Invariant Bianchi type I models in \( f(R,T) \) Gravity

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Abstract

In this paper, we search the existence of invariant solutions of Bianchi type I space-time in the context of \( f(R,T) \) gravity. The exact solution of the Einstein’s field equations are derived by using Lie point symmetry analysis method that yield two models of invariant universe for symmetries \( X^{(1)} \) and \( X^{(3)} \). The model with symmetries \( X^{(1)} \) begins with big bang singularity while the model with symmetries \( X^{(3)} \) does not favour the big bang singularity. Under this specification, we find out a set of singular and non singular solution of Bianchi type I model which present several other physically valid features within the framework of \( f(R,T) \).

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1 Introduction

To explain the observed late time acceleration of the universe, one may assume that at large scale, general relativity breaks down and a more general action for the gravitational field needs to be invoked. In this paper, we investigate \( f(R,T) \) gravity, which is a modification of general relativity in which the geometrical part of the Einstein-Hilbert action is different. Apart from the Ricci scalar \( R \) in the Lagrangian, there is also an arbitrary function of the trace \( T \) of the energy momentum tensor. This theory has attracted much attention in the recent past, and various aspects of the theory has been studied. It is possible to explain the dark energy and the observed late time acceleration of the universe within this theory. Harko et al [1] have obtained the equation of motion of the test particle and gravitational field equations in the metric formalism. Marzakulov [2] presented the point like Lagrangian for \( f(R,T) \) gravity. Latter on Houndjo [3] considered the \( f(R,T) \) gravity model that satisfies the local tests and transition of matter from the dominated era to the accelerated phase of universe. Recently Yadav [4] has investigated anisotropic string cosmological model in \( f(R,T) \) gravity. In this paper, our aim is to study invariant Bianchi type I cosmological models in \( f(R,T) \) gravity by using Lie point symmetry analysis method. One can refers the detail for Lie point symmetry analysis methods.
and optimal system in the references [3]−[15]. We note that relate to \( f(R, T) \) gravity in anisotropic space-time, there are lot of work available in the literature based on different mathematical as well as physical issues [16]−[18]. In the present model our motivation is to study the invariant model of universe under \( f(R, T) \) gravity and to observe different physical features of the model.

In the recent years, there has been considerable interest in anisotropic space-time that have consistency with CMB observations. Bianchi type I model is the most simplest model of anisotropy universe whose spatial section are flat but the expansion rate are direction dependent. In the literature, to study the possible effects of anisotropy in the early universe, many authors (Yadav et al [19], Akarsu and Kinc [20], Kumar and Singh [21], Yadav and Saha [22], Yadav [23], Sahoo et al [24]) have investigated Bianchi type I model from different point of view. Recently, Sahoo et al [24] have investigated Bianchi type I magnetized strange quark model with big rip singularity in \( f(R, T) \) and found that the model begins with big bang and ends with big rip. In 2010, Saha and Visinescu [25] and Saha et al [26] have studied Bianchi type I string cosmological model in presence of magnetic flux and the study reveals that the presence of cosmic strings do not allow the anisotropic universe into an isotropic one. In this paper, we confine ourselves to study the Bianchi type I model in the framework of \( f(R, T) \) with different approach i. e. Lie point symmetry analysis method.

The outline of present study is therefore as follows: In section 2, the basic mathematical formalism of \( f(R, T) \) gravity has been given. Thereafter in section 3, we provide the anisotropic metric and field equations for invariant model of universe in \( f(R, T) \) gravity. The section 4 include the description of optical system and symmetry analysis method that leads the exact solution of the model. The solution of the field equations are given in section 5. Several physical properties of the model viz expansion scalar, deceleration parameter and energy density, are discussed in section 6. Finally, some concluding remarks are given in section 7.

2 Some Basics of \( f(R, T) \) Gravity

The \( f(R, T) \) theory of gravity is the generalization or modification of general relativity (GR). The action for this theory is given by [1][7]:

\[
S = \int \sqrt{-g} \left( \frac{f(R, T)}{16\pi G} + L_m \right) dx^4, \tag{1}
\]

where \( f(R, T) \) is an arbitrary function of the Ricci scalar \( R \) and the trace \( T \) of energy momentum tensor \( T_{\mu\nu} \) while \( L_m \) is the usual matter Lagrangian. It is worth mentioning that if we replace \( f(R, T) \) with \( f(R) \) leads to the action of GR. The energy momentum tensor \( T_{\mu\nu} \) is defined as [13]:

\[
T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} L_m)}{\delta g^{\mu\nu}}. \tag{2}
\]
Here we assume that the dependence of matter Lagrangian is merely on the metric tensor \( g_{\mu\nu} \) rather than its derivatives. In this case, we obtain

\[ T_{\mu\nu} = L_m g_{\mu\nu} - \frac{\delta L_m}{\delta g^{\mu\nu}}. \]  

