Real-time non-equilibrium dynamics of quantum glassy systems

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Abstract

We develop a systematic analytic approach to aging effects in quantum disordered systems in contact with an environment. Within the closed-time path-integral formalism we include dissipation by coupling the system to a set of independent harmonic oscillators that mimic a quantum thermal bath. After integrating over the bath variables and averaging over disorder we obtain an effective action that determines the real-time dynamics of the system. The classical limit yields the Martin-Siggia-Rose generating functional associated to a colored noise. We apply this general formalism to a prototype model related to the $p$ spin-glass. We show that the model has a dynamic phase transition separating the paramagnetic from the spin-glass phase and that quantum fluctuations depress the transition temperature until a quantum critical point is reached. We show that the dynamics in the paramagnetic phase is stationary but presents an interesting crossover from a region controlled by the classical critical point to another one controlled by the quantum critical point. The most characteristic property of the dynamics in a glassy phase, namely aging, survives the quantum fluctuations. In the sub-critical region the quantum fluctuation-dissipation theorem is modified in a way that is consistent with the notion of effective temperatures introduced for the classical case. We discuss these results in connection with recent experiments in dipolar quantum spin-glasses and the relevance of the effective temperatures with respect to the understanding of the low temperature dynamics.
I. INTRODUCTION

In the last few years, great progress has been made towards the understanding of the out of equilibrium dynamics of classical systems with slow dynamics and aging effects. Experiments in several glassy materials such as polymer glasses, spin-glasses, orientational glasses, simple liquids like glycerol and gels have shown that all these systems share an extremely slow dynamics at low temperatures. Although some aspects of the dynamic evolution as the precise scaling laws or the dependence on cooling rates vary from system to system, all the examples mentioned above are characterized by the existence of a non-equilibrium low temperature phase with aging effects. This means that measurements show a strong dependence upon the time elapsed since its preparation by a thermal quench or annealing from a high temperature phase. Needless to say, the standard equilibrium approach is not suited to describe such a phenomenology.

In order to describe the spin-glass case, a number of theoretical methods have been exploited, ranging from scaling arguments based on coarsening ideas; phenomenological models founded on particular assumptions on the structure of phase space; analytical solutions to models with infinite-range interactions; and numerical simulations of finite and infinite dimensional models. All of these approaches have succeeded in capturing some of the effects observed experimentally.

Importantly enough, the solution of simplified models with infinite range interactions has been fruitful for several reasons:

(i) It has shown that indeed a large family of such models do capture aging effects that resemble, at least qualitatively, the experimental observations.

(ii) It has provided a general framework to analyze the non-equilibrium dynamics with new predictions, like scaling laws for the time-delayed correlation function and asymptotic violations of the fluctuation-dissipation theorem, that are not restricted to the infinite dimensional case. Some of these have been confirmed by numerical studies of more realistic models and will be probed experimentally soon.

(iii) It has allowed to extend the analytical connection between mean-field theories of disordered systems and the mode-coupling theory of super-cooled liquids to the low temperature glassy phase. This is most welcome since many of the similarities observed in the behavior of real glasses of very different origin can now be understood theoretically.

(iv) Problems of a more general nature such as the dynamics of manifolds in a random quenched environment, relevant to the description of e.g. the pinning of the flux lattice by disorder in dirty high $T_c$ superconductors can be studied in a similar way. The dependence on the internal dimension of the manifold can be included in the formalism and the analysis yields, among other results, dynamic scaling laws for the displacements with aging effects.

The studies we have mentioned so far are mostly concerned with the analysis of classical systems. This treatment is in general justified since, in many cases, the critical temperature at which the transition to the “glassy phase” occurs is sufficiently high as to make quantum effects irrelevant. Nevertheless, in many cases of practical interest, the critical temperature can be lowered by tuning an external parameter. Experimental realizations of this situation
are given by insulating magnetic materials such as LiHo$_x$Y$_{1-x}$F$_4$ and randomly mixed hydrogen-bonded ferro-antiferroelectric crystals of the Rb$_{1-x}$(NH$_4$)$_x$H$_2$PO$_4$ type (known as RADP).

Theoretical studies addressing the effects of quantum fluctuations in disordered media have been considered over the last twenty years. A variety of techniques including replica theory, renormalization group, and Monte Carlo simulations have been employed in order to understand the static properties and critical behavior of quantum systems such as quantum rotors, transverse field Ising models in finite dimensions, and mean-field systems like the Sherrington-Kirkpatrick model in a transverse field.

From the study of classical glassy systems described in the first paragraph, one knows that at low temperatures their dynamics becomes so slow that the systems are not able to reach thermal equilibrium at any experimentally accessible time. The same occurs regarding numerical simulations of large systems. It is then plausible that even at very low temperatures, say $T = 0$, with only quantum fluctuations driving the dynamics, the materials and/or models will need an unaffordable long time to reach equilibrium and that the relevant evolution will be characterized by non-equilibrium effects. Indeed, it is well-known that certain glasses in the limit of very low temperatures have aging effects and hence the approaches based on the assumption of equilibration have a restricted domain of application. Instead, one should necessarily start from a quantum dynamic description in order to obtain sensible information about the systems. To the best of our knowledge this problem has not been theoretically tackled, from a microscopic point of view, yet.

The question then arises as to how do quantum fluctuations modify the real time dynamics of glassy systems. In this paper, we treat the case of a disordered model in contact with a thermal quantum bath.

Our first aim is to present a general framework that allows us to study the real time dynamics of a system with quantum and thermal fluctuations starting from an arbitrary initial condition. In the case of classical models this is most easily done within the Langevin approach. One generally starts by postulating that the dynamics of the microscopic variables is given by a stochastic Langevin equation. The noise and friction terms account for the bath-system interaction. A path-integral formulation of the generating functional, known as the Martin-Siggia-Rose formalism (MSR), yields an elegant and simple formulation which is particularly well suited for problems with disorder. However, the generalization of this procedure to the quantum case is far from being straightforward. The Langevin equation does not seem to be a good starting point since, being phenomenological, it cannot be quantized in an obvious way. In other words, it is not clear how to make the Langevin approach compatible with the intrinsic microscopic dynamics of a quantum system.

The approach we use here results from the combination of different techniques which have been intensively discussed in the literature and successfully applied to a wide range of phenomena. Essentially, it consists of the Schwinger and Keldysh, closed-time path-integral formalism applied to a system in contact with a thermal bath, combined with the integration of the bath variables leading to the Feynman and Vernon influence functional. A subsequent integration over the disorder gives the quantum analogue of the MSR generating functional for disordered systems.

Though the formalism allows us to consider general cooling procedures or/and various
temperature variations, we shall not study involved thermal histories here but concentrate instead on a quench from a configuration associated to the disordered phase.

For the sake of definiteness we focus on a system whose classical counterpart, the so-called “p-spin spherical spin-glass model”, has been studied in great detail. This model is described by a potential energy

\[ V[\sigma, J] = - \sum_{i_1 < \ldots < i_p}^N J_{i_1 \ldots i_p} \sigma_{i_1} \ldots \sigma_{i_p} \]  

with \( J_{i_1, \ldots, i_p} \) quenched independent Gaussian variables and \( p \geq 2 \) a parameter.

Let us attempt a short review of the behavior of the classical \( p \)-spin models.

In its Ising version it was first introduced by Derrida in relation with the random energy model. Several studies of its static properties, in particular using replicas, followed. Kirkpatrick and Thirumalai studied the equilibrium dynamics, à la Sompolinsky, and remarked that the dynamic equations, in the high temperature phase, are those appearing as \( F_{p-1} \) models in the mode-coupling theory of super-cooled liquids developed to describe the glass transition.

The spherical version, where a time-dependent Lagrange multiplier is introduced to enforce the spherical constraint \( \sum_{i=1}^N \sigma_i^2 = N \), was first studied by Jones, Kosterlitz and Thouless for \( p = 2 \). Its statics is fully solvable either by using replicas – a replica symmetric ansatz is exact – or through a direct calculation. Its dynamics is also fully solvable and it renders explicit the fact that the model is formally related to the \( O(N) \) model in the large \( N \) limit and dimension \( d = 3 \). In the classical limit, it is then a toy model for ferromagnetic domain-growth. In the quantum case one feels tempted to relate it to coarsening in quantum systems.

The analysis of the spherical model for \( p \geq 3 \) is due to Crisanti and Sommers, who also studied the equilibrium dynamics à la Sompolinsky in collaboration with Horner. In the classical limit and at the static level, this model is the prototype spin-glass solved by a Parisi one-step replica symmetric ansatz. Still at the classical level and in the large \( N \) limit, one derives exact evolution equations that exhibit a first order – discontinuous – dynamic transition at \( T_d \). Above \( T_d \), as in the Ising case, one recovers the mode-coupling equation for model \( F_{p-1} \). Below \( T_d \) it is well-known that the dynamic approach used in Ref. is incorrect. The dynamics in the low-temperature phase was solved in Ref. showing that the dynamics of mean-field spin-glasses strikingly resembles that of real systems. In the spin-glass phase the evolution is intrinsically out of equilibrium and the systems age in similar way to the one observed in experiments.

In order to define the model at the quantum level, we need to specify the dynamics of the variables \( \sigma \). For simplicity, we consider a kinetic term of the form

\[ K = \frac{m}{2} \sum_{i=1}^N \dot{\sigma}_i^2 \]  

where the dot denotes a time-derivative. Calling \( z \) the Lagrange multiplier enforcing the spherical constraint, the model we here consider is then described by the action
Our main goals are to detect a dynamic transition from a disordered phase to an ordered glassy phase and to characterize the dynamics in these phases. With this purpose, we pay particular attention to the departure of correlation and response functions from stationarity and, we search for one of the hallmarks of a glassy phase, namely, violations of the fluctuation-dissipation theorem. We also intend to discuss our results in view of previous and possible experimental tests.

The paper is organized as follows. In Section II we discuss the closed-time path-integral formalism of Keldysh and Schwinger and the coupling of the quantum system to a quantum heat bath. We obtain a generating functional with an effective action that takes into account the effect of the coupling to the bath. In three Appendices we give details of the Keldysh Schwinger formalism (Appendix A), we recall the quantum FDT (Appendix B) and we discuss the classical limit (Appendix C). Section III is devoted to the derivation of the dynamic equations through a saddle-point approximation of the generating functional. The method is similar to the one used in the classical case when manipulating the Martin-Siggia-Rose generating functional. The general scenario for the real-time dynamics of this system is discussed in Section IV. It is inspired by the behavior of the classical counterpart but some generalizations are necessary. In Section V the equations and their solution in the paramagnetic phase are studied with special emphasis set on the analysis of the out of phase susceptibility close to the transition line. In Section VI the dynamic equations in the glassy phase are studied. The non-equilibrium dynamics is shown to survive quantum fluctuations in a finite region of the phase diagram. Further details of the approach to solve the equations are given in Appendix D. The modification of FDT needed to construct an ansatz to solve these equations and its relation to effective temperatures are discussed in Section VII. Finally, in Section VIII we present a summary of our results, our conclusions and some ideas as to how to continue investigating along the lines here proposed. A short account of some of these results has appeared in Ref. [52].

II. CLOSED-TIME PATH-INTEGRAL FORMALISM

The closed-time path-integral (CTP) formalism of Keldysh and Schwinger\textsuperscript{26} for treating the non-equilibrium dynamics of a quantum system has been reviewed several times in the literature\textsuperscript{53–58}. In order to set the notation we briefly present it in Section II A. We then consider the situation in which the system under study is coupled to a quantum thermal bath and derive the effective action following Feynman and Vernon\textsuperscript{40} (Section II B). In Section II C we finally address the case of a disordered system and we obtain the effective action from which the dynamic equations are derived in Section III.
A. General framework

The basic ingredient of this formulation is the “in-in” generating functional from which, after derivation with respect to the external sources $\xi^+_i, \xi^-_j$, the different Green functions can be obtained. Let us consider a model with degrees of freedom described by a field $\phi(t) = (\phi_1(t), \ldots, \phi_N(t))$ in the Heisenberg picture. Denoting $\hat{\rho}(t_o)$ the density matrix at the initial time $t_o$ (which from now on we set to zero), the generating functional is defined as

$$Z[\xi^+, \xi^-] = \text{Tr} \left[ T^* \exp \left( -\frac{i}{\hbar} \int_0^\infty dt \xi^- (t) \phi(t) \right) T \exp \left( \frac{i}{\hbar} \int_0^\infty dt \xi^+ (t) \phi(t) \right) \hat{\rho}(0) \right]$$

(2.1)

where $\xi^+(t)$ and $\xi^-(t)$ are the $N$-vector external sources and a scalar product between $N$ vectors is assumed. The symbols $T$ and $T^*$ are the time and anti-time ordering operators:

$$T(A(t_1)B(t_2)) = \theta(t_1 - t_2) A(t_1) B(t_2) + \theta(t_2 - t_1) B(t_2) A(t_1) , \quad (2.2)$$

$$T^*(A(t_1)B(t_2)) = \theta(t_2 - t_1) A(t_1) B(t_2) + \theta(t_1 - t_2) B(t_2) A(t_1) . \quad (2.3)$$

Defining ensemble averages in the usual way

$$\langle A(t) \rangle = \frac{\text{Tr}(A(t) \hat{\rho}(0))}{\text{Tr} \hat{\rho}(0)} , \quad (2.4)$$

one obtains

$$\frac{\delta^2 \ln Z}{\delta \xi^+_i(t) \delta \xi^+_j(t')} \bigg|_{\xi=0} = -\frac{1}{\hbar^2} \langle T(\phi_i(t) \phi_j(t')) \rangle \quad (2.5)$$

which is directly related to the Feynman causal propagator. In the same way,

$$C_{ij}(t, t') \equiv \frac{1}{2} \langle \phi_i(t) \phi_j(t') + \phi_j(t') \phi_i(t) \rangle \quad (2.6)$$

$$= \frac{\hbar^2}{2} \left[ \frac{\delta^2}{\delta \xi^+_i(t) \delta \xi^-_j(t')} + \frac{\delta^2}{\delta \xi^+_j(t') \delta \xi^-_i(t)} \right] \ln Z \bigg|_{\xi=0} \quad (2.7)$$

where $C_{ij}(t, t')$ is the symmetric correlation function ($C_{ij}(t, t') = C_{ji}(t', t)$).

The response function $R_{ij}(t, t')$ is defined as the variation of the averaged field $\langle \phi_i \rangle$ with respect to the “strength” $f_j$ of a perturbation that modifies the potential energy according to $V \rightarrow V - f \phi$:

$$R_{ij}(t, t') \equiv \frac{\delta \langle \phi_i(t) \rangle}{\delta f_j(t')} \bigg|_{f=0} . \quad (2.8)$$

In linear-response theory, it can be expressed in terms of the averaged commutator:

$$R_{ij}(t, t') = \frac{i}{\hbar} \theta(t - t') \langle [\phi_i(t), \phi_j(t')] \rangle \quad (2.9)$$

$$= \frac{\hbar}{i} \left[ \frac{\delta^2}{\delta \xi^+_i(t) \delta \xi^+_j(t')} + \frac{\delta^2}{\delta \xi^+_j(t') \delta \xi^+_i(t)} \right] \ln Z \bigg|_{\xi=0} . \quad (2.10)$$
One of the main advantages of this formalism is that the generating functional \( \langle \xi^+, \xi^- \rangle \) admits a path-integral representation which is a slight modification of the usual “in-out” Feynman path integral. This will allow us to make formal manipulations similar to the ones performed in the study of the dynamics of classical systems via the Martin-Siggia-Rose (MSR) formalism. In a short-hand notation, the generating functional reads

\[
\mathcal{Z}[\xi^+, \xi^-] = \int \mathcal{D}\phi^+ \mathcal{D}\phi^- \exp \left[ \frac{i}{\hbar} \left( S[\phi^+] - S[\phi^-] + \int dt \, \xi^+(t) \phi^+(t) - \int dt \, \xi^-(t) \phi^-(t) \right) \right] \times \langle \phi^+ | \hat{\rho}(0) | \phi^- \rangle \tag{2.11}
\]

where \( S[\phi] \) is the action of the system and \( \langle \phi^+ | \hat{\rho}(0) | \phi^- \rangle \) stands for the matrix element of the density matrix which has support only at \( t = 0 \) (see Appendix A for a proof of this equation). Hereafter, we omit the limits of the time-integrals that, unless otherwise stated, go from \( t_o = 0 \) to \( \infty \).

