ABSTRACT. Researchers have formalized reinforcement learning (RL) in different ways. If an agent in one RL framework is to run within another RL framework’s environments, the agent must first be converted, or mapped, into that other framework. Whether or not this is possible depends on not only the RL frameworks in question and but also how intelligence itself is measured. In this paper, we lay foundations for studying relative-intelligence-preserving mappability between RL frameworks. We define two types of mappings, called weak and strong translations, between RL frameworks and prove that existence of these mappings enables two types of intelligence comparison according to the mappings preserving relative intelligence. We investigate the existence or lack thereof of these mappings between: (i) RL frameworks where agents go first and RL frameworks where environments go first; and (ii) twelve different RL frameworks differing in terms of whether or not agents or environments are required to be deterministic. In the former case, we consider various natural mappings between agent-first and environment-first RL and vice versa; we show some positive results (some such mappings are strong or weak translations) and some negative results (some such mappings are not). In the latter case, we completely characterize which of the twelve RL-framework pairs admit weak translations, under the assumption of integer-valued rewards and some additional mild assumptions.

1. INTRODUCTION

If we changed the rules, would the wise become fools? In reinforcement learning (RL), agents and environments interact. When an agent interacts with an environment, the agent and the environment take turns. On the agent’s turn, the agent chooses an action (or rather, a probability distribution over a set of actions according to which an action is selected at random). The action so chosen is thereupon transmitted to agent and environment. On the environment’s turn, the environment chooses a percept to send to the agent in response (or, as before, the environment chooses a probability distribution over a set of percepts according to which a percept is selected at random). The percept so chosen is likewise transmitted to agent and environment. Each percept includes a numerical reward and an observation. The agent’s objective is to learn to act in its environment in order to maximize its rewards.

This is all simple enough, but there are many different ways to formally represent RL. These different representations can be organized according to answers to key questions, such as who goes first, what actions are permitted, what observations are allowed, and how numerical rewards are issued. Implicit in treatments of RL is that answers to these questions are inconsequential. Is this so? If answers to
such questions are inconsequential to problems for reinforcement learning, then evaluations of agent performance — measures of their relative intelligence — should be invariant with respect to transformations on the different RL representations which answer them.

The present paper develops techniques for understanding the extent to which scales for measuring agent intelligence are invariant to different RL representations. Its analysis elaborates on what means for one agent to perform better, or worse, relative to another agent. To this end, the method we use is based on the idea proposed in [1] that agents can be compared by considering them to compete in a certain “election.”

A primary objective of this paper is to get a foothold on mapping out RL frameworks using two notions of RL translation, distinguishing between so-called weak and strong translations from one framework to another (§3). For any two frameworks, such translations may or may not exist from one to the other. We establish the existence or non-existence of such translations for certain frameworks of special interest (§4).

This paper is limited in scope, its objective being but to begin mapping out a jungle of variations of reinforcement learning. In particular, we confine our attention to variations answering (1) whether the agent or the environment goes first (§5), and (2) whether agents or environments (or both) are required to be deterministic (§6).

2. Reinforcement Learning in Abstract

In this section, we tackle the challenge of effectively defining the subject matter under study. Our goal is to outline an abstract conception of reinforcement learning framework to facilitate comparison between deterministic and non-deterministic RL, as well as between agent-first and percept-first RL. These two dichotomies are a natural testing ground for studying mappings (or lack thereof) between RL frameworks. And they are both important in their own right, too

- Theoretical RL often assumes genuine randomness. But in practice, RL is often implemented using pseudo-random numbers generators, which are not truly random. Thus, maps between deterministic and non-deterministic RL play an under-appreciated role in the interface between theory and practice.
- It might be presumed to be inconsequential whether agents go first or whether environments go first, but our results demonstrate that, warranting caution.

We choose a formulation of RL not for maximal abstraction. The developments of this paper lay the groundwork for studying every more exotic RL abstracta.

**Definition 2.1.** Fix a background set \( \mathcal{A} \) of actions (with \( |\mathcal{A}| > 1 \)), a background set \( \mathcal{P} \) of percepts (with \( |\mathcal{P}| > 0 \)), and a function \( R : \mathcal{P} \to \mathbb{R} \) that assigns a real-valued reward to each percept. We assume \( \mathcal{A} \cap \mathcal{P} = \emptyset \). We usually write actions as \( x \) (or \( x_i \)) and percepts as \( y \) (or \( y_i \)).

At first glance Definition 2.1 might seem biased toward RL formalizations with a fixed action-set and percept-set (as opposed to RL formalizations where the action-set and percept-set depend on the environment). But our definition of RL frameworks below will not require that individual environments recognize the full \( \mathcal{A} \) or
\( \mathcal{P} \). Thus, \( \mathcal{A} \) can be taken to be the universe of all actions over all the environments one has in mind, even if no particular environment recognizes that full set. Likewise for \( \mathcal{P} \). Often in the literature, percepts are reward-observation pairs \( (r, o) \) (or \(^1\) observation-reward pairs \( (o, r) \)) where \( r \in \mathbb{R} \) (or \( r \in \mathbb{Q} \)), and where \( o \) comes from some observation-set, and implicitly \( R((r, o)) = r \) (or \( R((o, r)) = r \)). By abstracting the function \( R \) out, we avoid committing to particular implementation details (a similar device is used in [8]).

**Definition 2.2.** A **history** is a finite (possibly empty) sequence of the form \((s_0, \ldots, s_n)\), each \( s_i \in \mathcal{A} \cup \mathcal{P} \), which alternates between actions and percepts. We consider the histories to form a tree, with root \( \langle \rangle \) the empty history and each \((s_0, \ldots, s_{n+1})\) a child of \((s_0, \ldots, s_n)\). We extend \( R \) to histories by defining \( R(h) = R(y) \) if \( h \) ends with percept \( y \in \mathcal{P} \), \( R(h) = 0 \) if \( h = \langle \rangle \) or \( h \) ends with an action \( x \in \mathcal{A} \).

For example, if \( x_1, x_2 \in \mathcal{A} \) and \( y_1, y_2 \in \mathcal{P} \) then \( \langle \rangle, \langle x_1 \rangle, x_1y_1x_2, x_1y_1x_2y_2, \langle y_1 \rangle, y_1x_1, y_1x_1y_2, \) and \( y_1x_1y_2x_2 \) are histories. But \( x_1x_2 \) is not a history, since it does not alternate between actions and percepts.

Let \( \bowtie \) denote concatenation.

In the following definition, we define agents (resp. environments) to be functions that output probability distributions on sets of actions (resp. percepts). At first glance this might seem biased toward non-deterministic RL formalizations. But one can simply consider a deterministic agent to be an agent whose output (in response to input \( h \)) assigns probability 1 to some action (dependant on \( h \)); likewise, one can consider a deterministic environment to be an environment whose outputs assign probability 1 to some percept.

**Definition 2.3.** An **RL pre-framework** is a triple \( \mathcal{F} = (\mathcal{A}, \mathcal{E}, \mathcal{H}) \) where:

1. \( \mathcal{H} \) is an infinite leafless subforest of the tree of histories (its nodes are the histories of \( \mathcal{F} \)) which includes \( \langle \rangle \).
2. One of the following is true:
   - For all nonempty \( h \in \mathcal{H} \), \( h \) ends with an action iff \( h \) has even length (if so, \( \mathcal{F} \) is **environment-first**, and we write \( \mathcal{H}_E \) for the set of even-length histories in \( \mathcal{H} \) (including \( \langle \rangle \)), \( \mathcal{H}_A \) for the set of odd-length histories in \( \mathcal{H} \)).
   - For all nonempty \( h \in \mathcal{H} \), \( h \) ends with an action iff \( h \) has odd length (if so, \( \mathcal{F} \) is **agent-first**, and we write \( \mathcal{H}_A \) for the set of even-length histories in \( \mathcal{H} \) (including \( \langle \rangle \)), \( \mathcal{H}_E \) for the set of odd-length histories in \( \mathcal{H} \)).
3. \( \mathcal{A} \) is a nonempty set of functions (called **agents**) with domain \( \mathcal{H}_A \), such that for each \( \pi \in \mathcal{A} \), for each \( h \in \mathcal{H}_A \), \( \pi(h) \) is a finitely-supported probability distribution on \( \{ x \in \mathcal{A} : h \bowtie x \in \mathcal{H} \} \).
4. \( \mathcal{E} \) is a nonempty set of functions (called **environments**) with domain \( \mathcal{H}_E \), such that for each \( \mu \in \mathcal{E} \), for each \( h \in \mathcal{H}_E \), \( \mu(h) \) is a finitely-supported probability distribution on \( \{ y \in \mathcal{P} : h \bowtie y \in \mathcal{H} \} \).

\(^1\)For example Legg and Veness [14] explicitly implement percepts in both of these two forms, separately, in order to experimentally check whether some empirical results depend on which form is used.
(5) For every \( \pi \in A \), for every finite set \( S_1 \subseteq H_A \), for every infinite set \( S_2 \subseteq H_A \), there exists some \( \rho \in A \) such that \( \rho(h) = \pi(h) \) for all \( h \in S_1 \) but \( \rho(h) \neq \pi(h) \) for some \( h \in S_2 \).

Note that for any RL pre-framework \( \mathcal{F} = (A, E, H) \), condition 2 of Definition 2.3 implies \( H_A \cap H_E = \emptyset \).

Condition 5 of Definition 2.3 is needed to later ensure that identity functions and inclusion functions are translations. The condition is a sort of free-will condition, excluding systems where finitely many past decisions force infinitely many future decisions. Indeed, the condition would make sense for RL even if \( S_2 \) was a singleton and \( S_1 \) was empty (we require \( S_2 \) infinite in order to weaken Condition 5 for sake of greater generality). For example in chess, if Black puts White in check and thereby forces White’s next move, then in some sense White’s turn in which White is forced to make that move is not really White’s turn at all. Chess could be modified so that Black moves White’s piece for White in that situation and then proceeds directly to Black’s next move, all in one turn, and this would not really alter the game of chess at all.

**Definition 2.4.** Suppose \( \mathcal{F} = (A, E, H) \) is an RL pre-framework, \( \pi \in A \) is an agent, and \( \mu \in E \) is an environment.

1. For all \( t \in \mathbb{N} \), let \( V_{\mu,t}^\pi \) (the expected total reward when \( \pi \) interacts with \( \mu \) for time \( t \)) be the expected value of \( \sum_{i=0}^{t} R(h_i) \), the sum of the rewards in the \( H \)-path \((h_0, \ldots, h_t)\) randomly generated as follows:
   - (a) \( h_0 = \langle \rangle \).
   - (b) If \( h_i \in H_A \) then \( h_{i+1} = h_i \bowtie x \), where \( x \in A \) is randomly generated based on the probability distribution \( \pi(h_i) \).
   - (c) If \( h_i \in H_P \), then \( h_{i+1} = h_i \bowtie y \) where \( y \in \mathcal{P} \) is randomly generated based on the probability distribution \( \mu(h_i) \).
2. We define \( V_{\mu}^\pi = \lim_{t \to \infty} V_{\mu,t}^\pi \) (when the limit converges to a finite real number). We understand \( V_{\mu}^\pi \) to be the expected total reward when \( \pi \) interacts with \( \mu \).

**Definition 2.5.** An **RL framework** is an RL pre-framework \( \mathcal{F} = (A, E, H) \) satisfying the following additional condition. For every environment \( \mu \in E \), for every agent \( \pi \in A \), \( V_{\mu}^\pi \) exists.

