Perturbative approach to an exactly solved problem: the Kitaev honeycomb model

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We analyze the gapped phase of the Kitaev honeycomb model perturbatively in the isolated-dimer limit. Our analysis is based on the continuous unitary transformations method which allows one to compute the spectrum as well as matrix elements of operators between eigenstates, at high order. The starting point of our study consists in an exact mapping of the original honeycomb spin system onto a square-lattice model involving an effective spin and a hardcore boson. We then derive the low-energy effective Hamiltonian up to order 10 which is found to describe an interacting-anyon system, contrary to the order 4 result which predicts a free theory. These results give the ground-state energy in any vortex sector and thus also the vertex gap, which is relevant for experiments. Furthermore, we show that the elementary excitations are emerging free fermions composed of a hardcore boson with an attached spin- and phase-operator string. We also focus on observables and compute, in particular, the spin-spin correlation functions. We show that they admit a multiplaquette expansion that we derive up to order 6. Finally, we study the creation and manipulation of anyons with local operators, show that they also create fermions, and discuss the relevance of our findings for experiments in optical lattices.

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I. INTRODUCTION

Elementary particles can be classified in two categories according to the value of their spin. Half-integer spin particles obey Fermi-Dirac statistics and are called fermions whereas integer-spin particles obey Bose-Einstein statistics and are known as bosons. However, some quantum objects may obey other (fractional) statistics describing nontrivial braiding as initially suggested by Leinaas and Myrheim more than thirty years ago, and by Wilczek in the eighties. Despite numerous theoretical works, these so-called anyons are still waiting for a direct observation, and recent experimental proposals are very promising (see Ref. 4).

In the last years, anyons have drawn much attention because of their interest for topological quantum computation. In this perspective, several models have been proposed among which the celebrated toric code is a spin-1/2 system whose elementary excitations behave as semions. However, the experimental realization of this system is rather tricky since it involves four-spin interactions. Here, we shall focus on another system originally proposed by Kitaev which only involves two-spin interactions. This model is very rich since it contains Abelian and non-Abelian anyonic as well as fermionic excitations. Thus, it has been the subject of many recent studies concerning the spectrum, the correlation functions and the entanglement, or the quench dynamics. Let us also mention several extensions among which the analysis of time-reversal symmetry breaking terms which may give rise to a chiral spin liquid.

Furthermore, this model is susceptible to be realized in various experimental systems such as polar molecules, ultracold atoms or Josephson junctions. It thus constitutes an appealing candidate for the observation of anyons. Nevertheless, the presence of fermions in the spectrum may spoil the detection process; a point completely missed in a recent proposal (see Ref. 33 for explanation and Ref. 16 for details).

The goal of the present paper is to investigate the gapped phase of the Kitaev honeycomb model. Indeed, in his remarkable seminal paper, Kitaev mainly focuses on the special subspace of the Hilbert space to which the ground state belongs to and the low-energy spectrum of other subspaces has only been discussed lately. Our aim is to bridge this gap by providing a high-order perturbative analysis, in the isolated-dimer limit, of the spectrum as well as some interesting results about the creation and the manipulation of anyons which is of relevance for experiments. Part of our results have already been given in two short papers and the present paper may be considered as an extended and detailed version of these works. However many other results are presented here among which the interplay between fermions and anyons under string operations discussed in Sec. IX.

This paper is organized as follows. In the next section, we introduce the model as well as its main properties. In particular, we discuss the importance of the boundary conditions and insist on the role played by conserved quantities and the constraints resulting from them. In Sec. III, we show how to map the Kitaev model involving spins on the honeycomb lattice onto an effective spin and hardcore boson on a square lattice. This mapping is the starting point of the perturbation theory presented in this work. In Sec. IV, we explain how to diagonalize the Hamiltonian order by order using the perturbative continuous unitary transformation (PCUT) method.
The study of the low-energy (zero-quasiparticle) sector is the subject of Sec. V, a large part of which is devoted to a pictorial (and hopefully pedagogical) analysis and construction of the eigenstates of the toric code model which naturally emerges from this problem. There, we give the perturbative expansion form of the ground state energy for any vortex configuration. The effective low-energy theory is found to be described by interacting anyons contrary to the lowest-order result which predicts free anyons. Sec. VI focuses on the study of the one-quasiparticle subspace, where the physics is shown to be that of a particle hopping in a magnetic field with zero or half a flux quantum per elementary plaquette. The demonstration of the fermionic nature (known from exact solutions) of the quasiparticles is brieﬂy sketched in Sec. VII. In Sec. VIII, we provide some checks of our results by analyzing simple vortex conﬁgurations which allow for an exact solution. The spin-spin correlation functions and the manipulation of anyons are tackled in Sec. IX, which is devoted to the renormalization of observables. Finally, we discuss several issues and give some perspectives. Technical details as well as all relevant coefﬁcients involved in the perturbative expansions are gathered in appendices.

In what follows, we tried to be as pedagogical as possible and always favored simple demonstrations on concrete examples rather than lengthy proofs for general situations. We hope that it will help the reader to understand the richness of this model.

II. THE MODEL

A. Hamiltonian and boundary conditions

The model considered in this work is a spin-1/2 system proposed by Kitaev in which spins are located at the vertices of a honeycomb lattice. Since the honeycomb lattice is topologically equivalent to the brick-wall lattice, we shall always represent it as shown in Fig. 1a. In this lattice, one distinguishes three types of links (x, y, and z) to which one associates three different couplings and interactions. The Hamiltonian of the system is

$$H = - \sum_{\alpha = x, y, z} \sum_{\alpha - \text{links}} J_{\alpha} \sigma_i^\alpha \sigma_j^\alpha, \quad (1)$$

where $\sigma_i^\alpha$ are the usual Pauli matrices at site $i$. In the following we assume, without loss of generality, that $J_x \geq 0$ for all $\alpha$ and $J_y \geq J_x, J_z$.

We will either work with an inﬁnite system and open boundary conditions (a plane), or with a ﬁnite (or inﬁnite) system and periodic boundary conditions (a torus). In the latter case and for reasons that will become clearer in the following (in particular, see Sec. VB), we shall restrict ourselves to the periodic boundary conditions (PBC) depicted in Fig. 2. The number of sites $N_s$ is $N_s = 2(2p)^2 = 8p^2$, with $p \in \mathbb{N}$ ($p = 1$ in Fig. 2a). Let us anticipate what follows and mention that these boundary conditions are such that the lattice of z-dimers (Fig. 1b) can be bi-colored as shown in Fig. 2b).

B. Conserved quantities

A remarkable property of Hamiltonian (1), is that its elementary operators $K_{ij} = \sigma_i^\alpha \sigma_j^\alpha$ commute with plaquette operators $W_p$ so that $[H, W_p] = 0$. For the plaquette $p$ shown in Fig. 1a, such an operator is deﬁned as

$$W_p = K_{12}K_{23}K_{34}K_{45}K_{56}K_{61} = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z. \quad (2)$$

Let us mention that in the expression of $W_p$ in terms of the $K$’s, one could have started at any site instead of site 1 and/or one could have taken the product of $K$’s anti-clockwise instead of clockwise. Furthermore, the expression in terms of $\sigma$’s could also be written $W_p =$

![Fig. 1](image1.png)

![Fig. 2](image2.png)
FIG. 3: (color online). Illustration of the conserved plaquette quantity $W_p$. The thick yellow (lightest gray) line delimitates the plaquette $p$. The thick red (gray), green (light gray) or blue (dark gray) segments represent the Pauli matrices $\sigma_i$. $\prod_i^{p} \sigma_i^{\text{out}(i)}$ where $i$ runs over the set of six spins around the plaquette $p$, and where the notation out$(i)$ means the “outgoing” direction at site $i$, with respect to the plaquette’s contour. An illustration of the $W_p$ operator is given in Fig. 3.

Since $W^2 = I$, the eigenvalues of the plaquette operators are $w_p = \pm 1$. Note that $[W_p, W'_p] = 0$, as can be shown from the usual Pauli matrices algebra. As a consequence, $H$ and the $W_p$’s can be diagonalized simultaneously. Following Kitaev, we will call a vortex sector a subspace of the Hilbert space with a given map of the $w_p$’s. By definition a vortex is a plaquette for which $w_p = -1$, so that for example, the vortex-free sector is defined by $w_p = +1$ for all $p$’s.

In fact, all loop operators made of “outgoing spins” (see Figs. 4 and 5) are conserved and all commute with each other, which can be verified in the same way as for the $W_p$’s. However, not all of them can be set independently to $\pm 1$. Some relations among them arise from the following fact (which can be checked by studying all possible cases): the product of $W_p$ and a nearby loop operator $L$ gives a new loop operator $L' = W_p L$, as illustrated on a particular example in Fig. 4.

As an illustration of other relations involving loop operators around the torus, with the loops of Fig. 5 one has

$$L_a = \prod_{n=1}^{6} W_{a_n} ,$$

$$L'_b = L_b \prod_{n=1}^{8} W_{b_n} ,$$

$$L_d = -L_d L_b L_c ,$$

where we have denoted, for example, $L_a = \prod_{p' \in C} \sigma_i^{\text{out}(i)}$. The minus sign in the last equation above comes from the crossing of $L_b$ and $L_c$. In the three expressions above, the product of plaquette operators could also have been taken over the complementary set of plaquettes. Indeed, on the torus the relations among loop operators yield the following constraint

$$\prod_{p' \in \text{all}} W_{p'} = I ,$$

showing in particular that the number of vortices has to be even in a system with PBC.

From examples shown in Fig. 5, one can deduce that all $W_p$’s [except one, because of Eq. (6)], $L_b$ and $L_c$ can...
be set independently to ±1, which then imposes all other conserved quantities.

C. Some results from the exact solution

The above discussed local conserved quantities are not sufficient to fully diagonalize the Hamiltonian. Indeed, if \( N_s \) is the number of sites, then there is a total of \( N = N_s/2 \) plaquettes, but only \( N - 1 \) independent ones. With the two cycles around the torus, this gives \( N + 1 \) independent conserved quantities, which is obviously smaller than \( N_s \).

However, \( H \) has a crucial property: it can be transformed into a free Majorana fermions Hamiltonian and is thus exactly solvable. Let us also mention that another solution based on the Jordan-Wigner transformation maps the spin Hamiltonian \( H \) onto a spinless fermions system with \( p \)-wave pairing\(^{9,11,12} \).

As shown by Kitaev\(^1\), the ground state of \( H \) lies in the vortex-free sector and the phase diagram contains, in this sector, two phases: a gapped phase for \( J_z > J_x + J_y \) and a gapless phase for \( J_z < J_x + J_y \). In the gapped phase the low-energy excitations are Abelian anyons (semons) whereas in the gapless phase, the low-energy excitations are fermionic. The gapless phase acquires a gap in the presence of a magnetic field and then contains gapped non-Abelian anyon excitations. The phase diagram has also been investigated in other vortex configurations such as the vortex-full sector and similar phases have been obtained. More precisely, one has a gapped phase for \( J_z^2 > J_x^2 + J_y^2 \) and a gapless phase in the opposite case\(^14\).

Our goal here is to determine the low-energy spectrum for any vortex configuration. Of course, one may use the fermionic Hamiltonian mentioned above but, it can only be exactly diagonalized for translation-invariant configuration. Here, we follow an alternative route by focusing on the isolated-dimer limit \( J_z \gg J_x, J_y \).

III. MAPPING ONTO AN EFFECTIVE SPIN BOSON PROBLEM

A. Mapping of the Hamiltonian

The very first step of our analysis consists in mapping the four possible states of the two spins of a \( z \)-dimer onto those of an effective spin and a hardcore boson. More precisely, denoting \( |↑⟩ \) (\(|↓⟩\)) the eigenstate of \( \sigma_z \) with eigenvalue +1 (−1), an isolated \( z \)-dimer can be in one of the two low-energy states \( \{ |↑↑⟩, |↓↓⟩ \} \) with energy \(-J_z\), or in one of the two high-energy states \( \{ |↑↓⟩, |↓↑⟩ \} \) with energy \(+J_z\). Keeping in mind that our aim is to perform a perturbation theory in the limit \( J_z \gg J_x, J_y \), it is natural to interpret the change from a ferromagnetic to an antiferromagnetic configuration as the creation of a particle with an energy cost \( 2J_z \). By construction, such a particle is a hardcore boson. The remaining degree of freedom can be described by a spin \( 1/2 \), indicating which of the two configurations is realized. There are many possible parametrizations but here we choose the following

\[
|↑↑⟩ = |↑⟩|0⟩, |↓↓⟩ = |↓⟩|0⟩, |↑↓⟩ = |↑⟩|↓⟩, |↓↑⟩ = |↓⟩|↑⟩.
\]

The left (right) spin is the one of the black (white) site of the dimer \(|↑⟩\rangle = |↑⟩|↑⟩, |↓⟩⟩ = |↓⟩|↓⟩\), etc. Double arrows represent the state of the effective spin which is here the same as the state of the left (black) spin.

Within such a mapping, effective spins and hardcore bosons live on the effective square lattice of \( z \)-dimers (see Fig. 1). This lattice is shown again in Fig. 2b, together with the PBC, which are such that it can be bi-colored. In what follows, the sites of the effective lattice will be denoted with bold letters, such as \( i \).

Let us now write the Hamiltonian (1) in this language. Therefore, we first translate the action of the spin operators in the effective-spin boson (ESB) formalism. It is easy to check that one has

\[
\sigma_{x,i} = \tau_{x,i} (b_i^+ + b_i), \quad \sigma_{y,i} = \tau_{y,i} (b_i^+ + b_i), \quad \sigma_{z,i} = \tau_{z,i} (b_i^+ - b_i), \quad \sigma_{x,0} = \tau_{x,0} (b_1^+ + b_2^+), \quad \sigma_{y,0} = \tau_{y,0} (b_1^+ - b_2^+), \quad \sigma_{z,0} = \tau_{z,0} (1 - 2b_1^+ b_2^+).
\]

The operators \( \tau_{α,i} (α = x, y, z) \) are the Pauli matrices acting on the effective spin at site \( i \), while \( b_i^\dagger \) and \( b_i^\dagger \) are hardcore bosonic annihilation and creation operators, satisfying the usual on-site anticommutation relation \( \{b_i^\dagger, b_j\} = \mathbb{I} \) (and operators on different sites commute). Setting once for all \( J_z = 1/2 \) so that creating a boson costs an energy \( 1 \) in the isolated-dimer limit, the Hamiltonian (1) reads

\[
H = -\frac{N}{2} + Q + T_0 + T_{+2} + T_{-2},
\]

where \( N \) is the number of \( z \)-dimers (or, equivalently, of square plaquettes), and

\[
Q = \sum_i b_i^\dagger b_i, \quad T_0 = -\sum_i \left( J_x t_{i,i+n_1}^x + J_y t_{i,i+n_2}^y + \text{h.c.} \right),
\]

\[
T_{+2} = -\sum_i \left( J_x v_{i,i+n_1}^x + J_y v_{i,i+n_2}^y \right) = (T_{-2})^\dagger.
\]

These operators are built from local hopping and pair creation operators

\[
\begin{align*}
t_{i,i+n_1}^x &= b_i^\dagger b_{i+n_1}^\dagger \tau_{x,i+n_1}, & \quad t_{i,i+n_2}^z &= -ib_i^\dagger b_{i+n_2}^\dagger \tau_{x,i+n_2}^z, \\
v_{i,i+n_1}^x &= b_i^\dagger b_{i+n_1}^\dagger \tau_{x,i+n_1}, & \quad v_{i,i+n_2}^z &= ib_i^\dagger b_{i+n_2}^\dagger \tau_{x,i+n_2}^z.
\end{align*}
\]

We emphasize that the mapping (8) explicitly breaks the symmetry between white and black sites of the original brick-wall lattice. This is responsible for the apparent breaking of symmetry between the \( x/n_1 \) and \( y/n_2 \) directions in Eq. (13). However, for all the physically observable results, this symmetry remains intact (see for example the series expansion of eigenenergies in Appendix C).
Note however that the $n_1 + n_2$ and $n_1 - n_2$ directions are not equivalent, as can be seen from the underlying brick-wall lattice.

