Entanglement entropy of Bell-network states in LQG: Analytical and numerical results

Eugenio Bianchi,\textsuperscript{a} Pietro Donà,\textsuperscript{a} Ilya Vilensky\textsuperscript{b}

\textsuperscript{a}Institute for Gravitation and the Cosmos \& Physics Department, Penn State, University Park, PA 16802, USA
\textsuperscript{b}Florida Atlantic University, 777 Glades Road, Boca Raton, FL 33431, USA

E-mail: ebianchi@gravity.psu.edu, pxd81@psu.edu, ilya.vilensky@fau.edu

ABSTRACT: Bell-network states are loop-quantum-gravity states that glue quantum polyhedra with entanglement. We present an algorithm and a code that evaluates the reduced density matrix of a Bell-network state and computes its entanglement entropy. In particular, we use our code for simple graphs to study properties of Bell-network states and to show that they are non-typical in the Hilbert space. Moreover, we investigate analytically Bell-network states on arbitrary finite graphs. We develop methods to compute the Rényi entropy of order $p$ for a restriction of the state to an arbitrary region. In the uniform large-spin regime, we determine bounds on the entanglement entropy and show that it obeys an area law. Finally, we discuss the implications of our results for correlations of geometric observables.

Contents

1 Introduction 2
2 Entanglement entropy and Rényi entropy in LQG 4
3 Bell-network states and vector geometries 5
4 Large-spin asymptotic analysis of the entanglement entropy 6
5 Large-spin numerical analysis of the entanglement entropy 10
6 Discussion 19
A Diagrammatic method to compute $f_p$: a detailed example 22
1 Introduction

Elementary quanta of space—also known as quantum polyhedra [1]—are a characterizing feature of loop quantum gravity (LQG) [2–5]. These microscopic degrees of freedom are ultra-local and associated with the \( N \) nodes of a finite graph \( \Gamma \). The collective state of the system is an element of the kinematical Hilbert space of the theory which, at fixed spin-network graph \( \Gamma \) and spins \( j_\ell \), has the tensor-product structure

\[
\mathcal{H}_{\Gamma j_\ell} = \bigotimes_{n \in \Gamma} \mathcal{H}_n .
\]  

(1.1)

Here \( \mathcal{H}_n \) is the Hilbert space of the \( SU(2) \) intertwiner associated to each node \( n \), the quantization of a classical polyhedron [1, 6–9]. This decomposition is the basis of the geometric picture of quantum space: an LQG state is a many-body state of quantum polyhedra with the adjacency relations given by the connectivity of the graph and areas given by the spins [10]. A generic state in the space \( \mathcal{H}_{\Gamma j_\ell} \) is a linear superposition of quantum polyhedra,

\[
|s\rangle = \sum_{i_1 \ldots i_N} c_{i_1 \ldots i_N} |i_1\rangle \cdots |i_N\rangle .
\]  

(1.2)

The factorized orthonormal basis associated to the tensor product (1.1), denoted \( |\Gamma, j_\ell, i_n\rangle = |i_1\rangle \cdots |i_N\rangle \), is called the spin-network basis. This is a basis of simultaneous eigenstates of a maximal commuting set of operators that are \textit{ultralocal}, i.e., each operator measures a geometric property of a single quantum polyhedron such as its volume or the dihedral angle between two faces [1, 6–15]. As a result, spin-network basis states are factorized over polyhedra: they are un-entangled. On the other hand a typical state of this many-body system has the form (1.2) and represents entangled polyhedra.

The connectivity of the graph \( \Gamma \), together with the factorized structure of the Hilbert space \( \mathcal{H}_{\Gamma j_\ell} \), allows us to define regions of the graph and their associated Hilbert space. Specifically, we call \( A \) a region if the set of nodes \( n \in A \) is path connected with respect to the graph structure. The associated Hilbert space is \( \mathcal{H}_A = \bigotimes_{n \in A} \mathcal{H}_n \) and, denoting \( \overline{A} \) the complement of \( A \), we have a bipartition of the Hilbert space as the tensor product

\[
\mathcal{H}_{\Gamma j_\ell} = \mathcal{H}_A \otimes \mathcal{H}_{\overline{A}} .
\]  

(1.3)

Given this structure, it is immediate to define the entanglement entropy \( S_A \) of a pure state \( |s\rangle \) restricted to the subsystem \( A \). Let us assume that \( 1 \ll \dim \mathcal{H}_A \leq \sqrt{\dim \mathcal{H}_{\Gamma j_\ell}} \). In this case, Page’s result [16] states that a typical state in the Hilbert space \( \mathcal{H}_{\Gamma j_\ell} \) has entanglement entropy

\[
S_A(\text{typical}) \approx \log(\dim \mathcal{H}_A) - \frac{1}{2}(\dim \mathcal{H}_A)^2 / \dim \mathcal{H}_{\Gamma j_\ell} .
\]  

(1.4)

This result indicates that the restriction \( \rho_A \) of a typical pure state in \( \mathcal{H}_{\Gamma j_\ell} \) is close to being maximally mixed when the subsystem \( A \) is small. It is instructive to consider the case of a graph \( \Gamma \) that is dual to an equal-area triangulation. In this case, all spins are assumed to
be equal to $j_0$ and the dimension of the Hilbert space of a node—a
equal-area quantum
tetrahedron—is simply $\dim \mathcal{H}_n = 2j_0 + 1$. The expectation value of the volume of a node
is $v(j_0) = \text{Tr}(\hat{V}_n \rho_n) = \frac{1}{2j_0+1} \sum_k v_k$, where $v_k$ are the eigenvalues of the volume
operator $\hat{V}_n$ and $\rho_n$ is the maximally mixed state. When expressed in terms of these parameters, the
entanglement entropy of a typical state in $\mathcal{H}_{\Gamma j_0}$ is

$$S_A(\text{typical}) \approx \frac{\log(2j_0 + 1)}{v(j_0)} V_A - \frac{1}{2} \exp \left( - \frac{\log(2j_0 + 1)}{v(j_0)} (V_\Gamma - 2V_A) \right).$$

This is a volume law for the entanglement entropy of a typical state in the Hilbert space $\mathcal{H}_{\Gamma j_0}$.

On the other hand, it is known that taking into account the dynamics—and in particular
constraints such as the selection of an eigenstate of an Hamiltonian with local interactions
[17–19]—selects states that are non-typical in the Hilbert space and leads to a behavior of
the entanglement entropy that deviates qualitatively from Page’s law for typical states.

In this paper, we focus on Bell-network states and show that—instead of a volume law—
their entanglement entropy obeys an area law,

$$S_A(\text{Bell-network}) \approx \frac{\log(2j_0 + 1)}{a(j_0)} \text{Area}_A + \ldots ,$$

where $a(j_0) = 8\pi \gamma G \hbar \sqrt{j_0(j_0 + 1)}$ is the area eigenvalue of a boundary link.

Bell-network states are defined using squeezed vacuum techniques that enforce prescribed
correlations. In particular, Bell-network states [20] have correlations that reduce the general
twisted geometry [21, 22] at adjacent nodes to vector geometries [23, 24] by introducing Bell-
like correlation in the normals to faces of adjacent polyhedra. The structure of correlations
is well illustrated by the Bell state for two spin-1\!/2 particles,

$$|\mathcal{B}\rangle = \frac{|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2}{\sqrt{2}} = \sqrt{2} \int \frac{d\vec{n}}{4\pi} |\vec{n}\rangle_1 |\vec{n}\rangle_2 ,$$

which is given by a uniform superposition of back-to-back spins. For a given graph $\Gamma$ and
assignment $j_\ell$ of spins, there is a unique Bell-network state in $\mathcal{H}_{\Gamma j_\ell}$, here denoted $|\Gamma, j_\ell, \mathcal{B}\rangle$.
Its expansion coefficients in the basis (1.2) are given by the $SU(2)$ symbol of the graph.

