Noether Theorem and Nilpotency Property of the (Anti-)BRST Charges in the BRST Formalism: A Brief Review

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Abstract: In some of the physically interesting gauge systems, we show that the application of the Noether theorem does not lead to the deduction of the Becchi-Rouet-Stora-Tyutin (BRST) and anti-BRST charges that obey precisely the off-shell nilpotency property despite the fact that these charges are (i) derived by using the off-shell nilpotent (anti-)BRST symmetry transformations, (ii) found to be the generators of the above continuous symmetry transformations, and (iii) conserved w.r.t. the time-evolution due to the Euler-Lagrange equations of motion derived from the Lagrangians/Lagrangian densities (that describe the dynamics of these suitably chosen physical systems). We propose a systematic method for the derivation of the off-shell nilpotent (anti-)BRST charges from the corresponding non-nilpotent Noether (anti-)BRST charges. To corroborate the sanctity and preciseness of our proposal, we take into account the examples of (i) the one (0 + 1)-dimensional (1D) system of a massive spinning (i.e. SUSY) relativistic particle, (ii) the D-dimensional non-Abelian 1-form gauge theory, and (iii) the Abelian 2-form and the St"uckelberg-modified version of the massive Abelian 3-form gauge theories in any arbitrary D-dimension of spacetime. Our present endeavor is a brief review where some decisive proposals have been made and a few novel results have been obtained as far as the nilpotency property is concerned.

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1 Introduction

It is an undeniable truth that the symmetries of all kinds (e.g. global, local, discrete, continuous, spacetime, internal, etc.) have played a decisive role in the realm of theoretical physics as they have provided a set of deep insights into various working aspects of the physical systems of interest. The basic principles behind the local gauge symmetries and diffeomorphism symmetries provide the precise theoretical descriptions of the standard model of particle physics and theory of gravity (i.e. general theory of relativity and (super)string theories). The Becchi-Rouet-Stora-Tyutin (BRST) formalism \cite{1-4} is applied fruitfully to the gauge theories as well as the diffeomorphism invariant theories. Some of the salient features of BRST formalism are \((i)\) it covariantly quantizes the gauge theories which are characterized by the existence of the first-class constraints on them in the terminology of Dirac’s prescription for the classifications of constraints (see, e.g. \cite{5, 6} for details), \((ii)\) it is consistent with the Dirac quantization scheme because the physicality criteria with the nilpotent and conserved (anti-)BRST charges imply that the physical states, in the total quantum Hilbert space, are those that are annihilated by the operator form of the first-class constraints of the gauge theories (see, e.g. \cite{7-9} for details), \((iii)\) it maintains the unitarity and quantum gauge (i.e. BRST) invariance at any arbitrary order of perturbative computations of a given physical process that is allowed by an interacting gauge theory, and \((iv)\) it has deep connections with some of the key ideas behind differential geometry and its (anti-)BRST transformations resemble that of the \(\mathcal{N} = 2\) supersymmetry transformations. There is a decisive difference, however, between the key properties associated with the (anti-)BRST symmetries and the \(\mathcal{N} = 2\) supersymmetries in the sense that the nilpotent BRST and anti-BRST symmetries are absolutely anticommuting in nature but the nilpotent \(\mathcal{N} = 2\) supersymmetries are not. To sum-up, we note that the application of the BRST formalism is physically very useful and mathematically its horizon is quite wide.

The purpose of our present endeavor is related with the derivations of the conserved (anti-)BRST charges by exploiting the theoretical potential of Noether’s theorem and a thorough study of their nilpotency property. We take into account diverse examples of physically interesting models of gauge theories and demonstrate that the Noether theorem does not always lead to the derivation of conserved and nilpotent (anti-)BRST charges. One has to apply specific set of theoretical tricks and techniques to obtain the nilpotent versions of the (anti-)BRST charges from the Noether conserved (anti-)BRST charges which are found to be non-nilpotent. In simple examples, we show that one equation of motion is good enough to convert the non-nilpotent Noether conserved (anti-)BRST charges into the conserved and nilpotent (anti-)BRST charges. For instance, in the case of a 1D massive spinning (i.e. SUSY) relativistic particle (see, e.g. \cite{10-13} and references therein), the Euler-Lagrange (EL) equations of motion (EoM) w.r.t. the “gauge” and “supergauge” variables are sufficient to convert a non-nilpotent set of (anti-)BRST charges into nilpotent of order two (cf. Sec. 2). However, in the case of a D-dimensional non-Abelian 1-form theory, the Gauss divergence theorem and EL-EoM w.r.t. the gauge field are needed to obtain the nilpotent (anti-)BRST charges from the conserved and non-nilpotent Noether (anti-)BRST

\footnote{It is found that the Noether conserved (anti-)BRST charges are the generators of the off-shell nilpotent (anti-)BRST symmetry transformations from which they are derived by using the Noether theorem.}
charges. These simple examples, we have purposely chosen so that it becomes clear that the celebrated Noether theorem does not lead to the derivation of nilpotent (anti-)BRST charges where (i) the non-trivial (anti-)BRST invariant Curci-Ferrari (CF) type restrictions exist, and (ii) a set of coupled (but equivalent) Lagrangians/Lagrangian densities respect the off-shell nilpotent (anti-)BRST symmetry transformations. It is worthwhile to mention here that the limiting cases of the above two examples are the free scalar relativistic particle and D-dimensional Abelian 1-form gauge theory where there is existence of a single Lagrangian/Lagrangian density. In these cases, the Noether conserved (anti-)BRST charges are off-shell nilpotent automatically because the CF-type restriction is trivial. In fact, it is observed that the trivial CF-type restriction of the scalar relativistic particle is the limiting case of the non-trivial CF-type restriction of the spinning relativistic particle and the trivial CF-type restriction of the Abelian 1-form gauge theory is the limiting case of the non-trivial CF-condition of non-Abelian 1-form gauge theory.

One of the central issue we address in our present investigation is the cases of the BRST approach to higher p-form ($p = 2, 3, ...$) gauge theories\footnote{The higher p-form ($p = 2, 3, ...$) gauge fields (and corresponding theories) are important because such fields appear in the quantum excitations of the (super)string theories (see, e.g. [14] for details).} where there is always existence of (i) a set of coupled (but equivalent) Lagrangian densities, and (ii) a set of (anti-)BRST invariant CF-type restrictions. In such cases, the Noether theorem always leads to the derivation of conserved (anti-)BRST charges which are found to be non-nilpotent. In fact, the expressions for these charges are quite complicated and the EL-EoMs are too many because of the presence of too many fields in the theory (cf. Sec. 5 below for details).

We make a systematic proposal which enables us to obtain the off-shell nilpotent (anti-)BRST charges from the conserved Noether (anti-)BRST charges which are found to be non-nilpotent. One of the key ingredients of our proposal is the observation that one has to start with the EL-EoM w.r.t. gauge field of the p-form gauge theories where, most of the time, the Gauss divergence theorem is required to be applied (before we exploit the potential and power of the EL-EoM w.r.t. the gauge fields). To be precise, all the D-dimensional ($D \geq 2$) higher p-form gauge theories require the application of the Gauss divergence theorem before we exploit the potential of EL-EoM w.r.t. the gauge field. After this, it is the interplay amongst (i) the application of the EL-EoMs, (ii) use of the (anti-)BRST transformations, and (iii) the requirement of Gauss’s divergence theorem that lead to the derivation of the nilpotent versions of the (anti-)BRST charges from the conserved Noether (anti-)BRST charges which are found to be non-nilpotent (cf. Secs. 4, 5, 6).

The theoretical contents of our present endeavor are organized as follows. In Sec. 2, we exploit the theoretical potential and power of Noether’s theorem in the context of a gauge system of 1D spinning relativistic particle to deduce the explicit expressions for the conserved (anti-)BRST charges $Q_{(a) b}$ and establish that they are not off-shell nilpotent. We focus, in our Sec. 3, on the D-dimensional non-Abelian 1-form gauge theory (without any interaction with matter fields) and demonstrate that the Noether conserved (anti-)BRST charges, once again, are non-nilpotent to begin with. We pinpoint the specific EL-EoMs that have to be used to make these (anti-)BRST charges off-shell nilpotent. Our Sec. 4 is devoted to the discussion of Noether’s theorem in the context of D-dimensional (anti-)BRST invariant Abelian 2-form theory and discuss the nitty-gritty details of the nilpotency
property. The theoretical content of our Sec. 5 is concerned with the (anti-)BRST invariant coupled (but equivalent) Lagrangian densities of the (anti-)BRST invariant modified massive Abelian 3-form gauge theory and our discussion is centered around the property of the off-shell nilpotency of the (anti-)BRST charges. Finally, in Sec. 6, we make some concluding remarks and comment on the future prospects of our present investigation.

In our Appendix A, we discuss the St"uckelberg-modified D-dimensional massive Abelian 2-form theory and deduce the off-shell nilpotent versions of the (anti-)BRST charges \([Q^{(1)}_{(ab)}]\) from the non-nilpotent Noether conserved (anti-)BRST charges \([Q_{(ab)}]\).

Convention and Notations for the 1D Massive Spinning (SUSY) Relativistic Particle and D-dimensional Abelian 2-Form as well as 3-Form Theories: We follow the convention of the left-derivatives w.r.t. all the fermionic variables/fields of our theory in the computations of the canonical conjugate momenta and the Noether conserved currents. The flat metric tensor \(\eta_{\mu \nu} = \text{diag} (+1, -1, -1, \ldots)\) is chosen for the D-dimensional flat Minkowskian space so that the dot product between two non-null vectors \(P^\mu\) and \(Q^\nu\) is denoted by \(P \cdot Q = \eta_{\mu \nu} P^\mu Q^\nu = P_0 Q_0 - P_i Q_i\) where the Greek indices \(\mu, \nu, \lambda, \ldots = 0, 1, 2, \ldots D - 1\) correspond to the time and space directions and the Latin indices \(i, j, k, \ldots = 1, 2, \ldots D - 1\) stand for the space directions only. Throughout the whole body of our text, the nilpotent (anti-)BRST symmetry transformations carry the symbol \(s_{(ab)}\) and the corresponding conserved (anti-)BRST charges are denoted by \(Q_{(ab)}\) for all kinds of the Abelian systems that have been chosen for our present discussion. For our discussions on the D-dimensional non-Abelian gauge theory, we shall adopt different convention in Sec. 3.

2 Preliminary: (Anti-)BRST Charges and Nilpotency for a Massive Spinning Relativistic Particle

We begin with the following coupled (but equivalent) (anti-)BRST invariant (see, e.g. [12, 13]) Lagrangians that describe the dynamics of a one \((0 + 1)\)-dimensional massive spinning (i.e. supersymmetric) relativistic particle in the D-dimensional target space, namely:

\[
L_b = L_f + b^2 + b (\dot{e} + 2 \bar{\beta} \beta) - i \dot{\bar{c}} \bar{c} + \bar{\beta}^2 \beta^2 + 2 i \chi (\beta \dot{\bar{c}} - \bar{\beta} \dot{c}) - 2 e (\bar{\beta} \dot{\beta} + \gamma \chi) \\
+ 2 \gamma (\beta \dot{\bar{c}} - \bar{\beta} c) + m (\bar{\beta} \dot{\beta} - \dot{\beta} \bar{\beta} + \gamma \chi) - \dot{\gamma} \psi_5, \tag{1}
\]

\[
L_{\bar{b}} = L_f + \bar{b}^2 - \bar{b} (\dot{e} - 2 \bar{\beta} \beta) - i \dot{c} \bar{c} + \beta^2 \bar{\beta}^2 + 2 i \chi (\beta \dot{c} - \bar{\beta} \dot{\bar{c}}) + 2 e (\dot{\beta} \beta - \gamma \chi) \\
+ 2 \gamma (\beta \dot{c} - \bar{\beta} \bar{c}) + m (\beta \dot{\beta} - \dot{\beta} \beta + \gamma \chi) - \dot{\gamma} \psi_5, \tag{2}
\]

where \(L_f\) is the first-order Lagrangian for our system [10]

\[
L_f = p_\mu \dot{x}^\mu + \frac{i}{2} (\psi_\mu \dot{\psi}^\mu - \psi_5 \dot{\psi}_5) - \frac{e}{2} (p^2 - m^2) + i \chi (p_\mu \psi^\mu - m \psi_5). \tag{3}
\]

In the above, the target space canonical conjugate quantities \((x_\mu(\tau), p^\mu(\tau))\) are the bosonic coordinates \((x_\mu)\) and canonical momenta \((p^\mu)\), respectively, with the Greek indices \(\mu =
0, 1, 2, ..., $D - 1$ corresponding to the D-dimensional flat Minkowskian target space. The trajectory of the spinning particle is parameterized by $\tau$ and the generalized velocities: $x_\mu = (d x^\mu / d \tau)$, $\dot{\psi}_\mu = (d \psi^\mu / d \tau)$ are defined w.r.t. it. The pair of fermionic ($\psi_\mu^2 = 0$, $\psi_\mu \psi_\nu + \psi_\nu \psi_\mu = 0$, $\psi_\mu \bar{\psi}_\mu = 0$, $\bar{\psi}_\mu \psi_\mu + \bar{\psi}_\mu \bar{\psi}_\mu = 0$, etc.) variables ($\psi_\mu, \bar{\psi}_\mu$) are introduced in the theory to (i) maintain the SUSY gauge symmetry transformations, and (ii) incorporate the mass-shell condition $p^2 - m^2 = 0$ where $m$ is the rest mass of the particle. The variable $e(\tau)$ and $\chi(\tau)$ are the superpartners of each-other where $e(\tau)$ is the einbein variable and fermionic ($\chi^2 = 0$) variable $\chi(\tau)$ is its superpartner and both of them behave as the “gauge” and “supergauge” variables. We have incorporated a pair of variables $(b, \bar{b})$ as the bosonic Nakanishi-Lautrup type auxiliary variables which participate in defining the CF-type restriction\[\text{(anti-)}BRST symmetry transformations (4) and (5) are absolutely anticommuting (i.e. $\{ s_b, s_{ab} \} = 0$) in nature on the submanifold of the Hilbert space of quantum variables which is defined by the (anti-)BRST invariant (i.e. $s_{(ab)}[b + \bar{b} + 2 \bar{\beta} \beta] = 0$) CF-type restriction: $b + \bar{b} + 2 \bar{\beta} \beta = 0$. The off-shell nilpotent $[s_{(a)b}] = 0$ (anti-)BRST symmetry transformations $s_{(a)b}$ [cf. Eqs. (4), (5)] are said to be perfect symmetry transformations for the Lagrangians $L_{(b)}$, respectively, because of our observations in (6) and (7) where no EL-EoMs and/or CF-type restriction(s) are used for their validity. As a consequence, the action integrals $S_1 = \int_{-\infty}^{\infty} d \tau L_b$ and $S_2 = \int_{-\infty}^{\infty} d \tau L_{\bar{b}}$ remain invariant for the physically well-defined variables that vanish-off as $\tau \to \pm \infty$. The application of the (anti-)BRST symmetry transformations (4) and (5) are absolutely anticommuting (i.e. $\{ s_b, s_{ab} \} = 0$) in nature on the submanifold of the Hilbert space of quantum variables which is defined by the (anti-)BRST invariant (i.e. $s_{(ab)}[b + \bar{b} + 2 \bar{\beta} \beta] = 0$) CF-type restriction: $b + \bar{b} + 2 \bar{\beta} \beta = 0$. The off-shell nilpotent $[s_{(a)b}] = 0$ (anti-)BRST symmetry transformations $s_{(a)b}$ [cf. Eqs. (4), (5)] are said to be perfect symmetry transformations for the Lagrangians $L_{(b)}$, respectively, because of our observations in (6) and (7) where no EL-EoMs and/or CF-type restriction(s) are used for their validity.
Noether’s theorem yields the following explicit expressions for the conserved (anti-)BRST charges \(Q_{(a)b}\) for our 1D SUSY system of a relativistic spinning particle, namely:

\[
Q_{ab} = \frac{c}{2} (p^2 - m^2) + \beta (p_\mu \psi^\mu - m \psi_5) - \bar{b} \dot{\beta} - 2 \bar{b} \beta \dot{\chi} - i m \bar{\beta} \gamma - \beta^2 \dot{c} - 2 \beta \bar{\beta}^2 \chi,
\]

\[
Q_b = \frac{c}{2} (p^2 - m^2) + \beta (p_\mu \psi^\mu - m \psi_5) + b \dot{c} + 2 b \beta \dot{\chi} - i m \beta \gamma + \bar{\beta}^2 \dot{c} + 2 \beta \bar{\beta}^2 \chi,
\]

where we have used the following explicit relationships

\[
\begin{align*}
\psi & = \dot{\chi} + \beta \chi, \\
\bar{\psi} & = \dot{\chi} - \beta \chi,
\end{align*}
\]

for the derivations of conserved (anti-)BRST charges \(Q_{(a)b}\).

