The effect of fractional damping and time-delayed feedback on the stochastic resonance of asymmetric SD oscillator

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Received: 25 June 2021 / Accepted: 24 November 2021 / Published online: 13 January 2022
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Abstract This paper proposes the stiffness nonlinearities and asymmetric SD (smooth and discontinuous) oscillator under time-delayed feedback control with the fractional derivative damping. With the effect of displacement and velocity feedback, the oscillator can be adjusted to the desired vibration state, and then the stochastic resonance (SR) is achieved. This article discusses the contribution of various system parameters and time-delayed feedback to SR, especially which induced by fractional order. It should be noted that this paper provides effective guidance for fault diagnosis and weak signal detection, energy harvesting, vibration isolation and vibration reduction.

Keywords SD oscillator · Fractional damping · Time-delayed feedback · Stochastic resonance

1 Introduction

Fault diagnosis, signal detection and vibration control have been active fields of research over the past decades. There are various methods for weak signal detection and feature extraction for early mechanical faults; two effective methods are as follows:

The first one is detecting weak signals and extracting key features such as amplitude and frequency, by using SR or it is rich dynamical behavior transition with the help of filter which constructed by nonlinear dynamic systems. Chen and Varshney [1], Kumar and Jha [2], Lai et al. [3], Liu et al. [4] and Mba et al. [5] done an excellent work in weak signal detection by SR. The paper [6] has made a comprehensive summary of the principle and limitations of SR. Besides, the work on fault diagnosis and signal detection by Lu et al. [7] has given a complete summary of such method. For signal detection using the dynamical behaviors transfer, these papers [8,9] make a preliminary discussion with duffing oscillator. This paper is based on the methods of TVICM and TV that are proposed by us [10,11]. Furthermore, we obtain a whole set of procedures for signal detection and feature extraction.

The second method is using deep neural networks (DNN) for fault diagnosis on the basis of certain fault samples and most of them are fault classification [12–14].

The combination of two methods is an inevitable trend in the future. It is well-known that strengthening the effective components in the preprocessed signal before diagnosis and detection, which is the goal that all methods want to achieve. SR is an effective method at present, and it can transfer the energy of noise to the
effective signal under the state of resonance and lay a good foundation for further work.

Many methods are proposed to vibration control, which can be divided into three categories as engineering measures: reducing disturbance, preventing resonance and taking vibration isolation [15–19]. For example, many researchers have investigated based on the vibration control technology of the time-delayed feedback [20–23]. The phenomenon of time delay in finite signal transmission is common [24,25], which may induce great changes in the dynamic characteristics of the system. Thus, by studying the effect of time delay, we can not only understand the operating mechanism of the system, but also obtain unexpected gains by actively making use of it [26–28]. For example, the effective control of time delay on the vibration of SD oscillator [29] lays the foundation for the vibration energy harvesting. The stochastic perturbation is unavoidable in vibration control, so the study on controlling the vibration of the system reasonably by adjusting the random parameters (the noise and so on) is more and more important. Such as by adjusting the values of the parameters, the system can reach the state of SR.

As is known to all, SR is a physical phenomenon, which exists in nonlinear systems. Benzi et al. [30,31] have carried out the work on SR in the context of simulating the transition of Earth’s climate between ice ages and periods of relative warmth with a period of about 100,000 years in the past 700,000 years. Nicolis and Nicolis [32] proposed to use the bistable potential climate model to simulate the two states of quaternary glacial and warm climate period, and established the corresponding stochastic differential equation, using adiabatic approximation conditions to solve differential equations, and then drew the same conclusion as Benzi.

The SD oscillator with nonlinear stiffness was proposed by Cao et al. [33]. The nonlinear stiffness of SD oscillator [34,35] makes its dynamic behavior specially abundant, even under the circumstances of single degree of freedom. The phenomena of snap-through buckling in engineering were revealed by Thompson and Hunt [36], who put forward that the SD oscillator was developed based on the shallow arch model. It is necessary for the engineering practice needs to study on nonlinear dynamics of oscillator under nonlinear stiffness time-delayed feedback control [29]. Moreover, the principal resonance of the quasi-zero-stiffness SD oscillator is studied, and the resonance of negative-stiffness SD oscillator is analyzed. The optimal time delay of changing the control intensity can be obtained according to the proposed optimization standard.

In this paper, the SD oscillator is further modified on the basis [29]. As the SD oscillator with asymmetric horizontal stiffness is proposed, fractional derivative damping in the Captou sense is introduced, which makes the model have more practical significance. As a matter of fact, in the system of viscoelastic materials, the damping term is not only related to the current state, but also depends on the previous. The work [37] proves that for the materials with memory properties, the fractional-order models can better describe its dynamic behavior than the integer-order models. There are many types of fractional derivatives, such as Caputo, Riemann–Liouville and Grünwald–Letnikov. In this paper, we adopt the Caputo fractional derivative to analyze the model as it can accurately describe and reflect practical physical problems, besides the nonlocality that plays a significant role in various fields. Therefore, more and more research studies complete with the help of it. In order to construct a filter based on SD oscillator for fault diagnosis and signal detection, the SR of the nonlinear stiffness SD oscillator subjected to time-delayed feedback, asymmetric parameters and fractional damping is studied under the influence of harmonic forcing and Gaussian white noise.