In the following it will be useful to write the correlation and response in terms of the fields \( \phi^+, \phi^- \):

\[
C_{ij}(t, t') = \frac{1}{2} \langle \phi_i^+(t) \phi_j^-(t') + \phi_j^+(t') \phi_i^-(t) \rangle , \tag{2.12}
\]

\[
R_{ij}(t, t') = \frac{i}{\hbar} \langle \phi_i^+(t) \left( \phi_j^+(t') - \phi_j^-(t') \right) \rangle . \tag{2.13}
\]

Notice that as a result of having two external sources (one related to the evolution with time ordering and the other with anti-time ordering) a doubling of degree of freedom is necessary when writing the path-integral. As we explicitly show in Appendix C, the doubling of degrees of freedom characteristic of this description is intimately linked to the introduction of Lagrange multipliers in the MSR description of the classical model.

The time-integration in Eq. (2.11) can be interpreted as being closed, going forward from \( t_0 = 0 \) to \( t = \infty \) and then backwards from \( t = \infty \) to \( t_0 = 0 \). This motivates the name of the method.

**B. Coupling to a heat bath**

Up to now, the discussion has been completely general and it applies to systems with an arbitrary number of degrees of freedom. We now consider the coupling of the system of interest to a thermal quantum bath, assumed to be in equilibrium. A convenient way to deal with the interaction between the system and the bath in the path-integral formalism is due to Feynman and Vernon. In this approach one starts by describing the system plus bath by \( \phi = (\sigma, v^a) \) with \( \sigma = (\sigma_1, \ldots, \sigma_N) \) denoting the variables of the system and \( v^a = (v_a^1, \ldots, v_{N_b}^a) \), \( a = 1, \ldots, N_b \), denoting the variables of the bath. The total action is

\[
S[\phi] = S_s[\sigma] + S_b[v^a] + S_{sb}[\sigma, v^a] \tag{2.14}
\]

where \( S_s \) is the action characterizing the system (and eventually depending upon disorder), \( S_b \) is the action for the bath, and \( S_{sb} \) contains the system-bath interaction terms. For definiteness we assume that our system interacts linearly with the bath which we model as a set of independent harmonic oscillators,
\[ S[\phi] = S_s[\sigma] + \frac{1}{2} \sum_{a=1}^{N_b} M_a \left( \dot{v}_a^2 - \omega_a^2 v_a^2 \right) - \sum_{a=1}^{N_b} C_a v_a \sigma. \] (2.15)

\( M_a \) are their masses, \( \omega_a \) their frequencies and \( C_a \) the coupling constants between the oscillators and the system. Of course, more general couplings between bath and system are possible but this is the simplest choice that allows us to go a bit farther analytically. If we further assume that the initial density matrix \( \hat{\rho}(0) \) is factorizable as

\[ \hat{\rho}(0) = \hat{\rho}_s(0) \times \hat{\rho}_b(0), \] (2.16)

where \( \hat{\rho}_b(0) \) is the Boltzman distribution for the bath variables at equilibrium at a temperature \( k_B T = 1/\beta \), one can integrate out the bath variables, obtaining in this way an effective thermal action \( S_{T}[\sigma^+, \sigma^-] \), that enters the Feynman-Vernon influence functional:

\[ S_{T}[\sigma^+, \sigma^-] = -\int dt \int dt' \left( \sigma^+(t) - \sigma^-(t) \right) \eta(t-t') \left( \sigma^+(t') + \sigma^-(t') \right) + i \int dt \int dt' \left( \sigma^+(t) - \sigma^-(t) \right) \nu(t-t') \left( \sigma^+(t') - \sigma^-(t') \right). \] (2.17)

The noise and dissipative kernels \( \nu \) and \( \eta \) are given by

\[ \nu(t-t') = \int_0^\infty d\omega I(\omega) \coth \left( \frac{1}{2} \beta \hbar \omega \right) \cos(\omega(t-t')), \] (2.18)

\[ \eta(t-t') = -\theta(t-t') \int_0^\infty d\omega I(\omega) \sin(\omega(t-t')). \] (2.19)

In these equations, \( I(\omega) \) is the spectral density of the bath:

\[ I(\omega) \equiv \sum_{a=1}^{N_b} \delta(\omega - \omega_a) \frac{C_a^2}{2M_a \omega_a}. \] (2.20)

The system-bath interaction manifests via the appearance of two quadratic and non-local terms (2.17) in the effective action. As it can be most easily seen in the classical limit presented in Appendix \( \mathcal{C} \) they can be related to the noise and dissipative terms of the associated Langevin formulation of the classical dynamics.

In addition, it is easy to see that the kernels \( \nu \) and \( \eta \) are related to the correlation and response functions of the bath. In fact \( 4\eta(t-t') \) is the response of the bath while \( -2\hbar \nu(t-t') \) is the correlation of the bath. They are related in a way dictated by the fluctuation-dissipation theorem (FDT):

\[ 4\eta(\omega) = \frac{1}{\hbar} \lim_{\epsilon \to 0^+} \int \frac{d\omega'}{\pi} \frac{1}{\omega - \omega' + i\epsilon} \tanh \left( \frac{\beta \hbar \omega'}{2} \right) 2\hbar \nu(\omega'). \] (2.21)

(See Appendix \( \mathcal{B} \) for the derivation and properties of the quantum FDT.) This equation is the quantum counterpart of the fluctuation-dissipation relation of classical relaxational dynamics, that fixes the dissipative coefficient in terms of the strength of the noise correlation of the Langevin equation. It expresses the fact that the bath is assumed to be in equilibrium at the initial time \( t_0 = 0 \) and that it stays in equilibrium at all subsequent times \( t > 0 \).
It is important to notice that the assumption of thermal equilibrium of the bath does not imply that the system – that is in contact with it – is in equilibrium at the initial time nor that it will reach equilibrium at a given equilibration time $t_{\text{eq}}$. It is known that “mean-field” classical disordered models with relaxational dynamics – the $p$ spin-glass model is an example — do not reach equilibrium at any time if the limit of $N$ going to infinity is taken at the outset of the calculation. It is one of the main aims of this paper to show that this result carries through to the quantum case. As a consequence of the out of equilibrium behavior, we shall find that the FDT does not hold between response and correlation of the system in a sub-critical region of the phase diagram.

The integrals in the $\eta$ and $\nu$ definitions are, in general, ill-defined (divergent) and might need to be regularized. A way to regularize them is to observe that one expects $I(\omega)$ to decrease to zero for large $\omega$ and introduce an explicit cut-off:

$$I(\omega) \sim 0 \quad \text{for} \quad \omega > \Lambda.$$  \hspace{1cm} (2.22)

By modifying $I(\omega)$ in this way, we regularize the kernels $\eta$ and $\nu$ at the same time preserving the FDT relation for the bath variables; in other words, we do not break the assumption of equilibrium of the bath.

Different environments are characterized by different behaviors of $I(\omega)$ such that for $\omega < \Lambda$:

$$I(\omega) \sim \omega^a \quad \text{and} \quad \begin{cases} a = 1 & \text{Ohmic} \\ a < 1 & \text{Subohmic} \\ a > 1 & \text{Superohmic} \end{cases}.$$  \hspace{1cm} (2.23)

A typical example is

$$I(\omega) = \frac{M\gamma_0}{\pi} \omega^{a-1} \exp\left( -\frac{\omega}{\Lambda} \right).$$  \hspace{1cm} (2.24)

where $M\gamma_0$ is a constant that plays the role of a friction coefficient.

Throughout this paper, we concentrate in the Ohmic case. The kernel $\eta$ is

$$\eta(t-t') = \theta(t-t') \frac{M\gamma_0}{\pi} \frac{d}{d(t-t')} \left( \frac{\Lambda}{1 + (\Lambda(t-t'))^2} \right).$$  \hspace{1cm} (2.25)

In the limit where the cut-off tends to infinity it becomes

$$\lim_{\Lambda \to \infty} \eta(t-t') = \theta(t-t') \ M\gamma_0 \delta'(t-t').$$  \hspace{1cm} (2.26)

Of particular importance is the zero temperature limit of the kernel $\nu$:

$$\lim_{T \to 0} \nu(t-t') = \frac{M\gamma_0}{\pi} \frac{\Lambda^2}{1 + (\Lambda(t-t'))^2}.$$  \hspace{1cm} (2.27)

One sees that the quantum fluctuations yield a non-trivial kernel even in the absence of thermal fluctuations. In the zero temperature limit the characteristic time goes to infinity and the long-time decay of $\nu$ becomes power-law:
\[ \lim_{t - t' \to \infty} \lim_{T \to 0} \nu(t - t') = -\frac{M \gamma_o}{\pi} (t - t')^{-2}. \quad (2.27) \]

In the classical limit, for any temperature, \( \nu \) becomes

\[ \lim_{\hbar \to 0} \lim_{\Lambda \to \infty} 2\hbar \nu(t - t') = 4M \gamma_o k_B T \frac{\Lambda}{1 + \Lambda^2 (t - t')^2}. \quad (2.28) \]

Note that for a finite cut-off \( \Lambda \), the classical limit of the kernel \( \nu \) also has a rather slow, power-law decay at large time differences. If, next, the cut-off is sent to infinity one recovers a delta function

\[ \lim_{\Lambda \to \infty} \lim_{\hbar \to 0} 2\hbar \nu(t - t') = 4M \gamma_o k_B T \delta(t - t') \quad (2.29) \]

characteristic of e.g. a white thermal noise. The classical limit of these terms is further discussed in Appendix C where we show that the dynamic equations of the corresponding classical model correspond to a friction coefficient \( \Gamma_o^{-1} \equiv 2M \gamma_o \).

The coupling to the quantum thermal bath introduces three time-scales into the problem. First, the inverse cut-off \( 1/\Lambda \) characterizes the memory of the bath. Second, the coupling strength \( M \gamma_o \) yields the inverse relaxational characteristic time. Third, \( (\beta \hbar)^{-1} \) accounts for the relative importance of quantum to thermal effects.

### C. CTP for disordered systems

The CTP formalism results to be well suited for problems with disorder. Let us suppose that the system under consideration is described by the action,

\[ S_s[\sigma, J] = S_0[\sigma] - \int dt V[\sigma, J] \quad (2.30) \]

where \( S_0 \) is the disordered independent part of the action and \( V[\sigma, J] \) is the potential energy that depends upon random couplings collectively denoted by \( J \). As usual in disordered systems, we concentrate on quantities that are averaged over the disorder distribution \( P[J] \).

We wish to consider the real-time dynamics of the system starting from a random initial condition at time \( t_o = 0 \) when it is set in contact with the environment (that is itself in equilibrium at a constant temperature \( T \)). Given that the initial condition is chosen to be random, it is not correlated with the disorder. Therefore, the density operator \( \hat{\rho}(0) \) does not depend upon disorder and the generating functional without sources

\[ \mathcal{Z} = \text{Tr} [\hat{\rho}(0)] \quad (2.31) \]

is also independent of disorder.

This property is equivalent to the independence of the classical Martin-Siggia-Rose (MSR) generating functional without sources upon disorder. As in the classical case, it allows us to write dynamic equations for random initial conditions without having to compute the average over disorder of \( \ln \mathcal{Z}[\xi^+, \xi^-, J] \) and hence without resorting to the use of replicas. We are then interested in the averaged generating functional:
\[ Z[\xi^+, \xi^-] = \int dJ \mathcal{P}[J] Z[\xi^+, \xi-, J], \]  

(2.32)

from which any averaged operator can be computed as

\[ \langle \sigma(t) \rangle = \frac{\delta \ln Z[\xi^+, \xi-, J]}{\delta \xi^+(t)} \bigg|_{\xi=0} = \frac{1}{Z[0, 0, J]} \frac{\delta Z[\xi^+, \xi-, J]}{\delta \xi^+(t)} \bigg|_{\xi=0}, \]  

(2.33)

with all sources set to zero and \( \sigma^+ = \sigma^- = \sigma \). Here and in what follows, the overline represents an average over the disorder.

In many cases of interest, the integration over the disorder can be performed explicitly,

\[ \int dJ \mathcal{P}[J] \exp \left[ -\frac{i}{\hbar} \int dt \left( V[\sigma^+, J] - V[\sigma^-, J] \right) \right] = \exp \left[ -\frac{i}{\hbar} V_0[\sigma^+, \sigma^-] \right] \]  

(2.34)

and it introduces a non-linear interaction term that is, in general, non-local both in time and in the “space” indices \( i, j \).

A rather general way of introducing randomness is to consider a model with a Gaussian random potential energy term \( V[\sigma, J] \) depending upon the variables \( \sigma = (\sigma_1, \ldots, \sigma_N) \) and correlated as

\[ V[\sigma, J]V[\sigma', J] = -\frac{N}{2} \mathcal{V} \left( \frac{(\sigma - \sigma')^2}{N} \right). \]  

(2.35)

The variables \( \sigma \) may represent spins leading to a spin-glass model or a position in an \( N \)-dimensional space becoming the problem of the dynamics and/or diffusion of a particle in a random potential. Since we are interested in following the real time dynamics of such systems, \( \sigma \) will be a time-dependent \( N \)-vector.

For the sake of definiteness, we shall solve a particular realization of the random potential that in the classical case defines the so-called \( p \) spin-glass:

\[ V[\sigma, J] = -\sum_{i_1 < \ldots < i_p} J_{i_1 \ldots i_p} \sigma_{i_1} \ldots \sigma_{i_p}. \]  

(2.36)

\( p \) is a parameter, \( p \geq 2 \). The interaction strengths are quenched independent random variables with a Gaussian distribution

\[ P[J] = \left( \frac{N^{p-1}}{\sqrt{2\pi p!}} \right) \exp \left( -\frac{N^{p-1}}{2J^2p!} \sum_{i_1 \neq \ldots \neq i_p} (J_{i_1 \ldots i_p})^2 \right) \Rightarrow (J_{i_1 \ldots i_p})^2 = \frac{\tilde{J}^2 p!}{2N^{p-1}}. \]  

(2.37)

This corresponds to a random potential correlation \( \mathcal{V}(x) = -1/2(1 - x/2)^p \) and it leads to an effective, non-local in time, interaction

\[ V_0[\sigma^+, \sigma^-] = \frac{\tilde{J}^2 N i}{4\hbar} \int dt \int dt' \left[ \left( \frac{1}{N} \sigma^+(t)\sigma^+(t') \right)^p + \left( \frac{1}{N} \sigma^-(t)\sigma^-(t') \right)^p \right. \]

\[ - \left. \left( \frac{1}{N} \sigma^+(t)\sigma^-(t') \right)^p - \left( \frac{1}{N} \sigma^-(t)\sigma^+(t') \right)^p \right]. \]  

(2.38)
In its spherical version, \(\sigma_i, i = 1, \ldots, N\) are continuous dynamic variables \(-\sqrt{N} < \sigma_i < \sqrt{N}, \forall i\), that satisfy the global spherical constraint

\[
\frac{1}{N} \sigma^2(t) = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^2(t) = 1,
\]

at each instant. We enforce this constraint by introducing a time-dependent Lagrange multiplier \(z(t)\).

The disordered-independent part of the action is

\[
S_0[\sigma, J] = \int dt \left[ \frac{m}{2} \dot{\sigma}^2 - \frac{z}{2} (\sigma^2 - N) \right].
\]

As explained in the Introduction, for simplicity we have chosen a kinetic term that dictates a microscopic dynamics with second derivatives in time. Other choices, for example leading to first time-derivatives, are also possible. The derivation of the dynamic equations that we explain in Section III applies with only minor modifications.

