Broadcast Channels with Privacy Leakage Constraints

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Abstract

The broadcast channel (BC) with one common and two private messages with leakage constraints is studied, where leakage refers to the normalized mutual information between a message and a channel symbol string. Each private message is destined for a different user and the leakage to the other receiver must satisfy a constraint. This model captures several scenarios concerning secrecy, i.e., when both, either or neither of the private messages are secret. Inner and outer bounds on the leakage-capacity region are derived. Without leakage constraints the inner bound recovers Marton’s region and the outer bound reduces to the UVW-outer bound. The bounds match for semi-deterministic (SD) and physically degraded (PD) BCs, as well as for BCs with a degraded message set. The leakage-capacity regions of the SD-BC and the BC with a degraded message set recover past results for different secrecy scenarios. A Blackwell BC example illustrates the results and shows how its leakage-capacity region changes from the capacity region without secrecy to the secrecy-capacity regions for different secrecy scenarios.

Index Terms

Broadcast channel, Marton’s inner bound, Privacy Leakage, Secrecy, Physical-layer Security.

I. Introduction

Confidential and non-confidential messages are often transmitted over the same channel. However, the underlying principles for constructing codes without and with secrecy are different. Without secrecy constraints, codes should use all available channel resources to reliably convey information to the destinations. The presence of confidential messages, on the other hand, requires that some resources are allocated to preserve secrecy. We study relationships between the coding strategies and the fundamental limits of communication with and without secrecy. To this end we incorporate secret and non-secret transmissions over a two-user broadcast channel (BC) by considering the BC with privacy leakage constraints (Fig. 1).

Information theoretic secrecy was introduced by Shannon [1] who studied communication between a source and a receiver in the presence of an eavesdropper. Wyner modeled secret communication over noisy channels when he...
introduced the degraded wiretap channel (WTC) and derived its secrecy-capacity region [2]. Csiszár and Körner [3] extended Wyner’s result to a general BC where the source also transmits a common message to both users. The development of wireless communication, whose inherent open nature makes it vulnerable to security attacks, has inspired a growing interest in understanding the fundamental limits of secure communication.

Multiuser settings with secrecy were extensively treated in the literature. Broadcast and interference channels with two confidential messages were studied in [4], where inner and outer bounds on the secrecy-capacity region of both problems were derived. The secrecy-capacity region for the semi-deterministic (SD) BC was established in [5]. The opposite case, i.e., when the message of the deterministic user is kept secret from the deterministic user was derived in [6]. Secret communication over multiuser channels was considered in [8], where the authors derive inner and outer bounds on the rate-equivocation region of the relay-BC (RBC) with one or two confidential messages. Gaussian multiple-input multiple-output (MIMO) BCs and WTCs were studied in [9]–[14], while [15]–[17] focused on BCs with an eavesdropper as an external entity from which all messages are kept secret.

We study a two-user BC over which a common message for both users and a pair of private messages, each destined for a different user, are transmitted. A limited amount of rate of each private message may be leaked to the opposite receiver. The leaked rate is quantified as the normalized mutual information between the message of interest and the channel output sequence at the opposite user. Setting either leakage to zero or infinity reduces the problem to the case where the associated message is confidential or non-confidential, respectively. Thus, our problem setting captures as special cases four scenarios concerning secrecy, i.e., when both, either or neither of the private messages are secret. We derive inner and outer bounds on the leakage-capacity region of the BC. The bounds are tight for SD-BCs, physically degraded (PD) BCs, and BCs with a degraded message set, thus characterizing their leakage-capacity regions.

Various past results are captured as special cases. By taking the leakage thresholds to infinity, our inner bound recovers Marton’s inner bound with a common message [18], which is optimal for every BC with a known capacity region. Making the leakage constraint inactive in our outer bound recovers the UVW-outer bound [19] or the New-Jersey outer bound [20]. These bounds are at least as good as previously known bounds (see [21], [22] and [23]). The leakage-capacity region of the SD-BC reduces to each of the regions in [5]–[7] and [24] by discarding the
common message and choosing the leakage constraints appropriately. The capacity result also recovers the optimal regions for the BC with confidential messages \[3\] and the BC with a degraded message set (without secrecy) \[25\].

Our code construction splits each private message into a public and a private part. The public parts along with the common message constitute a public message that is decoded by both users, and therefore, each public part is leaked to the opposite receiver by default. The codebooks of the private parts are double-binned to allow joint encoding and to control the amount of rate leaked from each private part. The bin sizes are chosen to satisfy the total leakage constraints. Our coding scheme is essentially a Marton code with an additional layer of bins, whose sizes correspond to the amount of leakage; the larger these extra bins are, the smaller the leakage. The resulting achievable region is simplified using the Fourier-Motzkin elimination for information theory (FME-IT) software \[26\]. The outer bound is established by using telescoping identities \[27\]. A Blackwell BC (BWC) \[28\], \[29\] illustrates the results and visualizes the transition of the leakage-capacity region from the capacity region without secrecy to the secrecy-capacity regions for different secrecy scenarios.

This paper is organized as follows. In Section II we describe the BC with privacy leakage constraints. In Section III we state inner and outer bounds on the leakage-capacity region and characterize the leakage-capacity regions for the SD-BC, the BC with a degraded message set and the PD-BC. Section IV discusses past results that are captured within our framework. In Section V we study a BWC example and visualise the results, while Section VI provides proofs. Finally, Section VII summarizes the main achievements and insights of this work.

II. NOTATIONS AND PROBLEM DEFINITION

We use the following notations. Given two real numbers \(a, b\), we denote by \([a:b]\) the set of integers \(\{n \in \mathbb{N} | a \leq n \leq b\}\). We define \(\mathbb{R}_+ = \{x \in \mathbb{R} | x \geq 0\}\). Calligraphic letters denote discrete sets, e.g., \(\mathcal{X}\), while the cardinality of a set \(\mathcal{X}\) is denoted by \(|\mathcal{X}|\). \(\mathcal{X}^n\) stands for the \(n\)-fold Cartesian product of \(\mathcal{X}\). An element of \(\mathcal{X}^n\) is denoted by \(x^n = (x_1, x_2, \ldots, x_n)\), and its substrings as \(x^j_i = (x_i, x_{i+1}, \ldots, x_j)\); when \(i = 1\), the subscript is omitted. We define \(x^{n\setminus i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)\). Whenever the dimension \(n\) is clear from the context, vectors (or sequences) are denoted by boldface letters, e.g., \(x\). Random variables are denoted by uppercase letters, e.g., \(X\), with similar conventions for random vectors. \(X^n_j\) represents the sequence of random variables \((X_i, X_{i+1}, \ldots, X_j)\), while \(X^n\) stands for \(X^n\). The probability of an event \(A\) is denoted by \(\mathbb{P}(A)\), while \(\mathbb{P}(A|B)\) denotes conditional probability of \(A\) given \(B\). We use \(1_A\) to denote the indicator function of \(A\). Probability mass functions (PMFs) are denoted by the capital letter \(P\), with a subscript that identifies the random variable and its possible conditioning. For example, for two jointly distributed random variables \(X\) and \(Y\), let \(P_X, P_{X|Y}\) and \(P_{X|Y}\) denote, respectively, the PMF of \(X\), the joint PMF of \((X, Y)\) and the conditional PMF of \(X\) given \(Y\). In particular, when \(X\) and \(Y\) are discrete, \(P_{X|Y}\) represents the stochastic matrix whose elements are given by \(P_{X|Y}(x|y) = \mathbb{P}(X = x|Y = y)\). We omit the subscripts if the arguments of the PMF are lowercase versions of the random variables. If the entries of \(X^n\) are drawn in an independent and identically distributed (i.i.d.) manner according to \(P_X\), then for every \(x \in \mathcal{X}^n\) we have \(P_{X^n}(x) = \prod_{i=1}^n P_X(x_i)\) and we write \(P_{X^n}(x) = P^n_x(x)\). Similarly, if for every \((x, y) \in \mathcal{X}^n \times \mathcal{Y}^n\) we have \(P_{Y^n|X^n}(y|x) = \prod_{i=1}^n P_{Y|X}(y_i|x_i)\), we write \(P_{Y^n|X^n}(y|x) = P^n_{y|x}(y|x)\). The empirical PMF \(\nu_x\) of a sequence
\( \mathbf{x} \in \mathcal{X}^n \) is
\[
\nu_x(a) \triangleq \frac{N(a|x)}{n} \tag{1}
\]
where \( N(a|x) = \sum_{i=1}^{n} \mathbb{1}_{\{x_i = a\}} \). We use \( \mathcal{T}_e(n)(P_X) \) to denote the set of letter-typical sequences of length \( n \) with respect to the PMF \( P_X \) and the non-negative number \( \epsilon \) [30, Ch. 3], [31], i.e., we have
\[
\mathcal{T}_e(n)(P_X) = \left\{ \mathbf{x} \in \mathcal{X}^n : |\nu_x(a) - P_X(a)| \leq \epsilon P_X(a), \ \forall a \in \mathcal{X} \right\}. \tag{2}
\]

The BC with privacy leakage constraints is illustrated in Fig. 1. The channel has one sender and two receivers. The sender randomly chooses a triple \((m_0, m_1, m_2)\) of indices uniformly and independently from the set \([1 : 2^{nR_0}] \times [1 : 2^{nR_1}] \times [1 : 2^{nR_2}]\) and maps them to a sequence \( \mathbf{x} \in \mathcal{X}^n \), which is the channel input. The sequence \( \mathbf{x} \) is transmitted over a BC with transition probability \( P_{Y_1,Y_2|X} \). If the channel transition matrix factors as \( \mathbb{1}_{\{Y_1 = f(x)\}} P_{Y_2|X} \) or \( P_{Y_1|X} P_{Y_2|Y_1} \), we call the BC SD or PD, respectively. The output sequence \( \mathbf{y}_j \in \mathcal{Y}_j^n \), where \( j = 1, 2 \), is received by decoder \( j \). Decoder \( j \) produces a pair of estimates \((\hat{m}_0^{(j)}, \hat{m}_j)\) of \((m_0, m_j)\).

**Definition 1 (Code Description)** A \((n, 2^{nR_0}, 2^{nR_1}, 2^{nR_2})\) code for the BC with leakage constraints is denoted by \( \mathcal{C}_n \) and has:
1) Three message sets \( \mathcal{M}_j = [1 : 2^{nR_j}], \ j = 0, 1, 2. \)
2) A stochastic encoder that is described by a matrix \( P_{X^n|\mathcal{M}_0,\mathcal{M}_1,\mathcal{M}_2} \) of conditional PMFs on \( \mathcal{X}^n \).
3) Two decoding functions, \( \psi_j : \mathcal{Y}_j^n \to \mathcal{M}_0 \times \mathcal{M}_j, \ j = 1, 2. \)

**Definition 2 (Error Probability)** The average error probability for the \((n, 2^{nR_0}, 2^{nR_1}, 2^{nR_2})\) code \( \mathcal{C}_n \) is
\[
\mathcal{P}_e(\mathcal{C}_n) = \mathbb{P}\left( (\hat{M}_0^{(1)}, \hat{M}_0^{(2)}, \hat{M}_1, \hat{M}_2) \neq (M_0, M_0, M_1, M_2) \mid \mathcal{C}_n \right) \tag{3a}
\]
where \((\hat{M}_0^{(j)}, \hat{M}_j) = \psi_j(Y_j^n), \ j = 1, 2. \) The average error probability for receiver \( j = 1, 2 \) is
\[
\mathcal{P}_{e,j}(\mathcal{C}_n) = \mathbb{P}\left( (\hat{M}_0^{(j)}, \hat{M}_j) \neq (M_0, M_j) \mid \mathcal{C}_n \right). \tag{3b}
\]

The information leakage at receivers 1 and 2 are measured by \( \mathbb{L}_1(\mathcal{C}_n) = \frac{1}{n} I(M_1; Y_2^n \mid \mathcal{C}_n) \) and \( \mathbb{L}_2(\mathcal{C}_n) = \frac{1}{n} I(M_2; Y_1^n \mid \mathcal{C}_n) \), respectively.

