Dynamical Systems approach to Saffman-Taylor fingering. 
A Dynamical Solvability Scenario

E. Pauné, F.X. Magdaleno and J. Casademunt 
Departament d’Estructura i Constituents de la Matèria 
Universitat de Barcelona, Av. Diagonal, 647, E-08028-Barcelona, Spain 

A dynamical systems approach to competition of Saffman-Taylor fingers in a Hele-Shaw channel is developed. This is based on the global study of the phase space structure of the low-dimensional ODE’s defined by the classes of exact solutions of the problem without surface tension. Some simple examples are studied in detail. A general proof of the existence of finite-time singularities for broad classes of solutions is given. Solutions leading to finite-time interface pinch-off are also identified. The existence of a continuum of multifinger fixed points and its dynamical implications are discussed. The main conclusion is that exact zero-surface tension solutions taken in a global sense as families of trajectories in phase space spanning a sufficiently large set of initial conditions, are unphysical because the multilinger fixed points are nonhyperbolic, and an unfolding of them does not exist within the same class of solutions. Hyperbolicity (saddle-point structure) of the multifinger fixed points is argued to be essential to the physically correct qualitative description of finger competition. The restoring of hyperbolicity by surface tension is discussed as the key point for a generic Dynamical Solvability Scenario which is proposed for a general context of interfacial pattern selection.

I. INTRODUCTION

The Saffman-Taylor (ST) problem \cite{24} has played a central role for several decades as a prototype system in the study of interfacial pattern formation \cite{11}, particularly concerning the issue of pattern selection \cite{12,13}. Despite its elongated existence, the problem continues to pose new challenges with the focus now on its dynamical aspects \cite{15}. In this sense, the ST problem is becoming instrumental once more in gaining insights into possibly generic behavior, due to its relative simplicity in the context of morphologically unstable interfaces in nonequilibrium systems.

Full understanding of the analytical mechanisms leading to steady state selection by surface tension as a singular perturbation in the problem was not completely achieved until the late eighties \cite{10,13} and the resulting scenario, usually referred to as Microscopic Solvability (MS) \cite{9,10}, has currently become a paradigm for many other systems for instance in free dendritic growth \cite{1,3,23}. Such solvability analysis, however, is strictly static, in the sense that it is concerned with the existence and linear stability of stationary solutions. The importance of dynamics in the process of selection was pointed out in Refs. \cite{26,27} where it was argued that the Saffman-Taylor finger solution was not the universal attractor of the problem if the displacing fluid has a non-negligible viscosity. More recently, the traditional MS scenario of selection has not been exempt of some controversy in connection with the dynamics of the zero surface tension problem \cite{28,34}. The singular effects of surface tension on the dynamics have been pointed out as a rather subtle and challenging issue \cite{35,37} and the possibility of some extension of the MS scenario of selection to the dynamics has been suggested \cite{12,38,24}. In any case, the study of the dynamics of morphologically unstable interfaces in the context of Laplacian growth or, more generally, of diffusion-limited growth of interfaces in nonequilibrium conditions, has been rather elusive to analytical treatment due to the highly nonlinear and nonlocal character of the equations. Simplified models of Laplacian growth in the absence of surface tension have thus been studied in some detail \cite{10,13} and very recently a new impulse to the problem of Laplacian growth without surface tension, with focus on DLA-like self-similar growth, is developing after new analytical insights \cite{12} leading to somewhat controversial conclusions \cite{10}. For the viscous fingering problem, however, the basic question remains as to what extent the zero surface tension problem does capture the physics of the fingering dynamics. The direct motivation of this question is the existence of large classes of explicit time dependent solutions of the zero surface tension which are nonsingular and span a great variety of possible interface morphologies. To what extent these solutions are qualitatively or quantitatively describing the dynamical behavior or real systems is an outstanding, nontrivial question. As we will see, in some cases they may evolve to the correct asymptotic solution with the wrong dynamics. In others, despite the fact that the solutions are unstable to arbitrary perturbations, they may accurately describe the time evolution of systems with nonzero surface tension. The difficulty of the question lies to a large extent in the precise way to formulate it.

The present paper expands and elaborates in depth several aspects which were first pointed out in Ref. \cite{38}. Our central objective is to contribute to elucidate the role
of surface tension in the dynamics of finger competition of the ST problem in the channel geometry, beyond the obvious role of keeping the interface stable and free of singularities and the selection of a stationary state. That is to elucidate the qualitative differences between the nonsingular solutions of the zero surface tension problem and the real problem with a possibly small but nonzero surface tension. Our main contribution is the development of a new approach to the issue in terms of the ideas and concepts of the theory of Dynamical Systems (DS). With this general point of view, we will study in detail some specific classes of solutions of the zero surface tension problem. As a byproduct, this will provide some interesting results in the context of Laplacian growth concerning the interplay of finger widths and screening effects. However the most important issue will be the discussion of the physical deficiencies of the nonsingular solutions of the zero surface tension problem. This will be addressed with the aim at the maximum possible generality. As we will see, the comparison of the problem with and without surface tension is necessarily qualitative in nature, so it is important to pose questions in a framework which is at the same time qualitative and mathematically precise. Such framework is the theory of Dynamical Systems. The use of this conceptual tool will help us formulate precise questions to which we can give an answer. From the above results and within this spirit we will reformulate the issue of a possible extension to dynamics of the MS scenario of steady state selection, and suggest a possible answer to that.

Common understanding of the finger competition process (sometimes referred to as finger coalescence) leading to the selected steady state is usually based on qualitative screening arguments. In some cases these have been shown to be too naive, in accord to the recent findings of stationary finger solutions with coexisting unequal fingers, as we will discuss in more detail below. In a first effort to develop precise conceptual tools for a quantitative and qualitative characterization of fingering dynamics, a topological approach was developed in Refs. to address the nontrivial effects of viscosity contrast in the dynamics of finger competition and their long time asymptotics. The basic insight was to focus on topological properties of the physical velocity field in the bulk, quantified by the existence of topological defects. In this paper we will develop a different global viewpoint, which focuses on topological properties of the flow in the (infinite-dimensional) phase space of interface configurations, rather than the (two-dimensional) velocity field.

In this paper we will analyze in detail some exact solutions of the Saffman-Taylor problem with zero surface tension. Hereinafter we will refer to this case as the idealized problem, as opposed to the regularized one, which will denote the problem with a small but finite surface tension. Is is known that the idealized ST problem is ill-posed as an initial-value problem. Nevertheless, one of the crucial facts that makes the ST problem attractive from an analytical point of view is the existence of rather broad classes of exact time-dependent solutions of the problem without surface tension. Some classes of those solutions are known to develop finite-time singularities in the form of cusps and are thus of no interest to the physics of viscous fingering, but still a remarkably broad class of solutions is free of singularities and therefore physically acceptable, in principle. The basic question is then what would be the effect of a small but finite surface tension to those solutions. This question was first raised in Ref. where it was shown that for some classes of initial conditions, the effect of surface tension as a perturbation could be considered as basically regular, while for other initial conditions the singular character of the perturbation showed up dramatically in the dynamics. In other configurations, such as for circular geometry, surface tension has also been shown to behave as a regular perturbation. Indeed, in view of the morphological diversity which is included in the known nonsingular solutions, one may be tempted to believe that, since such solutions remain smooth for all the time evolution, they should stay close to the solutions of the regularized problem as $d_0 \to 0$. Siegel and Tanveer have shown that, contrarily to what happens in other more familiar singular perturbations in physics, in the case of Hele-Shaw flows that is not the case, and, in general, the idealized and the regularized solutions differ from each other at order one time. In the remarkable contribution of Refs., however, only simple examples of single-finger evolutions are considered, so the extent to which those conclusions can be extrapolated to multifinger configurations still requires a careful analysis. Furthermore, even though the idealized and the regularized solutions differ significantly after a time of order unity which is basically independent of surface tension, one could still argue that the qualitative evolution is basically unaffected by surface tension if the finger width is not too different from the selected one in the regularized case. Therefore, the possibility that some classes of solutions or some particular dynamic mechanisms are basically insensitive to surface tension remains open. This question directly motivated the work of Ref. which first made use of a dynamical systems approach and which we will here pursue further. The crucial aspect to be exploited is the fact that the integrable classes of initial conditions define finite-dimensional invariant manifolds of the full (infinite-dimensional) problem, so it makes sense to study the resulting low-dimensional dynamical systems and compare them with properly defined finite-dimensional subsets of the regularized problem. With this analysis we will clarify in what precise sense the nonsingular exact solutions of the idealized ST problem are, in general, unphysical. Once settled the unphysical nature of a broad class of solutions, a natural question to address in whether a selection principle is associated to the surface tension regularization, which can be understood as a dynamical generalization of the MS scenario. We will address this point in the light of our results and discuss how and in what sense such dynamical MS could be formulated.
The rest of the paper is organized as follows. In Sect. II the equations describing Hele-Shaw flows in channel geometry are recalled, together with the conformal mapping formulation used to study the problem. The finger competition phenomenon is described and the relevant quantities to characterize finger competition are presented. In Sect. III the dynamical systems approach to the problem is introduced. In Sect. IV the minimal class presented in Ref. [28] is revisited. The dynamics of this class, namely, its phase portrait is studied in detail. The comparison of this phase portrait with the known topology of the physical problem reveals that the minimal class dynamics is unphysical. The main reason for this unphysical behavior is the existence of a continuum of fixed points. In Sect. V and VI various generalizations of the minimal class are introduced. We pay a special attention to the perturbation of the minimal class that removes the continuum of fixed points but keeps the dimensionality of the phase space unchanged. These solutions contain the main general features of zero surface tension, and it is shown that they do not describe in general the correct dynamics for finite surface tension. The main reason is that the dynamical system that they define lacks the saddle-point structure of the multifinger fixed point, necessary to account for the observed finite surface tension. In Sect. VII we discuss the precise role of zero surface tension solutions and their relevance to an understanding of the dynamics of Hele-Shaw flows. In this section a Dynamical Solvability Scenario is proposed and discussed as a generalization of MS theory. Finally, in Sect. VIII we summarize our main results and conclusions.

II. FORMULATION OF THE PROBLEM AND CHARACTERIZATION OF FINGER COMPETITION

Consider a Hele-Shaw cell of width \( W \) in the \( y \)-direction and infinite length in the \( x \)-direction, with a small gap \( b \) between the plates. The fluid flow in this system is effectively two dimensional and the velocity \( \mathbf{v} \) obeys Darcy’s law

\[
\mathbf{v} = -\frac{b^2}{12\mu} \nabla p
\]

(1)

where \( p \) is the fluid pressure and \( \mu \) is the viscosity. We define a velocity potential \( \varphi = -\frac{b^2}{12\mu} p \), and assuming that the fluid is incompressible (\( \nabla \cdot \mathbf{v} = 0 \)) we obtain the bulk equation to be the Laplace equation

\[
\nabla^2 \varphi = 0.
\]

(2)

This must be supplemented with the two boundary conditions

\[
\varphi|_\Gamma = d_0 \kappa|_\Gamma
\]

(3)

where \( \Gamma \) means that the quantity is evaluated on the interface, \( v_n \) is the normal component of the velocity of the interface, \( \kappa \) is the curvature, \( \mathbf{n} \) is the unit vector normal to the interface and \( d_0 \) is a dimensionless surface tension defined by \( d_0 = \frac{\pi^2 \kappa}{12 \mu V_\infty^2 W} \), where \( V_\infty \) is the fluid velocity at infinity. Eq. (3) is the Laplace pressure jump at the interface due to local equilibrium, written in terms of the velocity potential. Here we consider the case with the non-viscous fluid displacing the viscous one, therefore the pressure in the non-viscous fluid is constant, and we choose it to be zero. Eq. (4) is the continuity condition, that states that the interface follows the motion of the fluid. We assume periodicity at the sidewalls of the channel. Except for configurations symmetric with respect to the center axis of the channel, periodic boundary conditions define different dynamics from the more physical case of rigid sidewalls (with no-flux through them).

Strictly speaking our case describes an infinite periodic array of unit channels. We will argue that nothing essential is lost with respect to competition in a rigid-wall channel, while the analysis is significantly simplified.