The \( f(R,T) \) gravity field equations are obtained by varying the action \( S \) with respect to the metric tensor \( g_{\mu\nu} \)

\[ f_R (R,T) R_{\mu\nu} - \frac{1}{2} f (R,T) g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) f_R (R,T) = \kappa T_{\mu\nu} - f_T (R,T) \left( T_{\mu\nu} + \Theta_{\mu\nu} \right), \]

where \( \nabla_\mu \) denotes the covariant derivative and \( \Box = \nabla^\mu \nabla_\mu \), \( f_R (R,T) = \frac{\partial f (R,T)}{\partial R} \), \( f_T (R,T) = \frac{\partial f (R,T)}{\partial T} \) and \( \Theta_{\mu\nu} = g^{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}} \). Contraction of (4) yields

\[ R f_R (R,T) + 3 \Box f_R (R,T) - 2 f (R,T) = \kappa T - (T + \Theta) f_T (R,T), \]

where \( \Theta = \Theta^\mu_\mu \). This is an important equation because it provides a relationship between Ricci scalar \( R \) and the trace \( T \) of energy momentum tensor. Using matter Lagrangian \( L_m \), the standard matter energy-momentum tensor is derived as

\[ T_{\mu\nu} = (p + \rho) u_\mu u_\nu - p g_{\mu\nu}, \]

where \( u_\mu = \sqrt{g_{\mu\mu}} (1,0,0,0) \) is the four-velocity in co-moving coordinates and \( \rho \) and \( p \) denotes energy density and pressure of the fluid respectively. Perfect fluids problems involving energy density and pressure are not any easy task to deal with. Moreover, there does not exist any unique definition for matter Lagrangian. Thus we can assume the matter Lagrangian as \( L_m = -p \) which gives

\[ \Theta_{\mu\nu} = -p g_{\mu\nu} - 2 T_{\mu\nu}, \]

and consequently the field equation (4) takes the form

\[ R f_R (R,T) + 3 \Box f_R (R,T) - 2 f (R,T) = \kappa T - (T + \Theta) f_T (R,T), \]

It is mentioned here that these field equations depend on the physical nature of matter field. Many theoretical models corresponding to different matter contributions for \( f(R,T) \) gravity are possible. However, Harko et. al. gave three classes of these models

\[ f(R,T) \begin{cases} R + 2 f'(T), \\ f_1(R) + f_2(T), \\ f_1(R) + f_2(R) f_3(T). \end{cases} \]

In this paper we focused to the first class, i.e., \( f(R,T) = R + 2 f(T) \). For this model the field equations become

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \left[ \kappa + 2 f'(T) \right] T_{\mu\nu} + \left[ f(T) + 2 p f'(T) \right] g_{\mu\nu}, \]

where prime represents derivative with respect to \( T \).
3 The metric and field equations

The line element of the Bianchi type I space-time is given by

$$ds^2 = dt^2 - A^2 dx^2 - B^2 dy^2 - C^2 dz^2,$$  \hspace{1cm} (11)

where $A = A(x, t)$, $B = B(x, t)$ and $C = C(x, t)$ are functions of $x$ and $t$.

Using Equation (11), we get following five independent field equations

$$\lambda p - (3 \lambda + 8 \pi) \rho = \frac{1}{A^2} \left[ \frac{C''}{C} + \frac{B' C'}{BC} + \frac{B''}{B} - \frac{A' C'}{AC} - \frac{A' B'}{AB} \right] - \frac{\dot{B} \dot{C}}{BC} - \frac{\dot{A} \dot{C}}{AC} - \frac{\dot{A} \dot{B}}{AB},$$  \hspace{1cm} (12)

$$(3 \lambda + 8 \pi) p - \lambda \rho = \frac{B' C'}{A^2 BC} - \frac{\ddot{C}}{C} - \frac{\dot{B} \dot{C}}{BC} - \frac{\ddot{B}}{B},$$  \hspace{1cm} (13)

$$(3 \lambda + 8 \pi) p - \lambda \rho = \frac{1}{A^2} \left[ \frac{C''}{C} - \frac{A' C'}{AC} \right] - \frac{\ddot{C}}{C} - \frac{\dot{A} \dot{C}}{AC} - \frac{\ddot{A}}{A},$$  \hspace{1cm} (14)

$$(3 \lambda + 8 \pi) p - \lambda \rho = \frac{1}{A^2} \left[ \frac{B''}{B} - \frac{A' B'}{AB} \right] - \frac{\ddot{B}}{B} - \frac{\dot{A} \dot{B}}{AB} - \frac{\ddot{A}}{A},$$  \hspace{1cm} (15)

$$\frac{\dot{B}'}{B} + \frac{\dot{C}''}{C} = \frac{\dot{A}}{A} \left( \frac{B'}{B} + \frac{C''}{C} \right).$$  \hspace{1cm} (16)

Here $A' = \frac{dA}{dx}$, $\dot{A} = \frac{dA}{dt}$, $G = c = 1$ and $f(T) = \lambda T$.

The scalar expansion ($\Theta$), shear scalar ($\sigma^2$) and proper volume $V$ are respectively obtained from following well known expressions[29]:

$$\Theta = u^i_{\ i} = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C},$$  \hspace{1cm} (17)

$$\sigma^2 = \frac{1}{2} \sigma_{ij} \sigma^{ij} = \frac{\Theta^2}{3} - \frac{\dot{A} \dot{B}}{AB} - \frac{\dot{A} \dot{C}}{AC} - \frac{\dot{B} \dot{C}}{BC},$$  \hspace{1cm} (18)

$$u_i = u_{i;j} u^j = (0, 0, 0, 0),$$  \hspace{1cm} (19)

$$V = \sqrt{-g} = ABC,$$  \hspace{1cm} (20)

where $g$ is the determinant of the metric (11).

