Multiinstantons in curvilinear coordinates.

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Abstract

The 'tHooft's $5N$-parametric multinasstanton solution is generalized to curvilinear coordinates. Expressions can be simplified by the gauge transformation that makes $\eta$-symbols constant in the vierbein formalism. This generates the compensating addition to the gauge potential of pseudoparticles. Typical examples (4-spherical, $2+2$- and $3+1$-cylindrical coordinates) are studied and explicit formulae presented for reference. Singularities of the compensating field are discussed. They are irrelevant for physics but affect gauge dependent quantities.

Introduction

Instantons form an essential nonperturbative element of nonabelian gauge theories applied in physics of strong and weak interactions. But despite a significant knowledge that was acquired since the pioneering paper [1] the topic is not exhausted yet. Among others the question of the role of instantons in quark confinement remains unsettled. Probably the problem will not be solved unless new methods and ideas appear. Reviews of the current state of affairs can be found in [2, 3].

Usually instantons are studied in the infinite 4-dimensional euclidean space parameterized by Cartesian coordinates. A demand for non-Cartesian coordinates first came from investigation of anomalies and index theorems on manifolds with boundaries. Adaptation of existing results (see [4, 5]) to practical problems requires explicit formulae.

A promising direction would be to examine pseudoparticles in confinement models such as the QCD-string or a MIT-bag. However Cartesian coordinates are not the best choice for such objects. For example strings looks more natural in $(2+2)$-cylindrical coordinates while a $(3+1)$-frame may be convenient for the bag. Studies of pseudoparticles against nonuniform backgrounds may require even more imagination.

A notable progress in the instanton physics was associated with exact multiinstanton solutions. The general solution [6] proved too complicated and many physicists preferred the convenient 'tHooft's Ansatz [7]. Most of instanton-based models start from pseudoparticles in singular gauge that are the specific case of this solution. An advantage of the Ansatz is that it makes possible to calculate exact Green functions in multipseudoparticle field [8]. Besides it provides a way to study QCD at nonzero temperature [9, 10] and quark density [11].

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The purpose of the present paper is to develop a simple approach to 'tHooft's multipseudoparticle solution in curvilinear coordinates. This offers a chance to benefit from it's advantages preserving spatial symmetry of other objects. This would simplify calculations and widen the research field.

A basic component of the Cartesian 'tHooft's solution are the so-called $\eta$-symbols ($\eta$-tensors) that project (anti-)selfdual tensors onto the $SU(2)$ gauge group, [12]. Being transformed to other coordinates $\eta$-symbols lose their simple form. Our idea is to consider instead of them the $\xi$-symbols that are literal analogues of $\eta$'s in the vierbein formalism. The $\xi$-symbols can be made constant by a gauge transform related to the metric tensor. The resulting $5N$-parametric gauge potential is the sum of the 'tHooft's multiinstanton and the compensating field. The latter may be found by a simple calculation. After discussing general aspects we illustrate the idea by an example and derive explicit formulae for two typical cases. It turns out that the compensating potential may exhibit singularities that affect gauge dependent quantities.

The paper has the following structure. We start from reminding the basics: Sect. 1 introduces the 'tHooft's Ansatz while Sect. 2 deals with curvilinear coordinates. Section 3 is dedicated to the compensating gauge connection. First we calculate the compensating gauge potential and then discuss the relation between $\xi$- and $\eta$-symbols. After that we turn to pseudoparticles in non-Cartesian frame in Sect. 4. We calculate vector potentials and gauge field strengths for a general $N$-instanton configuration and for an instanton in singular and regular gauges\(^1\).

Section 5 presents a detailed study of an instanton in 4-spherical coordinates. We calculate explicitly the (gauged) pseudoparticle fields in singular and regular gauges and obtain the same result provided that the instanton is at the origin. This may serve an indication in favour of the approach. The calculation of the Chern-Simons number reveals that being gauge dependent it feels the singularity of the compensating field. In Section 6 we compute the compensating connection for $(2+2)$- and $(3+1)$-cylindrical coordinates. The results are summarized in the Conclusion.

1 Instantons in singular and regular gauges.

This section reviews basic facts on instantons in 4-dimensional euclidean space. For time being we shall focus on Cartesian coordinates and make no distinction between upper and lower indices. We shall consider a Yang-Mills theory with the $SU(2)$ gauge group. The action of the gauge field is:

\[
S = \int d^4x \frac{(F^a_{\mu\nu})^2}{4g^2} = \frac{1}{4g^2} \int d^4x \left( \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc} A^b_\mu A^c_\nu \right)^2.
\]

Vector potential is denoted by $A^a_\mu$ with $\mu = 1, \ldots, 4$ and $a = 1, 2, 3$ being the Lorentz and group indices respectively. Throughout the paper we shall use for gauge fields the matrix notation. The covariant derivative in fundamental representation is:

\[
D_\mu = \partial_\mu - i\hat{A}_\mu = \partial_\mu - \frac{i}{2} \tau^a A^a_\mu,
\]

\(^1\) More precisely our solutions coincide with those in Cartesian coordinates
where $\tau^a$ are Pauli matrices and hats indicate matrices. Later we shall introduce the Levi-Civita and spin connections that are absent in Cartesian coordinates. Commutator of the covariant derivatives gives the field strength:

$$\hat{F}_{\mu\nu} = \frac{1}{2} \tau^a F^a_{\mu\nu} = i [D_\mu, D_\nu].$$

(3)

Classical equations for $A^a_\mu$ have instanton, or pseudoparticle, solutions. Instantons are characterized by their topological properties: their field is selfdual, (a), and has the unit topological charge, (b):

$$\hat{F}^{I}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} \hat{F}^{I}_{\lambda\sigma}, \quad (a); \quad \frac{1}{32\pi^2} \int d^4x \epsilon_{\mu\nu\lambda\sigma} \text{tr} \hat{F}^{I}_{\mu\nu} \hat{F}^{I}_{\lambda\sigma} = 1, \quad (b).$$

(4)

In addition to instantons there exist anti-instantons. Anti-instanton field $\hat{F}^{A}_{\mu\nu}$ is antiselfdual and has the topological charge $-1$, i.e. the signs of the right hand side of the equations (4) must be reversed. From here on we shall speak mostly about instantons indicating generalization to anti-instantons if necessary.

Instanton field depends on gauge. The two most popular are regular and singular gauges. The fields of instantons of radius $\rho$ centered at $x^0$ are (up to uniform gauge rotations):

$$\hat{A}^{+}_\mu \big|_{\text{reg}} = \frac{\hat{\eta}^{+}_\mu}{2} \partial_\nu \ln \Pi_{\text{reg}}(x, x^0) = \frac{\hat{\eta}^{+}_\mu}{2} \partial_\nu \ln \left[ (x - x^0)^2 + \rho^2 \right]; \quad (5a)$$

$$\hat{A}^{+}_\mu \big|_{\text{sing}} = -\frac{\hat{\eta}^{+}_\mu}{2} \partial_\nu \ln \Pi_{\text{sing}}(x, x^0) = -\frac{\hat{\eta}^{+}_\mu}{2} \partial_\nu \ln \left[ 1 + \frac{\rho^2}{(x - x^0)^2} \right]. \quad (5b)$$

Here $\hat{\eta}^{+}_\mu = \tau^a \eta^a_\mu$ and $\hat{\eta}^{-}_\mu = \tau^a \bar{\eta}^a_\mu$ are the matrix versions of the 'tHooft’s $\eta$-symbols defined by:

$$\eta(\bar{\eta})^a_{\mu\nu} = -\eta(\bar{\eta})^a_{\nu\mu} = \left\{ \begin{array}{ll} \epsilon^{a\mu\nu} & \text{for } \mu, \nu = 1, 2, 3; \\ (-)\delta^{a\mu} & \text{for } \nu = 4. \end{array} \right.$$

(6)

The $\eta$-symbols are antisymmetric in Lorentz indices and selfdual or antiselfdual respectively, i.e.

$$\frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} \hat{\eta}^{+}_{\lambda\sigma} = \hat{\eta}^{+}_{\mu\nu}; \quad \frac{1}{2} \epsilon_{\mu\nu\lambda\sigma} \hat{\eta}^{-}_{\lambda\sigma} = -\hat{\eta}^{-}_{\mu\nu}. \quad (7)$$

Their properties can be found in [12] or obtained directly from the matrix representation (9) below.

