MIMO Integrated Sensing and Communication with Extended Target: CRB-Rate Tradeoff

Haocheng Hua*, Xianxin Song*, Yuan Fang*, Tony Xiao Han†, and Jie Xu*

*School of Science and Engineering and Future Network of Intelligence Institute, The Chinese University of Hong Kong (Shenzhen), Shenzhen, China
†The 2012 Lab, Huawei, Shenzhen, China

Email: haochenghua@link.cuhk.edu.cn, xianxinsong@link.cuhk.edu.cn, fangyuan@cuhk.edu.cn, tony.hanxiao@huawei.com, xujie@cuhk.edu.cn

Abstract—This paper studies a multiple-input multiple-output (MIMO) integrated sensing and communication (ISAC) system, in which a multi-antenna communication user (CU) at the same time. We investigate the fundamental tradeoff between the estimation Cramér-Rao bound (CRB) for sensing and the data rate for communication, by characterizing the Pareto boundary of the achievable CRB-rate (C-R) region. Towards this end, we formulate a new MIMO rate maximization problem by optimizing the transmit covariance matrix at the BS, subject to a new form of maximum CRB constraint together with a maximum transmit power constraint. We derive the optimal transmit covariance solution in a semi-closed form, by first implementing the singular-value decomposition (SVD) to diagonalize the communication channel and then properly allocating the transmit power over these subchannels for communication and other orthogonal subchannels (if any) for dedicated sensing. It is shown that the optimal transmit covariance is of full rank, which unifies the conventional rate maximization design with water-filling power allocation and the CRB minimization design with isotropic transmission. Numerical results are provided to validate the performance achieved by our proposed optimal design, in comparison with other benchmark schemes.

I. INTRODUCTION

Recently, integrated sensing and communication (ISAC) has been recognized as a candidate technology towards future sixth-generation (6G) cellular networks to enable various environment-aware intelligent applications (see, e.g., [1] and the references therein), in which wireless signals and cellular infrastructures are reused for both sensing and communication functionalities. Motivated by the great success of multi-antenna or multiple-input multiple-output (MIMO) techniques in wireless communications [2], [3] and radar sensing [4]–[6] independently, MIMO ISAC has attracted particular research interests, in which the multiple antennas can be exploited to provide spatial multiplexing and diversity gains to increase the communication rate and reliability [2], [3], as well as the waveform/spatial diversity gains to enhance the sensing accuracy and resolution [4]–[7]. In the literature, there have been some prior works [8]–[14] studying multi-antenna ISAC designs to optimize the sensing and communication performance. However, these existing works mainly focused on practical waveform and beamforming approaches that are generally suboptimal in achieving the performance limits for sensing and/or communication.

In ISAC systems, it is essential to understand the performance tradeoffs between radar sensing and communication from detection/estimation and information theories [15]. This not only helps reveal the fundamental performance limits, but also guides practical ISAC system designs. On one hand, the Cramér-Rao bound (CRB) serves as a fundamental performance metric for radar estimation, which defines the variance lower bound by any unbiased estimators. On the other hand, the channel capacity acts as a fundamental performance metric for wireless communication, which captures the rate upper bound by any practical modulation and coding schemes. Therefore, how to characterize the fundamental CRB-rate (C-R) tradeoff for MIMO ISAC systems is becoming an important problem to be tackled. To our best knowledge, only one recent work [16] studied the so-called C-R region for a point-to-point MIMO ISAC system with one sensing target, which is defined as the set containing all C-R pairs that can be simultaneously achieved by sensing and communication. However, [16] only obtained two boundary points on the C-R region, namely the CRB-minimization and rate-maximization points, at which the CRB is minimized and the rate is maximized, respectively. Unfortunately, [16] failed to characterize the whole boundary of the C-R region, especially the boundary points between the above two. This thus motivates our work to fill in such a research gap.

In particular, this paper considers a point-to-point MIMO ISAC system with an extended target, in which a multi-antenna base station (BS) sends unified wireless signals to estimate an extended target from the echo and communicate with a multi-antenna communication user (CU) at the same time. We aim to reveal the fundamental C-R tradeoff of this system, by characterizing the Pareto boundary of the C-R region. The main results are listed as follows.

• First, to characterize the C-R-region boundary between
the CRB-minimization and rate-maximization points, we formulate a new CRB-constrained rate maximization problem, in which the data rate for MIMO communication is maximized by optimizing the transmit covariance matrix, subject to a new form of maximum CRB constraint and the maximum transmit power constraint.