Calculating the entanglement entropy $S_A$ of a region $A$ of a Bell-network state is non-
trivial. In this paper, we present analytical and numerical methods to compute it. We will
work only with finite graphs having a finite number of nodes. The problem is structured in
the same way as the standard entanglement entropy computation in many-body quantum
systems, where one considers a state (e.g. the ground state of a specific Hamiltonian) and
then computes the entanglement entropy for various subsystems [16–19]. In this work, we
present a numerical code [25] that, for a given graph, first evaluates the expansion of the
Bell-network state on a factorized basis, and then computes the entanglement entropy of
various subsystems. We present explicit versions of the code adapted to different graphs and
subregions. We consider: the dipole graph, the pentagram graph with subregions containing
one or two nodes, and the hexagram graph with subregions containing one or two nodes (either connected or disconnected).

To identify qualitative features of the behavior of the entanglement entropy for any Bell-network state and arbitrary subsystem, we employ analytical methods which provide good approximations under the assumption of uniformly large spins. Under a homogeneous rescaling of the spins of the state $j_\ell \to \lambda j_\ell$, we derive a bound for the leading order in $\lambda \gg 1$ of the entanglement entropy of a region $A$,

$$(|\partial A| - c_A) \log \lambda \leq S_A \leq (|\partial A| - 3) \log \lambda,$$

(1.8)

where $c_A$ is a half-integer and $|\partial A|$ is the number of links that cross the boundary of $A$. In this regime the bound implies an area-law behavior for $S_A$. We determine also the behavior of the Rényi entropy of order $p$ of any Bell-network state and arbitrary region. When compared to our numerical data, we find good agreement within our approximation.

The expectation that entanglement in the degrees of freedom of the gravitational field is a necessary condition for the emergence of a classical spacetime is shared by various approaches to nonperturbative quantum gravity [26–34]. The result that Bell-network states satisfy an area law supports the conjecture that entanglement can be used as a probe of semiclassicality in quantum gravity [28].

The paper is structured as follows. In Section 2 we give an elementary introduction to entropic inequalities for the entanglement entropy and the Rényi entropy, together with their application to LQG. In Section 3 we review the definition of Bell-network states and their relation to vector geometries. In Section 4 we present the large-spin asymptotic analysis of the Rényi entropy and the entanglement entropy for a Bell-network state on a generic graph. In Section 5 we present our code and compare the numerical results to our analytical asymptotic formulae for some specific graphs. We conclude with a discussion of our results.

## 2 Entanglement entropy and Rényi entropy in LQG

A quantum system composed of two subsystems $A$ and $\bar{A}$ has a Hilbert space given by the tensor product:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}.$$  

(2.1)

Given a state $|\psi\rangle$ in $\mathcal{H}$, the reduced density matrix of the subsystem $A$ is defined by the partial trace over its complement $\bar{A}$

$$\rho_A = \text{Tr}_{\bar{A}} |\psi\rangle\langle\psi|.$$  

(2.2)

The entanglement entropy of the subsystem $A$ is defined as the von Neumann entropy of the reduced density matrix

$$S_A = -\text{Tr} (\rho_A \log \rho_A).$$  

(2.3)
It is also useful to define the Rényi entropy of order $p$, defined as
\[
R_A^{(p)} = -\frac{1}{p-1} \log \text{Tr} \rho_A^p ,
\] (2.4)
with $p \geq 0$. The limit $p \to 1$ reproduces the entanglement entropy, as can be easily shown:
\[
\lim_{p \to 1} R_A^{(p)} = -\lim_{\epsilon \to 0} \frac{1}{\epsilon} \log \text{Tr} \left( \rho_A \left( 1 + \epsilon \log \rho_A \right) \right)
= -\lim_{\epsilon \to 0} \frac{1}{\epsilon} \log \left( 1 + \epsilon \text{Tr} \rho_A \log \rho_A \right) = -\text{Tr} \left( \rho_A \log \rho_A \right) = S_A .
\] (2.5)

On the other hand, the limit $p \to 0$ of the Rényi entropy reproduces the maximum entropy $\mathcal{S}_A^{\text{max}} = \log(\dim \mathcal{H}_A)$,
\[
\lim_{p \to 0} R_A^{(p)} = \log \text{Tr} \lim_{p \to 0} \rho_A^p = \log(\dim \mathcal{H}_A) = \mathcal{S}_A^{\text{max}} .
\] (2.6)

The Rényi entropies satisfy the useful inequality
\[
R_A^{(p)} \leq R_A^{(p')} \quad \text{for} \quad p > p'
\] (2.7)
with the equality corresponding to the maximally mixed state $\rho_A = \mathbb{1}/\dim \mathcal{H}_A$ for which $R_A^{(p)} = \log(\dim \mathcal{H}_A)$. Considering the limits $p \to 0, 1$, these inequalities provide an upper and a lower bound on the entanglement entropy,
\[
R_A^{(p)} \leq S_A \leq \log(\dim \mathcal{H}_A) \quad \text{with} \quad p > 1 .
\] (2.8)

This relation is instrumental in our analysis.

The structures discussed above apply immediately to states in the LQG Hilbert space $\mathcal{H}_{\Gamma j_\ell} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ with fixed graph $\Gamma$, $N$ nodes and fixed spins $j_\ell$. In this case, a state of this many-body system is a quantum geometry consisting of $N$ entangled polyhedra.

A similar decomposition of the Hilbert space has been used for the investigation of entanglement in the intertwiner degrees of freedom in [35] and in [36], where a class of area-law states with spin 1/2 is studied. The decomposition discussed in these works differs from the edge-mode decomposition of [37–43], where the origin of an area law is not the entanglement between quantum polyhedra but the correlations of non-observable degrees of freedom arising from splitting links that cross the boundary of the region.

3 Bell-network states and vector geometries

The LQG Hilbert space at fixed graph $\Gamma$ and spins $j_\ell$ consist of a collection of quantum polyhedra, one for each node. Geometrical quantities like angles, areas, volumes, and shapes of these polyhedra are quantum operators in this Hilbert space. To glue two polyhedra together, we need to impose the matching of the shape of the face shared between the two. At
the quantum level, due to the uncertainty relations of shape operators, we cannot require two shape eigenstates to coincide, but we can impose gluing as expectation values. Moreover, we can require correlations between two adjacent quantum polyhedra so that also the fluctuations of the shape of two adjacent faces are correlated. Bell-network states [20, 31, 32] are a specific proposal that uniformly maximizes correlations of all neighboring polyhedra on a given graph. They are given by the formula

$$|\Gamma, j_\ell, B\rangle = \frac{1}{\sqrt{Z}} \sum_{i_n} \mathcal{A}_\Gamma(j_\ell, i_n) \bigotimes_{n} |i_n\rangle , \quad (3.1)$$

where $Z$ is a normalization and the amplitude $\mathcal{A}$ is the $SU(2)$-symbol of the graph $\Gamma$, i.e.,

$$\mathcal{A}_\Gamma(j_\ell, i_n) = \sum_{\{m\}} \prod_{n} [i_n]^{m_1 \cdots m_{F_n}} . \quad (3.2)$$

Here the intertwining tensors $[i_n]^{m_1 \cdots m_{F_n}}$ are contracted according to the connectivity of the graph $\Gamma$ and $F_n$ is the number of faces of the quantum polyhedron in the node $n$.

These states have an appealing geometrical interpretation: for large spins, they describe a uniform superposition of vector geometries, a collection of polyhedra glued together by requiring that the normals of adjacent faces are back-to-back, even though in general the faces don’t have the same shape. This class of geometries plays an essential role in the study of the asymptotic behavior of topological $BF$ $SU(2)$ spin foam vertex amplitudes [24, 44–48]. Bell-network states have built-in short-range correlations and are expected to satisfy an area law for the entanglement entropy. Proving that the area law arises at the gauge-invariant level is not immediate as it requires us to control correlations in the intertwiner degrees of freedom. Here we present the explicit computations necessary to determine this behavior.