B. Conserved quantities

Let us now rephrase the conserved operators discussed in Sec. II B in the effective language. Using the notations depicted in Fig. 1b, as well as the mapping (8), the plaquette operators transform into

$$W_p = (-1)^{b_L b_L + b_D b_D} \tau_L^x \tau_U^x \tau_R^y \tau_D^y,$$ (14)

Note that $(-1)^{b_L b_L} = 1 - 2b_L b_L$. In the same vein, for the cycles around the torus shown in Figs. 5b-5c, which are reproduced for the effective lattice in Figs. 6b-6c, one has

$$L_b = \prod_{i \in C_b} (-1)^{b_i^b b_i^b} \tau_i^b \tau_{i+1}^b \tau_i^b \tau_{i+1}^b,$$

and

$$L_c = \prod_{i \in C_c} \tau_i^c.$$ (since there is an even number of sites on the contour with the PBC chosen here), as well as $L_d = \prod_{i \in C_d} \tau_i^d$. The expression for $L_d$ (see Fig. 5d), namely $L_d = \prod_{i \in C_d} \omega_i$, is a bit more complicated, but it should be clear from Fig. 6d what the $\omega_i$’s are. Finally, for the contour shown in Fig. 6a (which is in correspondence with Fig. 5a), one has

$$\prod_{i \in C_a} W_p = \prod_{i \in C_a} \omega_i = L_a$$

and with $p \subset C_a$ meaning the plaquettes $p$ enclosed in the contour $C_a$. With these notations, one can easily check that Eq. (5) still holds.

The elementary hopping and pair creation operators, namely $t_i^x$ and $v_i^x$ with $i$ and $j$ nearest neighbors, have a very remarkable property: they all commute with the $W_p$’s, as well as with any other loop operator

$$[t_i^x, W_p] = [v_i^x, W_p] = [t_i^x, \mathcal{L}] = [v_i^x, \mathcal{L}] = 0.$$ (15)

The original spin problem on the honeycomb lattice is thus mapped onto a quadratic hardcore boson problem on an effective square lattice, with conserved plaquette and loop operators. Let us underline that this mapping is exact and just provides an alternative description of the spin problem. The resulting Hamiltonian (9) remains difficult to diagonalize (except, of course, if one remembers that the model can be fermionized), since (i) bosons are hard core which prevents the use of a Bogoliubov transformation, (ii) bosonic and spin degrees of freedom are correlated. The conserved plaquette operators will of course be useful in simplifying and solving the problem as recently underlined in Ref. 15.

IV. PERTURBATION THEORY IN THE GAPPED PHASE

A. Effective Hamiltonian from PCUTs

The starting point of the present perturbation theory is the isolated-dimer limit, namely $J_x = J_y = 0$. In this limit, the spectrum is made of equidistant and degenerate levels separated by an energy gap $\Delta = 2J_z = 1$. To compute the perturbative spectrum, there are of course several methods among which the Green’s function formalism initially used by Kitaev. However, if this approach is efficient to obtain the first nontrivial (nonconstant) correction, it becomes tricky to implement at higher orders.

Here, following Ref. 13, we use an alternative approach based on continuous unitary transformations (CUTs) conjointly proposed by Wegner and Glazek and Wilson. We refer the interested reader to Ref. 37 for a recent pedagogical introduction. Its perturbative version denoted PCUTs is especially well-suited to the problem at hand. This technique is detailed in several works. Let us simply mention that the CUTs method requires the choice of a generator that drives the flow of the operators. All the results given here have been obtained with the so-called quasiparticle number-conserving generator first proposed by Mielke for finite matrices and generalized to many-body systems by Knetter and Uhrig.

The latter have computed the perturbative expansion for any Hamiltonian of the form

$$H = Q + T_{-2} + T_{-1} + T_0 + T_{+1} + T_{+2},$$ (16)

provided two hypothesis are satisfied:

- the unperturbed Hamiltonian $Q$ has an equidistant
spectrum bounded from below:

- the perturbing Hamiltonian $\sum_{n=-2}^{+2} T_n$ is such that $[Q, T_n] = n T_n$.

Clearly, the Hamiltonian (9) meets these two criteria (up to a constant term) noting that in the present case, one has $T_{\pm 1} = 0$. Here, we have included the "small" parameters, namely $J_x$ and $J_y$, in the definition of the $T_n$ operators, which is not the convention usually adopted in the CUTs community.

The CUTs method together with the quasiparticle number-conserving generator unitarily transforms the Hamiltonian (16) into an effective Hamiltonian $H_{\text{eff}} = U^\dagger H U$ commuting with $Q$, $U$ being a unitary operator. We give the first terms of the expansion up to order 4 in Appendix A. As can be seen in Table 1, the number of terms appearing in the perturbative expansion quickly increases with the order. For instance, at order 2, the effective Hamiltonian reads in our case

$$H_{\text{eff}} = -\frac{N}{2} + Q + T_0 - \frac{1}{2} T_{-2} T_{+2} + \frac{1}{2} T_{+2} T_{-2},$$

whereas at order 10, there are more than $10^4$ operators to consider.

Writing $H_{\text{eff}}$ this way is only the very first part of the job since one next has to (i) determine its action in each subspace of a given quasiparticle (QP) number $q$; (ii) diagonalize $H_{\text{eff}}$ in each of these subspaces. This is the object of the next sections: we first study the lowest-energy states ($q = 0$ QP) which is the main contribution of our work; then we turn to the $q = 1$ QP states and recover the high-energy gap from the QP dispersion; we end by $q \geq 2$ QP states, whose properties determine the statistics of the QPs, and we will see that the QPs behave as fermions which are, furthermore, non-interacting. This fact is at the origin of a tremendous simplification of the effective Hamiltonian. Indeed, we found that $H_{\text{eff}}$ can be written, at all orders and in the thermodynamical limit, as

$$H_{\text{eff}} = E_0 + \mu Q - \sum_{\{p_1,..,p_n\}} C_{p_1,..,p_n} W_{p_1},.., W_{p_n}$$

$$- \sum_{\{j_1,..,j_n\}} D_{j_1,..,j_n} S_{j_1,..,j_n} b_{j_1}^\dagger b_{j_n}.$$ 

We shall discuss each term in detail in the following sections, but let us mention that $E_0$, $\mu$, the $C$'s and the $D$'s are coefficients whose series expansion are computed. The $S$ operators are strings of spin operators $\tau_j^y$ and of phase factors $(-1)^{b_j^\dagger b_j}$ on the cluster $\{j_1,..,j_n\}$. This very special form of multi-particle terms [remember $(-1)^{b_j^\dagger b_j} = 1 - 2h_j b_j^\dagger b_j$, leading to phase factors and spin-strings only, is responsible for the emergence of fermions in the model.

For a finite-size system with PBC, new terms appear in the effective Hamiltonian. They involve loop operators around the torus, and appear at a minimal order being the linear size $2p$ of the lattice. Such loop operators are associated to contours as the ones shown in Figs. 5 and 6, namely $L_b$, $L_c$ and $L_b'$ for the contours $C_b$, $C_c$ and $C_b'$. The presence of such loop operators in the effective Hamiltonian shows that the eigenstates of the Hamiltonian are also eigenstates of these loop operators. Their effect is to lift the degeneracies between states (which for each energy is at least four in the thermodynamical limit, since some of the excitations are Abelian semions and the genus of a torus is 1, see Ref. 5). We shall not dive into the details of such finite-size corrections, since our approach allows us to directly tackle with the most interesting thermodynamical limit. However, let us make a remark about a numerical check of this statement for small system sizes. For a torus whose linear size is strictly smaller than 4, the loop operator terms around the torus dominate the expansion over the $W_p$'s, and for a size of 4 both types of terms start contributing at the same order. One should thus not be surprised to find a ground-state for $p = 1$ which is not in the vortex-free sector$^{15}$.

\section*{B. Counting of states}

Before we turn to a detailed analysis of each QP subspace, let us show that we do not miss any state using simple counting arguments. We have already seen in Sec. II, that one has $N + 2$ conserved $\mathbb{Z}_2$ quantities (two loop operators, and $N$ plaquette operators), with the constraint $\prod_{p\in \text{white}} W_p = 1$. There is in fact one more relation between the $W_p$'s, involving the number of bosons, which reads

$$\prod_{p\in \text{white}} W_p = (-1)^{\sum_i b_i^\dagger b_i} = (-1)^Q = \prod_{p\in \text{cyan (gray)}} W_p,$$

showing that the parity of the number of vortices living on white plaquettes (see Fig. 2) has to be the same as the parity of the number of bosons. The last equality simply comes from the previously mentioned constraint (6). The first equality can be checked using the expression (14) of the $W_p$'s. Indeed, for a site $i$ having a white plaquette on its left, and another one on its right, the product of the two associated $W_p$'s will give $\tau_i^y \times (-1)^{b_i^\dagger b_i} \tau_i^y = (-1)^{b_i^\dagger b_i}$. In the same way, for a site $i$ having a white plaquette above it, and another one under it, the product of the two associated $W_p$'s will give $\tau_i^x \times (-1)^{b_i^\dagger b_i} \tau_i^x = (-1)^{b_i^\dagger b_i}$. Let us note that (19) has a meaning in the two bases we are working in, the initial one and the unitarily transformed one. Indeed, in the initial basis, the Hamiltonian $H$ commutes with the parity operator $(-1)^Q$ ; in the rotated basis, $H_{\text{eff}}$ commutes with $Q$.

We thus see that in a subspace with a given number of QP’s, there are $N$ independent conserved $\mathbb{Z}_2$ quantities. Thus, $N$ being the number of effective spins $\tau_i$, there is no remaining effective spin degree of freedom once the $\mathbb{Z}_2$ quantities are chosen. As a conclusion, the $q$-QP sub-
space has dimension \( d_q = 2^N \binom{N}{q} \), with the usual notation for binomial coefficients. This shows that we miss no state, since

\[
\sum_{q=0}^{N} d_q = 2^N \sum_{q=0}^{N} \binom{N}{q} = 2^{2N} = 2^{N_c},
\]

(20)

where \( N_c \) is the total number of spins in the brick-wall lattice.

This discussion furthermore sheds light on the fact that in order to compute eigenenergies, a perturbative expansion of the Kitaev model (as opposed to exact numerics) is really of interest only in the 0-QP subspace. Indeed, we have just seen that there are \( N \) independent \( \mathbb{Z}_2 \)-conserved quantities. It is thus clear that as soon as we will have written down the effective Hamiltonian in the 0-QP subspace, the Hamiltonian will already be diagonalizable in whatever the vortex configuration, although writing down the eigenstates of the \( \mathbb{Z}_2 \) quantities in the basis of effective spin operators still has to be done. However, in the 1-QP subspace, one will have to diagonalize an \( N \times N \) matrix (numerically in the case of a nonperiodic vortex configuration), which is identical to what one has to do when solving the problem exactly as Kitaev did. For \( q \geq 2 \), the perturbative expansion looks even more complicated than the exact solution, but this is an artifact, since we recover free fermions.

V. EFFECTIVE HAMILTONIAN IN THE 0-QP SUBSPACE

A. Effective Hamiltonian and eigenenergies

In the 0-QP sector, and in the thermodynamical limit the effective Hamiltonian (18) simplifies and reads

\[
H_{\text{eff}}|_{q=0} = E_0 - \sum_{\{p_1, \ldots, p_n\}} C_{p_1, \ldots, p_n} W_{p_1} \ldots W_{p_n},
\]

(21)

where \( \{p_1, p_2, \ldots, p_n\} \) denotes a set of \( n \) plaquettes and the \( W_p \)'s are the conserved plaquette operators introduced in Sec. II. Note than when restricted to the 0-QP sector they simplify to \( W_p|_{q=0} = \tau^+_y \tau^+_z \tau^+_y \tau^+_z \) [see Eq. (14)].

As mentioned at the end of the previous section, obtaining eigenenergies only requires a minimal amount of work, namely replacing each \( W_p \) by numbers \( w_p = \pm 1 \), and doing the same with loop operators, without forgetting about the constraints among these quantities. The perturbative expansion of the coefficients \( E_0 \) and \( C_{p_1, \ldots, p_n} \) are given in Appendix C. Let us note that \( \{p_1, p_2, \ldots, p_n\} \) does not need to be a linked cluster of plaquettes (as seen for \( C_{p,p+2n_1} \) that is nonvanishing at order 10), and that translational invariance of the Hamiltonian implies that the \( C_{p_1, \ldots, p_n} \) coefficients only depend on \( n - 1 \) relative positions of the plaquettes.

The lowest nontrivial order involving the \( W_p \)'s (order 4) has been derived by Kitaev\(^7\) (\( C_p = J_p^2 J_p^2 / 2 \)) and led him to identify the effective low-energy theory with the toric code\(^6\). One of the main results of our work is to show that, at order 6 and beyond, one obtains a multiplaquette expansion in the effective low-energy Hamiltonian. In other words vortices interact, though they remain static as they have to since the \( W_p \)'s are conserved. The interaction energies between vortices are not directly the \( C \) coefficients. One should write \( w_p = 1 - 2n_p \) where \( n_p \) is the number of vortices at plaquette \( p \) (0 or 1), then look at coefficients in the expansion in terms of the \( n_p \)'s. The results of such an analysis for two-vortex interaction energies in the case \( J_x = J_y = J \) are illustrated in Fig. 7 which shows that the interaction \( \langle ij \rangle \) lowers the energy and is therefore attractive, \( \langle ii \rangle \) is anisotropic even for \( J_x = J_y = J \) which is clear from the structure of the underlying brick-wall lattice, \( \langle iii \rangle \) decreases with the distance \( d \) between vortices as expected in a gapped system. Note that for a finite-size system with PBC, the two-vortex configurations with a central vortex and an-
other vortex at one of 1, 5, 6 and 7 sites (see Fig. 7) are forbidden since they violate the constraint (19). A one-vortex configuration is also forbidden since it violates the constraint (6). These configurations would be allowed in an infinite system, or in a finite system with open boundary conditions.

A most remarkable point which emerges from the analysis of $H_{\text{eff} | \tau = 0}$ is that its eigenstates are those of the $W_p$'s. They are thus the same at any order ($\geq 4$), and are those of the toric code\(^6\), although their eigenenergy changes with the perturbation order (we emphasize we are talking about eigenstates of $H_{\text{eff} | \tau = 0}$, not of the original Hamiltonian $H$). We graphically sketch the construction of these eigenstates in the next subsection, which will also prove to be useful for the ($q \geq 1$) QP sectors, and show explicitly that they obey anyonic, more precisely semionic, statistics. Our discussion of the toric code focuses on peculiarities related to our way of studying the problem, that is not restricted to the 0-QP subspace. For more details about the toric code model, we refer the interested reader to Refs. 5,6,7.