Definition 2.5 is flexible enough to encompass many variations of RL, such as:

- Variations where agents are deterministic but environments need not be (as in [11] or [18]).
- Variations where neither agent nor environment need be deterministic (as in [13]).
- Variations where each percept also includes a true-or-false flag indicating whether or not the percept signals the start of a new “episode” (as in [7] or [19])—simply assume a background function \( NewEp \) like the background function \( R \) from Definition 2.1.
- Variations where rewards are more restricted, e.g. to \( Q \) or (as in [13]) some finite subset of \( Q \)—just assume further restrictions on \( R \) in Definition 2.1.
• Variations where available actions vary from turn to turn (as in [19] or [9])—just restrict $H$ accordingly.
• Percept-first variations where available actions vary from environment to environment (as in [7])—this is a special case of the previous bullet.
• Variations where environments are Markov decision processes (as in most of [19])—just restrict $E$ accordingly.
• Variations where environments and/or agents must be computable—restrict $A$ and/or $E$ accordingly.

Other variations of RL will require, in future work, more general RL framework notions (but the ideas in this paper serve as a template for how one can explore intelligence preservation results in those more general frameworks). Some such variations include:

• RL with multiple reward-signals.
• RL with multiple agents [16, 5, 20, 15, 22, 10].
• Preference-based RL [21].
• RL with rewards from non-Archimedean number systems allowing infinitary or infinitesimal rewards (suggested by [2], implicitly suggested by [23], and conspicuously not ruled out by [18]). Or along the same lines, RL involving non-Archimedean probabilities [17].
• RL where the environment can secretly simulate the agent [3, 6, 4].
• RL where $V_\pi^\mu$ is allowed to diverge (and relative performance of agents in an environment is defined in various ways accordingly).

3. Translations

For any two RL-frameworks $\mathcal{F}$ and $\mathcal{G}$, if we are given an $\mathcal{F}$-agent, can we map it to a $\mathcal{G}$-agent? And if we apply said mapping to several $\mathcal{F}$-agents, can we be sure the mapping preserves their relative intelligence? For example, suppose a factory uses $\mathcal{G}$ internally, and a research lab publishes agents intended for $\mathcal{F}$. The factory imports the lab’s agents and maps them to $\mathcal{G}$. One day, the lab announces a new agent and advertises that it outperforms all its previous agents. The factory imports the new agent and maps it to $\mathcal{G}$. Is it guaranteed that the new agent, mapped into $\mathcal{G}$, still outperforms all the lab’s previous agents, mapped into $\mathcal{G}$? It depends on the mapping and on how intelligence is measured. We consider it desirable to map agents and measure intelligence in such a way as to guarantee that, yes, improvements to the agent in $\mathcal{F}$ will result in improvements post-mapping in $\mathcal{G}$.

In Section 4 we will discuss two ways of comparing the intelligence of agents: namely, by considering agents to compete in intelligence elections in which the voters are...

1. The set of environments, or
2. The set of equivalence classes of environments (two environments being equivalent if they vote the same way).

Such infinite-voter elections can be decided using ultrafilters (and in a certain technical sense, using ultrafilters is the only way to decide them subject to certain basic requirements [12]). In the present section, we will define so-called weak translations and strong translations from one RL framework to another. In Section 4 we will show that:
(1) Given a weak translation from a source RL framework to a destination RL framework, any relative intelligence comparator for the destination framework (based on elections where environments are voters) induces a relative intelligence comparator for the source framework (based on elections where environments are voters) such that the translation in question preserves relative intelligence.

(2) Likewise for strong translations and elections where environment equivalence classes are voters.

Before getting to the definitions of weak and strong translations, some preliminary definitions are needed.

**Definition 3.1.** Suppose \( \mathcal{F} = (A, E, H) \) and \( \mathcal{F}' = (A', E', H') \) are RL frameworks and \( \bullet^* : A \to A' \) maps agents \( \pi \in A \) to agents \( \pi^* \in A' \). For any \( h \in H'_A \), for any \( h_1, \ldots, h_n \in H_A \), we say \( \pi^*(h) \) is determined by \( \pi(h_1), \ldots, \pi(h_n) \) if the following condition holds: for all \( \rho \in A \), if \( \rho(h_i) = \pi(h_i) \) for \( i = 1, \ldots, n \), then \( \rho^*(h) = \pi^*(h) \).

**Definition 3.2.** (Pre-translations) Suppose \( \mathcal{F} = (A, E, H) \) and \( \mathcal{F}' = (A', E', H') \) are RL frameworks. A pre-translation from \( \mathcal{F} \) to \( \mathcal{F}' \) is a pair \( (\bullet^* : A \to A', \bullet_* : E' \to E) \) of functions such that:

1. For all \( \pi, \rho \in A \), for all \( \mu \in E' \), \( V^\pi^\mu \leq V^\pi^\rho \) iff \( V^\mu^\pi \leq V^\rho^\pi \).
2. For all \( \pi \in A \), for all \( h \in H'_A \), there exist \( h_1, \ldots, h_n \in H_A \) such that \( \pi^*(h) \) is determined by \( \pi(h_1), \ldots, \pi(h_n) \).
3. For every \( \pi \in A \), for every finite set \( S_1 \subseteq H'_A \) and every infinite set \( S_2 \subseteq H'_A \), there exists some \( \rho \in A \) such that \( \rho^*(h) = \pi^*(h) \) for all \( h \in S_1 \) but \( \rho^*(h) \neq \pi^*(h) \) for some \( h \in S_2 \).

In Definition 3.2, Condition 1 should be thought of through the eyes of \( \pi \) and \( \rho \). Intuitively, we think of \( \pi^* \) working by intercepting destination-framework histories from \( \mu \) and converting them into source-framework histories, plugging those into \( \pi \), and using the outputs from \( \pi \) to decide what \( \pi^* \) should do. Thus, as far as \( \pi \) is aware, \( \pi \) is interacting with a source-environment \( \mu_* \) (when in reality \( \pi \) is being used by \( \pi^* \) to interact with destination-environment \( \mu \)). Thus the relative performance of \( \pi \) and \( \rho \) in \( \mu_* \) should match that of \( \pi^* \) and \( \rho^* \) in \( \mu \). Condition 2 codifies the idea that, just as in an oracle machine in computability theory, \( \pi^* \) should be able to be computed on any input \( h \) by querying \( \pi \) on finitely many inputs \( h_1, \ldots, h_n \) (which depend on \( h \)). Condition 3 is a kind of free will condition, capturing the intuition that an agent’s future actions should not be completely determined by its past actions.

**Definition 3.3.** (Weak translations) Suppose \( \mathcal{F} = (A, E, H) \) and \( \mathcal{F}' = (A', E', H') \) are as in Definition 3.2. A pre-translation \( (\bullet^*, \bullet_*) \) from \( \mathcal{F} \) to \( \mathcal{F}' \) is a weak translation if \( \bullet_* \) is injective.

**Lemma 3.4.** (Weak translation reflexivity and transitivity)

1. For every RL framework \( \mathcal{F} = (A, E, H) \), \( (\text{id}_A, \text{id}_E) \) is a weak translation from \( \mathcal{F} \) to \( \mathcal{F} \), where \( \text{id}_A \) and \( \text{id}_E \) are the identity functions.
2. Let \( \mathcal{F} = (A, E, H), \mathcal{F}' = (A', E', H'), \mathcal{F}'' = (A'', E'', H'') \) be RL frameworks. If \( (\bullet^* : A \to A', \bullet_* : E' \to E) \) is a weak translation from \( \mathcal{F} \) to \( \mathcal{F}' \) and \( (\bullet^* : A' \to A'', \bullet_* : E'' \to E') \) is a weak translation from \( \mathcal{F}' \) to \( \mathcal{F}'' \), then \( (\bullet^* \circ \bullet^* : A \to A'', \bullet_* \circ \bullet_* : E'' \to E') \) is a weak translation from \( \mathcal{F} \) to \( \mathcal{F}'' \).
then \((\bullet : A \to A'', \bullet : E'' \to E)\) is a weak translation from \(\mathcal{F}\) to \(\mathcal{F}''\), where \(\pi = (\pi^*)^\circ\) and \(\mu = (\mu_\pi)\) for all \(\pi \in A, \mu \in E\).

**Proof.** (1) Straightforward. To verify Condition 3 of Definition 3.2, use Condition 5 of Definition 2.3.

(2) Straightforward. □

Before proceeding to the definition of strong translations, we will prove a technical lemma about weak translations which will be useful for establishing negative results about weak translation existence. First, we need a preliminary definition.

**Definition 3.5.** Let \(\mathcal{F} = (A, E, H)\) be an RL framework.

- For any \(h, g \in H\), we write \(h \subseteq g\) if and only if \(h\) is a proper initial segment of \(g\). We write \(h \subseteq g\) if either \(h \subseteq g\) or \(h = g\).
- If \(\pi, \rho \in A\) and \(h \in H\), we say that \(\pi\) agrees with \(\rho\) up to \(h\) if the following requirement holds: for all \(h_0 \in H_A\), if \(h_0 \subset h\) then \(\pi(h_0) = \rho(h_0)\).

**Lemma 3.6.** Suppose \(\mathcal{F} = (A, E, H)\) and \(\mathcal{F}' = (A', E', H')\) are RL frameworks and \((\bullet^* : A \to A', \bullet : E' \to E)\) is a pre-translation from \(\mathcal{F}\) to \(\mathcal{F}'\). For each \(\pi \in A\), for each infinite path \(Q\) through \(H'\), for each \(h \in Q\), there exists some \(\rho \in A\) and some \(h^+ \in Q \cap H_A\) such that:

1. \(h \subset h^+\).
2. \(\pi^* (h^+) \neq \rho^*(h^+)\).
3. \(\pi^*(h^+) \# \rho^*(h^+)\).

**Proof.** Let \(S_1 = \{g \in Q \cap H'_A : g \subseteq h\}\), \(S_2 = \{g \in Q \cap H'_A : g \not\subseteq h\}\). By Definition 3.2 Condition 3, there exists \(\rho \in A\) such that \(\rho^*(g) = \pi^*(g)\) for all \(g \in S_1\) but \(\rho^*(h^+) \# \pi^*(h^+)\) for some \(h^+ \in S_2\). Let \(h^+\) be the shortest history in \(S_2\) such that \(\rho^*(h^+) \# \pi^*(h^+)\) (some such \(h^+\) must exist because \(h^+\) is a candidate). By minimality of \(h^+, \pi^*\) agrees with \(\rho^*\) up to \(h^+\). □

We now proceed toward defining strong translations. First, we need some preliminary machinery. In the following definition, we define an equivalence relation according to which two environments are equivalent if they assign the same relative performance to all pairs of agents.

**Definition 3.7.** Suppose \(\mathcal{F} = (A, E, H)\) is an RL framework and \(\mu, \nu \in E\).

1. For any \(\pi, \rho \in A\), we say that \(\mu\) and \(\nu\) agree about \(\pi\) vs. \(\rho\) if the following condition holds: \(V^\pi_\mu \leq V^\pi_\rho\) iff \(V^\nu_\nu \leq V^\nu_\nu\).
2. We say \(\mu\) is equivalent to \(\nu\), and write \(\mu \sim_{\mathcal{F}} \nu\) (or \(\mu \sim \nu\) if \(\mathcal{F}\) is clear from context) if for all \(\pi, \rho \in A\), \(\mu\) and \(\nu\) agree about \(\pi\) vs. \(\rho\).

**Lemma 3.8.** For any RL framework \(\mathcal{F} = (A, E, H)\), \(\sim_{\mathcal{F}}\) is an equivalence relation on \(E\).

**Proof.** Straightforward. □

**Definition 3.9.** If \(\mathcal{F} = (A, E, H)\) is an RL-framework, we write \([E]\) for the set of \(\sim\)-equivalence classes of \(E\). For every \(\mu \in E\), we write \([\mu]\) for the \(\sim\)-equivalence class containing \(\mu\).
**Definition 3.10.** (Strong Translations) By a strong translation from \( \mathcal{F} = (A, E, H) \) to \( \mathcal{F}' = (A', E', H') \), we mean a pre-translation \((\bullet^*, \bullet_*)\) from \( \mathcal{F} \) to \( \mathcal{F}' \) such that:

- For all \( \mu, \nu \in E' \), if \( \mu_* \sim_\mathcal{F} \nu_* \), then \( \mu \sim_\mathcal{F} \nu \).

