On Relations among Fourier Coefficients and Sum-functions*

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ABSTRACT: We generalize five theorems of Leindler on the relations among Fourier coefficients and sum-functions under the more general \( NBV \) condition.

Keywords. Fourier coefficients; sum-functions; \( NBV \) condition.

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§1. Introduction

Let \( f(x) \) be a \( p \) power integrable function of period \( 2\pi \), in symbol, \( f \in L^p, p \geq 1 \). Define

\[
\omega_p(f, h) := \sup_{|t| \leq h} \| f(x + t) - f(x) \|_p,
\]

and

\[
\omega^*_p(f, h) := \sup_{0 < t \leq h} \| f(x + t) + f(x - t) - 2f(x) \|_p,
\]

where \( \| \cdot \|_p \) denotes the usual \( L^p \) norm.

Denote by \( E_n^{(p)}(f) \) the best approximation of order \( n \) of \( f \) in \( L^p \). The Lipschitz class \( \Lambda_p \) and the Zygmund class \( \Lambda^*_p \) are defined by

\[
\Lambda_p := \{ f \in L^p : \omega_p(f, h) = O(h) \}
\]

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and

\[ \Lambda^*_p := \{ f \in L^p : \omega^*_p(f, h) = O(h) \} \]

respectively.

Leindler [5] introduced a class of sequences, a natural extension of monotone decreasing sequences, named as \textit{RBVS}. Namely, a sequence \( C := \{c_n\} \) of nonnegative numbers tending to zero is called of "rest bounded variation", written as \( C \in RBVS \), if it satisfies

\[
\sum_{n=m}^{\infty} |\Delta c_n| \leq K(C)c_m
\]

for all \( m = 1, 2, \cdots \), where \( K(C) \) is a constant only depending upon \( C \) and \( \Delta c_n = c_n - c_{n+1}, \ n = 1, 2, \cdots \).

Leindler [6] pointed out that \( RBVS \) and the well known quasi-monotone sequences (\textit{CQMS}) are not comparable. Very recently, Le and Zhou [2] defined a new condition named as \textit{GBV} condition to include both the \textit{RBV} and quasi-monotone conditions. In the special real case, the \textit{GBV} condition can be stated as follows: Let \( A := \{a_n\} \) be a sequence of nonnegative numbers, if

\[
\sum_{m=n}^{2n} |\Delta a_n| \leq K(A)a_n
\]

for all \( n = 1, 2, \cdots \), then we say \( A \) satisfies the \textit{GBV} condition, briefly, write \( A \in GBVS \). Many important classic results in Fourier analysis could be generalized by replacing the monotonicity of coefficients by \textit{RBV} or \textit{GBV} condition. For example, readers could refer to [11] for more information. Recently, we [8] further introduced a new kind of sequences named as \textit{NBVS}. In the real case, the \textit{NBVS} can be defined as follows. Let \( A := \{a_n\} \) be a sequence of nonnegative numbers, if

\[
\sum_{m=n}^{2n} |\Delta a_n| \leq K(A)(a_n + a_{2n})
\]

for all \( n = 1, 2, \cdots \), then we say \( A \) satisfies the \textit{NBV} condition, briefly, write \( A \in NBVS \). As we know, the following embedding relations

\[
RBVS \cup CQMS \subset GBVS \subset NBVS
\]
holds. Furthermore, as we mentioned in [8], NBVS can be regarded as a “two-sided” monotonicity condition.

For convenience, throughout the paper, we use $K$ to indicate a positive constant which may depend upon $p$ and $A$, its value may be different even in the same line.

§2. Main Results

In this paper, we will establish the following results on the relations among Fourier coefficients and the sum-functions. All of them were proved for RBVS by Leindler [7], and Theorem 1 was proved for GBVS by Zhou and Le [10].

**Theorem 1.** Let $A \in NBVS$ be such for a fixed $p$, $1 < p < \infty$, that

$$
\sum_{n=1}^{\infty} n^{p-2} a_n^p < \infty. \quad (2.1)
$$

If $f$ is the sum of either of the series

$$
\sum_{n=1}^{\infty} a_n \cos nx \quad \text{or} \quad \sum_{n=1}^{\infty} a_n \sin nx, \quad (2.2)
$$

then

$$
\omega_p(f, n^{-1}) \leq K_1 n^{-1} \left\{ \sum_{\nu=1}^{n-1} \nu^{2p-2} a_{\nu}^p \right\}^{1/p} + K_2 \left\{ \sum_{\nu=n}^{\infty} \nu^{p-2} a_{\nu}^p \right\}^{1/p}. \quad (2.3)
$$

**Theorem 2.** Let $1 < p < \infty$, $1 \leq r < \infty$, and $\lambda(x)$, $x \geq 1$, be a positive monotone function with $K_1 \lambda(2^n) \leq \lambda(2^{n+1}) \leq K_2 \lambda(2^n)$, where $K_1 > 0$. Write $A = \{a_n\} \in NBVS$, $f(x) = \sum_{k=1}^{\infty} a_k \cos kx \in L^p$. Then

$$
\sum_{n=1}^{\infty} \lambda(n)n^{r,p} a_n^p \leq K I(f, \lambda, r, p) := 
K \int_{0}^{1} \lambda \left( \frac{1}{t} \right) t^{r-2-\frac{r}{p}} \left( \int_{0}^{\pi} |f(x+t)+f(x-t)-2f(x)|^p dx \right)^{r/p} dt. \quad (2.4)
$$

