Integration of geometric rough paths

Danyu Yang

Abstract

We build a connection between rough path theory and noncommu-
tative algebra, and interpret the integration of geometric rough paths as
an example of a non-abelian Young integration. We identify a class of
slowly-varying one-forms, and prove that the class is stable under basic
operations. In particular rough path theory is extended to allow a natural
class of time varying integrands.

Consider two topological groups $G_1$ and $G_2$, and a differentiable function $f : G_1 \to G_2$. For a time interval $[S, T]$ and a differentiable path $X : [S, T] \to G_1$, the integration of the exact one-form $df$ along $X$ can be defined as:

$$\int_{t=S}^{T} df dX_r = \int_{X=S}^{X=T} df = f(X_S)^{-1} f(X_T) \in G_2.$$ 

When $f$ and $X$ are only continuous, $\int_{t=S}^{T} df dX_r$ can be defined as $f(X_S)^{-1} f(X_T)$.

Consider a time-varying exact one-form $(df_t)_t$ with $f_t : G_1 \to G_2$ indexed by $t \in [0, 1]$, and $X : [0, 1] \to G_1$. If the following limit exists in $G_2$:

$$\lim_{|D| \to 0, D \subseteq [0, 1]} \int_{r=t_0}^{t_1} df_{t_0} dX_r \int_{r=t_1}^{t_2} df_{t_1} dX_r \cdots \int_{r=t_{n-1}}^{t_n} df_{t_{n-1}} dX_r$$

where $D = \{t_k\}_{k=0}^n, 0 = t_0 < \cdots < t_n < 1, n \geq 1$ with $|D| := \max_{k=0}^{n-1} |t_{k+1} - t_k|$, then the integral $\int_{r=0}^{1} df_r dX_r$ is defined to be the limit.

We reinterpret the integration of Lipschitz one-forms along geometric rough paths developed by Lyons [14] as an integration of time-varying exact one-forms along group-valued paths. The interpretation is in the language of the Malvenuto–Reutenauer Hopf algebra of permutations introduced in [17] [18].
1 Background: rough path theory

In [25] Young proved that, for $x: [0, 1] \to \mathbb{C}$ of finite $p$-variation and $y: [0, 1] \to \mathbb{C}$ of finite $q$-variation, $p \geq 1$, $q \geq 1$, $p^{-1} + q^{-1} > 1$, the Stieltjes integral

$$\int_{t=0}^{1} x_t dy_t$$

is well defined as the limit of Riemann sums. The definition of $p$-variation goes back to Wiener [24]:

$$\|x\|_{p-\text{var},[0,1]} := \sup_{D \subset [0,1]} \left( \sum_{k,t_k \in D} \left\| x_{t_{k+1}} - x_{t_k} \right\|^p \right)^{\frac{1}{p}}$$

where the supremum is over all finite partitions $D = \{t_k\}_{k=0}^n$, $0 = t_0 < \cdots < t_n = 1$, $n \geq 1$. The condition given by Young is sharp: the Riemann-Stieltjes integral $\int x dy$ does not necessarily exist when $p^{-1} + q^{-1} = 1$ [25]. In [12], Lyons demonstrated that similar obstacle exists in stochastic integration, and one needs to consider $x$ and $y$ as a “pair” “in a fairly strong way”.

From a different perspective, Chen [2, 3, 4] investigated the iterated integration of one-forms and developed a theory of cohomology for loop spaces. One of the major objects he studied is noncommutative formal series with coefficients the iterated integrals of the coordinates of a path. For a time interval $[S, T]$, let $x: [S, T] \to \mathbb{R}^d$ be a smooth path, and let $X_1, \ldots, X_d$ be noncommutative indeterminates. Consider the formal power series:

$$\theta(x) := 1 + \sum_{n \geq 1, i_j \in \{1, \ldots, d\}} w_{i_1 \cdots i_n} X_{i_1} \cdots X_{i_n}$$

where

$$w_{i_1 \cdots i_n} := \int_{S<u_1<\cdots<u_n<T} dx_{i_1}^{u_1} \cdots dx_{i_n}^{u_n}.$$ 

The space of paths in $\mathbb{R}^d$ have an associative multiplication given by concatenation, and the set of formal series have an associative multiplication that is the bilinear extension of the concatenation of finite sequences. Chen [2] proved that $\theta$ is an algebra homomorphism from paths to formal series and satisfies $\theta(x)^{-1} = \theta(x)$ where $x$ denotes the path given by running $x$ backwards. Based on Chen [3] $\theta$ takes values in a group whose elements are algebraic exponentials of Lie series, and the multiplication in the group is given by Campbell-Baker-Hausdorff formula.

In [14], Lyons developed the theory of rough paths. He observed that, in controlled systems, both the driving path and the solution path evolve in a group (denoted by $G$) rather than in a vector space. Lyons also identified a family of metrics on group-valued paths such that the Itô map that sends a

\footnote{at least when $x$ and $y$ have no common jumps.}
driving path to the solution path is continuous. Elements in $G$ are algebraic exponentials of Lie series, and a geometric $p$-rough path is a continuous path in $G$ with finite $p$-variation.

For a continuous bounded variation path, there exists a canonical lift of the path to a geometric 1-rough path given by the sequence of indefinite iterated integrals. The sequence of definite iterated integrals is called the signature of the continuous bounded variation path.

The following is Definition 1.1 Hambly and Lyons [8].

**Definition 1 (Signature)** Let $\gamma$ be a path of bounded variation on $[S, T]$ with values in a vector space $V$. Then its signature is the sequence of definite iterated integrals

$$X_{S,T} = (1 + X_{S,T}^1 + \cdots + X_{S,T}^k + \cdots)$$

$$= \left(1 + \int_{S<u<T} d\gamma_u + \cdots + \int_{S<u_1<\cdots<u_k<T} d\gamma_{u_1} \otimes \cdots \otimes d\gamma_{u_k} + \cdots\right)$$

regarded as an element of an appropriate closure of the tensor algebra $T(V) = \bigoplus_{n=0}^{\infty} V^\otimes n$.

The signature is invariant under reparametrisations of the path.

Following [8] $X_{S,T}$ is also denoted by $S(\gamma)$ where $S$ is the signature mapping that sends a continuous bounded variation path to the sequence of definite iterated integrals.

**Notation 2** Denote by $S : \gamma \mapsto S(\gamma)$ the signature mapping.

The signature provides an efficient and effective description of the information encoded in paths, and two paths with the same signature have the same effect on all controlled systems. An important problem in the theory of rough path is the ‘uniqueness of signature’ problem: to describe the kernel of the signature mapping. This problem goes back to Chen [4] where he proved that the map $\theta$ (signature) provides a faithful representation for a family of paths that are irreducible, piecewise regular, and continuous. In [8] Hambly and Lyons established quantitative estimates of a path in terms of its signature, and proved the uniqueness of signature result for continuous bounded variation paths. In [1] Boedihardjo, Geng, Lyons and Yang extended the result to weak geometric rough paths over Banach spaces. There are also progresses in the direction of machine learning, where rough path is introduced as a new feature set for streamed data [13, 9].

**Notation 3** For a vector space $V$, let $G(V)$ denote the set of group-like elements in the tensor algebra $T(V)$.

Based on Corollary 3.3 [21], elements in $G(V)$ are algebraic exponentials of Lie series, and $G(V)$ is a group.

A time interval is an interval of the form $[S, T]$ for $0 \leq S \leq T < \infty$. For a time interval $J$ and a Banach space $V$, let $BV(J, V)$ denote the set of continuous
bounded variation paths $J \to V$. Based on Chen [2], the signature of paths in $BV(J, V)$ form a subgroup of $G(V)$. For $x \in BV(J_1, V)$ and $y \in BV(J_2, V)$, let $x \ast y$ denote the concatenation of $x$ with $y$, and let $\overleftarrow{x}$ denote the time-reversing of $x$. Then the following identities hold:

$$S(x \ast y) = S(x) S(y) \quad \text{and} \quad S(\overleftarrow{x}) = S(x)^{-1}.$$  

Let $V$ be a Banach space. Suppose the tensor powers of $V$ are equipped with admissible norms Definition 1.25 p.20 [15]. Following Definition 2.1 [1], we equip $G(V)$ with the metric

$$d(a, b) := \max_{k \in \mathbb{N}} \| \pi_k (a^{-1} b) \|^p$$

for $a, b \in G(V)$, where $\pi_k$ denotes the projection of $T(V)$ to $V^\otimes k$. Based on Definition 1.2.2 [14] for a time interval $J$, $\omega : \{(s, t) \mid s \leq t, s \in J, t \in J\} \to \mathbb{R}^+$ is a control if $\omega$ is continuous, super-additive and vanishes on the diagonal.

The following definition is based on Definition 1.2.2 Lyons [14].

Definition 4 (Geometric Rough Paths) For a time interval $J$ and a Banach space $V$, $X : J \to G(V)$ is called a geometric $p$-rough path for some $p \geq 1$, if there exists a control $\omega : \{(s, t) \mid s \leq t, s \in J, t \in J\} \to \mathbb{R}^+$ such that

$$\|X\|_{p-\text{var},[s,t]}^p \leq \omega(s, t)$$

for every $s \leq t$, where

$$\|X\|_{p-\text{var},[s,t]}^p := \sup_{D \subset [s,t]} \sum_{k, t_k \in D} d(X_{t_k}, X_{t_{k+1}})^p,$$

with the supremum over all $D = \{t_k\}_{k=0}^n \subset [s,t], s = t_0 < \cdots < t_n = t, n \geq 1$.

