A Generalized Framework for Virtual Substitution

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January 22, 2015

Abstract
We generalize the framework of virtual substitution for real quantifier elimination to arbitrary but bounded degrees. We make explicit the representation of test points in elimination sets using roots of parametric univariate polynomials described by Thom codes. Our approach follows an early suggestion by Weispfenning, which has never been carried out explicitly. Inspired by virtual substitution for linear formulas, we show how to systematically construct elimination sets containing only test points representing lower bounds.

1 Introduction

After Tarski’s seminal paper [23] there has been considerable research on quantifier elimination for real closed fields. One research line lead to complexity results and asymptotically fast algorithms [2, 11, 19, 24]. Unfortunately, these algorithms turned out not to be feasible in practice [12].

From a practical point of view, the invention of quantifier elimination by cylindrical algebraic decomposition (CAD) was an important step [6]. Several enhancements [7, 18] of the original procedure combined with an efficient implementation [4] made it possible to apply CAD-based algorithms to real-world problems to some extent [13]. One principal drawback of all CAD-based algorithms is the fact that parameters significantly contribute to the theoretical complexity [5].

Virtual substitution is an alternative approach, particularly strong for formulas with low degrees of the quantified variables, which is not sensitive to the number of parameters. Similarly to CAD there exist efficient implementations [9]. The original description of the method as well as first improvements and implementations were limited to linear formulas [24, 17]. The next important step focused on formulas up to total degree two of the quantified variables [27]. That publication furthermore suggested in a very abstract way to extend the procedure to arbitrarily large degree bounds, and mentioned Thom’s lemma [3] as a possibility for distinguishing real polynomial roots. In another publication Weispfenning made precise how to perform virtual substitution up to degree three, without using Thom codes, however [26].
Recently, virtual substitution has been playing some role in satisfiability modulo theory solving [1, 8]. There appears to be considerable interest in higher degrees [16], focusing on the software aspect rather than on theoretical foundations.

The present work picks up the original idea of Thom codes for generalizing virtual substitution to arbitrary but fixed degree bounds, developing theoretically precise foundations and a rigorous framework. Our original contributions are the following:

1. We describe an encoding of parametric polynomial roots based on Thom’s lemma. We prove that this encoding uniquely determines a root of a parametric univariate polynomial.

2. For a given encoding we specify formal necessary and sufficient conditions for the existence of a corresponding root. This allows to discard redundant elimination terms.

3. Our encoding allows to easily identify roots representing lower in contrast to upper bounds of relevant intervals.

4. We generally reduce the size of elimination sets by considering exclusively lower bounds of relevant intervals. This improves even the well-known elimination sets for the quadratic case.

The plan of the paper is as follows: In Section 2 we describe our root encoding based on Thom’s lemma. In Section 3 we introduce our framework by specifying what elimination terms look like and how they are substituted. We prove the correctness of the framework. In Section 4 we reduce the size of elimination sets by considering exclusively lower bounds of relevant intervals. In Section 5 we discuss our framework in the context of alternative approaches in the literature. We furthermore point at its compatibility with various generalizations of quantifier elimination by virtual substitution. In Section 6 we discuss practical issues related to possible implementation strategies.

2 Parametric Roots

Let \( p \in \mathbb{Z}[y_1, \ldots, y_m][x] \) with \( \deg p = n \), and let \( \alpha_1, \ldots, \alpha_m \in \mathbb{R} \). Denote by \( (\alpha_1, \ldots, \alpha_m) : \mathbb{Z}[y_1, \ldots, y_m] \to \mathbb{R} \) the evaluation homomorphism in postfix notation, i.e., we have \( p(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}[x] \). Consider arbitrary \( f \in \mathbb{R}[x] \). For \( \xi \in \mathbb{R} \) we define the sign sequence of length \( k \) at \( \xi \) as follows:

\[
\text{sgn}_\xi(f, k) = (\text{sgn}(f(\xi)), \text{sgn}(f'(\xi)), \ldots, \text{sgn}(f^{(k-1)}(\xi))).
\]

A sign sequence \( \sigma = (s_1, \ldots, s_n) \in \{-1, 0, 1\}^n \) is consistent with \( p \) if there exist \( \alpha_1, \ldots, \alpha_m, \xi \in \mathbb{R} \) such that the following conditions hold:

\begin{align*}
(C_1) \quad & \deg p(\alpha_1, \ldots, \alpha_m) > 0, \\
(C_2) \quad & p(\alpha_1, \ldots, \alpha_m)(\xi) = 0, \\
(C_3) \quad & \text{sgn}_\xi(p(\alpha_1, \ldots, \alpha_m)', n) = \sigma.
\end{align*}
positive degree, i.e., $\Gamma(p, s) \neq 0$.

### Lemma 1

Let $p \in \mathbb{Z}[y_1, \ldots, y_m][x]$ with $\deg p = n > 0$. Let $\bar{s} \neq (0, \ldots, 0)$ be a sign sequence of length $n$, and let $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$. The following are equivalent:

(i) $\Gamma(p, \bar{s})$ holds for $\alpha_1, \ldots, \alpha_m$.

(ii) There exists a unique $\xi \in \mathbb{R}$ such that conditions (C1) and (C2) hold.

### Proof

Assume (i). It is easy to see that there exists $\xi \in \mathbb{R}$ such that (C1) and (C2) hold. Denote $p(\alpha_1, \ldots, \alpha_m)$ by $p_n$. Then $s \neq (0, \ldots, 0)$ implies that at least one of the polynomials $p_n, \ldots, p_n(s)$, is not identically zero. Hence, $p_n$ is of positive degree, i.e., (C1) holds as well. The uniqueness of $\xi$ follows directly from Thom’s little lemma [3].

Assume (ii). Then it is easy to see that $\alpha_1, \ldots, \alpha_m$ satisfy $\Gamma(p, \bar{s})$, even with a unique $x$. 

### Table 1: Consistent sign sequences and guards

| $\bar{s}$ | $\Gamma(p, \bar{s})$ | $f$ |
|----------|----------------------|-----|
| $(-1, -1)$ | $a < 0 \land 4ac - b^2 < 0$ | $-x^2 + x$ |
| $(-1, 0)$ | $a = 0 \land b < 0$ | $-x + 1$ |
| $(-1, 1)$ | $a > 0 \land 4ac - b^2 < 0$ | $x^2 - 3x + 2$ |
| $(0, -1)$ | $a < 0 \land 4ac - b^2 = 0$ | $-x^2 + 2x - 1$ |
| $(0, 1)$ | $a > 0 \land 4ac - b^2 = 0$ | $x^2 - 2x + 1$ |
| $(1, -1)$ | $a < 0 \land 4ac - b^2 < 0$ | $-x^2 + 3x + 2$ |
| $(1, 0)$ | $a = 0 \land b > 0$ | $x - 1$ |
| $(1, 1)$ | $a > 0 \land 4ac - b^2 < 0$ | $x^2 - x$ |

It follows that $(0, \ldots, 0)$ is not consistent with any $q \in \mathbb{Z}[y_1, \ldots, y_m][x]$. The idea is that $\bar{s}$ uniquely describes a real root $\xi$ of $p(\alpha_1, \ldots, \alpha_m)$.

