No end of tricks: electrons in the fractional quantum Hall regime

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Abstract

In the strong magnetic field fractional quantum Hall regime, electrons in a two-dimensional electron system are confined to their lowest Landau level. Because of the macroscopic Landau level degeneracy nearly all physical properties at low energies and temperatures are then entirely determined by electron-electron interactions. The properties of these non-Fermi-liquid electronic systems continue to surprise. In this article we briefly survey some recent advances in the physics of the fractional Hall regime.

1 Introduction

Electrons in metals and doped semiconductors typically form Fermi liquid states with qualitative physical properties identical to those of a non-interacting fermion gas state. Exceptions can occur when the electrons are strongly correlated. The degree of correlation in typical electronic systems depends on the ratio of the interaction strength to the electronic Fermi energy or bandwidth. Consequently interactions and correlations frequently play an essential role, for example in giving rise to insulating rather than metallic behavior, in the physics of narrow-band systems.

The role of interactions is particularly important for two-dimensional electron systems (2DES) in a strong magnetic field. In a magnetic field classical electrons move in circular cyclotron orbits with angular frequency \( \omega_c = eB/mc \). (Here \( e \) is the magnitude of the electron charge, \( B \) is the magnetic field strength, \( m \) is the electron mass.) When the cyclotron motion is treated quantum mechanically [1], allowed kinetic energy values for a single electron in a magnetic field are quantized; \( \epsilon_n = \hbar \omega_c (n + 1/2) \), where \( n \) is an integer. The kinetic energy of a cyclotron orbit is independent of the location of the center of the orbit and, correspondingly, there are many allowed single-particle states with a given quantized kinetic energy. (The Landau level degeneracy
$N_\phi = AB/\Phi_0$ where $A$ is the area of the system and $\Phi_0 = hc/e$ is the magnetic flux quantum. $\nu \equiv N/N_\phi$ is the Landau level filling factor. A macroscopically degenerate set of single-particle states with a common kinetic energy eigenvalue is known as a Landau level. The fractional quantum Hall regime occurs when the number of electrons is smaller than the number of single-particle states available in the lowest kinetic energy Landau level. In this regime the number of many-particle states which are degenerate in the absence of interactions diverges exponentially with system size. The Landau level in most senses plays the role of a band with zero energetic width. The perturbative treatment of interactions, which we use to understand Fermi liquid behavior in typical electronic systems, fails utterly.

All the physics discussed in this survey occurs in the strong magnetic field limit in systems for which electron-electron interactions dominate over electron-disorder interactions [2]. The realization of such systems required the development of modulation doping techniques for the creation of high-mobility two-dimensional layers in semiconductors. Since these systems have become available, experiments have revealed a startlingly rich variety of correlation effects, many of which are still only partially understood. The fractional quantum Hall regime has proved to be a wonderful laboratory in which a wide variety of strong correlation behaviors, most notably the fractional quantum Hall effect [3,4] itself, occur in well characterized experimental systems. This article discusses several phenomena which have received recent attention [5]. It is directed primarily toward researchers in other subfields of solid state physics.

This article is organized as follows. In Section II we present a capsule summary of some essential features of fractional quantum Hall physics which will be helpful in appreciating subsequent sections. One consequence of interactions which can occur in electronic systems and which does occur in the quantum Hall regime is ferromagnetism. In Section III we summarize recent work on the unique properties of quantum Hall ferromagnets. Interactions in the quantum Hall regime can also lead to energy gaps at the Fermi energy. When the chemical potential lies in one of these gaps there are always states localized at the edge of the system. Recent work on the unusual physics of these effectively one-dimensional ‘edge-state’ systems is summarized in Section IV. Interaction and correlation effects in an electronic systems often lead to suppressed tunneling near zero bias. These effects are very strong in the fractional Hall regime and are not yet completely understood. Recent work on this topic is outlined in Section V. One important topic, composite fermion behavior of electrons in the quantum Hall regime, has been covered [6] in a previous Highlights issue of Solid State Communications and will be omitted here. (That report is partially updated elsewhere in this volume [7].) We conclude in Section VI with some brief closing remarks.
The perfect degeneracy of a Landau level appears to make the kinetic energy portion of the Hamiltonian irrelevant for fractional quantum Hall physics. This appearance is deceiving however, since the restriction of the Hilbert space to single-particle orbitals with the minimum kinetic energy constrains correlations contained in the many-particle wavefunction [8]. Without this constraint the electrons would behave classically and, in their ground state, would form a triangular lattice Wigner crystal. Indeed electron crystallization was expected before experimental studies of the fractional Hall regime became possible. (The Wigner crystal state does in fact occur when the Landau level is nearly empty.) An important property of the constrained Hilbert space is that it contains only one relative motion state of a pair of electrons for each relative angular momentum [9]. The Haldane pseudopotential [10] $V_M$ is the interaction energy of a pair of electrons with relative angular momentum $M$; $\{V_M\}$ completely specify the projection of the Hamiltonian on to the lowest Landau level. Low energy states in the many-particle Hilbert space are those for which pairs of electrons are unlikely to be in the small $M$ relative angular momentum states in which the repulsive interactions are stronger.

