Distributed trajectory tracking and formation control without velocity measurements by the notion of prior bounded local neighborhood synchronization error

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Abstract
This paper investigates the robust consensus tracking and formation control problems of multiple second-order systems having exogenous disturbances and no velocity measurements. To account for the input saturation constraint in controller design, a novel notion of local neighborhood synchronization error is proposed, which is obtained using generalized saturation functions and can be regarded as a nonlinear variation of the well-known linear local neighborhood synchronization error. An important property of the notion is proved and then a continuous distributed controller is designed using it. To improve the robustness of the controller with respect to exogenous disturbance, a disturbance estimator-based design and a simple parameter mapping for parameter tuning are proposed. The resulting error system is proven to be small-signal $L_2$ stable and input-to-output stable. In particular, the synchronization errors and tracking errors converge asymptotically to zero if the disturbances converge to some constants. By the parameter mapping, the steady-state synchronization errors and tracking errors can be made arbitrarily small. The control scheme is finally modified to adapt to formation control applications by adding the desired position deviation from the leader’s trajectory. The performance of the scheme is demonstrated by the simulation results.

Keywords
Robust consensus tracking, formation control, bounded control, uncertainty and disturbance estimator

Introduction
In recent years, distributed coordination of multi-vehicle systems (MVSs) has attracted much attention due to its practical significance. Two important types of cooperative control problems, the synchronized trajectory tracking problems and the distributed formation control problems, have been studied in the past decade. In a synchronized trajectory tracking problem, a given reference signal is required to be tracked by each vehicle with local interaction. The objective of a formation control is to ensure each vehicle involved in the group stay in a fixed pattern (or a geometric shape). It has been extended to allow these vehicles to follow a certain dynamic pattern, and representative works on the two topics include, but are not limited to previous works.¹⁻⁷

However, the synchronized trajectory tracking and formation control under a given input constraint without using velocity measurements have not received much attention. On one hand, as far as the robust synchronized position tracking problem of MVS is concerned, the velocities of each vehicle and its neighbors are often assumed to be available in most existing works.⁸⁻¹¹ On the other hand, these works¹¹⁻¹⁴ have considered the input constraint in designing controllers without using velocity signals. The works¹²,¹³ focus on the problem of synchronized velocity tracking rather than position tracking.

Motivated by the above observations, the synchronized position tracking and formation control problems of multiple second-order systems are considered, and a robust distributed solution is aimed to develop...
without using any velocity signals under the input constraint. The communication topology among the group of agents is modeled by an undirected graph, and only partial agents in the group have access to the desired position trajectory information. The main technical contributions of this paper are threefold:

1. The notion of prior bounded local neighborhood synchronization error (LNSE) is introduced, which can be regarded as a nonlinear variation of the standard LNSE used in previous works.8,15,16 A lemma to show the relationship between the convergence of position tracking errors and the convergence of the prior bounded LNSEs is proposed. This work has been included in our previous conference papers14 and 17.

2. Using the notion of prior bounded LNSEs, a robust control scheme for synchronized trajectory tracking of multiple second-order systems without velocity measurements is designed. The scheme includes a tracker which can stabilize the nominal system globally asymptotically, and an uncertainty and disturbance estimator (UDE) to attenuate the effect of input disturbances. The scheme results in a continuous control signals for each agent and is different from the variable-structure schemes proposed in Khoo et al.8 and Li et al.18 The extension of the scheme for formation control application is also discussed.

3. An ingenious parameter mapping for UDE design is introduced and a simple parameter tuning approach is developed to reduce steady-state control errors via a single-parameter tuning.

The rest of this paper is organized as follows. Some preliminaries are reviewed and then the problem is formulated in section “Preliminaries and problem formulation.” Section “Main results” introduces the extension of LNSE, followed by some supporting lemmas and the main results of this paper. Simulation examples are shown to demonstrate the effectiveness of the control scheme proposed in section “Simulation results.” Section “Conclusion” concludes this paper.

Preliminaries and problem formulation
Notation and mathematical preliminaries
\( G(V, \mathcal{E}, A) \) is used to denote a weighted graph of order \( n \), \( V = \{1, 2, \ldots, n\} \), \( \mathcal{E} \subseteq V \times V \) denote the set of nodes and the set of edges, respectively. All the node indices in graph \( G \) are expressed as a finite set \( \mathcal{I} = \{1, \ldots, n\} \). \( (s_i, s_j) \) denotes an edge of \( G \) where \( s_i, s_j \) are the nodes of agent \( i \), \( j \). Adjacent matrix \( A_n \) is defined as \( a_{ii} = 0 \) for all \( i \in \mathcal{I} \), \( a_{ij} > 0 \), if \( (s_i, s_j) \in \mathcal{E} \) and \( j \neq i \), otherwise \( a_{ij} = 0 \) when the topology of the graph \( G \) is undirected. The Laplacian matrix of an undirected \( G \) is denoted by \( L = [l_{ij}] \in \mathbb{R}^{n \times n} \), where \( l_{ii} = -\alpha_i, \) if \( i \neq j \) for \( i, j \in \mathcal{I} \), and \( l_{ij} = \sum_{1 \leq j \neq i} a_{ij} \) for \( i \in \mathcal{I} \). Thus, both the weighted adjacency matrix \( A_n \) and the Laplacian matrix \( L_n \) are symmetric when \( G \) is undirected. Considering the vector \( x \in \mathbb{R}^n \) and the matrix \( C = [c_{ij}] \in \mathbb{R}^{m \times n}, \) \( \|X\|_2 = (\sum_{i=1}^n |x_i|^2)^{1/2} \) and \( \|X\|_\infty = \max_{1 \leq i \leq n} |x_i| \) are defined. Then, we have

\[
\|CX\|_\infty \leq \|C\|_\infty \|X\|_\infty
\]  