The shear tensor is

$$\sigma_{ij} = u_{(ij)} + u_{(i} u_{j)} - \frac{1}{3} \Theta (g_{ij} - u_i u_j) = u_{ij} + \frac{1}{2} (u_{i;k} u^{k} u_{j} + u_{j;k} u^{k} u_{i}) - \frac{1}{3} \Theta (g_{ij} - u_i u_j).$$  \hspace{1cm} (21)
and the non-vanishing components of the $\sigma^j_i$ are

$$
\begin{align*}
\sigma^0_0 &= 0, & \sigma^1_1 &= \frac{1}{3} \left( \frac{2 \dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C} \right), \\
\sigma^2_2 &= \frac{1}{3} \left( \frac{2 \dot{B}}{B} - \frac{\dot{C}}{C} - \frac{\dot{A}}{A} \right), & \sigma^3_3 &= \frac{1}{3} \left( \frac{2 \dot{C}}{C} - \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right).
\end{align*}
$$

(22)

The field equations (12)-(16) constitute a system of five highly non-linear partial differential equations (NLPDE) with five unknowns parameters namely, $A$, $B$, $C$, $\rho$ and $p$. The equations (12)-(16) can be transform to the following NLPDE.

$$
E_1 = \frac{1}{A^2} \left[ \frac{B''}{B} - \frac{B'C'}{BC} + \frac{A'B'}{AB} \right] + \frac{\dot{C}}{C} - \frac{\dot{A}}{A} + \frac{\dot{B}C}{BC} - \frac{\dot{A}B}{AB} = 0,
$$

(23)

$$
E_2 = \frac{1}{A^2} \left[ \frac{C''}{C} - \frac{B''}{B} - \frac{A'C'}{AC} + \frac{A'B'}{AB} \right] + \frac{\dot{B}}{B} - \frac{\dot{C}C}{AC} + \frac{\dot{A}B}{AB} = 0,
$$

(24)

$$
E_3 = \frac{\dot{B}}{B} + \frac{\dot{C}}{C} - \frac{\dot{A}}{A} \left( \frac{B'}{B} + \frac{C'}{C} \right) = 0,
$$

(25)

where

$$
8 (4 \pi + \lambda) (2 \pi + \lambda) p(x, t) = \lambda \left[ \frac{\dot{A}C'}{AC} + \frac{\dot{B}C'}{BC} - \frac{1}{A^2} \left( \frac{C''}{C} + \frac{B'C'}{BC} - \frac{A'C'}{AC} \right) \right]
$$

$$
-(8 \pi + 2 \lambda) \left[ \frac{\dot{A}B}{AB} + \frac{1}{A^2} \left( \frac{A'B'}{AB} - \frac{B''}{B} \right) \right] - (8 \pi + 3 \lambda) \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right),
$$

(26)

$$
8 (4 \pi + \lambda) (2 \pi + \lambda) \rho(x, t) = (8 \pi + 3 \lambda) \left[ \frac{\dot{A}C'}{AC} + \frac{\dot{B}C'}{BC} - \frac{1}{A^2} \left( \frac{C''}{C} + \frac{B'C'}{BC} - \frac{A'C'}{AC} \right) \right]
$$

$$
+(8 \pi + 2 \lambda) \left[ \frac{\dot{A}B}{AB} + \frac{1}{A^2} \left( \frac{A'B'}{AB} - \frac{B''}{B} \right) \right] - \lambda \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right).
$$

(27)

4 Optimal system and Symmetry analysis method

The classical method for finding the solution of equations (23)-(27) is a separation method by taking $A(x, t) = A_1(x) A_2(t)$, $B(x, t) = B_1(x) B_2(t)$ and $C(x, t) = C_1(x) C_2(t)$ [27, 28]. In order to obtain an exact and new solution of equations (23)-(27), here we have used the symmetry analysis method which is a powerful method for solving NLPDE. In refs.
the authors have described optimal system and the symmetry analysis method Bianchi-I in detail. Following, Yadav and Ali [7] and Ali et al [9], we acquire an optimal system of one-dimensional subalgebras which is spanned as following:

\[
\begin{align*}
X^{(1)} &= X_1 + a_3 X_3 + a_5 X_5 + a_6 X_6, \\
X^{(2)} &= X_1 + a_4 X_4 + a_5 X_5 + a_6 X_6, \\
X^{(3)} &= X_2 + a_3 X_3 + a_5 X_5 + a_6 X_6, \\
X^{(4)} &= X_2 + a_4 X_4 + a_5 X_5 + a_6 X_6, \\
X^{(5)} &= X_3 + a_5 X_5 + a_6 X_6, \\
X^{(6)} &= X_4 + a_5 X_5 + a_6 X_6, \\
X^{(7)} &= X_5 + a_6 X_6, \\
X^{(8)} &= X_6.
\end{align*}
\]

(28)

5 Invariant solutions

In the case of symmetries \(X^{(5)}\) or \(X^{(6)}\) or \(X^{(7)}\) or \(X^{(8)}\), then \(a_1 = a_2 = 0\) that leads the metric functions \(A, B\) and \(C\) are functions of \(x\) only. For this reason, we shall consider the invariant solutions associated with the optimal systems of symmetries \(X^{(1)}, X^{(2)}, X^{(3)}\) and \(X^{(4)}\) only. Among these symmetries, only \(X^{(1)}\) gives the model of physical universe while the model with \(X^{(3)}\) does not favour the big bang singularity. In this paper, we have purposely omitted the models with symmetries \(X^{(2)}\) and \(X^{(4)}\) because this prescription represent non-realistic model of universe.

**Solution (I):** The symmetries \(X^{(1)}\) has the characteristic equations:

\[
\frac{dx}{x} = \frac{dt}{a_3 t} = \frac{dA}{(a_3 - 1) A} = \frac{dB}{a_5 B} = \frac{dC}{a_6 C}.
\]

(29)

Then the Invariant transformations take the following form:

\[
\xi = \frac{x^a}{t}, \quad A(x, t) = x^{a-1} \Psi(\xi), \quad B(x, t) = x^b \Phi(\xi), \quad C(x, t) = x^c \Omega(\xi),
\]

(30)

where \(a = a_3, b = a_5\) and \(c = a_6\) are an arbitrary constants.

