Solvability Conditions and General Solution of a System of Matrix Equations Involving $\eta$-Skew-Hermitian Quaternion Matrices

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1. Introduction

Throughout this article, the symbol $\mathbb{R}$ represents the field of real numbers and $\mathbb{C}$ represents the field of complex numbers. The quaternion algebra is represented by $\mathbb{H}$ and is defined as

$$\mathbb{H} = \{ h_0 + h_1i + h_2j + h_3k \mid i^2 = j^2 = k^2 = ijk = -1, h_0, h_1, h_2, h_3 \in \mathbb{R} \}.$$ 

The collection of matrices of size $m \times n$ with entries from $\mathbb{H}$ is denoted by $\mathbb{H}^{m \times n}$. The rank and the conjugate transpose of $A \in \mathbb{H}^{m \times n}$ are denoted by $r(A)$ and $A^*$, respectively. The Moore–Penrose inverse of $A \in \mathbb{H}^{m \times n}$ is denoted by $A^+ = W$, which uniquely satisfies the following system of matrix equations:

$$A W A = A, \ W A W = W, (A W)^* = A W, (W A)^* = W A.$$ 

The Moore–Penrose inverse and the other generalized inverses of matrices are extensively studied in the literature; see [1,2] and the references therein. The left and the right projectors with respect to $A$ are denoted by $L_A = I - A^+ A$ and $R_A = I - A A^+$, respectively. Clearly, $L_A = L_A^* = L_A^2$ and $R_A = R_A^* = R_A^2$. The concept of the quaternions was introduced by William Rowan Hamilton in [3]. Quaternions found tremendous applications in computation, geometry, and algebra [4–7]. The use of quaternions in the field of computer graphics was initiated by Shoemake in [8]. Applications of quaternion matrices in control engineering, mechanics, and other areas of applied sciences can be found in [9–12] among others.

Numerous problems in diverse areas of sciences and engineering can be modeled by linear and nonlinear matrix equations. The discrete-time Ricatti matrix equations [13], the Lyapunov matrix equation [14], and the Sylvester matrix equation [15–18] are widely used in control theory and stability analysis of control systems. For a detailed review of the
solutions of linear matrix equations and their applications, see [19–30] and the references therein.

Solutions of matrix equations with some special structures, properties, and symmetry have been studied by numerous researchers [31–38]. For instance, Khatri and Mitra in [39] and Groß in [40] derived some necessary and sufficient conditions for the existence of a Hermitian solution for an important class of matrix equations. Hermitian solutions of a split quaternion matrix equation were discussed in [41]. A matrix $A \in \mathbb{H}^{n \times n}$ is known to be $\eta$-Hermitian and $\eta$-skew-Hermitian [42–44] if $A = A^{\eta} = -\eta A^* \eta$ and $A = -A^{\eta} = \eta A^* \eta$, respectively, where $\eta \in \{i, j, k\}$. Applications of these matrices in different areas can be found in [43,45,46]. The singular value decomposition of the $\eta$-Hermitian matrix was derived in [42]. Yuan and Wang [44] evaluated the least squares $\eta$-Hermitian and $\eta$-skew-Hermitian solutions of

$$AXB + CXD = E$$  \hspace{1cm} (1)

Liu [47] explored the $\eta$-anti-Hermitian solutions for

$$BYB^{\eta} + CZC^{\eta} = D$$  \hspace{1cm} (2)

In [48], A. Rehman et al. computed the general solution of

$$A_4X - (A_4X)^{\eta} + B_4YB_4^{\eta} + C_4ZC_4^{\eta} = D_4$$  \hspace{1cm} (3)

over $\mathbb{H}$. An efficient iterative algorithm for computing the $\eta$-Hermitian and $\eta$-anti-Hermitian least squares solutions of the matrix equation

$$AXB + CYD = E$$  \hspace{1cm} (4)

over the skew field of quaternions was provided by Beik and Ahmadi-Asl [49].

Motivated by the recent findings and the applications of $\eta$-skew-Hermitian matrices, we study the $\eta$-skew-Hermitian matrices as solutions of

$$A_1X = D_1,$$
$$A_2Y = C_2, \quad YB_2 = C_3, \quad Y = -Y^{\eta},$$
$$A_3Z = C_4, \quad ZB_3 = C_5, \quad Z = -Z^{\eta},$$
$$A_4X - (A_4X)^{\eta} + B_4YB_4^{\eta} + C_6ZC_6^{\eta} = D_4,$$  \hspace{1cm} (5)

when this system is consistent.

