STRONGLY GAUDUCHON SPACES

LINGXU MENG AND WEI XIA

ABSTRACT. We define strongly Gauduchon spaces and the class \( \mathcal{J} \) which are generalizations of strongly Gauduchon manifolds in complex spaces. Comparing with the case of Kählerian, the strongly Gauduchon space and the class \( \mathcal{J} \) are similar to the Kähler space and the Fujiki class \( \mathcal{C} \) respectively. Some properties about these complex spaces are obtained, and the relations between the strongly Gauduchon spaces and the class \( \mathcal{J} \) are studied.

**Keywords:** strongly Gauduchon metric, strongly Gauduchon space, class \( \mathcal{J} \), topologically essential map.

**AMSC:** 32C15, 32C10, 53C55,

1. Introduction

The complex manifold with a strongly Gauduchon metric is an important object in non-Kähler geometry. In [14], D. Popovici first defined the strongly Gauduchon metric in the study of limits of projective manifolds under deformations. A *strongly Gauduchon* metric on a complex \( n \)-dimensional manifold is a hermitian metric \( \omega \) such that \( \partial \omega \) is \( \overline{\partial} \)-exact. A compact complex manifold is called a *strongly Gauduchon manifold*, if there exists a strongly Gauduchon metric on it.

**Proposition 1.1.** Let \( M \) be a compact complex manifold of dimension \( n \). Then the following is equivalent.

1. \( M \) is a strongly Gauduchon manifold.
2. There exists a strictly positive \( (n-1, n-1) \)-form \( \Omega \), such that \( \partial \Omega \) is \( \overline{\partial} \)-exact.
3. There exists a real closed \( (2n-2) \)-form \( \Omega \) whose \( (n-1, n-1) \)-component \( \Omega^{n-1,n-1} \) is strictly positive.

In [14], D. Popovici observed (1) and (3) are equivalent. “(1) \( \Rightarrow \) (2)” is obvious by the definition of strongly Gauduchon manifolds. Conversely, for any strictly positive \( (n-1, n-1) \)-form \( \Omega \), there exists a unique strictly positive \( (1, 1) \)-form \( \omega \), such that \( \omega^{n-1} = \Omega \) (see [13], page 280). So we have “(2) \( \Rightarrow \) (1)”. D. Popovici proved following two important theorems.
Theorem 1.2 ([14], Proposition 3.3). Let $M$ be a compact complex manifold. Then $M$ is a strongly Gauduchon manifold if and only if there is no nonzero positive current $T$ of bidegree $(1,1)$ on $M$ which is $d$-exact on $M$.

Theorem 1.3 ([15], Theorem 1.3). Let $f : M \to N$ be a modification of compact complex manifolds. Then $M$ is a strongly Gauduchon manifold if and only if $N$ is a strongly Gauduchon manifold.

On the other hand, in [9], A. Fujiki generalized the concept “Kähler” to general complex spaces. A kind of generalization is the Kähler space which is a complex space admitting a strictly positive closed $(1,1)$-form and the other kind is the Fujiki class $\mathcal{C}$ consisting of the reduced compact complex spaces which are the meromorphic images of a compact Kähler spaces. In [19] and [20], J. Varouchas proved that any reduced complex space in the Fujiki class $\mathcal{C}$ has a proper modification which is a compact Kähler manifold. Now, many authors use it as the definition of the Fujiki class $\mathcal{C}$. Inspired by the method of A. Fujiki and the theorem of J. Varouchas, we give two kinds of generalization of strongly Gauduchon manifolds to complex spaces: the strongly Gauduchon spaces and class $\mathcal{SG}$. In view of definitions of them, the strongly Gauduchon spaces (see Definition 2.2) is similar to the Kähler spaces and the class $\mathcal{SG}$ (see Definition 3.1) is similar to the Fujiki class $\mathcal{C}$.

In section 2, we study the properties of strongly Gauduchon spaces and give a method of constructing examples which are singular strongly Gauduchon spaces, but not in $\mathcal{B}$, where $\mathcal{B}$ is the set of reduced compact complex spaces which are bimeromorphomorphic to compact balanced manifolds.

In section 3, we study the class $\mathcal{SG}$ and propose a conjecture on the relation between strongly Gauduchon spaces and the class $\mathcal{SG}$ as follows.

Conjecture 1.4. Any strongly Gauduchon space belongs to class $\mathcal{SG}$.

We prove it in some special cases (see Theorem 3.9, 3.11, 3.12).

In section 4, we study a family of reduced complex spaces over a nonsingular curve and give a theorem on the total space being in $\mathcal{SG}$.

2. STRONGLY GAUDUCHON SPACES

First, we give a proposition about strongly Gauduchon manifolds which is similar to the case of balanced manifolds.

Proposition 2.1. Let $M$ and $N$ be compact complex manifolds of pure dimension.

(1) If $f : M \to N$ is a holomorphic submersion and $M$ is a strongly Gauduchon manifold, then $N$ is a strongly Gauduchon manifold.
(2) $M \times N$ is a strongly Gauduchon manifold, if and only if, $M$ and $N$ are both strongly Gauduchon manifolds.

Proof. Set $\dim M = m$, $\dim N = n$.

(1) Let $\Omega_M$ be a strictly positive $(m - 1, m - 1)$-form, such that $\partial \Omega_M = \bar{\partial} \alpha$, where $\alpha$ is a $(2m - 2)$-form on $M$. Define

$$\Omega_N := f^* \Omega_M.$$ 

By the proof of Proposition 1.9(ii) in [13], we know $\Omega_N$ is a strictly positive $(n - 1, n - 1)$-form. Obviously, $\partial \Omega_N = \bar{\partial} (f^* \alpha)$ is $\bar{\partial}$-exact. So $N$ is a strongly Gauduchon manifold.

(2) If $M \times N$ is a strongly Gauduchon manifold, then $M$ and $N$ are both strongly Gauduchon manifolds by (i).

