Achieving Differential Privacy with Matrix Masking in Big Data

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Abstract

Differential privacy schemes have been widely adopted in recent years to address issues of data privacy protection. We propose a new Gaussian scheme combining with another data protection technique, called random orthogonal matrix masking, to achieve $(\varepsilon, \delta)$-differential privacy (DP) more efficiently. We prove that the additional matrix masking significantly reduces the rate of noise variance required in the Gaussian scheme to achieve $(\varepsilon, \delta)$-DP in big data setting. Specifically, when $\varepsilon \to 0$, $\delta \to 0$, and the sample size $n$ exceeds the number $p$ of attributes by $\frac{n}{p} = O(ln(1/\delta))$, the required additive noise variance to achieve $(\varepsilon, \delta)$-DP is reduced from $O(ln(1/\delta)/\varepsilon^2)$ to $O(1/\varepsilon)$. With much less noise added, the resulting differential privacy protected pseudo data sets allow much more accurate inferences, thus can significantly improve the scope of application for differential privacy.
1 Introduction

In the past decades, much research interest has been devoted to the issue of how to share and publish data sets while protecting the privacy of individuals in the data set. The differential privacy (DP) \[13, 16, 14\] provides a quantitative measure for privacy loss in data release. In brief, DP guarantees that the distributions over potential outputs are statistically close for any two neighboring databases, and thus the chance of an individual being identified is low. Over the years, DP schemes have been widely adopted, through adding noises, to release data statistics with given privacy cost constraints. For instance, DP has been implemented in real world applications developed by Google \[18\], Microsoft \[11\] and the U.S. Census Bureau \[2\].

In recent literature, there has been an emphasis on linking DP to statistical concepts. Objective perturbation was applied to build differentially private linear regression and logistic regression model \[23, 7\]. Another well-known method, Johnson-Lindenstrauss transform, has been investigated to provide DP by mapping original data to a lower-dimensional space \[5, 21\]. The DP framework has been expanded to include hypothesis tests \[3, 6\], deep learning \[1\], network analysis \[19\] and Bayesian inference \[9\]. Most DP schemes focus on releasing some specific summary statistics rather than a whole pseudo data set (i.e., a data set with noise perturbations of its entries), which is the focus of this work. Most recently the concept of local differential privacy (LDP) \[12\] has come to the fore where users perturb their data and then send the perturbed data to an aggregator for summary and release. The aggregator does not know the raw user data. LDP provides a stronger privacy guarantee than the central DP approach where users directly share their raw data with a trusted aggregator that performs perturbation. A major limitation of LDP is that decentralized perturbation causes larger error in order to achieve the same privacy level.

In another thread of research, random orthogonal perturbation has been deployed for privacy protection \[8, 20, 22\] where a masked data set $AX$ is published, with $X$ denoting the original raw data set, and $A$ being a random orthogonal matrix. This pseudo data set $Y = AX$ allows users to use many standard data analysis methods since, for linear models, it has the same sufficient statistics as the raw data set. However, the random orthogonal matrix masking schemes do not satisfy the theoretical DP constraints.

In this paper, we investigate combining the random orthogonal matrix masking with noise addition methods to achieve DP on the released pseudo
data set. Compared to the traditional method of adding noise directly to the raw data set, we show that the matrix masking greatly reduces the magnitude of noise needed to achieve DP, thus allowing more accurate statistical inferences on the released pseudo data set. Moreover, data utility does not decrease significantly due to the property of the orthogonal masking matrix and the small magnitude of noise required for DP. This opens up many more applications for using differential privacy schemes.

2 Mathematical Setup for Releasing Pseudo-Data Set Satisfying Differential Privacy

Mathematically, we denote the raw data set as $X$, an $n \times p$ matrix. Technically, we assume that the elements of the raw data $X$ are bounded and we can scale them to within magnitude of one. Let $x_{ij}$ denote the $i$-th row and $j$-th column entry in $X$. That is,

$$|x_{ij}| \leq 1, \quad \text{for } 1 \leq i \leq n \text{ and } 1 \leq j \leq p. \quad (1)$$

Here we study the problem of releasing a pseudo data set, represented as an $n \times p$ matrix $Y$, which is a perturbed version of raw data $X$. A privacy preserving data releasing mechanism makes it hard to distinguish which of two raw data sets $X$ and $X'$, that differs only in one individual (one row of the matrix), leads to $Y$. Mathematically, the differential privacy (DP) requirements can be specified as follows.

**Definition 2.1.** Two data sets $X$ and $X'$ are neighbors if $\|X - X'| \leq 1$ and they differ only in one entry (one row).\(^1\) Here and following, $\| \cdot \|$ denotes the $L_2$-norm.

**Definition 2.2.** A data release mechanism $Y(X)$ satisfies the $(\varepsilon, \delta)$-differential privacy [15] if, for any set $S$ and any pair of neighbors $X$ and $X'$,

$$P[Y(X) \in S] \leq e^\varepsilon P[Y(X') \in S] + \delta.$$

\(^1\)Some literature defines the difference between two neighbors as one of them missing an entry that the other has [14]. Here we define neighbors as that they differ in one entry and the norm of the difference is bounded by one [17]. For technical simplicity, we use the $L_2$-norm in this paper. Other norms were also used in literature but various $L_p$-norms can be bounded with each other mathematically.
We will use the shorthand notation $(\varepsilon, \delta)$-DP to denote $(\varepsilon, \delta)$-differential privacy.

Commonly DP is achieved through adding noise to data. We consider this method as a baseline setting for comparison. That is, for the raw data matrix $X$, we add a noise matrix $C$ whose elements independently follow the identical Gaussian distribution $N(0, \sigma^2)$.

Setting (A) – Add noise to achieve $(\varepsilon, \delta)$-DP: release

$$Y = X + C \quad \text{where} \quad C \sim NI_{n \times p}(0, \sigma^2).$$

(2)

$NI_{n \times p}(0, \sigma^2)$ denotes an $n \times p$ matrix whose elements are independently identically distributed $N(0, \sigma^2)$ random variables.

In contrast, we propose to combine the noise addition with masking by a random orthogonal $n \times n$ matrix $A$ to achieve $(\varepsilon, \delta)$-DP. There are two settings to consider:

(B) Add noise then apply matrix masking: release

$$Y = A(X + C) \quad \text{where} \quad C \sim NI_{n \times p}(0, \sigma^2).$$

(3)

(C) Apply matrix masking first then add noise: release

$$Y = AX + C \quad \text{where} \quad C \sim NI_{n \times p}(0, \sigma^2).$$

(4)

We will show that under settings (B) and (C), with matrix masking, the noise magnitude $\sigma^2$ needed to achieve $(\varepsilon, \delta)$-DP can be much smaller than that needed under the noise only setting (A). First, it is obvious that an $(\varepsilon, \delta)$-DP scheme under setting (A) will remain $(\varepsilon, \delta)$-DP with the extra matrix masking in setting (B) or setting (C). We state this in the following lemma whose proof is provided in subsection A.1

**Lemma 2.3.** Assume that, for a given noise variance value $\sigma = \sigma_0$, the mechanism $Y(X)$ in setting (A) is $(\varepsilon, \delta)$-DP. Then for this value $\sigma = \sigma_0$, the mechanism $Y(X)$ is also $(\varepsilon, \delta)$-DP in setting (B) and in setting (C).

While the above Lemma states that the introduction of the extra matrix masking will never require a stronger noise variance condition to achieve $(\varepsilon, \delta)$-DP, we want to explore when the mechanisms in settings (B) and (C) with matrix masking will require a weaker condition to achieve $(\varepsilon, \delta)$-DP. To study this, we first state another mathematical condition for a release mechanism $Y(X)$ to satisfy $(\varepsilon, \delta)$-DP. (The proof is provided in subsection A.2)
Lemma 2.4. Assume that, given any data set $X, Y(X)$ follows a distribution that is absolutely continuous with respect to the Lebesgue measure. And let $p_{Y(X)}$ denote the probability density of $Y(X)$. Let
\[ \bar{S}_{X,X'} = \{ y : p_{Y(X)}(y) > e^\varepsilon p_{Y(X')}(y) \} . \]
Then the mechanism $Y(X)$ satisfies $(\varepsilon, \delta)$-DP if
\[ P[Y(X) \in \bar{S}_{X,X'}] \leq \delta \] (5)
for all neighboring $X$ and $X'$.
Also, if $P[Y(X) \in \bar{S}_{X,X'}] > \delta$ for any neighboring $X$ and $X'$, then the mechanism $Y(X)$ is not $(\varepsilon', \delta')$-DP for any $0 < \varepsilon' < \varepsilon$ and $\delta' = (1 - e^{\varepsilon'})\delta$.

We will show that it is easier to achieve condition (5) in settings (B) and (C) than in setting (A). Then Lemma 2.4 implies that it is easier to achieve $(\varepsilon, \delta)$-DP in settings (B) and (C) than in setting (A).

3 Main Analysis Results

The following Lemma gives the formulas for the density ratio $\frac{p_{Y(X)}(y)}{p_{Y(X')}(y)}$ under settings (A), (B) and (C), which allows the characterization of $\bar{S}_{X,X'}$.

