Abstract

Let $\Sigma_r$ be the symmetric group acting on $r$ letters, $K$ be a field of characteristic 2 and $\lambda$ and $\mu$ be partitions of $r$ in at most two parts. Denote the permutation module corresponding to the Young subgroup $\Sigma_\lambda$ in $\Sigma_r$ by $M^\lambda$, and the indecomposable Young module by $Y^\mu$. We give an explicit presentation of the endomorphism algebra $\text{End}_{K[\Sigma_r]}(Y^\mu)$, using the idempotents found by Doty, Erdmann and Henke in [1].
Endomorphism Rings of Some Young Modules

Jasdeep Singh Kochhar

December 9, 2014

1 Introduction

Permutation modules of symmetric groups coming from actions on set partitions are of central interest in the representation theory of symmetric groups, and they also provide a link with the representation theory of general linear groups, via Schur algebras.

Let $K$ be a field of prime characteristic $p$, and let $n$ and $r$ be positive integers. For each partition $\lambda$ of $r$ with at most $n$ parts, let $M^\lambda$ be the permutation module of the symmetric group $\Sigma_r$ of degree $r$ corresponding to set partition $\lambda$. The indecomposable summands of these modules are known as Young modules, where $Y^\lambda$ is the unique summand of $M^\lambda$ which contains the Specht module $S^\lambda$. The module $M^\lambda$ is in general a direct sum of Young modules $Y^\mu$, and if $Y^\mu$ occurs then $\mu \geq \lambda$, in the dominance order of partitions. The $p$-Kostka number $[M^\lambda : Y^\mu]$ is the number of indecomposable summands of $M^\lambda$ isomorphic to $Y^\mu$. We have

$$M^\lambda \cong \bigoplus_{\mu \geq \lambda} [M^\lambda : Y^\mu]Y^\mu$$

The paper [1] studies the endomorphism algebra of $M^\lambda$, denoted by $S_K(\lambda)$, when $K$ has characteristic 2 and $\lambda$ has two parts. In this case, $S_K(\lambda)$ is commutative, and its primitive idempotents are unique. The main result of [1] is the explicit construction of all primitive idempotents, establishing a natural one-to-one correspondence with the 2-Kostka numbers: The idempotent corresponding to $[M^\lambda : Y^\mu]$ generates the endomorphism algebra of the Young module $Y^\mu$.

Here we study the endomorphism algebra of $Y^\mu$, as the subalgebra of $S_K(\lambda)$ generated by the primitive idempotent constructed in [1]. We show that its algebra structure depends only on the dimension of the algebra but not on $\mu$. [The dimension of the endomorphism algebra of $Y^\mu$ for partitions with two parts is known, see [2].]

Moreover, we will give an explicit presentation by generators and relations: Any algebra is isomorphic to a quotient of a truncated polynomial ring

$$K[x_1, \ldots, x_k]/(x_i^2 : 1 \leq i \leq k),$$

\[\text{dim } K[x_1, \ldots, x_k]/(x_i^2 : 1 \leq i \leq k) = \begin{cases} k & \text{ if } \mu \text{ has two parts,} \\ 2^k & \text{ in all cases.} \end{cases}\]
for a precise description, see Theorem 2.7.

This may be surprising since the submodule structure of these Young modules can get more and more complicated for large $r$, as it can be for example seen from [3]. As a representative for a $t$-dimensional endomorphism algebra one can take the endomorphism algebra of $Y^{(t-1,t-1)}$; this module is isomorphic to $M^{(t-1,t-1)}$.

2 The algebra $S_K(\lambda)$

Doty and Giaquinto found presentations of Schur algebras $S_K(n,r)$ in terms of the universal enveloping algebras of Lie algebras $\mathfrak{gl}_n$. We assume $n = 2$, then based on their results in [4], the paper [1] determines basis and the multiplication formula for the endomorphism algebra of $M^{\lambda}$. We will summarise what we need, for details we refer to [1]. Let $\lambda = (\lambda_1, \lambda_2)$ be a partition of $r$. As in [1], we let $S_K(\lambda) := \text{End}_{K^\Sigma} (M^{\lambda})$ and we use basis and multiplication as in [1]:

2.1 The canonical basis for $S_K(\lambda)$

Definition 2.1. The algebra $S_K(\lambda)$ has basis

$$\{b(i) : 0 \leq i \leq \lambda_2\}$$

and the multiplication is given by

$$b(i) \cdot b(j) = \sum_{k=0}^{i} \binom{j+k}{i} \binom{j+k}{k} \binom{m+j+i}{i-k} b(j+k),$$

setting $b(a) = 0$ for $a > \lambda_2$. Here the binomial coefficients are taken modulo 2.