Conversely, let $M$ and $N$ be both strongly Gauduchon manifolds. Suppose $\omega_M$ and $\omega_N$ are strongly Gauduchon metrics on $M$ and $N$ respectively, such that $\partial \omega_M^{m-1} = \bar{\partial} \alpha$ and $\partial \omega_N^{n-1} = \bar{\partial} \beta$, where $\alpha$ and $\beta$ are $(2m - 2)$ and $(2n - 2)$-form on $M$ and $N$ respectively. We define a metric on $M \times N$

$$\omega := \omega_M + \omega_N$$

then

$$\omega^{m+n-1} := C_1 \omega_M^{m-1} \wedge \omega_N^n + C_2 \omega_M^m \wedge \omega_N^{n-1},$$

where $C_1, C_2$ are constants. So

$$\partial \omega^{m+n-1} = C_1 \partial \omega_M^{m-1} \wedge \omega_N^n + C_2 \omega_M^m \wedge \partial \omega_N^{n-1} = \bar{\partial} (C_1 \alpha \wedge \omega_N^n + C_2 \omega_M^m \wedge \beta)$$

is $\bar{\partial}$-exact on $M \times N$. Hence $\omega$ is a strongly Gauduchon metric on $M \times N$. □

We recall the definitions of forms and currents on complex spaces, following [12].

Let $X$ be a reduced complex space and $X_{\text{reg}}$ the set of nonsingular points on $X$. Obviously, $X_{\text{reg}}$ is a complex manifold.

Suppose that $X$ is an analytic subset of a complex manifold $M$. Set $I^{p,q}_X(M) = \{ \alpha \in A^{p,q}(M) \mid i^* \alpha = 0 \}$, where $i : X_{\text{reg}} \to M$ is the inclusion. Define $A^{p,q}(X) := A^{p,q}(M)/I^{p,q}_X(M)$. It can be easily shown that $A^{p,q}(X)$ does not depend on the embedding of $X$ into $M$. Hence, for any complex space $X$, we can define $A^{p,q}(X)$ through the local embeddings in $\mathbb{C}^N$. More precisely, we define a sheaf of germs $A^{p,q}_X$ of $(p, q)$-forms on $X$ and $A^{p,q}(X)$ as the group of its global sections. Similarly, we can also define $A^{p,q}_c(X)$ (the space of $(p, q)$-forms with compact supports), $A^k(X)$ and $A^k_c(X)$.

We can naturally define $\partial : A^{p,q}(X) \to A^{p+1,q}(X)$, $\bar{\partial} : A^{p,q}(X) \to A^{p,q+1}(X)$ and $d : A^k(X) \to A^{k+1}(X)$. 
If \( f : X \to Y \) is a holomorphic map between reduced complex spaces, then we can naturally define \( f^* : A^{p,q}(Y) \to A^{p,q}(X) \) such that \( f^* \) commutes with \( \partial, \bar{\partial}, d \).

When \( X \) is a subvariety of a complex manifold \( M \), we define the space of currents on \( X \)
\[
D^r(X) := \{ T \in D^r(M) \mid T(u) = 0, \forall u \in I_{X,c}^{2n-r}(M) \},
\]
where \( D^r(M) \) is the space of currents on \( M \) and \( I_{X,c}^{2n-r}(M) = \{ \alpha \in A^{2n-r}_c(M) \mid i^*\alpha = 0 \} \). We can define a space \( D^r(X) \) of the currents on any reduced complex space \( X \) as the case of \( A^r(X) \). Define
\[
D^{p,q}(X) := \{ T \in D^{p+q}(X) \mid T(u) = 0, \forall u \in A^{r,s}_c(M), (r, s) \neq (n-p, n-q) \}.
\]
A current \( T \) is called a \((p, q)\)-current on \( X \), if \( T \in D^{p,q}(X) \). If \( T \in D^r(X) \), we call \( r \) the degree. If \( T \in D^{p,q}(X) \), we call \((p, q)\) the bidegree. We also denote \( D^r_r(X) = D^{2n-r}_r(X) \) and \( D^r_{p,q}(X) = D^{p+n-r-n,q}(X) \). A current \( T \in D^{p,q}(X) \) is called real if for every \( \alpha \in A^{2n-2p}_c(X) \), \( T(\pi) = \overline{T(\alpha)} \).

If \( f : X \to Y \) is a holomorphic map of reduced compact complex spaces, we define \( f_* : D^r_r(X) \to D^r_r(Y) \) as \( f_*T(u) := T(f^*u) \) for any \( u \in A^r_r(Y) \).

A real \((p,p)\)-form \( \omega \) on \( X \) is called strictly positive, if there exist an open covering \( U = \{U_{\alpha}\} \) of \( X \) with an embedding \( i_{\alpha} : U_{\alpha} \to V_{\alpha} \) of \( U_{\alpha} \) into a domain \( V_{\alpha} \) in \( \mathbb{C}^{n_{\alpha}} \) and a strictly positive \((p,p)\)-form \( \omega_{\alpha} \) on \( V_{\alpha} \), such that \( \omega|_{U_{\alpha}} = i_{\alpha}^*\omega_{\alpha} \), for each \( \alpha \).

Now, we give a kind of generalization of strongly Gauduchon manifolds.

**Definition 2.2.** A purely \( n \)-dimensional reduced compact complex space \( X \) is called a strongly Gauduchon space, if there exists a strictly positive \((n-1, n-1)\)-form \( \Omega \), such that \( \partial \Omega \) is \( \bar{\partial} \)-exact.

