COCYCLE CONJUGACY OF FREE BOGOLJUBOV ACTIONS OF $\mathbb{R}$

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Abstract. We show that Bogoljubov actions of $\mathbb{R}$ on the free group factor $L(F_\infty)$ associated to sums of infinite multiplicity trivial and certain mixing representations are cocycle conjugate if and only if the underlying representations are conjugate.

1. Introduction

Recall that two actions $\beta_t, \gamma_t$ of $\mathbb{R}$ on a von Neumann algebra $M$ are said to be conjugate if $\beta_t = \alpha \circ \gamma_t \circ \alpha^{-1}$ for some automorphism $\alpha$ of $M$. The actions are said to be cocycle conjugate if there exists a strongly continuous one-parameter family of unitaries $u_t \in M$ and an automorphism $\alpha$ of $M$ so that

$$\beta_t(x) = \alpha(\text{Ad}_{u_t}(\gamma_t(\alpha^{-1}(x)))), \quad \forall x \in M;$$

in other words, $\beta_t$ and $\text{Ad}_{u_t} \circ \gamma_t$ are conjugate. Cocycle conjugacy is clearly a weaker notion of equivalence than conjugacy.

A consequence of cocycle conjugacy is an isomorphism between the crossed product von Neumann algebras $M \rtimes_\beta \mathbb{R}$ and $M \rtimes_\gamma \mathbb{R}$. This isomorphism takes $M$ to $M$ and sends the unitary $U_t \in L(\mathbb{R}) \subset M \rtimes_\beta \mathbb{R}$ implementing the automorphism $\beta_t$ to the unitary $u_t V_t$, where $V_t \in M \rtimes_\gamma \mathbb{R}$ is the implementing unitary for $\gamma_t$.

An important class of automorphisms of the free group factor $L(F_\infty)$ are so-called free Bogoljubov automorphisms, which are defined using Voiculescu’s free Gaussian functor. As a starting point, we write $L(F_\infty) = W^*(S_1, S_2, \ldots)$ where $S_j$ are an infinite free semicircular system. The closure in the $L^2$ norm of their real-linear span is an infinite dimensional real Hilbert space. Voiculescu proved that any automorphism of that Hilbert space extends to an automorphism of $L(F_\infty)$. In particular, any representation of $\mathbb{R}$ on an infinite dimensional Hilbert space canonically gives an action of $\mathbb{R}$ on $L(F_\infty)$.

Motivated by the approach in [1], we prove the following theorem, which states that for a large class of Bogoljubov automorphisms, cocycle conjugacy and conjugacy are equivalent to conjugacy of the underlying representations and thus gives a classification of such automorphisms up to cocycle conjugacy.

**Theorem.** Let $\pi_1, \pi'_1$ be two mixing orthogonal representations of $\mathbb{R}$, and assume that $\pi_1 \otimes \pi_1 \cong \pi_1, \pi'_1 \otimes \pi'_1 \cong \pi'_1$. Denote by $1$ the trivial representation of $\mathbb{R}$. Let

$$\pi = (1 \oplus \pi_1)^{\oplus \infty}, \quad \pi' = (1 \oplus \pi'_1)^{\oplus \infty},$$

and let $\alpha$ (resp., $\alpha'$) be the associated free Bogoljubov actions of $\mathbb{R}$ on $L(F_\infty)$. Then $\alpha$ and $\alpha'$ are cocycle conjugate iff the representations $\pi^{\oplus \infty}$ and $(\pi')^{\oplus \infty}$ are conjugate, i.e.

$$\pi^{\oplus \infty} = V(\pi')^{\oplus \infty} V^{-1}$$

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for some orthogonal isomorphism $V$ of the underlying real Hilbert spaces.

It is worth noting that the conjugacy class of a representation of $\mathbb{R}$ on a real Hilbert space is determined by the measure class of its spectral measure (a measure on $\mathbb{R}$ satisfying $\mu(-X) = \mu(X)$ for all Borel sets $X$) and a multiplicity function which is measurable with respect to that class (for the purposes of our Theorem, we may assume that this multiplicity function is identically infinite). Our Theorem then states that, for Bogoljubov actions satisfying the hypothesis of the Theorem, cocycle conjugacy occurs if and only if these measure classes are the same.

