly. In [LW3] complete boundedness of the ‘columns’ of the coefficient was identified as a sufficient condition for the solution to be strong. (When the noise dimension space is finite dimensional boundedness suffices.) In all the above cases the initial condition for the QSDE was given by an identity map amplified to the Fock space.

Parallel to the theory of quantum stochastic cocycles, Schürmann developed a theory of quantum Lévy processes on quantum groups, or more generally $*$-bialgebras, (see [Sch] and references therein). He showed that each quantum Lévy process satisfies a QSDE of a certain type, with initial condition given by the counit of the underlying $*$-bialgebra (see [5,10] below). The notion of quantum Lévy process was recently generalised to quantum stochastic convolution cocycle on a coalgebra in [LS] where it was shown that such objects arise as solutions of coaglaebraic quantum stochastic differential equations. Extension of the results of

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that paper to the context of compact quantum groups, or more generally operator space coalgebras ([LS]), was our motivation for analysing quantum stochastic differential equations on an operator space with nontrivial initial conditions. Results obtained here have also enabled the development of a dilation theory for completely positive convolution cocycles on a \( C^\ast \)-bialgebra ([S]).

The aim of this paper is to provide existence and uniqueness results for a class of quantum stochastic differential equations, under natural conditions, together with cocycle characterisation of solutions, The crucial role played by complete boundedness ([LW]) suggests that the main object for consideration as initial space should be an operator space. In general operator space theory is very useful for describing properties of coefficients, initial conditions and solutions of our equations (cf. [LW]). The main existence theorem is proved for coefficients with \( k \)-bounded columns and initial condition given by a \( k \)-bounded map, where \( k \) is the ‘noise dimension space’. (The term \( k \)-bounded means simply bounded if \( k \) is finite dimensional and completely bounded otherwise). Solutions are expressed in terms of iterated quantum stochastic integrals (cf. [LW]) and have \( k \)-bounded columns themselves (completely bounded columns if the coefficient has cb-columns and the initial condition is completely bounded). Due to our choice of test vectors (exponentials of step-functions with values in a given dense subspace of the noise dimension space) the results are explicitly basis-independent. As solutions of equations of the type considered are quantum stochastic cocycles, one may ask which cocycles satisfy a QSDE. Sufficient conditions for the cocycle to satisfy a QSDE weakly, established for the case of \( C^\ast \)-algebras in [LW], remain valid in the coordinate-free, operator space context of this paper. A new result here, informed by our recent theorem on convolution cocycles ([LS]), is the characterisation of cocycles on finite dimensional operator spaces which, together with a conjugate process, satisfy a QSDE strongly — namely, they are the locally Hölder-continuous processes with exponent 1/2 whose conjugate process enjoys the same continuity.

The plan of the paper is as follows. In Section 1 the notation is established and basic operator-space theoretic and quantum stochastic notions are introduced. There also a concept of finite localisability is discussed. Weak regularity is shown to be sufficient for uniqueness of weak solutions in Section 2 (cf. [LW]). Section 3 contains the main result on the existence of strong solutions of equations on operator spaces and elucidates their dependence on initial conditions. Although in the case of (algebraic) quantum Lévy processes the initial object is a vector space \( V \), rather than an operator space, the Fundamental Theorem on Coalgebras allows us to effectively work with finite-dimensional subspaces and thereby to circumvent the lack of analytic structure on \( V \) (cf. [Sch]). For this purpose, the version of the existence theorem for finitely localisable maps relevant for coalgebraic quantum stochastic differential equations is given in Section 4. Section 5 begins by recalling known facts on relations between quantum stochastic cocycles and quantum stochastic differential equations whose initial condition is given by the identity map on \( V \). It then gives new necessary and sufficient conditions for a conjugate pair of cocycles on a finite-dimensional operator space to satisfy a QSDE strongly and ends with an application of this result to the infinitesimal generation of quantum stochastic convolution cocycles.

**Notation.** For dense subspaces \( E \) and \( E' \) of Hilbert spaces \( H \) and \( H' \), \( \mathcal{O}(E;H') \) denotes the space of operators \( H \to H' \) with domain \( E \) and \( \mathcal{O}^\dagger(E,E') := \{ T \in \mathcal{O}(E;H') : \text{Dom} T^\ast \supset E' \} \). Thus \( \mathcal{O}^\dagger(E',E) \) is the conjugate space of \( \mathcal{O}(E,E') \) with conjugation \( T \mapsto T^\dagger := T'^\ast|_{E'} \). When \( H' = H \) we write \( \mathcal{O}(E) \) for \( \mathcal{O}(E;H) \). We view \( B(H;H') \) as a subspace of \( \mathcal{O}^\dagger(E,E') \) (via restriction/continuous linear extension). For vectors \( \zeta \in E \) and \( \zeta' \in H' \), \( \omega_{\zeta',\zeta} \) denotes the linear functional on
\( \mathcal{O}(E; H') \) given by \( T \mapsto (\zeta', T\zeta) \), extending a standard notation. We also use the Dirac-inspired notations \( \langle \zeta : \zeta \in E \rangle := \{ \zeta \} : \zeta \in E \) and \( [E] := \{ \langle \zeta : \zeta \in E \rangle \} \) where \( \langle \zeta : \zeta \in [h] \rangle := \mathcal{B}(C; h) \) and \( \langle \zeta : \zeta \in (h) \rangle := \mathcal{B}((h); C) \) are defined by \( \lambda \mapsto \lambda \zeta \) and \( \zeta' \mapsto \langle \zeta, \zeta' \rangle \) respectively — inner products (and all sesquilinear maps) here being linear in their second argument.

Tensor products of vector spaces, such as dense subspaces of Hilbert spaces, are denoted by \( \odot \); minimal/spatial tensor products of operator spaces by \( \odot_{sp} \); and ultraweak tensor products of ultraweakly closed spaces of bounded operators by \( \otimes \). The symbol \( \odot \) is used for Hilbert space tensor products and tensor products of completely bounded maps between operator spaces; the symbol \( \odot_{sp} \) is also used for the tensor product of unbounded operators, thus if \( S \in \mathcal{O}(E; H') \) and \( T \in \mathcal{O}(F; K') \) then \( S \odot T \in \mathcal{O}(E \odot F; H' \odot K') \). We also need ampliations of bra’s and kets: for \( \zeta \in h \) define

\[
E_{\zeta} := \operatorname{I}_{H} \otimes \langle \zeta \rangle \in \mathcal{B}(H \otimes h; H) \quad \text{and} \quad E_{\xi} := \operatorname{I}_{H} \otimes \langle \xi \rangle \in \mathcal{B}(H \otimes h; H),
\]

where the Hilbert space \( H \) is determined by context.

For a vector-valued function \( f \) on \( \mathbb{R}^{+} \) and subinterval \( I \) of \( \mathbb{R}^{+} \) \( f_{I} \) denotes the function on \( \mathbb{R}^{+} \) which agrees with \( f \) on \( I \) and vanishes outside \( I \). Similarly, for a vector \( \xi, \xi_{t} \) is defined by viewing \( \xi \) as a constant function. This extends the standard indicator function notation. The symmetric measure space over the Lebesgue measure space \( \mathbb{R}^{+} \) \( (\mathcal{G}_{\mathbb{R}}) \) is denoted \( \Gamma \), with integration denoted \( \int_{I} \cdots d\sigma \), thus \( \Gamma = \{ \sigma \subset \mathbb{R}^{+} : \# \sigma < \infty \} = \bigcup_{n \geq 0} \Gamma^{	ext{n}} \) where \( \Gamma^{	ext{n}} = \{ \sigma \subset \mathbb{R}^{+} : \# \sigma = n \} \) and \( \emptyset \) is an atom having unit measure. If \( \mathbb{R}^{+} \) is replaced by a subinterval \( I \) then we write \( \Gamma_{I} \) and \( \Gamma_{I}^{n} \), thus the measure of \( \Gamma_{I}^{n} \) is \( |I|^{n}/n! \) where \( |I| \) is the Lebesgue measure of \( I \). Finally, we write \( X \subset Y \subset \mathcal{G} \) to mean that \( X \) is a finite subset of \( Y \).

1. Preliminaries

Quantum stochastics \( \{\text{Par}, \text{Mey}\} \); we follow \[\text{L}]. Fix now, and for the rest of the paper, a complex Hilbert space \( k \) which we refer to as the noise dimension space, and let \( k \) denote the orthogonal sum \( C \oplus k \). Whenever \( c \in k, \hat{c} := \{c\} \in k \); for \( E \subset k, \hat{E} := \text{Lin}\{c : c \in E\} \) and when \( g \) is a function with values in \( k \), \( \hat{g} \) denotes the corresponding function with values in \( k \) defined by \( \hat{g}(s) := g(s) \). Let \( F \) denote the symmetric Fock space over \( L^{2}(\mathbb{R}^{+}; k) \). For any dense subspace \( D \) of \( k \) let \( S_{D} \) denote the linear span of \( \{d_{0,t} : d \in D, t \in \mathbb{R}^{+}\} \) in \( L^{2}(\mathbb{R}^{+}; k) \) (we always take these right-continuous versions) and let \( \mathcal{E}_{D} \) denote the linear span of \( \{\varepsilon(g) : g \in S_{D}\} \) in \( F \), where \( \varepsilon(g) \) denotes the exponential vector \( \{(n!)^{-\frac{1}{2}}g^{\otimes n}\}_{n \geq 0} \). The subscript \( D \) is dropped when \( D = k \). An exponential domain is a dense subspace of \( \mathfrak{h} \otimes F \), for a Hilbert space \( \mathfrak{h} \), of the form \( \mathcal{D} \odot \mathcal{E}_{D} \). We usually drop the tensor symbol and denote simple tensors such as \( v \otimes \varepsilon(f) \) by \( v \varepsilon(f) \).