We use conformal mapping techniques to formulate the problem [2]. We define a function \( f(\omega, t) \) that conformally maps the interior of the unit circle in the complex plane \( \omega \) into the viscous fluid in the physical plane \( z = x + iy \). We assume an infinite channel in the \( x \)-direction. The mapping \( f(\omega, t) \) must satisfy \( \partial_x f(\omega, t) \neq 0 \) inside the unit circle, \( |\omega| \leq 1 \), and moreover, \( h(\omega, t) = f(\omega, t) + \ln \omega \) must be analytic in the interior. We define the complex potential as the analytic function \( \Phi = \varphi + i \psi \), where the harmonic conjugate \( \psi \) of \( \varphi \) is the stream function. The width of the channel is \( W = 2\pi \) and the velocity of the fluid at infinity is \( V_\infty = 1 \). It can be shown that the evolution equation for the mapping \( f(\omega, t) \) reads

\[
\partial_t f(\omega, t) = \omega \partial_\omega f(\omega, t) A \left( \frac{\text{Re}(i \partial_\omega \Phi(e^{i\phi}, t))}{|\partial_\omega f(e^{i\phi}, t)|^2} \right)
\]

(5)

where \( A[g] \) is an integral operator that acts on a real function \( g(\phi) \) defined on the unit circle. \( A[g] \) has the form

\[
A[g] = \frac{1}{2\pi} \int_0^{2\pi} g(\theta) e^{i\theta} + \omega, d\theta,
\]

(6)

and on the unit circle \( \omega = e^{i\phi} \) it reads

\[
A[g]|_{\omega=e^{i\phi}} = g(\phi) + iH_\phi[g]
\]

(7)

where \( H_\phi[g] \) is the so-called Hilbert transform of \( g(\phi) \) defined by

\[
H_\phi[g] = \frac{1}{2\pi} P \int_0^{2\pi} g(\theta) \cot\left(\frac{\phi - \theta}{2}\right) d\theta
\]

(8)
where \( P \) stands for the principal value prescription. The complex potential \( \Phi \) satisfies
\[
\Phi(\omega, t) = -\ln \omega + d_0 A[\kappa]
\]
where the curvature \( \kappa \) given in terms of \( f(\omega, t) \) is
\[
\kappa = -\frac{1}{|\partial_{\omega} f|} \text{Im} \left[ \frac{\partial^2 f}{\partial \omega f} \right].
\]
The evolution equation (5) written on the unit circle \( \omega = e^{i\phi} \) is
\[
\text{Re}\{i\partial_{\phi} f(\phi, t) \partial_t f^*(\phi, t)\} = 1 - d_0 \partial_{\phi} H_0[\kappa]
\]
where \( f(\phi, t) \equiv f(\omega, t) \). In the zero surface tension case \( d_0 = 0 \) the integro-differential equation (11) reduces to a much simpler equation, and the evolution of \( f(\phi, t) \) for \( d_0 = 0 \) is then given by
\[
\text{Re}\{i\partial_{\phi} f(\phi, t) \partial_t f^*(\phi, t)\} = 1.
\]
The direct motivation of the present study is that, despite the fact that neglecting surface tension is in principle incorrect from a physical standpoint, the \( d_0 = 0 \) case can be solved explicitly in many cases \[1,48,50\] including solutions which, although being unstable, they exhibit a smooth and physically acceptable (nonsingular) behavior, quite similar to what is observed in experiments and simulations of the full problem. Before proceeding to the description of the general approach and its application to specific solutions, let us first introduce some ideas and definitions which will be helpful in future discussion. To quantify finger competition it is useful to define individual growth rates of fingers \[38\]. In simple situations like those considered in the paper, the growth rate of a finger can be simply defined (in the reference frame moving with the mean interface position) as the peak-to-peak difference of the stream function between the maximum and the minimum which are adjacent to the zero of the stream function located at (or near) the finger tip \[27\] (the definition can be generalized to more complicated situations). According to this definition, one assigns a nonzero growth rate to the finger if the tip advances at a velocity which is larger than the mean interface position. Looking at individual growth rates one can easily distinguish two different stages of the dynamics in the process of finger competition. A first stage characterized by the monotonously growth of all individual finger growth rates and a second one dominated by the redistribution of the total growth rate among the fingers. We call these two stages growth and competition regimes respectively. For a configuration of two different fingers, which is practically the only one addressed throughout this paper, during the growth regime the two fingers develop from small bumps of the initially flat interface, while the total growth rate \( \Delta \psi_T(t) \), defined as \( \Delta \psi_T(t) = \Delta \psi_1(t) + \Delta \psi_2(t) \), grows until it reaches a value close to its asymptotic one \( \Delta \psi_T(\infty) \). The decrease of the growth rate of one of the fingers signals the outcome of the competition regime: there is a redistribution of flux from one finger to the other one. We also define the existence of successful competition as the ability to completely suppress the growth rate of one finger. A finger is dynamically suppressed of the competition process when its growth rate \( \Delta \psi \) is reduced to zero.

### III. DYNAMICAL SYSTEMS APPROACH

The theory of Dynamical Systems is a mathematical discipline which studies ordinary differential equations or flows (and also difference equations or maps) with stress on geometrical and topological properties of solutions \[36\]. The approach is sometimes referred to as qualitative theory of differential equations. The focus is not on the study of individual solutions or trajectories of the differential equation, but on global properties of families of solutions. This point of view has become very fruitful in searching for universality in the context of nonlinear phenomena. A dynamical systems approach seems thus appropriate to study in a mathematically precise way, the qualitative properties of the dynamics of our problem, and the qualitative differences associated to the presence or absence of surface tension. One of the important concepts in dynamical systems theory is that of structural stability, which captures the physically reasonable requirement of robustness of the mathematical description to slight changes in the equations. Roughly speaking, a system is said to be structurally stable if slight perturbations of the equations yield a topologically equivalent phase space flow. Although the structural stability 'dogma' must be taken with some caution \[29\], a structurally unstable description of a physical problem must be seen in principle as suspect. When a dynamical system (DS) depends on a set of parameters, the bifurcation set is defined as those points in parameter space where it is structurally unstable. In this case the structural instability at an isolated point in parameter space is the property necessary for the system to change its qualitative behavior. At a bifurcation point, adding perturbations to the equations to make the system structurally stable is called an unfolding \[56\]. For dimension higher than two, the mathematical definition of structural stability is usually too stringent. For the purposes of the present discussion and most physical applications it is sufficient to consider the notion of hyperbolicity of fixed points, which in 2 dimensions is directly associated to structural stability through the Peixoto theorem \[57\]. A fixed point is hyperbolic when the linearized flow has no marginal directions, that is, all eigenvalues of the linearized dynamics are nonzero. We will see that the non-hyperbolicity of the double-finger fixed point (in general the \( n \)-equal-finger fixed point) and the non-existence of an unfolding of it within the known class of solutions is at the heart of the unphysical nature of this class of
solutions.

In the approach to the Saffman-Taylor problem with concepts of DS theory, there is, however, an important additional difficulty in the fact that our problem is infinite-dimensional and unbounded. In similar spatially extended systems, such as described by PDE’s, it is customary to project the dynamics onto effective low-dimensional dynamical systems based on the so-called center manifold reduction theorem \[54\]. This is possible near the instability threshold and, for truly early transient \[57\]. All the above techniques are thus of no use for our purpose of studying the strongly nonlinear dynamics of competing fingers in their way to the ST stationary solution.

The basic point that we will exploit here to gain some analytical insight into the dynamics of the ST problems as a dynamical system is the fact that all exact solutions known explicitly for the idealized problem \((d_0 = 0)\) are defined in terms of ODE’s for a finite number of parameters, and thus define finite-dimensional DS’s in the phase space defined by those parameters \[53\]. The complete ST problem, for any finite \(d_0\), defines a DS in an infinite dimensional phase space. We will refer to this DS as \(S^\infty(d_0)\). The limit \(d_0 \rightarrow 0\) defines a limiting DS which we will refer to as \(S^\infty(0^+)\), which, as we will see, is different from \(S^\infty(0)\).

The conformal mapping \(f(\omega, t)\) of the reference unit disk in the complex \(\omega\)-plane into the physical region occupied by the viscous fluid \(z = x + iy = f(\omega)\) has the form

\[
f(\omega, t) = -\ln \omega + h(\omega, t)
\]

where \(h(\omega, t)\) is an analytic function in the whole unit disk, and therefore has a Taylor expansion

\[
h(\omega, t) = \sum_{k=0}^{\infty} a_k(t)\omega^k
\]

which is convergent in the whole unit disk. Inserting Eq.\([14]\) into the equation for the mapping \(f(\omega, t)\) we find an infinite set of equations of the form

\[
\dot{a}_k = g_k(a_0, \ldots, a_k; d_0).
\]

In the co-moving frame of reference the precise form of the infinite set of equations \([15]\) is

\[
\dot{a}_0 = C_0
\]

\[
\dot{a}_k = C_k - kC_0a_k - \sum_{j=0}^{k-1} ja_jC_{k-j}
\]

where

\[
C_0 = \frac{1}{2\pi} \int_0^{2\pi} \nu(\theta, t) d\theta
\]

\[
C_k = \frac{1}{2\pi} \int_0^{2\pi} \nu(\theta, t) e^{ik\theta} d\theta
\]

and

\[
\nu(\theta, t) = \frac{\text{Re}[\omega \partial_\omega h(\omega, t)] - d_0 \text{Re}[\omega \partial_\omega A(\omega, t)]}{|\omega \partial_\omega f(\omega, t)|^2}_{\omega = e^{i\theta}}
\]

The infinite set of equations \([15]\) defines the DS \(S^\infty(d_0)\). In the special case of strictly zero surface tension, the DS \(S^\infty(0)\) can be explicitly solved for some classes of initial conditions. These classes define invariant manifolds of \(S^\infty(0)\) of finite dimension. In this context, finding explicit solutions implies identifying a specific analytic structure of \(h(\omega)\), with a finite number of parameters, which is preserved under the time evolution. If this condition is fulfilled, then a set of ODE’s for those parameters can be closed, and defines a certain DS on a finite-dimension space. For instance, for \(d_0 = 0\) the truncation of \(h(\omega)\) into a polynomial form is preserved by the time evolution, so Eqs.\([15]\) themselves remain a finite set of ODE’s. This simple case, however, is known to lead to finite-time singularities. The evolution is in general not defined after some finite time and cannot be considered as sufficiently well behaved as a DS’s. On the other hand, classes of solutions have been reported which are smooth (non-singular) for all the time evolution. The corresponding conformal mapping takes the general form

\[
h(\omega) = d(t) + \sum_{j=1}^{N} \gamma_j \ln(1 - \alpha_j(t)\omega)
\]

where \(\gamma_j\) are constants of motion with the restriction \(\sum_{j=1}^{N} \gamma_j = 2(1 - \lambda)\) where \(\lambda\) is the asymptotic filling fraction of the channel occupied by fingers. If all \(\gamma_j\) are real the evolution is free of finite-time singularities, and if any \(\gamma_j\) has an imaginary part then finite-time singularities may appear for some set of initial conditions (see Sect. \([57]\)). Although this form of the mapping contains all orders of the Taylor expansion of \(h(\omega)\), it defines finite dimension invariant manifolds, since the superposition of logarithmic terms of Eq.\([21]\) is preserved under the dynamics, this means that a closed set of ODE’s for the finite number of parameters \(\alpha_j(t)\) can be found. In addition, the region which is physically meaningful is that in which \(|\alpha_j| \leq 1\) (including the equal sign allows for the limiting case of infinite fingers, and makes the phase space compact). The DS defined by Eq.\([21]\) in the 2N-dimensional hyper-volume will be denoted as \(L_{2N}(\{\gamma_j\})\).

For the sake of discussion throughout this paper it is important to have in mind that modifying parameters \(\{\gamma_j\}\), which are constants of motion under the dynamics defined through Eq.\([12]\) corresponds to varying initial conditions in the phase space of \(S^\infty(0)\), while, from
the standpoint of the finite-dimensional DS's denoted by \( L^{2\mathcal{N}}(\gamma_j) \) it corresponds to changing the DS itself, that is, changing the ODE's obeyed by the dynamical variables. In this sense, \( \{\gamma_j\} \) label a set of DS's defined on a 2N-hyper-volume \(|\alpha_j| \leq 1\).

