On a certain subalgebra of $U_q(\hat{\mathfrak{sl}}_2)$ related to the degenerate $q$-Onsager algebra

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Abstract

In [4], it is discussed that a certain subalgebra of the quantum affine algebra $U_q(\hat{\mathfrak{sl}}_2)$ controls the type I TD-algebra of the second kind (the degenerate $q$-Onsager algebra). The subalgebra, which we denote by $U_q'(\hat{\mathfrak{sl}}_2)$, is generated by $e_0^+, e_1^+, k_1^\pm 1 (i = 0, 1), e_0^-$ missing from the Chevalley generators $e_i^\pm, k_i^\pm 1 (i = 0, 1)$ of $U_q(\hat{\mathfrak{sl}}_2)$. In this paper, we determine the finite-dimensional irreducible representations of $U'_q(\hat{\mathfrak{sl}}_2)$. Intertwiners are also determined.

Keywords. Degenerate $q$-Onsager algebra, quantum affine algebra, TD-algebra, augmented TD-algebra, TD-pair, Terwilliger algebra, P- and Q-polynomial association scheme.

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1 Introduction

Throughout this paper, the ground field is $\mathbb{C}$ and $q$ stands for a nonzero scalar that is not a root of unity. The symbols $\varepsilon, \varepsilon^*$ stand for an integer chosen from $\{0, 1\}$. Let $A_q = A_q^{(\varepsilon, \varepsilon^*)}$ denote the associative algebra with 1 generated by $z, z^*$ subject to the defining relations [4]

$$(TD) \quad \left\{ \begin{array}{l} [z, [z, [z, z^*]_q]_{q^{-1}}] = -\varepsilon(q^2 - q^{-2})^2[z, z^*], \\
[z^*, [z^*, [z^*, z]_q]_{q^{-1}}] = -\varepsilon^*(q^2 - q^{-2})^2[z^*, z], \end{array} \right. \quad (1)$$
where $[X, Y] = XY - YX$, $[X, Y]_q = qXY - q^{-1}YX$. This paper deals with a subalgebra of the quantum affine algebra $U_q(\widehat{sl}_2)$ that is closely related to $A_q$ in the case of $(\varepsilon, \varepsilon^*) = (1, 0)$. If $(\varepsilon, \varepsilon^*) = (0, 0)$, $A_q$ is isomorphic to the positive part of $U_q(\widehat{sl}_2)$. If $(\varepsilon, \varepsilon^*) = (1, 1)$, $A_q$ is called the $q$-Onsager algebra. If $(\varepsilon, \varepsilon^*) = (1, 0)$, $A_q$ may well be called the degenerate $q$-Onsager algebra.

The algebra $A_q$ arises in the course of the classification of TD-pairs of type I, which is a critically important step in the study of representations of Terwilliger algebras for P- and Q- polynomial association schemes [3]. For this reason, $A_q$ is called the TD-algebra of type I. Precisely speaking, the TD-algebra of type I is standardized to be the algebra $A_q$, where $q$ is the main parameter for TD-pairs of type I; so $q \neq \pm 1$ and $q$ is allowed to be a root of unity. In our case where we assume $q$ is not a root of unity, to classify the TD-pairs of type I is to determine the finite-dimensional irreducible representations $\rho: A_q \rightarrow \text{End}(V)$ with the property that $\rho(z), \rho(z^*)$ are both diagonalizable, and vice versa. Such irreducible representations of $A_q$ are determined in [4] via embeddings of $A_q$ into the augmented TD-algebra $T_q$. (In the case of $(\varepsilon, \varepsilon^*) = (1, 1)$, the diagonalizability condition of $\rho(z), \rho(z^*)$ can be dropped, because it turns out that this condition always holds for every finite-dimensional irreducible representation $\rho$ of the $q$-Onsager algebra $A_q$.) $T_q$ is easier than $A_q$ to study representations for, and each finite-dimensional irreducible representation $\rho: A_q \rightarrow \text{End}(V)$ with $\rho(z), \rho(z^*)$ diagonalizable can be extended to a finite-dimensional irreducible representation of $T_q$ via a certain embedding of $A_q$ into $T_q$.

The augmented TD-algebra $T_q = T_q^{(\varepsilon, \varepsilon^*)}$ is the associative algebra with 1 generated by $x, y, k^{\pm 1}$ subject to the defining relations

\[(\text{TD})_0 \left\{ \begin{array}{ll}
k k^{-1} = k^{-1}k = 1, \\
k^2 = q^2x, \\
k^{-1}y = q^{-2}y,
\end{array} \right. \tag{2}\]

and

\[(\text{TD})_1 \left\{ \begin{array}{ll}
[x, [x, [x, y]_q]_q^{-1}] = \delta(\varepsilon^*x^2k^2 - \varepsilon k^{-2}x^2), \\
[y, [y, [y, x]_q]_q^{-1}] = \delta(-\varepsilon^*k^2y^2 + \varepsilon y^2k^{-2}),
\end{array} \right. \tag{3}\]

where $\delta = -(q - q^{-1})(q^2 - q^{-2})(q^3 - q^{-3})q^4$. The finite-dimensional irreducible representations of $T_q$ are determined in [4] via embeddings of $T_q$ into the $U_q(sl_2)$-loop algebra $U_q(L(sl_2))$. 2
Let \( e_i^\pm, k_i^{\pm 1} (i = 0, 1) \) be the Chevalley generators of \( U_q(L(\mathfrak{sl}_2)) \). So the defining relations of \( U_q(L(\mathfrak{sl}_2)) \) are

\[
\begin{align*}
  k_0 k_1 &= k_1 k_0 = 1, \\
  k_i k_i^{-1} &= k_i^{-1} k_i = 1, \\
  k_i e_i^\pm k_i^{-1} &= q^{\pm 2} e_i^\pm, \\
  k_i e_j^\pm k_i^{-1} &= q^{\mp 2} e_j^\pm, \\
  [e_i^+, e_i^-] &= k_i - k_i^{-1}, \\
  [e_i^+, e_j^-] &= 0 \quad (i \neq j), \\
  [e_i^\pm, [e_i^\pm, e_j^\pm]]_{q^{-1}} &= 0 \quad (i \neq j).
\end{align*}
\tag{4}
\]

Note that if \( k_0 k_1 = k_1 k_0 = 1 \) is replaced by \( k_0 k_1 = k_1 k_0 \) in (4), we have the quantum affine algebra \( U_q(\widehat{\mathfrak{sl}}_2) : U_q(L(\mathfrak{sl}_2)) \) is the quotient algebra of \( U_q(\widehat{\mathfrak{sl}}_2) \) by the two-sided ideal generated by \( k_0 k_1 - 1 \). For a nonzero scalar \( s \), define the elements \( x(s), y(s), k(s) \) of \( U_q(L(\mathfrak{sl}_2)) \) by

\[
\begin{align*}
  x(s) &= -q^{-1}(q - q^{-1})^2 (se_0^+ + \varepsilon s^{-1} e_1^- k_1), \\
  y(s) &= \varepsilon^* se_0^- k_0 + s^{-1} e_1^+, \\
  k(s) &= sk_0.
\end{align*}
\tag{5}
\]

