LIMIT THEOREMS FOR CYLINDRICAL MARTINGALE PROBLEMS
ASSOCIATED WITH LÉVY GENERATORS

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ABSTRACT. We derive limit theorems for cylindrical martingale problems associated to Lévy
generators. Furthermore, we give sufficient and necessary conditions for the Feller property of
well-posed problems with continuous coefficients. As applications we derive a Skorokhod-type
existence result for weak solutions of infinite-dimensional jump-diffusion-type SDEs and local
boundedness conditions for limit theorems and the Feller property of weak solutions to semilinear
SPDEs driven by Wiener noise whose linearity is the generator of a compact semigroup.

1. INTRODUCTION

Cylindrical martingale problems (MPs) associated with Lévy generators can be considered as
the martingale formulation of (analytically and probabilistically) weak and mild solutions to (semi-
linear) stochastic (partial) differential equations (S(P)DEs) driven by Lévy noise. As in the classical
finite-dimensional case, the martingale formulation gives access to weak conditions for the strong
Markov property and Girsanov-type theorems, see [4, 19]. Another application of MPs, which was
impressively exploited by Stroock and Varadhan [29] and Jacod and Shiryaev [16] in the finite-
dimensional case, are limit theorems. In this article, we show the following: A sequence of solutions
to MPs whose initial laws converge weakly and whose coefficients converge uniformly on compact
sets to continuous coefficients can only converge weakly to a solution of the MP associated with the
limiting initial law and the limiting coefficients. Using results from [4], this observation transfers to
solution measures of weak and mild solutions to S(P)DEs. Moreover, we prove that under a uniqueness
and existence assumption on the limiting MP, the tightness assumption can be replaced by a
localized tightness condition, which is useful when one aims to verify tightness via boundedness or
moment conditions.

Let us mention two consequences of this observation. Following Stroock and Varadhan [29], we
say that a family of solutions to a well-posed MP form a Feller family (or correspond to a Feller
process) if it is weakly continuous w.r.t. their initial values. The limit theorem yields that a family
of well-posed MPs with continuous coefficients is Feller family if and only if a tightness property holds.
This generalizes results known for finite- and infinite-dimensional cases, see [19, 28, 29]. Moreover,
the limit theorem can be used to construct solutions to MPs from solutions of approximate MPs
and hence provides existence results in the spirit of Skorokhod’s theorem for SDEs.

We provide examples for these applications. First, we derive continuity and linear growth con-
ditions for the existence of weak solutions to SDEs of the type

\[ dY_t = Sb(Y_t)dt + S\sigma(Y_t)dW_t + \int S\nu(x,Y_t)(p-q)(dx,dt), \]

(1.1)

where \( W \) is a Brownian motion, \( p - q \) is a compensated random measure and \( S \) is a compact
operator. Second, we consider MPs corresponding to equations of the form

\[ dY_t = (AY_t + b(Y_t))dt + \sigma(Y_t)dW_t, \]

(1.2)
where $A$ is the generator of a compact $C_0$-semigroup. We adapt the compactness method from [11] to show that the localized version of tightness holds if the non-linearities $b$ and $\sigma$ satisfy local boundedness conditions. Consequently, well-posed diffusion-type MPs with continuous locally bounded non-linearities form a Feller family. This observation is, to the best of our current knowledge, new. In addition, we derive limit theorems either under a well-posedness assumption or under a linear growth conditions on the non-linearities $b$ and $\sigma$. An example for the generator of a compact semigroup is the Laplacian. Thus, our results apply, for instance, to non-linear stochastic heat equations.

We comment on related literature in infinite-dimensional frameworks. In the diffusion case, a limit theorem for diffusions under a tightness and local boundedness condition can be found in [14]. We extend the result by showing that the tightness is implied by the existence of a unique limit law. The limit theorem under the linear growth condition is known, see [27]. Our result confirms the conjecture from [27] that no moment assumption on the initial law is needed. In frameworks with jumps we are only aware of limit theorems for semimartingale cases, see [32]. Our existence result for SDEs of the type (1.1) seems to be new. It extends a result from [12], which applies to SDEs driven by Wiener noise. There is a vast literature on the Feller property of diffusion-type S(P)DEs. We mentioned two related papers: [20, 22]. In [20] the martingale formulation is used to identify the transition semigroup of the studied Cauchy problem, while the core argument is based on a perturbation result for semigroups. The approach in [22] is based on Growall’s lemma and Girsanov’s theorem.

The article is structured as follows: In Section 2 we formally introduce the MP, following the exposition given in [4]. In Section 3 we state our main results, in Section 4 we discuss the existence and let

Next, we introduce the parameters for the martingale problem:

(i) Let $A: D(A) \subseteq B \to B$ be a linear, densely defined and closed operator. Here, $D(A)$ denotes the domain of the operator $A$.

(ii) Let $b: B \to B$ be Borel and such that for all bounded sequences $(y_n^*)_{n \in \mathbb{N}} \subseteq B^*$ and all bounded sets $G \in \mathcal{B}(B)$ it holds that

$$\sup_{n \in \mathbb{N}} \sup_{x \in G} |\langle b(x), y_n^* \rangle| < \infty.$$
The set of solutions is denoted by $\mathcal{M}$. We are in the position to define the martingale problem. For all solutions coincide on $\mathbb{B}$ and such that for all bounded sequences $(g_n)_{n \in \mathbb{N}} \subset \mathbb{B}^*$, all bounded sets $G \in \mathcal{B}(\mathbb{B})$ and all $\epsilon > 0$ it holds that

$$\sup_{n \in \mathbb{N}} \sup_{x \in G} \int 1_{\{|y| \leq \epsilon\}} |\langle y, g_n \rangle|^2 K(x, dy) < \infty,$$

and $K(\cdot, \{0\}) = 0.$

Let $A^*$ be the Banach adjoint of $A$ and let $C_2^2(\mathbb{R}^d)$ be the set of twice continuously differentiable functions $\mathbb{R}^d \to \mathbb{R}$ with compact support. The set of test functions for our MP consists of cylindrical functions:

$$C \triangleq \{ g(\langle \cdot, y_1 \rangle, \ldots, \langle \cdot, y_n \rangle) : g \in C_c^\infty(\mathbb{R}^n), y_1, \ldots, y_n \in D(A^*), n \in \mathbb{N} \}.$$

If $g$ is twice continuously differentiable and $f = g(\langle \cdot, y_1 \rangle, \ldots, \langle \cdot, y_n \rangle)$, we write $\partial_i f$ for the partial derivative

$$(\partial_i g)(\langle \cdot, y_1 \rangle, \ldots, \langle \cdot, y_n \rangle)$$

and define $\partial_i^2 f$ in the same manner.

A bounded Borel function $h : \mathbb{B} \to \mathbb{B}$ is called truncation function if there exists an $\epsilon > 0$ such that $h(x) = x$ on the set $\{ x \in \mathbb{B} : \|x\| \leq \epsilon \}$. Throughout the article we fix a truncation function $h$.

For $f = g(\langle \cdot, y_1 \rangle, \ldots, \langle \cdot, y_n \rangle) \in C$ we set

$$Kf(x) \triangleq \sum_{i=1}^n (\langle x, A^* y_i \rangle + \langle b(x), y_i \rangle) \partial_i f(x) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \langle a(x) y_i, y_j \rangle \partial^2_{ij} f(x)$$

$$+ \int \left( f(x + y) - f(x) - \sum_{i=1}^n \langle h(y), y_i \rangle \partial_i f(x) \right) K(x, dy).$$

We are in the position to define the martingale problem.

**Definition 1.** We call a probability measure $P$ on $(\Omega, \mathcal{F})$ a solution to the martingale problem (MP) $(\mathbb{A}, b, a, K, \eta)$, if the following hold:

(i) $P \circ X_{t_{-1}}^{-1} = \eta$.

(ii) For all $f \in C$ the process

$$M^f \triangleq f(X) - f(X_0) - \int_0^t Kf(X_s-)ds,$$

is a local $P$-martingale.

The set of solutions is denoted by $\mathcal{M}(\mathbb{A}, b, a, K, \eta)$. We say that the MP has a unique solution, if all solutions coincide on $\mathcal{F}$. Moreover, we say that the MP is well-posed, if there exists a unique solution for all degenerated initial laws, i.e. for all $\eta = \delta_x$, $x \in \mathbb{B}$, where $\delta_x$ is the Dirac measure on $x \in \mathbb{B}$.

It can be proven that the set

$$\mathcal{D} \triangleq \{ g(\langle \cdot, y^* \rangle) : g \in C_2^2(\mathbb{R}), y^* \in D(A^*) \} \subset C$$

determines a MP.

In the next section we will show limit theorems for MPs and deduce that well-posed MPs with continuous coefficients $b, a$ and $K$ form a Feller family if and only if they satisfy a tightness condition. For a diffusion framework we derive explicit conditions on the coefficients in Section 5 below.
3. Limit Theorems for Cylindrical Martingale Problems

In this section we state our main results. We start with the limit theorem as described in the introduction. Let us stress that we implicitly assume that all coefficients to the MPs satisfy the assumptions introduced in the previous section.

**Theorem 1.** For all \( n \in \mathbb{N} \) let \( P^n \) be a solution to the MP \((A^n, b^n, a^n, K^n, \eta^n)\). Assume that the following hold:

(i) The map \( x \mapsto K f(x) \) is continuous for all \( f \in \mathcal{D} \). Here, \( K \) is defined as in (2.1).

(ii) Denote \( \mathcal{D}^n \) as in (2.3) with \( A^* \) replaced by \( (A^n)^* \). For all \( f \in \mathcal{D} \) there exists a sequence \( (f^n)_{n \in \mathbb{N}} \) with \( f^n \in \mathcal{D}^n \) such that for all \( m \in \mathbb{N} \)

\[
\sup_{n \in \mathbb{N}} \sup_{x \in \mathbb{B}} |f^n(x)| + \sup_{n \in \mathbb{N}} \sup_{\|x\| \leq m} |K^n f^n(x)| < \infty, \tag{3.1}
\]

and

\[
|f^n - f| + |K^n f^n - Kf| \to 0
\]

as \( n \to \infty \) uniformly on compact subsets of \( \mathbb{B} \). Here, \( K^n \) is defined as in (2.1) with \((A, b, a, K)\) replaced by \((A^n, b^n, a^n, K^n)\).

(iii) \( \eta^n \to \eta \) weakly as \( n \to \infty \).

If \( P \) is a probability measure on \((\Omega, \mathcal{F})\) such that \( P^n \to P \) weakly as \( n \to \infty \), we have \( P \in \mathcal{M}(A, b, a, K, \eta) \).

The proof is given in Section 6.1 below.

In some situations one is interested in whether solutions of MPs converge weakly to a given solution of a well-posed MP. Many criteria for tightness include boundedness or moment conditions. In such cases, it might be easier to consider a localized version of \((P^n)_{n \in \mathbb{N}}\). We introduce the stopping time

\[
\tau_\alpha(a) \triangleq \inf\{t \geq 0 : \|\alpha(t)\| > \alpha \text{ or } \|\alpha(t)\| \geq a\}, \quad a \geq 0, \quad \alpha \in \Omega, \tag{3.2}
\]

see [4, Proposition 2.1.5]. As the following theorem shows, if a good candidate for the limit of \((P^n)_{n \in \mathbb{N}}\) exists, it suffices to show tightness for the localized sequences \((P^n \circ X^{-1}_{\tau_m})_{n \in \mathbb{N}}\) and all \( m \in \mathbb{N} \). In Section 5 below we use this observation together with the compactness method from [11] to deduce weak conditions such that diffusion-type MPs form a Feller family.

