THE DUAL LAGRANGIAN FIBRATION OF KNOWN HYPER-KÄHLER MANIFOLDS

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Abstract. Given a Lagrangian fibration \( \pi : X \to P^n \) of a compact hyper-Kähler manifold of K3\(^{(0)}\), Kum\(_n\), OG10 or OG6-type, we construct a natural compactification of its dual torus fibration. Specifically, this compactification is given by a quotient of \( X \) by certain automorphisms acting trivially on the second cohomology and respecting the Lagrangian fibration. It is a compact hyper-Kähler orbifold with identical period mapping behavior as \( X \).

1. Introduction

Let \( Y \) be a compact Calabi–Yau manifold with a fixed Kähler class and \( \pi : Y \to B \) its Lagrangian fibration. A general fiber of \( \pi \) is a torus by the classical Arnold–Liouville theorem. Any torus has its dual, so one may wonder if we can systematically dualize general fibers of \( \pi \) to obtain a new fibration \( \tilde{\pi} \). The mirror symmetry conjecture in [SYZ96] predicts this should be possible for certain situations. More specifically, one expects there exists a “dual Lagrangian fibration” \( \tilde{\pi} : \tilde{Y} \to B \) satisfying: (1) \( \tilde{Y} \) is a compact Calabi–Yau orbifold and \( \tilde{\pi} \) is its Lagrangian fibration, and (2) the smooth fibers of \( \tilde{\pi} \) are dual tori to the smooth fibers of \( \pi \). When the Calabi–Yau manifold of interest is a K3 surface, there is a holomorphic variant of this question. The Kähler class and Lagrangian fibration are replaced into a holomorphic symplectic form and holomorphic elliptic fibration \( \pi : X \to B \). Unfortunately, elliptic curves are self-dual, so the original \( \pi : X \to B \) satisfies both the conditions (1–2) and the conjecture becomes rather uninteresting.

A compact hyper-Kähler manifold is a higher dimensional generalization of a K3 surface. It is a simply connected compact Kähler manifold with a unique global holomorphic symplectic form up to scale. Let \( \pi : X \to B \) be a holomorphic Lagrangian fibration of a compact hyper-Kähler manifold \( X \). By the same reasons for K3 surfaces, [GTZ13, §2] claimed \( \pi \) should be considered as self-dual if the following two conditions hold: (A) all the torus fibers of \( \pi \) are principally polarized abelian varieties, and (B) \( \pi \) admits at least one section. The role of the assumption (A) is to say \( \pi \) is fiberwise self-dual, and the role of (B) is to single out a uniform dualization of complex tori as a family. If the assumptions are dropped, there is a priori no reason why one should believe the existence of a good notion of a dual Lagrangian fibration \( \tilde{\pi} : \tilde{X} \to B \). The goal of this paper is to give, without the assumptions (A–B), one distinguished candidate of a dual Lagrangian fibration \( \tilde{\pi} \) that satisfies all the expected properties. Unfortunately, we were able to realize our strategy only for the currently known deformation types of hyper-Kähler manifolds (Theorem 1.1), but we believe similar results should hold in the most general set-up. Once the assumption (A) fails, the construction yields a compact hyper-Kähler orbifold \( \tilde{X} \) that is not homeomorphic to \( X \). The technical assumption (B) will be completely overcome.

Let again \( X \) be a compact hyper-Kähler manifold of dimension \( 2n \). A Lagrangian fibration of \( X \) in this paper will mean a holomorphic surjective morphism \( \pi : X \to B \) with connected fibers to a complex manifold \( B \) of \( 0 < \dim B < 2n \). By [Hwa08] and [GL14], the base \( B \) is necessarily
isomorphic to \( \mathbb{P}^n \). It is well-known that any smooth fiber of \( \pi \) is a complex Lagrangian subtorus of \( X \), so by restricting the Lagrangian fibration \( \pi \) to its smooth locus \( B_0 \subset B \) we get a torus fibration \( \pi_0 : X_0 \to B_0 \), a smooth proper family of complex tori.

The dual Lagrangian fibration \( \tilde{\pi} \) will be obtained by a suitable compactification of the “dual torus fibration” \( \pi_0 : \tilde{X}_0 \to B_0 \) which fiberwise dualizes the original torus fibration \( \pi_0 \). [Saw04] and [Nag05] proposed to define the dual torus fibration as the relative Picard scheme of \( \pi_0 \). While this definition behaves well when \( \pi_0 \) admits a section, it behaves slightly awkward when \( \pi_0 \) has no sections. We thus start with proposing a new definition of \( \tilde{\pi}_0 \). Recall the fact that all the torus fibers of \( \pi_0 \) are canonically polarized (e.g., Voisin’s argument in [Cam06, Prop 2.1]). That is, each torus fiber \( F \) of \( \pi_0 \) admits a natural isogeny \( F \to \hat{F} \) to its dual torus \( \hat{F} \). Let us denote the kernel of this isogeny by \( \mathrm{ker} \) and obtain an isomorphism \( \hat{F} \cong F/\mathrm{ker} \). The idea is to make this discussion global over the entire base \( B_0 \). In Theorem 3.1, we will attach a canonically polarized abelian scheme \( \hat{P}_0 \to B_0 \) to \( \pi_0 \) so that \( X_0 \) becomes a \( P_0 \)-torsor (this combines the results of Arinkin–Fedorov and van Geemen–Voisin). Let \( K_0 \) be the kernel of this canonical polarization \( \hat{P}_0 \to \hat{P}_0 \). It is a group scheme over \( B_0 \) acting on both \( P_0 \) and \( X_0 \). Take the \( K_0 \)-quotient of both spaces; on the one hand we recover the dual abelian scheme \( \hat{P}_0 \cong P_0/K_0 \), and on the other hand we obtain a new space

\[
\tilde{\pi}_0 : \tilde{X}_0 \to B_0 \quad \text{for} \quad \tilde{X}_0 = X_0/K_0.
\]

By construction, \( \tilde{X}_0 \) is a \( \hat{P}_0 \)-torsor, a smooth proper family of complex tori which are fiberwise dual to the original fibration \( \pi_0 \). This \( \tilde{\pi}_0 \) is our definition of the dual torus fibration. We will later see that if \( \pi_0 \) admits at least one section, then this \( \tilde{\pi}_0 \) becomes isomorphic to the relative Picard scheme of \( \pi_0 \).

It is important to notice that the group scheme \( K_0 \) is only a finite étale group scheme over \( B_0 \). One can think of this as the total space of a local system on \( B_0 \); there is a monodromy issue hiding on the background, and \textit{a priori} \( K_0 \) may not be a constant group scheme. We are now ready to state the main result of this paper.

**Theorem 1.1.** Let \( \pi : X \to B \) be a Lagrangian fibration of a compact hyper-Kähler manifold. Assume \( X \) is of \( K3^{[n]} \), \( \mathrm{Kum}_n \), \( \text{OG10} \) or \( \text{OG6} \)-type. Then

1. The kernel group scheme \( K_0 \to B_0 \) extends to a constant group scheme \( K \to B \) that acts on the entire Lagrangian fibration \( \pi : X \to B \). Moreover, \( K \) is a subgroup\(^1\) of the group \( \text{Aut}^c(X/B) = \{ f \in \text{Aut}(X) : \pi \circ f = \pi, \ f^* \text{ acts as the identity on } H^2(X, \mathbb{Z}) \} \).

2. The quotient

\[
\tilde{\pi} : \tilde{X} \to B \quad \text{for} \quad \tilde{X} = X/K
\]

compactifies the dual torus fibration \( \tilde{\pi}_0 \).

3. \( \tilde{X} \) is a compact hyper-Kähler orbifold and \( \tilde{\pi} \) is its Lagrangian fibration. Moreover, \( \tilde{X} \) has the same period mapping/deformation behavior as \( X \).

If \( X \) is of \( K3^{[n]} \) or \( \text{OG10} \)-type, then the group \( K \) (or the constant group scheme \( K \to B \)) is in fact trivial and these hyper-Kähler manifolds are self-dual. On the other hand, if \( X \) is of \( \text{Kum}_n \) or \( \text{OG6} \)-type, then \( K \) is nontrivial and \( \tilde{X} \) is not even homeomorphic to \( X \). We will provide explicit computations for the group \( K \) in Theorem 5.1 and Remark 6.2. Note also that \( \tilde{X} \) is a global quotient of \( X \) by automorphisms acting trivially on \( H^2(X, \mathbb{Z}) \). As a result, the second rational cohomology

\(^1\)We will frequently view a finite constant group scheme \( K \to B \) as a finite group, and vice versa. We will denote them by the same letter \( K \) if no confusions arise.
of \( \hat{X} \) and \( X \) are isometric as Beauville–Bogomolov quadratic spaces. The higher cohomology of \( \hat{X} \) may be strictly smaller than that of \( X \) by [Ogn20], but they are still tightly connected via their Looijenga–Lunts–Verbitsky (LLV) structures (see [LL97], [Ver95] and [GKLR22]). Finally, the singularities of \( \hat{X} \) are quotient singularities of high codimensions (\( \geq 4 \)), so they do not admit any symplectic resolutions. We briefly recall for reader’s convenience the notion of a singular hyper-Kähler variety and its Lagrangian fibration in Appendix A.

Remark 1.2. There were several previous results on the constructions of dual Lagrangian fibrations of compact hyper-Kähler manifolds. Especially, [Saw20, Thm 24] announced the construction of a dual Lagrangian fibration of certain \( \text{Kum}_n \)-type hyper-Kähler manifolds (without a proof). Although Sawon’s method is different from ours, it is isomorphic to our construction when \( \pi \) admits a section and the polarization type is \((1, \cdots , 1, n + 1)\). This can be shown by using the results in Section 5. [Saw04] and [Nag05] discussed a possible hyper-Kähler structure on a partial compactification of the relative Picard scheme of \( \pi_0 \). These are different to our direction because our dual torus fibration \( \pi_0 : \hat{X}_0 \to B_0 \) is not isomorphic to the relative Picard scheme when \( \pi_0 \) does not have any section. [MT07] and [Men14] introduced an explicit geometric construction of certain 4-dimensional Lagrangian fibered hyper-Kähler orbifolds, and realized their dual Lagrangian fibrations using the same construction. It would be interesting to find a connection between their results and our perspective. Finally, [Ver99] discussed certain self-dualities of hyper-Kähler manifolds at the level of cohomology.

There are two key ingredients for our proof of Theorem 1.1: the group \( \text{Aut}^\circ(X/B) \) and the notion of a polarization type. The definition of the group \( \text{Aut}^\circ(X/B) \) is inspired by the similar group \( \text{Aut}^\circ(X) \), which has already played an important role in the theory of hyper-Kähler manifolds. The two main properties of \( \text{Aut}^\circ(X) \) are its finiteness [Huy99] and deformation invariance [HT13]. The group \( \text{Aut}^\circ(X) \) is also computed for all known deformation types of hyper-Kähler manifolds (see [Bea83a], [BNWS11] and [MW17]). We provide similar results for the group \( \text{Aut}^\circ(X/B) \): it is finite abelian (Proposition 3.28) and deformation invariant (Theorem 2.3). We also compute \( \text{Aut}^\circ(X/B) \) for all known deformation types in Theorem 5.1. The idea of considering the polarization type of the fibers of \( \pi_0 \) has long been used, but only recently comprehensively studied by [Wie16, Wie18]. We relate the polarization type to the study of our group scheme \( K_0 \).

1.1. Structure of the paper. In Section 2, we prove the group \( \text{Aut}^\circ(X/B) \) is deformation invariant on the Lagrangian fibration \( \pi \). This is inspired by Hassett–Tschinkel’s proof of deformation invariance of \( \text{Aut}^\circ(X) \) in [HT13, Thm 2.1]. In Section 3, we start by attaching an abelian scheme \( P_0 \) to any Lagrangian fibration of a hyper-Kähler manifold: \( P_0 \) is the identity component of the relative automorphism scheme of \( \pi \). There exists a unique primitive polarization \( \lambda \) on \( P_0 \) so that we can define its kernel group scheme \( K_0 \). We then try to relate \( K_0 \) and \( \text{Aut}^\circ(X/B) \) in general. This section also discusses the notion of the polarization type of a Lagrangian fibration. In essence, the polarization type is the study of a single fiber of the group scheme \( K_0 \).

The goal of Section 5 is twofold. First, we compute the group \( \text{Aut}^\circ(X/B) \) for all currently known deformation types of hyper-Kähler manifolds. Second, we prove an inclusion \( K_0 \subset \text{Aut}^\circ(X/B) \) for special constructions of \( \text{Kum}_n \)-type hyper-Kähler manifolds. The material here will be mostly concrete computations. Section 4 introduces a slightly more systematic method to assist this computations. In Section 6, we prove the main result of this article: there exists a natural compactification of the dual torus fibration for all currently known deformation types of hyper-Kähler manifolds. In Section 7, we give an illustration of the geometry and cohomology of \( \hat{X} \) when \( X \) is of \( \text{Kum}_2 \)-type.
We provide two appendices. Appendix A contains various definitions of singular hyper-Kähler varieties appearing in the literature. In Appendix B, we discuss certain special quotients of compact hyper-Kähler manifolds. The quotient $\tilde{X} = X/K$ will be a special instance of this more general set-up.

1.2. Notation and conventions. In this paper, every hyper-Kähler manifold $X$ will be assumed to be compact but not necessarily projective unless stated explicitly. When $X$ further admits a Lagrangian fibration $\pi : X \to B$, it is helpful to keep in mind that $X$ is projective if and only if $\pi$ admits at least one rational multisection. Indeed, if $X$ is projective then a general scheme-theoretic fact says any smooth morphism between algebraic varieties admits an étale local section. The converse is [Saw99, Lem 2].

Assume $X$ has dimension $2n$. Any Lagrangian fibration $\pi : X \to B$ in this paper will always have the base $B = \mathbb{P}^n$ since we are assuming $B$ is smooth and $0 < \dim B < 2n$ (see [Hwa08] and [GL14]). The Beauville–Bogomolov form and the Fujiki constant of $X$ are a unique primitive symmetric bilinear form $q : H^2(X, \mathbb{Z}) \otimes H^2(X, \mathbb{Z}) \to \mathbb{Z}$ and a positive rational number $c_X$ satisfying the Fujiki relation

$$\int_X x^{2n} = c_X \cdot \frac{(2n)!}{2^n \cdot n!} \cdot q(x)^n \quad \text{for} \quad x \in H^2(X, \mathbb{Z}).$$

The Fujiki constant is computed for all currently known deformation types of hyper-Kähler manifolds: (1) $c_X = 1$ for $K3^{[n]}$ or OG10-type, and (2) $c_X = n + 1$ for $\text{Kum}_n$ or OG6-type (see [Bea83b] and [Rap07, Rap08]). In practice, we will mostly need a stronger version of the Fujiki relation, which follows from the polarization process

$$\int_X x_1 \cdots x_{2n} = c_X \sum_\sigma q(x_{\sigma(1)}, x_{\sigma(2)}) \cdots q(x_{\sigma(2n-1)}, x_{\sigma(2n)}) \quad \text{for} \quad x_i \in H^2(X, \mathbb{Z}).$$

Here $\sigma \in \mathfrak{S}_{2n}$ runs through all the $2n$-permutations but up to $2^n \cdot n!$ ambiguities inducing the same expression in the summation. The divisibility of $x \in H^2(X, \mathbb{Z})$ is defined to be a positive integer

$$\text{div}(x) = \gcd\{q(x, y) : y \in H^2(X, \mathbb{Z})\}. \quad (1.4)$$

The study of the full cohomology $H^*(X, \mathbb{Q})$ will need the notion of the LLV algebra $\mathfrak{g}$, introduced by Looijenga–Lunts [LL97] and Verbitsky [Ver95]. For its concrete computations we will follow the representation theoretic notation used in [GKLR22, §2–3].

Throughout, group schemes will be used both in algebraic and analytic context. A group scheme is a morphism $G \to S$ equipped with an identity section $S \to G$, a group law morphism $G \times_S G \to G$ and an inverse $G \to G$ satisfying the usual axioms (either in the algebraic or analytic setting). An abelian scheme $P \to S$ is an analytically proper connected commutative group scheme over $S$ with complex torus fibers. Any abelian scheme $P$ admits a dual abelian scheme $\hat{P}$. A polarization of an abelian scheme $P$ is a finite étale homomorphism $\lambda : P \to \hat{P}$ over $S$ such that for each fiber $F$, the restriction $\lambda|_F : F \to \hat{F}$ is of the form $x \mapsto [t^\alpha L \otimes L^{-1}]$ for an ample line bundle $L$ on $F$. Given a group scheme $G \to S$, an analytic torsor under $G$ (or analytic $G$-torsor) is a morphism $Y \to S$ equipped with a $G$-action, such that there exists an analytic covering $\tilde{S} = \bigsqcup_\alpha U_\alpha \to S$ where the base change $\tilde{Y} = Y \times_S \tilde{S}$ and $\tilde{G} = G \times_S \tilde{S}$ are $G$-equivariantly isomorphic over $\tilde{S}$. In the algebraic setting, one can use a different topology, e.g., étale topology to define an étale torsor. Our reference for the theory of abelian schemes is [MFK94], [BLR90] and [FC90]. For the notion of torsors, see [Mil80] or [BLR90].
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2. Deformation invariance of the $H^2$-trivial automorphisms

Let $X$ be a compact hyper-Kähler manifold. Consider the group of $H^2$-trivial automorphisms

$$\text{Aut}^\circ(X) = \ker \text{( Aut}(X) \to \text{O}(H^2(X,\mathbb{Z}), q), \quad f \mapsto f_*)$$

Here $\text{Aut}(X)$ is the group of biholomorphic automorphisms of $X$. Huybrechts \cite[Prop 9.1]{Huy99} together with Hassett–Tschinkel \cite[Thm 2.1]{HT13} proved that $\text{Aut}^\circ(X)$ is a finite group which is invariant under deformations of $X$.

Let us now further assume $X$ admits a Lagrangian fibration $\pi : X \to B$. We can restrict our attention to $H^2$-trivial automorphisms that respect the Lagrangian fibration

$$\text{Aut}^\circ(X/B) = \text{Aut}(X/B) \cap \text{Aut}^\circ(X). \quad (2.1)$$

Since $\text{Aut}^\circ(X)$ is finite, so is $\text{Aut}^\circ(X/B)$. In fact, we can further prove $\text{Aut}^\circ(X/B)$ is abelian: this will be showed later in Proposition 3.28. Notice that $\text{Aut}^\circ(X/B)$ not only depends on $X$ but also on the Lagrangian fibration $\pi : X \to B$. Hence, if $X$ admits two different Lagrangian fibrations then they may have different $\text{Aut}^\circ(X/B)$. In Section 5, we will compute $\text{Aut}^\circ(X/B)$ for all currently known deformation types of hyper-Kähler manifolds $X$. In Section 3, we will reinterpret $\text{Aut}^\circ(X/B)$ as global sections of the “translation automorphism scheme” $P_0 \to B_0$.

But before doing so, here we establish a more basic fact in this section; we prove $\text{Aut}^\circ(X/B)$ is deformation invariant on $\pi$. To make this more precise, we first need to define the notion of a family of Lagrangian fibered compact hyper-Kähler manifolds.

**Definition 2.2.** A family of Lagrangian fibered compact hyper-Kähler manifolds is a commutative diagram

$$\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\pi} & B \\
\downarrow p & & \downarrow q \\
S & \xleftarrow{q} & \end{array}$$

with the following conditions.

1. $p : \mathcal{X} \to S$ is a smooth proper family of compact hyper-Kähler manifolds of dimension $2n$ over a complex analytic space $S$.
2. $q : B \to S$ is the projectivization of a rank $n+1$ holomorphic vector bundle on $S$.
3. For all $t \in S$, the fiber $\pi : X_t \to B_t$ is a Lagrangian fibration.

Note that the second condition ensures $B$ is projective over $S$ and admits a relative ample line bundle $\mathcal{O}_{B/S}(1)$. It is also possible to consider a weaker version of this definition which only assumes $B \to S$ to be a $\mathbb{P}^n$-bundle. The obstruction for a $\mathbb{P}^n$-bundle to be the projectivization of a vector bundle lies in the analytic Brauer group $H^2(S, \mathcal{O}_S^\times)$. Thus, if $H^2(S, \mathcal{O}_S^\times) = 0$ (for example, when $S$
is a complex open ball) then the second axiom is in fact equivalent to the weaker one. Notice that the pullback $\mathcal{H} = \pi^*\mathcal{O}_{B/S}(1)$ can be considered as a family of line bundles $H_t$ on $X_t$. Therefore, Definition 2.2 induces a family of pairs $(X, H)$ where $H = \pi^*\mathcal{O}_{B}(1)$.

