Attractors, Bifurcations and Curvature in Multi-field Inflation

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Recent years have seen the introduction of various multi-field inflationary scenarios, in which the curvature and geodesics of the scalar manifold play a crucial role. We outline a simple description that unifies these different proposals and discuss their stability criteria. We demonstrate how the underlying dynamics is governed by an effective potential, whose critical points and bifurcations determine the late-time behaviour of the system, thus unifying hyperinflation, angular, orbital and side-tracked inflation. Interestingly, hyperinflation is shown to be a special case of side-tracked inflation. This equivalence relies on the enhanced isometries of the hyperbolic manifold and we provide the explicit coordinate transformation that maps the two models into each other.

**Introduction.** Inflation, the hypothesis of rapid accelerated expansion in the primordial Universe, provides an elegant solution to the flatness and horizon problem [1,2], and seeds the primordial Universe with quantum fluctuations whose predictions are in excellent agreement with recent observations [3]. It is often defined as a period of quasi-De Sitter expansion, and hence requires a small deviation from the scale invariant De Sitter space-time for a prolonged period of time.

This definition consists of two conditions. Firstly, the Hubble slow-roll parameter, $\epsilon = -\frac{1}{2} \frac{d}{dN} (\log H)$, must be smaller than one to inflate, and much smaller than one to have slow-roll inflation. We will concern ourselves with the implications of the second condition, i.e. for inflation to be prolonged (independently of the requirement that $\epsilon$ should be small). This can be translated into the requirement that the variation of $\epsilon$ as a function of e-folds, $\dot{\eta} = \frac{d\epsilon}{dN} \equiv \epsilon'$, should be small. Note that this is usually phrased as $\eta = (\log \epsilon)' = \dot{\eta}/\epsilon$ being small, and since inflation has $\epsilon < 1$ our condition for prolonged inflation is weaker (and hence more general).

Under the assumption of a two-derivative model consisting of gravity and $n$ scalar fields $\Phi^I$, both these quantities can be phrased in geometric terms for the scalar manifold metric $g_{IJ}(\Phi)$. The Hubble flow parameter can be written as

$$\epsilon = \frac{1}{2} v^I v^I, \quad v^I \equiv \frac{d\Phi^I}{dN},$$

and is set by the norm of the velocity of the scalar fields with respect to the natural clock during inflation, the number of e-folds $N$. The latter is related to cosmic time via the Hubble parameter

$$dN = H dt, \quad 3H^2 = \frac{1}{2} g_{IJ} \dot{\Phi}^I \dot{\Phi}^J + V.$$

The variation of the Hubble flow parameter reads

$$\dot{\eta} = v_I a^I, \quad a^I \equiv D_N v^I = \frac{dv^I}{dN} + \Gamma^I_{JK} v^J v^K,$$

in terms of the covariant (or generalised) acceleration. Importantly, the vanishing of the latter is related via the scalar field equation to the slow-roll condition,

$$a^I = -(3 - \epsilon)v^I - V^I/H^2,$$

where the RHS consists of the Hubble friction and the potential gradient terms.

Prolonged inflation requires the Hubble flow parameter to be approximately constant, translating into the (approximate) vanishing of the inner product between the velocity and covariant acceleration of the scalar fields. For a single field, this implies that the acceleration must be very small and that the unique manner to have prolonged single-field inflation is to impose the slow-roll condition, given by the separate vanishing of the two sides of the scalar field equation. Fast-roll inflation can be achieved by including higher-order terms as in e.g. DBI inflation [4].

In multi-field inflation, on the other hand, the inner product can be vanishing while both vectors $a^I$ and $v^I$ are not, allowing one to violate the slow-roll, slow-turn condition [5,9]. As we will outline, this requires an interplay between gradient terms and (generalised) centrifugal forces acting on the scalars orthogonal to the inflaton, which can be phrased in terms of an effective potential that can be linked to the Hubble parameter. As we will show, this is a common feature shared by numerous recent proposals [10,14]. While hyperinflation [15] might appear to be of a different nature, we will demonstrate that it can also be captured by our effective potential formalism. This goes beyond recent investigations that have pointed out similarities between the hyperinflation and sidetracked scenarios [17,19] in the context of geometrical destabilization [20,23]. We instead show that these models are actually identical.

