LIMITING BEHAVIOUR OF MOVING AVERAGE PROCESSES
GENERATED BY NEGATIVELY DEPENDENT RANDOM
VARIABLES UNDER SUB-LINEAR EXPECTATIONS

Mingzhou Xu 1  Kun Cheng 2  Wangke Yu 3
School of Information Engineering, Jingdezhen Ceramic University
Jingdezhen 333403, China

Abstract. Let \( \{Y_i, -\infty < i < \infty\} \) be a doubly infinite sequence of identically distributed, negatively dependent random variables under sub-linear expectations, \( \{a_i, -\infty < i < \infty\} \) be an absolutely summable sequence of real numbers. In this article, we study complete convergence and Marcinkiewicz-Zygmund strong law of large numbers for the partial sums of moving average processes \( \{X_n = \sum_{i=-\infty}^{\infty} a_i Y_{i+n}, n \geq 1\} \) based on the sequence \( \{Y_i, -\infty < i < \infty\} \) of identically distributed, negatively dependent random variables under sub-linear expectations, complementing the result of [Chen, et al., 2009. Limiting behaviour of moving average processes under \( \varphi \)-mixing assumption. Statist. Probab. Lett. 79, 105-111].

1. INTRODUCTION

[Peng(2007), Peng(2010), Peng(2019)] firstly introduced the important concepts of the sub-linear expectations space to study the uncertainty in probability. Inspired by the seminal works of [Peng(2007), Peng(2010), Peng(2019)], many scholars try to study the results under sub-linear expectations space, generalizing the corresponding ones in classic probability space. [Zhang(2015), Zhang(2016a), Zhang (2016b)] established Donsker’s invariance principle, exponential inequalities and Rosenthal’s inequality under sub-linear expectations. [Wu(2020)] obtained precise asymptotics for complete integral convergence under sub-linear expectations. Under sub-linear expectations, [Xu and Cheng(2022a)] investigated how small the increments of \( G \)-Brownian motion are. For more limit theorems under sub-linear expectations, the interested readers could refer to [Xu and Zhang(2019), Xu and Zhang(2020), Wu and Jiang(2018), Zhang and Lin(2018), Zhong and Wu(2017), Hu and Yang(2017), Chen(2016), Chen and Wu(2022), Zhang(2016c), Hu et al.(2014)Hu, Chen, and Zhang

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1Email: mingzhouxu@whu.edu.cn

2Email: chengkun0010@126.com

3Email: ywkyyyy@163.com
In classic probability space, \cite{Zhang and Ding(2017)} studied the complete moment convergence of the partial sums of moving average processes under some proper assumptions, \cite{Chen et al.(2009)} proved complete convergence of moving average processes under $\varphi$-mixing assumption. For references on complete convergence and complete moment convergence in linear expectation space, the interested reader could refer to \cite{Ko(2015)}, \cite{Meng et al.(2021)}, \cite{Hosseini and Nezakati(2019)}, \cite{Meng et al.(2022)} and references therein. Inspired by the work of \cite{Chen et al.(2009)}, we try to study complete convergence and Marcinkiewicz-Zygmund strong law of large numbers for the partial sums of moving average processes generated by negatively dependent random variables under sub-linear expectations, which complements the corresponding results in \cite{Meng et al.(2022)}.

We organize the rest of this paper as follows. We give necessary basic notions, concepts and relevant properties, and cite necessary lemmas under sub-linear expectations in the next section. In Section 3, we give our main results, Theorems 3.1-3.2, the proofs of which are presented in Section 4.

2. Preliminaries

As in \cite{Xu and Cheng(2021a)}, we use similar notations as in the work by \cite{Peng(2010)}, \cite{Peng(2019)}, \cite{Chen(2016)}, \cite{Zhang(2016b)}. Assume that $(\Omega, \mathcal{F})$ is a given measurable space. Suppose that $\mathcal{H}$ is a subset of all random variables on $(\Omega, \mathcal{F})$ such that $X_1, \ldots, X_n \in \mathcal{H}$ implies $\varphi(X_1, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l\text{-Lip}}(\mathbb{R}^n)$, where $C_{l\text{-Lip}}(\mathbb{R}^n)$ represents the linear space of (local lipschitz) function $\varphi$ fulfilling

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)(|x - y|), \forall x, y \in \mathbb{R}^n$$

for some $C > 0$, $m \in \mathbb{N}$ depending on $\varphi$.

**Definition 2.1.** A sub-linear expectation $\mathbb{E}$ on $\mathcal{H}$ is a functional $\mathbb{E} : \mathcal{H} \mapsto \bar{\mathbb{R}} := [-\infty, \infty]$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(a): Monotonicity: If $X \geq Y$, then $\mathbb{E}[X] \geq \mathbb{E}[Y]$;
(b): Constant preserving: $\mathbb{E}[c] = c, \forall c \in \mathbb{R}$;
(c): Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X], \forall \lambda \geq 0$;
(d): Sub-additivity: $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$ whenever $\mathbb{E}[X] + \mathbb{E}[Y]$ is not of the form $\infty - \infty$ or $-\infty + \infty$.