As usual, two Lagrangian fibrations $\pi : X \to B$ and $\pi' : X' \to B'$ are deformation equivalent if there exists a family of Lagrangian fibered compact hyper-Kähler manifolds $X/B/S$ over a connected union of 1-dimensional open disks $S$, realizing them as two fibers at $t, t' \in S$. Matsushita in [Mat16] proved such a deformation problem admits a local universal deformation.

We can now state the main theorem of this section.

**Theorem 2.3.** The group $\text{Aut}^G(X/B)$ is invariant under deformations of $\pi : X \to B$.

The rest of this section will be devoted to the proof of Theorem 2.3. The sketch of the proof is as follows. First, we descend the $\text{Aut}^G(X)$-action on $X$ to $B$ so that the Lagrangian fibration $\pi : X \to B$ becomes an equivariant morphism. This means we have a group homomorphism $\text{Aut}^G(X) \to \text{Aut}(B)$ whose kernel is precisely $\text{Aut}^G(X/B)$. Descending such an action is a nontrivial problem (this is quite similar to the result of [Bri11]), so we need to overcome this issue using the notion of a $G$-linearizability of line bundles. Next, we need to sheafify the discussions as we are interested in the deformation behavior of them. The result will follow from formal properties of the kernel of the sheaf homomorphism.

### 2.1. $G$-linearizability of a line bundle

Before we get into the proof of Theorem 2.3, let us recall the notion of $G$-linearizability of a line bundle on a complex manifold. For simplicity we only consider finite group actions. Our references are [Bri18, §3], [Dol03, §7] and [MFK94], but we need to take some additional care since these references only consider the algebraic setting.

Let $G$ be an arbitrary finite group and $\mathcal{X}$ be a complex manifold equipped with a holomorphic $G$-action. A $G$-linearized line bundle on $\mathcal{X}$ is a holomorphic line bundle $\mathcal{L}$ together with a collection of isomorphisms $\Phi_g : g^*\mathcal{L} \to \mathcal{L}$ for $g \in G$, satisfying the condition $\Phi_{gg'} = \Phi_{g'} \circ g^*\Phi_g$ for $g, g' \in G$. A $G$-invariant line bundle on $\mathcal{X}$ is a holomorphic line bundle $\mathcal{L}$ such that $g^*\mathcal{L} \cong \mathcal{L}$ for all $g \in G$ (without any condition). We denote by $\text{Pic}^G(\mathcal{X})$ and $\text{Pic}(\mathcal{X})^G$ the groups of $G$-linearized line bundles and $G$-invariant line bundles on $\mathcal{X}$ up to isomorphisms. The second group is precisely the $G$-invariant subgroup of $\text{Pic}(\mathcal{X})$.

There is a forgetful homomorphism $\text{Pic}^G(\mathcal{X}) \to \text{Pic}(\mathcal{X})^G$, which is neither injective nor surjective in general. To understand the obstruction to its surjectivity, one considers an exact sequence of abelian groups ([Dol03, Rmk 7.2] or [Bri18, Prop 3.4.5])

$$\text{Pic}^G(\mathcal{X}) \to \text{Pic}(\mathcal{X})^G \to H^2(G, \Gamma), \quad \Gamma = H^0(\mathcal{X}, \mathcal{O}_\mathcal{X}).$$

Both Dolgachev and Brion’s discussions are for algebraic varieties, but their proofs can be adapted to our analytic setting as well. With this exact sequence in hand, we have:

**Lemma 2.4.** Every $G$-invariant line bundle $\mathcal{H}$ on $\mathcal{X}$ is $G$-linearizable up to a suitable tensor power.

**Proof.** It is a general fact in the theory of group cohomology (for finite groups) that all the higher degree cohomologies $H^{2k}(G, \Gamma)$ are $|G|$-torsion for any $G$-module $\Gamma$ (e.g., [Ser79, Cor VIII.1]). Hence by the exact sequence above, the $|G|$-th tensor $\mathcal{H}^{|G|}$ vanishes in $H^2(G, \Gamma)$ and hence comes from $\text{Pic}^G(\mathcal{X})$. \(\square\)

For us, the importance of the $G$-linearizability of a line bundle comes from the induced $G$-action on the higher direct images of a linearized line bundle. If $\mathcal{L}$ is a $G$-linearized line bundle on $\mathcal{X}$ and
Lemma 2.8. \( \Aut_{G} \) acting on \( X \rightarrow S \) is a \( G \)-invariant holomorphic map, then we have a contravariant \( G \)-action on all the higher direct image sheaves

\[
g^* : R^k p_* L \rightarrow R^k p_* L, \quad (g \circ g')^* = g'^* \circ g^*. \]

Now assume further \( L \) is globally generated over \( S \) and \( p_* L \) is a vector bundle on \( S \). Then we have a \( G \)-action on \( \mathbb{P}_S(p_* L) \) making the holomorphic map \( X \rightarrow \mathbb{P}_S(p_* L) \) \( G \)-equivariant over \( S \). See [MFK94, Prop 1.7].

2.2. The automorphism sheaves and deformation invariance of the \( H^2 \)-trivial automorphisms. Suppose we have a smooth proper family of hyper-Kähler manifolds \( p : X \rightarrow B \rightarrow S \). Then we can consider it as a family of groups \( \Aut^e(X_t) \) for \( t \in S \). Similarly, given a family of Lagrangian fibered hyper-Kähler manifolds, we can define a family of groups \( \Aut^e(X_t/B_t) \):

**Definition 2.5.** Given a family of Lagrangian fibered hyper-Kähler manifolds \( p : X \rightarrow B \rightarrow S \), we define a sheaf of groups \( \Aut^e_{X/B/S} \) on \( S \) by

\[
\Aut^e_{X/B/S}(U) = \{ f : X_U \rightarrow X_U : U \text{-isomorphism such that } f^* : R^2 p_* L \rightarrow R^2 p_* L \text{ is the identity} \}. \]

By the work of Huybrechts and Hassett–Tschinkel, this sheaf is a local system of finite groups. We can consider it as a family of groups \( \Aut^e(X_t) \) for \( t \in S \). Similarly, given a family of Lagrangian fibered hyper-Kähler manifolds, we can define a family of groups \( \Aut^e(X_t/B_t) \):

For \( \Aut^e_{X/B/S} \) is a subsheaf of \( \Aut^e_{X/S} \), and our goal is to prove it is locally constant as well. The question is certainly local on the base \( S \), so we may assume \( S \) is a small open ball. Then \( \Aut^e_{X/S} \) becomes a constant sheaf, so we may consider it as an abstract finite group

\[
G = \Aut^e(X) \]

acting on \( X \rightarrow S \) fiberwise.

Consider the automorphism sheaf \( \Aut_{B/S} \) of the \( \mathbb{P}^n \)-bundle \( B \rightarrow S \). It is the sheaf of analytic local sections of the \( \text{PGL}(n+1, \mathbb{C}) \)-group scheme \( \Aut_{B/S} \rightarrow S \). Our first step is to realize the sheaf \( \Aut^e_{X/B/S} \) as the kernel of a certain homomorphism \( \Aut^e_{X/S} \rightarrow \Aut_{B/S} \).

**Proposition 2.6.** Assume \( S \) is an open ball. Then there exists a homomorphism of sheaves

\[
G = \Aut^e_{X/S} \rightarrow \Aut_{B/S} \tag{2.7} \]

whose kernel is \( \Aut^e_{X/B/S} \).

Equivalently, the proposition states that there exists a \( G \)-action on \( B \) making \( \pi : X \rightarrow B \) a \( G \)-equivariant morphism over \( S \). To prove the proposition, we need to use the \( G \)-linearizability of line bundles in the previous subsection. The following lemma proves every line bundle on \( X \) is \( G \)-invariant.

**Lemma 2.8.** \( G \) acts trivially on \( \text{Pic}(X) \).
Proof. We first claim \( G \) acts trivially on \( H^2(\mathcal{X}, \mathbb{Z}) \). Apply the Leray spectral sequence
\[
E_2^{p,q} = H^p(S, R^q p_* \mathbb{Z}) \Rightarrow H^{p+q}(\mathcal{X}, \mathbb{Z}).
\]
Noticing that \( R^0 p_* \mathbb{Z} = \mathbb{Z}, R^1 p_* \mathbb{Z} = 0 \), and \( S \) is an open ball, we obtain an isomorphism \( H^2(\mathcal{X}, \mathbb{Z}) \cong H^0(S, R^2 p_* \mathbb{Z}) \). This isomorphism respects the \( G \)-action as the Leray spectral sequence is functorial. Now \( G \) acts on \( H^2(X_t, \mathbb{Z}) \) trivially for any fiber \( X_t \), so \( G \) acts on \( R^2 p_* \mathbb{Z} \) trivially and the claim follows.

It is enough to prove the first Chern class map \( \text{Pic}(\mathcal{X}) \to H^2(\mathcal{X}, \mathbb{Z}) \) is injective. This homomorphism is induced by the exponential sequence \( 0 \to \mathbb{Z} \to O_X \to O_X^* \to 0 \), so it suffices to prove \( H^1(\mathcal{X}, O_X) = 0 \). Again, use the Leray spectral sequence
\[
E_2^{p,q} = H^p(S, R^q p_* O_X) \Rightarrow H^{p+q}(\mathcal{X}, O_X).
\]
This time, we have \( R^0 p_* O_X = O_S \) and \( R^1 p_* O_X = 0 \). This implies \( H^1(\mathcal{X}, O_X) = 0 \).

Consider the line bundle \( \mathcal{H} = \pi^* O_{B/S}(1) \) on \( \mathcal{X} \). Since Pic(\( \mathcal{X} \)) is \( G \)-invariant, we can apply Lemma 2.10 to \( \mathcal{H} \) and conclude \( \mathcal{H}^{\otimes m} \) is \( G \)-linearizable for some positive integer \( m \). As a result, we have a \( G \)-equivariant morphism \( \pi_m : \mathcal{X} \to B_m \) where \( B_m = \mathbb{P}_S(p_* \mathcal{H}^{\otimes m}) \) is the dual of the complete linear system associated to \( \mathcal{H}^{\otimes m} \). Consider the diagram
\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\pi} & B_m \\
p \downarrow & & \downarrow q \\
S & \xrightarrow{\eta_m} & B_m
\end{array}
\]

**Lemma 2.10.** The \( m \)-th relative Veronese embedding \( B \to B_m \) makes the diagram (2.9) commute.

Proof. Notice that the morphism \( \pi : \mathcal{X} \to B \) associated to \( \mathcal{H} \) has connected fibers. Hence, for any \( t \in S \), the fiber \( \pi : X_t \to B_t \) becomes the Iitaka fibration of the line bundle \( H_t \) (e.g., [Laz04, §2.1.B]). This in particular implies that any morphism \( \pi_m : X_t \to (B_m)_t \) associated to \( H_t^{\otimes m} \) factors through the Iitaka fibration \( \pi \), where the morphism \( B_t \to (B_m)_t \) precisely \( m \)-th Veronese embedding. In other words, the \( m \)-th relative Veronese embedding makes the diagram commute.

The equivariance of \( \pi_m \), with the diagram (2.9) implies \( \pi \) is equivariant, completing the proof of Proposition 2.6. We now present the proof of the main theorem.

**Proof of Theorem 2.3.** Let \( \mathcal{X} \to B \to S \) be a family of Lagrangian fibered hyper-Kähler manifolds over an open ball \( S \). The sheaves \( \text{Aut}^\circ_{\mathcal{X}/S} \) and \( \text{Aut}_{B/S} \) are represented by the constant group schemes
\[
\text{Aut}^\circ_{\mathcal{X}/S} \cong \bigcup_{f \in \text{Aut}^\circ(\mathcal{X})} S, \quad \text{Aut}_{B/S} \cong \text{PGL}(n+1) \times S.
\]
The homomorphism (2.7) becomes a morphism \( \alpha : \text{Aut}^\circ_{\mathcal{X}/S} \to \text{Aut}_{B/S} \). The desired sheaf \( \text{Aut}^\circ_{\mathcal{X}/B/S} \) is representable by \( \ker \alpha \) by Proposition 2.6.

To prove \( \ker \alpha \) is a constant subgroup scheme, it is enough to show the following: let \( S' \) be a connected component of \( \text{Aut}^\circ_{\mathcal{X}/S} \). Consider the restriction of \( \alpha \) followed by the projection
\[
\beta : S' \to \text{PGL}(n+1).
\]
Then we claim that either \( \beta(S') = \{ \text{id} \} \) or \( \beta(S') \not= \text{id} \). Notice that the image \( \beta(S') \) consists of \( |G| \)-torsion matrices in \( \text{PGL}(n+1) \). Since the set of \( |G| \)-torsion matrices is a disjoint union of \( \text{PGL}(n+1) \)-adjoint orbits (classified by eigenvalues), the connected set \( \beta(S') \) has to lie in a single
orbit. The adjoint orbit containing the identity matrix is a singleton set \{\text{id}\}. Hence the claim follows. □

3. ABELIAN SCHEMES ASSOCIATED TO LAGRANGIAN FIBRATIONS

The aim of this section is to associate a polarized abelian scheme to every Lagrangian fibered compact hyper-Kähler manifold, and to discuss its consequences. The following is the first main theorem of this section.

**Theorem 3.1.** Let \( \pi : X \to B \) be a Lagrangian fibration of a compact hyper-Kähler manifold and \( B_0 \subset B \) its smooth locus. Set \( X_0 = \pi^{-1}(B_0) \) so that it becomes a smooth proper family of complex tori over \( B_0 \).

1. There exists a unique projective abelian scheme \( \nu : P_0 \to B_0 \) making \( \pi : X_0 \to B_0 \) an analytic torsor under \( \nu \).

2. Moreover, the abelian scheme is simple and has a unique choice of a primitive polarization \( \lambda : P_0 \to \tilde{P}_0 \).

(3.2)

Here \( \tilde{P}_0 \to B_0 \) is the dual abelian scheme of \( P_0 \to B_0 \).

**Definition 3.3.** The abelian scheme \( \nu : P_0 \to B_0 \) in Theorem 3.1 is called the abelian scheme associated to \( \pi \).

Our statement is motivated by Arinkin–Fedorov’s result in [AF16, Thm 2], van Geemen–Voisin’s argument in [vGV16], and Sawon’s result in [Saw04]. The theorem combines and slightly generalizes these results. Before discussing the applications of this theorem, let us first present some examples.

**Example 3.4.** Let \( X \) be a smooth projective moduli of torsion coherent sheaves on a K3 surface with a fixed Mukai vector, so that it becomes a hyper-Kähler manifold of K3\(^{[n]}\)-type equipped with a Lagrangian fibration \( \pi : X \to B \) (see, e.g., [dCRS21]). In this case, it is known that the torus fibration \( \pi : X_0 \to B_0 \) is isomorphic to a relative Jacobian Pic\(_dC/B_0\) associated to a certain universal family \( C/B_0 \) of smooth curves on the K3 surface. Now Pic\(_dC/B_0\) is a torsor under the numerically trivial relative Jacobian Pic\(_0C/B_0\) [BLR90, Thm 9.3.1]. By the uniqueness assertion of Theorem 3.1, this is the associated abelian scheme \( P_0 \).

**Example 3.5.** When \( \pi : X \to B = \mathbb{P}^1 \) is an elliptic K3 surface, Theorem 3.1 is a weaker version of the relative Jacobian fibration construction of \( \pi \) (e.g., [Huy16, §11.4]). In this case, one may even construct a semi-abelian scheme \( P \to B \) over the entire base (Néron model) so that the smooth locus of \( \pi \) becomes a torsor under \( P \). Arinkin–Fedorov generalized this result to certain higher dimensional projective hyper-Kähler manifolds. A stronger version of Theorem 3.1 would potentially improve the arguments in this paper, but we will not discuss this further.

**Example 3.6.** For higher dimensional compact hyper-Kähler manifolds, one may still consider the relative Picard scheme Pic\(_0X_0/B_0\) → \( B_0 \). However, in general this is the dual of the abelian scheme \( P_0 \). This means we can consider \( P_0 \) as the “double Picard scheme” of the original \( X_0 \) [Saw04]. However, for us it will be more useful to consider \( P_0 \) as the identity component of the relative automorphism scheme of \( X_0/B_0 \). We will show this in Proposition 3.19.

**Example 3.7.** When \( \pi \) admits at least one rational section, then the abelian scheme \( P_0 \) is in fact isomorphic to \( X_0 \). This is because the rational section must be defined over \( B_0 \) by Remark 3.26, so that \( X_0 \) becomes a trivial \( P_0 \)-torsor. In some sense, Theorem 3.1 is thus a generalization of certain
properties of $X_0$ to the case where $\pi$ does not have any rational section. For example, one can study the Mordell–Weil group of $\nu$, generalizing the study of the Mordell–Weil group of $\pi$.

One application of Theorem 3.1 is a more systematic study of the polarization type of the torus fibers arising in $\pi$. The study of the polarization type of the torus fibers goes back to at least [Saw03], which in turn references an earlier idea of Mukai (see Proposition 5.3 in loc. cit.). However, to our knowledge, Wieneck’s series of papers [Wie16, Wie18] were the first work to consider the polarization type as an invariant attached to a Lagrangian fibration and study them in great details for K3$^{[a]}$ and Kum$_n$-type hyper-Kähler manifolds. Using Theorem 3.1, we can given an alternative definition of the polarization type.

**Definition 3.8.**  
(1) The **polarization scheme** of $\pi$ is the kernel $$K_0 = \ker \lambda$$ of the polarization (3.2).

(2) The **polarization type** of $\pi$ is an $n$-tuple of positive integers $(d_1, \cdots, d_n)$ with $d_1 | \cdots | d_n$ such that the fibers of the polarization scheme are isomorphic to $(\mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_n)^{\oplus 2}$.

The polarization scheme $K_0$ is a finite étale commutative group scheme over $B_0$. Hence its fibers are all isomorphic and the polarization type is well-defined. The polarization type will be an important ingredient for our method. We devote a short subsection 3.2 to collect its properties.

The second theme of this section is a relation between the group $\text{Aut}^\circ(X/B)$ and the polarization scheme $K_0$. We will see in Proposition 3.27 that every automorphism $f \in \text{Aut}^\circ(X/B)$ defines a global section of the abelian scheme $P_0 \to B_0$. If we consider $\text{Aut}^\circ(X/B)$ as a constant group scheme over $B_0$, this means we have a closed immersion of group schemes

$$\text{Aut}^\circ(X/B) \hookrightarrow P_0.$$  \hspace{1cm} (3.9)

We expect the image of this injective map will contain the polarization scheme $K_0$. This is a nontrivial claim; this would imply the polarization scheme is a constant group scheme and is extendable to $K \to B$ acting on the entire $X \to B$. We were not able to prove this claim in general, and a large part of this paper will be devoted to showing this for known deformation types of hyper-Kähler manifolds. The following propositions will be our technical tools for doing this. It will be convenient to introduce a temporary notation

$$K_0[a] = \ker(a \lambda : P_0 \to \tilde{P}_0),$$

a finite étale commutative group scheme over $B_0$.

**Proposition 3.10.** Let $\pi : X \to B$ and $\pi' : X' \to B'$ be two deformation equivalent Lagrangian fibrations of compact hyper-Kähler manifolds. Let $a$ be any positive integer. Then the inclusion (3.9) factors through

$$\text{Aut}^\circ(X/B) \hookrightarrow K_0[a]$$

if and only if the same holds for $\pi'$.

**Proposition 3.12.** Let $\pi : X \to B$ be a Lagrangian fibration of a compact hyper-Kähler manifold and $(d_1, \cdots, d_n)$ its polarization type. Assume we have an equality $c_X = d_1 \cdots d_n$. Then (3.11) holds for $a = \text{div}(h)$, where $h \in H^2(X, \mathbb{Z})$ is the class of $\pi^*\mathcal{O}_B(1)$ and its divisibility $\text{div}(h)$ is as defined in (1.4).
Note that the inclusion (3.11) has a different direction from our desired \( K_0 \hookrightarrow \text{Aut}^\circ(X/B) \). Our strategy will be to first show (3.11) for a certain value of \( a \), and then deduce the relation between two subgroup schemes \( K_0, \text{Aut}^\circ(X/B) \subset K_0[a] \). The first proposition says the inclusion (3.11) is deformation invariant on \( \pi \). The second proposition provides at least one such an integer \( a \), though this may not be the minimum possible value. The unfortunate assumption \( e_X = d_1 \cdots d_n \) will be satisfied for all known deformation types of hyper-Kähler manifolds, so it will not be a huge problem. See Theorem 3.22.

