A HILBERT $C^*$-MODULE FOR GABOR SYSTEMS

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Abstract. We construct Hilbert $C^*$-modules useful for studying Gabor systems and show that they are Banach Algebra under pointwise multiplication. For rational $ab < 1$ we prove that the set of functions $g \in L^2(\mathbb{R})$ so that $(g, a, b)$ is a Bessel system is an ideal for the Hilbert $C^*$-module given this pointwise algebraic structure. This allows us to give a multiplicative perturbation theorem for frames. Finally we show that a system $(g, a, b)$ yields a frame for $L^2(\mathbb{R})$ iff it is a modular frame for the given Hilbert $C^*$-module.

1. Intro

A Gabor system for a function $g \in L^2(\mathbb{R})$ is obtained by applying modulations and translations $(g(x - ka)e^{2\pi imb})$ to this function. In the study of these systems many results have been obtained by using the function-valued inner product

$$\langle f, g \rangle_1(x) = \sum_{k \in \mathbb{Z}} f(x - k)g(x - k).$$

Many of the results from Hilbert space theory can be reproduced in this setting. However, one of the most basic properties of a Hilbert space is lost. Namely, for $f, g, h \in L^2(\mathbb{R})$ the function $\langle f, g \rangle_1(x)h(x)$ need not be be in $L^2(\mathbb{R})$. This brings us to the study of Hilbert $C^*$-Modules (HCM). A Hilbert $C^*$-Module is the generalization of the Hilbert space in which the inner product maps into a $C^*$-algebra. By restricting to a subspace of $L^2(\mathbb{R})$ we produce a Hilbert $C^*$-Module for the inner product above containing the functions $g$ which are of most interest in the study of Gabor systems.

Our study of this HCM and its relation to Gabor systems is organized in the following manner. In section 2 we provide some of the known results involving Gabor systems and Gabor frames along with some of the basics regarding HCM’s. In section 3 we define our HCM’s and develop a list of basic properties which relate to Gabor systems. In section 4 we show that our HCM is a Banach algebra under pointwise multiplication and show that the set of functions with a finite upper frame bound form an ideal for this algebra. In addition we give a Balian-Low type theorem in the positive direction and show that one can apply an additive perturbation result of Christenson and Heil to obtain a multiplicative one in our Banach algebra. In section 5 we make the connection between modular frames for our HCM (defined by Frank and Larson) and Gabor frames for $L^2(\mathbb{R})$. We show that the system $(g, a, b)$ is a frame for $L^2(\mathbb{R})$ iff the translates of $g$, $\{T_{ka}g\}_{k \in \mathbb{Z}}$, form a modular frame for our HCM.
The authors would like to thank Michael Frank for his many helpful suggestions and references regarding Hilbert C*-modules. Throughout all sums are considered to be over Z or Z^α unless otherwise stated.

2. Preliminaries

In 1952 Duffin and Schaeffer defined frames:

**Definition 2.1.** A sequence \((f_n)_{n \in \mathbb{Z}}\) of elements of a Hilbert space \(H\) is called a frame if there are constants \(A, B > 0\) such that

\[
A\|f\|^2 \leq \sum_{n \in \mathbb{Z}} |\langle f, f_n \rangle|^2 \leq B\|f\|^2, \quad \text{for all } f \in H.
\]

For a function \(f \in L^2(\mathbb{R})\) and \(a, b \in \mathbb{R}\) define the operators:

- **Translation:** \(T_a f(x) = f(x - a)\),
- **Modulation:** \(E_b f(x) = e^{2\pi ibx} f(x)\),

**Definition 2.2.** If \(a, b \in \mathbb{R}\) and \(g \in L^2(\mathbb{R})\) we let \((E_{mb}T_{na}g) = g_{m,n}\) and call \(\{(E_{mb}T_{na}g)\}_{m,n \in \mathbb{Z}}\) a Gabor system (also called a Weyl-Heisenberg system) and denote it by \((g, a, b)\). If this system is a frame then we call it a Gabor frame. We denote by \((g, a)\) the family \((T_{na}g)_{n \in \mathbb{Z}}\).

Now we wish to define some important operators associated with a Gabor system \((g, a, b)\). Let \((e_{m,n})\) be an orthonormal basis for \(L^2(\mathbb{R})\). We call the operator \(T_g : L^2(\mathbb{R}) \to L^2(\mathbb{R})\) given by \(T_ge_{m,n} = E_{mb}T_{na}g\) the **preframe operator** associated with \((g, a, b)\). The adjoint, \(T_g^*\), is called the **frame transform** and for \(f \in L^2(\mathbb{R})\) and \(m, n \in \mathbb{Z}\) we have \(\langle T_g^* f, e_{m,n} \rangle = \langle f, T_g(e_{m,n}) \rangle = \langle f, (E_{mb}T_{na}g) \rangle\). Thus

\[
T_g^* f = \sum_{m,n} \langle f, E_{mb}T_{na}g \rangle e_{m,n} \quad \text{and} \quad \|T_g^* f\|^2 = \sum_{m,n} |\langle f, E_{mb}T_{na}g \rangle|^2 \quad \text{for all } f \in L^2(\mathbb{R}).
\]

