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CONFORMAL MAPPINGS OF STRETCHED POLYOMINOES ONTO HALF-PLANE

We give an algorithm for finding conformal mappings onto the upper half-plane and conformal modules of some types of polygons. The polygons are obtained by stretching along the real axis polyominoes i. e., polygons which are connected unions of unit squares with vertices from the integer lattice. We consider the polyominoes of two types, so-called the $P$-pentomino and the $L$-tetromino. The proofs are based on the Riemann-Schwarz reflection principle and uniformization of compact simply-connected Riemann surfaces by rational functions.

1 Introduction

Recently many authors have studied the computation of moduli of quadrilaterals and rings and capacities of condensers and ring domains, see, e. g., [1], [2], [3], [4], [5]). Special attention is devoted to the case of polygonal boundaries of domains.

It is very important for the theory of conformal and quasiconformal mappings to study behavior of the modules under various distortions of domains by quasiconformal maps, for example, under the stretching $f_H : x + iy \mapsto Hx + iy, H > 0$, with respect to the real axis (the problem by M.Vuorinen).

One more important direction of investigations is development of approximate methods of calculating of conformal modules and capacities such as method of decomposition of domain, finite element method and others (see, e. g., [2], [3], [4], [5], [6], and [7]).

Along with approximate methods, it is of interest obtaining exact analytic formulas for domains of special kind. The formulas could be used for checking results obtained by approximate methods.

One of the main model domains is so-called $L$-polygon, i. e., hexagon which is the difference of two rectangles with common vertex, in addition, one of them contains another. Such polygons were studied by many specialists, see , e. g., [2], [4], [5], [8]. We consider $L$-polygons of the form

$$P_{a_1,a}^{b_1,b} := ([0, a] \times [0, b]) \setminus ([0, a_1] \times [0, b_1]), \quad 0 < a_1 < a, \ 0 < b_1 < b.$$ (1)
Fixation of four vertices turns it to a quadrilateral. If we choose the vertices \(a_1, a, ib,\) and \(ib_1,\) then the module of the quadrilateral is four times less than the module of the doubly-connected domain \((-a, a) \times (-b, b) \setminus \left([-a_1, a_1] \times [-b_1, b_1]\right).\) Further, when we say about the module of an \(L\)-polygon we will have in mind exactly such choice of vertices.

In the paper we find conformal mappings of \(L\)-polygons of special type onto the upper half-plane. The polygons are obtained from polyominos by stretching. We recall that polyomino is a polygon which is the union of a finite number of squares of the same size; the intersection of two distinct squares, if nonempty, is their common size or vertex; see., e. g., [6]. For simplicity we consider unit squares with vertices from the integer lattice. For \(n = 3\) it is said about trimino, for \(n = 4\) the polyominos are called tetrominoes, and for \(n = 5\) they are pentominoes.

Making use the notation (1) let us denote \(S := P_{12}^{12},\) \(P := P_{12}^{13},\) and \(R := P_{13}^{23}.\) The figure \(S\) is a trimino. In connection with their similarity of form with Latin letters, the polygon \(P\) is called a \(P\)-pentomino, and \(R\) is said to be an \(L\)-tetromino.

A conformal mapping of the trimino \(S,\) stretched by the map \(f_H,\) onto the upper half-plane and its module are found in [8], [18]; we should note that a formula for the module for non-stretched trimino can be obtained from results of in [7], see also [3].

Here we find formulas for obtaining conformal mappings of the stretched figures \(P_H := f_H(P)\) and \(R_H := f_H(R)\) onto the upper half-plane and, as a corollary, their modules(Theorems 1 and 2).

The proofs use the Schwarz-Cristoffel symmetry principle and uniformization of simply-connected Riemann surfaces by rational functions.