A guard $\Gamma(p, \bar{s})$ for $p$ and $\bar{s} = (s_1, \ldots, s_n) \in \{-1, 0, 1\}^n$ is a quantifier-free equivalent in variables $y_1, \ldots, y_m$ of the Tarski formula

$$\exists x \left( p = 0 \land \bigwedge_{i=1}^{\left\lceil \bar{s} \right\rceil} (p^{(i)}(\alpha_i 0)) \right),$$

where

$$\sigma(s_i) = \begin{cases} 
"<" & \text{if } s_i = -1, \\
"\approx" & \text{if } s_i = 0, \\
"\approx" & \text{if } s_i = 1.
\end{cases}$$

Lemma 1 ensures that $\bar{s} \neq (0, \ldots, 0)$ is consistent with $p$ if and only if $\Gamma(p, \bar{s})$ is satisfiable. In the positive case, $\Gamma(p, \bar{s})$ gives a necessary and sufficient condition for $p$ to have a root that is described by $\bar{s}$.

As an example consider $p = ax^2 + bx + c$. Table 1 lists all sign sequences consistent with $p$ along with their respective guards. The polynomials $f$ in the table are obtained from $p$ by substituting for $a$, $b$, and $c$ suitable values satisfying the corresponding condition $\Gamma(p, \bar{s})$ in the table. They all have 1 as a root described by the corresponding $\bar{s}$, formally, $\text{sgn}_1(f', 2) = \bar{s}$.

Let $\bar{s}$ be consistent with $p$. We call the pair $(p, \bar{s})$ a parametric root of $p$. Recall from the definition of consistency of $\bar{s}$ with $p$ that there are particular
\( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \) such that \( \bar{s} \) uniquely describes a real root of \( p(\alpha_1, \ldots, \alpha_m) \).

Note that these \( \alpha_1, \ldots, \alpha_m \in \mathbb{R} \) need not be unique. The possible choices for \( \alpha_1, \ldots, \alpha_m \) are in fact described by \( \Gamma(p, \bar{s}) \).

The following lemma considers two such possible choices, and states that the signs of the obtained univariate polynomials in the neighborhoods of the corresponding roots are invariant with respect to the two choices.

**Lemma 2.** Let \( p \in \mathbb{Z}[y_1, \ldots, y_m][x] \) with \( \deg p = n > 0 \). Let \( \bar{s} = (s_1, \ldots, s_n) \) be consistent with \( p \), where \( \alpha_1, \ldots, \alpha_m, \xi \in \mathbb{R} \), and \( \beta_1, \ldots, \beta_m, \zeta \in \mathbb{R} \) are two possible sets of choices in \( \langle C_1 \rangle \langle C_2 \rangle \langle C_3 \rangle \). Denote \( p(\alpha_1, \ldots, \alpha_m) \) by \( p_\alpha \), and \( p(\beta_1, \ldots, \beta_m) \) by \( p_\beta \). Let \( \varepsilon \) be a positive infinitesimal number. Then the following hold:

(i) \( \sgn(p_\alpha(\xi - \varepsilon)) = \sgn(p_\beta(\xi - \varepsilon)) \)

(ii) \( \sgn(p_\alpha(\xi + \varepsilon)) = \sgn(p_\beta(\xi + \varepsilon)) \).

**Proof.** We show only (i), the proof of (ii) is similar. To start with, note that \( (C_2) \langle C_3 \rangle \) ensure that

\[
\sgn_{\xi}(p_\alpha, n) = \sgn_{\xi}(p_\beta', n) = \bar{s}.
\]

We show by induction that for \( i \in \{n, \ldots, 0\} \) the following holds:

\[
\sgn(p_\alpha^{(i)}(\xi - \varepsilon)) = \sgn(p_\beta^{(i)}(\xi - \varepsilon)).
\]

For \( i = n \), observe that \( p_\alpha^{(n)}, p_\beta^{(n)} \in \mathbb{R} \). From (I) it follows that \( \sgn(p_\alpha^{(n)}(\xi)) = \sgn(p_\beta^{(n)}(\xi)) = s_n \), and we can conclude that \( \sgn(p_\alpha^{(n)}(\xi - \varepsilon)) = \sgn(p_\beta^{(n)}(\xi - \varepsilon)) \).

Let now \( k \in \{n - 1, \ldots, 0\} \), and assume that \( \sgn(p_\alpha^{(k+1)}(\xi - \varepsilon)) = \sgn(p_\beta^{(k+1)}(\xi - \varepsilon)) \). We have to show that \( \sgn(p_\alpha^{(k)}(\xi - \varepsilon)) = \sgn(p_\beta^{(k)}(\xi - \varepsilon)) \). We distinguish cases. If \( s_k \neq 0 \), then \( (I) \) ensures that \( \sgn(p_\alpha^{(k)}(\xi)) = \sgn(p_\beta^{(k)}(\xi)) = s_k \neq 0 \).

If follows that \( \sgn(p_\alpha^{(k)}(\xi - \varepsilon)) = \sgn(p_\beta^{(k)}(\xi - \varepsilon)) \), because \( \varepsilon \) is infinitesimal. Assume now that \( s_k = 0 \), and distinguish three cases:

(a) \( \sgn(p_\alpha^{(k)}(\xi - \varepsilon)) = 0 \): Since \( \sgn(p_\alpha^{(k)}(\xi)) = s_k = 0 \), and \( \varepsilon \) is infinitesimal, \( p_\alpha^{(k)} \) is the zero polynomial. Thus, \( p_\alpha^{(k+1)} \) is the zero polynomial as well.

The induction hypothesis and our assumption \( s_k = 0 \) yield that \( p_\beta^{(k+1)} \) is the zero polynomial as well. This means that \( p_\beta^{(k)} \) is a constant polynomial.

On the other hand, \( \sgn(p_\beta^{(k)}(\xi)) = s_k = 0 \). Therefore, \( p_\beta^{(k)} \) is the zero polynomial, in particular \( \sgn(p_\beta^{(k)}(\xi - \varepsilon)) = 0 \).