By its definition, it is easy to see that \( X \) is a strongly Gauduchon space, if and only if, there exists a real closed \((2n-2)\)-form \( \Omega' \) on \( X \) whose \((n-1, n-1)\)-component \( \Omega'^{n-1,n-1} \) is strictly positive. Indeed, if \( \Omega \) is a strictly positive \((n-1, n-1)\)-form, such that \( \partial \Omega = \bar{\partial} \alpha \), where \( \alpha \) is a \((n, n-2)\)-form, then
\[
\Omega' := \Omega - \alpha - \bar{\alpha}
\]
is the desired form. Conversely, since \( \Omega' \) is real and \( d \)-closed, \( \partial \Omega'^{n-1,n-1} = -\bar{\partial} \Omega'^{n,n-2} \). Hence, \( \Omega := \Omega'^{n-1,n-1} \) is the desired form.

Obviously, strongly Gauduchon manifolds and compact balanced spaces are strongly Gauduchon spaces.

**Proposition 2.3.** Let \( X \) be a reduced compact complex space of pure dimension and \( M \) a compact complex manifold of pure dimension. If \( X \times M \) is a strongly Gauduchon space, then \( M \) is a strongly Gauduchon manifold.
Proof. Let $X_{\text{reg}}$ be the set of nonsingular points on $X$ and $\Omega$ a strictly positive $(n + m - 1, n + m - 1)$-form on $X \times M$, such that $\partial \Omega$ is $\overline{\partial}$-exact, where $n = \dim X$ and $m = \dim M$. Suppose $\pi : X_{\text{reg}} \times M \to M$ is the second projection. By the proof of Proposition 1.9(ii) in [13], we know $\pi^*(\Omega_{|X_{\text{reg}} \times M})$ is a strictly positive $(m - 1, m - 1)$-form on $M$. Obviously, $\partial \pi^*(\Omega_{|X_{\text{reg}} \times M})$ is $\partial$-exact. So $M$ is a strongly Gauduchon manifold. □

We know that, on a compact balanced manifold $M$, the fundamental class $[V]$ of any hypersurface $V$ is not zero in $H^2(M, \mathbb{R})$ (see [13], Corollary 1.7). It is equivalent to that, the current $[V]$ on $M$ defined by any hypersurface $V$ is not $d$-exact. For strongly Gauduchon spaces, we have following proposition.

**Proposition 2.4.** If $X$ is a strongly Gauduchon space, then the current $[V]$ defined by any hypersurface $V$ of $X$ is not $\partial \overline{\partial}$-exact.

**Proof.** Suppose $\dim X = n$. Let $\Omega$ be a strictly positive $(n - 1, n - 1)$-form on $X$ such that $\partial \Omega = \overline{\partial} \alpha$, where $\alpha$ is a $(2n - 2)$-form on $X$. If $[V] = \partial \overline{\partial} Q$ for some current $Q$ on $X$, then

$$[V](\Omega) = \int_V \Omega > 0.$$  

On the other hand,

$$[V](\Omega) = (\partial \overline{\partial} Q)(\Omega) = - Q(\overline{\partial} \partial \Omega) = - Q(\overline{\partial} \partial \alpha) = 0.$$  

It is a contradiction. □

**Proposition 2.5.** If $f : X \to Y$ is a finite holomorphic unramified covering map of reduced compact complex spaces of pure dimension, then $X$ is a strongly Gauduchon space if and only if $Y$ is a strongly Gauduchon space.

**Proof.** Set $n = \dim X = \dim Y$ and $d = \deg f$.

Let $X$ be a strongly Gauduchon space and $\Omega_X$ a strictly positive $(n - 1, n - 1)$-form on $X$ such that $\partial \Omega_X = \overline{\partial} \alpha_X$, where $\alpha_X$ is a $2(n - 1)$-form on $X$. For every $y \in Y$, we set $f^{-1}(y) = \{x_1, ..., x_d\}$, then there exists an open neighbourhood $V \subseteq Y$ of $y$, and open neighbourhoods $U_1, ..., U_d$ of $x_1, ..., x_d$ in $X$ respectively, which do not intersect with each other, such that $f^{-1}(V) = \bigcup_{i=1}^d U_i$ and the restriction $f|_{U_i} : U_i \to V$ is an isomorphism for $i = 1, ..., d$. We define two forms on $V$ as

$$\Omega_V := \Sigma_{i=1}^d (f|_{U_i})^*(\Omega_X|_{U_i})$$

$$\alpha_V := \Sigma_{i=1}^d (f|_{U_i})^*(\alpha_X|_{U_i})$$

If $V$ and $V'$ are two open subsets in $Y$ as above (possible for different points in $Y$) and $V \cap V' \neq \emptyset$, we can easily check $\Omega_V = \Omega_{V'}$ on $V \cap V'$. Hence we can construct a global $(n - 1, n - 1)$-form $\Omega_Y$ on $Y$ such that $\Omega_Y|_V = \Omega_V$.  

**Proof.** Let $X_{\text{reg}}$ be the set of nonsingular points on $X$ and $\Omega$ a strictly positive $(n + m - 1, n + m - 1)$-form on $X \times M$, such that $\partial \Omega$ is $\overline{\partial}$-exact, where $n = \dim X$ and $m = \dim M$. Suppose $\pi : X_{\text{reg}} \times M \to M$ is the second projection. By the proof of Proposition 1.9(ii) in [13], we know $\pi^*(\Omega_{|X_{\text{reg}} \times M})$ is a strictly positive $(m - 1, m - 1)$-form on $M$. Obviously, $\partial \pi^*(\Omega_{|X_{\text{reg}} \times M})$ is $\partial$-exact. So $M$ is a strongly Gauduchon manifold. □
By the same reason, we can define a global \(2(n-1)\)-form \(\alpha_Y\) on \(Y\) such that \(\alpha_Y\|_V = \alpha_Y\). Obviously, \(\Omega_Y\) is strictly positive and \(\partial\Omega_Y = \overline{\partial}\alpha_Y\). Therefore, \(Y\) is a strongly Gauduchon space.

