SYMBOLIC CALCULUS
AND CONVOLUTION SEMIGROUPS OF MEASURES
ON THE HEISENBERG GROUP

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Abstract. Let \( P \) be a symmetric generalised laplacian on \( \mathbb{R}^{2n+1} \). It is known that \( P \) generates semigroups of measures \( \mu_t \) on the Heisenberg group \( H^n \) and \( \nu_t \) on the Abelian group \( \mathbb{R}^{2n+1} \). Recall that the underlying manifold of the Heisenberg group is \( \mathbb{R}^{2n+1} \). Suppose that the negative defined function \( \psi(\xi) = -\hat{P}(\xi) \) satisfies some weight conditions and

\[
|D^\alpha \psi(\xi)| \leq c_0 \psi(\xi)(1 + ||\xi||)^{-|\alpha|}, \quad \xi \in \mathbb{R}^{2n+1}.
\]

We show that the semigroup \( \mu_t \) is a kind of perturbation of the semigroup \( \nu_t \). More precisely, we give pointwise estimates for the difference between the densities of \( \mu_t \) and \( \nu_t \) and we show that it is small with respect to \( t \) and \( x \).

As a consequence we get a description of the asymptotic behaviour at origin of the densities of a semigroup of measures which is an analogon of the symmetrized gamma (gamma-variance) semigroup on the Heisenberg group.

The main tools are an inverse theorem due to Beals and a calculus of symbols on a nilpotent Lie group specified to the Heisenberg group \( H^n \) which is very close to the standard pseudodifferential symbolic calculus on \( \mathbb{R}^n \).

1. Statement of the result

We work on the Heisenberg group \( H^n \), which is \( \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \) with the multiplication

\[
(x_1, x_2, x_3) \circ (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1 \cdot y_2).
\]

The convolution \( *_1 \) on \( H^n \) is given by

\[
f *_1 g(u) = \int_{\mathbb{R}^{2n+1}} f(u \circ v^{-1})g(v)dv, \quad f, g \in C_c^\infty(\mathbb{R}^{2n+1}).
\]

Apart from the convolution \( *_1 \) on the Heisenberg group \( H^n \) we also make use of the ordinary convolution \( *_0 \) on the Euclidean space \( \mathbb{R}^{2n+1} \).

Let us consider a functional \( P \) on \( C_c^\infty(\mathbb{R}^{2n+1}) \). Suppose that \( P \) is real and for every real-valued function \( f \in C_c^\infty(\mathbb{R}^{2n+1}) \) which achieves maximal value at zero, the condition \( \langle P, f \rangle \leq 0 \) holds. This is the generalised laplacian (GL) property. The functional \( P \) extends to the space of bounded and smooth functions on \( \mathbb{R}^{2n+1} \), in particular it can be regarded as a tempered distribution and still preserves (GL) property. Note that the definition of a generalised laplacian does not depend on the group structure. It is well known (see Duflo[9]) that for a given generalised laplacian \( P \) there exists exactly one convolution semigroup (with

\[2010 \text{ Mathematics Subject Classification.} \ 	ext{Primary 22E25; Secondary 43A30.} \]

Key words and phrases. semigroups of measures, Heisenberg group, symbolic calculus, multipliers, generalised laplacian, gamma-variance semigroup.
respect to $*_{0}$) of probabilistic measures $\nu_t$ on the Abelian group $\mathbb{R}^{2n+1}$ and exactly one convolution semigroup (with respect to $*_{1}$) of probabilistic measures $\mu_t$ on $\mathbb{H}^n$ for which $P$, is the generating functional. Let $\psi$ be the Abelian Fourier Transform of $-\hat{\mathcal{P}}$. Suppose that the generating functional $P$ satisfies some weight conditions and for every $\alpha \in \mathbb{N}^{2n+1}$,

$$|\partial^{\alpha} \psi(\xi)| \leq c_{\alpha} \psi(\xi)(1 + |\xi|)^{-|\alpha|}, \quad \xi \in (\mathbb{R}^{2n+1})^*.$$ 

We give pointwise estimates for the difference between the densities of $\mu_t$ and $\nu_t$ and we show that it is small with respect to the time variable $t$ and the space variable $x$. More precisely,

$$(1.1) \quad \mu_t = \nu_t - t^2 \partial_3 \nu_t *_{0} \sum_{|\alpha| = 1} (T_1^\alpha P *_{0} T_2^\alpha P) + r_t,$$

where

$$|r_t(x)| \leq c \min(t, t^{-1}) \|x\|^{-(2n+1)-2}.$$

As an example, let us consider the distribution $\Gamma$ on $\mathbb{R}^{2n+1}$

$$\langle \Gamma, f \rangle = f(0) + \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{f(x) - f(0)}{|x|^{2n+1}} K_{2n+1}(|x|) \, dx,$$

where $K$ is the modified Bessel function of the second kind. It is known that $\hat{\Gamma}(\xi) = -1 - \log(1 + |\xi|^2)$. It is easy to see that $\Gamma$ is a generalised laplacian and satisfies the above assumptions. $\Gamma$ is the generating functional of the symmetric gamma semigroup (also so-called gamma-variance semigroup) on the Abelian group $\mathbb{R}^d$ (for every $d \in \mathbb{N}$), whose densities $l_t$ are known (see Sikic-Song-Vondracek [25]). $\Gamma$ is also the generating functional of a semigroup of measures on the Heisenberg group $\mathbb{H}^n$. From the [1.1] and the formula for $l_t$ we can find a description of the asymptotic behaviour of the densities of the gamma-variance semigroup on the Heisenberg group

$$m_t(x) \sim c_t \|x\|^{2t-2n-1}, \quad 2t < 2n + 1, \quad |x| \to 0.$$

The main tools are inverse theorems due to Beals [3],[4]. Our method also relies on a calculus of symbols on a nilpotent Lie group (see Glowacki [15],[19], Core-Geller [7]) specified to the Heisenberg group $\mathbb{H}^n$ (see Howe [21], Folland [13], cf. Appelbaum-Cohen [1], Bahouri-Kammer-Gallagher [2], Glowacki [17],[14]). Such calculus is very close to the standard pseudodifferential symbolic calculus on $\mathbb{R}^n$ (see Beals [3], Stein [26]), but is parameter-dependent (cf. Jacob-Tokareev [[11]]).

The behaviour of densities of semigroups of measures on the Heisenberg group was investigated by many authors (see Glowacki-Hebisch [20], Dziubański [10]). For a general theory of semigroups of measures we refer to Berg-Forst [6] (the $\mathbb{R}^n$ case) and Hunt [23], Hulanicki [22], Duflo [9], Faraut [12] (the Lie group case).
2. Preliminaries

2.1. Notation. Let $X$ be a finite-dimensional Euclidean space with a fixed scalar product. The dual space $X^*$ will be identified with $X$ by means of the scalar product. We pick an auxiliary Euclidean norm $\| \cdot \|$ and fix an orthonormal basis $\{e_j\}_{j=1}^d$, where $d$ is the dimension of $X$. Thus the variable $x$ splits into $x = (x_1, \ldots, x_d)$. Similar notation will be applied to the variable $\xi \in X^*$ and multiindices $\alpha \in \mathbb{N}^d$. For $\alpha \in \mathbb{N}^d$ let $|\alpha| = \sum_{j=1}^d \alpha_j$. Let also

$$T_j f(x) = x_j f(x), \quad D_j f(x) = i \partial_j f(x) = i f'(x)e_j,$$

and

$$T^\alpha f(x) = x^\alpha f(x), \quad \partial^\alpha f(x) = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} f(x), \quad D^\alpha f(x) = D_1^{\alpha_1} \cdots D_d^{\alpha_d} f(x).$$

The space of smooth functions with compact support is denoted by $C^\infty_c(X)$. The Schwartz space $\mathcal{S}(X)$ with the family of seminorms

$$\| f \|_{(N)} = \max_{|\alpha| + |\beta| \leq N} \sup_{x \in X} |x^\alpha D^\beta f(x)|, \quad N \in \mathbb{N},$$

is a locally convex Fréchet space.

Let Lebesgue measures $dx, d\xi$ on $X$ and $X^*$ be normalized so that the relationship between a function $f \in \mathcal{S}(X)$ and its Abelian Fourier Transform $\hat{f} \in \mathcal{S}(X^*)$ is given by

$$\hat{f}(\xi) = \int_X e^{-i \xi x} f(x) dx, \quad f(x) = \int_{X^*} e^{i \xi x} \hat{f}(\xi) d\xi.$$  \hfill (2.1)

The Fourier transform extends by duality to the whole space of tempered distributions $\mathcal{S}'(X)$, the space of continuous, linear functionals on $\mathcal{S}(X)$. The pairing of a distribution $T$ with a smooth function $f$ is denoted by $\langle T, f \rangle$, whenever it makes sense. If $P \in \mathcal{S}'(X)$ and $\varphi \in \mathcal{S}(X)$ then $\varphi P$ denotes the distribution

$$\langle \varphi P, f \rangle = \langle P, \varphi f \rangle, \quad f \in \mathcal{S}(X).$$

The Dirac delta function is defined by $\langle \delta_0, f \rangle = f(0)$.

Every $a \in \mathcal{S}(X \oplus X^*)$ defines a linear map $A : \mathcal{S}(X) \to \mathcal{S}(X)$ by the Kohn-Nirenberg prescription

$$Af(x) = \int e^{i \xi x} a(x, \xi) \hat{f}(\xi) d\xi.$$

The space of the functions

$$\phi_{f,g}(x, \xi) = e^{-i \xi x} \overline{\hat{f}(\xi)} g(x),$$

where $f, g \in \mathcal{S}(X)$, is dense in $\mathcal{S}(X \oplus X^*)$. Therefore the weak version of the above definition

$$\langle Af, g \rangle_{L^2(X)} = \langle a, \phi_{f,g} \rangle,$$

make sense for any $a \in \mathcal{S}'(X \oplus X^*)$ and defines the linear operator which maps continuously $\mathcal{S}(X)$ into $\mathcal{S}'(X)$. The distribution $a$ is called a symbol of $A$ and $A$ is referred to as a pseudodifferential operator. The correspondence between the symbols $a \in \mathcal{S}'(X \oplus X^*)$ and the continuous linear operators $A$ from $\mathcal{S}(X)$ into $\mathcal{S}'(X)$ is bijective.
The following proposition shows the relationship between the estimates of the derivatives of a function and those of the derivatives of its Fourier transform.

