Interacting spinor and scalar fields in Bianchi type-I Universe filled with viscous fluid: exact and numerical solutions

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(Dated: April 22, 2019)

We consider a self-consistent system of spinor and scalar fields within the framework of a Bianchi type I gravitational field filled with viscous fluid in presence of a $\Lambda$ term. Exact self-consistent solutions to the corresponding spinor, scalar and BI gravitational field equations are obtained in terms of $\tau$, where $\tau$ is the volume scale of BI universe. System of equations for $\tau$ and $\varepsilon$, where $\varepsilon$ is the energy of the viscous fluid, is deduced. Some special cases allowing exact solutions are thoroughly studied.

PACS numbers: 03.65.Pm and 04.20.Ha
Keywords: Spinor field, Bianchi type I (BI) model, Cosmological constant

I. INTRODUCTION

The investigation of relativistic cosmological models usually has the energy momentum tensor of matter generated by a perfect fluid. To consider more realistic models one must take into account the viscosity mechanisms, which have already attracted the attention of many researchers [1, 2, 3, 4, 5, 6, 7, 8, 9, 10].

The nature of cosmological solutions for homogeneous Bianchi type I (BI) model was investigated by Belinsky and Khalatnikov [11] by taking into account dissipative process due to viscosity. In [12, 13] we reinvestigate the problem posed in [11] in presence of a $\Lambda$ term. Though that Murphy [9] claimed that the introduction of bulk viscosity can avoid the initial singularity at finite past, but Belinsky and Khalatnikov [11] showed that viscosity cannot remove the cosmological singularity but results in a qualitatively new behavior of the solutions near singularity. To eliminate the initial singularities a self-consistent system of nonlinear spinor and BI gravitational field was considered by us in a series of papers [14, 15, 16, 17]. For some cases we were able to find field configurations those were always regular. Recently it was found that the introduction of a spinor field into the system may explain the late time acceleration of the Universe, hence can be considered as an alternative to dark energy [18].

Given the importance of viscous fluid and spinor field to construct a more realistic model of the Universe, we in [19, 20] we introduced spinor field into the system and solved the system for some special choice of viscosity. The purpose of this paper is to study an interacting system of spinor and scalar fields within the scope of a Bianchi type I cosmological model filled with viscous fluid in presence of a $\Lambda$ term and clarify the role of viscosity and field interaction in the evolution of the Universe.

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II. DERIVATION OF BASIC EQUATIONS

In this section we derive the fundamental equations for the interacting spinor, scalar and gravitational fields from the action and write their solutions in term of the volume scale $\tau$ defined bellow (2.16). We also derive the equation for $\tau$ which plays the central role here.

We consider a system of nonlinear spinor, scalar and BI gravitational field in presence of perfect fluid given by the action

$$\mathcal{S}(g; \psi, \bar{\psi}) = \int \mathcal{L} \sqrt{-g} d\Omega$$

with

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_{ss} + \mathcal{L}_m.$$  \hspace{1cm} (2.2)

The gravitational part of the Lagrangian (2.2) is given by a Bianchi type I (BI hereafter) space-time, whereas $\mathcal{L}_{ss}$ describes the interacting spinor and scalar field lagrangian and $\mathcal{L}_m$ stands for the lagrangian density of viscous fluid.

A. Material field Lagrangian

We choose the interacting spinor and scalar field Lagrangian as

$$\mathcal{L}_{ss} = \frac{i}{2} \left[ \bar{\psi} \gamma^\mu \nabla_\mu \psi - \nabla_\mu \bar{\psi} \gamma^\mu \psi \right] - m \bar{\psi} \psi + \frac{1}{2} \phi, \alpha \phi^{\alpha}(1 + \lambda F),$$  \hspace{1cm} (2.3)

Here $m$ is the spinor mass, $\lambda$ is the coupling constant and $F = F(I, J)$ with $I = I_S = S^2 = (\bar{\psi} \psi)^2$ and $J = I_P = P^2 = (i \bar{\psi} \gamma^5 \psi)^2$. We would like to mention that there are 5 invariants constructed from bilinear spinor from. Using Fierz transformation it can be shown that among the five invariants only $I$ and $J$ are independent as all other can be expressed by them: $I_V = -I_A = I_S + I_P$ and $I_Q = 2(I_S - I_P)$ [21, 22, 23, 24]. Since the the bilinear identities appear to have been given in literature only partially and the relation between the invariants are given as problem [cf. eg. [24, 25]], we work them out in the appendix below. Therefore, the choice $F = F(I, J)$, describes the nonlinearity in the most general of its form [15]. Note that setting $\lambda = 0$ in (2.3) we come to the case with minimal coupling.

B. The gravitational field

As a gravitational field we consider the Bianchi type I (BI) cosmological model. It is the simplest model of anisotropic universe that describes a homogeneous and spatially flat space-time and if filled with perfect fluid with the equation of state $p = \zeta \epsilon$, $\zeta < 1$, it eventually evolves into a FRW universe [26, 27]. The isotropy of present-day universe makes BI model a prime candidate for studying the possible effects of an anisotropy in the early universe on modern-day data observations. In view of what has been mentioned above we choose the gravitational part of the Lagrangian (2.2) in the form

$$\mathcal{L}_g = \frac{R}{2\kappa},$$  \hspace{1cm} (2.4)

where $R$ is the scalar curvature, $\kappa = 8\pi G$ being the Einstein’s gravitational constant. The gravitational field in our case is given by a Bianchi type I (BI) metric

$$ds^2 = dt^2 - a^2 dx^2 - b^2 dy^2 - c^2 dz^2,$$

with $a, b, c$ being the functions of time $t$ only. Here the speed of light is taken to be unity.
C. Field equations

Let us now write the field equations corresponding to the action (2.1).