Following Ref. [38], the key idea is to look for the simplest of the DS defined above which contains the three physically relevant fixed points, namely, the planar interface (PI), the single finger ST solution (1ST) and the double Saffman-Taylor finger solution (2ST). We will call this minimal DS as \( L^2(\lambda, 0) \) or simply \( L^2(\lambda) \), since it has only one constant of motion, namely \( \lambda \). In Ref. [38] we proposed to compare the global flow properties in phase space of such DS with those of a corresponding two-dimensional dynamical system defined by the regularized problem. The latter was obtained by restricting \( S^\infty(d_0) \) to a one-dimensional set of initial conditions properly chosen in such a way that the invariant manifold of \( S^\infty(0) \) which defines \( L^2(\lambda) \) was tangent to \( S^\infty(0^+) \) at the PI fixed point \( L^2 d_0 \). The resulting DS \( S^2(d_0) \) was then shown to have a topological structure nonequivalent to that of \( L^2(\lambda) \). In particular in the limiting case, \( S^2(0^+) \) intersected \( L^2(1/2) \) not only at PI but also at the other fixed points 1ST and 2ST, and furthermore, at the two full trajectories connecting PI with 1ST and 2ST respectively. The basic conclusion was then that the regularized and the idealized problem were intrinsically different.

In this paper we will see that the conclusions drawn from that comparison do hold beyond that ‘minimal model’ analysis. The extent and interpretation to which those conclusions must be understood will be more precisely stated. We will also discuss on the implication of those results on the issue of dynamical selection.

### IV. THE TWO-FINGER MINIMAL MODEL

#### A. The model

The simplest exact time-dependent solution of Eq. [24] containing the three physically relevant fixed points: the planar interface (PI), the single Saffman-Taylor (1ST) fixed point and the double Saffman-Taylor (2ST) fixed point was introduced in Ref. [38] and reads

\[
\hat{f}(\omega, t) = - \ln \omega + d(t) + (1 - \lambda) \ln(1 - \alpha(t)\omega) \\
+ (1 - \lambda) \ln(1 + \alpha(t)\omega) \tag{22}
\]

where \( \lambda \) is a real-valued constant in the interval \([0, 1]\), \( \alpha(t) = \alpha'(t) + i\alpha''(t) \) and \( d(t) \) is real. The relevant phase space for a given \( \lambda \) is the first quadrant of the unit circle in the \((\alpha', \alpha'')\) space. The other three quadrants describe interface configurations that are equal or symmetrical to the interfaces contained in the first quadrant. The interface described by this mapping consists generically of two unequal fingers, axisymmetric and without overhangs. The axisymmetry of fingers simplifies the analysis reducing the number of variables of the problem, but it is important to remark that it plays no role in preventing finger competition, as seen in the regularized problem with identical symmetry. The case \( \alpha'(t) = 0 \) is equivalent to the time-dependent ST finger solution [1], and \( \alpha''(t) = 0 \) corresponds to the double time-dependent ST finger. For \( |\alpha(t)| \ll 1 \) the interface consists of a sinusoidal perturbation of the planar interface. \( d(t) \) affects the overall interface position but does not affect its shape, and it turns out to be irrelevant for the present discussion.

It is convenient to parameterize the phase space using the variables \( u = 1 - \alpha'^2 \) and \( r = (\alpha'^2 + \alpha''^2 - 1)/(\alpha'^2 - 1) \). Then, the new phase space is the square \([0, 1] \times [0, 1]\). With these variables \((u, r)\) the Saffman-Taylor single finger corresponds to the point \((0, 1)\), the double Saffman-Taylor finger corresponds to \((1, 0)\) and the planar interface to \((1, 1)\). Substituting the ansatz Eq. (22) in Eq. (23) we obtain that the temporal evolution in the new variables is given by

\[
\dot{u} = 2ru(1 - u) \frac{3r - 4 - gr(1 - ru)}{1 + gT_g(u, r)} \tag{23}
\]

\[
\dot{r} = 2r(1 - r) \frac{3r - 2(1 + ru) + g(1 - ru)(2 - r)}{1 + gT_g(u, r)} \tag{24}
\]

where

\[
T_g(u, r) = (1 - g)(2r + g(2r - 1)) - \frac{1}{2}(1 - g)^2 ru \\
- gur^2(1 + g(ru - 3)) \tag{25}
\]

with \( g = 1 - 2\lambda = \text{const.} \) The detailed study of the two-dimensional dynamical system defined by these equations will be the object of the following section.

#### B. Study of the dynamical system

The phase portrait of the dynamical system defined by Eqs. (23, 24) with \( \lambda = 1/2 \) is shown in Fig. 1. Its most interesting feature is the fact that the basin of attraction of the Saffman-Taylor single finger is not the whole phase space. The separatrix between the basin of attraction of the ST finger and the rest of the flow starts in the planar interface fixed point and ends in a new fixed point located at \( u^* = 0 \) and \( r^* = 2\lambda/(1 + \lambda) \). The flow in the region below the separatrix is not attracted to a single fixed point, but it evolves to a continuum of fixed points, the line \( r = 0 \). All these fixed points correspond to stationary solutions of the zero surface tension problem consisting of two unequal fingers advancing with the same velocity. The fingers have different width and tip position, the widths \( \lambda_{1, 2} \) given by

\[
\lambda_{1, 2} = \frac{\lambda}{2} \left[ 1 \pm \frac{2}{\pi} \text{cotg}^{-1} \sqrt{\frac{u}{1 - u}} \right] \tag{26}
\]

and the tip separation in the propagation direction \( x \) being
\[ \Delta x = (1 - \lambda) \ln \frac{1 + \sqrt{1 - u}}{1 - \sqrt{1 - u}} \] (27)

The complexity of the phase portrait shown in Fig. 1 is not at all evident with the original variables \((\alpha', \alpha'')\), as can be seen in Fig. 7a. Had we not used the parameterization \((u, r)\), the non-trivial structure of the ST limiting solution would have remained hidden. The points along the line \(u = 0\) (equivalent to the point \(\alpha' = 0\) and \(\alpha'' = 1\) in the original variables) can be assimilated to the Saffman-Taylor finger because the dominant finger has the asymptotic ST finger shape for the corresponding \(\lambda\). All these points differ only in the evolution of the second finger, which always evolves to a needle (a finger of zero width), but of different length. When a trajectory approaches the 1ST fixed point from the two finger region the second finger does not disappear but remains frozen, that is, with \(\lim_{t \to \infty} L_S/L_L = 0\) where \(L_S\) and \(L_L\) are the lengths of the short and the long finger respectively. At the other extreme of the \(u = 0\) line, the point \(u = r = 0\), the two fingers satisfy \(\lim_{t \to \infty} L_S/L_L = 1\), but the distance between the tips goes to infinity. The fixed point \((u^*, r^*)\) corresponds to a new type of asymptotic stationary solution of the zero surface tension problem, and it consists of two fingers with unequal positive velocities. Their length ratio satisfies \(\lim_{t \to \infty} L_S/L_L = 1/3\) for any \(\lambda\).

Although \(\lambda = 1/2\) is the physically selected value of the problem regularized with \(d_0 \to 0\), the study of values of \(\lambda \neq 1/2\) may be relevant to other purely Laplacian growth problems (without surface tension), such as DLA or needle-like growth. For fluid fingering problems it may also be relevant to situations where the selected finger width differs from 1/2 due to external perturbations (bubbles at the tip) or anisotropy. It is thus interesting to extend the analysis to arbitrary \(\lambda\).

The position of the fixed point \(r^*\) that separates the regions where the small finger has zero (upper) or non-zero (lower) flux (see Fig. 1). A crossing from the region with non-zero flux to the one with zero flux would be the characteristic signal of 'successful' competition in the sense defined in Section II. But for the physically relevant value of \(\lambda = 1/2\), no trajectory crosses this line from the lower region or, equivalently, fingers with finite growth rate keep it finite for all time. Therefore this minimal model does not exhibit successful competition, for \(\lambda = 1/2\).

The one-finger (resp. two-finger) region is above (below) the short-dashed line. For the region above the long-dashed line the secondary finger has zero growth rate while for the region below the secondary finger has finite growth rate. Note that there are trajectories crossing from below the line separating the zero and finite growth rate regions.

The position of the fixed point \(r^*\) that separates the flux depends monotonically on \(\lambda\), spanning the whole segment \((0,1)\). For \(\lambda > 1/2\), \(r^*\) approaches 1ST. Consequently the basin of attraction of 1ST is reduced. For \(\lambda < 1/2\) the behavior is the opposite: \(r^*\) departs further from 1ST and the basin of attraction of the single finger grows. In addition, as \(\lambda\) decreases, a critical value
\( \lambda = 1/3 \) is reached for which \( r^* \) crosses the line separating the zero and non-zero growth rate of the small finger. This crossing occurs at the point \( u = 0, r = 1/2 \). Therefore, for \( \lambda < 1/3 \), \( r^* \) will be located below that line, and some trajectories will cross the line from below: these trajectories exhibit successful competition, by definition. In Fig. 2 it is shown the phase portrait for \( \lambda = 1/10 \) and trajectories crossing the line from below are depicted. Despite this fact, such successful competition is rather anecdotic, since it appears in a region far away from the neighborhood of the double finger, where the two fingers have a similar flux. The trajectories crossing the line from below correspond to a second finger with a small growth rate, thus the amount of flux totally eliminated is relatively small and quantitatively not significant in the competition process compared to the regularized problem. In general, what we can say is that under zero surface tension dynamics, narrow fingers present 'stronger' competition than wide fingers. In Fig. 3 we show the evolution of the flux for a trajectory ending in a point of the continuum (case with \( \lambda = 1/2 \) and a trajectory exhibiting successful competition (case with \( \lambda = 1/10 \)).

C. Comparison with the regularized dynamics

We are interested in the comparison between the \( d_0 = 0 \) dynamics and the \( d_0 \neq 0 \) one. The dynamical system defined by the mapping Eq. (22) is referred to as \( L^2(\lambda) \). From now on we will restrict the analysis to most relevant case of \( \lambda = 1/2 \). We introduce the following construction in order to compare the dynamics with and without surface tension: consider a one dimensional set of initial conditions \( (t = 0) \) of the dynamical system \( (24 \, 25) \) surrounding the planar interface (PI) fixed point \( u = 1, r = 1 \). An example for this set is simply a quarter of circle of small radius centered at \((1, 1)\), given by \((1 - R \cos(\theta), 1 - R \sin(\theta))\) with \( 0 \leq \theta < \pi/2 \) and \( R \ll 0 \). For a fixed \( \lambda \), the mapping Eq. (22) applied to this set of initial conditions defines a continuous uni-parametric family of interfaces, where \( \theta \) is the parameter in this example. The evolution up to \( t \to \infty \) and backwards to \( t \to -\infty \) of the case with \( d_0 \) finite from this initial uni-parametric set spans a compact two-dimensional phase space (referred to as \( S^2(d_0) \) embedded in the infinite dimensional space of the problem with finite surface tension, \( S^\infty(d_0) \)). It is known from all experimental and numerical evidence that for finite surface tension the subspace \( S^2(d_0) \) must contain (at least) three fixed points. These three fixed points are the (unstable) planar interface (PI), the (stable) ST single finger (1ST') and a saddle fixed point that corresponds to the degenerate double ST finger (2ST'), where the primes denote the finite surface tension case. The Saffman-Taylor fixed point (1ST') is known to be the universal attractor for finite surface tension, so all the trajectories starting in the planar interface (PI) end up at 1ST'. The saddle fixed point ST' has an attracting manifold of lower dimension, defined by \( \alpha'' = 0 \), and will govern the dynamics of finger competition. We define the space \( S^2(0^+) \) as the limit of \( S^2(d_0) \) for \( d_0 \to 0 \).

In the neighborhood of the planar interface, namely the linear regime, the surface tension acts as a regular perturbation and the regularized problem for small \( d_0 \) converges regularly to the zero surface tension case. Thus, the manifolds \( L^2(1/2) \) and \( S^2(0^+) \) must be tangent at the PI fixed point \( u = 1, r = 1 \). Moreover, from selection theory we know that 1ST'\( \to 1\)ST and 2ST'\( \to 2\)ST in the limit \( d_0 \to 0 \), provided that \( \lambda = 1/2 \) is set for the zero surface tension manifold. Therefore, \( L^2(1/2) \) and \( S^2(0^+) \) must be tangent at 1ST and 2ST.

We have shown that the fixed points of the regularized space \( S^2(0^+) \) are contained in the zero surface tension one \( L^2(1/2) \), but the \( L^2(1/2) \) contains additional fixed points that are not part of the regularized problem \( S^2(0^+) \). As we have seen in the previous section, these additional fixed points are the continuum of two-finger
steady solutions. Therefore, it is important to stress that the flow for the two-dimensional subspace $S^2(0^+)$ defined above is not topologically equivalent to the flow $L^2(1/2)$ of the minimal model. More precisely, the $d_0 = 0$ dynamical system contains a new fixed point that separates the phase space in two basic regions: the set of points ending up at 1ST and the set not ending up at 1ST. Despite the fact that we do not know explicitly the trajectories of the regularized problem in the phase flow $S^2(0^+)$, we can conclude that the two cases are fundamentally different.

