Remarks on Schwinger Pair Production by Charged Black Holes

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(Dated: June 27, 2018)

Abstract

We introduce a canonical method for pair production by electromagnetic fields. The canonical
method in the space-dependent gauge provides pair-production rate even for inhomogeneous fields.
Further, the instanton action including all corrections leads to an accurate formula for the pair-
production rate. We discuss various aspects of the canonical method and clarify terminology for
pair production. We study pair production by charged black holes first by finding states of the
field equation that describe pair production and then by applying the canonical method.

PACS numbers: 12.20.-m, 04.70.-s, 04.62.+v

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I. INTRODUCTION

Strong quantum electrodynamics (QED) significantly differs from weak perturbative QED. Pairs of bosons and fermions are produced from external electric fields exceeding the critical strength $E_c = m^2 c^3/\hbar q$. Vacuum fluctuations polarize the vacuum and the instability therefrom leads to pairs of particles [1, 2]. This pair production of charged particles and antiparticles is a nonperturbative effect of QED. The recent development of X-ray free electron laser technology may make a test possible for pair production and attracts interests in this strong QED [3, 4]. Other natural phenomena associated with strong electromagnetic fields are astrophysical objects. For instance, pulsars have magnetic fields comparable with or above the critical strength, and the magnetic fields of magnetars exceed the critical strength by several order of magnitude. Likewise, charged black holes, especially the extremal ones, can have electric fields far greater than the critical strength. Recently, Ruffini et al. [5, 6, 7] proposed a mechanism for gamma rays burst through pair production and annihilation from charged black holes.

In this paper we introduce the recent canonical method for pair production [8, 9], which can be applied to inhomogeneous electromagnetic fields such as charged black holes. In fact, the electric fields by charged black holes are inhomogeneous in that the direction is radial and the field strength depends on the radial distance. The pair-production rate by the proper time method was derived for static uniform fields [1, 2]. However, this formula was used to calculate the pair-production rate by charged back holes [5, 6, 7, 10]. Thus a field theoretical method is needed to apply even to such inhomogeneous fields. Our canonical method is an extension of the early idea that particle production is related with tunneling processes in the space-dependent gauge [11, 12, 13, 14, 15]. In the space-dependent gauge of a static uniform electric field, each mode of the Klein-Gordon or Dirac or vector field equations has a potential barrier. It is the observation that the tunneling probability has relation with pair production and the no-tunneling probability is related with the vacuum persistence. Further, the tunneling probability is determined by the instanton action including all quantum corrections. Thus this canonical method can readily be applied to static inhomogeneous electric fields, in particular, charged black holes.

The organization of this paper is as follows. In Sec. II we review the vacuum polarization and pair production in proper time method. In Sec. III we use the time- and space-
dependent gauges to obtain the pair-production rate by a uniform electric field. We compare
the similarity and differences between two gauges. It is shown that particle production is
related with the tunneling probability and the vacuum persistence is determined by the no-
tunneling probability. In Sec. IV we discuss the spin effect on pair production and clarify
the terminology of pair-production probability, pair-production rate and the mean number
of pairs. In particular, the tunneling probability is given by the instanton action including
all quantum corrections. In Sec. V we study the Klein-Gordon equation first in Minkowski
spacetime and then in charged black hole background. It is shown that there are quasi-
stationary states describing purely outgoing waves with complex frequencies corresponding
to pair-production states. Finally, we apply the canonical method to calculate the pair-
production rates by charged black holes.

II. VACUUM POLARIZATION AND PAIR PRODUCTION

At the dawning of quantum field theory, Heisenberg and Euler calculated perturbatively
the effective action for uniform electromagnetic fields [1]. Later, Schwinger found the one-
loop effective action for fermions of spin 1/2 in uniform electromagnetic fields by using the
proper time method [2]:

$$\mathcal{L}_e = -\mathcal{F} - \frac{1}{8\pi^2} \int_0^\infty ds \frac{e^{-m^2 s}}{s^3} \left[ (es)^2 \mathcal{G} \frac{\text{Re} \cosh(esX) - 1 - \frac{2}{3}(es)^2}{\text{Im} \cosh(esX)} \right],$$

where

$$\mathcal{F} = \frac{1}{4} F_{\mu \nu} F^{\mu \nu} = \frac{1}{2} (B^2 - E^2), \quad \mathcal{G} = \frac{1}{4} F_{\mu \nu}^* F^{\mu \nu} = E \cdot B,$$

and

$$X = [2(\mathcal{F} + i\mathcal{G})]^{1/2} = X_r + iX_i.$$  

The one-loop effective action can be written in the form

$$\mathcal{L}_e = -\mathcal{F} - \frac{1}{8\pi^2} \int_0^\infty ds \frac{e^{-m^2 s}}{s^3} \left[ (es)^2 \mathcal{G} \coth(esX_r) \cot(esX_i) - 1 - \frac{2}{3}(es)^2 \mathcal{F} \right],$$