### III. MEAN-FIELD EQUATIONS

We now have all the elements to derive the equations which describe the dynamic evolution of a disordered quantum system. According to the discussion in Section II, the system is described by the generating functional

\[
\mathcal{Z}[\xi^+, \xi^-, J] = \int \mathcal{D}\sigma^+ \mathcal{D}\sigma^- \exp \left[ \frac{i}{\hbar} \left( S_{\text{eff}}[\sigma^+, \sigma^-] + \int dt \xi^+(t)\sigma^+(t) - \int dt \xi^-(t)\sigma^-(t) \right) \right]
\]

with

\[
S_{\text{eff}}[\sigma^+, \sigma^-] = S_0[\sigma^+] - S_0[\sigma^-] + S_I[\sigma^+, \sigma^-] - V_d[\sigma^+, \sigma^-].
\]

We remind that \(S_0\) is the disorder independent part of the action given by Eq. (2.40), \(S_I\) accounts for the the system-bath interaction, Eq. (2.17), and \(V_d\) is the effective potential which arises as a result of the integration over the disorder and is given by Eq. (2.38).

The quantum dynamic equations of motion follow from similar steps to those usually used to obtain the classical equations of motion in the Martin-Siggia-Rose formalism. We introduce macroscopic order parameters and derive, through a saddle-point point approximation of the KS generating functional (that becomes exact when \(N \to \infty\)), the dynamic equations of motion. Since we are interested in considering the dynamics of the system for large but finite times with respect to \(N\), out of equilibrium effects are expected. We shall therefore make no assumption about the time-dependence of the order-parameters.
A. Dynamic order parameters

The quadratic terms in the action can be condensed into one term by introducing the operator

\[ \mathcal{O}_p(t, t') = \begin{pmatrix} \mathcal{O}_p^{++}(t, t') & \mathcal{O}_p^{+-}(t, t') \\ \mathcal{O}_p^{-+}(t, t') & \mathcal{O}_p^{--}(t, t') \end{pmatrix} = \{ \mathcal{O}_p^{\alpha\beta}(t, t') \}, \]

\[ \mathcal{O}_p^{++}(t, t') = (m\partial_t^2 + z^+(t)) \delta(t - t') - 2i\nu(t - t') \]
\[ \mathcal{O}_p^{+-}(t, t') = 2\eta(t - t') + 2i\nu(t - t') \]
\[ \mathcal{O}_p^{-+}(t, t') = -2\eta(t - t') + 2i\nu(t - t') \]
\[ \mathcal{O}_p^{--}(t, t') = -(m\partial_t^2 + z^-(t)) \delta(t - t') - 2i\nu(t - t') \]  

in such a way that

\[ S_{\text{eff}}[\sigma^+, \sigma^-] = -\frac{1}{2} \int dt \int dt' \sigma^{\alpha}(t) \mathcal{O}_p^{\alpha\beta}(t, t') \sigma^{\beta}(t') - V_b[\sigma^+, \sigma^-] \]  

where Greek indices label \( \alpha = +, - \) and the sum convention is assumed. Introducing the identity

\[ 1 = \int \prod_{\alpha^\beta} DQ^{\alpha\beta} \delta \left( \frac{1}{N} \sigma^\alpha(t) \sigma^\beta(t') - Q^{\alpha\beta}(t, t') \right) \]
\[ \propto \int \prod_{\alpha^\beta} DQ^{\alpha\beta} D\lambda^{\alpha\beta} \exp \left( -\frac{i}{2\hbar} \lambda^{\alpha\beta} \left( \sigma^\alpha(t) \sigma^\beta(t') - NQ^{\alpha\beta}(t, t') \right) \right) \]  

into the generating functional, the full action can be rewritten as

\[ S_{\text{eff}}[\sigma^+, \sigma^-] = -\frac{1}{2} \int dt \int dt' \sigma^{\alpha}(t) \left( \mathcal{O}_p^{\alpha\beta}(t, t') + \lambda^{\alpha\beta}(t, t') \right) \sigma^{\beta}(t') + \frac{N}{2} \int dt \int dt' \sigma^{\alpha}(t) \lambda^{\alpha\beta}(t, t') Q^{\alpha\beta}(t, t') + \frac{N}{2} \int dt \left( z^+(t) - z^-(t) \right) + \frac{i\hbar^2 N}{4\hbar} \int dt \int dt' \left[ (Q^{++}(t, t'))^p + (Q^{+-}(t, t'))^p - (Q^{-+}(t, t'))^p - (Q^{--}(t, t'))^p \right]. \]  

The stationary-point values of the order parameters \( Q^{\alpha\beta}(t, t') \) with \( \alpha, \beta = +, - \) are related to the “physical” correlations and responses defined in Eqs. (2.12) and (2.13) as follows

\[ NQ^{++}(t, t') = \langle \sigma^+(t) \sigma^+(t') \rangle = N \left( C(t, t') - \frac{i\hbar}{2} (R(t, t') + R(t', t)) \right), \]  
\[ NQ^{+-}(t, t') = \langle \sigma^+(t) \sigma^-(t') \rangle = N \left( C(t, t') + \frac{i\hbar}{2} (R(t, t') - R(t', t)) \right), \]  
\[ NQ^{-+}(t, t') = \langle \sigma^-(t) \sigma^+(t') \rangle = N \left( C(t, t') - \frac{i\hbar}{2} (R(t, t') - R(t', t)) \right), \]  
\[ NQ^{--}(t, t') = \langle \sigma^-(t) \sigma^-(t') \rangle = N \left( C(t, t') + \frac{i\hbar}{2} (R(t, t') + R(t', t)) \right), \]
with

\[ NC(t, t') \equiv \frac{1}{2} \sum_{i=1}^{N} \langle \sigma_i^+(t)\sigma_i^-(t') + \sigma_i^-(t)\sigma_i^+(t') \rangle, \]

\[ NR(t, t') \equiv \frac{i}{\hbar} \sum_{i=1}^{N} \langle \sigma_i^+(t)\left(\sigma_i^-(t') - \sigma_i^-(t)\right) \rangle. \tag{3.11} \]

It is easy to check that these functions satisfy the identity

\[ Q^{++}(t, t') + Q^{--}(t, t') - Q^{+-}(t, t') - Q^{-+}(t, t') = 0. \tag{3.12} \]

The functional integration over \( \sigma_i^+(t) \) and \( \sigma_i^-(t) \) is now quadratic and can be performed. This amounts to replace the quadratic term in \( i/\hbar S_{\text{eff}} \) by

\[ -\frac{N}{2} \int dt \int dt' \text{Tr} \log \left( \frac{i}{\hbar} \text{Op}^{\alpha\beta}(t, t') + \frac{i}{\hbar} \lambda^{\alpha\beta}(t, t') \right). \tag{3.13} \]

\section*{B. Saddle-point evaluation}

At this stage, all terms in the action depend upon the “macroscopic” quantities \( \lambda^{\alpha\beta}, Q^{\alpha\beta} \) and \( z^\alpha \) and are proportional to \( N \). Since it is easier to write the equations in matrix notation, we encode \( \lambda^{\alpha\beta} \) and \( Q^{\alpha\beta} \) in two matrices

\[ \mathcal{L} = \begin{pmatrix} \lambda^{++} & \lambda^{+-} \\ \lambda^{--} & \lambda^{-+} \end{pmatrix}, \quad Q = \begin{pmatrix} Q^{++} & Q^{+-} \\ Q^{-+} & Q^{--} \end{pmatrix} \]

and we define

\[ F[Q](t, t') \equiv \begin{pmatrix} (Q^{++}(t, t'))^{p-1} & -(Q^{+-}(t, t'))^{p-1} \\ -(Q^{-+}(t, t'))^{p-1} & (Q^{--}(t, t'))^{p-1} \end{pmatrix}. \tag{3.14} \]

We denote with a cross the standard operational product

\[ \mathcal{A} \otimes \mathcal{B}(t, t') = \begin{pmatrix} \int dt'' A^{\gamma}(t, t'')B^{\gamma+}(t'', t') & \int dt'' A^{\gamma}(t, t'')B^{\gamma-}(t'', t') \\ \int dt'' A^{-\gamma}(t, t'')B^{\gamma+}(t'', t') & \int dt'' A^{-\gamma}(t, t'')B^{\gamma-}(t'', t') \end{pmatrix} \]

where a sum over \( \gamma \) is assumed. The saddle-point with respect to \( \lambda^{\alpha\beta}(t, t') \) yields

\[ \mathcal{L}(t, t') = \frac{\hbar}{i} Q^{-1}(t, t') - \mathcal{O} p(t, t'). \tag{3.15} \]

\footnote{At the classical level and in the Martin-Siggia-Rose language, this identity reduces to the condition \( \langle i\hat{s}(t)i\hat{s}(t') \rangle = 0 \) for all pairs of times \( t, t' \). In the classical case this is a requirement to be imposed upon the saddle-point two-point functions in order to preserve causality. In the quantum case, this condition emerges as a property of the Green functions.}
The matrix and time-operator inverse of $Q$ is denoted $Q^{-1}$. The saddle-point equation with respect to $Q^{\alpha\beta}(t, t')$ yields

$$L(t, t') = -\frac{ip\tilde{J}^2}{2\hbar} F[Q](t, t') \quad (3.16)$$

Equations (3.15) and (3.16) imply, in a compact matrix and time-operator notation,

$$i\frac{\hbar}{\hbar} O p \otimes Q = I - \frac{p\tilde{J}^2}{2\hbar^2} F[Q] \otimes Q \quad (3.17)$$

where $I$ is the identity: $I^{\alpha\beta}(t, t) = \delta^{\alpha\beta} \delta(t-t')$. The saddle-point with respect to $z^\alpha$ yields

$$i\frac{\hbar}{\hbar} = (O p + L)_{++}^{-1}(t, t) = i\frac{\hbar}{h} Q^{++}(t, t) \quad (3.18)$$

$$i\frac{\hbar}{\hbar} = (O p + L)_{--}^{-1}(t, t) = i\frac{\hbar}{h} Q^{--}(t, t) \quad (3.19)$$

and these equations lead, as expected, to the spherical constraint.

In the limit $N \to \infty$ one could also proceed as in Refs. [42, 62] and write the full action $S_{\text{eff}}$ in terms of a single variable. This is at the expense of modifying the thermal kernel and the interaction term in a self-consistent way, through the introduction of terms arising from the non-linear interactions (the vertex $\tilde{D}$ and the self-energy $\tilde{\Sigma}$, respectively).

This procedure is not of particular usefulness for the analysis of the model we treat in this paper since the single variable effective action is Gaussian. It does however become useful for dealing with quantum models whose single-spin effective action has higher order interaction terms. An example is the quantum Sherrington-Kirkpatrick model. One could then envisage to derive an effective quantum Langevin equation for the single variable and study this equation with an adequate numerical algorithm as the one developed by Eissfeller and Opper [64].

C. Dynamical equations for correlation and response

The dynamic equations for the auto-correlation and response follow from the set of equations (3.14)-(3.17) and the definitions of the dynamic order parameters given in Eqs. (3.7)-(3.10). More precisely, the equation of motion for the response function follows from the subtraction of the ++ and +− components of Eq. (3.17):

$$\left( m\partial_t^2 + z^+\right) R(t, t') + 4 \int_{t'}^t dt'' \eta(t-t'') R(t'', t') = \delta(t-t') - \frac{p\tilde{J}^2}{2i\hbar} \int_0^\infty dt'' \left[ (Q^{++}(t, t''))^{p-1} - (Q^{+-}(t, t''))^{p-1} \right] R(t'', t') \quad (3.20)$$

and the equation of motion for the auto-correlation function follows from the addition of the +− and −− components of Eq. (3.17):
\[
\left( m \partial_t^2 + \frac{1}{2} \left( z^+(t) + z^-(t) \right) \right) C(t, t') + \frac{i}{2} \left( z^+(t) - z^-(t) \right) \hbar (R(t', t) - R(t, t')) \\
- 2 \hbar \int_0^\infty \! dt'' \nu(t - t'') R(t', t'') + 4 \int_0^t \! dt'' \eta(t - t'') C(t'', t') \\
= - \frac{p \hbar^2}{2} \int_0^\infty \! dt'' \, \text{Im} \left[ (Q^{++}(t, t''))^{p-1} Q^{+-}(t'', t') - (Q^{+-}(t, t''))^{p-1} Q^{--}(t'', t') \right]. 
\]

Written in this way, Eq. (3.21) is complex. Its imaginary part yields

\[ z(t) \equiv z^+(t) = z^-(t) \, . \]  

Moreover, since the response is causal, products of advanced \( R(t, t'') \) and retarded \( R(t'', t') \) responses vanish identically for all \( t, t'' \):

\[ R(t, t'') R(t'', t) = 0 \quad \forall t, t'' \]  

and one can show that for any integer \( k > 0 \) and any constants \( c_1, c_2 \)

\[ [C(t, t') + c_1 R(t, t') + c_2 R(t', t)]^k = [C(t, t') + c_1 R(t, t')]^k + [C(t, t') + c_2 R(t', t)]^k - C^k(t, t') \, . \]  

Using this property one has

\[ (Q^{++}(t, t''))^{p-1} - (Q^{+-}(t, t''))^{p-1} = 2 \text{Im} \left[ C(t, t'') - \frac{i \hbar}{2} R(t, t'') \right]^{p-1} \]  

and

\[ \text{Im} \left[ (Q^{++}(t, t''))^{p-1} Q^{+-}(t'', t') - (Q^{+-}(t, t''))^{p-1} Q^{--}(t'', t') \right] = \\
2C(t', t') \text{Im} \left[ C(t, t'') - \frac{i \hbar}{2} R(t, t'') \right]^{p-1} - \hbar R(t', t'') \text{Re} \left[ C(t, t'') - \frac{i \hbar}{2} (R(t, t'') + R(t'', t)) \right]^{p-1} . \]  

We can identify the self-energy \( \Sigma \) and the vertex \( D \) as

\[ \Sigma(t, t') + 4 \eta(t - t') \equiv \bar{\Sigma}(t, t') \equiv - \frac{p \hbar^2}{h} \text{Im} \left[ C(t, t') - \frac{i \hbar}{2} R(t, t') \right]^{p-1} \, , \]  

\[ D(t, t') - 2 \hbar \nu(t - t') \equiv \bar{D}(t, t') \equiv \frac{p \hbar^2}{2} \text{Re} \left[ C(t, t') - \frac{i \hbar}{2} (R(t, t') + R(t', t)) \right]^{p-1} \, . \]  

For \( t \geq t' \), they can be encoded in a single complex equation:

\[ \bar{D}(t, t') + \frac{i \hbar}{2} \bar{\Sigma}(t, t') = \frac{p \hbar^2}{2} \left( C(t, t') + \frac{i \hbar}{2} R(t, t') \right)^{p-1} \, . \]  

Note that the total self-energy \( \Sigma \) and vertex \( D \) are real and have two contributions of different origin: one arises from the interaction of the system and the bath (\( \eta \) and \( \nu \)) and
one is caused by the non-linearities stemming from the average over disorder (that we called $\tilde{\Sigma}$ and $\tilde{D}$ in Eqs. (3.27) and (3.28)). If $p = 2$, $\tilde{\Sigma}$ and $\tilde{D}$ are identical to the classical ones. Instead, if $p \geq 3$, the non-linear terms acquire an explicit dependence upon $\bar{h}$.

The dynamic equations can then be written in a compact form

\begin{align}
(m\partial_t^2 + z(t)) R(t,t') &= \delta(t-t') + \int_0^\infty dt'' \tilde{\Sigma}(t,t'') R(t'',t') , \\
(m\partial_t^2 + z(t)) C(t,t') &= \int_0^\infty dt'' \tilde{\Sigma}(t,t'') C(t'',t') + \int_0^{t'} dt'' D(t,t'') R(t',t'') ,
\end{align}

that we call later the $R$-eq. and $C$-eq., respectively. It is important to realize that the self-energy $\tilde{\Sigma}(t,t')$ is proportional to the response function $R(t,t')$, which in turns implies

$$\tilde{\Sigma}(t,t') = \Sigma(t,t') = 0 \quad \text{for} \quad t < t'.$$

This means that the upper limit of integration in Eqs. (3.30) and (3.31) is $t$, which renders the equations explicitly causal. There are no more independent equations for $R$ and $C$. The other two equations that can be obtained from Eq. (3.17) are the equation for $R(t',t)$, that is equivalent to the $R$-eq above, and one equation that identically cancels by virtue of the identity (3.12).

