SURFACE LINKS WHICH ARE COVERINGS OVER THE STANDARD TORUS

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Abstract. We consider surface links in the 4-space or the 4-sphere which can be described in braid forms over the standard torus, which we call torus-covering-links. Torus-covering-links include spun $T^2$-knots, turned spun $T^2$-knots, symmetry-spun tori and torus $T^2$-knots. In this paper we give examples of interesting torus-covering-links, namely, torus-covering $T^2$-links which are ribbon but not symmetry-spun $T^2$-links, and a torus-covering $T^2$-knot whose triple point number is positive.

0. Introduction

Closed 2-manifolds embedded locally flatly in the Euclidean 4-space $\mathbb{R}^4$ or the 4-sphere $S^4$ are called surface links. It is known that any oriented surface link can be described in a form of the closure of a simple surface braid, i.e. in a braid form over the standard 2-sphere (cf. [12] [14]).

As surface knots of genus one which can be made from classical knots, there are spun $T^2$-knots, turned spun $T^2$-knots, symmetry-spun tori and torus $T^2$-knots. Consider $\mathbb{R}^4$ as obtained by rotating $\mathbb{R}^3$ around the boundary $\mathbb{R}^2$. Then a spun $T^2$-knot is obtained by rotating a classical knot (cf. [3]), a turned spun $T^2$-knot by turning a classical knot once while rotating it (cf. [3]), a symmetry-spun torus by turning a classical knot with periodicity rationally while rotating (cf. [20]), and a torus $T^2$-knot is a surface knot on the boundary of a neighborhood of a solid torus in $\mathbb{R}^4$ (cf. [10]). Symmetry-spun tori include spun $T^2$-knots, turned spun $T^2$-knots and torus $T^2$-knots. We call the link version of a symmetry-spun torus, a spun $T^2$-knot, and a turned spun $T^2$-knot a symmetry-spun $T^2$-link, a spun $T^2$-link and a turned spun $T^2$-link respectively. We enumerate several properties of symmetry-spun $T^2$-links.

(0.1) A symmetry-spun $T^2$-link is equivalent to a spun $T^2$-link or a turned spun $T^2$-link (cf. [20]).

(0.2) A spun $T^2$-link is ribbon and the turned spun $T^2$-link of a non-trivial classical link is not ribbon (cf. [3] [19]).

(0.3) A symmetry-spun $T^2$-link is pseudo-ribbon and has the triple point number zero (cf. [15]).

(0.4) A symmetry-spun $T^2$-link has a classical link group (cf. [20]).

Now we consider oriented surface links which can be described in braid forms over the standard torus, which we will define as torus-covering-links (see Definition 2.1). By definition, a torus-covering-link is associated with a torus-covering-chart, which is a chart on the standard torus. Torus-covering-links include symmetry-spun $T^2$-links (and spun $T^2$-links, turned spun $T^2$-links, and torus $T^2$-links). A torus-covering-link has no 2-knot component. Each component of a torus-covering-link
is of genus at least one. There is a natural question.

(Q) What difference from symmetry-spun $T^2$-links do torus-covering-links have?

In Section 2 we consider torus-covering-links in $S^4$, and from Section 3 throughout this paper we consider torus-covering $T^2$-links in $\mathbb{R}^4$, i.e. torus-covering-links in $\mathbb{R}^4$ whose each component is of genus one. Torus-covering $T^2$-links are associated with torus-covering-charts without black vertices.

In Section 2 we define torus-covering-links (Definition 2.1), and turned torus-covering-links (Definition 2.2).

In Section 3 we study link groups of torus-covering $T^2$-links. We show that “There are infinitely many 2-component torus-covering $T^2$-links whose link groups are not classical link groups.” (Theorem 3.3). We show its knot version as well: “There are infinitely many torus-covering $T^2$-knots whose knot groups are not classical knot groups.” (Theorem 3.4). This means that torus-covering $T^2$-links are not always symmetry-spun $T^2$-links. Concerning Theorem 3.5 we have a corollary: “There are infinitely many 2-component torus-covering $T^2$-links whose link groups are not 2-component 2-link groups.” (Corollary 3.6).

In Section 4 we show that the torus-covering-links given in Theorems 3.5 and 3.9 are ribbon, i.e. “There are infinitely many torus-covering $T^2$-links which are ribbon but not symmetry-spun $T^2$-links.” (Theorem 4.1).

Symmetry-spun $T^2$-links are pseudo-ribbon and by Theorem 4.1 the torus-covering $T^2$-links given in Theorems 3.6 and 3.9 are ribbon. Both of them have the triple point number zero. In Section 5 we show that “There is a torus-covering $T^2$-knot whose triple point number is positive.” (Theorem 5.1). Moreover we have “There is a torus-covering $T^2$-knot which is not (-)-amphicheiral.” (Corollary 5.2).

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1. Definitions and Preliminaries

**Definition 1.1.** A closed 2-manifold embedded locally flatly in $\mathbb{R}^4$ or $S^4$ is called a *surface link*. A surface link with one component is called a *surface knot*. A surface link whose each component is of genus zero (resp. one) is called a 2-link (resp. $T^2$-link). In particular a surface knot of genus zero (resp. one) is called a 2-knot (resp. $T^2$-knot).

An oriented surface link $F$ is trivial (or unknotted) if there is an embedded 3-manifold $M$ with $\partial M = F$ such that each component of $M$ is a handlebody.

Let $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ be a generic projection. Then a *surface diagram* of a surface link $F$ in $\mathbb{R}^4$ is the image $\pi(F)$ with additional crossing information at the singularity set. An oriented surface link $F$ is called pseudo-ribbon if there is a surface diagram of $F$ whose singularity set consists of double points, and ribbon if $F$ is obtained from a trivial 2-link $F_0$ by 1-handle surgeries along a finite number of mutually disjoint 1-handles attaching to $F_0$. By definition, a ribbon surface link is pseudo-ribbon.

Two surface links are equivalent if there is an ambient isotopy of $\mathbb{R}^4$ (or $S^4$) or an orientation-preserving self-homeomorphism of $\mathbb{R}^3$ (or $S^3$) which deforms one to the other.

**Definition 1.2.** A compact and oriented 2-manifold $S_B$ embedded properly and locally flatly in $D^2_t \times D^2_2$ is called a *braided surface* of degree $m$ if $S_B$ satisfies the following conditions:

(i) $\text{pr}_2|_{S_B} : S_B \to D^2_2$ is a branched covering map of degree $m$,
(ii) $\partial S_B$ is a closed $m$-braid in $D^2_t \times \partial D^2_2$, where $D^2_t, D^2_2$ are 2-disks, and $\text{pr}_2 : D^2_t \times D^2_2 \to D^2_2$ is the projection to the second factor.
A braided surface $S_B$ is called a surface braid if $\partial S_B$ is the trivial closed braid. Moreover, $S_B$ is called simple if every singular index is two. Two braided surfaces of the same degree are equivalent if there is a fiber-preserving ambient isotopy of $D^2_1 \times D^2_2$ rel $D^2_1 \times \partial D^2_2$ which carries one to the other.

There is a theorem which corresponds to Alexander’s theorem for classical oriented links.

**Theorem 1.3** ([12, 14]). Any oriented surface link in $\mathbb{R}^4$ can be deformed by an ambient isotopy of $\mathbb{R}^4$ to the closure of a simple surface braid.

There is a chart which represents a simple surface braid.

**Definition 1.4.** Let $m$ be a positive integer, and $\Gamma$ be a graph on a 2-disk $D^2_2$. Then $\Gamma$ is called a surface link chart of degree $m$ if it satisfies the following conditions:

(i) $\Gamma \cap \partial D^2_2 = \emptyset$.

(ii) Every edge is oriented and labeled, and the label is in $\{1, \ldots, m-1\}$.

(iii) Every vertex has degree 1, 4, or 6.

(iv) At each vertex of degree 6, there are six edges adhering to which, three consecutive arcs oriented inward and the other three outward, and those six edges are labeled $i$ and $i+1$ alternately for some $i$.

(v) At each vertex of degree 4, the diagonal edges have the same label and are oriented coherently, and the labels $i$ and $j$ of the diagonals satisfy $|i-j| > 1$ (Fig. 1.1).

A vertex of degree 1 (resp. 6) is called a black vertex (resp. white vertex). A black vertex (resp. white vertex) in a chart corresponds to a branch point (resp. triple point) in the surface diagram of the associated simple surface braid by the projection $p_2$.

A chart with a boundary represents a simple braided surface.

There is a notion of $C$-move equivalence between two charts of the same degree. The following theorem is well-known.

**Theorem 1.5** ([13, 14]). Two charts of the same degree are $C$-move equivalent if and only if their associated simple braided surfaces are equivalent.

2. Torus-covering-links

In this section, we greatly rely on [20]. We define torus-covering-links (Definition 2.1) and turned torus-covering-links (Definition 2.4). Throughout this paper, let $\sigma_1, \sigma_2, \ldots, \sigma_{m-1}$ be the standard generators of the braid group of degree $m$.