which has simple poles at $s_n = (n\pi)/(eX_i)$, $(n = 1, 2, \cdots)$. These poles contribute the
imaginary part

$$2\text{Im}\mathcal{L}_e = \frac{1}{4\pi} \sum_{n=1}^\infty \frac{e\mathcal{G}}{X_i s_n} e^{-m^2 s_n} \coth\left(\frac{X_r}{X_i n\pi}\right).$$
The real part of the one-loop effective action gives rise to the vacuum polarization

\[
\text{Re} L_e = -\mathcal{F} - \frac{1}{8\pi^2} \int_0^\infty ds \frac{e^{-m^2s}}{s^3} \left[ (es)^2 X_r X_i \left\{ \frac{1}{esX_i} - \sum_{k=1}^\infty \frac{2^{2k} |B_{2k}|}{(2k)!} \right\} \right.
\]

\[
\times (esX_i)^{2k-1} \coth(esX_r) \Bigg\} - 1 - \frac{2}{3} (es)^2 \mathcal{F} \Bigg]\]

\[
= -\mathcal{F} + \frac{2}{45} \left( \frac{e^2}{4\pi \hbar c} \right)^2 \frac{2(h/\hbar c)^3}{mc^2} \left[ (\mathbf{B}^2 - \mathbf{E}^2)^2 + 7(\mathbf{E} \cdot \mathbf{B})^2 \right] + \cdots
\]

(6)

For a pure electric field (\(\mathbf{B} = 0, X_r = 0, X_i = E\)), Eq. (5) leads to the pair-production rate

\[
2\text{Im} L_e = \frac{1}{4\pi^3} \sum_{n=1}^\infty \left( \frac{en}{n} \right)^2 e^{-\frac{n\pi m^2}{en}}.
\]

(7)

In the case of bosons, the pair-production rate becomes

\[
2\text{Im} L_b = \frac{1}{8\pi^3} \sum_{n=1}^\infty (-1)^{n+1} \left( \frac{en}{n} \right)^2 e^{-\frac{n\pi m^2}{en}}.
\]

(8)

These imaginary parts determine the vacuum persistence (vacuum-to-vacuum transition)

\[
|\langle 0, \text{out} | 0, \text{in} \rangle|^2 = e^{-2V T \text{Im} L_e},
\]

(9)

and thus lead to pair production of particle and antiparticle. It should be remarked that the pair-production rates (5), (7), (8) and the vacuum polarization (6) are valid only for uniform fields. These formulae may be used as long as fields vary slowly in region of interest.

There are many situations, such as charged black holes, where the fields are strong but not uniform. In the next sections, using a canonical method, we shall derive the pair-production rates valid even for inhomogeneous fields.

**III. CANONICAL METHOD FOR PAIR PRODUCTION**

Pair production of charged particles and antiparticles can also be understood in canonical theory. An advantage of the canonical method is that it has a direct meaning of pair production compared with the path integral methods. In the canonical method one directly solves either the Klein-Gordon or Dirac equation. An interesting point is that one can easily extend the result for bosons to fermions by taking into account the Pauli blocking effect due to spins. The Klein-Gordon equation for bosons is given by

\[
\left[ -\eta^{\mu\nu} \left( \frac{\partial}{\partial x^\mu} \right) \left( \frac{\partial}{\partial x^\nu} \right) + m^2 \right] \Phi(t, \mathbf{x}) = 0.
\]

(10)
In this section we focus on the uniform electric field, which is determined by either vector potential or scalar potential

$$E = -\frac{\partial}{\partial t} A - \nabla A_0.$$  \hfill (11)

A uniform electric field has a time-dependent vector potential

$$A = -Et \mathbf{k}$$  \hfill (12)

or space-dependent scalar potential

$$A_0 = -Ez.$$  \hfill (13)

**A. Time-dependent Gauge**

In the time-dependent gauge (12) with the electric field in the $z$-direction, the Klein-Gordon equation for spin-0 bosons now takes the form

$$\left[ \partial_t^2 - \partial_\perp^2 - (\partial_z + iqEt)^2 + m^2 \right] \Phi = 0.$$  \hfill (14)

The field is decomposed into Fourier mode, $\Phi = e^{i k \cdot x} \phi_k$ and results in a one-dimensional equation in the time-direction

$$\left[ -\partial_t^2 - (k_z + qEt)^2 - (m^2 + k_\perp^2) \right] \phi_k = 0.$$  \hfill (15)

