Relativistic treatment of Verlinde’s emergent force in Tsallis’ statistics

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Abstract

Following Chakrabarti, Chandrasekhar, and Naina [Physica A 389 (2010) 1571], we attempt a classical relativistic treatment of Verlinde’s emergent entropic force conjecture by appealing to a relativistic Hamiltonian in the context of Tsalli’s statistics. The ensuing partition function becomes the classical one for small velocities. We show that Tsallis’ relativistic (classical) free particle distribution at temperature $T$ can generate Newton’s gravitational force’s $r^{-2}$ distance’s dependence. If we want to repeat the concomitant argument by appealing to Renyi’s distribution, the attempt fails and one needs to modify the conjecture. Keywords: Tsallis’ and Renyi’s relativistic distributions, classical partition function, entropic force.

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1 Introduction

In 2011, Verlinde [1] put forward a conjecture that connects gravity to an entropic force. Gravity would then arise out of information regarding the positions of material bodies (‘it from bit’). This idea links a thermal gravity-treatment to ’t Hooft’s holographic principle. As a consequence, gravitation ought to be regarded as an emergent phenomenon. Verlinde’s conjecture attained considerable reception (just as an example, see [2]). For a superb overview on the statistical mechanics of gravitation, we recommend Padmanabhan’s work [3], and references therein.

Verlinde’s initiative originated works on cosmology, the dark energy hypothesis, cosmological acceleration, cosmological inflation, and loop quantum gravity. The literature is immense [4]. A relevant contribution to information theory is that of Guseo [5], who proved that the local entropy function, related to a logistic distribution, is a catenary and vice versa. Such invariance may be explained, at a deeper level, through the Verlindes conjecture on the origin of gravity, as an effect of the entropic force. Guseo puts forward a new interpretation of the local entropy in a system, as quantifying a hypothetical attraction force that the system would exert [5].

The present effort does not deal with any of these issues. What we will do is to show that a simple classical reasoning centered on Tsallis’ relativistic probability distributions proves Varlinde’s conjecture. For Renyis’s relativistic instance, one needs to modify the conjecture to achieve a similar result.

Our point of departure is Ref. [6], in which their authors studied a canonical ensemble of N particles for a classical relativistic ideal gas, and found its specific heat in the Tsallis-Mendes-Plastino (TMP) scenario [7]. We will not use here the TMP scenario. Inspired by [6], we appeal as well to our previous effort [8] for non-relativistic results and deal with Tsallis’ statistics with linear constraints as a priori information [7]. In addition to finding, for the first time ever, relativistic Verlinde-results in a Tsallis’context, we will, for the sake of completeness, register some advances regarding the relativistic Tsallis scenario with linear constraints for the ideal gas.
2 Tsallis’ relativistic partition function for the free particle

The celebrated and well-known Tsallis entropy is a generalization of Shannon’s one, that depends on a free real parameter \( q \) [7].

The \( q < 1 \) instance

We consider first the case \( q < 1 \). This case is not relevant to our Verlinde’s endeavor [8], but is a logical addition to the results of [6].

Tsallis’ relativistic \( q \)-partition function for \( N \)-free particles of mass \( m \) reads [6]

\[
Z = \frac{V}{N! h^{3N}} \int \left[ 1 + (1 - q)\beta(\sqrt{m^2 c^4 + p^2 c^2} - mc^2) \right] \frac{1}{\beta} \, p^2 \, dp. \tag{2.1}
\]

Using spherical coordinates and integrating over the angles the precedent integral we have

\[
Z = \frac{4\pi V}{N! h^{3N}} \int_0^\infty \left[ 1 + (1 - q)\beta(\sqrt{m^2 c^4 + p^2 c^2} - mc^2) \right] \frac{1}{\beta} \, p^2 \, dp. \tag{2.2}
\]

With the change of variables \( y^2 = p^2 + m^2 c^2 \) one now has

\[
Z = \frac{4\pi V}{N! h^{3N}} \int_{mc}^{\infty} y \sqrt{y^2 - m^2 c^2} \frac{1}{\beta} \left[ 1 + (1 - q)\beta(\sqrt{m^2 c^4 + y^2 c^2} - mc) \right] \frac{1}{\beta} \, dy. \tag{2.3}
\]