Lemma 3.1. Let $\tilde{G}(T) = e^{\text{tr}(T)}$, where $\text{tr}(\cdot)$ denotes the trace function of a square matrix; and $\tilde{G}(T) = \int e^{\text{tr}(AT)}d\mu(A)$, where $\mu(\cdot)$ is the measure for the uniform distribution on the group of $n \times n$ orthogonal matrices. Then we have
\[ \frac{p_{Y(X)}(y)}{p_{Y(X')}(y)} = e^{\frac{\|X\|^2 - \|X'\|^2}{2\sigma^2}} \frac{\tilde{G}(\frac{yX^T}{\sigma^2})}{\tilde{G}(\frac{y(X')^T}{\sigma^2})}, \text{ under setting (A),} \] (6)
\[ \frac{p_{Y(X)}(y)}{p_{Y(X')}(y)} = e^{\frac{\|X'\|^2 - \|X\|^2}{2\sigma^2}} \frac{\tilde{G}(\frac{yX^T}{\sigma^2})}{\tilde{G}(\frac{y(X')^T}{\sigma^2})}, \text{ under settings (B) and (C).} \] (7)

Comparing (6) and (7), the matrix masking averages the densities, before taking the ratio, over the set of all data points that differ only by multiplication of an orthogonal matrix. This makes the density ratio in equation (7) closer to 1 than those in equation (6) for large values of $\sigma^2$. Using this, we will obtain the bounds on $\sigma$ that can achieve (5).
3.1 Noise magnitude to achieve (5) under setting (A)

We first consider the condition when there is only additive noise with no matrix masking.

Denote $\Delta = X' - X$. And let $\bar{\gamma}_\delta$ denote the upper $\delta$-quantile of the standard Gaussian distribution $N(0, 1)$. The following Theorem provides a lower bound of $\sigma$ to achieve (5) under setting (A).

**Theorem 3.2.** Under setting (A), when $\delta < 1/2$ and $\varepsilon < 1$, a necessary condition for (5) to hold is

$$\sigma \geq \frac{\|\Delta\| \bar{\gamma}_\delta}{\varepsilon},$$

and a sufficient condition for (5) to hold is

$$\sigma \geq \frac{\|\Delta\| \bar{\gamma}_\delta}{\varepsilon}(1 + \frac{1}{2\bar{\gamma}_\delta^2}).$$

Hence for (5) to hold for every pair of neighboring $X$ and $X'$ (where $\|\Delta\| \leq 1$), it is necessary that

$$\sigma \geq \frac{\bar{\gamma}_\delta}{\varepsilon};$$

and it is sufficient that

$$\sigma \geq \frac{\bar{\gamma}_\delta}{\varepsilon}(1 + \frac{1}{2\bar{\gamma}_\delta^2}).$$

We note that the upper $\delta$-quantile of the standard Gaussian distribution $\bar{\gamma}_\delta$ is of order $O(\sqrt{\ln(1/\delta)})$. Thus Theorem 3.2 states that, for (5) to hold for every pair of neighboring $X$ and $X'$, $\sigma \geq \sigma_0$ for some $\sigma_0 = O(\sqrt{\ln(1/\delta)})$ which agrees with the order in the sufficient condition derived by [17]. Particularly, we have the following explicit bounds for small $\delta$ and $\varepsilon$ values.

**Corollary 3.3.** Under setting (A), when $\delta < 0.05$ and $\varepsilon < 1$, a sufficient condition for (5) to hold for every pair of neighboring $X$ and $X'$ is $\sigma > \frac{1.7\sqrt{\ln(1/\delta)}}{\varepsilon}$; a necessary condition for (5) to hold for every pair of neighboring $X$ and $X'$ is $\sigma \geq \frac{\sqrt{\ln(1/\delta)}}{\varepsilon}$. 

3.2 Noise magnitude to achieve (5) under settings (B) and (C)

In contrast, with matrix masking under settings (B) and (C), we have the following condition.

**Theorem 3.4.** Under settings (B) and (C), a sufficient condition for (5) to hold is

$$\sigma^2 \geq \frac{b + \sqrt{b^2 + 8np^2(n-p)\varepsilon}}{2(n-p)\varepsilon},$$  \hspace{1cm} (12)

where $b = (n-p)\sqrt{p} + 2p\gamma_{\delta, np}$ with $\gamma_{\delta, np}$ denoting the upper $\delta$-quantile of the $\chi^2$ distribution with $np$ degrees of freedom.

In light of Lemma 2.3, Theorem 3.2 and Theorem 3.4, we have the following sufficient condition for achieving differential privacy in settings (B) and (C).

**Corollary 3.5.** Under settings (B) and (C), a sufficient condition for the mechanism $Y(X)$ to be $(\varepsilon, \delta)$-DP is

$$\sigma \geq \min\left(\frac{\bar{\gamma}_{\delta}}{\varepsilon}(1 + \frac{1}{2\bar{\gamma}_{\delta}^2}), \sqrt{\frac{b + \sqrt{b^2 + 8np^2(n-p)\varepsilon}}{2(n-p)\varepsilon}}\right).$$  \hspace{1cm} (13)

Here the quantities $\bar{\gamma}_{\delta}$ and $b = (n-p)\sqrt{p} + 2p\gamma_{\delta, np}$ are the same quantities as given above in Theorem 3.2 and Theorem 3.4.

To understand the bound in (12), we notice that $\gamma_{\delta, np} \approx np$ for large values of $np$. We can further relax the condition to a simpler version as the following.

**Corollary 3.6.** Under settings (B) and (C), when $\varepsilon < 1$, a sufficient condition for (5) to hold for every pair of neighboring $X$ and $X'$ is

$$\sigma^2 \geq \max(2, \frac{4np^2}{(n-p)\varepsilon}, \frac{4p\gamma_{\delta, np}}{(n-p)\varepsilon}).$$  \hspace{1cm} (14)

Using the bound for $\gamma_{\delta, np}$ in Lemma A.3, below we obtain a more specific bound from (14).

**Corollary 3.7.** Under settings (B) and (C), when $\varepsilon < 1$ and $\delta < 1$, a sufficient condition for (5) to hold for every pair of neighboring $X$ and $X'$ is

$$\sigma \geq \sqrt{\frac{2np + 3ln(\frac{1}{\delta})}{n-p}}\frac{\sqrt{4p}}{\sqrt{\varepsilon}}.$$  \hspace{1cm} (15)
We compare the condition in Corollary 3.7 with the condition in Corollary 3.3 for setting (A). First, we consider the complexities in terms of $\varepsilon$ only, under any given configuration of other parameters. The lower bound on the additive noise variance $\sigma^2$ improves by a factor $\sqrt{\varepsilon}$, from $O(\frac{1}{\varepsilon})$ of setting (A) to $O(\frac{1}{\sqrt{\varepsilon}})$ of setting (B) or (C). Next, we consider all parameters, the lower bound on the noise variance improves from $O(\frac{\ln(\frac{1}{\delta})}{\varepsilon^2})$ in setting (A) to $O(\frac{\ln(\frac{1}{\delta})}{\varepsilon} \sqrt{\frac{pe}{(n-p)}})$ in setting (B) or (C). Let’s interpret $n$ as the number of data records and $p$ as the number of attributes in each record. When $\varepsilon \to 0$, $\delta \to 0$ and $\frac{p}{n} = O(ln(1/\delta))$, the required additive noise variance to achieve $(\varepsilon, \delta)$-DP is reduced from $O(ln(1/\delta)/\varepsilon^2)$ in setting (A) to $O(1/\varepsilon)$ in setting (B) or (C). Hence, the matrix masking in settings (B) and (C) provides significant advantage under strong privacy requirements (i.e., small values of $\varepsilon$ and $\delta$), particularly in big data applications where $n$ is large.

4 Discussion on the Implication of the Analysis Results

To further understand how much advantage is provided by the matrix masking in settings (B) and (C) over the simple additive noise only setting (A), we compare the bounds (13) versus (11) for various $\varepsilon$, $\delta$, $n$ and $p$ values in Table 1.

From Table 1, as expected, the advantage of applying the matrix masking grows as the values of $\varepsilon$ and $\delta$ decrease, or as the sample size $n$ increases. When $p = 1$, as shown in the last column of the table, the needed noise magnitude is reduced to 24.2% when $\varepsilon = 0.1, \delta = 0.01, n = 100$, and is reduced to 1.9% when $\varepsilon = 0.001, \delta = 0.001, n = 100$. Thus for applications with strict privacy requirements, the matrix masking drastically reduces the magnitude of additive noise needed, thus lead to much more accurate result at the same sample size for data analysis methods that are supported by matrix masking (such as linear models) [8, 20, 22], where the pseudo data matrix after orthogonal transformation has the same sufficient statistics as the raw data matrix before the transformation.

