Convexity and Some Geometric Properties

João Xavier da Cruz Neto¹ · Ítalo Dowell Lira Melo¹ · Paulo Alexandre Araújo Sousa¹

Abstract  The main goal of this paper is to present results of existence and nonexistence of convex functions on Riemannian manifolds, and in the case of the existence, we associate such functions to the geometry of the manifold. Precisely, we prove that the conservativity of the geodesic flow on a Riemannian manifold with infinite volume is an obstruction to the existence of convex functions. Next, we present a geometric condition that ensures the existence of (strictly) convex functions on a particular class of complete manifolds, and we use this fact to construct a manifold whose sectional curvature assumes any real value greater than a negative constant and admits a strictly convex function. In the last result, we relate the geometry of a Riemannian manifold of positive sectional curvature with the set of minimum points of a convex function defined on the manifold.

Keywords  Convex function · Geodesic flow · Conformal fields · Soul of a manifold

Mathematics Subject Classification 26B25 · 53C20
1 Introduction

The concept of convexity plays a very important role in optimization theory firstly, because many objective functions are convex in a sufficiently small neighborhood of a local minimum point, and secondly, because one can establish the convergence of numerical methods to estimate minimum points for convex functions.

There are numerous convex functions with many structural implications on Riemannian manifolds, and such functions form an important link between modern analysis and geometry. The existence of convex functions implies restrictions on the geometry or topology of a complete non-compact Riemannian manifold. For example, Bishop and O’Neill [1] proved that there is no nontrivial smooth convex function on a complete Riemannian manifold with finite volume; later, Yau [2] generalized this result to show that there is no nontrivial continuous convex function on a complete manifold with finite volume. Shiohama [3] proved a result relating the existence of strictly convex functions to the topology of the Riemannian manifold, showing that if a complete Riemannian manifold admits a strictly convex function, then the manifold has at most two ends.

In this paper, we obtain a geometric-topological restriction for the existence of nontrivial convex functions on complete non-compact Riemannian manifolds. More specifically, we prove that the conservativity of the geodesic flow on a Riemannian manifold with infinite volume implies that all convex functions on the manifold are constant. This fact generalizes, in a certain sense, the result proved by Yau [2].

In optimization, it is important to establish the convergence of numerical methods to find minimum points for convex functions. Usually, the technique used in the proof of convergence makes use of curvature. The work of Neto et al. [4] and Ferreira and Oliveira [5] pioneered the minimization of convex functions on Riemannian manifolds of nonnegative sectional curvature. In 2002, Ferreira and Oliveira [6] established a proximal point method in Hadamard manifolds (non-positive sectional curvature).

Under the assumption that the sectional curvature of the manifold is bounded from below, Wang et al. [7] established a convergence result for the cyclic subgradient projection algorithm. However, we have not found any examples of a Riemannian manifold whose sectional curvature changes sign endowed with a (explicit example of a) strictly convex function. In this work, we construct a Riemannian manifold whose sectional curvature assumes any real value greater than a negative constant and admits a strictly convex function.

Finally, we establish a result relating a geometric property of Riemannian manifolds of positive sectional curvature with the set of minimum points of a convex function defined on the manifold. Using an example, we illustrate how this result is useful in choosing the initial point for an iterative method that seeks the minimum of a convex function.

This paper is organized as follows. In Sect. 2, we introduce some notations, basic definitions and important properties of Riemannian manifolds. In Sect. 3, we prove that the conservativity of the geodesic flow of a Riemannian manifold with infinite volume implies that all convex functions on the manifold are constant. In Sect. 4, we construct a manifold whose sectional curvature assumes any real value greater than a negative constant and admits a strictly convex function. In Sect. 5, we prove that if
the set of minimum points of a convex function, defined on a Riemannian manifold of positive sectional curvature, is not empty, then we can minimize it using souls of the manifold.

2 Preliminaries

In this section, we present some basic results from Riemannian geometry. All the manifolds and vector fields considered in this paper are assumed to be differentiable (smooth). Let \( M \) be a (smooth) manifold. We denote the space of (smooth) vector fields over \( M \) by \( \mathfrak{X}(M) \). The tangent bundle of \( M \) will be denoted by \( TM \) and the ring of smooth functions over \( M \) by \( \mathcal{D}(M) \).

