Modified gravity with arbitrary coupling between matter and geometry

T. Harko

Department of Physics and Center for Theoretical and Computational Physics,
The University of Hong Kong, Pok Fu Lam Road, Hong Kong, P. R. China

(Dated: October 25, 2008)

Abstract

The field equations of a generalized $f(R)$ type gravity model, in which there is an arbitrary coupling between matter and geometry, are obtained. The equations of motion for test particles are derived from a variational principle in the particular case in which the Lagrange density of the matter is an arbitrary function of the energy-density of the matter only. Generally, the motion is non-geodesic, and takes place in the presence of an extra force orthogonal to the four-velocity. The Newtonian limit of the model is also considered. The perihelion precession of an elliptical planetary orbit in the presence of an extra force is obtained in a general form, and the magnitude of the extra gravitational effects is constrained in the case of a constant extra force by using Solar System observations.

PACS numbers: 04.50.Kd, 04.20.Cv, 04.20.Fy

*Electronic address: harko@hkucc.hku.hk
I. INTRODUCTION

Recent astrophysical observations have provided the astonishing result that around 95–96% of the content of the Universe is in the form of dark matter + dark energy, with only about 4–5% being represented by baryonic matter [1]. More intriguingly, around 70% of the energy-density is in the form of what is called "dark energy", and is responsible for the acceleration of the distant type Ia supernovae [2]. Hence, today’s models of astrophysics and cosmology face two fundamental problems, the dark energy problem, and the dark matter problem, respectively. Although in recent years many different suggestions have been proposed to overcome these issues, a satisfactory answer has yet to be obtained.

A very promising way to explain the observational data is to assume that at large scales the Einstein gravity model of general relativity breaks down, and a more general action describes the gravitational field. Theoretical models in which the standard Einstein-Hilbert action is replaced by an arbitrary function of the Ricci scalar $R$, first proposed in [3], have been extensively investigated lately. Cosmic acceleration can be explained by $f(R)$ gravity [4], and the conditions of viable cosmological models have been derived in [5]. In the context of the Solar System regime, severe weak field constraints seem to rule out most of the models proposed so far [6, 7], although viable models do exist [8, 9, 10, 11]. The possibility that the galactic dynamic of massive test particles can be understood without the need for dark matter was also considered in the framework of $f(R)$ gravity models [12, 13, 14, 15, 16]. For a review of $f(R)$ generalized gravity models see [17].

A generalization of the $f(R)$ gravity theories was proposed in [18] by including in the theory an explicit coupling of an arbitrary function of the Ricci scalar $R$ with the matter Lagrangian density $L_m$. As a result of the coupling the motion of the massive particles is non-geodesic, and an extra force, orthogonal to the four-velocity, arises. The connections with MOND and the Pioneer anomaly were also explored. The implications of the non-minimal coupling on the stellar equilibrium were investigated in [19], where constraints on the coupling were also obtained. An inequality which expresses a necessary and sufficient condition to avoid the Dolgov-Kawasaki instability for the model was derived in [20]. The relation between the model with geometry-matter coupling and ordinary scalar-tensor gravity, or scalar-tensor theories which include non-standard couplings between the scalar and matter was studied in [21].
problem in galaxies was also analyzed. In the specific case where both the action and the coupling are linear in $R$ the action leads to a theory of gravity which includes higher order derivatives of the matter fields without introducing more dynamics in the gravity sector \[22\]. The equivalence between a scalar theory and the model with the non-minimal coupling of the scalar curvature and matter was considered in \[23\]. This equivalence allows for the calculation of the PPN parameters $\beta$ and $\gamma$, which may lead to a better understanding of the weak-field limit of $f(R)$ theories. Different forms for the matter Lagrangian density $L_m$, and the resulting extra-force, were considered in \[24\], and it was shown that more natural forms for $L_m$ do not imply the vanishing of the extra-force. The impact on the classical equivalence between different Lagrangian descriptions of a perfect fluid was also analyzed. Similar couplings between gravitation and matter have also been considered as possible explanations for the accelerated expansion of the universe and of the dark energy in \[25\].