We have chosen the minimal model with the requirement of including the double degenerate ST finger fixed point because it is essential to the competition process. In fact, from a physical point of view it is clear that the 2ST fixed point must govern the crossover from the growth regime to the competition one and its saddle point structure is necessary to describe the crossover from growth to competition. This failure is not a particularity of the minimal model but a generic property of the zero surface tension problem, as we will see in the following sections.

The two-dimensional dynamical system (23,24,25) defines a structurally unstable flow, in application of the Peixoto theorem [56], due to the existence of the continuum of fixed points at the line $r = 0$. This structural instability is clearly related to the unphysical behavior of the minimal model, and its effects will be felt by solutions formally close to (23), as it will be shown in the following section. According to the theory of dynamical systems, the introduction of an arbitrarily small perturbation to the dynamical system equations (23,24,25) could act as an unfolding of the problem, suppressing the continuum of fixed points and replacing the non-hyperbolic 2-equal-finger fixed point by an isolated one with a saddle-point (hyperbolic) structure. In this way, the unfolding of the phase portrait defined by the minimal class (23) would have the same topology as the physical problem. Therefore, the natural step is to introduce a perturbation to the mapping (22) that should unfold the phase portrait and still be solvable. The next section is devoted to such perturbed ansatz.

V. EXTENSION WITHIN TWO DIMENSIONS: SEARCHING FOR AN UNFOLDING

A. Modified minimal model

A possible modification of the ansatz (22) which is solvable and preserves the two-dimensionality of the phase space is the following:

$$f(\omega, t) = -\ln \omega + d(t) + (1 - \lambda + i\epsilon) \ln(1 - \alpha(t)\omega) + (1 - \lambda - i\epsilon) \ln(1 + \alpha(t)^*\omega)$$

(28)

where $\epsilon$ is a real positive and is a constant of motion. If $\epsilon$ is set to zero then the minimal model Eq. (22) is retrieved. Solutions of this type have been studied before for instance in Refs. [49,62,33]. This mapping describes generically two unequal axisymmetric fingers, with the symmetry axis located in fixed channel positions separated a distance $\pi$, half the channel width. The main morphological difference between the interface described by the minimal class and the interface obtained from Eq. (28) is that the interface of the modified model may present overhangs. This can be understood from a geometrical point of view in the following manner: well-developed fingers are separated from each other by fjords of the viscous phase, and the width and orientation of these fjords is determined by the constant term $1 - \lambda + i\epsilon$. If the constant term is real, i.e. $\epsilon = 0$, the centerline of the fjords is parallel to the channel walls and the fingers do not present overhangs, but if the constant term has an imaginary part ($\epsilon \neq 0$) then the fjords form a finite angle with the walls [50]. As a result of the inclination of the fjords the fingers may present overhangs. An example of these solutions is shown in Fig.4, with a series of snapshots of the corresponding time evolution. The class of solutions Eq. (28) contains also the single finger Saffman-Taylor solution ($\alpha' = 0$) but, remarkably enough, the introduction of a finite $\epsilon$ has removed the double Saffman-Taylor finger solution.

![FIG. 4. Time evolution of a configuration with $\lambda = 1/2$ and $\epsilon = 0.1$.](image-url)
the same interfacial configurations and dynamics than the $\alpha' \geq 0$ region after a trivial transformation. For the minimal model the zeros $\omega_0$ of $\partial_\omega f(\omega, t)$ lay outside the unit circle, but for the modified minimal model Eq. (28) the situation is different. For $|\alpha| < 1$ a zero of $\partial_\omega f(\omega, t)$ can be inside the unit circle. The position of the zeros in this case is

$$\omega_0 = \frac{-i(\lambda \alpha'' - \epsilon \alpha') \pm \sqrt{(2\lambda - 1)|\alpha|^2 - (\lambda \alpha'' - \epsilon \alpha')^2}}{(2\lambda - 1)|\alpha|^2}$$  

(29)

for $\lambda \neq 1/2$ where $\alpha' = \Re \alpha$ and $\alpha'' = \Im \alpha$. For the particular value $\lambda = 1/2$ the position of the zero is

$$\omega_0 = \frac{1}{2i(\lambda \alpha'' - \epsilon \alpha')}.$$  

(30)

It can be shown that for any $\lambda$ and $\epsilon \neq 0$ a $\omega_0$ can be found such that $|\omega_0| < 1$ for some $|\alpha| < 1$. For instance, with $\lambda = 1/2$ the curve $|\omega_0(\alpha)| = 1$ is the line $\alpha'' = -1 + 2\epsilon\alpha'$, which clearly intersects the unit circle $|\alpha| = 1$, enclosing a region where $|\omega_0| < 1$. As a consequence of the presence of a zero inside the unit circle the parameter space $|\alpha| = 1$ contains unphysical regions, where the mapping Eq. (28) does not describe physically acceptable situations, with self-intersection of the interface associated to the fact that the mapping is not a single valued function. One of these regions is defined by the existence of a zero $\omega_0$ of $\partial_\omega f(\omega, t)$ inside the unit circle. In this region of phase space the interface crosses itself at one point, describing a single loop (see an example in Fig.5). Most remarkably a second unphysical region containing interfaces with two intersections cannot be so easily detected since, in this case, the zeros of $\partial_\omega f(\omega, t)$ lay outside the unit circle. Zero surface tension solutions displaying this feature were also reported in Ref. [3].

Fig. 6 shows a configuration with this double crossing.

FIG. 5. Interface with one crossing, with one zero inside the unit circle.

FIG. 6. Time evolution of a configuration with a double crossing of the interface, with $\lambda = \frac{1}{2}$ and $\epsilon = \frac{1}{2}$. The solid line corresponds to $t = 0$ with $\alpha = 0.85 + i0.4$ and the short-long dashed line to $t = 3.0$. The curves are plotted with its mean $x$ position shifted arbitrarily for better visualization.

Substituting the mapping Eq. (28) in Eq. (12) it can be checked that this ansatz is a solution and that $\epsilon$ is a constant preserved by the dynamics. The system of differential equations resulting from the substitution takes the form

$$d + 4 \Im[\alpha(1-\gamma)]\{d \Im[\alpha] + \Im[\gamma \alpha]\} + |\alpha|^2(2\lambda - 1) \times$$

$$\{d|\alpha|^2 + 2 \Re[\gamma \alpha^* \alpha]\} = 1 + |\alpha|^4 + 4 \Im[|\alpha|^2]$$  

(31)

$$2\{d \Im[\alpha] + \Im[\gamma \alpha]\} + 2d \Im[\alpha(1-\gamma)] +$$

$$2 \Im[\alpha(1-\gamma)](d|\alpha|^2 + \Re[\gamma \alpha^* \alpha]) +$$

$$2(2\lambda - 1)|\alpha|^2\{d \Im[\alpha] + \Im[\gamma \alpha]\} = 4(1+|\alpha|^2) \Im[|\alpha|]$$  

(32)

$$2\lambda d|\alpha|^2 + 2 \Re[\gamma \alpha^* \alpha] = 2|\alpha|^2$$  

(33)

where the time-dependence of $\alpha(t)$ and $d(t)$ has been dropped for sake of clarity and $\gamma = 1 - \lambda + i\epsilon$. Eqs.(31, 33) can be integrated explicitly and the corresponding solutions for the variables $d(t)$ and $\alpha(t) = \alpha'(t) + i\alpha''(t)$ take the form

$$\beta = \frac{d(t) - \ln \alpha(t) + (1 - \lambda - i\epsilon) \ln(1 - |\alpha(t)|^2)}{(1 - \lambda + i\epsilon) \ln(1 + |\alpha(t)|^2)}$$  

(34)

$$t + C = \frac{\lambda d(t) + (1-\lambda) \ln |\alpha(t)| - \epsilon \arctan \frac{\alpha''(t)}{\alpha'(t)}}{\alpha(t)}$$  

(35)

where $C$ is a real-valued constant and $\beta$ is a complex-valued constant. Notice that there is no apparent indication of the pathological situations described above in the form the explicit solutions above. The study of the dynamical systems defined by Eqs. (34, 35) is the object of the next section.

B. Study of the dynamical system

The addition of an imaginary part $i\epsilon$ to the constant $(1-\lambda)$ modifies dramatically the phase portrait of the minimal model, as can be seen in Fig.7, where the phase portraits for $\epsilon = 0$ (now in the variables $\alpha', \alpha''$)
$\epsilon = 0.1$ are depicted. The phase portrait of the modified minimal model is qualitatively different from the $\epsilon = 0$ one, as a direct consequence of the structural instability of the latter case [56], which implies that an arbitrary perturbation of the equations yields a flow which is not topologically equivalent (notice that a perturbation of an initial condition of the infinite-dimensional space of interface configurations is represented here as a perturbation of the equations themselves, that is, a displacement in the space of dynamical systems.) One could have expected that the introduction of this $\epsilon$ would have provided an unfolding of the phase portrait of the minimal model into a structurally stable one (within the integrable class of mappings), hopefully with the saddle-point connection between the unstable and the stable fixed points, as corresponds to the the physical case with $d_0 \neq 0$. The phase portrait of the regularized flow (which is obviously the natural unfolding) would be similar to that of $\epsilon = 0$ in Fig. 7a, except that all trajectories other than the line $\alpha'' = 0$ would end up symmetrically to the upper ST fixed point or the lower one. Notice that in this representation, the 1ST fixed point has been split in two 1ST(R) and 1ST(L), corresponding to whether the right or the left finger approaches the single finger attractor.

These two solutions correspond to having the ST finger located at two different positions (the symmetry axes of the fingers), owing to the translational invariance associated to the periodic boundary conditions. Since a $\pi$-shift in the transversal $y$-direction must yield an equivalent configuration, the identification of any point in the semicircle with the diametrically opposed which has reduced the actual phase space in half, implies also that the 1ST(R) and 1ST(L) must be topologically identified as the same point. Approaching one or the other thus means approaching from the left or from the right. With this identification the contact with the problem with rigid boundaries is clearer, since there, the attractor is clearly unique but the flow must also be symmetrically split into two parts, corresponding to whether the left or the right finger wins, owing to the symmetry of the system under parity (see a more detailed discussion in section VLD). Unfortunately, we must conclude that the resulting phase portrait for the modified minimal model does not provide the correct unfolding. This is particularly remarkable if one takes into account that, in two-dimensional systems, structurally stable dynamical systems are dense [54]. On the contrary, the perturbed equations contain finite-time singularities and, although they remove the continuum of double-finger fixed points, they also miss the equal-finger fixed point, which is an essential ingredient of the regularized flow.

In Fig. 8 we plot the phase portrait for $\epsilon = 0.5$ and the different regions of phase space. For any other $\epsilon$ the flow is topologically equivalent but the shape and size of the different regions varies smoothly. The line of finite-time singularities collapses towards the lower fixed point 1ST(L) in the limit $\epsilon \to 0$ as shown in Fig. 7b. In the absence of the 2ST fixed point, the splitting of flow is made possible by the existence of the line of finite-time singularities. Instead of a separatrix between the respective basins of attraction of 1ST(R) and 1ST(L), there is

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig7.png}
\caption{Phase portrait of the minimal model and the modified minimal model. $\lambda = \frac{1}{2}$ for both plots, the regions to the right of the dotted lines correspond to two-finger configurations (a) $\epsilon = 0$; note the continuum of fixed points (marked with a thick line) on $|\alpha| = 1$. (b) $\epsilon = 0.1$; the straight line in the lower left corner is a line of finite-time singularities and the two fingers have equal length on the dashed line.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig8.png}
\caption{(a) Phase portrait of the modified minimal model with $\lambda = 1/2$ and $\epsilon = 1/2$. (b) Plot of different regions of phase space of case (a). The grey regions correspond to single finger interfaces and the other regions to two finger interfaces. Regions IIa and IIb differ in which of the two fingers is larger. Regions III and IV are unphysical regions described in the text. The straight boundary of region III is a line of cusp singularities.}
\end{figure}
an intermediate, non-zero measure region, connected to the PI fixed point, whose evolution ends up at that singularity line, defined by the condition $|\omega_0| = 1$. Similarly to the finite-time singularities occurring for polynomial mappings, this line is reached in a finite time and is associated to the formation of a cusp at the interface. The evolution is not defined after that time. The flow in the region below the singularity line (region III of Fig.8b), defined by $|\omega_0| < 1$, is actually well defined although it describes evolution of unphysical interfaces which intersect themselves forming a loop, (see Fig.5). Their evolution originates and ends at different points of the singularity line. The region IV has double crossings of the interface (see Fig.6) and also originates at the singularity line but, remarkably enough, it evolves asymptotically towards the ST finger despite their unphysical double crossing at the tail of the finger. This double-crossing is removed in a finite time in some subregion of IV and it remains up to infinite time in another subregion. Incidentally, this clearly illustrates how dangerous it may be to infer a physically correct dynamics from the fact that the interface evolves asymptotically towards a single ST finger, and that zero surface tension solutions must be dealt with extreme care since smooth and apparently physical interfaces may contain elements that yield them physically unacceptable when the time evolution is considered either forward or backward.

