On Quantum Complexity

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The ETH ansatz for matrix elements of a given operator in the energy eigenstate basis results in a notion of thermalization for a chaotic system. In this context for a certain quantity - to be found for a given model - one may impose a particular condition on its matrix elements in the energy eigenstate basis so that the corresponding quantity exhibit linear growth at late times. The condition is to do with a possible pole structure the corresponding matrix elements may have. Based on the general expectation of complexity one may want to think of this quantity as a possible candidate for the quantum complexity. We note, however, that for the explicit examples we have considered in this paper, there are infinitely many quantities exhibiting similar behavior.

II. QUANTUM COMPLEXITY

Our main motivation to propose a candidate for the quantum complexity comes from the holographic setup in which it is believed that the holographic complexity exhibit a linear growth at late times [1,2]. Therefore, it what follows for a given quantum system we would like to define a quantity exhibiting such a linear growth.

It is worth noting that the late time linear growth is rather a special behavior which is relatively well understood in the context of holographic complexity. In general, it is not clear whether any definition of complexity should fulfill such a requirement. We note, however, that for the certain definition of complexity and under certain condition it is still possible to see the phase of linear growth at late times. We will come back to this point later when we compare our general expression to a particular definition of complexity.

To proceed, following our notation in the previous section let us consider the time dependent expectation value of an observable given by (1). To be more general, one may impose on the matrix elements of an observable in energy eigenstate basis so that the corresponding expectation value exhibits time growth even though the system has been already reached the thermal equilibrium.

Therefore the quantum expectation value of an observable satisfying ETH will approach its thermal equilibrium value given by the micro canonical average for long enough times.

Of course our main concern in this note is not to explore the thermalization process of the system. Actually the aim of the present letter is to understand the late time behavior of a certain observable when the system is in the thermal equilibrium.

More precisely, within the context of the ETH, we are interested in exploring a possible condition we may impose on the matrix elements of an observable in energy eigenstate basis so that the corresponding expectation value exhibits time growth even though the system has been already reached the thermal equilibrium.

I. INTRODUCTION

For chaotic systems with a finite entropy \(S\), complexity is expected to grow for exponentially large times in the entropy, long after thermal equilibrium has been reached [1,2]. For such systems the notion of thermalization may be described by the eigenstate thermalization hypothesis (ETH) which gives an understanding of how an observable thermalizes to its thermal equilibrium value [3,4](for review see [5]).

To be concrete let us consider a Hamiltonian, \(H\), whose eigenvalues and eigenstates are denoted by \(E\) and \(|E\rangle\), respectively. Given a general state \(|\psi\rangle\), the quantum expectation value of an operator, \(O\), at given time is

\[
\langle O(t) \rangle = \langle \psi | e^{itH} O e^{-itH} | \psi \rangle = \int dE_1 dE_2 \, e^{i(E_1 - E_2)} \, \langle \psi | E_1 | \psi \rangle \, \langle O | E_2 \rangle \langle E_2 | \psi \rangle.
\]

In the context of the thermalization of a quantum chaotic system one is typically interested in the equal time averages of observables. More precisely, we would like to find the time average of \(O\) over a time interval, which will be eventually sent to infinity.

According to the ETH, thermalization occurs at the level of individual eigenstates of the Hamiltonian. In fact setting

\[
\varepsilon = \frac{E_1 + E_2}{2}, \quad \omega = E_1 - E_2,
\]

the ETH states that the matrix elements of observables in energy eigenstate basis obey the following ansatz [3]

\[
\langle E_1 | O | E_2 \rangle = \tilde{O}(\varepsilon) \delta_{E_1, E_2} + e^{-S(\varepsilon, \omega)} R_{E_1, E_2}(\epsilon, \omega)
\]

where \(\tilde{O}(\varepsilon)\) is the micro canonical average of the corresponding operator, \(S\) is thermodynamical entropy of the system, \(R(\epsilon, \omega)\) is a smooth function of its arguments and \(R\) is unit variance random function with zero mean.

