On Fejér Type Inequalities via $(p, q)$-Calculus

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Abstract: In this paper, we use $(p, q)$-integral to establish some Fejér type inequalities. In particular, we generalize and correct existing results of quantum Fejér type inequalities by using new techniques and showing some problematic parts of those results. Most of the inequalities presented in this paper are significant extensions of results which appear in existing literatures.

Keywords: Fejér type inequalities; symmetric function; $(p, q)$-calculus; $(p, q)$-derivative; $(p, q)$-integral

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1. Introduction

Quantum calculus or $q$-calculus, the modern name of the study of calculus without limits, has been studied since the early eighteenth century. The famous mathematician Leonhard Euler (1707–1783) established $q$-calculus and, in 1910, F. H. Jackson [1] determined the definite $q$-integral called the $q$-Jackson integral. Many applications of quantum calculus appear in mathematics, such as number theory, orthogonal polynomials, combinatorics, basic hypergeometric functions, and in physics, such as mechanics, relativity theory and quantum theory, see for instance [2–10] and the references therein. Furthermore, the fundamental knowledge and also the fundamental theoretical concepts of quantum calculus are covered in the book by V. Kac and P. Cheung [11].

In 2013, J. Tariboon and S. K. Ntouyas [12] defined the $q$-derivative and $q$-integral of a continuous function on finite interval along with some studying of its significant properties. In addition, they firstly extended some inequalities to $q$-calculus, such as Cauchy–Bunyakovsky–Schwarz, Grüss, Grüss-Čebyšev, Hermite–Hadamard, Hölder, Ostrowski and Trapezoid inequalities by applying such definitions; see [13] for more details. Based on these results, there is much research on $q$-calculus; see [14–20] and the references cited therein.

In recent years, many interesting quantum integral inequalities on finite interval have been considered more generally in $(p, q)$-calculus, which was first considered by R. Chakrabarti and R. Jagannathan [21]. In 2016, M. Tunç and E. Göv [22,23] introduced the $(p, q)$-derivative and $(p, q)$-integral on finite interval while proving some properties, and gave several inequalities of integral via $(p, q)$-calculus. In addition, some more results of $(p, q)$-calculus appear in [24–33] and the references cited therein.

The function $f : [a, b] \rightarrow \mathbb{R}$ is called convex if

$$f(ax + (1-a)y) \leq af(x) + (1-a)f(y)$$

for all $x, y \in [a, b], a \in [0, 1]$, and $f$ is called concave provided that $-f$ is convex.
Let \( I \) be the interval or real numbers and \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a convex function on \( I \) with constants \( a < b \) in \( I \). The well-known inequality which is Hermite–Hadamard inequality \([34]\) is

\[
f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]  

(1)

In 2000, S. S. Dragomir et al. \([35,36]\) proved related result to the Hermite–Hadamard inequality, as in the following.

**Theorem 1.** Refs. \([35,36]\) If \( f : I \to \mathbb{R} \) is a twice differentiable function where \( a, b \in I \) with \( a < b \) and real constants \( m \) and \( M \) with \( m \leq f'' \leq M \), then

\[
m \frac{(b-a)^2}{12} \leq \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \leq M \frac{(b-a)^2}{12},
\]  

(2)

and

\[
m \frac{(b-a)^2}{24} \leq \frac{1}{b-a} \int_a^b f(x) \, dx - f\left(\frac{a+b}{2}\right) \leq M \frac{(b-a)^2}{24}.
\]  

(3)

The Hermite–Hadamard inequality and the Hermite–Hadamard–Fejér inequalities, which are famous inequalities for convex functions, have a deep relationship to its integral mean, see \([37–47]\) for more details and the references cited therein. A weighted generalization of inequality (1) was introduced by L. Fejér \([48]\), as in the following.

**Theorem 2.** Ref. \([48]\) If \( f : I \to \mathbb{R} \) is a convex function with constants \( a < b \) in \( I \), then

\[
f\left(\frac{a+b}{2}\right) \int_a^b w(x) \, dx \leq \frac{1}{b-a} \int_a^b f(x) w(x) \, dx \leq \frac{f(a) + f(b)}{2} \int_a^b w(x) \, dx,
\]

where \( w : [a, b] \to \mathbb{R} \) is integrable, nonnegative, and symmetric about \( x = \frac{a+b}{2} \), that is \( w(a + b - x) = w(x) \).

In \([49]\), N. Minculete and F. C. Mitroi introduced the inequalities which become important, as follows.

**Theorem 3.** Ref. \([49]\) Let \( f : I \to \mathbb{R} \) be a twice differentiable function with \( a < b \) in \( I \), such that \( m \leq f'' \leq M \). If \( \lambda \in [0, 1] \), then

\[
m \frac{\lambda(1-\lambda)}{2} (b-a)^2 \leq \lambda f(a) + (1-\lambda)f(b) - f_\lambda(a,b) \leq M \frac{\lambda(1-\lambda)}{2} (b-a)^2,
\]

and

\[
m \frac{(1-2\lambda)^2}{8} (b-a)^2 \leq f_\lambda(a,b) + f_\lambda(b,a) - f\left(\frac{a+b}{2}\right) \leq M \frac{(1-2\lambda)^2}{8} (b-a)^2,
\]

where \( f_\lambda(a,b) = f(\lambda a + (1-\lambda)b) \).

Some inequalities of Hermite–Hadamard–Fejér type for differentiable functions follow from Theorem 3.
Theorem 4. Ref. [49] Let \( f : I \to \mathbb{R} \) be a twice differentiable function with \( a < b \) in \( I \) such that \( m \leq f'' \leq M \). If \( w : [a, b] \to \mathbb{R} \) is integrable, nonnegative and symmetric about \( x = \frac{a + b}{2} \), then

\[
\frac{m}{2} \int_a^b (t-a)(b-t)w(t) \, dt \leq \frac{f(a) + f(b)}{2} \int_a^b w(t) \, dt - \int_a^b f(t)w(t) \, dt \leq \frac{M}{2} \int_a^b (t-a)(b-t)w(t) \, dt,
\]

and

\[
\frac{m}{8} \int_a^b (2t-a-b)^2w(t) \, dt \leq \int_a^b f(t)w(t) \, dt - f\left(\frac{a+b}{2}\right) \int_a^b w(t) \, dt \leq \frac{M}{8} \int_a^b (2t-a-b)^2w(t) \, dt.
\]

In \( q \)-calculus, some Fejér type inequalities for differentiable functions were established by W. Yang [50]. Moreover, Fejér type inequalities for fractional integrals were established by M. Z. Sarikaya [51].

In this paper, we propose to generalize and extend some Fejér-type inequalities in \( p, q \)-integral and fractional integral to \( (p, q) \)-integral. In particular, we correct existing results of quantum Fejér-type inequalities by using new techniques and showing some problematic parts of those results. The results presented here would extend some of those in existing literatures.

2. Preliminaries

In this section, we give fundamental concepts of \( (p, q) \)-calculus used in our work. We will use \( I = [a, b] \subseteq \mathbb{R} \), \( I^0 = (a, b) \), and \( p, q \) are constants with \( 0 < q < p \leq 1 \) throughout this paper.

