Static Surface Mode Expansion for the Electromagnetic Scattering From Penetrable Objects

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Abstract—We introduce longitudinal and transverse static surface modes and use them to solve the scattering problem from penetrable objects with arbitrary shapes. The longitudinal static modes are the eigenmodes, with zero surface-curvature, of the electrostatic integral operator that determines the tangential component of the electric field, as a function of the surface charge density. The transverse static modes are the eigenmodes, with zero surface-divergence, of the magnetostatic integral operator that determines the tangential component of the vector potential, as a function of the surface current density. These static modes are solely determined by the object's shape, thus, the same static basis can be used regardless of the operating frequency or material properties. We expand the unknown surface currents of the Poggio-Miller-Chang-Harrington-Wu–Tsai (PMCHWT) equation in terms of the static modes and solve it using Galerkin-projections. The static modes expansion allows for the regularization of the integral operators and also leads to a significant reduction in the number of unknowns compared to a discretization based on sub-domain basis functions. As a consequence, the CPU-time required for the numerical solution of the scattering problem from arrays of identical particles is significantly reduced by employing an expansion in terms of static modes of the isolated particle.

Index Terms—Computational electromagnetics, dielectric resonators, eigenvalues and eigenfunctions, electromagnetic scattering, integral equations, plasmonics, resonance, resonators.

I. INTRODUCTION

The analysis of the electromagnetic scattering from a collection of mutually coupled objects is of great importance for many applications, spanning from antenna arrays [1] to metasurfaces [2] and metasheets [3]. In this context, the use of integral formulations is appealing, since the unknowns are defined only within the objects’ volume or, if the objects are spatially piecewise homogeneous, on their boundary and internal interfaces. Additionally, these formulations naturally satisfy the radiation condition at infinity. Nevertheless, the corresponding discrete problem is characterized by dense matrices, and their inversion is usually associated with high computational burden, even when acceleration techniques such as fast multipole algorithms are implemented [4].

Accurate and efficient solutions of integral formulations heavily depend on the choice of basis functions. Two main categories of basis functions can be distinguished: sub-domain functions, which are nonzero only over a portion of the object, or entire-domain functions, which extend over the entire domain of the object. While sub-domain functions may have wider applicability and are more suitable for objects with irregular shapes and sharp corners, entire-domain functions are very appealing when multiple scattering problems are considered, where the electromagnetic system under investigation is a collection of mutually-coupled objects [5], [6], [7].

Representative examples of sub-domain functions include the Rao-Wilton–Glisson functions [8], loop/star functions [9], [10], [11], [12], loop/tree functions [13], Trintiniala–Ling functions [14], Buffa–Christiansen functions [15], higher order vector basis functions of Nédélec type [16], etc.

On the other hand, examples of entire-domain basis functions include the tangential vector spherical wave functions [17], [18] and the tangential vector spherical wave functions [19]. Specifically, the former functions have been exploited to carry out a systematic analysis of integral operators of potential theory [17] and to investigate stability and preconditioning of multiple traces formulations for electromagnetics [18].

Analytical entire-domain bases may be generated in coordinate systems where the Helmholtz equation is separable. A different strategy to generate entire domain basis functions even in irregular domains is to introduce a convenient auxiliary eigenvalue problem. This is done for instance with characteristic modes [20], [21], [22], [23], (see [5], [6] for arrays of perfectly conductive particles and [24] for perfectly conductive metasurfaces). The characteristic modes do not depend on the particular excitation conditions, and they are effective in the numerical solution of scattering problems from collections of objects of a given material at a fixed operating frequency. Nevertheless, characteristic modes do depend on the frequency, and their interesting properties are lost if they are used as basis at a frequency different from the one at which they are computed. Thus, they may not be the best choice when the scattering problem has to be solved at multiple frequencies since they have to be recalculated at each frequency.

In this article, we introduce a different set of entire domain basis functions that we call “static” surface current modes. These modes are the union of two sets: longitudinal and
transverse modes, exhibiting vanishing surface curl and surface divergence, respectively. We assemble these two sets by solving two auxiliary frequency-independent eigenvalue problems, involving Hermitian and positive-definite surface integral operators, having the static Green’s function as a kernel.

The representation of the surface current in terms of longitudinal (irrotational) and transverse (solenoidal) static modes implements a Helmholtz decomposition on a surface (e.g., [25]). In the context of electromagnetic problems, the idea of splitting a field into its irrotational and solenoidal components has been pioneered by Nédélec [26] and has emerged as a very powerful tool for theoretical analysis of integral formulations [17], [18], [27].

The static surface current modes are the low-frequency limit of the resonance modes of surfaces of finite conductivity [28]. Their volume counterparts have been presented in [29] and [30] and in [31] where they have already been used to expand the electromagnetic field [32], [33], [34], [35].

The static modes can be also considered as “high level expansion functions”, belonging to the same categories of the functions introduced in [36], [37], [38], and [39], which are typically used in the solution of integral equations in electrically large structure to reduce the number of unknowns. They are not bound to the conventional discretization limit of \( \lambda/20 \), to which the local basis function discretization is constrained. The proposed basis shares similarities with the one introduced by Vecchi et al. [40], [41] and is used in a hybrid spectral–spatial method for the analysis of printed antennas.

We expand the unknown surface currents of the Poggio-Miller–Chang–Harrington–Wu–Tsai (PMCHWT) equation in terms of the static surface modes and solve it using the Galerkin-projection scheme. The PMCHWT equation [21], [42], [43], [44] is a surface integral formulation of Maxwell’s equation, belonging to the class of single-trace methods [45]. We demonstrate the following advantages of employing the static modes: 1) regularization of the scattering integral operator; 2) “compression” of the matrix equation, which is associated with a significant reduction of the number of unknowns compared to sub-domain basis functions while maintaining solution accuracy; and 3) reduction of the CPU-time required for the solution of multiple scattering problems. The static mode basis can be also used in the context of multiple-trace methods, which were recently proposed for electromagnetic scattering from composite objects [18], [46], [47], [48], [49], [50].

The article is organized as follows: in Section II we introduce the static basis; in Section III we recall the PMCHWT formulation and we show that the static mode expansion regularizes the involved operator and that a proper rescaling of the unknowns makes this formulation immune from the low-frequency breakdown. In Section IV, we validate the introduced method in two resonant scattering problems, namely the scattering from a metal particle and high-permittivity particle in the visible/near-infrared spectral range. Lastly, we demonstrate the application of this method to solve a multiple-scattering problem. In Section VI we draw the conclusion.

II. Static Surface Current Modes

We denote with \( \Omega \) a bounded 3-D domain, whose boundary \( \partial \Omega \) is “sufficiently regular” [51]; \( \hat{n} \) is the normal to \( \partial \Omega \) pointing outward. A sufficiently smooth vector field \( \mathbf{j} \) defined on a regular surface \( \partial \Omega \) can be resolved into the sum of two components [25], [26], [27], [52]: an irrotational and nonsolenoidal vector field \( \mathbf{j}_l \) and a solenoidal and rotational (nonzero curl) vector field \( \mathbf{j}_s \). The vector fields \( \mathbf{j}_l \) and \( \mathbf{j}_s \) are orthogonal according to the scalar product

\[
\mathbf{C} \cdot \mathbf{D} = \int_{\partial \Omega} \mathbf{C}^*(\mathbf{r}) \cdot \mathbf{D}(\mathbf{r}) dS.
\]

In the following, we introduce a basis for each of the two components.

