CONVERGENCE OF A BLOW-UP CURVE FOR A SEMILINEAR WAVE EQUATION

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ABSTRACT. We consider a blow-up phenomenon for \( \partial_t^2 u_x - \varepsilon^2 \partial_x^2 u_x = F(\partial_t u_x) \). The derivative of the solution \( \partial_t u_x \) blows-up on a curve \( t = T_\varepsilon(x) \) if we impose some conditions on the initial values and the nonlinear term \( F \). We call \( T_\varepsilon \) blow-up curve for \( \partial_t^2 u_x - \varepsilon^2 \partial_x^2 u_x = F(\partial_t u_x) \). In the same way, we consider the blow-up curve \( t = \tilde{T}(x) \) for \( \partial_t^u = F(\partial_t u) \). The purpose of this paper is to show that, for each \( x \), \( T_\varepsilon(x) \) converges to \( \tilde{T}(x) \) as \( \varepsilon \to 0 \).

1. Introduction. We consider the following nonlinear wave equation:

\[
\begin{cases}
\partial_t^2 u_x - \varepsilon^2 \partial_x^2 u_x = F(2\partial_t u_x), & x \in \mathbb{R}, \ t > 0, \\
u_x(x, 0) = u_0(x), \ & \partial_t u_x(x, 0) = u_1(x), & x \in \mathbb{R}.
\end{cases}
\]

(1)

Here, \( u_x = u_x(x,t) : \mathbb{R} \times [0,\infty) \to \mathbb{R} \) is a real-valued unknown function and \( \varepsilon \) is a small parameter. The nonlinear term \( F \) is assumed to satisfy the following conditions:

(\( F_1 \)): \( F \in C^3(\mathbb{R}) \),

(\( F_2 \)): \( F(v), F'(v), F''(v) > 0 \) if \( v > 0 \).

We consider only a classical solution \( u_x \) of (1) in the space-time region \( K_{R^*, T^*} \), where

\[
B_{R^*} = \{ x \in \mathbb{R} \mid |x| < R^* \}, \ K^*_{x_0, T^*} = \left\{ (x, t) \in \mathbb{R}^2 \mid |x - x_0| < \varepsilon(t_0 - t), \ t > 0 \right\},
\]

\[
K_{R^*, T^*} = \bigcup_{x_0 \in B_{R^*}} K^*_{x_0, T^*}
\]

for arbitrarily fixed positive constants \( R^* \) and \( T^* \). We are concerned with the following function:

\[
T_\varepsilon(x) = \sup \left\{ t \in (0, T^*) \mid |\partial_t u_x(x, t)| < \infty \right\} \ \text{for} \ x \in B_{R^*}.
\]

(2)

We call \( T_\varepsilon \) the blow-up curve for the solution of (1). The aim of this paper is to show that the blow-up curve \( T_\varepsilon \) converges to the function \( \tilde{T} \), as \( \varepsilon \to 0 \), in the point-wise sense under some assumptions on \( u_0 \) and \( u_1 \). Herein, \( \tilde{T} \) is defined as

\[
\tilde{T}(x) = \sup \left\{ t \in (0, T^*) \mid |\partial_t v(x, t)| < \infty \right\} \ \text{for} \ x \in B_{R^*},
\]

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where, for $x \in B_{R^*}$, $v(x, t)$ denotes the solution of the ordinary differential equation
\[
\begin{align*}
\partial_t^2 v &= F(2\partial_t v), \quad t > 0, \\
v(x, 0) &= u_0(x), \quad \partial_t v(x, 0) = u_1(x).
\end{align*}
\]

Before stating our problems and the result, we briefly recall some known results about the blow-up curve of nonlinear wave equations. We consider
\[
\partial_t^2 u - \partial_x^2 u = F(u, \partial_t u, \partial_x u), \quad x \in \mathbb{R}, \quad t \geq 0
\]
and its blow-up curve $\hat{T}$ defined as
\[
\hat{T}(x) = \sup \{ t \in (0, T^*) \mid |u(x, t)| < \infty \} \quad \text{for} \quad x \in B_{R^*}.
\]
We only consider the case where $\hat{T}$ is the blow-up time of classical solution for $x \in B_{R^*}$.

In the case $F = |u|^{p-1}u$, the blow-up curve $\hat{T}$ is continuously differentiable if the initial functions are sufficiently large and sufficiently smooth. This result was first proved by Caffarelli and Friedman [2] in 1986 and was extended to more general initial functions by Merle-Zaag [6], [7], [8].

Godin [4] showed the similar results for $F = e^u$. Hamza and Zaag [5] applied Merle-Zaag’s strategy to a damped wave equation. Uesaka [13] showed the Lipschitz continuity of the blow-up curve for a system of nonlinear wave equations under suitable initial conditions. However, the differentiability of the blow-up curve still open at present.

