Graph Pattern Polynomials

Markus Bläser
Department of Computer Science, Saarland University, Saarland Informatics Campus, Saarbrücken, Germany
mblaeser@cs.uni-saarland.de
https://www-cc.cs.uni-saarland.de/mblaeser/

Balagopal Komarath
Saarland University, Saarland Informatics Campus, Saarbrücken, Germany
baluks@gmail.com
http://www-cc.cs.uni-saarland.de/bkomarath/

Karteek Sreenivasaiah
Department of Computer Science and Engineering, Indian Institute of Technology Hyderabad, India
karteek@iith.ac.in
http://www.iith.ac.in/~karteek

Abstract

Given a host graph $G$ and a pattern graph $H$, the induced subgraph isomorphism problem is to decide whether $G$ contains an induced subgraph that is isomorphic to $H$. We study the time complexity of induced subgraph isomorphism problems where the pattern graph is fixed. Nešetřil and Poljak gave an $O(n^{k\omega})$ time algorithm that decides the induced subgraph isomorphism problem for any $3k$ vertex pattern graph (the universal algorithm), where $\omega(p,q,r)$ is the exponent of $n^p \times n^q$ and $n^q \times n^r$ matrix multiplication and $\omega = \omega(1, 1, 1)$.

Algorithms that are faster than the universal algorithm are known only for a finite number of pattern graphs. In this paper, we obtain algorithms that are faster than the universal algorithm for infinitely many pattern graphs. More specifically, we show that there exists a family of pattern graphs $(H_{3k})_{k \geq 0}$ such that the induced subgraph isomorphism problem for $H_{3k}$ (on $3k$ vertices) has a $O(n^{\omega(k,k-1,k)})$ time algorithm when $k = 2^r, r \geq 1$. Note that for the currently known best matrix multiplication algorithms $\omega(k,k-1,k) < k\omega$.

This algorithm is obtained by a reduction to the multilinear term detection problem in a class of polynomials called graph pattern polynomials. We formally define this class of polynomials along with a notion of reduction between these polynomials that allows us to argue about the fine-grained complexity of isomorphism problems for different pattern graphs. Besides the aforementioned result, we obtain the following algorithms for induced subgraph isomorphism problems:

1. Faster than universal algorithm for $P_k$ ($k$-vertex paths) when $5 \leq k \leq 9$ and $C_k$ ($k$-vertex cycles) for $k \in \{5, 7, 9\}$. In particular, we obtain $O(n^\omega)$ time algorithms for $P_5$ and $C_5$ that are optimal under reasonable hardness assumptions.

2. Faster than universal algorithm for all pattern graphs except $K_k$ ($k$-vertex cliques) and $I_k$ ($k$-vertex independent sets) for $k \leq 8$.

3. Combinatorial algorithms (algorithms that do not use fast matrix multiplication) that take $O(n^{k-2})$ time for $P_k$ and $C_k$.

4. Combinatorial algorithms that take $O(n^{k-1})$ time for all pattern graphs except $K_k$ and $I_k$ for $k$.

Our notion of reduction can also be used to argue about hardness of detecting patterns within our framework. Since this method is used (explicitly or implicitly) by many existing algorithms (including the universal algorithm) for solving subgraph isomorphism problems, these hardness results show the limitations of existing methods. We obtain the following relative hardness results:
Graph Pattern Polynomials

1. Induced subgraph isomorphism problem for any pattern containing a $k$-clique is at least as hard as $k$-clique.
2. For almost all patterns, induced subgraph isomorphism is harder than subgraph isomorphism.
3. For almost all patterns, the subgraph isomorphism problem for any of its supergraphs is harder than subgraph isomorphism for the pattern.

2012 ACM Subject Classification Theory of computation $\rightarrow$ Probabilistic computation, Theory of computation $\rightarrow$ Problems, reductions and completeness

Keywords and phrases algorithms, induced subgraph detection, algebraic framework

1 Introduction

The induced subgraph isomorphism problem asks, given simple and undirected graphs $G$ and $H$, whether there is an induced subgraph of $G$ that is isomorphic to $H$. The graph $G$ is called the host graph and the graph $H$ is called the pattern graph. This problem is NP-complete (See [12], problem [GT21]). If the pattern graph $H$ is fixed, there is a simple $O(n|V(H)|)$ time algorithm to decide the induced subgraph isomorphism problem for $H$. We study the time complexity of the induced subgraph isomorphism problem for fixed pattern graphs on the Word-RAM model.

The earliest non-trivial algorithm for this problem was given by Itai and Rodeh [13] who showed that the number of triangles can be computed in $O(n^{\omega})$ time on $n$-vertex graphs, where $\omega$ is the exponent of matrix multiplication. Later, Nešetřil and Poljak [16] generalized this algorithm to count $K_{3k}$ in $O(n^{k\omega})$ time, where $K_{3k}$ is the clique on $3k$ vertices. Eisenbrand and Grandoni [8] extended this algorithm further to count $K_{3k+j}$ for $j \in \{0, 1, 2\}$ using rectangular matrix multiplication in $O(n^{\omega(k+\lceil j/2 \rceil,k,k+\lfloor j/2 \rfloor)})$ time. This algorithm uses fast matrix multiplication to achieve the speedup and in fact works for all pattern graphs on $3k+j$ vertices. Hence we call this algorithm the universal algorithm. It is reasonable to expect that one might be able to obtain faster algorithms for specific pattern graphs. However, algorithms faster than the universal algorithm are only known for finitely many pattern graphs.

Algorithms that do not use fast matrix multiplication, called combinatorial algorithms, have also been studied. No combinatorial algorithm that beats the trivial $O(n^k)$ time algorithm is known for detecting $k$-cliques in $n$-vertex graphs. However, improvements for certain pattern graphs such as $K_k - e$ has been shown by Virginia Williams (See [17], p.45). They show a combinatorial algorithm that decides the induced subgraph isomorphism problem for $K_k - e$ in time $O(n^{k-1})$. An $O(n^{k-1})$ combinatorial algorithm is also known for deciding induced subgraph isomorphism problem for $P_k$.

The use of algebraic methods has been particularly useful in finding fast combinatorial algorithms for detecting pattern graphs. Ryan Williams [18] gave a linear time algorithm for the (not necessarily induced) subgraph isomorphism problem for $P_k$. This was later generalized by Fomin, Lokshtanov, Raman, Saurabh, and Rao [11] to give $O(n^{tw(H)+1})$ time algorithms for the (not necessarily induced) subgraph isomorphism problem for $H$ in $n$ vertex graphs. These results use efficient constructions for homomorphism polynomials (defined later).

The question of whether improving algorithms for detecting a certain pattern implies faster algorithms for another pattern has also been studied. In particular, Nešetřil and Poljak show that improved algorithms for detecting $k$-cliques yield improved algorithms for all $k$-vertex pattern graphs. More precisely:
Theorem 1.1. (16) If the induced subgraph isomorphism problem for $K_k$ can be decided in $O(n^{f(k)})$ time for some $f(k)$, then the induced subgraph isomorphism problem for $H$ can be decided in time $O(n^{f(k)})$ time, where $H$ is any $k$-vertex pattern graph.

In this sense, the $k$-clique is a universal pattern.

Nešetřil and Poljak’s [16] algorithm can be easily modified to output the homomorphism polynomial no host graphs of $n$ vertices for the pattern $K_3^k$ in $O(n^{k \omega})$ time given $n$ as input. For cliques, counting (or detecting) homomorphisms and counting (or detecting) induced subgraph isomorphisms have the same complexity. It is unclear whether computing homomorphism polynomials efficiently for other pattern graphs help with the induced subgraph isomorphism problem for those pattern graphs.

Our Results

In this paper, we show that we can obtain algorithms that are faster than the universal algorithm for infinitely many pattern graphs.

Theorem 5.4. There exists a family of pattern graphs $(H_{3k})_{k \geq 0}$ where $H_{3k}$ is a $3k$-vertex graph such that the induced subgraph isomorphism problem for $H_{3k}$ has an $O(n^{\omega(k,k-1,k)})$ time algorithm for infinitely many $k$.

Here, $\omega(p,q,r)$ is the exponent of $n^p \times n^q$ and $n^q \times n^r$ matrix multiplication. The exponent of matrix multiplication is defined as $\omega = \omega(1,1,1)$ (See [3] for a more detailed introduction). The best known algorithm for $K_{3k}$ takes time $O(n^{k \omega})$ and the upper-bound on $\omega(k,k-1,k)$ is strictly smaller than the upper-bound on $k \omega$ for the currently known fastest matrix multiplication algorithms. If $\omega = 2$, then we have $\omega(k,k-1,k) = k \omega$ and the above algorithm does not improve upon the universal algorithm. However, the best upper-bound that we know is $\omega < 2.373$. It is known that current methods cannot show $\omega = 2$ (See [2, 1]).

We develop an algebraic framework to study algorithms for the induced subgraph isomorphism problems where we consider the size of the pattern graph a constant. The above algorithm is obtained using this framework. We show that the existing algorithms for natural pattern graphs such as $k$-paths and $k$-cycles can be improved by efficiently computing homomorphism polynomials for pattern graphs that are much sparser than $k$-cliques.

We obtain, in Theorem 6.4 and Theorem 6.8, the following faster (randomized, one-sided error) algorithms:

- Faster algorithms for induced subgraph isomorphism problem for $P_k$ for $5 \leq k \leq 9$.
- Faster algorithms for induced subgraph isomorphism problem for $C_k$ for $k \in \{5, 7, 9\}$.
- $O(n^{k-2})$ time combinatorial algorithm for induced subgraph isomorphism problem for $P_k$ and $C_k$.
- $O(n^{k-2})$ time deterministic combinatorial algorithms for computing the parity of the number of induced subgraphs isomorphic to $P_k$ and $C_k$ in $n$-vertex graphs.

Unfortunately, we do not know how to compute these homomorphism polynomials for smaller graphs using circuits of size smaller than that for homomorphism polynomials for $k$-cliques when $k$ is arbitrary. Therefore, we do not have an improvement similar to the one in Theorem 5.4 for paths or cycles.

---

³ For host $G$ and pattern $H$, a function $f : V(H) \rightarrow V(G)$ such that $\{u, v\} \in E(H) \implies \{f(u), f(v)\} \in E(G)$
Graph Pattern Polynomials

In light of Theorem 1.1, which shows that $k$-cliques are universal, we show that homomorphism polynomials for $K_k - e$, the $k$-vertex graph obtained by deleting an edge from $K_k$, are almost universal. We show that the arithmetic circuit complexity of $\text{Hom}_{K_k - e}$ can be used to unify many existing results. We show that if $\text{Hom}_{K_k - e}$ has $O(n^{f(k)})$ size circuits for some function $f(k)$, then:

1. (Theorem 7.4) The induced subgraph isomorphism problem for all $k$-vertex pattern graphs other than $K_k$ and $I_k$ can be decided by an $O(n^{f(k)})$ time algorithm, where $k$ is regarded as a constant and $f(k)$ is any function of $k$. ([17] gives a combinatorial algorithm for $K_k - e$, [10] gives an algorithm for $P_k$)

2. (Theorem 7.5) If there is an $O(t(n))$ time algorithm for counting the number of induced subgraph isomorphisms for a $k$-vertex pattern $H$, then the number of induced subgraph isomorphisms for all $k$-vertex patterns can be computed in $O(n^{f(k)} + t(n))$ time on $n$-vertex graphs. ([14] gives this result for $k = 4$ and [15] gives a weaker result similar to this one)

Even though these theorems are known for specific values of $f(k)$ as cited above. We believe that the connection to homomorphism polynomials for $\text{Hom}_{K_k - e}$ is new.

On the lower bounds front, we show in Theorem 8.3, Theorem 8.6 and Theorem 8.4 that within the framework that we develop:

1. The induced subgraph isomorphism problem for any pattern containing a $k$-clique or a $k$-independent set is at least as hard as the isomorphism problem for $k$-clique.
2. For almost all pattern graphs $H$, the induced subgraph isomorphism problem for $H$ is harder than the subgraph isomorphism problem for $H$.
3. For almost all pattern graphs $H$, the subgraph isomorphism problem for $H$ is easier than subgraph isomorphism problems for all supergraphs of $H$.

We note that only randomized algorithmic reductions are known for Part 2 of the above theorem and Part 3 is unknown. It is not clear whether our reductions imply algorithmic hardness for these problems.

Technique

The Homomorphism polynomial for a pattern graph $H$ denoted $\text{Hom}_{H,n}$ is a polynomial such that the monomials of the polynomial correspond one-to-one with homomorphisms from $H$ to an $n$-vertex graph. Similarly, we define the graph pattern polynomial families $I_H = (I_{H,n})_{n \geq 0}$ and $N_H = (N_{H,n})_{n \geq 0}$ that correspond to the induced subgraph isomorphism problem for $H$ and the (not necessarily induced) subgraph isomorphism problem for $H$ respectively. It can be shown that testing for subgraph isomorphism is equivalent to testing whether the homomorphism polynomial has multilinear terms because subgraph isomorphisms are exactly the injective homomorphisms. In fact, any polynomial family $f$ such that the multilinear terms of $f$ correspond to multilinear terms of $N_H$ is enough. This naturally leads to a notion of reduction between these graph pattern polynomial families (denoted by $\preceq$). For example, we say that $N_H \preceq \text{Hom}_{H}$. This notion of reduction allows us to compare the hardness of different pattern detection problems as well as construct new algorithms as follows:

Given $(G, H)$, decide whether there exists an injective $f : V(H) \to V(G)$ such that $\{u, v\} \in E(H) \implies \{f(u), f(v)\} \in E(G)$.
A subgraph isomorphism from \( \{ \phi \} \in \mathcal{E} \) consists of vertices \( v \in \mathcal{V} \) such that \( \phi(v) \) is a member of \( \mathcal{S} \) and \( \mathcal{E} \) contains a \( k \)-clique (or \( k \)-independent set) that is vertex-disjoint from all other \( k \)-cliques or \( k \)-independent sets. We show that this is not possible by showing that in this case \( \text{Hom}_H \) has circuits that are as small as circuits for \( f \).

