Differential Algebras on Digraphs and Generalized Path Homology

Shiquan Ren, Chong Wang

Abstract

The theory of path homology for digraphs was developed by Alexander Grigor’yan, Yong Lin, Yuri Muranov, and Shing-Tung Yau. In this paper, we generalize the path homology for digraphs. We prove that for any digraph $G$, any $t \geq 0$, any $0 \leq q \leq 2t$, and any $(2t+1)$-dimensional element $\alpha$ in the differential algebra on the set of the vertices, we always have an $(\alpha, q)$-path homology for $G$. In particular, if $t = 0$, then the $(\alpha, 0)$-path homology gives the weighted path homology for vertex-weighted digraphs.

1 Introduction

A digraph $G$ is a couple $(V, E)$ where $V$ is a set and $E$ is a subset of $V \times V$ such that for any $(u, v) \in E$, we have $u \neq v$. Throughout this paper, we assume that $V$ is finite. During the 2010’s, Alexander Grigor’yan, Yong Lin, Yuri Muranov, Shing-Tung Yau [1, 2, 3, 4] and Alexander Grigor’yan, Yuri Muranov, Shing-Tung Yau [5, 6] developed the theory of path complexes and path homology for digraphs. Recently, the authors in [7] generalized the path homology for digraphs and studied the weighted path homology for vertex-weighted digraphs.

Given a finite set $V$ and a non-negative integer $n \geq 0$, an $n$-path on $V$ is a sequence $v_0v_1 \ldots v_n$ of (not necessarily distinct) $n+1$ elements in $V$. Let $R$ be an integral domain such that 2 has a multiplicative inverse $\frac{1}{2}$. We let $\Lambda_n(V; R)$ be the free $R$-module generated by all the $n$-paths on $V$. For any element $v \in V$, the partial derivative $\frac{\partial}{\partial v}$ is an $R$-linear map from $\Lambda_n(V; R)$ to $\Lambda_{n-1}(V; R)$ whose explicit definition is given by (2.1).

We define the differential algebra $E^*(V; R)$ on $V$ as the exterior algebra generated by all the partial derivative operators of the elements in $V$. In this paper, for any digraph $G = (V, E)$, we prove the next theorem.

Theorem 1.1 (Main Result). Let $G = (V, E)$ be a digraph. Let $E^*(V; R)$ be the differential algebra of $V$. Then for any $t \geq 0$, any $\alpha \in E^{2t+1}(V; R)$, and any $0 \leq q \leq 2t$, we have a homology

$$H_n(G, \alpha, q; R), \quad n = 0, 1, \ldots, \quad (1.1)$$

for $G$. Moreover, for any $s \geq 0$ and any $\beta \in E^{2s}(V; R)$, we have a homomorphism

$$\beta_* : H_n(G, \alpha, q; R) \rightarrow H_n(G, \alpha, q - 2s; R), \quad n = 0, 1, 2, \ldots,$$

of the homology groups.

We call the homology in (1.1) the $(\alpha, q)$-path homology for $G$. Particularly, suppose $t = 0$. Then we have that $E^1(V; R)$ is the free $R$-module whose elements are of the form

$$\alpha = \sum_{v \in V} f(v) \frac{\partial}{\partial v}$$
where $f$ is any $R$-valued function on $V$. In this case, we have that $q = 0$. Moreover, the $(\alpha, 0)$-path homology for $G$ in Theorem 1.1 is the $f$-weighted path homology of the vertex-weighted digraph $(G, f)$ (cf. Example 2.4), which has been studied in [7]. In addition, if $f$ takes the constant value 1 for all $v \in V$, then the $(\alpha, 0)$-path homology in Theorem 1.1 is just the usual path homology of $G$ in [1, 2, 3, 4, 5, 6].

The remaining part of this paper is organized as follows. In Section 2, we give the differential algebra on a finite set $V$. In Section 3, we study the differential algebra for a digraph $G$ and prove Theorem 1.1. In Section 4, we give some examples for the $(\alpha, q)$-path homology for digraphs.

## 2 Differential Algebras on Finite Sets

In this section, we introduce the differential algebra $E^*(V)$ on any finite set $V$. For any element $\alpha$ in the differential algebra $E^*(V)$ with an odd homogeneous dimension $2t + 1$, we construct some chain complexes $\Lambda(V, \alpha, q)$ of paths on $V$ as well as some chain complexes $\mathcal{R}(V, \alpha, q)$ of regular paths on $V$, $0 \leq q \leq 2t$, with respect to $\alpha$.

Let $V$ be a finite set whose elements are called vertices. Let $n \geq 0$ be a non-negative integer. An elementary $n$-path on $V$ is an ordered sequence $v_0v_1 \ldots v_n$ of $n + 1$ vertices in $V$. Here for any $0 \leq i < j \leq n$, the two vertices $v_i$ and $v_j$ are not required to be distinct. Let $R$ be an integral domain such that $2$ has a multiplicative inverse $\frac{1}{2}$ in $R$. A formal linear combination of elementary $n$-paths on $V$ with coefficients in $R$ is called an $n$-path on $V$. Denote by $\Lambda_n(V; R)$ the free $R$-module of all $n$-paths; that is, any element in $\Lambda_n(V; R)$ is of the form

$$
\sum_{v_0, v_1, \ldots, v_n \in V} r_{v_0v_1 \ldots v_n} v_0v_1 \ldots v_n, \quad r_{v_0v_1 \ldots v_n} \in R.
$$

