COISOTROPIC LUTTINGER SURGERY AND SOME NEW SYMPLECTIC 6-MANIFOLDS WITH VANISHING CANONICAL CLASS

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Abstract. We introduce a surgery operation on symplectic manifolds called coisotropic Luttinger surgery, which generalizes Luttinger surgery on Lagrangian tori in symplectic 4-manifolds [11]. We use it to produce infinitely many distinct symplectic non-Kähler 6-manifolds $X$ with $c_1(X) = 0$ which are not of the form $M \times F$ for $M$ a symplectic 4-manifold and $F$ a closed surface.

1. Introduction

In this article we introduce a surgery operation on symplectic manifolds called coisotropic Luttinger surgery, which generalizes Luttinger surgery on Lagrangian tori in symplectic 4-manifolds [11]. We use it to produce infinitely many distinct symplectic non-Kähler 6-manifolds $X$ with $c_1(X) = 0$ which are not of the form $M \times F$ for $M$ a symplectic 4-manifold and $F$ a closed surface.

Theorem 1. Coisotropic surgery on 4-tori in $T^6$ produces an infinite family of pairwise non-homotopy equivalent closed symplectic 6-manifolds $X_n$ with $c_1(X_n) = 0$, Euler characteristic $\chi(X_n) = 0$, and Betti numbers satisfying $b_1(X_n) = 3$, $b_2(X_n) \leq 18$, and $b_3(X_n) \leq 32$. None of the manifolds $X_n$ are symplectomorphic to $M \times F$ for a symplectic 4-manifold $M$ and surface $F$.

Coisotropic Luttinger surgery has a very simple topological description which generalizes Dehn surgery in dimension 3. It is localized near a certain codimension two coisotropic submanifold. This makes it useful as method to produce related families of symplectic manifolds. In Theorem 1 the 4-tori on which the surgeries are performed are not symplectic, but rather products of Lagrangian with symplectic tori.

Symplectic manifolds $M$ with vanishing first Chern class are known as symplectic Calabi-Yau manifolds [13, 17]. The famous Kodaira-Thurston 4-manifold [14] provided the first non-Kähler example of such a manifold, and one can produce higher dimensional examples by taking its product with a torus.

Symplectic Calabi-Yau manifolds which do not admit Kahler structures have received attention recently (cf. [7, 8, 16, 15, 17]). In dimension 6 these manifolds were introduced

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by Smith, Thomas, and Yau in their paper on symplectic conifold transitions [13] motivated by their use in producing pairs exhibiting mirror symmetry. Their construction involves symplectically resolving singular complex projective 3-folds. Other known examples include certain nilmanifolds (e.g. [5]) (of which the Kodaira-Thurston manifold is an example), and the compelling constructions of Fine-Panov [7, 8] which are obtained from $S^2$ bundles over 4-dimensional hyperbolic orbifolds and $S^3$ bundles over hyperbolic 3-manifolds.

The examples we produce reconstruct a few of these previously known examples, although in this article we take our seed manifold to be $T^6$ and hence we do not produce simply connected examples. But the essential property that we exploit is that $T^6$ fibers in different ways.

In addition to its use in constructing the examples of Theorem 1, coisotropic Luttinger surgery applies in a wide range of contexts in symplectic topology. In particular, it extends to higher dimensions a codimension two symplectic surgery operation which, in concert with the symplectic sum operation [10], has already had significant impact in 4-dimensional smooth topology c.f. [3]. Moreover, coisotropic surgery is localized near a submanifold and so one can understand the change in homotopy invariants by standard Mayer-Vietoris arguments. We expect the process to have interesting applications outside the context of symplectic Calabi-Yau manifolds. We touch on some potential further applications in the last section.

2. Coisotropic Luttinger surgery

We describe the construction, which consists of removing $D^2 \times T^2 \times \Sigma$ from a symplectic $2n$-manifold and regluing by an appropriate diffeomorphism of the boundary.

Let $D^2_\epsilon$ denote the closed disk in $\mathbb{R}^2$ of radius $\epsilon$ and coordinates $x, y$, hence 1-forms $dx, dy$. Let $T^2 = S^1 \times S^1 = \mathbb{R}^2 / \mathbb{Z}^2$ denote the 2-torus with coordinates $e^{iz}, e^{iw}$ and its global 1-forms $dz, dw$ (descended from $\mathbb{R}^2$).

Suppose we are given a $2n - 4$ dimensional manifold $\Sigma$ and a family $\omega_{\Sigma,(d,t)}$ of symplectic forms on $\Sigma$ parametrized by $(d, t) \in D^2_\epsilon \times T^2$. Then $D^2_\epsilon \times T^2 \times \Sigma$ inherits the symplectic form

$$\omega = dx \ dz + dw \ dy + \omega_{\Sigma,(d,t)}.$$  

The parallel submanifolds $\{ (x, y) \} \times T^2 \times \Sigma, \ (x, y) \in D^2_\epsilon$ are cosiotropic with respect to $\omega$.

