Stability of generalized Turán number for linear forests

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Abstract

Given a graph $T$ and a family of graphs $\mathcal{F}$, the generalized Turán number of $\mathcal{F}$ is the maximum number of copies of $T$ in an $\mathcal{F}$-free graph on $n$ vertices, denoted by $ex(n, T, \mathcal{F})$. When $T = K_r$, $ex(n, K_r, \mathcal{F})$ is a function specifying the maximum possible number of $r$-cliques in an $\mathcal{F}$-free graph on $n$ vertices. A linear forest is a forest whose connected components are all paths and isolated vertices. Let $L_k$ be the family of all linear forests of size $k$ without isolated vertices. In this paper, we obtained the maximum possible number of $r$-cliques in $G$, where $G$ is $L_k$-free with minimum degree at least $d$. Furthermore, we give a stability version of the result. As an application of the stability version of the result, we obtain a clique version of the stability of the Erdős-Gallai Theorem on matchings.

Keywords: spanning linear forest, generalized Turán number, stability

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1 Introduction

Let $\mathcal{F}$ be a family of graphs. The Turán number of $\mathcal{F}$, denoted by $ex(n, \mathcal{F})$, is the maximum number of edges in a graph with $n$ vertices which does not contain any subgraph isomorphic to a graph in $\mathcal{F}$. When $\mathcal{F} = \{F\}$, we write $ex(n, F)$ instead of $ex(n, \{F\})$. The problem of determining Turán number for assorted graphs traces its history back to 1907, when Mantel showed that $ex(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$. In 1941, Turán [12] proved that if a graph does not contain a complete subgraph $K_r$, then the maximum number of edges it can contain is given by the Turán-graph, a complete balanced $(r - 1)$-partite graph.

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For a graph $G$ and $S, T \subseteq V(G)$, denote by $E_G(S, T)$ the set of edges between $S$ and $T$ in $G$, i.e., $E_G(S, T) = \{uv \in E(G): u \in S, v \in T\}$. Let $e_G(S, T) = |E_G(S, T)|$. If $S = T$, we use $e_G(S)$ instead of $e_G(S, S)$. For a vertex $v \in V(G)$, the degree of $v$, written as $d_G(v)$ or simply $d(v)$, is the number of edges incident with $v$. We use $d_T(v)$ instead of $e_G(S, T)$ when $S = \{v\}$. For any $U \subseteq V(G)$, let $G[U]$ be the subgraph induced by $U$ whose edges are precisely the edges of $G$ with both ends in $U$.

Let $G$ be a graph of order $n$, $P$ a property defined on $G$, and $k$ a positive integer. A property $P$ is said to be $k$-stable, if whenever $G + uv$ has the property $P$ and $d_G(u) + d_G(v) \geq k$, then $G$ itself has the property $P$. The $k$-closure of a graph $G$ is the (unique) smallest graph $G'$ of order $n$ such that $E(G) \subseteq E(G')$ and $d_{G'}(u) + d_{G'}(v) < k$ for all $uv \notin E(G')$. The $k$-closure can be obtained from $G$ by a recursive procedure of joining nonadjacent vertices with degree-sum at least $k$. In particular, if $G' = G$, we say that $G$ is stable under taking $k$-closure. Thus, if $P$ is $k$-stable and the $k$-closure of $G$ has property $P$, then $G$ itself has property $P$.

For a natural number $\alpha$ and a graph $G$, the $\alpha$-disintegration of a graph $G$ is the process of iteratively removing from $G$ the vertices with degree at most $\alpha$ until the resulting graph has minimum degree at least $\alpha + 1$ or is empty. The resulting subgraph $H = H(G, \alpha)$ will be called the $(\alpha + 1)$-core of $G$. It is well known that $H(G, \alpha)$ is unique and does not depend on the order of vertex deletion (for instance, see [11]). The matching number $\nu(G)$ is the number of edges in a maximum matching of $G$.

The $n$-vertex graph $H(n, k, a)$ is defined as follows. The vertex set of $H(n, k, a)$ is partitioned into three sets $A, B, C$ such that $|A| = a, |B| = k - 2a, |C| = n - k + a$, and the edge set of $H(n, k, a)$ consists of all edges between $A$ and $C$ together with all edges in $A \cup B$. Let $H^+(n, k, a)$ and $H^{++}(n, k, a)$ be the graph obtained by adding one edge and two independent edges in $C$ of $H(n, k, a)$, respectively. The number of $r$-cliques in $H(n, k, a)$ is denoted by $h_r(n, k, a) := \binom{k-a}{r} + (n-k+a)\binom{a}{r-1}$, where $h_r(n, k, 0) = \binom{k}{r}$.

A linear forest is a forest whose connected components are all paths and isolated vertices. Let $\mathcal{L}_k$ be the family of all linear forests of size $k$ without isolated vertices. In [13], Wang and Yang proved that $\text{ex}(n; \mathcal{L}_{n-k}) = \binom{n-k}{2} + O(k^2)$ when $n \geq 3k$. Later, Ning and Wang [10] completely determined the Turán number $\text{ex}(n; \mathcal{L}_k)$ for all $n > k$.

**Theorem 1.1** (Ning and Wang [10]). For any integers $n$ and $k$ with $1 \leq k \leq n - 1$, we have

$$
\text{ex}(n, \mathcal{L}_k) = \max \left\{ h_2(n, k, 0), h_2(n, k, \left\lfloor \frac{k-1}{2} \right\rfloor) \right\}.
$$

Given a graph $T$ and a family of graphs $\mathcal{F}$, the generalized Turán number of $\mathcal{F}$ is the maximum number of copies of $T$ in an $\mathcal{F}$-free graph on $n$ vertices, denoted by $ex(n, T, \mathcal{F})$. Note that $\text{ex}(n, K_2, \mathcal{F}) = \text{ex}(n, \mathcal{F})$. The problem to estimate generalized Turán number has received
a lot of attention. In 1962, Erdős [5] generalized the classical result of Turán by determining the exact value of $ex(n, K_r, K_t)$.

Figure 1: $H(n, k, a)$

Luo [9] determined the upper bounds on $ex(n, K_r, P_k)$ and $ex(n, K_r, C_{\geq k})$, where $C_{\geq k}$ is the family of all cycles with length at least $k$. In [8], Gerbner, Methuku and Vizer investigated the function $ex(n, T, F)$, where $kF$ denotes $k$ vertex disjoint copies of a fixed graph $F$. The systematic study of $ex(n, T, F)$ was initiated by Alon and Shikhelman [1]. Recently, Zhang, Wang and Zhou [14] determined the exact values of $ex(n, K_r, L_k)$ by using the shifting method.