**Definition 2.1.** Let $D^2$ be a 2-disk and $S^1 = [0, 1]/ \sim$ with $0 \sim 1$. First, embed $D^2 \times S^1 \times S^1$ into $S^4$ or $\mathbb{R}^4$ naturally, or more precisely, consider as follows (cf. [20], see also [3, 16]). Let $S^1 \times S^1$ be a standardly embedded torus in $S^4$ and let $D^2 \times S^1 \times S^1$ be a tubular neighborhood of $S^1 \times S^1$ in $S^4$. We can assume that its framing is canonical, i.e. the homomorphism induced by the inclusion map.
A torus-covering-link has no 2-knot component. Each component of a torus-covering-link is of genus at least one.

By definition, torus-covering-links include symmetry-spun $T^2$-links.

As we stated in Theorem 1.3 for two surface link charts of the same degree, their associated surface links are equivalent if the charts are C-move equivalent. It follows that if two torus-covering-charts of the same degree are C-move equivalent, then their associated torus-covering-links are equivalent.

A torus-covering-link is associated with a chart on the standard torus, i.e. with a chart $\Gamma_T$ on $I_3 \times I_4$ with $\Gamma_T \cap (I_3 \times \{0\}) = \Gamma_T \cap (I_3 \times \{1\})$ and $\Gamma_T \cap (\{0\} \times I_4) = \Gamma_T \cap (\{1\} \times I_4)$. Denote the classical braids associated with $\Gamma_T \cap (I_3 \times \{0\})$ and $\Gamma_T \cap (\{0\} \times I_4)$ by $a$ and $b$ respectively. We will call $\Gamma_T$ a torus-covering-chart with boundary braids $a$ and $b$. In particular, a torus-covering $T^2$-link is associated with a torus-covering-chart $\Gamma_T$ without black vertices, and the torus-covering $T^2$-link is determined from the boundary braids $a$ and $b$, which are commutative. We will call $\Gamma_T$ a torus-covering-chart without black vertices and with boundary braids $a$ and $b$.

By definition, torus-covering-links include symmetry-spun $T^2$-links.
Let $\Gamma_T$ be a torus-covering-chart of degree $m$ with the trivial boundary braids. Let $F$ be the surface link associated with the surface link chart obtained from $\Gamma_T$ by assuming it to be a surface link chart. Then the torus-covering-link associated with the torus-covering-chart $\Gamma_T$ is obtained from $F$ by applying $m$ trivial 1-handle surgeries.

**Example 2.2.** (2.2.1) Let $\Gamma_T$ be the torus-covering-chart of degree 2 without black vertices and with boundary braids $\sigma_1^2$ and $\varepsilon$ (the trivial braid). Then the torus-covering-knot associated with $\Gamma_T$ is the spun $T^2$-knot of a right-handed trefoil.

(2.2.2) Let $\Gamma_T$ be the torus-covering-chart of degree 2 without black vertices and with boundary braids $\sigma_1^2$ and $\sigma_1^3$. Then the torus-covering-knot associated with $\Gamma_T$ is the turned spun $T^2$-knot of a right-handed trefoil.

(2.2.3) Let $\Gamma_T$ be the torus-covering-chart of degree 2 without black vertices and with boundary braids $\sigma_2^3$ and $\sigma_1^1$. Then the torus-covering-knot associated with $\Gamma_T$ is a symmetry-spun torus.

Let $\Gamma_T$ be a torus-covering-chart on $I_3 \times I_4$, where $I_3 = I_4 = [-1, 1]$. Consider a map $\rho : I_3 \times I_4 \to I_3 \times I_4$ such that $\rho(t_1, t_2) = (-t_2, t_1)$. We call $\rho(\Gamma_T)$ the torus-covering-chart obtained from $\Gamma_T$ by rotating it by $\pi/2$. Let us denote by $\rho$ the $\pi/2$ rotation, which maps the torus-covering-link $S$ associated with $\Gamma_T$ to the torus-covering-link $\rho(S)$ associated with $\rho(\Gamma_T)$.

**Lemma 2.3** (cf. [20]). Let $S$ be the torus-covering-link associated with a torus-covering-chart $\Gamma_T$. Then the torus-covering-links $\rho^n(S)$ obtained from $S$ by the $n\pi/2$ rotation for $n \in \mathbb{Z}$ are all equivalent.

**Proof.** The torus-covering-links $\rho^n(S)$ are in $D^2 \times S^1 \times S^1$ in $S^4 = E^4 \cup_i D^2 \times S^1 \times S^1$, where $S^1 = [-1, 1]/ \sim$ with $-1 \sim 1$. We will show that $S$ and $\rho(S)$ are equivalent. Consider a homeomorphism $f : D^2 \times S^1 \times S^1 \to D^2 \times S^1 \times S^1$ such that $f((x, t_1, t_2) = (x, -t_2, t_1)$. The map $f|_{\partial D^2 \times S^1 \times S^1}$ can be considered as an orientation-preserving self-homeomorphism of $\partial E^4$. Since $A^f = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{array} \right) \in H$, $f|_{\partial D^2 \times S^1 \times S^1}$ can be extended to $\tilde{f} : E^4 \to E^4$. Hence there is an orientation-preserving self-homeomorphism of $S^4$ which deforms $S$ to $\rho(S)$.

**Remark.** Teragaito proved in [20] the same theorem in the symmetry-spun version. Lemma 7 in [20] corresponds to the $-\pi/2$ rotation.

We can consider **turned torus-covering-links**, which include turned spun $T^2$-links (cf. [3] [20]).

**Definition 2.4.** We use the notations of Definition [21]. Let $\sigma : \partial E^4 \to \partial E^4$ be the homeomorphism of a matrix $A^\sigma = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array} \right)$. Then $E^4 \cup_i D^2 \times S^1 \times S^1$ is homeomorphic to $S^4$. Let $\Gamma_T$ be a torus-covering-chart and let $S$ be the torus-covering-link associated with $\Gamma_T$ in $S^4 = E^4 \cup_i D^2 \times S^1 \times S^1$. Then we can consider the torus-covering-link obtained from $S$ by changing the identification map $i$ to $\sigma i$, which we will call the **turned torus-covering-link associated with** $(S, \Gamma_T)$ or $S$, and use the notation $\tau(S, \Gamma_T)$ or $\tau(S)$. That is, we define $\tau(S)$ as follows:

$$(E^4 \cup_i D^2 \times S^1 \times S^1 \cong S^4, S) = (S^4 = E^4 \cup_i D^2 \times S^1 \times S^1, \tau(S)).$$

Note that for a spun $T^2$-link $S$, $\tau(S)$ is the turned spun $T^2$-link. Let us denote by $\tau$ the map which maps $S$ to the turned torus-covering-link $\tau(S)$. 


By definition, we have the following lemmas.

**Lemma 2.5.** Let $S$ be the torus-covering-link associated with a torus-covering-chart $\Gamma_T$, and consider $\tau(S)$, the turned torus-covering-link associated with $S$. Then the torus-covering-chart $\Gamma_\tau(T)$ associated with $\tau(S)$ is as in Fig. 2.2.

**Proof.** Consider the inverse map $\sigma^{-1} : \partial E^4 \to \partial E^4$ of $\sigma$. Then $\sigma^{-1}(l \ s \ r) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} (l \ s \ r)$. The homeomorphism $\sigma^{-1}$ is decomposed as $1 \times \tau$ for a homeomorphism $\tau : S^1 \times S^1 \to S^1 \times S^1$ such that $\tau(s \ r) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} (s \ r)$. Hence, regarding torus-covering-charts as graphs on the standard torus, the torus-covering-chart $\Gamma_\tau(T)$ is $\tau(\Gamma_T)$, which is as in Fig. 2.2.

**Lemma 2.6** (cf. [3, 20]). For a torus-covering-link $S$, $\tau^2(S) = S$.

**Proof.** Since $A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \in H$, there is an orientation-preserving self-homeomorphism of $S^4$ which deforms $S$ to $\tau^2(S)$.

The equivalence class of $\tau(S)$ is not determined from the equivalence class of $S$. There is a pair of torus-covering-links $S$ and $S'$ with the associated torus-covering-charts $\Gamma_T$ and $\Gamma'_T$ such that $S$ and $S'$ are equivalent but $\tau(S, \Gamma_T) \neq \tau(S', \Gamma'_T)$ are not equivalent (Proposition 2.7). In Proposition 2.7 let $S' = \rho(S)$. By Lemma 2.3 $S$ and $S'$ are equivalent. By Proposition 2.7 $\tau(S')$ (i.e. $\tau \circ \rho(S)$) and $\rho \circ \tau(S)$ are not equivalent. However, by Lemma 2.3 $\rho \circ \tau(S)$ and $\tau(S)$ are equivalent. Hence $\tau(S')$ and $\tau(S)$ are not equivalent.

**Proposition 2.7.** There is a torus-covering-chart $\Gamma_T$ of degree 2 without black vertices and with boundary braids $\sigma^2_1$ and $e$. The torus-covering-knot $S$ associated with $\Gamma_T$ is the spun $T^2$-knot of a trefoil. Then the boundary braids of the torus-covering-chart $\tau \circ \rho(\Gamma_T)$ are $e$ and $\sigma^2_1$, and by Lemma 2.3 $\tau \circ \rho(S)$ is equivalent to $S$. On the other hand, by Lemma 2.3 again, $\rho \circ \tau(S)$ is equivalent to $\tau(S)$, which is the turned spun $T^2$-knot of the trefoil. By (0.2), $\tau \circ \rho(S)$ and $\rho \circ \tau(S)$ are not equivalent. □
3. Knot Groups and Link Groups

From now on throughout this paper, we consider torus-covering $T^2$-links in $\mathbb{R}^4$, i.e. the torus-covering-links in $\mathbb{R}^4$ associated with torus-covering-charts without black vertices. We can compute link groups of torus-covering $T^2$-links (Lemma 3.4).