The generalized saturation functions (GSFs) which will be used in control design are now defined.

Definition 1. Let \( \sigma : R \rightarrow R \) be a locally Lipschitz and nondecreasing function, such that11,19

\[
P1 : \quad x\sigma(x) > 0, \quad \forall x \neq 0
\]
\[
P2 : \quad |\sigma(x)| \leq \sigma, \quad \forall x \in R
\]

then \( \sigma(x) \) is said to be a GSF with bound \( \sigma \).

With the definition of GSF, the following properties are presented:

\[
P3 : \quad \text{Assume that there exists a constant } \alpha' \in (0, \infty), \alpha'(x) \text{ is positive and bounded by } \alpha' \text{, that is,}
\]

\[
0 < \alpha'(x) \leq \alpha', \forall x \in R
\]
\[
P4 : \quad \sigma(x) \text{ is globally Lipschitz, that is}
\]

\[
|\sigma(x_1) - \sigma(x_2)| \leq M|x_1 - x_2|, \quad \forall x_1, x_2 \in R
\]
\[
P5 : \quad \int_0^\infty \sigma^2(\tau) \, d\tau \rightarrow +\infty \quad \text{as } |x| \rightarrow +\infty
\]

Problem formulation
The mathematical model of the considered uncertain second-order system is

\[
\begin{align*}
\dot{r}_i &= v_i \\
\dot{v}_i &= \mu_i + d_i, \quad i \in \mathcal{I}
\end{align*}
\]  

(4)

where \( r_i \in R \) and \( v_i \in R \) are the position and velocity (generalized coordinates) of agent \( i \), respectively, \( \mu_i \in R \) is the control input which will be designed later and \( d_i \in R \) denotes the time-varying input disturbance.

The desired trajectory is \( r_i(t) \in R \) to be tracked by each agent, and \( \delta_i(t), i \in \mathcal{I} \) is used to denote the desired position deviation of agent \( i \) from the leader’s trajectory \( r_d \). \( (r_0, v_0) = (r(0), v(0)) \) denotes the initial state of agent \( i \). Under any initial condition \( (r_0, v_0) = (r(0), v(0)) \), the system is said to reach position tracking if

\[
\lim_{t \rightarrow \infty} (r_i(t) - r_d(t)) = 0
\]  

(5)

and the formation control is asymptotically achieved if

\[
\lim_{t \rightarrow \infty} (r_i(t) - \delta_i(t) - r_d(t)) = 0
\]  

(6)
The following three assumptions are set in this paper.

**Assumption 1.** For each \( i \in \mathcal{I} \), the disturbance \( d_i \) is a second-order continuous differentiable function in \( t \) almost everywhere and satisfies \( |d_i(t)| \leq d \) and \( |\dot{d}_i(t)| \leq \dot{d} \) for all \( t \geq 0 \), where \( d, \dot{d} \) are positive constants.

**Assumption 2.** The desired trajectory \( r_d(t) \) is a continuous differentiable function in \( t \), and its second-order time derivative in \( t \) satisfies \( \ddot{r}_d(t) = 0 \), which implies that \( \ddot{r}_d \) is bounded.

**Assumption 3.** \( G \) is undirected and connected, and the desired position trajectory information \( r_d(t) \) can be obtained by at least one agent.

### Main results

In the study by Li et al.\(^\text{15} \) and Das and Lewis,\(^\text{16} \) the LNSEs (which will be called as the standard LNSEs hereafter) are defined as

\[
e_i := \sum_{j=1}^{n} a_{ij} (r_i - r_j) + b_i (r_i - r_d), \quad i \in \mathcal{I} \tag{7}
\]

where \( a_{ij} \) denotes the \((i,j)\)th element of the adjacency matrix \( A_n \), \( b_i > 0 \) if agent \( i \) can obtain the information from \( r_j \) and if agent \( i \) is disconnected with \( r_d \), \( b_i = 0 \).

Let \( \tilde{r}_i = r_i - r_d \), \( \tilde{r}_i = r_i - r_j \). Then, \( \tilde{r}_j = \tilde{r}_i - \tilde{r}_j \) and equation (7) is equivalent to

\[
e_i = \sum_{j=1}^{n} a_{ij} \tilde{r}_j + b_i \tilde{r}_i, \quad i \in \mathcal{I} \tag{8}
\]

The Laplacian \( L \) and a diagonal matrix \( B = \text{diag}(b_1, \ldots, b_n) \in \mathbb{R}^{n \times n} \) are used to define a matrix \( H = L + B \in \mathbb{R}^{n \times n} \). Then, the following technical lemma (Lemma 1) is proposed.

