Information Leakage of Correlated Source Coded Sequences over Wiretap Channel

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Abstract—A new generalised approach for multiple correlated sources over a wiretap network is investigated. A basic model consisting of two correlated sources where each produce a component of the common information is initially investigated. There are several cases that consider wiretapped syndromes on the transmission links and based on these cases a new quantity, the information leakage at the source/s is determined. An interesting feature of the models described in this paper is the information leakage quantification. Shannon’s cipher system with eavesdroppers is incorporated into the two correlated sources model to minimize key lengths. These aspects of quantifying information leakage and reducing key lengths using Shannon’s cipher system are also considered for a multiple correlated source network approach. A new scheme that incorporates masking using common information combinations to reduce the key lengths is presented and applied to the generalised model for multiple sources.

I. INTRODUCTION

Keeping information secure has become a major concern with the advancement in technology. In this work, the information theory aspect of security is analyzed, as entropies are used to measure security. The system also incorporates some traditional ideas surrounding cryptography, namely Shannon’s cipher system and adversarial attackers in the form of eavesdroppers. In cryptographic systems, there is usually a message in plaintext that needs to be sent to a receiver. In order to secure it, the plaintext is encrypted so as to prevent eavesdroppers from reading its contents. This ciphertext is then transmitted to the receiver. Shannon’s cipher system (mentioned by Yamamoto [1]) incorporates this idea. The definition of Shannon’s cipher system has been discussed by Hanawal and Sundaresan [2]. In Yamamoto’s [1] development on this model, a correlated source approach is introduced. This gives an interesting view of the problem, and is depicted in Figure [1].

Correlated source coding incorporates the lossless compression of two or more correlated data streams. Correlated sources have the ability to decrease the bandwidth required to transmit and receive messages because a syndrome (compressed form of the original message) is sent across the communication links instead of the original message. A compressed message has more information per bit, and therefore has a higher entropy because the transmitted information is more unpredictable. The unpredictability of the compressed message is also beneficial in terms of securing the information.

The source sends information for the correlated sources, X and Y along the main transmission channel. A key $W_k$, is produced and used by the encoder when producing the ciphertext. The wiretapper has access to the transmitted codeword, $W$. The decoded codewords are represented by $\hat{X}$ and $\hat{Y}$. In Yamamoto’s scheme the security level was also focused on and found to be $\frac{1}{\mu} H(X^K, Y^K | W)$ (i.e. the joint entropy of $X$ and $Y$ given $W$, where $K$ is the length of $X$ and $Y$) when $X$ and $Y$ have equal importance, which is in accordance with traditional Shannon systems where the security is measured by the equivocation. When one source is more important than the other then the security level is measured by the pair of the individual uncertainties ($\frac{1}{\mu} H(X^K | W)$, $\frac{1}{\mu} H(Y^K | W)$).

In practical communication systems links are prone to eavesdropping and as such this work incorporates wiretapped channels, more specifically the Wiretap Channel II. The mathematical model for this Wiretap Channel is given by Rouayheb et al. [3], and can be explained as follows: the channel between a transmitter and receiver is error-free and can transmit $n$ symbols $Y = (y_1, \ldots, y_n)$ from which $\mu$ bits can be observed by the eavesdropper and the maximum secure rate can be shown to equal $n - \mu$ bits. The security aspect of wiretap networks have been looked at in various ways by Cheng et al. [4], and Cai and Yeung [5], emphasising that it is of concern to secure these type of channels.

Villard and Piantanida [6] also look at correlated sources and wiretap networks: A source sends information to the receiver and an eavesdropper has access to information correlated to the source, which is used as side information. There is a second encoder that sends a compressed version of its own correlation observation of the source privately to the receiver. Here, the authors show that the use of correlation decreases

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the required communication rate and increases secrecy. Villard et al. [7] explore this side information concept further where security using side information at the receiver and eavesdropper is investigated. Side information is generally used to assist the decoder to determine the transmitted message. An earlier work involving side information is that by Yang et al. [8]. The concept can be considered to be generalised in that the side information could represent a source. It is an interesting problem when one source is more important and Hayashi and Yamamoto [9] consider it in another scheme, where only $X$ is secure against wiretappers and $Y$ must be transmitted to a legitimate receiver. They develop a security criterion based on the number of correct guesses of a wiretapper to retrieve a message. In an extension of the Shannon cipher system, Yamamoto [10] investigated the secret sharing communication system.

In this case, we generalise a model for correlated sources across a wiretap channel and the security aspect is explored by quantifying the information leakage and reducing the key lengths when incorporating Shannon’s cipher system.

This paper initially describes a two correlated source model across wiretapped links, which is detailed in Section II. In Section III, the information leakage is investigated and proven for this two correlated source model. The information leakage is quantified to be the equivocation subtracted from the total obtained uncertainty; the actual bounds achieved are depicted in Table I. In Section IV the two correlated sources model is looked at according to Shannon’s cipher system and the summarised equivocations and rates are shown in Table I. The notation contained in the tables will be clarified in the following sections. The proofs for this Shannon cipher system aspect are detailed in Section V. Section VI details the extension of the two correlated source model where multiple correlated sources in a network scenario is investigated. There are two subsections here; one quantifying information leakage for the Slepian-Wolf scenario and the other incorporating Shannon’s cipher system where key lengths are minimized and a masking method to save on keys is presented. Section VII explains how the models detailed in this paper is a generalised model of Yamamoto’s [10] model, and further offers comparison to other models. The future work for this research is detailed in Section VIII and the paper is concluded in Section IX.

II. MODEL

The independent, identically distributed (i.i.d.) sources $X$ and $Y$ are mutually correlated random variables, depicted in Figure 2. The alphabet sets for sources $X$ and $Y$ are represented by $\mathcal{X}$ and $\mathcal{Y}$ respectively. Assume that $(X^K, Y^K)$ are encoded into two syndromes $(T_X$ and $T_Y)$. We can write $T_X = F(X^K)$ and $T_Y = F(Y^K)$ where $T_X$ and $T_Y$ are the syndromes of $X$ and $Y$. Here, $T_X$ and $T_Y$ are characterised by $(V_X, V_{CX}) = F'(T_X)$ and $(V_Y, V_{CY}) = F'(T_Y)$. The Venn diagram in Figure 3 easily illustrates this idea where it is shown that $V_X$ and $V_Y$ represent the private information of sources $X$ and $Y$ respectively and $V_{CX}$ and $V_{CY}$ represent the common information between $X^K$ and $Y^K$ generated by $X^K$ and $Y^K$ respectively.

The correlated sources $X$ and $Y$ transmit messages (in the form of syndromes) to the receiver along wiretapped links. The decoder determines $X$ and $Y$ only after receiving all of $T_X$ and $T_Y$. The common information between the sources are transmitted through the portions $V_{CX}$ and $V_{CY}$. In order to decode a transmitted message, a source’s private information and both common information portions are necessary. This aids in security as it is not possible to determine, for example $X$ by wiretapping all the contents transmitted along $X$’s channel only. This is different to Yamamoto’s [10] model as here the common information consists of two portions. The aim is to keep the system as secure as possible and these following sections show how it is achieved by this new model. We assume that the function $F$ is an ideal one-to-one process and is reversible, which means based on $T_X$ and $T_Y$ we can completely retrieve $X^K$ and $Y^K$. Furthermore, it reaches the Slepian-Wolf bound, $H(T_X, T_Y) = H(X^K, Y^K)$. Here, we note that the lengths of $T_X$ and $T_Y$ are not fixed, as it depends on the encoding process and nature of the Slepian-Wolf codes. The process is therefore not ideally one-to-one and reversible and is another difference between our model and Yamamoto’s [10] model.

The code described in this section satisfies the following inequalities for $\delta > 0$ and sufficiently large $K$.

\begin{align}
Pr\{X \neq G(V_X, V_{CX}, V_{CY})\} &\leq \delta \quad (1) \\
Pr\{Y \neq G(V_Y, V_{CX}, V_{CY})\} &\leq \delta \quad (2) \\
H(V_X, V_{CX}, V_{CY}) &\leq H(X) + \delta \quad (3)
\end{align}

Figure 2. Correlated source coding for two sources

\[
\begin{align*}
&x^K \quad \text{Encoder} \quad y^K \\
&\downarrow \quad \text{Decoder} \quad \uparrow \\
&x^K, y^K
\end{align*}
\]

Figure 3. The relation between private and common information
It can intuitively be seen from (3) and (4) that the private information and common information produced by each source should contain much as possible. The second common information is defined as the rate of the attainable maximum core V_c such that if we lose V_c then the uncertainty of X and Y becomes H(V_c).

Here, we consider the common information that V_{CX} and V_{CY} represent.

We begin demonstrating the relationship between the common information portions by constructing the prototype code (W_X, W_Y, W_{CX}, W_{CY}) as per Lemma 1.

**Lemma 1:** For any $\epsilon_0 \geq 0$ and sufficiently large $K$, there exists a code $W_X = F_X(X^K)$, $W_Y = F_Y(Y^K)$, $W_{CX} = F_{CX}(X^K)$, $W_{CY} = F_{CY}(Y^K)$, $\hat{X}^K, \hat{Y}^K = G(W_X, W_Y, W_{CX}, W_{CY})$, where $W_X \in I_{M_X}$, $W_Y \in I_{M_Y}$, $W_{CX} \in I_{M_{CX}}$, $W_{CY} \in I_{M_{CY}}$ for $I_{M_a}$, which is defined as $\{0, 1, \ldots, M_a - 1\}$, that satisfies,

$$I(X;Y).$$

Yamamoto \[1\] also defined two kinds of common information. The first common information is defined as the rate of the attainable minimum core V_c (i.e. V_{CX}, V_{CY} in this model) by removing each private information, which is independent of the other information, from (X^K, Y^K) as much as possible. The second common information is defined as the rate of the attainable maximum core V_c such that if we lose V_c then the uncertainty of X and Y becomes H(V_c).

