CURVATURES ON THE ABBENA-THURSTON MANIFOLD

Ju-Wan Han, Hyun Woong Kim and Yong-Soo Pyo*

Abstract. Let $H$ be the 3-dimensional Heisenberg group, $(G = H \times S^1, g)$ a product Riemannian manifold of Riemannian manifolds $H$ and $S^1$ with arbitrarily given left invariant Riemannian metrics respectively, and $\Gamma$ the discrete subgroup of $G$ with integer entries. Then, on the Riemannian manifold $(M := G/\Gamma, \Pi^*g = \bar{g})$, $\Pi : G \to G/\Gamma$, we evaluate the scalar curvature and the Ricci curvature.

1. Introduction

Recently, Park (cf. [7, 8]) investigated various differential geometric properties on the three dimensional Heisenberg group with an arbitrarily given left invariant Riemannian metric. And, the geometric properties on the Abbena-Thurston manifold have been also studied by many geometrician (cf. [1, 4, 5, 6]).

In this paper, we evaluate the scalar curvature and the Ricci curvature on the Abbena-Thurston manifold $(G/\Gamma, \bar{g})$ with an arbitrarily given $G$-invariant Riemannian metric $\bar{g}$.

Let $H$ be the 3-dimensional Heisenberg group, $(G = H \times S^1, g)$ a product Riemannian manifold of Riemannian manifolds $H$ and $S^1$ with arbitrarily given left invariant Riemannian metrics respectively, $\Gamma$ the discrete subgroup of $G$ with integer entries, and $\Pi : G \to G/\Gamma = M$ the natural projection. First of all, we completely classify $G$-invariant Riemannian metrics on the Abbena-Thurston manifold $G/\Gamma = M$. And then, on the Riemannian manifold $(M := G/\Gamma, \Pi^*g = \bar{g})$ with an arbitrarily given $G$-invariant Riemannian metric $\bar{g}$, we obtain the scalar curvature, homogeneous Riemannian manifold, Ricci curvature.

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curvature (cf. Theorem 3.1) and evaluate the Ricci curvature (cf. Theorem 3.2).

2. The Abbena-Thurston manifold \((G/\Gamma, \bar{g})\) with a \(G\)-invariant metric \(\bar{g}\)

Let \(G\) be the closed connected subgroup of \(GL(4, \mathbb{C})\) defined by

\[
\begin{pmatrix}
1 & a_{12} & a_{13} & 0 \\
0 & 1 & a_{23} & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & e^{2\pi ia}
\end{pmatrix}
| a_{12}, a_{23}, a_{13}, a \in \mathbb{R}
\]

\((i = \sqrt{-1})\).

i.e., \(G = H \times S^1\) is the product of the Heisenberg group \(H\) and \(S^1\). Let \(\Gamma\) be the discrete subgroup of \(G\) with integer entries and \(M = G/\Gamma\).

Denote by \(x, y, z, t\) coordinates on \(G\), say for \(A \in G\),

\(x(A) = a_{12}, y(A) = a_{23}, z(A) = a_{13}, t(A) = a\).

If \(L_B\) is the left translation by an element \(B \in G\), we have

\[
L_B^* dx = dx, \quad L_B^* dy = dy, \\
L_B^* (dz - xdy) = dz - xdy, \quad L_B^* dt = dt.
\]

In particular, these forms are invariant under the action of \(\Gamma\); let \(\Pi : G \rightarrow M\), then there exist 1-forms \(\alpha_1, \alpha_2, \alpha_3\) and \(\alpha_4\) on \(M\) such that

\[
dx = \Pi^* \alpha_1, \quad dy = \Pi^* \alpha_2, \quad dz - xdy = \Pi^* \alpha_3\quad \text{and}\quad dt = \Pi^* \alpha_4.
\]