**Lemma 3.11.** Strong translation is reflexive and transitive, in the same sense as in Lemma 3.4.

*Proof.* Straightforward. \( \square \)

**Lemma 3.12.** Suppose \((\bullet^*, \bullet_*)\) is a strong translation from \( \mathcal{F} = (A, E, H) \) to \( \mathcal{F}' = (A', E', H') \). Overloading the symbol \( \bullet_* \) (this will cause no confusion in practice), we define, for each \( \mu \in E' \), \( [\mu]_* = [\mu_*] \). The resulting function \( \bullet_* : [E'] \to [E] \) is well-defined, and is injective.

*Proof.* For well-definedness, we must show that for all \( \mu, \nu \in E' \), if \( [\mu_*] \neq [\nu_*] \) then \( [\mu] \neq [\nu] \). Fix such \( \mu, \nu \in E' \). Since \( [\mu_*] \neq [\nu_*] \), \( \mu_* \neq \nu_* \). Thus there exist \( \pi, \rho \in E \) such that \( \mu_* \) and \( \nu_* \) disagree about \( \pi \) vs. \( \rho \). Assume \( V^\pi_\mu \leq V^\rho_\mu \) and \( V^\pi_\nu > V^\rho_\nu \) (the other case is similar). By Definition 3.2 Condition 1, \( V^\pi_\mu < V^\rho_\mu \) and \( V^\pi_\nu > V^\rho_\nu \).

Thus \( \mu \) and \( \nu \) disagree about \( \pi^* \) vs. \( \rho^* \), so \( \mu \neq \nu \) and \( [\mu] \neq [\nu] \).

For injectivity, assume \( [\mu] \neq [\nu] \), so \( \mu \neq \nu \). Then \( \mu_* \neq \nu_* \) by Definition 3.10, so \( [\mu_*] \neq [\nu_*] \). \( \square \)

4. Comparing Intelligence Using Ultrafilters

In order to compare intelligence in such a way that the mathematics is tractable, we will consider environments (or environment equivalence classes) to be voters who vote in elections to rank agents against each other. When two agents \( \pi \) and \( \rho \) are to be compared in general (in aggregate over the whole space of all environments), then an environment \( \mu \) (or equivalence class \([\mu]\)) will cast its vote based on which agent performs best in \( \mu \). This reduces the intelligence comparison problem to the problem of deciding which sets of environments should be considered winning blocs.

This basic idea first appeared in [1]. It is known [12] that, subject to some basic requirements, choices of winning blocs correspond exactly to ultrafilters on the set of voters, with free ultrafilters corresponding exactly to non-dictatorial winning-bloc choices (this does not contradict Arrow’s Impossibility Theorem, because that theorem only applies when the voter-set is finite).

**Definition 4.1.** Suppose \( \mathcal{F} = (A, E, H) \) is an RL framework.

- If \( \mathcal{U} \) is an ultrafilter on \( E \), we define a comparator \( \leq_{\mathcal{U}} \) on \( A \) as follows: for all \( \pi, \rho \in A \), \( \pi \leq_{\mathcal{U}} \rho \) iff \( \{ \mu \in E : V^\pi_\mu \leq V^\rho_\mu \} \in \mathcal{U} \).
- If \( \mathcal{U} \) is an ultrafilter on \([E]\), we define a comparator \( \leq_{\mathcal{U}} \) on \( A \) as follows: for all \( \pi, \rho \in A \), \( \pi \leq_{\mathcal{U}} \rho \) iff \( \{ [\mu] \in [E] : V^\pi_\mu \leq V^\rho_\mu \} \in \mathcal{U} \).

**Definition 4.2.** Suppose \( \mathcal{F} = (A, E, H) \) and \( \mathcal{F}' = (A', E', H') \) are RL frameworks, \( \mathcal{U} \) is either an ultrafilter on \( E \) or on \([E]\), and \( \mathcal{U}' \) is either an ultrafilter on \( E' \) or on \([E']\). Suppose \((\bullet^*, \bullet_*)\) is a pre-translation from \( \mathcal{F} \) to \( \mathcal{F}' \). We say that \((\bullet^*, \bullet_*)\) preserves relative intelligence as measured by \( \mathcal{U} \mathcal{U}' \) if the following condition holds: for all \( \pi, \rho \in A \), \( \pi \leq_{\mathcal{U} \mathcal{U}'} \rho \) iff \( \pi^* \leq_{\mathcal{U}' \mathcal{U}} \rho^* \).
**Definition 3.3.** If $\mu \mapsto \mu_*$ is any function and $X$ is a subset of its domain, we write $X_*$ for $\{\mu_* : \mu \in X\}$.

The following lemma is the reason why we required injectivity in the definition of weak translations (Definition 3.3). We omit the proof of this lemma which can be established using standard ultrafilter techniques.

**Lemma 4.4.** (The Injection Lemma) Suppose $E'$ is a set, $\mathcal{U}'$ is an ultrafilter on $E'$, and $\mu \mapsto \mu_*$ is an injection from $E'$ to $E$. There exists an ultrafilter $\mathcal{U}$ on $E$ satisfying the following requirements:

1. $(E')_* \in \mathcal{U}$.
2. For all $Y \subseteq E'$, if $Y_* \in \mathcal{U}$, then $Y \in \mathcal{U}'$.

Using Lemma 4.4, the following theorem guarantees existence of ultrafilter-based intelligence comparators on a source RL framework such that relative intelligence is preserved by a translation to a destination RL framework.

**Theorem 4.5.** (Preservation Theorem) Let $\mathcal{F} = (A, E, H)$, $\mathcal{F}' = (A', E', H')$ be RL-frameworks, $(\bullet^*, \bullet_*)$ a pre-translation from $\mathcal{F}$ to $\mathcal{F}'$.

1. Assume $(\bullet^*, \bullet_*)$ is a weak translation (Definition 3.3). If $\mathcal{U}'$ is an ultrafilter on $E'$ and $\mathcal{U}$ is an ultrafilter on $E$ with the properties of Lemma 4.4, then $(\bullet^*, \bullet_*)$ preserves relative intelligence as measured by $\mathcal{U}, \mathcal{U}'$.
2. Assume $(\bullet^*, \bullet_*)$ is a strong translation (Definition 3.10). If $\mathcal{U}'$ is an ultrafilter on $[E']$ and $\mathcal{U}$ is an ultrafilter on $[E]$ with the properties of Lemma 4.4, then $(\bullet^*, \bullet_*)$ preserves relative intelligence as measured by $\mathcal{U}, \mathcal{U}'$.

**Proof.** We prove (1), the proof of (2) is similar. Let $\pi, \rho \in A$. We must show that $\pi \leq \mathcal{U} \rho$ iff $\pi^* \leq \mathcal{U}' \rho^*$. 

$(\Rightarrow)$ Assume $\pi \leq \mathcal{U} \rho$. Let $Y = \{\mu \in E' : V_\mu^\pi \leq V_\mu^\rho\}$. In order to show $\pi^* \leq \mathcal{U}' \rho^*$, we must show $Y \in \mathcal{U}'$.

By Definition 3.2 Condition 1,

$$Y = \{\mu \in E' : V_\mu^\pi \leq V_\mu^\rho\},$$

or in other words,

$$Y = \{\mu : \mu \in E' \text{ and } V_\mu^\pi \leq V_\mu^\rho\}.$$

Thus

$$Y_* = \{\mu_* : \mu \in E' \text{ and } V_\mu^\pi \leq V_\mu^\rho\}.$$

In other words, for an environment $\nu$ to be in $Y_*$, two conditions are necessary and sufficient: $\nu$ must be $\mu_*$ for some $\mu \in E'$, and $\nu$ must satisfy $V_\nu^\pi \leq V_\nu^\rho$. So the above equation can be rewritten

$$Y_* = \{\mu_* : \mu \in E' \} \cap \{\nu \in E : V_\nu^\pi \leq V_\nu^\rho\}.$$

The left intersectand is $E'_*$, which is in $\mathcal{U}$ by Lemma 4.4 Condition 1. The right intersectand, $\{\nu \in E : V_\nu^\pi \leq V_\nu^\rho\}$, is in $\mathcal{U}$ because $\pi \leq \mathcal{U} \rho$. So by $\cap$-closure of $\mathcal{U}$, $Y_* \in \mathcal{U}$. By Lemma 4.4 Condition 2, $Y \in \mathcal{U}'$, as desired.

$(\Leftarrow)$ By rewriting our proof of $(\Rightarrow)$ with “$\leq$” changed to “$\nleq$” throughout, we obtain a proof that if $\pi \nleq \mathcal{U} \rho$ then $\pi^* \nleq \mathcal{U}' \rho^*$.

□
5. Translating between action- and percept-first RL

In this section, we will explore translatability between action-first and percept-first RL. For simplicity, we will consider the following concrete RL frameworks.

**Definition 5.1.** (Maximum action- and percept-first RL frameworks)

- Let $H^{AP}$ be the set of all histories which are empty or begin with an action.
- Let $H^{PA}$ be the set of all histories which are empty or begin with a percept.

The **maximum action-first RL framework** is the RL framework $F_{AP} = (A^{AP}, E^{AP}, H^{AP})$ where:

- $A^{AP}$ is the set of all functions $\pi$ with domain $H^{AP}$ such that for all $h \in H^{AP}$, $\pi(h)$ is a finitely-supported probability distribution on $A$.
- $E^{AP}$ is the set of all functions $\mu$ with domain $H^{AP}$ such that:
  - For all $h \in H^{AP}$, $\mu(h)$ is a finitely-supported probability distribution on $P$.
  - For all $\pi \in A^{AP}$, $V^\pi_\mu$ converges.

The **maximum percept-first RL framework** is the RL framework $F_{PA} = (A^{PA}, E^{PA}, H^{PA})$ where:

- $A^{PA}$ is the set of all functions $\pi$ with domain $H^{PA}$ such that for all $h \in H^{PA}$, $\pi(h)$ is a finitely-supported probability distribution on $A$.
- $E^{PA}$ is the set of all functions $\mu$ with domain $H^{PA}$ such that:
  - For all $h \in H^{PA}$, $\mu(h)$ is a finitely-supported probability distribution on $P$.
  - For all $\pi \in A^{PA}$, $V^\pi_\mu$ converges.

We omit the straightforward proof that $F_{AP}$ and $F_{PA}$ are RL frameworks, but we note that to show they satisfy Condition 5 of Definition 2.3 requires our background assumption that $|A| > 1$.

5.1. From percept-first RL to action-first RL.

**Proposition 5.2.** (Strongly translating $F_{PA}$ to $F_{AP}$) Let $y_0 \in P$ be arbitrary. For every $\pi \in A^{PA}$, define $\pi^* \in A^{AP}$ by

$$\pi^*(h) = \pi(y_0 \sim h).$$

For every $\mu \in E^{AP}$, define $\mu_\mu^* \in E^{PA}$ by

$$\mu_\mu^*(y|\emptyset) = \begin{cases} 1 & \text{if } y = y_0, \\ 0 & \text{otherwise}, \end{cases}$$

$$\mu_\mu^*(y \sim h) = \mu(h).$$

Then $(\cdot^*, \cdot_\mu^*)$ is a strong translation from $F_{PA}$ to $F_{AP}$.

**Proof.** Preliminary Claim: $(\cdot^*, \cdot_\mu^*)$ is a pre-translation. Conditions 2 & 3 of Definition 3.2 are obvious (using our background assumption that $|A| > 1$). Condition 1 will require some work.