2.2 Notation

For an integer $a$ with $p$-adic expansion $a = \sum_{i=0}^{s} a_i p^i$, where $0 \leq a_i < p - 1$ for all $i$, we write $a = [a_0, a_1, \ldots, a_s]$. We also have for non-negative integers $m$ and $n$, with $p$-adic expansions $m = [m_0, m_1, \ldots]$ and $n = [n_0, n_1, \ldots]$ (respectively) that

$$\binom{m}{n} \equiv_p \prod_{i \geq 0} \binom{m_i}{n_i}.$$

We also have the following in a field of positive characteristic:

Lemma 2.2. [1, Lemma 3.7] Let $[i_0, i_1, \ldots]$ be the $p$-adic decomposition of $i$ and write $i = [i_0, i_1, \ldots]$. Then $b(i) = \prod_{t \geq 0} b(i_t \cdot p^t)$ in $S_K(\lambda)$.

It can be proved then that the algebra $S_K(\lambda)$ can be generated by the elements $b(p^0), b(p^1), \ldots, b(p^t)$, where $t$ is the unique natural number such that $p^t < \lambda_2 \leq p^{t+1}$. For the case when $p = 2$, this is immediate as for $i$ with binary expansion $[i_0, i_1, \ldots]$, the coefficients $i_t$ are 0 or 1.
2.3 The idempotents $e_{m,g}$

For $\lambda, \mu \vdash r$, define $m := \lambda_1 - \lambda_2$ and $g := \lambda_2 - \mu_2$, and so from $m$ and $g$, we can completely determine $\lambda$ and $\mu$. It is known (see [2]) that over a field of characteristic $p$, $Y^\mu$ is a direct summand of $M^\lambda$ if and only if

$$B(m, g) := \begin{pmatrix} m + 2g \\ g \end{pmatrix} \not\equiv_p 0.$$

In [1], the binary expansion of $(m+2g)$ is used to construct an element of $S_K(\lambda)$, denoted $e_{m,g}$. We begin by defining the sets $I_{m,g}, J_{m,g}$ as follows:

$$I_{m,g} := \{ u : g_u = 0 \text{ and } (m + 2g)_u = 1 \},$$

$$J_{m,g} := \{ u : g_u = 1 \text{ and } (m + 2g)_u = 1 \}.$$

Then for a natural number $t$, define elements of $S_K(\lambda)$ by:

$$e_{m,g} := \prod_{u \in J_{m,g}} b(2^u) \prod_{u \in I_{m,g}} 1 - b(2^u)$$

$$e_{m,g \leq t} := \prod_{u \in J_{m,g}, u \leq t} b(2^u) \prod_{u \in I_{m,g}, u \leq t} 1 - b(2^u)$$

We can immediately form a correspondence of these factors to the factors in the binary expansion of $B(m, g)$, as described below:

| Factor of $e_{m,g}$ | $B(m, g)_u$ | $\binom{1}{g}$ | $\binom{1}{u}$ | $\binom{0}{u}$ | $\binom{0}{u}$ |
|---------------------|-------------|---------------|---------------|---------------|---------------|

and we see that $e_{m,g} = 0$ if and only if $\binom{0}{1}$ occurs as column in the binary expansion of $B(m, g)$, which happens if and only if $B(m, g)$ equals 0 modulo 2.

In [1], it is proved that these elements are the primitive orthogonal idempotents in $S_K(\lambda)$, i.e. we have the following:

**Theorem 2.3 (Idempotent Theorem).** [1] Fix $m \geq 0$. The set of elements $e_{m,g}$, with $B(m, g) \not\equiv 0$, modulo 2, and $m + 2g \leq r$ give a complete set of primitive orthogonal idempotents for $S_K(\lambda)$

We then also have the following:

**Theorem 2.4. [1, Theorem 7.1]** Let $\lambda, \mu$ be partitions of $r$, such that $Y^\mu$ is a direct summand of $M^\lambda$. Define

$$m := \lambda_1 - \lambda_2 \text{ and } g := \lambda_2 - \mu_2.$$

Then $e_{m,g}$ is the projection of $M^\lambda$ onto $Y^\mu$. 