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associated with an operator $O$ as follows

$$A_O(\beta, t) \equiv \langle O(t) \rangle_\beta = \langle \psi | e^{-\frac{i}{\hbar}Ht} O e^{-\frac{i}{\hbar}Ht} | \psi \rangle.$$  

(4)

Using the completeness condition of the energy eigenstates, $\int dE |E\rangle \langle E| = 1$, one finds

$$A_O(\beta, t) = \int_0^\infty dE_1 dE_2 e^{-\frac{i}{\hbar}(E_1 + E_2)\pi t} e^{\frac{i}{\hbar}(E_1 - E_2)\pi t} \times \rho_\psi(E_1, E_2) A(E_1, E_2),$$  

(5)

where

$$\rho_\psi(E_1, E_2) = \langle E_1 | \psi \rangle \langle \psi | E_2 \rangle = \rho(E_1) \rho(E_2),$$

$$A(E_1, E_2) = \langle E_1 | O | E_2 \rangle,$$  

(6)

and $\rho(E) = \langle E | \psi \rangle$ is the density of state.

As far as the time dependence of the corresponding quantum object is concerned, as we will see, the main role is played by the function $A$ given in the equation (5) ($A$-function) which is essentially the matrix elements of the operator $O$ in the energy eigenstates. In particular one would expect that for a typical operator the $A$-function follows the ETH ansatz and therefore the long time average of $A_O$ approaches that of micro canonical average of the corresponding operator.

In our case, however, since we are interested in the late time behavior of the quantum object, we will not perform the long time average and instead will look for a possible procedure from which the late time behavior of $A_O$ may be read. Actually we would like to see whether there is a condition under which the corresponding quantum quantity, $A_O$, keeps growing with time even though the whole system is reached thermal equilibrium.

More precisely, in what follows we will explore a possible condition one may put on the $A$-function so that the corresponding quantum object $A_O$ exhibits linear growth at late times. To proceed, since we are interested in the late time behavior, it useful to rewrite the expression (5) in terms of variables defined in [2]

$$A_O(\beta, t) = \int \infty d\varepsilon e^{-\beta \varepsilon} \int \infty d\omega e^{\omega t} \rho(\varepsilon + \frac{\omega}{2}) \rho(\varepsilon - \frac{\omega}{2}) \times A(\varepsilon, \omega),$$  

(7)

and then study the behavior of $A(\varepsilon, \omega)$ in the limit of $\omega \to 0$.

Actually, as it is evident form the above expression, the time dependence of $A_O$ should be read from the $\omega$- integral. Indeed, due to the simple factor of $e^{\omega t}$ in the integrand, using the Cauchy’s residue theorem with the assumption that the density of state $\rho(\varepsilon \pm \omega/2)$ is a smooth function in the limit of $\omega \to 0$, in order to get a non-trivial time dependence, the $A$-function must have a pole structure of order of $n$ for $n \geq 2$. In particular, for the case of a double pole structure where the $A$-function has the following limiting behavior

$$A(\varepsilon, \omega) = - \frac{a(\varepsilon)}{\omega^2} + \text{local terms}, \quad \text{for } \omega \to 0,$$  

(8)

with a positive smooth function $a(\varepsilon)$, one finds that the quantum object $A_O$ exhibits a linear growth at late times

$$A_O(\beta, t) = C_0 + \int \infty d\varepsilon e^{-\beta \varepsilon} \rho^2(\varepsilon) a(\varepsilon) (2\pi t),$$  

(9)

where $C_0$ is a time independent constant that is a function of $\beta$. It is worth noting that for poles of higher order, one generically gets power low time dependent behavior. We will come back to this point later.

Motivated by holographic complexity, having found a quantum object exhibiting a linear growth at late times, it is tempting to identify the corresponding quantum object, $A_O$, as the quantum complexity. To be precise, we would like to define complexity as follows.

For a quantum system the quantum complexity is defined by (8) for a particular operator $O$-to be found for a given system so that the associated $A$-function exhibits a double pole structure in the limit of $E_1 \to E_2$.

$$A(E_1, E_2) \approx - \frac{a(E_1, E_2)}{(E_1 - E_2)^2} + \text{local terms}$$  

(10)

where $a(E_1, E_2)$ is a smooth positive function.

Of course for a given quantum system and a given state, a priori, it is not obvious how to find the operator $O$ that results in the desired double pole structure for $A$-function. Moreover, in general, the corresponding quantity may not be given in terms of local operators.