B. The Toric Code in a nutshell

1. Mapping to the toric code

As we have seen in the previous sections, the eigenstates of the effective Hamiltonian in the 0-QP subspace are the eigenstates of the $W_p$'s and of the $L$'s. We recall that in this subspace, the plaquette operators read $W_p | q = 0 = \tau_u^2 \tau_v^2 \tau_w^2 \tau_t^2$ (see Eq. (14) and Fig. 1b). A similar simplification occurs for the $L$'s. As mentioned by Kitaev\(^7\) (for the Hamiltonian at order 4), the effective Hamiltonian could be studied directly, but it is much easier to visualize the eigenstates by performing some spin rotations, and bring the Hamiltonian to the one of the toric code (generalized by multi-vortex terms). Thanks to the special PBC we have chosen, the lattice sites can be bi-colored in black and white as illustrated in Fig. 8. Then, one performs a different rotation on the two kinds of sites

\[
\begin{align*}
\tau_x^x &= s^y, & \tau_y^y &= s^z, & \tau_z^z &= s^x, \\
\tau_o^x &= -s^y, & \tau_o^y &= s^x, & \tau_o^z &= s^z.
\end{align*}
\]

This way, a cyan (gray) (resp. white) plaquette such as $m$ (resp. $e$) in Fig. 8a transforms into a plaquette (star) term $B_m = (-1)^{b_i^1 b_i^1 + b_i^1 b_i^0} s_i^x s_i^y s_i^z s_o^z$ (resp. $A_e = (-1)^{b_i^1 b_i^1 + b_i^1 b_i^0} s_i^x s_i^y s_i^z s_o^z$), as shown with thick (red) lines in Fig. 8b. We have kept track of the phases involving boson numbers because our construction will be needed for ($q \geq 1$)-subspaces, but it is clear that they can be dropped in the 0-QP subspace. Let us mention that the distinction between plaquette and star terms is purely conventional. The letters $m$ and $e$ refer to the magnetic and electric vocabulary also used by Kitaev\(^7\), although we emphasize there is absolutely no difference between an $A$ and a $B$ operator, which are both disguised $W$ operators. Up to an additive constant term, the effective Hamiltonian in the 0-QP subspace, and at order 4 finally reads (with $J_{\text{eff}} = J_2^2 J_y^2 / 2$)

\[
H_{\text{eff} | q = 0} = - J_{\text{eff}} \left( \sum_e A_e + \sum_m B_m \right).
\]

We work with this lowest (nontrivial) order Hamiltonian, because the eigenstates of $H_{\text{eff} | q = 0}$ remain the same whatever the order in perturbation. One should simply remember that the eigenstates of $H$ also have to be eigenstates of $L$ operators. If the PBC are not of the type we use (see Fig. 2), the sites can usually not be bi-colored and the rotations (22) cannot be performed, which makes the construction much more complicated, and we shall refer the interested readers to Ref. 41.

2. Construction of the ground state(s)

As a warming up, let us construct a ground-state of $H_{\text{eff} | 0}$, i.e., an eigenstate of all $B_m$ and $A_e$ operators...
with eigenvalue 1 (there are actually four of these). An eigenstate of all $B_m$’s is for example the “reference” state $|\uparrow\rangle$ where all spins $s$ point in the $+z$-direction, such that for all $i$ one has $s_i|\uparrow\rangle = |\uparrow\rangle$. This state is not an eigenstate of the $A_e$ operators yet, but a simple projection yields the desired state

$$2^{N/4-1/2} \prod_e \left( \frac{I + A_e}{2} \right) |\uparrow\rangle \otimes |0\rangle_b,$$  

(24)

whose normalization follows from the number $N/2$ of $e$’s and the property $\prod_e A_e = \hat{1}$ [see Eq. (19)]. The state $|0\rangle_b$ indicates that there is no quasiparticle, i.e., no hardcore boson. A graphical interpretation can be given of the state (24): it is an equal-weight superposition of multi-loop configurations produced by the $A_e$ operators, as the ones shown in Fig. 9.

One next has to get an eigenstate of two independent loop operators, which we choose to be $L_b$ and $L_c$ (see Figs. 5 and 6) and which, from now on, will be denoted $L^x$ and $L^y$, with eigenvalues $I^x$ and $I^y$. The expressions of these operators in the $s$-spin language are given in Fig. 10.

Note that in the 0-QP subspace, one could also have used other conserved loop operators which are products of $s^x$ or of $s^y$ on the contours defining $L^x$ and $L^y$. Such operators resemble more the ones used by Kitaev, but they are conserved only in the 0-QP subspace (in contrast to $L^x$ and $L^y$), and so will not prove to be very useful in the following. As can be seen in Fig. 10, $L^x$ and $L^y$ perform spin flips, with respect to $|\uparrow\rangle$ on their associated contours. The four ground-states of (23) are then obtained with another projection and proper normalization

$$\begin{align*}
\{|w_p = 1, I^x, I^y\rangle_0 = 2^{N/4 + 1/2} & \times \left( \frac{I + I^x L^x}{2} \right) \left( \frac{I + I^y L^y}{2} \right) \prod_e \left( \frac{I + A_e}{2} \right) |\uparrow\rangle \otimes |0\rangle_b.
\end{align*}$$

(25)

These four states are equal-weight (in absolute value) superposition of all possible multi-loop configurations, produced by the $A_e$ operators as in Fig. 9, as well as the $L^x$ and $L^y$ operators, as illustrated in Fig. 11 for $L^x$.

Let us note that the preceding construction relies on the $|\uparrow\rangle$ state and the fact that it is an eigenstate of the $B_m$’s, etc. However, one could also have started with a state $|\Rightarrow\rangle$ where all spins point in the $x$-direction, which is an eigenstate of the $A_e$’s and then follow a similar route.

3. Construction of excited states

We now have to see how to construct excited states, i.e., states containing vortices ($e$ or $m$) but still no quasiparticle.

Constructing a state with some $B_m$’s being minus one (“magnetic vortices”) is easy, once one has noticed that $s_i^x$ anticommutes with two $B_m$’s, and thus changes their values to their opposite. Since $s_i^x$ commutes with all $A_e$’s, as well as with $L^x$ and $L^y$ when $i$ does not belong to the corresponding contours, $s_i^x \{|w_p = 1, I^x, I^y\rangle_0$ is an eigenstate of the effective Hamiltonian, with two vortices living on the plaquettes touching the bond to which $i$ belongs. (Note that since $s_i^x$ anticommutes with $L^x$ and $L^y$ when $i$ belongs to the corresponding contours, one should use a string of $s_i^x$ going around the torus without crossing $L^x$ and $L^y$ instead of $s_i^x$). The corresponding state is again an equal-weight (in absolute value) superposition of states, but now with all possible open strings joining the created vortices, as well as all possible closed loops. This is illustrated in Fig. 12.

Creating “electric” vortices is as easy, since one can replace $\prod_e (\frac{I + a_e}{2})$ in Eq. (25) by $\prod_e (\frac{I + a_e - a_e}{2})$, with $a_e = \pm 1$, respecting the constraint $\prod_e a_e = \hat{1}$ [see Eq. (19)]. Such a change can also be obtained via the action of $s_i^z$ operators. Indeed, each $s_i^z$ operator anticommutes with two $A_e$’s, and thus changes their values to their opposite. The fluctuation of the strings induced by $s_i^z$ operators is however hard to see with the construction we have given, which relies on the reference state $|\uparrow\rangle$. To see this, one should construct states from the reference state $|\Rightarrow\rangle$ where all spins point in the $x$ direction, and then use projectors involving $B_m$’s instead of $A_e$’s. This is
not useful for our purpose so we let the interested reader doing it on his own.

4. The statistics of vortices

For completeness, let us now show that "magnetic" and "electric" vortices behave as semions with respect to each other. This is done by first creating a pair of "magnetic" vortices, then a pair of "electric" vortices, and finally by moving one of the "magnetic" vortices around one of the "electric" vortices as shown in Fig. 13 (one could also do the contrary, but then one should work with the reference state \(| \Rightarrow \rangle\) to see things more easily). With the notations of this figure (see also its caption), let us consider the state \(| \psi \rangle = ZX|\{w_p = 1\}, {l^x, l^y}\rangle_0\) with two e and m vortices. Then the repeated application of spin-flips along the loop \(X'\) (in any direction) moves the-downmost m vortex around the leftmost e vortex. The resulting state is \(X'|\psi\rangle\). But as \(Z\) and \(X'\) have one (and only one) common site, they anticommute, whereas \(X\) and \(X'\) commute, so that \(X'|\psi\rangle = -ZX X'|\{w_p = 1\}, {l^x, l^y}\rangle_0\). Now, \(X'\) which is a product of \(s^z_i\) operators forming a closed loop is nothing but a product of \(A_i\)'s operators (the ones enclosed in the loop). As \(|\{w_p = 1\}, {l^x, l^y}\rangle_0\) is an eigenstate of the \(A_i\)'s with eigenvalue one, we finally obtain that \(X'|\psi\rangle = -|\psi\rangle\) : braiding a magnetic vortex around an electric vortex yields a nontrivial phase of \(\pi\) \((-1 = e^{i\pi}\)) which proves the semionic statistics.

Let us mention that the magnetic vortices behave as bosons among themselves, and so do the electric vortices. This is easily seen by noticing that creating and moving \(m\) vortices for example, only requires \(s^x\) operators, which all commute with one another. To end this discussion about the statistics of vortices, let us also remark that a compound object made of an electric and a magnetic vortex is a fermion (see Ref. 5).

VI. EFFECTIVE HAMILTONIAN IN THE 1-QP SUBSPACE

A. Form of the Hamiltonian

The spectrum we obtained in the 0-QP subspace gives the lowest eigenenergies for each configuration of the \(W_p\)'s. In this section, we explain how to compute the high-energy spectrum for states with one quasiparticle, for each \(W_p\)'s configuration, and how to build the associated eigenstates. This is achieved by diagonalizing \(H_{\text{eff}}\) in the 1-QP subspace (whose dimension is \(d_1 = N^2\)), see the end of Sec. IV). In this subspace, the effective Hamiltonian (18) reads

\[
H_{\text{eff}}|q=1 = E_0 + \mu - \sum_{\{p_1, \ldots, p_n\}} C_{p_1, \ldots, p_n} W_{p_1} \cdots W_{p_n} - \sum_{\{j_1, \ldots, j_n\}} D_{j_1, \ldots, j_n} S_{j_1, \ldots, j_n} b^\dagger_{j_1} b_{j_1},
\]

where the second sum is performed over all non self-retracing paths of length \(n\) starting at \(j_1\) and ending at site \(j_n\), with possibly \(j_n = j_1\) when working at order 4 or higher. This is the reason why we give the expansion up to this order but we would like to emphasize that obtaining orders up to 10 for \(H_{\text{eff}}|q=1\) is of the same complexity as for \(H_{\text{eff}}|q=0\). Self-retracing paths are renormalizing the chemical potential \(\mu\). Note that a hopping process of one quasiparticle around a loop is nothing but the product of the \(W_p\)'s enclosed in the loop, as can be easily checked. This explains why at order 4, one obtains some terms proportional to \(b^\dagger_{j_1} b_{j_1} W_p\), where the plaquette \(p\) shares site \(i\) (see Appendix D).

From now on \((q \geq 1)\) the phase factors appearing in the \(W_p\)'s [see Eq. (14)] must be taken into account. The
operators $S$ have a structure similar to that of the $W_p$'s, except that they are open string operators. They involve $\tau^0_j$ as well as phase factors $(-1)^{b_{ij}b_j}$ as follows

$$S_{j_1, \ldots, j_n} = \varphi_{j_1, \ldots, j_n} T_{j_{n-1}}^{j_1} \cdots T_{j_1}^{j_2},$$

where the $\varphi_{j_1, \ldots, j_n}$ are phase factors which reduce to the identity in the 1-QP subspace, and will be discussed later on (see Sec. VII). The two-site $T_i^j$ operators are built from the same $\tau^0_j$ operators as the hoppings $t_i^j$, namely

$$T_i^{i+n_1} = \tau^x_{i+n_1} = (T_i^{i+n_1})^\dagger, \quad (28)$$

$$T_i^{i+n_2} = -i\tau^y_{i+n_2}\tau^z_i = (T_i^{i+n_2})^\dagger. \quad (29)$$

Note that in the 1-QP subspace, one can also write the hopping term of the Hamiltonian as

$$S_{j_1, \ldots, j_n} b_j^\dagger b_{j_1} |_{q=1} = t_{j_{n-1}}^{j_2} \cdots t_{j_1}^{j_2}. \quad (30)$$

**B. Construction of a 1-QP basis**

As can be seen when looking at the form of the hopping operators, bosonic and spin degrees of freedom are coupled so that one has to tackle a polaron-like problem. However, we shall now show that since all hopping operators $t_i^j$ commute with all $W_p$'s as well as with all loop operators $L$, the 1-QP problem is equivalent to that of one particle hopping in a static magnetic field.

As a first step, we build a basis of the 1-QP subspace. We denote by $|\{w_p\}, L^x, L^y\rangle_0$ a state of the 0-QP subspace, which is an eigenstate of the $W_p$'s and of $L^x$ and $L^y$, and built as explained in Sec. V B. We choose as the origin $O$ the site we have already denoted with a large (magenta) filled circle (see Figs. 2 and 6, as well as Fig. 14). Let us then consider the state $|\{w_p\}, L^x, L^y; O\rangle_1 = b_O^\dagger|\{w_p\}, L^x, L^y\rangle_0$ belonging to the 1-QP subspace, and with one QP at the origin. From formula (14), it is clear that adding a particle at the origin changes the value of two plaquettes, as illustrated in Fig. 14a for the action of $b_O^\dagger$ on the ground state, which is the reason why we made a distinction between $w_p$ and $w'_p$. Note that all this is perfectly consistent with Eq. (19), as well as with the conclusion of Ref. 42. Indeed, in this paper, Levin and Wen showed that fermions are always created in pairs, and this is the case here since a bound object of an "electric" vortex and a "magnetic" vortex is a fermion (see Sec. V B) and our quasiparticles will turn out to be fermions (see Sec. VII).

Other states $|\{w_p\}, L^x, L^y; \hat{i}\rangle_1$ with a particle at site $\hat{i}$ are obtained by applying an operator $S_{O, \ldots, j_1} b_O^\dagger b_j |\{w_p\}, L^x, L^y\rangle_0$ in order to make the particle hop, without affecting the conserved $Z_2$ quantities. Note that

$$S_{O, \ldots, j_1} b_O^\dagger b_j |\{w_p\}, L^x, L^y; O\rangle_1 = S_{O, \ldots, j_1} b_O^\dagger |\{w_p\}, L^x, L^y\rangle_0. \quad (31)$$

**C. Hamiltonian in the 1-QP basis**

Let us now consider the effective Hamiltonian at order 1, for which the hopping part is nothing but $T_0$, and study its action on a state $|\{w_p\}, L^x, L^y; \hat{i}\rangle_1$. From the way the states have been built, it is obvious that $t_{\hat{i}+n_2}^{\hat{i}+n_2}|\{w_p\}, L^x, L^y; \hat{i}\rangle_1 = |\{w_p\}, L^x, L^y; \hat{i} + \mathbf{n}_2\rangle_1$, and for the same reason and the fact that $T_{\hat{i}}$ is unitary, $t_{\hat{i}+n_2}^{\hat{i}+n_2}|\{w_p\}, L^x, L^y; \hat{i}\rangle_1 = |\{w_p\}, L^x, L^y; \hat{i} - \mathbf{n}_2\rangle_1$. We then turn to the hopping term $t_{\hat{i}+n_1}^{\hat{i}+n_1}$ and study its action on the state $|\{w_p\}, L^x, L^y; \hat{i}\rangle_1$. In other words, we wish to compute the matrix element

$$A_{\hat{i}+n_1}^{\hat{i}+n_1} = 1|\{w_p\}, L^x, L^y; \hat{i} + \mathbf{n}_1|_{\hat{i}+n_1}^i |\{w_p\}, L^x, L^y; \hat{i}\rangle_1. \quad (32)$$

All needed states are represented in Fig. 15: $|\{w_p\}, L^x, L^y; \hat{i}\rangle_1$ in (a), $t_{\hat{i}+n_1}^{\hat{i}+n_1}|\{w_p\}, L^x, L^y; \hat{i}\rangle_1$ in (b) and $|\{w_p\}, L^x, L^y; \hat{i} + \mathbf{n}_1\rangle_1$ in (c). Using the notations of Fig. 15 ($S_\alpha$ is the oriented product of spin operators $T_i^j$ on the contour shown in (a), starting at the origin, the same for $S_0$ and $S_\tau$, but for $S_\delta$ the product starts and ends at the particle’s position), it is easy to see that $S_{\delta}^{\hat{i}+n_1} |\{w_p\}, L^x, L^y; \hat{i}\rangle_1 = |\{w_p\}, L^x, L^y; \hat{i} + \mathbf{n}_1\rangle_1$. Then,
where the PBC play a role.