Definition 1. Refs. [22,23] Let \( f : I \to \mathbb{R} \) be a continuous function. The \( (p, q) \)-derivative of the function \( f \) at \( x \) on \( [a, b] \) is

\[
aD_{p,q}f(x) = \begin{cases} 
  \frac{f(px + (1-p)a) - f(qx + (1-q)a)}{(p-q)(x-a)} & \text{if } x \neq a; \\
  \lim_{x \to a} aD_{p,q}f(x) & \text{if } x = a.
\end{cases}
\]

A function \( f \) is called \( (p, q) \)-differentiable on \( I \) if for each \( x \in I \) there exists \( aD_{p,q}f(x) \). If \( a = 0 \) in Definition 1, then \( aD_{p,q}f = D_{p,q}f \), where \( D_{p,q}f \) is

\[
D_{p,q}f(x) = \begin{cases} 
  \frac{f(px) - f(qx)}{(p-q)(x)} & \text{if } x \neq 0; \\
  \lim_{x \to 0} D_{p,q}f(x) & \text{if } x = 0.
\end{cases}
\]

Furthermore, if \( p = 1 \), then \( aD_{p,q}f = aD_qf \), which is the \( q \)-derivative of the function \( f \).

Example 1. For \( x \in I \) and a natural number \( n \), if \( f(x) = (x-a)^n \), then

\[
aD_{p,q}f(x) = [n]_{p,q}(x-a)^{n-1},
\]

where \([n]_{p,q} = \frac{p^n - q^n}{p - q}\).
**Definition 2.** Refs. [22,23] Let \( f : I \rightarrow \mathbb{R} \) be a continuous function. The \((p, q)\)-integral of the function \( f \) for \( x \in I \) is defined to be

\[
\int_a^x f(t) \, dq_{p,t} = (p-q)(x-a) \sum_{n=0}^{\infty} \frac{q^n}{p^n+1} f\left(\frac{q^n}{p^n+1}x + \left(1 - \frac{q^n}{p^n+1}\right)a\right).
\]

Furthermore, for \( c \in (a, x) \), the \((p, q)\)-integral is defined to be

\[
\int_a^c f(t) \, dq_{p,t} = \int_a^x f(t) \, dq_{p,t} - \int_a^c f(t) \, dq_{p,t}.
\]

If \( \int_a^x f(t) \, dq_{p,t} \) exists for each \( x \in I \), then we say \( f \) is \((p, q)\)-integrable on \( I \). Observe Definition 2 reduces to the \( q \)-integral of the function \( f \) when \( a = 0 \) and \( p = 1 \).

**Example 2.** Define a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) by \( f(x) = kx \) for \( x \in I \) where \( k \in \mathbb{R} \). Then

\[
\int_a^x f(t) \, dq_{p,t} = \int_a^x kt \, dq_{p,t} = (p-q)(x-a) \sum_{n=0}^{\infty} \frac{q^n}{p^n+1} \left[k\left(\frac{q^n}{p^n+1}x + \left(1 - \frac{q^n}{p^n+1}\right)a\right)\right] = k(x-a)(x-a(1-p-q)) \frac{p+q}{p+q}.
\]

**Theorem 5.** Refs. [22,23] Let \( f : I \rightarrow \mathbb{R} \) be a continuous function. We have

(i) \( \int_a^x aD_{p,q}f(t) \, dq_{p,t} = f(x) - f(a) \);

(ii) \( \int_a^c aD_{p,q}f(t) \, dq_{p,t} = f(x) - f(c) \), for \( c \in (a, x) \).

**Theorem 6.** Refs. [22,23] If \( f, g : I \rightarrow \mathbb{R} \) are two continuous functions and \( \alpha \in \mathbb{R} \), then for \( x \in I \),

(i) \( \int_a^x [f(t) + g(t)] \, dq_{p,t} = \int_a^x f(t) \, dq_{p,t} + \int_a^x g(t) \, dq_{p,t} \);

(ii) \( \int_a^x \alpha f(t) \, dq_{p,t} = \alpha \int_a^x f(t) \, dq_{p,t} \);

(iii) \( \int_a^x f(pt + (1-p)a) \, dq_{p,t} = \int_a^x f((t)g(t)) \, dq_{p,t} = \int_a^x g(pt + (1-p)a) \, dq_{p,t} \).

**Lemma 1.** Ref. [50] Let \( f : I \rightarrow \mathbb{R} \) be a twice differentiable function such that \( m \leq f'' \leq M \). It follows that

\[
m\frac{\lambda(1-\lambda)}{2}(b-a)^2 \leq (1-\lambda)f(a) + \lambda f(b) - f\lambda(b,a) \leq M\frac{\lambda(1-\lambda)}{2}(b-a)^2. \tag{6}
\]

3. **Main Results**

In 2017, W. Yang [50] obtained some Fejér-type quantum integral inequalities. Unfortunately, there are many mistakes in the proofs. Many \( q \)-integrals are calculated incorrectly. Besides, the results of lemma and theorems are also wrong. Here, we will show the errors of Lemma 3 in [50].

**Statement 1** (Lemma 3, [50]). If \( f : I \rightarrow \mathbb{R} \) is a twice \( q \)-differentiable function with \( aD_q^2 f \) \( q \)-integrable on \( I \), then

\[
\int_a^b (x-a)(b-x)aD_q^2 \ f(x) \ dq_{q} x = (qb - a)f_q(b,a) + (b - qa)f(a) - (1 + q) \int_a^b f_q(x,a) \ dq_{q} x, \tag{7}
\]

and
\[
\int_a^b [(x - a)^2 + (x - b)^2] D_q^2 f(x) \, a \, d_q x = (b - a)^2 \left( a D_q f(b) - a D_q f(a) \right) - 2(qb - a) f_q(b, a) - 2(b - qa) f(a) + 2(1 + q) \int_a^b f_q(x, a) \, a \, d_q x.
\tag{8}
\]

**Example 3.** Let a function \( f : [1, 2] \to \mathbb{R} \) be defined by \( f(x) = x \). It follows that \( f \) satisfies the conditions of Lemma 1. The left side of Equality (7) and (8) become

\[
\int_1^2 (x - 1)(2 - x) D_q^2 x \, 1 \, d_q x = 0 \quad \text{and} \quad \int_1^2 [(x - 1)^2 + (x - 2)^2] D_q^2 f(x) \, a \, d_q x = 0,
\]

respectively. The right side of Equality (7) becomes

\[
(q(2) - (1)) f_q(2, 1) + (2 - q(1)) f(1) - (1 + q) \int_1^2 f_q(x, 1) \, 1 \, d_q x = q^2 - q,
\tag{9}
\]

and the right side of Equality (8) becomes

\[
(2 - 1)^2 \left( a D_q f(2) - a D_q f(1) \right) - 2(2q - 1) f_q(2, 1) - 2(2 - q) f(1) + 2(1 + q) \int_1^2 f_q(x, 1) \, 1 \, d_q x
\]

\[
= -2(2q - 1)(q + 1) - 2(2 - q)(1 + q) + 2(1 + q) \int_1^2 f_q(x, 1) \, 1 \, d_q x
\]

\[
= 2q - 2q^2.
\tag{10}
\]

Since \( q \in (0, 1) \), Equalities (9) and (10) are not equal to 0. Therefore, Equality (7) and (8) are not correct.