**Definition 2.1** An \( r \)-covariant tensor field \( \omega \) on a Riemannian manifold \( M \) is a \( \mathcal{D}(M) \)-multilinear mapping

\[
\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \mathcal{D}(M).
\]

A couple \( (M, \langle \cdot, \cdot \rangle) \), where \( M \) is a manifold and \( \langle \cdot, \cdot \rangle \) is a smooth metric (inner product on each tangent space varying smoothly on \( M \)), is called a Riemannian manifold and \( \langle \cdot, \cdot \rangle \) is a Riemannian metric. The metric induces a map \( f \in \mathcal{D}(M) \to \nabla f \in \mathfrak{X}(M) \) that associates a gradient to each \( f \). Let \( D \) be the Levi-Civita connection associated with \( (M, \langle \cdot, \cdot \rangle) \). The differential of a vector field \( X \in \mathfrak{X}(M) \) is the \( \mathcal{D}(M) \)-linear operator \( A_X : \mathfrak{X}(M) \to \mathfrak{X}(M) \) given by \( A_X(Y) := D_Y X \). To each point \( p \in M \), we assign the linear map \( A_X(p) : T_p M \to T_p M \) defined by \( A_X(p)v = D_v X \). In particular, if \( X = \nabla f \), then \( A_X(p) \) is the Hessian of \( f \) at \( p \) and is denoted by Hess \( f \).

Given a vector field \( V \in \mathfrak{X}(M) \) on a Riemannian manifold \( M \) and an \( r \)-covariant tensor field \( \omega \), the Lie derivative of \( \omega \) with respect to \( V \) is defined by

\[
(\mathcal{L}_V \omega)(X_1, \ldots, X_r) := V(\omega(X_1, \ldots, X_r)) - \sum_{i=1}^r \omega(X_1, \ldots, [V, X_i], \ldots, X_r).
\]

For instance, if \( \omega = \langle \cdot, \cdot \rangle \), then \( (\mathcal{L}_V \langle \cdot, \cdot \rangle)(X, Y) = \langle DX V, Y \rangle + \langle X, DY V \rangle \). We say that \( V \in \mathfrak{X}(M) \) is a conformal vector field iff there exists some \( \phi \in \mathcal{D}(M) \), which is called the conformal factor of \( V \), such that

\[
\mathcal{L}_V \langle \cdot, \cdot \rangle = 2\phi \langle \cdot, \cdot \rangle.
\]

An interesting case of a conformal vector field \( V \) occurs when \( DX V = \phi X \) for all \( X \in \mathfrak{X}(M) \). In this case, we say that \( V \) is closed.

Now, let \( \pi : TM \to M \) be the projection map. For any normal neighborhood \( U \) of a point \( p \in M \), there is a canonical map \( \tau : \pi^{-1}(U) \to T_p M \) defined as follows: For \( Z = (q, w) \in \pi^{-1}(U) \), the image \( \tau(Z) \) is obtained by a parallel translation of \( w \) along the unique geodesic arc in \( U \) joining the point \( q = \pi(Z) \) to \( p \). The
connection map corresponding to \( D \) is a map \( \kappa : T(TM) \to TM \) that induces, for any \( Z = (p, z) \in TM \), a linear map of \( T_Z(TM) \) into \( T_{\pi(Z)}M \). This map is defined as follows: Let \( A \in T_Z(TM) \) and \( \sim Z : t \to Z(t) \) be a path in \( TM \) representing \( A \) at \( t = 0 \); then,

\[
\kappa(A) := \lim_{t \to 0} \frac{\tau(\sim Z(t)) - z}{t}.
\]

The induced Riemannian metric \( G \) in \( TM \) (Riemann-Sasaki metric) is determined by the rule

\[
G(A, B) := \langle d\pi(A), d\pi(B) \rangle + \langle \kappa(A), \kappa(B) \rangle,
\]

where \( A, B \in T_Z(TM) \) and \( Z \in TM \). For more details on the Riemann-Sasaki metric, see, for instance, Kowalski [8].

To define the soul of a Riemannian manifold, one first requires the following concepts: A Riemannian submanifold \( S \) of \( M \) is said to be totally geodesic iff all geodesics in \( S \) are also geodesics in \( M \), and is said to be totally convex iff, for all points \( p, q \) in \( S \), all geodesics joining \( p \) to \( q \) are contained in \( S \).

**Definition 2.2** Let \( M \) be a complete manifold and \( S \subset M \) be a compact totally convex, totally geodesic submanifold such that \( M \) is diffeomorphic to the normal bundle of \( S \). The submanifold \( S \) is called a soul of \( M \).

In general, the soul is not uniquely determined, but any two souls of \( M \) are isometric. Gromoll and Meyer [9] proved that when the sectional curvature of \( M \) is positive, the soul \( S \) of \( M \) consists of a single point (simple point) and \( M \) is diffeomorphic to a Euclidean space. A point \( p \in M \) is said to be simple iff there are no geodesic loops in \( M \) that are closed at \( p \). Gromoll and Meyer proved that the set of simple points in \( M \) is open, implying that the set of souls cannot consist of a single point [9].