The manifold \(M = G/\Gamma\) is referred to as the Abbena-Thurston manifold. On \(G\), the vector fields

\[
(2.1) \quad \nu_1 := \frac{\partial}{\partial x}, \quad \nu_2 := \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \quad \nu_3 := \frac{\partial}{\partial z}, \quad \nu_4 := \frac{\partial}{\partial t}
\]

are dual to \(dx, dy, dz - xdy, dt\), and are left invariant. We denote by \(g\) the Lie algebra of all left invariant vector fields on \(G\). Let \(< , >\) be an inner product on the Lie algebra \(g\) such that

\[
(2.2) \quad < \nu_a, \nu_a > = k_a^2 \quad (a = 1, 2, 3, 4), \quad < \nu_1, \nu_2 > = k_1 k_2 \cos \varphi_1, \\
< \nu_2, \nu_3 > = k_2 k_3 \cos \varphi_2, \quad < \nu_3, \nu_1 > = k_3 k_1 \cos \varphi_3, \\
< \nu_4, \nu_a > = 0 \quad (a = 1, 2, 3), \quad (1 - \cos^2 \varphi_1 - \cos^2 \varphi_2 - \cos^2 \varphi_3 + 2 \cos \varphi_1 \cos \varphi_2 \cos \varphi_3) > 0,
\]

where each \(k_a\) is positive constant and \(0 < \varphi_1, \varphi_2, \varphi_3 < \pi\). Let \(g\) be the left invariant Riemannian metric on \(G(= H \times S^1)\) which corresponds to
the above inner product on \( g \). For simplicity, here and from now on in this paper, we use the following notations:

\[
    g_{ab} := \langle v_a, v_b \rangle, \\
    \lambda := (1 - \cos^2 \varphi_1 - \cos^2 \varphi_2 - \cos^2 \varphi_3 + 2 \cos \varphi_1 \cos \varphi_2 \cos \varphi_3)^{\frac{1}{2}}.
\]

Then

\[
    |(g_{ab})_{a,b}|^{\frac{1}{2}} = k_1 k_2 k_3 k_4 \lambda.
\]

So, the space of all Riemannian metrics on the product Riemannian manifold \( G = H \times S^1 \) of Riemannian manifolds \( H, S^1 \) with left invariant Riemannian metrics respectively is given by

\[
    \{ (k_1, k_2, k_3, k_4, \varphi_1, \varphi_2, \varphi_3) \mid \text{all } k_a > 0, 0 < \varphi_1, \varphi_2, \varphi_3 < \pi, (1 - \cos^2 \varphi_1 - \cos^2 \varphi_2 - \cos^2 \varphi_3 + 2 \cos \varphi_1 \cos \varphi_2 \cos \varphi_3) > 0 \}.
\]

Now we normalize left invariant Riemannian metrics by putting \( k_1 = 1 \), and put

\[
    \mathcal{M} := \{ (1, k_2, k_3, k_4, \varphi_1, \varphi_2, \varphi_3) \mid k_2, k_3, k_4 > 0, 0 < \varphi_1, \varphi_2, \varphi_3 < \pi, (1 - \cos^2 \varphi_1 - \cos^2 \varphi_2 - \cos^2 \varphi_3 + 2 \cos \varphi_1 \cos \varphi_2 \cos \varphi_3) > 0 \}.
\]

And we have from (2.1)

\[
    [v_1, v_2] = v_3, \quad [v_1, v_3] = [v_2, v_3] = 0, \quad [v_a, v_4] = 0 \quad (a = 1, 2, 3, 4).
\]

In general, the Riemannian connection \( \nabla \) on a Riemannian manifold \((M, g)\) is given by (cf. [2, 3])

\[
    2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) + g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X) \quad (X, Y, Z \in \mathfrak{X}(M)).
\]

On the other hand, for each Riemannian metric \( g \) on \( G \) which corresponds to \((k_1, k_2, k_3, k_4, \varphi_1, \varphi_2, \varphi_3) \in \mathcal{M}\), there exists a \( G \)-invariant Riemannian metric \( \bar{g} \) on \( M = G/\Gamma \) such that \( \Pi^* g = \bar{g} \).

3. Curvatures on the Abbena-Thurston manifold \((G/\Gamma, \bar{g})\)

We retain the notations as in Section 2.
From now on, all the calculations on \((M := H / \Gamma, \bar{g})\) will be done on \(G(= H \times S^1)\) and its Lie algebra \(g\). In fact, because \(M\) is a homogeneous space, the curvature is the same in all its points, and \(M\) is locally isomorphic to \(G\) (cf. \([1, 5, 9]\)).