We extend Luttinger surgery [11] as follows. Suppose that $(X, \omega_X)$ is a symplectic $2n$-manifold and $\Sigma$ is a closed $2n - 4$-dimensional smooth manifold. Suppose one is given an embedding

$$e : D^2_\epsilon \times T^2 \times \Sigma \hookrightarrow X$$

so that the pulled back symplectic form $e^*(\omega_X)$ satisfies

$$e^*(\omega_X) = \omega$$

for $\omega$ the form defined in Equation (1). Fix an integer $k$ and let

$$\phi_k : (D^2_\epsilon \setminus D^2_{\epsilon/3}) \times T^2 \times \Sigma \to (D^2_\epsilon \setminus D^2_{\epsilon/3}) \times T^2 \times \Sigma$$

denote the diffeomorphism given in polar coordinates on $D^2_\epsilon \setminus D^2_{\epsilon/3}$ by

$$\phi_k(re^{i\theta}, e^{iz}, e^{iw}, \sigma) = (re^{i\theta}, e^{iz}, e^{i(w+k\theta)}, \sigma).$$
Lemma 2. The symplectic form \( \phi_k^*(\omega) \) extends to a symplectic form on \( D^2_\epsilon \times T^2 \times \Sigma \)

Proof. One computes
\[
(\phi_k)_*(\frac{\partial}{\partial x}) = \frac{\partial}{\partial x}, \quad (\phi_k)_*(\frac{\partial}{\partial y}) = \frac{\partial}{\partial y} + k\frac{\partial}{\partial \theta}, \quad (\phi_k)_*(\frac{\partial}{\partial z}) = \frac{\partial}{\partial z}, \quad (\phi_k)_*(\frac{\partial}{\partial w}) = \frac{\partial}{\partial w}.
\]

Hence
\[
\phi_k^*(dr) = dr, \quad \phi_k^*(d\theta) = d\theta, \quad \phi_k^*(dz) = dz, \quad \text{and} \quad \phi_k^*(dw) = dw + kd\theta.
\]

Switching back to Cartesian coordinates on \( D^2_\epsilon \) yields
(2) \( \phi_k^*(dx) = dx, \quad \phi_k^*(dy) = dy, \quad \phi_k^*(dz) = dz, \quad \text{and} \quad \phi_k^*(dw) = dw + \frac{k}{x^2+y^2}(x dy - y dx).

Hence
\[
\phi_k^*(\omega) = \omega - \frac{k}{x^2+y^2} y \, dx \, dy.
\]

Fix a radially symmetric smooth function \( f : D^2_\epsilon \rightarrow \mathbb{R} \) which equals 0 for \( \sqrt{x^2+y^2} \leq \frac{\epsilon}{3} \) and \( \frac{1}{x^2+y^2} \) for \( \sqrt{x^2+y^2} \geq \frac{2\epsilon}{3} \). Then the 2-form
\[
\alpha = -kyf(x,y) \, dx \, dy
\]
on \( D^2_\epsilon \times T^2 \times \Sigma \) is closed since it is pulled back from a 2-form on \( D^2_\epsilon \). Thus
(3) \( \tilde{\omega} = \omega + \alpha = \omega - kyf \, dx \, dy \)
is closed. It agrees with \( \phi_k^*(\omega) \) on \( (D^2_\epsilon \setminus D^2_{2\epsilon/3}) \times T^2 \times \Sigma \) and agrees with \( \omega \) on \( D^2_{\epsilon/3} \times T^2 \times \Sigma \).

On \( (D^2_{\epsilon/3} \setminus D^2_{\epsilon/3}) \times T^2 \times \Sigma \), \( \alpha^2 = 0 \) and \( \omega \wedge \alpha = \omega \Sigma \wedge \tau \) and so \( \tilde{\omega}^n = \omega^n + \omega^n-\alpha \wedge \omega \Sigma \) and hence \( \tilde{\omega} \) is non-degenerate.

\[ \square \]

Construct a new manifold \( X' \) as the union with identifications
\[
X' = ((X \setminus e(D^2_{\epsilon/3} \times T^2 \times \Sigma)) \cup (D^2_\epsilon \times T^2 \times \Sigma))/\sim
\]
where the points \( e(re^{i\theta}, e^{iz}, e^{iw}, \sigma) \in X \) and \( \phi_k(re^{i\theta}, e^{iz}, e^{iw}, \sigma) \in D^2_\epsilon \times T^2 \times \Sigma \) are identified provided \( \frac{2\epsilon}{3} \leq r \leq \epsilon \). Lemma 2 shows that the symplectic form on \( X \setminus e(D^2_{\epsilon/3} \times T^2 \times \Sigma) \) extends to a symplectic form on \( X' \).

Since this construction depends on an coisotropic submanifold instead of a Lagrangian submanifold, we say \( X' \) is obtained from \( X \) by \( \frac{1}{k} \) coisotropic Luttinger surgery along \( T^2 \times \Sigma \subset X \). If \( k = 0 \) then clearly \( X' = X \).

As a smooth manifold, \( X' \) can be described as the manifold obtained by removing \( D^2 \times T^2 \times \Sigma \) from \( X \) and regluing using the restriction of \( \phi_k \) to the boundary:
(4) \( \psi_k : S^1 \times T^2 \times \Sigma \rightarrow S^1 \times T^2 \times \Sigma, \quad \psi_k(e^{i\theta}, e^{iz}, e^{iw}, \sigma) = (e^{i\theta}, e^{iz}, e^{i(w+k\theta)}, \sigma). \)

The following proposition is well known in the case when \( n = 2 \), that is, for Luttinger surgery on 4-manifolds.