We do note that the advantage of the matrix masking is reduced as the number of features $p$ in data set increases in the above table, since the bound (13) increases as $p$ increases (this can be observed clearly from (15)) while
Table 1: Comparison of $\sigma$ bounds (11) for setting (A) versus (13) for settings (B/C).

| $\varepsilon$ | $\delta$ | $p$ | $n$ | Setting (A) necessary (10) | Setting (A) sufficient (11) | Settings (B) and (C) sufficient (13) | Ratio of (13)/(11) |
|---------------|----------|-----|-----|---------------------------|----------------------------|----------------------------------|------------------|
| 0.1           | 0.01     | 1   | 100 | 23.3                      | 25.4                       | 6.2                              | 0.242            |
|               |          |     | 1000| 23.3                      | 25.4                       | 5.7                              | 0.225            |
|               |          |     | 10000| 23.3                       | 25.4                       | 5.6                              | 0.220            |
| 5             | 100      |     | 23.3 | 25.4                       | 25.1                       | 9.88                            |                 |
|               |          |     | 1000 | 23.3                      | 25.4                       | 23.4                             | 0.922            |
|               |          |     | 10000| 23.3                       | 25.4                       | 23.0                             | 0.907            |
| 20            | 100      |     | 23.3 | 25.4                       | 25.4                       | 25.4                             | 1.000            |
|               |          |     | 1000 | 23.3                      | 25.4                       | 25.4                             | 1.000            |
|               |          |     | 10000| 23.3                       | 25.4                       | 25.4                             | 1.000            |
| 0.001         | 1        | 100 | 30.9 | 32.5                       | 6.4                        | 0.196                           |                 |
|               |          |     | 1000 | 30.9                      | 32.5                       | 5.8                              | 0.178            |
|               |          |     | 10000| 30.9                       | 32.5                       | 5.6                              | 0.173            |
| 5             | 100      |     | 30.9 | 32.5                       | 25.7                       | 0.789                           |                 |
|               |          |     | 1000 | 30.9                      | 32.5                       | 23.6                             | 0.726            |
|               |          |     | 10000| 30.9                       | 32.5                       | 23.1                             | 0.710            |
| 20            | 100      |     | 30.9 | 32.5                       | 32.5                       | 32.5                             | 1.000            |
|               |          |     | 1000 | 30.9                      | 32.5                       | 32.5                             | 1.000            |
|               |          |     | 10000| 30.9                       | 32.5                       | 32.5                             | 1.000            |
| 0.01          | 0.01     | 100 | 232.6| 254.1                       | 19.4                       | 0.066                            |                 |
|               |          |     | 1000 | 232.6                     | 254.1                       | 18.0                             | 0.071            |
|               |          |     | 10000| 232.6                      | 254.1                       | 79.3                             | 0.312            |
| 5             | 100      |     | 232.6 | 254.1                       | 74.1                       | 0.291                           |                 |
|               |          |     | 1000 | 232.6                     | 254.1                       | 72.8                             | 0.287            |
|               |          |     | 10000| 232.6                      | 254.1                       | 72.8                             | 0.287            |
| 20            | 100      |     | 232.6 | 254.1                       | 254.1                       | 254.1                             | 1.000            |
|               |          |     | 1000 | 232.6                     | 254.1                       | 254.1                             | 1.000            |
|               |          |     | 10000| 232.6                      | 254.1                       | 254.1                             | 1.000            |
| 0.001         | 0.01     | 100 | 2326.3| 2541.3                      | 61.2                       | 0.024                            |                 |
|               |          |     | 1000 | 2326.3                    | 2541.3                      | 56.7                             | 0.022            |
|               |          |     | 10000| 2326.3                     | 2541.3                      | 55.4                             | 0.022            |
| 5             | 100      |     | 2326.3 | 2541.3                      | 250.8                       | 0.099                            |                 |
|               |          |     | 1000 | 2326.3                    | 2541.3                      | 234.2                             | 0.092            |
|               |          |     | 10000| 2326.3                     | 2541.3                      | 230.2                             | 0.091            |
| 20            | 100      |     | 2326.3 | 2541.3                      | 1639.0                       | 0.409                            |                 |
|               |          |     | 1000 | 2326.3                    | 2541.3                      | 916.5                             | 0.361            |
|               |          |     | 10000| 2326.3                     | 2541.3                      | 901.1                             | 0.356            |
| 0.001         | 0.01     | 100 | 3090.2 | 3252.0                       | 63.4                       | 0.019                            |                 |
|               |          |     | 1000 | 3090.2                    | 3252.0                       | 57.4                              | 0.018            |
|               |          |     | 10000| 3090.2                     | 3252.0                       | 55.6                              | 0.017            |
| 5             | 100      |     | 3090.2 | 3252.0                       | 256.4                       | 0.079                            |                 |
|               |          |     | 1000 | 3090.2                    | 3252.0                       | 235.9                             | 0.073            |
|               |          |     | 10000| 3090.2                     | 3252.0                       | 230.7                             | 0.071            |
| 20            | 100      |     | 3090.2 | 3252.0                       | 1651.2                       | 0.323                            |                 |
|               |          |     | 1000 | 3090.2                    | 3252.0                       | 919.9                             | 0.283            |
|               |          |     | 10000| 3090.2                     | 3252.0                       | 902.2                             | 0.277            |
(11) is invariant to the changes in $p$. However, it is not clear whether the dependence of (13) on $p$ is intrinsic to the problem or is due to proof techniques we used. A future research topic is to study whether the bound (13) for the matrix masking settings can be further improved. In any case, the current bound still demonstrates the advantage of matrix masking in the case of big data $n \gg p$. For example, when $p = 20$, for the case of $\varepsilon = 0.001$ and $\delta = 0.001$, the matrix masking reduces the needed noise magnitude to about 28% when $n \geq 1000$.

For the matrix masking settings (B) and (C), as we have pointed out above, (13) provides a lower bound on $\sigma$ which may be further improved in the future. In contrast, the bound (11) for setting (A) without matrix masking cannot be improved much better since the necessary condition (10) is very close. As shown in Table 1 column 5 versus column 6, the necessary condition (10) is less than 10% away from the sufficient condition (11) in all cases.

We also like to remark on a practical implication of the data collection procedure on setting (B). For applications of central differential privacy, there is a central data collector/aggregator who has the access to the raw data $X$ and is in charge of producing the perturbed data set for public release. In that case, our results have shown that setting (B) and setting (C) requires the same noise magnitude. Hence, both approaches work equally well for the central data collector/aggregator. However, the privacy protection in such a central setting is crucially dependent on the trustworthiness of the central data collector/aggregator to protect the raw data set access. In recent years, there have been interest in designing data collection procedures that also protect the privacy of each data provider (a row in $X$) from the data collector/aggregator so that the data collector only gets the perturbed data set and never sees the raw data set $X$. Without matrix masking, this may be achieved by collecting $X + C$ in setting (A) with local noise addition. The $i$-th data provider, $i = 1, \ldots, n$, adds Gaussian noise $C_i$ to its raw data $X_i$ (the $i$-th row of $X$) locally before sending the data to the collector, where the perturbed data from all providers are aggregated. We can also design an end-to-end privacy preserving data collection procedure in setting (B) by combining the local noise addition with the triple-matrix masking scheme [10] which enables the collection of a masked pseudo data set $AX$ from all data providers without allowing any party in the process to gain more useful information other than the published final $AX$. Using the triple-matrix masking scheme, the data collector can collect the pseudo
data set $A(X + C)$ in setting (B) without revealing extra information to any parties in the data collection process including the data collector him/herself. Compared to using local noise addition only without matrix masking, which results in $X + C$ in setting (A), the data collection through the triple-matrix masking scheme provides end-to-end privacy protection in setting (B) with a much lower added noise magnitude. More accurate statistical analysis can be conducted on the pseudo data set with lower added noise.

5 Proofs of Main Theorems

5.1 Proof of Lemma 3.1

Proof of Lemma 3.1. Under setting (A),

$$Y(X) = y \iff X + C = y \iff C = y - X.$$  

So the density $p_Y(X)(y) = p_Y(y - X)$ where the density $p_Y$ is the multivariate Gaussian distribution density. That is,

$$p_Y(X)(y) = \left( \frac{1}{\sqrt{2\pi \sigma}} \right)^n e^{-\frac{|y - X|^2}{2\sigma^2}}; \quad p_Y(X')(y) = \left( \frac{1}{\sqrt{2\pi \sigma}} \right)^n e^{-\frac{|y - X'|^2}{2\sigma^2}}$$

Hence the density ratio is

$$\frac{p_Y(X)(y)}{p_Y(X')(y)} = \frac{e^{-\frac{|y - X|^2}{2\sigma^2}}}{e^{-\frac{|y - X'|^2}{2\sigma^2}}}.$$  

Notice that for a $n \times p$ matrix $X$, $|X|^2 = tr(X^TX) = tr(XX^T)$. Hence

$$|y - X|^2 = tr[(y - X)(y - X)^T] = |y|^2 + |X|^2 - 2tr(yX^T).$$

And the above density ratio expression is simplified to equation (6) as

$$\frac{p_Y(X)(y)}{p_Y(X')(y)} = e^{-\frac{|y|^2 + |X|^2 - 2tr(yX^T)}{2\sigma^2}} \cdot \frac{G(yX^T/\sigma^2)}{G(y(X^T)/\sigma^2)}.$$  

On the other hand, under setting (B) $Y = A(X + C)$,

$$Y(X) = y \iff A(X + C) = y \iff C = A^T y - X.$$  

Given $A$,

$$p_{C}(A^T y - X) = \left(\frac{1}{\sqrt{2\pi\sigma}}\right)^n e^{-\frac{\|A^T y - X\|^2}{2\sigma^2}} = \left(\frac{1}{\sqrt{2\pi\sigma}}\right)^n e^{-\frac{\|Y\|^2 + \|X\|^2 - 2tr(A^T y X^T)}{2\sigma^2}},$$

where the last equality used the fact that $\|A^T y\|^2 = \|Y\|^2$ for orthogonal $A$. Since $A$ is a random orthogonal matrix, the density $p_{Y(X)}(y)$ would be the above Gaussian density integrated over $d\mu(A)$, so

$$p_{Y(X)}(y) = \int_A \left(\frac{1}{\sqrt{2\pi\sigma}}\right)^n e^{-\frac{\|Y\|^2 + \|X\|^2 - 2tr(A^T y X^T)}{2\sigma^2}} d\mu(A).$$

In contrast to (6), the density ratio under setting (B) becomes

$$\frac{p_{Y(X)}(y)}{p_{Y(X')}(y)} = \frac{\int e^{-\frac{\|y\|^2 + \|X\|^2}{2\sigma^2}} d\mu(A)}{\int e^{-\frac{\|y\|^2 + \|X\'^2}{2\sigma^2}} d\mu(A)} = \frac{\int e^{-\frac{\|y\|^2 + \|X\|^2 - 2tr(A^T y X^T)}{2\sigma^2}} d\mu(A)}{\int e^{-\frac{\|y\|^2 + \|X\'^2 - 2tr(A^T y X^T)}{2\sigma^2}} d\mu(A)}.$$

Notice that, when $A$ is uniformly distributed on the group of $n \times n$ orthogonal matrices, $A^T$ is also uniformly distributed on this group. Hence the above expression is equivalent to (7).