The following points are pertinent as far as the equations (10) and (11) are concerned. First, even though some of the auxiliary variables (e.g. \(b, \bar{b}, \chi\)) transform [cf. Eqs. (4), (5)] under the (anti-)BRST symmetry transformations, we have not used their contributions because their “time” derivative is not present in the Lagrangians \(L_b\) and \(L_{\bar{b}}\). Second, we have not taken into account the contributions of \(\beta\) and \(\bar{\beta}\) in (11) and (10), respectively, because \(s_b \bar{\beta} = 0\) and \(s_b \beta = 0\). Third, the expressions \(X\) and \(Y\) are the quantities that are present in the square brackets of (6) and (7). Fourth, the direct applications of the EL-EoMs ensure that \(Q_{(a)b}\) are conserved quantities [13]. Finally, the conserved charges \(Q_{(a)b}\) are the generators for the continuous (anti-)BRST symmetry transformations (4) and (5) because it can be checked that the following is true, namely:

\[
s_r \Phi = -i \{\Phi, Q_r\}_{\pm}, \quad r = b, ab,
\]

where the \((\pm)\) signs (as the subscript) on the square bracket, on the r.h.s., denote that the square bracket is an (anti)commutator for the given variable \(\Phi\) being fermionic/bosonic in nature. Here \(\Phi\) denotes the generic variable of \(L_b\) and \(L_{\bar{b}}\). In other words, we have:

\[
\Phi = x_\mu, \phi_\mu, \psi_\mu, \psi_5, b, \bar{b}, e, \chi, c, \bar{c}, \beta, \bar{\beta}, \gamma.
\]

Using the principle behind the continuous symmetries and their corresponding generators [cf. Eq. (12)], we have the following

\[
s_b Q_b = -i \{Q_b, Q_b\}, \quad s_a Q_{ab} = -i \{Q_{ab}, Q_{ab}\}, \quad s_{ab} Q_{ab} = -i \{Q_{ab}, Q_{ab}\},
\]

where the l.h.s. of both the entries can be directly computed by using the (anti-)BRST symmetry transformations [cf. Eqs. (4), (5)] and the explicit expressions for the Noether conserved (anti-)BRST charges (8) and (9). It is interesting to point out that the explicit computations of \(s_b Q_b\) and \(s_a Q_{ab}\) are as follows:

\[
s_b Q_b = i \beta^2 \left[ \bar{b} + \frac{1}{2} (p^2 - m^2) + 2 \gamma \chi + 2 \bar{\beta} \beta \right] \equiv -i \{Q_b, Q_b\},
\]

\[
s_{ab} Q_{ab} = i \beta^2 \left[ \frac{1}{2} (p^2 - m^2) - \bar{b} + 2 \gamma \chi - 2 \beta \bar{\beta} \right] \equiv -i \{Q_{ab}, Q_{ab}\}.
\]
It is obvious, from the above expressions, that the (anti-)BRST charges $Q_{(a)b}$ are not off-shell nilpotent. However, these non-nilpotent conserved charges can be made off-shell nilpotent if (i) we use the EL-EoMs w.r.t. the “gauge” and “supergauge” variables $e(\tau)$ and $\chi(\tau)$, respectively, which are, in some sense, superpartners of each-other, (ii) we apply the principle that the off-shell nilpotent continuous (anti-)BRST transformations are generated by the conserved (anti-)BRST charges, and (iii) we apply the (anti-)BRST symmetry transformations at appropriate places.

We propose here a systematic method to obtain the off-shell nilpotent version $Q^{(1)}_b$ of the BRST charge $Q_b$. Our aim would be to obtain $Q^{(1)}_b$ from the non-nilpotent Noether BRST charge from $Q_b$ such that $s_b Q^{(1)}_b = - i \{ Q^{(1)}_b, Q^{(1)}_b \} = 0$. In other words, the l.h.s. (i.e. $s_b Q^{(1)}_b$) should be precisely equal to zero. Towards this goal in mind, first of all, we focus on the EL-EoMs that emerge out from $L_b$ w.r.t. $e(\tau)$ and $\chi(\tau)$ which are the “gauge” and “supergauge” variables. These, in their useful form, are as follows:

$$\frac{1}{2} (p^2 - m^2) = - \dot{b} - 2 (\tilde{\beta} \dot{\beta} + \gamma \chi),$$

$$p_\mu \psi^\mu - m \psi_5 = - i m \gamma + 2 i e \gamma - 2 (\beta \dot{\bar{c}} - \bar{\beta} \dot{c}). \quad (15)$$

In the second step, the substitutions of EL-EoMs w.r.t. “gauge” and “supergauge” variables in the appropriate terms of Noether conserved BRST charge $Q_b$. For instance, the substitutions of (15) lead to the modifications of the following terms:

$$\frac{c}{2} (p^2 - m^2) = - \dot{b} c - 2 c (\tilde{\beta} \dot{\beta} + \gamma \chi),$$

$$\beta (p_\mu \psi^\mu - m \psi_5) = - i m \beta \gamma + 2 i e \beta \gamma - 2 \beta (\beta \dot{\bar{c}} - \bar{\beta} \dot{c}). \quad (16)$$

In the third step, we observe whether the above “modified” terms add, subtract and/or cancel out with some of the terms of $Q_b$. For instance, we note that, in our present case, only the term which is added in (16) is “$- i m \beta \gamma$” from $Q_b$. The total sum of the expressions in (16) and this term is the following explicit expression:

$$\dot{b} c - 2 c (\tilde{\beta} \dot{\beta} + \gamma \chi) - 2 i m \beta \gamma + 2 i e \beta \gamma - 2 \beta (\beta \dot{\bar{c}} - \bar{\beta} \dot{c}). \quad (17)$$

In the fourth step, we apply the BRST transformations on (17) which yields:

$$- i \dot{b} \beta^2 + 2 i \beta^2 \gamma \chi - 2 i \beta^2 \tilde{\beta} \bar{\beta}. \quad (18)$$

In our fifth step, we keenly observe whether some of terms of the Noether conserved charge $Q_b$ should be modified so that the terms of (18) cancel out precisely when we apply the BRST symmetry transformations on them. In our present case, we observe luckily that

$$s_b [\beta^2 \dot{\bar{c}} + 2 \beta^2 \tilde{\beta} \bar{\beta}] = i \dot{b} \beta^2 + 2 i \beta^2 \gamma \chi + 2 i \beta^2 \tilde{\beta} \bar{\beta}, \quad (19)$$

which cancels out whatever we have obtained in (18). In the final step, we apply the BRST symmetry transformations on the left-over terms of the Noether conserved charge $Q_b$. It turns out that we have the following

$$s_b [b \dot{c} + 2 b \beta \chi] = - 2 i b \beta \dot{\beta} + 2 i b \beta \bar{\beta} = 0. \quad (20)$$
It is pertinent to point out that all the terms that cancel out due to the application of the BRST symmetry transformations should be present in the off-shell nilpotent version of the (anti-)BRST charges $Q_b^{(1)}$ . For instance, all the terms of (17) and the terms, on the l.h.s. of (19) and (20) in the square bracket, will be present in $Q_b^{(1)}$ . Ultimately, in our present case, we obtain the following off-shell nilpotent version of the BRST charge:

$$Q_b^{(1)} = b \dot{c} - b c + 2 \beta [i e \gamma + \tilde{b} \dot{c} - i m \gamma + b \chi + \beta \tilde{\beta} \chi] - \beta^2 \dot{c} - 2 c [\tilde{\beta} \dot{\gamma} + \gamma \chi].$$

(21)

At this stage, it is straightforward to note that the following observation is true, namely;

$$s_b Q_b^{(1)} = -i \{Q_b^{(1)}, Q_b^{(1)} \}, \quad \implies \quad [Q_b^{(1)}]^2 = 0.$$

(22)

In other words, we point out that the off-shell nilpotent version of the conserved BRST charge $[Q_b^{(1)}]$ is obtained from the non-nilpotent Noether conserved charge by using the EL-EoMs w.r.t. the “gauge” variable $e (\tau)$ and “supergauge” variable $\chi (\tau)$ and the application of the (anti-)BRST symmetry transformations at appropriate places.

Against the backdrop of the above paragraph, we note that, to obtain the off-shell nilpotent version of the conserved anti-BRST charge, we use the following EL-EoMs

$$\frac{1}{2} (p^2 - m^2) = \dot{b} + 2 (\tilde{\beta} \beta - \gamma \chi),$$

$$\left(p_{\mu} \psi^\mu - m \psi_5 \right) = 2 i e \gamma - i m \gamma - 2 (\beta \dot{c} - \tilde{\beta} \dot{c}).$$

(23)

that are derived from the perfectly anti-BRST invariant $L_b$ . We follow exactly the same steps as in the case of BRST charge $Q_b$ to obtain the off-shell nilpotent version of the anti-BRST $Q_{ab}^{(1)}$ . In fact, the substitution of (23) into the expression for $Q_{ab}$ [cf. Eq. (8)] at appropriate places leads to the following expression for $Q_{ab}^{(1)}$, namely;

$$Q_{ab} \longrightarrow Q_{ab}^{(1)} = \dot{b} \dot{c} - \tilde{b} \dot{c} + 2 \beta [i e \gamma - \beta \dot{c} - \tilde{b} \chi - i m \gamma - \beta \tilde{\beta} \chi]$$

$$+ \quad 2 \dot{c} (\tilde{\beta} \dot{\gamma} + \gamma \chi) + \beta^2 \dot{c}.$$ 

(24)

It is now straightforward to note that we have the following:

$$s_{ab} Q_{ab}^{(1)} = -i \{Q_{ab}^{(1)}, Q_{ab}^{(1)} \}, \quad \implies \quad [Q_{ab}^{(1)}]^2 = 0.$$

(25)

The above observation is nothing but the proof of the off-shell nilpotency of the anti-BRST charge $Q_{ab}^{(1)}$ where the l.h.s. is computed explicitly by using (4) and (24).

We end this section with the following remarks. First of all, we note that it is the EL-EoMs w.r.t. the “gauge” and “supergauge” variables that have been used and these have been singled out from the rest of the EL-EoM. This observation is one of the key ingredients of our proposal followed by the steps that have been discussed from Eq. (15) to Eq. (20). Second, in our present simple case of a 1D spinning relativistic particle, only a single step is good enough to enable us to obtain an off-shell nilpotent set of (anti-)BRST conserved charges. However, we shall see that, in the context of Abelian 2-form and 3-form gauge theories defined in any arbitrary dimension of spacetime, more steps will be

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Footnote: This is due to the fact that, as pointed out earlier, we wish to obtain $Q_b^{(1)}$ such that $s_b Q_b^{(1)} = 0$. 

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required to obtain the off-shell nilpotent set of conserved (anti-)BRST charges from the non-nilpotent forms of the (anti)BRST charges (that are derived directly from the applications of Noether’s theorem). Third, it is interesting to mention that, in the limiting case of the spinning relativistic particle when \( \beta = \bar{\beta} = \gamma = 0 \), we obtain the (anti-)BRST charges for the scalar relativistic particle from (8) and (9) as

\[
Q_{ab} \rightarrow Q_{ab}^{(sr)} = \frac{\epsilon}{2} (p^2 - m^2) - \bar{b} \dot{c}, \quad Q_b \rightarrow Q_b^{(sr)} = \frac{c}{2} (p^2 - m^2) + b \dot{c},
\]

which are off-shell nilpotent of order two because \( s_b Q_b^{(sr)} = 0 \) and \( s_{ab} Q_{ab}^{(sr)} = 0 \) due to: \( s_b c = 0, s_b \bar{b} = 0, s_b p_\mu = 0 \) and \( s_{ab} \bar{c} = 0, s_{ab} \bar{b} = 0, s_{ab} p_\mu = 0 \) which are true for the scalar relativistic particle. In the above equation, the superscript \( (sr) \) on the charges stand for the conserved and off-shell nilpotent (anti-)BRST charges for the scalar relativistic particle. Finally, it can be explicitly checked that the modified versions of the (anti-)BRST charges \( Q^{(1)}_{(ab)} \) are also conserved quantities if we use the proper EL-EoMs that are derived from the coupled (but equivalent) Lagrangians \( L_b \) and \( L_\bar{b} \) of our 1D massive SUSY gauge theory.

3 (Anti-)BRST Charges and Nilpotency: Arbitrary Dimensional non-Abelian 1-Form Gauge Theory

In this section, we show that the Noether conserved (anti-)BRST charges for the D-dimensional non-Abelian 1-form gauge theory are non-nilpotent. However, following our proposal, we can obtain the appropriate forms of the conserved and off-shell nilpotent expressions for the (anti-)BRST charges for our non-Abelian theory (without any interactions with the matter fields). We begin with the following coupled (but equivalent) Lagrangian densities (see, e.g. [9]) in the Curci-Ferrari gauge (see, e.g. [15, 16])

\[
\mathcal{L}_B = -\frac{1}{4} F^{\mu \nu} \cdot F_{\mu \nu} + B \cdot (\partial_\mu A^\mu) + \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) - i \partial_\mu \bar{C} \cdot D^{\mu} C, \\
\mathcal{L}_{\bar{B}} = -\frac{1}{4} F^{\mu \nu} \cdot F_{\mu \nu} - \bar{B} \cdot (\partial_\mu A^\mu) + \frac{1}{2} (B \cdot B + \bar{B} \cdot \bar{B}) - i D^{\mu} \bar{C} \cdot \partial_\mu C, 
\]

where the field strength tensor \( F_{\mu \nu} \equiv F^{\alpha}_{\mu \nu} T^a \) \((a = 1, 2, ..., N^2 - 1)\) has been derived from the non-Abelian 2-form: \( F^{(2)} = d A^{(1)} + i A^{(1)} \wedge A^{(1)} \) where the 1-form \( A^{(1)} = dx^\mu A^\mu \equiv d x^\mu A^\mu_a T^a \) defines the non-Abelian gauge field \( A^a_\mu \) so that we have: \( F^{\alpha}_{\mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + i f^{abc} A^b_\mu A^c_\nu \). For the \( SU(N) \) Lie algebraic space, we have the Lie algebra: \( [T^a, T^b] = f^{abc} T^c \) that is satisfied by the \( SU(N) \) generators \( T^a \) \((a = 1, 2, ..., N^2 - 1)\) where \( f^{abc} \) are the structure constants that can be chosen [17] to be totally antisymmetric in all the indices for the semi-simple Lie group \( SU(N) \). In this section, we adopt the dot and cross products only in the \( SU(N) \) Lie algebraic space where we have: \( S \cdot T = S^a T^a, (S \times T)^a = f^{abc} S^b T^c \) for the two non-null vectors \( S^a \) and \( T^a \) (in this space and \( a, b, c, ... = 1, 2, ..., N^2 - 1 \)). We have also taken into account the summation convention where the repeated indices are summed over and \( D_\mu C = \partial_\mu C + i (A_\mu \times C) \) and \( D_\mu \bar{C} = \partial_\mu \bar{C} + i (A_\mu \times \bar{C}) \) are the covariant derivatives in the adjoint representation of the \( SU(N) \) Lie algebra.
The above coupled (but equivalent) Lagrangian densities (27) respect the following off-shell nilpotent \( [s^2_{(ab)} = 0] \) (anti-)BRST symmetry transformations \([s_{(ab)}]\)

\[
\begin{align*}
s_{ab} A_\mu &= D_\mu C, \\
s_{ab} C &= -\frac{i}{2} (C \times C), \\
s_{ab} B &= 0,
\end{align*}
\]

\[
\begin{align*}
s_{ab} F_{\mu\nu} &= i (F_{\mu\nu} \times \bar{C}), \\
s_{ab} (\partial_\mu A^\mu) &= \partial_\mu D^\mu \bar{C}, \\
s_{ab} B &= i (B \times \bar{C}),
\end{align*}
\]

\[
\begin{align*}
s_b A_\mu &= D_\mu C, \\
s_b C &= -\frac{i}{2} (C \times C), \\
s_b \bar{C} &= i B, \\
s_b B &= 0,
\end{align*}
\]

\[
\begin{align*}
s_b \bar{B} &= i (\bar{B} \times C), \\
s_b (\partial_\mu A^\mu) &= \partial_\mu D^\mu \bar{C}, \\
s_b F_{\mu\nu} &= i (F_{\mu\nu} \times C),
\end{align*}
\]

(28)

because of the following observations

\[
\begin{align*}
s_b \mathcal{L}_B &= \partial_\mu (B \cdot D^\mu C), \\
s_b \mathcal{L}_\bar{B} &= -\partial_\mu (\bar{B} \cdot D^\mu \bar{C}), \\
s_{ab} \mathcal{L}_B &= \partial_\mu \left[ (B + (C \times \bar{C}) \cdot \partial^\mu C \right] - \{ B + B + (C \times \bar{C}) \} \cdot D_\mu \partial^\mu C, \\
s_{ab} \mathcal{L}_\bar{B} &= -\partial_\mu \left[ \{ \bar{B} + (C \times C) \} \cdot \partial^\mu \bar{C} \right] + \{ (B + \bar{B} + (C \times \bar{C}) \} \cdot D_\mu \partial^\mu \bar{C},
\end{align*}
\]

(29)

which establish that the action integrals \( S_1 = \int d^D x \mathcal{L}_B, S_2 = \int d^D x \mathcal{L}_\bar{B} \) remain invariant \((s_b S_1 = 0, s_{ab} S_2 = 0)\) under the BRST and anti-BRST symmetry transformations, respectively, because the physical fields vanish-off as \( x \rightarrow \pm \infty \) due to Gauss’s divergence theorem. If we confine our whole discussion on the submanifold of the total quantum Hilbert space of fields where the CF-condition: \( B + \bar{B} + (C \times \bar{C}) = 0 \) is respected \cite{18}, we note that both the Lagrangian densities respect both the nilpotent symmetries. In other words, we have: \( s_b \mathcal{L}_B = -\partial_\mu [B \cdot \partial^\mu C], s_{ab} \mathcal{L}_B = \partial_\mu [B \cdot \partial^\mu \bar{C}] \) on the above submanifold of Hilbert space of quantum fields. We christen the transformations: \( s_b \mathcal{L}_B = \partial_\mu [B \cdot D^\mu C], s_{ab} \mathcal{L}_B = -\partial_\mu [B \cdot D^\mu \bar{C}] \) as perfect symmetry transformations because we do not use any EL-EoMs and/or CF-type condition for their proof.