Then the mapping

\[
\varphi_s : \mathcal{T}_q \longrightarrow U_q(L(\mathfrak{sl}_2)) \quad (x, y, k \mapsto x(s), y(s), k(s))
\tag{6}
\]

gives an injective algebra homomorphism. If \((\varepsilon, \varepsilon^*) = (0, 0)\), the image \( \varphi_s(\mathcal{T}_q) \) coincides with the Borel subalgebra generated by \( e_i^+ \), \( k_i^{\pm 1} \) \((i = 0, 1)\). If \((\varepsilon, \varepsilon^*) = (1, 0)\), the image \( \varphi_s(\mathcal{T}_q) \) is properly contained in the subalgebra generated by \( e_0^+, e_i^\pm, k_i^{\pm 1} \) \((i = 0, 1)\), \( e_0^- \) missing from the generators; we denote this subalgebra by \( U_q'(L(\mathfrak{sl}_2)) \). Through the natural homomorphism \( U_q(\widehat{\mathfrak{sl}}_2) \longrightarrow U_q(L(\mathfrak{sl}_2)) \), pull back the subalgebra \( U_q'(L(\mathfrak{sl}_2)) \) and denote the pre-image by \( U_q'(\widehat{\mathfrak{sl}}_2) \):

\[
U_q'(\widehat{\mathfrak{sl}}_2) = \langle e_0^+, e_1^\pm, k_i^{\pm 1} \mid i = 0, 1 \rangle \subset U_q(\widehat{\mathfrak{sl}}_2).
\tag{7}
\]

In [4], it is shown that in the case of \((\varepsilon, \varepsilon^*) = (1, 0)\), all the finite-dimensional irreducible representations of \( \mathcal{T}_q \) are produced by tensor products of evaluation modules for \( U_q'(L(\mathfrak{sl}_2)) \) via the embedding \( \varphi_s \) of \( \mathcal{T}_q \) into \( U_q'(L(\mathfrak{sl}_2)) \). Using this fact and the Drinfel’d polynomials, we show in this paper that there are no finite-dimensional irreducible representations of \( U_q'(L(\mathfrak{sl}_2)) \)
and hence of \( U'_q(\hat{\mathfrak{sl}}_2) \) other than those afforded by tensor products of evaluation modules, if we apply suitable automorphisms of \( U'_q(L(\mathfrak{sl}_2)), U'_q(\hat{\mathfrak{sl}}_2) \) to adjust the types of the representations to be \((1, 1)\). Here we note that the evaluation parameters are allowed to be zero for \( U'_q(L(\mathfrak{sl}_2)), U'_q(\hat{\mathfrak{sl}}_2) \). Details will be discussed in Section 2, where the isomorphism classes of finite-dimensional irreducible representations of \( U'_q(\hat{\mathfrak{sl}}_2) \) are also determined. In Section 3, intertwiners will be determined for finite-dimensional irreducible \( U'_q(\hat{\mathfrak{sl}}_2) \)-modules.

Drinfel’d polynomials are not the main subject of this paper but the key tool for the classification of finite-dimensional irreducible representations of \( U_q(\hat{\mathfrak{sl}}_2), U'_q(\hat{\mathfrak{sl}}_2) \). They are defined in [4], directly attached to \( T_q \)-modules, not to \( U_q(\mathfrak{sl}_2) \)- or \( U'_q(\hat{\mathfrak{sl}}_2) \)-modules. (In the case of \((\epsilon, \epsilon^*) = (0, 0)\), they turn out to coincide with the original ones up to the reciprocal of the variable.) So if our approach is applied to the case of \((\epsilon, \epsilon^*) = (0, 0)\), finite-dimensional irreducible representations are naturally classified in the first place for the Borel subalgebra of \( U_q(\hat{\mathfrak{sl}}_2) \) and then for \( U_q(\hat{\mathfrak{sl}}_2) \) itself. This reverses the process adopted in [1] and will be briefly demonstrated in Section 2 as a warm-up for the case of \((\epsilon, \epsilon^*) = (1, 0)\), thus giving another proof to the classical result of Chari-Pressley [2].

2 Finite-dimensional irreducible representations of \( U'_q(\hat{\mathfrak{sl}}_2) \)

The subalgebra \( U'_q(\hat{\mathfrak{sl}}_2) \) of the quantum affine algebra \( U_q(\hat{\mathfrak{sl}}_2) \) is generated by \( e_0^+, e_1^\pm, k_i^{\pm 1} (i = 0, 1), e_0^- \) missing from the generators, and has by the triangular decomposition of \( U_q(\hat{\mathfrak{sl}}_2) \) the defining relations

\[
\begin{align*}
 k_0k_1 &= k_1k_0, \\
 k_ik_i^{-1} &= k_i^{-1}k_i = 1,
 k_0e_0^+k_0^{-1} &= q^2e_0^+, \\
 k_1e_0^+k_1^{-1} &= q^{-2}e_0^+,
 k_0e_0^+k_0^{-1} &= q^{-2}e_0^+,
 k_0e_0^+k_0^{-1} &= q^2e_0^+,
 [e_i^+, e_j^-] &= \frac{k_i - k_i^{-1}}{q - q^{-1}},
 [e_i^+, e_j^-] &= 0,
 [e_i^+, [e_i^+, e_j^+]_{q^{-1}}] &= 0 \quad (i \neq j).
\end{align*}
\]
Note that if \( k_0k_1 = k_1k_0 \) is replaced by \( k_0k_1 = k_1k_0 = 1 \) in (8), we have the defining relations for \( U_q'(L(\mathfrak{sl}_2)) \).

Let \( V \) be a finite-dimensional irreducible \( U_q'(\mathfrak{sl}_2) \)-module. Let us first observe that the \( U_q'(\mathfrak{sl}_2) \)-module \( V \) is obtained from a \( U_q'(L(\mathfrak{sl}_2)) \)-module by applying an automorphism of \( U_q'(\mathfrak{sl}_2) \). Since the element \( k_0k_1 \) belongs to the centre of \( U_q'(\mathfrak{sl}_2) \), \( k_0k_1 \) acts on \( V \) as a scalar \( s \) by Schur’s lemma. Since \( k_0k_1 \) is invertible, the scalar \( s \) is nonzero: \( k_0k_1 \mid_v = s \in \mathbb{C}^\times \). Observe that \( U_q'(\mathfrak{sl}_2) \) admits an automorphism that sends \( k_0 \) to \( s^{-1}k_0 \) and preserves \( k_1 \). Hence we may assume \( k_0k_1 \mid_v = 1 \). Then we can regard \( V \) as a \( U_q'(L(\mathfrak{sl}_2)) \)-module.

