Average size of the 2-Selmer group of Jacobians of monic even hyperelliptic curves

Arul Shankar and Xiaoheng Wang

May 7, 2014

Abstract

In [5], Manjul Bhargava and Benedict Gross considered the family of hyperelliptic curves over \( \mathbb{Q} \) having a fixed genus and a marked rational Weierstrass point. They showed that the average size of the 2-Selmer group of the Jacobians of these curves, when ordered by height, is 3. In this paper, we consider the family of hyperelliptic curves over \( \mathbb{Q} \) having a fixed genus and a marked rational non-Weierstrass point. We show that when these curves are ordered by height, the average size of the 2-Selmer group of their Jacobians is 6. This yields an upper bound of \( \frac{5}{2} \) on the average rank of the Mordell-Weil group of the Jacobians of these hyperelliptic curves.

Finally using an equidistribution result, we modify the techniques of [19] to conclude that as \( g \) tends to infinity, a proportion tending to 1 of these monic even-degree hyperelliptic curves having genus \( g \) have exactly two rational points—the marked point at infinity and its hyperelliptic conjugate.

Contents

1 Introduction 2

2 Orbit parameterization 4

2.1 Geometric orbits 4

2.2 Rational orbits via Galois cohomology 5

2.3 Connection to hyperelliptic curves 8

2.4 Integral orbits 9

3 Interpretation using pencils of quadrics 11

4 Orbit counting 13

4.1 Outline of the proof 14

4.2 Construction of fundamental domains 15

4.3 Averaging, cutting off the cusp, and estimation in the main body 16

4.4 A squarefree sieve 21

4.5 Compatibility of measures and local computations 22

5 Proof of the main results 24

5.1 The number of hyperelliptic curves in a large family having bounded height 24

5.2 The average size of the 2-Selmer group 25

6 Most monic even hyperelliptic curves have only two rational points 25
1 Introduction

In [5], Manjul Bhargava and Benedict Gross studied hyperelliptic curves over \( \mathbb{Q} \) with a rational Weierstrass point. Any such curve of genus \( g \) can be given as the smooth projective model of the affine curve defined by

\[
y^2 = x^{2n+1} + c_2 x^{2n-1} + \cdots + c_{2n+1},
\]

where \( c_i \in \mathbb{Q} \) and the given rational Weierstrass point lies above \( x = \infty \). If we further assume that \( c_i \in \mathbb{Z} \) and that there is no prime \( p \) such that \( p^{2i} \) divides \( c_i \) for all \( i \), then such an expression is unique. The height \( H \) of such a curve \( C \) is defined by

\[
H(C) = \max \{|c_k|^{(2n+1)/(2n+2k)}\}^{(2n+1)/(2n+2k)}. 
\]

Bhargava and Gross showed:

**Theorem 1.1** ([5, Theorem 1.1]) When all hyperelliptic curves of fixed genus \( n \geq 1 \) over \( \mathbb{Q} \) having a rational Weierstrass point are ordered by height, the average size of the 2-Selmer group of their Jacobians is 3.

As an immediate corollary, they obtained that the average rank of the Mordell-Weil groups of the Jacobians of such curves is at most 3/2. For a concise summary of their results and the techniques used in the proofs, see [15].

In this paper, we consider hyperelliptic curves of genus \( n \geq 2 \) over \( \mathbb{Q} \) with a marked rational non-Weierstrass point that we will denote by \( \infty \). Any such curve \( C \) also has a second rational point \( \infty' \), namely the conjugate of \( \infty \) under the hyperelliptic involution. In other words, \( \infty' \) is the unique point in \( C(\bar{\mathbb{Q}}) \) such that

\[
h_0(\mathcal{O}_C(\infty + \infty')) = 2.
\]

By studying \( H_0(C, k \cdot (\infty + \infty')) \), one can show that \( C \) can be given as the smooth projective model of the affine curve defined by

\[
y^2 = x^{2n+2} + c_2 x^{2n} + \cdots + c_{2n+2} 
\]

(1)

where \( c_i \in \mathbb{Q} \) and the points \( \infty, \infty' \) lie above \( x = \infty \). If we further assume that \( c_i \in \mathbb{Z} \) and that there is no prime \( p \) such that \( p^{2i} \) divides \( c_i \) for all \( i \), then such an expression is unique. We analogously define the height \( H \) of \( C \) by

\[
H(C) = \max \{|c_k|^{(2n+1)/(2n+2k)}\}^{(2n+1)/(2n+2k)}. 
\]

Recall that the 2-Selmer group \( \text{Sel}_2(J) \) of the Jacobian \( J = \text{Jac}(C) \) of \( C \) is a finite subgroup of the Galois cohomology group \( H^1(\mathbb{Q}, J[2]) \), which is defined by local conditions and fits into an exact sequence

\[
0 \to J(\mathbb{Q})/2J(\mathbb{Q}) \to \text{Sel}_2(J) \to \text{III}_2(\mathbb{Q}, J) \to 0,
\]

where \( \text{III}_2(\mathbb{Q}, J) \) denotes the Tate-Shafarevich group of \( J \) over \( \mathbb{Q} \).

The main result of this paper is:

**Theorem 1.2** When all hyperelliptic curves of fixed genus \( n \geq 2 \) over \( \mathbb{Q} \) having a marked rational non-Weierstrass point are ordered by height, the average size of the 2-Selmer group of their Jacobians is 6.

More precisely, we show that

\[
\lim_{X \to \infty} \frac{\sum_{H(C) < X} \#\text{Sel}_2(\text{Jac}(C))}{\sum_{H(C) < X} 1} = 6,
\]

where \( C \) ranges over all hyperelliptic curves of the form (1). In fact, we prove that the same result remains true even when we average over any subset of hyperelliptic curves \( C \) defined by a finite set of congruence conditions on the coefficients \( c_2, c_3, \ldots, c_{2n+2} \).
We impose the condition that \( n \geq 2 \) because every point on a genus 1 curve is a Weierstrass point. We will show in Proposition 5.3 that the class \((\infty') - (\infty)\) is not divisible by 2 in \( J(\mathbb{Q}) \) for a 100% of hyperelliptic curves with a marked rational non-Weierstrass point. Therefore we expect the 2-Selmer groups of these Jacobians to have, on average, one extra generator compared to the Jacobians of hyperelliptic curve with one marked Weierstrass point. In other words: given Theorem 1.1, we expect Theorem 1.2 to be true. Now when \((\infty') - (\infty)\) is not divisible by 2 in \( J(\mathbb{Q}) \), the average 2-rank of the 2-Selmer group minus 1 is at most 3/2. This follows because \(|\text{Sel}_2(J)|/2\) is at least 1 and the average is 3 as \( C \) runs through hyperelliptic curves with a marked rational non-Weierstrass point. Therefore we obtain the following result.

**Corollary 1.3** When all hyperelliptic curves of fixed genus \( n \geq 2 \) over \( \mathbb{Q} \) having a marked rational non-Weierstrass point are ordered by height, the average rank of the 2-Selmer group of their Jacobians is at most 5/2. Thus the average rank of the Mordell-Weil groups of their Jacobians is at most 5/2.

In [5], Bhargava and Gross also used a method of Chabauty [11], [12] to show that when \( g \geq 2 \), a positive proportion of hyperelliptic curves of genus \( g \) with a rational Weierstrass point have at most 3 rational points; and when \( g \geq 3 \), a majority of such curves have at most 20 rational points. (These hyperelliptic curves having genus \( g \) correspond to the affine equation \( y^2 = x^{2g+1} + \cdots + c_{2g+1} \).) In [19], Poonen and Stoll used Chabauty’s method and the results of [5] to show that a positive proportion of odd degree hyperelliptic curves having a fixed genus \( g \geq 3 \) have exactly one rational point – the Weierstrass point at infinity – and that this proportion tends to 1 as \( g \) tends to infinity. Analogously, we show that in our case, a positive proportion of even degree hyperelliptic curves of genus \( g \geq 10 \) have exactly two rational points – the marked non-Weierstrass point \( \infty \) at infinity and its image \( \infty' \) under the hyperelliptic involution. We also show that as \( g \) tends to infinity, this proportion tends to 1. More precisely, we prove the following theorem:

**Theorem 1.4** The proportion of monic even degree hyperelliptic curves having genus \( g \geq 4 \) that have exactly two rational points is at least \( 1 - (48g + 120)2^{-g} \).

To prove Theorem 1.2, we follow the same strategy as [7], [8] and [5]. Let \((U, Q)\) denote the split quadratic space of dimension \( 2n + 2 \) over \( \mathbb{Q} \) and let \( V \) denote the space of operators \( T \) on \( U \) self-adjoint with respect to \( Q \). For any monic separable polynomial \( f(x) \) of degree \( 2n + 2 \), let \( J_f \) denote the Jacobian of the hyperelliptic curve defined by the affine equation \( y^2 = f(x) \), and let \( V_f \) denote the subscheme of \( V \) consisting of self-adjoint operators \( T \) with characteristic polynomial \( f(x) \). In Section 2, we obtain a bijection between \( \text{Sel}_2(J_f) \) and locally soluble orbits of the conjugation action of \( \text{PSO}(U) \) on \( V_f \). This parameterization step can be viewed as an example of Arithmetic Invariant Theory. Although not strictly needed, the arithmetic theory of pencils of quadrics as developed in [22] can be used to give a very nice geometric interpretation of solubility. More precisely, a self-adjoint operator \( T \in V_f(\mathbb{Q}) \) is soluble if and only if there exists a rational \( n \)-plane \( X \) that is isotropic with respect to the following two quadrics:

\[
Q(v) = \langle v, v \rangle_Q \\
Q_T(v) = \langle v, Tv \rangle_Q,
\]

where \( \langle \cdot, \cdot \rangle_Q \) is the bilinear form associated to \( Q \). A self-adjoint operator \( T \in V_f(\mathbb{Q}) \) is locally soluble if and only if such an \( n \)-plane exists locally everywhere.

In Section 4, we count the number of locally soluble orbits using techniques of Bhargava developed in [1]. We count first the number of integral orbits soluble at \( \mathbb{R} \) by counting the number of integral points inside a fundamental domain for the action of \( \text{PSO}(U)(\mathbb{R}) \) on \( V(\mathbb{R}) \). We break up this fundamental domain into a compact part and a cusp region where separate estimations are required. The compact part of the fundamental domain will contribute to, on average, four Selmer elements. The cusp region corresponds to the two “obvious” classes: 0 and \((\infty') - (\infty)\). The second step is a sieve to the locally soluble orbits by imposing infinitely many congruence conditions. For this the uniformity estimates of [3] are needed.

In Section 5, we combine the results from previous sections to prove Theorems 1.2. Finally in Section 6, we modify the methods of [19] to prove Theorem 1.4.
2 Orbit parameterization

Let \( k \) be a field of characteristic not 2 and let \((U, Q)\) be the (unique) split quadratic space over \( k \) of dimension \( 2n + 2 \) and discriminant 1. Let \( f(x) \) be a monic polynomial of degree \( 2n + 2 \) with no repeated roots and splitting completely over \( k^s \). In this section, we study the action of \( \text{PSO}(U) \) on self-adjoint operators of \( U \) with characteristic polynomial \( f(x) \) via conjugation. More precisely, let \( \langle v, w \rangle_Q = Q(v + w) - Q(v) - Q(w) \) denote the bilinear form associated to \( Q \). For any linear operator \( T : U \rightarrow U \), its adjoint \( T^* \) is defined via the following equation:

\[
(Tv, w)_Q = \langle v, T^*w \rangle_Q, \quad \forall v, w \in U.
\]

Let \( V \) denote the \( k \)-scheme

\[
V = \{ T : U \rightarrow U | T = T^* \},
\]

and \( V_f \) the \( k \)-scheme

\[
V_f = \{ T : U \rightarrow U | T = T^*, \det(xI - T) = f(x) \}.
\]

The group scheme

\[
\text{SO}(U) := \{ g \in \text{GL}(U) | gg^* = I, \det(g) = 1 \}
\]

acts on \( V_f \) via \( g \cdot T = gTg^{-1} \). The center \( \mu_2 \leq \text{SO}(U) \) acts trivially. Hence we obtain a faithful action of

\[
G = \text{PSO}_{2n+2} := \text{PSO}(U) = \text{SO}(U)/\mu_2.
\]

To study the orbits of these actions, we first work over the separable closure \( k^s \) of \( k \) in §2.1 and show that \( G(k^s) \) acts transitively on \( V_f(k^s) \) for separable polynomials \( f \). In §2.2, we work over \( k \) and classify the \( G(k) \)-orbits on \( V_f(k) \) using Galois cohomology. In §2.3, we consider the Jacobian \( J \) of the hyperelliptic curve given by the equation \( g^2 = f(x) \) and obtain a bijection between \( G(k) \setminus V_f(k) \) and a subset of \( H^1(k, J[2]) \). The most difficult part of this section will be to show that this subset contains the image of \( J(k)/2J(k) \) in \( H^1(k, J[2]) \). Finally, in §2.4, we work over \( \mathbb{Z}_p \) and describe the integral orbits \( G(\mathbb{Z}_p) \setminus V(\mathbb{Z}_p) \).

2.1 Geometric orbits

**Proposition 2.1** The group \( G(k^s) \) acts transitively on \( V_f(k^s) \). For any \( T \in V_f(k) \), the stabilizer subscheme \( \text{Stab}_G(T) \) is isomorphic to \( (\text{Res}_{L/k} \mu_2)_{N=1}/\mu_2 \), where \( L = k[x]/f(x) \) is an etale \( k \)-algebra of dimension \( 2n + 2 \).

**Proof:** Fix any \( T \) in \( V_f(k) \). Since \( T \) is regular semi-simple, its stabilizer scheme in \( \text{GL}(U) \) is a maximal torus. It contains and hence equals to the maximal torus \( \text{Res}_{L/k} \mathbb{G}_m \). For any \( k \)-algebra \( K \), we have

\[
\text{Stab}_{O(U)}(T)(K) = \{ g \in (K[T]/f(T))^\times | g^*g = 1 \}.
\]

Since \( T = T^* \) and \( g \) is a polynomial in \( T \), we have \( g = g^* \). Thus,

\[
\text{Stab}_{O(U)}(T) \cong \text{Stab}_{\text{GL}(U)}(T)[2] \cong \text{Res}_{L/k}\mu_2,
\]

\[
\text{Stab}_{\text{SO}(U)}(T) \cong (\text{Res}_{L/k}\mu_2)_{N=1},
\]

\[
\text{Stab}_{\text{PSO}(U)}(T) \cong (\text{Res}_{L/k}\mu_2)_{N=1}/\mu_2.
\]

Since \( T \) is self-adjoint, there is an orthonormal basis \( \{u_1, \ldots, u_{2n+2}\} \) for \( U \) consisting of eigenvectors of \( T \) with eigenvalues \( \lambda_1, \ldots, \lambda_{2n+2} \). If \( T' \) is another elements of \( V_f(k^s) \), then there is an orthonormal basis \( \{u'_1, \ldots, u'_{2n+2}\} \) of \( U \) consisting of eigenvectors of \( T' \) with eigenvalues \( \lambda_1, \ldots, \lambda_{2n+2} \). Let \( g \in \text{SL}(U)(k^s) \) be an operator sending \( u_i \) to \( \pm u'_i \). Then \( g \in \text{SO}(U)(k^s) \) and the image of \( g \) in \( \text{PSO}(U)(k^s) \) sends \( T \) to \( T' \). \( \square \)
2.2  Rational orbits via Galois cohomology

Our first aim is to show that $V_f(k)$ is non-empty. Indeed, one can view $L = k[x]/f(x)$ as a $2n + 2$ dimensional $k$-vector space with a power basis $\{1, \beta, \ldots, \beta^{2n+1}\}$ where $\beta \in k[x]/f(x)$ is the image of $x$. We define the binary form $<, >$ on $L$ as follows:

$$< \lambda, \mu > := \text{coefficient of } \beta^{2n+1} \text{ in } \lambda \mu = \text{Tr}_{L/k}(\lambda \mu / f'(\beta)).$$

This form is split since the $n + 1$ plane $Y = \text{Span}\{1, \beta, \ldots, \beta^n\}$ is isotropic. Its discriminant is 1 as one can readily compute using the above power basis. By the uniqueness of split quadratic spaces of fixed dimension and discriminant 1, there exists an isometry between $(L, <, >)$ and $(U, (\ , \ )_Q)$, well defined up to post composition by elements in $O(U)(k)$. Let $\cdot \beta : L \to L$ denote the linear map given by multiplication by $\beta$. Then $\cdot \beta$ is self-adjoint with characteristic polynomial $f(x)$, and hence yields an element in $V_f(k)$ well-defined up to $O(U)(k)$ conjugation. In what follows, we fix an isometry $\iota : L \to U$ thus yielding a fixed element $T_f \in V_f(k)$.

Given $T \in V_f(k)$ there exists $g \in G(k^s)$ such that $T = gTfg^{-1}$, since there is a unique geometric orbit (see Proposition 2.1). For any $\sigma \in \Gal(k^s/k)$, the element $g^{-1} \sigma g$ also conjugates $T_f$ to $T$ and hence $g^{-1} \sigma g \in \Stab_G(T_f)(k^s)$. The 1-cochain $c_T$ given by $(c_T)_\sigma = g^{-1} \sigma g$ is a 1-cocycle whose image in $H^1(k, G)$ is trivial. This defines a bijection

$$G(k) \backslash V_f(k) \leftrightarrow \ker(H^1(k, \Stab_G(T_f)) \to H^1(k, G))$$

$$T \leftrightarrow c_T.$$  \hspace{1cm} (2)

See [4, Proposition 1] for more details.

Distinguished oribtcs

We call a self-adjoint operator $T \in V_f(k)$ distinguished if it is $PO(U)(k)$-equivalent to $T_f$. Since the $PO(U)(k)$-orbit of $T_f$ might break up into two $PSO(U)(k)$-orbits, there might exist two distinguished $PSO(U)(k)$-orbits in contrast to the odd hyperelliptic case. As $\Stab_{PO(U)}(T_f) \simeq \Res_{L/k}\mu_2 / \mu_2$, we have the following diagram of exact rows:

$$\begin{array}{c}
\Res_{L/k}\mu_2 / \mu_2(k) \\
\downarrow \sim \\
\mu_2(k)
\end{array} \xleftarrow{N} \begin{array}{c}
\mu_2(k) \\
\downarrow \\
H^1(k, \Stab_{PSO(U)}(T_f))
\end{array} \xrightarrow{\sim} \begin{array}{c}
H^1(k, \Stab_{PSO(U)}(T_f)) \\
\downarrow \\
H^1(k, \Stab_{PO(U)}(T_f))
\end{array} \xrightarrow{\sim} \begin{array}{c}
H^1(k, \Stab_{PO(U)}(T_f))
\end{array} \xrightarrow{\sim} \begin{array}{c}
H^1(k, PO(U))
\end{array}.$$  \hspace{1cm} (3)

Therefore a self-adjoint operator $T \in V_f(k)$ is distinguished if and only if

$$c_T \in \ker(H^1(k, \Stab_{PSO(U)}(T_f)) \to H^1(k, \Stab_{PO(U)}(T_f))).$$

Since $H^1(k, PO(U)) \to H^1(k, PO(U))$ is injective, every class in the above kernel corresponds to a $PSO(U)(k)$-orbit.