Other related work

Curticapean, Dell, and Marx [5] showed that algorithms that count homomorphisms can be used to count subgraph isomorphisms. Williams, Wang, Williams, and Yu [19] gave \( O(n^w) \) time algorithms for the induced subgraph isomorphism problems for four vertex pattern graphs, except for \( I_4 \) and \( K_4 \). Floderus, Kowaluk, Lingas, and Lundell [10] invented a framework that gives \( O(n^{k-1}) \) combinatorial algorithms for induced subgraph isomorphism problems for many pattern graphs on \( k \) vertices.

Floderus, Kowaluk, Lingas, and Lundell [9] showed reductions between various induced subgraph isomorphism problems. They proved that induced subgraph isomorphism problem for \( H \) when \( H \) contains a \( k \)-clique (or \( k \)-independent set) that is vertex-disjoint from all other \( k \)-cliques (or \( k \)-independent sets) is at least as hard as the induced subgraph isomorphism problem for \( K_k \). They also proved that detecting an induced \( C_4 \) is at least as hard as detecting a \( K_3 \). This is the only example known where a pattern is harder than another pattern that is not a subgraph. Hardness results are also known for arithmetic circuits computing homomorphism polynomials. Austin, Kashi, and Kubjas [3] proved that tensor networks (a restricted form of arithmetic circuits) computing homomorphism polynomials for \( k \)-cliques require \( \Omega(n^{2k/3}) \) time. Durand, Mahajan, Malod, Rugy-Altherre, and Saurabh [7] proved that homomorphism polynomials for certain pattern families are complete for the class \( \text{VP} \), the algebraic analogue of the class \( \text{P} \). This is the only known polynomial family that is complete for \( \text{VP} \) other than the canonical complete family of universal circuits.

2 Preliminaries

For a polynomial \( f \), we use \( \text{deg}(f) \) to denote the degree of \( f \). A monomial is called multilinear, if every variable in it has degree at most one. We use \( \text{ML}(f) \) to denote the multilinear part of \( f \), that is, the sum of all multilinear monomials in \( f \). An arithmetic circuit computing a polynomial \( P \in K[x_1, \ldots, x_n] \) is a circuit with \( +, \times \) gates where the input gates are labelled by variables or constants from the underlying field and one gate is designated as the output gate. The size of an arithmetic circuit is the number of wires in the circuit. For indeterminates \( x_1, \ldots, x_n \) and a set \( S = \{ s_1, \ldots, s_p \} \subseteq \{ 1, \ldots, n \} \) of indices, we write \( x_S \) to denote the product \( x_{s_1} \cdots x_{s_p} \).

An induced subgraph isomorphism from \( H \) to \( G \) is an injective function \( \phi : V(H) \rightarrow V(G) \) such that \( \{ u, v \} \in E(H) \iff \{ \phi(u), \phi(v) \} \in E(G) \). Any function from \( V(H) \) to \( V(G) \) can be extended to unordered pairs of vertices of \( H \) as \( \phi(\{ u, v \}) = \{ \phi(u), \phi(v) \} \). A subgraph isomorphism from \( H \) to \( G \) is an injective function \( \phi : V(H) \rightarrow V(G) \) such that \( \{ u, v \} \in E(H) \implies \{ \phi(u), \phi(v) \} \in E(G) \). Two subgraph isomorphisms or induced subgraph isomorphisms are considered different only if the set of edges in the image are different. A graph homomorphism from \( H \) to \( G \) is a function \( \phi : V(H) \rightarrow V(G) \) such that \( \{ u, v \} \in E(H) \implies \{ \phi(u), \phi(v) \} \in E(G) \). Unlike isomorphisms, we consider two distinct
functions that yield the same set of edges in the image as distinct graph homomorphisms. We define \( \phi(S) = \{ \phi(s) : s \in S \} \).

We write \( H \sqsubseteq H' \) (\( H \sqsupseteq H' \)) to specify that \( H \) is a subgraph (supergraph) of \( H' \). The number \( tw(H) \) stands for the treewidth of \( H \). The graph \( K_n \) is the complete graph on \( n \) vertices labelled using \( [n] \). We use the fact that \( \# \text{aut}(H) = 1 \) for almost all graphs in many of our results. In this paper, we will frequently consider graphs where vertices are labelled by tuples. A vertex \((i, p)\) is said to have label \( i \) and colour \( p \). An edge \( \{(i_1, p_1), (i_2, p_2)\} \) has label \( \{i_1, i_2\} \) and colour \( \{p_1, p_2\} \). We will sometimes write this edge as \( \{(i_1, i_2), \{p_1, p_2\}\} \). Note that both \( \{(i_1, p_1), (i_2, p_2)\} \) and \( \{(i_2, p_1), (i_1, p_2)\} \) are written as \( \{(i_1, i_2), \{p_1, p_2\}\} \). But the context should make it clear which edge is being rewritten.

3 A Motivating Example: Induced-\( P_4 \) Isomorphism

In this section, we sketch a one-sided error, randomized \( O(n^2) \) time algorithm for the induced subgraph isomorphism problem for \( P_4 \) to illustrate the techniques used to derive algorithms in this paper.

We start by giving an algorithm for the subgraph isomorphism problem for \( P_4 \). Consider the following polynomial:

\[
N_{P_4,n} = \sum_{(p, q, r, s): p < s} y_p y_q y_r y_s x_{(p, q)} x_{(q, r)} x_{(r, s)}
\]

where the summation is over all quadruples over \( [n] \) where all four elements are distinct. Each monomial in the above polynomial corresponds naturally to a \( P_4 \) in an \( n \)-vertex graph. The condition \( p < s \) ensures that each path has exactly one monomial corresponding to it.

Given an \( n \)-vertex host graph \( G \) and an arithmetic circuit for \( N_{P_4,n} \), we can construct an arithmetic circuit for the polynomial \( N_{P_4,n}(G) \) on the \( y \) variables obtained by substituting \( x_v = 0 \) when \( e \notin E(G) \) and \( x_v = 1 \) when \( e \in E(G) \). The polynomial \( N_{P_4,n}(G) \) can be written as \( \sum_X a_X y_X \) where the summation is over all four vertex subsets \( X \) of \( V(G) \) and \( a_X \) is the number of \( P_4 \)s in the induced subgraph \( G[X] \). Therefore, we can decide whether \( G \) has a subgraph isomorphic to \( P_4 \) by testing whether \( N_{P_4,n}(G) \) is identically 0. Since the degree of this polynomial is a constant \( k \), this can be done in time linear in the size of the arithmetic circuit computing \( N_{P_4,n} \).

However, we do not know how to construct a \( O(n^2) \) size arithmetic circuit for \( N_{P_4,n} \). Instead, we construct a \( O(n^2) \) size arithmetic circuit for the following polynomial called the walk polynomial:

\[
\text{Hom}_{P_4,n} = \sum_{\phi: P_4 \to \phi} \prod_{v \in V(P_4)} z_{v, \phi(v)} y_{\phi(v)} \prod_{e \in E(P_4)} x_{\phi(e)}
\]

This polynomial is also called the homomorphism polynomial for \( P_4 \) because its terms are in one-to-one correspondence with graph homomorphisms from \( P_4 \) to \( K_4 \). As before, we consider the polynomial \( \text{Hom}_{P_4,n}(G) \) obtained by substituting for the \( x \) variables appropriately. The crucial observation is that \( \text{Hom}_{P_4,n}(G) \) contains a multilinear term if and only if \( N_{P_4,n}(G) \) is not identically zero. This is because the multilinear terms of \( \text{Hom}_{P_4,n} \) correspond to injective homomorphisms from \( P_4 \) which in turn correspond to subgraph isomorphisms from \( P_4 \). More specifically, each \( P_4 \) corresponds to two injective homomorphisms from \( P_4 \) since \( P_4 \) has two automorphisms. Therefore, we can test whether \( G \) has a subgraph isomorphic to...
We want to detect whether \( \text{Hom}_{P_4,n}(G) \) has a multilinear term. We can construct a \( O(n^2) \) size arithmetic circuit for the polynomial \( p_4 = \text{Hom}_{P_4,n} \) inductively as follows:

\[
\begin{align*}
    p_{1,v} &= y_v, v \in [n] \\
    p_{i+1,v} &= \sum_{u \in [n]} p_{i,u} y_v x_{\{u,v\}}, v \in [n], i \geq 1 \\
    p_4 &= \sum_{v \in [n]} p_{4,v}
\end{align*}
\]

The above construction works for any \( k \) and not just \( k = 4 \). This method is used by Ryan Williams [13] to obtain an \( O(2^k(n + m)) \) time algorithm for the subgraph isomorphism problem for \( P_k \).

In fact, the above method works for any pattern graph \( H \). Extend the definitions above to define \( N_{H,n} \) and \( \text{Hom}_{H,n} \) in the natural fashion. Then, we can test whether an \( n \)-vertex graph \( G \) has a subgraph isomorphic to \( H \) by testing whether \( N_{H,n}(G) \) is identically zero which in turn can be done by testing whether \( \text{Hom}_{H,n}(G) \) has a multilinear term. Therefore, the complexity of subgraph isomorphism problem for any pattern \( H \) is as easy as constructing the homomorphism polynomial for \( H \). This method is used by Fomin et. al. [11] to obtain efficient algorithms for subgraph isomorphism problems.

We now turn our attention to the induced subgraph isomorphism problem for \( P_4 \). We note that the induced subgraph isomorphism problem for \( P_k \) is much harder than the subgraph isomorphism problem for \( P_k \). The subgraph isomorphism problem for \( P_k \) has a linear time algorithm as seen above but the induced subgraph isomorphism problem for \( P_k \) cannot have \( n^{o(k)} \) time algorithms unless \( \text{FPT} = \text{W}[1] \). We start by considering the polynomial:

\[
I_{P_4,n} = \sum_{(p,q,r,s):p < s} y_p y_q y_r y_s x_{\{p,q\}} x_{\{r,s\}} (1 - x_{\{p,r\}})(1 - x_{\{p,s\}})(1 - x_{\{q,s\}})
\]

The polynomial \( I_{P_4,n}(G) \) can be written as \( \sum_X y_X \) where the summation is over all four vertex subsets of \( V(G) \) that induces a \( P_4 \). Notice that all coefficients are 1 because there can be at most 1 induced-\( P_4 \) on any four vertex subset. By expanding terms of the form \( 1 - x_s \) in the above polynomial, we observe that we can rewrite \( I_{P_4,n} \) as follows:

\[
I_{P_4,n} = N_{P_4,n} - 4N_{C_4,n} - 2N_{K_3+e,n} + 6N_{K_4-e,n} + 12N_{K_4,n}
\]

Since the coefficients in \( I_{P_4,n}(G) \) are all 0 or 1, it is sufficient to check whether \( I_{P_4,n}(G) \) (mod 2) is non-zero to test whether \( I_{P_4,n}(G) \) is non-zero. From the above equation, we can see that \( I_{P_4,n} = N_{P_4,n} \) (mod 2). Therefore, instead of working with \( I_{P_4,n} \) (mod 2), we can work with \( N_{P_4,n} \) (mod 2). We have already seen that we can use \( \text{Hom}_{P_4,n}(G) \) to test whether \( N_{P_4,n}(G) \) is non-zero. However, this is not sufficient to solve induced subgraph isomorphism. We want to detect whether \( N_{P_4,n}(G) \) is non-zero modulo 2. Therefore, the multilinear terms of \( \text{Hom}_{P_4,n}(G) \) has to be in one-to-one correspondence with the terms of \( N_{P_4,n}(G) \). We have to divide the polynomial \( \text{Hom}_{P_4,n}(G) \) by 2 before testing for the existence of multilinear terms modulo 2. However, since we are working over a field of characteristic 2, this division is not possible. We work around this problem by starting with \( \text{Hom}_{P_4,n'} \) for \( n' \) slightly larger than \( n \) and we show that this enables the “division” by 2.

The reader may have observed that instead of the homomorphism polynomial, we could have taken any polynomial \( f \) for which the multilinear terms of \( f(G) \) are in one-to-one
correspondence with $N_{P_4,n}(G)$. This observation leads to the definition of a notion of reduction between polynomials. Informally, $f \preceq g$ if detecting multilinear terms in $f(G)$ is as easy as detecting multilinear terms in $g(G)$. Additionally, for the evaluation $f(G)$ to be well-defined, the polynomial $f$ must have some special structure. We call such polynomials graph pattern polynomials.

On first glance, it appears hard to generalize this algorithm for $P_4$ to sparse pattern graphs on an arbitrary number of vertices (For example, $P_k$) because we have to argue about the coefficients of many $N_*$ polynomials in the expansion. On the other hand, if we consider the pattern graph $K_k$, we have $I_{K_k} = \text{Hom}_{K_k}$. In this paper, we show that for many graph patterns sparser than $K_k$, the induced subgraph isomorphism problem is as easy as constructing arithmetic circuits for homomorphism polynomials for those patterns (or patterns that are only slightly denser).

4 Graph pattern polynomial families

We will consider polynomial families $f = (f_n)$ of the following form: Each $f_n$ will be a polynomial in variables $y_1, \ldots, y_n$, the vertex variables, and variables $x_1, \ldots, x_{\binom{n}{2}}$, the edge variables, and at most linear in $n$ number of additional variables. The degree of each $f_n$ will usually be constant.