The quantum Hall effect is a transport anomaly which occurs in the fractional quantum Hall regime. It is characterized by dissipationless current flow in the limit of zero temperature. The occurrence of the quantum Hall effect can be related to the occurrence of an anomaly in a thermodynamic property, the compressibility. The compressibility of a system of interacting particles is proportional to the derivative of the chemical potential with respect to density. It can happen that at zero temperature the chemical potential has a discontinuity at a density $n^*$: the energy to add a particle to the system ($\mu^+$) differs, at this density, from the energy to remove a particle from the system ($\mu^-$). The system is then said to be incompressible. In an incompressible system a finite energy is required to create unbound positive and negative charges that are capable of carrying current through the bulk. For this reason incompressible systems are as a rule insulating at zero temperature. Paradoxically, incompressibility is precisely the condition required for the quantum Hall effect to occur. The twist is that in the case of the quantum Hall effect, the density $n^*$ at which the incompressibility occurs must depend on magnetic field.

The relationship between incompressibility and the transport anomalies that give the quantum Hall effect its name has been explained from several connected points of view [11] which are all consistent with the following conclusions. In the limit of zero temperature the dissipative conductivity ($\sigma_{xx}$) at density $n^*$ vanishes and the Hall conductivity approaches the value

$$\sigma_{xy} = \frac{(e^2}{\hbar})\Phi_0(\partial n^*/\partial B). \quad (1)$$
For incompressibilities that occur at fixed Landau level filling factor $\nu$, $\sigma_{xy} = (e^2/h)\nu$. We will see in the following sections that the magnetic field dependence of $n^*$ is directly responsible for a number of unusual properties at nearby densities. (The quantum Hall effect also occurs at weaker magnetic fields when many Landau levels are occupied. The gap in this case is usually due to kinetic energy quantization and, since $\nu$ is then always an integer, the transport anomaly is referred to as the integer quantum Hall effect. This survey is limited to the fractional Hall regime or, taking account of the electronic spin degree of freedom, to the filling factor range $\nu < 2$.)

The largest of the chemical potential gaps responsible for the fractional quantum Hall effect occurs when the Landau level filling factor $\nu = 1/3$. The origin of this gap is readily understood. It turns out that the largest Landau level filling factor at which it is possible [1] to form many-body states which completely avoid pairs with relative angular momentum $M = 0$ and $M = 1$, and hence avoid the most repulsive interactions, is $\nu = 1/3$. The many-body states which satisfy this constraint are the ones discovered by Laughlin [4] in his seminal theoretical work on this topic. When $\nu = 1/3$, an added electron must form a pair state with relative angular momentum $M = 1$ and it interacts more strongly with the electron fluid than a removed electron. The resulting chemical potential gap occurs at fixed Landau level filling factor and therefore at a density which is proportional to magnetic field strength. From transport experiments it is known that chemical potential gaps occur at other filling with the largest gaps occurring when $\nu = n/(2n + 1)$ or $\nu = (n + 1)/(2n + 1)$. The composite fermion picture [12,13,6] provides an intuitively appealing way of understanding why gaps occur at these particular filling factors. The existence of chemical potential gaps at certain filling factors is the non-perturbative interaction effect in the fractional Hall regime which is most dramatically manifested in transport experiments. It is not by any means the only one. It seems that every new class of experiments and every theoretical advance leads to unexpected conclusions. In the following sections we discuss some recent surprises.

3 Quantum Hall Ferromagnets

At first site the phrase ‘quantum Hall ferromagnet’ appears to be an oxymoron since ferromagnetism refers to spontaneous magnetization in the absence of an external magnetic field while the quantum Hall effect occurs in a 2DES in the extreme strong magnetic field limit. To understand why the terminology is sensible it is necessary to consider the relevant energy scales for the case of the semiconductors in which 2DESs are realized. For a free-electron system in a magnetic field, the Zeeman splitting of spin-levels $g\mu_B B$ and the Landau level separation $\hbar\omega_c$ are identical, apart from small relativistic corrections. Elec-
trons in states near the conduction band minimum of a semiconductor behave like free electrons except that band effects renormalize the electron mass $m^*$ and the g-factor. In the case of the GaAs system, where the quantum Hall effect is most often studied, band effects increase the Landau level separation by a factor of $\sim 20$ and reduce the Zeeman splitting by a factor of $\sim 4$. As a result for typical experimental situations, the Landau level separation (in temperature units) is $\approx 150$K, and the characteristic scale for electron-electron interactions is $\approx 100$K while the Zeeman splitting is only $\approx 2$K. We call a system a quantum Hall ferromagnet if the electronic spins in an incompressible ground state align in the absence of Zeeman coupling. The small typical Zeeman coupling plays an important role in determining measurable properties, but it is still useful to treat the system as a ferromagnet in the presence of a small symmetry breaking field.