If $\sum_{n=1}^{\infty} |\Delta a_n| < \infty$ and $\lambda(x)$ satisfies the additional conditions

$$
\sum_{n=1}^{m} \lambda(n)n^{r,p-r} \leq K \lambda(m)m^{r,p-r+1} \quad (2.5)
$$
and
\[ \sum_{n=m}^{\infty} \lambda(n)n^{r\left(\frac{1}{p}-3\right)} \leq K\lambda(m)m^{1+r\left(\frac{1}{p}-3\right)}, \quad (2.6) \]
then
\[ I(f, \lambda, r, p) \leq K \sum_{n=1}^{\infty} \lambda(n)a_{n}^{r}. \quad (2.7) \]

**Theorem 3.** Let \(1 < p < r\) and \(\{\varphi_{n}\}\) be a nonnegative nondecreasing sequence satisfying \(\varphi_{n}^{2} \leq K\varphi_{n}\) for all \(n\). Define
\[ \Phi(x) := \sum_{n=1}^{x} n^{\frac{r}{p}-2} \varphi_{n}, \]
where \(\varphi(x) := \varphi_{n}\) if \(x \in (n-1, n)\). Write \(A = \{a_{n}\} \in \text{NBVS}\), \(f(x) = \sum_{k=1}^{\infty} a_{k} \cos kx \in L^{p}\). Then the statements
\[ \sum_{n=1}^{\infty} \varphi_{n}n^{r-2}a_{n}^{r} < \infty, \]
\[ \sum_{n=1}^{\infty} \varphi_{n}n^{r+s+p-2} \left( \sum_{k=1}^{n} k^{(s+1)p-2}a_{k}^{p} \right)^{r/p} < \infty \text{ for any } s > \frac{1}{p} - \frac{1}{r}, \]
\[ \sum_{n=1}^{\infty} \varphi_{n}n^{\frac{r}{p}-2} \left( \sum_{k=n}^{\infty} k^{p-2}a_{k}^{p} \right)^{r/p} < \infty, \]
\[ \sum_{n=1}^{\infty} \varphi_{n}n^{\frac{r}{p}-2} \left( \sum_{k=n}^{\infty} k^{p-2}a_{k}^{p} \right)^{r} < \infty, \]
\[ \sum_{n=1}^{\infty} \varphi_{n}n^{\frac{r}{p}-2} \left( \omega_{p}\left(1, \frac{1}{n}\right) \right)^{r} < \infty, \]
\[ \sum_{n=1}^{\infty} \varphi_{n}n^{\frac{r}{p}-2} \left( E_{n}^{(p)}(f) \right)^{r} < \infty, \]
\[ \int_{0}^{\pi} |f(x)|^{r-\frac{r}{p}+1} \Phi(|f(x)|)dx < \infty, \]
\[ \int_{0}^{\pi} |f(x)|^{r} \varphi(|f(x)|)dx < \infty, \]
\[ \int_{0}^{\pi} |f(x)|^{r} \varphi\left(\frac{1}{x}\right)dx < \infty \]
and
\[ \int_{0}^{\pi} \varphi\left(\frac{1}{t}\right) t^{\frac{r}{p}} \left( \int_{0}^{\pi} |f(x+t)+f(x-t)-2f(x)|^{p}dx \right)^{r/p} < \infty \]
are equivalent.
Theorem 4. If $1 < p < \infty$ and $A \in NBV S$, then
\[
\sum_{n=1}^{\infty} n^{2p-2} a_n^p < \infty \tag{2.8}
\]
is a necessary and sufficient condition that a sum-function of either of the series (2.2)
(i) belongs to $\Lambda_p$, or
(ii) is equivalent to an absolutely continuous function whose derivative belongs to $L^p$.

Theorem 5. If $f \in L^p$, $1 < p < \infty$, $A \in NBV S$ and $f$ is a sum-function of
either the series (2.2), then $f \in \Lambda_p^*$ implies that
\[
\omega_p(f, h) \leq Kh\log|h|^{1/p}. \tag{2.9}
\]
The proofs of the above results will be proceeded as in a way as those of Leindler [7], only necessary modifications will be noted.

§3. Lemmas

Lemma 1. Let $1 < p < \infty$, $1 \leq r < \infty$ and $\lambda(x)$, $x \geq 1$, be a positive
monotone function with $K_1 \lambda(2^n) \leq \lambda(2^{n+1}) \leq K_2 \lambda(2^n)$, where $K_1 > 0$. Let $a_n \geq 0$,
$n = 1, 2, \cdots$, and write $f(x) = \sum_{k=1}^{\infty} a_k \cos kx \in L^p$. Then
\[
\sum_{n=1}^{\infty} \lambda(n)n^{-r} \left( \sum_{k=[n/2]}^{2n} a_k \right)^r \leq KI(f, \lambda, r, p). \tag{3.1}
\]
If $\sum_{n=1}^{\infty} |\Delta a_n| < \infty$ and $\lambda(x)$ satisfies (2.5) and (2.6), then
\[
I(f, \lambda, r, p) \leq K \sum_{n=1}^{\infty} \lambda(n) \left( \sum_{k=n}^{\infty} |\Delta a_k| \right)^r. \tag{3.2}
\]

Proof. The second result can be found directly in [4], while the first can be proved
by the same argument as of [4].