It follows that the preframe operator is bounded if and only if \((g_{m,n})\) has a finite upper frame bound \(B\). Finally we define the **frame operator associated with** \((g, a, b)\):

\[
S(f) = T_gT_g^*(f) = \sum_{m,n \in \mathbb{Z}} \langle f, E_{mb}T_{na}g \rangle (E_{mb}T_{na}g).
\]

It is known that if \((g, a, b)\) is a frame then \(S\) is a positive, invertible operator. We will be interested in when there is a finite upper frame bound for a Gabor system.

**Definition 2.3.** Let \(\{\psi_n\}_{n \in \mathbb{Z}}\) be an orthonormal basis for a Hilbert space \(H\). A Bessel sequence is any sequence \(\{\phi_n\}_{n \in \mathbb{Z}}\) (i.e. not necessarily one generated by the operators \(E_{mb}\) and \(T_{na}\)) so that \(T(\sigma) = \sum_{n \in \mathbb{Z}} \langle \sigma, \psi_n \rangle_H \phi_n\) defines a bounded operator from \(H\) to \(H\). For fixed \(a\) and \(b\) we denote the set of functions \(g\) so that \(\{E_{mb}T_{na}g\}\) is a Bessel sequence by \(X_Z\).
We will use a bracket product notation compatible with Gabor systems. We define the \textit{\(a\)-inner product} for \(f, g, \in L^2(\mathbb{R})\) and \(a \in \mathbb{R}^+\):

\[
\langle \, f, g \, \rangle_a = \langle \, f, g \, \rangle_a(x) = \sum_k f(x - ka)g(x - ka) \quad \text{and} \quad \|f\|_a(x) = \sqrt{\langle \, f, f \, \rangle_a(x)}.
\]

For a detailed development of this \(a\)-inner product which includes two forms of a Riesz Representation we refer the reader to [2] but here we give a brief list of some of the properties which will be useful later.

**Proposition 2.4.** For \(f, g \in L^2(\mathbb{R})\)

(a) \(\langle \, f, f \, \rangle_a \geq 0 \) and \( = 0 \) iff \(f = 0\)

(b) \(\langle \, f, g \, \rangle_a = \langle \, g, f \, \rangle_a\)

(c) \(\langle \, \phi f, g \, \rangle_a = \phi \langle \, f, g \, \rangle_a\) for all periodic functions such that \(f, \phi f \in L^2(\mathbb{R})\)

(d) \(\langle \, f, h + g \, \rangle_a = \langle \, f, h \, \rangle_a + \langle \, f, g \, \rangle_a\)

(e) \(| \langle \, f, g \, \rangle_a |^2 \leq \|f\|_a\|g\|_a \)

(f) \(\langle \, f, g \, \rangle_a \in L^1(0, a]\)

(g) \(\langle \, f, g \, \rangle_a\) is \(a\)-periodic on \(\mathbb{R}\)

(h) \(\|f\|_{L^2(\mathbb{R})} = \int_a^0 \langle \, f, f \, \rangle_a dx\)

(i) \(\|f + g\|_a \leq \|f\|_a + \|g\|_a\)

While we see that the \(a\)-inner product has many of the same properties as the usual inner product, there is one major difference as we mentioned in the introduction. Consider \(f, g, h \in L^2(\mathbb{R})\) where

\[
f(x) = g(x) = \frac{1}{x^3}1_{[0,1]}, \quad h(x) = 1_{[0,1]}.
\]

Then \(\langle \, f, g \, \rangle_1(x)h(x) = \frac{1}{x^4}1_{[0,1]}\) and this function is clearly not in \(L^2(\mathbb{R})\).

We say that an operator \(L : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})\) is \(a\)-factorable if \(L(\phi f) = \phi L(f)\) for all bounded \(a\)-periodic functions \(\phi\). The operators associated with Gabor systems are \(\frac{1}{b}\)-factorable and hence have the following representations which we refer to as \textit{compressions} since we compress the modulation into the \(\frac{1}{b}\)-inner product.

**Proposition 2.5.** [2] If \(\|g\|_{\frac{1}{b}} \leq B \) a.e., then the frame operator and the preframe operator have the following compressions

\[
T_g(f) = \sqrt{\frac{1}{b}} \sum_k \langle \, f, e_k \, \rangle_{\frac{1}{b}} T_{ka}g, \quad T^*_g(f) = \sqrt{\frac{1}{b}} \sum_k \langle \, f, T_{ka}g \, \rangle_{\frac{1}{b}} e_k \quad \text{and}
\]

\[
S_g(f) = \frac{1}{b} \sum_k \langle \, f, T_{ka}g \, \rangle_{\frac{1}{b}} T_{ka}g
\]

where \(e_k = T_{\frac{1}{b}}1_{[0,\frac{1}{b})}\).
Definition 2.6. For \( \lambda > 0 \) the Zak transform of a function \( f \in L^2(\mathbb{R}) \) is

\[
Z_\lambda(f)(t,v) = \lambda^{1/2} \sum_{k \in \mathbb{Z}} f(\lambda(t - k))e^{2\pi ikv}, \quad t,v \in \mathbb{R}.
\]

We may interpret this as a unitary operator from \( L^2(\mathbb{R}) \) to \( L^2[0,1]^2 \). Since we are frequently interested in the case where \( \lambda = 1 \), we will also use the notation \( Z_1 = Z \).

