The Hypermultiplet in N = 2 Superspace

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Abstract

Global N = 2 supersymmetry in four dimensions with a Fayet-Sohnius hypermultiplet and a complex central charge is studied in N = 2 superspace. It is shown how to construct the complete expansion of the hypermultiplet with respect to the central charge. In addition the low-energy effective action is discussed and it is shown that the ‘kernel’ of the Lagrangian only needs an integration over a ‘small’ superspace to construct a supersymmetric action.

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In this letter, a covariant formulation of $N = 2$ superspace with global $N = 2$ supersymmetry and a complex central charge is studied. Because of the interesting physical properties of $N = 2$ supersymmetric theories like, for instance, electric-magnetic duality [1,2] they have been extensively studied during the past years [3,4,5,6]. The supersymmetry generators $\Delta_A$ of any supersymmetric theory obey a graded Lie algebra

$$[\Delta_A, \Delta_B] = \Delta_A \Delta_B - (-)^{ab} \Delta_B \Delta_A = T_{AB}^C \Delta_C.$$  

(1.1)

The torsions $T_{AB}^C$ obey certain constraints depending on the number of supersymmetry generators and the structure group considered. It is possible to perform a change of basis in such a way that the supersymmetry generators become derivatives in superspace and the momentum operator becomes the usual partial derivative of four-dimensional Minkowski space. In this new basis the supersymmetry algebra is

$$[D_A, D_B] + i T_{AB}^C D_C = 0$$

(1.2)

In $N = 2$ supersymmetric theories central charges can occur in the supersymmetry algebra [7]. The corresponding $N = 2$ supersymmetry multiplets may be charged with respect to this central charge. For instance, the hypermultiplet [8] carries central charge, whereas the vector multiplet does not [10, 11]. In the context of the electric-magnetic duality the central charge plays a crucial rôle because the electric and the magnetic charges can appear as a complex central charge in the $N = 2$ supersymmetry algebra [12].

Hypermultiplets are of special interest in the analysis of electric-magnetic duality and non-perturbative dynamics, because in low-energy effective actions for $N = 2$ supersymmetric Yang-Mills theories the monopoles and dyons are massive hypermultiplets [2,3]. In a free field theory they saturate the BPS bound on-shell

$$M^2 \geq |Z|^2.$$  

(1.3)

Here $M$ denotes the mass of the soliton and $Z$ its central charge. These results of $N = 2$ supersymmetric gauge theories have been already applied to string theory [4,5,6]. Moreover, because black holes are the solitons of quantum gravity, hypermultiplets also occur in the discussion of extreme black holes [13,14]. It is essential in all these considerations that the hypermultiplet is a $N = 2$ supersymmetric multiplet with spin $\leq 1/2$ describing electrically and magnetically charged matter in four dimensions. However, the difficult off-shell structure of the hypermultiplet doesn’t play any rôle.

The discussion of supersymmetric quantum field theories in the framework of a superspace formulation is an old and successful approach. Although the $N = 1$ superspace formulation is established and the general couplings are well known, things are different in theories with $N \geq 2$. For instance, Berkovits and Siegel recently discussed superspace effective actions of heterotic and Type II superstrings [15] and found contradictions with the standard folklore [13] that Type II string loops are counted by just a hypermultiplet.
All these results and considerations motivate trying to gain a better understanding of $N = 2$ superspace, the hypermultiplet and the central charge on general grounds. One goal in the present paper is to give the off-shell expansion of the hypermultiplet in the presence of a complex central charge in a covariant formulation of $N = 2$ superspace.

In 1978 Sohnius wrote down a superfield representation of the hypermultiplet in the presence of one central charge and a remarkable action formula in superspace [9]. In the present paper his approach is the starting point. It is shown that in the presence of a complex central charge an additional constraint must hold to keep the minimal field content of the Fayet-Sohnius hypermultiplet. In particular it is shown how to construct the complete expansion of the hypermultiplet with respect to the central charge. In addition the low-energy effective action is discussed and it is shown that the ‘kernel’ of the Lagrangian only needs an integration over a ‘small’ superspace to construct a supersymmetric action.

The letter is organized as follows: First the $N = 2$ supersymmetry algebra and $N = 2$ superspace is introduced. Then the hypermultiplet is discussed at the superfield and the component level in great detail. A discussion of the hypermultiplet exists also in the harmonic superspace formalism [16], but this approach is not considered here. Moreover there exists another approach to consider central charge transformations of $N = 2$ scalar multiplets [18].