Lastly, under setting (C), $Y = AX + C$, $Y(X) = y \leftrightarrow C = y - AX$. Hence

$$p_{Y(X)}(y) = \int_A p_{C}(y - AX) d\mu(A) = \int_A \left(\frac{1}{\sqrt{2\pi\sigma}}\right)^n e^{-\frac{\|y - AX\|^2}{2\sigma^2}} d\mu(A).$$

Since, $tr[y(AX)^T] = [y X^T A^T] = tr(A^T y X^T)$, the above integral becomes

$$p_{Y(X)}(y) = \int_A \left(\frac{1}{\sqrt{2\pi\sigma}}\right)^n e^{-\frac{\|y\|^2 + \|X\|^2 - 2tr(A^T y X^T)}{2\sigma^2}} d\mu(A).$$

That is, the density is the same as the density in setting (B). Hence the density ratio is also the same as given in (7).

**5.2 Proof of Theorem 3.2 in setting (A) unmasked release**

*Proof of Theorem 3.2.* We solve the set $\hat{S}_{X,X'} = \{y : p_{Y(X)}(y) > e^\epsilon p_{Y(X')}(y)\}$. From Lemma 3.1, under setting (A), the density ratio is given by (6)

$$\frac{p_{Y(X)}(y)}{p_{Y(X')}(y)} = e^{\frac{\|y\|^2 + \|X\|^2}{2\sigma^2}} \frac{\tilde{G}(\frac{y X^T}{\sigma^2})}{\tilde{G}(\frac{y X'^T}{\sigma^2})}.$$
where \( \bar{G}(T) = e^{tr(T)} \).

We want to further simplify (6). Recall that we denote \( \Delta = X' - X \) so that \( X' = X + \Delta \). We further denote

\[
X = \begin{pmatrix}
X_1 \\
X_2 \\
\vdots \\
X_n
\end{pmatrix}, \quad \Delta = \begin{pmatrix}
\Delta_1 \\
\Delta_2 \\
\vdots \\
\Delta_n
\end{pmatrix}, \quad y = \begin{pmatrix}
y_1 \\
y_2 \\
\vdots \\
y_n
\end{pmatrix}, \quad C = \begin{pmatrix}
C_1 \\
C_2 \\
\vdots \\
C_n
\end{pmatrix}.
\]

Notice that here \( \| \Delta_2 \| = \ldots = \| \Delta_n \| = 0 \). Hence \( tr(y \Delta^T) = \sum_{k=1}^n y_k \Delta_k^T = y_1 \Delta_1^T \) and

\[
\|X + \Delta\|^2 - \|X\|^2 = \|\Delta\|^2 + 2tr(X \Delta^T) = \|\Delta_1\|^2 + 2X_1 \Delta_1^T.
\]

Thus (6) is simplified to

\[
\frac{p_Y(X)(y)}{p_Y(X')(y)} = e^{\frac{\| \Delta_1 \|^2 + 2y_1 \Delta_1^T}{2\sigma^2}} \frac{\bar{G}(yX^T)}{\bar{G}(y(X+\Delta)^T)} = e^{\frac{\| \Delta_1 \|^2 + 2y_1 \Delta_1^T}{2\sigma^2}} e^{-\frac{y_1 \Delta_1^T}{\sigma^2}}. \tag{16}
\]

Hence the density ratio exceeding \( e^\varepsilon \) is equivalent to

\[
\| \Delta_1 \|^2 - 2(y_1 - X_1) \Delta_1^T > 2\sigma^2 \varepsilon \quad \Leftrightarrow \quad -2(y_1 - X_1) \Delta_1^T > 2\sigma^2 \varepsilon - \| \Delta_1 \|^2.
\]

Hence \( \bar{S}_{X,X'} = \bar{S}_{X,X+\Delta} = \{ y : -(y_1 - X_1) \Delta_1^T > \sigma^2 \varepsilon - \| \Delta_1 \|^2 \} \). Notice that \( (Y_1 - X_1) = C_1 \) follows a \( p \)-dimensional Gaussian distribution with zero mean and variance \( \sigma^2 I_p \) where \( I_p \) denotes the \( p \times p \) identity matrix. Hence \( -(Y_1 - X_1) \Delta_1^T \) follows the Gaussian distribution with zero mean and variance \( \sigma^2 \| \Delta_1 \|^2 \). Thus \( -(\frac{Y_1 - X_1) \Delta_1^T}{\sigma\| \Delta_1 \|} \) follows the standard Gaussian distribution \( N(0,1) \). Hence condition (5) is equivalent to

\[
P[Y(X) \in \bar{S}_{X,X+\Delta}] \leq \delta
\]

\[
\Leftrightarrow P[-(Y_1 - X_1) \Delta_1^T > \sigma^2 \varepsilon - \| \Delta_1 \|^2 / 2] \leq \delta
\]

\[
\Leftrightarrow P[-\frac{(Y_1 - X_1) \Delta_1^T}{\sigma\| \Delta_1 \|} > \frac{\sigma \varepsilon}{\| \Delta_1 \|} - \frac{\| \Delta_1 \|^2}{2\sigma}] \leq \delta
\]

\[
\Leftrightarrow \bar{\gamma}_\delta \leq \frac{\sigma \varepsilon}{\| \Delta_1 \|} - \frac{\| \Delta_1 \|^2}{2\sigma} = \frac{\sigma \varepsilon}{\| \Delta_1 \|} - \frac{\| \Delta_1 \|}{2\sigma}.
\tag{17}
\]

Here the last line comes from the fact that \( P[-\frac{(Y_1 - X_1) \Delta_1^T}{\sigma\| \Delta_1 \|} > \bar{\gamma}_\delta] = \delta \).

Since \( \bar{\gamma}_\delta \leq \frac{\sigma \varepsilon}{\| \Delta \|} - \frac{\| \Delta \|}{2\sigma} \leq \frac{\sigma \varepsilon}{\| \Delta \|} \) always, a necessary condition for (17) to hold is

\[
\bar{\gamma}_\delta \leq \frac{\sigma \varepsilon}{\| \Delta \|} \quad \Leftrightarrow \quad \sigma \geq \frac{\| \Delta \| \bar{\gamma}_\delta}{\varepsilon}.
\]

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Thus we finished the proof for (8).

Now \( \sigma \geq \frac{\|\Delta\|_{\mathcal{H}}}{\epsilon} \) is necessary for (5) to hold for a given pair of neighboring \( X \) and \( X' \) that differs by \( \|\Delta\| \leq 1 \). Since there exists pairs of \( X \) and \( X' \) with \( \|\Delta\| = 1 \), for (5) to hold for every pair of neighboring \( X \) and \( X' \), a necessary condition is that
\[
\sigma \geq \frac{\bar{\gamma}_\delta}{\epsilon},
\]
i.e., condition (10) holds.

What remains to be proven is the sufficient condition. Particularly, we want to prove that \( \sigma \geq \frac{\|\Delta\|_{\mathcal{H}}}{\epsilon}(1 + \frac{1}{2\gamma_\delta}) \) is sufficient for (17). Since \( \frac{\sigma_\epsilon}{\|\Delta\|} - \frac{\|\Delta\|_{\mathcal{H}}}{2\sigma} \) is an increasing function of \( \sigma \), when \( \sigma \geq \frac{\|\Delta\|_{\mathcal{H}}}{\epsilon}(1 + \frac{1}{2\gamma_\delta}) \), we have
\[
\frac{\sigma_\epsilon}{\|\Delta\|} - \frac{\|\Delta\|_{\mathcal{H}}}{2\sigma} \geq \frac{\epsilon}{\|\Delta\|} \frac{\|\Delta\|_{\mathcal{H}}}{\epsilon}(1 + \frac{1}{2\gamma_\delta}) - \frac{\|\Delta\|_{\mathcal{H}}}{2\sigma}(1 + \frac{1}{2\gamma_\delta})
= \bar{\gamma}_\delta(1 + \frac{1}{2\gamma_\delta}) - \frac{\|\Delta\|_{\mathcal{H}}}{2\sigma}(1 + \frac{1}{2\gamma_\delta})
= \bar{\gamma}_\delta + \frac{1}{2\gamma_\delta} - \frac{\|\Delta\|_{\mathcal{H}}}{2\sigma}(1 + \frac{1}{2\gamma_\delta})
= \bar{\gamma}_\delta + \frac{1}{2\gamma_\delta}(1 - \frac{\epsilon}{1 + \frac{1}{2\gamma_\delta}})
\geq \bar{\gamma}_\delta,
\]
where the last line is due to \( \epsilon \leq 1 \). This is (17). Thus (9) is sufficient for (17) which then implies (5).

