Instantons and the 5D $U(1)$ gauge theory with extra adjoint

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Received 31 December 2008, in final form 2 January 2009
Published 14 July 2009
Online at stacks.iop.org/JPhysA/42/304024

Abstract

In this paper, we compute the partition function of 5D supersymmetric $U(1)$ gauge theory with extra adjoint matter in general $\Omega$ background. It is well known that such partition functions encode very rich topological information. We show in particular that unlike the case with no extra matter, the partition function with extra adjoint at some special values of the parameters directly reproduces the generating function for the Poincare polynomial of the moduli space of instantons. We compare our results with those recently obtained by Iqbal et al (Refined topological vertex, cylindric partitions and the $U(1)$ adjoint theory, arXiv:0803.2260), who used the so-called refined topological vertex method.

PACS numbers: 11.15.−q, 11.30.Pb

1. Introduction

Recent progress in understanding non-perturbative phenomena in supersymmetric Yang–Mills theories due to direct multi-instanton calculations is quite impressive. Two main ideas played an essential role in all these developments. First was the realization that the supersymmetric Yang–Mills action induced to the moduli space of instantons can be represented as an integral over a closed, equivariant with respect to the diagonal part of the gauge group form [1]. This observation leads to a crucial simplification reducing the SYM path integral to an integral over the sub-manifold of the moduli space of instantons which is stable with respect to the diagonal part of the gauge group. The next brilliant idea, which became the corner stone of all further developments, was suggested by Nekrasov in [2]. The idea was to generalize the theory involving into the game in equal footing with the already mentioned global diagonal gauge transformations also the diagonal part of the (Euclidean) space-time rotations. This
is so crucial because the subset of the instanton moduli space invariant under this combined group action consists only of a finite number of points.

In the case of the gauge group $U(N)$, this fixed point set is in one-to-one correspondence with the set of array of Young diagrams $\vec{Y} = (Y_1, \ldots, Y_N)$ with total number of boxes $|\vec{Y}|$ being equal to the instanton charge $k$. Thus, to calculate the path integral for the various ‘protected’ by super-symmetry physical quantities, one needs to know only the pattern of how the combined group acts in the neighborhoods of the fixed point. All this information can be encoded in the character of the group action in the tangent space at given fixed points. An elegant formula for this character which played a significant role in both physical and mathematical applications was proposed in [3] (see equation (2.1)). Let us note at once that combining space-time rotations with the gauge transformations besides giving a huge computational advantage due to the finiteness of the fixed point set has also a major physical significance generalizing the theory to the case with certain non-trivial graviphoton backgrounds [2]. In order to recover the standard flat space quantities (say the Seiberg–Witten prepotential of $\mathcal{N} = 2$ super-Yang–Mills theory), one should take the limit when the space-time rotation angles vanish. It is shown by Nekrasov and Okounkov [4] that in this limit the sum over the arrays of Young diagrams is dominated by a single array with specific ‘limiting shape’. As a result, it becomes possible to handle the entire instanton sum expressing all relevant quantities in terms of emerging Seiberg–Witten curve [5]. Note that only the entire sum but not its truncated part exhibits remarkable modular properties which allows one to investigate the rich phase structure of SYM theories. This is why all attempts to investigate the entire instanton sum also in a general case (i.e. keeping finite the space-time rotation angles) seem quite natural. Unfortunately, there was little progress till now in this direction besides the simplest case of the gauge group $U(1)$. Though the $U(1)$ 4D theory in flat background is trivial, the general 5D $U(1)$ theories compactified on a circle being rather non-trivial nevertheless in many cases admit full solution. In what follows, we investigate the partition function of 5D gauge theory with an extra adjoint hypermultiplet. It is not surprising that such partition functions encode very rich topological information. As a manifestation we argue that unlike the case with no extra matter, at some special values of the parameters this partition function directly reproduces the generating function of the Poincare polynomial for the moduli space of instantons. We check this conclusion explicitly computing the partition function in the case of the gauge group $U(1)$. We compare our result with that recently obtained by Iqbal et al [6] who used the refined topological vertex method [12] to find the same partition function and present our comments on discrepancies we found. See [13] for an earlier attempt to construct a refined version of the topological vertex.