Before stating Lemma 3.4, we will give the definition of Artin’s automorphism (Definition 3.3, cf. \[14\]). Let $D^2$ be a 2-disk, $\beta$ an $m$-braid in a cylinder $D^2 \times [0, 1]$, $Q_m$ the starting point set of $\beta$.

**Definition 3.1.** An isotopy of $D^2$ associated with $\beta$ is an ambient isotopy $\{\phi_t\}_{t \in [0, 1]}$ of $D^2$ such that

1. $\phi_0 = \text{id}$,
2. $\phi_t(Q_m) = \text{pr}_1(\beta \cap \text{pr}_2^{-1}(t))$ for $t \in [0, 1]$, where $\text{pr}_1 : D^2 \times [0, 1] \to D^2$ (resp. $\text{pr}_2 : D^2 \times [0, 1] \to [0, 1]$) is the projection to the first (resp. second) factor.

**Definition 3.2.** A homeomorphism of $D^2$ associated with $\beta$ is the terminal map $\psi = \phi_1 : D^2 \to D^2$ of an isotopy $\{\phi_t\}_{t \in [0, 1]}$ of $D^2$ associated with $\beta$.

Let $q_0$ be a point of $\partial D^2$. Identify the fundamental group $\pi_1(D^2 - Q_m, q_0)$ with the free group $F_m$ generated by the standard generator system of $\pi_1(D^2 - Q_m, q_0)$.

**Definition 3.3.** Artin’s automorphism of $F_m$ associated with $\beta$ is the automorphism of $F_m$ induced by a homeomorphism of $D^2$ associated with $\beta$. We denote it by $\text{Artin}(\beta)$.

We can obtain Artin’s automorphism (of the free group $F_m$ associated with an $m$-braid) algebraically by the following rules. Let $F_m = \langle x_1, x_2, \ldots, x_m \rangle$.

1. $\text{Artin}(\beta_1 \beta_2) = \text{Artin}(\beta_2) \circ \text{Artin}(\beta_1)$ for $m$-braids $\beta_1$ and $\beta_2$, and
2. $\text{Artin}(\sigma_j)(x_j) = \begin{cases} x_j & \text{if } j \neq i, i+1, \\ x_{i+1}x_{i+1}^{-1} & \text{if } j = i, \\ x_{i+1} & \text{if } j = i+1, \end{cases}$

and

$\text{Artin}(\sigma_j^{-1})(x_i) = \begin{cases} x_j & \text{if } j \neq i, i+1, \\ x_{i+1} & \text{if } j = i, \\ x_{i+1}x_{i+1}^{-1} & \text{if } j = i+1, \end{cases}$

where $i = 1, 2, \ldots, m - 1$ and $j = 1, 2, \ldots, m$.

**Lemma 3.4.** Let $\Gamma_T$ be a torus-covering-chart of degree $m$ without black vertices and with boundary braids $a$ and $b$. Let $S$ be the torus-covering-link associated with $\Gamma_T$. Then the link group of $S$ is obtained as follows:

$\pi_1(\mathbb{R}^4 - S) = \langle x_1, \ldots, x_m | x_j = \text{Artin}(\beta_j)(x_j), j = 1, 2, \ldots, m \rangle$,

where $\text{Artin}(\alpha) : F_m \to F_m$ (resp. $\text{Artin}(\beta)$) is Artin’s automorphism of the free group $F_m = \langle x_1, \ldots, x_m \rangle$ associated with the $m$-braid $\alpha$ (resp. $\beta$).

**Proof.** Let us denote by $D^2 \times S^1 \times S^1$ the neighborhood of the torus in which the torus-covering-link $S$ is embedded as in Definition 2.1, where $D^2$ is a 2-disk and $S^1 = [0, 1]/\sim$ with $0 \sim 1$. Let $x_0 = \{q_0\} \times \{0\} \times \{0\}$ be the base point, where $q_0 \in \partial D^2$ and $\{0\} \in S^1$. Then $\pi_1(D^2 \times S^1 \times S^1 - S, x_0)$ is given by

$\langle x_1, \ldots, x_m, s, t | x_j = s \cdot \text{Artin}(\alpha)(x_j) \cdot s^{-1}, x_j = t \cdot \text{Artin}(\beta)(x_j) \cdot t^{-1}, st = ts, j = 1, 2, \ldots, m \rangle$,

where $s$ (resp. $t$) is represented by the meridian (resp. longitude) of $\{q_0\} \times S^1 \times S^1$. 

Let $B_1^3$ be a 3-ball such that $(D^2 \times S^1) \cup B_1^3$ is a 3-ball and $(D^2 \times S^1) \cap B_0^3$ is homotopic to $S^1$. Let us denote the 3-ball $(D^2 \times S^1) \cup B_0^3$ by $E^3$. Put $X = E^3 \times S^1 - S$, $A = D^2 \times S^1 \times S^1 - S$ and $B = B_0^3 \times S^1$. We can assume that $B$ and $S$ are disjoint. Then $X = A \cup B$. We can also assume that the base point $x_0 = (q_0) \times \{0\} \times \{0\}$ is in $A \cap B$. Since $A$, $B$ and $A \cap B$ are connected, by van Kampen's theorem, $\pi_1(X, x_0)$ is given by

$$[\pi_1(A, x_0) * \pi_1(B, x_0)]/\text{Vk}(X, A, B, x_0),$$

where $\text{Vk}(X, A, B, x_0)$ is the van Kampen subgroup, i.e. the smallest normal subgroup of the free product $\pi_1(A, x_0) * \pi_1(B, x_0)$ containing the elements of the form $i_\ast(\delta)j_\ast(\delta^{-1})$, where $i : A \cap B \to A$, $j : A \cap B \to B$ are inclusion maps, and $\delta \in \pi_1(A \cap B, x_0)$. Since $A \cap B$ is homotopic to $S^1 \times S^1$ and include $(q_0) \times S^1 \times S^1$, $\pi_1(A \cap B, x_0)$ has a presentation $\langle s'', t'' | s''t'' = t''s'' \rangle$ such that $i_\ast(s'') = s$ and $j_\ast(t'') = t$, where $s$ and $t$ are two of the generators of $\pi_1(A, x_0)$ with the presentation (5.1). On the other hand, $\pi_1(B, x_0)$ has the presentation $\langle t' \rangle$ such that $j_\ast(t'') = 1$ and $j_\ast(t'') = t'$. Hence $\pi_1(X) = \pi_1(E^3 \times S^1 - S)$ is

$$\langle x_1, \ldots, x_m, t \mid x_j = \text{Artin}(a)(x_j), x_j = t \cdot \text{Artin}(b)(x_j) \cdot t^{-1}, j = 1, 2, \ldots, m \rangle.$$

Since $E^3 \times S^1$ is naturally embedded in $\mathbb{R}^4$, the longitude of $(q_0) \times S^1 \times S^1$ which represents $t$ bounds a disk in $\mathbb{R}^4$. Hence we have

$$\pi_1(\mathbb{R}^4 - S) = \langle x_1, \ldots, x_m \mid x_j = \text{Artin}(a)(x_j) = \text{Artin}(b)(x_j), j = 1, 2, \ldots, m \rangle.$$

Not every link group of torus-covering $T^2$-links is a classical link group, i.e. there are torus-covering $T^2$-links which are not symmetry-spun $T^2$-links (cf. (0.4)).

**Theorem 3.5.** Let $\Gamma_{T,n}$ be the torus-covering-chart of degree 4 without black vertices and with boundary braids $\sigma_1\sigma_3$ and $\Delta^{2n}$, where $\Delta = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$ (Garside’s $\Delta$), and $n$ is a positive integer. Let $S_n$ be the torus-covering-link associated with $\Gamma_{T,n}$. Then the link group of $S_n$ is not a classical link group. Moreover, $S_n$ and $S_m$ are not equivalent for $n \neq m$.

**Remark.** The torus-covering-link $S_n$ has two components, each of which is a trivial torus knot.

Before the proof, we give two known theorems.

**Theorem 3.6** ([17], [2] p.73, 11.3 Theorem). In the amalgamated product of groups $G_i \ (i = 1, 2)$ with the amalgamated subgroup $U$, there is in each class of equivalent words one, and only one, element in a normal form $f = u_1z_1z_2 \cdots z_t$. Here $u \in U$ and $z_j \ (j = 1, 2, \ldots, t)$ is a non-trivial right-handed coset representative of $U$ in some $G_i(j)$, taken from an arbitrarily prefixed selection of right-handed coset representatives which contain the trivial coset representative, and $G_{i(j)} \neq G_{i(j+1)}$ for $j = 1, 2, \ldots, t - 1$.

If the center of a classical knot group is non-trivial, then it is a torus knot (cf. [4]). There is a theorem concerning the center of a classical link group as follows.

**Theorem 3.7** ([5]). The statements listed below are equivalent.

1. The center of the group of a classical link $L$ is non-trivial.
2. The link group is isomorphic to one of the groups of type (a), (b), or (c).
   