If we regarded a partially filled Landau level as a giant open-shelled atom the familiar Hund’s rules from atomic physics would suggest that, in order to maximize the magnitude of its exchange energy, the ground state should have its spins aligned to the maximum degree consistent with the Pauli exclusion principle. In fact, a Hartree-Fock approximation would predict a ferromagnetic ground state for electrons in a partially filled Landau level at nearly any value of $\nu$. The Hartree-Fock approximation, in which many-electron states are approximated by single Slater determinants, often provides a simple and qualitatively correct picture of itinerant electron ferromagnetism at $T = 0$. In the fractional Hall regime, however, the truth is not so simple. For example, at certain filling factors it is known [14] that the interaction energy is minimized in a $S = 0$ state. There are nevertheless some filling factors at which the Hund’s rule picture of the ground state is, perhaps coincidentally, correct. We will focus our attention here on the $\nu = 1$ case which has been most extensively studied theoretically and experimentally. In this case the ground state is a $S = N/2$ spin-multiplet. The states in this multiplet are the only ones in the many-particle Hilbert space for which pairs of electrons with angular momentum $M = 0$ are completely avoided and that is why they minimize the interaction energy. An infinitesimal Zeeman coupling picks one state out of this multiplet and completely aligns all the electronic spins. The ground state has all majority spin orbitals in the lowest Landau level occupied and all minority spin orbitals empty. The same physics which is responsible for the ferromagnetic ground state at $\nu = 1$ gives rise to a chemical potential gap when the density crosses $n^* = B/\Phi_0$, as discussed above. The chemical potential gap makes a quantum Hall ferromagnet similar to an insulating ferromagnet at low temperatures and energies, although its itinerant character becomes important at higher temperatures and energies.

Quantum Hall ferromagnets are isotropic ferromagnets, i.e. the ground state moment can point in any direction. They are like ferromagnets in a Heisenberg spin model and unlike those in XY where the moment orientation is confined
Skyrmion excitations of this type are present in any two-dimensional isotropic magnetic system. Skyrmions in the quantum Hall effect have the unique property that they carry charge \([17] e\), and this is responsible for their enormous impact on physical properties. In a pioneering paper on quantum Hall ferromagnets Sondhi et al. pointed out [18] that in typical circumstances, Skyrmions should be the lowest energy charged excitations of quantum Hall ferromagnets at \(\nu = 1\). This work provided a satisfying explanation for earlier numerical work by Rezayi [19] which hinted tantalizingly at unusual spin magnetism near \(\nu = 1\). In the limit of large Skyrmion sizes, the energy of a Skyrmion can be calculated from a generalized non-linear sigma model [15,16,18]. Because of the spin-reversal at the center, any finite Zeeman coupling favors small Skyrmions. This tendency is countered predominantly by the Coulomb charging energy of a Skyrmion. For Zeeman coupling strengths typical of experimental systems, an accurate calculation of the energy of a Skyrmion and of the number of reversed spins each contains near its center requires a microscopic calculation. The required calculation was completed [20] by Fertig et al. who predicted approximately three reversed spins per quasiparticle near \(\nu = 1\) and suggested that the presence of Skyrmions would be manifested most clearly in measurements of the filling factor dependence of the ground state spin-polarization. Coincidentally, Barrett and Tycko [21] developed an optical pumping technique to achieve the first NMR studies of the two-dimensional electron gas. Their results for the low-temperature Knight shift, which is proportional to the ground state spin polarization are shown in Figure 1.
Fig. 1. Knight shift measurements by Barrett et al. for a two-dimensional electron gas near filling factor $\nu = 1$. The Knight shift is proportional to the spin-polarization of the electron system which is, as explained in the text, expected to be fully polarized for $\nu = 1$. The experiment is consistent with four reversed spins for each particle added to a full Landau level and three reversed spins for each particle removed from a full Landau level. The dashed line in this figure shows the Hund's rule prediction for the spin-polarization, the maximum value consistent with the Pauli exclusion principle. Full spin-polarization is possible when the filling factor $\nu < 1$; at larger filling factors minority spin levels must be occupied. The dependence of spin-polarization on filling factor expected for non-interacting electrons would be correct for interacting electrons if the Hartree-Fock approximation predictions were correct. (After Ref. [21]).