**Proposition 2.2** (cf. [26, Prop. 2, VI.4]). Let \( K \) be a distribution given by \( \hat{K} = m \) for a function \( m \) that is in \( C^\infty(X \setminus \{0\}) \). Suppose that, for every multiindex \( \alpha \) and some real number \( M < d = \dim X \), the function \( m \) satisfies
\[
|\partial_\alpha^\xi m(\xi)| \leq c_\alpha \|\xi\|^{-|\alpha|-M},\quad \alpha \in \mathbb{N}^d.
\]
Then \( K \) agrees with a smooth function \( K(x) \) away from the origin and satisfies
\[
|\partial_\alpha^x K(x)| \leq c'_\alpha \|x\|^{-d-|\alpha|+M},\quad \alpha \in \mathbb{N}^d,\quad \alpha \in \mathbb{N}^d.
\]

2.2. **Weights and weight functions.** A continuous function \( g : X \to \mathbb{R} \) is called a weight function if satisfies the following conditions
\[
a) \quad \left( \frac{g(x)}{g(y)} \right)^{\pm 1} \leq C \left( 1 + \frac{\|x - y\|}{g(x)} \right)^M,
\]
\[
b) \quad g(x) \geq 1,
\]
for positive constants \( C, M \). Condition a) says that \( g \) is self-tempered and implies that \( g \) is slowly varying. Condition b) is often called the uncertainty principle. We say that a positive function \( m \) on \( X \) is a weight for a weight function \( g \), if there are positive constants \( C, M \) such that
\[
\left( \frac{m(x)}{m(y)} \right)^{\pm 1} \leq C \left( 1 + \frac{\|x - y\|}{g(x)} \right)^M.
\]

For a given weight \( m \) and a weight function \( g \) let us denote by \( S_X(m,g) \) the class of all \( a \in C^\infty(X) \) satisfying the estimates
\[
|\partial_\alpha^x a(x)| \leq c_\alpha m(x)g(x)^{-|\alpha|},
\]
for all \( x \in X \) and every multiindex \( \alpha \in \mathbb{N}^{\dim X} \). The space \( S_X(m,g) \) endowed with the family of seminorms
\[
(2.3) \quad |a|_N = \sup_{|\alpha|=N} |\partial_\alpha^x a(x)| m(x)^{-1} g(x)^{|\alpha|}
\]
is a Fréchet space. Let us observe that if \( m, m' \) are equivalent weights (weights \( m, m' \) are equivalent, if \( \frac{m}{m'} \) and \( \frac{m'}{m} \) are bounded), then the spaces \( S_X(m,g) \) and \( S_X(m',g) \) are identical.

For a given weight \( m \) and a weight function \( g \) we denote by \( D_X(m,g) \) the space of all tempered distributions on \( X \) whose Fourier transform belong to \( S_X(m,g) \). It is easy to see that

**Fact 2.4.** If \( A \in D(m,g) \), then the Fourier transform of \( T^\alpha A \) is \( D^\alpha \hat{A} \). Therefore \( T^\alpha A \in D(mg^{-|\alpha|}, g) \). The Fourier transform of \( D^\alpha A \) is \( (-1)^{|\alpha|} T^\alpha \hat{A} \). Therefore, \( D^\alpha A \in D(m\rho^{|\alpha|}, g) \).
Theorem 2.7 (cf. Thms 2.7, 2.7' [4]).

For a weight function \( g \) we denote by \( L^2(g) \) the space of all linear operators on \( W \) whose symbols belong to \( S(m, g) \). Let \( A \) be the formal adjoint of \( A \in L^2(g) \). A has a unique continuous extension mapping \( S'(W) \) to itself, obtained as the formal adjoint of the restriction of \( A^* \) to \( S(W) \). Thus, we may consider \( A \) as being defined on any given subspace of \( S'(W) \).

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Remark 2.5. Let \( z \) be a complex number. Suppose that \( a \in S_X(m, g) \) and let \( a_z = a + z \). Then \( a_z \in S_X(|z| + m, g) \) and

\[
|\partial_x^\alpha a_z(x)| \leq c_\alpha m(x)g(x)^{-|\alpha|}, \quad \alpha \neq 0.
\]

In particular, the estimates are not dependent on \( z \). Similarly, if \( a \in S_X(m, g) \) and \(|a| \geq m \geq 1\), then \( \frac{1}{a} \in S_X \left( \frac{1}{m}, g \right) \) and \( \frac{1}{a_z} \in S_X \left( \frac{1}{m+|z|}, g \right) \), but

\[
|\partial_x^\alpha \frac{1}{a_z(x)}| \leq c_\alpha \frac{m(x)}{(|z| + m(x))^2}g(x)^{-|\alpha|}, \quad \alpha \neq 0.
\]

In both cases the quotient of the 'new' \((m \text{ and } \frac{m}{|z|+m})\) and the 'old' \((|z|+m \text{ and } \frac{1}{|z|+m})\), respectively) weight is equal to \( \frac{m}{|z|+m} \).

2.3. Symbolic calculus. Almost all in this subsection is based on [4]. However, some statements look different, because we use the Kohn-Nirenberg prescription in the definition of pseudodifferential operators instead of the Weyl prescription used in [4].

Let \( X \) be an \( n \)-dimensional Euclidean space and let \( W = X \oplus X^* \).

Proposition 2.6. If \( a \in S_W(m, g) \), then the symbol \( a \) is a continuous endomorphism of \( S(R^n) \).

Let \( a, b \in S(W) \) be symbols of \( A \) and \( B \). Then the symbol of \( AB \) is given by

\[
a\#b(x, \xi) = \int \int e^{i(x-y)\eta}a(x, \eta)b(y, \xi)d\eta dy
\]

for \((x, \xi) \in W\).

Theorem 2.7 (cf. Thms 2.7, 2.7' [4]). If \( m, m' \) are weights on \( W \) and \( a \in S_W(m, g) \), \( b \in S_W(m', g) \), then \( a\#b \in S_W(mm', g) \). Moreover, for every \( N > 0 \)

\[
a\#b - \sum_{|\alpha|<N} \frac{i^{-|\alpha|}}{\alpha!}(\partial_x^\alpha a)(\partial_x^\alpha b) = R_N(a, b) \in S_W(mm'g^{-2N}, g).
\]

For a given weight \( m \) and a weight function \( g \) we denote by \( L(m, g) \) the space of all linear operators on \( W \) whose symbols belong to \( S_W(m, g) \). Let \( A^* \) be the formal adjoint of \( A \in L(m, g) \). A has a unique continuous extension mapping \( S'(W) \) to itself, obtained as the formal adjoint of the restriction of \( A^* \) to \( S(W) \). Thus, we may consider \( A \) as being defined on any given subspace of \( S'(W) \).

For a given weight \( m \) for a weight function \( g \) let us define the Sobolev space

\[
H(m, g) = \{u \in S'(W) : Au \in L^2(R^n), A \in L(m, g)\}.
\]

Let \( H(m, g) \) have the weakest topology with respect to which each \( A \in L(m, g) \) is a continuous map from \( H(m, g) \) to \( L^2(R^n) \). This topology is determined by the seminorms \( N_A(u) = ||Au||_{L^2} \). There exists (cf. [4] Def. 3.3, Thm 3.3.) a norm \( || \cdot ||_{adm} \) for which

\[
H(m, g) = \{u \in S(W)' : ||u||_{adm} < \infty\}.
\]
Every such norm is called an admissible norm for $H(m, g)$. The following theorem gives some properties of the spaces $H(m, g)$.

**Theorem 2.8** (cf. [4, Thm 3.1]). Suppose $m, m'$ are weights for $g$.

a) If $A \in \mathcal{L}(m, g)$, then $A$ is continuous from $H(m'm, g)$ to $H(m', g)$.

b) $H(1, g) = L^2(\mathbb{R}^n)$

c) $\mathcal{S}(\mathbb{R}^n)$ is dense in $H(m, g)$.

d) If $A$ is in $\mathcal{L}(m, g)$ and $A$ is a topological isomorphism from $H(m'm', g)$ onto $H(m', g)$, then $A$ has an inverse belonging to $\mathcal{L}(m^{−1}, g)$.

The following proposition is a corollary from [4, Thm 3.7] and gives a criterion for determining admissible norms in the space $H(m, g)$.

**Proposition 2.9.** Let $m$ be a weight for a weight function $g$ such that $1 \leq m \leq g^M$ and suppose that $A$ is a operator in $\mathcal{L}(m, g)$ whose symbol $a$ satisfies $|a| \geq C^{-1}m$. Then the norm $\|u\|_A = \|u\|_{L^2} + \|Au\|_{L^2}$ is admissible for $H(m, g)$.

As a corollary we get that absolute value $|a| = m$ of the symbol $a$ of $A \in \mathcal{L}(m, g)$ is a weight for $g$, whenever $\| \cdot \|_A = \|u\|_{L^2} + \|Au\|_{L^2}$ is admissible norm for $H(m, g)$.

**Proposition 2.10** (cf. [17, Prop 1.22,]). Suppose that $A \in \mathcal{L}(m, g)$ satisfies conditions of Proposition 2.9 and $\mathcal{D}(A)$ is the domain of $A$ in $L^2(\mathbb{R}^n)$. If $A : \mathcal{D}(A) \to L^2(\mathbb{R}^n)$ is invertible, then $D(A) = H(m, g)$.