**Definition 3 (Achievable Rates)** Let \((L_1, L_2) \in \mathbb{R}_+^2 \). A rate triple \((R_0, R_1, R_2) \in \mathbb{R}_+^3 \) is \((L_1, L_2)\)-achievable if for any \( \epsilon, \xi_1, \xi_2 > 0 \) there is a sufficiently large \( n \) and a \((n, 2^{nR_0}, 2^{nR_1}, 2^{nR_2})\) code \( \mathcal{C}_n \) such that
\[
\mathcal{P}_e(\mathcal{C}_n) \leq \epsilon \tag{4a}
\]
\[
\mathbb{L}_1(\mathcal{C}_n) \leq L_1 + \xi_1 \tag{4b}
\]
\[
\mathbb{L}_2(\mathcal{C}_n) \leq L_2 + \xi_2. \tag{4c}
\]

\(^1\text{When clear from the context, we drop the conditioning on } \mathcal{C}_n.\)
The \((L_1, L_2)\)-leakage-capacity region \(C(L_1, L_2)\) is the closure of the set of the \((L_1, L_2)\)-achievable rates.

### III. Leakage-Capacity Results

We state an inner bound on the \((L_1, L_2)\)-leakage-capacity region \(C(L_1, L_2)\) of a BC with privacy leakage constraints. A proof of Theorem 1 is given in Section VI-A.

**Theorem 1 (Inner Bound)** Let \((L_1, L_2) \in \mathbb{R}_+^2\) and \(\mathcal{R}_i(L_1, L_2)\) be the closure of the union of rate triples \((R_0, R_1, R_2) \in \mathbb{R}_+^3\) satisfying:

\[
\begin{align*}
R_1 &\leq I(U_1; Y_1 | U_0) - I(U_1; U_2 | U_0) - I(U_1; Y_2 | U_0, U_2) + L_1 \quad (5a) \\
R_0 + R_1 &\leq I(U_0, U_1; Y_1) - I(U_1; U_2 | U_0) - I(U_1; Y_2 | U_0, U_2) + L_1 \quad (5b) \\
R_0 + R_1 &\leq I(U_0, U_1; Y_1) \quad (5c) \\
R_2 &\leq I(U_2; Y_2 | U_0) - I(U_1; U_2 | U_0) - I(U_2; Y_1 | U_0, U_1) + L_2 \quad (5d) \\
R_0 + R_2 &\leq I(U_0, U_2; Y_2) - I(U_1; U_2 | U_0) - I(U_2; Y_1 | U_0, U_1) + L_2 \quad (5e) \\
R_0 + R_2 &\leq I(U_0, U_2; Y_2) \quad (5f) \\
R_0 + R_1 + R_2 &\leq I(U_0, U_1; Y_1) + I(U_2; Y_2 | U_0) - I(U_1; U_2 | U_0) - I(U_1; Y_2 | U_0, U_2) + L_1 \quad (5g) \\
R_0 + R_1 + R_2 &\leq I(U_1; Y_1 | U_0) + I(U_0, U_2; Y_2) - I(U_1; U_2 | U_0) - I(U_2; Y_1 | U_0, U_1) + L_2 \quad (5h) \\
R_0 + R_1 + R_2 &\leq I(U_1; Y_1 | U_0) + I(U_2; Y_2 | U_0) - I(U_1; U_2 | U_0) - \min \{I(U_0; Y_1), I(U_0; Y_2)\} \quad (5i) \\
2R_0 + R_1 + R_2 &\leq I(U_0, U_1; Y_1) + I(U_0, U_2; Y_2) - I(U_1; U_2 | U_0) \quad (5j)
\end{align*}
\]

where the union is over all PMFs \(P_{U_0, U_1, U_2, X} P_{Y_1, Y_2 | X}\). The following inclusion holds:

\[\mathcal{R}_i(L_1, L_2) \subseteq C(L_1, L_2).\]  (6)

**Remark 1** The region \(\mathcal{R}_i(L_1, L_2)\) recovers Marton’s inner bound for BCs with a common message [18, Theorem 5]. By taking \(L_1, L_2 \to \infty\), the rate bounds in \((5a)-(5b), (5d)-(5e)\) and \((5g)-(5h)\) are redundant. The remaining bounds coincide with those defining Marton’s region. A full discussion on the special cases of \(\mathcal{R}_i(L_1, L_2)\) is given in Section IV-D.

Next, we state an outer bound on \(C(L_1, L_2)\). A proof of Theorem 2 is given in Section VI-B.

**Theorem 2 (Outer Bound)** Let \((L_1, L_2) \in \mathbb{R}_+^2\) and \(\mathcal{R}_o(L_1, L_2)\) be the closure of the union of rate triples \((R_0, R_1, R_2) \in \mathbb{R}_+^3\) satisfying:

\[
\begin{align*}
R_0 &\leq \min \{I(W; Y_1), I(W; Y_2)\} \quad (7a) \\
R_1 &\leq I(U; Y_1 | W, V) - I(U; Y_2 | W, V) + L_1 \quad (7b) \\
R_1 &\leq I(U; Y_1 | W) - I(U; Y_2 | W) + L_1 \quad (7c)
\end{align*}
\]
\[
\begin{align*}
R_0 + R_1 & \leq I(U;Y_1|W) + \min \{ I(W;Y_1), I(W;Y_2) \} \quad (7d) \\
R_2 & \leq I(V;Y_2|W,U) - I(V;Y_1|W,U) + L_2 \quad (7e) \\
R_2 & \leq I(V;Y_2|W) - I(V;Y_1|W) + L_2 \quad (7f) \\
R_0 + R_2 & \leq I(V;Y_2|W) + \min \{ I(W;Y_1), I(W;Y_2) \} \quad (7g) \\
R_0 + R_1 + R_2 & \leq I(U;Y_1|W,V) + I(V;Y_2|W) + \min \{ I(W;Y_1), I(W;Y_2) \} \quad (7h) \\
R_0 + R_1 + R_2 & \leq I(U;Y_1|W) + I(V;Y_2|W,U) + \min \{ I(W;Y_1), I(W;Y_2) \} \quad (7i)
\end{align*}
\]

where the union is over all PMFs \( P_{W,U,V}P_{X|U,V}P_{Y_1,Y_2|X} \). \( \mathcal{R}_O(L_1, L_2) \) is convex. The following inclusion holds:
\[
C(L_1, L_2) \subseteq \mathcal{R}_O(L_1, L_2).
\]

**Remark 2** The region \( \mathcal{R}_O(L_1, L_2) \) recovers the UVW-outer bound from [19] Bound 2, which is equivalent to the New-Jersey outer bound [20]. This follows by setting \( L_1, L_2 \to \infty \) into \( \mathcal{R}_O(L_1, L_2) \), which makes (7b)-(7d) and (7e)-(7f) inactive.

The inner and outer bounds in Theorems [1] and [2] are tight for SD-BCs and give rise to the following theorem. A proof of Theorem 3 is given in Section [VI-C].

**Theorem 3 (Leakage-Capacity for SD-BC)** Let \( (L_1, L_2) \in \mathbb{R}_+^2 \). The \( (L_1, L_2) \)-leakage-capacity region \( C_{SD}(L_1, L_2) \) of a SD-BC with privacy leakage constraints is the closure of the union of rate triples \( (R_0, R_1, R_2) \in \mathbb{R}_+^3 \) satisfying:

\[
\begin{align*}
R_1 & \leq H(Y_1|W,V,Y_2) + L_1 \quad (9a) \\
R_0 + R_1 & \leq H(Y_1|W,V,Y_2) + I(W;Y_1) + L_1 \quad (9b) \\
R_0 + R_1 & \leq H(Y_1) \quad (9c) \\
R_2 & \leq I(V;Y_2|W) - I(V;Y_1|W) + L_2 \quad (9d) \\
R_0 + R_2 & \leq I(W,V;Y_2) - I(V;Y_1|W) + L_2 \quad (9e) \\
R_0 + R_2 & \leq I(W,V;Y_2) \quad (9f) \\
R_0 + R_1 + R_2 & \leq H(Y_1|W,V,Y_2) + I(V;Y_2|W) + I(W;Y_1) + L_1 \quad (9g) \\
R_0 + R_1 + R_2 & \leq H(Y_1|W,V) + I(V;Y_2|W) + \min \{ I(W;Y_1), I(W;Y_2) \} \quad (9h) \\
2R_0 + R_1 + R_2 & \leq H(Y_1|W,V) + I(W,V;Y_2) + I(W;Y_1) \quad (9i)
\end{align*}
\]

where the union is over all PMFs \( P_{W,V,Y_1,X}P_{Y_2|X} \) for which \( Y_1 = f(X) \). \( C_{SD}(L_1, L_2) \) is convex.

**Remark 3** By taking \( L_j = 0 \), the SD-BC with leakage constraints is reduced to the corresponding BC in which \( M_j \) is secret. Similarly, setting \( L_j \to \infty \) results in the problem without a secrecy constraint on \( M_j \). All four cases of
the SD-BC concerning secrecy (i.e., when neither, either or both messages are secret) are solved and their solutions are retrieved from $C_{SD}(L_1, L_2)$ by inserting the appropriate values of $L_j$, $j = 1, 2$. This property of $C_{SD}(L_1, L_2)$ is discussed in Section IV-D

The inner and outer bounds in Theorems 1 and 2 also match when the message set is degraded, i.e., when there is only one private message. The leakage-capacity region of the BC where $M_2 = 0$ is defined only by the threshold $L_1$ and is stated next. The proof of Theorem 4 is given in Section VI-D.

Theorem 4 (Leakage-Capacity for BC with Degraded Message Set) Let $L_1 \in \mathbb{R}_+$. The $L_1$-leakage-capacity region $C_{DM}(L_1)$ of a BC with a degraded message set and a privacy leakage constraint is the closure of the union of rate pairs $(R_0, R_1) \in \mathbb{R}_+^2$ satisfying:

$$R_0 \leq I(W; Y_2)$$
$$R_1 \leq I(U; Y_1 | W) - I(U; Y_2 | Y_1) + L_1$$
$$R_0 + R_1 \leq I(W, U; Y_1) - I(U; Y_2 | Y_1) + L_1$$
$$R_0 + R_1 \leq I(U; Y_1 | W) + \min\{I(W; Y_1), I(W; Y_2)\}$$

where the union is over all PMFs $P_{W, U} P_{X | U} P_{Y_1, Y_2 | X}$. $C_{DM}(L_1)$ is convex.

Remark 4 The BC with a degraded message set and a privacy leakage constraint captures the BC with confidential messages [3] and the BC with a degraded message set [25]. The former is obtained by taking $L_1 = 0$, while $L_1 \rightarrow \infty$ recovers the latter. Setting $L_1 = 0$ or $L_1 \rightarrow \infty$ into $C_{DM}(L_1)$ recovers the capacity regions of these special cases (see Section IV-E for more details).

Corollary 5 (Leakage-Capacity for PD-BC) The $L_1$-leakage-capacity region $C_{PD}(L_1)$ of a PD-BC without a common message and transition probability $P_{Y_1 | X} P_{Y_2 | Y_1}$ is the closure of the union over the same domain as $C_{DM}(L_1)$ of rate pairs $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying (10a)-(10b) and (10d), while replacing $R_0$ with $R_2$ and noting that $\min\{I(W; Y_1), I(W; Y_2)\} = I(W; Y_2)$.

The proof of Corollary 5 is similar to that of Theorem 4 and is omitted.

Remark 5 Bounds on the cardinality of the auxiliary random variables in Theorems 1, 2, 3 and 4 can be derived using, e.g., the perturbation method [32, Appendix C] or techniques such as in [19] and [33]. The computability of the derived regions is not in the scope of this work.

Equivalently, one may consider the case where $M_1 = 0$
IV. Special Cases

A. Marton’s Inner Bound

Theorem 1 generalizes Marton’s region to the case with privacy leakage constraints, i.e., $R_I(\infty, \infty)$ recovers Marton’s region. Moreover, $R_I(L_1, L_2)$ is tight for every BC with a known capacity region.

B. UVW-Outer Bound

The New-Jersey outer bound was derived in [20] and shown to be at least as good as the previously known bounds. A simpler version of this outer bound was established in [19] and was named the UVW-outer bound. The UVW-outer bound is given by $R_O(\infty, \infty)$.