Table I

| Case | Secret | Leaked/Transmitted | Information Leakage |
|------|--------|-------------------|---------------------|
| 1    | X, Y   | T_X, T_Y          | $H(X) - H(Y) + \delta$ |
| 2    | X      | T_X, T_Y          | $H(X) - H(Y) + \delta$ |
| 3    | X, Y   | T_X               | $H(X) - H(Y) + \delta$ |
| 4    | Y      | T_X               | $H(X) - H(Y) + \delta$ |

Table II

| Case | Secret | Leaked/Transmitted | Security Level (Equivalence) | Rates |
|------|--------|-------------------|-----------------------------|-------|
| 1    | X, Y   | T_X, T_Y          | $\frac{1}{K} H(X^K, Y^K | W)$ | $R_X \geq H(X|Y) + \frac{1}{K} \log M_{CX}, R_Y \geq H(Y)$, $R_{kX} \geq h_X, R_{kY} \geq h_Y$ |
| 2    | X      | T_X               | $\frac{1}{K} H(X^K | W)$    | $R_X \geq H(X), R_Y \geq \frac{1}{K} \log M_{CY}, R_{kX} \geq h_X, R_{kY} \geq 0$ |
| 3    | X, Y   | T_X               | $(\frac{1}{K} H(X^K | W), \frac{1}{K} H(Y^K | W))$ | $R_X \geq H(X,Y), R_Y \geq H(X), R_{kX} \geq h_{XY}, R_{kY} \geq h_Y$ |
| 4    | Y      | T_X               | $\frac{1}{K} H(Y^K | W)$    | $R_X \geq H(Y), R_Y \geq H(Y), R_{kX} \geq h_Y$ |
| 5    | X      | T_X               | $\frac{1}{K} H(X^K | W)$    | $R_X \geq H(X), R_Y \geq H(Y), R_{kX} \geq h_X, R_{kY} \geq 0$ |

where $G$ is a function to define the decoding process at the receiver. It can intuitively be seen from (3) and (4) that X and Y are recovered from the corresponding private information and the common information produced by $X^K$ and $Y^K$.

Equations (3), (4) and (5) show that the private information and common information produced by each source should contain no redundancy. It is also seen from (7) and (8) that $V_Y$ is independent of $X^K$. Here, $V_X, V_Y, V_{CX}$ and $V_{CY}$ are disjoint, which ensures that there is no redundant information sent to the decoder.

To recover X the following components are necessary: $V_X$, $V_{CX}$ and $V_{CY}$. This comes from the property that $X^K$ cannot be derived from $V_X$ and $V_{CX}$ only and part of the common information between $X^K$ and $Y^K$ is produced by $Y^K$.

Yamamoto \[1\] proved that a common information between $X^K$ and $Y^K$ is represented by the mutual information $I(X;Y)$.
The codeword sets exist as C

We can see that (11) - (13) mean
\[ H(X, Y) - 3\epsilon_0 \leq \frac{1}{K}(H(W_X) + H(W_Y) + H(W_{CX})) \]
+ \[ H(W_{CY}) \]
\[ \leq H(X, Y) + 3\epsilon_0 \] (16)

Hence from (10), (16) and the ordinary source coding theorem, (W_X, W_Y, W_{CX}, W_{CY}) have no redundancy for sufficiently small \( \epsilon_0 \geq 0 \). It can also be seen that \( W_X \) and \( W_Y \) are independent of \( Y^K \) and \( X^K \) respectively.

Proof of Lemma 1:
As seen by Slepian and Wolf, mentioned by Yamamoto \[1\] there exist \( M_X \) codes for the \( P_{Y|X}(y|x) \) DMC (discrete memoryless channel) and \( M_Y \) codes for the \( P_{X|Y}(x|y) \) DMC. The codewords exist as \( C_{i_X}^X \) and \( C_{j_Y}^Y \), where \( C_{i_X}^X \) is a subset of the typical sequence of \( X^K \) and \( C_{j_Y}^Y \) is a subset of the typical sequence of \( Y^K \). The encoding functions are similar, but we have created one decoding function as there is one decoder at the receiver:

\[ f_{X_i} : I_{M_{CX}} \rightarrow C_{i_X}^X \] (17)

\[ f_{Y_j} : I_{M_{CY}} \rightarrow C_{j_Y}^Y \] (18)

\[ g : X^K, Y^K \rightarrow I_{M_{CX}} \times I_{M_{CY}} \] (19)

The relations for \( M_X, M_Y \) and the common information remain the same as per Yamamoto’s and will therefore not be proven here. The encoding and decoding is defined as follows:

**Encoding:** If \( X^K = x^K_l \in C_{i_X}^X \) and \( Y^K = y^K_m \in C_{j_Y}^Y \) then \( W_X = i, W_{CX} = \lfloor l/M \rfloor \) where \( M = M_{CX} + M_{CY}, W_{CY} = m \mod M \) and \( W_Y = j \). The one-to-one mapping between \( i, l \) and \( j, m \) at the encoders are arranged such that \( C_{i_X}^X \rightarrow C_{j_Y}^Y \), where \( l = m \). Here, \( \Pr \{ C_{i_X}^X \rightarrow C_{j_Y}^Y \} \leq 1 - \delta \) where \( \delta > 0 \) is small. Otherwise \( W_X = W_Y = W_{CX} = W_{CY} = 0 \).

**Decoding:** If \( W_X = i, W_{CX} \times M + W_{CY} = l, W_Y = j \), then \( \tilde{X}^K = x^K_l \) and \( \tilde{Y}^K = g(x^K_l, y^K_m) \).

Using this coding scheme, \( i, l \) and \( j \) are retrieved from \( x^K_l \) and \( y^K_m \). Then \( m \) is retrieved using \( l \) as there is a one-to-one mapping between \( l \) and \( m \), as shown by Yamamoto \[1\]. Here, we have ordered the codeword sets to make \( l = m \) in order to allow for both sources to have access to the common information. When the amount of information supplied by \( W_{CX} \) and \( W_{CY} \) is high enough, their entropy will be the same as that of \( W_C \). Both \( W_{CX} \) and \( W_{CY} \) contribute to finding the common information, which in this case can be referenced to \( l \). This proves that the code does exist and that \( W_X \) and \( W_Y \) are independent of \( Y \) and \( X \) respectively, as shown by Yamamoto \[1\].

The common information is important in this model as the sum of \( V_{CX} \) and \( V_{CY} \) represent a common information between the sources. The following theorem holds for this common information:

**Theorem 1:**
\[ \frac{1}{K}[H(V_{CX}) + H(V_{CY})] = I(X; Y) \] (20)

where \( V_{CX} \) is the common portion between \( X \) and \( Y \) produced by \( X^K \) and \( V_{CY} \) is the common portion between \( X \) and \( Y \) produced by \( Y^K \). The private portions for \( X \) and \( Y \) are represented as \( V_X \) and \( V_Y \) respectively. As explained in Yamamoto’s \[1\] Theorem 1, two types of common information exist (the first is represented by \( I(X; Y) \) and the second by \( \min(H(X), H(Y)) \)). We will develop part of this idea to show that the sum of the common information portions produced by \( X^K \) and \( Y^K \) in this new model is represented by the mutual information between the sources.

**Proof of Theorem 1:** The first part is to prove that \( H(V_{CX}) + H(V_{CY}) \geq I(X; Y) \), and is done as follows. We weaken the conditions (1) and (2) to
\[ \Pr \{ X^K, Y^K \neq G_{X,Y}(V_X, V_Y, V_{CX}, V_{CY}) \} \leq \delta_1 \] (21)

For any \( (V_X, V_Y, V_{CX}, V_{CY}) \in C(\delta) \), we have from (21) and the ordinary source coding theorem that
\[ H(X^K, Y^K) - \delta_1 \leq \frac{1}{K}[H(V_X, V_Y, V_{CX}, V_{CY}) \]
\[ \leq \frac{1}{K}[H(V_X) + H(V_Y) + H(V_{CX}) \]
\[ + H(V_{CY})] \] (22)

where \( \delta_1 \to 0 \) as \( \delta \to 0 \). From Lemma 1,
\[ \frac{1}{K}[H(V_Y|X^K)] \geq \frac{1}{K}[H(V_Y) - \delta] \] (23)

\[ \frac{1}{K}[H(V_X|Y^K)] \geq \frac{1}{K}[H(V_X) - \delta] \] (24)

From (22) - (24),
\[ \frac{1}{K}[H(V_{CX}) + H(V_{CY})] \geq H(X, Y) - \frac{1}{K}[H(V_X) \]
\[ - H(V_Y) - \delta_1 \]
\[ \geq H(X, Y) - \frac{1}{K}[H(V_X|Y) \]
\[ - \frac{1}{K}[H(V_Y|X)] - \delta_1 - 2\delta \] (25)

On the other hand, we can see that
\[ \frac{1}{K}[H(X^K, V_Y)] \leq H(X, Y) + \delta \] (26)

This implies that
\[ \frac{1}{K}[H(V_Y|X^K)] \leq H(Y|X) + \delta \] (27)

and
\[ \frac{1}{K}[H(V_X|Y^K)] \leq H(X|Y) + \delta \] (28)
From (25), (27) and (28) we get
\[
\frac{1}{K}[\text{H}(V_{CX}) + \text{H}(V_{CY})] \geq \text{H}(X,Y) - \text{H}(X|Y) - \text{H}(Y|X)
\]
\[
= \text{H}(V_{CY}) - \delta_1 - 4\delta
\]
\[
= I(X;Y) - \delta_1 - 4\delta
\]

It is possible to see from (13) that \(\text{H}(V_{CX}) + \text{H}(V_{CY}) \leq I(X;Y)\). From this result, (19) and (29), and as \(\delta_1 \to 0\) and \(\delta \to 0\) it can be seen that
\[
\frac{1}{K}[\text{H}(V_{CX}) + \text{H}(V_{CY})] = I(X;Y)
\]

This model can cater for a scenario where a particular source, say \(X\) needs to be more secure than \(Y\) (possibly because of eavesdropping on the \(X\) channel). In such a case, the \(\frac{1}{K}\text{H}(V_{CX})\) term in (29) needs to be as high as possible. When this uncertainty is increased then the security of \(X\) is increased. Another security measure that this model incorporates is that \(X\) cannot be determined from wiretapping only \(X\’s\) link.