3 Conservativity and Nonexistence of Convex Functions

Let \( M \) be a complete Riemannian manifold and \( \theta = (p, v) \in TM \), and denote by \( \gamma_{\theta}(t) \) the unique geodesic with initial conditions \( \gamma_{\theta}(0) = p \) and \( \gamma_{\theta}'(0) = v \). For a given \( t \in \mathbb{R} \), we define a diffeomorphism of the tangent bundle \( TM \)

\[
\varphi_t : TM \to TM
\]

as \( \varphi_t(\theta) := (\gamma_{\theta}(t), \gamma_{\theta}'(t)) \). The family of diffeomorphisms \( \varphi_t \) is in fact a flow (called a geodesic flow), that is, it satisfies \( \varphi_{t+s} = \varphi_t \circ \varphi_s \).

Denote by \( SM \) the unit tangent bundle of \( M \), that is, the subset of \( TM \) given by those pairs \( \theta = (p, v) \) such that \( v \) has norm one. Since geodesics travel with constant speed, we see that \( \varphi_t \) leaves \( SM \) invariant, that is, given \( \theta \in SM \) then for all \( t \in \mathbb{R} \), we have \( \varphi_t(\theta) \in SM \). So, \( \varphi_t \) preserves the Liouville measure of
the unit tangent bundle. The Liouville measure may be described as follows: Every inner product in a finite-dimensional vector space induces a volume element in that space, relative to which the cube spanned by any orthonormal basis has volume 1. In particular, the Riemannian metrics induces a volume element \( dv \) on each tangent space of \( M \). Integrating this volume element along \( M \), we get a volume measure \( dx \) on the manifolds itself. The Liouville measure of \( TM \) is given, locally, by the product \( \mu = dx \cdot dv \). If \( m \) denotes the Liouville measure restricted to the unit tangent bundle \( SM \), it is known (see, for instance, Paternain [10]) that \( m \) is invariant under the geodesic flow, i.e., the diffeomorphism \( \varphi_t : SM \to SM \) is a measure-preserving transformation for all \( t \in \mathbb{R} \). A transformation \( \varphi : SM \to SM \) is measure-preserving iff \( m(B) = m(\varphi^{-1}(B)) \) for all \( B \subset SM \). Equivalently, we say that the measure \( m \) is invariant under \( \varphi \).

**Definition 3.1** The geodesic flow is conservative with respect to the Liouville measure iff, given any measurable set \( A \subset SM \) with positive measure, for \( m \)-almost all \( \theta \in A \), there exists a sequence \( (t_n)_{n \in \mathbb{N}} \) in \( \mathbb{R} \) converging to \(+\infty\) such that \( \varphi_{t_n}(\theta) \in A \) for all \( t_n \).

In the class of Riemannian manifolds with finite volume, Poincaré’s recurrence theorem implies that the geodesic flow \( \varphi_t : SM \to SM \) is conservative. This property was used by Yau [2] to generalize the result of Bishop and O’Neill [1]. Yau proved that there is no nontrivial continuous convex function on a complete Riemannian manifold of finite volume. We emphasize that there are Riemannian manifolds of infinite volume whose geodesic flow is conservative (see remark below).

**Remark 3.1** A hyperbolic surface is a complete two-dimensional Riemannian manifold of constant curvature \(-1\). Every such surface has the unit disk as a universal cover and can be viewed as \( H/\Gamma \), where \( H \) is the unit disk equipped with the hyperbolic metric and \( \Gamma \) is the covering group of isometries of \( H \). Nicholls [11] proved that “The geodesic flow on the hyperbolic surface \( H/\Gamma \) is conservative and ergodic if and only if the Poincaré series of \( \Gamma \) diverges at \( s = 1 \).” Such surfaces are said to be divergent. In [12], Hopf proved that geodesic flows on hyperbolic surfaces of infinite area are either totally dissipative or conservative and ergodic. Thus, divergent surfaces with infinite area are examples of Riemannian manifolds with infinite volume whose geodesic flow is conservative.

**Lemma 3.1** If the geodesic flow \( \varphi_t : SM \to SM \) is conservative with respect to the Liouville measure, then, for \( m \)-almost all \( \theta \in SM \), there exists a sequence \( (t_n)_{n \in \mathbb{N}} \) in \( \mathbb{R} \) converging to \(+\infty\) such that \( \varphi_{t_n}(\theta) \to \theta \).