When (11) \( \sigma^2 \geq \frac{\gamma_\delta}{\epsilon^2}(1 + \frac{1}{2\gamma_\delta})^2 \) holds and \( \|\Delta\| \leq 1 \), then (9) \( \sigma \geq \frac{\|\Delta\|_{\mathcal{H}}}{\epsilon}(1 + \frac{1}{2\gamma_\delta}) \) holds also. Thus (11) is sufficient for (5) to hold for every pair of neighboring \( X \) and \( X' \) with \( \|\Delta\| \leq 1 \).

\[\square\]

5.3 Proof of Theorem 3.4 in settings (B) and (C)

Proof of Theorem 3.4. Under settings both (B) and (C), the density ratio is given by (7) by Lemma 3.1,
\[
\frac{p_{Y|X}(y)}{p_{Y|X'}(y)} = e^{\frac{|x|^2 - |x'|^2}{2\sigma^2}} \frac{\tilde{G}(\frac{x^T}{\sigma^2})}{\tilde{G}(\frac{x'^T}{\sigma^2})},
\]
where
\[
\tilde{G}(T) = \int e^{tr(\mathbf{AT})}d\mu(\mathbf{A}).
\]
The main issue in the proof is to bound the ratio \( \frac{\tilde{G}(\frac{yX^T}{\sigma^2})}{\tilde{G}(\frac{yX'^T}{\sigma^2})} \) for neighboring \( X \) and \( X' \). For this, we first simplify the integral in \( \tilde{G}(\frac{yX^T}{\sigma^2}) \) to a lower dimensional integral that involves only the first \( p + 1 \) rows of \( A \).

We apply a QR decomposition on the submatrix consisting of all rows in \( X \) except the first row, so that

\[
X = \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} X_1 \\ R \end{pmatrix} = Q X,
\]

where \( Q \) is a \((n-1) \times (n-1)\) orthogonal matrix and \( R \) is an \((n-1) \times p\) upper triangular matrix. Hence \( Q \) is a \( n \times n \) orthogonal matrix, and \( X \) has zero vectors as the last \( n - p - 1 \) rows while its first row \( X_1 = X_1 \) is unchanged from \( X \). Then

\[
tr(AyX^T) = tr(AyX'^TQ^T) = tr[Q^T AyX^T].
\]

Since the Haar measure is invariant under multiplication by an orthogonal matrix, \( \tilde{A} = Q^T A \) also follows the uniform distribution on the group of orthogonal matrices. Applying the change of variable \( \tilde{A} = Q^T A \) in the above integral, with \( A_i \) and \( X_i \) denote the \( i \)-th rows of \( A \) and \( X \) respectively, we have

\[
\tilde{G}(\frac{yX^T}{\sigma^2}) = \int e^{\frac{tr[\tilde{A}yX^T]}{\sigma^2}} d\mu(\tilde{A}) = \int e^{\frac{tr[AyX^T]}{\sigma^2}} d\mu(A) = \int e^{\frac{\sum_{i=1}^{p+1} A_i y X_i^T}{\sigma^2}} d\mu(A).
\]

Here the second equality use the fact that \( \tilde{A} \) and \( A \) follows the same uniform distribution. Notice that the summation at the end only adds up to \( i = p+1 \), instead of up to \( i = n \), since the last \( n - p - 1 \) rows of \( X \) are all zero vectors.

Since \( X \) and \( X' \) only differs in the first row, applying the same QR decomposition on \( X' \), then \( \overline{X} \) and \( \overline{X}' \) have the same last \( n - 1 \) rows. Also, \( \overline{X}_1 = X_1 \). Hence

\[
\frac{\tilde{G}(\frac{yX^T}{\sigma^2})}{\tilde{G}(\frac{y(X')^T}{\sigma^2})} = \int e^{\frac{A_1 y X_1^T}{\sigma^2}} \frac{\sum_{i=2}^{p+1} A_i y X_i^T}{\sigma^2} d\mu(A) \times \int e^{\frac{A_1 y (X_1')^T}{\sigma^2}} \frac{\sum_{i=2}^{p+1} A_i y X_i'^T}{\sigma^2} d\mu(A).
\]

Notice that the integrand involve only the first \( p + 1 \) rows of matrix \( A \), and the difference between the denominator and the numerator is only in the part of integral involving the first row \( A_1 \).
Denote $A_{p-} = (A_2^T, ..., A_{p+1}^T)^T$ as the $p \times n$ matrix consists of the second to the $(p + 1)$-th rows for $A$. Then, condition on a given $A_{p-}$, $A_1$ is uniformly distributed over the $(n - p - 1)$-dimensional unit sphere within $(n - p)$-dimensional linear subspace that is perpendicular to the span of $A_{p-}$. We denote $G_{p-}$ as the $(n - p)$-dimensional linear subspace that is perpendicular to $A_{p-}$.

For further simplification of (18), we will use the following Lemma (whose proof is in subsection A.6). In the following, when not specified, the vectors such as $b$ and $v$ are $n$-dimensional row vectors.

**Lemma 5.1.** Let $S$ denote a $q$-dimensional linear subspace of the $n$-dimensional Euclidean space $\mathbb{R}^n$. Let $\text{proj}_S(v)$ denotes the projection of a vector $v$ onto the subspace $S$. Let $\mu_q(\cdot)$ denote the measure for the uniform distribution over the $(q - 1)$-dimensional unit sphere (the surface of the $q$-dimensional unit ball) within the subspace $S$. Then for any function $g(\cdot)$,

$$
\int g(bv^T) \mu_q(b) = \int_{u=-1}^{1} g(\|\text{proj}_S(v)\|u) \frac{1}{c_q}(1 - u^2)^{\frac{q-3}{2}} du
$$

where $c_q = \int_{u=-1}^{1} (1 - u^2)^{\frac{q-3}{2}} du = \frac{\Gamma(\frac{q}{2})\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q+2}{2})}$. Here $\Gamma(\cdot)$ is the Gamma function.

Let $v_1 = \text{proj}_{G_{p-}}(X_1y^T)$ and $v'_1 = \text{proj}_{G_{p-}}(X'_1y^T)$, applying Lemma 5.1 to equation (18), we have

$$
\frac{G_q(\frac{v_{1}^T}{\sigma^2})}{G_q(\frac{v'_1}{\sigma^2})} = \frac{\int_{u=-1}^{1} \frac{||v_1||^2_u}{\sigma^2} \frac{1}{c_q-p+1} (1-u^2)^{\frac{n-p-2}{2}} du}{\int_{u=-1}^{1} \frac{||v'_1||^2_u}{\sigma^2} \frac{1}{c_q-p+1} (1-u^2)^{\frac{n-p-2}{2}} du} \frac{\sum_{i=1}^{p+1} A_i X_i^T}{\sigma^2} d\mu(A_{p-})
$$

$$
= \int G_{n-p}(\frac{||v_1||^2}{\sigma^2}) e^{\frac{\sum_{i=1}^{p+1} A_i X_i^T}{\sigma^2}} d\mu(A_{p-})
$$

$$
= \int G_{n-p}(\frac{||v'_1||^2}{\sigma^2}) e^{\frac{\sum_{i=1}^{p+1} A_i X_i^T}{\sigma^2}} d\mu(A_{p-})
$$

(19)

where

$$
G_q(t) = \int_{u=-1}^{1} e^{tu}(1 - u^2)^{\frac{q-3}{2}} du.
$$

(20)

We further bound the density ratio using the following Lemma 5.2 on the ratio of function $G_q(\cdot)$, whose proof is in subsection A.7.
Lemma 5.2. When \( t > 0 \), the derivative of \( G_q(t) \) satisfies \( 0 < G_q'(t) < \frac{q}{q} G_q(t) \). Thus \( G_q(t) \) is an increasing function when \( t \geq 0 \), and

\[
\frac{G_q(t_2)}{G_q(t_1)} \leq e^{\frac{t_2^2 - t_1^2}{2t_1}} \text{ for any } t_1 > 0 \text{ and } t_2 > 0.
\] (21)

Using (21), we have

\[
\frac{G_{n-p}(\|v_1\|^2)}{G_{n-p}(\|v'_1\|^2)} \leq e^{rac{\|v_1\|^2 - \|v'_1\|^2}{2(n-p)}} \leq e^{rac{\|v_1\|^2 + \|v'_1\|^2}{2(n-p)^2}}.
\]

Since \( v_1 \) and \( v'_1 \) are projections of \( X_1y^T \) and \( X'_1y^T \) respectively, \( \|v_1\|^2 \leq \|X_1y^T\|^2 \) and \( \|v'_1\|^2 \leq \|X'_1y^T\|^2 \). The above expression is then simplified to

\[
\frac{G_{n-p}(\|v_1\|^2)}{G_{n-p}(\|v'_1\|^2)} \leq e^{rac{\|X_1y^T\|^2 + \|X'_1y^T\|^2}{2(n-p)^2}} \leq e^{rac{\|X_1y\|^2 + \|X'_1y\|^2}{2(n-p)^2}} \leq e^{rac{\|X_1y\|^2}{2(n-p)^2}},
\] (22)

where the last inequality is due to

\[
\|X_1\|^2 = \sum_{j=1}^{p} x_{1j}^2 \leq \sum_{j=1}^{p} 1 = p, \quad \|X'_1\|^2 = \sum_{j=1}^{p} (x'_{1j})^2 \leq \sum_{j=1}^{p} 1 = p.
\]