Let \( x \) be given by \( y = mcx \). We have then

\[
Z = \frac{4\pi V m^3 c^3}{N! h^{3N}} \int_1^{\infty} x \sqrt{x^2 - 1} \left[ 1 + (1 - q)\beta mc^2(x - 1) \right] \frac{1}{\beta} \, dx. \tag{2.4}
\]

With \( s \) defined as \( x = s + 1 \) we obtain:

\[
Z = \frac{4\pi V m^3 c^3}{N! h^{3N}} \int_0^{\infty} \left( s^\frac{3}{2} + s^\frac{1}{2} \right) (s + 2)^\frac{1}{2} \left[ 1 + (1 - q)\beta mc^2 s \right] \frac{1}{\beta} \, ds, \tag{2.5}
\]
or

\[ Z = \frac{4\pi V m^3 e^3}{N! h^3 N} \left[ (1 - q) \beta mc^2 \right] \frac{1}{\sqrt{7}} \int_0^\infty \frac{1}{s^{3/2} (s + 2)^{3/2}} \left[ s + \frac{1}{(1 - q) \beta mc^2} \right] \frac{1}{\sqrt{7}} \, ds + \]

\[ \frac{4\pi V m^3 e^3}{N! h^3 N} \left[ (1 - q) \beta mc^2 \right] \frac{1}{\sqrt{7}} \int_0^\infty \frac{1}{s^{3/2} (s + 2)^{3/2}} \left[ s + \frac{1}{(1 - q) \beta mc^2} \right] \frac{1}{\sqrt{7}} \, ds. \quad (2.6) \]

Appealing to reference [9] we have now a result in terms of Hyper-geometric functions \( F \) and Beta functions \( B \), namely,

\[ Z = \frac{4\pi V m^3 e^3}{N! h^3 N} \left[ (1 - q) \beta mc^2 \right] \frac{1}{\sqrt{7}} \left[ \frac{B \left( \frac{5}{2}, \frac{1}{1 - q} - 3 \right)}{\beta mc^2 (1 - q)} \right. \]

\[ F \left( -\frac{1}{2}, \frac{5}{2}, \frac{1}{1 - q} - \frac{1}{2}; 1 - \frac{1}{2 \beta mc^2 (1 - q)} \right) + \]

\[ B \left( \frac{3}{2}, \frac{1}{1 - q} - 2 \right) F \left( -\frac{1}{2}, \frac{3}{2}, \frac{1}{1 - q} - \frac{1}{2}; 1 - \frac{1}{2 \beta mc^2 (1 - q)} \right) \]. \quad (2.7) \]

For \( \beta mc^2 >> 1, mc^2 >> k_B T \), we are in the non-relativistic case and have

\[ Z = \frac{2\pi V}{N! h^3 N} \left[ \frac{2m}{\beta (1 - q)} \right] \Gamma \left( \frac{3}{2} \right) \Gamma \left( \frac{1}{1 - q} - \frac{3}{2} \right) \frac{1}{\Gamma \left( \frac{1}{1 - q} \right)}. \quad (2.8) \]

**The case \( q > 1 \)**

Let us now consider gravitationally relevant [8] case \( q > 1 \). We have for the partition function

\[ Z = \frac{4\pi V m^3 e^3}{N! h^3 N} \int_0^\infty \left( s^3 + s^1 \right) (s + 2)^{3/2} \left[ 1 - (q - 1) \beta mc^2 s \right] \frac{1}{s^{1/2}} \, ds. \quad (2.9) \]

Integrating on the angles we have again

\[ Z = \frac{4\pi V m^3 e^3}{N! h^3 N} \int_0^\infty \left( s^3 + s^1 \right) (s + 2)^{3/2} \left[ 1 - (q - 1) \beta mc^2 s \right] \frac{1}{s^{1/2}} \, ds, \]

\[ (2.10) \]
or

\[ Z = \frac{4\pi V m^3 c^3}{N!h^{3N}} [(q-1)\beta mc^2]^{\frac{1}{q-1}} \int_0^1 s^{\frac{3}{2}} (s + 2)^{\frac{1}{2}} \left[ \frac{1}{(q-1)\beta mc^2} - s \right]^{\frac{1}{q-1}} ds + \]

\[ \frac{4\pi V m^3 c^3}{N!h^{3N}} [(q-1)\beta mc^2]^{\frac{1}{q-1}} \int_0^1 s^{\frac{1}{2}} (s + 2)^{\frac{1}{2}} \left[ \frac{1}{(q-1)\beta mc^2} - s \right]^{\frac{1}{q-1}} ds. \]