Lemma 2([9]). Let $1 < p < \infty$, $\{a_n\} \in NBV S$, and $f$ be the sum of either
of the series (2.2), then $f \in L^p$ if and only if (2.1) holds.
Lemma 3([3]). Let $\alpha_n \geq 0$ and $\lambda_n \geq 0$ be given, $\nu_1 < \cdots < \nu_n < \cdots$ denote the indices for which $\lambda_{\nu_n} > 0$, and $N$ denote the number of positive terms of the sequence $\lambda_n$, provides this number is finite, or in the contrary case set $N = \infty$. Set $\nu_0 = 0$ and if $N < \infty$ then $\nu_{N+1} = \infty$. We have the following inequalities:

$$\sum_{n=1}^{\infty} \lambda_n \left( \sum_{k=1}^{n} \alpha_k \right)^p \leq p^p \sum_{n=1}^{\infty} \lambda_{\nu_n}^{1-p} \left( \sum_{k=\nu_n}^{\infty} \lambda_k \right)^p \left( \sum_{k=\nu_n-1}^{\infty} \alpha_k \right)^p \quad \text{(3.3)}$$

$$\sum_{n=1}^{\infty} \lambda_n \left( \sum_{k=1}^{n} \alpha_k \right)^p \leq p^p \sum_{n=1}^{\infty} \lambda_{\nu_n}^{1-p} \left( \sum_{k=1}^{\nu_n} \lambda_k \right)^p \left( \sum_{k=\nu_n}^{\infty} \alpha_k \right)^p \quad \text{(3.4)}$$

Lemma 4([9]). If $A := \{a_n\} \in NBVS$, then for all $n \geq 1$, it holds that

$$I := \sum_{k=n}^{\infty} |\Delta a_k| \leq C(A) \left( a_n + a_{2n} + a_{4n} + \sum_{k=n}^{\infty} \frac{a_k}{k} \right).$$

Lemma 5. If $A := \{a_n\} \in NBVS$, then

$$n^{-p} \sum_{m=1}^{n-1} m^{-2} \left( \sum_{\nu=1}^{m} \nu^{2} |\Delta a_{\nu}| \right)^p \leq K \left( n^{-p} \sum_{\nu=1}^{n-1} \nu^{2p-2} a_{\nu}^{p} + \sum_{\nu=n}^{\infty} \nu^{p-2} a_{\nu}^{p} \right).$$

Proof. Write

$$N(x) := \frac{\log x}{\log 2}, \quad x > 0.$$ 

Since $m/4 \leq 2^{[N(m/2)]} \leq m/2$ for $m \geq 2$, then

$$\sum_{\nu=1}^{m} \nu^{2} |\Delta a_{\nu}| \leq \sum_{k=1}^{[N(m/2)]} \sum_{\nu=2^{k-1}}^{2^{k}} \nu^{2} |\Delta a_{\nu}| + \sum_{k=[m/4]+1}^{m} \nu^{2} |\Delta a_{\nu}| =: J_1 + J_2.$$ 

By the definition of $NBVS$, we get

$$a_{2^k} \leq \sum_{i=s}^{2^k} |\Delta a_i| + a_s \leq \sum_{i=s}^{2s} |\Delta a_i| + a_s \leq K(a_s + a_{2s}),$$

and

$$a_{2^{k-1}} \leq \sum_{i=2^{k-1}}^{s-1} |\Delta a_i| + a_s \leq K(a_{[s/2]} + a_s)$$

for all $2^{k-1} \leq s \leq 2^k$, and hence deduce that

$$J_1 \leq \sum_{k=1}^{[N(m/2)]} 2^{2k} \sum_{\nu=2^{k-1}}^{2^k} |\Delta a_{\nu}| \leq K \sum_{k=1}^{[N(m/2)]} 2^{2k} \left( a_{2^{k-1}} + a_{2^k} \right)$$

$$\leq K \sum_{k=1}^{[N(m/2)]} 2^{k} \sum_{s=2^{k-1}}^{2^k} \left( a_{[s/2]} + a_s + a_{2s} \right) \leq K \sum_{k=1}^{m} k a_k.$$
For any \( \lfloor m/4 \rfloor + 1 \leq s \leq 2\lfloor m/4 \rfloor + 2 \), if \( s \) is an even number, it follows from the definition of \( \text{NBV} \ S \) that

\[
|a_{\lfloor m/4 \rfloor + 1}| \leq \sum_{k=\lfloor m/4 \rfloor + 1}^{s-1} |\Delta a_k| + a_s \leq \sum_{k=\lfloor s/2 \rfloor}^{s} |\Delta a_k| + a_s \leq K(a_s + a_{\lfloor s/2 \rfloor}).
\]

If \( s \) is an odd number, then

\[
|a_{\lfloor m/4 \rfloor + 1}| \leq \sum_{k=\lfloor m/4 \rfloor + 1}^{s-1} |\Delta a_k| + a_s \leq \sum_{k=\lfloor s/2 \rfloor}^{s-1} |\Delta a_k| + a_s \leq K(a_s + a_{s-1} + a_{\lfloor s/2 \rfloor}).
\]

Therefore, in any case,

\[
m^2|a_{\lfloor m/4 \rfloor + 1}| \leq K \sum_{s=\lfloor m/4 \rfloor + 1}^{2\lfloor m/4 \rfloor + 2} s(a_s + a_{s-1} + a_{\lfloor s/2 \rfloor}) \leq K \sum_{k=1}^{m} ka_k. \quad (3.5)
\]