**Proof.** It follows from the fact that if $A \in \mathcal{L}(m, g)$ satisfies conditions of Proposition 2.9 then there exists $B \in \mathcal{L}(\frac{1}{m}, g)$, such that $C = AB - I \in \mathcal{L}(\frac{1}{m}, g)$. For instance, let $B$ the operator with the symbol $\frac{1}{a}$, where $a$ is the symbol of $A$. □

3. **Calculus of symbols on the Heisenberg group**

3.1. **Heisenberg group.** We consider a $2n + 1$-dimensional Euclidean space

$$ \mathfrak{h} = X_1 \oplus X_2 \oplus X_3, $$

where $X_1 = V$, $X_2 = V^*$, $X_3 = Z$ and $\dim V = n$, $\dim Z = 1$. The space $\mathfrak{h}$ is a Lie algebra with the commutator

$$ [(x_1, x_2, x_3), (y_1, y_2, y_3)] = (0, 0, x_1 \cdot y_2' - x_2 \cdot y_1), $$

and simultaneously a group with the multiplication

$$ (x_1, x_2, x_3) \circ (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + x_1 \cdot y_2). $$
The group is a model of the Heisenberg group $H^n$. Note that we identify the Heisenberg group and its Lie algebra, in particular, we will write $h$ rather than $H^n$ for the Heisenberg group in question.

The variable $x \in h$ splits into $x = (x_1, x_2, x_3)$ and $x_k = (x_{k1}, ..., x_{kd_k})$, $k = 1, 2, 3$, $d_k = \dim X_k$. Similar notation will be applied to the variable $\xi \in h^*$ and multiindices $\alpha$. Let $|\alpha| = \sum_{k=1}^{3} |\alpha_j| = \sum_{k=1}^{3} d_k \alpha_k$. Let also

$$T^\alpha_k f(x) = x_k^{\alpha_k} f(x), \quad \partial_k^{\alpha_k} f(x) = \partial_{k1}^{\alpha_k1} ... \partial_{kd_k}^{\alpha_k d_k} f(x), \quad D^\alpha_k f(x) = D_{k1}^{\alpha_k1} ... D_{kd_k}^{\alpha_k d_k} f(x).$$

Then $T^\alpha = T_1^{\alpha_1} T_2^{\alpha_2} T_3^{\alpha_3}$ and $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3}$.

The Heisenberg group $h$ is endowed with a family of dilations $\{\delta_t\}_{t>0}$, which are group automorphisms

$$\delta_t(w, \rho) = (tw, t^2 \rho).$$

The numbers $p_1 = 1$, $p_2 = 2$ are exponents of homogeneity associated with $W = V \oplus V^*$ and $Z$. The number $Q = \dim W + 2 \dim Z = 2n + 2$ is called the homogeneous dimension of $h$.

Notice that $[h, h] = Z$ and $Z$ is a central subalgebra corresponding to the eigenvalue 2 of the dilations. Then $W$ may be also identified with a quotient Lie algebra $h/Z$. The homogeneous dimension of $W$ is $2n$. Note also that $h^* = W^* \oplus Z^*$.

Let us also denote by $d(\alpha)$ the homogeneous length of multiindices, i.e.

$$d(\alpha) = p_1(|\alpha_1| + |\alpha_2|) + p_2 |\alpha_3|.$$

A (normalized) Lebesgue measure on the vector space $h$ is the (normalized) Haar measure on the group $h$. The Fourier transform on $h$ is given by (2.1).

The convolution $\ast_1$ on $h$ is given by

$$f \ast_1 g(u) = \int_h f(u \circ v^{-1}) g(v) dv.$$

Apart from the convolution $\ast_1$ on the Heisenberg group $h$ we also make use of the ordinary convolution $\ast_0$ on the Euclidean space $R^{2n+1}$

$$f \ast_0 g(u) = \int_{R^d} f(u - v) g(v) dv.$$

More generally, we define convolutions indexed by the parameter $\theta \in [0, 1]$:

$$f \ast_{\theta} g(u) = \int_{R^{2n+1}} f(u_1 - v_1, u_2 - v_2, u_3 - v_3 - \theta v_1 w_2) g(v_1, v_2, v_3) dv_1 dv_2 dv_3.$$

Notice that $\ast_{\theta}$ is a convolution with respect to the $\theta$-multiplication on the vector space $R^{2n+1}$

$$(u_1, u_2, u_3) \circ_{\theta} (v_1, v_2, v_3) = (w_1 + v_1, w_2 + v_2, w_3 + v_3 + \theta w_1 v_2).$$

$R^{2n+1}$ with the operation $\circ_{\theta}$ is a Lie group which is isomorphic with the Heisenberg group $h$ for $\theta \neq 0$. Such group will be denoted by $h_\theta$. For $\theta = 0$ the group is isomorphic with the Abelian group $h_0$. 
It follows immediately from (3.1) that for smooth functions \( f, g \)
\[
f *_1 g = f *_0 g + \int_0^1 (1 - \theta) \sum_{j=1}^n \partial_3 T_{1j} f *_0 T_{2j} g \, d\theta.
\]
By the Taylor formula can get a more general form.

**Lemma 3.2.** If \( f, g \in L^1(\mathfrak{h}) \) are smooth, then

\[
(3.3) \quad f *_1 g = \sum_{k=0}^{N-1} \frac{1}{k!} \sum_{|\alpha|=k} \partial_3^k T_1^\alpha f *_0 T_2^\alpha g + \int_0^1 \frac{(1 - \theta)^N}{N!} \sum_{|\alpha|=N} \partial_3^N T_1^\alpha f *_0 T_2^\alpha g \, d\theta.
\]

It is directly checked that for \( k = 1, 2 \)
\[
T_{1j}(f *_1 g) = T_{1j} f *_1 g + f *_1 T_{1j} g
\]
and
\[
T_3(f *_1 g) = T_3 f *_1 g + f *_1 T_3 g + \sum_{j=1}^n T_{1j} f *_1 T_{2j} g.
\]

In general, we have

**Lemma 3.4** (cf. Lemma 1.7 [15], [3]). For every \( f, g \in S \) and every multiindex \( \gamma \neq 0 \)

\[
(3.5) \quad T_\gamma(f *_1 g) = T_\gamma f *_1 g + f *_1 T_\gamma g + \sum_{d(\alpha) + d(\beta) = d(\gamma) \atop 0 < d(\alpha) < d(\gamma)} c_{\alpha\beta} T_\alpha f *_1 T_\beta g,
\]

for some real \( c_{\alpha\beta} \).

The family \( \pi^\lambda \) of unitary operators on the Hilbert space \( L^2(\mathbb{R}^n) \) indexed by the real parameter \( \lambda \neq 0 \) and given by

\[
\pi_h^{\pm\lambda} f(x) = \exp(-i\lambda^{1/2} \eta \cdot x) \exp(-i \pm \lambda z) f(x \pm \lambda^{1/2} y), \quad (\eta, \gamma, z) = h \in \mathfrak{h}, \lambda > 0
\]
is the family of all infinite-dimensional irreducible representations of the Heisenberg group \( \mathfrak{h} \) in \( \mathcal{B}(L^2(\mathbb{R}^n)) \). \( \pi^\lambda \) is called the Schrödinger representation with the Planck constant \( \lambda \).

For a function \( F \in L^1(\mathfrak{h}) \), the formula

\[
(3.6) \quad \pi_F^\lambda f(x) = \int_{\mathfrak{h}} F(h) \pi_h^\lambda f(x) \, dh, f \in L^2(\mathbb{R}^n),
\]
defines a bounded operator on \( L^2(\mathbb{R}^n) \) with the norm \( \|\pi_F^\lambda\| \leq \|F\|_{L^1} \). If \( F_1, F_2 \in L^1(\mathfrak{h}) \) then
\( \pi_{F_1 *_1 F_2} = \pi_{F_1} \pi_{F_2} \).

Formula (3.6) can be rewritten in the weak form
\[
\langle \pi_F^\lambda f, g \rangle_{L^2(\mathbb{R}^n)} = \langle F, \varphi_{f,g}^\lambda \rangle,
\]
where the function \( \varphi_{f,g}^\lambda \) on \( \mathfrak{h} \) is given by
\[
\varphi_{f,g}^\lambda(h) = \int \pi_h^\lambda f(x) \overline{g(x)} \, dx, \quad f, g \in S(\mathbb{R}^n), \lambda \neq 0.
\]
Thus, the definition of $\pi^\lambda_P$

$$\langle \pi^\lambda_P f, g \rangle_{L^2(\mathbb{R}^n)} = \langle P, \varphi_{f,g} \rangle,$$

make sense for any distribution $P$ with compact support.

It is directly checked that if $F$ is a Schwartz function on $\mathfrak{h}$, then the operator $\pi^\pm_\lambda F$ is a pseudodifferential operator via the Kohn-Nirenberg prescription with the symbol $\hat{F}(\lambda^{\frac{1}{2}}\xi, \pm\lambda^{\frac{1}{2}}x, \pm\lambda)$, (where $\lambda > 0$), i.e.

$$\pi^\pm_\lambda f(x) = \int e^{ix\xi} \hat{F}(\lambda^{\frac{1}{2}}\xi, \pm\lambda^{\frac{1}{2}}x, \pm\lambda) \hat{f}(\xi) \, d\xi. \quad (3.7)$$

3.2. $\mathcal{S}$-convolvers on $\mathfrak{h}$. The convolution of a tempered distribution $T$ with a Schwartz function $f$ on a nilpotent Lie group $G$ is defined by

$$T \ast f(x) = \langle T, \tilde{f}_x \rangle,$$

where $\tilde{f}_x(y) = f(xy^{-1})$. $\tilde{T}$ denotes the distribution

$$\langle \tilde{T}, f \rangle = \langle T, \tilde{f} \rangle.$$

We say that a distribution $T \in \mathcal{S}'(G)$ is a left $\mathcal{S}$-convolver on a nilpotent Lie group $G$ if $T \ast f \in \mathcal{S}(G)$ whenever $f \in \mathcal{S}(G)$. We define the space of right $\mathcal{S}$-convolvers in a similar way. $T$ is called an $\mathcal{S}$-convolver if it is both left and right $\mathcal{S}$-convolver. Notice that the convolution $T \ast S$ of two left $\mathcal{S}$-convolvers is well-defined

$$(T \ast S) \ast f = T \ast (S \ast f), \quad \langle T \ast S, f \rangle = \langle T, f \ast \tilde{S} \rangle,$$

and it is a left $\mathcal{S}$-convolver. The same is true for right $\mathcal{S}$-convolvers.

In the Abelian case there is a simple characterization of $\mathcal{S}$-convolvers.