Let \( V \) be a finite-dimensional irreducible \( U_q'(L(\mathfrak{sl}_2)) \)-module. For a scalar \( \theta \), set \( V(\theta) = \{ v \in V \mid k_0v = \theta v \} \). So if \( V(\theta) \neq 0 \), \( \theta \) is an eigenvalue of \( k_0 \) and \( V(\theta) \) is the corresponding eigenspace of \( k_0 \). For an eigenvalue \( \theta \) and an eigenvector \( v \in V(\theta) \), it holds that \( e_0^+v \in V(q^2\theta) \) by the relation \( k_0e_0^+ = q^2e_0^+k_0 \) and \( e_1^+v \in V(q^{\pm 2}\theta) \) by \( k_0e_1^+ = q^{\pm 2}e_1^+k_0 \). We have

\[
e_0^+V(\theta) \subseteq V(q^2\theta), \quad e_1^+V(\theta) \subseteq V(q^{\pm 2}\theta).
\]

If \( \dim V = 1 \), then \( e_0^+V = 0, e_1^+V = 0 \) by (9) and \( k_0 \mid_V = \pm 1 \) by \( [e_1^+, e_1^-] = (k_1 - k_1^{-1})/(q - q^{-1}) = (k_0^{-1} - k_0)/(q - q^{-1}) \). Such a \( U_q'(L(\mathfrak{sl}_2)) \)-module \( V \) is said to be trivial. Assume \( \dim V \geq 2 \). Choose an eigenvalue \( \theta \) of \( k_0 \) on \( V \). Then \( \sum_{i \in \mathbb{Z}} V(q^{\pm 2i}\theta) \) is invariant under the actions of the generators by (9), and so we have \( V = \sum_{i \in \mathbb{Z}} V(q^{\pm 2i}\theta) \) by the irreducibility of the \( U_q'(L(\mathfrak{sl}_2)) \)-module \( V \). Since \( V \) is finite-dimensional, there exists a positive integer \( d \) and a nonzero scalar \( s_0 \) such that the eigenspace decomposition of \( k_0 \) is

\[
V = \bigoplus_{i=0}^d V(s_0q^{2i-d}).
\]

We want to show that \( s_0 = \pm 1 \) holds in (10).

Consider the subalgebra of \( U_q'(L(\mathfrak{sl}_2)) \) generated by \( e_1^\pm, k_1^{\pm 1} \) and denote it by \( \mathcal{U} : \mathcal{U} = \langle e_1^\pm, k_1^{\pm 1} \rangle \). Regard \( V \) as a \( \mathcal{U} \)-module. Since \( \mathcal{U} \) is isomorphic to the quantum algebra \( U_q(\mathfrak{sl}_2) \), \( V \) is a direct sum of irreducible \( \mathcal{U} \)-modules, and for each irreducible \( \mathcal{U} \)-submodule \( W \) of \( V \), the eigenvalues of \( k_1 = k_0^{-1} \) on \( W \) are either \( \{ q^{2i-\ell} \mid 0 \leq i \leq \ell \} \) or \( \{ -q^{2i-\ell} \mid 0 \leq i \leq \ell \} \) for some nonnegative integer \( \ell \). In particular, if \( \theta \) is an eigenvalue of \( k_0 \), so is \( \theta^{-1} \). The collection of such eigenvalues gives rise to the eigenspace decomposition of (10). We obtain \( s_0 = \pm 1 \). Observe that \( U_q'(L(\mathfrak{sl}_2)) \) admits an automorphism that sends \( k_i \) to
\(-k_i (i = 0, 1) \text{ and } e_1^\pm \text{ to } -e_1^\pm.\) Hence we may assume \(s_0 = 1\) in (10). Note that in this case, \(k_1\) has the eigenvalues \(\{s_1q^{2i-\ell} \mid 0 \leq i \leq \ell\}\) with \(s_1 = 1.\) Such an irreducible module or the irreducible representation afforded by such is said to be of type \((1, 1)\), indicating \((s_0, s_1) = (1, 1)\). We conclude that the determination of finite-dimensional irreducible representations for \(U_q'(\mathfrak{sl}_2)\) is, via automorphisms, reduced to that of type \((1, 1)\) for \(U_q'(L(\mathfrak{sl}_2)).\)

In the rest of this section, we shall introduce evaluation modules for \(U_q'(L(\mathfrak{sl}_2))\) and show that every finite-dimensional irreducible representation of type \((1, 1)\) of \(U_q'(L(\mathfrak{sl}_2))\) is afforded by a tensor product of them. For \(a \in \mathbb{C}\) and \(\ell \in \mathbb{Z}_{\geq 0},\) let \(V(\ell, a)\) denote the \((\ell + 1)\)-dimensional vector space with a basis \(v_0, v_1, \ldots, v_\ell.\) Using (8), it can be routinely verified that \(U_q'(L(\mathfrak{sl}_2))\) acts on \(V(\ell, a)\) by

\[
\begin{align*}
  k_0 v_i &= q^{2i-\ell} v_i, \\
  k_1 v_i &= q^{\ell-2i} v_i, \\
  e_0^+ v_i &= a q^i [i + 1] v_{i+1}, \\
  e_1^+ v_i &= [\ell - i + 1] v_{i-1}, \\
  e_1^- v_i &= [i + 1] v_{i+1},
\end{align*}
\]

where \(v_{-1} = v_{\ell+1} = 0\) and \([t] = [t]_q = (q^t - q^{-t})/(q - q^{-1}).\) This \(U_q'(L(\mathfrak{sl}_2))-\)module \(V(\ell, a)\) is irreducible and called an evaluation module. The basis \(v_0, v_1, \ldots, v_\ell\) of the \(U_q'(L(\mathfrak{sl}_2))-\)module \(V(\ell, a)\) is called a standard basis. The vector \(v_0\) is called the highest weight vector. Note that the evaluation parameter \(a\) is allowed to be zero. Also note that if \(\ell = 0, V(\ell, a)\) is the trivial module. We denote the evaluation module \(V(\ell, 0)\) by \(V(\ell)\), allowing \(\ell = 0,\) and use the notation \(V(\ell, a)\) only for an evaluation module with \(a \neq 0\) and \(\ell \geq 1.\)

The \(U_q(\mathfrak{sl}_2)\)-loop algebra \(U_q'(L(\mathfrak{sl}_2))\) has the coproduct \(\Delta : U_q'(L(\mathfrak{sl}_2)) \to U_q'(L(\mathfrak{sl}_2)) \otimes U_q'(L(\mathfrak{sl}_2))\) defined by

\[
\begin{align*}
  \Delta(k_i^{\pm 1}) &= k_i^{\pm 1} \otimes k_i^{\pm 1}, \\
  \Delta(e_i^+ k_i) &= k_i \otimes e_i^+ k_i + e_i^- k_i \otimes 1, \\
  \Delta(e_i^- k_i) &= k_i \otimes e_i^- k_i + e_i^+ k_i \otimes 1.
\end{align*}
\]

The subalgebra \(U_q'(L(\mathfrak{sl}_2))\) is closed under \(\Delta.\) Thus given a set of evaluation modules \(V(\ell), V(\ell_i, a_i) (1 \leq i \leq n)\) for \(U_q'(L(\mathfrak{sl}_2))\), the tensor product

\[
V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)
\]
becomes a $U'_q(L(sl_2))$-module via $\Delta$. Note that by the coassociativity of $\Delta$, the way of putting parentheses in the tensor product of (13) does not affect the isomorphism class as a $U'_q(L(sl_2))$-module. Also note that if $\ell = 0$ in (13), then $V(0)$ is the trivial module and the tensor product of (13) is isomorphic to $V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ as $U'_q(L(sl_2))$-modules. Finally we allow $n = 0$, in which case we understand that the tensor product of (13) means $V(\ell)$.

With the evaluation module $V(\ell, a)$, we associate the set $S(\ell, a)$ of scalars $aq^{-\ell+1}, aq^{-\ell+3}, \ldots, aq^{\ell-1}$:

$$S(\ell, a) = \{aq^{2i-\ell+1} | 0 \leq i \leq \ell - 1\}. \quad (14)$$

The set $S(\ell, a)$ is called a $q$-string of length $\ell$. Two $q$-strings $S(\ell, a), S(\ell', a')$ are said to be in general position if either

(i) the union $S(\ell, a) \cup S(\ell', a')$ is not a $q$-string, or

(ii) one of $S(\ell, a), S(\ell', a')$ includes the other.