In addition to the equivalence of the coupled Lagrangian densities (27) from the point of view of the (anti-)BRST symmetry considerations, we note that the absolute anticommutativity (i.e. \( \{ s_b, s_{ab} \} = 0 \)) property of the (anti-)BRST symmetry transformations is satisfied if and only if we invoke the sanctity of the CF-condition: \( B + \bar{B} + (C \times \bar{C}) = 0 \). This becomes obvious when we observe that the following are true, namely:

\[
\begin{align*}
\{ s_b, s_{ab} \} A_\mu &= i D_\mu \left[ B + \bar{B} + (C \times \bar{C}) \right], \\
\{ s_b, s_{ab} \} F_{\mu\nu} &= -F_{\mu\nu} \times \left[ B + \bar{B} + (C \times \bar{C}) \right].
\end{align*}
\]

(30)

Thus, it is clear that the absolute anticommutativity properties: \( \{ s_b, s_{ab} \} A_\mu = 0 \) and \( \{ s_b, s_{ab} \} F_{\mu\nu} = 0 \) are true if and only if \( B + \bar{B} + (C \times \bar{C}) = 0 \). We further note that \( \{ s_b, s_{ab} \} \Phi = 0 \) where \( \Phi = C, \bar{C}, B, \bar{B} \) is the generic field of the theory (besides \( A_\mu \) and \( F_{\mu\nu} \)). Thus, the absolute anticommutativity property (i.e. \( \{ s_b, s_{ab} \} \Phi = 0 \)) is automatically satisfied for the fields: \( B, \bar{B}, C, \bar{C} \) due to the off-shell nilpotent (anti-)BRST symmetry transformations (28). It is very interesting to point out that the straightforward equivalence \( (\mathcal{L}_B = \mathcal{L}_\bar{B}) \) of both the Lagrangian densities (27) of our theory leads to

\[
(\partial_\mu A^\mu) \cdot \left[ B + \bar{B} + (C \times \bar{C}) \right] = 0,
\]

(31)

modulo a total spacetime derivative. The above observation establishes the fact that both the Lagrangian densities \( \mathcal{L}_B \) and \( \mathcal{L}_\bar{B} \) of equation (27) are coupled in the sense that the
Nakanishi-Lautrup auxiliary fields $B$ and $\bar{B}$ are not free but these specific fields are restricted to obey $B + \bar{B} + (C \times \bar{C}) = 0$ (which is nothing but the CF-condition [18]). This condition, for the $SU(N)$ non-Abelian gauge theory, is physically sacrosanct because it is an (anti-)BRST invariant (i.e. $s_{(a)b}[B + \bar{B} + (C \times \bar{C})] = 0$) quantity which can be verified by using the (anti-)BRST symmetry transformations (28).

The perfect symmetry invariance of the Lagrangian densities $\mathcal{L}_B$ and $\mathcal{L}_{\bar{B}}$ under the infinitesimal, continuous and off-shell nilpotent $[s_{(a)b}^2 = 0]$ BRST and anti-BRST symmetry transformations, respectively, leads to the derivation of the following expressions for the conserved Noether currents by using the (anti-)BRST symmetry transformations (28).

The conservation law $\partial_{\mu} J^\mu_{(r)} = 0$ (with $r = \bar{B}, B$) can be proven by exploiting the power and potential of the EL-EoMs. For the proof of $\partial_{\mu} J^\mu_{(B)} = 0$, we have to use the following EL-EoMs w.r.t. the gauge field $A_{\mu}$ and (anti-)ghost fields $(\bar{C}) C$, namely;

$$D_{\mu} F^{\mu\nu} + \partial^\nu \bar{B} + (\bar{C} \times \partial^\nu C) = 0, \quad \partial_{\mu} (D^\mu \bar{C}) = 0, \quad D_{\mu} (\partial^\mu C) = 0,$$

(34)

that are derived from $\mathcal{L}_B$. In exactly similar fashion, for the proof of $\partial_{\mu} J^\mu_{(\bar{B})} = 0$, we utilize the following EL-EoMs w.r.t. $A_{\mu}$, $C$ and $\bar{C}$, namely;

$$D_{\mu} F^{\mu\nu} - \partial^\nu B - (\partial^\nu \bar{C} \times C) = 0, \quad \partial_{\mu} (D^\mu C) = 0, \quad D_{\mu} (\partial^\mu \bar{C}) = 0,$$

(35)

which are derived from the Lagrangian density $\mathcal{L}_B$. Ultimately, we claim that the Noether currents (33) are conserved and they lead to the derivation of conserved (anti-)BRST charges for our D-dimensional non-Abelian 1-form gauge theory. Following the sacrosanct prescription of Noether theorem, we derive the expressions for the conserved (anti-)BRST charges $Q_{(B)B} = \int d^{D-1}x J^0_{(B)B}$ as follows:

$$Q_B = - \int d^{D-1}x \left[ F^{0i} \cdot D_i \bar{C} + \bar{B} \cdot D_0 \bar{C} + \frac{1}{2} (\bar{C} \times \bar{C}) \cdot \hat{\bar{C}} \right],$$

$$Q_{\bar{B}} = \int d^{D-1}x \left[ B \cdot D_0 C - F^{0i} \cdot D_i C + \frac{1}{2} \hat{C} \cdot (C \times C) \right].$$

(36)

A few comments, at this juncture, are in order. First of all, it can be checked that $\partial_0 Q_B = 0$ and $\partial_0 Q_{\bar{B}} = 0$ where we have to use the EL-EoMs from $\mathcal{L}_B$ and $\mathcal{L}_{\bar{B}}$, and (ii) the above
conserved (anti-)BRST charges are the generators of all the symmetry transformations (28) provided we use the canonical (anti-)commutators by deriving the explicit expressions for the canonical conjugate momenta from $\mathcal{L}_B$ and $\mathcal{L}_L$. Using the principle behind the continuous symmetry transformations and their generators as the Noether conserved charges, we note that the expressions (36) lead to the following

$$s_{ab} Q_B = - \int d^{D-1}x \left[ i (F^{0i} \times \bar{C}) \cdot D_i \bar{C} - \frac{i}{2} (\bar{C} \times \bar{C}) \cdot \hat{B} \right] \neq 0,$$

$$s_b Q_B = \int d^{D-1}x \left[ - i (F^{0i} \times C) \cdot D_i C + \frac{i}{2} \hat{B} \cdot (C \times C) \right] \neq 0,$$

which are not equal to zero. In other words, we note that: $s_{ab} Q_B = -i \{ Q_B, Q_B \} \neq 0$ and $s_b Q_B = -i \{ Q_B, Q_B \} \neq 0$. Hence, the expressions for the Noether (anti-)BRST charges (36) are not off-shell nilpotent (i.e. $Q_B^2 \neq 0$, $Q_B^2 \neq 0$) of order two.

Following the proposal mentioned in the context of 1D massive spinning relativistic particle, in the first step, we have to find out the EL-EOMs w.r.t. the non-Abelian gauge field and substitute it in the non-nilpotent Noether conserved charges $Q_B$ and $\bar{Q}_B$. In our present case, it can be done only after the application of the Gauss divergence theorem so that we have the following for the first term in $Q_B$ and the second term $Q_B$, namely:

$$\int d^{D-1}x \left[ - F^{0i} \cdot (\partial_i \bar{C} + i A_i \times \bar{C}) \right] \equiv + \int d^{D-1}x (\partial_i F^{0i}) \cdot \bar{C}$$

$$- i \int d^{D-1}x \left[ F^{0i} \cdot (A_i \times \bar{C}) \right], \quad (38)$$

$$\int d^{D-1}x \left[ - F^{0i} \cdot (\partial_i C + i A_i \times C) \right] = + \int d^{D-1}x (\partial_i F^{0i}) \cdot C$$

$$- i \int d^{D-1}x \left[ F^{0i} \cdot (A_i \times C) \right], \quad (39)$$

where we can use the following EL-EoM w.r.t the gauge field, namely:

$$(\partial_i F^{0i}) \cdot \bar{C} = \dot{\hat{B}} \cdot \bar{C} + (\bar{C} \times \dot{\bar{C}}) \cdot \bar{C} - i (A_i \times F^{0i}) \cdot \bar{C},$$

$$(\partial_i F^{0i}) \cdot C = -\dot{\hat{B}} \cdot C + (\dot{\bar{C}} \times C) \cdot C - i (A_i \times F^{0i}) \cdot C. \quad (40)$$

In the second step, we have to find out if there are addition, subtraction and/or cancellations with the rest of the terms of the conserved Noether charges $Q_B$ and $\bar{Q}_B$ in (36). At this stage, taking the help of: $+(\bar{C} \times \dot{\bar{C}}) \cdot \bar{C} = +(\bar{C} \times \dot{\bar{C}}) \cdot \bar{C}$ and $-(\dot{\bar{C}} \times C) \cdot \bar{C} = -\dot{\bar{C}} \cdot (C \times C)$, we have the following expressions for $Q_{(B)}^{(1)}$ from the expressions for $Q_B$ and $\bar{Q}_B$, namely:

$$Q_B \rightarrow Q_{(B)}^{(1)} = \int d^{D-1}x \left[ \dot{\hat{B}} \cdot \bar{C} - \hat{B} \cdot D_0 \bar{C} + \frac{1}{2} (\bar{C} \times \bar{C}) \cdot \bar{C} \right],$$

$$Q_B \rightarrow Q_{(B)}^{(1)} = \int d^{D-1}x \left[ B \cdot D_0 C - \hat{B} \cdot C - \frac{1}{2} \dot{\bar{C}} \cdot (C \times C) \right], \quad (41)$$

where (i) a cancellation has taken place between $[-i (A_i \times F^{0i}) \cdot \bar{C}]$ and $[-i F^{0i} (A_i \times C)]$ in the expression for $Q_B$, (ii) in the expression for $\bar{Q}_B$, there has been a cancellation between
\[ -i (A_i \times F^{0i}) \cdot C \] and \[ -i F^{0i} \cdot (A_i \times C) \], (iii) the pure ghost terms have been added to yield \[ + \frac{1}{2} (\bar{C} \times \bar{C}) \cdot \dot{\bar{C}} \] in \( Q_B \) and \(-\frac{1}{2} \bar{C} \cdot (C \times C) \) in the expression for \( Q_B \), and (iv) the final contributions, after the applications of the Gauss divergence theorem [cf. Eqs. (38), (39)] and the equation of motion (40), are as follows

\[
\bar{B} \cdot \bar{C} + \frac{1}{2} (\bar{C} \times \bar{C}) \cdot \dot{\bar{C}}, \quad -\bar{B} \cdot C - \frac{1}{2} \dot{\bar{C}} \cdot (C \times C),
\]

in \( Q_B \) and \( Q_B \), respectively. In the third step, we have to apply the anti-BRST and BRST symmetry transformations on (42), respectively. It turns out that we have the following:

\[
s_{ab} \left[ \bar{B} \cdot \bar{C} + \frac{1}{2} (\bar{C} \times \bar{C}) \cdot \dot{\bar{C}} \right] = 0, \quad s_b \left[ -\bar{B} \cdot C - \frac{1}{2} \dot{\bar{C}} \cdot (C \times C) \right] = 0.
\]

Thus, our all the relevant steps terminate here and we have the final expression for the (anti-)BRST charges as quoted in (41) because it is elementary to check that the left-over terms of (36) are (anti-)BRST invariant: \( s_{ab} (\bar{B} \cdot D_0 \bar{C}) = 0, \quad s_b (B \cdot D_0 C) = 0 \). It is straightforward now to observe that the above expressions for the conserved (anti-)BRST charges are off-shell nilpotent \([(Q^{(1)}_{(B)B})^2 = 0]\) of order two. To corroborate this statement, we have the following observations related with the modified version of \( Q^{(1)}_{(B)B} \):

\[
s_{ab} Q^{(1)}_B = -i \{ Q^{(1)}_B, Q^{(1)}_B \} = 0 \quad \Rightarrow \quad [Q^{(1)}_B]^2 = 0, \\
s_b Q^{(1)}_B = -i \{ Q^{(1)}_B, Q^{(1)}_B \} = 0 \quad \Rightarrow \quad [Q^{(1)}_B]^2 = 0,
\]

where the l.h.s. of the above equation can be computed explicitly by taking the help of (28) and (34). Thus, the (anti-)BRST charges \( Q^{(1)}_{(B)B} \) are off-shell nilpotent.

We end this section with the following useful remarks. First, we observe that the Noether conserved (anti-)BRST charges \( Q_B \) and \( Q_B \) are not off-shell nilpotent of order two as was the case with the massive spinning (i.e. SUSY) relativistic particle. Second, to obtain the conserved and off-shell nilpotent expressions for the (anti-)BRST charges \( Q^{(1)}_B \) and \( Q^{(1)}_B \), we have, first of all, taken the help of Gauss’s divergence theorem and, then, used only the equation of motion for the gauge field from the coupled Lagrangian densities \( L_B \) and \( L_B \). This observation is similar to our observation in the context of the 1D massive spinning relativistic particle (modulo Gauss’s divergence theorem). Finally, we observe that the Abelian limit (i.e. \( f^{abc} = 0 \) plus no dot and/or cross products plus no covariant derivative, etc.) of the Noether conserved charges \( \bar{Q}_B \) and \( Q_B \) from (31) are:

\[
Q_B \longrightarrow Q_{ab} = - \int d^{D-1}x \left[ F^{0i} \partial_i \bar{C} - B \bar{C} \right], \\
Q_B \longrightarrow Q_b = \int d^{D-1}x \left[ F_0^{0i} \partial_i C - B \bar{C} \right],
\]

where \( Q_{(a)b} \) are the (anti-)BRST charges for the free Abelian 1-form gauge theory. It is elementary exercise to note that the above expressions for the charges are off-shell nilpotent \( (s_b Q_b = - i \{ Q_b, Q_b \} = 0, \quad s_{ab} Q_{ab} = - i \{ Q_{ab}, Q_{ab} \} = 0) \) where we have to apply the analogues of the (anti-)BRST transformations (28) on \( Q_{(a)b} \) for the Abelian 1-form theory which are:

\[
s_b F_0^{0i} = 0, \quad s_b C = 0, \quad s_b B = 0 \quad and \quad s_{ab} F_0^{0i} = 0, \quad s_{ab} \bar{C} = 0, \quad s_{ab} \bar{B} = 0.
\]
4 (Anti-)BRST Charges and Nilpotency: Arbitrary Dimensional Abelian 2-Form Gauge Theory

We begin with the (anti-)BRST invariant coupled (but equivalent) Lagrangian densities for the free D-dimensional Abelian 2-form gauge theory as follows (see, e.g. [8, 19] for details)

\[ \mathcal{L}_B = \frac{1}{12} H^{\mu \nu \kappa} H_{\mu \nu \kappa} + B^\mu \left( \partial^\nu B_{\nu \mu} - \partial_\mu \phi \right) + B \cdot B + \partial_\mu \bar{\beta} \partial^\mu \beta \]

\[ + \left( \partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu \right) (\partial^\mu C^\nu) + (\partial \cdot C - \lambda) \rho + (\partial \cdot \bar{C} + \rho) \lambda, \quad (46) \]

\[ \mathcal{L}_{\bar{B}} = \frac{1}{12} \bar{H}^{\mu \nu \kappa} H_{\mu \nu \kappa} + \bar{B}^\mu \left( \partial^\nu B_{\nu \mu} + \partial_\mu \phi \right) + \bar{B} \cdot \bar{B} + \partial_\mu \beta \partial^\mu \beta \]

\[ + \left( \partial_\mu C_\nu - \partial_\nu C_\mu \right) (\partial^\mu \bar{C}^\nu) + (\partial \cdot C - \lambda) \rho + (\partial \cdot C + \rho) \lambda, \quad (47) \]

where the totally antisymmetric tensor \( H_{\mu \nu \lambda} = \partial_\mu B_{\nu \lambda} + \partial_\nu B_{\lambda \mu} + \partial_\lambda B_{\mu \nu} \) is derived from the Abelian 3-form \( H^{(3)} = dB^{(2)} \equiv [(dx^\mu \wedge dx^\nu \wedge dx^\lambda) / 3!] H_{\mu \nu \lambda} \). Here the Abelian 2-form \( B^{(2)} = [(dx^\mu \wedge dx^\nu) / 2!] B_{\mu \nu} \) is antisymmetric (\( B_{\mu \nu} = -B_{\nu \mu} \)) tensor gauge field and \( d = dx^\mu \partial_\mu \) (with \( d^2 = 0 \)) is the exterior derivative. The gauge-fixing term for the gauge field has its origin in the co-exterior derivative of differential geometry (see, e.g. [20-23]) as it is straightforward to check that \( \delta B^{(2)} = -\ast d \ast B^{(2)} \equiv (\partial^\nu B_{\mu \nu}) dx^\mu \) where \( \ast \) is the Hodge duality operator on the flat D-dimensional spacetime manifold. A derivative on a scalar field (\( \phi \)) has been incorporated into the gauge-fixing term on dimensional ground. The Nakanishi-Lautrup type auxiliary vector fields \( B_\mu \) and \( \bar{B}_\mu \) are restricted to obey the CF-type restriction: \( B_\mu - \bar{B}_\mu - \partial_\mu \phi = 0 \) (see, e.g. [24]). Here the fermionic \( (\bar{C}_\mu^2 = 0, \bar{C}_\mu = 0, C_\mu C_\nu + C_\nu C_\mu = 0, \bar{C}_\mu \bar{C}_\nu + \bar{C}_\nu \bar{C}_\mu = 0, C_\mu \bar{C}_\nu + \bar{C}_\nu C_\mu = 0, \) etc.) vector (anti-)ghost fields \( (\bar{C}_\mu) C_\mu \) carry the ghost numbers \((-1) + 1\), respectively, and the bosonic (anti-)ghost fields \( (\bar{\beta}) \beta \) are endowed with the ghost numbers \((-2) + 2\), respectively. The auxiliary (anti-)ghost fields \( (\rho) \lambda \) are fermionic \( (\rho^2 = \lambda^2 = 0, \rho \lambda + \lambda \rho = 0, \) etc.) in nature and they also carry the ghost numbers \((-1) + 1\), respectively, due to the fact that \( \lambda = \frac{1}{2} (\partial \cdot C) \) and \( \rho = -\frac{1}{2} (\partial \cdot \bar{C}) \). The (anti-)ghost fields are invoked to maintain the unitarity in the theory.