Since Lemma 1 is used in the proof of Theorems 9 and 10 in [50], there are errors in those theorems. Now, we show that Theorem 9 in [50] is not correct.

**Statement 2 (Theorem 9, [50]).** Let \( f : I \to \mathbb{R} \) be a twice \( q \)-differentiable function with \( a D_q^2 f \) \( q \)-integrable on \( I \), such that \( m \leq a D_q^2 f \leq M \). It follows that

\[
\frac{mq^2(b - a)^3}{(1 + q)^2(1 + q + q^2)} \leq (qb - a) f_q(b, a) + (b - qa) f(a) - (1 + q) \int_a^b f_q(x, a) \, a \, d_q x \leq \frac{Mq^2(b - a)^3}{(1 + q)^2(1 + q + q^2)},
\tag{11}
\]

and

\[
\frac{m(1 + q + q^2)(b - a)^3}{2(1 + q)^2(1 + q + q^2)} \leq (b - a)^2 \left( a D_q f(b) - a D_q f(a) \right) - 2(qb - a) f_q(b, a) - 2(b - qa) f(a) + 2(1 + q) \int_a^b f_q(x, a) \, a \, d_q x \leq \frac{M(1 + q + q^2)(b - a)^3}{2(1 + q)^2(1 + q + q^2)}.
\tag{12}
\]

**Example 4.** Let a function \( f : [1, 2] \to \mathbb{R} \) be defined by \( f(x) = x \). Since \( a D_q^2 f(x) = a D_q^2 x = 0 \), we obtain \( m \leq 0 \) and \( M \geq 0 \). It follows that \( f \) satisfies the conditions in Theorem 2 with \(-1 \leq a D_q^2 f \leq 1\). Then, we have

\[
\frac{(-1)q^2(2 - 1)^3}{(1 + q)^2(1 + q + q^2)} = \frac{-q^2}{(1 + q)^2(1 + q + q^2)}.
\tag{13}
\]
Theorem 7. Let \( f \) be a twice differentiable function such that \( m < f'' \leq M \). It follows that

\[
\frac{m(b-a)^2}{(p+q)(p^2 + pq + q^2)} \leq \left[ f(a) + f(b) - \frac{1}{p(b-a)} \int_a^b f(x) + f(a+b-x) \right]_{a \leq \lambda \leq b} d\lambda \leq \frac{M(b-a)^2}{(p+q)(p^2 + pq + q^2)}.
\] (16)

and

\[
\frac{m(b-a)^2}{4} \left[ 1 - \frac{4p}{p+q} + \frac{4p^2}{p^2 + pq + q^2} \right] \leq \left[ f(a) + f(b) - \frac{1}{p(b-a)} \int_a^b f(x) + f(a+b-x) \right]_{a \leq \lambda \leq b} d\lambda \leq \frac{4p^2}{4} \left[ 1 - \frac{4p}{p+q} + \frac{4p^2}{p^2 + pq + q^2} \right].
\] (17)

Proof. Taking \((p,q)\)-integral for Inequality (6) with respect to \( \lambda \) over \([0,p]\) yields

\[
\frac{m(b-a)^2}{2} \int_0^p \lambda(1-\lambda) d\lambda \leq \int_0^p f(a) f_0^p (1-\lambda) d\lambda d\lambda - \int_0^p f(a) d\lambda d\lambda \leq \frac{M(b-a)^2}{2} \int_0^p \lambda(1-\lambda) d\lambda.
\] (18)

Using direct computation and variable changing in (18), we have
\[ \frac{mp^2q^2(b-a)^2}{2(p+q)(p^2+pq+q^2)} \leq \frac{pqf(a) + p^2f(b)}{p+q} - \frac{1}{b-a} \int_a^{pb+(1-p)a} f(x) \, ad_{p,q}x \]

\[ \leq \frac{Mpq^2q^2(b-a)^2}{2(p+q)(p^2+pq+q^2)}. \]  

Similarly, using \((p,q)\)-integration on the first inequality of Theorem 3 with respect to \(\lambda\) over \([0, p]\), we obtain

\[ \frac{m(b-a)^2}{2} \int_0^p \lambda(1 - \lambda) \, dpq\lambda \leq f(a) \int_0^p \lambda \, dpq\lambda + f(b) \int_0^p (1 - \lambda) \, dpq\lambda - \int_0^p f(a,b) \, dpq\lambda \]

\[ \leq \frac{M(b-a)^2}{2} \int_0^p \lambda(1 - \lambda) \, dpq\lambda. \]  

Using direct computation and variable changing in (20), we obtain

\[ \frac{mp^2q^2(b-a)^2}{2(p+q)(p^2+pq+q^2)} \leq \frac{p^2f(a) + p^2f(b)}{p+q} - \frac{1}{b-a} \int_a^{pb+(1-p)a} f(a+b-x) \, ad_{p,q}x \]

\[ \leq \frac{Mpq^2q^2(b-a)^2}{2(p+q)(p^2+pq+q^2)}. \]

Inequality (16) comes from (19) and (21).

Next, using \((p,q)\)-integration on the second inequality of Theorem 3 with respect to \(\lambda\) over \([0, p]\), we obtain

\[ \frac{m(b-a)^2}{8} \int_0^p (1-2\lambda)^2 \, dpq\lambda \leq \int_0^p \frac{f(a,b) + f(b,a)}{2} \, dpq\lambda - \int_0^p f\left( \frac{a+b}{2} \right) \, dpq\lambda \]

\[ \leq \frac{M(b-a)^2}{8} \int_0^p (1-2\lambda)^2 \, dpq\lambda. \]

Changing the variable, we have

\[ \frac{m(b-a)^2}{8} \left[ p - \frac{4p^2}{p+q} + \frac{4p^3}{p^2+pq+q^2} \right] \]

\[ \leq \frac{1}{2(b-a)} \int_a^{pb+(1-p)a} [f(x) + f(a+b-x)] \, ad_{p,q}x - pf\left( \frac{a+b}{2} \right) \]

\[ \leq \frac{M(b-a)^2}{8} \left[ p - \frac{4p^2}{p+q} + \frac{4p^3}{p^2+pq+q^2} \right], \]

which implies Inequality (17). This completes the proof of theorem. \(\Box\)

**Remark 1.** \(i\) If \(p = 1\), then Theorem 7 reduces to Theorem 7 in [50].

\(ii\) If \(p = 1\) and \(q \rightarrow 1\), then Inequality (16) reduces to (2), and Inequality (17) reduces to (3).