   (a) $(\mathbb{Z} * \cdots * \mathbb{Z}) \times \mathbb{Z},$  
   
   (b) $(\mathbb{Z} * \cdots * \mathbb{Z}) * \mathbb{Z} ((\mathbb{Z} \times \mathbb{Z}) * \mathbb{Z} \mathbb{Z}),$
(c) \((\mathbb{Z} \ast \cdots \ast \mathbb{Z}) \times \mathbb{Z}) \ast \mathbb{Z} \ast \mathbb{Z} (\mathbb{Z} \ast \mathbb{Z} \ast \mathbb{Z})\),

where \(m\) is the number of components of \(L\), \(Z\) is an infinite cyclic group, and \(Z = \langle h \rangle\) is a “special” infinite cyclic group which is the center of the link group except when the link group is that of a Hopfian link of type \((\alpha)\). In case \((b)\) the amalgamation concerning the last factor \(Z = \langle h \rangle\) is given by \(h = q^n\) for an integer \(\alpha > 1\). In case \((c)\) the last factor \(\mathbb{Z} \ast \mathbb{Z}\) is the group of a torus \((\alpha, \beta)\)-knot.

Proof of Theorem 3.6. By Lemma 3.4 the link group \(G_n\) of \(S_n\) is computed as follows. Let \(x_1, \ldots, x_4\) be the generators. The relations concerning the vertical boundary braid \(\sigma_1\sigma_2\) are \(x_1 = x_2\) and \(x_3 = x_4\). The other relations concerning the horizontal boundary braid \(\Delta^{2n}\) are

\[
\begin{align*}
x_1 &= (x_1x_2x_3x_4)^a x_1 (x_1x_2x_3x_4)^{-n}, \\
x_2 &= (x_1x_2x_3x_4)^a x_2 (x_1x_2x_3x_4)^{-n}, \\
x_3 &= (x_1x_2x_3x_4)^a x_3 (x_1x_2x_3x_4)^{-n}, \\
x_4 &= (x_1x_2x_3x_4)^a x_4 (x_1x_2x_3x_4)^{-n}.
\end{align*}
\]

Putting \(a = x_1 = x_2\) and \(b = x_3 = x_4\), we have

\[
G_n = \langle a, b \mid (a^2b^2)^n b = b(a^2b^2)^n, (a^2b^2)^n a = a(a^2b^2)^n \rangle.
\]

Let \(Z_1\) be the subgroup of \(G_1\) generated by \(\{a^2, b^2\}\), and \(Z_n\) be the subgroup of \(G_n\) generated by \(h_n = (a^2b^2)^n\) for \(n > 1\). Note that \(Z_n\) consists of central elements. We will show that \(Z_n\) is the center of \(G_n\) for \(n \geq 1\). We can show as follows: Let \(N\) be a normal subgroup of \(G\). If the center of \(G/N\) is trivial, then \(N\) is contained in the center of \(G\). Hence it suffices to show that the center of the quotient group \(G_n/Z_n\) is trivial. If \(n = 1\), then \(G_1/Z_1 = \mathbb{Z}/2 * \mathbb{Z}/2\), whose center is trivial. If \(n > 1\), then \(G_n/Z_n = \langle a, b \mid (a^2b^2)^n = 1 \rangle\), which is an amalgamated product \(\langle a \rangle \ast_{\{\}} \langle b, x \mid x^n = 1 \rangle\), where \(U = \langle a^2 \rangle = \langle xb^{-2} \rangle = Z\) and the amalgamation is given by \(a^2 = xb^{-2}\). Put \(H_1 = \langle a \rangle\) and \(H_2 = \langle b, x \mid x^n = 1 \rangle\). We can take \(\{1, a\}\) as a set of right-handed coset representatives of \(U\) in \(H_1\). Take a set of right-handed coset representatives of \(U\) in \(H_2\), and denote it by \(C\). Let \(h\) be a central element of \(G_n/Z_n = H_1 \ast_U H_2\). By Theorem 3.6 \(h\) has a normal form \(h = ua^\delta c_1a_2 \cdots c_\epsilon a_\delta\), where \(u \in U\) and \(c_1, \ldots, c_\epsilon \in C - \{1\}\) and \(\delta, \epsilon \in \{0, 1\}\). Since \(ah = ha\), we have \(ua^\delta c_1a_2 \cdots c_\epsilon a_\delta = ua^\delta c_1a_2 \cdots c_\epsilon a_\delta a\). Since \(u = au\) in the amalgamated product \(H_1 \ast_U H_2\), it follows that \(ua^\delta c_1a_2 \cdots c_\epsilon a_\delta = ua^\delta c_1a_2 \cdots c_\epsilon a_\delta a\), hence \(ac_1a_2 \cdots ac_\epsilon a = c_1a_2 \cdots c_\epsilon a a\) as elements in \(H_1 \ast_U H_2\).

If \(t > 0\), then \(ac_1a_2 \cdots ac_\epsilon a = c_1a_2 \cdots c_\epsilon a a\) are in distinct normal forms, which is a contradiction. Hence \(t = 0\) and we have \(h = ua^\delta = a^k\) for an integer \(k\). Since \(hb = bh\), we have \(a^k b = ba^k\). If \(k = 1\), then we have \(ab = ba\). In this case, if \(b\) is not in \(U\), then we can take \(b\) as a non-trivial right-handed coset representative of \(U\) in \(H_2\). It follows that then \(ab\) and \(ba\) are in distinct normal forms, which is a contradiction. If \(k = 2l + 1\) (resp. \(k = 2l\)) for a non-zero integer \(l\), then we have \(a^k b = uab \text{ and } ba^k = ca\) (resp. \(a^k b = ub \text{ and } ba^k = c\)), where in both cases \(u = a^2 \in U\) and \(c = b(xb^{-2})^l\). In these cases, if neither \(b\) nor \(c\) is in \(U\) and \(a^k b\) and \(ba^k\) have distinct normal forms \(uab\) and \(ca\) (resp. \(ub\) and \(c\)), which is a contradiction. Then it follows that \(k = 0\) and hence \(h = 1\), i.e. the center of \(G_n/Z_n\) is trivial.

It remains to show that for non-zero integer \(l\), neither \(b\) nor \(c = b(xb^{-2})^l\) is in \(U\) and we can take \(b\) and \(c\) as distinct right-handed coset representatives of \(U\) in \(H_2\). The group \(H_2 = \langle b, x \mid x^n = 1 \rangle\) is the free product of \(\langle b \rangle\) and
(x | x^n = 1). Hence by Theorem 3.3 again, every element of \( H_2 \) has a normal form \( b^{n_1}x^{m_1}b^{n_2}x^{m_2} \cdots b^{n_r}x^{m_r} \), where \( n_1, \ldots, n_t \in \mathbb{Z} \setminus \{0\}, m_1, \ldots, m_t \in \{1, 2, \ldots, n - 1\} \), and \( \delta, \epsilon \in \{0, 1\} \). Hence we can see that \( b \) has a normal form \( b \), and \( c = b(x^{x^{x^{l}}}) \) has a normal form \( b(x^{x^{l}}) \), \( b(x^{x^{l}} - 1) \) if \( l > 0 \) (resp. \( l < 0 \), where \( l_0 = \|l\| \), a positive integer. On the other hand, an element of \( U = (x^{x^{l}}) \) in \( H_2 \) has a normal form \( (x^{x^{l}})^{m+1} \) or \( (b^{x^{x^{l}}})^m \), where \( m \) is a non-negative integer. Hence by the uniqueness of normal forms, we can see that neither \( b \) nor \( c \) is in \( U \). Similarly, for a non-negative integer \( m \), an element of \( Ub \) has a normal form \( (x^{x^{l}})^{m+1} \) or \( (b^{x^{x^{l}}})^m \), \( (b^{x^{x^{l}}})^m \) or \( (b^{x^{x^{l}}})^m \) in \( \mathbb{Z} \), \( \mathbb{Z} \) or \( \mathbb{Z} \), a positive integer. By the uniqueness of normal forms, we can see that \( Ub \neq Uc \). Thus for non-zero integer \( l \), neither \( b \) nor \( c = b(x^{x^{l}}) \) is in \( U \) and we can take \( b \) and \( c \) as distinct right-handed coset representatives of \( U \) in \( H_2 \), and it follows that the center of \( G_n / Z_n \) is trivial.

Consider the abelianization map \( \phi : G_n \to G_n / [G_n, G_n] = \mathbb{Z} \times Z \) and put \( \pi = \phi(a) \) and \( b = \phi(b) \), which are the basis. Since \( \phi(a^2) = \pi^2 \) \( (\phi(b^2) = \pi^2 \), the center \( Z_1 \) is a rank-2 free abelian group. The image \( \phi(h_n) \) is \( \pi x \). Hence the center \( Z_n \) for \( n > 1 \) is an infinite cyclic group.

Next we will show that \( G_n \) is not a classical link group. By Theorem 3.7, we will show that \( G_n \) is neither of type (a), (b) nor (c) in Theorem 3.7. Since the torus-covering-link \( S_n \) consists of two components, it follows that \( m \) in Theorem 3.7 is two. We will first prove this for \( n > 1 \). Note that \( \mathbb{Z} \) in Cases (b) and (c) in Theorem 3.7 is the center \( Z_n \).

