Quantitative Assessment of the Toner and Tu Theory of Polar Flocks

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We present a quantitative assessment of the Toner and Tu theory describing the universal scaling of fluctuations in polar phases of dry active matter. Using large scale simulations of the Vicsek model in two and three dimensions, we find the overall phenomenology and generic algebraic scaling predicted by Toner and Tu, but our data on density correlations reveal some qualitative discrepancies. The values of the associated scaling exponents we estimate differ significantly from those conjectured in 1995. In particular, we identify a large crossover scale beyond which flocks are only weakly anisotropic. We discuss the meaning and consequences of these results.

Two seminal papers, both published in this journal in 1995, can be argued to mark the birth of active matter physics. In [1], Vicsek and collaborators introduced their simple model for collective motion, where XY spins fly at constant speed along their magnetic direction. In [2], Toner and Tu (TT) wrote down fluctuating hydrodynamic equations for this flying XY model and performed a dynamic renormalization group calculation of its ordered phase, concluding, among other things, that such polar flocks possess true long-range orientational order even in two space dimensions (2D). In other words, flying spins defy the famous Mermin-Wagner theorem [3].

But the TT papers remain influential even though they subjected to local alignment in the absence of any surrounding fluid [15]. In particular, the TT theory (and related works by Ramaswamy et al.) predicted what has become one of the most popular features in active matter studies, the presence, in orientational-ordered phases, of “giant number fluctuations” where the variance of the number of particles in sub-systems of increasing size scales faster than the mean [16–24].

Over the years, numerous numerical and experimental works have tried to verify the TT results, but the evidence presented has been restricted to a limited range of scales [25] and/or isotropic measures averaged over all spatial directions that cannot resolve individual scaling exponents [19, 23, 26], resulting in exponent values that could only be deemed compatible with the TT predictions. This situation was satisfactory as long as the TT theory was believed, as claimed in the early papers [2, 27], to be ‘exact at all orders’ in 2D, the dimension of choice of most works. However, Toner himself realized in 2012 [28] that this is not actually true, and that a number of important terms has been overlooked, invalidating most claims of exactness. The remarkable result of true long-range order in 2D remains valid, as well as the overall structure of the theory, but scaling exponent values, and other important features, had to be ‘revisited’.

From then on, belief in the TT results became reliant on the partial numerical evidence mentioned above. In spite of this situation, not much further work was devoted to gauge the accuracy of the TT predictions (see however [29, 30]), and a full-fledged, quantitative evaluation of the TT theory is still missing.

In this Letter, we present large-scale numerical simulations of the Vicsek model designed to study the 2D and 3D anisotropic space-time correlations functions at the heart of TT theory. Our results largely confirm its qualitative validity, but our estimates of exponent values clearly differ from the conjectured ones. In particular, we find that anisotropy is weak, possibly vanishing. Moreover, the behavior of density correlations shows qualitative discrepancies with the theory. We discuss their origin, as well as the theoretical consequences of the hyperscaling relations that we find numerically satisfied.

We start with a synthetic account of the TT theory. The hydrodynamic equations written by Toner and Tu govern a conserved density ρ and a velocity field v:

\[ \partial_t \rho + \nabla \cdot (\rho v) = 0, \]  
\[ \partial_t v + \lambda_1 (v \cdot \nabla) v + \lambda_2 (\nabla \cdot v) v + \lambda_3 |v|^2 v - \nabla P + D_0 \nabla^2 v + D_1 \nabla (\nabla \cdot v) + D_2 (v \cdot \nabla)^2 v + f \]  