Conversely, suppose \(\Omega_Y\) is a strictly positive \((n-1, n-1)\)-form on \(Y\), such that \(\partial\Omega_Y\) is \(\overline{\partial}\)-exact on \(Y\). For all \(x \in X\), there is an open neighbourhood \(U\) of \(x\) in \(X\), an open neighbourhood \(V\) of \(f(x)\) in \(Y\), such that \(f|_U: U \to V\) is an isomorphism. \((f^*\Omega_Y)|_U = (f|_U)^*(\Omega_Y|_V)\) is obviously strictly positive on \(U\), so is \(f^*\Omega_Y\) on \(X\). Obviously, \(f^*\Omega_Y\) is \(\overline{\partial}\)-exact on \(X\). Therefore, \(X\) is a strongly Gauduchon space. \(\square\)

3. The class \(\mathcal{IG}\)

Now, we give the other generalization of strongly Gauduchon manifolds.

**Definition 3.1.** A reduced compact complex space \(X\) of pure dimension is called in class \(\mathcal{IG}\), if it has a desingularization \(\tilde{X}\) which is a strongly Gauduchon manifold.

If one desingularization of \(X\) is a strongly Gauduchon manifold, then every desingularization of \(X\) is a strongly Gauduchon manifold. Indeed, if \(X_1 \to X\) and \(X_2 \to X\) are two desingularizations of \(X\), then there exists a bimeromorphic map \(f: X_1 \to X_2\). Let \(\Gamma \subseteq X_1 \times X_2\) be the graph of \(f\), and \(p_1: \Gamma \to X_1, p_2: \Gamma \to X_2\) the two projections on \(X_1, X_2\), respectively. Then \(p_1, p_2\) are modifications. If \(\tilde{\Gamma}\) is a desingularization of \(\Gamma\), then \(\tilde{\Gamma} \to X_1\) and \(\tilde{\Gamma} \to X_2\) are modifications of compact complex manifolds. By Theorem \([13]\) we know that \(X_1\) is a strongly Gauduchon manifold if and only if \(\tilde{\Gamma}\) is a strongly Gauduchon manifold, and then if and only if \(X_2\) is a strongly Gauduchon manifold. Hence Definition 2.1 is not dependent on the choice of the desingularization of \(X\). So, if \(X \in \mathcal{IG}\) is nonsingular, then \(X\) is a strongly Gauduchon manifold.

Using the same method as above, we can prove the following proposition.

**Proposition 3.2.** The class \(\mathcal{IG}\) is invariant under bimeromorphic maps.

Obviously, strongly Gauduchon manifolds and the normalizations of complex spaces in class \(\mathcal{IG}\) are in class \(\mathcal{IG}\). Recall that a reduced compact complex space \(X\) is called in class \(\mathcal{B}\), if it has a desingularization \(\tilde{X}\) which is a balanced manifold, referring to \([8]\). Then complex spaces in class \(\mathcal{B}\) are in class \(\mathcal{IG}\).

**Proposition 3.3.** If \(X\) and \(Y\) are reduced compact complex spaces, then \(X \times Y\) is in the class \(\mathcal{IG}\) if and only if \(X\) and \(Y\) are both in the class \(\mathcal{IG}\).

**Proof.** If \(f: \tilde{X} \to X\) and \(g: \tilde{Y} \to Y\) are desingularizations, then \(f \times g: \tilde{X} \times \tilde{Y} \to X \times Y\) is a desingularization of \(X \times Y\). By Proposition \([24]\)(ii),
we know that $\tilde{X} \times \tilde{Y}$ is a strongly Gauduchon manifold if and only if $\tilde{X}$ and $\tilde{Y}$ are both strongly Gauduchon manifolds. So we get this proposition easily.

Using this proposition, we can construct some examples of complex spaces in $\mathcal{I}$ which are neither strongly Gauduchon manifolds nor in class $\mathcal{B}$. If $Y$ is a singular reduced compact complex space in class $\mathcal{B}$ and $Z$ is a compact strongly Gauduchon manifold but not a balanced manifold, then $Y \times Z$ is in $\mathcal{I}$, but it is neither a strongly Gauduchon manifold nor in $\mathcal{B}$. Indeed, $Y \times Z$ is singular, so it is not a strongly Gauduchon manifold. By Proposition 3.3, $Y \times Z \in \mathcal{I}$. Assume $Y \times Z \in \mathcal{B}$, by [8], Proposition 2.3, we know $Z \in \mathcal{B}$. Since $Z$ is nonsingular, $Z$ is balanced, which contradicts the choice of $Z$. Hence we get the following relations

$$C \subseteq B \subseteq S_G,$$

where $C$ is the Fujiki class and the first "$\subseteq$" is proved in [8], Section 2.

If $X$ is a reduced compact complex space of pure dimension, then $X \in \mathcal{I}$ if and only if every irreducible component of $X$ is in $\mathcal{I}$. Indeed, if let $\tilde{X}_1, \ldots,\tilde{X}_r$ be the desingularizations of $X_1, \ldots, X_r$, all the irreducible components of $X$, then the disjoint union $\tilde{X} := \tilde{X}_1 \sqcup \ldots \sqcup \tilde{X}_r$ is a desingularization of $X$. Hence the conclusion follows since $\tilde{X}$ is a strongly Gauduchon manifold if and only if $\tilde{X}_1, \ldots, \tilde{X}_r$ are all strongly Gauduchon manifolds.

In the following, we need the definition of a smooth morphism, referring to [5], (0.4). A surjective holomorphic map $f : X \to Y$ between reduced complex spaces is called a smooth morphism, if for all $x \in X$, there is an open neighbourhood $W$ of $x$ in $X$, an open neighbourhood $U$ of $f(x)$ in $Y$, such that $f(W) = U$ and there is a commutative diagram

$$\begin{array}{ccc}
W & \xrightarrow{f|_W} & U \\
g \downarrow & & \\
\Delta^r \times U & \xrightarrow{pr_2} & \\
\end{array}$$

where $r = \dim X - \dim Y$, $g$ is an isomorphism (i.e., biholomorphic map), $pr_2$ is the second projection, and $\Delta^r$ is a small polydisc. Moreover, if $\dim X = \dim Y$, a smooth morphism is exactly a surjective local isomorphism.