4 Large-spin asymptotic analysis of the entanglement entropy

Given a Bell-network state on a general graph $\Gamma$ with fixed spins $j_\ell$ (3.1), we consider a region $A$ containing a certain number of nodes $N_A$. We denote $|\partial A|$ the number of links crossing the boundary of $A$, see Figure 1. The density matrix of $A$ is defined by tracing away the intertwiners in $\bar{A}$, i.e.,

$$\rho_A = \text{Tr}_{\bar{A}} |\Gamma, j_\ell, B\rangle \langle \Gamma, j_\ell, B| = \frac{1}{Z} \sum_{i_n, i'_n} M(i_n, i'_n) \bigotimes_{n \in A} |i_n\rangle \langle i'_n| . \quad (4.1)$$

The normalization $Z$ guarantees that $\text{Tr} \rho = 1$. The matrix $M(i_n, i'_n)$ is defined as

$$M(i_n, i'_n) = \sum_{k_n \in \bar{A}} \mathcal{A}_{\bar{\Gamma}}(j_\ell, i_n, k_n) \mathcal{A}_\Gamma(j_\ell, i'_n, k_n) , \quad (4.2)$$

where the sum is over the intertwiners $k_n$ associated to the nodes contained in $\bar{A}$, the complement of $A$. The normalization factor is easily expressed in terms of the matrix $M$ as
Figure 1: Example of a general spin network on a graph $\Gamma$. We shaded in green a region $A$ which contains $N_A = 4$ nodes and $|\partial A| = 7$ boundary links (marked with a cross). The region $A$ determines a subsystem $\mathcal{H}_A = \bigotimes_{n \in A} \mathcal{H}_n$.

$Z = \text{Tr}M$. The trace of the density matrix raised to a power $p$ can also be expressed in terms of $M$ as $\text{Tr} \rho_A^p = \text{Tr} M^p / (\text{Tr} M)^p$. This formula allows us to compute the Rényi entropies in terms of the matrix $M$:

$$R_A^{(p)} = -\frac{1}{p-1} \log \text{Tr} \rho_A^p = -\frac{1}{p-1} (\log \text{Tr} M^p - p \log \text{Tr} M) \, .$$

(4.3)

The ingredients needed in the formula are the traces $\text{Tr} M^p$. For instance, $\text{Tr} M$ can be written in terms of $SU(2)$-symbols as

$$\text{Tr} M = \sum_{i_n,k_n} A_{\Gamma} (j_{\ell}, i_n, k_n) A_{\Gamma} (j_{\ell}, i_n, k_n) = Z \, .$$

(4.4)

Clearly, these quantities can be computed using $SU(2)$ recoupling theory. However, no closed expression is available and one has to resort to numerical or symbolic codes, as the ones discussed in the next section. Here we are interested in the behavior of the Rényi entropies under uniform rescaling of the spins. To this end we introduce a reformulation in terms of auxiliary variables familiar in spin foam calculations. This reformulation allows us to estimate the value of the Rényi entropies analytically in the large-spin regime using a saddle point approximation.

To write the Rényi entropies in spinfoam-like variables, we express each of the $p \cdot N$ sums over intertwiners appearing in $\text{Tr} M^p$ as an integral over $SU(2)$ Wigner matrices,

$$\sum_i i_{m_1}\ldots i_{m_F} i_{n_1}\ldots i_{n_F} = \int_{SU(2)} dg D_{m_1}^{(j_1)}(g) \cdots D_{m_F}^{(j_F)}(g) \, .$$

(4.5)

The indices are contracted according to the connectivity of the graph (3.2) and result into $SU(2)$ characters. In particular, there is one $SU(2)$ character for each link crossing the boundary of $A$, and $p$ characters for each link completely inside or outside the region $A$. In total, there are $pL - (p-1)|\partial A|$ characters. By introducing a resolution of the identity in
terms of $SU(2)$ coherent states $|\vec{j}, \vec{n}\rangle$ [49], each $SU(2)$ character $\chi^{(j)} = \text{Tr}D^{(j)}$ can be expressed as an integral over a unit vector $\vec{n} \in S^2$,

$$\chi^{(j)}(g) = (2j + 1) \int_{S^2} d\vec{n} \langle j\vec{n} | g | j\vec{n} \rangle = (2j + 1) \int_{S^2} d\vec{n} \exp \left( 2j \log \langle \frac{1}{2} \vec{n} | g | \frac{1}{2} \vec{n} \rangle \right).$$  \hspace{1cm} (4.6)

The trace $\text{Tr}M^p$ can then be written as an integral over $SU(2)$ group elements $g_e$ and over unit vectors $\vec{n}_f$ as

$$\text{Tr}M^p = \int d\vec{n}_f dg_e e^{f_p(j_l, \vec{n}_f, g_e)}.$$ \hspace{1cm} (4.7)

The function $f_p(j_l, \vec{n}_f, g_e)$ is linear in $j_l$ and can be determined using diagrammatic techniques as illustrated for a specific example in the Appendix A.

In the large spin limit, the integral (4.7) can be evaluated using a saddle point approximation. Under a uniform rescaling of all the spins $j_l \rightarrow \lambda j_l$, the function scales as $f_p(j_l, \vec{n}_f, g_e) \rightarrow \lambda f_p(j_l, \vec{n}_f, g_e)$, and the leading order in $\lambda$ of the logarithm of the trace $\text{Tr}M^p$ is given by:

$$\log (\text{Tr}M^p) = \left( \#S^2\text{integrals} - \frac{1}{2} \#\text{Hessian}(f_p) \right) \log \lambda + O(1).$$ \hspace{1cm} (4.8)

The first term, the number of integrals over coherent states is due to the dimensional factor $2j + 1$ in (4.6). The second term, $\#\text{Hessian}$, is the rank of the Hessian of $f_p$ and it can be expressed in terms of the number of $SU(2)$ integrals, the number of $S^2$ integrals, the number of symmetries of $f_p$ and the dimension of the space of solutions of the system of saddle point equations (denoted $\#\text{space of solutions}$). If multiple saddle points exist, the dominant saddle point is characterized by the largest space of solutions. Therefore,

$$\#\text{Hessian}(f_p) = + 3 \times \#SU(2)\text{integrals}$$ \hspace{1cm} (4.9)

$$+ 2 \times \#S^2\text{integrals}$$ \hspace{1cm} (4.10)

$$- \#\text{symmetries}$$ \hspace{1cm} (4.11)

$$- \#\text{space of solutions}(f_p).$$ \hspace{1cm} (4.12)

The symmetries of $f_p$ are related to right and left $SU(2)$ multiplication of the integration group elements $g_e$. It can be shown that the function $f_p$ always has $2p$ $SU(2)$ symmetries, resulting in $\#\text{symmetries} = 3 \times 2p$. As already discussed, the number of $SU(2)$ integrals are $p$ times the total number $N$ of nodes of the graph $\Gamma$. Therefore,

$$\log (\text{Tr}M^p) = - \frac{1}{2} (3pN - 6p - \#\text{space of solutions}(f_p)) \log \lambda + O(1).$$ \hspace{1cm} (4.13)

The number $\#\text{space of solutions}$ is given by the total number of independent variables (2 per unit vector), minus the number of independent critical point equations and a global rotation. In the case $p = 1$, the number of independent critical point equations coming from $f_1$ is $3(N - 1)$ (i.e., $N - 1$ vectorial equations), resulting in $\#\text{space of solutions}(f_1) =$
2L − 3 − 3(N − 1) = 2L − 3N. This number appears in [24] where it is derived in the context of the asymptotic analysis of spin foam amplitudes of topological theories.

In the general case, denoting

\[ C_A^{(p)} = \# \text{redundant critical-point equations of } f_p , \]  

we find that the number of independent critical point equations of \( f_p \) is \( 3pN − C_A^{(p)} − 3 \), corresponding to \( pN \) total vectorial equations minus the number of redundant equations \( (C_A^{(p)}) \) and a global rotation.