**Theorem 1.2** (Zhang, Wang and Zhou [14]). For any $r \geq 2$ and $n \geq k + 1$,

$$ex(n, K_r, L_k) = \max \left\{ h_r(n, k, 0), h_r(n, k, \lfloor \frac{k - 1}{2} \rfloor) \right\}.$$ 

Let $N_r(G)$ denote the number of $r$-cliques in $G$. When $T = K_r$, $ex(n, K_r, F)$ is a function specifying the maximum possible number of $r$-cliques in an $F$-free graph on $n$ vertices. We extend Theorem 1.2 as follows.

**Theorem 1.3.** Let $G$ be an $L_k$-free graph on $n$ vertices with minimum degree $d$ and $d \leq \lfloor \frac{k - 1}{2} \rfloor$. Then

$$N_r(G) \leq \max \left\{ h_r(n, k, d), h_r(n, k, \lfloor \frac{k - 1}{2} \rfloor) \right\}.$$ 

The graphs $H(n, k, d)$ and $H(n, k, \lfloor \frac{k - 1}{2} \rfloor)$ show that this bound is sharp.

Many extremal problems have the property that there is a unique extremal example, and moreover any construction of close to maximum size is structurally close to this extremal example. In [7], Füredi, Kostochka, and Luo studied the maximum number of cliques in non-$\ell$-hamiltonian graphs, where the property non-$\ell$-hamiltonian is $(n + \ell)$-stable. Actually, they not only asked to determine the maximum number of cliques in graphs having a stable property $P$, but also asked to prove a stability version of it. Motivated by the question proposed by Füredi, Kostochka, and Luo [7], we give the following result which is the stability version of Theorem 1.3.
Theorem 1.4. Let $G$ be an $L_k$-free graph on $n$ vertices with minimum degree at least $d$. If $n > k^5$, $r \leq \lfloor \frac{k-3}{2} \rfloor$ and

$$N_r(G) > \max \left\{ h_r(n,k,d), h_r\left(n, k, \left\lfloor \frac{k-5}{2} \right\rfloor \right) \right\},$$

then

(i) $G$ is a subgraph of the graph $H(n,k,\lfloor \frac{k-1}{2} \rfloor)$, $H(n,k,\lfloor \frac{k-3}{2} \rfloor)$ or $H^+(n,k-1,\lfloor \frac{k-3}{2} \rfloor)$ if $k$ is odd;

(ii) $G$ is a subgraph of the graph $H(n,k,\lfloor \frac{k-1}{2} \rfloor)$, $H(n,k,\lfloor \frac{k-3}{2} \rfloor)$, $H^+(n,k-1,\lfloor \frac{k-3}{2} \rfloor)$ or $H^{++}(n,k-2,\lfloor \frac{k-3}{2} \rfloor)$ if $k$ is even.

In 1959, Erdős and Gallai [6] determined the maximum number of edges in an $n$-vertex graph with $\nu(G) \leq k$.

Theorem 1.5 (Erdős-Gallai Theorem [6]). Let $G$ be a graph on $n$ vertices. If $\nu(G) \leq k$, then

$$e(G) \leq \max \{h_2(n,2k+1,0), h_2(n,2k+1,k)\}.$$ 

In [4], Duan et al. extended Erdős-Gallai Theorem as follows.

Theorem 1.6 (Duan et al. [4]). If $G$ is a graph with $n \geq 2k+2$ vertices, minimum degree $d$, and $\nu(G) \leq k$, then

$$N_r(G) \leq \max \{h_r(n,2k+1,d), h_r(n,2k+1,k)\}.$$ 

As an application of our result, we give the stability version of Theorem 1.6 for $2 \leq r \leq k-1$.

Theorem 1.7. Let $G$ be a graph on $n$ vertices with $\delta(G) \geq d$ and $\nu(G) \leq k$. If $r \leq k-1$, $n > (2k+1)^5$ and

$$N_r(G) > \max \{h_r(n,2k+1,d), h_r(n,2k+1,k-2)\},$$

then $G$ is a subgraph of $H(n,2k+1,k)$ or $H(n,2k+1,k-1)$.

2 The maximum number of cliques in $L_k$-free graphs with given minimum degree

The closure technique, which is initiated by Bondy and Chvátal [2] in 1976, played a crucial role in the proof of Theorem 1.3. In [10], Ning and Wang proved the property $L_k$-free is $k$-stable.
Lemma 2.1 (10). Let $G$ be a graph on $n$ vertices. Suppose that $u, v \in V(G)$ with $d(u) + d(v) \geq k$. Then $G$ is $\mathcal{L}_k$-free if and only if $G + uv$ is $\mathcal{L}_k$-free.

Proof of Theorem 1.3. Suppose, by way of contradiction, that $G$ is an $\mathcal{L}_k$-free graph with $N_r(G) > \max \{ h_r(n, k, d), h_r(n, k, \left\lceil \frac{k-1}{2} \right\rceil) \}$. Let $G'$ be the $k$-closure of $G$. Then Lemma 2.1 implies that $G'$ is $\mathcal{L}_k$-free. Obviously, $\delta(G') \geq \delta(G) = d$. Let $H_1$ denote the $\left\lceil \frac{k-1}{2} \right\rceil$-core of $G'$, i.e., the resulting graph of applying $\left\lceil \frac{k-1}{2} \right\rceil$-disintegration to $G'$.

Claim 1. $H_1$ is nonempty.

Proof. Suppose $H_1$ is empty. Since one vertex is deleted at each step during the process of $\left\lceil \frac{k-1}{2} \right\rceil$-disintegration, that destroys at most $\binom{k-1}{r-1}$ cliques of size $r$. The number of $K_r$'s contained in the last $\left\lceil \frac{k-1}{2} \right\rceil$ vertices is at most $\binom{k-1}{r}$. Therefore,

$$N_r(G') \leq \left( \left\lceil \frac{k+1}{r} \right\rceil \right) + \left( n - \left\lceil \frac{k+1}{2} \right\rceil \right) \binom{k+1}{r} \binom{k-1}{r-1}$$

$$= h_r\left( n, k, \left\lceil \frac{k-1}{2} \right\rceil \right)$$

$$\leq \max \left\{ h_r(n, k, d), h_r\left( n, k, \left\lceil \frac{k-1}{2} \right\rceil \right) \right\},$$

contradicting to the assumption of $G'$, the claim follows. $\square$

Claim 2. $H_1$ is a clique.

Proof. Note that $d_{G'}(u) \geq \left\lceil \frac{k+1}{2} \right\rceil$ for any vertex $u$ in $H_1$. Since $G'$ is closed under taking $k$-closure, $H_1$ is a clique. $\square$

Let $t = |V(H_1)|$. Now we estimate the range of $t$.