Finally, we would like to mention a special consequence of the above discussion when \( \pi \) admits a rational section. This may give the readers some more ideas on these objects. Recall from Example 3.7 that the existence of a rational section implies \( X_0 \cong P_0 \). Therefore, (3.9) implies the existence of certain torsion rational sections of the Lagrangian fibration
\[
\text{Aut}^\circ(X/B) \subset \text{MW}(X/B).
\]
For example, let us consider the case when \( X \) is of Kum\(_n\)-type. We will prove in Theorem 5.1 that the order of \( \text{Aut}^\circ(X/B) \) is at least \((n+1)^2\). Thus any Lagrangian fibration of a Kum\(_n\)-type hyper-Kähler manifold \emph{must} have at least \((n+1)^2\) torsion rational sections (once it admits a single torsion rational section), and the dual hyper-Kähler orbifold \( X \) is precisely the quotient of \( X \) by these special torsion rational sections. To our knowledge, this phenomenon has not been observed before and it became one of our original motivations. See also [Sac20, §3–5] for some related ideas on the Mordell–Weil group and birational automorphisms defined by torsion rational sections.

### 3.1. Abelian scheme associated to a Lagrangian fibration

In this subsection, we present the proof of Theorem 3.1. Note again that we are assuming neither \( X \) is projective nor \( \pi \) has a rational section.

Recall that every smooth closed fiber \( F \) of \( \pi \) is a complex torus (holomorphic Arnold–Liouville theorem). In fact, \( F \) is necessarily an abelian variety as observed by Voisin [Cam06, Prop 2.1]. It would be helpful for us to review this fact. The key idea is the following cohomological lemma, which has been discovered several times independently in [Voisin92, Ogu99, Mat16] and recently generalized into higher degree cohomologies by Shen–Yin and Voisin [SY22].

**Lemma 3.13.** Let \( F \) be any smooth fiber of \( \pi \) and \( h \in H^2(X, \mathbb{Z}) \) the cohomology class of \( \pi^*\mathcal{O}_B(1) \). Then the restriction map
\[
-|_F : H^2(X, \mathbb{Z}) \to H^2(F, \mathbb{Z})
\]
has \( \ker(-|_F) = h^\perp \). Consequently, it has \( \text{im}(-|_F) \cong \mathbb{Z} \).

**Corollary 3.14** (Voisin). The image of the restriction map \(-|_F\) is generated by an ample class of \( F \). As a result, \( F \) is an abelian variety.

**Proof.** Say \( y \) is an integral generator of Lemma 3.13. Choose any Kähler class \( \omega \in H^2(X, \mathbb{R}) \) and consider its restriction \( \omega|_F \), a Kähler class on \( F \). It has to be a nonzero real multiple of \( y \). This means, up to sign, \( y \) has to be a Kähler class on \( F \). Hence \( y \) is an integral Kähler class, so it is ample. \( \square \)

We caution the reader to be aware that the ample generator \( y \) of the image of the restriction map need not be primitive (see Proposition 3.23). One reasonable choice of a polarization on an abelian variety fiber \( F \) is a unique \emph{primitive} ample class in \( H^2(F, \mathbb{Z}) \) parallel to \( y \). Theorem 3.1 is essentially a more global way to formulate this over the whole base \( B_0 \).

We divide the proof of Theorem 3.1 into three parts: (1) an explicit construction of the polarized abelian scheme \( P_0 \), (2) proving such a construction makes \( X_0 \) a torsor under \( P_0 \), and finally (3) its
uniqueness. The uniqueness should be a more general fact about arbitrary torsors, at least in the algebraic case (see Moret-Bailly’s answer in [MB]). The construction of $P_0$ works for any proper family of complex tori. The uniqueness of the polarization is the only part that needs the fact $X_0$ is obtained from a Lagrangian fibered hyper-Kähler manifold $X$.

The proof of the construction part closely follows [vGV16], but for completeness we reproduce their argument here.

**Proof of Theorem 3.1, construction.** Apply the global invariant cycle theorem (for proper maps between compact Kähler manifolds [Del71]) and Lemma 3.13 to obtain

$$H^0(B_0, R^2\pi_*\mathbb{Q}) = \text{im}(H^2(X, \mathbb{Q}) \to H^2(F, \mathbb{Q})) \cong \mathbb{Q}. $$

Hence, there exists a unique homomorphism $(R^2\pi_*\mathbb{Q})^\vee \to \mathbb{Q}$ of local systems on $B_0$ up to scalar. This is a homomorphism of $\mathbb{Q}$-VHS: fiberwise, Corollary 3.14 proves the image of $H^2(X, \mathbb{Q}) \to H^2(F, \mathbb{Q})$ is an ample class. Restrict it to the morphism of $\mathbb{Z}$-VHS $(R^2\pi_*\mathbb{Z})^\vee \to \mathbb{Z}$. The morphism can be uniquely determined once we assume it to be primitive and represents an ample class on each fiber. Finally, use the fact that $\pi: X_0 \to B_0$ is a family of complex tori (abelian varieties) and obtain an isomorphism $R^2\pi_*\mathbb{Z} = \wedge^2 R^1\pi_*\mathbb{Z}$. The result is a primitive polarization

$$(R^1\pi_*\mathbb{Z})^\vee \otimes (R^1\pi_*\mathbb{Z})^\vee \to \mathbb{Z}. \tag{3.15}$$

We have constructed a weight $-1$ $\mathbb{Z}$-VHS $(R^1\pi_*\mathbb{Z})^\vee$ equipped with a polarization (3.15). Now use a formal equivalence of categories between polarized weight $-1$ $\mathbb{Z}$-VHS and that of polarized abelian schemes (e.g., [Del72, 5.2] [Del71, 4.4]). This constructs our desired abelian scheme $\nu: P_0 \to B_0$ with a unique primitive polarization $\lambda: P_0 \to P_0$ over $B_0$. To prove $P_0$ is simple, we may prove the corresponding VHS $R^1\pi_*\mathbb{Q}$ is simple. This is tacitly proved in [vGV16] and later explicitly stated in [Voi18, Lem 4.5]. The idea is that if $R^1\pi_*\mathbb{Q}$ splits as a direct sum $\mathcal{V}_1 \oplus \mathcal{V}_2$ of two VHS, then each of them has their own polarizations, forcing $h^0(B_0, R^2\pi_*\mathbb{Q}) \geq h^0(B_0, \wedge^2 \mathcal{V}_1) + h^0(B_0, \wedge^2 \mathcal{V}_2) \geq 2$. We omit the details here.

**Proof of Theorem 3.1, torsor.** Consider an analytic open covering $\{B_i: i \in I\}$ of $B_0$ so that over each $B_i$, the restriction of the Lagrangian fibration $\pi: X_i \to B_i$ admits at least one holomorphic section $s_i: B_i \to X_i$. Considering $s_i$ as a zero section, $\pi: X_i \to B_i$ becomes an abelian scheme. Hence by the equivalence of abelian schemes and $(R^1\pi_*\mathbb{Z})^\vee$, $\nu$ and $\pi$ are isomorphic over $B_i$ by $\phi_i: X_i \to P_i$ sending $s_i$ to the zero section of $P_i$.

Now use the isomorphism $\phi_i$ to transform the group law $+: P_i \times_{B_i} P_i \to P_i$ into a $P_i$-action on $X_i$. That is, we define a group action morphism by

$$\rho_i: P_i \times_{B_i} X_i \to X_i, \quad (p_i, x_i) \mapsto \phi_i^{-1}(\phi_i(x_i) + p_i).$$

We want to patch $\rho_i$ together to define a group action $\rho: P_0 \times_{B_0} X_0 \to X_0$ over the entire $B_0$. To do so, we need to check whether the definitions of $\rho_i$ and $\rho_j$ coincides over the intersection $B_{ij} = B_i \cap B_j$, i.e.,

$$\phi_i^{-1}(\phi_i(x_{ij}) + p_{ij}) = \phi_j^{-1}(\phi_j(x_{ij}) + p_{ij}) \quad \text{for all} \quad (p_{ij}, x_{ij}) \in P_{ij} \times_{B_{ij}} X_{ij}. \tag{3.16}$$

Over $B_{ij}$, one has a transition function $\phi_j \circ \phi_i^{-1}: P_{ij} \to X_{ij} \to P_{ij}$, an automorphism of $P_{ij}$. Recall that the isomorphisms $\phi_i$ and $\phi_j$ are constructed by choosing the zero sections $s_i$ and $s_j$, and the corresponding isomorphisms $\phi_i: X_{ij} \cong P_{ij}$ and $\phi_j: X_{ij} \cong P_{ij}$ are as abelian schemes. From it, we notice the automorphism $\phi_j \circ \phi_i^{-1}: P_{ij} \to P_{ij}$ is a translation automorphism. The translation is
by \( \phi_j \circ \phi_i^{-1}(0) \), the difference of the two zero sections. With this, we have a sequence of identities

\[
\phi_j(x_{ij}) + p_{ij} = \phi_j \circ \phi_i^{-1}(\phi_i(x_{ij})) + p_{ij} = (\phi_i(x_{ij}) + \phi_j \circ \phi_i^{-1}(0)) + p_{ij} = (\phi_i(x_{ij}) + p_{ij}) + \phi_j \circ \phi_i^{-1}(0) = \phi_j \circ \phi_i^{-1}(\phi_i(x_{ij}) + p_{ij}).
\]

This proves (3.16). Hence \( \rho_i \) patches together and defines a morphism \( \rho : P_0 \times_{B_0} X_0 \rightarrow X_0 \). The group action axioms are all easily verified. Also, \( X_0 \) is clearly a \( P_0 \)-torsor by construction.

**Proof of Theorem 3.1, uniqueness.** Let \( \nu : P_0 \rightarrow B_0 \) be a (not necessarily projective) abelian scheme so that \( \pi \) becomes a torsor under \( \nu \). We claim \( R^1 \nu_* \mathbb{Z} \cong R^1 \pi_* \mathbb{Z} \) as VHS over \( B_0 \). Consider the group scheme action map

\[
P_0 \times_{B_0} X_0 \xrightarrow{\rho} X_0 \xrightarrow{\pi} B_0.
\]

From the diagram, we have a pullback morphism between the VHS \( \rho^* : R^1 \pi_* \mathbb{Z} \rightarrow R^1 \mu_* \mathbb{Z} \). The latter VHS is isomorphic to the direct sum \( R^1 \nu_* \mathbb{Z} \oplus R^1 \pi_* \mathbb{Z} \) by the Künneth formula (e.g., [Ive86, VII.2.7]) and decomposition theorem for smooth proper morphisms. Hence composing with the first projection, we obtain a morphism \( R^1 \pi_* \mathbb{Z} \rightarrow R^1 \nu_* \mathbb{Z} \). Now over a small analytic open subset \( U \subset B_0 \), fix any holomorphic section of \( \pi : X_U \rightarrow U \) so that we can identify \( P_U \) and \( X_U \). Hence \( \rho \) becomes the addition operation of the abelian scheme \( X_U \times_U X_U \rightarrow X_U \). With this description, the pullback morphism is fiberwise \( \rho^* : H^1(F, \mathbb{Z}) \rightarrow H^1(F, \mathbb{Z}) \oplus H^1(F, \mathbb{Z}), x \mapsto (x, x) \). Hence the morphism \( R^1 \pi_* \mathbb{Z} \rightarrow R^1 \nu_* \mathbb{Z} \) is an isomorphism over \( U \), and the claim follows. \( \square \)

**Remark 3.17.**

(1) A posteriori, one has an interpretation of Corollary 3.14 in terms of Theorem 3.1. The abelian scheme \( \nu \) is projective, and \( \nu \) and \( \pi \) are fiberwise isomorphic. Hence the smooth fibers of \( \pi \) are projective, even when the hyper-Kähler manifold \( X \) is not.

(2) Theorem 3.1 is also related to Oguiso’s result [Ogu09] in the following sense. Consider the generic fiber \( P_L \rightarrow \text{Spec} \ L \) of \( \nu : P_0 \rightarrow B_0 \). It is an abelian variety over \( L \). Since \( P_0 \) has a unique polarization (up to scalar), \( P_L \) has a unique polarization. Now ampleness is an open condition in \( \text{NS}(P_L)_{\mathbb{R}} \), so the uniqueness of the polarization implies \( \rho(P_L) = 1 \).

(3) If we further assume \( X \) is projective, then the discussion becomes algebraic and hence the smooth morphism \( \pi : X_0 \rightarrow B_0 \) admits étale local sections. Thus \( \pi \) becomes an étale torsor under \( \nu \).

The unique abelian scheme \( P_0 \) above should be considered as the identity component of the automorphism scheme of \( \pi \) (see [AF16, §8.3] and Remark 3.20 below). Define a sheaf of relative automorphisms acting by translations on each fibers by

\[
\text{Aut}_{X_0/B_0}^{\text{tr}}(U) = \{ f : X_U \rightarrow X_U : U \text{-automorphism acting by translation on each fibers} \}. \tag{3.18}
\]

**Proposition 3.19.** The abelian scheme \( \nu : P_0 \rightarrow B_0 \) represents \( \text{Aut}_{X_0/B_0}^{\text{tr}} \).

**Proof.** This almost follows from the definition. Let us temporarily denote by \( \overline{P_0} \) the sheaf of analytic local sections of \( \nu \). Since \( P_0 \) acts on \( X_0 \) by fiberwise translation, we have a sheaf homomorphism \( \overline{P_0} \rightarrow \text{Aut}_{X_0/B_0}^{\text{tr}} \). The homomorphism is injective because the \( P_0 \)-action is effective. Now \( X_0 \) was in fact a torsor under \( P_0 \). Over a small analytic open subset \( U \subset B_0 \), \( X_U \rightarrow U \) admits a section so it becomes an abelian scheme isomorphic to \( P_U \rightarrow U \). Hence the set of sections \( P_U(U) = X_U(U) \) consists of precisely the translation automorphisms of \( X_U \). This proves the homomorphism \( \overline{P_0} \rightarrow \text{Aut}_{X_0/B_0}^{\text{tr}} \) is surjective stalkwise. Thus it is an isomorphism. \( \square \)
Remark 3.20. We have later learned that the full relative automorphism sheaf \( \text{Aut}_{X/B} \) of the Lagrangian fibration is representable by an analytic group scheme \( \text{Aut}_{X/B} \to B \). This is essentially a consequence of the existence of the Hilbert scheme of \( X \times_B X \to B \). See \cite[Thm 5.23]{Nit05} for the case when \( X \) is projective. The proof for the non-projective case roughly goes as follows. Since we are assuming \( B \) is smooth, \( \pi \) is flat by the miracle flatness theorem (e.g., \cite[§3.20]{Fis76}).

Deduce from \cite{Pou69} the existence of the Hilbert scheme \( \text{Hilb}_{X \times_B X} \to B \), an (infinite) disjoint union of complex spaces proper over \( B \). Imitate the proof of \cite[Thm 1.10]{Kol96} to show a morphism \( \text{Aut}_{X/B} \to \text{Hilb}_{X \times_B X} \) sending an automorphism to its graph is an open subfunctor. This proves \( \text{Aut}_{X/B} \) is representable by an open subspace of a complex space \( \text{Hilb}_{X \times_B X} \).

Therefore, \( \nu : P_0 \to B_0 \) is really the identity component of the group scheme \( \text{Aut}_{X_0/B_0} \to B_0 \) in a precise sense.

3.2. Polarization type and divisibility of \( \pi^*O_B(1) \). The purpose of this subsection is to study two numerical invariants associated to a Lagrangian fibered hyper-Kähler manifold and study their relations: they are the polarization type of \( \pi \) in Definition 3.8 and the divisibility of the line bundle \( \pi^*O_B(1) \). Throughout, we will write \( h \in H^2(X, \mathbb{Z}) \) for the first Chern class of \( \pi^*O_B(1) \) and \( \text{div}(h) \) for the divisibility (1.4) of \( h \).

The polarization type of \( \pi \) is an \( n \)-tuple of positive integers \( (d_1, \ldots, d_n) \) with \( d_1 | \cdots | d_n \) such that each fiber of the polarization scheme \( K_0 \) is isomorphic to \( (\mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_n)^{\oplus 2} \). Since we are assuming the polarization \( \lambda : P_0 \to P_0 \) is primitive, we always have \( d_1 = 1 \). The polarization type was already computed for all currently known deformation types of hyper-Kähler manifolds.

The computations were based on its original definition in \cite{Wie16}. Therefore, to use the previous results we first need to show our definition is equivalent to the original one.

Lemma 3.21. The polarization type in Definition 3.8 is equivalent to the definition in \cite{Wie16}.

Proof. Recall our definition of the polarization is constructed as the primitive morphism \( (R^2\pi_*\mathbb{Z})^\vee \to \mathbb{Z} \) of local systems, or equivalently a primitive morphism \( \mathbb{Z} \to R^2\pi_*\mathbb{Z} \). Fix any smooth fiber \( F \) of \( \pi \).

As a local system, \( R^2\pi_*\mathbb{Z} \) is identified with a \( \mathbb{Z} \)-module \( H^2(F, \mathbb{Z}) \) with a monodromy \( \pi_1(B_0) \)-action. In this setting, the primitive morphism \( \mathbb{Z} \to R^2\pi_*\mathbb{Z} \) of local systems corresponds to a primitive homomorphism \( \mathbb{Z} \to H^2(F, \mathbb{Z}) \) of \( \pi_1(B_0) \)-modules. In other words, our definition of the polarization type is equivalent to the primitive polarization type of a single smooth fiber \( F \) coming from the image of \( H^2(X, \mathbb{Q}) \to H^2(F, \mathbb{Q}) \) by the global invariant cycle theorem. This was precisely how Wieneck defined the polarization type.

We can now use the previous results on computations of the polarization type of \( \pi \). The following theorem collects all possible polarization types that can occur for known deformation types of hyper-Kähler manifolds. For \( \text{K3}^{[n]} \) and \( \text{Kum}_n \) types the computations are done by \cite{Wie16, Wie18}. For \( \text{OG10} \) and \( \text{OG6} \), the computations are contained in \cite{MO22} and \cite{MR21}, respectively.

Theorem 3.22 (\cite{Wie16, Wie18}, \cite{MO22}, \cite{MR21}). Let \( \pi : X \to B \) be a Lagrangian fibered compact hyper-Kähler manifold. Then the polarization type of \( \pi \) is

\[
\begin{cases}
(1, \cdots, 1) & \text{if } X \text{ is of } \text{K3}^{[n]} \text{-type}; \\
(1, 1, 1, 1) & \text{if } X \text{ is of } \text{OG10} \text{-type}; \\
(1, \cdots, 1, d_1, d_2) & \text{if } X \text{ is of } \text{Kum}_n \text{-type}; \text{ and} \\
(1, 2, 2) & \text{if } X \text{ is of } \text{OG6} \text{-type}.
\end{cases}
\]

When \( X \) is of \( \text{Kum}_n \) type, we set \( d_1 = \text{div}(h) \) in \( H^2(X, \mathbb{Z}) \) and \( d_2 = \frac{n+1}{d_1} \).
It is also important for us that the polarization type is deformation invariant on $\pi$ (see [Wie16, Thm 1.1]). We will later recover this result in Corollary 3.32. Observe in Theorem 3.22 that we have an equality $c_X = d_1 \cdots d_n$ for all known deformation types of hyper-Kähler manifolds. In this sense, we expect the polarization type should be considered as a refinement of the Fujiki constant $c_X$. This is also related to the non-primitiveness of the image of the restriction homomorphism $H^2(X, \mathbb{Z}) \to H^2(F, \mathbb{Z})$.

**Proposition 3.23.** Assume we have an equality $c_X = d_1 \cdots d_n$. Then the image of the restriction homomorphism $H^2(X, \mathbb{Z}) \to H^2(F, \mathbb{Z})$ in Lemma 3.13 is generated by $a\theta$, where $a = \text{div}(h)$ and $\theta$ is a primitive ample class representing the canonical polarization of $F$.

**Proof.** Choose a cohomology class $x \in H^2(X, \mathbb{Z})$ with $q(h, x) = a$. By Lemma 3.13, the class $x|_F \in H^2(F, \mathbb{Z})$ must be a positive integer multiple of the primitive polarization class $\theta$. Set $x|_F = b\theta$. Now the claim directly follows from the Fujiki relation

$$d_1 \cdots d_n = \frac{1}{n!} \int_F \theta^n = \frac{1}{n!} \int_X h^n(\frac{1}{b}x)^n = c_X \cdot q(h, \frac{1}{b}x)^n = c_X \left(\frac{a}{b}\right)^n.$$  

Though not used in this paper, the divisibility of $h$ is also related to the existence of a rational section of $\pi$. We end this subsection with the following observation.

**Proposition 3.24.** Assume $c_X = d_1 \cdots d_n$ and $\pi$ admits at least one rational section. Then $\text{div}(h) = 1$ or 2.