**Lemma 1.** Under Assumption 3, \( \text{rank}(H) = n \) (Lemma 1.6 in Ren and Cao\(^\text{20} \)).

Equation (7) can be written in the compact form

\[
\begin{bmatrix}
e_1 \\
e_2 \\
\vdots \\
e_n
\end{bmatrix}
= \begin{bmatrix}
\tilde{r}_1 \\
\tilde{r}_2 \\
\vdots \\
\tilde{r}_n
\end{bmatrix}
= H
\begin{bmatrix}
e_1 \\
e_2 \\
\vdots \\
e_n
\end{bmatrix}
\tag{9}
\]

Clearly, there is a linear relationship between the vector \( [e_1, e_2, \ldots, e_n]^T \) and the vector \( [\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_n]^T \). Thus, the larger the magnitude of \( \tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_n \), the larger the magnitude of \( e_1, e_2, \ldots, e_n \). For a controller using the standard LNSEs \( e_i \), an actuator saturation phenomenon will arise at certain time instants when the magnitudes of some \( \tilde{r} \) are large. Motivated by this observation, a prior bounded LNSE is introduced here, which will be directly used to design a distributed prior bounded controller in the following section.

By the GSFs defined above, equation (8) is modified as

\[
e_i := \sum_{j=1}^{n} a_{ij} \sigma(\tilde{r}_j) + b_i \sigma(\tilde{r}_i), \quad i \in \mathcal{I} \tag{10}
\]

where \( a_{ij} \) and \( b_i \) are the same as in equation (8), and \( \sigma(\cdot) \) is a GSF which is defined in Definition 1. With the property P2 of GSF, the modified \( e_i \) is prior bounded by

\[
|e_i(t)| \leq \left( \sum_{j=1}^{n} a_{ij} + b_i \right) \bar{\sigma}, \quad i \in \mathcal{I}, \forall t \in \mathbb{R} \tag{11}
\]

Then, the following technical lemma (Lemma 2) is proposed.

**Lemma 2.** If \( \lim_{t \to +\infty} e_i(t) = 0 \), and the communication topology satisfies Assumption 3, then \( \lim_{t \to +\infty} \tilde{r}_i(t) = 0 \), where \( e_i(t) \) is defined in equation (10).

**Proof.** Since a GSF \( \sigma(\cdot) \) is an odd function, \( \sigma(0) = 0 \). By the well-known Lagrange Mean Value Theorem, \( \sigma(\alpha) = \sigma'(\alpha_0) \alpha \) and \( \sigma(-\alpha) = -\sigma'(\alpha_0) \alpha \), where \( 0 < \alpha < \alpha \). By this result, equation (10) can be written as

\[
e_i = \sum_{j=1}^{n} a_{ij} \sigma'(\tilde{r}_j) \tilde{r}_j + b_i \sigma'(\tilde{r}_i) \tilde{r}_i, \quad i \in \mathcal{I} \tag{12}
\]

where \( 0 < |\tilde{r}_i| < |\tilde{r}_j| \) and \( 0 < |\tilde{r}_j| < |\tilde{r}_i| \). Furthermore, \( e_i(i \in \mathcal{I}) \) defined in equation (12) can be written in the following compact form

\[
[e_1, e_2, \ldots, e_n]^T = H_f [\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_n]^T \tag{13}
\]

with the coefficient matrix \( H_f \) given by

\[
H_f = \begin{pmatrix}
\sum_{j=1}^{n} a_{1j} f_{j1} + b_1 f_{10} & \cdots & l_{1n} f_{1n} \\
l_{21} f_{21} & \cdots & l_{2n} f_{2n} \\
\vdots & \ddots & \vdots \\
l_{n1} f_{n1} & \cdots & \sum_{j=1}^{n} a_{nj} f_{nj} + b_n f_{n0}
\end{pmatrix}
\tag{14}
\]

where

\[
f_{ij} = \sigma'(\tilde{r}_{j0}) f_{j0} = \sigma'(\tilde{r}_j), \quad i, j \in \mathcal{I} \tag{15}
\]

With property P3, \( f_{ij} = \sigma'(\tilde{r}_{j0}) > 0 \), \( f_{n0} = \sigma'(\tilde{r}_n) > 0 \), \( i \in \mathcal{I} \), for any \( r_{j0} \in \bar{R} \) and \( \tilde{r}_{j0} \in \bar{R} \). Despite that the Laplacian matrix \( H_f \) is time varying, the corresponding graph \( G_f \) has the same topology structure as \( G \), with different weights at any time instant. Thus, \( \text{rank}(H_f) = \text{rank}(H) = n \) under Assumption 3 (by
Lemma 1), and the asymptotic convergence of $(\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_n)^T$ to the origin implies that of $(e_1, e_2, \ldots, e_n)^T$.

