A NEUMANN PROBLEM INVOLVING THE $p(x)$-LAPLACIAN WITH $p = \infty$
IN A SUBDOMAIN

YIANNIS KARAGIORGOS AND NIKOS YANNAKAKIS

Abstract. In this paper we study a non-homogeneous Neumann problem, where the $p(x)$-
Laplacian is involved and $p = \infty$ in a subdomain. By considering a suitable sequence $p_k$
of bounded variable exponents such that $p_k \to p$ and replacing $p$ with $p_k$ in the original
problem, we prove the existence of a solution $u_k$ for each of those intermediate ones. We
show that the limit of the $u_k$ exists and after giving a variational characterization of it, in
the part of the domain where $p$ is bounded, we show that it is a viscosity solution in the part
where $p = \infty$. Finally, we formulate the problem of which this limit function is a solution in
the viscosity sense.

1. Introduction

Consider the following Neumann problem

\begin{equation}
\begin{cases}
-\Delta_{p(x)} u(x) = 0, & x \in \Omega \\
|\nabla u(x)|^{p(x)-2} \frac{\partial u}{\partial n}(x) = g(x), & x \in \partial\Omega
\end{cases}
\end{equation}

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth (at least of class $C^1$) domain and $N \geq 2$.

$$\Delta_{p(x)} u := \text{div}(|\nabla u|^{p(x)-2} \nabla u)$$

is the $p(x)$-Laplacian operator which is the variable exponent version of the $p$-Laplacian. Also,
g \in C(\Omega)$ and satisfies $\int_{\partial\Omega} g = 0$. Note that this latter condition is necessary, since otherwise
problem (1.1) has no solution.

The variable exponent $p$ satisfies the following hypothesis

\begin{equation}
p|_D = \infty
\end{equation}

where $D$ is a compactly supported subdomain of $\Omega$, with Lipschitz boundary.

Moreover, $p \in C^1(\overline{\Omega} \setminus D)$ with

\begin{equation}
p^+ := \sup_{\overline{\Omega}} p(x) < \infty
\end{equation}

and

\begin{equation}
p_- := \inf_{\overline{\Omega}} p(x) > N
\end{equation}

In the literature, most of the times the variable exponent $p(\cdot)$ is assumed to be bounded. Recently, the limits $p(x) \to \infty$ have been studied in several problems where the $p(x)$-Laplacian
is involved. See for instance [19] or [22] and the references therein. On the other hand, when
$p$ is constant the limits $p \to \infty$ in problems with the $p$-Laplacian were first studied in [4], in
which the physical motivation was given as well. On both cases the notion of infinity Laplacian arises naturally as the limit case.

In [22] the authors considered problem (1.1) and studied the limits as $p_n(x) \to \infty$ uniformly in $\Omega$, where $(p_n)_n$ was a sequence of variable exponents. J.J. Manfredi, et.al in [18] considered condition (1.2) for the first time to study the Dirichlet problem with Lipschitz boundary conditions. To the best of our knowledge this is the first time that condition (1.2) is considered in a Neumann problem involving the $p(x)$-Laplacian.

To find out what a solution of (1.1) might be, we follow the same strategy that is used in [18]. To be more specific we consider a sequence of bounded variable exponents $p_k$ such that

\[ p_k(x) = \min\{p(x), k\} \]

Then $p_k(x) \to p(x)$ as $k \to \infty$, while for $k > p^+$ we have that

\[ p_k(x) = \begin{cases} p(x), & x \in \overline{\Omega} \setminus D \\ k, & x \in D \end{cases} \]

**Remark 1.1.** Note that for $k > p^+$, the boundary of the set $\{x : p(x) > k\}$ coincides with the boundary of $D$ and so is independent of $k$. Due to this fact we have no problems when passing to the limit as $k \to \infty$.

If we replace $p$ with $p_k$ in problem (1.1) we have the intermediate boundary value problems.

\[
\begin{cases}
-\Delta_{p_k(x)} u(x) = 0, & x \in \Omega \\
|\nabla u(x)|^{p_k(x)-2} \frac{\partial u}{\partial n}(x) = g(x), & x \in \partial \Omega
\end{cases}
\]

(1.k)

Using standard methods we prove the existence of a unique weak solution $u_k$, for problem (1.k), that is also a viscosity solution. From the Arzelà-Ascoli theorem, we then show that the uniform limit of $(u_k)$ exists. We call this uniform limit $u_\infty$ and show that it satisfies a variational characterization in the set

\[
S = \left\{ u \in W^{1,p}(\Omega) : u|_{\overline{\Omega} \setminus D} \in W^{1,p(x)}(\Omega \setminus \overline{D}), \|\nabla u\|_{L^\infty(D)} \leq 1 \text{ and } \int_\Omega u = 0 \right\}
\]

and that it is infinity harmonic in $D$; that is, satisfies the equation $\Delta_{\infty} u = 0$, in the viscosity sense (see Definition 2.8), where

\[
\Delta_{\infty} u := \sum_{i,j=1}^N \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j}.
\]

**Remark 1.2.** Note that the infinity Laplace operator is in non-divergence form and the notion of weak solution does not make sense in this case. To give a meaning to a solution of the equation $\Delta_{\infty} u = 0$ that is not $C^2$ we need the notion of viscosity solution.

**Remark 1.3.** The condition $\int_\Omega u = 0$ in the definition of $S$, plays a crucial role in the proof of the existence and uniqueness of the solutions $u_k$ and also in their uniform boundedness.

**Remark 1.4.** In the Dirichlet case things are different. The existence of $u_\infty$ as a uniform limit of the sequence $(u_k)$ depends on the Lipschitz constant of the boundary condition and on the geometry of $D$ in $\Omega$. For reference see [18, 24] and [10].

The main results of this paper are Theorems 4.1 and 4.2. On the first one, we give a variational meaning to $u_\infty$ in $\Omega \setminus D$, where $p(\cdot)$ is bounded and next we prove that $u_\infty$ is infinity harmonic in $D$, where $p = \infty$. On the second one, we formulate the problem (as a limit case) of which $u_\infty$ is a solution in the viscosity sense.
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Partial Differential Equations involving the $p(x)$-Laplacian appear in a variety of applications. In [6] the authors proposed a framework for image restoration based on a variable exponent Laplacian. This was the starting point for the research on the connection between PDE’s with variable exponents and image processing. Recently there has been quite a rapid progress in this direction\footnote{1The reader can visit the website http://www.helsinki.fi/~pharjule/varsob/index.shtml for further details.} Other applications that use variable exponent type Laplacians are elasticity theory and the modelling of electrorheological fluids (see [23]).