A. Longitudinal Static Modes

The longitudinal static surface current modes (called longitudinal static modes in the following for brevity) are nontrivial solutions to the eigenvalue problem:

\[
\mathbf{T}_l^0 \{ \mathbf{j}_l^0 \}(\mathbf{r}) = \gamma_l^0 \mathbf{j}_l^0 \quad \text{on} \quad \partial \Omega
\]

where

\[
\mathbf{T}_l^0 [\mathbf{w}] (\mathbf{r}) = \hat{n} \times \nabla \times \int_{\partial \Omega} g_0 (\mathbf{r} - \mathbf{r}') \nabla_S' \cdot \mathbf{w}(\mathbf{r}') dS'.
\]

\( \nabla_S' \) denotes the surface divergence, and \( g_0 \) is the homogeneous static Green’s function

\[
g_0 (\mathbf{r} - \mathbf{r}') = \frac{1}{4\pi} \frac{1}{| \mathbf{r} - \mathbf{r}' |}.
\]

Apart from a multiplicative factor, the integral operator (3) determines the tangential component of the static electric field generated by a surface charge density distribution. Its spectrum has the following properties (see [29]).

1) The eigenvalues \( \{ \gamma_l^0 \} \) and the modes \( \{ \mathbf{j}_l^0 \} \) depend on the shape of the object, but are independent of the object material, and of the frequency of operation.
2) The eigenvalues are real and positive.
3) The modes are orthonormal with respect to the scalar product (1).

For a spherical surface of unit radius, the eigenvalues have the analytical expression

\[
\gamma_l^0 = \frac{n(n + 1)}{(2n + 1)}, \quad n = 1, 2, 3, \ldots
\]

In order to provide the expression of the corresponding modes, we introduce a spherical reference system. The spherical coordinates of the point with position vector \( \mathbf{r} \) are \( (r, \theta, \phi) \) (with \( 0 \leq r < \infty \), \( 0 \leq \theta < \pi \) and \( 0 \leq \phi < 2\pi \)). The basis for the 3-D vector space is the set \( \{ \hat{r}, \hat{\theta}, \hat{\phi} \} \), where \( \hat{r} \) is the radial unit vector, \( \hat{\theta} \) is the polar unit vector, and \( \hat{\phi} \) is the azimuthal unit vector. The \( 2n + 1 \) modes corresponding to the degenerate eigenvalue \( \gamma_n^0 \) are the vector spherical harmonics \( \mathbf{W}_m^n (\theta, \phi) \) [17], [18] where \( m \) is an integer such that \( -n \leq m \leq n \)

\[
\mathbf{j}_{mn}^l = \mathbf{W}_m^n \times \mathbf{X}_n^m = \frac{1}{\sqrt{n(n + 1)}} \nabla Y_n^m.
\]
The spherical harmonic $Y^m_n(\theta, \phi)$ of degree $n$ and order $m$ is given by

$$Y^m_n(\theta, \phi) = C_{mn} P^m_n(\cos \theta) e^{im\phi}$$

where $P^m_n(\cos \theta)$ is the associated Legendre polynomial of degree $n$ and order $m$, and $C_{mn}$ is a normalization coefficient. The spherical harmonics are orthogonal. We normalize them in such a way that

$$\int_{\partial \Omega} |Y^m_n(\theta, \phi)|^2 dS = 1.$$  

The normalization coefficient $C_{mn}$ is equal to (e.g., [17])

$$C_{mn} = \frac{\sqrt{2n + 1}}{4\pi} \frac{1}{(n + |m|)!}.$$  

The degree $n$ also determines the multipolar order of the vector spherical harmonics (e.g., $n = 1$ for a dipole, $n = 2$ for a quadrupole, etc.). Thus, (5) shows that longitudinal modes associated with larger eigenvalues $\gamma_n^\perp$ are characterized by higher multipolar order.

### B. Transverse Static Modes

The transverse static surface current modes (called \textit{transverse static modes} in the following for brevity) are nontrivial solutions to the eigenvalue problem:

$$\mathbf{T}\cdot\mathbf{j}_h \equiv \mathbf{J}_h = \gamma^\perp_h \mathbf{J}_h^\perp,$$

with

$$\mathbf{T}\cdot\mathbf{w}(r) = -\hat{n} \times \hat{n} \times \int_{\partial \Omega} g_0(r-r') \mathbf{w}(r') dS'.$$  

Apart from a multiplicative factor, the integral operator (11) determines the static vector potential generated by a surface current distribution. Its spectrum has the following properties, that can be derived using the standard methods of eigenvalue problems, analogously to [31] and [53].

1. The eigenvalues $\gamma^\perp_n$ and the modes $\mathbf{j}_h^\perp$ depend on the shape of the object, but are independent of the material and of the frequency of operation.
2. The eigenvalues are real and positive.
3. The modes $\mathbf{j}_h^\perp$ are orthonormal with respect to the scalar product (1).
4. For a spherical surface of unit radius, the eigenvalues have the analytical expression

$$\gamma^\perp_n = \frac{1}{(2n + 1)}, \quad n = 1, 2, 3, \ldots$$

The 2$^n + 1$ modes corresponding to the degenerate eigenvalue $\gamma^\perp_n$ are the vector spherical harmonics $\mathbf{X}^m_n(\theta, \phi)$ [17], [18] where $m$ is an integer such that $-n \leq m \leq n$.

$$\mathbf{j}_h^\perp = \mathbf{X}^m_n = \frac{1}{\sqrt{n(n+1)}} \nabla Y^m_n \times \mathbf{r}.$$  

The degree $n$ also determines the multipolar order of the vector spherical harmonics (e.g., $n = 1$ for a dipole, $n = 2$ for a quadrupole, and so on). Thus, (12) shows that transverse modes associated with smaller eigenvalues $\gamma^\perp_n$ are characterized by higher multipolar order.

### C. Computation of the Static Modes for Arbitrary Shapes

Let us introduce a surface triangulation of $\partial \Omega$, with $N_p$ vertices, $N_I$ elements, and $N_E$ edges. We represent the static modes in terms of convenient sub-domain basis functions, namely the loop and star functions [12], [54]. Specifically, any longitudinal mode $\mathbf{j}_h^\parallel$ is expanded in terms of (nonsolenoidal) star basis functions $\mathbf{j}_{h,p}^\star$ with coefficients $\alpha^\perp_{h,p}$. Dually, any transverse mode $\mathbf{j}_h^\perp$ is expanded in terms of (solenoidal) loop basis functions $\mathbf{j}_h^\circ$ with coefficients $\alpha^\parallel_{h,q}$

$$\mathbf{j}_h^\parallel = \sum_{p=1}^{N_p-1} \alpha^\perp_{h,p} \mathbf{j}_{h,p}^\star, \quad \mathbf{j}_h^\perp = \sum_{q=1}^{N_E} \alpha^\parallel_{h,q} \mathbf{j}_h^\circ.$$  

The loop functions are divergence-free, thus they correctly represent the transverse static modes. Instead, the star functions are not curl-free (they are often denoted as quasi-curl [55]), thus they only approximately represent the longitudinal static modes. Both star and loop functions admit a linear representation in terms of Rao–Wilton–Glisson basis functions [8].