4
2.4 The Algebra \( \text{End}_{K[\Sigma_r]}(Y^\mu) \)

We have that the algebra \( \text{End}_{K[\Sigma_r]}(Y^\mu) \) is generated by the non-zero elements of

\[ \{ e_{m,g}b(2^k) \} , \]

where \( e_{m,g} \) is the idempotent of \( M^\lambda \) corresponding to \( Y^\mu \). For a minimal set of generators we prove the following lemma:

**Lemma 2.5** (Involvement Lemma). If \( b(2^i) \) is involved in \( e_{m,g} \), then either \( e_{m,g}b(2^i) = e_{m,g} \) or \( e_{m,g}b(2^i) = 0 \).

We therefore have that the generators of our algebra are the \( b(2^i) \) not involved in \( e_{m,g} \), which is precisely the set

\[ \{ e_{m,g}b(2^s) : (m + 2g)_s = 0 \} \]

As \( (m + 2g) \) is non-zero, the generators of the algebra therefore satisfy the conditions of the following lemma ([1, 2.4]):

**Lemma 2.6** (Orthogonality Lemma). Suppose the \( s^{th} \) column in the binary expansion of \( (m + 2g) \) is zero. Then \( e_{m,g}b(2^s)^2 = 0 \).

We therefore have that the generators of the endomorphism algebra all have square zero and so the algebra is isomorphic to a quotient of the form

\[ \bigotimes_{i=1}^{k} K[x_1] / (x_i^2) \cong K[x_1, \ldots, x_k] / (x_i^2 : i = 1, \ldots, k) \]

We also have the following stronger result:

**Theorem 2.7** (Dimension Theorem). Let \( \lambda \) be a partition of \( r \). Given an Endomorphism Algebra of a Young module, \( A := \text{End}_{K[\Sigma_r]}(Y^\mu) \), of dimension \( n \), we have that \( A \) is isomorphic to the algebra:

\[ K[x_1, \ldots, x_k] / (R) , \]

where

\[ R := \{ x_i^2 \} \cup \{ x_r, x_2 \ldots x_r, x_k : r_i \neq r_j \text{ and } \sum_i 2^{r_i} - 1 + 2^{k-1} > n - 1 \} , \]

and \( k \) is the unique non-negative integer such that \( 2^{k-1} < n \leq 2^k \).

We give proofs of the Involvement Lemma and the Dimension Theorem in Section 4. We also have the following result, which is an extension of [1, 2.4]:

**Theorem 2.8** (Basis Theorem). An Endomorphism Algebra of a Young Module, \( Y^\mu \), has basis given by the non-zero elements in

\[ \{ e_{m,g}b(i) : i = [i_0, i_1, \ldots] \text{ and } i_j = 1 \text{ only if } (m + 2g)_j = 0 \} , \]

where \( m, g \) are such that \( e_{m,g} \) is the projection of \( M^\lambda \) onto \( Y^\mu \).
Proof. We first show the given set spans the algebra. This is clear by Theorem 5.1.5, as if \((m+2g)j = 1\), then \(b(2j)\) is involved in \(e_{m,g}\) and therefore \(e_{m,g}b(2j) = e_{m,g}\) or 0, and so can be expressed as a linear combination of the elements in the set.

It therefore remains to show that this set is linearly independent. Suppose that \(\sum i \alpha_i e_{m,g}b(i) = 0\), where \(b(i)\) is as in the set above. Then

\[
i = [i_0, i_1, \ldots] \text{ and } i_j = 1 \text{ only if } (m+2g)_i = 0.
\]

Note that:

- For \(i \neq j\), and \(e_{m,g}b(i), e_{m,g}b(j) \neq 0\), the binary expansions of \(i\) and \(j\) are unequal. We therefore see that the \(b(k)\) involved in \(e_{m,g}b(i)\) and \(e_{m,g}b(j)\) are distinct, else some \(b(k)\) has two different binary expansions, which is a contradiction.

- From the multiplication formula of the \(b(i)\), we see that the structure constants are in \(\mathbb{Z}\) and so we can reduce the coefficients modulo 2 and so the \(\alpha_i\) are 0 or 1.