## 2 \(P\)-pentomino

Let \(H > 0.\) Consider the polygon \(P_H = ABCEFL\) where \(A = i,\) \(B = 3i,\) \(C = 2H + 3i,\) \(E = 2H,\) \(F = H,\) and \(L = H + i\) (Fig. 1). Let us denote by \(\text{Mod}(P_H)\) its module, i. e., the extremal length of the family of curves joining its sides \(AB\) and \(EF.\)

Let us map the rectangle \(ABCD\) onto the upper half-plane. First we map \(DMGL\) onto the part of the unit disk lying in the first quarter of the plane \(w = \varphi + i\psi,\) i. e., the domain \(\{ |w| < 1, \varphi > 0, \psi > 0 \}\) by the function \(f\) so that the vertices \(L, G, M,\) and \(D\) are mapped on \(0, i, 1,\) and \(a;\) here
\( a \in (0, 1) \). It is easy to see that \( a \) is defined from the relations

\[
a = \sqrt{\lambda}, \quad 2 \frac{K(\lambda)}{K(\lambda')} = H. \tag{2}
\]

Here

\[
K(\lambda) = \int_0^1 \frac{d\xi}{\sqrt{(1 - \xi^2)(1 - \lambda^2 \xi^2)}}
\]

is the elliptic integral of the first kind and \( \lambda' = \sqrt{1 - \lambda^2} \).

By the reflection principle \( f \) can be extended to the rectangle \( ABCD \). Next we extend \( f \) through the segment \( LD \) to the rectangle \( DEFL \). As a result, we obtain the conformal mapping of \( P_H \) onto the domain \( \Sigma \) represented on Fig. 1. It is obtained by gluing \( \{|w| < 1, \varphi > 0, \psi < 0]\) and the upper half-plane along the segment \([0, a]\).

Now we find a conformal mapping of \( \Sigma \) onto the half-plane. To do this we map the quarter of the unit disk \( \{ |\zeta| < 1, \xi > 0, \eta < 0\} \) in the \( \zeta \)-plane, \( \zeta = \xi + i\eta \), onto \( \Sigma \) by the function \( h \) such that the points \(-i, 0, \) and \( 1 \) are mapped on themselves; in addition, the point \( h(1) \) lies on the lower edge of the slit along the segment \([a, 1]\). Meanwhile some points \( c, \alpha_0 \in (0, 1) \) are mapped on \( \infty \) and \( a \); besides, \( c < \alpha_0 \).
The function $h$ is extended by symmetry to a conformal map of the unit disk onto the three-sheeted Riemann surface which is obtained from $\Sigma$ by gluing the domains symmetric to it with respect to the coordinate axes. It is easy to see that, in addition, points on the unit circle are mapped on points on the unit circle, and the extended function has a zero of the third order at the origin and two simple poles at a pair of symmetric points $z = \pm c$.

Further we extend $h$ by symmetry to the extended complex plane. The extended function maps the Riemann sphere onto a 5-sheeted Riemann surface $\mathcal{R}$. In addition, by symmetry principle, it has zeroes at the points $\zeta = \pm 1/c$ and a pole of the third order at the infinity. It follows that $h$ is a rational function and has the form

$$h(\zeta) = \zeta^3 \frac{1 - c^2 \zeta^2}{\zeta^2 - c^2}. $$

Because of $\mathcal{R}$ has four branched points over the points $\pm a$, the derivative $h'(\zeta)$ vanishes at the points $\pm \alpha_0$ and $\pm 1/\alpha_0$.

Now we will find connections between $\alpha_0$ and $c$, and an equation to find $\alpha_0$ if the value $a$ is fixed. We have $h(\alpha_0) = a$, $h'(\alpha_0) = 0$. Therefore, taking into account the equality

$$\frac{h'(\zeta)}{h(\zeta)} = \frac{3 \zeta - \frac{2 \zeta}{\zeta^2 - c^2} + \frac{2 \zeta}{\zeta^2 - 1/c^2}}{\zeta^3 \frac{1 - c^2 \zeta^2}{\zeta^2 - c^2}} \tag{3}$$

we obtain

$$3 - \frac{2 \alpha_0^2}{\alpha_0^2 - c^2} + \frac{2 \alpha_0^2}{\alpha_0^2 - 1/c^2} = 0, \tag{4}$$