Proposition 3. If \( X' \) is obtained from \( X \) by \( \frac{1}{k} \) coisotropic Luttinger surgery along \( T^2 \times \Sigma \subset X \), then the Euler characteristic is unchanged, \( \chi(X') = \chi(X) \). When \( \dim(X) = 4\ell \), the signature is unchanged, \( \sigma(X') = \sigma(X) \). The fundamental group of \( X' \) is the quotient of \( \pi_1(X \setminus (T^2 \times \Sigma)) \) by the normal subgroup generated by the circle \( \psi_k(\partial D^2_\epsilon \times \{p\}) \).
Proof. Using the Mayer-Vietoris sequence one sees that the Euler characteristic is unchanged, \(\chi(X') = \chi(X)\). When \(n\) is even Novikov additivity shows that the signature is unchanged, \(\sigma(X') = \sigma(X)\).

Give \(T^2 \times \Sigma\) a handlebody structure with handles of index \(0, 1, \cdots, 2n - 3\) and a single \(2n - 2\)-handle. The product \(D^2 \times T^2 \times \Sigma\) has a handlebody structure obtained by taking the product of \(D^2\) with each handle, and in particular has a single \(2n - 2\)-handle. Turning the handle decomposition upside down shows that \(D^2 \times T^2 \times \Sigma\) is obtained from \(X \setminus (D^2_e \times T^2 \times \Sigma)\) by attaching a single 2-handle along the attaching circle \(\psi_k(\partial D^2_e \times (1, 1, \sigma_0))\), and then adding handles of index greater than 2.

The Seifert-Van Kampen theorem implies that
\[
\pi_1(X') = \pi_1(X \setminus (D^2_e \times T^2 \times \Sigma)) / \left( \psi_k(\partial D^2_e \times \{p\}) \right).
\]

Calculating \(\pi_1(X \setminus T^2 \times \Sigma)\) in terms of \(\pi_1(X)\) and the embedding \(T^2 \times \Sigma \subset X\) can be a challenge, since \(T^2 \times \Sigma\) has codimension two in \(X\). In our main application below we will content ourselves with the easier task of computing \(H_1(X \setminus T^2 \times \Sigma)\) and then \(H_1(X')\).

3. Producing symplectic 6-manifolds with \(c_1 = 0\)

Consider \(X = T^6 = T^2 \times T^2 \times T^2\), the 6-torus. Endow \(X\) with the symplectic form \(\omega_X = dx_1 dy_1 + dx_2 dy_2 + dx_3 dy_3\). We can find four disjoint embeddings of \(T^2 \times T^2\) in \(X\) with the properties we need to apply the construction of the previous section:
\[
e_1(e^{iz}, e^{iw}, \sigma_1, \sigma_2) = (1, e^{iz}, e^{iw}, 1, \sigma_1, \sigma_2)
e_2(e^{iz}, e^{iw}, \sigma_1, \sigma_2) = (i, e^{iz}, 1, e^{iw}, \sigma_1, \sigma_2)
e_3(e^{iz}, e^{iw}, \sigma_1, \sigma_2) = (-1, e^{iz}, \sigma_1, \sigma_2, e^{iw}, 1)
e_4(e^{iz}, e^{iw}, \sigma_1, \sigma_2) = (-i, e^{iz}, \sigma_1, \sigma_2, 1, e^{iw}).
\]

These are disjoint since their first coordinates are different. Note that \(e_i(T^2 \times (\sigma_1, \sigma_2))\) is isotropic and \(e_i((r, s) \times T^2)\) is symplectic.

For \(\epsilon > 0\) small, extend \(e_i\) to \(D^2_e \times T^2 \times T^2\) by
\[
e_1(x, y, e^{iz}, e^{iw}, e^{is_1}, e^{is_2}) = (e^{ix}, e^{iz}, e^{iw}, e^{iy}, e^{is_1}, e^{is_2}),
e_2(x, y, e^{iz}, e^{iw}, e^{is_1}, e^{is_2}) = (ie^{-ix}, i^{iz}, e^{-iy}, e^{iw}, e^{is_1}, e^{is_2}),
e_3(x, y, e^{iz}, e^{iw}, e^{is_1}, e^{is_2}) = (-ie^{-ix}, e^{iz}, e^{is_1}, e^{is_2}, e^{iw}, e^{iy}),
e_4(x, y, e^{iz}, e^{iw}, e^{is_1}, e^{is_2}) = (-ie^{-iz}, e^{iz}, e^{is_1}, e^{is_2}, e^{-iy}, e^{iw}).
\]

Then
\[
e_i^*(dx_1 dy_1 + dx_2 dy_2 + dx_3 dy_3) = dx \ dz + dw \ dy + d\sigma_1 \ d\sigma_2 = \omega
\]
for each \(i\).