The above equation describes a particle moving over an upside down oscillator potential with positive energy. This equation is an analog of fields in a time-dependent spacetime background. We may apply the canonical method by Parker and DeWitt to calculate the amount of particle creation [16, 17, 18]. The idea is that an incoming positive frequency solution at the past infinity scatters by the potential and separates into an outgoing positive and negative frequency solutions at the future infinity. The negative frequency solution leads to particle creation. Indeed, Eq. (15) has an incoming positive frequency solution at $t \to -\infty$

$$\phi_{k, in}(t) = \frac{1}{(8qE)^{1/4}} \left[ i \sqrt{\kappa} W(-a_{k_\perp}, \tau) + \frac{1}{\sqrt{\kappa}} W(-a_{k_\perp}, -\tau) \right],$$  \hfill (16)

where

$$a_{k_\perp} = \frac{k_\perp^2 + m^2}{2qE}, \quad \tau = \sqrt{2qEt}, \quad \kappa = \sqrt{1 + e^{-2\pi a_{k_\perp}} - e^{-\pi a_{k_\perp}}}.$$  \hfill (17)
The incoming wave can be written in another form
\[
\phi_{k,\text{in}}(t) = \frac{1}{(8qE)^{1/4}} \left[ \frac{i}{2} (\kappa - \frac{1}{\kappa}) E(-a_{k,\perp}, \tau) + \frac{i}{2} (\kappa + \frac{1}{\kappa}) E^*(-a_{k,\perp}, \tau) \right].
\] (18)

In fact, the incoming wave is a superposition of the outgoing positive and negative frequency solution
\[
\phi_{k,\text{in}}(t) = \mu_k \phi_{k,\text{out}}(t) + \nu_k \phi_{k,\text{out}}^*,
\] (19)
where \( \phi_{k,\text{out}} \) is the outgoing positive frequency solution given by
\[
\phi_{k,\text{out}}(t) = \frac{1}{(8qE)^{1/4}} E^*(-a_{k,\perp}, \tau),
\] (20)
and
\[
\mu_k = \frac{i}{2} (\kappa + \frac{1}{\kappa}), \quad \nu_k = \frac{i}{2} (\kappa - \frac{1}{\kappa}).
\] (21)

Equation (19) is equivalent to the Bogoliubov transformation between the operators in two asymptotic regions
\[
\hat{b}_{k,\text{out}} = \mu_k \hat{b}_{k,\text{in}} + \nu_k^* \hat{b}_{k,\text{out}}^\dagger,
\] (22)
The number of created particles in pairs in the future infinity is given by
\[
\langle 0, \text{in} | \sum_k \hat{b}_{k,\text{out}}^\dagger \hat{b}_{k,\text{out}} | 0, \text{in} \rangle = \sum_{k,\perp} |\nu_k|^2 = \sum_{k,\perp} e^{-2\pi a_{k,\perp}}.
\] (23)
Here the mean number of created pairs in the transverse mode \( k,\perp \) is \( \mathcal{N}_{k,\perp} = |\nu_k|^2 \leq 1 \). Then the vacuum persistence or vacuum-to-vacuum transition is given by
\[
|\langle 0, \text{out} | 0, \text{in} \rangle|^2 = \prod_{k,\perp} \frac{1}{|\mu_k|^2},
\] (24)
where
\[
\mu_k = \left(1 + e^{-2\pi a_{k,\perp}}\right)^{1/2}.
\] (25)
Thus the vacuum persistence is related with the imaginary part of the effective action as
\[
|\langle 0, \text{out} | 0, \text{in} \rangle|^2 = \exp\left[ - \sum_{k,\perp} \ln(1 + e^{-2\pi a_{k,\perp}}) \right] = \exp\left[ -2VT \text{Im} \mathcal{L}_e^b \right]
\] (26)
Now the imaginary part of the effective action for bosons per volume per time is given by
\[
2\text{Im} \mathcal{L}_e^b = \frac{1}{VT} \sum_{k,\perp} \ln(1 + e^{-2\pi a_{k,\perp}}) = \frac{1}{(2\pi)^d} \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{eE}{n}\right)^{(d+1)/2} \frac{(a_{k,\perp})^n}{eE}.
\] (27)
We may interpret Eq. (19) quantum mechanically. In the space with $x = -t$, $\phi_{k,\text{out}}$ corresponds to an incident wave and $\phi^*_{k,\text{out}}$ to a reflected one. Thus the transmission coefficient gives the vacuum persistence

$$\left|\frac{1}{\mu_k}\right|^2 = 1 - \frac{\nu_k}{\mu_k} = \frac{1}{1 + e^{-\pi a_k}} = |\langle 0_{k\perp}, \text{out}|0_{k\perp}, \text{in} \rangle|^2$$

(28)

and the mean number of created pairs is given by the reflection coefficient

$$|\nu_k|^2 = e^{-\pi a_k} = N_{k\perp}.$$  

(29)

B. Space-dependent Gauge

In the space-dependent gauge, i.e. the Coulomb gauge (13), the Klein-Gordon equation, after the mode-decomposition $\Phi = e^{i(k_{\perp} \cdot x_{\perp} - \omega t)} \phi_{\omega k_{\perp}}$, separates into the mode equation

$$\left[-\partial_z^2 - (\omega + qE z)^2 + (m^2 + k_{\perp}^2)\right] \phi_{\omega k_{\perp}} = 0.$$  

(30)