The (not necessarily induced) subgraph isomorphism polynomial family $N_H = \{N_{H,n}\}_{n \geq 0}$ for a fixed pattern graph $H$ on $k$ vertices and $\ell$ edges is a family of multilinear polynomials of degree $k + \ell$. The $n^{th}$ polynomial in the family, defined over the vertex set $[n]$, is the polynomial on $n + \binom{n}{2}$ variables given by (1):

$$N_{H,n} = \sum_{\phi: V(H) \subseteq V(K_n)} y_{\phi(V(H))} x_{\phi(E(H))}$$

(1)

When context is clear, we will often omit the subscript $n$ and simply write $N_H$. Given a (host) graph $G$ on $n$ vertices, we can substitute values for the edge variables of $N_{H,n}$ depending on the edges of $G$ ($x_e = 1$ if $e \in E(G)$ and $x_e = 0$ otherwise) to obtain a polynomial $N_{H,n}(G)$ on the vertex variables. The monomials of this polynomial are in one-to-one correspondence with the $H$-subgraphs of $G$. i.e., a term $a_{y_1} \cdots y_v$ where $a$ is a positive integer, indicates that there are $a$ subgraphs isomorphic to $H$ in $G$ on the vertices $v_1, \ldots, v_k$. Therefore, to detect if there is an $H$-subgraph in $G$, we only have to test whether $N_{H,n}(G)$ has a multilinear term.

The induced subgraph isomorphism polynomial family $I_H = \{I_{H,n}\}_{n \geq 0}$ for a pattern graph $H$ over the vertex set $[n]$ is defined in (2).

$$I_{H,n} = \sum_{\phi: V(H) \subseteq V(K_n)} y_{\phi(V(H))} x_{\phi(E(H))} \prod_{e \in E(H)} (1 - x_{\phi(e)})$$

(2)

If we substitute the edge variables of $I_{H,n}$ using a host graph $G$ on $n$ vertices, then the monomials of the resulting polynomial $I_{H,n}(G)$ on the vertex variables are in one-to-one correspondence with the induced $H$-subgraphs of $G$. In particular, all monomials have coefficient 0 or 1 because there can be at most one induced copy of $H$ on a set of $k$ vertices. This implies that to test if there is an induced $H$-subgraph in $G$, we only have to test whether $I_{H,n}(G)$ has a multilinear term and we can even do this modulo $p$ for any prime $p$. Also, note that $I_H$ is simply $I_{P_4}$ where all the edge variables $x_e$ are replaced by $1 - x_e$. 
The homomorphism polynomial family \( \text{Hom}_H = (\text{Hom}_{H,n})_{n \geq 0} \) for pattern graph \( H \) over the vertex set \([n] \) is defined in [3].

\[
\text{Hom}_{H,n} = \sum_{\phi : H \rightarrow K_n} \prod_{v \in V(H)} z_{v,\phi(v)} \prod_{e \in E(H)} x_{\phi(e)}
\]  

(3)

The variables \( z_{a,v} \)'s are called the homomorphism variables. They keep track how the vertices of \( H \) are mapped by the different homomorphisms in the summation. We note that the size of the arithmetic circuit computing \( \text{Hom}_{H,n} \) is independent of the labelling chosen to define the homomorphism polynomial.

The induced subgraph isomorphism polynomial for any graph \( H \) and subgraph isomorphism polynomials for supergraphs of \( H \) are related as follows:

\[
I_{H,n} = \sum_{H' \supseteq H} (-1)^{e(H') - e(H)} \#_{\text{sub}}(H, H') N_{H',n}
\]  

(4)

Here \( e(H) \) is the number of edges in \( H \) and \( \#_{\text{sub}}(H, H') \) is the number of times \( H \) appears as a subgraph in \( H' \). The sum is taken over all supergraphs \( H' \) of \( H \) having the same vertex set as \( H \). Equation [4] is used by Curticapean, Dell, and Marx [5] in the context of counting subgraph isomorphisms.

**Example 4.1.** Let \( P_3 \) be the path on 3 vertices and let \( K_3 \) be the triangle.

\[
N_{P_3,3} = y_1 y_2 y_3 (x_{1,2} x_{2,3} + x_{1,3} x_{2,3} + x_{1,2} x_{1,3})
\]

\[
I_{P_3,3} = y_1 y_2 y_3 (x_{1,2} x_{2,3} (1 - x_{1,3})
+ x_{1,3} x_{2,3} (1 - x_{1,2})
+ x_{1,2} x_{1,3} (1 - x_{2,3}))
\]

\[
= N_{P_3,3} - 3 N_{K_3,3}
\]

For any fixed pattern graph \( H \), the degree of polynomial families \( N_H, I_H, \) and \( \text{Hom}_H \) are bounded by a constant depending only on the size of \( H \). Such polynomial families are called constant-degree polynomial families.

**Definition 4.2.** A constant-degree polynomial family \( f = (f_n) \) is called a graph pattern polynomial family if the \( n^{th} \) polynomial in the family has \( n \) vertex variables, \( \binom{n}{2} \) edge variables, and at most \( cn \) other variables, where \( c \) is a constant, and every non-multilinear term of \( f_n \) has at least one non-edge variable of degree greater than 1.

It is easy to verify that \( I_H, N_H, \) and \( \text{Hom}_H \) are all graph pattern polynomial families. For a graph pattern polynomial \( f \), we denote by \( f(G) \) the polynomial obtained by substituting \( x_e = 0 \) if \( e \not\in E(G) \) and \( x_e = 1 \) if \( e \in E(G) \) for all edge variables \( x_e \). Note that for any graph pattern polynomial \( f \), we have \( \text{ML}(f(G)) = \text{ML}(f)(G) \). This is because any non-multilinear term in \( f \) has to remain non-multilinear or become 0 after this substitution.

**Definition 4.3.** 1. A constant degree polynomial family \( f = (f_n) \) has circuits of size \( s(n) \) if there is a sequence of arithmetic circuits \( (C_n) \) such that \( C_n \) computes \( f_n \) and has size at most \( s(n) \).
2. $f$ has uniform $s(n)$-size circuits, if on input $n$, we can construct $C_n$ in time $O(s(n))$ on a Word-RAM.

We now define a notion of reducibility among graph pattern polynomials.

**Definition 4.4.** A substitution family $\sigma = (\sigma_n)$ is a family of mappings

$$\sigma_n : \{y_1, \ldots, y_m, x_1, \ldots, x_{(2)}^n, u_1, \ldots, u_{m(n)}\} \rightarrow K[y_1, \ldots, y_m, x_1, \ldots, x_{(2)}^n, v_1, \ldots, v_{r(n)}]$$

mapping variables to polynomials such that:

1. $\sigma$ maps vertex variables to constant-degree monomials containing one or more vertex variables or other variables, and no edge variables.
2. $\sigma$ maps edge variables to polynomials with constant-size circuits containing at most one edge variable and no vertex variables.
3. $\sigma$ maps other variables to constant-degree monomials containing no vertex or edge variables and at least one other variable.

$\sigma_n$ naturally extends to $K[y_1, \ldots, y_m, x_1, \ldots, x_{(2)}^n, u_1, \ldots, u_{m(n)}]$.

**Definition 4.5.** A substitution family $\sigma = (\sigma_n)$ is constant-time computable if given $n$ and a variable $z$ in the domain of $\sigma_n$, we can compute $\sigma_n(z)$ in constant-time on a Word-RAM. (Note that an encoding of any $z$ fits into on cell of memory.)

**Definition 4.6.** Let $f = (f_n)$ and $g = (g_n)$ be graph pattern polynomial families. Then $f$ is reducible to $g$, denoted $f \preceq g$, via a constant time computable substitution family $\sigma = (\sigma_n)$ if for all $n$ there is an $m = O(n)$ and $q = O(1)$ such that

1. $\sigma_m(ML(g_m))$ is a graph pattern polynomial and
2. $ML(\sigma_m(g_m)) = v_{[q]}ML(f_n)$. (Recall that $v_{[q]} = v_1 \cdots v_q$.)

For any prime $p$, we say that $f \preceq g \pmod{p}$ if there exists an $f' = f \pmod{p}$ such that $f' \preceq g$.

Property 1 of Definition 4.6 and Properties 1 and 3 of Definition 4.4 imply that $\sigma_m(g_m)$ is a graph pattern polynomial because Properties 1 and 3 of Definition 4.4 ensure that non-multilinear terms remain so after the substitution. It is easy to see that $\preceq$ is reflexive via the identity substitution. We can also assume w.l.o.g. that the variables $v_1, \ldots, v_q$ are fresh variables introduced by the substitution family $\sigma$.

What is the difference between $\sigma_m(ML(g_m))$ and $ML(\sigma_m(g_m))$ in the Definition 4.6? Every monomial in $ML(\sigma_m(g_m))$ also appears in $\sigma_m(ML(g_m))$, however, the latter may contain further monomials that are not multilinear.

**Proposition 4.7.** $\preceq$ is transitive.

**Proof.** Let $f \preceq g$ via $\sigma$ and $g \preceq h$ via $\tau$. Assume that $f_n$ is written as a substitution instance of $g_n$ by $\sigma$ and $g_m$ is written as a substitution instance of $h_m$ by $\tau$ for some linearly bounded functions $m$ and $r$. Let $\sigma_m(g_m(n))$ and $\tau_{r(m)}(h_m(n))$ have $u_1, \ldots, u_p$ and $v_1, \ldots, v_q$, respectively, as other variables that are multiplied with the multilinear terms. We can assume w.l.o.g. that these two sets of other variables are disjoint.

Define $\sigma'$ as $\sigma$ extended to $v_i$ by $\sigma'_n(v_i) = v_i$ for all $i$ and $n \in \mathbb{N}$. We claim that $f \preceq h$ via the family $(\sigma'_m(n))\circ \tau_{r(m)}(n))$. We need to verify the two properties of Definition 4.6.

---

Footnote: Since we are dealing with fine-grained complexity, we have to be precise with the encoding of the circuit. We assume an encoding such that evaluating the circuit is linear time and substituting for variables with polynomials represented by circuits is constant-time.
Property 1: \( \sigma'_{m(n)}(\tau_{r(m(n))}(ML(h_{r(m(n))}))) = \sigma'_{m(n)}(v_{[q]}ML(g_{m(n)}) + h') \) where \( h' \) is a graph pattern polynomial containing only non-multilinear terms. Now, we have \( h'' = \sigma''_{m(n)}(v_{[q]}ML(g_{m(n)}) = v_{[q]}\sigma_{m(n)}(ML(g_{m(n)})) \) because \( ML(g_{m(n)}) \) cannot contain \( v_i \) and \( \sigma'_{m(n)}(v_i) = v_i \) for \( i \in [q] \). This implies that \( h'' \) is a graph pattern polynomial because \( \sigma_{m(n)}(ML(g_{m(n)})) \) is a graph pattern polynomial. Also, \( \sigma'_{m(n)}(h') \) is a graph pattern polynomial containing only non-multilinear terms by Properties 1 and 3 of Definition 4.4 proving that \( (\sigma'_{m(n)} \circ \tau_{r(m(n))})(ML(h_{r(m(n))})) \) is a graph pattern polynomial.

Property 2 is proved as follows:

\[
ML((\sigma'_{m(n)} \circ \tau_{r(m(n))})(h_{r(m(n))})) = ML(\sigma'_{m(n)}(\tau_{r(m(n))}(h_{r(m(n))}))) = ML(\sigma'_{m(n)}(v_{[q]}ML(g_{m(n)}) + h')) = ML(v_{[q]}\sigma_{m(n)}(ML(g_{m(n)}))) = v_{[q]}ML(\sigma_{m(n)}(ML(g_{m(n)}))) = v_{[q]}u_{[p]}ML(f_n)
\]

Note that the term \( h' \) vanishes, since \( \sigma_{m(n)} \) does not introduce new multilinear monomials and also \( ML(.) \) is a linear operator. The same happens in the second-last line, we did not write the additional term in the equation, since it vanishes anyway.

We also have \( r(m(n)) = O(n) \) and \( p + q = O(1) \). It is easy to verify that \( (\sigma'_{m(n)} \circ \tau_{r(m(n))}) \) is a constant-time computable substitution family.

Efficient algorithms are known for detecting multilinear terms of small degree with non-zero coefficient modulo primes. We state two such theorems that we use in this paper.

**Theorem 4.8.** Let \( k \) be any constant and let \( p \) be any prime. Given an arithmetic circuit of size \( s \), there is a randomized, one-sided error \( O(s) \)-time algorithm to detect whether the polynomial computed by the circuit has a multilinear term of degree almost \( k \) with non-zero modulo \( p \) coefficient.

**Theorem 4.9.** Let \( k \) be any constant. Given an arithmetic circuit of size \( s \) computing a polynomial of degree \( k \) on \( n \) variables, there is a deterministic \( O(s + n^{k/2}) \)-time algorithm to compute the parity of the sum of coefficient of multilinear terms.

An important algorithmic consequence of reducibility is stated in Proposition 4.10.

**Proposition 4.10.** Let \( p \) be any prime. Let \( f \) and \( g \) be graph pattern polynomial families. Let \( s(n) \) be a polynomially-bounded function. If \( f \leq g \) and \( g \) has size uniform \( s(n) \)-size arithmetic circuits, then we can test whether \( f_n(G) \) has a multilinear term with non-zero coefficient modulo \( p \) in \( O(s(n)) \) (randomized one-sided error) time for any \( n \)-vertex graph \( G \).