2. Preliminaries on conjugacy of automorphisms

2.1. Crossed products. If $M$ is a type $\Pi_1$ factor and $\alpha_t : \mathbb{R} \to \text{Aut}(M)$ is a one-parameter group of automorphisms, the crossed product $M \rtimes_\alpha \mathbb{R}$ is of type $\Pi_\infty$ with a canonical trace $\text{Tr}$. Furthermore, the crossed product construction produces in a canonical way a distinguished copy of the group algebra $L(\mathbb{R})$ inside the crossed product algebra. We denote this copy by $L_\alpha(\mathbb{R})$. The relative commutant $L_\alpha(\mathbb{R})' \cap M \rtimes_\alpha \mathbb{R}$ is generated by $L_\alpha(\mathbb{R})$ and the fixed point algebra $M^\alpha$.

2.2. Conjugacy of actions. Recall that if $\beta_t$ and $\gamma_t$ are cocycle conjugate, each choice of a cocycle conjugacy produces an isomorphism $\Pi_{\gamma,\beta}$ of the crossed product algebras $M \rtimes_\beta \mathbb{R}$ and $M \rtimes_\gamma \mathbb{R}$. Note that $\Pi_{\gamma,\beta}$ does not necessarily map $L_\beta(\mathbb{R})$ to $L_\gamma(\mathbb{R})$. In fact, as we shall see, this is rarely the case even if we compare the image $\Pi_{\gamma,\beta}(L_\beta(\mathbb{R}))$ with $L_\gamma(\mathbb{R})$ up to a weaker notion of equivalence, $\preceq_{M \rtimes_\beta \mathbb{R}}$ which was introduced by Popa in the framework of his deformation-rigidity theory. Indeed, in parallel to Theorem 3.1 in $[1]$, we show that, very roughly, conjugacy of $\Pi_{\gamma,\beta}(L_\beta(\mathbb{R}))$ and $L_\gamma(\mathbb{R})$ inside the crossed product is essentially equivalent (up to compressing by projections) to conjugacy of the actual actions by an inner automorphism of $M$.

Theorem 1. Let $M$ be a tracial von Neumann algebra with a fixed faithful normal trace $\tau$. Suppose $\alpha, \beta : \mathbb{R} \to \text{Aut}(M)$ are two trace-preserving actions of $\mathbb{R}$ on $M$ which are cocycle conjugate, and suppose that the only finite-dimensional $\alpha$-invariant subspaces of $L^2(M)$ are those on which $\alpha$ acts trivially. Fix any $q \in M^\beta$ a nonzero projection. The following are equivalent:

(a) There exists a nonzero projection $r \in L_\beta(\mathbb{R})$ such that

$$\Pi_{\alpha,\beta}(L_\beta(\mathbb{R})qr) \preceq_{M \rtimes_\alpha \mathbb{R}} L_\alpha(\mathbb{R})$$

(b) There exists a nonzero partial isometry $v \in M$ such that $v^*v \in qM^\beta q$, $vv^* \in M^\alpha$, and for all $x \in M$,

$$\alpha_t(vvx^*) = v\beta_t(x)v^*.$$ 

Proof. To see that (a) implies (b), take $r$ as in (a), so that $\Pi_{\alpha,\beta}(L_\beta(\mathbb{R})qr) \preceq_{M \rtimes_\alpha \mathbb{R}} L_\alpha(\mathbb{R})$, and take $w_t \in M$ with $\text{Ad} w_t \circ \alpha_t = \beta_t$.

First, we claim that there’s a $\delta > 0$ for which there exist $x_1, ..., x_k \in qM$ with

$$\sum_{i,j=1}^k |\tau(x_i^* w_t \alpha_t(x_j))|^2 \geq \delta$$
for all $t$. Suppose for a contradiction that no such $\delta$ exists. Then we can find a net $(t_i)_{i \in I}$ such that

$$\lim_{i} \tau(x^* w_{t_i} \alpha_{t_i}(y)) = 0$$

for any $x, y \in qM$.

But then for any $p, p'$ finite trace projections in $L_\alpha(\mathbb{R})$, $s, s' \in \mathbb{R}$, and $x, y \in M$, we have (in the 2-norm from the trace on $M \rtimes_\alpha \mathbb{R}$):

$$\|E_{L_\alpha(\mathbb{R})}(p \lambda_\alpha(s) x^* \Pi_{\alpha,\beta}(\lambda_\beta(t_i) q)y \lambda_\alpha(s') p')\|_2 = \|\lambda_\alpha(s)^* p E_{L_\alpha(\mathbb{R})} (x^* q \Pi_{\alpha,\beta}(\lambda_\beta(t_i) qy) p' \lambda_\alpha(s'))\|_2$$

$$= \|\lambda_\alpha(s)^* p E_{L_\alpha(\mathbb{R})} (x^* w_{t_i} \alpha_{t_i}(qy) p' \lambda_\alpha(s' + t_i))\|_2$$

where the last equality follows from the fact that $(qx)^* w_{t_i} \alpha_{t_i}(qy) \in M$, so

$$E_{L_\alpha(\mathbb{R})}(x^* w_{t_i} \alpha_{t_i}(qy)) = \tau((qx)^* w_{t_i} \alpha_{t_i}(qy)),$$

and the latter term goes to zero by supposition for any $x, y \in M$.