This equation describes a particle moving under the potential barrier in contrast with the over-barrier in the time-dependent gauge. The tunneling wave function is given in terms of the complex cylindrical function

$$\phi_{\omega k_{\perp}}(\xi) = cE(a_{k_{\perp}}, \xi),$$

(31)

where

$$a_{k_{\perp}} = \frac{k_{\perp}^2 + m^2}{2qE}, \quad \xi = \sqrt{\frac{2}{qE}}(\omega + qE z).$$

(32)

This wave function has the proper asymptotic forms at $z \to -\infty$

$$\phi_{\omega k_{\perp}}(\xi \to -\infty) = A_{k_{\perp}} \sqrt[4]{|\xi|} e^{-\frac{\xi^2}{4}} + B_{k_{\perp}} \sqrt[4]{|\xi|} e^{\frac{\xi^2}{4}},$$

(33)

and at $z \to +\infty$

$$\phi_{\omega k_{\perp}}(\xi \to +\infty) = C_{k_{\perp}} \sqrt[4]{|\xi|} e^{\frac{\xi^2}{4}}.$$  

(34)

Here the coefficients are

$$A_{k_{\perp}} = ic\sqrt{1 + e^{2\pi a_{k_{\perp}}}}, \quad B_{k_{\perp}} = ice^{\pi a_{k_{\perp}}}, \quad C_{k_{\perp}} = c.$$  

(35)

Then the tunneling probability is

$$P_{k_{\perp}} = \left|\frac{C_{k_{\perp}}}{A_{k_{\perp}}}\right|^2 = \frac{1}{1 + e^{2\pi a_{k_{\perp}}}},$$

(36)
and the probability for no-tunneling

\[ P_{k\perp}^{nb} = \frac{|B_{k\perp}|^2}{A_{k\perp}^2} = \frac{1}{1 + e^{-2\pi a_{k\perp}}}. \tag{37} \]

Note that the instanton action for tunneling

\[ S_{k\perp} = \frac{1}{2} \oint dz \sqrt{(m^2 + k^2_{\perp}) - (\omega + qEz)^2} = \pi a_{k\perp}, \tag{38} \]

after taking into account multi-instantons and multi-anti-instantons contributions, leads to the total tunneling probability

\[ P_{k\perp}^b = \sum_{n=1}^{\infty} (-1)^{n+1} e^{-2nS_{k\perp}} = \frac{1}{1 + e^{2S_{k\perp}}}, \tag{39} \]

and the total probability for no-tunneling

\[ P_{k\perp}^{nb} = \sum_{n=0}^{\infty} (-1)^n e^{-2nS_{k\perp}} = \frac{1}{1 + e^{-2S_{k\perp}}}. \tag{40} \]

The idea of Refs. 8, 9, 11, 12, 13, 14, 15 is that the tunneling probability is related with pair production and the no-tunneling probability with the vacuum-to-vacuum transition. As in the case of the time-dependent gauge, the vacuum persistence is given by

\[ |\langle 0, \text{out} | 0, \text{in} \rangle|^2 = \prod_{k\perp} P_{k\perp}^{nb} = \exp[-2VT\text{Im} \mathcal{L}_e^b]. \tag{41} \]

Finally, the imaginary part of the effective action for bosons per volume per time is given by

\[ 2\text{Im} \mathcal{L}_e^b = \frac{1}{VT} \sum_{k\perp} \ln(1 + e^{-2S_{k\perp}}) = \frac{1}{(2\pi)^d} \sum_{n=1}^{\infty} (-1)^{n+1} \left( \frac{eE}{n} \right)^{(d+1)/2} e^{-n\pi m^2/eE}. \tag{42} \]

In the fermion case, fermion pairs in the same state are prohibited by the Pauli exclusion principle, so the tunneling probability is limited to

\[ P_{k\perp}^f = e^{-2\pi a_{k\perp}}, \tag{43} \]

and the no-tunneling probability to

\[ P_{k\perp}^{nf} = 1 - e^{-2\pi a_{k\perp}}. \tag{44} \]

Thus the imaginary part of the effective action for fermions per volume per time is given by

\[ 2\text{Im} \mathcal{L}_e^f = -\frac{2}{VT} \sum_{k\perp} \ln(1 - e^{-2S_{k\perp}}) = \frac{2}{(2\pi)^d} \sum_{n=1}^{\infty} \left( \frac{eE}{n} \right)^{(d+1)/2} e^{-n\pi m^2/eE}. \tag{45} \]
IV. SOME REMARKS ON CANONICAL METHOD

In this section we discuss some issues of pair production in the canonical method. In Sec. III we derived the pair-production rate of spin-0 bosons and spin-1/2 fermions. Here we introduce a recent result extending the canonical method to spin particles [9]. We also clarify the different terminology for particle production in literature. Finally, we show that the canonical method based on the instanton action can readily be generalized to inhomogeneous fields.