For closed surfaces with no handles, the number of linearly independent loop functions is $N_p − 1$, while the number of linearly independent star functions is $N_I − 1$ (see [55] for a discussion on the linear independence of loop and star basis functions). Thus, the numerical auxiliary eigenvalue problem for longitudinal static modes [associated with problem (2)] is

$$\mathbf{T}_0^{**} \mathbf{J}_h = \gamma^\parallel_h \mathbf{R}^{**} \mathbf{J}_h^\parallel$$

where $(\mathbf{T}_0^{**})_{pq} = \langle \mathbf{j}_p^\parallel, \mathbf{T}_0^\parallel \mathbf{j}_q^\parallel \rangle$, $(\mathbf{R}^{**})_{pq} = \langle \mathbf{j}_p^\parallel, \mathbf{j}_q^\parallel \rangle$, and $\mathbf{J}_h^\parallel = [\alpha^\parallel_{h,1}, \alpha^\parallel_{h,2}, \ldots, \alpha^\parallel_{h,N_p-1}]^T$. The numerical auxiliary eigenvalue problem for transverse static modes [associated with problem (10)] is

$$\mathbf{T}^{\perp\circ\circ} \mathbf{J}_h^\circ = \gamma^\perp_h \mathbf{R}^{\perp\circ\circ} \mathbf{J}_h^\circ$$

where $(\mathbf{T}^{\perp\circ\circ})_{pq} = \langle \mathbf{j}_p^\circ, \mathbf{T}_0^\perp \mathbf{j}_q^\circ \rangle$, $(\mathbf{R}^{\perp\circ\circ})_{pq} = \langle \mathbf{j}_p^\circ, \mathbf{j}_q^\circ \rangle$, and $\mathbf{J}_h^\circ = [\alpha^\parallel_{h,1}, \alpha^\parallel_{h,2}, \ldots, \alpha^\parallel_{h,N_I-1}]^T$.

Since the loop and star functions are not orthogonal [55], the matrices $\mathbf{R}^{**}$ and $\mathbf{R}^{\perp\circ\circ}$ are not identity matrices, thus (15) and (16) are \textit{generalized} eigenvalue problems. The involved matrices are real, symmetric, and positive definite. Thus, efficient numerical algorithms for the eigenvalue calculation do apply, such as the Cholesky factorization [56]. Moreover, the matrices properties also determine the orthogonality, at the discrete level, of any pair of longitudinal modes, and any pair of transverse modes. The numerical integration of shape functions times the Green’s functions or its gradient are evaluated using the techniques introduced by Graglia [57].

### III. PMCHWT Surface Integral Equation

A linear, homogeneous, isotropic material occupies the 3-D domain $\Omega$. The material has permittivity $\varepsilon^r(\omega)$, permeability $\mu^r(\omega)$ and it is surrounded by a background medium with permittivity $\varepsilon^\infty(\omega)$ and permeability $\mu^\infty(\omega)$. The object is illuminated by a time harmonic electromagnetic field $\mathbf{R} \cdot [\mathbf{E}_{inc}(r) \cdot e^{j\omega t}]$. The equivalent electric $\mathbf{j}_e$ and magnetic $\mathbf{j}_m$ surface current densities, defined on $\partial \Omega$, are
The solutions of the following surface integral problem formulated by PMCHWT [21], [42], [43]:
\[ \mathbf{Z} \mathbf{J} = \mathbf{F} \]  
(17)
where
\[ \mathbf{Z} = \begin{pmatrix} \zeta^{-} \mathbf{T}_{-} + \zeta^{+} \mathbf{T}_{+} & \mathbf{K}_{-} + \mathbf{K}_{+} \\ -(\mathbf{K}_{-} + \mathbf{K}_{+}) & \mathbf{T}_{-}^{-1} \zeta^{-} + \mathbf{T}_{+}^{-1} \zeta^{+} \end{pmatrix} \]  
(18)
\[ \mathbf{J} = [j_{r}, j_{m}]^{T}, \quad \mathbf{F} = [e_{0}, h_{0}]^{T}, \]
(19)
\[ e_{0} = -\hat{n} \times \hat{n} \times \mathbf{E}_{inc} |_{\partial \Omega}, \quad h_{0} = -\hat{n} \times \hat{n} \times \mathbf{H}_{inc} |_{\partial \Omega}. \]
(20)
The operators \( \mathbf{K}_{\pm} \) and \( \mathbf{T}_{\pm} \) are the MFIE and EFIE integral operators
\[ \mathbf{T}_{\pm}(\mathbf{r}) = j k^{\pm} \hat{n} \times \int_{\partial \Omega} g^{\pm}(\mathbf{r} - \mathbf{r}') w(\mathbf{r}') dS' \]
\[ + \frac{1}{j k^{\pm}} \hat{n} \times \int_{\partial \Omega} \nabla g^{\pm}(\mathbf{r} - \mathbf{r}') \nabla_{\mathbf{r}} \cdot w(\mathbf{r}') dS' \]  
(21a)
\[ \mathbf{K}_{\pm}(\mathbf{r}) = \hat{n} \times \int_{\partial \Omega} w(\mathbf{r}') \times \nabla' \bar{g}^{\pm}(\mathbf{r} - \mathbf{r}') dS' \]  
(21b)
g\( ^{\pm} \) is the homogeneous space Green’s function of the region \( \Omega_{\pm} \), i.e.,
\[ g^{\pm}(\mathbf{r} - \mathbf{r}') = \frac{e^{-j k^{\pm} |\mathbf{r} - \mathbf{r}'|}}{4 \pi |\mathbf{r} - \mathbf{r}'|} \]  
(22)
k\( ^{\pm} \) and \( \zeta^{\pm} = \sqrt{\mu^{\pm} / e^{\pm}}. \)

A. Galerkin Equations
We aim to solve the PMCHWT (17). To do this, we represent the equivalent electromagnetic and surface currents in terms of the transverse static modes \( \{ j_{p}^{\pm} \}_{p=1,\ldots,N_{\perp}} \) associated with the first \( N_{\perp} \) eigenvalues \( \gamma_{p}^{\perp} \) (sorted in descending order), and in terms of the longitudinal static modes \( \{ j_{q}^{\parallel} \}_{q=1,\ldots,N_{\parallel}} \) associated with the first \( N_{\parallel} \) eigenvalues \( \gamma_{q}^{\parallel} \) (sorted in ascending order), namely
\[ \mathbf{j}_{r}(\mathbf{r}) = \sum_{p=1}^{N_{\perp}} \alpha_{p}^{\perp} j_{p}^{\perp}(\mathbf{r}) + \sum_{q=1}^{N_{\parallel}} \alpha_{q}^{\parallel} j_{q}^{\parallel}(\mathbf{r}) \]
(23)
\[ \mathbf{j}_{m}(\mathbf{r}) = \sum_{p=1}^{N_{\perp}} \beta_{p}^{\perp} j_{p}^{\perp}(\mathbf{r}) + \sum_{q=1}^{N_{\parallel}} \beta_{q}^{\parallel} j_{q}^{\parallel}(\mathbf{r}). \]
The choice of sorting the longitudinal eigenvalues accordingly to an ascending order and the transverse eigenvalues accordingly to a descending order guarantees that low-index eigenvalues are associated with modes of lower-order multipoles (dipole, quadrupole, octupole, and so on). Therefore, we define the unknown block vectors
\[ \mathbf{J}_{r} = (\mathbf{\alpha}^{\perp} | \mathbf{\alpha}^{\parallel})^{T}, \quad \mathbf{J}_{m} = (\mathbf{\beta}^{\perp} | \mathbf{\beta}^{\parallel})^{T} \]  
(24)
with \( \mathbf{\alpha}^{\perp} = [\alpha_{1}^{\perp}, \alpha_{2}^{\perp}, \ldots, \alpha_{N_{\perp}}^{\perp}]^{T} \) and with \( \mathbf{\beta}^{\parallel} = [\beta_{1}^{\parallel}, \beta_{2}^{\parallel}, \ldots, \beta_{N_{\parallel}}^{\parallel}]^{T} \) and \( a = \perp, \parallel \).