The above coupled Lagrangian densities \( \mathcal{L}_B \) and \( \mathcal{L}_{\bar{B}} \) respect the following perfect off-shell nilpotent \( [s^{(2)}_{(a)b} = 0] \) (anti-)BRST symmetry transformations \([s^{(a)b}]\), namely;

\[ s_{ab} B_\mu = -(\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu), \quad s_{ab} \bar{C}_\mu = -\partial_\mu \bar{\beta}, \quad s_{ab} C_\mu = \bar{B}_\mu, \]

\[ s_{ab} \phi = \rho, \quad s_{ab} \beta = -\lambda, \quad s_{ab} B_\mu = \partial_\mu \rho, \quad s_{ab} [\rho, \lambda, \bar{\beta}, B_\mu, H_{\mu \nu \kappa}] = 0, \]

\[ s_{ab} B_{\mu \nu} = -(\partial_\nu C_\mu - \partial_\mu C_\nu), \quad s_{ab} C_\mu = -\partial_\mu \beta, \quad s_{ab} \bar{C}_\mu = -B_\mu, \]

\[ s_{ab} \phi = \lambda, \quad s_{ab} \bar{\beta} = -\rho, \quad s_{ab} B_\mu = -\partial_\mu \lambda, \quad s_{ab} [\rho, \lambda, \beta, B_\mu, H_{\mu \nu \kappa}] = 0, \quad (48) \]

due to our observations that:

\[ s_{ab} \mathcal{L}_B = -\partial_\mu \left[ (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) \bar{B}_\nu - \rho \bar{B}^\mu + \lambda \partial^\mu \bar{\beta} \right], \]

\[ s_{ab} \mathcal{L}_{\bar{B}} = -\partial_\mu \left[ (\partial^\mu C^\nu - \partial^\nu C^\mu) B_\nu + \rho \partial^\mu \beta + \lambda B^\mu \right]. \quad (49) \]
Thus, it is crystal clear that the action integrals \( S_1 = \int d^D x \mathcal{L}_B \) and \( S_2 = \int d^D x \mathcal{L}_B \) remain invariant (i.e. \( s_0 S_1 = 0, s_0 S_2 = 0 \)) for the physical fields that vanish-off as \( x \to \pm \infty \) due to Gauss’s divergence theorem. It should be noted that the above (anti-)BRST symmetry transformations are absolutely anticommuting only on the submanifold of the Hilbert space of quantum fields where the CF-type restriction: \( B_\mu - \bar{B}_\mu - \partial_\mu \phi = 0 \) is satisfied. This statement can be corroborated by the following observation

\[
\{s_b, s_{ab}\} B_{\mu \nu} = \partial_\mu (B_\nu - \bar{B}_\nu) - \partial_\nu (B_\mu - \bar{B}_\mu),
\]

which establishes that \( \{s_b, s_{ab}\} B_{\mu \nu} = 0 \) if and only if we invoke the sanctity of CF-type restriction \( B_\mu - \bar{B}_\mu - \partial_\mu \phi = 0 \). It can be checked that the absolute anticommutativity property \( \{s_b, s_{ab}\} \Psi = 0 \) is satisfied automatically if we use (48) for the generic field \( \Psi = \mathcal{C}_\mu, \bar{\mathcal{C}}_\mu, \phi, \lambda, \rho, \beta, \bar{\beta}, B_\mu, \bar{B}_\mu \) of \( \mathcal{L}_B \) and \( \mathcal{L}_B \).

The above infinitesimal, continuous and off-shell nilpotent (anti-)BRST symmetry transformations lead to the derivations of the Noether conserved currents

\[
J^\mu_{(a)} = \rho B^\mu - (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) B_\nu - (\partial^\mu C^\nu - \partial^\nu C^\mu) \partial_\nu \bar{\beta} - \lambda \partial^\nu \bar{\beta} - \mathcal{H}^{\mu \nu \kappa} (\partial_\nu \bar{C}_\kappa),
\]
\[
J^\mu_{(b)} = (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) \partial_\nu \beta - (\partial^\mu C^\nu - \partial^\nu C^\mu) B_\nu - \lambda B^\mu - \rho \partial^\nu \beta - \mathcal{H}^{\mu \nu \kappa} (\partial_\nu \mathcal{C}_\kappa),
\]

where it is quite straightforward to check (see, e.g. [19] for details) that the conservation law \((\partial_\mu J^\mu_{(r)}, r = ab, b)\) is true provided we use the EL-EoMs from the coupled (but equivalent) (anti-)BRST invariant Lagrangian densities \( \mathcal{L}_B \) and \( \mathcal{L}_B \), respectively. The conserved Noether (anti-)BRST charges \( Q_r = \int d^{D-1} x J^\mu_{(r)} (r = ab, b) \) are as follows

\[
Q_{ab} = \int d^{D-1} x \left[ \rho B^0 - \partial^0 C^i - \partial^i C^0 \right] B_i - \left( \partial^0 C^i - \partial^i C^0 \right) \partial_i \bar{\beta} - \lambda \partial^0 \bar{\beta} - \mathcal{H}^{0ij} (\partial_i \mathcal{C}_j),
\]
\[
Q_b = \int d^{D-1} x \left[ (\partial^0 \bar{C}^i - \partial^i \bar{C}^0) \partial_i \beta - (\partial^0 C^i - \partial^i C^0) B_i - \lambda B^0 - \rho \partial^0 \beta - \mathcal{H}^{0ij} (\partial_i \mathcal{C}_j) \right],
\]

which are the generators for the infinitesimal, continuous and off-shell nilpotent \([s^2_{(ab)} = 0]\) (anti-)BRST symmetry transformations (48). At this stage, it is worthwhile to point out that the following observations are true, namely;

\[
s_{ab} Q_{ab} = - i \{Q_{ab}, Q_{ab}\} = \int d^{D-1} x \left[ - (\partial^0 \bar{B}^i - \partial^i \bar{B}^0) \partial_i \bar{\beta} \right] \neq 0,
\]
\[
s_b Q_b = - i \{Q_b, Q_b\} = \int d^{D-1} x \left[ - (\partial^0 B^i - \partial^i B^0) \partial_i \beta \right] \neq 0,
\]

when we apply the principle behind the continuous symmetry transformations and their generators as the conserved Noether charges. The above observations establish that the Noether conserved charges \( Q_b \) and \( Q_{ab} \) are not off-shell nilpotent (i.e. \( Q_b^2 \neq 0, Q_{ab}^2 \neq 0 \)).

At this juncture, we follow our proposal to obtain the off-shell nilpotent versions \([Q_{(1)ab}^{(1)}]\) of the (anti-)BRST conserved charges from the conserved Noether (anti-)BRST charge \( Q_{(ab)}^{(1)} \) [cf. Eq. (52)] which are found to be non-nilpotent [cf. Eq. (53)]. Our objective is to prove the validity of \( s_{(ab)}^{(1)} Q_{(ab)}^{(1)} = - i \{Q_{(ab)}^{(1)}, Q_{(ab)}^{(1)}\} = 0 \) by computing precisely the l.h.s. (i.e. \( s_{(ab)} Q_{(ab)}^{(1)} \)). In the first step, we have to use the equation of motion w.r.t. the gauge
field. Towards this goal in mind, first of all, we note that the last terms of $Q_b$ and $Q_{ab}$ can be re-expressed as follows

$$- \int d^{D-1}x \, H^{0ij} \partial_i C_j = - \int d^{D-1}x \, \partial_i [H^{0ij} C_j] + \int d^{D-1}x \, (\partial_i H^{0ij}) C_j,$$

$$- \int d^{D-1}x \, H^{0ij} \partial_i C_j = - \int d^{D-1}x \, \partial_i [H^{0ij} C_j] + \int d^{D-1}x \, (\partial_i H^{0ij}) C_j,$$

where the first-term will be zero due to Gauss’s divergence theorem for the physical fields (which vanish-off as $x \to \mp \infty$) and we can apply the first step of our proposal where the equations of motion w.r.t. the gauge field from $L_B$ and $L_B$ are as follows:

$$\partial_i H^{0ij} = (\partial^0 B^i - \partial^i B^0), \quad \partial_i H^{0ij} = (\partial^0 \bar{B}^i - \partial^i \bar{B}^0).$$

(55)

Thus, the last terms of $Q_{ab}$ and $Q_b$ are as follows:

$$\int d^{D-1}x \, (\partial_i H^{0ij}) C_j = \int d^{D-1}x \, [(\partial^0 \bar{B}^i - \partial^i \bar{B}^0) \bar{C}_i],$$

$$\int d^{D-1}x \, (\partial_i H^{0ij}) C_j = \int d^{D-1}x \, [(\partial^0 B^i - \partial^i B^0) C_i].$$

(56)

We remark here that the terms in (56) here emerged out from the EL-EoMs w.r.t. the gauge field and they are sacrosanct. As a consequence, they will be present in the off-shell nilpotent version of $Q^{(1)}_{(a)b}$. In the second step, we apply the (anti-)BRST symmetry transformations on the above terms (modulo integration) which lead to the following:

$$s_{ab} [(\partial^0 \bar{B}^i - \partial^i \bar{B}^0) \bar{C}_i] = -(\partial^0 \bar{B}^i - \partial^i \bar{B}^0) \partial_i \bar{\beta},$$

$$s_{b} [(\partial^0 B^i - \partial^i B^0) C_i] = -(\partial^0 B^i - \partial^i B^0) \partial_i \beta.$$  

(57)

In the third step, we have to modify\footnote{It will be noted that, in the cases of the 1D massive spinning relativistic particle and D-dimensional non-Abelian 1-form gauge theory, the original term of the Noether conserved charges are good enough to ensure that: $s_b Q^{(1)}_b = 0, s_{ab} Q^{(1)}_{ab} = 0, s_b Q^{(1)}_B = 0$ and $s_{ab} Q^{(1)}_B = 0.$} appropriate terms of $Q_{ab}$ and $Q_b$ so that (57) cancels out when we apply the (anti-)BRST symmetry transformations on them. Towards this goal in mind, first of all, we focus on the BRST charge $Q_b$ and explain clearly the third step of our proposal. It can be seen that the first term of $Q_b$ can be re-written as

$$(\partial^0 \bar{C}^i - \partial^i \bar{C}^0) \partial_i \beta = 2 (\partial^0 \bar{C}^i - \partial^i \bar{C}^0) \partial_i \beta - (\partial^0 \bar{C}^i - \partial^i \bar{C}^0) \partial_i \beta.$$  

(58)

It is clear that if we apply the BRST symmetry transformations on the second term of the above equation, it will serve our purpose and cancel out the second entry in (57). At this stage, we comment that the second term of (58) and (56) will be present in the off-shell nilpotent version ($Q^{(1)}_b$) of the BRST charge. This is due to the fact that our central aim is to show that $s_b Q^{(1)}_b = 0$. Now we focus on the first term of (57) which can be re-written in an appropriate form:

$$2 (\partial^0 \bar{C}^i - \partial^i \bar{C}^0) \partial_i \beta = \partial_i [2 (\partial^0 \bar{C}^i - \partial^i \bar{C}^0) \beta] - 2 \partial_i [\partial^0 \bar{C}^i - \partial^i \bar{C}^0] \beta.$$  

(59)
The above terms are inside the integration w.r.t. $d^{D-1}x$. Hence, the first term of the above equation will vanish due to the Gauss divergence theorem. The second term can be written, using the following equations of motion, derived from $\mathcal{L}_B$, as:

\[
\Box \tilde{C}_\mu - \partial_\mu (\partial \cdot \tilde{C}) - \partial_\mu \rho = 0,
\]

\[
\implies -2 \partial_i [\partial^i \tilde{C}^0 - \partial_i \tilde{C}^0] \beta = 2 \beta \partial^0 \rho \equiv 2 \beta \dot{\rho}. \quad (60)
\]

In the fourth step, we have to apply the BRST symmetry transformations on $(2 \beta \dot{\rho})$ which turns out to be zero. We comment that this term will be present in the off-shell nilpotent version $(Q_b^{(1)})$ of the BRST charge due to our objective to show that $s_b Q_b^{(1)} = 0$. Hence, all the steps of our proposal terminate at this stage. As a consequences, we have the following off-shell nilpotent version of the BRST charge:

\[
Q_b \longrightarrow Q_b^{(1)} = \int d^{D-1}x \left[ (\partial^0 B^i - \partial^i B^0) \tilde{C}_i - (\partial^0 \tilde{C}^i - \partial_i \tilde{C}^0) \partial_i \beta \right.
\]

\[
+ 2 \beta \dot{\rho} - (\partial^0 C^i - \partial_i C^0) B_i - \lambda B^0 - \rho \dot{\lambda}, \quad (61)
\]

where we have taken into account the appropriate term from (56), the second term from (58) and the r.h.s. of (60). It is straightforward now to check that

\[
s_b Q_b^{(1)} = -i \{Q_b^{(1)}, Q_b^{(1)}\} = 0 \implies [Q_b^{(1)}]^2 = 0, \quad (62)
\]

which proves the off-shell nilpotency of the modified version [i.e. $Q_b^{(1)}$] of the non-nilpotency Noether conserved charges $Q_b$. It is worthwhile to mention that $s_b [\lambda B^0 + \rho \dot{\lambda}] = 0$. Hence, these terms [i.e. $- (\lambda B^0 + \rho \dot{\lambda})$] remain intact and they are present in the expression for $Q_b^{(1)}$. We follow exactly the above prescription of our proposal in the case of the non-nilpotent Noether anti-BRST charge $Q_{ab}$ to obtain its modified off-shell nilpotent version as

\[
Q_{ab} \longrightarrow Q_{ab}^{(1)} = \int d^{D-1}x \left[ (\partial^0 \tilde{B}^i - \partial^i \tilde{B}^0) \tilde{C}_i + (\partial^0 \tilde{C}^i - \partial_i \tilde{C}^0) \partial_i \beta \right.
\]

\[
- (\partial^0 \tilde{C}^i - \partial_i \tilde{C}^0) \tilde{B}_i + \rho \tilde{B}^0 - 2 \tilde{\lambda} - \lambda \tilde{\beta}, \quad (63)
\]

which satisfies the following relationship:

\[
s_{ab} Q_{ab}^{(1)} = -i \{Q_{ab}^{(1)}, Q_{ab}^{(1)}\} = 0 \implies [Q_{ab}^{(1)}]^2 = 0. \quad (64)
\]

The above observations establish the sanctity of our proposal that enables us to obtain precisely the off-shell nilpotent versions of the (anti-)BRST charges $[Q_{(a)b}^{(1)}]$ from the conserved Noether (anti-)BRST charges $[Q_{(a)b}]$ which are non-nilpotent.

We end this section with the following remarks. First, in the case of the 1D massive spinning particle, only the EL-EoMs w.r.t. the “gauge” and “supergauge” variables were good enough to convert the non-nilpotent Noether (anti-)BRST charges into the off-shell nilpotent versions of the (anti-)BRST conserved charges. Second, to obtain the off-shell nilpotent versions of the (anti-)BRST conserved charges in the context of the D-dimensional non-Abelian 1-form theory, we required Gauss’s divergence theorem plus the EL-EoMs with
respect to the gauge field. Third, in the context of the D-dimensional Abelian 2-form theory, we invoked twice the Gauss divergence theorem and the EL-EoMs w.r.t. the gauge field and fermionic (anti-)ghost fields ($\bar{C}_\mu$) $C_\mu$ to obtain the off-shell nilpotent versions of the (anti-)BRST charges from the conserved Noether non-nilpotent (anti-)BRST charges. In all the above examples, we have also used the strength of the (anti-)BRST symmetry transformations, at appropriate places, so that we could obtain the off-shell nilpotent versions of the (anti-)BRST charges from the Noether conserved charges which are non-nilpotent.