In practical calculations it is convenient to make use of the two hermitian conjugated sets of $2 \times 2$ matrices (latin indices stand for the three “spatial” dimensions):

$$\tau_\mu = (\tau^a, i); \quad \tau^{\dagger}_\mu = (\tau^a, -i). \quad (8)$$

The following equations relate $\tau_\mu$, $\tau^{\dagger}_\mu$ to $\hat{\eta}$-symbols:

$$\tau^{\dagger}_\mu \tau_\nu = \delta_{\mu\nu} + i \hat{\eta}^{+}_{\mu\nu}; \quad \tau_\mu \tau^{\dagger}_\nu = \delta_{\mu\nu} + i \hat{\eta}^{-}_{\mu\nu}; \quad (9)$$

Separating the symmetric and antisymmetric parts of these expressions we obtain:

$$\tau^{\dagger}_{\mu [\nu} = i \tau^{\dagger}_{\mu \nu}; \quad \tau^{\dagger}_{\mu [\nu} = i \hat{\eta}^{+}_{\mu\nu}; \quad \tau^{\dagger}_{\mu \nu]} = i \hat{\eta}^{-}_{\mu\nu}. \quad (10)$$

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Expression (5b) can be obtained from (5a) by means of the gauge transform:

$$\hat{A}_\mu^I(x)|_{\text{sing}} = \hat{N}_+^{-1} \hat{A}_\mu^I(x)|_{\text{reg}} \hat{N}_+ + i \hat{N}_+^{-1} \partial_\mu \hat{N}_+,$$

(11)

with the matrices $\hat{N}_+ = x_\lambda \tau^\lambda / x$ and $\hat{N}_- = x_\lambda \tau^\lambda / x$ ($\tau$ and $\tau^\dagger$ being swapped for anti-instantons, $\hat{N}_- = \hat{N}_+^{-1}$).

Another way to derive (5b) is to carry out the inversion, $x_\mu \rightarrow x_\mu \rho^2 / x^2$. However the latter changes the orientation of the coordinate system and converts instanton to anti-instanton. This is corrected by the $\hat{\eta}^\pm \rightarrow \hat{\eta}^\mp$ replacement.

Singular gauge has several practical advantages. Gauge potentials fall rapidly and pseudoparticles are almost independent. Generalization to multipseudoparticle configurations is done simply by adding extra pieces to $\Pi(x)$:

$$\Pi_{\text{sing}}(x) \rightarrow \Pi_N(x) = 1 + \sum_{i=1}^{N} \frac{\rho_i^2}{(x - x_i)^2},$$

(12)

$\rho_i$ and $x_i$ being radii and positions of pseudoparticles. This solution bears the name of the ’tHooft’s Ansatz, [7]. It depends on $5N$ parameters $\{x_i, \rho_i\}$ and does not allow for independent gauge rotations of instantons. But the lack of generality is balanced by the extreme handiness.

2 Curvilinear coordinates.

In order to distinguish curvilinear coordinates from the Cartesian ones we shall denote them by $q^\alpha$, $q^\beta$ etc. The metric tensor is now $g_{\alpha\beta}(q)$,

$$ds^2 = dx_\mu^2 = g_{\alpha\beta}(q) \, dq^\alpha \, dq^\beta.$$  

(13)

The metric $g_{\alpha\beta}(q)$ and it’s inverse $g^{\alpha\beta}(q) = [g_{\alpha\beta}(q)]^{-1}$ are used for raising and lowering indices: $A_\alpha = g_{\alpha\beta} A^\beta$; $A^\alpha = g^{\alpha\beta} A_\beta$.

Let us start from fields that are singlet with respect to the gauge group. Covariant derivatives of vectors are taken with the help of the Levi-Civita connection $\Gamma_{\beta\gamma}^\alpha$:

$$D_\alpha A^\beta = \partial_\alpha A^\beta + \Gamma_{\alpha\delta}^\beta A_\delta; \quad \text{and} \quad D_\alpha A_\beta = \partial_\alpha A_\beta - \Gamma_{\alpha\beta}^\delta A_\delta.$$  

(14)

The latter is unambiguously defined by the condition that the metric tensor is covariantly constant, $D_\alpha g_{\beta\gamma} = 0$.

$$\Gamma_{\beta\gamma}^\alpha = \frac{1}{2} g_{\alpha\delta} \left( \frac{\partial g_{\delta\beta}}{\partial q^\gamma} + \frac{\partial g_{\delta\gamma}}{\partial q^\beta} - \frac{\partial g_{\beta\gamma}}{\partial q^\delta} \right).$$  

(15)

The metric $g_{\alpha\beta}$ may be decomposed into vierbeins $e^\alpha_a$ (from here on we reserve latin indices for the latter). Let $\pi_{ab}$ be the flat euclidean metric, $\pi^{ab} = \pi_{ab} = \text{diag} (1, 1, 1, 1)$. Then

$$g_{\alpha\beta}(q) = \pi_{ab} e^\alpha_a(q) e^b_b(q); \quad \text{while} \quad \pi^{ab} = g^{\alpha\beta}(q) e^\alpha_a(q) e^b_b(q).$$  

(16)

One may convert spatial indices into vierbein ones, $A^a = e^a_\alpha A^\alpha$. Raising and lowering of the latter is performed by means of the tensors $\pi$: $A_\alpha = \pi_{\alpha\beta} A^\beta$ and $A^\alpha = \pi^{\alpha\beta} A_\beta$. The inverse of the vierbein is $e^\alpha_a$:

$$e^\alpha_a e^b_b = \delta^\alpha_b; \quad \text{and} \quad e^\alpha_a e^\alpha_b = \delta^\alpha_\beta.$$  

(17)
Covariant derivatives of quantities with vierbein indices are defined in terms of the spin connection $R^a_{\alpha b}$:

$$D_{\alpha} A^a = \partial_{\alpha} A^a + R^a_{\alpha b} A^b; \quad \text{and} \quad D_{\alpha} A_a = \partial_{\alpha} A_a - A_b R^b_{\alpha a}. \quad (18)$$

Vierbeins are covariantly constant and this fixes the spin connection. Solving the equation $D_{\alpha} e^a_{\beta} = \partial_{\alpha} e^a_{\beta} + \Gamma_{\alpha \gamma}^\beta e^\gamma_a - e^a_{\beta} R^b_{\alpha a} = 0$ we obtain:

$$R^a_{\alpha b} = e^a_{\beta} \partial_{\alpha} e^\beta_b + e^a_{\beta} \Gamma_{\alpha \gamma}^\beta e^\gamma_b = e^a_{\beta} (D_{\alpha} e^\beta_b). \quad (19)$$

The matrices $R$ are antisymmetric with respect to the exchange $a \leftrightarrow b$. (This follows from $e^a_{\mu} e^\mu_b = \delta^a_b = \text{const.})$ Sometimes it is convenient to expand $R^a_{\alpha b}$ in terms of generators of the $O(4)$ group, $(L_{mn})^a_b$:

$$R^a_{\alpha b} = -\frac{i}{2} B^{mn}_{\alpha} (L_{mn})^a_b, \quad \text{where} \quad (L_{mn})^a_b = -i (\delta^a_m \pi_n - \delta^a_n \pi_m). \quad (20)$$

The coefficients $B^{mn}_{\alpha}$ are antisymmetric, $B^{mn}_{\alpha} = -B^{mn}_{\alpha}$,

$$B^{mn}_{\alpha} = \frac{i}{2} \text{tr} \hat{R}_{\alpha} (\hat{L}^{mn})^a_b = \frac{i}{2} (\hat{R}_{\alpha})^a_b (\hat{L}^{mn})^b_a. \quad (21)$$

The last thing is to extend the spin connection and covariant derivative to spin-$\frac{1}{2}$ fields. We would like matrices $\gamma_a$ with latin indices to be covariantly constant. It is easy to see that if we define $\gamma$-matrices and their commutators so that

$$\{\gamma_a, \gamma_b\} = 2\delta_{ab}, \quad \text{and} \quad \sigma_{ab} = -\frac{i}{2} [\gamma_a, \gamma_b], \quad (22)$$

then the matrices $\sigma_{mn}/2$ are rotation generators for spin-$\frac{1}{2}$ fields and

$$D_{\alpha} \gamma_a = \partial_{\alpha} \gamma_a + \frac{i}{2} B^{mn}_{\alpha} \left\{ \gamma_b (L_{mn})^b_a - \frac{1}{2} [\sigma_{mn}, \gamma_a] \right\} = 0. \quad (23)$$

The last relation can be easily checked directly.

### 3 The compensating gauge connection

#### 3.1 Definition and properties of the compensating connection

Now let us turn to gauge fields in curvilinear coordinates. One may define $\tau_a$ and $\tau^\dagger_a$ matrices with vierbein indices by analogy with (8). We shall show that it is possible to introduce a compensating gauge field such that either $\tau_a \tau^\dagger_b$ or $\tau^\dagger_a \tau_b$ is covariantly constant. The compensating field turns out to be a pure gauge provided that the space is flat.

