Connectivity of Triangulation Flip Graphs in the Plane∗†

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Abstract

Given a finite point set $P$ in general position in the plane, a full triangulation is a maximal straight-line embedded plane graph on $P$. A partial triangulation on $P$ is a full triangulation of some subset $P'$ of $P$ containing all extreme points in $P$. A bistellar flip on a partial triangulation either flips an edge, removes a non-extreme point of degree 3, or adds a point in $P \setminus P'$ as vertex of degree 3. The bistellar flip graph has all partial triangulations as vertices, and a pair of partial triangulations is adjacent if they can be obtained from one another by a bistellar flip. The goal of this paper is to investigate the structure of this graph, with emphasis on its connectivity.

For sets $P$ of $n$ points in general position, we show that the bistellar flip graph is $(n-3)$-connected, thereby answering, for sets in general position, an open questions raised in a book (by De Loera, Rambau, and Santos) and a survey (by Lee and Santos) on triangulations. This matches the situation for the subfamily of regular triangulations (i.e., partial triangulations obtained by lifting the points and projecting the lower convex hull), where $(n-3)$-connectivity has been known since the late 1980s through the secondary polytope (Gelfand, Kapranov, Zelevinsky) and Balinski’s Theorem.

Our methods also yield the following results: (i) The bistellar flip graph can be covered by graphs of polytopes of dimension $n-3$ (products of secondary polytopes). (ii) A partial triangulation is regular, if it has distance $n-3$ in the Hasse diagram of the partial order of partial subdivisions from the trivial subdivision. (iii) All partial triangulations are regular iff the partial order of partial subdivisions has height $n-3$. (iv) There are arbitrarily large sets $P$ with non-regular partial triangulations, while every proper subset has only regular triangulations, i.e., there are no small certificates for the existence of non-regular partial triangulations (answering a question by F. Santos in the unexpected direction).

Keywords. triangulation, flip graph, graph connectivity, associahedron, subdivision, convex decomposition, flippable edge, flip complex, regular triangulation, bistellar flip graph, secondary polytope, polyhedral subdivision.

∗This is a full version of [15] in Proceedings of the 36th Annual International Symposium on Computational Geometry (SoCG’20). We plan to extend this full version also with the material on edge flip graphs of full triangulations from [14] in Proceedings of the 31st Annual ACM-SIAM Symposium on Discrete Algorithms (SODA’20).
†This research started at the 11th Gremo’s Workshop on Open Problems (GWOP), Alp Sellamatt, Switzerland, June 24-28, 2013, motivated by a question posed by Filip Morić. Research was supported by the Swiss National Science Foundation within the collaborative DACH project Arrangements and Drawings as SNSF Project 200021E-171681, and by IST Austria and Berlin Free University during a sabbatical stay of the second author. We thank Michael Joswig, Jesús De Loera, and Francisco Santos for helpful discussions on the topics of this paper, and Daniel Bertschinger for carefully reading an earlier version and for many helpful comments.
1 Introduction

Throughout this paper we let $P$ denote a finite planar point set in general position (no three points on a line) with $n \geq 3$ points. The set of extreme points of $P$ (i.e., the vertices of the convex hull of $P$) is denoted by $\text{xtr}P$, and $P^\circ := P \setminus \text{xtr}P$ denotes the set of inner (i.e., non-extreme) points in $P$. We consistently use $h = h(P) := |\text{xtr}P|$ and $n^\circ = n^\circ(P) := |P^\circ| = n - h$.

We let $E_{\text{hull}} = E_{\text{hull}}(P) \subseteq \binom{P}{2}$ denote the $h$ edges of the convex hull of $P$.

For graphs $G = (P', E)$, $P' \subseteq P$, $E \subseteq \binom{P'}{2}$, on $P'$ we often identify edges $\{p, q\}$ with their corresponding straight line segments $pq$. We let $VG := P'$ and $EG := E$.

**Definition 1.1 (plane)** A graph $G$ on $P$ is plane if no two straight line segments corresponding to edges in $EG$ cross (i.e., they are disjoint except for possibly sharing an endpoint).

**Definition 1.2 (full, partial triangulation)** A full triangulation of $P$ is a maximal plane graph $T = (P, E)$. A partial triangulation of $P$ is a full triangulation $T = (P', E)$ with $\text{xtr}P \subseteq P' \subseteq P$ (hence $E_{\text{hull}} \subseteq ET$). Points in $V^*T := P^\circ \cap VT$ are called inner points. Points in $P^\circ \setminus V^*T$ are called skipped in $T$. Edges in $E^*T := ET \setminus E_{\text{hull}}$ are called inner edges. Edges in $E_{\text{hull}}$ are called boundary edges. $\mathcal{T}_{\text{part}}(P)$ denotes the set of all partial triangulations of $P$.

**Convention 1** From now on, we will mostly use “triangulation” for “partial triangulation”.

![Figure 1: Edge flips and point flips](image)

**Definition 1.3 (bistellar flip)** Let $T$ be a triangulation of $P$. An edge $e \in E^*T$ is called flippable in $T$ if removal of $e$ in $T$ creates a convex quadrilateral face $Q$, when $T[e]$ is the triangulation with the other diagonal $\tau$ of $Q$ added instead of $e$, i.e., $VT[e] := VT$ and $ET[e] := ET \setminus \{e\} \cup \{\tau\}$; we call this an edge flip.

A point $p \in P^\circ$ is called flippable in $T$ if $p \in P^\circ \setminus V^*T$ or if $p \in V^*T$, of degree 3 in $T$. (a) If $p \in P^\circ \setminus V^*T$ then $T[p]$ is the triangulation with $p$ added as a point of degree 3 (there is a unique way to do so); we call this a point insertion flip. (b) If $p \in V^*T$ of degree 3 in $T$ then $T[p]$ is obtained by removing $p$ and its incident edges; we call this a point removal flip.

![Figure 2: Bistellar flip graphs for 5 points](image)

Whenever we write $T[x]$ for a triangulation $T$, then $x$ is either a flippable point in $P^\circ$ or a flippable edge in $E^*T$, and we write $T[x, y]$ short for $(T[x])[y]$, etc. The bistellar flip graph of $P$ is the graph with vertex set $\mathcal{T}_{\text{part}}(P)$ and edge set $\{\{T, T[x]\} \mid T \in \mathcal{T}_{\text{part}}(P), x \text{ flippable in } T\}$.
Figure 3: Sets of 6 points with isomorphic bistellar flip graphs of triangulations. (Points indicated by crosses are points in $P$ skipped in the corresponding triangulation.)

The bistellar flip graph is connected (this follows easily from the connectedness of the edge flip graph of full triangulations, as established by Lawson in 1972 [9]). Here, we investigate how well connected the bistellar flip graph is. We refer to standard texts like [3, 7] for basics like the definition of $k$-vertex connectivity and Menger’s Theorem. Our main result is:

**Theorem 1.4** Let $P$ be a set of $n \geq 3$ points in general position in the plane. Then the bistellar flip graph of $P$ is $(n - 3)$-vertex connected. (This is tight: Any triangulation of $P$ that skips all inner points has degree $(n - 3)$ in the bistellar flip graph.)

This answers (for points in general position) a question by De Loera, Rambau, and Santos in 2010 [5, Exercise 3.23], and by Lee and Santos in 2017 [10, pg. 442]. A corresponding result, $\lceil \frac{n}{2} - 2 \rceil$-connectedness of the edge flip graph of full triangulations, is proved in [14].

A particular way of obtaining a triangulation of a point set $P$ is to vertically lift the points to $\mathbb{R}^3$ such that no 4 points are coplanar, and then to project the lower convex hull of the lifted points back into the plane. Triangulations obtained in this way are called regular triangulations (e.g., [5]). It is well known that point sets may have non-regular triangulations, see Sec. 6.2.

Furthermore, we study the partially ordered set of partial subdivisions of $P$ (see Def. 3.1 below, and, e.g., [5]), in which triangulations are the minimal elements. We introduce the notions of slack (Def. 3.2), perfect coarsenings (Def. 4.1), and perfect coarseners (Def. 4.2), and we prove the so-called Coarsening Lemma 4.6. We consider these our main contributions besides Thm. 1.4. Together with a sufficient condition for the regularity of partial triangulations and subdivisions (Thm. 6.1 and Regularity Preservation Lemma 6.10, Sec. 6), these yield several
other results on the structure of the bistellar flip graph and regular triangulations (see abstract); in particular, they allow us to settle, in an unexpected direction, another question by F. Santos [13] regarding the size of certificates for the existence of non-regular triangulations in the plane (Thm. 7.10, Sec. 7.3).

If \( P \) is in convex position, full, partial, and regular triangulations coincide. It is well-known that there is an \((n - 3)\)-dimensional convex polytope, the associahedron, whose vertices correspond to the triangulations of \( P \) and whose edges correspond to flips (Fig. 4, see [4] for a historical account). A classical theorem of Balinski [2], which asserts that the graph of any \( d \)-dimensional polytope is \( d \)-connected, immediately implies that the graph of the associahedron is \((n - 3)\)-connected. More generally, for arbitrary sets in the plane, it is known that there is an \((n - 3)\)-dimensional polytope, the secondary polytope defined by Gelfand et al. [8], whose vertices correspond to the regular triangulations of \( P \) and edges correspond to bistellar flips; again, Balinski’s Theorem implies \((n - 3)\)-connectivity. Our result extends this to arbitrary triangulations of arbitrary sets in general position in the plane.

Figure 4: The flip graph of the convex hexagon, the graph of the order 5 associahedron.

**Approach and Intuition.** There is evidence that the bistellar flip graph of partial triangulations does not exhibit a polytopal structure as we see it with regular triangulations [5]. Still, the intuition behind our approach is to “pretend” that such a structure exists, at least locally for the small dimensional features. This will become clearer below, and is made more explicit in Sec. 5 where we consider the link of a triangulation (related to the vertex figure in a polytope) and Sec. 7.1 where it shown how the bistellar flip graph can be covered by polytopal structures.

2 Preliminaries, Terminology, and Notation

**Definition 2.1 (legal graph; region)** For a graph \( G = (P', E) \), \( P' \subseteq P \), we let \( V^{bg} G \) be the points in \( P' \) which are isolated in \( G \), called bystanders in \( G \). \( G \) is called legal if it is plane, if \( E_{\text{hull}}(P) \subseteq E G \) (hence \( \text{extr} P \subseteq P' \)), and if the graph \((V G \setminus V^{bg} G, E G)\) is 2-edge connected.

Let \( G \) be a legal graph. Similar to triangulations, we define \( E^{\circ} G := E G \setminus E_{\text{hull}} \) and \( V^{\circ} G := V G \cap P^o \). Moreover, we let \( V^{inv} G := V^{\circ} G \setminus V^{bg} G \) (the involved points). Bounded faces of \((V G \setminus V^{bg} G, E G)\) are called regions of \( G \), i.e., these are bounded connected components in the complement of the straight line embedding of \( G \), ignoring its bystanders. \( R G \) denotes the set of regions of \( G \).

Regions of legal graphs are bounded simply connected polygonal open sets, pairwise disjoint. We state the following well-known facts for ease of reference.