For an exponential domain \( \mathcal{D} = \mathcal{D} \odot \mathcal{E}_{D} \subset \mathfrak{h} \otimes F \) and Hilbert space \( \mathfrak{h}' \), \( \mathbb{P}(D; \mathfrak{h} \otimes F) \) denotes the space of (equivalence classes of) weakly measurable and adapted functions \( X : \mathbb{R}^{+} \rightarrow \mathcal{O}(D; \mathfrak{h}' \otimes F) \): \( t \mapsto (\xi, X_{t}\xi) \) is measurable (\( \xi' \in \mathfrak{h} \otimes F, \xi \in \mathcal{D} \); 

\[
\langle u' \varepsilon(g'), X_{t} u \varepsilon(g) \rangle = \langle u' \varepsilon(g'_{0,t}), X_{t} u \varepsilon(g_{0,t}) \rangle \langle u' \varepsilon(g'_{t,\infty}), u \varepsilon(g_{t,\infty}) \rangle
\]

\((u \in \mathcal{D}, g \in S_{D}, u' \in \mathfrak{h}, g' \in S, t \in \mathbb{R}^{+})\), with processes \( X \) and \( X' \) being identified if, for all \( \xi \in D, X_{t}\xi = X'_{t}\xi \) for almost all \( t \in \mathbb{R}^{+} \). If \( D' \) is an exponential domain in \( \mathfrak{h}' \otimes F \) then \( \mathbb{P}^{1}(D, D') \) denotes the space of \( \mathcal{O}^{1}(D, D') \)-valued processes. Thus \( \mathbb{P}^{1}(D', D) \) is the conjugate space of \( \mathbb{P}^{1}(D, D') \) with conjugation defined pointwise: \( X_{t}^{\ast} = (X_{t})^{\ast} |_{D'} \).
Let $F \in \mathbb{P}(\mathcal{D} \oplus \mathcal{D} ; \mathcal{H} \otimes \mathbb{K} \otimes \mathcal{F})$ be quantum stochastically integrable (L). Then the process $(X_t = \int_0^t F_s d\Lambda_s)_{t \geq 0} \in \mathbb{P}(\mathcal{D} \oplus \mathcal{D} ; \mathcal{H}' \otimes \mathcal{F})$ satisfies

$$\langle v' \xi(g'), X_t v \xi(g) \rangle = \int_0^t ds \langle v' g'(s) \xi(g'), F_s v \hat{g}(s) \xi(g) \rangle \quad (1.1)$$

$$\|X_t v \xi(g)\|^2 \leq C(g, t)^2 \int_0^t ds \|F_s v \hat{g}(s) \xi(g)\|^2 \quad (1.2)$$

$(v \in \mathcal{D}, g \in \mathcal{S}, v' \in \mathcal{H}', g' \in \mathcal{S}, t \in \mathbb{R}_+)$ for a constant $C(g, t)$ which is independent of $F$ and $v$. These are known as the Fundamental Formula and Fundamental Estimate of quantum stochastic calculus. We also need basic estimates for sums of iterated integrals. Thus let $L = (L_n \in \mathcal{O}(\mathcal{D} \oplus \mathcal{D} ; \mathcal{H} \otimes \mathbb{K} \otimes \mathcal{F}))_{n \geq 0}$ satisfy the growth condition

$$\forall \gamma \in \mathbb{R}_+, \forall v \in \mathcal{D}, \forall F \in \mathcal{D}, \forall \beta \in \mathcal{B} \sum_{n \geq 0} \frac{\gamma^n}{\sqrt{n!}} \max\{\|L_n v \otimes \zeta_1 \otimes \cdots \otimes \zeta_n\| : \zeta_1, \ldots, \zeta_n \in F\} < \infty.$$ 

Then the iterated quantum stochastic integrals of the $L_n$ sum to a process $(\Lambda(L))_{t \geq 0}$ satisfying (for all $v \in \mathcal{D}, g \in \mathcal{S}, v' \in \mathcal{H}', g' \in \mathcal{S}$)

$$\langle v' \xi(g'), \Lambda_t(L) v \xi(g) \rangle = e^{(g,g')} \int_{\Gamma_{[0,t]}} d\sigma \langle v' \pi_\sigma(g) \xi(g), L_{\#\sigma} v \pi_\sigma(g) \xi(g) \rangle \quad (1.3)$$

$$\|\Lambda_t(L) v \xi(g)\| \leq \|v \xi(g)\| \sum_{n \geq 0} C(g, T)^n \left\{ \int_{\Gamma_{[0,t]}} d\sigma \|L_n v \pi_\sigma(g)\|^2 \right\}^{1/2} \quad (1.4)$$

$$\|\Lambda_t(L) - \Lambda_r(L) v \xi(g)\| \leq \|v \xi(g)\| \sum_{n \geq 0} C(g, T)^{n+1} \left\{ \int_r^t ds \int_{\Gamma_{[0,s]}} \int d\omega \|L_n v \pi_{\sigma}(g)\|^2 \right\}^{1/2}, \quad (1.5)$$

for $0 \leq r \leq t \leq T$, where

$$\pi_\sigma(g) := \hat{g}(s_n) \otimes \cdots \otimes \hat{g}(s_1) \text{ for } \sigma = \{s_1 < \cdots < s_n \} \in \Gamma,$$

with $\pi_\emptyset(0) := 1$.

**Forms and maps.** Let $V$ and $V'$ be vector spaces and let $E$ and $E'$ be dense subspaces of Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$. For any sesquilinear map $\phi$ defined on $E' \times E$ and vectors $\zeta' \in E'$ and $\zeta \in E$ we write $\phi_{\zeta,\zeta'}$ for the value of $\phi$ at $(\zeta', \zeta)$. We shall be invoking the following natural relations:

$$SL(E', E; L(V; V')) \supset L(E; L(V; V' \otimes |\mathcal{H}'|)) \quad (1.6)$$

$$\supset L(V; V' \otimes \mathcal{O}(E; \mathcal{H}')). \quad (1.7)$$

In case $\mathcal{H}$ is finite dimensional the inclusion (1.6) is an equality. In case $V'$ is finite dimensional the inclusion (1.7) is an equality. More generally the following observation is relevant here.

**Lemma 1.1.** Let $\chi \in L(E; L(V; V' \otimes |\mathcal{H}'|))$ satisfy the localising property:

$$\forall x \in V \exists \psi \in V' \text{ finite dimensional subspace of } V' \forall \xi \in E \chi_{\psi}(x) \in \psi \otimes |\mathcal{H}'|.$$ 

Then $\chi \in L(V; V' \otimes \mathcal{O}(E; \mathcal{H}')).$

**Proof.** Straightforward. \[\square\]
Definition. Let \( \chi \in L(E; L(V; V \otimes |H^r|)) \) for a vector space \( V \), pre-Hilbert space \( E \) and Hilbert space \( H^r \). A subspace \( V_1 \) of \( V \) localises \( \chi \) if it satisfies
\[
\chi_{|_\zeta}(V_1) \subset V_1 \otimes |H| \quad (\zeta \in E);
\]
\( \chi \) is finitely localisable if
\[
V = \bigcup \{ V_1 : V_1 \text{ localises } \chi \text{ and } \dim V_1 < \infty \}.
\]

Remark. By Lemma [12] if \( \chi \) is finitely localisable then it belongs to \( L(V; V \otimes \mathcal{O}(E; H^r)) \), and localisation by \( V_1 \) translates to
\[
\chi(V_1) \subset V_1 \otimes \mathcal{O}(E; H^r).
\]

Apart from the case of finite dimensional \( V \), the example we have in mind is that of a coalgebra \( \mathcal{C} \) with coproduct \( \Delta \). In this context all maps of the form \( \chi = (\text{id}_\mathcal{C} \otimes \varphi) \circ \Delta \), where \( \varphi \in L(\mathcal{C}; \mathcal{O}(E)) \), are finitely localisable. This follows from the Fundamental Theorem on Coalgebras.

Matrix spaces. For the general theory of operator spaces and completely bounded maps we refer to [ER] and [Pis2]. For an operator space \( Y \) in \( B(H; H^r) \) and Hilbert spaces \( h \) and \( h^r \) define
\[
Y \otimes_M B(h; h^r) := \{ T \in B(H \otimes h; H^r \otimes h^r) : B(H; H^r)B(h; h^r) : \Omega_{\zeta,\zeta'}(T) \in Y \} \quad (1.8)
\]
where \( \Omega_{\zeta,\zeta'} \) denotes the slice map \( \overline{\omega_{\zeta,\zeta'}} : T \mapsto E' T E_{\zeta'} \). For us the relevant cases are \( Y \otimes_M B(h) \) and \( Y \otimes_M \{ h \} \), referred to respectively as the \( h \)-matrix space over \( Y \) and the \( h \)-column space over \( Y \). (Previous notations: \( M(h; Y)_h \) and \( C(h; Y)_h \).)

Matrix spaces are operator spaces which lie between the spatial tensor product \( Y \otimes sp B(h; h^r) \) and the ultraweak tensor product \( \overline{\otimes} B(h; h^r) \) (\( \overline{\otimes} \) denoting the ultraweak closure of \( Y \)). They arise naturally in quantum stochastic analysis where a topological state space is to be coupled with the measure-theoretic noise — if \( Y \) is a \( C^* \)-algebra then typically the inclusion \( Y \otimes sp B(h) \subset Y \otimes_M B(h) \) is proper and \( Y \otimes_M B(h) \) is not a \( C^* \)-algebra. Completely bounded maps between concrete operator spaces lift to completely bounded maps between corresponding matrix spaces: for \( \phi \in CB(Y; Y') \) there is a unique map \( \Phi : Y \otimes_M B(h; h^r) \rightarrow Y' \otimes_M B(h; h^r) \) satisfying
\[
\Omega_{\zeta,\zeta'} \circ \Phi = \phi \circ \Omega_{\zeta,\zeta'} \quad (\zeta \in h, \zeta' \in h^r).
\]
This map is completely bounded and is denoted \( \phi \otimes_M \text{id}_{B(h,h^r)} \). A variant on this arises when \( Y' \) has the form \( X \otimes_M B(K; K') \):
\[
\phi^{h,h'} := \tau \circ (\phi \otimes_M \text{id}_{B(h,h^r)}) \quad (1.9)
\]
where \( \tau \) is the flip on the second and third tensor components, so that
\[
\phi^{h,h'}(Y \otimes_M B(h; h^r)) \subset X \otimes_M B(h; h^r) \otimes_M B(K; K').
\]
When \( h' = h \) we write \( \phi^h \).

Tensor-extended composition. We develop a short-hand notation which will be useful here. Let \( U, V \) and \( W \) be operator spaces and \( V \) a vector space. If \( \phi \in L(V; U \otimes sp V \otimes sp W) \) and \( \psi \in CB(V; V') \) then we compose in the obvious way:
\[
\psi \circ \phi := (\text{id}_U \otimes \psi \otimes \text{id}_W) \circ \phi \in L(V; U \otimes sp V' \otimes sp W). \quad (1.10)
\]
Ambiguity is avoided provided that the context dictates which tensor component the second-to-be-applied map \( \psi \) should act on. This works nicely for matrix-spaces too. Thus if \( \phi \in L(V; Y \otimes_M B(h; h^r)) \) and \( \psi \in CB(Y; Y') \) (or \( \psi \in B(Y; Y') \) if both \( h, h' \) are finite-dimensional), where \( Y \) and \( Y' \) are concrete operator spaces, then
\[
\psi \circ \phi := (\psi \otimes_M \text{id}_{B(h,h^r)}) \circ \phi \in L(V; Y' \otimes_M B(h; h^r)).
\]

The following elementary inequality will be needed in Section 3.
Lemma 1.2. Let $\psi \in B(X;Y)$ and $\phi_1, \ldots, \phi_n \in B(X;X \otimes_M |H|)$ for concrete operator spaces $X$ and $Y$ and finite dimensional Hilbert space $H$. Then
\[
\|\psi \circ \phi_1 \cdots \circ \phi_n\| \leq (\dim H)^{n/2} \|\psi\| \|\phi_1\| \cdots \|\phi_n\|.
\]

Proof. Let $(e_i)$ be an orthonormal basis for $H$, and for a multi-index $i = (i_1, \ldots, i_n)$ let $e(i)$ denote $e_{i_1} \otimes \cdots \otimes e_{i_n}$. Then, by a 'partial Parseval relation' (recall the 'E notation' introduced in (0.1))
\[
\|\psi \circ \phi_1 \cdots \circ \phi_n(\psi)d\| = \sum_i \|E(i)(\psi \circ \phi_1 \cdots \circ \phi_n)(\psi)d\|^2 \quad (\psi \in X, d \in H)
\]
where $h$ is the Hilbert space on which the operators of $Y$ act. The result therefore follows since, for any unit vectors $d_1, \ldots, d_n \in H$,
\[
\|E^{d_1 \otimes \cdots \otimes d_n} \psi \circ \phi_1 \cdots \circ \phi_n\| = \|\psi \circ E^{d_1} \phi_1 \circ \cdots \circ E^{d_n} \phi_n\| \\
\leq \|\psi\| \|\phi_1\| \cdots \|\phi_n\|.
\]

The following variant on tensor-extended composition will also be useful. For $\psi \in L(V; O(E \otimes K; K \otimes K'))$ where $V$ is a linear space, $E$ and $E'$ are dense subspaces of Hilbert spaces $H$ and $H'$ and $K$ and $K'$ are further Hilbert spaces,
\[
\omega_{E',E} \psi := E'\psi(\cdot)E, \quad \psi \in L(V; O(E'; K)).
\]

Thus $\omega_{E',E} \psi \in L(V; O(E'; K))$.