**Proof.** If $\pi$ admits a rational section, then $X_0 \cong P_0$ becomes a projective abelian scheme (Example 3.7). By the general theory of abelian schemes, twice a polarization is always associated to a line bundle (e.g., [MFK94, Prop 6.10] or [FC90, Def I.1.6]). This means $2\theta \in H^2(F, \mathbb{Z})$ is contained in the image of $\text{Pic}(X) \subset H^2(X, \mathbb{Z}) \to H^2(F, \mathbb{Z})$. By Proposition 3.23, this implies $\text{div}(h) = 1$ or 2.

If $X$ is of K3$^{[\nu]}$ or Kum$_n$-type then its Lagrangian fibration $\pi : X \to B$ may have $\text{div}(h) > 2$. In such cases, $\pi$ (and any of its deformation) would never admit any rational section and the notion of the $P_0$-torus is necessary.

### 3.3. The polarization scheme and $H^2$-trivial automorphisms.

We present the proof of Proposition 3.10 and 3.12 in this section.

**Lemma 3.25.** Any rational section of $\nu : P_0 \to B_0$ can be uniquely extended to an honest section.

**Proof.** Assume $s : B_0 \dasharrow P_0$ is a rational section undefined at $b \in B_0$. Let $S \subset P_0$ be the closure of the image of $s$, so that we obtain a proper birational morphism $\nu_S : S \to B_0$. Since $B_0$ is smooth and $s$ is undefined at $b$, the fiber $S_b = (\nu_S)^{-1}(b)$ is a uniruled variety (e.g., [Kol96, Thm VI.1.2]). This means an abelian variety $\nu^{-1}(b)$ contains a uniruled variety $S_b$. Contradiction. See [BLR90, Cor 8.4.6] for an alternative proof.

**Remark 3.26.** The same argument applies to $\pi$ and proves the following: any rational section of $\pi$ is necessarily defined over $B_0$.

**Proposition 3.27.** Every $H^2$-trivial automorphism in $\text{Aut}_\circ(X/B)$ defines a global section of $P_0 \to B_0$. That is, we have a closed immersion of group schemes

$$\text{Aut}_\circ(X/B) \hookrightarrow P_0.$$
Proof. Recall from Proposition 3.19 that \( P_0 \) is the abelian scheme representing the translation automorphism sheaf \( \underline{\text{Aut}}^\text{tr}_{X_0/B_0} \). Hence our goal is to prove \( \text{Aut}^\circ(X/B) \) acts on \( \pi : X_0 \to B_0 \) by fiberwise translation automorphisms. Consider the quotient \( \bar{X} = X/ \text{Aut}^\circ(X/B) \) with a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & \bar{X} \\
\downarrow & & \\
B & \xleftarrow{\bar{\pi}} & \cdot
\end{array}
\]

We first claim \( p \) is étale on general fibers over \( B \). Let \( S \subset X \) be the ramified locus of \( p \). It has codimension \( \geq 2 \) because \( p \) is quasi-étale by Proposition B.2. Let \( b \in B \) be a general point, so that the fibers \( F = X_b \) and \( \bar{F} = \bar{X}_b \) are both smooth. Observe the ramification locus of \( p : F \to \bar{F} \) is precisely \( S \cap F \), which is of codimension \( \geq 2 \) since \( b \) is general. The purity of the branch locus theorem forces \( p : F \to \bar{F} \) to be étale.

Now we have a finite étale quotient \( p : F \to \bar{F} = F/ \text{Aut}^\circ(X/B) \) between smooth projective varieties. Its Galois group \( \text{Aut}^\circ(X/B) \) acts on \( F \) by fixed point free automorphisms. Since \( F \) and \( \bar{F} \) are both abelian varieties ([Sch20, Thm 3]), this means \( \text{Aut}^\circ(X/B) \) acts on \( F \) by translations. The conclusion is that on a general fiber of \( \pi \), the group \( \text{Aut}^\circ(X/B) \) acts by translation. Finally, by Proposition 3.19 this means \( \text{Aut}^\circ(X/B) \) defines a rational section of \( \nu : P_0 \to B_0 \). By Lemma 3.25, the rational section must be defined over the entire \( B_0 \) and becomes an honest section. Hence \( \text{Aut}^\circ(X/B) \) acts by translations over the entire \( B_0 \).

\[ \square \]

An immediate byproduct is that \( \text{Aut}^\circ(X/B) \) is abelian.

**Proposition 3.28.** \( \text{Aut}^\circ(X/B) \) is a finite abelian group.

We next understand the behavior of the polarization \( \lambda \) under deformations of \( \pi \). A related result is Wieneck’s deformation invariance of the polarization type of \( \pi \) [Wie16, Thm 1.1]. Recall that the polarization scheme \( K_0 \) was defined to be the kernel of the polarization \( \ker(\lambda) \) for each positive integer \( a \):

\[
0 \to K_0[a] \to P_0 \xrightarrow{a\lambda} P_0 \to 0 .
\]

Here the morphism \( a\lambda \) is a composition \( P_0 \to P_0 \to P_0 \) of the multiplication by \( a \) endomorphism and \( \lambda \). Since the abelian scheme \( P_0 \) was associated to the VHS \( (R^1\pi_*\mathbb{Z})^\vee \), there is a VHS version of this sequence

\[
0 \to (R^1\pi_*\mathbb{Z})^\vee \xrightarrow{a\lambda} R^1\pi_*\mathbb{Z} \to K_0[a] \to 0 .
\]  

(3.29)

The cokernel \( K_0[a] \) is a local system of finite abelian groups on \( B_0 \) and is related to \( K_0 \) as follows:

\[
K_0[a] \text{ is a sheaf of analytic sections of the group scheme } K_0[a] \to B_0, \text{ and } K_0[a] \text{ is the total space of the local system } K_0[a] .
\]

Therefore, we can relate \( K_0[a] \) to either abelian schemes or variation of Hodge structures. This technical flexibility will be useful to describe deformation behaviors of \( K_0[a] \).

**Lemma 3.30.** Let \( \mathcal{X} \xrightarrow{\pi} \mathcal{B} \xrightarrow{\varphi} \Delta \) be a family of Lagrangian fibered compact hyper-Kähler manifolds over a complex open disc \( \Delta \). Set \( B_0 \subset \mathcal{B} \) the smooth locus of \( \pi \). Then for each positive integer \( a \), there exists a finite étale group scheme \( K_0[a] \) over \( B_0 \) parametrizing the group schemes \( K_a \) over \( (B_0)_t \) for all \( t \in \Delta \).
Proof. Let \( \chi_0 = \pi^{-1}(B_0) \) be the preimage of \( B_0 \), so that the restriction \( \pi_0 : \chi_0 \to B_0 \) of \( \pi \) is a smooth proper family of abelian varieties. Consider the local system \( R^2\pi_0^*Z \) on \( B_0 \). Our first claim is \( H^0(B_0, R^2\pi_0^*Z) \cong Z \). Denoting by \( j : B_0 \to B \) an open immersion, it is enough to prove \( H^0(\Delta, q_*j_*R^2\pi_0^*Z) \cong Z \). Notice that \( q_*j_*R^2\pi_0^*Z \) is a constructible sheaf, because \( R^2\pi_0^*Z \) is a local system, its pushforward by \( j_* \) is a constructible sheaf on \( B \) (e.g., [KS90, Ex VIII.10]), and again its pushforward by \( q_* \) is a constructible sheaf on \( \Delta \). For each \( t \in S \), we have a Lagrangian fibered hyper-Kähler manifold \( \pi : X_t \to B_t \) and we may apply our previous discussions

\[
H^0((B_0)_t, R^2\pi_*Z) \cong Z.
\]

This proves every fiber of \( q_*j_*R^2\pi_0^*Z \) is isomorphic to \( Z \). In this setting, we will formally prove the sheaf has \( \mathbb{Z} \) global sections in Lemma 3.31. This proves the claim \( H^0(B_0, R^2\pi_0^*Z) \cong Z \).

We have a unique primitive morphism \( (R^2\pi_0^*Z)^\vee \to \mathbb{Z} \) of local systems on \( B_0 \). As \( \pi_0 \) is a family of abelian varieties, we have an isomorphism \( R^2\pi_0^*Z = \wedge^2 R^1\pi_0^*Z \). This gives us a unique primitive morphism of local systems (in fact, a polarization of VHS by Corollary 3.14) \( (R^1\pi_0^*Z)^\vee \otimes (R^1\pi_0^*Z)^\vee \to \mathbb{Z} \) over \( B_0 \). Consider the scalar multiple \( a \) of it and induce a morphism \( (R^1\pi_0^*Z)^\vee \to R^1\pi_0^*Z \) whose cokernel \( K[0][a] \) is a local system on \( B_0 \), parametrizing the family of local systems \( K[0][a] \) on each \( (B_0)_t \) in (3.29). The total space of \( K[0][a] \) gives our desired finite étale group scheme \( K[0][a] \to B_0 \).

Lemma 3.31. Let \( F \) be a constructible sheaf on a complex open disc \( \Delta \). If every fiber of \( F \) is isomorphic to \( \mathbb{Z} \), then we have \( H^0(\Delta, F) \cong \mathbb{Z} \).

Proof. Since \( F \) is constructible, \( F|_U \) is a local system on a complement \( U \) of a finite set of points \( t_1, \cdots, t_k \in \Delta \). Let \( U_i \subset U \) be a small punctured disc around \( t_i \). The restriction \( F|_{U_i} \) is determined by the representation

\[
\rho_i : \mathbb{Z} \cong \pi_1(U_i) \to \text{Aut}(\mathbb{Z}) = \{\pm 1\}.
\]

We have only two possibilities \( \rho_i(1) = \pm 1 \) for each \( i \). Suppose we have \( \rho_i(1) = -1 \) for some \( i \). Consider the total space \( f : \text{Et}(F) \to \Delta \) of the entire constructible sheaf \( F \) (space étalé). The map \( f \) is holomorphic and étale, i.e., a local isomorphism. The condition \( \rho_i(1) = -1 \) geometrically translates to the fact that \( f^{-1}(U_i) \) consists of a single copy of \( U_i \) (the zero section) and infinite number of two-sheeted coverings of the punctured disc \( U_i \). By the very assumption, the preimage \( f^{-1}(t_i) = \{p_1, p_2, \cdots\} \) should be isomorphic to \( \mathbb{Z} \). Since \( f \) is a local isomorphism, there should be an open disc neighborhood of each \( p_1 \in \text{Et}(F) \). Along the two-sheeted coverings of \( U_i \) in \( f^{-1}(U_i) \), this cannot happen. Therefore, the only possibility is that all \( p_i \) are the non-Hausdorff points filling in the unique punctured disc component in \( f^{-1}(U_i) \) (i.e., the zero section). Hence we obtain at least \( \mathbb{Z} \) global sections around the zero section and we are done.

The remaining case is when \( \rho_i(1) = 1 \) for all \( i \). This means \( F|_U \) is a constant sheaf \( \mathbb{Z} \). The reader should be aware that this does not imply \( F \) is a constant sheaf \( \mathbb{Z} \) on \( \Delta \). This can be again conveniently seen in the total space \( f : \text{Et}(F) \to \Delta \). Although \( f \) is a local homeomorphism, it is not a covering space unless \( \text{Et}(F) \) is Hausdorff. Indeed, the fibers \( f^{-1}(t_i) \) can consist of non-Hausdorff points in \( \text{Et}(F) \) and this gives us a classification of such a constructible sheaf \( F \). In any case, there are always \( \mathbb{Z} \) global sections.

Lemma 3.30 in particular recovers [Wie16, Thm 1.1].

Corollary 3.32. The polarization type of \( \pi \) is invariant under deformations of \( \pi \). \( \square \)

The following final observation is elementary but nontrivial. We match its notation to our original discussion.
Lemma 3.33. Let $\mathcal{P}_0 \to \mathcal{B}_0$ be an abelian scheme over a complex manifold $\mathcal{B}_0$ and $a\lambda : \mathcal{P}_0 \to \mathcal{P}_0$ a polarization with $\mathcal{K}_0[a] = \ker(a\lambda)$. Assume there exists a torsion section $f : \mathcal{B}_0 \to \mathcal{P}_0$. If $f(\mathcal{B}_0) \cap \mathcal{K}_0[a] \neq \emptyset$ then $f(\mathcal{B}_0) \subset \mathcal{K}_0[a]$.

Proof. The statement is topological and local on the base $\mathcal{B}_0$, so we may assume $\mathcal{B}_0$ is a complex open ball $S$ and $\mathcal{P}_0 \to \mathcal{B}_0$ is homeomorphic to the topological constant group scheme $(\mathbb{R}/\mathbb{Z})^{2n} \times S \to S$. In this setting, the kernel $\mathcal{K}_0[a]$ is a constant subgroup scheme and the torsion section $f$ is a constant section. Hence $f(S) \cap \mathcal{K}_0[a] \neq \emptyset$ if and only if $f(S) \subset \mathcal{K}_0[a]$. $\square$

Proof of Proposition 3.10. Consider a one-parameter family of Lagrangian fibered hyper-Kähler manifolds $X \to \mathcal{B} \to \Delta$ over a complex disc $\Delta$. By Lemma 3.30, there exists a notion of a family of abelian schemes $\mathcal{P}_0 \to \mathcal{B}_0$ and a family of finite étale group schemes $\mathcal{K}_0[a] \subset \mathcal{P}_0$. Proposition 3.27 proves we have a closed immersion $\text{Aut}\circ(X/B) \hookrightarrow \mathcal{P}_0$ for a single fiber. In fact, the argument applies to the entire family and produces $\text{Aut}\circ(X/B)$ global sections of $\mathcal{P}_0 \to \mathcal{B}_0$, or equivalently an embedding

$$\text{Aut}\circ(X/B) \hookrightarrow \mathcal{P}_0.$$ Since $\text{Aut}\circ(X/B)$ is finite, the global sections are torsion. Suppose we had $\text{Aut}\circ(X/B) \hookrightarrow \mathcal{K}_0[a]$ for the original Lagrangian fibration over $0 \in \Delta$. Then this forces $\text{Aut}\circ(X/B) \hookrightarrow \mathcal{K}_0[a]$ over the entire $\Delta$ by Lemma 3.33. The claim follows. $\square$

Proof of Proposition 3.12. Recall from Proposition 3.23 that the restriction map $H^2(X,\mathbb{Z}) \to H^2(F,\mathbb{Z})$ has a rank 1 image generated by the class $a\theta$, where $a = \text{div}(h)$ and $\theta$ is the primitive ample class corresponding to our polarization $\lambda : F \to \tilde{F}$. The preimage of $a\theta \in H^2(F,\mathbb{Z})$ under this restriction homomorphism is precisely $S = \{x \in H^2(X,\mathbb{Z}) : q(x,h) = a\}$. By Proposition 3.10, the claim is invariant under deformations of $\pi$. We may thus deform $\pi$ and assume $\text{Pic}(X) \cap S \neq \emptyset$. In other words, we may assume the composition $\text{Pic}(X) \subset H^2(X,\mathbb{Z}) \to H^2(F,\mathbb{Z})$ is generated by $a\theta$.

The assertion $\text{Aut}\circ(X/B) \hookrightarrow \mathcal{K}_0[a] = \ker(a\lambda)$ is equivalent to $a\lambda(\text{Aut}\circ(X/B)) = 0$. The latter equality may be verified fiberwise, so we may concentrate on a single fiber $F = \nu^{-1}(b) = \pi^{-1}(b)$. Let $L$ be any line bundle on $X$ such that its image under $\text{Pic}(X) \to H^2(F,\mathbb{Z})$ is $a\theta$. This means the polarization $a\lambda$ can be described as

$$a\lambda : F \to \tilde{F}, \quad t_x \mapsto [t_x^*(L|_F) \otimes L^{-1}|_F].$$

If we assume $t_x = f|_F$ is from a global $H^2$-trivial automorphism $f \in \text{Aut}\circ(X/B)$, then we have a sequence of identities

$$t_x^*(L|_F) = (f|_F)^*(L|_F) = (f^*L)|_F \cong L|_F,$$

where the last isomorphism follows from the fact $f$ acts on $\text{Pic}(X) \subset H^2(X,\mathbb{Z})$ trivially. This proves $a\lambda$ sends $\text{Aut}\circ(X/B)$ to 0 and the claim follows. $\square$

4. The Minimal Split Covering and $H^2$-Trivial Automorphisms

This section discusses an explicit construction of certain $H^2$-trivial automorphisms. This will be conveniently used in the next section when we describe the $\text{Aut}\circ(X)$-action explicitly for certain examples of Kum_n-type hyper-Kähler manifolds. Recall that the group $\text{Aut}\circ(X)$ is computed for all known deformation types of hyper-Kähler manifolds: Beauville [Ben83], for Kum^3_n-type, Boisserie–Nieper-Wi̇skirchen–Sarti [BNWS11] for Kum_n-type, and Mongardi–Wandel [NW17] for OG10 and OG6-type. The strategy is to compute the group for a specific choice of a complex structure and then use Hassett–Tschinkel’s deformation equivalence [HT13, Thm 2.1]. Unfortunately, this
argument doesn’t tell us how $\text{Aut}^0(X)$ exactly acts on $X$ for the deformations. The goal of this section is to introduce Proposition 4.5 to partially resolve this problem.

Throughout the section, we stick to the following setting. Let $M$ be a projective holomorphic symplectic manifold, not necessarily irreducible. By Beauville–Bogomolov decomposition theorem, $M$ must admit a finite étale covering $X \times T \to M$, called a split covering, where $X$ is a finite product of projective hyper-Kähler manifolds and $T$ is an abelian variety. In fact, Beauville in [Bea83a, §3] also considered the smallest possible minimal covering. A minimal split covering of $M$ is the smallest possible split covering of $M$, in the sense that every split covering factors through it. The minimal split covering of $M$ always exists and is unique up to a (non-unique) isomorphism. Moreover, it is a Galois covering. We refer to Beauville’s original paper for more details about minimal split coverings.

Meanwhile, Kawamata [Kaw85, Thm 8.3] proved that if $M$ is a K-trivial smooth projective variety then its Albanese morphism $\text{Alb} : M \to \text{Alb}(M)$ has to be an étale fiber bundle. More concretely, there exists an isogeny $\phi : T \to \text{Alb}(M)$ of abelian varieties such that the base change of $\text{Alb}$ becomes a trivial fiber bundle over $T$. We obtain a cartesian diagram

$$
\begin{array}{ccc}
X \times T & \xrightarrow{\Phi} & M \\
\downarrow \text{pr}_2 & & \downarrow \text{Alb} \\
T & \xrightarrow{\phi} & \text{Alb}(M)
\end{array}
$$

where $X$ is a fiber of the Albanese morphism. In particular, one sees $\Phi : X \times T \to M$ becomes a split covering of $M$.

Combining the two discussions, we obtain:

**Proposition 4.2.** Let $M$ be a projective holomorphic symplectic manifold and $\text{Alb} : M \to \text{Alb}(M)$ its Albanese morphism, an étale fiber bundle by Kawamata. Assume $X = \text{Alb}^{-1}(0)$ is a projective hyper-Kähler manifold. Then there exists a unique isogeny $\phi : T \to \text{Alb}(M)$ of abelian varieties such that the morphism $\Phi$ in the fiber diagram (4.1) becomes the minimal split covering of Beauville.

**Proof.** Use Kawamata’s result to construct an isogeny $\phi' : T' \to \text{Alb}(M)$ trivializing the Albanese map as in (4.1). Since $\phi'$ is a finite Galois covering, $\phi'$ is also a finite Galois covering with $\text{Gal}(\phi') \cong \text{Gal}(\phi')$. The first lemma in [Bea83a, §3] claims $\text{Aut}(X \times T') = \text{Aut}(X) \times \text{Aut}(T')$. Hence the $\text{Gal}(\phi')$-action on $X \times T'$ is by $(f, a)$ where $f$ and $a$ are automorphisms on $X$ and $T$, respectively. The isomorphism $\text{Gal}(\phi') \to \text{Gal}(\phi')$ is by the second projection $(f, a) \mapsto a$. Since $\text{Gal}(\phi')$ is the kernel of the isogeny $\phi'$, the automorphisms $a$ must be translations of $T'$.

Now consider the homomorphism $\text{Gal}(\phi') \to \text{Aut}(X)$ by $(f, a) \mapsto f$. Set $H$ by the kernel of it; it consists of elements of the form $(\text{id}_X, a)$. Under the isomorphism $\text{Gal}(\phi') \cong \text{Gal}(\phi')$, we can consider it as a subgroup of $\text{Gal}(\phi')$, so there exists a Galois covering $T' \to T = T'/H$ corresponding to it. Let $\phi : T \to \text{Alb}(M)$ be the morphism factorizing $\phi'$. We have a cartesian diagram

$$
\begin{array}{ccc}
X \times T' & \xrightarrow{\Phi} & M \\
\downarrow \text{pr}_2 & & \downarrow \text{Alb} \\
T' & \xrightarrow{\phi} & \text{Alb}(M)
\end{array}
$$

By construction, $\text{Gal}(\Phi)$ consists of automorphisms $(f, a)$ with no $(\text{id}_X, a)$ (i.e., the $\text{Gal}(\phi)$-action on $X$ is effective). But this means $\Phi$ is precisely Beauville’s minimal split covering [Bea83a, §3]. The uniqueness of $\phi$ follows from the uniqueness of the minimal split covering. \qed
The proposition in particular proves that the minimal split covering can be always realized by an isogeny \( \phi : T \to \text{Alb}(M) \) and the base change (4.1).