**Lemma 2.2** For a full triangulation \( T \) of \( P \), \(|ET| = |E^cT| + h = 3n - 3 - h = 3n - 3 + 2h\) and \(|RT| = 2n - 2 - h = 2n - 2 + h\) (recall that the unbounded face is not a region).
Definition 2.3 (locked) In a legal graph $G$ on $P$, an edge $e \in E_G$ is locked at endpoint $p$ if the angle obtained at $p$ (between the edges adjacent to $e$ at $p$) after removal of $e$ exceeds $\pi$.

An edge in a triangulation is flippable if it is locked by none of its endpoints. Edges locked at a common endpoint $p$ have to be consecutive around $p$. There can be at most 3 edges locked at a given point $p$, and 3 edges can be locked at $p$ only if $p$ has degree 3.

Given a legal graph $G$, we consider partial orientations $\bar{G}$: These assign orientations to some (not all) of the edges in $E_G$, with no edge oriented in both directions, and with the boundary edges not oriented. We need the following [14, Lemma 5.1(i)]:

Lemma 2.4 (Unoriented Edges Lemma) Let $G$ be a legal graph with $\nabla^{by} G = \emptyset$, $N := |V_G|$, and $D := 3N - 3 - h - |E_G|$, i.e., the number of edges missing in $G$ towards a full triangulation of $V_G$. For $\bar{G}$ a partial orientation of $G$, let $C_i$ be the number of inner points of $\bar{G}$ with indegree $i$ and suppose $C_i = 0$ for $i \geq 4$. Then the number of unoriented inner edges is at least $N - 3 - C_3 - D$.

To indicate, how this can be useful in our context, consider $G = T$, $T$ a triangulation, i.e., $D = 0$. Orient every locked inner edge to the endpoint where it is locked. Then $C_3 = 0$ for $i \geq 4$, $C_3$ is exactly the number of inner points of degree 3, and the inner unoriented edges are exactly the unlocked, i.e., flippable edges. It follows that there are $C_3$ point removal flips, at least $N - 3 - C_3$ edge flips, and obviously $n - N$ point insertion flips. Altogether, there are at least $n - 3$ flips.

3 Partial Subdivisions – Slack and Order

We now define partial subdivisions, which form a poset in which the partial triangulations of $P$ are the minimal elements.

Definition 3.1 (full and partial subdivision) A partial subdivision $S$ on $P$ is a legal graph with all of its regions convex. For a region $r$ of $S$, let $V_r := r \cap V_S$ ($r$ the closure of $r$). $S_{\text{triv}} = S_{\text{triv}}(P) := (P, \mathbb{E}_{\text{hull}})$ is called the trivial subdivision of $P$. If $V_S = P$ and $\nabla^{by} S = \emptyset$, then $S$ is called a full subdivision on $P$.

Convention 2 From now on, we will mostly use “subdivision” for “partial subdivision”.

$V_S$ is essential in the definition of a subdivision, it is not simply the set of endpoints of edges in $S$, there are also bystanders; e.g., for $T$ a triangulation of $P$, all graphs $(P', ET)$, $VT \subseteq P' \subseteq P$, are subdivisions of $P$, all different. $V_S$ partitions into boundary points, involved points, and bystanders, i.e., $V_S = xtrP \cup \mathbb{V}^{inv}S \cup \mathbb{V}^{by}S$. Moreover there are the skipped points, $P \setminus V_S$.

A first important example of a subdivision is obtained from a triangulation $T$ and an element $x$ flippable in $T$, i.e., $\{T, T[x]\}$ is an edge of the bistellar flip graph:

$$T_{\pm x} := (VT \cup VT[x], ET \cap ET[x])$$

If $x = e$ is a flippable edge, then $T_{\pm e}$ has one convex quadrilateral region $Q$; all other regions are triangular. We obtain $T$ and $T[e]$ from $T_{\pm e}$ by adding one or the other of the 2 diagonals of $Q$ to $T_{\pm e}$. If $x = p$ is a flippable point, then $T_{\pm p}$ is almost a triangulation, all regions are triangular, except that $p \in VT_{\pm p}$ is a bystander. We obtain $T$ and $T[p]$ by either removing this point from $T_{\pm p}$ or by adding the three edges from $p$ to the points of the triangular region in which $p$ lies. The subdivision $T_{\pm x}$ is close to a triangulation and, in a sense, represents the flip between $T$ and $T[x]$. To formalize and generalize this we introduce the following notion:
Definition 3.2 (slack) Given a subdivision $S$ of $P$, we call a region of $S$ active if it is not triangular or if it contains at least one point in $VS$ (necessarily a bystander) in its interior.

For a region $r$ of $S$, we define its slack $sl_r := |Vr| - 3$. The slack of $S$, $sl_S$, is the sum of slacks of its regions.

Lemma 3.3 For a subdivision $S$ with $s$ bystanders we have

$$sl_S = 3(|VS| - s) - 3 - h - |ES| + s = 3|VS| - 3 - h - |ES| - 2s.$$  

Proof. The slack of a region $r$ equals the number of edges it takes to triangulate $r$ (ignoring bystanders) plus the number of bystanders. Thus, $sl_S$ is the number of edges it takes to triangulate $(VS \setminus V^{by}S, ES) \cup V^{by}S$. Now the claim follows from Lemma 2.2.

Observation 3.4 Let $S$ be a subdivision. (i) $sl_S = 0$ iff $S$ is a triangulation iff $S$ has no active region. (ii) $sl_S = 1$ iff $S$ has exactly one active region of slack 1; this region is either a convex quadrilateral, or a triangular region with one bystander in its interior. (iii) $sl_S = 2$ iff $S$ has either (a) exactly two active regions, both of slack 1, or (b) exactly one active region of slack 2, where this region is either a convex pentagon, or a convex quadrilateral with one bystander in its interior, or a triangular region with two bystanders in its interior (Fig. 2).

![Figure 5: Hasse diagram of the partial order ≤ for a set of 5 points.](image)

Definition 3.5 (coarsening, refinement) For subdivisions $S_1$ and $S_2$ of $P$, $S_2$ coarsens $S_1$, in symbols $S_2 \trianglerighteq S_1$, if $VS_2 \supseteq VS_1$, and $ES_2 \subseteq ES_1$. We also say that $S_1$ refines $S_2$, ($S_1 \triangleleft S_2$).

The example in Fig. 5 hides some of the intricacies of the partial order $\leq$; e.g., in general, it is not true that all paths from a triangulation to $S_{triv}$ have the same length $n - 3$. $S_{triv}$ is the unique coarsest (maximal) element. The triangulations (i.e., subdivisions of slack 0) are the minimal elements.

Definition 3.6 (set of refining triangulations) For a subdivision $S$ of $P$ we let $T_{part}(S) \triangleq \{T \in T_{part}(P) \mid T \leq S\}$.

Note that $T_{part}(S_{triv}) = T_{part}(P)$ and for $x$ flippable in $T$, $T_{part}(T_{\pm x}) = \{T, T[x]\}$.

Observation 3.7 (i) Any subdivision $S$ of slack 1 of $P$ equals $T_{xx}$ for some triangulation $T \leq S$ and some $x$ flippable in $T$. (ii) Let $S$ be a subdivision of slack 2 of $P$. If there are exactly 2 active regions in $S$ (of slack 1 each), then $T_{part}(S)$ has cardinality 4, spanning a 4-cycle in the bistellar flip graph of $P$ (Fig. 6). If there is exactly one active region in $S$ (of slack 2), then $T_{part}(S)$ has cardinality 5, spanning a 5-cycle (Fig. 2).
Figure 6: A subdivision $S$ with two active regions of slack 1 each with $\mathcal{T}_{\text{part}}(S)$ spanning a 4-cycle.

The 4- and 5-cycles mentioned in Obs. 3.7 are called elementary cycles in [11].

**Lemma 3.8** Any proper refinement of a subdivision of slack 2 has slack at most 1.

**Proof.** Let $s\text{l}S' = 2$ and let $S$ be a proper refinement of $S'$. For a refinement we add $m$ edges, thereby involving $s'$ bystanders, and we remove $s''$ bystanders (some of these parameters may be 0, but not all, since the refinement is assumed to be proper). We have $s\text{l}S = s\text{l}S' - (m - 2s' + s'')$ (easy consequence of Lemma 3.3) and want to show $m - 2s' + s'' > 0$.

Since $s\text{l}S' = 2$, $S'$ has at most two bystanders and thus $s' \leq 2$. If $s' = 0$, then $m - 2s' + s'' > 0$ holds, since some of the three parameters have to be positive. If $s' = 1$, we observe that we need at least three edges to involve a bystander and $m - 2s' \geq 3 - 2 \cdot 1$. If $s' = 2$, we need at least 5 edges to involve two bystanders and $m - 2s' \geq 5 - 2 \cdot 2$.

For $D \geq 3$, a proper refinement of a subdivision of slack $D$ can have slack $D$ or even higher (Fig. 7). The proof fails, since we can involve three bystanders with 6 edges.

Figure 7: 8 points, with a subdivision of slack 6, a refinement of $S_{\text{triv}}$ of slack $8 - 3 = 5$.

Intuitively, as briefly alluded to at the end of Sec. 1, one can think of the subdivisions as the faces of a higher-dimensional geometric structure behind the bistellar flip graph, with the slack playing the role of dimension, somewhat analogous to the secondary polytope for regular triangulations. (For the edge flip graph of full triangulations, an analogous higher-dimensional *flip complex* is treated in [12, 11], and provides a similar geometric intuition for the arguments in [14].) The following lemma shows that -- for slack at most 2 -- we have the property corresponding to the fact that faces of dimension $d$ are either equal, or intersect in a common face of smaller dimension (possibly empty).

**Lemma 3.9** (i) For subdivisions $S_1$ and $S_2$ of slack 2, $\mathcal{T}_{\text{part}}(S_1) \cap \mathcal{T}_{\text{part}}(S_2)$ is either (a) empty, (b) equals $\{T\}$ for some triangulation $T$, (c) equals $\{T, T[x]\}$ for some triangulation $T$ and some flippable element $x$, or (d) $S_1 = S_2$.

(ii) Let $x$ and $y$ be two distinct flippable elements in triangulation $T$. If there is a subdivision $S$ of slack 2 with $\{T[x], T, T[y]\} \subseteq \mathcal{T}_{\text{part}}(S)$, then this $S$ is unique.

**Proof.** If $\mathcal{T}_{\text{part}}(S_1) \cap \mathcal{T}_{\text{part}}(S_2)$ contains some triangulation, then we easily see that $S_1 \land S_2 := (V_{S_1} \cap V_{S_2}, E_{S_1} \cup E_{S_2})$ is a subdivision, and $\mathcal{T}_{\text{part}}(S_1 \land S_2) = \mathcal{T}_{\text{part}}(S_1) \cap \mathcal{T}_{\text{part}}(S_2)$. 