The rest of the paper is organized as follows. In Section 2, we present some known facts and findings that are crucial in the establishment of our key result. Some practical conditions that are necessary and sufficient for the existence of the general solution $(X, Y, Z)$ to (5), where $Y$ and $Z$ are $\eta$-skew-Hermitian, are given in Section 3. Section 3 also contains some particular cases of system (5) that recover few well-known results as corollaries. In Section 4, we provide an algorithm and a practical example to validate our main result. The conclusion of the article is provided in Section 5.

2. Preliminaries

We start with known results that are essential for developing our main result.

Lemma 1 ([50]). Let $D \in \mathbb{H}^{s \times t}$, $E \in \mathbb{H}^{s \times k}$ and $F \in \mathbb{H}^{l \times t}$ be given. Then,

1. $r(D) + r(R_D E) = r(E) + r(R_E D) = r \begin{bmatrix} D & E \end{bmatrix}$.
2. $r(D) + r(FL_D) = r(F) + r(DL_E) = r \begin{bmatrix} D & F \end{bmatrix}$.
3. $r(E) + r(F) + r(R_E DL_F) = r \begin{bmatrix} D & E & 0 \end{bmatrix}$. 
Lemma 2 ([51]). Let $A_1$ and $A_2$ be known matrices of suitable sizes. Then, $A_1 X = A_2$ is solvable if and only if $A_2 = A_1 A_1^* A_2$. In this case, the general solution is

$$X = A_1^* A_2 + L_{A_1} U_1,$$

where $U_1$ is any arbitrary matrix of suitable size.

Lemma 3 ([47]). Let $G$ and $H$ be given matrices over $\mathbb{H}$. Then, the system $G X = H$ will have an $\eta$-Skew-Hermitian solution if and only if we have $R_G H = 0$ and $G H^\eta_\ast = -G^\eta_\ast$. In this case, the $\eta$-Hermitian general solution is given by the formula

$$X = G^\eta - (G^\eta H)^{\eta_\ast} + G^\eta G(G^\eta H)^{\eta_\ast} + L_G V L_G^\eta_\ast,$$

where $V = -V^\eta_\ast$ is any arbitrary matrix with a suitable size.

Lemma 4 ([52]). Let $B \in \mathbb{H}^{m \times n}$ be given. Then,

1. $(B^\eta)^\dagger = (B^\eta)^\eta_\ast, (B^\eta_\ast)^\dagger = (B^\eta)^\eta_\ast$.
2. $r(B) = r(B^\eta_\ast) = r(B^\eta) = r(B^\eta B^\eta_\ast) = r(B^\eta_\ast B^\eta)$.
3. $(B B^\eta)^\eta_\ast = (B_\eta)^\eta_\ast B^\eta_\ast = (B^\eta)^\eta_\ast B^\eta_\ast$.
4. $(L_B)^{\eta_\ast} = -\eta(L_B)\eta = (L_B)^\eta = L_B = R_{B^\eta}$.
5. $(R_B)^{\eta_\ast} = -\eta(R_B)^\eta = (R_B)^\eta = L_B = R_{B^\eta}$.

Lemma 5 ([53]). For matrices $C$, $B$, and $A$, we have

1. $(A^* A)^\dagger A^* A = A^* (A A^* )^\dagger$.
2. $L_A = L_A^2 = L_A^\ast, R_A = R_A^2 = R_A^\ast$.
3. $L_A (B L_A)^\dagger = (B L_A)^\dagger, (R_A C)^\dagger = (R_A C)^\dagger$.

Lemma 6 ([48]). Let $A_4$, $B_4$, $C_4$ and $D_4 = D_4^\eta_\ast$ be matrices, as given in (3). Suppose that

$$A = R_{A_4} B_4, B = R_{A_4} C_4, C = R_{A_4} D_4 (R_{A_4})^{\eta_\ast}, M = R_A B, S = BL_M.$$

Then, the equivalent conditions are the following:

1. The linear system in (3) possesses a triplet solution of the form $(X, Y, Z)$, and the matrices $Z\ast$ and $Y$ are $\eta$-skew-Hermitian.

2. $R_M R_A C = 0, R_A C R_B^\eta_\ast = 0$.

