Undecidability in binary tag systems and the Post correspondence problem for four pairs of words

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Abstract

Since Cocke and Minsky proved 2-tag systems universal, they have been extensively used to prove the universality of numerous computational models. Unfortunately, all known algorithms give universal 2-tag systems that have a large number of symbols. In this work, tag systems with only 2 symbols (the minimum possible) are proved universal via an intricate construction showing that they simulate cyclic tag systems. Our simulation algorithm has a polynomial time overhead, and thus shows that binary tag systems simulate Turing machines in polynomial time.

We immediately find applications of our result. We reduce the halting problem for binary tag systems to the Post correspondence problem for 4 pairs of words. This improves on 7 pairs, the previous bound for undecidability in this problem. Following our result, only the case for 3 pairs of words remains open, as the problem is known to be decidable for 2 pairs. As a further application, we find that the matrix mortality problem is undecidable for sets with five $3 \times 3$ matrices and for sets with two $15 \times 15$ matrices. The previous bounds for the undecidability in this problem was seven $3 \times 3$ matrices and two $21 \times 21$ matrices.

1 Introduction

Introduced by Post [30], tag systems have been used to prove Turing universality in numerous computational models, including some of the simplest known universal systems [1, 9, 19, 21, 26, 20, 33, 34, 35, 36, 37]. Many universality results rely either on direct simulation of tag systems or on a chain of simulations the leads back to tag systems. Such relationships between models means that improvements in one model often has applications to many others. The results in [39] are a case in point, where an exponential improvement in the time efficiency of tag systems had the domino effect of showing that many of the simplest known models of computation [1, 9, 19, 20, 21, 22, 26, 33, 34, 35, 36, 37] are in fact polynomial time simulators of Turing machines. Despite being central to the search for simple universal systems for 50 years, tag systems have not been the subject of simplification since the early sixties.

In 1961, Minsky [25] solved Post’s longstanding open problem by showing that tag systems, with deletion number 6, are universal. Soon after, Cocke and Minsky [8] proved that tag systems with deletion number 2 (2-tag systems) are universal. Later, Hao Wang [38] showed that 2-tag systems with even shorter instructions were universal. The systems of both Wang, and Cocke and Minsky use large alphabets and so have a large number of rules. Here we show that tag systems with only 2 symbols, and thus only 2 rules, are universal. Surprisingly, one of our two rules is trivial. We find immediate applications of our result. Using Cook’s [9] reduction
of tag systems to cyclic tag systems, it is a straightforward matter to give a binary cyclic tag system program that is universal and contains only two 1 symbols. We also use our binary tag system construction to improve the bound for the number of pairs of words for which the Post correspondence problem \[31\] is undecidable, and the bounds for the simplest sets of matrices for which the mortality problem \[29\] is undecidable.

The search for the minimum number of word pairs for which the Post correspondence problem is undecidable began in the 1980s \[7, 28\]. The best result until now was found by Matiyasevich and Sénizergues, whose impressive 3-rule semi-Thue system \[23, 24\], along with a reduction due to Claus \[7\], showed that the problem is undecidable for 7 pairs of words. Improving on this undecidability bound of 7 pairs of words seemed like a challenging problem. In fact, Blondel and Tsitsiklis \[4\] stated in their survey “The decidability of the intermediate cases \(3 \leq n \leq 6\) is unknown but is likely to be difficult to settle”. We give the first improvement on the bound of Matiyasevich and Sénizergues in 17 years: We reduce the halting problem for our binary tag system to the Post correspondence problem for 4 pairs of words. This leaves open only the case for 3 pairs of words, as the problem is known to be decidable for 2 pairs \[14, 16\].

A number of authors \[3, 6, 15, 17, 29\], have used undecidability bounds for the Post correspondence problem to find simple matrix sets for which the mortality problem is undecidable. The matrix mortality problem is, given a set of \(d \times d\) integer matrices, decide if the zero matrix can be expressed as a product of matrices from the set. Halava et al. \[17\] proved the mortality problem undecidable for sets with seven \(3 \times 3\) matrices, and using a reduction due Cassaigne and Karhumäki \[6\] they also showed the problem undecidable for sets with two \(21 \times 21\) matrices. Using our new bound, and applying the reductions used in \[6, 15\], we find that the matrix mortality problem is undecidable for sets with five \(3 \times 3\) matrices and for sets with two \(15 \times 15\) matrices. In addition, by applying reductions due to Halava and Hirvensalo \[18\], we improve on previous undecidability bounds for a number of decision problems in sets that consist of two matrices. These new bounds include a set with two \(7 \times 7\) matrices for which the scalar reachability problem is undecidable.

We complete our introduction by recalling some decidability results and open problems for tag systems. Stephen Cook \[10\] proved that the reachability problem, and hence the halting problem, is decidable for non-deterministic 1-tag systems. More recently, De Mol \[12\] has shown that the reachability (and thus halting) problem is decidable for binary 2-tag systems, a problem which Post \[32\] claimed to have solved but never published. In the 1920s Post \[32\] gave a simple binary 3-tag system \((0 \rightarrow 00, 1 \rightarrow 1101)\) whose halting problem is still open \[13\]. De Mol \[11\] reduced the well know Collatz problem to the halting problem for a remarkably simple 2-tag system that has 3 rules. The simple tag systems of Post and De Mol suggest that improving on existing decidability results would be quite difficult.

2 Preliminaries

We write \(c_1 \vdash c_2\) if a configuration \(c_2\) is obtained from \(c_1\) via a single computation step. We let \(c_1 \vdash^t c_2\) denote a sequence of \(t\) computation steps. The length of a word \(w\) is denoted \(|w|\), and \(\varepsilon\) denotes the empty word. We let \(|v|\) denote the encoding of \(v\), where \(v\) is a symbol or a word. We use the standard binary modulo operation \(a = m \mod n\), where \(a = m - ny\), \(0 \leq a < n\), and \(a, m, n, y\) are integers.
We use the term round first 5 steps of Post’s [32] binary tag system with deletion number 3 and the rules 0
s = 3 and so the word w is entered with shift 3 = (0 + 3) mod 4, and the word r has a shift change of 3 so the word v is entered with shift 2 = (3 + 3) mod 4.

Figure 1: Four computation steps of a tag system with deletion number β = 4 on the word w = qrv. Here q = q0q1q2q3q4, r = r0r1r2r3r4r5r6r7r8, v = v0v1v2v3v4v5, and q, r, and v are tag system symbols, and for simplicity we assume that all symbols append the empty word (i.e. all rules have the form wi → ε).

2.1 Tag systems

Definition 1. A tag system consists of a finite alphabet of symbols Σ, a finite set of rules R : Σ → Σ* and a deletion number β ∈ N, β ≥ 1.

The tag systems we consider are deterministic. The computation of a tag system acts on a word w = w0w1 . . . w|w|−1 (here wi ∈ Σ) which we call the dataword. The entire configuration is given by w. In a computation step, the symbols w0w1 . . . wβ−1 are deleted and we apply the rule for w0, i.e. a rule of the form w0 → w0,1w0,2 . . . w0,e, by appending the word w0,1w0,2 . . . w0,e (here w0,j ∈ Σ). A dataword (configuration) w′ is obtained from w via a single computation step as follows:

\[ w0w1 . . . wβ . . . w|w|−1 wβ . . . w|w|−1w0,1w0,2 . . . w0,e \]

where w0 → w0,1w0,2 . . . w0,e ∈ R. A tag system halts if |w| < β. As an example we give the first 5 steps of Post’s [32] binary tag system with deletion number 3 and the rules 0 → 00 and 1 → 1101 on the input 0101110.

0101110 ⊢ 111000 ⊢ 0001101 ⊢ 110100 ⊢ 1001101 ⊢ 11011101 ⊢ . . .

We use the term round to describe the \( \lfloor \frac{|w|}{β} \rfloor \) or \( \lceil \frac{|w|}{β} \rceil \) computation steps that traverse the word w exactly once. We say a symbol w0 is read if and only if at the start of a computation step it is the leftmost symbol (i.e. the rule w0 → w0,0w0,1 . . . w0,e is applied), and we say a word w = w0w1 . . . w|w|−1 is entered with shift z < β if w z is the leftmost symbol that is read in w. For example, in Figure 1 the words q, r, and v are entered with shifts of 0, 3, and 2 respectively. We let \( \frac{w}{z} \) denote the sequence of symbols that is read during a single round on w when it is entered with shift z, and we call \( \frac{w}{z} \) a track of w. If w = w0w1 . . . w|w|−1, then

\[ \frac{w}{z} = w_zw_{z+β}w_{z+2β}w_{z+3β} . . . , w_{z+lβ} \]

where \( |w| − β ≤ z + lβ < |w| \). For example, in Figure 1 we have \( \frac{w}{0} = q0q1q3r7v2 \). A word w has a shift change of 0 ≤ s < β if |w| = yβ − s where y > 0 is a natural number.

Lemma 1. Given a tag system T with deletion number β and the word rv ∈ Σ*, where the word r has a shift change of s and |v| ≥ β, after one round of T on r entered with shift z the word v is entered with shift (z + s) mod β.

Before we give the proof of Lemma 1 we note that Figure 1 gives examples of the shift change caused by reading a word: In Figure 1 the word q is entered with shift 0 and has shift change of s = 3 and so r is entered with shift 3 = (0 + 3) mod 4, and the word r has a shift change of 3 so the word v is entered with shift 2 = (3 + 3) mod 4.

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Proof. Recall that when \( r \) is entered with shift \( z \) then \( r_z \) is the leftmost symbol read in \( r \). In Equation (1) the sequence of symbols read in \( r \) and the leftmost symbol read in \( v \) are given in bold. The rightmost symbol read in \( r \) is \( r_{z+l} \) for some \( l \in \mathbb{N} \) and the next symbol read is \( v_q \) in \( v \). It follows that symbols \( r_{z+l} \) to \( v_{q-1} \) are deleted in the computation step when \( r_{l+1} \) is read. Since \( \beta \) symbols are deleted at each computation step, from Equation (1) we get
\[
\frac{r_0 \ldots r_{z-1} r_z r_{z+1} \ldots r_{z+\beta-1} r_z + r_{z+\beta} r_{z+1} \ldots r_{z+l} r_{z+1} r_{z+l+1} \ldots r_{y\beta-s-1} v_0 \ldots v_{q-1} v_q v_{q+1} \ldots}{\beta}
\]
(1)

Lemma 2. Given a tag system \( T \) with deletion number \( \beta \) and the word \( w \), where \( |w| = y\beta - s \) with \( y \in \mathbb{N} \) and \( 0 < s < \beta \), one round of \( T \) on \( w \) entered with shift \( z < \beta \) reads \( \left\lceil \frac{|w|}{\beta} \right\rceil \) symbols if \( z < \beta - s \), and \( \left\lceil \frac{|w|}{\beta} \right\rceil = y - 1 \) symbols if \( z \geq \beta - s \).
Proof. In Equation (2) the sequence of symbols read when \( w \) is entered with shift \( \beta - s - 1 \) is given in bold. This bold sequence has length \( y \). It is fairly straightforward to see that if we enter \( w \) with a shift \( \beta - s \) we read \( w \) symbols if we enter \( w \) with a shift \( \geq \beta - s \) we read \( y - 1 \) symbols.
\[
w_0 w_1 \ldots w_{\beta-s-2} w_{\beta-s-1} w_{\beta-s} w_{\beta-s-2} w_{\beta-s-1} w_{\beta-s} w_{\beta-s-2} w_{\beta-s-1} \ldots
\]
(2)

2.2 Cyclic tag systems

Definition 2. A cyclic tag system \( C = \alpha_0, \ldots, \alpha_{p-1} \), is a list of words \( \alpha \in \{0,1\}^* \) called appendants.

A configuration of a cyclic tag system consists of (i) a marker that points to a single appendant \( \alpha_m \) in \( C \), and (ii) a word \( w = w_1 \ldots w_{|w|} \in \{0,1\}^* \). We call \( w \) the dataword. Intuitively the list \( C \) is a program with the marker pointing to instruction \( \alpha_m \). In the initial configuration the marker points to appendant \( \alpha_0 \) and \( w \) is the binary input word.