**Theorem 2.** Suppose that (i) – (iii) in Theorem 1 hold. If the MP \((A, b, a, K)\) is well-posed and for all \( m \in \mathbb{N} \) the sequence \((P^n \circ X^{-1}_{\tau_m})_{n \in \mathbb{N}}\) is tight, then the MP \((A, b, a, K, \eta)\) has a unique solution \( P \) and \( P^n \to P \) weakly as \( n \to \infty \).

The proof is given in Section 6.2 below. Tightness of stochastic processes is frequently studied. Sufficient and necessary conditions in various settings can be found in [4, 17, 24, 32]. A frequently used criterion is the following version of Aldous’s tightness criterion, see [30, Theorem 6.8] and [21, Corollary p. 120] for proofs.

**Proposition 1.** Let \((P^n)_{n \in \mathbb{N}}\) be a sequence of probability measures on \((\Omega, \mathcal{F})\) such that the following hold:

(i) For all \( t \in \mathbb{R}_+ \) the sequence \((P^n \circ X^{-1}_t)_{n \in \mathbb{N}}\) is tight.

(ii) For all \( \epsilon > 0, M \in \mathbb{N} \) and all sequences \((\rho_n, h_n)_{n \in \mathbb{N}}\), where \((\rho_n)_{n \in \mathbb{N}}\) is a sequence of \((F^n_t)_{t \geq 0}\)-stopping times such that \(\sup_{n \in \mathbb{N}} \rho_n \leq M\) and \((h_n)_{n \in \mathbb{N}} \in (0, \infty)\) is a sequence such that \( h_n \to 0 \) as \( n \to \infty \), we have

\[
\lim_{n \to \infty} P^n\left(\|X_{\rho_n + h_n} - X_{\rho_n}\| \geq \epsilon\right) = 0.
\]

Then, \((P^n)_{n \in \mathbb{N}}\) is tight.

We illustrate an application of this criterion in Section 4 below. Next, we collect consequences of Theorem 1. We start with an existence result, which does not require any uniqueness assumption.
Theorem 3. Let always, we identify Hilbert spaces, ⟨·⟩ denote the corresponding scalar product by \(\sigma\) [16, Section II.1.a]; any Polish space with its Borel \(q = W\) This fact is noteworthy, because if for all \(\text{Theorem 3.2}\). \[P\]

Remark 1. If \(\text{Remark 1.}\) If \(\text{Remark 1.}\) Then, the following are equivalent:

\[(a)\] The family \(\{P_n, x \in \mathcal{B}\}\) is Feller, i.e. \(x \mapsto P_n\) is weakly continuous, which means that \(P_n \rightarrow P\) weakly as \(n \rightarrow \infty\) whenever \(x_n \rightarrow x\) as \(n \rightarrow \infty\).

\[(b)\] For all sequences \((x_n)n \in \mathbb{N}\) with \(x_n \rightarrow x \in \mathcal{B}\) as \(n \rightarrow \infty\), the sequence \((P_{x_n})n \in \mathbb{N}\) is tight.

Corollary 2. Assume the following:

\[(i)\] The map \(x \mapsto \mathcal{K}f(x)\) is continuous for all \(f \in \mathcal{D}\).

\[(ii)\] For all \(x \in \mathcal{B}\) the MP \((A, b, a, K, \varepsilon)\) has a unique solution \(P_x\).

Then, the following are equivalent:

\[(a)\] The family \(\{P_n, x \in \mathcal{B}\}\) is Feller, i.e. \(x \mapsto P_n\) is weakly continuous, which means that \(P_n \rightarrow P\) weakly as \(n \rightarrow \infty\) whenever \(x_n \rightarrow x\) as \(n \rightarrow \infty\).

\[(b)\] For all sequences \((x_n)n \in \mathbb{N}\) with \(x_n \rightarrow x \in \mathcal{B}\) as \(n \rightarrow \infty\), the sequence \((P_{x_n})n \in \mathbb{N}\) is tight.

Corollary 1. In addition to the assumptions \((i) - (iii)\) from Theorem \(\text{Corollary 1}\) if the sequence \((P_n)n \in \mathbb{N}\) is tight, then the MP \((A, b, a, K, \eta)\) has a solution.

As a second corollary, we obtain a characterization of the Feller property of MPs. It can be viewed as a generalization of Corollary 4.4 to a setup including jumps.

4. Existence of Weak Solutions to Jump-Diffusion SDEs

In this section we apply our results in a semimartingale setting. Namely, we give linear growth conditions for the existence of weak solutions to jump-diffusion SDEs of the type

\[dY_t = Sb(Y_t)dt + S\sigma(Y_t)dW_t + \int S\varepsilon(Y_t)(p - q)(dx, dt),\] (4.1)

where \(W\) is a Brownian motion, \(p - q\) is a compensated random measure and \(S\) is a compact operator.

We assume that \(\mathcal{B}\) is a separable Hilbert space. Moreover, let \((E, \mathcal{E})\) be a Blackwell space (see Section II.1.a); any Polish space with its Borel \(\sigma\)-field is Blackwell, \(\mathbb{K}\) and \(\mathbb{H}\) be two separable Hilbert spaces, \(S \in L(\mathbb{K}, \mathcal{B})\) be a compact operator, \(Q \in L(\mathbb{H}, \mathcal{H})\) be a trace class operator and \(q = dt \otimes F\) be the compensator of a random measure on \(\mathbb{R}_+ \times E\) (see Theorem II.1.8). As always, we identify \(\mathcal{B}, \mathbb{K}\) and \(\mathbb{H}\) with their (topological) duals. For any of these Hilbert spaces we denote the corresponding scalar product by \(\langle \cdot, \cdot \rangle\) and the norm by \(\|\cdot\|\).

Theorem 3. Let \(b: \mathcal{B} \rightarrow \mathbb{K}, \sigma: \mathcal{B} \rightarrow L(\mathbb{H}, \mathcal{K})\) and \(v: E \times \mathcal{B} \rightarrow \mathbb{K}\) be Borel maps such that the following hold:

\[(i)\] For all \(y \in E\) the maps

\[x \mapsto Sb(x), S\sigma(x)Q^{\frac{1}{2}}, S\varepsilon(y, x)\]

are continuous. For \(S\sigma Q^{\frac{1}{2}}\) we refer to continuity in the Hilbert-Schmidt norm \(\|\cdot\|_{\text{HS}}\).

\[(ii)\] There exists a Borel function \(\gamma: E \rightarrow \mathbb{R}_+\) such that \(\int \gamma^2(y)F(dy) < \infty\) and a constant \(L \in (0, \infty)\) such that for all \(y \in E\) and \(x \in \mathcal{B}\)

\[\|b(x)\| + \|\sigma(x)Q^{\frac{1}{2}}\|_{\text{HS}} \leq L(1 + \|x\|),\]

\[\|v(y, x)\| \leq \gamma(y)(1 + \|x\|).\]
Then, for all Borel probability measures \( \nu \) on \( \mathbb{K} \) there exists a solution to the MP \((0, \mu, a, K, \eta)\), where \( \eta \equiv \nu \circ S^{-1} \) and for all \( x \in \mathbb{B} \)

\[
\begin{align*}
\mu(x) &\triangleq Sb(x) + \int (h(Sv(y, x)) - Sv(y, x)) F(dy), \\
a(x) &\triangleq S\sigma(x)Q(S\sigma(x))^*, \\
K(x, G) &\triangleq \int 1_{G \setminus \{0\}}(Sv(y, x)) F(dy), \quad G \in \mathcal{B}(\mathbb{B}),
\end{align*}
\]

and \((S\sigma(x))^*\) denotes the adjoint of \(S\sigma(x)\).

We prove this theorem in Section 5.3 below. Lipschitz conditions for the existence of a (pathwise) unique solutions to the SDE (4.1) can be found in [11, 23]. A version of Theorem 3 for SDEs driven by Wiener noise can be found in [12].

Let us comment shortly on the proof and the intuition behind the structure of the equation. The argument is based on the classical strategy behind Skorokhod’s existence result for finite-dimensional SDEs. Namely, we construct an approximation sequence, verify its tightness and use Proposition 1. In infinite dimensional cases part (i) of Proposition 1 is the more difficult part. This is because it does not suffice to prove a moment bound, since closed balls are not compact by Proposition 1. This idea is closely related to the fact that for some stochastic equations on infinite dimensional spaces one can only find a càdlàg solution on an enlarged space.

Because \( S \) is compact, the moment bound for \((P^n)_{n \in \mathbb{N}}\) implies the tightness of \((P^n \circ X_t)_{n \in \mathbb{N}}\), i.e. part (i) in Proposition 1. This idea is closely related to the fact that for some stochastic equations on infinite dimensional spaces one can only find a càdlàg solution on an enlarged space.

5. A Diffusion Setting

In this section we discuss the diffusion case as an important special case of our setting.

5.1. The Setup. We slightly adjust our setup. Let \( \Omega \) be the space of all continuous functions \( \mathbb{R}_+ \to \mathbb{B} \), where \( \mathbb{B} \) is assumed to be a separable Hilbert space. As usual, we identify \( \mathbb{B} \) with its (topological) dual and equip \( \Omega \) with the local uniform topology. We set \( X, \mathcal{F} \) and \( F = (\mathcal{F}_t)_{t \geq 0} \) as in Section 2. Also in this case, \( \mathcal{F} \) is the Borel \( \sigma \)-field on \( \Omega \). Furthermore, we define \( \tau_a \) as in Section 2. Due to the continuous paths of \( X \) we have

\[
\tau_a = \inf\{t \geq 0 : \|X_t\| \geq a\}.
\]

For diffusions the coefficient \( K \) is not relevant and we remove it from all notations. The MP is defined as in Definition 1. The Theorems 1 and 2 also hold in this modified setting.

Let \( b^n \) and \( a^n \) be as follows:

(a) The coefficients \( b^n : \mathbb{B} \to \mathbb{B} \) are Borel such that for all bounded subsets \( G \) of \( \mathbb{B} \)

\[
\sup_{n \in \mathbb{N}} \sup_{x \in G} \|b^n(x)\| < \infty.
\]

(b) The coefficients \( a^n : \mathbb{B} \to S^+(\mathbb{B}, \mathbb{B}) \) have decompositions \( a^n = \sigma^n Q(\sigma^n)^* \), where \( \sigma^n : \mathbb{B} \to L(\mathbb{H}, \mathbb{B}) \) is Borel and \( Q \in L(\mathbb{H}, \mathbb{H}) \) is of trace class. For all bounded sets \( G \in \mathcal{B}(\mathbb{B}) \)

\[
\sup_{n \in \mathbb{N}} \sup_{x \in G} \|\sigma^n(x)Q^\frac{1}{2}\|_{HS} < \infty.
\]
We recall that a $C_0$-semigroup $(S_t)_{t \geq 0}$ is called compact if $S_t$ is a compact operator for all $t > 0$. Note that a $C_0$-semigroup with generator $A$ is compact if and only if it is continuous on $(0, \infty)$ in the uniform operator topology and the resolvent of $A$ is compact, see [23, Theorem 3.3, p. 48]. In particular, by [23, (2.5), p. 235], an analytic semigroup (see [25, Definition 5.1, p. 60]) whose generator has a compact resolvent is compact.

In the following we assume that $A$ is the generator of a compact $C_0$-semigroup.

5.2. A Tightness Condition. Next, we study tightness of the sequence $(P^n \circ X_{\Lambda_\infty}^{-1})_{n \in \mathbb{N}}$ when $P^n \in \mathcal{M}(A, b^n, a^n, \eta^n)$. A proof for the following proposition can be found in Section 6.4 below.

**Proposition 2.** Let $m \in [0, \infty]$. Assume that either $m < \infty$ or that there exists a constant $K > 0$ such that for all $x \in \mathbb{B}$ and $n \in \mathbb{N}$

$$
\|b^n(x)\| + \|\sigma^n(x)Q^\frac{1}{2}\|_{HS} \leq K(1 + \|x\|).
$$

If $P^n \in \mathcal{M}(A, b^n, a^n, \eta^n)$ and $(\eta^n)_{n \in \mathbb{N}}$ is tight, then $(P^n \circ X_{\Lambda_\infty}^{-1})_{n \in \mathbb{N}}$ is tight.