**Theorem 8.** Let \(f : I \rightarrow \mathbb{R}\) be a twice \((p,q)\)-differentiable function such that \(m \leq f'' \leq M\). If \(w : I \rightarrow \mathbb{R}\) is \((p,q)\)-integrable on \(I\), nonnegative and symmetric about \(x = (a+b)/2\), then

\[ m \int_a^{pb+(1-p)a} (x-a)(b-x)w(x) \, ad_{p,q}x \]

\[ \leq [f(a)+f(b)] \int_a^{pb+(1-p)a} w(x) \, ad_{p,q}x - \int_a^{pb+(1-p)a} f(x) + f(a+b-x)w(x) \, ad_{p,q}x \]

\[ \leq M \int_a^{pb+(1-p)a} (x-a)(b-x)w(x) \, ad_{p,q}x, \]

and
\[
\frac{m}{4} \int_a^{pb+ (1-p)a} (2x - a - b)^2 w(x) \, adp,q x \leq \int_a^{pb+ (1-p)a} [f(x) + f(a + b - x)] w(x) \, adp,q x - 2f \left( \frac{a + b}{2} \right) \int_a^{pb+ (1-p)a} w(x) \, adp,q x
\]
\[\leq \frac{M}{4} \int_a^{pb+ (1-p)a} (2x - a - b)^2 w(x) \, adp,q x. \tag{23}\]

**Proof.** Multiplying Inequality (6) by \(w_\lambda(b, a)\), we get
\[
m \lambda (1 - \lambda) (b - a)^2 w_\lambda(b, a) \leq [(1 - \lambda)f(a) + \lambda f(b) - f_\lambda(b, a)] w_\lambda(b, a) \leq M \lambda (1 - \lambda) (b - a)^2 w_\lambda(b, a). \tag{24}\]
Taking \((p, q)\)-integral for Inequality (24) with respect to \(\lambda\) over \([0, p]\) yields
\[
\frac{m(b - a)^2}{2} \int_0^p \lambda (1 - \lambda) w_\lambda(b, a) \, d_p,q \lambda \leq \int_0^p [(1 - \lambda)f(a) + \lambda f(b) - f_\lambda(b, a)] w_\lambda(b, a) \, d_p,q \lambda \leq \frac{M(b - a)^2}{2} \int_0^p \lambda (1 - \lambda) w_\lambda(b, a) \, d_p,q \lambda. \tag{25}\]
Using directly computation and variable changing in (25), we obtain
\[
\frac{m}{2} \int_a^{pb+ (1-p)a} \frac{(x - a)(b - x)w(x)}{b - a} \, adp,q x \leq f(a) \int_a^{pb+ (1-p)a} \frac{(b - x)w(x)}{(b - a)^2} \, adp,q x + f(b) \int_a^{pb+ (1-p)a} \frac{(x - a)w(x)}{(b - a)^2} \, adp,q x \tag{26}
- f_\lambda^{pb+ (1-p)a} \frac{f(x)w(x)}{(b - a)^2} \, adp,q x
\]
\[
\leq \int_0^p \lambda (1 - \lambda) w_\lambda(b, a) \, d_p,q \lambda. \]

Similarly, multiplying the first inequality of Theorem 3 by \(w_\lambda(b, a)\), and subsequently taking \((p, q)\)-integral on the obtained inequality with respect to \(\lambda\) over \([0, p]\) yield
\[
\frac{m(b - a)^2}{2} \int_0^p \lambda (1 - \lambda) w_\lambda(b, a) \, d_p,q \lambda \leq f(a) \int_a^{pb+ (1-p)a} \frac{(b - x)w(x)}{(b - a)^2} \, adp,q x + f(b) \int_a^{pb+ (1-p)a} \frac{(x - a)w(x)}{(b - a)^2} \, adp,q x \tag{27}
- f_\lambda^{pb+ (1-p)a} \frac{f(x)w(x)}{(b - a)^2} \, adp,q x
\]
\[
\leq \int_0^p \lambda (1 - \lambda) w_\lambda(b, a) \, d_p,q \lambda. \]

From (27), we change the variable and apply the symmetry of \(w(x)\), it follows that
\[
\frac{m}{2} \int_a^{pb+ (1-p)a} \frac{(x - a)(b - x)w(x)}{b - a} \, adp,q x \leq \frac{m}{2} \int_a^{pb+ (1-p)a} \frac{(b - x)w(x)}{(b - a)^2} \, adp,q x + \frac{m}{2} \int_a^{pb+ (1-p)a} \frac{(x - a)w(x)}{(b - a)^2} \, adp,q x \tag{28}
- \frac{M}{2} \int_a^{pb+ (1-p)a} \frac{(x - a)(b - x)w(x)}{b - a} \, adp,q x
\]
\[
\leq \int_0^p (1 - 2\lambda)^2 w_\lambda(b, a) \, d_p,q \lambda. \]
Then, we obtain Inequality (22) from (26) and (28). Next, multiplying the second inequality of Theorem 3 by \(w_\lambda(b, a)\), and subsequently taking \((p, q)\)-integral on the obtained inequality with respect to \(\lambda\) over \([0, p]\) yields
\[
\frac{m(b - a)^2}{8} \int_0^p (1 - 2\lambda)^2 w_\lambda(b, a) \, d_p,q \lambda \leq \frac{1}{2} \left[ \int_0^p f_\lambda(b, a) w_\lambda(b, a) \, d_p,q \lambda + \int_0^p f_\lambda(b, a) w_\lambda(b, a) \, d_p,q \lambda \right] - f \left( \frac{a + b}{2} \right) \int_0^p w_\lambda(b, a) \, d_p,q \lambda \tag{29}
\leq M \frac{(b - a)^2}{8} \int_0^p (1 - 2\lambda)^2 w_\lambda(b, a) \, d_p,q \lambda.
By using the change of the variable of (29), we get
\[
\frac{m}{8} \int_a^{p_b+(1-p)a} \frac{(2x - a - b)^2 w(x)}{b - a} \, d_{p,q} x \\
\leq \frac{1}{2(b - a)} \left[ \int_a^{p_b+(1-p)a} f(x) w(x) \, d_{p,q} x + \int_a^{p_b+(1-p)a} f(a + b - x) w(x) \, d_{p,q} x \right] \\
- f \left( \frac{a + b}{2} \right) \int_a^{p_b+(1-p)a} \frac{w(x)}{b - a} \, d_{p,q} x \\
\leq M \int_a^{p_b+(1-p)a} \frac{(2x - a - b)^2 w(x)}{b - a} \, d_{p,q} x,
\]
which implies Inequality (23). The proof of the theorem is complete. \(\square\)

**Remark 2.**
(i) If \(p = 1\), then Theorem 8 reduces to Theorem 8 in [50].
(ii) If \(p = 1\) and \(q \to 1\), then Inequality (22) reduce to (4) and Inequality (23) reduce to (5).

