Four-qubit entangled symmetric states with positive partial transpositions

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We solve the open question of the existence of four-qubit entangled symmetric states with positive partial transpositions (PPT states). We reach this goal with two different approaches. First, we propose a half-analytical-half-numerical method that allows to construct multipartite PPT entangled symmetric states (PPTESS) from the qubit-qudit PPT entangled states. Second, we adapt the algorithm allowing to search for extremal elements in the convex set of bipartite PPT states [J. M. Leinaas, J. Myrheim, and E. Ovrum, Phys. Rev. A 76, 034304 (2007)] to the multipartite scenario. With its aid we search for extremal four-qubit PPTESS and show that generically they have ranks (5,7,8). Finally, we provide an exhaustive characterization of these states with respect to their separability properties.

Introduction.– Entanglement has become an important notion in modern physics [1]. This striking feature of composite physical systems not only fundamentally distinguishes classical and quantum theories, but it has also developed into a key resource for various applications. For instance, it allows for quantum teleportation [2], quantum cryptography [3], and is a prerequisite for another important resource in quantum information theory (QIT)—nonlocal correlations [4]. Deciding, then, if a given quantum state is entangled (i.e., if it is not a mixture of products of states representing individual subsystems [5]) has become one of the most important problems (the so-called separability problem) in QIT and, even if simple to formulate, it is one of the hardest to solve [6].

Due to the recent achievements in experimental implementations of various many-body states such as, for instance, the four-qubit bound entangled Smolin state [7], the six-qubit Dicke state states [8], or the eight-qubit Greenberger–Horne–Zeilinger (GHZ) state [9], the separability problem in quantum systems consisting of more than two constituents has gained importance. Here, the problem becomes even more complicated because one wants to answer not only the simple question of whether a particular state is entangled, but also what sort of entanglement it has (see Ref. [10]). Various approaches have been proposed to detect and characterize entanglement in such systems (see, e.g., Refs. [11–14] and a recent review [15]).

With this paper we fit into the above line of research and start a general program of characterization of entanglement properties and correlations of an important class of multipartite states—the so-called symmetric states1. These states have already been investigated (see, e.g., Refs. [16, 18–21]). More attention, however, has been devoted to pure states, while entanglement properties of mixed states are mostly unstudied. In particular, it remains uncertain if there exist four-qubit entangled symmetric states with all partial transpositions positive or, in other words, whether the separability condition based on partial transposition is necessary and sufficient in this case. It is known that PPT symmetric states of three-qubits are all separable [16], and existence of such states of five and six qubits has recently been reported [19, 20].

The main aim of the paper is to fill in this gap by showing, contrary to common belief, that there exist four-qubit PPTESSs. Then, we thoroughly study the entanglement properties of four-qubit PPT symmetric states.

Preliminaries and general entanglement properties of four-qubit symmetric states.– Let us start from a couple of definitions that will frequently be used throughout the paper. Consider a product Hilbert space $H_{d,N} = (\mathbb{C}^d)^{\otimes N}$ and a convex set $D$ of $N$-partite states $\rho$ acting on $H_{d,N}$. By $r(\rho)$, $K(\rho)$, and $R(\rho)$ we will be denoting the rank, kernel, and range of $\rho$. Also, the notations $(f_1, \ldots, f_{d-1})$ and $(e_1, \ldots, e_N)$ will be used to denote a vector $|f\rangle \in \mathbb{C}^d$ and a pure product vector from $H_{d,N}$, respectively.

We say that $\rho$ is fully separable if it can be written in the following form [5]:

$$\rho = \sum_{i} p_i \rho_{A_1}^i \otimes \cdots \otimes \rho_{A_N}^i, \quad p_i \geq 0, \quad \sum_{i} p_i = 1, \quad (1)$$

where $A_1, \ldots, A_N$ denote parties and $\rho_{A_i}^i$ are density matrices representing respective subsystems.

Then, splitting the parties $A_1, \ldots, A_N$ into two disjoint subsets $S$ and $\bar{S}$, we say that $\rho$ is PPT with respect to the bipartition $S|\bar{S}$ if and only if $\rho_{S|\bar{S}}^S \geq 0$ (note that the partial transpositions with respect to $S$ and $\bar{S}$ are equivalent definitions are also considered in such systems (see, e.g., Ref. [17] and references therein).

\footnote{In other words, states describing bosonic systems consisting of finite number of two-dimensional subsystems. Here we assume the usual definition of separability, with respect to the Hilbert space being a product of single-particle Hilbert spaces. Other...}
Roughly speaking, for a generic symmetric PPT state $\rho$ and its partial conjugation with respect to $S$ is in $R(\rho^T)$ for all $S$. Edge states are an important tool in the characterization of PPT entangled states because every PPT state can be decomposed into a convex combination of an edge state and a separable state, or, more precisely, edges states are those from which no separable states can be subtracted without losing the PPT property. Thus, they are crucial for the full characterization of entanglement in PPT states. More attention has been devoted to edge states in bipartite and three-partite systems (see Refs. [23, 24]), while little is known about them in $N$-partite systems. A very convenient way to classify and characterize edge states is to use their ranks together with the rank of all the relevant partial transposes, i.e., $(r(\rho), r(\rho^{T_A}), \ldots, r(\rho^{T_N})), r(\rho^{T_A} \otimes T_B), \ldots)$.

We can now pass to the $N$-qubit symmetric states. Let us focus on $H_{2,N}$ and denote by $S_N$ and $P_N$ the symmetric subspace of $H_{2,N}$ and a projector onto $S_N$. Recall, that $S_N$ is spanned by the unnormalized vectors $|E_N^N\rangle = \{|0, N-i\rangle, \{i, i\rangle\}$, where $\{|0, N-i\rangle, \{i, i\rangle\}$ is a symmetric vector consisting of $i$ ones and $N-1$ zeros. For further benefits, let us notice that $\dim S_N = N + 1$ and hence $S_N$ is isomorphic to $\mathbb{C}^{N+1}$. We then call a density matrix $\rho$ acting on $H_{2,N}$ symmetric iff $R(\rho) \subseteq S_N$.

In the case of symmetric states the number of relevant partial transpositions defining the set of PPT states can be significantly reduced. Clearly, positivity of a partial transposition with respect to a particular $S$ implies positivity of partial transposition with respect to all subsets $S$ with the same number of parties. Together with the equivalence of some of partial transpositions under the full transposition, this results in only $[N/2]$ of relevant partial transpositions. For concreteness, we choose them to be $T_A$, $T_{A_2}$, $T_{A_3}$, $T_{A_2} \otimes T_{A_4}$, etc. In the particular case of $N = 4$ there are only two of them, which, breaking the general notation, we will be denoting by $T_A$ and $T_{AB}$. Consequently, the set of four-qubit PPT symmetric states $D^\text{PPT}_{\text{sym}}$ is an intersection of three sets $D^\text{sym}$, $D^\text{sym}$, and $D^\text{sym}$. Accordingly, one has in this case only three relevant ranks $(r(\rho), r(\rho^{T_A}), r(\rho^{T_{AB}}))$, which, for the sake of simplicity, we call three-rank of $\rho$ and denote as $\bar{r}(\rho)$. Notice also that the fact that $\rho$ is symmetric imposes non-trivial bounds on $r(\rho)$, $r(\rho^{T_A})$, and $r(\rho^{T_{AB}})$. First of all, $\dim S_3 = 5$ implies $r(\rho) \leq 5$. Then, since $S_3$ is isomorphic to $\mathbb{C}^3$, while $S_2$ to $\mathbb{C}^3$, $r(\rho^{T_A}) \leq 8$ and $r(\rho^{T_{AB}}) \leq 9$.