(Case (a)) If \( G_n \) is of type (a), then \( G_n = \mathbb{Z} \times \mathbb{Z} \), and it follows that \( G_n \) is commutative. On the other hand, there is a natural epimorphism

\[
(3.2) \quad f : G_n \to \mathbb{Z}/2 \ast \mathbb{Z}/2 = \langle a' \rangle \ast \langle b' \rangle,
\]

where \( a' = f(a) \) and \( b' = f(b) \), which are the basis. Since \( \mathbb{Z}/2 \ast \mathbb{Z}/2 \) is not commutative, \( G_n \) is not commutative. This is a contradiction.

(Case (b)) If \( G_n \) is of type (b), then \( G_n = (\mathbb{Z} \times \mathbb{Z}_n) \ast \mathbb{Z}_n \mathbb{Z} = ((k) \times \langle h_n \rangle) \ast \mathbb{Z}_n \langle q \rangle \), where the amalgamation is given by \( h_n = q^n \) for an integer \( \alpha > 1 \). Put \( b_n' = f(h_n) \) and \( q' = f(q) \) for the natural epimorphism \( f \) of (3.2). Since \( h_n = (a^2b^2)^n \), we see that \( b_n' = 1 \). Since \( h_n = q^n \), it follows that \( q'^\alpha = 1 \). If \( q' = 1 \), then we have \( f(G_n) = (f(k)) \), which is generated by at most one generator. However, \( f(G_n) \) is generated by two generators \( a' \) and \( b' \). Hence \( q' \) is non-trivial. Consider the abelianization map \( \phi' : f(G_n) \to \mathbb{Z}/2 \times \mathbb{Z}/2 \), and put \( \pi' = \phi'(a') \) and \( b' = \phi'(b') \), which are the basis. Since \( \alpha > 1 \) and a non-trivial element of \( f(G_n) = \mathbb{Z}/2 \ast \mathbb{Z}/2 \) has order \( 2 \) or \( \infty \), we see that \( \alpha = 2, i.e. q'^2 = 1 \). Then it follows that \( \phi'(q') = \pi' \) or \( \pi' = b' \). However, since \( h_n = q'^2 \) and \( \phi(h_n) = \pi' \pi'^{-2} \), we have \( \phi(q) = \pi' \pi'^{-2} \). Then it follows that \( \phi'(q') = 1 \) or \( \pi' b' \). This is a contradiction.

(Case (c)) If \( G_n \) is of type (c), then \( G_n = (\mathbb{Z} \times \mathbb{Z}_n) \ast \mathbb{Z}_n \langle z \rangle \langle z \rangle \), where \( \mathbb{Z} \times \mathbb{Z}_n \) is a classical knot group of a torus \( (\alpha, \beta) \)-knot, i.e. \( \mathbb{Z} \times \mathbb{Z} \mathbb{Z} = \langle x, y | x^\alpha = y^\beta \rangle \) for coprime positive integers \( \alpha \) and \( \beta \). Then similarly to Case (b), we have the following argument. Since \( h_n = (2b^2)^n \), we see that \( b_n' = 1 \), \( \alpha > 1 \) for the natural epimorphism \( f \) of (3.2). Since \( h_n = x^\alpha = y^\beta \), it follows that \( x'^\alpha = y'^\beta = 1 \), where \( x' = f(x) \) and \( y' = f(y) \). If \( x' = y' = 1 \), then \( f((x, y | x^\alpha = y^\beta)) = 1 \) and it follows that \( f(G_n) \) is generated by at most one generator. However, \( f(G_n) \) is generated by two generators \( a' \) and \( b' \). Hence we can assume that \( x' \) is non-trivial.

Since any element of \( f(G_n) = \mathbb{Z}/2 \ast \mathbb{Z}/2 \) has order \( 2 \) or \( \infty \), it follows that \( \alpha = 2 \) and \( \phi'(x') = \pi' \) or \( \pi' \), where \( \phi' : f(G_n) \to \mathbb{Z}/2 \times \mathbb{Z}/2 \) is the abelianization map.
Corollary 3.8. Hence we have a corollary.

In Theorem 3.5 are not 2-component 2-link groups. However, since $G$ is a link group, then $G_1$ is of type (a). The rest of the proof is the same as in Case (a).

Now we will show that $S_n$ and $S_m$ are not equivalent for $n \neq m$. It suffices to show in the case when $n > 1$ and $m > 1$. Since the abelianization of $G_n/Z_n$ is $\mathbb{Z} \times \mathbb{Z}/2n$, $G_n$ is not isomorphic to $G_m$ for $n \neq m$. □

It is known that if the center of a $\mu$-component 2-link group with $\mu > 1$ is non-trivial, then the center must be a torsion group (Corollary 2 of Chapter 3 in [9]). Hence we have a corollary.

Corollary 3.8. The link groups of the 2-component torus-covering $T^2$-links given in Theorem 3.3 are not 2-component 2-link groups.

Proof. In the proof of Theorem 3.3 we have shown that the center of $G_n$, where $n > 0$, is non-trivial and torsion free. □

We can consider the knot version of Theorem 3.3.

Theorem 3.9. Let $\Gamma_{T,n}$ be the torus-covering-chart of degree 4 without black vertices and with boundary braids $\sigma_1\sigma_3$ and $\Delta^{2n+1}$, where $\Delta$ is Garside’s $\Delta$ and $n$ is a positive integer. Let $S_n$ be the torus-covering-knot associated with $\Gamma_{T,n}$. Then the knot group of $S_n$ is not a classical knot group. Moreover, $S_n$ and $S_m$ are not equivalent for $n \neq m$.

Remark. Note that $S_0$ is a trivial torus knot.

Proof. By Lemma 3.4 the knot group $G_n$ of $S_n$ is computed as follows. Let $x_1, x_2, x_3, x_4$ be the generators. Then the relations concerning the vertical boundary braid $\Delta^{2n+1}$ are

\[
\begin{align*}
x_1 &= (x_1x_2x_3x_4)^n x_1x_2x_3x_4^{-1}x_3^{-1}x_2^{-1}(x_1x_2x_3x_4)^{-n}, \\
x_2 &= (x_1x_2x_3x_4)^n x_1x_2x_3x_4^{-1}x_1^{-1}(x_1x_2x_3x_4)^{-n}, \\
x_3 &= (x_1x_2x_3x_4)^n x_1x_2x_3x_4^{-1}(x_1x_2x_3x_4)^{-n}, \\
x_4 &= (x_1x_2x_3x_4)^n x_1(x_1x_2x_3x_4)^{-n}.
\end{align*}
\]

Putting $a = x_1 = x_2$ and $b = x_3 = x_4$, we have

\[G_n = \langle a, b | b(a^2b^2)^n = (a^2b^2)^n a, a(a^2b^2)^{n+1} = (a^2b^2)^{n+1} b \rangle.\]

Let $Z_n$ be the subgroup of $G_n$ generated by $h_n = (a^2b^2)^{2n+1}$, which is a central element. We show that $Z_n$ is the center of $G_n$. Consider the quotient group $G_n' = G_n/Z_n$, which is $\langle a, b, x | x = a^2b^2, ba^n = x^n a, ax^{n+1} = x^{n+1} b, x^{2n+1} = 1 \rangle$. By eliminating $b$ by $b = x^n ax^{-n}$, we have $G_n' = \langle a, x | x^{2n+1} = (a^2 x^n)^2 = 1 \rangle$, which is an amalgamated product $\langle a \rangle *_U \langle x, y | x^{2n+1} = 1, y^2 = 1 \rangle$, where $U = \langle a^2 \rangle = \langle yx^{n+1} \rangle = \mathbb{Z}$ and the amalgamation is given by $a^2 = yx^{n+1}$. By an argument similar to the proof of Theorem 3.3 we can show that the center of $G_n'$ is trivial. Thus $Z_n$ is the center. Considering the abelianization map of $G_n$, we see that $Z_n$ is an infinite cyclic group.

If $G_n$ is a classical knot group, then it is a torus knot group. Let $G_{p,q}$ be the $(p, q)$-torus knot group isomorphic to $G_n$, where $p$ and $q$ are coprime positive integers. Let $Z_{p,q}$ be the center of $G_{p,q}$. The abelianization of $G_n$ is $\mathbb{Z}/4(2n+1)$. On the other hand, the abelianization of $G_{p,q} = G_{p,q}/Z_{p,q}$ is $\mathbb{Z}/p \times \mathbb{Z}/q$. Since they are isomorphic, we have $p,q = 4(2n+1)$. Since $G_n'$ has an element of order
2n + 1, and the order of a non-trivial torsion element of $G'_{p,q}$ is a divisor of p or q, it follows that 2n + 1 is a divisor of p or q. Hence we can determine coprime positive integers p and q to be p = 4 and q = 2n + 1.

For any element w of order 2 in $G'_{2,2n+1} = \mathbb{Z}/4 \ast \mathbb{Z}(2n + 1)$, w can be written as $w = w^2$ for some element $w'$ of order 4. Since y in $G_n'$ is of order 2, there is an element $y'$ with $y = y'^2$, and we have $G_n' = \langle a, x, y' \mid x^{2n+1} = 1, y'^4 = 1, a^2x^n = y'^2 \rangle$. Let $N_{2n+1,v}$ be the normal subgroup of $G'_{2,2n+1}$ generated by an element v of order 2n+1. Since the quotient group $G'_{2,2n+1}/N_{2n+1,v}$ does not depend on the choice of v, we will denote it by $G'_{2,2n+1}/N_{2n+1}$. Let $N_m$ be the normal subgroup of $G_n'$ generated by x. Since x has order 2n + 1, we see that $G_n'/N_m$ is isomorphic to $G'_{2,2n+1}/N_{2n+1}$. We have $G_n'/N_m = \langle a, y' \mid y'^4 = 1, a^2 = y'^2 \rangle$, which is non-abelian. On the other hand, $G_{2n+1}'/N_{2n+1} = \mathbb{Z}/4$. This is a contradiction. Since the abelianization of $G_n'$ is $\mathbb{Z}/4(2n + 1)$, it follows that $S_n \neq S_m$ for $n \neq m$. □

4. Ribbon torus-covering $T^2$-links

In this section we show Theorem 4.1.