Using the conventions of [19] the $N = 2$ supersymmetry generators $\Delta_A = \{P_m, Q^i_A, Q^{ij}, Z, \bar{Z}, \}$ obey the following algebra

\[
\begin{align*}
\{Q^i_A, Q^{ij}_B\} & = 2 \delta^i_j \sigma^m_{\alpha\bar{\alpha}} P_m \\
\{Q^i_A, Q^j_B\} & = 2 g^{ij} \varepsilon_{\alpha\beta} Z \\
\{\bar{Q}_{\dot{a}i}, \bar{Q}_{\dot{b}j}\} & = -2 g_{ij} \varepsilon_{\dot{a}\dot{b}} \bar{Z} \\
\end{align*}
\]

(1.4)

All the other graded commutators vanish. Performing the change of basis the supersymmetry algebra in the new basis $D_A = \{\partial_m, D^i_{\alpha}, \bar{D}_{\dot{a}j}, \partial_z, \partial_{\bar{z}}\}$ is

\[
\begin{align*}
\{D^i_{\alpha}, D^j_{\beta}\} & = -2i \delta^i_j \sigma^m_{\alpha\bar{\alpha}} \partial_m \\
\{D^i_{\alpha}, \bar{D}^j_{\dot{b}}\} & = -2i g^{ij} \varepsilon_{\alpha\beta} \partial_z \\
\{\bar{D}^i_{\dot{a}}, \bar{D}^j_{\dot{b}}\} & = -2i g_{ij} \varepsilon_{\dot{a}\dot{b}} \partial_{\bar{z}} \\
\end{align*}
\]

(1.5)

Note that $\partial_{\bar{z}} = - \partial^+ \bar{z}$ holds in these conventions. Again all the other graded commutators vanish. This follows from the $N = 2$ torsion constraints:

\[
\begin{align*}
T^m_{\alpha\dot{a} j} = 2 \sigma^m_{\alpha\bar{\alpha}} \delta^i_j & \quad T^{zij} = 2 \varepsilon_{\alpha\beta} g^{ij} & \quad T^{\dot{a}\dot{b}jk} = 2 \varepsilon^{\dot{a}\dot{b}} g^{jk} \\
\end{align*}
\]

(1.6)

\[\text{1} \text{ The so-called ‘relaxed hypermultiplet’ [17] is not considered here.}\]

\[\text{2} \text{ The following conventions concerning the internal } SU(2)_R \text{ symmetry of } N = 2 \text{ supersymmetry have been used: } Q^i_{\alpha} = \bar{Q}_{\dot{a} i}, g^{12} = 1, g^{ij} = -g^{ji} = -g_{ij}, g^{ij} g_{jk} = \delta^i_k \]
Here the index $z$ is used as an internal index and all other torsions vanish. The grading of the generators follows from their index-structure: $|P_m| = |m| = |Z| = |z| = 0$ and $|Q^i_\alpha| = |\alpha| = |\bar{Q}_{\dot{\alpha}j}| = |\dot{\alpha}| = 1$. By the use of the $N = 2$ superspace coordinates $z^M = (x^m, \theta^\alpha_i, \bar{\theta}^{\dot{\alpha}j}, z, \bar{z})$ an explicit representation of the supersymmetry generators can be given with $\partial^i_\alpha = \frac{\partial}{\partial \theta^\alpha_i}$ and $\bar{\partial}_{\dot{\alpha}j} = \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}j}}$:

$$
Q^i_\alpha = \partial^i_\alpha - \sigma^m_{\alpha \dot{\alpha}} P_m \bar{\theta}^{\dot{\alpha}i} + g^{ij} \varepsilon_{\alpha \beta} \theta^\beta_j Z
$$

$$
\bar{Q}_{\dot{\alpha}j} = - \bar{\partial}_{\dot{\alpha}j} + \theta^\alpha_j \sigma^m_{\alpha \dot{\alpha}} P_m + g_{jk} \varepsilon_{\dot{\alpha} \beta} \bar{\theta}^{\dot{\beta}k} \bar{Z}
$$

(1.7)