### III. INFORMATION LEAKAGE

In order to determine the security of the system, a measure for the amount of information leaked has been developed. This is a new notation and quantification, which emphasizes the novelty of this work. The obtained information and total uncertainty are used to determine the leaked information. Information leakage is indicated by \(L_{P}^{\delta}\). Here \(P\) indicates the source/s for which information leakage is being quantified, \(P = \{S_1, \ldots, S_n\}\) where \(n\) is the number of sources (in this case, \(n = 2\)). Further, \(Q\) indicates the syndrome portion that has been wiretapped, \(Q = \{V_1, \ldots, V_m\}\) where \(m\) is the number of codewords (in this case, \(m = 4\)).

The information leakage bounds are as follows:
\[
L_{X,V_{Y}}^{X} \leq \text{H}(X^K) - \text{H}(V_{CX}) - \text{H}(V_{CY}) + \delta
\]  
\[
L_{V_{CX},V_{CY}}^{X} \leq \text{H}(X^K) - \text{H}(V_{X}) - \text{H}(V_{CY}) + \delta
\]  
\[
L_{V_{CX},V_{CY},V_{Y}}^{X} \leq \text{H}(X^K) - \text{H}(V_{X}) - \text{H}(V_{CY}) + \delta
\]  

These bounds developed in (31) - (34) are proven in the next section.

The proofs for the above mentioned information leakage inequalities are now detailed. First, the inequalities in (6) - (9) will be proven, so as to prove that the information leakage equations hold.

**Lemma 2:** The code \((V_X, V_{CX}, V_{CY}, V_Y)\) defined at the beginning of Section I, describing the model and (1) - (5) satisfy (6) - (9). Then the information leakage bounds are given by (31) - (34).

**Proof for (6):**
\[
\frac{1}{K}\text{H}(X^K|V_X, V_{Y})
\]
\[
= \frac{1}{K}[\text{H}(X^K, V_{CX}, V_{CY}) - \text{H}(V_{CX}, V_{CY})]
\]
\[
= \frac{1}{K}[\text{H}(X^K, V_{CY}) - \text{H}(V_{CX}, V_{CY})]
\]
\[
= \frac{1}{K}[\text{H}(X^K|V_{CY}) + \text{H}(V_{X}|V_{CY})] - \frac{1}{K}[\text{H}(V_{CX}|V_{CY}) + \text{H}(V_{CX}|V_{CY})]
\]
\[
= \frac{1}{K}[\text{H}(X^K|V_{CY}) + \text{H}(V_{X}|V_{CY}) - \text{H}(V_{CX}|V_{CY})]
\]
\[
= \frac{1}{K}[\text{H}(X^K) + \text{H}(V_{CX}) - \text{H}(V_{CX})] - \delta
\]
\[
= \frac{1}{K}[\text{H}(X^K) + \text{H}(V_{CY})] - \delta
\]

where (35) holds because \(V_X\) is a function of \(X^K\) and (36) holds because \(X\) is independent of \(V_Y\) and \(X\) is independent of \(V_Y\).

**Proof for (7):**
\[
\frac{1}{K}\text{H}(X^K|V_{CX}, V_{CY})
\]
\[
= \frac{1}{K}[\text{H}(X^K, V_{CX}, V_{CY}) - \text{H}(V_{CX}, V_{CY})]
\]
\[
= \frac{1}{K}[\text{H}(X^K, V_{CY}) - \text{H}(V_{CX}, V_{CY})]
\]
\[
= \frac{1}{K}[\text{H}(X^K|V_{CY}) + \text{H}(V_{X}|V_{CY})] - \frac{1}{K}[\text{H}(V_{CX}|V_{CY}) + \text{H}(V_{CX}|V_{CY})]
\]
\[
= \frac{1}{K}[\text{H}(X^K) + \text{H}(V_{CX}) - \text{H}(V_{CX})] - \delta
\]
\[
= \frac{1}{K}[\text{H}(X^K) - \text{H}(V_{CX})]
\]
\[
\geq \frac{1}{K}[\text{H}(V_{CX}) + \text{H}(V_{CY})] - \delta
\]

where (38) holds because \(V_{CX}\) is a function of \(X^K\) and
[39] holds because $X$ is independent of $V_{CY}$ and $V_{CX}$ is independent of $V_{CY}$.

The proof for $H(X|V_{CX}, V_{CY}, V_Y)$ is similar to that for $H(X|V_{CX}, V_{CY})$, because $V_Y$ is independent of $X$.

Proof for (8):

$$\frac{1}{K} H(X^K|V_{CX}, V_{CY}, V_Y)$$

$$= \frac{1}{K} H(X^K|V_{CX}, V_{CY})$$

$$= \frac{1}{K} [H(X^K, V_{CX}, V_{CY}) - H(V_{CX}, V_{CY})]$$

$$= \frac{1}{K} [H(X^K, V_{CX}, V_{CY}) - H(V_{CX}, V_{CY})]$$

$$= \frac{1}{K} [H(X^K, V_{CX}, V_{CY}) - H(V_{CX}, V_{CY})]$$

$$= \frac{1}{K} [H(X^K, V_{CX}, V_{CY}) - H(V_{CX}, V_{CY})]$$

$$= \frac{1}{K} [H(X^K, V_{CX}, V_{CY}) - H(V_{CX}, V_{CY})]$$

$$= \frac{1}{K} [H(X^K, V_{CX}, V_{CY}) - H(V_{CX}, V_{CY})]$$

$$= \frac{1}{K} [H(X^K, V_{CX}, V_{CY}) - H(V_{CX}, V_{CY})]$$

$$= \frac{1}{K} [H(X^K, V_{CX}, V_{CY}) - H(V_{CX}, V_{CY})]$$

where (41) holds because $V_Y$ and $X^K$ are independent, (42) holds because $V_{CX}$ is a function of $X^K$ and (43) holds because $X^K$ is independent of $V_{CY}$ and $V_{CX}$ is independent of $V_{CY}$.

For the proof of (9), we look at the following probabilities:

$$\Pr \{X_T, V_{CX} \neq G(T_X) \} \leq \delta$$

$$\Pr \{Y_T, V_{CY} \neq G(T_Y) \} \leq \delta$$

$$\frac{1}{K} H(X^K|T_Y)$$

$$\leq \frac{1}{K} H(X^K, V_{CY}, V_Y) + \delta$$

$$= \frac{1}{K} [H(X^K, V_{CY}, V_Y) - H(V_{CY}, V_Y)] + \delta$$

$$= \frac{1}{K} [H(X^K, V_{CY}, V_Y) - H(V_{CY}, V_Y)] + \delta$$

$$= \frac{1}{K} [H(X^K, V_{CY}, V_Y) - H(V_{CY}, V_Y)] + \delta$$

$$= \frac{1}{K} [H(X^K, V_{CY}, V_Y) - H(V_{CY}, V_Y)] + \delta$$

$$= \frac{1}{K} [H(X^K, V_{CY}, V_Y) - H(V_{CY}, V_Y)] + \delta$$

where (47) holds from (46), (48) holds because $V_{CY}$ and $V_Y$ are independent. Furthermore, (49) holds because $V_{CY}$ and $V_Y$ are independent and $X^K$ and $V_Y$ are independent.

Following a similar proof to those done above in this section, another bound for $H(X^K|V_{CY}, V_Y)$ can be found as follows:

$$\frac{1}{K} H(X^K|V_{CY}, V_Y)$$

$$= \frac{1}{K} [H(X^K, V_{CY}, V_Y) - H(V_{CY}, V_Y)]$$

$$= \frac{1}{K} [H(X^K, V_{CY}, V_Y) - H(V_{CY}, V_Y)]$$

$$= \frac{1}{K} [H(X^K, V_{CY}, V_Y) - H(V_{CY}, V_Y)]$$

$$= \frac{1}{K} [H(X^K, V_{CY}, V_Y) - H(V_{CY}, V_Y)]$$

$$= \frac{1}{K} [H(X^K, V_{CY}, V_Y) - H(V_{CY}, V_Y)]$$

$$= \frac{1}{K} [H(X^K, V_{CY}, V_Y) - H(V_{CY}, V_Y)]$$

$$= \frac{1}{K} [H(X^K, V_{CY}, V_Y) - H(V_{CY}, V_Y)]$$

$$= \frac{1}{K} [H(X^K, V_{CY}, V_Y) - H(V_{CY}, V_Y)]$$

where (51) and (52) hold for the same reason as (48) and (49) respectively.

Since we consider the information leakage as the total information obtained subtracted from the total uncertainty, the following hold for the four cases considered in this section:

$$L_{V_{CX}, V_{CY}}^{X^K} = H(X^K) - H(X|V_{CX}, V_{CY})$$

$$\leq H(X^K) - H(V_{CX}) - H(V_{CY}) + \delta$$

which proves (31).

$$L_{V_{CX}, V_{CY}}^{X^K} = H(X^K) - H(X|V_{CX}, V_{CY})$$

$$\leq H(X^K) - H(V_{CX}) - H(V_{CY}) + \delta$$

which proves (32).

$$L_{V_{CX}, V_{CY}}^{X^K} = H(X^K) - H(X|V_{CX}, V_{CY})$$

$$\leq H(X^K) - H(V_{CX}) - H(V_{CY}) + \delta$$

which proves (33).