Below is the main theorem of this paper. It classifies the isomorphism classes of the finite-dimensional irreducible $U'_q(L(sl_2))$-modules of type $(1,1)$.

**Theorem 1.** The following (i), (ii), (iii) holds.

(i) A tensor product $V = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ of evaluation modules is irreducible as a $U'_q(L(sl_2))$-module if and only if $S(\ell_i, a_i), S(\ell_j, a_j)$ are in general position for all $i, j \in \{1, 2, \ldots, n\}$. In this case, $V$ is of type $(1,1)$.

(ii) Consider two tensor products $V = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n), V' = V(\ell') \otimes V(\ell'_1, a'_1) \otimes \cdots \otimes V(\ell'_m, a'_m)$ of evaluation modules and assume that they are both irreducible as a $U'_q(L(sl_2))$-module. Then, $V, V'$ are isomorphic as $U'_q(L(sl_2))$-modules if and only if $\ell = \ell', n = m$ and $(\ell_i, a_i) = (\ell'_i, a'_i)$ for all $i$ ($1 \leq i \leq n$) with a suitable reordering of the evaluation modules $V(\ell_1, a_1), \ldots, V(\ell_n, a_n)$.

(iii) Every non-trivial finite-dimensional irreducible $U'_q(L(sl_2))$-module of type $(1,1)$ is isomorphic to some tensor product $V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ of evaluation modules.
Discard the evaluation module \( V(\ell) \) from the statement of Theorem 1 and replace \( U'_q(L(sl_2)) \) by \( U_q(L(sl_2)) \) or \( B \), where \( B \) is the Borel subalgebra of \( U_q(L(sl_2)) \) generated by \( e_i^+, k_i^{\pm 1} \) \( (i = 0, 1) \). Then it precisely describes the classification of the isomorphism classes of finite-dimensional irreducible modules of type \((1, 1)\) for \( U_q(L(sl_2)) \) \([2]\) or \( B \) \([1]\). There is a one-to-one correspondence of finite-dimensional irreducible modules of type \((1, 1)\) between \( U_q(L(sl_2)) \) and \( B \): every finite-dimensional irreducible \( U_q(L(sl_2)) \)-module of type \((1, 1)\) is irreducible as a \( B \)-module and every finite-dimensional irreducible \( B \)-module of type \((1, 1)\) is uniquely extended to a \( U_q(L(sl_2)) \)-module.

This sort of correspondence of finite-dimensional irreducible modules exists between \( U'_q(L(sl_2)) \) and \( T_q \) via the embedding \( \varphi \) of (6), where \( T_q \) is the augmented TD-algebra with \((\varepsilon, \varepsilon^*) = (1, 0)\), and this gives a proof of Theorem 1. The key to our understanding of the correspondence is the following two lemmas. Let \( \mathcal{U} \) denote the quantum algebra \( U_q(sl_2) \): \( \mathcal{U} \) is the associative algebra with 1 generated by \( X^\pm, K^{\pm 1} \) subject to the defining relations

\[
\begin{align*}
K K^{-1} &= K^{-1} K = 1, \\
K X^\pm K^{-1} &= q^{\pm 2} X^\pm, \\
[X^+, X^-] &= \frac{K - K^{-1}}{q - q^{-1}}.
\end{align*}
\] (15)

**Lemma 1.** [4, Lemma 7.5] Let \( V \) be a finite-dimensional \( \mathcal{U} \)-module that has the following weight-space \((K\text{-eigenspace})\) decomposition:

\[
V = \bigoplus_{i=0}^{d} U_i, \quad K |_{U_i} = q^{2i-d} \quad (0 \leq i \leq d).
\]

Let \( W \) be a subspace of \( V \) as a vector space. Assume that \( W \) is invariant under the actions of \( X^+ \) and \( K \):

\[
X^+ W \subseteq W, \quad KW \subseteq W.
\]

If it holds that

\[
\dim (W \cup U_i) = \dim (W \cup U_{d-i}) \quad (0 \leq i \leq d),
\]

then \( X^- W \subseteq W \), i.e., \( W \) is a \( \mathcal{U} \)-submodule.

**Lemma 2.** If \( V \) is a finite-dimensional \( \mathcal{U} \)-module, the action of \( X^- \) on \( V \) is uniquely determined by those of \( X^+, K^{\pm 1} \) on \( V \).
\textbf{Proof.} The claim holds if \( V \) is irreducible as a \( \mathcal{U} \)-module. By the semi-simplicity of \( \mathcal{U} \), it holds generally. \hfill \Box

As a warm-up for the proof of Theorem 1, we shall demonstrate how to use these lemmas in the case of the corresponding theorem \cite{2} for \( U_q(L(\mathfrak{sl}_2)) \). We want, and it is enough, to show part (iii) of the theorem for \( U_q(L(\mathfrak{sl}_2)) \) by using the classification of finite-dimensional irreducible \( \mathcal{B} \)-modules, since the parts (i), (ii) are well-known in advance of \cite{2} and the finite-dimensional irreducible \( \mathcal{B} \)-modules are classified in \cite{4} without using the part (iii) in question.

Let \( V \) be a finite-dimensional irreducible \( U_q(L(\mathfrak{sl}_2)) \)-module of type \((1,1)\). Then \( V \) has the weight-space decomposition

\[ V = \bigoplus_{i=0}^{d} U_i, \quad k_0 |_{U_i} = q^{2i-d} \quad (0 \leq i \leq d). \]

Regard \( V \) as a \( \mathcal{B} \)-module. Let \( W \) be a minimal \( \mathcal{B} \)-submodule of \( V \). Note that \( W \) is irreducible as a \( \mathcal{B} \)-module. We want to show \( W = V \), i.e., \( V \) is irreducible as a \( \mathcal{B} \)-module. Since the mapping \((e_1^+)^{d-2i} : U_i \to U_{d-i}\) is a bijection and \( W \cap U_i \) is mapped into \( W \cap U_{d-i} \) by \((e_1^+)^{d-2i}\), we have \( \dim (W \cap U_i) \leq \dim (W \cap U_{d-i}) \) \((0 \leq i \leq [d/2])\). Similarly from the bijection \((e_1^+)^{d-2i} : U_{d-i} \to U_i\), we get \( \dim (W \cap U_{d-i}) \leq \dim (W \cap U_i) \). Thus it holds that

\[ \dim (W \cap U_i) = \dim (W \cap U_{d-i}) \quad (0 \leq i \leq d). \]

Consider the algebra homomorphism from \( \mathcal{U} \) to \( U_q(L(\mathfrak{sl}_2)) \) that sends \( X^+, X^-, K^{\pm 1} \) to \( e_0^+, e_0^-, k_0^{\pm 1} \), respectively. Regard \( V \) as a \( \mathcal{U} \)-module via this algebra homomorphism. Then \( X^+W \subseteq W, KW \subseteq W \). Since \( \dim (W \cap U_i) = \dim (W \cap U_{d-i}) \) \((0 \leq i \leq d)\), we have by Lemma 1 that \( X^-W \subseteq W \), i.e., \( e_0^-W \subseteq W \). Similarly, Lemma 1 can be applied to the \( \mathcal{U} \)-module \( V \) via the algebra homomorphism from \( \mathcal{U} \) to \( U_q(L(\mathfrak{sl}_2)) \) that sends \( X^+, X^-, K^{\pm 1} \) to \( e_1^+, e_1^-, k_1^{\pm 1} \), respectively, in which case the weight-space decomposition of the \( \mathcal{U} \)-module \( V \) is \( V = \bigoplus_{i=0}^{d} U_{d-i}, K|_{U_{d-i}} = q^{2i-d} \) \((0 \leq i \leq d)\). Consequently, we get \( X^-W \subseteq W \), i.e., \( e_1^-W \subseteq W \). Thus \( W \) is \( U_q(L(\mathfrak{sl}_2)) \)-invariant and we have \( W = V \) by the irreducibility of the \( U_q(L(\mathfrak{sl}_2)) \)-module \( V \). We conclude that every finite-dimensional irreducible \( U_q(L(\mathfrak{sl}_2)) \)-module of type \((1,1)\) is irreducible as a \( \mathcal{B} \)-module.