Passing to the separability properties of PPT qubit symmetric states, it is known that for $N = 2$ and $N = 3$ all are separable. While the first case directly follows from the results of Ref. [25], for the second one, one uses the fact that $S_2$ is isomorphic to $\mathbb{C}^3$ and therefore $\rho$ can be seen as a PPT qubit-qutrit state. Again, the results of Ref. [25] apply here. Finally, it was shown in Ref. [16] that if $\rho$ is a symmetric $N$-qubit state and $r(\rho) \leq N$, then it takes the form (1). The first nontrivial, and so far unsolved case appears for $N = 4$. All PPT four-qubit symmetric states with $r(\rho) \leq 4$ are separable, however, it has not been known whether the same holds for $r(\rho) = 5$. In other words, it remains uncertain whether there are no PPT entangled four-qubit symmetric states and the partial transposition provides a necessary and sufficient criterion in this case. Our main aim is to show that this is not the case and there do exist examples of PPT entangled states supported on $S_1$.

Before getting to the construction, let us first discuss separability properties of four-qubit PPT symmetric states and single out all instances with respect to the three-rank when there are edge states. All the theorems proven below are left with sketches of proofs, while their detailed versions may be found in appendix A and Ref. [26].

Together with the already mentioned results of Ref. [16] we have the following theorem.

**Theorem 1.** Let $\rho$ be a four-qubit PPT symmetric state. If either $r(\rho) \leq 4$, or $r(\rho^{T_A}) \leq 4$, or $r(\rho^{T_{AB}}) \leq 3$ then $\rho$ is separable, while if $r(\rho^{T_A}) \leq 6$, or $r(\rho^{T_{AB}}) \leq 6$, then generic $\rho$ of such ranks is separable.

**Proof.** The cases of $r(\rho) \leq 4$ and of $r(\rho^{T_A}) \leq 4$ are proven in Ref. [16]. The remaining ones follow from the results of Refs. [27, 28], which say that a PPT state $\rho$ acting and supported on $\mathbb{C}^M \otimes \mathbb{C}^N$ ($M \leq N$) of rank $r(\rho) \leq N$ is separable. In the case of $r(\rho^{T_A}) \leq 3$ one treats $\rho^{T_{AB}}$ as a PPT state acting on $(\mathbb{C}^3)^{\otimes 2}$. In the cases of $r(\rho^{T_A}) \leq 6$ and of $r(\rho^{T_{AB}}) \leq 6$ one sees $\rho_T^{T_A}$ and $\rho_T^{T_{AB}}$ as bipartite PPT states acting and generically supported on $\mathbb{C}^2 \otimes \mathbb{C}^6$.

**Theorem 2.** Generic four-qubit PPT symmetric states of a given three-rank different from $(5, 7, 7)$ and $(5, 7, 8)$ are not edge.

**Proof.** Roughly speaking, for a generic symmetric PPT state $\rho$ of a particular three-rank $\bar{r}(\rho)$, except for $(5, 7, 7)$ and $(5, 7, 8)$, we find a symmetric product vector $|e\rangle \otimes 4 \in H_{2,4}$ in the support of $\rho$ such that $|e\rangle \otimes 4 \in R(\rho^{T_A})$, and $|e, e, e, e, e\rangle \in R(\rho^{T_{AB}})$. Clearly, due to theorem 1 most of the cases with respect to $\bar{r}(\rho)$ are already ruled out, $(5, 8, 9)$ is trivial, and those that need to be treated separately are $(5, 8, 7), (5, 8, 8), (5, 7, 9)$ (see appendix A).

Although using the above method we cannot prove that generic states of ranks $(5, 7, 7)$ are not edge, it is conjectured to be the case. More importantly, there is an indication that they are generically separable, but this will be studied elsewhere [26]. Also, exploiting the methods below we obtained examples of edge PPT states of ranks $(5, 7, 8)$; all the found examples of PPT states of ranks $(5, 7, 7)$ were separable.
Finally, let us prove that any entangled element of $D^\text{sym}_{4\text{-PPT}}$ can be decomposed in terms of at most six vectors of Schmidt rank two, i.e., entangled vectors that can be written as a sum of two fully product vectors.

**Theorem 3.** Any entangled four-qubit symmetric state $\rho$ can be written as

$$
\rho = \sum_{k=1}^{K} [A_k(a_k, b_k) \otimes^4 + B_k(a_k, -b_k) \otimes^4],
$$

where $K \leq 6$, $(a_k, b_k) \in \mathbb{C}^2$, $A_k, B_k \in \mathbb{C}$, and by $|\psi\rangle$ we denote a projector onto $|\psi\rangle$.

**Proof.** Applying a nonsingular transformation $V(a, b) \otimes^2 = (a^2, b^2)$ with $a, b \in \mathbb{C}$ to the last two qubits of a four-qubit symmetric PPT entangled state $\rho$, one brings it to a three-qubit PPT state $\sigma$ acting on $S_2 \otimes \mathbb{C}^2$. The latter is clearly separable and therefore can be written as a convex combination of at most six product rank-one projections [29], i.e.,

$$
\sigma = \sum_{k=1}^{K} |e_k]\langle e_k| \otimes |f_k]\langle f_k|,
$$

where the form of $|e_k\rangle \in S_2$ can determined from the orthogonality to $K(\sigma)$ and is given by $|e_k\rangle = A_k(a_k^2 a_k^* b_k^2 + B_k(a_k^2 a_k^* b_k^2))$, with $\mathbb{C}^2 \ni |f_k\rangle = [a_k^2, b_k^2]$ with $a_k, b_k, A_k, B_k \in \mathbb{C}$. To obtain (2) and complete the proof, one applies another full rank transformation $W(a, b) \otimes (a^2, b^2) = (a, b)^{\otimes 3}$ $a, b, \in \mathbb{C}$ to the last two qubits of $\sigma$. \hfill \Box

Using the methods developed in Ref. [29], one can obtain similar decomposition in which vectors $(a_k, -b_k)$ are replaced by $(0, 1)$ or $(1, 0)$ (see appendix A for the proof).

**Constructing four-qubit PPT entangled symmetric states.** We start by introducing a class of qubit-qudit PPT entangled states being a direct generalization of the $2 \otimes 4$ PPT entangled states introduced by Horodecki [30] (see Ref. [31] for generalizations of $3 \otimes 3$ Horodecki states). To this end, consider the density matrices

$$
\rho^d_{\text{insep}} = \frac{2}{2d-1} \sum_{i=0}^{d-2} |\Psi_i\rangle\langle \Psi_i| + \frac{1}{2d-1} |1, 0\rangle\langle 1, 0|,
$$

where $|\Psi_i\rangle = (1/\sqrt{2})(|0, i\rangle + |1, i+1\rangle)$ ($i = 0, \ldots, d - 2$).