2. The $U(1)$ theory with adjoint matter

The weight decomposition of the torus action on the tangent space at the fixed point $\vec{Y} = (Y_1, \ldots, Y_N)$ is given by [3]

$$\chi = \sum_{a, \beta = 1}^{N} e_\beta e^{-1}_a \left\{ \sum_{s \in Y_a} (T_1^{-y_\beta(s)} T_2^{s_\beta(s)+1}) + \sum_{s \in Y_\beta} (T_1^{y_\alpha(s)+1} T_2^{-s_\alpha(s)}) \right\},$$

(2.1)

where $e_1, \ldots, e_N$ are the elements of (complexified) maximal torus of the gauge group $U(N)$, $T_1, T_2$ belong to the maximal torus of the (Euclidean) space-time rotations and $s_\alpha(s)$

3 Roughly speaking the main technical difference between 4D and 5D cases is that in the former case the above-mentioned combined group enters into the game in the infinitesimal level while in the latter case the main roles are played by the finite group elements.
each term here could be thought of as a trace over the space of local holomorphic forms, with to give the super-trace over the globally defined holomorphic forms. We see that \( Z \) of the torus action. In addition, an extra \( U(N) \) \( \langle \rangle \) acts trivially on bosonic directions while on each grassmannian coordinate it acts in its fundamental representation. Then (2.3) is the super-trace of the extended torus action on the super-tangent space at given fixed point. The corresponding 5D partition function now reads

\[
Z = \sum_{\vec{r}} \prod_{a,\beta=1}^{N} \prod_{s \in Y_{a}} \left( 1 - e_{a} e_{a}^{-1} T_{1}^{-l_{\beta}(s)} T_{2}^{-a_{\beta}(s)+1} \right) \prod_{s \in Y_{a}} \left( 1 - e_{a} e_{a}^{-1} T_{1}^{l_{\beta}(s)+1} T_{2}^{-a_{\beta}(s)} \right) \]

Each term here could be thought of as a trace over the space of local holomorphic forms, with parameter \( T_{m} \) counting the degrees of forms. Hence, the sum over the fixed points is expected to give the super-trace over the globally defined holomorphic forms. We see that \( Z_{\text{adj}} \) is an extremely rich quantity from both physical and mathematical points of view. It is interesting to note that at special values of the parameters \( Z_{\text{adj}} \) directly reproduces the generating function for the Poincaré polynomial of the moduli space of \( U(N) \) instantons. Indeed, following [8] part (3.3) let us assume that \( T_{2} \gg T_{a} \gg \cdots \gg T_{a_{N}} \gg T_{1} > 0 \). It is easy to see that in the limit when all these parameters go to zero each fraction under the products in (2.4) tends to \( T_{m} \) or 1 depending on whether we have a negative weight direction or not (see the classification of negative directions in [8], proof of corollary 3.10). We will see this explicitly in the simplest case \( N = 1 \) when the moduli space of instantons coincides with the Hilbert scheme of points on \( \mathbb{C}^{2} \).

From now on we will restrict ourselves to the simplest case of the \( U(1) \) gauge group, when the partition function could be computed in a closed way. The partition function of the pure \( \mathcal{N} = 2 \) \( U(1) \) theory has the form [9]

\[
Z = \sum_{\vec{r}} \prod_{s \in Y} \left( 1 - T_{1}^{-l_{\beta}(s)} T_{2}^{-a_{\beta}(s)+1} \right) \left( 1 - T_{1}^{l_{\beta}(s)+1} T_{2}^{-a_{\beta}(s)} \right)
\]
where $R_\lambda$ is the irreducible representation of the symmetric group given by the Young diagram $\lambda$.

Now let us turn to the $U(1)$ theory with adjoint matter. Doing low instanton calculations using (2.4) is straightforward and gives

$$\log Z_{adj} = \sum_{n=1}^{\infty} \frac{q^n (1 - T_m^a T_n^a)(1 - T_m^a T_n^b)}{n(1 - T_1^a)(1 - T_2^a)}. \tag{2.8}$$

which is equivalent to the following highly non-trivial combinatorial identity:

$$Z_{adj} = \sum_{n=1}^{\infty} \frac{q^n (1 - T_m^a T_n^a)(1 - T_m^a T_n^b)}{n(1 - T_1^a)(1 - T_2^a)} \tag{2.9}$$

Indeed, calculations with Mathematica code up to 10 instantons further convinced us that this formula is indeed correct. Note that the 4D limit of this identity with a particular choice of the graviphoton background $\epsilon_1 = -\epsilon_2$ boils down to the Burnside’s theorem

$$\sum_{|\lambda|=n} (\dim R_\lambda)^2 = n!, \tag{2.6}$$

where $R_\lambda$ is the irreducible representation of the symmetric group given by the Young diagram $\lambda$.