which implies

$$\alpha_0^4 - \frac{1}{3} (5c^2 + 1/c^2)\alpha_0^2 + 1 = 0. \tag{5}$$

On the other hand, from (4) it follows that

$$\frac{2 \alpha_0^2}{\alpha_0^2 - c^2} = 3 - \frac{2 \alpha_0^2}{\alpha_0^2 - 1/c^2} = \frac{3 - 5 \alpha_0^2}{1 - \alpha_0^2 c^2} \tag{6}$$

and

$$h(\alpha_0) = \frac{\alpha_0^3 (1 - \alpha_0^2 c^2)}{\alpha_0^2 - c^2} = \frac{\alpha_0 (3 - 5 \alpha_0^2 c^2)}{2},$$

whence, subject to $h(\alpha_0) = a$, it follows that

$$\alpha_0 (3 - 5 \alpha_0^2 c^2) = 2a. \tag{6}$$
From (6) we find
\[ c^2 = \frac{3\alpha_0 - 2a}{5\alpha_0^3}. \] (7)

Substituting (7) to (5) we have
\[ \alpha_0^4 - \frac{1}{3} \left( \frac{3\alpha_0 - 2a}{\alpha_0^3} + \frac{5\alpha_0^3}{3\alpha_0 - 2a} \right) \alpha_0^2 + 1 = 0. \]

After simplification we obtain to the following equation to find \( \alpha_0 \):
\[ 2\alpha_0^6 - 3a\alpha_0^5 + 3a\alpha_0 - 2a^2 = 0. \] (8)

After finding \( \alpha_0 \) from (8) we get the value \( c \) by (7). Then we find the points \( \alpha_1, \alpha_2, \) and \( \alpha_3 \) lying on the positive part of the abscissa axis in the \( \zeta \)-plane, which correspond to the vertices \( C, B, \) and \( A \) of \( P_H \). They satisfy the relations \( h(\alpha_1) = 1/a, h(\alpha_2) = -1/a, h(\alpha_3) = -a \). Therefore, the points can be got from the following equations
\[ c^2 x^5 - x^3 + (1/a)x^2 - (1/a)c^2 = 0, \] (9)
\[ c^2 x^5 - x^3 - (1/a)x^2 + (1/a)c^2 = 0, \] (10)
and
\[ c^2 x^5 - x^3 - ax^2 + ac^2 = 0. \] (11)

After finding \( \alpha_2 \) and \( \alpha_3 \) we can get the module of \( P_H \) from the condition
\[ \text{Mod}(P_H) = \frac{K(\mu)}{K(\mu')}, \] (12)
where
\[ \mu = \frac{1 + \alpha_2^2}{1 - \alpha_2^2} \cdot \frac{1 - \alpha_3^2}{1 + \alpha_3^2}. \] (13)

Here \( K(\mu) \) is the elliptic integral of the first kind, see, e. g., [11].

Really, let us map \( \{ |\zeta| < 1, \xi > 0, \eta < 0 \} \) onto the upper half-plane by the function
\[ \omega = \left( \frac{1 - \varpi^2}{1 + \varpi^2} \right)^2. \]
The points \( \alpha_k \) change under the mapping to
\[ \beta_k = [(1 - \alpha_k^2)/(1 + \alpha_k^2)]^2, \quad 1 \leq k \leq 3, \] (14)
and the points from the arc of the unit circle are mapped on points of the negative part of the real axis. It follows that the module of $P_H$ equals the extremal length of the family of curves lying in the upper half-plane and joining the segment $[\beta_2, \beta_3]$ with the negative part of the real axis, i.e.,

$$\frac{K(\mu)}{K(\mu')} \quad \mu = \frac{\beta_2}{\beta_3}.$$ 

It is easy to see that under the mapping the points $L, F, E,$ and $C$ change to $1, \infty, 0,$ and $\beta_1$.