For some hopping processes, the matrix element not involving a product of plaquettes in between these two loop operators.

Furthermore a calculation on Pauli matrices shows that the action of $S_d$ on the state $|\{w_p\},l^x,l^y;i+n_1\rangle_1$ is the same as the product of the plaquette operators encircled by the closed contour of $S_d$, which on the example of Fig. 15 reads $W_{p_1}W_{p_2}W_{p_3}$. We finally obtain

$$A^{i+n_1}_i = \prod_{p \in \mathbb{Z}_{2}} w_p, \tag{34}$$

where the product has to be taken over all encircled plaquettes $\mathcal{E}_{i,i+n_1}$ as illustrated on a particular example in Fig. 15. The case of a hopping in the $-n_1$ direction can of course be deduced from the above matrix elements by hermitian conjugation.

For some hopping processes, the matrix element not only involves a product of $w_p$’s but also a loop operator around the torus. This is illustrated in Fig. 16 for a hopping in the $n_2$ direction, starting from site $i = i_x n_1 + (2p-1)n_2$ ($i_x = 2$ and $p = 2$ in the figure). In this case, one has $S_d = \mathcal{L}_{i_x}^{n_2}$ (i. e., the loop operator in the $y$-direction, around the torus, going through the sites $i = i_x n_1 + p n_2$ where $p$ takes all possible values), so that

$$A^{i+n_2}_i = 1|\{w_p\},l^x,l^y,i\rangle \langle i^{i+n_2} \{w_p\},l^x,l^y,i|_1 = l^y. \tag{35}$$

As explained in Secs. II and III the value of $l^y_{i_x}$ is determined from the one of $l^y = l^y_1$ and from the value of the plaquettes in between these two loop operators.

All the above examples lead to the following conclusion. The matrix elements of the effective Hamiltonian at lowest order, in the 1-QP subspace, are the same as what one would obtain for a particle with hopping amplitudes $-J_x$ and $-J_y$ in the $n_1$ and $n_2$ direction of the square lattice, in a magnetic field whose (reduced) fluxes in plaquettes or cycles around the torus are $\Phi/\Phi_0 = 0$ or $\Phi/\Phi_0 = 1/2$ (where $\Phi_0$ is the flux quantum). This comes from the fact that, for example, hopping around a plaquette $p$ gives a phase factor 1 for the two hoppings in the $\pm n_2$ directions, and an overall $w_p$ for the hoppings in the $\pm n_1$ directions. The overall contribution is then $w_p$, which takes value $w_p = \exp[2\pi i \Phi/\Phi_0]$. This analysis can be extended to the case of hoppings of the kind represented in Fig. 16 where the PBC play a role.

When tackling higher-order corrections, hoppings become longer-ranged as seen in Eq. (26), but the above considerations still apply because of Eq. (30). It is then easy to compute the 1-QP spectrum for a given map of the $\mathbb{Z}_2$ conserved quantities. As already explained, when the map does not possess translational invariance, one can only compute the spectrum numerically. When the $w^p$’s are translationally invariant, an analytic solution is available, and for example, in the vortex-free subspace, the dispersion relation obtained at order 2 (see Appendix D) is

$$E^{\text{free}}(k_x,k_y) = 1 - 2|J_x \cos(k_x) + J_y \cos(k_y)| \tag{36} + 2[J_x \sin(k_x) + J_y \sin(k_y)]^2,$$
where the wave vector \((k_x, k_y)\) belongs to \([-\pi, \pi] \times [-\pi, \pi]\). The gap in this sector is then obtained by minimizing \(E_{\text{free}}\) which yields \(\Delta_{\text{free}} = 1 - 2(J_x + J_y)\). Note that, in this case, the perturbative result at order 1 coincides with the nonperturbative result obtained by Kitaev\(^\dagger\) (see also Sec. VIII) and one recovers the transition point at \(J_x + J_y = 1/2 = J_z\).

Results for other sectors (vortex-full, or one vortex every two plaquettes) can also be obtained. We mainly used them to check the validity of the coefficients we computed perturbatively, as explained in Sec. VIII.

As a final remark about the 1-QP subspace, let us mention a difference with what is obtained when using exact fermionization methods. With these methods, the low-energy subspace already contains many fermions, and one then considers fermionic excitations on top of this complicated vacuum to reach high-energy states. In our approach, the low-energy states are really empty of fermions, and the excitations are only made of one particle, which can thus be qualified of Landau quasiparticle.

VII. EFFECTIVE HAMILTONIAN IN THE \((q \geq 2)\)-QP SUBSPACE

Let us now turn to multi-particle states with the aim of showing how the Fermi statistics can be recovered from hardcore bosons with a string of spin and phase operators. We shall not give many details here, since our approach becomes cumbersome when studying multi-particle states, and because one knows from exact solutions that one has to recover free fermions.

A. Phase factors

To obtain Fermi statistics, the phase factors appearing in the string operators \(S\) [see Eq. (18)] are of utmost importance. These factors do neither appear at the end sites of the string operators \(S\) [so in particular not at all for nearest-neighbor hoppings which are simply the \(t^z_2\) operators of Eq. (13)], nor at points of turning back, but only at intermediate sites between two truly different other sites. The six possibilities are shown in Fig. 17, where the phase factors occur only for the topmost hoppings [phase factors for hoppings not represented in the figure can be inferred from hermitian conjugation and Eq. (28)]. In the figure, oriented thick (cyan) lines represent the string operators \(S\), and the sites \(j_2\) marked with a large dot are the ones involving a phase factor \((-1)^{b_j} b_{j_2}^\dagger b_{j_2}\). As an example, the string operator associated to a three-site hopping as the one shown top left in Fig. 17 reads

\[
S_{j,j+n_1,j+2n_1} = (-1)^{b_j} b_{j_2} b_{j_2}^\dagger T_{j+n_1, j} T^{j+2n_1, j} T^{j+n_1, j}. \tag{37}
\]

In fact, in \(S_{j,j,j_2,j_3,...}\), a phase factor appears at the intermediate site \(j_2\) if \(T_{j_2}^2\) and \(T_{j_2}^{3}\) commute, and does not appear if they anticommute.

B. Fermionic creation operators

Rigorously, it is impossible to introduce creation/annihilation operators for single fermions, because fermions should always be created/annihilated in pairs. In fact, after choosing a site \(O\) as an origin, and after choosing a reference path from site \(O\) to site \(i\) (as was done in Sec. VI), the operator (running on this reference path) \(c_i^\dagger \sim S_{O,...,i} b_i^\dagger\) can be considered as a fermionic creation operator at site \(i\), once the origin \(O\) has been sent to infinity (using the same trick as when constructing a Dirac monopole in electrodynamics). It should be clear from arguments similar to those of Sec. VI that such an operator creates a high-energy (spinless) fermion at site \(i\), but also creates (or destroys if there is already one) one low-energy fermion made of two vortices, top and right of site \(O\), as in Fig. 14. It however commutes with all other \(W_p\) operators, except with these two.

The fermionic anticommutation relations between fermion operators at sites \(i\) and \(j\) can be checked by exhausting all possible crossings of two reference paths \(O,...,i\) and \(O,...,j\).

C. Multi-particle basis and effective Hamiltonian

From there on, one can construct a multi-particle basis of the Fock-space, as was done for the one-particle basis, by successively creating fermions at some sites (after having decided for an ordering of these sites).

It can then be shown, as was done in the 1-QP subspace, that the Hamiltonian is nothing but a hopping Hamiltonian of free fermions in a magnetic field, whose flux per plaquette is zero or half the flux quantum \((w_p = \pm 1)\). The phase factors, apart from ensuring...
and whose correspond-

FIG. 18: (color online). Illustration of the exchange of two particles discussed in the text, for \( j = i + n_2, k = i + n_1 \) and \( l = i - n_2 \).

proper Fermi statistics, also yield the correct expressions for the \( W_p \)'s or product of \( W_p \)'s, which involve both \( \tau \)'s and phase factors, and which appear for hoppings around closed paths.

D. An alternative picture

As was suggested by Levin and Wen in Ref. 42, the statistics of the effective quasiparticles can be probed with a simple argument. It relies on exchanging two of these quasiparticles by using hoppings from the Hamiltonian only, and doing so in such a way that a hopping on a bond between two sites as occurred exactly once in each direction, in order to capture phases coming from the statistics only (and not, e.g., from a magnetic Aharonov-Bohm-like phase).

Let us thus consider the exchange process of two particles initially sitting at sites \( j \) and \( l \) (no other particle is present), as depicted in Fig. 18 and whose corresponding operator sequence is (using only hopping operators arising at lowest order)

\[
\tau_{i_l}^j \tau_{i_j}^k \tau_{i_j}^l \tau_{i_l}^k = -1,
\]

or, equivalently, \( \tau_{i_l}^j \tau_{i_j}^k = -\tau_{i_l}^k \tau_{i_j}^j \). The sign in the latter identity confirms that the quasiparticles made of a hardcore boson and an effective spin-1/2 obey fermionic statistics.

VIII. SIMPLE CHECKS FROM SIMPLE VORTEX CONFIGURATIONS

As shown by Kitaev\(^7\), the spectrum of the Hamiltonian (1) can be computed exactly by mapping the spin system onto free Majorana fermions. The main drawback of this mapping is that one has first to work in a fixed vortex-sector and, in a second step, perform the symmetrization procedure involving all equivalent gauge sectors. An alternative route\(^9,11,12\) consists in using the Jordan-Wigner transformation which maps the problem onto free spinless fermions with \( p \)-wave pairing. However, in both approaches and as is often the case, only periodic configurations allow one to obtain analytical expressions of the spectrum. In the following, we use Kitaev’s approach (Majorana fermions) to compute the spectrum in several simple periodic configurations characterized by a filling factor \( \nu \).

Actually, diagonalizing the Majorana fermion Hamiltonian on this honeycomb lattice\(^7\) is completely equivalent to analyzing the problem of a free particle on this lattice in a transverse magnetic field\(^43\) with a flux per plaquette which can take only two values corresponding to \( w_p = \pm 1 \) (see Appendix F for details). The ground-state energy is then obtained by filling all levels with negative energy which amounts, in a bipartite lattice for which the spectrum is symmetric, to consider half-filling.

For the three cases considered here, we compute the exact spectrum (still assuming \( J_z \geq J_x, J_y \geq 0 \)). Then, we perform the perturbative expansion of the ground-state energy up to order 10. This provides some simple checks of the results given in Sections V.

A. Vortex-free configuration \( \nu = 0 \)

This configuration defined by \( w_p = +1 \) for all \( p \)'s is of special interest since, in the thermodynamical limit, the ground state of \( H \) lies in this sector. This is a direct consequence of Lieb’s theorem for flux phases\(^44\). The spectrum, in this sector, is simply obtained since it is equivalent to compute the spectrum of a free particle in zero field. The system being periodic with 2 sites per unit cell (see Fig. 1), the single-particle spectrum consists of two bands given by the roots of the following characteristic polynomial

\[
P^{\nu=0}(\varepsilon) = \varepsilon^2 - f(\mathbf{q})^2,
\]

where for all \( \mathbf{q} \) in the reciprocal lattice,

\[
f(\mathbf{q})^2 = 4 \left\{ J_x^2 + J_y^2 + J_z^2 + 2 \left[ J_x J_y \cos (\mathbf{q} \cdot \mathbf{n}_1 - \mathbf{n}_2) + J_y J_z \cos (\mathbf{q} \cdot \mathbf{n}_2) + J_x J_z \cos (\mathbf{q} \cdot \mathbf{n}_1) \right] \right\}.
\]

The ground-state energy per plaquette is thus given, in the thermodynamical limit, by

\[
e^{\nu=0} = -\frac{1}{8\pi^2} \int_0^\pi dq_x \int_0^\pi dq_y \left| f(\mathbf{q}) \right|.
\]

As already found by Kitaev, at the isotropic point \( J_x = J_y = J_z = 1 \), one has \( e^{\nu=0} \simeq -1.5746 \).

The gap is given by the minimum, in modulus, of \( P^{\nu=0} \)'s roots, i.e., \( \min_\mathbf{q} | f(\mathbf{q}) | \). Thus, one obtains

\[
\Delta^{\nu=0} = 2(J_z - J_x - J_y).
\]
Setting $J_z = 1/2$, and considering the perturbative limit $J_z \gg J_x, J_y$, one obtains the following expansion for the ground-state energy at order 10

$$e_0^\nu=0 = -1 \frac{1}{2} - J^2 - \frac{3 J^4}{4} - \frac{5 J^6}{2} - \frac{875 J^8}{64} - \frac{3087 J^{10}}{32}.$$ (43)

For simplicity, we have set here $J_x = J_y = J$. This result can be easily recovered by setting $w_p = +1$ for all $p$'s in Eq. (21) and using the coefficients given in Appendix C.

One can also check directly the one-particle spectrum by expanding $f(q)$ in the same limit and by comparing it with the one-particle spectrum in the vortex-free sector obtained in Sec. VI.

B. Vortex-full configuration $\nu = 1$

The vortex-full sector is defined by $w_p = -1$ for all $p$'s. In the “particle in a field” language, this problem corresponds to a magnetic flux per plaquette which is half a flux quantum. With the gauge choice shown in Fig. 19, the system is periodic with 4 sites per unit cell. The single-particle spectrum thus consists of four bands given by the roots of the characteristic polynomial

$$P^{\nu=1}(\varepsilon) = \varepsilon^4 - 8 \varepsilon^2 (J_x^2 + J_y^2 + J_z^2) + 16 g(q)^2,$$ (44)

where for all $q$ in the reciprocal lattice,

$$g(q)^2 = J_x^4 + J_y^4 + J_z^4 - 2 \left\{ J_x^2 J_y^2 \cos(2q \cdot n_1) + J_x^2 J_z^2 \cos(q \cdot (n_1 - n_2)) - J_y^2 J_z^2 \cos(q \cdot (n_1 + n_2)) \right\}.$$

The vectors $n_1 = (1, 0)$ and $n_2 = (0, 1)$ are defined in Fig. 19. The ground-state energy per plaquette is given, in the thermodynamical limit, by

$$e_0^\nu = -\frac{\sqrt{2}}{8\pi^2} \int_{-\pi}^{\pi} dq_x \int_{-\pi}^{\pi} dq_y \sqrt{J_x^2 + J_y^2 + J_z^2} + |g(q)|,$$ (46)

Once again, for $J_x = J_y = J_z = 1$, this expression gives $e_0^{\nu=1} \approx -1.5077$ in agreement with Kitaev’s results.