A similar discussion leads to

\[
m^2|a_{2\lfloor m/4 \rfloor + 2}| \leq K \sum_{k=\lfloor m/4 \rfloor + 1}^{4\lfloor m/4 \rfloor} s(a_s + a_{s-1} + a_{\lfloor s/2 \rfloor}) \leq K \sum_{k=1}^{m} ka_k, \quad (3.6)
\]

and

\[
m^2|a_{\lfloor m/2 \rfloor}| \leq K \sum_{k=\lfloor m/2 \rfloor}^{m} s(a_s + a_{s-1} + a_{\lfloor s/2 \rfloor}) \leq K \sum_{k=1}^{m} ka_k. \quad (3.7)
\]

Set

\[
m^* =: \begin{cases} m, & \text{m is even,} \\ m - 1, & \text{m is odd.} \end{cases}
\]

By (3.5)-(3.7), we deduce that

\[
J_2 \leq \sum_{\nu=\lfloor m/4 \rfloor + 1}^{2\lfloor m/4 \rfloor + 2} \nu^2|\Delta a_\nu| + \sum_{\nu=\lfloor m/2 \rfloor}^{m^*} \nu^2|\Delta a_\nu| + m^2(a_m + a_{m+1}) \\
\leq Km^2 \left(a_{\lfloor m/4 \rfloor + 1} + a_{2\lfloor m/4 \rfloor + 2} + a_{m^* / 2} + a_{m^*} + a_m + a_{m+1}\right) \\
\leq Km^2 \left(a_{\lfloor m/4 \rfloor + 1} + a_{2\lfloor m/4 \rfloor + 2} + a_{\lfloor m/2 \rfloor} + a_{m-1} + a_m + a_{m+1}\right) \\
\leq K \sum_{k=1}^{m} ka_k + Km^2 (a_{m-1} + a_m + a_{m+1}).
\]
Combining all the estimates for $J_1$ and $J_2$ with the fact (see [7])

$$n^{-p} \sum_{m=1}^{n-1} m^{-2} \left( \sum_{\nu=1}^{m} \nu a_{\nu} \right)^p \leq Kn^{-p} \sum_{\nu=1}^{n-1} \nu^{2p-2} a_{\nu}^p,$$

we see that

$$n^{-p} \sum_{m=1}^{n-1} m^{-2} \left( \sum_{\nu=1}^{m} \nu^2 |\Delta a_{\nu}| \right)^p \leq K n^{-p} \sum_{m=1}^{n-1} m^{-2} \left( \sum_{k=1}^{m} k a_k \right)^p$$

$$+ Kn^{-p} \sum_{m=1}^{n-1} m^{2p-2} (a_{m-1}^p + a_m^p + a_{m+1}^p)$$

$$\leq K n^{-p} \sum_{\nu=1}^{n-1} \nu^{2p-2} a_{\nu}^p + Kn^{-p} \sum_{m=1}^{n-1} m^{2p-2} a_{m+1}^p$$

$$\leq K n^{-p} \sum_{\nu=1}^{n-1} \nu^{2p-2} a_{\nu}^p + Kn^{p-2} a_n^p$$

$$\leq K n^{-p} \sum_{\nu=1}^{n-1} \nu^{2p-2} a_{\nu}^p + K \sum_{\nu=n}^\infty \nu^{p-2} a_{\nu}^p.$$

**Lemma 6.** If $A := \{a_n\} \in NBVS$, then

$$n^{-p} \sum_{m=1}^{n-1} m^{p-2} \left( \sum_{\nu=m+1}^{n} \nu |\Delta a_{\nu}| \right)^p \leq K \left( n^{-p} \sum_{\nu=1}^{n-1} \nu^{2p-2} a_{\nu}^p + \sum_{\nu=n}^\infty \nu^{p-2} a_{\nu}^p \right).$$

**Proof.** First assume that $n \geq 8(m+1)$. By noting that $m + 1 \leq 2^{\left[N(m+1)\right]+1} \leq 2(m+1)$ and $n/4 \leq 2^{\left[N(n/2)\right]} \leq n/2$, we can split $\sum_{k=m+1}^{n} \nu |\Delta a_{\nu}|$ into

$$\sum_{\nu=m+1}^{n} \nu |\Delta a_{\nu}| \leq \sum_{k=\left[N(m+1)\right]+3}^{N(n/2)} \sum_{\nu=2^k-1}^{2^k} \nu |\Delta a_{\nu}| + \sum_{\nu=\left[n/4\right]+1}^{n} \nu |\Delta a_{\nu}| + \sum_{\nu=m+1}^{2^{\left[N(m+1)\right]+3}-1} \nu |\Delta a_{\nu}|$$

$$=: H_1 + H_2 + H_3.$$

Similar to what we have done for $J_1$ in the proof of Lemma 5, we get

$$H_1 \leq K \sum_{\nu=m+1}^{n} a_{\nu}.$$
Since $n \geq 8(m + 1)$, then setting

$$N^* =: \left\lceil \frac{1}{2}([n/4] + 1) \right\rceil \geq m + 1,$$

again similar to $J_2$, we have

$$H_2 \leq K \sum_{k=N^*}^{n} a_k + Kn (a_{n-1} + a_n + a_{n+1})$$

$$\leq K \sum_{k=m+1}^{n} a_k + Kn (a_{n-1} + a_n + a_{n+1}).$$

At the same time, it is evident that

$$H_3 \leq \sum_{\nu=m+1}^{8(m+1)} \nu|\Delta a_\nu| \leq Km(a_{m+1} + a_{2(m+1)} + a_{4(m+1)} + a_{8(m+1)}).$$