**Proposition 3.8** (cf. [8, Section 1]). On $\mathbb{R}^n$ left $\mathcal{S}$-convolvers and right $\mathcal{S}$-convolvers coincide and $T$ is a $\mathcal{S}$-convolver on $\mathbb{R}^n$ if and only if $\hat{T}$ and all its derivatives are $C^\infty$ functions bounded by polynomials.

The class of left (right) $\mathcal{S}$-convolvers is flexible in the following sense.

**Proposition 3.9** (cf. [8, Proposition 2.5.]). On nilpotent Lie groups the space of left (right) $\mathcal{S}$-convolvers is closed under addition, scalar multiplication, convolution, differentiation and multiplication by polynomials.

**Remark 3.10.** If $A \in \mathcal{D}(m,g)$, then $A = A_0 + F$, where $A_0$ is a compactly supported distribution and $F$ is a Schwartz function. Therefore, $A$ is an $\mathcal{S}$-convolver.

Now, we extend some results from previous subsection to the class of $\mathcal{S}$-convolvers on the Heisenberg group.
Proposition 3.11. The formula (3.5) extends to the space of $S$-convolvers, i.e. if $A, B$ are $S$-convolvers, then for every multiindex $\alpha \in \mathbb{N}^{2n+1}$,

$$\tag{3.12} T^\gamma (A \ast_1 B) = T^\gamma A \ast_1 B + A \ast_1 T^\gamma B \sum_{d(\alpha) + d(\beta) = d(\gamma)} c_{\alpha\beta} T^\alpha A \ast_1 T^\beta B.$$  

Proof. The reasoning is based on the proof of [18, Lemma 2.9]. We prove (3.12) by induction on the length of $\gamma$. Let $d(\gamma) = 1$, i.e. $T^\gamma = T_{kj}$, $k = 1, 2$ and $1 \leq j \leq n$. Suppose at first that $B$ is a Schwartz function. Then,

$$\langle T_{kj}(A \ast_1 B), f \rangle = \langle A \ast_1 B, T_{kj} f \rangle = \langle A, T_{kj} f \ast_1 \tilde{B} \rangle.$$  

By (3.5), it is equal to

$$\langle A, T_{kj}(f \ast_1 \tilde{B}) - f \ast_1 T_{kj} \tilde{B} \rangle = \langle T_{kj} A, f \ast_1 \tilde{B} \rangle - \langle A, f \ast_1 T_{kj} \tilde{B} \rangle.$$  

As $T_{kj} \tilde{B} = -T_{kj} B$ for $k = 1, 2$, the first step is done, when $B$ is a Schwartz function. If $B$ is an $S$-convolver we can repeat the same reasoning using the just proven formula

$$T_{kj}(B \ast_1 f) = T_{kj} B \ast_1 g + f \ast_1 T_{kj} f, \quad f \in \mathcal{S}(\mathfrak{h}),$$

instead of (3.5).

Now, let $T^\gamma = T_3$. If $B$ is a Schwartz function, then

$$\langle T_3(A \ast_1 B), f \rangle = \langle A \ast_1 B, T_3 f \rangle = \langle A, T_3 f \ast_1 \tilde{B} \rangle = \langle A, T_3(f \ast_1 \tilde{B}) - f \ast_1 T_3 \tilde{B} - T_1 f \ast_1 T_2 \tilde{B} \rangle = \langle T_3 A, f \ast_1 \tilde{B} \rangle - \langle A, f \ast_1 (T_3 \tilde{B} - T^1 T^2 \tilde{B}) \rangle - \langle T_1 A, f \ast_1 T_2 \tilde{B} \rangle.$$  

As $T_3 \tilde{B} = -T_3 B + \sum_{j=1}^n T_{1j} T_{2j} B$, we get that

$$\tag{3.13} T_3(A \ast_1 B) = T_3 A \ast_1 B + A \ast T_3 B + \sum_{j=1}^n T_{1j} A \ast_1 T_{2j} B,$$

whenever $B$ is a Schwartz function. Similarly to the case $T^\gamma = T_{kj}$, for $k = 1, 2$, we get that (3.13) is valid also when $B$ is an $S$-convolver.

Let $\gamma \neq 0$ and suppose that (3.12) is true for all $d(\delta) < d(\gamma)$. Then $T^\gamma = T^\kappa T^\delta$, where $T^\kappa = T_{kj}$ for some $k = 1, 2, 3$ and $1 \leq j \leq d_k$. By induction hypothesis, the first step, and (3.13),

$$T^\gamma (A \ast_1 B) = T^\kappa (T^\delta A \ast_1 B + A \ast_1 T^\delta B) \sum_{d(\alpha) + d(\beta) = d(\delta)} c_{\alpha\beta} T^\alpha A \ast_1 T^\beta B) = T^\kappa A \ast_1 B + A \ast_1 T^\kappa B + U_\gamma (A, B),$$
where
\[ U_G(A, B) = T^\delta A *_1 T^\kappa B + A_\kappa *_1 T^\delta B \]
\[ + \sum_{d(\alpha)+d(\beta)=d(\gamma)} c_{\alpha\beta} \sum_{0<d(\alpha)<d(\beta)} c_{\tau} T^\tau(T^\alpha A *_1 T^\tau T^\beta B). \]

To complete the proof one only needs to note that \( d(\iota) < d(\gamma) \) and apply the induction hypothesis to the expressions \( T^\gamma(T^\alpha A * T^\tau T^\beta B). \)

□

It is easy to see that, for every \( \theta \neq 0 \), the spaces of \( S \)-convolvers on \( h_\theta \) coincide. In analogy with the previous Proposition we can also extend Lemma 3.2.

**Proposition 3.14.** Suppose that \( A, B \) are \( S \)-convolvers on the Heisenberg group \( h \). Then, for every \( N > 0 \)
\[ A *_1 B = \sum_{k=0}^{N-1} \frac{1}{k!} \sum_{|\alpha|=k} \partial^k_T T^\alpha_1 A *_0 T^\alpha_2 B + R_N(A, B), \]
where \( \alpha \in \mathbb{N}^n \) and
\[ R_N(A, B) = \int_0^1 \frac{(1 - \theta)^N}{N!} \sum_{|\alpha|=N} \partial^N_T T^\alpha_1 A *_\theta T^\alpha_2 B \ d\theta. \]

**Proof.** The proof is based on Lemma 3.2 and is similar to the proof of Proposition 3.11. □

From Proposition 3.11 and the relation \( A *_1 B = B *_1 A = \delta \) (if it holds) one can get the explicit formula for \( T^\gamma B \). In the simplest cases we have
\[ T_{kj}B = -B * T_{kj}A * B \]
for \( k = 1, 2 \) and \( 1 \leq j \leq n \) and
\[ T_{31}B = -B * T_{31}A * B - \sum_{j=1}^{n} B * A_{1j} * B_{2j}. \]

In general, we have

**Corollary 3.16.** If \( A, B \) are \( S \)-convolvers and \( A *_1 B = B *_1 A = \delta \), then, for every \( \gamma \in \mathbb{N} \),
\[ T^\gamma B = -B *_1 T^\gamma A *_1 B + \sum_{d(\alpha)+d(\beta)=d(\gamma)} c_{\alpha\beta} B *_1 T^\alpha A *_1 T^\beta B. \]

By direct calculation,
\[ T_{kj}T_{li}B = -B *_1 T_{kj}T_{li}A *_1 B + B *_1 T_{kj} *_1 B * T_{li}A *_1 B + B *_1 T_{li} *_1 B *_1 T_{kj}A *_1 B. \]

Iterating the formula (3.17), we can express distribution \( T^\gamma B \) as sums of convolution products having as factors \( B \) and terms of the form \( T^\gamma A \). More precisely,
Corollary 3.18. If $A, B$ are $\mathcal{S}$-convolvers and $A *_1 B = B *_1 A = \delta$, then for every $\gamma \in \mathbb{N}$, there are finite sequences of multiindices $(\alpha^i)_i \in \mathbb{N}^{2n+1}$ such that

$$T^\gamma B = \sum_{i \in \text{d}(\alpha^i) = \text{d}(\gamma)} c_{(\alpha^i)_i, \gamma} B *_1 \ldots (T^{\alpha^i} A *_1 B) _* 1 \ldots .$$

Moreover, $\sum \alpha^i \leq \gamma$.

Proof. Let $\gamma > 0$. The first element $-B *_1 T_\gamma A *_1 B$ on the right-hand side of (3.17) is of the form as in the sum in (3.19). Any other element $B *_1 T^\alpha A *_1 T^\beta B$ satisfies $|\beta| < |\gamma|$. If we write $T^\beta B$ as in Corollary 3.16 we get again the element of the form as in the sum in (3.19) and the sum with $T^\beta B$ for $|\delta| < |\beta|$. After finite number of such steps we get the sum of the form (3.19). $\square$

The Gårding space

$$H^\infty = \{ \pi^\lambda \phi : \phi \in \mathcal{S}(\mathfrak{h}), g \in L^2(\mathbb{R}^n) \}$$

is dense in $L^2(\mathbb{R}^n)$. If $A$ is an $\mathcal{S}$-convolver, we define $\pi^\lambda_A$ on $H^\infty$ by

$$\pi^\lambda_A f = \pi^\lambda_A (\pi^\lambda \phi g) := \pi^\lambda_A \phi g, \quad f \in H^\infty.$$  

Remark 3.20. As a consequence we get that the operator of the form $\pi^\pm \lambda_A$ is a pseudodifferential operator with the symbol (cf. (3.7))

$$\hat{A}(\lambda^\pm \xi, \pm \lambda^\pm x, \pm \lambda), \quad \lambda > 0.$$  

Moreover, if $A, B$ are $\mathcal{S}$-convolvers, then $\pi^\lambda_A * B = \pi^\lambda_A * \pi^\lambda_B$.

By $\text{Op}(T)$ we shall denote the linear convolution operator

$$\mathcal{S}(\mathfrak{h}) \ni f \mapsto T * f \in C^\infty(\mathfrak{h}).$$  

$T$ is called an $L^2$-convolver on $\mathfrak{h}$ if $\text{Op}(T)$ extends to a bounded endomorphism of $L^2(\mathfrak{h})$. By the Calderon-Vaillencourt Theorem if $T \in \mathcal{D}(1, g)$, then $T$ is an $L^2$-convolver.