The number \( C_A^{(p)} \) can be computed case-by-case for a given graph \( \Gamma \), region \( A \) and power \( p \). However, a general closed formula is not available. We note that \( C_A^{(p)} \) has two important properties: \( C_A^{(p)} \) is an integer and is bounded from below by

\[ C_A^{(p)} \geq 6 \left( p - 1 \right) , \]  

(4.15)

which is the number of equations that are redundant because of symmetries\(^1\). The resulting expression is

\[ \# \text{space of solutions} (f_p) = 2(2L − (p − 1)|\partial A|) − 3 − 3pN + C_A^{(p)} + 3. \]  

In summary, at the leading order in \( \lambda \), the Rényi entropy of a Bell-network state is

\[ R_A^{(p)} = - \frac{1}{2(p − 1)} \left( 2(pL − (p − 1)|\partial A|) − 3pN + C_A^{(p)} − p(2L − 3N) \right) \log \lambda + O(1) = \]

\[ = \left( |\partial A| − \frac{C_A^{(p)}}{2(p − 1)} \right) \log \lambda + O(1) . \]  

(4.16)

Using the properties of Rényi entropy, we can characterize the dependence of \( C_A^{(p)} \) on the order \( p \). From the set of inequalities (2.8) we find

\[ \frac{C_A^{(p)}}{2(p − 1)} \leq \frac{C_A^{(p+1)}}{2p} \Rightarrow C_A^{(p)} < \frac{p}{p − 1}C_A^{(p)} \leq C_A^{(p+1)} . \]  

(4.17)

The parameter \( C_A^{(p)} \) is monotonically increasing in \( p \) and, since \( R_A^{(p)} \) is positive, for each \( p > 1 \) is bounded from above by \( 2(p − 1)(|\partial A| − 3) \).

As discussed in Section 2 the entanglement entropy can be obtained from the limit of \( p \to 1 \) of the Rényi entropy of order \( p \). At the leading order in \( \lambda \gg 1 \),

\[ S_A = \lim_{p \to 1} R_A^{(p)} \approx \lim_{p \to 1} \left( |\partial A| − \frac{C_A^{(p)}}{2(p − 1)} \right) \log \lambda = (|\partial A| − c_A) \log \lambda \]  

(4.18)

\(^1\)We note that there are \( 2p \) symmetries \( f_p (j_\ell, \vec{n}_f, \tilde{g}_e) = f_p (j_\ell, \vec{n}_f, g_e) \) with \( \tilde{g}_e = hg_e \) or \( \tilde{g}_e = g_e h \) for some \( e \), with \( h \in SU(2) \). For \( h \) close to the identity, these symmetries result in a linear constraint \( \delta_h f_p (j_\ell, \vec{n}_f, \tilde{g}_e) = 0 \) on the critical point equations.
where we denoted as $c_A$ the limit $\lim_{p \to 1} \frac{C_A^{(p)}}{\frac{2}{(p-1)}}$. This limit exists and is finite since the entanglement entropy of a system with a finite number of degrees of freedom is a well-defined quantity. From the properties of $C_A^{(p)}$, we also find that

$$R_A^{(p)} \approx \left( |\partial A| - \frac{C_A^{(2)}}{2} \right) \log \lambda \leq (|\partial A| - 3) \log \lambda. \quad (4.19)$$

Note that the inequalities here are understood as asymptotic statements holding at the leading order in $\lambda \gg 1$. Combining the limit of this inequality (that is guaranteed to exist by the monotonicity of $C_A^{(p)}$) with the most strict of the inequality in (2.8) we can determine that at the leading order in $\lambda$ the entanglement entropy of a region $A$ of a Bell-network state is bounded from below and above by

$$\left( |\partial A| - \frac{C_A^{(2)}}{2} \right) \log \lambda \leq S_A \leq (|\partial A| - 3) \log \lambda. \quad (4.20)$$

Hence, the entanglement entropy of a Bell-network state satisfies an area law.

The explicit computation of $C_A^{(p)}$ requires the analysis of critical points equations for a given graph $\Gamma$ and subsystem $A$. In the special case of a general graph $\Gamma$ and a subsystem $A$ containing one single node, we can prove that $C_A^{(p)} = 6(p - 1)$, which is also the minimum allowed value of $C_A^{(p)}$. In this case, at leading order, $R_A^{(p)}$ is independent of $p$ and (4.18) implies that $S_A = (|\partial A| - 3) \log \lambda$.

5 Large-spin numerical analysis of the entanglement entropy

We provide a numerical code to compute the entanglement entropy and the Rényi entropy of any order of the density matrix of a Bell-network state restricted to a region $A$. In this section, we provide three explicit examples with Bell-network states defined on different graphs: the dipole graph, the pentagram graph, and the hexagram graph. We consider subsystems defined by regions $A_i$ containing one or two nodes. The code for the pentagram and the hexagram graph are available in [25].

The algorithm implemented in our C code is illustrated in the panel below (See Algorithm 1). The key ingredient of the calculation is the precomputation of all the $\{6j\}$ and $\{9j\}$ symbols needed for the evaluation of the symbol of the graph. To efficiently perform this task we employ the wigxjpf library and its extension fastwigxj developed in [50] and previously employed in LQG computations [24, 51].
Algorithm 1 Numerical algorithm for the evaluation of the Bell-network state entropy

1: Precompute the \( \{6j\} \) symbols with \texttt{wigxjpf} if needed
2: Precompute the \( \{9j\} \) symbols with \texttt{wigxjpf} if needed
3: Load the symbol tables in the memory
4: \textbf{for} each \( j_\ell \) \textbf{do}
5: \hspace{1em} \textbf{for} each \( i_n \) \textbf{do}
6: \hspace{2em} Assemble the matrix \( (A_\Gamma)_{ik} = A_\Gamma (j_\ell, i_n, k_n) \) with the intertwiners \( i_n, k_n \) in \( A, \bar{A} \)
7: \hspace{2em} Compute the matrix \( M_{ii'} = (A_{\bar{\Gamma}}^T \cdot A_\Gamma)_{ii'} \)
8: \hspace{2em} Normalize it to obtain the density matrix \( (\rho_A)_{ii'} = M_{ii'}/Tr(M) \)
9: \hspace{2em} Find its eigenvalues \( \rho_A \rightarrow \nu_i \)
10: \hspace{2em} Compute the entanglement entropy \( S_A = -\sum_i \nu_i \log \nu_i \)
11: \hspace{2em} Compute the Rényi entropy \( R_A^{(p)} = -\log \sum_i \nu_i^p \)

Figure 2: Graphs and regions considered in our numerical analysis.
The range of applicability of our numerical code is limited to spins up to $O(20)$ because of two factors. First, we need a considerable amount of RAM to keep accessible all $\{6j\}$ and $\{9j\}$ symbols required in the computation. For example, the computation of any symbol with spins up to 25 (both integers and half-integers) requires $\approx 15$ GB of available RAM, while its extension to spin 30 requires $\approx 40$ GB of available RAM, not commonly available on ordinary Laptop Computers. This obstacle can be possibly circumvented performing a selection of the symbols prepared and loaded by fastwigxj. At the present stage, we are not selecting symbols and therefore we need to load all of them in the memory. Second, we use an array of double-precision floating-point numbers to store the symbols. Compilers generally limit the size of this array to the amount of available RAM. For example, the array of symbols for the hexagram graph occupies $(2^j + 1)^6 \cdot 8$ Byte. We executed our code on a machine with 16 GB of RAM, therefore our maximum spin was limited by a hard cutoff at about $j_{\text{max}} \approx \frac{1}{2} \left( \sqrt[6]{16 \text{ GB}}/(8 \text{ Byte}) - 1 \right) \approx 17$.

In the following, we report the numerical computation of the entanglement entropy and the Rényi entropy of order two for a set of specific cases.