For all $\sigma \in A^{PA}$ and $t \in \mathbb{N}$, for all $h_0, \ldots, h_t \in H^{AP}$, define:

- $m^\sigma_\mu(h_0, \ldots, h_t)$ is the probability that $(h_0, \ldots, h_t)$ is the random $H^{AP}$-path randomly generated as in the definition of $V^\sigma_\mu$ (Definition 2.4).
\[ m_{\pi}(\langle \rangle, y_0 \sim h_0, \ldots, y_t \sim h_t) \] is the probability that \( (\langle \rangle, y_0 \sim h_0, \ldots, y_t \sim h_t) \) is the random \( H^{PA} \) path randomly generated as in the definition of \( V_{\pi,t+1}^{\sigma} \) (Definition 2.4).

First show, by induction on \( t \), that for all \( t \in \mathbb{N} \), for all \( h_0, \ldots, h_t \in H^{AP} \), the following equation (*) holds:

\[ m_{\sigma}(h_0, \ldots, h_t) = m_{\sigma}(\langle \rangle, y_0 \sim h_0, \ldots, y_t \sim h_t). \]

Since \( \mu_*(y_0|\langle \rangle) = 1 \), in computing \( V_{\mu_*,t+1}^{\sigma} \) we may ignore all histories not of the form \( (\langle \rangle, y_0 \sim h_0, \ldots, y_t \sim h_t) \) (any history not of this form must have probability 0 and so contributes nothing to \( V_{\mu_*,t+1}^{\sigma} \)). By (*), the probability of \( (h_0, \ldots, h_t) \) being randomly generated using \( \sigma^* \) and \( \mu \) is the same as the probability of \( (\langle \rangle, y_0 \sim h_0, \ldots, y_t \sim h_t) \) being randomly generated using \( \sigma \) and \( \mu_* \). It follows that \( V_{\mu_*,t+1}^{\sigma} = V_{\mu_*}^{\sigma^*} + R(h_0) \) and so \( V_{\mu_*}^{\sigma^*} = V_{\mu_*}^{\sigma^*} + R(y_0) \). Since this holds for all \( \sigma \in A^{PA} \), Condition 1 of Definition 3.2 follows. This proves the Preliminary Claim.

To show \( (\bullet^*, \bullet_* \circ) \) is strong, suppose \( \mu, \nu \in E^{AP} \) are such that \( \mu \not\sim \nu \). We must show \( \mu_* \not\sim \nu_* \). Since \( \mu \not\sim \nu \), there exist \( \pi, \rho \in A^{AP} \) such that \( \mu \) and \( \nu \) disagree about \( \pi \) vs. \( \rho \) (Definition 3.7).

Case 1: \( V_{\mu}^\pi \leq V_{\mu}^\rho \) but \( V_{\nu}^\pi > V_{\nu}^\rho \). Define \( \pi^\times, \rho^\times \in A^{PA} \) by

\[ \pi^\times(y \sim h) = \pi(h), \]
\[ \rho^\times(y \sim h) = \rho(h). \]

By similar reasoning as in the Preliminary Claim, it is easy to check that \( V_{\mu_*}^{\pi^\times} = V_{\mu_*}^\pi + R(y_0), V_{\mu_*}^{\rho^\times} = V_{\mu_*}^\rho + R(y_0), V_{\nu_*}^{\pi^\times} = V_{\nu_*}^\pi + R(y_0) \), and \( V_{\nu_*}^{\rho^\times} = V_{\nu_*}^\rho + R(y_0) \). Thus \( V_{\mu_*}^{\pi^\times} \leq V_{\mu_*}^{\rho^\times} \) but \( V_{\nu_*}^{\pi^\times} > V_{\nu_*}^{\rho^\times} \). So \( \mu_* \) and \( \nu_* \) disagree about \( \pi^\times \) vs. \( \rho^\times \), showing \( \mu_* \not\sim \nu_* \).

Case 2: \( V_{\mu}^\pi > V_{\mu}^\rho \) but \( V_{\nu}^\pi \leq V_{\nu}^\rho \). Similar to Case 1. \( \square \)

If \( (\bullet^*, \bullet_* \circ) \) is a weak (resp. strong) translation, \( \bullet_* \) is not unique in general, i.e., except in degenerate cases, one can find \( \bullet_* \neq \bullet \), such that \( (\bullet^*, \bullet_* \circ) \) is also a weak (resp. strong) translation. However, if an environment \( \mu \) is sufficiently non-trivial, then it is sometimes possible to prove that \( \mu_*(h) \) is forced to have some particular value for some particular history \( h \). We give an example of this type of reasoning, because we feel it might help in future to address some of the open questions we state below.

**Example 5.3.** Fix \( y_0 \in P \) and let \( \bullet^* : A^{PA} \to A^{AP} \) be as in Proposition 5.2 (i.e., \( \pi^*(h) = \pi(y_0 \sim h) \) for all \( \pi \in A^{PA} \)). Suppose \( \mu \in E^{AP} \) is such that there exist \( \pi, \rho \in A^{PA} \) such that \( V_{\mu}^{\pi^*} \neq V_{\mu}^{\rho^*} \). Then for any \( \bullet_* : E^{AP} \to E^{PA} \) such that \( (\bullet^*, \bullet_* \circ) \) is a pre-translation, \( \mu_*(y_0|\langle \rangle) = 1 \).

**Proof.** Define \( \sigma \in A^{PA} \) by

\[ \sigma(y \sim h) = \begin{cases} \pi(y \sim h) & \text{if } y = y_0, \\ \rho(y \sim h) & \text{otherwise.} \end{cases} \]

Clearly \( \sigma^* = \pi^* \). Thus \( V_{\mu}^{\sigma^*} = V_{\mu}^{\pi^*} \neq V_{\mu}^{\rho^*} \), so by Definition 3.2 Condition 1, \( V_{\mu_*}^{\pi^*} = V_{\mu_*}^{\rho^*} \). Now, when \( \sigma \) interacts with \( \mu_* \), there are two possibilities:
The initial percept is \( y_0 \), in which case \( \sigma \) acts as \( \pi \). This occurs with probability \( \alpha = \mu_1(y_0) \).

- The initial percept is \( \not= y_0 \), in which case \( \sigma \) acts as \( \rho \). This occurs with probability \( 1 - \alpha \).

Thus by basic probability

\[
V_{\mu_1}^\pi = \alpha V_{\mu_1}^\pi + (1 - \alpha)V_{\mu_1}^\rho.
\]

The above sum equals \( V_{\mu_1}^\pi \) (since \( V_{\mu_1}^\rho = V_{\mu_1}^\pi \)), so, since \( V_{\mu_1}^\pi \neq V_{\mu_1}^\rho \), all the weight must be on the first summand, i.e., \( \alpha = 1 \).

\[\Box\]

**Proposition 5.5.** (A pre-translation from \( \mathcal{F}_{AP} \) to \( \mathcal{F}_A \) which is not strong) Let \( x_0 \in A \) be arbitrary. For every \( \pi \in AP_A \), define \( \pi^* \in AP \) by

\[
\pi^*(x|\langle \rangle) = \begin{cases} 1 & \text{if } x = x_0, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
\pi^*(x \sim h) = \pi(h).
\]

For every \( \mu \in EAP \), define \( \mu_* \in PAP \) by

\[
\mu_*(h) = \mu(x_0 \sim h).
\]

Then \( (\cdot^*, \cdot_*) \) is a pre-translation from \( \mathcal{F}_{AP} \) to \( \mathcal{F}_A \). But assuming there is some percept with reward 0 and some percept with reward \( \not= 0 \), then \( (\cdot^*, \cdot_*) \) is not a strong translation.

**Proof.** That \( (\cdot^*, \cdot_*) \) is a pre-translation is similar to the Preliminary Claim in the proof of Proposition 5.2. To see non-strongness, using the assumption that there exists some percept with reward 0 and some percept with reward \( \not= 0 \), construct \( \mu, \nu \in EAP \) such that:

- (\( \mu \) always gives reward 0) For all \( h \in H_E^{AP} \), there is some \( y \in P \) such that \( R(y) = 0 \) and \( \mu(y|h) = 1 \).
- (\( \nu \) gives a nonzero reward immediately after the agent performs any initial action except \( x_0 \)) For all \( x \in A \) with \( x \not= x_0 \), there is some \( y \in P \) such that \( R(y) \neq 0 \) and \( \nu(y|x) = 1 \).
- (\( \nu \) gives reward 0 for the agent taking initial action \( x_0 \)) There is some \( y \in P \) such that \( R(y) = 0 \) and \( \nu(y|\langle x_0 \rangle) = 1 \).
- (\( \nu \) gives reward 0 in all other circumstances) For all \( h \in H_E^{AP} \) with length \( > 1 \), there is some \( y \in P \) such that \( R(y) = 0 \) and \( \nu(y|h) = 1 \).

Let \( \pi, \rho \in AP \) be any agents such that \( \pi(x_0|\langle \rangle) = 1 \) and \( \rho(x_1|\langle \rangle) = 1 \) for some \( x_1 \not= x_0 \) (possible since \( |A| > 1 \)). Clearly \( V_\mu^\pi = V_\rho^\rho = 0 \) while \( 0 = V_\nu^\pi \neq V_\nu^\rho \) (showing \( \mu \not\sim \nu \)), but for all \( \pi' \in AP_A \), \( V_{\mu_*}^{\pi'} = V_{\nu_*}^{\pi'} = 0 \), which implies \( \mu_* \sim \nu_* \).

\[\Box\]

5.2. From action-first RL to percept-first RL. Surprisingly, mapping from action-first RL to percept-first RL seems trickier than the other direction. We will start by showing that an action-first-to-percept-first version of Proposition 5.2 fails to work.

**Proposition 5.5.** (A map from \( \mathcal{F}_{AP} \) to \( \mathcal{F}_{P_A} \) which is not a pre-translation) Fix any \( x_0 \in A \) and define \( \cdot^* : AP \to AP_A \) by \( \pi^*(h) = \pi(x_0 \sim h) \). Assuming there is some \( y \in P \) with \( R(y) = 0 \) and some \( y \in P \) with \( R(y) \neq 0 \), there can be no \( \cdot_* : PAP \to EAP \) such that \( (\cdot^*, \cdot_*) \) is a pre-translation.
Proof. Assume there is some such \( \bullet \). Let \( x_1 \in \mathcal{A} \), \( x_1 \neq x_0 \). Let \( \pi \in A^{AP} \) be the agent that always takes action \( x_0 \), i.e., \( \pi(x_0|h) = 1 \) for all \( h \in H_A^{AP} \). Let \( \rho \in A^{AP} \) be the agent that always takes action \( x_1 \). Clearly \( \pi^* \) also always takes action \( x_0 \), and \( \rho^* \) also always takes action \( x_1 \). Since there is some \( y \in \mathcal{P} \) with \( R(y) = 0 \) and there is some \( y \in \mathcal{P} \) with \( R(y) \neq 0 \), it follows that there exists an environment \( \mu \in E^{PA} \) such that \( V_{\mu^*}^\pi \neq V_{\mu^*}^\rho \) (for example, \( \mu \) could be an environment that only ever gives a nonzero reward when the agent plays \( x_0 \) for the first time (if ever)). Define \( \sigma \in A^{AP} \) by
\[
\sigma(x_0|\langle \rangle) = \sigma(x_1|\langle \rangle) = \frac{1}{2};
\]
\[
\sigma(x \sim h) = \begin{cases} 
\pi(x \sim h) & \text{if } x = x_0, \\
\rho(x \sim h) & \text{otherwise.}
\end{cases}
\]
Clearly \( \sigma^* = \pi^* \). Now, when \( \sigma \) interacts with \( \mu_\bullet \), it is easy to show that
\[
V_{\mu_\bullet}^\sigma = \frac{1}{2}V_{\mu_\bullet}^\pi + \frac{1}{2}V_{\mu_\bullet}^\rho.
\]
Since \( V_{\mu_\bullet}^\pi \neq V_{\mu_\bullet}^\rho \), Definition 3.2 Condition 1 forces \( V_{\mu_\bullet}^\pi \neq V_{\mu_\bullet}^\rho \), so it follows that \( V_{\mu_\bullet}^\sigma \neq V_{\mu_\bullet}^\pi \). By Definition 3.2 Condition 1, \( V_{\mu^*}^\pi \neq V_{\mu^*}^\rho \), contrary to \( \sigma^* = \pi^* \). \( \square \)

**Definition 5.6.** For every even-length sequence \( h \) we define the local reverse \( \bar{h} \) to be the result of transposing each \( h_{2i}, h_{2i+1} \) in \( h \), or, more formally, by induction:
- If \( h = \langle \rangle \) then \( \bar{h} = \langle \rangle \).
- If \( h = g \sim \langle a, b \rangle \) then \( \bar{h} = \bar{g} \sim \langle b, a \rangle \).