Definition 3. A computation step is deterministic and acts on a configuration in one of two ways:

- If \( w_1 = 0 \) then \( w_1 \) is deleted and the marker moves to appendant \( \alpha_{(m+1 \mod p)} \).
- If \( w_1 = 1 \) then \( w_1 \) is deleted, the word \( \alpha_m \) is appended onto the right end of \( w \), and the marker moves to appendant \( \alpha_{(m+1 \mod p)} \).

A cyclic tag system completes its computation if (i) the dataword is the empty word or (ii) it enters a repeating sequence of configurations.

As an example we give first 6 steps of the cyclic tag system \( C = 001, 01, 11 \) on the input word 101. In each configuration \( C \) is given on the left with the marked appendant highlighted in bold font.

\[
\begin{align*}
001, 01, 11 & \vdash 001, 01, 11 & 01001 & \vdash 001, 01, 11 & 1001 & \vdash 001, 01, 11 & 00111 \\
& \vdash 001, 01, 11 & 0111 & \vdash 001, 01, 11 & 111 & \vdash 001, 01, 11 & 1111 & \vdash \ldots
\end{align*}
\]
Cyclic tag systems were introduced by Cook [9] and used to prove the cellular automaton Rule 110 universal. We gave an exponential improvement in the time efficiency of cyclic tag systems to show:

**Theorem 1 ([27]).** Let $M$ be a single-tape deterministic Turing machine that computes in time $t$. Then there is a cyclic tag system $C$ that simulates the computation of $M$ in time $O(t^3 \log t)$.

Given a cyclic tag system $C = \alpha_0, \ldots, \alpha_{p-1}$ we can construct another cyclic tag system $C'$ by concatenating an arbitrary number of copies of the program for $C$. The system $C'$ simulates step for step the computation of $C$. For example take $C = 001, 01, 11$ given above, if we define $C' = 001, 01, 11, 001, 01, 11$ then $C'$ will give the same sequence of computation steps for any computation of $C$.

## 3 Simulating cyclic tag systems with binary tag systems

In Theorem 2, the tag system $T_C$ simulates an arbitrary cyclic tag system with a program of length $3k + 2$ where $k \in \mathbb{N}$. We do not lose generality with this restriction since the cyclic tag systems given by the construction in [27] satisfy this condition and are Turing universal.

**Theorem 2.** Let $C = \alpha_0, \alpha_1, \ldots, \alpha_{3k+1}$ with $k \in \mathbb{N}$ be a cyclic tag system that runs in time $t$. Then there is a binary tag system $T_C$ that simulates the computation of $C$ in time $O(t^2)$.

### 3.0.1 Cyclic tag system $C'$ and binary tag system $T_C$

Given the program $C = \alpha_0, \alpha_1, \ldots, \alpha_{3k+1}$, we can give a cyclic tag system $C' = (\alpha_0, \alpha_1, \ldots, \alpha_{3k+1})^q$ of length $q(3k + 2)$ that simulates $C$ step for step when given the same input dataword as $C$ (see the last paragraph of Section 2.2). The value $q \in \mathbb{N}$ is chosen so that $q(3k + 2) = 3x - 2$ for $x \in \mathbb{N}$, such that $0 = x \mod 2$ and $r < \frac{x}{3} - 7$, where $r$ is the length of the longest appendant in $C$.

We construct a binary tag system $T_C$ that simulates the computation of $C'$. The deletion number of $T_C$ is $\beta$, its alphabet is $\{b, c\}$, and its rules are of the form $b \rightarrow b$ and $c \rightarrow u$, where $u \in \{b, c\}^*$. The binary word $u$ encodes the entire program of $C'$ and is defined by Tables 2 to 4. We will explain how to read these tables later in Section 3.

### 3.0.2 Encoding used by $T_C$

The cyclic tag system symbols 0 and 1 are encoded as the binary words $\langle 0 \rangle = b^4ub^2u^{-1}b^2ub^{2x-8}$ and $\langle 1 \rangle = b^{10}(ubb)^{\frac{x}{3}-7}u^{\frac{x}{3}+7}b^2ub^{x+2}$ respectively. We refer to $\langle 0 \rangle$ and $\langle 1 \rangle$ as objects.

**Definition 4.** An arbitrary input dataword $w_1 w_2 \ldots w_n \in \{0, 1\}^*$ to a cyclic tag system is encoded as the $T_C$ input dataword $\langle w_1 \rangle \langle w_2 \rangle \ldots \langle w_n \rangle$.

During the simulation we make use of two extra objects: the binary words $\langle \epsilon \rangle = b^2ub^{3x-2}$ and $\langle \epsilon' \rangle = b^4ub^2u^{-2}b^2ub^{2x-8}$. An arbitrary (not necessarily input) cyclic tag system dataword $w_1 w_2 \ldots w_l \in \{0, 1\}^*$ is encoded as

\[
\langle w_1, z \rangle \langle \langle \epsilon \rangle, \langle \epsilon' \rangle \rangle^* \langle w_2 \rangle \langle \langle \epsilon \rangle, \langle \epsilon' \rangle \rangle^* \langle w_3 \rangle \ldots \langle \langle \epsilon \rangle, \langle \epsilon' \rangle \rangle^* \langle w_l \rangle \langle \langle \epsilon \rangle, \langle \epsilon' \rangle \rangle^*
\] (3)

where $\langle w_1, z \rangle$ denotes the word given by an object $\langle w_1 \rangle \in \{\langle 0 \rangle, \langle 1 \rangle\}$ with its leftmost $z < \beta$ symbols deleted. This implies that $\langle w_1 \rangle$ is entered with the shift value $z$ from Table 1. Finally, each appendant $\alpha_m = \sigma_1 \sigma_2 \ldots \sigma_v$ of $C'$ is encoded via Equations (4), (5) or (6), (where $\sigma_i \in \{0, 1\}$).

\[
\langle \alpha_m \rangle = \langle \sigma_1 \rangle \langle \sigma_2 \rangle \ldots \langle \sigma_v \rangle \langle \epsilon \rangle^{x-v+1}
\] (4)

\[
\langle \alpha_m' \rangle = \langle \sigma_1 \rangle \langle \sigma_2 \rangle \ldots \langle \sigma_j \rangle \langle \epsilon \rangle \langle \sigma_{j+1} \rangle \ldots \langle \sigma_v \rangle \langle \epsilon \rangle^{x-v}
\] (5)

\[
\langle \alpha_m'' \rangle = \langle \sigma_1 \rangle \langle \sigma_2 \rangle \ldots \langle \sigma_v \rangle \langle \epsilon \rangle^{j-v} \langle \epsilon' \rangle \langle \epsilon \rangle^{x-j}
\] (6)
Table 1 covers all possible shift values. To see this note that when 
\( z \) when entered with shift \( z \), the shift change for \( \langle 1 \rangle \) and \( \langle 0 \rangle \) is \( z_1 \), the shift change for \( \langle \epsilon' \rangle \) is \( z_2 \), the deletion number of \( \mathcal{T}_C \) is \( \beta \), and \( 3x - 2 \) is the number of appendants in \( \mathcal{C}' \). The value of \( x \) is given in Section 3.0.1.

\[
\begin{align*}
|\langle \epsilon \rangle| &= (3x + 1)\beta, & |\langle \epsilon \rangle| &= u + 3x, & |u| &= (3x + 1)\beta - 3x, & |\langle 1 \rangle| &= |\langle 0 \rangle| = (x + 1)|u| + 2x, \\
|\langle 1 \rangle| &= |\langle 0 \rangle| = (x + 1)((3x + 1)\beta - 3x) + 2x, & z_1 &= 3x^2 + x, & z_1(3x - 2) = \beta, \\
|\langle \epsilon' \rangle| &= x|u| + 2x, & |\langle \epsilon' \rangle| &= x((3x + 1)\beta - 3x) + 2x, & z_2 &= 3x^2 - 2x, \\
z &= ((z_1m + z_2d) \mod \beta), & 0 \leq m < 3x - 2, & 0 \leq d < 3x + 1
\end{align*}
\]

Table 1: Length of objects and shift change values. The shift change for \( \langle 1 \rangle \) and \( \langle 0 \rangle \) is \( z_1 \), the shift change for \( \langle \epsilon' \rangle \) is \( z_2 \), the deletion number of \( \mathcal{T}_C \) is \( \beta \), and \( 3x - 2 \) is the number of appendants in \( \mathcal{C}' \). The value of \( x \) is given in Section 3.0.1.

\[
\begin{align*}
(i) \quad \langle 1, z \rangle \langle a_2 \rangle \ldots \langle a_h \rangle \quad |\langle 1 \rangle| &= \left\lceil \frac{|\langle 1 \rangle|}{\beta} \right\rceil & \langle a_2, z + 1 \rangle \langle a_3 \rangle \ldots \langle a_h \rangle \langle \alpha_m \rangle \\
(ii) \quad \langle 0, z \rangle \langle a_2 \rangle \ldots \langle a_h \rangle \quad |\langle 0 \rangle| &= \left\lceil \frac{|\langle 0 \rangle|}{\beta} \right\rceil & \langle a_2, z + 1 \rangle \langle a_3 \rangle \ldots \langle a_h \rangle \langle \epsilon \rangle^{x+1} \\
(iii) \quad \langle \epsilon', z \rangle \langle a_2 \rangle \ldots \langle a_h \rangle \quad |\langle \epsilon' \rangle| &= \left\lceil \frac{|\langle \epsilon' \rangle|}{\beta} \right\rceil & \langle a_2, z + 2 \rangle \langle a_3 \rangle \ldots \langle a_h \rangle \langle \epsilon \rangle^x \\
(iv) \quad \langle \epsilon, z \rangle \langle a_2 \rangle \ldots \langle a_h \rangle \quad |\langle \epsilon \rangle| &= \left\lceil \frac{|\langle \epsilon \rangle|}{\beta} \right\rceil & \langle a_2, z \rangle \langle a_3 \rangle \ldots \langle a_h \rangle \langle \epsilon \rangle
\end{align*}
\]

Figure 2: Objects \( \langle 1 \rangle \), \( \langle 0 \rangle \), \( \langle \epsilon \rangle \) and \( \langle \epsilon' \rangle \) being read by \( \mathcal{T}_C \) when entered with shift \( z \), where \( a_i \in \{ \langle \epsilon \rangle, \langle \epsilon' \rangle, \langle 0 \rangle, \langle 1 \rangle \} \), and \( z = (z_1m + z_2d) \mod \beta \). In (i) and (ii) \( z < \beta - z_1 \), in (iii) \( z < \beta - z_2 \), and in (iv) \( z < \beta \). The encoded appendant \( \langle \alpha_m \rangle \) is given in Equation (4), and the values \( z_1 \), \( z_2 \), \( m \), and \( d \) are given in Table 1.

### 3.0.3 Lengths of objects and shift values

The sequence of symbols that is read in a word is determined by the shift value with which it is entered (see for example Figure 5 (i)). So in the simulation we use the shift value for algorithm control flow. In Table 1 we give the length of objects \( \langle 0 \rangle \), \( \langle 1 \rangle \), \( \langle \epsilon \rangle \), and \( \langle \epsilon' \rangle \), and their shift change values. Recall from Section 2.1 that an object of length \( y\beta - s \) has a shift change of \( s \), where \( s < \beta \) and \( \beta \) is the deletion number. So, from the object lengths \( |\langle 1 \rangle| = |\langle 0 \rangle| \) and \( |\langle \epsilon' \rangle| \) we get the respective shift change values of \( z_1 \) and \( z_2 \) in Table 1. From Lemma 1, the shift an object is entered with is determined by the shift change of the objects previously read in the dataword. So when we have a dataword containing only \( \langle 1 \rangle \), \( \langle 0 \rangle \), \( \langle \epsilon \rangle \) and \( \langle \epsilon' \rangle \) objects, we enter objects with shifts of the form \( z = ((z_1m + z_2d) \mod \beta) \) (see Table 1). The range of values for \( m \) and \( d \) in Table 1 covers all possible shift values. To see this note that when \( m = 3x - 2 \), then \( z_1m = \beta \) giving \( 0 = z_1m \mod \beta \), and when \( d = 3x + 1 \), then \( z_2d = \beta \) giving \( 0 = z_2d \mod \beta \).

### 3.1 The simulation algorithm

Here we give a high level picture of our algorithm using Figures 2 and 3. Following this, in Sections 3.1.2 and 3.1.3, the lower level details are then given.