Then, analogously to [30], for any $d \geq 2$, we define

$$
\rho_{d,b} = \frac{[(2d-1)b\rho^d_{\text{insep}} + |\Phi_0\rangle\langle \Phi_0|]}{(2d-1)b + 1}
$$

(0 \leq b \leq 1) (5)

with $|\Phi_0\rangle = (1/\sqrt{2})|0\rangle(\sqrt{1 - b}|0\rangle + \sqrt{1 + b}|d - 1\rangle)$. For $d = 4$, Eq. (5) gives the original states of Ref. [30].

Let us briefly characterize this class, and show that it maintains the separability and PPTness properties of the original $2 \otimes 4$ Horodecki state (see appendix B for more details). First, one checks that for $d \geq 2$ and $b \in [0, 1]$, $\rho^d_{a,b} = (I_d \otimes U)\rho_{d,b}(I_d \otimes U^\dagger)$, where $U$ is an anti-diagonal unitary operation consisting of unities, and therefore $\rho_{d,b}$ is PPT. Second, following the argumentation of Ref. [30], one can show (cf. appendix B) that for $d \geq 4$ and $b \in (0, 1)$, the states (5) are PPT entangled, and more importantly, they are edge, while for $b = 0$ or $b = 1$, or $d = 2, 3$ separable. Third, one has that $r(\rho_{d,b}) = r(\rho^d_{a,b}) = d + 1$.

With the aid of the states $\rho_{d,b}$ we can construct PPT symmetric entangled states. We will do that in few steps. First, we apply a full-rank transformation $F = \mathbb{I}_d - y|0\rangle\langle d - 1|$, where $y = \sqrt{(1 - b)/(1 + b)}$ to the second subsystem of $\rho_{d,b}$ so that the product vectors in the range of the resulting (unnormalized) state $\rho'_{d,b} = (\mathbb{I} \otimes F)\rho_{d,b}(\mathbb{I} \otimes F^\dagger)$ are given by

$$
(1, \alpha) \otimes (a^{d-1}, \ldots, \alpha, 1) \quad (\alpha \in \mathbb{C})
$$

(6)

(0, 1) \otimes (1, 0, \ldots, 0).

(7)

The above form of the product vectors in $R(\rho'_{d,b})$ is a key feature here because it allows one for a simple mapping of our states to many-qubit symmetric states.

Second, to the same subsystem, we apply a nonsingular $d' \times d$ ($d' < d$) matrix

$$
F_2 = \sum_{i=0}^{d'} \sum_{j=0}^{d'} \gamma_{ji}|i + j\rangle,
$$

where $\gamma_i (i = 0, \ldots, d' - d')$ are some complex parameters. By doing so, we obtain another class of PPT states $\rho''_{d',b} = (\mathbb{I} \otimes F_2)\rho'_{d,b}(\mathbb{I} \otimes F_2^\dagger)$ which act on $\mathbb{C}^2 \otimes \mathbb{C}^{d'}$ with $d < d'$ but have additional $d - d' + 1$ parameters $\gamma_i$. The transformation $F_2$ is chosen in such a way that it allows to introduce additional parameters preserving the form of the product vectors in the range of the resulting states. Precisely, the product vectors in $R(\rho''_{d',b})$, although living in a smaller-dimensional Hilbert space, are of the form (6). Noticeably, since we are only using local operations, the resulting states are also edge.

Third, we apply yet another local operation, denoted $V$, which maps the local vectors $(\alpha^{d-1}, \ldots, \alpha, 1)$ to $(1, \alpha)^{(d-1)}$. Clearly, $V$ is of full rank because in both vectors the same powers of $\alpha$ appear. By applying $V$ to the second subsystem of $\rho''_{d',b}$ we simply obtain $(N = d')$-qubit symmetric states $\omega_N$, which by the very construction, have all one-particle partial transpositions positive.

As a result the above three filters allow us to construct a family of many-qubit symmetric entangled states from the generalized Horodecki states, which are our starting point for searching for PPT entangled symmetric states. However, the states $\omega_N$ have in general nonpositive partial transpositions except for the single-partite ones. To overcome this, we can consider another state $\omega_{N,\lambda} = \omega_N + \lambda P_N$, where $P_N$ stands for the projector onto the symmetric subspace $S_N$. Clearly there exists the smallest $\lambda$, denoted $\lambda_*$, such that $\omega_{N,\lambda_*}$ is PPT. Although it keeps the rank of the state constant, this operation, however, inevitably increases the ranks of the
partial transpositions most probably destroying the entanglement of the resulting states. To lower them we can search for symmetric product vectors \(|e|^{\otimes N} \in H_{2,N}\) such that their respective partial conjugations belong to all the ranges of the relevant partial transpositions \(R(\omega_{\Lambda}^{T_N,i}) (i = A, AB, \ldots)\). Every such vector can be removed from \(\omega_{\Lambda}^{N,i}\), i.e., we consider a state \(\tilde{\omega}_{\Lambda}^{N,i} = \omega_{\Lambda}^{N,i} - \mu P_{|e|^{\otimes N}}\), where \(P_{|e|^{\otimes N}}\) denotes a projector onto \(|e|^{\otimes N}\). By properly choosing \(\mu\) and the product vector \(|e|^{\otimes N}\) provided they exist, we can lower some of the ranks of partial transpositions of \(\omega_{\Lambda}^{N,i}\).

Let us now follow the above general recipe and get the aforementioned four-qubit symmetric PPT entangled states. To this end, we take \(\rho_{5,6}^{b} = (1 \otimes F)\rho_{5,6}^{b}(1 \otimes F^{t})\) [cf. Eq. (5)], and, following the above description, apply the local filter \(F_{2}\), which is now \(4 \times 4\) matrix with two parameters \(\gamma_{1}\) and \(\gamma_{2}\), and subsequently, the next filter \(V\). This results in a family of four-qubit symmetric states \(\omega_{4,\Lambda}^{A,B,\gamma_{1},\gamma_{2}}\) parameterized by \(b\) and \(\gamma_{i}\) (\(i = 1, 2\)). In order to get a particular example of four-qubit PPTESS, let us put \(b = 1/2\) and \(\gamma_{1} = \gamma_{2}^{-1} = 1/\sqrt{2}\), which leads us to the state \(\omega_{4,\Lambda}^{\gamma_{1},\gamma_{2}}\) such that \(\omega_{4,\Lambda}^{A,B} \geq 0\), while \(\omega_{4,\Lambda}^{N} \geq 0\). To “cover” the negative eigenvalues of \(\omega_{4,\Lambda}^{T_{A,B}}\), we consider \(\omega_{4,\Lambda}^{\gamma_{1},\gamma_{2}}\). The minimal \(\Lambda\) for which \(\omega_{4,\Lambda}^{N} \geq 0\) is \(\omega_{4,\Lambda}^{N} \geq 0\). However, the three-rank of \(\omega_{4,\Lambda}^{\gamma_{1},\gamma_{2}}\) is \((5,8,8)\). We can then lower the second rank by subtracting a product vector \(|e|^{\otimes 3} \in R(\omega_{4,\Lambda}^{\gamma_{1},\gamma_{2}})\) such that \(|e|^{\otimes 3} \in R(\omega_{4,\Lambda}^{\gamma_{1},\gamma_{2}})\) and \(|e^{*}, e^{*}, e, e\rangle \in R(\omega_{4,\Lambda}^{\gamma_{1},\gamma_{2}})\). Again, exploiting numerics, we find that there exists such a vector \(|e\rangle = (1, \alpha)\), where \(\alpha \approx 7 + 38.520916\). Then, one checks that for \(\mu \approx 0.64625\), \(\omega_{4,\Lambda}^{\gamma_{1},\gamma_{2}} - \mu P_{|e|^{\otimes 3}}\) after normalization is the expected example of four-qubit symmetric PPT entangled state with three-rank \((5,7,8)\). Using the algorithm below one can check that the state is extremal, and thus also edge. It should be noticed that the above choice of parameters \(\gamma_{1}, b\) was made for simplicity, but other choices can also lead to PPT entangled states (e.g., \(\gamma_{1} = 3/8, \gamma_{2} = 11/23, b = 1/6\)).

