BICROSSED PRODUCTS FOR FINITE GROUPS

A. L. AGORE, A. CHIRVĂSITU, B. ION, AND G. MILITARU

Dedicated to Freddy Van Oystaeyen on the occasion of his 60th birthday.

Abstract. We investigate one question regarding bicrossed products of finite groups which we believe has the potential of being approachable for other classes of algebraic objects (algebras, Hopf algebras). The problem is to classify the groups that can be written as bicrossed products between groups of fixed isomorphism types. The groups obtained as bicrossed products of two finite cyclic groups, one being of prime order, are described.

Introduction

The bicrossed product construction is a generalization of the semidirect product construction for the case when neither factor is required to be normal: a group \( E \) is the internal bicrossed product of its subgroups \( H \) and \( G \) if \( HG = E \) and their intersection is trivial. Groups with this property (but allowing for nontrivial intersection) have been in the literature for a quite long time under the terminology permutable groups \([7, 8]\) or groups that admit an exact factorization (see e.g. \([3, 14]\)).

The bicrossed product construction itself is due to Zappa \([13]\). It was rediscovered by Szép \([11]\) and yet again by Takeuchi \([12]\). The terminology bicrossed product is taken from Takeuchi, other terms referring to this construction used in the literature are knit product and Zappa-Szép product. Bicrossed product constructions were subsequently introduced and studied for other structures: algebras, Hopf algebras, Lie algebras, Lie groups, locally compact quantum groups, groupoids. For Hopf algebras, in particular, structural results are still missing and objects obtained from such constructions form a considerable proportion of the known examples (see e.g. \([3, 14]\)). Assume for simplicity that \( k \) is a field of characteristic zero. Let \( E \) be a finite group that is a bicrossed product of the groups \( H \) and \( G \). A noncommutative noncocommutative Hopf algebra \( k[H] \ast \# k[G] \) that is both semisimple and cosemisimple can be constructed \([12]\). This is the easiest way to construct semisimple cosemisimple finite dimensional Hopf algebras. For this reason we decided to investigate some aspects of the bicrossed product construction in its original finite group setting.

Our main question, going back to Ore \([8]\) asks for the description of all groups which arise as bicrossed products of two fixed groups. Little progress has been made on this question. In this respect we would like to mention the result of Wielandt \([15]\) establishing that from

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two finite nilpotent groups of coprime orders one always obtains a solvable group and the
work of Douglas [2] on finite groups expressible as bicrossed products of two finite cyclic
groups. Finding all matched pairs between two finite cyclic groups seems to be still an
open question, even though J. Douglas [2] has devoted four papers and over two dozen
theorems to the subject. In fact, solving this problem does not provide an answer to the
classification of all associated bicrossed products and does not indicate whether a bicrossed
product could not be obtained more easily as a semidirect product. In Section 2 we will
give a complete answer to this question for the case of two finite cyclic groups, one of them
being of prime order. As it turns out, if a group is isomorphic to a bicrossed product of two
finite cyclic groups, one of them being of prime order then it is isomorphic to a semidirect
product between the same cyclic groups. We would like also to mention some interesting
recent investigations [4], [6] into the corresponding question at the level of algebras.

1. Preliminaries

1.1. Definitions and notation. Let us fix the notation that will be used throughout the
paper. Let \( H \) and \( G \) be two groups and \( \alpha : G \times H \to H \) and \( \beta : G \times H \to G \) two maps.
We use the notation
\[
\alpha(g, h) = g \triangleright h \quad \text{and} \quad \beta(g, h) = g \triangleleft h
\]
for all \( g \in G \) and \( h \in H \). The map \( \alpha \) (resp. \( \beta \)) is called trivial if \( g \triangleright h = h \) (resp. \( g \triangleleft h = g \))
for all \( g \in G \) and \( h \in H \). If \( \alpha : G \times H \to H \) is an action of \( G \) on \( H \) as group automorphisms
we denote by \( H \rtimes \alpha G \) the semidirect product of \( H \) and \( G \): \( H \rtimes \alpha G = H \times G \) as a set with
the multiplication given by
\[
(h_1, g_1) \cdot (h_2, g_2) := (h_1(g_1 \triangleright h_2), g_1g_2)
\]
for all \( h_1, h_2 \in H \), \( g_1, g_2 \in G \).
The opposite group structure on \( H \) will be denoted by \( H^{\text{op}} : H^{\text{op}} = H \) as a set with the
multiplication \( h_1^{\text{op}}h_2 = h_2h_1 \) for all \( h_1, h_2 \in H \).