The double-crossing removal in some of the above solutions has some implications in the general study of topological singularities associated to interface pinch-off in fluid systems (for a recent review see Ref. [64] and, in the context of Hele-Shaw flows, see for instance Ref. [33]). Consider the stable Saffman-Taylor problem, in which the viscous fluid displaces the inviscid one. The planar interface is stable in this case and is the attractor of the dynamics. The conformal mapping obeys then an equation formally equivalent to Eq. (12), that applies to the unstable Hele-Shaw flow, with the only difference that time is reversed, $t$ is substituted by $-t$, in Eq. (12). Therefore, the dynamics of the stable case is obtained from the unstable one simply by a time reversal. As a consequence, the double-crossing removal we observed in the original problem encompasses a prediction of a finite-time interface pinch-off in the stable configuration of the problem, for some class of initial conditions. A similar pinch-off phenomenon for zero surface tension was detected numerically by Baker, Siegel and Tanveer [33] for other types of mapping singularities. Our result provides a very simple example of exactly solvable finite-time pinch-off. Notice that there is no singularity of the interface shape or velocity at the interface contact, so one could presume that surface tension may not affect significantly the phenomenon in this case, although this is an open question yet.

The evolution of a trajectory ending up in the cusp line cannot be continued beyond the impact time $t_0$ of the zero $\omega_0$ with the unit circle $|\omega| = 1$ because the cusp line attracts the flow from both sides, but the flow is indeed well defined in the interior of region III, where we have $d\omega/dt < 0$ in opposition to the physical region where $d\omega/dt > 0$, so, in a sense, the temporal evolution inside the singularity region is reversed. As a matter of fact, the graph $\alpha''(\alpha')$ can be obtained and is smooth in all regions of phase space. Defining $\alpha = re^{i\theta}$, its substitution in Eq. (34) yields after some algebra

$$
\frac{d\theta}{dr} = \frac{4r \cos \theta - (1 - \lambda)(1 - r^2) \sin \theta + \epsilon(1 + r^2) \cos \theta}{1 - r^2 + (2\lambda - 1)r^4 + 2\lambda r^2 \cos 2\theta + 2\epsilon r^2 \sin 2\theta}
$$

and from this expression the trajectory can be obtained also in region III. The fact that the modified minimal model does not yield an unfolding of the minimal one is more deeply stressed by the fact that the field of directions defined by the above graph, even removing the singularities through a proper time reparametrization and time reversal in region III, is still a structurally unstable flow.

It is well known that the zero-surface tension problem is extremely sensitive to initial conditions: given a zero-surface tension solution at $t = 0$, another one can be found which is as close as desired to the interface of the first solution, and the evolution of the two solutions be completely different in general. This is a consequence of the ill-posedness of the initial-value problem [28], which is most clearly manifest in polynomial mappings, which may arbitrarily approximate any initial condition, possibly one which does not develop singularities, but themselves always develop cusps. However, it is illustrative to see several striking examples of sensitivity to initial conditions within the class of logarithmic mappings which are much less predictive a priori.
Example 1. Consider two initial conditions \((\alpha_1', \alpha_1'')\) and \((\alpha_2', \alpha_2'')\) close to the PI fixed point, with \(|\alpha_1|, |\alpha_2| \ll 1\), which differ only in nonlinear orders of their mode amplitudes [60].

![Image](https://via.placeholder.com/150)

**FIG. 10.** Evolution of two interfaces initially equal to linear order (see text), with \(\lambda = 1/3\) and \(\epsilon = 0.1\). \(\alpha(0) = 0.04619398 - i0.01913417\) for the solid line and \(\alpha(0) = -0.04619398 - i0.00527598\) for the dashed line. Upper left plot \(t = 0\), upper right plot \(t = 2.0\), lower left plot \(t = 4.0\) and lower right plot \(t = 6.0\).

One can easily choose \((\alpha_2', \alpha_2'')\) (with \(\alpha_2' \alpha_2'' < 0\), that is, considering not only the semi-circle \(\alpha' > 0\) but the whole unit circle) such that the time evolution will be completely different from the evolution of the original initial condition, even though the two initial conditions where equivalent to linear order. In Fig.10 we show an explicit example. While the two initial conditions for the interface configuration cannot be distinguished in the scale of the plot, the final outcome is dramatically different. One of the evolutions is an example of successful competition, where the finger in the initial condition is eventually approaching the ST solution, with a small secondary finger (not present in the initial condition) which is generated but screened out by the leading one. The other evolution is quite surprising since the secondary finger grows to the point of taking over and winning the competition.

Example 2. A similar situation is found if one compares two initial conditions equivalent to linear order up to a parity transformation. Pairs of initial conditions of this type, with the same values of \(\lambda\) and \(\epsilon\), can easily be found within the same semicircular phase space, and since the dynamics is indeed symmetric under mirror reflection, one should not expect, in principle, a very different behavior, even though such points are not close to each other in phase space. Fig.11 shows an example in which one of the evolutions is smooth, with a leading finger and a small one being generated, and the other generates a cusp in finite time. As in the first example, no signature of the different fate of the system could apparently be seen in the initial conditions. In both cases the extremely small differences associated to higher orders in the mode amplitudes have thus been crucial. The sensitivity to initial conditions of these examples is more striking for decreasing values of \(\epsilon\), since the time in which the two evolutions stay close to each other increases as \(O(-\ln \epsilon)\). For instance, given an initial condition \(\alpha_0\) close to PI, the difference between the \(\epsilon = 0\) interface and the \(\epsilon \to 0\) one will remain of \(O(\epsilon)\) for a time of \(O(-\ln \epsilon)\). Later on in the evolution the differences between the two interfaces will be of \(O(1)\): the asymptotic shape of the \(\epsilon = 0\) case will be two unequal fingers while the shape of the \(\epsilon \to 0\) will be a single Saffman-Taylor finger. Similarly, for two initial conditions symmetrical to linear order such as in Example 2, with \(\epsilon \to 0\), the differences between their interfaces will remain symmetric to \(O(\epsilon)\) for a time of \(O(-\ln \epsilon)\), but later they will lose symmetry and finally both will end up at the same fixed point, say the right one, even though one of the two evolutions has been favoring the other one, say the left one, for a long time (up to well developed fingers).

![Image](https://via.placeholder.com/150)

**FIG. 11.** Evolution of two interfaces symmetric to linear order (see text), with \(\lambda = 1/3\), \(\epsilon = 0.1\), \(\alpha(0) = 0.02724 + i0.03104\) for the solid line and \(\alpha(0) = 0.02724 - i0.04193\) for the dashed line. The upper plot corresponds to \(t = 0\) and the lower to \(t = 4.19\), when a cusp develops.

Example 3. In Fig.12 we illustrate the effect of changing \(\epsilon\) in initial conditions which are equivalent to linear order. Notice that in one case a cusp is generated at the secondary finger. In others the small fingers rapidly overcomes the large one, while for the smallest \(\epsilon\) the initial finger seems to lead the competition. Remarkably, in
of the initially small finger is higher than the flux of
the other finger and finally the flux of the initially large
finger reaches zero: it has been suppressed from the com-
petition. This is opposite to what it would expect that for well
developed fingers the larger growth rate (and larger length too)
wins the competition. For the (a) case the finger which ini-
tially has larger growth rate (and larger length too) wins the compe-
tition. For the (b) case the finger which initially has lower
growth rate and surpasses the first one, which is eventu-
al suppressed from the competition process. This is in-
herent to the evolution with the regularized dyna-
mics (small surface tension).

All the above examples have been chosen to empha-
size the caution that is required when trying to use exact
solutions to approximate the dynamics of the problem.
A direct comparison of these solutions with numerical
integration for very small surface tension would be re-
quired in order to make a more quantitative assessment
of the issue. This will be presented elsewhere [55]. In any
case, it must also be stated that the class of logarithmic
solutions does provide also qualitatively correct evolu-
tions, not only of single finger configurations as stated
in [38], but also with two-finger configurations showing
successful competition. An example of this is plotted in
Fig.4. Starting from the planar interface, during the lin-
ear regime a bump starts to grow, followed generically
by a second bump as the evolution enters the non-linear
regime. The two fingers keep on growing for some time,
until later on in the evolution, one of the fingers is dy-
namically eliminated of the competition process and the
other finger approaches asymptotically the ST finger so-
lution. This general scenario is illustrated in figure 13a,
where the individual growth rates of the two fingers $\Delta \psi_1$ and $\Delta \psi_2$ are plotted versus time, for two different initial
conditions.

For other initial conditions as generic as the previous
one, however, the following phenomenon is observed: the
small finger (with initially zero flux) of a configuration
with two significantly unequal fingers increases its flux
while the flux of the large finger decreases, until the flux
this last case the smaller finger will also take over after
a much longer time.

\[ \Delta \psi(t) = 0, \quad t = 0, \quad (b) \ t = 2.4, \quad (c) \ t = 3.5, \quad (d) \ t = 5.0. \]
singularity \cite{53}). The winning finger with the regularized dynamics is thus the losing finger with the zero surface tension one.

Again, in the limit $\epsilon \to 0$ these phenomena appear even more dramatically, as a consequence of the structural instability of the minimal model. In this limit, for a $O(-\ln \epsilon)$ time we will observe two unequal fully developed fingers advancing with a fixed tip distance, but eventually the presence of finite $\epsilon$ will 'activate' the competition process and one of the two fingers will reduce its growth rate until fully suppressed from the competition. If $\alpha''(0) > 0$ the suppressed finger will be the small one, but if $\alpha''(0) < 0$ the dynamically suppressed finger will be the large one.

C. Comparison with the regularized dynamics.

In order to compare the $d_0 = 0$ dynamics with the physical case of $d_0 \neq 0$, as we have done for the $\epsilon = 0$ case, we use the construction introduced in Sect. IV.C. Consider a one dimensional set of initial conditions ($t = 0$) of the form Eq. \((28)\), surrounding the planar interface (PI) fixed point $\alpha' = 0, \alpha'' = 0$, for a fixed $\lambda$ and $\epsilon$. We choose the points of this set infinitesimally close to $\alpha = 0$, and therefore the interface is in the linear regime.

The $d_0 \neq 0$ time evolution from $t = -\infty$ to $t = \infty$ of this set spans a compact two dimensional phase space and defines a dynamical system $S^2(d_0)$, embedded in the infinite dimensional $S^\infty(d_0)$. We define the dynamical system $S^2(0^+)$ as the limit of $S^2(d_0)$ for $d_0 \to 0$, intersecting the one dimensional set at $t = 0$.

Since in the linear regime the regularized problem for vanishingly small $d_0$ converges regularly to the $d_0 = 0$ solution, then the dynamical system $L^2(\lambda, \epsilon)$ of $d_0 = 0$ and $S^2(0^+)$ must be tangent at the PI fixed point, $\alpha = 0$. According to solvability theory the selection of the fixed ST finger in the limit $d_0 \to 0$ is $\lambda = 1/2$. Then, $L^2(1/2, \epsilon)$ and $S^2(0^+)$ must intersect at $1ST(R)$, $\alpha' = 0, \alpha'' = 1$ and $1ST(L)$, $\alpha' = 0, \alpha'' = -1$. In addition, from Ref. \cite{27} it follows that the single-finger solution with $d_0 \to 0$ converges uniformly to the the time-dependent Saffman-Taylor finger solution with $\lambda = 1/2$. This implies that $L^2(1/2, \epsilon)$ and $S^2(0^+)$ intersect not only in the fixed points that they have in common, but also on the line $\alpha' = 0$ (that corresponds to the time-dependent ST solution). For the dynamical system $S^2(0^+)$, the basins of attraction of $1ST(R)$ and $1ST(L)$ are two-dimensional and finite, and therefore there must be at least one separatrix trajectory between the two basins. This separatrix must end at a saddle fixed point (which does not exist in the phase portrait of the $d_0 = 0$ solution). It is reasonable to assume that this fixed point is the double ST finger fixed point (2ST). Thus, the topology of the flow defined by the dynamical system with $d_0 = 0$, $L^2(1/2, \epsilon)$ is different from the flow of the dynamical system $d_0 \to 0$, $S^2(0^+)$: the flow for the regularized problem contains a trajectory and a fixed point that it is not contained in the flow defined by the modified minimal model, the trajectory starting at the planar interface PI fixed point and ending up at the 2ST fixed point. The phase flow of the modified minimal model with $d_0 = 0$ is qualitatively different from the phase flow of the regularized problem, $d_0 \to 0$, and therefore the solution Eq. \((28)\) is unphysical in a global sense, what is to say, when a sufficiently large set of initial conditions (spanning evolutions towards $1ST(R)$ and $1ST(L)$) is considered simultaneously.