Infinity harmonic functions (in the classical sense) were first studied by G. Arronson (see [1, 2]). Arronson studied the connection between infinity harmonic functions and optimal Lipschitz extensions, but only for $C^2$ functions. When the viscosity theory appeared, Crandall, Evans and Gariepy (see [8] or the survey paper [3]) used viscosity solutions to prove that the connection still holds. Note that, infinity harmonic functions appear in several applications such as optimal transportation (see [11, 14]), image processing (see [5]) and tug of war games (see [21]).

2. Preliminaries

In this section we give some basic properties of the variable exponent Lebesgue and Sobolev spaces. For details the interested reader should refer to [17], [13] and [9].

Let $L^{p(\cdot)}(\Omega)$ be the space of real valued measurable functions in $\Omega$ and $p: \Omega \to [1, \infty]$ a measurable function. We define the variable exponent Lebesgue space as

\[
L^{p(\cdot)}(\cdot) = \left\{ u \in L^0(\Omega) : \int_{\Omega} |\lambda u(x)|^{p(x)} \, dx < \infty, \text{ for some } \lambda > 0 \right\}
\]

equipped with the norm

\[
\| u \|_{L^{p(\cdot)}(\cdot)} := \| u \|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} \, dx \leq 1 \right\}.
\]

The variable exponent Sobolev space is defined by

\[
W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : \nabla u \in L^{p(\cdot)}(\Omega, \mathbb{R}^N) \right\}
\]

with norm

\[
\| u \|_{W^{1,p(\cdot)}(\cdot)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} + \left| \nabla u(x) \right|^{p(x)} \, dx \leq 1 \right\}.
\]

The spaces $(L^{p(\cdot)}(\Omega), \| \cdot \|_{p(\cdot)})$, $(W^{1,p(\cdot)}(\Omega), \| \cdot \|_{1,p(\cdot)})$ are Banach spaces and if

\[
1 < p_- := \text{ess inf}_{x \in \Omega} p(x) \leq p^+ := \text{ess sup}_{x \in \Omega} p(x) < \infty,
\]

they are also separable and reflexive.

When $p$ is constant, it is well known that smooth functions are dense in $W^{1,p}(\Omega)$. This is no longer true when we are dealing with the variable exponent spaces, see [12, 20, 9]. In fact, we have to consider additional conditions for the variable exponent. The most prevalent is the so called log-$\text{Hölder}$ continuity, i.e, there exists $C > 0$, such that

\[
|p(x) - p(y)| \leq \frac{C}{\log(e + \frac{1}{|x - y|})}, \quad \text{for } x, y \in \Omega.
\]
However, it turns out that we can have the density of smooth functions in some cases of discontinuous exponents (see [9, section 9.3]). In our case with the variable exponent $p_k$ as defined in Section 1, the following holds.

**Proposition 2.1.** The space $C^\infty(\Omega)$ is dense in $W^{1,p_k}(\Omega)$.

**Proof.** This is straightforward, if we use Theorem 9.3.5 of [9, p. 298] with $\Omega_1 = \Omega \setminus \overline{D}$ and $\Omega_2 = D$, where each of $\Omega_i$, $i = 1, 2$ has Lipschitz boundary. □

**Proposition 2.2.** Let $p: \Omega \to \mathbb{R}$ be a measurable function. The dual space of $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is the space $(L^{q(\cdot)}(\Omega), \|\cdot\|_{q(\cdot)})$ where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ and the variable exponent version of Hölder inequality holds, namely

$$\int_{\Omega} |u(x)v(x)| \, dx \leq 2\|u\|_{p(\cdot)}\|v\|_{q(\cdot)}, \quad \text{for all } u \in L^{p(\cdot)}(\Omega), v \in L^{q(\cdot)}(\Omega).$$

The next proposition is very important in the proof of the existence of a solution for problem (1.k) (see Lemma 3.2).

**Proposition 2.3.** There exists $C > 0$ such that the following Poincaré type inequality holds

$$\|u\|_{1,p_k} \leq C\|\nabla u\|_{L^{p_k}}, \quad \text{for all } u \in W^{1,p_k}(\Omega) \text{ s.t } \int_{\Omega} u = 0.$$

**Proof.** Apply Theorem 8.2.17 in [9, p. 256] with $D_1 = \Omega \setminus D$, $D_2 = D$. Then,

$$(p_k|_{D_i})_- := \inf_{x \in D_i} p(x) \geq p_- > N$$

and

$$(p_k|_{\partial D_i})_- = k > N.$$

□

**Remark 2.4.** In our case the variable exponent $p_k(\cdot)$ for $k > p^+$ satisfies,

$$p_k(\cdot) \geq (p_k)_- \geq p_- > N$$

so inequality (2.1) holds and the norms $\|u\|_{1,p_k(\cdot)}, \|\nabla u\|_{p_k(\cdot)}$ are equivalent in the set

$$\left\{ u \in W^{1,p_k(\cdot)}(\Omega) : \int_{\Omega} u(x) \, dx = 0 \right\}.$$

**Proposition 2.5.** Let $p$ be a variable exponent such that $p_- > N$. Then the following holds

(i) $W^{1,p(\cdot)}(\Omega) \hookrightarrow W^{1,p_-}(\Omega) \hookrightarrow C(\overline{\Omega})$.

(ii) If $q \in C(\partial\Omega)$, the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\partial\Omega)$, is compact and continuous.

For reference, see [13] [17] [9] for (i) and [25] Proposition 2.6 for (ii).

**Remark 2.6.** In our case, we have that $(p_k)_- > N$ and $p_k|_{\partial\Omega} = p \in C(\partial\Omega)$. Thus from (ii) of Proposition 2.5, we have that

$$W^{1,p_k(\cdot)}(\Omega) \hookrightarrow L^{p_k(\cdot)}(\partial\Omega)$$

**Proposition 2.7.** Let $u \in L^{p(\cdot)}(\Omega)$, then we have
Next, we recall the definition of a viscosity solution. First, we give the classical definition and then the one that involves a transmission condition. For reference see [7] and [18].