We find the finite-dimensional approximation of the PMCHWT problem by substituting (23) in (17) and by projecting along the same set of modes, accordingly to a Galerkin projection scheme
\[ \mathbf{Z} \mathbf{J} = \begin{pmatrix} \mathbf{E}_{0} \\ \mathbf{H}_{0} \end{pmatrix} \]  
(25)
diagonalize the static operators $\mathbf{T}_{0}^{\perp}$ and $\mathbf{T}_{0}^{\parallel}$

\[
\left(\mathbf{T}_{\pm}^{\parallel}\right)_{pq} = \langle j_{p}^{\pm}, \mathbf{T}_{\pm}^{\parallel} j_{q}^{\pm}\rangle = \frac{\gamma_{\pm}}{jk} \delta_{p,q} - jk^{\pm} \langle j_{p}^{\pm}, \mathbf{T}_{0}^{\pm} j_{q}^{\pm}\rangle + \langle j_{p}^{\pm}, \mathbf{T}_{d\pm} j_{q}^{\pm}\rangle
\]

\[
\left(\mathbf{T}_{\pm}^{\pm}\right)_{pq} = \langle j_{p}^{\pm}, \mathbf{T}_{\pm}^{\pm} j_{q}^{\pm}\rangle = \langle j_{p}^{\pm}, \mathbf{T}_{d\pm} j_{q}^{\pm}\rangle
\]

\[
\left(\mathbf{T}_{\pm}^{\perp}\right)_{pq} = \langle j_{p}^{\pm}, \mathbf{T}_{\pm}^{\perp} j_{q}^{\pm}\rangle = \langle j_{p}^{\pm}, \mathbf{T}_{d\pm} j_{q}^{\pm}\rangle
\]

(38a)

(38b)

(38c)

(38d)

where $\delta_{p,q}$ is the Kronecker delta, and we have exploited the fact that $\langle j_{p}^{\pm}, \mathbf{T}_{0}^{\pm} j_{q}^{\pm}\rangle = \langle j_{p}^{\pm}, \mathbf{T}_{0}^{\parallel} j_{q}^{\pm}\rangle = 0$, $\langle j_{p}^{\pm}, \mathbf{T}_{0}^{\perp} j_{q}^{\pm}\rangle = 0$, $\langle j_{p}^{\pm}, \mathbf{T}_{d\pm} j_{q}^{\pm}\rangle = \gamma_{\pm} \delta_{p,q}$ and $\langle j_{p}^{\pm}, \mathbf{T}_{d\pm} j_{q}^{\pm}\rangle = 0$. The numerical computation of the terms $\langle j_{p}^{\pm}, \mathbf{T}_{d\pm} j_{q}^{\pm}\rangle$, with $a, b = \parallel, \perp$ is straightforward since their kernels are regular functions. These terms are the only ones depending on the operating frequency. The decomposition of Green’s function into the sum of a static term and of a regular difference term relieves us from the task of computing almost all the integrals with (integrable) singularity, which usually results in longer computational time compared to their regular counterpart. There are however two exceptions: $\langle j_{p}^{\pm}, \mathbf{T}_{0}^{\parallel} j_{q}^{\pm}\rangle$ and $\langle j_{p}^{\pm}, \mathbf{T}_{0}^{\perp} j_{q}^{\pm}\rangle$, with $a, b = \parallel, \perp$. These terms are frequency-independent, thus when the calculation of the scattering response of an object for multiple frequencies of the exciting field is required, they can be conveniently precalculated and stored away, while only the regular terms must be calculated at any frequencies.

B. Low-Frequency Analysis

Surface integral formulations may suffer from ill-conditioning due to low-frequency breakdown. The low-frequency breakdown phenomenon manifests when the operating wavelength is much larger than the dimension of the object [59], and originates from the different frequency-scaling of the terms associated with the vector and the scalar potentials. This is a common scenario that may be encountered in several applications, including metamaterials and electromagnetic bandgap (EBG) structures, or in the analysis of interconnects and packaging. This problem has been addressed by using quasi-Helmholtz decompositions, such as loop/star [54], [55], [59], loop-tree [13], [60], tree-cotree [10], or null-piny [61] decompositions, followed by a basis rearrangement [62], [63]. It is also worth pointing out that some formulations are immune from this problem, such as the N-Müller formulation [64], or the formulation obtained by augmenting the traditional EFIE by including charge as extra unknown [65], [66].

In this section, we summarize the behavior of the PMCHWT at very low frequencies. In this limit, the frequency dependence of the elements of the matrix $Z$ follows different scaling laws, which are easily determined by following [63], [67]:

\[
\mathbf{T}_{0}^{\pm} = \begin{pmatrix} \mathbf{T}_{0}^{\parallel} & \mathbf{T}_{0}^{\perp} \\ \mathbf{T}_{0}^{\perp} & \mathbf{T}_{0}^{\parallel} \end{pmatrix} \begin{pmatrix} \omega \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Gamma} \end{pmatrix} \begin{pmatrix} \mathbf{T}_{0}^{\parallel} & \mathbf{T}_{0}^{\perp} \\ \mathbf{T}_{0}^{\perp} & \mathbf{T}_{0}^{\parallel} \end{pmatrix}^{-1}
\]

(39)

\[
\mathbf{K}_{0}^{\pm} = \begin{pmatrix} \mathbf{K}_{0}^{\parallel} & \mathbf{K}_{0}^{\perp} \\ \mathbf{K}_{0}^{\perp} & \mathbf{K}_{0}^{\parallel} \end{pmatrix} \begin{pmatrix} \omega \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Gamma} \end{pmatrix} \begin{pmatrix} \mathbf{K}_{0}^{\parallel} & \mathbf{K}_{0}^{\perp} \\ \mathbf{K}_{0}^{\perp} & \mathbf{K}_{0}^{\parallel} \end{pmatrix}^{-1}
\]

(40)

where $\mathbf{\Gamma}$ and $\mathbf{\Gamma}^{\dagger}$ are diagonal matrices, $\mathbf{\Gamma}_{\parallel} = \text{diag} \{\gamma_{\parallel 1}, \gamma_{\parallel 2}, \ldots, \gamma_{\parallel N}\}$, $\mathbf{\Gamma}_{\perp} = \text{diag} \{\gamma_{\perp 1}, \gamma_{\perp 2}, \ldots, \gamma_{\perp N}\}$, and

\[
T_{2}^{\perp} = \frac{1}{8\pi} \int_{\mathcal{S}_{2}} \mathbf{j}_{0}^{\perp}(\mathbf{r}) \cdot \int_{\mathcal{S}_{2}} \mathbf{r} - \mathbf{r}^{'}, \mathbf{r}^{'} \mid \mathbf{j}_{0}^{\perp}(\mathbf{r}) dSdS^{'}
\]