**Remark 2.** Because $\tau_\infty(\omega) = \infty$ for all $\omega \in \Omega$, the Proposition 2 includes a tightness criterion for the global sequence $(P^n)_{n \in \mathbb{N}}$ as well as for the localizations $(P^n \circ X_{\Lambda_\infty}^{-1})_{n \in \mathbb{N}}$.

For $m = \infty$ the previous proposition is known, see [27].

5.3. Corollaries. In view of our results from Section 5, Proposition 2 has the following consequences:

**Corollary 3.** Let $b : \mathbb{B} \to \mathbb{B}$ and $a : \mathbb{B} \to S^+(\mathbb{B}, \mathbb{B})$ be such that for all $y^* \in D(A^*)$ the maps

$$
x \mapsto \langle b(x), y^* \rangle, \langle a(x)y^*, y^* \rangle
$$

are continuous and

$$
\langle b^n, y^* \rangle \to \langle b, y^* \rangle, \quad \langle a^n y^*, y^* \rangle \to \langle ay^*, y^* \rangle
$$

as $n \to \infty$ uniformly on compact subsets of $\mathbb{B}$. Furthermore, take $P^n \in \mathcal{M}(A, b^n, a^n, \eta^n)$. If for all $x \in \mathbb{B}$ the MP $(A, b, a, \varepsilon_x)$ has a unique solution $P_x$, then $P^n \to \int P_x \eta(dx)$ weakly as $n \to \infty$ whenever $\eta^n \to \eta$ weakly as $n \to \infty$.

This observation is a generalization of a classical finite-dimension result [21, Theorem 11.1.4] and it extends the related infinite-dimensional result [19, Lemma 4.3].

We can replace the existence and uniqueness assumption in the previous corollary by a linear growth condition as the following corollary shows.

**Corollary 4.** Assume that there is a constant $K > 0$ such that for all $x \in \mathbb{B}$ and $n \in \mathbb{N}$

$$
\|b^n(x)\| + \|\sigma^n(x)Q^\frac{1}{2}\|_{HS} \leq K(1 + \|x\|).
$$

Let $b : \mathbb{B} \to \mathbb{B}$ and $a : \mathbb{B} \to S^+(\mathbb{B}, \mathbb{B})$ be such that for all $y^* \in D(A^*)$ the maps

$$
x \mapsto \langle b(x), y^* \rangle, \langle a(x)y^*, y^* \rangle
$$

are continuous and

$$
\langle b^n, y^* \rangle \to \langle b, y^* \rangle, \quad \langle a^n y^*, y^* \rangle \to \langle ay^*, y^* \rangle
$$

as $n \to \infty$ uniformly on compact subsets of $\mathbb{B}$ and $P^n \in \mathcal{M}(A, b^n, a^n, \eta^n)$. If $\eta^n \to \eta$ weakly as $n \to \infty$, then there exists a solution $P$ to the MP $(A, b, a, \eta)$ such that $P^n \to P$ weakly as $n \to \infty$.

The main part of the previous corollary is known and given in [27, Theorem 2.1] under a slightly different continuity and convergence assumption and a moment assumption on the initial law. In [25, Remark 2.2] it is conjectured that the assumption on the initial law is not necessary. Corollary 4 confirms this conjecture in a comparable setup.

Finally, we obtain continuity conditions for the Feller property of well-posed problems.

**Corollary 5.** Suppose that $b \equiv b^n$ and $a \equiv a^n$ are such that for all $y^* \in D(A^*)$ the maps

$$
x \mapsto \langle b(x), y^* \rangle, \langle a(x)y^*, y^* \rangle
$$

are continuous. If for all $x \in \mathbb{B}$ the MP $(A, b, a, \varepsilon_x)$ has the unique solution $P_x$, then $\{P_x, x \in \mathbb{B}\}$ forms a Feller family.
This observation generalizes the classical result for the finite-dimensional case given in [22, Corollary 11.1.5] and it extends the infinite-dimensional result [13, Corollary 4.4].

**Example 1.** Assume that \( \mathbb{B} = \mathbb{H} \), that \( \sigma \) is the identity operator on \( \mathbb{B} \) and that \( b \triangleq Qb^\sigma \), where \( b^\sigma : \mathbb{B} \to \mathbb{B} \) is a Borel function which is of linear growth, i.e.,
\[
\|b^\sigma(x)\| \leq \text{const. } (1 + \|x\|).
\]
Furthermore, assume that \( x \mapsto (b(x), y) \) is continuous for all \( y \in \mathbb{B} \). In other words, the MP \( (A, b, a) \) corresponds to the Cauchy problem
\[
dY_t = (AY_t + b(Y_t))dt + dW_t,
\]
where \( W \) is a \( Q \)-Brownian motion. In this case, we have the following:

**Lemma 1.** The MP \( (A, b, a) \) is well-posed.

We sketch a proof in Section 5.5 below. The previous lemma can be compared to [3, Theorem 13], where a similar observation is shown for cylindrical driving noise. Let \( P_x \) be the unique solution to the MP \( (A, b, a, \varepsilon_x) \). The following is a consequence of Corollary 5 and Lemma 1.

**Corollary 6.** The family \( \{P_x, x \in \mathbb{B}\} \) is Feller.

In [22] it is shown that the map \( x \mapsto E_x[f(X_t)] \) is weakly continuous for all bounded continuous functions \( f \). The proof is based on the observation that the Feller property is preserved by Girsanov’s theorem together with the fact that Ornstein-Uhlenbeck families are Feller.

### 5.4. Examples for \( \mathbb{B} \) and \( A \)

**Example 2.** Let \( \mathcal{O} \) be a bounded domain in \( \mathbb{R}^d \) with smooth boundary. We take \( \mathbb{B} \triangleq L^2(\mathcal{O}) \), where \( L^2(\mathcal{O}) \) is defined as usual. For \( i, j = 1, \ldots, d \) let \( \gamma^{ij}, \phi^k, \psi : \text{cl}(\mathcal{O}) \to \mathbb{R} \) be sufficiently smooth. We assume that there exists a constant \( C > 0 \) such that
\[
\sum_{i=1}^d \sum_{j=1}^d \gamma^{ij} \xi_i \xi_j \geq C|\xi|^2,
\]
for all \( \xi = (\xi^1, \ldots, \xi^d) \in \mathbb{R}^d \). Define the differential operator
\[
A \triangleq \sum_{i=1}^d \sum_{j=1}^d \gamma^{ij} \partial^2 \xi_j + \sum_{k=1}^d \phi^k \partial_k + \psi
\]
on
\[
D(A) \triangleq H^2(\mathcal{O}) \cap H^1_0(\mathcal{O}),
\]
where the Sobolev spaces on the r.h.s. are defined as usual. We note that \( A \) generates an analytic semigroup, see [7, Appendix A.5.2] or [23, Theorem 2.7, p. 211], and that the resolvent of \( A \) is compact, see [2, Remark A.28]. Thus, \( A \) generates a compact \( C_0 \)-semigroup. Using the conditions from previous sections, one can formulate limit theorems and criteria for the Feller property. In particular, choosing \( \gamma^{ij} \equiv 1_{\{i=j\}} \) and \( \phi^i \equiv \psi \equiv 0 \), we see that the solution measures to the non-linear stochastic heat equation
\[
dY_t = (AY_t + b(Y_t)) dt + dW_t
\]
forms a Feller family if the non-linearity \( b \) satisfies the assumptions from Example 1. Here, \( \Delta \) denotes the Laplacian.

**Example 3.** (Example 1) Let \( \mathbb{V} \hookrightarrow \mathbb{B} \hookrightarrow \mathbb{B}^* \hookrightarrow \mathbb{V}^* \) be a Gelfand triple of Hilbert spaces with compact embedding \( \mathbb{V} \hookrightarrow \mathbb{B} \). Let \( A \in L(\mathbb{V}, \mathbb{V}^*) \) and assume that there exists a constant \( C > 0 \) such that
\[
\langle u, Au \rangle \geq C\|u\|_{\mathbb{V}}^2
\]
for all \( u \in \mathbb{V} \). The restriction of \( A \) to an operator taking values in \( \mathbb{B} \) has compact resolvent and generates an analytic semigroup. Thus, it generates a compact \( C_0 \)-semigroup. As in the previous example, if the non-linearities satisfy the assumption presented in the previous sections, limit theorems and criteria for the Feller property can be formulated.
6. Proofs

6.1. Proof of Theorem 1

For \( \alpha \in \Omega \) we introduce the following sets:

\[
V(\alpha) \triangleq \{ t > 0 : \tau_{\alpha}(\alpha) < \tau_{+}(\alpha) \},
\]

\[
V'(\alpha) \triangleq \{ t > 0 : \alpha(\tau_{\alpha}(\alpha)) \neq \alpha(\tau_{+}(\alpha) -) \text{ and } \| \alpha(\tau_{\alpha}(\alpha)) - \| = t \}.
\]

We stress that \( \tau_{+} \) is well-defined, since \( t \mapsto \tau_{t} \) is increasing. Due to [3, Problem 13, p. 151] and [10, Proposition VI.2.11], the map \( \alpha \mapsto \tau_{\alpha}(\alpha) \) is continuous at each point \( \alpha \) such that \( \alpha \notin V(\alpha) \).

Furthermore, using [9, Theorem 3.6.3, Remark 3.6.4] instead of [16, Theorem VI.1.14 b)], we can argue as in the proof of [16, Proposition IX.1.17] that the set

\[
\{ t \geq 0 : P(t \in V \cup V') > 0 \}
\]

is at most countable. Therefore, we can choose \( \lambda_m \in [m - 1, m] \) such that

\[
P(\lambda_m \in V \cup V') = 0.
\]

We summarize:

\[
\alpha \mapsto \tau_{\lambda_m}(\alpha), X_{\lambda \wedge \tau_{\lambda_m}(\alpha)}(\alpha) \text{ are continuous up to a } P\text{-null set.}
\]

Next, we show that the process \( M_{t \wedge \tau_{\lambda_m}} \) is a \( P\)-martingale for all \( f \in \mathcal{D} \). Fix an \( f \in \mathcal{D} \) and let \( (f^{n})_{n \in \mathbb{N}} \) be such that \( f^{n} \in \mathcal{D}^{n} \), \( f^{n} \to f \) and \( \mathbb{K}^{n} f^{n} \to \mathbb{K} f \) as \( n \to \infty \) uniformly on compact subsets of \( \mathbb{B} \), see hypothesis (ii). Define \( M^{f,n} \) as in [8] with \( f \) replaced by \( f^{n} \) and \( \mathbb{K} f \) replaced by \( \mathbb{K}^{n} f^{n} \).

Due to [4, Corollary 4.6], the process \( M^{f,n}_{t \wedge \tau_{\lambda_m}} \) is a \( P^{n}\)-martingale. We claim the following: There exists a set \( D \subseteq \mathbb{R}_{+} \), which is dense in \( \mathbb{R}_{+} \), such that

(a) for any bounded sequence \( (t_{n})_{n \in \mathbb{N}} \subseteq D \) and any \( t \in D \) we have

\[
\sup_{n \in \mathbb{N}} \sup_{\alpha \in \Omega} \left| M^{f}_{t \wedge \tau_{\lambda_m}(\alpha)}(\alpha) \right| + \sup_{n \in \mathbb{N}} \sup_{\alpha \in \Omega} \left| M^{f,n}_{t \wedge \tau_{\lambda_m}(\alpha)}(\alpha) \right| < \infty.
\]

(b) for all \( t \in D \) the map \( \alpha \mapsto M^{f}_{t \wedge \tau_{\lambda_m}(\alpha)}(\alpha) \) is continuous up to a \( P\)-null set.