**Definition 4.3.** We call \( \phi : T \to \text{Alb}(M) \) in Proposition 4.2 the minimal isogeny trivializing the Albanese morphism \( \text{Alb} : M \to \text{Alb}(M) \). It is unique up to a (non-unique) isomorphism.

In fact, the proof of Proposition 4.2 is saying more about an arbitrary isogeny \( \phi' \).

**Corollary 4.4.** Notation as in Proposition 4.2. Let \( \phi' : T' \to \text{Alb}(M) \) be any isogeny trivializing the Albanese morphism. Then

1. \( \phi' \) factors through the minimal isogeny \( \phi \).
2. There exists a canonical \( \text{Gal}(\phi') \)-action on \( X \).
3. The isogeny \( \phi' \) is minimal if and only if the \( \text{Gal}(\phi') \)-action on \( X \) is effective.

**Proof.** All of these can be directly deduced from the proof of Proposition 4.2. Recall \( \text{Gal}(\Phi') \to \text{Gal}(\phi') \), \( (f, a) \mapsto a \) is an isomorphism. Therefore, \( f = f_a \) is uniquely determined by \( a \), and this defines \( \text{Gal}(\phi') \to \text{Aut}(X), a \mapsto f_a \).

Now we can state the main result of this section. The ideas here were already contained in [Bea83a, Bea83b].

**Proposition 4.5.** Notation as in Proposition 4.2 and 4.4. Then \( \text{Gal}(\phi') \) acts on \( X \) by \( H^2 \)-trivial automorphisms. That is, we have a canonical homomorphism

\[
\text{Gal}(\phi') \to \text{Aut}^s(X),
\]

which is injective if and only if \( \phi' \) is minimal.

**Proof.** By Corollary 4.4, we may assume \( \phi' = \phi \) is minimal and \( \text{Gal}(\phi) \subset \text{Aut}(X) \). The content of the proposition is that it is further a subgroup of \( \text{Aut}^s(X) \).

Consider the diagram (4.1). Our first step is to equip \( T \)-actions on all the four spaces to make the diagram \( T \)-equivariant. Equip a \( T \)-action on \( T \) by translation, and on \( X \times T \) only on the second factor again by translation. The \( T \)-action on \( \text{Alb}(M) \) is by translation via the morphism \( \phi \): if \( a \in T \) and \( z \in \text{Alb}(M) \) then we define \( a.z = z + \phi(a) \).

To equip a \( T \)-action on \( M \), we claim the \( T \)-action on \( X \times T \) descends to \( M \) via \( \Phi \). The descent works if the Gal(\( \Phi \))-action on \( X \times T \) commutes with the \( T \)-action. Recall from the discussions in Proposition 4.2 that \( \text{Gal}(\Phi) \) acts on \( X \times T \) by \( (f, a) \) where \( f \) is an automorphism of \( X \) and \( a \) is a translation of \( T \). Let \( b \in T \) and \( (x, t) \in X \times T \). Then we have a sequence of identities

\[
b.((f, a).\langle x, t \rangle) = (f(x), t + a + b) = (f, a).\langle b(x, t) \rangle.
\]

This proves the \( T \)-action and Gal(\( \Phi \))-action commutes, yielding the descent \( T \)-action on \( M \). The conclusion is that \( \text{Alb} \) becomes automatically \( T \)-equivariant (and hence the diagram (4.1) becomes \( T \)-equivariant).

By definition, the stabilizer the \( T \)-action on \( \text{Alb}(M) \) is precisely \( \ker \phi = \text{Gal}(\phi) \). Since the Albanese map \( \text{Alb} : M \to \text{Alb}(M) \) is \( T \)-equivariant, this induces a Gal(\( \phi \))-action on the fiber \( \text{Alb}^{-1}(0) = X \). One easily shows this coincides with our previous Gal(\( \phi \))-action on \( X \). Notice that any \( T \)-action on \( M \) is isotopic to the identity map because \( T \) is path connected. In particular, \( T \) acts on \( M \) trivially at the level of cohomology \( H^*(M, \mathbb{Q}) \). The embedding \( X \subset M \) is Gal(\( \phi \))-equivariant, so we have a Gal(\( \phi \))-equivariant restriction homomorphism

\[
H^2(M, \mathbb{Q}) \to H^2(X, \mathbb{Q}).
\]
Hence it suffices to prove this restriction homomorphism is surjective.

The question now became topological. Deform the complex structure of the hyper-Kähler manifold $X$ very generally so that $H^2(X, \mathbb{Q})$ becomes a simple $\mathbb{Q}$-Hodge structure (we will have to lose the projectiveness of $X$). The complex structure of $M$ can be correspondingly chosen in a way that the finite covering map $\Phi : X \times T \to M$ becomes holomorphic. Therefore, the Hodge structure morphism $H^2(M, \mathbb{Q}) \to H^2(X, \mathbb{Q})$ is either 0 or surjective. We only need to rule out the former possibility.

To prove it is nonzero, consider any global holomorphic symplectic form $\sigma$ on $M$. Pulling it back to $X \times T$ gives a global holomorphic symplectic form on $X \times T$. But $H^{2,0}(X \times T) = H^{2,0}(X) \oplus H^{2,0}(T)$ by Kunneth. If $\sigma$ was 0 in the $H^{2,0}(X)$-component then this would mean $\sigma$ doesn’t contain any 2-forms along the tangent direction of $X$, violating $\sigma$ is a symplectic form. Hence $\sigma|_X$ cannot be 0. The claim follows.

\textit{Remark 4.6.} An alternative way to state the results in this section is as follows. Any isogeny $\phi' : T' \to \text{Alb}(M)$ trivializing the Albanese morphism defines a group homomorphism $\text{Gal}(\phi') \to \text{Aut}^\circ(X)$. The image of this homomorphism is independent on the choice of $\phi'$, which we denote by

$$\text{Aut}'(X) \subset \text{Aut}^\circ(X).$$

It is a finite abelian group, isomorphic to $\text{Gal}(\phi)$ for a minimal isogeny $\phi$, and is deformation invariant on $X$. For example, we will later see that when $X$ is of Kum$_n$-type then

$$\text{Aut}'(X) \cong (\mathbb{Z}/n + 1)^{\oplus 4}, \quad \text{Aut}^\circ(X) \cong \mathbb{Z}/2 \times (\mathbb{Z}/n + 1)^{\oplus 4}.$$

Our main result can be more directly stated with this definition. See Remark 6.2.

5. THE $H^2$-TRIVIAL AUTOMORPHISMS AND POLARIZATION SCHEME FOR GENERALIZED KUMMER VARIETIES

The goal of this section is an explicit computation of the group $\text{Aut}^\circ(X/B)$ and the polarization scheme $K_0$ for certain Lagrangian fibrations of Kum$_n$-type hyper-Kähler manifolds. Since the group $\text{Aut}^\circ(X/B)$ will be of interest for all known deformation types of hyper-Kähler manifolds, we state the result in a more general form. The following is the first main theorem of this section.

\textbf{Theorem 5.1.} Let $\pi : X \to B$ be a Lagrangian fibration of a compact hyper-Kähler manifold.

\begin{enumerate}
    \item $\text{Aut}^\circ(X/B) \cong \begin{cases} 
    \{\text{id}\} & \text{if } X \text{ is of K3}^{[n]} \text{ or OG10-type,} \\
    (\mathbb{Z}/2)^{\oplus 4} & \text{if } X \text{ is of OG6-type.}
    \end{cases}$
    
    \item Assume $X$ is of Kum$_n$-type and $(1, \cdots , 1, d_1, d_2)$ is the polarization type of $\pi$ in Theorem 3.22. Then
    
    $\text{Aut}^\circ(X/B) \cong \begin{cases} 
    (\mathbb{Z}/2)^{\oplus 5} & \text{if } n = 3 \text{ and the polarization type is } (1, 2, 2), \\
    (\mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2)^{\oplus 2} & \text{otherwise.}
    \end{cases}$
\end{enumerate}

Notice that the bigger group $\text{Aut}^\circ(X)$ is already trivial for K3$^{[n]}$ and OG10-types (see [Bea83a] and [MW17]), so the theorem is clear in these cases. For Kum$_n$ and OG6-types, recall from Theorem 2.3 that $\text{Aut}^\circ(X/B)$ is deformation invariant on $\pi$. By [Wie18, §6.28], every Lagrangian fibration of a Kum$_n$-type hyper-Kähler manifold is deformation equivalent to the moduli of sheaves construction, which will be recalled in Section 5.1. By [MR21], every Lagrangian fibration of an OG6-type hyper-Kähler manifold is deformation equivalent to each other. Therefore, Theorem 5.1 follows from the following more concrete results.
Proposition 5.2. Let $\pi : X \to B$ be a Lagrangian fibration of a $\text{Kum}_a$-type hyper-Kähler manifold, obtained by the moduli of sheaves construction from a triple $(S, l, s)$ in Definition 5.7. Let $(d_1, d_2)$ be the polarization type of the ample class $l$. Then

$$\text{Aut}^\circ(X/B) \cong \begin{cases} (\mathbb{Z}/2)^{\oplus 5} & \text{if } n = 3 \text{ and } d_1 = d_2 = 2, \\ (\mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2)^{\oplus 2} & \text{otherwise.} \end{cases}$$

Proposition 5.3 (Mongardi–Wandel). Let $\pi : X \to B$ be a Lagrangian fibration of an OG6-type hyper-Kähler manifold, obtained by the moduli of sheaves construction. Then

$$\text{Aut}^\circ(X/B) \cong (\mathbb{Z}/2)^{\oplus 4}.$$  

We note that the latter computation for OG6-type was already done by Mongardi–Wandel in [MW17, §5], as an intermediate step for their computation of the larger group $\text{Aut}^\circ(X) \cong (\mathbb{Z}/2)^{\oplus 8}$. Thus proving Proposition 5.2 will be enough to conclude Theorem 5.1. As mentioned before, the proof will be done by an explicit computation. In fact, the computation can be carried out further and calculates the polarization scheme $K_0$ as well. This is the second main result of this section.

Proposition 5.4. Let $\pi : X \to B$ be a Lagrangian fibration of a $\text{Kum}_a$-type hyper-Kähler manifold, obtained by the moduli of sheaves construction from a triple $(S, l, s)$ in Definition 5.7. Let $(d_1, d_2)$ be the polarization type of the ample class $l$.

1. If $n = 3$ and $d_1 = d_2 = 2$, then

$$K_0 \hookrightarrow \text{Aut}^\circ(X/B) \hookrightarrow K_0[2].$$

2. Otherwise, we have

$$K_0 = \text{Aut}^\circ(X/B).$$

Contrary to the previous sections, all the discussions in this section will be algebraic. In particular, algebraic Chern classes and Chow groups will be used. Given a coherent sheaf $E$ on a smooth projective variety $S$, we denote by

$$c_i(E) \in H^{2i}(S, \mathbb{Z}), \quad \tilde{c}_i(E) \in \text{CH}^i(S)$$

the $i$-th numerical (i.e., cohomological) and algebraic Chern classes of $E$, respectively.

5.1. Moduli of coherent sheaves on an abelian variety. In this subsection, we recall the construction of $\text{Kum}_a$-type hyper-Kähler manifolds obtained from certain moduli spaces of sheaves on abelian varieties. We will mostly follow [Yo01].

Let $S$ be an abelian surface and $l \in \text{NS}(S)$ an ample cohomology class with $\int_S l^2 = 2n + 2$. Fix a nonzero class $s \in H^4(S, \mathbb{Z})$ so that we have a primitive Mukai vector

$$v = (0, l, s) \in H^*_{\text{even}}(S, \mathbb{Z}). \quad (5.5)$$

Then the moduli space $M$ of stable sheaves on $S$ with Chern character $v$, with respect to a $v$-generic ample line bundle, becomes a smooth projective holomorphic symplectic variety of dimension $\langle v, v \rangle + 2 = 2n + 4$. Denote by $\text{Pic}^l_S$ a connected component of the Picard scheme of $S$ with numerical first Chern class $l$. Yoshioka proved the Albanese variety of $M$ is isomorphic to $S \times \text{Pic}^l_S$, so that we can define the Albanese morphism $\text{Alb} : M \to S \times \text{Pic}^l_S$.

To be more precise, we first need to choose a specific reference line bundle $L_0$ and coherent sheaf $E_0$ on $S$. We choose a line bundle $L_0$ a symmetric ample line bundle in $\text{Pic}^l_S$ (there are precisely 16 of them). Fix a smooth curve $i : C_0 \hookrightarrow S$ in the linear system $|L_0|$ and define a reference coherent sheaf by $E_0 = i_* D$ for a line bundle $D$ on $C_0$ with degree $s + n + 1$. The Riemann–Roch
computation gives \( \text{ch}(E_0) = v \) and \( c_1(E_0) = c_1(L_0) \). Say \( \Sigma : \text{CH}^2(S) \to S(\mathbb{C}) \) is the summation map. The composition

\[
\Sigma \circ i_* \circ c_1 : \text{Pic}^*_{C_0} \to \text{CH}^1(C_0) \to \text{CH}^2(S) \to S(\mathbb{C})
\]

is surjective by Lemma 5.23. Due to this fact, we may choose an appropriate line bundle \( D \) on \( C_0 \) to further assume \( \Sigma(c_2(E_0)) = 0 \) (this again uses Riemann–Roch). Once choosing these reference points, the Albanese morphism can be explicitly described by

\[
\text{Alb} : M \to S \times \text{Pic}^l_S, \quad [E] \mapsto (\zeta(E), \tilde{c}_1(E)), \quad (5.6)
\]

where we define \( \zeta(E) = \Sigma(\tilde{c}_2(E)) \). It sends the reference point \([E_0]\) to the origin \((0,[L_0])\) of \( S \times \text{Pic}^l_S \). The morphism becomes an étale trivial fiber bundle with \( \text{Kum}_n \)-type projective hyper-Kähler manifold fibers. We will work with the central fiber

\[
X = \text{Alb}^{-1}(0,[L_0])
\]

Due to our choice of the Mukai vector \( v \) in (5.5), the above construction further comes with a Lagrangian fibration. Consider a connected component \( \tilde{B} \) of the Chow variety of effective divisors on \( S \) with numerical first Chern class \( l \). Le Potier [LP93] constructed a morphism

\[
\text{Supp} : M \to \tilde{B}, \quad [E] \mapsto [\text{Fitt}_0 E],
\]

where \( \text{Fitt}_0 E \) is the Fitting support of a coherent sheaf \( E \). Finally, consider the Poincaré line bundle \( \mathcal{P} \) on \( S \times \text{Pic}^l_S \), the universal family of line bundles with the numerical Chern class \( l \). Denote by \( r : S \times \text{Pic}^l_S \to \text{Pic}^l_S \) the second projection. Then by Riemann–Roch, \( r_*\mathcal{P} \) is a vector bundle of rank \( n + 1 \). Its projectivization is a Zariski locally trivial \( \mathbb{P}^n \)-bundle

\[
\text{LB} : \tilde{B} \to \text{Pic}^l_S, \quad [C] \mapsto [\mathcal{O}_S(C)].
\]

Gathering all the morphisms together, one easily checks we have a commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{(\zeta, \text{Supp})} & S \times \tilde{B} \\
\text{Alb} \downarrow & & \downarrow \text{id} \times \text{LB} \\
S \times \text{Pic}^l_S & & 
\end{array}
\]

This is an isotrivial family of Lagrangian fibered hyper-Kähler manifolds in the sense of Definition 2.2. Setting \( B = \text{LB}^{-1}([L_0]) = [L_0] \cong \mathbb{P}^n \), we obtain a Lagrangian fibration \( \pi : X \to B \).

**Definition 5.7.** Let \((S,l)\) be a degree \( n + 1 \) polarized abelian surface with a polarization type \((d_1,d_2)\) and \( s \in H^4(S,\mathbb{Z}) \) any nonzero class with \( \gcd(d_1,s) = 1 \). Then the above construction \( \pi : X \to B \) is called the *moduli construction of \text{Kum}_n\)-type* associated to the triple \((S,l,s)\). It is a Lagrangian fibration of a projective hyper-Kähler manifold of \( \text{Kum}_n\)-type to a projective space.

5.2. **Describing the \( H^2\)-trivial automorphisms of \( \text{Kum}_n\)-type moduli constructions.** Recall that [BNWS11] and [HT13] proved that any \( \text{Kum}_n \)-type hyper-Kähler manifold have the group of \( H^2\)-trivial automorphisms

\[
\text{Aut}^0(X) \cong \mathbb{Z}/2 \ltimes (\mathbb{Z}/n + 1)^\oplus 4.
\]

The goal of this subsection is to explicitly describe such automorphisms for the moduli construction. Note that describing such automorphisms is about \( X \) itself but not about the Lagrangian fibration \( \pi : X \to B \). Hence, the Lagrangian fibration plays no role in this subsection.
Recall that we have fixed the origin \( [L_0] \in \text{Pic}_S^1 \), a symmetric ample line bundle on \( S \). By the general theory of abelian varieties, there exists a dual ample line bundle \( \tilde{L}_0 \) on the dual abelian variety \( \tilde{S} \) (see [BL04, §14.4]). The ample line bundles \( L_0 \) and \( \tilde{L}_0 \) induce polarization isogenies
\[
\varphi : S \to \tilde{S}, \quad \tilde{\varphi} : \tilde{S} \to S,
\]
making their compositions the multiplication endomorphisms
\[
[n + 1] : S \xrightarrow{\varphi} \tilde{S} \xrightarrow{\tilde{\varphi}} S, \quad [n + 1] : \tilde{S} \xrightarrow{\tilde{\varphi}} S \xrightarrow{\varphi} \tilde{S}.
\]
(5.8)
Since \( L_0 \) has a polarization type \((d_1, d_2)\), the dual line bundle \( \tilde{L}_0 \) has a polarization type \((d_1, d_2)\) as well. In particular, we have an isomorphism
\[
\ker \tilde{\varphi} \cong (\mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2)^{\oplus 2}.
\]
(5.9)
A closed point \( x \) on \( S \) defines a translation automorphism by \( x \). Our notation for the translation automorphism is
\[
t_x : S \to S, \quad y \mapsto y + x.
\]
A closed point \( \xi \) on \( \tilde{S} \) represents a numerically trivial line bundle on \( S \). Considering \( \xi \) both as a closed point on \( \tilde{S} \) and a line bundle on \( S \) can possibly lead to a confusion. Thus, we will write
\[
P_\xi : \text{numerically trivial line bundle on } S \text{ corresponding to } \xi \in \tilde{S}.
\]
With these notation in mind, we can explicitly realize the \( \text{Aut}^\phi(X) \)-action for the moduli of sheaves construction \( X \).

**Proposition 5.10.** Let \( X \) be a \( \text{Kum}_n \)-type moduli construction associated to a triple \((S, l, s)\) in Definition 5.5. Then

1. We have an isomorphism
   \[
   \text{Aut}^\phi(X) = \{\pm 1\} \ltimes \{(x, \xi) \in S[n + 1] \times \tilde{S}[n + 1] : \varphi(x) = 0, \tilde{\varphi}(\xi) = sx\}.
   \]
2. With the above identification, the \( \text{Aut}^\phi(X) \)-action on \( X \) is defined by
   \[
   (1, x, \xi), [E] = [t_x^*E \otimes P_\xi], \quad (-1, x, \xi), [E] = [t_x^*([-1]^*E) \otimes P_\xi],
   \]
where \([-1] : S \to S\) is the multiplication by \(-1\) automorphism on \( S \).

The rest of this subsection is devoted to the proof of Proposition 5.10. To start, we note that Yoshioka has already computed an explicit trivialization of Albanese morphism \( \text{Alb} : M \to S \times \text{Pic}_S^1 \). Yoshioka’s trivialization is obtained by the base change \([n + 1] : S \times \text{Pic}_S^1 \to S \times \text{Pic}_S^1\), which is a degree \((n + 1)^8\) isogeny. As we will see in a moment, this is not a minimal isogeny in the sense of Definition 4.3. Using the methods in Section 4, we first prove the morphism
\[
\phi : S \times \text{Pic}_S^1 \to S \times \text{Pic}_S^1, \quad (y, [L]) \mapsto (sy - \tilde{\varphi}(L \otimes L_0^{-1}), [L_0 \otimes P_{\phi(y)}])
\]
(5.11)
is the minimal isogeny trivializing the Albanese morphism.