(41)

\[
K_{2}^{\perp} = -\frac{1}{8\pi} \int_{\mathcal{S}_{2}} \mathbf{j}_{0}^{\perp}(\mathbf{r}) \cdot \int_{\mathcal{S}_{2}} \mathbf{r} - \mathbf{r}^{'}, \mathbf{r}^{'} \times \mathbf{j}_{0}^{\perp}(\mathbf{r}) dSdS^{'}
\]

(42)

Thus, by using (39) and (40) it easy to prove that, in the static limit, the discrete matrix $Z$ approaches the matrix $Z_{0}$ which exhibits the following frequency dependence:

\[
Z_{0} = \begin{pmatrix} \mathcal{O}(\omega) & \mathcal{O}(\omega^{2}) & \mathcal{O}(\omega^{3}) & \mathcal{O}(1) \\ \mathcal{O}(\omega^{3}) & \mathcal{O}(\omega^{4}) & \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(\omega) & \mathcal{O}(1) & \mathcal{O}(\omega^{3}) & \mathcal{O}(\omega^{4}) \\ \mathcal{O}(1) & \mathcal{O}(3) & \mathcal{O}(\omega^{3}) & \mathcal{O}(\omega^{4}) \end{pmatrix}
\]

(43)

The excitation vector associated with a plane wave exhibits the following dependencies [63]:

\[
\left[\mathbf{E}_{0}^{\parallel}, \mathbf{E}_{0}^{\perp}, \mathbf{H}_{0}^{\parallel}, \mathbf{H}_{0}^{\perp}\right]^{T} = \left[\mathcal{O}(\omega), \mathcal{O}(1), \mathcal{O}(\omega), \mathcal{O}(\omega^{3})\right]^{T}
\]

(44)

Following [63], we introduce the rearrangement and scaling of the basis

\[
\mathbf{\tilde{Z}} = \mathbf{D}_{1} \mathbf{Z} \mathbf{D}_{2}
\]

(45)

where $\mathbf{D}_{1} = \text{diag}[k_{-1}^{-1}, k_{1}^{-1}, k_{-1}^{-1}, k_{1}^{-1}]$, $\mathbf{D}_{2} = \text{diag}[k_{1}, k_{1}, k_{1}, k_{1}]$, $k_{1}$ is the $N_{\parallel} \times N_{\parallel}$ identity matrix, and $k_{1}$ is the $N_{\parallel} \times N_{\parallel}$ identity matrix. After the above rearrangement the matrix $\mathbf{\tilde{Z}}$ is well-behaved.

IV. RESULTS AND DISCUSSION

A. Sphere

We first consider the scattering from a sphere. This problem has an analytical solution [68], [69]. Fig. 1 shows the convergence of the numerical eigenvalues $\{\gamma_{\parallel k}\}$ and $\{\gamma_{\perp k}\}$ toward their analytical counterpart, given by (5) and (12), as a function of the triangular mesh density. From this point forward, we will consider the static basis calculated using a triangular mesh with $N_{p} = 1000$ nodes and $N_{t} = 1996$ triangles. The first eight longitudinal modes are shown in Fig. 2, the first eight transverse modes are shown in Fig. 3.

In the discrete problem, the orthogonality between any pair of longitudinal modes and between any pair of transverse modes is always guaranteed, because the matrices $\mathbf{T}^{\perp \parallel}$ and $\mathbf{T}^{\times \times}$ are real and symmetric. This property is indeed verified at the numerical level with machine precision. Although we expect the mutual product between a transverse and a longitudinal static mode to be nearly zero, this is only approximately the case because the sub-domain basis functions used to represent the longitudinal modes, namely the “star” functions, are not rigorously curl-free [55]. Thus, it is clarifying to calculate the
“mutual” Gram matrix $G$, whose occurrences are defined as $g_{hk} = \langle j_h^\parallel, j_k^\parallel \rangle$. The maximum occurrence of the mutual Gram matrix is 0.024 for a sphere with the considered surface mesh, assuming $\|j_h^\parallel\| = \|j_k^\parallel\| = 1$, $\forall h$.

1) Gold Sphere: We initiate our discussion with the scattering problem from a gold sphere of radius $R = 100$ nm operating within the visible and near-infrared spectral range. We describe the gold permittivity by interpolating experimental data [70]. At these frequencies, a metal nano-object may undergo plasmonic resonances, which have an electrostatic origin [29]. The sphere is excited by a linearly polarized plane wave of wavelength $\lambda$.

The scattering efficiency, defined as the scattering cross section normalized by the geometrical cross section $G$ (which is $G = \pi R^2$ [69] in this case) is plotted in Fig. 4 as a function of $\lambda$. We compare the results of different solutions that were obtained by incrementally increasing the number of modes employed in the expansion (23) while keeping $N^\parallel = N^\perp$. The reference Mie solution [69] is included for reference. For $N^\parallel = N^\perp = 5$, the numerical solution is in good agreement with the reference solution only in the long-wavelength regime, while it slightly deviates as $\lambda$ approaches $R$. Increasing the number of modes to $N^\parallel = N^\perp = 10$ delivers good agreement over the investigated spectral. In this latter case, the inversion of a $40 \times 40$ matrix is required at each frequency.

In Fig. 5, we show the condition number of the PMCHWT problem with and without the rearrangement of the basis described in Section III-B as a function of the sphere radius $R$, at $\lambda = 620$ nm. It is apparent that, without the basis rearrangement, the condition number exponentially increases, which is symptomatic of the low-frequency breakdown problem. By rearranging the basis, the condition number is constant over the whole investigated range of radii $R$.

We now present a more comprehensive error analysis. In particular, we define the relative error in the scattering efficiency as

$$\epsilon [\sigma_{\text{sca}}] = \frac{\sigma_{\text{sca}} - \bar{\sigma}_{\text{sca}}}{\bar{\sigma}_{\text{sca}}}$$

where $\bar{\sigma}_{\text{sca}}$ is the reference solution obtained by solving the PMCHWT problem applying the boundary element method with $N^\ominus = 999$ loop functions and $N^\ast = 1995$ star functions, associated with the same mesh used for the calculation of static modes. In Fig. 6(a), we plot $\epsilon [\sigma_{\text{sca}}]$ as a function of the wavelength $\lambda$, where the number of static modes is varied keeping $N^\parallel = N^\perp$. We note that for $N^\parallel = N^\perp \geq 10$ the achieved error is lower than 0.02 all over the investigated spectral range. The error only slowly decreases if the number of modes $N^\parallel = N^\perp$ is increased from 15 to 25.

We proceed to investigate the error in the evaluation of the equivalent surface currents, which are directly related to the total electric field on the object’s surface. The relative error is defined as

$$\epsilon [J] = \frac{\| J - \tilde{J} \|_2}{\| \tilde{J} \|_2}$$

where $\tilde{J}$ is the reference loop/star solution and $\| \cdot \|_2$ is the Euclidean norm. In Fig. 6(b), we show $\epsilon [J]$ as a function of $\lambda$. It’s evident that achieving a prescribed error necessitates more static basis functions than Fig. 6(a).