Definition 1.1. A matched pair of groups is a quadruple \( \Lambda = (H, G, \alpha, \beta) \) where \( H \) and \( G \)
are groups, \( \alpha : G \times H \to H \) is a left action of the group \( G \) on the set \( H \), \( \beta : G \times H \to G \) is
a right action of the group \( H \) on the set \( G \) such that the following compatibility conditions hold:
\[
\begin{align*}
g \triangleright (h_1h_2) &= (g \triangleright h_1)((g \triangleleft h_1) \triangleright h_2) \quad &\text{(1)} \\
(g_1g_2) \triangleleft h &= (g_1 \triangleleft (g_2 \triangleright h))(g_2 \triangleleft h) \quad &\text{(2)}
\end{align*}
\]
for all \( g, g_1, g_2 \in G \), \( h, h_1, h_2 \in H \).
A morphism \( \varphi : (H_1, G_1, \alpha_1, \beta_1) \to (H_2, G_2, \alpha_2, \beta_2) \) between two matched pairs consists of
a pair of group morphisms \( \varphi_H : H_1 \to H_2 \), \( \varphi_G : G_1 \to G_2 \) such that
\[
\varphi_H \circ \alpha_1 = \alpha_2 \circ (\varphi_G \times \varphi_H), \quad \varphi_G \circ \beta_1 = \beta_2 \circ (\varphi_G \times \varphi_H)
\]
Remark 1.2. Let \( \Lambda = (H, G, \alpha, \beta) \) be a matched pair of groups. Then
\[
g \triangleright 1 = 1 \quad \text{and} \quad 1 \triangleleft h = 1
\]
for all \( g \in G \) and \( h \in H \).
Let \( H \) and \( G \) be groups and \( \alpha : G \times H \to H \) and \( \beta : G \times H \to G \) two maps. Let 
\[ H_{\alpha \bowtie \beta} G = H \bowtie G := H \times G \] as a set with an binary operation defined by the formula:
\[
(h_1, g_1) \cdot (h_2, g_2) = (h_1(g_1 \triangleright h_2), (g_1 \triangleleft h_2)g_2)
\]
for all \( h_1, h_2 \in H, \ g_1, g_2 \in G \).

The main motivation behind the definition of matched pair is the following result (we refer to [5, section IX.1] for the proof).

**Theorem 1.3.** Let \( H \) and \( G \) be groups and \( \alpha \) and \( \beta \) two maps as above. Then \( H_{\alpha \bowtie \beta} G \) is a group with unit \((1, 1)\) if and only if \((H, G, \alpha, \beta)\) is a matched pair. Moreover, a morphism between two matched pairs induces a morphism between the corresponding groups.

If \((H, G, \alpha, \beta)\) is a matched pair the group \( H \bowtie G \) is called the bicrossed product (or the Zappa-Szép product) of \( H \) and \( G \). The inverse of an element of the group \( H \bowtie G \) is given by the formula
\[
(h, g)^{-1} = \left(g^{-1} \triangleright h^{-1}, (g \triangleleft (g^{-1} \triangleright h^{-1}))^{-1}\right)
\]
for all \( h \in H \) and \( g \in G \). Also, remark that \( H \times \{1\} \cong H \) and \( \{1\} \times G \cong G \) are subgroups of \( H \bowtie G \) and every element \((h, g)\) of \( H \bowtie G \) can be written uniquely as a product of an element of \( H \times \{1\} \) and of an element of \( \{1\} \times G \) as follows:
\[
(h, g) = (h, 1) \cdot (1, g)
\]
Conversely, one can see that this observation characterizes the bicrossed product. Again, we refer to [5, 12] for the details.

**Theorem 1.4.** Let \( E \) be a group \( H, G \leq E \) be subgroups such that any element of \( E \) can be written uniquely as a product of an element of \( H \) and an element of \( G \). Then there exists a matched pair \((H, G, \alpha, \beta)\) such that
\[
\theta : H \bowtie G \to E, \quad \theta(h, g) = hg
\]
is group isomorphism.