One can find many more embeddings of \(T^2 \times T^2\) by precomposing \(e_i\) by a diffeomorphism \(\tau : T^2 \times T^2 \to T^2 \times T^2\) of the form
\[
\tau(e^{iz}, e^{iw}, \sigma_1, \sigma_2) = (e^{i(pz + qw)}, e^{i(rz + sw)}, \sigma_1, \sigma_2)
\]
for integers $p, q, r, s$ satisfying $ps - qr = 1$. Identify $\tau$ with the corresponding matrix in $SL(2, \mathbb{Z})$. Precomposing $c_i$ by $\tau \in SL(2, \mathbb{Z})$ and extending over $D_4^2 \times T^2 \times T^2$ yields another embedding with $(c_i \circ \tau)^* (dx_1dy_1 + dx_2dy_2 + dx_3dy_3) = \omega$, since $\det \tau = 1$.

Choose a surgery parameter $k_1$ and a matrix $\tau_i \in SL(2, \mathbb{Z})$ for each embedding $c_i$. Applying the coisotropic surgery procedure to $T^6$ yields a family of 6-dimensional symplectic manifolds $X_{k,\tau}$, indexed by $(k, \tau) = (k_1, k_2, k_3, k_4; \tau_1, \tau_2, \tau_3, \tau_4)$ in the infinite set $\mathbb{Z}^4 \times (SL(2, \mathbb{Z}))^4$. These are not all symplectically distinct; for example $SL(2, \mathbb{Z})^3$ (and even $Sp(6, \mathbb{Z})$) acts on this collection via its action on $T^6 = T^2 \times T^2 \times T^2$. But there are infinitely many distinct manifolds in this family. The following theorem is our main result, which immediately implies Theorem 4 promised in the introduction, by taking $d_1 = 0, d_2 = n, d_3 = 1$ and $d_4 = 1$.

**Theorem 4.** For $(k, \tau) \in \mathbb{Z}^4 \times (SL(2, \mathbb{Z}))^4$, the closed symplectic manifolds $X_{k,\tau}$ satisfy $c_1(X_{k,\tau}) = 0$. The first homology $H_1(X_{k,\tau})$ is the quotient of $\mathbb{Z}^6 = (x_1, x_2, x_3, x_4, x_5, x_6)$ by the subgroup generated by

$$k_1(x_1 + s_1x_3), k_2(x_2 + s_2x_4), k_3(x_3 + s_3x_5), k_4(x_4 + s_4x_6).$$

Hence any abelian group of the form

$$\mathbb{Z}^2 \oplus \mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2 \oplus \mathbb{Z}/d_3 \oplus \mathbb{Z}/d_4$$

with $d_1, d_2, d_3, d_4$ non-negative integers can be realized as $H_1(X_{k,\tau})$ for an appropriate $(k, \tau)$. If $b_1(X_{k,\tau})$ is odd then $X_{k,\tau}$ admits no Kähler structure. If $b_1(X_{k,\tau}) \leq 3$ then $X_{k,\tau}$ is not symplectomorphic to the product $M \times F$ of a symplectic 4-manifold and a surface. Finally, $b_2(X_{k,\tau}) \leq 15 + b_1(X_{k,\tau})$ and $b_3(X_{k,\tau}) \leq 32$.

The proof will take up the remainder of this section, and follows from Theorem 5 Proposition 7 and Theorem 8.

We begin with the calculation of the first Chern class.

**Theorem 5.** The symplectic 6-manifolds $X_{\tau,f}$ satisfy $c_1(X_{\tau,f}) = 0$.

**Proof.** Fix $k \in \mathbb{Z}$. We make use of the function $f : D_4^2 \to \mathbb{R}$ which equals 0 for $\sqrt{x^2 + y^2} \leq \frac{\tau}{3}$ and $\frac{1}{x^2 + y^2}$ for $\sqrt{x^2 + y^2} \geq \frac{2\tau}{3}$. In terms of this function, define an almost complex structure $J_k$ acting on 1-forms on $D_4^2 \times T^2 \times T^2$ by

$$J_k(dx) = -dz,$$

$$J_k(dz) = dx,$$

$$J_k(dy) = dw - kyd x + kxf dy,$$

$$J_k(dw) = -(1 + k^2x^2f^2) dy + k^2xyf^2 dx - kyd x - kxf dw,$$

$$J_k(d\sigma_1) = -d\sigma_2,$$

$$J_k(d\sigma_2) = d\sigma_1.$$

It is routine to check that $J_k^2 = -1$ and that $J_k$ is compatible with the symplectic form

$$\omega_k = dx dz + dw dy + d\sigma_1 d\sigma_2 - kyd x dy = \omega - kyd x dy.$$
Thus

\[ s_k = (dx - iJ_k(dx))(dw - iJ_k(dw))(d\sigma_1 - iJ_k(d\sigma_1)) \]
\[ = (dx + idz)(dw + idy)(d\sigma_1 + id\sigma_2) + kF(xdez + iydxz - xdydz)(d\sigma_1 + id\sigma_2) \]

is a section of \((3, 0)\) forms. This is nowhere zero since the coefficient of \(dx dw\) is 1, and hence pointwise spans the canonical bundle of \((D_2^2 \times T \times T, J_k)\).