**Proposition 5.12.** The base change \((5.11)\) is the minimal isogeny trivializing the Albanese morphism \( \text{Alb} : M \to S \times \text{Pic}_S^1 \) in the sense of Definition 4.3.

**Proof.** Start from Yoshioka’s diagram [Yos01, §4.1] trivializing the Albanese morphism, which is a cartesian diagram
\[
\begin{array}{ccc}
X \times (S \times \text{Pic}_S^1) & \xrightarrow{\Phi'} & M \\
\downarrow \text{pr}_2 & & \downarrow \text{Alb} \\
S \times \text{Pic}_S^1 & \xrightarrow{[n + 1]} & S \times \text{Pic}_S^1
\end{array}
\]
(5.13)
Here $\Phi' : X \times (S \times \Pic^d_S) \to M$ is defined to be a Galois étale morphism
\[
\Phi'([E], y, [L]) = \left( t^*_{\varphi(L \otimes L_0^{-1})} E \otimes (L \otimes L_0^{-1})^{\otimes s} \otimes P_{-\varphi(y)} \right).
\]
Note that our convention differs by sign to Yoshioka’s original paper, because Yoshioka’s dual line bundle $\hat{L}_0$ differs to ours by sign.

The Galois group of the base change $[n+1]$ is the group of $(n+1)$-torsion points $S[n+1] \times \hat{S}[n+1] \cong (\mathbb{Z}/n + 1)^{\mathbb{Q}}$. By Proposition 4.5, it acts on $X \times (S \times \Pic^d_S)$ by translation on the second factor
\[
(x, \xi, ([E], y, [L]) = ([E], y + x, [L \otimes P_t^d]).
\]
One computes the descent of this action to $M$ via $\Phi'$:
\[
(x, \xi, [E]) = \left( t^*_{\varphi(L \otimes L_0^{-1})} E \otimes P_{s \xi - \varphi(x)} \right).
\]
This is the $S[n+1] \times \hat{S}[n+1]$-action on $X = \Alb^{-1}(0, [L_0])$ in Proposition 4.5. One sees this action is not an effective action, and the kernel of the action is precisely
\[
\{(x, \xi) \in S[n+1] \times \hat{S}[n+1] : \varphi(\xi) = 0, \ s\xi - \varphi(x) = 0\}.
\]

To kill the kernel and obtain an effective action, take a Galois quotient corresponding to the kernel (via the Galois correspondence). This is an isogeny $\psi : S \times \Pic^d_S \to S \times \Pic^d_S$ defined by
\[
\psi(y, [L]) = (\varphi(L \otimes L_0^{-1}), (L \otimes L_0^{-1})^{\otimes s} \otimes P_{-\varphi(y)}).
\]

One can check the morphism $\phi$ in (5.11) is precisely the isogeny making $\phi \circ \psi = [n + 1]$ (here one needs to use (5.8), but we omit the computation). The result is a factorization of (5.13) into the minimal isogeny
\[
\begin{array}{ccc}
X \times (S \times \Pic^d_S) & \xrightarrow{\Phi} & X \times (S \times \Pic^d_S) \\
\downarrow \psi & & \downarrow \psi \\
S \times \Pic^d_S & \xrightarrow{\Phi} & S \times \Pic^d_S
\end{array}
\]
Here our new morphism $\Phi$, Beauville’s minimal split covering of $M$, turns out to have a better form than the original $\Phi'$:
\[
\Phi([E], y, [L]) = \left( t^*_{\varphi(L \otimes L_0^{-1})} E \otimes (L \otimes L_0^{-1}) \right). \tag{5.14}
\]
The claim follows. \hfill \Box

Again thanks to Proposition 4.5, we have a canonical, effective and $H^2$-trivial $\Gal(\phi)$-action on $X$. The Galois group $\Gal(\phi)$ is captured by the kernel of $\varphi$, so we have
\[
\Gal(\phi) = \{(x, \xi) \in S[n+1] \times \hat{S}[n+1] : \varphi(x) = 0, \ \varphi(\xi) = s\xi\}. \tag{5.15}
\]
This explains the isomorphism in Proposition 5.10. The $\Gal(\phi)$-action on the fiber $X$ is obtained via the description of $\Phi$ in (5.14). This explains how we obtained the group action in Proposition 5.10.

We can compute $\Gal(\phi)$ more explicitly.

**Lemma 5.16.** $\Gal(\phi) \cong (\mathbb{Z}/n + 1)^{\mathbb{Q}}$.

**Proof.** Let us compute the group (5.15) explicitly. The expression involves the abelian surfaces $S$ and its dual $\hat{S}$, their $(n + 1)$-torsion points and their polarization isogenies $\varphi$ and $\hat{\varphi}$. Therefore, the expression is independent on the complex structure on $S$ and the question is topological. We may fix polarization bases $H_1(S, \mathbb{Z}) = \mathbb{Z}\{e_1, \cdots, e_4\}$ and $H_1(\hat{S}, \mathbb{Z}) = \mathbb{Z}\{\hat{e}_1, \cdots, \hat{e}_4\}$ so that we can
identify $S = (\mathbb{R}/\mathbb{Z})\{e_1, \cdots, e_4\}$ and \(\tilde{S} = (\mathbb{R}/\mathbb{Z})\{e'_1, \cdots, e'_4\}\). The polarization isogenies with respect to them are

$$\varphi = \begin{pmatrix} 0 & 0 & d_1 & 0 \\ 0 & 0 & 0 & d_2 \\ -d_1 & 0 & 0 & 0 \\ 0 & -d_2 & 0 & 0 \end{pmatrix}, \quad \tilde{\varphi} = \begin{pmatrix} 0 & 0 & d_2 & 0 \\ 0 & 0 & 0 & -d_1 \\ d_2 & 0 & 0 & 0 \\ 0 & d_1 & 0 & 0 \end{pmatrix}. \quad (5.17)$$

Writing the coordinate \((a_1, \cdots, a_4)\) for \(S = (\mathbb{R}/\mathbb{Z})^4\) and \((b_1, \cdots, b_4)\) for \(\tilde{S} = (\mathbb{R}/\mathbb{Z})^4\), one explicitly computes

$$\text{Gal}(\phi) = \{(a_i, b_i)_{i=1}^4 \in \left(\frac{\mathbb{Z}}{n+1}\mathbb{Z}\right)^{\oplus 8} : d_1a_1 = 0, d_2b_3 = sa_1, \cdots \} \cong A^{\oplus 4},$$

where the abelian group \(A\) is defined by

$$A = \{(a, b) \in (\mathbb{Z}/n + 1)^{\oplus 2} : d_1a = 0, d_2b = sa\}.$$

Notice that \(\gcd(d_1, s) = 1\) by the very assumption we had in Definition 5.7. Now \(A \cong \mathbb{Z}/n + 1\) by the following simple computational lemma, and the desired isomorphism is proved. \(\square\)

**Lemma 5.18.** Let \(p, q, s\) be nonzero integers. Set \(m = pq\) and assume either \(\gcd(p, s) = 1\) or \(\gcd(q, s) = 1\). Then the abelian group

$$A = \{(a, b) \in (\mathbb{Z}/m)^{\oplus 2} : pa = 0, qb = sa\}$$

is isomorphic to \(\mathbb{Z}/m\).

**Proof.** The group \(A\) is realized by the kernel of a homomorphism \(f : (\mathbb{Z}/m)^{\oplus 2} \to (\mathbb{Z}/m)^{\oplus 2}\), \(f = (\begin{smallmatrix} p & 0 \\ -s & q \end{smallmatrix})\). Adjusting the bases of both the domain and codomain (i.e., performing elementary row and column operations), the matrix can be transformed into \((\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}) = (\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix})\). Here one needs the assumption \(\gcd(p, s) = 1\) or \(\gcd(q, s) = 1\) to apply the Euclidean algorithm. The claim follows. \(\square\)

We have described \(\text{Gal}(\phi) \cong (\mathbb{Z}/n + 1)^{\oplus 4}\)-action on \(X\) acting trivially on \(H^2\). Since \(\text{Aut}^\circ(X) \cong \mathbb{Z}/2 \times (\mathbb{Z}/n + 1)^{\oplus 4}\), we still need an additional \(\mathbb{Z}/2\)-part to describe. Fortunately, this is not hard to guess. Construct an involution \(\iota\) on \(X \times (S \times \text{Pic}\^1_S)\) by

$$\iota([E], y, [L]) = ([\neg 1]^*E], -y, [-1]^*[L]).$$

Because we are not relying on the general theory anymore, we need to check \(\iota\) acts on \(M\). We omit the typical Chern class computation.

The involution does not commute with the \(S \times \tilde{S}\)-action on \(X \times (S \times \text{Pic}\^1_S)\), and this is the reason why \(\mathbb{Z}/2\) should act on \((\mathbb{Z}/n + 1)^{\oplus 4}\) nontrivially and leads to the semi-direct product. The action descends to \(M\) as a satisfying form

$$\iota([E]) = [-1]^*E].$$

To check \(\iota\) acts on the fiber \(X = \text{Alb}^{-1}(0, [L_0])\), we need to check \(\iota([\neg 1]^*E] = 0\) and \(\tilde{\iota}_1([-1]^*E) = \tilde{\iota}_1(L_0)\) for all \([E] \in X\). The former follows from definition and the latter follows from the fact that \(L_0\) is a symmetric line bundle. It remains to prove \(\iota\) acts on the second cohomology of \(X\) as the identity. We have already proved in Proposition 4.5 that \(H^2(M, \mathbb{Q}) \to H^2(X, \mathbb{Q})\) is surjective. Hence we only need to prove \(\iota\) acts on \(H^2(M, \mathbb{Q})\) as the identity. This follows because \(\iota\) is induced from the automorphism \([-1]\) on \(S\), \([-1]\) acts on \(H^2(S, \mathbb{Q})\) trivially and finally the Hodge structure \(H^2(M, \mathbb{Q})\) is obtained by a tensor construction of \(H^2(S, \mathbb{Q})\) by [28]. This exhausts the entire \(\text{Aut}^\circ(X)\)-action description on \(X\) and hence completes the proof of Proposition 5.10.
5.3. Automorphisms respecting the Lagrangian fibration. With Proposition 5.10 at hand, the proof of Proposition 5.2 becomes fairly straightforward. Any $H^2$-trivial automorphism is of the form

$$f = (\pm 1, x, \xi) \quad \text{for} \quad x \in \ker \varphi, \quad \xi \in \overline{S}[n + 1] \quad \text{with} \quad \varphi(\xi) = sx.$$

Let us first consider automorphisms of the form $f = (1, x, \xi)$. It acts on $X$ by $f : [E] \mapsto [t_*E \otimes P\xi]$. Recall $\pi : X \to B$ was by definition the (Fitting) support map $\text{Supp} : [E] \mapsto [\text{Fitt}_0 E]$. The support of $t_*E \otimes P\xi$ is $\text{Supp} E - x$, so $f$ respects $\pi$ if and only if $\text{Supp} E = \text{Supp} E - x$. This means $x = 0$ and $\xi \in \ker \varphi$. Therefore, such automorphisms form a group $\ker \varphi$, which is isomorphic to $(\mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2)_{\oplus 2}$ by (5.9).

We next consider automorphisms of the form $f = (-1, x, \xi)$. A similar argument shows $f$ respects $\pi$ if and only if $\text{Supp} E = [-1]^* \text{Supp} E - x$ for all $[E] \in X$. In other words, we have $D = [-1]^* D - x$ for all $D \in [L_0]$. Fix any $\frac{1}{2}x \in S$ with $2 \cdot (\frac{1}{2}x) = x$. Then this condition is equivalent to every $D \in [t_*x^* L_0]$ being a symmetric divisor. In particular, $t_*\frac{1}{2^*} L_0$ is a symmetric line bundle. We have chosen $L_0$ to be a symmetric line bundle, so this implies $\frac{1}{2}x$ is a 2-torsion point, or $x = 0$. The condition now becomes that every $D \in [L_0]$ is symmetric.

**Lemma 5.19.** Let $S$ be an abelian surface and $L_0$ a symmetric ample line bundle on it. Then every divisor in the complete linear system $|L_0|$ is symmetric if and only if $L_0$ has a polarization type $(1, 1), (1, 2)$ or $(2, 2)$.

**Proof.** Assume $L_0$ has one of the three given polarization types. When $L_0$ is a principal polarization, $|L_0|$ consists of a single symmetric divisor. When $L_0$ is twice a principal polarization, the statement is proved in [BL04, Thm 4.8.1]. When $L_0$ has a polarization type $(1, 2)$, the statement can be found in [Bar87, Prop 1.6].

Conversely, let us assume every divisor in $|L_0|$ is symmetric. Denote by $H^0(S, L_0)_\pm$ the $\pm 1$-eigenspaces of the involution $[-1]^*$ on $H^0(S, L_0)$. Every divisor in $|L_0|$ is symmetric if and only if either $H^0(S, L_0)_+ = 0$ or $H^0(S, L_0)_- = 0$. The dimensions of $H^0(S, L_0)_\pm$ are computed in [BL04, Ex 4.12.11]: if we let the polarization type of $L_0$ to be $(d_1, d_2)$ then

$$h^0(L_0)_\pm = \frac{1}{2} h^0(L_0), \quad \frac{1}{2} h^0(L_0) \pm 2^{1-s} \quad \text{or} \quad \frac{1}{2} h^0(L_0) \mp 2^{1-s},$$

where $0 \leq s \leq 2$ is an integer where $d_1, \ldots, d_s$ are odd and $d_{s+1}$ is even. There are three possibilities making $h^0(L_0)_+ = 1$ or $h^0(L_0)_- = 0$:

1. $h^0(L_0) = 1$ and $s = 2$;
2. $h^0(L_0) = 2$ and $s = 1$; or
3. $h^0(L_0) = 4$ and $s = 0$.

Using $h^0(L_0) = d_1 d_2$, it is easy to check these are the desired three cases in the statement. \[\square\]

From the lemma, there are only three possible polarization types of $L_0$. We have assumed from the very beginning that $\int_S t^2 = 2n + 2 = 2d_1 d_2$ and $n \geq 2$. The first two cases are thus excluded. The only possible case is when $n = 3$ and $d_1 = d_2 = 2$. This completes the proof of Proposition 5.2.

5.4. The polarization scheme of generalized Kummer varieties. This subsection will be devoted to the proof of Proposition 5.4. Let us keep assume $\pi : X \to B$ is a Kum$_n$-type moduli construction. The computations in this subsection are highly influenced by [Wic18, §6]. Recall from §5.1 that we had a Fitting support morphism $\text{Supp} : M \to \tilde{B}$ over $\text{Pic}^l_S$. Fix a point $[L_0] \in \text{Pic}^l_S$ and consider the fibers of $M$ and $\tilde{B}$ over it. We obtain a morphism

$$\text{Supp} : Y \to B,$$
where $B = |L_0|$ is a complete linear system and $Y \subset M$ consists of torsion coherent sheaves $E$ on $S$ with $\text{ch}(E) = v$ and $\hat{c}_1(E) = \hat{c}_1(L_0)$. The $\text{Kum}_r$-type hyper-Kähler manifold $X$ is obtained by a fiber of the isotrivial fiber bundle $\epsilon : Y \to S$.

Consider the universal family $C \to B$ of curves on $S$ parametrizing effective divisors in $B = |L_0|$. Since $L_0$ is ample, by Bertini there exists a Zariski dense open subset $B_0 \subset B$ parametrizing smooth curves. The restriction of the universal family $C_0 \to B_0$ becomes a smooth projective family of curves. The following lemma is standard and we omit its proof.

**Lemma 5.20.** The morphism $\text{Supp}: Y_0 = \text{Supp}^{-1}(B_0) \to B_0$ is isomorphic to the relative Picard scheme of the universal family of curves $\text{Pic}_{C_0/B_0} \to B_0$ for $d = s + n + 1$.

The lemma in particular says $Y_0 \to B_0$ is a torsor under the numerically trivial relative Picard scheme

$$J_0 = \text{Pic}_{C_0/B_0}^0 \to B_0.$$ 

Since $C_0/B_0$ is a smooth projective family of curves, its relative Picard scheme $J_0$ is a canonically principally polarized abelian scheme. As standard, we will call it a relative Jacobian of the family $C_0/B_0$. The following lemma is standard and we omit its proof.

**Lemma 5.21.** There exists a short exact sequence of abelian schemes over $B_0$:

$$0 \to P_0 \to J_0 \to S \times B_0 \to 0.$$ 

(5.22)

**Proof.** The universal family $C_0 \to B_0$ is a subvariety of the product $i : C_0 \hookrightarrow S \times B_0$. This induces a pullback morphism $i^* : \tilde{S} \times B_0 \to J_0$ between their relative Picard schemes over $B_0$. The morphism $J_0 \to S \times B_0$ can be constructed by the dual of it. Fiberwise, it is the morphism $J_C \to S$ induced by the universal property of the Albanese morphism applied to $i : C \hookrightarrow S$.

We prove the kernel of the morphism $J_0 \to S \times B_0$ is $P_0$. The claim can be verified fiberwise. Fix a closed point $[C] \in B_0$ corresponding to a smooth curve $i : C \to S$. Over it, a closed point of $Y_0$ (resp., $J_0$) is represented by a degree $d$ line bundle $L$ on $C$ (resp., degree 0 line bundle $M$ on $C$). The $J_0$-action on $Y_0$ is given by $[i^*M],[i^*L] = [i^*(L \otimes M)]$. Recall that $X$ is a fiber of the morphism $\epsilon : Y \to S$. Hence the abelian scheme $P_0$ consists of translation automorphisms of $J_0$ invariant under the morphism $\epsilon$. The definition of $\epsilon$ in (5.6). A Riemann–Roch computation gives us

$$\epsilon([i^*(L \otimes M)]) = \epsilon([i^*L]) - \Sigma \epsilon(M),$$

where $\epsilon(M) \in H^1(S)$, $i^* : H^1(S) \to H^2(S)$ and $\Sigma : H^2(S) \to S(\mathbb{C})$ is a summation map. This proves the $[i^*M]$-action on the fiber of $Y_0$ is $\epsilon$-invariant if and only if $\Sigma \epsilon(M) = 0$. The claim follows by the following lemma, which is already proved in [Wie18, (6.8)].

**Lemma 5.22.** The morphism $J_C \to S$ sends a closed point $[M] \in J_C(\mathbb{C})$ to $\Sigma \epsilon(M) \in S(\mathbb{C})$.

The dual of (5.22) is automatically (e.g., [BL04, Prop 2.4.2]) a short exact sequence of abelian schemes

$$0 \to \tilde{S} \times B_0 \to J_0 \to \tilde{P}_0 \to 0.$$ 

(5.24)

In particular, $P_0$ and $\tilde{S} \times B_0$ are both abelian subschemes of a bigger abelian scheme $J_0$. The following proposition describes the polarization scheme $K_0$ more explicitly for the moduli constructions.
Proposition 5.25. We have the following two additional descriptions of the polarization scheme \( K_0 \) as a \( B_0 \)-group scheme:

\[
K_0 = P_0 \cap (\tilde{S} \times B_0) = \ker(\tilde{\varphi} \times \text{id} : \tilde{S} \times B_0 \to S \times B_0).
\]

Proof. Fiberwise at a closed point \([C] \in B_0\), the sequences (5.22) and (5.24) are short exact sequences of abelian varieties

\[
0 \to F \to J_C \to S \to 0, \quad 0 \to \tilde{S} \to J_C \to \tilde{F} \to 0.
\]

Here \( F = \nu^{-1}([C]) \) is a fiber of \( P_0 \) and \( J_C \) is the Jacobian of the curve \( C \). The two abelian subvarieties \( F \) and \( \tilde{S} \) of the principally polarized abelian variety \( J_C \) are the so-called complementary abelian subvarieties (see [Wie18, §6.4] and [BL04, §12.1]). In this case, we have an equality ([BL04, Cor 12.1.4])

\[
\ker(F \to J_C \to \tilde{F}) = F \cap \tilde{S} = \ker(\tilde{S} \to J_C \to S).
\]

We will soon prove in Lemma 5.26 that the composition \( \tilde{S} \to J_C \to S \) is precisely the polarization isogeny \( \tilde{\varphi} \) regardless of the choice of a closed point \([C] \in B_0\). Given this, we obtain a sequence of identities of group schemes

\[
\ker(P_0 \to J_0 \to \tilde{P}_0) = P_0 \cap (\tilde{S} \times B_0) = \ker(\tilde{\varphi} \times \text{id} : \tilde{S} \times B_0 \to S \times B_0).
\]

From the last description and (5.9), this group scheme is a constant group scheme with fibers \((\mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2)^{\oplus 2}\). The first description is describing the polarization scheme \( K_0 \); combine the uniqueness of the polarization in Theorem 3.1 and the computation of polarization types in Theorem 3.22. The claim follows. \( \square \)

Lemma 5.26. The composition \( \tilde{S} \to J_C \to S \) is the polarization isogeny \( \tilde{\varphi} \).