**Background evolution.** We split up the scalar fields

$$\Phi^I = (\phi, \chi^I),$$

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where $\phi$ is defined as the inflationary direction while the orthogonal $\chi^i$ are (approximately) constant during inflation. Coordinate systems with a transitively acting isometry can provide a natural basis for this velocity decomposition. Moreover, for simplicity of the discussion we will assume the metric to be diagonal w.r.t. the above split. This coordinate choice leads to

\[ v' = (v, 0, \ldots), \quad a' = \left( \frac{d\mathcal{r}}{d\mathcal{N}} + \Gamma^\phi_{\phi\phi} v^2, \Gamma^i_{\phi\phi} v^2 \right), \quad (6) \]

evaluated on the particular inflationary solution. This construction is always possible for any given trajectory parametrized by different initial conditions; we have in mind that there will be a late-time attractor and the coordinate system is adapted to that solution.

In this coordinate system, there is a particularly striking separation of the consequences of prolonged inflation. Along the inflationary direction $\phi$, this is similar to the single-field case:

\[ \ddot{\phi} + \Gamma^\phi_{\phi\phi} \dot{\phi}^2 + 3H \dot{\phi} + V_{\phi} = 0. \quad (7) \]

This implies that the inflationary direction is subject to the usual slow-roll slow-turn conditions with $D_\phi \dot{\phi} \approx 0$.

Turning to the orthogonal direction, the situation is strikingly different. By adapting our coordinates we have defined these as stationary, with a vanishing velocity $\dot{\chi}^i \approx 0$. Remarkably, they can still have a non-vanishing acceleration, but only when deviating away from a geodesic. This will introduce a (generalized) centrifugal force that has to be balanced by a potential gradient: for the stationary directions Eqs. (4) read

\[ V_{\text{eff}}^{\chi^i} \equiv \ddot{V}^{\chi^i} + \Gamma^i_{\phi\phi} v^2 H^2 = 0. \quad (8) \]

which we will refer to as the effective gradient along the $i$’th direction in field space.

These conditions should be seen as algebraic field equations for the stationary fields $\chi^i$, that will adapt their values to balance the centrifugal and potential forces acting on them. Therefore, at a given moment during inflation, i.e. for a particular value of $\phi$, one can view Eq. (8) as the gradient of an effective potential, whose extrema fix the values of these fields, akin to moduli stabilisation. When both terms above vanish separately, one has slow-roll slow-turn for all fields, but this is by no means necessary in the multi-field case; as we will see, negative curvature tends to induce non-geodesic motion.

Moreover, there is an attractive interpretation of the above condition provided the dependence of the inflaton velocity $\dot{\phi}$ on the orthogonal fields is proportional to $G^{\phi\phi}$. Such a dependence is suggested by the slow-roll approximation \cite{7} and will be further justified in subsequent examples. With this modulus dependence, the effective potential as defined above coincides with the total energy as captured by the Hubble parameter \cite{2} as a function of the orthogonal field values $\chi^i$. In other words, the space-time metric and the inflaton field are assumed as a fixed time-dependent background, and the orthogonal fields are subject to an energy extremization condition of the form

\[ \partial_i \left( \frac{1}{2} G_{\phi\phi}(\phi, \chi^i)(\dot{\phi})^2 + V(\phi, \chi^i) \right) = 0. \quad (9) \]

With the above proviso ($\dot{\phi} \propto G^{\phi\phi}$), the moduli dependence of the first terms is given by the inverse metric $G^{\phi\phi}$. For negative curvature manifolds, this coefficient decreases as one moves away from the geodesic solution with $\partial_i G_{\phi\phi} = 0$, opening the door for a competition between an increasing potential and decreasing kinetic energies. This intuitively explains geometric destabilization \cite{20, 23} as a simple competition of energy contributions.

**Stability.** The stability conditions for any background solution in general are determined by the eigenvalues of the full stability matrix spanned by the fields and their velocities. In order to obtain closed-form expressions we will restrict ourselves to two dimensions. For clarity we will consider the following form for the metric (any 2D metric can be diagonalized)

\[ ds^2 = G_{\chi\chi}(\phi) d\chi^2 + G_{\phi\phi}(\chi) d\phi^2. \quad (10) \]

It turns out that in the two-field cases of interest in this paper, i.e. $\epsilon' \ll 1$ and an almost frozen $\chi$ field, the stability criteria are set by the expansion of the effective potential (equivalently the Hubble parameter) at quadratic order, $\partial_i V_{\text{eff}}^{\chi}$ and an algebraic restriction on $G_{\chi\chi}$ (we refer the reader to Ref. \cite{27} for more details).