In their integrated form as Schwinger-Dyson equations the dynamic equations read:

\begin{align}
R(t,t') &= G_o(t,t') + \int_t^t dt'' \int_{t'}^{t''} dt''' G_o(t,t'') \Sigma(t'',t''' ) R(t''',t') , \\
C(t,t') &= \int_0^t dt'' \int_0^{t'} dt''' R(t,t'') D(t'',t''') R(t',t''') ,
\end{align}

with the propagator given by

$$G_o^{-1}(t,t') \equiv m\partial_t^2 + z(t) .$$

Real and imaginary parts of Eqs. (3.18) and (3.19) combined with Eq. (3.15) imply the equal-times conditions

$$C(t,t) = 1 ,$$
$$R(t,t) = 0 .$$

In addition, from Eq. (3.20) one obtains that the first derivative of the response function is discontinuous at equal times:

\begin{align}
\lim_{t' \to t^-} \partial_{t} R(t,t') &= \frac{1}{m} , \\
\lim_{t' \to t^+} \partial_{t} R(t,t') &= 0 ,
\end{align}

while from Eq. (3.21) one obtains that the correlation is continuous:

\begin{align}
\lim_{t' \to t^-} \partial_{t} C(t,t') = \lim_{t' \to t^+} \partial_{t} C(t,t') = 0 .
\end{align}
The equation for \( z(t) = z^+(t) = z^-(t) \) can be derived from the Schwinger-Dyson equation (3.33), by imposing the spherical constraint through the evaluation at \( t = t' \). Multiplying operationally by \( G^{-1}_o \) one obtains

\[
z(t) = \int_0^t dt'' \left[ \Sigma(t, t'') C(t, t'') + D(t, t'') R(t, t'') \right] + m \int_0^t dt'' \int_0^t dt''' \left( \partial_t R(t, t''') \right) D(t'', t''') \left( \partial_t R(t, t''') \right) .
\]  

(3.38)

The last term is a consequence of having a kinetic term with second derivatives. Had we chosen a first derivative term in the kinetic energy the last term would have not appeared. It can be easily identified with minus the second-derivative of the correlation at equal times by taking the limit \( t' \to t^- \) in Eq. (3.31). Thus

\[
z(t) = \int_0^t dt'' \left[ \Sigma(t, t'') C(t, t'') + D(t, t'') R(t, t'') \right] - m \left. \frac{\partial^2}{\partial t^2} C(t, t') \right|_{t' \to t^-} .
\]  

(3.39)

In conclusion, Eqs. (3.30), (3.31) and (3.39) are the complete set of equations that determines the dynamics of the system. In the following we make a time reparametrization

\[
\hat{t} = M \gamma_o t
\]  

(3.40)

that implies

\[
\hat{C} = C \quad \hat{R} = \frac{1}{M \gamma_o} R
\]  

(3.41)

and

\[
\hat{m} = (M \gamma_o)^2 m , \quad \hat{h} = M \gamma_o \hat{h} , \quad \hat{J} = \tilde{J} , \quad \hat{\beta} = \beta \quad \hat{\Lambda} = \frac{\Lambda}{M \gamma_o} .
\]  

(3.42)

The system of units we use is

\[
[\hat{C}] = [\hat{R}] = [\hat{h}] = 1 , \\
[\hat{m}] = [\hat{\beta}] = [1/\tilde{J}] = [1/\hat{\Lambda}] = [\hat{t}] .
\]  

(3.43)

Hereafter we write the equation in reparametrized variables and drop all hats.

**IV. SOLUTIONS TO THE MODE COUPLING EQUATIONS**

As already mentioned, the set of integro-differential equations (3.30), (3.31) and (3.39) is causal – as in the classical case – and one can attempt a numerical solution by a simple algorithm. The correlation \( C(t, t') \) and response \( R(t, t') \) are two time quantities, that is, they depend on \( t \) (which physically corresponds to the time of observation) and \( t' \) (which corresponds to the age of the system). In a normal situation, one expects that after an equilibration time \( t_{eq} \) (in general model dependent) all two-time functions will depend only on the time difference \( t - t' \). In other words, all two-time functions become time-translation
invariant (TTI). Under these circumstances, to which we refer as equilibrium evolution, response and correlation are not independent quantities but are linked via the fluctuation-dissipation theorem (FDT).

Nevertheless, it is also known that certain systems never reach (at least for experimentally accessible time-scales) an equilibrium dynamics. For arbitrary large $t'$ and in a certain region of the parameter space, although one-time quantities, such as the energy-density, tend to an asymptotic limit, two-time quantities, such as the correlation and response, depend both on $t$ and $t'$ in a non-trivial way. This is indeed the typical situation for a glassy system. The question we would like to explore is whether Eqs. (3.30), (3.31) and (3.39) encode a non-equilibrium evolution as described above. We know that indeed this is the case for classical disordered systems. As shown in Ref. [52], this kind of equations describe, at least partially, the nonequilibrium dynamics of glassy systems, notably *aging* effects of spin-glasses. The salient feature of these solutions is that, below a critical temperature $T_d \equiv T_c(\bar{h} = 0)$, and for large $t'$, there exist at least two time scales: for $t - t'$ small with respect to a characteristic time $\mathcal{T}(t')$, that depends upon the age of the system, the dynamics is similar to an equilibrium evolution (stationary regime) but for $t - t'$ large with respect to $\mathcal{T}(t')$ the dynamics becomes extremely slow with aging effects. In this regime, correlation and response are related in a manner that violates FDT.

In this paper we show that this also happens in the quantum problem in a region of parameter space. In what follows we concentrate on the case $p \geq 3$. As in the classical case, the $p = 2$ model is connected to $\mathcal{O}(N)$ models and has a different behavior. In Fig. 1 we present a schematic phase diagram $(T, h)$ for $p = 3$. As we show below, the dynamics in the disordered ("paramagnetic") and ordered ("spin-glass") phases are characterized by "equilibrium" and "out-of-equilibrium" effects, respectively. The dynamics for the quantum system in the glassy phase has a similar pattern to the classical one, in the sense that there are at least two time-sectors in which the evolution is of a different nature. Inspired by the classical problem, we propose that also in the quantum case the weak-ergodicity breaking and the weak long-term memory properties hold. These two assumptions are based in part on the insight coming from the numerical solution of the full equations.

A. The correlation function and the weak-ergodicity breaking scenario

The *weak-ergodicity breaking scenario* states that, for $t \geq t'$, the correlation function decays in such a way that

$$\lim_{t' \to \infty} C(t, t') = q + C_{st}(t - t')$$  \hspace{1cm} (4.1)

$$\lim_{t - t' \to \infty} C_{st}(t - t') = 0 \quad \Rightarrow \quad \lim_{t' \to \infty} \lim_{t - t' \to \infty} C(t, t') = q$$  \hspace{1cm} (4.2)

$$\lim_{t \to \infty} C(t, t') = 0.$$  \hspace{1cm} (4.3)

In these equations $q$ is the Edwards-Anderson order parameter that characterizes the spin-glass phase. This means that for large $t \geq t'$ such that $t - t'$ is small with respect to $\mathcal{T}(t')$, the correlation function first decays from 1 to $q$ in a time-translational invariant manner. $\mathcal{T}(t')$ is a growing function of $t'$ whose precise form depends on the model. The correlation
FIG. 1. The schematic phase diagram for the $p = 3$ model.

goes further below $q$ to eventually reach 0 in a manner that depends both upon $t$ and $t'$ (the aging effect). This behavior suggests the presence of at least two time-sectors in which the dynamics is stationary and non-stationary, respectively.

We then write the correlation as the sum of a stationary and an aging contribution:

$$C(t, t') = C_{st}(t - t') + C_{ag}(t, t').$$  \hspace{1cm} (4.4)

The matching conditions at equal times between $C_{st}$ and $C_{ag}$ are

$$C(t, t) = 1 \quad \Rightarrow \quad C_{st}(0) + C_{ag}(t, t) = 1$$  \hspace{1cm} (4.5)

with

$$C_{st}(0) = 1 - q \quad \quad C_{ag}(t, t) = q.$$  \hspace{1cm} (4.6)

Together with Eq. (4.2), they ensure that in the two-time sector in which $C_{st}$ decays from $1 - q$ to 0, $C_{ag}$ is just a constant $q$. Instead, in the time-sector in which $C_{ag}$ decays from $q$ to 0, $C_{st}$ vanishes identically.

The name weak-ergodicity breaking\cite{9,11} reflects the fact that for short time-differences the system behaves as if it would be trapped in some region of phase space of “size” $q$ – suggesting ergodicity breaking. However, it is always able to escape this region in a time-scale $T(t')$ that depends upon its age $t'$. Hence, trapping is gradual and ergodicity breaking is weak.
We recall that in the classical and purely relaxational case (Langevin dynamics) the correlation functions are monotonic with respect to both times \( t \) and \( t' \). However, in the presence of inertial terms there can be oscillations and the decay can be non-monotonic. This will depend upon the relative value of the mass \( m \) with respect to the other parameters in the problem. For the parameters we study in this paper, the oscillations appear only in the stationary regime, the aging dynamics having a monotonic decay towards zero. This is relevant since it will allow us to use the general properties of monotonic correlation functions proven in Ref. [12] to find the two-time scaling of \( C_{AG}(t, t') \).

**B. The response function and the weak long-term memory scenario**

Regarding the response function, we propose

\[
R(t, t') = R_{st}(t - t') + R_{ag}(t, t')
\]

with

\[
R_{st}(t - t') \equiv \lim_{t' \to \infty} R(t, t') .
\]

When a system is in equilibrium, the response is simply related to the correlation via FDT. Here we need to extend this relation to a non-equilibrium quantum situation. As in the classical case we assume that the dynamics in the stationary regime is like an equilibrium dynamics and satisfies FDT. We then assume that \( R_{st}(\tau) \) and \( C_{st}(\tau) \) are related by

\[
R_{st}(\omega) = -\frac{2}{\hbar} \lim_{\epsilon \to 0^+} \int d\omega' \frac{1}{2\pi} \frac{1}{\omega - \omega' + i\epsilon} \tanh \left( \frac{\beta \hbar \omega'}{2} \right) C_{st}(\omega)
\]

(see Appendix B). In contrast, in the time-sector in which the dynamics is non-stationary and manifestly non-equilibrium, FDT will be modified. In Section VII we suggest an extension of FDT to quantum non-equilibrium systems with slow dynamics, that we later check in the particular model under study.

The **weak long-term memory scenario** states that the system keeps a weak memory of what happened in the past. More precisely, the response function tends to zero when times get far apart,

\[
\lim_{t \to \infty} R(t, t^*) = 0 \quad \forall \text{ fixed } t^* ,
\]

and its integral over a finite time-interval also vanishes

\[
\lim_{t \to \infty} \int_0^{t^*} dt'' R(t, t'') = 0 \quad \forall \text{ fixed } t^* 
\]

but, its integral over an interval that grows with time gives a finite contribution:

\[
\lim_{t - t^* \to \infty} \int_{t - t^*}^{t} dt'' R(t, t'') < \lim_{t \to \infty} \int_0^{t} dt'' R(t, t'') \quad \forall \text{ fixed } t^* .
\]

Note that the property (4.10) implies
An important quantity that reflects the decay of the response function is the two-time susceptibility defined as the integral of the response over the time-interval \([t_w, t]\):

\[
\chi(t, t_w) = f \int_{t_w}^{t} ds \, R(t, s) \tag{4.14}
\]

\((f\) is the strength of the applied perturbation, see Eq. (2.8)). In the classical case, a useful description of the departure from FDT is obtained by plotting \(\chi(t, t_w)\) vs \(C(t, t_w)\) for several \(t_w\) using \(\tau \equiv t - t_w\) as a parameter. Since the classical FDT implies \(\chi(t, t_w) = 1/T (C(t, t_w) - C(t, 0))\), violations of the theorem manifest as departures from a straight line plot. In the quantum case, the relation between \(R\) and \(C\) is more involved and a priori one does not expect these plots to be useful. As we show in Section \(\text{[VII]}\), it turns out that even for a quantum system these parametric plots are relevant.

**V. DYNAMICS IN THE PARAMAGNETIC PHASE**

In the paramagnetic phase (see Fig. [I]) one expects that, after a short non-equilibrium transient, a time-translational invariant (TTI) solution establishes. We show that a TTI ansatz implies that FDT holds between \(R\) and \(C\) and between \(\tilde{\Sigma}\) and \(\tilde{D}\). By solving numerically the full set of equations (3.30), (3.31) and (3.39) we show the correctness of this ansatz and later discuss some properties of the solutions.

After the short initial transient, one-time quantities such as \(z(t)\) should reach a limit \(z_\infty\). We evaluate it as

\[
\begin{align*}
 z_\infty &\equiv \lim_{t \to \infty} z(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[ \Sigma(\omega)C(\omega) + D(\omega)R(\omega) \right] - m \partial_t^2 C(t - t') \bigg|_{t' \to t \to \infty} .
\end{align*}
\tag{5.1}
\]

The Fourier-transform of Eqs. (3.30) and (3.31) yields the quantum mode-coupling equations:

\[
\begin{align*}
 R(\omega) &= \frac{1}{-m\omega^2 + z_\infty + 4\eta(\omega) - \Sigma(\omega)} , \\
 C(\omega) &= (2\hbar\nu(\omega) + \tilde{D}(\omega))|R(\omega)|^2 .
\end{align*}
\tag{5.3}
\]

Thus,

\[
\begin{align*}
 \text{Im } R(\omega) &= -\text{Im}(\tilde{\Sigma}(\omega) - 4\eta(\omega))|R(\omega)|^2 , \\
 \text{Re } R(\omega) &= \left( -m\omega^2 + z_\infty - \text{Re } (\tilde{\Sigma}(\omega) - 4\eta(\omega)) \right) |R(\omega)|^2 .
\end{align*}
\tag{5.5}
\]

If we now assume that \(\tilde{\Sigma}\) and \(\tilde{D}\) are related by FDT, it is easy to check that Eqs. (5.2), (5.3) and (5.4) imply that \(R\) and \(C\) are also related by FDT. The proof is completed by showing that FDT between \(R\) and \(C\) implies FDT between \(\tilde{\Sigma}\) and \(\tilde{D}\). This is easier to do in the time-domain, by verifying that Eq. (B.8) holds between \(\tilde{\Sigma}\) and \(\tilde{D}\) if it does between \(R\) and \(C\).
FIG. 2. The auto-correlation $C(\tau + t_w, t_w)$ as a function of the time-difference $\tau$ for $t_w = 2, 4, 8, 16$ for the classical model in the paramagnetic phase, $T = 3$ and $\bar{h} = 0$. The curves for $t_w = 8$ and $t_w = 16$ are superposed. As in all following figures, $p = 3, \Lambda = 5, \tilde{J} = 1$ and $m = 1$.

FIG. 3. The same parameters as in Fig. 2. The response function $R(\tau + t_w, t_w)$ as a function of the time-difference $\tau$ for $t_w = 2, 4, 8, 16$. TTI establishes for $t_w \geq 8$. In the inset, the integrated response $\chi(\tau + t_w, t_w)$ vs. $C(\tau + t_w, t_w)$ in a parametric plot for waiting time $t_w = 4$ and $\tau$ in $[0, 28]$. The classical FDT prediction is represented by the straight line of slope $-1/T = -1/3$; the $\chi$ vs. $C$ curve coincides with it showing that the classical FDT holds.
FIG. 4. The auto-correlation $C(\tau + t_w, t_w)$ as a function of the time-difference $\tau$ for $t_w = 2, 4, 8, 16$ for the quantum model in the paramagnetic phase, $T = 0$ and $\hbar = 6$. The curves for $t_w = 8$ and $t_w = 16$ fall on top of each other demonstrating TTI.