Now consider the class of finite-dimensional irreducible \( \mathcal{B} \)-modules \( V \), where \( V \) runs through all tensor products of evaluation modules that are
irreducible as a $U_q(L(\mathfrak{sl}_2))$-module:

$$V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n).$$

Then it turns out that the Drinfel’d polynomials $P_V(\lambda)$ of the irreducible $\mathcal{B}$-modules $V$ exhaust all that are possible for finite-dimensional irreducible $\mathcal{B}$-modules of type $(1, 1)$, as shown in [4] by the product formula

$$P_V(\lambda) = \prod_{i=1}^{n} P_{V(\ell_i, a_i)}(\lambda),$$

$$P_{V(\ell_i, a_i)}(\lambda) = \prod_{\zeta \in S(\ell_i, a_i)} (\lambda + \zeta).$$

Since the Drinfel’d polynomial $P_V(\lambda)$ determines the isomorphism class of the $\mathcal{B}$-module $V$ of type $(1, 1)$ [4, the injectivity part of Theorem 1.9'], there are no other finite-dimensional irreducible $\mathcal{B}$-modules of type $(1, 1)$. This means that every finite-dimensional irreducible $\mathcal{B}$-module of type $(1, 1)$ comes from some tensor product of evaluation modules for $U_q(L(\mathfrak{sl}_2))$. Let $V$ be a finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$-module of type $(1, 1)$. Then $V$ is irreducible as a $\mathcal{B}$-module and so there exists an irreducible $U_q(L(\mathfrak{sl}_2))$-module $V' = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ such that $V, V'$ are isomorphic as $\mathcal{B}$-modules. By Lemma 2, $V, V'$ are isomorphic as $U_q(L(\mathfrak{sl}_2))$-modules. This completes the proof of part (iii) of the theorem for $U_q(L(\mathfrak{sl}_2))$. The proof of Theorem 1 can be given by an argument very similar to the one we have seen above for the case of $U_q(L(\mathfrak{sl}_2))$. We prepare two more lemmas for the case of $U'_q(L(\mathfrak{sl}_2))$ to make the point clearer. Set $(\varepsilon, \varepsilon^*) = (1, 0)$ and let $\mathcal{T}_q$ be the augmented TD-algebra defined by (TD)$_0$, (TD)$_1$ in (2), (3). For $s \in \mathbb{C}^\times$, let $\varphi_s$ be the embedding of $\mathcal{T}_q$ into $U'_q(L(\mathfrak{sl}_2))$ given by (5), (6).

**Lemma 3.** Let $V_1, V_2$ be finite-dimensional irreducible $U'_q(L(\mathfrak{sl}_2))$-modules. If $V_1, V_2$ are isomorphic as $\varphi_s(\mathcal{T}_q)$-modules for some $s \in \mathbb{C}^\times$, then $V_1, V_2$ are isomorphic as $U'_q(L(\mathfrak{sl}_2))$-modules.

**Proof.** By (5), $\varphi_s(\mathcal{T}_q)$ is generated by $s e_0^+ + s^{-1} e_1^-$, $e_1^+$ and $k_i^{1 \pm 1}$ ($i = 0, 1$). Since $(e_1^+, k_1^{1 \pm 1})$ is isomorphic to the quantum algebra $U_q(\mathfrak{sl}_2)$, the action of $e_1^+$ on $V_i$ ($i = 1, 2$) is uniquely determined by those of $e_1^+, k_1^{1 \pm 1} \in \varphi_s(\mathcal{T}_q)$ by Lemma 2. Apparently the action of $e_0^+$ on $V_i$ ($i = 1, 2$) is uniquely determined by those of $s e_0^+ + s^{-1} e_1^-$, $e_1^-$, $k_1$, and hence by that of $\varphi_s(\mathcal{T}_q)$. So the action of $U'_q(L(\mathfrak{sl}_2))$ on $V_i$ ($i = 1, 2$) is uniquely determined by that of $\varphi_s(\mathcal{T}_q)$. \qed
Lemma 4. Let $V$ be a finite-dimensional irreducible $U_q'(L(sl_2))$-module of type $(1, 1)$. Then there exists a finite set $\Lambda$ of nonzero scalars such that $V$ is irreducible as a $\varphi_s(T_q)$-module for each $s \in \mathbb{C}^\times - \Lambda$.

Proof. For $s \in \mathbb{C}^\times$, regard $V$ be a $\varphi_s(T_q)$-module. Let $W$ be a minimal $\varphi_s(T_q)$-submodule of $V$. It is enough to show that $W = V$ holds if $s$ avoids finitely many scalars. By (10) with $s_0 = 1$, the eigenspace decomposition of $k_1 = k_0^{-1}$ on $V$ is $V = \bigoplus_{i=0}^d U_{d-i}$, $k_1 |_{U_{d-i}} = q^{2i-d}$ $(0 \leq i \leq d)$. The subalgebra $\langle e_1^\pm, k_1 \pm 1 \rangle$ of $U_q'(L(sl_2))$ is isomorphic to the quantum algebra $U = U_q(sl_2)$ in (15) via the correspondence of $e_1^\pm, k_1 \pm 1$ to $X^\pm, K^\pm$. The element $(e_1^+)^{d-2i}$ maps $U_{d-i}$ onto $U_i$ bijectively $(0 \leq i \leq \lfloor d/2 \rfloor)$. Also $(e_1^- k_1)^{d-2i}$ maps $U_i$ onto $U_{d-i}$ bijectively $(0 \leq i \leq \lfloor d/2 \rfloor)$.

The element $(e_1^+)^{d-2i}$ belongs to $\varphi_s(T_q)$. So $(e_1^+)^{d-2i}W \subseteq W$. Since the mapping $(e_1^+)^{d-2i} : U_{d-i} \rightarrow U_i$ is a bijection, we have $\dim(W \cap U_{d-i}) \leq \dim(W \cap U_i)$ $(0 \leq i \leq \lfloor d/2 \rfloor)$.