Letting $n$ run over all non-negative integers, we have a graded $R$-module

$$
\Lambda_*(V; R) = \bigoplus_{n=0}^{\infty} \Lambda_n(V; R).
$$

For any $v \in V$, we define the partial derivative of $\Lambda_*(V; R)$ with respect to $v$ to be a graded $R$-linear map

$$
\frac{\partial}{\partial v} : \Lambda_n(V; R) \to \Lambda_{n-1}(V; R), \quad n \geq 0,
$$

by letting

$$
\frac{\partial}{\partial v}(v_0v_1 \ldots v_n) = \sum_{i=0}^{n} (-1)^i \delta(v, v_i)v_0 \ldots \hat{v_i} \ldots v_n. \quad (2.1)
$$

Here in (2.1) we use the notation that for any vertices $u, v \in V$, $\delta(u, v) = 1$ if $u = v$ and $\delta(u, v) = 0$ if $u \neq v$. We extend (2.1) linearly over $R$. Particularly, if $v_0, v_1, \ldots, v_n$ are distinct vertices in $V$, then

$$
\frac{\partial}{\partial v}(v_0v_1 \ldots v_n) = (-1)^i v_0 \ldots \hat{v_i} \ldots v_n
$$

if $v_i = v$ for some $0 \leq i \leq n$; and

$$
\frac{\partial}{\partial v}(v_0v_1 \ldots v_n) = 0
$$

if $v_i \neq v$ for any $0 \leq i \leq n$. 

2
Lemma 2.1. For any \( u, v \in V \), we have
\[
\frac{\partial}{\partial u} \circ \frac{\partial}{\partial v} = -\frac{\partial}{\partial v} \circ \frac{\partial}{\partial u}.
\] (2.2)

Proof. Since both \( \frac{\partial}{\partial u} \) and \( \frac{\partial}{\partial v} \) are \( R \)-linear, it follows that both \( \frac{\partial}{\partial u} \circ \frac{\partial}{\partial v} \) and \( \frac{\partial}{\partial v} \circ \frac{\partial}{\partial u} \) are \( R \)-linear as well. Hence in order to prove the identity (2.2) as linear maps from \( \Lambda_n(V; R) \) to \( \Lambda_{n-1}(V; R) \), we only need to verify the identity (2.2) on an elementary \( n \)-path \( v_0v_1 \ldots v_n \). By the definition (2.1), we have
\[
\frac{\partial}{\partial u} \circ \frac{\partial}{\partial v}(v_0v_1 \ldots v_n) = \frac{\partial}{\partial u} \left( \sum_{j=0}^{n} (-1)^j \delta(v, v_j) v_0 \ldots \widehat{v_j} \ldots v_n \right)
= \sum_{j=0}^{n} (-1)^j \delta(v, v_j) \frac{\partial}{\partial u}(v_0 \ldots \widehat{v_j} \ldots v_n)
= \sum_{j=0}^{n} (-1)^j \delta(v, v_j) \sum_{i=0}^{j-1} (-1)^i \delta(u, v_i)(v_0 \ldots \widehat{v_i} \ldots \widehat{v_j} \ldots v_n)
+ \sum_{j=0}^{n} (-1)^j \delta(v, v_j) \sum_{i=j+1}^{n} (-1)^{i-1} \delta(u, v_i)(v_0 \ldots \widehat{v_i} \ldots \widehat{v_j} \ldots v_n)
= \sum_{0 \leq i < j \leq n} (-1)^{i+j} \delta(u, v_i) \delta(v, v_j)(v_0 \ldots \widehat{v_i} \ldots \widehat{v_j} \ldots v_n)
+ \sum_{0 \leq i < j \leq n} (-1)^{i+j-1} \delta(u, v_i) \delta(v, v_j)(v_0 \ldots \widehat{v_j} \ldots \widehat{v_i} \ldots v_n).
\]

Similarly,
\[
\frac{\partial}{\partial v} \circ \frac{\partial}{\partial u}(v_0v_1 \ldots v_n) = \sum_{0 \leq i < j \leq n} (-1)^{i+j} \delta(u, v_i) \delta(v, v_j)(v_0 \ldots \widehat{v_i} \ldots \widehat{v_j} \ldots v_n)
+ \sum_{0 \leq i < j \leq n} (-1)^{i+j-1} \delta(u, v_i) \delta(v, v_j)(v_0 \ldots \widehat{v_j} \ldots \widehat{v_i} \ldots v_n).
\]

Therefore, for any elementary \( n \)-path \( v_0v_1 \ldots v_n \) on \( V \), we have
\[
\frac{\partial}{\partial u} \circ \frac{\partial}{\partial v}(v_0v_1 \ldots v_n) + \frac{\partial}{\partial v} \circ \frac{\partial}{\partial u}(v_0v_1 \ldots v_n) = 0.
\]
Consequently, by the \( R \)-linear property of \( \frac{\partial}{\partial u} \circ \frac{\partial}{\partial v} \) and \( \frac{\partial}{\partial v} \circ \frac{\partial}{\partial u} \), we obtain (2.2). \( \square \)

By Lemma 2.1, it makes sense that we denote \( \frac{\partial}{\partial u} \circ \frac{\partial}{\partial v} \) as \( \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v} \) for convenience.