The two bounds for $H(V_{CY}, V_Y)$ are given by (50) and (53). From (50):

$$L_{V_{CY}, V_{CY}}^{X^K} \geq H(X^K) - [H(X) - H(V_{CY}) + \delta]$$

$$\geq H(V_{CY}) - \delta$$

and from (53):

$$L_{V_{CY}, V_{CY}}^{X^K} \leq H(X^K) - (H(V_X) + H(V_{CX}) - \delta)$$

$$\leq H(X^K) - H(V_X) - H(V_{CX}) + \delta$$

Combining these results from (57) and (58) gives (34).
IV. SHANNON’S CIPHER SYSTEM

Here, we discuss Shannon’s cipher system for two independent correlated sources (depicted in Figure 4). The two source outputs are i.i.d random variables \( X \) and \( Y \), taking on values in the finite sets \( \mathcal{X} \) and \( \mathcal{Y} \). Both the transmitter and receiver have access to the key, a random variable, independent of \( X^K \) and \( Y^K \) and taking values in \( I_{M_k} = \{0, 1, 2, \ldots, M_k - 1\} \). The sources \( X^K \) and \( Y^K \) compute the ciphertexts \( X' \) and \( Y' \), which are the result of specific encryption functions on the plaintext from \( X \) and \( Y \) respectively. The encryption functions are invertible, thus knowing \( X' \) and the key, \( X^K \) can be retrieved.

The mutual information between the plaintext and ciphertext should be small so that the wiretapper cannot gain much information about the plaintext. For perfect secrecy, this mutual information should be zero, then the length of the key should be at least the length of the plaintext.

![Diagram of Shannon cipher system for two correlated sources](image)

The encoder functions for \( X \) and \( Y \), \((E_X \) and \( E_Y \) respectively) are given as:

\[
E_X : X^K \times I_{M_{kX}} \rightarrow I_{M'_{X}} = \{0, 1, \ldots, M'_{X} - 1\} \quad I_{M'_{CX}} = \{0, 1, \ldots, M'_{CX} - 1\}
\]

\[
E_Y : Y^K \times I_{M_{kY}} \rightarrow I_{M'_{Y}} = \{0, 1, \ldots, M'_{Y} - 1\} \quad I_{M'_{CY}} = \{0, 1, \ldots, M'_{CY} - 1\}
\]

The decoder is defined as:

\[
D_{XY} : (I_{M'_{X}}, I_{M'_{Y}}, I_{M'_{CX}}, I_{M'_{CY}}) \times I_{M_{X}} \times I_{M_{Y}} \rightarrow X^K \times Y^K
\]

The encoder and decoder mappings are below:

\[
W_1 = F_{E_X}(X^K, W_{kX}) \quad (62)
\]

\[
W_2 = F_{E_Y}(Y^K, W_{kY}) \quad (63)
\]

\[
\hat{X}^K = F_{D_X}(W_1, W_2, W_{kX}) \quad (64)
\]

\[
\hat{Y}^K = F_{D_Y}(W_1, W_2, W_{kY}) \quad (65)
\]

or

\[
(\hat{X}^K, \hat{Y}^K) = F_{D_{XY}}(W_1, W_2, W_{kX}, W_{kY}) \quad (66)
\]

The following conditions should be satisfied for cases 1 - 4:

\[
\frac{1}{K} \log M_X \leq R_X + \epsilon \quad (67)
\]

\[
\frac{1}{K} \log M_Y \leq R_Y + \epsilon \quad (68)
\]

\[
\frac{1}{K} \log M_{kX} \leq R_{kX} + \epsilon \quad (69)
\]

\[
\frac{1}{K} \log M_{kY} \leq R_{kY} + \epsilon \quad (70)
\]

\[
\Pr\{\hat{X}^K \neq X^K\} \leq \epsilon \quad (71)
\]

\[
\Pr\{\hat{Y}^K \neq Y^K\} \leq \epsilon \quad (72)
\]

\[
\frac{1}{K} H(X^K | W_1) \leq h_X - \epsilon \quad (73)
\]

\[
\frac{1}{K} H(Y^K | W_2) \leq h_Y - \epsilon \quad (74)
\]

where \( R_X \) is the the rate of source \( X \)'s channel and \( R_Y \) is the rate of source \( Y \)'s channel. Here, \( R_{kX} \) is the rate of the key channel at \( X^K \) and \( R_{kY} \) is the rate of the key channel at \( Y^K \). The security levels, which are measured by the total and individual uncertainties are \( h_{XY} \) and \( (h_X, h_Y) \) respectively.

The cases 1 - 5 are:

Case 1: When \( T_X \) and \( T_Y \) are leaked and both \( X^K \) and \( Y^K \) need to be kept secret.

Case 2: When \( T_X \) and \( T_Y \) are leaked and \( X^K \) needs to be kept secret.

Case 3: When \( T_X \) is leaked and both \( X^K \) and \( Y^K \) need to be kept secret.

Case 4: When \( T_X \) is leaked and \( Y^K \) needs to be kept secret.

Case 5: When \( T_X \) is leaked and \( X^K \) needs to be kept secret. where \( T_X \) is the syndrome produced by \( X \), containing \( V_{CX} \) and \( V_X \) and \( T_Y \) is the syndrome produced by \( Y \), containing...
The admissible rate region for each case is defined as follows:

**Definition 1a:** \( \mathcal{R}(h_{XY}) = \{(R_X, R_Y, R_{kX}, R_{kY}, h_{XY}) : (R_X, R_Y, R_{kX}, R_{kY}, h_{XY}) \text{ is admissible for case 1}\} \) \hspace{1cm} (76)

**Definition 1b:** \( \mathcal{R}(h_X) = \{(R_X, R_Y, R_{kX}, R_{kY}, h_X) : (R_X, R_Y, R_{kX}, R_{kY}, h_X) \text{ is admissible for case 2}\} \) \hspace{1cm} (77)

**Definition 1c:** \( \mathcal{R}(h_{XY}) = \{(R_X, R_Y, R_{kX}, R_{kY}, h_{XY}) : (R_X, R_Y, R_{kX}, R_{kY}, h_{XY}) \text{ is admissible for case 3}\} \) \hspace{1cm} (78)

**Definition 1d:** \( \mathcal{R}(h_Y) = \{(R_X, R_Y, R_{kX}, R_{kY}, h_Y) : (R_X, R_Y, R_{kX}, R_{kY}, h_Y) \text{ is admissible for case 4}\} \) \hspace{1cm} (79)

**Definition 1e:** \( \mathcal{R}(h_X) = \{(R_X, R_Y, R_{kX}, R_{kY}, h_X) : (R_X, R_Y, R_{kX}, R_{kY}, h_X) \text{ is admissible for case 5}\} \) \hspace{1cm} (80)

Theorems for these regions have been developed:

**Theorem 2:** For \( 0 \leq h_{XY} \leq H(X,Y) \),
\[ \mathcal{R}_1(h_{XY}) = \{(R_X, R_Y, R_{kX}, R_{kY}) : R_X \geq H(X|Y) + \frac{1}{K} \log M_{CX}, R_Y \geq H(Y), R_{kX} \geq h_X \text{ and } R_{kY} \geq h_Y\} \] \hspace{1cm} (81)

**Theorem 3:** For \( 0 \leq h_X \leq H(X) \),
\[ \mathcal{R}_2(h_X) = \{(R_X, R_Y, R_{kX}, R_{kY}) : R_X \geq H(X), R_Y \geq \frac{1}{K} \log M_{CY}, R_{kX} \geq h_X \text{ and } R_{kY} \geq 0\} \] \hspace{1cm} (82)

**Theorem 4:** For \( 0 \leq h_X \leq H(X) \) and \( 0 \leq h_Y \leq H(Y) \),
\[ \mathcal{R}_3(h_X, h_Y) = \{(R_X, R_Y, R_{kX}, R_{kY}) : R_X \geq H(X,Y), R_Y \geq H(X,Y), R_{kX} \geq h_X \text{ and } R_{kY} \geq h_Y\} \] \hspace{1cm} (83)

When \( h_X = 0 \) then Case 4 is reduced to Case 3 and when \( h_Y = 0 \) then Case 5 is reduced to Case 3. Hence, Corollary 1 and 2 follow:

**Corollary 1:** \( \mathcal{R}_4(h_Y) = \mathcal{R}_3(0, h_Y) \)

**Corollary 2:** \( \mathcal{R}_5(h_X) = \mathcal{R}_3(h_X, 0) \)

The security levels, which are measured by the total and individual uncertainties \( h_{XY} \) and \( (h_X, h_Y) \) respectively give an indication of the level of uncertainty in knowing certain information. When the uncertainty increases then less information is known to an eavesdropper and there is a higher level of security.

**V. PROOF OF THEOREMS 2 - 4**

We construct a code based on the prototype code \( (W_X, W_Y, W_{CX}, W_{CY}) \) in Lemma 1. In order to include a key in the prototype code, \( W_X \) is divided into two parts as per the method used by Yamamoto [1]:

\[ W_{X1} = W_X \mod M_{X1} \in I_{M_{X1}} = \{0, 1, 2, \ldots, M_{X1} - 1\} \] \hspace{1cm} (84)

\[ W_{X2} = \frac{W_X - W_{X1}}{M_{X1}} \in I_{M_{X2}} = \{0, 1, 2, \ldots, M_{X2} - 1\} \] \hspace{1cm} (85)

where \( M_{X1} \) is a given integer and \( M_{X2} \) is the ceiling of \( M_{X}/M_{X1} \). The \( M_{X}/M_{X1} \) is considered an integer for simplicity, because the difference between the ceiling value and the actual value can be ignored when \( K \) is sufficiently large. In the same way, \( W_Y \) is divided:

\[ W_{Y1} = W_Y \mod M_{Y1} \in I_{M_{Y1}} = \{0, 1, 2, \ldots, M_{Y1} - 1\} \] \hspace{1cm} (86)

\[ W_{Y2} = \frac{W_Y - W_{Y1}}{M_{Y1}} \in I_{M_{Y2}} = \{0, 1, 2, \ldots, M_{Y2} - 1\} \] \hspace{1cm} (87)