A set function $V : \mathcal{F} \mapsto [0, 1]$ is named to be a capacity if

(a): $V(\emptyset) = 0, V(\Omega) = 1$;
(b): $V(A) \leq V(B), A \subset B, A, B \in \mathcal{F}$.

Moreover, if $V$ is continuous, then $V$ should obey

(c): $V(A_n) \uparrow V(A)$, if $A_n \uparrow A$.
(d): $V(A_n) \downarrow V(A)$, if $A_n \downarrow A$. 

Gao and Xu(2011), Kuczmaszewska(2020), Xu and Cheng(2021c), Xu and Cheng(2021b), Xu and Cheng(2021a), Xu and Cheng(2022a), Xu and Cheng(2022 b) and references therein.
A capacity $V$ is called to be sub-additive if $V(A + B) \leq V(A) + V(B)$, $A, B \in \mathcal{F}$.

In this article, given a sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, set $\mathbb{V}(A) := \inf\{\mathbb{E}[\xi] : I_A \leq \xi, \xi \in \mathcal{H}\}$, $\forall A \in \mathcal{F}$ (see (2.3) and the definitions of $\mathbb{V}$ above (2.3) in [Zhang(2016a)]. $\mathbb{V}$ is a sub-additive capacity. Define

$$C_V(X) := \int_{0}^{\infty} V(X > x)dx + \int_{-\infty}^{0} (V(X > x) - 1)dx.$$  

Suppose that $X = (X_1, \cdots, X_m)$, $X_i \in \mathcal{H}$ and $Y = (Y_1, \cdots, Y_n)$, $Y_i \in \mathcal{H}$ are two random vectors on $(\Omega, \mathcal{H}, \mathbb{E})$. $Y$ is named to be negatively dependent to $X$, if for each Borel-measurable function $\psi_1$ on $\mathbb{R}^m$, $\psi_2$ on $\mathbb{R}^n$, we have $\mathbb{E}[\psi_1(X)\psi_2(Y)] \leq \mathbb{E}[\psi_1(X)]\mathbb{E}[\psi_2(Y)]$ whenever $\psi_1(X) \geq 0$, $\mathbb{E}[\psi_2(Y)] \geq 0$, $\mathbb{E}[\psi_1(X)\psi_2(Y)] < \infty$, $\mathbb{E}[\psi_1(X)] < \infty$, $\mathbb{E}[\psi_2(Y)] < \infty$, and either $\psi_1$ and $\psi_2$ are coordinatewise nondecreasing or $\psi_1$ and $\psi_2$ are coordinatewise nonincreasing (see Definition 2.3 of [Zhang(2016a)], Definition 1.5 of [Zhang(2016b)], Definition 2.5 in [Chen(2016)]). $\{X_n\}_{n=1}^{\infty}$ is called a sequence of negatively dependent random variables, if $X_{n+1}$ is negatively dependent to $(X_1, \cdots, X_n)$ for each $n \geq 1$.

Assume that $X_1$ and $X_2$ are two $n$-dimensional random vectors defined, respectively, in sub-linear expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$. They are named identically distributed if for every Borel-measurable function $\psi$ such that $\psi(X_1) \in \mathcal{H}_1, \psi(X_2) \in \mathcal{H}_2$,

$$\mathbb{E}_1[\psi(X_1)] = \mathbb{E}_2[\psi(X_2)],$$  

whenever the sub-linear expectations are finite. $\{X_n\}_{n=1}^{\infty}$ is called to be identically distributed if for each $i \geq 1$, $X_i$ and $X_1$ are identically distributed.

In this article we suppose that $\mathbb{E}$ is countably sub-additive, i.e., $\mathbb{E}(X) \leq \sum_{n=1}^{\infty} \mathbb{E}(X_n)$, whenever $X \leq \sum_{n=1}^{\infty} X_n$, $X, X_n \in \mathcal{H}$, and $X \geq 0$, $X_n \geq 0$, $n = 1, 2, \ldots$. Write $S_n = \sum_{i=1}^{n} X_i$, $n \geq 1$. Let $C$ represent a positive constant which may differ from place to place. $I(A)$ or $I_A$ stand for the indicator function of $A$.

As discussed in [Zhang(2016b)], by the definition of negative dependence, if $X_1, X_2, \ldots, X_n$ are negatively dependent random variables and $f_1, f_2, \ldots, f_n$ are all non increasing (or non decreasing) functions, then $f_1(X_1), f_2(X_2), \ldots, f_n(X_n)$ are still negatively dependent random variables.

We cite the following inequalities under sub-linear expectations.