**Lemma 2.** If \(f : I \to \mathbb{R}\) is a twice \((p,q)\)-differentiable function with \(a D^2_{p,q} f(x)\) integrable on \(I\), then
\[
\int_a^{p_b+(1-p)a} (x - a)(b - x) a D^2_{p,q} f(x) \, d_{p,q} x \\
= \frac{q}{p^2} (b - a) f_{pq}(b,a) + \frac{1}{p} (b - a) f(a) - \left( \frac{p + q}{p^3} \right) \int_a^{p_b+(1-p)a} f_{q^2}(x,a) \, d_{p,q} x,
\]
and
\[
\int_a^{p_b+(1-p)a} [(x - a)^2 + (x - b)^2] a D^2_{p,q} f(x) \, d_{p,q} x \\
= (b - a)^2 f_{pq}(b,a) - a D_{p,q} f(a) - \frac{2q}{p^2} (b - a) f_{pq}(b,a) - \frac{2}{p} (b - a) f(a) \\
+ 2 \left( \frac{p + q}{p^3} \right) \int_a^{p_b+(1-p)a} f_{q^2}(x,a) \, d_{p,q} x.
\]

**Proof.** Using \((p,q)\)-integration by parts yields
\[
\int_a^{p_b+(1-p)a} (x - a)(b - x) a D^2_{p,q} f(x) \, d_{p,q} x \\
= \left[ \frac{1}{p^2} (x - a) (pb + (1-p)a - x) a D_{p,q} f(x) \right]_a^{p_b+(1-p)a} \\
- \int_a^{p_b+(1-p)a} a D_{p,q} f(qx + (1 - q)a) a D_{p,q} \left( \frac{1}{p^2} (x - a) (pb + (1-p)a - x) \right) \, d_{p,q} x
\]
\[
= - \frac{1}{p^2} \int_a^{p_b+(1-p)a} \frac{[pb + qa - (p + q)x] a D_{p,q} f(x,a)}{a} \, d_{p,q} x \\
= - \frac{1}{p^2} \left\{ \left[ (pb + qa - (p + q) a) \left( \frac{x - (1-p)a}{p} \right) \right] f_q(x,a) \right\}^{p_b+(1-p)a}_a \\
- \int_a^{p_b+(1-p)a} f_q(qx + (1 - q)a, a) a D_{p,q} \left( \frac{pb + qa - (p + q) (x - (1-p)a)}{p} \right) \, d_{p,q} x
\]
\[
= - \frac{1}{p^2} \left\{ \{q(a-b) f_{pq}(b,a) - p(b-a) f(a)\} + \left( \frac{p + q}{p} \right) \int_a^{p_b+(1-p)a} f_{q^2}(x,a) \, d_{p,q} x \right\}
\]
\[
= \frac{q}{p^2} (b - a) f_{pq}(b,a) + \frac{1}{p} (b - a) f(a) - \left( \frac{p + q}{p^3} \right) \int_a^{p_b+(1-p)a} f_{q^2}(x,a) \, d_{p,q} x,
\]
which is Inequality (30).
Next, we prove Inequality (31). Using \((p, q)\)-integration by parts, we obtain

\[
\int_a^{p+(1-p)a} (x - a)^2 a D_{p,q}^2 f(x) \, d_{p,q} x \\
= \left[ \frac{(x - a)^2}{p} a D_{p,q} f(x) \right]_a^{p+(1-p)a} - \int_a^{p+(1-p)a} a D_{p,q} f(qx + (1 - q)a) D_{p,q} \left( \frac{x - a}{p} \right)^2 \, d_{p,q} x \\
= (b - a)^2 a D_{p,q} f_p(b, a) - \frac{1}{p^2} \int_a^{p+(1-p)a} (p + q)(x - a) D_{p,q} f(x, a) \, d_{p,q} x \\
= (b - a)^2 a D_{p,q} f_p(b, a) - \frac{p + q}{p^2} \int_a^{p+(1-p)a} (x - a) D_{p,q} f(x, a) \, d_{p,q} x \\
= (b - a)^2 a D_{p,q} f_p(b, a) - \frac{p + q}{p^2} \int_a^{p+(1-p)a} f_{p,q}(x, a) \, d_{p,q} x, \\
\]

and

\[
\int_a^{p+(1-p)a} (x - b)^2 a D_{p,q}^2 f(x) \, d_{p,q} x \\
= \left[ \frac{(x - (pb + (1-p)a))^2}{p} a D_{p,q} f(x) \right]_a^{p+(1-p)a} - \int_a^{p+(1-p)a} a D_{p,q} f(qx + (1 - q)a) D_{p,q} \left( \frac{x - (pb + (1-p)a)}{p} \right)^2 \, d_{p,q} x \\
= -(b - a)^2 a D_{p,q} f(a) - \frac{1}{p^2} \int_a^{p+(1-p)a} \left[(p + q)(x - a) - 2p(b - a)\right] D_{p,q} f(x, a) \, d_{p,q} x \\
= -(b - a)^2 a D_{p,q} f(a) - \frac{p + q}{p^2} \int_a^{p+(1-p)a} \left[(p + q)(x - a) - 2p(b - a)\right] f_{p,q}(x, a) \, d_{p,q} x, \\
\]

Adding (32) and (33), we obtain

\[
\int_a^{p+(1-p)a} \left[(x - a)^2 + (x - b)^2\right] a D_{p,q}^2 f(x) \, d_{p,q} x \\
= (b - a)^2 \left(a D_{p,q} f_p(b, a) - a D_{p,q} f(a)\right) - \frac{2q}{p^2} (b - a) f_{p,q}(b, a) - \frac{2}{p} (b - a) f(a) \\
+ 2 \left(\frac{p + q}{p^3}\right) \int_a^{p+(1-p)a} f_{q,q}(x, a) \, d_{p,q} x, \\
\]

which is Inequality (31). Thus the proof is completed. \(\Box\)

Taking \(p = 1\) in Lemma 2 yields the correct result of Statement 1.

**Corollary 1.** If \(f : I \to \mathbb{R}\) is a twice \(q\)-differentiable function where \(a D_{q}^2 f\) \(q\)-integrable on \(I\), then
\[ f_a^b (x - a)(b - x) a D_q \frac{1}{2} f(x) a d_q x = q(b - a)f_q(b,a) + (b - a)f(a) - (1 + q) \int_a^b f_q(x,a) a d_q x, \] (34)

and

\[ f_a^b [(x - a)^2 + (x - b)^2] a D_q \frac{1}{2} f(x) a d_q x = (b - a)^2 \left( a D_q f(b) - a D_q f(a) \right) - 2q(b - a)f_q(b,a) - 2(b - a)f(a) + 2(1 + q) \int_a^b f_q(x,a) a d_q x. \] (35)

**Remark 3.** From Example 3, the left side of Equality (34) and (35) become

\[ \int_1^2 (x - 1)(2 - x) 1 D_q^2 x 1 d_q x = 0 \quad \text{and} \quad \int_1^2 [(x - 1)^2 + (x - 2)^2] 1 D_q^2 f(x) a d_q x = 0, \]

respectively. The right side of Equality (34) becomes

\[ q(2 - 1)f_q(2,1) + (2 - 1)f(1) - (1 + q) \int_1^2 f_q(x,1) 1 d_q x = q(q + 1) + 1 - (1 + q) \int_1^2 f_q^2(x,1) 1 d_q x = q^2 + q + 1 - (1 + q) \left[ q^2 \left( \frac{1}{1 + q} \right) + 1 \right] = 0, \]

and the right side of Equality (35) becomes

\[ (2 - 1)^2 \left( 1 D_q f(2) - 1 D_q f(1) \right) - 2q(2 - 1)f_q(2,1) - 2(2 - 1)f(1) + 2(1 + q) \int_1^2 f_q^2(x,1) 1 d_q x = -2q(q + 1) - 2(1) + 2(1 + q) \left[ q^2 \left( \frac{1}{1 + q} \right) + 1 \right] = 0, \]

which shows the result appearing in Corollary 1.