Now note that linear combinations of terms of the form $x \lambda_\alpha(s)p$ (resp. $y \lambda_\beta(s') p'$) as above are dense in $L^2(M \rtimes_\alpha \mathbb{R}, Tr)$, so by approximating $\Pi_{\alpha,\beta}(r) a$, (resp. $\Pi_{\alpha,\beta}(r) b$) with such sums for any $a, b \in M \rtimes_\alpha \mathbb{R}$, it follows from the above estimate that

$$\|E_{L_\alpha(\mathbb{R})}(a^* \Pi_{\alpha,\beta}(\lambda_\beta(t_i) qr) b)\|_2 \to 0.$$

But this contradicts $\Pi_{\alpha,\beta}(L_\beta(\mathbb{R}) qr) \prec M \rtimes_\alpha \mathbb{R} L_\alpha(\mathbb{R})$, so the $\delta > 0$ of our above claim exists.

We can thus find $\delta > 0$, $x_1, ..., x_k \in qM$ such that $\sum_{i,j=1}^k |\tau(x_i^* w_{t_i} \alpha_{t_i}(x_j))|^2 \geq \delta$ for all $t$.

Let us now consider the space $B(L^2(M))$ of bounded operators on $L^2(M, \tau)$ and the rank-one orthogonal projection $e_\tau$ onto $1 \in L^2(M, \tau)$. We can identify $(B(L^2(M)), e_\tau)$ with the basic construction $(M, e_\tau)$ for $C \subset M$, so that $e_\tau$ is the Jones projection. Let $\hat{\tau}$ be the usual trace on $B(L^2(M))$, satisfying $\hat{\tau}(xe_\tau y) = \tau(xy)$ for all $x, y \in M$. Finally, for a finite-rank operator $Q = \sum_i y_i e_\tau z_i^*$ with $y_i, z_i \in M$, let

$$T_M(Q) = \sum y_i z_i^* \in M.$$

Then $T_M$ extends to a normal operator-valued weight from the basic construction to $M$ satisfying $\hat{\tau} = \tau \circ T_M$ (i.e. $T_M$ is the pull-down map).

Consider now the positive element

$$X = \sum_{i=1}^k x_i e_\tau x_i^*,$$

together with the following normal positive linear functional on $(M, e_\tau)$:

$$\psi(T) = \sum_{i=1}^k \hat{\tau}(e_\tau x_i^* T x_i e_\tau).$$

Note that $T_M(X) = \sum_{i=1}^k x_i x_i^* \in M$, so in particular $\|T_M(X)\| < \infty$. 

For every $t \in \mathbb{R}$, we have:
\[
\psi(\beta_t(X)) = \sum_{i,j} \hat{\tau}(e_{\tau} \alpha_t(x_i) e_{\tau} \alpha_t(x_j)^* w_t^* e_{\tau} x_{i}) \\
= \sum_{i,j} |\tau(x_i^* w_t \alpha_t(x_j))|^2 \geq \delta > 0.
\]

Now consider $K$, the ultraweak closure of the convex hull of $\{\beta_t(X) : t \in \mathbb{R}\}$ inside $q(M, e_{\tau})q$. Note that by normality of $\psi$, $\psi(x) \geq \delta$ for any $x \in K$.

Since $K$ is convex and $\|\cdot\|_2$-closed, there exists a unique $X_0 \in K$ of minimal 2-norm. But since the 2-norm is invariant under $\beta$, we must have that $\|\beta_t(X_0)\|_2 = \|X_0\|_2$ for all $t$, so by uniqueness of the minimizer, $X_0$ is itself fixed by the extended $\beta$ action (and nonzero since $\psi(X_0) \geq \delta$). Also, by ultraweak lower semicontinuity of $T_M$, we know that $\|T_M(X_0)\| \leq \|T_M(X)\| < \infty$.