A change of basis leads to the following representation of the spinorial derivatives in superspace

$$
D^i_\alpha = \partial^i_\alpha + i \sigma^m_{\alpha \dot{\alpha}} \partial_m \bar{\theta}^{\dot{\alpha}i} - i g^{ij} \varepsilon_{\alpha \beta} \theta^\beta_j \partial_z
$$

$$
\bar{D}_{\dot{\alpha}j} = - \bar{\partial}_{\dot{\alpha}j} - i \theta^\alpha_j \sigma^m_{\alpha \dot{\alpha}} \partial_m - i g_{jk} \varepsilon_{\dot{\alpha} \beta} \bar{\theta}^{\dot{\beta}k} \partial_{\bar{z}}
$$

(1.8)

These spinorial derivatives commute with the standard supersymmetry generators $Q^i_\alpha$:

$$
\{D^i_\alpha, Q^j_\beta\} = \{D^i_\alpha, \bar{Q}^j_{\dot{\beta}}\} = \{\bar{D}_{\dot{\alpha}i}, Q^j_\beta\} = \{\bar{D}_{\dot{\alpha}i}, \bar{Q}^j_{\dot{\beta}}\} = 0.
$$

(1.9)

Now the concept of integration in $N = 2$ superspace can be introduced. With the definition

$$
\int d\theta^\alpha_i = 0 \quad \int d\theta^\alpha_i \theta^\alpha_i = 1
$$

(1.10)

one finds the well known result that integration and differentiation give the same result for Grassmann variables:

$$
\int d\theta^\alpha_i = \frac{1}{4} D^\alpha_i
$$

(1.11)

Analogous one finds the following identities in superspace

$$
\int d\bar{\theta}^\dot{\alpha}_i = - \frac{1}{4} \bar{D}_{\dot{\alpha}i}
$$

$$
\int d^2 \theta^{\alpha j} = \frac{1}{12} D^\alpha_i D^j_\alpha
$$

$$
\int d^2 \bar{\theta}^{\dot{\alpha} j} = \frac{1}{12} \bar{D}_{\dot{\alpha}j} \bar{D}^{\dot{\alpha} i}
$$

(1.12)

To work in a basis independent way a vielbein $e^A_M(z)$ with $A \sim (a, \alpha i, \dot{\alpha} \dot{i}, z, \bar{z})$ and its inverse can be introduced.
Using $D_A = e_A^M \frac{\partial}{\partial z^M}$ the exterior derivative is $d = e^A D_A$. In particular there is also an explicit representation of the vielbein

$$e_A^M = \begin{pmatrix}
\delta^m_a & 0 & 0 & 0 & 0 \\
i \sigma^m_{\alpha \dot{\alpha}} \theta^{\dot{\alpha}} & \delta^\alpha_\mu \delta^\mu_j & 0 & i \varepsilon_{\alpha \beta} \theta^\beta_j g^{ji} \\
 i g^{ij} \theta^\alpha_j \sigma^m_{\alpha \dot{\alpha}} \varepsilon^{\dot{\alpha} \dot{\beta}} & 0 & \delta^\dot{\alpha}_{\dot{\mu}} \delta^\mu_i & 0 & i \bar{\theta}^{\dot{\alpha} i} \\
 0 & 0 & 0 & \delta^z_i & 0 \\
 0 & 0 & 0 & \delta^\bar{z}_i & 0 \\
 0 & 0 & 0 & \delta^z & 0 \\
 0 & 0 & 0 & \delta^{\bar{z}} & 0
\end{pmatrix}$$

(1.14)

Its inverse is given in this specific basis as

$$e_M^A = \begin{pmatrix}
\delta_m^a & 0 & 0 & 0 & 0 \\
-i \sigma^a_{\mu \dot{\mu}} \bar{\theta}^{\mu j} & \delta^\alpha_\mu \delta^\mu_j & 0 & 0 & 0 \\
-i g^{ik} \theta^\mu_k \sigma^a_{\nu \dot{\nu}} \varepsilon^{\dot{\nu} \mu} & 0 & \delta^\dot{\mu}_{\dot{\alpha}} \delta^\alpha_i & 0 & -i \bar{\theta}^{\dot{\mu} j} \\
0 & 0 & 0 & \delta^z_i & 0 \\
0 & 0 & 0 & \delta^{\bar{z}}_i & 0 \\
0 & 0 & 0 & \delta^z & 0 \\
0 & 0 & 0 & \delta^{\bar{z}} & 0
\end{pmatrix}$$