Variation of (2.1) with respect to spinor field \( \psi (\bar{\psi}) \) gives spinor field equations

\[
\begin{align*}
\gamma_{\mu} \nabla_{\mu} \psi - m \psi + \mathcal{D} \psi + \bar{\mathcal{D}} \gamma_{5} \psi &= 0, \\
\nabla_{\mu} \bar{\psi} \gamma^{\mu} + m \bar{\psi} - \mathcal{D} \bar{\psi} - \bar{\mathcal{D}} \gamma_{5} \bar{\psi} &= 0,
\end{align*}
\]

(2.6a)

where we denote

\[
\mathcal{D} = \frac{\lambda}{2} \phi \phi^{\alpha} \frac{\partial F}{\partial s}, \quad \mathcal{G} = \frac{\lambda}{2} \phi \phi^{\alpha} \frac{\partial F}{\partial p}.
\]

Varying (2.1) with respect to scalar field we find

\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\nu}} (\sqrt{-g} g_{\nu\mu} (1 + \lambda F) \phi_{,\mu}) = 0.
\]

(2.7)

Variation of (2.1) with respect to metric tensor \( g_{\mu\nu} \) gives the Einstein’s field equation. For the BI space-time (2.5) on account of the \( \Lambda \) term this system has the form

\[
\begin{align*}
\ddot{b} + \frac{\dot{c}}{c} + \frac{\dot{b}}{b} &= \kappa T_{1}^{1} + \Lambda, \\
\ddot{c} + \frac{\dot{a}}{a} + \frac{\dot{c}}{c} &= \kappa T_{2}^{2} + \Lambda, \\
\ddot{a} + \frac{\dot{b}}{b} + \frac{\dot{a}}{a} &= \kappa T_{3}^{3} + \Lambda, \\
\dot{a} \dot{b} + \dot{c} \dot{a} &= \kappa T_{0}^{0} + \Lambda,
\end{align*}
\]

(2.8a)

where over dot means differentiation with respect to \( t \) and \( T_{\mu}^{\nu} \) is the energy-momentum tensor of the material field given by

\[
T_{\mu}^{\nu} = \frac{i}{4} g^{\rho\nu} \left( \bar{\psi} \gamma_{\mu} \nabla_{\nu} \psi + \bar{\psi} \gamma_{\nu} \nabla_{\mu} \psi - \nabla_{\mu} \bar{\psi} \gamma_{\nu} \psi - \nabla_{\nu} \bar{\psi} \gamma_{\mu} \psi \right) + (1 - \lambda F) \phi_{,\mu} \phi^{\rho} - \delta_{\mu}^{\rho} \mathcal{L} + T_{\mu}^{\nu}_{m}.
\]

(2.9)

Here \( T_{\mu}^{\nu}_{m} \) is the energy-momentum tensor of a viscous fluid having the form

\[
T_{\mu}^{\nu}_{m} = (\epsilon + p') u_{\mu} u^{\nu} - p' \delta_{\mu}^{\nu} + \eta g^{\nu\beta} [u_{\mu;\beta} + u_{\beta;\mu} - u_{\mu} u^{\alpha} u_{\beta;\alpha} - u_{\beta} u^{\alpha} u_{\mu;\alpha}]
\]

(2.10)

where

\[
p' = p - \left( \xi - \frac{2}{3} \eta \right) u_{\mu}^{\mu}
\]

(2.11)

Here \( \epsilon \) is the energy density, \( p \) - pressure, \( \eta \) and \( \xi \) are the coefficients of shear and bulk viscosity, respectively. In a comoving system of reference such that \( u^{\mu} = (1, 0, 0, 0) \) we have

\[
\begin{align*}
T_{0 m}^{0} &= \epsilon, \\
T_{1 m}^{1} &= -p' + 2 \eta \frac{\dot{a}}{a}, \\
T_{2 m}^{2} &= -p' + 2 \eta \frac{\dot{b}}{b}, \\
T_{3 m}^{3} &= -p' + 2 \eta \frac{\dot{c}}{c},
\end{align*}
\]

(2.12a)
In the Eqs. (2.6) and (2.9) $\nabla_\mu$ is the covariant derivatives acting on a spinor field as \cite{28, 29}

$$\nabla_\mu \psi = \frac{\partial \psi}{\partial x^\mu} - \Gamma_\mu \psi, \quad \nabla_\mu \bar{\psi} = \frac{\partial \bar{\psi}}{\partial x^\mu} + \bar{\psi} \Gamma_\mu, \quad (2.13)$$

where $\Gamma_\mu$ are the Fock-Ivanenko spinor connection coefficients defined by

$$\Gamma_\mu = \frac{1}{4} \gamma^\sigma \left( \Gamma_\mu^\nu \gamma^\nu - \partial_\mu \gamma^{\sigma} \right). \quad (2.14)$$

For the metric (2.5) one has the following components of the spinor connection coefficients

$$\Gamma_0 = 0, \quad \Gamma_1 = \frac{1}{2} \dot{a}(t) \bar{\gamma}^1 \gamma^0, \quad \Gamma_2 = \frac{1}{2} \dot{b}(t) \bar{\gamma}^2 \gamma^0, \quad \Gamma_3 = \frac{1}{2} \dot{c}(t) \bar{\gamma}^3 \gamma^0. \quad (2.15)$$

The Dirac matrices $\gamma^\mu(x)$ of curved space-time are connected with those of Minkowski one as follows:

$$\gamma^0 = \bar{\gamma}^0, \quad \gamma^1 = \bar{\gamma}^1 / a, \quad \gamma^2 = \bar{\gamma}^2 / b, \quad \gamma^3 = \bar{\gamma}^3 / c$$

with

$$\bar{\gamma}^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \bar{\gamma}^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \bar{\gamma}^5 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix},$$

where $\sigma_i$ are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that the $\gamma$ and the $\sigma$ matrices obey the following properties:

$$\gamma^i \gamma^j + \gamma^j \gamma^i = 2 \eta^{ij}, \quad i, j = 0, 1, 2, 3$$

$$\bar{\gamma}^i \gamma^5 + \gamma^5 \bar{\gamma}^i = 0, \quad (\gamma^5)^2 = I, \quad i = 0, 1, 2, 3$$

$$\sigma^i \sigma^k = \delta_{jk} + i \epsilon_{jkl} \sigma^l, \quad j, k, l = 1, 2, 3$$

where $\eta_{ij} = \{1, -1, -1, -1\}$ is the diagonal matrix, $\delta_{jk}$ is the Kronekar symbol and $\epsilon_{jkl}$ is the totally antisymmetric matrix with $\epsilon_{123} = +1$.