The element $(e_1^- k_1)^{d-2i}$ does not belong to $\varphi_s(T_q)$, but $(e_1^+ - s^2 e_1^- k_1)^{d-2i}$ does. By (9), $(e_1^+ - s^2 e_1^- k_1)^{d-2i}$ maps $U_i$ to $U_{d-i} (0 \leq i \leq \lfloor d/2 \rfloor)$. We want to show it is a bijection if $s$ avoids finitely many scalars. Identify $U_{d-i}$ with $U_i$ as vector spaces by the bijection $(e_1^-)^{d-2i}$ between them. Then it makes sense to consider the determinant of a linear map from $U_i$ to $U_{d-i}$. Set $t = s^{-2}$ and expand $(e_1^+ + te_1^- k_1)^{d-2i}$ as

$$t^{d-2i}(e_1^- k_1)^{d-2i} + \text{lower terms in } t.$$  

Each term of the expansion gives a linear map from $U_i$ to $U_{d-i}$. So the determinant of $(e_1^+ + te_1^- k_1)^{d-2i} |_{U_i}$ equals

$$t^{(d-2i)\dim U_i} \det(e_1^- k_1)^{d-2i} |_{U_i} + \text{lower terms in } t,$$  

and this is a polynomial in $t$ of degree $(d-2i)\dim U_i$, since $\det(e_1^- k_1)^{d-2i} |_{U_i} \neq 0$. Let $\Lambda_i$ be the set of nonzero $s$ such that $t = s^{-2}$ is not a root of the polynomial in (16). Then if $s \in \mathbb{C}^\times - \Lambda_i$, $(e_1^+ + s^2 e_1^- k_1)^{d-2i}$ maps $U_i$ to $U_{d-i}$ bijectively.

Set $\Lambda = \bigcup_{i=0}^{\lfloor d/2 \rfloor} \Lambda_i$. Choose $s \in \mathbb{C}^\times - \Lambda$. Then the mapping $(e_1^+ + s^2 e_1^- k_1)^{d-2i} : U_i \rightarrow U_{d-i}$ is a bijection for $0 \leq i \leq \lfloor d/2 \rfloor$. Since $(e_1^+ + s^2 e_1^- k_1)$ belongs to $\varphi_s(T_q)$, we have $(e_1^+ + s^2 e_1^- k_1)^{d-2i}W \subseteq W$ and so $\dim(W \cap U_i) \leq \dim(W \cap U_{d-i})$. Since we have already shown $\dim(W \cap U_{d-i}) \leq \dim(W \cap U_i)$, we obtain $\dim(W \cap U_i) = \dim(W \cap U_{d-i}) (0 \leq i \leq \lfloor d/2 \rfloor)$. Therefore by Lemma 1, we have $e_1^i W \subseteq W$. Since $(e_1^+ + s^2 e_1^- k_1)W \subseteq W$, the inclusion
Proof of Theorem 1. We use the classification of finite-dimensional irreducible $\mathcal{T}_q$-modules in the case of $(\varepsilon, \varepsilon^*) = (1, 0)$ [4, Theorem 1.18]:

(i) A tensor product $V = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ of evaluation modules is irreducible as a $\varphi_s(\mathcal{T}_q)$-module if and only if 

$$s^{-2} \notin S(\ell_i, a_i)$$

for all $i \in \{1, \ldots, n\}$ and $S(\ell_i, a_i), S(\ell_j, a_j)$ are in general position for all $i, j \in \{1, \ldots, n\}$.

(ii) Consider two tensor products $V = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$, $V' = V(\ell') \otimes V(\ell'_1, a'_1) \otimes \cdots \otimes V(\ell'_m, a'_m)$ of evaluation modules and assume that they are both irreducible as a $\varphi_s(\mathcal{T}_q)$-module. Then, $V, V'$ are isomorphic as $\varphi_s(\mathcal{T}_q)$-modules if and only if $\ell = \ell'$, $n = m$ and $(\ell_i, a_i) = (\ell'_i, a'_i)$ for all $i \in \{1, \ldots, n\}$ with a suitable reordering of the evaluation modules $V(\ell_1, a_1), \ldots, V(\ell_n, a_n)$.

(iii) Every finite-dimensional irreducible $\mathcal{T}_q$-module $V$ ($\dim V \geq 2$) is isomorphic to a $\mathcal{T}_q$-module $V' = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ on which $\mathcal{T}_q$ acts via some embedding $\varphi_s : \mathcal{T}_q \rightarrow U'_q(L(\mathfrak{sl}_2))$.

Part (i) of Theorem 1 follows immediately from the part (i) above, due to Lemma 4. Part (ii) of Theorem 2 follows immediately from the part (ii) above, the ‘if’ part due to Lemma 3 (and Lemma 4) and the ‘only if’ part due to Lemma 4.

We want to show part (iii) of Theorem 1. Let $V$ be a finite-dimensional irreducible $U'_q(L(\mathfrak{sl}_2))$-module of type $(1, 1)$. By Lemma 4, there exists a nonzero scalar $s$ such that $V$ is irreducible as a $\varphi_s(\mathcal{T}_q)$-module. By the part (iii) above, the $\mathcal{T}_q$-module $V$ via $\varphi_s$ is isomorphic to some $\mathcal{T}_q$-module $V'' = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ via some embedding $\varphi_{s'}$ of $\mathcal{T}_q$ into $U'_q(L(\mathfrak{sl}_2))$. Since $k_0$ has the same eigenvalues on $V, V'$, we have $s = s'$ and so $V, V''$ are isomorphic as $\varphi_s(\mathcal{T}_q)$-modules. By Lemma 3, $V, V''$ are isomorphic as $U'_q(L(\mathfrak{sl}_2))$-modules. This completes the proof of Theorem 1.

3 Intertwiners

In this section, we show that for $\ell, m \in \mathbb{Z}_{\geq 0}, a \in \mathbb{C}^\times$, there exists an intertwiner between the $U'_q(L(\mathfrak{sl}_2))$-modules $V(\ell, a) \otimes V(m), V(m) \otimes V(\ell, a)$, i.e.,
a nonzero linear map \( R \) from \( V(\ell, a) \otimes V(m) \) to \( V(m) \otimes V(\ell, a) \) such that

\[
R \Delta(\xi) = \Delta(\xi)R \quad (\forall \xi \in U'_q(L(\mathfrak{sl}_2))).
\]  

(17)

If such an intertwiner \( R \) exists, then it is routinely concluded that \( V(\ell, a) \otimes V(m) \) is isomorphic to \( V(m) \otimes V(\ell, a) \) as \( U'_q(L(\mathfrak{sl}_2)) \)-modules and any other intertwiner is a scalar multiple of \( R \); since \( V(m) \otimes V(\ell, a) \) is irreducible as a \( U'_q(L(\mathfrak{sl}_2)) \)-module by Theorem 1.

Using the theory of Drinfel’d polynomials [4] for the augmented TD-algebra \( T_q = T_q(\varepsilon, \varepsilon^*) \) with \( (\varepsilon, \varepsilon^*) = (1, 0) \), we shall firstly show that \( V(\ell, a) \otimes V(m) \) is isomorphic to \( V(m) \otimes V(\ell, a) \) as \( U'_q(L(\mathfrak{sl}_2)) \)-modules. We shall then construct an intertwiner explicitly.

Let us denote the \( U'_q(L(\mathfrak{sl}_2)) \)-modules \( V(\ell, a) \otimes V(m) \), \( V(m) \otimes V(\ell, a) \) by \( V, V' \):

\[
V = V(\ell, a) \otimes V(m), \quad V' = V(m) \otimes V(\ell, a).
\]

Recall we assume that the integers \( \ell, m \) and the scalar \( a \) are nonzero. Let us denote a standard basis of the \( U'_q(L(\mathfrak{sl}_2)) \)-module \( V(\ell, a) \) (resp. \( V(m) \)) by \( v_0, v_1, \ldots, v_\ell \) (resp. \( v'_0, v'_1, \ldots, v'_m \)) in the sense of (11). Recall \( V(m) \) is an abbreviation of \( V(m, 0) \) and the action of \( U'_q(L(\mathfrak{sl}_2)) \) on \( V, V' \) are via the coproduct \( \Delta \) of (12).