**Proof.** Assume that \( f_n \) is reducible to \( g_m \) where \( m = O(n) \). Since \( s(n) \) is polynomially bounded, we have \( size(g_m) = O(s(n)) \). Apply the substitution \( \sigma_m \) to \( g_m \) to obtain \( g' \). Let \( u_1, \ldots, u_r \) be the other variables of \( g' \). We claim that testing whether the polynomial \( g'(G) \) has a multilinear term is equivalent to testing whether \( f_n(G) \) has a multilinear term. We have \( u_{[r]}ML(f_n) = ML(g') \). Since both \( f_n \) and \( g' \) are graph pattern polynomials, we have \( u_{[r]}ML(f_n(G)) = u_{[r]}ML(g'(G)) = ML(g'(G)) \). Therefore, testing whether the polynomial \( f_n(G) \) has a multilinear term of degree at most \( k \), where \( k \) is some constant, reduces to testing whether \( g'(G) \) has a multilinear term of degree \( k + r = O(1) \). Since \( g' \) has \( O(s(n)) \) size circuits, this can be done in \( O(s(n)) \) (randomized one-sided error) time. ▶
On the other hand, if we only have \( f \equiv g \pmod{p} \) for some specific prime \( p \), then it is only possible to test for multilinear terms in \( f \) that have non-zero coefficients modulo \( p \) for that prime \( p \).

**Corollary 4.11.** Let \( f \equiv g \pmod{p} \) and \( g \) has \( s(n) \) size circuits where \( s(n) \) is polynomially bounded. Then we can test whether \( f_n(G) \) has a multilinear term with non-zero coefficient modulo \( p \) in \( O(s(n)) \) time for any \( n \)-vertex graph \( G \).

More relaxed notions of reduction allowing an increase of \( \text{polylog}(n) \) factors in size or allowing multilinear terms to be multiplied by arbitrary sets of other variables could also be useful to obtain better algorithms. We do not pursue this because we could not find any reductions that make use of this freedom.

The following result allows efficient construction of \( \text{Hom}_H \) when \( H \) has small treewidth.

**Theorem 4.12.** (Implicit in [3], Also used in [11] and [7]) \( \text{Hom}_H \) can be computed by \( O(n^{tw(H)+1}) \) size arithmetic circuits for all graphs \( H \).

## 5 Pattern graphs easier than cliques

In this section, we describe a family \( H_{3k} \) of pattern graphs such that the induced subgraph isomorphism problem for \( H_{3k} \) has an \( O(n^{\omega(k,k-1,k)}) \) time algorithm when \( k = 2^\ell, \ell \geq 1 \). Note that for the currently known best algorithms for fast matrix multiplication, we have \( \omega(k,k-1,k) < k\omega \). Therefore, these pattern graphs are strictly easier to detect than cliques.

The pattern graph \( H_{3k} \) is defined on \( 3k \) vertices and we consider the canonical labelling of \( H_{3k} \) where there is a \((3k-1)\)-clique on vertices \( \{1, \ldots, 3k-1\} \) and the vertex \( 3k \) is adjacent to the vertices \( \{1, \ldots, 2k-1\} \).

**Lemma 5.1.** \( I_{H_{3k}} = N_{H_{3k}} \pmod{2} \) when \( k = 2^\ell, \ell \geq 1 \)

**Proof.** We show that the number of times \( H_{3k} \) is contained in any of its proper supergraphs is even if \( k \) is a power of 2. The graph \( K_{3k} \) contains \( 3k\binom{2k-1}{2k-1} \) copies of \( H_{3k} \). This number is even when \( k \) is even. The graph \( K_{3k} - e \) contains \( 2\binom{2k-2}{2k-1} \) copies of \( H_{3k} \). This number is always even. The remaining proper supergraphs of \( H_{3k} \) are the graphs \( K_{3k-1} + (2k+i)e \), i.e., a \((3k-1)\)-clique with \( 2k+i \) edges to a single vertex, for \( 0 \leq i < k-2 \). There are \( m_i = \binom{2k+i}{2k-1} \) copies of the graph \( H_{3k} \) in these supergraphs. We observe that the numbers \( m_i \) are even when \( k = 2^\ell, \ell \geq 1 \) by Lucas’ theorem. Lucas’ theorem states that \( \binom{p}{q} \) is even if and only if in the binary representation of \( p \) and \( q \), there exists some bit position \( i \) such that \( q_i = 1 \) and \( p_i = 0 \). To see why \( m_i \) is even, observe that in the binary representation of \( 2k-1 \), all bits 0 through \( \ell \) are 1 and in the binary representation of \( 2k+i \), \( 0 \leq i < k-2 \), at least one of those bits is 0. 

**Lemma 5.2.** \( N_{H_{3k}} \preceq \text{Hom}_{H_{3k}} \)

**Proof.** We start with \( \text{Hom}_{H_{3k}} \) over the vertex set \([n] \times [3k]\) and apply the following substitution.
\[
\sigma(z_{a,(v,a)}) = z_a \tag{1}
\]
\[
\sigma(z_{a,(v,b)}) = z_a^b, a \neq b \tag{2}
\]
\[
\sigma(y_{(v,a)}) = y_v \tag{3}
\]
\[
\sigma(x_{(a,a),(v,b)}) = 0, \text{if } a, b \in \{1, \ldots, 2k-1\} \text{ and } a < b \text{ and } u > v \tag{4}
\]
\[
\sigma(x_{(a,a),(v,b)}) = 0, \text{if } a, b \in \{2k, \ldots, 3k-1\} \text{ and } a < b \text{ and } u > v \tag{5}
\]
\[
\sigma(x_{(a,a),(v,b)}) = x_{\{u,v\}}, \text{otherwise} \tag{6}
\]

Rule 3 ensures that in any surviving monomial, all vertices have distinct labels. Rule 4 ensures that the vertices coloured 1, \ldots, 2k – 1 are in increasing order and Rule 5 ensures that the vertices coloured 2k, \ldots, 3k – 1 are in increasing order.

Consider an \( H_{3k} \) labelled using \([n] \) where the vertices in the \((3k – 1)\)-clique are labelled \( v_1, \ldots, v_{3k-1} \) and the remaining vertex is labelled \( v_{3k} \) which is connected to \( v_1 < \ldots < v_{2k-1} \). Also, \( v_{2k} < \ldots < v_{3k-1} \). We claim that the monomial corresponding to this labelled \( H_{3k} \) (say \( m \)) is uniquely generated by the monomial \( m' = \prod_{1 \leq i < 3k} z_i(v_{(i)})^w \) in \( \text{Hom}_{H_{3k}} \). Note that the vertices and edges in the image of the homomorphism is determined by the map \( i \mapsto (v_1, i) \). The monomial \( w \) is simply the product of these vertex and edge variables. It is easy to see that this monomial yields the required monomial under the above substitution. The uniqueness is proved as follows: observe that in any monomial \( m'' \) in \( \text{Hom}_{H_{3k}} \) that generates \( m \), the vertex coloured \( 3k \) must be \( v_{3k} \). This implies that the vertices coloured \( 1, \ldots, 2k – 1 \) must be the set \( \{v_1, \ldots, v_{2k-1}\} \). Rule 4 ensures that vertex coloured \( i \) must be \( v_i \) for \( 1 \leq i \leq 2k – 1 \). Similarly, the vertices coloured \( 2k, \ldots, 3k – 1 \) must be the set \( \{v_{2k}, \ldots, v_{3k-1}\} \) and Rule 5 ensures that vertex coloured \( i \) must be \( v_i \) for \( 2k \leq i \leq 3k – 1 \) as well. But then the monomials \( m' \) and \( m'' \) are the same.

\[\blacktriangleright\textbf{Lemma 5.3.} \text{Hom}_{H_{3k}} \text{ can be computed by arithmetic circuits of size } O(n^{o(k,k-1,k)}) \text{ for } k > 1.\]

\[\blacktriangleright\textbf{Proof.} \text{Consider } H_{3k} \text{ labelled as before. We define the sets } S_{1,k,2k,3k-1} = \{1, \ldots, k, 2k, \ldots, 3k-1\}, \text{ } S_{k+1,3k-1} = \{k+1, \ldots, 3k-1\}, \text{ } S_{k+1,2k-1} = \{k+1, \ldots, 2k-1\}, \text{ and } S_{1,2k-1} = \{1, \ldots, 2k-1\}. \text{ We also define the tuples } V_{1,k} = (v_1, \ldots, v_k), V_{2k,3k-1} = (v_{2k}, \ldots, v_{3k-1}), \text{ and } V_{k+1,2k-1} = (v_{k+1}, \ldots, v_{2k-1}) \text{ for any set } v_i \text{ of } 3k – 1 \text{ distinct vertex labels. The algorithm also uses the matrices defined below. The dimensions of each matrix are specified as the superscript. All other entries of the matrix are 0.} \]

Graph Pattern Polynomials

In this section, we will prove that the time complexity of the induced subgraph isomorphism problem for \(K_{3k}\) is upper bounded by the circuit complexity of \(K_{3k}\). Let \(U_i, V_i, k, \ell\) be defined similarly to the tuples \(U, V, k, \ell\) in the previous section. Consider a homomorphism of \(H_{3k}\) defined as \(\phi : i \mapsto u_i\). The monomial corresponding to this homomorphism is uniquely generated as follows. Let \(U_i\) be defined similarly to the tuples \(V_i, k, \ell\) in the summation and consider the monomial generated by the product \(A_{U_i, k, V_i, k, \ell, 3k-1}\) after replacing the variable \(x_{(v_i, u_i)}\) by \((DE)_{U_i, k, V_i, k, \ell, 3k-1}\) and taking the monomial \(A_{U_i, k, V_i, k, \ell, 3k-1}\) from that entry. It is easy to verify that this generates the required monomial. For uniqueness, observe that this is the only way to generate the required product of the homomorphism variables.

Computing \(ABC\) can be done using \(O(n^\omega(k, k-1))\) size circuits. Computing \(DE\) can be done using \(O(n^\omega(k, 1))\) size circuits. The top level sum contributes \(O(n^k)\) gates. This proves the lemma.

We conclude this section by stating our main theorem.

**Theorem 5.4.** The induced subgraph isomorphism problem for \(H_{3k}\) has an \(O(n^\omega(k, k-1))\) time algorithm when \(k = 2^\ell, \ell \geq 1\).

## 6 Algorithms for induced paths and cycles

In this section, we will prove that the time complexity of the induced subgraph isomorphism problems for paths and cycles are upper bounded by the circuit complexities of the homomorphism polynomials for \(P_k\) and \(K_k - P_{k-1}\) respectively. Using this we derive efficient algorithms for induced subgraph isomorphism problem for \(P_k\) for \(k \in \{5, 6, 7, 8, 9\}\) and \(C_k\) for \(k \in \{5, 7, 9\}\). We also obtain efficient combinatorial algorithms for the induced subgraph isomorphism problem for \(P_k\) for all \(k\) and \(C_k\) when \(k\) is odd.

The proof has two main steps: First, we show that the induced subgraph isomorphism polynomials for these patterns are reducible to the aforementioned homomorphism polynomials (Lemmas 6.1 and 6.2). Then, we prove that these homomorphism polynomials can be computed efficiently (Theorems 6.4 and 6.8).
We describe how to compute $\text{Hom}_{\overline{P_k}}$ using arithmetic circuits of size $O(n^\omega)$. We start by defining the following matrices.

1. An $O(n^{\omega})$-time algorithm for induced subgraph isomorphism problem for $P_5$ in $n$-vertex graphs.
2. An $O(n^{\omega(2.1.1)})$-time algorithm for induced subgraph isomorphism problem for $P_6$ in $n$-vertex graphs.
3. An $O(n^{k-2})$-time combinatorial algorithm for induced subgraph isomorphism problem for $P_k$ in $n$-vertex graphs.
4. An $O(n^{k-2})$-time deterministic combinatorial algorithm for computing the parity of the number of induced subgraphs isomorphic to $P_k$ in $n$-vertex graphs.

Proof. We will prove that for any proper super-graph $H$ of $\overline{P_k}$, the number $\#\text{sub}(\overline{P_k}, H)$ is even. Observe that this number is the same as the number of ways to extend a proper labelled subgraph of $P_k$ to some labelled $P_k$. Let $H$ be an arbitrary proper subgraph of $P_k$. Let $2 \leq \ell \leq k$ be the number of connected components in $H$ out of which $0 \leq s \leq \ell$ of them consists only of a single vertex. Then the number of ways to extend $H$ to a $P_k$ is $\ell!2^{s-\ell}/2$. We can extend $H$ to a $P_k$ by ordering the connected components from left to right and then connecting the endpoints from left to right. There are $\ell!$ ways to order $\ell$ components and $2$ ways to place all components with more than one vertex. Out of these, a configuration and its reverse will lead to the same labelled $P_k$. Since $\ell \geq 2$, this number is even if $\ell > s$. Otherwise, this number is $k!/2$ because $\ell = s$ implies that there are $k$ components. This is even when $k \geq 4$. We conclude that $I_{\overline{P_k}} = N_{\overline{P_k}} \pmod 2$.