In case $n < 8(m + 1)$, then it clearly holds that

$$\sum_{\nu=m+1}^{n} \nu|\Delta a_\nu| \leq Km(a_{m+1} + a_{2(m+1)} + a_{4(m+1)} + a_{8(m+1)}).$$

Altogether, all the above estimates lead to that

$$n^{-p} \sum_{m=1}^{n-1} m^{p-2} \left( \sum_{\nu=m+1}^{n} \nu|\Delta a_\nu| \right)^p \leq Kn^{-p} \sum_{m=1}^{n-1} m^{p-2} \left( \sum_{\nu=m+1}^{n} a_\nu \right)^p$$

$$+ Kn^{p-1} (a_{n-1}^p + a_n^p + a_{n+1}^p) + Kn^{-p} \sum_{m=1}^{n-1} m^{2p-2} \sum_{j=1}^{8} a_j^{p(m+1)}.$$

Obviously,

$$a_n \leq \sum_{i=k}^{n-1} |\Delta a_i| + a_k \leq K(a_k + a_{2k})$$

for $[n/2] + 1 \leq k \leq n$, which implies that

$$a_n \leq Kn^{-1} \sum_{k=[n/2]+1}^{n-1} (a_k + a_{2k}) \leq Kn^{-1} \sum_{k=[n/2]+1}^{2n-2} a_k,$$

so that applying Hölder’s inequality yields that

$$n^{p-1}a_n^p \leq Kn^{-1} \left( \sum_{k=[n/2]+1}^{2n-2} a_k \right)^p \leq Kn^{p-2} \sum_{k=[n/2]+1}^{2n-2} a_k^p.$$
\[ \begin{align*}
\leq K \left( n^{-p} \sum_{k=n}^{n-1} k^{2p-2} a_k^p + \sum_{k=n}^{2n-2} k^{p-2} a_k^p \right) & \leq K \left( n^{-p} \sum_{k=1}^{n-1} k^{2p-2} a_k^p + \sum_{k=n}^{\infty} k^{p-2} a_k^p \right). \tag{3.9} \\
\end{align*} \]

Similarly,

\[ \begin{align*}
n^{p-1} a_{n-1}^p & \leq K \left( n^{-p} \sum_{k=1}^{n-1} k^{2p-2} a_k^p + \sum_{k=n}^{\infty} k^{p-2} a_k^p \right), \tag{3.10} \\
n^{p-1} a_{n+1}^p & \leq K \left( n^{-p} \sum_{k=1}^{n-1} k^{2p-2} a_k^p + \sum_{k=n}^{\infty} k^{p-2} a_k^p \right). \tag{3.11} \\
\end{align*} \]

Since (see [7]),

\[ \begin{align*}
n^{-p} \sum_{m=1}^{n-1} m^{p-2} \left( \sum_{\nu=m+1}^{n} a_{\nu} \right)^p & \leq K n^{-p} \sum_{\nu=1}^{n-1} \nu^{2p-2} a_{\nu}^p, \\
\end{align*} \]

with the estimates (3.9)-(3.11), Lemma 6 will be completed if we can verify that

\[ \begin{align*}
\Delta =: n^{-p} \sum_{m=1}^{n-1} \nu^{2p-2} a_{\nu}^p \leq K n^{-p} \sum_{\nu=1}^{n-1} \nu^{2p-2} a_{\nu}^p + \sum_{\nu=n}^{\infty} \nu^{p-2} a_{\nu}^p. \tag{3.12} \\
\end{align*} \]

Indeed, we prove (3.12) by the following way:

\[ \begin{align*}
\Delta & = n^{-p} \sum_{j=1}^{\lfloor n/j \rfloor - 1} \nu^{p-2} a_{\nu(j+1)}^p + n^{-p} \sum_{j=1}^{\lfloor n/j \rfloor} \sum_{\nu=1}^{n-1} \nu^{2p-2} a_{\nu(j+1)}^p \\
& \leq K \left( n^{-p} \sum_{\nu=1}^{n-1} \nu^{2p-2} a_{\nu}^p + n^{p-2} a_n^p + \sum_{j=1}^{n} \sum_{\nu=n}^{\infty} \nu^{p-2} a_{\nu}^p \right) \\
& \leq K \left( n^{-p} \sum_{\nu=1}^{n-1} \nu^{2p-2} a_{\nu}^p + \sum_{j=1}^{n} \sum_{\nu=n}^{\infty} \nu^{p-2} a_{\nu}^p \right) \\
& \leq K \left( n^{-p} \sum_{\nu=1}^{n-1} \nu^{2p-2} a_{\nu}^p + \sum_{\nu=n}^{\infty} \nu^{p-2} a_{\nu}^p \right). \\
\end{align*} \]