We consider the exterior algebra
\[
E^*(V; R) = E^*_R \left( \frac{\partial}{\partial v} | v \in V \right).
\] (2.3)
Precisely, (2.3) is a graded \( R \)-module
\[
E^*(V; R) = \bigoplus_{k=1}^{\infty} E^k(V; R).
\]
Here for each \( k \geq 1 \), \( E^k(V; R) \) is the \( R \)-module consisting of all the formal linear combinations
\[
\sum_{v_1, v_2, \ldots, v_k \in V} r_{v_1v_2 \ldots v_k} \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_2} \wedge \cdots \wedge \frac{\partial}{\partial v_k}, \quad r_{v_1v_2 \ldots v_k} \in R.
\]
Since \( \partial V \), we have an exterior product

\[ \Lambda^k(V; R) \times E^l(V; R) \rightarrow E^{k+l}(V) \]

such that for any \( \alpha \in E^k(V; R) \) and any \( \beta \in E^l(V; R) \), the anti-commutative law

\[ \alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha \]  \hspace{1cm} (2.4)

is satisfied. Furthermore, we have the distributive law

\[
(\alpha_1 + \alpha_2) \wedge \beta = \alpha_1 \wedge \beta + \alpha_2 \wedge \beta,
\]

and the associative law

\[ (\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma) \]

for any positive integers \( k, l, s, \) and any \( \alpha, \alpha_1, \alpha_2 \in E^k(V; R) \), any \( \beta, \beta_1, \beta_2 \in E^l(V; R) \); and the associative law

\[ (\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma) \]

for any positive integers \( k, l, s, \) and any \( \alpha, \beta, \gamma \in E^k(V) \).

By Lemma 2.1, we see that \( E^*(V; R) \) gives all the \( R \)-linear combinations of the partial derivatives on \( \Lambda_s(V; R) \) as well as all the \( R \)-linear combinations of the compositions of the partial derivatives. We call \( E^*(V; R) \) the differential algebra on \( V \) with coefficients in \( R \). For any element \( \alpha \in E^*(V; R) \), if there exists \( k \geq 1 \) such that \( \alpha \in E^k(V; R) \), then we say that \( \alpha \) is of homogeneous dimension \( k \).

Particularly, we take \( k = l = 2t + 1 \) for the integers \( t \geq 0 \) and take \( \alpha = \beta \) in (2.4). Then (2.4) implies the next lemma.

**Lemma 2.2.** For any non-negative integer \( t \) and any \( \alpha \in E^{2t+1}(V; R) \), we have

\[ \alpha \wedge \alpha = 0. \]

**Proof.** Take \( k = l = 2t + 1 \) and \( \alpha = \beta \) in (2.4). Then we have

\[ 2 \alpha \wedge \alpha = 0. \]

Since 2 is assumed to have a multiplicative inverse \( \frac{1}{2} \) in \( R \), the lemma follows. \( \square \)

Consider a fixed positive odd integer \( 2t + 1 \) and a fixed element \( \alpha \in E^{2t+1}(V; R) \). We recall that \( \alpha \) is a graded linear map

\[ \alpha : \Lambda_n(V; R) \rightarrow \Lambda_{n-(2t+1)}(V; R), \quad n \geq 0. \]

For each \( 0 \leq q \leq 2t \) and each \( n \geq 0 \), we use the following notation

\[ \Lambda_n(V, \alpha, q; R) = \Lambda_{n(2t+1)+q}(V; R) \]

and the following notation

\[ \alpha_{n,q} : \Lambda_n(V, \alpha, q; R) \rightarrow \Lambda_{n-1}(V, \alpha, q; R). \]
By Lemma 2.2

\[ \text{Im} \alpha_{n+1,q} \subseteq \text{Ker} \alpha_{n,q} \]

for each \( 0 \leq q \leq 2t \) and each \( n \geq 0 \). Thus for each fixed \( 0 \leq q \leq 2t \), \( \alpha_{n,q} \) is a boundary operator for each \( n \geq 0 \). Consequently, we have a chain complex

\[ \Lambda_\ast(V, \alpha, q; R) = \{ \Lambda_n(V, \alpha, q; R), \alpha_{n,q} \}_{n \geq 0} \]

for each fixed \( 0 \leq q \leq 2t \). Let \( k \in \mathbb{Z} \). For any \( 0 \leq q \leq 2t \), we use the notation

\[ \Lambda_n(V, \alpha, q + k(2t + 1); R) = \Lambda_{n+k}(V, \alpha; q; R), \quad n \geq 0. \]

In other words, for any \( p \in \mathbb{Z} \), we use the notation

\[ \Lambda_n(V, \alpha, p; R) = \Lambda_{n+k}(V, \alpha; q; R), \quad n \geq 0, \]

where we write \( p \) as its unique expression

\[ p = k(2t + 1) + q, \quad 0 \leq q \leq 2t, \quad k \in \mathbb{Z}. \]

On the other hand, let \( s \) be a non-negative integer. Let \( \beta \in E^{2s}(V; R) \). It follows from (2.3) that for any \( t \geq 0 \) and any \( \alpha \in E^{2t+1}(V; R) \), we have

\[ \beta \wedge \alpha = \alpha \wedge \beta. \]

Therefore, for any \( t \geq 0 \), any \( \alpha \in E^{2t+1}(V; R) \), any \( 0 \leq q \leq 2t \), and any \( n \geq 0 \), the following diagram commutes

\[ \begin{array}{ccc}
\Lambda_n(V, \alpha, q; R) & \xrightarrow{\alpha} & \Lambda_{n-1}(V, \alpha, q; R) \\
\beta & & \beta \\
\Lambda_n(V, \alpha, q-2s; R) & \xrightarrow{\alpha} & \Lambda_{n-1}(V, \alpha, q-2s; R). 
\end{array} \]

Consequently, for any \( t \geq 0 \), any \( \alpha \in E^{2t+1}(V; R) \), and any \( 0 \leq q \leq 2t \), it follows that \( \beta \) gives a chain map

\[ \beta : \Lambda_\ast(V, \alpha, q; R) \longrightarrow \Lambda_\ast(V, \alpha, q - 2s; R). \]

Summarizing the above two paragraphs, the next proposition follows.