**Theorem 9.** Let \( f : I \to \mathbb{R} \) be a twice \((p,q)\) - differentiable function where \( a D_{p,q}^2 f(p,q) \)-integrable on \( I \) with \( m \leq a D_{p,q}^2 f \leq M \). It follows that

\[ \frac{mp^3q^2(b - a)^3}{(p+q)(p^2+pq+q^2)} \leq \frac{q(b - a)f_{pq}(b,a) + (b - a)f(a) - \left( \frac{p+q}{p^2} \right) \int_a^b f_{pq+1} a d_{p,q} x}{(1 + q) \left[ q^2 \left( \frac{1}{1 + q} \right) + 1 \right]} \]

and

\[ \frac{mp^3q^3(b - a)^3}{(p+q)(p^2+pq+q^2)} \leq \frac{a D_{p,q} f_{pq}(b,a) - a D_{p,q} f(a) - 2q(b - a)f_q(b,a) - 2(b - a)f(a) + 2(1 + q) \int_a^b f_q^2(x,a) a d_{p,q} x}{(1 + q) \left[ q^2 \left( \frac{1}{1 + q} \right) + 1 \right]} \]

**Proof.** Since \( m \leq a D_{p,q}^2 f \leq M \), it follows that

\[ m(x - a)(b - x) \leq (x - a)(b - x) a D_{p,q}^2 f(x) \leq M(x - a)(b - x), \quad \forall x \in I. \] (38)
Take \((p, q)\)-integral for Inequality (38) with respect to \(x\) from \(a\) to \(pb + (1 - p)a\), we obtain
\[
m\int_a^{pb + (1 - p)a} (x - a)(b - x) \, ad_{p,q}x \leq \int_a^{pb + (1 - p)a} (x - a)(b - x)D_p^2 f(x) \, ad_{p,q}x \leq M\int_a^{pb + (1 - p)a} (x - a)(b - x) \, ad_{p,q}x.
\] (39)

Applying Inequality (30) in Lemma 2 and
\[
\int_a^{pb + (1 - p)a} (x - a)(b - x) \, ad_{p,q}x = \frac{p^2q^2(b - a)^3}{(p + q)(p^2 + pq + q^2)}
\]
into (39), we get
\[
\frac{mp^2q^2(b - a)^3}{(p + q)(p^2 + pq + q^2)} \leq \frac{q}{p^2}(b - a)f_{pq}(b,a) + \frac{1}{p}(b - a)f(a) - \left(\frac{p + q}{p^3}\right)\int_a^{pb + (1 - p)a} f^2_{pq}(x,a) \, ad_{p,q}x
\]
\[
\leq \frac{Mmp^2q^2(b - a)^3}{(p + q)(p^2 + pq + q^2)},
\]
which implies Inequality (36). From \(m \leq aD_{pq}^2f \leq M\), we have
\[
m(x - a)^2 \leq (x - a)^2D_{pq}^2f(x) \leq M(x - a)^2,
\] (40)
and
\[
m(x - b)^2 \leq (x - b)^2D_{pq}^2f(x) \leq M(x - b)^2,
\] (41)
for all \(x \in I\). Taking \((p, q)\)-integral on (40) and (41) with respect to \(x\) from \(a\) to \(pb + (1 - p)a\), we obtain
\[
m\int_a^{pb + (1 - p)a} (x - a)^2 \, ad_{p,q}x \leq \int_a^{pb + (1 - p)a} (x - a)(b - x)D_p^2 f(x) \, ad_{p,q}x \leq M\int_a^{pb + (1 - p)a} (x - a)^2 \, ad_{p,q}x,
\] (42)
and
\[
m\int_a^{pb + (1 - p)a} (x - b)^2 \, ad_{p,q}x \leq \int_a^{pb + (1 - p)a} (x - b)(b - x)D_p^2 f(x) \, ad_{p,q}x \leq M\int_a^{pb + (1 - p)a} (x - b)^2 \, ad_{p,q}x,
\] (43)
respectively. By directly computation, we obtain
\[
\int_a^{pb + (1 - p)a} (x - a)^2 \, ad_{p,q}x = \frac{p^3(b - a)^3}{p^2 + pq + q^2},
\] (44)
and
\[
\int_a^{pb + (1 - p)a} (x - b)^2 \, ad_{p,q}x = \frac{(p^3q + pq^3)(b - a)^3}{(p + q)(p^2 + pq + q^2)}.
\] (45)
Substituting (44) into (42), we get
\[
\frac{mp^3(b - a)^3}{p^2 + pq + q^2} \leq \int_a^{pb + (1 - p)a} (x - a)^2D_p^2 f(x) \, ad_{p,q}x \leq \frac{Mmp^3(b - a)^3}{p^2 + pq + q^2}.
\] (46)
And substituting (45) into (43), we get
\[
\frac{m(p^3q + pq^3)(b - a)^3}{(p + q)(p^2 + pq + q^2)} \leq \int_a^{pb + (1 - p)a} (x - b)^2D_p^2 f(x) \, ad_{p,q}x \leq \frac{M(p^3q + pq^3)(b - a)^3}{(p + q)(p^2 + pq + q^2)}.
\] (47)
Adding (46) and (47), we obtain
\[
\frac{m(p^4 + 2p^3q + pq^3)(b - a)^3}{(p + q)(p^2 + pq + q^2)} \leq \int_a^{pb + (1 - p)a} [(x - a)^2 + (x - b)^2]D_p^2 f(x) \, ad_{p,q}x \leq \frac{M(p^4 + 2p^3q + pq^3)(b - a)^3}{(p + q)(p^2 + pq + q^2)}.
\] (48)
Substituting Equality (31) into (48), we get Inequality (37). This completes the proof. □

Taking \( p = 1 \) in Theorem 9 yields the correct result of Statement 2.