(b) \( \sgn(p_\alpha^{(k)}(\xi - \varepsilon)) = 1 \): Since \( \varepsilon \) is positive and infinitesimal and \( p_\alpha^{(k)}(\xi - \varepsilon) > p_\alpha^{(k)}(\xi) \), it follows that \( p_\alpha^{(k)} \) is decreasing at \( \xi - \varepsilon \). Therefore, we have \( \sgn(p_\alpha^{(k+1)}(\xi - \varepsilon)) = -1 \). By the induction hypothesis, it follows that \( \sgn(p_\beta^{(k+1)}(\xi - \varepsilon)) = -1 \). Therefore, \( p_\beta^{(k+1)} \) is decreasing at \( \xi - \varepsilon \). Finally, our assumption \( s_k = 0 \) ensures that \( \sgn(p_\beta^{(k)}(\xi)) = s_k = 0 \). Since \( \varepsilon \) is infinitesimal, we have \( p_\beta^{(k)}(\xi - \varepsilon) > 0 \), i.e., \( \sgn(p_\beta^{(k)}(\xi - \varepsilon)) = 1 \).

(c) \( \sgn(p_\alpha^{(k)}(\xi - \varepsilon)) = -1 \): Similar to (b). \( \square \)
Consider a parametric root \((p, \bar{s})\), and let \(\alpha_1, \ldots, \alpha_m \in \mathbb{R}\) be such that \(\Gamma(p, \bar{s})\) holds. By Lemma 1 there exists a unique \(\xi \in \mathbb{R}\) such that \([C_1]\) holds. We define the left sign of \((p, \bar{s})\) as \(\text{sgn}(p, \bar{s}) = \text{sgn}(p(\alpha_1, \ldots, \alpha_m)/(\xi - \varepsilon))\). Similarly, the right sign of \((p, \bar{s})\) is defined as \(\text{sgn}(p, \bar{s}) = \text{sgn}(p(\alpha_1, \ldots, \alpha_m)/(\xi + \varepsilon))\). This is well-defined by Lemma 2. Notice that \([C_1]\) ensures that both \(\text{sgn}(p, \bar{s})\) and \(\text{sgn}(p, \bar{s})\) cannot be zero.

The left and the right sign of \((p, \bar{s})\) can be computed as follows:

1. Find \(\alpha_1, \ldots, \alpha_m \in \mathbb{R}\) satisfying \(\Gamma(p, \bar{s})\).

2. Compute an isolating interval \([l, r]\) of the root \(\xi\) of \(p(\alpha_1, \ldots, \alpha_m)\) identified by \(\bar{s}\) such that \(p(\alpha_1, \ldots, \alpha_m)(l) \neq 0\) and \(p(\alpha_1, \ldots, \alpha_m)(r) \neq 0\). Then \(\text{sgn}(p, \bar{s}) = \text{sgn}(p(\alpha_1, \ldots, \alpha_m)(l))\) and \(\text{sgn}(p, \bar{s}) = \text{sgn}(p(\alpha_1, \ldots, \alpha_m)(r))\).

### 3 Elimination Terms and Elimination Sets

Let \(n \in \mathbb{N}\setminus\{0\}\), and let \(\varphi\) be an \(\land\lor\)-combination of atomic formulas \(\{p_i, q_i\}_{i \in I}\), where \(p_i \in \mathbb{Z}[y_1, \ldots, y_m][x]\) and \(q_i \in \{\land, \lor, <, \geq, >\}\). Assume that \(\deg p_i \leq n\) for all \(i \in I\). We say that \(\varphi\) is of degree at most \(n\) in \(x\). In this section we are going to describe a method for eliminating \(\exists x\) from the formula \(\exists x\varphi\).

Our method is not self-contained. It depends on an algorithm \(\mathcal{A}\) that is capable of eliminating a single existential quantifier from formulas of degree \(n\) in \(x\), which have a very particular shape. We are going to describe these formulas along with the definition of virtual substitution of elimination terms. Later in Section 6 we will show that it is even sufficient to consider only finitely many such formulas. Since the real numbers admit effective quantifier elimination, it is clear that such an algorithm \(\mathcal{A}\) exists. The key challenge with the approach proposed here is going to be to find short quantifier-free equivalents, possibly including considerable human intelligence at least for post-processing.

We start with the description of the set \(E_i\) of elimination terms generated by a particular atomic formula \((p_i, q_i, 0)\) in \(\varphi\):

\[
E_i = \begin{cases} 
\{(p_i, \bar{s}) \mid (p_i, \bar{s}) \text{ is a parametric root of } p_i\} & \text{if } q_i \in \{\land, \lor, \geq\}, \\
\{(p_i, \bar{s}) + \varepsilon \mid (p_i, \bar{s}) \text{ is a parametric root of } p_i\} & \text{if } q_i \in \{\land, \lor, >\}.
\end{cases}
\]

Consider an atomic formula \((q \varphi 0)\), where \(q = b_nx^n + \cdots + b_1x + b_0 \in \mathbb{Z}[y_1, \ldots, y_m][x]\). The virtual substitution \((q \varphi 0)[x/r]\) of an elimination term \(r = (p, \bar{s})\) for \(x\) into \((q \varphi 0)\) is the quantifier-free formula computed by \(\mathcal{A}\) for

\[
\exists x \left( p = 0 \land \bigwedge_{i=1}^{\mid \bar{s} \mid} (p^{(i)}(s_i) 0) \land (q \varphi 0) \right).
\]

Let \(N\) and \(P\) be the sets of all sign sequences \(\bar{I}\) consistent with \(q\) such that \(\text{sgn}(q, \bar{I}) < 0\) and \(\text{sgn}(q, \bar{I}) > 0\), respectively. Furthermore, let be \(\nu((p, \bar{s}), (q, \bar{I}))\) the quantifier-free formula computed by \(\mathcal{A}\) for

\[
\exists x \left( p = 0 \land \bigwedge_{i=1}^{\mid \bar{s} \mid} (p^{(i)}(s_i) 0) \land q = 0 \land \bigwedge_{i=1}^{\mid \bar{I} \mid} (q^{(i)}(t_i) 0) \right).
\]
In these terms, the virtual substitution \((q \varrho 0)[x // r + \epsilon]\) of \(r + \epsilon\) for \(x\) is defined as follows:

\[
\begin{align*}
(q = 0)[x // r + \epsilon] & : \quad b_n = 0 \land \cdots \land b_0 = 0 \\
(q \neq 0)[x // r + \epsilon] & : \quad b_n \neq 0 \lor \cdots \lor b_0 \neq 0 \\
(q < 0)[x // r + \epsilon] & : \quad (q < 0)[x // r] \lor \bigvee_{\tau \in N} \nu((p, \overline{s}), (q, \overline{t})) \\
(q > 0)[x // r + \epsilon] & : \quad (q > 0)[x // r] \lor \bigvee_{\tau \in P} \nu((p, \overline{s}), (q, \overline{t})) \\
(q \leq 0)[x // r + \epsilon] & : \quad (q < 0)[x // r + \epsilon] \lor (q = 0)[x // r + \epsilon] \\
(q \geq 0)[x // r + \epsilon] & : \quad (q > 0)[x // r + \epsilon] \lor (q = 0)[x // r + \epsilon].
\end{align*}
\]

Note that, in contrast to [27, Section 3], our definition of virtual substitution of \(r + \epsilon\) is not recursive. The deeply nested subformulas introduced with the recursive definition of \(\nu\) for quadratic quantifier elimination in [27, Section 3] are a considerable obstacle for simplification. A good choice of \(\mathcal{A}\) might help to overcome this.

