Resolution Limits of Noisy 20 Questions Estimation

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Abstract

We establish fundamental limits on estimation accuracy for the noisy 20 questions problem with measurement dependent noise and introduce optimal non-adaptive procedures that achieve this limits. The minimal achievable resolution is defined as the absolute difference between the estimated and the true values of the target random variable, given a finite number of queries constrained by the excess-resolution probability. Inspired by the relationship between the 20 questions problem and the channel coding problem, we derive non-asymptotic bounds on the minimal achievable resolution. Furthermore, applying the Berry-Esseen theorem to our non-asymptotic bounds, we obtain a second-order asymptotic approximation to finite blocklength performance, specifically the achievable resolution of optimal non-adaptive query procedures with a finite number of queries subject to the excess-resolution probability constraint. We specialize our second-order results to measurement dependent versions of several channel models including the binary symmetric, the binary erasure and the binary Z- channels. Our results are then extended to adaptive query procedures, establishing a lower bound on the resolution gain associated with adaptive querying.

I. INTRODUCTION

The noisy 20 questions problem (cf. [1]–[7]) arises when one aims to accurately estimate an arbitrarily distributed random variable $S$ by successively querying an oracle and using noisy responses to form an estimate $\hat{S}$. A central goal in this problem is to find optimal query strategies that yield a good estimate $\hat{S}$ for the unknown target random variable $S$.

Depending on the framework adopted to design queries, the 20 questions problem is either adaptive or non-adaptive. In adaptive query procedures, the design of a subsequent query depends on all previous queries and noisy responses to these queries from the oracle. In non-adaptive query procedures, all the queries are designed independently in advance. For example, the bisection policy [5 Section 4.1] is an adaptive query procedure and the dyadic policy [5 Section 4.2] is a non-adaptive query procedure. Compared with adaptive query procedures, non-adaptive query procedures have the advantage of lower computation cost, parallelizability and no need for feedback. Depending on whether or not the noisy responses depend on the queries, the noisy 20 questions problem is classified into two categories: querying with measurement independent noise (e.g., [5], [6]); and querying with measurement dependent noise (e.g., [7], [8]). As argued in [8], measurement dependent noise better models practical applications. For example, for target localization in a sensor network, the noisy response to each query can depend on the size of the query region. Another example is in human query systems where personal biases abut the state may affect the response.

In earlier works on the noisy 20 questions problem, e.g., [5], [9], [10], the queries were designed to minimize the entropy of the posterior distribution of the target variable $S$. As pointed out in later works, e.g., [6], [8], [11], other accuracy measures, such as the resolution and the quadratic loss are often better criteria, where the resolution is defined as the absolute difference between $S$ and its estimate $\hat{S}$, $|S - \hat{S}|$, and the quadratic loss is $(\hat{S} - S)^2$.

Motivated by the scenario of limited resources, computation and response time, we obtain new results on the non-asymptotic tradeoff among the number of queries $n$, the achievable resolution $\delta$ and the excess-resolution probability $\epsilon$ of optimal adaptive and non-adaptive query procedures for noisy 20 questions estimation of an arbitrarily distributed random variable $S$ taking values in the alphabet $S = \{0, 1\}$. Our results apply to both adaptive and non-adaptive querying. For the case of adaptive query procedures, we derive achievable non-asymptotic and second-order asymptotic bounds on optimal resolution. We define the gain of adaptivity as the logarithm of the ratio between achievable resolutions of optimal non-adaptive and adaptive query procedures. This gain over non-adaptivity can be interpreted as the additional number of bits in the binary expansion of the target variable extracted by optimal adaptive querying where the logarithm is base 2. We numerically evaluate a lower bound on this gain for measurement dependent versions of binary symmetric, binary erasure and binary Z- channels.

Our contributions for non-adaptive querying are as follows. First, we derive non-asymptotic bounds of optimal non-adaptive query procedures for any number of queries $n$ and any excess-resolution probability $\epsilon$. To do so, inspired by [8], we use the connection of the 20 questions problem with measurement dependent noise to channel coding with states and borrow ideas from finite blocklength analyses for channel coding [12] (see also [13]). In particular, we adopt the change-of-measure technique of [14] in the achievability proof to handle the measurement dependent noise.

Secondly, applying the Berry-Esseen theorem, under mild conditions on the measurement dependent noise, we obtain second-order asymptotic approximation to the achievable resolution of optimal non-adaptive query procedures with finite number of queries. As a corollary of our result, we establish a phase transition for optimal non-adaptive query procedures. This means that, if one is allowed to make an infinite number of queries, regardless of the excess-resolution probability, the average number
of bits (in the binary expansion of the target variable) extracted per query by optimal non-adaptive query procedures remains the same.

Furthermore, we specialize our second-order analyses to measurement dependent versions of three channel models: the binary symmetric, the binary erasure and the binary Z- channels. Similarly to our proofs for the measurement dependent channels, the second-order asymptotic approximation to the achievable resolution of optimal non-adaptive query procedures for measurement independent channels can be obtained. To compare the performances of optimal non-adaptive query procedures for measurement dependent and measurement independent channels, we contrast the minimal achievable resolution of both scenarios. Intuitively, the optimal non-adaptive query procedure for a measurement dependent channel should have a lower resolution compared to its measurement independent counterpart. We verify this intuition for the asymmetric binary erasure and binary Z channels. However, for the binary symmetric channel, we find that the achievable resolution of optimal non-adaptive query procedures for the measurement independent channel can in fact achieve a better resolution when the crossover probability is large. We provide plausible explanations for this counter-intuitive phenomenon.

II. Problem Formulation

Notation

Random variables and their realizations are denoted by upper case variables (e.g., $X$) and lower case variables (e.g., $x$), respectively. All sets are denoted in calligraphic font (e.g., $\mathcal{X}$). Let $X^n := (X_1, \ldots, X_n)$ be a random vector of length $n$. We use $\Phi^{-1}(\cdot)$ to denote the inverse of the cumulative distribution function (cdf) of the standard Gaussian. We use $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ to denote the sets of real numbers, non-negative real numbers and integers respectively. Given any two integers $(a, b) \in \mathbb{N}^2$, we use $[a : b]$ to denote the set of integers $\{a, a+1, \ldots, b\}$ and use $[a]$ to denote $[1 : a]$. The set of all probability distributions on a finite set $\mathcal{X}$ is denoted as $\mathcal{P}(\mathcal{X})$ and the set of all conditional probability distributions from $\mathcal{X}$ to $\mathcal{Y}$ is denoted as $\mathcal{P}(\mathcal{Y}|\mathcal{X})$. Furthermore, we use $\mathcal{F}(\mathcal{S})$ to denote the set of all probability distribution functions on a continuous set $\mathcal{S}$. All logarithms are base $e$ unless otherwise noted. Finally, we use $\mathbb{I}(\cdot)$ to denote the indicator function.

A. Noisy 20 Questions Problem

Let $S$ be a continuous random variable defined on the unit interval $S = [0, 1]$ with arbitrary probability density function (pdf) $f_S \in \mathcal{F}(S)$. In the noisy 20 questions problem, a player aims to accurately estimate the value of the target random variable $S$ by posing a sequence of queries $\mathcal{A}^n = (A_1, \ldots, A_n) \subseteq [0, 1]^n$ to an oracle knowing $S$. After receiving the queries, the oracle finds binary answers $\{X_i = \mathbb{1}(S \in A_i)\}_{i \in [n]}$ and passes these answers through a measurement dependent channel with transition matrix $P_{Y^n|X^n}^{A^n} \in \mathcal{P}(\mathcal{Y}^n|[0, 1]^n)$ yielding noisy responses $Y^n = (Y_1, \ldots, Y_n)$. Given the noisy responses $Y^n$, the player uses a decoding function $g: \mathcal{Y}^n \rightarrow S$ to obtain an estimate $\hat{S}$ of the target variable $S$. Throughout the paper, we assume that the alphabet $\mathcal{Y}$ for the noisy response is finite.

A query procedure for the noisy 20 questions problem consists of the queries $\mathcal{A}^n \subseteq [0, 1]^n$ and the decoder $g: \mathcal{Y}^n \rightarrow S$. In general, these procedures can be classified into two categories: non-adaptive and adaptive querying. In a non-adaptive query procedure, the player needs to first determine the number of queries $n$ and then design all the queries $\mathcal{A}^n$ simultaneously. In contrast, in an adaptive query procedure, the design of queries is done sequentially and the number of queries is a variable. In particular, when designing the $i$-th query, the player can use the previous queries and the noisy responses from the oracle to these queries, i.e., $\{X_j, Y_j\}_{j \in [i-1]}$, to formulate the next query $A_i$. Furthermore, the player needs to choose a stopping criterion, which may be random, determining the number of queries to make.

We illustrate the difference between non-adaptive and adaptive query procedures in Figure [1]. In subsequent sections, we clarify the notion of the measurement dependent channel with concrete examples and present specific definitions of non-adaptive and adaptive query procedures.

B. The Measurement Dependent Channel

In this subsection, we describe succinctly the measurement dependent channel scenario [8], also known as a channel with state [15] Chapter 7]. Given a sequence of queries $\mathcal{A}^n \subseteq [0, 1]^n$, the channel from the oracle to the player is a memoryless channel whose transition probabilities are functions of the queries. Specifically, for any $(x^n, y^n) \in \{0, 1\}^n \times \mathcal{Y}^n$,

$$P_{Y^n|X^n}^{A^n}(y^n|x^n) = \prod_{i \in [n]} P_{Y|X}^{A_i}(y_i|x_i),$$

(1)

where $P_{Y|X}^{A_i}$ denotes the transition probability of the channel which depends on the $i$-th query $A_i$. Given any query $\mathcal{A} \subseteq [0, 1]$, define the size $|\mathcal{A}|$ of $\mathcal{A}$ as its Lebesgue measure, i.e., $|\mathcal{A}| = \int_{t \in \mathcal{A}} dt$. Throughout the paper, we assume that the measurement dependent channel $P_{Y|X}^{\mathcal{A}}$ depends on the query $\mathcal{A}$ only through its size, i.e., $P_{Y|X}^{\mathcal{A}}$ is equivalent to a channel with state $P_{Y|X}^q$, where the state $q = |\mathcal{A}| \in [0, 1]$. 