We set all spins equal $j_{\ell} = \frac{1}{2}$ and rescale them with a parameter $\lambda$ so that $j_{\ell} \to \lambda/2$. We will compare the numerical results to the analytical bounds derived in (4.20). We note that in all cases considered, the number of nodes in the region $A$ is small. As a result, the bound $R^{(0)}_A = \log \dim H_A$ on the entanglement entropy $S_A$ is tighter than the bound $(|\partial A| - 3) \log \lambda$.

5.1 The dipole graph

The dipole Bell-network state takes a simple form, computed explicitly in [20]. In this case, analytical computations of the entropy are possible and useful for checking some of the properties derived before. We consider a dipole graph with four links ($\Gamma_2$, see Figure 2a):

$$\langle \Gamma_2, j_{\ell}, B \rangle = \frac{1}{\sqrt{\dim H_1}} \sum_{i \in H_1} |i\rangle_1 |i\rangle_2 , \quad (5.1)$$

where $H_1$ is the intertwiner space of a node. The two intertwiners are maximally entangled. Choosing a region $A$ that contains a single node, the reduced density matrix is:

$$\rho_A = \text{Tr}_2 (|\Gamma_2, j_{\ell}, B \rangle \langle \Gamma_2, j_{\ell}, B |) = \frac{1}{\dim H_1} . \quad (5.2)$$

The resulting state is maximally mixed, and all the entropies are maximal and equal to

$$S_A = R_A^{(p)} = \log \dim H_1 . \quad (5.3)$$

We can verify the asymptotic formula (4.16) from the exact computation in the limit of large spins. In the four valent case $|\partial A| = 4$ the asymptotic estimate reduces to $S_A = R_A^{(p)} = \log \lambda$ which can also be obtained from the uniform rescaling of (5.3). A similar conclusion can be reached for a dipole graph with an arbitrary number of links greater or equal to three.
Figure 3: [Graph $\Gamma_5$, entropy of a single node]. The figure shows our numerical results on the Bell-network state with pentagram graph $\Gamma_5$ and equal spins $j_\ell = \lambda/2$. The entanglement entropy $S_A$ is denoted by blue dots, the Rényi entropy $R_A^{(2)}$ by red diamonds. The lower bound asymptotic estimate is shown as a solid orange line (the $O(1)$ is fitted using the numerical data). The upper bound estimate given by the maximal entropy is shown as a dashed orange line. We show also the bounds (4.20) as a yellow band. The inset shows the 20 data points with the largest spins.

5.2 The pentagram graph

The pentagram Bell-network state (see the graph $\Gamma_5$ in Figures 2b, and 2c) is a superposition of the intertwiner states weighted by the symbol of the graph, the $\{15j\}$ symbol,

$$|\Gamma_5, j_\ell, B\rangle = \frac{1}{\sqrt{Z}} \sum_{i_n} \{15j\} (j_\ell, i_n) |i_1\rangle |i_2\rangle |i_3\rangle |i_4\rangle |i_5\rangle .$$

One node subsystem of $\Gamma_5$. Choosing a region $A$ containing a single node, our asymptotic formula reduces to

$$S_A \approx R_A^{(p)} \approx \log \lambda$$

at the leading order. Note that $S_A$ and $R_A^{(p)}$ can differ by $O(1)$ terms. We used our code to compute the entanglement entropy and Rényi entropy of order two for all equal spins $j_\ell = \lambda/2$ and the scale parameter up to $\lambda \leq 50$. We report the numerical results in Figure 3. The plot shows clearly that $S_A$, $R_A^{(2)}$ and the maximal entropy $R_A^{(0)} = \log (\lambda + 1) \approx \log \lambda + O(\lambda^{-1})$
Figure 4: [Graph $\Gamma_5$, entropy of two connected nodes]. The figure shows our numerical results on the Bell-network state with pentagram graph $\Gamma_5$ and equal spins $j_\ell = \lambda / 2$. The entanglement entropy $S_{AB}$ is denoted by blue dots, the Rényi entropy $R^{(2)}_{AB}$ by red diamonds. The lower bound asymptotic estimate is shown as a solid orange line (the $O(1)$ is fitted using the numerical data). The upper bound estimate given by the maximal entropy is shown as a dashed orange line. We show also the bounds (4.20) as a yellow band. Using the 10 largest-spin data points, the entanglement entropy $S_{AB}$ is fitted by $a \log(\lambda) + b + c\lambda^{-1}$ on the last 10 data points obtaining $a \approx 1.94$, $b \approx -0.30$ and $c \approx 3.19$.

differ only by a constant contribution. We interpret this $O(1)$ difference as an indication that the restriction of the Bell-network state is not maximally mixed and the state is not typical in the Hilbert space.

**Two nodes subsystem of $\Gamma_5$.** If we choose a subsystem $AB$ containing two nodes, our asymptotic estimates reduce to

$$R^{(2)}_{AB} = \frac{3}{2} \log \lambda + O(1)$$

and to an asymptotic band for the entanglement entropy given by

$$\frac{3}{2} \log \lambda \leq S_{AB} \leq 3 \log \lambda.$$  

For this specific configuration, the bound from above given by the maximal entropy $S_{AB} \leq 2 \log \lambda$ is tighter. The results of the computation of the entanglement entropy and Rényi
entropy of order two for the case of all equal spins $j_{\ell} = \frac{\lambda}{2}$ and the scale parameter up to $\lambda \leq 44$ are reported in Figure 4.

5.3 The hexagram graph

The hexagram Bell-network state ($\Gamma_6$, see Figures 2d, 2e, and 2f) is a superposition of the intertwiner states weighted by the symbol of the graph, the $\{18j\}$ symbol,

$$|\Gamma_6, j_{\ell}, B\rangle = \frac{1}{\sqrt{Z}} \sum_{i_n} \{18j\} (j_{\ell}, i_n^\ast) |i_1\rangle |i_2\rangle |i_3\rangle |i_4\rangle |i_5\rangle |i_6\rangle .$$  \hspace{1cm} (5.8)

One node subsystem of $\Gamma_6$. Choosing a region $A$ containing a single node, our asymptotic formula reduces to

$$S_A = R_A^{(p)} = \log \lambda + O(1).$$  \hspace{1cm} (5.9)

Note that $S_A$ and $R_A^{(p)}$ can differ by $O(1)$ terms. As already done for the graph $\Gamma_5$, we compute the entanglement entropy and Rényi entropy of order two for the hexagram $\Gamma_6$ with all equal spins $j_{\ell} = \frac{\lambda}{2}$ and scale parameter up to $\lambda \leq 34$. We report the results in Figure 5. The plot shows that the $O(1)$ contributions differ for $S_A$, $R_A^{(2)}$ and the maximal entropy $R_A^{(0)} = \log (\lambda + 1) \approx \log \lambda + O(\lambda^{-1})$. We interpret this difference as an indication that the subsystem is not maximally mixed and the Bell-network state is not typical in the Hilbert space.

Two connected nodes subsystem of $\Gamma_6$. Choosing the subsystem $AB$ consisting of two connected nodes, our asymptotic formula reduces to

$$R_{AB}^{(2)} = \frac{3}{2} \log \lambda + O(1).$$  \hspace{1cm} (5.10)

The asymptotic band with upper and lower bounds on the entanglement entropy given by

$$\frac{3}{2} \log \lambda \leq S_{AB} \leq 3 \log \lambda$$  \hspace{1cm} (5.11)

For this specific configuration, the maximal entropy $S_{AB} \leq 2 \log \lambda$ provides a tighter upper bound. The results of the numerical computation of the entanglement entropy and Rényi entropy of order two for the case of all equal spins $j_{\ell} = \frac{\lambda}{2}$ and the scale parameter up to $\lambda \leq 34$ are reported in Figure 6.