It is worth mentioning that there are other quantities which could also exhibit linear growth at late times. These include, for example, the spectrum form factor that has a linear growth known as ramp phase (see for example [3]). We note, however, that the linear growth

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1 In what follows we will consider a quantum system with continuous spectrum. Therefore, our study is more appropriate for gravity or holographic field theories.

2 It is worth noting that the analytic continuation of $\omega$ to the complex plane evolves the closing of the contour through in the upper half plane where $\omega \to \infty$. It is then important to make sure that the integrand decays along this contour. Due to the presence of the the exponential factor, $e^{\omega t}$, the appropriate decay for whole integrand occurs for a density which is bounded in this limit. This should be also the case for the finite part of the $A$-function. Although in general, it is hard to explicitly show the desired behavior for these functions, in what follows we will assume that this is the case. Indeed our insight about this assumption comes for the JT-gravity where analytic form of the integrand is known (see the explicit example in the next section) and it is possible to see, rather explicitly, that it has the desired property. We would like to thank the referee for his/her comment on this point.
we have found in [9] must not be confused with that of the ramp phase mentioned above. Actually, the ramp phase is a consequence of subleading connected part of the density-density correlation, though in our case the linear growth at late times occurs at leading disconnected level which is, indeed, a distinctive behavior associated with the complexity.

Another point we would like to mention is that the quantum object defined in (5) is non-local. Our motivation to look for a non-local object as the complexity has mainly come from holography. Indeed the holographic complexity is conjectured to be proportional to the volume of an extremal hypersurface extending all the way behind the horizon of a black hole. For two sided eternal black hole the complexity is given by the Einstein-Rosen bridge connecting two boundaries [1, 2]. In particular in two dimensions it is given by the quantum geodesic length connecting two boundaries [14–16] which is a non-local object.

Therefore we would expect that quantum complexity should be defined in terms of a non-local quantum object. Although in general it might be difficult to understand the natural of non-locality for the quantum object [9], it is possible to partially understand this for the case where our definition of complexity reduces to that of Krylov complexity [8].

To further explore our proposal for complexity, in the next section we will present an explicit example in which one could identify a proper operator $O$, that results in a linear growth for $A_C$.

III. EXPLICIT EXAMPLE

Let us consider a quantum system with the following Hamiltonian

$$H = \frac{p^2}{2} + 2\mu e^{-x} + 2e^{-2x}. \tag{11}$$

Then the corresponding Schrödinger equation is

$$\left(-\frac{d^2}{dx^2} + 4\mu e^{-x} + 4e^{-2x}\right) \psi(x) = 2E\psi(x). \tag{12}$$

The eigenstate wave functions of the above equation are given in terms of the Whittaker function of the second kind with imaginary order

$$\psi_{\mu,E}(x) = e^{x/2} W_{\mu,\frac{i}{2}(4e^{-x})}. \tag{13}$$

Actually this Hamiltonian is used to study different aspects of two dimensional JT gravity (see e.g. [7, 9–11]). For general $\mu \neq 0$ it corresponds to JT gravity with end of the world brane whose tension is given by $\mu$. For the particular value of $\mu = \frac{1}{2}$ it may also be considered as supersymmetric version of JT gravity [12]. We note that this model could be also thought of as a Liouville quantum mechanics describing Sachdev-Ye-Kitaev Model [13].

Using this Hamiltonian the complexity of JT gravity has been also studied in [14–16]. Of course in what follows the relevance of this quantum system to the two-dimensional JT gravity is not important for us, and we will consider it as a one dimensional quantum system.

The orthogonality condition for the eigenstates $\psi_{\mu,E}(x)$ is [17]

$$\int_0^\infty dx \psi_{\mu,E_1}(x) \psi_{\mu,E_2}(x) = \frac{\delta(E_1 - E_2)}{\rho_\mu(E_1)}, \tag{14}$$

where

$$\rho_\mu(E) = \left| \Gamma \left( \frac{1}{2} + \mu + i\sqrt{2E} \right) \right|^2 \frac{\sinh 2\pi\sqrt{2E}}{4\pi^2}, \tag{15}$$