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(i) If (a) does not apply, let $S := S_1 \land S_2$, a subdivision with $\mathcal{T}_{\text{part}}(S) = \mathcal{T}_{\text{part}}(S_1) \cap \mathcal{T}_{\text{part}}(S_2)$. If $s \mathcal{I} S = 0$ we have property (b), if $s \mathcal{I} S = 1$ we have property (c). In the remaining case $s \mathcal{I} S \geq 2$, $S$ is a refinement of $S_1$ and of $S_2$. Lemma 3.8 tells us that $S$ cannot be a proper refinement of $S_1$, hence $S = S_1$; similarly, $S = S_2$, hence $S_1 = S_2$.

(ii) Suppose $S_1$ and $S_2$ are subdivisions of slack 2 with $\{T[x], T, T[y]\} \subseteq \mathcal{T}_{\text{part}}(S_1) \cap \mathcal{T}_{\text{part}}(S_2)$. Since options (a-c) above cannot apply, we are left with $S_1 = S_2$.

Two edges incident to a vertex of a polytope may span a 2-face, or not; same here:

**Definition 3.10 (compatible elements)** Two distinct flippable elements $x, y \in V^o T \cup E^o T$ are called compatible in $T$, in symbols $x \odot y$, if there is a subdivision $T_{\pm x,y} \preceq T$ of slack 2, s.t. $\{T[x], T, T[y]\} \subseteq \mathcal{T}_{\text{part}}(T_{\pm x,y})$. (Note that $T_{\pm x,y}$ is unique, by Lemma 3.9(ii).) Otherwise, $x$ and $y$ are called incompatible in $T$, in symbols $x \not\odot y$.

This needs some time to digest. In particular, if two flippable edges $e$ and $f$ share a common endpoint of degree 4, then they are compatible (Fig. 8 bottom left), quite contrary to the situation for full triangulations as treated in [14]. The configurations of 2 flippable but incompatible are shown in Fig. 8, rightmost examples: (a) Two flippable edges $e$ and $f$ whose removal creates a nonconvex pentagon and whose common endpoint $q$ has degree at least 5. (b) A flippable edge $e$ and a flippable point $p$ of degree 3 whose removal creates a nonconvex quadrilateral region whose reflex point $q$ has degree at least 5 in the triangulation.

![Figure 8: Compatible elements (with overlapping incident regions, all contained in a 5-cycle, see Fig. 2, and incompatible elements (two rightmost, where $q$ is assumed to have degree at least 5). Shaded areas are unions of incident regions of flippable elements (not the active region in $T_{\pm x,y}$).)](image)

What is essential for us is that whenever $x$ and $y$ are compatible in a triangulation $T$, then there is a cycle of length 4 or 5 containing $\{T[x], T, T[y]\}$, and therefore, apart from the path $(T[x], T, T[y])$, there exists a $T$-avoiding $T[x]-T[y]$-path of length 2 or 3.

**Observation 3.11** Let $T \in \mathcal{T}_{\text{part}}(P)$. (i) A skipped point $p \in P^o \setminus V^o T$ is compatible with every flippable element of $T$. (ii) Any two flippable points $p, q \in P^o$ are compatible.

## 4 Coarsening Partial Subdivisions

As in [14] for full triangulations, the existence of many coarsenings is essential for our connectivity result. In order to motivate the definitions below, note that for full subdivisions (as employed in [14]), if $S_1 \preceq S_2$, then $(S_1, S_2)$ is an edge in the Hasse-diagram of the partial order $\leq$ iff $s \mathcal{I} S_2 = s \mathcal{I} S_1 + 1$. For partial subdivisions, this is not the case (Fig. 9).

**Definition 4.1 (direct, perfect coarsening)** Let $S_1$ and $S_2$ be subdivisions. (i) We call $S_2$ a direct coarsening of $S_1$ (and $S_1$ a direct refinement of $S_2$), in symbols $S_1 \prec_{\text{dir}} S_2$, if $S_1 \preceq S_2$ and any subdivision $S$ with $S_1 \preceq S \preceq S_2$ satisfies $S \in \{S_1, S_2\}$ (equivalently, if $(S_1, S_2)$ is an edge in the Hasse diagram of $\leq$). (ii) We call $S_2$ a perfect coarsening of $S_1$ ($S_1$ a perfect
Figure 9: $slS_1 = 2$, $slS_2 = 3$, $slS_3 = 3$. Note that $S_2 \prec_{\text{dir}} S_3$ but $S_2 \not\prec S_3$, and that $S_1 \leq S_3$ with $slS_3 = slS_1 + 1$ but $S_1 \neq S_3$.

Figure 10: A subdivision, edges are oriented to endpoints where locked (not what we called a partial orientation, since some edges are doubly oriented). Removing the three edges incident to $p_0$ does not yield a subdivision, since a reflex angle occurs at $p_1$ and $p_2$. The edges incident to $\{p_0, p_1, p_2\}$ are not looked outside this set, but removing all incident edges creates a reflex angle at point $q$.

refinement of $S_2$, in symbols $S_1 \prec_1 S_2$, if $S_1 \prec_{\text{dir}} S_2$ and $slS_2 = slS_1 + 1$. (iii) $\prec_1^*$ is the reflexive transitive closure of $\prec_1$.

The reflexive transitive closure of $\prec_{\text{dir}}$ is exactly $\preceq$, while $\prec_1^* \subseteq \preceq$ and, in general, the inclusion is proper.

To motivate the upcoming definitions, let us discuss a few possibilities of coarsenings, direct coarsenings and perfect coarsenings. There are the simple operations of removing an unlocked edge, and of adding a point $p \in P \setminus VS$ as a bystander. For a triangulation, we can isolate a point of degree 3. How does this generalize to subdivisions? Removing the edges incident to a point of degree 3 does not work if some incident edge might be locked at its other endpoint (e.g., $p_0$ in Fig.10). If, however, no edge incident to a given point $p$ (of any degree) is locked at the respective other endpoint, then we can isolate this point for a coarsening $S'$. Unless $p$ has degree 3, $S'$ is not a direct coarsening of $S$, though. If $p$ has degree at least 4, one of the incident edges, say $e$, is not locked at $p$, thus not locked at all, and therefore, $S \preceq S'' \preceq S'$ for $S'' := \langle VS, ES \setminus \{e\} \rangle$. Finally, suppose we want to isolate all points in a set $U$ of points for obtaining a coarsening $S'$. For this to work, it is necessary that no edge $e$ connecting $U$ with the outside is locked at the endpoint of $e$ not in $U$. However, this is not a sufficient condition, because several edges connecting $U$ with a point not in $U$ can collectively create a reflex vertex by their removal (e.g., $U = \{p_0, p_1, p_2\}$ in Fig.10). Moreover, for $S \prec_{\text{dir}} S'$ to hold, $U$ cannot be incident to unlocked edges, and no nonempty subset of $U$ can be suitable for such an isolation operation.

**Definition 4.2 (prime, perfect coarsener; increment)** Let $S$ be a subdivision and let $U \subseteq VS \cap P^o$. (i) $U$ is called a coarsener, if (a) $U$ is incident to at least one edge in $S$, and (b) removal of the set $E_U$ of all edges incident to $U$ in $S$ yields a subdivision. (ii) If $U$ is a coarsener, the increment of $U$, $\text{inc}U$, is defined as $|E_U| - 2|U|$. (iii) $U$ is called a prime coarsener, if (a) $U$ is a coarsener, (b) $U$ is a minimal coarsener, i.e., no proper subset of $U$ is a coarsener, and (c) all edges incident to $U$ are locked. (iv) $U$ is called a perfect coarsener, if (a) $U$ is a prime coarsener, and (b) $\text{inc}U = 1$.

The following observation, a simple consequence of Lemma 3.3, explains the term “increment”.

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Figure 11: Prime coarseners, all perfect, except for the rightmost one (with \(\text{inc} = 0\)).

**Observation 4.3** Let \(S\) be a subdivision with coarsener \(U\), and let \(S'\) be the subdivision obtained from \(S\) by removing all edges incident to \(U\). Then \(\text{sl} S' = \text{sl} S + \text{inc} U\).

**Observation 4.4**
(i) Every subdivision \(S\) with \(E^o S \neq \emptyset\) has a coarsener (the set \(V^o S\)).
(ii) If \(U_1\) and \(U_2\) are coarseners, then \(U_1 \cap U_2\) is a coarsener, unless there is no edge of \(S\) incident to \(U_1 \cap U_2\).
(iii) If \(U_1\) and \(U_2\) are prime coarseners, then \(U_1 = U_2\) or \(U_1 \cap U_2 = \emptyset\).
(iv) If \(U\) is a prime coarsener, then the subgraph of \(S\) induced by \(U\) is connected.

The following observation lists all ways of obtaining direct and perfect coarsenings.

**Observation 4.5** Let \(S = (V, E)\) and \(S'\) be subdivisions.
(i) \(S'\) is a direct coarsening of \(S\) iff it is obtained from \(S\) by one of the following.
- Adding a single point. For \(p \in P \setminus V\), \(S' = (V \cup \{p\}, E)\) (with \(\text{sl} S' = \text{sl} S + 1\)).
- Removing a single unlocked edge. For \(e \in E\), not locked by either of its two endpoints, \(S' = (V, E \setminus \{e\})\) (with \(\text{sl} S' = \text{sl} S + 1\)).
- Isolating a prime coarsener. For \(U\) a prime coarsener in \(S\), \(S'\) is obtained from \(S\) by removal of the set, \(E_U\), of all edges incident to points in \(U\), i.e., \(S' = (V, E \setminus E_U)\) (with \(\text{sl} S' = \text{sl} S + \text{inc} U\)).
(ii) \(S'\) is a perfect coarsening of \(S\) iff it is obtained from \(S\) by adding a single point, removing a single unlocked edge, or by isolating a perfect coarsener.

**Lemma 4.6 (Coarsening Lemma)** Every subdivision of slack \(D\) has at least \(n - 3 - D\) perfect coarsenings (i.e., direct coarsenings of slack \(D + 1\)).

*Proof.* We start with the case \(D = 0\), i.e., we have a triangulation \(T\) and we want to show that there are at least \(n - 3\) direct coarsenings of slack 1. Let \(N := |VT|\). We orient inner locked edges to their locking endpoints (recall that in a triangulation there is at most one such endpoint for each inner edge). Let \(C_i, i \in \mathbb{N}_0\), be the number of points \(p \in V^o T\) with indegree \(i\). The number of unoriented, thus unlocked edges is at least \(N - 3 - C_3\) (Lemma 2.4).

There are \(n - N\) subdivisions obtained from \(T\) by adding a single point, there are at least \(N - 3 - C_3\) subdivisions obtained from \(T\) by removing a single unlocked edge, and there are \(C_3\) direct coarsenings obtained from \(T\) by isolating an inner point of degree 3. Adding up these numbers gives at least \(n - 3\) perfect coarsenings of \(T\).

We let \(S\) be a subdivision of slack \(D \geq 1\) assuming the assertion holds for slack less than \(D\).

**Case 1.** There is a bystander \(p_0 \in VS \cap P^o\). Then \((VS \setminus \{p_0\}, ES)\) is a subdivision of slack \(D - 1\) of \(P \setminus \{p_0\}\) with at least \((n - 1) - 3 - (D - 1) = n - 3 - D\) perfect coarsenings of slack.
D. For each such perfect coarsening $S'$, the subdivision $(VS' \cup \{p_0\}, ES')$ is a direct coarsening of $S$ of slack $D + 1$, thus a perfect coarsening.