**Theorem 4.1.** The torus-covering $T^2$-links given in Theorems 3.5 and 3.9 are ribbon but not symmetry-span $T^2$-links.

**Remark.** Theorem 4.1 means that there are torus-covering $T^2$-links which can be described by surface link charts without white vertices, but any torus-covering-chart describing which has white vertices.

Before the proof, we prove the following Proposition 4.3. First we will give the definition of ribbon singularities and certain conditions for boundary braids.

Let $B^4$ be a disjoint union of a finite number of handlebodies. The image of $B^3$ into $\mathbb{R}^4$ by an immersion $\phi$ is called a 3-ribbon if the singularity set consists of ribbon singularities, i.e. the self-intersection of $\phi(B^3)$ consists of a finite number of mutually disjoint 2-disks, and for each 2-disk $D_i$, the preimage $\phi^{-1}(D_i)$ consists of a pair of 2-disks $D_i'$, $D_i''$ such that $D_i' \cap D_i'' = \emptyset, D_i' \subset \text{Int} B^3$ and $\partial D_i'' = D_i' \cap \partial B^3$. An oriented surface link is ribbon if and only if it bounds a 3-ribbon (cf. [21]).

Let us consider a torus-covering-chart of degree $nm$. Let $\Gamma_T$ be a torus-covering-chart of degree $nm$ without black vertices and with boundary braids $a$ and $b$. Let $S$ be the torus-covering $T^2$-link in $D^2 \times S^1 \times S^1$ associated with $\Gamma_T$, where $D^2$ is a 2-disk and $S^1 = [0, 1]/\sim$ with $0 \sim 1$. Let $a_i$ (resp. $b_i$) be the i-th string of a (resp. b), where $i = 1, 2, \ldots , nm$. A cylinder is $D^2_i \times [0, 1]$, where $D^2_i$ is a 2-disk. Let us consider the following conditions.

(R1) There are embedded cylinders $N_1(a), N_2(a), \ldots , N_m(a)$ (resp. $N_1(b), N_2(b), \ldots , N_m(b)$) in $D^2 \times [0, 1] \times \{0\}$ (resp. $D^2 \times \{0\} \times [0, 1]$) such that they are mutually disjoint in $D^2 \times [0, 1] \times \{0\}$ (resp. $D^2 \times \{0\} \times [0, 1]$) and $N_j(a) \cap a = a_{n(j-1)+1} \cup a_{n(j-1)+2} \cup \cdots \cup a_{nj}$ (resp. $N_j(b) \cap b = b_{n(j-1)+1} \cup b_{n(j-1)+2} \cup \cdots \cup b_{nj}$), where $j = 1, 2, \ldots , m$.

By regarding each $N_j(a)$ (resp. $N_j(b)$) as a string, we have new classical m-braids. Let us call the new m-braid obtained from a (resp. b) the tubular braid associated with a (resp. b) and denote it by $R(a)$ (resp. $R(b)$).

(R2) The associated tubular braid $R(a)$ is the trivial m-braid.

Let $R_1(a), R_2(a), \ldots , R_m(a)$ be the n-braids associated with the vertical boundary braid $a$ such that $R_j(a)$ corresponds to the j-th string of the trivial m-braid $R(a)$, where $j = 1, 2, \ldots , m$. In other words, $R_j(a)$ is the n-braid $a \cap N_j(a)$ in the cylinder $N_j(a)$.

(R3) The closed braid associated with $R_j(a)$ is a trivial knot, where $j = 1, 2, \ldots , m$. 


Let $N'\sigma_1$ be a $2n$-braid defined as follows:

$$N'\sigma_1 = \sigma_n(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)(\sigma_{n+1}\sigma_{n+2}\cdots\sigma_{2n-1}) \cdot \sigma_n(\sigma_{n-1}\sigma_{n-2}\cdots\sigma_2)(\sigma_{n+1}\sigma_{n+2}\cdots\sigma_{2n-2}) \cdot \cdots \cdot \sigma_n(\sigma_{n-1})(\sigma_{n+1}) \cdot \sigma_n.$$ 

Then for an $m$-braid $\beta$, let us define the $nm$-braid $N\beta$ as follows.

1. $N(\beta'\beta'') = N\beta' \cdot N\beta''$, for $m$-braids $\beta'$ and $\beta''$, and
2. $N\sigma_j = \iota_n^{m-j-1}(N'\sigma_1)$, where $j = 1, 2, \ldots, m - 1$.

Then under the conditions (R1), (R2) and (R3), the horizontal boundary braid $b$ can be written uniquely as follows.

$$(4.1) \quad b = NR(b) \cdot \iota^{n(m-1)}_0(\beta_1) \cdot \iota^{n(m-2)}_n(\beta_2) \cdots \cdot \iota^{0}_{n(m-1)}(\beta_m),$$

where $\beta_1, \beta_2, \ldots, \beta_m$ are $n$-braids.

**Example 4.2.** Let $\Gamma_T$ be the torus-covering-chart of degree 4 without black vertices and with boundary braids $a = \sigma_1\sigma_3$ and $b = (\sigma_1\sigma_2\sigma_3)^2$. Note that $b$ is equivalent to $\Delta^2$, where $\Delta$ is Garside’s $\Delta$ (cf. Theorem 3.5). Then $\Gamma_T$ satisfies the conditions (R1), (R2) and (R3). The associated tubular braids are 2-braids $R(a)$ and $R(b)$. The 2-braids $R_1(a)$ and $R_2(a)$ are $\sigma_1$, which are the braids associated with the vertical boundary braid $a$ such that $R_1(a)$ (resp. $R_2(a)$) corresponds to the first (resp. second) string of the trivial 2-braid $R(a).$ See Fig. 4.1. Note that the closed braid associated with $R_j(a)$ ($j = 1, 2$) is a trivial knot. Since $R(b) = \sigma_1^4$, we have $NR(b) = N\sigma_1^2(= (\sigma_2\sigma_1)\sigma_2)^2$, and the horizontal boundary braid $b$ can be written uniquely as follows: $b = NR(b) \cdot \iota_0^1(\beta_1) \cdot \iota_2^0(\beta_2)$, where $\beta_1$ and $\beta_2$ are 2-braids $\sigma_1^2$ as in Fig. 4.2.
Figure 4.2. The boundary braid $b$

**Proposition 4.3.** Let $\Gamma_T$ be a torus-covering-chart of degree $nm$ without black vertices and with boundary braids $a$ and $b$ such that the boundary braids $a$ and $b$ satisfy the conditions (R1), (R2) and (R3). Then the torus-covering $T^2$-link associated with $\Gamma_T$ is ribbon.

**Proof.** Let $S$ be the torus-covering $T^2$-link associated with $\Gamma_T$, and we will use the notations in (R1), (R2), (R3) and (4.1). Let us prepare other notations.

For a braid $\beta$, let $\text{cl}(\beta)$ be the closed braid associated with $\beta$. Let $E^3 = E^2 \times [0, 1]$ be a 3-ball which contain $D^2 \times S^1$, where $E^2$ is a 2-disk. The torus-covering-link $S$ is embedded in $D^2 \times S^1 \times S^1$ in $E^3 \times S^1$, and $E^3 \times S^1$ is naturally embedded in $\mathbb{R}^4$.

(1) We consider the motion picture of $S$ along $S^1$, i.e. we consider $S$ as $\cup_{t \in S^1} S_t$ in $E^3 \times S^1$, where $S_t = S \cap (E^3 \times \{t\})$. We can assume that $S^1 = \{0, 2\nu + 2\}$. Divide $S^1$ into $2\nu + 2$ intervals $J_1^1, J_2^1, J_3^1, \ldots, J_{\nu}^1$ with $J_1^1 = [2k - 2, 2k - 1]$ and $J_2^1 = [2k - 1, 2k]$, where $k = 1, 2, \ldots, \nu + 1$. The associated tubular braid $R(b)$ in $D^2 \times \{0\} \cup \{2\nu + 2\}$ is $E^3 \times [0, 2\nu + 2]$. Let $R(b) = \sigma_1^{i_1} \sigma_2^{i_2} \ldots \sigma_{\nu}^{i_{\nu}}$ such that $R(b) \cap (D^2 \times \{0\}) = \sigma_{i_k}^1$, where $i_k \in \{1, 2, \ldots, m - 1\}$ and $k = 1, 2, \ldots, \nu$. Then by (4.1) the horizontal boundary braid $b$ can be written uniquely as follows. $b = N\sigma_{i_1}^{i_1} N\sigma_{i_2}^{i_2} \cdots N\sigma_{i_\nu}^{i_\nu} \cdot \iota_0^0(\beta_1) \cdot \iota_n^{n(m-1)}(\beta_2) \cdots \iota_n^{n(m-1)}(\beta_m)$, where $\beta_1, \ldots, \beta_m$ are $n$-braids.