**Proposition 2.3.** For any non-negative integer \( t \), any \( \alpha \in E^{2t+1}(V; R) \), and any \( 0 \leq q \leq 2t \), we have a chain complex \( \Lambda_\ast(V, \alpha, q; R) \) given by

\[ \cdots \xrightarrow{\alpha_{n+1,q}} \Lambda_{n(2t+1)+q}(V; R) \xrightarrow{\alpha_{n,q}} \Lambda_{(n-1)(2t+1)+q}(V; R) \xrightarrow{\alpha_{n-1,q}} \cdots \]

\[ \cdots \xrightarrow{\alpha_{2,q}} \Lambda_{(2t+1)+q}(V; R) \xrightarrow{\alpha_{1,q}} \Lambda_{q}(V; R) \xrightarrow{\alpha_{0,q}} 0. \]

Moreover, for any non-negative integer \( s \) and any \( \beta \in E^{2s}(V; R) \), \( \beta \) induces a chain map from \( \Lambda_\ast(V, \alpha, q; R) \) to \( \Lambda_\ast(V, \alpha, q - 2s; R) \). \( \square \)

In the next example, we consider the particular case \( t = 0 \) in Proposition 2.3, which gives the usual boundary operator \( \partial_s \) (cf. [11, 12, 13, 14]) on \( \Lambda_\ast(V; R) \) as well as the weighted boundary operator \( \partial_{2,q}^l \) (cf. [17]) on \( \Lambda_\ast(V; R) \). We calculate in the example without using Proposition 2.3 in order to corroborate Proposition 2.3.
Example 2.4. Let $V$ be a finite set. Let $f : V \to R$ be any $R$-valued function on $V$. We consider a formal linear combination

$$\partial f^* = \sum_{v \in V} f(v) \frac{\partial}{\partial v}$$

with coefficients in $R$. Then for each $n \geq 0$,

$$\partial_n f^* : \Lambda_n(V; R) \to \Lambda_{n-1}(V; R)$$

is an $R$-linear map. Moreover, with the help of Lemma 2.1, we have

$$\partial_n f^* \circ \partial_{n+1} f^* = \frac{1}{2} \left( \sum_{v \in V} f(v) \frac{\partial}{\partial v} \right) \wedge \left( \sum_{u \in V} f(u) \frac{\partial}{\partial u} \right)$$

$$= \frac{1}{2} \sum_{u,v \in V} f(v) f(u) \left( \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial u} + \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v} \right)$$

$$= 0.$$

Furthermore, for any $v_0 v_1 \ldots v_n \in \Lambda_n(V; R)$, we have

$$\partial_n f^* (v_0 v_1 \ldots v_n) = \sum_{v \in V} f(v) \frac{\partial}{\partial v} (v_0 v_1 \ldots v_n)$$

$$= \sum_{v \in V} f(v) \sum_{i=0}^{n-1} (-1)^i \delta(v, v_i) v_0 \ldots \hat{v}_i \ldots v_n$$

$$= \sum_{i=0}^{n} (-1)^i \left( \sum_{v \in V} f(v) \delta(v, v_i) \right) v_0 \ldots \hat{v}_i \ldots v_n$$

$$= \sum_{i=0}^{n} (-1)^i f(v_i) v_0 \ldots \hat{v}_i \ldots v_n.$$

Thus we see that $\partial f^*$ is the $f$-weighted boundary operator studied in [7]. In particular, letting $f$ be the constant function $1$ taking the value $1 \in R$, then $\partial_* = \partial_1$ is the usual (unweighted) boundary operator adopted in [1, 2, 3, 4, 5].

In the next example, we consider the particular case $t = 1$ in Proposition 2.3. We calculate without using Proposition 2.3 in order to corroborate Proposition 2.3.

Example 2.5. Let $V$ be a finite set. Let $\alpha \in E^3(V; R)$. Then we can write $\alpha$ in the form

$$\alpha = \sum_{v,u,w \in V} f(v, u, w) \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial w}$$

where $f : V \times V \times V \to R$ is an $R$-valued function on the 3-fold Cartesian product of $V$. For each $n \geq 0$, $\alpha$ gives an $R$-linear map

$$\alpha_n : \Lambda_n(V; R) \to \Lambda_{n-3}(V; R).$$
Moreover, with the help of Lemma \([2.1]\) we have that for each \(n \geq 0\),

\[
\alpha_n \circ \alpha_{n+1} = \sum_{v,u,w,v',w'\in V} f(v,u,w)f(v',u',w') \left( \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial w} \wedge \frac{\partial}{\partial v'} \wedge \frac{\partial}{\partial u'} \wedge \frac{\partial}{\partial w'} \right) + \frac{1}{2} \left( \sum_{v,u,w,v',w'\in V} f(v,u,w)f(v',u',w') \left( \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial w} \wedge \frac{\partial}{\partial w'} \wedge \frac{\partial}{\partial u'} \wedge \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial w} \right) \right) = 0.
\]

We see that \(\alpha\) is a boundary operator.

For any \(n \geq 0\), an elementary \(n\)-path \(v_0v_1\ldots v_n\) on \(V\) is called regular if \(v_{i-1} \neq v_i\) for each \(1 \leq i \leq n\). Let \(\mathcal{R}_n(V; R)\) be the free \(R\)-module consisting of all the formal linear combinations of the regular elementary \(n\)-paths on \(V\), with coefficients in \(R\). By the argument in \([2]\), we have that for any \(v \in V\), \(\mathcal{R}_n(V; R)\) induces a graded \(R\)-linear map

\[
\frac{\partial}{\partial v} : \mathcal{R}_n(V; R) \rightarrow \mathcal{R}_{n-1}(V; R), \quad n \geq 0.
\]

It follows that for each positive integer \(k\) and each element \(\alpha \in E^k(V; R)\), \(\alpha\) is a graded linear map

\[
\alpha : \mathcal{R}_n(V; R) \rightarrow \mathcal{R}_{n-k}(V; R), \quad n \geq 0.
\]