- Next, we derive the optimal transmit covariance solution to the CRB-constrained rate maximization problem in a semi-closed form. Towards this end, we first implement the singular-value decomposition (SVD) to diagonalize the communication channel, and accordingly transform the transmit covariance optimization problem into an equivalent power allocation problem over these subchannels for communication and other orthogonal subchannels (if any) for dedicated sensing. Then, we obtain the optimal power allocation solution by the Lagrange duality method. It is shown that the optimal transmit covariance is of full rank, which unifies the conventional rate maximization design with water-filling power allocation and the CRB minimization design with isotropic transmission.

- Finally, we present numerical results to evaluate the C-R-region boundary achieved by the optimal transmit covariance by considering two cases with rank-deficient and full-column-rank communication channels, respectively, as compared with other benchmark schemes.

**Notations:** Boldface letters refer to vectors (lower case) or matrices (upper case). For a square matrix \( S \), \( \text{tr}(S) \) denotes its trace, and \( S \succeq 0 \) means that \( S \) is positive semidefinite. For an arbitrary-size matrix \( M \), \( \text{det}(M) \), \( \text{rank}(M) \), \( M^H \), and \( M^t \) denote its determinant, rank, conjugate transpose, and transpose, respectively. \( \mathbb{R}^{x \times y} \) and \( \mathbb{C}^{x \times y} \) denote the spaces of real and complex matrices, respectively. \( \mathbb{E}\{\cdot\} \) denotes the statistical expectation. \( ||x|| \) denotes the Euclidean norm of a complex vector \( x \). \( |z| \) and \( |z|^* \) denote the magnitude and the conjugate of a complex number \( z \), respectively. For a real number \( x \), \( (x)^T = \max(x, 0) \). \( \text{diag}(x_1, \ldots, x_n) \) denotes a diagonal matrix with diagonal elements \( x_1, \ldots, x_n \).

II. SYSTEM MODEL

We consider a MIMO ISAC system, in which a BS communicates with a CU and simultaneously estimates an extended target, as shown in Figs. 1(a) and 1(b) with monostatic and bistatic sensing, respectively. There are \( M > 1 \) transmit antennas at the BS transmitter (Tx), \( N_s \) receive antennas at the BS receiver (Rx), and \( N_c > 1 \) antennas at the CU.

![Illustration of the MIMO ISAC system](image)

**Fig. 1.** Illustration of the MIMO ISAC system.

Let \( x(n) \) denote the unified transmit signal at symbol \( n \) for both communication and sensing. It is assumed that \( x(n) \) is a circular symmetric complex Gaussian (CSCG) random vector with zero mean and covariance matrix \( Q = \mathbb{E}\{x(n)x^H(n)\} \succeq 0 \), i.e., \( x(n) \sim \mathcal{CN}(0, Q) \). Let \( P \) denote the transmit power budget at the BS-Tx. Then we have the power constraint as

\[
\text{tr}(Q) = \mathbb{E}\{|x(n)|^2\} \leq P. \tag{1}
\]

First, we consider the point-to-point MIMO communication. Let \( H_c \in \mathbb{C}^{N_c \times M} \) denote the channel matrix from the BS-Tx to the CU, whose rank is denoted by \( r = \text{rank}(H_c) \leq \min(N_c, M) \). The received signal by the CU at symbol \( n \) is

\[
y_c(n) = H_c x(n) + z_c(n), \tag{2}
\]

where \( z_c(n) \) denotes the noise at the CU receiver that is a CSCG random vector with zero mean and covariance \( \sigma_c^2 I_{N_c} \), i.e., \( z_c(n) \sim \mathcal{CN}(0, \sigma_c^2 I_{N_c}) \). In this case, the achievable rate (in bps/Hz) of the MIMO channel with \( Q \) is

\[
R(Q) = \log_2 \det \left( I_{N_c} + \frac{1}{\sigma_c^2} H_c Q H_c^H \right) \tag{3}
\]

It is assumed that the channel matrix \( H_c \) is perfectly known at the BS-Tx, such that it can design \( Q \) based on \( H_c \) to optimize the achievable rate \( R(Q) \).