Another somewhat special elimination term is \(-\infty\). It is not generated by any atomic formula, but will generally occur in every elimination set. The virtual substitution \((q \varrho 0)[x // -\infty]\) of \(-\infty\) for \(x\) is defined as follows:

\[
\begin{align*}
(q = 0)[x // -\infty] & : \quad b_n = 0 \land \cdots \land b_0 = 0 \\
(q \neq 0)[x // -\infty] & : \quad b_n \neq 0 \lor \cdots \lor b_0 \neq 0 \\
(q < 0)[x // -\infty] & : \quad \mu_<(q, n) \lor \cdots \lor \mu_<(q, 0) \\
(q > 0)[x // -\infty] & : \quad \mu_>(q, n) \lor \cdots \lor \mu_>(q, 0) \\
(q \leq 0)[x // -\infty] & : \quad (q < 0)[x // -\infty] \lor (q = 0)[x // -\infty] \\
(q \geq 0)[x // -\infty] & : \quad (q > 0)[x // -\infty] \lor (q = 0)[x // -\infty],
\end{align*}
\]

where for \(k \in \{0, \ldots, n\}\) we define

\[
\begin{align*}
\mu_<(q, k) & : \quad b_n = 0 \land \cdots \land b_{k+1} = 0 \land (-1)^k b_k < 0 \\
\mu_>(q, k) & : \quad b_n = 0 \land \cdots \land b_{k+1} = 0 \land (-1)^k b_k > 0.
\end{align*}
\]

As usual, the key idea of virtual substitution is to map not terms to terms, but atomic formulas to quantifier-free formulas, which gives us the freedom to describe its results in the Tarski language. That mapping on atomic formulas naturally induces virtual substitution on arbitrary quantifier-free formulas.

In the sequel we prove that the terms generated by all atomic formulas occurring in \(\varphi\) together with \(-\infty\) constitute an elimination set \(E\) for \(\exists x \varphi\) in the following sense:

\[
\mathbb{R} \models \exists x \varphi \iff \bigvee_{e \in E} \varphi[x // e].
\]

The following lemma is an immediate consequence of our definition of virtual substitution:

**Lemma 3.** Let \((q \varrho 0)\) be an atomic formula, where \(q \in \mathbb{Z}[y_1, \ldots, y_m][x]\). Consider a parametric root \((p, \overline{s})\). Assign values \(\alpha_1, \ldots, \alpha_m \in \mathbb{R}\) to the variables \(y_1, \ldots, y_m\). Then the following hold:
(i) If \((q \not\equiv 0)[x \parallel (p, \bar{\sigma})]\) holds, then \(\Gamma'(p, \bar{\sigma})\) holds as well.

(ii) If \((q \not\equiv 0)[x \parallel (p, \bar{\sigma}) + \varepsilon]\) holds, and \(q \in \{<, >\}\), then \(\Gamma'(p, \bar{\sigma})\) holds as well. \(\square\)

**Lemma 4.** Consider \(\{p_k, q_k\}_{k \in K}\), where \(K\) is finite, \(p_k, q_k \in \mathbb{Z}[y_1, \ldots, y_m][x]\), and \(q_k \in \{=, \neq, <, \leq, >, \geq\}\). Assign values \(\alpha_1, \ldots, \alpha_m \in \mathbb{R}\) to the variables \(y_1, \ldots, y_m\). Consider a parametric root \((p, \bar{\sigma})\).

(i) There exists \(\vartheta \in \mathbb{R}\) such that for all \(k \in K\) the following holds: If \((p_k q_k 0)[x \parallel (p, \bar{\sigma})]\) holds, then \(\vartheta\) satisfies \((p_k q_k 0)\).

(ii) There exists \(\zeta \in \mathbb{R}\) such that for all \(k \in K\) the following holds: If \((p_k q_k 0)[x \parallel (p, \bar{\sigma})]\) holds, then \(\zeta\) satisfies \((p_k q_k 0)\).

**Proof.** To begin the proof of (i), note that if there is no \(k \in K\) such that \((p_k q_k 0)[x \parallel (p, \bar{\sigma})]\) holds, there is nothing to prove. Assume w.l.o.g. that \((p_1 q_1 0)[x \parallel (p, \bar{\sigma})]\) holds. Lemma \(3\) implies that \(\Gamma(p, \bar{\sigma})\) holds as well. Consequently, Lemma \(1\) ensures that there is a unique \(\xi \in \mathbb{R}\) such that \((C_1)[(C_3)]\) hold. Let now \(k \in K\) and assume that \((p_k q_k 0)[x \parallel (p, \bar{\sigma})]\), i.e.,

\[
\exists x \left( p = 0 \land \bigwedge_{i=1}^{[\tau]} (p^{(i)} \sigma(s_i) 0) \land (p_k q_k 0) \right),
\]

holds. The uniqueness of \(\xi\) satisfying conditions \((C_1)[(C_3)]\) now ensures that \((p_k q_k 0)\) holds for \(\xi\). This proves (i).

To prove (ii) we first assume that there is some \(l \in K\) such that \((p_l < 0)[x \parallel (p, \bar{\sigma}) + \varepsilon]\) or \((p_l > 0)[x \parallel (p, \bar{\sigma}) + \varepsilon]\) holds. Again, Lemma \(3\) ensures that \(\Gamma(p, \bar{\sigma})\) holds, and Lemma \(1\) ensures that there is a unique \(\xi, \zeta \in \mathbb{R}\) such that conditions \((C_1)[(C_3)]\) hold. Pick \(\zeta'\) from the open interval \([\xi, \xi']\), where

\[
\zeta' = \min \{\delta \mid j \in K \land \deg p_j > 0 \land p_j(\delta) = 0 \land \delta > \zeta\}.
\]

Let now \(k \in K\), and assume that \((p_k q_k 0)[x \parallel (p, \bar{\sigma}) + \varepsilon]\) holds. If \(q_k\) is \(\neq\), then \(p_k\) is identically zero, so \((p_k = 0)\) trivially holds for \(\zeta\). If \(q_k\) is \(\neq\), then there are two cases to consider:

(a) In the first case \((p_k < 0)[x \parallel (p, \bar{\sigma})]\) holds. Since \(\xi \in \mathbb{R}\) is unique satisfying conditions \((C_1)[(C_3)]\), we deduce that \((p_k < 0)\) holds at \(\xi\). The choice of \(\zeta\) now guarantees that there is no root of \(p_k\) in \([\xi, \zeta]\), so we obtain that \((p_k < 0)\) holds at \(\zeta\), as well.