(c) for all \( t \in D \) and all compact sets \( K \subseteq \Omega \) we have

\[
\sup_{\alpha \in K} \left| M^{f,n}_{t \wedge \tau_{\lambda_m}(\alpha)}(\alpha) - M^{f}_{t \wedge \tau_{\lambda_m}(\alpha)}(\alpha) \right| \to 0
\]

as \( n \to \infty \).

Before we check these properties, we show that they imply that the process \( M^{f} \) is a local \( P\)-martingale. Let \( t \in D \) and \( k : \Omega \to \mathbb{R} \) be bounded and continuous. In this case, (a) and (b) imply that the map

\[
\alpha \mapsto k(\alpha) M^{f}_{t \wedge \tau_{\lambda_m}(\alpha)}(\alpha)
\]

is bounded and continuous up to a \( P\)-null set. Therefore, \( P^{n} \to P \) weakly as \( n \to \infty \) yields that

\[
E^{P^{n}} \left[ k M^{f}_{t \wedge \tau_{\lambda_m}} \right] \to E^{P} \left[ k M^{f}_{t \wedge \tau_{\lambda_m}} \right]
\]

as \( n \to \infty \). Fix \( \varepsilon > 0 \) and denote

\[
c \triangleq \max \left( \sup_{\omega \in \Omega} |k(\omega)|, \sup_{\omega \in \Omega} |k(\omega) M^{f}_{t \wedge \tau_{\lambda_m}(\omega)}(\omega)|, \sup_{n \in \mathbb{N}, \omega \in \Omega} |k(\omega) M^{f,n}_{t \wedge \tau_{\lambda_m}(\omega)}(\omega)|, 1 \right).
\]

Because \( (P^{n})_{n \in \mathbb{N}} \) is tight, we find a compact set \( K \subseteq \Omega \) such that

\[
\sup_{n \in \mathbb{N}} P^{n}(K^{c}) \leq \frac{\varepsilon}{4c}.
\]

Using (c) we find an \( N \in \mathbb{N} \) such that for all \( n \geq N \)

\[
E^{P^{n}} \left[ k M^{f,n}_{t \wedge \tau_{\lambda_m}} - k M^{f}_{t \wedge \tau_{\lambda_m}} \right] \leq 2c \sup_{m \in \mathbb{N}} P^{m}(K^{c}) + c \sup_{\omega \in K} |M^{f,n}_{t \wedge \tau_{\lambda_m}(\omega)}(\omega) - M^{f}_{t \wedge \tau_{\lambda_m}(\omega)}(\omega)| \leq \varepsilon + \frac{\varepsilon}{2} = \varepsilon.
\]
Therefore, using the triangle inequality, we obtain that

$$E^{P^n}[kM^{f,n}_{\lambda \wedge \tau m}] \to E^{P}[kM^{f}_{\lambda \wedge \tau m}]$$

as $n \to \infty$.

Take $s, t \in \mathbb{R}_+$ with $s < t$. Since $D$ is dense in $\mathbb{R}_+$ we find two sequences $(s_n)_{n \in \mathbb{N}}, (t_n)_{n \in \mathbb{N}} \subset D$ with $s_n \searrow s$ and $t_n \nearrow t$ as $n \to \infty$. For any bounded, continuous, and $\mathcal{F}_s$-measurable function $k: \Omega \to \mathbb{R}$ we have

$$E^{P}[k\left(M^{f}_{\lambda \wedge \tau m} - M^{f}_{s \wedge \tau m}\right)] = \lim_{k \to \infty} E^{P}[k\left(M^{f}_{t \wedge \tau m} - M^{f}_{s \wedge \tau m}\right)] = \lim_{k \to \infty} \lim_{n \to \infty} E^{P^n}[k\left(M^{f,n}_{t \wedge \tau m} - M^{f,n}_{s \wedge \tau m}\right)] = 0,$$

by the dominated convergence theorem, which we can apply due to (a), the right-continuity of $k$ is a $1$-martingale.

Therefore, using the triangle inequality, we obtain that

$$\nu(M^{f}_{\lambda \wedge \tau m}) \leq \nu(M^{f}_{s \wedge \tau m})$$

as $s \to t$. Hence, the dominated convergence theorem and the right-continuity of $M^{f}_{\lambda \wedge \tau m}$ yield that

$$\nu(M^{f}_{\lambda \wedge \tau m} \cap F) \leq \nu(M^{f}_{s \wedge \tau m} \cap F)$$

for all $F \subset \mathbb{B}$ for arbitrary closed sets $F$.

Finally, using monotone class arguments, we conclude that

$$E^{P}[M^{f}_{\lambda \wedge \tau m} \mathbf{1}_F] = E^{P}[M^{f}_{s \wedge \tau m} \mathbf{1}_F]$$

for all $F \subset \mathcal{F}_s$, which implies that $M^{f}_{\lambda \wedge \tau m}$ is a $P$-martingale.

Since $\lambda_m \in [m-1, m]$, we have $\lambda_m \nearrow \infty$ as $m \to \infty$ and therefore also $\tau_{\lambda m}(\alpha) \nearrow \infty$ as $m \to \infty$ for all $\alpha \in \Omega$. In other words, $M^f$ is a local $P$-martingale.
One shows as in the proof of [4, Lemma 4.7] that MPs are determined by the test functions in $\mathcal{D}$. Thus, to conclude that $P \in \mathcal{M}(A, b, a, K, \eta)$ it remains to show that $P \circ X_0^{-1} = \eta$. Because $\alpha \mapsto \alpha(0)$ is continuous, the continuous mapping theorem implies that

$$\eta^n = P^n \circ X_0^{-1} \to P \circ X_0^{-1}$$

weakly as $n \to \infty$. The uniqueness of the limit yields the identity $P \circ X_0^{-1} = \eta$, see hypothesis (iii). Thus, $P \in \mathcal{M}(A, b, a, K, \eta)$ and the theorem is proven.

It remains to check (a) – (c). The finiteness of the first part term in (a) follows similar to the proof of [4, Lemma 4.5]. The second term is finite due to the assumption (3.1). Next, we check (b). Set

$$D \triangleq \{ t \geq 0 : P(X_{t \wedge \tau_m} \neq X_{(t \wedge \tau_m)^-}) = 0 \}.$$ 

By [4, Lemma 3.7.7], the complement of $D$ in $\mathbb{R}_+$ is countable. Thus, $D$ is dense in $\mathbb{R}_+$. For each $t \in D$ set

$$U_t \triangleq \{ \alpha \in \Omega : \alpha(t \wedge \tau_m(\alpha)) \neq \alpha((t \wedge \tau_m(\alpha))^-) \},$$

which is a $P$-null set by the definition of $D$. Let $N \in \mathcal{F}$ be a $P$-null set such that the maps $\alpha \mapsto \tau_{\lambda_m}(\alpha), X_{\tau_{\lambda_m}(\alpha)}$ are continuous at all $\alpha \not\in N$, see [6.2]. Take $t \in D$ and $\alpha \not\in N \cup U_t$.

Recalling [12, VI.2.3], we see that the functions $\omega \mapsto \omega(t \wedge \tau_{\lambda_m}(\omega))$ and $\omega \mapsto \omega(0)$ are continuous at $\alpha$. Thus, $M_{t \wedge \tau_{\lambda_m}}$ is continuous at $\alpha$ if the map

$$\omega \mapsto I_{t \wedge \tau_{\lambda_m}}(\omega) \triangleq \int_0^{t \wedge \tau_{\lambda_m}(\omega)} Kf(\omega(s^-))ds$$

is continuous at $\alpha$. Because the set of all $\alpha > 0$ such that $\tau_{\alpha}$ is not continuous at $\alpha$ is at most countable (see [12, Lemma VI.2.10, Proposition VI.2.11]) and $\tau_{\lambda_m}(\alpha) \uparrow \infty$ as $a \to \infty$, we find a $\tilde{\lambda}_m < \infty$ such that $\lambda_m \leq \tilde{\lambda}_m, \tau_{\lambda_m}$ is continuous at $\alpha$ and $\tau_{\lambda_m} > t$. Let $(\alpha_k)_{k \in \mathbb{N}} \subset \Omega$ be such that $\alpha_k \to \alpha$ as $k \to \infty$. Due to the continuity of $\tau_{\lambda_m}$ at $\alpha$, there exists an $N \in \mathbb{N}$ such that $\tau_{\lambda_m}(\alpha_k) > t$ for all $k \geq N$. It follows as in the proof of [4, Lemma 4.7] that there exists a constant $C > 0$ such that $\sup_{\|x\| \leq \tilde{\lambda}_m} |Kf(x)| \leq C$. Now, for all $k \geq N$ we have

$$|I_{t \wedge \tau_{\lambda_m}(\alpha)}(\alpha) - I_{t \wedge \tau_{\lambda_m}(\alpha_k)}(\alpha_k)|$$

$$\leq |I_{t \wedge \tau_{\lambda_m}(\alpha)}(\alpha) - I_{t \wedge \tau_{\lambda_m}(\alpha_k)}(\alpha_k)| + |I_{t \wedge \tau_{\lambda_m}(\alpha_k)}(\alpha_k) - I_{t \wedge \tau_{\lambda_m}(\alpha_k)}(\alpha_k)|$$

$$\leq |I_{t \wedge \tau_{\lambda_m}(\alpha)}(\alpha) - I_{t \wedge \tau_{\lambda_m}(\alpha_k)}(\alpha_k)| + \sup_{x \in K_t} |K^n f^n(x) - Kf(x)| \to 0$$

as $n \to \infty$. Therefore, (c) holds and the proof is complete.

\[ \square \]

6.2. Proof of Theorem [24] Due to the assumption that the MP $(A, b, a, K)$ is well-posed, [4, Theorem 3.2] yields that the MP $(A, b, a, K, \eta)$ has a unique solution $P$. We show that $P^n \to P$ weakly as $n \to \infty$. It is well-known that $P^n \to P$ weakly as $n \to \infty$ if and only if each subsequence of $(P^n)_{n \in \mathbb{N}}$ has a further subsequence which converges weakly to $P$, see, e.g., [2, Theorem 2.6]. If we show that $(P^n)_{n \in \mathbb{N}}$ is tight, then Prohorov’s theorem yields that any subsequence of $(P^n)_{n \in \mathbb{N}}$ has a weakly convergent subsequence, and Theorem [11] together with the uniqueness of $P$, yields that this subsequence converges weakly to $P$. Thus, it suffices to prove that $(P^n)_{n \in \mathbb{N}}$ is tight.
We define the following modulus of continuity:

\[ w'(\alpha, \theta, N) \doteq \inf \{ t_i \} \sup_{s,t \in \{ t_i \}} \| \alpha(s) - \alpha(t) \|, \]

where \{ t_i \} ranges over all partitions of the form 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n \leq N with \( \min_{1 \leq i < n} (t_i - t_{i-1}) \geq \theta \) and \( n \geq 1 \). Now, recall the following fact (see [3, Corollary 3.7.4]):

**Fact 1.** A sequence \((\mu^n)_{n \in \mathbb{N}}\) of probability measures on \((\Omega, \mathcal{F})\) is tight if and only if the following hold:

(a) For all \( t \in \mathbb{Q}_+ \) and \( \epsilon > 0 \) there exists a compact set \( C(t, \epsilon) \subseteq \mathbb{B} \) such that

\[ \limsup_{n \to \infty} \mu^n(X_t \notin C(t, \epsilon)) \leq \epsilon. \]

(b) For all \( \epsilon > 0 \) and \( t > 0 \) there exists a \( \delta > 0 \) such that

\[ \limsup_{n \to \infty} \mu^n(w'(X, \delta, t) \geq \epsilon) \leq \epsilon. \]