Plug (22) into (19), we have

\[
\tilde{G}(\frac{y^TX}{\sigma^2}) \leq \frac{\int e^{\frac{\|X'_1\|^2 - \|X\|^2}{2\sigma^2}} G_{n-p}(\|v'_1\|^2) e^{\frac{\sum_{i=1}^{p} A_iy^TX_i^T}{\sigma^2}} d\mu(A_{p-})}{\int G_{n-p}(\|v'_1\|^2) e^{\frac{\sum_{i=1}^{p+1} A_iy^TX_i^T}{\sigma^2}} d\mu(A_{p-})} = e^{\frac{\|X\|^2}{2\sigma^2}} e^{\frac{\|X_1y\|^2}{2(n-p)^2}}.
\]

Plug this into (7),

\[
\frac{pY(x)(y)}{pY(x')(y)} \leq e^{\frac{\|X'_1\|^2 - \|X\|^2}{2\sigma^2}} e^{\frac{\|X\|^2}{2(n-p)^2}}.
\] (23)

To further simplify (23), we notice that

\[
\|X'_1\|^2 - \|X\|^2 = \|X'_1\|^2 - \|X_1\|^2 = (\|X'_1\| - \|X_1\|) (\|X'_1\| + \|X_1\|) \\
\leq (1)(\sqrt{p} + \sqrt{p}) = 2\sqrt{p},
\]

\[
\|y\|^2 = \|y - X + X\|^2 \leq 2(\|y - X\|^2 + \|X\|^2) \leq 2(\|y - X\|^2 + np).
\]
Plug these into (23), we have
\[
\frac{p_Y(x)(y)}{p_Y(x')(y)} \leq e^{\frac{\sqrt{p(n-p)\sigma^2 + 2p\|y - X\|^2 + np}}{(n-p)\sigma^4}} = e^{\frac{\sqrt{p(n-p)\sigma^2 + 2p\|y - X\|^2 + np}}{(n-p)\sigma^4}}.
\]

With this bound, we are ready to derive the sufficient condition for achieving (5). From this density ratio bound, it is clear that
\[
S_{x,x'}^C = \{ y : p_Y(x)(y) \leq e^c p_Y(x')(y) \} \supseteq \{ y : \frac{\sqrt{p(n-p)\sigma^2 + 2p\|y - X\|^2 + np}}{(n-p)\sigma^4} \leq \varepsilon \}. 
\tag{24}
\]

Notice that \( \frac{\|y(x)-x\|^2}{\sigma^2} \) follows the \( \chi^2 \) distribution with \( np \) degrees of freedom, and we denote the upper-\( \delta \) quantile of this distribution by \( \gamma_{\delta, np} \). Denote the set
\[
D_\sigma = \{ y : \|y - X\|^2 \leq \gamma_{\delta, np} \sigma^2 \}
\]
so that \( P[Y(X) \in D_\sigma] = 1 - \delta \). Thus if \( D_\sigma \subseteq S_{x,x'}^C \) by (24). Then \( p_Y(x)(y) > e^c p_Y(x')(y) \) can at most be of the probability \( P[Y(X) \notin D_\sigma] = \delta \), and (5) is achieved. Thus (25) is a sufficient condition for (5).

We further find a condition on \( \sigma \) that can ensure (25). Notice that for any \( y \in D_\sigma \),
\[
\frac{\sqrt{p(n-p)\sigma^2 + 2p\|y - X\|^2 + np}}{(n-p)\sigma^4} \leq \frac{\sqrt{p(n-p)\sigma^2 + 2p(\gamma_{\delta, np} \sigma^2 + np)}}{(n-p)\sigma^4}.
\]
Hence \( \frac{\sqrt{p(n-p)\sigma^2 + 2p(\gamma_{\delta, np} \sigma^2 + np)}}{(n-p)\sigma^4} \leq \varepsilon \) would imply (25).

We note that
\[
\frac{\sqrt{p(n-p)\sigma^2 + 2p(\gamma_{\delta, np} \sigma^2 + np)}}{(n-p)\sigma^4} = \varepsilon \iff \sqrt{p(n-p)\sigma^2 + 2p(\gamma_{\delta, np} \sigma^2 + np)} = \varepsilon (n-p)\sigma^4 \iff \varepsilon (n-p)\sigma^4 - \sigma^2 [\sqrt{p(n-p) + 2p\gamma_{\delta, np}^2} - 2n\sigma^2] = 0,
\]
which is a quadratic equation for \( \sigma^2 \). Let \( a = (n-p)\varepsilon, b = (n-p)\sqrt{p} + 2p\gamma_{\delta, np}^2, c = 2n\sigma^2 \). Then the solution to this quadratic equation is \( \frac{b + \sqrt{b^2 + 4ac}}{2a} \).

Denote the larger of the two roots as
\[
\sigma_0^2 = \frac{b + \sqrt{b^2 + 4ac}}{2a} = \frac{b + \sqrt{b^2 + 8n\sigma^2 (n-p)\varepsilon}}{2(n-p)\varepsilon}. \tag{26}
\]
Then, when $\sigma^2 \geq \sigma_0^2$, $\sqrt{p(n-p)\sigma^2 + 2p(\gamma_{\delta, np}\sigma^2 + np)}$ is decreasing in $\sigma^2$ thus
\[
\frac{\sqrt{p(n-p)\sigma^2 + 2p(\gamma_{\delta, np}\sigma^2 + np)}}{(n-p)\sigma^4} \leq \frac{\sqrt{p(n-p)\sigma_0^2 + 2p(\gamma_{\delta, np}\sigma_0^2 + np)}}{(n-p)\sigma_0^4} = \varepsilon.
\]

Since condition (12) is exactly $\sigma^2 \geq \sigma_0^2$, it ensures that $\sqrt{p(n-p)\sigma^2 + 2p(\gamma_{\delta, np}\sigma^2 + np)} \leq \varepsilon$ which in turn implies (25). Hence (12) is also a sufficient condition for (5).

\[\square\]

5.4 Proof of Corollary 3.6

Proof of Corollary 3.6. From the above proof of Theorem 3.4, we only need to show that (14) implies $\frac{\sqrt{p(n-p)\sigma^2 + 2p(\gamma_{\delta, np}\sigma^2 + np)}}{(n-p)\sigma^4} \leq \varepsilon$ because it would then imply (25), a sufficient condition for (5).

Suppose that (14) holds, i.e., $\sigma^2 \geq \max(2, \frac{4n\delta^2}{(n-p)\varepsilon}, \frac{4p\gamma_{\delta, np}}{(n-p)\varepsilon})$. Since $\sigma^2 \geq 2$,
\[
\frac{\sqrt{p(n-p)\sigma^2 + 2p(\gamma_{\delta, np}\sigma^2 + np)}}{(n-p)\sigma^4} \leq \frac{\sqrt{p(n-p)\sigma^2 + 2p(\gamma_{\delta, np}\sigma^2 + np)}}{(n-p)\sigma^4} \leq \frac{\sqrt{p(n-p)\sigma^2 + 2p(\gamma_{\delta, np}\sigma^2 + np)}}{(n-p)\sigma^4} \leq \frac{\sqrt{p(n-p)\sigma^2 + 2p(\gamma_{\delta, np}\sigma^2 + np)}}{(n-p)\sigma^4} \leq \frac{\sqrt{p(n-p)\sigma^2 + 2p(\gamma_{\delta, np}\sigma^2 + np)}}{(n-p)\sigma^4} \leq \epsilon.
\]

Since $\sigma^2 \geq \max(2, \frac{4n\delta^2}{(n-p)\varepsilon}, \frac{4p\gamma_{\delta, np}}{(n-p)\varepsilon})$ implies $\sigma^2 \geq \frac{4p\max(np, \gamma_{\delta, np})}{(n-p)\varepsilon}$ thus $\frac{4p\max(np, \gamma_{\delta, np})}{(n-p)\varepsilon} \leq \epsilon$, we have
\[
\frac{\sqrt{p(n-p)\sigma^2 + 2p(\gamma_{\delta, np}\sigma^2 + np)}}{(n-p)\sigma^4} \leq \frac{4p\max(np, \gamma_{\delta, np})}{(n-p)\sigma^2} \leq \epsilon.
\]
This finishes the proof. \[\square\]

A Proofs of technical results

A.1 Proof of Lemma 2.3

Proof of Lemma 2.3. Under setting (A), $Y(X)$ satisfies $(\varepsilon, \delta)$-DP Definition 2.2. That is, for any set $S$ and any pair of neighbors $X$ and $X'$,
\[
P[(X + c) \in S] \leq e^{\varepsilon}P[(X' + C) \in S] + \delta.
\]
Let \( \nu_\sigma(\cdot) \) denote the multivariate Gaussian density for \( C \sim NI_{n \times p}(0, \sigma^2) \), and \( \mathbb{I}(\mathcal{E}) \) denote the indicator variable that event \( \mathcal{E} \) occurs. Then the above expression becomes

\[
\int \mathbb{I}([X + c] \in S) d\nu_\sigma(c) \leq e^{\varepsilon} \int \mathbb{I}([X' + c] \in S) d\nu_\sigma(c) + \delta, \tag{27}
\]

for any set \( S \) and any pair of neighbors \( X \) and \( X' \).