**Proof** Because \( SM \) is a manifold, there is a countable basis \( \{V_i\}_{i \in \mathbb{N}} \) for the topology of \( SM \) such that \( m(V_i) < +\infty \) for all \( i \). As the geodesic flow is conservative, there exists, for all \( i \), a proper subset \( U_i \subset V_i \) with \( m(U_i) = m(V_i) \) for which, if \( \theta \in U_i \), there exists a sequence \( (t_n)_{n \in \mathbb{N}} \) in \( \mathbb{R} \) converging to \(+\infty\) such that \( \varphi_{t_n}(\theta) \in V_i \). Note that \( m(U_i) = 0 \), where \( U_i := \bigcup_{i \in \mathbb{N}} (V_i \setminus U_i) \), and if \( \theta \in SM \setminus \sim U \), there exists a sequence \( (t_n)_{n \in \mathbb{N}} \) in \( \mathbb{R} \) converging to \(+\infty\) such that \( \varphi_{t_n}(\theta) \to \theta \). This concludes the proof. \( \square \)
Remark 3.2 Neto et al. [13] considered the case where $M$ is a complete non-compact Riemannian manifold with finite volume, then $m(SM) < +\infty$ and they used the Poincaré recurrence theorem to get a characterization of the $C^1$ monotone vector fields.

In order to prove our first result, we need the following lemma.

Lemma 3.2 If the geodesic flow $\varphi_t : SM \to SM$ is conservative with respect to the Liouville measure, then for $m$-almost all $\theta = (p, v) \in SM$, there are sequences $(t_n)_{n \in \mathbb{N}}, (s_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ converging to $+\infty$ such that $\varphi_{t_n}(\theta) \to \theta$ and $\varphi_{-s_n}(\theta) \to \overline{\theta} = (p, -v)$.

Proof Consider the map $T : SM \to SM$ defined by $T(p, v) := (p, -v)$, it is not hard to see that $T$ is an isometry with respect to Sasaki metric whose Riemannian volume coincides with the Liouville measure. Consider the set

\[
\Omega := \{ \theta = (p, v) \in SM : \exists (t_n)_{n \in \mathbb{N}} \subset \mathbb{R} \text{ such that } t_n \to +\infty \text{ and } \varphi_{t_n}(\theta) \to \theta \}.
\]

Since $m(SM \setminus \Omega) = 0$ and $T$ is an isometry, it follows that $m(SM \setminus T(\Omega)) = 0$. Therefore,

\[
m(SM \setminus (\Omega \cap T(\Omega))) = m((SM \setminus \Omega) \cup (SM \setminus T(\Omega))) = 0.
\]

Note that if $\theta = (p, v) \in \Omega \cap T(\Omega)$, then $\theta = (p, v), \overline{\theta} = (p, -v) \in \Omega$, since $\varphi_t(\overline{\theta}) = \varphi_{-t}(\theta)$ and we are done. $\square$

A function $f : M \to \mathbb{R}$ on a Riemannian manifold $M$ is convex iff its restriction to every geodesic in $M$ is a convex function along the geodesic, i.e., iff for every geodesic segment $\gamma : [a, b] \to \mathbb{R}$ and every $t \in [0, 1],$

\[
f(\gamma((1-t)a + tb)) \leq (1-t)f(\gamma(a)) + tf(\gamma(b)).
\]

A convex function $f$ is strictly convex iff this inequality is strict whenever $t \in ]0, 1[.$

A convex function is always continuous. If $f$ is smooth, it is known that $f$ is (strictly) convex provided its Hessian is positive semidefinite (definite), or equivalently, if $(f \circ \gamma)' \geq 0$ ($> 0$) for every geodesic $\gamma : I \subset \mathbb{R} \to M$.

Theorem 3.1 Let $M$ be a connected complete Riemannian manifold. If the geodesic flow $\varphi_t : SM \to SM$ is conservative with respect to the Liouville measure, then all convex functions $f : M \to \mathbb{R}$ are constant.

Proof Suppose that $f : M \to \mathbb{R}$ is a convex function. Note that if $\gamma(t)$ is any geodesic in $M$ with

\[
\lim_{t_n \to +\infty} f(\gamma(t_n)) = \lim_{s_n \to -\infty} f(\gamma(s_n)) \in \mathbb{R}
\]

for sequences $(t_n), (s_n) \subset \mathbb{R}$, then $f$ is constant on the geodesic $\{ \gamma(t) : t \in \mathbb{R} \}$.\[ Springer\]
As the geodesic flow on $M$ is conservative, then Lemma 3.2 implies that for $m$-almost all points $\theta = (p, v) \in SM$, there are sequences $(t_n)_{n \in \mathbb{N}}, (s_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ converging to $+\infty$ such that $\varphi_{t_n}(\theta) \to \theta$ and $\varphi_{-s_n}(\theta) \to \theta = (p, -v)$.