This paper is organized as follows: In Sect. 2, the stochastic delay differential equation (SDDE) with fractional order is given to describe the nonlinear SD oscillator, in order to analyze it was reduced to the Langevin equation after dimensionless processing. In Sect. 3, the effects of time-delayed feedback control on the nonlinear SD oscillator and the contribution of fractional-order damping and asymmetric parameter in the equation to SR are discussed. In Sect. 4, conclusion and discussion are provided and put forward the prospect of the future work.

2 Time-delay-controlled asymmetrical SD oscillator driven by fractional derivatives with stiffness nonlinearities

In 2006, Cao [38] proposed a SD oscillator, and studied its complex dynamical behavior, then found the “strange hand” attractor (SD attractor). Subsequently, a large number of scholars have studied the abundant dynamical behavior of SD oscillator [39–41].
Based on the SD oscillator [29], the dynamic model of asymmetrical SD oscillator, which is driven by fractional derivatives with stiffness nonlinearities and controlled by delayed feedback, is established, as shown in Fig. 1.

The equation of motion of the system has the following form:

$$M\ddot{X} + CD^q X + K_1 X + K_2 X \left(1 - \frac{L_1}{\sqrt{X^2 + l_1^2}}\right) + K_2 X \left(1 - \frac{\rho L_1}{X^2 + l_2^2}\right) = -K_1 \dot{\xi} + Mg$$

where $X$ is the displacement of the loaded mass $M$, $K_1$ is a vertical linear stiffness spring, and $C$ is the coefficient of a viscoelastic damping; $K_2$ is the stiffness of a pair of oblique springs, which produces the nonlinear stiffness mechanism, and unstretched length $L_1$ reveals the asymmetry of the system. $L_2 = \rho L_1$ and the length $l_1$, $l_2$ while compressed the direction in the horizontal; $\dot{\xi}$ represents the free length of the oblique of spring over the origin, it represents the midpoint of the pivots of the supporting ends of the oblique springs; $Mg$ is the gravity of the loaded mass; $\Upsilon_d X(T - \sigma) + \Upsilon_r \dot{X}(T - \sigma)$ is the signal of delayed feedback control, where $\Upsilon_d$ is the displacement feedback intensities and $\Upsilon_r$ is velocity feedback intensities; $\sigma > 0$ is the time-delay feedback; the purpose of designing mechanical actuator is to realize delayed feedback control; $F$ is amplitude of harmonic exciting force and $F_{th}$ is the environmental base excitation; $\Omega$ is frequency of harmonic exciting force; $D^q x$ is the Caputo-type fractional derivative [37,42] and represents as following form:

$$D^q x = \frac{1}{\Gamma(1 - q)} \int_0^t \dot{x}(\sigma) (t - \sigma)^{q - 1} d\sigma, \quad (0 < q \leq 1)$$

where $t \in [0, d]$ as the initial torque of a physical oscillator is $t = 0$, and $\Gamma(\cdot)$ is a Gamma function.

For brevity, the oblique springs are horizontal; at the same time, the unforced mass in the position of static equilibrium that $X = 0$, the constant force $K_1 \dot{\xi} - Mg$ can be neglected, and define dimensionless parameters (see Table 1)

$$x = \frac{X}{L_1}, \quad \alpha_1 = \frac{l_1}{L_1}, \quad \alpha_2 = \frac{l_2}{L_2}, \quad L_2 = \rho L_1$$

$$t = \omega_1 T, \quad \omega_1 = \frac{K_1}{M}, \quad \gamma = \frac{\omega_2^2}{\omega_1^2}$$

$$2c = \frac{C}{M \omega_1}, \quad \omega_1 = \frac{K_2}{M}, \quad f = \frac{F}{ML \omega_1^2}, \quad \omega = \frac{\Omega}{\omega_1}$$

$$\eta(t) = \frac{F_{th}}{ML \omega_1^2}$$

$$\mu = \frac{\Upsilon_d}{M \omega_1^2}, \quad \nu = \frac{\Upsilon_r}{M \omega_1}, \quad \tau = \omega_1 \sigma$$

and by the transformation of variables, we can obtain the equation

$$\ddot{x} + 2c D^q x + x + \gamma x \left(1 - \frac{1}{\sqrt{x^2 + \alpha_1^2}}\right) + \gamma x \left(1 - \frac{1}{\sqrt{(\frac{x}{\alpha_2})^2 + \alpha_2^2}}\right) = f \cos(\omega t) + \eta(t) + \mu x(t - \tau) + \nu \dot{x}(t - \tau)$$

where $\eta(t)$ is Gaussian white noise, which satisfies

$$E[\eta(t)] = 0, \quad E[\eta(t)\eta(t - s)] = 2\kappa \delta(s)$$

where $\kappa$ denotes the intensities of Gaussian white noise $\eta(t)$, and $\kappa^2 = 2\pi K$ ($K$ is a constant power spectral density), $\delta(s)$ is the Dirac function, and $s$ is the correlation time of Gaussian white noise $\eta(s)$.