**Lemma 2.2.** (See Lemma 4.5 (iii) of [Zhang(2016a)]) If $\mathbb{E}$ is countably sub-additive under sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, then for $X \in \mathcal{H}$,

$$\mathbb{E}|X| \leq C_V(|X|).$$  

**Lemma 2.3.** (See Theorem 2.1 of [Zhang(2016b)]) Assume that $p > 1$ and $\{Y_n; n \geq 1\}$ is a sequence of negatively dependent random variables under sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Then for each $n \geq 1$, there exists a positive constant $C = C(p)$ depending on $p$
such that for $p \geq 2$,
\[
E \left[ \max_{0 \leq i \leq n} \left| \sum_{j=1}^{i} Y_j \right|^p \right] 
\leq C \left\{ \sum_{i=1}^{n} E|Y_i|^p + \left( \sum_{i=1}^{n} EY_i^2 \right)^{p/2} + \left( \sum_{i=1}^{n} \left[ (-E(-Y_i))^+ + (E(Y_i))^+ \right] \right)^p \right\}. \tag{1}
\]

3. Main Results

Our main results are the following.

**Theorem 3.1.** Assume that $h$ is a function slowly varying at infinity, $1 \leq p < 2$, and $r > 1$. Let $\{X_n, n \geq 1\}$ be a moving average process based on a sequence of $\{Y_i, -\infty < i < \infty\}$ of negatively dependent random variables, identically distributed as $Y$ under sub-linear expectation space $(\Omega, \mathcal{H}, E)$. Suppose that $E(Y) = E(-Y) = 0$ and $C_V(|Y|^p h(|Y|^p)) < \infty$. Then
(i) $\sum_{n=1}^{\infty} n^{-2} h(n) \mathbb{V} \left\{ \max_{1 \leq k \leq n} |S_k| \geq \varepsilon n^{1/p} \right\} < \infty$, for all $\varepsilon > 0$,
and
(ii) $\sum_{n=1}^{\infty} n^{-2} h(n) \mathbb{V} \left\{ \sup_{k \geq n} |S_k/k^{1/p}| \geq \varepsilon \right\} < \infty$, for all $\varepsilon > 0$.

The following theorem investigates the case $r = 1$.

**Theorem 3.2.** Assume that $h$ is a function slowly varying at infinity and $1 \leq p < 2$. Suppose that $\sum_{i=-\infty}^{\infty} |a_i|^{1/\theta} < \infty$, where $\theta \in (0, 1)$ if $p = 1$ and $\theta = 1$ if $1 < p < 2$. Assume that $\{X_n, n \geq 1\}$ is a moving average process based on a sequence of $\{Y_i, -\infty < i < \infty\}$ of negatively dependent random variables, identically distributed as $Y$ under sub-linear expectation space $(\Omega, \mathcal{H}, E)$. Suppose that $E(Y) = E(-Y) = 0$ and $C_V(|Y|^p h(|Y|^p)) < \infty$.

Then
\[
\sum_{n=1}^{\infty} \frac{h(n)}{n} \mathbb{V} \left\{ \max_{1 \leq k \leq n} |S_k| \geq \varepsilon n^{1/p} \right\} < \infty, \text{ for all } \varepsilon > 0.
\]

In particular, the conditions that $EY = E(-Y) = 0$, $C_V(|Y|^p) < \infty$ and $\mathbb{V}$ is continuous imply the following Marcinkiewicz-Zygmund strong law of large numbers, $S_n/n^{1/p} \to 0$ a.s. $\mathbb{V}$, i.e.,
\[
\mathbb{V} \left\{ \Omega \setminus \left\{ \lim_{n \to \infty} S_n/n^{1/p} = 0 \right\} \right\} = 0.
\]

**Remark 3.3.** Theorem 3.2 complements Theorem 1 for independent, identically distributed random variables under sub-linear expectations in [Zhang and Lin(2018)].

4. Proofs of Main Results

We first present some lemmas.

**Lemma 4.1.** Suppose $r > 1$, and $1 \leq p < 2$. Then for any $\varepsilon > 0$,
\[
\sum_{n=1}^{\infty} n^{-2} h(n) \mathbb{V} \left\{ \sup_{k \geq n} |S_k/k^{1/p}| \geq \varepsilon \right\} \leq \sum_{n=1}^{\infty} n^{-2} h(n) \mathbb{V} \left\{ \sup_{k \geq n} |S_k| \geq \left( \varepsilon/2^{2/p} \right) n^{1/p} \right\}.
\]
Proof. We see that
\[ \sum_{n=1}^{\infty} n^{r-2} h(n) \mathbb{V} \left\{ \sup_{k \geq n} \left| S_k/k^{1/p} \right| \geq \varepsilon \right\} = \sum_{m=1}^{\infty} \sum_{n=2^{m-1}}^{2^{m-1}} n^{r-2} h(n) \mathbb{V} \left\{ \sup_{k \geq n} \left| S_k/k^{1/p} \right| \geq \varepsilon \right\} \]
\[ \leq C \sum_{m=1}^{\infty} \mathbb{V} \left\{ \sup_{k \geq 2^{m-1}} \left| S_k/k^{1/p} \right| \geq \varepsilon \right\} \sum_{n=2^{m-1}}^{2^{m-1}} 2^{m(r-2)} h(2^m) \]
\[ \leq C \sum_{m=1}^{\infty} 2^{m(r-1)} h(2^m) \mathbb{V} \left\{ \sup_{k \geq 2^{m-1}} \left| S_k/k^{1/p} \right| \geq \varepsilon \right\} \]
\[ = C \sum_{m=1}^{\infty} 2^{m(r-1)} h(2^m) \mathbb{V} \left\{ \max_{1 \leq k < 2^l} |S_k| \geq \varepsilon 2^{(l-1)/p} \right\} \]
\[ \leq C \sum_{l=1}^{\infty} 2^{l(r-1)} h(2^l) \mathbb{V} \left\{ \max_{1 \leq k < 2^l} |S_k| \geq \varepsilon 2^{(l-1)/p} \right\} \]
\[ \leq C \sum_{l=1}^{\infty} \sum_{n=2^l}^{2^{l+1}-1} n^{r-2} h(n) \mathbb{V} \left\{ \max_{1 \leq k \leq n} |S_k| \geq \varepsilon / 2^{2/p} n^{1/p} \right\} \]
\[ \leq C \sum_{n=1}^{\infty} n^{r-2} h(n) \mathbb{V} \left\{ \max_{1 \leq k \leq n} |S_k| \geq \varepsilon / 2^{2/p} n^{1/p} \right\} . \]