Obviously, if $f : X \to Y$ is a smooth morphism and $Y$ is a complex manifold, then $X$ must also be a complex manifold and $f$ is a submersion between complex manifolds.

**Proposition 3.4.** Let $f : X \to Y$ be a smooth morphism of reduced compact complex spaces. If $X \in \mathcal{I}$, then $Y \in \mathcal{I}$.
Proof. Suppose \( p : \tilde{Y} \to Y \) is a desingularization. Consider the following Cartesian diagram

\[
\begin{array}{ccc}
\tilde{X} := X \times_Y \tilde{Y} & \xrightarrow{\tilde{f}} & \tilde{Y} \\
q \downarrow & & \downarrow p \\
X & \xrightarrow{f} & Y
\end{array}
\]

where \( X \times_Y \tilde{Y} = \{(x, \tilde{y}) \in X \times \tilde{Y} | f(x) = p(\tilde{y})\} \), \( q \) is the projection to \( X \), and \( \tilde{f} \) is the projection to \( \tilde{Y} \). We can prove that \( \tilde{f} \) is a submersion of complex manifolds and \( q \) is a modification, referring to [8], Claim 1 and 2 in the proof of Proposition 2.4. Since \( X \in \mathcal{G} \), \( \tilde{X} \) is a strongly Gauduchon manifold, so is \( \tilde{Y} \) by Proposition 2.1(i), hence \( Y \in \mathcal{G} \). \( \square \)

**Proposition 3.5.** If \( f : X \to Y \) is a finite unramified covering map of reduced compact complex spaces, then \( X \in \mathcal{G} \) if and only if \( Y \in \mathcal{G} \).

Proof. Suppose \( p : \tilde{Y} \to Y \) is a desingularization. Consider the Cartesian diagram (1). We know that \( \tilde{f} \) is a surjective local isomorphism and \( q \) is a modification. Since \( \tilde{Y} \) is locally compact, by [11], Lemma 2, \( \tilde{f} \) is a finite covering map in topological sense. Moreover, since \( \tilde{f} \) is a local isomorphism (in analytic sense), \( \tilde{f} \) is a finite unramified covering map (in analytic sense). By Proposition 2.5 we know \( \tilde{X} \) is a strongly Gauduchon manifold if and only if \( \tilde{Y} \) is a strongly Gauduchon manifold. Hence \( X \in \mathcal{G} \) if and only if \( Y \in \mathcal{G} \). \( \square \)

We generalize Theorem 3.5 (2) and Theorem 3.9 (2) in \([1]\) as follows.

**Proposition 3.6.** Let \( f : X \to Y \) be a smooth morphism of reduced compact complex spaces, and \( n = \text{dim } X > m = \text{dim } Y \geq 2 \). If \( Y \in \mathcal{B} \) and there exists a point \( y_0 \) in \( Y \) such that the current \( [f^{-1}(y_0)] \) is not d-exact on \( X \), then \( X \in \mathcal{G} \).

Proof. Choose a desingularization \( p : \tilde{Y} \to Y \) such that \( \tilde{Y} \) is a compact balanced manifold. Considering the Cartesian diagram (1), we know that \( \tilde{f} \) is a surjective local isomorphism and \( q \) is a modification.

For every \( \tilde{y} \in p^{-1}(y_0) \), the current \( [\tilde{f}^{-1}(\tilde{y})] \) can not be written as \( dQ \) for any current \( Q \) of degree \( 2m - 1 \) on \( \tilde{X} \). If not, since \( \tilde{f}^{-1}(\tilde{y}) = f^{-1}(y_0) \times \{\tilde{y}\} \), we have 

\[
[f^{-1}(y_0)] = q_*[\tilde{f}^{-1}(\tilde{y})] = q_*(dQ) = dq_*Q,
\]

which contradicts the assumption.

Now suppose \( \tilde{y}' \) is any point in \( \tilde{Y} \). Then the fundamental classes \( [\tilde{y}] = [\tilde{y}'] \) in \( H^{2m}(\tilde{Y}, \mathbb{R}) \). Since \( \tilde{f} \) is smooth,

\[
[\tilde{f}^{-1}(\tilde{y}')] = \tilde{f}^*[\tilde{y}'] = \tilde{f}^*[\tilde{y}] = [\tilde{f}^{-1}(\tilde{y})]
\]
in $H^{2m}(\tilde{X}, \mathbb{R})$, where $\tilde{y} \in p^{-1}(y_0)$ and $\tilde{f}^* : H^{2m}(\tilde{Y}, \mathbb{R}) \to H^{2m}(\tilde{X}, \mathbb{R})$ is the pull back of $\tilde{f}$. Hence for every $\tilde{y} \in \tilde{Y}$, the current $[\tilde{f}^{-1}(\tilde{y})]$ is not $d$-exact on $\tilde{X}$. By [1], Theorem 3.5 (2) and Theorem 3.9 (2), $X$ is a strongly Gauduchon manifold, hence $X \in \mathcal{SG}$. □

Next we consider the relation between strongly Gauduchon spaces and class $\mathcal{SG}$. From definitions of them, the relation between strongly Gauduchon spaces and class $\mathcal{SG}$ is similar to that of Kähler spaces and Fujiki class $\mathcal{C}$. Moreover, in the nonsingular case, we know that a modification of a strongly Gauduchon manifold is also a strongly Gauduchon manifold, by Theorem 1.3. So we think the following also hold.

**Conjecture 3.7.** Any strongly Gauduchon space belongs to class $\mathcal{SG}$.

We can prove it in some extra conditions. First, we recall a theorem and several notations.