In addition, if \(k = 2t + 1\) is odd, then we have \(\alpha \wedge \alpha = 0\). Let \(t \geq 0\) and \(\alpha \in E^{2t+1}(V; R)\). Let \(0 \leq q \leq 2t\). For any \(n \geq 0\), we use the notation

\[
\mathcal{R}_n(V, \alpha, q; R) = \mathcal{R}_{n(2t+1)+q}(V; R), \quad n \geq 0.
\]

Let \(p \in \mathbb{Z}\). More generally, we use the notation

\[
\mathcal{R}_n(V, \alpha, p; R) = \mathcal{R}_{n+k}(V, \alpha, q; R), \quad n \geq 0.
\]

Here we write \(p\) as its unique expression

\[
p = k(2t+1) + q, \quad 0 \leq q \leq 2t, \quad k \in \mathbb{Z},
\]

and write \(\mathcal{R}_{n-k}(V, \alpha, p; R) = 0\) for all \(n \geq 0\). Similar with Proposition \([2.3]\), the next proposition follows.

**Proposition 2.6.** For any non-negative integer \(t\), any \(\alpha \in E^{2t+1}(V; R)\), and any \(0 \leq q \leq 2t\), we have a chain complex \(\mathcal{R}_*(V, \alpha, q; R)\) given by

\[
\cdots \xrightarrow{\alpha_{n+1,q}} \mathcal{R}_{n+1}(V; R) \xrightarrow{\alpha_{n,q}} \mathcal{R}_n(V; R) \xrightarrow{\alpha_{n-1,q}} \cdots
\]

\[
\cdots \xrightarrow{\alpha_{2,q}} \mathcal{R}_{2t+1}(V; R) \xrightarrow{\alpha_{1,q}} \mathcal{R}_{2t}(V; R) \xrightarrow{\alpha_{0,q}} \mathcal{R}_q(V; R) \rightarrow 0.
\]

Moreover, for any non-negative integer \(s\) and any \(\beta \in E^{2s}(V; R)\), \(\beta\) induces a chain map from \(\mathcal{R}_*(V, \alpha, q; R)\) to \(\mathcal{R}_*(V, \alpha, q - 2s; R)\). \(\blacksquare\)
3 Differential Algebras on Digraphs

In this section, for any element $\alpha$ in the differential algebra $E^*(V)$ with an odd homogeneous dimension $2t + 1$ and any digraph $G = (V, E)$, we construct some chain complexes $\Omega(G, \alpha, q)$ of allowed paths on $G$, $0 \leq q \leq 2t$, with respect to $\alpha$. Then we prove Theorem 3.1.

Let $V$ be a finite set. A digraph $G$ is a couple $(V, E)$ where $E$ is a subset of $\{(x, y) \in V \times V \mid x \neq y\}$. Note that $E$ consists of some ordered pairs of distinct vertices in $V$. The elements of $E$ are called directed edges.

Let $G = (V, E)$ be a digraph. Let $n \geq 0$. A regular elementary $n$-path $v_0 v_1 \ldots v_n$ on $V$ is called allowed if for any $1 \leq i \leq n$, we have $(v_{i-1}, v_i) \in E$. Let $A_n(G; R)$ be the free $R$-module generated by all the allowed regular elementary $n$-paths on $G$. Then $A_n(G; R)$ is a sub-$R$-module of $\mathcal{R}_n(V; R)$. We consider the graded $R$-module

$$A_\ast(G; R) = \bigoplus_{n=0}^{\infty} A_n(G; R).$$

Note that $A_\ast(G; R)$ may not be closed under the operations of the elements in $E^*(V; R)$. That is, for any $k \geq 1$ and any $\alpha \in E^k(V; R)$, we have a graded $R$-linear map

$$\alpha : A_n(G; R) \rightarrow \mathcal{R}_{n-k}(V; R), \quad n \geq 0.$$

Here for each $n \geq 0$, the image of $A_\ast(G; R)$ under the map $\alpha$ may not be contained in $A_{n-k}(G; R)$.

Now we fix $k \geq 1$ and $\alpha \in E^k(V; R)$. For each $n \geq 0$, we define

$$\Omega_n(G, \alpha; R) = A_n(G; R) \cap \alpha^{-1}(A_{n-k}(G; R)).$$

Here $\alpha^{-1}(A_{n-k}(G; R))$ denotes the pre-image of $A_{n-k}(G; R)$ under the map $\alpha$. Then we have a graded $R$-linear map

$$\alpha : \Omega_n(G, \alpha; R) \rightarrow \Omega_{n-k}(G, \alpha; R), \quad n \geq 0. \quad (3.1)$$

For differentiation, we denote the $\alpha$ in (3.1) as $\alpha_{n,q}$ for $n = mk + q$ where $0 \leq q \leq k - 1$. We take $k$ to be an odd number $2t + 1$ with $t \geq 0$. Let $\alpha \in E^{2t+1}(V; R)$. Let $0 \leq q \leq 2t$. For any $n \geq 0$, we use the notation

$$\Omega_n(V, \alpha, q; R) = \Omega_n(2t+1)+q(V, \alpha; R), \quad n \geq 0.$$

Let $p \in \mathbb{Z}$. More generally, we use the notation

$$\Omega_n(V, \alpha, p; R) = \Omega_{n+k}(V, \alpha, q; R), \quad n \geq 0.$$

Here we write $p$ as its unique expression

$$p = k(2t + 1) + q, \quad 0 \leq q \leq 2t, \quad k \in \mathbb{Z},$$

and write $\Omega_{-n}(V, \alpha, p; R) = 0$ for all $n \geq 0$. We have the following proposition.