Lemma 4.2. Let \( Y \) be a random variable with \( C_V (|Y|^p h(|Y|^p)) < \infty \), where \( r \geq 1 \), and \( p \geq 1 \). Write \( Y' = -n^{-1/p} I \{ Y < -n^{-1/p} \} + Y I \{ Y \leq n^{1/p} \} + n^{1/p} I \{ Y > n^{1/p} \} \). If \( q > rp \), then
\[ \sum_{n=1}^{\infty} n^{r-1-q/p} h(n) \mathbb{E} |Y'|^q \leq C C_V (|Y|^p h(|Y|^p)) . \]

Proof. Since \( r - q/p < 0 \), by Lemma 2.2 and similar proof of Lemma 2.2 in Zhong and Wu (2017), we see that
\[ \sum_{n=1}^{\infty} n^{r-1-q/p} h(n) \mathbb{E} |Y'|^q \leq \sum_{n=1}^{\infty} n^{r-1-q/p} h(n) C_V \{|Y'|^q\} \]
\[ \leq \sum_{n=1}^{\infty} n^{r-1-q/p} h(n) \int_0^{n^{1/p}} \mathbb{V} \{|Y'|^q > x^q\} q x^{q-1} dx \]
\[ \leq C \int_1^{\infty} y^{r-1-q/p} h(y) \left[ \int_0^{y^{1/p}} \mathbb{V} \{|Y'|^q > x^q\} x^{q-1} dx \right] dy \]
\[ \leq C \int_0^{1} \mathbb{V} \{|Y'| > x\} dx \int_1^{\infty} y^{r-1-q/p} h(y) dy . \]
\begin{align*}
&+ C \int_1^\infty \mathbb{V} \{ |Y| > x \} x^{q-1} \int_x^\infty y^{r-1-q/p} h(y) dy dx \\
&\leq C + C \int_1^\infty \mathbb{V} \{ |Y| > x \} h(x^p) x^{r_p-1} dx \\
&\leq CC_V (|Y|^p h(|Y|^p)) < \infty.
\end{align*}

\[ \square \]

In the rest of this paper, let \( \frac{1}{2} < \mu < 1 \), \( g(y) \in C_{l,Lip}(\mathbb{R}) \), such that \( 0 \leq g(y) \leq 1 \) for all \( y \) and \( g(y) = 1 \) if \( |y| \leq \mu \), \( g(y) = 0 \), if \( |y| > 1 \). And \( g(y) \) is a decreasing function for \( y \geq 0 \). The next lemma presents an important fact in the proofs of Theorems 3.1 and 3.2.

**Lemma 4.3.** Assume that \( h \) is a function slowly varying at infinity and \( p \geq 1 \). Assume that \( \{X_n, n \geq 1\} \) is a moving average process based on a sequence of \( \{Y_i, -\infty < i < \infty\} \) of negatively dependent random variables, identically distributed as \( Y \) with \( \mathbb{E}(Y) = \mathbb{E}(-Y) = 0 \), \( C_V (|X|^p) < \infty \) under sub-linear expectation space \( (\Omega, \mathcal{H}, \mathbb{E}) \). For any \( \varepsilon > 0 \), write

\[ I := \sum_{n=1}^\infty n^{-2} h(n) \mathbb{V} \left\{ \max_{1 \leq k \leq n} \left| \sum_{i=-\infty}^{i=k} a_i \sum_{j=i+1}^{i+k} Y_j'' \right| \geq \varepsilon n^{1/p}/2 \right\}, \]

and

\[ II := \sum_{n=1}^\infty n^{-2} h(n) \mathbb{V} \left\{ \max_{1 \leq k \leq n} \left| \sum_{i=-\infty}^{i=k} a_i \sum_{j=i+1}^{i+k} (Y_j' - \mathbb{E}[Y_j']) \right| \geq \varepsilon n^{1/p}/4 \right\}, \]

where

\[ Y_j' = -n^{1/p} I \{ Y_j < -n^{-1/p} \} + |Y_j| I \{ |Y_j| \leq n^{1/p} \} + n^{1/p} I \{ Y_j > n^{1/p} \}, \]

\[ Y_j'' = Y_j - Y_j' = (Y_j + n^{1/p}) I \{ Y_j < -n^{1/p} \} + (Y_j - n^{1/p}) I \{ Y_j > n^{1/p} \}. \]