C. Liu-Marić-Spasaljević-Yates Inner Bound for BCs with Secrecy

In [4] an inner bound on the secrecy-capacity region of a BC with two confidential messages (each destined for one of the receivers and kept secret from the other) was characterized as the set of rate pairs $(R_1, R_2) \in \mathbb{R}^2_+$ satisfying:

\begin{align*}
R_1 &\leq I(U_1; Y_1 | U_0) - I(U_1; U_2 | U_0) - I(U_1; Y_2 | U_0, U_2) \quad (11a) \\
R_2 &\leq I(U_2; Y_2 | U_0) - I(U_1; U_2 | U_0) - I(U_2; Y_1 | U_0, U_1) \quad (11b)
\end{align*}

where the union is over all PMFs $P_{U_0,U_1,U_2,P_X|U_1,U_2,P_{Y_1,Y_2}|X}$. This inner bound is tight for SD-BCs [5] and MIMO Gaussian BCs [11]. Setting $R_0 = 0$ into $R_I(0, 0)$ recovers [11].

D. SD-BCs with and without Secrecy

The SD-BC without a common message, i.e., when $R_0 = 0$, is solved when both, either or neither private messages are secret (see [5]–[7] and [24], respectively). Setting $L_j = 0$, for $j = 1, 2$, reduces the SD-BC with privacy leakage constraints to the problem where $M_j$ is secret. Taking $L_j \to \infty$ results in a SD-BC without a secrecy constraint on $M_j$. We use Theorem 3 to obtain the leakage-capacity region of the SD-BC without a common message.

**Corollary 6 (Leakage-Capacity for SD-BC without Common Message)** Let $(L_1, L_2) \in \mathbb{R}^2_+$. The $(L_1, L_2)$-leakage-capacity region $C_{SD}^0(L_1, L_2)$ of a SD-BC with privacy leakage constraints and without a common message is the closure of the union over the domain stated in Theorem 3 of rate pairs $(R_1, R_2) \in \mathbb{R}^2_+$ satisfying:

\begin{align*}
R_1 &\leq H(Y_1 | W, V, Y_2) + L_1 \quad (12a) \\
R_1 &\leq H(Y_1) \quad (12b) \\
R_2 &\leq I(V; Y_2 | W) - I(V; Y_1 | W) + L_2 \quad (12c) \\
R_2 &\leq I(W; V; Y_2) \quad (12d)
\end{align*}
\begin{align}
R_1 + R_2 &\leq H(Y_1|W,V,Y_2) + I(V;Y_2|W) + I(W;Y_1) + L_1 \\
R_1 + R_2 &\leq H(Y_1|W,V) + I(V;Y_2|W) + \min \{I(W;Y_1), I(W;Y_2)\}. \tag{12f}
\end{align}

1) Neither Message is Secret: If \( L_1, L_2 \to \infty \), the SD-BC with leakage reduces to the classic case without secrecy \([24]\), for which the capacity region is the closure of the union of rate pairs \((R_1, R_2) \in \mathbb{R}_+^2 \) satisfying:

\begin{align}
R_1 &\leq H(Y_1) \tag{13a} \\
R_2 &\leq I(V;Y_2) \tag{13b} \\
R_1 + R_2 &\leq H(Y_1|V) + I(V;Y_2) \tag{13c}
\end{align}

where the union is over all PMFs \( P_{Y_1,X} P_{Y_2|X} \) for which \( Y_1 = f(X) \). The region \((13)\) coincides with \( C_{SD}^0(\infty, \infty) \) by first noting that the bound

\begin{align}
R_1 + R_2 &\leq H(Y_1|W,V) + I(V;Y_2|W) + I(W;Y_1) \tag{14}
\end{align}

is redundant because if for some PMF \( P_{W,V,X} P_{Y_1,Y_2|X} \) \((14)\) is active, then setting \( \tilde{W} = 0 \) and \( \tilde{V} = (W,V) \) achieves a larger region. Removing \((14)\) from \( C_{SD}^0(\infty, \infty) \) and setting \( \tilde{V} = (W,V) \) recovers \((13)\). This agrees with the discussion in Section IV-A since Marton’s inner bound is tight for the SD-BC.

2) Only \( M_1 \) is Secret: The SD-BC in which \( M_1 \) is a secret is obtained by taking \( L_1 = 0 \) and \( L_2 \to \infty \). The secrecy-capacity region was derived in \([7, \text{Corollary 4}]\) and is the closure of the union over the same domain as \((13)\) of rate pairs \((R_1, R_2) \in \mathbb{R}_+^2 \) satisfying:

\begin{align}
R_1 &\leq H(Y_1|V,Y_2) \tag{15a} \\
R_2 &\leq I(V;Y_2) \tag{15b}
\end{align}

\( C_{SD}^0(0, \infty) \) and \((15)\) match by dropping

\begin{align}
R_1 + R_2 &\leq H(Y_1|W,V,Y_2) + I(V;Y_2|W) + I(W;Y_1) \tag{16}
\end{align}

based on arguments similar to those used to remove \((14)\) from \( C_{SD}^0(\infty, \infty) \), and setting \( \tilde{V} = (W,V) \).

\textbf{Remark 6} The optimal code for the SD-BC with a secret message \( M_1 \) relies on double-binning the codebook of \( M_1 \), while \( M_2 \) is transmitted at maximal rate and no binning of its codebook is performed. Referring to the bounds in Section VI-A inserting \( L_1 = 0 \) and \( L_2 \to \infty \) into our code construction results in \((29a)\) and \((38b)\) becoming inactive since \((37b)\) is the dominant constraint. Furthermore, \( L_1 = 0 \) combined \((26c)\) implies that the public message consists of a portion of \( M_2 \) only. Keeping in mind that the public message is decoded by both receivers, unless \( R_{10} = 0 \) (i.e., unless the public message contains no information about \( M_1 \)) the secrecy constraint will be violated.
3) Only $M_2$ is Secret: The SD-BC in which $M_2$ is secret is obtained by taking $L_1 \to \infty$ and $L_2 = 0$. The secrecy-capacity region is the closure of the union of rate pairs $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying:

\begin{align*}
R_1 &\leq H(Y_1) \\
R_1 &\leq H(Y_1|W) + I(W;Y_2) \\
R_2 &\leq I(V;Y_2|W) - I(V;Y_1|W)
\end{align*}

where the union is over all PMFs $P_{W,V,Y_1,X|Y_2,X}$ for which $Y_1 = f(X)$ [6, Theorem 1]. Using Corollary 6, $C_{SD}^0(\infty,0)$ is the union over the same domain as (17) of rate pairs $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying:

\begin{align*}
R_1 &\leq H(Y_1) \\
R_2 &\leq I(V;Y_2|W) - I(V;Y_1|W) \\
R_1 + R_2 &\leq H(Y_1|W,V) + I(W;V;Y_2)
\end{align*}

The second bound on $R_1 + R_2$ in $C_{SD}^0(\infty,0)$ is redundant since it follows by adding (18a) and (18b). Both (17) and (18) describe the secrecy-capacity region of the SD-BC with a secret message $M_2$. In Appendix A we prove the equivalence by using bidirectional inclusion arguments. By symmetry, the effect of $L_1 \to \infty$ and $L_2 = 0$ on the coding scheme in Section VI-A is analogous to the one described in Section IV-D2.

4) Both Messages are Secret: Taking $L_1 = L_2 = 0$ recovers the SD-BC where both messages are secret. The secrecy-capacity region for this case was found in [5, Theorem 1] and is the closure of the union of rate pairs $(R_1, R_2) \in \mathbb{R}_+^2$ satisfying:

\begin{align*}
R_1 &\leq H(Y_1|W,V,Y_2) \\
R_2 &\leq I(V;Y_2|W) - I(V;Y_1|W)
\end{align*}

where the union is over all PMFs $P_{W,V,Y_1,X|Y_2,X}$ for which $Y_1 = f(X)$. The region in (19) coincides with $C_{SD}^0(0,0)$. Restricting the union in $C_{SD}^0(0,0)$ to encompass only PMFs that satisfy the Markov relation $W-V-X$ does not shrink the region. This is since in the proof of Theorem 2 we define $V_q \triangleq (M_2,W_q)$, and therefore, $X_q - V_q - W_q$ forms a Markov chain for every $q \in [1:n]$. The optimality of PMFs in which $X-V-W$ is a Markov chain follows.

Remark 7 The coding scheme that achieves (19) uses double-binning for the codebooks of both private messages. To preserve confidentiality, the rate bounds of each message includes the penalty term $I(U_1;U_2|V)$ (without the confidentiality constraints, Marton’s coding scheme [34] requires only that the sum-rate has that penalty term). This is evident from our scheme by setting $L_1 = L_2 = 0$ in (37b), (38b) and (26c), which results in (29a) being redundant.
E. BCs with One Private Message

Consider the BC with leakage constraints in which $M_2 = 0$; its leakage-capacity region $C_{DM}(L_1)$ is stated in Theorem 4. We show that $C_{DM}(L_1)$ recovers the secrecy-capacity region the BC with confidential messages \cite{3} and the capacity region of the BC with a degraded message set (without secrecy) \cite{25}.

1) BCs with Confidential Messages: The secrecy-capacity region of the BC with confidential messages was derived in \cite{3} and is the union over the same domain as in Theorem 4 of rate pairs $(R_0, R_1) \in \mathbb{R}^2_+$ satisfying:

\begin{align*}
R_0 &\leq I(W; Y_1) \quad (20a) \\
R_0 &\leq I(W; Y_2) \quad (20b) \\
R_1 &\leq I(U; Y_1|W) - I(U; Y_2|W). \quad (20c)
\end{align*}

Inserting $L_1 = 0$ into the result of Theorem 4 yields $C_{DM}(0)$ which is the union over the same domain as (20) of rate pairs $(R_0, R_1) \in \mathbb{R}^2_+$ satisfying:

\begin{align*}
R_0 &\leq I(W; Y_2) \quad (21a) \\
R_1 &\leq I(U; Y_1|W) - I(U; Y_2|W) \quad (21b) \\
R_0 + R_1 &\leq I(W, U; Y_1) - I(U; Y_2|W). \quad (21c)
\end{align*}

The regions (20) and (21) are equal and a proof of the equality is given in Appendix B. Inserting $L_1 = 0$ and $U_2 = 0$ into the code construction in Section VI-A reduces it to a superposition code in which the outer codebook (that is associated with the confidential message) is binned.

Remark 8 The BC with confidential messages captures the WTC by setting $M_0 = 0$. Thus, the WTC is also a special case of the BC with privacy leakage constraints.

2) BCs with a Degraded Message Set: If $L_1 \to \infty$, we get the BC with a degraded message set \cite{25}. Inserting $L_1 \to \infty$ into $C_{DM}(L_1)$ and setting $U = X$ recovers the capacity region which is the union of rate pairs $(R_0, R_1) \in \mathbb{R}^2_+$ satisfying:

\begin{align*}
R_0 &\leq I(W; Y_2) \quad (22a) \\
R_0 + R_1 &\leq I(X; Y_1|W) + I(W; Y_2) \quad (22b) \\
R_0 + R_1 &\leq I(X; Y_1) \quad (22c)
\end{align*}

where the union is over all PMFs $P_{V, X}P_{Y_1, Y_2|X}$. In fact, (22) is an alternative characterization of the capacity region of the BC with a degraded message set, as described in \cite[Theorem 7]{18} and \cite[Eq. (8.1)]{25}. 
V. Example

Suppose the channel from the transmitter to receivers 1 and 2 is the BWC without a common message as illustrated in Fig. 2 [28], [29]. Using Corollary 6, the \((L_1, L_2)\)-leakage-capacity region of a deterministic BC (DBC) is the following.

**Corollary 7 (Leakage-Capacity Region for DBC)** The \((L_1, L_2)\)-leakage-capacity region \(C_D(L_1, L_2)\) of the DBC with privacy leakage constraints and no common message is the union rate pairs \((R_1, R_2)\) satisfying:

\[
R_1 \leq \min \{H(Y_1), H(Y_1|Y_2) + L_1\} \quad (23a)
\]

\[
R_2 \leq \min \{H(Y_2), H(Y_2|Y_1) + L_2\} \quad (23b)
\]

\[
R_1 + R_2 \leq H(Y_1, Y_2) \quad (23c)
\]

where the union is over all input PMFs \(P_X\).