**Proposition 3.1.** For any $t \geq 0$, any $\alpha \in E^{2t+1}(V; R)$, and any $0 \leq q \leq 2t$, we have a chain complex $\Omega_\ast(G, \alpha, q; R)$ given by

$$\cdots \xrightarrow{\alpha_{n+1,q}} \Omega_n(2t+1)+q(G, \alpha; R) \xrightarrow{\alpha_{n,q}} \Omega_{n-1}(2t+1)+q(G, \alpha; R) \xrightarrow{\alpha_{n-1,q}} \cdots$$

$$\cdots \xrightarrow{\alpha_{2,q}} \Omega_{2t+1}+q(G, \alpha; R) \xrightarrow{\alpha_{1,q}} \Omega_q(G, \alpha; R) \xrightarrow{\alpha_{0,q}} 0.$$
Moreover, for any non-negative integer \( s \geq 0 \) and any \( \beta \in E^{2s}(V; R) \), \( \beta \) induces a chain map from \( \Omega_*(G, \alpha, q; R) \) to \( \Omega_*(G, \alpha, q - 2s; R) \).

Proof. We restrict the boundary operators of the chain complex \( \mathcal{R}_*(V, \alpha, q; R) \) (given in Proposition 2.6) to the graded sub-\( R \)-module \( \Omega_*(2n+1+q)(G, \alpha; R) \), \( n \geq 0 \). With the help of Proposition 4.1, we obtain that \( \Omega_*(G, \alpha, q; R) \) is a sub-chain complex of \( \mathcal{R}_*(V, \alpha, q; R) \). We obtain the first assertion of the proposition.

Let \( s \geq 0 \) and \( \beta \in E^{2s}(V; R) \). Since \( \Omega_*(G, \alpha, q; R) \) is a sub-chain complex of \( \mathcal{R}_*(V, \alpha, q; R) \), by restricting the chain map \( \beta \) in Proposition 2.6 to \( \Omega_*(G, \alpha, q; R) \), it follows that there is a chain map

\[
\beta : \Omega_*(V, \alpha, q; R) \rightarrow \Omega_*(V, \alpha, q - 2s; R).
\]

We obtain the second assertion of the proposition. \( \square \)

Now we prove the main result Theorem 4.1

Proof of Theorem 4.1 Let \( G = (V, E) \) be a digraph. Let \( t \geq 0 \), \( \alpha \in E^{2t+1}(V; R) \), and \( 0 \leq q \leq 2t \). Take the homology groups of the chain complex \( \Omega_*(G, \alpha, q; R) \) and denote the homology groups as \( H_*(G, \alpha, q; R) \).

In addition, let \( s \geq 0 \) and \( \beta \in E^{2s}(V; R) \). By Proposition 5.1, \( \beta \) is a chain map from \( \Omega_*(G, \alpha, q; R) \) to \( \Omega_*(G, \alpha, q - 2s; R) \). Hence \( \beta \) induces a homomorphism

\[
\beta_* : H_*(G, \alpha, q; R) \rightarrow H_*(G, \alpha, q - 2s; R)
\]

of homology groups. The theorem follows. \( \square \)

4 Examples

We give some examples for the \((\alpha, q)\)-path homology for digraphs. Throughout this section, we take \( R \) to be the real numbers \( \mathbb{R} \).

Example 4.1. Consider the digraph \( G = (V, E) \) where \( V = \{v_0, v_1, v_2, v_3, v_4, v_5\} \) and \( E = \{v_0v_1, v_0v_2, v_0v_3, v_1v_4, v_2v_3, v_2v_4, v_5v_3, v_5v_4\} \) (cf. [2, Section 4.6 and Figure 8]). We have

\[
\begin{align*}
\mathcal{A}_0(G; \mathbb{R}) &= \text{Span}\{v_0, v_1, v_2, v_3, v_4, v_5\}, \\
\mathcal{A}_1(G; \mathbb{R}) &= \text{Span}\{v_0v_1, v_0v_2, v_0v_3, v_1v_4, v_2v_3, v_2v_4, v_5v_3, v_5v_4\}, \\
\mathcal{A}_2(G; \mathbb{R}) &= \text{Span}\{v_0v_1v_3, v_0v_1v_4, v_0v_2v_3, v_0v_2v_4\}.
\end{align*}
\]

(1) Let \( t = 1 \), \( \alpha \in E^5(V; \mathbb{R}) \), and \( q = 0 \). Then we write \( \alpha \) in the following form

\[
\alpha = \sum_{i=0}^{5} f(v_i) \frac{\partial}{\partial v_i}.
\]

We have

\[
\alpha(v_i, v_j) = f(v_i)v_j - f(v_j)v_i,
\]

for any allowed elementary 1-path \( v_i, v_j \in \mathcal{A}_1(G) \) and

\[
\alpha(v_i, v_j, v_k) = f(v_i)v_jv_k - f(v_j)v_iv_k + f(v_k)v_iv_j
\]
for any allowed elementary 2-path $v_i v_j v_k \in \mathcal{A}_2(G)$. It follows that

$$
\Omega_0(G, \alpha, 0; \mathbb{R}) = \mathcal{A}_0(G; \mathbb{R}), \quad \Omega_1(G, \alpha, 0; \mathbb{R}) = \mathcal{A}_1(G; \mathbb{R}).
$$

**Case 1.** either $f(v_1) \neq 0$ or $f(v_2) \neq 0$. Then we have

$$
\Omega_2(G, \alpha, 0; \mathbb{R}) = \text{Span}\{f(v_2)v_0v_1v_3 - f(v_1)v_0v_2v_3, f(v_2)v_0v_1v_4 - f(v_1)v_0v_2v_4\}.
$$