Following our proposal, the quantum complexity is given by the equation (5) whose $A$-function, using the coordinate system, is

$$A(E_1, E_2) = \int_0^\infty dx dx' \psi_{\mu,E_1}(x)\psi_{\mu,E_2}(x')f(x, x'), \tag{16}$$

where $f(x, x') = \langle x'|O|x'\rangle$. Motivated by the result of [15] we will consider $f(x, x') = \delta(x - x')x$ by which the the above $A$-function reads

$$A(E_1, E_2) = \int_0^\infty dx \psi_{\mu,E_1}(x)\psi_{\mu,E_2}(x)x. \tag{17}$$

Actually since the function $f$ may be interpreted as matrix elements of the operator in the coordinate basis, the above choice corresponds to the matrix elements of position operator that is obviously diagonal leading to a delta function. On the other hand since the wave function satisfies the Schrödinger equation, in this case, essentially the $A$-function is the average of the position operator.

By making use of the explicit expression for the wave function in terms of the Whittaker function, it is then straightforward to study the pole structure of the $A$-function. Indeed, using the variables defined in (2) and in the limit of $E_1 \to E_2$ one finds [15]

$$A(\varepsilon, \omega) = -\frac{\sqrt{2\varepsilon}}{2\pi\rho_\mu(\varepsilon)} \frac{1}{\omega^2} + \text{local terms}. \tag{18}$$

Therefore from the equation (5) one can find the late time behavior as follows

$$A(\beta, t) = C_0 + \int_0^\infty d\varepsilon e^{-\beta\varepsilon} \rho_\mu(\varepsilon)\sqrt{2\varepsilon} t \tag{19}$$

that is the linear growth, as expected.

If one recalls that the Hamiltonian (11) describes two dimensional JT-gravity it is possible to identify exactly the quantity given in the equation (5) computes. Indeed in this case it can be interpreted as the quantum expectation value of the geodesic length (wormhole) connecting two boundaries of a two sided 2d black hole (or a geodesic length connecting the boundary and an end of
the world brane) \cite{15, 16}. This means that the function $f(x, x')$ is just the (regularized) geodesic length. Therefore it is expected to get a linear growth behavior at late times.

As another example we note that in the context of random matrix model and its connection with chaos we are typically dealing with matter two point functions whose matrix elements in energy eigenstate basis have the following general form \cite{18}

$$O_{E_1, E_2} = \frac{\Gamma(\Delta + i(\sqrt{E_1} - \sqrt{E_2}))\Gamma(\Delta + i(\sqrt{E_1} + \sqrt{E_2}))}{\Gamma(2\Delta)} \quad (20)$$

where $\Delta$ is the dimension of the corresponding matter field. From this expression one may define an $A$-function as follows

$$A(E_1, E_2) = -\lim_{\Delta \to 0} \frac{d}{d\Delta} O_{E_1, E_2} \quad (21)$$

$$= -2\Gamma(i(\sqrt{E_1} - \sqrt{E_2}))\Gamma(i(\sqrt{E_1} + \sqrt{E_2}))^2. \quad (22)$$

It is then easy to see that in the limit of $E_1 \to E_2$ this $A$-function exhibits a double pole structure

$$A(E_1, E_2) = -\frac{4\pi\sqrt{\varepsilon}}{\sinh(2\pi\sqrt{\varepsilon})\omega^2} + \text{local terms}. \quad (22)$$

In fact, recalling the relation between random matrix model and two-dimensional JT gravity, the above expression corresponds to the case of $\mu = 0$ in \cite{15}.

It is also interesting to look at the rate of the complexity growth

$$\frac{dA_f(t)}{dt} = \int_0^\infty d\varepsilon e^{-\beta\varepsilon} \rho_\mu(\varepsilon) \sqrt{2\varepsilon}, \quad (23)$$

which may be compared with the Lloyd’s bound \cite{19}. Actually, in the context of holographic complexity in which the complexity may be computed using CA conjecture \cite{20} the rate of the complexity growth turns out to be twice of the energy of the system, saturating the Lloyd’s bound \cite{19}. In the present case, at low energies where $\rho_\mu(\varepsilon) \sim \sqrt{2\varepsilon}$ the above expression may be thought of as the average of energy in a canonical ensemble.

