Width-amplitude relation of Bernstein-Greene-Kruskal solitary waves

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Inequality width-amplitude relations for three-dimensional Bernstein-Greene-Kruskal solitary waves are derived for magnetized plasmas. Criteria for neglecting effects of nonzero cyclotron radius are obtained. We emphasize that the form of the solitary potential is not tightly constrained, and the amplitude and widths of the potential are constrained by inequalities. The existence of a continuous range of allowed sizes and shapes for these waves makes them easily accessible. We propose that these solitary waves can be spontaneously generated in turbulence or thermal fluctuations. We expect that the high excitation probability of these waves should alter the bulk properties of the plasma medium such as electrical resistivity and thermal conductivity.

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Coherent structures with nonuniform charge densities are ubiquitous in plasma systems. Laboratory experiments have shown that such structures can be generated by applying voltage pulses,2,2 voltage jumps,3 intense laser,4 or plasma beam injections.5 Increasing numbers of space-borne observations have revealed frequent appearance of electrostatic solitary structures in space plasmas (6, 7 and references therein) including regions where magnetic reconnection occurs (8, 9). Solitary waves can efficiently transport energy, momentum and charge, and are one of the building blocks in a deterministic description of turbulence (10, 11). The study of their allowed parameter space is crucial in establishing their relevance to real systems. Most solitary waves, such as those for shallow water (Korteweg-de Vries solitons) and those that describe crystal dislocations (sine-Gordon solitons), have a strict one-to-one mapping between their widths, amplitudes and velocities.12 However, in collisionless plasmas, the widths and amplitudes of the electrostatic solitary waves that exhibit vortex structures in phase space are not tightly constrained (13, 14). Although the width-amplitude relation of these so-called Bernstein-Greene-Kruskal (BGK) electron solitary waves in three dimensions (3D) have been studied by a few authors (15, 16), the results were limited either by incomplete analysis of the width-amplitude relation or by incomplete solutions (13, 17, 18), and the broad impact has not been recognized. In this letter, we derive inequality width-amplitude relation for 3D BGK solitary waves. The description is valid for both electron and ion modes. The inequality width-amplitude relation dictates a continuous range of allowed sizes and amplitudes for these waves, and thus makes them easily accessible. We propose that the continua of allowed existence range enable BGK solitary waves to be spontaneously generated in thermal fluctuations as well as turbulence, and is responsible for their ubiquitous presence in widely different classes of collisionless plasmas.

To construct exact nonlinear solutions that are localized in 3D, we use the BGK approach that was formulated for 1D nonlinear Vlasov-Poisson equations, but extend the Poisson equation to 3D. We construct azimuthally symmetric solutions in the limit of infinite magnetic field, and then tune down the magnetic field to obtain the criteria for neglecting effects of nonzero cyclotron radius. One key step in the BGK approach is to separate particles that are trapped in the potential and those that are passing. We prescribe the potential form and the passing particle distribution, solve for the trapped particle distribution, and derive the physical parameter range. This approach is much easier than to prescribe the passing and trapped particle distributions to solve for the potential (13, 18), and so allows us to explore the solution space much more fully.

We consider two species of charge carriers (electrons and one type of ions), each with charge $q_s$, mass $m_s$, and thermal energy $T_s$. The background magnetic field $B$ is along the $\hat{z}$ direction. In the strong field limit, particles move only along $B$ with velocity $v$, and the distribution functions $f_s$ satisfy the following Vlasov equations,

$$\frac{\partial f_s(r,v)}{\partial t} + \frac{q_s}{m_s} \frac{\partial \Phi(r)}{\partial z} \frac{\partial f_s(r,v)}{\partial v} = 0,$$

where $\Phi(r)$ is the electrostatic potential. It can be easily shown by chain rules that any $f(r_L,w)$ is a solution to Eqs. (1) since its dependence on $z$ and $v$ is only through the particle energy $w = m_s v^2/2 + q_s \Phi(r)$. Such distribution functions and the potential are further constrained by the Poisson equation,

$$-\nabla^2 \Phi(r) = \frac{2}{\sqrt{2\pi m_s}} \int_0^{\infty} dw \int_{q_1 \Phi(r)}^{\infty} \frac{4\pi n_s q_s f_s(r_L,w)}{\sqrt{w - q_s \Phi(r)}} ,$$

where the velocity space integral has been converted to energy space integral, and $f_s(r_L,w)$ is normalized so that $n_s$ is the particle density in the unperturbed region where charge neutrality gives $\sum_s n_s q_s = 0$. Species 1 is defined to be the one which involves trapping ($\min[q_1 \Phi(r)] < 0$),
and species 2 does not. The distribution function \( f_1 \) is further divided into passing and trapped components, \( f_p \) and \( f_{tr} \). The second term in Eq. 11 is nonlinear as \( \Phi \) is a functional of the particle distributions and vice versa. In physical terms, the system is nonlinear because plasma particles collectively determine the mean-field potential, and the potential in turn determines how particles distribute themselves. It is the presence of this nonlinear term that admits solitary wave solutions which exhibit localized structures in potentials and distribution functions.