In the remainder of this proof, we show that \((P^n)_{n \in \mathbb{N}}\) satisfies (a) and (b) in Fact 1. We start with a few preparations. In what follows let \( m \in \mathbb{N} \) be arbitrary. Denote \( P^{n,m} \doteq P^n \circ X^{1}_{\lambda_{m}} \) and fix \( t \geq 0 \). Let \( Q^m \) be an accumulation point of \((P^{n,m})_{n \in \mathbb{N}}\). Using the same arguments as in the proof of Theorem 1, we find a \( \lambda_m \in [m - 1, m] \) such that for all \( f \in \mathcal{D} \) the process \( M^\lambda_{t \wedge \tau_{m}} \) is a \( Q^m \)-martingale and \( Q^m \circ X^{1}_{ \theta} = \eta \). Due to [3, Proposition 4.13], the assumption that the MP \((A, b, a, K)\) is well-posed implies that

\[ Q^m = P \text{ on } \mathcal{F}_{\tau_{m}}. \]

We note that the choice of \( \lambda_m \) depends on \( Q^m \), see (6.1). However, for any accumulation point of \((P^{n,m})_{n \in \mathbb{N}}\) we find an appropriate \( \lambda_m \) in the interval \([m - 1, m]\). Thus, any accumulation point of \((P^{n,m})_{n \in \mathbb{N}}\) coincides with \( P \) on \( \mathcal{F}_{\tau_{m-1}} \). Due to [3, Problem 13, p. 151] and [14, Lemma 15.20], the map

\[ \alpha \mapsto M^\alpha_{t}(\alpha) \doteq \sup_{s \in [0,t]} \| \alpha(s) \| \]

is upper semicontinuous. Thus, the set

\[ \{ \tau_{m-1} \leq t \} = \{ M^\alpha_{t} \geq m - 1 \} \]

is closed in the Skorokhod topology. We deduce from the Portmanteau theorem that

\[ \limsup_{n \to \infty} P^{n,m}(\tau_{m-1} \leq t) \leq P(\tau_{m-1} \leq t). \] \hspace{1cm} (6.4)

Fix \( \epsilon > 0 \). Since \( P(\tau_{m-1} \leq t) \searrow 0 \) as \( m \to \infty \), we find an \( m^o \in \mathbb{N}_{\geq 2} \) such that

\[ P(\tau_{m^o-1} \leq t) \leq \frac{\epsilon}{2}. \] \hspace{1cm} (6.5)

Because \((P^{n,m^o-1})_{n \in \mathbb{N}}\) is tight, we deduce from Fact 1 that there exists a compact set \( C(t, \epsilon) \subseteq \mathbb{B} \) such that

\[ \limsup_{n \to \infty} P^n(X_{t \wedge \tau_{m^o-1}} \notin C(t, \epsilon)) \leq \frac{\epsilon}{2}. \] \hspace{1cm} (6.6)

Due to Galmarino’s test, see [14, Lemma III.2.43], we have \( \tau_{m^o-1} = \tau_{m^o-1} \circ X^{1}_{\lambda_{m}} \). Thus, we obtain

\[ P^n(X_{t} \notin C(t, \epsilon)) = P^n(X_{t \wedge \tau_{m^o-1}} \notin C(t, \epsilon), \tau_{m^o-1} > t) + P^n(X_{t} \notin C(t, \epsilon), \tau_{m^o-1} \leq t) \leq P^n(X_{t \wedge \tau_{m^o-1}} \notin C(t, \epsilon)) \]

From this, (6.3), (6.5) and (6.6), we deduce that

\[ \limsup_{n \to \infty} P^n(X_{t} \notin C(t, \epsilon)) \leq \epsilon. \]

This proves that the sequence \((P^n)_{n \in \mathbb{N}}\) satisfies (a) in Fact 1.
Next, we show that \((P^n)_{n \in \mathbb{N}}\) satisfies (b) in Fact 1. Let \(\epsilon, t \) and \(m^\circ\) be as before. Because \((P^{n,m^\circ-1})_{n \in \mathbb{N}}\) is tight there exists a \(\delta > 0\) such that

\[
\limsup_{n \to \infty} P^n\left(w'(X_{\wedge \tau_{m^\circ-1}}, \delta, t) \geq \epsilon \right) \leq \frac{\epsilon}{2}.
\] (6.7)

On the set \(\{\tau_{m^\circ-1} > t\}\) we have \(w'(X, \delta, t) = w'(X_{\wedge \tau_{m^\circ-1}}, \delta, t)\). Thus, using (6.4), (6.5) and (6.7), we obtain

\[
\limsup_{n \to \infty} P^n(w'(X, \delta, t) \geq \epsilon) \leq \limsup_{n \to \infty} P^n(w'(X_{\wedge \tau_{m^\circ-1}}, \delta, t) \geq \epsilon) + \limsup_{n \to \infty} P^{n,m^\circ} (\tau_{m^\circ-1} \leq t) \leq \epsilon.
\]

In other words, \((P^n)_{n \in \mathbb{N}}\) satisfies (b) in Fact 1 and the proof is complete. \(\square\)

6.3. Proof of Theorem 3. For each \(n \in \mathbb{N}\) let \(\phi^n : \mathbb{R}^n \to [0, 1]\) be the standard mollifier on \(\mathbb{R}^n\), see, e.g., [13, p. 147]. Recall that \(\phi^n\) is supported on the Euclidean unit ball. Moreover, let \(\psi^n : \mathbb{R} \to [0, 1]\) be a cutoff function, i.e. a smooth function such that \(\psi^n = 1\) on \([-n, n]\) and \(\psi^n(x) = 0\) for \(x \notin [-n - 1, n + 1]\). We denote by \((\epsilon_n)_{n \in \mathbb{N}}\) an orthonormal basis of \(\mathbb{B}\). Moreover, we fix a sequence \((\epsilon_n)_{n \in \mathbb{N}} \subset (0, 1)\) such that \(\epsilon_n \leq \frac{1}{n}\) for all \(n \in \mathbb{N}\). Define

\[
\theta_n x \triangleq \langle (x, e_1), \ldots, (x, e_n) \rangle, \quad x \in \mathbb{B},
\]

and set

\[
v^n(y, x) \triangleq \frac{1}{\epsilon_n} \int \phi^n\left(z - \theta_n x \epsilon_n\right) \psi^n\left(\left(\sum_{i=1}^n z^i e_i\right)\right) v\left(y, \sum_{i=1}^n z^i e_i\right) \, dz, \quad y \in E, x \in \mathbb{B}.
\]

We define \(b^n\) and \(\sigma^n\) in the same manner. Next, we check properties of \(v^n, b^n\) and \(\sigma^n\).

Lemma 2. For all \(m \in \mathbb{N}\) and all \(x, z \in \mathbb{B}\) with \(\|x\| \leq m, \|z\| \leq m\) there exist constants \(L = L(n, m, \gamma, F) \in (0, \infty)\) and \(l = l(n, m) \in (0, \infty)\) such that

\[
\int \|v^n(y, x) - v^n(y, z)\|^2 F(dy) \leq L\|x - z\|^2
\]

and

\[
\|b^n(x) - b^n(z)\| + \|\sigma^n(x)Q^{\frac{\gamma}{2}} - \sigma^n(z)Q^{\frac{\gamma}{2}}\|_{HS} \leq l\|x - z\|.
\]

Proof: We only prove the case \(v^n\). For \(b^n\) and \(\sigma^n\) the claim follows in the same manner. Fix \(m \in \mathbb{N}\). Let \(y \in E\) and \(x, z \in \mathbb{B}\) with \(\|x\|, \|z\| \leq m\). Denote \(G_{n,m} \triangleq \{u \in \mathbb{R}^n : \|u\| \geq m + \epsilon_n\}\). Here, \(\|\cdot\|\) denotes the Euclidean norm. For all \(u \in G_{n,m}\) we have

\[
\|u - \theta_n x\| \geq \|u\| - \|\theta_n x\| \geq \|u\| - m \geq \epsilon_n.
\]

Because \(\phi^n\) is smooth with compact support it is Lipschitz continuous. We denote the corresponding Lipschitz constant by \(L_n\). Furthermore, we have for all \(y \in E\) and \(x \in \mathbb{B}\)

\[
\psi^n(\|x\|)\|v(y, x)\| \leq \psi^n(\|x\|)\gamma(y) (1 + \|x\|) \leq \gamma(y)(2 + n).
\]

Now, we obtain

\[
\|v^n(y, x) - v^n(y, z)\|
\]

\[
\leq \frac{1}{\epsilon_n} \int G_{n,m} \phi^n\left(u - \theta_n x \epsilon_n\right) - \phi^n\left(u - \theta_n z \epsilon_n\right) \psi^n\left(\left(\sum_{i=1}^n u^i e_i\right)\right) \|v\left(y, \sum_{i=1}^n u^i e_i\right)\| \, du
\]

\[
\leq \frac{1}{\epsilon_n} \int G_{n,m} \phi^n\left(u - \theta_n x \epsilon_n\right) - \phi^n\left(u - \theta_n z \epsilon_n\right) \psi^n\left(\left(\sum_{i=1}^n u^i e_i\right)\right) \|v\left(y, \sum_{i=1}^n u^i e_i\right)\| \, du
\]

\[
\leq \frac{\gamma(y)(2 + n) L_n |G_{n,m}|}{\epsilon_n^{n+1}} \|x - z\|.
\]

Thus, we have

\[
\int \|v^n(y, x) - v^n(y, z)\|^2 F(dy) \leq \frac{1}{\epsilon_n^{2(n+1)}} \int \gamma^2(y) F(dy) \left(\frac{L_n (2 + n) |G_{n,m}|}{\epsilon_n^{n+1}}\right)^2 \|x - z\|^2.
\]
This completes the proof. □

Lemma 3. There exists a constant \( l \in (0, \infty) \) such that for all \( y \in E \) and \( x \in B \)
\[
\|v^n(y, x)\|^2 \leq l_\gamma^2(y)(1 + \|x\|^2), \quad \|b^n(x)\|^2 + \|\sigma^n(x)Q^{\frac{1}{2}}\|_{\text{HS}}^2 \leq l(1 + \|x\|^2).
\]

Proof: For all \( u \in \mathbb{R}^n \) with \( \|x\| \leq 1 \) the triangle and the Cauchy-Schwarz inequality yield that
\[
\left\| \sum_{k=1}^{n} \epsilon_n u^k e_i \right\| \leq \sum_{k=1}^{n} \epsilon_n |u^k| \leq \epsilon_n \sqrt{n} \|u\| \leq 1.
\]
Thus, we obtain that for all \( y \in E \) and \( x \in B \)
\[
\|v^n(y, x)\| \leq \int \phi^n(u) \psi^n \left( \left\| \sum_{k=1}^{n} (\epsilon_n u^k e_i + \langle x, e_i \rangle e_i) \right\| \right) \|v(y, x, n)\| \|u\| \, du
\]
\[
\leq \gamma(y) \left( 2 + \sum_{k=1}^{n} \langle x, e_i \rangle \right) \int \phi^n(u) \, du
\]
\[
= \gamma(y) \left( 2 + \sum_{k=1}^{n} \langle x, e_i \rangle \right).
\]
Note that
\[
\left( 2 + \sum_{k=1}^{n} \langle x, e_i \rangle \right)^2 \leq 8 + 2 \sum_{k=1}^{n} \langle x, e_k \rangle^2
\]
\[
= 8 + 2 \sum_{k=1}^{n} \langle x, e_k \rangle^2
\]
\[
\leq 8(1 + \|x\|^2).
\]
Thus, we have
\[
\|v^n(y, x)\|^2 \leq 8 \gamma^2(y)(1 + \|x\|^2).
\]
A similar argument applies for \( b^n \) and \( \sigma^n \). □

Lemma 4. For all \( y \in E \) we have
\[
\|Sv^n(y, \cdot) - Sv(y, \cdot)\| + \|Sb^n - Sb\| + \|S\sigma^nQ^{\frac{1}{2}} - S\sigma Q^{\frac{1}{2}}\|_{\text{HS}} \to 0
\]
as \( n \to \infty \) uniformly on compact subsets of \( B \).