Take a nonzero spectral projection $e$ of $X_0$. Then $e$ is still $\beta$-invariant and satisfies $\|T_M(e)\| < \infty$. But this means that $\hat{\tau}(e) = \tau(T_M(e)) < \infty$, so $e$ must be a finite rank projection, since $\hat{\tau}$ corresponds to the usual trace $Tr$ on the trace-class operators in $B(L^2(M), \tau)$.

Now since $e_{\tau}$ has central support 1 in $\langle M, e_{\tau} \rangle$ (and because $e_{\tau}$ is minimal), we have that there exists $V$ a partial isometry in $\langle M, e_{\tau} \rangle$ such that $V^* V = f \leq e$ and $VV^* = e_{\tau}$. We remark that $f$ remains $\beta$-invariant, since $e$ was finite rank, and our finite-dimensional invariant subspaces are all fixed by the action. Note also that $e \leq q$ (since $X_0 \in q(M, e_{\tau})q$), so that $V = \overline{Vq} = e_{\tau} V$.

Applying the pull-down lemma, we see that:
\[
V = e_{\tau} V = e_{\tau} (T_M(e_{\tau} V)) = e_{\tau} T_M(V).
\]
Set $v = T_M(V)$ and note that because $\|T_M(V^* V)\| \leq \|T_M(e)\| < \infty$, we have $v \in M$, and $V = e_{\tau} v$.

Since $e_{\tau} \langle M, e_{\tau} \rangle e_{\tau} = \mathbb{C} e_{\tau}$, and since $V$ is left-supported by $e_{\tau}$, we have that for each $t$ there exists a $\lambda_t \in \mathbb{C}$ such that $\lambda_t e_{\tau} = V w_t \alpha_t(V^* V)$. Note that since $V w_t \alpha_t(V^* V) w_t^* V^* = V \beta_t(e) V^* = V V^* = e_{\tau}$, the last equality of the previous sentence implies that $\lambda_t \lambda_t = 1$. We also have:
\[
e_{\tau} \lambda_t \alpha_t(v) = \lambda_t e_{\tau} \alpha_t(v) = \lambda_t e_{\tau} \alpha_t(V) \\
= V w_t \alpha_t(V^* V) = V \beta_t(e) w_t = V w_t \\
= e_{\tau} v w_t.
\]

Thus, applying the pull-down map, we have that $\lambda_t \alpha_t(v) = v w_t$, and, replacing $v$ by its polar part if necessary, we’ve found a partial isometry in $M$, conjugation by which intertwines the actions. We have for any $x \in M$:
\[
\alpha_t(v x v^*) = \alpha_t(v) \alpha_t(x) \alpha_t(v^*) = \overline{\lambda_t v w_t \alpha_t(x) w_t^* v^*} = w_t^* v \lambda_t = v \beta_t(x) v^*.
\]

Furthermore, with some applications of $\alpha_t(v) = \overline{\lambda_t v w_t}$, we see that $\beta_t(v^* v) = w_t \alpha_t(v^* v) w_t^* = w_t^* w_t \alpha_t(v^* v) \lambda_t = w_t^* v \lambda_t = v^* v$, and
\[
\alpha_t(v v^*) = (\overline{\lambda_t v w_t})(w_t^* v \lambda_t) = v v^*.
\]

so we’ve found the promised intertwiner.
Conversely, assume that we have \( v \in M \) satisfying \( v^*v \in qM^\beta q \), \( vv^* \in M^\alpha \), and \( \alpha_t(vxv^*) = v\beta_t(x)v^* \) for all \( x \in M \). Take \( w_t \in M \) with \( \text{Ad} \, w_t \circ \alpha_t = \beta_t \). Then, as above, we have \( vw_t = \lambda_t \alpha_t(v) \), for some \( \lambda_t \in \mathbb{T} \). Multiplying both sides by \( \overline{\lambda_t} \) and absorbing this factor into \( w_t \), we may assume without loss of generality that \( \lambda_t = 1 \) for all \( t \), so we have \( vw_t = \alpha_t(v) \).