Case 2. There is no bystander in $S$. Again we employ a partial orientation of $S$. The choice of the orientation is somewhat more intricate and we will proceed in three phases (Fig. 12). We keep the invariant that the unoriented inner edges are exactly the unlocked inner edges.

In a first phase, we orient all locked inner edges to all of their locking endpoints, i.e., we temporarily allow edges to be directed to both ends (to be corrected in the third phase); edges directed to both endpoints are called mutual edges. We can give the following interpretation to an edge directed from $p$ to $q$: If we decide to isolate $p$ (i.e., remove all incident edges of $p$) for a coarsening of $S$, then $q$ becomes a reflex point of some region and we have to isolate $q$ as well (i.e., every coarsener containing $p$ must contain $q$ as well). In particular, if $\{p, q\}$ is a mutual edge, then either both or none of the points $p$ and $q$ will be isolated. In fact, if we consider the graph $G$ on $V^o S$ with all mutual edges in the current orientation, then in any coarsening of $S$ either all points in a connected component of $G$ are isolated, or none.

A connected component $K$ of $G$ is called a candidate component, (a) if all edges connecting $K$ with points outside are directed towards $K$, (b) no point in $K$ is incident to an unoriented edge, (c) all points in $K$ have indegree 3, and (d) the mutual edges in $K$ do not form any cycle (i.e., they have to form a spanning tree of $K$). It follows that if $K$ has $k$ points then the number of edges is $3k - (k - 1) = 2k + 1$. The term “candidate” refers to the fact that removing all edges incident to $K$ seems like a direct coarsening step with incrementing the slack by 1 (Lemma 3.3); however, while individual edges connecting $K$ to the rest of the graph are not locked at their endpoints outside $K$, some of these edges collectively may actually create a reflex vertex in this way (see $K$ and $q$ in Fig. 12 (left)). So $K$ is only a candidate for a perfect coarsener.

![Figure 12: Orientation after phase 1, with candidate components shaded (left); after phase 2 (middle), with the connected components of $G^*$; after phase 3 (right), with unoriented edges bold (each of these can be removed for a coarsening of slack 1 larger), and with the candidate components with a leader shaded (perfect coarseners).](image)

We start the second phase of orienting edges further. In the spirit of our remarks about candidate components of $G$, suppose $q$ is an inner point outside a candidate $K$ of $G$ (thus all edges connecting $q$ to $K$ are directed from $q$ to $K$), such that removing the edges connecting $q$ to $K$ creates a reflex angle at $q$. Then we orient one (and only one) of the edges connecting $q$ to $K$, say $\{p, q\}$, also to $q$ (thereby making this edge mutual). We call all the edges connecting

\[1\text{The reader might be worried that } q \text{ now joins the candidate component while possibly not having indegree} \]
to q, except for \{p,q\}, the witnesses of the extra new orientation of \{p,q\} from p to q. We successively proceed orienting edges, with the graph G of mutual edges evolving in this way (and candidate components growing or disappearing). The process will clearly stop at some point when the second phase is completed. We freeze G and denote it by G*.

Before we start the third phase, let us make a few crucial observations:

(i) If p,q are inner points in the same connected component of G*, then any coarsener contains both or none (i.e., if a connected component is a coarsener, then it is prime). This holds after phase 1, and whenever we expand a connected component, it is maintained.

(ii) During the second phase, an edge can be witness only once, and it is and will never be directed to the endpoint where it witnesses. Why? (a) Before it becomes a witness, it connects different connected components of G, after that it is and stays in a connected component of G. (b) Before it becomes a witness, it is not directed to the endpoint to which it witnesses an orientation, after that it is and stays in a connected component of G and can therefore not get an extra direction. (An unoriented edge can never get an orientation and it can never be a witness.)

(iii) If we remove, conceptually, for each incoming edge of a point q the witnesses (which direct away from q) for the orientation of this edge to q, then among remaining incident edges, all the incoming edges are locked at q (an incoming edge that was oriented already in the first phase to q has no witness). In particular, the indegree of q cannot exceed 3, and if q is incident to some not ingoing edge which is not a witness for any edge incoming at q, then the indegree of q is at most 2. (We might generate incoming edges to a point q that are not consecutive around q.)

(iv) If an unoriented edge e connects two points of the same connected component of G*, then both endpoints have indegree at most 2 (recall that this edge e cannot be a witness at its endpoints). If an edge e is directed from a connected component K of G* to a point outside K, then the tail of this edge e has indegree at most 2 (recall that e cannot be a witness at all, since its endpoints are in different connected components if G*).

(v) A candidate component K of G* is a perfect coarsener. It is a coarsener (otherwise, we would have expanded it further), it is a prime coarsener (see (i) above) and \text{inc} K = 1 (we have argued before that a candidate component increases the slack by exactly 1).

The third phase will make sure that each mutual edge loses exactly one direction. Our goal is to have in every connected component K of G* at most one point with indegree 3. To be more precise, only candidate components have exactly one point with indegree 3, others don’t. Consider a connected component K.

(a) If the mutual edges form cycles in K, choose such a cycle c and keep for each edge on c one orientation so that we have a directed cycle, counterclockwise, say. All other mutual edges in K keep the direction in decreasing distance in G* to c, ties broken arbitrarily. This completed, no point in K has indegree 3, since there is always a mutual edge incident that decreases the distance to c and the incoming direction of this edge will be removed.

(b) If K has points of indegree at most 2, choose one such point p with indegree at most 2, orient all mutual edges in K in decreasing distance in G* to p, ties broken arbitrarily. Again, this completed, no point in K will have indegree 3.

(c) If none of the above applies, the mutual edges of K form a spanning tree and all points in K have indegree 3. Moreover, all edges connecting K with points outside are directed

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2 The reader will correctly observe that our approach is very conservative towards prime coarseners, but by what we observed and by what will follow, since we are interested only in perfect coarseners, we can afford to leave alone connected components other than the candidate components.

3 as required in a candidate component. Fine, this just means that the enlarged component is not a candidate component, i.e., we have lost a candidate component.
towards $K$ and no edge within $K$ is unoriented (violation of these forces property form a point of indegree at most 2). So this is a candidate component. We choose an arbitrary point $p$ in $K$, call it the leader of $K$, and for all mutual edges keep the orientation of decreasing distance in $G^*$ to $p$ (ties cannot occur, mutual edges form a tree). Now the leader $p$ is the only point of $K$ with indegree 3, all other points in $K$ have indegree exactly 2.

Phase 3 is completed. Let us denote the obtained partial orientation on $S$ as $\vec{S}^*$. It has identified certain connected components of $G^*$ which have a leader of indegree 3. In fact, every point of indegree 3 after phase 3 is part of a perfect coarsener (probably of size 1).

We can now describe a sufficient supply of perfect coarsenings of $S$. Let $N := |VS|$ and let $C_3$ be the number of points of indegree 3 in $\vec{S}^*$. We know that there are at least $N - 3 - D - C_3$ unoriented inner edges (Lemma 2.4).

(I) There are $n - N$ perfect coarsenings obtained by adding a single point $p \in P \setminus VS$.

(II) There are at least $N - 3 - D - C_3$ perfect coarsenings obtained by removing a single unoriented inner edge in $\vec{S}^*$.

(III) And there are $C_3$ perfect coarsenings obtained by isolating all points in a candidate component in $G^*$ (with a leader of indegree 3).

In this way we have identified at least $n - 3 - D$ perfect coarsenings.

Here are two immediate implications which we will need later: The first in the connectivity proof in Sec. 5 and the second for the result about covering of the bistellar flip graph by $(n - 3)$-polytopes in Sec. 7.

**Corollary 4.7** Let $T$ be a triangulation. (i) $T$ has at least $n - 3$ flippable elements. (ii) For every $x$ flippable in $T$ there are at least $n - 4$ elements compatible with $x$.

Part (i) of the corollary was proved, without general position assumption, in [6, Thm. 2.1].

**Corollary 4.8** For every subdivision $S'$ with $slS' \leq n - 3$ there is a subdivision $S$ with $S' \prec_1 S$ and $slS = n - 3$.

## 5 Link of a Triangulation – Proof of $(n - 3)$-Connectivity

To complete the proof of the connectivity bound for the bistellar flip graph, we need two further ingredients. The first is the following variant of Menger’s Theorem [14, Lemma 3.1].

**Lemma 5.1 (Local Menger)** Let $k \geq 2$ be an integer and let $G$ be a connected simple undirected graph. Then $G$ is $k$-vertex connected iff $G$ has at least $k + 1$ vertices and for any pair of vertices $u$ and $v$ at distance 2 there are $k$ pairwise internally vertex disjoint $u$-$v$-paths.

The second ingredient are *links* of triangulations, which are graphs that represent the compatibility relation among flippable elements (Def. 3.10). Recall that if $x$ is a flippable element in a triangulation $T$ then $T_{\pm x}$ denotes the subdivision with $T_{\text{part}}(T_{\pm x}) = \{T, T[x]\}$, and if $y$ is compatible with $x$, denoted $x \diamond y$, then $T_{\pm x,y}$ denotes the unique coarsening of slack 2 of $T$ with $\{T[x], T, T[y]\} \subseteq T_{\text{part}}(T_{\pm x,y})$ (Def. 3.10).

**Definition 5.2 (link)** For $T \in T_{\text{part}}(P)$, the link of $T$, denoted $LkT$, is the edge-weighted graph with vertices $FT := \{x \in V^oT \cup E^+T \mid x \text{ flippable in } T\}$ and edge set $\{\{x, y\} \in (FT)^2 \mid x \diamond y\}$. The weight of an edge $\{x, y\}$ is $|T_{\text{part}}(T_{\pm x,y})| - 2$ (which is 2 or 3).

We will see that it is enough to prove $(n - 4)$-vertex connectivity of all links. Again, the intuition can be explained for polytopes: Recall that for a vertex $v$ in a $d$-polytope $P$, its vertex figure is the $(d - 1)$-polytope $P'$ obtained by intersecting $P$ with a hyperplane that separates $v$ from the remaining vertices of the polytope. Vertices of $P'$ correspond to edges of $P$, edges in the graph
of \( P' \) correspond to 2-faces of \( P \). There is a natural way of mapping paths in the graph of \( P' \) to paths in the graph of \( P \). This can be easily made an inductive proof of Balinski’s Theorem, as mentioned in Sec. 1 (using the Local Menger Lemma 5.1). We follow exactly this line of thought in our setting, except that we will not need induction – the link is a dense graph which directly yields \((n-4)\)-vertex connectivity.

Note that, indeed, the following lemma implies that the complement of the link is sparse, hence the link is dense.

**Lemma 5.3** The complement of \( \text{Lk} T \) has no cycle of length 4, i.e., if \((x_0, x_1, x_2, x_3)\) are flippable elements in \( T \), then there exists \( i \in \{0, 1, 2, 3\} \) such that \( x_i \circ x_{i+1} \mod 3 \).