In all the previous studies of the models with matter-geometry coupling, the matter part in the coupling was represented by the Lagrangian density of the matter, while the geometric part was considered to be an arbitrary function of the Ricci scalar. We may call this class of models as modified gravity models with linear matter-geometry coupling. However, more general models, in which the matter part in the coupling is an arbitrary function of the Lagrangian density of the matter, can also be constructed, and they represent the natural generalization of the models with linear matter coupling. It is the purpose of this Letter to present the field equations of a generalized gravity model, in which there is an arbitrary coupling between matter and geometry. In this class of models the energy-momentum tensor of the matter is generally not conserved. The equations of motion of the test particles are also obtained, by using a variational principle, in the particular case in which the Lagrange function of the matter is a function of the density of the matter only. In this model the motion is non-geodesic. The study of the Newtonian limit shows that the matter-geometry coupling induces a supplementary acceleration of the test particles. This acceleration may be responsible for the constancy of the galactic rotation curves, which is usually attributed to the presence of the dark matter. As an observational test of the model we consider the perihelion precession of an elliptic planetary orbit in the presence of the extra force. The precession angle is derived in a general form, and from the observed value of the perihelion precession of the planet Mercury the magnitude of the extra acceleration is constrained.

The present Letter is organized as follows. The field equations of the model are derived
in Section II. The equations of motion of the test particles and their Newtonian limit are considered in Section III for a particular model in which the matter Lagrangian is a function of the density only. We discuss and conclude our results in Section IV.

II. GRAVITATIONAL FIELD EQUATIONS IN $f(R)$ TYPE MODELS WITH ARBITRARY COUPLING BETWEEN MATTER AND GEOMETRY

The most general action for a $f(R)$ type modified gravity involving an arbitrary coupling between matter and geometry is given, in a system of units with $8\pi G = c = 1$, by

$$S = \int \left[ \frac{1}{2} f_1(R) + G(L_m) f_2(R) \right] \sqrt{-g} d^4x,$$

(1)

where $f_i(R)$, $i = 1, 2$ are arbitrary functions of the Ricci scalar $R$, while $G(L_m)$ is an arbitrary function of the matter Lagrangian density $L_m$. The only requirement for the functions $f_i$, $i = 1, 2$ and $G$ is to be analytical function of the Ricci scalar $R$ and $L_m$, respectively, that is, they must possess a Taylor series expansion about any point. When $f_1(R) = R$, $f_2(R) = 1$ and $G(L_m) = L_m$, we recover standard general relativity. With $f_2(R) = 1$ and $G(L_m) = L_m$ we obtain the $f(R)$ generalized gravity models. The case $G(L_m) = 1 + \lambda L_m$, $\lambda = \text{constant}$, corresponds to the (linear) coupling between matter and geometry, considered in [18] - [24].

We define the energy-momentum tensor of the matter as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} L_m)}{\delta g^{\mu\nu}}.$$

(2)

By assuming that the Lagrangian density $L_m$ of the matter depends only on the metric tensor components, and not on its derivatives, we obtain $T_{\mu\nu} = L_m g_{\mu\nu} - 2 \partial L_m / \partial g^{\mu\nu}$.

Varying the action with respect to the metric tensor $g_{\mu\nu}$, we obtain the field equations of the model as

$$F_1(R) R_{\mu\nu} - \frac{1}{2} f_1(R) g_{\mu\nu} + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) F_1(R) = -2 G(L_m) F_2(R) R_{\mu\nu} - 2 (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) G(L_m) F_2(R) - f_2(R) \left[ K(L_m) L_m - G(L_m) \right] g_{\mu\nu} + f_2(R) K(L_m) T_{\mu\nu},$$

(3)

where we denoted $F_i(R) = df_i(R)/dR$, $i = 1, 2$ and $K(L_m) = dG(L_m)/dL_m$, respectively. For $G(L_m) = L_m$ and by rescaling the function $f_2(R)$ so that $f_2(R) \to 1 + \lambda f_2(R)$, we reobtain the field equations proposed in [18].
By contracting the field equations given by Eq. (3) we obtain the scalar equation

$$3 \Box [F_1(R) + 2G(L_m)F_2(R)] + [F_1(R) + 2G(L_m)F_2(R)] R -$$

$$2f_1(R) + 4f_2(R) [K(L_m) L_m - G(L_m)] = K(L_m) f_2(R) T,$$

(4)

where $T = T^\mu_\mu$. By taking the covariant divergence of Eq. (3), with the use of the mathematical identity $\nabla^\mu [a'(R) R_{\mu\nu} - a(R) g_{\mu\nu}/2 + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) a(R)] \equiv 0$, where $a(R)$ is an arbitrary function of the Ricci scalar $R$ and $a'(R) = da/dR$, we obtain

$$\nabla^\mu T_{\mu\nu} = \nabla^\mu \ln [f_2(R) K(L_m)] \{L_m g_{\mu\nu} - T_{\mu\nu}\} = 2\nabla^\mu \ln [f_2(R) K(L_m)] \frac{\partial L_m}{\partial g_{\mu\nu}}. \quad (5)$$