Proof. Denote by \( i : C \to S \) the closed immersion. At the level of first homologies, the composition \( \tilde{S} \to J_C \to S \) becomes a Hodge structure homomorphism

\[
H_1(\tilde{S}, \mathbb{Z}) = H^1(S, \mathbb{Z}) \xrightarrow{i^*} H^1(C, \mathbb{Z}) \xrightarrow{1} H^3(S, \mathbb{Z}) = H_1(S, \mathbb{Z}).
\]

Hence the composition is \( i_* \circ i^* \), which is the multiplication map by \( c_1(\mathcal{O}_S(C)) \in H^2(S, \mathbb{Z}) \). Because we have chosen \([C]\) in a complete linear system \([L_0]\), it is a multiplication by \( c_1(L_0) = l \).

Therefore, the question reduces to the following claim: the dual polarization \( \tilde{\varphi} : \tilde{S} \to S \) is given by \( l \cup - : H^1(S, \mathbb{Z}) \to H^3(S, \mathbb{Z}) \). Again choose polarization bases \( H_1(S, \mathbb{Z}) = \mathbb{Z}\{e_1, \ldots, e_4\} \) and \( H_1(\tilde{S}, \mathbb{Z}) = H^1(S, \mathbb{Z}) = \mathbb{Z}\{e_1^*, \ldots, e_4^*\} \) as in Lemma 5.16. The polarization isogenies \( \varphi \) and \( \tilde{\varphi} \) have the matrix forms (5.17). The ample class \( l \) is the skew-symmetric bilinear map \( \varphi : H_1(S, \mathbb{Z}) \otimes H_1(S, \mathbb{Z}) \to \mathbb{Z} \) considered as an element of \( H^2(S, \mathbb{Z}) \). Hence it is \( l = d_1 e_1^* \wedge e_3^* + d_2 e_2^* \wedge e_4^* \).

We can now explicitly compute the map \( l \cup - : H^1(S, \mathbb{Z}) \to H^3(S, \mathbb{Z}) \):

\[
e_1^* \mapsto d_2 e_1^* \wedge e_2^* \wedge e_3^*, \quad e_2^* \mapsto -d_1 e_1^* \wedge e_2^* \wedge e_3^*,
\]

\[
e_3^* \mapsto -d_2 e_1^* \wedge e_3^* \wedge e_4^*, \quad e_4^* \mapsto d_1 e_1^* \wedge e_3^* \wedge e_4^*.
\]

The Poincaré duality \( H_1(S, \mathbb{Z}) = H^3(S, \mathbb{Z}) \) yields the basis of \( H^3(S, \mathbb{Z}) \):

\[
\{e_2^* \wedge e_3^* \wedge e_4^*, \quad -e_1^* \wedge e_3^* \wedge e_4^*, \quad e_1^* \wedge e_2^* \wedge e_3^*, \quad -e_1^* \wedge e_2^* \wedge e_4^*\}.
\]

With respect to it, the matrix form of the multiplication coincides with precisely the matrix form of \( \tilde{\varphi} \) above. (Compare this lemma with [Wie18, Lem 6.14].) \( \square \)
Proof of Proposition 5.4. Recall from §5.3 the complete description of $\text{Aut}^\circ(X/B)$. Let us assume $n \neq 3$ or $(d_1, d_2) \neq (2, 2)$, so that every automorphism $f \in \text{Aut}^\circ(X/B)$ is of the form $(1, 0, \xi)$ for $\xi \in \ker \varphi$. It acts on $Y$ by $[E] \mapsto [E \otimes P_2]$, where $P_2$ is the numerically trivial line bundle on $S$ represented by $\xi \in \ker \varphi \subset S$. On $Y_0$, closed points are of the form $[E] = [L_i]$ where $L$ is a line bundle on a smooth curve $i : C \to S$. Hence $f$ acts on it by

\[ f.[i_*L] = [i_*L \otimes P_2] = [i_*(L \otimes i^*P_2)]. \]

This means the global section of $J_0 \to B_0$ defined by $f$ represents a line bundle $[i^*P_2]$ over $[C] \in B_0$. The inclusion $\tilde{S} \times B_0 \subset J_0$ was by definition the pullback morphism of line bundles. Hence $f$ defines in fact a global section $\xi = [P_2] \in \tilde{S}$ of the constant group scheme $\tilde{S} \times B_0$. This coincides with the description of the polarization scheme $K_0$ in Proposition 5.25, proving the desired $K_0 = \text{Aut}^\circ(X/B)$.

The proof for the exceptional case $n = 3$ and $d_1 = d_2 = 2$ goes identical. The only difference is that the automorphisms $f \in \text{Aut}^\circ(X/B)$ of the form $(1, 0, \xi)$ consist of an index 2 subgroup of $\text{Aut}^\circ(X/B)$. So this case proves $K_0 \subset \text{Aut}^\circ(X/B)$ as an index 2 subgroup. The second inclusion $\text{Aut}^\circ(X/B) \subset K_0[2]$ follows from Proposition 3.12 since we have $\text{div}(h) = d_1 = 2$. □

6. THE DUAL LAGRANGIAN FIBRATION OF A COMPACT HYPER-KÄHLER MANIFOLD

Combining the previous results, we can prove the polarization scheme extends to a constant subgroup scheme of $\text{Aut}^\circ(X/B)$ over $B$ for known hyper-Kähler manifolds.

Theorem 6.1. Let $\pi : X \to B$ be a Lagrangian fibration of a compact hyper-Kähler manifold of $K3^{[n]}$, $\text{Kum}_n$, $\text{OG}10$ or $\text{OG}6$-type. Then the polarization scheme $K_0 \to B_0$ uniquely extends to a constant group scheme $K \to B$ that is a subgroup scheme of the constant group scheme $\text{Aut}^\circ(X/B)$.

Proof. When $X$ is of $K3^{[n]}$ or $\text{OG}10$-type, both the polarization scheme $K$ and the global sections defined by $\text{Aut}^\circ(X/B)$ are the zero section of the abelian scheme $P_1$. Hence the claim is trivial. When $X$ is of $\text{OG}6$-type, lattice theory forces $\text{div}(h) = 1$ as shown in [MR21, Lem 7.1]. Proposition 3.12 applies and we get an inclusion $\text{Aut}^\circ(X/B) \hookrightarrow K_0$. Combining Theorem 3.22 and 5.1, the inclusion is forced to be an equality fiberwise. Hence we get the global equality $K_0 = \text{Aut}^\circ(X/B)$. In particular, $K_0$ extends over $B$ to a constant group scheme $\text{Aut}^\circ(X/B)$.

Assume $X$ is of $\text{Kum}_n$-type and the polarization type of $\pi$ is not $(1, 2, 2)$. In this case, Proposition 5.4 together with Proposition 3.10 implies an equality of group schemes $K_0 = \text{Aut}^\circ(X/B)$. The remaining case is when $X$ is of $\text{Kum}_3$-type and the polarization type of $\pi$ is $(1, 2, 2)$. In this case, we have $\text{div}(h) = 2$ so Proposition 3.10 guarantees $\text{Aut}^\circ(X/B) \subset K_0[2]$, where $K_0[2] = \ker(2\Lambda)$ is slightly bigger than $K_0$. Both $\text{Aut}^\circ(X/B)$ and $K_0 = 2 : K_0[2]$ contained in $K_0[2]$ are invariant under deformations, so the inclusion $K_0 \subset \text{Aut}^\circ(X/B)$ in Proposition 5.4 is preserved under deformation. The claim follows. □

Remark 6.2. We may state Theorem 6.1 in the following simpler way: we have an equality of group schemes

\[ K_0 = \text{Aut}'(X/B) \quad (:= \text{Aut}^\circ(X/B) \cap \text{Aut}'(X)), \]

where $\text{Aut}'(X) \subset \text{Aut}^\circ(X)$ is a group defined in Remark 4.6. For most of the known examples of Lagrangian fibered hyper-Kähler manifolds, we have $\text{Aut}'(X/B) = \text{Aut}^\circ(X/B)$. There is a single known example where the inclusion $\text{Aut}'(X/B) \subset \text{Aut}^\circ(X/B)$ is strict, when $X$ is of $\text{Kum}_3$-type and $\pi$ has the polarization type $(1, 2, 2)$. In this case, $\text{Aut}'(X/B) \cong (\mathbb{Z}/2)^{\oplus 4}$ and $\text{Aut}^\circ(X/B) \cong (\mathbb{Z}/2)^{\oplus 5}$.

A direct consequence of this theorem is a promised compactification of the dual torus fibration $\tilde{\pi} : \tilde{X}_0 \to B_0$. 

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Theorem 6.3. Let \( \pi : X \rightarrow B \) be a Lagrangian fibration of a compact hyper-Kähler manifold of K\(^{[n]}\), Kum\(_n\), OG10 or OG6-type. Then

\[ \tilde{\pi} : \tilde{X} \rightarrow B \quad \text{for} \quad \tilde{X} = X/K \]

defines a compactification of the dual torus fibration \( \tilde{\pi} : \tilde{X}_0 \rightarrow B_0 \).

Proof. As explained in the introduction, we have defined the dual torus fibration by \( \tilde{X}_0 = X_0/K_0 \). For known deformation types, Theorem 6.1 proved that \( K_0 \) extends to a constant group scheme \( K \) over \( B \) acting on \( X \). Therefore, the group scheme quotient \( X_0/K_0 \rightarrow B_0 \) can be compactified into \( X/K \rightarrow B \). Since \( K \rightarrow B \) is a constant group scheme, the quotient \( X/K \) may be considered either as a group scheme quotient over \( B \) or a finite group quotient over \( C \).

When \( X \) is of K\(^{[n]}\) or OG10-type, \( \tilde{X} \) is identical to \( X \) and there is nothing more to say. Let us study more on the space \( \tilde{X} \) when \( X \) is of Kum\(_n\) or OG6-type. Being a quotient by \( H^2 \)-trivial automorphisms, \( \tilde{X} \) inherits many interesting properties from \( X \). We provide an appendix B to collect their properties in a more general setup; the following proposition is a direct consequence of this more general discussion. For definitions of a primitive symplectic orbifold and irreducible symplectic variety used in the following proposition, see Appendix A.

Proposition 6.4. Keep the notation from Theorem 6.3, and assume \( X \) is either of Kum\(_n\) or OG6-type. Then

1. \( \tilde{X} \) is a compact primitive symplectic orbifold and also an irreducible symplectic variety.
2. \( \tilde{X} \) does not admit a symplectic resolution.
3. \( \tilde{X} \) is simply connected. It has the Fujiki constant \( c_X = 1/c_{\tilde{X}} \).
4. \( H^2(\tilde{X}, \mathbb{Q}) \) and \( H^2(X, \mathbb{Q}) \) are Hodge isomorphic and Beauville–Bogomolov isometric.
5. The LLV algebras and Mumford–Tate algebras of \( X \) and \( \tilde{X} \) are isomorphic.
6. The pullback \( H^*(\tilde{X}, \mathbb{Q}) \rightarrow H^*(X, \mathbb{Q}) \) is an injective map of LLV structures.
7. If \( X \rightarrow \text{Def}(X) \) is the universal deformation of \( X \), then \( \tilde{X}/K \rightarrow \text{Def}(X) \) is the (locally trivial) universal deformation of \( \tilde{X} \).

Proof. Everything is a direct consequence of Proposition B.2 and B.3. Only the first three items need further explanations. For the first and second items, it is enough to show \( \text{codim} X'^f \geq 4 \) for all \( f \in K \setminus \{ \text{id} \} \). We will see later in Lemma 7.5 that the fixed loci of \( H^2 \)-trivial automorphisms deform when \( X \) deform. Hence we may prove this for any model in the deformation class on \( X \). For OG6, the fixed loci are computed in [MW17, §6]; they are either K3 surfaces or points. For Kum\(_n\), the fixed loci are computed in [Ogn20, Lem 3.5], and similarly one can deduce their codimension is always \( \geq 4 \). For the third item, simply notice the group \( K \) has order \( c_{\tilde{X}}^2 \) in all cases.

The proposition shows \( \tilde{X} \) has quotient singularities when \( X \) is of Kum\(_n\) or OG6-type. Therefore, \( \tilde{X} \) cannot be homeomorphic to \( X \). We call the corresponding \( \tilde{X} \) in each case the dual Kummer variety and dual OG6, respectively.

Finally, the proposition shows in particular the local deformation behavior and period domains of \( X \) and \( \tilde{X} \) are identical. Therefore, one can still apply the method in [GTZ13, §2] at the level of period domains and obtain similar conclusions for all known deformation types of hyper-Kähler manifolds. One subtlety here is that the quotient construction works for any deformation \( X' \) of \( X \), even if \( X' \) does not admit any Lagrangian fibration; the quotient \( X'/K \) is still well-defined because we have considered \( K \) as an abstract subgroup of the group \( \text{Aut}^0(X) \). The local universal deformation space of the Lagrangian fibration \( \pi : X \rightarrow B \) is a hyperplane \( \text{Def}(X, H) \subset \text{Def}(X) \) (see [Mat16]). Once
we choose a deformation $X'$ by respecting the Lagrangian fibration $[\pi' : X' \to B'] \in \text{Def}(X, H)$, we can say $\tilde{\pi}' : X'/K \to B'$ is the dual Lagrangian fibration of $\pi' : X' \to B'$.

7. Example: the dual Kummer fourfolds

To illustrate the geometry of dual Lagrangian fibrations more concretely, we focus on the simplest nontrivial case of Theorem 6.3: when $X$ is of Kum$_2$-type. Throughout, we let $X$ to be a Kum$_2$-type hyper-Kähler fourfold and $\pi : X \to B = \mathbb{P}^2$ its Lagrangian fibration. We will use all the results in previous sections without mentioning them explicitly. We write for simplicity $K = \text{Aut}^0(X/B)$.

There exist isomorphisms $\text{Aut}^0(X) \cong \mathbb{Z}/2 \ltimes (\mathbb{Z}/3)^{\oplus 4}$ and $K \cong (\mathbb{Z}/3)^{\oplus 2}$. Again for simplicity, we call $f \in \text{Aut}^0(X)$ a translation if $f$ does not contain a $\mathbb{Z}/2$-part, and call $f$ an involution if $f$ has a nontrivial $\mathbb{Z}/2$-part. If $f \neq \text{id}$ is a translation (resp., involution) then it has order 3 (resp., order 2). There are precisely 81 translations and 81 involutions. The subgroup $K \subset \text{Aut}^0(X)$ consists of 9 translations respecting the Lagrangian fibration. We define the dual Kummer fourfold by $\tilde{\pi} : \tilde{X} = X/K \to B$. The main result of this section will be Proposition 7.2. It collects some more precise geometric and cohomological descriptions of $\tilde{X}$. Similar method may apply to the OG6-type and higher dimensional Kum$_n$-types.

We will use the notion of the LLV structure to describe the cohomology of $\tilde{X}$. To do so, we first need to review the LLV structure of the generalized Kummer fourfolds (following [GKL-R22]). Recall that the Beauville–Bogomolov quadratic space of $X$ (and hence $\tilde{X}$) is isomorphic to $(H^2(X, \mathbb{Q}), q) \cong U^{\oplus 4} \oplus \langle -6 \rangle$ where $U$ denotes the hyperbolic plane $(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$. For simplicity, we denote $\tilde{V} = H^2(X, \mathbb{Q})$ and its Mukai completion by

$$(V, q) = (\tilde{V}, q) \oplus U \quad (\cong U^{\oplus 4} \oplus \langle -6 \rangle).$$

Set $\mathfrak{g} \cong \mathfrak{so}(V, q)$ and $\tilde{\mathfrak{g}} \cong \mathfrak{so}(\tilde{V}, q)$ the LLV algebra and reduced LLV algebra of $X$ (and $\tilde{X}$). It is a split semisimple $\mathbb{Q}$-Lie algebra. Associated to any dominant weight $\mu \in \mathfrak{g}$, there exists an irreducible $\mathfrak{g}$-module $V_\mu$ over $\mathbb{Q}$. The LLV structure of $X$ is explicitly computed in [LL97, (4.7)]; we have an isomorphism of $\mathfrak{g}$-modules

$$H^*(X, \mathbb{Q}) \cong V_{(2)} \oplus 80\mathbb{Q} \oplus V_{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})}.$$  

It is sometimes convenient to consider the reduced LLV structure on the fixed degree cohomologies $H^k(X, \mathbb{Q})$. The LLV structure restricts to the reduced LLV structure on the middle cohomology: there exists an isomorphism of $\mathfrak{g}$-modules

$$H^4(X, \mathbb{Q}) \cong \tilde{V}_{(2)} \oplus 81\mathbb{Q}.$$  

Let us now state the main result of this section.

**Proposition 7.2.** Let $\pi : X \to B$ be a Lagrangian fibration of a Kum$_2$-type hyper-Kähler fourfold and $\tilde{\pi} : \tilde{X} \to B$ its dual fibration. Then (in addition to Proposition 6.4)

1. $\tilde{X}$ has precisely 36 isolated cyclic quotient singularities of type $\frac{1}{2}(1, 1, 2, 2)$.
2. $\tilde{X}$ (and any of its deformation) contains 9 smooth K3 surfaces. Each of them passes through 4 singularities of $\tilde{X}$. The image of each K3 surface by $\tilde{\pi}$ is a line in $B = \mathbb{P}^2$.
3. The LLV decomposition of the cohomology of $\tilde{X}$ is

$$H^*(\tilde{X}, \mathbb{Q}) \cong V_{(2)} \oplus 8\mathbb{Q} \oplus V_{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})}.$$
The rest of the section will be devoted to the proof of Proposition 7.2. The following proposition claims any Lagrangian fibered Kum\textsubscript{2}-type hyper-Kähler manifolds are deformation equivalent. The idea originates from the results of Markman (e.g., [Mar14, Prop 1.7]).

**Lemma 7.3.** Any Lagrangian fibration of a Kum\textsubscript{2}-type hyper-Kähler fourfold \( \pi: X \to B \) is deformation equivalent to each other.

**Proof.** The polarization type of \( \pi \) is \((1, 3)\) by Theorem 3.22. Setting \( h \in H^2(X, \mathbb{Z}) \) to be an associated cohomology class of \( \pi^*\mathcal{O}_B(1) \), its divisibility \( \text{div}(h) \) is 1 by [Wie18, Thm 1.1]. We can now imitate the method of [MR21, §7]. The lattice theory result in [MR21, Lem 2.6] forces any two primitive isotropic elements \( h, h' \) with divisibility 1 in \( H^2(X, \mathbb{Z}) \) are monodromy equivalent. We can imitate the proof of [MR21, Thm 7.2] (or the proof of [Mar14, Prop 1.7]) and show that any pairs \( (X, H) \) with a primitive isotropic \( H \) with divisibility 1 are deformation equivalent. This proves the claim. □

Thanks to this proposition, we can often specialize our discussion to a single model. Our explicit model for Lagrangian fibered Kum\textsubscript{2}-type hyper-Kähler manifolds is the following example presented in [Mat15, §2]. Let \( E \) and \( E' \) be elliptic curves and \( S = E' \times E \) be an abelian surface. Consider a commutative diagram

\[
\begin{array}{ccc}
S^{[3]} & \xrightarrow{(\text{Sop}_{\text{pr}_1}, \text{pr}_2)} & (E' \times E)^{(3)} \\
\text{Alb} & & \downarrow \Sigma \\
E' \times E & \xleftarrow{\text{id} \times \Sigma} &
\end{array}
\]

(7.4)

where \( \text{pr}_1: S^{[3]} \to (E')^{(3)} \) and \( \text{pr}_2: S^{[3]} \to E^{(3)} \) are the coordinate projections and \( \Sigma \) are the summation maps. By the discussion we had in Section 5, this is an isotrivial family of Lagrangian fibered hyper-Kähler manifolds of Kum\textsubscript{2}-type. The advantage of this construction to the moduli construction in Section 5 is that this gives us an honest generalized Kummer fourfold, and thus we can use the computational results in [HT13] and [Ogu20].

Let us also recall some known facts about the fixed loci of \( H^2 \)-trivial automorphisms.

**Lemma 7.5.** Let \( X \) be a compact hyper-Kähler manifold and \( G \subset \text{Aut}_{\mathbb{C}}(X) \) any subgroup. If \( X' \) is deformation equivalent to \( X \) then \( (X')^G \) is deformation equivalent to \( X^G \).