Proof. Let $f = N_{\overline{P_k}}$ and $g = \text{Hom}_{\overline{P_k}}$. We fix the labelling of $\overline{P_k}$ where the vertices of the complementary $P_k$ are labelled $1, 2, \ldots, k$ with $1$ and $k$ as the endpoints and for every other vertex $i$, the neighbours are $i - 1$ and $i + 1$. Start with $g$ over the vertex set $[n] \times [k]$ and use the following substitution.

$$
\sigma(z_{a,(v,a)}) = z_a \quad (1)
$$

$$
\sigma(z_{a,(v,b)}) = z_a^2, \text{ if } a \neq b \quad (2)
$$

$$
\sigma(y_{(v,a)}) = y_v \quad (3)
$$

$$
\sigma(x_{(u,p),(v,q)}) = 0, \text{ if } \{p,q\} \notin E(\overline{P_k}) \text{ or if } p = 1 \text{ and } q = k \text{ and } u > v \quad (4)
$$

$$
\sigma(x_{(u,p),(v,q)}) = x_{(u,v)}, \text{ otherwise} \quad (5)
$$

The resulting polynomial $g'$ satisfies $ML(g') = z_1 \cdots z_k ML(f_n)$ as required. The reduction works because there is exactly one non-trivial automorphism for $\overline{P_k}$ and that automorphism maps $1$ to $k$. The monomial corresponding to one of these automorphisms become $0$ because of $u > v$ where $u$ has colour $1$ and $v$ has colour $k$.

Theorem 6.3. If $\text{Hom}_{\overline{P_k}}$ can be computed by circuits of size $n^{f(k)}$, then there is an $O(n^{f(k)})$ time algorithm for the induced subgraph isomorphism problem for $P_k$ on $n$-vertex graphs.

Theorem 6.4. The following algorithms exist

1. An $O(n^{\omega})$-time algorithm for induced subgraph isomorphism problem for $P_5$ in $n$-vertex graphs.
2. An $O(n^{\omega(2.1.1)})$-time algorithm for induced subgraph isomorphism problem for $P_6$ in $n$-vertex graphs.
3. An $O(n^{k-2})$-time combinatorial algorithm for induced subgraph isomorphism problem for $P_k$ in $n$-vertex graphs.
4. An $O(n^{k-2})$-time deterministic combinatorial algorithm for computing the parity of the number of induced subgraphs isomorphic to $P_k$ in $n$-vertex graphs.

Proof. 1. We describe how to compute $\text{Hom}_{\overline{P_k}}$ using arithmetic circuits of size $O(n^\omega)$. We start by defining the following matrices.
We show how to compute $A_{i,j}^n = x_{(i,j),i \neq j}$

$B_{i,i}^n = y_i z_{3,i}$

$C_{i,i}^n = y_i z_{4,i}$

$D_{i,i}^n = y_i z_{5,i}$

Consider the labelled $P_5$ in Figure 1. Then we can write

$\text{Hom}_{P_5} = \sum_{i,j \in [n], i \neq j} z_{1,i} z_{2,j} x_{(i,j)} y_i y_j (ABA)_{i,j} (ACADA)_{i,j}$

Clearly, this can be implemented using $O(n^\omega)$ size circuits. We will now prove that this circuit correctly computes the polynomial $\text{Hom}_{P_5}$. Consider a homomorphism $\phi : j \mapsto i_j$. Consider the monomial generated by $i = i_1, j = i_2$ in the outer sum, the monomial $A_{i_1,i_3} B_{i_2,i_3} A_{i_3,i_2}$ in the product $(ABA)_{i_1,i_2}$, and the monomial $A_{i_1,i_4} C_{i_4,i_4} A_{i_4,i_5} D_{i_5,i_5} A_{i_5,i_2}$ in the product $(ACADA)_{i_1,i_2}$. This monomial corresponds to the homomorphism $\phi$ and one can observe that this is the only way to generate this monomial.

On the other hand, any monomial in the computed polynomial is generated as described above and therefore corresponds to a homomorphism.

2. We show how to compute $\text{Hom}_{P_5}$ using arithmetic circuits of size $O(n^{\omega(2,1,1)})$. We define the following matrices.

$A_{i,j,k}^{n \times 2} = z_2 x_{(2,i)} y_i x_{(2,j)} x_{(2,k)}$

$B_{j,k}^{n \times 2} = z_5 x_{(5,j)} y_j x_{(5,k)}$

$C_{i,k}^{n \times 2} = x_{(i,k)}$

$D_{j,k}^{n \times 2, p} = y_p z_4 x_{(4,j)} x_{(4,k)}$

$E_{p,k}^{n \times 2, j} = x_{(p,k)}$

$F_{q,i}^{n \times 2, k} = y_q z_4 x_{(4,q)} x_{(4,k)}$

$G_{i,k}^{n \times 2} = x_{(i,k)}$

Compute the matrix products $ABC, DE, \text{and } FG$. The output of the circuit is $\sum_i (ABC)_{i,i}^n$ after substituting for the variables as follows. Replace each $x_{((1,j),(6,k)),(5,\ell)}$ with $D E_{(j,k),\ell}$ and each $x_{((1,j),(6,k)),(2,i)}$ with $F G_{(j,k),i}$. Replace each $x_{((1,j),(6,k)),(3,p)}$ with $x_{(j,p)} x_{(k,p)}$ and each $x_{((1,j),(6,k)),(4,q)}$ with $x_{(j,q)} x_{(k,q)}$.

Consider the labelling of $P_6$ in Figure 2. After substituting for all variables as mentioned above, the monomials of $(ABC)_{i,i}$ correspond to homomorphisms from this labelled $P_6$ to $K_n$ that maps vertex 2 to $i$. Therefore, the circuit correctly computes $\text{Hom}_{P_6}$. 

\begin{figure}[h]
\centering
\begin{tikzpicture}
\draw (0,0) -- (1,1);
\draw (1,1) -- (2,0);
\draw (2,0) -- (3,1);
\draw (3,1) -- (4,0);
\draw (4,0) -- (5,1);
\end{tikzpicture}
\caption{A labelled $P_5$}
\end{figure}
We observe that \( \text{tw}(\overline{P_k}) = k - 3 \) and therefore using Theorem 4.12 we can compute \( \text{Hom}_{\overline{P_k}} \) using \( O(n^{k-2}) \) size circuits.

4. Consider the substitution in the proof of Lemma 6.2 and replace rules (1) and (2) by the following rules.

\[
\sigma(z_{a,(v,a)}) = 1 \quad (1')
\]
\[
\sigma(z_{a,(v,b)}) = 0 \quad (2')
\]

The multilinear part of the resulting polynomial \( f \) is the same as \( N_{\overline{P_k}} \) and hence has degree-\( k \). Therefore, we only have to compute the parity of the sum of coefficients of the multilinear terms of \( f(G) \). By Theorem 4.9 this can be done in \( O(n^{k-2}) \) time.

We remark that by computing homomorphism polynomials for \( \overline{P_k} \) for \( k = 7, 8, 9 \) using small-size circuits, we can obtain the following algorithms for the induced subgraph isomorphism problem for paths: An \( O(n^2\omega) \) time algorithm for \( P_7 \), an \( O(n^\omega(3,3,2)) \) time algorithm for \( P_8 \), and an \( O(n^\omega(3,3,2)) \) time algorithm for \( P_9 \). All these algorithms are faster than the corresponding algorithms for \( k \)-cliques.

Lemma 6.5. \( I_{C_k} = N_{C_k} + N_{\overline{P_k}} + N_{K_k-P_{k-1}} \pmod{2} \) for odd \( k \geq 5 \).

Proof. We claim that the only proper supergraphs of \( C_k \) containing it an odd number of times are \( \overline{P_k} \) and \( K_k - P_{k-1} \). There is exactly one way to extend a \( P_k \) or a \( P_{k-1} + v \) to a \( C_k \). Let \( H \) be a proper subgraph of \( C_k \) other than these two graphs. Assume that \( H \) has \( 2 \leq \ell \leq k \) connected components out of which \( 0 \leq s \leq \ell \) are single vertices. Then there are \( m = \ell!2^{\ell-s}/2\ell \) ways to extend \( H \) to \( C_k \). If \( \ell > s \), then \( m \) is even because \( (\ell - 1)! \) is even when \( \ell \geq 3 \) and when \( \ell = 2 \) the number \( s \) is 0 and \( m = 2 \). If \( \ell = 2 \) and \( s = 1 \), then \( H = P_{k-1} + v \). If \( \ell = s \), then \( m = \ell!/2\ell = (\ell - 1)!/2 \). But \( \ell = s \) implies that \( \ell = k \) and therefore \( m = (k - 1)!/2 \) which is even when \( k \geq 5 \).

Lemma 6.6. 1. \( N_{C_k} \leq \text{Hom}_{K_k-P_{k-1}} \pmod{2} \) for odd \( k \geq 5 \).
2. \( N_{\overline{P_k}} \leq \text{Hom}_{K_k-P_{k-1}} \) for \( k \geq 5 \).
3. \( N_{K_k-P_{k-1}} \leq \text{Hom}_{K_k-P_{k-1}} \) for \( k \geq 5 \).
4. \( I_{C_k} \leq \text{Hom}_{K_k-P_{k-1}} \pmod{2} \) for odd \( k \geq 5 \).

Proof. We start with \( \text{Hom}_{K_k-P_{k-1}} \) over the vertex set \([n] \times [k]\) in all cases and apply the following substitutions.
1. Fix the labelling of $\overline{C_k}$ where the complementary $C_k$ is labelled $1, \ldots, k$ such that the vertex 1 has neighbours 2 and $k$ and $k$ has neighbours 1 and $k - 1$ and every other vertex $i$ has $i + 1$ and $i - 1$ as its neighbours. The crucial observation is that $\overline{C_k}$ has $2k$ automorphisms and if we only select automorphisms where the label of the vertex coloured 1 is strictly less than the label of the vertex coloured 3, then we select exactly $k$ automorphisms. This allows us to compute a polynomial family $h$ such that $k.N_{\overline{C_k}} \leq h$ and $k.N_{\overline{C_k}} = N_{\overline{C_k}}$ (mod 2).

$$\sigma_1(z_{a,(v,a)}) = z_a$$
$$\sigma_1(z_{a,(v,b)}) = z_a^2, \text{ if } a \neq b$$
$$\sigma_1(y_{(v,a)}) = y_v$$
$$\sigma_1(x_{((u,p),(v,q))}) = 0, \text{ if } p = 1 \text{ and } q = 3 \text{ and } u > v$$
$$\sigma_1(x_{((u,p),(v,q))}) = 1, \text{ if } p = 1 \text{ and } q = 2 \text{ or } p = 1 \text{ and } q = k$$
$$\sigma_1(x_{((u,p),(v,q))}) = x_{(u,v)}, \text{ otherwise}$$

2. Fix the labelling of $\overline{P_k}$ where the complementary $P_k$ is $1 \cdot 2 \cdots k$.

$$\sigma_2(z_{a,(v,a)}) = z_a$$
$$\sigma_2(z_{a,(v,b)}) = z_a^2, \text{ if } a \neq b$$
$$\sigma_2(y_{(v,a)}) = y_v$$
$$\sigma_2(x_{((u,p),(v,q))}) = 0, \text{ if } p = 1 \text{ and } q = k \text{ and } u > v$$
$$\sigma_2(x_{((u,p),(v,q))}) = 1, \text{ if } p = 1 \text{ and } q = 2$$
$$\sigma_2(x_{((u,p),(v,q))}) = x_{(u,v)}, \text{ otherwise}$$

3. Fix the labelling of $K_k - P_{k-1}$ where the complementary $P_{k-1} + v$ is $1 \cdot 2 \cdot 3 \cdots k$.

$$\sigma_3(z_{a,(v,a)}) = z_a$$
$$\sigma_3(z_{a,(v,b)}) = z_a^2, \text{ if } a \neq b$$
$$\sigma_3(y_{(v,a)}) = y_v$$
$$\sigma_3(x_{((u,p),(v,q))}) = 0, \text{ if } p = 2 \text{ and } q = k \text{ and } u > v$$
$$\sigma_3(x_{((u,p),(v,q))}) = x_{(u,v)}, \text{ otherwise}$$

4. We prove that $k.N_{\overline{C_k}} + N_{\overline{P_k}} + N_{K_k - P_{k-1}} \leq \text{Hom}_{K_k - P_{k-1}}$. Start with $\text{Hom}_{K_k - P_{k-1}}$ over the vertex set $[n] \times [k] \times [3]$ and apply the following substitution.

$$\sigma(z_{a,(v,b,i)}) = \sigma_1(z_{a,(v,b)})$$
$$\sigma(y_{(v,a,i)}) = \sigma_1(y_{(v,a)})$$
$$\sigma(x_{((u,p,i),(v,q,j))}) = 0, \text{ if } i \neq j$$
$$\sigma(x_{((u,p,i),(v,a,i))}) = \sigma_1(x_{((u,p),(v,q))}), \text{ otherwise}$$

Rule 3 ensures that only the monomials where every vertex is indexed by the same element in [3] survive. The other rules ensure that any monomial $m$ indexed by $i \in [3]$ are mapped to $\sigma_i(m')$, where $m'$ is the same as $m$ but with $i$ removed.
The proof of correctness of these reductions is the same as the argument in Theorem 8.3. In addition, the condition that $u > v$ when $u$ is coloured 1 and $v$ is coloured $k$ rules out one out of two automorphisms for $P_k$ in part 2 and the condition that $u > v$ when $u$ is coloured 2 and $v$ is coloured $k$ rules out one out of two automorphisms for $K_k - P_{k-1}$ in part 3.

Theorem 6.7. If $\text{Hom}_{K_k - P_{k-1}}$ can be computed by circuits of size $n^{f(k)}$, then there is an $O(n^{f(k)})$ time algorithm for induced subgraph isomorphism problem for $C_k$ on $n$-vertex graphs for odd $k \geq 5$.