Let \( \mathcal{U} \) denote the subalgebra of \( U'_q(L(\mathfrak{sl}_2)) \) generated by \( e_1^\pm, K_1^\pm \). The subalgebra \( \mathcal{U} \) is isomorphic to the quantum algebra \( U_q(\mathfrak{sl}_2) \). Let \( V(n) \) denote the irreducible \( \mathcal{U} \)-module of dimension \( n + 1 \): \( V(n) \) has a standard basis \( x_0, x_1, \ldots, x_n \) on which \( \mathcal{U} \) acts as

\[
\begin{cases} 
  k_1 x_i = q^{n-2i} x_i, \\
  e_1^+ x_i = [n-i+1] x_{i-1}, \\
  e_1^- x_i = [i+1] x_{i+1},
\end{cases}
\]

(18)

where 

\[
[t]_q = (q^t - q^{-t})/(q - q^{-1}), \quad x_{-1} = x_{n+1} = 0. \quad \text{We call } x_n \text{ (resp. } x_0) \text{ the lowest (highest) weight vector: } k_1 x_n = q^{-n} x_n, e_1^- x_n = 0 \quad (k_1 x_0 = q^n x_0, e_1^+ x_0 = 0). \quad \text{Note that } V(\ell, a) \text{ is isomorphic to } V(\ell) \text{ as } \mathcal{U} \text{-modules.}
\]

By the Clebsch-Gordan formula, \( V = V(\ell, a) \otimes V(m) \) is decomposed into the direct sum of \( \mathcal{U} \)-submodules \( \tilde{V}(n) \) \( (|\ell - m| \leq n \leq \ell + m, \ n \equiv \ell + m \mod 2) \), where \( \tilde{V}(n) \) is the unique irreducible \( \mathcal{U} \)-submodule of \( V \) isomorphic to \( V(n) \).

With \( n = \ell + m - 2\nu \), we have

\[
V = V(\ell, a) \otimes V(m) = \bigoplus_{\nu=0}^{\min\{\ell, m\}} \tilde{V}(\ell + m - 2\nu).
\]

(19)
Let $\tilde{x}_n$ denote a lowest weight vector of the $U$-module $\tilde{V}(n)$. So

$$\begin{align*}
\Delta(k_1)\tilde{x}_n &= q^{-n}\tilde{x}_n, \\
\Delta(e_{-1})\tilde{x}_n &= 0.
\end{align*}$$

(20)

Since $V$ has a basis $\{v_{\ell-i} \otimes v_{m-j}' \mid 0 \leq i \leq \ell, 0 \leq j \leq m\}$ and $k_1$ acts on it by $\Delta(k_1)(v_{\ell-i} \otimes v_{m-j}') = q^{-(\ell+m)+2(i+j)}v_{\ell-i} \otimes v_{m-j}'$, the lowest weight vector $\tilde{x}_n$ of $\tilde{V}(n)$ can be expressed as

$$\tilde{x}_n = \sum_{i+j=\nu} c_j v_{\ell-i} \otimes v_{m-j}' \quad (n = \ell + m - 2\nu).$$

(21)

Solving $\Delta(e_{-1})\tilde{x}_n = 0$ for the coefficients $c_j$, we obtain

$$\frac{c_j}{c_{j-1}} = -q^{m-2j+2} \frac{[\ell - \nu + j]}{[m - j + 1]}$$

and so with a suitable choice of $c_0$

$$\tilde{x}_n = \sum_{j=0}^{\nu} (-1)^j q^{j(m-j+1)}[\ell - \nu + j]![m - j]! v_{\ell-\nu+j} \otimes v_{m-j}'.$$

(22)

where $n = \ell + m - 2\nu$ and $[t]! = [t][t-1] \cdots [1]$.

**Lemma 5.** $\Delta(e_{0}^+)\tilde{x}_n = aq\tilde{x}_{n+2}$.

**Proof.** By (12), we have $\Delta(e_{0}^+) = e_{0}^+ \otimes 1 + k_0 \otimes e_{0}^+$. By (11), the element $e_{0}^+$ vanishes on $V(m)$ and acts on $V(\ell, a)$ as $aqe_{-1}^-$. Since $e_{-1}^-v_{\ell-\nu+j} = [\ell - (\nu - 1) + j]v_{\ell-(\nu-1)+j}$, the result follows from (22), using $v_{\ell+1} = 0$. $\square$

**Corollary 1.** Any nonzero $U'_q(L(\mathfrak{sl}_2))$-submodule of $V(\ell, a) \otimes V(m)$ contains $\tilde{x}_{\ell+m}$, the lowest weight vector of the $U$-module $V(\ell, a) \otimes V(m)$.

We are ready to prove our second main result.

**Theorem 2.** The $U'_q(L(\mathfrak{sl}_2))$-modules $V(\ell, a) \otimes V(m)$, $V(m) \otimes V(\ell, a)$ are isomorphic for every $\ell, m \in \mathbb{Z}_{>0}$, $a \in \mathbb{C}^\times$.

**Proof.** Let $T_q = T_q^{(\varepsilon, \varepsilon^*)}$ be the augmented TD-algebra with $(\varepsilon, \varepsilon^*) = (1, 0)$. Let $\varphi_a : T_q \rightarrow U'_q(L(\mathfrak{sl}_2))$ denote the embedding of $T_q$ into $U'_q(L(\mathfrak{sl}_2))$ given
in (6). By Theorem 5.2 of [4], the Drinfel’d polynomial \( P_V(\lambda) \) of the \( \varphi_s(\mathcal{T}_q) \)-module \( V = V(\ell, a) \otimes V(m) \) turns out to be

\[
P_V(\lambda) = \lambda^m \prod_{i=0}^{\ell-1} (\lambda + aq^{2i-\ell+1}). \tag{23}
\]

(Note that the parameter \( s \) of the embedding \( \varphi_s \) does not appear in \( P_V(\lambda) \).

So the polynomial \( P_V(\lambda) \) can be called the Drinfel’d polynomial attached to the \( U'_q(\mathfrak{sl}_2) \)-module \( V \).)

Let \( W \) be a minimal \( U'_q(\mathfrak{sl}_2) \)-submodule of \( V \). By Corollary 1, \( W \) contains the lowest and hence highest weight vectors of \( V \). In particular, the irreducible \( U'_q(\mathfrak{sl}_2) \)-module \( W \) is of type \((1, 1)\). By Lemma 4, there exists a finite set \( \Lambda \) of nonzero scalars such that \( W \) is irreducible as a \( \varphi_s(\mathcal{T}_q) \)-module for any \( s \in \mathbb{C}^\times - \Lambda \). By the definition [4, (25)], the Drinfel’d polynomial \( P_W(\lambda) \) of the irreducible \( \varphi_s(\mathcal{T}_q) \)-module \( W \) coincides with \( P_V(\lambda) \):

\[
P_W(\lambda) = P_V(\lambda). \tag{24}
\]

By Theorem 1, \( V' = V(m) \otimes V(\ell, a) \) is irreducible as a \( U'_q(\mathfrak{sl}_2) \)-module. So by Lemma 4, there exists a finite set \( \Lambda' \) of nonzero scalars such that \( V' \) is irreducible as a \( \varphi_s(\mathcal{T}_q) \)-module for any \( s \in \mathbb{C}^\times - \Lambda' \). By Theorem 5.2 of [4], the Drinfel’d polynomial \( P_{V'}(\lambda) \) of the irreducible \( \varphi_s(\mathcal{T}_q) \)-module \( V' \) coincides with \( P_V(\lambda) \):

\[
P_{V'}(\lambda) = P_V(\lambda). \tag{25}
\]