We study the space-independent solutions to the spinor and scalar field equations (2.6) so that $\psi = \psi(t)$ and $\varphi = \varphi(t)$. Here we define

$$\tau = abc = \sqrt{-g} \quad (2.16)$$

Under this assumption from (2.7) for the scalar field we find

$$\varphi = C \int [\tau(1 + \lambda F)]^{-1} dt, \quad C = \text{const.} \quad (2.17)$$

The spinor field equation (2.6a) in account of (2.13) and (2.15) takes the form

$$i \bar{\gamma}^0 \left( \frac{\partial}{\partial t} + \frac{\dot{t}}{2 \tau} \right) \psi - m \psi + \mathcal{D} \psi + \mathcal{G} \gamma^5 \psi = 0. \quad (2.18)$$
Setting $V_j(t) = \sqrt{\tau} \psi_j(t), \quad j = 1, 2, 3, 4,$ from (2.18) one deduces the following system of equations:

$$
\dot{V}_1 + i(m - D) V_1 - \mathcal{D} V_3 = 0, \quad (2.19a)
$$

$$
\dot{V}_2 + i(m - D) V_2 - \mathcal{D} V_4 = 0, \quad (2.19b)
$$

$$
\dot{V}_3 - i(m - D) V_3 + \mathcal{D} V_1 = 0, \quad (2.19c)
$$

$$
\dot{V}_4 - i(m - D) V_4 + \mathcal{D} V_2 = 0. \quad (2.19d)
$$

From (2.6a) we also write the equations for the invariants $S, P$ and $A = \bar{\psi} \gamma^5 \gamma^0 \psi$

$$
\dot{S}_0 - 2 \mathcal{D} A_0 = 0, \quad (2.20a)
$$

$$
\dot{P}_0 - 2(m - D) A_0 = 0, \quad (2.20b)
$$

$$
\dot{A}_0 + 2(m - D) P_0 + 2 \mathcal{D} S_0 = 0, \quad (2.20c)
$$

where $S_0 = \tau S, \quad P_0 = \tau P$, and $A_0 = \tau A$. The Eq. (2.20) leads to the following relation

$$
S^2 + P^2 + A^2 = C^2_1 / \tau^2, \quad C^2_1 = \text{const.} \quad (2.21)
$$

Giving the concrete form of $F$ from (2.19) one writes the components of the spinor functions in explicitly and using the solutions obtained one can write the components of spinor current:

$$
j^\mu = \bar{\psi} \gamma^\mu \psi. \quad (2.22)
$$

The component $j^0$

$$
 j^0 = \frac{1}{\tau} [V_1^* V_1 + V_2^* V_2 + V_3^* V_3 + V_4^* V_4], \quad (2.23)
$$

defines the charge density of spinor field that has the following chronometric-invariant form

$$
\rho = (j^0 \cdot j^0)^{1/2}. \quad (2.24)
$$

The total charge of spinor field is defined as

$$
Q = \int_{-\infty}^{\infty} \rho \sqrt{-g} dx dy dz = \rho \tau \mathcal{V}, \quad (2.25)
$$

where $\mathcal{V}$ is the volume. From the spin tensor

$$
S^{\mu \nu \epsilon} = \frac{1}{4} \bar{\psi} \{ \gamma^\epsilon \sigma^{\mu \nu} + \sigma^{\mu \nu} \gamma^\epsilon \} \psi. \quad (2.26)
$$

one finds chronometric invariant spin tensor

$$
S^{i,j,0}_{\text{ch}} = (S_{ij,0} S^{ij,0})^{1/2}, \quad (2.27)
$$

and the projection of the spin vector on $k$ axis

$$
S_k = \int_{-\infty}^{\infty} S^{i,j,0}_{\text{ch}} \sqrt{-g} dx dy dz = S^{i,j,0}_{\text{ch}} \tau \mathcal{V}. \quad (2.28)
$$
Let us now solve the Einstein equations. To do it we first write the expressions for the components of the energy-momentum tensor explicitly:

\[
T^0_0 = mS + \frac{C^2}{2\tau^2(1 + \lambda F)} + \epsilon \equiv \tilde{T}^0_0, \tag{2.29a}
\]

\[
T^1_1 = \mathcal{D}S + \mathcal{G}P - \frac{C^2}{2\tau^2(1 + \lambda F)} - p' + 2\eta \frac{\dot{a}}{a} \equiv \tilde{T}^1_1 + 2\eta \frac{\dot{a}}{a}, \tag{2.29b}
\]

\[
T^2_2 = \mathcal{D}S + \mathcal{G}P - \frac{C^2}{2\tau^2(1 + \lambda F)} - p' + 2\eta \frac{\dot{b}}{b} \equiv \tilde{T}^2_1 + 2\eta \frac{\dot{b}}{b}, \tag{2.29c}
\]

\[
T^3_3 = \mathcal{D}S + \mathcal{G}P - \frac{C^2}{2\tau^2(1 + \lambda F)} - p' + 2\eta \frac{\dot{c}}{c} \equiv \tilde{T}^3_1 + 2\eta \frac{\dot{c}}{c}. \tag{2.29d}
\]

In account of (2.29) from (2.8) we find the metric functions [15]

\[
a(t) = Y_1 \tau^{1/3} \exp \left[ \frac{X_1}{3} \int \frac{e^{-2\xi \int \eta dt}}{\tau(t)} dt \right], \tag{2.30a}
\]

\[
b(t) = Y_2 \tau^{1/3} \exp \left[ \frac{X_2}{3} \int \frac{e^{-2\xi \int \eta dt}}{\tau(t)} dt \right], \tag{2.30b}
\]

\[
c(t) = Y_3 \tau^{1/3} \exp \left[ \frac{X_3}{3} \int \frac{e^{-2\xi \int \eta dt}}{\tau(t)} dt \right], \tag{2.30c}
\]

with the constants \(Y_i\) and \(X_i\) obeying

\[Y_1 Y_2 Y_3 = 1, \quad X_1 + X_2 + X_3 = 0.\]

As one sees from (2.30a), (2.30b) and (2.30c), for \(\tau = t^n\) with \(n > 1\) the exponent tends to unity at large \(t\), and the anisotropic model becomes isotropic one.

Further we will investigate the existence of singularity (singular point) of the gravitational case, which can be done by investigating the invariant characteristics of the space-time. In general relativity these invariants are composed from the curvature tensor and the metric one. In a 4D Riemann space-time there are 14 independent invariants. Instead of analyzing all 14 invariants, one can confine this study only in 3, namely the scalar curvature \(I_1 = R\), \(I_2 = R_{\mu\nu\mu\nu}^R\), and the Kretschmann scalar \(I_3 = R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}\). At any regular space-time point, these three invariants \(I_1, I_2, I_3\) should be finite. One can easily verify that

\[I_1 \propto \frac{1}{\tau^2}, \quad I_2 \propto \frac{1}{\tau^4}, \quad I_3 \propto \frac{1}{\tau^6}.\]

Thus we see that at any space-time point, where \(\tau = 0\) the invariants \(I_1, I_2, I_3\), as well as the scalar and spinor fields become infinity, hence the space-time becomes singular at this point.