**Proof.** Recall that all \( p \in P^e \setminus V^e T \) are flippable and compatible with every flippable element (Obs. 3.11), hence let us assume \( \{x_0, x_1, x_2, x_3\} \subseteq V^e T \cup E^e T \). Moreover, if \( p, q \in V^e T \) are two distinct points flippable in \( T \), then \( p \circ q \). Hence, we assume that no two consecutive elements in the cyclic sequence \((x_0, x_1, x_2, x_3)\) are points; w.l.o.g. let \( x_0 = e \) and \( x_2 = f \) be edges.

![Figure 13: Intersections of boundaries of territories of two flippable edges.](image)

For an inner edge \( e \) in a triangulation \( T \), we define its territory \( \text{terre} = \text{terr}_T e \), as the interior of the closure of the union of the two regions in \( T \) incident to \( e \). Obviously, \( e \) is flippable in \( T \) iff the quadrilateral \( \text{terr}_T e \) is convex. Note that for an element \( x \) to be incompatible with edge \( e \), \( x \) must appear on the boundary of \( \text{terre} \), and analogously elements incompatible with \( f \) must appear on the boundary of \( \text{terr} f \).

We show that there is at most one flippable element in the intersection of the boundaries of \( \text{terre} \) and \( \text{terr} f \) (Fig. 13). This is obvious, if \( \text{terre} \cap \text{terr} f \) is empty or a single point (recall that \( \overline{A} \) denotes the closure of \( A \subseteq \mathbb{R}^2 \)). If this intersection is an edge and its two endpoints, we observe that among any edge and its two incident points, at most one element can be flippable (inner degree 3 points cannot be adjacent and cannot be incident to a flippable edge). This covers already all possibilities if \( \text{terre} \) and \( \text{terr} f \) are disjoint (since they are convex). Finally, \( \text{terre} \cap \text{terr} f \) can be a triangle, in which case the common boundary consists of the common endpoint of \( e \) and \( f \), clearly not flippable, and an edge with its two endpoints; again, only one of these three can be flippable.

**Lemma 5.4** Given a triangulation \( T \) with \( x \) and \( y \) flippable elements, \( x \neq y \), every \( x\cdot y \)-path of weight \( w \) in \( \text{Lk} T \) induces a \( T \)-avoiding \( T[x] \cdot T[y] \)-path of length \( w \) in the bistellar flip graph. Interior vertex disjoint \( x\cdot y \)-paths in the link induce interior vertex disjoint \( T[x] \cdot T[y] \)-paths.

**Proof.** Given an \( x\cdot y \)-path in \( \text{Lk} T \), we replace every edge \( \{z', z''\} \) on this path by \( \text{path}_T(z', z'') = (T[z'], \ldots, T[z'']) \) (of length 2 or 3) which draws its (1 or 2) interior vertices from \( T_{\text{part}}(T_{\pm z', z''}) \setminus \{T[z'], T[z'']\} \) (Fig. 14); these vertices must have distance 2 from \( T \) in the flip graph, while \( T[z'] \) and \( T[z''] \) have distance 1. In the resulting \( T[x] \cdot T[y] \)-path, all interior vertices adjacent to \( T \) (i.e., of the form \( T[z] \)) are distinct from interior vertices at other paths by assumption on the initial paths in the link. For vertices at distance 2, suppose \( T_1 \in T_{\text{part}}(T_{\pm z'_1, z''_1}) \) coincides
with $T_2 \in \mathcal{T}_{\text{part}}(T_{z_1',z_1''})$, both at distance 2 from $T$. Since $sl(T_{z_1',z_1''}) = sl(T_{z_2',z_2''}) = 2$, we have that $\mathcal{T}_{\text{part}}(T_{z_1',z_1''}) \cap \mathcal{T}_{\text{part}}(T_{z_2',z_2''})$ either (a) equals $\{T\}$, (b) equals $\{T, T[z]\}$ for some $z$, or (c) $T_{z_1',z_1''} = T_{z_2',z_2''}$ (Lemma 3.9). In (a-b) $T_{z_1',z_1''}$ and $T_{z_2',z_2''}$ cannot possibly share a vertex at distance 2 from $T$. Thus (c) holds. $T_{z_1',z_1''} = T_{z_2',z_2''}$ implies $\{z_1', z_1''\} = \{z_2', z_2''\}$. 

**Lemma 5.5** The link $LkT$ satisfies: (i) There are at least $n - 3$ vertices. (ii) Every vertex has degree at least $n - 4$. (iii) Every pair of non-adjacent vertices has at least $n - 4$ connecting interior vertex disjoint paths (all of length at most 3). (iv) It is $(n - 4)$-vertex connected.

**Proof.** (i) $x$ is a vertex in $LkT$ iff $x$ is flippable in $T$ iff $T_{\pm x}$ is a subdivision of slack 1, a perfect coarsening of $T$. Lemma 4.6 ensures the existence of at least $n - 3$ such perfect coarsenings.

(ii) Let $x$ be a vertex of $LkT$. $T_{\pm x}$, a subdivision of slack 1, has at least $n - 4$ perfect coarsenings of slack 2 (Lemma 4.6). Each such coarsening equals $T_{x,y}$ for some $x \neq y$, i.e., $y$ is a neighbor of $x$ in $LkT$.

(iii) Let $x$ and $y$ be non-adjacent vertices of $LkT$, i.e., $x \neq y$. Let $z_1, z_2, \ldots, z_\ell$ be all flippable elements in $T$ compatible with both $x$ and $y$. Each such element $z_i$ gives rise to a path $(x, z_i, y)$ of length 2 in the link. If $\ell \geq n - 4$, we are done. Otherwise, there is an extra supply of elements $x_1, x_2, \ldots, x_{n-4-\ell}$ compatible with $x$ but not with $y$, and elements $y_1, y_2, \ldots, y_{n-4-\ell}$ compatible with $y$ but not with $x$. For all $i = 1, 2, \ldots, n - 4 - \ell$, $y_i \neq x$, $x \neq y$, and $y \neq x_i$. By Lemma 5.3, $x_i \diamond y_i$, hence we have a path $(x, x_i, y)$ of length 3 in the link. Obviously, these paths of length 3 are pairwise internally vertex disjoint, and also internally vertex disjoint from all $x$-$y$-paths of length 2 (interior vertices on both two paths are connected to $x$ and $y$, interior vertices on the constructed length 3 paths are not).

(iv) We apply the Local Menger Lemma 5.1. Indeed, $LkT$ has at least $(n - 4) + 1 = n - 3$ vertices (see (i)), and every pair of vertices at distance 2 has at least $n - 4$ internally vertex disjoint paths (see (iii)). Hence, the link is $(n - 4)$-vertex connected.

### 5.1 $(n - 3)$-Connectivity of the Bistellar Flip Graph – Proof of Thm. 1.4

**Proof of Thm. 1.4.** If $n \leq 4$, $(n - 3)$-vertex connectivity can be easily checked according to the definition of $k$-vertex connectivity. For $n \geq 5$, we employ the Local Menger Lemma 5.1. Thus (apart from the presence of at least $n - 2$ vertices), we have to show that for any triangulation $T$ and flippable elements $x$ and $y$, there are at least $n - 3$ internally vertex disjoint $T[x]$-$T[y]$-paths in the bistellar flip graph. We know that in $LkT$ has at least $n - 4$ internally vertex disjoint $x$-$y$-paths (Menger’s Theorem, [3, 7]). Therefore, there are at least $n - 4$ interior vertex disjoint $T[x]$-$T[y]$-paths disjoint from $T$ (Lemma 5.4). Together with the path $(T[x], T, T[y])$, the claim is established.

### 6 Regular Subdivisions by Successive Perfect Refinements

Suppose $h = 3$ and consider stacked triangulations of $P$, i.e., we start with the triangulation $(xtrP, E_{null})$, and then we successively add points in $P^3$ by connecting a new point to the three...
vertices of the triangle where it lands in (Fig. 15). It is easily seen that this yields regular
triangulations. The result of this section is the following sufficient condition for the regularity
of a subdivision (Def. 6.2 below), which can be seen as a generalization of the regularity of
stacked triangulations (Fig. 16). The condition is not necessary, see Sec. 6.2.

**Theorem 6.1** If $S \prec_{1} S_{\text{triv}}$ for a subdivision $S$, then $S$ is a regular subdivision.

In other words, all subdivisions, in particular, all triangulations in the $\prec_{1}$-lower closure of
$S_{\text{triv}}$ are regular. This condition will eventually allow us to show the covering of the bistellar
flip graph by graphs of $(n - 3)$-polytopes. The proof of Thm. 6.1 stretches out over several
definitions and lemmas. Before we give a brief outline of this proof shortly (Sec. 6.3), we first
introduce some notions.

### 6.1 Height functions, liftings, and regular subdivisions

**Definition 6.2 (linear, compliant, realizing height function; regular subdivision)** A
height function on $A \subseteq \mathbb{R}^{2}$ is a vector $\omega \in \mathbb{R}^{A}$, $p \mapsto \omega_{p}$. For $p = (x_{p}, y_{p}) \in A$, we let
$p^{(\omega)} := (x_{p}, y_{p}, \omega_{p})$, and for $B \subseteq A$, we set $B^{(\omega)} := \{p^{(\omega)} \mid p \in B\}$. We say that $\omega$ is linear on
$B \subseteq A$, if there exist $a, b,$ and $c$ in $\mathbb{R}$ such that $\omega_{p} = ax_{p} + by_{p} + c$ for all $p \in B$, i.e., if $B^{(\omega)}$ is
coplanar.

Let $S$ be a subdivision.

(i) A height function $\omega$ on $VS$ is linear on $S$ if it is linear on $VS$. $\Lambda(S)$ denotes the set of
linear height functions on $S$ and for $A \subseteq VS$, $\Lambda_{A}(S)$ denotes the set of height functions on
$S$ linear on $A$.

(ii) A height function $\omega$ on $VS$ complies with $S$, (or is $S$-compliant), if for every region $r$ of
$S$, $\omega$ is linear on $VR$ (including bystanders). Let $\Gamma(S)$ be the set of $S$-compliant height
functions.

(iii) A height function $\omega$ on $VS$ realizes $S$, if $S$ is the projection of the lower convex hull of
$VS^{(\omega)}$, with all points of $VS$ (also the bystanders) appearing on the boundary of this lower
convex hull.

(iv) $S$ is called regular if there is a height function realizing $S$.

Compliant height functions constitute a relaxation of realizable height functions (and of
linear height functions): Every realizing height function (and every linear height function) is
compliant. All height functions on a triangulation $T$ are compliant, i.e., $\Gamma(T) = \mathbb{R}^{VT}$. For the trivial subdivision $S_{triv}$, the compliant height functions are exactly the linear height functions, i.e., $\Gamma(S_{triv}) = \Lambda(S_{triv})$.

**Lemma 6.3** Let $S$ be a subdivision.

(i) $\Lambda(S)$ is a linear subspace of $\mathbb{R}^{VS}$ of dimension $\dim \Lambda(S) = 3$. More generally, for every $B \subseteq VS$ with $|B| \geq 3$, $\Lambda_B(S)$ is a linear subspace of $\mathbb{R}^{VS}$ of dimension $|VS| - (|B| - 3)$.