These results can be understood as follows. For $N = N_\phi$ the ground state has $S_z = S = N/2$ as discussed above. For $N = N_\phi \pm 1$, the ground state contains a single charged Skyrmion. The Skyrmion can be introduced by changing the total electron number or, in what is the typical experimental situation, by changing the magnetic field strength and hence $N_\phi$. In a quantum description [22,23] the number of reversed spins per skyrmion is quantized so that, when Skyrmion-Skyrmion interactions can be neglected, we expect that the component of the total spin along the direction of the Zeeman field is

$$S_z = N/2 - (K + \theta)|N - N_\phi|$$

Here, in reflection of a particle-hole symmetry which exists in the fractional Hall regime [24], $\theta = 1$ for $N > N_\phi$ and $\theta = 0$ for $N < N_\phi$. $|N - N_\phi|$ is the number of Skyrmions or antiskyrmions present in the system. The Knight shift measurements [21] of Barrett et al. are consistent with $K = 3$ and with the Hartree-Fock calculations [20] of Fertig et al. For non-interacting electrons, or
with interactions treated in the Hartree-Fock approximation, \( K = 0 \) so that \( S_z \) always has the maximum value allowed by the Pauli exclusion principle. There seems to be little doubt that the elementary charged excitations of quantum Hall ferromagnets are Skyrmion-like objects that carry large spin quantum numbers. Recent transport [25] and optical [26] experiments add additional support to this conclusion.

For large enough \( |N - N_φ| \) the Skyrmion-like objects will eventually interact strongly. When the density of Skyrmions is low and the temperature is low, Skyrmions are expected to form a triangular lattice crystal similar to the Wigner crystal state formed by electrons in the limit of very strong magnetic fields. The large range of filling factors over which \( S_z \) falls linearly with \( \nu - 1 \) indicates that Skyrmions tolerate crowding with amazing alacrity. This property has been explained [27] by Brey et al. who have proposed that at higher densities Skyrmions will form a square lattice. The basis of this proposal, supported by detailed microscopic calculations, is the observations that neighboring Skyrmions repel each other less strongly when they have opposite orientations. Dense Skyrmions will thus tend to form a bipartite lattice, most likely square, with long range order in the Skyrmion orientations. At a finite temperature this state has both quasi-long-range translational and magnetic order, leaving open the possibility of one or two continuous Kosterlitz-Thouless or first order phase transitions at finite temperature. Recent anomalies [28] seen in heat capacity studies of 2DESs in the quantum Hall regime near \( \nu = 1 \) may reflect such a phase transition.

The finite temperature properties of quantum Hall ferromagnets are equally interesting. The simplest situation, which occurs when \( \nu = 1 \) and no Skyrmions are present in the ground state, has received the greatest attention to date. In Figure 2 we show measurements of the temperature dependence of the magnetization at \( \nu = 1 \) by Goldberg[29] and collaborators as determined using polarization dependent optical absorption measurements. Similar results were obtained earlier in the NMR studies of Barrett and Tycko [21]. According to the Mermin-Wagner theorem, long range spontaneous magnetic order is impossible in two-dimensional systems at any finite temperature. The thermal suppression of the magnetization at very weak Zeeman coupling is expected to exhibit quantum critical behavior. Most of the thermal magnetization suppression occurs at temperatures low enough that the system is completely characterized by its spin stiffness energy, \( \rho_s \). In Figure 2 the measured magnetization is compared with theoretical approximations to the temperature dependence of magnetization calculated by Read and Sachdev [31] using the appropriate continuum quantum field theory model. The field theory description used for the long-wavelength physics is appropriate for any two-dimensional ferromagnet with a magnetic correlation length which is long compared to microscopic lengths and which has no low energy excitations other than those associated with slow smooth magnetic fluctuations. The validity of the continuum quan-
Fig. 2. Temperature dependence of the spin-polarization as determined from polarization dependent optical absorption measurements. The open circles and closed squares are experimental data. The solid and long-dashed lines show two different approximate theoretical results calculated by Read and Sachdev in Ref. [31] using a field-theoretical model with a realistically chosen spin-stiffness parameter. The short-dashed line is a theoretical result obtained in Ref. [32] using an approximation which accounts for the itinerant character of the two-dimensional electrons, important at higher temperatures. (After Ref. [29].)

At higher temperatures the magnetization curves are no longer universal and the itinerant nature of the underlying electronic degrees of freedom becomes important, for example in screening the Coulomb interactions responsible for the spin stiffness. The need for a simultaneous treatment of collective spin and constituent fermionic degrees of freedom produces a fundamental difficulty. Lack of progress on this front has blocked the development of a completely satisfactory theory of itinerant electron magnetism. Quantum Hall ferromagnets present a, possibly especially simple, system in which to compare theoretical treatments of itinerant electron ferromagnets with experiment. The $\nu = 1$ quantum hall ferromagnet is in many senses similar to a strong ferromagnetic ground state in a single-band itinerant electron system. In Figure 2 we show theoretical results for the magnetization obtained in an approximation [32] which does not account for the spin-wave interactions important in the quantum critical regime but does account for the emergence, at high temperatures,
of the underlaying fermionic degrees of freedom upon which the spin-waves are built. Evidently this theory overestimates the magnetization at high temperatures. Possible sources of this discrepancy include an inadequate treatment of screening, which certainly reduces the spin stiffness at higher temperatures, and the neglect of the spin-wave interactions important at lower temperatures and Zeeman energies and included in the field theoretical approximations. Progress in identifying a major culprit and success in making appropriate improvements to the theory could have important implications for theories of itinerant electron magnetism. Understanding the temperature dependence at nearby filling factors [21] presents an especially interesting challenge to theory.