Lemma 3.21. Let $A$ be an $\mathcal{S}$-convolver on the Heisenberg group $\mathfrak{h}$ such that $A \in \mathcal{D}(m, g)$ for some $m \leq cg$. Suppose that $B$ is an $L^2$-convolver and we have $A *_1 B = B *_1 A = \delta_0$. Then $B$ is also an $\mathcal{S}$-convolver on $\mathfrak{h}$.

Proof. By the Sobolev lemma, it is enough to show that for every $\gamma$, $T^\gamma (B * f) \in L^2$, whenever $f \in \mathcal{S}$. From the proof of Proposition 3.11 it is not to hard to see that the formula (3.12) is also valid for convolution of any tempered distribution and a Schwartz function, in particular

$$T^\gamma (B *_1 f) = T^\gamma B *_1 f + B *_1 T^\gamma f \sum_{d(\alpha) + d(\beta) = d(\gamma)} c_{\alpha \beta} T^\alpha B *_1 T^\beta f.$$  

Moreover, (3.12) is valid for convolution of compactly supported distribution and tempered distribution. By Remark 3.10 and the relation $A *_1 B = B *_1 A = \delta$, we get

$$T^\gamma B = \sum_{\text{d}(\alpha^i + \gamma) = \text{d}(\gamma)} c_{(\alpha^i)_i, \gamma} B *_1 \ldots *_1 (T^{\alpha^i} A *_1 B) *_1 \ldots .$$
From the assumptions, we have that for $|\gamma| \neq 0$, the operators $\text{Op}(T^\gamma A), \text{Op}(B)$ are bounded on $L^2$. By (3.22) and (3.23) we conclude that $T^\gamma Bf \in L^2$. □

3.3. Calculus of symbols on $\mathfrak{h}$. Let $m$ be a weight for a weight function $g$ on the vector space $\mathfrak{h}$. Let $S(m, g) = S_0(m, g)$ and $\mathcal{D}(m, g) = \mathcal{D}_0(m, g)$. For a function $f$ on $\mathfrak{h}$ let us denote the parameter-dependent functions $f^{(\lambda)}, f(\lambda)$ on $W$ given by

$$f_{(\pm \lambda)}(y, \eta) = f(\lambda^{\frac{1}{2}} y, \pm \lambda^{\frac{1}{2}} \eta, \lambda), \quad f^{(\pm \lambda)}(w) = |\lambda|^{-\frac{1}{2}} f(\lambda^{\frac{1}{2}} y, \pm \lambda^{\frac{1}{2}} \eta, \lambda), \lambda > 0.$$ 

Note that a weight function $g$ on the Heisenberg group $\mathfrak{h}$ can be considered as a family $g^{\lambda}$ of parameter-dependent weight functions on $W$.

Lemma 3.24 (cf. [2 Prop. 1.20]). Suppose that $m$ is a weight for a weight function $g$ on the Heisenberg group and $g(\lambda)(w) \geq |\lambda|^\frac{1}{2}$. Then, for every $\lambda$, $m(\lambda)$ is a weight for $g(\lambda)$ on the space $W$. The relevant constants are independent of $\lambda$.

Proof. Without loss of generality we assume that $\lambda > 0$. The uncertainty principle is an assumption of lemma. Since $g$ is self-tempered

$$\left(\frac{g(\lambda^{\frac{1}{2}} w, \lambda)}{g(\lambda^{\frac{1}{2}} v, \lambda)}\right)^{\pm 1} \leq C \left(1 + \frac{\|\lambda^{\frac{1}{2}} (w - v), 0\|}{g(\lambda^{\frac{1}{2}} w, \lambda)}\right)^M,$$

which exactly means that $g^{(\lambda)}$ is self-tempered on $W$. □

Lemma 3.25. If $A \in \mathcal{D}_0(m, g)$, then $\pi^\lambda_A \in \mathcal{L}(m(\lambda), g^{(\lambda)})$ for every $\lambda \neq 0$. Seminorms (2.3) of symbol of $\pi^\lambda_A$ do not depend on the parameter $\lambda$.

Proof. The symbol of $\pi^\lambda_A$ is given by

$$a^\lambda(w) = \hat{A}(\lambda^{\frac{1}{2}} w, \lambda).$$

By differentiating with respect to $w$ we get

$$|D^\alpha a^\lambda(w)| = |\lambda^{|\alpha|} (D^\alpha \hat{A})(\lambda^{\frac{1}{2}} w, \lambda)|$$

$$\leq c_\alpha \lambda^{\frac{|\alpha|}{2}} m(\lambda^{\frac{1}{2}} w, \lambda) (g(\lambda^{\frac{1}{2}} w, \lambda))^{-|\alpha|} = c_\alpha m(\lambda)(g^{(\lambda)}(w))^{-|\alpha|},$$

for every $\alpha \in \mathbb{N}^{2n}$. The constant $c_\alpha$ does not depend on the parameter $\lambda$, because it is the same constant as in the estimate of $\hat{A}$ on $\mathfrak{h}$. □

By Fact 2.4, we get that for every multiindex $\alpha \in \mathbb{N}^{2n+1}$, $\pi^\lambda_{T^\gamma A} \in \mathcal{L}(m(\lambda)(g^{(\lambda)}))^{-|\alpha|}, g^{(\lambda)}).$ We have a characterization of $\mathcal{D}(m, g)$.

Lemma 3.26. A tempered distribution $A$ belongs to the class $D(m, g)$ if and only if for every $\gamma \geq 0$, the operator $\pi^\lambda_{T^\gamma A} \in \mathcal{L}(m(\lambda)(g^{(\lambda)}))^{-|\gamma|}, g^{(\lambda)}).$
Remark 3.29. The Proposition 3.27 is valid also for the $\gamma \geq 0$.

The implication follows from the previous corollary. Now, suppose that for every $\gamma \geq 0$ we have $\pi^{\lambda}_{T_3}A \in L(m(\lambda)(g(\lambda))^{-|\gamma|}, g(\lambda))$. The symbol of $\pi^{\lambda}_{T_3}A$ is $(D^{\tau}_{3}\hat{A})(\lambda^{\frac{1}{2}}w, \lambda)$, so

$$|(D^{\alpha}_{w}D^{\gamma}_{3}\hat{A})(\lambda^{\frac{1}{2}}w, \lambda)| \leq m(\lambda^{\frac{1}{2}}w, \lambda)|\lambda|^{\frac{1}{2}}g(\lambda^{\frac{1}{2}}w, \lambda)^{-|\alpha|-\gamma}.$$ 

By the change of variables,

$$|D^{\alpha}_{w}D^{\gamma}_{3}\hat{A}(w, \lambda)| \leq m(w, \lambda)g(w, \lambda)^{-|\alpha|-\gamma},$$

which implies our claim. \hfill \Box

Proposition 3.27. Suppose that $A \in D(m, g)$, $B \in D(m', g)$. The map

$$(f, g) \mapsto f *_1 g$$

extends from $S(h) \times S(h) \to S(h)$ to a continuous map $D(m, g) \times D(m', g) \to D(mm', g)$. 

Proof. Let $\gamma \geq 0$. Combining Fact 2.4 with Lemma 3.25 we get that for every $\alpha, \beta \in \mathbb{N}^{2n+1}$

$$\pi^{\lambda}_{T_\alpha}A \in L(m(\lambda)(g(\lambda))^{-|\alpha|}, g(\lambda)), \quad \pi^{\lambda}_{T_\beta}B \in L(m(\lambda)(g(\lambda))^{-|\beta|}, g(\lambda)).$$

By (3.12) we have

$$\pi^{\lambda}_{T_3}(A*_{1}B) = \sum_{d(\alpha)+d(\beta)=d(\gamma)} c_{\alpha\beta} \pi^{\lambda}_{T^{\alpha}A} \pi^{\lambda}_{T^{\beta}B}. \tag{3.28}$$

By symbolic calculus on $W$, every element of the sum on the right-hand side of (3.28) is in the class $L(m(\lambda)m'(\lambda)(g(\lambda))^{-|\alpha|-|\beta|}, g(\lambda))$, where $d(\alpha)+d(\beta)=d(\gamma)$. From the inequality

$$|\alpha| + |\beta| \geq \frac{1}{2}(d(\alpha)+d(\beta)) = \frac{1}{2}d(\gamma) = \gamma,$$

we get that $\pi^{\lambda}_{T_3} \in L(m(\lambda)m'(\lambda)(g(\lambda))^{-\gamma}, g(\lambda))$, which completes the proof. \hfill \Box

Remark 3.29. The Proposition 3.27 is valid also for the 'θ-Heisenberg group' $h_\theta$, i.e. if $A \in D(m, g)$, $B \in D(m', g)$, then the map

$$(f, g) \mapsto f *_{\theta} g$$

extends from $S(h) \times S(h) \to S(h)$ to a continuous map $D(m, g) \times D(m', g) \to D(mm', g)$. This is uniform in $\theta \in [0, 1]$.

Proposition 3.30. The map

$$(f, g) \mapsto f *_{1} g - \sum_{k=0}^{N-1} \frac{1}{k!} \sum_{|\alpha|=k} \partial^{\alpha}_{k} T^{\alpha}_{1} f *_{0} T^{\alpha}_{2} g$$

extends from $S(h) \times S(h) \to S(h)$ to a continuous map $D(m, g) \times D(m', g) \to D(mm' \rho^{N}g^{-2N}, g)$. 