The gap is again given by the minimum, in modulus, of $P^{\nu=1}$’s roots

$$\Delta^{\nu=1} = 2 \left( J_z - \sqrt{J_x^2 + J_y^2} \right).$$ (47)

in agreement with results given in Ref. 8.

As for $\nu = 0$, setting $J_z = 1/2$, and considering the perturbative limit $J_z \gg J_x, J_y$, one obtains the following expansion for the ground-state energy at order 10

$$e_0^\nu=1 = -1 \frac{1}{2} - J^2 + \frac{3 J^4}{4} - \frac{3 J^6}{2} + \frac{149 J^8}{64} - \frac{547 J^{10}}{32}.$$ (48)

For simplicity, we have also set here $J_x = J_y = J$. This result can be easily recovered by setting $w_p = -1$ for all $p$ in Eq. (21) and using the coefficients given in Appendix C.

C. Vortex-half configuration $\nu = 1/2$

Let us now consider the vortex-half configuration shown in Fig. 20 which is made of alternating vortex-free and vortex-full rows. With the gauge choice shown in this figure, the system is periodic with 8 sites per unit cell. The 8 bands of the single-particle spectrum are given from the roots of the following characteristic polynomial

$$P^{\nu=1/2}(\varepsilon) = \varepsilon^8 - 16 \varepsilon^6 (J_x^2 + J_y^2 + J_z^2) + 32 \varepsilon^4 \left\{ 3 (J_x^2 + J_y^2 + J_z^2) + 4 (J_x^2 J_y^2 + J_x^2 J_z^2 + J_y^2 J_z^2) - 2 J_x^2 J_y^2 \cos(q \cdot n_1) \right\}$$

$$-256 \varepsilon^2 \left\{ J_x^6 + J_y^6 + J_z^6 + (J_x^2 + J_y^2)(J_y^2 + J_z^2)(J_z^2 + J_x^2) - 2 J_x^2 J_y^2 (J_y^2 + J_z^2) \cos(q \cdot n_1) \right\}$$

$$+256 \left\{ J_x^4 J_y^4 + J_x^4 J_z^4 + J_y^4 J_z^4 + 4 J_x^2 J_y^2 J_z^2 \cos(q \cdot n_1) + 2 J_x^2 J_y^2 \cos(2q \cdot n_1) - J_y^2 J_z^2 \cos(2q \cdot n_2) \right\},$$

FIG. 19: (color online). A possible gauge choice realizing the vortex-full lattice $\nu = 1$. The thin (bold) links are associated to $u_{jk} = +1 (u_{jk} = -1)$ where $j$ belongs to the black sublattice and $k$ to the white one (see Appendix F). The eigenvalue of the plaquette operator is then simply given by $w_p = \prod_{(j,k) \in p} u_{jk}$.
The vectors $\mathbf{n}_1 = (1, 0)$ and $\mathbf{n}_2 = (0, 1)$ are defined in Fig. 20. Note that since the hexagonal lattice is bipartite, the single-particle spectrum is even and, consequently, all characteristic polynomials are functions of $\varepsilon^2$. Thus, even in this vortex-half configuration, one can get analytical expressions for the 8 bands since, practically, one only has to find the roots of a fourth-order polynomial.

At the isotropic point, one obtains the ground-state energy per plaquette $e_0^{\nu=1/2} \simeq -1.5227$. The gap is given by the minimum, in modulus, of $P^{\nu=1/2}$'s roots

$$\Delta^{\nu=1/2} = 2 \left( J_z - \sqrt{J_x^2 + J_y^2} \right). \quad (49)$$

It is worth noting that the gap, in this sector is exactly the same as the one in the vortex-full sector $\Delta^{\nu=1}$ [see Eq. (47)].

Expanding the negative roots of $P^{\nu=1/2}$ at order 10 and integrating them out as in the previous sector, one gets for $J_x = J_y = J$

$$e_0^{\nu=1/2} = -\frac{1}{2} - J^2 - \frac{J^4}{4} + \frac{3J^6}{2} - \frac{411J^8}{64} - \frac{211J^{10}}{32}. \quad (50)$$

Finally, one may also consider another vortex-half configuration rotated as shown in Fig. 21. The corresponding characteristic polynomial is straightforwardly obtained from $P^{\nu=1/2}$ by the permutation $J_x \rightarrow J_y$, $J_y \rightarrow J_z$, $J_z \rightarrow J_x$. However, since the perturbation is performed in the limit $J_z \gg J_x, J_y$ it leads to a different expression for the expanded ground-state energy. In this case, one gets for $J_z = J_y = J$

$$e_0^{\nu=1/2} = -\frac{1}{2} - J^2 - \frac{J^4}{4} + \frac{J^6}{4} + \frac{109J^8}{64} + \frac{59J^{10}}{16}. \quad (51)$$

Once again, both expressions (50) and (51) can be recovered from Eq. (21) using the coefficients given in Appendix C.

The various results obtained for $\nu = 0, 1, 1/2$ provide (partial) checks of the coefficients given in Appendices C and D and show the power of the PCUTs to compute high-order expansion for the spectrum. In the next section, we shall show that this method is also an efficient tool to tackle more complex problematics.

IX. OBSERVABLES

One of the advantages of the CUTs method is that it allows one to obtain the effective form of any observables and to compute its matrix elements in the eigenbasis of the Hamiltonian. The aim of this section is two-fold. First, we compute perturbatively the spin-spin correlations and show that they admit a plaquette-operator expansion similar to that of the spectrum. The second part of this section is dedicated to the most fundamental problem of local spin operations onto the ground state. Following Ref. 16, we show that single-spin operations create anyons but also fermions. We compute the spectral weights of various states stemming from such operations and we also analyze the action of string operations which allow for manipulation of anyons. Finally, we give a procedure to derive the operators which create anyons without fermions and show that they involve tricky superpositions of multi-spin operators.

A. Spin-spin correlation functions

The Hamiltonian (1) is invariant under the time-reversal symmetry since it is a quadratic function of the spin operators. Thus, any expectation value of an odd number of spin operators vanishes (such as the magnetization $\langle \sigma_i^z \rangle$). Note also that the absence of odd cycles ensures that the eigenstates do not break this symmetry.$^{7,26,27}$

In addition, the only nonvanishing spin correlators are those involving products of $\sigma_i^x \sigma_j^z$ on a-dimers.$^{12,17,18}$ In this section, we focus on the spin-spin correlation functions and their expression in the 0-QP sector. More pre-
cisesly, we consider the following operators $C_{i,j}^{\alpha\alpha} = \sigma_i^x \sigma_j^x$ where $(i,j)$ is an $\alpha$-dimer. To compute these quantities, we proceed in a way similar to what we have already done to derive the effective Hamiltonian:

- we express the observable in the ESB language;
- we compute its effective form perturbatively as explained in Appendix B (see also Ref. 45 for a detailed discussion);
- we project it out in the sector of interest.

Using the ESB form of the spin operators (8), we straightforwardly achieve the first step mentioned above for the three correlation functions

$$\sigma_i^x \sigma_{i+n_1}^x = (b_i^\dagger + b_i) \tau_i^x (b_{i+n_1}^\dagger + b_{i+n_1}), \quad \text{(52)}$$

$$\sigma_i^y \sigma_{i+n_2}^y = \tau_i^y (b_{i+n_2}^\dagger + b_{i+n_2}), \quad \text{(53)}$$

$$\sigma_i^z \sigma_{i+n_1}^z = 1 - 2b_i^\dagger b_i = (-1)^{b_i^\dagger b_i}. \quad \text{(54)}$$

To avoid any ambiguity, we keep track of the type of sites (● or ○) but we are working, at this stage, on the effective square lattice. Next, we turn to the second step using the perturbative expansion described in Appendix B. In the present case, we pushed the calculation up to order 6 and, finally, we focus on the 0-QP sector.

As one expects, the effective form of the spin-spin correlation function is similar to that of the effective Hamiltonian. This is due to the fact that, in the low-energy sector, $W_p$’s are the only degrees of freedom. Thus, we obtain, an expansion in terms of the plaquette operators

$$C_{i,j}^{\alpha\alpha} |q=0\rangle = a^{\alpha\alpha} - \sum_{\{p_1,\ldots,p_n\}} b^{\alpha\alpha}_{p_1,\ldots,p_n} W_{p_1} W_{p_2} \cdots W_{p_n}. \quad \text{(55)}$$

The coefficients $a^{\alpha\alpha}$ and $b^{\alpha\alpha}_{p_1,\ldots,p_n}$ are given in Appendix E up to order 6. Here again, we can see that these correlation functions involve interactions between connected or disconnected plaquettes.

As a simple check of our expression, one can easily compute the nonperturbative correlation function in the vortex-free and the vortex-full sector thanks to the Hellman-Feynman theorem. Indeed, in these sectors, all sites are equivalent so that one readily gets the expectation value

$$\langle C_{i,j}^{\alpha\alpha} |q=0\rangle_{\nu} = C_{i,j}^{\alpha\alpha \nu} |q=0\rangle = -\frac{\partial c_{ij}^{\nu}}{\partial J_{\nu}}, \quad \text{(56)}$$

for both cases $\nu = 0, 1$ for which the ground-state energies are given in Eqs. (41) and (46). As in the previous section, the subscript $\nu$ indicates that we consider the ground state of the sector with filling factor $\nu = 0, 1$.

Then, expanding these expressions (before derivation), setting $J_z = 1/2$ and for simplicity $J_x = J_y = J$, one gets

$$C_{i,j}^{zz |q=0\rangle_{\nu}} = 1 - 2J^2 - \frac{J_{\nu}^4}{2} - 25J^6, \quad \text{(57)}$$

$$C_{i,j}^{xx |q=0\rangle_{\nu}} = J + \frac{3J^3}{2} + \frac{15J_5}{2} = C_{i,j}^{yy |q=0\rangle_{\nu}}, \quad \text{(58)}$$

for the vortex-free sector and

$$C_{i,j}^{zz |q=0\rangle_{\nu}} = 1 - 2J^2 + \frac{3J_{\nu}^4}{2} - 15J^6, \quad \text{(59)}$$

$$C_{i,j}^{xx |q=0\rangle_{\nu}} = J + \frac{3J^3}{2} + \frac{9J_5}{2} = C_{i,j}^{yy |q=0\rangle_{\nu}}, \quad \text{(60)}$$

for the vortex-full sector. As can be checked, these results can be recovered using the coefficients given in Appendix E and Eq. (55). We emphasize that, as for the spectrum, our expressions allow us to investigate arbitrary vortex configurations such as sparse vortex ones, recently studied numerically 14,23.

B. Creation of anyons

Let us now analyze the action of a single-spin operation onto the ground state and following Ref. 16, let us focus on $\sigma_i^z$. As for the correlation functions, one first has to write this operator in the ESB formalism which is, again, straightforward since $\sigma_i^z = \tau_i^z$. Then, one computes its renormalization under the unitary transformation $U$ which “diagonalizes” the Hamiltonian. Finally, one can compute any matrix element of this observable between any eigenstates.

At order 0, the observable is not renormalized and one has $U^1 \tau_i^z U = \tau_i^z$. When this operator acts onto the ground state which is in the vortex-free sector, it thus simply flips the two plaquettes as shown in Fig. 22. In other words, it creates two anyons and nothing else.

At order 1, one gets

$$U^1 \tau_i^z U = \tau_i^z [1 + (J_x v_i^{-n_1} + J_y v_i^{n_2} + \text{h.c.})], \quad \text{(61)}$$

FIG. 22: (color online). Behavior of the spectral weights $I_n^z$ for fermion numbers $n = 0, 2, 4$, as a function of the coupling $J = J_x = J_y$ for $J_z = 1/2$. Gray plaquettes in the insets show the positions $p_1$ and $p_2$ at which the anyons are created under the action of $\tau_i^z$. 

As before, the expressions allow us to investigate arbitrary vortex configurations such as sparse vortex ones, recently studied numerically 14,23.
showing that things are more complex since pairs of particles (fermions) are created. It means that, at this order, $\tau^z_i$ couples the 0-QP subspace of the vortex-free sector with the 2-QP subspace of the two-vortex sector discussed above. To have a physical quantitative picture of such processes, let us compute the spectral weights defined as

$$I^n_z = \sum_k |\langle \{p^+_i, p^-_j\}, n, k| \tau^z_i |0\rangle|^2,$$  \hspace{1cm} (62)

where $|\{p\}, n, k\rangle$ denotes the eigenstate of $H$ in a sector given by an anyon configuration $w_p = -1$, and $n$ high-energy quasiparticles with quantum numbers $k$. Here, the plaquettes $p^+_i$ and $p^-_j$ are as indicated in the inset of Fig. 22. This quantity measures the weight of all n-anyon contributions obtained by the action of $\tau^z_i$ onto the ground state $|0\rangle$ which contains no fermion and no anyon. As it should, these spectral weights satisfy the sum rule $\sum_n I^n_z = 1$. At order 6, one gets

$$I^z_0 = 1 - \left( J^2_x + J^2_y \right) - \frac{3}{2} \left( J^4_x + J^4_y \right) - 4 J^2_x J^2_y - \frac{7}{2} \left( J^6_x + J^6_y \right) - \frac{43}{2} \left( J^4_x J^4_y + J^2_x J^2_y \right),$$ \hspace{1cm} (63)

$$I^z_2 = \left( J^2_x + J^2_y \right) + \frac{3}{2} \left( J^4_x + J^4_y \right) + 4 J^2_x J^2_y + \frac{7}{2} \left( J^6_x + J^6_y \right) + \frac{43}{2} \left( J^2_x J^2_y + J^4_x J^4_y \right),$$ \hspace{1cm} (64)

which shows the importance of the two-anyon states for increasing couplings as can be seen in Fig. 22. Note that the sum rule is fulfilled here implying $I^z_n = 0$ at order 6. Actually, one may consider representative curves in Fig. 22 as almost converged since order 8 corrections would bring very small corrections.

To summarize, one must realize that local spin operations onto the ground state create anyons (here two) but also give rise to fermionic excitations whose weight increases significantly with the perturbation.

### C. Manipulation of anyons

Another important question concerns the manipulation of the anyons which, as shown above, may be created by local spin operations. Such an issue is of special interest for experiments aiming at braiding anyons. This topic has been the subject of a recent controversy with Zhang et al.\textsuperscript{33,40} who completely neglected the existence of fermions in this model. Following Jiang et al.\textsuperscript{30} who proposed an ingenious protocol to detect anyons statistics, we wish to compute the action of a string operator onto the ground state.

For simplicity, we consider here the operator $S = \prod_{a=1,m} \sigma^{a\,\dagger}_{i,j}$ along a horizontal line of the original brick-wall lattice (see Fig. 23 with $m = 3$ for notations).

At order 0, it is simple to see that $S$ first creates two anyons and make one of them jump in the direction of the string so that, at the end, one eventually has one anyon at plaquette 1, another anyon at the plaquette $m + 1$, and no fermion.