(b) In the second case

\[
\exists x \left( p = 0 \land \bigwedge_{i=1}^{[\tau]} (p^{(i)} \sigma(s_i) 0) \land p_k = 0 \land \bigwedge_{i=1}^{[\tau]} (p_k^{(i)} \sigma(t_i) 0) \right)
\]

holds for some \(\tilde{\zeta}\), and \(\text{sgn}((p_k, \tilde{\zeta})) = -1\). Again, from the uniqueness of \(\xi \in \mathbb{R}\) satisfying \((C_1)[(C_3)]\) we deduce that \(p_k = 0\) holds at \(\xi\). Finally, since the right sign of \((p_k, \tilde{\zeta})\) is negative, and there is no root of \(p_k\) in \([\xi, \zeta]\), so we conclude that \(p_k < 0\) holds at \(\zeta\).
If \( \varrho_k \) is “>,” then the proof is done similarly as for “<.” If \( \varrho_k \) is “\( \not= \)” we apply the lemma either to \( (p_k > 0)[x // (p, \overline{s}) + \varepsilon] \) or to \( (p_k < 0)[x // (p, \overline{s}) + \varepsilon] \) to obtain that \( (p_k \not= 0) \) holds for \( \zeta \). When \( \varrho \in \{\leq, \geq\} \) the proof is similar.

To finish the proof of (ii), we assume that for every \( l \in K \) the formulas \( (p_l < 0)[x // (p, \overline{s}) + \varepsilon] \) and \( (p_l > 0)[x // (p, \overline{s}) + \varepsilon] \) do not hold. Let \( \zeta \) be any real number. Let now \( k \in K \), and assume that \( (p_k \varrho_k 0)[x // (p, \overline{s}) + \varepsilon] \) holds. Our assumption together with the definition of \( [x // (p, \overline{s}) + \varepsilon] \) imply that \( (p_k \varrho_k 0)[x // (p, \overline{s}) + \varepsilon] \) is equivalent to \( (p_k = 0)[x // (p, \overline{s}) + \varepsilon] \), which shows that \( (p_k \varrho_k 0) \) holds for \( \zeta \).

**Lemma 5.** Let \( (q \varrho 0) \) be an atomic formula, where \( q \in \mathbb{Z}[y_1, \ldots, y_m][x] \), and \( \varrho \in \{\not=, \not<, \not>, >\} \). Consider a parametric root \((p, \overline{s})\). Assign values \( \alpha_1, \ldots, \alpha_m \in \mathbb{R} \) to the variables \( y_1, \ldots, y_m \). Assume that \( \Gamma(p, \overline{s}) \) holds, and denote the unique real number satisfying conditions \([C_1], [C_3]\) by \( \xi \). Then the following hold:

(i) If \( \xi \) satisfies \( (q \varrho 0) \), then \( (q \varrho 0)[x // (p, \overline{s})] \) holds.

(ii) Let \( \xi' \in \mathbb{R} \) be such that all \( \xi \in [\xi, \xi'] \) satisfy \( (q \varrho 0) \). Then \( (q \varrho 0)[x // (p, \overline{s}) + \varepsilon] \) holds.

**Proof.** Since \( (q \varrho 0) \) holds for \( \xi \in \mathbb{R} \), and \( \xi \) is unique such that \([C_1], [C_3]\) hold, we obtain that

\[
(p = 0 \land \bigwedge_{i=1}^{|\overline{s}|} (p^{(i)} \sigma(s_i) 0 \land (q \varrho 0))
\]

holds for \( \xi \). By the definition of \([x // (p, \overline{s})]\), we see that \( (q \varrho 0)[x // (p, \overline{s})] \) holds. This proves (i).

To prove (ii), we begin with the case when \( \varrho \) is “\( =\)” Since \( (q = 0) \) holds in a non-empty interval, \( q \) is the zero polynomial, so \( (q = 0)[x // (p, \overline{s}) + \varepsilon] \) follows. If \( \varrho \) is “\( \not=\)” \( (q \not= 0) \) holds in a non-empty interval, so \( q \) is a non-zero polynomial. Therefore, \( (q \not= 0)[x // (p, \overline{s}) + \varepsilon] \) holds. If \( \varrho \) is “\( <\)” there are two possibilities. If \( (q < 0) \) holds at \( \xi \), then by (i) we have that \( (q < 0)[x // (p, \overline{s})] \) holds, so \( (q < 0)[x // (p, \overline{s}) + \varepsilon] \) holds, as well. If \( (q < 0) \) does not hold for \( \xi \), then \( (q = 0) \) holds for \( \xi \), because \( (q < 0) \) holds for all \( \xi \in [\xi, \xi'] \). Therefore, there is a Thom code \( \tilde{7} \) identifying \( \xi \) as a root of \( q \), and the right sign of \( (q, \tilde{7}) \) is negative. This ensures that \( (q < 0)[x // (p, \overline{s}) + \varepsilon] \) holds in this case, as well. The proof for the case when \( \varrho \) is “\( >\)” is similar. Finally, the cases \( \varrho \) is “\( <\)” or “\( >\)” boil down to cases which we have just proven. Finally, we conclude that (ii) holds. \( \square \)

**Theorem 6 (Correctness of the Elimination Set).** Consider a formula \( \varphi \) of degree at most \( n \) in \( x \), which is an \( \land \lor \)-combination of atomic formulas \( \{p_i, \varrho_i, 0\}_{i \in I} \), where \( p_i \in \mathbb{Z}[y_1, \ldots, y_m][x] \), and \( \varrho_i \in \{\not=, \not<, \not>, >\} \). Let \( E_i \) be the set of elimination terms generated by \( (p_i, \varrho_i, 0) \) as described above. Then the following is an elimination set for \( \exists x \varphi \):

\[
E = \bigcup_{i \in I} E_i \cup \{-\infty\}.
\]

**Proof.** We have to show that

\[
\mathbb{R} \models \exists x \varphi \iff \bigvee_{e \in E} \varphi[x // e].
\]

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Assign values $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ to variables $y_1, \ldots, y_m$. The sets $S_i \subseteq \mathbb{R}$ satisfying $(p_i, \varrho_i, 0)$—and therefore also the set $S \subseteq \mathbb{R}$ satisfying $\varphi$—are now finite unions of pairwise disjoint (closed, half-closed, open, semi-infinite, or infinite) intervals. We show that $\bigvee_{e \in E} \varphi[x/e]$ holds if and only if $S \neq \emptyset$. Since $\alpha_1, \ldots, \alpha_m$ are arbitrary real values, this will imply the theorem.

First assume that $\bigvee_{e \in E} \varphi[x/e]$ holds. Thus, $\varphi[x/e]$ holds for some $e \in E$. If $\varphi[x/\infty]$ holds, then the set of values satisfying $\varphi$ is unbounded from below, i.e., non-empty.