In what follows, we write the equation for \(\tau\) and study it in details.

Summation of Einstein equations (2.8a), (2.8b), (2.8c) and (2.8d) multiplied by 3 gives

\[
\tau = \frac{3}{2} \kappa \left( T^0_0 + \tilde{T}^1_1 \right) \tau + 3\kappa \eta \tau + 3\Lambda \tau, \tag{2.31}
\]

which can be rearranged as

\[
\tau - \frac{3}{2} \kappa \xi \tau = \frac{3}{2} \kappa \left( mS + \mathcal{D}S + \mathcal{G}P + \epsilon - p \right) \tau + 3\Lambda \tau. \tag{2.32}
\]
For the right-hand-side of (2.32) to be a function of $\tau$ only, the solution to this equation is well-known [30].

On the other hand from Bianchi identity $G_{\mu;\nu}^\nu = 0$ one finds

$$T_{\mu;\nu}^\nu = T_{\mu;\nu}^\nu + \Gamma_{\rho \nu}^\nu T_{\rho}^\mu - \Gamma_{\mu \nu}^\nu T_{\rho}^\nu = 0,$$

which in our case has the form

$$\frac{1}{\tau} (\tau T_0^0) - \frac{\dot{a}}{a} T_1^1 - \frac{\dot{b}}{b} T_2^2 - \frac{\dot{c}}{c} T_3^3 = 0.$$  \hfill (2.34)

This equation can be rewritten as

$$\dot{T}_0^0 = \frac{\dot{a}}{a} \left( T_1^1 - T_0^0 \right) + 2\eta \left( \frac{\dot{a}^2}{a^2} + \frac{\dot{b}^2}{b^2} + \frac{\dot{c}^2}{c^2} \right).$$  \hfill (2.35)

Recall that (2.20) gives

$$(m - D) S_0 - \mathcal{G} \dot{P}_0 = 0.$$  \hfill (2.36)

In view of that after a little manipulation from (2.33) we obtain

$$\dot{\varepsilon} + \frac{\dot{\tau}}{\tau} \omega - \left( \frac{4}{3} \eta \right) \frac{\tau^2}{\varepsilon^2} + 4\eta (\kappa T_0^0 + \Lambda) = 0,$$  \hfill (2.37)

where

$$\omega = \varepsilon + p,$$  \hfill (2.38)

is the thermal function. Let us now in analogy with Hubble constant introduce the quantity $H$, such that

$$\frac{\dot{\tau}}{\tau} = \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} = 3H.$$  \hfill (2.39)

Then (2.32) and (2.36) in account of (2.29) can be rewritten as

$$\dot{H} = \frac{K}{2} (3\xi H - \omega) - (3H^2 - \kappa \varepsilon - \Lambda) + \frac{K}{2} (mS + \mathcal{D}S + \mathcal{G}P),$$

$$\dot{\varepsilon} = 3H (3\xi H - \omega) + 4\eta (3H^2 - \kappa \varepsilon - \Lambda) - 4\eta \kappa \left[ mS + \frac{C^2}{2\tau^2(1 + \lambda F)} \right].$$  \hfill (2.39)

Thus, the metric functions are found explicitly in terms of $\tau$ and viscosity. To write $\tau$ and components of spinor field as well and scalar one we have to specify the function $F$. In the next section we explicitly solve Eqs. (2.19) and (2.39) for some concrete value of $F$.

### III. SOME SPECIAL SOLUTIONS

In this section we first solve the spinor field equations for some special choice of $F$, which will be given in terms of $\tau$. Thereafter, we will study the system (2.39) in details and give explicit solution for some special cases.

#### A. Solutions to the spinor field equations

As one sees, introduction of viscous fluid has no direct effect on the system of spinor field equations (2.19). Viscous fluid has an implicit influence on the system through $\tau$. A detailed analysis of the system in question can be found in [15]. Here we just write the final results.
1. Case with $F = F(I)$

Here we consider the case when the nonlinear spinor field is given by $F = F(I)$. As in the case with minimal coupling from (2.20a) one finds

$$S = C_0 / \tau, \quad C_0 = \text{const.} \quad (3.1)$$

For components of spinor field we find [15]

$$\psi_1 (t) = C_1 / \sqrt{\tau} e^{-i \beta}, \quad \psi_2 (t) = C_2 / \sqrt{\tau} e^{-i \beta},$$

$$\psi_3 (t) = C_3 / \sqrt{\tau} e^{i \beta}, \quad \psi_4 (t) = C_4 / \sqrt{\tau} e^{i \beta}, \quad (3.2)$$

with $C_i$ being the integration constants and are related to $C_0$ as $C_0 = C_1^2 + C_2^2 - C_3^2 - C_4^2$. Here $\beta = \int (m - \mathcal{H}) dt$.

For the components of the spin current from (2.22) we find

$$j^0 = 1 / \tau [C_1^2 + C_2^2 + C_3^2 + C_4^2], \quad j^1 = 2 / a \tau [C_1 C_4 + C_2 C_3] \cos (2 \beta),$$

$$j^2 = 2 / b \tau [C_1 C_4 - C_2 C_3] \sin (2 \beta), \quad j^3 = 2 / c \tau [C_1 C_3 - C_2 C_4] \cos (2 \beta),$$

whereas, for the projection of spin vectors on the $X, Y$ and $Z$ axis we find

$$S_{23,0} = C_1 C_2 + C_3 C_4 / bc \tau, \quad S_{31,0} = 0, \quad S_{12,0} = (C_1^2 - C_2^2 + C_3^2 - C_4^2) / 2ab \tau.$$

Total charge of the system in a volume $\mathcal{V}$ in this case is

$$Q = [C_1^2 + C_2^2 + C_3^2 + C_4^2] \mathcal{V}. \quad (3.3)$$

Thus, for $\tau \neq 0$ the components of spin current and the projection of spin vectors are singularity-free and the total charge of the system in a finite volume is always finite. Note that, setting $\lambda = 0$, i.e., $\beta = mt$ in the foregoing expressions one get the results for the linear spinor field.