By using different method of proof, they showed that
\[
|\hat{T}(x_1) - \hat{T}(x_2)| < |x_1 - x_2| \quad \text{for} \quad x_1, x_2 \in B_{R^*}.
\]
The property (5) is closely related to the fact that the solution has the properties similar to the solution of the ordinary differential equation $\partial_t^2 u = F(u, \partial_t u)$, near the blow-up curve almost everywhere under suitable initial conditions. We obtain this ordinary differential equation by removing the space derivatives from $\partial_t^2 u - \partial_x^2 u = F(u, \partial_t u, \partial_x u)$.

In contrast, Ohta and Takamura [10] reported some examples such that there are $x_1, x_2 \in B_{R^*}$ that satisfy
\[
|\hat{T}(x_1) - \hat{T}(x_2)| = |x_1 - x_2|.
\]
for $F = a(\partial_t u)^p + \beta(\partial_x u)^q$. They only consider the case $a = 1, \beta = -1, p = q = 2$. One of the reasons such blow-up curve exists may be that the solution near the blow-up curve cannot have the properties similar to the solution of the corresponding ordinary differential equation because the nonlinear term involves the space derivative. However, the exact reason has not yet been proven. Beside this, they also showed that the solution blows up at only $(x, t) = (0, 0)$ under suitable assumptions on initial functions. They also reported an example such that $\hat{T} \in C^\infty$.

In this manner, we can find a new structure of the blow-up curve for $F = a(\partial_t u)^p + \beta(\partial_x u)^q$.

As the first step to the study, Sasaki [12] considered the case $F = |\partial_t u|^{p-1}\partial_t u$ and $\varepsilon = 1$. She considered that the blow-up curve $T_1$ defined by (2) since the blow-up of classical solution is equivalent to the blow-up of $\partial_t u_1$. She showed $T_1 \in C^1(B_{R^*})$ when $(\partial_t \pm \partial_x)u_1(x, 0)$ are sufficiently large and sufficiently smooth. This result is
proved in the similar way as Caffarelli and Friedman [2]. However, the viewpoints concerning the initial functions are different from their results.

In this paper, we consider the nonlinear term involves \( \partial_t u_t \) for the same motivation as [12].

Next, we state the results for the semilinear wave equation with small spatial velocity. Friedman and Oswald [3] considered the following semilinear wave equation:

\[
\begin{aligned}
\partial_t^2 u_t - \varepsilon^2 \partial_x^2 u_t &= \hat{F}(u_t), \\
u_t(x,0) &= u_0(x), \quad \partial_t u_t(x,0) = u_1(x), \quad x \in \mathbb{R},
\end{aligned}
\]

Here, \( \hat{F} \) is a function such that \( \hat{F}(u) \approx u^p \) when \( u \) is large enough. Moreover, they considered the blow-up curve \( \hat{T}_{\varepsilon} \) defined as

\[
\hat{T}_{\varepsilon}(x) = \sup \{ t \geq 0 \mid u_t(x,t) < \infty \} \quad \text{for} \quad x \in \mathbb{R}.
\]

They also considered

\[
\bar{T}(x) = \sup \{ t \geq 0 \mid v(x,t) < \infty \} \quad \text{for} \quad x \in \mathbb{R}.
\]

\( \bar{T} \) is the blow-up curve in the sense of (4) of the solution for the following equation. They consider the case \( \bar{T}(0) < \infty \). For \( x \in \mathbb{R} \), \( v(x,t) \) denotes the solution of the ordinary differential equation

\[
\begin{aligned}
\partial_t^2 v &= \hat{F}(v), \\
v(x,0) &= u_0(x), \quad \partial_t v(x,0) = u_1(x).
\end{aligned}
\]

Then, they showed \( \hat{T}_{\varepsilon} \in C^1(B_{R^*}) \) and there exist positive constants \( R^* \) and \( C \) which are independent of \( x \) and satisfy

\[
|\hat{T}_{\varepsilon}(x) - \bar{T}(x)|, \quad |\hat{T}'_{\varepsilon}(x) - \bar{T}'(x)| \leq C\varepsilon^2 \quad \text{for} \quad x \in B_{R^*}.
\]

under suitable assumptions on initial functions. To show (8), they used the explicit expression of the blow-up rate

\[
C_1(\hat{T}_{\varepsilon}(x) - t)^{-2/(p-1)} \leq u_t(x,t) \leq C_2(\hat{T}_{\varepsilon}(x) - t)^{-2/(p-1)}
\]

if \( t \) is sufficiently close to \( \hat{T}_{\varepsilon}(x) \) for \( x \in B_{R^*} \). Here, \( C_1 \) and \( C_2 \) are positive constants.

**Remark 1.** Friedman and Oswald [3] also showed that \( \hat{T}_{\varepsilon} \to \bar{T} \) as \( \varepsilon \to 0 \), in the point-wise sense without using (9).