Before we present some of the basics regarding Hilbert \( C^* \)-modules we give two very important results for \((g,1,1)\) systems. The first is due to Daubechies and the second was proved independently by Balian and Low. A nice treatment of both can be found in [6].

Theorem 2.7. [4] The system \((g,1,1)\) yields a frame with frame bounds \( A \) and \( B \) iff

\[
A \leq |Z(f)(t,v)| \leq B \quad \text{for a.e.} \quad (t,v) \in [0,1]^2
\]

Theorem 2.8. (Balian-Low) If the system \((g,1,1)\) yields a frame, then

\[
\|tg(t)\|_{L^2(\mathbb{R})} \|\gamma \hat{g}(\gamma)\|_{L^2(\mathbb{R})} = +\infty
\]

Now we give some of the basics regarding Hilbert \( C^* \)-modules. We refer the reader to [11] for a thorough treatment of this subject. Essentially, a Hilbert \( C^* \)-module is the generalization of a Hilbert space in which the inner product takes values in a \( C^* \)-algebra instead of the complex numbers.

Definition 2.9. Let \( A \) be a \( C^* \)-algebra. An inner product \( A \)-module is an \( A \)-module \( \mathcal{M} \) with a mapping \( \langle \cdot, \cdot \rangle_A : \mathcal{M} \times \mathcal{M} \to A \) so that for all \( x,y,z \in \mathcal{M} \) and \( a \in A \)

a) \( \langle x,x \rangle_A \geq 0 \) (as an element of \( A \));
b) \( \langle x,x \rangle_A = 0 \) iff \( x = 0 \);
c) \( \langle x,y \rangle_A = \langle y,x \rangle^*_A \);
d) \( \langle ax,x \rangle_A = a \langle x,x \rangle_A \);
e) \( \langle x + y,x \rangle_A = \langle x,z \rangle_A + \langle y,z \rangle_A \).

If the space \( \mathcal{M} \) is complete with respect to the norm \( \|x\|^2_A = \|\langle x,x \rangle_A\|_A \), then we say \( \mathcal{M} \) is a Hilbert \( C^* \)-module with respect to \( A \) or more simply a Hilbert \( A \)-module. When referring to a Hilbert \( C^* \)-module we will also use the abbreviation HCM.

We provide a very simple example.

Example 2.10. Let \( A \) be a \( C^* \)-algebra. The \( A \)-valued inner product defined by \( \langle a,b \rangle_A = ab^* \) makes \( A \) a Hilbert \( A \)-module.

3. The HCM

Given the way we have defined the \( a \)-inner product and the Hilbert \( C^* \)-module it is natural to try to use the \( a \)-inner product to turn \( L^2(\mathbb{R}) \) into a HCM. Unfortunately, the \( a \)-inner product maps \( L^2(\mathbb{R}) \) into \( L^1[0,a] \) which is not a \( C^* \)-algebra. For this reason we consider the subspace of \( L^2(\mathbb{R}) \) consisting of those functions which are mapped into \( L^\infty[0,a] \). This
yields the following Lebesgue-Bochner space.

For \( b > 0 \) let \( L^\infty_b(\ell_2) \) be defined to be the set of measurable functions \( f : \mathbb{R} \rightarrow \mathbb{C} \) for which the norm

\[
\| f \|^2_{L^\infty_b(\ell_2)} = \text{esssup}_{[0,\frac{1}{b}]} \sum_k |f(x - \frac{k}{b})|^2
\]

is finite. Proposition 2.4 gives us that \( L^\infty_b(\ell_2) \) is a Hilbert \( C^* \)-Module (HCM) over the \( C^* \)-algebra \( L^\infty[0,\frac{1}{b}] \) with \( C^* \)-valued inner product and \( C^* \)-valued norm

\[
\langle f, g \rangle_{\frac{1}{b}}(x) = \sum_k f(x - \frac{k}{b})g(x - \frac{k}{b}) \quad \text{and} \quad \| f \|_{\frac{1}{b}}(x) = \left( \sum_k |f(x - \frac{k}{b})|^2 \right)^{1/2}.
\]

We begin with a few basic remarks:

1. Weak-* convergence. Because we are constructing our HCM with Gabor systems in mind we will only require the \( C^* \)-valued inner product to converge weak-* in \( L^\infty[0,1] \). We motivate this by considering the case \( a = b = 1 \). In this case we have the classification Theorem 2.7. With this theorem it is easy to see that the system \((g,1,1)\) with \( g = \sum_{k=0}^\infty 1_{[k+\frac{1}{2k+1},k+\frac{1}{2k+1}]} \) produces a Gabor frame because \( |Z(g)(t,v)| = 1 \) everywhere. However the series \( \langle g, g \rangle_{\frac{1}{b}} \) does not converge in norm in \( L^\infty[0,1] \). If we only require that the series converge in the weak* topology we avoid this problem. This assumption results in frames being what Frank and Larson [7] have termed nonstandard modular frames. We will address this more in section 5.