Let $\{h_{k,u}\}_{u \in [0,1]}$ be an isotopy of $D^2$ associated with $N\sigma_{i_k}^{i_k}$ for $k = 1, 2, \ldots, \nu$ (cf. Definition 3.1). Then let $\{h_{k,u}\}_{u \in [0,1]}$ be an isotopy of $E^3$ such that $f_{u\mid D^2 \times S^1} = h_{k,u} \times \text{id}_{S^1}$ and $f_{u\mid E^3 - D^2 \times S^1} = \text{id}_{E^3 - D^2 \times S^1}$, where $u \in [0,1]$ and $k = 1, 2, \ldots, \nu$. Similarly, let $\{h_{k,u}\}_{u \in [0,1]}$ be an isotopy of $D^2$ associated with $\iota_n^{n(m-1)}(\beta_1) \iota_n^{n(m-2)}(\beta_2) \cdots \iota_n^{n(m-1)}(\beta_m)$.

Then let $\{g_{u}\}_{u \in [0,1]}$ be an isotopy of $E^3$ such that $g_{u\mid D^2 \times S^1} = h'_{u} \times \text{id}_{S^1}$ and $g_{u\mid E^3 - D^2 \times S^1} = \text{id}_{E^3 - D^2 \times S^1}$, where $u \in [0,1]$.

Then by definition of $S$, we can assume that the motion picture of $S$, i.e. $S = \cup_{t \in S^1} S_t$, where $S^1 = \cup_{k=1}^{\nu+1}(J_1^k \cup J_2^k)$, is as follows. Identify $E^3 \times \{t\}$, where $t \in S^1$. Put $R^k_j = R_j(a)$ for $j = 1, 2, \ldots, m$ and $S_0 = \text{cl}(\iota_0^{n(m-1)}(R^1_1) \cdot \iota_n^{n(m-2)}(R^1_2) \cdots \iota_n^{n(m-1)}(R^m_0))$. Then

$$S_t = \begin{cases} f_{k uomini}\iota_k^{n(m-1)}(R^1_k) \iota_n^{n(m-2)}(R^1_2) \cdots \iota_n^{n(m-1)}(R^m_0) \end{cases}$$

if $t \in J_{i_k}^k = [2k - 2, 2k - 1]$, $k = 1, 2, \ldots, \nu$;
if $t \in J_{i_k}^k = [2k - 2, 2k - 1]$, $k = 1, 2, \ldots, \nu$;
if $t \in J_{i_k}^k = [2k - 2, 2k - 1]$, $k = 1, 2, \ldots, \nu$.

Put

$$R^k_j = \begin{cases} R^k_j \iota_k^{n(m-1)}(R^1_k) \iota_n^{n(m-2)}(R^1_2) \cdots \iota_n^{n(m-1)}(R^m_0) \end{cases}$$

where $k = 0, 1, 2, \ldots, \nu$ and $j = 1, 2, \ldots, m$, and put

$$R^k = \iota_k^{n(m-1)}(R^1_k) \iota_n^{n(m-2)}(R^1_2) \cdots \iota_n^{n(m-1)}(R^m_0),$$
where \( k = 0, 1, 2, \ldots, \nu \). Then \( R^{k-1} \cdot N\sigma_{i_k}^e = N\sigma_{i_k}^e \cdot R^k \) holds for \( k = 1, 2, \ldots, \nu \).

The ambient isotopy \( \{ f_{k,u} \} \) is an ambient isotopy which deforms \( \text{cl}(R^{k-1}) \) to \( \text{cl}(R^k) \).

Let \( R^{\nu+1} = R^0 \). Note that

\[
S_\nu = \text{cl}(R^\nu)
\]

if \( t \in J_2^\nu = [2k - 1, 2k] \), where \( k = 1, 2, \ldots, \nu, \nu + 1 \).

Since \( f^t \) has no black vertices, \( ab = ba \) holds. Since \( a = R^0 \) and \( a \cdot NR(b) = NR(b) \cdot R^\nu \), it follows that \( R^0_j / \beta_j = \beta_j R^0_j \) holds for \( j = 1, 2, \ldots, m \). The ambient isotopy \( \{ g_u \} \) is an ambient isotopy which deforms \( S_{2\nu} \) to \( S_{2\nu+1} = S_0 \) as follows:

\[
S_{2\nu} = \text{cl}(R^\nu) = \text{cl}(\nu(n-1)(R^0_0) \cdot \nu(n-2)(R^0_2) \cdots \nu(0)(R^0_m)) = \text{cl}(\nu(n-1)(\beta_1 R^0_1 \beta^{-1}_1) \cdot \nu(n-2)(\beta_2 R^0_2 \beta^{-1}_2) \cdots \nu(0)(\beta_m R^0_m \beta^{-1}_m)) = \text{cl}(\nu(n-1)(R^0_0) \cdot \nu(n-2)(R^0_2) \cdots \nu(0)(R^0_m)) = \text{cl}(R^0) = S_0.
\]

(2) Next we will construct a 3-ribbon \( B \) such that \( \partial B = S \).

Let \( p : E^3 = E^2 \times [0, 1] \rightarrow E^2 \) be the projection to the first factor. For two given subsets \( V, W \) of \( E^3 \), we say \( V \) is over \( W \) with respect to \( p \) if for any point \( x \) in \( p(V) \cap p(W) \), any pair of its preimages \((x, v) \in V \) and \((x, w) \in W \) satisfies that \( v > w \), where \( x \in E^2 \) and \( v, w \in [0, 1] \).

For \( t \in S^1 \), we identify \( E^3 \times \{ t \} \) with \( E^3 \). Take mutually disjoint 2-disks \( D_1^j, D_2^j, \ldots, D_m^j \) in \( E^3 \times \{ 2 \} \) such that \( \partial D_j^j = \text{cl}(R_j^0) \) and \( D_0^j \) is over \( \partial D_j^{j+1} \) with respect to the projection \( p \) for \( j = 1, 2, \ldots, m - 1 \) as follows, where \( k = 0, 1, 2, \ldots, \nu \). We take such 2-disks inductively for \( k = 0, 1, 2, \ldots, \nu \) as follows. First, since \( \text{cl}(R_0^0), \text{cl}(R_0^2), \ldots, \text{cl}(R_m^0) \) are trivial knots by (R3), we can take such 2-disks \( D_0^0, D_0^2, \ldots, D_m^0 \). Then define \( D_j^1 = f_{j,1}(D_j^0) \) if \( j \neq i_1 - 1, i_1 \), where \( j = 1, 2, \ldots, m \), and \( D_{i_1-1}^1 = f_{i_1,1}(D_{i_1}^0) \). Then take a 2-disks \( D_i^1 \) with \( \partial D_i^1 = \text{cl}(R_i^0) \) which satisfies the conditions that (i) \( D_i^1 \cap D_j^1 = \emptyset \) if \( j \neq i_1, 1, i_1 \), and (ii) \( D_i^1 \) is over \( D_{i_1+1}^1 \) with respect to \( p \) for \( j = 1, 2, \ldots, m - 1 \). Define \( D_j^2 = f_{j,2}(D_j^1) \) if \( j \neq i_2 - 1, i_2 \), where \( j = 1, 2, \ldots, m \), and \( D_{i_2-1}^2 = f_{i_2,1}(D_{i_2}^1) \). And then take a 2-disks \( D_i^2 \) with \( \partial D_i^2 = \text{cl}(R_i^2) \) which satisfies the conditions that (i) \( D_i^2 \cap D_j^2 = \emptyset \) if \( j \neq i_2, \) where \( j = 1, 2, \ldots, m \), and (ii) \( D_j^2 \) is over \( D_{j+1}^2 \) with respect to \( p \) for \( j = 1, 2, \ldots, m - 1 \). Repeating this process, we have 2-disks \( D_1^k, D_2^k, \ldots, D_m^k \) for \( k = 0, 1, 2, \ldots, \nu \). Note that we have

\[
D_j^k = \begin{cases} f_{j,1}(D_k^{j-1}) & \text{if } j \neq i_k - 1, i_k, \\ f_{i_k,1}(D_{i_k}^{k-1}) & \text{if } j = i_k - 1, \end{cases}
\]

where \( j = 1, 2, \ldots, m \) and \( k = 1, 2, \ldots, \nu \).