On the other hand, Bellout and Friedman [1] considered the following damped wave equation:

\[
\begin{aligned}
\varepsilon \partial_t^2 u_t - \partial_x^2 u_t + \partial_t u_t &= F(u_t), \\
u_t(x,0) &= u_0(x), \quad \partial_t u_t(x,0) = u_1(x), \quad x \in \mathbb{R},
\end{aligned}
\]

Here, \( F \) is a function satisfying

1. \( F \in C^2 \).
2. \( F(v)(v \in \mathbb{R}), \quad F''(v) \geq 0 \ (v \geq 0) \) and \( F'(v) > 0 \ (v > 0) \).
3. \( \int_1^\infty \frac{1}{F(v)} dv < \infty \).
Moreover, they assumed $u_0$ and $u_1$ are sufficiently large and sufficiently smooth. Then, they showed that, for $x \in \mathbb{R}$,
$$
\hat{T}_\varepsilon(x) \to T_0 \quad \text{as} \quad \varepsilon \to 0.
$$

Here, $\hat{T}_\varepsilon$ is defined by (7) and $T_0$ is the blow-up time of
$$
\partial_t u - \partial_x^2 u = F(u).
$$

It is interesting that the method of Bellout and Friedman [1] looks quite similar to that of Nakagawa [9] where the convergence of the numerical blow-up time $T_h$ to the real blow-up time of the semilinear heat equation
$$
\partial_t u - \partial_x^2 u = u^2
$$
with the zero Dirichlet boundary value condition. The numerical blow-up time $T_h$ is defined using the finite difference method to (10).

Saito and Sasaki [11] applied Nakagawa’s strategy to a nonlinear wave equation. In this paper, we consider (1); the nonlinearity $F$ involves $\partial_t u$. We shall derive the convergence results for the blow-up curve. We apply the arguments in Saito and Sasaki [11] and Sasaki [12] to (1).

To state our main results, we rewrite (1) as follows. We define the functions $\phi_\varepsilon$ and $\psi_\varepsilon$ by
$$
\phi_\varepsilon = (\partial_t + \varepsilon \partial_x)u_\varepsilon, \quad \psi_\varepsilon = (\partial_t - \varepsilon \partial_x)u_\varepsilon.
$$

Then, (1) can be rewritten as
$$
\begin{align*}
D_{\varepsilon, -}\phi_\varepsilon &= F(\phi_\varepsilon + \psi_\varepsilon), & x \in \mathbb{R}, \quad t > 0, \\
D_{\varepsilon, +}\psi_\varepsilon &= F(\phi_\varepsilon + \psi_\varepsilon), & x \in \mathbb{R}, \quad t > 0, \\
\phi_\varepsilon(x, 0) &= g_\varepsilon(x), & x \in \mathbb{R}, \\
\psi_\varepsilon(x, 0) &= h_\varepsilon(x), & x \in \mathbb{R},
\end{align*}
$$

where
$$
D_{\varepsilon, \pm} = (\partial_t \pm \varepsilon \partial_x), \quad g_\varepsilon = u_1 \pm \varepsilon \partial_x u_0 \quad \text{and} \quad h_\varepsilon = u_1 - \varepsilon \partial_x u_0.
$$

We now make the following assumptions on the initial functions. Let $C_g$ and $C_h$ be positive constants which are independent of $\varepsilon$ and satisfy

(F3): \[
\int_{C_g + C_h}^{\infty} \frac{1}{2F(v)} dv < T^*. \]

Moreover, we assume

(I1): $g_\varepsilon > C_g$ and $h_\varepsilon > C_h$ in $B_{R^* + T^*}$.

(I2): $g_\varepsilon, h_\varepsilon \in C^3(B_{R^* + T^*})$.

(I3): There is a positive constant $\varepsilon_0$ such that
$$
F(C_g + C_h) \geq (2 + \varepsilon_0)(|\partial_x g_\varepsilon| + |\partial_x h_\varepsilon|) \quad \text{in} \quad B_{R^* + T^*}.
$$

for $0 \leq \varepsilon \leq 1$.

We now state the main results of this paper. Hereinafter, we assume $0 < \varepsilon \leq 1$.

\textbf{Theorem 1.1.} Let $R^*$ and $T^*$ be positive constants and assume (F1)–(F3) and (I1)–(I3). Then, there exists a unique Lipschitz continuous function $T_\varepsilon$ and a unique ($C^{2,1}(\Omega_\varepsilon)$)$^2$ solution $(\phi_\varepsilon, \psi_\varepsilon)$ of (11) such that

$$
0 < T_\varepsilon(x) < T^* \quad (x \in B_{R^*}) \tag{12}
$$

$$
(\phi_\varepsilon + \psi_\varepsilon)(x, t) \to \infty \quad \text{as} \quad t \to T_\varepsilon(x) \quad (x \in B_{R^*}) \tag{13}
$$

for $0 < \varepsilon \leq 1$. Here, $\Omega_\varepsilon = \{(x, t) \in \mathbb{R}^2 \mid x \in B_{R^*}, \ 0 < t < T_\varepsilon(x)\}$. 
Moreover, for any $x \in B_{R^*}$,

$$|T_{\varepsilon}(x) - \tilde{T}(x)| \to 0 \quad \text{as} \quad \varepsilon \to 0$$

where

$$\tilde{T}(x) = \int_{2u_1(x)}^{\infty} \frac{1}{2F(v)} dv.$$ 

Remark 2. The proof of (12) and (13) is based on the arguments in Caffarelli and Friedman [2] or Sasaki [12]. Moreover, the proof of (14) is an analogue of Bellout and Friedman [1], Nakagawa [9] or Saito and Sasaki [11].