Given any $\eta \in \{0, \min(q, 1-q)\}$ and any subsets $\mathcal{A}$, $\mathcal{A}^+$ and $\mathcal{A}^-$ of $[0,1]$ with sizes $|\mathcal{A}| = q$, $|\mathcal{A}^+| = q + \eta$ and $|\mathcal{A}^-| = q - \eta$, we assume the measurement dependent channel is continuous in the sense that there exists a constant $c(q)$ depending on $q$ only such that

$$\max \left\{ \left\| \log \frac{P_{Y|X}^A}{P_{Y|X}^\mathcal{A}} \right\|_\infty, \left\| \log \frac{P_{Y|X}^{A^+}}{P_{Y|X}^\mathcal{A}} \right\|_\infty \right\} \leq c(q)\eta. \quad (2)$$

Some examples of measurement dependent channels satisfying the continuous constraint in (2) are as follows.

**Definition 1.** Given any $A \subseteq [0,1]$, a channel $P_{Y|X}^A$ is said to be a measurement dependent Binary Symmetric Channel (BSC) with parameter $\nu \in [0,1]$ if $X = \mathcal{Y} = \{0,1\}$ and

$$P_{Y|X}^A(y|x) = (\nu|A|)^{1(y \neq x)}(1 - \nu|A|)^{1(y = x)}, \quad \forall \ (x, y) \in \{0,1\}^2. \quad (3)$$

This definition generalizes [8] Theorem 1], where the authors considered a measurement dependent BSC with parameter $\nu = 1$.

**Definition 2.** Given any $A \subseteq [0,1]$, a measurement dependent channel $P_{Y|X}^A$ is said to be a measurement dependent Binary Erasure Channel (BEC) with parameter $\tau \in [0,1]$ if $X = \mathcal{Y} = \{0,1\}$ and $\mathcal{Y} = \{0,1,\epsilon\}$ and

$$P_{Y|X}^A(y|x) = \left( \frac{1 - \tau|A|}{2} \right)^{1(y \neq \epsilon)}(\tau|A|)^{1(y = \epsilon)} \quad (4)$$

**Definition 3.** Given any $A \subseteq [0,1]$, a measurement dependent channel $P_{Y|X}^A$ is said to be a measurement dependent Z-channel with parameter $\zeta \in [0,1]$ if $X = \mathcal{Y} = \{0,1\}$ and

$$P_{Y|X}^A(y|x) = (1 - \zeta|A|)^{1(y = x = 1)}(\zeta|A|)^{1(y = 0, x = 1)}. \quad (5)$$

Each of these measurement dependent channels will be considered in the sequel.

### C. Non-Adaptive Query Procedures

A non-adaptive query procedure with resolution $\delta$ and excess-resolution constraint $\epsilon$ is defined as follows.

**Definition 4.** Given any $n \in \mathbb{N}$, $\delta \in \mathbb{R}_+$ and $\epsilon \in [0,1]$, an $(n, \delta, \epsilon)$-non-adaptive query procedure for the noisy 20 questions problem consists of

- $n$ independent queries $(\mathcal{A}_1, \ldots, \mathcal{A}_n) \subseteq [0,1]^n$,
- and a decoder $g : \mathcal{Y}^n \to \mathcal{S}$

such that the excess-resolution probability satisfies

$$P_\epsilon(n, \delta) := \sup_{f_\delta \in \mathcal{F}(\mathcal{S})} \Pr\{|g(Y^n) - S| \geq \delta\} \leq \epsilon. \quad (6)$$
We remark that the definition of the excess-resolution probability with respect to $\delta$ is inspired by rate-distortion theory [16, 17]. Our formulation differs slightly from that of [8] where the authors constrained the $s$-dependent maximum excess-resolution probability, where $s \in S$ is the target variable.

Motivated by practical applications where the number of queries are limited (e.g., due to the high cost of queries and low-delay requirement), we are interested in the following non-asymptotic fundamental limit on achievable resolution $\delta^*$:

$$\delta^*(n, \varepsilon) := \inf \{ \delta : \exists (n, \delta, \varepsilon) - \text{non-adaptive - procedure} \}. \quad (7)$$

Note that $\delta^*(n, \varepsilon)$ denotes the minimal resolution one can achieve with probability at least $1 - \varepsilon$ using a non-adaptive query procedure with $n$ queries. In other words, $\delta^*(n, \varepsilon)$ is the achievable resolution of optimal non-adaptive query procedures tolerating an excess-resolution probability of $\varepsilon \in [0, 1]$. Dual to (7) is the sample complexity, determined by the minimal number of queries required to achieve a resolution $\delta$ with probability at least $1 - \varepsilon$, i.e.,

$$n^*(\delta, \varepsilon) := \inf \{ n : \exists (n, \delta, \varepsilon) - \text{non-adaptive - procedure} \}. \quad (8)$$

One can easily verify that for any $(\delta, \varepsilon) \in \mathbb{R}_+ \times [0, 1]$,

$$n^*(\delta, \varepsilon) = \inf \{ n : \delta^*(n, \varepsilon) \leq \delta \}. \quad (9)$$

Thus, in this paper, we only focus on the fundamental limit $\delta^*(n, \varepsilon)$.

### D. Adaptive Query Procedures

An adaptive query procedure with resolution $\delta$ and excess-resolution constraint $\varepsilon$ is defined as follows.

**Definition 5.** Given any $(l, \delta, \varepsilon) \in \mathbb{R}_+^3 \times [0, 1]$, an $(l, M, \varepsilon)$-adaptive query procedure for the noisy 20 questions problem consists of

- a sequence of adaptive queries where for each $i \in \mathbb{N}$, the design of query $A_i$ is based on all previous queries $\{A_j\}_{j \in [i-1]}$ and the noisy responses $Y^{i-1}$ from the oracle
- a sequence of decoding functions $g_i : \mathcal{Y}^i \to S$ for $i \in \mathbb{N}$
- a random stopping time $\tau$ depending on noisy responses $\{Y_i\}_{i \in \mathbb{N}}$ such that under any pdf $f_S$ of the target random variable $S$, the average number of queries satisfies

$$\mathbb{E}[\tau] \leq l, \quad (10)$$

such that the excess-resolution probability satisfies

$$P_{e,a}(l, \delta) := \sup_{f_S \in \mathcal{F}(S)} \Pr\{|g(Y^\tau) - S| > \delta\} \leq \varepsilon. \quad (11)$$

Similar to (7), given any $(l, \varepsilon) \in \mathbb{R}_+ \times [0, 1]$, we can define the fundamental resolution limit for adaptive querying as follows:

$$\delta^a(l, \varepsilon) := \inf \{ \delta \in \mathbb{R}_+ : \exists (l, \delta, \varepsilon) - \text{adaptive query procedure} \}, \quad (12)$$

with analogous definition of mean sample complexity (cf. [9])

$$l^*(\delta, \varepsilon) := \inf \{ l \in \mathbb{R}_+ : \exists (l, \delta, \varepsilon) - \text{adaptive query procedure} \}. \quad (13)$$

### III. Non-Adaptive Query Procedures

#### A. Non-Asymptotic Bounds

We first present an upper bound on the error probability of optimal non-adaptive query procedures. Given any $(p, q) \in [0, 1]^2$, let $P_{\mathcal{Y}|\mathcal{X}}$ be the marginal distribution on $\mathcal{Y}$ induced by the Bernoulli distribution $P_{\mathcal{X}} = \text{Bern}(p)$ and the measurement dependent channel $P_{\mathcal{Y}|\mathcal{X}}$. Furthermore, define the following information density

$$\iota_{p,q}(x, y) := \log \frac{P_{\mathcal{Y}|\mathcal{X}}(y|x)}{P_{\mathcal{Y}}(y)}, \quad \forall (x, y) \in \mathcal{X} \times \mathcal{Y}. \quad (14)$$

Correspondingly, for any $(x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n$, we define

$$\iota_{p}(x^n, y^n) := \sum_{i \in [n]} \iota_{p,p}(x_i, y_i) \quad (15)$$

as the empirical mutual information between $x^n$ and $y^n$. 

Theorem 1. Given any \((n,M) \in \mathbb{N}^2\), for any \(p \in [0,1]\) and any \(\eta \in \mathbb{R}_+\), there exists an \((n,\frac{1}{M},\varepsilon)\)-non-adaptive query procedure such that
\[
\varepsilon \leq 4n \exp(-2M\eta^2) + \exp(n\eta(p)) \mathbb{E}[\min\{1, M \Pr\{i_p(X^n; Y^n) \geq i_p(X^n; Y^n) | X^n, Y^n\}\}],
\]
where \((X^n, \hat{X}^n, Y^n)\) is distributed as \(P_X^n(X^n)P_{\hat{X}|X}^n(\hat{X}^n)(P_{Y|X}^n(Y^n|X^n))\) with \(P_X\) defined as the Bernoulli distribution with parameter \(p\) (i.e., \(P_X(1) = p\)).

The proof of Theorem 1 is given in Appendix A. When one considers the measurement independent channel, it is straightforward to verify that for any \(p \in [0,1]\), there exists an \((n,\frac{1}{M},\varepsilon)\)-non-adaptive query procedure such that
\[
\varepsilon \leq \mathbb{E}[\min\{1, M \Pr\{i_p(X^n; Y^n) \geq i_p(X^n; Y^n) | X^n, Y^n\}\}],
\]
where \((X^n, \hat{X}^n, Y^n)\) is distributed as \(P_X^n(X^n)P_{\hat{X}|X}^n(\hat{X}^n)(P_{Y|X}^n(Y^n|X^n))\) and \(P_{Y|X}\) is the measurement independent channel. Comparing the measurement dependent case (17) with the measurement dependent case (16), the non-asymptotic upper bound (16) in Theorem 1 differs from (17) in two aspects: an additional additive term and an additional multiplicative term \(\exp(n\eta(p))\). As is made clear in the proof of Theorem 1, the additive term \(4n \exp(-2M\eta^2)\) results from the atypicality of the measurement dependent channel and the multiplicative term \(\exp(n\eta(p))\) appears due to the change-of-measure we use to replace the measurement dependent channel \(P_{Y|X}^n\), with the measurement independent channel \(P_{Y|X}^n\).

We next provide a non-asymptotic converse bound to complement Theorem 1. For simplicity, for any query \(A \subseteq [0,1]\) and any \((x,y) \in X \times Y\), we use \(i_A(x,y)\) to denote \(i_{A[n]}(x,y)\).