(ii) $\Gamma(S) = \bigcap_{r \in RS} \Lambda_Vr(S)$ (RS the set of regions of $S$).

(iii) $\Gamma(S)$ is a linear subspace of $\mathbb{R}^{VS}$ with $\Gamma(S) \supseteq \Lambda(S)$ and $\dim \Gamma(S) \geq |VS| - \text{sl}S$.

**Proof.** (i) is obvious and (ii) holds directly by definition. Now, with $\text{sl}r = |Vr| - 3$ and $\text{sl}S = \sum_{r \in RS} (|Vr| - 3)$, assertion (iii) is an immediate consequence of (i), (ii), and the fact that intersecting linear subspaces of co-dimension $d_1$ and $d_2$ yields a subspace of co-dimension at most $d_1 + d_2$:

$$|VS| - \dim \Gamma(S) \leq \sum_{r \in RS} (|VS| - \dim \Lambda_Vr(S)) = \sum_{r \in RS} (|Vr| - 3) = \text{sl}S$$

We see that if $\text{sl}S < |VS| - 3$, there are always compliant height functions not in $\Lambda(S)$. In order to extract among those a realizable height function we consider mountains and valleys in the lifting given by a compliant height function.

**Definition 6.4 (ω-lifting; ω-labeling)** Let $\omega \in \Gamma(S)$. The $\omega$-lifting of $S$ is the unique piecewise linear function $f$ on the convex hull of $VS$, that is linear on every region $r$ of $S$, and $f|VS = \omega$.

We call $e \in E^cS$ a mountain, a valley, or flat in the $\omega$-lifting, depending on whether the derivative of function $f$ decreases, increases, or remains constant, respectively, as one traverses the $f$-lifted edge from one side to the other (at a mountain, the function is locally strictly concave, at a valley it is locally strictly convex). The $\omega$-labeling of $S$ assigns $\oplus$, $\ominus$, and $0$ to each inner edge of $S$, depending on whether the lifted edge is a mountain, a valley, or flat, respectively.

![Figure 17: Valleys (left) and mountains (right).](image)

**Observation 6.5** Let $\omega \in \Gamma(S)$.

(i) $\omega$ is linear on $S$ iff the $\omega$-labeling of $S$ is constant 0.

(ii) $\omega$ realizes $S$ iff the $\omega$-labeling of $S$ is constant $\ominus$.

### 6.2 Mother of examples

In order to understand the subtleties of whether a subdivision is regular or not, we should briefly discuss the mother of examples, see [5]. For this consider the configuration in Fig. 18. Whether or not the displayed subdivisions are regular or not depends on how exactly the three dashed lines (as indicated in $S$) meet.
(a) If the three dashes lines meet in a common point, then $S$ is a regular subdivision, but none of $T'$ and $T''$ is regular.

(b) If the three dashes lines do not meet in a common point, then $S$ is not a regular subdivision, but one of $T'$ and $T''$ is regular, the other one not.

Figure 18: Exactly one of $S$, $T'$, and $T''$ is regular. Which one depends on how the dashed lines meet.

The example allows us to clarify a few points.

– The condition in Thm. 6.1 for regularity is not necessary (consider Case (b) with $T'$ regular, and note $T' \not≺_1 S_{triv}$). In fact, this is inherently so, since the condition in Thm. 6.1 depends only on the order type of the point set. In fact, note that a perturbation of the point set does not change the order type of the set, but it affects how the dashed lines meet and, therefore, whether subdivisions are regular or not.

– The condition in Thm. 6.1 cannot be generalized to: If $S \prec_1 S'$ and $S'$ is regular, then $S$ is regular. In fact, adding a single edge in a subdivision may switch from regular to non-regular (Case (a)). The right generalization will be given in the Regularity Preservation Lemma 6.10 below.

6.3 Outline of proof of Thm. 6.1

It is easy to see that if $p$ is an inner point of degree 3 in a triangulation $T$, then for any height function $\omega$ (which, as we observed, is $T$-compliant), the $\omega$-labeling assigns the same value to the three edges incident to $p$. We will generalize this observation for an $S$-compliant height function $\omega$ in two ways:

(A) If $p \in V^o S$, then the $\oplus$-labeled and $\ominus$-labeled edges incident to $p$ cannot be separated by a line through $p$, unless all these edges are 0-labeled (Lemma 6.7). (In particular, this forces the $\omega$-labeling to be constant on the edges incident to an inner point of degree 3 in any subdivision.)

(B) If $K$ is a perfect coarsener of $S$, then the $\omega$-labeling assigns the same label to all the edges $E_K$ incident to a perfect coarsener $K$ (Lemma 6.9).

Here is another simple observation about an inner point $p$ of degree 3 in a triangulation $T$. Removing the three edges incident to $p$ (while keeping $p$ as a bystander) yields a subdivision $S$ with $T \prec_1 S$. Obviously, if $S$ is a regular subdivision, then $T$ is a regular triangulation (this was behind our observation about stacked triangulations at the beginning of this section): Given a height function $\omega$ realizing $S$, we can always perturb $p$ downwards (decrease $\omega_p$ by a sufficiently small value $\epsilon$), obtaining a height function that realizes $T$. Again, this allows an appropriate generalization:

(C) Suppose $S$ is a regular subdivision with $\dim \Gamma(S) = |VS| - sl S$. Then every perfect refinement $S'$ of $S$, i.e., $S' \prec_1 S$, is regular, and, moreover, $\dim \Gamma(S') = |VS'| - sl S'$ (Regularity Preservation Lemma 6.10).

Since $S_{triv}$ is regular and $\dim \Gamma(S_{triv}) = 3 = |VS_{triv}| - sl S_{triv}$, this immediately yields an inductive argument for Thm. 6.1. For the proof of (C), we consider the perfect coarsener $K$ of $S'$ whose isolation leads to $S$, and a height function $\omega_1$ that realizes $S$. First, we show that $\dim \Gamma(S') > \dim \Gamma(S)$, and, therefore, a height function $\omega' \in \Gamma(S') \setminus \Gamma(S)$ exists. According to (B), the
\(\omega\)'-labeling assigns the same label to all edges incident to \(K\), and since \(\omega' \notin \Gamma(S)\), this label cannot be 0. Hence, either \(\omega\) or \(-\omega\) assigns constant \(\ominus\), and it can be used for a controlled perturbation \(\omega_1 + \varepsilon\omega'\) which realizes \(S'\).

We will now carefully work out these steps.

6.4 Valid \(\{\oplus, \ominus, 0\}\)-edge labelings

**Definition 6.6** Let \(S\) be a subdivision. Given a labeling \(\alpha : E^o S \to \{\oplus, \ominus, 0\}\), we call an inner point \(p \in V^o S\) \(\alpha\)-pointed, if \(\alpha\) is not constant 0 on the edges incident to \(p\), and if there is a line through \(p\) that has all \(\oplus\)-labeled edges incident to \(p\) strictly on one side and all \(\ominus\)-labeled edges incident to \(p\) strictly on the other side of this line. (We do not require that both \(\oplus\) and \(\ominus\)-labeled edges incident to \(p\) exist.)

We call \(\alpha\) a valid labeling of \(E^o S\) if no point in \(V^o S\) is \(\alpha\)-pointed (Fig. 19).

![Figure 19: Patterns prohibited in valid labelings.](image)

For example, for an inner point of degree 3 in a subdivision, a valid labeling must assign the same label to its three incident edges. We can now prove (A) above.

**Lemma 6.7** Let \(\omega\) be a height function compliant with subdivision \(S\). Then the \(\omega\)-labeling of \(S\) is a valid labeling of \(E^o S\).

**Proof.** Let \(p \in V^o S\) and suppose there is a line \(\ell\) through \(p\) that has all \(\oplus\)-labeled edges incident to \(p\) on one side, and all \(\ominus\)-labeled edges incident to \(p\) on the other side. Sweep a vertical plane \(h\) parallel to \(\ell\) in \(\mathbb{R}^3\) over \(p\) and observe its intersection with the \(\omega\)-lifting \(f\). On the side of the \(\oplus\)-labeled edges, this intersection must be a locally concave function, on the side of the \(\ominus\)-labeled edges a locally convex function. Consequently, it has to be locally linear at the point when \(h\) contains \(p\) and \(\ell\). Now it follows that \(f\) must be locally linear around \(p\) and all edges incident to \(p\) must be flat. \(\square\)

**Lemma 6.8** Let \(K\) be a perfect coarsener in a subdivision \(S\). In every valid \(\{\oplus, \ominus, 0\}\)-labeling of \(E^o S\), the edges \(E_K\) incident to \(K\) all get the same label.

**Proof.** We plan to prove the following.

**Claim.** With reference to the orientation process in the proof of Lemma 4.6, after the second phase, in any valid labeling, the edges incident to a candidate component of \(G^*\) get the same label.

It is not obvious from the proof of Lemma 4.6 that every perfect coarsener is identified by the three phase process. In order to close this gap (from the claim to the assertion of the lemma), isolate \(K\) in \(S\) obtaining a subdivision \(S'\) with a region \(r\) containing the points in \(K\). Let \(V_r\) be the vertices of this regions, i.e., \(V_r = V_r \setminus V^{by} S'\). Now consider the subgraph \(S_r\) of \(S\) induced by \(V_r \cup K\). \(K\) is a perfect coarsener of \(S_r\) whose isolation yields the trivial subdivision of \(V_r \cup K\). It is the only coarsener of \(S - r\) and \(|S_r| = |V S_r| - 4\). Therefore, the procedure in the proof of Lemma 4.6 must identify \(K\) as a candidate component after the second phase.

We establish the claim by showing the following invariant in the process during the second phase:
(a) For every candidate component $K$, the edges $E_K$ incident to $K$ obtain the same label in any valid labeling.

(b) An edge gaining a new orientation in the second phase and its witnesses obtain the same label in any valid labeling.

At the end of the first phase, a point with indegree 3 in $S$ has actually degree 3 in $S$, and therefore any valid labeling must give the same label to all incident edges. Since in a candidate component $K$, all points have indegree 3 in $S$ (i.e., at this point, have degree 3 in $S$) and since a connected component is connected [sic!], it easily follows that all edges incident to a candidate component must have the same label in any valid labeling.

During the second phase, a newly oriented edge and its witnesses are part of the edges incident to a candidate component. Hence, given (a), invariant (b) is maintained after an orientation step of phase 2. We are left to show that (a) is preserved. Consider a point $p$ of a candidate component. It must have indegree 3, all incident edges are either ingoing or witnesses for an ingoing edge (otherwise, indegree 3 is impossible); we know that each bundle of an ingoing edge and its witnesses have the same label, and such a bundle can be separated from the other incident edges by a line through the given point $p$ (this is why the edge was oriented in phase 2). Hence, due to a simple consideration, any valid labeling must assign the same label to all incident edges. (The simple consideration: Suppose a single bundle is labeled $\oplus$, then this bundle can be separated from the other two bundles by a line, contradiction. Suppose exactly two bundles are labeled $\oplus$, then the remaining bundle can be separated from these two $\oplus$-labeled bundles by a line, which is a contradiction. Hence, if any incident edge is labeled $\oplus$, then all incident edges must be labeled $\oplus$. Similarly, for $\ominus$.) This completes the proof of the claim, and thus of the lemma.