It is convenient to turn back to the spin connection $B^{mn}_{\alpha}$ and use the following representation of $\gamma$-matrices, (check 22):

$$\gamma_a = \left( \begin{array}{cc} 0 & \tau_a^\dagger \\ \tau_a & 0 \end{array} \right); \quad \gamma_a \gamma_b = \left( \begin{array}{cc} \tau_a \tau_b^\dagger & 0 \\ 0 & \tau_a^\dagger \tau_b \end{array} \right); \quad \sigma_{ab} = \left( \begin{array}{cc} \hat{\xi}_{ab}^- & 0 \\ 0 & \hat{\xi}_{ab}^+ \end{array} \right); \quad (24)$$

where

$$\hat{\xi}_{ab}^+ = -i \tau^\dagger_{[a} \tau_b^\dagger_{]} \quad \text{and} \quad \hat{\xi}_{ab}^- = -i \tau_{[a} \tau^\dagger_{b]} \quad (25)$$
are the vierbein analogues of the Cartesian $\hat{n}_{\mu\nu}$ and $\hat{n}_{\mu\nu}$. One may define $\hat{\xi}$-symbols with coordinate indices as follows:

$$\hat{\xi}^+_{\alpha\beta} = e^a_{\alpha} e^b_{\beta} \hat{\xi}^+_{ab} \quad \text{and} \quad \hat{\xi}^-_{\alpha\beta} = e^a_{\alpha} e^b_{\beta} \hat{\xi}^-_{ab}, \quad (26)$$

In order to escape confusion we shall use only $\hat{\xi}_{ab}$ that are just constant numerical matrices. From the relation (23) applied to the block-diagonal matrix $\gamma_{\alpha} \gamma_{\beta}$ it is easy to deduce for the separate blocks that:

\[
D_{\alpha} \tau^+_{\alpha\beta} = \partial_{\alpha} \tau^+_{\alpha\beta} + i \frac{1}{2} B^{mn}_{\alpha} \tau^+_{\alpha\beta} (\delta^c_a (L_{mn})^d_b + (L_{mn})^c_a \delta^d_b) - i \left[ A^+_{\alpha}, \tau^+_{\alpha\beta} \right] = 0, \quad (27a)
\]

\[
D_{\alpha} \tau^-_{\alpha\beta} = \partial_{\alpha} \tau^-_{\alpha\beta} + i \frac{1}{2} B^{mn}_{\alpha} \tau^-_{\alpha\beta} (\delta^c_a (L_{mn})^d_b + (L_{mn})^c_a \delta^d_b) - i \left[ A^-_{\alpha}, \tau^-_{\alpha\beta} \right] = 0, \quad (27b)
\]

where the vector-potentials $A^+_{\alpha}$ and $A^-_{\alpha}$ are:

\[
A^+_{\alpha} = \frac{1}{4} B^{mn}_{\alpha} \xi^+_{mn} = \frac{i}{4} \tau^+_{\alpha\beta} R^{ab}_{\alpha\beta} \tau_{ab}, \quad (28a)
\]

\[
A^-_{\alpha} = \frac{1}{4} B^{mn}_{\alpha} \xi^-_{mn} = \frac{i}{4} \tau^-_{\alpha\beta} R^{ab}_{\alpha\beta} \tau_{ab}, \quad (28b)
\]

Thus the gauge connections $A^+_{\alpha}$ and $A^-_{\alpha}$ are projections of the selfdual and antiselfdual parts of the spin connection $R^{ab}_{\alpha}$ onto the $SU(2)$ gauge group. In general only one of the bilinears $\tau^+_{\alpha\beta}$, $\tau^-_{\alpha\beta}$ can be made covariantly constant by the appropriate choice of the compensating field. The equations (28) are the most straightforward way to calculate $A^\pm$. (Note that together with $\hat{\xi}^+_{ab}$ the corresponding $\hat{\xi}^-_{ab}$, (26) becomes covariantly constant as well.)

In flat euclidean space either of the potentials $A^+_{\alpha}$ and $A^-_{\alpha}$ is the pure gauge, i.e. the corresponding field strengths $\hat{F}^\pm_{\alpha\beta} = \partial_{\alpha} A^\pm_{\beta} - \partial_{\beta} A^\pm_{\alpha} - i \left[ A^\pm_{\alpha}, A^\pm_{\beta} \right]$ are zero. In order to show that let us again resort to the help of the spin connection. Note that matrices $\gamma_{\alpha} = e^a_{\alpha} \gamma_{a}$ are covariantly constant as well:

\[
D_{\alpha} \gamma_{\beta} = \partial_{\alpha} \gamma_{\beta} - \gamma_{\gamma} \Gamma^{\gamma}_{\alpha\beta} - \frac{i}{4} \left[ B^{mn}_{\alpha} \sigma_{mn}, \gamma_{\beta} \right] = 0, \quad (29)
\]

where $D_{\alpha}$ is the full covariant derivative. It follows from (29) that

\[
\left[ D_{\alpha}, D_{\beta} \right] \gamma_{\gamma} = -i \left[ \hat{G}_{\alpha\beta}, \gamma_{\gamma} \right] - \gamma_{\delta} R^\delta_{\alpha\beta} = 0. \quad (30)
\]

Here $R^\delta_{\gamma\alpha\beta}$ is Riemann curvature tensor and $\hat{G}_{\alpha\beta}$ is the commutator of covariant derivatives with respect to the spin connection $B^{mn}_{\alpha}$:

\[
\hat{G}_{\alpha\beta} = i \left[ \partial_{\alpha} - i \frac{1}{4} B^{mn}_{\alpha} \sigma_{mn}, \partial_{\beta} - i \frac{1}{4} B^{kl}_{\beta} \sigma_{kl} \right] = \left( \begin{array}{cc} \hat{F}_{\alpha\beta}^+ & 0 \\ 0 & \hat{F}_{\alpha\beta}^- \end{array} \right); \quad (31)
\]

It is not a problem to rewrite the second term in (30) as $\gamma_{\delta} R^\delta_{\gamma\alpha\beta} = i \frac{1}{4} R^{\delta \kappa}_{\alpha\beta} \left[ \sigma_{\delta \kappa}, \gamma_{\gamma} \right]$.

Separating diagonal blocks of the matrices $\hat{G}_{\alpha\beta}$ and $\sigma_{\delta \kappa}$ one obtains the relation between the compensating field strength and the Riemann curvature of the space:

\[
\hat{F}_{\alpha\beta}^+ = -\frac{1}{4} R_{\alpha\beta} \gamma_{\delta} \hat{\xi}^+_\delta, \quad (32a)
\]

\[
\hat{F}_{\alpha\beta}^- = -\frac{1}{4} R_{\alpha\beta} \gamma_{\delta} \hat{\xi}^-_\delta. \quad (32b)
\]

Thus $\hat{F}_{\alpha\beta}^+ = \hat{F}_{\alpha\beta}^- = 0$ provided that $R_{\alpha\beta} \gamma_{\delta} = 0$. Simple changes of variables $x^\mu \rightarrow q^\alpha$ do not generate curvature and both compensating fields are pure gauges.
3.2 Relation between \( \hat{\xi}_{ab} \) and \( \hat{\eta}_{\mu\nu} \) symbols

As soon as the vector-potentials \( \hat{A}_\alpha^+ \) and \( \hat{A}_\alpha^- \) are pure gauges they may be represented as

\[
\hat{A}_\alpha^+(q) = i \Omega_+^{-1}(q) \partial_\alpha \Omega_+(q) \quad \text{or} \quad \hat{A}_\alpha^-(q) = i \Omega_-^{-1}(q) \partial_\alpha \Omega_-(q). \tag{33}
\]

This defines the \( 2 \times 2 \) matrices \( \Omega_+ \) and \( \Omega_- \) up to the left multiplication by a constant nondegenerate matrix \( U \): \( \Omega \rightarrow U \Omega \). However the freedom may be eliminated by requiring that \( \Omega \) gauge rotated \( \hat{\eta}_{\mu\nu} \) into \( \xi_{\alpha\beta} \).