Claim 3. $\left\lfloor \frac{k+3}{2} \right\rfloor \leq t \leq k - d$.

Proof. As $H_1$ is a clique and $d_{H_1}(u) \geq \left\lceil \frac{k+1}{2} \right\rceil$ for any vertex $u$ in $H_1$, we get $t \geq \left\lfloor \frac{k+3}{2} \right\rfloor$. If $t \geq k - d + 1$, then $d_{G'}(u) \geq d_{H_1}(u) = t - 1 \geq k - d$ for any vertex $u$ in $H_1$. Let $v$ be any vertex in $V(G') \setminus V(H_1)$. Notice that $d_{G'}(v) \geq d_{G}(v) \geq d$ and $d_{G'}(u) + d_{G'}(v) \geq k - d + d = k$. Since $G'$ is the $k$-closure of $G$, $v$ is adjacent to $u$. Then $G'$ contains a $P_{k+1}$, which is a contradiction. Thus $\left\lfloor \frac{k+3}{2} \right\rfloor \leq t \leq k - d$. $\square$

Let $H_2$ be the $(k+1-t)$-core of $G'$. Since $t \geq \left\lfloor \frac{k+3}{2} \right\rfloor$, we obtain $k+1-t \leq \left\lceil \frac{k+1}{2} \right\rceil$. Therefore, $H_1 \subseteq H_2$.

Claim 4. $H_1 \neq H_2$.

Proof. Suppose $H_1 = H_2$. Then $|V(H_2)| = t$. Since each step during the process of $(k-t)$-disintegration destroys at most $\binom{k-t}{r-1}$ cliques of size $r$, we have $N_r(G') \leq \binom{k-t}{r} + (n-t) \binom{k-t}{r-1} = h_r(n, k, k - t)$. Note that $d \leq k - t \leq \left\lfloor \frac{k-3}{2} \right\rfloor$ from Claim 3. By the convexity of $h_r(n, k, k - t)$,
we have $N_r(G') \leq \max \{ h_r(n, k, \lfloor \frac{k-3}{2} \rfloor) \} \leq \max \{ h_r(n, k, \lfloor \frac{k-1}{2} \rfloor) \}$, a contradiction. Thus the claim follows.

By Claim 4, $H_1$ is a proper subgraph of $H_2$. This implies that there are non-adjacent vertices $u$ and $v$ such that $u \in V(H_1)$ and $v \in V(H_2) \setminus V(H_1)$. We have $d_{G'}(u) + d_{G'}(v) \geq t - 1 + (k + 1 - t) = k$. As $G'$ is stable under taking $k$-closure, $u$ must be adjacent to $v$. We obtained a contradiction.

It is easy to see that graphs $H(n, k, d)$ and $H(n, k, \lfloor \frac{k-1}{2} \rfloor)$ are $\mathcal{L}_k$-free. Then either $H(n, k, d)$ or $H(n, k, \lfloor \frac{k-1}{2} \rfloor)$ obtains the bound. The theorem is proved.

3 Stability on $\mathcal{L}_k$-free graphs

3.1 Proof of Theorem 1.4

Let $G$ be a graph on $n$ vertices. If there are at least $s$ vertices in $V(G)$ with degree at most $q$, then we say $G$ has $(s, q)$-Pósa property. If $G$ has $(s, q)$-Pósa property and $n \geq s + q$, then we can check that

$$N_v(G) \leq \binom{n-s}{r} + s \binom{q}{r-1}.$$

The following two lemmas show the relationship between the $k$-stable property and the Pósa property. With the help of these two lemmas, we can approximate the structure of $k$-closure of a graph.

Lemma 3.1. Let $n \geq k + 1$. Assume property $P$ is $k$-stable and the complete graph $K_n$ has the property $P$. Suppose $G$ is a graph on $n$ vertices with minimum degree at least $d$. If $G$ does not have property $P$, then there exists an integer $q$ with $d \leq q \leq \frac{k-1}{2}$ such that $G$ has $(n - k + q, q)$-Pósa property.

Proof. Let $G'$ be the $k$-closure of $G$ and $d_{G'}(v_1), d_{G'}(v_2), \ldots, d_{G'}(v_n)$ be the degree sequence of $G'$ such that $d_{G'}(v_1) \geq d_{G'}(v_2) \geq \cdots \geq d_{G'}(v_n)$. Clearly, $G'$ is not a complete graph. Otherwise $G'$ has property $P$, so does $G$, a contradiction.

Let $v_i$ and $v_j$ be two non-adjacent vertices in $G'$ with $1 \leq i < j \leq n$ and $d_{G'}(v_i) + d_{G'}(v_j)$ as large as possible. Obviously, $d_{G'}(v_i) + d_{G'}(v_j) \leq k - 1$. Let $S$ be the set of vertices in $V(G) \setminus \{v_i\}$ which are not adjacent to $v_i$ in $G$. By the choice of $v_j$, we have $d_{G'}(v) \leq d_{G'}(v_j)$ for any $v \in S$. Then

$$|S| = n - 1 - d_{G'}(v_i) \geq n - k + d_{G'}(v_j).$$

There are at least $n - k + d_{G'}(v_j)$ vertices in $V(G')$ with degree at most $d_{G'}(v_j)$. Let $q = d_{G'}(v_j)$. Then $G'$ has $(n - k + q, q)$-Pósa property. Moreover, since $d_{G'}(v_i) \geq d_{G'}(v_j)$ and
\[ d_{G'}(v_i) + d_{G'}(v_j) \leq k - 1, \text{ it follows that } q = d_{G'}(v_j) \leq \frac{k-1}{2}. \] Since \( G \) is a subgraph of \( G' \) and
\[ d_{G'}(v_j) \geq \delta(G') \geq \delta(G) \geq d, \]
we complete the proof. \qed

The following lemma gives a structural characterization of graphs with Pósa property.

**Lemma 3.2.** Suppose \( G \) has \( n \) vertices and is stable under taking \( k \)-closure. Let \( q \) be the minimum integer such that \( G \) has \((n-k+q,q)\)-Pósa property and \( q \leq \frac{k-1}{2} \). If \( T \) is the set of vertices in \( V(G) \) with degree at least \( k - q \) and \( T' = V(G) \setminus T \), then \( G[T,T'] \) is a complete bipartite graph.