Distinguished $PSO(U)(k)$-orbits in $V_f(k)$ are unique if and only if the norm map $N : \Res_{L/k}\mu_2 / \mu_2(k) \to \mu_2(k)$ is surjective. Therefore, [18, Lemma 11.2] immediately implies the following result.

Proposition 2.2 The set of distinguished elements in $V_f(k)$ consists of a single $PSO(U)(k)$-orbit if and only if one of the following conditions is satisfied:

(1) $f(x)$ has a factor of odd degree in $k[x]$.

(2) $n$ is even and $f(x)$ factors over some quadratic extension $K$ of $k$ as $h(x)\overline{h}(x)$, where $h(x) \in K[x]$ and $\overline{h}(x)$ is the $\Gal(K/k)$-conjugate of $h(x)$.

Otherwise, the set of distinguished elements in $V_f(k)$ consists of two $PSO(U)(k)$-orbits. Condition (2) is equivalent to saying that $n$ is even, and $L$ contains a quadratic extension $K$ of $k$.  \hspace{1cm} (4)
To give a more explicit description of distinguished orbits, we have the following result, the proof of which is deferred to Section 3.

**Proposition 2.3** A self-adjoint operator $T \in V_f(k)$ is distinguished if and only if there exists a $k$-rational $n$-plane $X \subset U$ such that $\text{Span}\{X, TX\}$ is an isotropic $n + 1$ plane.

After a change of basis, we may take the matrix $A$ with 1’s on the anti-diagonal and 0’s elsewhere as a Gram matrix for $Q$. We express this basis as

$$\{e_1, \ldots, e_{n+1}, f_{n+1}, \ldots, f_1\}$$

where

$$\langle e_i, f_j \rangle_Q = \delta_{ij}, \quad \langle e_i, e_j \rangle_Q = 0 = \langle f_i, f_j \rangle_Q. \quad (4)$$

We call this the standard basis. Then the above proposition yields the following explicit description of distinguished elements which will be useful in Section 4.

**Proposition 2.4** A self-adjoint operator in $V_f(k)$ is distinguished if and only if its $\text{PSO}(U)(k)$-orbit contains an element $T$ whose matrix $M$, with respect to the standard basis, satisfies

$$AM = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & * & * \\
0 & 0 & \cdots & 0 & * & * & * \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & * & \vdots & \vdots & 0 \\
0 & * & \cdots & * & \cdots & * & 0 \\
* & * & \cdots & * & * & \cdots & * \\
* & * & \cdots & * & * & \cdots & * \\
\end{pmatrix}. \quad (5)$$

**Proof:** The forward direction follows from an argument identical to the proof of [5, Proposition 4.4]. For the backwards direction, suppose $AM$ has the form in (5). Then

$$Te_i \in \text{Span}\{e_1, \ldots, e_{n+1}\} = \text{Span}\{e_1, \ldots, e_{n+1}\}, \quad \text{for } i = 1, \ldots, n. \quad (6)$$

Let $X$ be the $n$-plane $\text{Span}\{e_1, \ldots, e_n\}$. Since $T$ is self-adjoint, its eigenspaces are pairwise orthogonal. Since $Q$ is non-degenerate, none of the eigenvectors of $T$ is isotropic. As a result, no isotropic linear space is $T$-stable. Therefore by (6),

$$\text{Span}\{X, TX\} = \text{Span}\{e_1, \ldots, e_{n+1}\}.$$ 

By Proposition 2.3, $T$ is distinguished. □

**Remaining orbits**

We start by describing the set of $O(U)(k)$-orbits on $V_f(k)$. Recall that $\text{Stab}_{O(U)}(T_f) \simeq \text{Res}_{L/k}\mu_2$. The set

$$\ker(H^1(k, \text{Stab}_{O(U)}(T_f)) \to H^1(k, O(U)))$$

consists of elements $\alpha \in H^1(k, \text{Res}_{L/k}\mu_2) \simeq L^x/L^x2$ whose image in $H^1(k, O(U))$ is trivial. For any $\alpha \in L^x/L^x2$, lift it arbitrarily to $L^x$ and consider the following bilinear form on $L$:

$$\langle \lambda, \mu \rangle_\alpha = \text{coefficient of } \beta^{2n+1} \text{ in } \alpha \lambda \mu = \text{Tr}_{L/k}(\alpha \lambda \mu / f'(\beta)).$$

We claim that $\alpha$ maps to 0 in $H^1(k, O(U))$ if and only if $\langle \cdot, \cdot \rangle_\alpha$ is split with discriminant 1. Indeed, let $\iota : (L, \langle \cdot, \cdot \rangle) \to (U, \langle \cdot, \cdot \rangle_Q)$ denote the isometry used to define $T_f$. Now $\langle \cdot, \cdot \rangle_\alpha$ is split with discriminant 1 if and only if there exists $g \in O(U)(k^*)$ such that the following composite map is defined over $k$:

$$(L, \langle \cdot, \cdot \rangle) \xrightarrow{\sqrt{\gamma}} k^* (L, \langle \cdot, \cdot \rangle) \xrightarrow{\gamma} k^* (U, \langle \cdot, \cdot \rangle_Q) \xrightarrow{\gamma^*} (U, \langle \cdot, \cdot \rangle_Q), \quad (7)$$

6
where the subscripts below the arrows indicate the fields of definition and where the last map is the standard action of $g \in O(U(k^2))$. Unwinding the definitions ([22, Proposition 2.13]), we see that this is equivalent to the image of $\alpha$ mapping to 0 in $H^1(k, O(U))$. We have therefore shown the following result.

**Theorem 2.5** There is a bijection between $O(U)(k)$-orbits on $V_f(k)$ and classes $\alpha \in (L^\times/L^\times 2)_{N=1}$ such that $\langle , \rangle_\alpha$ is split.

To study $SO(U)(k)$- and $PO(U)(k)$-orbits, we note that all the maps in the following diagram are injections.

$$
\begin{array}{ccc}
H^1(k, SO(U)) & \longrightarrow & H^1(k, O(U)) \\
\uparrow & & \uparrow \\
H^1(k, PSO(U)) & \longrightarrow & H^1(k, PO(U))
\end{array}
$$

The horizontal maps are injective because $\det: O(U)(k) \rightarrow \mu_2(k)$ is surjective. The vertical maps are injective because the connecting homomorphism $PSO(U)(k) \rightarrow k^\times/k^\times 2$ is surjective. Indeed, for any $c \in k^\times$, the element in $PSO(U)(k)$ mapping to $c$ is the operator

$$
e_i \mapsto \sqrt{c}e_i, \quad f_i \mapsto \sqrt{c}f_i, \quad \forall i = 1, \ldots, n+1.
$$

Recall that $\text{Stab}_{SO}(T_f) \simeq (\text{Res}_{L/k} \mu_2)_{N=1}$. From the exact sequence

$$
1 \rightarrow (\text{Res}_{L/k} \mu_2)_{N=1} \rightarrow \text{Res}_{L/k} \mu_2 \xrightarrow{N} \mu_2 \rightarrow 1,
$$

we obtain the isomorphism

$$
\ker(H^1(k, (\text{Res}_{L/k} \mu_2)_{N=1}) \rightarrow H^1(k, \text{Res}_{L/k} \mu_2)) \simeq \text{coker}(\mu_2(L) \xrightarrow{N} \mu_2(k)).
$$

We see that each $O(U)(k)$-orbits breaks up into one or two $SO(U)(k)$-orbit depending on whether $f(x)$ has an odd degree factor or not, respectively.

We next describe the set of $PO(U)(k)$-orbits on $V_f(k)$. Each such orbit breaks up into either one or two $PSO(U)(k)$-orbits depending on whether the norm map $N: \text{Res}_{L/k} \mu_2(k) \rightarrow \mu_2(k)$ is surjective or not, respectively (see Proposition 2.2 for a more descriptive criterion). As the stabilizer subschema of $T_f$ in $PO(U)$ is $\text{Res}_{L/k} \mu_2(k)$, we have the following diagram of exact rows:

$$
\begin{array}{ccc}
H^1(k, \mu_2) & \longrightarrow & H^1(k, \text{Res}_{L/k} \mu_2) \\
\downarrow & & \downarrow \\
H^1(k, \mu_2) & \longrightarrow & H^1(k, O(U)) \\
\downarrow & & \downarrow \\
H^2(k, \mu_2) & \longrightarrow & H^2(k, O(U))
\end{array}
$$

Suppose

$$
e^U_T \in \ker(H^1(k, \text{Res}_{L/k} \mu_2/\mu_2) \rightarrow H^1(k, PO(U))).
$$

Since $e^U_T$ maps to 0 in $H^2(k, \mu_2)$, it is the image of some $\alpha \in L^\times/L^\times 2$ well-defined up to $k^\times/k^\times 2$. Since the map $H^1(k, O(U)) \rightarrow H^1(k, PO(U))$ is injective, the image of $\alpha$ in $H^1(k, O(U))$ is trivial. By Theorem 2.5, this is equivalent to the form $\langle , \rangle_\alpha$ being split with discriminant 1. Therefore, we have the following characterization of $PO(U)(k)$-orbits.

**Theorem 2.6** There is a bijection between $PO(U)(k)$-orbits and classes $\alpha \in (L^\times/L^\times 2k^\times)_{N=1}$ such that $\langle , \rangle_\alpha$ is split. The distinguished orbit corresponds to $\alpha = 1$. Two $O(U)(k)$-orbits corresponding to $\alpha_1, \alpha_2 \in (L^\times/L^\times 2)_{N=1}$ are $PO(U)(k)$-equivalent if and only if $\alpha_1$ and $\alpha_2$ have the same image in $(L^\times/L^\times 2k^\times)_{N=1}$.  

7
2.3 Connection to hyperelliptic curves

Let $C$ be the hyperelliptic curve of genus $n$ given by the affine equation $y^2 = f(x)$, and let $J$ denote its Jacobian. The curve $C$ has two rational points above infinity, denoted by $\infty$ and $\infty'$. Let $P_1, \ldots, P_{2n+2}$ denote the Weierstrass points of $C$ over $k^s$. These form the ramification locus of the map $x : C \to \mathbb{P}^1$. Let $D_0$ denote the hyperelliptic class obtained as the pullback of $O_{\mathbb{P}^1}(1)$. Then the group $J[2](k^s)$ is generated by the divisors classes $(P_i) - (P_j) - D_0$ for $i \neq j$ subject only to the condition that

$$2n+2 \sum_{i=1}^{2n+2} (P_i) - (n+1)D_0 \sim 0.$$ 

We have the following isomorphisms of group schemes over $k$:

$$J[2] \simeq (\text{Res}_{L/k}\mu_2)_{N=1}/\mu_2 \simeq \text{Stab}_G(T_f).$$ \hfill (8)

An explicit formula for this identification is given in [21, Remark 2.6].

In conjunction with (2), this identification yields a bijection

$$G(k)\backslash V_f(k) \longrightarrow \ker(H^1(k, J[2]) \to H^1(k, G)).$$

Thus $G(k)$-orbits on $V_f(k)$ can be identified with a subset of $H^1(k, J[2])$. Recall that we have the following descent exact sequence:

$$1 \to J(k)/2J(k) \to H^1(k, J[2]) \to H^1(k, J)[2] \to 1.$$ \hfill (9)

A $G(k)$-orbit in $V_f(k)$ is said to be soluble if it corresponds to a class in $H^1(k, J[2])$ which is in the image of the map from $J(k)/2J(k)$. The following theorem states that there is a bijection between soluble $G(k)$-orbits in $V_f(k)$ and elements of $J(k)/2J(k)$.

**Theorem 2.7** The following composite map is trivial:

$$J(k)/2J(k) \to H^1(k, J[2]) \to H^1(k, G).$$ \hfill (10)

Therefore, there is a bijection between soluble $G(k)$-orbits in $V_f(k)$ and elements of $J(k)/2J(k)$.

**Proof:** We only prove the theorem in the case when $k$ is a local field. For a complete proof, see §3. Combining the descent sequence (9) and the long exact sequence obtained by taking Galois cohomology of the short exact sequence

$$1 \to J[2] \to \text{Res}_{L/k}\mu_2/\mu_2 \xrightarrow{N} \mu_2 \to 1,$$

we get the following commutative diagram.

$$
\begin{array}{cccccc}
\langle (\infty') - (\infty) \rangle & \xrightarrow{\sim} & J(k)/2J(k) & \xrightarrow{\delta'} & L^x/L^x2k^x & \xrightarrow{N} & k^x/k^x2 \\
\mu_2(k) & \xrightarrow{N(\text{Res}_{L/k}\mu_2/\mu_2(k))} & H^1(k, J[2]) & \xrightarrow{\delta} & H^1(k, \text{Res}_{L/k}\mu_2/\mu_2) & \xrightarrow{N} & H^1(k, \mu_2)
\end{array}
$$ \hfill (11)

The map $\delta'$ is defined in [18] by evaluating $(x - \beta)$ on a given divisor class. As shown in [18], the first row is not exact: the image of $\delta'$ lands inside, generally not onto, $(L^x/L^x2k^x)_{N=1}$ with kernel the subgroup generated by the class $(\infty') - (\infty)$. Note that $(\infty') - (\infty) \in 2J(k)$ if and only if the norm map $N : \text{Res}_{L/k}\mu_2/\mu_2(k) \to \mu_2(k)$ is surjective which happens when there is a unique distinguished orbit.

To prove Theorem 2.7, it suffices to show that if $\alpha \in (L^x/L^x2k^x)_{N=1}$ lies in the image of $\delta'$, then $<, >_\alpha$ is split. We will prove this by explicitly writing down a $k$-rational $n + 1$ dimensional isotropic subspace in the special case when $k$ is a local field. For a complete and more conceptual proof using pencils of quadrics, see Section 3. Suppose $\alpha = \delta'(\langle D \rangle)$ for some $[D] \in J(k)/2J(k)$ of the form

$$[D] = (Q_1) + \cdots + (Q_m) - m(\infty) \mod 2J(k) \cdot \langle (\infty') - (\infty) \rangle,$$
where \( Q_1, \ldots, Q_m \in C(k^*) \) are non-Weierstrass non-infinity points and \( m \leq n + 1 \). When \( k \) is a local field, every \( |D| \in J(k)/2J(k) \) can be written in this form ([22, Lemma 3.8]). If we write \( Q_i = (x_i, y_i) \), then
\[
\alpha = (x_1 - \beta) \cdots (x_m - \beta) \quad \text{and} \quad \lambda < \lambda, \mu > \alpha = Tr_{L/k}((x_1 - \beta) \cdots (x_m - \beta)\lambda\mu/f'(\beta)).
\]
Write
\[
V = \prod_{1 \leq i < j \leq m} (x_i - x_j)
\]
for the Vandermonde polynomial, and for each \( i = 1, \ldots, m \), define
\[
q_i := \prod_{1 \leq j \leq m, j \neq i} (x_j - x_i), \quad a_i := V/q_i, \quad h_i(t) := \frac{f(t) - f(x_i)}{t - x_i}.
\]
For any \( j \geq 0 \), we define
\[
g_j(t) = \sum_{i=1}^{m} x_i^j a_i h_i(t)/y_i.
\]
Then the \( n + 1 \) plane \( Y \) defined below is \( k \)-rational and isotropic ([22, Lemma 2.44]):
\[
Y := \begin{cases} 
\text{Span}\{1, \beta, \ldots, \beta^n\}, & \text{if } m = 1; \\
\text{Span}\{1, \beta, \ldots, \beta^{n-m'}, g_0(\beta), \ldots, g_{m'-1}(\beta)\}, & \text{if } m = 2m' \text{ or } m = 2m' + 1.
\end{cases}
\]
This completes the proof of Theorem 2.7 in this special case. □

Suppose that \( k \) is a number field. Then the 2-Selmer group \( \text{Sel}_2(k, J) \) is the subgroup of \( H^1(k, J[2]) \) consisting of elements whose images in \( H^1(k_\nu, J[2]) \) lie in the image of \( J(k_\nu)/2J(k_\nu) \) for all completions \( k_\nu \) of \( k \). Since the group \( G = \text{PSO}_{2n+2} \) satisfies the Hasse principle, Theorem 2.7 implies that the following composite is also trivial:
\[
\text{Sel}_2(k, J) \to H^1(k, J[2]) \to H^1(k, G).
\]
A self-adjoint operator \( T \in V_f(k) \) is said to be locally soluble if \( T \) is soluble in \( V_f(k_\nu) \) for all completions \( k_\nu \) of \( k \). Equivalently, \( c_T \) lies in \( \text{Sel}_2(k, J) \). We have thus proven the following theorem:

**Theorem 2.8** ([22]) *Let \( k \) be a number field, and \( f \) a monic separable polynomial of degree \( 2n + 2 \) over \( k \). There is a bijection between locally soluble \( G(k) \)-orbits on \( V_f(k) \) and elements in \( \text{Sel}_2(k, J) \), where \( J \) is the Jacobian of the hyperelliptic curve given by the equation \( y^2 = f(x) \).*

### 2.4 Integral orbits

Let \( f(x) \in \mathbb{Q}[x] \) be a degree \( 2n + 2 \) monic separable polynomial, let \( C \) be the corresponding hyperelliptic curve, and \( J \) its Jacobian. We have seen that elements in the 2-Selmer group of \( J \) are in bijection with locally soluble \( G(\mathbb{Q}) \)-orbits in \( V_f(\mathbb{Q}) \). In this section, our aim is to show that when \( f \) has integral coefficients, every locally soluble \( G(\mathbb{Q}) \)-orbit in \( V_f(\mathbb{Q}) \) contains an integral representative.