2. Regularity and uniqueness

For this section fix a complex vector space $V$ and exponential domains $\mathcal{D} = \mathcal{D} \otimes \mathcal{E}_D$ and $\mathcal{D}' = \mathcal{D}' \otimes \mathcal{E}_{D'}$ in $\mathfrak{h} \otimes \mathcal{E}$ and $\mathfrak{h}' \otimes \mathcal{E}$ respectively. A map $V \rightarrow \mathbb{P}(D; \mathfrak{h}' \otimes \mathcal{E})$ is called a process on $V$. We are interested in such processes which are linear and denote the collection of these by $\mathbb{P}(V; \mathfrak{h}' \otimes \mathcal{E})$. Also define
\[
\mathbb{P}^+(V; \mathcal{D}; \mathcal{D}') := \{ k \in \mathbb{P}(V; \mathcal{D}; \mathfrak{h}' \otimes \mathcal{E}) : k(V) \subset \mathbb{P}^+(\mathcal{D}; \mathcal{D}') \},
\]
and for such a process $k$ its conjugate process $k^+ \in \mathbb{P}^+(V^*; \mathcal{D}'^*; \mathcal{D})$ is defined by $k^+(x^*) = k(x)^+$. A process $k$ on $V$ is $(\mathcal{D}', \mathcal{D})$-pointwise weakly continuous if $s \rightarrow (\omega_{E',E} \circ k_s)(x)$ is measurable for all $x \in \mathcal{D}$ and $E \in V$; it is $(\mathcal{D}', \mathcal{D})$-weakly regular if, for some norm on $V$, the following set is bounded
\[
\|x\|^{-1}(\omega_{E',E} \circ k_s)(x) : x \in V \setminus \{0\}, s \in [0, t]\}
\]
(2.1) we drop the $(\mathcal{D}', \mathcal{D})$ and refer simply to weakly continuous and weakly regular processes. If $V$ already has a norm then weak regularity refers to that norm. We denote the spaces of such processes which are also linear by $\mathbb{P}_{wc}(V; \mathcal{D}; \mathcal{D}')$ and $\mathbb{P}_{wc}(V; \mathcal{D}; \mathcal{D}')$ respectively.

A weaker notion of regularity tailored to the coefficient of a quantum stochastic differential equation is also relevant to the uniqueness question. Thus let $\phi \in SL(\mathcal{D}', \mathcal{D}; L(V))$ (sesquilinear maps). For each $R \subset \subset V$, $F \subset \subset \mathcal{D}$ and $F' \subset \subset \mathcal{D}'$ define the following subspace of $V$
\[
V_{F'} : = \text{Lin} \{ \phi(z_1) \cdots \circ \phi(z_n) : n \in \mathbb{Z}_+, z \in R, \phi(z_1), \ldots, \phi(z_n) \in F', \zeta_1, \ldots, \zeta_n \in \mathcal{F} \}
\]
(with the convention that an empty product in $L(V)$ equals id$_V$), and for $f, f' \in \mathbb{S}$ write $F'_f$ and $F_f$ for $\text{Ran } f|_{[0,t]}$ and $\text{Ran } f'|_{[0,t]}$ respectively.
Definition. A process \( k : \mathcal{V} \rightarrow \mathbb{P}(\mathcal{D}; \mathcal{h}' \otimes \mathcal{F}) \) is \((\mathcal{D}', \mathcal{D})\)-weakly regular locally with respect to \( \phi \) if \( V_{\mathcal{F}_t, \mathcal{F}_t}^\phi \) has a norm for which the following is finite:

\[
C_{\mathcal{E}_t, \mathcal{R}_t, \mathcal{E}_t}^{k, \phi, t} = \sup \left\{ \| z \|^{-1} |\omega_{\mathcal{E}_t, \mathcal{E}_t} \circ k_s(z) | : z \in V_{\mathcal{F}_t, \mathcal{F}_t}^\phi \setminus \{0\}, s \in [0, t] \right\}
\tag{2.2}
\]

where \( \mathcal{E}_t \subset \mathcal{V}, \xi = u\varepsilon(f) \in \mathcal{D}, \xi' = v\varepsilon(f') \in \mathcal{D}', t \in \mathbb{R}_+ \).

We shall refer to such norms as regularity norms and let \( \mathbb{P}_{\text{wr}}(V : \mathcal{D}, \mathcal{D}') \) denote the space of such processes which are linear.

**Proposition 2.1.** Let \( k \in \mathbb{P}_{\text{wc}}(V : \mathcal{D}, \mathcal{D}') \).

(a) Let \( \phi \in SL(\mathcal{D}', \mathcal{D}; L(\mathcal{V})) \) and suppose that \( \phi \) satisfies

\[
\dim V_{\mathcal{F}_t, \mathcal{F}_t}^\phi < \infty \quad (R \subset \subset \mathcal{V}, f \in \mathcal{S}_\mathcal{D}, f' \in \mathcal{S}_{\mathcal{D}'}, t \in \mathbb{R}_+).
\]

Then \( k \in \mathbb{P}_{\text{wr}}(V : \mathcal{D}, \mathcal{D}') \).

(b) Suppose that \( V \) is a Banach space and \( \omega_{\mathcal{E}_t, \mathcal{E}_t} \circ k_t \) is bounded for each \( \xi' \in \mathcal{D}', \xi \in \mathcal{D}, t \in \mathbb{R}_+ \). Then \( k \in \mathbb{P}_{\text{wr}}(V : \mathcal{D}, \mathcal{D}') \).

**Proof.** Let \( \xi = u\varepsilon(f) \in \mathcal{D}, \xi' = v\varepsilon(f') \in \mathcal{D}' \) and \( t \in \mathbb{R}_+ \).

(a) In this case let \( R \subset \subset \mathcal{V} \) and consider the \( l^1 \)-norm on \( V_{\mathcal{F}_t, \mathcal{F}_t}^\phi \) determined by a choice of basis: \( \| \sum_{i=1}^d \lambda_i e_i \| := \sum_{i=1}^d |\lambda_i| \). By linearity

\[
C_{\mathcal{E}_t, \mathcal{R}_t, \mathcal{E}_t}^{k, \phi, t} \leq \sup \left\{ \| \xi' \|, k_s(e_i)\xi \| : 0 \leq s \leq t, i = 1, \ldots, d \right\},
\]

which is finite by weak continuity.

(b) In this case the family of bounded linear functionals \( \{ \omega_{\mathcal{E}_t, \mathcal{E}_t} \circ k_s : 0 \leq s \leq t \} \) is pointwise bounded, by weak continuity, and so the Banach-Steinhaus Theorem applies. \( \square \)

In particular, if \( V \) is finite dimensional then, once equipped with a norm, Part (b) applies.

**Corollary 2.2.** If \( V \) is finite dimensional then

\[
\mathbb{P}_{\text{wc}}(V : \mathcal{D}, \mathcal{D}') \subset \mathbb{P}_{\text{wr}}(V : \mathcal{D}, \mathcal{D}').
\]

**Quantum stochastic differential equations.** Now let \( \phi \in SL(\mathcal{D}', \mathcal{D}; L(\mathcal{V})) \) and \( \kappa \in L(\mathcal{V}; \mathcal{W}) \) where \( \mathcal{W} \) is a subspace of \( \mathcal{O}(\mathcal{D}; \mathcal{h}') \) for example \( B(\mathcal{h}; \mathcal{h}') \). A process \( k : \mathcal{V} \rightarrow \mathbb{P}(\mathcal{D}; \mathcal{h}' \otimes \mathcal{F}) \) is a \((\mathcal{D}', \mathcal{D})\)-weak solution of the quantum stochastic differential equation

\[
dk_t = k_t \bullet d\lambda_\phi(t), \quad k_0 = \iota \circ \kappa
\tag{2.3}
\]

where \( \iota \) denotes ampliation \( \mathcal{O}(\mathcal{D}; \mathcal{h}') \rightarrow \mathcal{O}(\mathcal{D}; \mathcal{h}' \otimes \mathcal{F}) \), if \( k \) is \((\mathcal{D}', \mathcal{D})\)-pointwise weakly continuous and

\[
\langle \xi', k_t(x)\xi \rangle - \langle \nu', \kappa(x)\nu \rangle = \int_0^t ds \langle \xi', k_s(\phi^\varepsilon(s)(x))\xi \rangle
\tag{2.4}
\]

\((\xi = u\varepsilon(g) \in \mathcal{D}, \xi' = v\varepsilon(g') \in \mathcal{D}', x \in \mathcal{V}, s \in \mathbb{R}_+) \).

**Remark.** Suppose that \( \mathcal{W} \) is a subspace of \( \mathcal{O}(\mathcal{D}, \mathcal{D}') \) and \( \mathcal{D}' = \mathcal{D}' \otimes \mathcal{E}_{\mathcal{D}'} \). If a \((\mathcal{D}', \mathcal{D})\)-weak solution \( k \) of the equation \((2.3)\) is \( \mathbb{P}(\mathcal{D}, \mathcal{D}') \)-valued then the conjugate process \( k^* : \mathcal{V}' \rightarrow \mathbb{P}(\mathcal{D}', \mathcal{D}) \) is a \((\mathcal{D}, \mathcal{D}')\)-weak solution of the quantum stochastic differential equation \((2.3)\) with \( \phi \) and \( \kappa \) replaced by \( \phi^\dagger \in SL(\mathcal{D}, \mathcal{D}; L(\mathcal{V}')) \) and \( \kappa^\dagger \in L(\mathcal{V}'; \mathcal{W}') \) respectively.
A process $k \in P(V : \mathcal{D}; h' \otimes \mathcal{F})$ is a $\mathcal{D}$-strong solution of the quantum stochastic differential equation (2.3) if there is a process $K \in P(V : \mathcal{D} \circ \mathcal{D} \otimes \mathcal{E}_D; h' \otimes \hat{k} \otimes \mathcal{F})$ which is pointwise quantum stochastically integrable and satisfies

$$\omega_{\xi', \xi} \cdot K_t = k_t \circ \phi_{\xi} (\xi' \in \hat{D}, \xi \in \hat{D}, t \in \mathbb{R}^+),$$

and

$$k_t(x) = \kappa(x) \circ I + \int_0^t K_s(x) \, d\Lambda_s \quad (x \in V, t \in \mathbb{R}^+).$$

In particular, strong solutions are (pointwise strongly) continuous. In view of the First Fundamental Formula (1.1), any strong solution is a $(\mathcal{D}, \hat{\mathcal{D}})$-weak solution. Conversely, if $k$ is a $(\mathcal{D}', \hat{\mathcal{D}})$-weak solution, with $\mathcal{D}' \subset \mathcal{D}$, then $K$ is a pointwise quantum stochastically integrable process satisfying (2.3) for all $t \in \mathbb{R}^+$.