**Corollary 2.** If \( f : I \to \mathbb{R} \) is a twice \( q \)-differentiable function with \( \lambda D_q^2 f \) \( q \)-integrable on \( I \) such that \( m \leq \lambda D_q^2 f \leq M \), then

\[
\frac{mq^2(b-a)^3}{(1+q)(1+q+q^2)} \leq q(b-a)f_q(b,a) + (b-a)f(a) - (1+q) \int_a^b f_q^2(x,a) \, d\lambda q x
\]

and

\[
\frac{m(1+2q+q^3)(b-a)^3}{(1+q)(1+q+q^2)} \leq (b-a)^2(\lambda D_q f(b) - \lambda D_q f(a)) - 2q(b-a)f_q(b,a)
\]

\[
-2(b-a)f(a) + 2(1+q) \int_a^b f_q^2(x,a) \, d\lambda q x
\]

\[
\leq \frac{M(1+2q+q^3)(b-a)}{(1+q)(1+q+q^2)}.
\]

**Remark 4.** From Example 4, \( f \) satisfies the conditions of Corollary 2 with \( -1 \leq \lambda D_q^2 f \leq 1 \). Then we have

\[
\frac{(-1)q^2(2-1)^3}{(1+q)(1+q+q^2)} = \frac{-q^2}{(1+q)(1+q+q^2)},
\]

and

\[
\frac{(1)q^2(2-1)^3}{(1+q)(1+q+q^2)} = \frac{q^2}{(1+q)(1+q+q^2)}.
\]

Also,

\[
q(b-a)f_q(b,a) + (b-a)f(a) - (1+q) \int_a^b f_q^2(x,a) \, d\lambda q x = 0.
\]

As we seen, from (49) to (51) and for \( q \in (0,1) \) we write

\[
\frac{-q^2}{(1+q)(1+q+q^2)} \leq 0 \leq \frac{q^2}{(1+q)(1+q+q^2)}.
\]

For instance, choose \( q = \frac{1}{2} \), we have

\[
\frac{-\left(\frac{1}{4}\right)}{\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{2} + \frac{1}{4}\right)} \leq 0 \leq \frac{\left(\frac{1}{4}\right)}{\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{2} + \frac{1}{4}\right)}.
\]

That is,

\[
-\frac{2}{21} \leq 0 \leq \frac{2}{21},
\]

which shows the result described in Corollary 2.
Theorem 10. If \( f : I \rightarrow \mathbb{R} \) is a twice \((p, q)\)-differentiable function with \( aD^2_{p,q}f(p, q)\)-integrable on \( 1 \) such that \( m \leq aD^2_{p,q}f \leq M \), then

\[
\left| \left( \frac{q}{p^2}(b-a)f_{pq}(b,a) + \frac{1}{p}(b-a)f(a) - \left( \frac{p+q}{p^3} \right) f_{aq}^{pb+(1-p)a}f_{aq}(x, a) a_{pq} x \right) \right| \leq \frac{p(b-a)^3}{M-m}.
\]  

\((52)\)

Proof. We observe that

\[
\sup_{x \in [a, pb+(1-p)a]} (x-a)(b-x) = p(1-p)(b-a)^2 \leq \frac{(b-a)^2}{4}, \quad \text{for } 0 < p < \frac{1}{2},
\]

\[
\sup_{x \in [a, pb+(1-p)a]} (x-a)(b-x) = \frac{(b-a)^2}{4}, \quad \text{for } \frac{1}{2} \leq p \leq 1,
\]

and

\[
\inf_{x \in [a, pb+(1-p)a]} (x-a)(b-x) = 0.
\]

Consequently,

\[
0 \leq (x-a)(b-x) \leq \frac{(b-a)^2}{4}, \quad \text{for } x \in [a, pb+(1-p)a].
\]

Substituting \( b, f(x), \) and \( g(x) \) in Theorem 9 in [23] by \( pb + (1-p)a, (x-a)(b-x), \) and \( aD^2_{p,q}f(x), \) respectively, we obtain

\[
\left| \frac{1}{(pb + (1-p)a-a)} \int_a^{pb+(1-p)a} (x-a)(b-x)aD^2_{p,q}f(x) \, a_{pq}x \right|
\]

\[
- \frac{1}{(pb + (1-p)a-a)^2} \left( \int_a^{pb+(1-p)a} (x-a)(b-x) \, a_{pq}x \right) \left( \int_a^{pb+(1-p)a} aD^2_{p,q}f(x) \, a_{pq}x \right)
\]

\[
\leq \frac{1}{4} \left( \frac{(b-a)^2}{4} \right) (M-m).
\]

By Equality \((31)\) in Lemma 2, we obtain

\[
\left| \frac{1}{p(b-a)} \left( \frac{q}{p^2}(b-a)f_{pq}(b,a) + \frac{1}{p}(b-a)f(a) - \left( \frac{p+q}{p^3} \right) f_{aq}^{pb+(1-p)a}f_{aq}(x, a) a_{pq} x \right) \right|
\]

\[
- \frac{q^2(b-a)^2(aD_{p,q}f_p(b,a) - aD_{p,q}f(a))}{(p+q)(p^2+pq+q^2)} \leq \frac{(b-a)^2}{16} (M-m),
\]

which implies Inequality \((52)\). This completes the proof. \( \square \)

Taking \( p = 1 \) in Theorem 10 yields the correct result of Theorem 10 in [50].

Corollary 3. If \( f : I \rightarrow \mathbb{R} \) is a twice \( q \)-differentiable function with \( aD^2_qf \) \( q \)-integrable on \( 1 \) and \( m \leq aD^2_qf \leq M \), then

\[
\left| \left( q(b-a)f_q(b,a) + (b-a)f(a) - (1+q) \int_a^b f_{aq}(x, a) a_{dq} x \right) \right|
\]

\[
- \frac{q^2(b-a)^2(aD_qf(b) - aD_qf(a))}{(1+q)(1+q+q^2)} \leq \frac{(b-a)^3}{16} (M-m).
\]
Remark 5. If $p = 1$ and $q \to 1$, then Theorem 10 reduces to the result obtained in [36].

Lemma 3. Let $\phi, \varphi : I \to \mathbb{R}$ be two continuous and $(p,q)$-differentiable functions on $I^0$. If $a D_{p,q} \varphi(x) \neq 0$ on $I^0$ and $m \leq a D_{p,q} \phi(x) / a D_{p,q} \varphi(x) \leq M$ on $I^0$, then

$$
m \left( p(b - a) \int_a^b (p+1)(p) \varphi^2(x) \, ad_{p,q} x - \left( \int_a^b (p+1)(p) \varphi(x) \, ad_{p,q} x \right)^2 \right) \leq p(b - a) \int_a^b (p+1)(p) \varphi(x) (x) \varphi(x) \, ad_{p,q} x - \left( \int_a^b (p+1)(p) \varphi(x) \, ad_{p,q} x \right) \left( \int_a^b (p+1)(p) \varphi(x) \, ad_{p,q} x \right)
$$

(53)

Proof. If $a D_{p,q} \varphi(x) > 0$, then $\varphi(x)$ is an increasing function with

$$
m_a D_{p,q} \varphi(x) \leq a D_{p,q} \varphi(x) \leq M a D_{p,q} \varphi(x),
$$

(54)

for all $x \in I^0$. For $a \leq x \leq y \leq b$, taking $(p, q)$-integral for Inequality (54) from $x$ to $y$ with respect to $x$ yields

$$
m(\varphi(y) - \varphi(x)) \leq \varphi(y) - \varphi(x) \leq M(\varphi(y) - \varphi(x)).
$$