Approximation of fractional derivatives [43] and displacement and velocity delay feedback with the
assumption that $\tau$ is finite \cite{44,45}

$$D^q x = \omega^q \cos \left( \frac{\pi q}{2} \right) x + \omega^{q-1} \sin \left( \frac{\pi q}{2} \right) \dot{x},$$

$$0 < q \leq 1$$

Then, we can get the reduced equation

$$\ddot{x} + a \dot{x} + bx + \gamma x \left( 1 - \frac{1}{\sqrt{x^2 + \alpha_1^2}} \right) + \gamma x \left( 1 - \frac{1}{\sqrt{\left( \frac{\omega}{\rho} \right)^2 + \alpha_2^2}} \right) = f \cos(\omega t) + \eta(t)$$

where

$$a = 2c \omega^{q-1} \sin \left( \frac{\pi q}{2} \right) \frac{\mu}{\omega} \sin(\omega \tau) - v \cos(\omega \tau)$$

$$b = 2c \omega^q \cos \left( \frac{\pi q}{2} \right) - \mu \cos(\omega \tau) - v \omega \sin(\omega \tau) + 1$$

The Pfaffian equation after the reduction is

$$\dot{x} = y,$$

$$\dot{y} = -ay - bx - \gamma x \left( 1 - \frac{1}{\sqrt{x^2 + \alpha_1^2}} \right)$$

$$- \gamma x \left( 1 - \frac{1}{\sqrt{\left( \frac{\omega}{\rho} \right)^2 + \alpha_2^2}} \right) + f \cos(\omega t) + \eta(t).$$

The restoring force of the dimensionless asymmetric SD oscillator dynamic model with stiffness nonlinearity is given as follows:

$$G(x) = bx + \gamma x \left( 1 - \frac{1}{\sqrt{x^2 + \alpha_1^2}} \right)$$

$$+ \gamma x \left( 1 - \frac{1}{\sqrt{\left( \frac{\omega}{\rho} \right)^2 + \alpha_2^2}} \right)$$

The deterministic potential of this system is

$$V(x) = \left( \frac{b}{2} + \gamma \right) x^2 - \gamma \sqrt{x^2 + \alpha_1^2}$$

$$- \rho \gamma \sqrt{(x^2 + (\rho \alpha_2)^2)^2}$$

and the equivalent stiffness of this system can be described as:

$$K(x) = b + 2\gamma \frac{\gamma}{\sqrt{x^2 + \alpha_1^2}} - \rho \gamma \frac{\rho \gamma}{\sqrt{(x^2 + (\rho \alpha_2)^2)^2}}$$

$$+ \frac{\gamma x^2}{(x^2 + \alpha_1^2)^2} + \frac{\rho \gamma x^2}{(x^2 + (\rho \alpha_2)^2)^2}$$

### 3 Delay-fractional derivative-controlled stochastic resonance

In this section, we use the joint probability density of velocity and displacement to discuss the influence of time delay, fractional order and other parameters on the signal-to-noise ratio (SNR) and seek for the maximum SNR, that is SR.
3.1 Joint quasi-stationary probability density function $P_e$ and effective potential energy $U_e$

According to the adiabatic approximation theory, the Fokker–Planck–Kolmogorov (FPK) equation is described by Eq. (9) which takes the form

$$
\frac{\partial}{\partial t}P(x, y, t) + \frac{\partial}{\partial x}(yP(x, y, t)) + \frac{\partial}{\partial y}(-ay - bx) - \gamma x \left(1 - \frac{1}{\sqrt{x^2 + \alpha_1^2}}\right) - \gamma x P(x, y, t) \\
\left(1 - \frac{1}{\sqrt{(\frac{y}{\rho})^2 + \alpha_2^2}}\right)P(x, y, t)
$$

$$
\frac{\partial^2}{\partial y^2}P(x, y, t) + \frac{\partial}{\partial y}(f \cos(\omega t)P(x, y, t)) = 0.
$$

From Eq. (13), $P_e(x, y, t)$ (where $x$ represents the displacement and $y$ is velocity) is the joint quasi-stationary probability density function, which is given as follows:

$$
P_e(x, y, t) = N \exp\left[-\frac{U_e(x, y, t)}{\pi \kappa}\right],
$$

where $N$ represents a normalization constant, the effective potential energy is denoted by $N = (\int \exp(P_e)dx)^{-1}$, and $U_e(x, y, t)$ has given

$$
U_e(x, y, t) = a\left(\frac{y^2}{2} + \frac{1}{2}bx^2 + \gamma x^2 - \gamma \sqrt{x^2 + \alpha_1^2}\right) \\
- \gamma \rho \sqrt{x^2 + (\rho \alpha_2)^2} - xf \cos(\omega t))
$$

When the fractional order $q = 0.46$ is fixed, the probability density function $P_e$ for different values of the time delay $\tau$ is shown in Fig. 2, and these figures both on the left and right not only validate the effectiveness of the analytical investigations, but also show that the time delay can make a difference in stochastic Hopf bifurcation of the system. Specifically, with the increase in time delay $\tau$, the probability density function $P_e$ shows different forms in Fig. 2a, c, e. The stochastic D-bifurcation is shown in Fig. 2a through Fig. 2c, due to the property of probability density $P_e$ changes from fixed point to non-trivial probability density [46]; the stochastic P-bifurcation is shown in Fig. 2c–e, as it can be observed from Fig. 2c that the probability density function $P_e$ exhibits a single-peak structure, while Fig. 2e presents a double-peak structure.

When the time delay $\tau = \frac{\pi}{\omega}$ is fixed, the probability density function $P_e$ for different values of the fractional-order $q$ is shown in Fig. 3, and these figures both on the left and right not only validate the effectiveness of the analytical investigations, but also show that the fractional order can induce the occurrence of the stochastic bifurcation. Specifically, with the increase in fractional-order $q$, the probability density function $P_e$ shows different forms in Fig. 3a, c, e. The stochastic D-bifurcation is shown in Fig. 3a through Fig. 3c, as the property of probability density $P_e$ changes from fixed point to non-trivial probability density [46]; the stochastic P-bifurcation is shown in Fig. 3c–e, as it can be observed from Fig. 2c that the probability density function $P_e$ exhibits a single-peak structure, while Fig. 3e presents a double-peak structure.