FIG. 5. The response $R(\tau + t_w, t_w)$ as a function of the time-difference $\tau$ for the same parameters as in Fig. 4. The waiting times are $t_w = 2, 4, 8, 16$ and TTI holds for $t_w \geq 8$. In the inset the integrated response $\chi(\tau + t_w, t_w)$ vs. the auto-correlation $C(\tau + t_w, t_w)$ in a parametric plot for $t_w = 8$. Since the system is in the paramagnetic phase with pure quantum fluctuations, the $\chi$ vs. $C$ plot is not expected to give us further information (see, however, Section VII).
A solution of this type exists only in a restricted region of the phase diagram. In order to demonstrate the existence of a dynamic transition, we solve the full dynamic equations numerically and find that at a critical line, both the TTI and FDT assumptions break down.

For the purpose of solving Eqs. (3.30), (3.31) and (3.39) numerically, we proceed as in the purely relaxational classical case. Notice though that the presence of non local kernels $\eta$ and $\nu$, that resemble the delta-function and appear convoluted with the correlation and response, renders the numerical solution harder. The larger the cut-off $\Lambda$, the smaller the iteration step $\delta$ we need to compute these integrals with a good precision. We found an acceptable data collapse for $\delta = 0.0025, 0.005, 0.01$ if $\Lambda = 5$. Note that the values of $\delta$ typically used in the purely relaxational case are at least one order of magnitude larger. This is the reason why the time-window we explore here is rather narrow though it suffices to show the trend of the solution.

Even after rescaling time and all parameters in the problem, there are still many free parameters left: $(\tilde{J}, p, m, \Lambda, \bar{h}, T)$. We here concentrate on the dependence upon only a small fraction of this set. We fix hereafter $p = 3, \tilde{J} = 1, m = 1$ and $\Lambda = 5$ and study the behavior upon the remaining parameters $T, \bar{h}$. We discuss the data for the correlation, response and integrated response for three points in the paramagnetic phase:

- $T = 3 > T_d, \bar{h} = 0$. This corresponds to the classical problem with colored noise and inertia. The classical critical temperature for the purely relaxational case is $T_d \sim 0.6$. In Fig. 2 we show the correlation $C(\tau + t_w, t_w)$ vs. the time-difference $\tau$ for $t_w = 2, 4, 8, 16$. In Fig. 3 we plot the response $R(\tau + t_w, t_w)$ vs $\tau$ for the same waiting times. In both cases the decay is very fast, typical of upper critical dynamics, and TTI quickly sets in (cfr. the curves for $t_w = 8$ and $t_w = 16$ that are indistinguishable in both plots). In the inset, we plot the integrated response $\chi(\tau + t_w, t_w)$ vs. $C(\tau + t_w, t_w)$ for $t_w = 4$ using $\tau$ as a parameter. The $\chi$ vs. $C$ plot very soon becomes a straight line of slope $-1/T = -1/3$ showing that the classical FDT is satisfied.

- $T = 0, \bar{h} = 6 > \bar{h}_c$. This corresponds to a strong quantum regime without thermal noise. The data for correlations and response in Figs. 4 and 5 are qualitatively similar to the ones for the classical problem at high temperatures in the sense that an equilibrium regime is quickly attained. The decay is fast and TTI holds. In the inset of Fig. 5 we display the $\chi$ vs. $C$ plot. In this strong quantum case at $T = 0$, the curve severely deviates from the classical expectation of a straight line of infinite slope. The quantum fluctuation-dissipation theorem is verified in Fig. 6 by comparing the response function obtained numerically with the convolution of the kernel $\tanh(\beta \hbar \omega/2)$ with the numerical correlation function, see Eq. (3.12). These two curves should be equal in equilibrium, as imposed by FDT, and we here check that this is indeed the case in the paramagnetic phase.

- $T = 2, \bar{h} = 1 > \bar{h}_c(T = 2)$. This is the quantum problem with both thermal and quantum fluctuations in the paramagnetic phase. In Fig. 7 we check FDT for these parameters and observe, in the $\chi$ vs. $C$ curve displayed in the inset, that the FDT becomes the classical.
FIG. 6. Check of FDT for $T = 0$ and $\bar{h} = 6$ (the paramagnetic phase). We compare the response $R(t, t - \tau)$ for total time fixed $t = 32$ and $\tau \in [0, t]$ obtained from the iteration of the dynamic equations, and the response computed from Eq. (B.12) that assumes FDT. The correlation values used in Eq. (B.12) are also obtained from the numerical algorithm.

We continue the study of the paramagnetic phase by analyzing the variation with $T$ and $\bar{h}$, close to the transition line, of the out of phase susceptibility

$$\chi''(\omega) = \text{Im} R(\omega).$$

(5.6)

This is the quantity that is most commonly measured experimentally. For instance, in the context of the study of quantum phase transitions, the dynamics of the randomly diluted, dipolar coupled, Ising magnet LiHo$_x$Y$_{1-x}$F$_4$ was studied by measuring $\chi''(\omega)$ among other quantities.

We follow two routes to the transition:

- $T = 1.3 > T_d$, $\bar{h} = 3, 2, 1, 0$. One approaches the classical model by decreasing the strength of quantum fluctuation at fixed temperature $T = 1.3 \sim 2 T_d$. In Fig. 8 we plot $\chi''(\omega)$ vs. $\omega$. In the curves for small $\bar{h}$ we observe a local minimum that is associated to the appearance of a plateau in the correlation function, at the value $q$, when approaching $T_d$. This is typical of “discontinuous transitions” as the one in the classical $p$-spin model (and in super-cooled-liquids). We here note that this feature survives the presence of small quantum fluctuations and progressively disappears when $\bar{h}$ increases. For $\bar{h} = 3$, the minimum has been substituted by a single peak.

- $T = 0.1 < T_d$, $\bar{h} = 6, 5.5, 5, 4.5, 4, 3.5$. One approaches the critical line $(T_c, h_c)$ from above by modifying the strength of the quantum fluctuations while keeping the
FIG. 7. The same comparison as in Fig. 6 for $T = 2$ and $h = 1$ (the paramagnetic phase). In the inset, the $\chi$ vs $C$ plot for $t_w = 16$. The straight line of slope $-1/2$ is the expectation for a classical FDT at this temperature. For $t_w$ larger than a characteristic time, the FDT becomes the classical one, as expected in a high-temperature situation.

Contrary to Fig. 8, there is no local minimum in any of the curves. This is in accordance with a decreasing and eventually vanishing plateau value $q$ when the parameters approach the quantum critical point. We shall further discuss the behavior of the model close to the transition line in Section VII B.

These results demonstrate the existence of a cross-over in the disordered phase, separating a region where the scaling laws are controlled by the quantum critical point $(0, h_c)$ from another region with different scaling laws controlled by the classical critical point $(T_d, 0)$.

Before concluding our analysis of the paramagnetic phase, let us compare our results with the situation encountered for the system $^{23}$LiHo$_x$Y$_{1-x}$F$_4$, which is believed to be an experimental realization of the transverse Ising model. Naturally, we do not expect to find

$^2$ Note however that experiments signal a rather different dependence of the dynamics on the
the same behavior in both models as we know that in the classical limit they are different. The classical transition being first order for the \( p \)-spin model and second order for the Ising case. Our analysis indicates that the dynamic transition of the \( p \)-spin model remains first order along the critical line but becomes second order at the quantum critical point. This is the mirror situation to what has been observed in the dipolar glass. In Ref. [23], it has been suggested that the quantum spin-glass transition may be first order. Figure 8 displays the effect of quantum fluctuations close to the classical critical point and looks rather similar to Fig. 2 in Ref. [23] which instead shows the dynamics close to the quantum critical point. Figure 9 looks similar to Fig. 4 in Ref. [23], where the dynamics close to the classical critical point is presented. This comparison indicates that this model belongs to a different class from the system \( \text{LiHo}_x\text{Y}_{1-x}\text{F}_4 \) and probably from the transverse field Ising model.

VI. DYNAMICS IN THE GLASSY PHASE

In this section we focus on the non-equilibrium dynamics observed in the glassy phase. In order to justify the scenario discussed in Section [IV], we first present the results from the numerical solution of the full equations. Next, we analyze these equations in the stationary and aging regimes and obtain a self-consistent relation that determines the Edwards-Anderson parameter.

A. Numerical results

The numerical solution shows a very different behavior below the critical line \( h_c(T_c) \). In order to illustrate the slow dynamics, the breakdown of TTI and the weak-memory of the system we choose the following parameters:

- \( T < T_d, h = 0 \). This is the classical, glassy model studied in Ref. [11] but with the addition of inertia. The effect of inertia amounts to the presence of oscillations around \( q \) but it does not change the qualitative behavior of the model. For example, as we shown analytically below, the value of \( q \) is not modified by inertia if \( m \) is small.

- \( T = 0, h = 1 \). This corresponds to pure quantum fluctuations. Figure 10 shows the correlation \( C(\tau + t_w, t_w) \) vs. \( \tau \) for the waiting times \( t_w = 0.5, 1, 2, 4, 8, 16 \). The curves have a stationary and an aging regime separated by the Edwards-Anderson parameter \( q \sim 0.72 \), see Eq. (4.2). The correlations decay to zero at far apart times, as in Eq. (4.3). This demonstrates the weak ergodicity breaking scenario.

In Fig. 11 we plot the response \( R(\tau + t_w, t_w) \) vs. \( \tau \) for the same waiting times. One observes a small time-difference regime, \( \tau \leq 5 \), where TTI rapidly establishes and a temperature history of the sample in the case of classical orientational glasses and classical spin-glasses with short-range interactions. This might give a warning on choosing the transverse Ising model to describe the dynamics of quantum dipolar glasses.
FIG. 8. $\chi''(\omega)$ vs. $\omega$ for $T = 1.3$ and $h = 0 (\Diamond)$, $h = 1 (+)$, $h = 2 (\square)$, $h = 3 (\times)$.

FIG. 9. $\chi''(\omega)$ vs. $\omega$ for $T = 0.1$ and $h = 3.5 (\ast)$, $h = 4 (\triangle)$, $h = 4.5 (\Diamond)$, $h = 5 (+)$, $h = 5.5 (\square)$, $h = 6 (\times)$. 
large time-difference regime, \( \tau > 5 \), where the response shows a decaying queue with a \( t_w \) dependence. The integration of this queue over a time-interval scaling with \( t_w \) will tell us about the violations of FDT in the quantum case (see Fig. 16 below). It also shows that the system has a weak long-term memory as described in Section IV.

- \( T = 0.5, \hbar = 1 \). The qualitative behavior is similar to that obtained for \( T = 0 \) and \( \hbar = 1 \) apart from the fact that thermal fluctuations slightly decrease the value of the Edwards-Anderson parameter and accelerate significantly the decay in the aging regime, as demonstrated in Fig. 12.

**B. The Lagrange multiplier \( z(t) \): a one-time quantity that reaches a limit**

The Lagrange multiplier \( z(t) \) is a one-time quantity, i.e. it depends only upon the total time \( t \). In this formalism we assume that all one-time quantities reach a well-defined limit asymptotically. In Appendix D we describe how we obtain the following equation for the asymptotic limit of \( z(t) \):

\[
\begin{align*}
z_\infty &= A_\infty + q \int_0^\infty d\tau' \, \Sigma_{ST}(\tau') + \tilde{D}_q \int_0^\infty d\tau' R_{ST}(\tau') \\
&\quad + \int_0^\infty d\tau' \left[ \Sigma_{ST}(\tau')C_{ST}(\tau') + D_{ST}(\tau')R_{ST}(\tau') \right] - m \partial_\tau^2 C_{ST}(\tau) \bigg|_{\tau \to 0}.
\end{align*}
\]  

(6.1)

Let us discuss each term in this expression. We called \( A_\infty \) the aging contribution:

\[
A_\infty = \lim_{t \to \infty} \int_0^t dt'' \left[ \tilde{\Sigma}_{AG}(t, t'')C_{AG}(t, t'') + \tilde{D}_{AG}(t, t'')R_{AG}(t, t'') \right].
\]  

(6.2)

The kernels \( \nu \) and \( \eta \), that are related to the bath, do not contribute to \( A_\infty \). This is so because we have assumed that for \( \tau > T(t') \) the kernels have already decayed to zero in such a way that they do not contribute to these integrals. More precisely, we are neglecting terms of the form

\[
\lim_{t \to \infty} \int_0^t dt'' A(t - t'') B(t, t'')
\]  

(6.3)

where \( A \) is either \( \nu \) or \( \eta \) and \( B \) is either \( C_{AG} \) or \( R_{AG} \).

The second and third terms come from the constant (non-zero) limit of the first decay of the correlation \( q \equiv \lim_{t-t' \to \infty} \lim_{t' \to \infty} C(t, t') \) and equivalently of the vertex

\[
\tilde{D}_q \equiv \lim_{t-t' \to \infty} \lim_{t' \to \infty} \tilde{D}(t, t').
\]  

(6.4)

In the model under study

\[
\tilde{D}_q = \frac{j^2 p}{2} q^{p-1}
\]  

(6.5)

if we use \( \lim_{\tau \to \infty} R_{ST}(\tau) \ll q \), a property of the weak long-term memory scenario.
FIG. 10. $C(\tau + t_w, t_w)$ vs. $\tau$ for $T = 0$, $h = 1$ and from bottom to top $t_w = 2.5, 5, 10, 20, 30, 40$. The weak ergodicity breaking scenario and aging are explicit.

FIG. 11. $R(\tau + t_w, t_w)$ vs. $\tau$ for $T = 0$, $h = 1$ and from top to bottom $t_w = 5, 10, 20, 30, 40$. The weak long-term memory scenario is explicit.
FIG. 12. $C(\tau + t_w, t_w)$ vs. $\tau$ for $T = 0.5$, $h = 1$ and from bottom to top $t_w = 4, 8, 16, 32$. The Edwards-Anderson parameter is slightly smaller and the decay in the aging regime is faster than in Fig. [10].

C. Dynamical equations in the stationary regime

If $(t, t')$ are such that $C(t, t') > q$, the discussion in Section [V] implies $C(t, t') = q + C_{\text{ST}}(t - t')$ and $R(t - t') = R_{\text{ST}}(t - t')$. The Schwinger-Dyson equation for $R$ in this time sector reads

$$\left(m \partial_\tau^2 + z_\infty\right) R_{\text{ST}}(\tau) = \delta(\tau) + \int_0^\tau d\tau' \Sigma_{\text{ST}}(\tau - \tau') R_{\text{ST}}(\tau')$$

(6.6)

and it keeps the same form as in the high-temperature regime, apart from the fact that the constant $z_\infty$ has contributions from the aging regime.

The Schwinger-Dyson equation for $C$ reads

$$\left(m \partial_\tau^2 + z_\infty\right) \left(q + C_{\text{ST}}(\tau)\right) = A_\infty + q \int_0^\infty d\tau' \Sigma_{\text{ST}}(\tau') + \tilde{D} q \int_0^\infty d\tau' R_{\text{ST}}(\tau')$$

$$+ \int_{-\infty}^\infty d\tau' \left[\Sigma_{\text{ST}}(\tau + \tau') C_{\text{ST}}(\tau') + D_{\text{ST}}(\tau + \tau') R_{\text{ST}}(\tau')\right].$$

(6.7)

One can now Fourier-transform both equations

$$R_{\text{ST}}(\omega) = \frac{1}{-m\omega^2 + z_\infty - \Sigma_{\text{ST}}(\omega)},$$

$$(-m\omega^2 + z_\infty) C_{\text{ST}}(\omega) + z_\infty q \delta(\omega) = \left(A_\infty + q \Sigma_{\text{ST}}(\omega) + \tilde{D} q R_{\text{ST}}(\omega)\right) \delta(\omega)$$

$$+ \Sigma_{\text{ST}}(\omega) C_{\text{ST}}(\omega) + D_{\text{ST}}(\omega) R_{\text{ST}}(-\omega).$$

(6.8)
The formal solution to the equation for $C_{st}$ is

$$C_{st}(\omega) = \left(-z_\infty q + A_\infty + q\Sigma_{st}(\omega) + \tilde{D}_q R_{st}(\omega)\right) \delta(\omega) R_{st}(\omega) + D_{st}(\omega)|R_{st}(\omega)|^2. \quad (6.9)$$

The first term on the right-hand-side has an imaginary and a real part. The imaginary part vanishes identically since, due to FDT, both $\text{Im} R_{st}(\omega)$ and $\text{Im} \Sigma_{st}(\omega)$ are proportional to $\tanh(\beta\hbar\omega/2)$ which is zero at $\omega = 0$. Concerning the real part of this first term, as we have assumed that $C_{st}(\tau)$ goes to zero for $\tau \to \infty$, we need to impose the self-consistent condition

$$-z_\infty q + A_\infty + q\Sigma_{st}(\omega = 0) + \tilde{D}_q R_{st}(\omega = 0) = 0. \quad (6.10)$$

This is the condition that fixes the Edwards-Anderson parameter. We shall find it again in the next section as the matching condition between the stationary and aging regimes.