Proof. By Proposition 3.14 and Proposition 3.27 it is enough to show that $R_{N}(A, B) \in D(mm' \rho^{N}g^{-2N}, g)$, where

$$R_{N}(A, B) = \int_{0}^{1} (1 - \theta)^{N} \sum_{|\alpha|=N} \partial^{\alpha}_{N} T^{\alpha}_{1} A_{\theta} *_{\theta} T^{\alpha}_{2} B d\theta.$$
By Fact 2.4 (which is valid also for $\mathfrak{h}_\theta$) combined with Remark 3.29 for every $|\alpha| = N$, $\partial^N T^A_1 A * \theta T^B_2 B \in D(m^a g^{2N}, g)$ uniformly with respect to the parameter $\theta$. Thus, $R_N(A, B) \in D(m^a g^{2N}, g)$. □

As a conclusion we get

Corollary 3.31. If $\hat{A} \in S(m_1, g)$ and $\hat{B} \in S(m_2, g)$, then $\hat{A} \ast_1 \hat{B} \in S(m_1 m_2, g)$ and

$$(3.32) \quad \hat{A} \ast_1 \hat{B} = \sum_{k=0}^{N-1} \frac{1}{k!} \sum_{|\alpha|=k} \lambda^k D^a_1 \hat{A} D^a_2 \hat{B} + R_N(\hat{A}, \hat{B}),$$

where $R_N(\hat{A}, \hat{B}) \in S(m_1 m_2 g^{-N}, g)$.

3.4. Inverses on $\mathfrak{h}$.

Lemma 3.33. Suppose that $A \in \mathcal{D}(m, g)$ is invertible on $L^2$, $|a| \geq m$ and $1 \leq m \leq g^{2N}$. Then $D(\pi^A_\lambda) = H(m(A), g^{(A)})$ and $\pi^A_\lambda : H(m(A), g^{(A)}) \to L^2$ is a topological isomorphism.

Proof. It follows from the fact that $\|u\|_{A, \lambda} = \|\pi^A_\lambda u\|_{L^2} + \|u\|_{L^2}$ is an admissible norm in $H(m(A), g_{\lambda})$. □

Lemma 3.34. Let $\pi^A_\lambda$ be a topological isomorphism from Lemma 3.33. $A \ast B = B \ast A = \delta$ and $B \in L^1$. Then $\pi^B_\lambda \in \mathcal{L}(m_{A, \lambda}, g^{(A)})$. The seminorms of symbol of $\pi^B_\lambda$ do not depend on the parameter $\lambda$.

Proof. We have that $\pi^A_\lambda \in \mathcal{L}(m_{A, \lambda}, g_{\lambda})$ is a topological isomorphism $H(m_{A, \lambda}, g^{(A)}) \to L^2$, $\|u\|_{A, \lambda} = \|\pi^A_\lambda u\|_{L^2} + \|u\|_{L^2}$ is an admissible norm in $H(m_{A, \lambda}, g_{\lambda})$ and the seminorms do not depend on the parameter $\lambda$. From Theorem 2.8 we have that, for every $\lambda \neq 0$, the operator $\pi^A_\lambda$ is in $S(m^{-1}_{A, \lambda}, g^{(A)})$, but it is not clear whether the seminorms of the symbols of $\pi^A_\lambda$ are independent of $\lambda$. We start by showing the estimate

$$(3.35) \quad |\tilde{B}_{(\lambda)}(w)| \leq c(m_{\lambda}(w))^{-1}.$$  

It is enough to show (cf. the proof of [4] Thm. 4.9) that

$$\|\pi^B_\lambda u\|_{A, \lambda} \leq C\|u\|_{L^2}, \quad u \in \mathcal{S},$$

where the constant $C$ is independent of $\lambda$. Since $\pi^A_\lambda$ is topological isomorphism and $\pi^A_\lambda \pi^A_\lambda = I$, we have

$$\|\pi^A_\lambda u\|_{A, \lambda} = (\|\pi^A_\lambda \pi^B_\lambda u\|_{L^2} + \|\pi^B_\lambda u\|_{L^2})$$

By the fact that $B \in L^1$ and the Plancherel Theorem we get that $\|\pi^B_\lambda\|_{L^2 \to L^2} \leq C$ and consequently

$$\|\pi^B_\lambda u\|_{A, \lambda} \leq C\|u\|_{L^2}.$$  

Thus (3.35) is proved.

Now, let $|\alpha| = 1$. We are going to show that

$$(3.36) \quad |D^a_\alpha \tilde{B}_{(\lambda)}(w)| \leq c(m_{\lambda}(w))^{-1} g^{(A)}(w)^{-1}.$$
We examine $\pi_{T^*B}^\lambda u$. By \textbf{3.15} and the relation $\pi_{T^*B}^\lambda = I$,
\[
\|\pi_{T^*B}^\lambda u\|_{A,\lambda} = \| - \pi_{T^*A}^\lambda \pi_{T^*B}^\lambda u\|_{A,\lambda} = \| \pi_{T^*A}^\lambda u\|_{L^2} + \| \pi_{T^*A}^\lambda \pi_{T^*B}^\lambda u\|_{L^2}.
\]
For every $\alpha \neq 0$, $\pi_{T^*A}^\lambda$ and $\pi_{B}^\lambda$ are bounded on $L^2$. By the Plancherel theorem their norms are independent of the parameter $\lambda$. Consequently $\|\pi_{T^*B}^\lambda u\|_{A,\lambda} \leq c\|u\|$ and for $|\alpha| = 1$
\[
|D^\alpha \hat{B}(\lambda)(w)| \leq c(m_\lambda(w))^{-1}.
\]
By \textbf{3.35} and the relation $\pi_{T^*B}^\lambda = \pi_{T^*A}^\lambda \pi_{T^*B}^\lambda$, we have
\[
|D^\alpha \hat{B}(\lambda)(w)| \leq |\hat{B}(\lambda)(w)|^2 |D^\alpha \hat{A}(\lambda)(w)| \leq c(m_\lambda(w))^{-1} g(\lambda)(w)^{-1}.
\]
Thus \textbf{3.36} is proved. The estimates for higher derivatives are obtained, by using \textbf{3.35} and Corollary \textbf{3.18}, in the same way, and we conclude that $\pi_{B}^\lambda \in \mathcal{L}(m^{-1}_\lambda, g(\lambda))$ with the relevant constants independent of the parameter $\lambda$. \hfill $\square$

From \textbf{18} we have the following lemma.

\textbf{Lemma 3.37} (cf. \textbf{18} Lemma 4.5]). Let $(a_u)_{u \in U}$ be a family symbols of $S(m, g)$ depending smoothly on $u \in U$. If the operators with symbols $a_u$ are invertible and the family of symbols $a_u^{-1\#}$ of inverses is bounded in $S(m^{-1}, g)$, then $a_u^{-1\#}$ also depends smoothly on $U$.

As a corollary from Lemma 3.36, we get

\textbf{Corollary 3.38.} The family $\hat{B}(\lambda)$ depends smoothly on the parameter $\lambda \in \mathbb{R}\setminus\{0\}$.

\textbf{Proof.} Let $\Lambda_k = (\frac{1}{k}, k)$. It is easy to see that
\[
c_k^{-1} \sup_{\lambda \in \Lambda_k} m_\lambda(w) \leq m_{(1)}(w) \leq c_k \inf_{\lambda \in \Lambda_k} m_\lambda(w),
\]
\[
c_k^{-1} \sup_{\lambda \in \Lambda_k} g(\lambda)(w) \leq g^{(1)}(w) \leq c_k \inf_{\lambda \in \Lambda_k} g(\lambda)(w).
\]
For every $\lambda \in \Lambda_k$ the family $(\hat{A}_\lambda)_{\lambda \in \Lambda_k}$ is bounded in $S(m_{(1)}, g^{(1)})$ and the family $(\hat{B}(\lambda))_{\lambda \in \Lambda_k}$ is bounded in $S(m^{-1}_{(1)}, g^{(1)})$. From Lemma \textbf{3.37} we get that $\hat{B}(\lambda)$ depends smoothly on the parameter $\lambda \in \Lambda_k$. Number $k$ is arbitrary, so the proof is complete. \hfill $\square$

\textbf{Proposition 3.39.} Suppose that $A \in \mathcal{D}(m, g)$ is invertible. Let $A \ast B = B \ast A = \delta$ and assume that $B \in L^1$. Then, the inverse $B$ belongs to $\mathcal{D}(m^{-1}, g)$.

\textbf{Proof.} By Lemma \textbf{3.34}
\[
(3.40) \quad |D_w^\alpha D_\lambda^\gamma \hat{B}(w, \lambda)| \leq c_\alpha m(w, \lambda)^{-1} g(w, \lambda)^{-|\alpha| - |\gamma|},
\]
for every $\alpha$ and $\gamma = 0$. By Corollary \textbf{3.18} we have that
\[
(3.41) \quad \pi_{T^*B}^\lambda = \sum_{d(\alpha^*) = d(\gamma)} c(\alpha_i), \gamma \pi_{T^*A}^\lambda \prod_i (\pi_{T^*A}^\lambda \pi_{T^*B}^\lambda).
\]
By symbolic calculus on \( W \), every element of the sum on the right-hand side of (3.41) is in the class \( L(m_{(\lambda)}^{-1}(g(\lambda)))^{-|\alpha|-|\beta|, g^{(\lambda)}} \), where \( d(\alpha) + d(\beta) = d(\gamma) \). From the inequalities

\[
|\alpha| + |\beta| \geq \frac{1}{2} (d(\alpha) + d(\beta)) = \frac{1}{2} d(\gamma) = \gamma, \quad \gamma_3 \geq \sum \alpha_3,
\]

we get that \( \sum |\alpha'| \geq \gamma \) and thus \( \pi_{T_3^B}^\lambda \in L(m_{(\lambda)}^{-1}(g(\lambda)))^{-\gamma, g^{(\lambda)}} \). The assertion follows from Lemma 3.26. \( \square \)

**Proposition 3.42.** Suppose \( A \in D(m, g) \) and \( A \) is invertible with respect to \( \ast_1 \) and \( \ast_0 \). Let \( B^0, B^1 \) be the distributions such that \( A \ast_1 B^1 = B^1 \ast_1 A = \delta \) and \( A \ast_0 B^0 = B_0 \ast_0 A = \delta \). Then

\[
B_1 = B_0 - \sum_{|\alpha|=1} \partial_3 (T_1^\alpha B_0 \ast_0 T_2^\alpha A \ast_0 B_0) + H,
\]

where \( H \in D(m^{-1}\rho^2 g^{-4}, g) \).