The requirement of the conservation of the energy-momentum tensor of matter, $\nabla^\mu T_{\mu\nu} = 0$, gives an effective functional relation between the matter Lagrangian density and the functions $f_2(R)$ and $K(L_m)$,

$$\nabla^\mu \ln [f_2(R) K(L_m)] \frac{\partial L_m}{\partial g_{\mu\nu}} = 0. \quad (6)$$

Thus, once the matter Lagrangian density is known, by an appropriate choice of the functions $G(L_m)$ and $f_2(R)$ one can construct, at least in principle, conservative models with arbitrary matter-geometry coupling.

### III. MODELS WITH ARBITRARY DENSITY-DEPENDENT MATTER LAGRANGIAN

As a specific case of generalized gravity models with arbitrary matter-geometry coupling, we consider the case in which the matter Lagrangian density is an arbitrary function of the energy density of the matter $\rho$ only, so that $L_m = L_m(\rho)$. Then the energy-momentum tensor of the matter is given by

$$T^{\mu\nu} = \rho \frac{dL_m}{d\rho} u^\mu u^\nu + \left(L_m - \rho \frac{dL_m}{d\rho}\right) g^{\mu\nu}, \quad (7)$$

where the four-velocity $u^\mu = dx^\mu/ds$ satisfies the condition $g^{\mu\nu} u_\mu u_\nu = 1$, and we have also used the relation $\delta \rho = (1/2) \rho (g_{\mu\nu} - u_\mu u_\nu) \delta g^{\mu\nu}$.

The energy-momentum tensor given by Eq. (7) can be written in a form similar to the perfect fluid case if we assume that the thermodynamic pressure $p$ obeys a barotropic equation of state, so that $p = p(\rho)$. The perfect-fluid type representation can be obtained by
assuming that the matter Lagrangian satisfies the equations

\[ \rho \frac{dL_m(\rho)}{d\rho} = \rho + \rho \Pi(\rho) + p(\rho), \tag{8} \]

and

\[ \rho \frac{dL_m(\rho)}{d\rho} - L_m(\rho) = p(\rho), \tag{9} \]

respectively, where \( \Pi(\rho) \) is an arbitrary function of the density. Substituting the term \( \rho dL_m(\rho)/d\rho \) from Eq. (8) into Eq. (9) gives the matter Lagrangian as \( L_m(\rho) = \rho + \rho \Pi(\rho) \). With this form of \( L_m \), Eq. (8) gives the following differential equation for \( \Pi(\rho) \),

\[ \rho^2 \frac{d\Pi(\rho)}{d\rho} = p(\rho), \tag{10} \]

with the general solution given by

\[ \Pi(\rho) = \int_0^\rho \frac{p}{\rho^2} d\rho = \int_0^\rho \frac{dp}{\rho} - \frac{p(\rho)}{\rho}. \tag{11} \]

Therefore the matter Lagrangian and the energy-momentum tensor can be written as

\[ L_m(\rho) = \rho \left( 1 + \int_0^\rho \frac{dp}{\rho} \right) - p(\rho), \tag{12} \]

and

\[ T^{\mu\nu} = [\rho + p(\rho) + \rho \Pi(\rho)] u^\mu u^\nu - p(\rho) g^{\mu\nu}, \tag{13} \]

respectively. From a physical point of view \( \Pi(\rho) \) can be interpreted as the elastic (deformation) potential energy of the body, and therefore Eq. (13) corresponds to the energy-momentum tensor of a compressible elastic isotropic system.