It is useful to define a projection of the trajectories from the regularized two-dimensional space $S^2(0^+)$ onto the $d_0 = 0 L^2(1/2, \epsilon)$ space in order to make possible not only the comparison between the topology of the two flows but also the comparison between individual trajectories. It is obvious that there are many different possible definitions, and the projection of the regularized flow $S^2(0^+)$ onto $L^2(1/2, \epsilon)$ is in that sense not unique, but the following discussion should not depend on the particular definition of the projection. One possible definition of a useful projection is the following \cite{55}: given a point $\zeta$ of $S^2(0^+)$ and a point $\alpha$ of $L^2(1/2, \epsilon)$, we define a 'distance' $D_\alpha(\zeta)$ between $\zeta$ and $\alpha$, that is, a distance between an interface within $S^2(0^+)$ and an interface within $L^2(1/2, \epsilon)$. One possible choice of $D_\alpha(\zeta)$ is the area enclosed or trapped between the two interfaces. Then, we define the projection $\alpha_\zeta$ of $\zeta$ (that corresponds to an interface solution of the problem with $d_0 \neq 0$) onto the $d_0 = 0$ phase space $L^2(1/2, \epsilon)$ as the value of $\alpha$ that minimizes the 'distance' $D_\zeta(\alpha)$ between $\alpha$ and $\zeta$, or equivalently, that minimizes the 'distance' between the two interfaces, with the restriction that the average position of the two interfaces is the same to ensure that the projection satisfies mass conservation. Then, once we have defined a projection $\alpha_\zeta$ of a point of $S^2(0^+)$ it is straightforward to obtain the projection of a particular trajectory $\zeta(t)$ in $S^2(0^+)$, applying the definition of $\alpha_\zeta$ to any point of the trajectory. This projection of a trajectory will be denoted by $\alpha_\zeta(t)$.

The projection on the $d_0 = 0$ phase portrait of the $d_0 \to 0$ separatrix trajectory as $\epsilon \to 0$ will approach the time-dependent 2ST trajectory for $\epsilon = 0$, the line $\alpha'' = 0$, but a broad set of initial conditions of $\epsilon \to 0$ located below the projection of the separatrix (that is, with $\alpha'' < 0$) will be attracted by the $1ST(R)$ fixed point under the $d_0 = 0$ dynamics instead of being attracted by the $1ST(L)$ fixed point as their position with respect the separatrix implies \cite{27}. Thus, the evolution under the $d_0 = 0$ dynamics of this set of initial conditions will be not only quantitatively but also qualitatively incorrect, in the sense that the finger that wins the competition is the one that in the regularized dynamics loses. An example of this was given in Fig.13b. More details will be presented elsewhere \cite{59}.

The motivation for the introduction of a finite $i\epsilon$ term to the minimal model Eq. \((22)\) was to obtain an unfolding of its non-hyperbolic fixed point structure, but in this section we have shown that the introduction of $i\epsilon$ has
failed to unfold the phase portrait to the expected physical topology, the saddle-point structure of the regularized problem. It has dramatically changed the topology of the flow obtained for \( \epsilon = 0 \), but the flow for \( \epsilon \neq 0 \) does not have the expected structurally stable flow of the physical problem: an unstable fixed point, two stable fixed points and one saddle fixed point. Instead of this, the evolution of Eq. (28) with \( \epsilon \neq 0 \) presents finite-time singularities for a non-zero measure set of initial conditions. This can be understood as a consequence of the absence of the 2ST saddle point, which controls the competition regime. Without this fixed point the separatrix trajectory necessary to separate the basins of attraction of ST(L) and ST(R) is not present and the only possible way to split the flow is through the existence of finite time singularities. This is not a particularity of the mapping Eq. (28) but a more general feature of \( d_0 = 0 \) solutions. Below we will prove that, within the N-logarithms class, finite \( \epsilon \) implies finite-time singularities in the evolution of a non-zero measure set of initial conditions (see Sect. VI C). Besides the existence of finite-time singularities we have seen that, unlike the case \( \epsilon = 0 \) solutions exhibiting successful competition are possible with \( \epsilon \neq 0 \) for \( \lambda = 1/2 \). However, part of those evolutions are unphysical in the sense the winning finger may differ from the one with the regularized dynamics.

**VI. GENERALIZATION TO HIGHER DIMENSIONS**

This section is devoted to the study of solutions that define a dynamical system of dimensionality greater than two. We will show that the discussion of previous sections is not peculiar of dimension 2 but is extendable to higher dimensions.

**A. Non-axisymmetric fingers**

The solutions that have been studied in the previous sections, Eq. (22) and Eq. (23), have two pole-like singularities \( \omega_{1,2} \) located at \( \omega_1 = 1/\alpha \) and \( \omega_2 = -1/\alpha^* \). The property \( \omega_1 = -\omega_2^* \) reduces the dimensionality of the dynamical system to two and also forces the axisymmetry of the fingers. If the singularities \( \omega_{1,2} \) are not related, then the phase space has one additional dimension and the fingers are not axisymmetric.

We will now study the ansatz

\[
 f(\omega, t) = -\ln \omega + d(t) + (1 - \lambda) \ln(1 - \alpha_1(t)\omega) \\
 + (1 - \lambda) \ln(1 - \alpha_2(t)\omega)
\]

where \( \alpha_j(t) = \alpha_j'(t) + i\alpha_j''(t) \) are two complex quantities satisfying \( |\alpha_j| < 1 \). This ansatz is an exact solution of Eq. (12) and it can be proved that the evolution is free of finite time singularities.

From the evolution equations for \( \alpha_{1,2} \) in the case \( \lambda = 1/2 \) it is found that the dynamics satisfy

\[
 \frac{\alpha_1'\alpha_2' - \alpha_1''\alpha_2''}{\alpha_1'\alpha_2'' + \alpha_1''\alpha_2'} = \text{Const.}
\]

Writing \( \alpha \) in polar coordinates, \( \alpha_j = r_j e^{i\theta_j} \), this constraint reduces to the simpler form

\[
 \theta_1 + \theta_2 = C.
\]

The constant \( C \) depends on the initial condition, but its value can be fixed arbitrarily using the property of rotational invariance of the ansatz Eq. (37): the transformation \( \alpha_1 \to \alpha_1 e^{i\phi} \) and \( \alpha_2 \to \alpha_2 e^{i\phi} \) is equivalent to a translation of the interface a distance \( \phi \) in the \( y \) axis direction. For the minimal class studied previously (Eq. (22)) the value of \( C \) was \( C = \pi \). The existence of the constraint Eq. (38) reduces the dimensionality of the problem from four to three, with variables \( r_1, r_2 \) and \( \theta_1 - \theta_2 \), implying that the dynamical system is actually three-dimensional.

In order to study the dynamical system defined by the substitution of the ansatz Eq. (17) into the evolution equation (12) we introduce the variables \( z = r e^{i\theta} \) and \( \rho = \alpha_1 \alpha_2 \). Moreover, we use the arbitrariness of \( C \) to choose the simpler value \( C = 0 \). Then, the dynamical system in the variables \( (r, \theta, \rho) \) reads

\[
 \dot{r} = 4r - r^2 + (2 + r^2) \cos(2\theta) - 3\rho^2 \\
 \dot{\rho} = 4\rho \left( 2 - \rho^2 + r^2 (\rho \cos(2\theta) - 1) \right) \\
 \dot{\theta} = -2\rho \sin(2\theta).
\]

With these variables the axisymmetric case studied in previous sections is contained as a the particular case \( \theta = 0 \). From the last equation it is immediate to see that \( \theta \) decreases monotonically with time (since, for \( C = 0, \rho > 0 \), and tends to zero for long times, \( \theta \to 0 \), converging to the axisymmetric case \( \theta = 0 \). Therefore, we can conclude that the axisymmetric case is attracting the dynamics of the non-axisymmetric one and, asymptotically, its dynamics reduces to the axisymmetric one.

Therefore, we have shown that the dynamics of the minimal model is not a particularity of a zero-measure subset but the general behavior of the non-axisymmetric neighborhood of the original axisymmetric class. This neighborhood has a non-zero measure within the three dimensional system. Similarly, it can be shown that the basin of attraction of the Saffman-Taylor finger for the ansatz Eq. (22) is also relatively small.

It is worth noting that non-axisymmetric fingers can be obtained also in 2d. A conformal mapping describing non-axisymmetric fingers is

\[
 f(\omega, t) = -\ln \omega + d(t) + (1 - \lambda + p + i\epsilon) \ln(1 - \alpha(t)\omega) \\
 + (1 - \lambda - p - i\epsilon) \ln(1 + \alpha^*(t)\omega)
\]

(41)
where $0 < p < 1 - \lambda$. However, the lack of axisymmetry caused by the introduction of $p$ does not change qualitatively the phase portraits obtained for $p = \epsilon = 0$, the continuum of fixed points present for $\epsilon = 0$ is not removed by the introduction of a finite $p$ and the finite-time singularities that appear for $\epsilon \neq 0$ are also present when $p \neq 0$. Therefore, the results obtained for 2d in the previous sections are not modified at all if we relax the condition of finger axisymmetry.

B. Perturbations which change finger widths

The second type of modification of the ansatz (22) we have studied is the following. Consider

$$f(\omega, t) = -\ln \omega + d(t) + (1 - \lambda) \ln(1 - \alpha_1(t)\omega) + (1 - \lambda) \ln(1 - \alpha_2(t)\omega) + 2(\lambda - \lambda_s) \ln(1 - \delta(t)\omega)$$

with initial conditions $\alpha_1(0) = -\alpha_2(0) = \alpha(0), 0 < \lambda, \lambda_s < 1$ and $|\delta(0)| \ll 1$. Note that for $\delta(0) = 0$ this ansatz reduces to the minimal model (23). From substitution of this ansatz into the evolution equation (13) it is obtained that Eq. (12) is a solution with $\lambda$ and $\lambda_s$ constants. From the dynamical equations it can be proved that the asymptotic configuration of this ansatz consists of one or two fingers, with asymptotic filling fraction equal to $\lambda_s$. But if $|\delta(0)| \ll |\alpha(0)|$ then the interface will be initially almost identical to the one obtained within the class (22) with the same $\alpha(0)$ and $\lambda$, and its evolution will remain close to the one obtained with the minimal class for a time that will increase with decreasing $|\delta(0)|$. Therefore, given a small enough $|\delta(0)|$, starting from the planar interface a configuration with one or two fingers (depending on the initial conditions) of total width $\lambda$ will develop. Later on, as $|\delta|$ grows and approaches 1, the total width will change from $\lambda$ to $\lambda_s$ for long enough time. The ansatz (12) thus describes an interface that changes the filling fraction of the fingers from $\lambda$ to $\lambda_s$. The same phenomenon will appear with any other of the solutions described in this paper (and in general in pole-like solutions) if a term of the type $2(\lambda - \lambda_s) \ln(1 - \delta(t)\omega)$ is added, and in particular it will appear in the single finger configurations obtained setting $\alpha(0)$ and $\delta(0)$ purely imaginary. Note that in this case if $\delta(0) = 0$ the ansatz describes exactly the time-dependent Saffman-Taylor finger. This changing-width phenomenon of $d_0 = 0$ solutions has been known for long [45], but it has been recently claimed [28] to be responsible of the known width selection observed with finite surface tension both experimentally and numerically. The idea was that, although solutions of arbitrary $\lambda$ exist in the absence of surface tension, these are unstable under some perturbations which trigger the evolution towards the $\lambda = 1/2$ solution. Since the present paper is basically emphasizing the unphysical dynamics of the idealized $(d_0 = 0)$ problem, in direct contradiction with Ref. [28], we feel compelled to briefly comment on this respect here. The basic argument of Ref. [28] is as follows, in terms of the parameterization of the interface used by the author: a term of the form $i\mu\varphi$ in the conformal mapping is always unstable under the substitution $i\mu\varphi \to \mu \ln(e^{i\varphi} - \epsilon)$. The introduction of such perturbation then leads to the $\mu = 0$ case, which corresponds to $\lambda = 1/2$. In Refs. [26,27] it was pointed out that, with the same degree of generality, equivalent perturbations exist which lead to any desired $\lambda$, and therefore the conclusion that $\lambda = 1/2$ is the only attractor is incorrect. It is argued [31] that the latter class of perturbations is different form the former since they increase the number of logarithmic terms in the conformal mapping and therefore modify the dimension of the subspace of solutions. This objection is somewhat misleading since such partitioning of classes of solutions in terms of the number of logarithms is arbitrary and not intrinsic. This can be seen by choosing a different reference region to conformally map the physical fluid. Instead of mapping it into the semi-infinite strip [26], the mapping into the interior of the unit circle avoids the confusion on the dimension of the subspace of solutions. Thus, the perturbation proposed in Ref. [28] is equivalent to choosing $\lambda_s = 1/2$ in the ansatz (12), but it is manifest in this formulation that there is nothing special with this particular choice of $\lambda_s$. Perturbations leading to any finger width $\lambda_s$ occur with the same genericity. Therefore the instability of the point $\delta = 0$ is not related to the steady state selection phenomenon.