Thus

$$
\dim \Omega_2(G, \alpha, 0; \mathbb{R}) = 2.
$$

It is direct that

$$
\alpha(f(v_2)v_0v_1v_3 - f(v_1)v_0v_2v_3) = f(v_2)f(v_0)v_1v_3 + f(v_2)f(v_3)v_0v_1 - f(v_1)f(v_0)v_2v_3 - f(v_1)f(v_3)v_0v_2,
$$

$$
\alpha(f(v_2)v_0v_1v_4 - f(v_1)v_0v_2v_4) = f(v_2)f(v_0)v_1v_4 + f(v_2)f(v_4)v_0v_1 - f(v_1)f(v_0)v_2v_4 - f(v_1)f(v_4)v_0v_2.
$$

**Subcase 1.1.** $f(v_i) \neq 0$. Then $\alpha(f(v_2)v_0v_1v_3 - f(v_1)v_0v_2v_3)$ and $\alpha(f(v_2)v_0v_1v_4 - f(v_1)v_0v_2v_4)$ are linearly independent. Thus

$$
\dim \alpha(\Omega_2(G, \alpha, 0; \mathbb{R})) = 2.
$$

Hence

$$
\dim H_2(G, \alpha, 0; \mathbb{R}) = \dim \Omega_2(G, \alpha, 0; \mathbb{R}) - \dim \alpha(\Omega_2(G, \alpha, 0; \mathbb{R})) = 0.
$$

**Subcase 1.2.** $f(v_0) = 0$.

**Subcase 1.2.1.** either $f(v_3) \neq 0$ or $f(v_4) \neq 0$. Then

$$
\dim \alpha(\Omega_2(G, \alpha, 0; \mathbb{R})) = 1.
$$

Hence

$$
\dim H_2(G, \alpha, 0; \mathbb{R}) = 1.
$$

**Subcase 1.2.2.** $f(v_3) = f(v_4) = 0$. Then

$$
\dim \alpha(\Omega_2(G, \alpha, 0; \mathbb{R})) = 0.
$$

Hence

$$
\dim H_2(G, \alpha, 0; \mathbb{R}) = 2.
$$

**Case 2.** $f(v_1) = f(v_2) = 0$. Then

$$
\Omega_2(G, \alpha, 0; \mathbb{R}) = \mathcal{A}_2(G; \mathbb{R}).
$$

Thus

$$
\dim \Omega_2(G, \alpha, 0; \mathbb{R}) = 4.
$$
It is direct that
\[
\alpha(v_0v_1v_3) = f(v_0)v_1v_3 + f(v_3)v_0v_1,
\]
\[
\alpha(v_0v_1v_4) = f(v_0)v_1v_4 + f(v_4)v_0v_1,
\]
\[
\alpha(v_0v_2v_3) = f(v_0)v_2v_3 + f(v_3)v_0v_2,
\]
\[
\alpha(v_0v_2v_4) = f(v_0)v_2v_4 + f(v_4)v_0v_2.
\]

**Subcase 2.1.** \( f(v_0) \neq 0 \). Then \( \alpha(v_0v_1v_3), \alpha(v_0v_1v_4), \alpha(v_0v_2v_3), \) and \( \alpha(v_0v_2v_4) \) are linearly independent. Thus

\[\dim \alpha(\Omega_2(G, \alpha, 0; \mathbb{R})) = 4.\]

Therefore,

\[\dim H_2(G, \alpha, 0; \mathbb{R}) = \dim \Omega_2(G, \alpha, 0; \mathbb{R}) - \dim \alpha(\Omega_2(G, \alpha, 0; \mathbb{R})) = 0.\]

**Subcase 2.2.** \( f(v_0) = 0 \).

**Subcase 2.2.1.** either \( f(v_3) \neq 0 \) or \( f(v_4) \neq 0 \). Then

\[\dim \alpha(\Omega_2(G, \alpha, 0; \mathbb{R})) = 2.\]

Hence

\[\dim H_2(G, \alpha, 0; \mathbb{R}) = 2.\]

**Subcase 2.2.2.** \( f(v_3) = f(v_4) = 0 \). Then

\[\dim \alpha(\Omega_2(G, \alpha, 0; \mathbb{R})) = 0.\]

Hence

\[\dim H_2(G, \alpha, 0; \mathbb{R}) = 4.\]

Summarizing all the above cases, we obtain all the possibilities of \( H_2(G, \alpha, 0; \mathbb{R}) \). Similarly, we can also calculate \( H_0(G, \alpha, 0; \mathbb{R}) \) and \( H_1(G, \alpha, 0; \mathbb{R}) \). We omit the details for the calculation.

(2). Let \( t = 1, \alpha \in E^3(V; \mathbb{R}) \), and \( q = 0, 1, 2 \). Then we can write \( \alpha \) in the following form

\[
\alpha = \sum_{0 \leq i < j < k \leq 5} f(v_i, v_j, v_k) \frac{\partial}{\partial v_i} \wedge \frac{\partial}{\partial v_j} \wedge \frac{\partial}{\partial v_k}.
\]

Suppose in addition that for any \( 0 \leq i < j < k \leq 5 \), \( f(v_i, v_j, v_k) \neq 0 \). Then we have

\[\alpha(A_0(G; \mathbb{R})) = \alpha(A_1(G; \mathbb{R})) = \alpha(A_2(G; \mathbb{R})) = 0.\]

Hence

\[\Omega_0(G, \alpha, q; \mathbb{R}) = A_q(G; \mathbb{R})\]

for any \( q = 0, 1, 2 \) and

\[\Omega_i(G, \alpha, q; \mathbb{R}) = 0\]
for any \( i \geq 1 \) and any \( q = 0, 1, 2 \). Therefore,

\[
\dim H_0(G, \alpha, q; \mathbb{R}) = \dim A_q(G; \mathbb{R})
\]

for any \( q = 0, 1, 2 \); explicitly,

\[
\dim H_0(G, \alpha, 0; \mathbb{R}) = 6, \quad \dim H_0(G, \alpha, 1; \mathbb{R}) = 8, \quad \dim H_0(G, \alpha, 2; \mathbb{R}) = 4.
\]

Moreover,

\[
\dim H_i(G, \alpha, q; \mathbb{R}) = 0
\]

for any \( i \geq 1 \) and any \( q = 0, 1, 2 \).