On the other hand, if one works with a non-normalized situation by dropping $1/Z$ factor, one could evaluate the rate of the complexity growth in the macro canonical ensemble by making use of the inverse Laplace transformation. In this case the corresponding rate is given by $\rho(E_0)/\sqrt{2E_0}$ which at low energies results in $\sim 2E_0$, reminding of the Lloyd’s bound. Here $E_0$ is the energy of the macro canonical ensemble.

In general for $\mu = 0$ the integral may be performed exactly to find

$$\frac{dA_f(t)}{dt} = \frac{2e^{-2\pi^2}}{\sqrt{2\pi}\beta} + \left(\frac{1}{2\pi} + \frac{4\pi^2}{\beta^2}\right)\text{Erf}\left(\frac{\sqrt{2\pi}}{\sqrt{\beta}}\right), \quad (24)$$

which at low temperatures goes as $\sim \beta^{-1/2}$ while at high temperatures it is $\sim \beta^{-1}$. Although for general $\mu$ the full expression for the rate of the complexity growth may not be written explicitly, asymptotic behaviors at low and high temperatures are the same as that of $\mu = 0$.

IV. DISCUSSIONS

In this letter we have defined a quantum object associated with a given operator in a chaotic system. We have demonstrated that under certain condition the corresponding quantum object exhibits linear growth at late times, much longer than the system reaches the thermal equilibrium.

We have shown that for a given operator $\mathcal{O}$-to be found for given system- if its matrix elements in the energy eigenstates exhibit a double pole structure at late times \cite{10}, the corresponding quantum object defined it in the equation (5) will have the linear growth at late times, which could be interpreted as the quantum complexity. Of course for a given state in a given system, a priori, it is not clear how to find the operator $\mathcal{O}$ with the above desired property. It is not even clear if it is a local operator.

In the context of thermalization of quantum system, one generally assumes that matrix elements follow the ETH ansatz. Though, as we have seen, in order to get a non-trivial time dependence at late times, the $A$-function should have poles of order of $n$ with $n \geq 2$. For $n > 2$ one generally gets power low growth at late times. For $n = 2$ it is a linear growth. Since having a linear growth at late times (at leading order) is a signature of the complexity \cite{1} which is expected to be the fastest growth, one may propose a hypothesis that the double pole structure is the highest pole structure the $A$-function could have.

It is worth mentioning that for a given chaotic model there could be several $f$’s (matrix elements in coordinate basis) that give double pole structure for $A$-function which result in the late time linear growth. For the explicit example we have presented in the previous section it is straightforward to see that for any functions in the form of $f(x, x') = \delta(x - x')x^m$, with integer $m$, one finds double pole structure. Moreover form the matrix elements \cite{20} it is easy to construct several $A$-functions with the desired property. They can be obtained by taking $\Delta \to 0$ limit of $m$th $\Delta$-derivative of the matrix elements \cite{20}.

This is very similar to the observation made in \cite{21} in the context of the holographic complexity where it was shown that there are infinite class of gravitational observables in asymptotically Anti-de Sitter space which living on codimension one slices of the geometry, that exhibit universal features as that in complexity. Namely they grow linearly in time at late times.

An other interesting feature of complexity is that it saturates at the very late times given by the exponential of the entropy of the system. It is then natural to see how the saturation could occur in this context.

To address this question we note that the the den-
term \( \rho(E_1, E_2) \) appearing in the expression of the quantum object \( \mathcal{A}_\mathcal{O} \) has the following general form

\[
\rho(E_1, E_2) = \rho(E_1)\rho(E_2) + \rho_c(E_1, E_2),
\]

where \( \rho_c \) represents the connected term meaning that it cannot be written in a factorized form of \( g_1(E_1)g_2(E_2) \) with \( g_{1,2} \) being arbitrary functions. Clearly form the first factorized term the above function reduces to that of \([5]\). The connected terms could have either perturbative or non-perturbative origins which may have generally non-trivial pole structure that could result in the saturation phase at very late times.

Actually this is a well known structure which has been seen in the literature for the spectrum form factor of chaotic models such as JT-gravity in which the pole structure of \( \rho(E_1, E_2) \) results in the ramp phase. Of course for the spectrum form factor there is no an \( A \)-function and the whole time dependence is controlled by the density-density correlator. On the other hand, for the holographic complexity of JT-gravity where there is an \( A \)-function, the connected part of \( \rho(E_1, E_2) \), which has non-trivial pole structure at late times is, indeed, responsible for the saturation phase \([14\text{-}16]\).