The three-dimensional surfaces of the effective potential energy $U_e$ as functions of $x$ and $q$ (or $\rho$, $\tau$, $\mu$, $\nu$, $\kappa$) are shown in Fig. 4. From Fig. 4a, c it could be noticed that with the increase in the order $q$ of fractional derivative and the time delay $\tau$, the potential barriers increases obviously at first and then decreases. The potential barriers are basically the same as the increase in displacement feedback strength $\mu$, asymmetric proportion $\rho$ and velocity feedback intensity $v$, as shown in Fig. 4b, d, e.

To put it simply, based on the effective potential of the FPK equation, the physical meaning of noise delay and fractional-assisted frequency hopping mechanism is explained. In the next section, we will analyze and extract a function which plays the role of activation energy to identify the SNR.
3.2 Signal-to-noise ratio

By assuming condition of \((1 + 2\gamma)\alpha_1\alpha_2 < \gamma(\alpha_1 + \alpha_2)\) and \(\gamma \neq -\frac{1}{2}\), three singular points \(x_{\pm}, x_0\) of the asymmetrical SD oscillator are obtained when lack of the delayed feedback control [47]. The eigenvalues of the linearization matrix at the equilibria \(x_{\pm}\) and \(x_0\) are \(\beta_{1,2}(\pm)\) and \(\lambda_{1,2}(0)\), respectively. Therefore, we can analyze SNR further.

By researching SNR from the view of Kramers rate, we can calculate the propagation rate of noise-driven motion in Eq. (1) [48]. For the exact expression of escape rate, for reference [49]. Fundamentally, we can start with the Kolmogorov’s backward equation, which is equivalent to (15). Eventually, it results in the escape rate \(R_\pm(t)\) for

\[
R_\pm(t) = \frac{\sqrt{\beta_1 \beta_2}}{2\pi} \sqrt{-\frac{\lambda_1}{\lambda_2}} \exp\left[\frac{U_e(x_{\pm}, t) - U_e(x_0, t)}{\pi \kappa}\right].
\]

The mean-first-passage time (MFPT) over the potential barrier is calculated from the following definition:

\[
\text{MFPT}_{R,L} = \frac{1}{R_\pm(t)}. \tag{17}
\]
Let us substitute Eq. (15) into Eq. (16); then, the probability transition rate can be obtained as

\[ R_{\pm}(t) = \frac{\sqrt{\beta_1 \beta_2}}{2\pi} \sqrt{-\frac{\lambda_1}{\lambda_2}} \exp \left[ \frac{a}{\pi \kappa} \right] \]

\[ \times \left( \frac{1}{2} b x_{\pm s}^2 + \frac{\gamma}{2} \left( x_{\pm s}^2 + \alpha_1^2 \right) \right) \]

\[ - \gamma \rho \left( x_{\pm s}^2 + \alpha_1^2 \right) \]

(18)

\[ R_{\pm}(t) \text{ can be expressed in the form of Taylor series} \]

\[ R_{\pm}(t) = R_0 (1 \mp x_{\pm s} f \cos(\omega t)) \]

\[ + \frac{1}{2} x_{\pm s}^2 f^2 \cos^2(\omega t) \pm \cdots \]

(19)

where

\[ R_0 = \frac{\sqrt{\beta_1 \beta_2}}{2\pi} \sqrt{-\frac{\lambda_1}{\lambda_2}} \exp \left[ \frac{a}{\pi \kappa} \left( \frac{1}{2} b x_{\pm s}^2 + \gamma |x_{\pm s}|^2 \right) - \gamma \sqrt{|x_{\pm s}|^2 + \alpha_1^2} \right] - \gamma \rho \sqrt{|x_{\pm s}|^2 + (\rho \alpha_2)^2} \]

\[ - \gamma \rho \sqrt{|x_{\pm s}|^2 + (\rho \alpha_2)^2 - \gamma \alpha_1 - \gamma \rho^2 \alpha_2} \]

(20)

Further, the output spectrum of the system can be obtained

\[ S(\Omega) = S_1(\Omega) + S_2(\Omega) \]

(21)
where $S_1(\Omega)$ is the signal of the output power spectra and $S_2(\Omega)$ is noise of the output power spectra, and their expressions can be written as follows:

$$S_1(\Omega) = \frac{\pi x_{\pm}^2 (R_0 a|x_{\pm}|f)^2}{2(R_0^2 + \omega^2)} \left[ \delta(\Omega - \omega) + \delta(\Omega + \omega) \right]$$

$$S_2(\Omega) = [1 - \left(\frac{R_0 a|x_{\pm}|f}{2(R_0^2 + \omega^2)}\right)^2] \frac{2x_{\pm}^2 R_0}{2(R_0^2 + \Omega^2)}$$

(22)

According to the definition of output power SNR, the SNR of fraction-delay-controlled asymmetric SD oscillator can be obtained as

$$\text{SNR} = \frac{\int_0^{\infty} S_1(\Omega) d\Omega}{S_2(\Omega = \omega)} = \frac{\pi (R_0 a|x_{\pm}|f)^2 (R_0^2 + \omega^2)}{4 R_0^2 (R_0^2 + \omega^2) - 2 R_0^2 (R_0 a|x_{\pm}|f)^2}$$

(23)

REMARK: There is a minor error in formula (41) in Ref. [29], and it should be

$$\frac{\pi (R_0 a|x_{\pm}|f)^2 (R_0^2 + \omega^2)}{4 R_0 (R_0^2 + \omega^2) - 2 R_0 (R_0 a|x_{\pm}|f)^2}.$$ But this small error will not have a great impact on the SNR analysis, as shown in Figs. 13 and 14 of Ref. [29], it does not affect the shape of the SNR.