§4. Proofs

**Proof of Theorem 1.** By Lemma 2, we know that the condition (2.1) is both necessary and sufficient for \( f \in L^p, p > 1 \). We only need to treat the cosine series case, the other could be done similarly. Assume that \( h = \pi/2n \). By the symmetry of \( f \), it is clear that

\[ \omega_p(f, h) \leq K \sup_{0 < t \leq h} \left\{ \int_0^{\pi/n} |f(x + t) - f(x)|^p dx \right\}^{1/p} \]

\[ + \left\{ \int_{\pi/n}^\pi |f(x + t) - f(x)|^p dx \right\}^{1/p} := K \sup_{0 < t \leq h} (I_1 + I_2). \]
As the way done by Leindler [7], we have
\[
\frac{1}{2} I_1 \leq t \left\{ \int_0^{\pi/n} \left( \sum_{\nu=1}^{n-1} \nu a_{\nu} \right)^p \, dx \right\}^{1/p} + K \left\{ \sum_{m=n}^{\infty} \int_{3\pi/2m}^{3\pi/(2m+1)} \left| \sum_{\nu=n}^{\infty} a_{\nu} \cos \nu x \right|^p \, dx \right\}^{1/p} =: I_{11} + I_{12}.
\]

Applying Hölder’s inequality leads to
\[
I_{11} \leq Kn^{-1} \left\{ \sum_{\nu=1}^{n-1} \nu^{2p-2} a_{\nu}^p \right\}^{1/p}.
\]

By Abel’s transformation and Lemma 4, we have
\[
\left| \sum_{\nu=n}^{\infty} a_{\nu} \cos \nu x \right| \leq \sum_{\nu=n}^{m} a_{\nu} + (m + 1) \sum_{\nu=m+1}^{\infty} |\Delta a_{\nu}|
\]
\[
\leq K \left( \sum_{\nu=n}^{m} a_{\nu} + m(a_m + a_{2m} + a_{4m}) + m \sum_{\nu=m+1}^{\infty} \frac{a_{\nu}}{\nu} \right).
\]

Setting \( \lambda_m = m^{-2} \) and \( \alpha_m = 0 \) for \( m < n \) and \( \alpha_m = a_m \) for \( m \geq n \), we get
\[
\sum_{m=n}^{\infty} m^{-2} \left( \sum_{\nu=n}^{m} \alpha_{\nu} \right)^p = \sum_{m=1}^{\infty} m^{-2} \left( \sum_{\nu=1}^{m} \alpha_{\nu} \right)^p
\]
\[
\leq K \sum_{m=1}^{\infty} m^{p-2} \alpha_m^p = K \sum_{m=n}^{\infty} m^{p-2} a_m^p.
\]

by (3.3). Again setting
\[
\nu_1 = n, \ \nu_2 = n + 1, \ldots, \nu_j = n + j, \ldots,
\]
\[
\lambda_1 = \lambda_2 = \cdots = \lambda_{\nu_1-1} = 0, \ \lambda_{\nu_j} = \nu_j^{p-2}, \ j = 1, 2, \cdots,
\]

with (3.4), we get
\[
\sum_{m=n}^{\infty} m^{p-2} \left( \sum_{\nu=m}^{\infty} \frac{a_{\nu}}{\nu} \right)^p = \sum_{j=1}^{\infty} \lambda_j \left( \sum_{k=j}^{\nu_j} \frac{a_k}{k} \right)^p
\]
\[
\leq p^p \sum_{j=1}^{\infty} \lambda_j^{1-p} \left( \sum_{k=1}^{\nu_j} \lambda_k \right)^p \left( \sum_{k=\nu_j}^{\nu_{j+1}-1} \frac{a_k}{k} \right)^p
\]
\[
= p^p \sum_{m=n}^{\infty} m^{(p-2)(1-p)} \left( \sum_{k=n}^{m} k^{p-2} \right)^p \left( \frac{a_m}{m} \right)^p
\]
\[
\leq p^p \sum_{m=n}^{\infty} m^{p-2} a_m^p. \quad (4.1)
\]
Thus, by Lemma 5 and Lemma 6, we obtain that

\[ I_{12}^p \leq K \sum_{m=n}^\infty m^{-2} \left( \sum_{\nu=n}^m a_\nu \right)^p + K \sum_{m=n}^\infty m^{p-2} a_m^p + K \sum_{m=n}^\infty m^{p-2} \left( \sum_{\nu=m}^{\infty} a_\nu \right)^p \]

\[ \leq K \sum_{m=n}^\infty m^{p-2} a_m^p. \]

Let \( D_\nu(x) \) be the Dirichlet Kernel. Following the way of Leindler [7], we see that

\[ I_2 \leq \left\{ \int_{\pi/n}^{\pi} \left| \sum_{\nu=1}^n \Delta a_\nu [D_\nu(x \pm t) - D_\nu(x)] \right|^p dx \right\}^{1/p} \]

\[ + \left\{ \int_{\pi/n}^{\pi} \left| \sum_{\nu=n+1}^\infty \Delta a_\nu [D_\nu(x \pm t) - D_\nu(x)] \right|^p dx \right\}^{1/p} := I_{21} + I_{22}, \]

and

\[ I_{21}^p \leq K \sum_{m=1}^{n-1} \int_{\pi/(m+1)}^{\pi/m} \sum_{\nu=1}^n \left| \Delta a_\nu [D_\nu(x \pm t) - D_\nu(x)] \right|^p dx \]

\[ \leq K p \left\{ \sum_{m=1}^{n-1} m^{-2} \left( \sum_{\nu=1}^m \nu^2 |\Delta a_\nu| \right)^p + \sum_{m=1}^{n-1} m^{p-2} \left( \sum_{\nu=m+1}^\infty \nu |\Delta a_\nu| \right)^p \right\}. \]