2 An algebra of permutations

The group-like elements in the tensor algebra is the group of characters of the shuffle algebra p.54 Theorem 3.2 Reutenauer [21]. The shuffle product can be expressed in terms of permutations based on the Malvenuto–Reutenauer Hopf algebra (denoted by MR) introduced in [17, 18]. MR is a Hopf algebra of permutations and is noncommutative.

We first review the shuffle Hopf algebra and its dual space based on Section 1.5 Reutenauer [21]. Let $A$ be a (possibly infinite) set, and let $K$ be a commutative $\mathbb{Q}$-algebra. Let $K(A)$ denote the set of non-commutative polynomials on $A$ over $K$. Let $A^*$ denote the free monoid generated by $A$ that is the set of finite sequences of elements in $A$ including the empty sequence denoted by $e$. The operation on $A^*$ is given by concatenation:

$$(a_1 \cdots a_n)(a_{n+1} \cdots a_{n+m}) := a_1 \cdots a_{n+m}$$
for $a_i \in A$, with $e$ the identity element. There is a natural embedding $A \hookrightarrow A^*$. Based on Rees [20] the shuffle product $sh : K\langle A \rangle \otimes K\langle A \rangle \to K\langle A \rangle$ is the $K$-bilinear map that can be defined recursively by

$$sh \circ (w_1 a_1 \otimes w_2 a_2) := (sh \circ (w_1 a_1 \otimes w_2)) a_2 + (sh \circ (w_1 \otimes w_2 a_2)) a_1$$

for $w_1 \in A^*, a_i \in A$, where $wa$ denotes the concatenation of $w \in A^*$ with $a \in A$, and $sh \circ (e \otimes w) = sh \circ (w \otimes e) := w$ for $w \in A^*$. The product $sh$ is associative with unit $u(k) := ke$ for $k \in K$. Let $\delta' : K\langle A \rangle \to K\langle A \rangle \otimes K\langle A \rangle$ denote the deconcatenation coproduct that is the $K$-linear map given by

$$\delta'(a_1 \cdots a_n) := \sum_{k=0}^n a_1 \cdots a_k \otimes a_{k+1} \cdots a_n$$

for $a_i \in A, n \geq 1$, and $\delta'(e) := e \otimes e$. The counit $\epsilon$ is the projection of $K\langle A \rangle$ to the space spanned by $e \in A^*$. $(K\langle A \rangle, sh, u, \delta', \epsilon)$ is a Hopf algebra p.31 [21] which we call the shuffle Hopf algebra.

Let $K\langle\langle A\rangle\rangle$ denote the set of formal series $s = \sum_{w \in A^*} (s, w) w$ on $A$ over $K$. The concatenation product $conc : K\langle\langle A\rangle\rangle \otimes K\langle\langle A\rangle\rangle \to K\langle\langle A\rangle\rangle$ is the $K$-bilinear map given by

$$(conc \circ (s \otimes t), w) := \sum_{uv = w} (s, u) (t, v)$$

for $w \in A^*$. The map $\delta$ on $K\langle\langle A\rangle\rangle$ is the homomorphism of the concatenation algebra given by $\delta(a) := e \otimes a + a \otimes e$ for each $a \in A$, and is of the explicit form p.25 [21]

$$\delta(s) = \sum_{w_1, w_2 \in A^*} (s, sh \circ (w_1 \otimes w_2)) w_1 \otimes w_2$$

for $s \in K\langle\langle A\rangle\rangle$. $K\langle\langle A\rangle\rangle$ is nearly a Hopf algebra, but the map $\delta$ does not necessarily take values in $K\langle\langle A\rangle\rangle \otimes K\langle\langle A\rangle\rangle$ and the sum in $\delta(s) = \sum s^{(1)} \otimes s^{(2)}$ can be infinite p.38 [21].

There is a duality between $K\langle\langle A\rangle\rangle$ and $K\langle A \rangle$ given by

$$(s, p) := \sum_{w \in A^*} (s, w) (p, w)$$

for $s \in K\langle\langle A\rangle\rangle$ and $p \in K\langle A \rangle$. Then the following duality holds p.26 [21]:

$$(s, sh \circ (p \otimes q)) = (\delta s, p \otimes q)$$

$$(conc \circ (s \otimes t), p) = (s \otimes t, \delta' p)$$

for $s, t \in K\langle\langle A\rangle\rangle$ and $p, q \in K\langle A \rangle$.

**Definition 5 (Group-like Elements)** The group-like elements in $K\langle\langle A\rangle\rangle$ is the set of $s \in K\langle\langle A\rangle\rangle$ that satisfies $\delta s = s \otimes s$.

**Notation 6** Let $G(A)$ denote the set of group-like elements in $K\langle\langle A\rangle\rangle$.  

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Based on Theorem 3.2 [21] the group-like elements in $K \langle \langle A \rangle \rangle$ are algebraic exponentials of Lie series, and they form a group in $K \langle \langle A \rangle \rangle$. Based on the duality between $sh$ and $\delta$, group-like elements in $K \langle \langle A \rangle \rangle$ are the set of characters of the shuffle algebra on $K(A)$. Theorem 3.2 [21]: $s \in G(A)$ iff $(s, p) (s, q) = (s, sh \circ (p \otimes q))$ for every $p, q \in K(A)$.

MR is a $\mathbb{Z}$-Hopf algebra on permutations $S := \bigcup_{n \geq 0} S_n$. Let $\text{End}(K \langle A \rangle)$ denote the $K$-module of linear endomorphisms of $K \langle A \rangle$. Based on Proposition 1.10 [21] $\text{End}(K \langle A \rangle)$ becomes a $K$-associative algebra with the convolution product $\ast'$ given by

$$f \ast' g = sh \circ (f \otimes g) \circ \delta'$$

for $f, g \in \text{End}(K \langle A \rangle)$. There is an embedding of permutations $\mathbb{Z}S$ in $\text{End}(K \langle A \rangle)$ given by

$$\sigma \cdot (a_1 \cdots a_n) := a_{\sigma 1} \cdots a_{\sigma n}$$

for $\sigma \in S_n$ and $a_i \in A$. Based on [18], the product in MR is the $\mathbb{Z}$-bilinear map $\ast' : \mathbb{Z}S \times \mathbb{Z}S \to \mathbb{Z}S$ and is of the explicit form

$$\sigma \ast' \rho := sh \circ (\sigma \otimes \bar{\rho})$$

for $\sigma \in S_n$ and $\rho \in S_m$, where permutations are considered as words with $\bar{\rho}(i) := n + \rho(i)$. The product $\ast'$ is associative with identity element $\lambda \in S_0$.

The coproduct on MR $\triangle' : \mathbb{Z}S \to \mathbb{Z}S \otimes \mathbb{Z}S$ is the $\mathbb{Z}$-linear map given by

$$\triangle' := (st \otimes st) \circ \delta'$$

where ‘st’ denotes the unique increasing map that sends a sequence of $k$ non-repeating integers to $\{1, 2, \ldots, k\}$ for $k \geq 1$. The counit is the projection of $\mathbb{Z}S$ to the space spanned by $\lambda \in S_0$. Based on [17, 18, 19] MR is a Hopf algebra that is self-dual, free and cofree, so is neither commutative nor cocommutative.

For $f \in \text{End}(K \langle A \rangle)$, define the adjoint map $f^* \in \text{End}(K \langle A \rangle)$ as

$$(f^* s, p) := (s, fp)$$

for $s \in K \langle A \rangle$ and $p \in K(A)$. As a sub-algebra of $\text{End}(K \langle A \rangle)$, MR induces a sub-algebra of $\text{End}(K \langle A \rangle)$. Proposition 7 below states that the group-like elements in $K \langle \langle A \rangle \rangle$ (denoted by $G(A)$) is a group of characters of MR: for $s \in G(A)$,

$$\hat{s} : (\mathbb{Z}S, \ast') \to (K \langle A \rangle, \text{conc})$$

$$\sigma \mapsto \sigma^* s$$

is an algebra homomorphism. Proposition 7 is closely related to the cotensor algebra p.248 Ronco [22].

**Proposition 7** For $s \in G(A)$, define a $\mathbb{Z}$-linear map $\hat{s} : \mathbb{Z}S \to K \langle \langle A \rangle \rangle$ by

$$\hat{s}(\sigma) := \sum_{w \in A^*} (s, \sigma \cdot w) w.$$
Then

\[ \text{conc} \circ (\hat{s}(\sigma) \otimes \hat{s}(\rho)) = \hat{s}(\sigma \ast \rho) \]

for \( \sigma, \rho \in ZS \).

**Proof.** Since \( \sigma \ast \rho := \text{sh} \circ (\sigma \otimes \rho) \circ \delta' \),

\[ (\sigma \ast \rho) \cdot w = \text{sh} \circ \left( (\sigma \cdot u) \otimes (\rho \cdot v) \right) \]

for \( \sigma \in S_n, \rho \in S_m, w \in A^*, w = uv, |u| = n, |v| = m \), where \( |u| \) denotes the number of letters in \( u \in A^* \). Since \( s \in G(A) \) is a character of the shuffle algebra on \( K\langle A \rangle \),

\[ (s, (\sigma \ast \rho) \cdot w) = (s, \text{sh} \circ ((\sigma \cdot u) \otimes (\rho \cdot v))) = (s, \sigma \cdot u) (s, \rho \cdot v) . \]

The statement follows. ■

Based on the bidendriform algebra structure of the shuffle algebra Loday and Ronco [11] Loday [10], consider \( \succ \) that is an abstract iterated integration.