Here all coefficients can in principle depend on ρ and |v|, the pressure P is expressed as a series in the density, and f is an additive noise with zero mean and variance Σ delta-correlated in space and time. To obtain the quantities of interest hereafter, i.e., correlation functions of density and transverse velocity fluctuations, Eqs. are linearized around the homogeneous ordered solution:
\[ \rho = \rho_0 + \delta \rho \text{ and } v = (v_0 + \delta v_0) \hat{e}_i + \delta v \perp, \text{ with } \rho_0 \text{ the global density and } v_0 = \sqrt{\alpha/\beta}. \] (Hereafter subscripts \( i \) and \( \perp \) refer respectively to directions longitudinal and transverse to global order.) After enslaving the fast field \( \delta v_0 \), the Fourier-transformed slow fluctuations read, in the small \( q = |q| \) limit \[28\]:

\[
\langle |\delta \rho(\omega, q)|^2 \rangle = \frac{\rho_0^2 \Sigma}{S(\omega, q)} q^2, \tag{2a}
\]

\[
\langle |\delta v \perp(\omega, q)|^2 \rangle = \frac{\Sigma(\omega - v_2 q \parallel)^2}{S(\omega, q)} + \frac{\Sigma(d - 2)}{S_T(\omega, q)}, \tag{2b}
\]

where \( S(\omega, q) = [(\omega - c_+ (\theta q)^2 + \varepsilon^2(q))[(\omega - c_-(\theta q)^2 + \varepsilon^2(q)], S_T(\omega, q) = (\omega - c_T(\theta q)^2 + \varepsilon^2(q)] \), with \( \theta q \) the angle between global order and \( q_\perp = |q| \). The definitions of \( v_2, \gamma, c_+ , c_T \), \( \pm \varepsilon \) which are unimportant for the following discussion, can be found in \[28\].

Eq. (2a) implies the existence of propagative sound modes, or density waves, whose dispersion relations follow \( \omega \mp (q) = c_\pm (q)/q \). This endows density fluctuations \( \langle |\delta \rho(\omega, q)|^2 \rangle \) with two sharp peaks in \( \omega \) centered in \( c_\pm (q) \) and of respective widths \( \varepsilon \pm (q) \). The two terms of the rhs of Eq. (2b) correspond respectively to transverse velocity fluctuations parallel and perpendicular to \( q_\perp \). The first term represents correlations of \( v_\parallel = \delta v \perp \cdot \hat{q}_\perp \), which behave like the density fluctuations. The second term denotes the fluctuations of \( v_T = \delta v \perp - v_\parallel \hat{q}_\perp \), which exist only for \( d > 2 \), and yields a third peak centered in \( c_T(q) \), of width \( \varepsilon_T(q) \).

Since \( \varepsilon_\pm , c_T(q) \) essentially scale as \( q^2 \) in the small wavenumber limit \[28\], the equal-time correlation functions are easily obtained by integrating Eqs. (2) over \( \omega \).

The resulting expressions, presented in \[28\], imply that \( \langle |\delta v \perp(\omega, q)|^2 \rangle \approx q^{-2} \) when \( q \to 0 \), whose primary consequence is the absence of long-range order in \( d \leq 2 \).

However, nonlinearities in Eqs. (1) are relevant perturbations for all \( d \leq d_c = 4 \) \[28\]. Correlation functions in the nonlinear theory are then given by Eqs. (2) using the renormalized noise variance and sound modes dampings

\[
\Sigma^* = q^2 f_{\Sigma}(q_\parallel/q_\perp^2), \quad \varepsilon^* \perp = q^2 f_{\mp}(q_\parallel/q_\perp^2), \quad (3)
\]

with \( \zeta \equiv d - 1 + 2\chi + \xi, \quad f_{\Sigma \pm}(x) = 0(1) \) for \( x \to 0 \), \( f_{\Sigma}(x) \sim x^{(d-\zeta)/\xi} \) and \( f_{\mp}(x) \sim x^{\zeta/\xi} \) when \( x \to \infty \), while the sound speeds \( c_{\pm/T}(q) \) remain those given by the linear theory.