(1.15)

Moreover, in this basis there is torsion, which is in general defined as the exterior derivative of the vielbein $de^A = i T^A = \frac{1}{2} e^C e^B i T_{BC}^A$. In our specific basis this leads to

$$
\begin{align*}
d e^a &= e^a_i e^{\dot{\alpha} j} i T_{ai}^{\alpha j} = e^a_i e^{\dot{\alpha} j} 2 i \sigma^a_{\alpha \alpha} \delta^i_j \\
d e^z &= \frac{1}{2} e^\alpha_i e^\beta_j i T_{\alpha \beta}^{zi} = e^\alpha_i e^\beta_j i \varepsilon_{\alpha \beta} g^{ij} \\
d e^{\bar{z}} &= \frac{1}{2} e_\dot{\alpha} j e_\dot{\beta} k i T^{\dot{\alpha} \dot{\beta} z j} = e_\dot{\alpha} j e_\dot{\beta} k i \varepsilon^\dot{\alpha} \dot{\beta} g^{jk}
\end{align*}
$$

(1.16)

Let us now discuss the hypermultiplet in the framework of this $N = 2$ superspace. The $8_B + 8_F$ hypermultiplet

$$\phi_i \sim ( A_i | \chi_\alpha \bar{\psi}_{\dot{\alpha}} || F_i )$$

(1.17)

contains two complex scalars, two Weyl fermions and two complex auxiliary fields - even if the central charge is complex. It obeys the following constraint:

$$D^i_\alpha (i \phi^j) = 0 \quad \bar{D}_{\dot{\alpha}} (i \phi^j) = 0$$

(1.18)
In addition the following reality constraint must hold to eliminate superfluous component fields if the central charge is complex:

\[(\partial_z + \partial_{\bar{z}}) \phi_i = 0 \quad (\partial_z + \partial_{\bar{z}}) \bar{\phi}^i = 0 \quad (1.19)\]

This constraint is the only possibility to extend the hypermultiplet of Fayet and Sohnius \[8,9\] to the case of a complex central charge without introducing new component fields and preserving supersymmetry. Hence the phase of the central charge is irrelevant for the hypermultiplet of Fayet and Sohnius. Eq. (1.19) becomes trivial if the central charge is real. Moreover (1.19) yields the following power expansion of the hypermultiplet in terms of the internal coordinates \(z\) and \(\bar{z}\)

\[\phi_i = \sum_{k=0}^{\infty} \alpha_{k,i}(x, \theta, \bar{\theta}) (z - \bar{z})^k. \quad (1.20)\]

In the following we will determine all coefficients \(\alpha_{k,i}(x, \theta, \bar{\theta})\) and we will show that only the first coefficients need to be considered in a renormalizable low-energy effective theory in four dimensions.

Using (1.18) and the algebra one can calculate the following superfield identities:

**Superfield identities for \(\phi_i\):**

\[D^i_\alpha \phi_j = \frac{1}{2} \delta^i_j D^k_\alpha \phi_k\]
\[\bar{D}_{\dot{\alpha} i} \phi_j = -\frac{1}{2} g_{ij} \bar{D}_{\dot{\alpha} k} g^{kl} \phi_l\]
\[D^i_\alpha D^j_\beta \phi_j = -4 i \varepsilon_{\alpha \beta} g^{ij} \partial_{\bar{z}} \phi_j\]
\[\bar{D}_{\dot{\alpha} i} \bar{D}_{\dot{\beta} j} g^{jk} \phi_k = -4 i \sigma^m_{\alpha \dot{\beta}} \partial_m g^{ij} \phi_j\]
\[D^i_\alpha D^j_\beta (\bar{\phi}) = -4 i \sigma^m_{\alpha \dot{\beta}} \partial_m \phi_i\]
\[\partial_z D^i_\alpha \phi_i = -\sigma^m_{\alpha \alpha} \partial_m \varepsilon^{\alpha \beta} \bar{D}_{\dot{\beta} k} g^{ij} \phi_j\]
\[\partial_{\bar{z}} \bar{D}_{\dot{\alpha} i} g^{ij} \phi_j = -\sigma^m_{\alpha \dot{\alpha}} \partial_m \varepsilon^{\alpha \beta} D^i_\beta \phi_i\]