**Notation 8** Denote \( \succ \colon (K\langle A \rangle \times K\langle A \rangle) \setminus (Ke \times Ke) \to K\langle A \rangle \) as the \( K \)-bilinear map given by

\[ (a_1 \cdots a_n) \succ (a_{n+1} \cdots a_{n+m}) := \text{sh} \circ (a_1 \cdots a_n) \otimes (a_{n+1} \cdots a_{n+m-1}) \]

for \( a_i \in A, m \geq 1 \), where \( wa \) denotes the concatenation of \( w \in A^* \) with \( a \in A \); \( (a_1 \cdots a_n) \succ e := 0 \in K \) for \( a_i \in A, n \geq 1 \).

In defining the integration of geometric rough paths, Lyons considered an ordered shuffle to define almost multiplicative functionals p.285 Definition 3.2.2 [14]. The ordered shuffle can be viewed as iterated applications of \( \succ \) as defined in Notation 2.3 [22].

The following Lemma helps to prove that indefinite integrals of one-forms along geometric rough paths are geometric rough paths. Consider \( p_1, \ldots, p_n \in K\langle A \rangle \) that satisfy \( (p_i, e) = 0, i = 1, \ldots, n \), where \( p = \sum_{w \in A^*} (p, w) w \) and \( e \) is the empty sequence in \( A^* \). Define

\[ m_\succ (p_1) := p_1 \]
\[ m_\succ (p_1, \cdots, p_n) := (\cdots (p_1 \succ p_2) \succ \cdots \succ p_{n-1}) \succ p_n. \]

**Lemma 9** Let \( p_1, \ldots, p_{n+m} \in K\langle A \rangle \) satisfy \( (p_i, e) = 0, i = 1, \ldots, n + m \). Then

\[
sh \circ (m_\succ (p_1, \cdots, p_n) \otimes m_\succ (p_{n+1}, \cdots, p_{n+m}))
= \sum_{\rho \in 1_{n} \ast 1_{m}} m_\succ (p_{\rho(1)}, \cdots, p_{\rho(n+m)}).
\]
Proof. Based on the multi-linearity of $m_\cdot (\cdots)$, we assume $p_i = w_i$ for $w_1 \in A^\ast, |w_i| \geq 1, i = 1, \ldots, n + m$. Based on the order of the image of the last element of $(w_i)_{i=1}^n$ in $m_\cdot (w_1, \cdots, w_n)$, elements in $m_\cdot (w_1, \cdots, w_n)$ can be divided into $\binom{n+m}{n!}$ subsets indexed by elements in $1_n \ast 1_m$. Indeed, denote $l_0 := 0, l_k := \sum_{j=1}^k |w_i|, k = 1, \ldots, n + m$. For each $\rho \in 1_n \ast 1_m$, let $E(\rho)$ denote the set of $\alpha \in S_{l_n+m}$ that satisfy $\alpha (l_k + i) < \alpha (l_k + i + 1), i = 1, \ldots, |w_{k+1}| - 1, k = 0, \ldots, n + m - 1$ and $\alpha (l_{\rho(k)}) < \alpha (l_{\rho(k+1)})$, $k = 1, \ldots, n + m - 1$. Then

\[
sh \circ (m_\cdot (w_1, \cdots, w_n) \otimes m_\cdot (w_{n+1}, \cdots, w_{n+m})) = \sum_{\rho \in 1_n \ast 1_m} \sum_{\alpha \in E(\rho)} \alpha \cdot (w_1 \cdots w_{n+m}) = \sum_{\rho \in 1_n \ast 1_m} m_\cdot (w_{\rho(1)}, \cdots, w_{\rho(n+m)})
\]

where in the second line $w_1 \cdots w_{n+m} \in A^\ast$ denotes the concatenation of $w_i$. ■

The operation $\triangleright : K(A) \times K(A) \rightarrow K(A)$ induces an operation on $\text{End} (K(A))$ given by $f \triangleright g := f \circ (f \otimes g) \circ \delta'$. $\triangleright$ is closed on permutations \text{[22]}. \par

Notation 10 Let $\triangleright : (\mathbb{Z}S \times \mathbb{Z}S) \setminus (\mathbb{Z}S_0 \times \mathbb{Z}S_0) \rightarrow \mathbb{Z}S$ denote the Z-bilinear map given by $\sigma \triangleright \rho := \sigma \triangleright \tilde{\rho}$ for $\sigma \in S_n, \rho \in S_m$, where permutations are viewed as words with $\tilde{\rho}(i) := n + \rho(i)$. \par

MR is actually a bidendriform algebra Loday \text{[10]}. With $(1) \in S_1$, define $I : \mathbb{Z}S \rightarrow \mathbb{Z}S, \sigma \mapsto \sigma \triangleright (1)$. $I$ can be viewed as an abstract integration. Since $\Delta' (1) = (1) \otimes \lambda + \lambda \otimes (1)$ for $\lambda \in S_0$, based on p.248 Definition 1.2 (b) Ronco \text{[22]}, for $\sigma \in \mathbb{Z}S$, $\Delta' I (\sigma) = I (\sigma) \otimes \lambda + \sum_{\Delta' \sigma = \sigma(1) \otimes \sigma(2)} \sigma(1) \otimes I (\sigma(2))$.

Proposition 11 below helps to prove that slowly-varying one-forms are closed under iterated integration (Proposition 30). Let $1_n$ denote the identity element in $S_n, n \geq 1$, and $l_0 := \lambda \in S_0$. $G(A)$ denotes the group-like elements in $K(\langle A \rangle)$. \par

Proposition 11 For $s \in G(A)$, define $\hat{s} : \mathbb{Z}S \rightarrow K(\langle A \rangle)$ as in Proposition 7. Then

\[
\text{conc} \circ (s \otimes t) (1_{n_1} \triangleright 1_{n_2}) = \hat{s} (1_{n_1} \triangleright 1_{n_2}) - \sum_{k_1 = 0, \ldots, n_1}^{k_2 = 0, \ldots, n_2 - 1} \rho_{k_2, n_1 - k_1} \cdot \left( \text{conc} \circ \left( \hat{s} (1_{k_1} \ast 1_{k_2}) \otimes \hat{s} (1_{n_1 - k_1} \triangleright 1_{n_2 - k_2}) \right) \right)
\]
for \( s, t \in G(A) \) and \( n_1 \geq 0, n_2 \geq 1 \), where \( \sigma \cdot (a_1 \cdots a_n) := a_{\sigma 1} \cdots a_{\sigma n} \) for \( \sigma \in S_n \) and \( a_i \in A; \rho \in S_{n_1+k_1} \) and \( \rho \) is given by changing the order of two subsequences \((k_1 + 1, \ldots, k_1 + k_2)\) \((k_1 + k_2 + 1, \ldots, n_1 + n_2 + 1)\) in \((1, \ldots, n_1 + n_2)\).

**Remark 12** The equality in Proposition 11 can be formally expressed as

\[
\left( \int x^{n_1} dx^{n_2} \right) |_{x=s}^{t} = \left( \int (sx)^{n_1} d(x)^{n_2} \right) |_{x=1}^{t}
\]

and is a change of variable formula for abstract iterated integration.

**Proof.** For \( w \in A^* \) let \(|w|\) denote the number of letters in \( w \). Based on the bidendriform algebra structure of shuffle algebra and Definition 1.2 (b) \cite{[22]},

\[
\delta' (u \succ v) = (u \succ v) \otimes e + \sum_{|v| \geq 1} (u(1) \shuffle v(1)) \otimes (u(2) \succ v(2))
\]

for \( u, v \in A^*, |v| \geq 1 \), where \( u(1) \otimes u(2) := \delta' (u) \), \( v(1) \otimes v(2) := \delta'(v) \) and \( \shuffle \) denotes the shuffle product. Then based on the definition \( 1_{n_1} \succ 1_{n_2} := \succ \circ (1_{n_1} \otimes 1_{n_2}) \circ \delta', \) for \( w \in A^*, w = uv, |u| = n_1, |v| = n_2, \)

\[
\begin{align*}
 & (\text{conc} \circ (s \otimes t), (1_{n_1} \succ 1_{n_2}) \cdot w) \\
 = & \ (\text{conc} \circ (s \otimes t), u \succ v) \\
 = & \ (s \otimes t, \delta' (u \succ v)) \quad \text{(duality between conc and } \delta') \\
 = & \ (s, u \succ v) + \sum_{|v| \geq 1} (s, u(1) \shuffle v(1)) \ (t, u(2) \succ v(2)) \\
 = & \ (s, (1_{n_1} \succ 1_{n_2}) \cdot w) \\
 & \quad + \sum_{|v| \geq 1} \left( s, \left(1^{1}_{u(1)} 1_{v(1)} \right) \cdot (u(1)v(1)) \right) \left( t, \left(1^{1}_{u(2)} 1_{v(2)} \right) \cdot (u(2)v(2)) \right)
\end{align*}
\]

for \( s, t \in G(A) \). 

Although expressed in terms of shuffles/permutations, core arguments in this section (in particular Proposition 11) can be applied to a general bidendriform algebra.

### 3 Integration of geometric rough paths

We first consider an example that is simple and important.

For two Banach spaces \( V \) and \( U \), let \( L(V, U) \) denote the set of continuous linear mappings from \( V \) to \( U \). Consider a polynomial one-form \( p : V \to L(V, U) \) that is a polynomial taking values in \( L(V, U) \). For \( v, w, v_0 \in V \),

\[
p (v) (w) = \sum_{k=0}^n (D^k p) (v_0) (v - v_0)^{\otimes k} \frac{1}{k!} (w),
\]
where \( p(v) \in L(V, U) \) and \( p(v) (w) \in U \). The value of \( p(v) (w) \) does not depend on \( v_0 \).