(2.11)

By recourse to [9] we now obtain

\[ Z = \frac{2\pi V}{N!h^{3N}} \left[ \frac{2m}{\beta m(q-1)} \right]^{\frac{3}{2}} \left[ B\left(\frac{5}{2}, \frac{1}{q-1} + 1\right) \right. \times \]

\[ F\left(-\frac{1}{2}, \frac{5}{2}, \frac{7}{2} + \frac{1}{q-1}; -\frac{1}{2\beta mc^2(q-1)}\right) + \]

\[ B\left(\frac{3}{2}, \frac{1}{q-1} + 1\right) F\left(-\frac{1}{2}, \frac{3}{2}, \frac{5}{2} + \frac{1}{q-1}; -\frac{1}{2\beta mc^2(q-1)}\right) \].

(2.12)

For \( \beta mc^2 >> 1 \), the classic case, the partition function reads

\[ Z = \frac{2\pi V}{N!h^{3N}} \left[ \frac{2m}{\beta (q-1)} \right]^{\frac{3}{2}} \Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{1}{q-1} + 1\right) \frac{\Gamma\left(\frac{1}{q-1} + 1\right)}{\Gamma\left(\frac{1}{1-q} + \frac{5}{2}\right)}, \]

(2.13)

which is the usual non relativistic Tsalli’s partition function for \( q > 1 \) already obtained in [8]. Figure 1 displays the graph of the function \( H(T) \) given by

\[ Z = \frac{2\pi V}{N!h^{3N}} \left[ \frac{2m}{\beta (q-1)} \right]^{\frac{3}{2}} H(T), \]

(2.14)

for \( q = \frac{4}{3} \), the specific \( q \)-value needed for gravitational considerations [8]. It tells us that \( Z \) is always positive, as it should be.
3 Tsallis’ relativistic mean energy of the free particle

Case $q < 1$

Let us now calculate the average energy corresponding, firstly in the case $q < 1$. For it we have

$$Z < U > = \frac{V}{N!h^3N} \int [\sqrt{m^2c^4 + p^2c^2} - mc^2] \times$$

$$\left[ 1 + (1 - q) \beta (\sqrt{m^2c^4 + p^2c^2} - mc^2) \right]^{\frac{1}{q-1}} d^4p,$$

or

$$Z < U > = \frac{V}{N!h^3N} \int [\sqrt{m^2c^4 + p^2c^2}] \times$$

$$\left[ 1 + (1 - q) \beta (\sqrt{m^2c^4 + p^2c^2} - mc^2) \right]^{\frac{1}{q-1}} d^4p - mc^2Z. \quad (3.16)$$

With changes in the variables similar to those made for the partition function, we obtain here

$$Z < U > = \frac{4\pi V m^4C^5}{N!h^3N} \int_0^\infty x^{\frac{3}{2}} (x + 1)(\sqrt{x + 2} \times$$

$$\left[ 1 + (1 - q) \beta mc^2 x \right]^{\frac{1}{q-1}} dx. \quad (3.17)$$

This last equation can be rewritten as

$$Z < U > = \frac{4\pi V m^4C^5}{N!h^3N} \beta mc^2(1 - q) \int_0^\infty x^{\frac{3}{2}} (x + 1)(\sqrt{x + 2} \times$$

$$\left[ x + \frac{1}{(1 - q)\beta mc^2} \right]^{\frac{1}{q-1}} dx. \quad (3.18)$$

Returning again to reference [9], we obtain for $< U >$

$$< U > = \sqrt{2} \frac{4\pi V m^4c^5}{N!h^3NZ} \left[ \frac{1}{\beta mc^2(1 - q)} \right]^{\frac{3}{2} + \frac{1}{q-1}} \int \frac{B \left( \frac{7}{2}, \frac{1}{1-q} - 4 \right)}{\beta mc^2(1 - q)} \times$$
\[ F \left( -\frac{1}{2}, \frac{7}{2}, 1 - q, -\frac{1}{2}, 1 - \frac{1}{2\beta mc^2(1 - q)} \right) + \\
B \left( \frac{5}{2}, \frac{1}{1 - q} - 3 \right) F \left( -\frac{1}{2}, \frac{5}{2}, 1 - q, -\frac{1}{2}, 1 - \frac{1}{2\beta mc^2(1 - q)} \right) \right]. \] (3.19)