**Theorem 3.8** ([2], Theorem 1.5). Let $M$ be a complex manifold of dimension $n$, $E$ a compact analytic subset and $\{E_i\}_{i=1, \ldots, s}$ all the $p$-dimensional irreducible components of $E$. If $T$ is a $\partial \bar{\partial}$-closed positive $(n-p, n-p)$-current on $M$ such that $\text{supp} T \subseteq E$, then there exist constants $c_i \geq 0$ such that $T - \sum_{i=1}^s c_i [E_i]$ is supported on the union of the irreducible components of $E$ of dimension greater than $p$.

For a compact complex manifold $M$, the *Bott-Chern cohomology group* of degree $(p, q)$ is defined as

$$H^{p,q}_{BC}(M) := \text{Ker} \left( \frac{d : A^{p,q}(M) \to A^{p+q+1}(M)}{\partial \bar{\partial} A^{p-1,q-1}(M)} \right),$$

and the *Aeppli cohomology group* of degree $(p, q)$ is defined as

$$H^{p,q}_{A}(M) := \frac{\text{Ker} \left( \partial \bar{\partial} : A^{p,q}(M) \to A^{p+1,q+1}(M) \right)}{\partial A^{p-1,q}(M) + \bar{\partial} A^{p,q-1}(M)}.$$

It is well known that all these groups can also be defined by means of currents of corresponding degree. For every $(p, q) \in \mathbb{N}^2$, the identity induces a natural map

$$i : H^{p,q}_{BC}(M) \to H^{p,q}_{A}(M).$$

In general, the map $i$ is neither injective nor surjective. If $M$ satisfies $\partial \bar{\partial}$-lemma, then for every $(p, q) \in \mathbb{N}^2$, $i$ is an isomorphism, referring to [6], Lemma 5.15, Remarks 5.16, 5.21.

**Theorem 3.9.** Let $X$ be a strongly Gauduchon space. If it has a desingularization $\tilde{X}$ such that $i : H^{1,1}_{BC}(\tilde{X}) \to H^{1,1}_{A}(\tilde{X})$ is injective, then $X \in \mathcal{SG}$.
Proof. Set dim \( X = n \). Suppose \( \pi : \tilde{X} \to X \) is the desingularization. We need to prove that \( \tilde{X} \) is a strongly Gauduchon manifold. By Theorem 1.2, it suffices to prove that if \( T \) is a positive \((1, 1)\)-current on \( \tilde{X} \) which is \( d \)-exact, then \( T = 0 \).

Let \( E \subseteq \tilde{X} \) be the exceptional set of \( \pi \), \( \Omega \) the real closed \((2n - 2)\)-form on \( X \) whose \((n - 1, n - 1)\)-part \( \Omega^{n-1,n-1} \) is strictly positive. Since \( T \) is \( d \)-exact, we have \( T(\pi^{*}\Omega) = 0 \). On the other hand, since \( T \) is a \((1, 1)\)-current, we have \( T(\pi^{*}\Omega) = T(\pi^{*}\Omega^{n-1,n-1}) = \int_{\tilde{X}} T \wedge \pi^{*}\Omega^{n-1,n-1} \) and \( \pi^{*}\Omega^{n-1,n-1} \) is strictly positive on \( \tilde{X} - E \), so we obtain \( \text{supp}T \subseteq E \).

By Theorem 3.8 for \( p = n - 1 \), we obtain \( T = \sum c_{i}[E_{i}] \), where \( c_{i} \geq 0 \) and \( E_{i} \) are the \((n - 1)\)-dimensional irreducible components of \( E \). Since \( T \) is \( d \)-exact, \( \sum c_{i}[E_{i}] = [T]_{\tilde{X}} = 0 \) in \( H_{2n-2}(\tilde{X}, \mathbb{R}) \). Beacuse \( i \) is injective, we know \( [T]_{BC} = 0 \) in \( H_{2n-2}^{1}(\tilde{X}) \). So, there is a real 0-current \( Q \) on \( \tilde{X} \), such that \( T = i\partial\bar{\partial}Q \). Since \( T \geq 0 \), \( Q \) is plurisubhamonic. By maximum principle, \( Q \) is a constant, hence \( T = 0 \). \( \square \)

Lemma 3.10 (\cite{8}, Lemma 3.6). Let \( f : X \to Y \) be a modification between reduced compact complex spaces of dimension \( n \). If \( Y \) is normal and the betti number \( b_{2n-1}(Y) = 0 \), then there is a exact sequence

\[
0 \longrightarrow H_{2n-2}(E, \mathbb{R}) \overset{i^{*}}{\longrightarrow} H_{2n-2}(X, \mathbb{R}) \overset{f_{*}}{\longrightarrow} H_{2n-2}(Y, \mathbb{R})
\]

where \( E \) is the exceptional set of \( f \), \( i : E \to X \) is the inclusion. Moreover, \( H_{2n-2}(E, \mathbb{R}) = \oplus j \mathbb{R}[E_{j}] \), where \( \{E_{j}\}_{j} \) are all the \((n - 1)\)-dimensional irreducible components of \( E \) (possibly there exist some other components of dimension \( < n - 1 \) in \( E \)).

Theorem 3.11. If \( X \) is a normal strongly Gauduchon space of dimension \( n \) with the betti number \( b_{2n-1}(X) = 0 \), then \( X \in \mathcal{G} \).