Using (2) one calculates that over \((D_2^2 \setminus D_{2e/3}^2) \times T \times T\)

\[ (8) \]
\[ \phi_k^*(J_0) = J_k, \phi_k^*(\omega) = \omega_k, \text{ and } \phi_k^*(s_0) = s_k. \]

Now \(T^6\) is endowed with the symplectic form \(\omega_X = dx_1 dy_1 + dx_2 dy_2 + dx_3 dy_3\), compatible almost complex structure \(J_X(dx_1) = -dy_1\), and nowhere zero section of its canonical bundle \(s_X = (dx_1 + idy_1)(dx_2 + idy_2)(dx_3 + idy_3)\).

For each \(i = 1, 2, 3, 4\), \(e_i^*(s_X) = s_0, e_i^*(J_X) = J_0,\) and \(e_i^*(s_X) = s_0\). Using (5) it follows that when all the \(\tau_i\) are the identity, the almost complex structures \(J_k\) and the sections \(s_k\) over \(D_2^2 \times T \times T\) and the restrictions of \(J_X\) and \(s_X\) to the complement of \(\bigcup_1 e_i(D_{2e/3}^2 \times T \times T)\) in \(T^6\) patch together over \(e_i((D_2^2 \setminus D_{2e/3}^2) \times T \times T)\) to give an almost complex structure \(\tilde{J}\) compatible with \(\tilde{\omega}\) and a nowhere zero section of the associated canonical bundle of \(X_{k, \tau}\).

For more general \(\tau = (\tau_1, \tau_2, \tau_3, \tau_4)\), observe that the extension of \(\tau_i\ to a symplectomorphism \(\tau_i: D_2^2 \times T \times T \rightarrow D_2^2 \times T \times T\) by the formula

\[ (9) \]
\[ \tau_i(x, y, e^{iz}, e^{iw}, \sigma_1, \sigma_2) = (x, y, e^{(p_i z + q_i w)}, e^{(r_i z + s_i w)}, \sigma_1, \sigma_2) \]

induces a linear change of coordinate 1-forms

\[ \tau_i^*(dz) = p_i dz + r_i dw, \quad \tau_i^*(dw) = q_i dz + s_i dw \]

(and \(\tau_i^*(dx) = dx, \tau_i^*(dy) = dy, \tau_i^*(d\sigma_1) = d\sigma_1, \tau_i^*(d\sigma_2) = d\sigma_1\)). The argument extends by replacing all occurrences of \(dz\) and \(dw\) by \(\tau_i^*(dz)\) and \(\tau_i^*(dw)\) in the definitions of \(J_k, \omega_k,\) and \(s_k\). We leave the details to the reader.

Thus the tangent bundle of \(X_{k, \tau}\) admits an almost complex structure compatible with its symplectic form and a nowhere zero section of the associated canonical bundle of \((3, 0)\) forms. Thus \(c_1(X_{k, \tau}) = 0\), as asserted. \(\square\)

It is not necessarily true that the result of coisotropic Luttinger surgery along a 4-torus in a symplectic 6-manifold \(X\) with \(c_1(X) = 0\) yields a manifold with vanishing first Chern class in general, see e.g. [1]. The important point in the preceding proof is that the non-vanishing section \(s_X\) of the canonical bundle of \(T^6\) over each \(D_2^2 \times T \times T\) coincides with the “coordinate” section \(s_0 = (dx + idz)(dw + idy)(d\sigma_1 + id\sigma_2)\) via the embedding \(e_i\). In the general case one may need to interpolate between the restriction of a given section of the canonical bundle of \(X\) to the coordinate section over the neighborhood of the 4-torus. This interpolation leads in general to the addition of “rim” 4-cycles, supported near the boundary of \(D_2^2 \times T \times T\), to the divisor of the canonical class of the surgered symplectic manifold.

The following lemma will be used in the proof of Theorem 8.

**Lemma 6.** Let \(N\) denote the union of the four tubular neighborhoods of \(e_i(T^4)\) in \(T^6\), where \(e_i: T^4 \rightarrow T^6\) are the embeddings of Equation (2). Then the inclusion

\[ H_1(T^6 \setminus N) \rightarrow H_1(T^6) \cong \mathbb{Z}^6 \]
is an isomorphism. Moreover, $H_2(T^6 \setminus N) \cong \mathbb{Z}^{17}$.

**Proof.** Thicken the embeddings $e_i : T^4 \rightarrow T^6$ to (closed) tubular neighborhoods $e_i : D^2_i \times T^4 \rightarrow T^6$ and denote the union of these tubular neighborhoods by $N$. The excision and Kunneth theorems give isomorphisms

$$
\bigoplus_{i=1}^4 H_{n-2}(e_i(T^4)) \cong H_{n-2}(\bigcup e_i(T^4)) \otimes H_2(D^2_i, S^1) \cong H_n(N, \partial N) \cong H_n(T^6, T^6 \setminus N).
$$

This correspondence assigns to an $(n-2)$-cycle $\gamma \subseteq e_i(T^4)$ the product $\gamma \times (D^2_i, S^1)$. The connecting homomorphism $H_n(T^6, T^6 \setminus N) \rightarrow H_{n-1}(T^6 \setminus N)$ takes $\gamma \times (D^2_i, S^1)$ to $\gamma \times S^1$.