(3)

$$r \begin{bmatrix} D_4 & A_4 & B_4 & C_4 \\ A_4^{\eta_\ast} & 0 & 0 & 0 \end{bmatrix} = r \begin{bmatrix} A_4 & B_4 & C_4 \end{bmatrix} + r(A_4),$$

$$r \begin{bmatrix} D_4 & A_4 & B_4 \\ A_4^{\eta_\ast} & 0 & 0 \end{bmatrix} = r \begin{bmatrix} A_4 & B_4 \end{bmatrix} + r \begin{bmatrix} A_4 & C_4 \end{bmatrix}. $$
In this case, \( X, Y^\eta = -Y, \) and \( Z = -Z^\eta \) can be given as follows:

\[
X = A_4^1[D_A - B_4 Y B_4^{\eta^*} - C_4 Z C_4^{\eta^*}] - \frac{1}{2} A_4^1[D_A - B_4 Y B_4^{\eta^*} - C_4 Z C_4^{\eta^*}] (A_4^1)^{\eta^*} A_4^{\eta^*} + U_2^{\eta^*} (A_4^1)^{\eta^*} A_4^{\eta^*} + A_4^1 U_2 A_4^{\eta^*} - L_{A_4} U_1, \\
Y = Y^{\eta^*} = A_4^1 C (A_4^1)^{\eta^*} - \frac{1}{2} A_4^1 B M^1 C [I + (B_4^*)^{\eta^*} S_6^*] (A_4^1)^{\eta^*} - \frac{1}{2} A_4^1 I + S B^* \\
\times C (M^1)^{\eta^*} B = (A^1)^{\eta^*} - A_4^1 S U_6^* S^* - L_{A_4} U_4 + U_4^* L_{A_4}^{\eta^*}, \\
Z = Z^{\eta^*} = \frac{1}{2} M^1 C (B^*)^{\eta^*} [I + (S_4^1 S)^{\eta^*}] + \frac{1}{2} (I + S_4^1 S) B^* C (M^1)^{\eta^*} + L_M U_6^{\eta^*} + U_5 L_M U_5 + L_B U_5^{\eta^*} + L_M L_U U_3 - (L_M L_U U_3)^{\eta^*},
\]

where \( U_1, \cdots, U_5 \) and \( U_6 = -U_6^{\eta^*} \) are arbitrary matrices.

Lemma 6 has the main contribution to proving the key theorem of this paper.

3. Main Results

In this section, we present our main results. Our main result provides necessary and sufficient conditions for the system of matrix Equation (5) to possess a triplet solution \((X, Y, Z)\) with \( Y = -Y^{\eta^*} \) and \( Z = -Z^{\eta^*} \). The formula for computing the general solution is also given provided that the solution exists. Afterwards, we discuss some special cases of our main theorem and recover some known results from the literature as corollaries.

**Theorem 1.** Suppose that \( A_1, \cdots, A_4, B_2, B_3, B_4, C_2, \cdots, C_6, D_1, \) and \( D_4 = -D_4^{\eta^*} \), are given matrices. Suppose that

\[
A_5 = \begin{bmatrix} A_2 \\ B_2^{\eta^*} \end{bmatrix}, \quad A_6 = \begin{bmatrix} A_3 \\ B_3^{\eta^*} \end{bmatrix}, \quad B_5 = \begin{bmatrix} C_2 \\ C_3^{\eta^*} \end{bmatrix}, \quad B_6 = \begin{bmatrix} C_4 \\ C_5^{\eta^*} \end{bmatrix},
\]

\[
A_7 = A_4 L_{A_4}, \quad A_8 = B_4 L_{A_4}, \quad B_8 = C_6 L_{A_6}, \\
C_8 = D_4 - A_4 (A_4^1 D_1) + [A_4 (A_4^1 D_1)]^{\eta^*} - B_4 [A_4^1 B_5] (A_4^1 D_1)^{\eta^*} \\
+ A_4^3 A_5 (A_4^1 D_1)^{\eta^*} B_8^{\eta^*} - C_6 [A_4^3 B_5 B_8^{\eta^*} - (A_4^3 B_5)^{\eta^*} + A_4^3 A_5 (A_4^1 D_1)^{\eta^*} C_6^{\eta^*}, \\
A = R_{A_2} A_8, \quad B = R_{A_2} B_8, \quad C = R_{A_2} C_6 (R_{A_7})^{\eta^*}, \quad M = R_{A_2} B, \quad S = B L_{M}.
\]

Then, the statements given below are equivalent:

1. The linear system (5) has a solution of the form \((X, Y, Z)\) with \( Y = -Y^{\eta^*} \) and \( Z = -Z^{\eta^*} \).
2. The matrices involved in (5) must satisfy:

\[
R_{A_4} D_1 = 0, \quad R_{A_4} B_8 = 0, \quad A_5 B_5^{\eta^*} = -B_5 A_5^{\eta^*}, \quad R_{A_6} B_6 = 0.
\]

\[
A_6 B_6^{\eta^*} = -B_6 A_6^{\eta^*}, \quad R_{A_4} C R_6^{\eta^*} = 0, \quad R_{M} R_{A_2} C = 0.
\]