5 (Anti-)BRST Charges and Nilpotency: Arbitrary Dimensional St"uckelberg-Modified Massive Abelian 3-Form Gauge Theory

We begin our present section with the (anti-)BRST invariant coupled (but equivalent) Lagrangian densities for the St"uckelberg-modified massive Abelian 3-form gauge theory (see, e.g. [25] for details)

$$\mathcal{L}_B = C \mathcal{L}_S + (\partial_\mu A^{\mu\lambda}) B_{\lambda\nu} - \frac{1}{2} B_{\mu\nu} B^{\mu\nu} + \frac{1}{2} B^{\mu\nu} \left[ \partial_\mu \phi_\nu - \partial_\nu \phi_\mu \mp m \Phi_{\mu\nu} \right]$$

\[ - (\partial_\mu \Phi^{\mu\nu}) B_\nu + \frac{1}{2} B^{\mu\nu} B_\mu + \frac{1}{2} B^{\mu\nu} \left[ \pm m \phi_\mu - \partial_\mu \phi \right] + \frac{m^2}{2} \bar{C}_{\mu\nu} C^{\mu\nu} \]

\[ + \left( \partial_\mu \bar{C}_{\nu\lambda} + \partial_\nu \bar{C}_{\lambda\mu} + \partial_\lambda \bar{C}_{\mu\nu} \right) (\partial^\mu C^{\nu\lambda}) \pm m (\partial_\mu \bar{C}^{\mu\nu}) C_\nu \mp m \bar{C}_\nu (\partial^\mu C_{\mu\nu}) \]

\[ + \left( \partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu \right) (\partial^\mu \bar{C}_\nu) - \frac{1}{2} \left[ \pm m \beta^\mu - \partial^\mu \beta \right] \left[ \pm m \beta_\mu - \partial_\mu \beta \right] \]

\[ - (\partial_\mu \bar{B}_\nu - \partial_\nu \bar{B}_\mu) (\partial^\mu \bar{B}_\nu) - \partial_\mu \bar{B}_2 (\partial^\mu \bar{C}_2) - m^2 \bar{C}_2 \bar{C}_2 + [(\partial \cdot \beta) \mp m \beta] B_2 \]

\[ - \left[ (\partial \cdot \phi) \mp m \phi \right] B_1 - [(\partial \cdot \beta) \mp m \beta] B_2 + \left[ \partial_\nu \bar{C}^{\nu\mu} + \partial^\mu \bar{C}_1 \mp \frac{m}{2} \bar{C}_2 \right] f_\mu \]

\[ - 2 F^\mu f_\mu - 2 F f - \left[ \partial_\nu \bar{C}^{\mu\nu} + \partial^\mu C_1 \mp \frac{m}{2} C_2 \right] F_\mu + \left[ \frac{1}{2} (\partial \cdot C) \mp m C_1 \right] F \]

\[ - \left[ \frac{1}{2} (\partial \cdot \bar{C}) \mp m \bar{C}_1 \right] f - B_2 B_2 - \frac{1}{2} B_1^2, \] (65)

\[ \mathcal{L}_{\bar{B}} = C \mathcal{L}_S - (\partial_\mu A^{\mu\lambda}) \bar{B}_{\lambda\nu} - \frac{1}{2} \bar{B}_{\mu\nu} \bar{B}^{\mu\nu} + \frac{1}{2} \bar{B}^{\mu\nu} \left[ \partial_\mu \phi_\nu - \partial_\nu \phi_\mu \pm m \Phi_{\mu\nu} \right] \]

\[ + (\partial_\mu \Phi^{\mu\nu}) \bar{B}_\nu - \frac{1}{2} \bar{B}^{\mu\nu} \bar{B}_\mu + \frac{1}{2} \bar{B}^{\mu\nu} \left[ \pm m \phi_\mu - \partial_\mu \phi \right] + \frac{m^2}{2} \bar{C}_{\mu\nu} C^{\mu\nu} \]

\[ + \left( \partial_\mu \bar{B}_{\nu\lambda} + \partial_\nu \bar{B}_{\lambda\mu} + \partial_\lambda \bar{B}_{\mu\nu} \right) (\partial^\mu \bar{C}^{\nu\lambda}) \pm m (\partial_\mu \bar{C}^{\mu\nu}) C_\nu \pm m \bar{C}_\nu (\partial^\mu C_{\mu\nu}) \]

\[ + \left( \partial_\mu \bar{B}_\nu - \partial_\nu \bar{B}_\mu \right) (\partial^\mu \bar{B}_\nu) - \frac{1}{2} \left[ \pm m \beta^\mu - \partial^\mu \beta \right] \left[ \pm m \beta_\mu - \partial_\mu \beta \right] \]

**The Lagrangian densities $\mathcal{L}_B$ and $\mathcal{L}_{\bar{B}}$ are coupled and equivalent due to the existence of six (anti-)BRST invariant CF-type restrictions on our theory which have been systematically derived using the augmented version of superfield approach to BRST formalism in [25].**
where $L$ is the St"{u}ckelberg-modified classical Lagrangian density that incorporates into it the totally antisymmetric tensor gauge field $(A_{\mu\nu\lambda})$ and antisymmetric $(\Phi_{\mu\nu} = -\Phi_{\nu\mu})$ St"{u}ckelberg field $(\Phi_{\mu\nu})$ along with their kinetic terms as follow [25]:

$$
L_S = \frac{1}{24} H^{\mu\nu\lambda} H_{\mu\nu\lambda} - \frac{m^2}{6} A^{\mu\nu\lambda} A_{\mu\nu\lambda} + \frac{m}{3} A^{\mu\nu\lambda} \Sigma_{\mu\nu\lambda} - \frac{1}{6} \Sigma^{\mu\nu\lambda} \Sigma_{\mu\nu\lambda}.
$$

(67)

In the above, the kinetic term with the tensor field $H_{\mu\nu\lambda}$ for the gauge field $A_{\mu\nu\lambda}$ owes its origin to the exterior derivative $d = dx^\mu \partial_\mu (\mu = 0, 1...D - 1)$ because the Abelian 4-form is: $H^{(4)} = d A^{(3)} = [(d x^\mu \land d x^\nu \land d x^\lambda \land d x^\rho)/4!] H_{\mu\nu\lambda\rho}$ where the Abelian 3-form totally antisymmetric gauge field is defined through: $A^{(3)} = [(d x^\mu \land d x^\nu \land d x^\rho)/3!] A_{\mu\nu\lambda}$. In addition, the Abelian 3-form $\Sigma^{(3)} = [(d x^\mu \land d x^\nu \land d x^\lambda)/3!] \Sigma_{\mu\nu\lambda}$ is defined through the Abelian St"{u}ckelberg 2-form $\Phi^{(2)} = [(d x^\mu \land d x^\rho)/2!] \Phi_{\mu\nu}$ as: $\Sigma^{(3)} = d \Phi^{(2)}$ which implies that $\Sigma_{\mu\nu\lambda} = \partial_\mu \Phi_{\nu\lambda} + \partial_\nu \Phi_{\mu\lambda} + \partial_\lambda \Phi_{\mu\nu}$. It is also evident that $H_{\mu\nu\lambda}$ is equal to

$$
H_{\mu\nu\lambda} = \partial_\mu A_{\nu\lambda} - \partial_\nu A_{\mu\lambda} + \partial_\lambda A_{\mu\nu} - \partial_\nu A_{\mu\lambda}.
$$

(68)

which is invoked in the definition of the kinetic term for the gauge field $A_{\mu\nu\lambda}$.

In the (anti-)BRST invariant Lagrangian densities $L_B$ and $L_{\bar{B}}$, we have the bosonic auxiliary fields ($B_{\mu\nu}, B_{\mu\beta}, B_{\beta}, B_2, B$) and a fermionic set of auxiliary fields is ($F_\mu, F_\mu, \bar{f}_\mu, f_\mu, F, f, \bar{f}, f$) out of which the two bosonic auxiliary fields ($B, B_2$) carry the ghost numbers ($-2, +2$), respectively, and a set of fermionic auxiliary fields ($F_\mu, \bar{f}_\mu, f_\mu, F$) carry the ghost number ($-1$). On the other hand, a fermionic set of auxiliary fields ($\bar{F}_\mu, f_\mu, f, \bar{F}$) is endowed with ghost number (+1). To maintain the unitarity in the theory, we need the fermionic set of ghost fields ($C_{\mu\nu}, C_{\mu\nu}, C_{\mu\bar{\nu}}, C_{\bar{\mu}}, C_{\bar{\nu}}, C_{\bar{\lambda}}, C_2, C_1$) as well as the bosonic set of (anti-)ghost fields ($\beta_{\mu\nu}, \beta_{\mu\bar{\nu}}, \beta_{\bar{\mu}}, \beta_{\bar{\nu}}$) where the latter (anti-)ghost fields carry the ghost numbers ($-2, +2$). To be precise, the set of bosonic anti-ghost fields ($\bar{\beta}_{\mu\nu}, \bar{\beta}$) and the ghost fields ($\beta_{\mu\nu}, \beta_{\bar{\mu}}$) are endowed with ($-2$) and ($+2$) ghost numbers, respectively. The fermionic set of (anti-)ghost fields ($\bar{C}_{\mu\nu}$) carry the ghost numbers ($-3, +3$, respectively, and all the rest of the fermionic (anti-)ghost fields: ($\bar{C}_{\mu}, \bar{C}_{1}$) and ($C_{\mu}, C_1$) carry the ghost numbers ($-1$) and ($+1$), respectively. We have the bosonic vector and scalar fields ($\phi_{\mu}, \phi$), too, in our theory which appear in the gauge-fixing terms.

The following off-shell nilpotent $[s^2_{(a)b} = 0]$ anti-BRST transformations $[s_{(a)b}]$

$$
\begin{align*}
    s_{ab} A_{\mu\nu\lambda} &= \partial_\mu \bar{C}_{\nu\lambda} + \partial_\nu \bar{C}_{\mu\lambda} + \partial_\lambda \bar{C}_{\mu\nu}, \quad \text{s}_{ab} \bar{C}_{\mu\nu} = \partial_\mu \bar{\beta}_\nu - \partial_\nu \bar{\beta}_\mu, \\
    s_{ab} B_{\mu\nu} &= -(\partial_\mu F_\nu - \partial_\nu F_\mu) \equiv (\partial_\mu \bar{f}_\nu - \partial_\nu \bar{f}_\mu), \quad \text{s}_{ab} C_{\mu\nu} = \bar{B}_{\mu\nu}, \\
    s_{ab} \Phi_{\mu\nu} &= \pm m \bar{C}_{\mu\nu} - (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu), \quad \text{s}_{ab} F_{\mu} = -\partial_\mu B_2, \\
    s_{ab} \bar{C}_\mu &= \pm m \bar{\beta}_\mu - \partial_\mu \bar{\beta}, \quad \text{s}_{ab} \phi_{\mu} = \bar{f}_\mu, \quad \text{s}_{ab} \bar{\beta}_\mu = \partial_\mu \bar{C}_2,
\end{align*}
$$

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On the other hand, under the infinitesimal, continuous and off-shell nilpotent \((s_{ab}^2 = 0)\) anti-BRST symmetry transformations \(s_{ab}\):

\[
s_{ab} \mathcal{L}_B = \partial_\mu \left[ B^{\mu \nu} \bar{f}_\nu - \left( \partial^\nu \bar{C}^{\nu \lambda} + \partial^\nu C^{\lambda \mu} + \partial^\lambda C^{\mu \nu} \right) \bar{B}_{\nu \lambda} + B \partial^\mu \bar{C}_2 \right.
- B_2 \bar{F}^\mu - B_1 \bar{f}^\mu - \left( \partial^\nu \beta^\mu - \partial^\nu \bar{\beta}^\mu \right) \bar{F}_\nu + \frac{1}{2} \left( \pm m \bar{\beta}^\mu - \partial^\mu \beta \right) \bar{F}
- \left( \partial^\mu \bar{C}^{\mu \nu} - \partial^\nu \bar{C}^{\nu \mu} \right) \bar{B}_\nu \pm m B^{\mu \nu} \bar{C}_\nu - \frac{1}{2} B^\mu \bar{f}
\pm m C^{\mu \nu} \left( \pm m \bar{\beta}_\nu - \partial_\nu \beta \right) \pm m \left( \partial^\mu \beta^\nu - \partial^\nu \bar{\beta}^\mu \right) C_\nu \bigg].
\]

On the other hand, under the infinitesimal, continuous and off-shell nilpotent \((s_b^2 = 0)\) BRST symmetry transformations \(s_b\):

\[
s_b A_{\mu \nu \lambda} = \partial_\mu C_{\nu \lambda} + \partial_\lambda C_{\mu \nu} + \partial_\nu C_{\lambda \mu}, \quad s_b C_{\mu \nu} = \partial_\mu \beta_\nu - \partial_\nu \beta_\mu,
\]

\[
s_b B_{\mu \nu} = \pm \left( \partial_\mu F_\nu - \partial_\nu F_\mu \right), \quad s_b \bar{C}_{\mu \nu} = B_{\mu \nu},
\]

\[
s_b \Phi_{\mu \nu} = \pm m C_{\mu \nu}, \quad s_b \bar{B}_{\mu} = \pm \partial_\mu f, \quad s_b \bar{f}_{\mu} = - \partial_\mu B_1, \quad s_b \bar{\beta}_{\mu} = F_\mu,
\]

\[
s_b C_{\mu} = B_\mu, \quad s_b \bar{C}_2 = B_2, \quad s_b \bar{C}_1 = - B_1,
\]

\[
s_b \phi = f, \quad s_b \beta = \pm m C_2, \quad s_b \bar{\beta} = F,
\]

\[
s_b \bar{F} = \pm m B, \quad s_b \bar{f} = \pm m B_1, \quad s_b C_1 = - B,
\]

\[
s_b \left[ H_{\mu \nu \lambda}, B_{\mu \nu}, B_{\mu}, f_{\mu}, F_{\mu}, F, f, B, B_1, B_2, C_2 \right] = 0,
\]

the Lagrangian density \(\mathcal{L}_B\) transforms to a \(total\) spacetime derivative

\[
s_b \mathcal{L}_B = \partial_\mu \left[ \left( \partial^\nu \bar{C}^{\nu \lambda} + \partial^\nu C^{\lambda \mu} + \partial^\lambda C^{\mu \nu} \right) \bar{B}_{\nu \lambda} + B^{\mu \nu} f_\nu - B_2 \partial^\mu C_2 \right.
- B_1 f^\mu + B F^\mu - \left( \partial^\nu \beta^\mu - \partial^\nu \bar{\beta}^\mu \right) F_\nu + \frac{1}{2} \left( \pm m \beta^\mu - \partial^\mu \beta \right) F
+ \left( \partial^\mu \bar{C}^{\mu \nu} - \partial^\nu \bar{C}^{\nu \mu} \right) \bar{B}_\nu \pm m B^{\mu \nu} C_\nu - \frac{1}{2} B^\mu \bar{f}
\pm m \bar{C}^{\mu \nu} \left( \pm m \beta_\nu - \partial_\nu \beta \right) \pm m \left( \partial^\mu \beta^\nu - \partial^\nu \bar{\beta}^\mu \right) C_\nu \bigg],
\]

which establishes the fact that the action integral \(S_2 = \int d^{D-1} x \mathcal{L}_B\) respects the above infinitesimal, continuous and nilpotent BRST symmetry transformations \(s_b\).
According to Noether’s theorem, the above observations imply that the BRST conserved currents can be derived, by exploiting the standard theoretical formula, as [26]:

\[
J_{(b)}^\mu = H^{\mu\nu\lambda} (\partial_\nu C_{\lambda \beta}) \pm \frac{m}{2} \left[ \pm m \beta^\mu - \partial^\mu \beta^\nu \right] C_2 \pm m C^{\mu\nu} (\pm m \beta_\nu - \partial_\nu \beta^\nu) + (\partial^\mu C^{\nu\lambda}) \\
+ \partial^\nu C^{\lambda\mu} + \partial^\lambda C^{\mu\nu} B_{\nu\lambda} - (\partial^\nu \bar{C}^{\lambda\mu} + \partial^\lambda \bar{C}^{\mu\nu} + \partial^\lambda \bar{C}^{\nu\mu}) (\partial_\nu \beta_\lambda - \partial_\lambda \beta_\nu) + m C^{\mu\nu} B_\nu \\
- B_1 f^\mu + [\pm m A^{\mu\nu\lambda} - \Sigma^{\mu\nu\lambda}] [\pm m C_{\nu\lambda} - (\partial_\nu C_{\lambda \rho} - \partial_\lambda C_{\nu \rho})] - B_2 \bar{F}^\mu C_2 + B^{\mu\nu} F_\nu \\
- (\partial^\mu \bar{\beta}^\nu - \partial^\nu \beta^\mu) (\partial_\nu C_2) - (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) F_\nu + (\partial^\mu C^{\nu\omega} - \partial^\nu C^{\omega\mu}) B_\nu + B F^\mu \\
- \frac{1}{2} B^\mu f + \frac{1}{2} \left( \pm m \beta^\mu - \partial^\mu \beta^\nu \right) F - (\partial^\mu C^{\nu\omega} - \partial^\nu C^{\omega\mu}) (\pm m \beta_\nu - \partial_\nu \beta^\nu). \tag{73}
\]

In exactly similar fashion, we obtain the precise expression for the conserved \((\partial_\mu J_{(ab)}^\mu = 0)\) anti-BRST Noether current \(J_{(ab)}^\mu\) [26]:

\[
J_{(ab)}^\mu = H^{\mu\nu\lambda} (\partial_\nu \bar{C}_{\lambda \beta}) \pm \frac{m}{2} \left[ \pm m \beta^\mu - \partial^\mu \beta^\nu \right] C_2 \mp m C^{\mu\nu} (\pm m \bar{\beta}_\nu - \partial_\nu \bar{\beta}^\nu) - (\partial^\mu \bar{C}^{\nu\lambda}) \\
+ \partial^\nu \bar{C}^{\lambda\mu} + \partial^\lambda \bar{C}^{\mu\nu} B_{\nu\lambda} - (\partial^\nu C^{\lambda\mu} + \partial^\lambda C^{\mu\nu} + \partial^\lambda C^{\nu\mu}) (\partial_\nu \bar{\beta}_\lambda - \partial_\lambda \bar{\beta}^\nu) \pm m C^{\mu\nu} B_\nu \\
- \bar{f}^\mu B_1 + [\pm m A^{\mu\nu\lambda} - \Sigma^{\mu\nu\lambda}] [\pm m \bar{C}_{\nu\lambda} - (\partial_\nu \bar{C}_\lambda - \partial_\lambda \bar{C}_\nu)] + B \partial^\mu \bar{C}_2 + B^{\mu\nu} \bar{F}_\nu \\
- (\partial^\mu \bar{\beta}^\nu - \partial^\nu \beta^\mu) (\partial_\nu \bar{C}_2) - (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) \bar{F}_\nu - (\partial^\mu C^{\nu\omega} - \partial^\nu C^{\omega\mu}) \bar{B}_\nu + B \bar{F}^\mu \\
- \frac{1}{2} B^\mu \bar{f} + \frac{1}{2} \left( \pm m \bar{\beta}^\mu - \partial^\mu \bar{\beta}^\nu \right) \bar{F} + (\partial^\mu C^{\nu\omega} - \partial^\nu C^{\omega\mu}) (\pm m \bar{\beta}_\nu - \partial_\nu \bar{\beta}^\nu). \tag{74}
\]

The conservation law \((\partial_\mu J_{(ab)}^\mu = 0)\) can be checked by using the EL-EoMs that have been derived in our earlier work [26]. In these proofs, the algebra is a bit involved but it is quite straightforward to check that \(\partial_\mu J_{(r)}^\mu = 0\) with \(r = b, ab\).