**Proof.** Assume that \( G[T,T'] \) is not a complete bipartite graph. Choose two non-adjacent vertices \( u \in T \) and \( v \in T' \) such that \( d(u) + d(v) \) is as large as possible. Clearly, \( d(u) + d(v) \leq k - 1 \) and \( T \) forms a clique in \( G \) as \( G \) is stable under taking \( k \)-closure. Now denote by \( S \) the set of vertices in \( V \setminus \{u\} \) which are not adjacent to \( u \) in \( G \). Clearly, for any \( v' \in S \), \( d(v') \leq d(v) \) and
\[ |S| = n - 1 - d(u) \geq n - k + d(v). \]

Since \( d(u) \geq k - q \) and \( d(u) + d(v) \leq k - 1 \), \( d(v) \leq q - 1 \). Let \( q' = d(v) \leq q - 1 \). We have at least \( n - k + q' \) vertices in \( V(G) \) with degree at most \( q' \). Then \( G \) has \((n-k+q',q')\)-Pósa property with \( q' < q \), which contradicts the minimality of \( q \). The lemma follows. \qed

Let \( g(k,\Delta) \) be the maximum number of edges in a graph such that the size of linear forests is at most \( k \) and the maximum degree is at most \( \Delta \). The following lemma estimates the upper bound of \( g(k,\Delta) \).

**Lemma 3.3.** For \( k \geq 1 \) and \( \Delta \geq 3 \),

(i) \( g(k,2) \leq \frac{3}{2}k \).

(ii) \( g(k,\Delta) \leq k(\Delta - 1) \).

**Proof of (i).** Let \( G \) be an \( L_{k+1} \)-free graph with \( e(G) = g(k,2) \) and \( \Delta(G) \leq 2 \). Clearly, \( g(1,2) = 1 \) and \( g(2,2) = 3 \). Now suppose that \( k \geq 3 \). Since the maximum degree is at most \( 2 \), each nontrivial component is either a path or a cycle. We claim that each component with at least 3 vertices is a cycle. If not, we add an edge between the two ends of the path and the resulting graph is still \( L_{k+1} \)-free, which contradicts the maximality of \( G \). If there is a component consisting of exactly one edge, we replace this edge and a component \( C_\ell \) in \( G \) with \( C_{\ell+1} \). Then the resulting graph is still \( L_{k+1} \)-free and the number of edges is equal to \( g(k,2) \). Therefore, we can further assume that each nontrivial component of \( G \) is a cycle.

Let \( C_{k_1}, \ldots, C_{k_t} \) be the nontrivial components of \( G \). Then \( k = (k_1 - 1) + \cdots + (k_t - 1) \) and \( e(G) = k_1 + \cdots + k_t = k + t \). Note that \( k_i - 1 \geq 2 \), we have \( t \leq \frac{k}{2} \). Thus \( g(k,2) = e(G) \leq \frac{3}{2}k \). \qed
Proof of (ii). We use induction on $k$. It is easy to check that $g(1, \Delta) = 1$ and $g(2, \Delta) = \Delta$. Thus lemma holds for $k = 1, 2$. Suppose that the lemma holds for all $k' < k$. Let $G$ be an $\mathcal{L}_{k+1}$-free graph with $\Delta(G) \leq \Delta$. Let $P = v_0v_1 \cdots v_t$ be the longest path in $G$ and $B = V(G) \setminus V(P)$. Then $G[B]$ is $\mathcal{L}_{k+1-t}$-free and $e(G[B]) \leq (k-t)(\Delta - 1)$ by the induction hypothesis.

Since $P$ is the longest path in $G$, $d_B(v_0) = d_B(v_t) = 0$ and $d_B(v_i) \leq \Delta - 2$ for $1 \leq i \leq t - 1$. Thus,

$$e(G[V(P)]) + e_G[V(P), B]) = \frac{1}{2} \left( \sum_{i=0}^{t} d_G(v_i) + \sum_{i=0}^{t} d_B(v_i) \right) \leq \frac{1}{2} \left( (t+1)\Delta + (t-1)(\Delta - 2) \right) = t(\Delta - 1) + 1$$

The equality holds only if $d_G(v_0) = \cdots = d_G(v_t) = \Delta$, $d_B(v_1) = \cdots = d_B(v_{t-1}) = \Delta - 2$ and $d_B(v_0) = d_B(v_t) = 0$ hold simultaneously, which is impossible. Therefore, $e(G[V(P)]) + e_G[V(P), B]) \leq t(\Delta - 1)$. Moreover, we have

$$e(G) = e(G[B]) + e(G[V(P)]) + e_G[V(P), B]) \leq (k-t)(\Delta - 1) + t(\Delta - 1) \leq k(\Delta - 1).$$

\[ \square \]

Remark. The graph consisting of $k/3$ pairwise disjoint $K_4$'s shows the bound in Lemma 3.3 (ii) is sharp when $3$ divides $k$ and $\Delta = 3$.

For integers $m, l, r$, the following combinatorial identity is well-known.

$$\binom{m + l}{r} = \sum_{j=0}^{r} \binom{m}{j} \binom{l}{r-j} \quad (3.1)$$

The following lemma bounds the number of $r$-cliques by the number of edges.

Lemma 3.4 ([3]). Let $r \geq 3$ be an integer, and let $x \geq r$ be a real number. Then, every graph with exactly $\left(\frac{x}{2}\right)$ edges contains at most $\left(\frac{x}{r}\right)$ cliques of order $r$.

For two disjoint vertex sets $T$ and $T'$ of $G$, we use $N^i_r(T, T')$ and $\overline{N}^i_r(T, T')$ to denote the number of $r$-cliques in $G[T, T']$ that contain exactly $i$ vertices and at least $i$ vertices in $T'$, respectively.

Proof of Theorem 1.4. Let $G'$ be the $k$-closure of $G$. Then $G'$ is $\mathcal{L}_k$-free from Lemma 2.1. By Lemma 3.1 there exists an integer $q$ with $d \leq q \leq \left\lceil \frac{k-1}{2} \right\rceil$ such that $G'$ has $(n-k+q, q)$-Pósa property. Furthermore, we assume $q$ is as small as possible. Then either $q = \left\lfloor \frac{k-1}{2} \right\rfloor$ or
\( q = \left\lfloor \frac{k-3}{2} \right\rfloor \). Otherwise, \( d \leq q \leq \left\lfloor \frac{k-5}{2} \right\rfloor \) implies that \( N_r(G) \leq \binom{k-q}{r} + (n-k+1)\binom{q}{r-1} = h_r(n,k,q) \leq \max \left\{ h_r(n,k,d), h_r(n,k,\left\lfloor \frac{k-5}{2} \right\rfloor) \right\} \), a contradiction.

(i) \( k \) is odd.

