3-coloring triangle-free planar graphs with a precolored 9-cycle

ILKYOO CHOI\textsuperscript{1}, Jan Ekstein\textsuperscript{2}, Přemysl Holub\textsuperscript{2}, Bernard Lidický\textsuperscript{1}

University of Illinois at Urbana-Champaign, USA
University of West Bohemia, Czech Republic

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A graph $G$ is \textit{$k$-colorable} if there is a function $f$ where
- for each vertex $v$: $f(v) \in [k]$
- for each edge $xy$: $f(x) \neq f(y)$

A graph $G$ is \textit{$k$-critical} if
- $G$ is not $(k - 1)$-colorable
- for each subgraph $H$: $H$ is $(k - 1)$-colorable
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A graph $G$ is $C$-critical for $k$-coloring if
- for each edge $e$, there is a $k$-coloring $f_e$ of $V(C)$ where
  - $f_e$ extends to $G - e$
  - $f_e$ does not extend to $G$
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**Observation**

If $G$ is $(k + 1)$-critical, then $G$ is $\emptyset$-critical for $k$-coloring.
Observation
There exists a 3-coloring of $V(C)$ that extends to $G_1 - e$ but does not extend to $G_1$.

Observation
For every cut $C$ and every $e \in V(G_1)$ exists a 3-coloring of $V(C)$ that extends to $G_1 - e$ but does not extend to $G_1$.

4-critical
- not 3-colorable
- each subgraph is 3-colorable
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Definition

A graph $G$ is $C$-critical for $k$-coloring if for each $e \in E(G)$, there exists a $k$-coloring $f_e$ of $V(C)$ that extends to $G - e$ but does not extend to $G$. 
**Definition**

A graph $G$ is **$C$-critical** for $k$-coloring if for each $e \in E(G)$, there exists a $k$-coloring $f_e$ of $V(C)$ that extends to $G - e$ but does not extend to $G$. 

![Diagram of a graph with edges $e_1$ and $e_2$ and a cycle $C$.]
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\[ C \]

\[ e_1 \quad e_2 \]

\[ \varphi_1 \]

\[ \varphi_2 \]

\[ 1 \quad 2 \quad 3 \quad 1 \quad 3 \]

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Observation

*If $G$ is $(k + 1)$-critical, then $G$ is $\emptyset$-critical for $k$-coloring.*
- Why \textit{C-critical}? Which \textit{C} is a good choice?
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\begin{itemize}
  \item simplifying graphs on surfaces
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- Why \( C \)-critical? Which \( C \) is a good choice?
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- Simplifying graphs on surfaces

- Precolored tree

- Interior of a cycle
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- simplifying graphs on surfaces

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**Theorem (Grötzsch 1959, Aksenov 1974)**

*If $G$ is a plane graph of girth 4, then a pre-coloring of either a 4-cycle or a 5-cycle extends to 3-coloring of $G$.**
Theorem (Grötzsch 1959, Aksenov 1974)

*If* $G$ *is a plane graph of girth* $4$, *then a pre-coloring of either a 4-cycle or a 5-cycle extends to 3-coloring of* $G$.

Focus: plane graphs that are $C$-critical for 3-coloring where $C$ is a cycle.
Theorem (Grötzsch 1959, Aksenov 1974)

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Goal: Characterize all $C$-critical plane graphs of girth 4.
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Easier goal: Characterize all $C$-critical plane graphs of girth 5.
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Theorem (Grötzsch 1959, Aksenov 1974)

*If* $G$ *is a plane graph of girth* 4, *then a pre-coloring of either a* 4-cycle *or a* 5-cycle *extends to 3-coloring of* $G$.

Focus: *plane* graphs that are $C$-critical for 3-coloring where $C$ is a cycle.

Goal: Characterize all $C$-critical plane graphs of girth 4. *STILL OPEN!*

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- $|C| \leq 11$ by Thomassen 2003 and Walls 1999
- $|C| = 12$ by Dvořák–Kawarabayashi 2011
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- $|C| \leq 16$ by Dvořák–Lidický 2013+
$|C| \leq 10$
Goal: Characterize all $C$-critical plane graphs of girth $4$. STILL OPEN!
Goal: Characterize all $C$-critical plane graphs of girth 4. *STILL OPEN!*

Known characterizations:

- $|C| \in \{4, 5\}$ by Aksenov 1974
- $|C| = 6$ by Gimbel–Thomassen 1997
- $|C| = 6$ by Aksenov–Borodin–Glebov 2003
- $|C| = 7$ by Aksenov–Borodin–Glebov 2004
- $|C| = 8$ by Dvořák–Lidický 2013+
- $|C| = 9$ by C.–Ekstein–Holub–Lidický 2014+
**Theorem (Aksenov 1974)**

*If $G$ is a *plane* graph of girth 4, then a pre-coloring of either a 4-cycle or a 5-cycle extends to a 3-coloring of $G$.*
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If $G$ is a plane graph of girth 4, then a pre-coloring of either a 4-cycle or a 5-cycle extends to a 3-coloring of $G$.