Next, we consider the MIMO radar sensing over a particular coherent processing interval (CPI) with \( L > M \) symbols or radar pulses. Let \( L = \{1, \ldots, L\} \) denote the set of symbols in the CPI, and \( X = [x(1), \ldots, x(L)] \in \mathbb{C}^{M \times L} \) denote the transmitted signals over the CPI. Suppose that \( H_s \in \mathbb{C}^{N_s \times M} \) denotes the target response matrix from the BS-Tx to the target at the BS-Rx. Accordingly, the received echo signal \( Y_s \) at the BS-Rx is

\[
Y_s = H_s X + Z_s, \tag{4}
\]

where \( Z_s \in \mathbb{C}^{N_s \times L} \) denotes the noise matrix at the BS-Rx, with each element being a CSCG random variable with zero mean and variance \( \sigma_s^2 \). In particular, we consider the case with an extended target, which is modelled as the combination of a large number of \( K \) distributed point-like scatterers. Suppose that the target is located at a fixed location during the CPI. In this case, \( H_s \) is expressed as [13]

\[
H_s = \sum_{k=1}^{K} \alpha_k b(\phi_k) a^T(\theta_k), \tag{5}
\]

where \( \alpha_k \) denotes the reflection coefficient of the \( k \)-th scatterer, \( \theta_k \) and \( \phi_k \) denote its associated angle of departure (AoD) and angle of arrival (AoA) at the BS, and \( a(\theta_k) \) and \( b(\phi_k) \) denote the corresponding transmit and receive steering vectors, respectively. The objective of sensing is to estimate the target response matrix \( H_s \), which contains \( MN_s \) complex parameters. In this case, the CRB matrix for estimating \( H_s \) is given by [13]

\[
\text{CRB}(Q) = J(Q)^{-1} \tag{6}
\]

which is a complex matrix with dimension \( MN_s \times MN_s \), with the \( (M(i-1)+j) \)-th diagonal element representing the lower bound of variance for unbiasedly estimating the \((i, j)\)-element of \( H_{s, ij} \), \( i \in \{1, \ldots, N_s\}, j \in \{1, \ldots, M\} \). In (6), \( J(Q) \) is the
Fisher information matrix given by [13]
\[ J(Q) = \frac{1}{\sigma^2} X^T X^T \otimes I_{N_s} \approx \frac{L}{\sigma^2} Q^T \otimes I_{N_s}, \]
where \( \frac{1}{L} XX^H \) is approximated as \( Q \) by assuming that \( L \) is sufficiently large [13]. Based on the CRB matrix \( \text{CRB}(Q) \) in (6), we use its trace as the sensing performance metric for estimating \( H_s \) [17], [18], i.e.,
\[ \text{CRB}(Q) = \text{tr}(\text{CRB}(Q)) = \text{tr}(J(Q)^{-1}) = \frac{\sigma^2 L N_s}{\sigma^2} \text{tr}(Q^{-1}). \]

The BS-Tx can design \( Q \) to optimize \( \text{CRB}(Q) \) for estimation.

III. C-R REGION CHARACTERIZATION

This section characterizes the C-R region to reveal the fundamental tradeoff between the data rate \( R(Q) \) in (3) for communication and the estimation CRB \( \text{CRB}(Q) \) in (8) for sensing. To start with, we define the C-R region, which is a set containing all C-R pairs that can be simultaneously achievable by the ISAC system under the given transmit power constraint. Mathematically, the C-R region with power budget \( P \) is defined as
\[ C_{C-R}(P) \triangleq \{(\Gamma, \tilde{R}) : \Gamma \geq \frac{\sigma^2 L}{\sigma^2} \text{tr}(Q^{-1}), \tilde{R} \leq \log_2 \det(I_{N_c} + \frac{1}{\sigma^2} H_c Q H_c^H), \text{tr}(Q) \leq P, Q \succeq 0\}. \]

(9)

In this case, revealing the optimal tradeoff between communication rate \( R(Q) \) and estimation CRB \( \text{CRB}(Q) \) corresponds to finding the Pareto boundary of the C-R region \( C_{C-R}(P) \) in (9). Towards this end, we first introduce two boundary points corresponding to rate maximization and CRB minimization, respectively.