**Superfield identities for \(\bar{\phi}^i\):**

\[\bar{D}_{\dot{\alpha} i} \bar{\phi}^j = \frac{1}{2} \delta^i_j \bar{D}_{\dot{\alpha} k} \bar{\phi}^k\]
\[D^i_\alpha \bar{\phi}^j = -\frac{1}{2} g^{ij} D^k_\alpha g_{kl} \bar{\phi}^l\]
\[D^i_\alpha D^j_\beta g_{jk} \bar{\phi}^k = -4 i \varepsilon_{\alpha \beta} \partial_z \bar{\phi}^i\]
independent four-dimensional hypermultiplet component fields: α

\[ \phi^i = -4i \varepsilon_{\alpha \beta} g_{ij} \partial_z \bar{\phi}^j \]

\[ D^i \bar{D}^j \bar{\phi}^i = -4i \sigma_{\alpha \beta}^m \partial_m \bar{\phi}^j \]

\[ \bar{D}_{\dot{\alpha}} \bar{D}^j \bar{\phi}^i = -4i g_{ij} \sigma_{\beta \dot{\alpha}}^m \partial_m \bar{\phi}^j \]

\[ \partial_z \bar{D}_{\dot{\alpha}} \phi^i = -\sigma_{\alpha \dot{\alpha}}^m \partial_m \varepsilon^\beta \partial_\beta \bar{D}^i g_{ij} \bar{\phi}^j \]

\[ \partial_z D^i \phi_{\dot{\alpha}} = -\sigma_{\alpha \dot{\alpha}}^m \partial_m \varepsilon^\beta \partial_\beta \bar{D}_{\dot{\alpha}} \bar{\phi}^i \]

(1.22)

These identities yield the important result

\[ \Box \phi_i = \partial_z \partial_z \phi_i \quad \Box \bar{\phi}^i = \partial_z \partial_z \bar{\phi}^i. \]

(1.23)

Hence for any massive hypermultiplet the BPS bound (1.3) is saturated on-shell. Moreover we find by the use of (1.23) for the coefficients \( \alpha_{k,i} \) of the power expansion (1.20) the following recursion formula

\[ k (k - 1) \alpha_{k,i} + \Box \alpha_{k-2,i} = 0 \quad k \geq 2 \]

(1.24)

As a consequence of (1.24) we can construct the complete power expansion of the hypermultiplet in superspace from the coefficients \( \alpha_{0,i} \) and \( \alpha_{1,i} \). To do so we define the independent four-dimensional hypermultiplet component fields:

\[ \phi_i = A_i (x) \]

\[ D^i \phi_{\dot{\alpha}} = 2\sqrt{2} \chi_{\dot{\alpha}} (x) \]

\[ \bar{D}_{\dot{\alpha}} g^i \phi_j = -2\sqrt{2} \bar{\psi}_{\dot{\alpha}} (x) \]

\[ \partial_z \phi_i = F_i (x) \]

\[ \partial_z \phi_{\dot{\alpha}} = -F_{\dot{\alpha}} (x) \]

(1.25)

\[ \bar{\phi}^i = \bar{A}^i (x) \]

\[ \bar{D}_{\dot{\alpha}} \bar{\phi}^i = 2\sqrt{2} \bar{\chi}_{\dot{\alpha}} (x) \]

\[ D^i g_{ij} \bar{\phi}^j = 2\sqrt{2} \psi_{\alpha} (x) \]

\[ \partial_z \bar{\phi}^i = -\bar{F}^i (x) \]

\[ \partial_z \bar{\phi}_{\dot{\alpha}} = \bar{F}_{\dot{\alpha}} (x). \]

(1.26)