Now, with Lemma 6.7 we immediately get property (B).

**Lemma 6.9** If $\omega$ is an $S$-compliant height function, then the $\omega$-labeling is constant on any set of edges incident to a perfect coarsener of $S$.

### 6.5 Final step: The Regularity Preservation Lemma

**Lemma 6.10 (Regularity Preservation)** Let $S_1$ be a regular subdivision with $\dim \Gamma(S_1) = |V S_1| - sl S_1$. If $S_0 \prec S_1$, then $S_0$ is regular and $\dim \Gamma(S_0) = |V S_0| - sl S_0$.

**Proof.** Let $\omega_1 \in \mathbb{R}^{V S_1}$ be a height function that realizes $S_1$.

**Case 1.** $S_1$ is obtained from $S_0$ by adding a single point $p \in P^o \setminus V^o S_1$. Then $\omega_1|_{V S_0}$ realizes $S_0$ and $S_0$ is regular. We have $\Gamma(S_0) = \{ \omega|_{V S_0} \mid \omega \in \Gamma(S_1) \}$. For $\omega \in \Gamma(S_1)$, the value of $\omega_p$ ($p$ the added point) is determined by $\omega|_{V S_0}$, i.e., $\dim \Gamma(S_1) = \dim \Gamma(S_0)$. Therefore,

$$\dim \Gamma(S_0) = |V S_1| - sl S_1 = (|V S_0| + 1) - (sl S_1 + 1) = |V S_0| - sl S_0.$$

**Case 2.** $S_1$ is obtained from $S_0$ by removing a single unlocked edge or by isolating a perfect coarsener in $S_0$. We have $V S_1 = V S_0$. The set $E^* := ES_0 \setminus ES_1$ is either a single unlocked edge or the set $E_K$ of edges incident to a perfect coarsener $K$ in $S_0$. Let $r^*$ be the region in $S_1$ generated by the removal of the edges in $E^*$.

We have $\Gamma(S_0) \supseteq \Gamma(S_1)$ and (with Lemma 6.3(iii) and $sl S_0 = sl S_1 - 1$)

$$\dim \Gamma(S_0) \geq |V S_0| - sl S_0 = |V S_1| - sl S_1 + 1 = \dim \Gamma(S_1) + 1.$$

Therefore, there must exist $\omega^* \in \Gamma(S_0) \setminus \Gamma(S_1)$, a height function that is not flat on $r^*$ and there is an edge in $E^*$ that is not flat in the $\omega^*$-lifting of $S_0$. In that case, all edges in $E^*$ are mountains or all are valleys (trivially true if $|E^*| = 1$, otherwise by Lemma 6.9). Let us suppose that the $\omega^*$-labeling is constant $\ominus$ on $E^*$ (if not switch to $-\omega'$). Now, for any sufficiently small
positive $\varepsilon \in \mathbb{R}$, the height function $\omega_0 := \omega_1 + \varepsilon \omega'$ is compliant with $S_0$ and all inner edges in $S_0$ are valleys in the $\omega_0$-lifting of $S_0$ ($\varepsilon$ has to be small enough such that all valleys in the $\omega_1$-lifting remain valleys in $\omega_0$; this is a familiar operation, see [5, Lemma 2.3.16]). This establishes that $S_0$ is regular.

We are left to show that $\dim \Gamma(S_0) = |V S_0| - \text{sl } S_0$, or, equivalently, $\dim \Gamma(S_0) = \dim \Gamma(S_1) + 1$.

This holds, if for any two $\omega', \omega''$ in $\Gamma(S_0)$ \( \setminus \Gamma(S_1) \) there exists $a \in \mathbb{R}$ and $\omega \in \Gamma(S_1)$ such that $\omega' = a \omega'' + \omega$. Suppose that all edges in $E^*$ are valleys in $\omega'$, and all edges in $E^*$ are mountains in $\omega''$ (switch signs, if necessary). Now consider $\omega_t := (1 - t) \omega' + t \omega''$, $t \in [0, 1]$. There must be a value $t$ in $(0, 1)$ where some edge in $E^*$ is flat in the $\omega_t$-lifting, but then all edges have to be flat and $\omega_t \in \Gamma(S_1)$ (Lemma 6.9). We have shown $\omega' = -\frac{1}{1-t} \omega'' + \frac{1}{1-t} \omega_t$ with $\omega_t \in \Gamma(S_1)$.

**Proof of Thm. 6.1.** If $S \prec_1 S_{\text{triv}}$ then there is a sequence

$$S = S_0 \prec_1 S_1 \prec_1 \cdots \prec_1 S_\ell = S_{\text{triv}}.$$ 

$S_{\text{triv}}$ is regular, $\text{sl } S_{\text{triv}} = n - 3$ and $\Gamma(S_{\text{triv}}) = \Lambda(S_{\text{triv}})$, of dimension $3 = n - \text{sl } S_{\text{triv}} = |V S_{\text{triv}}| - \text{sl } S_{\text{triv}}$. Along Lemma 6.10 we have an inductive argument that $S_0 = S$ is regular.

We immediately get, that successive perfect refinements of a subdivision $S^*$ fill the regions of $S^*$ with locally regular subdivisions.

**Definition 6.11 (restriction of subdivision)** Let $S \preceq S'$ be subdivisions, and let $r \in RS'$. Then the restriction of $S$ to $r$, $S|_r$, is the subgraph of $S$ induced by $V S \cap \overline{r}$ ($\overline{r}$ the closure of $r$).

**Corollary 6.12** Let $S \prec_1 S'$. For $r \in RS'$, we have $S|_r \prec_1 S_{\text{triv}}(V r)$ and $S|_r$ is a regular subdivision of $V r$.

Let us conclude this section with the remark, that successive perfect coarsening starting from a subdivision is a non-deterministic process, that may – even for the same subdivision – lead to $S_{\text{triv}}$ or not (Fig. 20).

**Figure 20:** Successive perfect coarsenings may lead to the trivial subdivisions (and thus imply regularity) or to another subdivision of slack $n - 3$.

## 7 Implications of Regularity Preservation

### 7.1 Covering the bistellar flip graph with polytopes

**Theorem 7.1** The edge set of the bistellar flip graph of $P$ can be covered by subgraphs isomorphic to 1-skeletons of $(n - 3)$-polytopes (products of secondary polytopes).

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Proof. Given an edge \( \{T, T[x]\} \) of the bistellar flip graph, let \( S \) be a subdivision with \( T_{\pm x} \prec_1^* S \) and \( \text{sl} S = n - 3 \) (Cor. 4.8). For every region \( r \) of \( S \), the subdivisions \( T|_r, T[x]|_r, \) and \( T_{\pm x}|_r \) are regular subdivisions of \( V_r \).

Now consider the product of polytopes (see [16])

\[
\prod_{r \in S} \Sigma\text{-poly}(V_r),
\]

where \( \Sigma\text{-poly}(A) \) denotes the secondary polytope of \( A \subseteq P \) [5]. The dimension of this product is \( \sum_r(|V_r| - 3) = \text{sl} S = n - 3 \). Its faces correspond to the refinements \( S' \) of \( S \) such that for each region \( r \) of \( S \), \( S'|_r \) is regular, i.e., this includes \( T \) and \( T[x] \) (as vertices), and \( T_{\pm x} \) (as edge) (Cor. 6.12). This completes the argument.

\[
\begin{align*}
\text{Figure 21: The bistellar flip graph of the mother-of-examples configuration.} \\
\text{Figure 22: The bistellar flip graph of the mother-of-examples configuration as the union of the} \\
\text{graphs of two 3-polytopes.}
\end{align*}
\]

7.2 Sets with all triangulations regular

We give characterizations of point sets for which all triangulations are regular (as, e.g., it is the case for point sets in convex position). In particular, we show that this can be easily read off the height of the partial order \( \preceq \). In a first step we prove that property to be equivalent to requiring that all subdivisions are regular.

Lemma 7.2 All subdivisions are regular iff all triangulations are regular.
Proof. The direction (⇒) is obvious.

For (⇐) it suffices to show that every non-regular subdivision $S$ with $slS > 0$ has a direct refinement which is not regular.

**Case 1.** $S$ has a bystander $p \in V^{by}S$. Clearly, the direct refinement $(VS \setminus \{p\}, ES)$ is not regular iff $S$ is not regular.

**Case 2.** $S$ has no bystander. Since $slS > 0$, $S$ must have an active region $r^*$ which is a $k$-gon for $k \geq 4$. Choose two crossing diagonals $e_0$ and $e_1$ in $r^*$ and consider the subdivisions $S_i := (VS, ES \cup \{e_i\}), i = 0, 1$. We want to show that if $S$ is not regular, then at least one of $S_0$ and $S_1$ is not regular. So let us suppose that, for $i = 0, 1$, $\omega_i \in \mathbb{R}^{VS}$ is a height function realizing $S_i$ (as a regular subdivision) and, for $t \in [0, 1]$, consider the convex combination $\omega_t := (1 - t)\omega_0 + t\omega_1$.

We say that a height function $\omega$ respects region $r$ in subdivision $S$, if $\omega$ is linear on $Vr$ and all points in $VS^{(0)} \setminus Vr^{(0)}$ lie strictly above the plane spanned by $Vr^{(0)}$. Clearly, $\omega$ realizes $S$ iff it respects all regions $r \in RS$. Moreover, if two height functions respect a region, then all convex combinations do.

It follows, that $\omega_t$ respects all regions in $RS$ except for $r^*$, since these are regions both in $S_0$ and $S_1$. We have that $e^{(0)}_1$ lies below $e^{(0)}_1$ (as segments in the lifting in $\mathbb{R}^3$), while $e^{(0)}_1$ lies below $e^{(0)}_1$ and, therefore, there must be a $t \in (0, 1)$, where $e^{(0)}_1$ and $e^{(0)}_1$ intersect (in the lifting in $\mathbb{R}^3$). For that value of $t$, $\omega_t$ is linear on $Vr^*$. Moreover, all edges in $S^*$ are valleys in the $\omega_t$-lifting, since these are valleys both in the $\omega_0$-lifting and the $\omega_1$-lifting (and that property is preserved for all convex combinations of $\omega_0$ and $\omega_1$). Hence, $\omega_t$ realizes $S$ and we have a contradiction.

We recall the definition of the height of an element in a partial order, and of the height of the partial order.

**Definition 7.3 (height)** The height of a subdivision $S$ (in the partial order $\preceq$) is recursively defined: (a) If $S$ is a triangulation, then $hS := 0$, and (b) if $S$ is not a triangulation, then $hS := 1 + \max_{S' \prec S} hS'$. (Equivalently, $hS$ is the size of the longest $\preceq$-chain ending in $S$ minus 1.) We let $h_{\max} = h_{\max}(P)$ be the maximum height of any subdivision of $P$ (i.e., $h_{\max} = h_{\text{triv}}$).