The curvature of this metric splits in two parts, $R = R^{(\phi)} + R^{(\chi)}$, parametrizing the derivative dependence on the two fields (there are no mixed derivatives $\partial_i \partial_j \chi$). In this case, the effective mass (defined as the linearization

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1 Note that the present construction differs from the adiabatic/entropic decomposition \cite{7, 20, 24, 26}, since the latter does not introduce a new coordinate system. Instead, the adiabatic direction is defined as $\sigma^2 = G_{\phi\phi} \dot{\phi}^2$.  

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FIG. 1. *Isometries of the scalar manifold induce natural trajectories for inflation, evolving along $\phi$ at fixed values of $\chi^i$. Dynamical bifurcations during inflation correspond to transitions between different trajectories.*
of $V_{\text{eff}}$, reads

$$M_{\text{eff}}^2 = V_{,\chi} + \epsilon H^2 R(\chi) + 3V_{,\chi}V_{,\chi} - 2\epsilon H^2$$ \hspace{1cm} (11)$$

in terms of the second derivative, the partial curvature and the turn rate that indicates the deviation from a geodesic trajectory. Note that this effective mass differs from the superhorizon mass of isocurvature fluctuations

$$\mu^2 = M_{\text{eff}}^2 - G V_{,\phi} V_{,\phi} + \epsilon H^2 R(\phi)$$ \hspace{1cm} (12)$$

completing the covariant derivative and the curvature. The two masses are equal when the metric has an isometry in $\phi$ and we will illustrate their difference through a specific example.

The effective mass distinguishes between physically different possibilities. For positive values, the stationary fields have stable background values corresponding to a minimum of the effective potential. This solution will function as a dynamical attractor, with the ratio $M_{\text{eff}}^2/H^2$ determining the convergence rate to the attractor. Similarly, for negative masses the background is unstable and hence will be a repeller. Finally, one can have vanishing masses. In this case the stability will depend on higher-order terms in the expansion around the background solution to Eq. (8). In the case when all higher-order terms also vanish, i.e. when the stationary condition is identically satisfied, this solution is neutrally stable and the effective potential has a flat direction.

A stable example is provided by two-field models of $\alpha$-attractors on the Poincaré disc, where the scalar potential takes a finite value at the boundary and depends on the angular direction. Such models will proceed for a prolonged number of e-folds along a slow-roll, slow-turn trajectory, giving rise to the universal predictions of $\alpha$-attractors for intermediate field-space curvature. For large negative curvature, a subsequent attractor emerges, proceeding predominately along the angular direction. Indeed one can check that the effective gradient $V_{\text{eff}}^\phi$ stabilizes the radius near the boundary of the Poincaré disc, leading to a late-time single-field attractor with non-vanishing turn rate, proceeding along a non-geodesic direction in field space.

Turning to a second example, it was recently pointed out that neutral stability can be achieved using the Hamilton-Jacobi formalism, where the scalar potential is given in terms of the Hubble parameter by $V = 3H^2 - 2H J H$. This formalism has an exact first-order solution for the scalar velocity $v^\rho = -2H J/H$ [22]. Upon adapting coordinates such that $H = H(\phi)$, one has a natural distinction between the inflationary and the stationary directions (as well as the $\dot{\phi} \propto G\dot{\phi}$ dependence). Such trajectories may be (strongly) turning, however, as the Hubble gradient may differ from the potential gradient. The latter will be non-vanishing if the metric along the inflationary direction $G\phi$ depends on the stationary directions, resulting in

$$V^\phi = -2\dot{\phi} G_\phi (H^\phi)^2$$ \hspace{1cm} (13)$$

which is equivalent to the vanishing of the effective gradient of Eq. (8). The latter is therefore identical to the vanishing of the effective gradient in the effective potential and hence Hubble parameter, which are directly related to the choice $H = H(\phi)$. This implies that the field space is spanned by adjacent trajectories. One thus has a convergence of the 2n-dimensional phase space of initial conditions to the n-dimensional hypersurface that fixes the fields’ velocity but not their positions.

The Hamilton-Jacobi class of models provides a clear illustration between the two (effective and isocurvature) mass notions in the absence of an isometry. The above discussion holds for any metric of the form of Eq. (10) and thus generates an infinite set of adjacent, non-isolated critical points for the moduli. One can check that the effective mass of Eq. (11) vanishes for such constructions, highlighting the flat directions, while the isocurvature mass will be proportional to the “missing” curvature term.

**Bifurcations.** For more general scalar potentials, the orthogonal directions will be stabilised at extrema of the effective potential of which there can be multiple; moreover, the number and stability properties of these solutions can change during inflation. The resulting bifurcations are elegantly captured by the effective potential. We will illustrate this using two characteristic examples from the recent literature.