The common information components \( W_{CX} \) and \( W_{CY} \) are already portions and are not divided further. It can be shown that when some of the codewords are wiretapped the uncertainties of \( X^K \) and \( Y^K \) are bounded as follows:

\[ \frac{1}{K} H(X^K|W_{X2}, W_Y) \geq I(X;Y) + \frac{1}{K} \log M_{CX} - \epsilon_0 \] \hspace{1cm} (88)

\[ \frac{1}{K} H(Y^K|W_X, W_{Y2}) \geq I(X;Y) + \frac{1}{K} \log M_{CY} - \epsilon_0 \] \hspace{1cm} (89)

\[ \frac{1}{K} H(X^K|W_X, W_{Y2}) \geq I(X;Y) - \epsilon_0 \] \hspace{1cm} (90)

\[ \frac{1}{K} H(X^K|W_X, W_{CY}) \geq \frac{1}{K} \log M_{CX} - \epsilon_0 \] \hspace{1cm} (91)
\[
\frac{1}{K} H(Y^K | W_X, W_Y, W_{CY}) \geq \frac{1}{K} \log M_{CX} - \epsilon_0
\]  
(92)

\[
\frac{1}{K} H(X^K | W_Y, W_{CY}) \geq H(X^K | Y^K) + \frac{1}{K} \log M_{CX} - \epsilon_0
\]  
(93)

\[
\frac{1}{K} H(Y^K | W_Y, W_{CY}) \geq H(Y^K | X^K) + \frac{1}{K} \log M_{CX} - \epsilon_0
\]  
(94)

where \( \epsilon_0 \to 0 \) as \( \epsilon_0 \to 0 \). The proofs for (88) - (94) are the same as per Yamamoto’s Lemma A1. The difference is that \( W_{CX}, W_{CY}, M_{CX} \) and \( M_{CY} \) are described as \( W_{C1}, W_{C2}, M_{C1} \) and \( M_{C2} \) respectively by Yamamoto. Here, we consider that \( W_{CX} \) and \( W_{CY} \) are represented by Yamamoto’s \( W_{C1} \) and \( W_{C2} \) respectively. In addition there are some more inequalities considered here:

\[
\frac{1}{K} H(Y^K | W_X, W_{CX}, W_{CY}, W_{Y2}) \geq \frac{1}{K} \log M_{Y1}
- \epsilon_0
\]  
(95)

\[
\frac{1}{K} H(Y^K | W_X, W_{CX}, W_{CY}) \geq \frac{1}{K} \log M_{Y1}
+ \frac{1}{K} \log M_{Y2} - \epsilon_0
\]  
(96)

\[
\frac{1}{K} H(X^K | W_{X2}, W_{CY}) \geq \frac{1}{K} \log M_{X1}
+ \frac{1}{K} \log M_{CX} - \epsilon_0
\]  
(97)

\[
\frac{1}{K} H(Y^K | W_{X2}, W_{CY}) \geq \frac{1}{K} \log M_{Y1}
+ \frac{1}{K} \log M_{Y2} + \frac{1}{K} \log M_{CX}
- \epsilon_0
\]  
(98)

The inequalities (95) and (96) can be proved in the same way as per Yamamoto’s Lemma A2, and (97) and (98) can be proved in the same way as per Yamamoto’s Lemma A1.

**Proof of Theorem 2:** Suppose that \((R_X, R_Y, R_{KX}, R_{KY}) \in R_1 \) for \( h_{XY} \leq H(X,Y) \). Without loss of generality, we assume that \( h_X \leq h_Y \). Then, from (81)

\[
R_X \geq H(X^K | Y^K) + \frac{1}{K} \log M_{CX},
\]  
(99)

\[
R_Y \geq H(Y^K)
\]

\[
R_{KX} \geq h_X, R_{KY} \geq h_Y
\]  
(100)

The case when \( h_X > I(X;Y) \) is first considered. For this case, \( M_{X1}, M_{Y1}, M_{CX} \) and \( M_{CY} \) is defined as:

\[
M_{X1} = 2^{K h(X;Y)}
\]  
(101)

\[
M_{Y1} = 2^{K(h_Y - I(X;Y))}
\]  
(102)
1 \log M_{kX} = 1 \log (\log M_{X1} + \log M_{CX}) \\
= \log (X|Y) + h_X - H(X|Y) \\
= h_X \\
\leq R_{kX} \quad (114)

where (113) comes from (101) and (103).

\frac{1}{K} \log M_{kY} = \frac{1}{K} (\log M_{CY} + \log M_{Y1}) \\
= I(X; Y) + h_Y - I(X; Y) \\
= h_Y \\
\leq R_{kY} \quad (116)

where (115) comes from (101) and (104).

It can also be seen from (105) and (106) that

\frac{1}{K} H(X^K|W_X, W_Y) = \frac{1}{K} H(X|W_{X1} \oplus W_{kX}, W_{X2}, W_{CX} \oplus W_{kCX}, W_{Y1} \oplus W_{kY1}, W_{Y2}, W_{CY} \oplus W_{kCY}) \\
= \frac{1}{K} H(X^K|W_{X2}, W_{Y2}) \\
\geq I(X; Y) + \frac{1}{K} \log M_{X1} - \epsilon_0' \\
= I(X; Y) + H(X|Y) - \epsilon_0' \\
= \frac{1}{K} H(X) - \epsilon_0' \\
\geq h_X - \epsilon_0' \quad (118)

where (117) holds because \( W_{X1} \) and \( W_{CX} \) are covered by the keys \( W_{kX1} \) and \( W_{kCX} \). Equations (10) - (16) imply that \( W_{X1}, W_{X2}, W_{Y1} \) and \( W_{Y2} \) have almost no redundancy and they are mutually independent.

Similarly,

\frac{1}{K} H(Y^K|W_X, W_Y) \geq h_Y - \epsilon_0' \quad (119)

Therefore \( (R_X, R_Y, R_{kX}, R_{kY}, h_X, h_Y) \) is admissible from (110) - (116), (118) and (119).

Next, the case when \( h_X \leq I(X; Y) \) is considered. If \( h_Y > I(X; Y) \) then we define \( M_{CX}, M_{CY}, M_{X1}, M_{X2}, M_Y, W_X, W_Y, W_{kX} \) and \( W_{kY} \) as follows:

\[ M_{CX} = 2^{Kh_X} \quad (120) \]

\[ M_{CY} = 2^{K(h_Y - H(Y|X))} \quad (121) \]

\[ M_{Y1} = 2^{KH(Y|X)} \quad (122) \]

\[ M_{Y2} = 2^{K(H(Y) - h_X)} \quad (123) \]

\[ W_X = (W_{X1}, W_{X2}, W_{CX} \oplus W_{kCX}) \quad (124) \]

\[ W_Y = (W_{CY} \oplus W_{kCY}, W_{Y1} \oplus W_{kY1}) \quad (125) \]

\[ W_{kX} = W_{kCX} \quad (126) \]

\[ W_{kY} = (W_{kCY}, W_{kY1}) \quad (127) \]

where \( \emptyset \) represents an empty sequence. In this case we have that

\[ \frac{1}{K} \log M_{kX} = \frac{1}{K} \log M_{CX} \\
= h_X \\
\leq R_{kX} \quad (128) \]

\[ \frac{1}{K} \log M_{kY} = \frac{1}{K} \log M_{CY} + \frac{1}{K} \log M_{Y1} \\
= h_Y - H(Y|X) + H(Y|X) \\
= h_Y \\
\leq R_{kY} \quad (129) \]

Here, \( R_{kY} \geq 0 \) shows that no enciphering on the \( Y \) channel is necessary. We also assume that \( H(X) = h_X \) and \( H(Y) = h_Y \).

The equivocations of the wiretapper are given by:

\[ \frac{1}{K} H(X^K|W_X, W_Y) \]

\[ = \frac{1}{K} H(X^K|W_{X1}, W_{X2}, W_{CY}, W_{Y1}) \]

\[ \geq \frac{1}{K} \log M_{CX} - \epsilon_0' \]

\[ = h_X - \epsilon_0' \quad (130) \]

where (130) results from (92).

\[ \frac{1}{K} H(Y^K|W_X, W_Y) \]

\[ = \frac{1}{K} H(Y^K|W_{X1}, W_{X2}, W_{CY}, W_{Y1}) \]

\[ \geq \frac{1}{K} \log M_{CX} + \frac{1}{K} \log M_{Y2} - \epsilon_0' \]

\[ = h_X + H(Y) - h_X - \epsilon_0' \]

\[ = h_Y - \epsilon_0' \quad (131) \]

If \( h_Y \leq I(X; Y) \), then we define \( W_X, W_Y \) and \( W_{kY1} \) as follows:

\[ M_{CX} = 2^{Kh_X} \quad (133) \]

\[ M_{CY} = 2^{K(h_Y - H(Y|X))} \quad (121) \]

\[ M_{Y1} = 2^{KH(Y|X)} \quad (122) \]

\[ M_{Y2} = 2^{K(H(Y) - h_X)} \quad (123) \]

\[ W_X = (W_{X1}, W_{X2}, W_{CX} \oplus W_{kCX}) \quad (124) \]

\[ W_X = (W_{X1}, W_{X2}, W_{CX} \oplus W_{kCX}) \quad (135) \]
The keys and uncertainties are bounded by:

\[
\frac{1}{K} \log M_{kX} = \frac{1}{K} \log M_{CX} = h_X \leq R_{kX}
\]

\[
\frac{1}{K} \log M_{kY} = \frac{1}{K} \log M_{Y1} = h_Y \leq R_{kY} + \epsilon_0
\]

From (128) - (132) and (139) - (143) (where (142) results from (95)).

where (142) results from (95).

From (128) - (132) and (139) - (143) \((R_X, R_Y, R_{kX}, R_{kY}, h_X, h_Y)\) is admissible in the case where \(\min (h_X, h_Y) \leq I(X;Y)\).