The maps \( \alpha \) and \( \beta \) play in fact a symmetric role.

**Proposition 1.5.** Let \( \Lambda = (H, G, \alpha, \beta) \) be a matched pair of groups. Then

(i) \( \tilde{\Lambda} = (G, H, \tilde{\alpha}, \tilde{\beta}) \), where \( \tilde{\alpha} \) and \( \tilde{\beta} \) are given by
\[
\tilde{\alpha} : H \times G \to G, \quad \tilde{\alpha}(h, g) = \left(\beta(g^{-1}, h^{-1})\right)^{-1}
\]
\[
\tilde{\beta} : H \times G \to H, \quad \tilde{\beta}(h, g) = \left(\alpha(g^{-1}, h^{-1})\right)^{-1}
\]
for all \( h \in H \) and \( g \in G \) is a matched pair of groups.

(ii) The map
\[
\chi : \left(H_{\alpha \bowtie \beta} G\right)^{\text{op}} \to G_{\alpha \bowtie \beta} H, \quad \chi(h, g) = (g^{-1}, h^{-1})
\]
is a group isomorphism. In particular,
\[
\xi : H_{\alpha \bowtie \beta} G \to G_{\alpha \bowtie \beta} H, \quad \xi(h, g) = \left(g \triangleleft (g^{-1} \triangleright h^{-1}), (g^{-1} \triangleright h^{-1})^{-1}\right)
\]
is a group isomorphism.

Proof. The proof is a straightforward verification. \qed

Remark 1.6. Let $H$ and $G$ be two groups as above and let $\beta : G \times H \to G$ be the trivial action. Then $(H,G,\alpha,\beta)$ is a matched pair if and only if the map $\alpha : G \times H \to H$ is an action of $G$ on $H$ as group automorphisms. In this case the bicrossed product is the semidirect product $H \rtimes G$.

Assume now that the map $\alpha$ is the trivial action. We obtain from (8) that $\tilde{\beta}$ is trivial. Keeping in mind that the bicrossed product $G \tilde{\bowtie} H$ is a semidirect product of $H$ and $G$, we can invoke Proposition 1.5 (ii) to conclude that $H \bowtie G \cong G \rtimes \tilde{\alpha} H$.

1.2. Universality properties. Let $\Lambda = (H,G,\alpha,\beta)$ be a matched pair of groups. We associate to $\Lambda$ two categories such that the bicrossed product of $H$ and $G$ becomes an initial object in one of them and a final object in the other.

Define the category $\Lambda \mathcal{C}$ as follows: the objects of $\Lambda \mathcal{C}$ are pairs $(X,(u,v))$ where $X$ is a group, $u : H \to X$, $v : G \to X$ are group morphisms and $(i_H,i_G)$ and $(\pi_H,\pi_G)$ are the canonical inclusions and projections of $H$ and $G$ inside their bicrossed product.

Define the category $\mathcal{C}_\Lambda$ as follows: the objects of $\mathcal{C}_\Lambda$ are pairs $(X,(u,v))$ where $X$ is a group, $u : X \to H$, $v : X \to G$ are two maps such that the following two compatibility condition holds:

$$u(xy) = u(x)(v(x) \bowtie u(y)), \quad v(xy) = (v(x) \bowtie u(y))v(y)$$

for all $x, y \in X$. A morphism in $\mathcal{C}_\Lambda$

$$f : (X,(u,v)) \to (X',(u',v'))$$

is a morphism of groups $f : X \to X'$ such that $u' \circ f = u$ and $v' \circ f = v$. It can be checked that $(H \bowtie G, (i_H,i_G))$ is an object in $\Lambda \mathcal{C}$, where $i_H$ and $i_G$ are the canonical inclusions of $H$ and $G$ inside their bicrossed product.

Proposition 1.7. Let $\Lambda = (H,G,\alpha,\beta)$ be a matched pair of groups. Then

(i) $(H \bowtie G, (i_H,i_G))$ is an initial object of $\Lambda \mathcal{C}$.