2. Proposition 2.4 (h) yields the following inequality which shows that \( L^\infty_b(\ell_2) \) embeds continuously in \( L^2(\mathbb{R}) \)

\[
\| f \|^2 = \int_0^{\frac{1}{b}} |\langle f, f \rangle_{\frac{1}{b}}| dx \leq \sup_{[0,1]} |\langle f, f \rangle_{\frac{1}{b}}| = \| f \|^2_{L^\infty_b(\ell_2)}.
\]

Finally, since \( L^\infty_b(\ell_2) \) contains continuous functions of compact support, we see it is norm dense in \( L^2(\mathbb{R}) \).

3. \( L^\infty_b(\ell_2) \neq L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \). We illustrate this for \( L^\infty_1(\ell_2) \) and note that it is easily generalized. Let \( f_k = 1_{[k,k+\frac{1}{k}]} \) and \( f = \sum_{k=0}^1 f_k \). Then \( f \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) but \( f \notin L^\infty_1(\ell_2) \) for \( \langle f, f \rangle_1(x) \) is unbounded near zero.

4. We interpret a basic result about Gabor frames in this space. A necessary condition for \( \{g_{m,n}\}_{m,n \in \mathbb{Z}} \) to be a Gabor frame in \( L^2(\mathbb{R}) \) with upper frame bound \( B \) is that \( \text{esssup}_{[0,\frac{1}{b}]} \sum_k |g(x - \frac{k}{b})|^2 \leq B \) a.e.. In other words, if \( g_{m,n} \) is a frame for \( L^2(\mathbb{R}) \) with upper frame bound \( B \) then \( g \) in the \( B \)-ball of \( L^\infty_1(\ell_2) \).
The Wiener amalgam spaces $W(L, G)$ are defined using the norm of two spaces where the local behavior is determined by $(L)$ and the global behavior by $(G)$. For example,

$$W(L^p, \ell^q) = \left\{ f : \| f \|_{W(L^p, \ell^q)} = \left( \sum_k \| f \cdot e_k \|_p^q \right)^{\frac{1}{q}} < \infty \right\}.$$ 

We will refer to $W(\mathbb{R}) = W(L^\infty, \ell^1)$ as the Wiener Algebra. The Segal algebra may be viewed as a specific Wiener amalgam space:

$$S_0(\mathbb{R}) = W(C_0, \ell^1) = \{ f \in W(L^\infty, \ell^1) : f \text{ is continuous} \}.$$ 

These algebras have been used effectively in the study of Gabor systems by Benedetto, Heil, Walnut, Feichtinger, Zimmerman and Strohmer (See chapters 2, 3 and 8 of [B]). From the inequality

$$\| g \|_{L^\infty(\ell^2)} \leq \| g \|_{W(L^\infty, \ell^2)} \leq \| g \|_{W(L^\infty, \ell^1)}$$

we conclude that the Segal algebra and the Weiner algebra embed in $L^\infty(\ell^2)$. That is, for any $g$ in $S_0(\mathbb{R})$ we have

$$\| g \|_{L^\infty(\ell^2)} \leq \| g \|_{W(\mathbb{R})} \leq \| g \|_{S_0(\mathbb{R})}$$

One may view the Zak transform here as the analogue of the Fourier transform when $a = b = 1$. We give the usual notation for this but add the representation with respect to the inner product for our HCM. We should point out that although these transforms agree for $t, v \in [0, 1]^2$, the latter is periodic in $t$ and $v$ while the former yields only a quasi periodicity.

$$Z(f)(t, v) = \sum_k f(t + k)e^{-2\pi ikv} = \sum_k \langle f, e_k \rangle_1(t)e^{-2\pi ikv}$$

where $e_k = T_k 1_{[0, 1]}$.

$$\hat{f}(v) = \int_{\mathbb{R}} f(t)e^{-2\pi itv}dt = \int_0^1 Z(f)(t, v)e^{-2\pi itv}dt$$

The Fourier transform is not a bounded linear operator on $L^\infty_1(\ell_2)$. Indeed, if we let $f(x) = 1_{[0, 1]} \frac{1}{x^\frac{1}{4}}$ it can be shown that the absolute value of the Fourier transform is bounded by the function

$$g(x) = \begin{cases} 2 & x \in [-1, 1] \\ \frac{2}{x^\frac{1}{4}} & \text{otherwise} \end{cases}.$$ 

The function $g(x)$ is clearly in $L^\infty_1(\ell_2)$ which implies $\hat{f}$ is also. However, since the original function $f$ is not even bounded, it is clearly not in $L^\infty_1(\ell_2)$. Thus by the properties of the Fourier transform we see that $\hat{f}$ is not in $L^\infty_1(\ell_2)$.
4. Bessel Systems

Before we examine the role Bessel systems play in our HCM’s we show that these HCM’s are Banach algebras under pointwise multiplication.