Physics closely related to that discussed above occurs in systems with two 2D layers close together, even at stronger fields where the electronic system is always fully spin polarized.[33] The role of the the spin quantum number is assumed by the quantum number which specifies the layer in which an electron resides. The role of the Zeeman energy is assumed by the tunneling gap between symmetric and antisymmetric single-particle states. There are, however, a number of important differences between the spin and double-layer cases. It turns out that, when the double-layer system is described using a pseudospin language, the effective magnetic interaction in the double-layer case is anisotropic leading to an easy-plane ordered state. The broken symmetry in this ordered state results in inter-layer phase coherence in the absence of interlayer tunneling. The topologically and electrically charged excitations gradually deform with increasing anisotropy from skyrmions to vortex-antivortex pairs. The physics of double-layer systems is further enriched by the possibility of disturbing the system with a magnetic field parallel to the electronic planes. In the pseudospin description such a field gives rise, in a system with non-zero symmetric-antisymmetric splitting, to an effective Zeeman field which rotates as a function of the perpendicular planar coordinate and at a critical in-plane field strength causes a phase transition. The mathematical description of this phase transition is identical to the description of commensurate-incommensurate phase transitions. This phase transition was first discovered in measurements[34] of the transport properties of double-layer systems. This experimental work actually predates the experimental studies of quantum Hall ferromagnets surveyed in the preceding paragraphs. Further progress in unraveling the richer physics of double-layer quantum Hall ferromagnets has been hampered, however, by slower headway in the development of pertinent experimental probes. It appears that progress has accelerated recently, however, with the demonstration that optical probes can measure the pseudospin moment [30].
Fig. 3. Log-log plot of the current-voltage \( (I - V) \) characteristic for tunneling from bulk-doped \( n^+ \) GaAs into the edge of a two-dimensional electron system with \( \nu = 1/3 \). Results are shown for two different samples with different densities for which \( \nu = 1/3 \) occurs at \( B = 13.4T \) (crosses), and \( B = 10.8T \) (solid circles). The solid curves represent fits to the universal form predicted theoretically by Kane and Fisher in Ref. [39]. (After Ref. [37].)

4 Edge Excitations of an Incompressible Quantum Hall Fluid

When the chemical potential lies in a gap a finite fractional Hall system has no gapless excitations [35] in the bulk but it can have gapless excitations localized at its edge, and in fact must [36] have such excitations whenever the Hall conductivity is non-zero. In Figure 3 we show some recent experimental results obtained by Chang [37] on the tunneling density-of-states for a fractional Hall system at filling factor \( \nu = 1/3 \). What is plotted here is the measured tunneling current between an effectively metallic contact and the edge of the quantum Hall system. Assuming that there are no anomalies in the density-of-states of the contact, the current is proportional to the tunneling density-of-states integrated over an energy interval \( eV \) starting from the chemical potential. A finite tunneling density-of-states would lead to a tunneling current \( I \propto V \) and a tunneling conductance which approaches a constant at low-temperatures. Instead Chang’s experiments show a tunneling current \( I \propto V^\alpha \) (in the low temperature limit) and a tunneling conductance proportional to \( T^{\alpha - 1} \) where \( \alpha \) is \( \approx 2.7 \). These results are in substantial agreement with earlier theoretical predictions by Wen [38] and Kane and Fisher [39]. Evidently the density-of-states is strongly suppressed at energies near the Fermi energy in the \( \nu = 1/3 \)
The edge of a two-dimensional fractional Hall system constitutes a one-dimensional electron system. At energies well below the chemical potential gap, the incompressible fractional Hall system can be described using a chiral Luttinger liquid picture [41] which is an adaptation of the Luttinger liquid theory [40] for low-energy properties of one-dimensional electron systems. Low-energy excitations of the electronic system are expected to involve modulations of the edge which are slow on a microscopic length scale. The simplest version of the chiral Luttinger liquid picture starts from the assumption that quantum states at the edge are completely characterized by the one-dimensional density \( [n(x)] \) obtained by integrating the change in the two-dimensional density from its ground state form along a coordinate perpendicular to the edge. (We’ll comment further on this assumption, which is not always valid, later.) Assuming translational invariance along the edge and expanding around the ground state to describe low-energy excitations gives

\[
E[n] = E_0 + \frac{1}{2} \int dx' \int dx \, \delta n(x') \alpha(|x - x'|) \delta n(x)
\]  