Let us rewrite the condition of \( \hat{\xi}_{ab} \) being covariantly constant, (27), in Cartesian coordinates. It will read (presently we may drop the \( \pm \)-superscripts):

\[
D_\lambda \hat{\xi}_{\mu\nu}(x) = D_\lambda \frac{\partial q^a}{\partial x^\mu} \frac{\partial q^b}{\partial x^\nu} e_\alpha^a e_\beta^b \hat{\xi}_{ab} = \partial_\lambda e_\mu^a e_\nu^b \hat{\xi}_{ab} + e_\mu^a e_\nu^b \left[ \Omega^{-1} \partial_\lambda \Omega, \hat{\xi}_{ab} \right] = 0, \tag{34}
\]

that is equivalent to

\[
\Omega^{-1} \left[ \partial_\lambda \left( \Omega e_\mu^a e_\nu^b \hat{\xi}_{ab} \Omega^{-1} \right) \right] \Omega = 0 \quad \text{or} \quad \left( \Omega e_\mu^a e_\nu^b \hat{\xi}_{ab} \Omega^{-1} \right) = \text{const.} \tag{35}
\]

We conclude that \( \Omega e_\mu^a e_\nu^b \hat{\xi}_{ab} \Omega^{-1} \) is a constant tensor that projects selfdual (or antiselfdual respectively) antisymmetric tensors onto the \( SU(2) \) group (see Appendix). Hence it must be equal (up to a gauge rotation) to the corresponding \( \hat{\eta}_{\mu\nu} \). Uniform gauge rotations do not affect the compensating fields (33) and we may fix \( \Omega \) up to a phase factor \( e^{i\alpha} \) by demanding that:

\[
e_\mu^a e_\nu^b \hat{\xi}_{ab} = \Omega^{-1} \hat{\eta}_{\mu\nu} \Omega \quad \text{or} \quad \Omega \hat{\xi}_{ab} = e_\alpha^a e_\beta^b \hat{\eta}_{\mu\nu} \Omega. \tag{36}
\]

The question is whether there are solutions to these equations. One may figure out the two conditions. First, the both sides of (36) must be normalized in the same way. In order to prove this we take the square of the first equation:

\[
e_\mu^a e_\nu^b \hat{\xi}_{ab} e_\alpha^c e_\beta^d \hat{\xi}_{cd} = \Omega^{-1} \hat{\eta}_{\mu\nu} \hat{\eta}^{\mu\nu} \Omega. \tag{37}
\]

This results into identity \( \hat{\xi}_{ab} \hat{\xi}_{ab} = \hat{\eta}_{\mu\nu} \hat{\eta}^{\mu\nu} = 12 \). Thus the first condition is fulfilled.

However this does not guarantee existence of solutions. It is necessary that both sides of the equations were of same duality. This is the case if the two sets \( \tau_\alpha \) and \( \tau_\mu e_\alpha^\mu \) differ by a rotation and/or an even permutation.\(^2\) (Another way is to say that the change of variables must respect parity.) Gauge transformations do not interfere with duality and it suffice to check the equivalence of (36) to their duals,

\[
\frac{1}{2} \epsilon^{\mu\nu\lambda\sigma} e_\lambda^a e_\sigma^b \hat{\xi}_{ab}^\pm = \pm \Omega^{-1} \hat{\eta}_{\mu\nu}^\pm \Omega \quad \text{or} \quad \Omega \hat{\xi}_{ab}^\pm = \pm \frac{1}{2} e_{abcd} e^{\mu\nu} e^{\epsilon\sigma} \hat{\eta}_{\epsilon\sigma}^\pm \Omega. \tag{38}
\]

A well-known example of a parity violating procedure is inversion \( q_\alpha = x_\alpha / x^2 \) that may be implemented to derive the singular gauge, (5b) from the regular one, (5a). In order to restore duality altered by the inversion the substitution \( \eta_{\mu\nu} \rightarrow -\hat{\eta}_{\mu\nu} \) is necessary.

\(^2\)Although gauge transformation rotate the traceless Pauli \( \tau \)-matrices into each other, they do not affect the \( \tau_4 \)-matrix. Hence permutations can not be reduced to gauge transforms.
4 Instanton with the compensating field.

A naive way to transform an instanton to curvilinear coordinates would be simply to change variables: \( x^\mu \rightarrow q^\alpha \) and \( \hat{A}_\mu \rightarrow \hat{A}_\alpha = \hat{A}_\mu (\partial x^\mu / \partial q^\alpha) \). It is more convenient however to accompany this by one of the previously discussed gauge rotations. Let us begin with the singular gauge, (5b), that contains the antiselfdual symbol \( \hat{\eta}_{\mu\nu} \). This dictates the choice of the matrix \( \Omega_- \) for the transformation. The combined change of variables and gauge transform give:

\[
\hat{A}_\alpha^\Omega \big|_{\text{sing}} (q) = \Omega_+^{-1}(q) \frac{\partial x^\mu}{\partial q^\alpha} \hat{A}_\mu^I \big|_{\text{sing}} (q) \Omega_-(q) + i \Omega_+^{-1}(q) \frac{\partial \Omega_-}{\partial q^\alpha}, \tag{39}
\]

With the help of the relations (33) and (36) this may be immediately reduced to

\[
\hat{A}_\alpha^\Omega \big|_{\text{sing}} (q) = -\frac{1}{2} e^a_\alpha \hat{\xi}^-_{ab} e^{b\beta} \partial_\beta \ln \Pi_{\text{sing}}(q) + \hat{A}_\alpha^- . \tag{40}
\]

On the other hand instanton field in the regular gauge depends on \( \hat{\eta}_{\mu\nu}^+ \) and the matrix \( \Omega_+ \) must be employed:

\[
\hat{A}_\alpha^\Omega \big|_{\text{reg}} (q) = \frac{1}{2} e^a_\alpha \hat{\xi}^+_{ab} e^{b\beta} \partial_\beta \ln \Pi_{\text{reg}}(q) + \hat{A}_\alpha^+ . \tag{41}
\]

An interesting feature of the expressions (40, 41) is that they do not contain \( \Omega \) explicitly. One needs only \( \hat{A}_\alpha^\pm \) which may be found directly from (28). That is much easier than solving (36) for \( \Omega \). From here on we shall omit the superscript \( \Omega \) in \( \hat{A}_\alpha^\Omega(q) \). In fact if the sign of the topological charge is not crucial one may apply the formulae (40, 41) and (28) without checking duality properties of \( \hat{\xi} \) and \( \hat{\eta}_{\mu\nu} e^\mu_a e^\nu_b \).

The duality equation, (4a), in the non-Cartesian frame takes the form

\[
\hat{F}_{\alpha\beta} = \frac{\sqrt{g}}{2} \epsilon_{\alpha\beta\gamma\delta} \hat{F}^{\gamma\delta} \quad \text{or} \quad \hat{F}_{ab} = \frac{1}{2} \epsilon_{abcd} \hat{F}^{cd}, \tag{42}
\]

where \( g = \det ||g_{\alpha\beta}|| \). The topological charge is, (compare to (4b)),

\[
q = \frac{1}{32\pi^2} \int \epsilon_{\alpha\beta\gamma\delta} \text{tr} \hat{F}^{\alpha\beta} \hat{F}^{\gamma\delta} d^4q = \frac{1}{32\pi^2} \int \epsilon_{abcd} \text{tr} \hat{F}^{ab} \hat{F}^{cd} \sqrt{g} d^4q. \tag{43}
\]

We shall calculate the field strength in two ways. First we shall derive a general expression that works in multiinstanton case as well. Then we shall demonstrate that for one pseudoparticle the formulae do simplify both in regular and singular gauges.

First of all let us notice that we may keep the Levi-Civita connection \( \Gamma_{\alpha\beta}^\gamma \) in the covariant derivative since it will drop out of the final result. For brevity we shall denote the first addends in the right hand sides of (40, 41) by \( \hat{A}_\alpha^I \).

\[
\hat{F}_{\alpha\beta}(q) = D_\alpha \left( \hat{A}_\alpha^I + \hat{A}_\alpha^+ \right) - D_\beta \left( \hat{A}_\beta^I + \hat{A}_\beta^+ \right) + i \left[ \hat{A}_\alpha^I + \hat{A}_\alpha^+, \hat{A}_\beta^I + \hat{A}_\beta^+ \right], \tag{44}
\]

where \( D_\alpha \) is the full covariant derivative:

\[
D_\alpha \left( \hat{A}_\beta^I + \hat{A}_\beta^+ \right) = \partial_\alpha \left( \hat{A}_\beta^I + \hat{A}_\beta^+ \right) - i \left[ \hat{A}_\alpha^I + \hat{A}_\alpha^+, \hat{A}_\beta^I + \hat{A}_\beta^+ \right] - \Gamma_{\alpha\beta}^\gamma \left( \hat{A}_\gamma^I + \hat{A}_\gamma^+ \right). \tag{45}
\]
The compensating field $\hat{A}_\alpha^\pm$ is a pure gauge and does not contribute to $\hat{F}_{\alpha\beta}$. Thus it’s only role is to ensure that the factor $e^a_\alpha \xi_{ab} e^b_\beta$ is covariantly constant. Hence covariant derivatives act only on the logarithms:

$$
\hat{F}_{\alpha\beta}^I|_{\text{sing}} = \frac{1}{2} e^a_\alpha \xi_+^\pm_{ab} e^b_\beta \partial_a \ln \Pi_{\text{sing}} - \frac{1}{2} e^a_\alpha \xi_-^\pm_{ab} e^b_\beta \partial_a \ln \Pi_{\text{sing}} - i [\hat{A}^I_{\alpha}, \hat{A}^I_{\beta}],
$$

and do not contain gauge terms:

$$
D_\alpha \partial_\gamma \ln \Pi(q) = \partial_\alpha \partial_\gamma \ln \Pi(q) - \Gamma^\gamma_{\alpha\gamma} \partial_\delta \ln \Pi(q).
$$