Theorem 2. Set \((n,\delta,\varepsilon) \in \mathbb{N} \times \mathbb{R}_+ \times [0,1]\). Any \((n,\delta,\varepsilon)\)-non-adaptive query procedure satisfies the following. For any \(\beta \in (0,\frac{1}{\varepsilon})\) and any \(\kappa \in (0,1-\varepsilon-2\beta)\),
\[
-\log \delta \leq -\log \beta - \log \kappa + \sup_{A^n \in [0,1]^n} \sup_{t \in [n]} \{t \Pr\{\sum_{i \in [n]} i_A(X_i; Y_i) \leq t\} \leq \varepsilon + 2\beta + \kappa\}.
\]

The proof of Theorem 2 is given in Appendix A. The proof of Theorem 2 is decomposed into two steps: i) we use the result in [8] which states that the excess-resolution probability of any non-adaptive query procedure can be lower bounded by the error probability associated with channel coding over the measurement dependent channel with universal message distribution, minus a certain term depending on \(\beta\); and ii) we apply the non-asymptotic converse bound for channel coding [13, Proposition 4.4] by realizing that, given a sequence of queries, the measurement dependent channel is simply a time varying channel with deterministic states at each time point.

We remark that the non-asymptotic bounds in Theorems 1 and 2 hold for any number of queries and any measurement dependent channels satisfying (2). As we shall see in the next subsection, these non-asymptotic bounds lead to the second-order asymptotic result, which provides a tight approximation to the finite blocklength fundamental limit \(\delta^*(n,\varepsilon)\).

B. Second-Order Asymptotic Approximation

In this subsection, we present the second-order asymptotic approximation to the achievable resolution \(\delta^*(n,\varepsilon)\) of optimal non-adaptive query procedures after \(n\) queries subject to a worst case excess-resolution probability of \(\varepsilon \in [0,1]\).

Given measurement dependent channels \(\{P_{Y|X}^q\}_{q \in [0,1]}\), the channel “capacity” is defined as
\[
C := \max_{q \in [0,1]} \mathbb{E}[i_{q,q}(X; Y)],
\]
where \((X,Y) \sim \text{Bern}(q) \times P_{Y|X}^q\).

Let the capacity-achieving set \(\mathcal{P}_{ca}\) be the set of optimizers achieving (19). Then, for any \(\varepsilon \in [0,1]\), define the following “dispersion” of the measurement dependent channel
\[
V_{\varepsilon} := \begin{cases} \min_{q \in \mathcal{P}_{ca}} \text{Var}[i_{q,q}(X; Y)] & \text{if } \varepsilon < 0.5, \\ \max_{q \in \mathcal{P}_{ca}} \text{Var}[i_{q,q}(X; Y)] & \text{if } \varepsilon > 0.5. \end{cases}
\]

The case of \(\varepsilon < 0.5\) will be the focus of the sequel of this paper. We assume that the channel capacity \(C < \infty\) and also assume that for any \(q \in \mathcal{P}_{ca}\), the third absolute moment of \(i_{q,q}(X; Y)\) is finite. Under these assumptions, we obtain the second-order asymptotic result.

Theorem 3. For any \(n \in \mathbb{N}\) and \(\varepsilon \in (0,1)\), the achievable resolution \(\delta^*(n,\varepsilon)\) of optimal non-adaptive query procedures satisfies that
\[
-\log \delta^*(n,\varepsilon) = nC + \sqrt{nV_{\varepsilon}} \Phi^{-1}(\varepsilon) + O(\log n),
\]
where the term \(O(\log n)\) is lower bounded by \(-\frac{1}{2} \log n\) and upper bounded by \(\log n + O(1)\).
Fig. 2. Illustration of the phase transition of non-adaptive query procedures for the noisy 20 questions problem with measurement dependent noise, using the example of a measurement dependent BSC with parameter $\nu = 0.2$. On the one hand, when the resolution decay rate is strictly greater than the capacity $C$, then as the number of the queries $n \to \infty$, the excess-resolution probability tends to one. On the other hand, when the resolution decay rate is strictly less than the capacity $C$, then the excess-resolution probability vanishes as the number of the queries increases.

The proof of Theorem 3 is provided in Appendix C. We make the following remarks.

First, Theorem 3 implies a phase transition in machine learning [18], [19], which we interpret in Figure 2. As a corollary of Theorem 3, for any $\varepsilon \in (0, 1)$,

$$\lim_{n \to \infty} -\frac{1}{n} \log \delta^*(n, \varepsilon) = C.$$  \hspace{1cm} (22)

which takes the form of a strong converse [20]–[22] to the channel coding theorem. The result in (22) indicates that tolerating a smaller, or even vanishing excess-resolution probability, does not improve the asymptotic achievable resolution decay rate of optimal non-adaptive query procedures.

Another remark is that Theorem 3 refines [8, Theorem 1] in several directions. First, Theorem 3 is a second-order asymptotic result which provides good approximation for the finite blocklength performance while [8, Theorem 1] only characterizes the asymptotic resolution decay rate with vanishing worst-case excess-resolution probability, i.e., $\lim_{\varepsilon \to 0} \lim_{n \to \infty} \delta^*(n, \varepsilon)$. Second, our results hold for any measurement dependent channel satisfying (2) while [8, Theorem 1] only considers a special case of the measurement dependent BSC with parameter $\nu = 1$.

Finally, we remark that any $s \in S = [0, 1]$ has the binary expansion

$$s = b_0 + \sum_{j \in \mathbb{N}} b_j 2^{-j},$$  \hspace{1cm} (23)

by letting

$$b_0 = 1(s = 1), \quad b_k = 1\left\{s - b_0 - \sum_{j \in [k-1]} b_j 2^{-j} \geq 2^{-k}\right\}, \quad \forall \ k \in \mathbb{N}.$$  \hspace{1cm} (24)

We can thus interpret the result in Theorem 3 as follows: using optimal non-adaptive query procedures, after $n$ queries, with probability of at least $1 - \varepsilon$, one can extract the first $\lfloor -\log_2 \delta^*(n, \varepsilon) \rfloor$ bits of the binary expansion of the target variable $S$.

C. Case of Measurement Dependent BSC

In the following, we specialize Theorem 3 to different measurement dependent channels. We first consider a measurement dependent BSC. Given any $\nu \in (0, 1]$ and any $q \in [0, 1]$, let $\beta(\nu, q) := q(1 - \nu q) + (1 - q)\nu q$. For any $(x, y) \in \{0, 1\}^2$, the information density of a measurement dependent BSC with parameter $\nu$ is

$$i_{q, \nu q}(x, y) = 1(x \neq y) \log(\nu q) + 1(x = y) \log(1 - \nu q) - 1(y = 1) \log(\beta(\nu, q)) - 1(y = 0) \log(1 - \beta(\nu, q)).$$  \hspace{1cm} (25)
We observe an interesting phenomenon. When \( \nu < h \) where \( h \) is the maximum value of \( C(\nu, q) \) over \( q \in [0, 1] \) is the capacity of the measurement dependent BSC with parameter \( \nu \) and the values of \( q \) achieving this maximum consists of the set of capacity-achieving parameters \( \mathcal{P}_{ca} \). For \( \nu = 0.2 \) and \( \nu = 0.5 \), \( \mathcal{P}_{ca} \) is singleton and for \( \nu = 1 \), \( \mathcal{P}_{ca} \) contains two elements.

Then the mean and variance of the information density are respectively

\[
C(\nu, q) := \mathbb{E}[i_{q,\nu}(X, Y)] = h_b(\beta(\nu, q)) - h_b(\nu q),
\]

\[
V(\nu, q) := \text{Var}[i_{q,\nu}(X, Y)] = \nu q(1 - \nu q) \left( \log \frac{\nu q}{1 - \nu q} \right)^2 + \beta(\nu, q)(1 - \beta(\nu, q)) \left( \log \frac{\beta(\nu, q)}{1 - \beta(\nu, q)} \right)^2 - \beta(\nu, q)(1 - 2\nu q) \log \frac{\nu q}{1 - \nu q} \log \frac{\beta(\nu, q)}{1 - \beta(\nu, q)},
\]

where \( h_b(p) = -p \log(p) - (1 - p) \log(1 - p) \) is the binary entropy function. The capacity of the measurement dependent BSC with parameter \( \nu \) is thus

\[
C(\nu) = \max_{q \in [0, 1]} C(\nu, q),
\]

Depending on the value of \( \nu \in (0, 1] \), the set of capacity-achieving parameters \( \mathcal{P}_{ca} \) may or may not be singleton (cf. Figure 3). In particular, for any \( \nu \in (0, 1) \), the capacity-achieving parameter \( q^* \) is unique. When \( \nu = 1 \), there are two capacity-achieving parameters \( q_1^* \) and \( q_2^* \) where \( q_1^* + q_2^* = 1 \). It can be verified easily that \( V(1, q_1^*) = V(1, 1 - q_1^*) \). As a result, for any capacity-achieving parameter \( q^* \) of the measurement dependent BSC with parameter \( \nu \in (0, 1] \), the dispersion of the channel is

\[
V(\nu) = V(\nu, q^*).
\]

**Corollary 4.** Set any \( \nu \in (0, 1) \). If the channel from the oracle to the player is a measurement dependent BSC with parameter \( \nu \), then Theorem 3 holds with \( C = C(\nu) \) and \( V_\varepsilon = V(\nu) \) for any \( \varepsilon \in (0, 1) \).

We make the following observations.

First, if we let \( \nu = 1 \) and take \( n \to \infty \), then for any \( \varepsilon \in (0, 1) \),

\[
\lim_{n \to \infty} \frac{-\log^* (n, \varepsilon)}{n} = \max_{q \in [0, 1]} \left( h_b(\beta(1, q)) - h_b(q) \right).
\]

This is a strengthened version of [8] Theorem 1) with strong converse.