#### 3.1.1 Algorithm overview

Figures 2 and 3 give arbitrary examples that cover all possible cases for reading each of the four objects. In both figures \( \vdash^y \) denotes the \( y \) computation steps that read the entire leftmost object in the dataword on the left and produce the new dataword on the right. For example, in Figure 2 (ii) when \( \langle 0 \rangle \) is read it appends \( \langle \epsilon \rangle^{x+1} \) in \( \left\lceil \frac{|\langle 0 \rangle|}{\beta} \right\rceil \) computation steps. There are two cases for reading \( \langle 0 \rangle \), \( \langle 1 \rangle \), and \( \langle \epsilon' \rangle \) objects with each case determined by the number of symbols read in the object (see Lemma 2). There is only one case for reading \( \langle \epsilon \rangle \) as \( \left\lceil \frac{|\langle \epsilon \rangle|}{\beta} \right\rceil \). The objects \( \langle \epsilon \rangle \) and \( \langle \epsilon' \rangle \)
are garbage objects that have no effect on the simulation. To see this note from Figures 2 and 3 that $\langle \epsilon \rangle$ and $\langle \epsilon' \rangle$ objects append only more garbage objects, and as we will see in Section 3.1.2 the shift change caused by reading an $\langle \epsilon \rangle$ or an $\langle \epsilon' \rangle$ does not effect algorithm control flow. The garbage objects are introduced to simulate deletion as our binary tag system has no rule that appends the empty word $\epsilon$.

When $\mathcal{T}_c$ reads an $\langle 1 \rangle$ or an $\langle 0 \rangle$ object as shown in (i) and (ii) of Figures 2 and 3 it simulates a computation step where $C'$ reads a 1 or a 0. At the beginning of the simulated computation step the currently marked appendant $\alpha_m$ is encoded by the shift value $z = (z_1 m + z_2 d) \mod \beta$. In (i) and (ii) $z \geq \beta - z_1$, and in (iii) $z \geq \beta - z_2$.

The encoded appendant $\langle \alpha'_m \rangle$ is given in Equations (5) and (6), and the values $z_1$, $z_2$, $m$, and $d$ are given in Table 1.

Figure 3: Objects $\langle 0 \rangle$, $\langle 1 \rangle$ and $\langle \epsilon' \rangle$ being read by $\mathcal{T}_c$ when entered with shift $z$, where $a_i \in \{\langle \epsilon \rangle, \langle \epsilon' \rangle, \langle 0 \rangle, \langle 1 \rangle\}$, and $z = (z_1 m + z_2 d) \mod \beta$. In (i) and (ii) $z \geq \beta - z_1$, and in (iii) $z \geq \beta - z_2$. The encoded appendant $\langle \alpha'_m \rangle$ is given in Equations (5) and (6), and the values $z_1$, $z_2$, $m$, and $d$ are given in Table 1.

In both cases above there is one further step that is needed to complete the simulation of the Definition 3 computation step. Note from Equation (3) that between a pair of encoded cyclic tag system symbols $\langle w_1 \rangle$ and $\langle w_2 \rangle$ is a word of the form $\{\langle \epsilon \rangle, \langle \epsilon' \rangle\}^*$. So after $\langle w_1 \rangle$ is read, the word $\{\langle \epsilon \rangle, \langle \epsilon' \rangle\}^*$ is read placing the object $\langle w_2 \rangle$ at the left end of the dataword. This completes the simulated computation step as $\mathcal{T}_c$ is now ready to begin reading the next encoded symbol $\langle w_2 \rangle$. At the end of the next section we see that reading the garbage objects $\langle \epsilon \rangle$ and $\langle \epsilon' \rangle$ does not change the appendant encoded in the shift which means that $\langle w_2 \rangle$ is entered with a shift value encoding the correct appendant $\alpha_{((m+1) \mod (3x-2))}$.

We have not yet described how reading the objects $\langle 1 \rangle$, $\langle 0 \rangle$, $\langle \epsilon \rangle$, and $\langle \epsilon' \rangle$ append the appendants shown in Figures 2 and 3. To do so we must give the sequence of symbols read in each object when entered with shift $z$. The word $u$ that appears in the objects $\langle 1 \rangle$, $\langle 0 \rangle$, $\langle \epsilon \rangle$, and $\langle \epsilon' \rangle$ is defined such that the $u$ words read in these objects append the appendants shown in Figures 2 and 3. The $b$ symbols that appear in each object are used to control the shift with which we enter each $u$ within an object and thus control the sequence of symbols read in each $u$ (see Figure 5 (i)). For example, if we enter an $\langle \epsilon \rangle = b^2 u b^{3x-2}$ with shift $z$ then the leftmost pair of $b$ symbols cause the $u$ to be entered with shift $z - 2$ and we read track $u_{[z-2]}$ (in this case $s = z - 2$ in Figure 5 (i)). If we assign track $u_{[z-2]}$ a value that will append an $\langle \epsilon \rangle$, then when $\langle \epsilon \rangle$ is entered with shift $z$ the word $u$ is entered with shift $z - 2$ and an $\langle \epsilon \rangle$ gets appended as shown in Figure 2 (iv). So using
From Lemma 1, when \langle \text{or track} \rangle read in a word depends on the shift with which the word is entered (see Figure 5 (i)).

The word subwords of \( u \) of 0 it too does not change the appendant encoded in the shift.

\[
\begin{align*}
(\text{i}) & \ b^{10}(cbb)^{\beta} u^{\alpha} + 7b^2ub^x+2 \rightarrow b^{10}(ubb)^{\beta} u^{\alpha} + 7b^2ub^x+2 \\
(\text{ii}) & \ b^4cb^2c^{\beta}^{-1}b^2cb^{2x-8} \rightarrow b^4ub^2u^{\beta} u^{\alpha} + 1b^2ub^{2x-8} \\
(\text{iii}) & \ b^2cb^{3x-2} \rightarrow b^2ub^{3x-2} \\
(\text{iv}) & \ b^4cb^2c^{\beta}^{-2}b^2cb^{2x-8} \rightarrow b^4ub^2u^{\beta} u^{\alpha} + 1b^2ub^{2x-8}
\end{align*}
\]

Figure 4: Sequence of symbols read to append \langle 1 \rangle (i), \langle 0 \rangle (ii), \langle \epsilon \rangle (iii), and \langle \epsilon' \rangle (iv). Rules \( b \rightarrow b \) or \( c \rightarrow u \) are applied to each symbol in sequence on the left to give the object it appends on the right.

the \( b \) symbols in each object we control the tracks read in each \( u \) within the object so that the correct appendant gets appended when the object is read. The details of reading objects and \( u \) subwords are given in Section 3.1.3.

3.1.2 Encoding the marked appendant of \( C' \) in the shift of \( T_C \)

From Table 1 the shift change when reading an \langle 0 \rangle or an \langle 1 \rangle object is \( z_1 \). So from Lemma 1, when an \langle 0 \rangle or an \langle 1 \rangle object is entered with shift \((z_1m + z_2d) \mod \beta\), the next object immediately to its right is entered with shift \((z_1(m + 1) + z_2d) \mod \beta\) as shown in (i) and (ii) of Figures 2 and 3. This shift change of \( z_1 \) simulates that the marked appendant changes from \( \alpha_m \) to \( \alpha((m+1) \mod (3x-2)) \) (the length of the program for \( C' \) is \( 3x-2 \)). If \( C' \) is at the marked appendant \( \alpha_m \) and then reads \( 3x-2 \) symbols, it traverses its entire circular program and returns to appendant \( \alpha_m \). Notice from Table 1 that \( z_1(3x-2) = \beta \), and so if we read \( 3x-2 \) of the \langle 0 \rangle and \langle 1 \rangle objects, then the total shift change is \( 0 = z_1(3x-2) \mod \beta \). Since the shift change value is 0, the encoding of the marked appendant remains unchanged after reading \( 3x-2 \) of the \langle 0 \rangle and \langle 1 \rangle objects, correctly simulating a traversal of the entire circular program of \( C' \). In Lemma 6 it is proved that if \( m_i \neq m_j \) then \((z_1m_i + z_2d_i) \mod \beta \neq (z_1m_j + z_2d_j) \mod \beta\) for all \( 0 \leq d_i, d_j < 3x + 1 \). This shows that each shift value \( z = (z_1m + z_2d) \mod \beta \) encodes one and only one appendant \( \alpha_m \). So reading \( \langle \epsilon \rangle \), with its shift change of \( z_2 \), moves from one shift value that encodes \( \alpha_m \) to another shift value that also encodes \( \alpha_m \) (as shown in (iii) of Figures 2 and 3). In other words, reading \( \langle \epsilon \rangle \) does not change the value of the appendant encoded in the shift. Finally, since \( \langle \epsilon \rangle \) has a shift change value of 0 it too does not change the appendant encoded in the shift.

3.1.3 Reading objects and defining the word \( u \)

The word \( u \) is defined via Tables 2 to 4 such that when each object is read it appends the correct appendant as shown in Figures 2 and 3. We will take the case of reading an \langle 1 \rangle entered with shift \( z < \beta - z_1 \) and show that it appends \langle \alpha_m \rangle as illustrated in Figure 2 (i). We will then explain how the method used to verify this case can be applied to verify the remaining cases in Figures 2 and 3.

Here we show that when \langle 1 \rangle is entered with shift \( z < \beta - z_1 \) the sequence of symbols read in the \( u \) subwords of \langle 1 \rangle append \langle \alpha_m \rangle = \langle \sigma_1 \rangle \langle \sigma_2 \rangle \ldots \langle \sigma_n \rangle \langle \epsilon \rangle^{x-v+1} \). Recall that the sequence of symbols (or track) read in a word depends on the shift with which the word is entered (see Figure 5 (i)). From Lemma 1, when \langle 1 \rangle = \( b^{10}(ubb)^{\beta} u^{\alpha} + 7b^2ub^x+2 \) is entered with shift \( z \) then the leftmost \( b^{10} \) causes the leftmost \( u \) to be entered with shift \( (z - 10) \mod \beta \), and because each \( u \) has a shift change of \( 3x \), following the \( ubb \) subword the second \( u \) is entered with shift \( (z + (3x - 2) - 10) \mod \beta \), the third \( u \) with shift \( (z + 2(3x - 2) - 10) \mod \beta \), and so on (as shown in Figure 5 (ii)). Now we define the track read in each \( u \) in Figure 5 (ii) so that it appends a single object from \{ \langle 0 \rangle, \langle 1 \rangle, \langle \epsilon \rangle, \langle \epsilon' \rangle \} at the right end of the dataword. The tracks that append each object are given in Figure 4. For example in Figure 4 (iii) we see that applying the rules \( b \rightarrow b \) and \( c \rightarrow u \) to the sequence \( b^2cb^{3x-2} \) appends the object \langle \epsilon \rangle = b^2ub^{3x-2} \). Note from Figure 4 that the number of symbols read
(i) \( u = u_0 \ldots u_{\alpha - 1} u_\beta u_{\alpha + 1} \ldots u_{\alpha + \beta - 1} u_\beta u_{\alpha + \beta} \ldots u_s(3s + 1) \beta \)

\[ u[s] = u_\beta u_{\alpha + 1} u_{\alpha + 2} \ldots u_s(3s + 1) \beta \]

(\( \alpha, \beta, \beta - 10 \) to append each object is either of length \( 3s + 1 \) or \( 3s \). Note from Table 1 and Lemma 2 that when \( u \) is read as \( 3s + 1 \) or \( 3s \), and each \( u \) word that is read can append a single object from \( \{ \langle e \rangle, \langle b \rangle, (0), \langle 1 \rangle \} \). For example, in Figure 5 (i) if we wish \( u \) to append \( \langle e \rangle = b^2b^2u^3u^{-2} \) when entered with shift \( s \) then we define \( u[s] = b^2c^b^{2x-2} \).