For a time interval \([S, T]\) and Banach space \( V \), let \( BV ([S, T], V) \) denote the set of continuous bounded variation paths \([S, T] \to V\).

Let \( x \in BV ([S, T], V) \) satisfy \( x_S = 0 \). Then

\[
\int_{r=S}^{T} p (x_r) \, dx_r = \sum_{k=0}^{n} \left( D^k p \right)(0) \int_{r=S}^{T} \frac{(x_r)^{\otimes k}}{k!} \, dx_r
\]

\[
= \sum_{k=0}^{n} \left( D^k p \right)(0) \int_{S < u_1 < \cdots < u_{k+1} < T} dx_{u_1} \otimes \cdots \otimes dx_{u_{k+1}}
\]

\[
= : \sum_{k=0}^{n} \left( D^k p \right)(0) X_{S, T}^{k+1}
\]

where the first equality is the Taylor expansion of \( p \) and the second equality is based on the symmetry of \( D^k p \). As a result, the classical integral \( \int p(x) \, dx \) is expressed as a finite linear combination of iterated integrals of \( x \).

Let \( G(V) \) denote the set of group-like elements in the tensor algebra \( T(V) \). Based on Chen [2], the signature of continuous bounded variation paths form a subgroup of \( G(V) \).

**Notation 13** For a polynomial one-form \( p : V \to L(V, U) \) of degree \( n \), define \( f_p : G(V) \to U \) as

\[
f_p (g) := \sum_{k=0}^{n} \left( D^k p \right)(0) g^{k+1}
\]

where \( g = \sum_{k \geq 0} g^k \) with \( g^k \in V^{\otimes k} \).

A polynomial one-form \( p : V \to L(V, U) \) can be lifted to an exact one-form \( df_p \) for \( f_p : G(V) \to U \) defined at (2). Indeed, for \( x \in BV ([S, T], V) \), if denote \( X_t := \exp (x_S) S (x_{|[S,t]} \) for each \( t \), then

\[
\int_{r=S}^{T} p (x_r) \, dx_r = f_p (X_T) - f_p (X_S).
\]

When \( x_S = 0 \), \( f (X_S) = 0 \) and the equality holds based on the calculation above. When \( x_S \neq 0 \), by connecting 0 and \( x_S \) with a straight line and using the additive property of integrals, the equality \( \int_{r=S}^{T} p (x_r) \, dx_r = f_p (X_T) - f_p (X_S) \) still holds. Then based on the fundamental theorem of calculus and the change of variable formula

\[
f_p (X_T) - f_p (X_S) = \int_{X_S}^{X_T} df_p = \int_{r=S}^{T} df_p dX_r.
\]
As a result, the polynomial one-form $p : V \to L(V, U)$ is lifted to an exact one-form $df_p$ for $f_p : G(V) \to U$ such that

$$
\int_{r=S}^{T} p(x_r) dx_r = \int_{r=S}^{T} df_p dX_r
$$

for each $x \in BV([S, T], V)$, where $X_t := \exp (x_S) S (x_{[S,t]})$. 

The lifting of a path to a rough path simplifies the formulation of integration, and is necessary. A rough path can be viewed as a basis of controlled systems, and integration/differential equation can be viewed as a transformation between bases of controlled systems. The metric on rough path space can be considerably weaker ($p$-variation, $p < \infty$) than the metric needed to define iterated integrals ($p$-variation, $p < 2$). When the metric is weaker than $p < 2$, the basis systems $1, \ldots, [p]$ are selected to postulate iterated integrals and satisfy an abstract 'integration by parts formula' (defines a character of shuffle algebra for each fixed time). The algebraic structure is important to interpret the limit behavior of controlled systems. For example, physical Brownian motion in a magnetic field can be described by Brownian motion with a 'non-canonical' Lévy area [6].

Consider the lifting of the classical integral $\int p(x) dx$ to $f_p : G(V) \to U$. The function $f_p$ takes values in the Banach space $U$ same as the integral $\int p(x) dx$. The full rough integral is a mapping between paths in groups, and we need to lift $f_p : G(V) \to U$ to a function $F_p : G(V) \to G(U)$ such that

$$
S \left( \int_{r=S}^{T} p(x_r) dx_r \right) = F_p (X_S)^{-1} F_p (X_T)
$$

for each $x \in BV([S, T], V)$, where $\int_{r=S}^{T} p(x_r) dx_r \in BV([S, T], U)$ denotes the integral path and $X_t := \exp (x_S) S (x_{[S,t]})$. Lyons defined the lift of $f_p$ to $F_p$ p.285 Definition 3.2.2 [14]. The lift can also be interpreted as iterated integrals of controlled paths p.101 Theorem 1 Gubinelli [7].

We interpret the lifting of $f_p$ to $F_p$ in the language of the Malvenuto–Reutenauer Hopf algebra of permutations (denoted by MR) [17, 18]. Let $S_n$ denote the symmetric group of order $n$ for $n \geq 1$, and $S_0 = \{\lambda\}$. MR is a Hopf algebra on $\mathbb{Z}S$ with $S := \cup_{n=0}^{\infty} S_n$. The product on MR $\ast' : \mathbb{Z}S \times \mathbb{Z}S \to \mathbb{Z}S$ is the $\mathbb{Z}$-bilinear map given by

$$
\sigma \ast' \rho := \sigma \shuffle \bar{\rho}
$$

for $\sigma \in S_n$ and $\rho \in S_m$, where permutations are considered as words with $\bar{\rho}(i) := n + \rho(i)$ and $\shuffle$ denotes the shuffle product. For example,

$$
(1) \ast' (21) = 1 \shuffle 32 = (132) + (312) + (321).
$$

Based on Ronco [22], there is a natural operation $\succ : (\mathbb{Z}S \times \mathbb{Z}S) \setminus (\mathbb{Z}S_0 \times \mathbb{Z}S_0) \to \mathbb{Z}S$ that is the $\mathbb{Z}$-bilinear map given by

$$
\sigma \succ \rho := (\sigma \shuffle \bar{\rho}([m-1])) \bar{\rho}(m)
$$
for $\sigma \in S_n, \rho \in S_m, m \geq 1$, where permutations are considered as words with 
$\bar{\rho}(i) := n + \rho(i), \bar{\rho}([m - 1])$ denotes the word consisting the first $m - 1$ letters in $\bar{\rho}$, and $wi$ denotes the concatenation of $w$ with $i$. For example,

$$(1) \triangleright (312) = (1 \sqcup 42) 3 = (1423) + (4123) + (4213).$$

Set $\sigma \triangleright \lambda := 0 \in \mathbb{Z}$ for $\sigma \in S_n, n \geq 1, \lambda \in S_0$.

The operation $\triangleright$ can be viewed as an abstract iterated integration. For $\rho_i \in ZS, i = 1, \ldots, n$, denote p.252 Notation 2.3

$$m_\triangleright (\rho_1, \ldots, \rho_n) := (\cdots ((\rho_1 \triangleright \rho_2) \triangleright \rho_3) \cdots) \triangleright \rho_n.$$ 

For a (possibly infinite) set $A$ and a commutative $\mathbb{Q}$-algebra $K$, let $K\langle A \rangle$ resp. $K\langle A \rangle$ denote the ring of non-commutative formal series resp. polynomials on $A$ over $K$. There is a bilinear action of $ZS$ on $K\langle A \rangle$ given by

$$\sigma \cdot a_1 \cdots a_n := a_{\sigma 1} \cdots a_{\sigma n}$$

for $a_i \in A, \sigma \in S_n$. Let $A^*$ denote the free monoid generated by $A$ that is the set of sequences $a_1 \cdots a_n$ of elements in $A$ including the empty sequence $e$ with the operation of concatenation. Let $G(A)$ denote the set of group-like elements in $K\langle A \rangle$ that is the set of characters of the shuffle algebra on $K\langle A \rangle$: $s \in G(A)$ iff $(s, p \sqcup q) = (s, p) (s, q)$ for $p, q \in K\langle A \rangle$. For $s \in G(A)$, define $\tilde{s} : ZS \rightarrow K\langle A \rangle$ as

$$\tilde{s}(\sigma) := \sum_{w \in A^*} (s, \sigma \cdot w) w$$

for $\sigma \in ZS$. Based on Proposition

$$\text{conc} \circ (\tilde{s}(\sigma) \otimes \tilde{s}(\rho)) = \tilde{s}(\sigma \ast \rho)$$

for $\sigma, \rho \in ZS$, where $\text{conc}$ denotes the concatenation product.

Let $V$ and $U$ be two Banach spaces. For a polynomial one-form $p : V \rightarrow L(V, U)$ of degree $n$, let $(D^k p)(0) \in L(V \otimes (k+1), U)$ denote the $k$th derivative of $p$ evaluated at $0 \in V$. Let $1_k$ denote the identity element in $S_k, k \geq 1$. For $l = 1, 2, \ldots, n$

$$\sigma_l := \sum_{\substack{k_i = 0, \ldots, n, \allowbreak i = 1, \ldots, l}} (D^{k_1} p)(0) \otimes \cdots \otimes (D^{k_l} p)(0) m_\triangleright (1_{k_1+1}, \ldots, 1_{k_l+1}).$$

Let $G(V)$ be the set of group-like elements in the tensor algebra $T(V)$. For simplicity, we assume that $V$ has a (possibly infinite) basis given by a set $A$, and let $G(V)$ be the set of group-like elements in $K\langle A \rangle$.