Let \(\langle , \rangle\) be an inner product on the Lie algebra \(g\) of \(G(= H \times S^1)\) which corresponds to \((1, k_2, k_3, \varphi_1, \varphi_2, \varphi_3)(\in \mathfrak{M})\). Let \(g\) be the left invariant Riemannian metric on \(G\) which is induced by the inner product \(\langle , \rangle\).

Putting

\[
\begin{align*}
  d_1 &:= v_1, \quad d'_2 := v_2 - \langle v_1, v_2 \rangle v_1, \\
  d_2 &:= < d'_2, d'_2 >^{-1/2} d'_2, \\
  d'_3 &:= v_3 - < d_1, v_3 > d_1 - < d_2, v_3 > d_2, \\
  d_3 &:= < d'_3, d'_3 >^{-1/2} d'_3, \quad d_4 := < v_4, v_4 >^{-1/2} v_4, \\
\end{align*}
\]

we have an orthonormal frame

\[
\{d_1, d_2, d_3, d_4\}
\]

on \((G, g)\). We have from (2.2) and (3.1)

\[
\|d'_2\|_h = k_2 \sin \varphi_1, \quad \|d'_3\|_h = k_3 \lambda (\sin \varphi_1)^{-1}.
\]

By the help of (2.2), (3.1) and (3.2), we obtain

\[
\begin{align*}
  v_1 &= d_1, \quad v_2 = k_2 \cos \varphi_1 d_1 + k_2 \sin \varphi_1 d_2, \\
  v_3 &= k_3 \cos \varphi_3 d_1 + k_3 (\cos \varphi_2 - \cos \varphi_3 \cos \varphi_1)(\sin \varphi_1)^{-1} d_2 \\
  &\quad + k_3 \lambda (\sin \varphi_1)^{-1} d_3, \\
  v_4 &= k_4 d_4.
\end{align*}
\]
By virtue of (2.1), (2.2), (3.1), (3.2) and (3.3), we get
\[
[d_1, d_2] = k_2^{-1}k_3\{(\sin \varphi_1)^{-1}\cos \varphi_3 d_1
+ (\cos \varphi_2 - \cos \varphi_3 \cos \varphi_1)(\sin \varphi_1)^{-2}d_2
+ \lambda(\sin \varphi_1)^{-2}d_3\},
\]
\[
[d_2, d_3] = k_2^{-1}k_3 \cos \varphi_3\{\lambda^{-1}\cos \varphi_3 d_1
+ (\lambda \sin \varphi_1)^{-1}(\cos \varphi_2 - \cos \varphi_3 \cos \varphi_1)d_2
+ (\sin \varphi_1)^{-1}d_3\},
\]
\[
[d_3, d_1] = k_3(k_2 \lambda \sin \varphi_1)^{-1}(\cos \varphi_2 - \cos \varphi_3 \cos \varphi_1)
\{\cos \varphi_3 d_1 + (\sin \varphi_1)^{-1}(\cos \varphi_2 - \cos \varphi_3 \cos \varphi_1)d_2
+ \lambda(\sin \varphi_1)^{-1}d_3\},
\]
\[
[d_a, d_4] = 0 \quad (a = 1, 2, 3, 4).
\]

Let $\nabla$ be the Riemannian connection on $(G, g)$. And, let $R$ be the curvature tensor field on $(G, g)$, that is,
\[
R(X, Y)Z = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})(Z) \quad (X, Y, Z \in \mathfrak{X}(G)).
\]