To append the sequence of objects \( \langle \alpha_m \rangle = \langle \sigma_1 \rangle \langle \sigma_2 \rangle \ldots \langle \sigma_v \rangle \langle e \rangle^{x-v+1} \) when reading an \( (1) \), the track read in the \( i + 1 \)th \( u \) from the left in \( (1) \) appends the \( i + 1 \)th object from the left in \( \langle \alpha_m \rangle \), when \( 0 \leq i \leq x \). Thus in Figure 5 (ii), for \( 0 \leq i < v \) if \( \sigma_{i+1} = 0 \), then track \( z+i(3s-2)-10 = b^4c^b^2e^{x-1}b^2c^b^2x^{-8} \) is read causing the word \( (0) = b^4u^2b^2u^{-1}b^2u^2b^x{-8} \) to be appended, and if \( \sigma_{i+1} = 1 \), then track \( |z+i(3s-2)-10| = b^4(ubb)^{x-7}c^b^2b^2x^{-2} \) is read causing the word \( (1) = b^4(ubb)^{x-7}c^b^2b^2x^{-2} \) to be appended. There is an exception when \( i = 0 \) and \( z = 0 \) as we have \( \beta - 10 = (z + i(3s - 2) - 10) \) mod \( \beta \). In this special case, tracks have the from \( u = b^4(ubb)^{x-7}c^b^2b^2x^{-2} \) with one less \( \beta \) than usual. From Lemma 2 and Table 1, when \( u \) is entered with shift \( \beta - 10 \) only \( |u_{\beta-10}| = 3x \) (instead of \( 3x + 1 \)) symbols are read. When \( u \) is entered with shift \( \beta - 10 \) then \( z = 0 \), and so the leftmost \( b \) in the \( (1) \) is read and provides the first \( b \) in the sequence \( b^4b^2c^b^2x^{-1}b^2c^b^2x^{-8} \) that prints an \( (0) \), or provides the first \( b \) in the sequence \( b^4(ubb)^{x-7}c^b^2b^2x^{-2} \) which gives \( \beta - 3x + 2 = (z + 3x^2 - x + 2) \). In this special case, tracks have the from \( u = b^4(ubb)^{x-7}c^b^2b^2x^{-2} \) with one less \( \beta \) than usual. From Lemma 2 and Table 1, when \( u \) is entered with shift \( \beta - 2x \) or \( 2x \) symbols are read. When \( u \) is entered with shift \( \beta - 2x \) then the \( b^2x^2 \) at the end of the \( (1) \) is read and provides the last \( b \) in the sequence \( b^4b^2c^b^2x^{-3} \) that prints an \( (\epsilon) \). The \( u \) tracks in this paragraph show that when an \( (1) \) is entered with shift \( z < \beta - z_1 \) then the column \( \langle \alpha \rangle \) is appended at the right end of the dataword as shown in Figure 2 (i). The \( u \) tracks given above have the same values as the bottom eight \( u \) tracks in Table 2.

Using the same method as in the previous two paragraphs one can show that the \( u \) tracks given for each of the objects \( \langle \epsilon \rangle, \langle 0 \rangle \) and \( \langle \epsilon \rangle \) in Table 2 will cause the correct appendant to be appended as shown in Figure 2. Note that once the shift values for the \( u \) tracks have been determined (as in Figure 5 (ii)) we can use these shift values to determine the special cases. When \( u \) is entered with a shift \( \beta - 3x \) then from Lemma 2 any \( 3x \) (instead of \( 3x + 1 \)) symbols are read and we have a
| Object track | Tracks read in $u$ | Values for $z$, $i$ and $\sigma_{i+1}$ |
|--------------|------------------|---------------------------------|
| $\langle \epsilon \rangle = (b^2 c b^{3x-2})_{[z]}$ | $u_{[\beta-2]} = b c b^{3x-2}$ | $i = 0, \ z = 0$ |
| | $u_{[z-2]} = b^2 c b^{3x-2}$ | $i = 0, \ 0 < z - 2 < \beta - 3x$ |
| | $u_{[\beta-2]} = b^2 c b^{3x-3}$ | $i = 0, \ \beta - 3x \leq z - 2 < \beta - 2$ |
| | $u_{[\beta-4]} = b c b^{3x-2}$ | $i = 0, \ z = 0$ |
| | $u_{[\beta-4]} = b^2 c b^{3x-2}$ | $i = 0, \ 0 < z < \beta - z_2$ |
| | $u_{[z+3x-6]} = b^2 c b^{3x-2}$ | $1 \leq i < x - 1, \ 0 \leq z < \beta - z_2$ |
| | $u_{[z+3x(x-1)-8]} = b^2 c b^{3x-2}$ | $i = x - 1, \ 0 \leq z \leq \beta - z_2 - 2x$ |
| | $u_{[\beta-2x-8]} = b^2 c b^{3x-3}$ | $i = x - 1, \ z = \beta - z_2 - x$ |
| $\langle \epsilon' \rangle = (b^2 c b^{3x-2})_{[z]}^x$ | $u_{[\beta-4]} = b c b^{3x-2}$ | $i = 0, \ z = 0$ |
| | $u_{[\beta-4]} = b^2 c b^{3x-2}$ | $i = 0, \ 0 < z < \beta - z_1$ |
| | $u_{[z+3x-6]} = b^2 c b^{3x-2}$ | $1 \leq i < x, \ 0 \leq z < \beta - z_1$ |
| | $u_{[z+3x^2-8]} = b^2 c b^{3x-2}$ | $i = x, \ 0 \leq z \leq \beta - z_1 - 2x$ |
| | $u_{[\beta-2x-8]} = b^2 c b^{3x-3}$ | $i = x, \ z = \beta - z_1 - x$ |
| $\langle 0 \rangle = (b^2 c b^{3x-2})_{[z]}^{x+1}$ | $u_{[\beta-10]} = b^3 c b^2 c^{x-1} b^2 c b^{2x-8}$ | $i = 0, \ z = 0, \ \sigma_1 = 0$ |
| | $u_{[\beta-10]} = b^0 (c b b)^{\frac{z}{2} - 7} c^x + 7 b^2 c b^{x+2}$ | $i = 0, \ z = 0, \ \sigma_1 = 1$ |
| | $[z+i(3x-2)-10] = b^1 c b^2 c^{x-1} b^2 c b^{2x-8}$ | $0 \leq i < v, \ 0 \leq z < \beta - z_1, \ (i, z) \neq (0, 0, \sigma_{i+1} = 0)$ |
| | $[z+i(3x-2)-10] = b^1 (c b b)^{\frac{z}{2} - 7} c^x + 7 b^2 c b^{x+2}$ | $0 \leq i < v, \ 0 \leq z < \beta - z_1, \ (i, z) \neq (0, 0, \sigma_{i+1} = 1)$ |
| | $[z+i(3x-2)-10] = b^2 c b^{3x-2}$ | $0 \leq z < \beta - z_1$ |
| | $[z+i(3x-2)-10] = b^2 c b^{3x-2}$ | $0 \leq z < \beta - z_1$ |
| $\langle 1 \rangle = \langle \alpha_m \rangle_{[z]}$ | $u_{[\beta-3x+2]} = b^2 c b^{3x-3}$ | $i = x, \ z = \beta - z_1 - x$ |

Table 2: Tracks read in each object. Here $\langle \epsilon \rangle$ is entered with shift $z < \beta$, $\langle \epsilon' \rangle$ is entered with a shift $z < \beta - z_2$, $\langle 0 \rangle$ and $\langle 1 \rangle$ are entered with a shift $z < \beta - z_1$. The values $z$, $z_1$, and $z_2$ are given in Table 1, and $\langle \alpha_m \rangle$, $\sigma_{i+1}$ and $v$ are given in Equation (4). The value $i$ indexes the position of the $u$ subword within the object being read (see Figure 5 (ii)). Here the mod $\beta$ is dropped from the underscripts in $u$ tracks as all of the underscript terms above are $0 \leq i$ and $< \beta$.
| Object track | Tracks read in \( u \) | Values for \( z, i, j, \) and \( \sigma_{i+1} \) |
|--------------|------------------------|---------------------------------------------|
| \( \langle \epsilon' \rangle = \langle z \rangle \) | \( u \) \mod \beta = b^2 cb^{3x-2} \) | \( i = 0, \quad \beta - z_2 - 4 \leq z - 4 < \beta - 3x \) |
| \( (b^2 cb^{3x-2})^j \) | \( u \) \mod \beta = b^2 cb^{3x-2} \) | \( j = 0, \quad \beta - 3x \leq z - 4 < \beta - 4 \), \( \beta < z + 3x \) |
| \( b^4 cb^2 c^{x-2} b^2 cb^{2x-8} \) | \( u \) \mod \beta = b^2 cb^{3x-2} \) | \( \beta > z + 3x \) |
| \( \langle 0 \rangle = \langle z \rangle \) | \( u \) \mod \beta = b^2 cb^{3x-2} \) | \( i = 0, \quad \beta - z_1 - 4 \leq z - 4 < \beta - 3x \) |
| \( (b^2 cb^{3x-2})^j \) | \( u \) \mod \beta = b^2 cb^{3x-2} \) | \( j = 0, \quad \beta - 3x \leq z - 4 < \beta - 4 \), \( \beta < z + 3x \) |
| \( b^4 cb^2 c^{x-2} b^2 cb^{2x-8} \) | \( u \) \mod \beta = b^2 cb^{3x-2} \) | \( \beta > z + 3x \) |

Table 3: Tracks read in \( \langle \epsilon' \rangle \) when entered with shift \( z \geq \beta - z_2 \), and tracks read in \( \langle 0 \rangle \) when entered with shift \( z \geq \beta - z_1 \). The values \( z, z_1 \), and \( z_2 \) are given in Table 1. The value \( i \) indexes the position of the \( u \) subword within the object being read (see Figure 5 (ii)). The value \( j \) gives the index of the \( u \) subword that appends \( \langle \epsilon' \rangle \) (see Figure 3). The \( \mod \beta \) is dropped from underscripts where the term is \( < \beta \).

Special case where the object track is missing a single \( b \). For the special cases in Table 2 (rows 1, 3, 4, 8, 9, 13, 14, 15, and 21) the missing \( b \) needed to complete the object track is provided by reading a \( b \) in a sequence of \( b \) symbols to the left or right of the \( u \) that is entered with shift \( \geq \beta - 3x \). For example, from row 4 of Table 2 when \( \langle \epsilon' \rangle = b^4 ub^2 u^{x-2} b^2 ub^{2x-8} \) is entered with shift \( z = 0 \) we read track \( u \mod \beta = b^2 cb^{3x-2} \) and since \( z = 0 \) we also read the leftmost \( b \) in \( \langle \epsilon \rangle \) which provides the leftmost \( b \) needed to complete the track \( b^2 cb^{3x-2} \) that appends an \( \langle \epsilon \rangle \). Note that in Table 2 there are no special cases (i.e. tracks of length \( 3x \)) for \( 1 \leq i < x - 1 \) when reading \( \langle \epsilon' \rangle \), and for \( 1 \leq i < x \) when reading \( \langle 0 \rangle \) and \( \langle 1 \rangle \). This is because when \( \langle \epsilon' \rangle \) is entered with shifts \( \beta - z_2 \), and when \( \langle 0 \rangle \) and \( \langle 1 \rangle \) are entered with shifts \( \beta - z_1 \), it is not possible to enter the \( u \) subwords at these positions with shifts \( \geq \beta - 3x \).