The proof of Corollary 7 is relegated to Appendix C. We parameterize the input PMF \(P_X\) in Corollary 7 as

\[
P_X(0) = \alpha, \quad P_X(1) = \beta, \quad P_X(2) = 1 - \alpha - \beta, \quad (24)
\]

where \(\alpha, \beta \in \mathbb{R}_+\) and \(\alpha + \beta \leq 1\) and describe the \((L_1, L_2)\)-leakage-capacity region of the BWC as:

\[
C_{BWC}(L_1, L_2) = \bigcup_{\alpha, \beta \in \mathbb{R}_+, \alpha + \beta \leq 1} \left\{ \begin{array}{l}
R_1 \leq \min \{H_b(\beta), (1 - \alpha)H_b\left(\frac{\beta}{1-\alpha}\right) + L_1\} \\
R_2 \leq \min \{H_b(\alpha), (1 - \beta)H_b\left(\frac{\alpha}{1-\beta}\right) + L_2\} \\
R_1 + R_2 \leq H_b(\alpha) + (1 - \alpha)H_b\left(\frac{\beta}{1-\alpha}\right)
\end{array} \right\}. \quad (25)
\]

Fig. 3 illustrates \(C_{BWC}(L_1, L_2)\) for three cases. In Fig. 3(a) \(L_2 \to \infty\) while \(L_1 \in \{0, 0.05, 0.1, 0.4\}\). The blue (inner) line corresponds to \(L_1 = 0\) and is the secrecy-capacity region of a BWC where \(M_1\) is secret [7, Fig. 5]. The red (outer) line corresponds to \(L_1 = 0.4\) (which is sufficiently large and can be thought of as \(L_1 \to \infty\)) and depicts the capacity region of the classic BWC. As \(L_1\) grows, the inner (blue) region converges to coincide with

![Fig. 2: Blackwell BC with privacy leakage constraints.](image_url)
Fig. 3: \((L_1, L_2)\)-leakage-capacity region of the BWC for three cases: (a) \(L_1 = L\) and \(L_2 \to \infty\); (b) \(L_1 \to \infty\) and \(L_2 = L\); (c) \(L_1 = L_2 = L\).

Fig. 4: The sum-rate capacity versus the allowed leakage for \(L_1 = L_2 = L\).

the outer (red) region. Fig. 3(b) considers the opposite case, i.e., where \(L_1 \to \infty\) and \(L_2 \in \{0, 0.05, 0.1, 0.4\}\), and is analogous to Fig. 3(a). In Fig. 3(c) we choose \(L_1 = L_2 = L\), where \(L \in \{0, 0.05, 0.1, 0.4\}\), and we demonstrate the impact of two leakage constraints on the region. When \(L = 0\), one obtains the secrecy-capacity region of the BWC when both messages are confidential \[5\]. In each case, the capacity region grows with \(L\) and saturates at the red (outer) region, for which neither message is secret. The variation of the sum of rates \(R_1 + R_2\) as a function of \(L\) is shown by the blue curve in Fig. 4; the red dashed vertical lines correspond to the values of \(L\) considered in Fig. 3.

The regions in Fig. 3 are a union of rectangles or pentagons, each corresponds to a different input PMF \(P_X\). In Fig. 5 we illustrate a typical structure of these rectangles and pentagons for a fixed \(P_X\) at the extreme values of \(L_1\) and \(L_2\). When both \(L_1\) and \(L_2\) are sufficiently large, the leakage constraints degenerate and the classic BWC is obtained. Its capacity region (the red (outer) line in, e.g., Fig. 3(c)) is a union of the pentagons depicted in Fig.
Fig. 5: The pentagons/rectangles whose union produces the capacity region of a BWC for different secrecy cases: The outer pentagon corresponds to the case without secrecy; the red and blue rectangles correspond to $L_1 = 0$ and $L_2 = 0$, respectively; the inner rectangle is associated with $L_1 = L_2 = 0$.

The secrecy-capacity region for $L_1 = 0$ and $L_2 \rightarrow \infty$ (depicted by the blue line in Fig. 3(a)) is a union of the red rectangles in Fig. 5. Similarly, when $L_2 = 0$ and $L_1 \rightarrow \infty$ the secrecy-capacity region is a union of the blue rectangles in Fig. 5. Finally, if $L_1 = L_2 = 0$ and both messages are secret, the secrecy-capacity region of the BWC is the union of the dark rectangles in Fig. 5, i.e., the intersection of the blue and the red regions. Fig. 5 highlights that as $L_1$ and/or $L_2$ decrease, the underlying pentagons/rectangles (the union of which produces the admissible rate region) shrink and results in a smaller region.

VI. PROOFS

A. Proof of Theorem 1

Fix $(L_1, L_2) \in \mathbb{R}_+^2$, a PMF $P_{U_0, U_1, U_2, X} P_{Y_1, Y_2 | X}$ and $\epsilon, \xi_1, \xi_2 > 0$.

**Codebook Generation:** Split each message $m_j$, $j = 1, 2$, into two sub-messages denoted by $(m_{j0}, m_{jj})$. The triple $m_p \triangleq (m_0, m_{10}, m_{20})$ is referred to as a *public message* while $m_{jj}$, $j = 1, 2$, serve as *private message* $j$. The rates associated with $m_{j0}$ and $m_{jj}$, $j = 1, 2$, are denoted by $R_{j0}$ and $R_{jj}$, while the corresponding alphabets are $\mathcal{M}_{j0}$ and $\mathcal{M}_{jj}$, respectively. The partial rates $R_{j0}$ and $R_{jj}$, $j = 1, 2$, satisfy

$$R_j = R_{j0} + R_{jj}$$
$$0 \leq R_{j0} \leq R_j$$
$$R_{j0} \leq L_j.$$  

The random variables $M_{j0}$ and $M_{jj}$ are independent and uniform over $\mathcal{M}_{j0}$ and $\mathcal{M}_{jj}$. We use the notations $M_p \triangleq (M_0, M_{10}, M_{20})$, $\mathcal{M}_p \triangleq \mathcal{M}_0 \times \mathcal{M}_{10} \times \mathcal{M}_{20}$ and $R_p \triangleq R_0 + R_{10} + R_{20}$. Note that $M_p$ is uniformly distributed over $\mathcal{M}_p$ and that $|\mathcal{M}_p| = 2^{nR_p}$. Moreover, let $(W_1, W_2)$ be a pair of independent random variables,
where \( W_j, j = 1, 2, \) is uniformly distributed over \( \mathcal{W}_j = [1 : 2^{nR_j}] \) and independent of \( (M_0, M_1, M_2) \) (which implies their independence of \( (M_p, M_{11}, M_{22}) \) as well).

Generate a public message codebook, denoted by \( C_{U_0} \), that comprises \( 2^{nR_p} \) \( u_0 \)-codewords \( u_0(m_p), m_p \in M_p \), each drawn according to \( P^n_{U_0} \) independent of all the other \( u_0 \)-codewords.

For every \( m_p \in M_p \), generate a codebook \( C_{U_j}(m_p), j = 1, 2 \) that comprises \( 2^{n(R_j + R'_j + R_{ij})} \) codewords \( u_j \), each drawn according to \( P^n_{U_j|U_0}(\cdot | u_0(m_p)) \) independent of all the other \( u_j \)-codewords. The \( u_j \)-codewords in \( C_{U_j}(m_p) \) are labeled as \( u_j(m_p, m_{jj}, i_{j}, w_{ij}) \), where \( (m_{jj}, i_{j}, w_{ij}) \in M_{jj} \times \mathcal{I}_{j} \times \mathcal{W}_{j} \) and \( \mathcal{I}_j = [1 : 2^{nR_j}] \). Based on this labeling, the codebook \( C_{U_j}(m_p) \) has a \( u_j \)-bin associated with every pair \( (m_{jj}, w_{ij}) \in M_{jj} \times \mathcal{W}_{j} \), each containing \( 2^{nR'_j} \) \( u_j \)-codewords.

The channel input sequence \( x \) that is associated with a triple \( (u_0, u_1, u_2) \) is generated according to \( P^n_{X|U_0,U_1,U_2}(\cdot | u_0, u_1, u_2) \).

**Encoding:** To transmit the message pair \( (m_1, m_2) \) the encoder transforms it into the triple \( (m_p, m_{11}, m_{22}) \), and draws \( W_j \) uniformly over \( \mathcal{W}_j, j = 1, 2 \). Then it searches for a pair of indices \( (i_1, i_2) \in \mathcal{I}_1 \times \mathcal{I}_2 \) such that

\[
\left( u_0(m_p), u_1(m_p, m_1, i_1, w_1), u_2(m_p, m_2, i_2, w_2) \right) \in \mathcal{T}_e^n(P_{U_0,U_1,U_2})
\]

where \( u_0(m_p) \in C_{U_0} \), \( u_j(m_p, m_{jj}, i_{j}, w_{ij}) \in C_{U_j}(m_p) \), and \( w_{ij} \) denotes the realization of \( W_j \). If the set of appropriate index pairs contains more than one element, the encoder chooses a pair uniformly over the set; if the set is empty, a pair is chosen uniformly over \( \mathcal{I}_1 \times \mathcal{I}_2 \). The sequence \( x \) that is associated with the chosen \( u_0 \)-, \( u_1 \)- and \( u_2 \)-codewords is transmitted over the channel.

**Decoding Process:** Decoder \( j, j = 1, 2 \), searches for a unique triple \( (\hat{m}_p, \hat{m}_{jj}, \hat{w}_{ij}) \in M_p \times M_{jj} \times \mathcal{W}_j \) for which there is an index \( \hat{i}_j \in \mathcal{I}_j \) such that

\[
\left( u_0(\hat{m}_p), u_j(\hat{m}_p, \hat{m}_{jj}, \hat{i}_j, \hat{w}_{ij}), y_j \right) \in \mathcal{T}_e^n(P_{U_0,U_j,Y_j})
\]

where \( u_0(\hat{m}_p) \in C_{U_0} \) and \( u_j(\hat{m}_p, \hat{m}_{jj}, \hat{i}_j, \hat{w}_{ij}) \in C_{U_j}(\hat{m}_p) \). Upon finding such a unique triple, \( (\hat{m}_0^{(j)}, \hat{m}_j) = (\hat{m}_0, (\hat{m}_{j0}, \hat{m}_{jj})) \) is declared as the decoded message pair; otherwise, an error is declared.

By standard error probability analysis (see Appendix [D]), reliability requires

\[
R'_1 + R'_2 > I(U_1; U_2|U_0) \tag{29a}
\]

\[
R_{11} + R'_1 + R_1 < I(U_1; Y_1|U_0) \tag{29b}
\]

\[
R_0 + R_{20} + R_1 + R'_1 + R_1 < I(U_0, U_1; Y_1) \tag{29c}
\]

\[
R_{22} + R'_2 + R_2 < I(U_2; Y_2|U_0) \tag{29d}
\]

\[
R_0 + R_{10} + R_2 + R'_2 + R_1 < I(U_0, U_2; Y_2). \tag{29e}
\]

The leakage analysis requires two properties in addition to reliability. Namely, for a fixed \( m_1 \in M_1 \) (respectively, \( m_2 \in M_2 \)) Decoder 2 (respectively, Decoder 1) should be able to decode \( W_1 \) (respectively, \( W_2 \)) with an arbitrarily
low error probability based on \((U_0, U_2, Y_2)\) (respectively, \((U_0, U_1, Y_1)\)). As shown in Appendix D this is possible provided that

\[
\hat{R}_1 < I(U_1; Y_2 | U_0, U_2) \quad (30a)
\]
\[
\hat{R}_2 < I(U_2; Y_1 | U_0, U_1). \quad (30b)
\]

**Leakage Analysis:** Next, we compute an upper bound on \(\mathbb{E}[L_1(C_n)]\) and \(\mathbb{E}[L_2(C_n)]\), where \(C_n\) denotes the random variable the represents the randomly generated codebook that adheres to the above scheme. By symmetry, only the analysis for the average rate-leakage of \(M_1\) to the 2nd receiver is presented. The corresponding derivation for \(M_2\) follows similar lines. Consider:

\[
H(M_1 | Y_2, C_n) \overset{(a)}{\geq} H(M_1 | U_0, U_2, Y_2, C_n) \\
= H(M_1, Y_2 | U_0, U_2, C_n) - H(Y_2 | U_0, U_2, C_n) \\
= H(M_1, U_1, Y_2 | U_0, U_2, C_n) - H(U_1 | M_1, U_0, U_2, Y_2, C_n) - H(Y_2 | U_0, U_2, C_n) \\
\overset{(b)}{\geq} H(U_1 | U_0, U_2, C_n) - H(U_1 | M_1, U_0, U_2, Y_2, C_n) - H(Y_2 | U_0, U_2, C_n) \\
\overset{(c)}{=} H(U_1 | U_0, C_n) - I(U_1; U_2 | U_0, C_n) - H(U_1 | M_1, U_0, U_2, Y_2, C_n) - I(U_1; Y_2 | U_0, U_2, C_n) \\
(31)
\]

where (a) and (b) follow because conditioning cannot increase entropy, while (c) follows since \(Y_2 - (U_0, U_1, U_2, C_n) - M_1\) forms a Markov chain (this can be shown using functional dependence graphs [36]).