We do this by working over the field \( \mathbb{Q}_p \) and the ring \( \mathbb{Z}_p \). Specifically, we prove the following result:

**Proposition 2.9** *Let \( p \) be a prime and let \( f(x) = x^{2n+2} + c_1 x^{2n+1} + \cdots + c_{2n+2} \) be a monic separable polynomial in \( \mathbb{Z}_p[x] \) such that \( 2^i \mid c_i \) in \( \mathbb{Z}_p \) for \( i = 1, \ldots, 2n + 2 \). Then every soluble \( G(\mathbb{Q}_p) \)-orbit in \( V_f(\mathbb{Q}_p) \) contains an integral representative.*

Since the group \( G \) has class number \( 1 \) over \( \mathbb{Q} \), we immediately obtain the following corollary:

**Corollary 2.10** *Let \( f(x) = x^{2n+2} + c_1 x^{2n+1} + \cdots + c_{2n+2} \) be a monic separable polynomial in \( \mathbb{Z}[x] \) such that \( 2^i \mid c_i \) for \( i = 1, \ldots, 2n + 2 \). Then every locally soluble \( G(\mathbb{Q}) \)-orbit in \( V_f(\mathbb{Q}) \) contains an integral representative.*
We will also prove the following result, which will be important to us in §4.4:

**Proposition 2.11** Let $p$ be any odd prime, and let $f(x) \in \mathbb{Z}_p[x]$ be a degree $2n+2$ monic separable polynomial such that $p^2 \nmid \Delta(f)$. Then the $G(\mathbb{Z}_p)$-orbits in $V_f(\mathbb{Z}_p)$ are in bijection with soluble $G(\mathbb{Q}_p)$-orbits in $V_f(\mathbb{Q}_p)$. Furthermore, if $T \in V_f(\mathbb{Z}_p)$, then \( \text{Stab}_{G(\mathbb{Z}_p)}(T) = \text{Stab}_{G(\mathbb{Q}_p)}(T) \).

Let $p$ be a fixed prime. We start by considering the $O(U(\mathbb{Z}_p))$-orbits. A self-adjoint operator $T \in V_f(\mathbb{Q}_p)$ is integral if it stabilizes the self-dual lattice
\[
M_0 = \text{Span}_{\mathbb{Z}_p}\{e_1, \ldots, e_n, f_1, \ldots, f_1\}.
\]
In other words, $T$ is integral if and only if when expressed in the standard basis (4), its entries are in $\mathbb{Z}_p$. In general, a lattice $M$ is self-dual if the bilinear form restricts to a non-degenerate bilinear form: $M \times M \to \mathbb{Z}_p$. Since genus theory implies that any two self-dual lattices are $O(U)$-conjugate, the rational orbit of $T$ contains an integral representative if and only if $T$ stabilizes a self-dual lattice.

The action of $T$ on $U$ gives $U$ the structure of a $\mathbb{Q}_p[x]$-module, where $x$ acts via $T$. Since $T$ is regular, we have an isomorphism of $\mathbb{Q}_p[x]$-modules: $U \simeq \mathbb{Q}_p[x]/f(x) = L$. Suppose $T$ is integral, stabilizing the self-dual lattice $M_0$. The action of $T$ on $M_0$ realizes $M_0$ as a $\mathbb{Z}_p[x]/f(x)$-module. Write $R$ for $\mathbb{Z}_p[x]/f(x)$. Since $M_0$ is a lattice, we see that after the identification $U \simeq L$, $M_0$ becomes a fractional ideal $I$ for the order $R$. The split form $Q$ on $U$ gives a split form of discriminant 1 on $L$ for which multiplication by $\beta$ is self-adjoint. Any such form on $L$ is of the form $\langle r, r \rangle$, for some $r \in \mathbb{L}^\times$ with $N_{L/k}(r) \in k^{\times 2}$. The condition that $M_0$ is self-dual translates to saying $\alpha I^2 \subset R$ and $N(I)^2 = N(\alpha^{-1})$.

The identification $U \simeq L$ is unique up to multiplication by some element $c \in L^\times$, which transforms the data $(I, \alpha)$ to $(c \cdot I, c^{-2} \alpha)$. We call two pairs $(I, \alpha), (I', \alpha')$ equivalent if there exists $c \in L^\times$ such that $I' = c \cdot I$ and $\alpha' = c^{-2} \alpha$. Choosing a different integral representative $T$ in an integral orbit amounts to pre-composing the map $U \simeq L$ by an element of $O(U)(\mathbb{Z}_p)$ which does not change the equivalence class of the pair $(I, \alpha)$. Hence we have a well-defined map
\[
O(U)(\mathbb{Z}_p)/V_f(\mathbb{Z}_p) \to \text{equivalence classes of pairs } (I, \alpha).
\]

**Theorem 2.12** There is a bijection between $O(U)(\mathbb{Z}_p)$-orbits and equivalence classes of pairs $(I, \alpha)$ such that $\langle r, r \rangle$ is split, $\alpha \cdot I^2 \subset R$, and $N(I)^2 = N(\alpha^{-1})$. The image of $\alpha$ in $(L^\times/L^{\times 2})_{N=1}$ determines the rational orbit.

**Proof:** Given a pair $(I, \alpha)$ such that $\langle r, r \rangle$ is split and $\alpha I^2 = R$, there exists an isometry over $\mathbb{Q}_p$ from $(L, \langle r, r \rangle)$ to $(U, \langle r, r \rangle)$ that sends $I$ to the self-dual lattice $M_0$. The image of the multiplication by $\beta$ operator lies in $V_f(\mathbb{Z}_p)$. Any two such isometries differ by an element in $O(U)(\mathbb{Z}_p)$, hence we get a well-defined $O(U)(\mathbb{Z}_p)$-orbit. Along with (12), we have proved the first statement.

For the second statement, from the sequence of isometries (7), we see that since $\langle r, r \rangle$ is split, there exists $g \in O(U)(\mathbb{Q}_p)$ such that
\[
\sigma \sqrt{\alpha}/\sqrt{\alpha} = g^{-1} \sigma g, \quad \forall \sigma \in \text{Gal}(k^s/k).
\]
Here, the left hand side is viewed as an element of $\text{Stab}_{O(U)}(T_f)$. The rational orbit corresponding the pair $(I, \alpha)$ is therefore the rational orbit of $T = g T_f g^{-1}$. The rest follows formally from unwinding definitions. $\square$

Suppose the $O(U)(\mathbb{Z}_p)$-orbit of some $T \in V_f(\mathbb{Z}_p)$ corresponds to an equivalence class of pair $(I, \alpha)$. Then the stabilizer of $T$ in $\text{GL}(U)(\mathbb{Z}_p)$ is $\text{End}_R(I)^\times$. Moreover, just as the proof of Proposition 2.1, we have
\[
\text{Stab}_{O(U)}(T)(\mathbb{Z}_p) = \text{End}_R(I)^\times[2],
\]
\[
\text{Stab}_{SO(U)}(T)(\mathbb{Z}_p) = (\text{End}_R(I)^\times[2])_{N=1}.
\]
The stabilizers in the group $\text{PO}(U)(\mathbb{Z}_p)$ (and $\text{PSO}(U)(\mathbb{Z}_p)$) are slightly complicated because $\text{PO}(U)(\mathbb{Z}_p)$ contains $O(U)(\mathbb{Z}_p)$ as a subgroup with quotient $\mathbb{Z}_p^\times/\mathbb{Z}_p^{\times 2}$. We have the following exact sequences.

\[
1 \to \text{End}_R(I)^\times[2]/\mu_2 \to \text{Stab}_{O(U)}(T)(\mathbb{Z}_p) \to (R^{\times 2} \cap \mathbb{Z}_p^\times)/\mathbb{Z}_p^{\times 2} \to 1.
\]
In this section, we give geometric meanings to the notion of distinguished and soluble. For the proof of all statements below, see [22, Lemma 3.8], there exists non-Weierstrass non-infinit points $Q_1, \ldots, Q_m \in C(Q_p)$, with $m \leq n + 1$, such that

$$[D] = (Q_1) + \cdots + (Q_m) - m(\infty) \mod 2J(Q_p) \cdot ((\infty') - (\infty)).$$

Write each $Q_i = (x_i, y_i) \in C(O(Q_p^\times))$ then $\alpha = (x_1 - \beta) \cdots (x_m - \beta)$ is a lift of $\bar{\alpha}$ to $L^\times$. We claim that either the $O(U)(Q_p)$-orbit of $T$ corresponding to the image of $\alpha$ in $L^\times/L^\times$ has an integral representative, or $[D]$ can be expressed in the form $(14)$ with $m$ replaced by $m - 2$. Applying induction on $m$ completes the proof.

The claim follows verbatim from the proof of [5, Proposition 8.5]. We give a quick sketch here. Let $r(x) \in Q_p[x]$ be a polynomial of degree at most $m - 1$ such that for all $i$, $r(x_i) = y_i$ and let

$$p(x) = (x - x_1) \cdots (x - x_m) \in Z_p[x].$$

Now $p(x)$ divides $r(x)^2 - f(x)$ in $Q_p[x]$ and let $q(x)$ denote the quotient. By definition, $\alpha = (-1)^m P(\beta)$. If the polynomial $r(x) \in Z_p[x]$, then the ideal $I = (1, r(\beta)/\alpha)$ does the job. Note $\alpha T = (\alpha, r(\beta), q(\beta)).$ The integrality assumption of $r(x)$ is used to show that $r(\beta), q(\beta) \in R$. A computation of ideal norms shows that $N(I)^2 = N(\alpha)^{-1}$.

When $r(x)$ is not integral, a Newton polygon analysis on $f(x) - r(x)^2$ shows that $\text{div}(y - r(x)) - [D]$ has the form $D^* + E$ with $D^*, E \in J(Q_p)$ where $D^*$ can be expressed in $(14)$ with $m$ replaced by $m - 2$ and the $x$-coordinates of the non-infinit points in $E$ have negative valuation. The condition of divisibility on the coefficients of $f(x)$ ensures that $E \in 2J(Q_p). (\infty') - (\infty')$, or equivalently $(x - \beta)(E) \in L^\times Q_p. □$

**Proof of Proposition 2.11:** Once again, it suffices to work with $O(U)$-orbits instead of $PSO(U)$-orbits directly. The assumption on $\Delta(f)$ implies that $R$ is the maximal order. Hence there is a bijection between $O(U)(Z_p)$-orbits and $(R^\times/R^\times)^{N=1}$. Note over non-archimedean local fields, the splitting of the quadratic form is automatic from the existence of a self-dual lattice. Taking flat cohomology over $Spec(Z_p)$ of the sequence

$$1 \to \mu_2 \to O(U) \to PO(U) \to 1$$

gives:

$$1 \to O(U)(Z_p)/\pm 1 \to PO(U)(Z_p) \to Z_p^\times/Z_p^\times \to 1.$$ 

Hence $PO(U)(Z_p)$-orbits correspond bijectively to $(R^\times/R^\times)^{N=1}$. 

On the other hand, the assumption on $\Delta(f)$ implies that the projective closure $C$ of the hyperelliptic curve $C$ defined by affine equation $y^2 = f(x)$ over $Spec(Z_p)$ is regular. Since the special fiber of $C$ is geometrically reduced and irreducible, the Neron model $J$ of its Jacobian $J_{Q_p}$ is fiberwise connected ([6, §9.5 Theorem 1]) and its 2-torsion $J[2]$ is isomorphic to $(Res_{R/Z_p})_{N=1}/\mu_2$. Using diagram (11) after replacing $L, k, J$ by $R, Z_p, J$, we see that the vertical maps are all isomorphisms and $\delta'$ maps $J(Z_p)/2J(Z_p)$ surjectively to $(R^\times/R^\times)^{N=1}$. The Neron mapping property implies that $J(Z_p)/2J(Z_p) = J(Q_p)/2J(Q_p)$.

Suppose the $O(U)(Z_p)$-orbit of some $T \in V_f(Z_p)$ corresponds to an equivalent class of pair $(I, \alpha)$. Since $R$ is maximal, $End_R(I) = R$. Since $R^\times[2] = L^\times[2]$, we see from (13) that it remains to compare $(R^\times \cap Z_p^\times)/Z_p^\times$ with $(L^\times \cap Q_p^\times)/Q_p^\times$. These two sets are only nonempty when $L$ contains a quadratic extension $K'$ of $Q_p$. The condition $p^2 \nmid \Delta(f)$ implies that $K' = Q_{p}(\sqrt{u})$ can only be the unramified quadratic extension of $Q_p$. In other words, $u \in Z_p^\times$. Hence in this case $(L^\times \cap Q_p^\times)/Q_p^\times$ and $(R^\times \cap Z_p^\times)/Z_p^\times$ both are equal to the group of order 2 generated by the class of $u$. □

## 3 Interpretation using pencils of quadrics

In this section, we give geometric meanings to the notion of distinguished and soluble. For the proof of all the statements below, see [21, Section 2.2]. These geometric interpretations are not necessary if one wants only the average size of the 2-Selmer groups.
Let $k$ be a field of characteristic not 2 and let $f(x)$ be a monic separable polynomial of degree $2n + 2$. Let $T$ be a self-adjoint operator in $V_f(k)$ and let $C$ denote the hyperelliptic curve $y^2 = f(x)$. Let $\infty$ and $\infty'$ denote the two points above infinity. One has a pencil of quadrics in $U$ spanned by the following two quadrics:

\[
Q(v) = \langle v, v \rangle_Q \\
Q_T(v) = \langle v, T v \rangle_Q.
\]

This pencil is generic in the sense that there are precisely $2n + 2$ singular quadrics among $x_1 Q - x_2 Q_T$ for $[x_1, x_2] \in \mathbb{P}^1$, and that they are all simple cones. Its associated hyperelliptic curve $C'$ is the curve parameterizing the rulings of the quadrics in the pencil. A ruling of a quadric $Q_0$ is a connected component of the Lagrangian variety of maximal isotropic subspaces. When $Q_0$ is a simple cone, there is only one ruling. When $Q_0$ is non-degenerate, there are two rulings defined over $k(\sqrt{\det(Q_0)})$. To give a point on $C'$ is the same as giving a quadric in the pencil along with a choice of ruling. Therefore, the curve $C'$ is isomorphic non-canonically to the hyperelliptic curve

\[y^2 = \det(xQ - Q_T) = \det(Q) \det(xI - T) = f(x).
\]

Hence $C'$ is isomorphic to $C$ over $k$. We fix an isomorphism $C' \simeq C$ and denote by $Y_0$ the ruling on $Q$ that corresponds to $\infty \in C(k)$. Since $C$ has a rational point, the Fano variety $F_T$ of $n$-planes isotropic with respect to both quadrics is a torsor of $J$ of order dividing 2. In fact, it fits inside a disconnected algebraic group

\[J \cup F_T \cup \text{Pic}^1(C) \cup F_T',
\]

where $F_T' \simeq F_T$ as varieties. Using the point $\infty$, one obtains a lift of $F_T$ to a torsor of $J[2]$ by taking

\[F_T[2]_\infty = \{ X \in F_T | X + X = (\infty) \}
\]

\[= \{ X \text{-plane} | \text{Span}\{X, TX\} \text{ is an isotropic } n + 1 \text{ plane in the ruling } Y_0 \}.
\]

The second equality is [21, Proposition 2.32].

The group scheme $G = \text{PSO}(U)$ acts on the $k$-scheme

\[W_f = \{ (T, X) | T \in V_f, X \in F_T[2]_\infty \}
\]

via $g(T, X) = (gTg^{-1}, gX)$. Let $W_T$ denote the fiber above any fixed $T \in V_f(k)$. This action is simply-transitive on $k$-points ([21] Corollary 2.36). Hence for any $T \in V_f(k)$, the above action induces a simply-transitive action of $J[2] \simeq \text{Stab}_G(T)$ on the fiber $W_T = F_T[2]_\infty$.

**Theorem 3.1** ([21, Proposition 2.38], [22, Lemma 2.19]) *These two actions of $J[2]$ coincide as elements of $H^1(k, J[2])$,*

\[|F_T[2]_\infty| = |W_T| = \text{c}_T.
\]  

(15)

For hyperelliptic curves with a rational Weierstrass point, one can obtain all torsors of $J[2]$ using pencils of quadrics ([22, Proposition 2.11]). For hyperelliptic curves with no rational Weierstrass point but with a rational non-Weierstrass point, we do not recover all torsors of $J[2]$ using pencils of quadrics but we recover enough to study $\text{PSO}(U)\langle k \rangle$-orbits.

Suppose $T \in V_f(k)$. From (15), we see that there exists a $k$-rational $n$-plane $X$ such that $\text{Span}\{X, TX\}$ is an isotropic $n + 1$ plane if and only if either $|F_T[2]_\infty|$ or $|F_T[2]_{\infty'}|$ is trivial. Again by (15), this is equivalent to $c_T$ being in the image of the subgroup generated by $(\infty') - (\infty) \in J(k)/2J(k)$ under the descent map $J(k)/2J(k) \hookrightarrow H^1(k, J[2])$. Commutativity of the top left square in (11) implies that this is in turn equivalent to $c_T$ mapping to 0 in $H^1(k, \text{Stab}_{\text{PSO}(U)}(T))$. Finally, this is equivalent to $T$ being distinguished. We have therefore proved Proposition 2.3.

Since $|F_T[2]_\infty|$ maps to $|F_T|$ under the canonical map $H^1(k, J[2]) \to H^1(k, J)[2]$, we see that $T$ is soluble if and only if $F_T(k) \neq \emptyset$. This equivalence of solubility and the existence of rational points is the
main reason why the name “soluble” is used. Likewise, $T$ is locally soluble if and only if $F_T(k) \neq \emptyset$ at all places $\nu$.

We now give a complete proof for the claim that if $\alpha \in (L^x/L^{x^2}k^x)_{n=1}$ lies in the image of $\delta'$, then $<,>_{\alpha}$ is split. Consider instead the pencil of quadrics in $L$ spanned by the following two quadrics:

$$Q_\alpha(\lambda) = <\lambda, \lambda>_{\alpha}$$
$$Q'_\alpha(\lambda) = <\lambda, \beta\lambda>_{\alpha}.$$ 

This pencil is once again generic, its associated hyperelliptic curve $C_{\alpha}$ is smooth of genus $n$ isomorphic non-canonically to the hyperelliptic curve defined by affine equation

$$y^2 = \text{disc}(xQ_\alpha - Q'_\alpha) = N_{L/k}(\alpha)f(x).$$

Since $N_{L/k}(\alpha) \in k^{x^2}$, the curve $C_{\alpha}$ is isomorphic to $C$ over $k$. Fix any isomorphism $C'_\alpha \simeq C$. The Fano variety $F_\alpha$ of $n$-planes isotropic with respect to both quadrics is a torsor of $J$ of order dividing 2. There are two natural lifts of $F_\alpha$ to torsors of $J[2]$ by taking

$$F_\alpha[2]_\infty = \{X \in F| X + X = (\infty)\} \quad \text{or} \quad F_\alpha[2]_{\infty'} = \{X \in F| X + X = (\infty')\}.$$

As elements of $H^1(k, J[2])$, these two lifts map to the same class in $H^1(k, \text{Res}_{L/k}\mu_2/\mu_2)$. The class $\alpha$ also maps to a class in $H^1(k, \text{Res}_{L/k}\mu_2/\mu_2)$ as in (11). By [22, Proposition 2.27], these two classes coincide. When $\alpha = \delta'([D])$ comes from $J(k)/2J(k)$, one of these two lifts recovers $[D]$ and hence $F_\alpha(k) \neq \emptyset$. Pick any $X \in F_\alpha(k)$. If $X + X = (\infty)$, then $[D] = 0$, $\alpha = 1$ and $<,>$ is split. Otherwise, $\text{Span}\{X, (\infty) - X\}$ is a $k$-rational $n + 1$ plane isotropic with respect to $<,>_{\alpha}$.