**Searching for extremal PPT entangled four-qubit symmetric states.** We have just shown that four-qubit symmetric PPT entangled states exist. Clearly, there must then exist extremal entangled elements in the corresponding set of PPT states \(D_{PPT}^{sym}\). Our aim now is to search for such states and characterize them. For this purpose we adopt the algorithm for searching of extremal elements in the set of PPT states, originally proposed for bipartite systems [32] (see also Ref. [33]), to the multipartite scenario. Then, we will apply it to \(D_{PPT}^{sym}\).

Let us consider again the Hilbert space \(H_{4,N} = (\mathbb{C}^{d})^{\otimes N}\) and the set \(D_{PPT}\) of all PPT states acting on \(H_{4,N}\). Let \(\rho \in D_{PPT}\) and let \(\mathcal{P}\) and \(\mathcal{P}_{k}\) denote projectors onto, respectively, \(R(\rho)\) and \(R(\rho^{T_{k}})\) \((k = 1, \ldots, M)\), where \(M\) denotes the number of independent partial transpositions (recall that some of them are equivalent under the full transposition).

The state \(\rho\) is extremal in \(D_{PPT}\) iff it cannot be written as \(\rho = p\rho_{1} + (1 - p)\rho_{2}\) for some \(\rho_{1} \in D_{PPT}\) and \(0 < p < 1\). This is equivalent to say that \(\rho\) is not extremal iff there exists a density matrix \(\sigma \neq \rho\) such that \(R(\sigma) \subseteq R(\rho)\) and \(R(\rho^{T_{k}}) \subseteq R(\rho^{T_{k}})\) for all \(k\). One can even relax the assumption of \(\sigma\) being positive to be Hermitian and such Hermitian matrices are solutions to system of equations

\[
\hat{P}_{k}(h) = h \quad (k = 0, \ldots, M),
\]

where \(\hat{P}_{k} = \mathcal{P}(\cdot)\mathcal{P}_{k}(\cdot)\) and \(\hat{P}_{k}(\cdot) = [\mathcal{P}_{k}(\cdot)T_{k}\mathcal{P}_{k}(\cdot)_{k} (k = 1, \ldots, M)\). This system is equivalent to the single equation \(\hat{P}_{M} \circ \ldots \circ \mathcal{P}_{1} \circ \mathcal{P}(h) = h\). Clearly, \(\rho\) is a particular solution of the system (9) and, due to the above statements, is extremal iff it is its only solution, which gives necessary and sufficient condition for extremality [32, 33]. This also leads to a simple necessary criterion for extremality. Precisely, each equation in (9) imposes \(d^{2N} - [r(\rho^{T_{k}})]^{2}\) linear constraints on the matrix \(h\). The maximal number of conditions imposed by the system (9) is then \(\sum_{k}[d^{2N} - [r(\rho^{T_{k}})]^{2}]\). Then, a Hermitian matrix \(h\) has \(d^{2N}\) real parameters, and hence if

\[
\sum_{k=0}^{M} [r(\rho^{T_{k}})]^{2} \geq Md^{2N} + 1,
\]

the state \(\rho\) is not extremal.

All this induces a method of searching for the extremal states in \(D_{PPT}^{sym}\) [32]. Taking \(\rho \in D_{PPT}\), one solves the corresponding system (9). If the latter has a solution \(h \neq \rho\), the state \(\rho\) is not extremal. One then considers a family of matrices \(\rho(x) = (1 - x Trh)\rho + xh\) with \(x\) being in general a real parameter. Clearly, there is \(x = \alpha\) such that \(\rho_{2} = \rho(\alpha)\) is a still PPT state, however, either \(r(\rho_{2}) = r(\rho) - 1\) or \(r(\rho_{2}^{T_{k}}) = r(\rho^{T_{k}}) - 1\) for some \(k\). We can again apply the above procedure to \(\rho_{2}\), and in case it is not extremal get another PPT state \(\rho_{3}\) with at least one of the ranks diminished by one. We keep applying this procedure until we obtain an extremal state, which appears in a finite number of repetitions. If the resulting state is pure, it is separable, otherwise it is entangled. Notice that in order to get a particular extremal entangled state one has to properly choose the initial state which basically means that it should be of higher ranks, as for instance the maximally mixed state, and the directions which follow from solving Eqs. (9).

Let us apply the above algorithm to the symmetric states. In this case the left-hand side of (10) has to be modified as \(\rho = \rho_{PPT}(X = A, AB, \ldots)\) are supported on Hilbert spaces of different dimensions. In particular, for \(N = 4\) such analysis was done in Ref. [33] and it follows that states of ranks \((5, 7, 9), (5, 8, 8)\) cannot be extremal. Moreover, theorems (1) and 2 imply that generic states of ranks \((5, 7, 8)\) and \((5, r(\rho^{T_{k}}), r(\rho^{T_{k}}))\) with \(r(\rho^{T_{k}})\) or \(r(\rho^{T_{k}}) \leq 6\) also cannot be extremal. The natural candidates for extremal states have then ranks \((5, 7, 7)\) and \((5, 7, 8)\).

We applied the above algorithm to four-qubit PPT states and all the extremal examples we found have ranks...
(5,7,8). In 30 000 runs we generated 5760 unitarily nonequivalent extremal entangled states and all of them have ranks (5, 7, 8). As an initial state we took the projector \( P_3 \) onto \( S_3 \) (recall that the initial state has to be of rank five and due to theorem 1 must also have appropriately high ranks of \( P^X \) (X = A, AB)). At each step of the algorithm we used solutions of (9) chosen so that we could reach one of the three-ranks not excluded by theorem 2. We also got 24 240 states of ranks (5, 7, 7) in this way but they all are separable.

**Conclusion.** The main aim of this note was to solve the open question of the existence of four-qubit PPT entangled symmetric states. We have reached this goal by proposing a half-analytical-half-numerical method of constructing of such states. The analytical part of the method allows one to map a class of qubit-qudit PPT entangled states onto many-qubit entangled symmetric states. Then, using already-well-established methods of the theory of entanglement, and with the help of numerics, we have found the desired PPT entangled states.