From this last equation we obtain the mean energy expression for the non-relativistic case
\[ \langle U \rangle = \frac{3}{\beta[2 - 5(1 - q)]}. \] (3.20)

**Case q larger than one**

When \( q > 1 \) we have
\[ Z \langle U \rangle = \frac{4\pi V m^4 c^5}{N!h^3} \int_0^\infty x^{\frac{1}{2}}(x + 1)(\sqrt{x + 2} \times \\
\left[ 1 - (q - 1)\beta mc^2 x \right]^{\frac{1}{q - 1}} dx - mc^2 Z. \] (3.21)

Making a similar reasoning as for the case \( q < 1 \) we obtain
\[ \langle U \rangle = \sqrt{2} \frac{4\pi V m^4 c^5}{N!h^3 Z} \left[ \frac{1}{\beta mc^2(q - 1)} \right]^{\frac{1}{q - 1}} \left[ \frac{B \left( \frac{5}{2}, \frac{1}{q - 1} + 1 \right)}{\beta mc^2(q - 1)} \right] \times \\
F \left( -\frac{1}{2}, \frac{7}{2}, 1 - q, -\frac{1}{2}, 1 + \frac{9}{2}, 2\beta mc^2(q - 1) \right) + \\
B \left( \frac{5}{2}, \frac{1}{q - 1} + 1 \right) F \left( -\frac{1}{2}, \frac{5}{2}, 1 - q, -\frac{1}{2}, 1 + \frac{7}{2}, 2\beta mc^2(q - 1) \right) \right]. \] (3.22)

For \( \beta mc^2 >> 1 \) (the non-relativistic case) we obtain the result of [8], i.e.,
\[ \langle U \rangle = \frac{3}{\beta[2 + 5(q - 1)]}. \] (3.23)
4 Specific heat in the linear constraints Tsal- lis’ scenario

Let us now calculate the specific heat for the case \( q = \frac{4}{3} \), relevant for Verlinde-endeavors [8]. This was not done in [6]. We should first note, with respect to Hyper-geometric functions, that

\[
\frac{d}{dz} F(\alpha, \beta, \gamma; z) = -\alpha \beta F(\alpha + 1, \beta + 1, \gamma + 1; z). \tag{4.24}
\]

We now use the notation

\[
F_1 = F\left(-\frac{1}{2}, \frac{7}{2}, \frac{9}{2}; \frac{3k_B T}{2mc^2}\right), \quad F_2 = F\left(-\frac{1}{2}, \frac{5}{2}, \frac{7}{2}; \frac{3k_B T}{2mc^2}\right), \quad F_3 = F\left(-\frac{1}{2}, \frac{3}{2}, \frac{5}{2}; \frac{3k_B T}{2mc^2}\right), \tag{4.25}
\]

\[
F_4 = F\left(-\frac{1}{2}, \frac{1}{2}, \frac{9}{2}; \frac{3k_B T}{2mc^2}\right), \quad F_5 = F\left(-\frac{1}{2}, \frac{1}{2}, \frac{7}{2}; \frac{3k_B T}{2mc^2}\right), \quad F_6 = F\left(-\frac{1}{2}, \frac{1}{2}, \frac{5}{2}; \frac{3k_B T}{2mc^2}\right). \tag{4.26}
\]

Thus, we can write

\[
< \mathcal{U} > = 3k_B T \frac{3k_B T}{mc^2} B\left(\frac{7}{2}, 4\right) F_1 + B\left(\frac{5}{2}, 4\right) F_2 + B\left(\frac{3}{2}, 4\right) F_3, \tag{4.31}
\]

and, for the specific heat we have then

\[
C = \frac{\partial < \mathcal{U} >}{\partial T} = \frac{9k_B^2 T}{mc^2} B\left(\frac{7}{2}, 4\right) F_1 - \frac{21k_B T}{8mc^2} B\left(\frac{7}{2}, 4\right) F_4 - \frac{5}{8} B\left(\frac{5}{2}, 4\right) F_5 + \frac{3k_B T}{mc^2} B\left(\frac{5}{2}, 4\right) F_2 + B\left(\frac{3}{2}, 4\right) F_3 - \frac{3k_B T}{mc^2} B\left(\frac{5}{2}, 4\right) F_2 - \frac{15k_B T}{8mc^2} B\left(\frac{5}{2}, 4\right) F_5 - \frac{3}{8} B\left(\frac{3}{2}, 4\right) F_6. \tag{4.32}
\]