Putting the transformations (30) in the field Eqs. (23)-(25), we have

\[
\frac{a \xi}{\Psi^2} \left[ b \left( \frac{\Psi'}{\Psi} + \frac{\Omega'}{\Omega} \right) + (c - 2b) \frac{\Phi'}{\Phi} \right] = \frac{b(b-a-c)}{\Psi^2},
\]

(32)

\[
\xi^4 \left( \frac{\Psi''}{\Psi} + \frac{\Phi''}{\Phi} - \frac{\Phi' \Omega'}{\Phi \Omega} \right) + 2\xi^3 \left( \frac{\Phi'}{\Phi} - \frac{\Omega'}{\Omega} \right) + \frac{a^2 \xi^2}{\Psi^2} \left( \frac{\Phi' \Phi'}{\Phi^2} + \frac{\Phi' \Omega'}{\Phi \Omega} - \frac{\Phi''}{\Phi} \right)
\]

\[
\quad + \frac{a \xi}{\Psi^2} \left[ (b-a) \frac{\Psi'}{\Psi} + 2c \frac{\Omega'}{\Omega} + 2b \frac{\Phi'}{\Phi} \right] = \frac{(c-b) (b-a-c)}{\Psi^2},
\]

(33)
One can not solve equations (31)-(33) in general. So, in order to solve the problem completely, we have to choose the following transformations:

\[ \Psi(\xi) = \alpha_1 \xi^{\alpha_2}, \quad \Phi(\xi) = \beta_1 \xi^{\beta_2}, \quad \Omega(\xi) = \gamma_0 + \gamma_1 \xi^{\gamma_2}, \]  

(34)

where \( \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_0, \gamma_1 \) and \( \gamma_2 \) are arbitrary non-zero constants.

Substituting eq. (34) in eq. (31), we have

\[ \gamma_0 \left[ c \alpha_2 + (b + a \beta_2) (\alpha_2 - \beta_2) \right] + \gamma_1 \left[ (b + a \beta_2) (\alpha_2 - \beta_2) + (c + a \gamma_2) (\alpha_2 - \gamma_2) \right] \xi^{\gamma_2} = 0. \]  

(35)

The coefficients of \( \xi^{\gamma_2} \) and the absolute value must be equal zero. Solving the two resulting conditions with respect to \( a \) and \( b \), we have:

\[ a = \frac{c}{\alpha_2 - \gamma_2}, \quad b = \frac{\alpha_2 c}{\beta_2 - \alpha_2} - \frac{\beta_2 c}{\gamma_2 - \alpha_2}. \]  

(36)

where \( \gamma_2 \neq \alpha_2 \) and \( \beta_2 \neq \alpha_2 \).

Therefore the (32) can be written as

\[ Q_0 + Q_1 \xi^{2+2\alpha_2} + Q_2 \xi^{\gamma_1} + Q_3 \xi^{2+2\alpha_2+\gamma_2} = 0, \]  

(37)

where

\[ Q_0 = \frac{c^2 \alpha_2 \gamma_0}{(\alpha_2 - \beta_2)^2 (\alpha_2 - \gamma_2)} \left[ \beta_2 (1 - \gamma_2) + \alpha_2 (2 \beta_2 + 2 \gamma_2 - 3 \alpha_2 - 1) \right], \]

\[ Q_1 = \alpha_1^2 \alpha_2 \gamma_0 (1 + \alpha_2 + \beta_2), \]

\[ Q_2 = \frac{c^2 \alpha_2 \gamma_1}{(\alpha_2 - \beta_2)^2 (\alpha_2 - \gamma_2)} \left[ \beta_2 + \alpha_2 (2 \beta_2 + \gamma_2 - 3 \alpha_2 - 1) \right], \]

\[ Q_3 = \alpha_1^2 \gamma_1 (\alpha_2 - \gamma_2) (1 + \alpha_2 + \beta_2 + \gamma_2). \]  

(38)

From equation (37), we have two cases: (1): \( 2 + 2 \alpha_2 = 0 \) and (2): \( 2 + 2 \alpha_2 \neq 0 \).

In case (2), all \( Q_i, i = 0, 1, 2, 3 \) must be equal to zero. The equation \( Q_1 = 0 \) leads to \( \beta_2 = -\alpha_2 - 1 \) and the coefficient \( Q_4 \) becomes \( \alpha_1^2 \gamma_1 (\alpha_2 - \gamma_2) \gamma_2 \) which equal zero when \( \gamma_2 = \alpha_2 \) contradiction.

However, in case (1), i.e. \( \alpha_2 = -1 \).

The equation (37) transform to the following form:

\[ \gamma_0 (\gamma_2 + 1) \left[ \alpha_1^2 \beta_2 (\beta_2 + 1)^2 + c^2 (\beta_2 + 2) \right] + \gamma_1 \left[ \alpha_1^2 (\beta_2 + 1)^2 (\gamma_2 + 1)^2 + c^2 (\gamma_2 + \beta_2 + 2) \right] \xi^{\gamma_2} = 0. \]  

(39)

The equation (39) leads to

\[ c = \pm \frac{\alpha_1 \delta_0 (1 - \delta_0^2)}{1 + \delta_0^2}, \quad \beta_2 = \frac{-1 \pm \delta_0 \sqrt{1 + \delta_0^2 - \delta_0^4}}{1 + \delta_0^2}, \]  

(40)
where \( \gamma_2 = -\frac{2 \delta_0^2}{1 + \delta_0^2} \).