We have also characterized the four-qubit PPT symmetric states with respect to separability, edgeness and extremality properties. First, we have proven that generic states of a given three-rank different from (5, 7, 7) or (5, 7, 8) are not edge. Then, by adapting to the multipartite case an algorithm allowing to search for extremal PPT states [32], we have sought extremal four-qubit PPT entangled symmetric states. All the entangled states found in this way have ranks (5, 7, 8), while those with ranks (5, 7, 7) encountered in this way are separable (see Ref. [26] for more details).

Interestingly, all the methods presented in this paper can be applied to \( N \)-qubit symmetric states. For instance, we have shown that with our method one can obtain five-qubit and six-qubit PPT entangled symmetric states, confirming the findings of Refs. [19, 20]. Generalization of these findings to \( N \)-qubit symmetric Hilbert spaces is currently being studied and will be a subject of a forthcoming publication [26].

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Appendix A: Characterization of four-qubit symmetric states

Here we recall theorems 1, 2, and 3 and prove them in detail.

Theorem 1. Let $\rho$ be a four-qubit PPT symmetric state. If either $r(\rho) \leq 4$, or $r(\rho^{T_A}) \leq 4$, or $r(\rho^{T_{AB}}) \leq 3$ then $\rho$ is separable, while if $r(\rho^{T_A}) \leq 6$, or $r(\rho^{T_{AB}}) \leq 6$, then generic $\rho$ is separable.

Proof. Although the first two cases of $r(\rho) \leq 4$ and $r(\rho^{T_A}) \leq 4$ were already proven in Ref. [16] let us, for completeness, recall their proofs. First, due to the fact that $S_3 \cong \mathbb{C}^4$, one can always treat $\rho$ as a qubit-ququart PPT state. Then, the results of Ref. [27] say that any qubit-ququart PPT state of rank $r(\rho) \leq 4$ is separable. Replacing then $\rho$ by $\rho^{T_A}$ and following the same arguments, one proves the case of $r(\rho^{T_A}) \leq 4$.

In order to prove the case of $r(\rho^{T_{AB}}) \leq 3$, one considers a state $\sigma = \rho^{T_{AB}}$ and exploits the fact that $S_2$ is isomorphic to $\mathbb{C}^3$. Then, $\sigma$ is a two-qutrit PPT state such that $r(\sigma) \leq 3$, and it was shown in Ref. [28] that any two-qutrit PPT state of rank less or equal to three is separable.

In the case of $r(\rho^{T_A}) \leq 6$ let us define $\sigma = \rho^{T_A}$ and consider it as a bipartite state with respect to the partition $B|ACD$. Clearly, in such case $\sigma$ acts on $\mathbb{C}^2 \otimes \mathbb{C}^6$ and is of rank at most six. Provided that it is supported on $\mathbb{C}^2 \otimes \mathbb{C}^6$, which generically is the case, the results of Ref. [27] tell us that $\sigma$ is separable across $B|ACD$, i.e.,

$$\sigma = \rho^{T_A} = \sum_i p_i |e_i\rangle\langle e_i|_B \otimes |\psi_i\rangle\langle \psi_i|_ACD,$$  \hspace{1cm} (A1)

where, for the time being, $|\psi_i\rangle$ are entangled states from $\mathbb{C}^2 \otimes S_2$. On the other hand, the $BCD$ subsystem of $\sigma$ is still supported on the three-qubit symmetric subspace. As a result, any vector $|e_i\rangle\langle \psi_i|_ACD$ in the decomposition (A1) must obey $P_3|e_i\rangle\langle \psi_i|_ACD = |e_i\rangle\langle \psi_i|_ACD$, where $P_3$ is applied to $BCD$ subsystem. This, after some algebra, implies that $|\psi_{i\langle CD}| = |f_{i\langle}_A|e_i\rangle_C|e_i\rangle_D$ for some $|f_{i\langle}\rangle \in \mathbb{C}^2$, and hence $\sigma$, and accordingly $\rho = \sigma^{T_A}$ are fully separable.

To prove the last case of $r(\rho^{T_{AB}}) \leq 6$ one follows the same lines as before substituting $\rho^{T_{AB}}$ for $\rho^{T_A}$.

Theorem 2. Generic four-qubit PPT symmetric states of a given three-rank different from $(5,7,7)$ and $(5,7,8)$ are not edge.

Proof. We will show that in all relevant cases with respect to the three-rank, except for $(5,7,7)$ and $(5,7,8)$, there exists a symmetric product vector $|e\rangle^{\otimes 3} \in R(\rho)$ such that $|e^{*}\rangle^{\otimes 3} \in R(\rho^{T_A})$, and $|e^{*},e,e,e\rangle \in R(\rho^{T_{AB}})$. Clearly, many of the three-ranks can be ruled out with the aid of theorem 1, and the remaining ones are $(5,7,7), (5,7,8), (5,8,7), (5,8,8), (5,7,9),$ and $(5,8,9)$. The last one is trivial because all symmetric product vectors $|e\rangle^{\otimes 3}$ belong to $R(\rho)$ and their respective partial conjugations to $R(\rho^{T_A})$ and $R(\rho^{T_{AB}})$. In what follows we give a proof for the cases $(5,8,7), (5,8,8), (5,7,9)$.

In the case of $\rho = (5,7,9)$, $\rho$ and $\rho^{T_{AB}}$ are of full rank and therefore one has to find a product vector $|e^{*},e,e,e\rangle$ which is orthogonal to the only vector $|\Psi\rangle$ from $K(\rho^{T_A})$. To this end, let us write $|\Psi\rangle = |0\rangle|\psi_0\rangle + |1\rangle|\psi_1\rangle$ with $|\psi_0\rangle \in S_3$, and take $|e\rangle = (1,0)\langle 0\rangle (a \in \mathbb{C})$. The orthogonality condition then reads $V_3(a) + a^*W_3(a) = 0$ with $V_3$ and $W_3$ denoting polynomials of degree at most three over the complex field. In Ref. [29], this equation was shown to have generically (both polynomials $V_3$ and $W_3$ are of degree three) at least one solution. Consequently, generic four-qubit PPT states of of ranks $(5,7,9)$ are not edge.

In the case of $\rho = (5,8,8)$, $\rho$ and $\rho^{T_{AB}}$ are of full rank, and so one has to find a product vector $|e^{*},e,e,e\rangle$ with $|e\rangle \in \mathbb{C}^2$ orthogonal to the only vector $|\Psi\rangle$ from the kernel of $\rho^{T_{AB}}$. For this purpose, let us notice that $\rho^{T_{AB}} = G(\rho^{T_{AB}})^G$, where $G$ denotes an operator swapping subsystems $AB$ and $CD$. This means that $|\Psi\rangle$ enjoys the same symmetry, i.e., $G|\Psi\rangle = |\Psi^*\rangle$. As a result, one can express it as

$$|\Psi\rangle = \sum_{k=1}^{3} \lambda_k |e_k\rangle_{AB}|e_k\rangle_{CD},$$  \hspace{1cm} (A2)

where $\lambda_k \in \mathbb{R}$ and $|e_k\rangle$ are orthogonal symmetric two-qubit vectors. Exploiting the fact that $|\Psi\rangle \in K(\rho^{T_{AB}})$, one sees that

$$\frac{K}{k=1} \lambda_k (x^{*},y|\rho^{T_{AB}}|e_k,e_k) = \frac{K}{k=1} \lambda_k (e_k,y|x,e_k)$$

$$= \frac{K}{k=1} \lambda_k (e_k,x|\rho|e_k,y) = 0$$  \hspace{1cm} (A3)

holds for any pair of qubit vectors $|x\rangle$ and $|y\rangle$ with the second equality stemming from the fact that $\rho$ is symmetric and hence $\rho = \rho G$. This immediately implies that

$$\frac{K}{k=1} \lambda_k (e_k|\rho|e_k) = 0,$$  \hspace{1cm} (A4)

where the right-hand side is a two-qubit matrix acting on the $CD$ subspace, obtained by ”sandwiching” $\rho$ with $|e_k\rangle$s on the first two qubits.