We note, however, that in the present case, where we are dealing with a general formalism which is not directly related to the holography picture, it is not clear how the full expression of the connected term could be computed. Nonetheless, for a chaotic system, as far as the late time behavior is concerned, one would expect that the main contribution comes from the short range correlation which is given by the universal sine-kernel term \([22]\)

\[
\rho_c(\varepsilon, \omega) \approx -\frac{\sin^2(D\omega\rho(\varepsilon))}{(D\omega)^2}, \quad \text{for } \omega \ll 1.
\]

Here \( D \) is the dimension of Hilbert space which is given by the exponential of the entropy of the system. Therefore the whole late time behavior of the quantum object \( \mathcal{A} \) is described as follows: the double point structure of the \( A \)-function leads to linear growth at the leading disconnected part of the density-density correlation, while there is the saturation phase which can be described by subleading connected term given in the universal sine-kernel term multiplied by the double pole structure of the \( A \)-function. It is then easy to see that the saturation occurs at \( t \sim D \).

To be precise plugging the expression \([25]\) into \([5]\) and using \([26]\) and \([8]\) one arrives at

\[
\begin{align*}
\mathcal{A} = \text{Constant} & - \int_0^\infty d\varepsilon e^{-\beta\varepsilon}\rho^2(\varepsilon)a(\varepsilon) \\
& \int_{-\infty}^\infty d\omega \frac{e^{-i\omega\varepsilon}}{\omega^2} \left( 1 - \frac{\sin^2(\rho(\varepsilon)D\omega)}{\rho(\varepsilon)D\omega} \right).
\end{align*}
\]

From this expression one observes that at late times when \( \omega \sim \frac{1}{D} \rightarrow 0 \) and for \( \rho \omega \gg 1 \) essentially the first term in the bracket on the r.h.s of \([27]\) dominates leading to a linear growth, while for \( \rho \omega \ll 1 \) which is the case at \( t \sim D \), the second term starts dominating that essentially cause the whole integral to approach zero leading to a constant complexity which is the saturation phase. For more details see \([14\text{-}16]\).

It is worth noting that due to the particular form of the quantum object \( \mathcal{A}_\mathcal{O} \) in which the \( A \)-function and \( \rho(E_1, E_2) \) appear in a product form, there is an alternative way to think about the saturation phase. Indeed the saturation may occur via an ETH-like behavior of \( A \)-function at very late times. In this case the disconnected part of density matrix is enough to see the saturation phase.

As a final comment we note that the structure we have presented in this letter has similar features as that of the Krylov complexity \([23\text{-}25]\). Actually, it can be shown that for a particular case our proposal for complexity reduces to that of Krylov complexity. In this case the \( A \)-function is given by the matrix elements in energy basis of a \textit{label operator} of an orthonormal and ordered basis (Krylov basis). More precisely, denoting the orthonormal ordered basis by \( \{|n\} \) the label operator is defined by \( \ell = \sum_n n \langle n | n \rangle \). Therefore one gets \( A^L(0,t) = \langle \ell(t) \rangle \) with \( \ell(t) = e^{-iHt}\ell e^{iHt} \) and the \( A \)-function reads \( A = \langle \ell | \ell | E_2 \rangle \). Although the linear growth in the Krylov complexity is rather a special case which may occur when the Lanczos coefficients saturate to a constant\(^3\), it is possible to perform explicit computations (at least in the continuum limit) to show that the saturation of Lanczos coefficients corresponds to the double pole structure of the \( A \)-function at late times (for more details see \([8]\)). Therefore our proposal would appropriately reproduce the linear growth of the Krylov complexity known in the literature \([24]\). Moreover, in this case the saturation occurs due to an ETH-like behavior of the label operator.

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\footnote{We note that the late time linear growth in the context of Krylov complexity for generic chaotic systems has been first observed in \([24]\) where it was shown that the saturation of Lanczos coefficients to a constant results in the linear growth at sufficiently late times, much after the scrambling time.}

\[3\] I would like to thank Julian Sonner for pointing out this to me.
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