**Claim 1.** \( |T| = q \). Let \( T_1 \) be the set of vertices in \( V(G') \) with degree at least \( \frac{k+1}{2} \), i.e.,

\[
T_1 = \left\{ u \in V(G') : d_{G'}(u) \geq \frac{k+1}{2} \right\}.
\]

Then \( T_1 \) is a clique in \( G' \). Let \( T'_1 = V(G') \setminus T_1 \). By Lemma 3.2, \( G'[T_1,T'_1] \) is a complete bipartite graph. We will show that \( |T_1| = \frac{k-1}{2} \) or \( |T_1| = \frac{k-3}{2} \).

**Claim 1.** \( |T_1| \leq \frac{k-1}{2} \).

**Proof.** Otherwise, \( |T_1| \geq \frac{k+1}{2} \). Since \( G'[T_1,T'_1] \) is a complete bipartite graph, all vertices in \( T' \) with degree at least \( \frac{k+1}{2} \). It implies that \( T_1' \) is an empty set. Thus \( G' \) is a complete graph. Since \( n \geq k+1 \), \( G' \) contains a linear forest of size \( k \), a contradiction.

**Claim 2.** \( |T_1| \geq \frac{k-3}{2} \).

**Proof.** Otherwise, \( |T_1| \leq \frac{k-5}{2} \). Suppose \( |T_1| = \frac{k-1}{2} - t \), then \( 2 \leq t \leq \frac{k-1}{2} \). Since \( G'[T_1,T'_1] \) is a complete bipartite graph, the maximum degree of \( G'[T'_1] \) is at most \( t \). Moreover, \( G'[T'_1] \) is \( L_{2t+1} \)-free. Otherwise we will find a linear forest of size at least \( k \) in \( G' \). By Lemma 3.3, \( e(T') \leq g(2t, t) \leq 2t(t-1) \) when \( t \geq 3 \) and \( e(T') \leq g(2t, t) \leq 6 \) when \( t = 2 \). Suppose \( uv \in E(G'[T'_1]) \). Since the degrees of \( u \) and \( v \) are at most \( \frac{k-1}{2} \), \( u \) and \( v \) have at most \( \frac{k-3}{2} \) common neighbors. Thus the edge \( uv \) is contained in at most \( \binom{r}{r-2} \) \( r \)-cliques.

If \( t = 2 \), then

\[
N_r(G') = N_r(T_1) + N_r^1(T_1,T'_1) + N_r^\geq 2(T_1,T'_1)
\]

\[
\leq \binom{k-5}{r} + \left( n - \frac{k-5}{2} \right) \binom{k-5}{r-1} + 6 \binom{k-3}{r-2}
\]

\[
= \binom{k-5}{r} + \left( n - \frac{k+5}{2} \right) \binom{k-5}{r-1} + 5 \binom{k-5}{r-1} + 6 \binom{k-3}{r-2}
\]

\[
< \binom{k-5}{r} + \left( n - \frac{k+5}{2} \right) \binom{k-5}{r-1}
\]

\[
= h_r \left( n, k, \left\lfloor \frac{k-5}{2} \right\rfloor \right),
\]

where the last inequality follows from (3.1), a contradiction.
If \(3 \leq t \leq \frac{k-1}{2}\), then

\[
N_r \left( G' \right) = N_r(T_1) + N_r^1 \left( T_1, T'_1 \right) + N_r^{\geq 2} \left( T_1, T'_1 \right)
\]

\[
\leq \left( \frac{k-1}{2} - t \right) + \left( n - \frac{k-1}{2} + t \right) \left( \frac{k-1}{r-1} \right) + 2t(t-1) \left( \frac{k-3}{r-2} \right)
\]

\[
\leq \left( \frac{k-7}{2} \right) + \left( n - \frac{k-7}{2} \right) \left( \frac{k-7}{r-1} \right) + \frac{(k-1)(k-3)}{2} \left( \frac{k-3}{r-2} \right)
\]

\[
= \left( \frac{k-7}{2} \right) + \left( n - \frac{k+5}{2} \right) \left( \left( \frac{k-5}{r-1} \right) - \left( \frac{k-7}{r-2} \right) \right) + 6 \left( \frac{k-7}{r-1} \right) + \frac{(k-1)(k-3)}{2} \left( \frac{k-1}{r-2} \right)
\]

\[
< \left( \frac{k+5}{2} \right) + \left( n - \frac{k+5}{2} \right) \left( \frac{k-5}{r-1} \right)
\]

\[
= h_r \left( n, k, \left\lfloor \frac{k-5}{2} \right\rfloor \right),
\]

where the third inequality follows from (3.1), \(n > k^5\) and \(r \leq \left\lfloor \frac{k-3}{2} \right\rfloor\), a contradiction. \(\square\)

By Claim 1 and Claim 2, we have \(|T_1| = \frac{k-1}{2}\) or \(|T_1| = \frac{k-3}{2}\). When \(|T_1| = \frac{k-3}{2}\), since \(G' \left[ T_1, T'_1 \right]\) is a complete bipartite graph and all the vertices in \(T'_1\) have degree at most \(\frac{k-1}{2}\), it follows that all vertices in \(T'_1\) have degree at most one in \(G' \left[ T'_1 \right]\). Therefore, \(G' \left[ T'_1 \right]\) consists of independent edges and isolated vertices. We claim there are at most two edges in \(G' \left[ T'_1 \right]\). Otherwise, one can find \(P_{k-2} \cup 3P_2\) in \(G'\), a contradiction. Thus, \(G' \subseteq H^+ \left( n, k-1, \left\lfloor \frac{k-3}{2} \right\rfloor \right)\). When \(|T_1| = \frac{k-1}{2}\), since \(G' \left[ T_1, T'_1 \right]\) is a complete bipartite graph and vertices in \(T'_1\) have degree at most \(\frac{k-1}{2}\), it follows that \(T'_1\) forms an independent set of \(G'\). Then \(G'\) is isomorphic to \(H(n, k, \left\lfloor \frac{k-3}{2} \right\rfloor)\).

**Case 2.** \(q = \frac{k-3}{2}\).

Let \(T_2\) be the set of vertices in \(V(G')\) with degree at least \(\frac{k+3}{2}\), i.e.,

\[
T_2 = \left\{ u \in V \left( G' \right) : d_{G'}(u) \geq \frac{k+3}{2} \right\}.
\]

Then \(T_2\) is a clique in \(G'\). Let \(T'_2 = V(G') \setminus T_2\). By Lemma 3.2, \(G' \left[ T_2, T'_2 \right]\) is a complete bipartite graph. We will show that \(|T_2| = \frac{k-3}{2}|\).