Exponents \( \chi, \xi \) and \( z \) and scaling functions \( f_{\Sigma \pm}(x) \) are universal. The roughness exponent \( \chi \) rules how the variance of density and density fluctuations varies with length scales. Fluctuations vanish asymptotically when \( \chi < 0 \), insuring long-range polar order. Toner and Tu’s calculations proved that this is true for \( d = 2 \) and 3, while in linear theory, where \( \chi = 1 - d/2 \), fluctuations diverge and order is destroyed in 2D. The anisotropy exponent \( \xi \) measures the difference in scaling along and transversally to global order. TT theory predicts that fluctuations scale anisotropically for \( d < d_c = 4 \) \((\xi < 1)\) while in mean-field \( \xi = 1 \). Finally, the dynamical exponent \( z \) measures how the lifetime of sound modes scales with system size. At the linear level \( z = 2 \), which corresponds to a diffusive damping, while \( z < 2 \) is expected for \( d < 4 \) according to TT theory. In their first publications \[27\], Toner and Tu claimed an exact computation of these exponents in \( d = 2 \), and found \( \chi = (3 - 2d)/3, \quad \xi = z/2 = (d + 1)/5 \) (see TT95 numbers in Table I). In his later “reanalysis” of the theory \[28\], Toner realized that additional relevant nonlinearities were missed, so that the above exponent values could only be exact, even in \( d = 2 \), under the conjecture of the asymptotic irrelevance of these terms.

We now turn to our numerical assessment of TT theory. We use the standard discrete-time Vicsek model for efficiency. Particles \( i = 1, \ldots, N \) with position \( r_i \) and orientation \( \hat{e}_i \) move at constant speed \( v_0 \) and align their velocities with current neighbors \( j \):

\[
\mathbf{\hat{e}}_{i}^{t+1} = \vartheta \left( \left( \mathbf{\hat{e}}_{j}^{t} \right)_{j \neq i} + \eta \mathbf{\xi}_{i}^{t} \right), \quad r_{i}^{t+1} = r_{i}^{t} + v_{0} \mathbf{\hat{e}}_{i}^{t+1}, \tag{4}
\]

where \( \vartheta[\mathbf{u}] = \mathbf{u}[\mathbf{u}] \langle \mathbf{u} \rangle_{j \neq i} \) is the average over all particles \( j \) within unit distance of \( i \) (including \( i \)), and \( \mathbf{\xi}_{i}^{t} \) are uncorrelated random vectors uniformly distributed on the unit circle(2D)/sphere(3D) \[31\]. Square domains of linear size \( L \) containing \( N = \rho_0 L^d \) particles, with \( N \) ranging from a few million to a few billion were considered. For numerical efficiency, small speed and weak noise were avoided. We used \( v_0 = 1, \eta = 0.5 \) (2D) and 0.45 (3D) with \( \rho_0 = 2 \), parameter values in the homogeneous ordered phase, but not too deep inside. Fluctuation fields \( \delta \rho \) and \( \delta v \perp \) were obtained by coarse-graining over boxes of unit linear length. The associated correlation functions were simply obtained by computing the square norm of the fields’ Fourier transform.

In finite systems with periodic boundary conditions, the direction of order diffuses slowly (the diffusion constant \( \sim 1/N \) \[32\]). To estimate quantities scaling anisotropically like those defined by Eqs. (2), one then needs, before averaging in time, to rotate a copy of the system at each measure so that global order remains along a chosen direction. Moreover data have then to