Let \( F: \overline{\Omega} \times \mathbb{R}^N \times \mathbb{S}^N \rightarrow \mathbb{R} \), where \( \mathbb{S}^N \) is the space of the \( N \)-dimensional symmetric matrices. \( F \) is said to be degenerate elliptic, if for each \( X, Y \in \mathbb{S}^N \) with \( X \geq Y \), (i.e \( \langle X \xi, \xi \rangle \geq \langle Y \xi, \xi \rangle \), for \( \xi \in \mathbb{R}^N \) ) then

\[
F(x, \xi, X) \leq F(x, \xi, Y).
\]

For the next definitions by \( D^2 u \) we denote the Hessian matrix of \( u \).

**Definition 2.8.** Let \( F: \Omega \times \mathbb{R}^N \times \mathbb{S}^N \rightarrow \mathbb{R} \) be degenerate elliptic operator and \( u: \Omega \rightarrow \mathbb{R} \).

(i) Let \( u \) be a lower semicontinuous function in \( \Omega \). We say that \( u \) is a viscosity supersolution of \( F(x, \nabla u, D^2 u) = 0 \) in \( \Omega \), if for every \( \varphi \in C^2(\Omega) \) such that \( u - \varphi \) attains its strict minimum at \( x_0 \in \Omega \) with \( u(x_0) = \varphi(x_0) \), we have \( F(x_0, \nabla u(x_0), D^2 u(x_0)) \geq 0 \).

(ii) Let \( u \) be an upper semicontinuous function in \( \Omega \). We say that \( u \) is a viscosity subsolution of \( F(x, \nabla u, D^2 u) = 0 \) in \( \Omega \), if for every \( \varphi \in C^2(\Omega) \) such that \( u - \varphi \) attains its strict maximum at \( x_0 \in \Omega \) with \( u(x_0) = \varphi(x_0) \), we have \( F(x_0, \nabla u(x_0), D^2 u(x_0)) \leq 0 \).

(iii) We say that \( u \in C(\Omega) \) is a viscosity solution of \( F(x, \nabla u, D^2 u) = 0 \) in \( \Omega \) if is both a viscosity supersolution and subsolution.

**Definition 2.9.** Let \( F_1, F_2: \overline{\Omega} \times \mathbb{R}^N \times \mathbb{S}^N \rightarrow \mathbb{R} \), \( B_1: \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \) and \( B_2: \partial \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \) be degenerate elliptic operators and \( u: \overline{\Omega} \rightarrow \mathbb{R} \). Consider the following problem

\[
F_1(x, \nabla u, D^2 u) = 0, \quad \text{in} \quad \Omega \setminus \overline{D}
\]

\[
F_2(x, \nabla u, D^2 u) = 0, \quad \text{in} \quad D
\]

with transmission condition

\[
B_1(x, u, \nabla u) = 0, \quad \text{on} \quad \partial D
\]

and boundary condition

\[
B_2(x, u, \nabla u) = 0, \quad \text{on} \quad \partial \Omega.
\]

(i) Let \( u \) be a lower semicontinuous function in \( \overline{\Omega} \). We say that \( u \) is a viscosity supersolution of the problem (2.3)-(2.6), if for every \( \varphi \in C^2(\overline{\Omega}) \) such that \( u - \varphi \) attains its strict minimum at \( x_0 \in \overline{\Omega} \) with \( u(x_0) = \varphi(x_0) \), we have

- if \( x_0 \in \Omega \setminus \overline{D} \), then \( F_1(x_0, \nabla \varphi(x_0), D^2 \varphi(x_0)) \geq 0 \).
- If \( x_0 \in D \), then \( F_2(x_0, \nabla \varphi(x_0), D^2 \varphi(x_0)) \geq 0 \).
We say that $\Omega$ with $S$

There exists a unique weak solution $u$ of $\Omega$

Due to the fact that the norms $\|u\|_{1,pk}^k$ are equivalent in $S_k$, we may suppose

\[ \int_{\Omega} |\nabla u|^{p_k(x)} - 2 \nabla u \cdot \nabla v dx = \int_{\partial \Omega} g v dS, \quad \text{for all } v \in W^{1,p_k}(\Omega). \]

Lemma 3.2. There exists a unique weak solution $u_k$ to problem (1.k), which is the unique minimizer of the functional

$$ I_k(u) = \int_{\Omega} \frac{|\nabla u|^{p_k(x)}}{p_k(x)} dx - \int_{\partial \Omega} gu dS $$

in the set

$$ S_k = \left\{ u \in W^{1,p_k}(\Omega) : \int_{\Omega} u = 0 \right\}. $$

Proof. First we show that $I_k$ is coercive and weakly lower semicontinuous, so attains its minimum in $S_k$. Let $\|u\|_{1,p_k} \to \infty$. To obtain coercivity we need to show that $I_k(u) \to \infty$. Due to the fact that the norms $\|u\|_{1,p_k}, |\nabla u|_{p_k}$ are equivalent in $S_k$, we may suppose
that \(|\nabla u|_{p_k(x)} > 1|.
From the \(\varepsilon\)-Young inequality, the embeddings \(W^{1,p_-}(\Omega) \hookrightarrow L^{p_-}(\partial\Omega)\), \(W^{1,p_k(x)}(\Omega) \hookrightarrow W^{1,p_-}(\Omega)\) and (i) of Proposition 2.7 we have that,

\[
I_k(u) = \int_{\Omega} \frac{\nabla u_{p_k(x)}}{p_k(x)} \cdot dx - \int_{\partial\Omega} g u dS
\]

\[
\geq \frac{1}{p_-} \int_{\Omega} |\nabla u|_{p_k(x)}^{p_-} \cdot dx - \frac{1}{\varepsilon} \left(\|g\|_{L^{p_-}} - \varepsilon\|u\|_{L^{p_-}}(\partial\Omega)\right)
\]

\[
\geq \frac{1}{p_-} \|u\|_{1,p_k(x)}^{p_-} - \frac{1}{\varepsilon} \|\nabla u\|_{L^{p_-}} - C\varepsilon\|u\|_{L^{p_-}}(\partial\Omega)
\]

\[
\geq \tilde{C}\|u\|_{1,p_k(x)}^{p_-} - \frac{1}{\varepsilon} \|\nabla u\|_{L^{p_-}} - C\varepsilon\|u\|_{L^{p_-}}^2(\partial\Omega)
\]

where \(p_-\) is the conjugate exponent of \(p_-\).