4 Orbit counting

In this section, we let the monic polynomial $f$ vary and count the average number of locally soluble orbits of the action of $G(\mathbb{Q})$ on $V_f(\mathbb{Q})$. We redefine $V$ to be the following scheme over $\mathbb{Z}$:

$$V = \{T : U \to U|T = T^*, \text{Trace}(T) = 0\} \simeq \mathbb{A}_\mathbb{Z}^{2n^2 + 5n + 2}.$$ 

For any ring $R$, we shall think of elements in $V(R)$ as $B = AT$, where $A$ is the matrix with 1’s on the anti-diagonal and 0’s elsewhere and where $T$ is a $(2n + 2) \times (2n + 2)$ matrix with coefficients in $R$ such that $\text{Trace}(T) = 0$ and $T = T^*$. Thus, elements $B \in V(R)$ are symmetric matrices with anti-trace 0. This change of perspective is only to simplify notation in what follows. The group scheme $G = \text{PSO}_{2n+2}$ acts on $V$ by $g \cdot B := gBg^t$. The ring of polynomial invariants for this action is generated by the coefficients $c_2, \ldots, c_{2n+2}$ of the polynomial $\text{det}(Ax - By)$. We define the scheme $S$ to be:

$$S = \text{Spec} \mathbb{Z}[c_2, \ldots, c_{2n+2}].$$

The map $\pi : V \to S$ is given by the coefficients of the characteristic polynomial; we call $\pi(B)$ the invariant of $B$.

A point $c = (c_2, \ldots, c_{2n+2}) \in S(\mathbb{R})$ corresponds to a monic polynomial

$$f_c(x) := f(x) = x^{2n+2} + c_2x^{2n} + \cdots + c_{2n+2}.$$ 

We define its height $H(f)$ by

$$H(f) := H(c) := \max\{|c_k|^{d/k}\}_{k=2}^{2n+2},$$

where $d = (2n + 2)(2n + 1) = \deg H$ is the “total degree” of the discriminant of $f$. The height of $B \in V(\mathbb{R})$ is defined to be the height of $\pi(B)$, and the height of the hyperelliptic curve $C(c)$ given by $y^2 = f(x)$ is defined to be $H(c)$.

For each prime $p$, let $\Sigma_p$ be a closed subset of $S(\mathbb{Z}_p)\{\Delta = 0\}$ whose boundary has measure 0. Let $\Sigma_\infty$ be the set of all $c \in S(\mathbb{R})\{\Delta = 0\}$ such that the corresponding polynomial $f$ has $m$ distinct pairs of
complex conjugate roots, where \( m \) belongs to a fixed subset of \( \{0, \ldots, n+1\} \). To such a collection \( (\Sigma_\nu)_\nu \), we associate the family \( F = \Sigma C \) of hyperelliptic curves (with a marked rational non-Weierstrass point), where \( C(c) \in F \) if and only if \( c \in \Sigma_\nu \) for all places \( \nu \). Such a family is said to be defined by congruence conditions.

Given a family \( F \) that is defined by congruence conditions, let \( \text{Inv}(F) \subset S(\mathbb{Z}) \) denote the set 
\[ \{ c(C) : C \in F \} \]
of invariants. We denote the \( p \)-adic closure of \( \text{Inv}(F) \) in \( S(\mathbb{Z}_p) \) by \( \text{Inv}_p(F) \). We say that a family \( F \) defined by congruence conditions is large at \( p \) if \( \text{Inv}_p(F) \) contains every element \( c \in S(\mathbb{Z}_p) \) such that \( p^2 \nmid \Delta(c) \). Finally, we say that \( F \) and \( \text{Inv}(F) \) are large if \( F \) is large at all but finitely many primes. An example of a large subset of \( S(\mathbb{Z}) \) is the set
\[ F_0 = \{ (c_2, \ldots, c_{2n+2}) \in S(\mathbb{Z}) | p^k \mid c_k, \forall k = 2, \ldots, 2n + 2 \}. \]

Another example is the set of elements in \( S(\mathbb{Z}) \) having squarefree discriminant.

Our goal is to prove the following theorem:

**Theorem 4.1** The average number of locally soluble orbits for the action of \( G(\mathbb{Q}) \) on \( V_f(\mathbb{Q}) \) as \( f \) runs through any large subset of \( S(\mathbb{Z}) \), when ordered by height, is 6.

In view of the correspondence (in Theorem 2.8) between locally soluble orbits and 2-Selmer elements, the above result immediately implies the following strengthening of Theorem 1.2:

**Theorem 4.2** When all hyperelliptic curves over \( \mathbb{Q} \) of genus \( n \) with a marked rational non-Weierstrass point in any large family are ordered by height, the average size of the 2-Selmer group of their Jacobians is 6.

### 4.1 Outline of the proof

We now give an outline of the proof of Theorem 4.1. Let \( F \) be a large subset of \( S(\mathbb{Z}) \). Since the curve \( C(c_2, \ldots, c_{2n+2}) \) is isomorphic to \( C(u^2c_2, \ldots, u^{2n+2}c_{2n+2}) \), for any \( u \in \mathbb{Q} \), we may assume that \( 2^{4i} \mid c_i \) for every \( (c_2, \ldots, c_{2n+2}) \in \text{Inv}(F) \). Hence by Corollary 2.10, it suffices to determine the average number of locally soluble \( G(\mathbb{Q}) \)-equivalence classes on \( V_f(\mathbb{Z}) \) as \( f \) runs through \( F \).

As a first step, we count the number of \( \mathbb{R} \)-soluble \( G(\mathbb{Z}) \)-orbits of \( V(\mathbb{Z}) \) having bounded height and non-zero discriminant. An element in \( V(\mathbb{Z}) \) is reducible if either the discriminant of its characteristic polynomial is 0 or it is distinguished; otherwise it is called irreducible. Apart from a negligible number of invariants \( c \in S(\mathbb{Z}) \) (Proposition 5.3), there will always be 2 distinguished orbits having invariant \( c \). Let \( V(\mathbb{R})^{\text{sol}} \) denote the set of \( \mathbb{R} \)-soluble elements of \( V(\mathbb{R}) \). To estimate the number of irreducible orbits having bounded height, we construct in Section 4.2 a fundamental domain for the action of \( G(\mathbb{Z}) \) on \( V(\mathbb{R})^{\text{sol}} \).

The difficulty in estimating the number of lattice points in this fundamental domain is that it is not compact, but rather has cusps going to infinity. We handle these cusps by averaging this fundamental domain over a bounded subset of \( G(\mathbb{R}) \), and breaking it up into two pieces, namely, the main body and the cusp region. We show in Section 4.3 that the cusp region has small volume and negligibly many irreducible elements while the main body has a small number of reducible elements. Hence, using Proposition 4.3, we obtain:

\[ \#(V(\mathbb{Z})^{\text{irr}} \cap V(\mathbb{R})^{\text{sol}}), \quad \text{Vol}(G(\mathbb{Z}) \backslash V(\mathbb{R})^{\text{sol}}), \]

where \( V(\mathbb{Z})^{\text{irr}} \) denotes the set of irreducible elements, \( V(\mathbb{R})^{\text{sol}} \) denotes the set of points in \( V(\mathbb{R}) \) that are \( \mathbb{R} \)-soluble and have height less than \( X \), and the above volume is taken with respect to Euclidean measure \( \nu \) on \( V \) normalized so that \( V(\mathbb{Z}) \subset V(\mathbb{R}) \) has co-volume 1. In other words, the number of irreducible integral orbits that are soluble at \( \mathbb{R} \) of height less than \( X \) is asymptotic to the volume of a fundamental domain for the action of \( G(\mathbb{Z}) \) on \( V(\mathbb{R})^{\text{sol}} \).

Fix \( \tau \) and \( \mu \) to be Haar measures on \( G(\mathbb{R}) \) and \( S(\mathbb{R}) \), respectively, induced from left-invariant differential top forms over \( \mathbb{Q} \) where \( \mu \) is normalized such that \( S(\mathbb{Z}) \subset S(\mathbb{R}) \) has co-volume 1. For suitably “nice” morphisms \( \delta : G \times S \to V \), there exists a fixed rational constant \( J \) such that

\[ \delta^*d\nu = J \cdot d\tau \wedge d\mu. \]

Here, \( J \) is independent of \( \delta \).
Let $S(\mathbb{R})_{<X}$ denote the set of invariants $c \in S(\mathbb{R})$ of height less than $X$. For any place $\nu$ of $\mathbb{Q}$, let $a_{\nu}$ be the ratio

$$a_{\nu} = \frac{|J(\mathbb{Q}_{\nu})|/2|J(\mathbb{Q}_{\nu})|}{|J(\mathbb{Q}_{\nu})|}. \quad (18)$$

Here $J$ is the Jacobian of any hyperelliptic curve of genus $n$. The above quotient depends only on $\mathbb{Q}_{\nu}, n$ ([20, Lemmas 5.7, 5.14]) and satisfies the product formula $\prod_{\nu} a_{\nu} = 1$. We use (17) to compute the right hand side of (16) obtaining:

$$\#(V(\mathbb{Z})^{\text{irr}} \cap V(\mathbb{R})_{<X}^{\text{sol}}) \sim \operatorname{Vol}(G(\mathbb{Z}) \backslash V(\mathbb{R})_{<X}^{\text{sol}})$$

$$= |J| \cdot a_{\infty} \cdot \tau(G(\mathbb{Z}) \backslash G(\mathbb{R})) \cdot \mu(S(\mathbb{R})_{<X}). \quad (19)$$

To prove Theorem 4.1, we need to instead count $G(\mathbb{Q})$-equivalence classes of locally soluble elements of $V(\mathbb{Z})$ having invariants in $\operatorname{Inv}(F)$. We accomplish this via a sieve in Section 4.4. The vital ingredient for this sieve is a uniformity estimate proved in [3]. The sieving factor at the finite places are computed in Section 4.5 to be

$$|J|_{p} \cdot a_{p} \cdot \tau(G(\mathbb{Z})_{p}) \cdot \mu_{p}, \quad (20)$$

where $\mu_{p}$ is the local density of $\operatorname{Inv}(F)$ at $p$, namely, $\mu_{p} := \mu(\operatorname{Inv}_{p}(F))/\mu(S(\mathbb{Z})_{p})$. The analogous local density at $\infty$ is given by $\mu_{\infty} := \mu(\operatorname{Inv}_{\infty}(F)_{<X})/\mu(S(\mathbb{R})_{<X})$.

Therefore, we finally obtain:

$$\#(G(\mathbb{Q}) \backslash V_{F}^{\text{irr}}(\mathbb{Q})_{<X}) \sim |J| \cdot a_{\infty} \cdot \tau(G(\mathbb{Z}) \backslash G(\mathbb{R})) \cdot \mu(S(\mathbb{R})_{<X}) \cdot \mu_{\infty} \prod_{p} (|J|_{p} \cdot a_{p} \cdot \tau(G(\mathbb{Z})_{p}) \cdot \mu_{p})$$

$$\sim \tau_G \mu(S(\mathbb{R})_{<X}) \mu_{\infty} \prod_{p} \mu_{p}, \quad (21)$$

where $G(\mathbb{Q}) \backslash V_{F}^{\text{irr}}(\mathbb{Q})_{<X}$ denotes a set of representatives for the $G(\mathbb{Q})$-equivalence classes of locally soluble irreducible elements in $V(\mathbb{Q})$ having invariants in $\operatorname{Inv}(F)$ and height bounded by $X$.

We will show that up to a negligible quantity, the number of hyperelliptic curves $C : y^2 = f(x) \in F$ is equal to $\mu(S(\mathbb{R})_{<X}) \mu_{\infty} \prod_{p} \mu_{p}$. Furthermore, for 100% of these curves, the set $V_{f}(\mathbb{Q})$ contains two distinct distinguished orbits. Thus, the average number of locally soluble orbits for the action of $G(\mathbb{Q})$ on $V_{f}(\mathbb{Q})$ is equal to $2 + \tau_G = 6$.

### 4.2 Construction of fundamental domains

Let $V(\mathbb{R})^{\text{sol}}$ denote the set of $\mathbb{R}$-soluble elements in $V(\mathbb{R})$ having nonzero discriminant. We partition $V(\mathbb{R})^{\text{sol}}$ into $n + 2$ sets as follows,

$$V(\mathbb{R})^{\text{sol}} = \bigcup_{m=0}^{n+1} V(\mathbb{R})^{(m)},$$

where $V(\mathbb{R})^{(m)}$ consists of elements $B \in V(\mathbb{R})^{\text{sol}}$ such that the polynomial corresponding to $\pi(B)$ has $m$ pairs of complex conjugate roots (and $2n + 2 - 2m$ real roots). In this section, our goal is to describe convenient fundamental domains for the action of $G(\mathbb{Z})$ on $V(\mathbb{R})^{(m)}$ for $m \in \{0, \ldots, n + 1\}$.

#### Fundamental sets for the action of $G(\mathbb{R})$ on $V(\mathbb{R})^{\text{sol}}$

First, we construct convenient fundamental sets for the action of $G(\mathbb{R})$ on $V(\mathbb{R})^{(m)}$. Let $S(\mathbb{R})^{(m)}$ denote the set of elements $c \in S(\mathbb{R}) \{ \Delta = 0 \}$ such that the corresponding polynomial has $m$ pairs of complex conjugate roots. There exists an algebraic section $\kappa : S \to V$ defined over $\mathbb{Z}[1/2]$ such that every element in the image of $S(\mathbb{R}) \{ \Delta = 0 \}$ under $\kappa$ is distinguished [22, Section 3.1]. The number of $\mathbb{R}$-soluble $G(\mathbb{R})$-orbits in $V_{f}(\mathbb{R})$, for $c \in S(\mathbb{R})^{(m)}$, depends only on $m$. We denote it by $\tau_{m}$. There exist elements $g_{1}, \ldots, g_{\tau_{m}} \in \operatorname{GL}(U)(\mathbb{R})$ such that the set

$$R^{(m)} := \bigcup_{i} g_{i} \kappa(S(\mathbb{R})^{(m)}) g_{i}^{-1} \quad (22)$$

15
is a fundamental set for $G(\mathbb{R}) \setminus V(\mathbb{R})^{(m)}$. Indeed, since $L := \mathbb{R}[x]/f_c(x)$ is independent of $c \in S(\mathbb{R})^{(m)}$, an element $g \in \text{GL}(U)(\mathbb{R})$ that conjugates $\kappa(c_0)$ to a $G(\mathbb{R})$-orbit corresponding to a class $\alpha \in (L^x/L^{x2}\mathbb{R}^x)_{N=1}$ does so for every $c \in S(\mathbb{R})^{(m)}$.

We now construct our fundamental set $R^{(m)}$ for $G(\mathbb{R}) \setminus V(\mathbb{R})^{(m)}$ to be

$$R^{(m)} := \mathbb{R}_{>0} \cdot \{B \in R^{(m)} : H(B) = 1\}.$$  

(23)

The reason we use the set $R^{(m)}$ instead of $R^{(m)}$ is that the size of the coefficients of each element in $R^{(m)}$ having height $X$ is bounded by $O(X^{1/d})$, where $d = (2n + 2)(2n + 1)$ is the degree of the height function. This follows because the elements in $R^{(m)}$ having height 1 lie in a bounded subset of $V(\mathbb{R})$.

**Fundamental domains for the action of $G(\mathbb{Z})$ on $G(\mathbb{R})$**

We now describe Borel’s construction [10] of a fundamental domain $\mathcal{F}$ for the left action of $G(\mathbb{Z})$ on $G(\mathbb{R})$. Let $G(\mathbb{R}) = NTK$ be the Iwasawa decomposition of $G(\mathbb{R})$. Here, $N \subset G(\mathbb{R})$ denotes the set of unipotent lower triangular matrices, $T \subset G(\mathbb{R})$ denotes the set of diagonal matrices, and $K \subset G(\mathbb{R})$ is a maximal compact subgroup. Then a fundamental domain $\mathcal{F}$ for the action of $G(\mathbb{Z})$ on $G(\mathbb{R})$ may be expressed in the following form:

$$\mathcal{F} := \{utk : u \in N'(t), t \in T', k \in K\} \subset N'T'K$$

where $N' \subset N$ is a bounded set, $N'(t) \subset N'$ is a measurable set depending on $t \in T'$, and $T' \subset T$ is given by

$$T' := \{\text{diag}(t_1^{-1}, t_2^{-1}, \ldots, t_{n+1}^{-1}, t_{n+1}, \ldots, t_1) : t_1/t_2 > c, \ldots, t_n/t_{n+1} > c, t_n t_{n+1} > c\},$$

for some constant $c > 0$.

**Fundamental domains for the action of $G(\mathbb{Z})$ on $V(\mathbb{R})^{\text{sol}}$**

For $h \in G(\mathbb{R})$, we regard $\mathcal{F}h \cdot R^{(m)}$ as a multiset, where the multiplicity of $B$ in $\mathcal{F}h \cdot R^{(m)}$ is given by $\#\{g \in \mathcal{F} : B = gh \cdot R^{(m)}\}$. The $G(\mathbb{Z})$-orbit of any $B \in V(\mathbb{R})$ is represented $\#\text{Stab}_{G(\mathbb{R})}(B)/\#\text{Stab}_{G(\mathbb{Z})}(B)$ times in this multiset $\mathcal{F}h \cdot R^{(m)}$.

The group $\text{Stab}_{G(\mathbb{Z})}(B)$ is nontrivial only for a measure 0 set in $V(\mathbb{R})^{(m)}$. Indeed, $G(\mathbb{Z})$ is countable and every element $g \in G(\mathbb{Z})$ only fixes a measure 0 set in $V(\mathbb{R})$. (Later on, in Proposition 4.8, we will show that the number of $G(\mathbb{Z})$-orbits on $V(\mathbb{Z})$ having a nontrivial stabilizer in $G(\mathbb{Z})$ is negligible.) The size $\#\text{Stab}_{G(\mathbb{Z})}(B)$ is constant over $B \in V(\mathbb{R})^{(m)}$. We denote it by $\#J^{(m)}[2](\mathbb{R})$. Therefore, the multiset $\mathcal{F}h \cdot R^{(m)}$ is a cover of a fundamental domain for $G(\mathbb{Z})$ on $V(\mathbb{R})^{(m)}$ of degree $\#J^{(m)}[2](\mathbb{R})$.