Now we can formulate

**Theorem 1.** Let $a \in (0, 1)$ is defined by (2), $\alpha_0$ is a unique on $(0, 1)$ root of the equation (8), and $c$ is given by (7). Let $\alpha_1, \alpha_2,$ and $\alpha_3$ are unique roots of the equations (9), (10), and (11) in $(0, 1)$, and $\beta_k, 1 \leq k \leq 3,$ are defined by (14). Then $\text{Mod}(P_H)$ can be calculated by (12), and the conformal map of the upper half-plane onto $P_H$ can be expressed via the Schwarz-Cristoffel integral

$$z(\zeta) = -\frac{3i}{l} \int_0^\zeta \frac{\sqrt{t-1}}{\sqrt{t(t-\beta_1)(t-\beta_2)(t-\beta_3)}} dt + 2H$$

where

$$l = \int_0^{\beta_1} \frac{\sqrt{|t-1|}}{\sqrt{|t(t-\beta_1)(t-\beta_2)(t-\beta_3)|}} dt.$$ 

The proof of Theorem 1 follows from the reasoning above. We only mention some qualifying details.

The equation (8) has a unique root in $(0, 1)$. Really, let $g(x) := 2x^6 - 3ax^5 + 3ax - 2a^2$.

Then

$$g'(x) = 12x^5 - 15ax^4 + 3a = 12(1-a)x^5 + 3a(4x^5 - 5x^4 + 1) > 0, \quad 0 < x < 1,$$

since $a < 1$ and $4x^5 - 5x^4 + 1 > 0, \quad 0 < x < 1$. Thus, $g$ increases on $[0, 1]$. In addition, $h(0) < 0$ and $h(1) > 0$, therefore, the equation $g(x) = 0$ has a unique solution $\alpha_0$ in $(0, 1)$.

Now we prove that $c$, defined by (7), satisfies $0 < c < \alpha_0$. We have

$$g\left(\frac{2a}{3}\right) = -\frac{160}{243}a^6 < 0.$$
Since \( g \) increases on \((0, 1)\) and \( g(\alpha_0) = 0 \), we have \( \frac{2a}{3} < \alpha_0 \) and from (7) we obtain \( c > 0 \). To prove that \( c < \alpha_0 \) we note that (8) can be written in the form \( \alpha_0^5(2\alpha_0 - 3a) + a(3\alpha_0 - 2a) = 0 \), therefore, \( a(3\alpha_0 - 2a) = \alpha_0^5(3a - 2\alpha_0) < 5a\alpha_0^4 \). From the latter inequality we obtain

\[
c = \frac{3\alpha_0 - 2a}{5a\alpha_0^2} < \alpha_0.
\]

At last, we prove that every of equations (9), (10), and (11) has a unique solution on \((0, 1)\). From (3), taking into account (4) and the equality \( h'(\alpha_0) = 0 \), we have

\[
\frac{h'(x)}{h(x)} = \frac{3(x^4 - \frac{1}{3}(5c^2 + 1/c^2)x^2 + 1)}{x(x^2 - c^2)(x^2 - 1/c^2)} = \frac{3(x^2 - \alpha_0^2)(x^2 - 1/\alpha_0^2)}{x(x^2 - c^2)(x^2 - 1/c^2)}.
\]

From (16) we deduce that \( h \) is negative and decreases in \((0, c)\). It decreases on \((c, \alpha_0)\) from \(+\infty\) to \( a \in (0, 1) \) and increases on \((\alpha_0, 1)\) from \( a \) to \( 1 \). Therefore, since \( 0 < a < 1 \), every of the equations \( h(\alpha_1) = 1/a \), \( h(\alpha_2) = -1/a \), and \( h(\alpha_3) = -a \) has a unique solutions in \((0, 1)\). But they are equivalent to (9), (10), and (11).

Now we give some examples of calculating the module of \( P_H \) and the accessory parameters of the Schwarz-Cristoffel integral (15) using the package Mathematica.