Theorem 6.8. The following algorithms exist

1. An $O(n^\omega)$-time algorithm for induced subgraph isomorphism problem for $C_5$ in $n$-vertex graphs.
2. An $O(n^{k-2})$-time combinatorial algorithm for induced subgraph isomorphism problem for $C_k$ in $n$-vertex graphs, where $k \geq 5$ is odd.
3. An $O(n^{k-2})$-time deterministic combinatorial algorithm for computing the parity of the number of induced subgraphs isomorphic to $C_k$ in $n$-vertex graphs, where $k \geq 5$ is odd.

Proof. 1. We describe how to compute $\text{Hom}_{K_5 - P_4}$ using arithmetic circuits of size $O(n^\omega)$. We start by defining the following matrices.

- $A_{i,j}^{n \times n} = x_{(i,1),(j,3)}$, $i \neq j$
- $B_{i,j}^{n \times n} = x_{(i,3),(j,2)}$, $i \neq j$
- $F_{i,j}^{n \times n} = x_{(i,j)}$, $i \neq j$
- $D_{i,i}^{n \times n} = y_i z_{3,i}$
- $C_{i,i}^{n \times n} = y_i z_{4,i}$
- $D_{i,i}^{n \times n} = y_i z_{5,i}$

Consider the labelled $K_5 - P_4$ in Figure 3. Compute the matrix products $FCF$, $FDF$, and $ABE$. Compute the polynomial $\sum_{i,j \in [n], i \neq j} z_{1,i} z_{2,j} y_i y_j x_{(i,j)} (ABE)_{i,j}$ and replace $x_{(i,1),(j,3)}$ with $(FCF)_{i,j}$ and replace $x_{(i,3),(j,2)}$ with $(FDF)_{i,j}$. It is easy to see that the resulting polynomial is $\text{Hom}_{K_5 - P_4}$ for this labelled $K_5 - P_4$ and the circuit has size $O(n^\omega)$.

2. $tw(K_k - P_{k-1}) = k - 3$.

3. The proof is similar to the proof of Part 4 of Theorem 6.4.

We remark that by computing homomorphism polynomials for $K_k - P_{k-1}$ for $k = 7, 9$ using small-size circuits, we can obtain an $O(n^{2\omega})$ time algorithm for induced subgraph isomorphism for $C_7$ and an $O(n^{\omega(3.3,2)})$ time algorithm for induced subgraph isomorphism for $C_9$. These algorithms are faster than the corresponding algorithms for $k$-cliques.
7 Algorithms for almost all induced patterns

In this section, we prove a result that is similar in spirit to Theorem 1.1 in [16] which states that the time complexity of induced subgraph isomorphism problem for $K_k$ upper bounds that of any $k$-vertex pattern graph. We show that the circuit complexity of $\text{Hom}_{K_k,e}$ upper bounds the time complexity of the induced subgraph isomorphism problem for all $k$-vertex pattern graphs $H$ except $K_k$ and $I_k$. The algorithms obtained from this statement can be obtained from known results. However, we believe that restating these upper-bounds in terms of circuits for $K_k-e$ homomorphism polynomials may give new insights to improve these algorithms.

The key idea is that an efficient construction of homomorphism polynomial for $K_k-e$ enables efficient construction of homomorphism polynomials for all smaller graphs. First, we prove the following technical result.

\textbf{Proposition 7.1.} If $N_H \preceq f$ and $f$ is a graph pattern polynomial family with uniform $s(n)$-size circuits, then $\text{Hom}_H$ has uniform $O(s(n))$-size circuits.

\textbf{Proof.} We can assume w.l.o.g. that $H$ does not have isolated vertices. Let $H$ have $k$ nodes and let $K_k^n$ be the complete $k$-partite graph with $n$ nodes in each partition. The nodes of $K_k^n$ are of the form $(i, \kappa)$, $1 \leq i \leq n$, $1 \leq \kappa \leq k$. Let $\sigma$ be a family of substitutions realizing $N_H \preceq f$. Consider $N_{H,kn}$. We know that $ML(\sigma_m(f_m)) = v[q] N_{H,kn}$ for some $m = O(n)$ and $q = O(1)$. Since $H$ does not contain isolated vertices, there is a function $g$ that maps $V(H)$ to $E(H)$ such that the image of $f(v)$ for any $v$ is an edge incident on $v$. Now we define the substitution $\tau$ on the edge and vertex variables:

$$
\tau(x(i,\kappa),(j,\mu)) = \begin{cases} 
Y_{i,\kappa,\mu}x(i,j) & \text{if } i \neq j, \\
0 & \text{if } i = j \text{ or } \{\kappa,\mu\} \notin E(H), 
\end{cases}
$$

$$
\tau(y(i,\kappa)) = \hat{a}_\kappa,
$$

where the variables $\hat{a}$ are fresh variables that we need for book-keeping and we define:

$$
Y_{i,\kappa,\mu} = z_{i,\kappa,\mu} y_{i,\kappa,\mu}
$$

if $g(\kappa') = \{\kappa,\mu\}$

$$
Y_{i,\kappa,\mu} = 1
$$

if $g(\kappa') \neq \{\kappa,\mu\}$

Every embedding of $H$ into $K_k^n$ such that each node of $H$ goes into another part will contribute a term that is multilinear in the $\hat{a}_\kappa$-variables in $\tau(N_{H,kn})$. The substitution also ensures that the colours of edges correspond to edges in $H$ and labels of adjacent vertices are different. It is easy to see that these embeddings correspond to homomorphisms to $K_n$.

We have proved that the part of $\tau(N_{H,kn})$ multilinear in $\hat{a}$ variables is,

$$
\hat{a}_{V(H)} \sum_{\phi:H \to e K_n, v \in V(H)} \prod_{e \in E(H)} z_{v,\phi(v)} y_{\phi(v)} \prod_{e \in E(H)} x_{\phi(e)} = \hat{a}_{[k]} \text{Hom}_{H,n}.
$$

Furthermore the part of $\tau(\sigma_m(f_m))$ multilinear in $\hat{a}$ and $v_i$ variables is $v[q] \hat{a}_{[k]} \text{Hom}_{H,n}$ since every non-multilinear term stays non-multilinear under $\tau$. Therefore, we get an exact computation for $\text{Hom}_{H,n}$ by differentiating the circuit with respect to $v_1, \ldots, v_q, \hat{a}_1, \ldots, \hat{a}_k$ once and then setting all variables $v_i$ for all $i$ and $\hat{a}_1, \ldots, \hat{a}_k$ to 0. Note that each differentiation
will increase the circuit size by a constant factor and we differentiate a constant number of
times. This operation is linear-time in the size of the circuit.

The above result can be interpreted in two different ways: (1) Homomorphism polynomials
are the best graph pattern polynomials or (2) Efficient constructions for homomorphism
polynomials can be obtained by obtaining efficient constructions for any pattern family f
such that \( N_H \leq f \).

\[ \text{Lemma 7.2.} \] Let \( k > 2 \). If \( H \neq K_k \) is a \( k \)-vertex graph, then \( 2N_H \leq \text{Hom}_{K_k-e} \).

\textbf{Proof.} The proof of this claim is similar to the proof of Theorem. Let \( M \) be the labelling
of \( K_k - e \) using \([k]\) such that vertices 1 and \( k \) are not adjacent. Let \( L \) be a labelling of \( H \)
using \([k]\) such that 1 and \( k \) are not adjacent. Therefore, the labelled graph \( L \) is a subgraph
of the labelled graph \( M \). Let \( q_1, \ldots, q_e \) be the edges of \( L \) and \( q_{e+1}, \ldots, q_m \) be the non-edges
of \( L \). Let \( S \) be the set of all labellings of \( H \). For each labelling \( L' \) in \( S \), associate a permutation
with \( L' \) such that applying it to \( L' \) yields \( L \). Let \( P \) be the set of all such permutations.

We partition \( P \) into \( P_1 \) and \( P_2 \) as follows: A permutation \( \phi \in P_1 \) if given a sequence of \( k \)
numbers, we can determine whether the sequence is consistent with \( \phi \), i.e., the \( k \)th
smallest element in the sequence is at position \( \phi(i) \), without comparing the first and last elements
in the sequence. Otherwise, \( \phi \in P_2 \). We start with the \( \text{Hom}_{K_k-e} \) polynomial over the vertex
set \([n] \times [k] \times P \) and apply the following substitution.

\[
\begin{align*}
\sigma_H(y_{v,p,\phi}) &= y_v \\
\sigma_H(x_{(v_1,p_1,\phi),(v_2,p_2,\phi)}) &= 0, \text{ if } \phi \neq \phi' \\
\sigma_H(x_{(v_1,p_1,\phi),(v_2,p_2,\phi)}) &= 0, \phi^{-1}(p_1) < \phi^{-1}(p_2) \land v_1 > v_2 \\
\sigma_H(x_{(v_1,p_1,\phi),(v_2,p_2,\phi)}) &= \begin{cases} 
{x_{v_1,v_2}}, & \{p_1,p_2\} \in E(L) \\
1, & \{p_1,p_2\} \in E(M) \setminus E(L) \\
0, & \text{otherwise}
\end{cases} \\
\sigma_H(z_{1,(v_1,1,\phi)}) &= \begin{cases} 
u_1, & \phi \in P_2 \\
u_1, & \phi \in P_1 \\
\end{cases} \\
\sigma_H(z_{i,(v,i,\phi)}) &= u_i, i > 1 \\
\sigma_H(z_{i,(v,j,\phi)}) &= u^2_i, i \neq j
\end{align*}
\]

First, we state some properties satisfied by the surviving monomials. Rule 1 ensures that
all vertices have different labels. Rule 2 ensures that all variables in a surviving monomial
are indexed by the same permutation. Rules 6 and 7 ensure that all vertices have different
colours. Let \( \tau = (1 \ k)(2) \cdots (k-1) \). Consider an arbitrary surviving monomial indexed
by a permutation \( \phi \). If \( \phi \in P_1 \), then Rule 3 ensures that the vertices of the monomial are
consistent with \( \phi \). Assume that the vertices are \((v_1,1,\phi),\ldots,(v_k,k,\phi)\) and they are not consistent
with \( \phi \). This is possible only if \( \phi^{-1}(1) < \phi^{-1}(k) \) and \( v_1 > v_k \) or \( \phi^{-1}(1) > \phi^{-1}(k) \)
and \( v_1 < v_k \). Since \( \phi \in P_1 \), there exists an \( i' \) such that \( \phi^{-1}(1) < \phi^{-1}(i') < \phi^{-1}(k) \) or
\( \phi^{-1}(1) > \phi^{-1}(i') > \phi^{-1}(k) \). Therefore, we have that the vertices are inconsistent at either
\( \{1,i'\} \) or \( \{i',k\} \), a contradiction. If \( \phi \in P_2 \), then Rule 3 ensures that the vertices are
consistent with \( \phi \) or \( \tau \circ \phi \). To see this, observe that, by Rule 3, the inconsistency with

\footnote{Note that unlike in the Baur-Strassen theorem, we only compute one derivative!}
\( \phi \) can only occur \( \{1, k\} \). This implies that the vertices are consistent with \( \tau \circ \phi \) because 
\((\tau \circ \phi)^{-1}(1) = \phi^{-1}(k) \) and 
\((\tau \circ \phi)^{-1}(k) = \phi^{-1}(1) \) removing the inconsistency at \( \{1, k\} \) and 
for all other \( i \), we have 
\((\tau \circ \phi)^{-1}(i) = \phi^{-1}(i) \) preserving consistency at all other points.

Consider a labelled \( H \), say \( L' \), labelled using \( v_1 < \cdots < v_k \) with associated permutation \( \phi \). Let \( \psi : v_i \mapsto i \). Let \( e_1, \ldots, e_m \) be the edges and non-edges of \( L' \) such that 
\[ e_i = \psi^{-1}(\phi^{-1}(q_i)) \] for all \( i \). We split the proof into two cases: If \( \phi \in P_1 \), the monomial 
\[ z_1(v_{a-1(1)}, 1, \phi) \cdots z_1(v_{a-I(k)}, k, \phi) \] (A monomial in \( \text{Hom}_{K_k-e} \) is completely determined by the homomorphism variables and we will not specify the other variables for brevity) uniquely generates the monomial in \( N_H \) that corresponds to \( L' \). If \( \phi \in P_2 \), then there are two cases to consider depending on whether the permutation \( \tau \) is in \( \text{Aut}(L) \) or not. If \( \tau \not\in \text{Aut}(L) \), then the monomials 
\[ z_1(v_{a-1(1)}, 1, \phi) \cdots z_1(v_{a-I(k)}, k, \phi) \] and 
\[ z_1(v_{a-1(1)}, 1, \tau \phi) \cdots z_1(v_{a-I(k)}, k, \tau \phi) \] are the only two monomials that yield the required monomial. If \( \tau \in \text{Aut}(L) \), then the monomials 
\[ z_1(v_{a-1(1)}, 1, \phi) \cdots z_1(v_{a-I(k)}, k, \phi) \] and 
\[ z_1(v_{a-1(1)}, 1, \phi) \cdots z_1(v_{a-I(k)}, k, \phi) \] are the only two monomials that yield the required monomial. ▶

The above lemma shows that, as expected, the polynomial \( \text{Hom}_{K_k-e} \) is strong enough to compute every other graph homomorphism except that of \( K_k \). This allows us to parameterize many existing results in terms of the size of the arithmetic circuits computing \( \text{Hom}_{K_k-e} \).

\[ \mathbf{Theorem} \ 7.3. \] If there are uniform \( O(n^{s(k)}) \) size circuits for \( \text{Hom}_{K_k-e} \), then the number of subgraph isomorphisms for any \( k \)-vertex \( H \neq K_k \) can be computed in \( O(n^{s(k)}) \) time on \( n \)-vertex graphs.