The commutator can be calculated with the help of the identity

$$
\left[\xi_{ab}, \xi_{cd}\right] = -2i \left(\pi_{bc} \xi_{ad} + \pi_{ad} \xi_{bc} - \pi_{ac} \xi_{bd} - \pi_{bd} \xi_{ac}\right),
$$

that follows from the properties of the $SU(2)$ generators but may be proved directly. Denoting for brevity $(\partial \Pi)^2 = g^{\alpha\beta} \partial_\alpha \Pi \partial_\beta \Pi$ we obtain:

$$
\hat{F}_{\alpha\beta}^I|_{\text{sing}} = \frac{(\partial \Pi)^2}{2\Pi^2} \left(g_{\alpha\gamma} - \frac{2 \partial_\alpha \Pi \partial_\gamma \Pi}{(\partial \Pi)^2}\right) e^a_\alpha \xi_+^\pm \xi_{ab} e^b_\beta \left(g_{\delta\beta} - \frac{2 \partial_\beta \Pi \partial_\delta \Pi}{(\partial \Pi)^2}\right) + \\
+ \frac{D_\alpha \partial_\gamma \Pi}{2\Pi} e^a_\alpha \xi_+^\pm \xi_{ab} e^b_\beta \frac{D_\beta \partial_\delta \Pi}{2\Pi}.
$$

Mind that so long we have not referred to the specific form of $\Pi(q)$ and to the instanton number in particular. Thus the formula works for the multi-instanton solutions as well.

In the regular gauge (this automatically implies that the topological charge is unity) the formula looks more compact (note that now $\Pi = \Pi_{\text{reg}}$):

$$
\hat{F}_{\alpha\beta}^I|_{\text{reg}} = \frac{(\partial \Pi)^2}{2\Pi^2} e^a_\alpha \xi_+^\pm_{ab} e^b_\beta - \frac{D_\alpha \partial_\gamma \Pi}{2\Pi} e^a_\alpha \xi_+^\pm \xi_{ab} e^b_\beta - e^a_\alpha \xi_-^\pm_{ab} e^b_\beta \frac{D_\beta \partial_\delta \Pi}{2\Pi}.
$$

However it may be significantly simplified if we start from $\hat{F}_{\mu\nu}^I|_{\text{sing}}$ in Cartesian coordinates,

$$
\hat{F}_{\mu\nu}^I|_{\text{reg}} = - \frac{2 \hat{\eta}^+_{\mu\nu}}{(r^2 + \rho^2)^2},
$$

with $r^2 = x^2_\alpha$ and transform it to $q$-coordinates. Obviously the transformation affects only $\hat{\eta}^+_{\mu\nu}$. According to the $\Omega_+$ version of (36) the result is:

$$
\hat{F}_{\alpha\beta}^I|_{\text{reg}} = - \frac{2 e^a_\alpha \xi_+^\pm_{ab} e^b_\beta}{(r^2 + \rho^2)^2}, \quad \text{or} \quad \hat{F}_{ab}^I|_{\text{reg}} = - \frac{2 \xi_+^\pm_{ab}}{(r^2 + \rho^2)^2}.
$$

The last version is explicitly selfdual.

The expression (50) in the one instanton case can be simplified too. Once more we shall start from (52). Remember that Cartesian pseudoparticle in singular gauge can be obtained from that in the regular gauge by means of the transformation (11). This
results into $\hat{F}_{\mu\nu}^{I} |_{\text{sing}} = \hat{N}^{-1}_{+} \hat{F}_{\mu\nu}^{I} |_{\text{reg}} \hat{N}^{+}_{+}$. Now we should carry out the simultaneous change of variables and $\Omega_{-}$ gauge transform:

$$
\hat{F}_{\alpha\beta}^{I} |_{\text{sing}} = -\frac{\partial x^{\mu}}{\partial q^{\alpha}} \frac{\partial x^{\nu}}{\partial q^{\beta}} \Omega_{-}^{-1} \hat{N}^{-1}_{+} \frac{2\hat{\eta}_{\mu\nu}^{+}}{(r^{2} + \rho^{2})^{2}} \hat{N}^{+}_{+} \Omega_{-} .
$$

It is convenient to rewrite $\hat{N}$-matrices as: $\hat{N}^{+}_{+} = \tau^{a}_{\alpha} \partial_{\alpha} r$ and $\hat{N}^{-1}_{+} = \tau^{\dagger}_{a} \partial_{a} r$ (note that the Cartesian $g_{\mu\nu} = \delta_{\mu\nu}$). After a little algebra with the use of (36) one obtains:

$$
\hat{F}_{\alpha\beta}^{I} |_{\text{sing}} = -\hat{N}^{-1}_{+} \frac{2\hat{\xi}_{\alpha\beta}^{+}}{(r^{2} + \rho^{2})^{2}} \hat{N}^{+}_{+} ; \quad \hat{N}^{+} = \tau^{a}_{\alpha} \epsilon^{a\alpha} \partial_{\alpha} r ; \quad \hat{N}^{-1}_{+} = \tau_{a}^{\dagger} \epsilon^{a\alpha} \partial_{a} r .
$$

We note that all connections have dropped out giving the compact results (53, 55). Unfortunately for multipseudoparticle solutions one still has to apply the clumsy formula (50). Later in the Sect. 5.2 we shall describe the case when matrices $\hat{N}^{+}_{+}$ become degenerate (i.e. $\hat{N}^{+}_{+} = -i$) so that (53) and (55) coincide.

It is obvious that all the reasoning may be literally repeated for anti-instantons. The only difference is that one has to interchange $\hat{\xi}_{\alpha\beta}^{+} \leftrightarrow \hat{\xi}_{\alpha\beta}^{-}$ and $\tau_{a}^{\dagger} \leftrightarrow \tau_{a}$. The matrices $\hat{N}^{+}_{+}, \hat{N}^{-1}_{+}$ must be substituted by $\hat{N}^{-}_{-} = \hat{N}^{-1}_{+}$ and $\hat{N}^{-1}_{-} = \hat{N}^{+}_{+}$. Certainly the signs of the right hand sides of the equations (42) must be reversed as well.

## 5 Instanton in $O(4)$-spherical coordinates.

In order to show how the general theory works we shall apply it to the familiar case. Let us derive explicit formulae for one instanton placed at the origin of the 4-dimensional spherical coordinates. The example happens to be instructive and reveals two unexpected features. First, it turns out that (in this particular setting) our prescription converts pseudoparticles both in singular and regular gauges to the same form. In a sense this is an evidence in favor of the approach. Second, the compensating vector potential exhibits a singularity. From the first sight the singularity of the field that is a pure gauge is of no importance. However in presence of a pseudoparticle the singularity comes to life and contributes to the Chern-Simons number.

We begin from a general description of the 4-dimensional spherical coordinates and then focus on the pseudoparticles. Finally we present the calculation of the Chern-Simons number.