2. Case with $F = F(J)$

Here we consider the case with $F = F(J)$. In this case we assume the spinor field to be massless. Note that, in the unified nonlinear spinor theory of Heisenberg, the massive term remains absent, and according to Heisenberg, the particle mass should be obtained as a result of quantization of spinor prematter [31]. In the nonlinear generalization of classical field equations, the massive term does not possess the significance that it possesses in the linear one, as it by no means defines total energy (or mass) of the nonlinear field system. Thus without losing the generality we can consider massless spinor field putting $m = 0$. Then from (2.20b) one gets

$$P = D_0 / \tau, \quad D_0 = \text{const.} \quad (3.4)$$
In this case the spinor field components take the form

\[ \psi_1 = \frac{1}{\sqrt{\tau}}(D_1 e^{i\sigma} + iD_3 e^{-i\sigma}), \quad \psi_2 = \frac{1}{\sqrt{\tau}}(D_2 e^{i\sigma} + iD_4 e^{-i\sigma}), \]

\[ \psi_3 = \frac{1}{\sqrt{\tau}}(iD_1 e^{i\sigma} + D_3 e^{-i\sigma}), \quad \psi_4 = \frac{1}{\sqrt{\tau}}(iD_2 e^{i\sigma} + D_4 e^{-i\sigma}). \]

(3.5)

The integration constants \( D_i \) are connected to \( D_0 \) by\( D_0 = 2(D_1^2 + D_2^2 - D_3^2 - D_4^2) \). Here we set \( \sigma = \int \mathcal{G} \, dt \).

For the components of the spin current from (2.22) we find

\[ j^0 = \frac{2}{\tau} [D_1^2 + D_2^2 + D_3^2 + D_4^2], \quad j^1 = \frac{4}{a\tau} [D_2 D_3 + D_1 D_4] \cos(2\sigma), \]

\[ j^2 = \frac{4}{b\tau} [D_2 D_3 - D_1 D_4] \sin(2\sigma), \quad j^3 = \frac{4}{c\tau} [D_1 D_3 - D_2 D_4] \cos(2\sigma), \]

whereas, for the projection of spin vectors on the \( X, Y \) and \( Z \) axis we find

\[ S^{23,0} = \frac{2(D_1 D_2 + D_3 D_4)}{b c \tau}, \quad S^{31,0} = 0, \quad S^{12,0} = \frac{D_1^2 - D_2^2 + D_3^2 - D_4^2}{2 a b \tau} \]

We see that for any nontrivial \( \tau \) as in previous case the components of spin current and the projection of spin vectors are singularity-free and the total charge of the system in a finite volume is always finite.

### B. Determination of \( \tau \)

In this subsection we simultaneously solve the system of equations for \( \tau \) and \( \epsilon \). Since setting \( m = 0 \) in the equations for \( F = F(I) \) one comes to the case when \( F = F(J) \), we consider the case with \( F \) being the function of \( J \) only. Let \( F \) be the power function of \( S \), i.e., \( F = S^n \). As it was established earlier, in this case \( S = C_0 / \tau \), or setting \( C_0 = 1 \) simply \( S = 1 / \tau \). for simplicity we also set \( C = 1 \). Evaluating \( \mathcal{G} \) in terms of \( \tau \) we then come to the following system of equations

\[ \dot{\tau} = \frac{3}{2} \xi \tau + \frac{3 \kappa}{2} \left( \frac{m}{\tau} + \frac{\lambda n}{2(\lambda + \tau^n)} \right) \tau - 3 \Lambda \tau, \]

(3.6a)

\[ \dot{\epsilon} = -\frac{\epsilon}{\tau} \omega + \left( \xi + \frac{4}{3} \eta \right) \frac{\tau^2}{\tau^2} - 4 \eta \left[ \frac{m}{\tau} + \frac{\tau^{n-2}}{2(\lambda + \tau^n)} + \Lambda \right], \]

(3.6b)

or in terms of \( H \)

\[ \dot{\tau} = 3 H \tau, \]

(3.7a)

\[ H = \frac{1}{2} (3 \xi \tau - \omega) - (3 H^2 - \epsilon - \Lambda) + \frac{\kappa}{2} \left[ \frac{m}{\tau} + \frac{\lambda n}{2(\lambda + \tau^n)} \right], \]

(3.7b)

\[ \dot{\epsilon} = 3 H (3 \xi + \omega) + 4 \eta (3 H^2 - \epsilon - \Lambda) - 4 \eta \left[ \frac{m}{\tau} + \frac{\tau^{n-2}}{2(\lambda + \tau^n)} \right]. \]

(3.7c)

Here \( \eta \) and \( \xi \) are the bulk and shear viscosity, respectively and they are both positively definite, i.e.,

\[ \eta > 0, \quad \xi > 0. \]

(3.8)
They may be either constant or function of time or energy. We consider the case when

$$\eta = Ae^\alpha, \quad \xi = Be^\beta,$$

with $A$ and $B$ being some positive quantities. For $p$ we set as in perfect fluid,

$$p = \zeta \epsilon, \quad \zeta \in (0, 1].$$

Note that in this case $\zeta \neq 0$, since for dust pressure, hence temperature is zero, that results in vanishing viscosity.

The system (3.7) without spinor field have been extensively studied in literature either partially [9, 32, 33] or as a whole [11]. Here we try to solve the system (3.6) for some particular choice of parameters.

1. Case with bulk viscosity

Let us first consider the case with bulk viscosity alone setting coefficient of shear viscosity $\eta = 0$. In this case from (2.31) and (2.35) we find the following relation

$$\kappa \tilde{T}_0^0 = 3H^2 - \Lambda + C_{00}, \quad C_{00} = \text{const.}$$

We also demand the coefficient of bulk viscosity be inverse proportional to expansion, i.e.,

$$\xi \theta = 3\xi H = C_2, \quad C_2 = \text{const.}$$

Inserting $\eta = 0$, (3.12) and (3.10) into (3.7c) one finds

$$\epsilon = \frac{1}{1 + \zeta} [C_2 - C_3 / \tau^{1+\zeta}].$$

Then from (3.6a) we get the following equation for determining $\tau$:

$$\tau = \frac{3}{2} \kappa m + 3 \left[ \frac{C_2}{2} \kappa + \Lambda \right] \tau + \frac{3\kappa (1 - \zeta)}{2(1 + \zeta)} C_2 \tau^{1+\zeta} - C_3 + \frac{3\kappa m n}{4} \frac{\tau^{n-1}}{\left( \lambda + \tau^n \right)^2} \equiv \mathcal{F}(q, \tau),$$

where $q$ is the set of problem parameters. As one sees, the right hand side of the Eq. (3.14) is a function of $\tau$, hence can be solved in quadrature [30]. We solve the Eq. (3.14) numerically. It can be noted that the Eq. (3.14) can be viewed as one describing the motion of a single particle. Sometimes it is useful to plot the potential of the corresponding equation which in this case is

$$\mathcal{U}(q, \tau) = -2 \int \mathcal{F}(q, \tau) d\tau.$$ 

The problem parameters are chosen as follows: $\kappa = 1, \ m = 1, \ \lambda = 0.5, \ \zeta = 1/3, \ n = 4, \ C_2 = 2$ and $C_3 = 1$. Here we consider the cases with different $\Lambda$, namely with $\Lambda = -2, 0, 1$, respectively. The initial value of $\tau$ is taken to be a small one, whereas, the first derivative of $\tau$, i.e., $\dot{\tau}$ at that point of time is calculated from (3.11). In Fig. [1] we have illustrated the potential corresponding to Eq. (3.14). It can be immediately seen that independent of the sign of $\Lambda$ we have always expanding universe. But as is seen from Fig. [2] a positive $\Lambda$ results in accelerated mode of expansion, while the negative one causes deceleration.
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2. Case with bulk and shear viscosities