**Tracker design**

Equation (4) with $d_i=0$ is reduced to a nominal system

\[
\begin{align*}
\dot{\tilde{r}}_i &= v_i \\
\dot{v}_i &= u_i, \ i \in \mathcal{I}
\end{align*}
\]  

(16)

where $u_i$ denotes the control input of nominal systems (equation (16)), which are required to satisfy the following constraint

\[
\|u_i\|_\infty \leq \bar{u}, \ i \in \mathcal{I}
\]  

(17)

where $\bar{u}$ denotes a known upper bound for all $u_i$.

The following distributed controller is proposed for equation (16) using equation (10)

\[
\begin{align*}
\dot{u}_i &= -k_i e_i + k_i^2 \tilde{z}_i, \ i \in \mathcal{I} \\
\tilde{z}_i &= -k_i^3 \sigma(\tilde{z}_i) - e_i, \ i \in \mathcal{I}
\end{align*}
\]  

(18)

(19)

where $\tilde{z}_i \in \mathbb{R}^m$ is the state of the dynamic system (equation (19)) and its initial state $\tilde{z}_i(0)$ can be selected arbitrarily; $k_i^1, k_i^2$ and $k_i^3$ are positive real gains.

Then, the following inequality can be obtained by substituting equation (19) into equation (18) for $i \in \mathcal{I}$

\[
\|u_i\|_\infty \leq \left( k_i^2 k_i^3 + (k_i^1 + k_i^2) \left( b_i + \sum_{j=1}^n a_{ij} \right) \right) \bar{\sigma}
\]  

(20)

the inequality shows the control input $u_i$ given by equation (18) is bounded.

Based on the preparations above, the following theorem is ready to be presented now.

**Theorem 1.** Consider the control input given by equation (18) with equation (19) of the $n$-dimensional nominal system (equation (16)). Under Assumptions 2 and 3, that is, the graph $\mathcal{G}$ is an undirected and connected graph, the desired position trajectory $r_d$ is connected at least one agent, and $\tilde{r}_d$ is bounded by a known upper bound. If the control gains of equations (18) and (19) for each $i, j \in \mathcal{I}$ satisfy

\[
\left( k_i^2 k_j^3 + (k_i^1 + k_j^2) \left( b_i + \sum_{j=1}^n a_{ij} \right) \right) \bar{\sigma} \leq \bar{u}
\]  

(21)

then, the following results have been obtained: (a) $\|u_i\|_\infty \leq \bar{u}$; (b) $\lim_{t \to \infty} \tilde{z}_i(t) = 0$, $\lim_{t \to \infty} \dot{\tilde{z}}_i(t) = 0$, $\lim_{t \to \infty} \tilde{r}_i = 0$ and $\lim_{t \to \infty} (v_i(t) - \dot{r}_d(t)) = 0$ for all $i \in \mathcal{I}$, that is, the objective (equation (5)) is achieved.

**Proof.** From equations (20) and (21), result (a) can be readily proved. For proof (result (b)), Lyapunov-based approach is used, and the Lyapunov function candidate is presented in the following

\[
V = V_1 + V_2 + V_3 + V_4
\]  

(22)

Each term of $V$ is as follows

\[
V_1 = \frac{1}{2} \sum_{i=1}^n \left( v_i - \dot{r}_d - k_i^1 \tilde{z}_i \right)^2
\]  

(23)

\[
V_2 = \frac{1}{2} \sum_{i=1}^n k_i^1 k_i^2 \dot{\tilde{z}}_i
\]  

The time derivative of $V_i, i = 1, \ldots, 4$ is as follows

\[
\dot{V}_1 = -\sum_{i=1}^n k_i^1 (v_i - \dot{r}_d - k_i^1 \tilde{z}_i)
\]  

(24)

\[
\dot{V}_2 = -\sum_{i=1}^n k_i^1 k_i^2 \tilde{z}_i (\frac{\sigma(\tilde{z}_i)}{\sigma(\tilde{z}_i)}) + b_i \sigma(\tilde{z}_i))
\]  

(25)

\[
\dot{V}_3 = \sum_{i=1}^n k_i^1 b_i (v_i - \dot{r}_d) \sigma(\tilde{z}_i)
\]  

(26)

\[
\dot{V}_4 = \frac{1}{2} \sum_{i=1}^n \sum_{i=1}^n k_i^1 a_{ij} (v_i - v_j) \sigma(\tilde{z}_j)
\]  

(27)

Since $\mathcal{G}$ is undirected, $a_{ij} = a_{ji}$ for $i, j \in \mathcal{I}$. Using the results of Ren and Cao,\textsuperscript{20} the following is obtained

\[
\sum_{i=1}^n \sum_{j=1}^n k_i^1 a_{ij} ((v_i - \dot{r}_d) - (v_j - \dot{r}_d)) \sigma(\tilde{z}_j)
\]  

(28)

with which equation (27) becomes

\[
\dot{V}_4 = \sum_{i=1}^n \sum_{j=1}^n k_i^1 a_{ij} (v_i - \dot{r}_d) \sigma(\tilde{z}_j)
\]  