By imposing the condition of the conservation of the matter current, \( \nabla_\nu (\rho u^\nu) = 0 \), and with the use of the identity \( u^\nu \nabla_\nu u^\mu = d^2 x^\mu/ds^2 + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda \), from Eq. (5) we obtain the equation of motion of a test particle in the modified gravity model as

\[ \frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda = f^\mu, \tag{14} \]

where

\[ f^\mu = -\nabla_\nu \ln \left\{ f_2(R) K [L_m(\rho)] \frac{dL_m(\rho)}{d\rho} \right\} (u^\nu u^\mu - g^{\mu\nu}). \tag{15} \]

The extra-force \( f^\mu \), generated due to the presence of the coupling between matter and geometry, is perpendicular to the four-velocity, \( f^\mu u_\mu = 0 \). The equation of motion Eq. (14) can be obtained from the variational principle

\[ \delta S_p = \delta \int L_\rho ds = \delta \int \sqrt{Q} \sqrt{g_{\mu\nu} u^\mu u^\nu} ds = 0, \tag{16} \]
where $S_p$ and $L_p = \sqrt{Q} \sqrt{g_{\mu\nu} u^\mu u^\nu}$ are the action and the Lagrangian density for the test particles, respectively, and

$$\sqrt{Q} = f_2(R) K [L_m (\rho)] \frac{dL_m (\rho)}{d\rho}. \quad (17)$$

To prove this result we start with the Lagrange equations corresponding to the action (16),

$$\frac{d}{ds} \left( \frac{\partial L_p}{\partial u^\lambda} \right) - \frac{\partial L_p}{\partial x^\lambda} = 0. \quad (18)$$

Since $\partial L_p/\partial u^\lambda = \sqrt{Q} u^\lambda$ and $\partial L_p/\partial x^\lambda = (1/2) \sqrt{Q} g_{\mu\nu,\lambda} u^\mu u^\nu + (1/2) Q_{,\lambda}/Q$, a straightforward calculation gives the equations of motion of the particle as

$$\frac{d^2 x^\mu}{ds^2} + \Gamma_{\nu\lambda}^\mu u^\nu u^\lambda + (u^\mu u^\nu - g^{\mu\nu}) \nabla_\nu \ln \sqrt{Q} = 0. \quad (19)$$

By simple identification with the equation of motion of the modified gravity model with arbitrary matter-geometry coupling, given by Eq. (14), we obtain the explicit form of $\sqrt{Q}$ as given by Eq. (17).

The variational principle (16) can be used to study the Newtonian limit of the model. In the limit of weak gravitational fields, $ds \approx \sqrt{1 + 2\phi - \vec{v}^2} dt \approx (1 + \phi - \vec{v}^2/2) dt$, where $\phi$ is the Newtonian potential and $\vec{v}$ is the usual tridimensional velocity of the particle. By representing the function $\sqrt{Q}$ as

$$\sqrt{Q} = f_2(R) K [L_m (\rho)] \frac{dL_m (\rho)}{d\rho} = 1 + U \left( R, L_m (\rho), \frac{dL_m (\rho)}{d\rho} \right), \quad (20)$$

where $U << 1$, the equations of motion of the particle can be obtained from the variational principle

$$\delta \int \left[ U \left( R, L_m (\rho), \frac{dL_m (\rho)}{d\rho} \right) + \phi - \frac{\vec{v}^2}{2} \right] dt = 0, \quad (21)$$

and are given by

$$\vec{a} = -\nabla \phi - \nabla U = \vec{a}_N + \vec{a}_E, \quad (22)$$

where $\vec{a}_N = -\nabla \phi$ is the usual Newtonian gravitational acceleration, and $\vec{a}_E = -\nabla U$ is a supplementary acceleration induced due to the coupling between matter and geometry.