\[ \mathbf{Proof.} \] For all \( k \)-vertex \( H \neq K_k \), we have \( 2N_H \leq \text{Hom}_{K_k-e} \). For all \( H \) on less than \( k \) vertices, we have \( N_H \leq I_{K_k} \leq \text{Hom}_{K_k-e} \). Therefore, for all graphs \( H \neq K_k \) on at most \( k \) vertices, we can construct \( O(n^{s(k)}) \) size circuits that compute \( 2\text{Hom}_H \). We know that the number of subgraph isomorphisms for \( H \) can be expressed as a linear combination of the number of homomorphisms for \( H \) and the number of homomorphisms for graphs on less than \( k \) vertices [5]. ▶

\[ \mathbf{Theorem} \ 7.4. \] If there are uniform \( O(n^{s(k)}) \) size circuits for \( \text{Hom}_{K_k-e} \), then the induced subgraph isomorphism problem for all \( k \)-vertex pattern graphs except \( K_k \) and \( I_k \) have an \( O(n^{s(k)}) \) time algorithm.

\[ \mathbf{Proof.} \] We will show how to decide induced subgraph isomorphism for \( H \neq K_k \) in \( O(n^{s(k)}) \) time. Now, choose a prime \( p \) such that \( p \) divides the number of occurrences of \( H \) in \( K_k \). The number of induced subgraph isomorphisms modulo \( p \) for \( H \) can be expressed as a linear combination of the number of subgraph isomorphisms modulo \( p \) of \( k \)-vertex graphs except \( K_k \) and can be computed in \( O(n^{s(k)}) \) time. It is known that the induced subgraph isomorphism problem for \( H \) is randomly reducible to this problem [19]. ▶

\[ \mathbf{Theorem} \ 7.5. \] If there are uniform \( O(n^{s(k)}) \) size circuits for \( \text{Hom}_{K_k-e} \) and if there is an \( O(t(n)) \) time algorithm for counting the number of induced subgraph isomorphisms for a \( k \)-vertex pattern \( H \), then the number of induced subgraph isomorphisms for all \( k \)-vertex patterns can be computed in \( O(n^{s(k)} + t(n)) \) time on \( n \)-vertex graphs.

\[ \mathbf{Proof.} \] We know that \( t_H = \sum_{H' \neq H} a_{H'} n_{H'} \), where all \( a_{H'} \neq 0 \), \( i_H \) is the number of induced subgraph isomorphisms from \( H \) to \( G \) and \( n_{H'} \) is the number of subgraph isomorphisms from \( H \) to \( G \). Furthermore, we can compute \( n_{H'} \) for all \( H' \neq K_k \) in \( O(n^{s(k)}) \) time. Therefore, if we can compute \( i_H \) in \( t(n) \) time, we can compute \( n_{K_k} \) in \( O(n^{s(k)} + t(n)) \) time. ▶

The following corollary follows by observing that \( \text{tw}(K_k - e) = k - 2 \).
Corollary 7.6. All $k$-vertex pattern graphs except $K_k$ and $I_k$ have an $O(n^{k-1})$ time combinatorial algorithm for deciding induced subgraph isomorphism on $n$-vertex graphs.

Corollary 7.7. For $k \in \{4, 5, 6, 7, 8\}$, the induced subgraph isomorphism problem for any $k$-vertex pattern graph $H$ except $K_k$ and $I_k$ can be decided faster than currently known best clique algorithms.

Proof. The polynomial family $Hom_{K_n-e}$ can be computed by uniform arithmetic circuits of size $O(n^{\omega((\frac{\log 2}{\log \log 2}) \cdot \frac{\log 2}{\log \log 2})})$ for all $k$. The construction is similar to the other constructions for homomorphism polynomials using fast matrix multiplication in this paper.

8 Reducions between patterns

The following proposition is analogous to the obvious fact that the complexity of the induced subgraph isomorphism problem is the same for any pattern and its complement.

Proposition 8.1. $I_H \leq I_{\overline{H}}$ for all graphs $H$.

Proof. Use the substitution that maps $x_e$ to $1 - x_e$ for any edge variable $x_e$ and maps any vertex variable to itself.

It is known that $\# aut(H) = 1$ for almost all graphs $H$. Therefore, the following proposition can be interpreted as stating that the homomorphism polynomial is harder than the subgraph isomorphism polynomial for almost all pattern graphs $H$. This is used in [11] to obtain algorithms for subgraph isomorphism problems.

Proposition 8.2. $\# aut(H) N_H \leq Hom_{11}$ for all graphs $H$.

Proof. Let $H$ be a $k$ vertex graph labelled using $[k]$. Use the substitution $\sigma(z_{a,v}) = u_a$ for all $a \in V(H), v \in V(G)$ and $\sigma(w) = w$ for all the other variables $w$ in $Hom_H$ over the vertex set $[n]$. We have $\# aut(H), u_{[k]} \cdot ML(N_H) = ML(\sigma(Hom_H)) = \sigma(ML(Hom_H))$. Consider an arbitrary automorphism $\phi$ of $H$. For every monomial $m = y_{v_1} \cdots y_{v_k}, x_{e_1} \cdots x_{e_j}$ in $N_H$, there are exactly $\# aut(H)$ monomials $m_\phi = z_{(\phi(v_1), v_1)} \cdots z_{(\phi(v_k), v_k)} y_{v_1} \cdots y_{v_k}, x_{e_1} \cdots x_{e_j}$ in $Hom_H$ that satisfy $\sigma(m_\phi) = u_{[k]} m$. This proves Properties 1 and 2 of the reduction. It is easy to see that the reduction satisfies the other properties too.

Intuitively, the subgraph isomorphism problem should become harder when the pattern graph becomes larger. However, it is not known whether this is the case. Nevertheless, we can show this hardness result holds for subgraph isomorphism polynomials for almost all pattern graphs.

Theorem 8.3. If $H \subseteq H'$, then $\# aut(H) N_H \leq N_{H'}$.

Proof. Let $|V(H)| = k$ and $|V(H')| = k + \ell$ for some $\ell \geq 0$. Choose a labelling $L$ of the vertices of $H'$ such that the vertices of an $H$ in $H'$ are labelled $1, \ldots, k$. Consider the polynomial $N_H$, over the vertex set $\{[n] \times [k]\} \cup \{(n + i, k + i) : 1 \leq i \leq \ell\}$. Substitute for the variables as follows:

\[
\begin{align*}
\sigma(y_{(i,p)}) &= \begin{cases} 
 y_i u_p, & \text{for all } i \in [n], p \in [k] \\
 u_p, & \text{otherwise}
\end{cases} \\
\sigma(x_{(i_1,p_1),(i_2,p_2)}) &= \begin{cases} 
 x_{(i_1,i_2)} & \text{if } \{p_1, p_2\} \in E(H) \\
 1 & \text{if } \{p_1, p_2\} \in E(H') \setminus E(H) \\
 0 & \text{otherwise}
\end{cases}
\end{align*}
\]
We say that a monomial in $N_{H'}$ survives if the monomial does not become non-multilinear or 0 after the substitution. First, we will prove that all surviving monomials correspond to $H'$-subgraphs where the labels and colours of vertices are different and the colours of edges are the same as in the labelling $L$. Rule 1 ensures that the colours and labels of all vertices in the surviving monomials are different. Rule 2 ensures that there is a one-to-one correspondence between the edges $\{p_1, p_2\}$ in the labelling $L$ and the edge variables $x_{\{(i_1, p_1), (i_2, p_2)\}}$. To see this, observe that each monomial in $N_{H'}$ has $|E(H')|$ edge variables. Since all vertices in a surviving monomial have different colours, all edges in the monomial must have different colours. Since any edge variable that has a colour not in the labelling $L$ is set to 0, the colours of edges must be in one-to-one correspondence with the edges in the labelling $L$. This proves the all surviving monomials are of the form $y(u_1, 1) \cdots y(u_k, k) \prod_i y(n+i, k+i) x_1(q_1) \cdots x_{e_m, q_m} w$ for $u_1, \ldots, u_k \in [n]$, where $w$ is the product of edge variables with colour $\{p, q\}$ such that $\{p, q\}$ is an edge in $H'$ but not in $H$ in the labelling $L$, $u_1, \ldots, u_k$ are all different, and $q_1, \ldots, q_m$ are edges in $H$ in the labelling $L$. Note that the product $w$ is determined uniquely by $u_1, \ldots, u_k$.

We claim that for each monomial $y_{ST}$ in $N_H$ over the vertex set $[n]$ there are $\#aut(H)$ monomials $y_{ST}u[k]$ in $\sigma(N_{H'})$. Consider an arbitrary monomial $y_{ST} = y_{1} \cdots y_{m} x_{1} \cdots x_{e_m}$ in $N_H$ where $m = |E(H)|$. The monomials in $N_{H'}$ that yield $y_{ST}u[k]$ after the substitution are exactly the monomials $y_{w_1, 1} \cdots y_{w_k, k} \prod_i y(n+i, k+i) x_{e'_1, q_1} \cdots x_{e'_m, q_m} w$ where $w$ is the product of edge variables with colour $\{p, q\}$ such that $\{p, q\}$ is an edge in $H'$ but not in $H$ in the labelling $L$, $\{w_1, \ldots, w_k\} = \{v_1, \ldots, v_k\}$, and $\{e_1, \ldots, e_m\} = \{e'_1, \ldots, e'_m\}$. But this monomial corresponds to the automorphism $\phi : v_i \mapsto w_i$. Since $w$ is uniquely determined given $w_1, \ldots, w_k$, the number of such monomials is $\#aut(H)$. Also, each surviving monomial yields a monomial in $N_{H'}$.

Additionally, each non-multilinear term in the polynomial obtained after the substitution contains at least one vertex or other variable with degree more than one. This proves the theorem.

The following theorem states that the induced subgraph isomorphism polynomial is harder than the subgraph isomorphism polynomial for almost all graphs.

**Theorem 8.4.** $\#aut(H)N_{H} \preceq I_{H}$ for all graphs $H$.

**Proof.** Observe that $I_{H} = N_{H} + \sum_{H' \supset H} a_{H'} N_{H'}$. Let $k$ be the number of vertices in $H$ and fix some labelling of $H$ using $[k]$. Now consider the polynomial $I_{H}$ over the vertex set $[n] \times [k]$ and apply the following substitution.

\[
\sigma(y_{(i, p)}) = y_{i} u_{p}
\]

\[
\sigma(x_{(i_1, p_1), (i_2, p_2)}) = \begin{cases} x_{(i_1, i_2)} & \text{if } \{p_1, p_2\} \in E(H) \\ 0 & \text{otherwise} \end{cases}
\]

Now observe that any monomial in $N_{H'}$ for $H' \supset H$ must vanish because it will have at least one more edge than $H$. By the same argument as in the proof of Theorem 8.3 we conclude that there are exactly $\#aut(H)$ monomials in $N_{H}$ over $[n] \times [k]$ that yield the monomial $y_{ST}u[k]$ after the substitution for any monomial $y_{ST}$ in $N_{H}$ over $[n]$. 

We now prove the analogue of Theorem 1.1 in [16] which states that $k$-clique is harder than any other $k$-vertex pattern graph.

**Theorem 8.5.** For any $k$-vertex graph $H$, $I_{H} \preceq I_{K_k}$. 
We know that with vertices are different and all vertices are indexed by the same permutation. We will prove that for the term \( I_k \) over the vertex set \([n] \times [k] \times P\):

\[
\begin{align*}
\sigma(y_{(v,p,\phi)}) &= y_v u_p \quad (1) \\
\sigma(x_{((v_1,p_1,\phi), (v_2,p_2,\phi'))}) &= 0 \text{ if } \phi \neq \phi' \text{ or } p_1 = p_2 \text{ or } v_1 = v_2 \quad (2) \\
\sigma(x_{((v_1,p_1,\phi), (v_2,p_2,\phi'))}) &= 0 \text{ if } \phi^{-1}(p_1) < \phi^{-1}(p_2) \text{ and } v_1 > v_2 \quad (3) \\
\sigma(x_{((v_1,p_1,\phi), (v_2,p_2,\phi'))}) &= \begin{cases} x_{(v_1,v_2)} & \text{if } \{p_1,p_2\} \in E(L) \\ 1 - x_{(v_1,v_2)} & \text{if } \{p_1,p_2\} \notin E(L) \end{cases} \quad (4)
\end{align*}
\]

The first two rules ensure that in any surviving monomial, the labels and colours of all vertices are different and all vertices are indexed by the same permutation.

We can extend the correspondence between labellings of \( H \) and permutations to arbitrary labellings (as opposed to labellings using \([k]\)). Given a labelling of \( H \) using \( v_1 < \cdots < v_k \), we can obtain a labelling \( L' \) of \( H \) using \([k]\) by replacing each \( v_i \) by \( i \) for all \( i \). The permutation associated with the labelling \( M \) is the same as the permutation associated with labelling \( L' \).

Consider an arbitrary labelling \( M \) of \( H \) using \( v_1 < \cdots < v_k \) where each \( v_i \in [n] \). Let \( L' \in S \) be the labelling corresponding to the labelling \( M \) such that \( \psi : v_i \mapsto i \) is the permutation that maps \( i \) to \( L' \). Let \( \phi \in P \) be the permutation associated with \( L' \). For convenience, we denote the edges and non-edges of \( M \) by \( e_1, \ldots, e_m \) such that \( e_i = \psi^{-1}(\phi^{-1}(q_i)) \) for all \( i \).