(29)

It is obtained that $\dot{V}$ in equation (29) is negative semi-definite since $k_i^1 > 0, k_i^2 > 0, k_i^3 > 0$ for any $i \in \mathcal{I}$. Thus, $v_i - \dot{r}_d, \tilde{z}_i, \tilde{r}_i$ and $\tilde{r}_d$ are all globally bounded. Since $\dot{r}_d$ is bounded due to the Assumption 2, $v_i, \ i \in \mathcal{I}$,
are also globally bounded. Then, $e_i$ and $\xi_i$ are global bounded using relationship (equation (19)). Moreover, due to the expression (equation (29)), the largest invariant set of the states is

$$S_i = \{(v_i, \xi_i, r_i, \bar{r}_i)| \xi_i = 0\}, i \in \mathcal{I} \quad (30)$$

Since $V$ given by equation (24) is radially unbounded, LaSalle’s invariance principle is applied to obtain

$$\lim_{t \to \infty} \xi_i(t) = 0, i \in \mathcal{I} \quad (31)$$

Then, the boundedness of $\xi_i$ can be derived due to the globally boundedness of $\bar{r}_i, v_i, \xi_i$ and $\dot{\xi}_i$ for $i \in \mathcal{I}$. By applying Barbálat Lemma, the following is obtained

$$\lim_{t \to \infty} \xi_i(t) = 0, i \in \mathcal{I} \quad (32)$$

From equations (19), (31) and (32)

$$\lim_{t \to \infty} e_i(t) = 0, i \in \mathcal{I} \quad (33)$$

Under the dynamic system (equation (19)), this implies that the conditions of Lemma 2 are satisfied. Thus, $\lim_{t \to \infty} \bar{r}_i = 0$. With the relationship $\bar{r}_i(t) = u_i(t)$, equation (20) and $\bar{r}_i(t) = 0$, the signals $\bar{r}_i(t), i \in \mathcal{I}$ are bounded. Then, $\lim_{t \to \infty} (v_i(t) - \bar{r}_i(t)) = 0$ by applying Barbálat Lemma. The objectives of point (b) are all proved now.

Remark 1. By Assumption 2, the second-order derivatives of $r_d$ are assumed to be 0. However, if $r_d \neq 0$, the controller for equation (16) can be modified to be $u_i = \bar{r}_i - e_i + \xi_i, i \in \mathcal{I}$. Since $\bar{r}_d$ is needed for each agent, Assumption 3 is not satisfied by this modification. For this purpose, the interpolation approach can be considered to let $r_d$ be linear approximated with enought short interval. For a piecewise linear polynomial curve, its second-order derivative is zero almost everywhere and Assumptions 2 and 3 are trivially satisfied.

UDE-based robustness improvement

For equation (4) with $\dot{d}_i \neq 0$, with the controller $u_i$ given by equations (18) and (19), the following tracking controller is proposed

$$\mu_i = u_i - \dot{d}_i, i \in \mathcal{I} \quad (34)$$

where $\dot{d}_i$ is the estimation of $d_i$.

The disturbance estimation error $\bar{d}_i$ is defined as

$$\bar{d}_i = d_i - \dot{d}_i, i \in \mathcal{I} \quad (35)$$

and $\bar{d}$ can be written as

$$\bar{d} = [\bar{d}_1, \ldots, \bar{d}_n]^T \in \mathbb{R}^n \quad (36)$$

The following UDE proposed in Zhu et al.$^{21}$ are used for all $i \in \mathcal{I}$ in this paper

$$\dot{d}_i = G_{\theta}(s)(r_i(t) - r_i(0)) - G_{\theta}(s) \int_{0}^{t} u_i(\tau) d\tau \quad (37)$$

where $G_{\theta}(s)$ and $G_{\theta}(s)$ are given by

$$G_{\theta}(s) = \frac{s}{a_1 s^2 + a_0}, G_{\theta}(s) = \frac{1}{a_1 s^2 + a_0}, a_0 > 0, a_1 > 0 \quad (38)$$

The transfer function from $\dot{d}_i$ to $d_i$ is

$$G(s) = \frac{a_1 s^2 + a_0}{a_1 s^2 + a_0 s + 1} \quad (39)$$

It is practically important to find a single design parameter which is closely related with the closed-loop steady-state performance. In particular, to make parameter tuning easy, it is expected that the relationship is as simple as possible. To this end, the following parameter mapping is proposed

$$\begin{cases} a_0 = \frac{a_1}{\varepsilon^2} \\ a_1 = \frac{a_0}{\varepsilon} \end{cases} \quad (40)$$

where $\varepsilon > 0, a_1 > 0$ and $a_2 > 0$ are three auxiliary design parameters. In this paper, the design parameters $a_0$ and $a_1$ are determined from $\varepsilon, a_1$ and $a_2$, as shown in Figure 1.