The final equation for $C_{st}(\omega)$ is

$$C_{st}(\omega) = D_{st}(\omega)|R_{st}(\omega)|^2. \quad (6.11)$$

One can check that these calculations are consistent with the results from $z_\infty$. Actually, the integrals in $z$-eq. involving the stationary parts can be evaluated with the help of the equations for $R_{st}$ and $C_{st}$, Eqs. (5.8) and (5.11), and yield once again Eq. (5.10).

Similarly to the high-temperature case one can now show that FDT for $\Sigma_{st}$ and $\tilde{D}_{st}$ implies FDT for $R_{st}$ and $C_{st}$. The remainder of the proof, i.e. to show that FDT between $R_{st}$ and $C_{st}$ implies FDT between $\Sigma_{st}$ and $\tilde{D}_{st}$ depends only upon the from of $\Sigma_{st}$ and $\tilde{D}_{st}$ as functions of $R_{st}$ and $C_{st}$ and is not modified from the one discussed in Section [1].

### D. Dynamical equations in the aging regime

If we now choose the times $t, t'$ to be well-separated so as to have $C(t, t') = C_{AG}(t, t') \leq q$ and $R(t, t') = R_{AG}(t, t')$, the weak-ergodicity breaking and weak long-term memory hypotheses allow us to throw the second time derivatives on the left-hand-side. We assume that their contribution is much weaker than the one of each of the integral terms on the right-hand-side. This is an assumption that we have to verify at the end of the calculation, once the solution for $C_{AG}$ and $R_{AG}$ is known. It corresponds to the over-damped limit:

$$m\ddot{\delta}^2 C_{AG} \ll \text{Terms in the RHS}_C(t, t'),
\quad m\ddot{\delta}^2 R_{AG} \ll \text{Terms in the RHS}_R(t, t'). \quad (6.12)$$

Using the approximation described in Appendix [D], the $R$-eq in the aging regime becomes

$$z_\infty R_{AG}(t, t') = \Sigma_{AG}(t, t') \int_0^\infty d\tau' R_{st}(\tau') + R_{AG}(t, t') \int_0^\infty d\tau' \Sigma_{st}(\tau')
+ \int_0^t dt'' \Sigma_{AG}(t, t''') R_{AG}(t'''', t') \quad (6.13)$$

and we call it the $R_{AG}$-eq. Similarly, the $C$-eq becomes

$$z_\infty C_{AG}(t, t') = C_{AG}(t, t') \int_0^\infty d\tau' \Sigma_{st}(\tau') + \tilde{D}_{AG}(t, t') \int_0^\infty d\tau' R_{st}(\tau')
+ \int_0^t dt'' \Sigma_{AG}(t, t''') C_{AG}(t''', t') + \int_0^t dt'' \tilde{D}_{AG}(t, t''') R_{AG}(t'''', t) \quad (6.14)$$

and we call it the $C_{AG}$-eq. In all integrals we approximated $\Sigma_{AG}(t, t') \sim \tilde{\Sigma}_{AG}(t, t')$ and $D_{AG}(t, t') \sim \tilde{D}_{AG}(t, t')$ since $\eta(t - t')$ and $\nu(t - t')$ are very rapidly decreasing functions.
E. The Edwards-Anderson parameter

The Edwards-Anderson parameter \( q \) is determined self-consistently from the matching of \( \lim_{t \to \infty} C_{AG}(t, t) = \lim_{t' \to \infty} \lim_{t'' \to \infty} C(t, t') = q \). Taking the limit \( t' \to t^- \) in the \( R_{AG} \)-eq and \( C_{AG} \)-eq one obtains

\[
 z_\infty R_{AG}(t, t) = \tilde{\Sigma}_{AG}(t, t) \int_0^\infty d\tau' R_{ST}(\tau') + R_{AG}(t, t) \int_0^\infty d\tau' \Sigma_{ST}(\tau') ,
\]

\[
 z_\infty q = A_\infty + q \int_0^\infty d\tau' \Sigma_{ST}(\tau') + \tilde{D}_{AG}(t, t) \int_0^\infty d\tau' R_{ST}(\tau') .
\]

The first equation admits the solution \( R_{AG}(t, t) = 0 \) since \( \tilde{\Sigma}_{AG}(t, t) \) is proportional to \( R_{AG}(t, t) - \) see Eq. (3.27). This corresponds to the high-temperature solution where there is no aging regime. Here we concentrate on the other possibility, that is to say when

\[
 z_\infty = \tilde{\Sigma}_{AG}(t, t) R_{AG}(t, t) \int_0^\infty d\tau' R_{ST}(\tau') + \int_0^\infty d\tau' \Sigma_{ST}(\tau') .
\]

In actual fact, we can further approximate this equation by using one of the assumptions already used all over this section: that the response becomes smaller and smaller as time passes – though its integral over an infinite interval gives a finite contribution. If we neglect all terms that are proportional to \( R_{AG}(t, t) \) with respect to terms that are proportional to \( q \), only the first term in the power expansions of \( \tilde{\Sigma} \) and \( \tilde{D} \) survive:

\[
 \left( \tilde{\Sigma}/R \right)_q \equiv \lim_{t \to \infty} \frac{\tilde{\Sigma}_{AG}(t, t)}{R_{AG}(t, t)} \quad \tilde{D}_q \equiv \lim_{t \to \infty} \tilde{D}_{AG}(t, t)
\]

that in our model become

\[
 \left( \tilde{\Sigma}/R \right)_q = \frac{\tilde{J}^2 p(p-1)}{2} q^{p-2} \quad \tilde{D}_q = \frac{\tilde{J}^2 p}{2} q^{p-1} ,
\]

in accord with the large \( \tau \) limit of the stationary values (6.3). Equations (6.15) and (6.16) become

\[
 z_\infty = \left( \tilde{\Sigma}/R \right)_q \int_0^\infty d\tau' R_{ST}(\tau') + \int_0^\infty d\tau' \Sigma_{ST}(\tau') ,
\]

\[
 z_\infty q = A_\infty + q \int_0^\infty d\tau' \Sigma_{ST}(\tau') + \tilde{D}_q \int_0^\infty d\tau' R_{ST}(\tau') .
\]

The second equation is the same as the one arising from the end of the stationary regime, Eq. (6.10). We now study these equations with the help of Eqs. (6.8) and (6.11) for the stationary functions. Using

\[
 \int_0^\infty d\tau R_{ST}(\tau) = R_{ST}(\omega = 0) = \frac{1}{z_\infty - \Sigma_{ST}(\omega = 0)} ,
\]

one has
FIG. 13. The $\bar{h}$ dependence of the Edwards-Anderson parameter $q$ that is read from the plateau in the log-log plot of the auto-correlation $C(\tau + t_w, t_w)$ vs. $\tau$ for $t_w = 32$. From top to bottom $\bar{h} = 0.1, 0.5, 1.2$ and the temperature is always $T = 0$.

\[
1 = \left(\frac{\bar{\Sigma}}{R}\right)_q R_{st}^2(\omega = 0).
\] (6.23)

We remind that the factor $R_{st}^2(\omega = 0)$ can be written in terms of the stationary correlation function using FDT; therefore this is a closed equation for the correlation that gives the Edwards-Anderson parameter $q$. In the case of the $p$-spin model it reads

\[
1 = \frac{\bar{j}^2 p(p - 1)}{2} q^{p-2} \left( \frac{1}{\bar{h}} P \int_{-\infty}^{\infty} \frac{d\omega'}{\omega'} \tanh \left( \frac{\beta \bar{h} \omega'}{2} \right) C_{st}(\omega') \right)^2.
\] (6.24)

In the classical case, the integral can be readily computed and the final equation for $q$ is

\[
\frac{\bar{j}^2 p(p - 1)}{2} q^{p-2}(1 - q)^2 = T^2,
\] (6.25)

that coincides with the result for the purely relaxational dynamics.\[\text{For } p \geq 3 \text{ fixed, } q \text{ is a function of temperature; it equals one at } T = 0 \text{ and tends to } q_d \equiv q(T_d) > 0 \text{ at the dynamic critical temperature } T_d \equiv T_c(h = 0).\]

If $h \neq 0$, the Edwards-Anderson parameter $q$ depends upon $h$ and $T$, $q(T, h)$. It decreases when thermal and/or quantum fluctuations increase. One observes $q(0, 0) = 1$. At the critical line $(h_c, T_c)$, $q(T_c, h_c) \neq 0$ as long as $T \neq 0$. At the quantum critical point the nature of the transition changes and $q(h_c, 0) = 0$. We shall explain this result after introducing the modification of the quantum FDT needed to complete the solution. Figure 13 shows the tendency of $q(T = 0, h)$ for $h = 0.1, 0.5, 1.2$.\[\text{For } p \geq 3 \text{ fixed, } q \text{ is a function of temperature; it equals one at } T = 0 \text{ and tends to } q_d \equiv q(T_d) > 0 \text{ at the dynamic critical temperature } T_d \equiv T_c(h = 0).\]
VII. FLUCTUATION-DISSIPATION RELATIONS OUT OF EQUILIBRIUM

One of the important characteristics of the dynamic evolution in equilibrium is the fluctuation-dissipation theorem (FDT) that establishes a model independent relationship (in the sense that only involves $T$ and $\hbar$ but no other parameter in the model) between correlation and response functions. As recalled in Appendix B, this result can be easily obtained, for the quantum and classical cases, under the hypothesis of equilibrium. In general, such a relation is not expected to hold in a non equilibrium situation and we can then say that FDT is “violated” (of course what is violated is not the theorem but the hypothesis under which the result is derived). It is nevertheless surprising that in the glassy phase of certain classical models, the violation of FDT takes such a simple form that it is possible to formulate a generalized version of this relation through the introduction of an effective temperature $T_{\text{eff}}$. In this way the FDT takes the standard equilibrium form but with a model dependent effective temperature which is different from the bath temperature $T$ (in general $T_{\text{eff}}$ is higher or equal than $T$ and it can take different values in different time-sectors). We would like to explore in this section how these ideas carry over to the quantum case.

A. General considerations

The standard quantum fluctuation-dissipation theorem states

$$R(t-t') = \frac{2i}{\hbar} \theta(t-t') \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \exp(-i\omega(t-t')) \tanh\left(\frac{\beta \hbar \omega}{2}\right) C(\omega)$$

with

$$C(\omega) \equiv 2 \text{Re} \int_{0}^{\infty} d\tau \exp(i \omega \tau) C(\tau).$$

(7.1)

(7.2)

Guided by the classical case, we introduce the effective temperature

$$T_{\text{eff}}(t,t') \equiv \frac{T}{X(t,t')}$$

(7.3)

and we propose, as a generalization of FDT to the out of equilibrium case, the following relation

$$R(t,t') = \frac{2i}{\hbar} \theta(t-t') \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \exp(-i\omega(t-t')) \tanh\left(\frac{X(t,t') \beta \hbar \omega}{2}\right) C(t,\omega)$$

(7.4)

with

$$C(t,\omega) \equiv 2 \text{Re} \int_{0}^{t} ds \exp(i \omega (t-s)) C(t,s)$$

(7.5)

and $X(t,t')$ a function of both times $t$ and $t'$. Evidently, Eqs. (7.4) and (7.5) reduce to Eqs. (7.1) and (7.2) when $t \to \infty$ if the evolution is TTI and the factor $X$ is set to one.
Let us assume, for reasons that are intimately related to the time-separation discussed in Section IV, that 
\[ X(t, t') = \begin{cases} 1 & \text{if } t - t' \leq \mathcal{T}(t') \Rightarrow C(t, t') > q \\ X_{\text{Ag}}(T, \hbar) & \text{if } t - t' > \mathcal{T}(t') \Rightarrow C(t, t') \leq q \end{cases} \] (7.6)

For times in the stationary regime \( t - t' \leq \mathcal{T}(t') \) this implies:
\[ R_{\text{st}}(t - t') = \frac{2i}{\hbar} \theta(t - t') \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \exp(-i\omega(t - t')) \tanh\left(\frac{\beta\hbar\omega}{2}\right) C_{\text{st}}(\omega) \] (7.7)

with
\[ C_{\text{st}}(\omega) \equiv 2\text{Re} \int_{0}^{\infty} d\tau \exp(i\omega\tau)C_{\text{st}}(\tau) . \] (7.8)

If we suppose that (even at \( T = 0 \)) \( \beta X_{\text{Ag}}(T, \hbar) \) is finite in the glassy phase, it is then clear that in the aging sector the Fourier integral is dominated by \( \omega \sim 0 \). After using
\[ \tanh\left(\frac{X_{\text{Ag}}\beta\omega}{2}\right) \sim \frac{X_{\text{Ag}}(T, \hbar)\beta\omega}{2} \] (7.9)

we obtain
\[ R_{\text{Ag}}(t, t') = \beta X_{\text{Ag}}(T, \hbar) \frac{\partial}{\partial t'} C_{\text{Ag}}(t, t') . \] (7.10)

For a standard quantum system in equilibrium at finite temperature, there is a characteristic time, that depends upon temperature and the strength of quantum fluctuations, beyond which the evolution becomes classical. Concerning FDT, this means that the classical relation establishes with \( X(t, t') = 1 \). This is clear, for instance, for the model under consideration at finite temperature in the paramagnetic phase (see the inset in Fig. 3). If our assumptions are correct, the situation in the glassy phase is different. For small times compared with \( \mathcal{T}(t_w) \), now a waiting-time dependent characteristic time, correlation and response are related by the standard (quantum) FDT. Instead, for time differences larger than \( \mathcal{T}(t_w) \), they are related in the same way as in the classical case \textit{but} with a \( T \) and \( \hbar \) dependent effective temperature that is different from the one of the environment.

In general, the proposal given in Eq. (7.10) will allow us to reduce the \( C_{\text{Ag}} \)-eq. and \( R_{\text{Ag}} \)-eq equations to single equation for \( C_{\text{Ag}} \). It is simple to check that, at least for the model under consideration, this equation has the same structure as the one for the classical problem. All dependence on \( \hbar \) enters through \( q \), \( X_{\text{Ag}} \) and \( R_{\text{st}} \). Hence, once the explicit form of \( X_{\text{Ag}} \) is known, the aging equations are solved by the same ansatz as in the classical case:

\[ C_{\text{Ag}}(t, t') = q^{-1}\left(\frac{h(t')}{h(t)}\right) , \] (7.11)

\[ R_{\text{Ag}}(t, t') = \frac{X_{\text{Ag}}}{T} q\partial_t^{-1}\left(\frac{h(t')}{h(t)}\right) . \] (7.12)
Note that the general arguments introduced in Ref. [12] to characterize the behavior of monotonic correlation functions apply to the correlations in the aging regime and justify the solution (7.11)-(7.12). Even if the inertia might make the correlation oscillate, it does only in the stationary regime since it oscillates around $q$, and it becomes monotonic in the aging regime.

In Eq. (7.6), we assumed that $X(t, t')$ can take only two values in view of the model we study. However, in the classical case, a more general situation arises in models with multiple time-scales, for which the factor measuring the violation of FDT is a non-constant function of the correlation in the aging regime. We expect the same structure to carry through to quantum problems whose classical counterparts are of the multi-scale type.