Two disconnected nodes subsystem of $\Gamma_6$. Choosing a subsystem $AD$ consisting of two disconnected nodes, our asymptotic formula reduces again to

$$R_{AD}^{(2)} = \frac{3}{2} \log \lambda + O(1).$$  \hspace{1cm} (5.12)

The asymptotic band providing upper and lower bounds for the entanglement entropy is now

$$\frac{3}{2} \log \lambda \leq S_{AD} \leq 5 \log \lambda .$$  \hspace{1cm} (5.13)
Figure 5: [Graph $\Gamma_6$, entropy of a single node]. The figure shows our numerical results on the Bell-network state with hexagram graph $\Gamma_6$ and equal spins $j_\ell = \lambda/2$. The entanglement entropy $S_A$ is denoted by blue dots, the Rényi entropy $R^{(2)}_A$ by red diamonds. The lower bound asymptotic estimate is shown as a solid orange line (the $O(1)$ term is fitted using the numerical data). The upper bound estimate given by the maximal entropy is shown as a dashed orange line. We show also the bounds (4.20) as a yellow band. The inset shows the 20 data points with the largest spins.

Again, as the subsystem still consists of a small number of nodes, the maximal entropy $S_{AD} \leq 2 \log \lambda$ provides a tighter upper bound. The results of our numerical computations of the entanglement entropy and Rényi entropy of order two for the case of all equal spins $j_\ell = \frac{\lambda}{2}$ and the scale parameter up to $\lambda \leq 34$ are reported in Figure 7.
Figure 6: [Graph $\Gamma_6$, entropy of two connected nodes]. The figure shows our numerical results on the Bell-network state with hexagram graph $\Gamma_6$ and equal spins $j_\ell = \lambda/2$, for a region consisting of two connected nodes. The entanglement entropy $S_{AB}$ is denoted by blue dots, the Rényi entropy $R^{(2)}_{AB}$ by red diamonds. The lower bound asymptotic estimate is shown as a solid orange line (the $O(1)$ is fitted using the numerical data). The upper bound estimate given by the maximal entropy is shown as a dashed orange line. We show also the bounds (4.20) as a yellow band. Using the 10 largest-spin data points, the entanglement entropy $S_{AB}$ is fitted by $a \log(\lambda) + b + c\lambda^{-1}$ on the last 10 data points obtaining $a \approx 1.85$, $b \approx -0.08$ and $c \approx 2.23$. 

\[ S_{AB} \ldots \quad S_{AB} \text{ bound} \]
\[ R^{(2)}_{AB} \quad R^{(2)}_{AB} \text{ bound} \]
\[ S_{\text{max}} \text{ bound} \]
Figure 7: [Graph Γ₆, entropy of two disconnected nodes]. The figure shows our numerical results on the Bell-network state with hexagram graph Γ₆ and equal spins \( j_\ell = \lambda/2 \), for a region consisting of two disconnected nodes. The entanglement entropy \( S_{AD} \) is denoted by blue dots, the Rényi entropy \( R_{AD}^{(2)} \) by red diamonds. The lower bound asymptotic estimate is shown as a solid orange line (the \( O(1) \) is fitted using the numerical data). The upper bound estimate given by the maximal entropy is shown as a dashed orange line. We show also the bounds (4.20) as a yellow band. Using the 10 largest-spin data points, the entanglement entropy \( S_{AD} \) is fitted by \( a \log(\lambda) + b + c \lambda^{-1} \) on the last 10 data points obtaining \( a \approx 1.86 \), \( b \approx 0.09 \) and \( c \approx 1.97 \).
6 Discussion

We studied the entanglement entropy of Bell-network states, numerically and analytically. Bell-network states were introduced in [20] as states that glue quantum polyhedra with entanglement. For given graph $\Gamma$ and spin assignment $j_\ell$, there is a unique Bell-network state $|\Gamma, j_\ell, B\rangle$ defined by the $SU(2)$ symbol of the graph. Computing their entanglement entropy allows us to put information-theoretic bounds on correlations of shapes of adjacent polyhedra.

On the numerical side, we presented a code for evaluating the reduced density matrix of a Bell-network state and its entropy. We use the code to evaluate the entropy for small graphs containing up to six nodes. We consider various subsystems as described in Figure (2). At fixed graph, we studied spins ranging from $1/2$ to approximately 20. The numerical results show that Bell-network states are non-typical in the Hilbert space: We find that, at large spins, their entropy remains below by a term $O(1)$ with respect to the one of the maximally mixed state.

While the specific code used here is adapted to small graphs and large spins, a similar procedure can be adopted for general graphs.

On the analytical side, we developed methods for computing the Rényi entropy of order $p$ for arbitrary graph and generic region. For spins that are uniformly large, these methods provide reliable bounds on the entanglement entropy. The Rényi entropy is computed using techniques borrowed from spinfoam asymptotics [23, 24]: We write the trace of powers of the reduced density matrix as an integral over unit vectors and $SU(2)$ group elements,

$$
\text{Tr} \left( \rho_A^p \right) = \frac{\int d\vec{n} \, dg \, e^{f_p(j,\vec{n},g)}}{\left( \int d\vec{n} \, dg \, e^{f_1(j,\vec{n},g)} \right)^p},
$$

where $f_p(j, \vec{n}, g)$ is a linear function of all spins $j_\ell$. The integral is then evaluated with saddle-point techniques under a uniform rescaling of the spins $j_\ell \rightarrow \lambda j_\ell$ with $\lambda \gg 1$. At the leading order in $\lambda$, we find that the Rényi entropy of order $p$ is

$$\hat{R}_A^{(p)} = \left( |\partial A| - \frac{C_A^{(p)}}{2(p-1)} \right) \log \lambda + O(1),
$$

where $|\partial A|$ is the number of links that cross the boundary of the region $A$. The constant $C_A^{(p)}$ is an integer that counts the number of redundant critical point equations for the ‘action’ $f_p$. While there is no general closed formula, the number $C_A^{(p)}$ can be computed explicitly for a given graph and region, as we have done in the cases that we have studied. Moreover, using our results on the asymptotics of the Rényi entropy, we have shown that the entanglement entropy, at leading order in $\lambda$, scales logarithmically as

$$
\left( |\partial A| - \frac{C_A^{(2)}}{2} \right) \log \lambda \leq S_A \leq (|\partial A| - 3) \log \lambda.
$$
This result shows that, asymptotically, the entanglement entropy of Bell-network states scales linearly with the number of links $|\partial A|$ that cross the boundary of the region $A$. This result can be understood as an area law for Bell-network states. To clarify this point, let us consider a graph dual to a tessellation of 3-space and a region $A$. The area of a face dual to a link $\ell$ is $a(\ell) = 8\pi G h \gamma \sqrt{j_\ell(j_\ell + 1)}$. Under a rescaling $j_\ell \rightarrow \lambda j_\ell$ with $\lambda \gg 1$, the area of the boundary of the region $A$ can be written as

$$\text{Area}_A = \sum_{\ell \in \partial A} a(\lambda j_\ell) = \langle a(\lambda j_\ell) \rangle |\partial A|,$$

where $|\partial A|$ is the number of boundary links and $\langle a(\lambda j_\ell) \rangle$ is the average area of a face. Therefore the entanglement entropy of a Bell-network state takes the form

$$S_A(\Gamma, \lambda, B) \approx \log \lambda \langle a(\lambda j_\ell) \rangle \text{Area}_A.$$  \hspace{1cm} (6.5)

The origin of this area law is the entanglement between the shapes of quantum polyhedra.