### 5.1 4-dimensional spherical coordinates.

Spherical coordinates make a natural choice for problems involving single euclidean pseudoparticle. The set of spherical coordinates includes radius and three angles: $q^{\alpha} = (r, \theta, \phi, \chi)$. The polar axis is aligned with $x^{1}$ and

$$
\begin{align*}
&x^{1} = r \cos \chi; \quad x^{2} = r \sin \chi \sin \theta \cos \phi; \quad x^{3} = r \sin \chi \sin \theta \sin \phi; \quad x^{4} = r \sin \chi \cos \theta .
\end{align*}
$$

The metric tensor and the vierbein are:

$$
\begin{align*}
g_{\alpha\beta} = \text{diag}(1, r^{2} \sin^{2} \chi, r^{2} \sin^{2} \chi \sin^{2} \theta, r^{2}); \quad e^{\alpha}_{a} = \text{diag}(1, r \sin \chi, r \sin \chi \sin \theta, r) .
\end{align*}
$$
It is convenient to introduce the matrix notation for the Levi-Civita symbols, \( \hat{\Gamma}_\alpha = ||\Gamma^\gamma_{\alpha\beta}||. \) The standard calculation gives:

\[
\hat{\Gamma}_\theta = \begin{pmatrix}
0 & -r \sin^2 \chi & 0 & 0 \\
\frac{1}{r} & 0 & 0 & \cot \chi \\
0 & 0 & \cot \theta & 0 \\
0 & -\frac{1}{2} \sin 2\chi & 0 & 0
\end{pmatrix}; \quad \hat{\Gamma}_\chi = \begin{pmatrix}
0 & 0 & 0 & -r \\
0 & \cot \chi & 0 & 0 \\
0 & 0 & \cot \chi & 0 \\
\frac{1}{r} & 0 & 0 & 0
\end{pmatrix};
\]

(58)

\[
\hat{\Gamma}_\phi = \begin{pmatrix}
0 & 0 & -r \sin^2 \chi \sin^2 \theta & 0 \\
0 & 0 & \frac{1}{r} \sin \theta & 0 \\
\frac{1}{r} \cot \theta & 0 & 0 & \cot \chi \\
0 & -\frac{1}{2} \sin 2\chi \sin^2 \theta & 0 & 0
\end{pmatrix}; \quad \hat{\Gamma}_r = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \frac{1}{r} & 0 & 0 \\
0 & 0 & \frac{1}{r} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix};
\]

By means of the relation (19) we may find the spin connection. Sticking once again to the matrix notation, \( \hat{R}_\alpha = ||R^a_{\alpha\beta}||, \) one obtains:

\[
\hat{R}_\theta = \begin{pmatrix}
0 & -\sin \chi & 0 & 0 \\
\sin \chi & 0 & 0 & \cos \chi \\
0 & 0 & 0 & 0 \\
0 & -\cos \chi & 0 & 0
\end{pmatrix}; \quad \hat{R}_\chi = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}; \quad \hat{R}_r = 0.
\]

(59)

The compensating gauge potentials \( \hat{A}_+ \) and \( \hat{A}_- \) depend on the particular choice of \( \tau \)-matrices. Let us remind that the antisymmetrized products \( \hat{\xi}_{ab} = \hat{\xi}_{ab} \) and \( \hat{\xi}_{ab} = \hat{\xi}_{ab} \) must be of correct duality so that the equations (36) and (38) were equivalent. For example, if \( \tau_x, \tau_y, \tau_z \) stand for standard Pauli matrices we can take:

\[
\tau_a = (i, \tau_z, \tau_y, \tau_x); \quad \tau_a^\dagger = (-i, \tau_z, \tau_y, \tau_x).
\]

(60)

Convolutions of the \( \tau \)-matrices (60) with the spin connection, (28), result into the compensating gauge fields:

\[
\hat{A}_\theta^\pm = -\frac{\tau_y \cos \chi \mp \tau_z \sin \chi}{2}, \quad \hat{A}_\chi^\pm = \mp \frac{\tau_x}{2}; \quad \hat{A}_\phi^\pm = -\frac{\tau_y}{2} \cos \theta \mp \frac{\tau_x}{2} \sin \chi \sin \theta + \frac{\tau_z}{2} \cos \chi \sin \theta; \quad \hat{A}_r^\pm = 0;
\]

(61)

Either of these fields is a pure gauge generated by the corresponding unitary (\( \Omega_{\pm}^{-1} = \Omega_{\pm}^1 \)) matrix:

\[
\Omega_{\pm} = \begin{pmatrix}
\sin \frac{\theta}{2} \sin \frac{\phi \pm \chi}{2} & i \cos \frac{\theta}{2} \cos \frac{\phi \pm \chi}{2} & -\cos \frac{\theta}{2} \sin \frac{\phi \pm \chi}{2} & i \sin \frac{\theta}{2} \cos \frac{\phi \pm \chi}{2} \\
\cos \frac{\theta}{2} \sin \frac{\phi \pm \chi}{2} & i \sin \frac{\theta}{2} \cos \frac{\phi \pm \chi}{2} & \sin \frac{\theta}{2} \sin \frac{\phi \pm \chi}{2} & -i \cos \frac{\theta}{2} \cos \frac{\phi \pm \chi}{2}
\end{pmatrix}.
\]

(62)

The matrices \( \Omega_+ \) and \( \Omega_- \) for selfdual and antiselfdual cases differ by the sign of the polar angle \( \chi \).

The compensating connection is singular since neither \( \hat{A}_\theta^\pm \) nor \( \hat{A}_\phi^\pm \) go to zero near polar axes \( \chi = 0 \) and \( \theta = 0 \). As long as the field is a pure gauge this singularity is not observable. However as we shall see below in presence of physical fields it may tell on gauge variant quantities.
5.2 Instantons in 4-spherical coordinates.

We shall consider instantons in singular and regular gauges with their centers at the origin. The analysis reveals an amusing coincidence. Remember that instanton gauge potentials in these two cases involve different \( \hat{\xi}^+_{ab} \) and \( \hat{\xi}^-_{ab} \). Hence according to our prescription the compensating vector potentials are \( \hat{A}^+ \) and \( \hat{A}^- \) respectively. It turns out that if the \( \tau \)-matrices are taken in the form (60) then equation (40) for the singular gauge and that (41) for the regular one give the same result. Let us take \( \Pi_{\text{reg}} \) and \( \Pi_{\text{sing}} \) (5) in the form:

\[
\Pi_{\text{reg}}(r) = r^2 + \rho^2; \quad \text{and} \quad \Pi_{\text{sing}}(r) = 1 + \rho^2/r^2. \tag{63}
\]

Substituting those into the equations (40, 41) with the compensating potentials given by (61) we obtain:

\[
A^\pm_r = 0; \tag{64a}
\]

\[
A^\pm_\theta = -\frac{\tau_\nu}{2} \cos \chi \pm \frac{\tau_z}{2} \sin \chi \left( 1 - \frac{2\rho^2}{r^2 + \rho^2} \right); \tag{64b}
\]

\[
A^\pm_\phi = -\frac{\tau_\nu}{2} \cos \theta \pm \frac{\tau_y}{2} \sin \chi \sin \theta \left( 1 - \frac{2\rho^2}{r^2 + \rho^2} \right) + \frac{\tau_z}{2} \cos \chi \sin \theta; \tag{64c}
\]

\[
A^\pm_\chi = \pm \frac{\tau_z}{2} \left( 1 - \frac{2\rho^2}{r^2 + \rho^2} \right); \tag{64d}
\]

Note that the potentials \( A^\pm \) interpolate between \( \hat{A}^\pm \) at the origin and \( \hat{A}^\mp \) at infinity:

\[
\lim_{r \to 0} A^\pm_a(q) = \hat{A}^\pm_a(q); \quad \text{and} \quad \lim_{r \to \infty} A^\pm_a(q) = \hat{A}^\mp_a(q). \tag{65}
\]

The field strength is given by the expression (53). The comparison of the latter with (55) proves that as a rule regular and singular gauges are different. However in present case the matrices \( N^\pm = \mp i \) and the transform is trivial. The degeneracy is specific to our choice of coordinates. Note that we took \( q^1 = r \) and associated with it \( \tau_1 = i \). According to (55) these are necessary and sufficient conditions of the coincidence of the two gauges in curvilinear coordinates. Hence this coincidence is stable with respect to reparametrizations of the angles \( \theta, \phi, \chi \) and \( \tau \)-matrices as long as \( r \) and \( \tau_1 \) stay intact.