Strong solutions will be considered in subsequent sections. For now let $W = \mathcal{O}(\mathcal{D}; h')$.

**Theorem 2.3.** Let $\phi \in SL(\hat{D}' \hat{D}; L(V))$ and $\kappa \in L(V; W)$ and let $k$ be a $(\mathcal{D}', \hat{\mathcal{D}})$-weak solution of the quantum stochastic differential equation (2.3). If $k$ is weakly regular locally with respect to $\phi$ and is such that, for each $R \subset V$, $v \in (f) \in \mathcal{D}$, $u \in (f') \in \mathcal{D}'$, $t \in \mathbb{R}^+$ and $s \in [0, t]$, the map $\phi_{f(s)}^{t}$ is bounded on $V_{\mathcal{F}_s, R, \mathcal{F}_s}$ with respect to a corresponding regularity norm, then

(a) $k$ is linear, so that $k \in \mathbb{P}_{\text{wrt}}(V : \mathcal{D}, \mathcal{D}')$, and

(b) the equation (2.3) has no other such solutions.

**Proof.** Fix $\xi' = u \in (f') \in \mathcal{D}'$, $\xi = u \in (f) \in \mathcal{D}$ and $t \in \mathbb{R}^+$.

(a) Let $x, y \in V$ and $\lambda \in \mathbb{C}$; set $R = \{x, y, x + \lambda y\}, U = V_{\mathcal{F}_s, R, \mathcal{F}_s}$ with a regularity norm $\|\cdot\|$ and $C = 2C_{\mathcal{F}_s, R, \mathcal{F}_s}$; and define

$$\gamma_{\mathcal{F}_s}^\lambda(z', z) = \left\langle \xi', [k_{s}(z') + \lambda k_{s}(z) - k_{s}(z' + \lambda z)] \xi \right\rangle$$

for $z, z' \in U, s \in [0, t]$.

By the regularity assumption this satisfies

$$\|\gamma_{\mathcal{F}_s}^\lambda(z', z)\| \leq C(\|z'\| + |\lambda|\|z\|).$$

The linearity of $\kappa$ and each $\phi_{\xi}^{\lambda}$ yields the identity

$$\gamma_{\mathcal{F}_s}^\lambda(z', z) = \int_0^s dr \, \gamma_{\mathcal{F}_s}^\lambda\left(\phi_{f(r)}^{t}(z'), \phi_{f(r)}^{t}(z)\right).$$

Iterating this and using the boundedness assumption gives

$$\|\gamma_{\mathcal{F}_s}^\lambda(x, y)\| \leq \frac{t^n}{n!} C M^n (\|x\| + |\lambda|\|y\|),$$

where $M = \max \{\|\phi_{\xi}^{r}(z)\| : z \in U, \|z\| \leq 1, c' \in F_t, c \in F_t\}$. Thus $\gamma_{\mathcal{F}_s}^\lambda(x, y) = 0$. It follows that $k$ is linear.

(b) Let $\tilde{k}$ be another such solution. For $x \in V$ and $t \in \mathbb{R}^+$ define

$$\gamma_s(z) = \left\langle \xi', [k_{s}(z') - \tilde{k}_{s}(z)] \xi \right\rangle \quad (z \in V_{\mathcal{F}_s, \{x\}, \mathcal{F}_s}, s \in [0, t]).$$

Then

$$\|\gamma_s(z)\| \leq C \left(\max\{\|z\|, \|z\|\_\}\right),$$

where $C = C_{\mathcal{F}_s, \mathcal{F}_s}^{\lambda, \chi} + C_{\mathcal{F}_s, \mathcal{F}_s}^{\lambda, \chi}$ and $\|\cdot\|$ and $\|\cdot\|_\_\$ denote the corresponding regularity norms. Arguing as in (a) yields (b). □

The following two special cases are relevant for the case of coalgebraic ([LS]) and operator space (Section 3 of this paper) quantum stochastic differential equations respectively. The first applies in particular when $V$ is finite dimensional.
Corollary 2.4. Suppose that $\phi$ satisfies
\[ \dim V^{\phi}_{F_{\{x\}};F} < \infty \quad (F' \subset D', x \in V, F \subset D). \]
Then the quantum stochastic differential equation (2.3) has at most one $(D', D)$-weak solution. Moreover any such solution is necessarily linear.

Corollary 2.5. Suppose that $V$ is a Banach space and the sesquilinear map $\phi$ is $B(V)$-valued. Then the quantum stochastic differential equation (2.3) has at most one linear $(D', D)$-weak solution $k$ for which each $\omega'_{t,\xi} \circ k_t$ is bounded ($\xi' \in D', \xi \in D, t \in \mathbb{R}_+)$.

3. Existence and dependence on initial conditions

For this section let $V$ be an operator space (with conjugate operator space $V^\vee$ and conjugation $x \mapsto x^1$), let $Y$ be an operator space in $B(\mathfrak{h}; \mathfrak{h}')$, let $D = \mathfrak{h} \otimes \mathcal{E}_D$ and $D' = \mathfrak{h}' \otimes \mathcal{E}_{D'}$ for dense subspaces $D$ and $D'$ of $k$ and recall the notation (2.1). Then $\mathbb{P}(\mathcal{V} \to \mathcal{Y} : D'; D')$ denotes the following class of processes on $\mathcal{V}$:
\[ \{ k \in \mathbb{P}(\mathcal{V} : D'; \mathfrak{h}' \otimes \mathcal{F}) : \omega_{t,\varepsilon} \circ k_t(V) \subset Y \text{ for all } t', \varepsilon \in \mathcal{E}_{D'}; t \in \mathbb{R}_+ \}. \]

Recall that $k$-bounded means bounded if the noise dimension $k$ is finite dimensional and completely bounded otherwise. For operator spaces $V$ and $W$, we write $k-B(V; W)$ for the space of all linear $k$-bounded maps acting from $V$ to $W$, and give it the operator norm if $k$ is finite-dimensional and the cb-norm otherwise.

We consider the quantum stochastic differential equation (2.3)
\[ dk_t = k_t \cdot d\Lambda_\phi(t), \quad k_0 = \varepsilon \circ \kappa \]
where $\phi \in L(\hat{D}; k-B(V; CB(\hat{k}; V))) < SL(\hat{k}, \hat{D}; B(V))$ and $\kappa \in k-B(V; Y)$. Now ampliation is of bounded operators, so $i(Y) \subset Y \otimes M B(F)$. We say that $\phi$ has ‘$k$-bounded columns’ (cf. [LW3]). Note that $CB(\hat{k}; V) = k-B(\hat{k}; V)$ (topological isomorphism).

Theorem 3.1. Let $\phi \in L(\hat{D}; k-B(V; CB(\hat{k}; V)))$ and $\kappa \in k-B(V; Y)$. Then the quantum stochastic differential equation (2.3) has a $D$-strong solution $k \in \mathbb{P}(\mathcal{V} \to \mathcal{Y} : D; D')$, enjoying the following properties

(a) $k$ has $k$-bounded columns:

\[ k_{t,|e} \in k-B(V; Y \otimes M |F \rangle) \quad (t \in \mathbb{R}_+, \varepsilon \in \mathcal{E}_D). \]

(b) For each $\varepsilon \in \mathcal{E}_D$ the map

\[ \mathbb{R}_+ \to k-B(V; Y \otimes M |F \rangle), \quad s \mapsto k_{s,|e} \]

is locally Hölder-continuous with exponent $1/2$.

(c) If $\tilde{k}$ is a linear $(D_1', D_1)$-weak solution of (2.3), for exponential domains $D_1'$ and $D_1$ contained in $D'$ and $D$ respectively, then $\tilde{k}$ is a restriction of $k$:
\[ \tilde{k}_t(x) = k_t(x)|_{D_1'} \quad (x \in V, t \in \mathbb{R}_+). \]

(d) If $\phi$ has cb-columns and $\kappa$ is completely bounded then $k$ has cb-columns and $\textbf{(b)}$ holds with $CB(V; Y \otimes M |F \rangle)$ in place of $k-B(V; Y \otimes M |F \rangle)$.

Proof. Define a process $k \in \mathbb{P}(\mathcal{V} \to \mathcal{Y} : D; D')$ as follows: $k_t = \Lambda_\phi \circ v$ where
\[ v^e \in L(\hat{D}'^e; k-B(V; Y \otimes M |\hat{k}^e \rangle) \subset L(V; \mathcal{O}(\mathfrak{h} \otimes \hat{D}'^e; \mathfrak{h}' \otimes \hat{k}^e)) \quad (e \in \mathbb{Z}) \]
is defined by
\[ E^{\zeta_1, \ldots, \zeta_n}_{\zeta_1, \ldots, \zeta_n} v^n_{\zeta_1, \ldots, \zeta_n} = \kappa \circ \phi_{\zeta_1}^{\zeta_1} \circ \cdots \circ \phi_{\zeta_n}^{\zeta_n} \quad (\zeta_1, \ldots, \zeta_n \in \hat{D}, \zeta_1, \ldots, \zeta_n \in \hat{k}). \] (3.1)
Thus, in terms of any concrete realisation of $V$ in $B(\mathfrak{h})$ for a Hilbert space $\mathfrak{h}$,
\[ v^n_{\zeta_1, \ldots, \zeta_n} = \tau \circ (\kappa \cdot \phi_{\zeta_1} \cdots \phi_{\zeta_n} \circ \phi_{\zeta_1} \cdots \phi_{\zeta_n} \circ \phi_{\zeta_1} \cdots \phi_{\zeta_n}) \].
where \( \tau : Y \otimes_M \hat{k}^{\otimes n} \to Y \otimes_M \hat{k}^{\otimes n} \) denotes the tensor flip reversing the order of \( n \) copies of \( \hat{k} \). Therefore, if \( k \) is finite dimensional then Lemma \([12]\) implies that

\[
\|t^{(\tau \otimes \cdots \otimes \tau)}_{\kappa}\| \leq \|\kappa\| \left( \sqrt{\dim k} \max_{i} \|\phi(\zeta)\| \right)^n,
\]

whereas if \( \kappa \) is completely bounded and \( \phi \) has cb-columns then

\[
\|t^{(\tau \otimes \cdots \otimes \tau)}_{\kappa}\|_{cb} \leq \|\kappa\|_{cb} \left( \max_{i} \|\phi(\zeta)\|_{cb} \right)^n.
\]

It follows from \([1.4]\) and \([1.5]\) that \( k_{t,\epsilon}(V) \subset Y \otimes_M |F\) and \( k_{t,\epsilon} \) is bounded \( V \to Y \otimes_M |F\) \((\epsilon = \epsilon(g) \in \mathcal{E}_D, t \in \mathbb{R}_+)\), with

\[
\|k_{t,\epsilon}\| \leq \|\kappa\| \|\epsilon\| \sum_{n \geq 0} \frac{C^n}{\sqrt{n!}}, \quad \text{and}
\]

\[
\|k_{t,\epsilon} - k_{s,\epsilon}\| \leq \sqrt{t-s} \|\kappa\| \|\epsilon\| C \left(\max_{n \geq 0} \frac{C^n}{\sqrt{n!}} \right) \quad (0 \leq s \leq t \leq T),
\]

where \( C = C(g,T)\sqrt{C'} \max \{ \|\phi(\zeta)\| : \zeta = \epsilon(g) \in \operatorname{Ran} \hat{g} \}_{[0,T]} \), with \( \| \cdot \| \) and \( C' \) meaning \( \| \cdot \| \) and \( \dim k \) respectively, when \( k \) is finite-dimensional, but \( \| \cdot \|_{cb} \) and 1 otherwise. We have therefore shown that \( k \) satisfies \([3]\) and \([4]\) when \( k \) is finite dimensional.