We discuss also an application of our numerical results. For the case of a pentagram graph $\Gamma_5$, we find that the entanglement entropy of two adjacent nodes in $[\Gamma_5, \lambda/2, B]$ is smaller than the sum of the entropies of each node. Calling the two nodes $A$ and $B$, we have $S_A \approx \log \lambda$, $S_B \approx \log \lambda$ and $S_{AB} \approx 1.94 \log \lambda$, (See Figure 4). Therefore the mutual information $I(A,B)$ scales as

$$I(A,B) = S_A + S_B - S_{AB} \approx 0.06 \log \lambda \quad \text{(Bell-network state)}.$$  \hspace{1cm} (6.6)

This numerical result provides us with a tool to bound correlations of shapes of two adjacent polyhedra in a Bell-network state. Let us consider observables $O_A$ and $O_B$ which measure the shape of the quantum polyhedra $A$ and $B$. In order to have correlated fluctuations of shapes [52–56], the connected correlation function

$$G_{AB} = \langle O_A O_B \rangle - \langle O_A \rangle \langle O_B \rangle$$  \hspace{1cm} (6.7)

has to be non-vanishing. Remarkably, knowing the mutual information between $A$ and $B$ provides us with a bound on correlations [57, 58],

$$\frac{\langle \langle O_A O_B \rangle - \langle O_A \rangle \langle O_B \rangle \rangle^2}{4\|O_A\|^2\|O_B\|^2} \leq I(A,B).$$  \hspace{1cm} (6.8)

This relation is especially useful for bounded operator with known norm, as is the case for instance for the operator that measures the dihedral angle between two faces of a quantum polyhedron [13, 59, 60].

In the case of two quantum tetrahedra in the Bell-network state $[\Gamma_5, \lambda/2, B]$, our numerical result (6.6) tells us that the correlations between shapes are allowed to be non-vanishing.
at large spins. This result is to be contrasted to the case of the typical state in the Hilbert space $\mathcal{H}_{\Gamma_{3/2}}$ for which, using Page’s result (1.4), we find

$$I(A, B) = S_A + S_B - S_{AB} \sim \frac{1}{2\lambda} \quad \text{(Typical state).} \quad (6.9)$$

Therefore correlations in shapes of adjacent polyhedra are suppressed as $1/\lambda$ in a typical state, but unsuppressed in a Bell-network state.

The developments presented in this paper are part of an ongoing numerical revolution in the field [51, 61–64] and represent the first numerical results on the entanglement entropy of space in loop quantum gravity.

Acknowledgments

E.B. is supported by the NSF Grant PHY-1806428. P.D. is supported by the NSF grants PHY-1505411, PHY-1806356 and the Eberly research funds of Penn State. I.V. thanks Jonathan Engle and U.S. National Science Foundation for partial support under grants PHY-1505490 and PHY-1806290. We thank Simone Speziale for helpful insights at an early stage of this work and Bekir Baytaş for useful discussions on Bell-network states.
A Diagrammatic method to compute $f_p$: a detailed example

We report here the detailed construction of $\text{Tr}M^2$ for a pentagram graph and a region $A$ containing two nodes. Using the same graphical notation used in spin foams, each green box corresponds to a $SU(2)$ integral and each closed line corresponds to a $SU(2)$ character:

$$
\text{Tr}M^2 = \ldots
$$

We call $g_a$ and $\tilde{g}_a$ the group elements, $j_{ab}$ the spin of the closed line involving the group elements $g_a$ and $g_b$, $\vec{n}_{ab}$ and $\vec{m}_{ab}$ the unit vectors used to exponentiate the characters as in (4.6).

The function $f_2$ defined by (4.7), $\text{Tr}M^2 = \int d\vec{n}_f d\vec{g}_e e^{f_2(j_\vec{f}, \vec{n}_f, \vec{g}_e)}$, is given in this case by the expression

$$
f_2(j_\vec{f}, g_n, \vec{n}_f) = \sum_{a=1}^{3} j_{a4} \log \langle \vec{n}_{a4} | g_{a4} \tilde{g}_{a4} | \vec{n}_{a4} \rangle + \sum_{a=1}^{3} j_{a5} \log \langle \vec{n}_{a5} | g_{a5} \tilde{g}_{a5} | \vec{n}_{a5} \rangle + \\
\sum_{1 \leq a < b \leq 3} j_{ab} \log \langle \vec{n}_{ab} | g_b g_b^{-1} | \vec{n}_{ab} \rangle + \sum_{1 \leq a < b \leq 3} j_{ab} \log \langle \vec{m}_{ab} | g_b \tilde{g}_b^{-1} | \vec{m}_{ab} \rangle + \\
j_{45} \log \langle \vec{n}_{45} | g_{45} g_{45}^{-1} | \vec{n}_{45} \rangle + j_{45} \log \langle \vec{m}_{45} | \tilde{g}_{45} \tilde{g}_{45}^{-1} | \vec{m}_{45} \rangle.
$$
References

[1] E. Bianchi, P. Dona and S. Speziale, *Polyhedra in loop quantum gravity*, Phys. Rev. D83 (2011) 044035 [1009.3402].

[2] C. Rovelli, *Quantum Gravity*. Cambridge Monographs on Mathematical Physics. Univ. Pr., Cambridge, UK, 2004.

[3] A. Ashtekar and J. Lewandowski, *Background independent quantum gravity: A status report*, Class. Quant. Grav. 21 (2004) R53 [gr-qc/0404018].

[4] T. Thiemann, *Modern Canonical Quantum General Relativity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2007.

[5] C. Rovelli and F. Vidotto, *Covariant Loop Quantum Gravity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2014.

[6] A. Barbieri, *Quantum tetrahedra and simplicial spin networks*, Nucl. Phys. B518 (1998) 714–728 [gr-qc/9707010].

[7] J. C. Baez and J. W. Barrett, *The quantum tetrahedron in three-dimensions and four-dimensions*, Adv. Theor. Math. Phys. 3 (1999) 815–850 [gr-qc/9903060].

[8] E. Bianchi and H. M. Haggard, *Discreteness of the volume of space from bohr-sommerfeld quantization*, Phys. Rev. Lett. 107 (2011) 011301 [1102.5439].

[9] E. Bianchi and H. M. Haggard, *Bohr-sommerfeld quantization of space*, Phys. Rev. D86 (2012) 124010 [1208.2228].

[10] C. Rovelli and S. Speziale, *On the geometry of loop quantum gravity on a graph*, Phys. Rev. D82 (2010) 044018 [1005.2927].

[11] C. Rovelli and L. Smolin, *Discreteness of area and volume in quantum gravity*, Nucl. Phys. B442 (1995) 593–622 [gr-qc/9411005]. [Erratum: Nucl. Phys.B456,753(1995)].

[12] A. Ashtekar and J. Lewandowski, *Quantum theory of geometry. 2. Volume operators*, Adv. Theor. Math. Phys. 1 (1998) 388–429 [gr-qc/9711031].

[13] S. A. Major, *Operators for quantized directions*, Class. Quant. Grav. 16 (1999) 3859–3877 [gr-qc/9905019].

[14] B. Dittrich and S. Speziale, *Area-angle variables for general relativity*, New J. Phys. 10 (2008) 083006 [0802.0864].

[15] E. Bianchi, *The length operator in loop quantum gravity*, Nucl. Phys. B807 (2009) 591–624 [0806.4710].

[16] D. N. Page, *Average entropy of a subsystem*, Phys. Rev. Lett. 71 (Aug, 1993) 1291–1294.

[17] L. Vidmar, L. Hackl, E. Bianchi and M. Rigol, *Entanglement Entropy of Eigenstates of Quadratic Fermionic Hamiltonians*, Phys. Rev. Lett. 119 (2017), no. 2 020601 [1703.02979].

[18] L. Vidmar, L. Hackl, E. Bianchi and M. Rigol, *Volume Law and Quantum Criticality in the Entanglement Entropy of Excited Eigenstates of the Quantum Ising Model*, Phys. Rev. Lett. 121 (2018), no. 22 220602 [1808.08963].
[19] L. Hackl, L. Vidmar, M. Rigol and E. Bianchi, *Average eigenstate entanglement entropy of the XY chain in a transverse field and its universality for translationally invariant quadratic fermionic models*, 1812.08757.

[20] B. Baytas, E. Bianchi and N. Yokomizo, *Gluing polyhedra with entanglement in loop quantum gravity*, Phys. Rev. D98 (2018), no. 2 026001 [1805.05856].