If \( I < \infty \) and \( II < \infty \), then

\[ \sum_{n=1}^\infty n^{-2} h(n) \mathbb{V} \left\{ \max_{1 \leq k \leq n} |S_k| \geq \varepsilon n^{1/p} \right\} \leq I + II < \infty. \]

**Proof.** Observe that

\[ \sum_{k=1}^n X_k = \sum_{k=1}^n \sum_{i=-\infty}^n a_i Y_{i+k} = \sum_{i=-\infty}^n a_i \sum_{j=i+1}^{i+n} Y_j. \]

By \( \sum_{i=-\infty}^\infty |a_i| < \infty \), \( \mathbb{E}(Y_j) = \mathbb{E}(-Y_j) = 0 \), and Proposition 1.3.7 of [Peng(2019)], Lemma 2.2, we have

\[ n^{-1/p} \left| \sum_{i=-\infty}^\infty a_i \sum_{j=i+1}^{i+n} \mathbb{E} Y_j' \right| = n^{-1/p} \left| \sum_{i=-\infty}^\infty a_i \sum_{j=i+1}^{i+n} \mathbb{E} [Y_j' - Y_j] \right| \]

\[ \leq n^{-1/p} \sum_{i=-\infty}^\infty |a_i| \sum_{j=i+1}^{i+n} \mathbb{E} |Y_j - Y_j'| \leq C n^{-1/p} \mathbb{E} |Y_1'| \leq C n^{-1/p} \mathbb{E}(n^{-1/p} Y_1'' | Y_1'|)^{p-1} |Y_1'|^p \]

\[ \leq C n^{-1/p} \mathbb{E}(Y_1'| Y_1'|^p \left( 1 - g \left( \frac{|Y_1|}{n^{1/p}} \right) \right) \leq CC_V \left\{ |Y_1|^p \left( 1 - g \left( \frac{|Y_1|}{n^{1/p}} \right) \right) \right\} \]

\[ \leq CC_V \left\{ |Y_1|^p I \{ |Y_1| \geq \mu n^{1/p} \} \right\} \to 0, \ n \to 0. \]
Therefore for \( n \) sufficiently large, we see that

\[
n^{-1/p} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} \mathbb{E} Y_j' \right| < \varepsilon/4.
\]

Then

\[
\sum_{n=1}^{\infty} n^{-2} h(n) \mathbb{V} \left\{ \max_{1 \leq k \leq n} |S_k| \geq \varepsilon n^{1/p} \right\}
\leq C \sum_{n=1}^{\infty} n^{-2} h(n) \mathbb{V} \left\{ \max_{1 \leq k \leq n} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_j'' \right| \geq \varepsilon n^{1/p}/2 \right\}
\]

\[
+ \sum_{n=1}^{\infty} n^{-2} h(n) \mathbb{V} \left\{ \max_{1 \leq k \leq n} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} (Y_j' - \mathbb{E} Y_j') \right| \geq \varepsilon n^{1/p}/4 \right\}
\]

\[=: I + II.\]

Proof of Theorem 3.1. According to Lemma 4.1, it is enough to establish that (i) holds. By Lemma 4.3, we only need to establish that \( I < \infty \) and \( II < \infty \).

For \( I \), by Markov inequality under sub-linear expectations, Lemma 2.2 and similar proof of Lemma 2.2 in [Zhong and Wu(2017)], we obtain

\[
I \leq C \sum_{n=1}^{\infty} n^{-1-1/p} h(n) \mathbb{E} |Y''_1|
\leq C \sum_{n=1}^{\infty} n^{-1-1/p} h(n) C_Y \{ |Y''_1| \}
\leq C \sum_{n=1}^{\infty} n^{-1-1/p} h(n) \int_{0}^{\infty} \mathbb{V} \{ |Y''_1| > x \} \, dx
\leq C \sum_{n=1}^{\infty} n^{-1-1/p} h(n) \left[ \mathbb{V} \{ |Y| > n^{1/p} \} n^{1/p} + \int_{n^{1/p}}^{\infty} \mathbb{V} \{ |Y| > x \} \, dx \right]
\leq C \int_{1}^{\infty} x^{-1} h(x) \mathbb{V} \{ |Y| > x^{1/p} \} \, dx + C \int_{1}^{\infty} \mathbb{V} \{ |Y| > x \} \, dx \int_{1}^{x^{p}} y^{-1-1/p} h(y) \, dy
\leq C \int_{1}^{\infty} \mathbb{V} \{ |Y|^p h(|Y|^p) > x^p h(x) \} \, dx + C \int_{1}^{\infty} \mathbb{V} \{ |Y| > x \} \, dx \int_{1}^{x^{p}} \mathbb{V} \{ |Y| > x \} \, dx \int_{1}^{x^{p}} y^{-1-1/p} h(y) \, dy
\leq CC_V \{ |Y|^p h(|Y|^p) \} + C \int_{1}^{\infty} \mathbb{V} \{ |Y| > x \} \, dx \int_{1}^{x^{p}} y^{-1-1/p} h(x^{p}) \, dx
\leq CC_V \{ |Y|^p h(|Y|^p) \} < \infty.
\]
For $II$, by Markov inequality under sub-linear expectations, Hölder inequality, Lemma \ref{lem:holder}, we see that for any $q > 2$,