In a subsequent test, we assess the convergence of the static modes solution with a near-field excitation source. Here, an electric point dipole excites a gold sphere of radius $R = 100$ nm. The dipole is placed 30 nm away from the sphere’s surface and oriented as shown in the inset of Fig. 7. We evaluate the total scattered power $W_{\text{sca}}$ using an increasing number of longitudinal/transverse static modes ($N^\parallel = N^\perp = 5, 10, 15$) and compared it with the reference Mie solution. As soon as $N^\parallel = N^\perp = 15$, the two solutions become indistinguishable.

In Fig. 8, we investigate the two relative errors $\epsilon [W_{\text{sca}}]$ and $\epsilon [J]$, with the loop/star solution as a reference. Fig. 8(a) shows that the total scattered power evaluated using the static modes rapidly converges toward the loop/star solution. However, the convergence of the surface currents, shown in Fig. 8(b), is considerably slower due to the rapid spatial variations of the surface current near the exciting dipole. These variations necessitate higher-order static modes for accurate description.

2) High-Permittivity Sphere: We now consider a high permittivity sphere, assumed to be nondispersive in time with relative permittivity $\varepsilon_R = 16$. Sub-wavelength objects of sufficiently high permittivity may support scattering resonances, which have a magnetostatic origin [31], [34].

In Fig. 9, we calculate the scattering efficiency $\sigma_{\text{sca}}$ as a function of the wavelength $\lambda$ of the exciting, linearly polarized, plane wave. We consider different solutions computed using an increasing number of static modes, maintaining $N^\parallel = N^\perp$. The
Fig. 2. Longitudinal static modes of a sphere, displayed in ascending order according to their static eigenvalue. The first eight modes are presented: the initial three correspond to $n = 1$ (electric dipole) and the subsequent five to $n = 2$ (electric quadrupole). The arrows indicate the direction of the surface current density field, while the varying colors denote the surface charge density.

Fig. 3. Transverse static modes of a sphere, displayed in ascending order according to their static eigenvalue. The first eight modes are presented: the initial three correspond to $n = 1$ (magnetic dipole) and the subsequent five to $n = 2$ (magnetic quadrupole). The arrows indicate the direction of the surface current density field, the colors denote the magnitude of the current density field.

Fig. 4. Scattering efficiency $\sigma_{\text{sca}}$ of a gold sphere with radius $R = 100 \text{ nm}$ under the excitation of a linearly polarized plane wave at wavelength $\lambda$. $\sigma_{\text{sca}}$ is calculated using the PMCHWT with an incrementing number of longitudinal and transverse static modes ($N^\parallel = N^\perp = 5, 10, 15$). The reference Mie solution is also provided for comparison, and is shown by the black dashed line.

Fig. 5. Comparison of the condition number with and without basis rearrangement and scaling, as a function of the size parameter $2\pi R/\lambda$, under the assumption of $N^\parallel = N^\perp = 15$. The study involves a gold sphere of varying radius $R$, excited by a linearly polarized plane wave at wavelength $\lambda = 620 \text{ nm}$.

Fig. 6. Error made in the evaluation of (a) scattering efficiency $\epsilon[\sigma_{\text{sca}}]$ and (b) equivalent surface currents $\epsilon[J]$ of a gold sphere $R = 100 \text{ nm}$ by using $N^\parallel = N^\perp = 5, 10, 15, 25$ static modes corresponding to 20, 40, 60, 100 degrees of freedom. The reference solutions use the loop/star method with 999 loop and 1996 star functions, totaling 5998 degrees of freedom. In both cases, a PMCHWT formulation is used. The sphere is excited by a linearly polarized plane wave as a function of the wavelength $\lambda$.

While the solution displays a good match with the Mie solution at lower frequencies when $N^\parallel = N^\perp = 5$, the accuracy of the solution deteriorates at higher frequencies and fails to describe some peaks of the scattering response. However, elevating the number of modes utilized to $N^\parallel = N^\perp = 15$ leads to a remarkable agreement throughout the entire examined spectrum.

We now quantify the errors in the scattering efficiency and in the surface currents of the static mode solution. Fig. 10(a) presents $\epsilon[\sigma_{\text{sca}}]$ as a function of $\lambda$, employing an increasing number of modes, ensuring $N^\perp = N^\parallel$. With $N^\perp = N^\parallel = 5$, the error remains within an acceptable range provided $\lambda$ significantly exceeds the object’s dimensions. However, the error sharply rises for wavelengths below 600 nm. Adopting $N^\perp = N^\parallel = 25$ results in a low error all over the investigated spectrum. The error only slightly exceeds 0.01 at the resonances. A further increase in the number of basis functions improves the convergence, particularly in the vicinity of the resonance peaks. In Fig. 10(b), we show $\epsilon[J]$ as a function of $\lambda$. Compared to panel (a), the relative error is higher. To verify the convergence of the static mode expansion, we also tested the analytic Mie solution [68], [69], shown with a black dashed line, serves as our reference.
Fig. 7. Plot of the power $W_{sca}$ scattered by a gold sphere with radius $R = 100 \text{ nm}$, which is excited by an electric point dipole. The dipole is located at a distance of $30 \text{ nm}$ from the surface of the particle, and oriented as shown in the inset. $W_{sca}$ is evaluated as a function of the wavelength $\lambda$ using the PMCHWT with an increasing number of longitudinal/transverse static modes ($N^\parallel = N^\perp = 5, 10, 15$). For comparison, the reference Mie solution is also shown with a black dashed line.

Fig. 8. Error made in the evaluation of (a) scattered power $W_{sca}$ and (b) equivalent surface currents $j$ of a gold sphere $R = 100 \text{ nm}$ by solving the PMCHWT using $N^\parallel = N^\perp = 5, 10, 30, 50, 100$ static modes. The reference values were obtained by solving the PMCHWT using $999$ loop and $1996$ star functions. The sphere is excited by a dipole, located at a distance of $30 \text{ nm}$ from the surface of the particle and oriented as shown in the inset of Fig. 7.

Fig. 9. Scattering efficiency $\sigma_{sca}$ of a dielectric sphere with $R = 100 \text{ nm}$ and $\varepsilon_R = 16$, excited by a linearly polarized plane wave at wavelength $\lambda$. The evaluation is performed using the PMCHWT with an increasing number of longitudinal and transverse static modes ($N^\parallel = N^\perp = 5, 15, 25$). The reference Mie solution is also shown for comparison (black dashed line).

Fig. 10. Error made in the evaluation of (a) scattering efficiency $\sigma_{sca}$ and (b) equivalent surface currents $j$ of a dielectric sphere with $\varepsilon_R = 16$ and $R = 100 \text{ nm}$. The evaluation is performed using the PMCHWT equation with the static mode basis assuming $N^\parallel = N^\perp = 5, 25, 55, 500$. The reference values are obtained by solving the PMCHWT and using the loop/star expansion. The sphere is excited by a linearly polarized plane wave as a function of the wavelength $\lambda$.