Thus, the metric functions are given by

\[
A(x, t) = \frac{\alpha_1}{x}, \quad B(x, t) = \beta_1 x^{\delta_0 + \sqrt{1 + \delta_0^2 - \delta_0^4}} t^{\frac{1 - \delta_0 \sqrt{1 + \delta_0^2 - \delta_0^4}}{1 + \delta_0^2}}, \quad C(x, t) = \left( \gamma_1 x^{\frac{2 \alpha_1 \delta_0^3}{1 + \delta_0^2}} + \gamma_2 t^{\frac{2 \delta_0^2}{1 + \delta_0^2}} \right) x^{-\alpha_1 \delta_0},
\]

(41)

The equations (41) and (11) lead to

\[
ds_1^2 = dt^2 - \frac{\alpha_1^2 t^2}{x^2} \, dx^2 + \beta_1^2 x^{\delta_0 + \sqrt{1 + \delta_0^2 - \delta_0^4}} t^{\frac{2 - 2 \delta_0 \sqrt{1 + \delta_0^2 - \delta_0^4}}{1 + \delta_0^2}} 
\, dy^2 + \left( \gamma_0 x^{\frac{2 \alpha_1 \delta_0^3}{1 + \delta_0^2}} + \gamma_1 t^{\frac{2 \delta_0^2}{1 + \delta_0^2}} \right)^2 x^{-2 \alpha_1 \delta_0} 
\, dz^2.
\]

(42)

**Solution (II):** The symmetries \( X^{(3)} \) has the characteristic equations:

\[
\frac{dx}{1} = \frac{dt}{a_3 t} = \frac{dA}{a_3 A} = \frac{dB}{a_5 B} = \frac{dC}{a_6 C}.
\]

(43)

Then the Invariant transformations take the following form:

\[
\xi = t \, \exp \left[ a \, x \right], \quad A(x, t) = \Psi(\xi) \, t, \quad B(x, t) = \Phi(\xi) \, t^b, \quad C(x, t) = \Omega(\xi) \, t^c,
\]

(44)

where \( a = -\frac{1}{a_3} \), \( b = \frac{a_5}{a_3} \) and \( c = \frac{a_6}{a_3} \) are arbitrary constants.

Putting the transformations (44) in the field Equations (23)-(25), we can get the following system of ordinary differential equations:

\[
\xi \left( \frac{\Omega''}{\Omega} + \frac{\Phi''}{\Phi} - \frac{\Psi' \Phi'}{\Psi \Phi} - \frac{\Psi' \Omega'}{\Psi \Omega} \right) + b \frac{\Phi'}{\Phi} + c \frac{\Omega'}{\Omega} = 0,
\]

(45)

\[
a^2 \xi \left[ \xi \left( \frac{\Omega''}{\Omega} - \frac{\Psi''}{\Psi} + \frac{\Phi' \Omega'}{\Phi \Omega} - \frac{\Psi' \Phi'}{\Psi \Phi} \right) + (b + 2 \, c) \frac{\Omega'}{\Omega} + (c - 1) \frac{\Phi'}{\Phi} + (b + 2) \frac{\Psi'}{\Psi} \right]
+ \frac{\xi}{\Psi^2} \left[ \xi \left( \frac{\Phi''}{\Phi} - \frac{\Omega''}{\Omega} + \frac{\Psi' \Phi'}{\Psi \Phi} - \frac{\Psi' \Omega'}{\Psi \Omega} \right) - \frac{\Phi'}{\Phi} \right] = \frac{(1 - c) \, (b + c)}{\Psi^2},
\]

(46)

\[
\xi \left[ \xi \left( \frac{\Phi''}{\Phi} - \frac{\Omega''}{\Omega} + \frac{\Psi' \Phi'}{\Psi \Phi} - \frac{\Psi' \Omega'}{\Psi \Omega} \right) - (1 + 2 \, c) \frac{\Omega'}{\Omega} + (1 + 2 \, b) \frac{\Phi'}{\Phi} + (b + c) \frac{\Psi'}{\Psi} \right]
+ \frac{a^2 \xi}{\Psi^2} \left[ \xi \left( \frac{\Omega''}{\Omega} - \frac{\Psi''}{\Psi} + \frac{\Phi' \Omega'}{\Phi \Omega} - \frac{\Psi' \Phi'}{\Psi \Phi} \right) + \frac{\Omega'}{\Omega} - \frac{\Phi'}{\Phi} \right] = \frac{c^2 - b^2}{\Psi^2}.
\]

(47)

One can not solve equations (31)-(33) in general. So, in order to solve the problem completely, we have to choose the following transformations:

\[
\Psi(\xi) = \alpha_1 \xi^{\alpha_2}, \quad \Phi(\xi) = \beta_1 \xi^{\beta_2}, \quad \Omega(\xi) = \gamma_0 + \gamma_1 \xi^{\gamma_2},
\]

(48)
where $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$, $\gamma_0$, $\gamma_1$ and $\gamma_2$ are arbitrary non-zero constants. Substituting \((48)\) in \((45)\), we have

$$
\gamma_0 (\beta_2 - \alpha_2 + b - 1) + \gamma_1 \left[ \beta_2 (\beta_2 - \alpha_2 + b - 1) + \gamma_2 (\gamma_2 - \alpha_2 + c - 1) \right] \xi^{\gamma_2} = 0. \tag{49}
$$