However, at higher orders, as previously, such an operation creates fermions. To quantify this phenomenon, we consider the probability $P = |\langle \{1, m + 1\}, 0|S|0\rangle|^2$ to find the final state in the lowest-energy state (no fermion) with anyons at plaquettes 1 and $(m + 1)$ which coincides with $I^z_0$ for $m = 1$. In Ref. 16, we computed this probability at order 2, but here, we go beyond and give the result at order 6

$$P = 1 - m(J^2_x + J^2_y) + \frac{m(m-4)}{2}(J^4_x + J^4_y) + \frac{m^2 - 12 m + 32}{6}(J^6_x + J^6_y) + \left(-\frac{m^3}{2} + 10 m^2 - 51 m + 20\right)(J^2_x J^2_y + J^4_x J^4_y).$$ \hspace{1cm} (65)

The main reason to perform this high-order calculation is that the above expression pleads in favor of an exponentiated form linear with $m$. Indeed, although we have no proof, we conjecture that $P$ can be recast into $\exp(A - mB)$ as suggested in footnote 4 in Ref.\textsuperscript{7}. It is indeed striking to see that the expression (65) which is a polynomial of the variable $m$, can be seen as the expansion of such a simple form with, at order 6

$$A = 3J^2_x J^2_y + 20(J^4_x J^4_y + J^2_x J^2_y), \hspace{1cm} (66)$$

$$B = J^2_x + J^2_y + 8J^2_x J^2_y + 2(J^4_x + J^4_y) + 48(J^4_x J^4_y + J^2_x J^2_y) + \frac{16}{3}(J^6_x + J^6_y). \hspace{1cm} (67)$$

Further, it is clear that $P$ is bounded by 0 and 1 for any $m$, which is clearly not the case if one considers (65). Let us also note that the fact that $U \tau^z_i U^\dagger$ is found to be proportional to $\tau^z_i$ [see Eq. (61)] strengthen the idea of an exponential form of this effective observable and hence for $S$.

We display the results at various order in Fig. 24 using the expanded form (65) and the exponential form. As can be clearly seen, the exponential form seems to be well-behaved. In addition, the order 6 expansion of $A$ and $B$ seems to provide an almost converged result.
when put in the exponential. Thus, we claim that one can use this form to obtain a very accurate value of \( \mathcal{P} \) which is known to be of primer interest for braiding experiments\textsuperscript{36,30,31,32}.

\section{D. Anyons without fermions}

As discussed previously, local or string spin operations create fermions. However, in experiments, one wishes to manipulate anyons without being spoiled by these fermions\textsuperscript{30}. In other words, the ideal operations would consist in exciting plaquettes (only) while remaining in the ground state of the corresponding vortex sector. In this section, we shall show, perturbatively, that it is possible to do so even if the form of such operators is hard to implement in realistic devices.

As an example, let us determine the operator \( \Omega_i \) creating two vortices at the left and right plaquettes of a given site \( i \) [see inset Fig. 22 (left)]. This operator must be such that \( \Omega_{\text{eff}} = U^\dagger \Omega_i U = \tau_i^z \) which indeed leads to \( J_0^i = 1 \). Note that this procedure is the inverse of what is usually done with CUTs since, here, we wish to compute the bare observable given the effective observable instead of the opposite.

Let us assume that this operator has a perturbative expansion, namely

\[
\Omega_i = \sum_{k \in \mathbb{N}} \Omega_i^{(k)},
\]

where \( \Omega_i^{(k)} \) contains all operators of order \( k \) and thus associated to \( J_l^z J_m^z \) (with \( l + m = k \)). At order 0, operators are not renormalized so that one obviously has \( \Omega_i^{(0)} = \tau_i^z \). The renormalization of \( \Omega_i \) under the unitary transformation \( U \) reads

\[
\Omega_{\text{eff}} = \sum_{k \in \mathbb{N}} \Omega_{\text{eff}}^{(k)} = \sum_{k \in \mathbb{N}} U^\dagger \Omega_i^{(k)} U = \sum_{k \in \mathbb{N}} \sum_{l \in \mathbb{N}} \Omega_{\text{eff}}^{(k),[l]},
\]

where \( \Omega_{\text{eff}}^{(k),[l]} \) is of order \( k + l \). Since, at order 0, one has \( \Omega_{\text{eff}} = \tau_i^z \), one must have, at each order \( r > 0 \)

\[
\sum_{k \in \mathbb{N}} \sum_{l \in \mathbb{N}} \Omega_{\text{eff}}^{(k),[l]} = 0
\]

where the sum is restricted to values of indices such that \( k + l = r \). At order 1, this leads to

\[
\Omega_{\text{eff}}^{(0),[1]} + \Omega_{\text{eff}}^{(1),[0]} = 0.
\]

Using Eq. (61) and the fact that \( \Omega_{\text{eff}}^{(k),[0]} = \Omega_i^{(k)} \), one then obtains

\[
\Omega_i^{(1)} = - \left( J_x \sigma_{i-n_1}^x + J_y \sigma_{i-n_2}^y + \text{h.c.} \right).
\]

Using the inverse mapping of Eq. (8)

\[
\tau_i^x = \sigma_{i}^x \sigma_{i-\sigma}^x,
\]

\[
\tau_i^y = \sigma_{i}^y \sigma_{i-\sigma}^y,
\]

\[
\tau_i^z = \sigma_{i}^z \sigma_{i-\sigma}^z,
\]

\[
b_i^l = \frac{1}{2} \left( \sigma_{i-\sigma}^x - i \sigma_{i-\sigma}^y \right),
\]

one finally gets, in the original spin language and at order 1,

\[
\Omega_i = \Omega_i^{(0)} + \Omega_i^{(1)},
\]

\[
= \sigma_{i-\sigma}^z + \frac{1}{2} \left[ J_x \left( \sigma_{i-n_1}^x \sigma_{i-n_2}^x + \sigma_{i-n_1}^x \sigma_{i-\sigma}^y - \sigma_{i-n_2}^x \sigma_{i-\sigma}^x \sigma_{i-\sigma}^y \right) + J_y \left( \sigma_{i-n_2}^y \sigma_{i-n_2}^y + \sigma_{i-n_2}^y \sigma_{i-\sigma}^x - \sigma_{i-n_2}^y \sigma_{i-\sigma}^x \sigma_{i-\sigma}^y \right) \right].
\]

This expression shows that to create anyons without fermions, one has to build a complex superposition of operators with fine-tuned coefficients. At order 1 considered here, such states require single and triple spin-flip operations but, of course, higher-order corrections would involve higher order spin-flip processes. Such constraints makes creation of anyons without fermions via local operations difficult experimentally\textsuperscript{16}. 

![FIG. 24: (color online). \( \mathcal{P} \) as a function of \( J_x = J_y = J \) computed for \( m = 25 \). Top : nonresummed (bare) expression (65) at order 2 [red (bottom)], 4 [green (top)] and 6 [blue (middle)]. Bottom : exponentiated form \( \exp(A - mB) \) at order 2 [red (top)], 4 [green (middle)] and 6 [blue (bottom)].](image)
X. CONCLUSION AND PERSPECTIVES

We have analyzed perturbatively the gapped phase of the Kitaev honeycomb model in the isolated-dimer limit using the continuous unitary transformations. We have thus derived the low-energy effective theory up to order 10 which has been found to describe an interacting anyon system. This result has to be contrasted with the order 4 result which predicts a free anyon system. We also showed that the excitations in each vortex sector obey fermionic statistics.

In a second step, we focused on the action of local spin operators onto the ground state and we have shown that they generate both anyons and fermions. We also gave the form of the operator which creates anyons without fermions. This operator involves multi-spin operators which may be hard to implement experimentally.

Of course, several questions remain open in this model. As explained by Kitaev, there exists a gapless phase which is associated to non-Abelian anyons. The influence of a magnetic field in this phase is certainly one of the most challenging questions and should reveal rich phenomena. Note that the effect of a magnetic field in the toric code already gives rise to a nontrivial phase diagram as recently discussed in Ref. 47,48.

Another interesting issue concerns the time evolution of local excitations. Indeed, in Sec. IX C, we discussed the effect of a string operator onto the ground state but we always considered static quantities. Although experimentally, successive spin operations may be performed on ”short” time scales, it would be of primer interest to compute the spreading of fermionic excitations during the braiding processes proposed to detect anyons.

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APPENDIX A: STRUCTURE OF THE EFFECTIVE HAMILTONIAN

As we have seen in Sec. III, when setting \( J_z = 1/2 \), the Hamiltonian (1) can be written as

\[
H = - \frac{N}{2} + Q + T_0 + T_{+2} + T_{-2},
\]

where \( N \) is the number of \( z \)-dimers, \( Q \) is the particle-number operator, \( T_0 \) contains the pure hopping operators which does not change the number of particles and, \( T_{+2} (T_{-2}) \) creates (annihilates) pairs of particles. The operators \( T_{0, \pm 2} \) are proportional to the small parameters from which the perturbation theory is performed.

The idea of the present approach is to transform the Hamiltonian (9) into an effective one which conserves the particle number. Of course, in general, this cannot be achieved exactly and, as often, one has to perform a perturbative expansion. To achieve this goal, a very powerful tool is the continuous unitary transformations method\(^{34}\). For the problem at hand, Knetter and Uhrig\(^{39}\) have developed a code which computes the coefficients of this expansion at high orders\(^{49}\). Practically, one must keep in mind that, at order 10 which is the maximum order considered in this paper, one already has more than \( 10^4 \) terms. We refer the interested reader to Ref. 39 for a detailed derivation and we give below, for illustration, the results up to order 4.

| Order | Operator \( O \) | Coefficient | \( q_{\text{min}} \) |
|-------|----------------|-------------|---------|
| 1     | \( T_0 \)      | 1           | 1       |
| 2     | \( T_{-2}T_{+2} \) | -1/2       | 0       |
| 2     | \( T_{+2}T_{-2} \) | 1/2        | 2       |
| 3     | \( T_{-2}T_0T_{+2} \) | 1/4       | 0       |
| 3     | \( T_0T_{-2}T_{+2} \) | -1/8      | 1       |
| 3     | \( T_{-2}T_{+2}T_0 \) | -1/8      | 1       |
| 3     | \( T_0T_{+2}T_{-2} \) | -1/8      | 2       |
| 3     | \( T_{+2}T_0T_{-2} \) | 1/4       | 3       |
| 4     | \( T_{+2}T_{-2}T_{+2} \) | -1/16     | 0       |
| 4     | \( T_{-2}T_0T_{+2} \) | -1/8      | 0       |
| 4     | \( T_{-2}T_2T_{-2}T_{+2} \) | 1/8       | 0       |
| 4     | \( T_0T_{-2}T_{+2} \) | 1/8       | 1       |
| 4     | \( T_{+2}T_{-2}T_{+2} \) | -1/32     | 1       |
| 4     | \( T_{-2}T_0T_{+2} \) | 1/8       | 1       |
| 4     | \( T_0T_{+2}T_{-2} \) | -1/16     | 1       |
| 4     | \( T_{-2}T_{+2}T_{-2} \) | -1/32     | 1       |
| 4     | \( T_{+2}T_{-2}T_0 \) | 1/32      | 2       |
| 4     | \( T_0T_{+2}T_{-2} \) | 1/16      | 2       |
| 4     | \( T_{+2}T_{-2}T_0 \) | 1/32      | 2       |
| 4     | \( T_{-2}T_0T_{+2} \) | -1/8      | 2       |
| 4     | \( T_0T_{+2}T_{-2} \) | -1/8      | 3       |
| 4     | \( T_{+2}T_0T_{+2} \) | 1/8       | 3       |
| 4     | \( T_{-2}T_{+2}T_{-2} \) | -1/16     | 4       |

TABLE I: Operators appearing in \( H_{\text{eff}} \) with its corresponding coefficient up to order 4, together with the \( q \)-particle subspace they start to act on.

The operators and corresponding coefficients are put in Table I, together with the lowest number \( q_{\text{min}} \) of particles such that the operator has, a priori, a nonzero action within the \( q \)-particle subspace for \( q \geq q_{\text{min}} \). \( q_{\text{min}} \) is found by requiring that the number of particles in the system is always positive, and by using the fact that \( T_0 \) projects out 0-particle states. Note that some terms may vanish for more subtle reasons. For example, the third order term \( T_{-2}T_0T_{+2} \) does not act on the 0-QP states. Indeed, \( T_{+2} \) creates a 2-QP state ; then \( T_0 \) makes one of the
particle hop; and finally $T_{-2}$ tries to annihilate two particles, but cannot, since these are not nearest-neighbor anymore, due to the hopping.

One can then directly write the effective Hamiltonian

$$H_{\text{eff}} = -\frac{N}{2} + Q + \sum_i c_i O_i,$$  \hspace{1cm} (A2)

where $O_i$ is the $i^{\text{th}}$ element of the column "operator" of Table I and $c_i$ the associated coefficient, the order being given by the first column. By construction, the effective Hamiltonian conserves the particle number and the energy states are ordered according to their quasiparticle number, the ground state being in the 0-QP sector. Furthermore, since $[H_{\text{eff}}, Q] = 0$, one may also rewrite the effective Hamiltonian in the following form

$$H_{\text{eff}} = \sum_{q \in \mathbb{N}} H_{\text{eff}}|_q,$$  \hspace{1cm} (A3)

where $H_{\text{eff}}|_q$ denotes the projection of $H_{\text{eff}}$ onto the $q$-QP sector. Note that it is not the decoupling used in the CUTFs community where usually one gathers all operators which contain exactly $q$ creation and $q$ annihilation operators and thus act on the $q'$-QP sector with $q' \geq q$.

Finally, one must analyze each sector defined by the number of quasiparticles and determine the action of each operator $O_i$ in the corresponding subspace. This is the nontrivial part of the job which depends on the problem under consideration. Let us emphasize that if each operator only starts to act in the $q_{\text{min}}$-QP sector, it has also, in general, a nontrivial action on the $q$-QP sectors for $q > q_{\text{min}}$.

APPENDIX B: PERTURBATIVE EXPANSION OF OBSERVABLES

In this appendix, we give the general perturbative expansion of any observable $\Omega$ obtained with the CUTFs using the quasiparticle number conserving generator. As is the case for the Kitaev model, we suppose that the Hamiltonian of the system can be casted in the following form

$$H = Q + T_{-2} + T_0 + T_{+2},$$  \hspace{1cm} (B1)

and satisfies the hypothesis given after Eq. (16). In this case, the flow equations obtained from the CUTFs method can be solved perturbatively\(^{45}\), and the effective observable can be written as:

$$\Omega_{\text{eff}} = \Omega + \sum_i c_i O_i,$$  \hspace{1cm} (B2)

where $O_i$ is the $i^{\text{th}}$ element of the column "operator" of Table II and $c_i$ the associated coefficient, the order being given by the first column.

| Order | Operator | Coefficient |
|-------|----------|-------------|
| 1     | $T_{-2} \Omega$ | $-\frac{1}{2}$ |
| 1     | $T_{+2} \Omega$ | $\frac{1}{2}$ |
| 1     | $\Omega T_{-2}$ | $\frac{1}{2}$ |
| 1     | $\Omega T_{+2}$ | $-\frac{1}{2}$ |
| 2     | $T_{-2} \Omega T_{-2}$ | $\frac{1}{8}$ |
| 2     | $T_{-2} \Omega T_{+2}$ | $\frac{1}{4}$ |
| 2     | $T_{+2} \Omega T_{-2}$ | $-\frac{1}{8}$ |
| 2     | $T_{+2} \Omega T_{+2}$ | $-\frac{1}{4}$ |
| 2     | $T_{0} T_{-2} \Omega$ | $-\frac{1}{4}$ |
| 2     | $T_{0} T_{+2} \Omega$ | $-\frac{1}{4}$ |
| 2     | $T_{+2} T_{-2} \Omega$ | $-\frac{1}{8}$ |
| 2     | $T_{+2} T_{0} \Omega$ | $\frac{1}{4}$ |
| 2     | $T_{+2} T_{+2} \Omega$ | $\frac{1}{8}$ |
| 2     | $T_{+2} \Omega T_{-2}$ | $\frac{1}{4}$ |
| 2     | $T_{+2} \Omega T_{+2}$ | $-\frac{1}{4}$ |
| 2     | $\Omega T_{-2} T_{-2}$ | $\frac{1}{4}$ |
| 2     | $\Omega T_{-2} T_{+2}$ | $-\frac{1}{8}$ |
| 2     | $\Omega T_{0} T_{-2}$ | $\frac{1}{4}$ |
| 2     | $\Omega T_{0} T_{+2}$ | $\frac{1}{4}$ |
| 2     | $\Omega T_{+2} T_{-2}$ | $-\frac{1}{8}$ |
| 2     | $\Omega T_{+2} T_{+2}$ | $-\frac{1}{4}$ |
| 2     | $\Omega T_{+2} T_{0}$ | $\frac{1}{4}$ |
| 2     | $\Omega T_{+2} T_{+2}$ | $\frac{1}{8}$ |

TABLE II: Operators appearing in $\Omega_{\text{eff}}$ with the corresponding coefficient up to order 2.