The method used earlier in this section can also be used to demonstrate that the \( u \) tracks for each object in Tables 3 and 4 will cause the appendants shown in Figure 3 to be appended when each object is read. Note from the captions of Figures 2 and 3 that the shift values differ between
Table 4: Track read in (1), when entered with shift $z > \beta - z_1$. The values $z$ and $z_1$ are given in Table 1, and $\langle \alpha_m \rangle$ and $v$ are given in Equations (5) and (6). The value $i$ indexes the position of the $u$ subword within the object being read (see Figure 5 (ii)). The value $j$ gives the index of the $u$ subword that appends $\langle \epsilon' \rangle$ (see Figure 3). The mod $\beta$ is dropped from underscripts where the term is $< \beta$.

| Object track | Tracks read in $u$ | Values for $z$, $i$, $j$, and $\sigma_{i+1}$ |
|--------------|-------------------|---------------------------------------------|
| $u_{[z+i(3x-2)-10]}$ | $b^4 c b^2 c^x - 1 b^2 c b^{2x} - 8$ | $0 \leq i < j$, $i < v$, $\sigma_{i+1} = 0$ |
| $u_{[z+i(3x-2)-10]}$ | $b^{10} (c b b)^{\beta-7} c^\beta b^{x+2}$ | $0 \leq i < j$, $i < v$, $\sigma_{i+1} = 1$ |
| $u_{[(z+i(3x-2)-10) \mod \beta]}$ | $b^4 c b^2 c^x - 1 b^2 c b^{2x} - 8$ | $j < i < v$, $\sigma_i = 0$ |
| $u_{[(z+i(3x-2)-10) \mod \beta]}$ | $b^{10} (c b b)^{\beta-7} c^\beta b^{x+2}$ | $j < i < v$, $\sigma_i = 1$ |

Table 1, and $\langle \alpha_m \rangle$ and $v$ are given in Equations (5) and (6). The value $i$ indexes the position of the $u$ subword within the object being read (see Figure 5 (ii)). The value $j$ gives the index of the $u$ subword that appends $\langle \epsilon' \rangle$ (see Figure 3). The mod $\beta$ is dropped from underscripts where the term is $< \beta$.

the two figures. From Table 1 and Lemma 2, the number of symbols read in each object in Figure 2 is $y \in \{[\frac{1}{2}]_\beta, [\frac{01}{2}]_\beta, [\frac{00}{2}]_\beta, [\frac{\epsilon}{2}]_\beta \}$ and the number of symbols read in each object in Figure 3 is $y - 1 \in \{[\frac{10}{2}]_\beta, [\frac{10}{2}]_\beta, [\frac{00}{2}]_\beta \}$ (there is only one case for reading $\langle \epsilon \rangle$ since $[\frac{10}{2}]_\beta = [\frac{\epsilon}{2}]_\beta$). Note from Figure 4, to append an $\langle \epsilon \rangle$ object we read a symbol sequence of length $3x + 1$, and to append an $\langle \epsilon' \rangle$ object we read a symbol sequence of length $3x$. So when we read $y - 1$ symbols (instead of $y$) in an object as shown in Figure 3, we include an $\langle \epsilon' \rangle$ object instead of one of the $\langle \epsilon \rangle$ objects as this gives an object track that is one symbol shorter than the tracks read in Figure 2. Recall that only $3x$ symbols are read when $u$ is entered with a shift $\geq \beta - 3x$, and so the location of the $\langle \epsilon' \rangle$ object in the sequence of objects that are appended depends on which $u$ is entered with a shift $\geq \beta - 3x$. In Tables 3 and 4 we introduce the variable $j$ to denote the position of the $u$ word within the object that appends the $\langle \epsilon' \rangle$ object. In rows 2, 6, 8 and 12 in Table 3 and rows 7 and 11 in Table 4 the range of shift values for which $\langle \epsilon' \rangle$ gets appended by $u_{[u]}$ is less than the usual
range of \( \beta - 3x \leq s < \beta \). The reason for this is that the object being read is entered with a shift \( z \) that does not give all values in the range \( \beta - 3x \leq s < \beta \) for these cases. For example, in row 12 of Table 3 we have \( \beta - 2x < z + 3x^2 - 8 < \beta \) because it is not possible to have \( z + 3x^2 - 8 \leq \beta - 2x \) when \( \langle 0 \rangle \) is entered with shift \( z \geq \beta - z_1 \).

In this section we provided a method for showing that \( u \) is defined via Tables 2 to 4 such that each object appends the correct sequence of objects as shown in Figures 2 and 3. In Lemma 4 we prove the correctness of \( u \) by show that we have not assigned more than one value to the same track in the word \( u \).

### 3.1.4 Complexity analysis

We give the time analysis for \( \mathcal{T}_C \) simulating the cyclic tag systems \( C \) that runs in time \( t \). During the simulation, for every \( 3x - 2 \) objects from \{\( \langle 0 \rangle, \langle 1 \rangle \} \) that are read, we enter one of these objects with shift \( \geq \beta - z_1 \) (this is because \( z_1(3x - 2) = \beta \) and \{\( \langle 0 \rangle, \langle 1 \rangle \} \) have a shift change of \( z_1 \)). From Figure 3 (i) and (ii), if we enter an \( \langle 0 \rangle \) or an \( \langle 1 \rangle \) object with shift \( \geq \beta - z_1 \), then there is a single \( \langle e' \rangle \) object in the sequence of objects that are appended. So after reading \( t \) objects from \{\( \langle 0 \rangle, \langle 1 \rangle \} \) to simulate \( t \) steps of \( C \), we have \( O(t) \) of the \( \langle e' \rangle \) objects in the dataword of \( \mathcal{T}_C \). For each object read from \{\( \langle e' \rangle, \langle 0 \rangle, \langle 1 \rangle \} \), a constant number (independent of the input) of the \( \langle e \rangle \) objects are appended, and so we have \( O(t) \) of the \( \langle e \rangle \) objects in the dataword of \( \mathcal{T}_C \). There are \( O(t) \) objects from \{\( \langle 0 \rangle, \langle 1 \rangle \} \) in the dataword, and so the space used by \( \mathcal{T}_C \) is \( O(t) \). Between each pair of objects \( \langle w_1 \rangle, \langle w_2 \rangle \) \{\( \langle 0 \rangle, \langle 1 \rangle \} \) that encode adjacent symbols in the dataword of \( C \) there are \( O(t) \) of the \( \langle e \rangle \) and \( \langle e' \rangle \) objects. So the word \( \langle w_1 \rangle \{\langle e \rangle, \langle e' \rangle \}^* \) that is read to simulate a computation step, as described in the second last paragraph Section 3.1.1, has \( O(t) \) objects. From Figures 2 and 3 it takes a constant number of steps to read each object, and thus reading \( \langle w_1 \rangle \{\langle e \rangle, \langle e' \rangle \}^* \) to simulate a single computation step takes time \( O(t) \). So, \( \mathcal{T}_C \) simulates a single step of \( C \) in time \( O(t) \), and \( t \) steps of \( C \) in time \( O(t^2) \).

### 3.2 Correctness of \( \mathcal{T}_C \)

Note that in addition to the proof of correctness given here, \( \mathcal{T}_C \) was implement in software and tested extensively. Below, the correctness of \( \mathcal{T}_C \) is proved by showing that it correctly simulates an arbitrary computation step of the cyclic tag system \( C' \). In the third, fourth and fifth paragraphs of Section 3.1.1 is an overview of how \( \mathcal{T}_C \) reads a word of the form \( \langle w_1, z \rangle \{\langle e \rangle, \langle e' \rangle \}^* \) (where \( w_1 \in \{1, 0\} \)) to simulate a computation step of \( C' \). Lemma 3 shows that \( \mathcal{T}_C \) reads \( \langle w_1, z \rangle \{\langle e \rangle, \langle e' \rangle \}^* \) to correctly simulate an arbitrary computation step of \( C' \). The \( \mathcal{T}_C \) dataword immediately before the simulated computation step is given by Equation (7) and the \( \mathcal{T}_C \) dataword immediately after the simulated computation step is given by Equation (8). In Equation (8) the next encoded symbol to be read, \( \langle w_2 \rangle \), is at the left end of the dataword, and so after the simulated computation step the dataword has the correct form to begin the simulation of the next computation step. It follows that \( \mathcal{T}_C \) correctly simulates the computation \( C' \). Recall that cyclic tag systems end their computation by entering a repeating sequence of configurations. While \( \mathcal{T}_C \) correctly simulates this repeating sequence of configurations, \( \mathcal{T}_C \) itself does not enter a repeating sequence as the number of garbage objects in the dataword increases with each simulated computation step.

In Lemma 3 the objects \( \langle 1 \rangle, \langle 0 \rangle, \langle e \rangle, \langle e' \rangle \) are defined in Section 3.0.2 and the values \( z_1, z_2, m \) and \( d \) can be found in Table 1. Equations (7) and (8) encode arbitrary datawords of \( C' \) in the manner described by Equation (3).
Lemma 3 (\(T_C\) simulates an arbitrary computation step of \(C'\)). Given a dataword of the form
\[
\langle w_1, z \rangle \{\langle \epsilon, \epsilon' \rangle \}^* \langle w_2 \rangle \{\langle \epsilon, \epsilon' \rangle \}^* \langle w_3 \rangle \ldots \{\langle \epsilon, \epsilon' \rangle \}^* \langle w_1 \rangle \{\langle \epsilon, \epsilon' \rangle \}^*
\]  
(7)
where \(z = (z_1 m + z_2 d) \mod \beta\) and \(w_1 \in \{0, 1\}\), then a single round of \(T_C\) on the word \(\langle w_1, z \rangle \{\langle \epsilon, \epsilon' \rangle \}^*\) gives a dataword of the form
\[
\langle w_2, z' \rangle \{\langle \epsilon, \epsilon' \rangle \}^* \langle w_3 \rangle \ldots \{\langle \epsilon, \epsilon' \rangle \}^* \langle w_1 \rangle \{\langle \epsilon, \epsilon' \rangle \}^* w' \{\langle \epsilon, \epsilon' \rangle \}^*
\]  
(8)
where \(z' = (z_1 (m + 1) + z_2 d') \mod \beta\) and \(0 \leq d' < 3x + 1\), and
\[
w' = \begin{cases} 
\langle \alpha_m \rangle & \text{if } \langle w_1 \rangle = \langle 1 \rangle \text{ and } z < \beta - z_1 \\
\langle \alpha'_m \rangle & \text{if } \langle w_1 \rangle = \langle 1 \rangle \text{ and } z \geq \beta - z_1 \\
\langle \epsilon \rangle^{x+1} & \text{if } \langle w_1 \rangle = \langle 0 \rangle \text{ and } z < \beta - z_1 \\
\langle \epsilon \rangle^j (\langle \epsilon' \rangle \langle \epsilon \rangle)^{x-j} & \text{if } \langle w_1 \rangle = \langle 0 \rangle \text{ and } z \geq \beta - z_1
\end{cases}
\]  
(9)

Proof. We begin by showing that to prove this lemma it is sufficient to verify that \(T_C\) behaves as described in Figures 2 and 3. In (i) and (ii) of Figures 2 and 3 each dataword on left gives one of the four possible cases for Equation (7). These four cases are given by the four possible values for the pair \((\langle w_1 \rangle, z)\) which also determine the four cases in Equation (9). Note that if \(T_C\) behaves as described in the (i) and (ii) of Figures 2 and 3, then the correct value for \(w'\) is appended for each of the four cases in Equation (9). So proving the correctness of (i) and (ii) in Figures 2 and 3 verifies that the correct value for \(w'\) is appended.

From (i) and (ii) of Figures 2 and 3, when \(\langle w_1 \rangle \in \{\langle 1 \rangle, \langle 0 \rangle\}\) is entered with shift \(z = (z_1 m + z_2 d) \mod \beta\), the object immediately to the right is entered with a shift of the form \((z_1 (m + 1) + z_2 d) \mod \beta\). So from Equation (7), we enter the leftmost object in the word \(\{\langle \epsilon, \epsilon' \rangle \}^* \langle w_2 \rangle\) with shift \((z_1 (m + 1) + z_2 d) \mod \beta\). From Table 1 an \(\langle \epsilon \rangle\) has a shift change of 0 and \(\langle \epsilon' \rangle\) has a shift change of \(z_2\), and so from Lemma 1 each object in the word \(\{\langle \epsilon, \epsilon' \rangle \}^* \langle w_2 \rangle\) is entered with a shift of the form \(z' = (z_1 (m + 1) + z_2 d') \mod \beta\). Here \(0 \leq d' < 3x + 1\) since \(0 = z_2(3x + 1) \mod \beta\.

From (iii) and (iv) in Figure 2 and (iii) in Figure 3, when \(\langle \epsilon \rangle\) and \(\langle \epsilon' \rangle\) objects are entered with a shift of the form \(z' = (z_1 (m + 1) + z_2 d') \mod \beta\), they append only \(\langle \epsilon \rangle\) and \(\langle \epsilon' \rangle\) objects. (From Section 3.0.3, the range of values for \(m\) and \(d\) cover all possible shift values, and thus shift values of the from \((z_1 (m + 1) + z_2 d') \mod \beta\) are covered by the cases in Figures 2 and 3). So proving the correctness of (iii) and (iv) in Figure 2 and (iii) in Figure 3 verifies that reading the word \(\{\langle \epsilon, \epsilon' \rangle \}^*\) in Equation (7) appends a word of the from \(\{\langle \epsilon, \epsilon' \rangle \}^*\).