[21] L. Freidel and S. Speziale, *Twisted geometries: A geometric parametrisation of su(2) phase space*, Phys. Rev. D82 (2010) 084040 [1001.2748].

[22] L. Freidel and S. Speziale, *From twistors to twisted geometries*, Phys. Rev. D82 (2010) 084041 [1006.0199].

[23] J. W. Barrett and C. M. Steele, *Asymptotics of relativistic spin networks*, Class. Quant. Grav. 20 (2003) 1341–1362 [gr-qc/0209023].

[24] P. Dona, M. Fanizza, G. Sarno and S. Speziale, *Su(2) graph invariants, regge actions and polytopes*, Class. Quant. Grav. 35 (2018), no. 4 045011 [1708.01727].

[25] The code presented in this paper is available at https://bitbucket.org/pietrodona/bellnetworkentropy.

[26] M. Van Raamsdonk, *Comments on quantum gravity and entanglement*, 0907.2939.

[27] M. Van Raamsdonk, *Building up spacetime with quantum entanglement*, Gen. Rel. Grav. 42 (2010) 2323–2329 [1005.3035]. [Int. J. Mod. Phys.D19,2429(2010)].

[28] E. Bianchi and R. C. Myers, *On the architecture of spacetime geometry*, Class. Quant. Grav. 31 (2014) 214002 [1212.5183].

[29] T. Jacobson, *Thermodynamics of space-time: The einstein equation of state*, Phys. Rev. Lett. 75 (1995) 1260–1263 [gr-qc/9504004].

[30] T. Jacobson, *Entanglement equilibrium and the einstein equation*, Phys. Rev. Lett. 116 (2016), no. 20 201101 [1505.04753].

[31] E. Bianchi, J. Guglielmon, L. Hackl and N. Yokomizo, *Squeezed vacua in loop quantum gravity*, 1605.05356.

[32] E. Bianchi, J. Guglielmon, L. Hackl and N. Yokomizo, *Loop expansion and the bosonic representation of loop quantum gravity*, Phys. Rev. D94 (2016), no. 8 086009 [1609.02219].

[33] G. Chirco, F. M. Mele, D. Oriti and P. Vitale, *Fisher metric, geometric entanglement and spin networks*, Phys. Rev. D97 (2018), no. 4 046015 [1703.05231].

[34] C. Cao, S. M. Carroll and S. Michalakis, *Space from hilbert space: Recovering geometry from bulk entanglement*, Phys. Rev. D95 (2017), no. 2 024031 [1606.08444].

[35] E. R. Livine, *Intertwiner entanglement on spin networks*, Phys. Rev. D97 (2018), no. 2 026009 [1709.08511].

[36] A. Feller and E. R. Livine, *Entanglement entropy and correlations in loop quantum gravity*, Class. Quant. Grav. 35 (2018), no. 4 045009 [1710.04473].

[37] W. Donnelly, *Entanglement entropy in loop quantum gravity*, Phys. Rev. D77 (2008) 104006 [0802.0880].
[38] W. Donnelly, *Decomposition of entanglement entropy in lattice gauge theory*, Phys. Rev. **D85** (2012) 085004 [1109.0036].

[39] N. Bodendorfer, *A note on entanglement entropy and quantum geometry*, Class. Quant. Grav. **31** (2014), no. 21 214004 [1402.1038].

[40] A. Hamma, L.-Y. Hung, A. Marciano and M. Zhang, *Area Law from Loop Quantum Gravity*, Phys. Rev. **D97** (2018), no. 6 064040 [1506.01623].

[41] F. Anz and G. Chirco, *Typicality in spin-network states of quantum geometry*, Phys. Rev. **D94** (2016), no. 8 084047 [1605.04946].

[42] C. Delcamp, B. Dittrich and A. Riello, *On entanglement entropy in non-Abelian lattice gauge theory and 3D quantum gravity*, JHEP **11** (2016) 102 [1609.04806].

[43] D. Gruber, H. Sahlmann and T. Zilker, *Geometry and entanglement entropy of surfaces in loop quantum gravity*, Phys. Rev. **D98** (2018), no. 6 066009 [1806.05937].

[44] J. W. Barrett, W. J. Fairbairn and F. Hellmann, *Quantum gravity asymptotics from the su(2) 15j symbol*, Int. J. Mod. Phys. **A25** (2010) 2897–2916 [0912.4907].

[45] J. W. Barrett, R. J. Dowdall, W. J. Fairbairn, H. Gomes and F. Hellmann, *Asymptotic analysis of the eprl four-simplex amplitude*, J. Math. Phys. **50** (2009) 112504 [0902.1170].

[46] M. Han and M. Zhang, *Asymptotics of spinfoam amplitude on simplicial manifold: Lorentzian theory*, Class. Quant. Grav. **30** (2013) 165012 [1109.0499].

[47] J. Engle, I. Vilensky and A. Zipfel, *Lorentzian proper vertex amplitude: Asymptotics*, Phys. Rev. **D94** (2016), no. 6 064025 [1505.06683].

[48] E. Bianchi, *Spinfoam gravity*, in *Loop Quantum Gravity: The First 30 Years* (A. Ashtekar and J. Pullin, eds.), pp. 97–124. WSP, 2017.

[49] A. M. Perelomov, *Generalized Coherent States and Their Applications*. 1986.

[50] H. T. Johansson and C. Forssen, *Fast and accurate evaluation of wigner 3j, 6j, and 9j symbols using prime factorisation and multi-word integer arithmetic*, SIAM J. Sci. Statist. Comput. **38** (2016) A376–A384 [1504.08329].

[51] P. Dona and G. Sarno, *Numerical methods for eprl spin foam transition amplitudes and lorentzian recoupling theory*, Gen. Rel. Grav. **50** (2018) 127 [1807.03066].

[52] E. Bianchi, L. Modesto, C. Rovelli and S. Speziale, *Graviton propagator in loop quantum gravity*, Class. Quant. Grav. **23** (2006) 6989–7028 [gr-qc/0604044].

[53] E. R. Livine and S. Speziale, *Group integral techniques for the spinfoam graviton propagator*, JHEP **11** (2006) 092 [gr-qc/0608131].

[54] E. Alesci, E. Bianchi and C. Rovelli, *Lgg propagator: Iii. the new vertex*, Class. Quant. Grav. **26** (2009) 215001 [0812.5018].

[55] E. Bianchi, E. Magliaro and C. Perini, *Lgg propagator from the new spin foams*, Nucl. Phys. **B822** (2009) 245–269 [0905.4082].

[56] E. Bianchi and Y. Ding, *Lorentzian spinfoam propagator*, Phys. Rev. **D86** (2012) 104040 [1109.6538].
[57] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.

[58] M. M. Wolf, F. Verstraete, M. B. Hastings and J. I. Cirac, *Area laws in quantum systems: Mutual information and correlations*, Phys. Rev. Lett. 100 (Feb, 2008) 070502.

[59] R. Penrose, *Angular momentum: an approach to combinatorial space-time*, Quantum Theory and Beyond, edited by T. Bastin (1971).

[60] R. Penrose, *On the nature of quantum geometry*, Magic without magic (1972) 333–354.

[61] B. Bahr and S. Steinhaus, *Numerical evidence for a phase transition in 4d spin foam quantum gravity*, Phys. Rev. Lett. 117 (2016), no. 14 141302 [1605.07649].

[62] B. Bahr, G. Rabuffo and S. Steinhaus, *Renormalization of symmetry restricted spin foam models with curvature in the asymptotic regime*, Phys. Rev. D98 (2018), no. 10 106026 [1804.00023].

[63] B. Dittrich, E. Schnetter, C. J. Seth and S. Steinhaus, *Coarse graining flow of spin foam intertwiners*, Phys. Rev. D94 (2016), no. 12 124050 [1609.02429].

[64] C. Delcamp and B. Dittrich, *Towards a phase diagram for spin foams*, Class. Quant. Grav. 34 (2017), no. 22 225006 [1612.04506].