Both of the irreducible \( \varphi_s(\mathcal{T}_q) \)-modules \( W, V' \) have type \( s \), diameter \( d = \ell + m \) and the Drinfel’d polynomial \( P_V(\lambda) \). By Theorem 1.9’ of [4], \( W \) and \( V' \) are isomorphic as \( \varphi_s(\mathcal{T}_q) \)-modules. By Lemma 3, \( W \) and \( V' \) are isomorphic as \( U'_q(\mathfrak{sl}_2) \)-modules. In particular, \( \dim W = \dim V' \). Since \( \dim V' = \dim V \), we have \( W = V \), i.e., \( V \) and \( V' \) are isomorphic as \( U'_q(\mathfrak{sl}_2) \)-modules. \( \Box \)

Finally we want to construct an intertwiner \( R \) between the irreducible \( U'_q(\mathfrak{sl}_2) \)-modules \( V, V' \). Regard \( V' = V(m) \otimes V(\ell, a) \) as a \( \mathcal{U} \)-module. By the Clebsch-Gordan formula, we have the direct sum decomposition

\[
V' = V(m) \otimes V(\ell, a) = \bigoplus_{\nu=0}^{\min\{\ell, m\}} \tilde{V}'(\ell + m - 2\nu), \tag{26}
\]

15
where \( \tilde{V}'(n) \) is the unique irreducible \( \mathcal{U} \)-submodule of \( V' \) isomorphic to \( V(n) \) \((n = \ell + m - 2\nu)\). Let \( \tilde{x}_n' \) be a lowest weight vector of the \( \mathcal{U} \)-module \( \tilde{V}'(n) \). By (22), we have

\[
\tilde{x}_n' = \sum_{j=0}^{\nu} (-1)^j q^{j(\ell-j+1)} [m - \nu + j]! [\ell - j]! v'_{m-\nu+j} \otimes v_{\ell-j}
\]  

(27)

up to a scalar multiple, where \( n = \ell + m - 2\nu \). It can be easily checked as in Lemma 5 that the lowest weight vectors \( \tilde{x}_n' \) \((n = \ell + m - 2\nu, 0 \leq \nu \leq \min\{\ell, m\}\)) are related by

\[
(e_1 \otimes 1) \tilde{x}_n' = \tilde{x}_{n+2}',
\]

(28)

where \( V' = V(m) \otimes V(\ell, a) \) is regarded as a \((\mathcal{U} \otimes \mathcal{U})\)-module in the natural way.

**Lemma 6.** \( \Delta(e_0^+) \tilde{x}_n' = -aq \cdot q^{n+2} \tilde{x}_{n+2}'. \)

**Proof.** We have \( \Delta(e_0^+) \tilde{x}_n' = aq \,(k_1^{-1} \otimes e_1^-) \tilde{x}_n' \), since \( \Delta(e_0^+) = e_0^+ \otimes 1 + k_0 \otimes e_0^+ \), and \( e_0^+ \) vanishes on \( V(m) \) and acts on \( V(\ell, a) \) as \( aq e_1^- \). Express \( k_1^{-1} \otimes e_1^- \) as

\[
k_1^{-1} \otimes e_1^- = (k_1^{-1} \otimes 1) (1 \otimes e_1^-) = (k_1^{-1} \otimes 1) (\Delta(e_1^-) - e_1^- \otimes k_1^{-1}) = (k_1^{-1} \otimes 1) \Delta(e_1^-) - k_1^{-1} e_1^- \otimes k_1^{-1} = (k_1^{-1} \otimes 1) \Delta(e_1^-) - q^2 (e_1^- \otimes 1) \Delta(k_1^{-1}).
\]

Since \( \Delta(e_1^-) \tilde{x}_n' = 0 \), \( \Delta(k_1^{-1}) \tilde{x}_n' = q^a \tilde{x}_n' \), the result follows from (28). \( \square \)

There exists a unique linear map

\[
R_n : V = V(\ell, a) \otimes V(m) \longrightarrow \tilde{V}'(n)
\]

that commutes with the action of \( \mathcal{U} \) and sends \( \tilde{x}_n \) to \( \tilde{x}_n' \). The linear map \( R_n \) vanishes on \( \tilde{V}(t) \) for \( t \neq n \) and affords an isomorphism between \( \tilde{V}(n) \) and \( \tilde{V}'(n) \) as \( \mathcal{U} \)-modules. If \( R \) is an intertwiner in the sense of (17), \( R \) can be expressed as

\[
R = \sum_{\nu=0}^{\min\{\ell, m\}} \alpha_\nu R_{\ell+m-2\nu},
\]

(29)

regarding \( R \) as an intertwiner for the \( \mathcal{U} \)-modules \( V, V' \). By (17), we have

\[
R \Delta(e_0^+) = \Delta(e_0^+) R.
\]

(30)

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Apply (30) to the lowest weight vector $\tilde{x}_n$ in (22). By Lemma 5, $\Delta(e_0^+) \tilde{x}_n = aq \tilde{x}_{n+2}$ and so with $n = \ell + m - 2\nu$, we have

$$R \Delta(e_0^+) \tilde{x}_n = aq \alpha_{\nu-1} \tilde{x}_{n+2}'.$$  \hspace{1cm} (31)

On the other hand, $R \tilde{x}_n = \alpha_\nu \tilde{x}_n' (n = \ell + m - 2\nu)$ and so by Lemma 6, we have

$$\Delta(e_0^+) R \tilde{x}_n = -aq \alpha_\nu q^{n+2} \tilde{x}_{n+2}'.$$  \hspace{1cm} (32)

By (31), (32), we have $\alpha_\nu/\alpha_{\nu-1} = -q^{-n+2} = -q^{-\ell-m+2(\nu-1)}$ and so

$$\alpha_\nu = (-1)^\nu q^{-\nu(\ell+m-\nu+1)}$$  \hspace{1cm} (33)

with a suitable choice of $\alpha_0$. An intertwiner exists by Theorem 2. If it exists, it has to be in the form of (29), (33). Thus we obtain our third main result.

**Theorem 3.** The linear map

$$R = \sum_{\nu=0}^{\min\{\ell,m\}} (-1)^\nu q^{-(\ell+m-\nu+1)} R_{\ell+m-2\nu}$$

is an intertwiner between the $U'_q(L(\mathfrak{sl}_2))$-modules $V(\ell, a) \otimes V(m)$, $V(m) \otimes V(\ell, a)$.

**References**

[1] G. Benkart, P. Terwilliger, Irreducible modules for the quantum affine algebra $U_q(\hat{\mathfrak{sl}}_2)$ and its Borel subalgebra, J. Algebra 282 (2004) 172-194.

[2] V. Chari, A. Pressley, Quantum affine algebras, Commun. Math. Phys. 142 (1991) 261-283.

[3] T. Ito, K. Tanabe, P. Terwilliger, Some algebra related to P- and Q-polynomial association schemes, in: Codes and Association Schemes (Piscataway NJ, 1999), Amer. Math. Soc., Providence RI, 2001, pp. 167-192.

[4] T. Ito, P. Terwilliger, The Augmented Tridiagonal Algebra, Kyushu J. Math. 64 (2010) 81-144.
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