If we choose \(\varepsilon > 0\) small enough such that \(\tilde{C} - C\varepsilon > 0\), we have that \(I_k(u) \to \infty\) as \(\|u\|_{1,p_k(x)} \to \infty\). Thus, \(I_k\) is coercive.

For the weak lower semicontinuity let \(u_n \rightharpoonup u\) in \(S_k\). Using the weak lower semicontinuity of the integral \(\int_{\Omega} \frac{|\nabla u|_{p_k(x)}}{p_k(x)} \cdot dx\) and the embedding \(S_k \hookrightarrow L^{p_-}(\partial\Omega)\) we obtain that \(I_k\) is weak lower semicontinuous. Hence, \(I_k\) attains its minimum in \(S_k\). The uniqueness is standard due to the strict convexity of \(I_k\). It remains to show that the unique minimizer \(u\) is also a unique weak solution of problem (1.k). Let \(v \in W^{1,p_k(x)}(\Omega)\) and set \(\tilde{v} = v - \frac{1}{|\Omega|} \int_\Omega v dx\). Then \(\tilde{v} \in S_k\) and using the fact that \(u\) minimizes \(I_k\) in \(S_k\), it is easy to see that \(u\) satisfies (3.1) and hence is a weak solution of problem (1.k). Due to the fact that a weak solution of problem (1.k) is a minimizer of \(I_k\) in \(S_k\), the proof is completed.

\(\square\)

The next Lemma is very useful since it gives an equivalent form of problem (1.k) that allows us to see that \(I_k(u) = 0\) is equivalent to the following problem

\[
\begin{cases}
-\Delta_{p_k(x)} u(x) = 0, & x \in \Omega \setminus \overline{D} \\
-\Delta_k u(x) = 0, & x \in D \\
|\nabla u(x)|^{k-2} \frac{\partial u}{\partial n}(x) = |\nabla u(x)|^{p(x)-2} \frac{\partial u}{\partial n}(x), & x \in \partial D \\
|\nabla u(x)|^{p(x)-2} \frac{\partial u}{\partial n}(x) = g(x), & x \in \partial\Omega.
\end{cases}
\]

\(\text{Lemma 3.3.}\) Problem (1.k) is equivalent to the following problem

\(\text{Proof.}\) Let \(k > p^+\), then \(C^\infty(\overline{\Omega})\) is dense in \(W^{1,p_k(x)}(\Omega)\) (see Proposition 2.1). If we take as a test function \(v \in C^\infty(\overline{\Omega})\), use integration by parts, Gauss-Green theorem and the fact that \(D\) is compactly supported in \(\Omega\), we conclude that the weak formulation of (3.2) is (3.1).

\(\square\)

The next Lemma is crucial in the proof of our main results. Note that the importance of the condition \(\int_{\Omega} u = 0\) is evident.

\(\text{Lemma 3.4.}\) Let \(u_k\) be a weak solution of problem (1.k). Then the sequence \((u_k)\) is equicontinuous and uniformly bounded.

\(\text{Proof.}\) If we multiply (1.k) by \(u_k\) and use integration by parts, we obtain

\[
\int_{\Omega} |\nabla u_k|_{p_k(x)} \cdot dx = \int_{\partial\Omega} g u_k dS \leq 2\|u_k\|_{L^{p_k(x)}(\partial\Omega)}\|g\|_{L^{p_k(x)}(\partial\Omega)} \leq C(\Omega, g)\|\nabla u_k\|_{L^{p_k(x)}}
\]
where we used the variable exponent version of Hölder’s inequality, (2.1) and the embedding \( W^{1,p_k(\cdot)}(\Omega) \hookrightarrow L^{p_k(\cdot)}(\partial \Omega) \) (see Remark 2.6). We consider the cases

- if \( \| \nabla u_k \|_{L^{p_k(\cdot)}} \leq 1 \), then \( \int_{\Omega} |\nabla u_k|^{p_k(x)} \, dx \leq 2C(\Omega, g) \)
- if \( \| \nabla u_k \|_{L^{p_k(\cdot)}} > 1 \), then from (i) of Proposition 2.7, we have

\[
\int_{\Omega} |\nabla u_k|^{p_k(x)} \, dx \leq C(\Omega, g)\| \nabla u_k \|_{L^{p_k(\cdot)}} = C\left(\| \nabla u_k \|_{L^{p_k(\cdot)}}^{p_k} \right)^{\frac{1}{p_k}} \leq C\left( \int_{\Omega} |\nabla u_k|^{p_k(x)} \, dx \right)^{\frac{1}{p_k}}.
\]

So, we end up with the following inequality

\[
(3.3) \quad \int_{\Omega} |\nabla u_k|^{p_k(x)} \, dx \leq C(\Omega, g, p_-),
\]

where \( C \) is independent of \( k \).

On the other hand, since \( p_- > N \) from Morrey’s inequality (see [10, p. 183]) we have

\[
|u_k(x) - u_k(y)| \leq C(N, p_-)|x - y|^{1 - \frac{N}{p_-}} \left( \int_{\Omega} |\nabla u_k|^{p_-} \, dx \right)^{\frac{1}{p_-}}, \quad \text{for all } x, y \in \Omega
\]

and

\[
\int_{\Omega} |\nabla u_k|^{p_-} \, dx = \int_{\Omega \cap \{|\nabla u_k| \leq 1\}} |\nabla u_k|^{p_-} \, dx + \int_{\Omega \cap \{|\nabla u_k| > 1\}} |\nabla u_k|^{p_-} \, dx \leq \Omega(\{ |\nabla u_k| > 1 \})
\]

\[
(3.4) \quad \leq C(\Omega, p_-) + \int_{\Omega} |\nabla u_k|^{p_k(x)} \, dx \leq C(\Omega, p_-, g),
\]

where in the last inequality we used the previous estimate for \( \int_{\Omega} |\nabla u_k|^{p_k(x)} \, dx \). From the above, we obtain that

\[
(3.5) \quad |u_k(x) - u_k(y)| \leq C(\Omega, N, p_-, g)|x - y|^{1 - \frac{N}{p_-}}, \quad \text{for all } x, y \in \Omega.
\]

Hence, the sequence \( (u_k) \) is equicontinuous in \( C(\Omega) \).