Multiplying the inequality above by $\varphi(y) - \varphi(x) \geq 0$, we have

$$
m(\varphi(y) - \varphi(x))^2 \leq (\varphi(y) - \varphi(x))(\varphi(y) - \varphi(x)) \leq M(\varphi(y) - \varphi(x))^2.
$$

(55)

Similarly, if $a D_{p,q} \varphi(x) < 0$, then we also obtain (55). Taking $(p, q)$-integral for Inequality (55) from $a$ to $b - (1 - p)a$ with respect to $x$ and $y$, we have

$$
m \int_a^b (p+1)(p) \int_a^b (p+1)(p) \varphi^2(x) \varphi^2(x) - \left( \int_a^b (p+1)(p) \varphi(x) \right)^2 ad_{p,q} x \, ad_{p,q} y
$$

\begin{align*}
\leq & \int_a^b (p+1)(p) \int_a^b (p+1)(p) (\varphi(y) - \varphi(x))(\varphi(y) - \varphi(x)) \, ad_{p,q} x \, ad_{p,q} y \\
\leq & M \int_a^b (p+1)(p) \int_a^b (p+1)(p) (\varphi(y) - \varphi(x))^2 \, ad_{p,q} x \, ad_{p,q} y. \\
\end{align*}

(56)

A direct calculation yields

$$
\int_a^b (p+1)(p) \int_a^b (p+1)(p) (\varphi(y) - \varphi(x))^2 \, ad_{p,q} x \, ad_{p,q} y = 2 \left[ p(b - a) \int_a^b (p+1)(p) \varphi^2(x) \, ad_{p,q} x - \left( \int_a^b (p+1)(p) \varphi(x) \right)^2 \right],
$$

(57)

and

$$
\int_a^b (p+1)(p) \int_a^b (p+1)(p) (\varphi(y) - \varphi(x))(\varphi(y) - \varphi(x)) \, ad_{p,q} x \, ad_{p,q} y
$$

\begin{align*}
= & 2 \left[ p(b - a) \int_a^b (p+1)(p) \varphi(x) \, ad_{p,q} x - \left( \int_a^b (p+1)(p) \varphi(x) \right)^2 \right]. \\
\end{align*}

(58)

Substituting (57) and (58) into (56), we obtain

$$
m \left( p(b - a) \int_a^b (p+1)(p) \varphi^2(x) \, ad_{p,q} x - \left( \int_a^b (p+1)(p) \varphi(x) \right)^2 \right) \leq p(b - a) \int_a^b (p+1)(p) \varphi(x) \, ad_{p,q} x - \left( \int_a^b (p+1)(p) \varphi(x) \right)^2 \leq M \left( p(b - a) \int_a^b (p+1)(p) \varphi^2(x) \, ad_{p,q} x - \left( \int_a^b (p+1)(p) \varphi(x) \right)^2 \right),
$$

which is Inequality (53). This completes the proof. \qed
Theorem 11. Let \( f : I \to \mathbb{R} \) be a twice \((p,q)\)-differentiable function with \( a Da_{p,q}^2 f (p,q) \)-integrable on \( I \) such that \( m \leq a Da_{p,q}^2 f \leq M \). It follows that
\[
\frac{mp^5 q(b - a)^3}{(p + q)^2(p^2 + pq + q^2)} \leq \frac{p(b - a)(q f_p(b,a) + pf(a))}{p + q} - f_a^{p^{b + (1-p)a}} f_q(x,a) a_{p,q} x \leq \frac{M p^5 q(b - a)^3}{(p + q)^2(p^2 + pq + q^2)}. \tag{59}
\]

Proof. Let
\[ \phi(x) = a Da_{p,q} f(x) \quad \text{and} \quad \varphi(x) = x - \left( a + \frac{b}{2} \right). \]

Then \( m \leq a Da_{p,q} \phi(x) / a Da_{p,q} \varphi(x) \leq M \) on \( I \). Lemma 3 yields
\[
m \left( p(b - a) f_a^{p^{b + (1-p)a}} \left( x - \left( a + \frac{b}{2} \right) \right)^2 a_{p,q} x - \left( f_a^{p^{b - (1-p)a}} \left( x - \left( a + \frac{b}{2} \right) \right) a_{p,q} x \right)^2 \right) \leq (61)
\]
\[
= p(b - a) f_a^{p^{b + (1-p)a}} \left( x - \left( a + \frac{b}{2} \right) \right)^2 a_{p,q} x - \left( f_a^{p^{b - (1-p)a}} \left( x - \left( a + \frac{b}{2} \right) \right) a_{p,q} x \right)^2 \leq M \left( p(b - a) f_a^{p^{b + (1-p)a}} \left( x - \left( a + \frac{b}{2} \right) \right)^2 a_{p,q} x - \left( f_a^{p^{b - (1-p)a}} \left( x - \left( a + \frac{b}{2} \right) \right) a_{p,q} x \right)^2 \right). \tag{60}
\]

A direct calculation shows that
\[
\int_a^{p^{b + (1-p)a}} \left( x - \left( a + \frac{b}{2} \right) \right)^2 a_{p,q} x = \frac{(p - q)^2(b - a)^2}{2(p + q)}, \tag{61}
\]
\[
\int_a^{p^{b + (1-p)a}} \left( x - \left( a + \frac{b}{2} \right) \right)^2 a_{p,q} x = \frac{(p - q)^2q^2 + 2p^2q + pq^3)(b - a)^3}{4(p + q)(p^2 + pq + q^2)}, \tag{62}
\]
and
\[
f_a^{p^{b + (1-p)a}} \left( x - \left( a + \frac{b}{2} \right) \right) a_{p,q} x = \frac{(b - a) f_p(b,a) + b - a}{2 f(a)} - \frac{1}{p} f_a^{p^{b - (1-p)a}} f_q(x,a) a_{p,q} x. \tag{63}
\]

Substituting (61)–(63) into (60), we obtain
\[
\frac{mp^5 q(b - a)^4}{(p + q)^2(p^2 + pq + q^2)} \leq \frac{p(b - a)^2(q f_p(b,a) + pf(a))}{p + q} - (b - a) \int_a^{p^{b + (1-p)a}} f_q(x,a) a_{p,q} x \leq \frac{M p^5 q(b - a)^4}{(p + q)^2(p^2 + pq + q^2)},
\]
which implies Inequality (59). The proof is complete. \( \square \)

Remark 6. If \( p = 1 \), then Theorem 11 reduces to the result obtained in [50].

4. Conclusions

We have established some inequalities of Fejér-type inequalities by using \((p,q)\)-integral, such as the trapezoid-like inequalities, the midpoint-like inequalities, the Fejér-like inequalities. In particular, we generalized and corrected existing results of quantum Fejér-type inequalities by using new techniques and showing some problematic parts of those results. Our work improves the results of Fejér-type quantum integral inequalities. By taking \( q \to 1 \) and \( p = 1 \), our results give classical inequalities. The \((p,q)\)-integral inequalities deduced in the present research are very general and helpful in error estimations involved in various approximation processes. With these contributions, we hope that these techniques and ideas established in this article will inspire the interest of readers in exploring the field of \((p,q)\)-integral inequalities.
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