Now suppose that \( \kappa \) is completely bounded and \( \phi \) has cb-columns. Then, identifying \( M_N(Y \otimes_M \hat{k}) = Y \otimes_M |k| \otimes_M M_N \) with \( M_N(Y) \otimes_M |k| = Y \otimes_M M_N \otimes_M |k| \) gives

\[
(k_{t,\epsilon})^{(N)} = \tilde{k}_{t,\epsilon} \quad (N \in \mathbb{N}, t \in \mathbb{R}_+, \epsilon \in \mathcal{E}_D),
\]

where \( \tilde{k} \) is the process arising from the above construction when \( \kappa \) and \( \phi \) are replaced by \( \kappa^{(N)} \) and \( \phi^{(N)} \), \( \phi^{(N)} \) being given by \((\phi^{(N)})(\zeta) = (\phi(\zeta))^{(N)}\). It follows that the above estimates apply with cb-norms on the left-hand side (as well as the right). This completes the proof of \([3]\), \([4]\) and \([5]\).

Recalling \([1.3]\) we next note that \( k \) enjoys the following useful ‘form representation’: for \( \epsilon = \epsilon(g) \in \mathcal{E}_D, \epsilon' = \epsilon(g') \in \mathcal{E} \) and \( t \in \mathbb{R}_+ \),

\[
e^{-\langle \sigma' \cdot g', \omega_{\epsilon',\epsilon} \rangle} k_t = \int_{\Gamma_{[0,t]}} d\sigma \nu^{\sigma'_g,\sigma' \cdot g}_t \quad (t \in \mathbb{R}_+)
\]

in \( B(V;Y) \) where

\[
u^{\sigma' \cdot g,\sigma}_t = \kappa \circ \phi^{\sigma'_{\sigma}(s_1)} \circ \cdots \circ \phi^{\sigma'_{\sigma}(s_n)} \quad \sigma = \{s_1 < \cdots < s_n\} \in \Gamma.
\]

Therefore

\[
\omega_{\epsilon',\epsilon} \cdot k_t - \langle \epsilon', \epsilon \rangle \kappa = \langle \epsilon', \epsilon \rangle \int_{\Gamma_{[0,t]}} d\sigma (1 - \delta_{\emptyset}(\sigma)) \nu^{\sigma'_{\sigma},\sigma}_t
\]

\[
= \langle \epsilon', \epsilon \rangle \int_0^t ds \int_{\Gamma_{[0,s]}} d\rho \nu^{\sigma'_{\rho},\rho}_{s,\rho} \circ \phi^{\sigma'_{\sigma}(s)}
\]

\[
= \langle \epsilon', \epsilon \rangle \int_0^t ds \int_{\Gamma_{[0,s]}} d\rho \nu^{\sigma'_{\rho},\rho}_{s,\rho} \circ \phi^{\sigma'_{\sigma}(s)}
\]

\[
= \int_0^t ds \omega_{\epsilon',\epsilon} \cdot \{k_s \circ \phi^{\sigma'_{\sigma}(s)}\},
\]

so \( k_s \) is a \((D,D')\)-weak solution of \([2.2]\).

Now define a process \( K \in \mathcal{P}(V \to Y \otimes_M |\hat{k}|) : h \circ \hat{D} \circ \mathcal{E}_D, h' \circ \hat{D}' \circ \mathcal{E} \) by

\[
K_{t,\zeta}(\phi) = k_{t,\zeta} \cdot \phi(\zeta) \quad (t \in \mathbb{R}_+, \zeta \in \hat{D}, \epsilon \in \mathcal{E}_D).
\]
Since it is (pointwise strongly) continuous, by part (b), \( K \) is quantum stochastically integrable. Moreover, since
\[
E_k^{\nu} K_{t,|\nu \otimes \nu|} = E_k^{\nu} k_{t,|\nu|} \circ \phi_{|\nu|} = k_{t,|\nu|} \circ \phi_{|\nu|},
\]
\( K \) also satisfies \((2.5)\). Therefore \( k \) is a \( \mathcal{D} \)-strong solution of \((2.3)\). Part (c) follows from the uniqueness result Corollary \((2.5)\). This completes the proof. \( \square \)

**Notation.** The process uniquely determined by \( \kappa \) and \( \phi \) in this theorem will be denoted \( k^{\kappa,\phi} \), extending the established notation \( k^{\phi} \) for the case \( Y = V \) and \( \kappa = \text{id}_V \).

**Corollary 3.2.** Let \( \phi \in k\cdot B(V; CB(T(\hat{k}); V)) \) and \( \kappa \in k\cdot B(V; Y) \). Then (for any exponential domains \( \mathcal{D} \) and \( \mathcal{D}' \)) the quantum stochastic differential equation \((2.3)\) has a unique \( \mathcal{D}, \mathcal{D}' \)-weakly regular weak solution \( k \in \mathcal{P}(V \rightarrow Y : \mathcal{D}, \mathcal{D}'); \) it is also a \( \mathcal{D} \)-strong solution.

Here \( T(\hat{k}) \) denotes the operator space of trace-class operators on \( \hat{k} \) and we are invoking the natural complete isometry \( CB(T(\hat{k}); V) = CB([\hat{k}]; CB([\hat{k}]; V)) \). If \( V \) is a concrete operator space then there is a natural completely isometric isomorphism between \( CB(T(\hat{k}); V) \) and \( V \otimes_M B(\hat{k}) \), so that \( \phi \) above may be viewed as a map in \( k\cdot B(V; V \otimes_M B(\hat{k})) \).

**Corollary 3.3.** Suppose that \( \phi \) has a conjugate \( \phi^{\dagger} \) in \( L(\hat{D}^D; k\cdot B(V^D; CB([\hat{k}]; V^D))) \). Then \( k^{\kappa,\phi} \in \mathcal{P}(V \rightarrow Y : \mathcal{D}, \mathcal{D}'; \kappa,\phi) \) and \((k^{\kappa,\phi})^{\dagger} = k^{\kappa,\phi^{\dagger}}\).

**Proof.** In view of the identity
\[
\hat{e}_G^{\sigma} \hat{g}^{\sigma'} = (v^{\sigma'})^{\dagger} \quad (g \in S_D, g' \in S_D', \sigma \in \Gamma),
\]
where \( \hat{u} \) is defined by \((2.4)\) with \( k^{\dagger} \) and \( \phi^{\dagger} \) in place of \( \kappa \) and \( \phi \), this follows from the form representations \((3.2)\) for \( k^{\kappa,\phi^{\dagger}} \) and \( k^{\kappa,\phi} \). \( \square \)

**Remarks.** (i) If \( U \) is a subspace of \( V \) invariant under each of the maps \( \phi_{|\zeta|}^{\dagger} \) (\( \zeta' \in \hat{k}, \zeta \in \hat{D} \)) then \( \omega_{\nu',\nu} \circ k_t(U) \subset \kappa(U) \) for all \( \nu', \nu \in \mathcal{E}_D, \nu \in \mathcal{E}_D \).

(ii) The identification \((3.2)\) extends as follows. If \( \phi \) has cb-columns and \( \kappa \) is completely bounded then \( h \)-matrix space liftings, of coefficient, initial condition and solution, are compatible:
\[
(k_{t,|\nu|})^h = k_{t,|\nu|}^\kappa \quad (3.5)
\]
where \( \kappa' = \kappa \otimes_M \text{id}_{B(h)} \) and \( \phi' \) is determined by \( \phi'_{|\zeta|} = (\phi_{|\zeta|})^h \). This follows easily from the equality
\[
(\kappa \bullet \phi_{|\zeta|} \bullet \cdots \bullet \phi_{|\zeta_n|})^h = \kappa' \bullet \phi'_{|\zeta_1|} \bullet \cdots \bullet \phi'_{|\zeta_n|}
\]
(in the notation \((1.9)\)) and the identity
\[
\Lambda^n(T \otimes L) = T \otimes \Lambda^n(L) \quad (T \in B(h), n \in \mathbb{Z}_+, L \in B(h; h') \otimes B(\hat{k}^\otimes n)).
\]

In the next result we consider the case where the operator space \( V \) is concrete itself, and so the process \( k^{\kappa,\phi} \) may be compared to the process \( k^{\phi} \).

**Proposition 3.4.** Let \( \kappa \) and \( \phi \) be as in Theorem \((3.2)\) and suppose that the operator space \( V \) is concrete. Then the following hold.

(a) \( \omega_{\nu',\nu} \circ k_{t,|\nu|}^{\kappa,\phi} = \kappa \circ (\omega_{\nu',\nu} \circ k_{t,|\nu|}^\phi) \quad (\nu' \in \mathcal{E}_D, \nu' \in \mathcal{E}, t \in \mathbb{R}^+) \).

(b) If \( \kappa \) is completely bounded then
\[
k_{t,|\nu|}^{\kappa,\phi} = \kappa \bullet k_{t,|\nu|}^{\phi} \quad (t \in \mathbb{R}^+, \nu \in \mathcal{E}_D).
\]
(c) If \( \kappa \) is completely bounded and the process \( k^\phi \) is completely bounded then \( k^{\kappa,\phi} \) is the completely bounded process given by

\[
k_t^{\kappa,\phi} = \kappa \bullet k_t^\phi \quad (t \in \mathbb{R}_+).
\]

Proof. (a) follows easily from (3.3); (b) and (c) are simple consequences of (a). \( \square \)

Remarks. Since the process \( k^{\kappa,\phi} \) depends linearly on \( \kappa \), the proposition implies that it also depends continuously on its initial condition — in various senses, depending on the regularity of the initial condition and process \( k^\phi \).