Remark 3. The function $\tilde{T}(x)$ is the blow-up time of $(\tilde{\phi} + \tilde{\psi})$ at each $x \in B_{R^*+T^*}$. Here, for $x \in B_{R^*+T^*}$, $(\tilde{\phi}, \tilde{\psi})$ denotes the solution of the system of the ordinary differential equations

$$\begin{cases}
\partial_t \tilde{\phi} = F(\tilde{\phi} + \tilde{\psi}), & t > 0, \\
\partial_t \tilde{\psi} = F(\tilde{\phi} + \tilde{\psi}), & t > 0, \\
\tilde{\phi}(x,0) = \tilde{\psi}(x,0) = u_1(x).
\end{cases}$$

Moreover, we can show $(\tilde{\phi}, \tilde{\psi}) \in C^{2,1}(\tilde{\Omega})$. Here,

$$\tilde{\Omega} = \{(x,t) \in \mathbb{R}^2 \mid x \in B_{R^*+T^*}, \ 0 < t < \tilde{T}(x)\}.$$ 

Remark 4. We notice that the equation (1) is equivalent to (11). Actually, if we set

$$u_\varepsilon(x,t) = u_0(x) + \frac{1}{2} \int_0^t (\phi_\varepsilon + \psi_\varepsilon)(x,s)ds,$$

then, $u_\varepsilon$ satisfies (1). It is apparent that $T_{\varepsilon}(x)$ is equivalently written as

$$T_{\varepsilon}(x) = \sup \{t \in (0,T^*) \mid \|\partial_t u_\varepsilon(x,t)\| < \infty\} = \sup \{t \in (0,T^*) \mid \|(\phi_\varepsilon + \psi_\varepsilon)(x,t)\| < \infty\}$$

for $x \in B_{R^*+T^*}$.

Moreover, we see that (15) is equivalent to

$$\begin{cases}
\partial_t^2 v = F(2\partial_t v), & t > 0, \\
v(0) = u_1(x).
\end{cases}$$

If we set

$$v(x,t) = u_1(x) + \frac{1}{2} \int_0^t (\tilde{\phi} + \tilde{\psi})(x,s)ds,$$

then, $v$ satisfies (16). Hence, $\tilde{T}(x)$ is the blow-up time of $\partial_t v$ at each $x \in B_{R^*+T^*}$.

The remainder of this paper is organized as follows. In Section 2, we present some properties of the solution of (11). In Section 3, we show that $(\phi_\varepsilon, \psi_\varepsilon)$ converges to $(\tilde{\phi}, \tilde{\psi})$. The discussion on the convergence is based on the arguments in Sasaki [12]. In Sections 4, we prove the convergence of the blow-up curve (14) by using the convergence of the solution. The proof is analogous to that of Saito and Sasaki [11].
Remark 5. Bellout and Friedman [1], Nakagawa [9], and Saito and Sasaki [11] used a similar idea for showing the convergence to the blow-up time.

Bellout and Friedman [1] and Nakagawa [9] showed the convergence to the blow-up time for the heat equation. Saito and Sasaki [11] showed it for the wave equation. Because the case in this study is more similar to that of Saito and Sasaki [11] than others, we apply their arguments.

Also, we may apply the arguments in Friedman and Oswald [3] to (11). Moreover, we may obtain the convergence rate if we assume \( F(v) \approx v^p \) by applying Friedman and Oswald [3].

2. Lemmas. In this section, we present some properties of the solution \((\phi_\varepsilon, \psi_\varepsilon)\) of (11) and the blow-up curve \(T_\varepsilon\). We omit the proof. This method of the proof is based on Caffarelli and Friedman [2] or Sasaki [12].

We consider the successive approximation such that \( \phi_{\varepsilon,0} \equiv C_g, \psi_{\varepsilon,0} \equiv C_h \) and

\[
\begin{aligned}
D_{\varepsilon,-} \phi_{\varepsilon,n+1} &= F(\phi_{\varepsilon,n} + \psi_{\varepsilon,n}), & (x, t) \in K_{R^*, T^*}^\varepsilon, \\
D_{\varepsilon,+} \psi_{\varepsilon,n+1} &= F(\phi_{\varepsilon,n} + \psi_{\varepsilon,n}), & (x, t) \in K_{R^*, T^*}^\varepsilon, \\
\phi_{\varepsilon,n+1}(x, 0) &= g_\varepsilon(x), & \psi_{\varepsilon,n}(x, 0) &= h_\varepsilon(x), & x \in B_{R^* + \varepsilon T^*},
\end{aligned}
\]

for \( n \in \mathbb{N} \cup \{0\} \). Then, we have

\[
\begin{aligned}
\phi_{\varepsilon,n+1}(x, t) &= g_\varepsilon(x + \varepsilon t) + \int_0^t F(\phi_{\varepsilon,n} + \psi_{\varepsilon,n})(x + \varepsilon(t-s), s) ds, & (x, t) \in K_{R^*, T^*}^\varepsilon, \\
\psi_{\varepsilon,n+1}(x, t) &= h_\varepsilon(x - \varepsilon t) + \int_0^t F(\phi_{\varepsilon,n} + \psi_{\varepsilon,n})(x - \varepsilon(t-s), s) ds,
\end{aligned}
\]

for \( n \in \mathbb{N} \cup \{0\} \).