Proof: Again, we only show the claim for \( v^n \). Fix \( y \in E \) and \( \varepsilon > 0 \) and let \( K \subset B \) be compact. We set
\[
G_n \triangleq \left\{ \sum_{k=1}^{n} (\epsilon_n u^k e_i + \langle x, e_i \rangle e_i) : x \in K, u \in \mathbb{R}^n \text{ with } \|u\| \leq 1 \right\}.
\]
and \( G \triangleq K \cup \left( \bigcup_{n \in \mathbb{N}} G_n \right) \). For all \( n \in \mathbb{N} \) the set \( G_n \) is compact in \( B \) as it is the image of the compact set \( \{ u \in \mathbb{R}^n : \|u\| \leq 1 \} \times K \) under the continuous map
\[
(u, x) \mapsto \sum_{k=1}^{n} (\epsilon_n u^k e_i + \langle x, e_i \rangle e_i).
\]
We claim that also the set \( G \) is compact in \( B \). To see this take a sequence \( (y_n)_{n \in \mathbb{N}} \subset G \). We have to show that \( (y_n)_{n \in \mathbb{N}} \) has a subsequence converging to an element in \( G \). There exists a sequence \( (k_n)_{n \in \mathbb{N}} \subset \mathbb{N} \) and two sequences \( (x_n)_{n \in \mathbb{N}} \subset \mathbb{B} \) and \( (u_n)_{n \in \mathbb{N}} \) with \( u_n \in \mathbb{R}^{k_n} \) and \( \|u_n\| \leq 1 \) such that
\[
y_n = \sum_{i=1}^{k_n} (\epsilon_{k_n} u_n^i e_i + \langle x_n, e_i \rangle e_i), \quad n \in \mathbb{N}.
\]
Suppose that \( k \triangleq \sup_{n \in \mathbb{N}} k_n < \infty \). Then, we have \((y_n)_{n \in \mathbb{N}} \subset \bigcup_{i=1}^{k} G_i \). Because \( \bigcup_{i=1}^{k} G_i \) is compact as the finite union of compact sets, the sequence \((y_n)_{n \in \mathbb{N}}\) has a subsequence converging to an element in \( \bigcup_{i=1}^{k} G_i \subset G \). Suppose now that \( k = \infty \). Because \( G \) is compact, passing eventually to a subsequence, we can assume that \( k_n \to \infty \) as \( n \to \infty \) and that there exists an \( x \in K \) such that \( x_n \to x \) as \( n \to \infty \). Now, we have

\[
\left\| \sum_{i=1}^{k_n} (\epsilon_{k_n} u_i^t e_i + \langle x_n, e_i \rangle e_i) - x \right\| \leq \sum_{i=1}^{k_n} |u_i^t| \epsilon_{k_n} + \left\| \sum_{i=1}^{k_n} (x_n - x, e_i) e_i - \sum_{i=k_n+1}^{\infty} \langle x, e_i \rangle e_i \right\|
\]

\[
\leq \frac{1}{\sqrt{k_n}} + \|x_n - x\| + \sqrt{\sum_{i=k_n+1}^{\infty} |\langle x, e_i \rangle|^2} \to 0
\]

as \( n \to \infty \). Thus, in the case \( k = \infty \) the sequence \((y_n)_{n \in \mathbb{N}}\) has a subsequence converging to a point in \( K \subset G \). We conclude the compactness of \( G \).

Because \( G \) is compact, the map \( G \ni x \to Sv(y, x) \) is uniformly continuous. In other words, there exists a \( \delta > 0 \) such that for all \( x, z \in G \) with \( \|x - z\| < \delta \) we have

\[
\|Sv(y, x) - Sv(y, z)\| \leq \varepsilon.
\]

Let \( \varepsilon \leq \frac{\delta}{2\sqrt{2}} \). Because compact sets are totally bounded, there exists an \( N_1 \in \mathbb{N} \) and points \( x_1, \ldots, x_{N_1} \in B \) such that

\[
K \subseteq \bigcup_{i=1}^{N_1} B_{e_i}(\varepsilon),
\]

where \( B_{e_i}(\varepsilon) \triangleq \{ x \in B : \|x - e_i\| \leq \varepsilon \} \). Take \( u \in \mathbb{R}^n \) with \( \|u\| \leq 1 \) and \( x \in K \). We find a \( k \in \{1, \ldots, N_1\} \) such that \( \|x - x_k\| \leq \varepsilon \). Thus, we have

\[
\left\| \sum_{i=1}^{n} (\epsilon_n u^i e_i + \langle x, e_i \rangle e_i) - x \right\| \leq \frac{1}{\sqrt{n}} + \sqrt{\sum_{i=n+1}^{\infty} |\langle x, e_i \rangle|^2}
\]

\[
\leq \frac{1}{\sqrt{n}} + \sqrt{2 \varepsilon^2 + 2 \max_{j=1,\ldots,N_1} \sum_{i=n+1}^{\infty} |\langle x, e_i \rangle|^2}
\]

\[
\leq \frac{1}{\sqrt{n}} + \frac{\delta}{2} + \sqrt{2 \max_{j=1,\ldots,N_1} \sum_{i=n+1}^{\infty} |\langle x, e_i \rangle|^2}.
\]

Therefore, we find an \( N_2 \in \mathbb{N} \) and for all \( n \geq N_2 \) we have

\[
\sup_{\|u\| \leq 1} \sup_{x \in K} \left\| \sum_{k=1}^{n} (\epsilon_n u^i e_i + \langle x, e_i \rangle e_i) - x \right\| < \delta.
\]

Because \( G \) is compact, there exists an \( N_3 \geq N_2 \) such that \( \psi_n(\|x\|) = 1 \) for all \( x \in G \) and \( n \geq N_3 \). Consequently, we obtain that for all \( n \geq N_3 \)

\[
\sup_{x \in K} \|Sv^n(y, x) - Sv(y, x)\| \leq \sup_{x \in K} \int_{\phi^n(u)} \|Sv(y, x) - \sum_{k=1}^{n} (\epsilon_n u^i e_i + \langle x, e_i \rangle e_i) - Sv(y, x)\| du \leq \varepsilon.
\]

This proves the claim.

**Lemma 5.** For all \( n \in \mathbb{N} \) the MP \((0, \bar{b}^n, \bar{a}^n, \bar{K}^n, \iota)\) has a solution \( \bar{P}^n \), where for all \( x \in K \)

\[
\bar{b}^n(x) \triangleq \bar{b}^n(Sx) + \int (h(v(y, Sx)) - v(y, Sx)) F(dy),
\]

\[
\bar{a}^n(x) \triangleq \sigma^n(Sx)Q(\sigma^n(Sx))^*,
\]

\[
\bar{K}^n(x, G) \triangleq \int_{G \setminus \{0\}} \psi^n(y, Sx) F(dy), \quad G \in \mathcal{B}(K).
\]
Proof: In \cite{10} global Lipschitz conditions for the existence of solutions of the SDE
\begin{equation}
    dY_t = b^n(Y_t)dt + \sigma^n(Y_t)dW_t + \int v^n(y, Y_t)(p - q)(dy, dt), \quad Y_0 = \zeta \sim \mu.
\end{equation}
are proven.\footnote{In fact, only deterministic initial values have been considered in \cite{10}. Due to \cite{4} Theorem 3.2 weak existence and uniqueness for all degenerated initial laws implies weak existence for all initial laws. Because pathwise uniqueness holds under Lipschitz conditions, a Yamada-Watanabe argument shows that also strong existence holds for arbitrary initial laws.} It is well-known that for homogeneous SDEs global Lipschitz conditions can be relaxed to local Lipschitz and linear growth conditions (see, e.g., \cite{26} Theorem 18.16)). Due to the Lemmata \cite{2} and \cite{13} the coefficients $b^n, \sigma^n$ and $v^n$ satisfy a local Lipschitz and a linear growth condition. Thus, the claim of the lemma follows from \cite{4} Theorem 3.13. For completeness, we sketch details of the argument. As in the proof of \cite{23} Lemma 34.11, we define for $m \in \mathbb{N}, y \in E$ and $x \in \mathbb{K}$
\begin{align*}
    b^{n,m}(x) &\triangleq b^n\left((1 \wedge \frac{m}{\|x\|_\infty})x\right), \\
    \sigma^{n,m}(x) &\triangleq \sigma^n\left((1 \wedge \frac{m}{\|x\|_\infty})x\right), \\
    v^{n,m}(y, x) &\triangleq v^n\left((1 \wedge \frac{m}{\|x\|_\infty})x\right).
\end{align*}
Recalling Lemma \cite{2} one can check that the coefficients $b^{n,m}, \sigma^{n,m}$ and $v^{n,m}$ satisfy a global Lipschitz condition, whose Lipschitz constant might depend on $m$. We conclude from \cite{10} Theorem 3.11] that for each $m \in \mathbb{N}$ there exists a solution process (see \cite{10} Definition 3.1) to the SDE
\begin{equation}
    dY^m_t = b^{n,m}(Y^m_t)dt + \sigma^{n,m}(Y^m_t)dW_t + \int v^{n,m}(y, Y^m_t)(p - q)(dy, dt), \quad Y^m_0 = \zeta.
\end{equation}
Define
\begin{equation}
    \tau_m := \inf(t \geq 0 : \|SX_t\| \geq m \text{ or } \|SX_t\| \geq m).
\end{equation}
We note that $b^{n,m}(x) = b^{n,m+1}(x), \sigma^{n,m}(x) = \sigma^{n,m+1}(x)$ and $v^{n,m}(y, x) = v^{n,m+1}(y, x)$ for all $x \in \mathbb{K}: \|x\| \leq m$ and $y \in E$. Thus, as in the proof of \cite{26} Lemma 18.15], the Lipschitz conditions imply that a.s. $Y^m_t = Y^m_{t+1}$ for $t \leq \tau_m \circ Y^{m+1} \circ \tau_m \circ Y^{m+1}$. Due to Galmarino's test (see, e.g., \cite{16} Lemma III.2.43), this yields that a.s. $\tau_m \circ Y^m = \tau_m \circ Y^{m+1}$. Next, we note that Lemma \cite{3} implies that $b^{n,m}, \sigma^{n,m}$ and $v^{n,m}$ satisfy a linear growth condition with a linear growth constant independent of $m$. Hence, an argument based on Doob's inequality and Gronwall's lemma (see the proof of Lemma \cite{8} below) yields that a.s. $\tau_m \circ Y^m \to \infty$ as $m \to \infty$. Finally, we conclude that the process
\begin{equation}
    Y_t \triangleq Y^1_t 1\{t < \tau_1 \circ Y^1\} + \sum_{k=2}^{\infty} Y^k_t 1\{\tau_{k-1} \circ Y^{k-1} \leq t < \tau_k \circ Y^k\}, \quad t \geq 0,
\end{equation}
solves the SDE \cite{10}. Now, \cite{4} Theorem 3.13] yields the claim. \hfill \Box

We stress that $P^n$ is a Borel probability measure on the Skorokhod space of càdlàg functions $\mathbb{R}_+ \to \mathbb{K}$. Define by $P^n(d\omega) \triangleq P^n((SX_t)_{t \geq 0} \in d\omega)$ a Borel probability measure on the Skorokhod space of càdlàg functions $\mathbb{R}_+ \to \mathcal{B}$.