Then, the transfer function (equation (39)) can be rewritten as

$$G(s) = \frac{\varepsilon (a_1 s + a_1)}{(a_1 s + a_0)(a_1 s + a_2)} \quad (41)$$

Note that $G(s)$ is stable and satisfies

$$\lim_{s \to 0} G(s) = 0 \quad (42)$$

With the parameter mapping, it is readily obtained that when $\varepsilon$ is sufficiently small, the ultimate bounds of $\dot{d}$ would be arbitrarily small. From equation (40), $1/a_1 = \varepsilon^2/a_2$ and $a_0/a_1 = a_1/a_2; \varepsilon \rightarrow 0$ implies that $1 \ll a_0/a_1 \ll 1/a_1$ or $a_1 \ll a_0 \ll 1$.

Lemma 3. Under Assumption 1, the (globally) ultimate boundedness of $\dot{d}(t)$ is achieved, furthermore (Lemma 5 in Zhu et al.$^{21}$)

![Figure 1. The approach to determine disturbance estimator parameters $a_0$ and $a_1$.](image-url)
1. If the time derivative of \( \dot{d}(t) \) additionally satisfies

\[
\lim_{t \to +\infty} \dot{d}(t) = 0 \quad (43)
\]

then for any initial condition

\[
\lim_{t \to +\infty} \ddot{d}(t) = 0 \quad (44)
\]

2. When equation (43) is not satisfied, the parameter \( \varepsilon \) can be set small enough to make \( \ddot{d}(t) \) ultimate bounded, that is, assume that there exist a number \( \varepsilon_0 > 0 \) and a time \( t_f > 0 \), for any \( \overline{d} > 0 \), whenever \( \varepsilon < \varepsilon_0 \), such that

\[
|\ddot{d}(t)| < \overline{d}, \forall t \geq t_f \quad (45)
\]

Lemma 3 implies that the steady-state errors can be controlled by a parameter tuning approach. From equations (4), (18), (34) and (35), the closed-loop error equations are obtained

\[
\ddot{r}_i = - (k_i^1 + k_i^2) e_i - k_i^3 k_i^4 \sigma(\xi_i) + \ddot{d}, i \in I \quad (46)
\]

The following analyses are focused on the closed-loop stability of equation (46).

**Lemma 4.** Consider the system (46) with \( e_i \) defined in equation (10) and \( \xi_i \) defined in equation (19). If the condition of Lemma 3 is satisfied and \( \lim_{t \to +\infty} \ddot{d}(t) = 0 \), then \( \ddot{r}_i(t) \) and \( v_i(t) - \ddot{r}_i(t) \) are bounded for all time and \( \lim_{t \to +\infty} F_i(t) = \lim_{t \to +\infty} (v_i(t) - \ddot{r}_i(t)) = 0 \).

**Proof.** Consider the same Lyapunov function candidate as equation (24). The derivative of \( V_i \), \( i = 1, \ldots, 4 \) are given as follows

\[
\begin{align*}
\dot{V}_1 &= -\sum_{i=1}^{n} k_1^1 (v_i - \ddot{r}_i - k_1^1 \xi_i) (e_i - \ddot{d}) \\
\dot{V}_2 &= \sum_{i=1}^{n} k_2^1 \xi_i (-k_3^1 \sigma(\xi_i) - e_i) \\
\dot{V}_3 &= \sum_{i=1}^{n} k_3^1 b_i \sigma(\ddot{r}_i) \ddot{r}_i \\
\dot{V}_4 &= \sum_{i=1}^{n} \sum_{j=1}^{n} k_4^1 a_{ij} \sigma(v_i - \ddot{r}_i, j), i \in I
\end{align*}
\]

Define \( \chi = [(v_1 - \ddot{r}_1 - k_1^2 \xi_1), \ldots, (v_n - \ddot{r}_n - k_1^2 \xi_n)]^T \).

Then, for all \( i \in I \)

\[
\dot{V} = -\sum_{i=1}^{n} k_1^2 \sum_{j=1}^{n} k_4^1 (v_i - \ddot{r}_i - k_1^2 \xi_i) \ddot{d}_i + \sum_{i=1}^{n} k_3^1 (v_i - \ddot{r}_i - k_1^2 \xi_i) \ddot{d}_i \leq \| V \| \| \ddot{V} \| \leq \sqrt{n} \sqrt{\dot{V}} \quad (48)
\]

Then it is readily to obtain the inequality \( dV/V \leq \sqrt{n} \dot{V} \). Then, \( V(t) \leq V(0) \leq 1/2 \sqrt{n} \). Thus, \( \ddot{r}_i(t), v_i(t) - \ddot{r}_i(t) \) and \( \zeta_i(t), i \in I \) cannot escape in finite time. With Lemma 1 in Abdessameud and Tayebi, \( \lim_{t \to +\infty} \xi_i(t) = 0 \). With a similar analysis in Theorem 1, the convergence of \( \ddot{r}_i(t) \) and \( v_i(t) - \ddot{r}_i(t) \) can be achieved.

For the general case with \( \lim_{t \to +\infty} \ddot{d}_i \neq 0, i \in I \) and \( \ddot{d}_i \) is viewed as input, the following results are obtained.