First, we maximize the achievable rate \( R(Q) \) by optimizing the transmit covariance \( Q \), i.e.,
\[ \max_{Q \succeq 0} \log_2 \det(I_{N_c} + \frac{1}{\sigma^2} H_c Q H_c^H) \quad \text{s.t.} \quad \text{tr}(Q) \leq P. \]

(10)

Based on the SVD, we have \( H_c = U_c \Sigma V_c^H \), where \( U_c \in \mathbb{C}^{N_c \times N_c} \) and \( V_c \in \mathbb{C}^{M \times M} \) with \( U_c^H U_c = U_c M^2 = I_{N_c} \) and \( V_c^H V_c = V_c V_c^H = I_M \), and \( \Sigma \in \mathbb{C}^{N_c \times M} \) is an all-zero matrix except the first \( r \) diagonal elements being the \( r \) non-zero singular values \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0 \). It has been well established in [2] that the optimal solution to the rate maximization problem (10) is given by \( Q^*_c = V_c A V_c^H \), where \( A = \text{diag}(p_{c,1}^*, \ldots, p_{c,r}^*, 0, \ldots, 0) \) denotes the water-filling power allocation matrix with its first \( r \) diagonal elements given by
\[ p_{c,i}^* = \left( \nu - \frac{\sigma^2 \lambda_i}{L} \right)^+, \forall i \in \{1, \ldots, r\}. \]

(11)

In (11), \( \nu \) is the water level that can be obtained based on \( \sum_{i=1}^r p_{c,i}^* = P \). At the obtained \( Q^*_c \), let \( R_{\max} = R(Q^*_c) = \sum_{i=1}^r \log_2(1 + \frac{\lambda_i p_{c,i}^*}{2}) \) and \( \text{CRB}_c = \text{CRB}(Q^*_c) \) denote the correspondingly achieved data rate and estimation CRB, respectively. As a result, we obtain the rate-maximization boundary point of the C-R region as \((\Gamma^*, R_{\max})\).

Remark 1. It is worth noting that if the rate-maximization transmit covariance \( Q^*_c \) is rank-deficient (i.e., \( \text{rank}(Q^*_c) < M \)), then it follows from (8) that \( \text{CRB}_c \rightarrow \infty \). This means that \( H_c \) is not estimable in this case due to the lack of degrees of freedom (DoF). Accordingly, the rate-maximization boundary point becomes \((\infty, R_{\max})\). It can be verified that this case happens when the communication channel matrix \( H_c \) is rank-deficient (i.e., \( \text{rank}(H_c) = r < M \)) or the transmit power is small (i.e., \( P < P_0 \triangleq \sum_{i=1}^M (\frac{\sigma^2}{N_i^2} - \frac{\sigma^2}{N_i^2}) \)).

Next, we minimize the CRB \( \text{CRB}(Q) \) by optimizing the transmit covariance \( Q \), i.e.,
\[ \min_{Q \succeq 0} \frac{\sigma^2 L N_s}{\sigma^2} \text{tr}(Q^{-1}) \quad \text{s.t.} \quad \text{tr}(Q) \leq P. \]

(12)

By checking the Karush-Kuhn-Tucker (KKT) conditions, the optimal solution to problem (12) is obtained as \( Q^*_s = \frac{P}{M} I_M \) [13]. As a result, the correspondingly achieved minimum CRB and data rate become \( \text{CRB}_\min = \frac{\sigma^2 N_s M^2}{PL} \) and \( R_s = \sum_{i=1}^r \log_2(1 + \frac{\lambda_i^2}{2 PM}) \), respectively. Therefore, we obtain the CRB-minimization boundary point as \((\bar{\Gamma}^*, R_s)\).

Based on the obtained two boundary points \((\text{CRB}_c, R_{\max})\) and \((\text{CRB}_s, R_s)\), it now remains to find the remaining Pareto boundary points between them for characterizing the whole C-R region. To find each boundary point, we propose to maximize the achievable rate \( R(Q) \) by optimizing \( Q \), subject to the maximum CRB constraint \( \text{CRB}(Q) \leq \bar{\Gamma} \) and the transmit power constraint in (1), where the CRB threshold \( \bar{\Gamma} \) is set as a constant such that \( \text{CRB}_\min \leq \bar{\Gamma} \leq \text{CRB}_c \). By defining \( \bar{\Gamma} = \frac{PF}{\sigma^2 N_s} \), the CRB-constrained rate maximization problem is formulated as

\[ \text{(P1)}: \quad \max_{Q \succeq 0} \log_2 \det(I_{N_c} + \frac{1}{\sigma^2} H_c Q H_c^H) \quad \text{s.t.} \quad \text{tr}(Q^{-1}) \leq \bar{\Gamma}, \quad \text{tr}(Q) \leq P. \]

(13a, 13b, 13c)

Note that problem (P1) is convex and can thus be solved optimally based on standard convex optimization techniques [19]. To gain insights, we derive its optimal solution in a semi-closed form in the next section.