**4.3 Averaging, cutting off the cusp, and estimation in the main body**

An element $B \in V(\mathbb{Q})$ is said to be irreducible if it has nonzero discriminant and it is not distinguished. For any $G(\mathbb{Z})$-invariant set $S \subset V(\mathbb{Z})^{(m)} := V(\mathbb{R})^{(m)} \cap V(\mathbb{Z})$, let $N(S; X)$ denote the number of irreducible $G(\mathbb{Z})$-orbits of $S$ that have height bounded by $X$, where each orbit $G(\mathbb{Z}) \cdot B$ is weighted by $1/\#\text{Stab}_{G(\mathbb{Z})}(B)$. The result of the previous section shows that we have

$$N(S; X) = \frac{1}{\#J^{(m)}[2](\mathbb{R}) \#\{\mathcal{F}h R^{(m)}(X) \cap S^{\text{irr}}\}}$$

for any $h \in G(\mathbb{R})$, where $R^{(m)}(X)$ denotes the elements in $R^{(m)}$ having height bounded by $X$ and $S^{\text{irr}}$ denotes the set of irreducible elements in $S$. Let $G_0$ be a bounded open $K$-invariant ball in $G(\mathbb{R})$. Averaging the above equation over $h \in G_0$ we obtain:

$$N(S; X) = \frac{1}{\#J^{(m)}[2](\mathbb{R}) \text{Vol}(G_0)} \int_{h \in G_0} \#\{\mathcal{F}h R^{(m)}(X) \cap S^{\text{irr}}\} dh,$$

for any Haar-measure $dh$ on $G(\mathbb{R})$, and where the volume of $G_0$ is computed with respect to $dh$. We use (24) to define $N(S; X)$ when $S$ is not $G(\mathbb{Z})$-invariant.
By an argument identical to the proof of [7, Theorem 2.5], we obtain

\[ N(S; X) = \frac{1}{\#(m)[2](\mathbb{R})\text{Vol}(G_0)} \int_{h\in\mathcal{F}} \#\{hG_0R^{(m)}(X) \cap S^{\text{irr}}\} \, dh. \]  \tag{25}

To estimate the number of integral points in the bounded region \(hG_0R^{(m)}(X)\), we use the following result of Davenport [13].

**Proposition 4.3** Let \(\mathcal{R}\) be a bounded, semi-algebraic multiset in \(\mathbb{R}^n\) having maximum multiplicity \(m\), and that is defined by at most \(k\) polynomial inequalities each having degree at most \(\ell\). Then the number of integral lattice points (counted with multiplicity) contained in the region \(\mathcal{R}\) is

\[ \text{Vol}(\mathcal{R}) + O(\max\{\text{Vol}(\mathcal{R}), 1\}), \]

where \(\text{Vol}(\mathcal{R})\) denotes the greatest \(d\)-dimensional volume of any projection of \(\mathcal{R}\) onto a coordinate subspace obtained by equating \(n-d\) coordinates to zero, where \(d\) takes all values from 1 to \(n-1\). The implied constant in the second summand depends only on \(n, m, k,\) and \(\ell\).

We can express any \(h\in\mathcal{F}\) as \(h = utk\), where \(u \in \mathcal{N}'\), \(t \in \mathcal{T}'\), and \(k \in K\). Since \(G_0\) is \(K\)-invariant, we have for any \(h \in \mathcal{F}\),

\[ hG_0R^{(m)}(X) = utkG_0R^{(m)}(X) = t(t^{-1}ut)G_0R^{(m)}(X). \]

By the descriptions of \(\mathcal{N}'\) and \(\mathcal{T}'\), we see that the set \(t^{-1}N't\) is bounded independent of \(t \in \mathcal{T}'\). The coordinates of elements in \(\mathcal{N}'\) are scaled by either \((t_i/t_{i+1})^{-1}\) for \(i = 1, \ldots, n\), or \((t_n t_{n+1})^{-1}\), which are bounded above by \(1/c'\). Therefore \((t^{-1}ut)G_0R^{(m)}(X)\) is a compact region where the coefficients of the elements inside are growing homogeneously in \(X\). It is the action of \(t \in \mathcal{T}'\) that stretches and compresses different coordinates.

As \(t\) grows in \(\mathcal{T}'\), the estimates on the number of integral points in \(hG_0R^{(m)}(X)\) obtained from Proposition 4.3 gets worse and worse. Indeed when \(t\) gets high enough (in the cusp of \(\mathcal{T}'\)), the top left entry \(b_{11}\) of every element in \(hG_0R^{(m)}(X)\) will be less than 1 in absolute value, at which point the error term in Proposition 4.3 dominates the main term. As \(t\) gets bigger, other entries start becoming less than 1 in absolute value and we get even worse estimates. To deal with this problem, we break \(V(\mathcal{R})\) up into two pieces: the main body, which contains all elements \(B \in V(\mathcal{R})\) with \(|b_{11}| \geq 1\); and the cusp region, which contains all elements \(B \in V(\mathcal{R})\) with \(|b_{11}| < 1\). As \(t\) gets bigger, more and more coefficients of the integral elements of \(hG_0R^{(m)}(X)\) will become 0. Using Proposition 2.4, we know that once enough entries of \(B\) are 0, it will become distinguished and thus reducible. In Proposition 4.5, we compute the number of irreducible integral points in the cusp region and in Proposition 4.7, we compute the number of reducible integral points in the main body. They are both negligible when compared to the number of integral points in the main region and as a result, we will prove the following result.

**Theorem 4.4**

\[ N(V(\mathbb{Z})^{(m)}; X) = \frac{1}{\#(m)[2](\mathbb{R})\text{Vol}(\mathcal{F} \cdot R^{(m)}(X))} + o(X^{\frac{\dim V}{d}}). \]

In §4.5, we show that \(\text{Vol}(\mathcal{F} \cdot R^{(m)}(X))\) grows on the order of \(X^{\frac{\dim V}{d}}\) so the error term is indeed smaller than the main term.

Let \(V(\mathbb{Z})(b_{11} = 0)\) denote the set of points \(B \in V(\mathbb{Z})\) such that \(b_{11} = 0\). Then we have the following proposition:

**Proposition 4.5** With notation as above, we have \(N(V(\mathbb{Z})(b_{11} = 0); X) = O_\epsilon(X^{\frac{\dim V - 1}{d} + \epsilon}). \)

**Proof:** It will be convenient to use the following parameters for \(T\): \(s_i = t_i/t_{i+1}\) for \(i = 1, \ldots, n\); and \(s_{n+1} = t_n t_{n+1}\). The condition for \(t \in \mathcal{T}'\) translates to \(s_i > c\) for all \(i\). We pick the following Haar measure \(dh\) on \(G(\mathbb{R}) = \mathcal{N}K\):

\[ dh = du \prod_{j=1}^{n-1} s_j^{j-2n-1} \cdot (s_n s_{n+1})^{-\frac{n(n+1)}{2}} d^x s_j dk \]

\[ = du \delta(s) d^x s \, dk, \]  \tag{26}
where \( du \) is a Haar measure on the unipotent group \( N \), \( dk \) is Haar measure on \( K \) normalized so that \( K \) has volume 1, \( \delta(s) \) denotes \( \prod_{j=1}^{n+1} s_{j-2} \), \( s_{j-n+1}^{-\frac{n-1}{2}} \), and \( ds \) denotes \( \prod_{j=1}^{n+1} ds_k \).

Then, since \( G_0 \) is \( K \)-invariant, (25) implies that

\[
N(V(Z)(b_{11} = 0); X) = O\left( \int_{h \in T} \# \{ hG_0R^{(m)}(X) \cap V(Z)(b_{11} = 0) \} dh \right)
= O\left( \int_{u \in N'} \int_{t \in T'} \# \{ uG_0R^{(m)}(X) \cap V(Z)(b_{11} = 0) \} \delta(s) ds du \right)
= O\left( \int_{t \in T'} \# \{ tG_0R^{(m)}(X) \cap V(Z)(b_{11} = 0) \} \delta(s) ds \right),
\]

where the final equality follows because \( N' \) has finite measure, \( uG_0R^{(m)}(X) = t(t^{-1}ut)G_0R^{(m)}(X) \), and the coefficients of \( t^{-1}ut \) are bounded independent of \( t \in T' \) and \( u \in N' \).

Let \( b_{ij}, i \leq j, (i,j) \neq (n+1, n+2) \) be the system of coordinates on \( V(\mathbb{R}) \), where \( b_{ij} \) is the \((i,j)\)’th entry of the symmetric matrix \( B \). To each coordinate \( b_{ij} \), we associate the weight \( w(b_{ij}) \) which records how an element \( s \in T \) scales \( b_{ij} \). For example,

\[
w(b_{11}) = s_1^{-2}, \quad w(b_{1,2n+3-i}) = 1, \quad w(b_{1,2n+2-i}) = s_i^{-1}, \quad i = 1, \ldots, n, \quad w(b_{n+1,n+1}) = s_n^{1-n}.
\]

Let \( C \) be an absolute constant such that \( CX^{\frac{d}{2}} \) bounds the absolute value of all the coordinates of elements \( B \in G_0R^{(m)}(X) \). If, for \( (s_1, \ldots, s_{n+1}) \in T' \), we have \( CX^{\frac{d}{2}} w(b_{i_0,2n+2-i_0}) < 1 \) for some \( i_0 \in \{1, \ldots, n+1\} \), then \( CX^{\frac{d}{2}} w(b_{ij}) < 1 \) for all \( i \leq i_0, j \leq 2n + 2 - i_0 \). Hence the top left \( i_0 \times (2n + 2 - i_0) \) block of any integral \( B \in tG_0R^{(m)}(X) = 0 \). Just as [5, Lemma 10.3] shows, any such \( B \) has zero discriminant. Therefore, to prove Proposition 4.5, we may assume

\[
s_i < \frac{X^{1/d}}{C}, \quad \text{i = 1, \ldots, n; } \quad s_{n+1} < \frac{X^{2/d}}{C^2}.
\]

We use \( T_X \) to denote the set of \( t = (s_1, \ldots, s_{n+1}) \in T' \) satisfying these bounds.

Let \( U_1 \) denote any subset of the coordinates \( b_{ij} \). Let \( V(\mathbb{R})(U_1) \) denote the subset of \( V(\mathbb{R}) \) consisting of elements \( B \) whose \((i,j)\) entry is less than 1 in absolute value when \( b_{ij} \in U_1 \) and whose \((i,j)\) entry is greater than 1 when \( b_{ij} \notin U_1 \). Let \( V(Z)(U_1) \) denote the set of integral points in \( V(\mathbb{R})(U_1) \). Then to prove Proposition 4.5, it suffices to show that

\[
N(V(Z)(U_1); X) = O(X^{\frac{\dim V - 1}{d}} + \epsilon),
\]

for every set \( U_1 \) containing \( b_{11} \).

Proposition 4.3 in conjunction with the argument used to justify (27) implies

\[
N(V(Z)(U_1); X) = O\left( \int_{t \in T_X} \text{Vol}(tG_0R^{(m)}(X) \cap V(\mathbb{R})(U_1)) \delta(s) ds \right)
= O\left( X^{\frac{\dim V - \# U_1}{d}} \int_{t \in T_X} \prod_{b_{ij} \notin U_1} w(b_{ij}) \delta(s) ds \right).
\]

Therefore to prove (29), we need to estimate:

\[
I(U_1, X) := X^{\frac{\dim V - \# U_1}{d}} \int_{t \in T_X} \prod_{b_{ij} \notin U_1} w(b_{ij}) \delta(s) ds,
\]
for every set $U_1$ containing $b_{11}$.

Note that if $i' \leq i$ and $j' \leq j$, then $w(b_{i'j'})$ has smaller exponents in all the $s_k$’s than $w(b_{ij})$. Thus, if a set $U_1$ contains $b_{ij}$ but not $b_{i'j'}$, then

$$I(U_1 \setminus \{b_{ij}\} \cup \{b_{i'j'}\}, X) \geq I(U_1, X).$$

Hence for the purpose of obtaining an upper bound for $I(U_1, X)$, we may assume that if $b_{ij} \in U_1$, then $b_{i'j'} \in U_1$ for all $i' \leq i$ and $j' \leq j$. If such a set $U_1$ contains any element on, or to the right of, the off-anti-diagonal, then every element in $V(\mathbb{Z})(U_1)$ has discriminant 0 and by definition $N(V(\mathbb{Z})(U_1); X) = 0$. Let $U_0$ denote the set of coordinates $b_{ij}$ such that $i \leq j$ and $i + j \leq 2n + 1$. In other words, $U_0$ contains every coordinate to the left of the off-anti-diagonal. Since every element in $V(\mathbb{Z})(U_0)$ is distinguished (by Proposition 2.4), hence reducible, it suffices to consider $I(U_1, X)$ for all $U_1 \subseteq U_0$.

To this end, as the product of the weights over all coordinates is 1, we define

$$I(U_1, X) = X^{-\frac{\#U_1}{d}} \int_{s_1, \ldots, s_n = c} X^{\frac{d}{2}} \int_{s_{n+1} = c} X^{\frac{d}{2}} \prod_{(i,j) \in U_1} w(b_{ij})^{-1} \prod_{k=1}^{n-1} s_k^{k(2n-1)} (s_n s_{n+1})^{-\frac{n(n+1)}{2}} d^xs. \quad (31)$$

To complete the proof of Proposition 4.5, it suffices to prove the following lemma:

**Lemma 4.6** Let $U_1$ be nonempty proper subset of $U_0$. Then

$$I(U_1, X) = O_r(X^{-\frac{1}{d} + r}).$$

If $U_1 = U_0$ or $U_1 = \emptyset$, then $I(U_1, X) = O(1)$.

**Proof:** The proof of this lemma is a combinatorial argument using induction on $n$. We first compute

$$I(U_0, X) = X^{-\frac{n(n+1)}{2d}} \int_{s_1, \ldots, s_n = c} X^{\frac{d}{2}} \int_{s_{n+1} = c} X^{\frac{d}{2}} \prod_{(i,j) \in U_0} w(b_{ij}) \cdot s_1 s_2^3 \cdots s_{n-1} s_n s_{n+1} d^xs = O(1). \quad (32)$$

This is expected since $V(\mathbb{Z})(U_0)$ contains all but negligibly many distinguished orbits (see Proposition 4.7).

Let $U'_1$ denote $U_0 \setminus U_1$, and define $I'_n(U'_1, X)$ to equal $I(U_1, X)$. Combining (31) with (32), we obtain

$$I'_n(U'_1, X) = I(U_1, X) = X^{-\frac{\#U'_1 - (n+1)}{d}} \int_{s_1, \ldots, s_n = c} X^{\frac{d}{2}} \int_{s_{n+1} = c} X^{\frac{d}{2}} \prod_{(i,j) \in U'_1} w(b_{ij}) \cdot s_1 s_2^3 \cdots s_{n-1} s_n s_{n+1} d^xs. \quad (33)$$

Write $U'_1 = U'_2 \cup U'_3$ where $U'_2$ is the set of coordinates $b_{ij}$ in $U'_1$ and $U'_3 = U'_1 \setminus U'_2$. Then we may express $I'_n(U'_1, X)$ as the following product:

$$\left(X^{-\frac{\#U'_2 - 2n}{d}} \int_{b_{ij} \in U'_2} w(b_{ij}) s_1 s_2^2 \cdots s_{n-1} s_n s_{n+1} d^xs\right) \left(X^{-\frac{\#U'_3 - (n+1)}{d}} \int_{b_{ij} \in U'_3} w(b_{ij}) s_2 s_3^3 \cdots s_{n-1} s_n s_{n+1} d^xs\right).$$

Note that the second term in the above expression is equal to $I'_{n-1}(\{b_{ij} : b_{i+1,j+1} \in U'_3\}, X)$ (which we denote by $I'_{n-1}(U'_3, X)$) and we may estimate it using induction. Denote the first term in the above expression by $J'_n(U'_2, X)$. A similar, but much simpler, induction argument implies

$$J'_n(U'_2, X) = O(X^{-\frac{1}{d}}),$$

unless $U'_2 = \emptyset$, in which case it is $O(1)$.

Therefore, if $U'_2$ is not empty, then the lemma follows by induction on $n$ (used to bound $I'_{n-1}(U'_3, X)$ by $O(1)$). If $U'_2$ is empty, then $U'_3$ must be nonempty since $U'_1$ is nonempty. If further $U'_3 \neq U_0 \setminus \{b_{11}, \ldots, b_{12n}\}$, then by induction, we have $I'_{n-1}(U'_3, X) = O_r(X^{-1/2} + r)$. The only remaining case is when $U_1 = \{b_{11}, \ldots, b_{12n}\}$, for which a direct computation yields the result. □

This concludes the proof of Proposition 4.5. □

We now have the following two propositions, whose proofs follow that of [2, Lemma 14].
Proposition 4.7 Let $V(\mathbb{Z})(\phi)_{\text{red}}$ denote the set of elements in $V(\mathbb{Z})$ with $b_{11} \neq 0$ that are not irreducible. Then
$$\int_{G_0} \#\{V(\mathbb{Z})(\phi)_{\text{red}} \cap Fg \cdot R^{(m)}(X)\} dg = o(X^{\frac{\dim V}{4}}).$$

Proposition 4.8 Let $V(\mathbb{Z})_{\text{bigstab}}$ denote the set of elements in $V(\mathbb{Z})$ which have a nontrivial stabilizer in $G(\mathbb{Z})$. Then
$$N(V(\mathbb{Z})_{\text{bigstab}}; X) = o(X^{\frac{\dim V}{4}}).$$

**Proof:** Observe that if $B \in V(\mathbb{Z})$ is reducible over $\mathbb{Z}$, then the image of $B$ in $V(\mathbb{F}_p)$ is reducible for all $p$. For any prime $p$, let $\phi_p$ denote the $p$-adic density of the set of elements of $V(\mathbb{Z}_p)$ that are reducible mod $p$. Then to prove Proposition 4.7, it suffices to show
$$\prod_p \phi_p = 0.$$ 

We show this by proving that $\phi_p$ is bounded above by some constant less than 1 when $p$ is large enough. For large enough $p$, there is a positive proportion $r_n$ (depending only on $n$) of polynomials of degree $2n+2$ over $\mathbb{F}_p$ that factors into two linear terms and an irreducible polynomial of degree $2n$. Suppose $f(x) \in \mathbb{Z}_p[x]$ with this reduction type over $\mathbb{F}_p$. Since it has a linear factor, Proposition 2.2 implies that there is one distinguished orbit. Since $H^1(\mathbb{F}_p, J) = 0$ by Lang’s theorem, every orbit is soluble. The number of orbits $\#J(\mathbb{F}_p)/J(\mathbb{F}_p)$ is equal to the size of the stabilizer $\#J[2](\mathbb{F}_p)$. Since $f(x)$ has a degree two factor, $\#J[2](\mathbb{F}_p) \geq 2$. Therefore at least 1/2 of the elements in $V_I(\mathbb{F}_p)$ are not distinguished. Hence for $p$ large enough, $\phi_p \leq 1 - \frac{1}{2}r_n < 1$.