As a further check, let us go back to the general case. In various domains of the variables $T_1, T_2$ we can represent (2.8) as an infinite product as we did in the special case in (2.10). Let us consider separately the cases:

(a) $|T_1| < 1, |T_2| < 1, |T_m^a q^n| < 1$
In this region (2.8) could be rewritten as

\[
Z_{\text{adj}} = \exp \left( \sum_{n=1}^{\infty} \sum_{k,j=0}^{\infty} \frac{q^{n}}{n} T_{i}^{n} T_{j}^{n} (T_{m} q)^{n}(1 - T_{m}^{n} T_{1}^{n})(1 - T_{m}^{n} T_{2}^{n}) \right).
\]

(2.11)

Performing summation over \( n \) we get

\[
Z_{\text{adj}} = \prod_{i,j,k=0}^{\infty} \frac{1 - q^{k+1} T_{m}^{i} T_{1}^{j} T_{2}^{j+1}}{(1 - q^{k+1} T_{m}^{i} T_{1}^{j} T_{2}^{j})(1 - q^{k+1} T_{m}^{i} T_{1}^{j+1} T_{2}^{j+1})}.
\]

(2.12)

(b) \( |T_{1}| > 1, |T_{2}| < 1, |T_{m} q| < 1 \)

In this region we expand (2.8) over \( 1/T_{1} \):

\[
Z_{\text{adj}} = \exp \left( \sum_{n=1}^{\infty} \sum_{k,j=0}^{\infty} \frac{-q^{n}}{n} T_{1}^{-n} T_{2}^{-n} (T_{m} q)^{n}(1 - T_{m}^{n} T_{1}^{n})(1 - T_{m}^{n} T_{2}^{n}) \right),
\]

which leads to

\[
Z_{\text{adj}} = \prod_{i,j,k=0}^{\infty} \frac{1 - q^{i+1} T_{m}^{i} T_{1}^{i-1} T_{2}^{i+1}}{(1 - q^{i+1} T_{m}^{i} T_{1}^{i} T_{2}^{i})(1 - q^{i+1} T_{m}^{i} T_{1}^{i} T_{2}^{i+1})}.
\]

(2.14)

Recently Iqbal, Kozçaz and Shabir [6] have computed the partition function of these \( U(1) \) adjoint theories using the refined topological vertex formalism [12]. And, since formula (2.8) was known to the present authors for quite a while, we performed detailed comparison with their results. To make contact with the formulæ of Iqbal et al we need the following dictionary: \( T_{m} = Q_{m}(t/q) \), \( T_{1} = 1/t, T_{2} = q, Q = Q(q/t) \). In terms of these variables, equations (2.12) and (2.14) take the forms

(a) \( |t| > 1, |q| < 1, |Q Q_{m}| < 1 \)

\[
Z_{\text{adj}} = \prod_{i,j,k=0}^{\infty} \frac{(1 - Q^{k} Q_{m}^{i} q^{i-j} t^{j}) (1 - Q^{k} Q_{m}^{i} q^{i-j} t^{j})}{(1 - Q^{k} Q_{m}^{i} q^{i-j} t^{j}) (1 - Q^{k} Q_{m}^{i} q^{i-j} t^{j})}.
\]

(2.15)

and

(b) \( |t| < 1, |q| < 1, |Q Q_{m}| < 1 \)

\[
Z_{\text{adj}} = \prod_{i,j,k=0}^{\infty} \frac{(1 - Q^{k} Q_{m}^{i} q^{i-j} t^{j}) (1 - Q^{k} Q_{m}^{i} q^{i-j} t^{j})}{(1 - Q^{k} Q_{m}^{i} q^{i-j} t^{j}) (1 - Q^{k} Q_{m}^{i} q^{i-j} t^{j})}.
\]

(2.16)