A. Spin and Scattering Processes

To treat the bosons with spin $\sigma \geq 1$ and fermions with $\sigma \geq 3/2$, one has to solve the Klein-Gordon or Dirac equation coupled with the vector field. Then the Klein-Gordon equation for spin $\sigma$-particles takes the form

$$\left[-\eta^{\mu\nu}\left(\frac{\partial}{\partial x^\mu} + iqA^\mu\right)\left(\frac{\partial}{\partial x^\nu} + iqA^\nu\right) + m^2 + 2i\sigma qE\right]\Phi(t, \mathbf{x}) = 0. \quad (46)$$

After mode-decomposition, it reduces to

$$\left[-\partial_z^2 - (\omega + qEz)^2 + (m^2 + k_{\perp}^2) + 2i\sigma qE\right]\phi_{\omega k_{\perp}} = 0. \quad (47)$$

The tunneling wave function is now given by

$$\phi_{\omega k_{\perp}}(\xi) = cE(a_{k_{\perp}}, \xi), \quad (48)$$

where

$$a_{k_{\perp}} = \frac{k_{\perp}^2 + m^2 + 2i\sigma qE}{2qE}, \quad \xi = \sqrt{\frac{2}{qE}}(\omega + qEz). \quad (49)$$

Denoting the asymptotic form of the wave function

$$\varphi_{\omega k_{\perp}}(\xi) = \sqrt{\frac{2}{|\xi|}}e^{-\frac{i}{4}\xi^2}, \quad (50)$$

we find the wave function at $\xi = -\infty$

$$\phi_{\omega k_{\perp}} = A_{k_{\perp}}\varphi_{\omega k_{\perp}} - B_{k_{\perp}}\varphi^*_{\omega k_{\perp}}, \quad (51)$$

and at $\xi = +\infty$

$$\phi_{\omega k_{\perp}} = C_{k_{\perp}}\varphi^*_{\omega k_{\perp}}. \quad (52)$$
In terms of group velocity, the flux conservation reads for bosons

\[ |A_{k\perp}|^2 = |B_{k\perp}|^2 + |C_{k\perp}|^2, \]  

(53)

and for fermions

\[ |A_{k\perp}|^2 + |C_{k\perp}|^2 = |B_{k\perp}|^2. \]  

(54)

Therefore, the reflection coefficient for bosons is

\[ \left| \frac{B_{k\perp}}{A_{k\perp}} \right|^2 = 1 - \left| \frac{C_{k\perp}}{A_{k\perp}} \right|^2, \]  

(55)

and for fermions

\[ \left| \frac{A_{k\perp}}{B_{k\perp}} \right|^2 = 1 - \left| \frac{C_{k\perp}}{B_{k\perp}} \right|^2. \]  

(56)

This result makes the difference of pair production between bosons and fermions.

**B. Pair Production Rate and Mean Number**

Different terminology for pair production has been used in literature. In some cases one has to take care of the different quantities. Let us denote

\[ w = 2\text{Im}\mathcal{L}_e \]  

(57)

per volume per time, then in the case of both electric and magnetic fields this quantity becomes

\[ w_b = \frac{(qE)(qB)}{2(2\pi)^2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} \text{cosech} \left( \frac{n\pi B}{E} \right) e^{-\frac{am^2}{qE}}, \]  

(58)

\[ w_f = \frac{(qE)(qB)}{(2\pi)^2} \sum_{n=1}^{\infty} \frac{1}{n} \text{coth} \left( \frac{n\pi B}{E} \right) e^{-\frac{am^2}{qE}}. \]  

(59)

Schwinger [2] used the term pair-production probability for \( w_b \) or \( w_f \), and Itzykson and Zuber [19] the pair-production rate, whereas Nikishov [11] used only the first term of the series to denote the pair-production rate.

The mean number of created pairs can be derived in the following way. Let \( P_0 \) denote the probability for the vacuum-to-vacuum transition and \( P_1 \) denote the probability for one-pair production, then for the boson case one has the probability conservation

\[ 1 = P_0(1 + P_1 + P_1^2 + P_1^3 + \cdots), \]  

(60)
and the mean number of created pairs

\[ \mathcal{N}^b = P_0(P_1 + 2P_1^2 + 3P_1^3 + \cdots). \quad (61) \]

Therefore, one finds the relations

\[ P_0 = \frac{1}{1 + \mathcal{N}^b} = P^{nb}, \quad (62) \]
\[ P_1 = \frac{\mathcal{N}^b}{1 + \mathcal{N}^b} = P^b. \quad (63) \]

In the case of fermions, one has the probability conservation

\[ 1 = P_0(1 + P_1) \quad (64) \]
due to the Pauli blocking, and the mean number of created pairs

\[ \mathcal{N}^f = P_0P_1. \quad (65) \]

Thus one obtains the relations

\[ P_0 = 1 - \mathcal{N}^f = P^{nf}, \quad (66) \]
\[ P_1 = \frac{\mathcal{N}^f}{1 - \mathcal{N}^f}. \quad (67) \]

C. Instanton Method

The space-dependent gauge has an advantage of readily being applicable to static inhomogeneous fields. An inhomogeneous electric field, for instance along the z-direction, has the time-dependent gauge

\[ A_z = \int E(z)dt = E(z)t. \quad (68) \]