(ii) $(H \bowtie G, (\pi_H,\pi_G))$ is a final object of $\mathcal{C}_\Lambda$.

Proof. (i) Let $(X,(u,v)) \in \Lambda \mathcal{C}$. We have to prove that there exists a unique morphism of groups $w : H \bowtie G \to X$ such that $w \circ i_H = u$ and $w \circ i_G = v$.

Assume that $w$ satisfies this condition. Then using (6) we have:

$$w((h,g)) = w((h,1) \cdot (1,g)) = w((h,1))w((1,g)) = (w \circ i_H)(h)(w \circ i_G)(g) = u(h)v(g)$$
for all \( h \in H \) and \( g \in G \) and this proves that \( w \) is unique.

If we define \( w : H \bowtie G \rightarrow X, \quad w(h, g) = u(h)v(g) \)

then

\[
w((h_1, g_1) \cdot (h_2, g_2)) = w(h_1(g_1 \triangleright h_2), (g_1 \triangleleft h_2)g_2) \\
= u(h_1)u(g_1 \triangleright h_2)v(g_1 \triangleleft h_2)v(g_2) \\
= u(h_1)v(g_1)u(h_2)v(g_2) \\
= w((h_1, g_1))w((h_2, g_2))
\]

showing that \( w \) is a morphism of groups.

Part (ii) follows by a similar argument. \( \square \)

Straightforward from Proposition 1.7 we obtain the description of morphisms between a group and a bicrossed product.

**Corollary 1.8.** Let \( E \) be a group and \((H, G, \alpha, \beta)\) a matched pair. Then

(i) \( w : H \bowtie G \rightarrow E \) is a group morphism if and only if there exist \( u : H \rightarrow E \) and \( v : G \rightarrow E \) group morphisms such that

\[
v(g)u(h) = u(g \triangleright h)v(g \triangleleft h), \quad \text{and} \quad w(h, g) = u(h)v(g)
\]

for all \( h \in H \) and \( g \in G \).

(ii) \( w : E \rightarrow H \bowtie G \) is a morphism of groups if and only if there exist \( u : E \rightarrow H \) and \( v : E \rightarrow G \) two maps such that

\[
u(xy) = u(x)(v(x) \triangleright u(y)), \quad v(xy) = (v(x) \triangleleft u(y))v(y)
\]

and \( w(x) = (u(x), v(x)) \) for all \( x, y \in E \).

**Remark 1.9.** Corollary 1.8 can be used to describe all morphisms or isomorphisms between two matched pairs \( H_\alpha \bowtie G \) and \( H_\beta \bowtie G \). However, the descriptions are rather technical and we will not include them here.

### 2. Bicrossed products between finite cyclic groups

As mentioned in the Introduction the question of describing all groups which arise as bicrossed products of two given groups was asked by Ore. The first and, by our knowledge, the only systematic study of this kind, for groups which arise as bicrossed products of two finite cyclic groups, was employed by J. Douglas in 1951. In his first paper on the subject [2, pag. 604] Douglas formulates the problem he wants to solve: describe all groups all whose elements are expressible in the form \( a^ib^j \) where \( a \) and \( b \) are independent elements of order \( n \) and, respectively, \( m \). What Douglas refers to as independent elements is in fact the condition that the cyclic groups generated by each of these elements have trivial intersection. Therefore the problem can be formulated as follows: describe all groups which arise as bicrossed products of two finite cyclic groups.

In what follows \( C_n \) and \( C_m \) will be two cyclic groups of orders \( n \) and \( m \). We denote by \( a \) and \( b \) a fixed generator of \( C_n \) and, respectively, \( C_m \). For any positive integer \( k \) we denote
by \( \mathbb{Z}_k \) the ring of residue classes modulo \( k \) and by \( S(\mathbb{Z}_k) \) the set of bijective functions from \( \mathbb{Z}_k \) to itself. Let \( \alpha : C_n \times C_n \to C_n \) and \( \beta : C_m \times C_n \to C_m \) be two actions. They are completely determined by two maps \( \theta \in S(\mathbb{Z}_n) \) and \( \phi \in \mathbb{Z}_m \) such that
\[
\alpha(b, a^x) = a^{\theta(x)}, \quad \beta(b^y, a) = b^{\phi(y)}
\]
for any \( x \in \mathbb{Z}_n, y \in \mathbb{Z}_m \). Douglas [2, Theorem I] obtained necessary and sufficient conditions on the pair of maps \((\theta, \phi) \in S(\mathbb{Z}_n) \times S(\mathbb{Z}_m)\) which would make \((C_n, C_m, \alpha, \beta)\) a matched pair. The functions \( \theta \) and \( \phi \) satisfying his conditions were called conjugate special substitutions.