1) Gold Rod: First, we investigate a gold [70] rod with $a = 100 \text{ nm}$, excited by a plane wave linearly polarized along the direction $(\hat{x} + \hat{y})/\sqrt{2}$ and propagating along the $\hat{z}$ axis. In Fig. 13, we plot the spectrum of the scattering efficiency $\sigma_{sca}$, obtained by increasing the number of modes $N^\parallel = N^\perp = 5, 10, 15$. We take as reference the loop/star solution, with $5984$ total degrees of freedom. Only five longitudinal and transverse modes (20 total degrees of freedom) are sufficient to achieve a sufficiently good agreement with the reference solution over the whole investigated spectrum, demonstrating a drastic reduction of the total number of unknowns.

In Fig. 14, we carry out a more systematic analysis of the error on the scattering cross section $\epsilon(\sigma_{sca})$ (a), and on the equivalent surface currents $\epsilon(j)$ (b). In comparison to the sphere case, these errors appear to be slightly higher. However, similar to our previous numerical experiments, we can
conclude that: 1) the errors associated with the surface currents are an order of magnitude higher than the errors related to scattering efficiency and 2) the rate of convergence is not uniform as a function of the number of employed modes and it slows down as $N^\parallel = N^\perp$ increases.

2) High-Permittivity Rod: Next, we examine a high-permittivity rod, assuming a constant relative permittivity over the investigated frequency spectrum, with a value of $\varepsilon_R = 16$. The scattering efficiency spectrum ($\sigma_{sca}$) is presented in Fig. 15, where it’s calculated using an increasing number of static modes. We use as reference the loop/star solution, with a total of 5984 degrees of freedom. If the wavelength is much larger than the dimension of the rod, $N^\perp = N^\parallel = 15$ suffices to accurately describe the scattering cross section. However, the precision deteriorates when the wavelength is comparable to the rod’s linear dimensions, especially near high-frequency resonance peaks. Only by increasing the number of modes $N^\perp = N^\parallel$ to 55, all the resonance peaks, including the high-frequency ones, are correctly described.

Fig. 16 offers a more quantitative analysis of the errors (a) $\epsilon[\sigma_{sca}]$ and (b) $\epsilon[j]$. We observe that, for larger wavelength an error below $\epsilon[\sigma_{sca}] < 0.01$ and $\epsilon[j] < 0.2$, can be achieved with $N^\perp = N^\parallel = 25$ modes. However, when the wavelength becomes comparable to the rod’s largest dimension, as many as 55 longitudinal and transverse static modes are necessary to contain the error. Even then, the precision deteriorates near the resonance peaks. To verify the convergence of the static mode expansion, we also tested with $N^\perp = N^\parallel = 500$. We obtained errors less than 0.0015 for $\sigma_{sca}$ and less than 0.0068 for $j$ across the entire investigated spectral range.
To conclude, while a higher number of modes is needed to accurately describe the unknown current densities compared to previous scenarios, the required count remains significantly lower than the number of loop/star functions needed to achieve comparable accuracy.

C. Solution of Multiple Scattering Problems

The use of static mode basis may be particularly convenient for numerically solving multiple scattering problems from arrays of objects of the same shape but varying in size and orientation. In this section, we examine finite-size periodic arrays of \( n \) particles, composed of spheres and triangular prisms. In both cases, the particles are placed at the nodes of a \( \sqrt{n} \times \sqrt{n} \) square grid of pitch 250 nm. These arrays are excited by a plane wave linearly polarized in the plane of the array, and propagating in the direction orthogonal to the array’s plane with a wavelength \( \lambda = 600 \) nm.

Table I displays the errors made in the evaluation of the scattering efficiency and of the surface currents via the static mode solution of electromagnetic scattering from several arrays of spheres. We assume the loop/star solution as a reference. The arrays consist of \( n = 1, 4, 9, 16, 25 \) spherical particles of radius \( R = 100 \) nm, and are excited by a plane-wave linear polarization along an array’s axis. The single sphere’s surface mesh has 200 nodes and 396 triangles, which corresponds to a maximum edge length of the mesh’s triangles \( \approx \lambda/17 \). A loop/star description results in 1188 \( n \) unknowns. We instead utilized \( N^\parallel = 10 \) longitudinal and \( N^\perp = 10 \) transverse modes to describe the electric and magnetic surface currents on each nanoparticle, leading to 40 \( n \) unknowns. The errors achieved consistently remain below 4\% for both the scattering efficiency and the currents.

In Fig. 17, we show the magnitude of the electric field on the particles’ surface of a 13 \( \times \) 13 finite size array calculated using the static mode expansion.

In addition, in Fig. 18, we show the magnitude of the electric field on the particles’ surface of a 13 \( \times \) 13 array made of equilateral triangular prisms with a base edge of 170 nm, a height of 153 nm, and a thickness of 50 nm. The plane wave exhibits linear polarization along the height of the base of the triangles, with unit intensity. The individual prism is characterized by a surface mesh with 182 nodes and 360 triangles. We considered \( N^\parallel = 10 \) longitudinal and \( N^\perp = 10 \) transverse modes to describe the electric and magnetic surface currents on each nanoparticle, achieving a reduced count of 6760 unknowns.
Fig. 19. (a) CPU-time (in seconds) of the different stages of the numerical solution of the scattering problem from a gold sphere with $R = 100$ nm excited by linearly polarized plane wave with $\lambda = 620$ nm as a function of the number of modes $N^\perp = N^\parallel$, for a given mesh $N_p = 500$, $N_t = 996$. (b) Ratio between the total CPU-time for the computation of the PMCHWT solution using the loop/star basis $t_{\text{loop/star}}$ and using the static mode basis $t_{\text{modes}}$. This ratio is evaluated as a function of the number of static modes $N^\perp = N^\parallel$ for different meshes with different number of nodes $N_p$.

V. COMPUTATIONAL COSTS

The execution of our numerical algorithm consists of four distinct stages: 1) the “assembly” stage, during which the matrices $T_{\pm}$, $K_{\pm}$ and of the vectors $E_0$, $H_0$ are constructed in the loop/star basis; 2) the “static mode generation” stage, where the longitudinal/transverse static modes of an isolated object are computed by solving the eigenvalue problems (15) and (16); 3) the “compression” stage, which involves transforming matrices $T_{\pm}$, $K_{\pm}$ from a loop/star representation into a static modes representation; and 4) the “inversion” stage, during which direct inversion of the compressed matrices is performed using LU decomposition. In the following, we provide a time-analysis of the various phases of the numerical algorithm for two test cases: an isolated sphere, and an array of spheres. The FORTRAN code used for this analysis operates on a single CPU (Intel Xeon-Gold 6140 M 2.3 GHz).

A. Isolated Sphere

Fig. 19(a) illustrates the CPU-time (in seconds) taken by each stage of the numerical solution for the scattering problem from the isolated gold sphere of Section IV-A1. This is measured against the number of longitudinal and transverse static modes $N^\perp = N^\parallel$, for a predetermined mesh ($N_p = 500$, $N_t = 996$). Within the examined range ($N^\parallel = N^\perp = 1, \ldots, 100$) the time taken for the “inversion” and the “static mode generation” stages remains always negligible if compared with the “assembly” and “compression” stages. The compression stage typically demands the most time, increasing linearly with the number of transverse/longitudinal modes used.

Fig. 19(b) shows the ratio of CPU-time needed for the PMCHWT solution using the loop/star basis ($t_{\text{loop/star}}$) to that needed for a solution using $N^\perp = N^\parallel$ static modes ($t_{\text{modes}}$). This analysis was repeated across various mesh densities, which are defined by the number of nodes $N_p$.

