\(\sigma\)-CONTINUOUS FUNCTIONS AND RELATED CARDINAL
CHARACTERISTICS OF THE CONTINUUM

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ABSTRACT. A function \(f : X \to Y\) between topological spaces is called \(\sigma\)-continuous (resp. \(\sigma\)-continuous) if there exists a (closed) cover \(\{X_n\}_{n \in \omega}\) of \(X\) such that for every \(n \in \omega\) the restriction \(f|X_n\) is continuous. By \(c_\sigma\) (resp. \(c_\sigma\)) we denote the largest cardinal \(\kappa \leq \mathfrak{c}\) such that every function \(f : X \to \mathbb{R}\) defined on a subset \(X \subset \mathbb{R}\) of cardinality \(|X| < \kappa\) is \(\sigma\)-continuous (resp. \(\sigma\)-continuous). It is clear that \(\omega_1 \leq c_\sigma \leq \kappa \leq \mathfrak{c}\). We prove that \(p \leq q_0 = c_\sigma = \min\{c_\sigma, b, q\} \leq c_\sigma \leq \min\{\text{non}(\mathcal{M}), \text{non}(\mathcal{N})\}\).

In this paper we introduce and study two cardinal characteristics of the continuum, related to \(\sigma\)-continuity.

A function \(f : X \to Y\) between two topological spaces is called \(\sigma\)-continuous (resp. \(\sigma\)-continuous) if there exists a countable (closed) cover \(\mathcal{C}\) of \(X\) such that for every \(C \in \mathcal{C}\) the restriction \(f|C\) is continuous.

The problem of finding a (Borel) function \(f : \mathbb{R} \to \mathbb{R}\) which is not \(\sigma\)-continuous is not trivial and was first asked by Lusin. This problem of Lusin was answered by Sierpiński [10], [17] (under CH), Keldyš [12], and later by Adyan, Novikov [11], van Mill and Pol [13], Jackson, Mauldin [10], Cichoń, Morayne, Pawlikowski, and Solecki [4], [18], Darji [5]. By a dichotomy [17] (under CH), Keldyš [12], and later by Adyan, Novikov [11], van Mill and Pol [13], Jackson, Mauldin [10], Cichoń, Morayne, Pawlikowski, and Solecki [4], [18], Darji [5]. By a dichotomy of Zapletal [21] (generalized by Pawlikowski and Sabok [15]), a Borel function \(f : X \to Y\) between metrizable analytic spaces is not \(\sigma\)-continuous if and only if \(f\) contains a topological copy of the Pawlikowski function \(P : (\omega + 1)^\omega \to \omega^\omega\) (which is the countable power of a bijection \(\omega + 1 \to \omega\)). Since the Pawlikowski function \(P\) is not \(\sigma\)-continuous, the family

\[I_P = \{X \subset (\omega + 1)^\omega : P|X\text{ is not }\sigma\text{-continuous}\}\]

is a proper \(\sigma\)-ideal of the compact metrizable space \((\omega + 1)^\omega\).

Let \(c_\sigma\) (resp. \(c_\sigma\)) be the largest cardinal \(\kappa \leq \mathfrak{c}\) such that every function \(f : X \to \mathbb{R}\) defined on a subset \(X \subset \mathbb{R}\) of cardinality \(|X| < \kappa\) is \(\sigma\)-continuous (resp. \(\sigma\)-continuous). It is clear that

\[\omega_1 \leq c_\sigma \leq \kappa \leq \mathfrak{c},\]

so \(c_\sigma\) and \(c_\sigma\) are typical small uncountable cardinals in the interval \([\omega_1, \mathfrak{c}]\).

In this paper we establish the relation of the cardinals \(c_\sigma\) and \(c_\sigma\) to some known cardinal characteristics of the continuum: non(\(\mathcal{M}\)), non(\(\mathcal{N}\)), \(p\), \(b\), \(q_0\), \(q\).

Let us define their definitions. By \(\omega\) we denote the smallest infinite cardinal, by \(2^\omega\) the Cantor cube \(\{0, 1\}^\omega\), and by \(\mathfrak{c}\) the cardinality of \(2^\omega\). Let \(\mathcal{M}\) be the ideal of meager sets in the real line, \(\mathcal{N}\) be the ideal of Lebesgue null sets in \(\mathbb{R}\), and \(\mathcal{K}_\sigma\) be the \(\sigma\)-ideal, generated by compact subsets of \(\omega^\omega\). For a family of sets \(\mathcal{I}\) with \(\bigcup \mathcal{I} \not\in \mathcal{I}\) let non(\(\mathcal{I}\)) = \(\min\{|A| : A \subset \bigcup \mathcal{I}, A \not\in \mathcal{I}\}\). The cardinal non(\(\mathcal{K}_\sigma\)) is usually denoted by \(b\), see [3].