The conserved Noether charges \(Q_{(ab)} = \int d^{D-1} x J_{(ab)}^0\) (that emerge out from the conserved currents) \(J_{(ab)}^\mu\) are as follows [26]:

\[
Q_{ab} = \int d^{D-1} x J_{(ab)}^0 \equiv \int d^{D-1} x \left[ H^{0ij\kappa} (\partial_i \bar{C}_{j\kappa}) \pm \frac{m}{2} \left[ \pm m \beta^{\kappa 0} - \partial^{\kappa 0} \beta^\nu \right] C_2 \\
- [\pm m C^{0i} - (\partial^0 C_i - \partial^i C^0)] (\pm m \bar{\beta}_i - \partial_i \bar{\beta}^\nu) + [\pm m C^{0i} - (\partial^0 C_i - \partial^i C^0)] B_i \\
- (\partial^0 \bar{C}_{ij} + \partial^i \bar{C}^{0j} + \partial^j \bar{C}^{0i}) B_{ij} - (\partial^0 C^{0ij} + \partial^i C^{0j0} + \partial^j C^{0i0}) (\partial_i \bar{\beta}_j - \partial_j \bar{\beta}_i) \\
- \bar{F}^0 B_1 + [\pm m A^{0ij} - \Sigma^{0ij}] [\pm m \bar{C}_{ij} - (\partial_i \bar{C}_j - \partial_j \bar{C}_i)] + B \partial^0 \bar{C}_2 - \frac{1}{2} \bar{B}^0 \bar{f} - B_2 \bar{F}^0 \\
+ \bar{B}^{0i} \bar{f}_i - (\partial^0 \beta^i - \partial^i \beta^0) (\partial_i \bar{C}_2) - (\partial^0 \beta^i - \partial^i \beta^0) \bar{F}_i + \frac{1}{2} \left( \pm m \bar{\beta}^{0i} - \partial^{0i} \bar{\beta}^\nu \right) \bar{F}, \tag{75}
\]

\[
Q_b = \int d^{D-1} x J_{(b)}^0 \equiv \int d^{D-1} x \left[ H^{0ij\kappa} (\partial_i \bar{C}_{j\kappa}) \pm \frac{m}{2} \left[ \pm m \beta^{\kappa 0} - \partial^{\kappa 0} \beta^\nu \right] C_2 \\
+ [\pm m \bar{C}^{0i} - (\partial^0 \bar{C}^i - \partial^i \bar{C}^0)] (\pm m \beta_i - \partial_i \beta^\nu) - [\pm m C^{0i} - (\partial^0 C^i - \partial^i C^0)] B_i \\
+ (\partial^0 \bar{C}^{ij} + \partial^i \bar{C}^{0j} + \partial^j \bar{C}^{0i}) B_{ij} - (\partial^0 C^{0ij} + \partial^i C^{0j0} + \partial^j C^{0i0}) (\partial_i \beta_j - \partial_j \beta_i) \\
- B_1 f^0 + [\pm m A^{0ij} - \Sigma^{0ij}] [\pm m C_{ij} - (\partial_i C_j - \partial_j C_i)] - B_2 \partial^0 C_2 - \frac{1}{2} B^0 f + B^{0i} f_i \\
- (\partial^0 \bar{\beta}^{i0} - \partial^i \beta^{00}) (\partial_i \bar{C}_2) - (\partial^0 \beta^i - \partial^i \beta^0) F_i + B F^0 + \frac{1}{2} \left( \pm m \beta^{00} - \partial^{00} \beta^\nu \right) F. \tag{76}
\]

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These conserved charges are the generators of all the (anti-)BRST symmetry transformations. However, they are not off-shell nilpotent of order two. Exploiting the principle behind the continuous symmetry transformations and their generators, we obtain the following

\[ s_b Q_b = -i \{ Q_b, Q_b \} = \int d^{D-1}x \left[ \pm \frac{m}{2} \left( \pm m F^0 - \partial^0 F \right) C_2 - (\partial^0 F^i - \partial^i F^0) (\partial_i C_2) \right. \\
- (\partial^0 B^{ij} + \partial^i B^{0j} + \partial^j B^{0i}) (\partial_i \beta_j - \partial_j \beta_i) \\
+ \left[ \pm m B^{0i} - (\partial^0 B^i - \partial^i B^0) \right] (\pm m \beta_i - \partial_i \beta) \neq 0, \]

\[ s_{ab} Q_{ab} = -i \{ Q_{ab}, Q_{ab} \} = \int d^{D-1}x \left[ \pm \frac{m}{2} \left( \pm m \bar{F}^0 - \partial^0 \bar{F} \right) \bar{C}_2 - (\partial^0 \bar{F}^i - \partial^i \bar{F}^0) (\partial_i \bar{C}_2) \right. \\
+ (\partial^0 \bar{B}^{ij} + \partial^i \bar{B}^{0j} + \partial^j \bar{B}^{0i}) (\partial_i \bar{\beta}_j - \partial_j \bar{\beta}_i) \\
- \left[ \pm m \bar{B}^{0i} - (\partial^0 \bar{B}^i - \partial^i \bar{B}^0) \right] (\pm m \bar{\beta}_i - \partial_i \bar{\beta}) \neq 0, \quad (77) \]

where the l.h.s. of the above equations have been explicitly computed by using the (anti-)BRST symmetry transformations [cf. Eqs. (69), (71)] and the expressions for the Noether conserved charges \( Q_{a,b} \) [cf. Eqs. (75), (76)]. The above observations establish that the conserved Noether (anti-)BRST charges \( Q_{a,b} \) are not off-shell nilpotent of order two.

We now follow our step-by-step proposal to obtain the off-shell nilpotent expressions for the (anti-)BRST charges \( Q_{(1)}^{(1)} \) where the off-shell nilpotency is proven by using the principle behind the continuous symmetry transformations and their generators as: \( s_b Q_b^{(1)} = -i \{ Q_b^{(1)}, Q_b^{(1)} \} = 0 \) and \( s_{ab} Q_{ab}^{(1)} = -i \{ Q_{ab}^{(1)}, Q_{ab}^{(1)} \} = 0 \). First of all, we focus on the BRST charge \( Q_b \). The first step is to use the equation of motion w.r.t. the gauge field. Towards this goal in mind, we note that the first term of \( Q_b \) can be re-expressed as:

\[ \int d^{D-1}x \left[ H^{0ijk} \partial_i C_{jk} \right] = \int d^{D-1}x \left[ \partial_i \left( H^{0ijk} C_{jk} \right) \right] - \int d^{D-1}x \left[ (\partial_i H^{0ijk}) C_{jk} \right]. \quad (78) \]

The first term on the r.h.s. of the above equation vanishes for the physical fields due to celebrated Gauss's divergence theorem. In the second term, we apply the following equation of motion w.r.t. the gauge field \( A_{\mu \nu \lambda} \), namely:

\[ \partial_\mu H^{\mu \nu \lambda} + m^2 A^{\mu \nu \lambda} \mp m \Sigma^{\mu \nu \lambda} + (\partial^\mu B^{\nu \lambda} + \partial^\nu B^{\lambda \mu} + \partial^\lambda B^{\mu \nu}) = 0 \]

\[ \implies - (\partial_i H^{0ijk}) C_{jk} = \mp m \left[ \pm m A^{0ij} - \Sigma^{0ij} \right] C_{ij} - (\partial^0 B^{ij} + \partial^i B^{0j} + \partial^j B^{0i}) C_{ij} \quad (79) \]

In the second step, we look for the terms of \( Q_b \) that cancel out with some of the terms that emerge out after the first step. In this context, we note that such an appropriate term is

\[ \left[ \pm m A^{0ij} - \Sigma^{0ij} \right] \left[ \pm m C_{ij} - (\partial_i C_j - \partial_j C_i) \right]. \quad (80) \]

The sum of (79) and (80) yields the following:

\[ \implies -(\partial^0 B^{ij} + \partial^i B^{0j} + \partial^j B^{0i}) C_{ij} - 2 \left[ \pm m A^{0ij} - \Sigma^{0ij} \right] (\partial_i C_j). \quad (81) \]
We comment here that, as pointed out earlier, one of the key ingredients of our proposal is the use of EL-EoM w.r.t. the gauge field. This step opens up a decisive door for (i) our further applications of Gauss’s divergence theorem, (ii) use of the appropriate EL-EoM, and (iii) the application of the BRST symmetry transformations ($s_b$) at appropriate places. In the third step, we focus on the second term of the r.h.s. which can be written as:

$$-2 \partial_i \left[ (\pm m A^{0ij} - \Sigma^{0ij}) C_j \right] + 2 \partial_i \left[ (\pm m A^{0ij} - \Sigma^{0ij}) \right] C_j. \quad (82)$$

The terms in (82) are inside the integration. Thus, the first term of (82) will vanish due to Gauss’s divergence theorem for physical fields. Using the following equation of motion w.r.t. the St"uckelberg field $\Phi_{\mu\nu}$ from the Lagrangian density $L_B$, we obtain:

$$\partial_\mu \Sigma^{\mu\nu} + m (\partial_\mu A^{\mu\nu}) + \frac{1}{2} (\partial^\nu B^\lambda - \partial^\lambda B^\nu) \mp \frac{m}{2} B^{\mu\lambda} = 0 \quad (83)$$

where the l.h.s. is nothing but the second term of (82). Using (83), we can re-write the whole sum of (81) as follows:

$$+ \left[ \pm m B^{0i} - (\partial^0 B^i - \partial^i B^0) \right] C_i - (\partial^0 B^{ij} + \partial^i B^{j0} + \partial^j B^{0i}) C_{ij}. \quad (84)$$

Within the framework of our proposal, the terms in (84) will be present in the off-shell nilpotent form of the BRST charge ($Q_b^{(1)}$) because they have come out from the use of EL-EoMs and, hence, will remain intact. This is also required due to the fact that, as pointed out earlier, we wish to obtain $s_b Q_b^{(1)} = 0$ which will automatically imply that $[Q_b^{(1)}]^2 = 0$.

In the fourth step, we apply the BRST symmetry transformation ($s_b$) on (84) which leads to the following explicit expression:

$$+ \left[ \pm m B^{0i} - (\partial^0 B^i - \partial^i B^0) \right] (\pm m \beta_i - \partial_i \beta) - (\partial^0 B^{ij} + \partial^i B^{j0} + \partial^j B^{0i}) (\partial_i \beta_j - \partial_j \beta_i). \quad (85)$$

In the fifth step, we modify some of the terms of $Q_b$ so that when $s_b$ acts on a part of them, there is precise cancellation between the ensuing result and (85). In this context, we modify the following terms from the explicit expression for $Q_b$ [cf. Eq. (76)], namely;

$$- (\partial^0 \bar{C}^{ij} + \partial^i \bar{C}^{j0} + \partial^j \bar{C}^{0i}) (\partial_i \beta_j - \partial_j \beta_i) = + (\partial^0 \bar{C}^{ij} + \partial^i \bar{C}^{j0} + \partial^j \bar{C}^{0i}) (\partial_i \beta_j - \partial_j \beta_i)$$

$$- 2 (\partial^0 \bar{C}^{ij} + \partial^i \bar{C}^{j0} + \partial^j \bar{C}^{0i}) (\partial_i \beta_j - \partial_j \beta_i),$$

$$+ \left[ \pm m \bar{C}^{0i} - (\partial^0 \bar{C}^i - \partial^i \bar{C}^0) \right] (\pm m \beta_i - \partial_i \beta) = 2 \left[ \pm m \bar{C}^{0i} - (\partial^0 \bar{C}^i \right.$$ 

$$- \partial^i \bar{C}^{0i}) (\pm m \beta_i - \partial_i \beta) - \left[ \pm m \bar{C}^{0i} - (\partial^0 \bar{C}^i - \partial^i \bar{C}^0) \right] (\pm m \beta_i - \partial_i \beta). \quad (86)$$

In the sixth step, we note that, in the above modified expressions, when we apply $s_b$ on a part of (86) as explicitly written below, namely;

$$s_b \left[ (\partial^0 \bar{C}^{ij} + \partial^i \bar{C}^{j0} + \partial^j \bar{C}^{0i}) (\partial_i \beta_j - \partial_j \beta_i)$$

$$- \left\{ \pm m \bar{C}^{0i} - (\partial^0 \bar{C}^i - \partial^i \bar{C}^0) \right\} (\pm m \beta_i - \partial_i \beta) \right]. \quad (87)$$

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we obtain the desired result

\[
(\partial^0 B^{ij} + \partial^i B^{j0} + \partial^j B^{0i}) (\partial_i \beta_j - \partial_j \beta_i) \\
- \left\{ \pm m B^{0i} - (\partial^0 B^i - \partial^i B^0) \right\} (\pm m \beta_i - \partial_i \beta),
\]

(88)

which precisely cancels out with whatever we have obtained in (85). At this juncture, we point out that, in the off-shell nilpotent version of the BRST charge \(Q_b^{(1)}\), the terms in the square bracket of (87), will be always [along with (84) (as pointed out earlier)] present. We now focus on the left-over terms of (86) which can be re-expressed as follows:

\[
\pm 2 m \left\{ \pm m \tilde{C}^{0i} - (\partial^0 \tilde{C}^i - \partial^i \tilde{C}^0) \right\} \beta_i \\
- 2 \left\{ \pm m \tilde{C}^{0i} - (\partial^0 \tilde{C}^i - \partial^i \tilde{C}^0) \right\} \partial_i \beta \\
- 4 \left[ \partial^0 \tilde{C}^{ij} + \partial^i \tilde{C}^{j0} + \partial^j \tilde{C}^{0i} \right] (\partial_i \beta_j).
\]

(89)

All the above terms are inside the integration. Hence, we can apply the following algebraic tricks on the last two terms of (89) to obtain the following

\[
-2 \left\{ \pm m \tilde{C}^{0i} - (\partial^0 \tilde{C}^i - \partial^i \tilde{C}^0) \right\} \partial_i \beta = \partial_i \left[ -2 \left\{ \pm m \tilde{C}^{0i} - (\partial^0 \tilde{C}^i - \partial^i \tilde{C}^0) \right\} \beta \right] \\
+ 2 \partial_i \left[ \pm m \tilde{C}^{0i} - (\partial^0 \tilde{C}^i - \partial^i \tilde{C}^0) \right] \beta \\
-4 \left[ \partial^0 \tilde{C}^{ij} + \partial^i \tilde{C}^{j0} + \partial^j \tilde{C}^{0i} \right] \partial_i \beta_j = \partial_i \left[ -4 \left\{ \partial^0 \tilde{C}^{ij} + \partial^i \tilde{C}^{j0} + \partial^j \tilde{C}^{0i} \right\} \beta_j \right] \\
+ 4 \partial_i \left[ \partial^0 \tilde{C}^{ij} + \partial^i \tilde{C}^{j0} + \partial^j \tilde{C}^{0i} \right] \beta_j.
\]

(90)

so that the total space derivative terms vanish due to Gauss’s divergence theorem for the physical fields and we are left with the following from the above equation (90):

\[
2 \partial_i \left[ \pm m \tilde{C}^{0i} - (\partial^0 \tilde{C}^i - \partial^i \tilde{C}^0) \right] \beta + 4 \partial_i \left[ \partial^0 \tilde{C}^{ij} + \partial^i \tilde{C}^{j0} + \partial^j \tilde{C}^{0i} \right] \beta_j.
\]