Lemma 2.1. Assume (F2) and (11). Then, we have

\[
\phi_{\varepsilon,n+1} \geq \phi_{\varepsilon,n} \geq C_g, \quad \psi_{\varepsilon,n+1} \geq \psi_{\varepsilon,n} \geq C_h \quad \text{in} \quad K_{R^*, T^*}^\varepsilon.
\]  

(17)

for \( n \in \mathbb{N} \cup \{0\} \).

Proof. We omit the proof. See [12].

We fix \((x, t) \in K_{R^*, T^*}^\varepsilon\). Applying Lemma 2.1, we have \( \{\phi_{\varepsilon,n}(x, t)\}_{n=0}^\infty \) and \( \{\psi_{\varepsilon,n}(x, t)\}_{n=0}^\infty \) are increasing sequences on \( n \). We define \( \phi_\varepsilon(x, t) \) and \( \psi_\varepsilon(x, t) \) to be

\[
\phi_\varepsilon(x, t) = \lim_{n \to \infty} \phi_{\varepsilon,n}(x, t) = \sup_{n \in \mathbb{N}} \phi_{\varepsilon,n}(x, t),
\]

(18)

\[
\psi_\varepsilon(x, t) = \lim_{n \to \infty} \psi_{\varepsilon,n}(x, t) = \sup_{n \in \mathbb{N}} \psi_{\varepsilon,n}(x, t).
\]

(19)

We define a function \( T_\varepsilon(x) \) by

\[
T_\varepsilon(x) = \sup \{ t \in (0, T^*) \mid (\phi_\varepsilon + \psi_\varepsilon)(x, t) < \infty \} \quad \text{for} \quad x \in B_{R^*},
\]

and let \( \Omega_\varepsilon \) be given by \( \Omega_\varepsilon = \{(x, t) \in \mathbb{R}^2 \mid x \in B_{R^*}, \ 0 < t < T_\varepsilon(x)\} \).

We state the following local existence lemma.

Lemma 2.2. Assume (F1)–(F2) and (I1)–(I3). Then, \( (\phi_\varepsilon, \psi_\varepsilon) \) is a unique \((C^{2,1})^2\) solution of (11).

Proof. We omit the proof. See [12].
Lemma 2.3. Under the assumptions of Theorem 1.1, Convergence to the solution of ODEs.

Remark 6. We can show

\[ D^n_0 \phi_{\varepsilon,n} \to D^n_0 \phi_\varepsilon, \quad D^n_0 \psi_{\varepsilon,n} \to D^n_0 \psi_\varepsilon \text{ as } n \to \infty \text{ locally uniformly in } \Omega_\varepsilon. \] (20)

where \( \alpha = 0, 1, 2 \) and \( D_0 = \cos \theta \partial_x + \sin \theta \partial_t \). We omit the proof. See [12].

Proposition 1. Under the assumptions of Theorem 1.1, there exist a unique Lipschitz continuous function \( T_\varepsilon \) and a unique \((C^{2,1}(\Omega_\varepsilon))\) solution \((\phi_\varepsilon, \psi_\varepsilon)\) of (11) such that

\[
F(\phi_\varepsilon + \psi_\varepsilon) \leq \partial_t(\phi_\varepsilon + \psi_\varepsilon) \leq \frac{2(1+\varepsilon_0)}{\varepsilon_0} F(\phi_\varepsilon + \psi_\varepsilon) \tag{21}
\]

\[
\partial_t \phi_\varepsilon \geq (1+\varepsilon_0) |\partial_x \phi_\varepsilon|, \quad \partial_t \psi_\varepsilon \geq (1+\varepsilon_0) |\partial_x \psi_\varepsilon|, \tag{22}
\]

\[
|T_\varepsilon(x_1) - T_\varepsilon(x_2)| \leq \frac{1}{1+\varepsilon_0}|x_1 - x_2| \quad (x_1, x_2 \in B_{R^*}) \tag{23}
\]

in \( \Omega_\varepsilon \).

Proof. We omit the proof. See [12]. \( \square \)

Moreover, we obtain the following lemma

Lemma 2.3. Under the assumptions of Theorem 1.1,

\[
|\bar{T}(x_1) - \bar{T}(x_2)| \leq \frac{1}{1+\varepsilon_0}|x_1 - x_2| \quad \text{for } x_1, x_2 \in B_{R^*+T^*}. \tag{24}
\]

Proof. We omit the proof. See [12]. \( \square \)

3. Convergence to the solution of ODEs. In this section, we show the following proposition.

Proposition 2. Let \( x_0 \in B_{R^*} \) and \( T \in (0, \bar{T}(x_0)) \). Under the assumptions of Theorem 1.1, there exist positive constants \( C \) and \( \bar{\varepsilon} \) depending only on \( T \) such that

\[
\sup_{t \in [0,T]} \left( |\phi_\varepsilon(x_0,t) - \tilde{\phi}(x_0,t)| + |\psi_\varepsilon(x_0,t) - \tilde{\psi}(x_0,t)| \right) \leq C\varepsilon, \tag{25}
\]

for any \( \varepsilon \in [0, \bar{\varepsilon}] \).