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Let $p$ be the smallest cardinality of a family $F$ of infinite subsets of $\omega$ such that every finite subfamily $E \subset F$ has infinite intersection and for any infinite set $I \subset \omega$ there exists a set $F \in F$ such that $I \setminus F$ is infinite.

It is known \[6\], \[20\], \[3\] that $p \leq b \leq \text{non}(\mathcal{M}) \leq \mathfrak{c}$, and $p = \mathfrak{c}$ is equivalent to Martin’s Axiom for $\sigma$-centered posets.

A subset $A \subset \mathbb{R}$ is called a $Q$-set if each subset of $A$ is of type $F_\sigma$ in $A$. Let
- $q_0$ be the smallest cardinality of a subset $A \subset \mathbb{R}$, which is not a $Q$-set, and
- $q$ be the smallest cardinal $\kappa$ such that every subset $X \subset \mathbb{R}$ of cardinality $|X| \geq \kappa$ is not a $Q$-set.

It is known \[2\] that

$$p \leq q_0 \leq \min\{b, \text{non}(\mathcal{N}), q\} \leq q \leq \log(c^+),$$

where $\log c^+ = \min\{\kappa : 2^\kappa > \mathfrak{c}\}$. More information on $Q$-sets and the cardinals $q_0$ and $q$ can be found in \[14\] §4 and \[2\]. The following theorem was proved on 12.09.2019 during the XXXIII International Summer Conference on Real Function Theory in Ustka at the Baltic sea.

**Theorem 1.** Let $X, Y$ be separable metrizable spaces. If $|X| < b$, then each Borel function $f : X \to Y$ is $\sigma$-continuous.

**Proof.** Embed the metrizable separable spaces $X, Y$ into Polish spaces $\bar{X}$ and $\bar{Y}$, respectively. Let $\{V_n\}_{n \in \omega}$ be a countable base of the topology of the Polish space $\bar{Y}$. Since the function $f$ is Borel, for every $n \in \omega$ the preimage $f^{-1}(V_n)$ is a Borel subset of $X$ and hence $f^{-1}(V_n) = X \cap B_n$ for some Borel set $B_n$ in the Polish space $\bar{X}$.

By \[11\] 13.5], there exists a continuous bijective map $g : \bar{Z} \to \bar{X}$ from a Polish space $\bar{Z}$ such that for every $n \in \omega$ the Borel set $g^{-1}(B_n)$ is open in $\bar{Z}$. Consider the subspace $Z = g^{-1}(X)$ of the Polish space $\bar{Z}$ and observe that the map $h = f \circ g|Z$ is continuous. Indeed, for any $n \in \omega$ the preimage $h^{-1}(V_n)$ of the basic open set $V_n$ equals $Z \cap g^{-1}(B_n)$ and hence is open in $Z$.

Since $|Z| = |g^{-1}(X)| = |X| < b$, the set $Z = g^{-1}(X)$ is contained in some $\sigma$-compact subset $A$ of $\bar{Z}$. Write the set $A$ as the countable union $A = \bigcup_{n \in \omega} A_n$ of compact sets $A_n$ in $\bar{Z}$.

For every $n \in \omega$ the image $K_n := g(A_n) \subset \bar{X}$ of $A_n$ is compact and the continuous bijective function $g|A_n : A_n \to K_n$ is a homeomorphism and so is the function $g|A_n \cap Z : A_n \cap Z \to K_n \cap X$. Then the restriction $f|K_n \cap X = f \circ g \circ (g|A_n \cap Z)^{-1}$ is continuous, and the function $f$ is $\sigma$-continuous.

We say that two subsets $A, B$ of a topological space $X$ can be separated by $\sigma$-compact sets if there are two disjoint $\sigma$-compact subsets $\bar{A}$ and $\bar{B}$ of $X$ such that $A \subset \bar{A}$ and $B \subset \bar{B}$.