(3)

\[
D_1 = A_1 A_4^1 D_1, \quad B_5 = A_5 A_5^1 B_5, \quad A_5 B_5^{\eta^*} = -B_5 A_5^{\eta^*}, \quad B_6 = A_6 A_6^1 B_6, \\
A_6 B_6^{\eta^*} = -B_6 A_6^{\eta^*}, \quad M M^1 R_{A_2} C = R_{A_2} C = R_{A_2} C (B^*)^{\eta^*} B^{\eta^*}.
\]
In this case, the general solution to the system (5) is given by

\[
X = A_1^2 D_1 + L_{A_1} X_1, \quad Y = A_2^2 B_2 - (A_2^k B_2)^\eta A_5^\eta + A_5^\eta A_5 - B_3 Z_1 B_3^\eta A_5^\eta + L_{A_5} Y_1 L_{A_5}^\eta, \quad Z = A_3^2 B_3 - (A_3^k B_3)^\eta A_6^\eta + A_6^\eta A_6 - B_4 Z_1 B_4^\eta A_6^\eta + L_{A_5} Z_1 L_{A_5}^\eta,
\]

with

\[
X_1 = A_4^\eta (C_8 - A_8 Y_1 A_8^\eta - B_8 Z_1 B_8^\eta) - \frac{1}{2} A_4^\eta (C_8 - A_8 Y_1 A_8^\eta - B_8 Z_1 B_8^\eta) (A_4^\eta)^\eta A_7^\eta - L_{A_4} U_1 + U_2^\eta (A_4^\eta)^\eta A_7^\eta + A_4^\eta U_2 A_7^\eta^\eta,
\]

\[
Y_1 = -Y_1^\eta = A^T C (A^\dagger)^\eta + \frac{1}{2} A^T B M^T C [I + (B^\dagger)^\eta S^\eta] (A^\dagger)^\eta - \frac{1}{2} A^T [I + S B^T] C (M^\dagger)^\eta B^\eta (A^\dagger)^\eta - A^T S W_2 S^\eta (A^\dagger)^\eta - L_{A^T} U + U_2^\eta L_{A^T}^\eta,
\]

\[
Z_1 = -Z_1^\eta = \frac{1}{2} M^T C (B^\dagger)^\eta [I + (S^\dagger S)^\eta] + \frac{1}{2} [I + S^T S] B^T C (M^\dagger)^\eta + L_{M^T} W_2 L_{M^T}^\eta - V L_{B^\eta}^\eta + L_{B^\eta} V^\eta + L_{M^T} W_1 - W_2^\eta (L_{M^T} S)^\eta,
\]

where $U_1$, $U_2$, $W_1$, $L$, $V$, and $W_2^\eta = -W_2$ are arbitrary matrices over $\mathbb{H}$.

**Proof.** Clearly, (2)$\iff$(3).
We can prove that (2) ⇐⇒ (4) as follows: By Lemma 1 and Lemma 4, we have

\[ R_{A_1} D_1 = 0 \iff \left[ \begin{array}{c} A_1 \\ D_1 \end{array} \right] = r(A_1), \]
\[ R_{A_6} B_6 = 0 \iff \left[ \begin{array}{c} A_5 \\ B_6 \end{array} \right] = r(A_5), \]
\[ R_{A_6} B_6 = 0 \iff \left[ \begin{array}{c} A_6 \\ B_6 \end{array} \right] = r(A_6), \]

\[ R_A C L_{\eta^*} = 0 \iff \left[ \begin{array}{c} C \\ A \\ 0 \end{array} \right] = r(A) + r(B), \]