5.3 Singularity and the Chern-Simons number.

It is well known that the topological charge density, (43), may be represented as a total divergence:

\[
\frac{\epsilon_{\alpha\beta\gamma\delta}}{32\pi^2} \text{tr} \hat{F}^{\alpha\beta} \hat{F}^{\gamma\delta} = \partial_\alpha K^\alpha = \partial_\alpha \frac{\epsilon_{\alpha\beta\gamma\delta}}{16\pi^2} \text{tr} \left( \hat{A}_\beta \hat{F}_{\gamma\delta} + \frac{2i}{3} \hat{A}_\beta \hat{A}_\gamma \hat{A}_\delta \right). \tag{66}
\]

According to the Gauss theorem the space integral of the LHS may be reduced to the surface integral of \( K^\alpha \) over the boundary. However the current \( K^\alpha \) is not gauge invariant. Thus the distribution of \( K^\alpha \) over the boundary depends on gauge even though the topological charge \( q = \oint K^\alpha dS_\alpha \) does not. A text book example is an instanton in the \( \hat{A}_4 = 0 \) gauge. Here the two nonzero contributions to \( q \) come from the hyperplanes \( x_4 = \pm \infty \), \( (i, j, k = 1, 2, 3) \):

\[
q(\hat{A}_4 = 0) = \oint K^\alpha dS_\alpha = -\int \frac{d^3\vec{x}}{16\pi^2} \epsilon^{ijk} \text{tr} \left( \hat{A}_i \hat{F}_{jk} + \frac{2i}{3} \hat{A}_i \hat{A}_j \hat{A}_k \right)|_{x_4=+\infty} - \int \frac{d^3\vec{x}}{16\pi^2} \epsilon^{ijk} \text{tr} \left( \hat{A}_i \hat{F}_{jk} + \frac{2i}{3} \hat{A}_i \hat{A}_j \hat{A}_k \right)|_{x_4=-\infty}. \tag{67}
\]
The integral $Q_0(x_4) = \int d^3 \vec{x} K^4(x_4)$ over a 3-dimensional manifold is called the Chern-Simons number. In the example above $Q_0(\pm \infty) = \pm \frac{1}{2}$ leading to $q = 1$. Thus the topological charge may be interpreted as the change of the Chern-Simons number.

Let us try to carry out the same procedure for the gauge field (64) in spherical coordinates. At the first sight there is a striking similarity with the $\hat{A}_\phi$ now $\mathcal{A}_r = 0$ and the instanton interpolates between $\hat{A}^+ \to \hat{A}^-$ at $r = \infty$. It would be nice to take the radius (or rather $\rho \ln r/\rho$) for the 4-th coordinate and to associate the Chern-Simons number $Q(r) = \oint_{S^3_r} K^r d\theta d\chi d\phi$ with the sphere $S^3_r$ of the radius $r$. The analogy would be complete provided that one had the contributions $Q(\infty) = \frac{1}{2}$ from the infinite sphere and $Q(0) = -\frac{1}{2}$ from the infinitesimal sphere at the origin. However the explicit calculation proves that this is not the case because the singular line $\theta = 0$ carries the Chern-Simons number as well.

The topological charge in 4-spherical coordinates is given by the integral over a half infinite parallelepiped $\{0 \leq r \leq \infty, 0 \leq \chi \leq \pi, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi\}$:

$$q = \frac{1}{32\pi^2} \int_0^\infty dr \int_0^\pi d\chi \int_0^\pi d\theta \int_0^{2\pi} d\phi \epsilon_{\alpha\beta\gamma\delta} tr \hat{F}^{\alpha\beta} \hat{F}^{\gamma\delta}.$$  

(68)

When applying the Gauss theorem one must take into account the entire boundary including lateral faces. The components of the Chern-Simons current are:

$$K^r = \frac{\sin \theta}{2\pi^2} \left(1 - \frac{2\rho^2}{r^2 + \rho^2}\right) \left(\frac{1}{8} + \frac{\rho^2 r^2 \sin^2 \chi}{(r^2 + \rho^2)^2}\right);$$  

(69a)

$$K^\chi = -\frac{\rho^2 r \sin 2\chi \sin \theta}{4\pi^2 (r^2 + \rho^2)^2};$$  

(69b)

$$K^\theta = -\frac{\rho^2 r \cos \theta}{4\pi^2 (r^2 + \rho^2)^2}; \quad K^\phi = 0.$$  

(69c)

Converting (68) to the surface integral and keeping only nonzero pieces we write:

$$q = \int_0^\pi d\chi \int_0^\pi d\theta \int_0^{2\pi} d\phi K^r \bigg|_{r=\infty}^{r=0} + \int_0^\infty dr \int_0^\pi d\chi \int_0^{2\pi} d\phi K^\theta \bigg|_{\theta=\pi}^{\theta=0}.$$

(70)

The first addend in the RHS is $Q(\infty) - Q(0)$ and the second comes from the integration along the “hyperstrips” $\theta = 0, \pi$. Each of the four addends contributes $\frac{1}{4}$ to the final value $q = 1$. We see that the singularity of the compensating gauge field is not absolutely harmless. The reason for it’s coming to being was that $K^\alpha$ is not a gauge invariant.

It is easy to see that $K^\theta = \epsilon^{\theta\phi\chi\gamma} tr \hat{A}_\phi \hat{F}_{\gamma\chi}/8\pi^2$. Multiplication of the singular gauge potential $\hat{A}_\phi$ by the instanton field $\hat{F}_{\gamma\chi}$ results into the singularity of $K^\theta$. Thus the singularity of the seemingly unobservable gauge field $\hat{A}^\pm$ leads to nonzero consequences in presence of the physical field $\hat{A}^\prime$. In general singular gauge transformations may generate singularities of gauge variant quantities. Obviously this is specific neither to the instanton nor to the current choice of the $O(4)$-spherical coordinates.

---

3In regular gauge, (5a), the Chern-Simons number is concentrated at infinity $q = Q_r(\infty) = 1$, in contrast to the singular one, (5b), where $q = -Q_s(0) = 1$. 

13
6 Cylindrical coordinates

Let us describe two other coordinate systems that are relevant to physics. We shall start from 2 + 2 cylindrical coordinates, i.e. the geometry of the thick euclidean QCD-string or vortex. Then we shall consider 3 + 1 cylindrical coordinates which are characteristic for the bag model or a glueball.

6.1 2+2 cylindrical coordinates.

These coordinates may be appropriate for objects with axial symmetry, such as strings, vortices or quark-antiquark pairs. We parametrize the $x^1x^2$-plane by polar coordinates $q^1 = r$ and $q^2 = \phi$ and leave $q^3,4 = x^3,4$:

$$x^1 = r \cos \phi; \quad x^2 = r \sin \phi; \quad x^3 = z; \quad x^4 = t.$$  \hspace{1cm} (71)

The metric tensor and the vierbein are respectively:

$$g_{\alpha\beta} = \text{diag} \left( 1, r^2, 1, 1 \right) \quad \text{and} \quad e^\alpha_\alpha = \text{diag} \left( 1, r, 1, 1 \right).$$  \hspace{1cm} (72)

Only three of the Levi-Civita symbols, (15), are not zero:

$$\Gamma^r_{\phi\phi} = -r \quad \text{and} \quad \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = r^{-1}.$$  \hspace{1cm} (73)

The direct calculation (see equation (19)) proves that the spin connection has only one nonzero component. Namely:

$$\hat{\mathbf{R}}_\phi = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \hat{\mathbf{R}}_r = \hat{\mathbf{R}}_z = \hat{\mathbf{R}}_t = 0.$$  \hspace{1cm} (74)

Convolutions of the $\tau$-matrices (60) with the spin connection $R$, (28), give the compensating fields that are singular at $r = 0$:

$$\hat{A}^\pm_\phi = \mp \frac{\tau_z}{2}; \quad \hat{A}^\pm_r = \hat{A}^\pm_z = \hat{A}^\pm_t = 0.$$  \hspace{1cm} (75)

Finally, for completeness we present explicitly the unitary gauge matrices $\Omega_+$ and $\Omega_-$. \hspace{1cm} (66)

$$\Omega_\pm = \frac{1}{\sqrt{2}} \begin{pmatrix} \exp \pm \frac{i \phi}{2} - \exp \mp \frac{i \phi}{2} \\ \exp \pm \frac{i \phi}{2} \exp \mp \frac{i \phi}{2} \end{pmatrix}; \quad \Omega^{-1}_\pm = \Omega^\dagger_\pm.$$  \hspace{1cm} (76)

The difference between the matrices $\Omega_+$ and $\Omega_-$ is in the sign of the polar angle $\phi$ (corresponding to the ‘left’ and ‘right’ rotations).

6.2 3+1 cylindrical coordinates.