\[
II \leq C \sum_{n=1}^{\infty} n^{r-2} h(n) n^{-q/p} E \max_{1 \leq k \leq n} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} (Y'_j - E[Y'_j]) \right|^q
\]

\[
\leq C \sum_{n=1}^{\infty} n^{r-2} h(n) n^{-q/p} E \left[ \sum_{i=-\infty}^{\infty} |a_i|^{1/q} \max_{1 \leq k \leq n} \left| \sum_{j=i+1}^{i+k} (Y'_j - E[Y'_j]) \right| \right]^q
\]

\[
\leq C \sum_{n=1}^{\infty} n^{r-2-q/p} h(n) \left( \sum_{i=-\infty}^{\infty} |a_i| \right)^{q-1} \sum_{i=-\infty}^{\infty} |a_i| E \max_{1 \leq k \leq n} \left| \sum_{j=i+1}^{i+k} (Y'_j - E[Y'_j]) \right|^q
\]

\[
\leq C \sum_{n=1}^{\infty} n^{r-2-q/p} h(n) \left[ n \left( |E[-Y'_1]| + |E[Y'_1]| \right) \right]^q
\]

\[
+ C \sum_{n=1}^{\infty} n^{r-2-q/p} h(n) \left( nE|Y'_1|^2 \right)^{q/2}
\]

\[
+ C \sum_{n=1}^{\infty} n^{r-1-q/p} h(n) E|Y'_1|^q
\]

\[= II_1 + II_2 + II_3.\]

To establish $II_1 < \infty$, in view of $E(-Y) = E(Y) = 0$, and $|E(X) - E(Y)| \leq E|X - Y|$, by Lemma \ref{lem:koenig}, we see that

\[
\left[ n \left( |E[-Y'_1]| + |E[Y'_1]| \right) \right]^q
\]

\[
\leq \left[ n \left( |E[-Y'_1] - E[-Y_1]| + |E[Y'_1] - E[Y_1]| \right) \right]^q
\]

\[
\leq C n^q \left( |E[Y'_1]| \left( 1 - g \left( \frac{|Y_1|}{n^{1/p}} \right) \right) \right)^q
\]

\[
\leq C n^q \left( C_V \left\{ |Y_1| \left( 1 - g \left( \frac{|Y_1|}{n^{1/p}} \right) \right) \right\} \right)^q
\]

\[
\leq C n^q \left( C_V \left\{ |Y_1| I\{|Y_1| > \mu n^{1/p} \} \right\} \right)^q
\]

\[
\leq C n^q \left( \int_0^\infty \mathbb{P}\left\{ |Y|^p h(|Y|^p) I\{|Y| > \mu n^{1/p} \} > x \mu n^{(r-1)/ph(n)} \right\} dx \right)^q
\]

\[
\leq C n^q \left( \frac{C_V \left\{ |Y|^p h(|Y|^p) \right\}}{n^{(r-1)/ph(n)}} \right)^q \ll C n^{q-qr+q/p} / h(n)^q.
\]

Hence

\[
II_1 \leq C \sum_{n=1}^{\infty} n^{r-2-q/p} h(n) n^{q-qr+q/p} / h(n)^q
\]

\[
\leq C \sum_{n=1}^{\infty} n^{r-2-q(r-1)} h(n)^{1-q} < \infty.
\]
To prove $II_2 < \infty$, we study two cases. If $rp < 2$, take $q > 2$, observe that in this case $r - 2 + q/2 - r q/2 < -1$. By Lemma \[2.2\] we see that

$$II_2 = C \sum_{n=1}^{\infty} n^{r-2-q/p} h(n) n^{q/2} (\mathbb{E}|Y_1|^2)^{q/2}$$

$$\leq C \sum_{n=1}^{\infty} n^{r-2-q/p+q/2} h(n) (\mathbb{E}|Y_1|^p|Y_1|^2)^{q/2}$$

$$\leq C \sum_{n=1}^{\infty} n^{r-2-q/p+q/2} h(n) (C_V (|Y|^p))^{q/2} n^{2 \frac{2-p}{p-q}}$$

$$\leq C \sum_{n=1}^{\infty} n^{r-2+q/2-r q/2} h(n) < \infty.$$

If $rp \geq 2$, take $q > pr$. Note in this case $\mathbb{E}|Y|^2 < C_V(|Y|^2) < \infty$. We see that

$$II_2 = C \sum_{n=1}^{\infty} n^{r-1-q/p} h(n) (\mathbb{E}|Y_1|^2)^{q/2}$$

$$\leq C \sum_{n=1}^{\infty} n^{r-1-q/p} h(n) < \infty.$$

By Lemma \[4.2\] we conclude that $II_3 < \infty$. The proof of Theorem \[3.1\] is complete. 