\[ \iff r \left[ \begin{array}{ccc} R_{A_7} C_8 L_{A_7} & r_{A_7} A_8 \\ B_8 \eta^* & r_{A_7} A_8 \end{array} \right] = r(R_{A_7} A_8) + r(R_{A_7} B_8) \]
\[ \iff r \left[ \begin{array}{ccc} C_8 \\ A_8 \\ 0 \\ 0 \end{array} \right] = r \left[ \begin{array}{c} A_7 \\ A_8 \end{array} \right] + r \left[ \begin{array}{c} A_7 \\ B_8 \end{array} \right] \]
\[ \iff r \left[ \begin{array}{ccc} B_4 - A_4 X_{01} - (A_4 X_{01})^* - B_4 Y_{01} B_4^\eta^* - C_6 Z_{01} C_6^\eta^* \\ L_{A_4}^\eta^* C_4 \\ L_{A_4}^\eta^* A_4^\eta^* \\ 0 \end{array} \right] \]
\[ = r(A_4 L_{A_4} + A_4 L_{A_4}) \]
\[ \iff r \left[ \begin{array}{ccc} B_4 \\ A_4 \\ 0 \\ 0 \end{array} \right] + r \left[ \begin{array}{ccc} C_4 \\ A_4 \\ 0 \\ 0 \end{array} \right] \]
\[ \iff r \left[ \begin{array}{ccc} D_4 - A_4 X_{01} - (A_4 X_{01})^* - B_4 Y_{01} B_4^\eta^* - C_6 Z_{01} C_6^\eta^* \\ L_{A_4}^\eta^* C_4 \\ L_{A_4}^\eta^* A_4^\eta^* \\ 0 \end{array} \right] \]
\[ = r \left[ \begin{array}{ccc} B_4 \\ A_4 \\ 0 \\ 0 \end{array} \right] + r \left[ \begin{array}{ccc} C_4 \\ A_4 \\ 0 \\ 0 \end{array} \right] \]

\[ R_M R_A C = 0 \iff \left[ \begin{array}{ccc} C \\ B \\ A \end{array} \right] = r \left[ \begin{array}{ccc} M \\ R_A C \end{array} \right] = r(M) \]
\[ \iff r \left[ \begin{array}{ccc} C \\ B \\ A \end{array} \right] = r \left[ \begin{array}{ccc} A \\ B \end{array} \right] \]
\[ \iff r \left[ \begin{array}{ccc} R_{A_7} C_8 R_{A_7}^\eta^* \\ R_{A_7} B_8 \\ R_{A_7} A_8 \end{array} \right] = r \left[ \begin{array}{ccc} A_8 \\ A_7 \end{array} \right] + r(A_7) \]
\[ \iff r \left[ \begin{array}{ccc} C_8 \\ B_8 \\ A_7 \\ 0 \\ 0 \end{array} \right] = r \left[ \begin{array}{ccc} A_8 \\ B_8 \\ A_7 \end{array} \right] + r(A_7) \]
\[ \iff r \left[ \begin{array}{ccc} F \\ C_6 L_{A_4} \\ B_4 L_{A_4} \\ A_4 L_{A_4} \end{array} \right] \]
\[ = r \left[ \begin{array}{ccc} B_4 \\ A_4 \\ 0 \\ 0 \end{array} \right] + r(A_4 L_{A_4}) \]
\[ \iff r \left[ \begin{array}{ccc} D_4 - A_4 X_{01} - (A_4 X_{01})^* - B_4 Y_{01} B_4^\eta^* - C_6 Z_{01} C_6^\eta^* \\ 0 \\ A_5 \\ 0 \end{array} \right] \]
\[ \iff r \left[ \begin{array}{ccc} D_4 - A_4 X_{01} - (A_4 X_{01})^* - B_4 Y_{01} B_4^\eta^* - C_6 Z_{01} C_6^\eta^* \\ 0 \\ A_5 \\ 0 \end{array} \right] \]
\[ \iff r \left[ \begin{array}{ccc} D_4 - A_4 X_{01} - (A_4 X_{01})^* - B_4 Y_{01} B_4^\eta^* - C_6 Z_{01} C_6^\eta^* \\ 0 \\ A_5 \\ 0 \end{array} \right] \]
To prove (1) \(\Rightarrow\) (2): If the linear system (5) is solvable, then by using Lemma 2 and Lemma 3, we note that \(X, Y,\) and \(Z\) are given by

\[
X = A_4^4 D_1 + L_{A_4} X_1, \quad (15)
\]

\[
Y = A_4^5 B_5 - (A_4^5 B_5)^{\eta^*} + A_4^5 A_5 (A_4^5 B_5)^{\eta^*} + L_{A_5} Y_1 L_{A_5}, \quad (16)
\]

\[
Z = A_4^6 B_6 - (A_4^6 B_6)^{\eta^*} + A_4^6 A_6 (A_4^6 B_6)^{\eta^*} + L_{A_6} Z_1 L_{A_6}, \quad (17)
\]

respectively, where \(Y_1 = -Y_1^{\eta^*}\) and \(Z_1 = -Z_1^{\eta^*}.\) Putting the values of (15)–(17) into

\[
A_4 X_1 - (A_4 X_1)^{\eta^*} + B_4 Y_1 B_4^{\eta^*} + C_4 Z_1 C_4^{\eta^*} = D_4,
\]

we obtain

\[
A_7 X_1 - (A_7 X_1)^{\eta^*} + A_8 Y_1 A_8^{\eta^*} + B_8 Z_1 B_8^{\eta^*} = C_8. \quad (18)
\]