We will prove that for the term \( t = y_{v_1} \cdots y_{v_k} x_{e_1} \cdots x_{e_m} (1 - x_{e_{t+1}}) \cdots (1 - x_{e_m}) \) in \( I(H) \) that encodes \( M \), there is a unique monomial \( s \) in \( I_{K_k} \) such that \( \sigma(s) = w_{|k|} t \). The monomial \( s = y_{(v_1,\phi(1)),\phi} \cdots y_{(v_k,\phi(k)),\phi} x_{(e_1,q_1,\phi)} \cdots x_{(e_m,q_m,\phi)} \). First of all, we have to prove that given \( v_i \) has colour \( \phi(i) \), the edges are coloured such that \( e_i \) gets colour \( q_i \). Start with an arbitrary \( q_i = (j,k) \). Then, \( e_i = \psi^{-1}(\phi^{-1}(j)) \) and \( \phi^{-1}(k) \) which has colour \( (j,k) \) as required. Also, we have \( \sigma(s) \neq 0 \) because if \( \phi^{-1}(\phi(i)) = i < j < \phi^{-1}(\phi(j)) \), then \( v_i < v_j \). Given that \( \sigma(s) \neq 0 \), it is easy to see that \( \sigma(s) = w_{|k|} t \) by applying rules 1 and 4.

Given an arbitrary surviving monomial \( r = y_{(v_1,1,\phi)} \cdots y_{(v_k,k,\phi)} x_{(e_1,q_1,\phi)} \cdots x_{(e_m,q_m,\phi)} \) in \( I_{K_k} \) such that \( \sigma(r) = w_{|k|} w \) for some \( w \), we claim that \( w \) encodes a labelling \( M \) of \( H \) where the permutation associated with \( M \) is \( \phi \). It is easy to see that \( w \) encodes some labelling of \( H \). Observe that for \( r \) to survive, the vertices \((v_i,i,\phi)\) for all \( i \) has to be consistent with \( \phi \), i.e., the vertex coloured \( \phi(i) \) must be the \( i^{th} \) smallest among all \( v_j \) by Rule 3. By the definition of \( \psi \), we have \((i,j) \in E'(L) \) if and only if \( \{\phi(i),\phi(j)\} \in E(L) \). By Rule 4, we also have if \( \phi(i),\phi(j) \in E(L) \) then \( x_{(v_i,p_i,\phi(i))} \) appears in the term \( w \) and otherwise \((1 - x_{(v_i,p_i,\phi(i))}) \) appears in \( w \). In other words, in the graph encoded by \( w \), the \( i^{th} \) smallest and \( j^{th} \) smallest vertices are connected if and only if the \( i^{th} \) smallest and \( j^{th} \) smallest vertices are connected in \( L' \). Therefore, the associated permutation is \( \phi \) as claimed. We can now prove that \( w_{|k|} t \) is uniquely generated from \( s \). Suppose for contradiction that the monomial \( s' = y_{(v_1',1,\phi')} \cdots y_{(v_k',k,\phi')} x_{(e_1',q_1,\phi')} \cdots x_{(e_m',q_m,\phi')} \) also satisfies \( \sigma(s') = w_{|k|} t \). Then, it must be that \( \{v_1', \ldots, v_k'\} = \{v_1, \ldots, v_k\} \) and \( \{e_1', \ldots, e_m'\} = \{e_1, \ldots, e_m\} \). We know that \( \phi = \phi' \) because the permutation in the monomial must correspond to the
labelling encoded by \( t \). But, \( \phi = \phi' \) implies \( v'_i = v_i \) for all \( i \) (Otherwise, the third rule ensures that at least one edge variable in \( s' \) becomes 0 under \( \sigma \)). But, if \( v'_i = v_i \) for all \( i \), then \( e_j = e'_j \) for all \( j \) contradicting \( s \neq s' \).

We have proved that \( ML(\sigma(I_{K_k})) = u_{|k|} I_H \). Observe that the polynomial obtained after the substitution cannot contain edge variables of degree more than one because of Rule 2. It is easy to see that the substitution satisfies the other properties. □

The theorem below shows that the induced subgraph isomorphism polynomial for any graph containing a \( k \)-clique or \( k \)-independent set is harder than the \( k \)-clique polynomial. An analogous hardness result is known for algorithms, only when the pattern \( H \) contains a \( k \)-clique (or \( k \)-independent set) that is disjoint from all other \( k \)-cliques (or \( k \)-independent sets) [10].

\[ \textbf{Theorem 8.6.} \text{ If } H \text{ contains a } k \text{-clique or a } k \text{-independent set, then } I_{K_k} \preceq I_H. \]

\textbf{Proof.} We will prove the statement when \( H \) contains a \( k \)-clique. The other part follows because if \( H \) contains a \( k \)-independent set, then the graph \( \overline{H} \) contains a \( k \)-clique and \( I_{K_k} \preceq I_{\overline{H}} \preceq I_H \).

Fix a labelling of \( H \) where the vertices of a \( k \)-clique are labelled using \([k]\) and the remaining vertices are labelled \( k + 1, \ldots, k + \ell \). Consider the polynomial \( I_H \) over the vertex set \( ([n] \times [k]) \cup \{(n + i, k + i) : 1 \leq i \leq \ell \} \) and apply the following substitution.

\[
\sigma(y_{(i,p)}) = \begin{cases} 
  y_i u_p & \text{if } i \in [n] \text{ and } p \in [k] \\
  u_p & \text{otherwise} 
\end{cases} \quad (1)
\]

\[
\sigma(x_{\{i_1,p_1\},\{i_2,p_2\}}) = \begin{cases} 
  x_{\{i_1,i_2\}} & \text{if } \{p_1,p_2\} \in E(K_k) \text{ and } p_1 < p_2 \text{ and } i_1 < i_2 \\
  1 & \text{if } \{p_1,p_2\} \in E(H) \setminus E(K_k) \\
  0 & \text{otherwise} 
\end{cases} \quad (2)
\]

Consider a \( k \)-clique on the vertices \( i_1, \ldots, i_k \in [n] \) on an \( n \)-vertex graph where \( i_1 < \cdots < i_k \). The monomial in \( I_{K_k} \) corresponding to this clique is generated uniquely from the monomial \( y_{(i_1,1)} \cdots y_{(i_k,k)} \prod_{j=1}^{k} y_{(n+i,j)} x_{\{i_1,i_2\}} \cdots x_{\{i_{k-1},i_k\}} \) \( w \) in \( I_H \), where \( w \) is the product of all edge variables corresponding to edges in \( H \) but not in \( K_k \). Note that Rules 1 and 2 ensure that in any surviving monomial, the labels and colours of all vertices are distinct and the colours of the edges must be the same as \( E(H) \). The product \( w \) is determined by \( i_1, \ldots, i_k \). This proves that \( ML(\sigma(I_H)) = u_{|k|} ML(I_{K_k}) \). It is easy to verify that the substitution satisfies the other properties. □

Theorem 8.6 is true with \( N_H \) or \( Hom_H \) instead of \( I_H \). In fact, the same proof works for \( N_H \). For \( Hom_H \), use the substitution in the proof of Theorem 8.6 along with \( z_{a,(v,a)} = u_a \) and \( z_{a,(v,b)} = u_b^2 \) when \( a \neq b \) for all homomorphism variables.

\section{Discussion}

Since the subgraph isomorphism and homomorphism polynomials for cliques have the same size complexity, there is no advantage to be gained by using homomorphism polynomials instead of subgraph isomorphism problem. How hard is it to obtain better circuits for \( Hom_{K_k} \)? As the following proposition shows, improving the size of \( Hom_{K_k} \) implies improving matrix multiplication.
Proposition 9.1. If $N_{K_n}$ (or $I_{K_n}$ or $\text{Hom}_{K_n}$) has $O(n^\tau)$-size circuits then the exponent of matrix multiplication $\omega \leq \tau$.

Proof. Let $G$ be the complete tripartite graph $T_n$ on $3n$-vertices with partitions of size $n$. The vertex set of $T_n$ is $[3] \times [n]$. Instead of substituting a 1 for every edge in $T_n$, we substitute the variables $a_{i,j}$ for edges $\{(1,i), (2,j)\}$, $b_{i,j}$ for edges $\{(2,i), (3,j)\}$, and $c_{i,j}$ for edges $\{(3,i), (1,j)\}$. The resulting polynomial is:

$$N' = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} y_{1,i}y_{2,j}y_{3,k} \cdot a_{i,j}b_{j,k}c_{k,i}$$

We substitute 1 for all vertex variables and obtain

$$N'' = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} a_{i,j}b_{j,k}c_{k,i}$$

$N''$ has $O(n^\tau)$-size circuits. It is well-known that $\omega \leq \tau$ follows from this, see e.g. [4]. ◀

It is interesting to know whether such connections exist for $k > 3$.

References

1. Josh Alman and Virginia Vassilevska Williams. Further limitations of the known approaches for matrix multiplication. In Anna R. Karlin, editor, 9th Innovations in Theoretical Computer Science Conference, ITCS 2018, January 11-14, 2018, Cambridge, MA, USA, volume 94 of LIPIcs, pages 25:1–25:15. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2018. URL: https://doi.org/10.4230/LIPIcs.ITCS.2018.25

2. Andris Ambainis, Yuval Filmus, and François Le Gall. Fast matrix multiplication: Limitations of the coppersmith-winograd method. In Rocco A. Servedio and Ronitt Rubinfeld, editors, Proceedings of the Forty-Seventh Annual ACM on Symposium on Theory of Computing, STOC 2015, Portland, OR, USA, June 14-17, 2015, pages 585–593. ACM, 2015. URL: http://doi.acm.org/10.1145/2746539.2746554

3. Per Austrin, Petteri Kaski, and Kaie Kubjas. Tensor network complexity of multilinear maps. CoRR, abs/1712.09630, 2017. URL: http://arxiv.org/abs/1712.09630

4. Markus Bläser. Fast matrix multiplication. Theory of Computing, Graduate Surveys, 5:1–60, 2013. URL: https://doi.org/10.4086/toc.gs.2013.005

5. Radu Curticapean, Holger Dell, and Dániel Marx. Homomorphisms are a good basis for counting small subgraphs. In Hamed Hatami, Pierre McKenzie, and Valerie King, editors, Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2017, Montreal, QC, Canada, June 19-23, 2017, pages 210–223. ACM, 2017. URL: http://doi.acm.org/10.1145/3055399.3055502

6. Josep Díaz, Maria J. Serna, and Dimitrios M. Thilikos. Counting k-colorings of partial k-trees. Theor. Comput. Sci., 281(1-2):291–309, 2002. URL: https://doi.org/10.1016/S0304-3975(02)00017-8

7. Arnaud Durand, Meena Mahajan, Guillaume Malod, Nicolas de Rugy-Altherre, and Nitin Saurabh. Homomorphism polynomials complete for VP. In 34th International Conference on Foundation of Software Technology and Theoretical Computer Science, FSTTCS 2014, December 15-17, 2014, New Delhi, India, pages 493–504, 2014. URL: https://doi.org/10.4230/LIPIcs.FSTTCS.2014.493
A Omitted Proofs

Proof. (Of Theorem 4.8) Suppose that the arithmetic circuit given as input computes a polynomial on $n$ variables. For each variable $x_i$, select $a \in \mathbb{Z}_p$ and $y \in \{y_1, \ldots, y_k\}$ uniformly at random and substitute $ay$ for $x_i$. Evaluate the circuit over $\mathbb{Z}_p[y_1, \ldots, y_k]/\langle y_1^2, \ldots, y_k^2 \rangle$ and accept if and only if the result is non-zero. The correctness of this algorithm can be proved by induction on the number of multilinear terms in the polynomial that are non-zero modulo $p$. \hfill $\blacksquare$
Proof. (Of Theorem 4.12) We will describe how to construct an arithmetic circuit of size $O(n^{t+1})$ for $Hom_H$ where $t = tw(H)$. The construction mirrors the algorithm in Theorem 3.1 in [6]. We start with a nice tree decomposition $D$ of $H$. Each gate in the circuit will be labelled by some node (say $p$) in $D$ and a partial homomorphism $\phi : V(H) \mapsto [n]$. The label is $I_p(\phi)$.

Let $p$ be a node in the tree decomposition $D$. Construct the circuit in a bottom-up fashion as follows:

$p$ is a start node with $X_p = \{a\}$ Add $n$ input gates labelled $I_p(\{(a,v)\})$ with the constant 1 as value for each $v \in [n]$.

$p$ is an introduce node Let $q$ be the child of $p$ and $X_p - X_q = \{a\}$. Add gates labelled $I_p(\phi \cup \{(a,v)\}) = I_q(\phi)$ for each $v \in [n]$. Since there are at most $O(n^{t+1})$ choices for $\phi \cup \{(a,v)\}$, there are at most $O(n^{t+1})$ gates.

$p$ is a join node Let $q_1$ and $q_2$ be the children of $p$. Add gates labelled $I_p(\phi) = I_{q_1}(\phi).I_{q_2}(\phi)$. Since there are at most $O(n^{t+1})$ choices for $\phi$, there are at most $O(n^{t+1})$ gates.

$p$ is a forget node Let $q$ be the child of $p$ such that $X_q - X_p = \{a\}$. Add gates $I_p(\phi) = \sum_{v \in [n]} z_{a,v}x_{v,u_1}\cdots x_{v,u_k}I_q(\phi \cup \{(a,v)\})$ where $\{v,u_i\}, 1 \leq i \leq k$ are the images of the edges incident on $a$ in partial homomorphism $\phi \cup \{(a,v)\}$. Note that there are $O(n)$ gates corresponding to the tuple $(p,\phi)$. Since $p$ is a forget node, there are at most $O(n^2)$ such tuples and therefore at most $O(n^{t+1})$ gates.

$\blacktriangleright$