(91)

In the above, we apply the following equation of motion:

\[
\partial_\mu [\partial^\mu \tilde{C}^\nu - \partial^\nu \tilde{C}^\mu] - \frac{1}{2} \partial^\nu F + m (\partial_\mu \tilde{C}^{\mu\nu}) \pm \frac{m}{2} F^\nu = 0, \\
\implies 2 \partial_i \left[ \pm m \tilde{C}^{0i} - (\partial^0 \tilde{C}^i - \partial^i \tilde{C}^0) \right] \beta = - (\pm m F^0 - \partial^0 F) \beta,
\]

(92)

which provides the concise value of the first term of (91). Exactly, in a similar fashion, the application of the following equation of motion

\[
\partial_\mu [\partial^\mu \tilde{C}^{\nu\lambda} + \partial^\nu \tilde{C}^{\lambda\mu} + \partial^\lambda \tilde{C}^{\mu\nu}] \pm \frac{m}{2} (\partial^\nu \tilde{C}^{\lambda\mu} - \partial^\lambda \tilde{C}^{\mu\nu}) \\
+ \frac{1}{2} (\partial^\nu F^\lambda - \partial^\lambda F^\nu) - \frac{m^2}{2} \tilde{C}^{\nu\lambda} = 0, \\
\implies 4 \partial_i \left[ \partial^0 \tilde{C}^{ij} + \partial^i \tilde{C}^{j0} + \partial^j \tilde{C}^{0i} \right] \beta_j = 2 (\partial^0 F^i - \partial^i F^0) \beta_i \\
+ 2 m \left[ \pm m \tilde{C}^{0i} - (\partial^0 \tilde{C}^i - \partial^i \tilde{C}^0) \right] \beta_i.
\]

(93)
leads to the alternative (but appropriate) form of the second term of (91). The sum of the r.h.s. of (92), (93) and the left-over term (i.e. the first term) of (89) is equal to:

$$2 (\partial^0 F^i - \partial^i F^0) \beta_i - (\pm m F^0 - \partial^0 F) \beta.$$  \hfill (94)

At this stage, we take the seventh step and apply the BRST symmetry transformations ($s_b$) on (94) which yields the following explicit expression, namely:

$$-2 (\partial^0 F^i - \partial^i F^0) (\partial_i C_2) + (\pm m F^0 - \partial^0 F) (\pm m C_2).$$  \hfill (95)

We emphasize that the term in (94) will be present in the off-shell nilpotent version of the BRST charge ($Q_b^{(1)}$). We take now the eighth step, and modify some of the terms of $Q_b$ so that when we apply the BRST transformations ($s_b$) on (94) which yields the following explicit expression, namely:

\begin{align*}
- (\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0) \partial_i C_2 & \pm \frac{1}{2} m (\pm m \bar{\beta}^0 - \partial^0 \bar{\beta}) C_2 \\
\equiv & \ 2 (\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0) \partial_i C_2 \mp m (\pm m \bar{\beta}^0 - \partial^0 \bar{\beta}) C_2 \\
& \pm \frac{3}{2} (\pm m \bar{\beta}^0 - \partial^0 \bar{\beta}) C_2 - 3 (\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0) \partial_i C_2.  \hfill (96)
\end{align*}

It is evident that if we apply the BRST symmetry transformations ($s_b$) on the first two terms of (96), the resulting expressions will cancel out the terms that are written in (95). Hence, along with (94), the above first two terms will be present in the off-shell nilpotent version of the BRST charge ($Q_b^{(1)}$) so that our central objective $s_bQ_b^{(1)} = 0$ can be fulfilled. At this juncture, we focus on the last term of (96) which can be re-written as

$$-3 (\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0) \partial_i C_2 = \partial_i [-3 (\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0) C_2] + 3 \partial_i [(\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0)] C_2. \hfill (97)$$

The above terms are inside the integral. Hence, the first term on the r.h.s. will vanish due to Gauss’s divergence theorem and only the second term on the r.h.s. of (97) will survive. Now, we apply the following EL-EoM

\begin{align*}
\Box \bar{\beta}_\mu - \partial_\mu (\partial \cdot \bar{\beta}) + \partial_\mu B_2 - \frac{m^2}{2} \bar{\beta}_\mu \pm \frac{m}{2} \partial_\mu \bar{\beta} & = 0 \\
\implies & \ 3 \partial_i [\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0] C_2 = 3 \bar{B}_2 C_2 \mp \frac{3}{2} m (\pm m \bar{\beta}^0 - \partial^0 \bar{\beta}) C_2, \hfill (98)
\end{align*}

which will replace the last term of (96). The substitution of (98) into (96) yields the following concise and beautiful result, namely;

$$\pm \frac{3}{2} m (\pm m \bar{\beta}^0 - \partial^0 \bar{\beta}) C_2 + 3 \partial_i [\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0] C_2 = 3 \bar{B}_2 C_2.$$  \hfill (99)

If we apply further the BRST symmetry transformation ($s_b$) on (99), it turns out to be zero. Here, all our steps terminate (according to our proposal). It is self-evident that the
r.h.s. of (99) will be part in \( Q_b^{(1)} \). Ultimately, the off-shell nilpotent version \( Q_b^{(1)} \), from the non-nilpotent Noether conserved charge \( Q_b \), is as follows

\[
Q_b \rightarrow Q_b^{(1)} = \int d^{D-1}x \left[ \left( \partial^0 C^{ij} + \partial^i C^{j0} + \partial^j C^{0i} \right) B_{ij} - \left( \pm m C^{0i} - (\partial^0 C^i - \partial^i C^0) \right) C_i \right]
\]

It will be noted that we have not touched several terms of the Noether conserved charge \( Q_b \) [cf. Eq. (76)] because these terms are BRST invariant. For instance, we have the following explicit observations:

\[
s_b \left[ (\partial^0 C^{ij} + \partial^i C^{j0} + \partial^j C^{0i}) B_{ij} \right] = 0, \quad s_b \left\{ \pm m C^{0i} - (\partial^0 C^i - \partial^i C^0) \right\} B_i = 0,
\]

\[
s_b \left[ (\pm m F^0 - \partial^0 F) \beta \right] = 0, \quad s_b (B_1 f^0) = 0, \quad s_b (B^0 f) = 0, \quad s_b (B^0 f_i) = 0,
\]

\[
s_b (B_2 \tilde{C}_2) = 0, \quad s_b (B F^0) = 0.
\]

It is straightforward to check that the following is true, namely;

\[
s_b Q_b^{(1)} = -i \left\{ Q_b^{(1)}, Q_b^{(1)} \right\} = 0 \implies [Q_b^{(1)}]^2 = 0,
\]

where the l.h.s. is computed directly by using the BRST symmetry transformations (71) on the expression for the modified version of the BRST charge \( Q_b^{(1)} \).

We follow the prescriptions and proposal outlined above to compute the exact expression for the off-shell nilpotent version of the anti-BRST charge \( Q_{ab}^{(1)} \) from the non-nilpotent conserved Noether anti-BRST charge \( Q_{ab} \) as follows:

\[
Q_{ab} \rightarrow Q_{ab}^{(1)} = \int d^{D-1}x \left[ \left( \partial^0 \tilde{B}^{ij} + \partial^i \tilde{B}^{j0} + \partial^j \tilde{B}^{0i} \right) \tilde{C}_{ij} - \left( \pm m \tilde{B}^{0i} - (\partial^0 \tilde{B}^i - \partial^i \tilde{B}^0) \right) \tilde{C}_i \right]
\]

It is now straightforward to check that:

\[
s_{ab} Q_{ab}^{(1)} = -i \left\{ Q_{ab}^{(1)}, Q_{ab}^{(1)} \right\} = 0 \implies [Q_{ab}^{(1)}]^2 = 0.
\]
The above observation proves that we have derived the off-shell nilpotent version \([Q^{(1)}_{ab}]\) of the non-nilpotent Noether conserved charge \(Q_{ab}\) in a precise and logical manner. Our final results are the expressions for \(Q^{(1)}_{(a)b}\) in (104) and (100).

6 Conclusions

In our present investigation, for a few physically interesting gauge systems, we have shown that wherever there is existence of the coupled (but equivalent) Lagrangians/Lagrangian densities due to the presence of the (anti-)BRST invariant CF-type restriction(s), we observe that the Noether theorem does not lead to the derivation of the off-shell nilpotent versions of the conserved (anti-)BRST charges \([Q_{(a)b}]\) within the framework of BRST formalism. These Noether conserved charges are found to be the generators for the infinitesimal, continuous and off-shell nilpotent (anti-)BRST symmetry transformations from which they are derived by exploiting the theoretical strength of Noether’s theorem. However, they are found to be non-nilpotent in the sense that \(s_b Q_b = -i \{Q_b, Q_b\} \neq 0\) and \(s_a Q_b = -i \{Q_a, Q_b\} \neq 0\) which can be verified by applying the (anti-)BRST transformations directly on the Noether conserved charges \(Q_{(a)b}\). In other words, by directly computing the explicit expressions for \(s_b Q_b\) and \(s_a Q_{ab}\), which are the l.h.s. of: \(s_b Q_b = -i \{Q_b, Q_b\}\) and \(s_a Q_{ab} = -i \{Q_a, Q_{ab}\}\), we show that these charges are not nilpotent [see, e.g. Eq. (77)]. However, a close and careful look at them demonstrate that \(s_b Q_b\) and \(s_a Q_{ab}\) are proportional to the EL-EoM. For instance, we are sure that, in the simple case of a BRST-invariant 1D massive spinning relativistic particle, the r.h.s. of (14) are zero due to the EL-EoMs (15) and (23) that are derived from \(L_b\) and \(L_{\bar{b}}\), respectively.

In the derivations of the off-shell nilpotent versions of the (anti-)BRST charges \([Q^{(1)}_{(a)b}]\), the crucial roles are played by (i) the Gauss divergence theorem, (ii) the appropriate EL-EoMs from the appropriate Lagrangian/Lagrangian density of a set of coupled (but equivalent) Lagrangians/Lagrangian densities, and (iii) the application of the (anti-)BRST symmetry transformations at appropriate places (cf. Sec. 5 for details). We lay emphasis on the fact that, for the D-dimensional (\(D \geq 2\)) higher \(p\)-form (\(p = 1, 2, 3, ...\)) gauge theories, it is imperative to first exploit the Gauss divergence theorem so that we can use the EL-EoMs w.r.t. the gauge field. For the 1D case of a spinning (i.e. SUSY) relativistic particle, we have shown that the Gauss divergence theorem, for obvious reasons, is not required at all and we directly use the EL-EoMs, right in the beginning, w.r.t. the “gauge” and “supergauge” variables. This is not the case with the rest of the examples considered in our present investigation which are connected with the D-dimensional (\(D \geq 2\)) (non-)Abelian \(p\)-form (\(p = 1, 2, 3, ...\)) massless and massive gauge theories.

The sequence of our proposal, to obtain the off-shell nilpotent versions of the (anti-)BRST charges from the non-nilpotent Noether (anti-)BRST charges, is as follows. In the first step, we apply the Gauss divergence theorem and take the help of EL-EoM w.r.t. the gauge field. This is the crucial and key first step of our proposal. In the next (i.e. second) step, we observe carefully whether there are any addition, subtraction and/or cancellation of the resulting terms (from the first step) with any of the terms of the non-nilpotent Noether conserved charges. The existing terms, after these two steps, will always be present in the
off-shell nilpotent version of the (anti-)BRST charges $Q_{(a)b}^{(1)}$. After the above two steps, we apply the (anti-)BRST symmetry transformations on the existing terms. In the third step, we modify some of the appropriate terms of the non-nilpotent Noether conserved (anti-)BRST charges and see to it that a part of these modified terms cancel precisely with the terms that have appeared after the application of the nilpotent (anti-)BRST transformations on the existing terms (after the first two steps of our proposal). The parts which participate in the above cancellation are also always present in the off-shell nilpotent versions of (anti-)BRST charges $[Q_{(a)b}^{(1)}]$. After this step, it is the interplay amongst the Gauss divergence theorem, appropriate EL-EoMs and application of the nilpotent (anti-)BRST symmetry transformations at appropriate places that lead to the derivation of the precise forms of the off-shell nilpotent versions\(^\dagger\) of the (anti-)BRST charges $[Q_{(a)b}^{(1)}]$ from the non-nilpotent Noether conserved (anti-)BRST charges (cf. Secs. 4 and 5 for details).

Before we end this section by pointing out our future directions of investigation in the next paragraph, it is worthwhile to point out the physical significance of the conserved and off-shell nilpotent (anti-)BRST charges in the context of a given gauge theory which is endowed with a set of non-trivial CF-type restrictions(s). The physicality condition $(Q_B^{(1)} | \text{phys} > = 0$, cf. Sec. 3) ensures that the operator form of the first-class constraints of the given gauge theory annihilate the physical state which is consistent with the requirements of the Dirac quantization condition for a theory endowed with constraints (see, e.g. [5, 6] for details). As has been shown by Weinberg [19], in the Fock-space, the gauge transformed states differ from their original counterparts by the BRST-exact states. Hence, if we have off-shell nilpotent BRST charge, their difference becomes trivial when we invoke the physicality condition on the states w.r.t. the BRST charge. This argument has been extended in our earlier research works (see, e.g. [28, 29]) where we have been able to prove that the 2D (non-)Abelian 1-form and 4D Abelian 2-form gauge theories are the tractable field-theoretic models for the Hodge theory. In the case of the 2D (non-)Abelian 1-form theories, we have been able to show that the BRST and co-BRST symmetry transformations “gauge away” both the d.o.f. of the gauge fields and these theories become a new kind of topological field theory (see, e.g. [30] for details).

We end this section with the final comment that our proposal is very general and it can be applied to any physical system where (i) the (anti-)BRST invariant non-trivial CF-type restrictions exist, and (ii) the coupled (but equivalent) Lagrangians/Lagrangian densities describe the dynamics of the above physical systems within the framework of BRST formalism.

\(^\dagger\)It is worthwhile to point out here that the number of fields and corresponding equations of motion become too many even in the case of the modified version of a massive Abelian 3-form gauge theory (see, e.g. [26, 27] for details) and it becomes very difficult to know which equation of motion will be picked-up, from amongst these total number of EL-EoMs, to render the non-nilpotent versions of the Noether conserved charges into the off-shell nilpotent versions of the conserved charges. Moreover, the expressions for the conserved Noether charges themselves become very complicated and cumbersome (see, e.g. [26] for details) when we discuss the modified massive higher $p$-form ($p = 2, 3, \ldots$) gauge theories.

\(^\dagger\dagger\)There are two ways by which $s_b Q_{b}^{(1)} = 0$ and $s_{ab} Q_{ab}^{(1)} = 0$ can be proven because of the integration present in the expressions for $Q_{(a)b}^{(1)}$. The first option is a clear-cut proof that the total integrand turns out to be zero when we apply $s_{(a)b}$ on it. The second option is the case when the integrand transforms to a total space derivative under the applications of $s_{(a)b}$. In our present endeavor, it is the first option that has been chosen where we have proven that $s_b Q_{b}^{(1)} = 0$ and $s_{ab} Q_{ab}^{(1)} = 0$. 

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ism. The systematic application of our proposal sheds light on the appropriate directions that should be followed in order to obtain the off-shell nilpotent versions of the (anti-)BRST charges from the non-nilpotent versions of the Noether conserved (anti-)BRST charges. We plan to extend our present ideas in the context of more challenging problems of physical interest [31] in the future.

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Conflicts of Interest

The authors declare that there are no conflicts of interest.

Data Availability

No data were used to support this study.

Appendix A: On the Massive Abelian 2-Form Theory

To supplement our discussions in Sec. 4, in this Appendix, we concisely mention the off-shell nilpotency property of the non-nilpotent Noether conserved (anti-)BRST charges for the St"uckelberg-modified massive Abelian 2-form gauge theory within the framework of BRST formalism. For this purpose, we begin with the generalized (i.e. St"uckelberg-modified) versions of the (anti-)BRST invariant Lagrangian densities of equations (46) and (47) for the D-dimensional massive Abelian 2-form theory (see, e.g. [32, 27] for details) as

\[
\mathcal{L}_b = \frac{1}{12} H_{\mu\nu\eta} H^{\mu\nu\eta} - \frac{1}{4} m^2 B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} \Phi_{\mu\nu} \Phi^{\mu\nu} + \frac{1}{2} m B_{\mu\nu} \Phi^{\mu\nu} - B^2
\]

\[
- B (\partial \cdot \phi + m \varphi) + B^\mu B_\mu + B^\mu (\partial'^\nu B_{\nu\mu} - \partial'_\mu \varphi + m \phi_\mu) - m^2 \bar{\beta} \beta
\]

\[
+ (\partial_\mu C_\nu - \partial_\nu C_\mu) (\partial'^\mu C'^\nu) - (\partial_\mu C - m C_\mu) (\partial'^\mu C - m C'^\mu) + \partial_\mu \bar{\beta} \partial'^\mu \beta
\]

\[
+ (\partial \cdot C + \rho + m C) \lambda + (\partial \cdot C - \lambda + m C) \rho,
\]

\[
(A.1)
\]

\[
\mathcal{L}_\bar{b} = \frac{1}{12} H_{\mu\nu\eta} H^{\mu\nu\eta} - \frac{1}{4} m^2 B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} \Phi_{\mu\nu} \Phi^{\mu\nu} + \frac{1}{2} m B_{\mu\nu} \Phi^{\mu\nu} - \bar{B}^2
\]

\[
+ \bar{B} (\partial \cdot \phi - m \varphi) + \bar{B}_\mu \bar{B}^\mu + \bar{B}^\mu (\partial'^\nu B_{\nu\mu} + \partial'_\mu \varphi + m \phi_\mu) - m^2 \bar{\beta} \beta
\]

\[
+ (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu) (\partial'^\mu \bar{C}'^\nu) - (\partial_\mu \bar{C} - m \bar{C}_\mu) (\partial'^\mu \bar{C} - m \bar{C}'^\mu) + \partial_\mu \bar{\beta} \partial'^\mu \beta
\]
\[ + (\partial \cdot \bar{C} + \rho + m \bar{C}) \lambda + (\partial \cdot C - \lambda + m C) \rho, \]  

where \( \Phi_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu \) is the field-strength tensor for the vector Stückelberg field \( \phi_\mu \).