Finding all pairs of conjugate special substitutions (or, equivalently, all matched pairs between two finite cyclic groups) seems to be still an open question. Furthermore, solving this problem does not provide an answer to the classification problem for the associated bicrossed products. In particular, it does not indicate whether a bicrossed product could not be obtained more easily as a semidirect product.

We investigate the structure of bicrossed products of two finite cyclic groups, one of which has prime order. We shall prove now our main result.

**Theorem 2.1.** Let \( p \) be a prime number, and \( m \geq 2 \) a positive integer. A bicrossed product between two cyclic groups of orders \( p \) and respectively \( m \) is isomorphic to a semidirect product between cyclic groups of the same orders \( p \) and \( m \).

**Proof.** Let \( G \) be a group, \( C_p \) and \( C_m \) two fixed cyclic subgroups of \( G \), of orders \( p \) and \( m \) respectively, such that \( G = C_p \bowtie C_m \). Let \( a, b \) be generators for \( C_p \) and \( C_m \) respectively. We can certainly assume that \( C_p \) and \( C_m \) are not normal in \( G \).

Let \( H = C_m \cap aC_m a^{-1} \). A subgroup of a finite cyclic group is uniquely determined by its order so \( aHa^{-1} = H \). Since \( C_m \) and \( aC_m a^{-1} \) are abelian, \( H \) is central in both. Also note that \( C_m \neq aC_m a^{-1} \), because we assumed \( C_m \) is not normal in \( G \). The centralizer \( C(H) \) of \( H \) in \( G \) must then strictly contain \( C_m \). Since \( C_m \) has index \( p \) in \( G \), it is maximal. It follows that \( C(H) = G \), that is \( H \) is a central subgroup of \( G \).

In fact, for any \( 1 \leq k \leq p - 1 \), \( H \) is the intersection between \( C_m \) and \( a^k C_m a^{-k} \). Indeed, \( a^k \) generates \( C_p \), so any subgroup of \( G \) normalized by \( a^k \) must also be normalized by \( a \).

We are now going to work in the quotient group \( G/H \). Let us denote by
\[
\pi : G \to G/H
\]
the canonical projection. For any \( 1 \leq k \leq p - 1 \), the groups \( \pi(C_m) \) and \( \pi(a^k C_m a^{-k}) \) intersect trivially. We can now apply a theorem of Frobenius [10, Theorem 9.11 and Exercise 9.9] for the group \( G/H \) to conclude that the subgroup \( \pi(C_p) \) is normal. This means that \( \pi(bab^{-1}) = \pi(a^t) \) for some \( 1 \leq t \leq p - 1 \), or, equivalently, that
\[
bab^{-1} = a^tc
\]
for some \( c \in H \). Raising this identity to power \( p \), we find that \( c^p = 1 \). We may assume that \( c \neq 1 \), otherwise \( C_p \) would be normal in \( G \). Also, \( t \neq 1 \) otherwise \( a^{-1}ba = cb \in C_m \), which means that \( C_m \) is normal in \( G \), contrary to our assumption. Hence, we can find an integer \( u \) such that \( ut - 1 \equiv 1 \mod p \).
With the notation $\tilde{a} = ae^u$, we then have $b\tilde{a}b^{-1} = \tilde{a}$. The proof is now finished: the subgroup of $G$ generated by $\tilde{a}$ is normal in $G$, has order $p$ and intersects $C_m$ trivially. □

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(A.L.A., A.C., G.M.) FAculty of Mathematics and Computer Science, University of Bucharest, Str. Academiei 14, RO-70109 Bucharest 1, Romania

E-mail address: ana.agore@fmi.unibuc.ro, chirivasitua@gmail.com, gigel.militaru@fmi.unibuc.ro

(B.I.) Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260