For $|C| \in \{4, 5\}$, NO graphs are $C$-critical for 3-coloring!

“nice” plane graph: has no separating 4-cycles or 5-cycles.
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Theorem (Gimbel–Thomassen 1997, Aksenov–Borodin–Glebov 2003)

*If* $G$ *is a “nice” plane graph of girth 4 bounded by a cycle* $C$ *of length 6, then* $G$ *is* $C$-*critical if and only if* $G$ *“looks like” below.*
Theorem (Aksenov–Borodin–Glebov 2004)

*If* $G$ *is a “nice” plane graph of girth 4 bounded by a cycle* $C$ *of length 7, then* $G$ *is* $C$-*critical* if and only if $G$ “looks like” a graph below.

![Diagram](image1)

Theorem (Dvořák–Lidický 2013+)

*If* $G$ *is a “nice” plane graph of girth 4 bounded by a cycle* $C$ *of length 8, then* $G$ *is* $C$-*critical* if and only if $G$ “looks like” a graph below.

![Diagram](image2)
Theorem (C.–Ekstein–Holub–Lidický 2014+)

If $G$ is a “nice” plane graph of girth 4 bounded by a cycle $C$ of length 9, then $G$ is $C$-critical if and only if $G$ “looks like” a graph below (2 more).
Theorem (C.–Ekstein–Holub–Lidický 2014+)

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**Theorem (Tutte 1954)**

A plane graph $G$ has a 3-coloring if and only if its dual $G^*$ has a nowhere-zero $\mathbb{Z}_3$-flow.
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![Diagram of 3-coloring triangle-free planar graphs with a precolored 9-cycle]
Proof idea:

**Theorem (Tutte 1954)**

A plane graph $G$ has a 3-coloring if and only if its dual $G^*$ has a nowhere-zero $\mathbb{Z}_3$-flow.

(In-edges - out-edges) of every face is a multiple of 3!
Theorem (Gimbel–Thomassen 1997, Aksenov–Borodin–Glebov 2003)

If $G$ is a “nice” plane graph of girth 4 bounded by a cycle $C$ of length 6, then $G$ is $C$-critical if and only if $G$ “looks like” below.
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If $G$ is a “nice” plane graph of girth 4 bounded by a cycle $C$ of length 6, then $G$ is $C$-critical if and only if $G$ “looks like” below.

$(\iff)$ Need to show:
- coloring does not extend to $G$
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![Diagram of the graph and coloring](image-url)
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- coloring does not extend to $G$  done!
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$(\implies)$ done!

Corollary (Dvořák–Král–Thomas 2014+)

If $G$ is a “nice” plane graph of girth 4 bounded by a cycle $C$ of length $c$ and is $C$-critical, then

\[
\begin{align*}
c = 6 & : \emptyset \\
c = 7 & : \{5\} \\
c = 8 & : \emptyset, \{6\}, \{5, 5\} \\
c = 9 & : \{7\}, \{5, 6\}, \{5, 5, 5\}, \{5\}
\end{align*}
\]

are the only possible multisets of faces of length at least 5.
Corollary (Dvořák–Král’–Thomas 2014+)

If $G$ is a “nice” plane graph of girth 4 bounded by a cycle $C$ of length 9 and is $C$-critical, then

$$\{7\}, \{5, 6\}, \{5, 5, 5\}, \{5\}$$

are the only possible multisets of faces of length at least 5.
**Corollary (Dvořák–Král’–Thomas 2014+)**

If $G$ is a “nice” plane graph of girth 4 bounded by a cycle $C$ of length 9 and is $C$-critical, then

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**Theorem (C.–Ekstein–Holub–Lidický 2014+)**

If $G$ is a “nice” plane graph of girth 4 bounded by a cycle $C$ of length 9 containing a 5-face and a 6-face, then $G$ is $C$-critical if and only if $G$ “looks like” a graph below.
Corollary (Dvořák–Král’–Thomas 2014+)

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3-coloring triangle-free planar graphs with a precolored 9-cycle

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