**B. Analysis of the $p \geq 3$ model**

We check, both analytically and numerically, the viability of our assumptions for the $p$-spin model. The factor $X_{AG}$ measuring the violation of FDT is determined by Eqs. (6.21) and (6.22)

$$0 = A_\infty - \frac{q}{R_{st}(\omega = 0)} + \tilde{D}_q R_{st}(\omega = 0). \tag{7.13}$$

Using Eq. (7.10) and the equivalent relation between $\tilde{\Sigma}_{AG}$ and $\tilde{D}_{AG}$, we obtain

$$A_\infty = \frac{X_{AG}}{T} \lim_{t \to \infty} \left( \tilde{D}_{AG}(t, t) C_{AG}(t, t) \right) = \frac{X_{AG}}{T} q \tilde{D}_q \tag{7.14}$$

and

$$X_{AG} = \frac{(p - 2)}{q} R_{st}(\omega = 0) = \sqrt{\frac{2(p - 2)^2}{p(p - 1)}} q^{-p/2}. \tag{7.15}$$

In the classical limit $X_{AG} = (p - 2)(1 - q)/q$ and the result in Ref. [11] is recovered. Note that both in the classical and quantum case, $X = 0$ if $p = 2$. Since the case $p = 2$ is formally connected to ferromagnetic domain growth (in the mean-field approximation) there is still no-memory in the quantum domain growth.

The classical critical point $(T_d, h = 0)$ is characterized by

$$X_{AG}(T_d, 0) = 1 \quad q(T_d, 0) \neq 0. \tag{7.16}$$

Following the critical line towards the quantum critical point $(T = 0, h_c)$, one obtains

$$X_{AG}(T_c, h_c) = 1 \quad q(T_c, h_c) = \left( \frac{2(p - 2)^2}{J^2 p(p - 1)} \right)^{1/p} T_c^{2/p} \tag{7.17}$$

and a finite zero-frequency stationary susceptibility $R_{st}(\omega = 0) < +\infty$. At the quantum critical point, $q$ vanishes and the zero-frequency stationary susceptibility diverges:

$$q(T_c \sim 0, h_c) \propto T_c^{2/p} \to 0 \quad \Rightarrow \quad R_{st}(\omega = 0) \propto T_c^{(2-p)/p} \to \infty. \tag{7.18}$$
FIG. 14. Check of the quantum FDT for the quantum model in the glassy phase when times take values in the stationary regime $C \geq q$. The parameters are $T = 0$ and $\bar{h} = 1$. Thin curve: $R(t, t - \tau)$ from the numerical algorithm with fixed $t = 32$; bold curve: $R(t, t - \tau)$ from Eqs. (7.7) and (7.8) using $C_{ST}(\tau) = C(t, t - \tau) - q$ with $C(t, t - \tau)$ from the numerical algorithm and $q \sim 0.7$. The curves coincide for $\tau < 7$.

(It is interesting to notice that the case $p = 2$ is clearly different.) On the other hand, if approaching the quantum critical point from below along the axis $(T = 0, \bar{h})$

$$T_{\text{eff}}(T = 0, \bar{h}) \sim (\bar{h}_c - \bar{h})^\alpha$$

then

$$q \sim (\bar{h}_c - \bar{h})^{2\alpha/p} \quad \Rightarrow \quad R_{\text{ST}}(\omega = 0) \sim (\bar{h}_c - \bar{h})^{\alpha(1-p/2)} .$$

We next check numerically the relation between response and correlation in the glassy phase, $T = 0, \bar{h} = 0.1 < \bar{h}_c$ corresponding to pure quantum fluctuations.

First, we use Fourier analysis to check the quantum FDT for short time differences. In Fig. 14 we compare two different ways of obtaining $R(t, t - \tau)$ for total time $t$ fixed and equal to 32. The thin curve is the result from the direct numerical solution of the dynamic equations and shows a fast evolution for short time differences and a tail that gives rise to the weak memory of the system. The bold curve is obtained from Eqs. (7.7) and (7.8) using $C_{ST}(t, t - \tau) = C(t, t - \tau) - q$ with $C(t, t - \tau)$ obtained numerically. We have estimated the Edwards-Anderson parameter as $q \sim 0.7$. One observes that for small time-differences the two curves coincide showing that FDT holds in this time-sector.

Second, we check the generalized FDT in the aging regime. For this purpose, it is convenient to use the $\chi$ vs. $C$ plots defined in Section IV. In Fig. 15 we observe, as predicted by Eqs. (7.7), (7.8) and (7.10), two behaviors according to the relative value of
FIG. 15. The $\chi$ vs $C$ plot for $T = 0$ and $\bar{h} = 0.1$. From bottom to top, different curves are associated to $t_w = 5, 10, 20, 30$. One observes how a constant asymptotic limit is approached for increasing $t_w$. The dots are a straight line of slope $-1/T_{\text{EFF}} = -X_{\text{AG}}/T = -0.6$ (the analytic result).

FIG. 16. The $\chi$ vs $C$ plot for $T = 0$ and $\bar{h} = 1$. The waiting times are $t_w = 5, 10, 20, 30$ and the dotted straight line has a slope $-1/T_{\text{EFF}} = -X_{\text{AG}}/T = -0.9$ (the analytic result).
FIG. 17. The $\chi$ vs. $C$ curves for the quantum model at $\hbar = 0.1$ and different temperatures. From top to bottom $T = 0.1, 0.2, 0.3, 0.4, 0.5, 0.8$. In the inset, the estimated effective temperature $T_{\text{EFF}} = T/X_{AG}$ as a function of temperature for the same $\hbar$. See the text for a discussion.

$C$ and $q$. The portion of the graph that is significant for the aging regime corresponds to $C < q$. As claimed in Eq. (7.10) the $\chi$ vs $C$ curve approaches a time-independent asymptotic limit that is a straight line of slope $-X_{AG}/T$. In this case, $T_{\text{EFF}} = 1.6$ and this shows that even a $T = 0$, $T_{\text{EFF}} \neq 0$.

The effective temperature depends on $T$ and $\hbar$ in a non trivial way. In Figure 16 we show the $\chi$ vs. $C$ plot for $\hbar = 1$ and $T = 0$. In this case $T_{\text{EFF}} = 1.1$.

In Fig. 17 we work at fixed $\hbar = 0.1$ and study the variation of $T_{\text{EFF}}$ with $T$. From the $\chi$ vs $C$ plots we determine how $T_{\text{EFF}}$ varies with $T$ by measuring the slopes of the different curves and by using Eq. (7.15). The six curves above the dotted straight line correspond to temperatures below $T_c$. From top to bottom $T = 0.1, 0.2, 0.3, 0.4, 0.5$. One notices that $T_{\text{EFF}} > T$ in all these curves. The curve below the dotted straight line corresponds to $T = 0.8 > T_c$. It has a slope $-1/T$ (indicated by the dots) and $T_{\text{EFF}} = T$. In the inset we plot the temperature dependence of the effective temperature. $T_{\text{EFF}}$ is linear in $T$ when the bath temperature is above $T_c$ but severely deviates from the linear behavior for temperatures below the critical line. Two curious features are that, in the glassy phase, $T_{\text{EFF}}$ increases when either $T$ or $\hbar$ decrease. The surprising increase of $T_{\text{EFF}}$ as $T$ decreases is also obtained in the classical limit.13
VIII. SUMMARY AND CONCLUSIONS

In this paper we presented a formalism that allows us to analytically study the behavior of quantum glassy models within the typical situation encountered in experiments, i.e. out of equilibrium.

Faced to the question as to how relevant quantum fluctuations are for the asymptotic real-time dynamics, one would have naively answered that they are irrelevant since quantum mechanics is not expected to play a relevant role in the large time sector, meaning large $t_w$ and for all $t \geq t_w$. We show in this paper that this would have been a wrong conclusion and that the dynamic behavior is much richer:

Quantum effects, together with temperature, determine where in parameter space a glassy phase exists. We show that sufficiently strong quantum fluctuations destroy the glassy phase at arbitrary low temperatures driving the system towards a paramagnetic phase. We predict then the existence of a dynamic quantum phase transition. This has no analogue in the classical model where glassiness is always present at $T = 0$.

Quantum fluctuations also dictate when, in the glassy phase, non-equilibrium effects manifest. For any waiting-time $t_w$ after preparation, there exists a characteristic time $\mathcal{T}(t_w)$ that determines the end of a stationary regime and the entrance into the aging regime. This means that for time differences $t - t_w < \mathcal{T}(t_w)$ the dynamics is stationary while for $t - t_w \geq \mathcal{T}(t_w)$ two-point functions have explicit waiting-time dependences. The characteristic time $\mathcal{T}(t_w)$ depends on temperature, the strength of quantum fluctuations and the waiting-time $t_w$.

The dynamics in the paramagnetic phase satisfies the fluctuation-dissipation theorem. In the glassy phase, we found that the quantum FDT is satisfied in the stationary regime and that a generalization is needed in the aging regime. This generalization defines a two-time function $X(t, t')$ that is consistent with the definition of an effective temperature $T_{\text{eff}}(t, t') = T/X(t, t')$ and that we later characterized, based on the numerical evidence for this model. When $T_{\text{eff}}$ is different from zero, the modification of the quantum FDT takes a similar form as the classical one but with a $T$ and $\hbar$ dependent effective temperature.

The characteristic time $\mathcal{T}(t_w)$ is reminiscent of a decoherence time in the sense that it separates the stationary regime in which the influence of quantum effects is very strong and explicit from the aging regime in which the influence of quantum effects is implicit, appearing through, e.g., the $\hbar$ dependence of $T_{\text{eff}}$.

One of the virtues of the approach we have followed is that it is close in spirit to the dynamic approach to classical systems. This encourages us to attempt to extend many of the recent advances in the understanding of classical glassy dynamics to the quantum problem. We mention below some of the questions that merit further attention.

The model here studied has a modification of the quantum FDT given by a piecewise function $X = 1$ if $C > q$ ($\mathcal{T}(t_w) > t - t_w$) and a constant $X_{\alpha}(T, \hbar) < 1$ if $C \leq q$ ($\mathcal{T}(t_w) \leq t - t_w$). It is known that in classical systems a more general situation can arise. Another family of models exists with the factor measuring the FDT violation given by $X = 1$ if $C > q$ and a non-constant function of the auto-correlation $X_{\alpha}(C) < 1$ if $C \leq q$. We expect the arguments put forward in Section VII to apply to this kind of system with
minor modifications. It would be of particular interest to see how this structure appears in a
concrete model as, for instance, the problem of a particle in a long-range correlated random
potential.

In the classical case, an intriguing connection between the factor that measures the FDT
violation in the non-equilibrium approach $X_{AG}(C)$ and the Parisi function $x(q)$,
$0 \leq q < q_{EA}$ of the replica analysis in equilibrium, has been noticed and checked in several
occasions. The structure of both functions is the same and, on top of this, their explicit
values coincide if an argument of “marginality” is used to determine $x(q)$ in the
equilibrium case. In Ref. [26], several quantum models related, but not equal, to the one
considered here were studied with a Gaussian variational method and replica theory in
the imaginary time Matsubara representation. In order to obtain a physical value for the
conductivity of the model, Giamarchi and Le Doussal used the marginality condition to fix
the replica breaking point parameter $x = x(q)$. A careful comparison of the dynamic
$X_{AG}$ with the static $x$ is now in order. Nevertheless, this cannot be readily done by
comparing the result in Ref. [26] with ours since the way in which both systems are
coupled to the environment are not necessarily equivalent. Thence, if a formal relation
between FDT-violations and the replica $x(q)$ exists at the quantum level is an interesting
question that deserves further study.

The last remark we wish to make about the FDT violations is that in the classical case
it has been related to the production of entropy during evolution. It would be desirable
to see whether such a connection also holds at the quantum level.

Concerning the formalism here developed, it can be extended in several directions. On
the one hand, one would like to study models with a spatial structure such as the one
associated to the motion of a manifold of internal dimension $d$ in a random quenched
potential. This quantum problem has many interesting applications and, in the case of an
embedding space with a large dimensionality can be treated, for example, within a dynamic
Gaussian variational approach. On the other hand, it would be extremely interesting to study
the opposite limiting case of low dimensions and analyze, for example, the real-time dynamics
of the Ising chain in a transverse field. In addition, one could investigate the effect induced
on the dynamics by different environments like considering sub or super Ohmic baths,
non-linear couplings and even baths of a completely different nature as those generated by spin
variables.

The Martin-Siggia Rose generating functional for classical stochastic processes takes a
particularly simple form if the time-space is enlarged to a superspace by introducing a
couple of Grassmann coordinates, and the variables of the problem are encoded in a superfield.
This can also be done for systems with disorder. The dynamic equations are then written
in terms of a single super-correlator that encodes the auto-correlation and response. The
equilibrium theorems, viz. causality, invariance under time-translations and the fluctuation-
dissipation theorem, follow from the symmetries of the MSR effective action written in
this way. The supersymmetric formulation of the classical dynamics has been helpful to
find a solution of mean-field like models with an internal dimension like the problem of the
manifold in a random media but, most importantly, it has highlighted the formal connection
between the out of equilibrium formalism and the replica approach to the statics. It would
be very interesting to identify a supersymmetric structure in the quantum problem, probably
at the level of the Keldysh-Schwinger generating functional. This could provide a means to establish a relation between the quantum dynamic formalism here presented and the quantum replica approach.

We believe that an analytical study of aging effects along the lines presented in this paper will be relevant in the understanding of realistic glassy systems. Indeed, even at very low temperatures, where quantum effects are important, a glass is a non-equilibrium system that evolves over many orders of magnitude in time. Aging effects, demonstrating this non-equilibrium evolution, have been observed with, for example, burning hole experiments in organic glasses (see Ref. [35] and references therein). In the context of magnetic disordered systems with explicit quenched disorder like LiHo$_x$Y$_{1-x}$F$_4$, a great effort has been devoted to the study of the quantum phase transition and the dynamics close to the quantum critical point. It would be interesting to perform dynamic measurements in the glassy phase of these compounds to search for aging effects and check if a scenario as the one here discussed with, for example, a two step relaxation of correlation functions and FDT violations is also present.

Another area of research where the relevance of quantum effects in the very low temperature dynamics is currently under study are systems of magnetic nano-particles such as γ-Fe$_2$O$_3$ particles (maghemite) embedded in a silica matrix. The search for a non-stationary evolution in samples at small concentrations (~0.3%) has given a negative result. However, in experiments performed at higher temperatures (where quantum effects are not expected to play an important role) with samples with a larger concentration (~17%) much of the phenomenology of spin-glasses has been recovered. It is then licit to ask whether there is a region in the temperature-concentration parameter-space where the systems exhibit aging driven by quantum fluctuations. In the area of magnetic nano-particles, a notion of an effective temperature greater than the temperature of the environment was advocated in Ref. [74] and it would be interesting to explore the connection with effective temperature defined in this paper.

We have chosen to comment on three kind of materials where effects like the ones obtained in this article might be observed. Surely enough, many other glassy systems with or without explicit disorder might exhibit a similar phenomenology.

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The generating functional \((2.1)\) is more conveniently expressed with the help of path-integrals:

\[
Z[\xi^+, \xi^-] = \int d\phi \langle \phi | T^* \exp \left( -\frac{i}{\hbar} \int_0^\infty dt \xi^-(t) \phi(t) \right) T \exp \left( \frac{i}{\hbar} \int_0^\infty dt \xi^+(t) \phi(t) \right) |\hat{\rho}(0)\rangle \phi \rangle.
\]

(A.1)

The two matrix elements are now expressed as usual Feynman path-integrals

\[
Z[\xi^+, \xi^-] = \int d\phi d\phi_1 d\phi_2 \langle \phi | T^* \exp \left( -\frac{i}{\hbar} \int_0^\infty dt \xi^-(t) \phi(t) \right) |\phi_1\rangle \\
\phi_1 | T \exp \left( \frac{i}{\hbar} \int_0^\infty dt \xi^+(t) \phi(t) \right) |\phi_2\rangle \langle \phi_2 | \hat{\rho}(0) |\phi \rangle. \tag{A.2}
\]

APPENDIX B: QUANTUM FLUCTUATION-DISSIPATION THEOREM

In this appendix we recall the quantum fluctuation-dissipation theorem. Proofs and descriptions of this theorem can be found in several textbooks. We express it in the time-domain in a form that we use in Section V to show that the dynamic equations are compatible with a TTI-FDT ansatz. We also write it in a mixed time-Fourier notation that gives us insight as to how extend it to the case of glassy non-equilibrium dynamics.