We set \( B_{x_0,t_\varepsilon}(t) = \{ x \in B_{R^*+T^*} \mid |x - x_0| < \varepsilon(t_\varepsilon - t) \} \).

To show (25), we consider the following successive approximation which is defined by \( \tilde{\phi}_0 \equiv C_g, \tilde{\psi}_0 \equiv C_h \) and for each \( x \in B_{R^*+T^*} \),

\[
\begin{align*}
\partial_t \tilde{\phi}_{n+1} &= F(\tilde{\phi}_n + \tilde{\psi}_n), \quad t > 0, \\
\partial_t \tilde{\psi}_{n+1} &= F(\tilde{\psi}_n + \tilde{\phi}_n), \quad t > 0, \\
\tilde{\phi}_{n+1}(x,0) &= u_1(x), \quad \tilde{\psi}_{n+1}(x,0) = u_1(x)
\end{align*}
\]

for \( n \in \mathbb{N} \cup \{0\} \). We obtain

\[
D^n_0 \tilde{\phi}_n \to D^n_0 \tilde{\phi}, \quad D^n_0 \tilde{\psi}_n \to D^n_0 \tilde{\psi} \text{ as } n \to \infty \text{ locally uniformly in } \tilde{\Omega} \tag{26}
\]

where \((\tilde{\phi}, \tilde{\psi})\) is the solution of (15) and

\[
\tilde{\Omega} = \left\{ (x,t) \in \mathbb{R}^2 \mid x \in B_{R^*}, \quad 0 < t < \bar{T}(x) \right\}.
\]

Here, \( \alpha = 0, 1, 2 \) and \( D_0 = \cos \theta \partial_x + \sin \theta \partial_t \).
Proof. Let \( x_0 \in B_{R'} \) and \( 0 < \varepsilon \leq 1 \). We split proof into 2 parts.

**Step 1.** First, we show (25) in the case \( t \) is small enough. Let \( C_0 \) be a positive constant satisfying \( \sup_{x \in B_{R'+\varepsilon T}, \varepsilon} |\partial_x u_0(x)| \leq C_0 \). Then, by using (23) and (24), there exists a positive constant \( t_\varepsilon \in (0, \bar{T}(x_0)) \) depending on \( \varepsilon \) such that

\[
\sup_{t \in [0, t_\varepsilon]} \sup_{x \in B_{R'+\varepsilon T}(x)} |(\phi_{\varepsilon,n} + \psi_{\varepsilon,n})(x,t)| \leq C_0 \quad \text{for } n \in \mathbb{N}. 
\tag{27}
\]

We notice that \( C_0 \) does not depend on \( \varepsilon \). Then,

\[
\begin{align*}
\sup_{x \in B_{R', t_\varepsilon}(t)} |(\phi_{\varepsilon,n+1} - \tilde{\phi}_{n+1})(x,t)| + \sup_{x \in B_{R', t_\varepsilon}(t)} |(\psi_{\varepsilon,n+1} - \tilde{\psi}_{n+1})(x,t)| \\
\leq \sup_{x \in B_{R', t_\varepsilon}(t)} |(u_1 + \varepsilon \partial_x u_0)(x + \varepsilon t) - u_1(x)| \\
+ \sup_{x \in B_{R', t_\varepsilon}(t)} |(u_1 - \varepsilon \partial_x u_0)(x - \varepsilon t) - u_1(x)| \\
+ \int_0^t 2 \sup_{x \in B_{R', t_\varepsilon}(s_1)} \left| F(\phi_{\varepsilon,n} + \psi_{\varepsilon,n})(x,s_1) - F(\tilde{\phi}_n + \tilde{\psi}_n)(x,s_1) \right| ds_1 \\
+ \int_0^t \sup_{x \in B_{R', t_\varepsilon}(t)} \left| F(\tilde{\phi}_n + \tilde{\psi}_n)(x + \varepsilon(t - s_1),s_1) - F(\tilde{\phi}_n + \tilde{\psi}_n)(x,s_1) \right| ds_1 \\
+ \int_0^t \sup_{x \in B_{R', t_\varepsilon}(t)} \left| F(\tilde{\phi}_n + \tilde{\psi}_n)(x - \varepsilon(t - s_1),s_1) - F(\tilde{\phi}_n + \tilde{\psi}_n)(x,s_1) \right| ds_1
\end{align*}
\]

for \( t \in [0, t_\varepsilon] \) and \( n \in \mathbb{N} \). By (27), (28) and (29), we have