Theorem 3 and 4 are proven in the same way with varying codewords and keys. The proofs follow:

**Theorem 3 proof:** Here, \(R_X \geq H(X), R_Y \geq \frac{1}{K} \log M_{CY}, R_{kX} \geq h_X\) and \(R_{kY} = 0\). For the case where \(h_X > I(X;Y)\), the definitions for \(M_{CX}, M_{CY}, M_{X1}, M_{Y1}, M_{Y2}, W_X, W_{kX}\) follow:

\[
M_{CX} = 2^{K I(X;Y)}
\]

\[
M_{CY} = 2^{K h_Y}
\]

\[
M_{X2} = 2^{K (h_X - I(X;Y))}
\]

\[
M_{Y1} = 2^{K (h_Y - H(Y|X))}
\]

\[
M_{Y2} = 2^{K H(Y|X)}
\]

\[
W_X = (W_{X1}, W_{X2} \oplus W_{kX2}, W_{CX} \oplus W_{kCX})
\]

\[
W_Y = W_{CY}
\]

\[
W_{kX} = (W_{kX2}, W_{kCX})
\]

\[
W_{kY} = \emptyset
\]

The keys and uncertainties are calculated as follows:

\[
\frac{1}{K} \log M_X = \frac{1}{K} (\log M_{X1} + \log M_{X2} + \log M_{CX})
\]

\[
\leq H(X|Y) + \frac{1}{K} M_{CX} + \epsilon_0
\]

\[
= H(X|Y) + I(X;Y) + \epsilon_0
\]

\[
= H(X) + \epsilon_0
\]

\[
\leq R_X + \epsilon_0
\]

\[
\frac{1}{K} \log M_Y
\]

\[
= \frac{1}{K} \log M_{CY}
\]

\[
= h_Y
\]

\[
\leq R_Y + \epsilon_0
\]

where (156) is as a result of (144) and (146).
\[
\frac{1}{K} H(Y^K | W_X, W_Y) \\
= \frac{1}{K} H(Y^K | W_{X1}, W_{X2}, W_{CX}, W_{CY}) \\
\geq \frac{1}{K} \log W_{Y1} + \frac{1}{K} \log W_{Y2} - \epsilon'_0
\] (160) \\
= h_Y - H(Y|X) + H(Y|X) - \epsilon'_0
\] (161)

where (160) results from (96). From (154) - (161), (R_X, R_Y, R_{kX}, R_{kY}, h_X) is admissible for h_X > I(X;Y). We now consider the case where h_X \leq I(X;Y), and define M_{CX} and M_{X1}, W_X, W_Y, W_{kX}, W_{kY} as follows:

\[ M_{CX} = 2^{K I(X;Y)} \] (162) \\
\[ M_{X1} = 2^{K (h_X - I(X;Y))} \] (163)

\[ W_X = (W_{X1} \oplus W_{kX1}, W_{X2}, W_{CX} \oplus W_{kCX}) \] (164) \\
\[ W_Y = W_{CY} \] (165) \\
\[ W_{kX} = (W_{kX1}, W_{kCX}) \] (166) \\
\[ W_{kY} = \emptyset \] (167)

The keys and uncertainties are calculated as follows:

\[ \frac{1}{K} \log M_{kX} = 0 \]
\[ \leq R_{kX} \] (168) \\
where (168) is as a result of (162) and (163).

\[ \frac{1}{K} \log M_{kY} = 0 \]
\[ \leq R_{kY} \] (170)

\[ \frac{1}{K} H(X^K | W_X, W_Y) \\
= \frac{1}{K} H(X^K | W_{X2}, W_{CY}) \\
\geq \frac{1}{K} \log M_{X1} + \frac{1}{K} \log M_{CX} - \epsilon'_0
\] (171) \\
= h_X - I(X;Y) + I(X;Y) - \epsilon'_0
\] (172) \\
= h_X - \epsilon'_0
\] (173)

where (171) results from (97) and (172) results from (162) and (163).
\[
\frac{1}{K} \log M_{kX} = \frac{1}{K} (\log M_{X1}) \\
= h_{XY} \tag{187}
\]
\[
\leq R_{kX} \tag{188}
\]

where (187) results from (177).

\[
\frac{1}{K} \log M_{kY} = \frac{1}{K} (\log M_{CY}) \\
= H(X,Y) \\
= h_{XY} \tag{189}
\]
\[
\leq R_{kY} \tag{188}
\]

\[
\frac{1}{K} H(X^K|W_X,W_Y) \\
= \frac{1}{K} H(X^K|W_{X2},W_{CX},W_{CY}) \\
\geq \frac{1}{K} \log M_{X1} - \epsilon'_0 \\
= h_{XY} - \epsilon'_0 \tag{190}
\]
\[
\frac{1}{K} H(Y^K|W_X,W_Y) \\
= \frac{1}{K} H(Y^K|W_{X2},W_{CX},W_{CY}) \\
\geq H(Y|X) - \epsilon'_0 \\
\geq h_{XY} - \epsilon'_0 \tag{191}
\]

The (185) - (191) show that \((R_X, R_Y, R_{kX}, R_{kY}, h_{XY})\) are admissible for \(h_{XY} \leq H(Y|X)\). Now, the case where \(h_{XY} > H(Y|X)\) and \(h_{XY} \geq H(Y)\) is considered. The following are definitions for \(M_{X1}, M_{X2}, W_X, W_Y, W_{kX}, W_{kY}\):

\[
M_{X1} = 2^{K(h_{XY} - H(X|Y))} \tag{192}
\]
\[
M_{X2} = 2^{K H(X|Y)} \tag{193}
\]
\[
M_{CY} = 2^{K H(X,Y)} \tag{194}
\]
\[
W_X = (W_{X1} \oplus W_{kX1}, W_{X2} \oplus W_{kX2}, W_{CX}) \tag{195}
\]
\[
W_Y = W_{CY} \tag{196}
\]
\[
W_{kX} = (W_{kX1}, W_{kX2}) \tag{197}
\]
\[
W_{kY} = W_{KCY} \tag{198}
\]

The keys and uncertainties are as follows:

\[
\frac{1}{K} \log M_{kX} \\
= \frac{1}{K} (\log M_{X1} + \log M_{X2}) \\
= h_{XY} - H(X|Y) + H(X|Y) \tag{199}
\]
\[
= h_{XY} \leq R_{kX} \tag{200}
\]

(199) results from (192) and (193).

\[
\frac{1}{K} \log M_{kY} \\
= \frac{1}{K} (\log M_{CY}) \\
= H(X,Y) \\
= h_{XY} \tag{201}
\]
\[
\leq R_{kY} \tag{200}
\]

(199) results from (192) and (193).

\[
\frac{1}{K} H(X^K|W_X,W_Y) \\
= \frac{1}{K} H(X^K|W_{X2},W_{CX},W_{CY}) \\
\geq \frac{1}{K} \log M_{X1} + \frac{1}{K} \log M_{X2} - \epsilon'_0 \\
= h_{XY} - H(X|Y) + H(X|Y) - \epsilon'_0 \tag{202}
\]
\[
= h_{XY} - \epsilon'_0 \tag{203}
\]

(202) results from (192) and (193).

\[
\frac{1}{K} H(Y^K|W_X,W_Y) \\
= \frac{1}{K} H(Y^K|W_{X2},W_{CX},W_{CY}) \\
\geq H(Y|X) - \epsilon'_0 \\
< h_{XY} - \epsilon'_0 \tag{204}
\]

The (200) - (204) show that \((R_X, R_Y, R_{kX}, R_{kY}, h_{XY})\) are admissible for \(h_{XY} > H(Y|X)\) and \(h_{XY} \geq H(Y)\). The last case is to consider when \(h_{XY} > H(Y|X)\) and \(h_{XY} < H(Y)\). The definitions for \(M_{X1}, M_{X2}, W_X, W_Y, W_{kX}, W_{kY}\) for this case are:

\[
M_{X1} = 2^{K(h_{XY} - I(X:Y))} \tag{205}
\]
\[
M_{CX} = 2^{K h_{XY}} \tag{206}
\]
\[
M_{CY} = 2^{K H(X,Y)} \tag{207}
\]
\[
W_X = (W_{X1} \oplus W_{kX1}, W_{X2} \oplus W_{kX2}, W_{CX} \oplus W_{kCX}) \tag{208}
\]
\[
W_{kX} = (W_{kX1}, W_{kCX}) \tag{209}
\]

\[
W_{kY} = W_{KCY} \tag{198}
\]
\[ W_{kY} = W_{kCY} \] (210)

There is no \( Y \) codeword. The keys and uncertainties are as follows:

\[
\frac{1}{K} \log M_{kX} = \frac{1}{K} (\log M_{XY} + \log M_{CY}) \\
= H(X,Y) \\
\leq h_{XY} - I(X;Y) + h_{XY} + \epsilon_0 \\
= 2h_{XY} + \epsilon_0
\] (211) (212)

where (211) is a result of (205) and (207). This result is larger than the combined security level \( h_{XY} \), which is less than the rate of \( X \)'s key channel, \( R_{kX} \).

\[
\frac{1}{K} \log M_{kX} = \frac{1}{K} (\log M_{XY}) \\
= H(X) \\
\leq h_{XY} + \epsilon_0
\] (213)

\[
\frac{1}{K} H(Y|X,W) = \frac{1}{K} H(Y|X,W_1) \\
\geq \frac{1}{K} \log M_{XY} + I(X;Y) - \epsilon'_0 \\
= h_{XY} - I(X;Y) + I(X;Y) - \epsilon'_0 \\
= h_{XY} - \epsilon'_0
\] (214)

\[
\frac{1}{K} H(Y|X,W) = \frac{1}{K} H(Y|X,W_2) \\
\geq H(Y) - \epsilon'_0 \\
> h_{XY} - \epsilon'_0
\] (215)

Equations (212) - (215) show that \( R_{X}, R_{Y}, R_{kX}, R_{kY}, h_{XY} \) are admissible for \( h_{XY} > H(Y|X) \) and \( h_{XY} < H(Y) \).

### VI. SCHEME FOR MULTIPLE SOURCES

The two correlated source model presented in Section II is generalised even further, and now concentrates on multiple correlated sources transmitting syndromes across multiple wiretapped links. This new approach represents a network scenario where there are many sources and one receiver. We consider the information leakage for this model for Slepian-Wolf coding and thereafter consider the Shannon’s cipher system representation.

\[ \text{Figure 5. Extended generalised model} \]

#### A. Information leakage using Slepian-Wolf coding

Here, Figure 5 gives a pictorial view of the new extended model for multiple correlated sources.