Proposition 4.1. The space $L^\infty_{1/\pi} (\ell_2)$ is a Banach algebra under pointwise multiplication.

Proof. The justification can be made with one inequality. Note that the inequality below is not derived by Cauchy-Shwarz, but by the fact that all the terms are positive and the left hand side is just the diagonal of the product of the sums on the right.

$$\sum_{k} |fg(x - \frac{k}{b})|^2 \leq \sum_{k} |f(x - \frac{k}{b})|^2 \sum_{k} |g(x - \frac{k}{b})|^2.$$ 

Unfortunately $L^\infty_{1/\pi} (\ell_2)$ is not a $C^*$-algebra. It is also clear that this Banach Algebra has no identity since $f(x) = 1_R$ is clearly not in $L^\infty_{1/\pi} (\ell_2)$.

Now we turn our attention to the functions that yield Bessel systems. We show that the functions which produce a finite upper frame bound for $a = b = 1$ form a Banach space which may be embedded in $L^\infty_{1/\pi} (\ell_2)$. For now let us refer to this as the Zak space, denoted $X_Z$. More formally we define the space,

$$X_Z = \{ f \mid f \in L^2(\mathbb{R}) \text{ and } \| f \|_{X_Z} = \text{esssup}_{t,v \in [0,1]} |Z(f)(t,v)| < \infty \}.$$ 

Since $Z$ is a unitary operator from $L^2(\mathbb{R})$ to $L^2[0,1]^2$ it is easy to see that $\| \cdot \|_{X_Z}$ is actually a norm. The essential inequality in the following proposition is well known and is usually stated in the form $G_0 = \sum_k |g(x - k)|^2 < B$. We prove it here for completeness and for the development of the Zak transform on $L^\infty_{1/\pi} (\ell_2)$.

Proposition 4.2. The Zak space embeds continuously in $L^\infty_{1/\pi} (\ell_2)$, that is

$$\| g \|_{L^\infty_{1/\pi} (\ell_2)} \leq \| g \|_{X_Z} \text{ for all } g \in X_Z.$$ 

Proof. To see that $X_Z$ embeds in $L^\infty_{1/\pi} (\ell_2)$ we look at what the Zak transform does on $L^\infty_1 (\ell_2)$. Let us define

$$L^\infty L^2[0,1]^2 = \left\{ F(t,v) \mid \sup_{t \in [0,1]} \left( \int_0^1 |F(t,v)|^2 dv \right)^{\frac{1}{2}} < \infty \right\}.$$ 

Now let $f \in L^\infty_1 (\ell_2)$ so

$$\| Z(f)(t,v) \|_{L^\infty L^2[0,1]^2} = \sup_{t \in [0,1]} \left( \int_0^1 \left| \sum_{k} \langle f, e_k \rangle_1 e^{-2\pi i k v} \right|^2 dv \right)^{\frac{1}{2}} \leq \| f \|_{L^\infty (\ell_2)}$$
It is easy to see from here that the Zak transform is an isometry between these two spaces. Further, since the $L^2[0,1]$ norm is bounded by the $L^∞[0,1]$ norm, we see that $\|g\|_{L^∞(ℓ_2)} \leq \|g\|_{X_Z}$ for $g \in X_Z$.

\[ \square \]

Our next example shows that $X_Z \neq L^∞_1(ℓ_2)$.

**Example 4.3.** There exists $g \in L^∞_1(ℓ_2)$ with $g \notin X_Z$

**Proof.** Consider the function $g = ∑_{n=1}^{∞} \frac{e_n}{n}$. Clearly $g \in L^∞_1(ℓ_2)$. Now Corollary 3.7 from [1] states that a positive real-valued function is in $X_Z$ iff $\sum_{k} |\langle g, T_k g \rangle_1| \leq ∞$. However computation shows that $\langle g, T_k g \rangle_1 = (1/k) ∑_{n=1}^{k} \frac{1}{n}$ for the above $g$. These are square summable but not summable.

As we mentioned a detailed study of the Balian and Low theorem (Theorem 2.8) can be found in [6]. In this study an amalgam version of the BLT theorem for the Segal algebra $S_0$ due to Heil is given.

**Theorem 4.4.** [9] Let $g \in L^2(ℝ)$. If $(g, 1, 1)$ forms a frame for $L^2(ℝ)$, then $g \notin S_0$ and $\hat{g} \notin S_0$

A direct Corollary of the continuous embedding yields the positive Balian-Low type result which in light of remark 7 from the previous section is non-trivial.