Now suppose that $\varphi[x/(p, \tau)]$ holds for some parametric root $(p, \tau)$. By Lemma 3(ii), there exists $\vartheta \in \mathbb{R}$ such that for any $i \in I$ we have: $(p_i, \varrho_i, 0)$ holds for $\vartheta$ whenever $(p_i, \varrho_i, 0)[x/(p, \tau)]$ holds. The fact that $\varphi$ is an $\land$-$\lor$-combination of atomic formulas ensures that $\varphi$ holds for $\vartheta$.

Assume that $\varphi[x/(p, \tau) + \varepsilon]$ holds for some parametric root $(p, \tau)$. By Lemma 3(ii), there exists $\zeta \in \mathbb{R}$ such that for any $i \in I$ we have: $(p_i, \varrho_i, 0)$ holds for $\zeta$ whenever $(p_i, \varrho_i, 0)[x/(p, \tau)]$ holds. Again, $\varphi$ is an $\land$-$\lor$-combination of atomic formulas, so $\varphi$ holds for $\zeta$.

We continue by assuming that $S$ is non-empty. To begin with, notice that if $S$ is unbounded from below, then $\varphi[x/\infty]$ holds. In the following we therefore assume that $S$ is bounded from below. Since $S$ is a finite union of pairwise disjoint (closed, half-closed, open, semi-infinite, or infinite) intervals, there exists infimum $\xi$ of $S$ such that $\xi$ is a root of some polynomial occurring in $\varphi$. There are two cases to consider.

In the first case $\xi \in S$. This implies that there is some $i \in I$ such that $(p_i, \varrho_i, 0)$ holds for $\xi$, and $\varrho_i \in \{=, <, >\}$. Moreover, $\xi$ is a root of $p_i$ with Thom code $\tau$. By the definition of $E_i$ we conclude that $(p_i, \tau) \in E_i$. Now Lemma 5(i) together with the fact that $\varphi$ is an $\land$-$\lor$ combination of atomic formulas ensure that $\varphi[x/(p_i, \tau)]$ holds.

In the second case $\xi \notin S$, so there is some $i \in I$ such that $(p_i, \varrho_i, 0)$, $\varrho_i \in \{\neq, <, >\}$, holds for all $\zeta \in [\xi, \xi']$, where

$$\xi' = \min\{\delta \mid j \in I \land \deg p_j > 0 \land p_j(\delta) = 0 \land \delta > \xi\}.$$ 

Furthermore, $\xi$ is a root of $p_i$ with Thom code $\tau$. By the definition of $E_i$ we have $(p_i, \tau) + \varepsilon \in E_i$. Now Lemma 5(ii) together with the fact that $\varphi$ is an $\land$-$\lor$ combination of atomic formulas ensure that $\varphi[x/(p_i, \tau) + \varepsilon]$ holds. This concludes the proof of the theorem.

We conclude this section with Algorithm 4, which eliminates a single existential quantifier. We explicitly point to places where the external quantifier elimination algorithm $A$ is used. The correctness of Algorithm 4 follows from Theorem 6.

## 4 Smaller Elimination Sets

In this section we are going to considerably reduce the size of our elimination sets by generalizing a well-known idea from the linear quantifier elimination [17] to arbitrary degrees: An elimination term $t$ is added to the elimination set for $\varphi$ only if $t$ possibly represents a lower bound of a satisfying interval of $\varphi$ for some choice of parameters $y_1, \ldots, y_m$. 
Consider a single atomic formula \((p \leq 0)\), where \(p \in \mathbb{Z}[y_1, \ldots, y_m][x]\). Let \(\alpha_1, \ldots, \alpha_m \in \mathbb{R}\), and assume that the polynomial \(p(\alpha_1, \ldots, \alpha_m)\) is of positive degree and has a real root \(\xi \in \mathbb{R}\). There are four possibilities how \(p(\alpha_1, \ldots, \alpha_m)\) can look in the neighborhood of \(\xi\), all of which are pictured in Figure 1. The corresponding satisfying sets of \((p \leq 0)\), i.e., only when \(p(\alpha_1, \ldots, \alpha_m)\) is positive on the left-hand side of \(\xi\). Consequently, if \(p(\alpha_1, \ldots, \alpha_m)\) is negative on the left-hand side of \(\xi\), then there is either another root \(\zeta\) smaller than \(\xi\), or \(p(\alpha_1, \ldots, \alpha_m)\) holds at \(-\infty\).

Similar ideas apply to all other relations, which motivates the following revised definition of the sets of elimination terms \(E'_i\) generated by \((p_i, \varrho_i, 0)\), where again \((p_1, \varnothing)\) is a parametric root of \(p_i\):

\[
\begin{align*}
(p_i = 0) & : \quad \{ r \mid r = (p_i, \varnothing) \} \\
(p_i \neq 0) & : \quad \{ r + \varepsilon \mid r = (p_i, \varnothing) \} \\
(p_i < 0) & : \quad \{ r + \varepsilon \mid r = (p_i, \varnothing) \land \text{sgn}(r) = -1 \} \\
(p_i > 0) & : \quad \{ r + \varepsilon \mid r = (p_i, \varnothing) \land \text{sgn}(r) = 1 \} \\
(p_i \leq 0) & : \quad \{ r \mid r = (p_i, \varnothing) \land \text{sgn}(r) = 1 \} \\
(p_i \geq 0) & : \quad \{ r \mid r = (p_i, \varnothing) \land \text{sgn}(r) = -1 \}.
\end{align*}
\]

**Theorem 7 (Correctness of the Smaller Elimination Set).** Theorem 6 remains correct for \(E'_i\) instead of \(E_i\).
Proof. We have to show that
\[ \mathbb{R} \models \exists x \varphi \iff \bigvee_{e \in E} \varphi[x/e]. \]

We proceed similarly as in the proof of Theorem 6. Assign values \( \alpha_1, \ldots, \alpha_m \in \mathbb{R} \) to variables \( y_1, \ldots, y_m \), respectively. Again, the sets \( S_i \subseteq \mathbb{R} \) satisfying \((p_i, \varrho_i, 0)\) and also the set \( S \subseteq \mathbb{R} \) satisfying \( \varphi \) are finite unions of pairwise disjoint (closed, half-closed, open, semi-infinite, or infinite) intervals. We show that \( \bigvee_{e \in E} \varphi[x/e] \) holds if and only if \( S \neq \emptyset \). This will imply the theorem.

If \( \bigvee_{e \in E} \varphi[x/e] \) holds, the proof of the fact that \( S \neq \emptyset \) is exactly the same as in the proof of Theorem 6.

Assume that \( S \) is non-empty. To begin with, notice that if \( S \) is unbounded from below, then \( \varphi[x/\infty] \) holds. In the following we therefore assume that \( S \) is bounded from below. Since \( S \) is a finite union of pairwise disjoint (closed, half-closed, open, semi-infinite, or infinite) intervals, there exists infimum \( \xi \) of \( S \) such that \( \xi \) is a root of some polynomial occurring in \( \varphi \). There are two cases to consider.