**Claim 3.** \(|T_2| \leq \frac{k-3}{2}\).

**Proof.** Otherwise, \(|T_2| \geq \frac{k-1}{2}|\). The fact \(G' \left[ T_2, T'_2 \right]\) is a complete bipartite graph implies that all vertices in \(T'_2\) have degree at least \(\frac{k-1}{2}\). Therefore \(G'\) has no vertex with degree less than or equal to \(\frac{k-3}{2}\), which contradicts to the fact that \(G'\) has \(n - k + \left( \frac{k-3}{2}, \frac{k-3}{2}\right)\)-Pósa property. \(\square\)

**Claim 4.** \(|T_2| \geq \frac{k-3}{2}|\).

**Proof.** Otherwise, \(|T_2| \leq \frac{k-5}{2}|\). Suppose \(|T_2| = \frac{k-1}{2} - t\), where \(2 \leq t \leq \frac{k-1}{2}\). Since \(G' \left[ T_2, T'_2 \right]\) is a complete bipartite graph, the maximum degree of \(G' \left[ T'_2 \right]\) is at most \(t + 1\). Moreover, \(G' \left[ T'_2 \right]\) is \(L_{2t+1}\)-free. Otherwise we will find a linear forest of size at least \(k\) in \(G'\).
When $t = 2$, since $G'[T_2, T'_2]$ is a complete bipartite graph, $G'[T_2']$ is $\mathcal{L}_5$-free with maximum degree at most 3. By Lemma 3.3, $e(T') \leq g(4, 3) < 10 = \binom{5}{2}$. Then we have $N_r(G'[T_2']) \leq \binom{5}{2}$ from Lemma 3.4. Thus the following inequality holds:

$$N_r(G') = N_r(T_2) + N^1_r(T_2, T'_2) + \sum_{i=2}^{5} N^i_r(T_2, T'_2) \leq \binom{k-5}{2} + \left(n - \frac{k - 5}{2}\right) \binom{k-5}{r-1} + \sum_{i=2}^{5} \binom{5}{i} \binom{k-5}{r-i} \leq \binom{k+5}{2} + \left(n - \frac{k + 5}{2}\right) \binom{k-5}{r-1} = h_r\left(n, k, \left\lfloor \frac{k-5}{2}\right\rfloor\right),$$

where the second equality follows from (3.1), a contradiction.

When $3 \leq t \leq \frac{k-1}{2}$, by Lemma 3.3, $e(T') \leq g(2t, t + 1) \leq 2t^2$. Note each edge in $G'[T_2']$ is contained in at most $(\frac{k-t}{r-2})$-r-cliques. Thus we have

$$N_r(G') = N_r(T_2) + N^1_r(T_2, T'_2) + N^{\geq 2}_r(T_2, T'_2) \leq \binom{k-1}{2} - t + \left(n - \frac{k-1}{2} + t\right) \binom{k-1}{r-1} + 2t^2 \binom{k-1}{r-2} \leq \binom{k-7}{2} + \left(n - \frac{k - 7}{2}\right) \binom{k-7}{r-1} + \frac{(k-1)^2}{2} \binom{k-1}{r-2} = \binom{k-7}{2} + \left(n - \frac{k + 5}{2}\right) \left[\binom{k-5}{r-1} - \binom{k-7}{r-2}\right] + 6 \binom{k-7}{r-1} + \frac{(k-1)^2}{2} \binom{k-1}{r-2} < \binom{k+5}{2} + \left(n - \frac{k + 5}{2}\right) \binom{k-5}{r-1} = h_r\left(n, k, \left\lfloor \frac{k-5}{2}\right\rfloor\right),$$

where the third inequality follows from (3.1), $n > k^5$ and $r \leq \left\lceil \frac{k-3}{2}\right\rceil$, a contradiction.

By Claim 3 and Claim 4, we have $|T_2| = \frac{k-3}{2}$. Then $G'[T_2']$ must be $\mathcal{L}_3$-free. Otherwise we can find a linear forest of size $k$. Moreover, each vertex in $G'[T_2']$ has degree at most two. Thus $G'[T_2']$ is a subgraph of $C_3 \cup (n - 3)K_1$ or $2P_2 \cup (n - 4)K_1$. It follows that $G'$ is a subgraph of $H\left(n, k, \left\lceil \frac{k-3}{2}\right\rceil\right)$ or $H^+\left(n, k - 1, \left\lceil \frac{k-3}{2}\right\rceil\right)$.

Combining the two cases above, we get that $G$ is a subgraph of $H\left(n, k, \left\lceil \frac{k-1}{2}\right\rceil\right)$, $H\left(n, k, \left\lceil \frac{k-3}{2}\right\rceil\right)$ or $H^+\left(n, k - 1, \left\lceil \frac{k-3}{2}\right\rceil\right)$.

(ii) $k$ is even.
Case 1. $q = \frac{k - 2}{2}$.

Let $T_1$ be the set of vertices in $V(G')$ with degree at least $\frac{k + 2}{2}$, i.e.,

$$T_1 = \left\{ u \in V(G') : d_{G'}(u) \geq \frac{k + 2}{2} \right\}.$$

Then $T_1$ is a clique in $G'$. Let $T'_1 = V(G') \setminus T_1$. By Lemma 3.2, $G'[T_1, T'_1]$ is a complete bipartite graph. We will show that $|T_1| = \frac{k - 2}{2}$ or $|T_1| = \frac{k - 4}{2}$.

Claim 5. $|T_1| \leq \frac{k - 2}{2}$.

Proof. Otherwise, $|T_1| \geq \frac{k}{2}$. The fact $G'[T_1, T'_1]$ is a complete bipartite graph implies that all vertices in $T'_1$ have degree at least $\frac{k}{2}$. Then $G'$ has no vertex with degree less than or equal to $\frac{k - 2}{2}$, which is a contradiction to the fact that $G'$ has $(n - k + \frac{k - 2}{2}, \frac{k - 2}{2})$-Pósa property.

Claim 6. $|T_1| \geq \frac{k - 4}{2}$.

Proof. Otherwise, $|T_1| \leq \frac{k - 6}{2}$. Suppose $|T_1| = \frac{k}{2} - t$, then $3 \leq t \leq \frac{k}{2}$. Since $G'[T_1, T'_1]$ is a complete bipartite graph, the maximum degree of $G'[T'_1]$ is at most $t$. Moreover, $G'[T'_1]$ is $L_{2t}$-free. Otherwise we will find a linear forest of size at least $k$ in $G'$. By Lemma 3.3, $e(T'') \leq g(2t - 1, t) \leq (2t - 1)(t - 1)$.