The component fields transform under supersymmetry transformations generated by the operator \( \delta = \xi^\alpha D^i + \bar{\xi}_{\dot{\alpha}} \bar{D}^i \bar{D}_{\dot{\alpha}} \) as follows:

\[ \delta A_i = \sqrt{2} \xi^\alpha \chi_{\dot{\alpha}} + \sqrt{2} \bar{\xi}_{\dot{\alpha}} \bar{\psi}^\dot{\alpha} \]

\[ \delta \chi_{\dot{\alpha}} = -\sqrt{2} i \xi^\beta g^i \varepsilon_{\beta \dot{\alpha}} F_j - \sqrt{2} i \bar{\xi}_{\dot{\beta}} g^{ij} \varepsilon_{\beta \dot{\alpha}} \sigma_{\alpha \dot{\alpha}}^m \partial_m A_j \]

\[ \delta \bar{\psi}_{\dot{\alpha}} = \sqrt{2} i \xi^\beta g^{ij} \sigma_{\beta \dot{\alpha}} \partial_m A_j - \sqrt{2} \bar{\xi}_{\dot{\beta}} g^{ij} F_j \]

\[ \delta F_i = -\sqrt{2} \xi^\beta \sigma_{\beta \dot{\alpha}} \partial_m \bar{\psi}^\dot{\alpha} + \sqrt{2} \bar{\xi}_{\dot{\beta}} \varepsilon_{\beta \dot{\alpha}} \sigma_{\alpha \dot{\alpha}} \partial_m \chi^\alpha \]

(1.27)
The transformations of the component fields with respect to the central charge are generated by the operator $\hat{\delta} = \xi^z \partial_z + \tilde{\xi}^\tilde{z} \partial_{\tilde{z}}$:

\[
\begin{align*}
\hat{\delta} A_i &= (\xi^z - \tilde{\xi}^\tilde{z}) F_i \\
\hat{\delta} \chi_\alpha &= (\xi^z - \tilde{\xi}^\tilde{z}) \sigma^m_{\alpha \dot{\alpha}} \partial_m \bar{\psi}^\dot{\alpha} \\
\hat{\delta} \bar{\psi}_{\dot{\alpha}} &= (\xi^z - \tilde{\xi}^\tilde{z}) \sigma^m_{\alpha \dot{\alpha}} \partial_m \chi^\alpha \\
\hat{\delta} F_i &= (\xi^z - \tilde{\xi}^\tilde{z}) \Box A_i
\end{align*}
\]

(1.28)

Thus the central charge of the hypermultiplet vanishes for a massless free field theory if the equations of motion hold.

From the definition of the independent component fields of the hypermultiplet and the supersymmetry algebra we find the two coefficients $\alpha_{0,i}$ and $\alpha_{1,i}$ of the power expansion (1.29):

\[
\begin{align*}
\alpha_{0,i} &= A_i + \sqrt{2} \theta^\alpha \chi_\alpha - \sqrt{2} \bar{\theta}_{\dot{\alpha} i} \bar{\psi}^\dot{\alpha} + 3 i \theta^\alpha \sigma^m_{\alpha \dot{\alpha}} \partial_m \bar{\psi}^\dot{\alpha} A_i \\
&\quad + \frac{3}{2} i \theta^\alpha g^{jk} \varepsilon_{\alpha \beta} \theta^\beta_k A_i + \frac{3}{2} i \bar{\theta}_{\dot{\alpha} j} g^{jk} \varepsilon_{\dot{\alpha} \dot{\beta}} \bar{\theta}^\dot{\beta}_k F_i + \cdots \\
\alpha_{1,i} &= F_i - \sqrt{2} \theta^\alpha \sigma^m_{\alpha \dot{\alpha}} \partial_m \bar{\psi}^\dot{\alpha} - \sqrt{2} \bar{\theta}_{\dot{\alpha} i} \varepsilon_{\dot{\alpha} \dot{\beta}} \sigma^m_{\beta \dot{\beta}} \partial_m \chi^\beta \\
&\quad + 3 i \theta^\alpha \sigma^m_{\alpha \dot{\alpha}} \partial_m \bar{\psi}^\dot{\alpha} F_i - \frac{3}{2} i \theta^\alpha g^{jk} \varepsilon_{\alpha \beta} \theta^\beta_k \Box A_i \\
&\quad + \frac{3}{2} i \bar{\theta}_{\dot{\alpha} j} g^{jk} \varepsilon_{\dot{\alpha} \dot{\beta}} \bar{\theta}^\dot{\beta}_k \Box A_i + \cdots.
\end{align*}
\]

(1.30)