We evaluate each term in (31) separately using the following lemmas.

**Lemma 8** For any \(\epsilon_1, \epsilon_2 > 0\), there is a sufficiently large \(n\) for which

\[
I(U_1; U_2 | U_0, C_n) \leq nI(U_1; U_2 | U_0) + n\epsilon_1 \quad (32a)
\]
\[
I(U_1; Y_2 | U_0, U_2, C_n) \leq nI(U_1; Y_2 | U_0, U_2) + n\epsilon_2. \quad (32b)
\]

**Lemma 9** For any \(\epsilon_3 > 0\) there is a sufficiently large \(n\) for which

\[
H(U_1 | M_1, U_0, U_2, Y_2, C_n) \leq n\epsilon_3. \quad (33)
\]

The proofs of Lemmas 8 and 9 are similar to those of [4] Lemmas 2 and 3. For completeness, we give the proofs in Appendix E. Next, let \(I_1\) denote the random variable that represents the choice of the index \(i_1 \in I_3\) and observe that

\[
H(U_1 | U_0 = u_0, C_n) = H(M_{11}, W_1, I_1, U_1 | U_0 = u_0, C_n) - H(M_{11}, W_1, I_1 | U_1, U_0 = u_0, C_n) \\
\overset{(a)}{=} H(M_{11}, W_1, I_1 | U_0 = u_0, C_n) - H(M_{11}, W_1, I_1 | U_1, U_0 = u_0, C_n)
\]


we conclude that there exists a specific code

where (a) follows by denoting

The bound in (37b) insures that an

arbitrarily small with

Inserting (36) into (31) and using Lemmas 8 and 9 yields

(c) follows by the symmetry of the random codebook, which implies that conditioned on

Based on (34) we have

\[
H(U_1 | U_0, C_n) \geq n(R_{11} + \tilde{R}_1 + R'_1) - n\epsilon_4.
\]

Inserting (36) into (31) and using Lemmas 8 and 9 yields

\[
H(M_1 | Y_2, C_n) \geq n(R_{11} + \tilde{R}_1 + R'_1 - \epsilon_4 - I(U_1; U_2 | U_0) - \epsilon_1 - \epsilon_3 - I(U_1; Y_2 | U_0, U_2) - \epsilon_2)
\]

\[
= (a) n(R_1 + \tilde{R}_1 + R'_1 - R_{10} - I(U_1; Y_2 | U_0, U_2) - I(U_1; U_2 | U_0) - \epsilon_5)
\]

\[
(b) \geq nR_1 - n(L_1 + \epsilon_5)
\]

where (a) follows by denoting \( \epsilon_5 = \sum_{i=1}^{4} \epsilon_i \) and using (26a) and (26c), while (b) follows by taking

\[
\tilde{R}_1 + R'_1 - R_{10} > I(U_1; Y_2 | U_0, U_2) + I(U_1; U_2 | U_0) - L_1
\]

\[
R'_1 + L_1 - R_{10} > I(U_1; U_2 | U_0).
\]

The bound in (37b) insures that an \( \tilde{R}_1 > 0 \) that satisfies (30a) and (37a) is feasible. Note that \( \epsilon_5 \) can be made arbitrarily small with \( n \), which implies that there is an \( n \) for which \( \mathbb{E}[L_1(C_n)] \leq L_1 + \xi_1 \). A similar analysis of the average rate leaked from \( M_2 \) to the 1st receiver shows that \( \mathbb{E}[L_2(C_n)] \leq L_2 + \xi_2 \) for sufficiently large \( n \) if

\[
\tilde{R}_2 + R'_2 - R_{20} > I(U_2; Y_1 | U_0, U_1) + I(U_1, U_2 | U_0) - L_2
\]

\[
R'_2 + L_2 - R_{20} > I(U_1; U_2 | U_0).
\]

By applying the Selection Lemma [37, Lemma 2.2] to the random variable \( C_n \) and the functions \( P_c, L_1 \) and \( L_2 \), we conclude that there exists a specific code \( C_n \) that satisfied \( 4 \). Finally, we apply FME on (29)–(30) and (37)–(38) while using (26c) and the non-negativity of the involved terms, to eliminate \( R_{j0}, R'_j \) and \( \tilde{R}_j \), for \( j = 1, 2 \). Since all the above linear inequalities have constant coefficients, the FME can be performed by a computer program, e.g.,
by the FME-IT algorithm [26]. This shows the sufficiency of (5).

**Remark 9** Applying FME on (29)-(30) and (37)-(38) gives the rate bounds (5) as well as the inequality

\[ I(U_1; Y_1 | U_0) + I(U_2; Y_2 | U_0) - I(U_1; U_2 | U_0) \geq 0. \]  

(39)

However, if (39) is active then \( P_{U_0,U_1,U_2} \) is not a good choice for code design. Setting \( U_1 = U_2 = 0 \) and keeping the same \( U_0 \) (a choice which always satisfies (39)) achieves a larger region than the one achieved by \( P_{U_0,U_1,U_2,X|P_{Y_1,Y_2}|X} \).

**B. Proof of Theorem 2**

We show that given an \((L_1, L_2)\)-achievable rate triple \((R_0, R_1, R_2)\), there is a PMF \( P_{W,U,V|X,V|P_{Y_1,Y_2}|X} \), for which (7) holds. Due to the symmetric structure of the rate bounds defining \( R_0(L_1, L_2) \), we present only the derivation of (7a)-(7d) and (7h). The other inequalities in (7) are established by similar arguments.

Since \((R_0, R_1, R_2)\) is \((L_1, L_2)\)-achievable, for every \( \epsilon, \xi_1, \xi_2 > 0 \) there is a sufficiently large \( n \) and a \((n, 2^n R_0, 2^n R_1, 2^n R_2)\) code for which (4) is satisfied. Fix \( \epsilon, \xi_1, \xi_2 > 0 \) and the corresponding \( n \). By Fano’s inequality we have

\[ H(M_0, M_j | Y^n_j) \leq 1 + n \epsilon R_j \triangleq n \epsilon_n^{(j)}. \]  

(40)

Define \( \epsilon_n = \max \{ \epsilon_n^{(1)}, \epsilon_n^{(2)} \} \). Next, by (4b), we write

\[ n(L_1 + \xi_1) \geq I(M_1; Y^n_2) \]

\[ = I(M_1; M_0, M_2, Y^n_2) - I(M_1; M_0, M_2 | Y^n_2) \]

\[ \geq I(M_1; Y^n_2 | M_0, M_2) - H(M_0, M_2 | Y^n_2) \]

\[ \geq I(M_1; Y^n_2 | M_0, M_2) - n \epsilon_n \]  

(41)

where (a) follows from the independence of \( M_1 \) and \((M_0, M_2)\) and the non-negativity of entropy, while (b) follows from (40). (41) implies

\[ I(M_1; Y^n_2 | M_0, M_2) \leq n L_1 + n (\xi_1 + \epsilon_n). \]  

(42)

Similarly, we have

\[ I(M_1; Y^n_2 | M_0) \leq n L_1 + n (\xi_1 + \epsilon_n). \]  

(43)

The common message rate \( R_0 \) satisfies

\[ n R_0 = H(M_0) \]

\[ \leq I(M_0; Y^n_1) + n \epsilon_n \]

\[ = \sum_{i=1}^{n} I(M_0; Y_{1,i} | Y_{1,i-1}^{(i-1)}) + n \epsilon_n \]
\[
\sum_{i=1}^{n} I(M_0, Y_{1,i}^{i-1}; Y_i) + n\epsilon_n \quad (44a)
\]
\[
\sum_{i=1}^{n} I(W_i; Y_i) + n\epsilon_n \quad (44b)
\]

where (a) follows by (40) and (b) follows by defining \( W_i \triangleq (M_0, Y_{1,i}^{i-1}, Y_{2,i+1}^n) \). By reversing the roles of \( Y_1^n \) and \( Y_2^n \) and repeating similar steps, we also have

\[
R_0 \leq \sum_{i=1}^{n} I(M_0, Y_{n}^n; Y_i) + n\epsilon_n \quad (45a)
\]
\[
\sum_{i=1}^{n} I(W_i; Y_i) + n\epsilon_n \quad (45b)
\]

For \( R_1 \), it follows that

\[
nR_1 = H(M_1| M_0, M_2)
\]
\[
\leq I(M_1; Y_1^n | M_0, M_2) - I(M_1; Y_2^n | M_0, M_2) + nL_1 + n\delta_n^{(1)}
\]
\[
= \sum_{i=1}^{n} \left[ I(M_1; Y_i, Y_{2,i+1}^n | M_0, M_2) - I(M_1; Y_{1,i}^{i-1}, Y_{2,i}^n | M_0, M_2) \right] + nL_1 + n\delta_n^{(1)}
\]
\[
\leq \sum_{i=1}^{n} \left[ I(U_i; Y_i, V_i) - I(U_i; Y_{2,i} | W_i, V_i) \right] + nL_1 + n\delta_n^{(1)}
\]

where:

(a) follows from (40) and (43);
(b) follows from a telescoping identity [22, Eqs. (9) and (11)];
(c) follows by the definition of \( (W_i, U_i) \).

\( R_1 \) is also upper bounded as

\[
nR_1 = H(M_1| M_0)
\]
\[
\leq I(M_1; Y_1^n | M_0) + nL_1 + n\delta_n^{(1)}
\]
\[
= \sum_{i=1}^{n} \left[ I(M_1; Y_i, Y_{1,i}^{i-1} | M_0) - I(M_1; Y_{1,i}^{i-1}, Y_{2,i}^n | M_0) \right] + nL_1 + n\delta_n^{(1)}
\]
\[
\leq \sum_{i=1}^{n} \left[ I(U_i; Y_i, V_i) - I(U_i; Y_{2,i} | W_i, V_i) \right] + nL_1 + n\delta_n^{(1)}
\]

where:

(a) follows from (40) and (43);
(b) follows from a telescoping identity;
(c) follows by the definition of \( (W_i, U_i) \).
For the sum $R_0 + R_1$, we have

$$n(R_0 + R_1) = H(M_0, M_1)$$

$$(a) \leq I(M_0, M_1; Y^n_1) + n\epsilon_n$$

$$= \sum_{i=1}^{n} I(M_0, M_1; Y_{1,i}|Y_{1}^{i-1}) + n\epsilon_n$$

$$(b) \leq \sum_{i=1}^{n} I(W_i, U_i; Y_{1,i}) + n\epsilon_n$$

(48)

where (a) follows from (40) and (b) follows by the definition of $(W_i, U_i)$. Moreover, consider

$$n(R_0 + R_1) = H(M_1|M_0) + H(M_0)$$

$$(a) \leq I(M_1; Y^n_1|M_0) + I(M_0; Y^n_2) + n\epsilon_n$$

$$\leq \sum_{i=1}^{n} \left[ I(M_1, Y^n_{2,i+1}; Y_{1,i}|M_0, Y_{1}^{i-1}) + I(M_0; Y_{2,i}|Y^n_{2,i+1}) \right] + n\epsilon_n$$

$$= \sum_{i=1}^{n} \left[ I(U_i; Y_{1,i}|W_i) + I(Y^n_{1,i}; Y_{2,i}|Y^n_{2,i+1}) + I(M_0; Y_{2,i}|Y^n_{2,i+1}) \right] + n\epsilon_n$$

$$(b) \leq \sum_{i=1}^{n} \left[ I(U_i; Y_{1,i}|W_i) + I(Y^n_{1,i}; Y_{2,i}|M_0, Y^n_{2,i+1}) + I(M_0; Y_{2,i}|Y^n_{2,i+1}) \right] + n\epsilon_n$$

$$(c) \leq \sum_{i=1}^{n} \left[ I(U_i; Y_{1,i}|W_i) + I(W_i; Y_{2,i}) \right] + n\epsilon_n$$

(49)

where (a) follows from (40), (b) follows from Csiszár’s sum identity, while (c) follows by the definition of $(W_i, U_i)$.