**Lemma 1.** The cardinal $q_0$ is equal to the largest cardinal $\kappa$ such that any two disjoint sets $A, B \subset 2^\omega$ of cardinality $\max\{|A|, |B|\} < \kappa$ can be separated by $\sigma$-compact sets in $2^\omega$.

**Proof.** Let $q_0'$ be is equal to the largest cardinal $\kappa$ such that any two disjoint sets $A, B \subset 2^\omega$ of cardinality $\max\{|A|, |B|\} < \kappa$ can be separated by $\sigma$-compact sets in $2^\omega$. Then any set $X \subset 2^\omega$ of cardinality $|X| < q_0'$ is a $Q$-set, which implies that $q_0' \leq q_0$. Assuming that $q_0' < q_0$, we can find two disjoint sets $A, B \subset 2^\omega$ of cardinality $|A \cup B| = q_0' < q_0$ such that $A$ and $B$ cannot be separated by $\sigma$-compact subsets of $2^\omega$. We can identify the Cantor cube $2^\omega$ with a subspace of the real line. Since $|A \cup B| < q_0$, the subset $X = A \cup B \subset 2^\omega \subset \mathbb{R}$ is a $Q$-set and hence $A$ is an $F_\sigma$-set in $X$. Then $A = K \cap X$ for some $\sigma$-compact set $K \subset 2^\omega$. Since the
Polish space $P = 2^\omega \setminus K$ is a continuous image of $\omega^\omega$, we can use the definition of the cardinal $b \geq q_0 > \vert B \vert$ and prove that $B$ is contained in a $\sigma$-compact subset $\Sigma$ of $P$. Then $K$ and $\Sigma$ are disjoint $\sigma$-compact sets separating the sets $A, B$ in $2^\omega$, which contradicts the choice of the sets $A, B$. This contradiction shows that $q_0' = q_0$. □

Having in mind the characterization of $q_0$ in Lemma 1 let us consider two modifications of $q_0$. Namely, let

- $q_1$ be the smallest cardinal $\kappa$ for which there exists a subset $A \subset 2^\omega$ of cardinality $\vert A \vert \leq \kappa$ and a family $\mathcal{B}$ of compact subsets of $2^\omega$ with $\vert \mathcal{B} \vert \leq \kappa$ such that the sets $A$ and $\bigcup \mathcal{B}$ are disjoint but cannot be separated by $\sigma$-compact sets in $2^\omega$;
- $q_2$ be the smallest cardinal $\kappa$ for which there exist families $\mathcal{A}, \mathcal{B}$ of compact subsets of $2^\omega$ with $\max\{\vert \mathcal{A} \vert, \vert \mathcal{B} \vert \} \leq \kappa$ such that the sets $\bigcup \mathcal{A}$ and $\bigcup \mathcal{B}$ are disjoint but cannot be separated by $\sigma$-compact sets in $2^\omega$.

This definition and Lemma 1 imply that

$$\omega_1 \leq q_2 \leq q_1 \leq q_0 \leq q \leq \log(c^+) \leq c.$$

The following theorem is the main result of this paper.

**Theorem 2.** $p \leq q_2 \leq q_1 \leq q_0 = c_\sigma = \min\{c_\sigma, b, q\} \leq c_\sigma \leq \min\{\non(M), \non(N), \non(I_F)\}$.

The proof of this theorem is divided into a series of lemmas.

**Lemma 2.** $p \leq q_2$.

**Proof.** Given any non-empty families $\mathcal{A}, \mathcal{B}$ of compact subsets of $2^\omega$ with $\max\{\vert \mathcal{A} \vert, \vert \mathcal{B} \vert \} < p$ and $(\bigcup \mathcal{A}) \cap (\bigcup \mathcal{B}) = \emptyset$, we shall prove that $\bigcup \mathcal{A}$ and $\bigcup \mathcal{B}$ can be separated by $\sigma$-compact subsets of $2^\omega$. Identify the Cantor cube $2^\omega$ with the set of branches of the binary tree $2^{<\omega} = \bigcup_{n \in \omega} 2^n$. For every $n \in \omega$ let $\text{pr}_n : 2^\omega \to 2^n$, $\text{pr}_n : x \mapsto x \upharpoonright n$, be the projection of $2^\omega$ onto $2^n$.