The dots stand for a finite series in Grassmann variables that is not needed in the following. This technique to determine the dependence on the central charge can be applied to any supersymmetric multiplet. Note that this expansion contains derivatives of the component field $F_i$. The dimension of the coefficients $\alpha_{k,i}$ increase linear in $k$: $\text{dim}(\alpha_{k,i}) = k + 1$. Thus it is not necessary to consider the full hypermultiplet in a renormalizable low-energy effective theory. By definition the low-energy effective theory is of second order in derivatives in the bosonic fields and of first order in the fermionic fields. Hence, to find the free action of a massive hypermultiplet in a low-energy effective theory in four dimensions it is enough to consider the approximation

\[ \phi_i \approx \alpha_{0,i}(x, \theta, \bar{\theta}) + \alpha_{1,i}(x, \theta, \bar{\theta}) (z - \bar{z}) + \frac{1}{2} \Box A_i(x) (z - \bar{z})^2. \]

(1.31)

Using (1.31) the Lagrangian of the hypermultiplet in $N = 2$ superspace is

\[ \mathcal{L} = -\frac{1}{2} \int d^2 \theta^i \bar{J} \bar{J} \mathcal{L}^i(\phi, \bar{\phi}) + \frac{1}{2} \int d^2 \bar{\theta}^i \bar{J} \mathcal{L}^i(\phi, \bar{\phi}) \]

(1.32)

with the ‘kernel’
\[ L_j^i(\phi, \bar{\phi}) = \phi_j (m + i \partial_z) \bar{\phi}^i + \bar{\phi}^i (m + i \partial_z) \phi_j. \] (1.33)

Projection to the lowest component \((\theta = \bar{\theta} = z = \bar{z} = 0)\) (1.32) yields up to total derivatives

\[ L = -\partial^m \bar{A}^i \partial_m A_i - i \chi^\alpha \sigma^m_{\alpha \dot{\alpha}} \partial_m \bar{\chi}^{\dot{\alpha}} - i \psi^\alpha \sigma^m_{\alpha \dot{\alpha}} \partial_m \bar{\psi}^{\dot{\alpha}} + \bar{F}^i F_i \]
\[ + m \psi^\alpha \chi_\alpha + m \bar{\psi}^{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} - i m A_i \bar{F}^i + i m \bar{A}^i F_i. \] (1.34)

Eliminating the auxiliary fields via their equations of motion \(\bar{F}^i = -i m \bar{A}^i\) and \(F_i = i m A_i\) we find the Lagrangian of a massive hypermultiplet in a free field theory. If the equation of motion hold we have for a massless hypermultiplet

\[ (\hat{\delta} \phi_i)_{\text{on-shell}} = 0. \] (1.35)

Thus, the power expansion (1.20) breaks down and the expansion of a massless hypermultiplet in \(N = 2\) superspace is simply

\[ \phi_i \text{ on-shell} = A_i + \sqrt{2} \theta^\alpha \chi_\alpha - \sqrt{2} \bar{\theta}^{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} + 3 i \theta^\alpha \sigma^m_{\alpha \dot{\alpha}} \partial_m \bar{\theta}^{\dot{\alpha}j} A_i \] (1.36)

To take only the lowest component of (1.32) is only allowed if the higher components of the Lagrangian are total derivatives, which don’t contribute to the action. This is equivalent to the fact that the action is invariant under supersymmetry and central charge transformations. To show this it is sufficient to study the following superspace expression:

\[ \check{\mathcal{L}} = \int d^2 \theta^j_i \phi_j \bar{\phi}^i - \int d^2 \bar{\theta}^i_i \phi_j \bar{\phi}^i = -i \phi_i \partial_z \bar{\phi}^i - i \bar{\phi}^i \partial_z \phi_i + \]
\[ \frac{1}{8} D^i_{\alpha} g_{ij} \bar{\phi}^j D^{\alpha \beta} \phi_k + \]
\[ \frac{1}{8} D^i_{\dot{\alpha}} g^{ij} \phi_j \bar{\phi}^{\dot{\beta}} D^{\dot{\beta} \dot{\kappa}} \phi_k \] (1.37)