From the two paragraphs above it follows that if Figures 2 and 3 are correct, then given a dataword of the form shown in Equation (7), \(T_C\) produces a dataword of the form shown in Equation (8). We complete the proof of this lemma by demonstrating the correctness of Figures 2 and 3. In each line of Figures 2 and 3 the shift with which object \(\langle a_2 \rangle\) is entered follows immediately from the shift change caused by reading the leftmost object (see Lemma 1 and Table 1). For example in Figure 2 (i) an (1) is entered with shift \(z\), and because (1) has a shift change of \(z_1\), the object \(\langle a_2 \rangle\) immediately to the right is entered with shift \(z + z_1\). Given that we can show that \(\langle a_2 \rangle\) is entered with the correct shift for each case in Figures 2 and 3, it only remains to show that when each object is read the correct appendant gets appended. The method demonstrated in Section 3.1.3 can be applied to show that the word \(u\) is defined via Tables 2 to 4 such that when each object is read it appends the appendants as shown in Figures 2 and 3. In Section 3.1.3 the
method was applied to only the case in Figure 3 (i), however in Section 3.1.3 it was also explained how to apply the method to the remaining cases in Figures 2 and 3 and so we do not give them here.

In the previous paragraph we have shown how to verify that in each object the tracks for \( u \) in Tables 3 to 4 will append the appendants shown in Figures 2 and 3. To complete our proof we show that all possible tracks read in \( u \) are given in Tables 3 to 4, and that one and only one value has been assigned to each track. Using the method in paragraph 2 of Section 3.1.3 and Figure 5 (ii), one can to verify that the set of all tracks for an object entered with shift \( z \) appears in Tables 2 to 4. The entire range of values for \( z \) (given in Section 3.0.3) is covered in Tables 2 to 4. So all possible \( u \) tracks are given in Tables 2 to 4 and from Lemma 4 each \( u \) track in Tables 2 to 4 is assigned one and only one value.

The remainder of this section contains lemmas used in the proof of Lemma 3 and in explanations in Section 3.1. Before each lemma we briefly explain its significance for our algorithm.

Section 3.1.3 shows how to verify that in each object the tracks for \( u \) in Tables 3 to 4 will append the appendants shown in Figures 2 and 3. Lemma 4 shows that there are no contradictions in Tables 2, 3 and 4.

**Lemma 4.** Each track of the form \( \frac{u}{[s]} \) in Tables 2, 3, and 4 has been assigned one and only one value.

**Proof.** Below we state each case followed by the rows from Tables 2, 3, and 4 to which the case applies and then we give the proof of that case.

Case 1: Tracks of the form \( \frac{u}{[s]} \) for \( 0 \leq i < \frac{x}{2} - 7 \) (rows 14 to 18 in Table 2 and rows 1 to 7 in Table 4). Tracks of this form are used to append the encoding of \( \alpha_m \) (see Tables 2 and 4). There are three forms of encoding for \( \alpha_m \) (see Equations (4) to (6)). The encoding that is used depends on the shift \( z \) with which \( 1 \) is entered. Equation (4) is used when \( z < \beta - z_1 \) and the choice between Equations (5) and (6) depend on whether \( j < v \) or \( j \geq v \) (see third column of Table 4). Note from rows 7, 9 and 11 of Table 4 that the value of \( j \) depends on the value of \( z \). So the shift value \( z \) determines which of the encodings in Equations (4) to (6) to choose for \( \alpha_m \), and from Lemma 6 the shift value \( z = ((z_1m + 2d) \mod \beta) \) encodes \( \alpha_m \) and only \( \alpha_m \). It follows that there is one and only one encoding associated with each \( z \). From Lemma 5 each track of the form \( \frac{u}{[s]} \) for \( 0 \leq i < \frac{x}{2} - 7 \) is entered if and only if an \( 1 \) is entered with shift \( z \) and the \( i + 1 \)th \( u \) is being read. This means that track \( \frac{u}{[s]} \) will be read if and only if we are appending the \( i + 1 \)th object in the encoding of appendant \( \alpha_m \) as described in the third paragraph of Section 3.1.3. Since there is one and only one encoding associated with each \( z \) it follows that each track of the form \( \frac{u}{[s]} \) for \( 0 \leq i < \frac{x}{2} - 7 \) is assigned one and only one value.

Case 2: Tracks of the form \( \frac{u}{[s]} \) where \( s < \beta - 3x \) and \( s \neq (z + i(3x - 2) - 10) \mod \beta \) for \( 0 \leq i < \frac{x}{2} - 7 \) (rows 2, 5, 6, 7, 10, 11, 12, 19 and 20 in Table 2, rows 1, 3, 5, 7, 9, and 11 in Table 3, and rows 8 and 10 in Table 4). All \( u \) tracks for this case append only \( \langle \epsilon \rangle \) objects. This can be easily verified by checking the above mentioned rows. It follows that each track from this case is assigned one and only one value.

Case 3: Tracks of the form \( \frac{u}{[s]} \) where \( s \geq \beta - 3x \) and \( s \neq (z + i(3x - 2) - 10) \mod \beta \) for \( 0 \leq i < \frac{x}{2} - 7 \) (rows 1, 3, 4, 8, 9, 13 and 21 in Table 2, and rows 2, 4, 6, 8, 10 and 12 in Table 3 and rows 9 and 11 in Table 4). The \( u \) tracks entered with shift \( \geq \beta - 3x \) either append \( \langle \epsilon' \rangle \), or
result in one of the special cases given by rows 1, 3, 4, 8, and 21 of Table 2. Here we omit rows 9 and 13 as they define the same tracks as rows 4 and 8. No two of the cases from rows 1, 3, 4, 8, and 21 in Table 2 have the same underscript and thus no conflicts occurs when comparing one special case with another special case. Note that to see the difference between the underscripts in these special cases one should keep in mind that \( x \geq 14 \). For example, to show that the row 8 underscript \( \beta - 2x - 8 \) is not equal to the row 21 underscript \( \beta - 3x + 2 \) we must have \( x > 10 \). Given that no conflicts occurs between the special cases, it only remains to show that these special cases have no conflicts with tracks that append \( ' \). The u tracks that append \( ' \) are given by rows 2, 4, 6, 8, 10 and 12 in Table 3 and rows 9 and 11 of Table 4. (We need not consider row 7 of Table 4 as this was covered by Case 1.) For these cases comparing the underscripts \( \mod x \) shows that the values in the underscripts of some of the special cases differ from those that append \( ' \). For example, in Table 2 the underscript in row 8 gives \( x - 8 = (\beta - 2x - 8) \mod x \) (since \( 0 = \beta \mod x \)) and in Table 3 the underscript in row 4 gives \( x - 6 = (z + 3xi - 6) \mod x \) (since \( 0 = z \mod x \)). It follows that there is no conflict between these two rows as they define tracks in \( u \) for two different shift values. Comparing the underscript values \mod x \ does not work for all cases as some underscript values may have different shift values but the same values \mod x \. In each of these cases by looking at the range of values in the third column of the tables one sees that the shift values in the underscripts do not coincide. For example, in row 8 of Table 2 we have a shift \( \beta - 2x - 8 \) and in row 6 of Table 3 we have a shift \( z + 3x(1 - x) - 8 \). From the third column of the tables in Table 4, we have the range of values \( \beta - 2x < z + 3x(x - 1) - 8 \). Using this method one finds for the remaining cases (i.e. those not covered by comparing underscripts \mod x \) that no underscript for a \( u \) track from rows 1, 3, 4, 8, and 21 has the same value as an underscript for a \( u \) track given by rows 2, 4, 6, 8, 10 and 12 in Table 3 and rows 9 and 11 of Table 4. So tracks of the form \( u \) are assigned one and only one value, where \( s \geq \beta - 3x \) and \( s \neq (z + i(3x - 2) - 10) \mod \beta \) for \( 0 \leq i < \frac{x}{2} - 7 \). 

Recall from paragraph 3 of Section 3.1.3, that when an \( \langle 1 \rangle \) is entered with shift \( z = (z_1m + z_2d) \mod \beta \), the \( i + 1^{th} u \) read appends the \( i + 1^{th} \) object from the left in the encoding of \( \sigma_m = \sigma_1\sigma_2\ldots\sigma_v \). So the shift value for the track read in the \( i + 1^{th} u \) from the left when \( \langle 1 \rangle \) is entered with shift \( z = (z_1m + z_2d) \mod \beta \) must be unique. From Section 3.0.1, \( v \leq r < \frac{x}{2} - 7 \) and so we need not concern ourselves with values where \( i \geq \frac{x}{2} - 7 \). From Figure 5 (ii), for \( 0 \leq i < \frac{x}{2} - 7 \) track \( \left[ (z + i(3x - 2) - 10) \mod \beta \right] \) is read in the \( i + 1^{th} u \) from the left when \( \langle 1 \rangle \) is entered with shift \( z \). For this reason, in Lemma 5 we show for \( 0 \leq k < \frac{x}{2} - 7 \) that \( \left[ (z' + k(3x - 2) - 10) \mod \beta \right] \) is read in and only if \( \langle 1 \rangle \) is entered with shift \( z' \) and we are reading \( k + 1^{th} u \) from the left.

**Lemma 5.** Let \( z = (z_1m + z_2d) \mod \beta \) be the shift with which we enter \( \langle 1 \rangle \), \( \langle \epsilon' \rangle \), \( \langle 0 \rangle \), and \( \langle 1 \rangle \) objects. Then track \( \left[ (z' + k(3x - 2) - 10) \mod \beta \right] \) is read if and only if \( \langle 1 \rangle \) is entered with shift \( z' \) and the \( k + 1^{th} u \) from the left is being read. Here \( 0 \leq k < \frac{x}{2} - 7 \) and \( z' = (z_1m' + z_2d') \mod \beta \).

**Proof.** From Figure 5 (ii) and paragraph 2 of Section 3.1.3, we know that if an \( \langle 1 \rangle \) is entered with shift \( z' = (z_1m' + z_2d') \mod \beta \), then the track read in the \( k + 1^{th} u \) from the left is \( \left[ (z' + k(3x - 2) - 10) \mod \beta \right] \) where \( 0 \leq k < \frac{x}{2} - 7 \). To complete the proof we show that for any arbitrary track \( \left[ (z' + k(3x - 2) - 10) \mod \beta \right] \) that \( s \neq (z' + k(3x - 2) - 10) \mod \beta \) when \( \left[ (z' + k(3x - 2) - 10) \mod \beta \right] \) is not read in the \( k + 1^{th} u \) from the left in an \( \langle 1 \rangle \) entered with shift \( z' \). The values for \( s \) when entering the objects \( \langle 1 \rangle \), \( \langle \epsilon' \rangle \), \( \langle 0 \rangle \), and \( \langle 1 \rangle \) with shift \( z \) are given by the underscripts of the \( u \) tracks in the middle column of Tables 2 to 4. In these tables the value \( s \) in \( \left[ (z' + k(3x - 2) - 10) \mod \beta \right] \) is of the form \( s = (z + y) \mod \beta \) (for example \( s = (z - 2) \mod \beta \).
in rows 1 to 3 of Table 2). So to show $s \neq (z' + k(3x - 2) - 10) \mod \beta$ we prove $((z + y) \mod \beta) \neq ((z' + k(3x - 2) - 10) \mod \beta)$, which we rewrite as $(z - z') \mod \beta \neq ((k(3x - 2) - 10 - y) \mod \beta)$. Because $0 = \beta \mod x$ and $0 = (z - z') \mod x$ for all $z, z' \in \{(z_1m + z_2d) \mod \beta\}$ it is sufficient to show that

$$0 \neq (k(3x - 2) - 10 - y) \mod x$$

(10)

In Tables 2 to 4 the value $i$ denotes the position of the $u$ word read in an object. For example, $i = 0$ is the leftmost $u$ in the object, $i = 1$ is the second $u$ from the left and so on. Below we use the value $i$ to give the cases for $u[i]$ in each object.

Case 1: Reading track $u[i]$ in $\langle 1 \rangle$ when $0 \leq i < \frac{x}{2} - 7$. From Tables 2 and 4 we have $s = ((z + i(3x - 2) - 10) \mod \beta$, and from the previous paragraph we have $y = i(3x - 2) - 10$. When we substitute this value for $y$ in Equation (10) we get the inequality $0 \neq ((k - i)(3x - 2)) \mod x$ which holds when $k \neq i$, $0 \leq k < \frac{x}{2} - 7$ and $0 \leq i < \frac{x}{2} - 7$. Note that here we do not consider the case $k = i$ as this implies that $z = z'$ which means all the requirements (as given in the lemma statement) for reading $u[i]$ have been met.