The coefficients of $\xi^{\gamma_2}$ and the absolute value must be equal zero in the above equation. Solving the two resulting conditions with respect to $b$ and $c$, we obtain

$$
b = 1 + \alpha_2 - \beta_2, \quad c = 1 + \alpha_2 - \gamma_2. \tag{50}
$$

Therefore the \((46)\) becomes

$$
\gamma_0 \beta_2 a^2 (\alpha_2 - \beta_2) + \gamma_0 \gamma_2 \alpha_1^2 (2 + 3 \alpha_2 - \gamma_2) \xi^{2 \alpha_2} + \gamma_1 \beta_2 a^2 (\alpha_2 - \beta_2 + \gamma_2) \xi^{\gamma_2} = 0. \tag{51}
$$

Because $\alpha_2 \neq 0$ and $\gamma_2 \neq 0$, then the absolute value in the above equation must be equal zero we get

$$
\alpha_2 = \beta_2, \tag{52}
$$

and the equation \((51)\) becomes:

$$
\gamma_0 \alpha_1^2 (2 + 3 \beta_2 - \gamma_2) \xi^{2 \beta_2} + \gamma_1 \beta_2 a^2 \xi^{\gamma_2} = 0. \tag{53}
$$

From equation \((53)\), we have two cases: \((1)\): $\gamma_2 = 2 \beta_2$ and \((2)\): $\gamma_2 \neq 2 \beta_2$. The case \((2)\) leads contradiction while case \((1)\) leads to $\gamma_0 = -\frac{\gamma_1 \beta_2 a^2}{(2 + \beta_2) \alpha_1^2}$. The equations \((48)\), \((44)\) and \((31)-(33)\) lead to

$$
\begin{cases}
A(x, t) = \alpha_1 t^{1+\beta_2} \exp[\tilde{a} x], \\
B(x, t) = \beta_1 t^{1+\beta_2} \exp[\tilde{a} x], \\
C(x, t) = \tilde{\gamma}_1 \left[ \beta_2 (2 + \beta_2) \alpha_1^2 t^{1+\beta_2} \exp[2 \tilde{a} x] - \tilde{a}^2 t^{1-\beta_2} \right],
\end{cases} \tag{54}
$$

where $\tilde{a} = a \beta_2$, $\tilde{\gamma}_1 = \frac{\gamma_1}{\beta_2 (2 + \beta_2) \alpha_1^2}$, $\alpha_1$, $\beta_1$ and $\beta_2$ are arbitrary constants.

Thus, the line element \((11)\) can be written in the following form:

$$
d s_3^2 = dt^2 - t^{2 (1+\beta_2)} \exp[2 \tilde{a} x] \left[ \alpha_1^2 dx^2 + \beta_1^2 dy^2 \right] + \tilde{\gamma}_1 \left[ \beta_2 (2 + \beta_2) \alpha_1^2 t^{1+\beta_2} \exp[2 \tilde{a} x] - \tilde{a}^2 t^{1-\beta_2} \right]^2 dz^2. \tag{55}
$$

## 6 Physical and geometrical properties of the models

For the Model \((42)\):

For the model \((42)\), when $\delta_0 = \frac{1}{2}$, the expressions of $p$ and $\rho$ are given by:

$$
p(x, t) = \frac{9 \gamma_1 \left[ (23 - 4 \sqrt{19}) \kappa + (86 - 28 \sqrt{19}) \lambda \right] x^{\frac{\alpha_1}{\gamma_2}} - 3 \gamma_2 \left[ 3 (4 \sqrt{19} - 23) \kappa + 2 (74 \sqrt{19} - 313) \lambda \right] t^{\frac{\gamma_1}{\gamma_2}}}{100 (\sqrt{19} - 2)^2 (\kappa^2 + 6 \kappa \lambda + 8 \lambda^2) t^2 \left( \gamma_1 x^{\frac{\alpha_1}{\gamma_2}} + \gamma_2 t^{\frac{\gamma_1}{\gamma_2}} \right)^2}, \tag{56}
$$

$$
\rho(x, t) = \frac{9 \gamma_1 \left[ (23 - 4 \sqrt{19}) \kappa + (86 - 28 \sqrt{19}) \lambda \right] x^{\frac{\alpha_1}{\gamma_2}} - 3 \gamma_2 \left[ 3 (4 \sqrt{19} - 23) \kappa + 2 (74 \sqrt{19} - 313) \lambda \right] t^{\frac{\gamma_1}{\gamma_2}}}{100 (\sqrt{19} - 2)^2 (\kappa^2 + 6 \kappa \lambda + 8 \lambda^2) t^2 \left( \gamma_1 x^{\frac{\alpha_1}{\gamma_2}} + \gamma_2 t^{\frac{\gamma_1}{\gamma_2}} \right)^2}, \tag{57}
$$

$\kappa$, $\lambda$ are arbitrary non-zero constants.
\[ \rho(x,t) = \frac{9\gamma_1 \left(17 - 16\sqrt{19}\right)\kappa + (74 - 55\sqrt{19})\lambda}{100 \left(\sqrt{19} - 2\right) \left(\kappa^2 + 6\kappa \lambda + 8\lambda^2\right)^2 \left(\gamma_1 x^{\frac{a_1}{\kappa}} + \gamma_2 t^{\frac{a_2}{\lambda}}\right)} \left(\frac{a_{x_1}}{\kappa} + \frac{a_{x_2}}{\lambda} \right) \left(\frac{a_{x_3}}{\kappa} + \frac{a_{x_4}}{\lambda} \right) t^2, \]

where \( \gamma_1, \gamma_2, a_1 \) and \( \lambda \) are arbitrary constants while \( \kappa = 8\pi \).