An estimation of the effect of the extra-force, generated by the coupling between matter and geometry, on the orbital parameters of the motion of the planets around the Sun can be obtained in a simple way by using the properties of the Runge-Lenz vector, defined as $\vec{A} = \vec{v} \times \vec{L} - \alpha \vec{e}_r$, where $\vec{v}$ is the velocity relative to the Sun, with mass $M_\odot$, of a planet.
of mass $m$, $\vec{r} = re\hat{e}$, is the two-body position vector, $\vec{p} = \mu\vec{v}$ is the relative momentum, $\mu = mM_\odot / (m + M_\odot)$ is the reduced mass, $\vec{L} = \vec{r} \times \vec{p} = \mu r^2 \dot{\theta} \hat{k}$ is the angular momentum, and $\alpha = GmM_\odot$ [27]. For an elliptical orbit of eccentricity $e$, major semi-axis $a$, and period $T$, the equation of the orbit is given by $(L^2 / \mu \alpha)^{-1} = 1 + e \cos \theta$. The Runge-Lenz vector can be expressed as $\vec{A} = \left(\vec{L}^2 / \mu r - \alpha \right) \vec{e} - \dot{r} \vec{e}_r$, and its derivative with respect to the polar angle $\theta$ is given by $d \vec{A} / d\theta = r^2 \left[ 6 \alpha^2 / mr^2 + m\vec{a}_E(r) \right] \vec{e}_\theta$, where we have also assumed that $\mu \approx m$. The change in direction $\Delta \phi$ of the perihelion with a change of $\theta$ of $2\pi$ is obtained as $\Delta \phi = (1 / \alpha e) \int_0^{2\pi} \left| \vec{L} \times d\vec{A} / d\theta \right| d\theta$, and it is given by

$$\Delta \phi = 24\pi^3 \left( \frac{a}{T} \right)^2 \frac{1}{1 - e^2} + \frac{L}{8\pi^3 me} \left( \frac{1}{(a/T)^3} \right)^{3/2} \int_0^{2\pi} a_E \left[ \frac{L^2 (1 + e \cos \theta)^{-1} / \alpha}{(1 + e \cos \theta)^2} \right] \cos \theta d\theta, \quad (23)$$

where we have used the relation $\alpha / L = 2\pi (a/T) / \sqrt{1 - e^2}$. The first term of this equation corresponds to the standard general relativistic precession of the perihelion of the planets, while the second term gives the contribution to the perihelion precession due to the presence of the coupling between matter and geometry.

As an example of the application of Eq. (23) we consider the case for which the extra-force may be considered as a constant, $a_E \approx$ constant, an approximation that could be valid for small regions of the space-time. With the use of Eq. (23) one finds for the perihelion precession the expression

$$\Delta \phi = \frac{6\pi GM_\odot}{a (1 - e^2)} + \frac{2\pi a^2 \sqrt{1 - e^2}}{GM_\odot} a_E, \quad (24)$$

where we have also used Kepler’s third law, $T^2 = 4\pi^2 a^3 / GM_\odot$. For the planet Mercury $a = 57.91 \times 10^{11}$ cm, and $e = 0.205615$, respectively, while $M_\odot = 1.989 \times 10^{33}$ g. With these numerical values the first term in Eq. (24) gives the standard general relativistic value for the precession angle, $(\Delta \phi)_GR = 42.962$ arcsec per century, while the observed value of the precession is $(\Delta \phi)_obs = 43.11 \pm 0.21$ arcsec per century [28]. Therefore the difference $(\Delta \phi)_E = (\Delta \phi)_obs - (\Delta \phi)_GR = 0.17$ arcsec per century can be attributed to other physical effects. Hence the observational constraints requires that the value of the constant $a_E$ must satisfy the condition $a_E \leq 1.28 \times 10^{-9}$ cm/s$^2$. This value of $a_E$, obtained from the solar
system observations, is somewhat smaller than the value of the extra-acceleration $a_0 \approx 10^{-8}$ cm/s$^2$, necessary to explain the “dark matter” properties, as well as the Pioneer anomaly [18]. However, it does not rule out the possibility of the presence of some extra gravitational effects acting at both the solar system and galactic levels, since the assumption of a constant extra-force is not correct on larger astronomical scales.

IV. CONCLUSIONS

In the present Letter we have considered a generalized gravity model with an arbitrary coupling between matter and geometry, described by the product of an arbitrary function of the Lagrange density of the matter, and an arbitrary function of the Ricci scalar. The proposed action represents the most general extension of the standard Hilbert action for the gravitational field, $S = \int \left[ R/2 + L_m \right] \sqrt{-g} d^4x$. The equations of motion corresponding to this model show the presence of an extra-force acting on test particles, and the motion is generally non-geodesic. The physical implications of such a force have been already analyzed in the framework of the generalized gravity model with linear coupling between matter and geometry, considered in [18], and the possible implications for the dark matter problem and for the explanation of the Pioneer anomaly have also been investigated. On the other hand, the field equations Eqs. (3) are equivalent to the Einstein equations of the $f(R)$ model in empty space-time, but differ from them, as well as from standard general relativity, in the presence of matter. Therefore the predictions of the present model could lead to some major differences, as compared to the predictions of standard general relativity, in several problems of current interest, like cosmology, gravitational collapse or the generation of gravitational waves. The study of these phenomena may also provide some specific signatures and effects, which could distinguish and discriminate between the various gravity models.