So it is difficult to apply the canonical method in the time-dependent gauge. Now, in the space-dependent Coulomb gauge, the mode-decomposed field equation takes the form

\[ \left[ -\partial^2_z + Q(z) \right] \phi(z) = 0. \quad (69) \]

Hence the problem reduces to a one-dimensional tunneling problem to which we may apply the idea of the canonical method. In general, there are two asymptotic regions with \( Q(z) < 0 \)
and a potential barrier $Q(z) > 0$ in between asymptotic regions. Then the no-tunneling probability for the vacuum persistence for bosons becomes

$$ P^b = \frac{1}{1 + e^{-2S}}, \quad (70) $$

and for fermions

$$ P^f = 1 - e^{-2S}, \quad (71) $$

where $S$ is the total sum of all order instanton contributions

$$ S = \sum_{k=0}^{\infty} S_{(2k)}. \quad (72) $$

The 0-loop or classical instanton action is

$$ 2S_{(0)} = \oint Q^{1/2}(z), \quad (73) $$

and $2S_{(2k)}$ is the $k$-loop correction to the instanton action.

V. PAIR PRODUCTION FROM CHARGED BLACK HOLES

We now turn to the pair production by charged black holes. As a charged black hole we consider the Reissner-Nordström black hole with the metric

$$ ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2d\Omega_2^2. \quad (74) $$

where

$$ g(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \quad (75) $$

Here $M$ and $Q$ are the mass and charge of the black hole. The charge $Q$ also generates an electric field in the radial direction given by the Coulomb potential

$$ A_0 = \frac{Q}{r}, \quad A_i = 0. \quad (76) $$

Then the Klein-Gordon equation in the black hole background takes the form

$$ \left[ -\frac{1}{g(r)} \left\{ \frac{\partial}{\partial t} - \frac{iQ}{r} \right\}^2 + \frac{1}{r^2} \frac{\partial}{\partial r} \left\{ r^2 g(r) \frac{\partial}{\partial r} \right\} - \frac{L^2}{r^2} - m^2 \right] \Phi = 0. \quad (77) $$

It is convenient to introduce the tortoise coordinate

$$ r^* = \int \frac{dr}{g(r)} = r + \frac{r_+^2}{(r_+ - r_-)} \ln(r - r_+) - \frac{r_-^2}{(r_+ - r_-)} \ln(r - r_-). \quad (78) $$
where \( r_{\pm} = M \pm \sqrt{M^2 - Q^2} \) are the outer and inner horizons. The field can be expanded into the spherical harmonics

\[
\Phi(t, x) = e^{i\omega t} \frac{\Psi_l(r)}{r} Y_{lm}(\theta, \varphi),
\]

where it is assumed \( \text{Re}(\omega) > 0 \) in all cases and \( \text{Im}(\omega) = 0 \) for the bounded states of stable motions and \( \text{Im}(\omega) > 0 \) for the unbounded states of unstable motions. Then the radial equations for the mode-decomposed field (79) take the form

\[
\left[ -\frac{d^2}{dr^2} + V_l(r) \right] \Psi_l(r) = \epsilon \Psi_l(r),
\]

where

\[
V_l(r) = g(r) \left( m^2 + \frac{l(l + 1)}{r^2} + \frac{2M}{r^3} - \frac{2Q^2}{r^4} \right) - \left( \omega - \frac{qQ}{r} \right)^2 + (\omega^2 - m^2),
\]

and

\[
\epsilon = \omega^2 - m^2.
\]

Note that Eq. (80) has the same form as the Regge-Wheeler equation for the axial perturbations of black holes [20, 21].

### A. Minkowski Spacetime

In the Minkowski spacetime with \( M = 0 \) and \( Q = 0 \) in \( g(r) \), the effective potential for the radial motion reduces to

\[
V_l(r) = \frac{2\omega qQ}{r} + \frac{l(l + 1) - q^2 Q^2}{r^2}.
\]

Note that the effective potential (83) of the Klein-Gorodn equation is the same as that of the non-relativistic system with a charge \( 2\omega Q \) and effective angular momentum \( l'(l' + 1) = l(l + 1) - q^2 Q^2 \), i.e.,

\[
l' = -\frac{1}{2} + \sqrt{\left( l + \frac{1}{2} \right)^2 - q^2 Q^2}.
\]

The characteristics of states depend on the relative magnitude of charge \( qQ \) and angular momentum \( l + 1/2 \). The states for \( |qQ| < l + 1/2 \) describe stable motions of bound states and the hydrogen-like atoms with \( Z < (l + 1/2)/\alpha \) belong to this class, where \( Z \) is the nucleus.
charge and $\alpha = e^2$ the fine structure constant [22, 23]. On the other hand, for $qQ > l + 1/2$, as in the case of heavy atoms with $Z > (l + 1/2)/\alpha$, the so-called 'centrifugal' potential (83) becomes attractive and the effective angular momentum (84) takes a complex value. Moreover, the electric field is sufficiently strong enough to produce pairs of charged particles. There are certain quasi-stationary or meta-stable states describing the pair production.