Notation 14 For a polynomial one-form $p : V \rightarrow L(V, U)$ of degree $n$, define $F_p : G(V) \rightarrow T(U)$ as

$$F_p (s) := 1 + \sum_{l=1}^{\infty} \tilde{s}(\sigma_l), s \in G(V),$$

for $n \geq 1$.
where \( \tilde{s}(\sigma_1) \in U^{\otimes l} \) is given by

\[
\tilde{s}(\sigma_1) = \sum_{k_i=0,\ldots,n} \sum_{i=1,\ldots,l} (D^{k_i} p)(0) \otimes \cdots \otimes (D^{k_i} p)(0) \tilde{s}(m_\nu(1_{k_1+1},\ldots,1_{k_l+1}))
\]

with \( \tilde{s} \) defined at (3).

The following Proposition proves that \( F_p \) takes values in group-like elements in \( T(U) \) (denoted by \( G(U) \)). The result helps to prove that the indefinite integral of a polynomial one-form along a geometric rough path is again a geometric rough path.

**Proposition 15** \( F_p : G(V) \rightarrow T(U) \) is a lift of \( f_p : V \rightarrow U \), and \( F_p \) takes values in \( G(U) \) the group-like elements in \( T(U) \).

**Proof.** \( F_p \) is a lift of \( f_p \) because \( f_p(s) = \tilde{s}(\sigma_1), s \in G(V) \). For \( \rho \in S_j \) and \( n_1,\ldots,n_j \geq 1 \), denote

\[
\rho \cdot m_\nu(1_{n_1},\ldots,1_{n_j}) := m_\nu(1_{n_\rho(i)},\ldots,1_{n_\rho(l)})
\]

with \( \overline{n_1}(i) := i, \overline{n_{i+1}}(i) := \sum_{r=1}^l n_r + i \). Based on Lemma 9 and Proposition 7 for \( s \in G(V) \), \( \tilde{s} \) defined at (3) satisfies

\[
\text{conc} \circ (\tilde{s}(m_\nu(1_{n_1},\ldots,1_{n_k})) \otimes \tilde{s}(m_\nu(1_{n_{k+1}},\ldots,1_{n_j})))
\]

\[
= \tilde{s}(m_\nu(1_{n_1},\ldots,1_{n_k}) \cdot m_\nu(1_{n_{k+1}},\ldots,1_{n_j}))
\]

\[
= \tilde{s}(1_k \cdot 1_j) \cdot m_\nu(1_{n_1},\ldots,1_{n_j}).
\]

Then based on the group structure of \( \{S_k\}_k \),

\[
\text{conc} \circ (\tilde{s}(\alpha \cdot \sigma_k) \otimes \tilde{s}(\rho \cdot \sigma_j)) = \tilde{s}((\alpha \cdot \rho) \cdot \sigma_{k+j})
\]

for \( \alpha \in S_k, \rho \in S_j, k \geq 1, j \geq 1 \), where

\[
\tilde{s}(\rho \cdot \sigma_j) := \sum_{k_i=0,\ldots,n} \sum_{i=1,\ldots,l} (D^{k_i} p)(0) \otimes \cdots \otimes (D^{k_i} p)(0) \tilde{s}(m_\nu(1_{k_1+1},\ldots,1_{k_l+1})).
\]

A polynomial one-form \( p : V \rightarrow L(V,U) \) can be lifted to an exact one-form \( dF_p \) for \( F_p : G(V) \rightarrow G(U) \) defined at (4). Indeed, for \( x \in BV([S,T],V) \), denote \( X_t := \exp(x_S) S(x_{|_{[S,T]}}) \) for each \( t \). Define \( y \in BV([S,T],U) \) by \( y_t := \int_{r=S}^t p(x_r) \, dx_r \), and define \( Y_t := S(y_{|_{[S,T]}}) \). Then

\[
Y_S^{-1} Y_T = F_p(X_S)^{-1} F_p(X_T) = \int_{r=S}^T dF_p dX_r.
\]

When \( x_S = 0 \), \( F_p(X_S) = 1 \) and the equality holds. When \( x_S \neq 0 \), by connecting 0 and \( x_S \) with a straight line and using Chen’s identity, the equality \( Y_S^{-1} Y_T = \)
\( F_p(X_S)^{-1} F_p(X_T) \) still holds. The lift of \( p \) to \( dF_p \) does not depend on the path \( x \).

For a general geometric rough path \( X : [S, T] \to G(V) \), the integral of \( p \) along \( X \) can be defined as

\[
\int_{r=S}^{T} p(x_r) \, dX_r := F_p(X_S)^{-1} F_p(X_T) =: \int_{r=S}^{T} dF_p \, dX_r,
\]

where \( x \) denotes the projection of \( X \) to a path in \( V \). This integration provides another interpretation of the almost multiplicative functional defined by Lyons in Definition 3.2.2 [14].

The integration of polynomial one-forms provides basic ingredients for the integration of general regular one-forms. Polynomials have nice approximative properties, and classically the smoothness of a function is expressed in terms of polynomials. Based on Stein (Chapter VI [23]), a function \( \theta \) on a closed subset \( F \subset \mathbb{R}^n \) is \( \text{Lip}(\gamma) \) for some \( \gamma \in (k, k+1] \) if there exists a family of functions \( \theta_j, j = 0, \ldots, k \), with \( \theta_0 = \theta \), so that if

\[
\theta_j(x) = \sum_{|j| \leq k} \frac{\theta^{j+1}(y)}{t!} (x - y)^j + R_j(x, y)
\]

then \( |\theta_j(x)| \leq M \) and \( |R_j(x, y)| \leq M |x - y|^\gamma |j| \) for all \( x, y \in F, |j| \leq k \).

Based on Lyons [14] Lipschitz one-forms are Lipschitz functions in the sense of Stein, taking values in continuous linear mappings.

The following is Definition 3.2.2 and Theorem 3.2.1 p.285 [14].

**Definition 16 (Lyons)** For any multiplicative functional \( X_{s,t} \) in \( \Omega G(V)^p \) define

\[
Y_{s,t} = \sum_{l_1, \ldots, l_i = 1}^{[p]} \theta^{l_1}_1(x_s) \otimes \cdots \otimes \theta^{l_i}_i(x_s) \sum_{\pi \in \Pi_i} \pi((X_{s,t}^{|\pi|}))
\]

**Theorem 17 (Lyons, Existence of Integral)** For any multiplicative functional \( X_{s,t} \) in \( \Omega G(V)^p \) and any one-form \( \theta \in \text{Lip}[\gamma - 1, \{X_u, u \in [s, t]\}] \) with \( \gamma > p \) the sequence \( Y_{s,t} = (1, Y_{s,t}^1, \ldots, Y_{s,t}^{[p]}) \) defined above is almost multiplicative and of finite \( p \)-variation; if \( X_{s,t} \) is controlled by \( \omega \) on \( J \) where \( \omega \) is bounded by \( L \), and the \( \text{Lip}[\gamma - 1] \) norm of \( \theta \) is bounded by \( M \), then the almost multiplicative and \( p \)-variation properties of \( Y \) are controlled by multiples of \( \omega \) which depend only on \( \gamma, p, L, M \).

The multiplicative functional associated with \( Y \) obtained in Theorem 17 is defined to be the integral of the one-form \( \theta \) along geometric rough path \( X \) Theorem 3.3.1 p.274 Definition 3.2.3 p.288 [14]. Denote the integral as \( \int \theta(x) \, dX \).

Based on Theorem 3.3.1 p.274 [14],

\[
\int_{r=0}^{1} \theta(x_r) \, dX_r := \lim_{|D| \to 0, D \subset [0,1]} Y_{t_0, t_1} \cdots Y_{t_{n-1}, t_n}.
\]
Based on the lift of polynomial one-form \( p \) to exact one-form \( dF_p \), the integral \( \int \theta(x) \, dX \) can be interpreted in terms of time-varying exact one-forms.

Let \( X : [0, 1] \to G(V) \) be a geometric \( p \)-rough path, and let \( \theta : V \to L(V,U) \) be a \( \text{Lip}(\gamma) \) one-form for \( \gamma > p - 1 \). Let \( x \) denote the projection of \( X \) to a path in \( V \). For \( x_s \in V \), define the polynomial one-form \( p_{x_s} : V \to L(V,U) \) as

\[
p_{x_s}(v)(w) = \sum_{k=0}^{[p]} g^k(x_s) \frac{(v - x_s)^{\otimes k}}{k!}(w)
\]

(7)

for \( v, w \in V \). For the polynomial one-form \( p_{x_s} \), define \( F_{p_{x_s}} \) as at (4). The almost multiplicative functional \( Y_{s,t} \) in Definition 16 can be expressed as

\[
Y_{s,t} = F_{p_{x_s}}(X_s)^{-1} F_{p_{x_t}}(X_t) =: \int_{r=s}^{t} dF_{p_{x_r}} \, dX_r
\]

(8)

for every \( s \leq t \). The equality (8) follows from a generalized Chen’s identity about the multiplicativity of rough path liftings of controlled paths/effects (see Section 5 for details). The generalized Chen’s identity can be proved based on the uniqueness of the continuous lifting (Proposition 10).