Remark 2. It is worth discussing the C-R region in the special case when the communication channel is of full column rank \( (r = M) \) and the transmit power is sufficiently large (i.e., \( P \rightarrow \infty \)). In this case, it is easy to show that \( Q^*_c = Q^*_s = \frac{P}{M} I_M \) and the two boundary points become identical (i.e., \((\text{CRB}_\min, R_s) = (\text{CRB}_c, R_{\max})\)). As a result, the C-R region can be obtained as a box without solving problem (P1), which is denoted by \( C_{C-R}(P) = \{(\bar{\Gamma}, \tilde{R}) : \bar{\Gamma} \geq \text{CRB}_\min, \tilde{R} \leq R_{\max}\} \).

IV. OPTIMAL SOLUTION TO PROBLEM (P1)

This section presents the optimal solution to (P1). First, recall that the SVD of \( H_c \) is \( H_c = U_c \Sigma V_c^H \). By defining
\[ \tilde{Q} \triangleq V_c^H Q V_c \quad \text{or} \quad \tilde{Q} \triangleq V_c \tilde{Q} V_c^H, \]

(14)
The optimal solution of (P1) can be equivalently reformulated as

\[
\begin{align*}
(P1.1): \quad & \max_{Q \succeq 0} \log_2 \det(1M + \frac{1}{\sigma_c^2} \Sigma^2 \tilde{Q}) \\
& \text{s.t. } \text{tr}(\tilde{Q}^{-1}) \leq \bar{\Gamma} \\
& \text{tr}(\tilde{Q}) \leq P.
\end{align*}
\]

(15a) - (15c)

where \( \Sigma \triangleq \Sigma^H \Sigma = \text{diag}(\lambda_1, ..., \lambda_r, 0, ..., 0) \in \mathbb{R}^{M \times M} \).

Here, (15a) follows from (13a) based on the fact that \( \det(I_N + \frac{1}{\sigma_c^2} \Sigma^H \Sigma) = \det(I_N) \).

(16) follows from the Hadamard inequality \([20]\) and the facts that \( \text{tr}(Q^{-1}) = \text{tr}((Q^{-1} - 1) = \text{tr}(Q)^{-1}) \) and \( \text{tr}(Q) = \text{tr}(Q^H V_c Q) = \text{tr}(Q) \) respectively. Next, we have the following proposition.

**Proposition 1.** The optimal solution to problem (P1.1) is a diagonal matrix with positive diagonal elements, i.e., \( Q = \text{diag}(p_1, p_2, ..., p_M) \), where \( p_i > 0, \forall i \in \{1, ..., M\} \).

**Proof.** First, it is evident that \( \tilde{Q} \) must be positive definite (or \( Q \succ 0 \)) in order for the maximum CRB constraint in (15b) to hold. Next, suppose that the optimal solution is a positive definite matrix \( Q^* \) that is not diagonal, and we construct an alternative solution \( Q^{**} = Q^* \circ I \), which is a diagonal matrix whose diagonal elements are identical to \( Q^* \). Then, we have

\[
\begin{align*}
\det(I_M + \frac{1}{\sigma_c^2} \Sigma^2 \tilde{Q}^*) & \leq \det(I_M + \frac{1}{\sigma_c^2} \Sigma^2 \tilde{Q}^{**}), \\
\text{tr}(\tilde{Q}^{**}) & = \text{tr}(\tilde{Q}^*) \leq P, \\
\text{tr}(\tilde{Q}^{**})^{-1} & \leq \text{tr}(\tilde{Q}^*)^{-1} \leq \bar{\Gamma},
\end{align*}
\]

(17) - (18)

where (16) follows from the Hadamard inequality \([20]\) and the inequality in (18) follows from \([21, \text{Lemma 1}] \). By combining (16), (17), and (18), it is clear that \( Q^{**} \) is also feasible for problem (P1.1) and achieves a no lower objective value than that by \( Q^* \). This contradicts the presumption that the non-diagonal matrix \( Q^* \) is optimal. This thus verifies that the optimal solution of \( Q \) to (P1.1) must be diagonal, i.e., \( Q = \text{diag}(p_1, p_2, ..., p_M) \), where \( p_i > 0, \forall i \in \{1, ..., M\} \).