It remains to show that the sequence \( (u_k) \) is uniformly bounded in \( \Omega \). Let \( k > p^+ \). Since we are assuming that \( \int_{\Omega} u_k = 0 \) and \( u_k \in C(\Omega) \), we may choose a point \( y \in \Omega \) such that \( u_k(y) = 0 \). Then, from (3.5) we get

\[
|u_k(x)| \leq C(\Omega, N, p_-, g)|x - y|^{1 - \frac{N}{p_-}} \leq C(\Omega, N, p_-, g)(\text{diam}(\Omega))^{1 - \frac{N}{p_-}} \leq C(\Omega, N, p_-, g)
\]

so \( (u_k) \) is uniformly bounded in \( \Omega \) and this concludes the proof.

**Proposition 3.5.** Let \( u \) be a continuous weak solution of (1.k). Then \( u \) is a solution of (1.k) in the viscosity sense.

**Proof.** We prove that \( u \) is a viscosity supersolution of (1.k). The proof that \( u \) is a viscosity subsolution is similar. Let \( x_0 \in \Omega \) and \( \varphi \in C^2(\Omega) \) such that \( u - \varphi \) attains its strict minimum at \( x_0 \) and \( (u - \varphi)(x_0) = 0 \). We want so show that \( -\Delta_{p_k(x_0)}(x_0) \geq 0 \). To argue by contradiction suppose that \( -\Delta_{p_k(x_0)}\varphi(x_0) < 0 \). We consider the following cases.
Let $x_0 \in \Omega \setminus \overline{D}$. Then $-\Delta_{p(x)}(x_0) < 0$ and by continuity there exists $r > 0$ such that $B_r(x_0) \subset \Omega \setminus \overline{D}$ and for every $x \in B_r(x_0)$ we have

$$-\Delta_{p(x)}(x) = -|\nabla \varphi(x)|^{p(x)-2} \Delta \varphi(x)$$

$$= - (p(x) - 2)|\nabla \varphi(x)|^{p(x)-4} \Delta_\infty \varphi(x)$$

$$= - |\nabla \varphi(x)|^{p(x)-2} \ln(|\nabla \varphi(x)|) \nabla \varphi(x) \cdot \nabla p(x) < 0.$$ 

Set

$$m = \inf_{x \in S(x_0,r)} (u - \varphi)(x) > 0 \text{ and } \hat{\varphi} = \varphi + \frac{m}{2}.$$ 

Then $\hat{\varphi}$ satisfies $\hat{\varphi}(x_0) > u(x_0)$ and $\hat{\varphi}(x) \leq u(x)$, for every $x \in S(x_0,r)$. Moreover,

$$-\Delta_{p(x)}\hat{\varphi}(x) < 0, \text{ for all } x \in B_r(x_0).$$ 

Multiplying by $(\hat{\varphi} - u)^+$ and integrating we get,

$$0 > \int_{B_r(x_0)} |\nabla \hat{\varphi}(x)|^{p(x)-2} \nabla \hat{\varphi}(x) \cdot \nabla (\hat{\varphi} - u)^+ dx$$

$$= \int_{B_r(x_0) \cap \{\hat{\varphi} > u\}} |\nabla \hat{\varphi}(x)|^{p(x)-2} \nabla \hat{\varphi}(x) \cdot \nabla (\hat{\varphi} - u) dx.$$ 

If we extend $(\hat{\varphi} - u)^+$ as zero outside of $B_r(x_0)$ and use it as a test function in the weak formulation of $-\Delta_{p(x)}u(x) = 0$, we obtain

$$0 = \int_{\Omega \setminus \overline{D}} |\nabla u(x)|^{p(x)-2} \nabla u(x) \cdot \nabla (\hat{\varphi} - u)^+ dx$$

$$= \int_{B_r(x_0) \cap \{\hat{\varphi} > u\}} |\nabla u(x)|^{p(x)-2} \nabla u(x) \cdot \nabla (\hat{\varphi} - u) dx.$$ 

By subtracting and using a well known inequality (see [16, p.51]) we conclude

$$0 > \int_{B_r(x_0) \cap \{\hat{\varphi} > u\}} (|\nabla \hat{\varphi}(x)|^{p(x)-2} \nabla \hat{\varphi}(x) - |\nabla u(x)|^{p(x)-2} \nabla \hat{u}(x)) \cdot \nabla (\hat{\varphi} - u) dx$$

$$\geq c \int_{B_r(x_0) \cap \{\hat{\varphi} > u\}} |\nabla \hat{\varphi} - \nabla u|^{p(x)} dx,$$ 

which is a contradiction.

- If $x_0 \in D$ then the proof is exactly the same, so $-\Delta_k \varphi(x_0) \geq 0$.
- Let $x_0 \in \partial D$. We need to show that

$$\max\{ -\Delta_{p(x)} \varphi(x_0), -\Delta_k \varphi(x_0),$$

$$|\nabla \varphi(x_0)|^{k-2} \frac{\partial \varphi}{\partial p}(x_0) - |\nabla \varphi(x_0)|^{p(x)-2} \frac{\partial \varphi}{\partial p}(x_0) \} \geq 0.$$ 

We argue again by contradiction. So, by continuity there exists $r > 0$ such that

$$-\Delta_{p(x)} \varphi(x) < 0, \text{ for all } x \in B_r(x_0) \cap \Omega \setminus \overline{D}$$

$$-\Delta_k \varphi(x) < 0, \text{ for all } x \in B_r(x_0) \cap D$$
and

\[ |\nabla \varphi(x)|^{k-2} \frac{\partial \varphi}{\partial \nu}(x) - |\nabla \varphi(x)|^{p(x)-2} \frac{\partial \varphi}{\partial \nu}(x) < 0, \quad \text{for all } x \in B_r(x_0) \cap \partial D. \]