Therefore suppose that \(\sum i \alpha_i e_{m,g}b(i) = 0\), and \(e_{m,g}b(i) = \sum k \beta_{i,k}b(k)\), where \(\beta_{i,k}\) is 0 or 1. If \(b(k)\) is involved in \(e_{m,g}b(i)\), then it is not involved in \(e_{m,g}b(j)\) if \(i \neq j\). We thus have that \(\sum \alpha_i \beta_{i,k}b(k) = 0\), where for each \(i\), there exists some \(k\) such that \(\beta_{i,k} \neq 0\). Therefore as the \(b(k)\) form a basis of \(S_K(\lambda)\), we have that \(\alpha_i \beta_{i,k} = 0\), and taking \(k\) as in the previous statement, we have that \(\beta_{i,k} \neq 0\) and so \(\alpha_i = 0\), for every \(i\). Therefore the \(e_{m,g}b(i)\) are linearly independent and so do indeed form a basis.

Example 2.9. Let \(\lambda = (t-1, t-1)\) for \(t \geq 1\), so that \(m = 0\). Take any partition \(\mu\) of \(2t\), say \(\mu = (\mu_1, \mu_2)\), and let \(g = \lambda_2 - \mu_2\). In this case,

\[
\binom{m+2g}{g} = \binom{2g}{g}
\]

and this is non-zero modulo 2 if and only if \(g = 0\). That is, the identity is a primitive idempotent in \(S_K(\lambda)\), and \(M^\lambda \cong Y^\lambda\) and its endomorphism algebra has dimension \(t\). [See also [5]]

3 The Algebra \(\text{End}_{K[\Sigma_n]}(Y^\mu)\)

At the end of this section, we will see that the generators of the Endomorphism Algebras have a notion of 'size', which dictates which generators (and their products) are zero. We therefore introduce the following order on products of generators \(x_i\):

We denote the set \(\{1, 2, \ldots, k\}\) by \(\mathbb{N}_k\).
Definition 3.1. Let $A$ be a commutative algebra with fixed generators \( \{x_1, \ldots, x_k\} \), such that these generators have square zero. Let $x = \prod_{i \in I} x_i$ be a non-zero element of $A$, where $i \in I \subset \mathbb{N}_k$, so that $x$ is a monomial in the generators. We also impose that $x$ has no repeated factors, so that $x$ is indeed non-zero. Then define the function $\phi$ as follows:

$$\phi(x) = \phi(\prod_{i \in I} x_i) = \sum_{i \in I} 2^i.$$ 

Definition 3.2. Let $A$ be a commutative algebra with fixed generators \( \{x_1, \ldots, x_k\} \), such that these generators have square zero. Let $x = \prod_{i \in I} x_i$, and $y = \prod_{j \in J} x_j$ be non-zero elements of $A$, where $I, J \subset \mathbb{N}_k$, so that $x$ and $y$ are monomials in the generators. We also impose that $x$ and $y$ have no repeated factors, so that $x$ and $y$ are indeed non-zero. Define the ordering $\preceq$ on $x$ and $y$ as follows:

$$x \preceq y \iff \phi(x) \leq \phi(y).$$

One can see that this order is in fact a total order.

We have that the algebra End$_{K[\Sigma_j]}(Y^\mu)$ is generated by the non-zero elements of

\( \{e_{m,g}b(2^k)\} \),

where $e_{m,g}$ is the idempotent of $M^\lambda$ corresponding to $Y^\mu$. For a minimal set of generators we prove the following lemma:

Lemma 3.3. If $b(2^i)$ is involved in $e_{m,g}$, then either $e_{m,g}b(2^i) = e_{m,g}$ or $e_{m,g}b(2^i) = 0$.

Proof. We proceed by induction on the index $i$. Assume first that $i = 0$. Suppose that $b(1)$ is involved in $e_{m,g}$. So either $e_{m,g}$ has a factor of either $b(1)$ or $1 + b(1)$. Recall that $b(1)^2 = mb(1)$ and so

- if the factor is $b(1)$, we know $b(1)^2 = m_0b(1)$. Then if $m_0 = 1$, we have that $e_{m,g}b(1) = e_{m,g}$ and if $m_0 = 0$, then $e_{m,g}b(1) = 0$.
- if the factor is $1 + b(1)$, we then have that $b(1)(1 + b(1)) = b(1) + m_0b(1)$. If $m_0 = 1$, we then have $e_{m,g}b(1) = 0$ and if $m_0 = 0$, then $e_{m,g}b(1) = e_{m,g}$,

and so the result holds for $i = 0$.