Hence,

\[
R_{A_7} [A_7 X_1 - (A_7 X_1)^{\eta^*} + A_8 Y_1 A_8^{\eta^*} + B_8 Z_1 B_8^{\eta^*}] R_{A_7}^{\eta^*} = R_{A_7} [C_8] R_{A_7}^{\eta^*}
\]

\[
\Rightarrow A Y_1 A^{\eta^*} + B Z_1 B^{\eta^*} = C
\]

\[
\Rightarrow R_A C R_B^{\eta^*} = 0
\]

and

\[
R_A B Z_1 B^{\eta^*} = R_A C \Rightarrow M Z_1 B^{\eta^*} = R_A C \Rightarrow R_M R_A C = 0.
\]

Thus, by Lemma 6, we conclude that the general solution of (18) can be expressed as shown in (12)–(14). \(\square\)

Now, we consider a few special cases of linear system (5).

We remark that, if we consider the corresponding matrices to be equal to zero in (5), then we obtain the general solution of (3), which is the main result of [48], that is, Theorem 1 recovers Lemma 6.

If we put \(A_4 = 0, A_1 = 0,\) and \(D_1 = 0\) into (5), then we get the main result of [47].

**Corollary 1.** Suppose that \(A_2, A_3, B_2, B_3, B_4, C_2, \ldots, C_6,\) and \(D_4 = -D_4^{\eta^*}\) are known matrices in (5) over \(\mathbb{H}.\) We denote

\[
A_5 = \begin{bmatrix} A_2 \\ B_2^{\eta^*} \end{bmatrix}, \quad A_6 = \begin{bmatrix} A_3 \\ B_3^{\eta^*} \end{bmatrix}, \quad B_5 = \begin{bmatrix} C_2 \\ C_3^{\eta^*} \end{bmatrix}, \quad B_6 = \begin{bmatrix} C_4 \\ C_5^{\eta^*} \end{bmatrix},
\]

\[
C = D_4 - B_4 [A_4 B_5 - (A_4 B_5)^{\eta^*} + A_4 A_5 (A_4 B_5)^{\eta^*}] B_4^{\eta^*} -
\]

\[
C_6 [A_4 B_6 - (A_4 B_6)^{\eta^*} + A_4 A_6 (A_4 B_6)^{\eta^*}] C_6^{\eta^*},
\]

\[
A = B_4 L_{A_5}, \quad B = C_6 L_{A_6}, \quad M = R_A B, \quad S = B L_M.
\]

Then, the statements given below are equivalent:
Corollary 2. Suppose that $A_2, A_3, B_2, B_3, C_2, \ldots, C_5$ are given matrices involved as coefficients in (5). We define

$$A_5 = \begin{bmatrix} A_2 \\ B_2^{\ast} \end{bmatrix}, \quad A_6 = \begin{bmatrix} A_3 \\ B_3^{\ast} \end{bmatrix}, \quad B_5 = \begin{bmatrix} C_2 \\ C_2^{\ast} \end{bmatrix}, \quad B_6 = \begin{bmatrix} C_4 \\ C_3^{\ast} \end{bmatrix}. $$

Then, the following statements are equivalent:

(1) The system

$$A_2 Y = C_2, \quad Y B_2 = C_3, \quad Y = -Y^{\ast},$$

$$A_3 Z = C_4, \quad Z B_3 = C_5, \quad Z = -Z^{\ast},$$

$$B_4 Y B_4^{\ast} + C_6 Z C_6^{\ast} = D_4,$$

has a solution of the form $(Y, Z)$, where $Y = -Y^{\ast}$ and $Z = -Z^{\ast}$.

(2)

$$R_{A_5} B_5 = 0, \quad A_5 B_5^{\ast} = -B_5 A_5^{\ast}, \quad R_{A_6} B_6 = 0, \quad A_6 B_6^{\ast} = -B_6 A_6^{\ast}.$$

(3)

$$B_5 = A_5 A_5^{\ast} B_5, \quad A_5 B_5^{\ast} = -B_5 A_5^{\ast}, \quad B_6 = A_6 A_6^{\ast} B_6, \quad A_6 B_6^{\ast} = -B_6 A_6^{\ast}.$$
(4)  
\[
\begin{bmatrix}
B_5 & A_5 \\
A_6 & B_6
\end{bmatrix} = r(A_5), \quad \begin{bmatrix}
A_5 & B_6 \\
A_6 & B_5
\end{bmatrix} = r(A_6),
\]
\[A_5B_5^\eta = -B_5A_5^\eta, \quad A_6B_6^\eta = -B_6A_6^\eta,
\]
\[
r\begin{bmatrix}
0 & A_6^\eta \\
A_5 & 0
\end{bmatrix} = r(A_5) + r(A_6).
\]