If $t = 3$, then

$$N_r(G') = N_r(T_1) + N^1_r(T_1, T'_1) + N_{\geq 2}^2(T_1, T'_1)$$

$$\leq \left( \frac{k - 6}{2} \right) + \left( n - \frac{k - 6}{2} \right) \left( \frac{k - 6}{2} \right) + 10 \left( \frac{k - 2}{2} \right)$$

$$= \left( \frac{k - 6}{2} \right) + \left( n - \frac{k + 6}{2} \right) \left( \frac{k - 6}{2} \right) + 6 \left( \frac{k - 6}{2} \right) + 10 \left( \frac{k - 2}{2} \right)$$

$$< \left( \frac{k + 6}{2} \right) + \left( n - \frac{k + 6}{2} \right) \left( \frac{k - 6}{2} \right)$$

$$= h_r \left( n, k, \left\lfloor \frac{k - 5}{2} \right\rfloor \right),$$

where the last inequality follows from (3.1).
If $4 \leq t \leq \frac{k}{4}$, then

$$N_{r}(G') = N_{r}(T_{1}) + N_{r}^{1} (T_{1}, T_{1}') + N_{r}^{2} (T_{1}, T_{1}')$$

$$\leq \left( \frac{k}{2} - t \right) + \left( n - \frac{k}{2} + t \right) \left( \frac{k}{2} - t \right) + (2t - 1)(t - 1) \left( \frac{k}{r} - 2 \right)$$

$$\leq \left( \frac{k-8}{2} \right) + \left( n - \frac{k-8}{2} \right) \left( \frac{k-8}{2} \right) + \frac{(k-1)(k-2)}{2} \left( \frac{k}{r} - 2 \right)$$

$$= \left( \frac{k-8}{2} \right) + \left( n - \frac{k+6}{2} \right) \left( \frac{k-6}{2} \right) - \left( \frac{k-8}{2} \right) r - 1 \right) + \frac{(k-1)(k-2)}{2} \left( \frac{k}{r} - 2 \right)$$

$$< \left( \frac{k+6}{2} \right) + \left( n - \frac{k+6}{2} \right) \left( \frac{k-6}{2} \right)$$

$$= h_{r} \left( n, k, \left\lfloor \frac{k-5}{2} \right\rfloor \right),$$

where the third inequality holds since $\frac{k}{4} - t \leq \frac{k}{2}$, $n > k^{5}$ and $r \leq \left\lfloor \frac{k-3}{2} \right\rfloor$, a contradiction. □

By Claim 5 and Claim 6, we have $|T_{1}| = \frac{k-4}{2}$ or $|T_{1}| = \frac{k-2}{2}$. When $|T_{1}| = \frac{k-4}{2}$, since $G'[T_{1}, T_{1}']$ is a complete bipartite graph, the maximum degree of $G'[T_{1}']$ is at most two. Moreover, $G'[T_{1}']$ is $L_{4}$-free. Therefore, $G'[T_{1}']$ (without isolated vertices) is a subgraph of $\{C_{4}, C_{3} \cup P_{2}, 3P_{2}\}$. Thus $G$ is a subgraph of $H(n, k, \left\lfloor \frac{k-3}{2} \right\rfloor)$, $H^{+}(n, k-1, \left\lfloor \frac{k-3}{2} \right\rfloor)$ or $H^{++}(n, k-2, \left\lfloor \frac{k-3}{2} \right\rfloor)$. When $|T_{1}| = \frac{k}{2} - 1$, since $G'[T_{1}, T_{1}']$ is a complete bipartite graph, $G'[T_{1}']$ is $L_{2}$-free, i.e. there is at most one edge in $G'[T_{1}']$. Thus, $G' \subseteq H(n, k, \left\lfloor \frac{k-1}{2} \right\rfloor)$.

**Case 2.** $q = \frac{k-4}{2}$.

Let $T_{2}$ be the set of vertices in $V(G')$ with degree at least $\frac{k+4}{2}$, i.e.,

$$T_{2} = \left\{ u \in V(G') : d_{G'}(u) \geq \frac{k+4}{2} \right\}.$$

Then $T_{2}$ is a clique in $G'$. Let $T_{2}' = V(G') \setminus T_{2}$. By Lemma 3.2, $G'[T_{2}, T_{2}']$ is a complete bipartite graph. We will show that $|T_{2}| = \frac{k-4}{2}$.

**Claim 7.** $|T_{2}| \leq \frac{k-4}{2}$.

**Proof.** Otherwise, $|T_{2}| \geq \frac{k-4}{2}$. The fact $G'[T_{2}, T_{2}']$ is a complete bipartite graph implies that all vertices in $T_{2}'$ have degree at least $\frac{k-2}{2}$. Therefore $G'$ has no vertex with degree less than or equal to $\frac{k-4}{2}$, which contradicts to the fact that $G'$ has $(n - k + \frac{k-4}{2}, \frac{k-4}{2})$-Pósa property. □

**Claim 8.** $|T_{2}| \geq \frac{k-4}{2}$.

**Proof.** Otherwise, $|T_{2}| \leq \frac{k-6}{2}$. Suppose $|T_{2}| = \frac{k}{2} - t$, then $3 \leq t \leq \frac{k}{2}$. Since $G'[T_{2}, T_{2}']$ is a complete bipartite graph, the maximum degree of $G'[T_{2}']$ is at most $t + 1$. Moreover, $G'[T_{2}']$ is $L_{2t}$-free. Otherwise, we will find a linear forest of size at least $k$ in $G'$.
When \( t = 3 \), since \( G'[T_2, T_2'] \) is a complete bipartite graph, \( G'[T_2] \) is \( L_6 \)-free with maximum degree at most 4. By Lemma 3.3 (ii), \( e(T_2') \leq 15 = \binom{6}{2} \). So \( N_r(G'[T_2]) \leq \binom{6}{r} \) from Lemma 3.4. Then we have

\[
N_r(G') = N_r(T_2) + N_r^1(T_2, T_2') + \sum_{i=2}^{6} N_r^i(T_2, T_2')
\]

\[
\leq \left( \frac{k-6}{r} \right) + \left( n - \frac{k-6}{2} \right) \left( \frac{k-6}{r-1} \right) + \sum_{i=2}^{6} \left( \frac{6}{i} \right) \left( \frac{k-3}{r-i} \right)
\]

\[
= \left( \frac{k+6}{2} \right) + \left( n - \frac{k+6}{2} \right) \left( \frac{k-6}{r-1} \right)
\]

\[
= h_r \left( n, k, \left\lfloor \frac{k-5}{2} \right\rfloor \right),
\]

where the second equality follows from (3.1), a contradiction. 