Second, when one considers the measurement independent BSC with parameter \( \nu \in (0, 1) \), then it can be shown that the achievable resolution \( \delta^*_m(n, \varepsilon) \) of optimal non-adaptive query procedures satisfies

\[
-\log_2 \delta^*_m(n, \varepsilon) = n(1 - h_b(\nu)) + \sqrt{n \nu (1 - \nu)} \log_2 \frac{1 - \nu}{\nu} \Phi^{-1}(\varepsilon) + O(\log n).
\]

To compare the performances of optimal non-adaptive query procedures under measurement dependent and measurement independent channels, we plot in Figure 4 the number of bits in the binary expansion of the target random variable \( S \) extracted after \( n \) queries, i.e., \( -\log_2 \delta^*(n, \varepsilon) \) and \( -\log_2 \delta^*_m(n, \varepsilon) \) for \( \varepsilon = 0.001 \) and different values of \( \nu \) (the \( O(\log n) \) term is ignored). We observe an interesting phenomenon. When \( \nu < 0.5 \), the performance of optimal query procedures under a measurement
We can then compare the performances of optimal query procedures for the measurement dependent and independent cases. For \( \tau \) where the second equality follows since for any \( q \),

\[
\mathbb{E}[q_{\tau,q}(x)] = \mathbb{E}[\mathbb{I}(y = x) \log(1 - q\tau) - \mathbb{I}(y = 1) \log(q(1 - q\tau)) - \mathbb{I}(y = 0) \log((1 - q)(1 - q\tau))].
\]  

(32)

Thus, the mean and variance of the information density are respectively

\[
C(\tau, q) := \mathbb{E}[q_{\tau,q}(X; Y)] = (1 - q\tau) h_b(q),
\]

\[
V(\tau, q) := \text{Var}[q_{\tau,q}(X; Y)] = (1 - q\tau) \left( h_b(q) \log(1 - q\tau) + q \log q \log(q(1 - q\tau)) \right) + (1 - q\tau)^2 h_b(q)^2.
\]  

(34)

The capacity of the measurement dependent BEC with parameter \( \tau \) is then given by

\[
C(\tau) = \max_{q \in [0,1]} C(\tau, q) = \max_{q \in [0,0.5]} (1 - q\tau) h_b(q),
\]  

(35)

where the second equality follows since for any \( \tau \), \( C(\tau, q) \) is decreasing in \( q \in [0.5, 1] \). We plot \( C(\tau, q) \) for different values of \( \tau \) in Figure 5. It can be verified that the capacity-achieving parameter for the measurement dependent BEC is unique and we denote it by \( q^* \). Thus, the dispersion of the channel is \( V(\tau, q^*) \).

Then we have the following result.

**Corollary 5.** Set any \( \tau \in [0, 1] \). If the channel from the oracle to the player is a measurement dependent BEC with parameter \( \tau \), then Theorem 3 holds with \( C = C(\tau) \) and \( V_\varepsilon = V(\tau, q^*) \) for any \( \varepsilon \in [0, 1] \).

The remarks for Corollary 4 concerning strong converse apply also to Corollary 5. We make several additional remarks. First, if one considers a measurement independent BEC with parameter \( \tau \in [0, 1] \), then the achievable resolution \( \delta^*_m(n, \varepsilon) \) of optimal non-adaptive query procedures satisfies

\[
-\log_2 \delta^*_m(n, \varepsilon) = n(1 - \tau) + \sqrt{n \tau(1 - \tau) \Phi^{-1}(\varepsilon)} + O(\log n).
\]  

(36)

We can then compare the performances of optimal query procedures for the measurement dependent and independent cases. For this purposes, in Figure 6, we plot \( -\log_2 \delta^*(n, \varepsilon) \) and \( -\log_2 \delta^*_m(n, \varepsilon) \) for \( \varepsilon = 0.001 \) and various values of \( \nu \) where again we
Fig. 5. Plot of $C(\tau, q)$, the mean of the information density, for the measurement dependent BEC with parameter $\tau$ for $q \in [0, 1]$. For each given $\tau$, the maximum value of $C(\tau, q)$ over $q \in [0, 1]$ is the capacity of the measurement dependent BEC. Note that the capacity-achieving parameter for measurement dependent BEC is unique for any $\tau \in (0, 1]$.

Fig. 6. Plot of the number of bits in the binary expansion of the target variable $S$ (cf. (23)) extracted by optimal non-adaptive query procedures for both measurement dependent ($-\log_2 \delta^*(n, \varepsilon)$) and measurement independent ($-\log_2 \delta^\text{mi}(n, \varepsilon)$) versions of BEC with parameter $\tau$.

Ignore the $O(\log n)$ term. As can be observed in Figure 6 for the considered cases, the performance of optimal querying under the measurement dependent channel is better than the counterpart in the measurement independent channel. The intuition is that in the measurement dependent channel, the probability of erasure is usually smaller than the probability of erasure in measurement independent channel. In particular, when $\tau = 1$, the measurement independent channel is daunted by noise and thus no useful information can be obtained from the measurement independent channel output. In contrast, in the measurement dependent case, optimal non-adaptive querying can still accurately extract a significant number of bits in the binary expansion of the target variable.

Second, if $\tau = 0$, i.e., a noiseless channel, we have that the achievable resolution of optimal non-adaptive query procedures satisfies

$$-\log_2 \delta^*(n, \varepsilon) = n + O(\log n).$$

Note that, interestingly, for the noiseless 20 questions problem, the achievable resolution of optimal non-adaptive querying has nothing to do with the target excess-resolution probability $\varepsilon \in [0, 1)$ for any number of queries $n$ (in contrast to the noisy 20 questions problem where the similar phenomenon occurs only when $n \to \infty$, c.f. (22)). This result implies that for the noiseless 20 questions problem, for any number of the queries $n \in \mathbb{N}$, the achievable resolution of optimal non-adaptive query procedures cannot be improved even if one tolerates a large excess-resolution probability $\varepsilon$. 
Thus, the mean and the variance of the information density are respectively

\[
C(\zeta,q) = \frac{\log(1 - q + \zeta)}{1 - q + \zeta}q^2 + \frac{\zeta}{1 - q + \zeta}q^2 + \frac{1}{1 - q + \zeta}q^2.
\]

Therefore, the capacity of the measurement dependent Z-channel with parameter \(\zeta\) is

\[
C(\zeta) = \max_{q \in [0,1]} C(\zeta,q).
\]

We plot \(C(\zeta,q)\) for different values of \(\zeta\) and \(q \in [0,1]\) in Figure 7. It can be verified that the capacity achievable parameter for the measurement dependent Z-channel is unique and we denote the optimizer as \(q^*\). Therefore, the dispersion of Z-channel is \(V(\zeta,q^*)\). Thus our second-order asymptotic result specializes to Z-channel as follows.

**Corollary 6.** Set any \(\zeta \in (0,1)\). If the channel from the oracle to the player is a measurement dependent Z-channel with parameter \(\zeta\), then Theorem 3 holds with \(C = C(\zeta,q^*)\) and \(V_\epsilon = V(\zeta,q^*)\) for any \(\epsilon \in (0,1)\).

When one considers a measurement independent Z-channel with parameter \(\zeta\), it can be easily shown that the achievable resolution \(\delta_{mi}^*(n,\epsilon)\) of optimal non-adaptive query procedures satisfies that

\[
-\log \delta_{mi}^*(n,\epsilon) = nC_{mi}(\zeta) + \sqrt{nV_{mi}(\zeta)}\Phi^{-1}(\epsilon) + O(\log n),
\]

where \(C_{mi}(\zeta)\) and \(V_{mi}(\zeta)\) are the capacity and dispersion of the Z-channel, i.e.,

\[
C_{mi}(\zeta) = \sup_{q \in [0,1]} h_b(q(1 - \zeta)) - qh_b(\zeta),
\]

\[
V_{mi}(\zeta) = q_{mi}^*(1 - q_{mi}^*) \left( \log(1 - q_{mi}^* + \zeta q_{mi}^*) \right)^2 + \zeta q_{mi}^*(1 - q_{mi}^*) \left( \log \frac{\zeta}{1 - q_{mi}^* + \zeta q_{mi}^*} \right)^2
\]

\[
+ (q_{mi}^* - \zeta q_{mi}^*)(1 - q_{mi}^* + \zeta q_{mi}^*) \left( \log \frac{1 - \zeta}{q_{mi}^*} \right),
\]

with \(q_{mi}^* \in [0,1]\) being the unique optimizer of \(C_{mi}(\zeta)\).

To compare the performances of optimal non-adaptive querying for the measurement dependent and measurement independent Z-channels with different parameters, in Figure 8 we plot \(-\log \delta_{mi}^*(n,\epsilon)\) and \(-\log \delta_{mi}^*(n,\epsilon)\) for \(\epsilon = 0.001\) and different values.
of $\zeta$. Once again, we find that optimal non-adaptive query procedures for the measurement dependent channel outperforms the counterpart for the measurement independent channel.

IV. ADAPTIVE QUERY PROCEDURES

In this section, we present fundamental limits of adaptive query procedures. We first present a non-asymptotic achievability bound and then derive an achievable second-order asymptotic result, which provides an approximation to the minimal achievable resolution of the adaptive query procedure in Appendix D.

To present our results, we need the following definitions. Let $X^\infty$ be a collection of $M$ random binary codewords $(X^\infty(1), \ldots, X^\infty(M))$, each with infinite length and let $x^\infty$ denote a realization of $X^\infty$. Furthermore, let $Y^\infty$ be another random vector with infinite length where each element takes values in $\mathcal{Y}$ and let $y^\infty$ be a realization of $Y^\infty$. Given any $p \in (0, 1)$, any $w \in [M]$ and any $n \in \mathbb{N}$, the joint distribution of $(X^n, Y^n)$ is

$$P_{X^n, Y^n}(x^n, y^n) = \prod_{i \in [n]} \left( \prod_{j \in [M]} \text{Bern}_p(x_j(i)) \right) P_{Y|X}^{\frac{1}{2}} \sum_{i \in [M]} x_j(i)(y_i|x_i(w)).$$ (45)

Finally, given any $\lambda \in \mathbb{R}_+$ and any $m \in [M]$, define the stopping time

$$\tau_m(x^\infty, y^\infty) := \inf\{n \in \mathbb{N} : t_q(x^n(m); y^n) \geq \lambda\}. \quad (46)$$

**Theorem 7.** Given any $(n, M) \in \mathbb{N}^2$, for any $p \in (0, 1)$ and $\lambda \in \mathbb{R}_+$, there exists an $(l, \frac{1}{M}, \varepsilon)$-adaptive query procedure such that

$$I \leq E[\tau_1(X^\infty, Y^\infty)], \quad (47)$$

$$\varepsilon \leq (M - 1) \Pr\{\tau_1(X^\infty, Y^\infty) \geq \tau_2(X^\infty, Y^\infty)\}. \quad (48)$$

The proof of Theorem 7 is similar to that of Theorem 1 and builds on [23], Theorem 3. For completeness, the proof of Theorem 7 is given in Appendix D.

Using Theorem 7, we obtain the following lower bound on the second-order asymptotic approximation to the achievable resolution of optimal adaptive querying. Recall the definition of the capacity $C$ of measurement dependent channels in (19).

**Theorem 8.** For any $(l, \varepsilon) \in \mathbb{R}_+ \times [0, 1)$, we have

$$- \log \delta^*_n(l, \varepsilon) \geq \frac{lC}{1 - \varepsilon} + O(\log l). \quad (49)$$

The proof of Theorem 8 is provided in Appendix E. We make several remarks.

First, to prove Theorem 8, we adapt the proof techniques in [23], [24] for the analyses of average delay and excess-resolution probability. Furthermore, we apply the change-of-measure technique to deal with the measurement dependent channel.