For example if \( s = \langle 1, 2, 3, 4 \rangle \) then \( \bar{s} = \langle 2, 1, 4, 3 \rangle \).

**Proposition 5.7.** (A weak translation \( \mathcal{F}^{AP} \) to \( \mathcal{R}^{PA} \)) For every \( \pi \in A^{AP} \), define \( \pi^* \in A^{PA} \) by
\[
\pi^*(h \sim y) = \pi(\bar{h}).
\]
For every \( \mu \in E^{PA} \), define \( \mu_\bullet \in E^{AP} \) by
\[
\mu_\bullet(h \sim x) = \mu(\bar{h}).
\]
Then \( (\bullet^*, \bullet_\bullet) \) is a weak translation from \( \mathcal{F}^{AP} \) to \( \mathcal{R}^{PA} \).

Proof. Clearly \( \bullet^* \) is injective. Conditions 2 & 3 of Definition 3.2 are obvious (using our background assumption that \( |\mathcal{A}| > 1 \)). Condition 1 requires work.

Fix \( \mu \in E^{PA} \), \( \sigma \in A^{AP} \). For every \( h \) of the form \( \langle x_1 y_1 \ldots x_i y_i \rangle \) with each \( x_i \in \mathcal{A} \), \( y_i \in \mathcal{P} \) (including \( h = \langle \rangle \) when \( i = 0 \)), define:
- \( m_\sigma(h) \) is the probability that \( h_{2t} = h \) if \( \langle h_0, \ldots, h_{2t} \rangle \) is randomly generated as in the definition of \( V_{\mu_\bullet}^{\sigma^*} \).
- \( m_{\sigma^*}(\bar{h}) \) is the probability that \( h_{2t} = \bar{h} \) if \( \langle h_0, \ldots, h_{2t} \rangle \) is randomly generated as in the definition of \( V_{\mu_\bullet}^{\sigma^*} \).

As in the proof of Proposition 5.2, it can be established that
\[
m_\sigma(h) = m_{\sigma^*}(\bar{h}).
\]
This implies \( V_{\mu_\bullet}^{\sigma^*} = V_{\mu_\bullet}^{\sigma^*} \) for all \( t \in \mathbb{N} \), thus \( V_{\mu_\bullet}^{\sigma} = V_{\mu_\bullet}^{\sigma^*} \). By arbitrariness of \( \sigma \) and \( \mu \), Condition 1 of Definition 3.2 follows. \( \square \)
We do not know whether the mapping in Proposition 5.7 is strong. For a different translation, we can explicitly show non-strongness.

**Proposition 5.8.** (A mapping from $A^AP$ to $A^PA$ which is not a strong translation) Define $\bullet^\times: A^AP \to A^PA$ by $\pi^\times(y \sim h) = \pi(h)$. Assuming there is some percept $y$ with $R(y) = 0$ and some percept $y$ with $R(y) \neq 0$, there does not exist any $\bullet: E^PA \to E^AP$ such that $(\bullet^\times, \bullet)$ is a strong translation from $\mathcal{F}_{AP}$ to $\mathcal{F}_{PA}$.

**Proof.** Assume, for sake of contradiction, that there is some such $\bullet: E^PA \to E^AP$. Let $y_0 \in P$ be arbitrary, and let $(\bullet^\times, \bullet)$ be the corresponding strong translation from Proposition 5.2. Let

$$\bullet^\dagger = \bullet^\times \circ \bullet^*: A^PA \to A^PA$$
$$\bullet = \bullet^\dagger \circ \bullet^*: E^PA \to E^PA.$$

By Lemma 3.11, $(\bullet^\dagger, \bullet)$ is a strong translation from $\mathcal{F}_{PA}$ to $\mathcal{F}_{PA}$. An easy computation shows that in general $\pi^\dagger(y \sim h) = \pi(y_0 \sim h)$ and thus $\pi^\dagger = \pi^\dagger$.

Claim 1: For all $\mu \in E^PA$, $\mu^\dagger \sim \mu^\dagger$. To see this, let $\pi, \rho \in A^PA$. By Definition 3.2 Condition 1,

$$V^\pi_\mu \leq V^\rho_\mu \text{ if } V^\pi_{\mu^\dagger} \leq V^\rho_{\mu^\dagger}$$
$$V^\rho_\mu \leq V^\pi_{\mu^\dagger} \text{ if } V^\pi_{\mu^\dagger} \leq V^\rho_{\mu^\dagger}$$

so $\mu^\dagger$ and $\mu^\dagger$ agree about $\pi$ vs. $\rho$.

Claim 2: There is some $\mu \in E^PA$ such that $\mu \not\sim \mu^\dagger$. Let $x_0, x_1 \in A$ be distinct. Using our assumption that there is some percept $y$ with $R(y) = 0$ and some percept $y$ with $R(y) \neq 0$, one can construct $\mu \in E^PA$ and $\pi, \rho \in A^PA$ satisfying the following conditions:

$\uparrow$ $\mu$ outputs initial percept $\neq y_0$.
$\uparrow$ $\mu$ gives the agent a nonzero reward if the agent takes initial action $x_0$.
$\uparrow$ $\mu$ gives reward 0 in all other cases.
$\uparrow$ $\pi$ always takes action $x_0$ if the first percept was $y_0$, or $x_1$ otherwise.
$\uparrow$ $\rho$ always takes action $x_1$.

Clearly $V^\pi_\mu = V^\rho_\mu$ (the first percept is not $y_0$, so $\pi$ and $\rho$ act the same). But

$$\pi^\dagger(x_0|y_1) = \pi^\dagger(x_0|y_0 \sim |) = \pi(x_0|y_0 \sim |) = 1, \text{ (Claim 1)}$$

i.e., $\pi^\dagger$ takes initial action $x_0$, so receives a nonzero reward from $\mu$ for doing so, whereas $\rho^\dagger = \rho$ does not. Thus $V^\pi_{\mu^\dagger} \neq V^\rho_{\mu^\dagger}$. By Definition 3.2 Condition 1, $V^\pi_{\mu^\dagger} \neq V^\rho_{\mu^\dagger}$, so since $V^\pi_{\mu^\dagger} = V^\rho_{\mu^\dagger}$, $\mu \not\sim \mu^\dagger$.

Claims 1 & 2 together contradict Definition 3.10. □

We do not know whether the mapping $\bullet^\times$ in Proposition 5.8 can be made into a weak translation.

If there is no strong translation from $\mathcal{F}_{AP}$ to $\mathcal{F}_{PA}$ (Open Question 8.2) then that would suggest an interesting asymmetry in reinforcement learning, since we showed that there is a strong translation from $\mathcal{F}_{PA}$ to $\mathcal{F}_{AP}$ (Proposition 5.2). Is it fundamentally harder to convert an action-first agent into a percept-first agent (in such a way as to preserve relative intelligence) than vice versa?

The following example offers some insight into why the answers to Open Questions 8.2 and 8.3 might be negative.
Example 5.9. (Some troublesome environments)

- Consider a percept-first environment which always rewards the agent for reacting immediately to the environment’s latest percept. As an intuitive example, the environment might consist of a military drill instructor giving the agent random drill instructions (“About face! Forward march! Halt!”) to be immediately obeyed. Percept-first agents of the form defined in Proposition 5.7 would always react to the percepts up to but not including the latest percept, and so would apparently tend to perform uniformly poorly, regardless of the underlying action-first agents’ intelligence.

- Consider a percept-first environment whose initial percept includes an English-text message declaring a certain cryptographic key and informing the agent that all subsequent percepts shall be encrypted using that key. All subsequent percepts are indeed so encrypted. Percept-first agents of the form defined in Proposition 5.8 would ignore the environment’s initial percept, and thus would have no hope of understanding later percepts (if the encryption method is strong). Such agents would tend to perform uniformly poorly, regardless of the underlying action-first agents’ intelligence.

Example 5.9 suggests weaknesses in the agent transformations of Propositions 5.7 and 5.8 due to the transformed agents not taking all available percepts into consideration when deciding how to act. Can we translate agents from $A^{AP}$ to $A^{PA}$ so that the translated agent takes all percepts into account? Proposition 5.5 was one negative result of this type. Here is another.

Proposition 5.10. Define $\bullet^* : A^{AP} \to A^{PA}$ by

$$\pi^*(x|h) = \sum_{x_0 \in A} \pi(x_0|\langle \rangle)\pi(x|x_0 \bowtie h).$$

Assuming there exists some $y \in P$ with $R(y) = 0$ and some $y \in P$ with $R(y) \neq 0$, then there is no $\bullet, : E^{PA} \to E^{AP}$ such that $(\bullet^*, \bullet,)$ is a pre-translation from $F_{AP}$ to $F_{PA}$.

Proof. Similar to Proposition 5.5.

6. Deterministic Reinforcement Learning

In this section, we consider the special case of deterministic RL. As a guiding question, suppose a certain research lab publishes deterministic RL agents designed for deterministic RL environments that give rewards from $\mathbb{N}$. But suppose a certain factory needs deterministic RL agents that can interact with deterministic RL environments that give rewards from all of $\mathbb{Z}$. Can the former agents be transformed into the latter type of agents in such a way as to preserve relative intelligence? We will prove a theorem (Theorem 6.7) which suggests the answer may be “no”.

Definition 6.1. A probability distribution is deterministic if it assigns probability 1 to one element of its domain and probability 0 to all others.

Definition 6.2. Let $F = (A, E, H)$ be an RL framework.

1. An agent $\pi \in A$ is deterministic if for all $h \in H_A$, $\pi(h)$ is deterministic.
(2) An environment \( \mu \in E \) is deterministic if for all \( h \in H_E \), \( \mu(h) \) is deterministic.

(3) \( \mathcal{F} \) is deterministic if all its agents and environments are deterministic.

If environment \( \mu \) is deterministic, we will abuse notation and write \( \mu(h) \) for the unique percept \( y \in \mathcal{P} \) such that \( \mu(y|h) = 1 \). We will abuse notation likewise for deterministic agents.

**Definition 6.3.** Suppose \( \mathcal{F} = (A, E, H) \) is a deterministic RL framework. For every agent \( \pi \in A \) and environment \( \mu \in E \), the infinite path in \( H \) determined by \( \pi \) and \( \mu \) is the infinite path \((h_0, h_1, \ldots)\) where:

- \( h_0 = \langle \rangle \).
- If \( h_i \in H_A \) then \( h_{i+1} = h_i \triangledown \pi(h_i) \).
- If \( h_i \in H_P \) then \( h_{i+1} = h_i \triangledown \mu(h_i) \).