Set

\[ m = \inf_{x \in S(x_0, r)} (u - \varphi)(x) > 0 \text{ and } \tilde{\varphi} = \varphi + \frac{m}{2}. \]

Then \( \tilde{\varphi} \) satisfies \( \tilde{\varphi}(x_0) > u(x_0) \) and \( \tilde{\varphi}(x) \leq u(x) \), for every \( x \in S(x_0, r) \). Multiplying the previous inequalities by \( (\tilde{\varphi} - u)^+ \), integrating by parts and adding, we have

\[
\int_{B_r(x_0) \cap (\Omega \setminus D)} |\nabla \tilde{\varphi}|^{p(x)-2} \nabla \tilde{\varphi} \cdot \nabla (\tilde{\varphi} - u)^+ dx + \int_{B_r(x_0) \cap D} |\nabla \tilde{\varphi}|^{k-2} \nabla \tilde{\varphi} \cdot \nabla (\tilde{\varphi} - u)^+ dx
\]

\[
< \int_{B_r(x_0) \cap \partial D} \left( |\nabla \tilde{\varphi}|^{k-2} \frac{\partial \varphi}{\partial \nu} - \nabla \tilde{\varphi} |^{p(x)-2} \frac{\partial \varphi}{\partial \nu} \right) (\tilde{\varphi} - u)^+ dS.
\]

On the other hand, we may extend \( (\tilde{\varphi} - u)^+ \) as zero outside \( B_r(x_0) \), take it as a test function in the weak formulation of (1.k) and reach a contradiction as we did in the previous case.

- Let \( x_0 \in \partial \Omega \). We need to show that

\[
\max \{ |\nabla \varphi(x_0)|^{p(x_0)-2} \frac{\partial \varphi}{\partial \nu}(x_0) - g(x_0), -\Delta p(x_0) \varphi(x_0) \} \geq 0.
\]

To contradiction, suppose that

\[
|\nabla \varphi(x_0)|^{p(x_0)-2} \frac{\partial \varphi}{\partial \nu}(x_0) - g(x_0) < 0
\]

and

\[-\Delta p(x_0) \varphi(x_0) < 0.
\]

Proceeding as before, we get

\[
\int_{B_r(x_0) \cap \{ \tilde{\varphi} > u \}} |\nabla \tilde{\varphi}|^{p(x)-2} \nabla \tilde{\varphi} \cdot \nabla (\tilde{\varphi} - u) dx < \int_{\partial \Omega \cap B_r(x_0) \cap \{ \tilde{\varphi} > u \}} (\tilde{\varphi} - u) g dS
\]

and

\[
\int_{B_r(x_0) \cap \{ \tilde{\varphi} > u \}} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla (\tilde{\varphi} - u) dx = \int_{\partial \Omega \cap B_r(x_0) \cap \{ \tilde{\varphi} > u \}} (\tilde{\varphi} - u) g dS
\]

which is a contradiction. Thus, \( u \) is a viscosity supersolution of (1.k). \( \square \)

4. Passing to the limit

Consider the following set

\[
S = \left\{ u \in W^{1,p}(\Omega) : u|_{\Omega \setminus \overline{D}} \in W^{1,p(\cdot)}(\Omega \setminus \overline{D}), \|\nabla u\|_{L_\infty(D)} \leq 1 \text{ and } \int_{\Omega} u = 0 \right\}.
\]
If \( v \in S \subset S_k \), we have
\[
I_k(v) = \int_{\Omega \setminus \partial D} \left| \frac{\nabla v}{p(x)} \right|^{p(x)} dx + \int_D \left| \frac{\nabla v}{k} \right|^k dx - \int_{\partial \Omega} g v dS
\]
\[
\leq \int_{\Omega \setminus \partial D} \left| \frac{\nabla v}{p(x)} \right|^{p(x)} dx + \left| \frac{D}{k} \right| - \int_{\partial \Omega} g v dS
\]
and passing to the limit as \( k \to \infty \), we have
\[
\liminf_k I_k(v) \leq \int_{\Omega \setminus \partial D} \left| \frac{\nabla v}{p(x)} \right|^{p(x)} dx - \int_{\partial \Omega} g v dS := I_\infty(v)
\]

The next theorem is the first main result of this paper. We give a variational characterization of the limit function \( u_\infty \) in \( \Omega \setminus \overline{D} \), where \( p^+ := \sup_{\Omega \setminus \overline{D}} p(x) < \infty \) and next we prove that \( u_\infty \) is infinity harmonic in \( D \) in the viscosity sense.

**Theorem 4.1.** Let \( u_k \) be the unique minimizer of \( I_k \) in \( S_k \). Then there exists a function \( u_\infty \in S \), such that \( u_\infty \) minimizes \( I_\infty \) in \( S \) and is also infinity harmonic in \( D \).

**Proof.** From Lemma 3.4 and the Arzelà - Ascoli theorem, there exists a subsequence of \((u_k)\) (denoted again \( u_k \)) and a function \( u_\infty \in C(\overline{\Omega}) \) such that
\[
u_k \to u_\infty, \quad \text{uniformly in } \overline{\Omega}.
\]

First we show that \( u_\infty \in S \). From the estimate (3.4) and the Poincaré-Wirtinger inequality in \( W^{1,p_-}(\Omega) \) (recall that \( \int_\Omega u_k = 0 \)) we have that \((u_k)\) is bounded in \( W^{1,p_-}(\Omega) \). Thus,
\[
u_k \rightharpoonup u_\infty, \quad \text{in } W^{1,p_-}(\Omega).
\]

To obtain that \( u_\infty \in W^{1,p^+}(\Omega \setminus \overline{D}) \) we use the estimate (3.3) for the integral \( \int_\Omega |\nabla u_k|^{p_+} dx \) and the inequality (2.1), to show that \((u_k)\) is bounded in \( W^{1,p^+}(\Omega \setminus \overline{D}) \). Thus,
\[
u_k \rightharpoonup u_\infty, \quad \text{in } W^{1,p^+}(\Omega \setminus \overline{D}).
\]