Lemma 6. For each $n \in \mathbb{N}$ the probability measure $P^n$ solves the MP $(0, \bar{b}^n, \bar{\sigma}^n, \bar{K}^n, \eta)$, where for all $x \in \mathbb{B}$
\begin{align*}
    \bar{b}^n(x) &\triangleq Sb^n(x) + \int (h(Sv^n(y, x)) - Sv^n(y, x))F(dy), \\
    \bar{\sigma}^n(x) &\triangleq S\sigma^n(x)Q(S\sigma^n(x))^*, \\
    \bar{K}^n(x, G) &\triangleq \int 1_{G \setminus \{0\}}(Sv^n(y, x))F(dy), \quad G \in \mathcal{B}(\mathbb{B}).
\end{align*}

Proof: The claim follows readily from \cite{4} Lemma 4.7]. \hfill \Box

Lemma 7. The sequence $(P^n)_{n \in \mathbb{N}}$ is tight.
Proof: We start with a moment bound. Let \( K \subset \mathbb{K} \) and \( G \subset \mathbb{B} \) be compact sets and denote \( Z \triangleq 1_{\{X_0 \in K\}} \) and \( Y \triangleq 1_{\{X_0 \in G\}} \). Here \( Z \) is a random variable on the Skorokhod space of càdlàg functions \( \mathbb{R}_+ \to \mathbb{K} \), while \( Y \) is a random variable on the Skorokhod space of càdlàg functions \( \mathbb{R}_+ \to \mathbb{B} \).

**Lemma 8.** For all \( T \in \mathbb{R}_+ \) the following hold:

(i) \( \sup_{n \in \mathbb{N}} \mathbb{E}^{n}\left[ \sup_{t \in [0,T]} \|X_t\|^2 Z \right] < \infty \).

(ii) \( \sup_{n \in \mathbb{N}} \mathbb{E}^{n}\left[ \sup_{t \in [0,T]} \|X_t\|^2 Y \right] < \infty \).

**Proof:** We only prove (i). The claim of (ii) follows in the same manner. For \( m \in \mathbb{N} \) let \( \tau_m \) be defined as in (3.2). Let \( (o_k)_{k \in \mathbb{N}} \) be an orthonormal basis of \( \mathbb{K} \). Using Doob’s and Hölder’s inequality, [16, Proposition II.1.28, Theorems II.1.33, II.2.34] we obtain that

\[
\mathbb{E}^{n}\left[ \sup_{t \in [0,T]} \|X_{t \wedge \tau_m}, o_k\|^2 Z \right] \leq 4 \left( \mathbb{E}^{n}\left[ \int_{0}^{T \wedge \tau_m} \langle \tilde{a}^n(X_s), o_k \rangle ds + T \mathbb{E}^{n}\left[ \int_{0}^{T \wedge \tau_m} \langle b^n(S X_s), o_k \rangle ds \right] \right) + \mathbb{E}^{n}\left[ \int_{0}^{T \wedge \tau_m} \langle x, o_k \rangle^2 K^n(X_s, dx) ds + Z(X_0, o_k)^2 \right] \right).
\]

Using Parseval’s identity, we obtain

\[
\mathbb{E}^{n}\left[ \sup_{t \in [0,T]} \|X_{t \wedge \tau_m}\|^2 Z \right] \leq 4 \left( \mathbb{E}^{n}\left[ \int_{0}^{T \wedge \tau_m} \text{trace} \tilde{a}^n(X_s) ds + T \mathbb{E}^{n}\left[ \int_{0}^{T \wedge \tau_m} \|b^n(S X_s)\|^2 ds \right] \right) + \mathbb{E}^{n}\left[ \int_{0}^{T \wedge \tau_m} \|x\|^2 K^n(X_s, dx) ds + Z\|X_0\|^2 \right] \right).
\]

Due to Lemma 3, we find two constants \( c_1, c_2 > 0 \), only depending on \( t, \gamma, F, S \) and \( T \) (in an increasing manner), such that

\[
\mathbb{E}^{n}\left[ \sup_{t \in [0,T]} \|X_{t \wedge \tau_m}\|^2 Z \right] \leq c_1 + c_2 \int_{0}^{T} \mathbb{E}^{n}\left[ \sup_{t \in [0,s]} \|X_{t \wedge \tau_m}\|^2 Z \right] ds.
\]

Gronwall’s lemma yields that

\[
\mathbb{E}^{n}\left[ \sup_{t \in [0,T]} \|X_{t \wedge \tau_m}\|^2 Z \right] \leq c_1 e^{c_2 T}.
\]

Finally, Fatou’s lemma yields that

\[
\mathbb{E}^{n}\left[ \sup_{t \in [0,T]} \|X_t\|^2 Z \right] \leq \liminf_{m \to \infty} \frac{\mathbb{E}^{n}\left[ \sup_{t \in [0,T]} \|X_{t \wedge \tau_m}\|^2 Z \right]}{R^2} \leq c_1 e^{c_2 T}.
\]

This completes the proof. \( \square \)

Fix \( \epsilon > 0 \). Because any Borel probability measure on a Polish space is tight (see, e.g., [9, Lemma 3.2.1]), we can choose the compact set \( K \subset \mathbb{K} \) such that \( \epsilon(K) \geq 1 - \frac{\epsilon}{2} \). Using Chebyshev’s inequality, we obtain that

\[
\bar{P}^{n}(\|X_t\| Z \leq R) \geq 1 - \frac{\sup_{n \in \mathbb{N}} \mathbb{E}^{n}[\|X_t\|^2 Z]}{R^2}.
\]

Due to Lemma 8 we find \( R^* > 0 \) such that

\[
\inf_{n \in \mathbb{N}} \bar{P}^{n}(\|X_t\| Z \leq R^*) \geq 1 - \frac{\epsilon}{2}.
\]

Set

\[
K_1 \triangleq \text{cl} \{ Sx : x \in \mathbb{B} \text{ and } \|x\| \leq R^* \} \subset \mathbb{B}.
\]
Because $S$ is compact, the set $K_1$ is compact in $\mathbb{B}$. We obtain

$$P^n(X_t \in K_1) \geq P^n(SX_t \in K_1, X_0 \in K) = P^n(SX_t \in K_1, X_0 \in K) \geq P^n(SX_t \in K_1) - 1 + \epsilon(K) \geq P^n(\|X_t\| \leq R^* - \frac{\epsilon}{2}) \geq 1 - \epsilon.$$

This shows that $\left(P^n \circ X_t^{-1}\right)_{t \in \mathbb{R}^+}$ is tight for all $t \in \mathbb{R}^+$. In other words, (i) in Proposition 1 holds. Let $M > 0$ and $(\rho_n)_{n \in \mathbb{N}}$ be a sequence of $(F^\rho)_{t \geq 0}$-stopping times such that $\sup_{n \in \mathbb{N}} \rho_n \leq M$. Moreover, let $(h_n)_{n \in \mathbb{N}} \subset \mathbb{R}^+$ be a sequence such that $h_n \to 0$ as $n \to \infty$. For $\epsilon > 0$ let $K_\epsilon \subset \mathbb{B}$ be compact and set $Y_\epsilon \triangleq 1 \{X_0 \in K_\epsilon\}$. Using classical results on time-changed semimartingales (see, e.g., [12, Section 10.1]), we can argue as in the proof of Lemma 8 and obtain that

$$E^n \left[\|X_{\tau \wedge (\rho_n + h_n)} - X_{\tau \wedge \rho_n}\|^2 Y_\epsilon\right] \leq 3 \left( E^n \left[Y_\epsilon \int_{\tau \wedge (\rho_n + h_n)}^{\tau \wedge \rho_n} \operatorname{trace} \tilde{a}^n(X_s) ds + Y_\epsilon \int_{\tau \wedge (\rho_n + h_n)}^{\tau \wedge \rho_n} \int \|x\|^2 \bar{K}^n(X_s, dx) ds \right] + E^n \left[h_n Y_\epsilon \int_{\tau \wedge (\rho_n + h_n)}^{\tau \wedge \rho_n} \|S \tilde{b}^n(X_s)\|^2 ds\right]\right).$$

Thus, due to the Lemmata 3 and 8 we find two constants $c_1, c_2 > 0$ (depending on $\epsilon$) such that

$$E^n \left[\|X_{\tau \wedge (\rho_n + h_n)} - X_{\tau \wedge \rho_n}\|^2 Y_\epsilon\right] \leq c_1 h_n + c_2 h_n^2.$$

Let $(\epsilon_k)_{k \in \mathbb{N}} \subset (0, \infty)$ be a sequence such that $\epsilon_k \searrow 0$ as $k \to \infty$ and $K_{\epsilon_k}$ satisfies

$$\inf_{n \in \mathbb{N}} \eta^n(K_{\epsilon_k}) \geq 1 - \epsilon_k.$$

Fix $\delta_1, \delta_2 > 0$. By Chebychev’s inequality, we have

$$P^n(\|X_{\tau \wedge (\rho_n + h_n)} - X_{\tau \wedge \rho_n}\| \geq \delta_1) \leq P^n(\|X_{\tau \wedge (\rho_n + h_n)} - X_{\tau \wedge \rho_n}\| Y_\epsilon_k \geq \delta_1) + \eta^n(K_{\epsilon_k}) \leq E^n \left[\|X_{\tau \wedge (\rho_n + h_n)} - X_{\tau \wedge \rho_n}\|^2 Y_\epsilon_k\right] + \epsilon_k \leq \frac{(c_1 h_n + c_2 h_n^2)}{\delta_1^2} + \epsilon_k.$$

We note that $c_1$ and $c_2$ depend on $k$. Choose $k \in \mathbb{N}$ such that $\epsilon_k < \frac{\delta_2}{2}$. From now on $k$ is fixed and the $c_1$ and $c_2$ are considered to be usual constants. Because $h_n \to 0$ as $n \to \infty$, we find an $N \in \mathbb{N}$ such that

$$c_1 h_n + c_2 h_n^2 < \frac{\delta_1^2 \delta_2}{2}$$

for all $n \geq N$. We conclude that

$$P^n(\|X_{\tau \wedge (\rho_n + h_n)} - X_{\tau \wedge \rho_n}\| \geq \delta_1) < \delta_2$$

for all $n \geq N$. In other words, (ii) in Proposition 1 holds. This completes the proof.

Due to the continuity assumptions on $b, \sigma$ and $\nu$ the map

$$x \mapsto K f(x)$$

is continuous for all $f \in \mathcal{D}$. Here, $K$ is defined as in [12.2] with $A = 0$ and $b$ replaced by $\mu$ as given in the statement of Theorem 3. Let $K \subset \mathbb{B}$ be compact and $f = g(\cdot, y^*) \in \mathcal{D}$. Then, using Taylor’s
Due to Proposition 8.4, the operator $G$

As usual, let $p > 7$. We obtain

Theorem 7.2. yields the bound

Let $\varepsilon > 0$ be fixed. Because $(\eta^n)_{n \in \mathbb{N}}$ is tight, there exists a compact set $B \subset \mathbb{B}$ such that

Let $\varepsilon > 0$ be fixed. Because $(\eta^n)_{n \in \mathbb{N}}$ is tight, there exists a compact set $B \subset \mathbb{B}$ such that

Thus, the dominated convergence theorem and Lemma 4 yield that also the third part converges to zero as $n \to \infty$. We conclude from Corollary 1 and Lemma 3 that the MP $(\mu, a, K, q)$ has a solution.

6.4. Proof of Proposition 2

For $T \in (0, \infty)$, let $\Omega^T$ be the space of continuous functions $[0, T) \to \mathbb{B}$, let $X^T$ denote the coordinate process on $\Omega^T$ and let $\mathcal{F}^T \triangleq \sigma(X^T, t \in [0, T])$.

Due to [3], Corollary 5, the sequence $(P^n \circ X_{\cdot \wedge \tau_m})_{n \in \mathbb{N}}$ is tight if and only if for all $T \in \mathbb{N}$ its restriction to $(\Omega^T, \mathcal{F}^T)$ is tight.