Second, it is interesting to see that the decay rate of the achievable resolution of the adaptive query procedure in Appendix D actually relies on the excess-resolution probability $\varepsilon \in [0, 1)$, which is in stark contrast with the case of non-adaptive query procedures in Theorem 3. This suggests that phase transition fails to hold for adaptive query procedures.
where \((n, \epsilon)\) with measurement dependent noise. For any Theorem 1, choosing the target resolution. The simulated and theoretical result appears since we have not accounted for the third-order term, which scales as \(O(\log n)\). For measurement independent channels satisfies

\[ G(n, \epsilon) = \log \delta^*(n, \epsilon) - \log \delta^*_n(n, \epsilon). \] (50)

Using Theorems 5 and 8, we have

\[ G(n, \epsilon) \geq \frac{nC\epsilon}{1 - \epsilon} - \sqrt{nV\epsilon}\Phi^{-1}(\epsilon) + O(\log n) =: G(n, \epsilon). \] (51)

Note that \(\Phi^{-1}(\epsilon) \leq 0\). To illustrate the adaptivity again, Figure 9, we plot \(G(n, \epsilon)\) for \(\epsilon = 0.001\) and three types of measurement dependent channels with various parameters. Note that adaptive query procedures gain over non-adaptive query procedures since for the former, one can make different number of queries with respect to different realization of the target variable, which is analogous to the reason why variable length code achieves better performance than fixed length code in channel coding [23, Theorem 1].

Another remark is that it is easy to verify that the achievable resolution \(\delta^*_{n, m}(l, \epsilon)\) of optimal adaptive query procedures for measurement independent channels satisfies

\[ -\log \delta^*_{n, m}(l, \epsilon) = \frac{lC_{mi}}{1 - \epsilon} + O(\log l), \] (52)

where \(C_{mi}\) is the capacity of the measurement independent channel. Using (49) and (52), one can then compare the performances of adaptive query procedures under measurement dependent and measurement independent channels, analogous to the non-adaptive cases. See Appendix 3 for the numerical results.

Finally, we comment on the challenge of deriving a matching converse result for optimal adaptive query procedures with measurement dependent channel models. As pointed in [8], under the measurement dependent channel, each noisy response \(Y_i\) depends not only on the target variable \(S\), but also on the previous queries \(A^{i-1}\) and noisy responses \(Y^{i-1}\). This strong dependency makes it difficult to directly relate the current problem to channel coding with feedback [25]. Indeed, under such a setting, the corresponding classical coding analogy is channel coding with feedback and state where the state has memory. Novel ideas and techniques are required for a converse proof for such a setting, which is left for future work.

V. NUMERICAL ILLUSTRATION

In this section, we numerically illustrate the minimal achievable resolution of both non-adaptive and adaptive query procedures over a measurement dependent BSC with parameter \(\nu = 0.4\). We consider the case where the target random variable \(S\) is uniformly distributed over the alphabet \(S = [0, 1]\) and set the target excess-resolution probability \(\epsilon = 0.1\).

In Figure 10 we plot the minimal achievable resolution of the non-adaptive query procedure for estimating \(S\), computed based on the random coding idea detailed in Appendix A, versus the theoretical value asserted in Corollary 4. The target resolution in the simulation is chosen to be the reciprocal of \(M\) such that

\[ \log M = nC(\nu) + \sqrt{nV(\nu)\Phi^{-1}(\epsilon)} - \frac{1}{2} \log n. \] (53)

Each point of the simulated result in Figure 10 is obtained by calculating the minimal achievable resolution associated with excess-resolution probability \(\epsilon\) as follows. For each number of queries \(n \in \{20, 30, \ldots, 70\}\), we estimate the minimal achievable resolution of the query procedure in Appendix A using \(10^4\) independent experiments. We believe the slight gap between simulated and theoretical result appears since we have not accounted for the third-order term, which scales as \(O(\log n)\), when choosing the target resolution.
Fig. 10. Minimal achievable resolution of non-adaptive query procedures for estimating a uniformly distributed target $S \in [0, 1]$. The red line corresponds to the theoretical value asserted in Corollary 4 where we neglect the $O(\log n)$ term and the blue dots denote the Monte Carlo Simulation of a non-adaptive query procedure based on random coding in Appendix A. The error bar for the simulated result denotes thirty standard deviations above and below the mean.

In Figure 11, we plot the minimal achievable resolution of the adaptive query procedure described in Appendix D versus the theoretical result in Theorem 8. Each point of the simulated result is obtained as follows. We use $\varepsilon = 0.1$ as a designed parameter. Given each $n \in \{20, 30, \ldots, 70\}$, we partition the unit interval into $M$ sub-intervals where $M$ satisfies

$$\log M = \frac{nC(\nu)}{1 - \varepsilon} - \log(n). \quad (54)$$

For each $n \in \{20, 30, \ldots, 70\}$, we run the adaptive query procedure in Appendix D independently for $10^4$ times and calculated the average stopping time $l_n$.

VI. CONCLUSION

We derived the minimal achievable resolution of non-adaptive query procedures for the noisy 20 questions problem where the channel from the oracle to the player is a measurement dependent discrete channel. Furthermore, we generalized our results to derive the achievable resolution of adaptive query procedures and discussed the intrinsic resolution gain due to adaptivity.

There are several avenues for future research. First, for adaptive query procedures, we only derived achievability results. It would be fruitful to apply novel techniques to derive a matching converse bound on the minimal achievable resolution of optimal adaptive query procedures. Second, we consider only discrete channel (finite output alphabet). It would be interesting to extend
We provide an illustration for this intuition in Figure 12. which is the index of the sub-interval where the target variable lies.

Hence, for each $i$, the $i$-th query is designed as follows:

$$A_i := \bigcup_{m \in [M]: x_i(m) = 1} S_m,$$

where $x_i(m)$ denotes the $i$-th element of the $m$-th codeword $x^n(m)$. By the above design, our $i$-th query to the oracle is whether the target variable $s$ lies in the union of sub-intervals with indices of the codewords whose $i$-th element are one. Hence, for each $i \in [n]$, the $i$-th element of each codeword can be understood as an indicator function for whether a particular sub-interval would be queried in $i$-th question, with one being positive and zero being negative.

Given any $s \in S$, using the query procedures in (56), the noiseless response from the oracle is $X^n = x^n(q(s))$, i.e., the $q(s)$-th codeword is the true answer to our $n$ queries. This observation was made in (58) and the intuition for this observation is quite straightforward. For each $i \in [n]$, given any $s \in S$ such that $q(s) = w \in [M]$, the noiseless response $X_i$ from the oracle is

$$X_i = 1(s \in A_i) = 1 \left( x \in \bigcup_{j \in [M]: x_i(j) = 1} S_j \right) = \begin{cases} 1 & \text{if } x_i(w) = 1, \\ 0 & \text{otherwise}. \end{cases}$$

We provide an illustration for this intuition in Figure 12.

The noiseless response $X^n$ from the oracle is corrupted by the measurement dependent channel $P_{Y^n|X^n}$ to produce a noisy response $Y^n$. Given $Y^n$, the player uses a two-step decoder to produce estimate $\hat{S}$ of the target variable $s$ as follows:

(i) the player first estimates $q(s)$ as $\hat{W}$ using a maximum mutual information density estimator, i.e.,

$$\hat{W} = \max_{w \in [M]} \mathbb{I}_p(x^n(w); Y^n).$$

(ii) the player then chooses the mean value in $\hat{W}$-th sub-interval as the final estimate, i.e.,

$$\hat{S} = g(Y^n) = \frac{\phi(Y^n)}{2M} = \frac{2\hat{W} - 1}{2M}.$$
is correct, i.e., \( \hat{W} = q(s) = w \). Since each pdf \( f_S \in \mathcal{F}(S) \) of the target random variable \( S \) induces a pmf \( P_W \in \mathcal{P}([M]) \) of the quantized random variable \( W \), for the above-mentioned non-adaptive query procedure, we have

\[
\sup_{f_S \in \mathcal{F}(S)} \Pr \left\{ |g(Y^n) - S| > \frac{1}{M} \right\} \leq \sup_{f_S \in \mathcal{F}(S)} \Pr \{ \hat{W} \neq q(S) \} \leq \sup_{P_W \in \mathcal{P}([M])} \Pr \{ \hat{W} \neq W \} \leq \sup_{P_W \in \mathcal{P}([M])} \sum_{w} P_W(w) \Pr \{ \exists \tilde{w} \in [M] : \tilde{w} \neq w, \ i_p(x^n(\tilde{w});Y^n) \geq \iota_p(x^n(w);Y^n) \} =: \sup_{P_W \in \mathcal{P}([M])} P_e(x,P_W), \tag{60}
\]

where the probability in \( \tag{60} \) is calculated according to the measurement dependent channel

\[
P_{Y^n|X^n}^{A}(y^n|x^n(w)) = \prod_{i \in [n]} P_{Y|X}^{A_i}(y_i|x_i(w)) = \prod_{i \in [n]} P_{Y|X}^{P}(y_i|x_i(w)), \tag{64}
\]

and \( \tag{65} \) follows from the design of queries in \( \tag{56} \) and in \( \tag{65} \), we define

\[
q_i(x) = \frac{1}{M} \sum_{j \in [M]} x_i(j). \tag{66}
\]

Note that \( P_e(x,P_W) \) is essentially the error probability of transmitting a message \( W \in [M] \) with pmf \( P_W \) over the measurement dependent channel \( P_{Y^n|X^n}^{A} \). Thus, to further bound \( P_e(x,P_W) \), we need to analyze the error probability of a channel coding problem over a codebook dependent channel where the channel output \( Y^n \) depends on all codewords \( x^n(1), \ldots, x^n(M) \). In contrast, in the classical channel coding problem, the channel output depends only on the channel input with respect to the message. However, as we shall see, using the change-of-measure technique and the assumption in \( \tag{2} \), with negligible loss in error probability, we can replace the measurement dependent channel with a memoryless channel \( (P_{Y|X})^n \).