In this case, the general solution of the form \((Y, Z)\) of the system (20) is represented as in Theorem 1 with

\[
Y_1 = -Y_1^\eta = -U + U_1^\eta,
\]
\[
Z_1 = -Z_1^\eta = W_2 - V + V_1^\eta + W_1^\eta,
\]
where \(U_1, U_2, W_1, U, V,\) and \(W_2^\eta = -W_2\) are arbitrary.

Similarly, by vanishing some matrices in (5), we obtain the general solution of (2).

**Corollary 3.** Suppose that \(B_4, C_4,\) and \(D_4 = -D_4^\eta\) are given matrices. We define

\[
C = D_4, \quad A = B_4, \quad B = C_6, \quad M = R_B C_6, \quad S = C_6 L_M.
\]

In this case, the following statements are equivalent:

1. The equation
   \[
   B_4 Y B_4^\eta + C_6 Z C_6^\eta = D_4,
   \]
   has a solution of the form \((Y, Z)\) with \(Y = -Y^\eta\) and \(Z = -Z^\eta\).

2. \(R_B D_4 R_C^\eta = 0, \quad R_M R_B D_4 = 0\).

3. \(MM^t R_B D_4 = R_B D_4 = R_B D_4 (C_6^t)^\eta C_6^\eta\).

4. \[
\begin{bmatrix}
D_4 & B_4 \\
C_6^\eta & 0
\end{bmatrix} = r(B_4) + r(C_6),
\]
\[
r\begin{bmatrix}
D_4 & C_6 \\
B_4 & 0
\end{bmatrix} = r\begin{bmatrix}
B_4 & C_6
\end{bmatrix}.
\]

In this case, the formula of the general solution \((Y, Z)\) for the system (21) can be expressed as in Theorem 1.

By making use of Theorem 1, we can establish the formula for computing the general solution of the following linear system.

**Corollary 4.** Suppose that \(A_1, A_4, B_4, C_6, D_1,\) and \(D_4 = -D_4^\eta\) are known coefficient matrices in (5). We define

\[
A_7 = A_4 L_{A_1}, \quad A_8 = B_4, \quad B_8 = C_6,
\]
\[
C_8 = D_4 - A_4 (A_1^t D_1) + [A_4 (A_1^t D_1)]^\eta,
\]
\[
A = R_A B_4, \quad B = R_A C_6, \quad C = R_A C_8 (R_A^t)^\eta, \quad M = R_A B, \quad S = BL_M.
\]

The following statements are equivalent:
(1) The system

\[ A_1 X = D_1, \]
\[ A_4 X - (A_4 X)^\eta + B_4 Y B_4^{\eta^*} + C_6 Z C_6^{\eta^*} = D_4, \]

has a triplet solution \((X, Y, Z)\) with \(Y = -Y^\eta\) and \(Z = -Z^\eta\).

(2)

\[ R_{A_1} D_1 = 0, \quad R_A C R_B^{\eta^*} = 0, \quad R_M R_A C = 0. \]

(3)

\[ D_1 = A_1 A_1^\dagger D_1, \quad MM^\dagger R_A C = R_A C = R_A C (B^\dagger)^{\eta^*} B^{\eta^*}. \]

(4)

\[
\begin{bmatrix}
D_4 & B_4 & A_4 & -D_4^{\eta^*} \\
A_4^{\eta^*} & 0 & 0 & A_4^{\eta^*} \\
C_6 & 0 & 0 & 0 \\
D_1 & 0 & A_1 & 0
\end{bmatrix}
\begin{bmatrix}
D_4 & C_6 & B_4 & A_4 & -D_1^{\eta^*} \\
A_4^{\eta^*} & 0 & 0 & A_4^{\eta^*}
\end{bmatrix}
= \begin{bmatrix}
B_4 & A_4 \\
0 & A_1
\end{bmatrix} + \begin{bmatrix}
C_6 & A_4 \\
0 & A_1
\end{bmatrix}.
\]

In this case, the system (22) has a general solution of the form

\[ X = A_1^\dagger D_1 + L_{A_1} X_1, \]
\[ Y = Y_1, \]
\[ Z = Z_1, \]

with \(X_1, Y_1,\) and \(Z_1\) represented as in (12), (13), and (14), respectively.