Consider a situation where there are many sources, which are part of the \( S \) set:

\[ S = \{S_1, S_2, \ldots, S_n\} \]

where \( i \) represents the \( i \)th source \( (i = 1, \ldots, n) \) and there are \( n \) sources in total. Each source may have some correlation between some other source and all sources are part of a binary alphabet. There is one receiver that is responsible for performing decoding. The syndrome for a source \( S_i \) is represented by \( T_{S_i} \), which is part of the same alphabet as the sources.

The entropy of a source is given by a combination of a specific conditional entropy and mutual information. In order to present the entropy we first define the following sets:

- The set, \( S \) that contains all sources: \( S = \{S_1, S_2, \ldots, S_n\} \).
- The set, \( S_t \) that contains \( t \) unique elements from \( S \) and \( S_t \subseteq S, S_t \in S, S_t \cup S'_t = S \) and \( |S_t| = t \)

Here, \( H(S_i) \) is obtained as follows:

\[
H(S_i) = H(S_i|S_i^c) + \sum_{t=2}^{n} (-1)^{t-1} \sum_{\text{all possible } S_t} I(S_t|S_t^c) \] (216)

Here, \( n \) is the number of sources, \( H(S_i|S_i^c) \) denotes the conditional entropy of the source \( S_i \) given \( S_i^c \) subtracted from the set \( S \) and \( I(S_t|S_t^c) \) denotes the mutual information between all sources in the subset \( S_t \) given the complement of \( S_t \). In the same way as for two sources, the generalised probabilities and entropies can be developed. It is then possible to decode the source message for source \( S_i \) by receiving all components related to \( S_i \). This gives rise to the following inequality for \( H(S_i) \) in terms of the sources:

\[
H(S_i|S_i^c) + \sum_{t=2}^{n} (-1)^{t-1} \sum_{\text{all possible } S_t} I(S_t|S_t^c) \leq H(S_i) + \delta \] (217)

In this type of model information from multiple links need to be gathered in order to determine the transmitted
information for one source. Here, the common information between sources is represented by the $I(S_i|S_j)$ term. The portions of common information sent by each source can be determined upfront and is an arbitrary allocation in our case. For example in a three source model where $X$, $Y$ and $Z$ are the correlated sources, the common information shared with $X$ and the other sources is represented as: $I(X; Y|Z)$ and $I(X; Z|Y)$. Each common information portion is divided such that the sources having access to it are able to produce a portion of it themselves. The common information $I(X; Y|Z)$ is divided into $V_{CX1}$ and $V_{CY1}$ where the former is the common information between $X$ and $Y$, produced by $X$ and the latter is the common information between $X$ and $Y$, produced by $Y$. Similarly, $I(X; Z|Y)$ consists of two common information portions, $V_{CX2}$ and $V_{CZ1}$ produced by $X$ and $Z$ respectively.

As with the previous model for two correlated sources, since wiretapping is possible there is a need to develop the information leakage for the model. The information leakages for this multiple source model is indicated in (218) and (219).

Remark 1: The leaked information for a source $S_i$ given the transmitted codewords $T_{S_i}$, is given by:

$$L_{T_{S_i}}^{S_i} = I(S_i; T_{S_i})$$  \hspace{1cm} (218)

Since we use the notion that the information leakage is the conditional entropy of the source given the transmitted information subtracted from the source’s uncertainty (i.e $H(S_i) - H(S_i|T_{S_i})$), the proof for (218) is trivial. Here, we note that the common information is the minimum amount of information leaked. Each source is responsible for transmitting its own private information and there is a possibility that this private information may also be leaked. The maximum leakage for this case is thus the uncertainty of the source itself, $H(S_i)$.

We also consider the information leakage for a source $S_i$ when another source $S_{j \neq i}$ has transmitted information. This gives rise to Remark 2.

Remark 2: The leaked information for a source $S_i$ given the transmitted codewords $T_{S_j}$, where $i \neq j$ is:

$$L_{T_{S_j}}^{S_i} = H(S_i) - H(S_i|T_{S_j})$$
$$= H(S_i) - (H(S_i) - I(S_i; T_{S_j}))$$
$$= I(S_i; T_{S_j})$$  \hspace{1cm} (219)

The information leakage for a source is determined based on the information transmitted from any other channel using the common information between them. The private information is not considered as it is transmitted by each source itself and can therefore not be obtained from an alternate channel. Remark 2 therefore gives an indication of the maximum amount of information leaked for source $S_i$, with knowledge of the syndrome $T_{S_j}$.

These remarks show that the common information can be used to quantify the leaked information. The common information provides information for more than one source and is therefore susceptible to leaking information about more than one source should it be compromised. This subsection gives an indication of the information leakage for the new generalised multiple correlated sources model when a source’s syndrome and other syndromes are wiretapped.

B. Information leakage for Shannon’s cipher system

This subsection details a novel masking method to minimize the key length and thereafter builds this multiple correlated source model on Shannon’s cipher system.

The new masking method encompasses masking the conditional entropy portion with a mutual information portion. By masking, certain information is hidden and it becomes more difficult to obtain the information that has been masked. Masking can typically be done using random numbers, however we eliminate the need for random numbers that represent keys and rather use a common information to mask with.

We make the following assumptions:

- The capacity of each link cannot be exhausted using this method.
- A common information is used to mask certain private information and can be used to mask multiple times. Further, private information that needs to be masked always exists in this method.

The allocation of common information for transmission are done on an arbitrary basis. The objective of this subsection is to minimize the key lengths while achieving perfect secrecy.

The private information for source $i$ is given by $H(S_i)$ according to (218), which is called $W_{S_i}$ and the common information associated with source $S_i$ is given by $W_{CS_i}$. First, choose a common information with which to mask. Then we take a part of $W_{S_i}$, i.e. $W_{S_i'}$, that has entropy equal to $H(W_{CS_i})$, and mask as follows:

$$W'_{S_i} \oplus W_{CS_i}$$ \hspace{1cm} (220)

When the two sequences are xor’ed the result is a single sequence that may look different to the originals. We then transmit the masked portion instead of the $W_{S_i}$ portion when transmitting $W_{S_i}$, thus providing added security. This brings in the interesting factor that there are many possibilities for a specific mutual information to mask conditional entropy portions. For example when considering three sources as before, it is possible to mask the private information for $X$, $Y$ and $Z$ with the common portion $I(X; Y; Z)$. If $Y$ is secure then this common information can be transmitted along $Y$’s channel, ensuring the information is kept secure. The ability to mask using the common information is a unique and interesting feature of this new model for multiple correlated sources. The underlying principle is that the secure link should transmit more common information after transmitting their private information.

We find that the lower bound for the channel rate when the masking approach is used is given by:

$$R_{i}^{M} \geq H(S_1, \ldots, S_n) - \sum_{t=2}^{n} \sum_{S_i \in \text{all possible } S_i} (t-1)I(S_i|S'_i)$$  \hspace{1cm} (221)

where $R_{i}^{M}$ is the $i$th channel rate when masking is used.

The method works theoretically but may result in some concern practically as there may be a security compromise
where common information is sent across non secure links. We see that if the $W_{CS}$ component used for masking has been compromised then the private portion it has masked will also be compromised. A method to overcome this involves using two common information parts for masking. Equation \(220\) representing the masking would become:

$$W_{S_i} \oplus W_{CS_i} \oplus W_{CS_j} \quad (222)$$

where $i \neq j$ and both $W_{CS_i}$ and $W_{CS_j}$ are common information associated with source $S_i$. This way, if only $W_{CS_j}$ is compromised then $W_{S_i}$ is not compromised as it is still protected by $W_{CS_i}$. Here, combinations of common information are used to increase the security. The advantage with (222) is that keys may be reused because common information may be shared by more than one source. Further, the method will not result in an increase in key length.

The Shannon’s cipher system for this multiple source model is now presented in order to determine the rate regions for perfect secrecy. The multiple sources each have their own encoder and there is a universal decoder. Each source has an encoder represented by:

$$E_i : S \times I_{W_{S_i}} \rightarrow I_{W_{CS_i}} = \{0, 1, \ldots, W_{S_i} - 1\}$$ $$I_{W_{CS_i}} = \{0, 1, \ldots, W_{CS_i} - 1\} \quad (223)$$

where $I_{MP_i}$ is the alphabet representing the private portion for source $S_i$ and $I_{MC_i}$ is the alphabet representing the common information for source $S_i$. The decoder at the receiver is defined as:

$$D : (I_{W_{CS_i}}, I_{W_{CS_j}}) \times I_{MK} \rightarrow S \quad (224)$$

The encoder and decoder mappings are below:

$$W_i = F_{E_i}(S_i, W_{ki}) \quad (225)$$

$$\hat{S}_i = F_{D_i}(W_i, W_{ki, W_{(kp)}}) \quad (226)$$

where $p = 1, \ldots, n, p \neq i$ and $W_{(kp)}$ represents the set of common information required to find $S_i$, and $\hat{S}_i$ is the decoded output.

The following conditions should be satisfied for the general cases:

\[ \frac{1}{K} \log W_{S_i} \leq R_i + \epsilon \quad (227) \]

\[ \frac{1}{K} \log M_{ki} \leq R_{ki} + \epsilon \quad (228) \]

\[ \Pr(\hat{S}_i \neq S_i) \leq \epsilon \quad (229) \]

\[ \frac{1}{K} H(S_i | W_i) \leq h_i - \epsilon \quad (230) \]

\[ \frac{1}{K} H(S_j | W_i) \leq h_j - \epsilon \quad (231) \]

where $R_i$ is the the rate of source $S_i$’s channel and $R_{ki}$ is the key rate of $S_i$. The security levels, for source $i$ and any other source $j$ are measured uncertainties $h_i$ and $h_j$ respectively.

The general cases considered are:

**Case 1:** When $T_{S_i}$ is leaked and $S_i$ needs to be kept secret.

**Case 2:** When $T_{S_i}$ is leaked and $S_i$ and/or $S_j$ needs to be kept secret.

The admissible rate region for each case is defined as follows:

**Definition 1a:** ($R_i, R_{ki}, h_i$) is admissible for case 1 if there exists a code $(F_{E_i}, F_{D_i})$ such that (227), (228), (229) hold for any $\epsilon \rightarrow 0$ and sufficiently large $K$.