For this purpose, we use random coding ideas \[27\]. Fix any Bernoulli distribution \( P_X \in \mathcal{P}(\{0,1\}) \) with parameter \( p \), i.e., \( P_X(1) = p \). Let \( X := X^n(1), \ldots, X^n(M) \) be \( M \) independent binary sequences, each generated i.i.d. according to \( P_X \). Furthermore, for any \( M \in \mathbb{N} \) and any \( p \in [0,1] \), given any \( \eta \in \mathbb{R}_+ \), define the following typical set of a collection of \( M \) binary codewords \( x = (x^n(1), \ldots, x^n(M)) \):

\[
T^n(M,p,\eta) := \left\{ x = (x^n(1), \ldots, x^n(M)) \in X^M : |q_i(x) - p| \leq \eta, \ \forall \ i \in [n] \right\}. \tag{67}
\]

For any \( x \in T^n(M,p,\eta) \), recalling the query procedure in \( \tag{56} \) and the condition in \( \tag{2} \), we have

\[
\log \left( \frac{P_{Y^n|X^n}^{A}(y^n|x^n)}{(P_{Y|X})^n(y^n|x^n)} \right) = \sum_{i \in [n]} \log \left( \frac{P_{Y|X}^{q_i(x)}(y_i|x_i)}{P_{Y|X}^{P}(y_i|x_i)} \right) \leq n\eta c(p). \tag{68}
\]

Note that given any \( w \in [M] \), the joint distribution of \( (X,Y^n) \) under the current query procedure is

\[
P_{X|Y^n}^{\text{mid},w}(x,y^n) = \left( \prod_{j \in [M]} P^n_X(x^n(j)) \right) \left( \prod_{i \in [n]} P_{Y|X}^{q_i(x)}(y_i|x_i(w)) \right). \tag{69}
\]

and furthermore, we need the following alternative joint distribution of \( (X,Y^n) \) to apply the change-of-measure idea

\[
P_{X|Y^n}^{\text{alt},w}(x,y^n) = \left( \prod_{j \in [M]} P^n_X(x^n(j)) \right) \left( \prod_{i \in [n]} P_{Y|X}^{P}(y_i|x_i(w)) \right). \tag{70}
\]

For any message distribution \( P_W \in \mathcal{P}([M]) \),

\[
\mathbb{E}_X[P_e(X,P_W)] \leq \Pr(X \notin T^n(M,P,\eta)) + \mathbb{E}_X[P_e(X,P_W)1(X \in T^n(M,p,\eta))] \leq 4n \exp(-2M\eta^2) + \mathbb{E}_X[P_e(X,P_W)1(X \in T^n(M,p,\eta))]. \tag{71}
\]

where \( \tag{72} \) follows from \( \tag{28} \) Lemma 22.
The second term in (72) can be further upper bounded as follows:

\[
\mathbb{E}_X [P_e(x, P_W) \mathbb{I}(X \in \mathcal{T}^n(M, p, \eta))] \\
= \sum_w P_W(w) \mathbb{E}_{P_{W^n|X^n}} [\mathbb{I}(X \in \mathcal{T}^n(M, p, \eta)) \mathbb{I}(\exists \bar{w} \in [M]: \bar{w} \neq w, \ t_p(X^n(\bar{w}); Y^n) \geq t_p(X^n(w); Y^n))]
\leq \exp(n \eta(p)) \sum_w P_W(w) \Pr \{\exists \bar{w} \in [M]: \bar{w} \neq w, \ t_p(X^n(\bar{w}); Y^n) \geq t_p(X^n(w); Y^n)\}
\leq \exp(n \eta(p)) \sum_w P_W(w) \sum_{\bar{w} \in [M]: \bar{w} \neq w} \Pr \{t_p(X^n(\bar{w}); Y^n) \geq t_p(X^n(w); Y^n)\}
= \exp(n \eta(p)) \mathbb{E}_{P_{WX^nY^n}} [\min \{1, M, \Pr \{t_p(X^n; Y^n) \geq t_p(X^n; Y^n)|X^n, Y^n\}\}]
\tag{76}
\]

where (73) follows from (68) and the change of measure technique, (75) follows from the union bound, (76) follows by noting that the codewords \(X^n(1), \ldots, X^n(M)\) are independent under \(P_{WX^nY^n}\) and applying the ideas leading to the random coding union bound (12) and furthermore

\[
P_{X^nY^n}(x^n, y^n) = \prod_{i \in [n]} P_X(x_i) P_{Y|X}(y_i|x_i).
\tag{77}
\]

Combining (73) and (76), we conclude that there exists a sequence of binary codewords \(x\) such that \(P_e(x, P_W)\) is upper bounded by the desired quantity for all message distributions \(P_W \in \mathcal{P}([M])\) and thus the proof is completed.

**B. Proof of Non-Asymptotic Converse Bound for Non-Adaptive Query Procedures (Theorem 2)**

We now focus on the converse proof. Consider any sequence of queries \(A^n \in \{0, 1\}^n\) and any decoding function \(g : \mathcal{Y}^n \rightarrow \mathcal{S}\) such that the worst-case excess-resolution probability with respect to \(\delta \in \mathbb{R}_+\) is upper bounded by \(\varepsilon \in (0, 1)\), i.e.

\[
\sup_{f_S} \Pr \{|g(Y^n) - S| > \delta\} \leq \varepsilon.
\tag{78}
\]

Let \(\beta \in \mathbb{R}_+\) be arbitrary such that \(\beta \leq \frac{1-\varepsilon}{2} \leq 0.5\) and let \(\bar{M} := \lceil \frac{\beta}{\varepsilon} \rceil\). Define the following quantization function

\[
q_{\beta}(s) := \lfloor s \bar{M} \rfloor, \quad \forall s \in \mathcal{S}.
\tag{79}
\]

In what follows, we consider uniformly distributed target variable \(S\) with pdf \(f_S^g\). Given queries \(A^n\), the noiseless response \(X^n\) is then a sequence of independent random variables where each \(X_i\) is a Bernoulli random variable with parameter equal to the size \(|A_i|\) of the query \(A_i\), i.e.,

\[
P_{X^n} = \prod_{i \in [n]} \text{Bern}(|A_i|).
\tag{80}
\]

The noisy response is the output of passing \(X^n\) into the measurement dependent channel \(P_{Y|X}^{A^n}\). Given \(Y^n\), the decoder \(g\) outputs estimate \(\hat{S} = g(Y^n)\) of the target variable \(S\). In subsequent analyses, for simplicity, let \(W := q_{\beta}(S)\) and \(\hat{W} := q_\beta(\hat{S})\). From the problem structure, we have the Markov chain \(W - S - X^n - Y^n - S - \hat{W}\). Furthermore, the joint distribution of \((W, S, X^n, Y^n, \hat{S}, \hat{W})\)

\[
P_{WSX^nY^nS\hat{W}}(w, s, x^n, y^n, \hat{s}, \hat{w}) = f_{\hat{S}}^g(s) \mathbb{I}(w = q_{\beta}(s)) \left( \prod_{i \in [n]} \mathbb{I}(x_i = \mathbb{I}(s \in A_i)) P_{Y|X}^{A_i}(y_i|x_i) \right) \mathbb{I}(\hat{s} = g(y^n), \hat{w} = q_{\beta}(\hat{s})).
\tag{81}
\]

In the following, unless stated otherwise, the probability of events is calculated according to the distribution in (81).

Inspired by (8), we have that

\[
\Pr\{\hat{W} \neq W\} = \Pr\{\hat{W} \neq W, |\hat{S} - S| > \delta\} + \Pr\{\hat{W} \neq W, |\hat{S} - S| \leq \delta\}
\leq \Pr\{|\hat{S} - S| > \delta\} + \Pr\{\hat{W} \neq W, |\hat{S} - S| \leq \delta\}
\leq \varepsilon + \Pr\{\hat{W} \neq W, |\hat{S} - S| \leq \delta\}
\leq \varepsilon + \frac{2\delta}{\bar{M}}
\leq \varepsilon + 2\beta \varepsilon^2
\]

where (84) follows from (78), (85) follows since i) only when \(S\) is within \(\delta\) to the boundaries (left and right) of the sub-interval with indices \(W = q_{\beta}(S)\) can the events \(\hat{W} \neq W\) and \(|S - \hat{S}| \leq \delta\) occur simultaneously, ii) \(S\) is uniformly distributed over \(S\)
where (92) follows from i) the change of measure technique which states that and ii) the result in [29, Eq. (37)].

To ease understanding of the critical step (85), we have provided a figure illustration in Figure 13.

Combining (78) and (86), we have that

$$\varepsilon \geq \Pr\{\hat{W} \neq W\} - 2\beta.$$

Note that given queries $A^n$, the probability $\Pr\{\hat{W} \neq W\}$ is the average error probability of channel coding with deterministic states when the distribution of the channel inputs is $P_{X^n}$ and the message $W$ is uniformly distributed over $[M]$. Therefore, we can use converse bounds for channel coding to bound achievable resolution $\delta$ (via $M$).

Similar as [13, Proposition 4.4], we have that for any $\kappa \in (0, 1 - \varepsilon - 2\beta)$,

$$\log M \leq \inf_{Q_{Y^n} \in \mathcal{P}(Y^n)} \sup_t \left\{ \Pr\left\{ \log \frac{P_{Y^n|X^n}A^n(Y^n|X^n)}{Q_{Y^n}} \leq t \right\} \leq \varepsilon + 2\beta + \kappa \right\} - \log \kappa \quad (88)$$

$$= \sup_t \left\{ \Pr\left\{ \sum_{i \in [n]} \log \frac{P_{Y^n|X^n}A^n(Y_i|X_i)}{P_{Y^n|X^n}A^n(Y_i)} \leq t \right\} \leq \varepsilon + 2\beta + \kappa \right\} - \log \kappa, \quad (89)$$

where (89) follows by choose $Q^n_{Y^n}$ being the marginal distribution of $Y^n$ induced distribution of $P_{X^n}A^n$ and the measurement dependent channel $P_{Y^n|X^n}A^n$. Since (89) holds for any sequence of queries and any decoder satisfying (78), recalling the definition of $\tilde{M}$ and the definition of $i_{A^n}(\cdot)$, we have

$$-\log \delta \leq -\log \beta - \log \kappa + \sup_{A^n} \Pr\left\{ M \leq \sum_{i \in [n]} i_{A^n}(X_i; Y_i) \leq \log M \right\} \leq \varepsilon + 2\beta + \kappa \quad (90)$$

C. Proof of Second-Order Asymptotics of Non-Adaptive Query Procedures (Theorem 3)

1) Achievability Proof: Invoking Theorem 1 with the capacity-achieving parameter $q \in \mathcal{P}_{ca}$, we have that for any $\eta \in \mathbb{R}_+$, there exists a non-adaptive query procedure with $n$ queries such that

$$P^n\left( \frac{1}{M} \right) \leq 4n \exp(-2M\eta^2) + \exp(n\eta q(\cdot))\mathbb{E}[\min\{1, M \Pr\{i_q(\tilde{X}^n, Y^n) \geq i_q(X^n; Y^n)\}\}]. \quad (91)$$

We first bound the expectation term in (91) as follows:

$$\mathbb{E}[\min\{1, M \Pr\{i_q(\tilde{X}^n, Y^n) \geq i_q(X^n; Y^n)\}\}]$$

$$\leq \Pr\left\{ M \exp(-i_q(X^n; Y^n)) \geq \frac{1}{\sqrt{n}} \right\} + \frac{1}{\sqrt{n}} \quad (92)$$

$$= \Pr\left\{ \log M - i_q(X^n; Y^n) \geq -\log \sqrt{n} \right\} + \frac{1}{\sqrt{n}} \quad (93)$$

$$= \Pr\left\{ \sum_{i \in [n]} i_q(X_i; Y_i) \leq \log M + \log(\sqrt{n}) \right\} + \frac{1}{\sqrt{n}}, \quad (94)$$

where (92) follows from i) the change of measure technique which states that

$$\Pr\{i_q(\tilde{X}^n; y^n) \geq t\} = \sum_{x^n} P^n_X(\tilde{x}^n) \mathbb{1}(i_q(\tilde{x}^n; y^n) \geq t) \leq \sum_{x^n} P^n_Y(\tilde{x}^n|y^n) \exp(-t) = \exp(-t), \quad (95)$$

and ii) the result in [29, Eq. (37)] saying that $\mathbb{E}[\min\{1, J\}] \leq \Pr\{J > \frac{1}{\sqrt{n}}\} + \frac{1}{\sqrt{n}}$ for any $n \in \mathbb{N}$.