Arguably the simplest setting that displays the bifurcation phenomenon is sidetracked inflation with quadratic potentials and negative curvature:

$$\text{d}s^2 = \left(1 + \frac{\chi^2}{L^2}\right) \text{d}\phi^2 + \text{d}\chi^2$$ \hspace{1cm} (14)$$

Inflation takes place along $\phi$ and is thus perfectly suited to the effective potential framework.

Let us first investigate the stability of the geodesic trajectory with $\rho = 0$. Particularly for quadratic potentials, both contributions to the isocurvature mass are approximately constant and read $\mu^2 = M^2 - 2m^2/(3L^2)$. Thus

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2 This can be seen as the cosmological analogue of the first-order equation that governs AdS critical points and BPS domain walls [25, 26].

3 In the special case of $\mu^2 = 0$, isocurvature modes grow on superhorizon scales at a constant rate. Combined with a constant and large turn-rate, they continuously seed the adiabatic modes outside the horizon, leading to predictions that mimic those of single-field models of inflation [14].
the curvature destabilizes the geodesic solution when $L < \sqrt{2m}/(\sqrt{3}M)$. However, for $\sqrt{3}ML \lesssim \sqrt{2m}$, subleading corrections to the isocurvature mass, consisting of the kinetic term for $\phi$ in the Hubble parameter, become important and lead to bifurcations. In particular $\mu^2(\chi = 0) < 0$ at large $\phi$ and it slowly increases as inflation proceeds along the geodesic, becoming positive at

$$\phi_{\text{cr}}^2 = \frac{4m^2}{3(2m^2 - 3L^2M^2)},$$

where we have assumed $\phi > 1$.

The subleading terms also determine the fate of the background trajectory when the geodesic solution is unstable. In addition to a local maximum, the subleading terms induce two minima in the effective potential at

$$\chi_{\pm}^2 = L \left( \frac{\sqrt{2m}}{\sqrt{3}M} - L \right),$$

for $\phi \gg \phi_{\text{cr}}$. The background trajectory will smoothly transit from the early non-geodesic trajectory at $\chi_{\pm}$ to the subsequent geodesic phase at $\chi = 0$. Fig. 2 shows the evolution of the effective gradient $V_{\phi\chi}^{\text{eff}}$ and its zero roots as $\phi$ evolves, resulting in a pitchfork bifurcation. Moreover, it is clear from the figure that the numerical trajectories converge to the geodesic solution somewhat later; this can be understood as inertia in the moduli system, and indeed the different trajectories only become geodesic when $\mu^2 \simeq H^2$ rather than 0.

A second example displaying a similar phenomenon is hyperinflation, originally formulated on the Poincaré disc

$$ds^2 = L^2 \sinh^2(\rho/L) \, d\theta^2 + d\rho^2,$$

with a spherically symmetric potential $V^{\phi\chi}$. Remarkably, inflation proceeds either along the radial gradient (for sufficiently shallow potentials) or along a spiralling trajectory (for sufficiently steep potentials). For the simple example with

$$V = \frac{1}{2} m^2 \rho^2,$$

the trajectory undergoes a transition between the two behaviours at $\rho = 2/(3L)$, after which any non-zero angular velocity will send the radial trajectory into a spiralling one. Remarkably, one can bring both these solutions to proceed along a single direction via the field redefinition

$$\cosh(\rho/L) = \cosh(\chi/L) \cosh(\phi/L),$$

$$\cot(\theta) = \coth(\chi/L) \sinh(\phi/L),$$

leading to

$$ds^2 = \cosh^2 \left( \frac{\chi}{L} \right) \, d\phi^2 + d\chi^2.$$  \hspace{1cm} (19)

This maps any spherically symmetric potential $V(\rho)$ onto a particular dependence $V(\phi, \chi)$, providing all the necessary ingredients for realizing sidetracked inflation along $\phi$.

Close to the geodesic solution along $\chi = 0$, the scalar potential reads (assuming $\phi > L$)

$$V = \frac{1}{2} m^2 \phi^2 + \frac{1}{2} m^2 \phi \chi^2.$$  \hspace{1cm} (20)

The inflaton-dependent mass for the orthogonal field provides a crucial difference with respect to the previous case, leading to a leading-term bifurcation at $\phi = 2/(3L)$ at which point the curvature bound is satisfied. At larger field values the geodesic solution is stable as the orthogonal field is strongly stabilised, while it becomes unstable at smaller field values. At this point, two new stable non-geodesic solutions come into existence, thus again making up a pitchfork bifurcation (see fig. 3).