On the other hand, taking into account Eq. (A2), there exists $|e\rangle \in \mathbb{C}^2$ such that $|e^{*},e,e,e\rangle \in R(\rho^{T_{AB}})$ iff

$$\langle e,e| \sum_k \lambda_k |e_k\rangle\langle e_k| |e,e\rangle = 0.$$  \hspace{1cm} (A5)

In order to show that such $|e\rangle$ exists, assume, in contrary, that Eq. (A3) does not hold for any $|e\rangle \in \mathbb{C}$. Then, its left-hand side must have the same sign for all $|e\rangle$, say
positive (as otherwise, from continuity, there would exist \( |e \rangle \) for which (A5) holds). Consequently,

\[
\langle e, e \rangle \left[ \sum_k \lambda_k |e_k\rangle \langle e_k| \right] |e, e\rangle > 0 \tag{A6}
\]

for any \( |e \rangle \in \mathbb{C}^2 \), which, owing to the fact that \( |e_k\rangle \) are symmetric, implies that \( W = \sum_k \lambda_k |e_k\rangle \langle e_k| \) is a two-qubit entanglement witness. Since all two-qubit witnesses are decomposable, we have \( W = P + Q_T^A \) with \( P, Q \geq 0 \). This, when substituted to Eq. (A4), implies that the two conditions

\[
\text{Tr}[(P \otimes |y\rangle \langle x|)\rho] = 0, \quad \text{Tr}[(Q \otimes |y\rangle \langle x|)\rho_T^A] = 0 \tag{A7}
\]

must be obeyed for any \( |x\rangle, |y\rangle \in \mathbb{C}^2 \), contradicting the fact that \( \rho \) and \( \rho_T^A \) are of full rank. Hence, this proof is general (not generic) meaning that there are no four-qubit symmetric edge states of ranks \((5,8,8)\).

Let us now pass to the most involving case of \( \mathfrak{r}(\rho) = (5,8,7) \). Here \( \rho \) and \( \rho_T^A \) are of full rank, while \( K(\rho_T^{AB}) \) has dimension two. Hence, to find a product vector \( |e\rangle \otimes 1 \in R(\rho) \) such that \( |e^*, e^*, e, e\rangle \in R(\rho_T^{AB}) \), one has to solve two equations \( |e^*, e^*, e, e\rangle \Psi_i = 0 \) \((i = 1,2)\), where \( |\Psi_i\rangle \) are two orthogonal vectors from \( K(\rho_T^{AB}) \). Exploiting again the identity \( \rho_T^{AB} = G(\rho)^{T_A} \sigma^1 \), it is fairly easy to see that they can be written as

\[
|\Psi_1\rangle = \sum_{k=1}^2 \lambda_k |e_k\rangle |f^*_k\rangle, \quad |\Psi_2\rangle = \sum_{k=1}^2 \lambda_k |f_k\rangle |e^*_k\rangle. \tag{A8}
\]

To see it explicitly, let us first notice that we can assume that one of \( |\Psi_i\rangle \) is of Schmidt rank two. If both of them are of rank three, then there exists a vector of Schmidt rank two in span\{\( |\Psi_1\rangle, |\Psi_2\rangle \}\}. On the other hand, if one of \( |\Psi_i\rangle \) \((i = 1, 2)\) is of rank one, i.e., is product with respect to the partition \( AB \big| CD \), \( \mathfrak{r}(\rho) = 4 \) contradicting the assumption that \( \rho \) is entangled. Assume then \( |\Psi_1\rangle \) is of rank two. Then either \( G|\Psi_1\rangle \) is linearly independent of \( |\Psi_1\rangle \) leading to Eq. (A8), or \( G|\Psi_1\rangle = \xi |\Psi_1\rangle \) for some \( \xi \in \mathbb{C} \). In the latter case, short algebra implies that \( |\Psi_1\rangle \) \((i = 1, 2)\) are not linearly independent contradicting the fact that they span two-dimensional kernel of \( \rho_T^{AB} \).

As a result, finding a vector \( |e^*, e^*, e, e\rangle \in R(\rho_T^{AB}) \) is equivalent to solving an equation

\[
V(\alpha^*) W(\alpha) + \bar{V}(\alpha^*) \bar{W}(\alpha) = 0, \tag{A9}
\]

where \( V, \bar{V} \) and \( W, \bar{W} \) are polynomials generically of degree two. A solution to this equation exists if and only if there exists \( z \in \mathbb{C} \) such that

\[
V(\alpha^*) = z \bar{V}(\alpha^*) \tag{A10}
\]

and

\[
\bar{W}(\alpha) = -z W(\alpha). \tag{A11}
\]

We have then brought a single equation of the fourth degree to two equations of the second degree. With the aid of the first one, we can determine \( \alpha^* \) as a function of \( z \). There are clearly at most two such solutions. Putting them to the second equation and getting rid of the square root, we arrive at

\[
(z^2)^2 W_4(z) + z W'_4(z) + W''_4(z) = 0, \tag{A12}
\]

where \( W_4, W'_4 \), and \( W''_4 \) stand for polynomials which are generically of fourth degree. In what follows, we will show that Eq. (A12) has at least one solution \( z = rs \) with \( |s| = 1 \), i.e., \( z^* = r/s \). To this end, let us consider two cases, when \( s = rx \) and \( s = x/r \). Putting all this to Eq. (A12), one gets the following equations

\[
\frac{1}{x} W_4(r^2 x) + \frac{1}{x} W'_4(r^2 x) + W''_4(r^2 x) = 0 \tag{A13}
\]

and

\[
\frac{1}{x} W_4(x) + \frac{1}{x} W'_4(x) + W''_4(x) = 0. \tag{A14}
\]

In the limit of \( r \to \infty \) the first equation has two roots \( s_{i}^{\infty} \to \infty \) \((i = 1,2)\), while the second one four roots \( s_{i}^{\infty} \to 0 \) \((i = 1,2,3,4)\). Then, in the limit of \( r \to 0 \), Eq. (A13) again has two roots \( s_{i}^{0} \to 0 \), while Eq. (A14) has four roots \( s_{i}^{0} \to \infty \) \((i = 1,2,3,4)\). Notice that, after the above substitution, (A12) is of sixth degree in \( s \) meaning that it can have at most six solutions with respect to \( s \). Consequently, by varying continuously \( r \) from zero to \( \infty \) we see that at least one of the roots \( s_{i} \), being a continuous function of \( r \), must go from zero to \( \infty \), and so there is a value of \( s \) such that \( |s| = 1 \). As a result, there is at least one \( z \in \mathbb{C} \) for which Eq. (A12) is fulfilled, and simultaneously at least one \( \alpha \in \mathbb{C} \) obeying (A9).

\[\Box \]

**Theorem 3.** Let \( \rho \) be an entangled symmetric PPT four-qubit state. Then it can be written as

\[
\rho = \sum_{k=1}^{K \leq 6} [A_k(a_k,b_k)^{\otimes 4} + B_k(a_k,-b_k)^{\otimes 4}], \tag{A15}
\]

where \( (a_k,b_k) \in \mathbb{C}^2 \) and \( A_k, B_k \) are some complex coefficients, and by \( |\psi\rangle \) we denote a projector onto \( |\psi\rangle \).