This expression is plotted in Figure 2. We see that the specific heat is always positive, as it happens in the non-relativistic case [8].
5 The relativistic, Tsallis entropic force

We arrive now at our main present goal. We specialize things now to $q = \frac{4}{3}$. Why do we select this special value $q = \frac{4}{3}$? There is a solid reason. This is because

$$S = \ln_q Z + Z^{1-q} \beta <U>.$$ 

Since the entropic force is to be defined as proportional to the gradient of $S$, there is a unique $q$-value for which the dependence on $r$ of the entropic force is $\sim r^{-2}$ when $\nu = 3$. Thus we obtain, for $q = 4/3$,

$$S = 3 - (3 - \beta <U>) Z^{-\frac{1}{3}}.$$ 

(5.1)

From (2.12) we can write

$$<Z> = a r^3,$$ 

(5.2)

from which it is obtained that

$$S = 3 - \frac{3 - \beta <U>}{a^\frac{3}{2} r}.$$ 

(5.3)

Following Verlinde [1] we define the entropic force as

$$\vec{F}_e = -\frac{\lambda}{\beta} \vec{\nabla}S,$$ 

(5.4)

where $\vec{\nabla}$ indicates the four-gradient in Minkowskian space.

$$\vec{F}_e = -\frac{\lambda}{\beta} \frac{3 - \beta <U>}{a^\frac{3}{2} r^2} \vec{e}_r,$$ 

where $\vec{e}_r$ is the radial unit vector. We see that $F_e$ acquires an appearance quite similar to that of Newton’s gravitational one, as conjectured by Verlinde in [1]. In Figures 3 and 4 the function $L = 3 - \beta <U>$ is plotted. We see that $L$ is always positive. This entails that the relativistic entropic force is purely gravitational.
6 The relativistic, Renyi’s entropic force

In Renyi’s approach to our problem [8] the entropy is

$$S = \ln Z + \ln[1 + (1 - \alpha)\beta < U >]\_+^{\frac{1}{\alpha}}. \quad (6.1)$$

For $\alpha = \frac{4}{3}$, the expression for the entropy is

$$S = \ln Z + \ln \left[1 - \beta \frac{< U >}{3}\right]_+^{-3}. \quad (6.2)$$

The second term on the right hand of (6.2) is independent of $r$. Additionally, from (5.2) we obtain

$$\ln Z = 3 \ln r + \ln a. \quad (6.3)$$

Here we need to derive the entropy with respect to the area, thus changing Verlindes conjecture. As in the non-relativistic case [8], we have then

$$\vec{F}_e = -\frac{\lambda}{\beta} \frac{\partial S}{\partial A} \vec{e}_r = -\frac{\lambda}{\beta} \frac{3}{8\pi r^2} \vec{e}_r. \quad (6.4)$$

This is again a gravitational expression for the entropic force.

7 Conclusions

We obtained here the relativistic partition function $Z$ of Tsalli’s theory with linear constraints, that adequately reduces itself to its non-relativistic counterpart for small velocities.

We do the same for the mean value of the energy $<U>$ for the relativistic Hamiltonian of the ideal gas.

We obtain the associated specific heat that turns out to be positive, as befits an ideal gas.

From $Z$ and $<U>$ we obtained the relativistic entropy $S$

We have presented two very simple relativistic classical realizations of Verlinde’s conjecture. The Tsallis treatment, for $q = 4/3$, seems to be neater, as the entropic force is directly associated to the gradient of Tsallis’ entropy $S_q$, which acts as a ”potential”, as Verlinde prescribes. This is not so in the Renyi instance, in which one has to modify Verlinde’s $F_e$ definition and derive $S$ with respect to the area.
Strictly speaking, Verlinde’s conjecture can be unambiguously proved for the Tsallis entropy with $q = 4/3$. The Renyi demonstration correspond to a modified version of Verlinde’s conjecture.
Of course, ours is a very preliminary, if significant, effort. A much more elaborate treatment would be desirable.
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Figure 1: H function
Figure 2: Specific heat
Figure 3: $L(T) = 3 - \beta <\mathcal{U}>$
Figure 4: Centered $L(T) = 3 - \beta <U>$