**Lemma 6.4.** Suppose \( \mathcal{F} = (A, E, H) \) is a deterministic RL framework, \( \pi \in A \), and \( \mu \in E \). Let \( Q = (h_0, h_1, \ldots) \) be the infinite path in \( H \) determined by \( \pi \) and \( \mu \). Then \( V^{\pi}_\mu = \sum_{i=0}^{\infty} R(h_i) \).

**Proof.** For each \( t \in \mathbb{N} \), \( V^{\pi}_{\mu,t} \) was defined (Definition 2.4) to be the expected value of \( \sum_{i=0}^{t} R(g_i) \) where \( g_0 = \langle \rangle \) and where each \( g_{i+1} \) is randomly generated based on the probability distribution \( \pi(g_i) \) (if \( g_i \in H_A \)) or \( \mu(g_i) \) (if \( g_i \in H_P \)). Since \( \pi \) and \( \mu \) are deterministic, it follows that \((g_0, \ldots, g_t) = (h_0, \ldots, h_t)\) with probability 1, thus \( V^{\pi}_{\mu,t} = \sum_{i=0}^{t} R(h_i) \). Since \( V^{\pi}_{\mu} = \lim_{t \to \infty} V^{\pi}_{\mu,t} \), the lemma follows. \( \square \)

We will prove a theorem suggesting that deterministic RL agents designed for deterministic RL with rewards from \( \mathbb{N} \) cannot be converted into deterministic RL agents designed for deterministic RL with rewards from \( \mathbb{Z} \) in such a way as to preserve relative intelligence. But we will state the theorem in greater generality. The following two definitions distil the key properties of \( \mathbb{Z} \) and \( \mathbb{N} \) which allow this negative result.

**Definition 6.5.** Let \( \mathcal{F} = (A, E, H) \) be a deterministic RL framework. We say \( \mathcal{F} \) has the weak descending rewards property if for every family \( \{h_i\}_{i=0}^{\infty} \) of histories in \( H_E \) such that \( h_i \not\subseteq h_j \) for all \( i \neq j \), there exists an environment \( \mu \in E \) such that each \( R(\mu(h_i)) > R(\mu(h_{i+1})) \) and \( R(\mu(h)) = 0 \) for all \( h \not\in \{h_0, h_1, \ldots\} \).

**Definition 6.6.** Let \( \mathcal{F} = (A, E, H) \) be an RL framework. We say \( \mathcal{F} \) has well-founded rewards if the following requirement holds for every environment \( \mu \in E \): there does not exist any infinite family \( \{\pi_i\}_{i=0}^{\infty} \) of agents in \( A \) such that \( \{V^{\pi_i}_{\mu}\}_{i=0}^{\infty} \) is a strictly descending sequence.

Definition 6.5 (resp. 6.6) should be thought of as a weak assumption about deterministic RL environments with rewards from \( \mathbb{Z} \) (resp. \( \mathbb{N} \)).

**Theorem 6.7.** (“Deterministic RL with well-founded rewards does not translate to deterministic RL with ill-founded rewards”) Let \( \mathcal{F} = (A, E, H) \) and \( \mathcal{F}' = (A', E', H') \) be deterministic RL frameworks. If \( \mathcal{F} \) has well-founded rewards and \( \mathcal{F}' \) has the weak descending rewards property, then there is no pre-translation from \( \mathcal{F} \) to \( \mathcal{F}' \).
Proof: For sake of contradiction, assume \((\bullet^* : A \to A', \bullet_* : E' \to E)\) is a pre-translation from \(\mathcal{F} \) to \(\mathcal{F}'\). Let \(\pi_0 \in A\) and \(\mu_0 \in E'\) be arbitrary. Let \(Q_0\) be the infinite path in \(H'\) determined by \(\pi_0^*\) and \(\mu_0\). Let \(h_0\) be any element of \(Q_0 \cap H'_A\).

Now, inductively, suppose we have defined agents \(\pi_0, \ldots, \pi_k \in A\) and histories \(h_0, \ldots, h_k \in Q_0 \cap H'_A\) such that for all \(0 \leq i < k\):

1. \(h_i\) is in the infinite path \(Q_i\) determined by \(\pi_i^*\) and \(\mu_0\).
2. \(h_i \subset h_{i+1}\).
3. \(\pi_i^*(h_i) \neq \pi_{i+1}^*(h_i)\).
4. \(\pi_{i+1}^*\) agrees with \(\pi_i^*\) up to \(h_i\) (cf. Definition 3.5).

By Lemma 3.6 (with \(h = h_k\)) there exists \(\pi_{k+1} \in A\) and \(h^+ \in Q_k \cap H'_A\) such that \(h_k \subset h^+\) and such that \(\pi_{k+1}^*(h^+) \neq \pi_k^*(h^+)\) and such that \(\pi_{k+1}^*\) agrees with \(\pi_k^*\) up to \(h^+\). So, letting \(h_{k+1} = h^+\), (1)-(4) all hold for \(\pi_0, \ldots, \pi_{k+1}\) and \(h_0, \ldots, h_{k+1}\). Thus, by induction, we may assume the existence of \(\pi_0, \pi_1, \ldots \in A\) and histories \(h_0, h_1, \ldots \in Q_0 \cap H'_A\) such that (1)-(4) hold for all \(k\).

By construction, \(\{h_i \sim \pi_i^*(h_i)\}_{i=0}^\infty\) is a family of histories in \(H'_{E'}\), no one of which is an initial segment of another. Since \(\mathcal{F}'\) has the weak descending rewards property, there is an environment \(\mu \in E'\) such that each \(R(\pi(h_i) \sim \pi_i^*(h_i))) > R(\pi(h_{i+1}^* \sim \pi_{i+1}^*(h_{i+1})))\) and \(R(\mu(h)) = 0\) whenever \(h \not\in \{h_i \sim \pi_i^*(h_i)\}_{i=0}^\infty\). It follows by Lemma 6.4 that each \(V^{\pi_i^*}_{\pi_i^*} = R(h_i \sim \pi_i^*)\), thus each \(V^{\pi_i^*}_\pi > V^{\pi_{i+1}^*}_{\pi_{i+1}}\). By Definition 3.2 Condition 1, each \(V^{\pi_i^*}_\pi > V^{\pi_{i+1}^*}_{\pi_{i+1}}\). This contradicts the assumption that \(\mathcal{F}\) has well-founded rewards.

\(\square\)

7. Converting Between Deterministic and Non-Deterministic RL

In this section, we consider four different types of RL corresponding to two binary possibilities: whether agents must be deterministic, and whether environments must be deterministic. This leads to \(4 \times 3 = 12\) questions about the ability to convert agents from one type of RL to another. For example: if a lab publishes deterministic agents intended to operate in possibly non-deterministic environments, can a factory convert them into possibly non-deterministic agents intended to operate in deterministic environments (while preserving relative intelligence)? We will consider the special case where rewards are integer-valued. In this special case, we are able to answer all twelve questions about existence of weak translations (subject to two mild additional assumptions).

One way to approach these questions might be to take the maximum RL frameworks from Definition 5.1 as RL frameworks with non-deterministic agents and non-deterministic environments, and restrict their agents and/or their environments to obtain RL frameworks with deterministic agents and/or environments. But we will take a more general approach. We will instead consider an arbitrary deterministic RL framework, whose agent-set and/or environment-set can be non-determinatized to obtain the other three RL frameworks.

**Definition 7.1.** Suppose \(\mathcal{F} = (A, E, H)\) is a deterministic RL framework.

1. For each \(h \in H_A\), the **available actions at** \(h\) in \(\mathcal{F}\) are
   \[ A_{\mathcal{F},h} = \{ x \in A : \exists \pi \in A \text{ such that } \pi(h) = x \}. \]

2. For each \(h \in H_E\), the **available percepts at** \(h\) in \(\mathcal{F}\) are
   \[ P_{\mathcal{F},h} = \{ y \in P : \exists \mu \in E \text{ such that } \mu(h) = y \}. \]
(3) By a **random agent based on** $\mathcal{F}$, we mean a function $\pi$ with domain $H_A$ such that the following hold:
   (a) For every $h \in H_A$, $\pi(h)$ is a finitely-supported probability distribution on $\{x \in A : h \rightarrow x \in H\}$.
   (b) $\pi(x|h) = 0$ whenever $\pi(x|h)$ is defined but $x \notin A_{\mathcal{F},h}$.
(4) By a **random environment based on** $\mathcal{F}$, we mean a function $\mu$ with domain $H_E$, such that the following hold:
   (a) For every $h \in H_E$, $\mu(h)$ is a finitely-supported probability distribution on $\{y \in P : h \rightarrow y \in H\}$.
   (b) $\mu(y|h) = 0$ whenever $\mu(y|h)$ is defined but $y \notin P_{\mathcal{F},h}$.
   (c) For every random agent $\pi$ based on $\mathcal{F}$, $V_\mu^\pi$ converges.
(5) By the **framework obtained from** $\mathcal{F}$ **by randomizing agents**, we mean the RL framework $\mathcal{F}^a = (A^a, E^a, H^a)$ where $E^a = E$, $H^a = H$, and $A^a$ is the set of random agents based on $\mathcal{F}$.
(6) By the **framework obtained from** $\mathcal{F}$ **by randomizing environments**, we mean the RL framework $\mathcal{F}^e = (A^e, E^e, H^e)$ where $A^e = A$, $H^e = H$, and $E^e$ is the set of random environments based on $\mathcal{F}$.
(7) By the **framework obtained from** $\mathcal{F}$ **by randomizing agents and environments**, we mean the RL framework $\mathcal{F}^{ae} = (A^{ae}, E^{ae}, H^{ae})$ where $A^{ae}$ is the set of random agents based on $\mathcal{F}$, $E^{ae}$ is the set of random environments based on $\mathcal{F}$, and $H^{ae} = H$.

**Lemma 7.2.** If $\mathcal{F} = (A, E, H)$ is a deterministic RL framework, then $\mathcal{F}^a$, $\mathcal{F}^e$, and $\mathcal{F}^{ae}$ really are RL frameworks.

**Proof.** Straightforward. \hfill $\square$

**Lemma 7.3.** Let $\mathcal{F} = (A, E, H)$ be a deterministic RL framework. Then:

1. There are weak translations from $\mathcal{F}^e$ to $\mathcal{F}$; $\mathcal{F}^a$; and $\mathcal{F}^{ae}$.
2. There are weak translations from $\mathcal{F}$ to $\mathcal{F}^a$, and from $\mathcal{F}$ to $\mathcal{F}^{ae}$.

**Proof.** We will prove there is a weak translation from $\mathcal{F}^e = (A^e, E^e, H^e)$ to $\mathcal{F} = (A, E, H)$; the other claims are similar. Let $\bullet^* : A^e \rightarrow A$ be the identity map ($A^e = A$) and let $\bullet_* : E \rightarrow E^e$ be the inclusion map from $E$ into $E^e$. It is straightforward to check that $(\bullet^*, \bullet_*)$ is a weak translation (for Condition 3 of Definition 3.2, use Condition 5 of Definition 2.3). \hfill $\square$

**Definition 7.4.** Let $\mathcal{F} = (A, E, H)$ be a deterministic RL framework. We say $\mathcal{F}$ has **rewards restricted to** $\mathbb{Z}$ if the following condition holds: for all $\pi \in A$, for all $\mu \in E$, $V_\mu^\pi \in \mathbb{Z}$.

The following is a weak constraint on RL environments (one which would certainly hold of all RL frameworks in practice). We will need it in order to obtain our first negative result (Proposition 7.10 below) about weak translatability from $\mathcal{F}$ into $\mathcal{F}^{ae}$.