\[
\begin{align*}
\sup_{x \in B_{R', t_\varepsilon}(t)} |\phi_{\varepsilon,n+1}(x,t) - \tilde{\phi}_{n+1}(x,t)| + \sup_{x \in B_{R', t_\varepsilon}(t)} |\psi_{\varepsilon,n+1}(x,t) - \tilde{\psi}_{n+1}(x,t)| \\
\leq 2C_0 \varepsilon (t + 1) + 4F'(C_0)C_0 t \varepsilon + \int_0^t 2F'(C_0) \left( \sup_{x \in B_{R', t_\varepsilon}(s_1)} |\phi_{\varepsilon,n}(x,s_1) - \tilde{\phi}_n(x,s_1)| \\
+ \sup_{x \in B_{R', t_\varepsilon}(s_1)} |\psi_{\varepsilon,n}(x,s_1) - \tilde{\psi}_n(x,s_1)| \right) ds_1 \\
\leq C_1 \varepsilon + 2F'(C_0) \int_0^t \left( C_1 \varepsilon + 2F'(C_0) \int_0^{s_1} \left( \sup_{x \in B_{R', t_\varepsilon}(s_1)} |\phi_{\varepsilon,n-1}(x,s_1) - \tilde{\phi}_{n-1}(x,s_1)| \\
+ \sup_{x \in B_{R', t_\varepsilon}(s_1)} |\psi_{\varepsilon,n-1}(x,s_1) - \tilde{\psi}_{n-1}(x,s_1)| \right) ds_2 \right) ds_1 \\
\leq C_1 \varepsilon \left( 1 + \int_0^t 2F'(C_0) ds_1 + \int_0^t \int_0^{s_1} (2F'(C_0))^2 ds_2 ds_1 \right)
\end{align*}
\]
Here, \( C_1 = 2C_0(T^* + 1) + 4F'(C_0)C_0T^* \).

By (20), (26), (23), (24) and letting \( n \to \infty \), there exists a positive constant \( C \) depending on \( C_0 \) and \( T^* \) such that

\[
\sup_{x \in B_{x_0, t_*}(t)} |\phi(x, t) - \tilde{\phi}(x, t)| + \sup_{x \in B_{x_0, t_*}(t)} |\psi(x, t) - \tilde{\psi}(x, t)| \leq C \varepsilon \quad (30)
\]

for \( t \in [0, t_*] \). In particular,

\[
|\phi(x_0, t) - \tilde{\phi}(x_0, t)| + |\psi(x_0, t) - \tilde{\psi}(x_0, t)| \leq C \varepsilon \quad (31)
\]

for \( t \in [0, t_*] \). We notice that (31) holds if \( t_* \) satisfies (27) and (28).

**Step 2.** In this stage, we show (25) holds for \( t \leq T \) if \( T < \tilde{T}(x_0) \).

Let \( T \in (0, \tilde{T}(x_0)) \) and define \( M \) and \( T^*_M \) by

\[
M = \sup_{(x, t) \in K^*_M(x_0, T)} (\tilde{\phi} + \tilde{\psi})(x, t) + \sup_{(x, t) \in K^*_M(x_0, T)} |\partial_x \tilde{\phi}(x, t)|
\]
\[
+ \sup_{(x, t) \in K^*_M(x_0, T)} |\partial_x \tilde{\psi}(x, t)| + \sup_{x \in B_{R^* + T^*}} |\partial_x u_0(x)|,
\]

\[
T^*_M = \sup \left\{ t \in (0, T^*) \mid \sup_{(x, s) \in K^*_M(x_0, t)} (\phi + \psi)(x, s) \leq 2M \right\}.
\]

From (24), we see that \( M < \infty \).

We show that there exists \( \varepsilon_M > 0 \) such that

\[
T \leq T^*_M \quad \text{for} \quad \varepsilon \in (0, \varepsilon_M]. \quad (32)
\]

We prove (32) by showing a contradiction. We assume that there exists \( \{\varepsilon_i\} \) such that

\[
0 < \varepsilon_i \leq 1 \quad (i \in \mathbb{N}), \quad \varepsilon_i \to 0 \quad \text{as} \quad i \to \infty \quad \text{and} \quad T^*_{M_i} < T,
\]

We notice that (27), (28) and (29) hold if we take \( t_* \) and \( C_0 \) as \( T^*_{M_i} \) and 2M respectively.

In view of Step 1, there exists \( C_* \) depending on \( M \) and \( T^* \) such that

\[
|\phi_{t_*}(x_0, t) - \tilde{\phi}(x_0, t)| + |\psi_{t_*}(x_0, t) - \tilde{\psi}(x_0, t)| \leq C_* \varepsilon_i
\]

for all \( t \in [0, T^*_{M_i}] \). We notice that \( C_* \) does not depend on \( \varepsilon_i \). Moreover, since \( T^*_{M_i} < T \), we have

\[
|\tilde{\phi} + \tilde{\psi}(x_0, t)| \leq M \quad \text{for} \quad t \in [0, T^*_{M_i}].
\]