From (2.3) and (3.4), we get
\[
\nabla_{d_1}d_1 = k_3 \cos \varphi_3\ (k_2 \sin \varphi_1)^{-1}
\{-d_2 + \lambda^{-1}(\cos \varphi_2 - \cos \varphi_3 \cos \varphi_1)d_3\},
\]
\[
\nabla_{d_1}d_2 = k_2^{-1}k_3\{(\sin \varphi_1)^{-1}\cos \varphi_3 d_1
+ (2\lambda)^{-1}(\sin^2 \varphi_3 - \cos^2 \varphi_3)d_3\},
\]
\[
\nabla_{d_1}d_3 = (2k_2 \lambda)^{-1}\{2k_3(\sin \varphi_1)^{-1}\cos \varphi_3(\cos \varphi_3 \cos \varphi_1
- \cos \varphi_2)d_1 + k_3(\cos^2 \varphi_3 - \sin^2 \varphi_3)d_2\},
\]
\[
\nabla_{d_2}d_2 = k_3(k_2 \sin \varphi_1)^{-1}(\cos \varphi_2 - \cos \varphi_3 \cos \varphi_1)
\{\sin \varphi_1)^{-1}d_1 - \lambda^{-1}\cos \varphi_3 d_3\},
\]
\[
\nabla_{d_2}d_3 = k_3(k_2 \lambda \sin \varphi_1)^{-1}\{(2 \sin \varphi_1)^{-1}
(2\lambda^2 + \sin^2 \varphi_1 \cos^2 \varphi_3 - \sin^2 \varphi_1 \sin^2 \varphi_3)d_1
+ \cos \varphi_3(\cos \varphi_2 - \cos \varphi_3 \cos \varphi_1)d_2\},
\]
\[
\nabla_{d_3}d_3 = k_3(k_2 \sin \varphi_1)^{-1}
\{(\cos \varphi_3 \cos \varphi_1 - \cos \varphi_2)\ d_1 + \cos \varphi_3 d_2\},
\]
\[
\nabla_{d_a}d_4 = 0 \quad (a = 1, 2, 3, 4).
\]
Putting $g(R(d_a, d_b) d_c, d_e) =: R^c_{abc}$, we have from (3.5)

\[
R_{212}^1 = k_3^2 (4 k_2^2 \lambda^2 \sin^2 \varphi_1)^{-1} (-3 + 3 \cos^2 \varphi_1 + 4 \cos^2 \varphi_2 + 4 \cos^2 \varphi_3 - 8 \cos \varphi_1 \cos \varphi_2 \cos \varphi_3),
\]

\[
R_{313}^1 = k_3^2 (4 k_2^2 \lambda^2 \sin^2 \varphi_1)^{-1} \{\sin^2 \varphi_1 - 4 (\cos \varphi_2 - \cos \varphi_3 \cos \varphi_1)^2\},
\]

\[
R_{323}^2 = k_3^2 (4 k_2^2 \lambda^2 \sin^2 \varphi_1)^{-1} (4 \cos^2 \varphi_3 \cos^2 \varphi_1 - 4 \cos^2 \varphi_3 - \cos^2 \varphi_1 + 1),
\]

\[
R_{223}^1 = -k_3^2 \cos \varphi_3 (k_2^2 \lambda \sin \varphi_1)^{-1},
\]

\[
R_{231}^1 = k_3^2 (\cos \varphi_3 \cos \varphi_1 - \cos \varphi_2)(k_2^2 \lambda \sin^2 \varphi_1)^{-1},
\]

\[
R_{123}^3 = k_3^2 \cos \varphi_3 (\cos \varphi_3 \cos \varphi_1 - \cos \varphi_2)(k_2^2 \lambda \sin^2 \varphi_1)^{-1},
\]

\[
R_{4ab}^c = R_{4ab}^c = 0.
\]

Let $\rho(= \rho_g)$ be the Ricci operator on $(G, g)$, that is,

\[
(3.7) \quad \rho(X, Y) = \sum_{a=1}^4 g(R(X, d_a) d_a, Y) \ (X, Y \in \mathfrak{X}(G)).
\]