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Fig. 5 SNR as a function of fractional-order $q$ or time-delay $\tau$ for various asymmetric ratio $\rho$. The other parameter values are $\gamma = 0.25, \alpha_1 = 0.21, \alpha_2 = 0.79, \omega = 0.5, \epsilon = 0.1, f = 0.1, \kappa = 0.01, \mu = 0.5, \nu = 0.1$

In Fig. 5a, under the particular parameter, the SNR decreases first, then increases, and finally decreases with the increase in fractional order $q$. When the SNR reaches the peak, the SR occurs in the system, which indicates that the order change of the fractional order $q$ will affect the occurrence and disappearance of SR, and our purpose is to adjust the order $q$ so that the system achieves maximum SR. We find that $q = 0$ is also an extreme point, and there is no damping term $\dot{x}$ in the system, that is, the system has no energy loss; and with the increase in $q$, it can be seen that the damping term plays a certain role from formula (6), which can be interpreted as the energy consumption of the system, resulting in the reduction in SNR. An interesting phenomenon worth mentioning is that when the order $q$ increases again, the SNR does not decrease but rises to a peak value. This will mean that by adjusting the order $q$, the time-delay-controlled asymmetrical SD oscillator driven by fractional derivatives can achieve ideal SR, laying a good foundation for the detection of weak signals.

Next, we can choose a fractional order $q = 0.7$ and analyze the SNR changes with time delay $\tau$. It can be seen that the appropriate time delay $\tau$ will make the SNR reach the peak, as shown in Fig. 5b. In addition, the effect of asymmetric ratio $\rho$ of asymmetric SD oscillator on SNR is also considered, presenting a non-monotone change. When time delay $\tau = 0.1, \rho = 0.3$ is the most appropriate, and when $q = 0.7, \rho = 0.2$ is optimal. In a word, $q, \tau$ and $\rho$ jointly restrict the change of SNR.

With the parameter of Fig. 5, according to Eq. (23), the SNR versus joint fractional-order $q$ and noise intensity $\kappa$ with $\tau = 0.1, \mu = 0.5, \nu = 0.1$ is shown in Fig. 6.

In Fig. 7a–c, under the particular parameter, the SNR increases with the increase in time delay $\tau$. Figure 7c shows that the maximum time delay $\tau$ will make the SNR reach the peak. The SR phenomenon occurs at the maximum of the SNR. Besides the SNR can also reach the peak under the change of displacement feedback strength $\mu$ and velocity feedback intensity $\nu$ in Fig. 7f, g. We can see the parameter of $\tau$ has the most obvious influence on the SNR, that is because the time delay...
delay $\tau$ affects the effect of the system on the non-linearization of the input signal. We find that there are many adjustable parameters for the SR in the system. In Fig. 8, we have studied the joint effects of displacement feedback strength $\mu$ and velocity feedback intensity $\nu$ on the SNR. Our aim is to use the mathematical optimal control method to find the appropriate parameters to make the SNR reach the maximum value.

3.3 Suggestion for application: weak signal detection and vibration energy harvesting

3.3.1 Weak signal detection

When add the harmonic weak signal containing Gaussian white noise $r \cos(\omega_1 t)$ to the original system and use the method of chaotic state “transient vacance” (namely “TV” method) to detect the frequency of weak signal, system will be period state if $\omega = \omega_1$, and if $\omega \neq \omega_1$ system will be in chaotic state. We can presume that the SR of the system contributes to the period state, and at that time, it is obviously to observe. Adjusting the parameters $\gamma = 0.9, \alpha_1 = 0.2, \alpha_2 = 0.1, \mu = 0.5, v = 0.4, \rho = 0.5, k = 0.1, c = 0.2, \text{ when } q = 0.7 \text{ and } \omega = \omega_1 = 1, \text{ the system reaches the state of SR. Comparing Fig. 9a, b, on the one hand, it obvious that when } q = 0.7, \text{ the periodic state is clearer than } q \neq 0.7, \text{ which makes the periodic state is easier to identify. On the other hand, when } q = 1, \text{ only with enough the chaotic threshold } f, \text{ the periodic state can be identified, see the left hand of the blue line which } f = 0.3 \text{ in Fig. 9a. But in Fig. 9b, when } q = 0.7,
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The value of $f$ can be little when the system reaches the periodic state, that is to say, weak signal detection more quickly and lower time cost when $q = 0.7$ than $q = 1$. Therefore, the fractional order on weak signal detection can be obviously reflected than the integer order, besides it is more efficient and takes less time.

The discovery of SR has completely subverted people’s perception of noise. Noise has always been considered to destroy the orderly behavior of the system and reduce the system performance, but now we find that the noise can be used to improve the output SNR of the system. When the system is adjusted to the state of SR, the noise energy is transferred to the weak signal, the SNR reach the maximum value.