These equations come rather close, but certainly do not coincide with those given in [6] at the end of the part 3.2. The reason for this discrepancy seems to us as follows. According to [6] the refined topological vertex method for the 5D \( U(1) \) theory with adjoint matter leads to (see equation (4.6) of [6]; below we omit the ‘perturbative part’ \( \prod_{i,j=0}^{\infty} (1 - Q_{m} q^{i} t^{j}) \))

\[
Z = \prod_{k=1}^{\infty} (1 - Q^{k} Q_{m}^{k})^{-1} \prod_{i,j=1}^{\infty} (1 - Q^{k} Q_{m}^{k-1} q^{-i} t^{-j}) (1 - Q^{k} Q_{m}^{k-1} q^{-i} t^{-j}) (1 - Q^{k} Q_{m}^{k} q^{i} t^{j}) (1 - Q^{k} Q_{m}^{k} q^{i} t^{j}) (1 - Q^{k} Q_{m}^{k} q^{i} t^{j}),
\]

(2.17)

where \( \rho_{i} = -i + 1/2. \) But four factors under the product over \( i, j \) have different, excluding each other, regions of convergence. Thus this infinite product should be treated very carefully. Unfortunately, the authors of [6] do not tell what analytic continuation procedure they have adopted to pass from their equation (4.6) to those presented at the end of the part 3.2, but we will demonstrate now that one perhaps the simplest approach directly leads to our conjectural
formula (2.8). We simply examine the product over each factor separately within its region of convergence and only after that continue analytically to a common region of the parameters. Thus, for the first factor in (2.17) we have
\[
\prod_{k=1}^{\infty} (1 - Q^k Q^k_m) = \exp \sum_{n,k=1}^{\infty} \frac{(Q Q_m)^{nk}}{n} = \exp \sum_{n=1}^{\infty} \frac{(Q Q_m)^n}{n(1 - (Q Q_m)^n)}.
\] (2.18)
For the next factor (assuming \(q < 1, t < 1\))
\[
\prod_{k,i,j=1}^{\infty} (1 - Q^k Q^k_m q^i t^j) = \exp \sum_{n,k,i,j=1}^{\infty} \frac{-Q^n Q^n_m q^{i-n} t^{j-n}}{n}
\] = \exp \sum_{n=1}^{\infty} \frac{-Q^n q^2 t^2}{(1 - (Q Q_m)^n)(1 - q^n)(1 - t^n)}.
\] (2.19)
Similarly for \(q > 1, t < 1\)
\[
\prod_{k,i,j=1}^{\infty} (1 - Q^k Q^k_m q^{-i} t^{-j}) = \exp \sum_{n=1}^{\infty} \frac{-Q^n Q^n_m q^{n-i} t^{n-j}}{n(1 - (Q Q_m)^{n})(1 - q^{-n})(1 - t^{-n})},
\] for \(q < 1, t > 1\)
\[
\prod_{k,i,j=1}^{\infty} (1 - Q^k Q^k_m q^{-i} t^{-j}) = \exp \sum_{n=1}^{\infty} \frac{-Q^n Q^n_m q^{n-i} t^{n-j}}{n(1 - (Q Q_m)^{n})(1 - q^{-n})(1 - t^{-n})},
\] and, finally for \(q > 1, t > 1\)
\[
\prod_{k,i,j=1}^{\infty} (1 - Q^k Q^k_m q^{-i} t^{-j}) = \exp \sum_{n=1}^{\infty} \frac{-Q^n Q^n_m q^{-i} t^{-j}}{n(1 - (Q Q_m)^{n})(1 - q^{-n})(1 - t^{-n})}.
\] (2.22)
Note that the rhs of above expressions are defined also outside of their initial convergence region. Combining all these together we get
\[
Z = \exp \sum_{n=1}^{\infty} \frac{(Q Q_m)^n (q^\frac{i}{t} t^\frac{i}{5} - Q^n_m)(q^\frac{i}{t} t^\frac{i}{5} - Q^{-n}_m)}{n(1 - (Q Q_m)^n)(1 - q^n)(1 - t^n)},
\] which in terms of the parameters \(q, T_1, T_2\) exactly coincides with our conjectural result (2.8).

Acknowledgments

It is a pleasure to thank R Flume, F Fucito and J-F Morales for discussions. RP would like to thank INFN for supporting a visit to the University of Rome II, ‘Tor Vergata’. This work was supported by the Institutional Partnership grant of Humboldt foundation of Germany and in part by the MIUR-PRIN contract 2007-SAT778. MS acknowledges support of RTN grant MRTN-CT-2004-503369.

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