Now let $i > 0$ and let the result hold for all $k < i$. Consider $e_{m,g}b(2^i)$ where $b(2^i)$ is involved in $e_{m,g}$. The element $e_{m,g}b(2^i)$ therefore has a factor of $b(2^i)^2$ or $b(2^i) + b(2^i)^2$. By [4, Lemma 4.2], we have that:

- $b(2^i)^2 = b(2^i)[m_i \cdot 1 + \sum_{k=v-1}^{i-1} b(2^k)^2]$
- $b(2^i) + b(2^i)^2 = b(2^i)[1 + m_i \cdot 1 + \sum_{k=v-1}^{i-1} b(2^k)^2]$
Clearly if $e_{m,g}b(2^i) = e_{m,g}$, then $e_{m,g}b(2^i)^2 = e_{m,g}$, and if $e_{m,g}b(2^i) = 0$, then $e_{m,g}b(2^i)^2 = 0$. Then by induction hypothesis we have that

$$e_{m,g}b(2^i) = e_{m,g} \cdot (\text{sum of 1's and 0's}).$$

This is equal to $e_{m,g}$ or zero as we are in a field of characteristic 2, and hence the result holds by induction.

We therefore have that the generators of our algebra are the $b(2^i)$ not involved in $e_{m,g}$, which is precisely the set

$$\{e_{m,g}b(2^s) : (m + 2g)_s = 0\}$$

As $(m + 2g)$ is non-zero, the generators of the algebra therefore satisfy the conditions of the following lemma:

**Orthogonality Lemma.** Suppose the $s^{th}$ column in the binary expansion of $(m + 2g)$ is zero. Then $e_{m,g}b(2^s)^2 = 0$.

We therefore have that the generators of the endomorphism algebra all have square zero and so the algebra is isomorphic to a quotient of the form

$$\bigotimes_{i=1}^k K[x_i]/\langle x_i^2 \rangle \cong K[x_1, \ldots, x_k]/\langle x_i^2 : i = 1, \ldots, k \rangle$$

We now prove the following result:

**Theorem 3.4.** Let $K$ be a field of characteristic 2. Let $A$ be the Endomorphism Algebra of a Young module with dimension $n$. Then $A$ is isomorphic as $K$-algebra to

$$K[x_1, \ldots, x_k]/\langle x_i^2 \prod_i x_i : |x| > n - (k + 1) \rangle,$$

and so any two Endomorphism Algebras of Young modules with the same dimension are isomorphic.

**Proof.** Let $k$ be the unique non-negative integer such that

$$2^{k-1} < n \leq 2^k.$$ 

As $A$ is the endomorphism algebra of a Young module, we have that its generators have square zero and therefore $A$ is a quotient of the algebra

$$\bigotimes_{i} K[x_i]/\langle x_i^2 : i = 1, \ldots, k \rangle \cong K[x_1, \ldots, x_k]/\langle x_1^2, \ldots, x_k^2 \rangle.$$ 

So suppose the generators of the algebra $A$ are:

$$\{e_{m,g}b(2^{a_1}), e_{m,g}b(2^{a_2}), \ldots\},$$

where $a_1 < a_2 < a_3 < \ldots$. 

8
We then construct a correspondence between the $e_{m, g}(2^a_i)$ and $x_i$ as follows:

$$
\begin{align*}
e_{m, g}(2^a_1) & \rightarrow x_1 \\
e_{m, g}(2^a_2) & \rightarrow x_2 \\
& \vdots 
\end{align*}
$$

and make this an algebra homomorphism by extending linearly and to products in the natural way. To check that such a map is well defined, we require that the relations satisfied by the $e_{m, g}(i)$ are also satisfied by their images under this map of algebras.

We first have that both algebras are commutative and so the commutativity relation is satisfied.

We then see that all the generators of $A$, $e_{m, g}(2^a_i)$, have square zero and the $x_i$ satisfy this relation by construction. We also have that the products of generators $e_{m, g}(i)$ involving the generator $e_{m, g}(2^a_k)$ with the $n - (k + 1)$ smallest $i$'s are the only such products which are non-zero. This is because the algebra has dimension $n$ and if $e_{m, g}(i) \neq 0$, then all $e_{m, g}(j)$ such that $j < i$ are non-zero. These are the products of the $x_i$ such that

$$
\{x_i^2\} \cup \{\prod_i x_i : |x| > n - (k + 1)\},
$$

where $|x|$ is the position of the product in the total order defined in Definition 2. Again by construction we have that these elements are zero. These are all the relations in $A$ and so the map is well defined.