(3)

where \( E_0 \) is the ground state density and \( \alpha(x) \) is a non-universal function. \( \alpha(x) \) must depend at a minimum on the details of the external potential which confines the electronic system to a finite area.) Note that, for excitations which change the total electron density, we have as a convenience chosen the zero of energy at the chemical potential in dropping a term proportional to \( \int dx \delta n(x) \). When expressed in terms of the Fourier components of \( n(x) \) Eq. 3 becomes

\[
E = E_0 + \frac{1}{2L} \sum_{q \neq 0} \alpha_q n_{-q} n_q.
\]

(4)

For the purposes of the discussion below we will assume that \( \alpha_q \) approaches a constant (to be denoted by \( \alpha \)) in the \( q \to 0 \) limit appropriate at low energies [42]. The energy above can be used as an effective Hamiltonian for low-energy long-wavelength excitations. It is quantized by imposing the following commutation relation on the density Fourier components:

\[
[n_{-q'}, n_q] = \frac{\nu q L}{2\pi} \delta_{q, q'}.
\]

(5)

This commutation relation is suggested by the Luttinger liquid theory of the one-dimensional electron gas. The innocent looking factor of \( \nu \) on its right hand side turns out to be responsible for the suppressed tunneling density-of-states when \( \nu \neq 1 \). As we explain below, it is required in order for the theory to imply the correct value of the quantized Hall conductance.
Eq. 5 and Eq. 4 imply that the quantized density-wave excitations at the edge are equivalent to a system of non-interacting bosons. The boson creation and annihilation operators are defined by

\[
a_q = \sqrt{\frac{2\pi}{\nu q L}} n_{-q} \tag{6}
\]

\[
a_q^\dagger = \sqrt{\frac{2\pi}{\nu q L}} n_q \tag{7}
\]

where \( q > 0 \) so that all the density waves travel in the same direction, hence the chiral designation for this Luttinger liquid system. The Hamiltonian is

\[
H = \sum_{q > 0} \hbar v q a_q^\dagger a_q \tag{8}
\]

so that the velocity of the edge waves is

\[
v = \frac{\alpha \nu}{2\pi \hbar} = \frac{\nu}{2\pi L \hbar} \frac{d^2 E_0}{dn^2} = \frac{\nu}{2\pi \hbar} \frac{d\mu}{dn} \tag{9}
\]

The justification of the chiral Luttinger liquid theory in the case of the fractional quantum Hall effect (\( \nu \neq 1 \)) is not as systematic as in the one-dimensional electron gas case. As explained above, Eq. 5 implies that low-energy excitations are chiral bosons. For \( \nu = 1/m \) where \( m \) is an odd integer the bosonization property can be established from microscopic theory [43], provided that the external potential which defines the edge of the system is sufficiently abrupt. For incompressible states at other values [43,41] of \( \nu \), and for smoother edges, the edge state electronic structure is more complicated. In general neither Eq. 4 nor Eq. 5 applies and more elaborate theories need to be developed [38,41,44]. When this simplest version of the chiral bosonization \( \textit{does} \) apply, the expectation of dispersionless propagation of low-energy edge density waves requires that the commutator \([n_{-q'}, n_q]\) be \( \propto q \delta_{q,q'} \). The constant of proportionality can then be determined by requiring that the rate of change of the equilibrium edge current with chemical potential be \( e\nu/\hbar \). It can be shown that this is equivalent to requiring that the theory reproduce the correct quantized value of the Hall conductivity. The change in equilibrium edge current is related to the change in equilibrium density by

\[
\delta I = ev\delta n \tag{10}
\]

When the chemical potential for the single edge system is shifted slightly from
its reference value (which we chose to be zero) the energy is given by

\[ E[n] = E_0 + \mu \delta n + \alpha \frac{(\delta n)^2}{2} \]  

(11)

Minimizing with respect to \( \delta n \) we find that

\[ \delta n = \frac{\delta \mu}{\alpha} \]  

(12)

so that

\[ \frac{\delta I}{\delta \mu} = \frac{e\nu}{\alpha} \]  

(13)

In order for this to be consistent with the quantum Hall effect \([\delta I = (e\nu/h)\delta \mu]\)

our theory must yield a edge phonon velocity given by

\[ v = \frac{\alpha}{2\pi\hbar} \cdot \nu. \]  

(14)

The factor of \( \nu \) appearing in this equation occurred in Eq. (9) because of the factor of \( \nu \) in Eq. (5). This factor is required if the theory is to reproduce the correct value of the quantized Hall conductance.