**Proof.** Let \( p: \mathcal{X} \to \text{Def}(X) \) be a universal deformation of \( X \). Since \( G \) acts fiberwise on \( p \), the morphism \( \mathcal{X}^G \to \text{Def}(X) \) gives a family of fixed loci \( (X_t)^G \). Because \( G \) is a finite group acting on a complex manifold \( \mathcal{X} \), its fixed locus \( \mathcal{X}^G \) is a complex manifold. Similarly, each \( (X_t)^G \) is a complex (symplectic) manifold. Hence \( \mathcal{X}^G \to \text{Def}(X) \) is a smooth proper family and the claim follows. □

**Lemma 7.6.** Let \( X \) be a Kum\textsubscript{2}-type hyper-Kähler manifold and \( f \in \text{Aut}_{\mathbb{C}}(X) \) its \( H^2 \)-trivial automorphism.

1. If \( f \) is an involution then its fixed locus \( X^f \) is a disjoint union of a K3 surface and 36 points.
2. If \( f \neq \text{id} \) is a translation then its fixed locus \( X^f \) consists of 27 points.

**Proof.** When \( X \) is an honest generalized Kummer fourfold the statements were proved in [Ogu20, Lem 3.5], [HT13, Thm 4.4] and [KM18, Thm 7.5]. If we deform \( X \) then the fixed locus \( X^f \) also deforms by Lemma 7.5. □
Following [HT13, Thm 4.4], any Kum2-type hyper-Kähler manifold must always contain 81 K3 surfaces obtained by the fixed loci of 81 involutions (this was first observed in [KV98, §6]). The 81 K3 surfaces are related by 81 translations, and represent the 81 trivial reduced LLV classes in (7.1). With these backgrounds, we can begin the proof of Proposition 7.2.

Proof of Proposition 7.2. Proposition B.2 says the singularity locus of \( \tilde{X} \) is the image of the set \( S = \bigcup_{f \in \mathcal{G} \setminus \{\text{id}\}} X^f \). The set \( X^f \) consists of 27 points for \( f \neq \text{id} \) by the lemma above. This means \( S \) consists of \( 27 \times 4 = 108 \) points. The quotient map \( p : X \to \tilde{X} \) identifies 3 points to a single point of \( \mathbb{Z}/3 \)-quotient singularity. This proves there are \( 108/3 = 36 \mathbb{Z}/3 \)-quotient singularities.\(^2\)

Any symplectic \( \mathbb{Z}/3 \)-quotient singularity must be of type \( \frac{1}{3}(1, 1, 2, 2) \). By [Pri67, Prop 6], the \( \mathbb{Z}/3 \)-action is locally biholomorphic to a symplectic linear action on \( \mathbb{C}^4 \) around 0. Its eigenvalues are either 1, \( \zeta \) and \( \zeta^2 \) where \( \zeta \) is the third primitive root of unity. In this case, 1 cannot arise because the fixed locus of the action should be the origin. Hence there are five possibilities of the linear action up to conjugate, and one easily checks \( \text{diag}(\zeta, \zeta, \zeta^2, \zeta^2) \) is the only symplectic linear map among them (for some symplectic form).

For the second item, notice first that the 81 K3 surfaces in \( X \) are identified into 9 K3 surfaces in \( \tilde{X} \). Let us deform the Lagrangian fibration and assume we are in the construction (7.4) (Lemma 7.3). The 81 K3 surfaces are explicitly described in this case by [HT13, Thm 4.4]. One explicitly computes each of 81 K3 surface passes through four points in \( \bigcup_{f \in \mathcal{G} \setminus \{\text{id}\}} X^f \), and the four points are not identified by the quotient map \( p \). Hence each of nine K3 surfaces in \( \tilde{X} \) passes through four singular points of \( \tilde{X} \). (Note: the nine K3 surfaces in \( \tilde{X} \) do intersect each others, but the intersections are smooth points in \( \tilde{X} \).) One finally checks the image of each K3 surface under \( \pi \) can be considered as a sublinear system in \( \mathbb{P}^2 \), so it is a line.

For the last item, recall \( H^*(\tilde{X}, \mathbb{Q}) = H^*(X, \mathbb{Q})^K \). The translations in \( K \) act trivially on \( H^2(X, \mathbb{Q}) \) by definition and trivially on \( H^3(X, \mathbb{Q}) \) by the computations in [Ogu20, §3]. Hence we only need to prove \( H^4(X, \mathbb{Q})^K \cong \hat{V}(2) \oplus 9\mathbb{Q} \). The Verbitsky component is preserved by \( K \), so \( \hat{V}(2) \) is \( K \)-invariant. Again recall the 81 trivial reduced LLV classes in \( H^4(X, \mathbb{Q}) \) were represented by 81 K3 surfaces which are bound by 81 translation automorphisms. Since only 9 of them survives in \( \tilde{X} \), the fourth cohomology is as desired.\( \square \)

Appendix A. Various notions of singular hyper-Kähler varieties

Many of the important properties of compact hyper-Kähler manifolds have been generalized to singular settings. There are several definitions of singular hyper-Kähler varieties in the current literature. To make our discussion less ambiguous, we collect some definitions and compare them. Our main references are [BL22], [Sch20] and [Men20].

If \( X \) is a normal complex space, then its sheaf of reflexive \( k \)-forms is defined to be the reflexive closure of the sheaf of \( k \)-forms \( \Omega^{[k]}_X = (\Omega^k_X)^{\vee\vee} \), or equivalently \( \Omega^{[k]}_X = j_*\Omega^k_{X_{\text{reg}}} \) where \( j : X_{\text{reg}} \hookrightarrow X \) is the smooth locus of \( X \). A quasi-étale morphism is a morphism étale outside of a codimension \( \geq 2 \) closed subvariety.

Definition A.1 ([BL22, Def 3.1], [Sch20, Def 1], [Men20, Def 3.1]). Let \( X \) be a compact normal Kähler space and \( \sigma \in H^0(X, \Omega^2_X) \) a reflexive 2-form.

(1) \((X, \sigma)\) is called a symplectic variety if \( X \) has rational singularities and \( \sigma \) is nondegenerate on \( X_{\text{reg}} \).

\(^2\)Our original computation was incorrect. This was pointed out in [BS22, Ex 3.6].
(2) $X$ is called a **primitive symplectic variety** if

$$H^0(X, \Omega^1_X) = 0, \quad H^0(X, \Omega^2_X) = \mathbb{C} \sigma,$$

and $(X, \sigma)$ is a symplectic variety.

(3) $X$ is called an **irreducible symplectic variety** if it is a primitive symplectic variety with the following condition: for any finite quasi-étale cover $f : X' \to X$, we have

$$H^0(X', \Omega^{2k+1}_{X'}) = 0, \quad H^0(X', \Omega^{2k}_{X'}) = \mathbb{C} \cdot f^* \sigma^k \quad \text{for} \quad k \geq 0.$$

(4) $X$ is called a **Namikawa symplectic variety** if it is a $\mathbb{Q}$-factorial and terminal primitive symplectic variety.

(5) $X$ is called a **primitive symplectic orbifold** if it is Namikawa symplectic with only finite quotient singularities.

We have a series of implications

$$\text{primitive symplectic orbifold } \Rightarrow \text{Namikawa symplectic } \Downarrow \quad \text{irreducible symplectic } \longrightarrow \text{primitive symplectic } \longrightarrow \text{symplectic}.$$ 

Eventually, the dual hyper-Kähler variety $\breve{X}$ in Theorem 1.1 will be both a primitive symplectic orbifold and an irreducible symplectic variety (Proposition 6.4). Hence all of the discussions here apply.

Many of the interesting properties of compact hyper-Kähler manifolds generalize to their singular analogues, especially to primitive symplectic varieties. We highlight some of their properties that will be useful to our discussion. Let $X$ be a primitive symplectic variety of dimension $2n$.

- The normalization of the singular locus $X_{\text{sing}}$ is again symplectic [Kal06]. In particular, $X_{\text{sing}}$ is always even dimensional.
- There exist a notion of the Beauville–Bogomolov form and Fujiki constant of $X$, so that the Fujiki relation (1.3) holds [Sch20, Thm 2] [BL22, Prop 5.20].
- $X$ is Namikawa symplectic if and only if it is $\mathbb{Q}$-factorial and codim $X_{\text{sing}} \geq 4$ [Nam01] [BL22, Thm 3.4].
- Every morphism $\pi : X \to B$ with connected fibers to a normal base $B$ (with $0 < \dim B < 2n$) is a Lagrangian fibration [Sch20, Thm 3]. That is, all the irreducible components of the fibers of $\pi$ are Lagrangian subvarieties of $X$.
- The Hodge structure $H^2(X, \mathbb{Z})$ is pure [Sch20, Thm 8] [BL22, Cor 3.5]. If $X$ is a primitive symplectic orbifold, then the full cohomology $H^*(X, \mathbb{Q})$ is a pure Hodge structure.
- There exists a universal locally trivial deformation $\mathcal{X} \to \text{Def}^{lt}(X)$ over a smooth complex germ $\text{Def}^{lt}(X)$ of dimension $h^{1,1}(X)$ [BL22, Thm 4.7]. If $X$ is Namikawa symplectic, then any deformation is automatically locally trivial [Nam06].
- The local Torelli theorem holds for $\text{Def}^{lt}(X)$. In fact, global Torelli theorem holds in a suitable form [BL22].

We will use these facts in Section 6 and Appendix B, without mentioning them explicitly.

**Appendix B. Quotient of a hyper-Kähler manifold by $H^2$-trivial automorphisms**

Let $X$ be a compact hyper-Kähler manifold and $\text{Aut}^\circ(X)$ the finite group of $H^2$-trivial automorphisms. Throughout the appendix, we always let $G \subset \text{Aut}^\circ(X)$.
to be any subgroup and write

$$p : X \to \bar{X} = X/G.$$  \hspace{1cm} (B.1)

The goal of this appendix is to gather basic geometric and cohomological properties of the quotient $\bar{X}$. Note that Lagrangian fibrations play no role in this appendix. The main results are Proposition B.2 and B.3.

**Proposition B.2.** Consider the quotient (B.1) of a compact hyper-Kähler manifold $X$.

1. The morphism $p$ is a finite quasi-étale symplectic quotient.
2. $\bar{X}$ is a $\mathbb{Q}$-factorial irreducible symplectic variety whose singularity locus is $p\left( \bigcup_{f \notin G \setminus \{\text{id}\}} X^f \right)$.
   
   If $\text{codim } X^f > 2$ for all $f \notin G \setminus \{\text{id}\}$, then $\bar{X}$ is also a primitive symplectic orbifold.
3. $\bar{X}$ is simply connected.
4. If $X \to \text{Def}(X)$ is the universal deformation of $X$ then the quotient $X/G \to \text{Def}(X)$ becomes the universal locally trivial deformation of $\bar{X}$.

The quotient $\bar{X} = X/G$ being an irreducible symplectic variety, its behavior is intimately related to its (second) cohomology. To talk about the precise cohomological behavior of $\bar{X}$, we first need to fix the Beauville–Bogomolov form; the Beauville–Bogomolov form is a priori only defined up to scalar. We define a symmetric bilinear form $q_{\bar{X}} : H^2(\bar{X}, \mathbb{Z}) \otimes H^2(\bar{X}, \mathbb{Z}) \to \mathbb{Z}$ by

$$q_{\bar{X}}(x, y) = q_X(p^*x, p^*y) \quad \text{for } x, y \in H^2(\bar{X}, \mathbb{Z}).$$

The reader should be aware that $q_{\bar{X}}$ may be a non-primitive bilinear form with this definition.

**Proposition B.3.** Notation as above.

1. $q_{\bar{X}}$ is a Beauville–Bogomolov form of $\bar{X}$. The Fujiki constant of $\bar{X}$ is $c_{\bar{X}} = c_X/|G|$.
2. The pullback
   
   $$p^* : H^2(\bar{X}, \mathbb{Z})/(\text{torsion}) \to H^2(X, \mathbb{Z})$$

   is an injective Hodge structure homomorphism and a Beauville–Bogomolov isometry. It is an isomorphism over $\mathbb{Q}$.
3. The LLV algebra of $X$ and $\bar{X}$ are canonically isomorphic. Denoting them by $\mathfrak{g}$, the pullback
   
   $$p^* : H^*(\bar{X}, \mathbb{Q}) \to H^*(X, \mathbb{Q})$$

   is an injective $\mathfrak{g}$-module homomorphism.
4. For all $k$, the special Mumford–Tate algebra of $H^k(\bar{X}, \mathbb{Q})$ is isomorphic to that of $H^2(X, \mathbb{Q})$.

   As a consequence, any $\mathfrak{g}$-module decomposition of $H^*(\bar{X}, \mathbb{Q})$ is a pure Hodge structure decomposition.

Note again that the subgroup $G \subset \text{Aut}^0(X)$ was taken arbitrary. Hence we have a family of irreducible symplectic varieties corresponding to each subgroup of $\text{Aut}^0(X)$. That is, we get a Galois correspondence between the subgroups $G \subset \text{Aut}^0(X)$ and the symplectic quotients $\bar{X} = X/G$ with the same rational Beauville–Bogomolov forms. In particular, their deformation behaviors are all identical.

The rest of this appendix is devoted to the proof of Proposition B.2 and B.3. Most of the proofs will be straightforward so we will be brief.

**Proof of Proposition B.2: Part 1.** Let us present the proof of the theorem without the second item. The second item will be proved separately in Part 2.
The group $G$ acts trivially on $H^2(X, \mathbb{Z})$, so it acts symplectically on $X$. Hence $p$ is a symplectic quotient. The ramified locus of $p$ is contained in the union of the fixed loci $\bigcup_{f \in G \setminus \{id\}} X^f$, which is of codimension $\geq 2$. This means $p$ is quasi-étale and the first item follows.

The third item is a direct consequence of the second item, because any irreducible symplectic variety is simply connected by [GGK19, Cor 13.3]. The last item again follows directly from [Fuj83, Thm 3.5, Lem 3.10]. Since $G$ acts on $X$ holomorphically and trivially on $H^2(X, \mathbb{Z})$, $X \to \text{Def}(X)$ equipped with a $G$-action has universal deformation of the pair $(X, G)$. Once we have a universal deformation of the pair $(X, G)$, the quotient $X/G \to \text{Def}(X)$ is the locally trivial universal family of $X/G$. \hfill \Box

**Lemma B.4.** Let $(X, \sigma)$ be a compact symplectic variety and $f : X' \to X$ a finite quasi-étale morphism. Then $(X', f^*\sigma)$ is a compact symplectic variety.

**Proof.** By [KM98, Prop 5.20] or [GKP16, Rmk 3.4], $X'$ is Gorenstein and canonical. Therefore, it has rational singularities by [KM98, Cor 5.24]. Now $f^*\sigma \in H^0(X', \Omega^2_{X'})$ is a symplectic form in codimension 1 as $f$ is étale in codimension 1. The claim follows. \hfill \Box

**Proof of Proposition B.2:** Part 2. We prove the second item here. As a finite quotient of a smooth variety $X$, the space $\tilde{X}$ is certainly $\mathbb{Q}$-factorial and has quotient singularities. Fix a point $x \in X$ and let $\tilde{x} = p(x)$. According to the Chevalley–Shephard–Todd theorem, the quotient $\tilde{X}$ is smooth at $\tilde{x}$ if and only if the stabilizer group $G_x$ acting on the tangent space $T_{\tilde{x}} \tilde{X}$ is generated by pseudoreflections (i.e., linear automorphisms on $T_{\tilde{x}} \tilde{X}$ with codimension 1 fixed loci). If $x \in X$ has a nontrivial stabilizer $G_x$, any nontrivial automorphism $f \in G_x$ is symplectic so has codimension $\geq 2$ fixed locus. This means $G_x$ cannot be generated by pseudoreflections. Therefore, $\tilde{X}$ is singular at $\tilde{x}$. If we further assume $\text{codim} X^f \geq 4$ for all nontrivial $f \in G$, then $X_{\text{sing}}^f \geq 4$ and $\tilde{X}$ becomes Namikawa symplectic.

To prove $\tilde{X}$ is irreducible symplectic, we follow the argument of Matsushita [Mat15, Lem 2.2]. Let $f : \tilde{Y} \to \tilde{X}$ be an arbitrary finite quasi-étale morphism. Consider the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{q} & X \\
\downarrow{g} & & \downarrow{p} \\
\tilde{Y} & \xrightarrow{f} & \tilde{X}
\end{array}
\]

where $Y$ is the normalization of the fiber product $X \times_{\tilde{X}} \tilde{Y}$. We claim $g$ and $q$ are finite quasi-étale. The finiteness is clear, so we concentrate on their quasi-étaleness. Notice that the quasi-étale property is stable under base change, so we need to prove the normalization in this case is quasi-étale. But notice that $X$ is smooth and $f$ is quasi-étale, so that $X \times_{\tilde{X}} \tilde{Y}$ is smooth in codimension 1. Hence the normalization of it is in fact isomorphism in codimension 1. This proves $g$ and $q$ are quasi-étale.

Now $X$ is smooth, $Y$ is normal, and $g : Y \to X$ is finite quasi-étale. By the Zariski–Nagata purity theorem of the branch locus (e.g., [Sta, Tag 0BMB]), this forces $g$ to be étale. The hyper-Kähler manifold $X$ is simply connected, so this means $Y$ must be a disjoint union of several isomorphic copies of $X$. Let us fix a connected component $Y_0$ of $Y$. It is a hyper-Kähler manifold isomorphic to $X$.

Consider the morphism $q$ restricted to the connected component $q : Y_0 \to \tilde{Y}$. It is a finite quasi-étale morphism. Note that the target $\tilde{Y}$ is canonical (Lemma B.4), so [GKPP11, Thm 4.3] guarantees the existence of a reflexive pullback $q^* : H^0(\tilde{Y}, \Omega^2_{\tilde{Y}}) \to H^0(Y_0, \Omega^2_{Y_0})$. Since $q$ is quasi-étale, this morphism is injective. But recall that $Y_0 \cong X$ is a hyper-Kähler manifold, so this forces...
\( \tilde{Y} \) to satisfy the dimension condition of the definition of irreducible symplectic varieties. This proves \( \tilde{X} \) is an irreducible symplectic variety. \( \square \)

**Proof of Proposition B.3.** The following sequence of identities proves \( q_{\tilde{X}} \) is the Beauville–Bogomolov form with the Fujiki constant \( c_{\tilde{X}} = c_{X}/|G| \): 

\[
\int_X x^{2n} = \frac{1}{|G|} \int_X (p^* x)^{2n} = \frac{c_X}{|G|} q_X (p^* x)^n = \frac{c_X}{|G|} q_{\tilde{X}}(x)^n.
\]

Since \( \tilde{X} \) is a compact Kähler orbifold, its rational singular cohomology admits a well-behaved pure Hodge structure (e.g., [PS08, §2.5]) and \( p^* : H^*(\tilde{X}, \mathbb{Q}) \rightarrow H^*(X, \mathbb{Q}) \) is an injective Hodge structure homomorphism with the image \( H^*(X, \mathbb{Q})^G \). In particular, \( p^* \) is an isomorphism in degree 2.

To prove \( H^*(\tilde{X}, \mathbb{Q}) = H^*(X, \mathbb{Q})^G \) is closed under the \( g \)-action, it is enough to prove the \( G \)-action and \( g \)-action on \( H^*(X, \mathbb{Q}) \) commutes. Recall that the LLV structure is diffeomorphism invariant. In other words, if \( f : X_1 \rightarrow X_2 \) is a diffeomorphism between two compact hyper-Kähler manifolds then we have \( f^* (L_x(\xi)) = L_{f^*x}(f^*\xi), \quad f^* (\Lambda_x(\xi)) = \Lambda_{f^*x}(f^*\xi) \) for any \( x \in H^2(X_2, \mathbb{Q}) \) and \( \xi \in H^*(X_2, \mathbb{Q}) \). Here \( L_x \) and \( \Lambda_x \) are Lefschetz and inverse Lefschetz operators associated to \( x \). If we set \( X_1 = X_2 = X \) and \( f \in G \) to be an \( H^2 \)-trivial automorphism then this means \( f^* \) commutes with the operators \( L_x \) and \( \Lambda_x \). That is, \( G \) commutes with \( g \).

To obtain the results about the Mumford–Tate algebras one imitates the method used in [GKLR22, §2] and deduces \( f \in g \) for \( f \) a Weil operator on the cohomology \( H^*(\tilde{X}, \mathbb{Q}) \) (which is the restriction of Weil operator on \( H^*(X, \mathbb{Q}) \)). This proves all the special Mumford–Tate algebra of \( H^k(\tilde{X}, \mathbb{Q}) \) are the same and even same for that of \( H^2(X, \mathbb{Q}) \).

\( \square \)

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