Consider the exact sequence of the pair

$$
\cdots \rightarrow H_{n+1}(T^6, T^6 \setminus N) \rightarrow H_n(T^6 \setminus N) \rightarrow H_n(T^6) \rightarrow H_n(T^6, T^6 \setminus N) \rightarrow \cdots
$$

Since $H_1(T^6, T^6 \setminus N) = 0$, $H_1(T^6 \setminus N) \rightarrow H_1(T^6)$ is surjective. The connecting homomorphism $\mathbb{Z}^4 \cong H_2(T^6, T^6 \setminus N) \rightarrow H_1(T^6 \setminus N)$ has image generated by the four meridians $\mu_i = e_i(p) \times S^1$.

But $\mu_i = 0 \in H_1(T^6 \setminus N)$ since they bound the punctured dual tori. More explicitly, $T_1 = S^2 \times \{1\} \times \{1\} \times S^1 \times \{1\} \times \{1\}$ is a 2-dimensional torus in $T^6$ which meets $e_i(T^4)$ in the meridian disk $e_1(D^2_i \times \{1\} \times \{1\})$ and is disjoint from $e_i(T^4)$ for $i = 2, 3, 4$. Thus $T_1 - e_1(D^2_i \times \{1\} \times \{1\})$ is a 2 chain in $T^6 \setminus e_1(T^4)$ with boundary $\mu_1$. Similar arguments show that all the $\mu_i$ are zero. Hence $H_1(T^6 \setminus N) \rightarrow H_1(T^6) \cong \mathbb{Z}^6$ is an isomorphism.

We claim the homomorphism

$$
H_3(T^6) \rightarrow H_3(T^6, T^6 \setminus N) \cong \bigoplus_{i=1}^4 H_1(e_i(T^4)) \cong \mathbb{Z}^{16}
$$

has rank 10.

First note that it has rank at most 10, since $H_3(T^6) \cong \mathbb{Z}^{20}$, and the coefficient in $H_3(T^6 \setminus N)$ is $\mathbb{Z}^{20}$, which fills in the 3-tori with first coordinate fixed lift to $H_3(T \setminus N)$: just choose their first coordinate distinct from $\pm 1, \pm i$.

Denote by $W_i, i = 1, 2, \cdots, 10$ the following representatives of the remaining ten coordinate 3-tori in $T^6$:

- $W_1 = \{(e^{ia}, e^{ib}, -1, e^{ic}, -1, -1) \mid a, b, c \in \mathbb{R}\}$
- $W_2 = \{(e^{ia}, e^{ib}, e^{ic}, -1, -1, -1) \mid a, b, c \in \mathbb{R}\}$
- $W_3 = \{(e^{ia}, e^{ib}, -1, -1, -1, e^{ic}) \mid a, b, c \in \mathbb{R}\}$
- $W_4 = \{(e^{ia}, e^{ib}, -1, -1, e^{ic}, -1) \mid a, b, c \in \mathbb{R}\}$
- $W_5 = \{(e^{ia}, -1, e^{ib}, -1, -1, e^{ic}) \mid a, b, c \in \mathbb{R}\}$
- $W_6 = \{(e^{ia}, -1, e^{ib}, -1, e^{ic}, -1) \mid a, b, c \in \mathbb{R}\}$
- $W_7 = \{(e^{ia}, -1, e^{ib}, e^{ic}, -1, -1) \mid a, b, c \in \mathbb{R}\}$
- $W_8 = \{(e^{ia}, -1, -1, e^{ib}, e^{ic} - 1) \mid a, b, c \in \mathbb{R}\}$
- $W_9 = \{(e^{ia}, -1, -1, e^{ib}, -1, e^{ic}) \mid a, b, c \in \mathbb{R}\}$
- $W_{10} = \{(e^{ia}, -1, -1, -1, e^{ib}, e^{ic}) \mid a, b, c \in \mathbb{R}\}$

These generate a rank 10 free abelian subgroup of $H_3(T^6)$, and intersect each of the $e_i(T^4)$ transversely. Thus the image of each of these ten 3-cycles in $H_3(T^6, T^6 \setminus N) \cong \bigoplus_{i=1}^4 H_1(e_i(T^4))$ is determined by taking its (transverse) intersection with $e_i(T^4)$. 
For example, $W_1$ misses $e_j(T^4)$ for $j \neq 1$ and intersects $e_1(T^4)$ transversely in the homologically essential circle $\{1\} \times S^1 \times (-1, 1, -1, 1) = e_1(S^1 \times (-1, -1, 1))$. Similarly, for $1 \leq i, j \leq 4$, $W_j$ misses $e_i(T^4)$ when $i \neq j$ and intersects $e_i(T^4)$ in a homologically essential circle.

For $5 \leq j \leq 10$, $W_j$ intersects exactly two of the $e_i(T^4)$. For example, $W_5$ is disjoint from $e_1(T^4)$ and $e_4(T^4)$ and intersects $e_2(T^4)$ in the circle $(i, -1, 1, -1, 1) \times S^1 = e_2((-1, -1, -1) \times S^1)$, and intersects $e_3(T^4)$ in the circle $(-1, -1) \times S^1 \times (-1, -1, 1) = e_3((-1, -1) \times S^1 \times -1)$.