Case 2: Reading track $u[i]$ in $\langle 1 \rangle$ when $\frac{x}{2} - 7 \leq i < x$. From Tables 2 and 4 we have $s = (z + 3xi - x + 4)$, and from paragraph 1 of this lemma we have $y = 3xi - x + 4$. When we substitute this value for $y$ in Equation (10) we get the inequality $0 \neq (k(3x - 2) - 3xi + x - 14) \mod x$ which holds for $0 \leq k < \frac{x}{2} - 7$.

Case 3: Reading track $u[i]$ in $\langle 1 \rangle$ when $i = x$. From Tables 2 and 4 we have $s = (z + 3x^2 - x + 2)$, and from paragraph 1 of this lemma we have $y = 3x^2 - x + 2$. When we substitute this value for $y$ in Equation (10) we get the inequality $0 \neq (k(3x - 2) - 3x^2 + x - 12) \mod x$ which holds for $0 \leq k < \frac{x}{2} - 7$.

Case 4: Reading track $u[i]$ in $\langle 0 \rangle$ or $\langle e' \rangle$ when $i = 0$. From Tables 2 and 3 we have $s = (z - 4)$, and from paragraph 1 of this lemma we have $y = -4$. When we substitute this value for $y$ in Equation (10) we get the inequality $0 \neq (k(3x - 2) - 6) \mod x$ which holds for $0 \leq k < \frac{x}{2} - 7$.

Case 5: Reading track $u[i]$ in $\langle 0 \rangle$ when $1 \leq i < x$ or in $\langle e' \rangle$ when $1 \leq i < x - 1$. From Tables 2 and 3 we have $s = (z + 3xi - 6)$, and from paragraph 1 of this lemma we have $y = 3xi - 6$. When we substitute this value for $y$ in Equation (10) we get the inequality $0 \neq (k(3x - 2) - 3xi - 4) \mod x$ which holds for $0 \leq k < \frac{x}{2} - 7$.

Case 6: Reading track $u[i]$ in $\langle 0 \rangle$ when $i = x$. From Tables 2 and 3 we have $s = (z + 3x^2 - 8)$, and from paragraph 1 of this lemma we have $y = 3x^2 - 8$. When we substitute this value for $y$ in Equation (10) we get the inequality $0 \neq (k(3x - 2) - 3x^2 - 2) \mod x$ which holds for $0 \leq k < \frac{x}{2} - 7$.

Case 7: Reading track $u[i]$ in $\langle e' \rangle$ when $i = x - 1$. From Tables 2 and 3 we have $s = (z + 3x(x - 1) - 8)$, and from paragraph 1 of this lemma we have $y = 3x(x - 1) - 8$. When we substitute this value for $y$ in Equation (10) we get the inequality $0 \neq (k(3x - 2) - 3x(x - 1) - 2) \mod x$ which holds for $0 \leq k < \frac{x}{2} - 7$.

Case 8: Reading track $u[i]$ in $\langle e \rangle$ when $i = 0$. From Table 2 we have $s = (z - 2)$, and from paragraph 1 of this lemma we have $y = -2$. When we substitute this value for $y$ in Equation (10) we get the inequality $0 \neq (k(3x - 2) - 8) \mod x$ which holds for $0 \leq k < \frac{x}{2} - 7$. □

The following lemma shows that each shift $z = (z_1m + z_2d) \mod \beta$ encodes one and only one appendant $a_m$. The variables in the Lemma statement are from Table 1.
Lemma 6. For each pair $z = ((z_1m + z_2d) \mod \beta)$ and $z' = ((z_1m' + z_2d') \mod \beta)$, if $m \neq m'$ then $z \neq z'$.

Proof. From the values in Table 1, we get $z_2(3x + 1) = z_1(3x - 2) = \beta$. Note that $0 \leq m < 3x - 2$ and $0 < d < 3x + 1$, and so we have $z_1m + z_2d < z_1(3x - 2) + z_2(3x + 1) = 2\beta$ (and similarly $z_1m' + z_2d' < 2\beta$). So if $z = z'$, then either $z_1m + z_2d = z_1m' + z_2d'$ or $z_1m + z_2d = z_1m' + z_2d' - \beta$ (here we can assume $z < z'$ as the argument is the same for $z' < z$). We rewrite these case as $z_2(d-d') = z_1(m'-m)$ and $z_2(d-d') = z_1(m'-m-3x+2)$. Note that $z_1(m'-m-3x+2) \neq 0$, and since $m \neq m'$ we also have $z_1(m'-m) \neq 0$, which means that for both cases $d \neq d'$. From the values in Table 1 we have $z_1 = x(3x + 1)$, and since $z_2$ and $3x + 1$ are relatively prime we get $0 \neq (z_2(d-d') \mod (3x + 1))$ for $d \neq d'$, $0 \leq d < 3x + 1$ and $0 \leq d' < 3x + 1$. It follows that the above equalities do not hold since $z_1(m'-m)$ and $z_1(m'-m-3x+2)$ are divisible by $3x + 1$ and $z_2(d-d')$ is not, and thus $z \neq z'$.

3.3 The halting problem for binary tag systems

Corollary 1. The halting problem for binary tag systems is undecidable.

Proof. In Theorem 2 the $u$ tracks at odd valued shifts are never read by $T_C$. So setting $u$ tracks at odd shifts to be sequences of all $b$ symbols causes no change in the simulation algorithm. $T_C$ simulates the cyclic tag system in [27] which has a special appendant $\alpha_h$ that is appended if and only if the Turing machine it simulates is halting. We can alter $T_C$ so that instead of appending $\alpha_h$ when $C$ halts, it appends an object of odd length so that all subsequent $u$ subwords are entered with an odd shift. This means that a single round on the tag system dataword changes everything to $b$ symbols. Now the rule $b \rightarrow b$, which appends one $b$ and deletes $\beta$ symbols, is repeated until number of symbols is $< \beta$ and the computation halts. So the computation halts if and only if the cyclic tag system is simulating a halting Turing machine.

4 The Post correspondence problem for 4 pairs of words

In Theorem 3 we show that the Post correspondence problem is undecidable for 4 pairs of words. Theorem 3 is proved by reducing the halting problem for the binary tag system given in Lemma 8 to the Post correspondence problem. The halting problem for the binary tag system in Lemma 8 is proved undecidability by simulating the cyclic tag system given in Lemma 7.

Definition 5 (Post correspondence problem). Given a set of pairs of words $\{(r_i,v_i)\}|r_i,v_i \in \Sigma^*, 0 \leq i \leq n\}$ where $\Sigma$ is a finite alphabet, determine whether or not there is a non-empty sequence $r_{i_1}r_{i_2} \ldots r_{i_l} = v_{i_1}v_{i_2} \ldots v_{i_l}$.

Lemma 7. Let $C = \alpha_0, \alpha_1, \ldots, \alpha_{p-1}$ be a cyclic tag system and let $w$ be an input dataword to $C$. Then there is a cyclic tag system $C_w$ that takes a single 1 as its input and simulates the computation of $C$ on $w$.

Proof. The binary dataword $w = w_1w_2w_3 \ldots w_n$ is encoded as $\langle w \rangle = w_10w_20w_30 \ldots w_n0$, and each binary appendant $\alpha_m = \sigma_1\sigma_2\sigma_3 \ldots \sigma_m$ in $C$ is encoded as $\langle \alpha_m \rangle = \sigma_10\sigma_20\sigma_30 \ldots \sigma_m0$. The program for $C_w$ is defined by the equation

$$C_w = \langle w \rangle, \langle \alpha_0 \rangle, \epsilon, \langle \alpha_1 \rangle, \epsilon, \langle \alpha_2 \rangle, \epsilon, \langle \alpha_3 \rangle, \ldots, \epsilon \langle \alpha_{p-1} \rangle$$
where $\epsilon$ is the empty word and $\langle \alpha_i \rangle$ is defined above. The configuration for $C_w$ at the start of the computation is given by

$$\langle w \rangle, \langle \alpha_0 \rangle, \epsilon, \langle \alpha_1 \rangle, \epsilon, \langle \alpha_2 \rangle, \ldots, \epsilon \langle \alpha_{p-1} \rangle$$

where the program is given on the left with the marked appendant $\langle w \rangle$ in bold, and the input dataword is a single 1 and is given on the right. After the first computation step we have

$$\langle w \rangle, \langle \alpha_0 \rangle, \epsilon, \langle \alpha_1 \rangle, \epsilon, \langle \alpha_2 \rangle, \ldots, \epsilon \langle \alpha_{p-1} \rangle$$

$w_1 w_2 w_3 0 \ldots w_n 0$

In the configuration above the encoding $\langle w \rangle = w_1 w_2 w_3 0 \ldots w_n 0$ of $w$ has been appended. Now the simulation of the first computation step of $C$ on $w$ begins. Every second appendant in $C_w$ is an $\langle \alpha_j \rangle$ appendant and every second symbol in the dataword of $C_w$ is a $w_i$ symbol. So $C_w$ on the input dataword 1 simulates the computation of $C$ on $w$. \hfill \Box

**Lemma 8.** The halting problem is undecidable for binary tag systems with deletion number $\beta$, alphabet $\{b, c\}$ and rules of the form $b \rightarrow b$ and $c \rightarrow u_1 \ldots u_1 b$ ($u_i \in \{b, c\}$), when given $u_\beta u_{\beta+1} \ldots u_1 b$ as input.

**Proof.** We use the tag system $T_C$ from Theorem 2 to construct a 2-symbol tag system $T_C'$ of the type mentioned in the lemma statement. From Lemma 7, we can assume without loss of generality that $T_C$ simulates cyclic tag systems whose input is a single 1.

Recall that $T_C$ has rules of the form $b \rightarrow b$ and $c \rightarrow u$ (where $u = u_0 \ldots u_l \in \{b, c\}^*$), and a deletion number $\beta$. In $T_C$ track $\vec{u}^{[0]} = u_0 u_\beta u_{2\beta} \ldots u_{3\beta}$ is never read, and so we can define $\vec{u}$ such that reading $u_\beta u_\beta u_\beta \ldots u_{3\beta}$ appends the word $\langle 1 \rangle' = b_{10} (ubb)_{10}^{\beta - 7} u_{10}^{\beta + 6} b_{10}^{\beta + 2}$. Note that $\langle 1 \rangle'$ is obtained from $\langle 1 \rangle = b_{10} (ubb)_{10}^{\beta - 7} u_{10}^{\beta + 6} b_{10}^{\beta + 2}$ by removing a single $u$ form the subword $u_{10}^{\beta + 7}$. Because each $u$ in the subword $u_{10}^{\beta + 7}$ appends a garbage object that has no effect on the computation, reading an $\langle 1 \rangle'$ simulates reading an $\langle 1 \rangle$. So given the input dataword $u_{10} (ubb)_{10}^{\beta - 7} u_{10}^{\beta + 6} b_{10}^{\beta + 2}$ the sequence $u_\beta u_\beta u_\beta \ldots u_{3\beta}$ is ready appending $\langle 1 \rangle'$ and the simulation of $C_w$ on input 1 is ready to begin.

Now we replace the rule $c \rightarrow u$ in $T_C$ with the rule $c \rightarrow u'$ (where $u' = u_1 \ldots u_1 b$) The new track that we defined above for appending $\langle 1 \rangle'$ is $\vec{u}^{[0]}$ with its first symbol $u_0$ deleted and for this reason we can delete $u_0$ from $u$ to give $u'$ as it is never read. The extra $b$ added at the right end of $u'$ means that $|u'| = |u|$ + 1. Following the replacement of $c \rightarrow u$ with $c \rightarrow u'$, we make a few minor changes which we detail below so that the simulation of $C_w$ on input 1 proceeds correctly.