In the first case \( \xi \in S \). Define the following sets of indices:

\[
\begin{align*}
I_1 &= \{ k \in I \mid \varrho_k \text{ is } = \land \deg p_k > 0 \land p_k(\xi) = 0 \}, \\
I_2 &= \{ k \in I \mid \varrho_k \text{ is } \leq \land \deg p_k > 0 \land p_k(\xi) = 0 \}, \\
I_3 &= \{ k \in I \mid \varrho_k \text{ is } \geq \land \deg p_k > 0 \land p_k(\xi) = 0 \}.
\end{align*}
\]

Since \( \xi \in S \), we have \((I_1 \cup I_2 \cup I_3) \neq \emptyset \).

If \( I_1 \neq \emptyset \), then there exists \( i \in I \) such that \( \xi \) is a root of \( p_i \) with Thom code \( \mathfrak{s} \). By the definition of \( E_i^s \), it follows that \((p_i, \varrho_i) \in E \). Now Lemma 5(i) together with the fact that \( \varphi \) is an \( \wedge \vee \) combination of atomic formulas ensure that \( \varphi[x/(p_i, \varrho_i)] \) holds.

Now assume that \( I_1 = \emptyset \). We show that there is either some \( i \in I_2 \) such that \( p_i \) is positive at \( \xi - \epsilon \), or some \( j \in I_3 \) such that \( p_j \) is negative at \( \xi - \epsilon \). Assume the opposite, i.e., for every \( k \in I_2 \cup I_3 \), the atomic formula \((p_k, \varrho_k, 0)\) holds at

![Figure 1: \( p(\alpha_1, \ldots, \alpha_m) \) near its root \( \xi \)](image-url)
Let $\zeta \in [\zeta', \xi]$, where

$$\xi' = \max\{\delta \mid l \in I \land \deg p_l > 0 \land p_l(\delta) = 0 \land \delta < \xi\}.$$ 

Since $I_4 = \emptyset$, the following implication holds for all $m \in I$: If $(p_m \varrho 0)$ holds for $\xi$, then it holds for $\zeta$. This together with the fact that $\varphi$ is an $\land \lor$-combination of atoms implies that $\varphi$ holds for $\zeta$; a contradiction. Therefore, there is either $i \in I_2$ such that $p_i$ is positive for $\xi - \epsilon$, or $j \in I_3$ such that $p_j$ is negative for $\xi - \epsilon$. In the first case $\xi$ is a root of $p_i$ with Thom code $\pi$, so the definition of $E'_j$ ensures that $(p_i, \pi) \in E'_i$. Now, Lemma 5(i) together with the fact that $\varphi$ is an $\land \lor$ combination of atoms ensure that $\varphi[x / (p_i, \pi)]$ holds. Finally, in the second case $\xi$ is a root of $p_j$ with Thom code $\tau$, so the definition of $E_j$ ensures that $(p_j, \tau) \in E_j$. Again, Lemma 5(ii) together with the fact that $\varphi$ is an $\land \lor$-combination of atoms ensure that $\varphi[x / (p_j, \tau)]$ holds.

We continue by assuming that $\xi \notin S$. Define the following sets of indices:

$$I_4 = \{k \in I \mid \varrho_k \text{ is } "\neq" \land \deg p_k > 0 \land p_k(\xi) = 0\},$$

$$I_5 = \{k \in I \mid \varrho_k \text{ is } "<" \land \deg p_k > 0 \land p_k(\xi) = 0\},$$

$$I_6 = \{k \in I \mid \varrho_k \text{ is } ">" \land \deg p_k > 0 \land p_k(\xi) = 0\}.$$

Since $\xi \notin S$ it follows that $(I_4 \cup I_5 \cup I_6) \neq \emptyset$. Define

$$\xi' = \min\{\delta \mid k \in I \land \deg p_k > 0 \land p_k(\delta) = 0 \land \delta > \xi\}.$$

If $I_4 \neq \emptyset$, then there exists $i \in I$ such that $(p_i \varrho_i 0)$ holds for all $\zeta \in [\zeta', \xi']$. Moreover, $\xi$ is a root of $p_i$ with Thom code $\pi$. By the definition of $E'_i$, it follows that $(p_i, \pi) + \epsilon \in E$. Now Lemma 5(ii) together with the fact that $\varphi$ is an $\land \lor$ combination of atomic formulas ensure that $\varphi[x / (p_i, \pi)]$ holds.

Now assume that $I_4 = \emptyset$. We show that there is either some $i \in I_5$ such that $p_i$ is negative at $\xi + \epsilon$ or some $j \in I_6$ such that $p_j$ is positive at $\xi + \epsilon$. Assume the opposite, i.e., for every $k \in I_5 \cup I_6$, the atomic formula $(p_k \varrho_k 0)$ does not hold at $\xi + \epsilon$. Since $I_4 = \emptyset$, the following implication holds for all $m \in I$: If $(p_m \varrho 0)$ holds in $[\xi, \xi']$, then it holds for $\xi$. This together with the fact that $\varphi$ is an $\land \lor$-combination of atomic formulas ensure that $\varphi$ holds for $\xi$; a contradiction. Therefore, there is either $i \in I_5$ such that $p_i$ is negative at $\xi + \epsilon$ or some $j \in I_6$ such that $p_j$ is positive at $\xi + \epsilon$. In the first case $\xi$ is a root of $p_i$ with Thom code $\pi$, so the definition of $E'_i$ ensures that $(p_i, \pi) + \epsilon \in E'_i$. Now Lemma 5(ii) together with the fact that $\varphi$ is an $\land \lor$-combination of atoms guarantee that $\varphi[x / (p_i, \pi) + \epsilon]$ holds. In the second case $\xi$ is a root of $p_j$ with Thom code $\tau$, so the definition of $E_j$ ensures that $(p_j, \tau) + \epsilon \in E_j$. Again, Lemma 5(ii) together with the fact that $\varphi$ is an $\land \lor$-combination of atoms ensure that $\varphi[x / (p_j, \tau) + \epsilon]$ holds. This finishes the proof of the theorem.

Theorem 7 ensures the correctness of Algorithm 2 which employs the idea of smaller elimination sets.

5 Relation to Other Work

The idea to use Thom’s lemma as a basis for a virtual substitution-based quantifier elimination method originally appeared as an outlook in Weispfenning’s
Algorithm 2: Elimination of $\exists x$ (smaller elimination sets)

**Input:** $\varphi$  
$\varphi$ is an $\land$-$\lor$-combination of atomic formulas $(p, q_i, 0)$, $i \in I$, where $q_i \in \{=, \neq, <, \leq, >, \geq\}$, and $p_i \in \mathbb{Z}[y_1, \ldots, y_m][x]$, with $\text{deg} \, p_i \leq n$.