Acknowledgments

I would like to thank to the anonymous referee, whose comments and suggestions helped me to significantly improve the manuscript. This work is supported by an RGC grant of the
government of the Hong Kong SAR.

[1] A. G. Riess et al., Astron. J. 116, 1009 (1998); S. Perlmutter et al., Astrophys. J. 517, 565 (1999); P. de Bernardis et al., Nature 404, 955 (2000); S. Hanany et al., Astrophys. J. 545, L5 (2000).

[2] P. J. E. Peebles and B. Ratra, Rev. Mod. Phys. 75, 559 (2003); T. Padmanabhan, Phys. Repts. 380, 235 (2003).

[3] H. A. Buchdahl, Mon. Not. Roy. Astron. Soc. 150, 1 (1970); R. Kerner, Gen. Rel. Grav. 14, 453 (1982); J. P. Duruisseau, R. Kerner and P. Eysseric, Gen. Rel. Grav. 15, 797 (1983); J. D. Barrow and A. C. Ottewill, J. Phys. A: Math. Gen. 16, 2757 (1983).

[4] S. M. Carroll, V. Duvvuri, M. Trodden and M. S. Turner, Phys. Rev. D 70, 043528 (2004).

[5] S. Capozziello, S. Nojiri, S. D. Odintsov and A. Troisi, Phys. Lett. B 639, 135 (2006); L. Amendola, D. Polarski and S. Tsujikawa, Phys. Rev. Lett. 98, 131302 (2007); S. Capozziello, S. Nojiri, S. D. Odintsov and A. Troisi, Phys. Lett. B 639, 135 (2006); S. Nojiri and S. D. Odintsov, Phys. Rev. D 74, 086005 (2006); M. Amarzguioui, O. Elgaroy, D. F. Mota and T. Multamaki, Astron. Astrophys. 454, 707 (2006); L. Amendola, R. Gannouji, D. Polarski and S. Tsujikawa, Phys. Rev. D 75, 083504 (2007); T. Koivisto, Phys. Rev. D 76, 043527 (2007); A. A. Starobinsky, JETP Lett. 86, 157 (2007); B. Li, J. D. Barrow and D. F. Mota, Phys. Rev. D 76, 044027 (2007); S. E. Perez Bergliaffa, Phys. Lett. B 642, 311 (2006); J. Santos, J. S. Alcaniz, M. J. Reboucas and F. C. Carvalho, Phys. Rev. D 76, 083513 (2007); G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov and S. Zerbini, JCAP 0502, 010 (2005); V. Faraoni, Phys. Rev. D 72, 061501 (2005); V. Faraoni, Phys. Rev. D 72, 124005 (2005); S. Nojiri and S. D. Odintsov, Int. J. Geom. Meth. Mod. Phys. 4, 115 (2007); L. M. Sokolowski, Class. Quantum Grav. 24, 3391 (2007); V. Faraoni, Phys. Rev. D 75, 067302 (2007); G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, L. Sebastiani and S. Zerbini, [gr-qc/0712.4017] (2007); C. G. Böhmer, L. Hollenstein and F. S. N. Lobo, Phys. Rev. D 76, 084005 (2007); S. Carloni, P. K. S. Dunsby and A. Troisi, Phys. Rev. D 77, 024024 (2008); S. Capozziello, R. Cianci, C. Stornaiolo and S. Vignolo, Class. Quant. Grav. 24, 6417 (2007); S. Nojiri, S. D. Odintsov and P. V. Tretyakov, Phys. Lett. B 651, 224 (2007); S. Nojiri and S. D. Odintsov, Phys. Lett. B 652, 343 (2007); S. Tsujikawa, Phys. Rev. D 77, 023507 (2008) ; K. N. Ananda, S. Carloni
and P. K. S. Dunsby, Phys. Rev. D 77, 024033 (2008).