Let us consider more general case. Following [12] we choose the shear viscosity being proportional to the expansion, namely,

$$\eta = -\frac{3}{2\kappa}H = -\frac{1}{2\kappa}\theta. \quad (3.16)$$

In absence of spinor field this assumption leads to

$$3H^2 = \kappa\epsilon + C_4, \quad C_4 = \text{const.} \quad (3.17)$$

It can be shown that the relation (3.17) in our case can be achieved only for massless spinor field with the nonlinear term being

$$F = (F_0S^2 - 1)/\lambda.$$

Equation for $\tau$ in this case has the form

$$\tau\ddot{\tau} - 0.5(1 - \zeta)\dot{\tau}^2 - 1.5\kappa\xi\tau\dot{\tau} - 3[\Lambda - 0.5(1 - \zeta)C_4 + \kappa/2F_0]\tau^2 = 0. \quad (3.18)$$

In case of $\xi = \text{const.}$ there exists several special solutions available in handbooks on differential equations. For that reason we rewrite this equation in terms of $H$:

$$\dot{H} = -1.5(1 + \zeta)H^2 + 1.5\kappa\xi H + [\Lambda - 0.5(1 - \zeta)C_4 + \kappa/2F_0]. \quad (3.19)$$

The solution of the foregoing equation can be written in quadrature as

$$\int \frac{dH}{AH^2 + BH + C} = t. \quad (3.20)$$
with \( A = -1.5(1 + \zeta), B = 1.5\kappa\xi \) and \( C = \Lambda - 0.5(1 - \zeta)C_4 + \kappa/2F_0 \). If the bulk viscosity is taken to be a constant one, i.e., \( \xi = \text{const.} \), then depending on the value of the discriminant \( B^2 - 4AC \) there exists three types of solutions, namely [34]:

\[
t = \begin{cases} 
\frac{1}{\sqrt{B^2 - 4AC}} \ln \left| \frac{2AH + B + \sqrt{B^2 - 4AC}}{2AH + B - \sqrt{B^2 - 4AC}} \right|, & B^2 > 4AC, \\
\frac{2}{\sqrt{4AC - B^2}} \arctan \frac{2AH + B}{\sqrt{4AC - B^2}}, & B^2 < 4AC, \\
\frac{2}{2AH + B}, & B^2 = 4AC.
\end{cases}
\]

(3.21)

Note that a detailed analysis of these solutions in absence of spinor and scalar field was given in [12]. We choose the problem parameters as follows: \( C_4 = 9, \zeta = 1/3 \) and \( \kappa = 2 \). Under this choice we find \( \delta = B^2 - 4AC = 9\xi^2 + 8(\Lambda - 2) \). After that we chose \( \Lambda \) positive, trivial or negative (in particular we chose \( \Lambda = (7/8, 0, -5/2) \)). The quantity \( \xi \) now is taken such a way that we have \( \delta = (\delta_1, \delta_2, \delta_3) \) for all values of \( \Lambda \) chosen above, whereas, \( \delta_1 > 0, \delta_2 = 0 \) and \( \delta_3 < 0 \). In Figs. 3, 4 and 5 we plot the evolution of \( \tau \) corresponding to a trivial, positive and negative value of \( \delta \), respectively. As is seen, the behavior of \( \tau \) mainly depends on \( \delta \) and independent to the sign of \( \Lambda \). Since a negative \( \delta \) gives non-periodic mode of evolution we plot the corresponding phase diagram in Fig. 6.

### IV. CONCLUSION

We consider the self consistent system of spinor, scalar and gravitational fields within the framework of Bianchi type-I cosmological model filled with viscous fluid. Solutions to the corresponding equations are given in terms of the volume scale of the BI space-time, i.e., in terms of \( \tau = abc \). The system of equations for determining \( \tau \), energy-density of the viscous fluid \( \epsilon \) and...
Hubble parameter $H$ has been worked out. Exact solution to the aforementioned system has been given only for some special choice of viscosity. It should be noted that the system (3.7) is far richer and allows a number of mathematically interesting results, though not all of them is physically realizable. Given this fact we plan to review this system and give a detailed analysis and qualitative solutions of the corresponding system in some of our future works.

V. APPENDIX

Let us now construct the invariants of spinor field. Since $\psi$ and $\psi^\ast$ (complex conjugate of $\psi$) has 4 component each, one can construct $4 \cdot 4 = 16$ independent bilinear combinations. They are scalar, pseudoscalar, vector, axial vector, and tensor denoted, respective, by

\begin{align*}
S &= \bar{\psi} \psi, \\
P &= i \bar{\psi} \gamma^5 \psi, \\
v^\mu &= (\bar{\psi} \gamma^\mu \psi), \\
A^\mu &= (\bar{\psi} \gamma^5 \gamma^\mu \psi), \\
Q^{\mu\nu} &= (\bar{\psi} \sigma^{\mu\nu} \psi),
\end{align*}

where $\sigma^{\mu\nu} = (i/2)[\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu]$ is the anti-symmetric tensor. Invariants, corresponding to the bilinear forms are

\begin{align*}
I_S &= S^2 = (\bar{\psi} \psi)^2, \\
I_P &= P^2 = (i \bar{\psi} \gamma^5 \psi)^2, \\
I_v &= v_\mu v^\mu = (\bar{\psi} \gamma^\mu \psi) g_{\mu\nu} (\bar{\psi} \gamma^\nu \psi), \\
I_A &= A_\mu A^\mu = (\bar{\psi} \gamma^5 \gamma^\mu \psi) g_{\mu\nu} (\bar{\psi} \gamma^5 \gamma^\nu \psi), \\
I_Q &= Q_{\mu\nu} Q^{\mu\nu} = (\bar{\psi} \sigma^{\mu\nu} \psi) g_{\mu\alpha} g_{\nu\beta} (\bar{\psi} \sigma^{\alpha\beta} \psi).
\end{align*}
\( \gamma \) matrices in the above expressions obey the following algebra
\[
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu \nu}
\]  
(5.3)

and are connected with the flat space-time Dirac matrices \( \bar{\gamma} \) in the following way
\[
g_{\mu \nu}(x) = e^a_{\mu}(x) e^b_{\nu}(x) \eta_{ab}, \quad \gamma_\mu(x) = e^a_{\mu}(x) \bar{\gamma}_a,
\]  
(5.4)

where \( \eta_{ab} = \text{diag}(1, -1, -1, -1) \) and \( e^a_{\mu} \) is a set of tetrad 4-vectors.