Proof of Theorem \[3.2\] By Lemma \[4.3\] we only need to establish that $I < \infty$ and $II < \infty$ with $r = 1$. For $I$, by Markov inequality under sub-linear expectations, $C_r$ inequality, Lemma \[2.2\] and the proof of Lemma 2.2 of \[Zhong and Wu(2017)\] (observe that $\theta < 1$), we see that

$$I \leq \sum_{n=1}^{\infty} n^{-1} h(n) n^{-\theta/p} \mathbb{E} \max_{1 \leq k \leq n} \sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Y_j^\theta$$

$$\leq C \sum_{n=1}^{\infty} h(n) n^{-\theta/p} \mathbb{E}|Y_1|^\theta$$

$$\leq C \sum_{n=1}^{\infty} h(n) n^{-\theta/p} C_V \left(|Y_1|^\theta \right)$$

$$\leq C \sum_{n=1}^{\infty} h(n) n^{-\theta/p} C_V \left(|Y|^\theta \mathbb{I}\{|Y| > n^{1/p}\} \right)$$

$$\leq C \sum_{n=1}^{\infty} n^{-\theta/p} h(n) \int_{0}^{\infty} \mathbb{V}\left\{|Y|^\theta \mathbb{I}\{|Y| > n^{1/p}\} > x \right\} dx$$

$$\leq C \int_{0}^{\infty} y^{-\theta/p} h(y) \int_{0}^{\infty} \mathbb{V}\left\{|Y|^\theta \mathbb{I}\{|Y| > y^{1/p}\} > x \right\} dxdy$$

$$\leq C \int_{0}^{\infty} y^{-\theta/p} h(y) \left[ \int_{0}^{y^{1/p}} + \int_{y^{1/p}}^{\infty} \right] \mathbb{V}\left\{|Y|^\theta \mathbb{I}\{|Y| > y^{1/p}\} > x \right\} dxdy$$
\[
\begin{align*}
\leq C \int_{1}^{\infty} \mathbb{V} \{ |Y| > y^{1/p} \} h(y) dy \\
+ C \int_{1}^{\infty} \mathbb{V} \{ |Y| > x \} \int_{1}^{x^{p/\theta}} y^{-\theta/p} h(y) dy dx \\
\leq CC_{\mathbb{V}} (|Y|^{p} h(|Y|^p)) + C \int_{1}^{\infty} \mathbb{V} \{ |Y| > x \} x^{p/\theta-1} h(x^{p/\theta}) dx
\end{align*}
\]

For \( II \), by Markov inequality under sub-linear expectations, H"{o}lder inequality, and Lemma 2.2, we see that

\[
II \leq C \sum_{n=1}^{\infty} n^{-1/2} h(n) n^{-2/p} \mathbb{E} \left[ \max_{1 \leq k \leq n} \left| \sum_{i=-\infty}^{\infty} a_i (Y'_i - \mathbb{E}[Y'_i]) \right| \right]^{2}
\]

\[
\leq C \sum_{n=1}^{\infty} n^{-1/2} h(n) n^{-2/p} \mathbb{E} \left[ \left( \sum_{i=-\infty}^{\infty} |a_i|^{1/2} \left( \left| \sum_{k=i+1}^{n} (Y'_k - \mathbb{E}[Y'_k]) \right| \right) \right)^2 \right]
\]

\[
\leq C \sum_{n=1}^{\infty} n^{-1/2} h(n) \left[ \mathbb{E}[|Y'_1|^2] + \left( \sum_{j=1}^{n} |\mathbb{E}(-Y'_j) + |\mathbb{E}(Y'_j)| \right) \right]^{2}
\]

\[
= C \sum_{n=1}^{\infty} n^{-2/p} h(n) \mathbb{E}[|Y'_1|^2] + C \sum_{n=1}^{\infty} n^{-1/2} h(n) \left( \sum_{j=1}^{n} |\mathbb{E}(-Y'_j) + |\mathbb{E}(Y'_j)| \right)^2
\]

\[
=: II_{1} + II_{2}.
\]