Eq. 11 can be thought as a set of 1D Vlasov equations in the \( \hat{z} \) direction for given \( r_{\perp} \). These parallel Vlasov equations are coupled by the perpendicular profile of the potential \( \Phi \) through Eq. 2. If \( \Phi \) is known, Eq. 2 reduces to a set of 1D integral equations parameterized by \( r_{\perp} \). For given \( f_2 \) and the passing distribution \( f_p \), the trapped distribution \( f_{tr} \) can be found by solving these integral equations. The important requirement for the solutions to be physical is that the trapped distribution \( f_{tr} \) so determined should be nonnegative. This leads to a self-consistent constraint on the form of the potential specified at the beginning. It turns out that neither the potential forms nor the passing distributions are tightly constrained. One can prescribe different localized potential functions or different passing particle distributions (as long as the distribution functions satisfy the Vlasov equation). As an illustrating example, the solitary potential is chosen to be an azimuthally symmetric double Gaussian,

\[
\Phi(r, z) = g\psi \exp \left(-z^2/2\delta_z^2 - r^2/2\delta_r^2\right),
\]

where \( g = -\text{sign}(q_1) \) in order for \( \Phi \) to trap species 1 particles, \( \psi \) is the potential amplitude and is positive, \( r = |r_{\perp}| \), \( \delta_z \) and \( \delta_r \) are the parallel and perpendicular widths. The distributions \( f_p \) and \( f_2 \) are of the Boltzmann type,

\[
\begin{align*}
    f_p(w) &= \sqrt{2m_1/\pi T_1} \exp(-w/T_1), \\
    f_2(w) &= \sqrt{2m_2/\pi T_2} \exp(-w/T_2).
\end{align*}
\]

By carrying out the integrals of \( f_p \) and \( f_2 \) in Eq. 2, we obtain the trapped particle density,

\[
n_{tr}(\Phi) = \Phi \left[ \frac{r^2}{\delta_r^2} \left( \frac{1}{\delta_r^2} - \frac{1}{\delta_z^2} \right) - \frac{2}{\delta_z^2} \right] - \frac{2}{\delta_z^2} \ln \left( \frac{\Phi}{g\psi} \right) - e^{-\Phi} \left[ 1 - \text{erf}(\sqrt{\Phi}) \right] + e^{-t\Phi},
\]

where \( t = q_2 T_1/(q_1 T_2) \). To simplify the expression, we have set the length unit to be \( \lambda_D = \sqrt{T_1/4\pi n_1 q_1^2} \), energy unit \( T_1 \) and charge unit \( q_1 \) (\( \Phi \) becomes strictly negative in units of \( T_1/q_1 \)). We have also rewritten the expression in terms of the potential, a crucial step for analytically solving the integral equation. We refer readers to Appendix B in Ref. 10 for the details of solving this Volterra type integral equation, and simply write down the result here,

\[
f_{tr}(r, w) = \frac{2\sqrt{-w}}{\pi} \left[ \frac{r^2}{\delta_r^2} \left( \frac{1}{\delta_r^2} - \frac{1}{\delta_z^2} \right) - \frac{2}{\delta_z^2} \right] - \frac{2}{\delta_z^2} \ln \left( \frac{4w}{-\psi} \right) + e^{-w} \left[ 1 - \text{erf}(\sqrt{-w}) \right] - \frac{e^{-w}}{\sqrt{\pi}} \sqrt{t} \text{erfi}(\sqrt{-tw}),
\]

where \( w < 0 \), and \( \text{erfi}(z) = \text{erf}(iz)/i \) is the complex error function which is a real function of its argument.

At this juncture, it is instructive to consider the small amplitude behavior of our solution and compare with previous results. Expand the RHS of Eq. 2 for small \( \Phi \). The leading term comes from the passing particle density and is of order \( \sqrt{\Phi} \), and the next order is linear in \( \Phi \). This means that for small potential amplitudes, the nonlinearity actually dominates over the linear response. In this case, even when \( T_i \sim T_e \) for electron-proton plasma, the trapped distribution \( f_{tr} \) can be solved keeping only the leading term, and it yields a constant \( \sqrt{2m_1/\pi} \), which demonstrates that the plasma can sustain the solitary structure by providing a uniform trapped particle distribution. This result, in the context of ion holes, sharply contrasts the previous result 13 which predicts that ion holes do not exist when \( T_e/T_i < 3.5 \).