**Proof.** First, let us introduce two linear transformations \( V: (\mathbb{C}^2)^{\otimes 2} \to \mathbb{C}^2 \) and \( W: (\mathbb{C}^2)^{\otimes 2} \to (\mathbb{C}^2)^{\otimes 3} \) defined as

\[
V[(a,b) \otimes (a,b)] = (a^2, b^2) \tag{A16}
\]

and

\[
W[(a,b) \otimes (a^2, b^2)] = (a, b)^{\otimes 3}, \tag{A17}
\]

respectively, with \( a, b \) being any complex numbers. Then, by \( V \) and \( W \) we denote maps that are defined through the adjoint actions of \( V \) and \( W \), i.e., \( \bar{X}(\cdot) = X(\cdot)^{\dagger} \) \((X = V, W)\).

The key feature of the two matrices \( V \) and \( W \) is that \( W_V BC |\psi\rangle = |\psi\rangle \) for any three-qubit symmetric \( |\psi\rangle \),
where BC denote qubits subject to V. The same holds when V is applied to any pair of qubits in |ψ⟩ and followed by a proper application of W. Accordingly, any N-qubit symmetric state ρ is left invariant under a proper application of V and W to any three-qubits. In particular, for a four-qubit symmetric state ρ, W_{BC} ⊙ V_{CD}(ρ) = ρ.

Let us now consider a four-qubit PPT symmetric state ρ. By applying V to the CD subsystem of ρ, we get a three-qubit state σ_{ABC} = V_{CD}(ρ) acting on S_{2} ⊗ C^{2}, where C' denotes the qubit resulting from the application of V. Clearly, the map V preserves positivity of partial transposition with respect to the first two parties, i.e., σ_{T_{AB}} ≥ 0. Since S_{2} is isomorphic to C^{3}, results of Ref. [25] imply that σ is separable across the cut AB|C' and so σ takes the form

$$\sigma = \sum_{k=1}^{K} |e_k⟩⟨e_k| \otimes |f_k⟩⟨f_k|, \quad (A18)$$

with |e_k⟩ ∈ C^{3} and |f_k⟩ ∈ C^{2} being in general unnormalized vectors from C^{3} and C^{2}, respectively, and K ≤ 6 [29].

By the very assumption ρ is entangled and therefore r(ρ) = 5, which together with the fact that r(V) = 2, mean that the rank of σ is also five. Therefore, K(σ) consists of a single vector |ϕ⟩ ∈ C^{3} ⊗ C^{2}, which, due to the fact that the range of σ is spanned by the vectors (a, b)^⊗2 ⊗ (a^{2}, b^{2}), takes the form |ϕ⟩ = |01⟩ - |20⟩. As a result, any product vector in Eq. (A18) has to be orthogonal to |ϕ⟩.

Putting |f_k⟩ = (a_{k}^{2}, b_{k}^{2}) with a_{k}, b_{k} ∈ C and solving the equation ⟨ϕ|e_k,f_k⟩ = 0 with respect to |e_k⟩ one finds that it can be written as |e_k⟩ = A_{k}(a_{k}^{2}, a_{k}b_{k}, b_{k}^{2}) + B_{k}(a_{k}^{2}, -a_{k}b_{k}, b_{k}^{2}) with some A_{k}, B_{k} ∈ C. Putting the above forms of |e_k⟩ and |f_k⟩ to Eq. (A18), one sees that σ can be written as

$$\sigma = \sum_{k=1}^{K} [(A_{k}(a_{k}^{2}, b_{k}^{2}) | e_k⟩ | f_k⟩ + B_{k}(a_{k}^{2}, -b_{k}^{2}) | e_k⟩ | f_k⟩) \otimes (a_{k}^{2}, b_{k}^{2})], \quad (A19)$$

where K ≤ 6 and |ψ⟩ denotes a projector onto |ϕ⟩. One completes the proof by applying W to the last two qubits of σ.

Utilizing the normal matrix approach to the separability problem [29], one can prove a bit different decomposition.

Theorem 4. Let ρ be an entangled symmetric PPT four-qubit state. Then it can be written as

$$ρ = \sum_{k=1}^{K} [A_k(a_k, b_k) | e_k⟩ | f_k⟩ + B_k(0, 1) | e_k⟩ | f_k⟩], \quad (A20)$$

where K ≤ 6, (a_k, b_k) ∈ C^{2}, and A_k, B_k are some complex coefficients, and by [ψ] we denote a projector onto |ψ⟩.

Proof. The proof exploits the method developed in Ref. [29]. First, one notices that any ρ can be written as a sum of rank-one matrices

$$ρ = \sum_{i=1}^{K} |Ψ_i⟩⟨Ψ_i|, \quad (A21)$$

where, in particular, |Ψ_i⟩ can be (unnormalized) eigenvectors of ρ, and K ≤ 6 (see the proof of theorem 3). On the other hand, ρ can always be expressed with the aid of the symmetric unnormalized basis {|(E_{μ}^4)_{μ=1}⟩} spanning S_{4} as

$$ρ = \sum_{μ,ν=1}^{5} ρ_{μν}|E_{μ}^4⟩⟨E_{ν}^4|, \quad (A22)$$

Both decompositions (A21) and (A22) are related via the so-called Gram system of ρ, i.e., a collection of K-dimensional vectors |v_{μ}⟩ = (1/(E_{μ}^4)⟨E_{μ}^4|)⟨Ψ_1|E_{μ}^4⟩, ..., ⟨Ψ_{K}|E_{μ}^4⟩)) (μ = 1, ..., 5), giving ρ_{μν} = ⟨v_{μ}|v_{ν}⟩. Putting the latter to (A22) with explicit forms of the vectors |v_{μ}⟩, one recovers (A21).