**Theorem 18** For a geometric \( p \)-rough path \( X : [0, 1] \to G(V) \) and a \( \text{Lip}(\gamma - 1) \) one-form \( \theta : V \to L(V,U) \) for \( \gamma > p \), let \( \int_{r=0}^{1} \theta(x_r) \, dX_r \) denote the integral defined by Lyons in [14]. Then with \( p_{x_s} \) defined at (7) and \( F_{p_{x_s}} \) defined at (4),

\[
\int_{r=0}^{1} \theta(x_r) \, dX_r = \lim_{|D| \to 0, D \subset [0,1]} \int_{r=t_0}^{t_1} dF_{p_{x_r}} \, dX_r \cdots \int_{r=t_{n-1}}^{t_n} dF_{p_{x_r}} \, dX_r
\]

(9)

\[
= \int_{r=0}^{1} dF_{p_{x_r}} \, dX_r
\]

where \( D = \{t_k\}_{k=0}^{n} \), \( 0 = t_0 < \cdots < t_n = 1 \), \( n \geq 1 \) with \( |D| := \max_{k} |t_{k+1} - t_k| \).

The equality is based on (8) and (9), the existence of the integral \( \int \theta(x) \, dX \) is obtained in Theorem 3.2.1 [14] i.e. Theorem 17.

There is a minor difference between geometric rough paths \( \Omega G(V)^p \) in [13] and that defined in Definition 1 (paths in Definition 4 are also called weak geometric rough paths). The integration \( \int_{r=0}^{1} dF_{p_{x_r}} \, dX_r \) in Theorem 18 can be applied to both classes of rough paths.

The integration of time-varying exact one-forms exists in a general setting. Consider two groups \( G_1 \) and \( G_2 \), and a path \( X : [0, 1] \to G_1 \). Suppose \( f_t : G_1 \to G_2 \) is a family of functions indexed by \( t \in [0, 1] \). If the limit exists in \( G_2 \):

\[
\lim_{|D| \to 0, D = \{t_k\}_{k=0}^{n} \subset [0,1]} \int_{r=t_0}^{t_1} df_{t_0} \, dX_r \int_{r=t_1}^{t_2} df_{t_1} \, dX_r \cdots \int_{r=t_{n-1}}^{t_n} df_{t_{n-1}} \, dX_r
\]

(10)
where \( f_{r}^{n+1}df_{r}dX_{r} := f_{r} \left( X_{r} \right)^{-1} f_{r} \left( X_{r+1} \right) \), then the integral \( \int_{r=0}^{1} df_{r}dX_{r} \) is defined to be the limit. A sufficient condition for the existence of the integral is given in Theorem 21 below based on “a non-commutative sewing lemma” by Feyel, de la Pradelle and Mokobodzki [5]. The integral can be viewed as an non-abelian analogue of Young’s integral [25].

The integration of time-varying exact one-forms can also be explained via reset of functions. For \( f : G_{1} \rightarrow G_{2} \), define the reset of \( f \) at \( a \in G_{1} \) as

\[
f_{a} : G_{1} \rightarrow G_{2} \\
g \mapsto f(a)^{-1}f(ag)
\]

The reset of functions are consistent with the integration of exact one-forms. For \( X : [0,1] \rightarrow G_{1} \), one has \( \int_{r=0}^{1} df_{r}dX_{r} = f(X_{0})^{-1}f(X_{1}) = f_{0} (X_{0}^{-1}X_{1}) \).

Consider a principal bundle \( P \) on \( G_{1} \) that associates each \( a \in G_{1} \) with the group of functions \( \{h|h: G_{1} \rightarrow G_{2}, h(1_{G_{1}}) = 1_{G_{2}}\} \). The reset of functions defines a parallel transportation on \( P \). For \( h \in P_{a}, a \in G_{1} \), the parallel translation of \( h \) is given by

\[
h_{b} := (h_{a})_{a^{-1}b}
\]

for \( b \in G_{1} \). The \( \{h_{b}|b \in G_{1}\} \) defined are consistent:

\[
( (h_{a})_{a^{-1}b} )_{b^{-1}c} = (h_{a})_{a^{-1}c}
\]

for \( b, c \in G_{1} \). For \( X : [0,1] \rightarrow G_{1} \), consider

\[
\beta \in (X, P_{X}) \text{ i.e. } \beta : X_{t} \mapsto \beta(X_{t}) \in P_{X_{t}}.
\]

Then the integration of time-varying exact one-forms (when exists)

\[
\int_{r=0}^{1} df_{r}dX_{r} := \lim_{|D| \rightarrow 0, D = \{ t_{i} \} \subset [0,1]} \int_{r=t_{0}}^{t_{1}} df_{t_{0}}dX_{r} \cdots \int_{r=t_{n-1}}^{t_{n}} df_{t_{n-1}}dX_{r}
\]

can be equivalently defined as

\[
\int_{r=0}^{1} \beta(X_{r})dX_{r} := \lim_{|D| \rightarrow 0, D = \{ t_{i} \} \subset [0,1]} \beta(X_{t_{0}}) \left( X_{t_{0},t_{1}} \right) \cdots \beta(X_{t_{n-1}}) \left( X_{t_{n-1},t_{n}} \right)
\]

where \( \beta(X_{t})\left(g\right) := f(X_{t})^{-1}f(X_{t}g) = f_{X_{t}}\left(g\right), g \in G_{1}, t \in [0,1], \) and \( \{\beta(X_{t})\}_{t} \) are compared after parallel translation.

For a fixed continuous path \( X : [0,1] \rightarrow G_{1} \), consider a condition on exact one-forms \( (df_{t})_{t} \) for \( f_{t} : G_{1} \rightarrow G_{2} \) that guarantees the existence of the integral \( \int_{r=0}^{1} df_{r}dX_{r} \). Theorem 21 below gives a condition that roughly states that, if one-step discrete approximations are comparable to two-steps discrete approximations up to a small error, then the integral exists as a limit of Riemann products. When \( X \) is a geometric \( p \)-rough path, the condition on \( (df_{t})_{t} \) can be further specified, and the condition can be viewed as an inhomogeneous analogue of Young’s condition p.264 [25]. Such a condition is closely related to the notion of weakly controlled paths introduced by Gubinelli [7]. The following is Definition 1 [7].

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Definition 19 (Gubinelli, weakly controlled paths) Fix an interval \( I \subseteq \mathbb{R} \) and let \( X \in \mathcal{C}^\gamma(I, V) \). A path \( Z \in \mathcal{C}^\gamma(I, V) \) is said to be weakly controlled by \( X \) in \( I \) with a reminder of order \( \eta \) if there exists a path \( Z' \in \mathcal{C}^{\eta-\gamma}(I, V \otimes V^*) \) and a process \( R_Z \in \Omega \mathcal{C}^\eta(I, V) \) with \( \eta > \gamma \) such that
\[
\delta Z^\mu = Z'^\mu \delta X^\nu + R_Z^\mu.
\]
If this is the case we will write \((Z, Z') \in \mathcal{D}^{\gamma, \eta}(X, V)\) and we will consider on the linear space \( \mathcal{D}^{\gamma, \eta}(X, V) \) the semi-norm
\[
\|Z\|_{\mathcal{D}^{\gamma, \eta}(X, V)} := \|Z'\|_{\infty, I} + \|Z'\|_{\eta-\gamma, I} + \|R_Z\|_{\eta, I} + \|Z\|_{\gamma, I}.
\]

For a fixed geometric rough path \( X \), we consider a class of integrable one-forms whose indefinite integrals are called effects (see Section 5 for more details). The set of effects of \( X \) is a subset of the paths controlled by \( X \). The relationship between controlled paths and effects is comparable to that between the integrand and the integral. In the integration of one-form \( \int \alpha(x) \, dX(t) \) is a controlled path; \( t \mapsto \int_{r=0}^t \alpha(x_r) \, dX_r \) is an effect so a controlled path. A controlled path can also be interpreted as a time-varying exact one-form, and its varying speed can be a little quicker than that of an effect. One benefit of working with effects is that basic operations (multiplication, composition with regular functions, integration, iterated integration) are continuous operations in the space of one-forms in operator norm. In particular, the lifting of an effect to a geometric rough path is continuous. In [16] effects (dominated paths) are employed to give a short proof of the unique solvability and stability of the solution to differential equations driven by rough paths, and differences between adjacent Picard iterations decay factorially in operator norm.

4 Integration of time-varying exact one-forms

For continuous \( x : [0, 1] \to \mathbb{C} \) of finite \( p \)-variation and continuous \( y : [0, 1] \to \mathbb{C} \) of finite \( q \)-variation \( p^{-1} + q^{-1} > 1, p \geq 1, q \geq 1 \), Young [25] defined the Stieltjes integral \( \int_{r=0}^t x_r \, dy_r \) as the limit of Riemann sums. Lyons [14] defined the integration of one-forms along geometric rough paths by constructing a multiplicative functional from an almost multiplicative functional.

In [5] Feyel, La Pradelle and Mokobodzki proved a non-commutative sewing lemma that constructs multiplicative functions taking values in a monoid with a distance.

Based on [5], let \( M \) be a monoid with a unit element \( I \), and \( M \) is complete under a distance \( d \) that satisfies
\[
d(xz, yz) \leq |z| d(x, y), \quad d(zx, zy) \leq |z| d(x, y)
\]
for \( x, y, z \in G_2 \), where \( z \mapsto |z| \) is a Lipschitz function on \( M \) with \( |I| = 1 \).

Suppose \( V : [0, T] \to \mathbb{R}^+ \) is a strong control function i.e. \( V(0) = 0 \), non-decreasing, and there exists a \( \theta > 2 \) such that for every \( t \)
\[
\nabla(t) := \sum_{n \geq 0} \theta^n V(t2^{-n}) < \infty.
\]
For example, \( V(t) = t^\alpha \) when \( \alpha > 1 \) is a strong control function.