In Fig. 10, the effect of noise intensity $\kappa$ on the periodic state of the system have been investigated. Fix the parameters as $q = 0.7$, $\gamma = 0.25$, $\alpha_1 = 0.21$, $\alpha_2 = 0.79$, $\mu = 0.5$, $\nu = 0.1$, $\rho = 0.2$, $c = 0.1$. In Fig. 10b, when $\kappa = 0.04$ and $\omega = \omega_1$, the system reaches the SR state, at that time the periodic state is clearer than the other values of $\omega$. In the left hand of the blue line $f = 0.3$, in Fig. 10b, the periodic state of the system can be easily observed. Figure 10a, b shows that the increase in $\kappa$ leads to the increase in the SNR. This illustrates that in the state of SR, the energy of the noise can be transferred to the signal which to be detected.

The effects of time delay $\tau$, you can refer to [29], which have been extensively studied by the author.

3.3.2 Vibration energy harvesting

As for vibration energy harvesting, the SR effect [34,50] should be effectively utilized. However, the development of practical SR precise control methods...
(a) $q=1, \omega = 1$ and fix $\omega = \{0.9, 0.95, 1, 1.05, 1.1, 1.15, 1.2\}$ from top to bottom, it shows when $\omega = \omega_1 = 1$ the system first reaches the periodic state.

(b) $q=0.7, \omega = 1$ and fix $\omega = \{0.9, 0.95, 1, 1.05, 1.1, 1.15, 1.2\}$ from top to bottom, it is obvious that when $\omega = \omega_1 = 1$ the system first reaches the periodic state.

**Fig. 9** Periodic state of the system with different parameter $q$
(a) $\kappa = 0.01$, $\omega = 1.2$ and fix $\omega = \{1.1, 1.15, 1.2, 1.25, 1.3, 1.35, 1.4\}$ from top to bottom, it shows when $\omega = \omega_1$ the system first reaches the periodic state.

(b) $\kappa = 0.04$, $\omega = 1.2$ and fix $\omega = \{1.1, 1.15, 1.2, 1.25, 1.3, 1.35, 1.4\}$ from top to bottom, it is obvious that when $\omega = \omega_1$ the system first reaches the periodic state.

Fig. 10 Periodic state of the system with different parameter $\kappa$
is a further challenge that needs to be overcome. In this paper, the application of SR with fractional damping and time-delayed feedback control in asymmetric SD oscillator is comprehensively analyzed and a mathematical model is established, which plays a key role in improving energy harvesting. The equation of motion controlled by the model is

$$M \ddot{X} + CD \dot{X} + \frac{dV(X)}{dX} + \theta \dot{\hat{V}} = \Upsilon_d X(T - \delta) + \Upsilon_\nu \dot{X}(T - \delta) - M \ddot{Y}$$

(24)

$$C_p \ddot{\hat{V}} + \frac{\hat{V}}{R} = \theta \dot{X}$$

It should be noted that $\theta$ is a linear electromechanical coupling coefficient; $\dot{\hat{V}}$ represents the base acceleration and $C_p$ represents the internal piezoelectric capacitance; $\hat{V}$ is the voltage measured on an equivalent resistive load, $R$; $V(X)$ is the function of the deterministic potential energy of mechanical asymmetric SD oscillator.

$$\frac{d(V(X))}{d(X)} = (b + 2\gamma)X - \gamma X \frac{1}{\sqrt{X^2 + \alpha_1^2}} - \rho \gamma X \frac{1}{\sqrt{X^2 + \rho \alpha_2^2}}$$

(25)

In Fig. 11, from the perspective of three-dimensional on the SNR as function of noise intensity and time-delayed feedback, obviously, the SR can be enhanced by adjusting the value of time delay $\tau$, fractional order $q$ and asymmetric parameter. The results show that under the optimal time delay, fractional order and asymmetric proportion $\rho$, the theoretical value of energy harvesting is optimized, and the significant improvement of energy collection is particularly important in real life and the production activity.

4 Conclusions and discussions

In this paper, fractional-order damping and time-delayed feedback are given to control the vibration of asymmetric SD oscillators with stiffness nonlinearities. And we mainly study the noise-induced SR of the controlled nonlinear stiffness SD oscillator and the effect of fractional damping on the SNR. Based on the theoretical research of this article, we have obtained the following interesting conclusions:

(i) Under the small noise intensity $\kappa$, with the time delay $\tau$ increasing, it can be found that the potential barriers has suffered a sharp increases first, then decreases slightly, finally increases again. Thus it is non-monotonic about the changes of the barriers. According to FPK equation, the SNR expression of an asymmetric SD oscillator with double-well potential is obtained. Besides, we have investigated the effect of harmonic forcing and Gaussian white noise on the SR characteristics of the controlled SD oscillator. In addition, we found an amazing phenomenon: when the system is in the state of SR, the energy of noise can be transferred to the weak signal, contributing to the detection of the weak signal.

(ii) Comparing with the article [29], the asymmetric SD oscillator and fractional-order damping are added in this paper, which enhance the value of SNR. This not only increases the number of the adjustable parameters, but also has more practical significance of the system. The design of adding fractional-order damping can explore the rich stochastic dynamic behavior of the system further and induce the occurrence of stochastic Hopf bifurcation. Besides, fractional-order damping can not only make the SR phenomenon of nonlinear stiffness SD oscillator weak, but also can enhance it. It can be clearly seen from the study that the maximum point of SNR changed with the different values of the fractional-order $q$. It should be highlighted that we first proposed that the SR in the oscillator can be induced by the fractional-order damping due to the occurrence of noise.

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(iii) In this study, we discuss the influence of parameters—fractional order $q$, time delay $\tau$, displacement feedback control intensity $\mu$, noise intensity $\kappa$, asymmetric ratio $\rho$ and velocity control intensity $v$—on vibration energy harvesting and weak signal detection. And we proposed that use the TV method to detect the frequency of weak signal.