To bound the sum $R_0 + R_1 + R_2$, we start by writing

$$H(M_1|M_0, M_2) \leq I(M_1; Y^n_1|M_0, M_2) + n\epsilon_n$$

$$(a) \leq \sum_{i=1}^{n} I(M_1; Y_{1,i}|M_0, M_2, Y_{1}^{i-1}) + n\epsilon_n$$

$$\leq \sum_{i=1}^{n} I(M_1, Y^n_{2,i+1}; Y_{1,i}|M_0, M_2, Y_{1}^{i-1}) + n\epsilon_n$$

$$(b) \leq \sum_{i=1}^{n} \left[ I(U_i; Y_{1,i}|W_i, V_i) + I(Y^n_{1,i}; Y_{2,i}|M_0, M_2, Y_{2}^{i-1}) \right] + n\epsilon_n$$

(50)

where (a) follows from (40), while (b) follows by the definition of $(W_i, U_i, V_i)$. Moreover, we have

$$H(M_2|M_0) \leq I(M_2; Y^n_2|M_0) + n\epsilon_n$$

$$(b) \leq \sum_{i=1}^{n} \left[ I(M_2; Y^n_{2,i}|M_0, Y_{1}^{i-1}) - I(M_2; Y^n_{2,i+1}|M_0, Y_1^{i}) \right] + n\epsilon_n$$
\[
\begin{align*}
(\text{c}) & \quad \sum_{i=1}^{n} \left[ I(M_2; Y_{2,i+1} | M_0, Y_1^{i-1}) + I(V_i; Y_{2,i} | W_i) - I(M_2; Y_{1,i}, Y_{2,i+1} | M_0, Y_1^{i-1}) \\
& \quad \quad + I(M_2; Y_{1,i} | M_0, Y_1^{i-1}) \right] + n\epsilon_n \\
(\text{d}) & \quad \sum_{i=1}^{n} \left[ I(V_i; Y_{2,i} | W_i) - I(V_i; Y_{1,i} | W_i) + I(M_2; Y_{1,i} | M_0, Y_1^{i-1}) \right] + n\epsilon_n \\
\end{align*}
\]

(51)

where:

(a) follows from (40);

(b) follows from a telescoping identity;

(c) follows from the mutual information chain rule and the definition of \((V_i, U_i)\);

(d) follows by the mutual information chain rule.

Combining (50) and (51) yields

\[
\begin{align*}
n(R_1 + R_2) & \leq \sum_{i=1}^{n} \left[ I(U_i; Y_{1,i} | W_i, V_i) + I(V_i; Y_{2,i} | W_i) - I(V_i; Y_{1,i} | W_i) + I(M_2, Y_{2,i+1}^{n} | Y_{1,i}, M_0, Y_1^{i-1}) \right] + 2n\epsilon_n \\
& = \sum_{i=1}^{n} \left[ I(U_i; Y_{1,i} | W_i, V_i) + I(V_i; Y_{2,i} | W_i) + I(Y_{2,i+1}^{n} | Y_{1,i}, M_0, Y_1^{i-1}) \right] + 2n\epsilon_n. \\
\end{align*}
\]

(52a)

By applying Csiszár’s sum identity on the last term in (52a), we have

\[
\begin{align*}
n(R_1 + R_2) & = \sum_{i=1}^{n} \left[ I(U_i; Y_{1,i} | W_i, V_i) + I(V_i; Y_{2,i} | W_i) + I(Y_{1,i}^{i-1}; Y_{2,i}, M_0, Y_{2,i+1}^{n}) \right] + 2n\epsilon_n. \\
\end{align*}
\]

(52b)

Combining (44a) with (52a) and (52b) with (52b) yields

\[
\begin{align*}
n(R_0 + R_1 + R_2) & \leq \sum_{i=1}^{n} \left[ I(U_i; Y_{1,i} | W_i, V_i) + I(V_i; Y_{2,i} | W_i) + I(W_i; Y_{1,i}) \right] + 3n\epsilon_n \\
& \quad \leq \sum_{i=1}^{n} \left[ I(U_i; Y_{1,i} | W_i, V_i) + I(V_i; Y_{2,i} | W_i) + I(W_i; Y_{2,i}) \right] + 3n\epsilon_n, \\
\end{align*}
\]

(53)

and

\[
\begin{align*}
n(R_0 + R_1 + R_2) & \leq \sum_{i=1}^{n} \left[ I(U_i; Y_{1,i} | W_i, V_i) + I(V_i; Y_{2,i} | W_i) + I(W_i; Y_{2,i}) \right] + 3n\epsilon_n, \\
\end{align*}
\]

(54)

respectively.

By repeating similar steps, we obtain bounds related to the remaining rate bounds in (7):

\[
\begin{align*}
nR_2 & \leq \sum_{i=1}^{n} \left[ I(V_i; Y_{2,i} | W_i, U_i) - I(V_i; Y_{1,i} | W_i, U_i) \right] + nL_2 + n\delta^{(2)}_n \\
& \quad \leq \sum_{i=1}^{n} \left[ I(V_i; Y_{2,i} | W_i) - I(V_i; Y_{1,i} | W_i) \right] + nL_2 + n\delta^{(2)}_n \\
& \quad \leq \sum_{i=1}^{n} I(W_i; V_i; Y_{2,i}) + n\epsilon_n \\
& \quad \leq \sum_{i=1}^{n} \left[ I(V_i; Y_{2,i} | W_i) + I(W_i; Y_{1,i}) \right] + n\epsilon_n \\
& \quad \leq \sum_{i=1}^{n} \left[ I(U_i; Y_{1,i} | W_i) + I(V_i; Y_{2,i} | W_i, U_i) + I(W_i; Y_{1,i}) \right] + 3n\epsilon_n \\
\end{align*}
\]

(55)

(56)

(57)

(58)

(59)
\[ n(R_0 + R_1 + R_2) \leq \sum_{i=1}^{n} \left[ I(U_i; Y_{1,i}|W_i) + I(V_i; Y_{2,i}|W_i, U_i) + I(W_i; Y_{2,i}) \right] + 3n\epsilon_n \]  

(60)

where \( \delta_n^{(2)} = 2\epsilon_n + \xi_2 \).

The bounds are rewritten by introducing a time-sharing random variable \( Q \) that is uniformly distributed over the set \([1 : n]\). For instance, the bound (46) is rewritten as

\[
R_1 \leq \frac{1}{n} \sum_{q=1}^{n} \left[ I(U_q; Y_{1,q}|W_q, V_q) - I(U_q; Y_{2,q}|W_q, V_q) \right] + L_1 + \delta_n^{(1)} \\
= \sum_{i=q}^{n} \mathbb{P}(Q = q) \left[ I(U_Q; Y_{1,Q}|W_Q, V_Q, Q = q) - I(U_Q; Y_{2,Q}|W_Q, V_Q, Q = q) \right] + L_1 + \delta_n^{(1)} \\
\leq I(U_Q; Y_{1,Q}|W_Q, V_Q, Q) - I(U_Q; Y_{2,Q}|W_Q, V_Q, Q) + L_1 + n\delta_n^{(1)}
\]  

(61)

Denote \( Y_1 \triangleq Y_{1,Q}, Y_2 \triangleq Y_{2,Q}, W \triangleq (W_Q, Q), U \triangleq (U_Q, Q) \) and \( V \triangleq (V_Q, Q) \). We thus have the bounds of (7) with small added terms such as \( \epsilon_n \) and \( \delta_n^{(1)} \). But for large \( n \) we can make these terms approach 0. The converse is completed by noting that since the channel is memoryless and without feedback, and because \( U_q = (M_1, W_q) \) and \( V_q = (M_2, W_q) \), the chain

\[
(Y_{1,q}, Y_{2,q}) - X_q - (U_q, V_q) - W_q
\]  

(62)

is Markov for every \( q \in [1 : n] \). This implies that \((Y_1, Y_2) - X - (U, V) - W\) is a Markov chain.

C. Proof of Theorem 3

To establish the direct part of Theorem 3 we show that \( C_{SD}(L_1, L_2) \subseteq R_I(L_1, L_2) \), which follows by setting \( U_0 = W, U_1 = Y_1 \) and \( U_2 = V \) in Theorem 1.

For the converse we show that \( R_{SD}(L_1, L_2) \subseteq C_{SD}(L_1, L_2) \). For every PMF \( P_{W,V,Y_1,X} P_{Y_2|X} \) for which \( Y_1 = f(X) \), we have the following chains of inequalities. The right-hand side (RHS) of (7b) is upper bounded by the RHS of (9a) since

\[
R_1 \leq I(U; Y_1|W, V) - I(U; Y_2|W, V) + L_1 \\
= H(Y_1|W, V) - H(Y_1|W, V, U) - I(U; Y_2|W, V) + L_1 \\
\overset{(a)}{\leq} H(Y_1|W, V) - I(Y_1; Y_2|W, V, U) - I(U; Y_2|W, V) + L_1 \\
= H(Y_1|W, V) - I(U, Y_1; Y_2|W, V) + L_1 \\
\overset{(b)}{\leq} H(Y_1|W, V, Y_2) + L_1
\]  

(63)

where (a) follows by the non-negativity of entropy and (b) follows because conditioning cannot increase entropy. Repeating similar steps while combining (7a) with (7b) yields (9b), i.e., we have

\[
R_0 + R_1 \leq H(Y_1|W, V, Y_2) + I(W; Y_1) + L_1.
\]  

(64)
Inequality (7d) implies (9c) since

\[ R_0 + R_1 \leq I(W;U;Y_1) \leq H(Y_1). \]  

The rate bound (9d) coincides with (7f), combining (7a) with (7d) implies (9e), while (9f) follows from (7g).

For the sum of rates, (9g) follows from (7g) and (65), while (9h) is implied by (7h) since

\[ I(U;Y_1|V,W) \leq H(Y_1|V,W). \]  

Finally, by combining (7a) and (7h) while using (66) we have

\[ 2R_0 + R_1 + R_2 \leq I(U;Y_1|W,V) + I(V;Y_2|W) + 2 \min \{ I(W;Y_1), I(W;Y_2) \} \leq H(Y_1|W,V) + I(W,V;Y_2) + I(W;Y_1), \]

which implies (9i). Dropping the rest of the bounds from (7) only increases the region and shows that \( R_O(L_1, L_2) \subseteq C_{SD}(L_1, L_2) \) (note that \( R_O(L_1, L_2) \) is described by a union over PMFs that satisfy the Markov relation \( X \rightarrow (U,V) \rightarrow W \), while in \( C_{SD}(L_1, L_2) \) this restriction is dropped). This characterizes \( C_{SD}(L_1, L_2) \) as the \((L_1, L_2)\)-leakage-capacity region of the SD-BC.

D. Proof of Theorem 4

The direct part of Theorem 4 follows by setting \( R_2 = 0, U_1 = U \) and \( U_2 = 0 \) in Theorem 1. For the converse we prove that \( R_O(L_1, L_2) \subseteq C_{DM}(L_1) \). Clearly, (10a), (10b) and (10d) coincide with (7a), (7c) and (7d), respectively. Inequality (10c) follows by merging (7a) and (7c).

VII. SUMMARY AND CONCLUDING REMARKS

We considered the BC with privacy leakage constraints. Under this model, all four scenarios concerning secrecy (i.e., when both, either or neither of the private messages are secret) become special cases and are recovered by properly setting the leakage thresholds. Inner and outer bounds on the leakage-capacity region were derived and shown to be tight for SD and PD BCs, as well as for BCs with a degraded message set. The coding strategy that achieved the inner bound relied on Marton’s coding scheme with a common message, but with an extra layer of binning. Each private message was split into a public and a private part and the codebooks of the private parts were double-binned. Taking into account that the rate of the public parts is always leaked, the sizes of the bins in the extra layer were chosen to satisfy the total leakage constraints. The outer bound was derived by leveraging telescoping identities.