Suppose \( \mu : \{(s, t) | 0 \leq s < t < T\} \to (M, d) \) is continuous, \( \mu(t, t) = I \) for every \( t \), and

\[
d(\mu(s, t), \mu(s, u) \mu(u, t)) \leq V(t - s) \quad (10)
\]

for every \( s \leq u \leq t \). \( u : \{(s, t) | 0 \leq s < t < T\} \to (M, d) \) is called multiplicative if \( u(s, t) = u(s, u) u(u, t) \) for every \( s \leq u \leq t \).

**Theorem 20 (Feyel, La Pradelle, Mokobodzki)** There exists a unique continuous multiplicative function \( u \) such that \( d(\mu(s, t), u(s, t)) \leq \text{Cst} \sqrt{V(t - s)} \) for every \( s \leq t \).

Let \( G_1 \) and \( G_2 \) be two groups, and suppose \( G_2 \) is complete under a distance \( d \) that satisfies \((9)\). Let \( X : [0, T] \to G_1 \). Suppose \( f_t : G_1 \to G_2 \) is a family of functions indexed by \( t \in [0, T] \). Define \( \mu : \{(s, t) | 0 \leq s < t < T\} \to (G_2, d) \) as

\[
\mu(s, t) := f_s(X_s)^{-1} f_s(X_t) \in G_2 \quad (11)
\]

for \( s \leq t \). Suppose \((10)\) holds for a strong control function \( V \).

Let \( u : \{(s, t) | 0 \leq s < t < T\} \to (G_2, d) \) denote the multiplicative function associated with \( \mu \) at \((11)\) obtained by Theorem 20.

**Theorem 21** Define \( \int_{r=0}^t df_dX_r : [0, T] \to G_2 \) as \( \int_{r=0}^t df_dX_r := u(0, t) \) for every \( t \). Then \( \int_{r=0}^t df_dX_r \) is the unique continuous path \( y : [0, T] \to G_2 \) such that \( y_0 = 1_{G_2} \) and \( d(y^{-1}y_r, \int_{r=0}^t df_dX_r) \leq \text{Cst} \sqrt{V(t - s)} \) for every \( s \leq t \).

Based on the definition, for \( X : [0, 1] \to G_1 \) and \( f : G_1 \to G_2 \), \( \int_0^1 df_dX_r := f(X_0)^{-1} f(X_1) \).

Based p.31 [5] Theorem 21 applies when \( G_2 \) is a group of elements beginning with 1 in the tensor algebra over a Banach space when tensor powers are equipped with admissible norms.

### 5 Effects of a geometric rough path

The set of effects of a geometric rough path is a subset of the paths controlled by the geometric rough path. Similar to controlled paths, effects are stable under basic operations. Integrals of one-forms and solution to differential equations are effects so are controlled paths.

Let \( V \) and \( U \) be two Banach spaces, and let \( G(V) \) be the set of group-like elements in the tensor algebra \( T(V) \). For \( p \geq 1 \), set \([p] := \max \{n | n \in \mathbb{N}, n \leq p\} \).

For \( k = 1, \ldots, [p] \), denote by \( L(V^\otimes k, U) \) the set of continuous linear operators from \( V^\otimes k \) to \( U \).

**Notation 22** Let \( E^U \) denote the vector bundle on \( G(V) \) that associates each \( a \in G(V) \) with the vector space:

\[
E^U_a := \left\{ \phi : G(V) \to U, \phi = \sum_{k=1}^{[p]} \phi_k, \phi_k \in L(V^\otimes k, U) \right\} \quad (12)
\]
where \( \phi^k(x) := \phi^k x^k \) for \( x \in G(V), x = \sum_{k=0} x^k, x^k \in V^{\otimes k} \).

\( E^U_a \) can be considered as the space of ‘polynomials’ up to degree-[\( p \)] that has no ‘constant’, and can be viewed as the polynomial approximation to the ‘tangent space’ of functions \( \{ f_a : G(V) \to U \} \) with \( f_a(x) := f(ax) - f(a), x \in G(V) \).

The parallel transportation on \( E^U \) is given by the reset of functions.

**Notation 23** For \( p_a \in E^U_a \) and \( b \in G(V) \), define \( (p_a)_{a^{-1}b} \in E^U_b \) as

\[
(p_a)_{a^{-1}b}(x) := p_a(a^{-1}bx) - p_a(a^{-1}b)
\]

for \( x \in G(V) \).

Then \( ((p_a)_{a^{-1}b})_{b^{-1}c} = (p_a)_{a^{-1}c} \) for \( a, b, c \in G(V) \).

For \( \phi \in E_a, \phi = \sum_{k=1}^{|p|} \phi^k \), denote

\[
\|\phi\| := \max_{k=1,...,|p|} \left\| \phi^k \right\| \text{ and } \|\phi\|_k := \left\| \phi^k \right\|
\]

where \( \left\| \phi^k \right\| \) denotes the norm of \( \phi^k \) as a linear operator.

**Definition 24 (Operator Norm)** Let \( X : [0,1] \to G(V) \) be a geometric \( p \)-rough path for some \( p \geq 1 \), and let \( U \) be a Banach space. Suppose

\[ \beta \in (X,E^U_X) \text{ i.e. } \beta : X_t \mapsto \beta(X_t) \in E^U_X. \]

For \( t \in [0,1] \) and \( a \in G(V) \), define

\[
(\beta(X_t))_a \in E^U_{X_t,a} \quad x \mapsto \beta(X_t)(ax) - \beta(X_t)(a), x \in G(V).
\]

For a control \( \omega \) and \( \theta > 1 \), define the operator norm

\[
\|\beta\|_\theta^\omega := \sup_{t \in [0,1]} \|\beta(X_t)\| + \max_{k=1,...,|p|} \sup_{0 \leq s < t \leq 1} \left| \beta(X_t) - (\beta(X_s))_{X_t^{-1}X_s} \right|_k.
\]

**Definition 25 (Slowly-Varying One-Form)** Let \( X : [0,1] \to G(V) \) be a geometric \( p \)-rough path for some \( p \geq 1 \), and let \( U \) be a Banach space. Then \( \beta \in (X,E^U_X) \) is called a slowly varying one-form, if there exists a control \( \omega \) and \( \theta > 1 \) such that \( \|\beta\|_\theta^\omega < \infty \).

For each \( t \in [0,1] \), \( \beta(X_t) \) can be viewed as a continuous linear mapping from monomials (components of rough paths) to the vector space \( U \).

For a control \( \omega \) and \( \theta > 1 \), the set of slowly varying one-forms along \( X \) with finite operator norm \( \|\beta\|_\theta^\omega \) form a Banach space.
Suppose $\beta \in (X, E_X U)$ is a slowly-varying one-form. Let $\beta(X_t)_{X^{-1}}$ be considered as functions $G(V) \to U$ indexed by $t \in [0, 1]$. Define

$$\int_{r=0}^{1} \beta(X_r) dX_r$$

$$= \int_{r=0}^{1} d \left( \beta(X_r)_{X^{-1}} \right) dX_r$$

$$= \lim_{|D| \to 0, D \subset [0,1]} \sum_{k,t_k \in D} \beta(X_{t_k}) (X_{t_k}, t_{k+1})$$

where the last equality is based on $\beta(X_{t_k}) \left( 1_{G(V)} \right) = 0$. The integral exists based on the slowly varying condition on $\beta$.

**Definition 26 (Effects)** Let $X : [0, 1] \to G(V)$ be a geometric $p$-rough path for some $p \geq 1$, and let $\beta \in (X, E_X U)$ be a slowly varying one-form. Then for $\xi \in U$, the integral path

$$t \mapsto \xi + \int_{r=0}^{t} \beta(X_r) dX_r, t \in [0, 1]$$

is called an effect of $X$.

**Theorem 27** Suppose $\beta \in (X, E_X U)$ is a slowly-varying one-form such that $\|\beta\|_\omega^\theta < \infty$ for a control $\omega$ and $\theta > 1$. Define $h : [0, 1] \to U$ as

$$h_t := \int_{r=0}^{t} \beta(X_r) dX_r, t \in [0, 1].$$

Then with control $\hat{\omega} := \omega + \|X\|_{p-var}^p$,

$$\|h_t - h_s - \beta_s(X_s)(X_{s,t})\| \leq C_{p,\theta, \hat{\omega}(0,T)} \|\beta\|_{\hat{\omega}}^\theta \hat{\omega}(s,t)^\theta$$

for every $s \leq t$, and

$$\|h\|_{p-var, [0,T]} \leq C_{p,\theta, \hat{\omega}(0,T)} \|\beta\|_{\hat{\omega}}^\theta.$$

The first estimate can be proved similarly to result 5 on p.254 Young [25]; the second estimate follows from the first.

### 6 Stability of Effects under Basic Operations

Consider the set of effects of a geometric rough path. Effects are closed under basic operations (multiplication, composition with regular functions, integration, iterated integration); the proof is similar to that for controlled paths as in
6.1 Composition with Regular Functions

The stability of effects under composition with regular functions follows from the fact that polynomials are closed under composition.