We leave the reader the straightforward check that the 10 cycles in $\oplus_{i=1}^4 H_1(e_i(T^4)) \cong \mathbb{Z}^{16}$ are linearly independent and span a summand. Thus the rank of $H_3(T^6) \to H^3(T^6, T^6 \setminus N)$ is 10 and its cokernel is $\mathbb{Z}^6$.

From the exact sequence (11) with $n = 2$ we obtain

$$0 \to \mathbb{Z}^6 \to H_2(T^6 \setminus N) \to H_2(T^6) \to \mathbb{Z}^4 \to 0.$$ 

Since $H_2(T^6) \cong \mathbb{Z}^{15}$ and $H_1(T^6 \setminus N)$ is free abelian, we conclude that $H_2(T^6 \setminus N) \cong \mathbb{Z}^{17}$. □

Lemma 5 says that $H_1(T^6 \setminus N) \to H_1(T^6)$ is an isomorphism and hence the six coordinate circles freely generate $H_1(T^6 \setminus N) \cong \mathbb{Z}^6$. Label these generators $x_1, \ldots, x_6$. As explicit curves in $T^6 \setminus N$, one can take $x_1 = S^1 \times (p,p,p,p,p)$, $x_2 = p \times S^1 \times (p,p,p,p)$, etc., where $p$ is a primitive eight root of unity.

**Proposition 7.** Given $(k, \tau) = (k_1, \tau_1; k_2, \tau_2; k_3, \tau_3; k_4, \tau_4) \in \mathbb{Z}^4 \times (SL(2, \mathbb{Z}))^4$, with

$$\tau_i = \begin{pmatrix} p_i & q_i \\ r_i & s_i \end{pmatrix},$$

then $H_1(X_{k,\tau})$ is the quotient of $\mathbb{Z}^6 = H_1(T^6 \setminus N) = \langle x_1, x_2, x_3, x_4, x_5, x_6 \rangle$ by the subgroup generated by

$$k_1(q_1x_2 + s_1x_3), k_2(q_2x_2 + s_2x_4), k_3(q_3x_2 + s_3x_5), k_4(q_4x_2 + s_4x_6).$$

Moreover, the Betti numbers of $X_{k,\tau}$ satisfy

$$b_2(X_{k,\tau}) \leq 15 + b_1(X_{k,\tau}) \text{ and } b_3(X_{k,\tau}) \leq 32.$$ 

**Proof.** The manifold $D^2 \times T^4$ has its usual handle decomposition with one 0-handle, four 1-handles, six 2-handles, four 3-handles, and one 4-handle. Turning it upside down one obtains the dual handle decomposition, showing that $D^2 \times T^4$ is obtained from $\partial(D^2) \times T^4 \times [0, 1]$ by adding one 2-handle along $\mu = \partial(D^2) \times \{p\} \times \{1\}$, then adding four 3-handles, six 4-handles, four 5-handles, and one 6-handle.

Hence $H_1(X_{k,\tau})$ is the quotient of $H_1(T^6 \setminus N)$ by the subgroup generated by the four circles along which the 2-handles are reattached in passing from $T^6 \setminus N$ to $X_{k,\tau}$.

From the formulas (14), (19), and (5) one sees that the 2-handle corresponding to $e_i$ is attached along $e_i \circ \tau_1 \circ \psi_{k_i}(\partial D^2_i)$. In terms of the generators $x_1 \ldots, x_6$ of $H_1(T^6 \setminus N)$, a simple
calculation shows that
\[
\begin{align*}
[e_1(\tau_1(\psi_{k_1}(\partial D^2)))] &= k_1(q_1x_2 + s_1x_3) \\
[e_2(\tau_2(\psi_{k_2}(\partial D^2)))] &= k_2(q_2x_2 + s_2x_4) \\
[e_3(\tau_3(\psi_{k_3}(\partial D^2)))] &= k_3(q_3x_2 + s_3x_5) \\
[e_4(\tau_4(\psi_{k_4}(\partial D^2)))] &= k_4(q_4x_2 + s_4x_6)
\end{align*}
\]
(11)

Thus \(H_1(X_{k,\tau})\) is isomorphic to the quotient of \(\mathbb{Z}^6\) by the four 1-cycles on the right side of Equation (11).

Lemma 6 shows that \(H_2(T^6 \setminus N) \cong \mathbb{Z}^{17}\). Attaching a 2-handle to a manifold increases the second Betti number if and only if the attaching circle has finite order in first homology. Hence if \(H_1(X_{k,\tau}) \cong \mathbb{Z}^{6-d} \oplus F\) for a finite abelian group \(F\), the second Betti number of \(T^6 \setminus N\) with the four 2-handles attached is \(17 + (4 - d) = 21 - d\). Attaching the sixteen 3-handles decreases the second Betti number further, and the 4-handles, 5-handles, and 6-handles do not change the second Betti number. Hence \(b_2(X_{k,\tau}) \leq 21 - d = 15 + b_1(X_{k,\tau})\). The Euler characteristic of \(X_{k,\tau}\) equals zero, and so
\[
0 = 2 - 2b_1(X_{k,\tau}) + 2b_2(X_{k,\tau}) - b_3(X_{k,\tau}) \leq 32 - b_3(X_{k,\tau}).
\]
Therefore \(b_3(X_{k,\tau}) \leq 32\).