Fig. 20. (a) CPU-time (in seconds) of the different stages of the numerical solution of the scattering problem from a gold sphere with $R = 100$ nm excited by linearly polarized plane wave with $\lambda = 620$ nm, as a function of the mesh density (number of number of nodes $N_p$), for a prescribed number of static modes $N^\perp = N^\parallel = 10$. (b) Ratio between the total CPU-time for the computation of the PMCHWT solution using the loop/star basis $t_{\text{loop/star}}$ and using the static mode basis $t_{\text{modes}}$. This ratio is evaluated as a function of the mesh density, for different number of static modes.

Fig. 21. CPU-time (in seconds) of the different stages of the numerical solution of the scattering problem from a square array of $n$ spheres (radius 100 nm, edge-edge distance 250 nm): assembly of the matrices using the loop/star basis; generation of the longitudinal/transverse static modes; compression, by passing from loop/star to static modes; direct inversion (LU); total time. The code is implemented in FORTRAN, and run on a single CPU.

Fig. 22. Total execution CPU-time (in seconds) of the numerical solution of the scattering problem from an array of $n$ spheres by using loop/star basis functions and by using the static modes basis. Both codes are implemented in FORTRAN and run on a single CPU. The two fitting curves are shown with a continuous line. The same sphere’s mesh ($N_p = 200$, $N_t = 396$) is used both for calculating the static modes and for the loop/star solution.
The same mesh was used to calculate both the static modes and the loop/star solution.

Fig. 20(a) presents the CPU-time (in seconds) of the different stages of the numerical solution of the scattering problem, but this time as a function of the mesh density, i.e., the number of nodes $N_p$, with a prescribed number of modes $N^\parallel = N^\perp = 10$. Within the investigated range ($N_p \leq 2100$), the time taken for the matrix inversion using LU decomposition remains negligible compared to the time required for the assembly of the matrices $K_{\pm}$ and $T_{\pm}$ and for their compression. Both assembly and compression times are proportional to $N_p^2$. Compression is still typically the most time-consuming stage, although for very dense meshes, the time spent for static mode calculation becomes comparable, as it is proportional to $N_p^3$.

Fig. 20(b) displays the ratio of $t_{\text{loop/star}}$ to $t_{\text{nodes}}$ as a function of the mesh density. This analysis was repeated for different numbers of modes. The same mesh was used for calculating the static modes and the loop/star solution. We found that the static mode solution is advantageous, i.e., $t_{\text{loop/star}} > t_{\text{nodes}}$, only for sufficiently dense meshes.

B. Particles’ Array

In Fig. 21, we show the CPU-time (in seconds) of the different stages of the PMCHWT solution of the electromagnetic scattering problem a finite-size periodic array of spheres, as a function of the number $n$ of spheres ($n = 1, 4, 9, 16, \ldots, 169$). The geometry and mesh are identical to those detailed in Section IV-C. We represent the electric and magnetic surface currents on each nanoparticle using $N^\parallel = 10$ longitudinal and $N^\perp = 10$ transverse modes, thus $40n$ unknowns.

In the investigated range ($n \leq 169$), the total computational time is always dominated by the compression time, which scales as $\propto n^2$. The assembly time, which also scales as $\propto n^2$, contributes significantly to the total execution time. The time required for the computation of the static modes of the isolated sphere, which are used as basis, is negligible and does not depend on the array’s size. The inversion time, which scales as $\propto n^3$, is negligible compared with the compression time if, as in our case, the dimensions of the arrays do not surpass a given number of particles ($n_{th}$). The value of $n_{th}$ can be estimated by extrapolating the curves in Fig. 21, and it is $n_{th} \approx 80,000$ particles.

Fig. 22 compares the total CPU-time required for the PMCHWT solution using the static modes basis with the corresponding time required for the loop/star solution, as a function of $n$. The same sphere’s mesh ($N_p = 200, N_c = 396$) is used both for calculating the static modes and for the loop/star solution. For $n > 4$, the static mode solution proves more advantageous, with the speed-up further increasing as $n$ increases.

To deduce the scaling laws, we fit the total computational time of the PMCHWT solution in terms of the static basis using the curve $t = 3n^2$, and in terms of the loop/star solution using the curve $t = 0.92n^3$. The ratio is $0.3 \, n$. Consequently, for a $5 \times 5$ array, the CPU-time required for the loop/star solution is 7.5 times greater than the time required for the static mode solution. For $n = 169$, we extrapolate that the static mode solution is 50 times faster. However, this could not be empirically verified as the time estimated for the loop/star solution computation was prohibitively long.

VI. Conclusion

In this article, we introduced a set of “static” surface current modes and used them to expand the unknown surface current densities in the surface integral equations governing the electromagnetic scattering problem from penetrable objects. We demonstrated the efficacy of the static mode expansion in the PMCHWT formulation [21], [42], [43]. We found several characteristics that make the use of static modes appealing:

1) The retarded Green’s function, constituting the kernel of integral operators recurring in surface integral formulations such as the PMCHWT, may be decomposed as the sum of the static Green’s function (with integrable singularity) and a proper difference (which is a regular function). The resulting integral operators containing the static Green’s function are diagonalized by the static modes, regularizing the overall problem [72].

2) The use of the static mode expansion, when combined with an appropriate rescaling and rearranging of the unknowns, makes the PMCHWT formulation immune from the low-frequency breakdown problem.

3) The static modes only depend on the shape of the object, making them universally applicable across varying operating frequencies and material properties of the object. This facilitates a unified, frequency-independent description of any scattering scenario involving objects of a given shape, outperforming other basis sets (like characteristic modes) that depend on frequency and materials.

As test cases, we considered the scattering from both metal and high-permittivity dielectric objects. We found that for objects of size smaller than the wavelength of operation, only a few modes are sufficient to accurately describe the emergent scattering response. The static mode expansion, thus, significantly cuts down the unknowns compared to loop/star or Rao-Wilton-Glisson function discretization, without compromising solution accuracy.

The most time-consuming stage of the PMCHWT numerical solution using static modes and direct inversion is typically the “compression” stage, which is the change of the basis used to represent the impedance matrix from the loop/star set to the static modes set. In the scattering problem from an isolated particle, the use of the static modes set is convenient in terms of total CPU-time, compared with the PMCHWT numerical solution using loop/star set and direct inversion, only when few static modes are needed to describe the surface currents. In contrast, the advantages of static modes are more pronounced in multiple scattering problems from particle arrays of $n$ objects with similar shapes but differing orientations and sizes. Even for small arrays, the static modes solution is faster than the loop/star solution with direct inversion, with this lead increasing linearly with particle count $n$.

Notwithstanding, for scattering problems involving object arrays, optimal CPU-time performance is usually achieved...
through fast factorization schemes, iterative solvers, and effective preconditioning [4]. Nevertheless, the compression of the matrix equations by static modes can be advantageous in several scenarios. For instance, when calculating the scattering from a specified particle array under numerous excitation conditions—a scenario where iterative solvers fall short, as the iterations need to be started anew for each right-hand side [37]. In addition, since implementing fast factorization schemes may be expensive in terms of human cost, static mode compression allows baseline implementations of the method of moments without fast factorization to address problems that would not be possible otherwise.

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Open Access funding provided by 'Università degli Studi di Napoli ”Federico II”’ within the CRUI CARE Agreement