In order to calculate the tunneling density-of-states we need to express the many-particle state produced when an electron is instantaneously added at the edge of a two-dimensional electron system in an incompressible state as a state of the bosonic edge wave system. The relationship between electron and boson operators is well-known from Luttinger liquid theory and is established by requiring the exact identity

\[ [n(x), \hat{\psi}^\dagger(x')] = \delta(x - x') \hat{\psi}^\dagger(x') \]  

(15)

to be reproduced by the effective low-energy theory. Here \( \hat{\psi}^\dagger(x') \) is the operator that creates an electron at position \( x' \) along the edge. This equation simply requires the electron charge density to increase by the required amount when an electron is added to the system. An elementary calculation [38,41,45] shows that in order to satisfy Eq. (15), the field operator must be given by

\[ \hat{\psi}^\dagger(x) = \sqrt{z} e^{i \nu^{-1} \phi(x)} \]  

(16)

where \( d\phi(x)/dx = n(x) \) and \( z \) is a constant that cannot be determined by the theory. The factor of \( \nu^{-1} \) in the argument of the exponential of Eq. (16)
is required because of the factor of $\nu$ in the commutator of density Fourier components that in turn was required to make the theory consistent with the fractional quantum Hall effect. When the exponential is expanded the $k^{th}$ order terms generate states with total boson occupation number $k$ and are multiplied in the fractional case by the factor $\nu^{-k}$; multi-phonon terms are increased in relative importance.

Eq. (16) has been carefully checked numerically [46]. The $\nu^{-1}$ factor leads to predictions of qualitative changes in a number of properties of fractional edges. The quantity that is most directly altered is the tunneling density-of-states. Consider the state created when an electron, localized on a magnetic length scale, is added to the ground state at the edge of a $N-$ electron system with $\nu = 1/m$:

$$\hat{\psi}^\dagger(0)|\Psi_0\rangle \sim \exp\left( - \sum_{n>0} a_n^\dagger \frac{\sqrt{n\nu}}{n\nu} \right) |\psi_0\rangle$$

$$= 1 + \frac{1}{\nu^{1/2}} \text{phonon term} + \frac{2}{\nu} \text{phonon terms} + \ldots$$  \hspace{1cm} (17)

The tunneling density-of-states at zero temperature is given by a sum over the ground and excited states of the $N+1$ particle system:

$$A(\epsilon) = \sum_n \delta(E_n - E_0 - \epsilon)|\langle \Psi_n |\hat{\psi}^\dagger(0) |\Psi_0\rangle|^2$$  \hspace{1cm} (18)

Because of the increased weighting of multiphonon states, which become more numerous at energies farther from the chemical potential, the spectral function is larger at larger $\epsilon - \mu$ in the fractional case. Explicit calculations [38,39] yield a spectral function at zero temperature that grows like $(\epsilon - \mu)^{\nu^{-1}-1}$. At finite temperature the tunneling density-of-states saturates to a finite value $\propto T^{\nu^{-1}-1}$.

The experimental results in Fig. 3 are consistent with the predicted energy dependence and with predictions for the temperature dependence of the voltage at which saturation is expected to occur. These observations evidently confirm the physics discussed in the preceding paragraphs.

It seems intuitively clear that the spectral function should be small at low-energies in the fractional case since the added electron will not share the very specific correlations common to all the low-energy states. It is marvelous that simply requiring the low-energy theory to be consistent with the fractional quantum Hall effect leads to a very specific prediction for the way in which this qualitative notion is manifested in the tunneling density-of-states at the edge of a $\nu = 1/m$ incompressible state. Similar predictions can be made for the richer behavior [39,41] expected in tunneling experiments at filling factors with more complicated edge electronic structures, for example at $\nu = 2/3$.

The experimental result shown in Fig. 3 is likely to be the harbinger of a
period of fruitful interaction between theoretical and experimental studies of fractional Hall edges with important consequences for our understanding of quasi one-dimensional electron systems more generally.

5 Tunneling Density of States; Spectral Function

In the preceding two sections we discussed experimental achievements leading to the confirmation of phenomena which had, at least in part, been anticipated theoretically. The possibility of performing experiments will no doubt in both cases lead to the further development and refinement of theoretical ideas. In this section we discuss experimental observations which were not, as far as we are aware, anticipated by theory. These experiments were made possible by the ability to make separate electrical contact to nearby two-dimensional electron systems, enabling measurements of the tunneling conductance from one bulk two-dimensional electron system to another. It was known from transport studies that the bulk tunneling density-of-states must be strictly zero near the Fermi energy at those discrete set of filling factors for which the ground state is incompressible and the transport anomalies of the fractional quantum Hall effect occur. Prior to the first experiments, most experts would probably have predicted a tunneling density-of-states which was depressed due to interactions only at energies quite close to the Fermi energy and only near incompressible filling factors. What was observed instead [47–49] is shown in Fig. 4. Clearly there is a strong suppression of the tunneling density-of-states (TDOS) near the Fermi energy over a wide range of \( \nu \) in the fractional quantum Hall regime. This suppression was also seen, although less distinctly, in earlier 3D to 2D tunneling experiments [50].