For Banach spaces $U$ and $W$, denote by $C^\gamma(U, W)$ the set of functions $\varphi : U \to W$ that are $[\gamma]$-times Fréchet differentiable with the $[\gamma]$th derivative $(\gamma - [\gamma])$-Hölder, uniformly on any bounded set. That is, for each $R > 0$,

\[
\|\varphi\|_{\gamma, R} := \max_{k=0,1,\ldots,[\gamma]} \|D^k \varphi\|_{\infty, R} + \|D^{[\gamma]} \varphi\|_{(\gamma-[\gamma]), \text{Hol}, R}
\]

where $\|\|_{\infty, R}$ resp. $\|\|_{(\gamma-[\gamma]), \text{Hol}, R}$ denote the uniform resp. Hölder norm on $\{u \in U : \|u\| \leq R\}$.

For $\phi(X_t) \in E^{U}_{X_t}$ and $l = 1, \ldots, [p]$, define the ‘truncated polynomial’:

\[
\prod_{l=1}^{[p]} \left( \phi(X_t) \otimes^l \right) \in E^{U}_{X_t}
\]

\[
x \mapsto \sum_{\sum_{l=0}^{[p]} k_l \leq [p]} \left( \phi^{k_1}(X_t) \otimes \cdots \otimes \phi^{k_{[p]}}(X_t) \right) \left( (1_{k_1} \ast \cdots \ast 1_{k_{[p]}}) \cdot (x^{k_1+\cdots+k_{[p]}}) \right)
\]

for $x \in G(V)$, where $\phi = \sum_{k=1}^{[p]} \phi^k$, $\phi^k \in L(V^\otimes k, U)$ and $x = \sum_{k=0}^{[p]} x^k$, $x^k \in V^\otimes k$.

**Proposition 28** Suppose $\beta_1 \in (X, E^U_X)$ is a slowly-varying one-form such that $\|\beta_1\|_{\varphi_1}^{\theta_1} < \infty$ for a control $\varphi_1$ and $\theta_1 > 1$. Denote $h_t := \int_0^t \beta_1(X_t) dX_t$, $t \in [0, 1]$. For $\varphi \in C^\gamma(U, W)$, $\gamma > p$, define $\beta \in (X, E^W_X)$ as

\[
\beta(X_t)(x) := \sum_{l=1}^{[p]} \prod_{k=1}^{[p]} \left( D^k \varphi \right)(h_t) \left( \prod_{l=1}^{[p]} \left( \beta_1(X_t) \otimes^l \right) \right)(x)
\]

for $x \in G(V)$. Then with $\omega := \omega_1 + \|X\|_{p-\text{var}}$ and $\theta := \min \left( \frac{\theta_1}{p}, \frac{\gamma + 1}{p} \right)$,

\[
\|\beta\|_{\theta}^{\omega} \leq C_{p, \theta, \omega(0,1)} \|\varphi\|_{\gamma, \|h\|_{\infty}} \max \left( \|\beta_1\|_{\theta_1}^{\omega_1}, \|\beta_1\|_{\theta_1}^{\omega_1} \right),
\]

where $\|h\|_{\infty} := \sup_{t \in [0,1]} \|h_t\|$, and

\[
\int_{0}^{t} \beta(X_r) dX_r = \varphi(h_t) - \varphi(h_0) \text{ for } t \in [0,1].
\]
Remark 29 Effects are closed under multiplication and form an algebra. For Banach spaces $U_i, i = 1, 2$, consider $\varphi : (U_1, U_2) \to U_1 \otimes U_2$ given by $(u_1, u_2) \mapsto u_1 \otimes u_2$. Then $\varphi$ is smooth and $D^3 \varphi \equiv 0$.

Proof. For $l = 1, \ldots, [p],$

$$(\phi \otimes^l)_a (x) = (\phi (a) + \phi_a (x)) \otimes^l - \phi \otimes^l (a)$$

for $x \in G (V)$.

We rescale $\varphi$ by $||\varphi||_{1, ||h||_\infty}^{-1}$ and assume $||\varphi||_{1, ||h||_\infty} = 1$. Denote $X_{s,t} := X_s^{-1} X_t$. For $s \leq t$,

$$\left( \beta (X_t) - (\beta (X_s)) _{X_{s,t}} \right) (x) = \sum_{l=1}^{[p]} \frac{[p]}{l!} \left( D^l \varphi (h_t) - \sum_{j=0}^{[p]-l} \frac{1}{j!} (D^{j+l} \varphi) (h_s) (h_t - h_s)^{\otimes^l} \right) \left( \prod_{[p]} (\beta_1 (X_s))^{\otimes^l} \right) (x)$$

$$+ \sum_{l=1}^{[p]} \sum_{j=0}^{[p]-l} \frac{1}{l!} \frac{1}{j!} (D^{j+l} \varphi) (h_s) \left( (h_t - h_s)^{\otimes^j} - \beta (X_s) (X_{s,t})^{\otimes^j} \right) \left( \prod_{[p]} (\beta_1 (X_s))^{\otimes^l} \right) (x)$$

$$+ \sum_{l=1}^{[p]} \frac{1}{l!} (D^l \varphi) (h_s) \prod_{[p]} \left( \beta_1 (X_s) (X_{s,t}) + \beta_1 (X_t) \right)^{\otimes^l} - \left( \beta_1 (X_s) (X_{s,t}) + (\beta_1 (X_s))_{X_{s,t}} \right)^{\otimes^l} (x)$$

for $x \in G (V)$. Then the estimate (13) follows from Theorem 27. Based on the comparison of local expansions,

$$\int_{r=0}^t \beta (X_r) dX_r = \varphi (h_t) - \varphi (0), t \in [0, 1].$$

6.2 Iterated Integration

Let $U_i, i = 1, 2$ be two Banach spaces. For $\phi_i \in E_X^{U_i}, i = 1, 2$, define the ‘truncated iterated integration’:

$$\prod_{[p]} (\phi_1 \otimes \phi_2) \in E_X^{U_1 \otimes U_2}$$

$$x \mapsto \sum_{k_1 \otimes k_2 \leq [p], \quad k_i = 1, \ldots, [p]} \left( \phi_1^{k_1} \otimes \phi_2^{k_2} \right) \left( (1_{k_1}) \otimes (1_{k_2}) \cdot (x^{k_1+k_2}) \right)$$

for $x \in G (V)$, where $\phi_i = \sum_{k=1}^{[p]} \phi_i^k, \phi_i^k \in L (V^{\otimes^k}, U_i), x = \sum_{k \geq 0} x^k, x^k \in V^{\otimes^k}$. 

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Proposition 30 For \( i = 1, 2 \), let \( \beta_i \in \left( X, E^{U_i}_X \right) \) be a slowly-varying one-form such that \( \|\beta_i\|_{\theta_i}^\omega < \infty \) for a control \( \omega_i \) and \( \theta_i > 1 \). Define \( \beta \in \left( X, E^{U_1 \otimes U_2}_X \right) \) as

\[
\beta(X_t) := \left( \int_{r=0}^{t} \beta_1(X_r) \, dX_r \right) \otimes \beta_2(X_t) + \prod_{\leq [p]} (\beta_1(X_t) \succ \beta_2(X_t))
\]

for \( t \in [0, 1] \). Then with \( \omega := \omega_1 + \omega_2 + \|X\|_{p-var}^p \) and \( \theta := \min(\theta_1, \theta_2) \),

\[
\|\beta\|_{\theta}^\omega \leq C_{p, \theta, \omega(1, 1)} \|\beta_1\|_{\theta_1}^{\omega_1} \|\beta_2\|_{\theta_2}^{\omega_2}.
\]

(14)

Remark 31 For \( t \in [0, 1] \), set \( \beta_2(X_t)(x) := x^1 \) for \( x \in G(V) \), \( x = \sum_{k \geq 0} x^k \), \( x^k \in V^\otimes k \). Then the \( \beta \) defined above corresponds to the integration of \( \beta_1 \), and integration is a continuous operation on slowly-varying one-forms.

Proof. Based on Proposition 11 for \( \phi^i \in E^{U_i}_{X_t} \), \( i = 1, 2 \) and \( a \in G(V) \),

\[
\left( (\phi^1)_a \succ (\phi^2)_a \right)(x) = (\phi^1 \succ \phi^2)_a(x) - (\phi^1)_a \otimes (\phi^2)_a(x)
\]

for \( x \in G(V) \). Denote \( X_{s,t} := X_s^{-1}X_t \). For \( s \leq t \),

\[
\left( \beta(X_t) - \beta(X_s) \right)_{X_{s,t}}(x)
\]

\[
= \int_{s}^{t} \beta_1(X_r) \, dX_r \otimes \left( \beta_2(X_t) - \beta_2(X_s) \right) (X_{s,t})(x)
\]

\[
+ \left( \int_{s}^{t} \beta_1(X_r) \, dX_r - \beta_1(X_s) \right) (X_{s,t})(x) \otimes \beta_2(X_t)(x)
\]

\[
+ \beta_1(X_s)(X_{s,t}) \otimes \left( \beta_2(X_t) - \beta_2(X_s) \right)_{X_{s,t}}(x)
\]

\[
+ \prod_{\leq [p]} \left( (\beta_1(X_t) \succ \beta_2(X_t)) - (\beta_1(X_s))_{X_{s,t}} \succ (\beta_2(X_s))_{X_{s,t}} \right)(x)
\]

for \( x \in G(V) \). Then the estimate (14) holds based on the definition of the operator norm and Theorem 27.

Acknowledgement

The author would like to express sincere gratitude to Prof. Terry Lyons for numerous inspiring discussions that eventually lead to this paper. The author also would like to thank Prof. Martin Hairer, Sina Nejad, Dr. Horatio Boedihardjo, Dr. Xi Geng, Dr. Ilya Chevyrev and Vlad Margarint for discussions and suggestions on (an earlier version of) the paper.
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