Writing $\varphi_t(\theta) = (\gamma_\theta(t), \gamma'_\theta(t))$, we obtain that for $m$-almost all $\theta = (p, v) \in SM$, there are sequences $(t_n), (s_n) \subset \mathbb{R}$ such that $t_n \to +\infty$, $s_n := -s_n \to -\infty$ and

$$\lim_{t_n \to +\infty} \gamma_{t_n}(\theta) = p = \lim_{s_n \to -\infty} \gamma_{s_n}(\theta).$$

Thus, as noted in the first paragraph of this proof, $f$ must be constant on $\gamma_{t_n}(t)$. Now, let $\theta = (p, v)$ be any point in $SM$ and $\{\theta_i\} = \{(p_i, v_i)\}$ be a sequence converging to $\theta$, where each $\theta_i$ satisfies the above property. Using the continuity of $f$ and the fact that the geodesic flow is continuous, it follows that $f$ is also constant on $\gamma_{t}(t)$. In particular, $f$ is locally constant, and by the connectedness, it follows that $f$ is constant. \hfill \Box

Remark 3.3 As in the classic results of Bishop and O’Neill [1] and Yau [2], which we generalize, Theorem 3.1 only considers convex functions on connected complete Riemannian manifolds whose counter domain is the real line $\mathbb{R}$. We emphasize that our result, as for those proved by Bishop-O’Neill and Yau, is not valid for convex functions that assume values on the extended real line $\mathbb{R} \cup \{+\infty\}$. For example, consider the indicator function $I_A : M \to \mathbb{R} \cup \{+\infty\}$ of a convex set $A \subset M$.

### 4 Convex Functions and the Sectional Curvature

The known examples of strictly convex functions are associated with the sign of the sectional curvature of the Riemannian manifold. For example:

- (Theorem 1 (a), Greene and Wu [14]) If $M$ is a complete non-compact Riemannian manifold of positive sectional curvature, then there exists a $C^\infty$ Lipschitz continuous strictly convex function on $M$ such that, for every $\lambda \in \mathbb{R}$, $f^{-1}(]-\infty, \lambda])$ is a compact subset of $M$.

- (Theorem 4.1, [1]) Let $M$ be a complete, simply connected Riemannian manifold of non-positive sectional curvature $K \leq 0$.
  (a) If $S$ is a closed, totally geodesic submanifold of $M$, then the $C^\infty$ function $f_S : M \to \mathbb{R}$ defined by $f_S(x) := d^2(x, S)$ is convex.
  (b) In (a), if $S$ is a single point $p$, then $f_p$ is strictly convex.

However, the sign change of the sectional curvatures does not obstruct the existence of convex functions. In the following example, we present a convex function defined on the tangent bundle of a Riemannian manifold. Choosing $M$ such that the sectional curvature of $TM$ changes sign, we have a Riemannian manifold whose sectional curvature changes sign and admits a convex function.

**Example 4.1** Let $(M, \langle \cdot, \cdot \rangle)$ be a complete Riemannian manifold and $(TM, G)$ be its tangent bundle, where $G$ is the Riemann-Sasaki metric induced on $TM$. A point in $TM$ is represented by an ordered pair $(p, v)$, where $p \in M$ and $v \in T_p M$. 

\Springer
Let us consider the kinetic energy \( E : TM \to \mathbb{R} \) defined by \( E(p, v) := \langle v, v \rangle \).

It is known (Theorem 3.6, pp. 205, Udriste [15]) that \( E \) is a \( C^\infty \) convex function on \((TM, G)\). Furthermore, the kinetic energy is not strictly convex. In fact, given a geodesic \( \gamma(t) \), we have that \( \beta(t) = (\gamma(t), \gamma'(t)) \) is a geodesic in \( TM \) and \( E(\beta(t)) \) is constant, implying that the function \( E : TM \to \mathbb{R} \) is not strictly convex.

In this context, the following question naturally arises:

**Question.** Does the change of sign in the curvature imply the nonexistence of strictly convex functions?

In the next theorem, we present a geometric condition that ensures the existence of (strictly) convex functions on a particular class of complete non-compact Riemannian manifolds. As a corollary, we obtain the (negative) answer to this question.

**Theorem 4.1** Let \( M \) be a complete non-compact Riemannian manifold and \( V \in \mathfrak{X}(M) \) be a closed conformal vector field with conformal factor \( \phi \). If \( \langle \nabla \phi, V \rangle \geq 0 \), then the energy function \( f = \frac{1}{2} \langle V, V \rangle \) is convex. In addition, if the vector field \( V \) and the function \( \phi \) never vanish, then \( f \) is a strictly convex function.