Now, by projecting the last party onto |0⟩ we get a three-qubit symmetric PPT state ̃ρ, which, as already stated, is separable. Then, according to Ref. [29], there exists a diagonal matrix M = diag[α_{1}, ..., α_{K}] such that |v_{μ}⟩ = M^{μ-1} |v_{1}⟩ (μ = 1, ..., 4). For convenience we can also put |v_{5}⟩ = M^{4} |v_{1}⟩ + |v⟩ with |v⟩ being some K-dimensional vector. Then, putting |v_{1}⟩ = (A_{1}, ..., A_{K}) and |v⟩ = (B_{1}, ..., B_{K}), one sees that

$$|Ψ_i⟩ = \sum_{μ=1}^{4} (E_{μ}^4)|v_{μ}⟩ = A_{i} \sum_{μ=1}^{4} α_{μ}^{i-1}|E_{μ}^4⟩ + B_{i}|E_{5}^4⟩ \quad (A23)$$

where the second equation follows from the explicit form of the vectors |v_{μ}⟩. Substituting vectors |Ψ_i⟩ to Eq. (A21), one gets (A20), which completes the proof.

Appendix B: Properties of the states ρ_{d,b}

Here we characterize the states (5) in more details. In particular, we prove that for d ≥ 4 and b ∈ [0,1] they are PPT entangled and edge.

Theorem 5. The states ρ_{d,b} are PPT for d ≥ 2 and b ∈ [0,1].

Proof. First, notice that we can rewrite (5) in the matrix form as

$$ρ_{d,b} = \frac{1}{(2d-1)b+1} \begin{pmatrix} C & bB_{U} \\ bB_{L} & b1_{d} \end{pmatrix}, \quad (B1)$$
where $B_U$ and $B_L$ are $d \times d$ dimensional matrices with entries 1 on the upper and lower diagonal, respectively. Further, $C$ is a $d \times d$ matrix given by

$$C = \frac{1}{2} \begin{pmatrix} 1 + b & 0 & \cdots & 0 & \sqrt{1 - b^2} \\ 0 & b & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & b & 0 \\ \sqrt{1 - b^2} & 0 & \cdots & 0 & 1 + b \end{pmatrix}. \tag{B2}$$

Clearly, the partial transposition of $\rho_{b,d}$ reads

$$\rho_{b,d}^{T_A} = \frac{1}{(2d - 1)b + 1} \begin{pmatrix} C & bB_L \\ bB_U & b\mathbb{1}_d \end{pmatrix}. \tag{B3}$$

Now, let us consider the unitary matrix $U = \text{antidiag}[1, \ldots, 1]$ (anti-diagonal matrix consisting of units). Straightforward calculations show that $UB_UU^\dagger = B_L, UBU^\dagger = B_U$, and $UCU^\dagger = C$. Consequently,

$$\rho_{b,d}^{T_A} = (I_2 \otimes U)\rho_{b,d}(I_2 \otimes U^\dagger), \tag{B4}$$

meaning that $\rho_{b,d}^{T_A} \geq 0$ iff $\rho_{b,d} \geq 0$. \hfill $\square$

**Theorem 6.** The states $\rho_{b,d}$ are entangled for $d \geq 4$ and $b \in (0, 1)$, while separable for $d = 2, 3$, or $b = 0$, or $b = 1$.

**Proof.** First, let us prove that for $d \geq 4$ and $b \in (0, 1)$, the states (5) are entangled. For this purpose, it suffices to use the necessary criterion for separability formulated in Ref. [30] — the range criterion. It says that if a given density matrix $\rho$ is separable then one is able to find product vectors $|e, f\rangle$ such that $|e^*, f\rangle$ span $R(\rho^{T_A})$. In what follows we show that none of the product vectors $|e, f\rangle$ in $R(\rho)$ is such that $|e^*, f\rangle \in R(\rho^{T_A})$.

All product vectors in the range of $\rho_{b,d}$ are given by

$$(1, \alpha) \otimes (\alpha^{d-1} + y, \alpha^{d-2}, \ldots, \alpha, 1) \quad (\alpha \in \mathbb{C}) \tag{B5}$$

$$(0, 1) \otimes (1, 0, \ldots, 0), \quad \tag{B6}$$

where $y = \sqrt{(1 - b)/(1 + b)}$. If we allow for infinite $\alpha$, the vector (B6) may be obtained from the class (B5). It is also worth mentioning that the above vectors span $R(\rho_{b,d})$.

On the other hand, all the vectors in the range of $\rho_{b,d}^{T_A}$ are given by

$$(a_1, \ldots, a_{d-1}, ya_1 + a_d; a_2, a_3, \ldots, a_d, b) \tag{B7}$$

with $a_1, \ldots, a_d, b \in \mathbb{C}$. Consequently, a product vector from the first class (B5), when partially conjugated with respect to the first subsystem, belongs to $R(\rho_{b,d}^{T_A})$, i.e., takes the form (B7), if and only if the conditions are satisfied: (i) $\alpha(1 - |\alpha|^2) = 0$, (ii) $\alpha^{d-2} = \alpha^*(\alpha^{d-1} + y)$, and (iii) $y(y + \alpha^{d-1}) = 1 + |\alpha|^2$. The first condition is satisfied if either $\alpha = 0$, which contradicts the third condition because $y \neq 1$, or $|\beta|^2 = 1$, which contradicts (ii) because $y \neq 0$. Along the same lines one checks that the vector (B6) is not of the form (B7).

In conclusion, the states $\rho_{b,d}$ are entangled for $d \geq 4$ and $b \in (0, 1)$.

Let us finally consider the missing cases of $d = 2, 3$ or $b = 1$ or $b = 0$. For $d = 2$ or $d = 3$, theorem 5 says that $\rho_{b,d}$ are PPT for any $b$. It is known [25] that all qubit-qubit and qubit-qutrit PPT states are separable.

For $b = 0$ it follows from Eq. (5) that $\rho_{d,0} = |\Phi_0\rangle\langle\Phi_0|$, which is separable, while $\rho_{d,1}$ can be written in the following separable form (cf. Ref. [30]):

$$\rho_{d,1} = \frac{1}{16\pi} \int_0^{2\pi} d\varphi \; P(\varphi) \otimes Q(\varphi), \tag{B8}$$

where $P(\varphi)$ and $Q(\varphi)$ are projectors onto $(1/\sqrt{2})(1, e^{i\varphi})$ and $(1/\sqrt{d})(1, e^{-i\varphi}, e^{-2i\varphi}, \ldots, e^{-(d-1)i\varphi})$, respectively. \hfill $\square$

**Theorem 7.** $r(\rho_{d,b}) = r(\rho_{d,b}^{T_A}) = d + 1$.

**Proof.** Direct check shows that the vectors

$$|\Psi_i\rangle = |0, i\rangle - |1, i + 1\rangle \quad (i = 1, \ldots, d - 2)$$

$$|\Psi_i\rangle = -\sqrt{1 + b}|00\rangle + \sqrt{1 - b}|0, d - 1\rangle + \sqrt{1 + b}|11\rangle. \tag{B9}$$

belong to the kernel of $\rho_{d,b}$ and the subspace they span has dimension $d - 1$. On the other hand, the product vector from $R(\rho_{d,b})$, given in Eqs. (B5) and (B6) span $(d + 1)$-dimensional subspace. Consequently, $r(\rho_{d,b}) = d + 1$ and, since $\rho_{d,b}^{T_A} = (I_2 \otimes U)\rho_{d,b}(I_2 \otimes U^\dagger)$, $r(\rho_{d,b}^{T_A}) = d + 1$. \hfill $\square$