By Lemma 4.2, we conclude that \( II_{1} < \infty \). By \( \mathbb{E}(Y'_j) = \mathbb{E}(-Y'_j) = 0, |\mathbb{E}(X) - \mathbb{E}(Y)| \leq \mathbb{E}|X - Y|, C_p \) inequality, and Lemma 2.2 we obtain

\[
II_{2} \leq C \sum_{n=1}^{\infty} n^{-1/2} h(n) \left[ \sum_{j=1}^{n} |\mathbb{E}(-Y'_j) + |\mathbb{E}(Y'_j)| \right]^{2}
\]

\[
\leq C \sum_{n=1}^{\infty} n^{-1/2} h(n) \left[ \sum_{j=1}^{n} |\mathbb{E}[Y'_j - Y'_j]| \right]^{2}
\]

\[
\leq C \sum_{n=1}^{\infty} n^{-1/2} h(n) \left[ \sum_{i=1}^{n} C_{\mathbb{V}} \left\{ |Y'_i|^2 \right\} \right]^{2}
\]

\[
\leq C \sum_{n=1}^{\infty} n^{-2/p} h(n) \left[ \int_{0}^{n^{1/p}} \mathbb{V} \{ |Y| > n^{1/p} \} dy + \int_{n^{1/p}}^{\infty} \mathbb{V} \{ |Y| > y \} dy \right]^{2}
\]

\[
\leq C \sum_{n=1}^{\infty} n^{1} h(n) \left[ \mathbb{V} \{ |Y| > n^{1/p} \} \right]^{2} + C \sum_{n=1}^{\infty} n^{-2/p} h(n) \left[ \int_{n^{1/p}}^{\infty} \mathbb{V} \{ |Y| > y \} dy \right]^{2}
\]
Therefore, 

\[
\begin{align*}
&\leq C \int_1^\infty x h(x) \mathbb{V}^2 \left\{ |Y| > x^{1/p} \right\} \, dx \\
&+ C \int_1^\infty x^{1-2/p} h(x) \, dx \int_y^\infty \mathbb{V} \left\{ |Y| > y \right\} \, dy \int_y^\infty \mathbb{V} \left\{ |Y| > z \right\} \, dz \\
&\leq C \int_1^\infty (x \mathbb{V} \{ |Y|^p h(|Y|^p) > x h(x) \}) \mathbb{V} \{ |Y|^p h(|Y|^p) > x h(x) \} \, dx \\
&+ C \int_1^\infty \mathbb{V} \left\{ |Y| > y \right\} dy \int_y^\infty \mathbb{V} \left\{ |Y| > z \right\} \, dz \int_1^\infty x^{1-2/p} h(x) \, dx \\
&\leq C \int_1^\infty \mathbb{V} \{ |Y|^p h(|Y|^p) > x h(x) \} \, dx \\
&+ C \int_1^\infty \mathbb{V} \left\{ |Y| > y \right\} dy \int_y^\infty \mathbb{V} \left\{ |Y| > z \right\} z^{2p-2} h(z) \, dz \\
&\leq CC_V (|Y|^p h(|Y|^p)) + C \int_1^\infty \mathbb{V} \left\{ |Y| > y \right\} dy \int_1^\infty \mathbb{E} \left\{ \frac{|Y|^p}{z^p} \right\} z^{2p-2} h(z) \, dz \\
&\leq CC_V (|Y|^p h(|Y|^p)) + C \int_1^\infty \mathbb{V} \left\{ |Y| > y \right\} C_V(|Y|^p) y^{p-1} h(y^p) \, dy \\
&\leq CC_V (|Y|^p h(|Y|^p)) + C \int_1^\infty \mathbb{V} \{ |Y|^p h(|Y|^p) > y^p h(y^p) \} \, dy y^p h(y^p) \\
&\leq CC_V (|Y|^p h(|Y|^p)) < \infty.
\end{align*}
\]

Now we will establish almost sure convergence under $\mathbb{V}$. By $\mathbb{E}(Y_1) = \mathbb{E}(-Y_1) = 0$ and $C_V(|Y|^p) < \infty$, we see that

\[
\sum_{n=1}^\infty n^{-1} \mathbb{V} \left\{ \max_{1 \leq k \leq n} |S_k| > \varepsilon n^{1/p} \right\} < \infty, \text{ for all } \varepsilon > 0.
\]

Therefore,

\[
\begin{align*}
\infty > \sum_{n=1}^\infty n^{-1} \mathbb{V} \left\{ \max_{1 \leq k \leq n} |S_k| > \varepsilon n^{1/p} \right\} \\
= \sum_{k=1}^\infty \sum_{n=2^{k-1}}^{2^k-1} n^{-1} \mathbb{V} \left\{ \max_{1 \leq k \leq n} |S_k| > \varepsilon n^{1/p} \right\} \\
\geq \frac{1}{2} \left\{ \max_{1 \leq m \leq 2^{k-1}} |S_m| > \varepsilon 2^{k/p} \right\}.
\end{align*}
\]

By Borel-Cantell lemma under sub-linear expectations (cf. [Chen et al.(2013)Chen, Wu, and Li] or Lemma 1 of [Zhang and Lin(2018)]), we see that

\[
2^{-k/p} \max_{1 \leq m \leq 2^k} |S_m| \to 0, \ \text{a. s. } \mathbb{V},
\]

which results in $S_n/n^{1/p} \to 0$, a. s. $\mathbb{V}$. \qed

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