**Proof.** By Proposition 3.30 \( A \ast_1 B^0 = A \ast_0 B^0 + (R_1(A, B^0) - R_2(A, B^0)) + R_2(A, B^0) \). Acting by \( \ast_1 \)-convolution with \( B^1 \) on the left we get

\[
B^1 = B^0 - B^1 \ast_1 ((R_1 - R_2) + R_2),
\]

Repeat it

\[
B^1 = B^0 - B^0 \ast_1 ((R_1 - R_2) + R_2) + B^1 \ast_1 R_1 \ast_1 R_1.
\]

Again by Proposition 3.30 we have

\[
B^0 - B^0 \ast_0 \partial_3 \sum_{|\alpha|=1} T_1^\alpha A T_2^\alpha B^0 + H,
\]

where

\[
H = +B^0 \ast_1 (R_1 - R_2) + B^1 \ast_1 R_1 \ast_1 R_1.
\]

By Proposition 3.30 we have that \( R_1(B^0, R_1 - R_2), B^0 \ast_1 (R_1 - R_2) \in D(m^{-1}\rho^2 g^{-4}, g) \), \( B^1 \ast_1 R_1 \ast_1 R_1 \in D(m^{-1}\rho^2 g^{-4}, g) \) and finally \( H \in D(m^{-1}\rho^2 g^{-4}, g) \). \( \square \)

4. **Semigroups of measures on the Heisenberg group**

4.1. **Generalised laplacians and semigroups of measures on a Lie group.** A family \( \mu_t \) of subprobabilistic measures on \( G \) which satisfies

(i) \( \mu_t \ast \mu_s = \mu_{t+s}, t, s > 0 \),

(ii) \( \lim_{t \to 0} (\mu_t, f) = f(e), f \in C_c^\infty(G) \),

is said to be a *(continuous) semigroup of measures.* If \( \mu_t \) is a continuous semigroup of measures on \( G \), then

\[
\langle P, f \rangle = \lim_{t \to 0} \frac{\langle \mu_t, f \rangle - f(e)}{t}
\]
defines a distribution $P \in S'(G)$ which is called the generating functional of the semigroup $\mu_t$. It follows directly from (4.1) that $P$ is real and satisfies the following maximal principle $(4.2)$

$$\langle P, f \rangle \leq 0$$

for real $f \in C_c^\infty(G)$ such that $f(e) = \sup_{g \in G} f(g)$. A real distribution which satisfies the maximum principle $(4.2)$ is called a *generalised laplacian*. Moreover, the following theorem is true.

**Proposition 4.3 ([9], Prop.4).** Suppose that $P$ is a generalised laplacian on $G$. Then, there exists exactly one (continuous) semigroup of measures such that $(4.1)$ holds.

Note that the definition of a generalised laplacian is independent of a group multiplication and that the semigroup generated by $P$ consists of symmetric measures if and only if $P = P^\ast$.

By a *cut-off function* we shall mean any nonnegative function $\varphi \in C_c^\infty(G)$ such that $\varphi \leq 1$ and $\varphi = 1$ in a neighborhood of $e$.

**Lemma 4.4 ([12] Prop. II.2).** Let $P$ be a generalised laplacian. Then, for any cut-off function $\varphi$ the distribution $(1 - \varphi)P$ is a bounded positive measure. Hence $P$ admits a decomposition $P = P_0 + \mu$, where $P_0$ is a compactly supported distribution and $\mu$ is a positive bounded measure.

Thus,

$$\langle \nu, f \rangle = \langle P, f \rangle, \ f \in C_c^\infty(G\{e\})$$

defines a measure, which is bounded outside any neighborhood of $e$. The measure $\nu$ is called the *Lévy measure* of $P$.

Our next proposition summarizes some well-known results (cf. [9], [24]) concerning semigroups in representations of the Heisenberg group.

**Proposition 4.5.** Let $P$ be a symmetric generalised laplacian. Then,

a) For real $\lambda \neq 0$ the operators $T^\lambda_t$ form a strongly continuous semigroup of operators in $B(L^2(\mathbb{R}^n))$. The generator, which we denote by $A^\lambda$, of $T^\lambda_t$ is the closure of the operator $\pi^\lambda_P$.

b) $A^\lambda$ is self-adjoint and $T^\lambda_t$ is analytic.

c) Let $R^\lambda_z$ be the resolvent of $A^\lambda$ (i.e. $R^\lambda_z(zi - A^\lambda) = (zi - A^\lambda)R^\lambda_z = I$). Then,

$$T^\lambda_t = \int_\Gamma e^{zt}R^\lambda_z \, dz,$$

where $\Gamma$ is a smooth curve given by

$$\Gamma(r) = \begin{cases} 
re^{-\varphi_0} & \text{ dla } r \geq t^{-1}, \\
re^{-\varphi_0} & \text{ dla } 0 \leq \varphi \leq \varphi_0, \\
t^{-1}e^{i\varphi} & \text{ dla } r \geq t^{-1},
\end{cases}$$

for any $\frac{\pi}{2} < \varphi_0 < \pi$. 

Lemma 4.6. Suppose that $|H(z)| \leq C(a + |z|)^{-k-1}$ for some natural $k \geq 0$ and a real $a > 0$. Then

$$\left| \int_{\Gamma} e^{zt} H(z) \, dz \right| \leq C t^k a^k (1 + at)^{-k-1}.$$ 

4.2. Pointwise estimates for semigroups on the Heisenberg group. Let $P$ be a symmetric generalised laplacian on $\mathbb{R}^{2n+1}$. Let $\psi = -\hat{P}$. Recall that $\rho(x) = (1 + \|x\|^2)^{\frac{k}{2}}$.

Assumptions 4.7. Suppose that

(i) $\psi = -\hat{P}$ is a weight for a weight function $\rho$ and $1 \leq \psi \leq \rho^M$ for some $M > 0$,
(ii) $P \in \mathcal{D}(\mathbb{R}^{2n+1}, \psi, \rho)$.

Observe that these assumptions guarantee that $P$ is an $\mathcal{S}$-convolver on every Lie group $\mathfrak{h}_\theta$, $\theta \in [0, 1]$. By Lemma 3.33 the Sobolev spaces $H(\psi, \rho)$ are the domains of the operators $A^\lambda = \pi^\lambda_\lambda$.

As explained above, $P$ is a generating functional of a semigroup of measures $\mu_t$ and $\nu_t$ on the Lie groups $\mathfrak{h}$ and $\mathfrak{e}$, respectively. The Fourier transform of $\nu_t$ is given by $\nu_t(x) = e^{-t\psi(x)}$. The explicit formula for the Fourier transform of $\mu_t$ is in general unknown. For every $\theta \in [0, 1]$, let $B^\theta$ be resolvent for $P$ with respect to the convolution $*$. In particular

$$B^\theta \ast_0 (z\delta - P) = (z\delta - P) \ast_0 B^\theta = \delta, \quad B^1 \ast_1 (z\delta - P) = (z\delta - P) \ast_1 B^1 = \delta.$$ 

It is clear that the assumptions on $P$ imply that the semigroups of measures $\mu_t, \nu_t$ and the resolvents are $L^1$ functions. By Proposition 3.39 the resolvents $B^\theta$, $\theta \in [0, 1]$ are in the class $\mathcal{D}((|z| + \psi)^{-1}, \rho)$, in particular, they are $\mathcal{S}$-convolvers on every Lie group $\mathfrak{h}_\theta$.

Proposition 4.8. Let $B^1_z$ be the resolvent for $P$. Then

$$|B^1_z| \leq c(|z| + \psi)^{-1}$$

and, for $\alpha \neq 0$,

$$|D^\alpha B^1_z| \leq c_\alpha \frac{\psi}{(|z| + \psi)^2} \rho^{-|\alpha|}, \quad \alpha \neq 0.$$ 

The constants in above estimates do not depend on $z$.

Proof. By Lemma 3.26, $A^\lambda_z = z + \pi^\lambda_z \in \mathcal{L}(|z| + \psi, \rho)$ and $\pi^\lambda_z$ is bounded relative to $|z| + \psi$. By the Beals theorem (Theorem 2.8d)), $R^\lambda_z \in \mathcal{L}((|z| + \psi)^{-1}, \rho)$. We will show that the inverse $R^\lambda_z$ is bounded with respect to $(|z| + \psi)^{-1}$. Here we follow Beals. It is enough (cf. the proof of [41 Thm 4.9]) to show that

$$\|R^\lambda_z u\|_{A^\lambda_z, \lambda} \leq C\|u\|_{L^2_z}, \quad u \in \mathcal{S}, \quad z \in \Sigma.$$ 

By definition,

$$\|u\|_{A^\lambda_z, \lambda}^2 \leq 2\|u\|_{A^\lambda_z, \lambda}^2 + 2|z|\|u\|_{L^2_z}^2.$$
Since $A^\lambda_2$ is topological isomorphism and $A^\lambda R^\lambda_2 = I + zR^\lambda_2$, we get

$$\|R^\lambda_2 u\|_{A_\nu,\lambda} \leq C(\|R^\lambda_2 u\|_{A_\nu,\mu} + |z|\|R^\lambda_2 u\|_{L^2})$$

$$\leq C'(\|A^\lambda R^\lambda_2 u\|_{L^2} + |z|\|R^\lambda_2 u\|_{L^2}) \leq C''(\|u\|_{L^2} + |z|\|R^\lambda_2 u\|_{L^2}).$$

The uniform estimates for resolvents

$$\|R^\lambda_2 u\|_{L^2} \leq M|z|^{-1}\|u\|_{L^2}$$

follow from the Plancherel theorem. Thus, $R^\lambda_2$ is bounded relative to $(|z|+\psi)^{-1}$. In particular, the symbol $r^\lambda_2$ satisfies

$$|r^\lambda_2| \leq C(|z| + \psi)^{-1}.$$

By a similar method as in the proof of Lemma 3.34 we get that $r^\lambda_2 \in S((|z| + \psi)^{-1},\rho_\lambda)$. The symbol $a^\lambda_2$ has better estimates for higher derivatives (cf. Remark 2.5), so from the proof of Lemma 3.34 we, in fact, get

$$|D^\alpha a^\lambda_2|^r(w) = c_{\alpha} \frac{\psi(\lambda)(w)}{(|z| + \psi(\lambda)(w))^2}\rho(\lambda)(w)^{-|\alpha|}.$$  

In the similar way as in the proof of the Proposition 3.39 we show also that, for every $\gamma > 0$, $\pi^\lambda_{\nu_t} B^1 \in \mathcal{L}(\frac{\psi^2}{(|z| + \psi)^2}(\rho(\lambda))^{-|\gamma|},\rho(\lambda))$ with the estimates independent of the parameter $z$. This completes the proof.