**Example 1.** For the non-stretched pentomino \((H = 1)\) we have

\[
\lambda = 3 - 2\sqrt{2} = 0.171572875253809, \quad a = \sqrt{2} - 1 = 0.414213562373095,
\]

\[
\alpha_0 = 0.277046760238506, \quad c^2 = 0.025524633877222, \quad \beta_1 = 0.896160135941632,
\]

\[
\alpha_1 = 0.165536032447626, \quad \beta_2 = 0.908495458702734, \quad \alpha_2 = 0.154876249272631,
\]

\[
\alpha_3 = 0.138335266800084, \quad \beta_3 = 0.926301104551506, \quad \mu = 0.990342209151293,
\]

\[
\Mod(P_H) = 2.137318917840447.
\]
Example 2. For $H = 2$ we obtain

$$
\lambda = \sqrt{2}/2 = 0.707106781186547, \quad a = 0.840896415253714,
$$

$$
\alpha_0 = 0.601898824534568, \quad c^2 = 0.113643234509673,
$$

$$
\alpha_1 = 0.415838661746455, \quad \beta_1 = 0.497227390863205,
$$

$$
\alpha_2 = 0.301418612412185, \quad \beta_2 = 0.694601063871823,
$$

$$
\alpha_3 = 0.290931295908172, \quad \beta_3 = 0.712214555130066,
$$

$$
\mu = 0.987557290912592, \quad \text{Mod}(P_H) = 2.056221831167256.
$$

Remark 1. Using the method of [18] we can obtain the following asymptotics:

$$
\text{Mod}(P_H) \sim H/2, \quad H \to \infty.
$$

We see that $a, \lambda \to 1, c^2 \to 1/5, \alpha_0, \alpha_1 \to 1, \alpha_2, \alpha_3 \to (3 - \sqrt{5})/2, \beta_1 \to 0,$

and $\beta_2, \beta_3 \to 1/9$ as $H \to \infty.$

For example, for $H = 10$ we have $\text{Mod}(P_H) = 5.398278068084735,$ and for $H = 14 \text{ Mod}(P_H) = 7.3982087317929.$

3 L-tetromino

Consider the polygon $R_H = ABCEFL$ where $A = 2i$, $B = 3i$, $C = 2H + 3i$, $E = 2H$, $F = H$, and $L = H + 2i$. Denote by $\text{Mod}(R_H)$ its module, i.e., the extremal length of the family of curves joining in $R_H$ its sides $AB$ and $EF$.

Let us map conformally the rectangle $QLMC$ onto the domain $\{|w| < 1, \phi > 0, \psi > 0\}$ by a function $f$ such that the vertices $L, Q, C,$ and $M$ are mapped on $0, i, 1,$ and $a$ where $a \in (0, 1)$. It is easy to see that $a$ is defined from the relations

$$
a = \sqrt{\lambda}, \quad 2 \frac{K(\lambda)}{K'(\lambda')} = H. \quad (17)
$$

Here, as above, $K(\lambda)$ is the elliptic integral of the first kind and $\lambda' = \sqrt{1 - \lambda^2}$.

As in the previous section, we extend $f$ by symmetry to $R_H$, and the extension maps conformally $R_H$ onto the domain is obtained by gluing the upper half of the unit disk and the quarter of the plane $\{|w| < 1, \phi > 0, \psi < 0\}$ along the segment $[0, a]$. The function $g(z) = (f(z))^2$ maps conformally $R_H$ onto the 2-sheeted Riemann surface $\tilde{R}$ which is a result of gluing the
Consider the function \( h \) which maps conformally the half-disk \( \{ |\zeta| < 1, \eta > 0 \} \) onto \( \tilde{R} \) such that it keeps 0 and maps \( \pm 1 \) on the points with affix 1 lying on the different sides of the slit. Next we extend \( h \) in the lower half-disk by symmetry through the segment \([ -1, 1 ]\). As a result we obtain the function in the unit disk which has a zero of the third order at the point \( \zeta = 0 \) and a simple pole at some point \( c \in (0, 1) \). In addition, the points of the unit circle are mapped to points of the unit circle. Further we extend \( h \) to the extended complex plane by symmetry. At last we conclude that \( h \) is a rational function of the form

\[
h(\zeta) = \frac{\zeta^3 (1 - c\zeta)}{\zeta - c}.
\]  

We may assume that \( c > 0 \). Actually, if \( c < 0 \), then we change \( h(\zeta) \) to

\[
h(-\zeta) = \frac{\zeta^3 (1 + c\zeta)}{\zeta + c}
\]

which has the same form as (18) but correspond to the parameter \((-c)\).