The volume of the universe is read as

\[ V = \alpha_1 \beta_1 x^{-1 - \left(\frac{9\gamma_2 + a_{x_1}}{\kappa}\right)} \left(\gamma_1 x^{\frac{a_1}{\kappa}} + \gamma_2 t^{\frac{a_2}{\lambda}}\right), \]

where \( \beta_1 \) is an arbitrary constant.

The expansion scalar, which determines the volume behavior of the fluid, is given by:

\[ \Theta = \frac{18 + \sqrt{19}}{10 \, t} + \frac{2\gamma_2}{5 \, t^2 \left(\gamma_1 x^{\frac{a_1}{\kappa}} + \gamma_2 t^{\frac{a_2}{\lambda}}\right)}, \]

The non-vanishing components of the shear tensor, \( \sigma_i^j \), are read as

\[
\begin{align*}
\sigma_0^0 &= 0, \\
\sigma_1^1 &= \frac{\gamma_1 (12 - \sqrt{19}) x^{\frac{a_1}{\kappa}} + \gamma_2 (8 - \sqrt{19}) t^{\frac{a_2}{\lambda}}}{30 \, t \left(\gamma_1 x^{\frac{a_1}{\kappa}} + \gamma_2 t^{\frac{a_2}{\lambda}}\right)}, \\
\sigma_2^2 &= \frac{2\gamma_1 (3 + \sqrt{19}) x^{\frac{a_1}{\kappa}} + 2\gamma_2 (1 + \sqrt{19}) t^{\frac{a_2}{\lambda}}}{30 \, t \left(\gamma_1 x^{\frac{a_1}{\kappa}} + \gamma_2 t^{\frac{a_2}{\lambda}}\right)}, \\
\sigma_3^3 &= -\frac{\gamma_1 (18 + \sqrt{19}) x^{\frac{a_1}{\kappa}} + 2\gamma_2 (10 + \sqrt{19}) t^{\frac{a_2}{\lambda}}}{30 \, t \left(\gamma_1 x^{\frac{a_1}{\kappa}} + \gamma_2 t^{\frac{a_2}{\lambda}}\right)}.
\end{align*}
\]

The shear scalar is given by

\[ \sigma^2 = \frac{(103 + 6\sqrt{19}) \gamma_1^2 x^{\frac{2\alpha_1}{\kappa}} + 2 \left(67 + 4\sqrt{19}\right) \gamma_1 \gamma_2 x^{\frac{\alpha_1}{\kappa}} t^{\frac{\alpha_2}{\lambda}} + (47 + 2\sqrt{19}) \gamma_2^2 t^{\frac{\alpha_2}{\lambda}}}{300 \, t^2 \left(\gamma_1 x^{\frac{a_1}{\kappa}} + \gamma_2 t^{\frac{a_2}{\lambda}}\right)^2}. \]

The deceleration parameter is determined as

\[ q = -3 \Theta^2 \left(\Theta_i^i u^i + \frac{1}{3} \Theta^2 \right) \\
= -3 \Theta^2 \left[ \frac{\Theta^2}{3} - \frac{(18 + \sqrt{19}) x^{\frac{a_1}{\kappa}}}{10 \, t^2 \left(\gamma_1 x^{\frac{a_1}{\kappa}} + \gamma_2 t^{\frac{a_2}{\lambda}}\right)} + \frac{2\gamma_2 x^{\frac{a_1}{\kappa}}}{25 \, t^2 \left(\gamma_1 x^{\frac{a_1}{\kappa}} + \gamma_2 t^{\frac{a_2}{\lambda}}\right)^3} \left(3 \gamma_1 x^{\frac{a_1}{\kappa}} + 5 \gamma_2 t^{\frac{a_2}{\lambda}}\right)\right]. \]
For model (42), the variation of pressure, energy density and volume are presented in Fig. 1. For the large $t$, the pressure and energy density approached towards zero and volume diversed towards infinity i.e. $p \rightarrow 0$, $\rho \rightarrow 0$ and $V \rightarrow \infty$ when $t \rightarrow \infty$. It has singularity at $t = 0$. In general the model has point type singularity at $t = 0$ because the scale factor and volume vanish at initial epoch. From eq. (62), it is evident that deceleration parameter is negative and decreasing function of time and hence the cosmic expansion is driven by big bang impulse.

For the Model (55):

The expressions of $p$ and $\rho$ for the model (55), are given by:

$$p(x,t) = \frac{K_1 + \left( \kappa + 3 \kappa \beta_2 + 6 \kappa \beta_2 \right) \left( 2 + 3 \beta_2 + \beta_2^2 \right) \beta_2 \alpha_1^2 \beta_2 \alpha_1^2 t^2 \beta_2 \exp[2 \tilde{a} x]}{\left( \kappa^2 + 6 \kappa \lambda + 8 \lambda^2 \right) \left[ \beta_2 \alpha_1^2 \beta_2 \alpha_1^2 (2 + \beta_2) \beta_2 \alpha_1^2 \beta_2 \alpha_1^2 t^2 \beta_2 \exp[2 \tilde{a} x] - \tilde{a}^2 \right]}, \quad (63)$$

$$\rho(x,t) = \frac{3(1 + \beta_2) \kappa + (8 + 2 \beta_2) \lambda}{\left( \kappa^2 + 6 \kappa \lambda + 8 \lambda^2 \right) \left[ \beta_2 \alpha_1^2 \beta_2 \alpha_1^2 (2 + \beta_2) \beta_2 \alpha_1^2 \beta_2 \alpha_1^2 t^2 \beta_2 \exp[2 \tilde{a} x] - K_2 \right]}, \quad (64)$$