**Corollary 4.9.** The error term $H_z$ from (3.42) is in the class $D(\frac{\psi^2}{(|z| + \psi)^2},\rho^{-n},\rho)$ and its seminorms do not depend on the parameter $z$.

**Proposition 4.10.** Suppose that a generalised laplacian $P$ satisfies Assumptions 4.7. Let $\mu_t$ and $\nu_t$ be semigroups of measures generated by $P$. Then

$$\hat{\mu}_t = \hat{\nu}_t - it^2 \lambda \hat{\nu}_t \sum_{|\alpha| = 1} \left( D^\alpha \hat{P} D^\alpha \hat{P} \right) + h_t,$$

where

$$|D^\alpha h_t| \leq c_{\alpha} \min (t^2 \psi^2, t^{-1} \psi^{-1}) \rho^{-2-|\alpha|}.$$  

**Proof.** By Proposition 4.5

$$\hat{\mu}_t = \int_\Gamma e^{zt} \hat{B}^1_2 dz.$$

Using Proposition 3.42 and Corollary 4.9 for the resolvent $B^1_2$, we get

$$\hat{\mu}_t = \int_\Gamma e^{zt} \hat{B}^1_2 dz - \int_\Gamma e^{zt} \hat{B}^0_2 dz \sum_{|\alpha| = 1} \left( i \lambda D^\alpha \hat{B}^0_2 D^\alpha \hat{P} \right) dz$$

$$+ \int_\Gamma e^{zt} R_2(B^1_2) dz$$

where $|D^\alpha R_2(B^1_2)| \leq c_{\alpha} \psi^2 \frac{\psi}{(|z| + \psi)} \rho^{-2-|\alpha|}$. For $|\alpha| = 1$,

$$D^\alpha \hat{B}^0_2 = - \frac{D^\alpha \hat{P}}{(z + \psi)^2}, \quad D^\alpha \hat{P} = -D^\alpha \hat{P}.$$
Thus,
\[
\hat{\mu}_t = \int e^{zt} (z + \psi)^{-1} \, dz - \sum_{|\alpha|=1} i\lambda D_1^\alpha \hat{\nu}_t^2 D_2^\alpha \hat{\nu}_t \int e^{zt} (z + \psi)^{-3} \, dz + \int e^{zt} R_N(B_2^1) \, dz.
\]

Using the fact that \(B_2^0\) is the resolvent of the semigroup \(\nu_t\), we have
\[
\hat{\mu}_t = \hat{\nu}_t - \hat{\nu}_t t^2 i\lambda \sum_{|\alpha|=1} D_1^\alpha \hat{\nu}_t^2 D_2^\alpha \hat{\nu}_t + \int e^{zt} R_N(B_2^1) \, dz.
\]

We have to estimate the above integral. By Lemma 4.6 we get, for every \(\alpha\),
\[
|D_1^\alpha \int e^{zt} R_2(B_2^1) \, dz| \leq c t^2 \psi^2 (1 + t\psi)^{-3} \rho^{-2-|\alpha|}.
\]

If \(t\psi \leq 1\), then \((1 + t\psi)^{-3} \leq 1\), and if \(t\psi > 1\), then \(t^2 \psi^2 (1 + t\psi)^{-3} \leq (t\psi)^{-1}\), which completes the proof. \(\square\)

**Theorem 4.11.** Suppose that a generalised laplacian \(P\) satisfies Assumptions (4.7). Let \(\mu_t\) and \(\nu_t\) be semigroups of measures generated by \(P\). Then
\[
\mu_t = \nu_t - t^2 \partial_3 \nu_t \ast_0 \sum_{|\alpha|=1} (T_1^\alpha P \ast_0 T_2^\alpha P) + k_{t,N},
\]
where
\[
|k_t(g)| \leq c' \min (t, t^{-1}) \|g\|^{-(2n+1)-N}.
\]

**Proof.** By applying the inverse Fourier transform to the formula of Proposition 4.10 we get
\[
\mu_t = \nu_t - t^2 \partial_3 \nu_t \ast_0 \sum_{|\alpha|=1} (T_1^\alpha P \ast_0 T_2^\alpha P) + \tilde{h}_t,
\]
and
\[
|k_t| \leq c \min (t^2 \psi^2, t^{-1} \psi^{-1}) \rho^{-N}.
\]

Thus by Proposition 2.2 we get
\[
|D_1^\alpha \tilde{h}_t(g)| \leq c' \min (t^2, t^{-1}) \|g\|^{-2n-1-2-|\alpha|}.
\]

\(\square\)

4.3. **Asymptotic behaviour of Gamma-variance semigroup on \(\mathfrak{h}\).** Let \((\gamma_t)_{t>0}\) be a standard gamma distribution on the positive semiaxis and let
\[
\hat{\gamma}_t(x) = \gamma_t \ast \tilde{\gamma}_t = 2^{1-2t} \Gamma(t)^{-2} \int_x^\infty (u^2 - x^2)^{t-1} e^{-u} \, du.
\]

The distribution with densities \(\hat{\gamma}_t\) is called the symmetric gamma semigroup (also so-called gamma-variance). The measures with densities \((\gamma_t)_{t>0}\) form a convolution semigroup and its
The multidimensional analogue is the semigroup generated by the distribution
\[ \hat{\Gamma}(f) = \lim_{\epsilon \to 0} \int_{|x| \geq \epsilon} \frac{f(x) - f(0)}{\|x\|^{2n+1}} K_{2n+1}(\|x\|) \, dx, \]
when \( K_{n}(v) \) is the modified Bessel function of the second kind (the McDonald function). The Lévy measure of this distribution is then \( \|x\|^{-\frac{2n+1}{2}} K_{2n+1}(\|x\|) \), and the Fourier Transform of the generating functional is \( -\log(1 + \|\xi\|^2) \). Thus, the transform of the semigroup is \( \hat{\Gamma}(\xi) = (1 + \|\xi\|^2)^{-t} \), and we get an explicit formula for the densities \( l_t \)
\[ l_t(x) = \pi^{-\frac{n}{2}} 2^{t-\frac{1}{2}} \Gamma(t)^{-1} \|x\|^{t-\frac{1}{2}} K_{t-\frac{1}{2}}(\|x\|). \]

In particular, for \( x \to 0 \) we have
\[ l_t(x) \sim \begin{cases} 
  c_t \|x\|^{2t-2n-1} & \text{dla } t < \frac{2n+1}{2}, \\
  c_t \log(\|x\|^{-1}) & \text{dla } t = \frac{2n+1}{2}, \\
  c_t & \text{dla } t > \frac{2n+1}{2}.
\end{cases} \]

The generalised laplacian \( P \) is also the generating functional of a semigroup of measures \( \mu_t \) on the Heisenberg group \( \mathfrak{h} \). It is easy to see that \( P \) satisfies the assumptions of Proposition 4.11. Thus, we can conclude

**Corollary 4.12.** We have the following asymptotic behaviour of the gamma-variance semigroup on the Heisenberg group
\[ \mu_t(g) \sim c_t \|g\|^{2n-2n-1}, \quad 2t < 2n + 1, |g| \to 0. \]

Let \( (\gamma_t)_{t>0} \) be a standard gamma distribution on the positive semiaxis and let
\[ \hat{\gamma}_t = \gamma_t \ast \hat{\gamma} = 2^{1-2t} \Gamma(t)^{-2} \int_{-\infty}^{\infty} (u^2 - x^2)^{-t} e^{-u} du. \]

The distribution with densities \( \gamma_t \) is called the symmetric gamma semigroup (also so-called gamma-variance). The measures with densities \( (\gamma_t)_{t>0} \) form a convolution semigroup and its multidimensional analogue is the semigroup generated by the distribution
\[ \hat{\Gamma}(f) = \lim_{\epsilon \to 0} \int_{|x| \geq \epsilon} \frac{f(x) - f(0)}{\|x\|^{2n+1}} K_{2n+1}(\|x\|) \, dx, \]
when \( K_{n}(v) \) is the modified Bessel function of the second kind (the McDonald function). The Lévy measure of this distribution is then \( \|x\|^{-\frac{2n+1}{2}} K_{2n+1}(\|x\|) \), and the Fourier Transform of the generating functional is \( -\log(1 + \|\xi\|^2) \). Thus, the transform of semigroup is \( \hat{\Gamma}(\xi) = (1 + \|\xi\|^2)^{-t} \), and we get an explicit formula for the densities \( l_t \)
\[ l_t(x) = \pi^{-\frac{n}{2}} 2^{t-\frac{1}{2}} \Gamma(t)^{-1} \|x\|^{t-\frac{1}{2}} K_{t-\frac{1}{2}}(\|x\|). \]

In particular, for \( x \to 0 \) we have
\[ l_t(x) \sim \begin{cases} 
  c_t \|x\|^{2t-2n-1} & \text{dla } t < \frac{2n+1}{2}, \\
  c_t \log(\|x\|^{-1}) & \text{dla } t = \frac{2n+1}{2}, \\
  c_t & \text{dla } t > \frac{2n+1}{2}.
\end{cases} \]
As a generalised laplacian $P$ is also the generator of a semigroup of measures $\mu_t$ on the Heisenberg group $h$. From Proposition 4.11 we can conclude

**Corollary 4.13.** We have following asymptotic behavior of the gamma-variance semigroup on the Heisenberg group

$$\mu_t(g) \sim c_t \|g\|^{2t-2n-1}, \quad 2t < 2n + 1, \|g\| \to 0.$$ 

**Acknowledgements.** The author is grateful to P.Głowacki for inspiring conversations on the subject of the present note and critical reading of the manuscript. He also thanks M.Preisner for his many useful suggestions.

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