Let us find the point \( \alpha_0 \in (0, c) \) which is mapped on the point \( \lambda = a^2 \),
the endpoint of the slit of $\tilde{R}$. We have
\[
\frac{h'(\zeta)}{h(\zeta)} = \frac{3}{\zeta} - \frac{1}{\zeta - c} - \frac{c}{1 - c\zeta}.
\] (19)

From $h'(\alpha_0) = 0$ and (19) it follows that
\[
\frac{3}{\alpha_0} - \frac{1}{\alpha_0 - c} - \frac{c}{1 - c\alpha_0} = 0
\] (20)
or
\[
\frac{3}{\alpha_0} - \frac{1 - c^2}{(\alpha_0 - c)(1 - c\alpha_0)} = 0.
\]
This implies
\[
\alpha_0^2 - \frac{2}{3} \left( \frac{2c + 1}{c} \right) \alpha_0 + 1 = 0.
\] (21)

On the other hand, from (18) and (20) we have
\[
\lambda = h(\alpha_0) = \alpha_0^3 \frac{1 - c\alpha_0}{\alpha_0 - c} = 3\alpha_0^2(1 - c\alpha_0) - c\alpha_0^3 = 3\alpha_0^2 - 4\alpha_0^3.
\]

Therefore,
\[
c = \frac{3\alpha_0^2 - \lambda}{4\alpha_0^3}.
\]

After substituting the obtained expression for $c$ in (21) we get
\[
\alpha_0^2 - \frac{2}{3} \left( \frac{3\alpha_0^2 - \lambda}{2\alpha_0^3} + \frac{4\alpha_0^3}{3\alpha_0^2 - \lambda} \right) \alpha_0 + 1 = 0
\]
or, after transformation,
\[
\alpha_0^6 - 3\lambda\alpha_0^4 + 3\lambda\alpha_0^2 - \lambda^2 = 0.
\] (22)

Now we find the points $\alpha_1 \in (-1, 0)$ and $\alpha_3 \in (0, 1)$ which are mapped on $\lambda$ and $1/\lambda$ under the map (18). The points can be found from the relations
\[
x^3 \frac{1 - cx}{x - c} = \lambda,
\]
\[
x^3 \frac{1 - cx}{x - c} = \frac{1}{\lambda},
\]
or, after simplification,

\[ cx^4 - x^3 + \lambda x - c\lambda = 0, \]  

(23)

\[ \lambda cx^4 - \lambda x^3 + x - c = 0. \]  

(24)

Denote \( \alpha_2 = c \). The desired module is equal to the extremal length of the family of curves joining the segments \( 1, \alpha_1 \) and \( \alpha_2, \alpha_3 \) in the unit upper half-disk. Let us map the half-disk onto the upper half-plane by the function

\[ \omega = \left( \frac{1 + \zeta}{1 - \zeta} \right)^2. \]

The points \( \alpha_k \) are changed to

\[ \beta_k = \left( \frac{1 + \alpha_k}{1 - \alpha_k} \right)^2. \]  

(25)

We need to calculate the extremal length of the curves joining the segments \([-\infty, \beta_1]\) and \([\beta_2, \beta_3]\) in the upper half-plane. It is equal to

\[ \text{Mod}(R_H) = \frac{2K(\mu)}{K(\mu')} \]  

(26)

where

\[ \mu = \frac{1 - A}{1 + A}, \quad A = \sqrt{\frac{\beta_1(\beta_3 - \beta_2)}{\beta_2(\beta_3 - \beta_1)}}. \]  

(27)

Thus, we have

**Theorem 2.** Let \( \lambda \) is defined from (17) and \( \alpha_0 \) is a unique solution of the equation (22). Let

\[ \alpha_2 = \frac{3\alpha_0^2 - \lambda}{4\alpha_0^3}, \]  

(28)