**Lemma 5.** Consider the closed-loop system (equation (46) with equation (10)), \( \ddot{d}_i \) is regarded as the input and \( (\ddot{r}_i, \ddot{r}, \ddot{\xi})^T \) is regarded as the state and output. Then system (equation (46)) is small-signal \( L_\infty \) stable and locally input-to-state stable (LISS).

**Proof.** The system state and output for equation (46) are denoted by \( s = (\ddot{r}_i, \ddot{r}, \ddot{\xi})^T \) and \( v = s \), respectively; then system (46) can be rewritten as follows

\[
\dot{s} = f(t, s, \ddot{d}_i), y = h(t, s, \ddot{d}_i), i \in I \quad (49)
\]

where

\[
\begin{align*}
\dot{s} &= f(t, s, \ddot{d}_i) = \left[ - (k_1^1 + k_2^1) e_i - k_3^1 k_4^1 \sigma(\xi_i) + \ddot{d}_i \right] \\
h(t, s, \ddot{d}_i) &= s
\end{align*}
\]

The Jacobian matrices of \( f(t, s, \ddot{d}_i) \) are computed as follows

\[
\frac{\partial f}{\partial s} = \left[ \begin{array}{ccc} 0 & 0 & 0 \\ -k_1^1 & -k_3^1 & -k_3^1 \\ b_i \frac{\partial \sigma(\ddot{r}_i)}{\partial s} & 0 & -k_3^1 \frac{\partial \sigma(\xi_i)}{\partial s} \\ b_i \frac{\partial \sigma(\ddot{r}_i)}{\partial s} & 0 & -k_3^1 \frac{\partial \sigma(\xi_i)}{\partial s} \end{array} \right] \quad (50)
\]

Define \( \chi = [(v_1 - \ddot{r}_1 - k_1^2 \xi_1), \ldots, (v_n - \ddot{r}_n - k_1^2 \xi_n)]^T \).

Then, for all \( i \in I \)

\[
\dot{V} = -\sum_{i=1}^{n} k_1^2 \sum_{j=1}^{n} k_4^1 (v_i - \ddot{r}_i - k_1^2 \xi_i) \ddot{d}_i + \sum_{i=1}^{n} k_3^1 (v_i - \ddot{r}_i - k_1^2 \xi_i) \ddot{d}_i \leq \| V \| \| \ddot{V} \| \leq \sqrt{n} \sqrt{\dot{V}} \quad (48)
\]

Under Assumption 1, \( f(t, s, \ddot{d}_i) \) is a continuously differentiable function; by applying the property P3 of GSFs, \( \partial f(t, s, \ddot{d}_i)/\partial s \) and \( \partial f(t, s, \ddot{d}_i)/\partial \ddot{d}_i \) are bounded uniformly in \( t \). According to Theorem 1, it is known that the globally asymptotically stable equilibrium point for system (equations (18) and (19)) is \( s = [0, 0, 0]^T \). Thus, the converse Lyapunov theorem is satisfied for system (equations (18) and (19)). Furthermore, since \( \| f(t, s, \ddot{d}_i) - f(t, s, 0) \| = \| \ddot{d}_i \| \) and \( \| h(t, s, \ddot{d}_i) \| = \| s \| \), all the conditions of small-signal \( L_\infty \) stable are satisfied. Thus, it is obtained that the system (46) is small-signal \( L_\infty \) stable. Furthermore, the system trajectory satisfies the following inequality.
globally bounded and (b) the objective (equation (5)) is reduced via a single-parameter tuning. This approach, the steady-state control errors can be minimized (or tracking errors is simple). Based on this approach to control the ultimate bounds of synchronization errors are only required to be positive real numbers to guarantee closed-loop stability is simple. The design parameters are expressed in 2D vectors. In the next section, the performance of the scheme for this paper is demonstrated in two-dimensional (2D) space; related parameters and variables are expressed in 2D vectors.

**Theorem 2.** Under all the assumptions defined in this paper, consider the closed-loop system (equation (46) with equations (4) and (10)). If all the conditions of Lemma 3–5 are satisfied and (\(\hat{r}_i, \tilde{r}_i, \xi_i\)) viewed as state and output, the small-signal \(L_{\infty}\) stable and LISS are both achieved for the system (equation (46)). Moreover, the state trajectories satisfy: (a) \(\tilde{r}_i, \hat{r}_i, \xi_i\) is globally bounded and (b) the objective (equation (5)) is achieved when \(\varepsilon\) is sufficiently small.

**Remark 2.** Compared with other relevant approaches, the advantages of the proposed approach in this paper are summarized in the following two aspects: (1) our approach yields continuous control signals and actuator chattering is avoided compared with sliding-mode control approaches. Since both the UDE and nominal tracker are continuous, the resulting control signal is continuous. However, a sign function is included in standard sliding-mode controllers, and actuator chattering issue cannot be avoided. (2) The proposed approach is simpler than most of the relevant approaches proposed in recent literature. The simplicity lies in the following three aspects. First, the controller is structurally simple and has a relative-lower order. It consists of two components: the nominal control and the output of UDE. Their roles are separated from each other. Second, the parameter condition for the closed-loop stability is simple. The design parameters are only required to be positive real numbers to guarantee the stability. Third, the parameter tuning approach to control the ultimate bounds of synchronization errors (or tracking errors) is simple. Based on this approach, the steady-state control errors can be reduced via a single-parameter tuning.