The results for the BC with leakage captures various past works. Large leakage thresholds reduce our inner and outer bounds to Marton’s inner bound [18] and the UVW-outer bound [19], respectively. The leakage-capacity region of the SD-BC without a common message recovers the capacity regions where both [5], either [6], [7], or neither [24] private message is secret. The result for the BC with a degraded message set and a privacy leakage constraint captures the capacity regions for the BC with confidential messages [3] and the BC with a degraded
message set (without secrecy) \textsuperscript{23}. Furthermore, our code construction for the inner bound is leakage-adaptive and recovers the best known codes for the aforementioned cases. A Blackwell BC example visualizes the transition of the leakage-capacity region from the capacity region without secrecy to the secrecy-capacity regions for different cases.

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\appendix

\section*{Equivalence of the Regions in (17) and (18)}

Denote the region in (17) by $C$ while the region in (18) is denoted by $C^0_{SD}(\infty, 0)$. The inclusion $C \subseteq C^0_{SD}(\infty, 0)$ follows since (18a)-(18b) coincide with (17a)-(17c), while for (18c) we have

$$H(Y_1|W,V)+I(W,V;Y_2)=H(Y_1|W)+I(W;Y_2)+I(V;Y_2|W)-I(V;Y_1|W) \overset{(a)}{\geq} R_1 + R_2. \tag{67}$$

Here (a) is due to (17b)-(17c).

To see that $C^0_{SD}(\infty, 0) \subseteq C$, let $(R_1, R_2) \in C^0_{SD}(\infty, 0)$ be a rate pair achieved by $(W,V,X)$. We show that there is a triple $(W^*, V^*, X^*)$ for which $(R_1, R_2) \in C$. First, suppose that (18b) holds with equality:

$$R_2 = I(V;Y_2|W) - I(V;Y_1|W). \tag{68}$$

By taking $W^* = W$, $V^* = T$ and $X^* = X$, (17a) and (17c) hold by (18a)-(18b), while (17b) is satisfied since

$$H(Y_1|W^*) + I(W^*;Y_2) \overset{(a)}{=} H(Y_1|W) + I(W;Y_2) + I(V;Y_2|W) - I(V;Y_1|W) - R_2 \geq R_1, \tag{69}$$

where (a) and (b) follow from (68) and (18b), respectively.

Next, assume that a strict inequality holds in (18b), i.e., there is a real number $\gamma > 0$, such that

$$R_2 = I(V;Y_2|W) - I(V;Y_1|W) - \gamma. \tag{70}$$

Define $W^* \triangleq (\Theta, \hat{W})$, where $\Theta$ is a binary random variable independent of $(W,V,X)$ that takes values in $\mathcal{O} = \{\theta_1, \theta_2\}$ with probabilities $\lambda > 0$ and $1 - \lambda$, respectively, and

$$\hat{W} = \begin{cases} W, & \Theta = \theta_1 \\ (W,V), & \Theta = \theta_2. \end{cases} \tag{71}$$
Furthermore, let
\[ \lambda = \frac{I(V; Y_2|W) - I(V; Y_1|W) - \gamma}{I(V; Y_2|W) - I(V; Y_2|W)}, \]
(72)

\( V^* = (W, V) \) and \( X^* = X \). Note that \( X^* - V^* - W^* \) forms a Markov chain and that (17a) follows from (18a). To see that (17b) also holds consider:

\[ I(V^*; Y_2|W^*) - I(V^*; Y_1|W^*) \]

where (a) follows from (73), (b) follows since (21a)-(21c) follows from (21a)-(21b), while (c) and (d) follow from (72) and (70), respectively. We conclude the proof by showing that (17b) also holds. This follows because

\[
H(Y_1|W^*) + I(W^*; Y_2) \overset{(a)}{=} H(Y_1|W^*) + I(W^*; Y_2) + I(V^*; Y_2|W^*) - I(V^*; Y_1|W^*) - R_2 \\
= H(Y_1|W^*, W^*) + I(W^*, V^*; Y_2) - R_2 \\
\overset{(b)}{=} H(Y_1|W, V) + I(W, V; Y_2) - R_2 \\
\overset{(c)}{\geq} R_1
\]
(74)

where (a) follows from (73), (b) follows since \( Y_1 - V^* - W^* \) forms a Markov chain and \( V^* = (W, V) \), while (c) follows from (17c).

**APPENDIX B**

**EQUVALENCE OF THE REGIONS IN (20) AND (21)**

Denote the region in (20) by \( C_{CK} \) while the region in (21) is denoted by \( C_{DM}(0) \). Since this proof mostly follows by arguments akin to those presented in Appendix A, we omit some of the detail. First, \( C_{CK} \subset C_{DM}(0) \) follows since (20b)-(20c) imply that (21a)-21b holds, while (21c) follows by combining (20a) and (20c).

For the opposite inclusion \( C_{DM}(0) \subset C_{CK} \), let \( (R_0, R_1) \in C_{DM}(0) \) be a rate pair achieved by \( (W, U, X) \). We construct a triple \( (W^*, U^*, X^*) \) that satisfies \( W^* - U^* - X^* = (Y_1, Y_2) \) for which \( (R_0, R_1) \in C_{CK} \). If (21b) holds with equality, i.e., if

\[ R_1 = I(U; Y_1|W) - I(U; Y_2|W), \]
(75)

then we take \( W^* = W, U^* = U \) and \( X^* = X \). With respect to this choice, (20b)-(20c) follows from (21a)-(21b), while (20a) is satisfied by combining (75) with (21c).

If, on the other hand, a strict inequality holds in (21b), i.e., we have

\[ R_1 = I(U; Y_1|W) - I(U; Y_2|W) - \gamma \]
(76)

where \( \gamma \) is a real and positive number, then we define \( W^* \triangleq (\Theta, \tilde{W}) \). Here \( \Theta \) is a binary random variable...
independent of \((W, U, X)\) as in Appendix A and

\[
\widetilde{W} = \begin{cases} 
W, & \Theta = \theta_1 \\
U, & \Theta = \theta_2 
\end{cases} 
\]  

(77)

Furthermore, set

\[
\lambda = \frac{I(U; Y_1|W) - I(U; Y_2|W) - \gamma}{I(U; Y_1|W) - I(U; Y_2|W)},
\]  

(78)

\(U^* = U\) and \(X^* = X\). Note that \((Y_1, Y_2) - X^* - U^* - W^*\) forms a Markov chain and consider the following.

\[
I(W^*; Y_2) = \lambda I(W; Y_2) + (1 - \lambda)I(U; Y_2) \overset{(a)}{=} I(W; Y_2) + (1 - \lambda)I(U; Y_2|W) \geq I(W; Y_2) \overset{(b)}{=} R_0
\]  

(79)

where (a) follows from (78) and (b) follows from (21a). Thus, (20b) is satisfied. To see that (20c) holds consider:

\[
I(U^*; Y_1|W^*) - I(U^*; Y_2|W^*) \overset{(a)}{=} \lambda \left[ I(U; Y_1|W) - I(U; Y_2|W) \right] \overset{(b)}{=} I(U; Y_1|W) - I(U; Y_2|W) - \gamma \overset{(c)}{=} R_1
\]  

(80)

where (a) follows from the definition of \((W^*, U^*)\), while (b) and (c) follow from (78) and (76), respectively. It remains to show that (20a) also holds. We begin by writing

\[
I(U^*; Y_2|W^*) \overset{(a)}{=} \lambda I(U; Y_2|W) \leq I(U; Y_2|W),
\]  

(81)

where (a) follows from the definition of \((W^*, U^*)\). Finally, (20a) follows because

\[
I(W^*; Y_1) \overset{(a)}{=} I(U^*; Y_1|W^*) - I(U^*; Y_2|W^*) + I(W^*; Y_1) - R_1
\overset{(b)}{=} I(W^*; U; Y_1) - I(U; Y_2|W) - R_1
\overset{(c)}{=} I(W; U; Y_1) - I(U; Y_2|W) - R_1
\overset{(d)}{=} R_0
\]  

(82)

where (a) follows from (80); (b) follows because \(U^* = U\) and by using (81); (c) follows since \(Y_1 - U - W^*\) and \(Y_1 - U - W\) form Markov chains, which implies that \(I(W^*; U; Y_1) = I(U; Y_1) = I(W; U; Y_1)\); (d) follows from (21c).

**APPENDIX C**

**PROOF OF COROLLARY 2**

The region \(\mathcal{C}_D(L_1, L_2)\) is obtained from \(\mathcal{C}_{SD}^0(L_1, L_2)\) by setting \(W = 0\) and \(V = Y_2\), which implies that \(\mathcal{C}_D(L_1, L_2) \subseteq \mathcal{C}_{SD}^0(L_1, L_2)\). For the converse, the RHS of (12a) is upper bounded by

\[
R_1 \leq H(Y_1|W; V, Y_2) + L_1 \leq H(Y_1|Y_2) + L_1.
\]  

(83)
For (12c) and (12d), respectively, we have

\[
I(V; Y_2|W) - I(V; Y_1|W) + L_2 \leq I(V; Y_1, Y_2|W) - I(V; Y_1|W) + L_2 \\
= I(V; Y_2|W, Y_1) + L_2 \\
\leq H(Y_2|Y_1) + L_2
\]  

(84)

and

\[
I(W, V; Y_2) \leq H(Y_2).
\]  

(85)

Finally, (23c) is implied by (12f) since

\[
R_1 + R_2 \leq H(Y_1|W, V) + I(V; Y_2|W) + \min \{I(W; Y_1), I(W; Y_2)\} \\
\leq H(Y_1|W, V) + I(W, V; Y_2) \\
\leq H(Y_1, Y_2|W, V) + I(W, V; Y_1, Y_2) \\
= H(Y_1, Y_2).
\]  

(86)

To complete the proof we drop (12c), which can only increase \( C_{SD}^{(0)}(L_1, L_2) \).

**APPENDIX D**

**ERROR PROBABILITY ANALYSIS FOR THEOREM I**

**Encoding errors:** Denote the transmitted messages by \((m_0, m_1, m_2) = (m_0, (m_{10}, m_{11}), (m_{20}, m_{22}))\) and the realizations of \(W_j\), for \(j = 1, 2\), by \(w_j\). An encoding error event is described as:

\[
\mathcal{E} = \bigcap_{(i_1, i_2) \in T_1 \times T_2} \left\{ (U_0(m_p), U_1(m_p, m_{11}, i_1, w_1), U_2(m_p, m_{22}, i_2, w_2) \notin T_{e(n)}(P_{U_0, U_1, U_2}) \right\}.
\]  

(87)

By the Multivariate Covering Lemma [35, Lemma 8.2], \(\mathbb{P}(\mathcal{E}) \to 0\) as \(n \to \infty\) if we have

\[
R_1' + R_2' > I(U_1; U_2|V).
\]  

(88)

**Decoding errors:** To account for decoding errors, define the events

\[
\mathcal{D}_j(m_p, m_{jj}, i_j, w_j) = \left\{ (U_0(m_p), U_1(m_p, m_{jj}, i_j, w_j), Y_1) \in T_{e(n)}(P_{U_0, U_1, Y_1}) \right\},
\]  

(89)

for \(j = 1, 2\).

**Leakage associated errors:** To satisfy the leakage constraints in (11b)-(11c), we also account for the following. Fix \(m_j \in M_j\), \(j = 1, 2\) and consider the event of an error in decoding \(u_j(m_p, m_{jj}, i_j, w_j)\) from \((u_0(m_p), u_j(m_p, m_{jj}, i_j, w_j), y_j)\), where \(\bar{j} = j + (-1)^{j+1}\). Since \((u_0(m_p), u_{\bar{j}}(m_p, m_{\bar{j}j}, i_{\bar{j}}, w_{\bar{j}}))\) define \((m_p, i_{\bar{j}})\)
and since \( m_{jj} \) is fixed, decoding \( u_j(m_p, m_{jj}, i_j, w_j) \) boils down to decoding \( w_j \). Define the events:

\[
\mathcal{L}_j(w_j|m_p, m_{11}, m_{22}, i_j, w_j) = \left\{ \left( U_0(m_p), U_j(m_p, m_{jj}, i_j, w_j), U_j(m_p, m_{jj}, i_j, w_j), Y_j \right) \in T^{(n)}(P_{U_0, U_j, Y_j}) \right\},
\]

for \( j = 1, 2 \).