**Corollary 4.5.** If the system $(g, 1, 1)$ is a frame then $g \in L^∞_1(ℓ_2)$ and $\hat{g} \in L^∞_1(ℓ_2)$

The proof follows directly from the theorem above and the fact that $(g, 1, 1)$ forms a frame if and only if $(\hat{g}, 1, 1)$ does. This, however, does not characterize the space $X_Z$. We showed in the proof of Proposition 4.2 that $Z$ is an isometry from $L^∞_1(ℓ_2)$ to $L^∞ L^2[0,1]^2$. Recall that $Z(\hat{g}) = e^{2πi tu} Z(g)(v,-t)$. It suffices to show that there exists a function $F(t, v) \in L^∞ L^2[0, 1]^2$ so that $F(v, -t) \in L^∞ L^2[0, 1]^2$ yet $F(t, v) \notin L^∞[0,1]^2$. The function $F(t, v) = v^{-γ}$ is such a function. Given that the Segal algebra (see remark 5 section 3) embeds in $L^∞_1(ℓ_2)$ we may interpret these results as follows. If $(g, 1, 1)$ forms a frame then $g, \hat{g} \in L^∞_1(ℓ_2) \setminus S_0$.

Now we show that the functions with finite upper frame bound have an algebraic connection to $L^∞_1(ℓ_2)$. We recall an algebraic term. If $M$ is a ring without identity we say the square ideal, $M^2$ is the ideal of the ring formed by taking finite sums of products from $M$. That is

$$M^2 = \left\{ \sum_{i=1}^{n} x_iy_i : x_i, y_i \in M \right\}.$$

The theorem below not only shows that the Zak space is an ideal in $L^∞_1(ℓ_2)$ but it contains the square ideal.
Theorem 4.6. For any \( f, g \in L_1^\infty(\ell_2) \) their product \( fg \) has a finite upper frame bound. In particular, the Zak space is an ideal for \( L_1^\infty(\ell_2) \).

Proof. Let \( f, g \) in \( L^\infty(\ell_2) \) and consider \( Z(fg) \).

\[
|Z(fg)(t,v)|^2 = \left| \sum_k \langle fg, e_k \rangle_1 (t)e^{-2\pi i k v} \right|^2 \leq \sum_k |f(t+k)|^2 \sum_k |g(t+k)|^2 \leq \|f\|_{L_1^\infty(\ell_2)} \|g\|_{L_1^\infty(\ell_2)}.
\]

As usual the argument may be mimicked in cases where \( ab \) is rational. To get the most general rational case one probably needs to use the Zak matrices of Zibulski-Zeevi which may be found in [5] section 1.5. We present the case where \( a = \frac{1}{2}, b = 1 \) and leave the rest to the reader.

Theorem 4.7. If \( f, g \) in \( L_1^\infty(\ell_2) \), then \( (fg, \frac{1}{2}, 1) \) is a Bessel system.

Proof. To see this we apply a standard technique when using the Zak transform. Applying the Zak transform to the frame operator of the system \((g, \frac{1}{2}, 1)\) we get:

\[
Z(S_g)(f) = Z(\sum_k \langle f, T_{\frac{1}{2}} g \rangle_1 T_{\frac{1}{2}} g)
\]

\[
= Z(\sum_k \langle f, T_k g \rangle_1 T_k g + \sum_k \langle f, T_k T_{\frac{1}{2}} g \rangle_1 T_k (T_{\frac{1}{2}} g))
\]

\[
= Z(\sum_k \langle f, T_k g \rangle_1 T_k g) + Z(\sum_k \langle f, T_k T_{\frac{1}{2}} g \rangle_1 T_k (T_{\frac{1}{2}} g))
\]

\[
= Z(f) \left( |Z(g)|^2 + |Z(T_{\frac{1}{2}} g)|^2 \right).
\]

This implies that the system \((g, \frac{1}{2}, 1)\) has a finite upper frame bound iff \( |Z(g)|^2 + |Z(T_{\frac{1}{2}} g)|^2 < B \). In view of this and the proposition above all we need to show is that \( |Z(T_{\frac{1}{2}} fg)| \) bounded. This follows easily from the fact that \( \|T_{\frac{1}{2}} f\|_{L_1^\infty(\ell_2)} = \|f\|_{L_1^\infty(\ell_2)} \)

One may interpret this theorem another way. We state it as a corollary in the case \( a = b = 1 \).

Corollary 4.8. For any \( g \in L_1^\infty(\ell_2) \) and \( x \in \mathbb{R} \) the operator

\[
\mathcal{V}_g^1(f)(t,v,x) = Z(fT_x g)(t,v)e^{-2\pi itv}
\]

is a bounded operator from \( L_1^\infty(\ell_2) \) to \( L^\infty(\mathbb{R}^3) \).