As we have a map that is surjective on the generators of the algebra, we have that it is surjective on the whole algebra.

To show that the map is injective, we argue by dimension. By definition, $A$ has dimension $n$. The polynomial algebra has basis vectors $1, x_1, x_2, \ldots, x_k$ and the $n - (k + 1)$ smallest products with respect to the total order of definition 2. Therefore there are $n$ non-zero vectors in its basis and it therefore has dimension $n$, and hence the result follows.

**Example 3.5.** In this example we look at the case $\text{End}_{K[\Sigma_r]}(M^{(n-1,n-1)})$ as it has dimension $n$. It is known (see [5, Theorem 2]) that $Y^{(n-1,n-1)} = M^{(n-1,n-1)}$, and so its endomorphism algebra has basis

$$
\{1, b(1), \ldots, b(n - 1)\}
$$

and so has dimension $n$. We see that in this case the generators of the algebra are $b(2^i)$ and therefore $A$ is isomorphic to a quotient of

$$
K[x_1, \ldots, x_k]/\langle x_i^2 \rangle,
$$

where $k$ is such that $2^{k-1} < n \leq 2^k$, which follows from the proof of the dimension theorem.
We therefore have that there are $2^{k-1}$ non-zero basis vectors of $A$ which span the proper subalgebra

$$K[x_1, \ldots, x_{k-1}] / \langle x_i^2 : i = 1, \ldots, k-1 \rangle,$$

and as $x_k$ is also a basis vector of $A$, we have that there are only $n - (2^{k-1} + 1)$ other possible basis vectors for $A$. If we label the generators of $A$ as follows:

$$b(2^i) \mapsto x_{i+1} \text{ for } i = 0, \ldots, k-1,$$

we see that the remaining $n - (2^{k-1} + 1)$ basis vectors will be the products of generators which involve $x_k$. As the dimension of $A$ is restricted only by the value of $n - 1$, we have that $b(i)$ is zero if and only if $i > n - 1$. Therefore we have that the monomials involving $x_k$ that are zero are given by the set:

$$R := \{ x_{r_1}x_{r_2} \ldots x_{r_l}x_k : r_i \neq r_j \text{ and } \sum_i 2^{r_i-1} + 2^{k-1} > n - 1 \},$$

and therefore $A = \text{End}_{K[\Sigma_r]}(Y^{r-1,r-1})$ is isomorphic to the algebra:

$$K[x_1, \ldots, x_k] / \langle \{ x_i^2 \} \cup R \rangle.$$

**Corollary 3.6 (Dimension Theorem).** Given an Endomorphism Algebra, $A$, of a Young module, $Y^\mu$, with dimension $n$, $A$ is isomorphic to the algebra

$$K[x_1, \ldots, x_k] / \langle \{ x_i^2 \} \cup R \rangle,$$

where $R$ is such that

$$R = \{ x_{r_1}x_{r_2} \ldots x_{r_l}x_k : r_i \neq r_j \text{ and } \sum_i 2^{r_i-1} + 2^{k-1} > n - 1 \}.$$

**Proof.** This follows from Theorem 3.4 and the preceding example. \qed

**References**

[1] Stephen Doty, Karin Erdmann, and Anne Henke. Endomorphism rings of permutation modules over maximal Young subgroups. *Journal of Algebra*, 307(1):377 - 396, 2007.

[2] Anne Henke. *Young Modules and Schur Subalgebras*. PhD thesis, Linacre College, University of Oxford, 1999.

[3] Karin Erdmann and Anne Henke. On Schur algebras, Ringel duality and symmetric groups. *Journal of Pure and Applied Algebra*, 169:175-199, 2002.

[4] Stephen Doty and Anthony Giaquinto. Presenting Schur Algebras as Quotients of the Universal Enveloping Algebra of $\mathfrak{gl}_2$. *Algebras and Representation Theory*, 7(1):1-17, 2004.

[5] Christopher C. Gill. Young module multiplicities and classifying the indecomposable Young permutation modules. *Journal of Algebra and its Applications*, 13(05):1350147 - 1 - 1350147 - 23, 2014.