Let \((i_1, i_2) \in \mathcal{I}_1 \times \mathcal{I}_2\) be the pair of indices that were originally chosen by the encoder and let \((I_1, I_2)\) be the corresponding random variables. Define

\[
\mathcal{K} = \left\{ (M_p, M_{11}, M_{22}, I_1, I_2, W_1, W_2) = (m_p, m_{11}, m_{22}, i_1, i_2, w_1, w_2) \right\}.
\]

By the union bound, the average error probability is bounded as

\[
\mathbb{E}[P_e(C_n)] \leq \mathbb{P}(\mathcal{E}) + \sum_{j=1}^{2} \left( 1 - \mathbb{P}(\mathcal{E}) \right) \left( \mathbb{P}\left( \bigcup_{j \neq m_p}^{p_1} \mathcal{D}_j(m_p, m_{jj}, i_j, w_j) | \mathcal{K}, \mathcal{E}^c \right) + \mathbb{P}\left( \bigcup_{j \neq m_p}^{p_2} \mathcal{D}_j(m_p, m_{jj}, i_j, w_j) | \mathcal{K}, \mathcal{E}^c \right) \right)
\]

\[
+ \mathbb{P}\left( \bigcup_{j \neq m_p}^{p_3} \mathcal{D}_j(m_p, \hat{m}_{jj}, \hat{i}_j, \hat{w}_j) | \mathcal{K}, \mathcal{E}^c \right) + \mathbb{P}\left( \bigcup_{j \neq m_p}^{p_4} \mathcal{D}_j(m_p, \hat{m}_{jj}, \hat{i}_j, \hat{w}_j) | \mathcal{K}, \mathcal{E}^c \right)
\]

\[
+ \mathbb{P}\left( \bigcup_{j \neq m_p}^{p_5} \mathcal{L}_j(w_j|m_1, m_2, i_j, w_j) | \mathcal{K}, \mathcal{E}^c \right) + \mathbb{P}\left( \bigcup_{j \neq m_p}^{p_6} \mathcal{L}_j(w_j|m_1, m_2, i_j, w_j) | \mathcal{K}, \mathcal{E}^c \right).
\]

Note that \( \{P_j^{[k]}\}_{k=1}^{4} \) correspond to decoding errors by Decoder \( j \), while \( \{P_j^{[k]}\}_{k=5}^{6} \) must be small in order to satisfy the leakage constraint on the message \( M_j \). We proceed with the following steps:

1) \( P_j^{[2]} \) and \( P_j^{[5]} \), for \( j = 1, 2 \), approach 0 as \( n \to \infty \) by the law of large numbers.

2) For \( P_j^{[3]} \), for \( j = 1, 2 \), consider:

\[
P_j^{[3]}(n) \leq \sum_{(\hat{m}_{jj}, \hat{i}_j, \hat{w}_j) \neq (m_{jj}, i_j, w_j)} 2^{-n \left( I(U_j; Y_j | U_0) - \delta_e \right)}
\]

\[
\leq 2^{n(R_j + R_j' + R_j) + 2^{-n \left( I(U_j; Y_j | U_0) - \delta_e \right)}}
\]

\[
= 2^{n \left( R_j + R_j' + R_j - I(U_j; Y_j | U_0) + \delta_e \right)}
\]

where (a) follows since for every \( (\hat{m}_{jj}, \hat{i}_j, \hat{w}_j) \neq (m_{jj}, i_j, w_j) \), \( U_j(m_p, \hat{m}_{jj}, \hat{i}_j, \hat{w}_j) \) is independent of \( Y_j \) while both of them are drawn conditioned on \( U_0(m_p) \). Moreover, \( \delta_e \to 0 \) as \( n \to \infty \). Hence, for the
probability $P_j^{[3]}$ to vanish as $n \to \infty$, we take:

$$R_{jj} + R'_j + \bar{R}_j < I(U_j; Y_j | U_0), \ j = 1, 2.$$  (93)

3) For $P_j^{[4]}$, $j = 1, 2$, we have

$$P_j^{[4]} \overset{(a)}{\leq} \sum_{(\tilde{m}_p, \tilde{m}_{jj}, \tilde{i}_j, \tilde{w}_j) \neq (m_p, m_{jj}, i_j, w_j)} 2^{-n(I(U_0, U_j; Y_j) - \delta_e)}$$

$$\leq 2^n(R_p + R_{jj} + R'_j + \bar{R}_j) 2^{-n(I(U_0, U_j; Y_j) - \delta_e)}$$

$$= 2^n(R_p + R_{jj} + R'_j + \bar{R}_j - I(U_0, U_j; Y_j) + \delta_e)$$

where (a) follows since for every $(\tilde{m}_p, \tilde{m}_{jj}, \tilde{i}_j, \tilde{w}_j) \neq (m_p, m_{jj}, i_j, w_j)$, $U_0(\tilde{m}_p)$ and $U_j(\tilde{m}_p, \tilde{m}_{jj}, \tilde{i}_j, \tilde{w}_j)$ are jointly typical with each other but independent of $Y_j$. Again, $\delta_e \to 0$ as $n \to \infty$, and therefore, we have that $P_j^{[4]} \to 0$ as $n \to \infty$ if

$$R_p + R_{jj} + R'_j + \bar{R}_j < I(U_0, U_j; Y_j), \ j = 1, 2.$$  (94)

4) By repeating similar steps to upper bound $P_j^{[2]}$, the obtained rate bound is redundant. This is since for every $\tilde{m}_p \neq m_p$ the codewords $U_0(\tilde{m}_p)$ and $U_j(\tilde{m}_p, m_{jj}, i_j, w_j)$ are independent of $Y_j$. Hence, to insure that $P_j^{[2]}$ vanishes to 0 as $n \to \infty$, we take

$$R_p < I(U_0, U_j; Y_j), \ j = 1, 2,$$  (95)

in which the RHS coincides with the RHS of (92), while the left-hand side (LHS) is with respect to $R_p$ only. Clearly, (94) is the dominating constraint.

5) For $P_j^{[6]}$, $j = 1, 2$, consider

$$P_j^{[6]} \overset{(a)}{\leq} \sum_{\tilde{w}_j \neq w_j} 2^{-n(I(U_j; Y_j | U_0, U_j) - \delta_e)}$$

$$\leq 2^n R_p 2^{-n(I(U_j; Y_j | U_0, U_j) - \delta_e)}$$

$$= 2^n(R_p - I(U_j; Y_j | U_0, U_j) + \delta_e)$$

where (a) follows since for every $\tilde{w}_j \neq w_j$, $U_j(m_p, m_{jj}, i_j, \tilde{w}_j)$ is independent of $Y_j$ while both of them are drawn conditioned on $(U_0(m_p), U_j(m_p, m_{jj}, i_j, w_j))$. Since $\delta_e \to 0$ as $n \to \infty$, to make $P_j^{[6]}$ decay to 0 as $n \to \infty$, we take

$$\bar{R}_j < I(U_j; Y_j | U_0, U_j), \ j = 1, 2.$$  (96)

Summarizing the above results, while substituting $R_p = R_0 + R_{10} + R_{20}$, we find that the RHS of (92) decays as the blocklength $n \to \infty$ if the conditions in (94), (95) are met.
A. Proof of Lemma 8

We prove (32a) only. The proof of (32b) follows similar lines. For every \((u_0, u_1, u_2) \in \mathcal{U}^n \times \mathcal{U}_1^n \times \mathcal{U}_2^n\) define

\[
\nu(u_0, u_1, u_2) = \begin{cases} 
1, & (u_0, u_1, u_2) \notin \mathcal{T}_\epsilon^n(P_{U_0,U_1,U_2}) \\
0, & \text{otherwise}
\end{cases}
\]

which we abbreviate as \(\nu\). The multi-letter mutual information term in the LHS of (32a) is expanded as follows

\[
I(U_1; U_2|U_0, C_n) \leq I(U_1; \nu; U_2|U_0, C_n)
\]

\[
= I(\nu; U_2|U_0, C_n) + I(U_1; U_2|U_0, \nu, C_n)
\]

\[
= I(\nu; U_2|U_0, C_n) + \sum_{j=0}^{1} \mathbb{P}(\nu = j)I(U_1; U_2|U_0, \nu = j, C_n).
\]

Note that

\[
\mathbb{P}(\nu = 1)I(U_1; U_2|U_0, \nu = 1, C_n) \leq \mathbb{P}((U_0, U_1, U_2) \notin \mathcal{T}_\epsilon^n(P_{U_0,U_1,U_2}))H(U_2|\nu = 1, C_n)
\]

\[
\leq n\mathbb{P}((U_0, U_1, U_2) \notin \mathcal{T}_\epsilon^n(P_{U_0,U_1,U_2})) \log |U_2|
\]

\[
\overset{(a)}{\leq} n\eta_1^{(1)} \log |U_2|.
\]

Here (a) follows by the properties the random code construction and \(\eta_1^{(1)}\) decreases as \(e^{-cn}\) for some constant \(c > 0\) [31] Lemma 5]. Furthermore, we have

\[
\mathbb{P}(\nu = 0)I(U_1; U_2|U_0, \nu = 0, C_n) \leq I(U_1; U_2|U_0, \nu = 0, C_n)
\]

\[
= \sum_{(u_0, u_1, u_2) \in \mathcal{T}_\epsilon^n(P_{U_0,U_1,U_2})} P(u_0, u_1, u_2) \log \left( \frac{P(u_1, u_2|u_0)}{P(u_1|u_0)P(u_2|u_0)} \right)
\]

\[
= \sum_{(u_0, u_1, u_2) \in \mathcal{T}_\epsilon^n(P_{U_0,U_1,U_2})} P(u_0, u_1, u_2) \log \left( \frac{2^{-nH(U_1, U_2|U_0)(1-\epsilon)}}{2^{-nH(U_1|U_0)(1+\epsilon)}2^{-nH(U_2|U_0)(1+\epsilon)}} \right)
\]

\[
\leq nI(U_1; U_2|U_0) + n\eta_2^{(2)}
\]

where \(\eta_2^{(2)} = 3\epsilon H(U_1, U_2|U_0)\). Inserting (99) and (100) into (98) yields

\[
I(U_1; U_2|U_0, C_n) \overset{(a)}{\leq} n\eta_1^{(1)} \log |U_2| + nI(U_1; U_2|U_0) + n\eta_2^{(2)} + 1
\]

\[
\overset{(b)}{=} nI(U_1; U_2|U_0) + n\epsilon_1
\]

where (a) follows since \(I(\nu; U_1|U_0) \leq H(\nu) \leq 1\), while (b) follows by setting \(\epsilon_1 = \eta_1^{(1)} \log |U_2| + \eta_2^{(2)} + \frac{1}{n}\).
B. Proof of Lemma

For a fixed \( m_1 \in \mathcal{M}_1 \), denote by \( \lambda(m_1) \) the error probability in decoding \( u_1(m_p, m_{11}, i_1, w_1) \) from \( (u_0(m_p), u_2(m_p, m_{22}, i_2, w_2), y_2) \). By the properties of \( C_n \) we have

\[
\lambda(m_1) \leq \eta_\epsilon^{(3)}
\]

where \( \eta_\epsilon^{(3)} \) decreases as \( e^{-\tilde{c}n} \) for some constant \( \tilde{c} > 0 \). By Fano’s inequality, we have

\[
H(U_1|M_1 = m_1, U_0, U_2, Y_2, C_n) \leq n\epsilon_3,
\]

where \( \epsilon_3 = \frac{1}{n} + \eta_\epsilon^{(3)} R_1 \), which implies

\[
H(U_1|M_1, U_0, U_2, Y_2, C_n) = \sum_{m_1 \in \mathcal{M}_1} 2^{-nR_1} H(U_1|M_1 = m_1, U_0, U_2, Y_2, C_n) \leq n\epsilon_3.
\]
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