Proof. First we point out that because we are considering the periodic extension of the Zak transform in \( t, v \in [0, 1]^2 \) we really only need \( [0, 1]^2 \times \mathbb{R} \) in place of \( \mathbb{R}^3 \). Given \( x \in \mathbb{R} \) we have

\[
\|T_x g\|_{L_1^\infty(\ell_2)} = \|g\|_{L_1^\infty(\ell_2)}.
\]

The result follows from the above inequality.
Let us point out that in many ways $V^1_g(f)(t,v,x)$ resembles the short time Fourier transform

$$V_g(f,x) = \int_R f(t)g(t-x)e^{-2\pi itv}dt$$

For this reason we term this operator the \textbf{windowed Zak transform}.

We have a simple corollary for producing frames in $L_1^\infty(\ell_2)$. Again we state it here for the case $a = b = 1$ but it is easily generalized to many rational cases.

\textbf{Corollary 4.9.} If $f \in L_1^\infty(\ell_2)$ and $\inf |Z(f^2)| > A$ then $(f^2,1,1)$ is a frame.

Finally we end the section by applying the following perturbation theorem of Christensen and Heil to produce frames.

\textbf{Theorem 4.10.} [3] Let $H$ be a Hilbert space, $\{f_i\}_{i=1}^\infty$ be a frame with bounds $A,B$ and let $\{g_i\}_{i=1}^\infty \subset H$. If there exists $R < A$ so that

$$\sum_{i=1}^\infty |\langle h, f_i - g_i \rangle_H|^2 \leq R \|h\|^2$$

for all $h \in H$,

then $\{g_i\}_{i=1}^\infty$ is a frame with bounds $A(1 - \sqrt{\frac{R}{A}})^2$ and $B(1 - \sqrt{\frac{R}{A}})^2$.

In the Gabor case this means that if $(g,a,b)$ is a frame then so is $(f,a,b)$ if the system $(f - g,a,b)$ has an upper frame bound less than that of $(g,a,b)$. We use this theorem and one of the inequalities above to produce a multiplicative perturbation result.

\textbf{Proposition 4.11.} Let $ab < 1$ and $ab$ rational. If $(g,a,b)$ produces a frame with bounds $A,B$ and if $(f - 1) \in L_1^\infty(\ell_2)$ with $\|f - 1\|_{L_1^\infty(\ell_2)}^2 \leq \frac{R}{B}$, then $(fg,a,b)$ is a frame.

\textbf{Proof.} As usual, to highlight the essential components of the argument we prove the case $a = b = 1$. Let us point out that $f - 1$ need not be in $L_1^\infty(\ell_2)$ even if $f$ is, since $1 \notin L_1^\infty(\ell_2)$. By the theorem above it is enough to show that $fg - g$ has a finite upper frame bound less than $A$. In view of Theorem 2.7 it is enough to show $|Z(fg - g)| < R$.

$$|Z(fg - g)(t,v)|^2 = \left| \sum_k \langle (f - 1)g, e_k \rangle_1(t)e^{-2\pi ikv} \right|^2 \leq \sum_k |(f - 1)(t + k)|^2 \sum_k |g(t + k)|^2 \leq \|f - 1\|_{L_1^\infty(\ell_2)}^2 \|g\|_{L_1^\infty(\ell_2)}^2 \leq \frac{R}{B} \cdot B.$$

Where the fact that $\|g\|_{L_1^\infty(\ell_2)} \leq B$ follows from Proposition 4.2. In this case we get frame bounds $A(1 - \sqrt{\frac{R}{A}})^2$ and $B(1 - \sqrt{\frac{R}{A}})^2$. 

$\square$
5. \(a\)-FRAMES AND MODULAR FRAMES

An immediate concern is whether the operators \(\mathcal{T}_g\) and \(\mathcal{T}_g^*\) are bounded operators on \(L_\mathbb{T}^\infty(\ell_2)\) if they are bounded operators on \(L^2(\mathbb{R})\). The answer is yes and follows from Proposition 5.9 of [2] which we state below.

**Proposition 5.1.** If \(L : L^2(\mathbb{R}) \to L^2(\mathbb{R})\) is an \(a\)-factorable operator, then \(L\) is bounded iff for all \(f \in L^2(\mathbb{R})\)

\[
\|L(f)\|_a(t) \leq \|L\| \|f\|_a(t)
\]

where \(\|L\| = \sup_{\|f\|_{L^2(\mathbb{R})} = 1} \|L(f)\|_{L^2(\mathbb{R})}\).

Given the compressed representations of \(\mathcal{T}_g\) and \(\mathcal{T}_g^*\) it is easy to see that they are \(\frac{1}{b}\)-factorable on \(L^2(\mathbb{R})\). Since the inequality in the proposition holds for all \(f \in L^2(\mathbb{R})\) it certainly holds for \(f \in L_\mathbb{T}^\infty(\ell_2)\) and we have our result by taking supremums on both sides.