Using the superspace identities for the hypermultiplet it can be shown that \(\delta \check{\mathcal{L}} = \partial_m \hat{K}^m\) and \(\hat{\delta} \check{\mathcal{L}} = \partial_m \hat{K}^m\) with

\[ K^m = \xi^\alpha_j \sigma^m_{\alpha \dot{\alpha}} \varepsilon^{\dot{\alpha} \dot{\beta}} \frac{i}{2} \left\{ \phi_i g^{ij} D^k_{\beta} \bar{\phi}^k - \bar{\phi}^j D_k^{\beta} \phi_i g_{kl} \right\} \]
\[ + \xi^{\dot{\alpha} \dot{\beta}} \sigma^{m \alpha \beta} \frac{i}{2} \left\{ g^{ij} \phi_j D^k_{\beta} g_{kl} \bar{\phi}^j + \bar{\phi}^i D_k^{\beta} \phi_i \right\} \]
\[ \hat{K}^m = (\bar{\xi}^z - \bar{\xi}) \frac{1}{8} \sigma^{m \beta \dot{\beta}} \left\{ D^i_{\beta} \phi_i \bar{D}_{\beta j} \bar{\phi}^j - \bar{D}^j_{\beta} \phi_i D_{\beta}^{i} g_{ij} \bar{\phi}^j \right\} \] (1.38)

\(^3\)By the use of the algebra the action is invariant under central charge transformations if it is invariant under supersymmetry transformations. For convenience we consider both cases.
holds. Therefore the action

$$S = \int d^4x \mathcal{L}(z^M)$$  \hspace{1cm} (1.39)$$

does not depend on the superspace coordinates $z^M$ and no integration over the internal coordinate $z$ is needed. This is remarkable because (1.37) and (1.32) depends on all superspace coordinates and is no ‘highest component’. The reason for this ‘phenomenon’ is that for extended supersymmetry it is possible to find invariants by integration over subspaces of the full superspace provided the kernel satisfies certain constraints [20,21].

Let us review this ‘phenomenon’ here briefly: It is already well known from $N = 1$ superspace that chiral kernels must be integrated over a chiral subspace to give the corresponding action. Furthermore in [21] it has been shown in the context of extended supersymmetry that for kernels with no central charge a classification for such superactions can be given. Moreover in [20] it is shown that there are kernels carrying central charge, which also only need an integration over a subspace of superspace to give a four dimensional action. This is the situation considered here and we have discussed this ‘phenomenon’ in the framework of a covariant $N = 2$ superspace formulation. To be more concrete, if we take the central charge to be real and identify $z$ with the coordinate of a fifth dimension, then (1.23) becomes the equation of motion of the hypermultiplet in five dimensions. Hence our Lagrangian (1.32) depends on the coordinates of five dimensional Minkowski space. The integration of this Lagrangian over four dimensions gives an action, which is independent of the additional dimension provided the five dimensional equation of motion of the hypermultiplet holds. Hence, if we start with an on-shell theory in five dimensions without central charge we can end up via dimensional reduction [22] with an off-shell theory in four dimensions with central charge. This is the result of [21]. From the point of view of a higher dimensional theory the generator of the complex central charge is related to partial derivatives with respect to the internal coordinates. For instance, one complex central charge corresponds to the momenta of the two-dimensional internal space. Thus the constraint (1.19), which eliminates the phase of the central charge, means that the hypermultiplet depends only on one of the two internal coordinates. From the point of view of a four dimensional theory it is therefore enough to consider the Fayet-Sohnius hypermultiplet only with one central charge. To keep things more general the starting point was, however, a four dimensional theory with complex central charge.

To conclude, the expansion of the hypermultiplet in $N = 2$ superspace has been given. In the whole discussion the rôle of the central charge has been emphasized and the connection to a low-energy effective theory has been discussed. In the presence of a complex central charge a new constraint has been introduced to preserve supersymmetry and the field content of the Fayet-Sohnius hypermultiplet. Then the superspace action in the context of a free field theory has been given. It has been pointed out that the two-dimensional superspace measure projects the kernel down to a very special
subspace of superspace, which leaves the action invariant under supersymmetry and central charge transformations, although the Lagrangian depends on all superspace coordinates. Therefore it is enough to consider the lowest component of the Lagrangian, because the action only depends on this part.

Finally the question arises for a matter coupled Yang-Mills theory. The easiest way to derive such a model is to gauge the central charge as in [11]. The expansion of the hypermultiplet with respect to the internal coordinates $z$ and $\bar{z}$ gets additional contributions in this case but the general construction remains the same. The symmetries of such an $N = 2$ Super Yang-Mills model are given by ‘special geometry’ and ‘Hyperkähler geometry’ [23].

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