**Definition 1b:** ($R_i, R_{ki}, R_{kj}, h_i, h_j$) is admissible for case 2 if there exists a code $(F_{E_i}, F_{D_i})$ such that (227), (228), (229) and (230) hold for any $\epsilon \rightarrow 0$ and sufficiently large $K$.

**Definition 2:** The admissible rate regions are defined as:

$$\mathcal{R}(h_i) = \{(R_i, R_{ki}, R_{kj}) : (R_i, R_{ki}, R_{kj}, h_i) \text{ is admissible for case 1}\} \quad (232)$$

$$\mathcal{R}(h_i, h_j) = \{(R_i, R_{ki}, R_{kj}, R_{kj}) : (R_i, R_{ki}, R_{kj}, h_i, h_j) \text{ is admissible for case 2}\} \quad (233)$$

The theorems developed for these regions follow:

**Theorem 6:** For $0 \leq h_i \leq I(S_i; S_r)_{S_i}$, $R_i \geq H(S_i)$, $R_{ki} \geq I(S_i; S_j)$

$$\mathcal{R}_1(h_i) = \{(R_i, R_{ki}) : R_i \geq H(S_i), R_{ki} \geq I(S_i; S_j)\} \quad (234)$$

**Theorem 7:** For $0 \leq h_i \leq H(S_i, S_j)$, $R_{ki} \geq I(S_i; S_j)$ and $R_{kj} \geq I(S_i; S_j)\}$

$$\mathcal{R}_2(h_i, h_j) = \{(R_i, R_{ki}, R_{kj}, h_j) : R_{ki} \geq I(S_i; S_j), R_{kj} \geq I(S_i; S_j)\} \quad (235)$$

The proofs for these theorems follow. The source information components are first identified. Assume the private portions of source $i$ and $j$ are given by $W_i$ and $W_j$ respectively.

**Theorem 6 proof:** Here, $R_i \geq H(S_i), R_{ki} \geq I(S_i; S_j)$. For the case where $h_i > I(S_i; S_j)$, the definitions for $W_{CS_i}$, $W_i$ and $W_{ki}$ follow:

$$W_{CS_i} = 2^{K \log I(S_i; S_j)} \quad (236)$$

$$W_i = (W_{Pi}, W_{ki, Ci}) \quad (237)$$

$$W_{ki} = W_{Ci} \quad (238)$$

The keys and uncertainties are calculated as follows:
The keys and uncertainties are calculated as follows:

\[ \frac{1}{K} \log M_i = \frac{1}{K} (\log W_{s_i} + \log W_{CS_i}) \]

From (239) - (242), \((R_i, R_{k_i}, h_i)\) is admissible for \(h_i > I(S_i; S'_i)\). We now consider the case where \(h_i \leq I(S_i; S'_i)\) and define \(W_{CS_i}, W_i\) and \(W_{k_i}\) as follows:

\[ W_{CS_i} = 2^{K_1(S_i; S'_i)} \]

\[ W_i = (W_{P_i}, W_{kC_i}) \]

\[ W_{k_i} = W_{C_i} \]

The keys and uncertainties are calculated as follows:

\[ \frac{1}{K} \log M_{k_i} = \frac{1}{K} (\log W_{s_i} + \log W_{CS_i}) \]

From (243) - (248), it is seen that \((R_i, R_{k_i}, h_i)\) is admissible for \(h_i \leq I(S_i; S'_i)\).

Theorem 7 is proven in a similar manner.

**Theorem 7 proof:** Here, \(R_i \geq H(S_i, S_j), R_j \geq H(S_i, S_j), R_{k_i} \geq I(S_i; S'_i)\) and \(R_{k_j} \geq I(S_i; S'_i)\). For the case where \(h_j \leq H(S_i; S_j)\), the definitions for \(W_{CS_i}, M_{C_j}\) \(W_i, W_{k_i}, W_j\) and \(W_{k_j}\) follow:

\[ W_{CS_i} = 2^{K_1(S_i; S'_i)} \]

\[ M_{C_j} = 2^{K_1(S_i; S'_i)} \]

\[ W_i = (W_{P_i}, W_{kC_i}) \]

\[ W_{k_i} = W_{C_i} \]

\[ W_{k_j} = W_{C_j} \]

The keys and uncertainties are calculated as follows:

\[ \frac{1}{K} \log M_i = \frac{1}{K} (\log W_{s_i} + \log W_{CS_i}) \]

From (249) - (262), it is seen that \((R_i, R_{k_i}, h_i)\) is admissible for \(h_i \leq I(S_i; S'_i)\).
The keys and uncertainties are calculated as follows:

\[ W_{C_i} = 2^{K I(S_i | S'_i)} \]  
(265)

\[ M_{C_j} = 2^{K I(S_i | S'_i)} \]  
(266)

\[ W_i = (W_{P_i}, W_{k_i C_i}) \]  
(267)

\[ W_{k_i} = W_{C_i} \]  
(268)

\[ W_j = (W_{P_j}, W_{k_j C_j}) \]  
(269)

\[ W_{k_j} = W_{C_j} \]  
(270)

The keys and uncertainties are calculated as follows:

\[ \frac{1}{K} \log M_{k_i} \]
\[ = \frac{1}{K} \log W_{C_i} \]
\[ = I(S_i | S'_i) \]
\[ \leq R_{k_i} + \epsilon_0 \]  
(271)

\[ \frac{1}{K} \log M_{k_j} \]
\[ \leq I(S_i; S_j) + \epsilon_0 \]
\[ \leq R_{k_j} + \epsilon_0 \]  
(272)

\[ \frac{1}{K} H(S_j | W_{P_i}, W_{C_i}) \]
\[ \leq H(S_j) - H(S_i) + \epsilon'_0 \]
\[ = H(S_i, S_j) - H(S_i) + \epsilon'_0 \]
\[ \leq h_j - H(S_i) \]  
(273)

From (272) - (245) it is seen that \((R_i, R_{ki}, R_j, R_{kj}, h_j)\) is admissible for \(h_j \leq H(S_i, S_j)\).

These theorems demonstrate the necessary rates required for perfect secrecy. The goal of the Shannon’s cipher aspect was to reduce the key lengths. The masking method explained in this section is able to use common information as keys and therefore minimise the key rates for the general cases.

The information leakage described in the Slepian-Wolf aspect indicates the common information that should be given added protection to ensure that even less information will be leaked. The new extended model presented here also incorporates a multiple correlated sources approach using Shannon’s cipher system, which is more practical than looking at two sources.

VII. COMPARISON TO OTHER MODELS

The two correlated sources model across a wiretap channel is a more generalised approach of Yamamoto’s [1] model. If we were to combine the links into one link, we would have the same situation as per Yamamoto’s [1]. From Section VI it is evident that the model can be implemented for multiple sources with Shannon’s cipher system. Due to the unique scenario incorporating multiple sources and multiple links, the new model is more secure as private information and common information from other link/s are required for decoding.

Further, information at the sources may be more secure in the new model because if one source is compromised then only one source’s information is known. In Yamamoto’s [1] method both source’s information is contained at one station and when that source is compromised then information about both sources are known. The information transmitted along the channels (i.e. the syndromes) do not have a fixed length as per Yamamoto’s [1] method. Here, the syndrome length may vary depending on the encoding procedure and nature of Slepian-Wolf codes, which is another feature of this generalised model.

The generalised model also has the advantage that varying amounts of the common information \(V_{CX}\) and \(V_{CY}\) (in the case of two sources) may be transmitted depending on the security of the transmission link and/or sources. For example, for two correlated sources, if \(Y’s\) channel is not secure we can specify that more of the common information is transmitted from \(X\). In this way we’re able to make better use of the transmission link’s security. For Yamamoto’s [1] method the common information was transmitted as one portion, \(V_C\).

In this model, information from more than one link is required in order to determine the information for one source. This gives rise to added security as even if one link is wiretapped it is not possible to determine the contents of a particular source. This is attributed to the fact that this model has separate common information portions, which is different to Yamamoto’s model.

Another major feature is that private information can be hidden using common information. Here, common information produced by a source may be used to mask its private codeword thus saving on key length. The key allocation is specified by general rules presented in Section VI. The multiple correlated sources model presents a combination masking scheme where more than one common information is used to protect a private information, which is a practical approach. This is an added feature developed in order to protect the system. This approach has not been considered in the other models mentioned in this section.
The work by Yang et al. [8] uses the concept of side information to assist the decoder in determining the transmitted message. The side information could be considered to be a source and is related to this work when the side information is considered as correlated information. Similar work with side information that incorporates wiretappers, by Villard and Piantanida [6] and Villard et al. [7] may be generalised in the sense that side information can be considered to be a source, however this new model is distinguishable as syndromes, which are independent of one another are transmitted across an error free channel in the new model. Further, to the author’s knowledge Shannon’s cipher system has not been incorporated into these models by Villard and Piantanida [6] and Villard et al. [7].

VIII. FUTURE WORK

This work has room for expansion and future work. It would be interesting to consider the case where the channel capacity has certain constraints (according to the assumptions in Section VI the channel capacity is enough at all times). In the new model, the channels are either protected by keys or not however this is limited and a real case scenario where there are varying security levels for the channels is an avenue for future work. Another aspect for expansion is to investigate the allocation of common information as keys to minimize additional keys with links having varying security levels and limited capacity.

IX. CONCLUSION

The information leakage for two correlated sources across a wiretap channel was initially considered. Knowing which components contribute most to information leakage aids in keeping the system more secure, as these terms can be made more secure. The information leakage for the two correlated source model was quantified and proven. Shannon’s cipher system was also incorporated for this model and channel and key rates to achieve perfect secrecy have been provided. The two correlated sources model has been extended for the network scenario where we consider multiple sources transmitting information across multiple links. The information leakage for this extended model is detailed. The channel and key rates are also considered for the multiple correlated source model when Shannon’s cipher system is implemented. A masking method is further presented to minimize key lengths and a combination masking method is presented to address its practical shortcoming.

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