C. Finite-time singularities within N-logarithms solutions

In this section we will prove that any solution of the N-logarithm class [54] that does not have only real constant parameters presents finite time singularities, that is, it contains a non-zero measure set of initial conditions which develop singularities at finite time. Consider a conformal mapping function $f(\omega, t)$

$$f(\omega, t) = -\ln \omega + d(t) + (A_1 + i\epsilon) \ln(1 - \alpha_1(t)\omega) + (A_2 - i\epsilon) \ln(1 - \alpha_2(t)\omega)$$

where $A_1 + A_2 = 2(1 - \lambda)$, $\epsilon > 0$ and $|\alpha_{1,2}| < 1$. The mapping $f(\omega, t)$ must satisfy $\partial_\omega f(\omega, t) \neq 0$ for $|\omega| \leq 1$. If any zero $\omega_0$ of $\partial_\omega f(\omega, t)$ hits the unit circle $|\omega| = 1$ then the interface develops a cusp. For the ansatz (13) $\partial_\omega f(\omega, t)$ reads

$$\partial_\omega f = \frac{1}{\omega} - \frac{(A_1 + i\epsilon)\alpha_1}{1 - \alpha_1\omega} - \frac{(A_2 - i\epsilon)\alpha_2}{1 - \alpha_2\omega}. $$

Thus, the position of the zero $\omega_0$ of $\partial_\omega f(\omega_0, t)$ is

$$\omega_0 = \frac{-(A_1+i\epsilon-1)\alpha_1+(A_2-i\epsilon-1)\alpha_2}{2\alpha_1\alpha_2(2\lambda-1)} \pm \sqrt{\frac{(A_1+i\epsilon-1)\alpha_1+(A_2-i\epsilon-1)\alpha_2)^2-4\alpha_1\alpha_2(2\lambda-1)}{2\alpha_1\alpha_2(2\lambda-1)}}$$

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If, for some value of $\alpha_{1,2}$, $|\alpha_{1,2}| \leq 1$, the zero $\omega_0$ is inside the unit circle, then the ansatz (43) will present finite time singularities for some initial conditions. Therefore, if $|\omega_0| < 1$ the interface will develop a cusp. Setting $\alpha_{1,2} = \omega_0 \gamma_{1,2}$ and $\theta_2 - \theta_1 = -2 \delta$ with $\delta \ll 1$ the position of the zero (keeping up to linear terms in $\delta$) is:

$$\omega_0 = e^{-i\theta_2} \frac{\lambda + (1 - \lambda)}{\alpha(2\lambda - 1)} + O(\delta^2)$$

and the modulus of the minus solution (the one with smaller modulus) reads

$$|\omega_0| = \frac{1}{\alpha} |1 - \frac{\epsilon\delta}{1 - \lambda} + O(\delta^2)|.$$  

In consequence, for $\alpha$ close to 1 we obtain $|\omega_0| < 1$, one of the zeros is inside the unit circle in a finite neighborhood of $\alpha_1 = \alpha_2 = e^{i\theta}$. Thus, the mapping (43) presents finite time singularities for some initial conditions independent of the value of $\epsilon$ and $\Lambda_{1,2}$, and the measure of this set is non-zero.

Now we consider a generic mapping with $N > 2$ logarithmic terms of the form:

$$f(\omega, t) = -\ln \omega + d(t) + \sum_{j=1}^{N} \gamma_j \ln(1 - \alpha_j(t)\omega)$$  

where $\gamma_j = \Lambda_j + i\Gamma_j$ are constants of motion with the restriction $\sum_{j=1}^{N} \gamma_j = 2(1 - \lambda)$. If we choose $\gamma_j = \gamma_1$ for $1 \leq j \leq k$ and $\gamma_j = \gamma_2$ for $k + 1 \leq j \leq N$ we recover the mapping (43). Therefore, the $N$-logarithm solution (48) contains initial conditions that develop a cusp with this subset of $\alpha_j$, but the dimension of this subset is lower than the dimension of the phase space, implying that the measure of the set is zero compared to the whole phase space. To prove that the subset that develops a cusp is finite we choose now the following values for $\alpha_j$: $\alpha_j = \alpha_1 + \eta_j$ for $1 \leq j \leq k$ and $\alpha_j = \alpha_2 + \eta_j$ for $k + 1 \leq j \leq N$, with $|\eta_j| \ll 1$, where $|\omega_0| < 1$ if $\eta_j = 0$. The equation $\partial_\omega f(\omega, t) = 0$ reads

$$\frac{1}{\omega} + \sum_{j=1}^{k} \frac{\gamma_j(\alpha_1 + \eta_j)}{1 - (\alpha_1 + \eta_j)\omega} + \sum_{j=k+1}^{N} \frac{\gamma_j(\alpha_2 + \eta_j)}{1 - (\alpha_2 + \eta_j)\omega} = 0.$$  

This equation (49) reduces to Eq. (48) if all $\eta_j = 0$ and it has $N$ zeros if $\eta_j \neq 0$. Defining $g(\omega) = \partial_\omega f(\omega, t)$ for $\eta_j = 0$ and $G(\omega, \bar{\eta}) = \partial_\omega f(\omega, t)$ for $\eta_j \neq 0$ then $G(\omega, \bar{\eta}) = g(\omega) + \delta G(\omega, \bar{\eta})$ where $|\delta G(\omega, \bar{\eta})| < K|\bar{\eta}|g(\omega)$ with $K$ and $R$ constants, and $g(\omega_0) = 0$. One zero $\omega_0^g$ of $G(\omega, \bar{\eta})$ can be written $\omega_0^g = \omega_0 + \delta \omega$, and assuming $|\delta \omega| < C|\bar{\eta}|$ with $C$ constant the substitution of $\omega_0^g$ in $G(\omega, \bar{\eta}) = 0$ yields

$$g(\omega_0) + \frac{\partial g}{\partial \omega} \bigg|_{\omega_0} \delta \omega + \delta G(\omega_0, \bar{\eta}) = 0.$$  

The position of the zero is then:

$$\omega_0' = \omega_0 - \frac{\delta G(\omega_0, \bar{\eta})}{\partial_\omega g(\omega_0)}$$

where $\frac{\partial g}{\partial \omega} \bigg|_{\omega_0} \neq 0$. Therefore, the zero $\omega_0'$ of Eq. (49) is inside a ball of radius $o(|\bar{\eta}|)$ centered in $\omega_0$. If $|\omega_0| < 1$, then choosing $|\bar{\eta}|$ small enough the zero will satisfy $|\omega_0'| < 1$: in a neighborhood of $\alpha_1, \alpha_2$ at least one zero of $\partial_\omega f(\omega, t)$ is inside the unit circle, and the dimension of this neighborhood will be the same of the phase space. So we can conclude that any mapping of the form (43) presents finite time singularities for some sets of initial conditions of non-zero measure, provided that at least one pair of $\gamma_j$ has a non zero imaginary part.

Thus, the requirement that a mapping function of the form (48) is free of finite time singularities for any initial condition $\alpha_j(0)$ is fulfilled if and only if $\Im[\gamma_j] = 0, j = 1, \ldots, N$. But this restriction implies (42) that for a wide range of initial conditions the asymptotic configuration is a N-finger interface with unequal fingers advancing at a constant speed, a situation fully analogous to the one discussed in Sect. [N]. Then, if a mapping of the form (48) with $\Im[\gamma_j] = 0$ is chosen, then the dynamical system $L^N(\gamma_j)$ will have non-hyperbolic fixed-points (continua of fixed points) and will lack the saddle-point structure of the regularized problem. In order to completely remove the continua of fixed points it is necessary to set $\Im[\gamma_j] \neq 0$, but in this case we will encounter finite-time singularities and the saddle-point structure will not be present anyway.

To sum up, we have shown that the features of the minimal model and its extensions that make them unphysical (in a global sense) are not specific of their low dimensionality. The features that make the solutions studied in previous sections ineligible as a physical description of low surface tension dynamics for a sufficiently large class of initial conditions, are also present within the much more general N-logarithm family of solutions, and the conclusions drawn in previous sections can be generalized to that class.

D. Rigid-wall boundary conditions

It is worth stressing here that the use of periodic boundary conditions throughout this study, as opposed to the physically more natural rigid-wall boundary conditions, is not essential to the basic discussion. In connection with the discussion of multifinger steady solutions, this point was raised in Ref. [67] and addressed in Ref. [68]. Here we will just recall that the choice of periodic boundary conditions is not only the simplest in terms of symmetry and dimensionality, but it is the relevant one if
one is interested in general mechanisms of finger competition in finger arrays. In this sense, the study of the two-finger configurations in this paper refers to an alternating mode of two-finger periodicity in an infinite array of fingers, in the spirit of Ref. [54]. For finite size-systems one can also argue that rigid-wall boundary conditions are included as a particular case of periodic boundary conditions in an enlarged system. That is, a channel with width $W$ with rigid walls in mathematically equivalent to a channel of width $2W$ with periodic boundary conditions where auxiliary channel of with $W$ is constructed as the mirror image of the physical one. The competition of two fingers in a channel with rigid walls at a distance $W$ is in practice equivalent to a four-finger problem with periodic boundary conditions in a double channel.

The only subtle point which we would like to stress is the apparent degeneracy of the single-finger attractor into a left ST finger and a right ST finger, as already pointed out in Sect. V B, and the possible relevance of this fact in connection with the saddle-point structure of the phase space flow. This degeneracy is inherited from the trivial continuous degeneracy associated to translation invariance in the transversal direction, when periodic boundary conditions are assumed. In fact an arbitrary shift in the transversal direction yields a physically equivalent configuration. When an initial condition is fixed, such continuous degeneracy is broken into two discrete spatial positions which are separated a distance $W/2$. The whole dynamical system is then invariant under translations of $W/2$. This is the reason why we only plotted a half of the disk in the phase portraits of section V. Technically, the resulting dynamical system must be defined ‘modulo-$W/2$’, that is, identifying any configuration with the resulting of a $W/2$ shift. In the phase space defined by the variables $(\alpha', \alpha'')$ one should identify any point with the resulting of a $\pi$ rotation. In this way the two single-finger attractors do correspond to the same fixed point. With this identification, the ST finger is not degenerate and the flow becomes topologically equivalent to the corresponding one in a channel with rigid-wall boundary conditions. The two-finger configurations have thus the same structure, regardless of the type of sidewall boundary conditions. The flow starts at the PI fixed point and ends up at the 1ST fixed point. Between them there is a saddle point corresponding to the 2ST fixed point. This separates the flow in two equivalent regions, namely ‘from the left’ and ‘from the right’ of the saddle point. With zero-surface tension, the case of rigid walls exhibits the same problems, namely the occurrence of a (nontrivial) continuum degeneracy of multifinger solutions, and the existence of finite-time singularities. The important point we want to stress is thus that all the general conclusions drawn in this paper are valid if rigid-wall boundary conditions are considered.