[6] T. Chiba, Phys. Lett. B 575, 1 (2003); A. L. Erickcek, T. L. Smith and M. Kamionkowski, Phys. Rev. D 74, 121501 (2006); T. Chiba, T. L. Smith and A. L. Erickcek, Phys. Rev. D 75, 124014 (2007). S. Nojiri and S. D. Odintsov, Phys. Lett. B 659, 821 (2008); S. Capozziello, A. Stabile and A. Troisi, Phys. Rev. D 76, 104019 (2007); S. Capozziello, A. Stabile and A. Troisi, Class. Quantum Grav. 25, 085004 (2008).

[7] G. J. Olmo, Phys. Rev. D 75, 023511 (2007).

[8] W. Hu and I. Sawicki, Phys. Rev. D 76, 064004 (2007).

[9] S. Nojiri and S. D. Odintsov, Phys. Rev. D 68, 123512 (2003); V. Faraoni, Phys. Rev. D 74, 023529 (2006); T. Faulkner, M. Tegmark, E. F. Bunn and Y. Mao, Phys. Rev. D 76, 063505 (2007); P. J. Zhang, Phys. Rev. D 76, 024007 (2007); S. Capozziello and S. Tsujikawa, arXiv:0712.2268 [gr-qc] (2007); C. S. J. Pun, Z. Kovacs and T. Harko, Phys. Rev. D 78, 024043 (2008).

[10] I. Sawicki and W. Hu, Phys. Rev. D 75, 127502 (2007).

[11] L. Amendola and S. Tsujikawa, Phys. Lett. B 660, 125 (2008).

[12] S. Capozziello, V. F. Cardone and A. Troisi, JCAP 0608, 001 (2006); S. Capozziello, V. F. Cardone and A. Troisi, Mon. Not. R. Astron. Soc. 375, 1423 (2007).

[13] A. Borowiec, W. Godlowski and M. Szydlowski, Int. J. Geom. Meth. Mod. Phys. 4 (2007) 183.

[14] C. F. Martins and P. Salucci, Mon. Not. R. Astron. Soc. 381, 1103 (2007).

[15] C. G. Boehmer, T. Harko and F. S. N. Lobo, Astropart. Phys. 29, 386 (2008).

[16] C. G. Boehmer, T. Harko and F. S. N. Lobo, JCAP 03, 024 (2008).

[17] T. P. Sotiriou and V. Faraoni, arXiv:0805.1726 (2008).

[18] O. Bertolami, C. G. Boehmer, T. Harko and F. S. N. Lobo, Phys. Rev. D 75, 104016 (2007).

[19] O. Bertolami and J. Paramos, arXiv:0709.3988 [astro-ph] (2007).

[20] V. Faraoni, Phys. Rev. D 76, 127501 (2007).

[21] T. P. Sotiriou and V. Faraoni, arXiv:0805.1249 (2008).

[22] T. P. Sotiriou, Phys. Lett. B 664, 225 (2008).

[23] O. Bertolami and J. Paramos, arXiv:0805.1241 (2008).

[24] Orfeu Bertolami, F. S. N. Lobo and J. Paramos, Phys. Rev. D78, 064036 (2008).

[25] A.D. Dolgov and M. Kawasaki, arXiv:astro-ph/0307442 (2003); S. Nojiri and S. Odintsov, Phys. Lett. B 599, 137 (2004); G. Allemandi, A. Borowiec, M. Francaviglia and S. Odintsov,
Phys. Rev. D 72, 063505 (2005); S. Mukohyama and L. Randall, Phys. Rev. Lett. 92, 211302 (2004); N. Deruelle, M. Sasaki and Y. Sendouda, arXiv:0803.2742 (2008).

[26] T. Koivisto, Class. Quant. Grav. 23, 4289 (2006).

[27] B. M. Barker and R. F. O’Connell, Phys. Rev. D 10, 1340 (1974); C. Duval, G. Gibbons, and P. Horvathy, Phys. Rev. D43, 3907 (1991).

[28] I. I. Shapiro, W. B. Smith, M. E. ASh and S. Herrick, Astron. J. 76, 588 (1971); I. I. Shapiro, C. C. Counselman and R. W. King, Phys. Rev. Lett. 36, 555 (1976).