Based on Proposition 1, (P1.1) is reformulated as

\[
\begin{align*}
(P1.2): \quad & \max_{\{p_i \geq 0\}} \sum_{i=1}^{r} \log_2 (1 + \frac{\lambda_i^2 p_i}{\sigma_c^2}) \\
& \text{s.t. } \sum_{i=1}^{M} \frac{1}{p_i} \leq \bar{\Gamma} \\
& \sum_{i=1}^{M} p_i \leq P.
\end{align*}
\]

(19a) - (19c)

Then, we find the optimal solution to (P1.2) as follows.

**Proposition 2.** The optimal solution to (P1.2) is denoted as

\[
p_i^{\text{opt}} = -t_1 + \sqrt{t_2 + t_3^2 + t_4^2} + t_2 + \sqrt{t_2^2 + t_3^2} \quad \forall i \in \{1, ..., r\},
\]

(20)

where

\[
t_1 = \frac{b_1}{3a} t_2 = \frac{27a^2 d_i - 9ab c + 2b^3}{54a^3}, t_3 = \frac{3ac - b^2}{9a^2},
\]

with \( a = \mu^{\text{opt}}, b_i = \sqrt{\mu^{\text{opt}} \sigma_i^2}, c = -\mu^{\text{opt}}, \) and \( d_i = -\mu^{\text{opt}} \sigma_i^2 \).

\[
p_i^{\text{opt}} = \sqrt{\mu^{\text{opt}} \sigma_i^2}, \quad i \in \{r + 1, ..., M\},
\]

(21)

Here, \( \mu^{\text{opt}} \) and \( \nu^{\text{opt}} \) are the optimal dual variables associated with the constraint in (19b) and (19c), respectively.

**Proof.** As problem (P1.2) is convex and satisfies the Slater’s condition, the strong duality holds between (P1.2) and its Lagrange dual problem \([19]\). The optimal solution to (P1.2) can be found by using the Lagrange duality method. Please refer to the technical report \([22, \text{Appendix A}] \) for details.

Finally, by combining (14) with Propositions 1 and 2, the optimal solution to (P1) is obtained as

\[
Q^{\text{opt}} = V_c Q^{\text{opt}^*} V_c^H,
\]

(22)

where \( Q^{\text{opt}} = \text{diag}(p_1^{\text{opt}}, ..., p_M^{\text{opt}}) \) with \( \{p_i^{\text{opt}}\} \) given in Proposition 2.

**A. Optimal Solution Structures**

To gain more insights, this subsection discusses the structure of the optimal transmit covariance solution \( Q^{\text{opt}} \) in (22). In particular, we express \( V_c \) as \( V_c = [V_c, \tilde{V}_c] \), where \( V_c \in \mathbb{C}^{M \times r} \) consists of the first \( r \) right singular vectors of the communication channel \( H_c \), and \( \tilde{V}_c \in \mathbb{C}^{M \times (M-r)} \) consists of the other \( M-r \) ones that span the null space of \( H_c \). In this case, the optimal transmit covariance solution \( Q^{\text{opt}} \) in (22) can be equivalently rewritten as

\[
Q^{\text{opt}} = V_c Q^{\text{opt}^*} V_c^H + \tilde{V}_c Q^{\text{opt}^*} \tilde{V}_c^H,
\]

(23)

where \( Q^{\text{opt}} = \text{diag}(p_1^{\text{opt}}, ..., p_r^{\text{opt}}) \) and \( Q^{\text{opt}} = \text{diag}(p_{r+1}^{\text{opt}}, ..., p_M^{\text{opt}}) \).

It is interesting to observe from (23) that the transmit covariance \( Q^{\text{opt}} \) is separated into two parts, including \( V_c Q^{\text{opt}^*} V_c^H \) lying in the range of \( H_c \) for both communication and sensing and \( \tilde{V}_c Q^{\text{opt}^*} \tilde{V}_c^H \) lying in the null space of \( H_c \) for dedicated sensing only. As the right singular matrix \( V_c = [V_c, \tilde{V}_c] \) actually diagonalizes the communication channel \( H_c \) into \( r \) parallel subchannels, it is clear that \( Q^{\text{opt}} = \text{diag}(p_1^{\text{opt}}, ..., p_r^{\text{opt}}) \) corresponds to the optimized power allocation over the \( r \) parallel communication subchannels, and \( Q^{\text{opt}} = \text{diag}(p_{r+1}^{\text{opt}}, ..., p_M^{\text{opt}}) \) corresponds to that over the other orthogonal \( M-r \) dedicated sensing subchannels.