We use the same technique to prove Proposition 4.8. For $p$ large enough, there is a positive proportion $r'_n$ (depending only on $n$) of polynomials of degree $2n+2$ over $\mathbb{F}_p$ that factors into a linear term and an irreducible polynomial of degree $2n+1$. If $B \in V_I(\mathbb{Z}_p)$ where $f(x)$ has this reduction type mod $p$, then $p$ does not divide the discriminant of $f(x)$. As a consequence, the hyperelliptic curve $y^2 = f(x)$ is smooth over $\text{Spec}(\mathbb{Z}_p)$ and the 2-torsion of its Jacobian $J[2]$ is a finite étale group scheme over $\text{Spec}(\mathbb{Z}_p)$. From the reduction type of $f(x)$ over $p$, we see that $\#J[2](\mathbb{F}_p) = \#J[2](\mathbb{F}_p) = 1$. Denote by $\phi_p$ the $p$-adic density of the set of elements of $V(\mathbb{Z}_p)$ with non-trivial stabilizer in $G(\mathbb{Q}_p)$. Then we have shown that $\phi_p \leq 1 - r'_n < 1$ for $p$ sufficiently large. This completes the proof. \(\square\)

We may now prove the main result of this section, which we state again for the convenience of the reader.

**Theorem 4.9**

$$N(V(\mathbb{Z})^{(m)}; X) = \frac{1}{\#J^{(m)}[2](\mathbb{R}) \text{Vol}(G_0)} \int_{h \in \mathcal{F}} \#\{hG_0R^{(m)}(X) \cap V(\mathbb{Z})^{\text{irr}}\} dh + o(X^{\frac{\dim V}{4}}).$$

**Proof:** Let $\mathcal{F}' \subset \mathcal{F}$ be the set consisting of $h \in \mathcal{F}$ such that the $b_{11}$-coefficient of any $B \in hG_0R^{(m)}(X)$ is less than 1 in absolute value. From (25), we see that $N(V(\mathbb{Z})^{(m)}; X)$ is equal to

$$\int_{h \in \mathcal{F}} \#\{hG_0R^{(m)}(X) \cap V(\mathbb{Z})^{\text{irr}}\} dh = \frac{1}{\#J^{(m)}[2](\mathbb{R}) \text{Vol}(G_0)} \left( \int_{h \in \mathcal{F} \setminus \mathcal{F}'} \#\{hG_0R^{(m)}(X) \cap V(\mathbb{Z})^{\text{irr}}\} dh + \int_{h \in \mathcal{F}'} \#\{hG_0R^{(m)}(X) \cap V(\mathbb{Z})^{\text{irr}}\} dh \right).$$

From Propositions 4.5 and 4.7, we obtain:

$$N(V(\mathbb{Z})^{(m)}; X) = \frac{1}{\#J^{(m)}[2](\mathbb{R}) \text{Vol}(G_0)} \int_{h \in \mathcal{F} \setminus \mathcal{F}'} \#\{hG_0R^{(m)}(X) \cap V(\mathbb{Z})\} dh + o(X^{\frac{\dim V}{4}}). \quad (34)$$

Note that $b_{11}$ has minimal weight among all the $b_{ij}$. Furthermore, the length of the projection of $hG_0R^{(m)}(X)$ onto the $b_{11}$-line is greater than 1 for $h \in \mathcal{F} \setminus \mathcal{F}'$ (by the definition of $\mathcal{F}'$). Therefore, for
Proposition 4.3 thus implies that

\[ N(V(Z)^{(m)};X) = \frac{1}{\# J^{(m)}[2](\mathbb{R}) \text{Vol}(G_0)} \int_{h \in \mathcal{F}\setminus \mathcal{F}'} \text{Vol}(hG_0R^{(m)}(X)) + O\left( \frac{\text{Vol}(hG_0R^{(m)}(X))}{X^{1/d_\nu(b_{11})}} \right) dh + o(X^{\frac{\dim W}{d}}). \]

Recall \( \mathcal{F}' \) is defined by the condition \( CX^\frac{2}{d} w(b_{11}) < 1 \). Therefore to be in \( \mathcal{F}' \), one of the \( s_i \) must be at least \( C X^\frac{d}{2} w \). Hence the volume of \( \mathcal{F}' \) is bounded by \( o(1) \). Moreover, since \( \int_{h \in \mathcal{F}\setminus \mathcal{F}'} 1/w(b_{11}) dh = O(1) \), we obtain

\[ N(V(Z)^{(m)};X) = \frac{1}{\# J^{(m)}[2](\mathbb{R}) \text{Vol}(G_0)} \int_{h \in \mathcal{F}} \text{Vol}(hG_0R^{(m)}(X)) dh + o(X^{\frac{\dim W}{d}}). \]

where the third equality follows because the volume of \( \mathcal{F}h \cdot R^{(m)}(X) \) is independent of \( h \). This concludes the proof of Theorem 4.4. \( \square \)

### 4.4 A squarefree sieve

In this section, we present versions of Theorem 4.4, where we count elements (and weighted elements) of \( V(Z) \) satisfying certain sets of congruence conditions.

**Theorem 4.10** Let \( L \) be a subset of \( V(Z) \) defined by finitely many congruence conditions on the coefficients of elements in \( V(Z) \). Then

\[ N(L \cap V(Z)^{(m)};X) = N(V(Z)^{(m)};X) \prod_p \nu_p(L) + o(X^{\frac{\dim W}{d}}), \]

where \( \nu_p(L) \) denotes the \( p \)-adic density of \( L \) in \( V(Z) \) and is equal to 1 for all but finitely many primes \( p \).

This theorem follows immediately from the proof of Theorem 4.4. (See [7, Theorem 2.11] for an analogous situation.)

The following weighted version of Theorem 4.4 also follows immediately:

**Theorem 4.11** Let \( p_1, \ldots, p_k \) be distinct prime numbers. For \( j = 1, \ldots, k \), let \( \phi_{p_j} : V(Z) \to \mathbb{R} \) be a \( G(Z) \)-invariant function on \( V(Z) \) such that \( \phi_{p_j}(B) \) depends only on the congruence class of \( B \) modulo some power \( p_j^{a_j} \) of \( p_j \). Let \( N_\phi(V^{(m)}(Z);X) \) denote the number of irreducible \( G(Z) \)-orbits of \( V^{(m)}(Z) \) having height bounded by \( X \), where each orbit \( G(Z) \cdot B \) is counted with weight \( \phi(B)/\# \text{Stab}_{G(Z)}(B) \); here \( \phi \) is defined by \( \phi(B) := \prod_{j=1}^k \phi_{p_j}(B) \). Then we have

\[ N_\phi(V^{(m)}(Z);X) = N(V^{(m)}(Z);X) \prod_{j=1}^k \int_{B \in V(Z_{p_j})} \tilde{\phi}_{p_j}(B) dB + o(X^{\frac{\dim W}{d}}), \]
However, in order to prove Theorem 4.1, we shall need a version of Theorem 4.11 in which we allow weights to be defined by certain infinite sets of congruence conditions. The technique for proving such a result involves using Theorem 4.11 to impose more and more congruence conditions. While doing so, we need to uniformly bound the error term. To this end, we have the following proposition proven in [3].

**Proposition 4.12** For each prime $p$, let $W_p$ denote the set of elements $B \in V(\mathbb{Z})$ such that $p^2 \mid \Delta(B)$. Then there exists $\delta > 0$ such that, for any $M > 0$, we have

$$\sum_{p > M} N(W_p; X) = O(X^{\frac{\dim V}{2}}/M^\delta),$$

where the implied constant is independent of $X$ and $M$.

To describe which weight functions on $V(\mathbb{Z})$ are allowed, we need the following definition:

**Definition 4.13** A function $\phi : V(\mathbb{Z}) \to [0, 1]$ is said to be defined by congruence conditions if there exist local functions $\phi_p : V(\mathbb{Z}_p) \to [0, 1]$ satisfying the following conditions:

1. For all $B \in V(\mathbb{Z})$, the product $\prod_p \phi_p(B)$ converges to $\phi(B)$.

2. For each prime $p$, the function $\phi_p$ is locally constant outside some closed set $S_p$ of measure 0.

Such a function is said to be acceptable if, for all sufficiently large $p$, we have $\phi_p(B) = 1$ whenever $p^2 \nmid \Delta(B)$.

Then we have the following theorem.

**Theorem 4.14** Let $\phi : V(\mathbb{Z}) \to [0, 1]$ be an acceptable function that is defined by congruence conditions via local functions $\phi_p : V(\mathbb{Z}_p) \to [0, 1]$. Then, with notation as in Theorem 4.11, we have

$$N_\phi(V^{(m)}(\mathbb{Z}); X) = N(V^{(m)}(\mathbb{Z}); X) \prod_p \int_{B \in V(\mathbb{Z}_p)} \phi_p(B) dB + o(X^{\frac{\dim V}{2}}).$$

Theorem 4.14 follows from Theorems 4.11 and Proposition 4.12 just as [7, Theorem 2.21] followed from [7, Theorem 2.12] and [7, Theorem 2.13].

### 4.5 Compatibility of measures and local computations

Let $F$ be a large family of hyperelliptic curves. Throughout this section and the next, we assume without loss of generality that $\text{Inv}^\infty(F) = S(\mathbb{R})^{(m)}$ for some fixed integer $m \in \{0, \ldots, n + 1\}$. To prove Theorem 4.1 we need to weight each locally soluble element $B \in V(\mathbb{Z})$ (having invariants in $\text{Inv}(F)$) by the reciprocal of the number of $G(\mathbb{Z})$-orbits in the $G(\mathbb{Q})$-equivalence class of $B$ in $V(\mathbb{Z})$. However, in order for our weight function to be defined by congruence conditions, we instead define the following weight function $w : V(\mathbb{Z}) \to [0, 1]$:

$$w(B) := \begin{cases} 
\frac{\#\text{Stab}_{G(\mathbb{Z})}(B')}{\#\text{Stab}_{G(\mathbb{Q})}(B')}^{-1} & \text{if } B \text{ is locally soluble and } \text{Inv}(B) \in \text{Inv}(F), \\
0 & \text{otherwise,}
\end{cases} \quad (37)$$

where the sum is over a complete set of representatives for the action of $G(\mathbb{Z})$ on the $G(\mathbb{Q})$-equivalence class of $B$ in $V(\mathbb{Z})$. We then have the following theorem:

**Theorem 4.15** Let $F$ be a large family of hyperelliptic curves. Then

$$\sum_{C \in F} \left( \#\text{Sel}_2(J(C)) - 2 \right) = N_w(V(\mathbb{Z})^{(m)}; X) + o(X^{\frac{\dim V}{2}}), \quad (38)$$

where $V(\mathbb{Z})^{(m)}$ is the set of all elements in $V(\mathbb{Z})$ whose invariants belong to $\Sigma_\infty = S(\mathbb{R})^{(m)}$. 

22
Proof: It follows from Proposition 5.3 that for a 100% of hyperelliptic curves $C(c) \in F$, the set $V_c(\mathbb{Q})$ has two distinguished orbits. Thus, Theorem 2.8 and Corollary 2.10 show that, up to an error of $o(X^{\frac{1}{26}})$, the left hand side of (38) is equal to

$$\#(G(\mathbb{Q}) \setminus V_F(\mathbb{Q}))_{y < X},$$

the number of $G(\mathbb{Q})$-equivalence classes of elements in $V(\mathbb{Z})$ that are locally soluble, have invariants in $\text{Inv}(F)$, and have height bounded by $X$. Given a locally soluble element $B \in V(\mathbb{Z})$ such that $\text{Inv}(B) \in F$, let $B_1 \ldots B_k$ denote a complete set of representatives for the action of $G(\mathbb{Z})$ on the $G(\mathbb{Q})$-equivalence class of $B$ in $V(\mathbb{Z})$. Then

$$\sum_{i=1}^{k} \frac{w(B_i)}{\#\text{Stab}_{G(\mathbb{Z})}(B_i)} = \frac{1}{\#\text{Stab}_{G(\mathbb{Q})}(B)} \left( \sum_{i=1}^{k} \frac{1}{\#\text{Stab}_{G(\mathbb{Z})}(B_i)} \right)^{-1} \sum_{i=1}^{k} \frac{1}{\#\text{Stab}_{G(\mathbb{Z})}(B_i)} = \frac{1}{\#\text{Stab}_{G(\mathbb{Q})}(B)}. \quad (39)$$

Therefore, the right hand side of (38) counts the number of $G(\mathbb{Q})$-equivalence classes of elements in $V(\mathbb{Z})$ that are locally soluble, have invariants in $F$, and have height bounded by $X$, such that the $G(\mathbb{Q})$-orbit of $B$ is weighted with $1/\#\text{Stab}_{G(\mathbb{Q})}(B)$ for all orbits. The theorem now follows since $\#\text{Stab}_{G(\mathbb{Q})}(B) = 1$ for all but negligible many $B \in V(\mathbb{Z})$ by Proposition 4.8. □

In order to demonstrate that $w$ is defined by congruence conditions, we need to express it as a local product of weight functions on $V(\mathbb{Z}_p)$. To this end, we define $w_p : V(\mathbb{Z}_p) \to [0,1]$

$$w_p(B) := \begin{cases} \left( \sum_{B'} \frac{\#\text{Stab}_{G(\mathbb{Z}_p)}(B')}{\#\text{Stab}_{G(\mathbb{Z}_p)}(B')} \right)^{-1} & \text{if } B \text{ is } \mathbb{Q}_p\text{-soluble and } \text{Inv}(B) \in \text{Inv}_p(F), \\ 0 & \text{otherwise,} \end{cases} \quad (40)$$

where the sum is over a set of representatives for the action of $G(\mathbb{Z}_p)$ on the $G(\mathbb{Q}_p)$-equivalence class of $B$ in $V(\mathbb{Z})$. Our next aim is to show that $w$ is an acceptable function that is defined by congruence conditions via the local functions $w_p$.

**Proposition 4.16** If $B \in V(\mathbb{Z})$ has nonzero discriminant, then $w(B) = \prod_p w_p(B)$. Furthermore, $w(b)$ is an acceptable function.

**Proof:** The first assertion of the proposition follows from the fact that $G$ has class number 1 over $\mathbb{Q}$; the proof is identical to that of [7, Proposition 3.6]. In order to prove that $m$ is acceptable, it therefore suffices to check that, for sufficiently large primes $p$, we have $w_p(B) = 1$ whenever $p^2 \nmid \Delta(B)$. This follows from Proposition 2.11. □

From Theorems 4.4 and 4.14, we have the following equality:

$$N_w(V(\mathbb{Z})^{(m)}, X) = \frac{1}{\#\mathcal{J}(m) |2| } \text{Vol}(F \cdot R^{(m)}(X)) \prod_p \int_{V(\mathbb{Z}_p)} w(B)dB + o(X^{\frac{1}{26}}). \quad (41)$$

For the rest of the section, our aim is to express $\text{Vol}(F \cdot R^{(m)}(X))$ and $\int_{V(\mathbb{Z}_p)} w(B)dB$ in more convenient forms. To this end, we have the following result that allows us to compute volumes of multisets in $V(K)$, for $K = \mathbb{R}$ and $\mathbb{Z}_p$. This result follows from [7, Proposition 3.11] and [7, Proposition 3.12].

**Proposition 4.17** Let $K$ be $\mathbb{R}$ or $\mathbb{Z}_p$ for some prime $p$, let $|.|$ denote the usual valuation on $K$, and let $s : S(K) \to V(K)$ be a continuous section. Then there exists a rational nonzero constant $\mathcal{J}$, independent of $K$ and $s$, such that for any measurable function $\phi$ on $V(K)$, we have

$$\int_{G(K) \cdot s(S(K))} \phi(B) d\nu(B) = |\mathcal{J}| \int_{s(S(K))} \int_{g \in G(K)} \phi(g \cdot s(c)) d\tau(g) d\mu(c),$$

$$\int_{V(K)} \phi(B) d\nu(B) = |\mathcal{J}| \int_{s(S(K))} \frac{1}{\#\text{Stab}_{G(K)}(B)} \int_{g \in G(K)} \phi(g \cdot B) d\tau(g) d\mu(c). \quad (42)$$
where we regard \(G(K) \cdot s(R)\) as a multiset, and \(\frac{V_c(K)}{G(K)}\) denotes a set of representatives for the action of \(G(K)\) on \(V_c(K)\).

We use Proposition 4.17 to compute \(\text{Vol}(\mathcal{F} \cdot R^{(m)}(X))\). If \(c \in R^{(m)}\) and \(J = J(C(c))\) is the Jacobian of the corresponding hyperelliptic curve, then the number of \(\mathbb{R}\)-soluble \(G(\mathbb{R})\)-orbits of \(V_c(\mathbb{R})\) is \(\#(J(\mathbb{R})/2J(\mathbb{R}))\). This number is a constant independent of \(c \in V(\mathbb{R})^{(m)}\), and we denote it by \(\#(J^{(m)}(\mathbb{R})/2J^{(m)}(\mathbb{R}))\). Thus, \(R^{(m)}\) contains \(\#(J^{(m)}(\mathbb{R})/2J^{(m)}(\mathbb{R}))\) elements having invariant \(c\) for every \(c \in S(\mathbb{R})^{(m)}\). Therefore, using the first equation of Proposition 4.17, we obtain:

\[
\frac{1}{\#J^{[m]/2}(\mathbb{R})} \text{Vol}(\mathcal{F} \cdot R^{(m)}(X)) = |\mathcal{J}| \frac{\#(J^{(m)}(\mathbb{R})/2J^{(m)}(\mathbb{R}))}{\#J^{[m]/2}(\mathbb{R})} \text{Vol}(\mathcal{F}) \text{Vol}(S(\mathbb{R})^{(m)})
\]

where \(a_\nu\) was defined in (18) for every place \(\nu\) of \(\mathbb{Q}\).

Next we compute \(\int_{V(\mathbb{Z}_p)} w_p(B) d\nu(B)\). Note that since \(w_p\) is \(G(\mathbb{Z}_p)\)-invariant, we have

\[
\int_{V(\mathbb{Z}_p)} w_p(B) d\nu(B) = |\mathcal{J}_p| \text{Vol}(G(\mathbb{Z}_p)) \int_{\mathcal{X}\text{Inv}_p(F)} \left( \sum_{B \in \mathcal{S} \mathcal{Z}_p} \frac{w_p(B)}{\#\text{Stab}_G(B)} \right) d\mu(c)
\]

The final equality follows from a computation similar to (39); namely, if \(J = J(C(c))\) and \(B_c\) is any element in \(V_c(\mathbb{Q}_p)\), we have by Proposition 2.9,

\[
\sum_{B \in \mathcal{S} \mathcal{Z}_p} \frac{w_p(B)}{\#\text{Stab}_G(B)} = \frac{\#G(\mathbb{Q}_p)\mathcal{V}_c(\mathbb{Q}_p)}{\#\text{Stab}_G(\mathbb{Q}_p)} = \frac{\#(J(\mathbb{Q}_p)/2J(\mathbb{Q}_p))}{\#J(\mathbb{Q}_p)} = a_p.
\]

Combining Theorem 4.15 with (41), (43), and (44), we obtain

\[
\sum_{G \in \mathcal{G}, H(C) \leq X} (\#\text{Sel}_2(J(C)) - 2) = |\mathcal{J}| a_\infty \text{Vol}(\mathcal{F}) \text{Vol}(S(\mathbb{R})^{(m)}) \prod_p |\mathcal{J}_p| a_p \text{Vol}(G(\mathbb{Z}_p)) \text{Vol}(\text{Inv}_p(F)) + o(X^{\frac{\dim V}{\dim S}})
\]

since \(a_\infty \prod_p a_p = 1\) by [20, Lemmas 5.7, 5.14], and \(|\mathcal{J}| \prod_p |\mathcal{J}_p| = 1\).