Thus, by Lemma 5 and Lemma 6, we obtain that

\[ I_{21}^p \leq K \left( n^{-p} \sum_{\nu=1}^{n-1} \nu^{2p-2} a_\nu^p + K_2 \sum_{\nu=n}^{\infty} \nu^{p-2} a_\nu^p \right). \]

In a way similar to the treatment of (3.9), we can easily deduce that

\[ n^{p-1} a_{j(n+1)}^p \leq Kn^{-p} \sum_{\nu=1}^{n-1} \nu^{2p-2} a_\nu^p + \sum_{\nu=n}^{\infty} \nu^{p-2} a_\nu^p, \quad j = 1, 2, 4, \]

with applying Lemma 4, we achieve that

\[ I_{22} \leq \left\{ \int_{\pi/2n}^{\pi+\pi/2n} \left| \sum_{\nu=n+1}^{\infty} |\Delta a_\nu| |D_\nu(x)| \right|^p dx \right\}^{1/p} \]

\[ \leq K \left\{ \sum_{\nu=n+1}^{\infty} |\Delta a_\nu| \left( \int_{\pi/2n}^{\pi} x^{-p} dx \right) \right\}^{1/p} \]

\[ \leq Kn^{1-p} \left( a_{n+1} + a_{2n+2} + a_{4n+4} + \sum_{k=n+1}^{\infty} \frac{a_k}{k} \right). \]

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\[
\leq Kn^{-1} \left( \sum_{\nu=1}^{n-1} \nu^{p-2} a_\nu^p \right)^{1/p} + K \left( \sum_{\nu=n}^{\infty} \nu^{p-2} a_\nu^p \right)^{1/p} \\
+ K \left( n^{p-1} \left( \frac{2n}{k} \sum_{k=n+1}^{2n} a_k^p \right)^p + n^{p-1} \left( \sum_{k=n}^{\infty} a_k^p \right)^p \right)^{1/p} \\
\leq Kn^{-1} \left( \sum_{\nu=1}^{n-1} \nu^{p-2} a_\nu^p \right)^{1/p} + K \left( \sum_{\nu=n}^{\infty} \nu^{p-2} a_\nu^p \right)^{1/p} \\
+ K \left( n^{-1} \left( \sum_{k=n}^{2n} a_k^p \right)^p + n^{p-2} \left( \sum_{k=n}^{\infty} a_k^p \right)^p \right)^{1/p} \\
\leq Kn^{-1} \left( \sum_{\nu=1}^{n-1} \nu^{p-2} a_\nu^p \right)^{1/p} + K \left( \sum_{\nu=n}^{\infty} \nu^{p-2} a_\nu^p \right)^{1/p} \\
+ K \left( \sum_{k=n}^{2n} k^{p-2} a_k^p + \sum_{\nu=n+1}^{\infty} \nu^{p-2} \left( \sum_{k=\nu}^{\infty} a_k^p \right)^p \right)^{1/p} \\
\leq Kn^{-1} \left( \sum_{\nu=1}^{n-1} \nu^{p-2} a_\nu^p \right)^{1/p} + K \left( \sum_{\nu=n}^{\infty} \nu^{p-2} a_\nu^p \right)^{1/p} \\
+ K \left( \sum_{k=n}^{2n} k^{p-2} a_k^p + \sum_{\nu=n+1}^{\infty} \nu^{p-2} \left( \sum_{k=\nu}^{\infty} a_k^p \right)^p \right)^{1/p} \\
\leq Kn^{-1} \left( \sum_{\nu=1}^{n-1} \nu^{p-2} a_\nu^p \right)^{1/p} + K \left( \sum_{\nu=n}^{\infty} \nu^{p-2} a_\nu^p \right)^{1/p} \\
\text{(by (4.1))}
\]

Altogether, the above estimates for \( I_1 \) and \( I_2 \) complete Theorem 1.

**Proof of Theorem 2.** By (3.8),

\[
\sum_{k=[n/2]}^{2n} a_k \geq Kn a_n,
\]

and combining (4.2) with (3.1) of Lemma 1, we have (2.4).

By estimate (3.2) of Lemma 1, (2.7) will be proved if the following inequality

\[
\sum_{n=1}^{\infty} \lambda(n) \left( \sum_{k=n}^{\infty} |\Delta a_k| \right)^r \leq K \sum_{n=1}^{\infty} \lambda(n)a_n^r
\]

holds. Furthermore, with the help of Lemma 4, what we really need to establish is
that
\[
\sum_{n=1}^{\infty} \lambda(n) \left( \sum_{k=n}^{\infty} \frac{a_k}{k} \right)^r \leq K \sum_{n=1}^{\infty} \lambda(n) a_n^r.
\]