(iv) The main purpose of this paper is to obtain sufficient effective signals to improve the output SNR by constructing the stiffness nonlinearities and asymmetric SD oscillator. When the system reach the state of SR, SNR is the maximum. Laying a good foundation for signal detection and fault diagnosis using the correspondence method, which is our pursued direction. Moreover, our research can be applied to vibration control and vibration energy harvesting, etc.

(v) We adopted the fractional derivative in the sense of Caputo in this article, because it is suitable for lot of physical problems, and many research studies have been done based on the Caputo-type fractional derivative. The definitions of fractional derivative are various but not unified at present.

(vi) In this paper, we assumed that the extraneous noise is independent of the variable $\dot{x}$, but it is also interesting to consider the situation of internal noise. For example, noise is related to the memory effect of fractional damping. In fact, the damping originates from the internal noise, which will undoubtedly make the study complicate, and it is the pursued direction for us.

**Funding** This work was supported by the National Natural Science Foundation of China [Nos. 11602151, 11872253] and Hebei Province Department of Education of [ZD2021335].

**Data availability** Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

**References**

1. Chen, H., Varshney, P.K.: Theory of the stochastic resonance effect in signal detection-part ii: variable detectors. IEEE Trans. Signal Process. 56(10), 5031–5041 (2008)

2. Kumar, S., Jha, R.K.: Weak signal detection using stochastic resonance with approximated fractional integrator. Circuits Syst. Signal Process. 38(3), 1157–1178 (2019)

3. Lai, Z., Liu, J., Zhang, H., Zhang, C., Zhang, J., Duan, D.: Multi-parameter-adjusting stochastic resonance in a standard tri-stable system and its application in incipient fault diagnosis. Nonlinear Dyn. 96(3), 2069–2085 (2019)

4. Liu, J., Hu, B., Yang, F., Zang, C., Ding, X.: Stochastic resonance in a delay-controlled dissipative bistable potential for weak signal enhancement. Commun. Nonlinear Sci. Numer. Simul. 85, 105245 (2020)

5. Mba, C.U., Makis, V., Marchesiello, S., Fasana, A., Garibaldi, L.: Condition monitoring and state classification of gearboxes using stochastic resonance and hidden Markov models. Measurement 126, 76–95 (2018)

6. Gammaitoni, L., Hanggi, P., Jung, P., Marchesoni, F.: Stochastic resonance. Rev. Mod. Phys. 70(1), 223 (1998)

7. Lu, S., He, Q., Wang, J.: A review of stochastic resonance in rotating machine fault detection. Mech. Syst. Signal Process. 116, 230–260 (2019)

8. Rashitchi, V., Nourazar, M.: Fpga implementation of a real-time weak signal detector using a duffing oscillator. Circuits Syst. Signal Process. 34(10), 3101–3119 (2015)

9. Zhihong, Z., Shaopu, Y.: Application of van der pol-duffing oscillator in weak signal detection. Comput. Electr. Eng. 41, 1–8 (2015)

10. Wang, Q., Zhang, X., Yang, Y.: The TVICMs method for weak signal detection based on a nonlinear stochastic delay differential system. Int. J. Non-Linear Mech. 126, 103557 (2020)

11. Wang, Q., Yang, Y., Zhang, X.: Weak signal detection based on Mathieu-Duffing oscillator with time-delay feedback and multiplicative noise. Chaos Solitons Fractals 137, 109832 (2020)

12. Grasso, M., Chatterton, S., Pennacchi, P., Colosimo, B.M.: A data-driven method to enhance vibration signal decomposition for rolling bearing fault analysis. Mech. Syst. Signal Process. 81, 126–147 (2016)

13. Zhao, R., Yan, R., Chen, Z., Mao, K., Wang, P., Gao, R.X.: Deep learning and its applications to machine health monitoring. Mech. Syst. Signal Process. 115, 213–237 (2019)

14. Ajagekar, A., You, F.: Quantum computing based hybrid deep learning for fault diagnosis in electrical power systems. Appl. Energy 303, 117628 (2021)

15. Zhang, Z., Zhang, Y.-W., Ding, H.: Vibration control combining nonlinear isolation and nonlinear absorption. Nonlinear Dyn. 100(3), 2121–2139 (2020)

16. Virgin, L., Santillan, S., Plaut, R.: Vibration isolation using extreme geometric nonlinearity. J. Sound Vib. 315(3), 721–731 (2008)

17. Donmez, A., Cigeroglu, E., Ozgen, G.O.: An improved quasi-zero stiffness vibration isolation system utilizing dry friction damping. Nonlinear Dyn. 101, 107–121 (2020)

18. Niu, MQ., Chen, LQ.: Nonlinear vibration isolation via a compliant mechanism and wire ropes. Nonlinear Dyn. (2021). https://doi.org/10.1007/s11071-021-06588-9

19. Tokarev, A., Valeev, A., Zотов, A.: Use of vibration isolation systems with negative stiffness on the basis of special shaped guides to reduce pump piping vibration. In: International Conference on Industrial Engineering, Springer, 2019, pp. 913–920
20. Ram, Y.M., Mottershead, J.E.: Receptance method in active vibration control. AIAA J. 45(3), 562–567 (2007)

21. Zhao, Y., Xu, J.: Using the delayed feedback control and saturation control to suppress the vibration of the dynamical system. Nonlinear Dyn. 67(1), 735–753 (2012)