If \( V = Y \) and the initial condition commutes with the coefficient operator, in the sense that \( \kappa \bullet \phi_\zeta = \phi_\zeta \circ \kappa \) (\( \zeta \in \tilde{D} \)), then \( \kappa \bullet \phi^{*n}_\eta = \phi^{*n}_\eta \circ \kappa \) (\( n \in \mathbb{Z}_+, \eta \in \tilde{D}^{\otimes n} \)) and so

\[
k_t^{\kappa,\phi} = k_t^\phi \circ \kappa \quad (t \in \mathbb{R}_+).
\]

Injectivity of the quantum stochastic operation \( \Lambda \) (LW[4], Proposition 2.3) implies that

\[
k^{\kappa,\phi} = k^{\kappa',\phi'} \quad \text{if and only if} \quad \kappa = \kappa' \quad \text{and} \quad \kappa \bullet \phi_\zeta = \kappa' \bullet \phi'_\zeta \quad (\zeta \in \tilde{D}).
\]

4. Localisable equations

In this section we consider the case where the source space is a vector space on which the coefficient map of the quantum stochastic differential equation is finitely localisable. Thus let \( V \) be a complex vector space, let \( D \) be a dense subspace of the noise dimensions space \( k \) and consider our quantum stochastic differential equation (2.3)

\[
dk_t = k_t \bullet d\Lambda_\phi(t), \quad k_0 = \iota \circ \kappa,
\]

where \( \phi \in L(\tilde{D}; L(V; V \circ [\hat{\kappa}])) \). We consider two cases. Recall that if \( \phi \) is finitely localisable then it necessarily belongs to \( L(V; V \circ \mathcal{O}(\tilde{D})) \); also recall the notation (2.1).

Theorem 4.1. Let \( \phi \in L(V; V \circ \mathcal{O}(\tilde{D})) \) be finitely localisable and let \( \kappa \in L(V; Y) \), where \( Y \) is an operator space in \( B(\mathfrak{h}; \mathfrak{h}') \). Set \( D = \mathfrak{h} \circ \mathcal{E}_D \). Then there is a process \( k \in \mathcal{P}(V \to Y : D, D'_e) \), which is a \( D \)-strong solution of (2.3) and enjoys the following further properties:

(a) \( k \) is \( L(V; Y \circ \mathcal{O}(\mathcal{E}_D)) \)-valued.

(b) The map \( s \mapsto k_s(x) \) is locally Hölder-continuous \( \mathbb{R}_+ \to Y \otimes_{sp} \mathcal{F} \) with exponent \( \frac{1}{2} \) \((x \in V, \varepsilon \in \mathcal{E}_D)\).

(c) If \( k \) is a \((D'_e, D_1)\)-weak solution of (2.3), where \( D_1 \) and \( D'_e \) are exponential domains contained in \( D \) and \( D'_e \) respectively, then \( \tilde{k} \) is a restriction of \( k \):

\[
\tilde{k}_t(x) = k_t(x)|_{D_1}.
\]

(d) For any subspace \( V_1 \) localising \( \phi \), \( k_t(V_1) \subset \kappa(V_1) \circ \mathcal{O}(\mathcal{E}_D) \quad (t \in \mathbb{R}_+) \).

Proof. Consider a finite dimensional subspace \( V_1 \) of \( V \) which localises \( \phi \) and let \( \kappa_1 \) and \( \phi_1 \) be the restrictions of \( \kappa \) and \( \phi \) to \( V_1 \). By endowing \( V_1 \) with operator space structure \( \kappa_1 \) becomes completely bounded and \( \phi_1 \) enjoys completely bounded columns. Theorem 3.1 therefore permits us to define a process \( k_1 \in \mathcal{P}(V_1 \to Y : D, D'_e) \) by \( k_1 = k^{\kappa_1,\phi_1} \). Now suppose that \( k_2 \in \mathcal{P}(V_2 \to Y : D, D'_e) \) is the process arising in this way from another finite dimensional subspace \( V_2 \) localising \( \phi \). Then the finite dimensional subspace \( V_3 := V_1 \cap V_2 \) also localises \( \phi \) and so gives rise to a third process \( k_3 \in \mathcal{P}(V_3 \to Y : D, D'_e) \). By the uniqueness part of Theorem 3.1 it follows that \( k_3 \) agrees with both \( k_1 \) and \( k_2 \) on \( V_3 \). The following prescription therefore gives a consistent definition of a process \( k \in \mathcal{P}(V \to Y : D, D'_e) \):

\[
k_t(x) = k_t^{\kappa_1,\phi_1}(x) \quad (t \in \mathbb{R}_+),
\]

where \( \kappa_1 \) and \( \phi_1 \) are the restrictions of \( \kappa \) and \( \phi \) to any finite dimensional subspace of \( V \) containing \( x \) which localises \( \phi \). That is \( k \) is a \( D \)-strong
solution of (2.3) satisfying properties (i), (ii), now follows easily from Theorem 5.1 and the subsequent remark. Observe that (iii) implies that for each \( s \geq 0 \) and \( \varepsilon \in \mathcal{E}_D \) the map \( k_{s,\varepsilon} \) takes values in \( Y \otimes |F| \). \( \square \)

**Remark.** Clearly the following weaker localisable property suffices: for all \( x \in V \) and \( F \subset \subset D \) there is a finite dimensional subspace \( V_1 \) of \( V \) containing \( x \) such that \( \phi(V_1) \subset V_1 \otimes |k| \) for all \( \zeta \in \hat{F} \); conclusion (ii) is then modified accordingly.

**Notation.** We again use the notation \( k^{\kappa,\phi} \) for the process obtained in the above theorem.

As before, \[ k^{\kappa,\phi} = k^{\kappa',\phi'} \] if and only if \( \kappa = \kappa' \) and \( \kappa \cdot \phi = \kappa' \cdot \phi' \).

**Corollary 4.2.** Suppose that \( \phi \in L(V; V \otimes \mathcal{O}^t(\hat{D}, \hat{D}')) \) for some dense subspace \( D' \) of \( k \). Then \( k^{\kappa,\phi} \in \mathbb{P}^t(V : D, D') \) where \( D' = h' \otimes \mathcal{E}_{D'} \) and \( (k^{\kappa,\phi})' = k^{\kappa',\phi'} \).

We next give a variant of the above existence theorem. Note that the definition of \( \mathbb{P}^t(V \rightarrow Y : D, D') \) extends in an obvious way if \( Y \) is replaced by \( W = \mathcal{O}(\hat{D}; h') \) and \( D \) by \( \mathcal{O} \otimes \mathcal{E}_D \).

**Theorem 4.3.** Let \( \phi \in L(V; V \otimes \mathcal{O}(\hat{D})) \) be finitely localisable, let \( \kappa \in L(V; W) \) and set \( D = \mathcal{O} \otimes \mathcal{E}_D \). Then the conclusions of Theorem 4.2 hold with \( Y \) replaced by \( W \) and (i), (ii) and (iii) replaced by

(a)’ \( s \mapsto k^\kappa_\phi(x) \xi \) is locally Hölder-continuous \( \mathbb{R}_+ \rightarrow h' \otimes F \) with exponent \( \frac{1}{2} \),

for all \( x \in V \) and \( \xi \in \mathcal{D} \).

**Proof.** For \( u \in \mathcal{D} \), Theorem 4.2 applies, with \( Y = |h'| \), to the quantum stochastic differential equation

\[ dk_t = k_t \cdot d\Lambda_s(t), \quad k_0 = \iota \circ \kappa_{|u|}; \]

Let \( l^\nu \in \mathbb{P}(V \rightarrow |h'| : \mathcal{E}_D, h' \otimes \mathcal{E}) \) be its \( \mathcal{E}_D \)-strong solution. For \( u, v \in \mathcal{D} \) and \( \lambda \in \mathbb{C} \), if \( g \in \mathcal{E}_D \) and \( \xi' = v' \xi(g') \in D' \) then the maps \( \gamma_s : V \rightarrow \mathbb{C} \) (\( s \in \mathbb{R}_+ \)) given by

\[ \gamma_s(x) = \left\langle \xi', \left[ l^\nu_s(x) + \lambda l^{v'}_s(x) - l^{v+\lambda v}_s(x) \right] \xi(g) \right\rangle \]

satisfy

\[ \gamma_t(x) = \int_0^t ds \gamma_s(\phi^\nu_{g'}(s)(x)) \quad (x \in V, t \in \mathbb{R}_+). \]

In view of finite localisability, iteration shows that \( \gamma \) is identically zero. If follows that \( k^\kappa_\phi(x)w\xi(g) := l^\nu_t(x)\xi(g) \quad (x \in V, u \in \mathcal{D}, g \in \mathbb{S}_D, t \in \mathbb{R}_+), \) defines a process \( k^\kappa_\phi \in \mathbb{P}(V \rightarrow W : \mathcal{D}, D') \) which is a \( \mathcal{D} \)-strong solution of (2.3); it is clear that it satisfies (a)’ and (ii) too. \( \square \)

## 5. Quantum Stochastic Cocycles

In this section we give a new result on the infinitesimal generation of quantum stochastic cocycles (cf. [LW2]). At the end we describe how the result may be applied to quantum stochastic convolution cocycles on a coalgebra ([LS]). Fix an operator space \( Y \) in \( B(h; h') \) and exponential domains \( D = h' \otimes \mathcal{E}_D \) and \( D' = h' \otimes \mathcal{E}_{D'} \).

The following notations for a process \( k \in \mathbb{P}(Y \rightarrow Y : D, D') \) prove useful:

\[ k_{\xi'}^{x'; g} := e^{-\langle g_{[0,1]} \cdot \xi_{[0,1]} \rangle_{\omega_{g_{[0,1]}, x_{g_{[0,1]}}}}}} \cdot k_t \quad (g' \in D', g \in \mathbb{S}_D, t \in \mathbb{R}_+) \]  \hspace{1cm} (5.1)

\[ k_{\xi'}^{x'; c} := k_{\xi'}^{x' \cdot c_{[0,1]}} \quad (c' \in D', c \in D). \]  \hspace{1cm} (5.2)
Thus $k^g_{t+\epsilon} \in L(Y)$ and the process is called initial space bounded if each map $k^g_{t+\epsilon}$ is bounded (cf. the condition of having bounded columns).

**Definition.** A process $k \in \mathbb{P}(Y \to Y : \mathcal{D}, \mathcal{D}')$ is a $(\mathcal{D}', \mathcal{D})$-weak quantum stochastic cocycle on $Y$ if it satisfies

$$k^g_{t+\epsilon} = k^g_t \circ k^g_{S^r \sigma^g} \quad (5.3)$$

for all $g' \in \mathcal{S}_{D'}$, $r, t \in \mathbb{R}_+$, and $g \in \mathcal{S}_D$, where $(S_t)_{t \geq 0}$ is the (isometric) right-shift semigroup on $L^2(\mathbb{R}_+; k)$.