Conclusions. Aligning our coordinate system with the inflationary trajectory, the orthogonal directions can be seen as moduli fields. In cases of interest, they are stabilised by their effective potential consisting of potential energy and generalized centrifugal forces due to non-geodesic motion. This can be interpreted as the minimisation of the total energy density given by the Hubble parameter as a function of the moduli. Moreover, as inflation proceeds, the stabilisation pattern can undergo pitchfork bifurcations, with a stable minimum becoming

\[ \text{FIG. 2. Left: The effective gradient of sidetracked inflation with } L = 0.0034, m = 1 \text{ and } M = 240 \text{ along the stationary direction } \chi \text{ for different values of } \phi, \text{ signaling the existence of one or three points of } V_{\phi\chi}^{\text{eff}} = 0. \text{ The stability of each is determined by the slope of the curve. Right: The corresponding bifurcation diagram. The black-dotted curve are the non-geodesic solutions to Eq. (3), while the colored curved correspond to numerical solutions of the background system.} \]

\[ \text{FIG. 3. Left: The effective potential gradient } V_{\phi\chi}^{\text{eff}} \text{ for hyper-inflation in the coordinates of Eq. (20) at different } \phi \text{-values with } m = 1 \text{ and } L = 0.01. \text{ Right: The corresponding bifurcation diagram. The black-dotted curve are the non-geodesic solutions to Eq. (3), while the colored curves correspond to numerical solutions of the background system.} \]
angular inflation has a unique minimum of narios of multi-field inflation in curved geometries. While angular inflation has a unique minimum of \( V_{eff} \) along the moduli direction, both sidetracked and hyperinflation exhibit dynamical pitchfork bifurcations when formulated in this framework. This instability is therefore intrinsically of the same nature; analyzing hyperinflation after the coordinate transformation of Eq. 19 makes it a special case of sidetracked inflation 11. This connects two models that were so far thought to be distinct, thus underlining the unifying nature of our approach. Moreover, it demonstrates that the conservation of angular moment is not essential to the bifurcation in hyperinflation; indeed, non-spherically symmetric potentials \( V(\rho, \theta) \) can give rise to the same regime 17.

While we have concerned ourselves mainly with the description of the attractors, it is also interesting to wonder about the approach to the attractor. In the \( \alpha \)-attractor set-up with a regular potential at the boundary of moduli space, inflation will (first) proceed along the gradient of the scalar potential, leading to a family of slow-roll slow-turn trajectories. In the context of the present formalism, this is a transient behaviour. For stronger curvatures, however, this phase is followed by angular inflation which can proceed for a large number of e-folds. The onset of the attractor is similar for sidetracked and hyperinflation; for instance, if the former one also has slow-roll slow-turn in the quadratic potential before arriving at the \( \rho = 0 \) attractor with the subsequent bifurcation. The present formalism captures in an elegant and universal way how to describe the latter attractor phases along a one-dimensional trajectory in phase space.

We have also highlighted the special role by the Hamilton-Jacobi construction of the scalar potential in this set-up. This always leads to a cancellation between the gradient and centrifugal forces, at any value of the moduli, and hence gives rise to a set of adjacent, neutrally stable critical points. The attractor is therefore a two-dimensional hypersurface in this special case. Moreover, the presence of such flat directions is signalled by the vanishing of the effective mass (by construction) and not necessarily of the isocurvature mass.

Another open question that we leave for future research is how the effective potential for the orthogonal directions determines the evolution of fluctuations, and hence predictions of inflation. The present formalism appears to be ideally suited for this analysis due to its decomposition, which carries over to the (adiabatic and isocurvature) fluctuations. It therefore should present a unified perspective on the recent perturbation analyses performed for hyperinflation 15, 16, 18. sidetracked 13, 34 and orbital inflation 10, 14. It would be interesting to investigate whether the analogy to hybrid inflation can be extended beyond the background evolution, providing distinct observational signatures 35, 37 for multi-field models exhibiting pitchfork bifurcations during inflation.

Acknowledgments. The authors gratefully acknowledge stimulating discussions with Ana Achúcarro, Cliff Burgess, Michele Cicoli, Sonia Paban, Robbie Rosati, and Vincent Vennin as well as financial support from the Dutch Organisation for Scientific Research (NWO).

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4 Here by “sidetracked” we refer to models that admit two approximate solutions in addition to an exact solution at the minimum of the “heavy” field potential.
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