If at time \(t'\) the system is characterized by a density functional \(\rho(t')\), the two-time correlation functions read

\[
C_{AB}(t, t') \equiv \langle A(t) B(t') \rangle = \frac{1}{Z} \text{Tr} [A(t) B(t') \rho(t')] \tag{B.1}
\]

where the time-dependent operators, in the Heisenberg representation, are defined as

\[
O(t) \equiv \exp \left( \frac{iHt}{\hbar} \right) O(0) \exp \left( -\frac{iHt}{\hbar} \right). \tag{B.2}
\]
The trace is defined in the usual way, \( \text{Tr}[\bullet] \equiv \sum_{\alpha} \langle \psi_{\alpha} | \bullet | \psi_{\alpha} \rangle \), with \( \{\psi_{\alpha}\} \) an orthonormal basis in Fock space. The normalization is given by \( Z \equiv \text{Tr}[\rho'(t')] \).

Since operators do not commute, the quantum two-time auto-correlation functions are not symmetric in times

\[
C_{AA}(t, t') = \langle A(t)A(t') \rangle \neq \langle A(t')A(t) \rangle .
\]

One thus defines the symmetrized and antisymmetrized correlation functions:

\[
C_{\{A,B\}}(t, t') = \frac{1}{2} \langle A(t)B(t') + B(t')A(t) \rangle ,
\]

\[
C_{[A,B]}(t, t') = \frac{1}{2} \langle A(t)B(t') - B(t')A(t) \rangle ,
\]

respectively.

In linear response theory \( R_{AB}(t, t') \) (see Eq. (2.8) for the definition) and the correlation \( C_{[A,B]}(t, t') \) are related by the Kubo formula

\[
R_{AB}(t, t') = \frac{i}{\hbar} \theta(t - t') \langle [A(t), B(t')] \rangle = \frac{2i}{\hbar} \theta(t - t') C_{[A,B]}(t, t') .
\]

If the system has reached equilibrium with a heat-bath at temperature \( T \) at time \( t' \), the density functional \( \rho(t') \) is just the Boltzmann factor \( \exp(-\beta H) \). It is then immediate to show that, in equilibrium, time-translation invariance holds

\[
C_{AB}(t, t') = C_{AB}(t - t') .
\]

In addition, Eq. (B.3) with \( \rho = \exp(-\beta H) \) imply the KMS condition

\[
C_{AB}(t, t') = C_{BA}(t', t + i\beta \hbar) = C_{BA}(-t - i\beta \hbar, -t') .
\]

Using now the KMS properties and assuming, for definiteness, that \( t > 0 \) it is easy to verify the following equation

\[
C_{\{A,B\}}(\tau) + \frac{i\hbar}{2} R_{AB}(\tau) = C_{\{A,B\}}(\tau^*) - \frac{i\hbar}{2} R_{AB}(\tau^*) ,
\]

where \( \tau = t + i\beta \hbar/2 \). This is a way to express FDT through an analytic continuation to complex times that we use in Section V to show that a TTI and FDT solution holds in the paramagnetic phase.

In terms of the Fourier transformed \( C_{AB}(\omega) \) defined by

\[
C_{AB}(t - t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \exp(-i\omega(t - t')) C_{AB}(\omega)
\]

the KMS relations read

\[
C_{AB}(\omega) = \exp(\beta \hbar \omega) C_{BA}(-\omega)
\]

and lead to the following relation between Fourier transforms of the correlation functions:
\[2C_{[A,B]}(\omega) = (1 - \exp(-\beta h\omega)) C_{AB}(\omega),
\]
\[2C_{(A,B)}(\omega) = (1 + \exp(-\beta h\omega)) C_{AB}(\omega),
\]
\[C_{[A,B]}(\omega) = \tanh \left( \frac{\beta h\omega}{2} \right) C_{\{A,B\}}(\omega).
\] (B.11)

Back in Eq. (B.3) this implies FDT
\[R_{AB}(t - t') = \frac{i}{\hbar} \theta(t - t') \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \exp(-i\omega(t - t')) \tanh \left( \frac{\beta h\omega}{2} \right) C_{\{A,B\}}(\omega)
\] (B.12)

Using
\[\int_{0}^{\infty} dt \exp(i\omega t) = \lim_{\epsilon \to 0^+} \frac{i}{\omega + i\epsilon} = \pi \delta(\omega) + iP_{\omega}
\] (B.13)
one has
\[R_{AB}(\omega) = -\frac{1}{\hbar} \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{1}{\omega - \omega' + i\epsilon} \tanh \left( \frac{\beta h\omega'}{2} \right) C_{\{A,B\}}(\omega')
\] (B.14)

from which we obtain the real and imaginary relations
\[\text{Im} R_{AB}(\omega) = \frac{1}{\hbar} \tanh \left( \frac{\beta h\omega}{2} \right) C_{\{A,B\}}(\omega'),
\]
\[\text{Re} R_{AB}(\omega) = -\frac{1}{\hbar} P \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{1}{\omega - \omega'} \tanh \left( \frac{\beta h\omega'}{2} \right) C_{\{A,B\}}(\omega').
\] (B.15)

If \(\beta h\omega/2 \ll 1\), \(\tanh(\beta h\omega/2) \sim \beta h\omega/2\) and Eq. (B.12) becomes the classical FDT:
\[R_{AB}(\tau) = -\frac{1}{T} \frac{dC_{AB}(\tau)}{d\tau}
\] (B.16)

with \(\tau = t - t'\).

In the zero temperature limit, and for \(A = B\), FDT reads
\[R(\omega) = -\frac{2}{\hbar} \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{1}{\omega - \omega' + i\epsilon} \text{sign}(\omega') C(\omega')
\]
\[R(t) = \frac{2}{\hbar} \theta(t) \int_{0}^{\infty} \frac{d\omega}{\pi} \sin(\omega t) C(\omega)
\] (B.17)

A finite integrated response
\[\int_{0}^{\infty} dt R(t) = R(\omega = 0) = \frac{2}{\hbar} \int_{0}^{\infty} \frac{d\omega'}{\pi} C(\omega') \frac{C(\omega')}{\omega'}
\] (B.18)

implies \(C(\omega = 0) = 0\). In the glassy model we study in the paper, below \(\hbar c(T_c)\), the response and correlation turn out to be the sum of two contributions. One is a stationary part that satisfies FDT. Since one expects it to yield a finite contribution to the susceptibility \(\chi(\tau + t_w, t_w)\), one must have:
\[\int_{0}^{\infty} d\tau R_{st}(\tau) \text{ Finite} \Rightarrow \int_{0}^{\infty} d\tau C_{st}(\tau) = 0.
\] (B.19)

Thus, the stationary part of the correlation function oscillates around zero or, in other words, the full correlation oscillates around \(q\) at zero temperature. This is observed in the numerical solution to the dynamic equations.
APPENDIX C: THE CLASSICAL LIMIT

The classical limit can be checked at every stage of the calculation. We shall here show how one can recover (i) the Martin-Siggia-Rose (MSR) action from the Keldysh-Shwinger (KS) action (3.2); (ii) the equations of motion, for the classical correlation and response from the equations of motion for their quantum partners.

This Appendix is intended for readers already familiar with the MSR formalism. We refer to Ref. [36] for a detailed description of the method.

Let us identify, in the limit \( \beta \hbar \to 0 \), the quantum linear combinations on the left with the classical variables on the right

\[
\sigma^+(t) - \sigma^-(t) \to \hat{s}(t), \quad \sigma^+(t) \to s(t) + \frac{\hbar}{2} \dot{s}(t), \\
\sigma^+(t) + \sigma^-(t) \to 2s(t), \quad \sigma^-(t) \to s(t) - \frac{\hbar}{2} \dot{s}(t).
\]  
(C.1)

In the classical limit \( \hbar \to 0 \), and when the cut-off tends to infinity, the kernels \( \eta \) and \( \nu \) become

\[
4\eta(t-t') = 4M\theta(t-t')\gamma_o\delta'(t-t'), \\
2\hbar \nu(t-t') = 4M\gamma_o k_BT\delta(t-t').
\]  
(C.2)

With these identifications, we establish a connection between the KS and the MSR actions.

The friction and thermal terms become

\[
\frac{\sigma^+(t) - \sigma^-(t)}{\hbar} \to \dot{s}(t), \quad \frac{\sigma^+(t) + \sigma^-(t)}{2} \to s(t), \quad \sigma^-(t) \to s(t) - \frac{\hbar}{2} \dot{s}(t).
\]  
(C.3)

By calling \( \Gamma_o^{-1} = 2M\gamma_o \) we recover two terms of the MSR action.

The kinetic and constraint terms reduce to

\[
\lim_{\hbar \to 0} \frac{i}{\hbar} S_0 = -2M\gamma_o \int_0^\infty dt i\dot{s}(t) d^2t s(t) + 2k_BT\gamma_o \int_0^\infty dt (i\dot{s}(t))^2.
\]  
(C.4)

By calling \( z^+(t) - z^-(t) = z_o(t) \) and \( z^+(t) + z^-(t) = z(t) \) the last two terms impose the spherical constraint and are usually included in the path-integral measure, while the second term is the one left in the MSR action:

\[
\frac{i}{\hbar} \int_0^\infty dt \frac{V}{\sigma^+} - \frac{V}{\sigma^-} = -\int_0^\infty dt \frac{i}{\hbar} \dot{s}(t)(m\partial_t^2 + z(t)) s(t).
\]  
(C.5)

Finally, the terms that depend upon disorder become

\[
\lim_{\hbar \to 0} \frac{i}{\hbar} \int_0^\infty dt V[\sigma^+, J] - V[\sigma^-, J] = -\int_0^\infty dt \frac{i}{\hbar} \frac{\delta V}{\delta s} s(t).
\]  
(C.6)
Putting everything together, in the large friction limit \( \Gamma_o^{-1} \equiv 2M\gamma_o \gg 1 \) such that one neglects the second derivative terms in the kinetic part of the action, one recovers the usual MSR action for the Langevin dynamics of the classical model in a thermal bath at temperature \( T \):

\[
\lim_{\bar{\hbar} \to 0} \frac{i}{\bar{\hbar}} S_{KS} = - \int dt \left[ i\hat{s}(t) \left( \Gamma_o^{-1} \frac{d}{dt} + z(t) \right) s(t) - k_B T \Gamma_o^{-1} (i\hat{s}(t))^2 + \frac{\delta V[s,J]}{\delta s(t)} i\hat{s}(t) \right].
\]

(C.7)

In order to study the classical limit of the dynamic equations (3.30) and (3.31) we recognize the classical limit of the correlation and response as follows:

\[
C(t,t') = \frac{1}{2} \langle \sigma^+(t) \sigma^-(t') + \sigma^-(t) \sigma^+(t') \rangle \to \langle s(t)s(t') \rangle,
\]

\[
R(t,t') = \frac{i}{\bar{\hbar}} \langle \sigma^+(t) \sigma^-(t') + \sigma^+(t) \sigma^-(t') \rangle \to \langle s(t)i\hat{s}(t') \rangle.
\]

(C.8)

In a similar way, it is now easy to verify that the classical dynamic equation for the \( p \)-spin model follow from the \( \bar{\hbar} \to 0 \) limit of Eqs. (3.30) and (3.31).

**APPENDIX D: TREATMENT OF INTEGRALS**

Integrals over time windows spanning the interval \([0, t] \) appear in the dynamic equations (3.30) and (3.31). In the large times limit we approximate them in the way we here describe.

1. **First type of integral**

Integrals of the form

\[
I_1(t) \equiv \int_0^t dt'' A(t,t'')B(t,t'')
\]

appear, for example, in \( z(t) \). The idea is to separate the integration time-interval as

\[
[0, t] = [0, \delta] \ U \ [\delta, t^-] \ U \ [t^-, t].
\]

If \( \delta \) is chosen to be a finite time, all functions can be approximated by \( A(t,0) \) that vanishes when \( t \to \infty \). Since the integration interval is finite, this term can be neglected. In the second interval the functions vary in the aging regime and in the third interval they vary in the stationary regime. Thus

\[
I_1(t) \sim \int_0^\delta dt'' A(t,t'')B(t,t'') + \int_\delta^t dt'' A_{AG}(t,t'')B_{AG}(t,t'')
\]

\[
+ \int_{t^-}^t dt'' \left[ \left( A_{ST}(t-t'') + \lim_{t-t'' \to \infty} \lim_{t'' \to \infty} A(t,t'') \right) \left( B_{ST}(t-t'') + \lim_{t-t'' \to \infty} \lim_{t'' \to \infty} B(t,t'') \right) \right].
\]

(D.3)
We assume that this separation is sharp and that we can neglect the corrections associated to mixing of the three regimes. In the third term we replaced \( A \) and \( B \) in terms of \( A_{\text{ST}}, B_{\text{ST}} \).

Since in all cases either \( A \) or \( B \) is proportional to the response, we can neglect the first term using the property (4.11) of the weak long-term memory scenario. We can then replace the lower limit of the second integral by 0 and the upper limit of the second integral by \( t \). In addition, assuming that \( B \) is proportional to the response,

\[
\lim_{t \to t''} \lim_{t' \to \infty} B(t, t'') = 0 .
\]  

(4.4)

This yields

\[
I_1(t) \sim \int_0^t dt'' A_{\text{AG}}(t, t'') B_{\text{AG}}(t, t''') + \left( \lim_{t \to t''} \lim_{t' \to \infty} A(t, t''') \right) \int_0^{t-t''} d\tau' B_{\text{ST}}(\tau') + \int_0^{t-t''} d\tau' A_{\text{ST}}(\tau') B_{\text{ST}}(\tau') .
\]  

(4.5)

2. Second type of integral

Another type of integrals is:

\[
I_2(t, t') \equiv \int_t^{t'} dt'' A(t, t'') B(t'', t') .
\]  

(4.6)

In particular, if \( B = 1 \), \( A(t, t'') = R(t, t'') \) and \( t' = 0 \), this integrals yields the susceptibility. Let us assume that \( t \) and \( t' \) are far apart; we start as for the first type of integrals, by dividing the time interval in three subintervals

\[
[t', t] = [t', t^+] \ U \ [t^+, t^-] \ U \ [t^-, t]
\]  

(4.7)

and by approximating the functions by their functional form inside each of the intervals:

\[
I_2(t, t') \sim \int_{t'}^{t^+} dt'' A_{\text{AG}}(t, t'') B(t'', t') + \int_{t^+}^{t} dt'' A_{\text{AG}}(t, t'') B_{\text{AG}}(t'', t') + \int_{t}^{t^-} dt'' A(t - t'') B_{\text{AG}}(t'', t') .
\]  

(4.8)

In the first term \( B(t - t'') \) can be replaced by \( B(t - t'') = B_{\text{ST}}(t - t'') + \lim_{t-t'' \to \infty} \lim_{t'' \to \infty} B(t, t'') \). The same applies to \( A \) in the last term. All functions vary fast in the stationary regime but very slowly in the aging regime. The next assumption is that functions in the aging regime that are convoluted with functions in the stationary regime, can be considered to be constant and taken out of the integral. That is to say

\[
I_2(t, t') \sim A_{\text{AG}}(t, t') \int_{t'}^{t^+} dt'' B(t'' - t') + \int_{t'}^{t} dt'' A_{\text{AG}}(t, t'') B_{\text{AG}}(t'', t') + B_{\text{AG}}(t, t') \int_{t^-}^{t} dt'' A(t - t'')
\]

\[
\sim A_{\text{AG}}(t, t') \int_{0}^{\infty} d\tau' B(\tau') + \int_{0}^{t} dt'' A_{\text{AG}}(t, t'') B_{\text{AG}}(t'', t') + B_{\text{AG}}(t, t') \int_{0}^{\infty} d\tau' A(\tau') ,
\]  

(4.9)
where we used $t - t^- \to \infty$ and $t'^+ - t' \to \infty$. Typically, $A(\tau')$ and $B(\tau')$ are proportional to the response function. Using FDT the integrals in the first and third term can then be performed in the classical limit or they can be expressed as functions of the correlation in the quantum case. The second term instead depends exclusively on the aging dynamic sector.

All other integrals can be evaluated, in the large-time limit, in a similar way.
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