Putting $\rho(d_a, d_b) =: \rho_{ab}$ $(a, b = 1, 2, 3, 4)$, we obtain from (3.6) and (3.7)

\[
\rho_{11} = k_3^2 (\cos^2 \varphi_3 - \sin^2 \varphi_3)(2 k_2^2 \lambda^2)^{-1},
\]

\[
\rho_{22} = k_3^2 \{\sin^2 \varphi_1 (2 \sin^2 \varphi_3 - 1) - 2 \lambda^2\}(2 k_2^2 \lambda^2 \sin^2 \varphi_1)^{-1},
\]

\[
\rho_{33} = k_3^2 (2 \lambda^2 - \sin^2 \varphi_1)(2 k_2^2 \lambda^2 \sin^2 \varphi_1)^{-1},
\]

\[
\rho_{12} = k_3^2 \cos \varphi_3 (\cos \varphi_2 - \cos \varphi_3 \cos \varphi_1)(k_2^2 \lambda^2 \sin^2 \varphi_1)^{-1},
\]

\[
\rho_{23} = k_3^2 (\cos \varphi_2 - \cos \varphi_3 \cos \varphi_1)(k_2^2 \lambda \sin \varphi_1)^{-1},
\]

\[
\rho_{31} = k_3^2 \cos \varphi_3 (k_2^2 \lambda \sin \varphi_1)^{-1},
\]

\[
\rho_{4a} = 0 \ (a = 1, 2, 3, 4).
\]

Generally for the Ricci operator $\rho$ on a Riemannian manifold $(M, g)$, the trace of $\rho$ is referred to as the scalar curvature on $(M, g)$.

In our situation, from (3.8) we get the following:

**Theorem 3.1.** Let $g$ be a left invariant Riemannian metric on $G(= H \times S^1)$ which corresponds to $(1, k_2, k_3, k_4, \varphi_1, \varphi_2, \varphi_3)(\in \mathfrak{M})$, and $\tilde{g}$ the $G$-invariant Riemannian metric on $M(= G/\Gamma)$ such that $\Pi^* \tilde{g} = g$. Then
the scalar curvature on \((M, \bar{g})\) is 
\[
-\frac{k_3^2}{2\lambda^2 k_2^2}.
\]

In general, for the Ricci curvature tensor field \(\text{Ric}\) of (0,2)-type in a Riemannian manifold \((M, g)\) and a nonzero vector \(x_p \in T_p(M)\),
\[
r(x_p) := \frac{\text{Ric}(x_p, x_p)}{\|x_p\|_g^2}.
\]
is said to be the Ricci curvature of \((M, g)\) with respect to \(x_p\). Moreover if \((M, g)\) has a constant Ricci curvature, then \((M, g)\) is said to be an Einstein manifold.

In our situation, we put \(S := (\rho_{ab})_{a,b} (\rho_{ab} = \rho(d_a, d_b))\). By a straightforward but lengthy computation, we get the characteristic equation of \(S\) from (3.8) as follows:
\[
(3.9) \quad |tI_4 - S| = t \left( t - \frac{k_3^2}{2k_2^2\lambda^2} \right) \left( t + \frac{k_3^2}{2k_2^2\lambda^2} \right)^2 = 0.
\]
Since \(S\) is real symmetric, by virtue of (3.9) we obtain the following

**Theorem 3.2.** Let \(g\) be a left invariant Riemannian metric on \(G(=H \times S^1)\) which corresponds to \((1, k_2, k_3, k_4, \varphi_1, \varphi_2, \varphi_3)\) \((\in \mathfrak{W})\), and \(\bar{g}\) the \(G\)-invariant Riemannian metric on \(M(= G/\Gamma)\) such that \(\Pi^* \bar{g} = g\). Then the Ricci curvature \(r\) on \((M, \bar{g})\) is estimated as follows;
\[
-\frac{k_3^2}{2k_2^2\lambda^2} \leq r \leq \frac{k_3^2}{2k_2^2\lambda^2}.
\]

**Remark 3.3.** By the help of (3.8), we obtain the following:

(* the product Riemannian manifold of the three dimensional Heisenberg group \(H\) and \(S^1\) with an arbitrarily given left invariant metric is not an Einstein manifold.

On the other hand, it is well known that three dimensional Einstein manifold is a space of constant curvature (cf. [3], p.293). Wolf (cf. [9]) showed the fact that three dimensional nilpotent Lie group is not a constant curvature space.

The above statement (*) follows from these facts.

More generally, the following theorem (cf. [4], Theorem 2.4) is well known;

**Let \(G\) be a nilpotent Lie group. Then, there does not exist any left invariant Einstein metric on \(G\).**

The above statement (*) also follows from this theorem.
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