## 5 Proof of the main results

In this section, we prove Theorem 4.1. Let \(F\) be a large family of hyperelliptic curves. As in the previous section, we assume without loss of generality that \(\text{Inv}_\infty(F)\) is \(S(\mathbb{R})^{(m)}\) for a fixed integer \(m \in \{0, \ldots, n+1\}\).

### 5.1 The number of hyperelliptic curves in a large family having bounded height

For any subset \(U\) of \(S(\mathbb{Z})\), let \(N(U; X)\) denote the number of elements in \(U\) having height bounded by \(X\). Our purpose in this section is to determine asymptotics for \(N(\text{Inv}(F); X)\) as \(X\) goes to infinity. To this end, we have the following uniformity estimate proved in [3].

**Proposition 5.1** For each prime \(p\), let \(U_p\) denote the set of elements \(c \in S(\mathbb{Z})\) such that \(p^2 \mid \Delta(c)\). Then there exists \(\delta > 0\) such that, for any \(M > 0\), we have

\[
\sum_{p > M} \frac{N(U_p; X)}{M^\delta} = O(X^{\frac{\dim V}{\dim S}} / M^\delta),
\]

where the implied constant is independent of \(X\) and \(M\).
Then we have the following theorem which follows from Propositions 4.3 and 5.1 just as [7, Theorem 2.21] followed from [7, Theorem 2.12] and [7, Theorem 2.13].

**Theorem 5.2** Let $F$ be a large family of hyperelliptic curves such that $\text{Inv}_\infty(F) = S(\mathbb{R})^{(m)}$. Then the number of hyperelliptic curves in $F$ having height bounded by $X$ is $\text{Vol}(S(\mathbb{R})^{(m)}) \prod_p \text{Vol}(\text{Inv}_p(F))$ up to an error of $o(X^{\frac{1}{100}})$.

Finally, we also need the following proposition:

**Proposition 5.3** Let $F$ be a large family of hyperelliptic curves. Then for a 100% of elements $C \in F$, the class $(\infty') - (\infty)$ is not divisible by 2 in $J(C)(\mathbb{Q})$.

**Proof:** By the proof of Theorem 2.7, the element $(\infty') - (\infty)$ is divisible by 2 in $J(C)(\mathbb{Q})$ if and only if $V_c(Q)$ has a unique $G(Q)$-distinguished orbit, where $c$ is the invariant of $C$. Since 100% of monic degree $2n + 2$ integral polynomials, when ordered by height, correspond to $S_n$-fields, the result follows from Proposition 2.2. □

### 5.2 The average size of the 2-Selmer group

Theorem 5.2, Proposition 5.3 and (45) imply that

$$
\lim_{X \to \infty} \frac{\sum_{C \in F, H(C) < X} (#\text{Sel}_2(J(C)) - 2)}{\sum_{C \in F, H(C) < X} 1} = \frac{\text{Vol}(F)\text{Vol}(S(\mathbb{R})^{(m)}) \prod_p (\text{Vol}(G(Z_p))\text{Vol}(\text{Inv}_p(F))}{\text{Vol}(S(\mathbb{R})^{(m)}) \prod_p \text{Vol}(\text{Inv}_p(F))}
$$

(46)

the Tamagawa number of $G$. Since the Tamagawa number of PSO is 4 ([16]), Theorem 4.1 follows.

### 6 Most monic even hyperelliptic curves have only two rational points

Poonen and Stoll used results from [5] and Chabauty’s method to show that a positive proportion of hyperelliptic curves over $\mathbb{Q}$ having genus $g \geq 3$ and a marked rational Weierstrass point have only one rational point, and that this proportion tends to one as $g$ tends to infinity ([19, Theorem 10.6]). In this section, we modify their argument to derive the analogous result for the family of hyperelliptic curves having a marked rational non-Weierstrass point, thereby proving Theorem 1.4.

Before starting the proof of Theorem 1.4, we sketch the proof of [19, Theorem 10.6]. Given a hyperelliptic curve $C$ with a marked Weierstrass point $\infty$ and Jacobian $J$, Poonen and Stoll considered the following diagram ([19, (6.1)]):

![Diagram of Sel2(J) and J(Q) relationships]

In this diagram, $C(\mathbb{Q})$ and $J(\mathbb{Q})$ are connected to $C(\mathbb{Q}_2)$ and $J(\mathbb{Q}_2)$, respectively, via the logarithm map $\rho$. The points $\mathbb{P}_2^g$ and $\mathbb{F}_2^g$ represent projective and finite fields, respectively, and the arrows indicate the flow of information between these spaces.
Above, \( C(\mathbb{Q}) \) and \( C(\mathbb{Q}_2) \) are embedded into \( J(\mathbb{Q}) \) and \( J(\mathbb{Q}_2) \) via the map \( P \mapsto (P) - (\infty) \). The map \( \rho \) is defined by taking the reduction modulo 2 of the primitive part of \( v \in \mathbb{Z}_2^g \) and taking its image under \( \mathbb{P} \). Note that the maps \( \rho \) and \( \mathbb{P} \) are only partially defined, and that the diagram is commutative on elements for which both maps are defined. Then the proof of [19, Theorem 10.6] follows from these four steps:

1. The image of \( C(\mathbb{Q}_2) \) in \( \mathbb{P}^{g-1}(\mathbb{F}_2) \) is usually small. More precisely, the average size of \( \rho \log(C(\mathbb{Q}_2)) \) is \( 6g + 9 \) ([19, Corollary 9.10]).
2. If \( \sigma \) is injective, then \( \rho \log(J(\mathbb{Q})) \subset \mathbb{P}\sigma(\text{Sel}_2(J)) \) ([19, Lemma 6.2]).
3. Restrict to a large family of hyperelliptic curves \( C \) such that the image \( \rho \log(C(\mathbb{Q}_2)) \) is constant in the family, say equal to \( J \). (That most hyperelliptic curves with genus \( g \) and a marked rational Weierstrass point are contained in a disjoint union of such large families is a consequence of [19, Lemma 8.3] and [19, Propositions 8.5 and 8.7].) Since [5, Theorem 1.1] implies that there are 2 non-trivial elements of \( \text{Sel}_2(J) \), on average over \( J \), and [5, Theorem 12.4] states that their images in \( \mathbb{P}_2^g \) are equidistributed, a proportion of at most \( #I^{1-g} \) curves \( C \) satisfy

\[
\rho \log(C(\mathbb{Q}_2)) \cap \mathbb{P}\sigma\text{Sel}_2(J) \neq \emptyset.
\]

This equidistribution further implies that the proportion of curves for which \( \sigma \) is not injective is at most \( 2^{1-g} \).
4. Therefore, aside from a set of density at most \( (1 + \#I)2^{1-g} \), all curves \( C \) in this large family satisfy

\[
C(\mathbb{Q}_2) \cap J(\mathbb{Q}) \subset J(\mathbb{Q}_2)_{\text{tors}},
\]

where \( J(\mathbb{Q}_2)_{\text{tors}} \) denotes the torsion elements in \( J(\mathbb{Q}_2) \). The proof is completed by showing that the density of curves \( C \) with \( J(\mathbb{Q})_{\text{tors}} \neq 0 \) is zero ([19, Proposition 8.4]).

We also embed \( C(\mathbb{Q}) \) and \( C(\mathbb{Q}_2) \) into \( J(\mathbb{Q}) \) and \( J(\mathbb{Q}_2) \) via the map \( P \mapsto (P) - (\infty) \) and normalize the log map to be surjective from \( J(\mathbb{Q}_2) \) to \( \mathbb{Z}_2^g \) as in [19]. The main difficulty in adapting their proof in our case is that the image of \( (\infty) - (\infty') \) in \( \mathbb{P}_2^g \) does not get equidistributed. Let \( v \in \mathbb{Z}_2^g \) denote the image of \( (\infty) - (\infty') \) under the log map. Let \( v_0 \) denote its primitive part and let \( v_0' \) denote the reduction modulo 2 of \( v_0 \). We use \( (\cdot) \) to denote the subgroup generated by \( \cdot \). We now consider the following modified version of [19, (6.1)]:

\[
\begin{array}{ccccccccc}
C(\mathbb{Q}) & \longrightarrow & C(\mathbb{Q}_2) & \longrightarrow & C(\mathbb{Q}_2) & \longrightarrow & \mathbb{P}_2^g & \longrightarrow & \mathbb{P}_2^g/(\langle v_0 \rangle) & \longrightarrow & \mathbb{F}_2^g/(\langle v_0 \rangle) & \longrightarrow & \mathbb{F}_2^g/(\langle v_0 \rangle) & \longrightarrow & \mathbb{F}_2^g/(\langle v_0 \rangle) \\
J(\mathbb{Q}) & \longrightarrow & J(\mathbb{Q}_2) & \longrightarrow & J(\mathbb{Q}_2) & \longrightarrow & \mathbb{Z}_2^g & \longrightarrow & \mathbb{Z}_2^g/\mathbb{Z}_2^g \cdot v_0 & \longrightarrow & \mathbb{Z}_2^g/\mathbb{Z}_2^g \cdot v_0 & \longrightarrow & \mathbb{Z}_2^g/\mathbb{Z}_2^g \cdot v_0 & \longrightarrow & \mathbb{Z}_2^g/\mathbb{Z}_2^g \cdot v_0 \\
J(\mathbb{Q})/2J(\mathbb{Q}) & \longrightarrow & J(\mathbb{Q}_2)/2J(\mathbb{Q}_2) & \longrightarrow & J(\mathbb{Q}_2)/2J(\mathbb{Q}_2) & \longrightarrow & \mathbb{F}_2^g & \longrightarrow & \mathbb{F}_2^g/(\langle v_0 \rangle) & \longrightarrow & \mathbb{F}_2^g/(\langle v_0 \rangle) & \longrightarrow & \mathbb{F}_2^g/(\langle v_0 \rangle) & \longrightarrow & \mathbb{F}_2^g/(\langle v_0 \rangle) \\
\text{Sel}_2(J) & \longrightarrow & \mathbb{P}_2(\mathbb{Q}_2) & \longrightarrow & \mathbb{P}_2(\mathbb{Q}_2) & \longrightarrow & \mathbb{F}_2^g & \longrightarrow & \mathbb{F}_2^g/(\langle v_0 \rangle) & \longrightarrow & \mathbb{F}_2^g/(\langle v_0 \rangle) & \longrightarrow & \mathbb{F}_2^g/(\langle v_0 \rangle) & \longrightarrow & \mathbb{F}_2^g/(\langle v_0 \rangle) \\
\end{array}
\]

Our version of Step 1 follows immediately from the proofs of [19, Proposition 5.4, Theorem 9.1], with the only difference being that in our case the expected size of \( C^\text{smooth}(\mathbb{F}_2) \) is bounded above by 4 instead of 3, where \( C \) denotes the minimal proper regular model of \( C \). The reason for this difference is that the \( \mathbb{Z}_2 \)-model of a random hyperelliptic curve has two smooth points \( \infty \) and \( \infty' \). The bound of 2 on the expected number of other smooth \( \mathbb{F}_2 \)-points follows from the arguments of [19, Lemma 9.5]. This yields the following proposition:
Proposition 6.1 Let $C$ range over hyperelliptic curves corresponding to elements in $\mathbb{Z}^2g+1\\setminus\{\Delta=0\}$ such that $(\infty)-(\infty') \notin J(\mathbb{Q})(2)$. Then $\rho' \log(C(\mathbb{Q}_v))$ is locally constant and its average size is at most $6g+14$.

For Step 2, we prove the analogous version of [19, Lemma 6.2]:

Lemma 6.2 Suppose $J(\mathbb{Q})_{\text{tors}} = 0$ and the kernel of $\sigma'$ in Sel$_2(J)$ is equal to the subgroup generated by the class of $d_0 = (\infty)-(\infty')$. Then $\rho' \log(\overline{J(\mathbb{Q})}) \subseteq \mathbb{P}\sigma'(\text{Sel}_2(J))$. Furthermore, if $g \in J(\mathbb{Q})$ has no image under $\rho'$, then there exist $m$ and $n$ such that $mg = nd_0$.

Proof: Since $\rho' \log$ is continuous and $\mathbb{P}^{g-2}(\mathbb{F}_2)$ is discrete, $\rho' \log(\overline{J(\mathbb{Q})}) = \rho' \log(J(\mathbb{Q}))$. Since $J(\mathbb{Q})_{\text{tors}} = 0$, we have $J(\mathbb{Q})/d_0 = F \oplus \mathbb{Z}^{-1}$, where $r$ is the rank of $J(\mathbb{Q})$ and $F$ is a finite abelian group such that any lift $g$ to $J(\mathbb{Q})$ of an element in $F$ satisfies $mg = nd_0$ for some integers $m$ and $n$. This implies that such a $g$ has no image under the partially defined map $\rho' \log$.

Let $h \in J(\mathbb{Q})$ be an element that does have an image under $\rho' \log$. Then the image of $h$ in $F \oplus \mathbb{Z}^{-1}$ is some $(t, h')$, where $t \in F$ and $h' \in \mathbb{Z}^{-1}$. Let $h_0$ denote the primitive part of $h'$. Then we have $\rho' \log(h) = \rho' \log(h_0)$ and furthermore, because the kernel of $\sigma'$ is equal to the subgroup generated by the class of $d_0$, the element $h_0$ has nonzero image under $\sigma'$. Therefore, we obtain $\rho' \log(h) = \mathbb{P}\sigma'(h_0)$ which proves the first assertion of the lemma.

For the second part, let $h \in J(\mathbb{Q})$ be an element that does not have an image under $\rho' \log$, and let the image of $h$ in $F \oplus \mathbb{Z}^{-1}$ be $(t, h')$, where $t \in F$ and $h' \in \mathbb{Z}^{-1}$. If $h' = 0$, then we are done. Otherwise, let $h_0$ denote the primitive part of $h'$. Since $h$ has no image under $\rho' \log$, neither does $h_0$, and we have $\log(h_0) \in \mathbb{Z}_g \cdot v_0$. This implies that the class of $h_0$ in Sel$_2(J)$ maps to 0 under $\sigma'$ contradicting our assumption that the kernel of $\sigma'$ is generated by the class of $d_0$. □

For Step 3, we start with the following analogue of [5, Theorem 12.4]; the proof is identical.

Theorem 6.3 Fix a place $\nu$ of $\mathbb{Q}$. Let $F$ be a large family of hyperelliptic curves $C$ with a marked non-Weierstrass point such that

(a) the cardinality of $\text{Sel}_2(C(\mathbb{Q}_v))/2\text{Sel}_2(C(\mathbb{Q}_v))$ is a constant $k$ for all $C \in F$; and

(b) the set $U_\nu(F) \subseteq V(\mathbb{Q}_v)$, defined to be the set of solvable elements in $V(\mathbb{Q}_v)$ having invariants in $\text{Inv}_\nu(F)$, can be partitioned into $k$ open sets $\Omega_i$ such that:

(i) for all $i$, if two elements in $\Omega_i$ have the same invariants, then they are $G(\mathbb{Q}_v)$-equivalent; and

(ii) for all $i \neq j$, we have $G(\mathbb{Q}_v)\Omega_i \cap G(\mathbb{Q}_v)\Omega_j = \emptyset$.

(5) For the groups $J(C(\mathbb{Q}_v))/2J(C(\mathbb{Q}_v))$ are naturally identified for all $C \in F$.) Then when elements $C \in F$ are ordered by height, the images of the non-distinguished elements (i.e., elements that do not correspond to either the identity or the class of $(\infty')-(\infty)$ in $J(C(\mathbb{Q}))$ under the map $\text{Sel}_2(J(\mathbb{Q})) \rightarrow J(C(\mathbb{Q}_v))/2J(C(\mathbb{Q}_v))$ are equidistributed.

Let $F$ be a large family of hyperelliptic curves corresponding to an open subset of $\mathbb{Z}^2g+1\\setminus\{\Delta = 0\}$ such that $F$ satisfies the hypothesis of Theorem 6.3 and the image of $\rho' \log(C(\mathbb{Q}_v))$ in $\mathbb{P}^{g-2}(\mathbb{F}_2)$ is constant for $C \in F$. We denote this image by $I$. We may further assume that the log maps are normalized such that the image $\nu$ of $(\infty)-(\infty)$ is constant throughout this family (cf. [19, Proposition 8.2]).

On average over the Jacobians $J$ of the curves in $F$, there are 4 non-distinguished elements in Sel$_2(J)$, and the images of these elements under $\sigma$ are equidistributed in $\mathbb{P}_2$. Therefore, a proportion of at least $1 - \#I2^3-g$ curves $C$ in $F$ satisfy

$$\rho' \log(C(\mathbb{Q}_v)) \cap \mathbb{P}\sigma'(\text{Sel}_2(J)) = \emptyset.$$ 

Furthermore, a proportion of at least $2^3-g$ curves fail to satisfy the conditions of Lemma 6.2 (we need the image of $\sigma$ to avoid both 0 and $v_0$). Say that a point $P \in C(\mathbb{Q})\\setminus\{\infty, \infty'\}$ is bad if there exist integers $m$ and $n$, not both zero, such that

$$m((P) - (\infty)) = n((\infty) - (\infty')).$$

27
Therefore, aside from a set of density at most \((1 + \#I)2^{3-g}\), all curves \(C \in F\) are such that every point \(P \in C(\mathbb{Q}) \setminus \{\infty, \infty'\}\) is bad.

We summarize the above discussion in the following theorem.

**Theorem 6.4** Suppose \(C\) is an even degree hyperelliptic curve of genus \(g\) over \(\mathbb{Q}\) satisfying the following three conditions:

1. \(J(\mathbb{Q})_{\text{tors}} = 0\),
2. \(\ker \sigma' = \langle (\infty) - (\infty') \rangle\),
3. \(\rho' \log(C(\mathbb{Q}_2)) \cap \mathbb{P} \sigma'(\text{Sel}(J)) = \emptyset\).

Then every point \(P \in C(\mathbb{Q}) \setminus \{\infty, \infty'\}\) is bad, i.e., there exist integers \(m\) and \(n\), not both 0, such that

\[ m((P) - (\infty)) = n((\infty) - (\infty')). \]

Moreover, the proportion of even degree hyperelliptic curves \(C\) of genus \(g\) over \(\mathbb{Q}\) satisfying the above three conditions is at least \(1 - (48g + 120)2^{-g}\).