In fact, if \( r = 1 \), by exchanging the order of summation and applying (2.5), we have
\[
\sum_{n=1}^{\infty} \lambda(n) \sum_{k=n}^{\infty} \frac{a_k}{k} = \sum_{n=1}^{\infty} \frac{a_n}{n} \sum_{k=1}^{n} \lambda(k) k^{1/p-1} k^{1-1/p}
\]
\[
\leq \sum_{n=1}^{\infty} \frac{a_n}{n} n^{-1/p} \sum_{k=1}^{n} \lambda(k) k^{1/p-1} \leq K \sum_{n=1}^{\infty} \lambda(n) a_n;
\]

if \( r > 1 \), by Hölder’s inequality and (2.5), we still have
\[
\sum_{n=1}^{\infty} \lambda(n) \left( \sum_{k=n}^{\infty} \frac{a_k}{k} \right)^r \leq \sum_{n=1}^{\infty} \lambda(n) \left( \sum_{k=n}^{\infty} \frac{1}{k^{1-1/p}} \right)^{r-1} \sum_{k=n}^{\infty} \frac{a_k^r}{k^{1+1/p}}
\]
\[
\leq K \sum_{k=1}^{\infty} \frac{a_k^r}{k^{1+1/p}} \sum_{n=1}^{k} \lambda(n) n^{p/r} \leq K \sum_{n=1}^{\infty} \lambda(n) a_n^r.
\]

**Proof of Theorem 3.** Most of the proof can be proceeded as the corresponding part of Leindler [7] word by word, we omit the details here.

**Proof of Theorem 4.** As Leindler [7] pointed out, what we need to do is to verify that (2.8) \( \Rightarrow \) (i) and that (ii) \( \Rightarrow \) (2.8). By applying Abel’s tranformation
\[
\sum_{\nu=n}^{\infty} \lambda_{\nu} u_{\nu} = \sum_{\nu=n}^{\infty} (\lambda_{\nu} - \lambda_{\nu+1}) \sum_{k=1}^{\nu} u_k - \lambda_n \sum_{k=1}^{n-1} u_k
\]
with \( \lambda_{\nu} = \nu^{-p} \) and \( u_{\nu} = \nu^{2p-2} a_{\nu}^p \), and also by (2.5), we evidently have
\[
\sum_{\nu=n}^{\infty} \nu^{p-2} a_{\nu}^p = \sum_{\nu=n}^{\infty} \nu^{-p} \nu^{2p-2} a_{\nu}^p \leq K n^{-p},
\]
thus, the second term in (2.3) is not larger than \( Kn^{-1} \). Altogether, by Theorem 1, it means that \( f \in \Lambda_p \).

Let \( f(x) \) be the sum function of, say, the series \( \sum_{n=1}^{\infty} a_n \sin nx \), and set
\[
F(x) := \int_0^x f(t) dt = \sum_{n=1}^{\infty} n^{-1} a_n (1 - \cos nx).
\]
An standard argument yields that
\[ F(\pi/(2n)) = 2 \sum_{k=1}^{\infty} \frac{a_k}{k} \sin \frac{k\pi}{4n} \geq \frac{K}{n} \sum_{k=\lfloor n/2 \rfloor}^{2n} a_k, \]
so that \( F(\pi/(2n)) \geq Ka_n \) by (4.2). Set
\[ G(x) := \int_0^x dt \int_0^t |f'(u)| du. \]
Obviously, \( F(x) \leq G(x) \). Hence, applying Hardy’s inequality ([12]) twice, we obtain
\[ \sum_{n=2}^{\infty} n^{2p-2} a_n^p \leq \sum_{n=2}^{\infty} n^{2p-2} G^p(\pi/(2n)) \leq K \sum_{n=2}^{\infty} n^{2p-2} G^p(\pi/n) \]
\[ \leq K \sum_{n=2}^{\infty} \frac{G(x)}{x} x^{-p} dx \leq K \int_0^\pi \left( \int_0^x |f'(t)| dt \right)^p x^{-p} dx \leq K \int_0^\pi |f'(x)|^p dx < \infty. \]

**Proof of Theorem 5.** Set \( T_{m,2n}(x) := \sum_{\nu=m}^{2n} \cos \nu x \), then ([1] or [7])
\[ I_{m,2n,t} := \int_{-\pi}^\pi (2f(x) - f(x+t) - f(x-t)) T_{m,2n}(x) dx = 4\pi \sum_{\nu=m}^{2n} a_\nu \sin^2 \frac{1}{2} \nu t. \] (4.3)
Taking \( t = \pi/n \) and \( m = \lfloor n/2 \rfloor \) in (4.3), then applying (4.2) again, we have
\[ \sum_{\nu=m}^{2n} a_\nu \sin^2 \frac{1}{2} \nu t \geq K \sum_{\nu=m}^{2n} a_\nu \geq na_n. \] (4.4)
On the other hand, we have
\[ \int_{-\pi}^\pi |T_{m,2n}(x)|^q dx \leq K \left\{ \int_0^{\pi/(2n)} n^q dx + \int_{\pi/(2n)}^\pi x^{-q} dx \right\} \leq Kn^{q-1}. \]
By Hölder’s inequality, it follows that
\[ I_{m,2n,\pi/n} \leq Kn^{1/p} \left\{ \int_{-\pi}^\pi \left| f \left( x + \frac{\pi}{n} \right) + f \left( x - \frac{\pi}{n} \right) - 2f(x) \right|^p dx \right\}^{1/p} \leq Kn^{1/p} \omega_p^*(f, \pi/n). \]
A combination of (4.3) and (4.4) leads to
\[ \omega_p^*(f, 1/n) \geq Kn^{1-1/p}a_n. \] (4.5)
Therefore, from that $\omega_p^*(f, 1/n) \leq Kn^{-1}$ and by (4.5), we get $a_n \leq Kn^{-2+1/p}$, whence by Theorem 1, it follows that

$$\omega_p(f, 1/n) \leq Kn^{-1}(\log n)^{1/p},$$

and (2.9) is done.

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