We consider the interesting case of pair production by the Coulomb field when $|qQ| > l + \frac{1}{2}$. The centrifugal potential becomes attractive rather than repulsive, and there may exist quasi-stationary states with complex frequencies $\omega$

$$\text{Re}(\omega) > 0, \quad \text{Im}(\omega) > 0.$$  

Then the wave function (79) decays in time and describes an outgoing wave for created particles by the Coulomb field. One may understand the quasi-stationary states by a simple analog of the quantum system having a potential well and finite potential barrier. In our case the real part of the effective potential consists of the attractive inverse square potential and a repulsive Coulomb barrier for $qQ > 0$. Therefore, the quasi-stationary state has a certain probability to tunnel the barrier and decays in time. Such radial wave function is found

$$\Psi_l(r) = N_l r^{l'+1} e^{i\sqrt{\epsilon}r} M(l' + 1 + i\lambda', 2l' + 2, -2i\sqrt{\epsilon}r),$$  

where $M$ is the confluent hypergeometric function and

$$\lambda' = \frac{\omega qQ}{\sqrt{\epsilon}}, \quad l' = -\frac{1}{2} + i\sqrt{q^2 Q^2 - (l + \frac{1}{2})^2}.$$  

From the asymptotic form

$$\Psi_l(r) = N_l r^{l'+1} e^{i\sqrt{\epsilon}r} \left[ e^{-ir(l'+1+i\lambda')} \frac{1}{\Gamma(l' + 1 - i\lambda') (-2i\sqrt{\epsilon}r)^{-l'-1-i\lambda'} + \frac{1}{\Gamma(l' + 1 + i\lambda') (-2i\sqrt{\epsilon}r)^{-l'-1+i\lambda'} e^{-2i\sqrt{\epsilon}r}} \right]$$  

we see that, assuming $\text{Re}(\sqrt{\epsilon}) > 0$, the first term corresponds to an incoming wave and the second term to an outgoing wave. We now require only outgoing waves and this condition is prescribed by

$$l' + 1 - i\lambda' = -n_r,$$

which makes first term vanish due to $\Gamma(-n_r) = \pm \infty$. Then we obtain the complex energy eigenvalues

$$\omega = m \left[ 1 + \left( \frac{qQ}{n - (l + \frac{1}{2}) + i\sqrt{q^2 Q^2 - (l + \frac{1}{2})^2}} \right)^2 \right]^{-1/2}.$$  

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We make a few comments. The consistency of quasi-stationary states requires that \( \text{Re}(\omega) > 0, \text{Im}(\omega) > 0 \) and \( \text{Re}(\sqrt{\epsilon}) > 0 \), which can be shown by a straightforward algebra. The result does not depend on the relative sign of charges, and Eq. (90) holds true also for the opposite charges, \( qQ < 0 \). The complex energy eigenvalues (90) are the analytical continuation of the real energy eigenvalues of bound states when \( qQ > l + \frac{1}{2} \). Moreover, the complex energy eigenvalues are quantized and indeed describe the quasi-stationary or meta-stable states that decay into infinity. More importantly, the pair production rate given by the imaginary part of frequency is proportional to \( m\sqrt{qQ} \). This implies that the stronger the electric field is, the more the pairs of charged particles are produced.

B. Charged Black Holes

Now the radial equation for each mode is described by Eq. (80). At sufficiently away from the black hole \( (r/M \gg 1) \), \( r^* \approx r \) and Eq. (81) becomes approximately

\[
V_l(r) \approx \frac{2(\omega qQ - M)}{r} + \frac{l(l + 1) - q^2Q^2}{r^2}.
\]

Hence, Eq. (80) is approximately the same as that of the Minkowski spacetime with the replacement of \( \omega \) by \( \omega qQ - M \). Therefore, we are able to obtain the wave function valid in the asymptotic regions

\[
\Psi_l(r) = Nr^{l'+1}e^{i\sqrt{\epsilon}r}M(l' + 1 + i\lambda_b, 2l' + 2, -2i\sqrt{\epsilon}r),
\]

where

\[
\lambda_b = \frac{\omega qQ - M}{\sqrt{\epsilon}}, \quad l' = -\frac{1}{2} + i\sqrt{q^2Q^2 - (l + \frac{1}{2})^2}.
\]

The outgoing wave is found by imposing the condition

\[
l' + 1 - i\lambda' = -n_r,
\]

which removes the incoming wave. Finally, after some algebra, we obtain the complex frequencies

\[
\omega = m \left[ 1 + \left( \frac{qQ}{F} \right)^2 \right]^{-1/2} + \frac{MqQ}{F^2 \left[ 1 + \left( \frac{qQ}{F} \right)^2 \right]},
\]

where

\[
F = n - \left(\frac{1}{2}\right) + i\sqrt{q^2Q^2 - \left( l + \frac{1}{2} \right)^2}.
\]
In the limit of \( M = 0 \), we recover the result (90) of the Minkowski spacetime.