Where we used again the pointwise convergence of \((u_k)\) to \( u_\infty \) in \( \overline{\Omega} \). Now let \( m > p_- \) and \( k > m \). Then from Hölder’s inequality and (3.3) we have
\[
\left( \int_D |\nabla u_k|^m dx \right)^\frac{1}{m} \leq |D|^{\frac{1}{m} - \frac{k}{p_+}} \left( \int_D |\nabla u_k|^k dx \right)^\frac{1}{k} \leq |D|^{\frac{1}{m} - \frac{k}{p_+}} \left( \int_\Omega |\nabla u_k|^{p_+} dx \right)^\frac{1}{k} \leq |D|^{\frac{1}{m} - \frac{k}{p_+}} (C(\Omega, g))^{\frac{k}{p_+}}.
\]

Take \( k \) large enough such that \( |D|^{\frac{1}{m} - \frac{k}{p_+}} (C(\Omega, g))^{\frac{k}{p_+}} \leq 2 |D|^{\frac{1}{m}} \) holds. Then we have
\[
(4.1) \quad \|\nabla u_k\|_{L^m(D)} \leq 2 |D|^{\frac{1}{m}}.
\]

From (4.1) and the Poincaré-Wirtinger inequality we have that \((u_k)\) is bounded in \( W^{1,m}(D) \). This fact together with the pointwise convergence of \((u_k)\) to \( u_\infty \), gives that \( u_k \rightharpoonup u_\infty \) in \( W^{1,m}(D) \).
Let \( m > p^+ \). From the weak lower semicontinuity of the integral and the Hölder inequality we have

\[
\|
abla u_\infty \|_{L^m(D, \mathbb{R}^N)} \leq \liminf_k \left( \int_D |\nabla u_k|^m \, dx \right)^{\frac{1}{m}} \\
\leq \liminf_k \left[ |D|^{\frac{1}{m} - \frac{1}{p}} \left( \int_D |\nabla u_k|^p \, dx \right)^{\frac{1}{p}} \right] \\
\leq \liminf_k \left[ |D|^{\frac{1}{m} - \frac{1}{p}} \left( C(\Omega, g) \right)^{\frac{1}{p}} \right] \\
= |D|^{\frac{1}{m}}.
\]

Thus, passing to the limit as \( m \to \infty \), we have \( \|
abla u_\infty \|_{L^\infty(D, \mathbb{R}^N)} \leq 1 \). The condition \( \int_\Omega u_\infty = 0 \) is immediately satisfied since \( \int_\Omega u_k = 0 \) for each \( k \). Thus, \( u_\infty \in S \).

It remains to show that \( u_\infty \) minimizes \( I_\infty \) in \( S \). To this end, let \( v \in S \). Then by the minimizing property of \( u_k \) in \( S_k \) and the weak lower semicontinuity of \( I_\infty \) we have

\[
I_\infty(u_\infty) \leq \liminf_k I_\infty(u_k) \leq \liminf_k I_k(u_k) \leq \liminf_k I_k(v) \leq I_k(v).
\]

Thus, \( u_\infty \) minimizes \( I_\infty \) in \( S \). To prove that \( u_\infty \) is infinity harmonic in \( D \) we use the fact that \( u_k \) is \( k \)-harmonic in \( D \) and \( u_k \to u_\infty \) uniformly in \( \Omega \) (see [4, Proposition 2.2], [15] or Theorem 2.8 in [16, p. 17]).

**Remark 4.2.** In general we do not have uniqueness of the minimizer of \( I_\infty \) in \( S \) since \( I_\infty \) is only defined in \( \Omega \setminus \overline{D} \). In the Dirichlet case (see [18]) the minimizing procedure is taking place in the set

\[
S' = \left\{ u \in W^{1,p-}(\Omega) : u|_{\Omega \setminus \overline{D}} \in W^{1,p(x)}(\Omega \setminus \overline{D}), \|\nabla u\|_{L^\infty(D)} \leq 1 \text{ and } u|_{\partial \Omega} = f \right\},
\]

where the function \( f \) is the Lipschitz boundary condition, and the corresponding functional \( I'_\infty \) is defined in \( S' \) as

\[
I'_\infty(u) := \int_{\Omega \setminus \overline{D}} \frac{\nabla u|_{p(x)}}{|p(x)} \, dx.
\]

If we demand that the minimizer of \( I'_\infty \) in \( S' \) is also infinity harmonic in \( D \), we can gain uniqueness. This is straightforward if we use the uniqueness of a solution for the Dirichlet problem with infinity Laplacian (see [15]) and the condition \( u|_{\partial \Omega} = f \). In our case, things are different. Indeed, let \( u_1, u_2 \) be minimizers of \( I_\infty \) in \( S \) that are also infinity harmonic in \( D \). Then, there exists some \( C \in \mathbb{R} \) such that

\[
u_2 = u_1 + C \quad \text{in} \quad \overline{\Omega} \setminus D,
\]

but we cannot say for sure that \( u_1 = u_2 \) in \( \Omega \).

In the following theorem we state the problem of which \( u_\infty \) is a viscosity solution. This arises naturally from Lemma 3.3 and Proposition 3.5 as a limit case.

**Theorem 4.3.** Let \( (u_k) \) be the sequence of solutions of problems (1.k) and \( u_\infty \) the uniform limit of a subsequence of \( (u_k) \). Then \( u_\infty \) is a viscosity solution of the following problem
Proof. Let \( x_0 \in \overline{\Omega} \), \( \varphi \in C^2(\overline{\Omega}) \) such that \( u_\infty - \varphi \) attains its strict minimum at \( x_0 \). We will show that \( u_\infty \) is a viscosity supersolution. The case of viscosity subsolution is exactly the same, so we omit the proof. We consider the cases.

- Let \( x_0 \in \Omega \setminus \overline{D} \). Since \( u_k \to u_\infty \) uniformly in \( \Omega \), we can find a sequence \((x_k)_{k \in \mathbb{N}} \) in \( \Omega \setminus \overline{D} \) such that \( x_k \to x_0 \) and \( u_k - \varphi \) attains its strict minimum at \( x_k \) (see for instance [4, Proposition 2.2] or Theorem 2.8 in [16, p. 17]). Since \( u_k \) is a viscosity solution of (1.1), we have that \( -\Delta_{p(x_k)} \varphi(x_k) \geq 0 \), for all \( k > p^+ \) and passing to the limit as \( k \to \infty \) we obtain that \( -\Delta_{p(x_0)} \varphi(x_0) \geq 0 \).