Combining these inequalities, we have

\[
|\phi_{t_*} + \psi_{t_*}(x_0, t)| \leq M + C_* \varepsilon_i \quad \text{for} \quad t \in [0, T^*_{M_i}].
\]

In particular, we have

\[
|\phi_{t_*} + \psi_{t_*}(x_0, t)| \leq \frac{3M}{2} \quad \text{for} \quad t \in [0, T^*_{M_i}]
\]

if \( 0 < \varepsilon_i \leq \frac{M}{2C_*} \). By Proposition 1, there exists \( \delta_0 > 0 \) such that
\[ |(\phi_{\varepsilon_i} + \psi_{\varepsilon_i})(x_0, t)| \leq 2M \quad \text{for} \quad t \in [0, T_{\delta_0}^\varepsilon + \delta] \]

if \( \delta \in (0, \delta_0] \). This contradicts the definition of \( T_{\delta_0}^\varepsilon \). This completes the proof. \( \square \)

4. Convergence of the blow-up time. In this section, we show the following proposition. In what follows, we assume \( 0 < \varepsilon \leq 1 \).

**Proposition 3.** Let \( x_0 \in B_{R^*} \). Under the assumptions of Theorem 1.1, we have
\[ T_{\varepsilon}(x_0) \to \tilde{T}(x_0) \quad \text{as} \quad \varepsilon \to 0. \]

**Proof.** First, we prove
\[ \tilde{T}(x_0) \leq \liminf_{\varepsilon \to 0} T_{\varepsilon}(x_0) \quad (33) \]
by showing a contradiction. We define \( T_* \) by \( T_* = \liminf_{\varepsilon \to 0} T_{\varepsilon}(x_0) \) and assume
\[ T_* < \tilde{T}(x_0). \]

Then, there exists a subsequence \( \{\varepsilon_i\}_i \) such that
\[ \varepsilon_i \to 0 \quad \text{as} \quad i \to \infty \quad \text{and} \quad T_{\varepsilon_i}(x_0) \leq T_* + \delta < \tilde{T}(x_0), \]
where \( \delta = (\tilde{T}(x_0) - T_*)/2. \)

We notice that
\[ \sup_{t \in [0, T_* + \delta]} |(\tilde{\phi} + \tilde{\psi})(x_0, t)| < \infty. \quad (34) \]

On the other hand, \( (\phi_{\varepsilon_i}, \psi_{\varepsilon_i}) \) satisfies
\[ |(\phi_{\varepsilon_i} + \psi_{\varepsilon_i})(x_0, t)| \to \infty \quad \text{as} \quad t \to T_{\varepsilon_i}(x_0). \]

(34) and (35) contradict to Proposition 2. Hence, we have (33).

Next, we prove
\[ \limsup_{\varepsilon \to 0} T_{\varepsilon}(x_0) \leq \tilde{T}(x_0) \quad (36) \]
by showing a contradiction. We define \( T^* \) by \( T^* = \limsup_{\varepsilon \to 0} T_{\varepsilon}(x_0) \) and assume
\[ \tilde{T}(x_0) < T^*. \]

From (F3), there exists \( R > 0 \) such that
\[ \int_R^\infty \frac{1}{F(v)} dv < h. \quad (37) \]

where \( h = (T^* - \tilde{T}(x_0))/2. \)

Below, we consider the constant \( R \) which satisfies (37). Then, there exists \( t' \in (0, \tilde{T}(x_0)) \) such that \( (\tilde{\phi} + \tilde{\psi})(x_0, t') > 2R \). By Proposition 2, there exist positive constants \( C' \) and \( \tilde{\varepsilon} \) which depend on \( t' \) such that
\[ |(\phi_{\varepsilon} + \psi_{\varepsilon})(x_0, t') - (\tilde{\phi} + \tilde{\psi})(x_0, t')| \leq C'\varepsilon \]
if we assume \( 0 < \varepsilon < \tilde{\varepsilon} \). Hence, we have
\[ (\phi_{\varepsilon} + \psi_{\varepsilon})(x_0, t') \geq R \]
if we assume \( \varepsilon < \max \left\{ \tilde{\varepsilon}, \frac{R}{C'} \right\} \). By using (22), there exist \( t_R \in (0, t'] \) such that
\[ (\phi_{\varepsilon} + \psi_{\varepsilon})(x_0, t_R^*) = R \quad \text{for} \quad 0 < \varepsilon < \max \left\{ \tilde{\varepsilon}, \frac{R}{C'} \right\}. \]
Moreover, we can take a subsequence \( \{\varepsilon_i\} \) such that
\[
\tilde{T}(x_0) + h < T_{\varepsilon_i}(x_0), \quad \varepsilon_i \to 0 \text{ as } i \to \infty.
\] (38)
By using (21),
\[
T_{\varepsilon_i}(x_0) = t_{R_i}^\varepsilon + T_{\varepsilon_i}(x_0) - t_{R_i}^\varepsilon \leq \tilde{T}(x_0) + \int_R^\infty \frac{1}{F(v)} dv < \tilde{T}(x_0) + h.
\]
This contradicts (38). Hence, we obtain (36). This completes the proof of Proposition 3.

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