Now let \( \lambda_t^\alpha \) (resp., \( \lambda_t^\beta \)) denote the canonical unitaries that implement the respective actions on \( M \) in the crossed product \( M \rtimes_{\alpha} \mathbb{R} \) (resp., \( M \rtimes_{\beta} \mathbb{R} \)). Then the relation \( vw_t = \alpha_t(v) \) implies \( v\Pi_{\alpha,\beta}(\lambda_t^\beta) = \lambda_t^\alpha v \). Furthermore, for any finite trace projection \( r \in L_\beta(\mathbb{R}) \), we have \( v\Pi_{\alpha,\beta}(qr) = vq\Pi_{\alpha,\beta}(r) = v\Pi_{\alpha,\beta}(r) \neq 0 \), so \( v^* \) is a partial isometry that witnesses \( \Pi_{\alpha,\beta}(L_\beta(\mathbb{R})q) \prec M \rtimes_{\alpha,\beta} L_\alpha(\mathbb{R}) \) (e.g., see condition (4) of Theorem F.12 in [2]). Thus, (b) implies (a).

\[ \square \]

3. Cocycle conjugacy of Bogoliubov Automorphisms

3.1. Free Bogoliubov automorphisms. Let \( \pi \) be an orthogonal representation of \( \mathbb{R} \) on a real Hilbert space \( H_\mathbb{R} \). Recall that Voiculescu’s free Gaussian functor associates to \( H_\mathbb{R} \) a von Neumann algebra

\[ \Phi(H_\mathbb{R}) = \{ s(h) : h \in H_\mathbb{R} \}'' \cong L(\mathbb{F}_{\dim H_\mathbb{R}}) \]

where \( s(h) = \ell(h) + \ell(h)^* \) and \( \ell(h) \) is the creation operator \( \xi \mapsto h \otimes \xi \) acting on the full Fock space \( \mathcal{F}(H) = \bigoplus_{n \geq 0} (H_\mathbb{R} \otimes \mathbb{C})^\otimes n \). Denoting by \( \Omega \) the unit basis vector of \( (H_\mathbb{R} \otimes \mathbb{C})^\otimes 0 = \mathbb{C} \), it is well-known that the vector-state

\[ \tau(\cdot) = \langle \Omega, \cdot \Omega \rangle \]

defines a faithful trace-state on \( \Phi(H_\mathbb{R}) \). Furthermore, \( \mathbb{R} \) acts on \( \mathcal{F}(H) \) by unitary transformations \( U_t = \bigoplus_{n \geq 0} (\pi \otimes 1)^\otimes n \), and conjugation by \( U_t \) leaves \( \Phi(H_\mathbb{R}) \) globally invariant thus defining a strongly continuous one-parameter family of free Bogoliubov automorphisms

\[ t \mapsto \alpha_t \in \text{Aut}(\Phi(H_\mathbb{R})). \]

Note that if \( \pi \) is such that \( \pi \otimes \pi \) and \( \pi \) are conjugate (as representations of \( \mathbb{R} \)), then the representation \( U_t \) is conjugate to \( 1 \oplus \pi^{\otimes \infty} \).

A complete invariant for the orthogonal representation \( \pi \) consists of the absolute continuity class \( \mathcal{C}_\pi \) of a measure \( \mu \) on \( \mathbb{R} \) satisfying the symmetry condition \( \mu(B) = \mu(-B) \) for all \( \mu \)-measurable sets \( B \) and a \( \mu \)-measurable multiplicity function \( n : \mathbb{R} \rightarrow \mathbb{N} \cup \{ +\infty \} \) satisfying \( n(x) = n(-x) \) almost surely in \( x \). In particular, assuming that \( \pi \cong \pi \otimes \pi \) (i.e., that for some (hence any) probability measure \( \mu \in \mathcal{C}_\pi \) that generates \( \mathcal{C}_\pi \), \( \mu \star \mu \) and \( \mu \) are mutually absolutely continuous), then the measure class \( \mathcal{C}_\pi \) is an invariant of \( \alpha \) up to conjugacy (since it can be recovered from \( U_t \), the unitary representation induced by \( \alpha_t \) on \( L^2(\Phi(H_\mathbb{R})). \)

Recall that the representation \( \pi \) is said to be mixing if for all \( \xi, \eta \in H_\mathbb{R} \), \( \lim_{|t| \to \infty} \langle \xi, \pi(t)\eta \rangle \to 0 \). This is equivalent to saying that for some (hence any) probability measure \( \mu \in \mathcal{C}_\pi \) that generates \( \mathcal{C}_\pi \), the Fourier transform satisfies \( \hat{\mu}(t) \to 0 \) whenever \( t \to \pm \infty \).

3.2. Operator-valued semicircular systems. The crossed product \( \Phi(H_\mathbb{R}) \rtimes_{\alpha} \mathbb{R} \) has a description in terms of so-called operator-valued semicircular systems (see [3, Examples 2.8, 5.2]). Decompose \( \pi = \bigoplus_{i \in I} \pi_i \) into cyclic representations \( \pi_i \) with cyclic vectors \( \xi_i \). Let \( \mu_i \) be the measure with Fourier transform \( t \mapsto \langle \xi_i, \pi_i(t)\xi_i \rangle \), and denote by \( \eta_i : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}) \) the completely positive map given by

\[ \eta_i(f)(x) = \int f(y) d\mu(x-y). \]
Lemma 2. Suppose that the proof is straightforward from the definition of mixing bimodules.