We fix $T \in \mathbb{N}$. Our strategy is to adapt the compactness method from [11]. For $m = \infty$ the claim is proven in [21] by a similar argument.

In what follows next, we also fix $n \in \mathbb{N}$ and work with the filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, P^n)$.

Note that we can find a Brownian motion $W$ with covariance $Q$ (possibly on an extension of $(\Omega, \mathcal{F}, \mathbf{F}, P^n)$) such that we have

We define

where $\mathcal{E}$ is the space of continuous functions $\mathbb{B} \to \mathbb{B}$ such that

Let $p > 2$ and $\theta \in (0, 1)$ such that $0 < \frac{1}{p} < \theta < \frac{1}{2}$. In the case $m = \infty$, using the linear growth condition, standard arguments based on Gronwall’s lemma (see [24, Lemma 3.1] or [7, Theorem 7.2]) yield the bound

where $C_{p,T} \in (0, \infty)$ is a constant independent of $n$. Note that

As usual, let $L^p([0, T], \mathbb{B})$ be the space of all Borel functions $f : [0, T] \to \mathbb{B}$ such that $\int_0^T ||f_s||^p ds < \infty$. For $\xi \in \left(\frac{1}{p}, 1\right]$, $t \in [0, T]$ and $f \in L^p([0, T], \mathbb{B})$, we define

Due to [3, Proposition 8.4], the operator $G_0$ is compact from $L^p([0, T], \mathbb{B})$ into $\Omega^T$. We define

$$Y_s^{n,m,0} \triangleq \int_0^s (s-r)^{-\theta} S_{s-r} \sigma^n(X_{r \wedge \tau_m})dW_r, \ s \in [0, 1].$$
As shown in the proof of [7, Proposition 7.3], there exists a constant \( \tilde{C}_{p,T} > 0 \) such that
\[
E^{P^n} \left[ 1_B(X_0) \int_0^T \| Y^n_{s,n,\theta} \|^p \, ds \right] \leq \tilde{C}_{p,T} \int_0^T E^{P^n} \left[ 1_B(X_0) \| \sigma^n(X_{s,\tau_m}) Q_\pi \|^p \right] \, ds
\]
\[
\leq \left\{ \tilde{C}_{p,T} T \left( \sup_{k \in \mathbb{N}} \sup_{\| x \| \leq m} \| \sigma^k(x) Q_\pi \|^p \right) , \quad m < \infty , \quad m = \infty . \right. \quad (6.10)
\]

By [7, Proposition 1], we have the following factorization formula
\[
\int_0^T S_{t-s} \sigma^n(X_{s,\tau_m}) \, dW_s = \frac{\sin(\pi \theta)}{\pi} (G_\theta Y^n_{s,n,\theta})(t), \quad t \in [0, T] .
\]
Thus, due to [7, Remarks A.1, A.2], for all \( t \in [0, T] \)
\[
\int_0^T S_{t-s} \sigma^n(X_{s,\tau_m}) \, dW_s (t \wedge \tau_m) = \frac{\sin(\pi \theta)}{\pi} (G_\theta Y^n_{s,n,\theta})(t \wedge \tau_m) .
\]
Consequently, we obtain
\[
X_{t,\tau_m} = S_{t,\tau_m} X_0 + (G_1 b^n(X,\tau_m)) (t \wedge \tau_m) + \frac{\sin(\pi \theta)}{\pi} (G_\theta Y^n_{s,n,\theta})(t \wedge \tau_m) , \quad t \in [0, T] .
\]
Define
\[
B_R \triangleq \left\{ u \in L^p([0,T], B) : \int_0^T \| u(s) \|^p \, ds \leq R \right\}
\]
and the sets
\[
\Lambda(R, \theta, m) \triangleq \left\{ \alpha \in \Omega^T : \alpha = (G_\theta u)(\cdot \wedge \tau_m(\lambda)) , \, u \in B_r , \, \lambda \in \Omega \right\} ,
\]
\[
\Lambda^*(m) \triangleq \left\{ \alpha \in \Omega^T : \alpha = S_{t,\tau_m(\lambda)} x , \, x \in B , \, \lambda \in \Omega \right\} ,
\]
and
\[
\Lambda'(R, m) \triangleq \left\{ \alpha \in \Omega^T : \alpha = \alpha^1 + \alpha^2 + \frac{\sin(\pi \theta)}{\pi} \alpha^3 , \, \alpha^1 \in \Lambda^*(m) , \, \alpha^2 \in \Lambda(R, 1, m) , \, \alpha^3 \in \Lambda(R, \theta, m) \right\} .
\]
We claim that the sets \( \Lambda^*(m) , \Lambda(R, 1, m) \) and \( \Lambda(R, \theta, m) \) are relatively compact. To show this, we recall the following version of the Arzelà-Ascoli theorem (see [18, Theorem A.2.1]):

**Fact 2.** Fix two metric spaces \( U \) and \( (V, \rho) \), where \( U \) is compact and \( V \) is complete. Let \( D \) be dense in \( U \). A set \( A \subset C(U, V) \) (the space of all continuous functions \( U \to V \) equipped with the uniform topology) is relatively compact if and only if for all \( t \in D \) the set \( \pi_t(A) \triangleq \{ \alpha(t) : \alpha \in A \} \subset V \) is relatively compact in \( V \) and \( A \) is equicontinuous, i.e. for all \( s \in U \) and \( \epsilon > 0 \) there exists a neighborhood \( N \subset U \) of \( s \) such that for all \( r \in N \) and \( \alpha \in A \)
\[
\rho(\alpha(r), \alpha(s)) < \epsilon .
\]
In that case, even \( \bigcup_{t \in U} \pi_t(A) \) is relatively compact in \( V \).

Let \( \xi \in \{ \theta, 1 \} \). The set \( \Lambda(R, \xi, m) \) is relatively compact if it is equicontinuous and for all \( t \in [0, T] \) the set
\[
C_t \triangleq \{ y \in B : y = \alpha(t) , \, \alpha \in \Lambda(R, \xi, m) \}
\]
is relatively compact in \( B \). We note that the set \( \Lambda(R, \xi, \infty) \) is relatively compact, because the operator \( G_\xi \) is compact. Thus, Fact 2 yields that the set
\[
C \triangleq \bigcup_{t \in [0, T]} \{ y \in B : y = \alpha(t) , \, \alpha \in \Lambda(R, \xi, \infty) \}
\]
is relatively compact in \( B \). For all \( t \in [0, T] \) we have \( C_t \subset C \) and hence the set \( C_t \) is relatively compact in \( B \). Let now \( t \in [0, T] \) and \( \epsilon > 0 \). Again due to Fact 2 there exists a \( \delta > 0 \) such that for all \( s \in [0, T] \) with \( |t - s| \leq \delta \) we have
\[
\| \alpha(t) - \alpha(s) \| < \frac{\epsilon}{2}.\]
for all $\alpha \in \Lambda(R, \xi, \infty)$. Note that
\[
\|\alpha(t \wedge \tau_m(\lambda)) - \alpha(s \wedge \tau_m(\lambda))\| = \begin{cases}
\|\alpha(\tau_m(\lambda)) - \alpha(s)\|, & t \geq \tau_m(\lambda) \geq s, \\
\|\alpha(t) - \alpha(\tau_m(\lambda))\|, & t \leq \tau_m(\lambda) \leq s, \\
\|\alpha(t) - \alpha(s)\|, & t, s \leq \tau_m(\lambda), \\
0, & t, s \geq \tau_m(\lambda).
\end{cases}
\]

In the first case, we have
\[
\|\alpha(\tau_m(\lambda)) - \alpha(s)\| \leq \|\alpha(\tau_m(\lambda)) - \alpha(t)\| + \|\alpha(t) - \alpha(s)\| < \epsilon,
\]
because $|t - \tau_m(\lambda)| \leq |t - s| \leq \delta$. In the second case, we have
\[
|t - \tau_m(\lambda)| = \tau_m(\lambda) - t \leq s - t \leq \delta,
\]
and hence
\[
\|\alpha(t) - \alpha(\tau_m(\lambda))\| < \frac{\epsilon}{2} \leq \epsilon.
\]
In the third and fourth case, the desired inequality holds trivially. Thus, the inequality
\[
\|\alpha(t) - \alpha(s)\| < \epsilon
\]
holds for all $\alpha \in \Lambda(R, \xi, m)$ and $s \in [0, T]$ with $|t - s| \leq \delta$. Thus, the set $\Lambda(R, \xi, m)$ is equicontinuous.

Now, Fact 2 yields that it is also relatively compact.

The relative compactness of $\Lambda^*(m)$ follows from the same argument, if we can show that the set $\Lambda^*(\infty)$ is relatively compact. Because $B$ is bounded and $S_x$ is compact for all $t > 0$, the set
\[
\{y \in \mathcal{B}: y = S_t x, x \in B\}, \quad t \in (0, T],
\]
is relatively compact in $\mathcal{B}$. Since $(S_t)_{t \geq 0}$ is a $C_0$-semigroup, [8, Lemma 5.2, p. 37] yields that the map
\[
[0, T] \times B \ni (t, x) \mapsto S_t x
\]
is uniformly continuous. From this observation we deduce that the set $\Lambda^*(\infty)$ is equicontinuous.

Thus, it is relatively compact by Fact 2. We conclude that also $\Lambda^*(m)$ is relatively compact.

Finally, it follows that $\Lambda'(R, m)$ is relatively compact. Using (6.9) and (6.10) together with Chebychev’s inequality, there exists a constant $\tilde{C}$ only depending on $p, T, m, \sigma$ and $b$ such that for all $R > 0$ we have
\[
P^n \left( X_0 \in B \quad \text{and} \quad \int_0^T \|b^n(X_{s \wedge \tau_m})\|^p ds \leq R \right) \quad \text{and} \quad \int_0^T \|Y^n_{s, m, \theta}\|^p ds \leq R ) \\
\geq \eta^n(B) - P^n \left( 1_{B}(X_0) \int_0^T \|b^n(X_{s \wedge \tau_m})\|^p ds > R \right) - P^n \left( 1_{B}(X_0) \int_0^T \|Y^n_{s, m, \theta}\|^p ds > R \right) \\
\geq 1 - \frac{\epsilon}{2} - \frac{1}{R} \left( E^{P^n} \left[ 1_{B}(X_0) \int_0^T \|b^n(X_{s \wedge \tau_m})\|^p ds \right] + E^{P^n} \left[ 1_{B}(X_0) \int_0^T \|Y^n_{s, m, \theta}\|^p ds \right] \right) \\
\geq 1 - \frac{\epsilon}{2} - \frac{\tilde{C}}{R}.
\]
Consequently, there exists an $R_x > 0$ such that
\[
\inf_{n \in \mathbb{N}} P^n \left( \left( X_{s \wedge \tau_m} \right)_{t \in [0, T]} \in \Lambda'(R_x, m) \right) \geq 1 - \varepsilon.
\]
We conclude that the restriction of $(P^n \circ X^{-1}_{s \wedge \tau_m})_{n \in \mathbb{N}}$ to $(\Omega^T, \mathcal{F}^T)$ is tight. Because $T \in (0, \infty)$ was arbitrary, this implies that $(P^n \circ X^{-1}_{s \wedge \tau_m})_{n \in \mathbb{N}}$ is tight.

**6.5. Sketch of the Proof for Lemma 1.** For all $x \in \mathcal{B}$ the MPs $(A, b, a, \varepsilon_x)$ and $(A, 0, a, \varepsilon_x)$ have solutions due to [11, Theorem 1]. In particular, using the mild formulation, we see that for all $x \in \mathcal{B}$ the MP $(A, 0, a, \varepsilon_x)$ satisfies uniqueness. Now, [4, Proposition 3.7] yields that for all $x \in \mathcal{B}$ also the MP $(A, b, a, \varepsilon_x)$ satisfies uniqueness. This shows the claim.
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