Let $[2^{<\omega}]^{<\omega}$ be the family of finite subsets of $2^{<\omega}$. For any disjoint compact subset $A, B \subset 2^\omega$ and $n \in \omega$ consider the set

$$\mathcal{F}_{A,B,n} = \bigcap_{a \in A} \{F \in [2^{<\omega}]^{<\omega} : F \cap \{a \upharpoonright i\} = \emptyset \} \cap \bigcap_{b \in B} \{F \in [2^{<\omega}]^{<\omega} : \vert F \cap \{b \upharpoonright i\} \vert \geq n \}.$$

We claim that this set is infinite. Indeed, since $A, B$ are compact disjoint sets in $2^\omega$, there exists $m \in \omega$ such that $\text{pr}_m(A) \cap \text{pr}_m(B) = \emptyset$. Then for every $k \geq n$ the set $F_k = \bigcup_{i=m}^{m+k} \text{pr}_i(B)$ belongs to the family $\mathcal{F}_{A,B,n}$.

We claim that the family

$$\mathcal{F} = \{\mathcal{F}_{A,B,n} : A \in \mathcal{A}, B \in \mathcal{B}, n \in \omega\}$$

is infinitely centered in the sense that each finite non-empty subfamily $\mathcal{E} \subset \mathcal{F}$ has infinite intersection. Indeed, write the family $\mathcal{E}$ as $\mathcal{E} = \{\mathcal{F}_{A_1, B_1, n_1} \}^{\bigcup}_{i=1}$ for some $A_1, \ldots, A_k \in \mathcal{A}$, $B_1, \ldots, B_k \in \mathcal{B}$, $n_1, \ldots, n_k \in \omega$. Consider the compact sets $A = \bigcup_{i=1}^{k} A_i$ and $B = \bigcup_{i=1}^{k} B_i$ and observe that they are disjoint as $A \cap B \subset (\bigcup \mathcal{A}) \cap (\bigcup \mathcal{B}) = \emptyset$. Let $n = \max_{1 \leq i \leq k} n_i$ observe that $\mathcal{F}_{A,B,n} \subset \bigcap \mathcal{E}$, which implies that the intersection $\bigcap \mathcal{E}$ is infinite.

Since $\vert \mathcal{F} \vert = \vert \omega \times \mathcal{A} \times \mathcal{B} \vert < p$, the family $\mathcal{F}$ has infinite pseudointersection $\{F_k\}_{k \in \omega} \subset [2^{<\omega}]^{<\omega}$, which means that for every $A \in \mathcal{A}, B \in \mathcal{B}$ and $n \in \omega$ the set $\{k \in \omega : F_k \notin \mathcal{F}_{A,B,n} \}$ is finite.

For every $n \in \omega$ consider the closed subset

$$K_n = \bigcap_{k \geq n} \{x \in 2^\omega : F_k \cap \{x \upharpoonright i\} = \emptyset \} = \bigcap_{k \geq n} \bigcap_{i \in \omega} \{x \in 2^\omega : x \upharpoonright i \notin F_k \}.$$
of $2^\omega$ and observe that \( \bigcup A \subset \bigcup_{n \in \omega} K_n \subset 2^\omega \setminus \bigcup B \). It follows that \( 2^\omega \setminus \bigcup_{n \in \omega} K_n \) is a Polish space containing the set \( \bigcup B \). The definition of the cardinal \( b \geq p > |B| \) ensures that \( \bigcup B \) is contained in some \( \sigma \)-compact subset \( \Sigma \) of the Polish space \( 2^\omega \setminus \bigcup_{n \in \omega} K_n \). Then \( \bigcup_{n \in \omega} K_n \) and \( \Sigma \) are two disjoint \( \sigma \)-compact sets separating the sets \( \bigcup A \) and \( \bigcup B \) in the Cantor cube \( 2^\omega \).

**Lemma 3.** \( c_\sigma = q_0 = \min\{c_\sigma, b, q\} \).

**Proof.** First we prove that \( c_\sigma \leq q_0 \). Fix a set \( X \subset \mathbb{R} \) of cardinality \( |X| = q_0 \), which is not a \( Q \)-set and hence contains a subset \( A \subset X \) which is not of type \( F_\sigma \) in \( X \). Consider the function \( f : X \to \{0, 1\} \subset \mathbb{R} \) defined by \( f^{-1}(1) = A \) and \( f^{-1}(x) = X \setminus A \). Assuming that the function \( f \) is \( \bar{\sigma} \)-continuous, we would conclude that \( A \) and \( B \) are \( F_\sigma \)-sets in \( X \), which contradicts the choice of \( X \). Consequently, \( c_\sigma \leq |X| = q_0 \).