We say that a monic even degree hyperelliptic curve \(C\) over \(\mathbb{Q}\) is good if \(C(\mathbb{Q})\) has no bad points. Then we have the following theorem:

**Theorem 6.5** A proportion of 100% of monic even degree hyperelliptic curves over \(\mathbb{Q}\) having fixed genus \(g \geq 4\) are good.

We work \(p\)-adically for some fixed prime \(p\). Suppose \(C\) is a monic even degree hyperelliptic curve with coefficients in \(\mathbb{Z}_p\). Let \(\ell : C(\mathbb{Q}_p) \to \mathbb{Z}_p^g\) denote the map sending \(P \in C(\mathbb{Q}_p)\) to \(\log((P) - (P'))\) where \(\tau\) denotes the hyperelliptic involution and \(\log\) is computed with respect to the differentials

\[ \{dx/y, xdx/y, \ldots, x^{g-1}dx/y\}. \]

We say a point \(P \in C(\mathbb{Q}_p) \setminus \{\infty, \infty'\}\) is bad if the \(\mathbb{Z}_p\)-lines spanned by \(\ell(P)\) and \(\ell(\infty)\) have nonzero intersections.

We thank Jacob Tsimerman for several conversations which led to the proof of the following theorem, from which Theorem 6.5 follows immediately.

**Theorem 6.6** Suppose \(g \geq 4\). The set \(U\) of elements in \(\mathbb{Z}_p^{2g+1} \setminus \{\Delta = 0\}\) corresponding to hyperelliptic curves \(C\) of genus \(g\) such that \(C(\mathbb{Q}_p) \setminus \{\infty, \infty'\}\) contains no bad points is dense. Furthermore, the \(p\)-adic closure of its complement has measure 0.

**Proof:** An element \(v \in \mathbb{Z}_p^{2g+1} \setminus \{\Delta = 0\}\) yields a hyperelliptic curve \(C\) along with a point \(\infty\). Let \(P \in C(\mathbb{Q}_p)\) be a point such that \(P \neq \infty\). The pair \((C, P)\) then corresponds to an element \(v' \in \mathbb{Z}_p^{2g+1} \setminus \{\Delta = 0\}\) such that \(v' \neq v\). Furthermore, as \(P \to \infty\) in \(C(\mathbb{Q}_p)\), we have \(v' \to v\). We say that a pair of points \((P, Q) \in C(\mathbb{Q}_p) \times C(\mathbb{Q}_p)\) is a bad pair if \(P \neq Q\), \(P \neq Q'\), and the \(\mathbb{Z}_p\)-lines spanned by \(\ell(P)\) and \(\ell(Q)\) have a nonzero intersection. We show in Lemma 6.7 that the number of bad pairs \((P, Q) \in C(\mathbb{Q}_p) \times C(\mathbb{Q}_p)\) is finite for any monic even degree hyperelliptic curve over \(\mathbb{Q}_p\). From this it follows that given a pair \((C, P)\) corresponding to \(v \in \mathbb{Z}_p^{2g+1} \setminus \{\Delta = 0\}\), there exist points \(P\) arbitrarily close to \(\infty\) such that \(P\) is not part of any bad pair. It thus follows that there exist points \(v' \in \mathbb{Z}_p^{2g+1} \setminus \{\Delta = 0\}\) (corresponding to such pairs \((C, P)\), arbitrarily close to \(v\), that correspond to hyperelliptic curves containing no bad points. Therefore, \(U\) is dense.

Let \(V\) denote the complement of \(U\) in \(\mathbb{Z}_p^{2g+1} \setminus \{\Delta = 0\}\). We claim that \(V\) is a \(p\)-adic subanalytic subset of \(M\). The theory of subanalytic sets is studied in great detail in [14]. We do not repeat the definition of subanalytic sets and instead remark that subanalytic sets are stable under projections onto coordinate hyperplanes and that sets defined by the vanishing and nonvanishing of analytic functions are subanalytic. Moreover, being subanalytic is a \((p\)-adic\) local property. The dimension of a subanalytic set is defined to be the maximal dimension of a \(p\)-adic manifold contained in it ([14, 3.15]). This notion of dimension behaves as
one expected: a 0-dimensional subanalytic set is finite; the dimension of the boundary $\bar{A}\setminus A$ of a subanalytic set $A$ is less than the dimension of $A$ ([14, 3.26]).

We now show that $V$ is a $p$-adic subanalytic subset of $M$. It suffices to check this locally. Restrict to an open subset $W$ of $\mathbb{Z}_p^{2g+1}\setminus \{\Delta = 0\}$ such that $\mathcal{C}^{\text{smooth}}(F_p)$ is constant (having size $k$) for curves $C$ corresponding to elements in $W$ where $C$ denote the minimal proper regular model of $C$. Then the moduli space of pairs $(C, P)$, where $C$ is a curve corresponding to an element in $W$ and $P$ is a point in $C(\mathbb{Q}_p)$, is isomorphic to $W \times \mathcal{C}^{\text{smooth}}(F_p) \times \mathbb{Z}_p$. The set of pairs $(C, P)$ corresponding to elements in this moduli space such that $P$ is a bad point of $C(\mathbb{Q}_p)$ is a subanalytic set of $W \times \mathcal{C}^{\text{smooth}}(F_p) \times \mathbb{Z}_p$ defined by $\ell(P), \ell(\infty) \neq 0$ and $\ell(P)/\ell(\infty)$. Since subanalytic sets are preserved by projections, this implies that $V \cap W$ is subanalytic in $W$, as necessary. We have already proven that $V$ does not contain any $p$-adic open ball of dimension $2g+1$ as its complement is dense. Hence its dimension as a subanalytic set ([14, 3.15]) is less than $\dim(\mathbb{Z}_p^{2g+1}\setminus \{\Delta = 0\}) = 2g+1$. Moreover, the dimension of $\bar{V}\setminus V$ is less than the dimension of $V$ ([14, 3.26]), where $\bar{V}$ denotes the $p$-adic closure of $V$. Therefore, the $p$-adic closure of $V$ has measure 0 as necessary. $\square$

We now have the following lemma which was assumed in the proof of Theorem 6.6.

**Lemma 6.7** Let $C$ be a monic even degree hyperelliptic curve with coefficients in $\mathbb{Z}_p$, having genus $g \geq 4$. Then the set of bad pairs $(P, Q) \in C(\mathbb{Q}_p) \times C(\mathbb{Q}_p)$ is finite.

**Proof:** Let $\Sigma$ denote the subset of $C(\mathbb{Q}_p) \times C(\mathbb{Q}_p)$ consisting of bad pairs $(P, Q)$. Then $\Sigma$ is subanalytic as it is defined by $\ell(P), \ell(Q) \neq 0$ and $\ell(P)/\ell(Q)$. We will show that the dimension of $\Sigma$ as a subanalytic set is zero which implies that $\Sigma$ is finite by [14, 3.26].

Let $P \in C(\mathbb{Q}_p)$ be any point. Restricting $\ell$ to a neighborhood $W_P$ around $P$ gives an analytic function $\ell_P : \mathbb{Z}_p \to \mathbb{Z}_p$. The main difficulty in proving this lemma is that it is difficult to explicitly compute the function $\ell_P$. However, for any $P'$ in the residue disk around $P$, $\ell_P(P')$ is the sum of $\ell_P(P)$ and twice a $p$-adic integral. Hence we can compute the derivative of $\ell_P$ using the fundamental theorem of calculus and obtain:

\[
\ell'_P(P') = \left( \frac{2}{y(P')}, \frac{2x(P')}{y(P')}, \ldots, \frac{2x(P')^{g-1}}{y(P')} \right), \quad \text{if } P' \notin \{\infty, \infty'\},
\]

\[
\ell'_P(P') = (0, 0, \ldots, \pm 2), \quad \text{if } P' \in \{\infty, \infty'\}.
\]

The second formula follows from applying a change of variable $t = 1/x, s = y/x^{g+1}$ and then using the fundamental theorem of calculus. One key fact to notice is that the projections of $\ell'_P(P')$ and $\ell'_Q(Q')$ onto any 2-dimensional coordinate hyperplane corresponding to two consecutive coordinates are $\mathbb{Q}_p$-parallel if and only if $P' = Q'$ or $P' = Q'\tau$. This observation yields the following lemma:

**Lemma 6.8** For a fixed point $P \in C(\mathbb{Q}_p)$, the set of points $Q \in C(\mathbb{Q}_p)$ such that $(P, Q)$ is a bad pair is finite.

**Proof:** Indeed, the intersection of $\mathbb{Q}_p, \ell(P)$ and $\ell(C(\mathbb{Z}_p))$ is a subanalytic set of dimension at most 1. Hence it is either finite or contains an open ball $B$. If it is finite, then we are done. Otherwise, the derivatives $\ell'(Q)$ are all parallel to $\ell(P)$ for $Q \in B$, and contradiction.

Let $(P, Q) \in C(\mathbb{Q}_p) \times C(\mathbb{Q}_p)$ be a bad pair. Since $\ell(P)$ and $\ell(Q)$ are $\mathbb{Q}_p$-parallel, there exists a coordinate, say $j$, for which both $\ell(P)$ and $\ell(Q)$ are nonzero and have the smallest $p$-adic valuation among all nonzero coordinates. Hence there exist small neighborhoods $W_P$ and $W_Q$ of $P$ and $Q$, respectively, such that the $j$-th coordinates of $\ell_P(P')$ and $\ell_Q(Q')$ are nonzero and have the smallest $p$-adic valuation among all nonzero coordinates for any $P' \in W_P$, and $Q' \in W_Q$. Moreover since $P \neq Q$ and $P \neq Q'$, we may further assume that $P' \neq Q'$ and $P' \neq Q'\tau$ for any $(P', Q') \in W_P \times W_Q$. For any $i = 1, \ldots, g$ and any vector $v \in \mathbb{Q}_p^g$, we write $v_i$ for the $i$-th coordinate of $v$ and write $v^{(i)}$ for the vector in $\mathbb{Q}_p^{g-1}$ obtained from $v$ by removing the $i$-th coordinate. For any $P' \in W_P$, write $f_P(P') \in \mathbb{Q}_p^{g-1}$ for the vector

\[
f_P(P') = (\frac{\ell_P(P'_1)}{\ell_P(P')_1}, \ldots, \frac{\ell_P(P'_g)}{\ell_P(P')_g})^{(j)}.
\]
Similarly define \( f_Q(Q') \) for \( Q' \in W_Q \). Then \((P', Q')\) is a bad pair if and only if \( f_P(P') = f_Q(Q') \). Let \( h : W_P \times W_Q \to \mathbb{Z}_p^{-1} \) denote the analytic function \( h(P', Q') = f_P(P') - f_Q(Q') \) and let \( S \) denote the vanishing locus of \( h \). Then \( S \) is an analytic subset of \( W_P \times W_Q \) and its projections to \( W_P \) and \( W_Q \) are subanalytic. Computing the partial derivatives of \( h \) at \((P, Q)\) gives

\[
h_P(P, Q) = \frac{1}{\ell_P(P)} \ell_P(P)^{(j)} - \frac{\ell_P(P)^{(j)}}{\ell_P(P)} \ell_P(P)^{(j)}, \quad h_Q(P, Q) = -\frac{1}{\ell_Q(Q)} \ell_Q(Q)^{(j)} + \frac{\ell_Q(Q)^{(j)}}{\ell_Q(Q)} \ell_Q(Q)^{(j)}.
\]

Hence if both of these partial derivatives are zero, then the vectors \( \ell_P(P)^{(j)}, \ell_P(Q)^{(j)}, \ell_Q(Q)^{(j)} \) are all \( \mathbb{Q}_p \)-parallel which leads to a contradiction since \( g \geq 4 \) and \( P \neq Q, Q' \). Note \( \ell_P(P)^{(j)}, \ell_Q(Q)^{(j)} \) are parallel because \( \ell_P(P) \) and \( \ell_Q(Q) \) are parallel. We assume without loss of generality that \( h_P(P, Q) \neq 0 \).

Let \( S_P \) denote the image of \( S \) under the projection map from \( W_P \times W_Q \) to \( W_P \). If the dimension of \( S_P \) as a subanalytic set is 0, then it is a finite set and Lemma 6.7 follows from Lemma 6.8. If the dimension of \( S_P \) as a subanalytic set is 1, then it contains an open ball. Replacing \( W_P \) by this open ball, we can assume that \( S_P = W_P \). Since \( h_P(P, Q) \neq 0 \), the implicit function theorem implies that there exists an analytic section \( W_P \to S \) and composing it with the second projection gives an analytic map \( s : W_P \to W_Q \) such that \((P', s(P'))\) is a bad pair for any \( P' \in W_P \). Let \( \alpha : W_P \to \mathbb{Q}_p^\times \) denote the analytic function such that

\[
\ell_Q(s(P')) = \alpha(P') \ell_P(P'),
\]

for any \( P' \in W_P \). The vanishing set of the derivative \( s' \) of \( s \) is analytic and hence is either finite or contains an open ball. In the latter case, \( s \) is constant on this open ball which contradicts Lemma 6.8. By replacing \( W_P \) by an open ball inside it, we may assume that \( s'(P') \neq 0 \) for any \( P' \in W_P \). Note that \( W_P \) might not contain \( P \) anymore. Differentiating (48) gives

\[
\ell_Q'(s(P')) = \alpha_1(P') \ell_P(P') + \alpha_2(P') \ell_P'(P'),
\]

with \( \alpha_1 = \alpha'/s' \) and \( \alpha_2 = \alpha/s' \). Differentiating (49) again shows that the vectors \( \ell''_Q(s(P')) \), \( \ell_P(P') \), \( \ell''_P(P') \), \( \ell''_P(P') \) are linearly dependent over \( \mathbb{Q}_p \) for any \( P' \in W_P \). Since \( P' \neq Q' \) and \( P' \neq Q'' \) for any \( P' \in W_P \) and \( Q' \in W_Q \) by assumption, we see that \( \ell''_Q(s(P')) \) and \( \ell''_P(P') \) are not parallel and hence \( \ell''_P(P') \) can be written as a linear combination of \( \ell''_Q(s(P')) \) and \( \ell''_P(P') \) by (49). Therefore, the vectors \( \ell''_Q(s(P')) \), \( \ell''_Q(s(P')) \), \( \ell''_P(P') \), \( \ell''_P(P') \) are linearly dependent for any \( P' \in W_P \).

Shrink \( W_P \) if necessary so that \( W_P \) does not contain \( \infty \) or \( \infty' \). This allows us to have a uniform formula for the derivative of \( \ell \). An elementary determinant computation (using the first 4 coordinates, which requires \( g \geq 4 \)) shows that the vectors \( \ell''_Q(Q') \), \( \ell''_Q(Q') \), \( \ell''_P(P') \), \( \ell''_P(P') \) are linearly dependent if and only if \( P' = Q' \) or \( P' = Q'' \) neither of which is true if \( Q' = s(P') \) and \( P' \in W_P \). This completes the proof of Lemma 6.7. \( \square \)

Theorem 1.4 follows from Theorem 6.4 and Theorem 6.5.

Acknowledgments

We are very grateful to Manjul Bhargava and Benedict Gross for suggesting this problem to us and for many helpful conversations. We are also very grateful to Bjorn Poonen for explaining Chabauty’s method to us and for helpful comments on earlier versions of the argument. We are extremely grateful to Cheng-Chiang Tsai, Jacob Tsimerman, and Ila Varma for several helpful conversations. The first author is grateful for support from NSF grant DMS-1128155. The second author is grateful for support from a Simons Investigator Grant and NSF grant DMS-1001828.

References

[1] M. Bhargava, The density of discriminants of quartic rings and fields, Ann. of Math. 162, 1031–1063.
[2] M. Bhargava, The density of discriminants of quintic rings and fields, Ann. of Math. (2), 172 (2010), no. 3, 1559–1591.

[3] M. Bhargava, The geometric squarefree sieve and unramified nonabelian extensions of quadratic fields, preprint.

[4] M. Bhargava and B. Gross, Arithmetic invariant theory (2012), arXiv/1206.4774.

[5] M. Bhargava and B. Gross, The average size of the 2-Selmer group of Jacobians of hyperelliptic curves having a rational Weierstrass point (2012), arXiv/1208.1007.

[6] S. Bosch, W. Lutkebohmert, and M. Raynaud, Néron models, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 21, Berlin, New York: Springer-Verlag, 1990.

[7] M. Bhargava and A. Shankar, Binary quartic forms having bounded invariants, and the boundedness of the average rank of elliptic curves, arXiv/1006.1002.

[8] M. Bhargava and A. Shankar, Ternary cubic forms having bounded invariants, and the existence of a positive proportion of elliptic curves having rank 0 (2010), arXiv/1007.0052.

[9] G. Bliss, A generalization of Weierstrass’ preparation theorem for a power series in several variables, Trans. Amer. Math. Soc. 13 (1912), no. 2, 133–145.

[10] A. Borel, Ensembles fondamentaux pour les groupes arithmétiques, Colloque sur la Théorie des Groupes Algébriques, Bruxelles (1962), 23–40.

[11] C. Chabauty, Sur les points rationnels des courbes algébriques de genre supérieur à l’unité, C. R. Acad. Sci. Paris 212 (1941), 882–885.

[12] R. Coleman, Effective Chabauty, Duke Math. J. 52 (1985) no.3, 765–770.

[13] H. Davenport, On a principle of Lipschitz, J. London Math. Soc. 26 (1951), 179–183. Corrigendum: “On a principle of Lipschitz ”, J. London Math. Soc. 39 (1964), 580.

[14] J. Denef and L. van den Dries, p-adic and Real Subanalytic Sets, Annals of Mathematics, Second Series, 128, No. 1 (1988), 79–138.

[15] B. Gross, Hanoi lectures on the arithmetic of hyperelliptic curves, Acta mathematica vietnamica 37 (2012), 579–588.

[16] R. P. Langlands, The volume of the fundamental domain for some arithmetical subgroups of Chevalley groups, 1966 Algebraic Groups and Discontinuous Subgroups, Proc. Sympos. Pure Math., Boulder, Colo. (1965), 143–148.

[17] V. Platonov and A. Rapinchuk, Algebraic groups and number theory. Translated from the 1991 Russian original by Rachel Rowen, Pure and Applied Mathematics 139, Academic Press, Inc., Boston, MA, 1994.

[18] B. Poonen and E. Schaefer, Explicit descent for Jacobians of cyclic covers of the projective line, J. Reine Angew. Math. 488 (1997), 141–188.

[19] B. Poonen and M. Stoll, Most odd degree hyperelliptic curves have only one rational point (2013), arXiv/1302.0061

[20] M. Stoll, Implementing 2-descent for Jacobians of hyperelliptic curves, Acta Arithmetica XCVIII.3 (2001), 245–277.

[21] X. Wang, Maximal linear spaces contained in the base loci of pencils of quadrics (2013), arXiv/1302.2385.

[22] X. Wang, Pencils of quadrics and Jacobians of hyperelliptic curves, Ph.D thesis, Harvard (2013).