### TABLE I. Exponent values conjectured by Toner and Tu in [2]

| \( d = 2 \) | \( d = 3 \) | \( d \geq 4 \) |
|--------|--------|--------|
| TT95 numerics | TT95 numerics | mean-field |
| \( \chi \) | \(-0.20 \) | \(-0.31(2) \) |
| \( \xi \) | \(-0.60 \) | \(0.95(2) \) |
| \( \zeta \) | \(0.80 \) | \( \approx 1 \) |
| \( z \) | \(1.00 \) | \(0.80 \) |
| 2 | \(0.80 \) | \(1.77(3) \) |
| \( 1+2/d \) | \(1.53 \) | \(1.59(3) \) |
be averaged over times longer than the timescale of this rotation. This is possible but costly and quickly becomes prohibitive for large systems. Forcing global order to remain along a given direction can be achieved by either applying an external field or by imposing reflecting side boundaries as in, e.g., [25] [29] [30]. This perturbs slightly the global behavior of the system, but allows for much shorter averaging times at equivalent sizes. All three protocols were tested, and we found that when used cautiously they yield identical results over the scales that can be explored by all (see [33] for details). Below, we only present data obtained using a channel with reflective walls.

Although we measured correlations in the whole \((q_\parallel, q_\perp)\) plane [34], exponents \(\chi\) and \(\xi\) can be estimated from just the longitudinal \((q_\perp = 0)\) and transverse \((q_\parallel = 0)\) directions. For velocity correlations, we have:

\[
\langle |\delta v_\perp(q)|^2 \rangle \sim \begin{cases} q_\perp^{-\chi} & \text{for } q_\perp \gg q_\parallel, \\ q_\parallel^{-\chi/\xi} & \text{for } q_\parallel \gg q_\perp^4 \end{cases}
\]

Our data in both 2D and 3D show that \(\langle |\delta v_\perp(q)|^2 \rangle\) scales cleanly at small values of \(q_\perp\) (lower sets of curves in Figs. [1]a,b)), with estimated values of \(\xi\) slightly but significantly different than those conjectured by Toner and Tu (see Table I). Behavior in the longitudinal direction is more surprising (upper set of curves in Figs. [1]a,b)). While from [27] a divergence for \(q_\parallel \to 0\) with an exponent \(-2\) is conjectured in both 2D and 3D, we observe, in 2D, a size-independent crossover from a power law with exponent \(-1.65\) at intermediate values of \(q_\parallel\) to one with a larger exponent \(-1.4\) at smaller \(q_\parallel\). The crossover scale \(\ell_c = 2\pi/q_\parallel \approx 100\), indicated by the purple dashed lines in our figures, is of the same order as typical sizes considered so far in other works [19, 25], which may explain why it has never been reported. Note further that our post-crossover estimate \(-1.4\) is not far from the \(-1.33\) value measured in the transverse direction, implying weak, possibly vanishing, anisotropy \((\xi \approx 0.95)\). In 3D the two correlation functions show approximately the same exponent above a scale \(\ell_c \approx 30\): scaling is isotropic (Fig. [1]b)). Overall, our measures lead to values of \(\chi\) and \(\xi\) in clear departure from those conjectured by Toner and Tu (see Table I). The density correlation function is expected to show the following longitudinal and transverse scalings [33]:

\[
\langle |\delta \rho(q)|^2 \rangle \sim \begin{cases} q_\perp^{-\chi} & \text{for } q_\perp \gg q_\parallel, \\ q_\parallel^{-2-\chi/\xi} & \text{for } q_\parallel \gg q_\perp^4 \end{cases}
\]

In the transverse direction, our data confirm that scaling takes place with the same exponent as for velocity correlations (Figs. [1]c,d), lower set of curves), albeit with more pronounced finite size effects, especially in 2D (compare insets of Fig. [1]a,b) and Fig. [1]c,d), see [33] for comments).

In 3D, the apparent exponent is slightly lower in absolute value than the one given by \(\langle |\delta v_\perp(q)|^2 \rangle\) \((-1.73\) vs. \(-1.77)\), but given the limited range of scaling available we cannot exclude that these two values are in fact the same asymptotically.