By choosing the \(\tau_i\) and \(k_i\) appropriately, one can ensure that \(H_1(X_{k,\tau})\) is isomorphic to
\[
\mathbb{Z}^2 \oplus \mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2 \oplus \mathbb{Z}/d_3 \oplus \mathbb{Z}/d_4
\]
for any 4-tuple of non-negative integers \(d_i\) (e.g. take \(\tau_i = \text{Id}\) and \(k_i = d_i\)). In particular, when an odd number of the \(d_i\) are non-zero, then the first Betti number is is odd and hence \(X_{k,\tau}\) cannot be Kähler.

To ensure that our construction produces new manifolds, we have the following.

**Theorem 8.** If \(H_1(X_{k,\tau})\) has rank 2 or 3, then \(X_{k,\tau}\) is not symplectomorphic to the product of any symplectic 4-manifold with a surface.

**Proof.** Choose an \(X = X_{k,\tau}\) such that the first Betti number of \(X\) satisfies \(b_1(X) = 2 + r\) for \(r = 0\) or 1. Lemma 6 shows that \(b_2(X) \leq 18\).

Suppose that \(X\) were symplectomorphic to \(M \times F\), for some symplectic 4-manifold \(M\) and closed oriented surface \(F\). Then \(0 = c_1(X) = \pi_1^*(c_1(M)) + \pi_2^*(c_1(F))\), where \(\pi_i\) denote the projections of \(M \times F\) to its two factors. The Kunneth theorem shows that \(\pi_1^* + \pi_2^* : H^2(M) \oplus H^2(F) \to H^2(M \times F)\) is injective, and hence \(c_1(M) = 0\) and \(c_1(F) = 0\). Thus \(F\) is a torus, \(F = T^2\), and \(M\) admits a Spin structure.

Rohlin’s theorem then shows that the signature \(\sigma(M) = b^+(M) - b^-(M)\) is a multiple of 16, and so
\[
b^-(M) = b^+(M) + 16n
\]
for some integer \(n\). The Kunneth theorem implies that \(b_1(M) = r\), and so the Euler characteristic is given by \(e(M) = 2 - 2r + b_2(M) = 2 - 2r + b^+(M) + b^-(M)\). Since \(0 = c_1(M)^2 = 2e(M) + 3\sigma(M) = 4 - 4r + 5b^+(M) - b^-(M)\),
\[
b^-(M) = 5b^+(M) + 4 - 4r
\]
and we conclude that

\[ b^+(M) = 4n + r - 1. \]

The symplectic form satisfies \( \omega^2_M > 0 \) and so \( b^+(M) \geq 1 \). Since \( r = 0 \) or \( 1 \), this implies that \( n \geq 1 \). Hence

\[ b_2(M) = b^+(M) + b^-(M) = 4b^+(M) + 4 - 4r = 24n + 2r - 2 \geq 22. \]

Then

\[ b_2(X) = b_2(M) + b_1(M)b_1(T^2) + b_2(T^2) \geq 23. \]

This contradicts the bound \( b_2(X) \leq 18 \) obtained above.

□

Arguments like those given in the proof of Theorem \( \Box \) can be used to show that \( X_{k,\tau} \) is not homotopy equivalent to the product of a symplectic 4-manifold with a surface of genus 2 or more for any \( (k, \tau) \). This leaves open the possibility that some \( X_{k,\tau} \) is homotopy equivalent or even diffeomorphic to \( M \times S^2 \).

We do not know if every \( X_{k,\tau} \) with even first Betti number admits a Kähler structure, but conjecture that most do not. The reason for this conjecture is that most \( X_{k,\tau} \) are not likely to satisfy the Hard Lefschetz Theorem (cf. \( \cite{2, 3} \)).

4. Concluding remarks

Coisotropic Luttinger surgery can be useful in other contexts. For example, an easy extension of Theorem \( \Box \) can be obtained by considering surgeries on 2n-dimensional tori. An interesting setting occurs when a symplectic manifold fibers in several different ways. In our examples we applied this to \( T^6 = T^2 \times T^2 \times T^2 \) and its three coordinate fibrations to the 4-torus. One could also start with a product of closed surfaces \( X = \Sigma_{g_1} \times \Sigma_{g_2} \times \cdots \times \Sigma_{g_n} \), which contains many coisotropic submanifolds of the form \( T^2 \times Z \) obtained as preimages of Lagrangian tori with respect to various projections of \( X \) to \( \Sigma_{g_i} \times \Sigma_{g_j} \).

One can produce symplectic 2n-manifolds with a wide range of possible homology groups and canonical classes by this method. Deriving more concrete homotopy or diffeomorphism information is more difficult, as establishing control of the fundamental group is always a challenge in codimension two surgery constructions, c.f. \( \cite{3} \).

Another promising direction is to apply the method to Lefschetz fibrations. For example, the \( K^3 \) surface, as a desingularization of \( T^4/\mathbb{Z}/2 \), has admits different elliptic fibrations. Thus \( K^3 \times T^2 \) contains submanifolds on which one can perform coisotropic Luttinger surgery. This should lead to examples with smaller first homology and perhaps even to simply connected examples.

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