Let $\mathcal{QSC}(Y : \mathcal{D}, \mathcal{D}')$ denote the collection of these. Also define

$$\mathcal{QSC}(Y : \mathcal{D}', \mathcal{D}) = \mathcal{QSC}(Y : \mathcal{D}, \mathcal{D}') \cap \mathbb{P}(Y \to Y : \mathcal{D}, \mathcal{D}');$$

if $k$ is in this class then $k^g_{\epsilon} = (k^g_{t+\epsilon})^\dagger$ and it is easily seen that the conjugate process $k^\dagger$ is a cocycle on $Y^\dagger$.

In case the process has cb-columns (each map $x \mapsto k_{t,[c]}(x)$ is completely bounded $Y \to V \otimes_M [\mathcal{F}]$) the cocycle relation is equivalent to

$$k_{r+t,[c]}(g_{(r_0, r+t)}^t) = k_{r,[c]}(g_{(r_0, r)}^t) \bullet k_{t,[c]}(g_{(r+r-t, r+t)}^t);$$

in case the process itself is completely bounded it simplifies further, to the more recognisable cocycle property:

$$k_{r+t} = k_t \bullet \sigma_r \bullet k_t$$

were $(\sigma_r)_{r \geq 0}$ is the CCR flow of index $k$ \(\left[\text{AV}\right]\).

**Lemma 5.1.** Let $k \in \mathbb{P}(Y \to Y : \mathcal{D}, \mathcal{D}')$ and define $P^{c', c} := (k_{t,[c]}^t)_{t \geq 0}$ ($c', c \in k$). Then the following are equivalent:

(i) $k \in \mathcal{QSC}(Y : \mathcal{D}, \mathcal{D}')$.

(ii) For all $c' \in \mathcal{D}'$ and $c \in \mathcal{D}$, $P^{c', c}$ is a one-parameter semigroup in $L(Y)$ and, for all $g' \in \mathcal{S}_{D'}$, $g \in \mathcal{S}_D$ and $t \in \mathbb{R}_+$, $k^g_{t \cdot \epsilon} = \ell^g_{t \cdot \epsilon}$ where

$$\ell^g_t = p^g_{t-t_0}(g(t_0)) \cdots p^g_{t_n-t_n}(g(t_n))$$

(5.4)

with $n \in \mathbb{Z}_+$, $t_0 = 0$, $t_{n+1} = t$ and $\{t_1 < \cdots < t_n\}$ being precisely the (possibly empty) union of the sets of points of discontinuity of $g'$ and $g$ in $[0, t]$.

(iii) For all $g' \in \mathcal{S}_{D'}$, $g \in \mathcal{S}_D$ and $t \in \mathbb{R}_+$,

$$k^g_{t \cdot \epsilon} = p^g_{t-t_0}(g(t_0)) \cdots p^g_{t_n-t_n}(g(t_n))$$

(5.5)

whenever $n \in \mathbb{Z}_+$ and $\{0 = t_0 \leq \cdots \leq t_{n+1} = t\}$ includes all the discontinuities of $g'_{[0,t]}$ and $g_{[0,t]}$.

**Proof.** Straightforward, see \(\left[\text{AV}\right]\). \hfill \Box

The one-parameter semigroups $\{P^{c', c} : c' \in \mathcal{D}', c \in \mathcal{D}\}$ in $L(Y)$ are referred to as the associated semigroups of $k$, $P^{0,0}$ as its Markov semigroup and \(\left[5.3\right]\) as its semigroup decomposition. If $k$ is initial space bounded and each semigroup is norm continuous $\mathbb{R}_+ \to B(Y)$ then the cocycle is called Markov-regular. When the cocycle is contractive, norm continuity of any of the associated semigroups (such as its Markov semigroup) implies Markov-regularity \(\left[\text{AV}\right]\). Proposition 5.4. In view of the semigroup decomposition, Markov-regular cocycles are necessarily both weakly regular and weakly continuous processes.

Now consider the quantum stochastic differential equation \(\left[2.3\right]\) where $\kappa = \text{id}_Y$:

$$dk_t = k_t \bullet d\Lambda_\kappa(t), \quad k_0 = t.$$  (5.6)
The following result is a coordinate-free counterpart to Proposition 5.2 of [LW2] in the operator space setting.

**Theorem 5.2.** Let \( \phi \in SL(\hat{D}, \hat{D}; B(Y)) \) and let \( k \in \mathcal{P}_{\phi_{\omega}(Y \to Y : \mathcal{D}, \mathcal{D})} \) be a \((\mathcal{D}, \mathcal{D}')\)-weak solution of the quantum stochastic differential equation (5.6). Then \( k \) is a Markov-regular quantum stochastic cocycle and the generators of its associated semigroups are given by

\[
\psi_{c', c} = \phi_{\omega}^c \quad (c' \in \mathcal{D}', c \in \mathcal{D}).
\]

**Proof.** Let \( \xi = v' \varepsilon(g) \in \mathcal{D}', \xi \in \mathcal{V}(g) \in \mathcal{D} \) and \( t \in \mathbb{R}_+ \). Define \( l_t^g: \mathcal{V}(g) \to \mathcal{V}(g) \) by (5.4) where \( P^{c', c} \) is the norm continuous semigroup in \( \mathcal{D}(Y) \) with generator \( \phi_{\omega}^c \).

Then \( m_t^g := k_t^g - l_t^g \) satisfies

\[
\langle v', m_t^g(x)v \rangle = \int_0^t ds \langle v', m_s^g(\phi_{\omega}^c(x))v \rangle.
\]

Iterating this gives

\[
\langle v', m_t^g(x)v \rangle = \int_0^1 ds_n \cdots \int_0^1 ds_1 \langle \omega_v, \xi \cdot k_{s_1} - \omega_{v', v} \cdot l_t^g \rangle(\phi_{\omega}^c)_{s_1} \cdots \phi_{\omega}^c_{s_n}(x).
\]

By \( \phi \)-weak regularity of \( k \) and norm continuity of \( l_t^g \), the integrand has a bound of the form \( C \|x\| M^n \) where the constants \( C \) and \( M \) are independent of \( n \). The identity \( k_t^g = l_t^g \) follows and so, by Lemma 5.1, \( k \) is a quantum stochastic cocycle with associated semigroups \( \{P_{c', c} : c' \in \mathcal{D}', c \in \mathcal{D}\} \). This completes the proof. \( \square \)

It follows from (5.4) that the associated semigroups are cb-norm continuous if and only if the sesquilinear map \( \phi \) is \( CB(Y) \)-valued.

**Remarks.** Note that, in this case, the ‘form representation’ of \( k \) (3.3) is given by:

\[
k_s^g = \int_{[0, t]} d\sigma v_s^g \quad (s \in [0, t])
\]

where \( v_s^g = \text{id}_V \) when \( \sigma = \emptyset \) and

\[
v_s^g = \phi_{\omega}^c(s_1) \circ \cdots \circ \phi_{\omega}(s_n) \quad \text{for} \quad \sigma = \{s_1 < \cdots < s_n\}.
\]

In particular, if \( k = k^\phi \) where \( \phi \in L(\hat{D}; k-B(Y; \mathcal{V} \otimes \hat{M})(\hat{D})) \) then

\[
v_s^g = \omega_{\gamma_{\sigma}(\phi, \pi_{\#}(\sigma)) \cdot v_{\#} \sigma},
\]

where \( v = v^\phi \) is defined by (5.1) with \( \kappa = \text{id}_V \), and the cocycle relation may be expressed as follows:

\[
\int_{[0, s]+} d\sigma v_s^g = \int_{[0, r]} dp \int_{[0, s]} d\tau v_p^g \circ v^S_{\tau} \circ v_{\#} \sigma.
\]

In this case the associated semigroup generators are given by

\[
\psi_{c', c} = \omega_{\gamma_{\sigma} \cdot \phi}.
\]

**Corollary 5.3.** Let \( \phi \in L(Y; Y \otimes \mathcal{O}(D)) \) and suppose that \( Y \) is finite dimensional. Then \( k^\phi \) is an \( L(Y; Y \otimes \mathcal{O}(D)) \)-valued Markov-regular quantum stochastic cocycle.

**Proof.** This follows from the theorem above and Theorem 5.1 since, for finite dimensional \( Y \), there are natural linear identifications

\[
L(Y; Y \otimes \mathcal{O}(E)) = L(E; L(Y; Y \otimes \mathcal{H})) = L(E; CB(Y; Y \otimes \mathcal{H})),
\]

for \((E, \mathcal{H})\) equal in turn to \((\hat{D}, \hat{K})\) and \((\mathcal{E}_D, \mathcal{F})\). \( \square \)
We now begin to develop converse results. The first is a coordinate-free counterpart to Theorem 5.6 of \([LW16]\) in the operator space setting.

**Theorem 5.4.** Let \(k \in \mathcal{QSC}^c(Y : D, D')\) and suppose that \(k\) is Markov-regular and the maps \(t \mapsto k_t(x)\xi\) and \(t \mapsto k_t(x)^*\xi^*\) \((x \in Y, \xi \in D, \xi^* \in D')\) are all continuous at 0. Then \(k\) is a \((D', D)\)-weak solution of the quantum stochastic differential equation \((5.9)\) for some \(\phi \in SL(\hat{D}', \hat{D}, B(Y))\).

**Proof.** Define a map as follows
\[
\phi : \hat{D}' \times \hat{D} \to B(Y), \left( \begin{pmatrix} z' \\ c' \end{pmatrix}, \begin{pmatrix} z \\ c \end{pmatrix} \right) \mapsto \begin{bmatrix} z' - 1 & 1 \\ \psi_{0,0} & \psi_{0,c} \end{bmatrix} \begin{bmatrix} z - 1 \\ 1 \end{bmatrix}
\]
where \(\{\psi_{c',c} : c' \in D', c \in D\}\) are the generators of \(k\)’s associated semigroups and, for \(x \in Y\), let \(\phi(x)\) denote the corresponding map \(\hat{D}' \times \hat{D} \to Y\). Markov-regularity implies that \(l^{(\phi,g)}\), given by \((5.3)\), satisfies
\[
l^{(\phi,g)}_t = \text{id}_Y + \int_0^t ds l^{(\phi,g)}_s \circ \psi_{c',c},
\]
where \(c' = g'(t-)\) and \(c = g(t-)\). But, by the semigroup decomposition, \(l^{(\phi,g)} = k^{(\phi,g)}\); since \(\phi_{c',c} = \psi_{c',c}\) it therefore suffices only to prove that \(\phi\) is sesquilinear.