The scaling of density fluctuations in the longitudinal direction is more subtle to analyse because it depends explicitly on \(q_\perp\) (see Eq. [6]). The behavior of \(\langle |\delta \rho(q)|^2 \rangle\) with \(q_\parallel\) for 3 fixed values of \(q_\perp^*\) is shown in Figs. [1]c,d) (upper sets of curves). One can identify three regimes below the crossover scale \(q_c = 2\pi/\ell_c\), which are most easily distinguished in 2D, but probably also present in 3D. For the smallest values of \(q_\parallel\), the functions reach a plateau, whose range of existence and amplitude respectively increases and decreases with \(q_\perp^*\). This behavior corresponds to the “transverse” regime where \(q_\parallel \ll q_\perp\).
Increasing $q_\parallel$ beyond this plateau, $\langle |\delta \rho(q)|^2 \rangle$ shows a second scaling behavior with $q_\perp^{-\mu}$-dependent amplitude, in qualitative agreement with Eq. (4). Finally, in 2D where sufficiently large systems can be studied, a third scaling region is observed, with slow (exponent $\sim -0.7$), $q_\parallel$-independent decay whose range increases when $q_\perp \to 0$. Such a regime is absent from the framework of TT theory.

The second regime also departs strikingly from the Toner-Tu results. In this region both 2D and 3D curves do not collapse when their amplitude is rescaled by $q_\perp^{-\mu}$ with $\mu = 2$, as predicted exactly by TT theory, but rather with $\mu \approx 1$ in 2D and 0.5 in 3D (Fig. 1(e,f)). Moreover, the collapsed curves do not decay with exponent $-2 - \zeta/\epsilon \approx -3.4$ (2D) and $-3.77$ (3D) as predicted by Eq. (4) using the values of $\chi$ and $\zeta$ determined from $\langle |\delta v_\perp(q)|^2 \rangle$. Rather, we find $-2.4$ in 2D and $-2.27$ in 3D. Our data therefore suggest that for $q_\parallel \gg q_\perp^{\chi}$, $\langle |\delta \rho(q)|^2 \rangle \sim q_\perp^{\mu - \zeta/\epsilon}$ with $\mu \approx 1$ in 2D and 0.5 in 3D.

In order to assess the dynamical exponent $z$, we now turn to the study of space-time correlations. As expected from Eqs. (2) and previous work in 2D [24, 30], both $\langle |\delta \rho(\omega, q)|^2 \rangle$ and $\langle |\delta v_\perp(\omega, q)|^2 \rangle$, as functions of $\omega$, show two asymmetric peaks that become symmetric in the transverse direction ($\theta_q = \pi/2$). In 3D, one observes the emergence of an additional third peak in $\langle |\delta v_\perp(\omega, q)|^2 \rangle$ coming from its component $v_T$. All these peaks are well fitted close to their maximum by Cauchy distributions of the type $H_{\epsilon,\delta,T}(q)/[1 + (\omega - \omega_{*,\epsilon,T}(q))^2/\Delta \omega_{*,\epsilon,T}(q)]^2$, where $H_{\epsilon,\delta,T}(q)$, $\omega_{*,\epsilon,T}(q)$ and $\Delta \omega_{*,\epsilon,T}(q)$ respectively account for their heights, positions and half-peak widths (see data in [33]). Since we have seen that density correlations seem more sensitive to finite-size effects, we now focus on velocity correlations for the quantitative characterization of the peaks. As expected peak positions $\omega_{*,\epsilon,T}(q)$ scale linearly with $q$ in the limit $q \to 0$, and the sound speeds $c_{*,\epsilon,T}(\theta_q)$ are given by the corresponding slopes. Perfect agreement is found with the linear theory, both in 2D and 3D [32].