For the diagonal metric such as Bianchi-I
\[
ds^2 = dt^2 - a(t)^2 dx^2 - b(t)^2 dy^2 - c(t)^2 dz^2.
\]

we have
\[
\gamma_0 = \bar{\gamma}_0, \quad \gamma_1 = a(t) \bar{\gamma}_1, \quad \gamma_2 = b(t) \bar{\gamma}_2, \quad \gamma_3 = c(t) \bar{\gamma}_3,
\]  
(5.5)

Flat space-time matrices \( \bar{\gamma} \) we will choose in the form:
\[
\bar{\gamma}^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \bar{\gamma}^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},
\]
\[
\bar{\gamma}^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \bar{\gamma}^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.
\]

Defining \( \gamma^5 \) as follows,
\[
\gamma^5 = -\frac{i}{4} E_{\mu \nu \sigma \rho} \gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\rho, \quad E_{\mu \nu \sigma \rho} = \sqrt{-g} e_{\mu \nu \sigma \rho}, \quad \varepsilon_{0123} = 1,
\]
\[
\gamma^5 = -i\sqrt{-g} \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -i\bar{\gamma}^0 \bar{\gamma}^1 \bar{\gamma}^2 \bar{\gamma}^3 = \bar{\gamma}^5,
\]

we obtain
\[
\bar{\gamma}^5 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.
\]

Note that \( \psi \) is a 4 component function given by,
\[
\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \quad \bar{\psi} = \psi^* \gamma^0 = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*),
\]  
(5.6)
Denoting $\vec{F}$ bilinear spinor form in Minkowski spacetime and $F$ in curve spacetime (in our case in BI) we find the following expressions for the non-trivial components of bilinear spinor form:

\[
\begin{align*}
\vec{S} & = (\psi^1 \psi_1 + \psi^2 \psi_2 - \psi^3 \psi_3 - \psi^4 \psi_4), \\
\vec{P} & = -i(\psi^1 \psi_3 + \psi^2 \psi_4 - \psi^3 \psi_1 - \psi^4 \psi_2), \\
\vec{V}^0 & = (\psi^1 \psi_1 + \psi^2 \psi_2 + \psi^3 \psi_3 + \psi^4 \psi_4), \\
\vec{V}^1 & = (\psi^1 \psi_4 + \psi^2 \psi_3 + \psi^3 \psi_2 + \psi^4 \psi_1), \\
\vec{V}^2 & = -i(\psi^1 \psi_4 - \psi^2 \psi_3 + \psi^3 \psi_2 - \psi^4 \psi_1), \\
\vec{V}^3 & = (\psi^1 \psi_3 - \psi^2 \psi_4 + \psi^3 \psi_1 - \psi^4 \psi_2), \\
\vec{A}^0 & = (\psi^1 \psi_3 + \psi^2 \psi_4 + \psi^3 \psi_1 + \psi^4 \psi_2), \\
\vec{A}^1 & = (\psi^1 \psi_2 + \psi^2 \psi_1 + \psi^3 \psi_4 + \psi^4 \psi_3), \\
\vec{A}^2 & = -i(\psi^1 \psi_2 - \psi^2 \psi_1 + \psi^3 \psi_4 - \psi^4 \psi_3), \\
\vec{A}^3 & = (\psi^1 \psi_1 - \psi^2 \psi_2 + \psi^3 \psi_3 - \psi^4 \psi_4), \\
\vec{Q}^{01} & = i(\psi^1 \psi_4 + \psi^2 \psi_3 - \psi^3 \psi_2 - \psi^4 \psi_1), \\
\vec{Q}^{02} & = (\psi^1 \psi_4 - \psi^2 \psi_3 - \psi^3 \psi_2 + \psi^4 \psi_1), \\
\vec{Q}^{03} & = i(\psi^1 \psi_3 - \psi^2 \psi_4 - \psi^3 \psi_1 + \psi^4 \psi_2), \\
\vec{Q}^{12} & = (\psi^1 \psi_1 - \psi^2 \psi_2 - \psi^3 \psi_3 + \psi^4 \psi_4), \\
\vec{Q}^{23} & = (\psi^1 \psi_2 + \psi^2 \psi_1 - \psi^3 \psi_4 - \psi^4 \psi_3), \\
\vec{Q}^{13} & = i(\psi^1 \psi_2 - \psi^2 \psi_1 - \psi^3 \psi_4 + \psi^4 \psi_3),
\end{align*}
\]

Using the above expressions we find

\[
\begin{align*}
I_s & = S^2 = (\psi^1 \psi_1)^2 + (\psi^2 \psi_2)^2 + (\psi^3 \psi_3)^2 + (\psi^4 \psi_4)^2 \\
& + 2(\psi^1 \psi_1 \psi^2 \psi_2 - \psi^3 \psi_3 \psi_1 \psi_4 - \psi^4 \psi_4 \psi_1 \psi_2) \\
& - \psi^2 \psi_2 \psi^3 \psi_3 - \psi^2 \psi_2 \psi^4 \psi_4 + \psi^3 \psi_3 \psi^4 \psi_4].
\end{align*}
\]

\[
\begin{align*}
I_p & = P^2 = -(\psi^1 \psi_3)^2 - (\psi^2 \psi_4)^2 - (\psi^3 \psi_1)^2 - (\psi^4 \psi_2)^2 \\
& - 2(\psi^1 \psi_1 \psi^2 \psi_2 - \psi^1 \psi_1 \psi^3 \psi_3 - \psi^1 \psi_1 \psi^4 \psi_4 \\
& - \psi^2 \psi_2 \psi^3 \psi_3 - \psi^2 \psi_2 \psi^4 \psi_4 + \psi^3 \psi_3 \psi^4 \psi_4].\end{align*}
\]