VII. DYNAMICAL SOLVABILITY. GENERAL DISCUSSION

A. The physics of zero-surface tension

The role of the zero surface tension solutions in the description of the dynamics of the nonzero but vanishingly small surface tension problem is now clearer. The $d_0 = 0$ dynamics is in general incorrect in a global sense, even if we choose solutions with the asymptotic width $\lambda$ given by selection theory. However, they have an important place in the description of the physical problem. It has been proved in Refs. [35–37] that the solutions with $d_0 = 0$ converge to the $d_0 \to 0$ during a time $O(1)$, before the impact with the unit circle of the so-called daughter singularity at time $t_d$. In practice this implies that the $d_0 = 0$ dynamics is not only correct in the linear regime (where $d_0$ acts as regular perturbation) but also quite deep into the nonlinear regime. After $t_d$ nothing can be said a priori: as we have shown in the present paper, there are regions of the $d_0 = 0$ phase space corresponding to smooth interfaces that are physically wrong, but other regions are a good description of the evolution with finite (but very small) surface tension. For instance, in the neighborhood of the time-dependent Saffman-Taylor finger (the line $\alpha' = 0$ in the solutions (22), (23)) the $d_0 = 0$ evolution is qualitatively correct for finite surface tension, and even quantitatively correct in the limit $d_0 \to 0$ (for $\lambda = 1/2$). However, a question remains open: given a $d_0 = 0$ evolution smooth for all time and consistent with the results of selection theory, is it the limit of a $d_0 \to 0$ evolution? This question can be explored numerically and is the subject of a forthcoming paper [55]. Generally speaking the conclusion is that exact solutions including evolution of two different fingers which are compatible with MS theory, that is, evolving to a single finger with the width predicted by selection theory, and which do not exhibit any kind of singularity in the interface shape, may be dramatically affected by surface tension. The outcome of the competition, that is, which one of the two competing fingers will survive at the end, when an infinitesimally small surface tension in introduced may be the opposite of that of the zero surface tension case. This may happen in situations where fingers are significantly different from each other and is not an instability of a particular trajectory, but a generic behavior in a finite (non-zero measure) range of initial conditions within the integrable class. For that region of phase space, it is clear that the dynamics of finger competition is completely wrong for the class of integrable solutions. Nevertheless, there is also a class of initial conditions which have a qualitatively correct evolution including ‘successful’ finger competition in the sense defined in sections above (this possibility was incorrectly excluded in Ref. [55], where the analysis was based on $\epsilon = 0$). Although strict convergence of the regularized solution to the idealized one may not occur in these cases, the
quantitative differences may be moderately small. Actual convergence of some type can only be expected at most when there is only one finger along the complete time evolution. In summary, according to this scenario there are basically four classes of initial conditions within the most general integrable solutions, once those a priori incompatible with selection theory are excluded, namely (i) finite-time singularities forward or backward (or both) in time; (ii) asymptotically correct ST finger with wrong dynamics (the incorrect finger wins); (iii) asymptotically correct ST finger with qualitatively correct evolution (the correct finger wins although shapes may differ during a transient); and (iv) (unphysical) evolution towards multifinger fixed points. It has to be added that, all of the above solutions plus those which are incompatible with selection theory are qualitatively and quantitatively correct in the limit of small surface tension, until a time of order one which is always in the deeply nonlinear regime.

As a general consideration it is worth remarking that fingers emerging from the instability of the planar interface when this is subject to noise are necessarily in the range of dimensionless surface tension of order one. A simple way to argue this point is that it is precisely surface tension which selects the size of the emerging fingers, since the fastest growing mode is that in which both stabilizing and destabilizing forces are of the same order. In these cases, surface tension is felt necessarily in the linear regime, and the usefulness of the zero surface tension solutions in the early stages of the evolution is obviously more limited.

Finally, from a physical point of view it is appropriate to recall that the presence of noise does modify the general picture of the fingering dynamics in the limit of small surface tension, as pointed out in Ref. [34]. Although the ST finger is the universal attractor of the problem, the linear basin of attraction decreases with dimensionless surface tension. In practice this implies that the interface approaches the ST finger but when it gets too close, noise triggers its nonlinear instability and the interface makes a long excursion (typically a tip splitting) before approaching again the ST finger. The considerations made in this paper concerning the limit of small surface tension thus imply that noise must be taken as sufficiently small. A careful discussion of the effects of noise, particularly in numerical simulation of very small surface tension will be presented elsewhere [55].

**B. A Dynamical Solvability Scenario**

In Ref. [24] we pointed out for the first time the dynamical implications of the MS analysis when extended to multifinger fixed points. The idea of the Dynamical Solvability Scenario (DSS) was already latent in that discussion. We pursued this extension of the steady state selection problem explicitly in Ref. [38], where we found that, in direct analogy to the single-finger case, the introduction of surface tension did select a discrete set of multifinger stationary states, in general with coexisting unequal fingers. Here we would like to discuss in what sense that analysis does provide a Dynamic Solvability Scenario.

Before doing that, let us briefly consider an alternative view of a possible DSS proposed by Sarkissian and Levine [24]. In Ref. [24], it was explicitly discussed with examples that exact solutions of the zero-surface tension problem did behave differently from numerical integration of the small surface tension problem. At the end, the authors speculated with the possibility that surface tension could play a selective role in the sense that it could basically pick up the physically correct evolutions out of the complete set of solutions without surface tension, in direct analogy with the introduction of a small surface tension selecting a unique finger width out of the continuum of stationary solutions. Since the class of nonsingular integrable solutions is indeed vast and infinite-dimensional, it is not unreasonable to expect that one could approximate any particular evolution with finite surface tension with one of those solutions for all time. However, as recently pointed out in Ref. [34], there is no simple way to determine which of those solutions is selected by any macroscopic construction. Furthermore, even if this were possible, one should still face the rather uncomfortable fact that the base of solutions defined by the superposition of logarithmic terms in the mapping, would itself correspond to unphysical (nonselected) solutions, as we have seen throughout this paper. Indeed, an initial condition defined exactly by a finite number of logarithms would have to be replaced in general by a solution with an infinite number of logarithms as the ‘selected’ solution which the (small) finite surface tension system tracks.

From a more general point of view, a dynamical selection principle understood as ‘selection of trajectories’ has an important shortcoming when considered within the perspective of a broader class of interfacial pattern forming systems. In fact, the solvability theory of steady state selection has turned into a general principle because its applicability to a large variety of systems, most remarkably in the context of dendritic solidification [6–9]. However, it is only for Laplacian growth problems that exact time-dependent solutions are known explicitly, so there would be no hope to extend the above DSS as a general principle to those other problems.

The DSS we propose here has a weaker form but it is susceptible of generalization to other interfacial pattern forming systems. The basic idea can be best expressed in similar words to those recently used by Golub and Langer [1] to describe solvability theory in a general context. They have nicely synthesized the singular role of surface tension in the language of dynamical systems as to ‘whether or not there exists a stable fixed point’ [1]. In this context, our DSS extends the (static) solvability scenario in the sense that the singular role of surface tension is precisely to guarantee the
existence of multifinger fixed points with a saddle-point (hyperbolic) structure. We have seen that the continuum of multifinger fixed points is directly related to a nonhyperbolic structure of the equal-finger fixed points. They imply directions in phase space were the flow is marginal, and this is so to all derivative orders. While in the traditional solvability scenario the introduction of surface tension does isolate a stable fixed point (a continuum of single-finger fixed points turns into a stable one and a discrete set of unstable ones), now it isolates multifinger saddle points out of continua of multifinger solutions, as discussed in Ref. (hyperbolic) structure. We have seen that the continuum of single-finger fixed points turns into a hyperbolic fixed point with stable and unstable directions, and a discrete set of unstable ones). Since the saddle fixed points are defined by the degenerate n-equal-finger solutions, the stable directions of the saddle-point are directly related to the stable directions of the single-finger fixed point, while the unstable directions correspond to all perturbations which break the n-periodicity of the equal-finger solution. The most important stable and unstable directions, however, are those depicted in the two-dimensional phase portraits discussed in the above section, namely the ‘growth’ direction connecting the planar interface and the n-finger fixed point, and the ‘competition’ direction connecting the n-finger fixed point to the single-finger fixed point [70]. Notice that arrays of fingers emerging from the morphological instability of the planar interface are relatively close of n-periodic solutions as long as the noise in the initial conditions is weak and white, which guarantees that the most unstable (fastest growing) mode dominates in the early nonlinear regime. In these conditions, the system feels the attraction to the corresponding n-equal finger fixed point. This stage is what we called ‘growth’. When the fingers are relatively large they start to feel the deviations form exact periodicity and start the ‘competition’ process.

Note that, despite the formal analogy to the single-finger solvability theory, the reference to a the restoring of multifinger hyperbolicity by surface tension as dynamical solvability scenario is fully justified. Indeed, the local structure of the multifinger fixed point has a dramatic impact on the global (topological) structure of the phase space flow, as we have seen in simple examples. The existence of a small but finite surface tension thus determines a global flow structure and it is in this sense that it ‘selects’ the dynamics of the system.

The possibility of extension of this analysis to other interfacial pattern forming problems relies on the existence of a continuum of unequal multifinger stationary solutions with zero surface tension. The fact that in the ST case the existence of those can be associated to a simple relationship between screening due to relative tip position and relative finger width (that is, a slower areal growth rate of the screened finger is compensated by its smaller width, resulting in an equal tip velocity), one is tempted to conjecture that similar classes of solutions must exist in other problems, for instance in the growth of needle crystals in the channel geometry [1]. Although this point should be more carefully addressed, it seems reasonable to expect that a DSS as presented above could be generalizable, to some extent, to other physical systems.

VIII. SUMMARY AND CONCLUSIONS

We have developed a Dynamical Systems approach to study the dynamics of the Saffman-Taylor problem, basing the analysis on the zero surface tension solutions. A minimal model has been analyzed, and from its phase flow we have concluded that it is unphysical. A detailed study of a perturbation of the minimal model within two dimensions has yielded the same conclusion. The unphysical behavior of zero surface tension solutions is a consequence of the non-hyperbolicity of the multifinger fixed points of the finite-dimensional dynamical system that they define, opposed to the saddle point structure of the regularized problem. Perturbations of the minimal model to higher dimensions confirm the generality of the conclusions reached in two-dimensional models. We have proved that the N-logarithms class of solutions presents finite-time singularities if the continua of fixed points are totally absent. From the analysis of zero surface tension solutions we conclude that they are unphysical in a global sense, when sufficiently large classes of initial conditions are considered simultaneously, because they lack the correct topology of the physical flow, structured in terms of a saddle-point connection between the unstable and the stable fixed points. This does not exclude that, for some sets of initial conditions, the zero surface tension dynamics might be correct, not only qualitatively but even quantitatively, but it is not possible in practice to know it a priori by any simple means. We have illustrated with several examples that although the asymptotic behavior may be correct (evolution towards a single ST finger) the intermediate dynamics may be completely wrong, or even physically meaningless, such as for the existence of interface crossings. We have also illustrated the sensitivity to initial conditions when approximating physically relevant situations with different integrable solutions. As a by-product we have also obtained some practical results concerning zero-surface tension dynamics which may be relevant to Laplacian growth problems, for instance in relation to the interplay of screening effects and finger widths. We have introduced precise definitions of ‘growth’ and ‘competition’. With the proper definition ‘successful’ competition, we can state for instance that, in the absence of surface tension, narrow fingers do compete more efficiently than wide ones. We have also found explicit solutions which lead to finite-time interface pinch-off in the stable configuration of the problem.

The detailed comparison of the dynamics with zero and non-zero but very small surface tension requires a
careful numerical study and can be analyzed in terms of the daughter singularities formalism developed in Refs. [36,37]. As a matter of fact it can be shown that the zero surface tension problem and the vanishingly small surface tension regularization differ dramatically even in regions where the former is nonsingular, in the sense that non-zero measure regions of phase space have a different outcome of the competition (namely, which one of two competing fingers survives) in the two cases. A detailed study of this point will be presented elsewhere [55].

Finally, we propose a Dynamical Solvability Scenario relevant in principle not only for viscous fingering problems but also of applicability to other pattern forming problems. Within this DSS the role of surface tension as a singular perturbation is to isolate multifinger saddle points out of the continua of multifinger fixed points, as shown previously in Ref. [39]. This extends the traditional solvability theory applied to steady state selection, where surface tension did also isolate a unique (stable) hyperbolic fixed point out of a continuum of nonhyperbolic ones. In that case the isolated fixed point was the global attractor of the problem. In the present extension, the introduction of surface tension does isolate a unique n-equal finger fixed point out of each continuum of n-finger fixed points, with both stable and unstable directions. By restoring this saddle point local structure the topology of the phase space flow is modified, so the introduction of surface tension has a deep impact on the global phase-space structure of the dynamics. It is in this sense that this scenario can be considered as a dynamical solvability theory.

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For simplicity of the discussion we abusively use the same notation to refer to the dynamical system and to the manifold on which it is defined.

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