**Proof** For every vector fields \( X, Y \in \mathfrak{X}(M) \), we have

\[ X(f) = \langle DX V, V \rangle = \langle \phi X, V \rangle, \]

then \( \nabla f = \phi V \). Now, note that

\[
\text{Hess } f(X, Y) = \langle DX \nabla f, Y \rangle = \langle DX(\phi V), Y \rangle \\
= \langle X, \nabla \phi \rangle \langle V, Y \rangle + \langle \phi DX V, Y \rangle \\
= \langle X, \nabla \phi \rangle \langle V, Y \rangle + \phi^2 \langle X, Y \rangle.
\]

Since both \( \text{Hess } f \) and the metric are symmetric tensors, we deduce

\[ \langle X, \nabla \phi \rangle \langle V, Y \rangle = \langle Y, \nabla \phi \rangle \langle V, X \rangle \]

for all \( X, Y \in \mathfrak{X}(M) \). So

\[ |V|^2 \cdot \nabla \phi = \langle V, \nabla \phi \rangle \cdot V. \]

Therefore,

\[ |V|^2 \cdot \text{Hess } f(X, X) = \langle V, \nabla \phi \rangle \langle V, X \rangle^2 + \phi^2 |V|^2 |X|^2 \geq \phi^2 |V|^2 |X|^2. \]

Since the set of the points of \( M \) where \( V \) vanishes is a discrete set (see, for instance, Montiel [16]), we conclude the proof of the theorem.

Let us now apply Theorem 4.1 to construct examples of strictly convex functions on Riemannian manifolds whose sectional curvature assumes any real value greater than some negative constant. We first describe some preliminaries that will be useful in understanding the construction.
Let \((B, \langle ., . \rangle_B)\) and \((P, \langle ., . \rangle_P)\) be Riemannian manifolds and \(g > 0\) be a positive smooth function on \(B\). Set \(M = B \times P\) (with the structure of a product manifold), and let \(\pi_B : M \to B\) and \(\pi_P : M \to P\) denote the canonical projections; we equip \(M\) with the warped metric \(\langle ., . \rangle\) given by
\[
\langle X, Y \rangle := \langle d\pi_B(X), d\pi_B(Y) \rangle_B + (g \circ \pi_B)^2 \langle d\pi_P(X), d\pi_P(Y) \rangle_P
\]
and denote the resulting Riemannian space by \(M = B \times_g P\). Note that the warped metric will be complete for any \(g\) as above if and only if \(B\) and \(P\) are complete.

In this case, \(B\) is called the base of \(M\) and \(P\) is the fiber. It is obvious that the fibers \(\{b\} \times P = \pi_B^{-1}(b)\) and the leaves \(B \times \{p\} = \pi_P^{-1}(p)\) are Riemannian submanifolds. Tangent vectors to the leaves are called horizontal and tangent vectors to the fibers are said to be vertical.

**Proposition 4.1** (Proposition 42 (5), O’Neill [17]) Let \(M = B \times_g P\) be a warped product with Riemannian curvature tensor \(R\), and let \(R^P\) be the lift to \(M\) of the Riemannian curvature tensor of \(P\). If \(U, V, W \in TM\) are vertical, then
\[
R(V, W)U = R^P(V, W)U - \frac{|\nabla g|^2}{g^2} (\langle V, U \rangle W - \langle W, U \rangle V).
\]
If \(U, V \in TM\) are orthonormal, and \(K\) and \(K^P\) denote the sectional curvature of \(M\) and \(P\), we get
\[
K(U, V) = \frac{1}{g^2} \left( K^P(U, V) - |\nabla g|^2 \right).
\]

An interesting class of spaces furnished with closed conformal vector fields is given by the subclass of warped product spaces that is formed when \(B = I \subset \mathbb{R}\) is an open interval.

If \(\partial_t = \nabla \pi_I\) is the standard unit vector field on \(I\), then \(V = (g \circ \pi_I)\partial_t\) is a closed conformal nowhere vanishing vector field on this manifold with conformal factor \(\phi = g' \circ \pi_I\). Therefore, \(\nabla \phi = (g'' \circ \pi_I)\partial_t\). In this case, the hypothesis of Theorem 4.1 is equivalent to
\[
0 \leq \langle V, \nabla \phi \rangle = \langle (g \circ \pi_I)\partial_t, (g'' \circ \pi_I)\partial_t \rangle = (g \circ \pi_I)(g'' \circ \pi_I).
\]

Thus, making use of Theorem 4.1, we conclude that the function \(f : M = I \times_g P \to \mathbb{R}\) defined by \(f = \frac{1}{2} \langle V, V \rangle\) is convex because \(g'' \geq 0\).