Assuming that \( c_\sigma < q_0 \), we can find a subset \( X \subset \mathbb{R} \) of cardinality \( |X| = c_\sigma < q_0 \) and a function \( g : X \to \mathbb{R} \) which is not \( \bar{\sigma} \)-continuous. The strict inequality \( |X| < q_0 \) implies that \( X \) is a \( Q \)-space. Consequently, each subset of \( X \) is of type \( F_\sigma \) in \( X \) and the function \( g \) is Borel. Since \( |X| < q_0 \leq b \), we can apply Theorem [11] and conclude that the function \( g \) is \( \bar{\sigma} \)-continuous, which contradicts the choice of \( g \). This contradiction completes the proof of the equality \( c_\sigma = q_0 \).

Since \( q_0 = c_\bar{\sigma} \leq c_\sigma \), the equality \( q_0 = \min\{c_\sigma, b, q\} \) will follow as soon as we prove that the strict inequality \( q_0 < \min\{b, q\} \) implies \( c_\sigma \leq q_0 \). Assuming that \( q_0 < \min\{b, q\} \), find a \( Q \)-set \( Y \subset \mathbb{R} \) of cardinality \( |Y| = q_0 \). By the definition of \( q_0 \) there exists a subset \( X \subset \mathbb{R} \) of cardinality \( |X| = q_0 \) which is not a \( Q \)-set. Let \( f : X \to Y \) be any bijection. We claim that either \( f \) or \( f^{-1} \) is not \( \sigma \)-continuous. To derive a contradiction, assume that both maps \( f \) and \( f^{-1} \) are \( \sigma \)-continuous. Then there exists a countable cover \( \mathcal{C} \) of \( X \) such that for every \( C \in \mathcal{C} \) the restriction \( f|C : C \to f(C) \) is a homeomorphism. By the Lavrentiev Theorem [11] 3.9], for every \( C \in \mathcal{C} \) the topological embedding \( f|C \) can be extended to a topological embedding \( f_C : \tilde{C} \to \mathbb{R} \) of some \( G_\delta \)-subset \( \tilde{C} \) of \( \mathbb{R} \). Since \( X \) is not a \( Q \)-set, there exists a subset \( A \subset X \) which is not of type \( F_\sigma \) in \( X \). On the other hand, for every \( C \in \mathcal{C} \) the subset \( f(C \cap A) \) is of type \( F_\sigma \) in the \( Q \)-space \( Y \). Then there exists a \( \sigma \)-compact set \( K_C \subset \mathbb{R} \) such that \( f(C \cap A) = Y \cap K_C \). By the Souslin Theorem [11] 14.2], the Borel set \( K_C \cap f_C(C) \) is a continuous image of \( \omega^\omega \). Using this fact and the definition of the cardinal \( b > |Y| \geq |f(C \cap A)| \), we can find a \( \sigma \)-compact set \( \Sigma_C \subset K_C \cap f_C(C) \) that contains the set \( f(C \cap A) \). Then

\[
  f_C(C \cap A) = f_C(C \cap A) \subset \Sigma_C \cap Y \subset f_C(C) \cap K_C \cap Y \subset K_C \cap Y = f(C \cap A)
\]

and hence \( f_C(C \cap A) = \Sigma_C \cap Y \). Then \( C \cap A = f_C^{-1}(\Sigma_C \cap Y) = f_C^{-1}(\Sigma_C) \cap X \). The continuity of the map \( f_C^{-1} : f_C(C) \to \tilde{C} \) and the inclusion \( \Sigma_C \subset f_C(C) \) imply that the preimage \( \Lambda_C = f_C^{-1}(\Sigma_C) \) is a \( \sigma \)-compact subset of the Polish space \( \tilde{C} \subset \mathbb{R} \) such that \( \Lambda_C \cap X = f_C^{-1}(\Sigma_C) \cap X = C \cap A \). Then the union \( \Lambda = \bigcup_{C \in \mathcal{C}} \Lambda_C \) is a \( \sigma \)-compact set in \( \mathbb{R} \) such that

\[
  \Lambda \cap X = \bigcup_{C \in \mathcal{C}} \Lambda_C \cap X = \bigcup_{C \in \mathcal{C}} C \cap A = A,
\]

which means that the set \( A \) is of type \( F_\sigma \) in \( X \). But this contradicts the choice of the set \( A \). This contradiction shows that one of the maps \( f \) or \( f^{-1} \) is not \( \sigma \)-continuous and hence \( c_\sigma \leq |A| = |B| = q_0 \). □