Lemma 3. Let \( (K_j : j \in J) \) be a family of kernel measures on \( \mathbb{R}^2 \) and let \( \eta_j \) be the associated completely positive maps on \( L^\infty(\mathbb{R}) \).

Assume that each \( K_j \) can be written as a sum of measures \( K_j = \sum_{i \in S(j)} K_j^{(i)} \) with \( K_j^{(i)} \) disjoint, and so that \( K_j^{(i)} \) is supported on the square \( I_j^{(i)} \times I_j^{(i)} \) for a finite interval \( I_j^{(i)} \). Finally, suppose that there exist measures \( \hat{K}_j^{(i)} \) on \( I_j^{(i)} \times I_j^{(i)} \), so that \( K_j^{(i)} \) is absolutely continuous with respect to \( \hat{K}_j^{(i)} \) and so that the associated completely positive map

\[
\eta_j^{(i)}(f) : x \mapsto \int f(y)d\hat{K}_j^{(i)}(x,y)
\]
defines a mixing $L^\infty(\mathcal{I}_j^{(i)})$-bimodule.

Let $X_j$ be $\eta_j$-semicircular variables over $L^\infty(\mathbb{R})$, and assume that $X_j$ are free with amalgamation over $L^\infty(\mathbb{R})$. Then the semifinite von Neumann algebra $M = W^*(L^\infty(\mathbb{R}), X_j : j \in J)$ is solid, in the sense that if $A \subset M$ is any diffuse abelian von Neumann subalgebra generated by its finite projections, then $A' \cap M$ is amenable.

**Proof.** Denote by $\hat{X}_j^{(i)}$ the $\hat{\eta}_j^{(i)}$-semicircular family, and assume that $\hat{X}_j^{(i)}$ are free with amalgamation over $L^\infty(\mathbb{R})$. Let $\hat{M} = W^*(L^\infty(\mathbb{R}), \{\hat{X}_j^{(i)} : j \in J, i \in S(j)\})$. Since $K_j = \sum_{j \in S(j)} K_j^{(i)}$ is a disjoint sum and $K_j^{(i)}$ is absolutely continuous with respect to $\hat{K}_j^{(i)}$, we conclude that $M \subset \hat{M}$ and moreover $M$ is in the range of a conditional expectation from $\hat{M}$. Thus is sufficient to prove that $\hat{M}$ is solid.

By freeness with amalgamation, we know that $\hat{M}$ is the amalgamated free product of the algebras $\hat{M}_j^{(i)} = W^*(L^\infty(\mathbb{R}), \hat{X}_j^{(i)})$. Thus by [3] Theorem [4.4], if $B \subset \hat{M}$ is an abelian algebra generated by its finite projections and $B' \cap \hat{M}$ is non-amenable, then $B \prec_{\hat{M}} \hat{M}_j^{(i)}$ for some $j \in J$ and $i \in S(j)$ and moreover it follows that $\hat{M}_j^{(i)}$ is not solid. But

$$\hat{M}_j^{(i)} \cong L^\infty(\mathbb{R} \setminus \mathcal{I}_j^{(i)}) \oplus W^*(L^\infty(\mathcal{I}_j^{(i)}), \hat{X}_j^{(i)})$$

and the (finite) von Neumann algebra $W^*(L^\infty(\mathcal{I}_j^{(i)}), \hat{X}_j^{(i)})$ is solid by [4] Corollary [4.2], which is a contradiction. \hfill \Box

**Corollary 4.** Suppose that $\pi$ is a mixing orthogonal representation of $\mathbb{R}$ on a real Hilbert space $H_\mathbb{R}$, and let $\alpha$ be the free Bogoljubov action on $\Phi(H_\mathbb{R})$ associated to $\pi$. Then the semifinite von Neumann algebra $M = \Phi(H_\mathbb{R}) \rtimes_\alpha \pi$ is solid: if $B \subset M$ is an diffuse abelian subalgebra generated by its finite projections, then $B' \cap M$ is amenable.