The shift change from reading the input word $u_\beta u_{\beta+1} \ldots u_1 b$ is $3x - 1$ (as $|u_\beta u_{\beta+1} \ldots u_1 b| = |u| - \beta + 1$ and the shift change for $u$ is $3\beta$). Recall that reading $u_\beta u_{\beta+1} \ldots u_1 b$ appends $\langle 1 \rangle'$, so after reading the input $u_\beta u_{\beta+1} \ldots u_1 b$ we enter $\langle 1 \rangle'$ with the shift value $3x - 1$. Above to get $u'$ we deleted the leftmost symbol $u_0$ from $u$ and so every track in $u$ is shifted one symbol to the left in $u'$. A shift of $-1$ corrects for this and so the shift change of $3x - 1$ is in fact equivalent to a shift change of $3x$. A shift of $3\beta$ simulates that the marked appendant is at $\alpha_1$ instead of $\alpha_0$. To see this note from Table 1 that $3x = (z_1 m + z_2 d) \mod \beta$ for $d = 3x$ and $m = 1$, which encodes that $\alpha_m = \alpha_1$ is the marked appendant. If we alter the simulated cyclic tag system by taking the last appendant in the program and placing it at the start of the list of appendants, then every appendant get shifted one place to the right in the circular program. Now when we enter $\langle 1 \rangle'$ with shift $3\beta$, we are simulating the marker at the correct encoded appendant.

There is another problem to overcome. The object $\langle 1 \rangle'$ has one less $u$ than $\langle 1 \rangle$ and so has a different shift change to $\langle 1 \rangle$. From the values in Table 1, the object $\langle 1 \rangle' = b_{10} (ubb)_{10}^{\beta - 7} u_{10}^{\beta + 6} b_{10}^{\beta + 2}$
In Equation (11), if \( x \) has a shift change of \( z_2 = 3x^2 - x \), and such a shift change value simulates no change in the marked appended (see the end of Section 3.1.2). Recall that we are simulating \( C_w \), and so from Lemma 7, when we read the encoded \( \langle 1 \rangle' \) we append the encoding of the dataword \( w \). If we change the dataword so that we encode the word \( 0w \) (instead of \( w \)), then the extra encoded 0 is read before we be read the encoding of \( w \). The shift change caused by the extra encoded 0 simulates the marker moving to the next appendant so that we enter the encoding of \( w \) with the correct shift. Now that we have successfully appended the encoding of \( w \) the remainder of the computation of \( T'_C \) is simulated step by step.

Finally, the technique from Corollary 1 can be used to modify the above system so that it halts if an only if it is simulating a halting Turing machine. Note that because the tracks from \( u \) are shifted one place to the left in \( u' \) when we apply the technique from Corollary 1 we set the even tracks (instead of the odd tracks) in \( u' \) to be sequences of all \( b \) symbols. This completes our construction of \( T'_C \).

**Theorem 3.** The Post correspondence problem is undecidable for 4 pairs of words.

**Proof.** We reduce the halting problem for the binary tag system \( T'_C \) in Lemma 8 to the Post correspondence problem for 4 pairs of words. The symbols \( b \) and \( c \) in \( T'_C \) are encoded as \( \langle b \rangle = 10^\beta \) and \( \langle c \rangle = 1 \) respectively, where \( \beta \) is the deletion number of \( T'_C \). The halting problem for \( T'_C \) reduces to the Post correspondence problem given by the 4 pairs of binary words

\[
\mathcal{P} = \{(1,1\langle u_1 \rangle \langle u_2 \rangle \ldots \langle u_l \rangle 10), (10^\beta 1, 110), (10^\beta, \epsilon), (1,0)\}
\]

where \( \epsilon \) is the empty word and \( u_i \in \{0,1\} \). Let \( w = w_{i_1}w_{i_2} \ldots w_{i_l} \) and \( v = v_{i_1}v_{i_2} \ldots v_{i_l} \), where each \( (w_i,v_i) \in \mathcal{P} \) and \( w \) is a prefix of \( v \). We will call the pair \((w,v)\) a configuration of \( \mathcal{P} \). Because \( T'_C \) has an initial input word that ends in a \( b \) and both of the rules of \( T'_C \) append words that end in a \( b \) an arbitrary dataword of \( T'_C \) has the form \( x_0x_1 \ldots x_rb \) encoded as \( \langle b \rangle^*b \). The arbitrary dataword \( x_0x_1 \ldots x_rb \) is encoded by a \( \mathcal{P} \) configuration of the form

\[
(w,v) = (w,w_0\\beta 1x_1 \ldots x_r)10^\beta
\]

In each configuration \((w,v)\), the unmatched part of \( v \) (given by \( w_0\\beta 1x_1 \ldots x_r \)) encodes the current dataword of \( T'_C \).

We must have \((1,1\langle u_1 \rangle \langle u_2 \rangle \ldots \langle u_l \rangle 10)\) as the leftmost pair \((w_{i_1},v_{i_1})\) in a match as having any other pair from \( \mathcal{P} \) as the leftmost pair will not give a match. Starting from the pair \((1,1\langle u_1 \rangle \langle u_2 \rangle \ldots \langle u_l \rangle 10)\), if \( u_1 = c \) we add the pair \((1,0)\) and this matches \( \langle c \rangle = 1 \) simulating the deletion of \( u_1 \). If, on the other hand, \( u_1 = b \) we add the pair \((10^\beta,\epsilon)\) followed by the pair \((1,0)\) and this matches \( \langle b \rangle = 10^\beta 1 \) simulating the deletion of \( u_1 \). So after matching \( \langle u_1 \rangle \) we have \((1\langle u_1 \rangle,1\langle u_1 \rangle \langle u_2 \rangle \ldots \langle u_l \rangle 100)\). We match \( \beta - 1 \) encoded \( T'_C \) symbols in this way to give \((w,v) = (1\langle u_1 \rangle \ldots \langle u_{\beta-1} \rangle, 1\langle u_1 \rangle \langle u_2 \rangle \ldots \langle u_l \rangle 10^\beta)\). The configuration is now of the form given in Equation (11) and the unmatched sequence \( \langle u_{\beta} \rangle \ldots \langle u_l \rangle 10^\beta \) in \( v \) encodes the input dataword to \( T'_C \) in Lemma 8.

A computation step of \( T'_C \) on the arbitrary dataword \( x_0x_1 \ldots x_rb \) is of one of the two forms:

\[
\begin{align*}
\text{cx}_1 \ldots x_rb & \quad \lor \quad x_{\beta-1} \ldots x_rbu_1 \ldots u_ib \\
\text{bx}_1 \ldots x_rb & \quad \lor \quad x_{\beta-1} \ldots x_rbb
\end{align*}
\]

The two forms of computation step given in Equations (12) and (13) are simulated as follows: In Equation (11), if \( x_0 = c \) then \( \langle x_0 \rangle = 1 \) and we add the pair \((1,1\langle u_1 \rangle \langle u_2 \rangle \ldots \langle u_l \rangle 10)\) to simulate
the $\mathcal{T}_c$ rule $c \to u_1 \ldots u_kb$, and this gives $(w_1, w_1(x_1) \ldots (x_\ell) 10^31(u_1)\langle u_2 \rangle \ldots \langle u_\ell \rangle 10)$. In Equation (11), if $x_0 = b$ then $\langle x_0 \rangle = 10^31$ and we add the pair $(10^31, 110)$ to simulate the $\mathcal{T}_c$ rule $b \to b$, and this gives $(w10^31, w10^31(x_1) \ldots (x_\ell) 10^3110)$. In both cases $(x_0 = c$ and $x_0 = b$) to complete the simulation of the computation step we continue to match the pairs $(10^3, \epsilon)$ and $(1,0)$ as we did in the previous paragraph to simulate the deletion of a further $\beta - 1$ tag system symbols. Simulating the deletion of $\beta - 1$ symbols adds a further $\beta - 1$ of the 0 symbols at the right end of the encoded dataword. So if $x_0 = c$ this gives $(w1(x_1) \ldots (x_\ell - 1), w1(x_1) \ldots (x_\ell) 10^31(u_1)\langle u_2 \rangle \ldots \langle u_\ell \rangle 10^3)$, with the unmatched part in this pair encoding the dataword on the right of Equation (12) after the computation step. Alternatively, if $x_0 = b$ we get $(w10^31(x_1) \ldots (x_\ell - 1), w10^31(x_1) \ldots (x_\ell) 10^3110^3)$, with the unmatched part in this pair encoding the dataword on the right of Equation (13) after the computation step. The simulated computation step is now complete.

We now explain how $\mathcal{P}$ simulates $\mathcal{T}_c$ halting with a matching sequence. In $\mathcal{T}_c$ the rule $b \to b$ deletes $\beta$ symbols and append a single $b$ reducing the number of symbols in the dataword by $\beta - 1$, and the rule $c \to u_1 \ldots u_kb$ deletes $\beta$ symbols and appends $(3x + 1)\beta - 3x$ symbols ($|w_1 \ldots w_b| = |w'| = |u|$, see Table 1 and Lemma 8) increasing the number of symbols in the dataword by $3x(\beta - 1)$. So, because the input dataword $u_\beta \ldots w_b$ is of length $3x(\beta - 1) + 1$ and the rules either increase the length by $3x(\beta - 1)$ or decrease it by $\beta - 1$, all datawords of $\mathcal{T}_c$ have lengths $\frac{y}{\beta - 1} + 1$, where $y \in \mathbb{N}$. From Corollary 1, $\mathcal{T}_c$ halts when the length of its final dataword (which consists entirely of $b$ symbols) is less than the deletion number $\beta$. So, when $\mathcal{T}_c$ halts we have $y(\beta - 1) + 1 < \beta$ which means the dataword is a single $b$. From Equation (11), this is encoded as the configuration $(w, v) = (w, w10^3)$. By appending the pair $(10^3, \epsilon)$ to $(w, v)$, we get the pair of matching sequences $(w10^3, w10^3)$ when $\mathcal{T}_c$ halts. Note that whenever there is choice of which pair to append, only the choice that follows the simulation as described above has the possibility to lead to a match (all other choices lead to a mismatch). Therefore, $\mathcal{P}$ has a matching sequence if and only if $\mathcal{T}_c$ halts.

\subsection{Undecidability in simple matrix semi-groups}

Undecidability bounds for the Post correspondence problem have been used by a number of authors \cite{2, 3, 5, 6, 15, 17, 18, 29} in the search for undecidable decision problems in simple matrix semi-groups. The undecidability of the Post correspondence problem for 7 pairs of words \cite{24} has been frequently used to find undecidability in simple matrix semi-groups. Theorem 3 results in an immediate improvement on many of these results. Here we will just describe improvements for two of these problems. Of the decision problems on simple matrix semi-groups the mortality problem has in particular received much attention.

**Definition 6** (Matrix mortality problem). Given a finite set of $d \times d$ integer matrices \{\(M_1, M_2, \ldots, M_{n-1}, M_n\)\}, is there a product $M_1M_2 \ldots M_k$ that produces the zero matrix, where \(1 \leq i \leq n\)?

To date the best known bounds for the undecidability of the matrix mortality problem are due to Halava et al. \cite{17}. Improving on the reduction of Paterson \cite{29}, they showed that the matrix mortality problem is undecidable for sets with seven $3 \times 3$ matrices. Cassaigne and Karhumäki \cite{6} showed that if the mortality problem is undecidable for a set of $n$ matrices of dimension $d \times d$ then the mortality problem is undecidable for a pair of $nd \times nd$ matrices. So an immediate corollary of the result given by Halava et al. is that the mortality problem is undecidable a set of two $21 \times 21$ matrices. By applying the reductions in \cite{15} and \cite{6} to $\mathcal{P}$ in Theorem 3 we get Corollary 2.
Corollary 2. The matrix mortality problem is undecidable for sets with five $3 \times 3$ matrices and for sets with two $15 \times 15$ matrices.

Halava and Hirvensalo [18] give undecidability results for sets that consist of a pair of matrices with remarkably small dimensions. One of the problems they tackle is the scalar reachability problem which they prove undecidable for a pair of $9 \times 9$ matrices.

Definition 7 (Scalar reachability problem). Given a finite set of $d \times d$ integer matrices $\{M_1, M_2, \ldots, M_{n-1}, M_n\}$, a vector $y \in \mathbb{Z}^d$, $x^T$ the transpose of vector $x \in \mathbb{Z}^d$, and a constant $e \in \mathbb{Z}$, is there a product $x^T M_1 M_2 \ldots M_k y = e$?

By applying the reductions in [18] to $\mathcal{P}$ in Theorem 3 we get Corollary 3.

Corollary 3. The scalar reachability problem is undecidable for two $7 \times 7$ matrices.

Acknowledgements:
This work was supported by Science Foundation Ireland, grant number 09/RFP/CMS2212 and by Swiss National Science Foundation grant number 200021-141029. I would like to thank Matthew Cook and Damien Woods for their comments and discussions, and Vesa Halava and Mika Hirvensalo for their advice on undecidability in simple matrix semi-groups.

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