\[
\begin{align*}
I_v & = (\vec{V}^0)^2 - (\vec{V}^1)^2 - (\vec{V}^2)^2 - (\vec{V}^3)^2 \\
& = (\psi^1 \psi_1)^2 + (\psi^2 \psi_2)^2 + (\psi^3 \psi_3)^2 + (\psi^4 \psi_4)^2 \\
& - (\psi^1 \psi_3)^2 - (\psi^2 \psi_4)^2 - (\psi^3 \psi_1)^2 - (\psi^4 \psi_2)^2 \\
& + 2(\psi^1 \psi_1 \psi^2 \psi_2 - \psi^1 \psi_1 \psi^3 \psi_3 - \psi^1 \psi_1 \psi^4 \psi_4 \\
& - \psi^2 \psi_2 \psi^3 \psi_3 - \psi^2 \psi_2 \psi^4 \psi_4 + \psi^3 \psi_3 \psi^4 \psi_4].
\end{align*}
\]

\[
I_s + I_p.
\]
Explicitly it can be presented as

\[ I_A = (\bar{A})^2 - (\bar{A})^2 - (\bar{A}^2)^2 - (\bar{A}^3)^2 \]

\[ = - (\psi_1^* \psi_1)^2 - (\psi_2^* \psi_2)^2 - (\psi_3^* \psi_3)^2 - (\psi_4^* \psi_4)^2 \]

\[ + (\psi_1^* \psi_3)^2 + (\psi_2^* \psi_4)^2 + (\psi_3^* \psi_1)^2 + (\psi_4^* \psi_2)^2 \]

\[ - 2[\psi_1^* \psi_1 \psi_2 \psi_2 - \psi_1^* \psi_1 \psi_3 \psi_3 - \psi_2^* \psi_2 \psi_3 \psi_3 - \psi_3^* \psi_3 \psi_4 \psi_4 - \psi_4^* \psi_4 \psi_3 \psi_3 - \psi_4^* \psi_4 \psi_4 \psi_4] \]

\[ = -(I_S + I_P). \quad (5.27) \]

\[ I_Q = 2[(\bar{Q}^{12})^2 + (\bar{Q}^{13})^2 + (\bar{Q}^{23})^2 - (\bar{Q}^{01})^2 - (\bar{Q}^{02})^2 - (\bar{Q}^{03})^2] \]

\[ = (\psi_1^* \psi_1)^2 + (\psi_2^* \psi_2)^2 + (\psi_3^* \psi_3)^2 + (\psi_4^* \psi_4)^2 \]

\[ + (\psi_1^* \psi_3)^2 + (\psi_2^* \psi_4)^2 + (\psi_3^* \psi_1)^2 + (\psi_4^* \psi_2)^2 \]

\[ + 2[\psi_1^* \psi_1 \psi_2 \psi_2 - \psi_1^* \psi_1 \psi_3 \psi_3 - \psi_2^* \psi_2 \psi_3 \psi_3 - \psi_3^* \psi_3 \psi_4 \psi_4 - \psi_4^* \psi_4 \psi_3 \psi_3 - \psi_4^* \psi_4 \psi_4 \psi_4] \]

\[ + \psi_4^* \psi_4 \psi_3 \psi_3 + \psi_4^* \psi_2 \psi_4 \psi_4 - \psi_1^* \psi_3 \psi_4 \psi_4 - \psi_2^* \psi_4 \psi_2 \psi_4 - \psi_4^* \psi_3 \psi_2 \psi_4 - \psi_4^* \psi_2 \psi_2 \psi_2 \psi_4] \]

\[ - 4(\psi_1^* \psi_1 \psi_3 \psi_3 + \psi_2^* \psi_2 \psi_2 \psi_4) = 2(I_S - I_P). \quad (5.28) \]

Thus we see that that the invariants \( I_V, I_A \) and \( I_Q \) can be expressed in terms of \( I_S \) and \( I_P \).

An alternative proof of \( I_V = -I_A = (I_S + I_P) \) and \( I_Q = 2(I_S - I_P) \) can be given using Feirz transformation. In doing so we recall that any 4x4 matrix \( \Gamma \) can be presented as a linear combination of \( \gamma^A \):

\[ \Gamma = \sum_A c_A \gamma^A, \quad c_A = \frac{1}{4} Tr \gamma^A \Gamma. \quad (5.29) \]

Explicitly it can be presented as

\[ \Gamma_{ik} = \frac{1}{4} \sum_A \Gamma_{lm} \gamma^A_{ml} \gamma_{ik}. \quad (5.30) \]

Here

\[ \gamma^A = \{ I, \gamma^5, \gamma^\mu, i\gamma^\mu \gamma^5, i\sigma^{\mu\nu} \}, \quad A = 1, 2..16. \quad (5.31) \]

The completeness condition gives

\[ \delta_{ij} \delta_{kl} = \frac{1}{4} \sum_A \gamma_{ik} \gamma^A_{kl}. \quad (5.32) \]

Multiplying the foregoing expression by \( \psi_i^a \psi_k^b \psi_m^c \psi_l^d \) one gets

\[ (\bar{\psi}^a \psi^d)(\bar{\psi}^c \psi^b) = \frac{1}{4} \sum_A (\bar{\psi}^a \gamma_A \psi^b)(\bar{\psi}^c \gamma^A \psi^d). \quad (5.33) \]

Replacing \( \psi^d \rightarrow \gamma^B \psi^d \) and \( \psi^b \rightarrow \gamma^C \psi^b \) and using the expression

\[ \gamma^A \gamma^B = \sum_R C_R \gamma^R, \quad C_R = \frac{1}{4} Tr \gamma^A \gamma^B \gamma_R, \quad (5.34) \]
one gets the other identities. Denoting \( I_S = (\bar{\psi}^a \psi^b)(\bar{\psi}^c \psi^d) \), \( I'_S = (\bar{\psi}^a \psi^d)(\bar{\psi}^c \psi^b) \) \( \cdots \) one finds \[ I'_S = I_S + I_P + I_V + I_Q + I_A, \] \[ I'_P = I_S + I_P - I_V + I_Q - I_A, \] \[ I'_V = 4I_S - 4I_P - 2I_V + 2I_A, \] \[ I'_Q = 6I_S + 6I_P - 2I_Q, \] \[ I'_A = 4I_S - 4I_P + 2I_V - 2I_A. \] (5.35)
(5.36)
(5.37)
(5.38)
(5.39)

After a little manipulations from the foregoing system one comes to desired relations between the invariants of the bilinear spinor fields.

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