We now introduce two definitions both of which are abstractions of frames. The first is that of an \(a\)-frame for \(L^2(\mathbb{R})\) and the second is the analogue of a frame in a Hilbert \(C^*\)-module.

**Definition 5.2.** [2] A sequence \(\{f_n\} \in L^2(\mathbb{R})\) is an \(a\)-frame for \(L^2(\mathbb{R})\) if there exists \(A, B\) such that for all \(f \in L^2(\mathbb{R})\)

\[
A \|f\|_a^2(x) \leq \sum_n |\langle f, f_n \rangle_a(x)|^2 \leq B \|f\|_a^2(x) \text{ a.e.}
\]

(5.1)

Since we are dealing with frames for different spaces we will add the term “modular” to Frank and Larson’s definition of a frame for a HCM.

**Definition 5.3.** [2] Let \(\mathcal{M}\) be a HCM over the unital \(C^*\)-algebra \(A\). A sequence \(\{x_n\} \in \mathcal{M}\) is said to be a modular frame for \(\mathcal{M}\) if there are real constants \(C, D > 0\) such that

\[
C \langle x, x \rangle_A \leq \sum_n \langle x, x_n \rangle_A \langle x_n, x \rangle_A \leq D \langle x, x \rangle_A
\]

for all \(x \in \mathcal{M}\). If the sum in the middle of the inequality always converges in norm this is referred to as a standard modular frame and if the middle sum converges weakly for some \(x \in \mathcal{M}\) we will call this a non-standard modular frame.

We point out here that our results are in a different direction than the “standard” results given by Frank and Larson. Since we only require the sum

\[
\sum_k f(x - \frac{k}{b})g(x - \frac{k}{b})
\]

to converge weakly in \(L^\infty[0, \frac{1}{b}]\), we are dealing with the nonstandard case. Most of their examination dealt with standard modular frames in HCM’s which were at worst countably generated. Neither of these conditions is met in our case. Now we give the result of Casazza and Lammers regarding the connection between \(a\)-frames and Gabor frames.
Theorem 5.4. Let $g_n(x) = g(x-na)$. Then $\{g_{m,n}\}_{m,n \in \mathbb{Z}}$ is a Gabor frame for $L^2(\mathbb{R})$ with frame bounds $A, B$ iff $\{g_n\}$ is a $\frac{1}{b}$-frame for $L^2(\mathbb{R})$.

This is somewhat surprising because this says that the frame inequality holds pointwise for the bracket product. We use this result to show the connections between Gabor frames and modular frames.

Theorem 5.5. For $g \in L^2(\mathbb{R})$, $\{g_{m,n}\}_{m,n \in \mathbb{Z}}$ is a Gabor frame for $L^2(\mathbb{R})$ iff $\{g_n\}_n$ is a non-standard modular frame for $L^\infty_\frac{1}{b}(\ell_2)$.

Proof. $\Rightarrow$ This follows directly from the theorem above since $L^\infty_\frac{1}{b}(\ell_2)$ is a subspace of $L^2(\mathbb{R})$.

$\Leftarrow$ For any $f \in L^2(\mathbb{R})$ consider $f_0 = \frac{f}{\|f\|_b^2}$. Then $\langle f_0, f_0 \rangle \leq 1$ and hence $f_0 \in L^\infty_\frac{1}{b}(\ell_2)$.

Since $\{g_n\}$ is a modular frame for $L^\infty_\frac{1}{b}(\ell_2)$,

$$A\|f_0\|_b^2(x) \leq \sum_n |\langle f_0, g_n \rangle_b^2(x)| \leq B\|f_0\|_b^2(x) \text{ a.e.}$$

$$A\frac{\|f\|_b^2(x)}{\|f\|_b^2(x)} \leq \frac{1}{\|f\|_b^2(x)} \sum_n |\langle f, g_n \rangle_b^2(x)| \leq B\frac{\|f\|_b^2(x)}{\|f\|_b^2(x)} \text{ a.e.}$$

$$A\|f\|_b^2(x) \leq \sum_n |\langle f, g_n \rangle_b^2(x)| \leq B\|f\|_b^2(x) \text{ a.e.}$$

So we are done by the previous theorem.

Finally we mention that there is a convolution for our HCM in the case $a = b = 1$. This is developed in much greater detail in [14]. Our convolution is simply the preframe operator $\mathcal{T}_g(f)$. That is, we define $g \star_1 f = \mathcal{T}_g(f)$ because

$$Z(\mathcal{T}_g(f)) = \sum_k \langle f, e_k \rangle_z Z(T_k g) = \sum_k \langle f, e_k \rangle_z e^{2\pi i k v} Z(g) = Z(g)Z(f).$$

The reason we call this a convolution is because the Zak transform turns it into multiplication. Clearly this does not necessarily map to an element in $L^2[0, 1]^2$ but from Holder’s inequality we see that $Z(\mathcal{T}_g(f)) \in L^1[0, 1]^2$.

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