**Proposition 3.** The optimal power allocation satisfies that

\[
p_1^{\text{opt}} \geq ... \geq p_r^{\text{opt}} \geq p_{r+1}^{\text{opt}} = ... = p_M^{\text{opt}} > 0.
\]

**Proof.** Please refer to \([22, \text{Appendix B}] \).

Proposition 3 shows that the power allocations over communication subchannels (i.e., \( \{p_i^{\text{opt}}\}_{i=1}^r \)) are monotonically increasing with respect to the subchannel gains \( \{\lambda_i^2\}_{i=1}^r \), which
is similar as the conventional water-filling power allocation in (11) for rate maximization. By contrast, the power allocations (i.e., \( \{p_{1i}^{\text{opt}}\}_{i=r+1}^M \)) are constant over dedicated sensing subchannels, similarly as that for CRB minimization (see (12)). As a result, the optimal power allocation for ISAC in Proposition 2 unifies the above two conventional power allocations for independent communication and sensing, respectively.

Finally, it is also interesting to discuss the optimal power allocation in the special case with \( P \to \infty \).

**Proposition 4.** When \( P \to \infty \), the optimal power allocation for problem (P1.2) is given by

\[
p_{1i}^{\text{opt}} = \begin{cases} \frac{1}{P} (P - \frac{(M-r)^2}{r}), & 1 \leq i \leq r \\ \frac{M-r}{P}, & r + 1 \leq i \leq M, \end{cases}
\]

in which the transmit power is split into two parts over communication and dedicated sensing subchannels, with equal power allocation within each part.

**Proof.** Please refer to [22, Appendix C].

V. NUMERICAL RESULTS

This section presents numerical results to validate the C-R region performance of the presented optimal transmit covariance, as compared to the following benchmark schemes.

- **Time switching:** The BS time switches the two transmit covariances \( Q^c \) and \( Q^e \) for rate maximization and CRB minimization, respectively. This design is only applicable when \( Q^c \) is of full rank, since otherwise CRB = CRB(\( Q^c \)) \( \to \infty \) follows (see Remark 1).

- **Power splitting with equal power allocation (EP):** Similarly as in (22), the BS sets the transmit covariance as \( Q^{\text{EP}} = V_cQ^cV_c^H \), in which \( Q^{\text{EP}} = \text{diag}(p_1^{\text{EP}}, \ldots, p_M^{\text{EP}}) \) denotes the power allocation. The BS splits the transmit power \( P \) into two parts, \( \beta P \) for the \( r \) communication subchannels and \( (1 - \beta)P \) for the \( M - r \) sensing subchannels, with \( 0 \leq \beta \leq 1 \) denoting the power splitting factor that is a parameter to be optimized. Following the equal power allocation, we have \( p_1^{\text{EP}} = \ldots = p_r^{\text{EP}} = \frac{\beta P}{r} \) and \( p_{r+1}^{\text{EP}} = \ldots = p_M^{\text{EP}} = \frac{(1 - \beta)P}{M - r} \). Notice that if \( r = M \), then we set \( \beta = 1 \).

- **Power splitting with strongest eigenmode transmission (SEM):** The BS sets \( Q^{\text{SEM}} = V_cQ^cV_c^H \), in which \( Q^{\text{SEM}} = \text{diag}(p_1^{\text{SEM}}, \ldots, p_M^{\text{SEM}}) \). The BS splits the transmit power \( P \) into two parts, \( \beta P \) for the dominant communication subchannel and \( (1 - \beta)P \) for the remaining \( M - 1 \) subchannels, with \( 0 \leq \beta \leq 1 \) to be optimized. In this case, we have \( p_1^{\text{SEM}} = \beta P \) and \( p_2^{\text{SEM}} = \ldots = p_M^{\text{SEM}} = \frac{(1 - \beta)P}{M - 1} \).