**Lemma 4.** \( c_\sigma \leq \min\{\text{non}(\mathcal{N}), \text{non}(\mathcal{M}), \text{non}(\mathcal{I}_F)\} \).
Proof. Choose a subset $X \subset (\omega + 1)^\omega$ of cardinality $|X| = \text{non}(\mathcal{I}_P)$ such that $P \setminus X$ is not $\sigma$-continuous. Being a subspace of the zero-dimensional compact metrizable space $(\omega + 1)^\omega$, the space $X$ can be embedded into the real line $\mathbb{R}$. Consequently, $\tau_\sigma \leq |X| = \text{non}(\mathcal{I}_P)$.

Let us recall that a Polish space is called perfect if it has no isolated points. A subset $A$ of a Polish space $X$ is called perfectly meager if for any perfect Polish subspace $P \subset X$ the intersection $P \cap A$ is meager in $P$. By a result of Grzegorek [7, 8] (see also [11, 5.4]), the real line $X$ contains a perfectly meager subset $A$ of cardinality $|A| = \text{non}(\mathcal{M})$. By the definition of the cardinal $\text{non}(\mathcal{M})$, there exists a non-meager set $B \subset \mathbb{R}$ of cardinality $|B| = \text{non}(\mathcal{M})$. Since $|B| = \text{non}(\mathcal{M}) = |A|$, there exists a bijective map $f : B \to A$. We claim that one of the maps $f$ or $f^{-1}$ is not $\sigma$-continuous. To derive a contradiction, assume that the maps $f$ and $f^{-1}$ are $\sigma$-continuous. Then we can find a countable cover $C$ of $B$ such that for every $C \in \mathcal{C}$ the restriction $f|C$ is a topological embedding. By the Lavrentiev Theorem [11, 3.9], for every $C \in \mathcal{C}$ the topological embedding $f|C$ can be extended to a topological embedding $f_C : \bar{C} \to \mathbb{R}$ of some $G_\delta$-subset $\bar{C}$ of $\mathbb{R}$. Let $U_C$ be the union of open countable subsets in $\bar{C}$. The hereditary Lindelöf property of the space $\bar{C} \subset \mathbb{R}$ implies that the open set $U_C$ is countable and hence meager in $\mathbb{R}$. On the other hand, the complement $P_C := \bar{C} \setminus U_C$ is a perfect Polish space. Since $f_C$ is a topological embedding, the image $f_C(P_C)$ is a perfect Polish subspace of $\mathbb{R}$. The perfect meagerness of $A$ ensures that the intersection $f_C(P_C) \cap A$ is a meager subset of the Polish space $f_C(P_C)$ and then the preimage $(f_C|P_C)^{-1}(A)$ is a meager subset of $P_C$ and of $\mathbb{R}$, too. Then the preimage $f_C^{-1}(A) \subset U_C \cup (f_C|P_C)^{-1}(A)$ is a meager subset of $\mathbb{R}$ and so is the set $C = (f_C|C)^{-1}(A)$. Then the set $B = \bigcup C$ is meager in $\mathbb{R}$, which contradicts the choice of $B$. This contradiction shows that one of the maps $f$ or $f^{-1}$ is not $\sigma$-continuous and hence $\tau_\sigma \leq |A| = |B| = \text{non}(\mathcal{M})$.

A subset $A \subset \mathbb{R}$ has universal measure zero if $\mu(A) = 0$ for any Borel continuous probability measure $\mu$ on $\mathbb{R}$. By a result of Grzegorek and Ryll-Nardzewski [1] (see also [11, 5.4]), the real line contains a subset $A$ of universal measure zero that has cardinality $|A| = \text{non}(\mathcal{N})$. By the definition of the cardinal $\text{non}(\mathcal{N})$, there exists a subset $B \notin \mathcal{N}$ of $\mathbb{R}$ with $|B| = \text{non}(\mathcal{N})$. Since $|B| = \text{non}(\mathcal{N}) = |A|$, there exists a bijective map $g : B \to A$. We claim that the map $g$ is not $\sigma$-continuous. To derive a contradiction, assume that $g$ is $\sigma$-continuous and find a countable cover $\mathcal{C}$ of $B$ such that for every $C \in \mathcal{C}$ the restriction $f|C$ is continuous. Dividing each set $C \in \mathcal{C}$ into countably many pieces, we can assume that $C$ is bounded in the real line.