Now choose $M$ such that

$$\log M = nC + \sqrt{n\Phi^{-1}(\varepsilon)} - \frac{1}{2} \log n, \quad (96)$$
and let
\[ \eta = \sqrt{\frac{\log M}{2M}} = O\left(\frac{\sqrt{n}}{\exp(nC/2)}\right). \]  
(97)

Thus, we have
\[ 4n \exp(-2M\eta^2) = \frac{4n}{M} = 4 \exp\left(- nC - \sqrt{nV_n\Phi^{-1}(\varepsilon)} + \frac{3}{2} \log n\right) = O(\exp(-nC)), \]  
(98)
and
\[ \exp(n\eta c(q)) = 1 + n\eta c(q) + o(n\eta c(q)) = 1 + O\left(\frac{n^{3/2}}{\exp(nC/2)}\right). \]  
(99)

Finally, applying the Berry-Esseen theorem to (94), we have that for any \( n \) sufficiently large,
\[ - \log \delta^*(n, \varepsilon) \geq nC + \sqrt{nV_n\Phi^{-1}(\varepsilon)} - \frac{1}{2} \log n. \]  
(101)

2) Converse Proof: We now proceed with the converse proof. Given any \( \varepsilon \in [0, 1) \), for any \( \beta \in (0, \frac{1-\varepsilon}{2}) \) and any \( \kappa \in (0, 1 - \varepsilon - \beta) \), from Theorem 2 we have
\[ - \log \delta^*(n, \varepsilon) \leq - \log \beta - \log \kappa + \sup_{\mathcal{A}^n} \operatorname{sup} \left\{ t \mid \Pr\left\{ \sum_{i \in [n]} t_{\mathcal{A}_i}(X_i; Y_i) \leq t \right\} \leq \varepsilon + 2\beta + \kappa \right\}. \]  
(102)

We first analyze the probability term in (102). Given any sequence of queries \( \mathcal{A}^n \), let
\[ C_{\mathcal{A}^n} := \frac{1}{n} \sum_{i \in [n]} \mathbb{E}[t_{\mathcal{A}_i}(X_i; Y_i)], \]  
(103)
\[ V_{\mathcal{A}^n} := \frac{1}{n} \sum_{i \in [n]} \operatorname{Var}[t_{\mathcal{A}_i}(X_i; Y_i)], \]  
(104)
\[ T_{\mathcal{A}^n} := \frac{1}{n} \sum_{i \in [n]} \mathbb{E}\left[ \left( t_{\mathcal{A}_i}(X_i; Y_i) - \mathbb{E}[t_{\mathcal{A}_i}(X_i; Y_i)] \right)^2 \right]. \]  
(105)

Assume that there exists \( V^- > 0 \) such that \( V^- \leq V_{\mathcal{A}^n} \). Applying the Berry-Esseen theorem \([30], [31]\), we have that
\[ \sup_{\mathcal{A}^n} \left\{ t \mid \Pr\left\{ \sum_{i \in [n]} t_{\mathcal{A}_i}(X_i; Y_i) \leq t \right\} \leq \varepsilon + 2\beta + \kappa \right\} \leq nC_{\mathcal{A}^n} + \sqrt{nV_{\mathcal{A}^n}\Phi^{-1}\left( \varepsilon + 2\beta + \kappa + \frac{6T_n}{\sqrt{nV^-}} \right)}. \]  
(106)

Let
\[ \beta = \kappa = \frac{1}{\sqrt{n}}. \]  
(107)

Using (102) and (106), we have
\[ - \log \delta^*(n, \varepsilon) \leq \log n + \sup_{\mathcal{A}^n} \left( nC_{\mathcal{A}^n} + \sqrt{nV_{\mathcal{A}^n}\Phi^{-1}\left( \varepsilon + 2\beta + \kappa + \frac{6T_n}{\sqrt{nV^-}} \right)} \right). \]  
(108)

For any sequence of queries \( \mathcal{A}^n \), we have
\[ C_{\mathcal{A}^n} \leq \sup_{\mathcal{A}\subseteq[0,1]} \mathbb{E}[t_{\mathcal{A}}(X; Y)] = \sup_{p \in [0,1]} \mathbb{E}[t_p(X; Y)] = C. \]  
(109)

Thus, when \( n \) is sufficiently large, for any \( \varepsilon \in [0, 1) \),
\[ - \log \delta^*(n, \varepsilon) \leq \log n + \sup_{\mathcal{A}^n : |\mathcal{A}| = q^*, \forall i \in [n]} \left( nC_{\mathcal{A}^n} + \sqrt{nV_{\mathcal{A}^n}\Phi^{-1}\left( \varepsilon + 2\beta + \kappa + \frac{6T_n}{\sqrt{nV^-}} \right)} \right) \]  
(110)
\[ = \log n + nC + \sqrt{nV_{\mathcal{A}^n}\Phi^{-1}\left( \varepsilon + 2\beta + \kappa + \frac{6T_n}{\sqrt{nV^-}} \right)} \]  
(111)
\[ = nC + \sqrt{nV_{\mathcal{A}^n}\Phi^{-1}\left( \varepsilon \right)} + \log n + O(1), \]  
(112)
where (112) follows from the Taylor’s expansion of \( \Phi^{-1}(\cdot) \) (cf. [13], Eq. (2.38)).
D. Proof of Non-Asymptotic Achievability Bound Using an Adaptive Query Procedure (Theorem 7)

The proof of the achievability non-asymptotic bound on the performances of non-adaptive query procedure is inspired by [23] Theorem 3] and is largely similar to the proof for non-adaptive query procedures in Appendix A. Thus, in the following, we only emphasize the differences.

Let \( x^n(1), \ldots, x^n(M) \) be a sequence of \( M \) binary codewords with infinite length. Then for any \( n \in \mathbb{N} \) and any \( m \in [M] \), let \( X^n(m) \) be the first \( n \) elements of \( X^n(m) \). The quantization of the target variable \( S \) and the design of the query procedure \( A_i \) is exactly the same as in Appendix A for non-adaptive query procedures. Thus, for any \( s \), the noiseless response from the oracle to the \( i \)-th query is \( X^n = x_i(q(s)) \).

In the following, we specify how the decoding is done, which consists of determining the stop time and designing decoding function. Let \( \lambda \in \mathbb{R}_+ \) be a fixed threshold.

Recall the definition of \( \tau_m(x^n, y^n) \) in (46). The stop time at the player is defined as

\[
\tau^*(x^n, y^n) := \min_{m \in [M]} \tau_m(x^n, y^n),
\]

and the player estimates \( w = q(s) \) as \( \hat{W} \) and outputs estimate \( \hat{S} = \frac{2\hat{W} - 1}{2M} \) where

\[
\hat{W} = \max\{m \in [M] : \tau_j(x^n, y^n) = \tau^*(x^n, y^n)\}.
\]

Given the above adaptive query procedure, we have that the worst case average stop time satisfies

\[
\sup_{f_S \in \mathcal{F}(S)} \mathbb{E}[\tau^*(x^n, Y^n)] = \sup_{f_S \in \mathcal{F}(S)} \int_{s \in [0,1]} f_S(s) \mathbb{E}[\tau^*(x^n, Y^n) | S = s] ds
\]

\[
= \sup_{P_W \in \mathcal{P}(\{M\})} \sum_{w \in [M]} P_W(w) \mathbb{E}[\tau^*(x^n, Y^n) | W = w]
\]

\[
\leq \sup_{P_W \in \mathcal{P}(\{M\})} \sum_{w \in [M]} P_W(w) \mathbb{E}[\tau^*(x^n, Y^n)]
\]

and the worst case excess-resolution probability with respect to \( \delta = \frac{1}{M} \) satisfies

\[
\sup_{f_S \in \mathcal{F}(S)} \text{Pr}\{|\hat{S} - S| > \delta\} \leq \sup_{P_W \in \mathcal{P}(\{M\})} \text{Pr}\{\hat{W} \neq W\}
\]

\[
\leq \sup_{P_W \in \mathcal{P}(\{M\})} \sum_{w \in [M]} P_W(w) \text{Pr}\{\tau_w(x^n, Y^n) \geq \tau^*(x^n, Y^n)\}.
\]

In the following, we will show that there exists binary codewords \( x^n = (x^n(1), \ldots, x^n(M)) \) such that the results in (117) and (119) are upper bounded by the desired result.

For this purpose, we use random coding idea as in [23] Proof of Theorem 3]. Let \( X^n := (X^n(1), \ldots, X^n(M)) \) be a sequence of \( M \) binary codewords with infinite length where each codeword is generated i.i.d. according to the Bernoulli distribution \( P_X \) with parameter \( p \in [0,1] \). Then we have that the joint distribution of \( (X^n, Y^n) \) for any \( w = q(s) \) is given as in (45).