When \( 4 \leq t \leq \frac{k}{2} \), by Lemma 3.3, \( e(T') \leq g(2t-1, t+1) \leq (2t-1)t \). Thus we have

\[
N_r(G') = N_r(T_2) + N_r^1(T_2, T_2') + N_r^{\geq 2}(T_2, T_2')
\]

\[
\leq \left( \frac{k-t}{r} \right) + \left( n - \frac{k-t}{2} + t \right) \left( \frac{k-t}{r-1} \right) + (2t-1)t \left( \frac{k}{r-2} \right)
\]

\[
\leq \left( \frac{k-8}{2} \right) + \left( n - \frac{k-8}{2} \right) \left( \frac{k-8}{r-1} \right) + \frac{k(k-1)}{2} \left( \frac{k}{r-2} \right)
\]

\[
= \left( \frac{k-8}{2} \right) + \left( n - \frac{k+6}{2} \right) \left[ \left( \frac{k-6}{r-1} - \frac{k-8}{r-2} \right) \right] + 7 \left( \frac{k-8}{r-1} \right) + \frac{k(k-1)}{2} \left( \frac{k}{r-2} \right)
\]

\[
< \left( \frac{k+6}{2} \right) + \left( n - \frac{k+6}{2} \right) \left( \frac{k-6}{r-1} \right)
\]

\[
= h_r \left( n, k, \left\lfloor \frac{k-5}{2} \right\rfloor \right),
\]

where the third inequality follows from (3.1), \( n > k^5 \) and \( r \leq \left\lfloor \frac{k-3}{2} \right\rfloor \), a contradiction. 

By Claim 7 and Claim 8, we have \( |T_2| = \frac{k-4}{2} \). Since \( G'[T_2, T_2'] \) is a complete bipartite graph, all vertices in \( T_2' \) have degree at least \( \frac{k-4}{2} \). The \( (n-k+\frac{k+4}{2}, \frac{k+4}{2}) \)-pósa property implies that there are at most 4 vertices in \( T_2' \) with degree great than 0. Thus \( G'[T_2'] \) is a subgraph of \( K_4 \cup (n-k+\frac{k+4}{2})K_1 \). Then \( G \subseteq H(n, k, \left\lfloor \frac{k+3}{2} \right\rfloor) \).

Combining the two cases above, we get that \( G \) is a subgraph of \( H(n, k, \left\lfloor \frac{k+4}{2} \right\rfloor), H(n, k, \left\lfloor \frac{k+3}{2} \right\rfloor), H^+(n, k-1, \left\lfloor \frac{k+3}{2} \right\rfloor) \) or \( H^{++}(n, k-2, \left\lfloor \frac{k+3}{2} \right\rfloor) \). The proof is finished. 

\( \square \)
4 The clique version of the stability of Erdős-Gallai Theorem

Notice that a linear forest with at least $2k + 1$ edges has a matching of size at least $k + 1$. A graph $G$ with $\nu(G) \leq k$ must be $\mathcal{L}_{2k+1}$-free. Combining Theorem 1.4 (i) and further discussions, we obtain Theorem 1.7.

**Proof of Theorem 1.7.** Let $G$ be a graph satisfying the conditions of Theorem 1.7. Then $G$ is $\mathcal{L}_{2k+1}$-free. By Theorem 1.4 (i), if $G \not\subseteq H^+(n, 2k, k-1)$, then $G$ is a subgraph of $H(n, 2k + 1, k)$ or $H(n, 2k + 1, k-1)$. Next we will show that if $G \subseteq H^+(n, 2k, k-1)$, then $G \not\subseteq H(n, 2k + 1, k-1)$.

If $G \subseteq H^+(n, 2k, k-1)$ and $G \subseteq H(n, 2k + 1, k-1)$, then we are done. Now we suppose that $G \subseteq H^+(n, 2k, k-1)$ and $G \not\subseteq H(n, 2k + 1, k-1)$.

Note that $H^+(n, 2k, k-1)$ can be viewed as a graph obtained from $H(n, 2k - 1, k-1)$ by adding two independent edges, say $x_1y_1$ and $x_2y_2$. If $G \subseteq H^+(n, 2k, k-1)$ but $G \not\subseteq H(n, 2k + 1, k-1)$, then $x_1y_1$ and $x_2y_2$ must be in $E(G)$. Let $G_1 = G - \{x_1, y_1, x_2, y_2\}$. Then $G_1 \subseteq H(n - 4, 2k - 1, k - 1)$ and

\[
N_r(G_1) > h_r(n, 2k + 1, k - 2) - 4 \binom{k - 1}{r - 1} - 2 \binom{k - 1}{r - 2} \geq \binom{k - 1}{r} + (n - k - 3) \binom{k - 2}{r - 1} \tag{4.1}
\]

Since $G_1 \subseteq H(n - 4, 2k - 1, k - 1)$, there exists an independent set $I$ satisfies $|I| = n - k - 3$ and $d_{G_1}(v) \leq k - 1$ for all $v \in I$. Suppose that there are $t$ vertices in $I$ with degree $k - 1$. Then $t \leq k - 2$. Otherwise, we can find a $(k - 1)$-matching $M$ in $G_1$. The $(k - 1)$-matching $M$ together with the edges $x_1y_1$ and $x_2y_2$ form a $(k + 1)$-matching in $G$, a contradiction.

**Case 1.** $t = 0$.

In this case, all vertices in $I$ have degree at most $k - 2$. Thus

\[
N_r(G_1) \leq \binom{k - 1}{r} + (n - k - 3) \binom{k - 2}{r - 1},
\]

contradicting to (4.1).

**Case 2.** $1 \leq t \leq k - 2$.

There are at most $k - 2 - t$ vertices in $I$ with degree $k - 2$. Otherwise, for any $S \subseteq V(G_1) \setminus I$, $|N(S)| \geq |S|$. By Hall’s Theorem, there exists a $(k - 1)$-matching $M$ in $G_1$. The $(k - 1)$-matching
\( M \) together with the edges \( x_1y_1 \) and \( x_2y_2 \) form a \((k+1)\)-matching in \( G \), a contradiction. Thus

\[
N_r(G_1) \leq \binom{k-1}{r} + t \binom{k-1}{r-1} + (k-2-t) \binom{k-2}{r-1} + (n-k-2) \binom{k-3}{r-1}
\]

\[
< \binom{k-1}{r} + (k-1) \binom{k-1}{r-1} + (n-k-3) \binom{k-3}{r-1}
\]

\[
< \binom{k-1}{r} + (n-k-3) \binom{k-2}{r-1}
\]

where the last inequality follows from \( n > (2k+1)^5 \), which is a contradiction to (4.1).

Thus \( G \subseteq H^+(n, 2k, k-1) \) implies \( G \subseteq H(n, 2k+1, k-1) \). That is, \( G \) is a subgraph of \( H(n, 2k+1, k) \) or \( H(n, 2k+1, k-1) \), completing the proof.

\[\square\]

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