Description of Projective Transvection of Projective Special Linear Group $PSL_2(K)$ Over a Field $K$

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Abstract. The purpose of this paper is to introduce one of the most important elements closely connected with the classical linear group namely projective special linear groups $SL_2(K)/Z$ of degree 2 over a field $K$, where $Z$ is the center of the special linear group $SL_2(K)$ consists of all scalar matrices whose determinants 1. This element called projective transvection which is an element of the projective special linear group $PSL_2(K)$ that is, a coset $gZ$ where $g$ is some element in $SL_2(K)$. In the present study, we will extend our conception about the projective transvection, so we can say this element is generating set of $PSL_2(K)$. For that reason, we need to describe this element for obtaining more conception about $PSL_2(K)$ especially to describe subgroups of $PSL_2(K)$ that have been generated by projective transvection.

1. Introduction

In this paper, we shall consider one important class of groups, linear (or matrix) groups which has numerous applications to many parts of mathematics, to science, in general, and to computer science, in particular. Study groups include the investigation of its quotient groups and its subgroups since this allows us to gain more information about the group itself and thereby to obtain extensive knowledge of the structure of the group. we will introduce one of the most important and central concepts of modern mathematics, which is an element that is called projective transvection. The precise description of this element gives us a comprehensive concept about studying subgroups of $PSL_2(K)$ generated by projective transvection of degree 2 over a field $K$ thereby developing the materials that are contained in [1], [2], and [3]. In [4], they proved the linear group is generated by transvections. In [1], Bashkirov has proved analogue results for arbitrary (infinite) fields. In [5], Cuypers and Steinbach, they proved Special linear groups generated by transvections. In [6], Humphries, he proved subgroups of $SL(3,Z)$ generated by transvections. In [7], McLaughlin, he proved Some groups generated by transvections. In [8], Steinbach, he proved Subgroups of classical groups generated by transvections. For those reasons, if we want to study a generation set of $PSL_2(K)$ should introduce the main tool for that of course, we mean projective transvection. The generating set for the group $SL_2(K)$ is formed by matrices which are known as transvections or more precisely elementary transvections. We give some initial information about these elements in the next section of the present study. We also regard transvections as a particular case of more general matrices or linear transformations, namely as a particular case of quadratic unipotent elements, such that all matrices $x$ satisfying the equation $(x - 1)^2 = 0$. 

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2. Preliminary and results

Let $s \in K^n$ (column) and $\psi \in \mathbb{K}$ (row). Then both product $s\psi, \psi s$ are defined. The product of $\psi$ and $s$, whereas $s\psi$ is an $n \times n$ matrix with entries from $K$. Assume $\psi s = 0$. Then the $n \times n$ matrix $g = I_n + s\psi$ is called a transvection. More precisely, we say that $g$ is the transvection corresponding to the pair $(s, \psi) \in K^n \times \mathbb{K}$. Each transvection is an invertible matrix. Indeed,

$$(I_n + s\psi)(I_n - s\psi) = I_n + s\psi - s\psi s\psi = I_n, \quad s\psi s\psi = 0,$$

since $\psi s = 0$. Similarly $(I_n - s\psi)(I_n + s\psi) = I_n$, therefore $(I_n + s\psi)^{-1} = I_n - s\psi$. Thus we have shown that each transvection is an element of the group $GL_n(K)$. It can be shown also that the determinant of each transvection is equal to 1, and so transvections belong to the special linear group $SL_n(K)$. Note that $I_n - s\psi$ is a transvection too.

The transvection $I_n + \alpha e_{ij}$, such that $e_{ij}$ a standard matrix unit which has 1 in its $(i, j)$ location and elsewhere is zero. This transvection is called an elementary transvection and denoted by $t_{ij}(\alpha)$. Thus by [9]

$$t_{ij}(\alpha) = I_n + \alpha e_{ij} \quad (i \neq j, \alpha \in K).$$

The determinant of each elementary transvection is 1, and so all elementary transvection are in the special linear group, $SL_n(K)$.

If $g$ is a transvection, that is if $g = I_n + s\psi$, where $s \in K^n, \psi \in \mathbb{K}$ and $\psi s = 0$, then

$$g - I_n = I_n + s\psi - I_n = s\psi,$$

and hence

$$(g - I_n)^2 = s\psi s\psi.$$

But since $\psi s = 0$, we obtain that

$$(g - I)^2 = s \cdot 0 \cdot \psi = 0_n,$$

where $0_n$ designates the zero $n$ by $n$ matrix.

Any matrix $x \in M_n(K)$ satisfying the condition $(x - I_n)^2 = 0$ this matrix is called a quadratic unipotent. Thus transvections give a particular example of quadratic unipotent matrices.

**Theorem 1.** ([6]) The group $SL(n, K)$ is generated by minimal of transvections $n$. In this case, the transvections $t_{12}(1), t_{23}(1), \ldots, t_{n-1,n}(1), t_{n,n}(1)$ generate $SL(n, K)$.

**Lemma 2.** ([10]) If $\alpha$ is an algebraic element over $k \neq GF(3)$, when $k$ be an infinite field of characteristic not equal 2, then the group are generated by all matrices

$$\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \alpha r & 1 \end{pmatrix}$$

coincides with the group $SL_2(k(\alpha r))$.

we introduce elements which are the main tool of our investigation, namely, projective transvections.