By the Kuratowski Theorem [11, 3.8], for every $C \in \mathcal{C}$ the continuous map $g|C$ can be extended to a continuous map $g_C : \bar{C} \to \mathbb{R}$ defined on some $G_\delta$-subset $\bar{C} \subset \mathcal{C}$ of $\mathbb{R}$. Let $\lambda$ be the Lebesgue measure on $\mathbb{R}$. For every $C \in \mathcal{C}$ consider the subset $A_C' = \{ a \in A : \lambda(g_C^{-1}(a)) > 0 \}$ and observe that it is at most countable. Then the set $A' = \bigcup_{C \in \mathcal{C}} A_C'$ is at most countable, too. Let $P = \bigcup_{C \in \mathcal{C}} g_C^{-1}(A')$ and $C' = \{ C \in \mathcal{C} : \lambda(\bar{C} \setminus P) > 0 \}$. For every $C \in C'$ the set $\bar{C} \setminus P$ has finite non-zero measure. So, we can define a Borel probability measure $\mu_C$ on $\mathbb{R}$ letting $\mu_C(X) = \lambda(g_C^{-1}(X) \setminus P)/\lambda(\bar{C} \setminus P)$ for any Borel subset $X \subset \mathbb{R}$. It is easy to see that the measure $\mu_C$ is continuous (which means that the measure of each singleton is zero). Since the set $A$ has universal measure zero, $0 = \mu_C(A) = \lambda(g_C^{-1}(A) \setminus P) = \lambda(B \cap \bar{C} \setminus P)$.

Then $\lambda(B \setminus P) \leq \sum_{C \in \mathcal{C}} \mu(B \cap C \setminus P) = 0$ and $\lambda(B) = \lambda(B \cap P) + \lambda(B \setminus P) = 0 + 0 = 0$. 
as \( B \cap P = g^{-1}(A') \) is at most countable. But the equality \( \lambda(B) = 0 \) contradicts the choice of \( B \). This contradiction shows that the map \( g \) is not \( \sigma \)-continuous and hence \( c_\sigma \leq |B| = \mathrm{non}(N) \).

Problem 1. Which of the strict inequalities
\[
c_\sigma < c_\sigma, \ p < q_2, \ q_2 < q_1, \ q_1 < q_0
\]
is consistent with ZFC?

Problem 2. Is the strict inequality \( q_0 < \min\{q, b\} \) consistent?

Problem 3. Is \( c_\sigma \leq b \)?

Remark 1. Let \( \kappa \) be a cardinal. A subset \( A \) of a topological space \( X \) is called a \( G_{<\kappa} \)-set if \( A \) can be written as the intersection \( A = \bigcap A \) of some family \( A \) of open sets in \( X \) with \( |A| < \kappa \). So, \( G_{\omega_1} \)-sets are exactly \( G_3 \)-sets.

For a cardinal \( \kappa \) let \( b_\kappa \) be the smallest cardinality of a subset \( A \) of a \( G_{<\kappa} \)-set \( G \) in \( \mathbb{R} \) such that \( A \) is not contained in a \( \sigma \)-compact subset of \( G \). It can be shown that \( b_\kappa \) is a non-increasing function on the variable \( \kappa \) such that
\[
b_{\omega_1} = b, \ b_{\epsilon} = \omega_1, \ b_p = p, \ b_{q_1} = q_1, \text{ and } q_1 = \max\{\kappa : \kappa \leq b_\kappa\}.
\]
By [19], for every \( A \in \mathcal{I}_p \) the image \( P(A) \) is a meager subset of \( \omega^{\omega} \). This implies that \( \mathrm{non}(\mathcal{I}_p) \leq \mathrm{non}(\mathcal{M}) \) and \( \mathrm{cov}(\mathcal{I}_p) \geq \mathrm{cov}(\mathcal{M}) \). By [19], it is consistent that \( \mathrm{cov}(\mathcal{I}_p) > \mathrm{cov}(\mathcal{M}) \).

Problem 4. Is the strict inequality \( \mathrm{non}(\mathcal{I}_p) < \mathrm{non}(\mathcal{M}) \) consistent?

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