**Output:** $\psi$  
$\psi$ is a quantifier-free formula equivalent to $\exists x \varphi$.

1. foreach $i \in I$ do  
2. $E'_i := \emptyset$  
3. foreach $\tau \in \{-1, 0, 1\}^n \setminus \{(0, \ldots, 0)\}$ do  
4. Use $A$ to compute $\Gamma(p_i, \tau)$.
5. if $\Gamma(p_i, \tau)$ is satisfiable then  
6. if $q_i$ is “$=$” then  
7. $E'_i := E'_i \cup \{(p_i, \tau)\}$
8. if $q_i$ is “$\neq$” then  
9. $E'_i := E'_i \cup \{(p_i, \tau) + \varepsilon\}$
10. if $q_i$ is “$<$” and $\text{sgn}(p_i, \tau) = -1$ then  
11. $E'_i := E'_i \cup \{(p_i, \tau) + \varepsilon\}$
12. if $q_i$ is “$>$” and $\text{sgn}(p_i, \tau) = 1$ then  
13. $E'_i := E'_i \cup \{(p_i, \tau) \}$
14. if $q_i$ is “$\leq$” and $\text{sgn}(p_i, \tau) = 1$ then  
15. $E'_i := E'_i \cup \{(p_i, \tau) \}$
16. if $q_i$ is “$\geq$” and $\text{sgn}(p_i, \tau) = -1$ then  
17. $E'_i := E'_i \cup \{(p_i, \tau) \}$
18. $\psi := \text{false}$  
19. foreach $e \in \{\{-\infty\} \cup \bigcup_{i \in I} E'_i\}$ do  
20. Use $A$ to compute $\varphi[x / e]$.
21. $\psi := \psi \lor \varphi[x / e]$  
22. return $\psi$

article on the quadratic case [27]. Weispfenning, too, used an external quantifier elimination algorithm $A$ for realizing the virtual substitution of elimination terms. Besides being way more explicit, there are some principal differences and novelties in our framework developed here.

The most important difference is the length of the sign sequences $\tau$ in parametric roots $(p, \tau)$, where $p \in \mathbb{Z}[y_1, \ldots, y_m][x]$ is of degree $n$. Weispfenning’s sequences are of length $n - 1$, while our framework uses sequences of length $n$. This implies that $\tau$ imposes stronger restrictions on the graph of the univariate polynomial $p(\alpha_1, \ldots, \alpha_m)$ after fixing parameters around the corresponding root, which we take advantage of to a considerable extent. For instance, Lemma 2 would not hold when using sequences of length $n - 1$. It follows that our notions of the left and the right sign make sense only when using sequences of length $n$, which is crucial for the correctness of our optimized elimination sets described in Section 4.

Weispfenning partitions his sign sequences into maximally consistent sets.
This requires the computation of a case distinction on the number of real roots of $p$. Our formulation of the Theorem 6 and Theorem 7 clearly exhibits that such a case distinction is not necessary. The deeper reason for this is our following insight: Given $n \in \mathbb{N} \setminus \{0\}$ there is $p \in \mathbb{Z}[y_1, \ldots, y_m][x]$, where $\deg p = n < m$ such that all $3^n - 1$ sign sequences (excluding the zero sequence) are consistent with $p$, and therefore need be included in the elimination sets.

Independently, Weispfenning very explicitly discussed the cubic case in [26]. This treatment does not rely on an external algorithm $\mathcal{A}$. However, it also does not use Thom codes at all, but relies on a thorough analysis and case distinction on the real type of the relevant, at most cubic, polynomials. As a matter of fact, many ideas from that work could be combined with our framework introduced here, either directly or for realizing $\mathcal{A}$.

Our framework is compatible with various generalizations of real quantifier elimination by virtual substitution. Positive quantifier elimination focuses on the special case where all variables are known to be strictly positive [21] [22]. It is straightforward to adjust our framework so that this assumption is taken into account during the construction of elimination sets, decreasing their size. Furthermore, the external algorithm $\mathcal{A}$ could take advantage of this assumption, possibly constructing shorter quantifier-free formulas.

Extended quantifier elimination keeps track of the elimination terms used and generates parametric sample solutions for $x$ guarded by quantifier-free conditions [25] [28]. The disjunction over these conditions is actually the regular quantifier elimination result. For details we refer the reader to our recent work [14]. There we show for fixed choices of parameters how to use post-processing to eliminate nonstandard symbols, like $-\infty$ and $\varepsilon$, from the sample solutions. Both extended quantifier elimination and our postprocessing are compatible with our proposed framework.

Generic quantifier elimination makes global assumptions on non-vanishing of parametric expressions during the elimination process. The result is correct only under these assumptions, which form part of the output. This saves case distinctions, which considerably increases efficiency and reduces the overall size of the output [10] [20]. In our framework we could assume that certain derivatives do not vanish, which would further decrease the size of the elimination sets.

6 Towards Practical Computations

Our theoretical discussion in Section 3 and Section 4 uses the external algorithm $\mathcal{A}$ online during the computation of $\Gamma(p, \bar{s})$, as well as during virtual substitution. However, it is a key idea of our framework to use in practice an offline approach. This is based on the following observation, which is not hard to see:

**Proposition 8** (Finiteness Property). Let $n \in \mathbb{N} \setminus \{0\}$. Then our framework requires only finitely many formulas to be computed by $\mathcal{A}$ to realize quantifier elimination for all formulas of degree at most $n$. \hfill $\square$

These finitely many formulas are essentially $\Gamma(p, \bar{s})$, $(q \circ 0)[x \gpar (p, \bar{s})]$, and $\nu((p, \bar{s}),(r, \bar{t}))$ for generic $p = u_n x^n + \cdots + u_0$, $q = v_n x^n + \cdots + v_0$, and for all possible choices of $g$, $\bar{s}$, and $\bar{t}$. Then, calls to $\mathcal{A}$ in our theoretical description can be replaced by substitutions into the $u_i$ and $v_j$. 

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Practical experiments have shown that formulas computed this way using, e.g., Qepcad [4] allow to generate such formulas for \( n = 2 \). However, using these results our framework cannot compete with the classical approach from [27]. Having computed part of the formulas for \( n = 3 \), we assume that Qepcad [4] is going to fail for efficiency reasons at \( n = 4 \) latest. Our vision is that researchers would create suitable sets of formulas for given degrees \( n \) combining automated methods with human intelligence. This can even lead to formulas of optimal size, as Lazard has demonstrated with his famous result on the quartic problem [15]. This way, independent research results in symbolic computation could push the limits of practically applicable quantifier elimination by virtual substitution towards increasingly higher degrees \( n \).

Acknowledgments

This research was supported in part by the German Transregional Collaborative Research Center SFB/TR 14 AVACS and by the ANR/DFG Programme Blanc Project STU 483/2-1 SMArT.

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