Proof. Suppose \( T \) is a positive \((1, 1)\)-current on \( \tilde{X} \) which is \( d \)-exact. As the proof in Theorem 3.9 we obtain

\[
T = \sum c_{i}[E_{i}]
\]

where \( c_{i} \geq 0 \), \( E_{i} \) are the \((n - 1)\)-dimensional irreducible components of \( E \). Since \( T \) is \( d \)-exact, \( \sum c_{i}[E_{i}] = [T]_{\tilde{X}} = 0 \) in \( H_{2n-2}(\tilde{X}, \mathbb{R}) \). By Lemma 3.10 we get \( c_{i} = 0 \) for all \( i \). \( \square \)
Theorem 3.12. Let $X$ be a compact strongly Gauduchon space. If it has a desingularization $\tilde{X}$ whose exceptional set has codimension $\geq 2$, then $X \in S_G$.

Proof. Suppose $\dim X = n$ and $T$ is a positive $(1,1)$-current on $\tilde{X}$ which is $d$-exact. As the proof in Theorem 3.9, we obtain $\text{supp} T \subseteq E$. By Theorem 3.8 for $p = n - 1$, we get $T = 0$ immediately. \hfill $\square$

4. Families of complex spaces over a nonsingular curve

In this section, we study families of complex spaces over a curve. It should be useful in the study of deformations and moduli spaces of complex spaces. The following definition is a generalization of the corresponding notion defined in [13].

Definition 4.1. Let $X$ be a reduced compact complex space of pure dimension $n$, and $f : X \to C$ a holomorphic map onto a nonsingular compact complex curve $C$. $f$ is called topologically essential, if for every $p \in C$, no linear combination $\sum_j c_j [F_j]$ is zero in $H_{2n-2}(X,\mathbb{R})$, where the $F_j$’s are all the irreducible components of the fibre $f^{-1}(p)$, $c_j \geq 0$ and at least one of the $c_j$’s is positive.

Note that, for any reduced compact complex space $X$ of pure dimension $n$ and the holomorphic map $f : X \to C$ onto a nonsingular compact complex curve $C$, $f$ is an open map by the open mapping theorem ([10], page 109). Hence for every $p \in C$, every irreducible component of $f^{-1}(p)$ has dimension $n - 1$ ([7], §3.10, Theorem).

Now, we can generalize [18], Theorem 4.1 as follows.

Theorem 4.2. Suppose $X$ is a purely $n$-dimensional compact normal complex space which admits a topologically essential holomorphic map $f : X \to C$ onto a nonsingular compact complex curve $C$, and $X$ has a desingularization $\pi : \tilde{X} \to X$, such that no nonzero nonnegative linear combination of hypersurfaces contained in the exceptional set of $\pi$ is zero in $H_{2n-2}(\tilde{X},\mathbb{R})$. If every nonsingular fiber of $f$ is a strongly Gauduchon manifold, then $X \in S_G$.

Proof. Set $\tilde{f} := f \circ \pi$. For every $p \in C$, set $f^{-1}(p) = \bigcup_i V_i$, where $V_i$ are all the irreducible components of $f^{-1}(p)$ which have dimension $n - 1$. Since $X$ is normal, $\text{codim} X_s \geq 2$, where $X_s$ is the set of singular points of $X$. So

$$\pi^{-1}(V_i) = \tilde{V}_i \cup \bigcup_j E_{ij},$$

where $\tilde{V}_i = \pi^{-1}(V_i - X_s)$ is the strict transform of $V_i$, and $E_{ij}$ are all irreducible components of $\pi^{-1}(V_i)$ contained in the exceptional set of $\pi$. It is
possible that some $E_{ij}$ are contained in other $E_{kl}$ or $\tilde{V}_k$. We denote any $E_{ij}$, which is not properly contained in other $E_{kl}$ or $\tilde{V}_k$, by $E_{ij}'$ and we denote any $E_{ij}$, which is properly contained in other $E_{kl}$ or $\tilde{V}_k$, by $E_{ij}''$ (i.e. there exists other $E_{kl}$ or $\tilde{V}_k$, such that $E_{ij}'' \subsetneq E_{kl}$ or $\tilde{V}_k$), then
\[
\tilde{f}^{-1}(p) = \bigcup_i (\tilde{V}_i \cup \bigcup_j E_{ij}')
\]
is the irreducible decomposition of $\tilde{f}^{-1}(p)$, hence $\text{codim}E_{ij}' = 1$.

We need the following two claims.

**Claim 1.** $\tilde{f}$ is topologically essential.

**Proof.** If not, we have
\[
\Sigma_i a_i[\tilde{V}_i] + \Sigma_{ij} b_{ij}[E_{ij}'] = 0,
\]
in $H_{2n-2}(\tilde{X}, \mathbb{R})$, for some $a_i, b_{ij} \geq 0$ and at least one of the $a_i$'s, $b_{ij}$'s is positive. Since $\pi(E_{ij}') \subseteq X_s$ has codimension $\geq 2$, $\pi_*[E_{ij}'] = 0$ in $H_{2n-2}(X, \mathbb{R})$. In $H_{2n-2}(X, \mathbb{R})$, $\pi_*[\tilde{V}_i] = [V_i]$, hence
\[
\Sigma_i a_i[\tilde{V}_i] = 0
\]
through $\pi_*$. Since $f$ is topologically essential, $a_i = 0$ for all $i$. So
\[
\Sigma_{ij} b_{ij}[E_{ij}'] = 0,
\]
in $H_{2n-2}(\tilde{X}, \mathbb{R})$, where $b_{ij} \geq 0$ and at least one of the $b_{ij}$'s is positive. It contradicts the assumption on $\tilde{X}$.