Accordingly, fix \(v' \in \mathfrak{h}', v \in \mathfrak{h}\) and \(x \in Y\) and note the identity
\[
\langle v', \phi_{\zeta'}(x)v \rangle = \lim_{t \to 0^+} t^{-1} \langle \alpha(t), \beta(t) \rangle
\]
where \(\zeta' = (\zeta', \zeta) \in \hat{D}'\), \(\zeta = (\zeta) \in \hat{D}\),
\[
\alpha(t) = (k^{(1)}_t(x^*) - x^* \otimes 1) \left( v' \otimes \{ (z' - 1)\varepsilon(0) + \varepsilon(c_i'|0,t) \} \right)
\]
and
\[
\beta(t) = v \otimes (z, c_{[0,t]}, (2!)^{-1/2}c_{[0,t]}\otimes 2, \ldots),
\]
Thus if \(\zeta = \zeta_i + \lambda \zeta_2\) for \(\zeta_i = (\zeta'_i) \in \hat{D}\) \((i = 1, 2)\) and \(\lambda \in \mathbb{C}\) then
\[
\langle v', (\phi_{\zeta'}(x) - \phi_{\zeta_1}(x) - \lambda \phi_{\zeta_2}(x))v \rangle = \lim_{t \to 0^+} \langle \alpha(t), \gamma(t) \rangle
\]
where
\[
\gamma(t) = t^{-1} v \otimes ((n!)^{-1/2} \{ c_{\otimes n} - (c_1)_{\otimes n} - (\lambda c_2)_{\otimes n} \} \otimes 1_{[0,t]}),
\]
Since \(\gamma\) is locally bounded and \(\alpha(t) \to 0\) as \(t \to 0\), by the continuity of the process \(k^1\), this shows that \(\phi(x)\) is linear in its second argument. A very similar argument, in which the roles of \(k\) and \(k^1\) are exchanged, shows that \(\phi(x)\) is conjugate linear in its first argument. The result follows. \(\square\)

**Remarks.** In view of Corollary \([2.4]\) \(k\) is the unique linear \((D', D)\)-weak solution of \((5.9)\). In particular, if either

(a) \(\phi \in L(\hat{D}; k_\bullet B(Y; Y \otimes_M \hat{k}))\), or

(b) \(Y\) is finite dimensional and \(\phi \in L(Y; Y \otimes \mathcal{O}(\hat{D}))\),

then \(k = k^\phi\) and so satisfies the equation strongly. If \(Y\) is a \(C^*\)-algebra and \(k\) is completely positive and contractive then (a) holds (by \([LW2]\), Theorem 5.4 and \([LW3]\), Theorem 2.4); it also holds if \(k\) is finite dimensional.

We now identify a necessary and sufficient condition for (b) to hold. To this end let \(\mathcal{QSC}_{\text{He}}(Y : D, D')\) denote the collection of cocycles \(k \in \mathcal{QSC}(Y : D, D')\) for which
\[
k_{t, \xi}(x)\) is bounded and \(s \mapsto k_{s, \xi}(x) \in V \otimes_M |F|\) is Hölder \(1/2\)-continuous at 0
\]
\[(5.9)\]
(t \in \mathbb{R}_+, \varepsilon \in \mathcal{E}_D, x \in Y). Let \( QSC^i(Y : D, D') \) denote the set of processes \( k \in \mathcal{P}^i(Y : D, D') \) such that both \( k \) and \( k^\dagger \) satisfy \([5,3]\).

**Lemma 5.5.** Let \( k \in QSC^i_{\Re}(Y : D, D') \) be Markov-regular, with resulting \( \phi \) (from Theorem \([5,4]\)) viewed as a linear map \( Y \to SL(\hat{D}', \hat{D}; Y) \). Then, for all \( x \in Y, \phi(x) \) is separately continuous in each argument.

**Proof.** Fix \( x \in Y \) and let \( \zeta' = (\zeta'_0) \in D' \) and \( \zeta = (\zeta) \in \hat{D} \). Then, in terms of the generators of the associated semigroups, \( \phi^\zeta(x) \) equals

\[
\phi^\zeta(x) = \lim_{t \to 0^+} \frac{1}{t} \langle (1, x), (\zeta) \rangle \langle (1, x), (\zeta) \rangle - \varepsilon = \varepsilon(\langle 0, 1 \rangle),
\]

and, for each \( \nu' \in h', \epsilon \in D \) and \( \nu \in h, \) setting \( C(x, \epsilon) = \sup \{ t^{1/2} \langle k_{t, \epsilon}(x) - x \otimes 1, \varepsilon \rangle : t \in [0, 1] \} \) where \( \varepsilon = \varepsilon(\langle 0, 1 \rangle), \)

\[
\langle (1, x), (\zeta) \rangle \langle (1, x), (\zeta) \rangle - \varepsilon = \varepsilon(\langle 0, 1 \rangle), \]

Then \( \| \phi^\zeta(x) - \phi^\zeta(y) \| \leq \| C(x, \epsilon) \| \| \varepsilon(\langle 0, 1 \rangle) \| \leq \| C(x, \epsilon) \| \| \varepsilon(\langle 0, 1 \rangle) \| .
\]

Thus \( \| \phi^\zeta(x) - \phi^\zeta(y) \| \leq \| C(x, \epsilon) \| \| \varepsilon(\langle 0, 1 \rangle) \| .
\]

**Remark.** If \( Y \) is finite dimensional then the continuity assumption introduced in \([5,3]\) is equivalent to Hölder-continuity at 0 of the map

\[
s \mapsto k_{s, \epsilon}(x) \in B(Y; Y \otimes M \langle F \rangle) \quad (\epsilon \in \mathcal{E}_D).
\]

If \( h \) is finite dimensional then this further reduces to the pointwise strong continuity condition

\[
s \mapsto k_{s, \epsilon}(x) \xi \in h' \otimes F \text{ is Hölder } \frac{1}{2} \text{-continuous at } 0 \quad (x \in Y, \xi \in D).
\]

We alert the reader to the fact that not all finite dimensional operator spaces can be concretely realised in \( B(H) \), in the sense of a completely isometric embedding, for a finite dimensional Hilbert space \( H \). For more on this point, and for details of an example given by the operator space spanned by the canonical unitary generators of the universal \( C^* \)-algebra of a free group \( F_n \) (\( n \geq 3 \)), we refer to \([Pimsa]\).

**Theorem 5.6.** Let \( k \in QSC^i_{\Re}(Y : D, D') \) and suppose that \( Y \) is finite dimensional. Then there is a \( \phi \in L(Y; Y \circ \circ O^i(\hat{D}', \hat{D})) \) such that \( k = k^\phi \).

**Proof.** Note first that, since \( Y \) is finite dimensional, the continuity assumption implies that \( k \) is Markov-regular. Let \( \phi \in L(Y; SL(\hat{D}', \hat{D}; Y)) \) be the map resulting from Theorem \([5,4]\). Choose an ordered basis \( \{x_1, \ldots, x_n\} \) of \( Y \) and for \( x \in Y, \zeta' \in D' \) and \( \zeta \in \hat{D} \), let \( \phi^\zeta(x)^i, i = 1, \ldots, n \), denote the components of \( \phi^\zeta(x) \), with respect to this basis. By Lemma \([5,5]\) each functional \( \phi^i : D' \times \hat{D} \to \mathbb{C} \) is sesquilinear and continuous in each argument; it is therefore given by an operator \( \phi^i(x) \in O^i(\hat{D}', \hat{D};)'\):

\[
\phi^\zeta(x)^i = \langle \zeta', \phi^i(x) \zeta \rangle \quad (\zeta' \in D', \zeta \in \hat{D}).
\]
Moreover, each map \( x \mapsto \phi^{(i)}(x) \) is clearly linear. Thus, setting
\[
\phi(x) = \sum_{i=1}^{n} x_i \otimes \phi^{(i)}(x)
\]
defines a linear map \( \phi : Y \to Y \otimes O^1(\hat{D}, \hat{D}') \). Therefore, by Corollary 5.3 \( \phi \) generates a stochastic cocycle. In view of the identity
\[
(\omega_{\phi} \circ \phi)(x) = \sum_{i=1}^{n} \phi_{\hat{D}}^n(x_i) \otimes \phi_{\hat{D}'}(x) = \psi_{\phi, \epsilon}(x)
\]
and Theorem 5.2, \( k \) has the same associated semigroups as the cocycle \( k^\phi \). Thus \( k = k^\phi \) and the proof is complete.

By finite localisability for a process \( k \in \mathbb{P}(Y \to Y : D, D') \) we mean finite localisability for each \( k_t \). Combining the above result with Corollary 5.3 and Theorem 4.3 straightforward localisation arguments allow us to summarize the new results of this section as follows.

**Corollary 5.7.**

(a) Let \( \phi \in L(Y; Y \otimes O(\hat{D})) \) be finitely localisable. Then \( k^\phi \in \mathcal{QSC}_{\text{loc}}(Y : D, D') \) and is finitely localisable, moreover if \( \phi \in L(Y; Y \otimes O^1(\hat{D}, \hat{D}')) \) then \( k^\phi \in \mathcal{QSC}_{\text{loc}}^{\pm}(Y : D, D') \).

(b) Conversely, let \( k \in \mathcal{QSC}_{\text{loc}}^{\pm}(Y : D, D') \) be finitely localisable. Then there is a finitely localisable map \( \phi \in L(Y; Y \otimes O^1(\hat{D}, \hat{D}')) \) such that \( k = k^\phi \).

**Application to coalgebraic cocycles.** Theorem 5.2 yields an alternative proof of the principal implication in Theorem 5.8 of [LS1] which states that if \( C \) is a coalgebra with coproduct \( \Delta \) and counit \( \epsilon \), then any Hölder-continuous quantum stochastic convolution cocycle \( l \in \mathcal{P}^1(C \to C; E_D, E_{D'}) \), with Hölder-continuous conjugate, satisfies a coalgebraic quantum stochastic differential equation
\[
dl t = l_t \ast_s d\Lambda_{\phi}(t), \quad l_0 = \epsilon \circ \epsilon, \quad (5.10)
\]
for some map \( \varphi \in L(C; O^1(\hat{D}, \hat{D}')) \). We end with a sketch of a proof of this. The Fundamental Theorem on Coalgebras and localisation arguments allow us to effectively assume that \( C \) is finite dimensional. Assuming this, linearly embed \( C \) into \( B(h) \), for some (finite dimensional) Hilbert space \( h \), and observe that the process \( k_t \in \mathcal{P}^1(C \to C; h \otimes E_D, h \otimes E_{D'}) \), defined by the formula
\[
k_t = (\text{id}_h \otimes l_t) \circ \Delta \quad (t \geq 0), \quad (5.11)
\]
is a Hölder-continuous quantum stochastic cocycle on \( C \). Theorem 5.6 then implies that \( k \) satisfies the quantum stochastic differential equation \((5.6)\) for some \( \phi \in L(C; C \otimes O^1(\hat{D}, \hat{D}')) \). Set
\[
\varphi = (\epsilon \circ \text{id}_{O^1(\hat{D}, \hat{D}')}) \circ \phi. \quad (5.12)
\]
It is then easily checked that the convolution cocycle \( l \) satisfies the coalgebraic quantum stochastic differential equation \((5.10)\).

**Remark.** The idea outlined here, of using correspondences such as \((5.11)\) and \((5.12)\) for moving between quantum stochastic cocycles and quantum stochastic convolution cocycles, or their respective stochastic generators, also works well in the analytic context of quantum stochastic convolution cocycles on operator space coalgebras. This enables application of known results for quantum stochastic cocycles to the development of a theory of quantum Lévy processes on compact quantum groups and the characterisation of their stochastic generators. This is done in the forthcoming paper [LS4] which also contains many examples. Dilation of completely
positive convolution cocycles on a $C^*$-bialgebra to $\ast$-homomorphic convolution cocycles is treated in [S]. The main results, in both the algebraic and $C^*$-algebraic cases, are summarized in [LS].

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