Let us denote \( f_{j,1}(D_k^{j-1}) \) by \( D_{i_k}^j \). Take an ambient isotopy \( \{ \phi_{k,u} \}_{u \in [0, 1]} \) which carries \( D_k^j \) to \( D_{i_k}^j \), where \( k = 1, 2, \ldots, \nu \), and let \( \{ \psi_u \}_{u \in [0, 1]} \) be an ambient isotopy which carries \( g_j(D_1^j \cup D_2^j \cup \cdots \cup D_m^j) \) to \( D_1^j \cup D_2^j \cup \cdots \cup D_m^j \). Since \( D_1^j \) (resp. \( D_0^j \)) is over \( D_{i_k+1}^1 \) (resp. \( D_{j+1}^2 \)) with respect to \( p \) for \( j = 1, 2, \ldots, m - 1 \), \( \{ \psi_u \} \) exists.
For $k = 1, 2, \ldots, \nu + 1$ and $j = 1, 2, \ldots, m$, define $D^k_j(t)$ for $t \in J^1_k$ and $t \in J^2_k$ as follows.

\[
D^k_j(t) = \begin{cases} 
   f_{k, t-(2k-2)}(D^{k-1}_j) & \text{if } t \in J^1_k = [2k-2, 2k-1], \ k = 1, 2, \ldots, \nu, \\
   D^k_j & \text{if } t \in J^2_k = [2k-1, 2k], \ j \neq i_k, \ k = 1, 2, \ldots, \nu, \\
   \varphi_{k, t-(2k-1)}(D^{k+1}_j) & \text{if } t \in J^2_k = [2k-1, 2k], \ j = i_k, \ k = 1, 2, \ldots, \nu, \\
   g_{t-2\nu}(D^k_j) & \text{if } t \in J^1_{\nu+1} = [2\nu, 2\nu+1], \ k = \nu + 1, \\
   \psi_{t-(2\nu+1)}(D^k_j) & \text{if } t \in J^2_{\nu+1} = [2\nu+1, 2\nu+2], \ k = \nu + 1,
\end{cases}
\]

where $j = 1, 2, \ldots, m$.

Then put

\[B_t = D^1(t) \cup \cdots \cup D^m(t),\]

where $t \in \cup_{k=1}^{\nu+1}(J_k^1 \cup J_k^2)$, and define

\[B = \cup_{t \in S} B_t = \cup_{j=1}^{\nu+1} \cup_{t \in J_j^1 \cup J_j^2} B_t.\]

We can assume that $D^k_j(t)$ is disjoint with $\cup_{j \neq i_k, k} D^k_j(t)$, where $t \in J^2_k$ and $k = 1, 2, \ldots, \nu$. By the construction, $(\cup_{j=1}^{\nu+1} \cup_{t \in J^1_j} B_t) \cup (\cup_{t \in J^2_{\nu+1}} B_t)$ has no singularities, and for $k = 1, 2, \ldots, \nu$, $\cup_{t \in J^2_k} B_t$ has $n$ singular 2-disks whose singularities are ribbon. Hence $B$ is a 3-ribbon such that $\partial B = S$. \qed

Proof of Theorem 5.1 By Proposition 4.3, the torus-covering $T^2$-links given in Theorems 3.5 and 3.9 are ribbon. \qed

5. Triple point numbers

The triple point number of a surface link $F$ is the minimum number of triple points in a surface diagram of $F$, for all the surface diagrams. Symmetry-spun $T^2$-links are pseudo-ribbon, and by Theorem 5.1 the torus-covering $T^2$-links given in Theorems 3.5 and 3.9 are ribbon. Both of them have the triple point number zero. In this section we give an example of a torus-covering $T^2$-knot whose triple point number is positive.

**Theorem 5.1.** Let $\Gamma_T$ be the torus-covering-chart of degree 4 without black vertices and with boundary braids $\sigma_1\sigma_2\sigma_3$ and $(\sigma_1\sigma_2\sigma_3)^4$. Then the triple point number of the torus-covering-knot $S$ associated with $\Gamma_T$ is positive.

We tri-color the surface diagram associated with $\Gamma_T$ and show that the quandle cocycle invariant associated with Mochizuki’s 3-cocycle does not have an integer value. Then any surface diagram of $S$ has at least four triple points (cf. [18]). We use the following facts. Let $F$ be an oriented surface link.

Let $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ be a generic projection. In the surface diagram $D = \pi(F)$, there are two intersecting sheets along each double point curve, one of which is higher than the other with respect to $\pi$. They are called the over sheet and the under sheet along the double point curve, respectively. In order to indicate crossing information of the surface diagram, we break the under sheet into two pieces missing the over sheet. This can be extended around a triple point. Around a triple point, the sheets are called the top sheet, the middle sheet, and the bottom sheet from the higher one. Then the surface diagram is presented by a disjoint union of compact surfaces which are called broken sheets. We denote by $B(D)$ the set of broken sheets of $D$.

A set $X$ with a binary operation $*: X \times X \to X$ is called a quandle if it satisfies the following conditions:

(i) for any $a \in X$, $a * a = a$,
(ii) for any $a, b \in X$, there exists a unique $c \in X$ such that $a = c * b$, and
The dihedral quandle of order 3, $R_3$, is the set $\{0, 1, 2\}$ with the binary operation $a \ast b = 2b - a \pmod{3}$. A tri-coloring for a surface diagram $D$ is a map $C : B(D) \rightarrow R_3$ such that $C(H_1) \ast C(H_2) = C(H'_1)$ along each double point curve of $D$, where $H_2$ is the over sheet and $H_1$ (resp. $H'_1$) is the under sheet such that the normal vector of $H_2$ points from (resp. toward) it. The image by $C$ is called the color.

At a triple point of $D$, there exist broken sheets $J_1$, $J_2$, $J_3 \in B(D)$ uniquely such that $J_1$ is the bottom sheet, $J_2$ is the middle sheet, $J_3$ is the top sheet and the normal vector of $J_2$ (resp. $J_3$) points from $J_1$ (resp. $J_2$). The color of the triple point is the triplet $(C(J_1), C(J_2), C(J_3)) \in R_3 \times R_3 \times R_3$. The sign of the triple point is positive or $+1$ (resp. negative or $-1$) if the triplet of the normal vectors of $J_1$, $J_2$, $J_3$ is right-handed (resp. left-handed).

If $D$ has a corresponding chart, this corresponds to the following (cf. Proposition 4.43 (3) in [5]). The color of the white vertex representing $\sigma_1^i \sigma_2^j \sigma_3^k$ is $(a, b, c)$, where $a$, $b$ and $c$ are the colors of the broken sheets of $D$ connected with the starting points of the $i$-th, $(i' + 1)$-th, and $(i' + 2)$-th strings of $\sigma_1^i \sigma_2^j \sigma_3^k$, where $i' = \min\{i, j\}$. The white vertex is positive (resp. negative) if $j > i$ (resp. $i > j$), i.e. if there is exactly one edge with the largest (resp. smallest) label oriented toward the white vertex.

**Remark.** For the quandle cocycle invariants, there is a general theory (cf. [6, 7]).

Proof of Theorem 5.1. Tri-color the surface diagram associated with $\Gamma_T$. Then the tri-coloring for the diagrams of the boundary braids $\sigma_1^i \sigma_2^j \sigma_3$ and $(\sigma_1^i \sigma_2^j \sigma_3)^k$ is as in Fig. 5.1 and 5.2, where $(a, b, c) = \{0, 1, 2\}$, $\{0\}$, $\{1\}$ or $\{2\}$, which are all the tri-colorings. Let us denote by $C$ the tri-coloring described by $(a, b, c)$. The part of the torus-covering-chart without black vertices and with boundary braids $\sigma_1$ and $(\sigma_1 \sigma_2 \sigma_3)^k$ has four white vertices. Denote these by $\tau_{11}^i$, $\ldots$, $\tau_{14}^i$ from left to right as in Fig. 5.3 The color of each white vertex is obtained from reading the colors along the dotted path in Fig. 5.2 and the sign is obtained from the labels and the orientations of the edges around the white vertex. Similarly, we have white vertices $\tau_{21}^i$, $i = 2, 3, 4$, $\tau_{31}^i$, $\ldots$, $\tau_{34}^i$, Fig. 5.3 shows the white vertices $\tau_{23}^i$, $\ldots$, $\tau_{24}^i$ and the colors when $i = 2$. The matrix which describes the sign and the color of each white vertex is as follows, where the $(i, j)$-element describes those of $\tau_{ij}^i$:
Figure 5.1. The tri-coloring for $\sigma_1 \sigma_2^2 \sigma_3$

Figure 5.2. The tri-coloring for $(\sigma_1 \sigma_2 \sigma_3)^4$

Figure 5.3. White vertices $\tau_{11}^1, \ldots, \tau_{14}^1$

Figure 5.4. White vertices $\tau_{11}^2, \ldots, \tau_{14}^2$ and the colors when $i = 2$
Then the Boltzmann weight $W_\theta(C)$ for the Mochizuki’s 3-cocycle $\theta$ and the tricoloring $C$ is $W_\theta(C) = \theta(c, b, c) \cdot \theta(a, c, b)^{-1} \cdot \theta(b, c, b)^{-1} \cdot \theta(b, c, a)$, and we have

$$W_\theta(C) = \begin{cases} t^2 & \text{if } \{a, b, c\} = \{0, 1, 2\} \\ 1 & \text{if } \{a, b, c\} = \{0\}, \{1\}, \{2\}. \end{cases}$$

Hence $\Phi_\theta(S)$, the quandle cocycle invariant for $S$ associated with Mochizuki’s 3-cocycle, is $\Phi_\theta(S) = 3 + 6t^2 \notin \mathbb{Z}$ in $\mathbb{Z}[t]/(t^3 = 1)$.

An oriented surface link $F$ is (-)-amphicheiral if $F = -F^*$, where $-F^*$ is the orientation-reversed mirror image of $F$. Symmetry-spun $T^2$-links are (-)-amphicheiral. Similarly to the proof of Theorem 5.1, using $-\Gamma_T^*$, we can see that $\Phi_\theta(-S^*) = 3 + 6t \neq \Phi_\theta(S)$, thus we have a corollary.

**Corollary 5.2.** The torus-covering $T^2$-knot of Theorem 5.1 is not (-)-amphicheiral.

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