where $\beta_2$, $\tilde{a}$ and $\lambda$ are an arbitrary constants while $\kappa = 8 \pi$, $K_1 = (\kappa + \kappa \beta_2 + 2 \lambda \beta_2) \left( 1 + 3 \beta_2 \right) \tilde{a}^2$ and $K_2 = \left( (3 + \beta_2) \kappa + (8 + 2 \beta_2) \lambda \right) (1 + 3 \beta_2) \tilde{a}^2$. The volume element is given by

$$V = \alpha_1 \beta_1 \tilde{\gamma}_1 \left[ \tilde{a}^2 - \beta_2 \alpha_1^2 (2 + \beta_2) \beta_2 \alpha_1^2 t^2 \beta_2 \exp[2 \tilde{a} x] \right] \tilde{a}^{3+ \beta_2} \exp[2 \tilde{a} x], \quad (65)$$

where $\beta_1$ and $\tilde{\gamma}_1$ are an arbitrary constants.

The expansion scalar, which determines the volume behavior of the fluid, is read as

$$\Theta = \frac{\tilde{a}^2 (3 + \beta_2) - 3 \beta_2 \alpha_1^2 (2 + 3 \beta_2 + \beta_2^2) \beta_2 \alpha_1^2 t^2 \beta_2 \exp[2 \tilde{a} x]}{t \left[ \tilde{a}^2 - \beta_2 \alpha_1^2 (2 + \beta_2) \beta_2 \alpha_1^2 t^2 \beta_2 \exp[2 \tilde{a} x] \right]}, \quad (66)$$
The non-vanishing components of the shear tensor, $\sigma^i_j$, are read as

$$
\left\{ \begin{array}{l}
\sigma^0_0 = 0, \\
\sigma^1_1 = \frac{2 \tilde{a}^2 \beta_2}{3t (\tilde{a}^2 - \beta_2 (\beta_2 + 2) \alpha^2_1 t^{2\beta_2} \exp[2\tilde{a} x])},
\end{array} \right.
\quad \sigma^2_2 = \sigma^1_1, \quad \sigma^3_3 = -2 \sigma^1_1.
$$

(67)

The shear scalar is given by

$$
\sigma^2 = \frac{4 \tilde{a}^4 \beta_2^2}{3 t^2 (\tilde{a}^2 - \beta_2 (\beta_2 + 2) \alpha^2_1 t^{2\beta_2} \exp[2\tilde{a} x])^2}.
$$

(68)

The deceleration parameter is obtained as

$$
q = -3 \Theta^2 \left( \Theta, u^i + \frac{1}{3} \Theta^2 \right) = -3 \Theta^2 \left[ \frac{\Theta^2}{3} \frac{t^{\beta_2-3}}{\gamma_1 \left[ \tilde{a}^2 - \beta_2 \alpha^2_1 (2 + \beta_2) t^{2\beta_2} \exp[2\tilde{a} x] \right]^2} \left( \tilde{a}^4 (\beta_2 - 3) + 2 \tilde{a}^2 \beta_2 \alpha^2_1 (2 + \beta_2) (3 + 2 \beta_2 - \beta_2^2) t^{2\beta_2} \exp[2\tilde{a} x] + 3 \beta_2^2 \alpha^4_1 (2 + 3 \beta_2 + \beta_2^2)^2 t^{4\beta_2} \exp[4\tilde{a} x] \right) \right].
$$

(69)

Figure 2: Variation of pressure (left panel) and energy density (middle panel) versus time for model (55):

For physically viable model, one have some certain criterion such as: (i) The energy density should be positive and decreases with time. (ii) The Volume of universe increases with increase in time. (iii) $\frac{\rho}{\Theta}$ should be vanish at larger time.

The Model (55) does not meet criterion (i) as stated above (Fig. 2). Here, in the model (55), the energy density increases with time. Therefore this model are not physically interesting and we omit the further physical and geometrical analysis of model (55).
7 Conclusion

In this paper, we have investigated an optimal system and invariant solutions of Bianchi type I space-time in the context of $f(R,T)$ Gravity. In $f(R,T)$ gravity, the cosmic acceleration may arise not only due to geometrical contribution of the matter but also depends on matter contents of the universe. In this gravity an extra acceleration is always present due to coupling between matter and geometry. We have derived the gravitational field equations for the fluid under consideration, corresponding to the $f(R,T)$ gravity models. On the basis of optimal systems of symmetries $X^{(1)}$ and $X^{(3)}$, we obtained two models \((42)\) and \((55)\) respectively. For model \((42)\), we note that at $t = 0$, the spatial volume vanishes while other parameter diverge. Thus the model \((42)\) starts expanding with big bang singularity at $t = 0$. This singularity is point type because the directional scale factors $A$, $B$ and $C$ vanish at $t = 0$. For model \((55)\), we observe that pressure and energy density are increasing function of time which is not in favour of standard cosmological model governed by big bang cosmology. Therefore, it is interesting to note that the present work deals singular as well as non singular model of universe under the specification of symmetries $X^{(1)}$ and $X^{(3)}$ respectively. However, to have consistency with observational data (from SN Ia, BAO and CMB) of standard cosmology, the model \((42)\) is suitable model of present universe. In other words, the solution presented here has potential to understand the features of observed universe. We also note that $\frac{x}{y} \to 0$ at $t \to \infty$. This means that anisotropy will be died out at larger time.

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