Using symmetry, we have that for any \( P_W \in \mathcal{P}(\{M\}) \),

\[
\mathbb{E}_{X^n}[\mathbb{E}[\tau^*(X^n, Y^n) | X^n]] \leq \sum_{w \in [M]} P_W(w) \mathbb{E}[\tau^*(X^n, Y^n)]
\]

\[
= \sum_{w \in [M]} P_W(w) \mathbb{E}[\tau_1(X^n, Y^n)]
\]

\[
= \mathbb{E}[\tau_1(X^n, Y^n)],
\]

and

\[
\mathbb{E}_{X^n}[\text{Pr}[^{\hat{W} \neq W}]] \leq \sum_{w \in [M]} P_W(w) \text{Pr}\{\tau_w(X^n, Y^n) \geq \tau^*(X^n, Y^n)\}
\]

\[
= \sum_{w \in [M]} P_W(w) \text{Pr}\{\tau_1(X^n, Y^n) \geq \tau^*(X^n, Y^n)\}
\]

\[
= \mathbb{P}\{\tau_1(X^n, Y^n) \geq \tau^*(X^n, Y^n)\}
\]

\[
\leq (M - 1) \mathbb{P}\{\tau_2(X^n, Y^n) \geq \tau^*(X^n, Y^n)\}.
\]
E. Proof of Achievable Second-Order Asymptotics for the Adaptive Query Procedure (Theorem 8)

Let \( q^* \in \mathcal{P}_{ca} \) be a capacity-achieving parameter for measurement dependent channels \( \{P_{Y|X}^q\}_{q \in [0,1]} \). From Theorem 7 we have that there exists an adaptive query procedure such that the average number of queries is upper bounded by

\[
l \leq \mathbb{E}[\tau_1(X^\infty, Y^\infty)]
\]

and the worst case excess-resolution probability is upper bounded by

\[
\sup_{f_S} P_{e,a}^n \leq (M - 1) \Pr \{ \tau_1(X^\infty, Y^\infty) \geq \tau_2(X^\infty, Y^\infty) \}.
\]

For subsequent analyses, let \( P_X \) be the Bernoulli distribution with parameter \( q^* \) and let \( P_{XY} \) be the following joint distribution

\[
\hat{P}_{XY}(x, y) := \sum_{\tilde{x}_1, \ldots, \tilde{x}_{M-1}} P_X(x)( \prod_{j \in [M-1]} P_X(\tilde{x}_j)) P_{Y|X}^{\tau}(x + \sum_{j \in [M-1]} \tilde{x}_j)(y|x).
\]

Furthermore, define the “mismatched” version of the capacity.

\[
C_1 := \mathbb{E}[\hat{P}_{XY}[\eta^*(X; Y)].
\]

Finally, for each \( n \in \mathbb{N} \), let

\[
U_n := \eta^*(X^n; Y^n) = \sum_{i \in [n]} \eta_{q^* \eta^*}(X_i; Y_i).
\]

It can be easily verified that \( \{U_n - nC_1\}_{n \in \mathbb{N}} \) is a martingale and for each \( n \in \mathbb{N} \), \( \mathbb{E}[U_n - nC_1] = 0 \). The optional stopping theorem [32, Theorem 10.10] implies that

\[
0 = \mathbb{E}[U_{\tau_1(X^\infty, Y^\infty)} - C_1 \tau_1(X^\infty, Y^\infty)] \leq \lambda + a_0 - C_1 \mathbb{E}[\tau_1(X^\infty, Y^\infty)],
\]

where \( a_0 \) is an upper bound on the information density \( U_1 \). Thus,

\[
\mathbb{E}[\tau_1(X^\infty, Y^\infty)] \leq \frac{\lambda + a_0}{C_1}.
\]

We then focus on upper bounding \((128)\). From \((134)\), we have that

\[
\Pr \{ \tau_1(X^\infty, Y^\infty) < \infty \} = 1,
\]

since otherwise the expectation value of \( \tau_1(X^\infty, Y^\infty) \) would be infinity.

Therefore, we have that for any \( \eta \in \mathbb{R}_+ \),

\[
\Pr \{ \tau_1(X^\infty, Y^\infty) \geq \tau_2(X^\infty, Y^\infty) \}
\]

\[
\leq \Pr \{ \tau_2(X^\infty, Y^\infty) < \infty \}
\]

\[
= \sum_{t \in \mathbb{N}} \mathbb{1}(t < \infty) \Pr \{ \tau_2(X^\infty, Y^\infty) = t \}
\]

\[
= \sum_{t \in \mathbb{N}} \mathbb{1}(t < \infty) \left\{ \Pr \{ \tau_2(X^\infty, Y^\infty) = t, X^t \in T^t(M, q^*, \eta) \} + \Pr \{ X^t \notin T^t(M, q^*, \eta) \} \right\}
\]

\[
\leq \sum_{t \in \mathbb{N}} \mathbb{1}(t < \infty) \left\{ \exp(t \eta_1(p)) \Pr \{ \tau_2(X^\infty, Y^\infty) = t \} + 4t \exp(-2M \eta^2) \right\},
\]

where \((139)\) follows from \((68)\) and the upper bound on the probability of atypicality similar to \((72)\) and in \((139)\), the distribution \( P_{X^\infty, Y^\infty} \) is a generalization of \( P_{\text{rand},1}^{\infty,1} \) (cf. \((70)\)) to an arbitrary length.

Given any \( l' \in \mathbb{R}_+ \), let

\[
\lambda = l'C_1 - a_0,
\]

\[
\log M = \lambda - \log l',
\]

\[
\eta := \sqrt{\frac{\log M}{2M}} = O \left( \frac{\sqrt{l'}}{\exp(l' C_1/2)} \right).
\]

Then we have that

\[
\mathbb{E}[\tau_1(X^\infty, Y^\infty)] \leq l'.
\]
As a result, similar to [99], we have that
\[
\Pr \{ \tau_1(X^\infty, Y^\infty) \geq \tau_2(X^\infty, Y^\infty) \}
\]
\[
= \sum_{t \in \mathbb{N}} \mathbb{I}(t < \infty) \left( 1 + O \left( l_{C_1}^2 \exp \left( -\frac{l C_1}{2} \right) \right) \right) \Pr \{ \tau_2(X^\infty, Y^\infty) = t \}
\]
\[
= \left( 1 + O \left( l_{C_1}^2 \exp \left( -\frac{l C_1}{2} \right) \right) \right) \lim_{t \to \infty} \Pr \{ \tau_2(X^\infty, Y^\infty) < t \}
\]
\[
= \left( 1 + O \left( l_{C_1}^2 \exp \left( -\frac{l C_1}{2} \right) \right) \right) \lim_{t \to \infty} \mathbb{E}_{\mathcal{P}_{X^\infty Y^\infty}} \left[ \exp(-U_t) \mathbb{I}(\tau_1(X^\infty, Y^\infty) < t) \right]\]
\[
\leq \left( 1 + O \left( l_{C_1}^2 \exp \left( -\frac{l C_1}{2} \right) \right) \right) \exp(-\lambda), \quad (148)
\]

where (146) follows from the change-of-measure technique, (147) follows similarly as [23, Eq. (113) to Eq. (117)] and (148) follows from the definition of \( \tau_1(X^\infty, Y^\infty) \) in [46]. Therefore, we have for \( l' \) sufficient large,
\[
(M-1) \Pr \{ \tau_1(X^\infty, Y^\infty) \geq \tau_2(X^\infty, Y^\infty) \}
\]
\[
\leq \left( 1 + O \left( l_{C_1}^2 \exp \left( -\frac{l C_1}{2} \right) \right) \right) M \exp(-\lambda) \quad (149)
\]
\[
= \left( 1 + O \left( l_{C_1}^2 \exp \left( -\frac{l C_1}{2} \right) \right) \right) \frac{1}{\tilde{p}^M}. \quad (150)
\]

Recall the definition of the “capacity” \( C \) of measurement dependent channels in [19]. Using the definition of \( C_1 \) in (130), we have that
\[
C_1 = \mathbb{E}_{\mathcal{P}_{X Y}} [h_q(X : Y)]
\]
\[
= \mathbb{E}_{\mathcal{P}_{X Y}} \left[ \frac{P_{XY}(X, Y)}{P_{X Y}(X, Y) h_q(X : Y)} \right]
\]
\[
\leq \exp(\eta c(p)) \mathbb{E}_{\mathcal{P}_{X Y}} [h_q(X : Y)] + 2 \exp(-2 M \eta^2) \quad (153)
\]
\[
= \exp(\eta c(p)) C + 2 \exp(-2 M \eta^2). \quad (154)
\]

where (153) follows from the change-of-measure technique and the result in (68). Given the choice of \( M \) and \( \eta \), we have
\[
C_1 = C + O(l' \exp(-l')). \quad (155)
\]

Thus, till now, we have constructed an \((l', \frac{1}{\tilde{p}}, \frac{1}{M})\)-adaptive query procedure for sufficiently large \( l' \). For any \( \varepsilon \in [0, 1) \), consider the following query procedure: with probability \( \frac{l' - 1}{l' - 1} \), we do not pose any query and with the remaining probability, we use the above-constructed \((l', \frac{1}{\tilde{p}}, \frac{1}{M})\)-adaptive query procedure. For \( l' \) sufficiently large, it is easy to verify that the combined adaptive query procedure is an \((l, \varepsilon, \delta)\)-adaptive query procedure where
\[
l = \left( 1 - \varepsilon l' - \frac{1}{l' - 1} \right) l' = l'^2(1 - \varepsilon) \left( 1 - \varepsilon l' + (1 - \varepsilon) \right) \approx (1 - \varepsilon) l', \quad (156)
\]
\[
- \log \delta \geq l'C_1 - a_0 - \log l' = l'C + O(\log l') = \frac{Cl}{1 - \varepsilon} + O(\log l). \quad (157)
\]

**F. Comparison the Performances of Adaptive Querying for Measurement Dependent and Measurement Independent Channels**

To compare the performances of optimal adaptive query procedures under measurement dependent and measurement independent channels, we plot in Figure 14 the number of bits in the binary expansion of the target random variable \( S \) extracted after \( n \) queries, i.e., \(-\log_2 \delta_0^*(n, \varepsilon)\) and \(-\log_2 \delta_{a, m}^*(n, \varepsilon)\) for \( \varepsilon = 0.001 \) (the \( O(\log n) \) term is ignored). Note that for measurement dependent noise channel, we plot only a lower bound of \(-\log_2 \delta_0^*(n, \varepsilon)\). The remarks for non-adaptive querying are still valid for adaptive querying.
Note that for measurement dependent channels, we plot only a lower bound (asserted in Theorem 8) of $-\log_2 \delta^*_a(n, \varepsilon)$ and measurement independent $-\log_2 \delta^*_a(n, \varepsilon)$ versions of three channels where we neglect the $O(\log l)$ term. Note that for measurement dependent channels, we plot only a lower bound (asserted in Theorem 8) of $-\log_2 \delta^*_a(n, \varepsilon)$.

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