**Definition 7.5.** Let $\mathcal{F} = (A, E, H)$ be a deterministic RL framework. We say $\mathcal{F}$ has **indicator environments** if the following condition holds: for every $h \in H_E$, there exists an environment $\mu \in E$ (called an **indicator environment for** $h$) such
that for all \( g \in H_E \),

\[
R(\mu(g)) = \begin{cases} 
1 & \text{if } g = h, \\
0 & \text{otherwise.}
\end{cases}
\]

**Lemma 7.6.** Let \( \mathcal{F} = (A, E, H) \) be a deterministic RL framework. If \( \mathcal{F} \) has indicator environments, then for every \( h \in H_E \), there exist environments \( \mu, \nu \in E \) such that \( R(\mu(h)) = 1 \) and \( R(\nu(h)) = 0 \).

*Proof.* Take \( \mu \) to be an indicator environment for \( h \), and take \( \nu \) to be an indicator environment for some \( h' \in H_E \) with \( h' \neq h \) (the fact that there exists some \( h' \neq h \) follows from Condition 5 of Definition 2.3).

**Lemma 7.7.** Let \( \mathcal{F} = (A, E, H) \) be a deterministic RL framework. Assume \( \mathcal{F} \) has indicator environments. Let \( I \) and \( J \) be two index sets, \( I \cap J = \emptyset \). Let \( \{h_i\}_{i \in I} \) and \( \{h_j\}_{j \in J} \) be families of histories in \( H_E^* \) and let \( \{y_i\}_{i \in I} \) be a family of percepts in \( P \), such that the following conditions hold:

1. For all \( i, i' \in I \), if \( i \neq i' \) then \( h_i \not\subseteq h_{i'} \).
2. \( \{h_i\}_{i \in I} \cap \{h_j\}_{j \in J} = \emptyset \).
3. For each \( j \in J \), \( y_j \in P_{\mathcal{F}, h_j} \) and \( R(y_j) = 0 \).

For every family \( \{p_i\}_{i \in I} \) of probabilities, there exist percept families \( \{y_i\}_{i \in I} \) and \( \{\hat{y}_i\}_{i \in I} \), and an environment \( \mu \in E^a^e \), such that:

1. For all \( i \in I \), \( R(y_i) = 1 \) and \( R(\hat{y}_i) = 0 \).
2. For all \( i \in I \), \( \mu(y_i|h_i) = p_i \) and each \( \mu(\hat{y}_i|h_i) = 1 - p_i \).
3. For all \( j \in J \), \( \mu(y_j|h_j) = 1 \).
4. For all \( h \not\in \{h_i\}_{i \in I} \cup \{h_j\}_{j \in J} \), there is a percept \( y \in P \) such that \( R(y) = 0 \) and \( \mu(y|h) = 1 \).

*Proof.* By Lemma 7.6, for every \( h \in H_E \) there exists some \( \mu_h \in E \) and some \( \nu_h \in E \) such that \( R(\mu_h(h)) = 1 \) and \( R(\nu_h(h)) = 0 \). By the axiom of choice, we may take \( \bullet \mapsto \mu_\bullet \) and \( \bullet \mapsto \nu_\bullet \) to be functions from \( H_E \) to \( E \). Let \( \mu \) assign to each \( h \in H_E \) the finitely-supported probability distribution

\[
\mu(y|h) = \begin{cases} 
p_i & \text{if } h = h_i \text{ for some } i \in I \text{ and } y = \mu_h(h), \\
1 - p_i & \text{if } h = h_i \text{ for some } i \in I \text{ and } y = \nu_h(h), \\
1 & \text{if } h = h_j \text{ for some } j \in J \text{ and } y = y_j, \\
0 & \text{if } h \not\in \{h_i\}_{i \in I} \cup \{h_j\}_{j \in J} \text{ and } y = \nu_h(h),
\end{cases}
\]
on \( \{y \in P : h \sim y \in H \} \). It is easy to check \( \mu \) is a random environment based on \( \mathcal{F} \) (Definition 7.1)—the only nontrivial thing to check is that \( V_\mu^\pi \) converges for every random agent \( \pi \) based on \( \mathcal{F} \); to see this, note that if \( y \) is any percept with \( R(y) \neq 0 \) then the only way for \( \mu(y|h) \) to be nonzero is for \( h \) to be one of the \( \{h_i\}_{i \in I} \) and for \( y \) to be \( \mu_h(h) \) (so \( R(y) = 1 \)), and (by condition C1) any random path generated by \( \pi \) and \( \mu \) can trigger at most one of these nonzero-reward percepts, and all together this implies \( V_\mu^\pi \) converges.

For each \( i \in I \), let \( y_i = \mu_{h_i}(h_i) \) and \( \hat{y}_i = \nu_{h_i}(h_i) \). By construction, \( \{y_i\}_{i \in I}, \{\hat{y}_i\}_{i \in I} \), and \( \mu \) together satisfy requirements (1)-(4). \( \square \)
Corollary 7.8. If deterministic RL framework $\mathcal{F} = (A, E, H)$ has indicator environments, then in $\mathcal{F}^{ae} = (A^{ae}, E^{ae}, H^{ae})$, there is an environment $\mu \in E^{ae}$ such that $\mu$ is deterministic and always gives reward 0 (i.e., $R(\mu(h)) = 0$ for all $h \in H_E^{ae}$).

Proof. By Lemma 7.7 with $I = J = \emptyset$. \qed

Lemma 7.9. Let $\mathcal{F} = (A, E, H)$ be an RL pre-framework and let $\pi \in A$, $\mu \in E$. There exists an infinite path $Q$ in $H$ such that for every $h \in Q$, for all $t \in \mathbb{N}$ such that $t$ is larger than the length of $h$, there is positive probability that $h$ is an initial segment of the random finite path $(h_0, \ldots, h_t)$ randomly generated as in Definition 2.4.

Proof. Let $H_0 \subseteq H$ be the minimum tree such that:

1. $\emptyset \in H_0$.
2. For all $h \in H_0 \cap H_A$, for all $x \in A$, if $\pi(x|h) > 0$ then $h \cap x \in H_0$.
3. For all $h \in H_0 \cap H_E$, for all $y \in P$, if $\mu(y|h) > 0$ then $h \cap y \in H_0$.

By König’s Lemma, $H_0$ has an infinite path $Q$, and by construction $Q$ has the desired properties. \qed

Proposition 7.10. Let $\mathcal{F} = (A, E, H)$ be a deterministic RL framework. If $\mathcal{F}$ has rewards restricted to $\mathbb{Z}$ and $\mathcal{F}$ has indicator environments, then there is no pre-translation of $\mathcal{F}$ into $\mathcal{F}^{ae}$.

Proof. For sake of contradiction, assume $(\bullet^* : A \rightarrow A^{ae}, \bullet_* : E^{ae} \rightarrow E)$ is a pre-translation.

By Corollary 7.8, we may pick some $\mu_0 \in E^{ae}$ such that $\mu$ is deterministic and $R(\mu_0(h)) = 0$ for every $h \in H_E^{ae}$. Let $\pi_0 \in A$ be arbitrary. By Lemma 7.9 there exists an infinite path $Q$ in $H^{ae}$ such that for every finite $h \subseteq Q$, there is positive probability, call it $P(h)$, of $h$ being an initial segment of the random infinite path obtained by randomly generating a path using $\pi_0$ and $\mu_0$.

Now inductively, suppose we’ve defined $\pi_0, \ldots, \pi_k \in A$ and $h_1 \subseteq \cdots \subseteq h_k$ in $Q \cap H_A^{ae}$ such that for all $0 < i \leq k$, $\pi_i^*(h_i)$ agrees with $\pi_0^*(h_i)$ up to $h_i$ but $\pi_i^*(h_i) \neq \pi_0^*(h_i)$. By Lemma 3.6, we may find $\pi_{k+1} \in A$ and $h_{k+1} \in Q \cap H_A^{ae}$ such that $h_k \subseteq h_{k+1}$ (if $k > 0$) and such that $\pi_{k+1}^*(h_{k+1})$ agrees with $\pi_k^*$ up to $h_{k+1}$ but $\pi_{k+1}^*(h_{k+1}) \neq \pi_k^*(h_{k+1})$. Thus, by induction, we may assume we have $\pi_0, \pi_1, \ldots \in A$ and $h_1 \subseteq h_2 \subseteq \cdots \subseteq h_k$ in $Q \cap H_A^{ae}$ with these properties for all $k$.

For each $i > 0$, let $x_i \in A$ be an action such that $\pi_i^*(x_i|h_i) > \pi_0^*(x_i|h_i)$ (there must be some such $x_i$ since $\pi_i^*(h_i)$ and $\pi_0^*(h_i)$ are unequal probability distributions on $A$). By simultaneous induction, choose probabilities $p_1, p_2, \ldots$ and define numbers $\Delta_1, \Delta_2, \ldots$ such that:

- $p_1 = 1$.
- Each $\Delta_i = p_i F(h_i)(\pi_i^*(x_i|h_i) - \pi_0^*(x_i|h_i))$.
- Each $0 < p_{i+1} < \min\{\Delta_j/2^{i-j} : 1 \leq j < i\}$.

By Lemma 7.7 (with $I = \{1, 2, \ldots\}$, $J = \{0, -1, -2, \ldots\}$, $h_j$ the $(-j)$th node of $Q \cap H_E^{ae}$ for all $j \in J$, and $y_j = \mu_0(h_j)$ for all $j \in J$) there is an environment $\mu \in E^{ae}$ and percepts $y_1, y_2, \ldots \in P$ such that each $R(y_i) = 1$, each $R(y_i) = 0$, each $\mu(y_i|h_i) \cap x_i = p_i$, and each $\mu(y_i|h_i \cap x_i) = 1 - p_i$, and such that for all
$j \in \{0, -1, -2, \ldots\}$, if $h$ is the $(-j)$th node of $Q \cap H_E^{\mu}$ then $\mu(\mu_0(h)|h) = 1$ (so that as long as the agent interacting with $\mu$ stays on $Q$, $\mu$ will also stay on $Q$).

Claim: For each $k \geq 1$, for each $\ell \not\in \{1, \ldots, k\}$ (including $\ell = 0$), $V^{\pi_{\ell+1}}_{\mu} > V^{\pi_{\ell}}_{\mu}$.

Fix such $k$ and $\ell$. To see the claim, first note that when an agent interacts with $\mu$, the only histories with nonzero probability of giving a nonzero reward are $\{h_i \sim x_i\}_{i=1}^{k}$. Now, $\pi_{\ell}^k$ agrees with $\pi_{\ell}^0$ up to $h_k$ (because they both agree with $\pi_{0}^{0}$ up to $h_k$). So $\pi_{\ell}^k$ and $\pi_{\ell}^0$ get the same expected reward from all reward-sources $\{h_i \sim x_i\}_{1 \leq i < k}$, thus for the purpose of comparing $V^{\pi_{\ell+1}}_{\mu}$ and $V^{\pi_{\ell}}_{\mu}$, we may ignore those reward-sources. The expected total reward which $\pi_{\ell}^k$ receives due to $h_k \sim x_k$ equals the product of:

- The probability that, when $\pi_{\ell}^{k}$ interacts with $\mu$, $h_k$ will be an initial segment of the result. Since $\pi_{\ell}^{k}$ agrees with $\pi_{\ell}^{0}$ up to $h_k$, and since $\mu$ agrees with $\mu_0$ on $P$, this probability is $F(h_k)$.
- The probability $\pi_{\ell}^{k}(x_k|h_k)$ of $\pi_{\ell}^{k}$ taking action $x_k$ in response to $h_k$.
- The probability $p_k$ of $\mu$ giving reward +1 in response to $\pi_{\ell}^{k}$ taking action $x_k$ in response to $h_k$.