3. Projective Transvection

An element $x \in PSL_2(k)$, that is, a coset

$$gZ = \{g, -g\},$$

where $g$ is some element in $SL_2(k)$ will be called a projective transvection if $x$ as a coset of $Z$ in $SL_2(k)$ has a representative which is a transvection of $SL_2(k)$.

Thus an element

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} Z \in PSL_2(k)$$

coincides with the group $PSL_2(k(\alpha r))$. 

we introduce elements which are the main tool of our investigation, namely, projective transvections.
is a projective transvection because the matrix
\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\]
is a transvection of the group $SL_2(k)$ (an elementary transvection). We have
\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix} Z = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, -\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}
= \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \right\}.
\]
Also
\[
\begin{pmatrix}
-1 & -1 \\
0 & -1
\end{pmatrix} Z = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} Z,
\]
and therefore the coset
\[
\begin{pmatrix}
-1 & -1 \\
0 & -1
\end{pmatrix} Z \in PSL_2(k)
\]
is also a projective transvection. Note, however, that the matrix
\[
\begin{pmatrix}
-1 & -1 \\
0 & -1
\end{pmatrix}
\]
is not a transvection. To see this, we recall that every transvection $g \in SL_2(K)$ is a quadratic unipotent element, that is, $(g - I)^2 = 0$. But
\[
\left( \begin{pmatrix}
-1 & -1 \\
0 & -1
\end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & +1 \end{pmatrix} \right)^2 = \begin{pmatrix}
-2 & -1 \\
0 & -2
\end{pmatrix}^2 = \begin{pmatrix}
4 & 4 \\
0 & 4
\end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]
because by our assumption the field $k$ is of characteristic not 2, and hence 4 \neq 0. Thus a coset
\[
g Z \in PSL_2(k) \ (g \in SL_2(k))
\]
can be a projective transvection even in the status $g$ is not a transvection. Also $PSL_2(k)$ contains elements which are not transvection at all. As a concrete example of such elements, we assume that char $k \neq 3$ (for instance, we can take $k = \mathbb{Q}$) and consider the coset
\[
\begin{pmatrix}
\frac{1}{2} & 0 \\
0 & 2
\end{pmatrix} Z = \left\{ \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -2 \end{pmatrix} \right\} \in PSL_2(k).
\]
Then
\[
\left( \begin{pmatrix}
\frac{1}{2} & 0 \\
0 & 2
\end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)^2 = \begin{pmatrix}
-\frac{1}{2} & 0 \\
0 & 1
\end{pmatrix}^2 = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix}
\]
which is distinct from the zero matrix because $1 \neq 0$ and char $k \neq 2, 3$; and
\[
\left( \begin{pmatrix}
-\frac{1}{2} & 0 \\
0 & -2
\end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)^2
\]
\[
\begin{pmatrix}
-\frac{3}{2} & 0 \\
0 & -3
\end{pmatrix}^2 = \begin{pmatrix}
\frac{9}{2} & 0 \\
0 & 3
\end{pmatrix}
\]

which is distinct from the zero matrix in view of our assumption \( \text{char } k \neq 2, 3 \). Thus both of

\[
\begin{pmatrix}
\frac{1}{2} & 0 \\
0 & 2
\end{pmatrix}, \begin{pmatrix}
-\frac{1}{2} & 0 \\
0 & -2
\end{pmatrix}
\]

are not quadratic unipotent elements and hence they are not transvections. This means that the coset

\[
\begin{pmatrix}
\frac{1}{2} & 0 \\
0 & 2
\end{pmatrix}Z
\]

is not projective transvection.

4. Intermediate Subgroups of PSL

In this section will be focus to describe intermediate subgroups of \( PSL_2 \) over various fields. First of all these results appeared in [1], when the author has described intermediate subgroups of \( SL_2 \) generated by transvection. This result is written as follows theorem.

**Theorem 3.** If \( k \) is a field of characteristic, not 2, and if \( K \) is an algebraic extension of \( k \), such that

\[
SL_2(k) \leq G \leq SL_2(K)
\]

then \( G \) contains as a normal subgroup the group \( SL_2(L) \), such that \( L \) is a subfield of \( K \) contains \( k \), \( (k \leq L \leq K) \). If we have \( G \) generated by the transvections, then \( G = SL_2(L) \).

The intention of the present study is to gain some additional results of theorem 3 with regard to the case of projective linear groups. In [11] the authors acquired analogous results when they described subgroups of the general linear group that contained the special linear group over semilocal commutative rings (see Theorems 2.3, 4.1, and 5.1).

By the previous study will be using it in the next study the main element of the present that called a projective transvection, we will describe intermediate subgroups generated by projective transvection lying between tow projective special linear groups over various fields. This description is given by the following theorem.

**Theorem 4.** Suppose \( K \) is an algebraic extension of \( k \), when \( k \) is a field of characteristic \( \neq 2 \) containing more than 9 element. If

\[
PSL_2(k) \leq X \leq PSL_2(K),
\]

let \( L \) be a subfield of \( K \) containing \( k \) such that the group \( PSL_2(L) \) as a normal subgroup contained in \( X \). If \( X \) is generated by projective transvections, then \( X = PSL_2(L) \).

5. Conclusion

In this paper, we have been extended our conception of transvection, elementary transvection, quadratic unipotent, and projective transvection. By these elements, we are able to describe intermediate subgroups between groups \( PSL_2(k) \) and \( PSL_2(K) \), when a field \( K \) of characteristic not equal 2 is an algebraic extension of a field \( k \).

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