\( \alpha_1 \) be a unique root of the equation (23) lying in \((-1, 0)\), and \( \alpha_3 \in (0, 1) \) be a unique root of (24) lying in \((0, 1)\). Define \( \beta_k, 1 \leq k \leq 3 \), by (25). Then \( \text{Mod}(R_H) \) can be found from (26), (27). The conformal map of the upper half-plane onto \( R_H \) is expressed via the Schwarz-Cristoffel integral

\[ z(\zeta) = \frac{3i}{l} \int_{\beta_3}^{\zeta} \frac{\sqrt{t - 1}}{\sqrt{t - \beta_1}(t - \beta_2)(t - \beta_3)} \, dt + 2H \]  

(29)
where
\[ l = \int_{\beta_3}^{+\infty} \frac{\sqrt{1-t}}{\sqrt{t} (t-\beta_1)(t-\beta_2)(t-\beta_3)} \, dt. \]

As in the case of Theorem 1, the main details of the proof are substantiated above; we only give some additional comments for explanation.

The equation (22) has a unique root in $(0, 1)$. Really, let
\[ r(x) := x^6 - 3\lambda x^4 + 3\lambda x^2 - \lambda^2. \]

Then
\[ r'(x) = 6x(x^4 - 2\lambda x^2 + \lambda) > 0, \quad x \in (0, 1), \]

since $\lambda \in (0, 1)$. Thus $h$ increases on $[0, 1]$. We have $r(0) = -\lambda^2 < 0$, $r(1) = 1 - \lambda^2 > 0$, therefore, the equation $r(x) = 0$ has a unique solution $\alpha_0$ in $(0, 1)$.

Now we show that $\alpha_2$, defined by (28), satisfies the inequality $0 < \alpha_2 < \alpha_0$. Since $r(x)$ increases on $(0, 1)$ and $r(\sqrt[3]{\lambda/3}) = -(8/27)\lambda^3 < 0$ we conclude that $\alpha_0 > \sqrt[3]{\lambda/3}$. From the latter inequality it follows that $\alpha_2 > 0$. From (28) we obtain $\lambda(3\alpha_0^2 - \lambda) = \alpha_0^4(3\lambda - \alpha_0^2) < 4\lambda\alpha_0^4$, therefore, $3\alpha_0^2 - \lambda < 4\lambda\alpha_0^4 \Rightarrow \alpha_2 < \alpha_0$.

At last, (23) has a unique solution in $(-1, 0)$ because it is equivalent to the equation $r(x) = \lambda$ and $r(x)$ decreases from 1 to 0 on the interval. The equation (24) has a unique solution in $(0, 1)$ because it is equivalent to $r(x) = 1/\lambda$ and $r(x)$ decreases on $(0, \alpha_2)$ from $+\infty$ to $\lambda$ and increases on $(\alpha_2, 1)$ from $\lambda$ to 1.

Now we consider examples.

**Example 3.** Non-stretched tetromino ($H = 1$):

\[ \lambda = 3 - 2\sqrt{2} = 0.171572875253809, \quad \alpha_0 = 0.245789017659106, \]
\[ \alpha_1 = -0.464160413208352, \quad \beta_1 = 0.133934441008549, \]
\[ \alpha_2 = 0.162705672169886, \quad \beta_2 = 1.928338552532678, \]
\[ \alpha_3 = 0.163434755042730, \quad \beta_3 = 1.934124519781231, \]
\[ A = 0.014941124900985, \quad \mu = 0.970557651996923, \]
\[ \text{Mod}(R_H) = 3.558625812230538. \]
Example 4. Let the coefficient of stretch $H = 2$. Then

$$
\lambda = \sqrt{2}/2 = 0.707106781186547, \quad \alpha_0 = 0.570342096440027,
$$

$$
\alpha_1 = -0.872092067525734, \quad \beta_1 = 0.004668104309033,
$$

$$
\alpha_2 = 0.362162999160609, \quad \beta_2 = 4.560775977283309,
$$

$$
\alpha_3 = 0.401188235613446, \quad \beta_3 = 5.475355432587552,
$$

$$
A = 0.013080991981584, \quad \mu = 0.974175821903443,
$$

$$
\text{Mod}(R_H) = 3.643277348370991.
$$

Remark 2. By the methods developed in [18] we can obtain the following asymptotics:

$$
\text{Mod}(R_H) \sim H, \quad H \to \infty.
$$

We see that $a, \lambda, \alpha_0 \to 1, \alpha_1 \to -1, \alpha_2 \to 1/2, \alpha_3 \to 1, \beta_1 \to 0, \text{ and } \beta_2 \to 9,$

$$
\beta_3 \to \infty, \quad A \to 0 \text{ as } H \to \infty.
$$

For example, for $H = 10$ we have $\text{Mod}(R_H) = 11.323877993993085$, and

for $H = 14 \text{ Mod}(R_H) = 15.323814523768869.$

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References

[1] R. Kühnau. The conformal module of quadrilaterals and of rings. In: Handbook of Complex Analysis: Geometric Function Theory, (ed. by R. Kühnau) Vol. 2. North Holland, Amsterdam: Elsevier, 2005, 99–129.

[2] D. Betsakos, K. Samuelsson, and M. Vuorinen. The computation of capacity of planar condensers. Publications De L’institut Mathematique, 75(89), 233–252, (2004).

[3] H. Hakula, A. Rasila, and M. Vuorinen. On moduli of rings and quadrilaterals: Algorithms and Experiments. Siam Journal on Scientific Computing, 33(1), 279–302, (2011).
[4] V. N. Dubinin and M. Vuorinen. On conformal moduli of polygonal quadrilaterals. Israel Journal of Mathematics, 171(1), 111–125, (2009).

[5] M. Vuorinen and X. Zhang. On exterior moduli of quadrilaterals and special functions. J. Fixed Point Theory Appl. 13(1), 215–230, (2013).

[6] Golomb, Solomon W. Polyominoes (2nd ed.). Princeton, New Jersey: Princeton University Press, 1994.

[7] F. Bowman. Introduction to elliptic functions with applications. English Universities Press Ltd., London, 1953.

[8] E. V. Borisova and S. R. Nasyrov. Asymptotics of the module of doubly-connected domain which is the difference of two homothetic rectangles. Trudi Matem. Tzentr. N. I. Lobachevskogo, 2011, 44, 62–63. (in Russian)

[9] L. V. Ahlfors. Lectures on quasiconformal mappings. D. Van Nostrand Company, Toronto–New York–London, 1966.

[10] S. R. Nasyrov. Asymptotics of the module of a rectangular frame under stretching it along a coordinate axis. Complex Analysis and Appl. Materiali VI Petrozavodskoi Mezhd. Konf. (1–7 July, 2012, Petrozavodsk, Petr. Univ), 51–53. (in Russian)

[11] G. D. Anderson, M. K. Vamanamurthy, M. Vuorinen. Conformal invariants, inequalities, and quasiconformal Maps. New York: Wiley, 1997.

[12] R. Kühnau. Zum konformen Modul eines Vierecks. Mitt. Math. Seminar Giesen, 211, 61–67, (1992). (in German)

[13] R. Kühnau. Der konforme Modul schmaler Vierecke. Math. Nachr., 175, 193–198, (1995). (in German)

[14] D. Gaier and W. K. Hayman. Moduli of long quadrilaterals and thick ring domains. Rend. Math. Appl., 10(7), 809–834, (1990).

[15] D. Gaier and W. K. Hayman. On the computation of modules of long quadrilaterals. Constr. Approx., 7, 453–467, (1991).

[16] R. Laugesen. Conformal mapping of long quadrilaterals and thick doubly connected domains. Constr. Approx., 1994, 10, 523-554, (1994).
[17] M. I. Falcão, N. Papamichael, and N. S. Stylianopoulos. Approximating the conformal maps of elongated quadrilaterals by domain decomposition. Constr. Approx., 17, 589-617, (2001).

[18] S. R. Nasyrov. Riemann–Schwarz reflection principle and asymptotics of modules of rectangular frames. arXiv:1305.6605 [math.CV].

[19] http://www.math.udel.edu/~driscoll/software/

[20] T. A. Driscoll and L. N. Trefethen. Schwarz-Christoffel Mapping. Cambridge Monographs on Applied and Computational Mathematics 8, Cambridge University Press, 2002.

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