4. Algorithm with a Numerical Example

In this section, we present an algorithm (Algorithm 1) and a numerical example to rectify and validate our main result. The numerical experiments were performed using MATLAB 2019 version 9.7 (R2019b).

**Algorithm 1: General solution algorithm**

(1) Input matrices \(A_1, \ldots, A_4, B_2, B_3, B_4, \ldots, C_6, D_1, D_4 = -D_4^{\eta^*}\) of admissible size over \(\mathbb{H}\) and \(\eta \in \{i, j, k\}\).

(2) Calculate \(A_5, \ldots, A_8, B_5, B_6, \ldots, C_8, A, B, C, M,\) and \(S\) with (6).

(3) Observe if (7) and (8) are all satisfied or not. If yes, then follow the next step.

(4) Find \(X, Y,\) and \(Z\) using equations (9)–(11) with the \(X_1, Y_1,\) and \(Z_1\) given in (12)–(14).
Example 1. Suppose that the following matrices are given:

\[
A_1 = \begin{bmatrix}
1 & i & k \\
2 & 1 & i+j \\
k & 4 & i+k 
\end{bmatrix},
D_1 = \begin{bmatrix}
i & 2+j & 3+k \\
0 & 3 & i+j \\
2 & 1 & j
\end{bmatrix},
A_2 = \begin{bmatrix}
i+j & 2 & 3+k \\
1 & 3 & i \\
j & i & k
\end{bmatrix},
\]

\[
A_3 = \begin{bmatrix}
i+j & 2 & 3+k \\
1 & 3 & i \\
j & i & k
\end{bmatrix},
B_2 = \begin{bmatrix}
i & k & 3 \\
j & 2 & 1 \\
1 & i & k
\end{bmatrix},
C_3 = \begin{bmatrix}
j & 2 & 3 \\
1 & k & 2 \\
1 & 2 & i
\end{bmatrix},
A_3 = \begin{bmatrix}
i & 2 & 3+k \\
1 & 3 & i \\
j & i & k
\end{bmatrix},
\]

\[
C_4 = \begin{bmatrix}
2 & i+j & 3 \\
1 & i & k \\
j & 2 & i+k
\end{bmatrix},
B_3 = \begin{bmatrix}
1 & 2 & 3 \\
j & 4 & i+k \\
i & k & 1
\end{bmatrix},
C_3 = \begin{bmatrix}
j & 2 & 3 \\
1 & k & 2 \\
1 & 2 & i
\end{bmatrix},
\]

\[
A_4 = \begin{bmatrix}
1 & i \\
j & 2 \\
3 & k
\end{bmatrix},
B_4 = \begin{bmatrix}
i & 1 & 3 \\
2 & j & 1 \\
1 & 3 & k
\end{bmatrix},
C_5 = \begin{bmatrix}
1 & k & 2 \\
i & j & 1 \\
1 & k & 1+i+j
\end{bmatrix},
\]

\[
D_4 = \begin{bmatrix}
1 & 3 & i+j \\
j & 1 & 2 \\
k & i+i+j
\end{bmatrix}.
\]

Now, we compute the general solution of (5) using Algorithm 1. After some calculations, we know that \( M = 0 \) and \( S = 0 \). We check that the matrices given above satisfy the Equations (7) and (8). Then, our system (5) has the \( j \)-skew-Hermitian solution \( X, Y, \) and \( Z \). The general solution of (5) can be represented as

\[
X = \begin{bmatrix}
1 & 2i+j & k \\
3 & k & i+j
\end{bmatrix},
Y = -Y^* = \begin{bmatrix}
0 & 1+i-k & -2 \\
-1-i+k & 0 & -i+2k \\
2 & i-2k & 0
\end{bmatrix},
Z = -Z^* = \begin{bmatrix}
0 & -2 & i-3k \\
2 & 0 & 5 \\
-i+3k & -5 & 0
\end{bmatrix}.
\]

5. Conclusions
We calculated some practical, necessary, and sufficient conditions for the system (5) in order to have a \( \eta \)-skew-Hermitian solution. We also gave the closed-form formula of a general solution when the solvability conditions are satisfied. Then, we considered the solvability of some particular well-known systems as applications of our main result. An algorithm and a numerical example were also given to endorse our results. In our algorithm, we needed to compute the Moore–Penrose inverse of some block matrices. Dealing with the high computational cost of the Moore–Penrose inverse of block matrices if the coefficient matrices involved in system (5) are large and sparse is an important research topic for future research. In addition, it would be of interest to study the least square solutions of system (5). Some efficient iterative algorithms for computing the least squares solutions of quaternion matrix equations are proposed in [49,54–56], among others.

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