Under setting (B), let \( \mu(\cdot) \) denote the density for the uniform distribution on the group of \( n \times n \) orthogonal matrices. We also denote \( AS = \{Ax : x \in S\} \) as the set whose elements are the elements in \( S \) multiplied by the matrix \( A \). Then

\[
P[Y(X) \in S] = \int \int \mathbb{I}[A(X + c) \in S] d\nu_\sigma(c) d\mu(A)
= \int \int \mathbb{I}([X + c] \in A^{-1}S) d\nu_\sigma(c) d\mu(A).
\]

Using (27), this implies that for any set \( S \) and any pair of neighbors \( X \) and \( X' \),

\[
P[Y(X) \in S] \leq \{e^{\varepsilon} \int \mathbb{I}([X' + c] \in A^{-1}S) d\nu_\sigma(c) + \delta\} d\mu(A)
= e^{\varepsilon} \int \mathbb{I}([X' + c] \in A^{-1}S) d\nu_\sigma(c) d\mu(A) + \int \delta d\mu(A)
= e^{\varepsilon} P[(X' + C) \in S] + \delta.
\]

Thus the mechanism \( Y(X) \) in setting (B) also satisfies \((\varepsilon, \delta)\)-DP Definition 2.2.

Under setting (C),

\[
P[Y(X) \in S] = \int \int \mathbb{I}[(AX + c) \in S] d\nu_\sigma(c) d\mu(A)
= \int \int \mathbb{I}([X + A^{-1}c] \in A^{-1}S) d\nu_\sigma(c) d\mu(A).
\]

For each fixed orthogonal matrix \( A \), when \( C \sim NI_{n \times p}(0, \sigma^2) \), \( AC \sim NI_{n \times p}(0, \sigma^2) \) also. Thus doing a change of variable \( C^* = AC \), \( C^* \) also have the density \( \nu_\sigma(\cdot) \). (27) implies that

\[
\int \mathbb{I}([X + c] \in S) d\nu_\sigma(c) = \int \mathbb{I}([X + A^{-1}c^*] \in S) d\nu_\sigma(c^*)
\leq e^{\varepsilon} \int \mathbb{I}([X' + A^{-1}c^*] \in S) d\nu_\sigma(c^*) + \delta, \tag{28}
\]

for any set \( S \) and any pair of neighbors \( X \) and \( X' \). Hence,

\[
P[Y(X) \in S] = \int \int \mathbb{I}([X + A^{-1}c] \in A^{-1}S) d\nu_\sigma(c) d\mu(A)
\leq \{e^{\varepsilon} \int \mathbb{I}([X' + A^{-1}c] \in A^{-1}S) d\nu_\sigma(c) + \delta\} d\mu(A)
= e^{\varepsilon} \int \mathbb{I}([X' + A^{-1}c] \in A^{-1}S) d\nu_\sigma(c) d\mu(A) + \int \delta d\mu(A)
= e^{\varepsilon} \int \mathbb{I}([AX' + c] \in S) d\nu_\sigma(c) d\mu(A) + \int \delta d\mu(A)
= e^{\varepsilon} P[(AX' + C) \in S] + \delta.
\]
Thus the mechanism $Y(X)$ in setting (C) also satisfies $(\varepsilon, \delta)$-DP Definition 2.2. \hfill \Box

A.2 Proof of Lemma 2.4

Proof of Lemma 2.4. When condition (5) holds, for any set $S$ and any pair of neighbors $X$ and $X'$,

$$P[Y(X) \in S] = P[Y(X) \in S \cap \tilde{S}_{X,X'}] + P[Y(X) \in S \cap \tilde{S}_{X,X'}]$$

$$\leq \int_{S \cap \tilde{S}_{X,X'}} p_Y(x)(y)dy + P[Y(X) \in \tilde{S}_{X,X'}]$$

$$\leq e^\varepsilon \int_{S \cap \tilde{S}_{X,X'}} p_Y(x)(y)dy + \delta$$

$$\leq e^\varepsilon \int_S p_Y(x)(y)dy + \delta$$

$$= e^\varepsilon P[Y(X') \in S] + \delta.$$

Thus the mechanism $Y(X)$ satisfies $(\varepsilon, \delta)$-DP Definition 2.2.

When condition (5) is violated, there exists a pair of neighboring $X$ and $X'$ such that $P[Y(X) \in \tilde{S}_{X,X'}] > \delta$. Let $S = \tilde{S}_{X,X'}$, then

$$P[Y(X') \in S] = \int_S p_Y(x)(y)dy \leq \frac{1}{e^\varepsilon} \int_S p_Y(x)(y)dy = \frac{1}{e^\varepsilon} P[Y(X) \in S].$$

Hence,

$$e^\varepsilon P[Y(X') \in S] + \delta' = e^\varepsilon P[Y(X') \in S] + (1 - \frac{\varepsilon'}{\varepsilon'}) \delta$$

$$\leq \frac{\varepsilon'}{\varepsilon'} P[Y(X) \in S] + (1 - \frac{\varepsilon'}{\varepsilon'}) \delta$$

$$< \frac{\varepsilon'}{\varepsilon'} P[Y(X) \in S] + (1 - \frac{\varepsilon'}{\varepsilon'}) P[Y(X) \in S]$$

$$= P[Y(X) \in S].$$

Thus the mechanism $Y(X)$ is not $(\varepsilon', \delta')$-DP. \hfill \Box

A.3 Proof of Corollary 3.3

Proof of Corollary 3.3. We will use an explicit bound (29) on the Gaussian quantile provided in the next Lemma A.1.

By Theorem 3.2, if (5) holds for every pair of neighboring $X$ and $X'$, it is necessary that

$$\sigma \geq \frac{\tilde{\gamma} \delta}{\varepsilon} > \sqrt{\frac{\ln(\frac{1}{\delta})}{\varepsilon}}.$$
where the last inequality comes from (29) $\sqrt{\ln(\frac{1}{\delta})} < \bar{\gamma}_\delta$.

Furthermore, when $\delta < 0.05$, $2\ln(\frac{1}{\delta}) > 2\ln(20) > 5$, thus $(1 + \frac{1}{2\ln(\frac{1}{\delta})}) < 1.2$. Hence

$$\sigma > \frac{1.7\sqrt{\ln(\frac{1}{\delta})}}{\varepsilon} > \frac{1.2\sqrt{2\ln(\frac{1}{\delta})}}{\varepsilon} \quad \Rightarrow \quad \sigma > \frac{\sqrt{2\ln(\frac{1}{\delta})}}{\varepsilon}(1 + \frac{1}{2\ln(\frac{1}{\delta})}) > \frac{\bar{\gamma}_\delta}{\varepsilon}(1 + \frac{1}{2\bar{\gamma}_\delta^2}),$$

where the last inequality comes from (29) $\sqrt{\ln(\frac{1}{\delta})} < \bar{\gamma}_\delta < \sqrt{2\ln(\frac{1}{\delta})}$. Hence by Theorem 3.2, condition (5) holds for every pair of neighboring $X$ and $X'$ (thus the release mechanism achieves $(\varepsilon, \delta)$-DP).

**Lemma A.1.** When $\delta < 0.05$,

$$\sqrt{\ln(\frac{1}{\delta})} < \bar{\gamma}_\delta < \sqrt{2\ln(\frac{1}{\delta})}$$

(29)

**Proof of Lemma A.1.** $\bar{\gamma}_\delta$ can be found using the Gaussian tail bound: for $Z \sim N(0, 1)$ and $t > 0$,

$$\frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} \left(1 - \frac{1}{t^3}\right) < P[Z > t] < \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi} t}.$$ 

Hence

$$\frac{e^{-\frac{\bar{\gamma}_\delta^2}{2}}}{\sqrt{2\pi} \bar{\gamma}_\delta \bar{\gamma}_\delta} - \frac{1}{\bar{\gamma}_\delta^2} = \frac{e^{-\frac{\bar{\gamma}_\delta^2}{2}}}{\sqrt{2\pi} \bar{\gamma}_\delta} \left(1 - \frac{1}{\bar{\gamma}_\delta^2}\right) < \frac{\delta}{\sqrt{2\pi} \bar{\gamma}_\delta} < \frac{e^{-\frac{\bar{\gamma}_\delta^2}{2}}}{\sqrt{2\pi} \bar{\gamma}_\delta}.$$ 

Taking the natural logarithm on all sides, we get

$$-\frac{\bar{\gamma}_\delta^2}{2} - \ln(\sqrt{2\pi}) - \ln(\bar{\gamma}_\delta) - \ln(\frac{\bar{\gamma}_\delta^2}{\bar{\gamma}_\delta^2 - 1}) < \ln(\delta) < -\frac{\bar{\gamma}_\delta^2}{2} - \ln(\sqrt{2\pi}) - \ln(\bar{\gamma}_\delta).$$ 

Since $\ln(\frac{1}{\delta}) = -\ln(\delta)$,

$$\frac{\bar{\gamma}_\delta^2}{2} + \ln(\sqrt{2\pi}) + \ln(\bar{\gamma}_\delta) < \ln\left(\frac{1}{\delta}\right) < \frac{\bar{\gamma}_\delta^2}{2} + \ln(\sqrt{2\pi}) + \ln(\bar{\gamma}_\delta) + \ln(\frac{\bar{\gamma}_\delta^2}{\bar{\gamma}_\delta^2 - 1}).$$

(30)
When $\delta < 0.05$, $\bar{\gamma}_\delta > \bar{\gamma}_{0.05} = 1.645 > 1$, hence $\ln(\sqrt{2\pi}) + \ln(\bar{\gamma}_\delta) > 0$. The left inequality in (30) implies that

$$\frac{\bar{\gamma}_\delta^2}{2} < \frac{\bar{\gamma}_{0.05}^2}{2} + \ln(\sqrt{2\pi}) + \ln(\bar{\gamma}_\delta) < \ln(\frac{1}{\delta}).$$

That is, $\bar{\gamma}_\delta < \sqrt{2\ln(\frac{1}{\delta})}$ which is the second half of (29).