22. Sun, X., Xu, J., Fu, J.: The effect and design of time delay in feedback control for a nonlinear isolation system. Mech. Syst. Signal Process. 87, 206–217 (2017)

23. Mohanty, S., Dwivedy, S.: Nonlinear dynamics of piezoelectric-based active nonlinear vibration absorber using time delay acceleration feedback. Nonlinear Dyn. 98(2), 1465–1490 (2019)

24. Liu, G.-P., Chai, S.C., Mu, J., Rees, D.: Networked predictive control of systems with random delay in signal transmission channels. Int. J. Syst. Sci. 39(11), 1055–1064 (2008)

25. Liu, L., Yang, A., Tu, X., Fei, M., Naem, W.: Distributed weighted fusion estimation for uncertain networked systems with transmission time-delay and cross-correlated noises. Neurocomputing 270, 54–65 (2017)

26. Jeung, E.-T., Oh, D.-C., Park, H.-B.: Delay-dependent control for time-delayed TS fuzzy systems using descriptor representation. Int. J. Control Autom. Syst. 2(2), 182–188 (2004)

27. Mahmoud, M.S., Ismail, A.: New results on delay-dependent control of time-delay systems. IEEE Trans. Autom. Control 50(1), 95–100 (2005)

28. Li, J., Chen, Z., Cai, D., Zhen, W., Huang, Q.: Delay-dependent stability control for power system with multiple time-delays. IEEE Trans. Power Syst. 31(3), 2316–2326 (2015)

29. Yang, T., Cao, Q.: Delay-controlled primary and stochastic resonances of the SD oscillator with stiffness nonlinearities. Mech. Syst. Signal Process. 103, 216–235 (2018)

30. Benzi, R., Sutera, A., Vulpiani, A.: The mechanism of stochastic resonance. J. Phys. A 14(11), L453 (1981)

31. Benzi, R., Ciliberto, S., Tripiccione, R., Baudet, C., Massaioli, F., Succi, S.: Extended self-similarity in turbulent flows. Phys. Rev. E 48(1), R29 (1993)

32. Nicolis, C., Nicolis, G.: Stochastic aspects of climatic transitions-additive fluctuations. Tellus A 33(3), 225–234 (1981)

33. Cao, Q., Wiercigroch, M., Pavlovskaia, E.E., Grebogi, C., Thompson, J.M.T.: Archetypal oscillator for smooth and discontinuous dynamics. Phys. Rev. E 74(4), 046218 (2006)

34. Litak, G., Friswell, M., Adhikari, S.: Magnetopiezoelectric energy harvesting driven by random excitations. Appl. Phys. Lett. 96(21), 214103 (2010)

35. Liu, W., Badel, A., Formosa, F., Wu, Y., Agbossou, A.: Novel piezoelectric bistable oscillator architecture for wideband vibration energy harvesting. Smart Mater. Struct. 22(3), 035013 (2013)

36. Thompson, J.M.T., Hunt, G.W.: A General Theory of Elastic Stability. Wiley (1973)

37. Podlubny, I.: Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications. Elsevier (1998)

38. Qingjie, C., Marian, W., Ekaterina, P.: Archetypal oscillator for smooth and discontinuous dynamics. Phys. Rev. E 74(4), 046218 (2006)

39. Rui-Lan, T., Qing-Jie, C., Zhi-Xin, L.: Hopf bifurcations for the recently proposed smooth-and-discontinuous oscillator. Chin. Phys. Lett. 27(7), 074701 (2010)

40. Chen, H., Xie, J.: Harmonic and subharmonic solutions of the SD oscillator. Nonlinear Dyn. 84(4), 2477–2486 (2016)

41. Benzi, R., Ciliberto, S., Tripiccione, R., Baudet, C., Massaioli, F., Succi, S.: Extended self-similarity in turbulent flows. Phys. Rev. E 48(1), R29 (1993)

42. Li, C., Cai, M.: Theory and Numerical Approximations of Fractional Integrals and Derivatives. SIAM (2019)

43. Zhu, W.Q., Huang, Z.L., Suzuki, Y.: Response and stability of strongly non-linear oscillators under wide-band random excitation. Int. J. Non-linear Mech. 36(8), 1235–1250 (2001)

44. Shen, Y., Yang, S., Xing, H., Ma, H.: Primary resonance of a duffing oscillator with two kinds of fractional-order derivatives. Int. J. Non-linear Mech. 47(9), 975–983 (2012)

45. Guo, Q., Sun, Z., Xu, W.: Bifurcations in a fractional biorhythmic biological system with time delay. Commun. Nonlinear Sci. Numer. Simul. 72, 318–328 (2019)

46. Arnold, L.: Random dynamical systems. In: Dynamical Systems, Springer, pp. 1–43 (1995)

47. Ruihan, T., Qiliang, W., Zhongjia, L., Xinwei, Y.: Dynamic analysis of the smooth-and-discontinuous oscillator under constant excitation. Chin. Phys. Lett. 29(8), 084706 (2012)

48. Reimann, P., Schmid, G., Hänggi, P.: Universal equivalence of mean first-passage time and Kramers rate. Phys. Rev. E 60(1), R1 (1999)

49. McNamara, B., Wiesenfeld, K.: Theory of stochastic resonance. Phys. Rev. A 39(9), 4854 (1989)

50. Zheng, R., Nakano, K., Hu, H., Su, D., Cartmell, M.P.: An application of stochastic resonance for energy harvesting in a bistable vibrating system. J. Sound Vib. 333(12), 2568–2587 (2014)