In recapitulation, we have made one step forward to obtain solutions in Eq. 7 for arbitrary \( t \) taking into account density perturbations of both species. We also obtained new understanding: nonlinearity dominates over linear response in the small amplitude limit – an unexpected result as one normally expects nonlinearity to be unimportant for small amplitudes. Hereafter, we shall use electron holes as the example to discuss the solution behavior and demonstrate how an inequality width-amplitude relation is obtained in the \( t \to 0 \) limit, the limit which all previous analytical solutions for electron holes were based on. This limit is also applicable to ion holes generated in laboratory experiments where \( T_i/T_e \sim 0.03 \) to 0.1 \( \times \). \( \times \).

We illustrate below how the tuning of the parameters (\( \psi, \delta_r, \delta_z \)) affects the trapped particle distribution. Fig. 4 plots \( f_{tr} \) (thick lines) and \( f_p \) (thin lines) at \( r = 0 \) and
the same expression. Upon re-arrangement, the resulting equality relation. For \( \delta \leq \delta_z \), the global minimum of \( f_{tr} \) occurs at maximum allowed \( r \). For a given \( w \leq 0 \), the maximum \( r \) at which a trapped particle with energy \( w \) can exist is the \( r_{\text{max}} \) that satisfies \(-w = \Psi(r_{\text{max}},0)\). Putting Eq. (3) into this condition, we obtain \( r_{\text{max}}^2 = -24 \Psi^2 \ln(-w/\Psi) \). Since the condition \( f_{tr}(r_{\text{max}},w) \geq 0 \) guarantees \( f_{tr}(r \leq r_{\text{max}},w) \geq 0 \) for \( r > \delta_z \), we replace \( r^2 \) in \( f_{tr}(r,w) \) by \( r_{\text{max}}^2 \). The global minimum of \( f_{tr}(r_{\text{max}},w) \) is \( f_{tr}(r_{\text{max}},w = -\Psi) \), hence the condition \( f_{tr}(r_{\text{max}},w = -\Psi) \geq 0 \) yields the width-amplitude relation for \( \delta_r > \delta_z \).

The conditions \( f_{tr}(r=0,w = -\Psi) \geq 0 \) when \( \delta_r \leq \delta_z \) and \( f_{tr}(r_{\text{max}},w = -\Psi) \geq 0 \) when \( \delta_r > \delta_z \) yield exactly the same expression. Upon re-arrangement, the resulting inequality is written as

\[
\delta_z \geq \sqrt{\frac{2(4\ln 2 - 1)}{\sqrt{\pi e^{\Psi}[1-\text{erf}(\sqrt{\Psi})]/\sqrt{\Psi} - 4/\delta_r^2}}}. \tag{8}
\]

Fig. 2 plots this inequality. Parameters lying on or above the shaded surface are allowed (\( \bigcirc \)), and those under the surface are forbidden (\( \times \)). Curves on the \( \delta_z = 0 \) plane are projections of the constant \( \delta_z \) contours to help visualization of the trend of the shaded surface. The shaded surface curves upward to infinity because the denominator in inequality (8) has to be positive, and that yields another relation between \( \delta_r \) and \( \Psi \) (given by \( \delta_r^2 > 4\sqrt{\pi}/(\sqrt{\pi e^{\Psi}[1-\text{erf}(\sqrt{\Psi})]) \) and is plotted as the asymptotic curve on \( \delta_z = 0 \) plane). These inequalities occur because we are free to place the global minimum of \( f_{tr} \) in a continuous range by correspondingly adjusting the amplitude and widths. A point on the shaded surface represents a parameter set that yields zero phase space density at \( w = -\Psi \), that is \((r = 0, z = 0, v = 0)\), the center of the solitary phase space structure. One example of the empty-centered distribution has been provided in Fig. 1. Lowering the amplitude or increasing the widths shifts a point on the surface to the region above, and Figs. 1a and 1b illustrate respectively the effects on the distribution functions.