There are additional fermionic (anti-)ghost fields \( (\bar{C}, C, \bar{C}, C) \) and bosonic (anti-)ghost fields \( (\beta, \beta') \) in our theory along with an additional scalar field \( \varphi \) with ghost numbers \((-1) + 1, (-2) + 2\) and zero, respectively. There are a couple of additional Nakanishi-Lautrup type fields \((\bar{B}, B)\), too. It will be noted that the massive Abelian 2-form field has the rest mass \( m \) which happens to be the mass of the rest of the fields of our theory \([32]\). The above Lagrangian densities \( \mathcal{L}_b \) and \( \mathcal{L}_\bar{b} \) respect the following off-shell nilpotent \([s^2_{(a)b} = 0]\) (anti-)BRST symmetry transformations \([s_{(a)b}], \) namely:

\[
s_{ab}B_{\mu\nu} = - (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu), \quad s_{ab}C_\mu = - \partial_\mu \bar{C}, \quad s_{ab}\phi_\mu = \partial_\mu \bar{C} - m \bar{C}_\mu, \\

s_{ab}\bar{C}_\mu = \bar{B}_\mu, \quad s_{ab}\beta = - \lambda, \quad s_{ab}\bar{C} = - m \bar{C}, \quad s_{ab}C = \bar{B}, \quad s_{ab}B = - m \rho, \\

s_{ab}\varphi = \rho, \quad s_{ab}B_\mu = \partial_\mu \rho, \quad s_{ab}[\bar{B}, \rho, \lambda, \bar{B}, H_{\mu\nu\kappa}] = 0, \\

s_b\bar{B}_{\mu\nu} = - (\partial_\mu C_\nu - \partial_\nu C_\mu), \quad s_bC_\mu = - \partial_\mu \bar{C}, \quad s_b\phi_\mu = \partial_\mu C - m C_\mu, \\

s_b\bar{C}_\mu = - B_\mu, \quad s_b\bar{B}_\mu = - \partial_\mu \lambda, \quad s_b[B, \rho, \lambda, B_\mu, H_{\mu\nu\kappa}] = 0. \tag{A.3} \]

The above (anti-)BRST symmetries are perfect symmetries for the Lagrangian densities \( \mathcal{L}_b \) and \( \mathcal{L}_\bar{b} \), respectively, because we observe that:

\[
s_{ab}\mathcal{L}_b = \partial_\mu \left[ (\partial^\mu \bar{C} + m \bar{C}^\mu) \bar{B} - (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) \bar{B}_\nu - \lambda \partial^\mu \bar{\beta} + \rho \bar{B}^\mu \right], \\

s_b\mathcal{L}_\bar{b} = - \partial_\mu \left[ (\partial^\mu C - m C^\mu) B + (\partial^\mu C^\nu - \partial^\nu C^\mu) B_\nu + \rho \partial^\mu \beta + \lambda B^\mu \right]. \tag{A.4} \]

The above results establish that the action integrals \( S_1 = \int d^D x \mathcal{L}_b \) and \( S_2 = \int d^D x \mathcal{L}_\bar{b} \) remain invariant under the (anti-)BRST symmetry transformations \([s_{(a)b}], \) respectively, due to Gauss’s divergence theorem.

There are a few comments in order here. First, we note that the kinetic term for the gauge field \((B_{\mu\nu})\), owing its origin to the exterior derivative (i.e. \( H^{(3)} = d B^{(2)} \)), remains invariant under the (anti-)BRST symmetry transformations \([s_{(a)b}], \) Second, we differ in some of our notations, a few terms in the coupled (but equivalent) Lagrangian densities and signs of the symmetry transformations from the earlier works \([32, 27]\). Finally, we note that the (anti-)BRST symmetry transformations \([s_{(a)b}], \) are off-shell nilpotent \([s^2_{(a)b} = 0]\) of order two and absolutely anticommuting (i.e. \( \{s_b, s_{ab}\} = 0 \) in nature, namely:

\[
\{s_b, s_{ab}\} B_{\mu\nu} = \partial_\mu(B_\nu - \bar{B}_\nu) - \partial_\nu(B_\mu - \bar{B}_\mu), \\
\{s_b, s_{ab}\} \Phi_\mu = \partial_\mu(B + \bar{B}) - m (B_\mu - \bar{B}_\mu), \tag{A.5} \]

provided we invoke the (anti-)BRST invariant

\[
s_{(a)b}[B_\mu - \bar{B}_\mu - \partial_\mu \varphi] = 0, \quad s_{(a)b}[B + \bar{B} + m \varphi] = 0, \tag{A.6} \]

CF-type restrictions of our theory for the proof of the absolute anticommutativity in \((A.5)\).
The Noether conserved (anti-)BRST charges have been computed in [32]. We quote here these expressions (with appropriate and correct signs) as (see, e.g. [32] for details):

\[ Q_{ab} = \int d^{D-1}x \left[ -H^{0ij} (\partial_i \bar{C}_j) - (\partial^0 \bar{C}^i - \partial^i \bar{C}^0) \bar{B}_i + (\partial^0 \bar{C} - m \bar{C}^0) \bar{B} ight. \\
- (\partial_i \bar{C} - m \bar{C}_i) (\Phi^{0i} - m B^{0i}) + m (\partial^0 C - m C^0) \beta \\
- (\partial^0 C^i - \partial^i C^0) (\partial_i \beta) - \lambda \partial^0 \beta + \rho B^0 \right]. \]

\[ Q_b = \int d^{D-1}x \left[ -H^{0ij} (\partial_i C_j) - (\partial^0 C^i - \partial^i C^0) B_i - (\partial^0 C - m C^0) B \\
- (\Phi^{0i} - m B^{0i}) (\partial_i C - m C_i) - m (\partial^0 \bar{C} - m \bar{C}^0) \beta \\
+ (\partial^0 \bar{C}^i - \partial^i \bar{C}^0) (\partial_i \beta) - \rho \partial^0 \beta - \lambda B^0 \right]. \tag{A.7} \]

These Noether conserved charges are the generators for all the (anti-)BRST symmetry transformations (A.3) which have been explicitly quoted in (A.3). At this stage, we observe that the above Noether conserved (anti-)BRST charges are non-nilpotent. This can be checked explicitly by applying the BRST symmetry transformations (\(s_b\)) on \(Q_b\) and anti-BRST symmetry transformation (\(s_{ab}\)) on \(Q_{ab}\) as illustrated below:

\[ s_b Q_b = -i \{Q_b, Q_b\} = - \int d^{D-1}x \left[ m (\partial^0 B + m B^0) \beta + (\partial^0 B^i - \partial^i B^0) \partial_i \beta \right] \neq 0, \]

\[ s_{ab} Q_{ab} = -i \{Q_{ab}, Q_{ab}\} = \int d^{D-1}x \left[ m (\partial^0 \bar{B} - m \bar{B}^0) \bar{\beta} - (\partial^0 \bar{B}^i - \partial^i \bar{B}^0) \partial_i \bar{\beta} \right] \neq 0. \tag{A.8} \]

Thus, we note that the Noether (anti-)BRST charges (i) are conserved quantities, (ii) are the generators [28] for the appropriate (anti-)BRST symmetry transformations, and (iii) are found to be not nilpotent of order two (i.e. \(Q_b^2 \neq 0, Q_{ab}^2 \neq 0\)).

In what follows, we follow a systematic method to convert the non-nilpotent Noether conserved (anti-)BRST charges (A.7) into their off-shell nilpotent versions. Since we have demonstrated our method in the context of the BRST charge for the St"uckelberg-modified massive Abelian 3-form theory (cf. Sec. 5), we derive the off-shell nilpotent version \([Q_{ab}^{(1)}]\) of the no-nilpotent Noether anti-BRST \(Q_{ab}\) following our proposal. First of all, we apply the Gauss divergence theorem which turns the first term of \(Q_{ab}\) [cf. Eq. (A.7)] as

\[ (\partial_i H^{0ij}) C_j + m (\Phi^{0i} - m B^{0i}) C_i = (\partial^0 B^i - \partial^i B^0) C_i, \tag{A.9} \]

where we have used the following EL-EoM that is derived from \(\mathcal{L}_b\)

\[ \partial_\mu H^{\mu\nu\lambda} + (\partial^\nu \bar{B}^\lambda - \partial^\lambda \bar{B}^\nu) - m (\Phi^{\nu\lambda} - m B^{\nu\lambda}) = 0. \tag{A.10} \]

We claim that the r.h.s. of (A.9) will be present in the off-shell nilpotent version of the anti-BRST charge \([Q_{ab}^{(1)}]\).

Now let us focus on the remaining part of the fourth term inside the integration. We note that, using the Gauss divergence theorem, we obtain

\[ - (\Phi^{0i} - m B^{0i}) \partial_i \bar{C} \equiv \partial_i [(\Phi^{0i} - m B^{0i}) \bar{C}]. \tag{A.11} \]
At this stage, we use the following equation of motion in the above:

\[ \partial_\mu \Phi^{\mu\nu} - \partial^\nu \vec{B} - m (\partial_\mu B^{\mu\nu}) + m \dot{\vec{B}}^\nu = 0. \]  

(A.12)

The appropriate substitution leads to the following

\[ \partial_i [(\Phi^{0i} - m B^{0i}) \vec{C}] = -\dot{\vec{B}} \vec{C} + m \vec{B}^0 \vec{C} \equiv -(\dot{\vec{B}} - m \vec{B}^0) \vec{C}. \]  

(A.13)

Ultimately, the sum of the first term and fourth term [i.e. sum of (A.10) and (A.13)] leads to the following two terms:

\[ + (\partial^0 \dot{B}^i - \partial^i \dot{B}^0) \vec{C}_i - (\dot{B} - m \vec{B}^0) \vec{C}. \]  

(A.14)

These two terms will stay in the off-shell nilpotent version of the anti-BRST charge \([Q_{ab}^{(1)}]\).

If we apply an anti-BRST symmetry transformations \(s_{ab}\) on (A.14), we obtain the following

\[ s_{ab} [(\partial^0 \dot{B}^i - \partial^i \dot{B}^0) \vec{C}_i - (\dot{B} - m \vec{B}^0) \vec{C}] = -(\partial^0 \dot{B}^i - \partial^i \dot{B}^0) \partial_i \vec{\beta} + m (\dot{B} - m \vec{B}^0) \vec{\beta}. \]  

(A.15)

We have to look, according to our proposal, whether the above terms in (A.15) can cancel and/or can be added and/or subtracted with any terms of \(Q_{ab}\) (i.e. Noether charge). We find that the following modifications of the fifth and sixth terms, namely;

\[ + m (\partial^0 C - m C^0) \vec{\beta} - (\partial^0 C^i - \partial^i C^0) \partial_i \vec{\beta} \equiv 2 m (\partial^0 C - m C^0) \vec{\beta}, \]

\[ -2 (\partial^0 C^i - \partial^i C^0) \partial_i \vec{\beta} - m (\partial^0 C - m C^0) \vec{\beta} + (\partial^0 C^i - \partial^i C^0) \partial_i \vec{\beta}, \]  

(A.16)

will enable us to observe that the anti-BRST symmetry transformations \(s_{ab}\) on the last two terms of the above equation, namely;

\[ s_{ab} [(\partial^0 C^i - \partial^i C^0) \partial_i \vec{\beta} - m (\partial^0 C - m C^0) \vec{\beta}] = (\partial^0 \dot{B}^i - \partial^i \dot{B}^0) \partial_i \vec{\beta} - m (\dot{B} - m \vec{B}^0) \vec{\beta}, \]  

(A.17)

cancels out with the ones that have been obtained in (A.15). Hence, we note that, in addition to the terms in (A.14), the terms in the square bracket of (A.17) will also stay in the off-shell nilpotent version \([Q_{ab}^{(1)}]\) of the anti-BRST charge.

We concentrate now on the left-over terms of (A.16) which are as follows

\[ 2 m (\partial^0 C - m C^0) \vec{\beta} - 2 (\partial^0 C^i - \partial^i C^0) \partial_i \vec{\beta}. \]  

(A.18)

Using the Gauss divergence theorem, we note that the last term can be written as

\[ 2 \partial_i (\partial^0 C^i - \partial^i C^0) \vec{\beta}, \]  

(A.19)

because both the terms of (A.18) are inside the integration. At this stage, we use the following EL-EoM derived from \(\mathcal{L}_b\), namely;

\[ \partial_\mu (\partial^\mu C^\nu - \partial^\nu C^\mu) + \partial^\nu \lambda - m (\partial^\nu C - m C^\nu) = 0, \]  

(A.20)

which leads to the following for the choice \(\nu = 0\), namely;

\[ \partial_i (\partial^0 C^i - \partial^i C^0) \vec{\beta} = 2 \dot{\lambda} \vec{\beta} - 2 m (\dot{C} - m C^0) \vec{\beta}. \]  

(A.21)
The above relationship plays an important and decisive role. For instance, the substitution of the above into equation (A.18) leads to

\[ 2m (\dot{C} - m C^0) \beta + 2 \partial_i (\partial^0 C^i - \partial^i C^0) \beta = 2 \lambda \beta. \] (A.22)

We find that if we apply an anti-BRST symmetry transformation \((s_{ab})\) on the r.h.s. of (A.22), it turns out to be zero. Thus, the theoretical tricks of our proposal terminate here. Hence, the term \((2 \lambda \beta)\) will stay in the off-shell nilpotent version of the \([Q_{ab}^{(1)}]\) of the anti-BRST charge. Finally, we have the off-shell nilpotent version of the anti-BRST charge \([Q_{ab}^{(1)}]\) (derived from the non-nilpotent Noether conserved anti-BRST charge) as follows

\[ Q_{ab} \longrightarrow Q_{ab}^{(1)} = \int d^{D-1}x \left[(\partial^0 B^i - \partial^i B^0) \bar{C}^i + (\bar{C}^0 - m \bar{C}^0) \bar{B}^0 + (\dot{C} - m \dot{C}^0) \bar{B} - (\partial^0 \bar{C}^i - \partial^i \bar{C}^0) \partial_i \beta \right], \] (A.23)

where, we note that some of the original terms of the Noether conserved anti-BRST charge \(Q_{ab}\) are present which are found to be the anti-BRST invariant quantities, namely;

\[ s_{ab}(\bar{\lambda} \beta) = 0, \quad s_{ab}(\rho \bar{B}^0), \quad s_{ab}[(\dot{C} - m \dot{C}^0) \bar{B}] = 0, \quad s_{ab}[(\partial^0 \bar{C}^i - \partial^i \bar{C}^0) \partial_i \beta] = 0. \] (A.24)

It is now straightforward to check that the following is true, namely;

\[ s_{ab}Q_{ab}^{(1)} = -i \{Q_{ab}^{(1)}, Q_{ab}^{(1)}\} = 0 \implies [Q_{ab}^{(1)}]^2 = 0, \] (A.25)

where the l.h.s. can be explicitly computed using the application of the anti-BRST symmetry transformation \((s_{ab})\) on the explicit expression (A.23).

We end this Appendix with the final remark that, following the theoretical tricks of our proposal, it is bit involve but straightforward algebra that leads to the deduction of the off-shell nilpotent \(([Q_{b}^{(1)}]^2 = 0)\) version of the BRST charge \([Q_{b}^{(1)}]\) from the non-nilpotent Noether BRST charge \((Q_{b})\) as follows:

\[ Q_{b} \longrightarrow Q_{b}^{(1)} = \int d^{D-1}x \left[ (\partial^0 B^i - \partial^i B^0) C^i + (\dot{B} + m B^0) C + m (\dot{C}^0 - m \dot{C}^0) \beta \right. \]

\[ - (\partial^0 \bar{C}^i - \partial^i \bar{C}^0) \partial_i \beta + 2 \dot{\rho} \beta - \rho \dot{\beta} - \lambda B^0 - (\partial^0 C^i - \partial^i C^0) B_i - (\dot{C} - m \dot{C}^0) B \right]. \] (A.26)

It is an elementary exercise to check that we have

\[ s_{b}Q_{b}^{(1)} = -i \{Q_{b}^{(1)}, Q_{b}^{(1)}\} = 0 \implies [Q_{b}^{(1)}]^2 = 0, \] (A.27)

where the l.h.s. is computed by an explicit application of \(s_{b}\) on the BRST charge \([Q_{b}^{(1)}]\) [cf. Eq. (A.26)] which establishes that we have obtained the off-shell nilpotent version of the BRST charge \([Q_{b}^{(1)}]\) from the non-nilpotent conserved Noether BRST charge \((Q_{b})\).
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