**Proof.** Our goal is to apply Lemma [3]. Fix any decomposition of $\pi$ into cyclic representations $(\pi_j : j \in J)$ with associated cyclic vectors $\xi_j$ in such a way that the spectrum of $\pi_j(t)$ is contained in the set $\exp(iI_j t)$ for a finite subinterval $I_j \subset \mathbb{R}$. Let us fix integers $n_j$ so that $I_j \subset [-n_j, n_j]$. Selecting a possibly different set of cyclic vectors and subrepresentations $\pi_j(t)$, we may assume that $\pi(t) = \bigoplus_j \pi_j(t)$, and that the spectrum of the infinitesimal generator of $\pi_j$ is contained in $I_j$. Denote by $\mu_j$ the measures with Fourier transform

$$\hat{\mu}_j(t) = \langle \xi_j, \pi(t) \xi_j \rangle.$$

By assumption that $\pi$ is mixing, $\lim_{t \to \pm\infty} \hat{\mu}_j(t) = 0$. Moreover, by construction, the support of $\mu_j$ is contained in $[-n_j, n_j]$.

Let $\eta_j : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ be the completely positive map given by convolution with $\mu_j$. Then $\eta_j$ has an associated kernel measure $K_j$ given by $dK_j(x, y) = \mu_j(x - y)$.

Let $K_j^{(i)}$ denote the restriction of $K_j$ to the region $[-4n_j + 1, 4n_j + i] \times [-4n_j + 1, 4n_j + i]$, $i \in \mathbb{Z}$, and let $\hat{K}_j^{(i)}$ be the restriction of $K_j$ to $[-4n_j + i, 4n_j + i] \times [-4n_j + 1, 4n_j + 1]$, $i \in \mathbb{Z}$, and let $\hat{K}_j^{(i)}$ be the restriction of $K_j$ to $[-4n_j + 1, 4n_j + i] \times [-4n_j + i, 4n_j + i]$. If we identify $[-4n_j + i, 4n_j + i]$ with the circle, then the completely positive map associated to $\hat{K}_j^{(i)}$ is given by convolution with the measure $\hat{\mu}_j^{(i)}$ whose Fourier transform is given by $k \mapsto \hat{\mu}_j^{(i)}(k/8n_j)$; since $\lim_{k \to \pm\infty} \hat{\mu}_j^{(i)}(k) = 0$, it follows that $\lim_{k \to \pm\infty} \hat{\mu}_j^{(i)}(k) = 0$, so that the hypothesis of Lemma [3] is satisfied. \hfill \Box
3.4. Cocycle conjugacy. We are now ready to prove the main result of this paper:

**Theorem 5.** Let \( \pi_1, \pi_1' \) be two mixing orthogonal representations of \( \mathbb{R} \), and assume that \( \pi_1 \otimes \pi_1 \cong \pi_1, \pi_1' \otimes \pi_1' \cong \pi_1' \). Denote by \( 1 \) the trivial representation of \( \mathbb{R} \). Let

\[
\pi = (1 \oplus \pi_1)^{\otimes \infty}, \quad \pi' = (1 \oplus \pi_1')^{\otimes \infty},
\]

and let \( \alpha \) (resp., \( \alpha' \)) be the corresponding free Bogoljubov actions of \( \mathbb{R} \) on \( L(\mathbb{F}_\infty) \). Then \( \alpha \) and \( \alpha' \) are cocycle conjugate iff \( \mathcal{C}_{\pi_1} \cong \mathcal{C}_{\pi_1'} \).

**Proof.** If \( \mathcal{C}_{\pi_1} = \mathcal{C}_{\pi_1'} \), then \( \pi \) and \( \pi' \) are conjugate representations and the associated Bogoljubov actions are conjugate; thus it is the opposite direction that needs to be proved. The proof will be broken into several steps.

If \( H_* \) is the representation space of \( \pi \), then \( H_* \cong H_0 \oplus H_1 \) corresponding to the decomposition \( \pi = 1^\infty \oplus \pi_1^{\otimes \infty} \). Let \( N = \Phi(H_*) \cong L(\mathbb{F}_\infty) \). Then \( N \) decomposes as a free product \( N \cong \Phi(H_0) \ast \Phi(H_1) \); moreover, the free Bogoljubov action is also a free product \( \alpha = 1 \ast \alpha_1 \). Note that the subalgebra \( L(\mathbb{F}_\infty) \cong \Phi(H_0) \subset N \) is fixed by the action \( \alpha \). In particular, the crossed product \( M = N \rtimes_\alpha \mathbb{R} \) decomposes as a free product:

\[
M = N \rtimes_\alpha \mathbb{R} \cong (\Phi(H_0) \otimes L(\mathbb{R})) \ast_{L(\mathbb{R})} (\Phi(H_1) \rtimes_{\alpha_1} \mathbb{R}.
\]