We note that in the limit of \( \delta_r \to \infty \), inequality (8) reduces to the width-amplitude relation for 1D BGK solitary waves. This limit of inequality (8) gives an upper bound for \( \delta_z \) that is valid for all finite \( \delta_r \). The resulting 1D inequality relation provides us a ground to understand the discrepancy of whether the width should increase \( \bigodot \) or decrease \( \bigtriangleup \) with the amplitude. Both results are contained in the inequality relation with that by Ref. \[21\] corresponding to the lower bounding curve since only empty-centered distributions were studied, and that by Ref. \[12\] contained in the region above the curve as the distributions only take finite values in the center of the phase space structure. To establish the validity of the above results in finite magnetic field, we need to know whether the effect of nonzero cyclotron radius would result in decoherence of the solitary structure, that is, how the distance of the in-
stantaneous particle guiding center to the symmetry axis would vary. Therefore, it is best to look at the particle motion projected onto the 2D $x - y$ plane perpendicular to the magnetic field. The motion of a charged particle inside the solitary structure is influenced by the uniform $B$ and the 3D inhomogeneous $E$ which upon projection becomes time-dependent $E_{2D}$. It can be shown that when the time variation scale of $E_{2D}$ is much smaller than the cyclotron frequency $\omega_c$, and the spatial variation scale ($L$) of $E_{2D}$ is much larger than the cyclotron radius $r_c$, the instantaneous guiding center would spiral around the infinite-field guiding center, and the solitary structure can be maintained (details of calculations will be presented elsewhere [21]). Since the time variation of $E_{2D}$ is characterized by the frequency ($\omega_b$) of the parallel trapped-particle bouncing in the potential, the condition can be written as

$$\omega_b/\omega_c \ll 1 \quad \rightarrow \quad \sqrt{m_e\psi/e/(B\delta_z)} \ll 1 \quad (9)$$

$$r_c/L \ll 1 \quad \rightarrow \quad \sqrt{2m_e\psi/e/(B\delta_r)} \ll 1, \quad (10)$$

where we have expressed on the right hand side of the arrows the condition in terms of familiar variables. For relevant numerical investigation of nonzero cyclotron radius effects, the readers are referred to Ref. [16].

We note that the size and the amplitude of BGK solitary waves do not have a lower cut-off within our theory. The underlying reason is that the screening of the charged core is accomplished by trapped particles which are part of the solitary structure itself. Debye screening is not involved in these self-consistent, self-sustained nonlinear objects. Their size can be well below the Debye radius as far as there are enough particles in the solitary wave to ensure the validity of the mean-field approach. Taking a Debye radius ($\lambda_D$) 100 m and a plasma density 5 cm$^{-3}$ (typical of the low altitude auroral ionosphere), a width of 0.01 $\lambda_D$ for the solitary potential allows $5 \times 10^6$ particles in the structure. Indeed, sub-Debye scale solitary waves have been observed [2].

It is the primary purpose of this paper to address the physical origin and significance of the inequality width-amplitude relation plotted in Fig. 2. The freedom to continuously adjust the global minimum of the trapped particle distribution is due to the collisionless nature of the plasma. Collisionlessness preserves the identity of trapped and passing particles as the energy of a particle is conserved. Collisions destroy energy conservation of individual particles, and consequently, do not allow the existence of trapped particle state nor the adjustment on the occupation number of the state. Therefore, the kinetic solitary waves have a continuum of allowed potential heights and widths, in great distinction to fluid solitons. Moreover, even with the same ambient plasma distribution, different functional forms for the solitary potential is allowed. The impact of this multitude of continua of allowed potentials is that these BGK states can be easily excited. In a system with certain fluctuation level and different fluctuation lengths, BGK states may be accessed easily since for a fixed amplitude there is a wide range of allowed widths. We therefore propose that in turbulent systems, BGK solitary waves can be spontaneously generated in the absence of two-stream or current-driven instabilities. The spontaneous generation of these coherent structures and their subsequent mutual interaction may dominate the transport properties of the turbulence. Because these solitary waves have many degrees of freedom, energy, momentum and charge are readily transferred between them, so that they can make important contributions to bulk properties of the plasma such as thermal transport and electrical resistivity. At the vicinity of the waves, passing particles are accelerated and, together with trapped particles, form counterstreaming beams (Fig. 1). Hence the velocity spread increases significantly and results in much higher average velocity spread (heating) of the plasma. Moreover, particle trapping will prevent particles from free acceleration by the applied electric field and regulate the electric current. The high excitation probability of BGK waves can thus lead to finite resistivity that is required for melting the frozen-in magnetic flux and facilitate reconnection to occur in collisionless plasmas.

In summary, we have derived inequality width-amplitude relation for 3D BGK solitary waves, and established their relevance in finite magnetic fields. The continuum of allowed potential sizes and shapes of the waves is due to their kinetic nature, and leads to their ubiquitous presence. We envision that BGK solitary waves can be spontaneously generated in thermal fluctuations as well as in turbulence.

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