**Theorem 7.4** The following five conditions are equivalent.

(i) All triangulations are regular.  
(ii) All subdivisions are regular.  
(iii) $h_{\max} = n - 3$.  
(iv) $h = sl$.

Proof. For (i) ⇔ (ii) see Lemma 7.2. For the rest we show the implication cycles

all subdivisions are regular $\Rightarrow$ (a) $h_{\max} = n - 3$ $\Rightarrow$ (b) $\preceq_{\text{dir}} = \preceq_1$ $\Rightarrow$ (c) all subdivisions are regular

and

$h_{\max} = n - 3$ $\Rightarrow$ (b) $\preceq_{\text{dir}} = \preceq_1$ $\Rightarrow$ (d) $h = sl$ $\Rightarrow$ (e) $h_{\max} = n - 3$.

(a) All subdivisions are regular $\Rightarrow h_{\max} = n - 3$. This is well known and discussed, e.g., in “Twelve proofs of non-regularity” in [5, Sec. 7.1.2, (6)]: On the one hand, if all subdivisions are regular, they all correspond to faces of the secondary polytope, an $(n - 3)$-polytope where every chain of proper faces (excluding the empty face and the polytope itself) has size at most $(n - 3)$. On the other hand, if $h_{\max} > n - 3$, that gives a chain of size exceeding $n - 3$ of non-trivial subdivisions.

Note here: There is a maximum chain of size $1 + h_{\max}$ in the $\preceq$-partial order. If we remove the trivial subdivision (not corresponding to a proper face), we get still a chain of length $h_{\max}$.  

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(b) \( h_{\text{max}} = n - 3 \Rightarrow \prec_{\text{dir}} \prec_{1} \). Consider a maximal chain \( S_0 \prec_{\text{dir}} S_1 \prec_{\text{dir}} \cdots \prec_{\text{dir}} S_m \); because of maximality, \( S_0 \) is a triangulation (of slack 0), and \( S_m = S_{\text{triv}} \) (of slack \( n - 3 \)). We know that \( s_l S_i \leq s_l S_{i-1} + 1 \) (Lemma 7.5 below) with equality iff \( S_{i-1} \prec_{\text{dir}} S_i \). It follows that \( m \geq n - 3 \). Moreover, if \( m = n - 3 \) (which is given if \( h_{\text{max}} = n - 3 \)), then \( S_{i-1} \prec_{\text{dir}} S_i \) for all \( i = 1, 2, \ldots, n - 3 \). Since every pair \( S' \prec_{\text{dir}} S \) is part of a maximal chain, the claim follows.

(c) \( \prec_{\text{dir}} \prec_{1} \Rightarrow \) all subdivisions are regular. Every subdivision \( S \) has a chain of direct coarsenings to \( S_{\text{triv}} \). If every direct coarsening is a perfect coarsening, this shows \( S \prec_{1} S_{\text{triv}} \) and therefore \( S \) is regular (Thm. 6.1).

(d) \( \prec_{\text{dir}} \prec_{1} \Rightarrow h = s_l \). For proving \( hS = s_l S \), we can proceed by induction on the height of \( S \), where the induction basis holds without assumptions. With the assumption of \( \prec_{\text{dir}} \prec_{1} \) and with the induction hypothesis

\[
  hS = 1 + \max_{S' \prec_{\text{dir}} S} hS' = 1 + \max_{S' \prec_{1} S} hS' = 1 + \max_{S' \prec_{1} S} s_l S' = 1 + \max_{S' \prec_{1} S} (s_l S - 1) = s_l S
\]

(e) \( h = s_l \Rightarrow h_{\text{max}} = n - 3 \). If \( h = s_l \), then \( h_{\text{max}} = hS_{\text{triv}} = s_l S_{\text{triv}} = n - 3 \).

More properties of coarseners. We derive two more properties of prime and perfect coarseners, Lemma 7.5 as just used in the proof of Thm. 7.4 above and Lemma 7.6 as needed in Sec. 7.3 below.

Lemma 7.5 \( \text{inc} U \leq 1 \) for every prime coarsener \( U \) in a subdivision \( S \).

Proof. Let \( r \) be the region in \( S' := S - E_U \) obtained by removing the edges \( E_U \) incident to \( U \). The subgraph of \( S \) induced by \( U \) is connected (Obs. 4.4(iv)), that is, all points of \( U \) have to lie in the same region of \( S' \). We consider the restriction \( S'|_r \) (Def. 6.11), a subdivision of \( VS|_r = VS \cap \overline{r} \). Isolating \( U \) in \( S'|_r \) yields \( S'|_r \), the trivial subdivision of \( VS|_r \). Therefore,

\[
  s_l S'|_r + \text{inc} U = s_l S'|_r = |VS|_r - 3,
\]

that is, \( s_l S'|_r = |VS|_r - 3 - \text{inc} U \) (Obs. 4.3).

Lemma 7.6 A prime coarsener \( U \) inducing a tree in its subdivision \( S \) is perfect.

Proof. Let \( k := |U| \) and let \( \ell \) be the number of edges in \( E_S \) that are incident to exactly one point in \( U \). We have \( |E_U| = (k - 1) + \ell \). Since every point in \( U \) has degree at least 3, we have \( |E_U| \geq \frac{3k + \ell}{2} \). Hence, \( (k - 1) + \ell \geq \frac{3k + \ell}{2} \), i.e., \( \ell \geq k + 2 \). Now \( \text{inc} U = |E_U| - 2|U| = (k - 1) + \ell - 2k = \ell - k - 1 \geq 1 \), i.e., by Lemma 7.5, \( \text{inc} U = 1 \).

7.3 Large minimal sets with non-regular triangulations

Observation 7.7 If \( P' \subseteq P \) and \( P' \) has non-regular triangulations, then \( P \) has non-regular triangulations.

“Given a set \( P \) with non-regular triangulations, is there always a small subset \( P' \) of \( P \) that witnesses this fact?”, a question asked by F. Santos, [13]; or equivalently, “How large can minimal sets \( P \) with non-regular triangulations be?” (“minimal” means that every proper subset of \( P \) has only regular triangulations). He gives such a minimal set of 8 points and conjectures that this is the largest example of such a minimal set. We will show that such sets exist of arbitrary size.

Definition 7.8 (twisted double-gon) A set \( P \) of \( n \) points in general position, \( n = 2k \) even, is called a twisted double-gon if the following holds.
Figure 23: A twisted double-gon of 10 points with subdivision $S^\Box$ (left). Subset $P^*$ (case $p_0^* = q_0$) of a twisted double-gon (right).

(I) $h(P) = k$.
Let $p_0, p_1, \ldots, p_{k-1}$ be a counter-clockwise numbering of $\text{xtr} P$ along the boundary of the convex hull of $P$.

(II) The set $Q := P^\circ$ of inner points is in convex position.
There is a numbering $q_0, q_1, \ldots, q_{k-1}$ of $Q$ following the order along the boundary of the convex hull of $Q$ such that for all $i$, $0 \leq i \leq k - 1$, it holds:

(III) $q_i$ is extreme in $P \setminus \{p_i\}$.
(IV) $q_i$ is extreme in $P \setminus \{p_{i-1}, q_{i-1}\}$.
(V) $q_i$ lies in the triangle $q_{i-1}p_iq_{i+1}$.

Fig. 23 indicates that such twisted double-gons exist\(^4\) for all even $n \geq 6$. For $n = 6$, this is the mother-of-examples configuration (Sec. 6.2). Here are a few simple observations.

Observation 7.9 Let $P$ be a twisted double-gon, with notation as in Def. 7.8.
(i) The graph $S^\Box := (P, E_{\text{hull}} \cup \{\{q_i, p_i\}, \{q_i, q_{i+1}\} | i = 0, 1, \ldots, k - 1\}$ is a subdivision (uses (II) and (V)).
(ii) If $q_i$ is involved in a subdivision of a subset of $P$ where it is not extreme, it is connected to $p_i$ (by (III)) and to one of $\{p_{i-1}, q_{i-1}\}$ (by (IV)).

Theorem 7.10 If $P$ is a twisted double-gon, then
(i) $P$ has a non-regular triangulation, and
(ii) any proper subset of $P$ has only regular triangulations.

Proof. (i) $Q$ is a prime coarsener of $S^\Box$ with increment 0. Hence, $\prec_{\text{dir}} \neq \prec_1$ and $P$ has non-regular triangulations (Thm. 7.4). Note that we do not claim that $S^\Box$ is a non-regular subdivision; in fact, this depends on the concrete coordinates of the point set $P$.

(ii) We remove $p_0$ or $q_0$ from $P$, we denote the resulting set by $P^*$ with $p_0^*$ the point among $p_0$ and $q_0$ remaining in $P^*$, see Fig. 23 (right). If we can show that all triangulations of $P^*$ are regular, then the proof is complete (by symmetry and Obs. 7.7).

It is enough to show that any prime coarsener $U$ of any subdivision $S$ of $P^*$ is perfect (i.e., $\text{inc} U = |E_U| - 2|U| \geq 1$), since then $\prec_{\text{dir}} = \prec_1$ (Thm. 7.4). So let us consider such a prime coarsener $U$, let $E_U^{\text{in}}$ be the edges in $S$ connecting two points in $U$, and let $E_U^{\text{out}}$ be the set of edges in $S$ connecting a point in $U$ to a point in $V S \setminus U$. We have $E_U = E_U^{\text{in}} \cup E_U^{\text{out}}$.

(a) The subgraph of $S$ induced by $U$ has to be connected (since $U$ is prime, Obs. 4.4(iv)), hence $|E_U^P| \geq |U| - 1$.

\(^4\)These are not double-circles, cf. [1]; double-circle have only regular triangulations.
(b) If the subgraph of $S$ induced by $U$ is a tree, then $U$ is a perfect coarsener (Lemma 7.6).

So let us assume that $U$ does not span a tree, which implies $|E_U^\text{in}| \geq |U|$. 

(a) Every point in $U$ has to connect to at least one point in $V \setminus S \setminus U$. This holds, since $U \subseteq Q \setminus \{q_0\}$ and $q_i \in U$, $i \geq 1$, has to connect to $p_i$ (by (III)). Hence, $|E_U^\text{out}| \geq |U|$. 

At this point we have already shown that $|E_U| = |E_U^\text{in}| + |E_U^\text{out}| \geq 2|U|$. Therefore, $\text{inc} U \geq 0$, and $U$ is perfect unless $|E_U^\text{in}| = |U|$ and $|E_U^\text{out}| = |U|$. 

(b) Let $j := \min\{i \in \mathbb{N} \mid i \geq 1, q_i \in U\}$. If $j = 1$, i.e., $q_1 \in U$, then $q_1$ connects to $p_1$ and $p_0^*$. Hence, $|E_U^\text{out}| \geq |U| + 1$. If $j \geq 2$, then $q_j$ has to connect to $p_j$ and $p_{j-1}$ (since $q_{j-1}$ is not available). Hence, $|E_U^\text{out}| \geq |U| + 1$ and $U$ has to be perfect.

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