**Claim 2.** For every $p \in C$, if $\tilde{f}^{-1}(p)$ is nonsingular, then it is a strongly Gauduchon manifold.

**Proof.** Since $\tilde{f}^{-1}(p) = \bigcup_i (\tilde{V}_i \cup \bigcup_j E_{ij}')$ is nonsingular, we have
\[
\tilde{V}_i \cap \tilde{V}_k = \emptyset, \quad \forall i \neq k;
\]
\[
\tilde{V}_i \cap E_{kl'} = \emptyset, \quad \forall i, k, l'.
\]
Since for any $i, j$, $E_{ij}$ is contained in some $E_{kl'}$ or $\tilde{V}_k$, we have $\tilde{V}_i \cap E_{ij} = \emptyset$. On the other hand, if $V_i \cap X_s \neq \emptyset$, then the intersection of $\tilde{V}_i$ and $\bigcup_j E_{ij}$ is not empty, which contradicts with $\tilde{V}_i \cap E_{ij} = \emptyset$. So for all $i$, $V_i \cap X_s = \emptyset$. Hence, the map
\[
\pi |_{\tilde{f}^{-1}(p)}: \tilde{f}^{-1}(p) \to f^{-1}(p)
\]
is an isomorphism. Since every nonsingular fiber of $f$ is a strongly Gauduchon manifold and $\tilde{f}^{-1}(p)$ is nonsingular, $\tilde{f}^{-1}(p)$ is a strongly Gauduchon manifold.

Now, by the Claim 1 and 2, $\tilde{X}$ is a strongly Gauduchon manifold according to [15], Theorem 4.1. Hence, $X \in \mathcal{G}$.
By the above theorem, we have the following corollary immediately.

**Corollary 4.3.** Suppose $X$ is a purely dimensional compact normal complex space which admits a topologically essential holomorphic map $f : X \rightarrow C$ onto a nonsingular compact complex curve $C$, and $X$ has a desingularization $\tilde{X}$ whose exceptional set has codimension $\geq 2$. If every nonsingular fiber of $f$ is a strongly Gauduchon manifold, then $X \in \mathcal{G}$.

**Corollary 4.4.** Let $X$ be a purely $n$-dimensional normal compact complex space which admits a topologically essential holomorphic map onto a nonsingular compact complex curve. If the betti number $b_{2n-1}(X) = 0$, then $X \in \mathcal{G}$.

**Proof.** By Lemma 3.10, we know that, for any desingularization $\pi : \tilde{X} \rightarrow X$, $\{[E_j]\}_j$ are linearly independent in $H_{2n-2}(\tilde{X}, \mathbb{R})$, where $\{E_j\}_j$ are all the $(n-1)$-dimensional irreducible components of the exceptional set of $\pi$. Using Theorem 4.2, we get this corollary immediately. $\square$

**Acknowledgements.** We would like to thank our supervisor Prof. Jixiang Fu for his constant encouragement and many helpful discussions.

**References**

[1] L. Alessandrini, Holomorphic submersions onto Kähler or balanced manifold, arXiv:1410.2396v1.
[2] L. Alessandrini and M. Andreatta, Closed transverse (p, p)-forms on compact complex manifolds, *Compositio Math.* 61 (1987), 181-200. Erratum ibid. 61 (1987), 143.
[3] L. Alessandrini and G. Bassanelli, Positive $\partial\bar{\partial}$-closed currents and non-Kähler Geometry, *J. Geom. Anal.* 2 (1992), 291-316.
[4] A. Borel and A. Haefliger, La classe d’homolgie fondamentale d’un espace analytique, *Bull. Soc. Math. France* 89 (1961), 461-513.
[5] P. Deligne, *Équations différentielles à points singuliers réguliers*, Lecture Notes in Math. 163, Springer-Verlag, Berlin-New York, 1970.
[6] P. Deligne; P. Griffiths; J. Morgan; D. Sullivan, Real homotopy theory of Kähler manifolds, *Invent. Math.* 29 (1975), no. 3, 245-274.
[7] G. Fischer, *Complex analytic geometry*, Lecture Notes in Math. 538, Berlin-Heidelberg-New York, Springer, 1976.
[8] J. Fu and L. Meng and W. Xia, Complex balanced spaces, *Internat. J. Math.* 26 (2015), no. 12, 1550105, 15 pp.
[9] A. Fujiki, Closedness of the Douady spaces of compact Kähler spaces, *Publ. RIMS, Kyoto Univ.* 14 (1978), 1-52.
[10] H. Grauert and R. Remmert, *Coherent analytic sheaves*, Grundlehren der math. Wiss. 265, Springer-Verlag, Berlin, 1984.
[11] Chung-Wu Ho, A note on proper maps, *Proc. Amer. Math. Soc.* 51 (1975), 237-241.
[12] J. King, The currents defined by analytic varieties, *Acta Math.* 127 (1971), 185-220.
[13] M. L. Michelsohn, On the existence of special metrics in complex geometry, *Acta Math.* 149 (1982), 261-295.
[14] D. Popovici, Limits of Projective Manifolds under Holomorphic Deformations, arXiv:0910.2032v1.

[15] D. Popovici, Stability of Strongly Gauduchon Manifolds under Modifications, J. Geom. Anal. 23 (2013), 653-659.

[16] D. Popovici, Deformation limits of projective manifolds: Hodge numbers and strongly Gauduchon metrics, Invent. Math. 194 (2013), 515-534.

[17] D. Popovici, Deformation openness and closedness of various classes of compact complex manifolds; examples, Ann.Sc. Norm. Super. Pisa Cl. Sci.(5) 13 (2014), 255-305.

[18] J. Xiao, On Strongly Gauduchon Metrics of Compact Complex Manifolds, J. Geom. Anal. 25 (2015), 2011-2027.

[19] J. Varouchas, Stabilité de la classe des variétés Kähleriennes par certains morphismes propres, Invent. Math. 77 (1984), 117-127.

[20] J. Varouchas, Kähler Spaces and Proper Open Morphisms, Math. Ann. 283 (1989), 13-52.

DEPARTMENT OF MATHEMATICS, NORTH UNIVERSITY OF CHINA, TAIYUAN, SHANXI 030051, P.R. CHINA
E-mail address: 20160012@nuc.edu.cn

INSTITUTE OF MATHEMATICS, FUDAN UNIVERSITY, SHANGHAI 200433, P.R. CHINA
E-mail address: 11110180005@fudan.edu.cn