This behavior for the tunneling conductance is in stark contrast with the behavior in identical systems in the absence of a strong magnetic field. In that case, momentum conservation and Fermi liquid theory, which implies quasiparticle peaks in the electronic spectral functions, conspire to produce large peaks in the tunneling current [51,52] at zero bias. On a qualitative level the very different results found for the non-Fermi liquid states of the strong magnetic field limit are easy to understand on the basis of the physical picture suggested in early experimental work [47]. As we have emphasized the ground state at any filling factor minimizes the interaction energy, subject to the constraint that all electrons lie in the lowest Landau level. Without this constraint the electrons would form a crystal. The ground state can still be thought of as a ‘quantum melted’ crystal in which positional order is lost because the fluctuations in electronic positions are too large. In the tunneling experiment an electron is removed from the ground state in one two-dimensional layer and added to another layer. The measurement tells us that the difference between the energy for adding an electron and removing an electron has a
Fig. 4. Low temperature tunneling I-V characteristics for bulk 2D to 2D tunneling in the fractional Hall regime. The traces are at magnetic fields separated by 0.25 Tesla and for this sample cover a range of filling factors from $\nu = 0.48$ to $\nu = 0.83$. The higher bias potential peak is due to Landau level mixing while the lower bias potential peak is due to tunneling within the lowest Landau level. The tunneling current is the convolution of the tunneling densities of states in the two layers at energies on opposite sides of the Fermi energy and separated by $eV$. This experiment shows that the tunneling current is strongly suppressed near zero bias and hence that the tunneling density-of-states is suppressed at energies near the Fermi energy over a broad range of filling factors. Features in the tunneling data associated with particular incompressible states, for example the one that occurs at $\nu = 2/3$, are weak. The inset shows the onset and peak values of the intra Landau level tunneling currents. (After Eisenstein et al. in Ref. [47].)

probability distribution which is relatively sharply peaked at a finite value and is extremely small values for energies close to zero. This is what would be expected if the ground state really were a crystal. In that case a suddenly removed electron would be removed from lattice site, or close to a lattice site if lattice vibrations were accounted for. A suddenly added electron could be inserted into the lattice at any point. However even the energetically most advantageous interstitial sites would would be less favorable than a lattice site. The net energy cost could be close to zero only if the lattice constant of the electron crystal were adjusted to allow for the creation of an additional lattice site for the added electron. The probability of finding the electronic system is such a convenient state is evidently very small and therefore the tunneling current at very small bias voltages must vanish very quickly.
There has been considerable theoretical interest in these experimental results and a number of different approaches [54,55] have been taken in an effort to put these ideas on a firmer footing and make more quantitative predictions which could be tested experimentally. At present there is no completely satisfactory theory. As is often the case, it is much easier to derive rigorous results for moments of a spectral function than for its full frequency dependence. Following Haussmann et al. [57] we define an effective energy gap in terms of the ratios of the first and zeroth moments of the tunneling I-V curve as follows:

\[ \Delta_{sr} = \frac{\int_0^\infty \epsilon I(\epsilon) d\epsilon}{\int_0^\infty I(\epsilon) d\epsilon}. \] (19)

In the limit of sharp peaks in I-V curve, \( \Delta_{sr} \) gives the position of the peak. It turns out that it is possible to derive [58,57] an exact relationship between \( \Delta_{sr} \) and the ground state energy at a particular filling factor:

\[ \Delta_{sr} = 2\left[\nu^2 \epsilon_0(\nu = 1) - \epsilon_0(\nu)\right] \nu(1-\nu) \] (20)

In this equation, \( \epsilon_0(\nu) \) is the ground state energy per state in the Landau level at filling factor \( \nu \). In the Hartree-Fock approximation the ground state energy per single-particle state is proportional to \( \nu^2 \), provided that translational symmetry is not broken, so Eq. 20 says that the gap is proportional to difference between the ground state energy in the Hartree-Fock approximation and the exact ground state energy, i.e., to the amount by which the ground state energy is lowered by correlations. This result is very much in accord with the physical picture given above. The ground state energy per electron is fairly accurately known and Eq. 20 is in good agreement with experimental results. Any complete theory of bulk 2D-2D tunneling in the fractional Hall regime will have to respect this exact identity.

It follows from Eq. 20 that the filling factor dependence of the ground state energy can be extracted from tunneling data, in much the same manner as it has previously been extracted from extrinsic photoluminescence data [56]. Cusps in the filling factor dependence of the ground state energy correspond to discontinuities in the chemical potential. The cusp at filling factor \( \nu = 2/3 \) is presumably responsible for features seen in the peak conductance voltage near \( B = 9 \) Tesla in Fig. 4; it will be interesting to see how much detailed information on the ground state energy will be extracted from tunneling experiments in the future.
6 Concluding Remarks

This brief survey highlights some active areas of electron-electron interaction physics in the fractional Hall regime. It seems certain that future experimental and theoretical activity will refine the present understanding of all these topics and, if the past is any guide, also lead to the discovery of phenomena which are not presently anticipated. I hope that any blurriness the reader notices in my snapshot of this tableau vivant inspires more intrigue than annoyance.

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