At order 6 considered in this paper for the correlation functions, there are several thousands of terms to consider. Once this effective form is derived, one then has to analyze it in the quasiparticle sector of interest as done for the effective Hamiltonian.

APPENDIX C: COEFFICIENTS OF THE PERTURBATIVE EXPANSION OF THE HAMILTONIAN IN THE 0-QP SECTOR

As explained in Sec. V, the effective Hamiltonian in the 0-QP sector schematically reads

$$H_{\text{eff}}|_{q=0} = E_0 - \sum_{\{p_1,...,p_n\}} C_{p_1,...,p_n} W_{p_1} W_{p_2} \cdots W_{p_n}.$$  \hspace{1cm} (C1)

where $\{p_1,p_2,...,p_n\}$ denotes a set of $n$ plaquettes and $W_p$ are conserved plaquette operators. The form of the effective Hamiltonian is translationally invariant (of course the configuration of the $w_p$’s need not be !), so that $C_{p_1,...,p_n}$ in fact only depends on relative coordinates of the plaquettes, and we will use (except for the one-plaquette coefficient) the notation $\tilde{C}_{p_2-p_1,...,p_n-p_1} = C_{p_1,...,p_n}$. Here, we give the perturbative expansion up
to order 10 of $E_0$ and the $\tilde{C}$'s in the limiting case $J_x, J_y \ll J_z$. Setting $J_z = 1/2$, one gets the following results.

### Constant term

\[
\frac{E_0}{N} = -\frac{1}{2} \frac{J_x^2 + J_y^2}{2} - \frac{J_x^4 + J_y^4}{8} - \frac{J_x^6 + J_y^6}{8} - \frac{25}{128} (J_x^8 + J_y^8) + \frac{9}{32} J_x^4 J_y^4 + \frac{49}{128} (J_x^{10} + J_y^{10}) + \frac{33}{64} (J_x^6 J_y^4 + J_x^4 J_y^6),
\]

where $N$ is the number of $z$-dimers.

### One-plaquette term

\[
C_p = \frac{1}{2} J_x^2 J_y^2 + \frac{1}{4} (J_x^4 J_y^2 + J_x^2 J_y^4) + \frac{5}{16} (J_x^6 J_y^2 + J_x^2 J_y^6) + \frac{1}{4} J_x^4 J_y^4 + \frac{35}{64} (J_x^8 J_y^2 + J_x^2 J_y^8) - \frac{59}{32} (J_x^6 J_y^4 + J_x^4 J_y^6).
\]

### Two-plaquette terms

\[
\tilde{C}_{n_1} = \frac{7}{8} J_x^4 J_y^2 - \frac{15}{16} J_x^4 J_y^4 + \frac{3}{4} J_x^6 J_y^2 + \frac{287}{128} J_x^4 J_y^4,
\]
\[
\tilde{C}_{n_2} = \frac{7}{8} J_x^2 J_y^4 - \frac{15}{16} J_x^2 J_y^6 + \frac{3}{4} J_x^4 J_y^6 + \frac{287}{128} J_x^2 J_y^6,
\]
\[
\tilde{C}_{n_1+n_2} = \frac{33}{8} J_x^4 J_y^4,
\]
\[
\tilde{C}_{n_1-n_2} = -\frac{1}{4} J_x^4 J_y^4 + \frac{3}{32} (J_x^2 J_y^4 + J_x^4 J_y^6),
\]
\[
\tilde{C}_{2n_1} = -\frac{143}{32} J_x^6 J_y^4,
\]
\[
\tilde{C}_{2n_2} = -\frac{143}{32} J_x^4 J_y^6,
\]
\[
\tilde{C}_{2n_1+2n_2} = \frac{715}{64} J_x^4 J_y^4,
\]
\[
\tilde{C}_{2n_2-n_2} = \frac{55}{64} J_x^6 J_y^4,
\]
\[
\tilde{C}_{n_1+2n_2} = \frac{715}{64} J_x^4 J_y^6,
\]
\[
\tilde{C}_{-n_1+2n_2} = \frac{55}{64} J_x^4 J_y^6.
\]

### Three-plaquette terms

\[
\tilde{C}_{n_1,2n_1} = \frac{33}{16} J_x^6 J_y^2 - \frac{143}{32} J_x^4 J_y^4 + \frac{143}{64} J_x^8 J_y^2,
\]
\[
\tilde{C}_{n_1,2n_2} = \frac{33}{16} J_x^6 J_y^2 - \frac{143}{32} J_x^4 J_y^4 + \frac{143}{64} J_x^8 J_y^2,
\]
\[
\tilde{C}_{n_1,n_1+n_2} = \frac{33}{16} J_x^4 J_y^4 - \frac{143}{128} (J_x^2 J_y^4 + J_x^4 J_y^6),
\]
\[
\tilde{C}_{n_1,n_1-n_2} = \frac{33}{16} J_x^4 J_y^4 - \frac{143}{128} (J_x^2 J_y^4 + J_x^4 J_y^6),
\]
\[
\tilde{C}_{n_1+2n_1} = \frac{9}{16} J_x^4 J_y^4 - \frac{275}{128} (J_x^2 J_y^4 + J_x^4 J_y^6),
\]
\[
\tilde{C}_{n_1+2n_2} = \frac{9}{16} J_x^4 J_y^4 - \frac{275}{128} (J_x^2 J_y^4 + J_x^4 J_y^6),
\]
\[
\tilde{C}_{n_1,n_1+2n_2} = \frac{715}{64} J_x^4 J_y^4,
\]
\[
\tilde{C}_{n_1,n_1+2n_2} = \frac{715}{64} J_x^4 J_y^4,
\]
\[
\tilde{C}_{n_1,2n_1+n_2} = \frac{715}{64} J_x^4 J_y^4,
\]
\[
\tilde{C}_{n_1,2n_1+2n_2} = \frac{715}{64} J_x^4 J_y^4,
\]
\[
\tilde{C}_{n_1,2n_2+n_2} = \frac{715}{64} J_x^4 J_y^4,
\]
\[
\tilde{C}_{n_1,n_1+n_2} = \frac{11}{16} J_x^4 J_y^4,
\]
\[
\tilde{C}_{n_1,n_1-2n_2} = \frac{11}{16} J_x^4 J_y^4,
\]
\[
\tilde{C}_{n_1,2n_1-n_2} = \frac{11}{16} J_x^4 J_y^4,
\]
\[
\tilde{C}_{n_1,2n_1-2n_2} = \frac{11}{16} J_x^4 J_y^4,
\]
\[
\tilde{C}_{n_1,2n_1+n_2} = \frac{11}{16} J_x^4 J_y^4,
\]
\[
\tilde{C}_{n_1,n_1+2n_2} = \frac{11}{16} J_x^4 J_y^4,
\]
\[
\tilde{C}_{n_1,n_1+2n_2} = \frac{11}{16} J_x^4 J_y^4.
\]

### Four-plaquette terms

\[
\tilde{C}_{n_1,n_1,n_1+2n_2} = \frac{33}{16} J_x^4 J_y^4,
\]
\[
\tilde{C}_{n_1,n_1,n_1-2n_2} = \frac{33}{16} J_x^4 J_y^4,
\]
\[
\tilde{C}_{n_1,n_1,2n_1-n_2} = \frac{55}{128} J_x^6 J_y^4,
\]
\[
\tilde{C}_{n_1,n_1,2n_1+2n_2} = \frac{55}{128} J_x^6 J_y^4,
\]
\[
\tilde{C}_{n_1,n_1,n_1+2n_2} = \frac{55}{128} J_x^6 J_y^4,
\]
\[
\tilde{C}_{n_1,n_1,n_1+2n_2} = \frac{55}{128} J_x^6 J_y^4.
\]
\[
\begin{align*}
\tilde{C}_{n_1,2n_1,3n_1} &= \frac{715}{128} j_x^6 j_y^2, \\
\tilde{C}_{n_2,2n_2,3n_2} &= \frac{715}{128} j_x^2 j_y^8, \\
\tilde{C}_{n_1,3n_1+2n_1,2n_2} &= \frac{715}{128} j_x^4 j_y^6, \\
\tilde{C}_{n_2,2n_2+2n_1,1n_2} &= \frac{715}{128} j_x^4 j_y^4, \\
\tilde{C}_{n_1,2n_1+2n_1+2n_2} &= \frac{715}{128} j_x^6 j_y^4, \\
\tilde{C}_{n_2,2n_2+2n_1+2n_2} &= \frac{715}{128} j_x^6 j_y^6.
\end{align*}
\]

As can be seen from these expansions, the number of interacting plaquettes increases with the order of the perturbation theory.

**APPENDIX D: COEFFICIENTS OF THE PERTURBATIVE EXPANSION OF THE HAMILTONIAN IN THE 1-QP SECTOR**

In the 1-QP sector, the effective Hamiltonian reads [see Eqs. (26)-(30)]

\[
H_{\text{eff}}|_{q=1} = H_{\text{eff}}|_{q=0} + \mu - \sum_{(j_1,\ldots,j_n)} D_{j_1,\ldots,j_n} \frac{t_{j_1}^n}{j_1} \ldots \frac{t_{j_n}^1}{j_n},
\]

where the sum is performed over all non self-retracing paths of length \( n \) starting at site \( j_1 \) and ending at site \( j_n \). The operators \( t_j \) are defined in Eqs. (10-13). Since the \( D_j \)'s do not depend on the initial site, we introduce \( D_{j_{n-1}} \ldots j_n = D_{j_{n-1}} \ldots j_n \). From the symmetries of the underlying lattice, it is clear that we can limit the analysis to processes involving a first jump in the direction +\( n_1 \) or +\( n_2 \).

We give below the perturbative expansion of \( \mu \) and the \( D_j \)'s in the limiting case \( J_x, J_y \ll J_z \) up to order 4 and set \( J_z = 1/2 \). Note that one could reach order 10 as for the 0-QP sector if needed. However, as explained in Sec. VI, it is simpler, in this sector, to use directly the Majorana formalism which is nonperturbative and requires a comparable numerical effort.

**Six-plaquette terms**

\[
\begin{align*}
\tilde{C}_{n_1,2n_1,n_2+2n_1+n_2} &= \frac{715}{128} j_x^6 j_y^4, \\
\tilde{C}_{n_2,2n_2,n_1+2n_2+n_2} &= \frac{715}{128} j_x^4 J_y^4.
\end{align*}
\]

**Five-plaquette terms**

\[
\begin{align*}
\tilde{C}_{n_1,2n_1,n_1+n_2,2n_1+n_2} &= \frac{715}{128} j_x^6 j_y^4, \\
\tilde{C}_{n_1,2n_1,-n_2,n_1-n_2} &= \frac{715}{128} j_x^6 j_y^4, \\
\tilde{C}_{n_2,2n_2,n_1+n_2,2n_1+n_2} &= \frac{715}{128} j_x^4 j_y^6, \\
\tilde{C}_{n_2,2n_2,-n_1,n_1-n_2} &= \frac{715}{128} j_x^4 j_y^6, \\
\tilde{C}_{n_1,2n_1,n_1+n_2,2n_1+n_2} &= \frac{715}{128} j_x^6 j_y^4, \\
\tilde{C}_{n_1,2n_1,-n_2,n_1-n_2} &= \frac{715}{128} j_x^6 j_y^4, \\
\tilde{C}_{n_2,2n_2,n_1+n_2,2n_1+n_2} &= \frac{715}{128} j_x^4 j_y^6, \\
\tilde{C}_{n_2,2n_2,-n_1,n_1-n_2} &= \frac{715}{128} j_x^4 j_y^6.
\end{align*}
\]

**Chemical potential**

\[
\mu = 1 + J_x^2 + J_y^2 + \frac{J_x^4 + J_y^4}{4}.
\]

**One-hopping terms**

\[
\begin{align*}
\tilde{D}_{n_1} &= J_x - \frac{1}{2} J_x^3 - \frac{1}{2} J_x J_y^2, \\
\tilde{D}_{n_2} &= J_y - \frac{1}{2} J_y^3 - \frac{1}{2} J_x^2 J_y.
\end{align*}
\]
### Two-hopping terms

\[
\begin{align*}
\tilde{D}_{n_1,n_1} &= \frac{1}{2} J_x^2 - \frac{5}{8} J_x^2 J_y^2 - \frac{1}{2} J_y^4, \\
\tilde{D}_{n_1,n_2} &= \frac{1}{2} J_x J_y - \frac{9}{16} J_x^3 J_y - \frac{9}{16} J_x J_y^3, \\
\tilde{D}_{n_1,-n_2} &= \frac{1}{2} J_x J_y, \\
\tilde{D}_{n_2,n_2} &= \frac{1}{2} J_y^2 - \frac{5}{8} J_x^2 J_y^2 - \frac{1}{2} J_y^4, \\
\tilde{D}_{n_2,n_1} &= \frac{1}{2} J_x J_y - \frac{9}{16} J_x J_y^3 - \frac{9}{16} J_x^3 J_y, \\
\tilde{D}_{n_2,-n_1} &= \frac{1}{2} J_x J_y.
\end{align*}
\]

### Three-hopping terms

\[
\begin{align*}
\tilde{D}_{n_1,n_1,n_1} &= \frac{1}{2} J_x^3, \\
\tilde{D}_{n_1,n_1,n_2} &= \frac{1}{2} J_x^2 J_y, \\
\tilde{D}_{n_1,n_1,-n_2} &= -\frac{1}{4} J_x J_y, \\
\tilde{D}_{n_1,n_2,n_1} &= \frac{1}{2} J_x^2 J_y, \\
\tilde{D}_{n_1,n_2,n_2} &= \frac{1}{2} J_x J_y^2, \\
\tilde{D}_{n_1,n_2,-n_1} &= \frac{1}{4} J_x J_y, \\
\tilde{D}_{n_1,-n_2,-n_1} &= 0, \\
\tilde{D}_{n_1,-n_2,-n_2} &= -\frac{1}{4} J_x J_y^2, \\
\tilde{D}_{n_1,-n_2,-n_1} &= -\frac{1}{4} J_x J_y, \\
\tilde{D}_{n_2,n_2,n_1} &= \frac{1}{2} J_y^3, \\
\tilde{D}_{n_2,n_2,n_2} &= \frac{1}{2} J_y^2 J_x, \\
\tilde{D}_{n_2,n_2,-n_1} &= -\frac{1}{4} J_x J_y^2, \\
\tilde{D}_{n_2,n_1,n_2} &= \frac{1}{2} J_x J_y^2, \\
\tilde{D}_{n_2,n_1,n_1} &= \frac{1}{2} J_y^3, \\
\tilde{D}_{n_2,n_1,-n_2} &= -\frac{1}{4} J_x J_y, \\
\tilde{D}_{n_2,-n_1,-n_2} &= 0, \\
\tilde{D}_{n_2,-n_1,-n_1} &= \frac{1}{4} J_x^2 J_y, \\
\tilde{D}_{n_2,-n_1,-n_2} &= \frac{1}{4} J_x J_y^2.
\end{align*}
\]