Now, consider the paraboloid of revolution \(P^2 := \{(x, y, z) : z = x^2 + y^2\}, B = \mathbb{R}\) and \(g(t) = e^t\). From Proposition 4.1, the manifold \(M^3 = \mathbb{R} \times e^t P^2\) has vertical sectional curvature
\[
K(t, x, y) = \frac{1}{e^{2t}} \left( \frac{4}{(1 + 4x^2 + 4y^2)^2} - e^{2t} \right).
\]
$M^3$ is a Riemannian manifold whose sectional curvature assumes any real value greater than $-1$ and, from Theorem 4.1, admits a strictly convex function given that $\phi(t, x, y) = |V(t, x, y)| = e^t \neq 0$ for all $t \in \mathbb{R}$.

We summarize the above construction with the following corollary.

**Corollary 4.1** There is a Riemannian manifold whose sectional curvature assumes any real value greater than some negative constant and admits a strictly convex function.

### 5 Soul of a Manifold and Minimization of Convex Functions

To prove our third result, we require the following lemma.

**Lemma 5.1** (Theorem 1 (a), [14]) If $M$ is a complete non-compact Riemannian manifold of everywhere positive sectional curvature, then there exists a $C^\infty$ Lipschitz continuous strictly convex function on $M$ such that for every $\lambda \in \mathbb{R}$, $f^{-1}([-\infty, \lambda])$ is a compact subset of $M$.

**Theorem 5.1** Let $M$ be a complete non-compact Riemannian manifold of positive sectional curvature and let $u : M \to \mathbb{R}$ be a convex function admitting a point $p_0 \in M$ such that $u(p_0) = \inf_M u > -\infty$. Then, there exists a sequence $(x_k)$ such that $x_k$ is a soul of $M$, $x_k \to p$, and $u(p) = \inf_M u$.

**Proof** Let $f$ be the function whose existence is assured by the previous lemma. As $f^{-1}([-\infty, \lambda])$ is a compact subset of $M$, for every $\lambda \in \mathbb{R}$, we have that $\inf_M f > -\infty$. Considering the function $g : M \to \mathbb{R}$ defined by $g := f - \inf_M f$, we have $g \geq 0$. We can assume, without loss of generality, that $u \geq 0$. For some fixed $k \in \mathbb{N}$, consider the function $h_k := k \cdot u + g$, and note that for $r \in \mathbb{R}$, we have

$$x \in h_k^{-1}([-\infty, r]) \iff h_k(x) \leq r \iff ku(x) + g(x) \leq r.$$

Then, $x \in h_k^{-1}([-\infty, r]) \Rightarrow g(x) \leq r \Rightarrow f(x) \leq r + \inf_M f$. Hence, we conclude that

$$h_k^{-1}([-\infty, r]) \subset f^{-1}([-\infty, r + \inf_M f]).$$

because $f^{-1}([-\infty, r + \inf_M f])$ being compact implies that $h_k^{-1}([-\infty, r])$ is also compact. On the other hand, as $h_k$ is strictly convex, there exists a unique point $x_k$ such that $h_k(x_k) = \inf_M h_k$. Now, fixing $p_0$ such that $u(p_0) = \inf_M u = 0$, we get

$$k \cdot u(x_k) + g(x_k) = h_k(x_k) \leq h_k(p_0) = g(p_0).$$

From the above inequality, it follows that $0 \leq g(x_k) \leq g(p_0)$. Thus, we have

$$x_k \in g^{-1}([-\infty, g(p_0)]) = f^{-1}([-\infty, g(p_0) + \inf_M f]).$$

$\square$ Springer
As this is a compact set, we have \( x_k \to p \in M \), or we can consider a subsequence of \( (x_k) \) if necessary. Therefore, from (5), we have

\[
0 \leq u(x_k) \leq \frac{g(p_0)}{k}
\]

and thus, we obtain \( u(p) = \inf_M u = 0 \). As \( h_k \) is a strictly convex function, we have that the set \( \{x_k\} \) is totally convex, and it follows from [Theorem 2, [9]] that every \( x_k \) is a soul of \( M \).

\( \square \)

In optimization, it is important to establish the convergence of numerical methods to find minimum points for convex functions. The result of Theorem 5.1 can be used to choose the initial point in the iterative process. If we know the region in which the souls are located, we can use this information to commence the iterative process and avoid starting at a point far from the set of minimizers. In the next result, we determine this region explicitly for the case when the manifold is a paraboloid.

**Corollary 5.1** Let \( u \) be a convex function defined on the paraboloid \( P^2 := \{(x, y, z) : z = x^2 + y^2\} \) such that the minimum set is nonempty. Then, there exists a minimizer \( p = (x_0, y_0, z_0) \) for \( u \) such that \( z_0 \leq \beta \), where \( \beta = \sqrt{\frac{3}{4}} \left( 1 + \mu_1^2 \right) \) and \( \mu_1 = \arctan \mu_1 = \frac{\pi}{2} \).