- Let \( x_0 \in D \). As before, we can find points \( x_k \) in \( D \) such that \( x_k \to x_0 \) and \( u_k - \varphi \) attains its strict minimum at \( x_k \). Since \( u_k \) is a viscosity solution of (1.1), we have that

\[
- (|\nabla \varphi(x_k)|^{k-2}\Delta \varphi(x_k) + (k-2)|\nabla \varphi(x_k)|^{k-4}\Delta \varphi(x_k)) \geq 0.
\]

If \( \nabla \varphi(x_0) = 0 \), then \( \Delta \varphi(x_0) = 0 \) and there is nothing to prove. Assume now that \( \nabla \varphi(x_0) \neq 0 \). Then \( \nabla \varphi(x_k) = 0 \), for large \( k \) and dividing the previous inequality with \((k-2)|\nabla \varphi(x_k)|^{k-4}\), we obtain

\[
-|\nabla \varphi(x_k)|^2 \Delta \varphi(x_k)\frac{2}{k-2} - \Delta \varphi(x_k) \geq 0.
\]

Passing to the limit as, \( k \to \infty \) we get

\[-\Delta \varphi(x_0) \geq 0.
\]

Thus, \( u_\infty \) is viscosity supersolution of \( -\Delta \varphi = 0 \) in \( D \).

- Let \( x_0 \in \partial D \). We need to show that,

\[
\max(\Delta_{p(x_0)} \varphi(x_0), \Delta \varphi(x_0)),
\]

\[
\mathrm{sgn}(|\nabla \varphi(x_0)| - 1)\mathrm{sgn}\left(\frac{\partial \varphi}{\partial \nu}(x_0)\right) \geq 0.
\]

Due to uniform convergence again we can find a sequence \((x_k) \) such that \( x_k \to x_0 \) and \( u_k - \varphi \) attains its strict minimum at \( x_k \). We consider the following cases.

(i) If infinitely many \( x_k \) belong to \( \Omega \setminus \overline{D} \), then for large \( k \), we have

\[-\Delta_{p(x_k)} \varphi(x_k) \geq 0\]

and passing to the limit as \( k \to \infty \), we obtain that

\[-\Delta_{p(x_0)} \varphi(x_0) \geq 0.
\]

(ii) If infinitely many \( x_k \) belong to \( D \), then for large \( k \), we have

\[-\Delta_k \varphi(x_k) \geq 0\]

and proceeding as we did in the second case, we have \(-\Delta \varphi(x_0) \geq 0 \).

(iii) If infinitely many \( x_k \) belong to \( \partial D \), then from Proposition 3.5 we get

\[
|\nabla \varphi(x_k)|^{k-2}\frac{\partial \varphi}{\partial \nu}(x_k) - |\nabla \varphi(x_k)|^{p(x_k)-2}\frac{\partial \varphi}{\partial \nu}(x_k) \geq 0.
\]
Hence,
\[ \frac{\partial \varphi}{\partial \nu}(x_k)(|\nabla \varphi(x_k)|^{k-p(x_k)} - 1) \geq 0. \]

Passing to the limit as \( k \to \infty \), we have that,

if \( |\nabla \varphi(x_0)| > 1 \), then \( \frac{\partial \varphi}{\partial \nu}(x_0) \geq 0 \)
and

if \( |\nabla \varphi(x_0)| < 1 \), then \( \frac{\partial \varphi}{\partial \nu}(x_0) \leq 0 \).

But this is the case when
\[ \text{sgn}(|\nabla \varphi(x_0)| - 1) \text{sgn}(\frac{\partial \varphi}{\partial \nu}(x_0)) \geq 0. \]

\[ \cdot \]

Let \( x_0 \in \partial \Omega \). We want to prove, that
\[ \max\{|\nabla \varphi(x_0)|^{p(x_0)-2} \frac{\partial \varphi}{\partial \nu}(x_0) - g(x_0), -\Delta_{p(x_0)} \varphi(x_0)\} \geq 0. \]

Again, we can find points \( x_k \in \Omega \) such that \( x_k \to x_0 \) and \( u_\infty - \varphi \) has a strict minimum at \( x_k \). If infinitely many \( x_k \) belong to \( \Omega \), then \( -\Delta_{p(x_k)} \varphi(x_k) \geq 0 \) and passing to the limit as \( k \to \infty \), we get \( -\Delta_{p(x_0)} \varphi(x_0) \geq 0 \). If infinitely many \( x_k \) belong to \( \partial \Omega \), from Proposition 3.5, we have
\[ \max\{|\nabla \varphi(x_k)|^{p(x_k)-2} \frac{\partial \varphi}{\partial \nu}(x_k) - g(x_k), -\Delta_{p(x_k)} \varphi(x_k)\} \geq 0. \]

If \( -\Delta_{p(x_k)} \varphi(x_k) \geq 0 \), as before we conclude that
\[ -\Delta_{p(x_0)} \varphi(x_0) \geq 0. \]

If,
\[ |\nabla \varphi(x_k)|^{p(x_k)-2} \frac{\partial \varphi}{\partial \nu}(x_k) - g(x_k) \geq 0, \]

passing to the limit as \( k \to \infty \) and since \( g \in C(\Omega) \), we conclude that
\[ |\nabla \varphi(x_0)|^{p(x_0)-2} \frac{\partial \varphi}{\partial \nu}(x_0) - g(x_0) \geq 0 \]
and this completes the proof.

\[ \square \]

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DEPARTMENT OF MATHEMATICS, NATIONAL TECHNICAL UNIVERSITY OF ATHENS, ZOGRAFOU CAMPUS, ATHENS 15780, GREECE

E-mail address: ykarag@math.ntua.gr

DEPARTMENT OF MATHEMATICS, NATIONAL TECHNICAL UNIVERSITY OF ATHENS, ZOGRAFOU CAMPUS, ATHENS 15780, GREECE

E-mail address: nyian@math.ntua.gr