### Four-hopping terms

\[
\begin{align*}
\tilde{D}_{n_1,n_1,n_1,n_1} &= \frac{5}{8} J_x^4, \\
\tilde{D}_{n_1,n_1,n_1,n_2} &= \frac{5}{8} J_x^3 J_y, \\
\tilde{D}_{n_1,n_1,n_1,-n_2} &= \frac{3}{16} J_x^3 J_y, \\
\tilde{D}_{n_1,n_1,n_2,n_1} &= \frac{5}{8} J_x^3 J_y, \\
\tilde{D}_{n_1,n_1,n_2,n_2} &= \frac{5}{8} J_x^2 J_y^2, \\
\tilde{D}_{n_1,n_1,-n_2,n_1} &= \frac{3}{16} J_x^3 J_y, \\
\tilde{D}_{n_1,n_1,-n_2,n_2} &= \frac{1}{4} J_x^3 J_y, \\
\tilde{D}_{n_1,-n_2,-n_1,n_1} &= \frac{1}{4} J_x^3 J_y, \\
\tilde{D}_{n_1,-n_2,-n_1,n_2} &= \frac{3}{16} J_x J_y^3, \\
\tilde{D}_{n_1,-n_2,-n_1,-n_2} &= \frac{3}{16} J_x^2 J_y^2, \\
\tilde{D}_{n_1,-n_2,-n_2,n_1} &= \frac{3}{16} J_x J_y^3, \\
\tilde{D}_{n_1,-n_2,-n_2,n_2} &= \frac{3}{16} J_x^2 J_y^2.
\end{align*}
\]
\begin{align*}
\tilde{D}_{n_2,n_2,n_2,n_2} &= \frac{5}{8} J^4_{x,y}, \\
\tilde{D}_{n_2,n_2,n_2,n_1} &= \frac{5}{8} J^3_{x,y}, \\
\tilde{D}_{n_2,n_2,n_1,n_1} &= \frac{3}{16} J^3_{x,y}, \\
\tilde{D}_{n_2,n_2,n_1,n_2} &= \frac{5}{8} J^3_{x,y}, \\
\tilde{D}_{n_2,n_2,n_1,n_1} &= \frac{5}{8} J^2_{x,y}, \\
\tilde{D}_{n_2,n_1,n_2,n_2} &= \frac{3}{16} J^4_{x,y}, \\
\tilde{D}_{n_2,n_1,n_1,n_1} &= \frac{1}{16} J^3_{x,y}, \\
\tilde{D}_{n_2,n_1,n_2,n_2} &= \frac{3}{16} J^2_{x,y}, \\
\tilde{D}_{n_2,n_1,n_1,n_2} &= \frac{1}{8} J^2_{x,y}, \\
\tilde{D}_{n_2,n_1,n_2,n_1} &= \frac{1}{8} J^2_{x,y}, \\
\tilde{D}_{n_2,n_1,n_1,n_1} &= 0, \\
\tilde{D}_{n_2,n_1,n_1,n_2} &= -\frac{3}{16} J^3_{x,y}, \\
\tilde{D}_{n_2,n_1,n_2,n_1} &= -\frac{3}{16} J^2_{x,y}, \\
\tilde{D}_{n_2,n_1,n_2,n_2} &= -\frac{3}{16} J^2_{x,y}.
\end{align*}

Additionally, there are some terms corresponding to processes where the particle hops one times around a plaquette. Note that the plaquette involved can be covered clockwise or anti-clockwise but the product of \( t_i^d \) leads exactly to the same operator \( b_i^d b_i^\dagger \).

\[ D_{n_1,n_2,n_1,n_2} = \frac{1}{4} J^2_{x,y}, \]
\[ D_{n_1,n_2,n_1,n_2} = 0, \]
\[ D_{n_1,n_2,n_1,n_1} = \frac{1}{4} J^2_{x,y}, \]
\[ D_{n_1,n_2,n_1,n_2} = 0. \]

**APPENDIX E: COEFFICIENTS OF THE SPIN-SPIN CORRELATION FUNCTION IN THE 0-QP SECTOR**

As discussed in Sec. IXA, the spin-spin correlation functions \( C^{\sigma_{i,j}}_{\alpha} = \sigma_i^\alpha \sigma_j^\beta \) computed on any eigenstate of \( H \) is nonvanishing only if \( \alpha = \beta \) and if \( i \) and \( j \) belong to the same dimer which is of \( \alpha \) type.

We give below the perturbative expansion of these correlation functions in the 0-QP sector.

1. Coefficients of \( C^{zz} \)

In the perturbative approach we use, note that a \( z \)-dimer in the honeycomb lattice becomes a single site \( i \) in the effective square lattice.

As for the Hamiltonian in the 0-QP sector [see Eq. (21)], we obtain an expansion which can be expressed only in terms of the plaquette operators, namely

\[ C^{zz}_{i} \big|_{q=0} = a^{zz} - \sum_{\{p_1,...,p_n\}} b^{zz}_{p_1,...,p_n} W_{p_1} W_{p_2} \cdots W_{p_n}. \quad (E1) \]

Below, we give the results up to order 6 and we index a plaquette \( p \) by a site \( i \) and an indice \( u,d,l,r \) according to notations given in Fig. 25.

**Constant term**

\[ a^{zz} = 1 - \left( J^2_x + J^2_y \right) - \frac{3}{4} (J^4_x + J^4_y) - \frac{5}{4} (J^6_x + J^6_y). \]
One-plaquette terms

\[ b_{(i,u)}^{zz} = -\frac{5}{4} J_x^2 J_y^2, \]
\[ b_{(i,d)}^{zz} = -\frac{5}{4} J_x^2 J_y^2, \]
\[ b_{(i,l)}^{zz} = -\frac{1}{4} J_x^2 J_y^2 + \frac{1}{2} (J_x^2 J_y^3 + J_x J_y^2), \]
\[ b_{(i,r)}^{zz} = -\frac{1}{4} J_x^2 J_y^2 + \frac{1}{2} (J_x^2 J_y^3 + J_x J_y^2), \]
\[ b_{(i+2n_1,l)}^{zz} = -\frac{21}{8} J_x^4 J_y^2, \]
\[ b_{(i+2n_1,r)}^{zz} = -\frac{21}{8} J_x^4 J_y^2, \]
\[ b_{(i-2n_1,r)}^{zz} = -\frac{21}{8} J_x^4 J_y^2, \]
\[ b_{(i-2n_1,l)}^{zz} = -\frac{21}{8} J_x^4 J_y^2, \]
\[ b_{(i-2n_1,u)}^{zz} = \frac{7}{8} J_x^4 J_y^2, \]
\[ b_{(i+2n_1,d)}^{zz} = \frac{7}{8} J_x^4 J_y^2, \]
\[ b_{(i+2n_1,u)}^{zz} = \frac{7}{8} J_x^4 J_y^2, \]
\[ b_{(i-2n_2,u)}^{zz} = \frac{7}{8} J_x^4 J_y^2. \]

Two-plaquette terms

\[ b_{(i,u),(i+2n_1,l)}^{zz} = -\frac{21}{8} J_x^4 J_y^2, \]
\[ b_{(i,u),(i+2n_1,r)}^{zz} = -\frac{21}{8} J_x^4 J_y^2, \]
\[ b_{(i,d),(i-2n_1,l)}^{zz} = -\frac{21}{8} J_x^4 J_y^2, \]
\[ b_{(i,d),(i-2n_1,r)}^{zz} = -\frac{21}{8} J_x^4 J_y^2, \]
\[ b_{(i,l),(i-2n_1,u)}^{zz} = \frac{7}{8} J_x^4 J_y^2, \]
\[ b_{(i,l),(i+2n_1,d)}^{zz} = \frac{7}{8} J_x^4 J_y^2, \]
\[ b_{(i,r),(i+2n_1,d)}^{zz} = \frac{7}{8} J_x^4 J_y^2, \]
\[ b_{(i,r),(i+2n_1,u)}^{zz} = \frac{7}{8} J_x^4 J_y^2, \]
\[ b_{(i,r),(i-2n_2,u)}^{zz} = \frac{7}{8} J_x^4 J_y^2. \]

2. Coefficients of \( C_{i,j}^{xx} \)

Contrary to \( z \)-dimers, \( x \)-dimers remain dimers perturbatively. Here again, one obtains an expansion in terms of plaquettes for these observables, in the 0-QP sector, which can be written as

\[ C_{i,i+n_1}^{xx} |_{q=0} = a^{xx} - \sum_{\{p_1,...,p_n\}} b_{p_1,...,p_n}^{xx} W_{p_1} W_{p_2} \cdots W_{p_n} \]  

(E3)

We give below the expansion of the coefficients up to order 5 (only odd orders are nonvanishing) and, as previously, we index a plaquette \( p \) by a site \( i \) and an index \( u, d, l, r \) according to the notations given in Fig. 25. In the following we consider a dimer located at \((i, i + n_1)\).

(E2)

Constant term

\[ a^{xx} = J_x + \frac{1}{2} J_x^3 + \frac{3}{4} J_x^5. \]

One-plaquette terms

\[ b_{(i,u)}^{xx} = \frac{7}{8} J_x^4 J_y^2, \]
\[ b_{(i,d)}^{xx} = \frac{7}{8} J_x^4 J_y^2, \]
\[ b_{(i,l)}^{xx} = -\frac{7}{8} J_x^4 J_y^2, \]
\[ b_{(i,r)}^{xx} = -\frac{7}{8} J_x^4 J_y^2, \]
\[ b_{(i,u),(i+2n_1,l)}^{xx} = \frac{7}{4} J_x^4 J_y^2, \]
\[ b_{(i,u),(i+2n_1,r)}^{xx} = \frac{7}{4} J_x^4 J_y^2, \]
\[ b_{(i,d),(i-2n_1,l)}^{xx} = -\frac{3}{4} J_x^4 J_y^2, \]
\[ b_{(i,d),(i-2n_1,r)}^{xx} = -\frac{3}{4} J_x^4 J_y^2, \]
\[ b_{(i,l),(i-2n_1,u)}^{xx} = \frac{3}{4} J_x^4 J_y^2, \]
\[ b_{(i,l),(i+2n_1,d)}^{xx} = \frac{3}{4} J_x^4 J_y^2, \]
\[ b_{(i,r),(i-2n_2,u)}^{xx} = \frac{3}{4} J_x^4 J_y^2. \]
Two-plaquette terms

\[ b_{(i,u), (i+2n_1, t)}^{xx} = \frac{7}{8} j_x^2 j_y^2, \]
\[ b_{(i,u), (i+2n_2, r)}^{xx} = \frac{7}{8} j_x^2 j_y^2, \]
\[ b_{(i,u), (i, i)}^{xx} = \frac{7}{8} j_x^2 j_y^2, \]
\[ b_{(i,r), (i, d)}^{xx} = \frac{7}{8} j_x^2 j_y^2, \]
\[ b_{(i,r), (i-2n_2, u)}^{xx} = \frac{7}{8} j_x^2 j_y^2. \]  

(E4)

3. Coefficients of \( C_{i,j}^{yy} \)

The correlation functions \( C_{i,j}^{yy} \) are straightforwardly obtained from \( C_{i,j}^{xx} \) by exchanging directions \( x \) and \( y \) as well as \( J_x \) and \( J_y \).

APPENDIX F: CORRESPONDENCE BETWEEN THE MAJORANA FERMION SPECTRUM AND A FREE-PARTICLE PROBLEM IN A MAGNETIC FIELD

As shown by Kitaev\(^7\), the spin Hamiltonian (1) can be mapped onto the following Majorana fermion Hamiltonian

\[ H = \frac{i}{4} \sum_{j,k} A_{jk} c_j c_k, \]  

(F1)

where \( A \) is a skew-symmetric matrix of size \( 2N \times 2N \) (\( N \) being the number of plaquette) and where the \( c_j \)'s are the (hermitian) Majorana operators which obey \( c_j^2 = 1 \) and \( c_j c_k = -c_k c_j \) if \( j \neq k \). The sum is performed over all sites \( j \) and \( k \) of the honeycomb (brickwall) lattice and

\[ A_{jk} = 2J_{u} u_{jk}, \]  

(F2)

if the link \((j, k)\) is of \( \alpha \)-type and \(0\) otherwise. The \( u_{jk} \)'s are antisymmetric \((u_{jk} = -u_{kj})\) and take the values \( \pm 1 \).

These numbers define the vortex configuration through:

\[ u_p = \prod_{(j,k) \in p} u_{jk} \]  

where \( j \) belongs to the black sublattice and \( k \) to the white one (see Fig. 1a). We refer the interested reader to Ref. 7 for details. In the very end, the whole spectrum of \( H \) can be obtained once one knows the spectrum of \( iA \), e. g. the ground-state energy per plaquette is given by

\[ e_0 = -\frac{1}{4N} \text{Tr}|iA|. \]  

(F3)

For a bipartite lattice such as the honeycomb lattice, we shall now show that the spectrum of \( iA \) is the same as the one-particle spectrum of the following Hamiltonian

\[ H' = -\frac{1}{2} \sum_{j,k} A_{jk} (a_j^\dagger a_k + \text{h.c.}), \]  

(F4)

where \( a_j^\dagger \) (\( a_j \)) are standard spinless fermion creation (annihilation) operators. The Hamiltonian \( H' \) describes free spinless fermions hopping in a honeycomb lattice in a magnetic field with a flux per plaquette which equals zero \((w_p = +1)\) or half a flux quantum \((w_p = -1)\). The spectra of \( H' \) (with one fermion) and of \( iA \) are identical provided

\[ A_{jk} = 2J_{u} u_{jk}', \]  

(F5)

with \( u_{jk}' = +u_{kj}' \). The choice of the \( u_{jk}' \) is as previously dictated by the flux configuration via \( w_p = \prod_{(j,k) \in p} u_{jk}' \) where, in this case, the \( u_{jk}' \) are not oriented but still take the value \( \pm 1 \).

To show this, consider an eigenstate \( |\psi\rangle \) of the matrix \( iA \) with energy \( E \) and let us denote \( \psi_j = (j|\psi\rangle \) its component on site \( j \). This state satisfies

\[ \sum_k iA_{jk} \psi_k = E\psi_j, \]  

(F6)

Since the honeycomb lattice is bipartite, we can always set \( u_{jk} = u_{jk}' \) if \( j \) is a white site (and \( k \) black), and \( u_{jk} = -u_{jk}' \) if \( j \) is a black site (and \( k \) white). Then, one can easily check that the state \( |\phi\rangle \) defined by \( \phi_j = -\psi_j \) if \( j \) is a black site and \( \phi_j = -i\psi_j \) if it is a white site, satisfies

\[ -\sum_k A_{jk} \phi_k = E\phi_j, \]  

(F7)

so that \( |\phi\rangle \) is an eigenstate of \( H' \) with the energy \( E \). This shows that \( H' \) (with one particle) and \( iA \) are isospectral. We insist on the fact that this correspondence only holds for a bipartite lattice but is no longer true in the presence of odd cycles.

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