Let us assume that \( \alpha \) and \( \alpha' \) are cocycle conjugate. Denote by \( A = L_\alpha(\mathbb{R}) \subset M \). Then \( N \rtimes_\alpha \mathbb{R} \cong N \rtimes_{\alpha'} \mathbb{R} \) and thus (up to this identification, which we fix once and for all) also \( L_{\alpha'}(\mathbb{R}) \subset M \). But the latter is impossible by Corollary [5, Theorem 4.4], since \( \alpha_1 \) comes from a mixing representation \( \pi_1 \). Thus it must be that \( L_{\alpha'}(\mathbb{R}) \cong \mathbb{R} \mathcal{M} \Phi(H_0) \otimes L(\mathbb{R}) \cong L(\mathbb{F}_\infty) \otimes L(\mathbb{R}) \) and thus \( L_{\alpha'}(\mathbb{R}) \prec_M L_{\alpha}(\mathbb{R}) \).

By Theorem [1] \( L_{\alpha'}(\mathbb{R}) \prec_M L_{\alpha}(\mathbb{R}) \) implies that there exists a nonzero partial isometry \( v \in N \) such that \( v^*v \in N_{\alpha'} \), \( vv^* \in N_\alpha \), and for all \( x \in N \), \( \alpha_1(xv^*) = v\alpha_1'(x)v^* \).

Let \( p = vv^* \in N_{\alpha'} \), and denote by \( \hat{\alpha} \) the restriction of \( \alpha \) to \( pNp \).

Let \( H_* = H_0 \oplus H_1 \) be as above. By replacing \( p \in \Phi(H_0) \) with a subprojection and modifying \( v \), we may assume that \( \tau(p) = 1/n \) for some \( n \). Then we can find partial isometries \( v_i \in \Phi(H_0), i \in \{1, \ldots, n\} \), such that \( v_i v_i^* = p \) for all \( i \) and \( \sum_i v_i^* v_i = 1 \).

Let \( \{s(h) : h \in H_1\} \) be a semicircular family of generators for \( \Phi(H_1) \). Then \( N \) is generated by \( \Phi(H_0) \cup \{s(h) : h \in H_1\} \), so that \( pNp \) is generated by \( p\Phi(H_0)p \) and \( \{v_i s(h) v_i^* : 1 \leq i, j \leq n, h \in H_1\} \) [6, Lemma 5.2.1].

For \( i, j \in \{1, \ldots, n\} \) and \( h \in H_1 \), denote \( S_{ij}(h) = \text{Re}(n^{1/2}v_i s(h) v_j^*), S'_{ij}(h) = \text{Im}(n^{1/2}v_i s(h) v_j^*) \). The normalization is chosen so that in the compressed \( W^* \)-probability space \( (pNp, n\tau|_{pNp}) \) these elements form a semicircular family [6, Prop. 5.1.7]. So, all together, \( pNp \) is generated \(*\)-freely by \( p\Phi(H_0)p \) and the semicircular family \( \{S_{ij}(h) : h \in H_1, 1 \leq i, j \leq n\} \cup \{S'_{ij}(h) : h \in H_1, 1 \leq i < j \leq n\} \). The action of the restriction \( \hat{\alpha}_t \) of \( \alpha_t \) to \( pNp \) is given, on these generators, as follows: \( \hat{\alpha}_t(x) = x \) for \( x \in p\Phi(H_0)p \); \( \hat{\alpha}_t(S_{ij}(h)) = S_{ij}(\tau_t(h)) \); \( \hat{\alpha}_t(S'_{ij}(h)) = S'_{ij}(\tau_t(h)) \).

From this we see that \( \hat{\alpha}_t \) is once again a Bogoljubov automorphism but corresponding to the representation \( 1^{\otimes \infty} \oplus (\pi_1^{\otimes \infty})^{\otimes n^2} \cong \pi \). Since by assumption \( \pi_1 \cong \pi_1 \otimes \pi_1 \), also \( \pi \cong \pi \otimes \pi \) and so conjugacy of \( \hat{\alpha}_t \) and \( \hat{\alpha}'_t \) implies equality of measure classes \( \mathcal{C}_\pi \) and \( \mathcal{C}_{\pi'} \) and thus of \( \mathcal{C}_{\pi_1} \) and \( \mathcal{C}_{\pi_1'} \). \( \square \)
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