In the simulation, the BS-Tx, the BS-Rx, and the CU are each equipped with a uniform linear array (ULA) with half wavelength spacing between consecutive antennas. We consider Rician fading for the communication channel, i.e.,

\[ H_c = \sqrt{\frac{K_c}{K_c + 1}} H_{c,\text{los}} + \sqrt{\frac{1}{K_c + 1}} H_{c,\text{r}} \]

where \( H_{c,\text{r}} \) is a CSCG random matrix with zero mean and unit variance, and \( H_{c,\text{los}} = a_c^\dagger(\theta_c)\alpha_c(\theta_c) \). Here, \( a_c(\theta_c) \) and \( \alpha_c(\theta_c) \) denote the steering vectors at the CU receiver and the BS-Tx, and \( \theta_c = \theta_c^e - \theta_c^s \) denote the AoA at the CU and the AoD at the BS-Tx, respectively. Furthermore, the noise power \( \sigma^2 \) at the CU and \( \sigma^2 \) at the BS-Rx are both normalized to be unity, the length of symbols in CPI is \( L = 200 \), and the number of antennas at the BS-Rx is \( N_s = 12 \).

First, we consider the scenario where the number of transmit antennas at the BS-Tx is \( M = 8 \), the number of antennas at CU is \( N_c = 6 \), the Rician factor is \( K_c = 100 \) and the power budget at the BS-Tx is \( P = 800 \) (29.3 dB). In this case, we have \( r < M \), such that \( Q^c \) is rank-deficient and CRB(\( Q^c \)) \( \to \infty \). Fig. 2 shows the resultant C-R regions achieved by the optimal design and other benchmark schemes. It is observed that the C-R-region boundary by the optimal design outperforms those by the power splitting designs with equal power allocation and strongest eigenmode transmission. It is also observed that when the CRB is low, the three designs achieve similar C-R-region boundaries. Furthermore, as \( \Gamma \) increases, the C-R-region boundary by the optimal design is observed to approach the capacity without sensing (i.e., \( R_{\text{max}} \)). This is consistent with the result in Remark 1.

Fig. 3 shows the optimal power allocation in the case with \( M = 8 \), \( N_c = 6 \), and CRB threshold \( \Gamma = 0.0152 \), as compared to the water-filling and equal power allocations for rate maximization and CRB minimization, respectively. It is observed that the proposed optimal power allocations over the first six communication subchannels are monotonically non-increasing, which are higher than the constant power allocated to the next two sensing subchannels. This is consistent with Proposition 3. It is also observed that the proposed optimal power allocations over the first five communication subchannels are lower than the corresponding water-filling

![Fig. 2. The C-R region in the case with M = 8 and r = N_c = 6.](image2)

![Fig. 3. The power allocation in the case with M = 8 and r = N_c = 6, and \( \Gamma = 0.0152 \).](image3)
Fig. 4. The rate versus the SNR in dB with $M = 8$ and $r = N_c = 6$, and $\Gamma = 0.1$.

Fig. 5. The C-R region in the case with $r = M = N_c = 6$.  

power allocations, as more power should be allocated to other subchannels for facilitating the sensing. By contrast, the proposed optimal power allocations over the last three subchannels are higher than the corresponding water-filling power allocations, in order to meet the sensing requirement.

Fig. 4 shows the rate versus the signal-to-noise ratio (SNR) (or equivalently the transmit power $P$) in the case with $M = 8$, $N_c = 6$, and $\Gamma = 0.1$. It is observed that the optimal design performs best over the whole SNR regime. In the high SNR regime, the rate achieved by the power splitting with equal power allocation is observed to approach that by the optimal design. This can be explained based on Proposition 4. In the low SNR, the power splitting with strongest eigenmode transmission is observed to approach the optimal design.

Next, we consider another scenario where $M = N_c = 6$, $K_c = 20$, and $P = 800$. In this case, we have $r = 6$, and $Q^*_c$ is of full rank (as $P > P_0$ in Remark 1), such that $\text{CRB}_{C^*}$ is finite. Fig. 5 shows the resultant C-R regions. It is observed that the boundary point $(\text{CRB}_{C^*}, R_{\text{max}})$ exists and the C-R-region boundary achieved by the optimal design outperforms other benchmark schemes. The C-R-region boundary by time switching is observed to outperform the other two power splitting designs when the CRB value becomes large.

VI. CONCLUSION

This paper investigated the fundamental performance trade-off between the estimation CRB and the communication data rate in a point-to-point MIMO ISAC system with an extended radar target. We characterized the complete Pareto boundary of the resultant C-R region, by proposing the semi-closed-form optimal transmit covariance solution to a new CRB-constrained rate maximization problem. Numerical results were provided to show the C-R-region boundary achieved by the optimal design as compared to other benchmark schemes. We hope that this paper can provide insights on revealing the fundamental limits of MIMO ISAC.

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