Peak widths, on the other hand, show non-trivial scaling: $\Delta \omega_{*,\epsilon,T}(q)$ correspond to the dampings $\epsilon_{+,-}(q)$ and thus from Eq. (3) are expected to scale as $q_\perp^{\epsilon/\epsilon}$ and $q_\perp^{\epsilon/\epsilon}$ in the longitudinal and transverse directions. We find rather good scaling in 2D for both longitudinal and transverse directions (Fig. 2(a)), with, in this last case, $z \approx 1.33$. In the longitudinal direction, we find weak evidence of a crossover at the same scale $\ell_c$ as for equal-time correlations. Below $\ell_c$, the estimated value of $z/\ell_c (1.65)$ is identical to that of $\zeta/\ell_c$ found below $\ell_c$ in Fig. 1(a). Beyond $\ell_c$, we unfortunately could not obtain much data, but the few points we have are compatible with a slope 1.4, i.e. the asymptotic value of $\zeta/\ell_c$ found from Fig. 1(a). In 3D, where the data is much more limited, we can nevertheless observe good scaling of the peak width $\Delta v_T(q)$ over almost a decade in the transverse direction $q_\parallel = \pi$, yielding the estimate $z \approx 1.77$ (see Fig. 2(b)), identical to our estimate of $\zeta$ from equal-time correlations. Results leading to similar values of $z$ in 2D and 3D are found from the scaling of the peaks heights, see [33] for details.

In summary, using the Vicsek model, we have measured independently the values of the three universal exponents $\chi$, $\zeta$ and $\epsilon$ that characterize long-range correlations of density and velocity fluctuations in polar flocks, and found them incompatible with those conjectured by Toner and Tu in [2] (see Table I). These differences indicate that at least some of the nonlinearities identified in [25] and neglected in the original calculation are indeed relevant asymptotically.

Our data suggest in particular the existence of a crossover scale beyond which $-i.e.$ at scales scarcely explored before—there is very little or vanishing anisotropy. Coming back to the popular giant number fluctuations, we find that $\zeta/d$, which governs their scaling, varies very little across scales, and takes values close to those predicted by Toner and Tu, (see Table I). This clarifies why previous studies focusing on this quantity could not challenge the Toner and Tu conjecture [19, 25, 26].

We find identical estimates, within our numerical accuracy, of $\zeta$ and $z$. In other words, the hyperscaling relation $z = d - 1 + 2\chi + \xi$ seems satisfied. If we take this numerical fact for granted, it implies that, somewhat counterintuitively, the vertices responsible for the departure from the 1995 TT results are not those coupling density and order. Moreover, $\zeta = z$ also implies that the noise variance $\Sigma$ does not renormalize, so that the dominant effective noise in the v-equation is indeed additive, as assumed in TT theory (see [33] for the simple arguments leading to these conclusions).

We also find qualitative discrepancies with TT theory in the longitudinal behavior of density correlations [36]. We have at present no full understanding of this, but, as explained in a forthcoming publication, the consideration
of a (conserved) additive noise in the density equation — something quite natural in the context of fluctuating hydrodynamic equations — leads to a modified form of Eq. \(q_⊥→0\) in the \(q_⊥→0\) sector. This change already occurs at the linear level and could account, upon renormalization, for the peculiar scaling regimes reported in Fig.1(c-f).

All in all, our numerical results, even though they clearly rule out the Toner and Tu 1995 predictions, call even more than before for a complete, possibly non-perturbative, renormalization group approach \[^{[7]}\].

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[33] See supplementary information at . . .
[34] In 3D we averaged over all directions of \(q_⊥\).
[35] A third intermediate scaling region with \(\langle|\delta\rho(q)|^2\rangle \sim q^{-2}q^{-2}q^{-2}\) is expected for \(q_⊥ \gg q_\parallel \gg q_\perp\) \[^{[27]}\], but given our estimate \(\xi \approx 1\) both in 2D and 3D, we expect it to be unobservable, at odds with the earlier results of Ref. \[^{[24]}\].
[36] Note that our 2D results would be in disagreement with TT theory under the hypothesis that the three scaling
regimes observed would correspond to those predicted (see Eq. (6) and the intermediate one mentioned in 35). In particular, the third scaling regime we identify is independent of $q_\perp$.

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