All together: $\pi_{\ell}^{k}$ derives $p_kF(h_k)\pi_{\ell}^{k}(x_k|h_k)$ expected total reward due to reward-source $h_k \sim x_k$. By similar reasoning, $\pi_{\ell}^{0}$ derives $p_kF(h_k)\pi_{\ell}^{0}(x_k|h_k)$ reward from $h_k \sim x_k$, but this equals $p_kF(h_k)\pi_{\ell}^{0}(x_k|h_k)$ since $\pi_{\ell}^{k}$ agrees with $\pi_{\ell}^{0}$ up to $h_k$. So from reward-source $h_k \sim x_k$, $\pi_{\ell}^{k}$ derives $p_kF(h_k)(\pi_{\ell}^{k}(x_k|h_k) - \pi_{\ell}^{0}(x_k|h_k)) = \Delta_k$ more reward than $\pi_{\ell}^{0}$ does. Can $\pi_{\ell}^{k}$ make up for the deficit via the remaining reward-sources $\{h_i \sim x_i\}_{i > k}$? No; even assuming $\pi_{\ell}^{k}$ managed to reach all the histories $\{h_i \sim x_i\}_{i > k}$, the total expected reward from all of them together would be bounded above by

$$\sum_{i > k} p_i < \sum_{i > k} \Delta_k/2^{i-k} = \Delta_k.$$  

This proves the claim.

Thus $V^{\pi_{\ell+1}}_{\mu} > V^{\pi_{\ell}}_{\mu}$, and yet each $V^{\pi_{\ell+1}}_{\mu} > V^{\pi_{\ell}}_{\mu}$. By Definition 3.2 Condition 1, $V^{\pi_{\ell+1}}_{\mu} > V^{\pi_{\ell}}_{\mu}$, and yet each $V^{\pi_{\ell+1}}_{\mu} > V^{\pi_{\ell}}_{\mu}$. But $V^{\pi_{\ell+1}}_{\mu}$, $V^{\pi_{\ell}}_{\mu}$, $\ldots$ are integers (since $\mathcal{F}$ has rewards restricted to $\mathbb{Z}$). Impossible. \hfill \Box

We will also establish (in Proposition 7.13) a negative result in the opposite direction, replacing one weak hypothesis (indicator environments) with a different weak hypothesis (the cutoff property, which we define as follows).

**Definition 7.11.** Let $\mathcal{F} = (A, E, H)$ be a deterministic RL framework. We say $\mathcal{F}$ has the **cutoff property** if the following condition holds. For all $\mu_0 \in E$, for all $h \in H_A$, for all $x \in A_{\mathcal{F}, h}$ (Definition 7.1), there exists $\mu \in E$ such that:

1. ("$\mu$ gives the agent a nonzero reward for taking action $x$ in response to $h$") $R(\mu(h \sim x)) \neq 0$.
2. ("The above-mentioned reward is the only possible nonzero reward after $h$") For all $h' \in H_E$ with $h \subset h'$ and $h' \neq h \sim x$, $R(\mu(h')) = 0$.
3. ("$\mu$ agrees with $\mu_0$ everywhere else") $\mu(h') = \mu_0(h')$ for all $h' \in H_E$ with $h \not\subset h'$.

**Lemma 7.12.** Let $\mathcal{F} = (A, E, H)$ be a deterministic RL framework. For all $\pi, \rho \in A^{ae}$ and $w \in \mathbb{R}$ with $0 \leq w \leq 1$, there exists $\sigma \in A^{ae}$ such that for all $\mu \in E^{ae}$, $V^{\pi}_{\mu} = wV^{\pi}_{\mu} + (1-w)V^{\rho}_{\mu}$.
Proof. Define numbers \( \pi_*(h) \) for every \( h \in H \) by induction on \( h \) so that:

- \( \pi_*(\langle \rangle) = 1. \)
- \( \pi_*(h_0 \land x) = \pi_*(h_0)\pi(x|h_0) \) for all \( x \in A \).
- \( \pi_*(h_0 \land y) = \pi_*(h_0) \) for all \( y \in \mathcal{P} \).

Intuitively, \( \pi_*(h) \) is the conditional probability that \( h \) would be an initial segment of the random path resulting from \( \pi \) interacting with an environment \( \mu \), given that \( \mu \) produces the percepts in question. Define \( \rho_*(h) \) similarly.

For all \( h \in H^A \) and all \( \{x \in A : h \land x \in H^A \} \), define

\[
\sigma(x|h) = \frac{w\pi_*(h \land x) + (1-w)\rho_*(h \land x)}{w\pi_*(h) + (1-w)\rho_*(h)}
\]

whenever the denominator is nonzero; when the denominator is zero, define \( \sigma(h) \) arbitrarily (but in such a way that \( \sigma(h) \) is a finitely-supported probability distribution on \( \{x \in A : h \land x \in H^A \} \) and \( \sigma(x|h) = 0 \) whenever \( \sigma(x|h) \) is defined but \( x \notin A_{\mathcal{F},h} \) (Definition 7.11)). It is straightforward to show \( \sigma \in E^A \).

By induction, one can show that for every \( \mu \in E^A \), for every \( h \in H^A \), if \( p_\pi \) (resp. \( p_\rho; p_\sigma \)) is the probability of \( h \) being an initial segment of a random path generated randomly using \( \pi \) and \( \mu \) (resp. using \( \rho \) and \( \mu \); using \( \sigma \) and \( \mu \)), then \( p_\sigma = w p_\pi + (1-w)p_\mu \). It follows that \( V_{\mu,t} = w V_{\mu,t} + (1-w)V_{\mu,t} \) for all \( t \in \mathbb{N} \), thus \( V_{\mu} = w V_{\mu} + (1-w)V_{\mu} \). \( \square \)

**Proposition 7.13.** Let \( \mathcal{F} = (A, E, H) \) be a deterministic RL framework. If \( \mathcal{F} \) has rewards restricted to \( \mathbb{Z} \) and \( \mathcal{F} \) has the cutoff property, then there is no pre-translation of \( \mathcal{F}^A \) into \( \mathcal{F} \).

**Proof.** For sake of contradiction, assume \((\bullet^* : A^A \to A, \bullet_* : E \to E^A)\) is a pre-translation.

Let \( \pi \in A^A \) and \( \mu_0 \in E \) be arbitrary; note that \( \pi^* \) and \( \mu_0 \) are deterministic. Thus we may consider the infinite path \( Q \) determined by \( \pi^* \) and \( \mu_0 \) in \( H \).

By Lemma 3.6 (with \( h = \langle \rangle \)) there is some \( \rho \in A^A \) and some \( h^+ \in Q \cap H^A \) such that \( \rho^* \) agrees with \( \pi^* \) up to \( h^+ \) but \( \rho^*(h^+) \neq \pi^*(h^+) \). Since \( \mathcal{F} \) has the cutoff property, there is some \( \mu \in E \) such that:

1. \( R(\mu(h^+ \land \pi^*(h^+))) \neq 0. \)
2. \( R(\mu(h')) = 0 \) for all \( h' \in H_E \) with \( h^+ \subset h', h' \neq h^+ \land \pi^*(h^+) \).
3. \( \mu(h') = \mu_0(h') \) for all \( h' \in H_E \) such that \( h^+ \not\subseteq h' \).

By (3) it follows that \( h^+ \land \pi^*(h^+) \) (resp. \( h^+ \land \rho^*(h^+) \)) is an initial segment of the path determined by \( \pi^* \) (resp. \( \rho^* \)) and \( \mu \). By (1) and (2) it follows that

\[
V^\pi_{\mu} = R(\mu(h^+ \land \pi^*(h^+))) + \sum_{h' \subseteq h^+} R(h') \\
\neq R(\mu(h^+ \land \rho^*(h^+))) + \sum_{h' \subseteq h^+} R(h') \\
= V^\rho_{\mu}. 
\]

Assume \( V^\pi_{\mu} > V^\rho_{\mu} \) (the case \( V^\pi_{\mu} < V^\rho_{\mu} \) is similar). Definition 3.2 Condition 1 ensures \( V^\pi_{\mu} > V^\rho_{\mu} \).

By Lemma 7.12, there exist agents \( \{\sigma_w\}_{0 \leq w \leq 1} \) in \( A^A \) such that each \( V^\pi_{\mu} = wV^\pi_{\mu} + (1-w)V^\rho_{\mu} \). Since \( V^\pi_{\mu} > V^\rho_{\mu} \), this implies that for all reals \( 0 < w < w' < 1 \),
Figure 1. Weak translations

\[ V_{\sigma}^w < V_{\mu}^{\sigma w} \], and thus (by Definition 3.2 Condition 1) \( V_{\mu}^{\sigma w} < V_{\mu}^{\sigma w} \). This is impossible because \( F \) has rewards restricted to \( \mathbb{Z} \) so each \( V_{\mu}^{\sigma w} \in \mathbb{Z} \). □

In the following theorem we answer the twelve questions which we described at the beginning of this section (assuming integer-valued rewards and the two additional weak constraints of indicator environments and the cutoff property).

**Theorem 7.14.** (See Figure 1) Let \( F = (A, E, H) \) be a deterministic RL framework with rewards restricted to \( \mathbb{Z} \), such that \( F \) has indicator environments and the cutoff property. For all \( G, G' \in \{ F, F^a, F^e, F^{ae} \} \) with \( G \neq G' \), the following are equivalent:

1. There is a weak translation from \( G \) to \( G' \).
2. \( G = F^e \) or \( G' = F^a \).

**Proof.** (2 ⇒ 1): By Lemma 7.3.

(1 ⇒ 2): Assume \( G \neq F^e \) and \( G' \neq F^a \).

Case 1: \( G = F \) and \( G' = F^{ae} \). We are done by Proposition 7.10.

Case 2: \( G = F^{ae} \) and \( G' = F \). We are done by Proposition 7.13.

Case 3: \( G = F \) and \( G' = F^e \). By Lemma 7.3 there is a weak translation from \( F^e \) to \( F^{ae} \). Thus, if there were a weak translation from \( F \) to \( F^e \), then by transitivity (Lemma 3.4), there would be a weak translation from \( F \) to \( F^{ae} \), contrary to Case 1.

The remaining 4 cases are diagram chases similar to Case 3. □

**8. Discussion**

We do not currently know whether a version of Theorem 7.14 holds for strong translations. To that end, consider the following (where \( A^{AP}, A^{PA}, E^{AP} \), and \( E^{PA} \) are as in Section 5).
**Open Question 8.1.** If $\mu, \nu \in E^{PA}$ and $\pi, \rho \in A^{PA}$ are such that $\mu$ and $\nu$ disagree about $\pi$ vs. $\rho$ (Definition 3.7), does that imply there exist deterministic $\pi', \rho' \in A^{PA}$ such that $\mu$ and $\nu$ disagree about $\pi'$ vs. $\rho'$? Similarly for $\mu, \nu \in E^{AP}$ and $\pi, \rho \in A^{AP}$.

Let $F_{PA}d$ be the restriction of $F_{PA}$ (Definition 5.1) to only deterministic agents and only deterministic environments with rewards limited to $\mathbb{Z}$. Suppose the percept-first part of Open Question 8.1 were answered affirmatively. Then a version of Lemma 7.3 (the positive part of Theorem 7.14), specialized to $F = F_{PA}d$, could be stated, with “weak” replaced by “strong”. The negative parts of Theorem 7.14 would all follow for $F = F_{PA}d$ ($F_{PA}d$ clearly has indicator environments and the cutoff property if the background reward function $R$ is not degenerate). Similar remarks apply for the action-first part of Open Question 8.1 and the corresponding restriction $F_{AP}d$ of $F_{AP}$.

**Open Question 8.2.** Is the weak translation of Proposition 5.7 strong? If not, is there any strong translation from $F_{AP}$ to $F_{PA}$?

**Open Question 8.3.** If $\bullet^\times$ is as in Proposition 5.8, is there some $\bullet^\times : E^{PA} \rightarrow E^{AP}$ which makes $\left(\bullet^\times, \bullet^\times\right)$ a weak translation?

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**Declaration of Competing Interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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