**Simulation results**

The position consensus tracking and formation control of a team of \(n\) agents are aimed in this paper. Numerical simulation is used to demonstrate the results of this paper. Suppose that the system is in 2D space and the number of agents is 4. For each case, the following simulation conditions are considered:

- The associated undirected communication digraph \(\hat{G}\) is shown in Figure 2 with \(b_1 = 0.5\) and \(b_i = 0\) for \(i = 2, 3, 4\), \(a_{ij} = 0.5\), \(i = 2, 3, 4\) if
the agents can access the information from neighbors, otherwise $a_{ii} = 0$, $i = 2, 3, 4$.

- The control gains $k_1 = k_2 = 1$, $k_3 = 0.5$, $i = 1, \ldots, 4$. The UDE parameters are: $\varepsilon = 0.1$, $\alpha_1 = \alpha_2 = 1$.
- The initial states of the four agents are as follows

$$r_1(0) = [1, 1]^T$$
$$r_2(0) = [-1, 1]^T$$

Figure 2. The communication graph used in this section.

Figure 3. Simulation results of Case 1.
The desired position signal is a ramp function, where $A$ and $B$ denote the scale factor, and $u(t)$ denotes the unit step function. Assume that $A = B = 0.5$. Note that the second derivative of a ramp function is 0, and this satisfies to the conditions of Assumption 2.

\[ r_3(0) = [-1, -1]^T \]
\[ r_4(0) = [1, -1]^T \]
\[ v_i(0) = [0, 1]^T, i = 1, ..., 4 \]

- The desired position signal is a ramp function, where $A$ and $B$ denote the scale factor, and $u(t)$ denotes the unit step function. Assume that $A = B = 0.5$. Note that the second derivative of a ramp function is 0, and this satisfies to the conditions of Assumption 2.

Choose GSF $\sigma(\cdot) = \tanh(\cdot)$. 

In the following, four simulation cases are considered based on the main results in this paper. MATLAB 2016a is used as the simulation tool and is assumed that the sampling time is fixed to be 0.001 s and the solver is ode3 (Bogacki–Shampine).
Case 1: position tracking without disturbance

Consider system (equation (16)) with controller (equations (18) and (19)) in 2D space with the simulation conditions. Figure 3 shows the results of this case. The following three points are obtained: each position trajectory \((r_{1i}, r_{2i})\) of the agent approaches the reference trajectory \(r_d(t)\) in 2D space; each position on the \(x\)- and \(y\)-axis of the agent also converges asymptotically to the reference trajectory; and the control signals are bounded.

Case 2: position tracking with UDEs to reject constant \(d_i\)

In this case, the position tracking with UDEs is considered to reject constant disturbances. The disturbance is given as

\[
d_i(t) = [2, 2]^T, i \in I, \forall t \geq 0
\]

It is readily to obtain that \(\lim_{t \to \infty} d_i \to 0\) is satisfied. Assume that \(a_1 = 0.01\) and \(a_0 = 0.5\). Apply

![Figure 5. Simulation results of Case 3.](image-url)
controllers (equation (34)) to system (equation (4)) with the above constant input disturbance. The simulation results are demonstrated in Figure 4. It is seen that the UDEs reject the disturbances effectively and the position tracking objective is achieved.

Case 3: formation control with $d_i = 0$

In this case, the desired position deviation is given as $\delta_i = [5, 5, 5, 5]^T$. Applying formation controller (equations (61) and (62)) to the system (equation (58)) without disturbances yields the simulation results (Figure 5). Note that both $\hat{\delta}$ and $\tilde{\delta}$ are ignored here. It is seen that the square formation is achieved and the controllers are converged to 0 as time increased.

Case 4: formation control with UDEs to reject constant $d_i$

In this case, the same UDEs as in Case 2 are used. Applying controller (equations (61) and (62)), the simulation results are shown in Figure 6. It is seen that the UDEs reject the disturbances effectively, and the square formation is achieved.
formation is achieved and the controllers are converged to 0 as time increased.

Remark 4. From the simulation examples, it is seen that the system trajectories suffer from quite obvious oscillations; the reason is twofold: (a) note that the control design in this paper is limited by input saturation constraint, the simulation results are compared in Zhu et al.\textsuperscript{21} (with no control input constraint) and in this paper and it is obvious that the control input constraint is the primary reason to cause system oscillations. (b) the velocity measurements are assumed to be unavailable in this paper; velocity information are not used to design a controller, and this will make system damping smaller.

Conclusion

In this paper, the synchronized trajectory tracking problem of a group of agents with multiple second-order dynamics without velocity measurements is studied first, taking into account the input saturation constraint. Then, the position tracking problem is extended to the formation problem by adding the desired position deviation of each agent from the leader’s trajectory. The effectiveness of the presented control schemes is illustrated by numerical simulation results. In future work, the effects of design parameters on the convergence rate are studied systematically.

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