Let $f(x) = -\frac{x^2}{2} + \ln(\sqrt{2\pi}) + \ln(x) + \ln(\frac{x^2}{x^2 - 1})$. Then the derivative

$$f'(x) = -x + \frac{1}{x} + \frac{-2}{x(x^2 - 1)} < 0 \quad \text{for all} \ x > 1.$$

Hence $f(x)$ is an decreasing function when $x > 1$. Therefore, when $\delta < 0.05$, $f(\bar{\gamma}_\delta) < f(\bar{\gamma}_{0.05}) = f(1.645) < 0$. Apply this to the right inequality in (30),

$$\ln(\frac{1}{\delta}) < \frac{\bar{\gamma}_\delta^2}{2} + \ln(\sqrt{2\pi}) + \ln(\bar{\gamma}_\delta) + \ln(\frac{\bar{\gamma}_\delta^2}{\bar{\gamma}_\delta^2 - 1}) = \bar{\gamma}_\delta^2 + f(\bar{\gamma}_\delta) < \bar{\gamma}_\delta^2.$$

This is the first half of (29).

A.4 Proof of Corollary 3.7

Proof of Corollary 3.7. We only need to prove that the condition (15) implies the condition (14) $\sigma^2 \geq \max(2, \frac{4np^2}{(n-p)\varepsilon}, \frac{4p\gamma_{\delta, np}}{(n-p)\varepsilon})$.

When (15) holds, firstly, we have

$$\sigma^2 \geq \frac{[2np + 3ln(\frac{1}{\delta})]4p}{(n-p)\varepsilon} \geq \frac{(2np)4p}{(n-p)\varepsilon} \geq (2p)4p \geq 2.$$

Secondly,

$$\sigma^2 \geq \frac{[2np + 3ln(\frac{1}{\delta})]4p}{(n-p)\varepsilon} \geq \frac{(2np)4p}{(n-p)\varepsilon} = \frac{8np^2}{(n-p)\varepsilon} \geq \frac{4np^2}{(n-p)\varepsilon}.$$ 

Lastly, using the bound (32) on the Chi-square quantile $\gamma_{\delta, np}$ provided in the Lemma A.3,

$$\sigma^2 \geq \frac{[2np + 3ln(\frac{1}{\delta})]4p}{(n-p)\varepsilon} \geq \frac{\gamma_{\delta, np}4p}{(n-p)\varepsilon}.$$
Taken together the above three expressions, we get (14)

\[ \sigma^2 \geq \max \left( 2, \frac{4np^2}{(n-p)\varepsilon}, \frac{4p\gamma_{\delta, np}}{(n-p)\varepsilon} \right). \]

This finishes the proof.

\[ \square \]

### A.5 Chi-square distribution Tail bound.

We first cite the following lemma which is the Lemma 8.1 in [4].

**Lemma A.2.** If \( \chi^2 \) follows a Chi-square distribution with \( n \) degrees of freedom and noncentral parameter \( \nu \) then for any \( x > 0 \),

\[ P[\chi^2 \geq (n + \nu^2) + 2\sqrt{(n + 2\nu^2)x + 2x}] \leq e^{-x}, \]
\[ P[\chi^2 \leq (n + \nu^2) - 2\sqrt{(n + 2\nu^2)x}] \leq e^{-x} \]

(31)

Using these bounds, we can find a simple bound for \( \gamma_{\delta, np} \) as in the following Lemma.

**Lemma A.3.**

\[ \gamma_{\delta, np} \leq 2np + 3\ln \left( \frac{1}{\delta} \right). \]  

(32)

**Proof of Lemma A.3.** Plug \( x = \ln \left( \frac{1}{\delta} \right) \) into (31) for the Chi-square distribution with \( np \) degrees of freedom and noncentral parameter \( \nu = 0 \),

\[ P \left[ \chi^2 \geq np + 2\sqrt{(np)\ln \left( \frac{1}{\delta} \right) + 2\ln \left( \frac{1}{\delta} \right)} \right] \leq e^{-\ln \left( \frac{1}{\delta} \right)} = e^{\ln(\delta)} = \delta. \]

Since by definition,

\[ P[\chi^2 \geq \gamma_{\delta, np}] = \delta, \]

we have

\[ \gamma_{\delta, np} \leq np + 2\sqrt{(np)\ln \left( \frac{1}{\delta} \right) + 2\ln \left( \frac{1}{\delta} \right)} \leq np + [(np) + \ln \left( \frac{1}{\delta} \right)] + 2\ln \left( \frac{1}{\delta} \right) = 2np + 3\ln \left( \frac{1}{\delta} \right). \]

\[ \square \]
A.6 Proof of Lemma 5.1

Proof of Lemma 5.1. Let \( e_1 = \frac{\text{proj}_S(v)}{\|\text{proj}_S(v)\|} \) be the unit vector along the direction of the projection for \( v \) onto the subspace \( S \). Then \( bv^T = be_1^T \|\text{proj}_S(v)\| \).

For the subspace \( S \), we can find unit vectors \( e_1, ..., e_q \) which are orthogonal to each other. They then form the base vectors for a coordinate systems of \( S \). Hence \( b = \sum_{i=1}^{q} \tilde{b}_i e_i \) with \( \tilde{b}_i = be_i^T \) being the \( i \)-th coordinate of \( b \) under this coordinate system. Hence

\[
bv^T = \tilde{b}_1 \|\text{proj}_S(v)\|. \]

Notice \( \tilde{b}_1 \) is the first coordinate of a random vector uniformly distributed over the \((q-1)\)-dimensional unit sphere, and its probability density is known to be \( \frac{1}{q_u} (1 - u^2)^{\frac{q-3}{2}} \). Thus

\[
\int g(bv^T) d\mu_q(b) = \int g(\|\text{proj}_S(v)\| \tilde{b}_1) d\mu_q(b) = \int_{u=-1}^{1} g(\|\text{proj}_S(v)\| u) \frac{1}{q_u} (1 - u^2)^{\frac{q-3}{2}} du. \]

\( \square \)

A.7 Proof of Lemma 5.2

Proof of Lemma 5.2.

\[
G'_q(t) = \int_{u=-1}^{1} e^{tu} u(1 - u^2)^{\frac{q-2}{2}} du = \int_{u=0}^{1} [e^{tu} - e^{-tu}] u(1 - u^2)^{\frac{q-2}{2}} du.
\]

Since \( e^{tu} - e^{-tu} > 0 \) for all positive \( tu \) values, the integrand is always positive in the last integral. Hence \( G'_q(t) > 0 \), when \( t > 0 \). Thus \( G_q(t) \) is an increasing function when \( t > 0 \).

Furthermore, using integral by parts,

\[
G'_q(t) = \int_{u=-1}^{1} e^{tu} u(1 - u^2)^{\frac{q-2}{2}} du = \int_{u=-1}^{1} e^{tu} \frac{q}{2} (1 - u^2)^{\frac{q-4}{2}} d(u^2) = \int_{u=-1}^{1} e^{tu} \frac{1}{q} d[(1 - u^2)^{\frac{q}{2}}] = \frac{e^{tu}}{q} (1 - u^2)^{\frac{q}{2}} |_{u=-1}^{1} - \int_{u=-1}^{1} \frac{1}{q} (1 - u^2)^{\frac{q}{2}} d(e^{tu}) = 0 - \int_{u=-1}^{1} \frac{1}{q} (1 - u^2)^{\frac{q}{2}} e^{tu} \frac{dq}{du} du = \frac{t}{q} \int_{u=-1}^{1} e^{tu} (1 - u^2)^{\frac{q}{2}} du \leq \frac{t}{q} \int_{u=-1}^{1} e^{tu} (1 - u^2)^{\frac{q-2}{2}} du = \frac{t}{q} G_q(t).
\]
For any $t_2 > t_1 > 0$, since $G_q(t) > 0$, we have

$$\ln G_q(t_2) - \ln G_q(t_1) = \int_{t_1}^{t_2} \frac{G_q'(t)}{G_q(t)} dt \leq \int_{t_1}^{t_2} \frac{t^q}{t^q} dt = \frac{t_2^q - t_1^q}{2q}.$$ 

Hence

$$\frac{G_q(t_2)}{G_q(t_1)} \leq e^{\frac{|t_2^q - t_1^q|}{2q}},$$

which is also true when $t_2 < t_1$ because $G_q(t)$ is increasing.

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