We make a few comments. There is a formal similarity of Eq. (80) and the Regge-Wheeler equation for the axial perturbations of the black hole [20, 21]. The boundary condition for the purely outgoing wave leads to a discrete spectrum of complex eigenfrequencies for both the quasi-stationary states and quasi-normal modes [24, 25, 26, 27]. However, the quasi-stationary states exist even in the Minkowski spacetime, whereas the quasi-normal modes vanish in the Minkowski spacetime. This is ascribed to the fact that the pair production is a universal phenomenon occurring in all spacetimes whereas the quasi-normal modes are a consequence of the spacetime curvature. More importantly, the decay rate, given by the imaginary part of the frequency, has an additional contribution in proportion to the mass of the hole. This implies that a charged black hole is a more efficient mechanism for Schwinger pair production.

C. Pair Production

Finally, we apply the canonical method in Secs. III and IV to calculate the pair-production rate from charged back holes. At the spatial infinity \( r \to +\infty \), \((r^* \to +\infty)\), the effective potential vanishes, \( V_l(r^* = \infty) = 0 \), and at the outer horizon \( r \to r_+ \), \((r^* \to -\infty)\), it has

\[
V_l(r_+) = \omega^2 - m^2 - \left(\omega - \frac{qQ}{r_+}\right)^2.
\]  

(97)

Thus there is a potential barrier between \( r^* = -\infty \) and \( r^* = +\infty \). We find the instanton action

\[
S_l(\omega; M, Q, m, q) = \int_{r_0^*}^{r_1^*} dr^* \sqrt{V_l(r^*)} - \epsilon = \int_{r_0^*}^{r_1^*} dr \frac{\sqrt{V_l(r^*)} - \epsilon}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}},
\]

(98)

where \( r_0^* \) and \( r_1^* \) are the turning points of \( V_l(r^*) \). Then the mean number of produced boson pairs is

\[
\mathcal{N}_l = e^{-2S_l(\omega)}.
\]

(99)

The mean number (99) is the leading term for the pair-production rate and was derived also in Ref. [28]. The pair-production rate for bosons per volume per time is given by

\[
2\text{Im} \mathcal{L}^b_e = (2\sigma + 1) \sum_{l, \omega} (2l + 1) \ln \left( 1 + e^{-2S_l(\omega)} \right)
\]
\[
= \frac{(2\sigma + 1)}{2\pi} \sum_{l=0, n=1}^{\infty} (-1)^{n+1} \frac{(2l + 1)}{n} \int d\omega e^{-2nS_l(\omega)},
\]

and for fermions by

\[
2\text{Im}L^f_e = -(2\sigma + 1) \sum_{l, \omega} (2l + 1) \ln \left(1 - e^{-2S_l(\omega)}\right)
\]

\[
= \frac{(2\sigma + 1)}{2\pi} \sum_{l=0, n=1}^{\infty} \frac{(2l + 1)}{n} \int d\omega e^{-2nS_l(\omega)}.
\]

VI. CONCLUSION

We studied the pair production by inhomogeneous electromagnetic fields, in particular, by charged black holes. We applied the recently developed canonical method to find the pair-production rate. For a static uniform electric field, the field equation can be solved either in the time-dependent gauge or in the space-dependent gauge. The concept of particle creation by an external field applies directly to the field equation in the time-dependent gauge. This time-dependent gauge, however, may not be practically applicable for inhomogeneous fields due to the mixed nature of potential. In the case of the space-dependent gauge, the mode-decomposed field equations have tunneling barriers, and the tunneling interpretation may be used for pair production. In this interpretation the tunneling probability is related with pair production and the no-tunneling probability with the vacuum persistence, i.e., the vacuum-to-vacuum transition. This canonical method can readily be applied to even inhomogeneous fields. Further, the instanton action including all order corrections leads to an accurate formula for the pair-production rate.

Finally, we studied pair production by charged black holes. The Klein-Gordon equation in a charged black hole background has the form of the Regge-Wheeler equation. For a strong electric field, the mode-decomposed field equations have the quasi-stationary states that describe the purely outgoing waves for produced pairs. Their complex frequencies determine the decaying rate for the quasi-stationary states. We also noted a remarkable similarity between the quasi-stationary states of created charged particles and the quasi-normal modes of the axial or polar perturbations of a black hole. The same boundary condition for purely outgoing waves is used in both cases and, as a consequence, both wave functions have a discrete spectrum of complex frequencies. We then applied the canonical method to find the pair-production rate in terms of the instanton action by charged black
holes.

Acknowledgments

We would like to thank R. Ruffini for many stimulating discussions and also appreciate the warm hospitality of ICRA, Pescara, Italy. The work of S.P. Kim was supported by the Korea Science and Engineering Foundation under Grant No. 1999-2-112-003-5 and the work of D.N. Page was supported by Natural Sciences and Engineering Council of Canada.

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