**Example 4.2.** Let \( V = \{v_0, v_1, v_2, v_3\} \) and \( E = \{v_0v_1, v_0v_2, v_0v_3, v_1v_2, v_1v_3, v_2v_3\} \).

Consider the digraph \( G = (V, E) \). Let \( \alpha \in E^3(V; \mathbb{R}) \) given by

\[
\alpha = \frac{\partial}{\partial v_0} \wedge \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_2}.
\]

We have \( \Omega_n(G, \alpha, 0, \mathbb{R}) = 0 \) for all \( n \geq 2 \) and

\[
\begin{align*}
\Omega_1(G, \alpha, 0; \mathbb{R}) &= \text{Span}\{v_0v_1v_2v_3\}, \\
\Omega_0(G, \alpha, 0; \mathbb{R}) &= \text{Span}\{v_0, v_1, v_2, v_3\}.
\end{align*}
\]

The boundary operator

\[
\alpha_{1,0} : \Omega_1(G, \alpha, 0; \mathbb{R}) \longrightarrow \Omega_0(G, \alpha, 0; \mathbb{R})
\]

is given by

\[
\begin{align*}
\alpha(v_0v_1v_2v_3) &= \frac{\partial}{\partial v_0} \wedge \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_2}(v_0v_1v_2v_3) \\
&= (-1)^2 \frac{\partial}{\partial v_0} \wedge \frac{\partial}{\partial v_1}(v_0v_1v_3) \\
&= (-1)^{2+1} \frac{\partial}{\partial v_0}(v_0v_3) \\
&= (-1)^{2+1+0}(v_3) \\
&= v_3.
\end{align*}
\]

Consequently,

\[
\ker(\alpha_{1,0}) = 0, \quad \text{Im}(\alpha_{1,0}) = \text{Span}\{v_3\}.
\]

Therefore,

\[
H_1(G, \alpha, 0; \mathbb{R}) = 0, \quad H_0(G, \alpha, 0; \mathbb{R}) = \text{Span}\{v_0, v_1, v_2\} \cong \mathbb{R}^3.
\]

Moreover,

\[
\begin{align*}
\Omega_1(G, \alpha, 2; \mathbb{R}) &= 0, \\
\Omega_0(G, \alpha, 2; \mathbb{R}) &= \text{Span}\{v_0v_1v_2, v_0v_1v_3, v_0v_2v_3, v_1v_2v_3\}
\end{align*}
\]

and

\[
\begin{align*}
\Omega_1(G, \alpha, 1; \mathbb{R}) &= 0, \\
\Omega_0(G, \alpha, 1; \mathbb{R}) &= \text{Span}\{v_0v_1, v_0v_2, v_0v_3, v_1v_2, v_1v_3, v_2v_3\}.
\end{align*}
\]
Consequently, it follows respectively that

\[ H_1(G, \alpha, 2; \mathbb{R}) = 0, \quad H_0(G, \alpha, 2; \mathbb{R}) \cong \mathbb{R}^4 \]

and

\[ H_1(G, \alpha, 1; \mathbb{R}) = 0, \quad H_0(G, \alpha, 1; \mathbb{R}) \cong \mathbb{R}^6. \]

Acknowledgements. The authors would like to express their deepest gratitude to Professor Yong Lin and Professor Jie Wu for their kind instruction and helpful guidance.

References

[1] Alexander Grigor’yan, Yong Lin, Shing-Tung Yau, Torsion of digraphs and path complexes, arXiv: 2012.07302v1, 2020.

[2] Alexander Grigor’yan, Yong Lin, Yuri Muranov, Shing-Tung Yau, Homologies of path complexes and digraphs, arXiv: 1207.2834, 2013.

[3] Alexander Grigor’yan, Yong Lin, Yuri Muranov, Shing-Tung Yau, Homotopy theory for digraphs, Pure and Applied Mathematics Quarterly, 10 (4), 619-674, 2014.

[4] Alexander Grigor’yan, Yong Lin, Yuri Muranov, Shing-Tung Yau, Cohomology of digraphs and (undirected) graph, Asian Journal of Mathematics, 15 (5), 887-932, 2015.

[5] Alexander Grigor’yan, Yuri Muranov, Shing-Tung Yau, Homologies of digraphs and K"{u}nneth formulas, Communications in Analysis and Geometry, 25, 969-1018, 2017.

[6] Alexander Grigor’yan, Yuri Muranov, Shing-Tung Yau, Path complexes and their homologies, Journal of Mathematical Sciences, 248 (5), 564-599, 2020.

[7] Shiquan Ren, Chong Wang, Weighted analytic torsions for weighted digraphs, arXiv: 2103.09552, 2021.

Shiquan Ren (corresponding author; first author)
Address: Yau Mathematical Sciences Center, Tsinghua University, Beijing, 100084, P. R. China
E-mail: srenmath@126.com

Chong Wang
Address: School of Mathematics, Renmin University of China, Beijing, 100872, P. R. China; School of Mathematics and Statistics, Cangzhou Normal University, Cangzhou, Hebei, 061000, P. R. China
E-mail: wangchong_618@163.com