**Proof** Let \( h : \mathbb{R}^3 \to \mathbb{R} \) be the projection on the third coordinate, i.e., \( h(x, y, z) := z \). We recall that a point \( p \in P^2 \) is simple iff there are no geodesic loops in \( M \) that are closed at \( p \). Following the terminology of Ling and Recht [18], \( p \) being a simple point (a soul) is equivalent to saying that \( p \) is not a vertex for any loop. Moreover, in [18], the authors showed that \( p \) is not a vertex for any loop if and only if \( h(p) < \beta \).

From Theorem 5.1, there is a sequence \( (p_k) \) of souls of \( P \) such that \( p_k \to p \), where \( u(p) = \min_P u \). As a consequence of the result proved by Ling and Recht, we have that \( h(p_k) < \beta \). Because \( h \) is a continuous function, we conclude that \( h(p) \leq \beta \). \( \square \)

6 Conclusions

In Theorem 3.1, we prove that the conservativity of the geodesic flow on a Riemannian manifold with infinite volume implies that all convex functions on the manifold are constant. This fact generalizes, in a certain sense, the result proved by Yau [2]. In Theorem 4.1, we presented a geometric condition that ensures the existence of (strictly) convex functions on a particular class of complete Riemannian manifolds. Finally, in Theorem 5.1, we establish a result relating a geometric property of Riemannian manifolds of positive sectional curvature with the set of minimum points of a convex function defined on the manifold.

**Acknowledgments** The authors would like to express their gratitude to referee for valuable suggestions on the improvement of the paper. João X. Cruz Neto and Paulo A. Sousa were partially supported by CNPq/Brazil Grant Nos. 305462/2014-8 and 304823/2013-9.
References

1. Bishop, R.L., O’Neill, B.: Manifolds of negative curvature. Trans. AMS 145, 1–49 (1969)
2. Yau, S.T.: Non-existence of continuous convex functions on certain Riemannian manifolds. Math. Ann. 207, 269–270 (1974)
3. Shiohama, K.: Convex sets and convex functions on complete manifolds. In: Proceedings of the international congress of mathematicians, Helsinki (1978)
4. Neto, J.X.C., Lima, L.L., Oliveira, P.R.: Geodesic algorithms in Riemannian geometry. Balkan J. Geom. Appl. 3(2), 89–100 (1998)
5. Ferreira, O.P., Oliveira, P.R.: Subgradient algorithm on Riemannian manifolds. J. Optim. Theory Appl. 97(1), 93–104 (1998)
6. Ferreira, O.P., Oliveira, P.R.: Proximal point algorithm on Riemannian manifolds. Optimization 51(2), 257–270 (2002)
7. Wang, X.M., Li, C., Yao, J.C.: Subgradient projection algorithms for convex feasibility on Riemannian manifolds with lower bounded curvatures. J. Optim. Theory Appl. 164(1), 202–217 (2015)
8. Kowalski, O.: Curvature of the induced Riemannian metric on the tangent bundle of a Riemannian manifold. J. fur die reine und angew. 250, 124–129 (1971)
9. Gromoll, D., Meyer, W.: On complete open manifolds of positive curvature. Ann. Math. 90, 75–90 (1969)
10. Paternain, G.P.: Geodesic Flows. Progress in Mathematics, vol. 180. Birkhäuser, Basel (1999)
11. Nicholls, P.: Transitivity properties of Fuchsian groups. Canad. J. Math. 28, 805–814 (1976)
12. Hopf, E.: Ergodentheorie. Chelsea, New York (1948)
13. Neto, J.X.C., Melo, I.D., Sousa, P.A.: Non-existence of strictly monotonous vector fields on certain Riemannian manifolds. Acta Math. Hung. 146, 240–246 (2015)
14. Greene, R.E., Wu, H.: $C^\infty$ convex functions and manifolds of positive curvature. Acta Math. 137(1), 209–245 (1976)
15. Udriste, C.: Convex Functions and Optimization Methods on Riemannian Manifolds. Mathematics and Its Applications, vol. 297. Kluwer, Dordrecht (1994)
16. Montiel, S.: Stable constant mean curvature hypersurfaces in some Riemannian manifolds. Comment. Math. Helv. 73, 584–602 (1998)
17. O’Neill, B.: Semi-Riemannian Geometry with Applications to Relativity. Academic Press, New York (2010)
18. Ling, D., Recht, L.: A theorem concerning the geodesics on a paraboloid of revolution. Bull. Am. Math. Soc. 47, 934–937 (1941)