This geometry is characteristic for objects that are spherically symmetrical in 3-dimensions. In addition to the famous MIT-bag and the glueball one may list the monopole and the caloron that is a periodic instanton chain along the 4th axis. It takes the place
of the instanton in thermal problems, [9]. Now we parametrize the “spatial” sector, i. e. $x^1, x^2, x^3$ by spherical coordinates $q^1 = r$, $q^2 = \theta$ and $q^3 = \phi$ leaving $q^4 = x^4$.

$$x^1 = r \sin \theta \cos \phi; \quad x^2 = r \sin \theta \sin \phi; \quad x^3 = r \cos \theta; \quad x^4 = t. \quad (77)$$

The metric tensor and the vierbein are well known:

$$g_{\alpha \beta} = \text{diag}(1, r^2, r^2 \sin^2 \theta, 1); \quad e^a_\alpha = \text{diag}(1, r, r \sin \theta, 1). \quad (78)$$

Since the transformation (77) affects only spatial sector the $t$-components, $\hat{\Gamma}_t = 0$, and $\Gamma$’s are essentially 3-dimensional:

$$\hat{\Gamma}_r = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{r} & 0 \\ 0 & 0 & \frac{1}{r} \end{pmatrix}; \quad \hat{\Gamma}_\theta = \begin{pmatrix} 0 & -r & 0 \\ \frac{1}{r} & 0 & 0 \\ 0 & 0 & \cot \theta \end{pmatrix}; \quad \hat{\Gamma}_\phi = \begin{pmatrix} 0 & 0 & -r \sin^2 \theta \\ 0 & 0 & -\frac{1}{2} \sin 2\theta \\ \frac{1}{\cot \theta} & 0 & 0 \end{pmatrix}. \quad (79)$$

The nonzero components of the spin connection are:

$$\hat{R}_\theta = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \hat{R}_\phi = \begin{pmatrix} 0 & 0 & -\sin \theta & 0 \\ 0 & 0 & -\cos \theta & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \hat{R}_r = \hat{R}_t = 0. \quad (80)$$

We shall use the same set of $\tau$-matrices as before, (60). This leads to the following compensating connections:

$$\hat{A}_\theta^\pm = \mp \frac{\tau_z}{2}; \quad \hat{A}_\phi^\pm = -\frac{\tau_x}{2} \cos \theta \mp \frac{\tau_y}{2} \sin \theta; \quad \hat{A}_r^\pm = \hat{A}_t^\pm = 0. \quad (81)$$

This vector potential is singular at $\theta = 0$. The gauge matrices generating the above connections are:

$$\Omega_{\pm} \frac{1}{\sqrt{2}} \begin{pmatrix} \exp \frac{i}{2} \left( \mp \theta - \phi - \frac{\pi}{2} \right) & -\exp \frac{i}{2} \left( \pm \theta - \phi - \frac{\pi}{2} \right) \\ \exp \frac{i}{2} \left( \mp \theta + \phi + \frac{\pi}{2} \right) & \exp \frac{i}{2} \left( \pm \theta + \phi + \frac{\pi}{2} \right) \end{pmatrix}; \quad \Omega_{\pm}^{-1} = \Omega_{\pm}^\dagger \quad (82)$$

We note again that $\Omega_+$ and $\Omega_-$ differ by the sign of the phase $\theta$.

**Conclusion**

Our purpose was to analyze exact multiinstanton solutions in non-Cartesian coordinates. We showed that ’t Hooft’s $5N$-parametric Ansatz can be economically generalized to curvilinear coordinates. The $\tilde{\eta}_{\mu\nu}$-symbols with coordinate indices are replaced by the $\hat{\xi}_{ab}$-symbols with the vierbein ones. The price for $\hat{\xi}_{ab}$ being constant is the appearance of the compensating gauge field. The origin of the latter is the coordinate dependent gauge transformation. Thus the proposed solution is a gauge rotated version of the original ’t Hooft’s Ansatz.

The compensating gauge potential may be obtained in a straightforward manner. The calculation proceeds in three steps.

1. Starting from the metrics $g_{\alpha\beta}$ one may find the Levi-Civita connections $\Gamma^\alpha_{\beta\gamma}$, (15).
2. Covariant differentiation of the vierbein $e^a_\alpha$ leads to the spin connection $R^a_\alpha\beta$, (19).

3. Finally the convolution of the spin connection with $\tau$-matrices gives the compensating gauge potential, (28).

The first two points are the standard calculation of the spin connection. The last one is nothing but projecting the selfdual (or antiselfdual) component of the antisymmetric tensor $R^{ab}_\alpha$ onto the $SU(2)$ gauge group.

The gauge potential of a pseudoparticle is a sum of the instanton part that is similar to 'tHooft’s formula and the compensating field, (40, 41). Having constant $\xi_{ab}$, (25), and covariantly constant $\hat{\xi}_{\alpha\beta}$-symbols, (26), notably simplifies calculations and is worth the appearance of the additive compensating background. Singularities of the compensating connection do not spoil physical quantities.

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**Appendix. Duality properties of the $\hat{\xi}_{\mu\nu}$-symbols.**

It was mentioned in Section 3.2 that $\Omega e^a_\mu e^b_\nu \hat{\xi}^+_{ab} \Omega^{-1}$ is a constant selfdual (antiselfdual for $\hat{\xi}_{ab}$) tensor, see (35). We shall prove the second part of this statement. First let us remind how coordinate changes $x^\mu \rightarrow q^\alpha$ affect the Levi-Civita antisymmetric pseudotensors. The change of variables leads to the substitution:

$$
\epsilon^{\mu\nu\lambda\sigma} \rightarrow E^{\alpha\beta\gamma\delta} = \frac{\partial q^\alpha}{\partial x^\mu} \frac{\partial q^\beta}{\partial x^\nu} \frac{\partial q^\gamma}{\partial x^\lambda} \frac{\partial q^\delta}{\partial x^\sigma} \epsilon^{\mu\nu\lambda\sigma} = \det \left| \frac{\partial q^\alpha}{\partial x^\mu} \right| \epsilon^{\alpha\beta\gamma\delta} = (\pm) \frac{\epsilon^{\alpha\beta\gamma\delta}}{\sqrt{g}},
$$

(83)

where $g = |\det \|g_{\alpha\beta}\||$ and $\epsilon^{\alpha\beta\gamma\delta}$ is the ordinary $\epsilon$-symbol ($\epsilon^{1234} = 1$). The sign of the last expression depends on the relative orientation of the $q$ and $x$ coordinate systems. We shall assume that the sign is plus.

The second formula relates $\epsilon$-symbol with coordinate (greek) and vierbein (latin) indices:

$$
\epsilon^{\alpha\beta\gamma\delta} e^a_\alpha e^b_\beta e^c_\gamma e^d_\delta = \det ||e^a_\alpha|| \epsilon^{abcd} = (\pm) \sqrt{g} \epsilon^{abcd} \quad \text{and} \quad \sqrt{g} \epsilon^{abcd} e^a_\alpha e^b_\beta e^c_\gamma e^d_\delta = (\pm) \epsilon^{\alpha\beta\gamma\delta}.
$$

(84)

Now the $(\pm)$-option is associated with the orientation of the vierbein. We shall stick to the plus sign again.

Let us turn to properties of $\Omega e^a_\mu e^b_\nu \hat{\xi}^+_{ab} \Omega^{-1}$. The gauge rotation respects duality and we may omit the $\Omega$, $\Omega^{-1}$ matrices. Let us calculate the dual of $e^a_\mu e^b_\nu \hat{\xi}^+_{ab}$:

$$
\frac{1}{2} \epsilon^{\mu\nu\lambda\sigma} e^a_\lambda e^b_\sigma \hat{\xi}^+_{ab} = \frac{1}{2} \epsilon^{\mu\nu\lambda\sigma} \frac{\partial q^\gamma}{\partial x^\lambda} \frac{\partial q^\delta}{\partial x^\sigma} \frac{\partial q^\alpha}{\partial x^\mu} \frac{\partial q^\beta}{\partial x^\nu} \hat{\xi}^{\gamma\delta}_{\sigma\tau}.
$$

(85)

With the help of the equation (83) one can show that:

$$
\frac{1}{2} \epsilon^{\mu\nu\lambda\sigma} e^a_\lambda e^b_\sigma \hat{\xi}^+_{ab} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} \frac{\partial q^\mu}{\partial x^\alpha} \frac{\partial q^\nu}{\partial x^\beta} \epsilon^{\gamma\delta}_{\sigma\tau} \hat{\xi}^+_{\sigma\tau}.
$$

(86)
and then using (84) and the identity $e^a_\alpha e^a_\beta = \delta^a_\beta$ one arrives at:

$$\frac{1}{2} e^{\alpha \beta \gamma \delta} \partial_\mu \partial_{\nu} e^c_\gamma e^d_\delta \hat{\xi}^+_{cd} = \frac{1}{2} \frac{\partial x^\mu}{\partial q^\alpha} e^a_\alpha \frac{\partial x^\nu}{\partial q^\beta} e^\beta e^{abcd} \hat{\xi}^+_{cd} = e^a_\mu e^b_\nu \hat{\xi}^+_{ab}. \quad (87)$$

This proves that $e^a_\mu e^b_\nu \hat{\xi}^+_{ab}$ is selfdual (and $e^a_\mu e^b_\nu \hat{\xi}^-_{ab}$ is antiselfdual) provided that the vierbein and the Cartesian coordinates are oriented in the same way.

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