Abstract. By iterative techniques, we present two fixed point theorems, whose modular formulations are relatively close to the Banach’s fixed point theorem in the normed spaces. The first result concerns the fixed point of the strongly $\rho$-contraction mappings. The second result deals with the fixed point of the strict $\rho$-contraction mappings where the modular satisfies the $\Delta_2$-condition. For the $\rho$-nonexpansive mappings, where the modular $\rho$ satisfies the regular growth condition, noted $T$, we present one result of the Schauder’s type, without boundedness conditions on the domain of these mappings.

A.M.S. Subject classifications: 46A80.47H10.

1 Introduction

By iterative techniques, we present two fixed point theorems, whose modular formulations are relatively close to the Banach’s fixed point theorem in the normed spaces. The first result concerns the fixed point of the strongly $\rho$-contraction mappings. As a consequence, we get an improved version of the theorem I-1[?], in particular, by the deletion of the hypothesis, the $\Delta_2$-condition and the Fatou property. The second result concerns the fixed point of the strict $\rho$-contraction mappings where the modular $\rho$ satisfies the $\Delta_2$-condition. With the last condition, the iterative techniques are to happen locally. For the $\rho$-nonexpansive mapping where $\rho$ satisfies the regular growth condition, noted $T$, we present one result of the Schauder’s type (i.e. $\overline{T(B)}^\rho$ is $\rho$-compact, with $B$ is the domain of $T$ ) without boundedness conditions on $B$ used in theorems 2.13 [2] and 2.5 [3].
2 I- Strongly \( \rho \)-contraction mappings

**Definition 2.1** Let \( X_\rho \) be a modular space.

A sequence \((x_n)_{n \in \mathbb{N}}\) in \( X_\rho \) is \( \rho \)-convergent to \( x \in X_\rho \) if: \( \exists c > 0 \) such that \( \rho(c(x_n - x)) \to 0 \) as \( n \to +\infty \).

\( X_\rho \) is \( \rho \)-complete if every \( \rho \)-Cauchy sequence \((x_n)_{n \in \mathbb{N}}\) in \( X_\rho \) is \( \rho \)-convergent, i.e. \( \exists c > 0 \) such that \( \rho(c(x_n - x_m)) \to 0 \) as \( n, m \to +\infty \), then, \( \exists x \in X_\rho \) such that \( \rho(c(x_n - x)) \to 0 \) as \( n \to +\infty \).

For example, Musielak-Orlicz space is \( \rho \)-complete in the sense of the above definition. (cite: jm).

The following result can be considered as the first approach of the Banach’s fixed point theorem in the normed spaces.

**Theorem 2.1** 1.1.Let \( X_\rho \) be a \( \rho \)-complete modular space, and \( B \subseteq X_\rho \) a \( \rho \)-closed subset of \( X_\rho \).

Let \( T : B \to B \) be a mapping such that:

\[ \exists c, k, l \in \mathbb{R}^+ \text{ with } c > l, \ k \in [0, 1[ \text{ and } \rho(c(Tx - Ty)) \leq k \rho(l(x - y)), \ \forall x, y \in B \quad (\ast) \]

Then \( T \) has a fixed point .

Remarks.

We note that if \( \rho(l(x - y)) < +\infty \), \( \forall x, y \in B \), then the fixed point is unique. The insertion of the constants \( c, k \) and \( l \) in \( (\ast) \) has been the field of application of this result and may be useful (see the study, by a fixed point theorem, of an integral equation of \( \rho \)-Lipschitz or perturbed integral equations in modular function space \( C^\varphi = C([0, A], L^\varphi) \) [7], [4])

we note that the contraction \( (\ast) \) is also valid for all constants \( c_0, l_0 \) and \( k_0 \) with \( l \leq l_0 < c_0 \leq c \) and \( 0 < k \leq k_0 < 1 \). Indeed:

\[
\rho(c_0(Tx - Ty)) \leq \rho(c(Tx - Ty)) \\
\leq k \rho(l(x - y)) \\
\leq k_0 \rho(l_0(x - y))
\]

Because \( \alpha \to \rho(\alpha x), \ (\alpha \in \mathbb{R}^+) \) is increasing.

If \( 1 \in [l, c] \), then \( T \) is a strict \( \rho \)-contraction because:

\[
\rho(Tx - Ty) \leq \rho(c(Tx - Ty)) \\
\leq k \rho(l(x - y)) \\
\leq k \rho(x - y)
\]

But, as \( c > l \), we have:

\[
\rho(\lambda(Tx - Ty)) \leq k \rho(\lambda(x - y))
\]

where \( \lambda = l \) or \( \lambda = c \).

With this last inequality, it can be said that \( T \) is a strict \( \rho \)-contraction. Hence, it can be said
that $T$ is a strongly $\rho$-contraction if $T$ satisfies $(\ast)$ in theorem 1.1. The supplementary condition $c > l$, in $(\ast)$, has permitted to delete the boundedness condition concerning the domain of $T$ in [2]-[3]- [5] where $T$ is a strict $\rho$-contraction. But, the hypothesis $c > l$, in theorem 1.1, is essential, and that is to apply constantly the inequality of the modular $\rho$ in the following proof.

Proof of the theorem 1.1.

Let $\alpha \in IR^+$ be the conjugate of $\tilde{\alpha}$, i.e. $\frac{1}{z} + \frac{1}{\alpha} = 1$. We assume without any loss of generality that:

$\exists x \in B$ such that $r = \rho(\alpha l(Tx - x)) < +\infty$. Then the sequence $\{T^n x\}_{n \in \mathbb{N}}$ is $\rho$-Cauchy. Indeed:

$$\rho(c(T^{n+m}x - T^m x)) \leq k\rho(l(T^{n+m-1}x - T^{m-1}x))$$
$$\leq k\rho(c(T^{n+m-1}x - T^{m-1}x))$$
$$\leq k^2\rho(l(T^{n+m-2}x - T^{m-2}x))$$

By induction, we deduce:

$$\rho(c(T^{n+m}x - T^m x)) \leq k^m\rho(l(T^n x - x))$$

Moreover,

$$\rho(l(T^n x - x)) = \rho(l(T^n x - Tx) + l(Tx - x))$$
$$= \rho\left(\frac{l}{c}c(T^n x - Tx) + \frac{\alpha l}{\alpha}(Tx - x)\right)$$
$$\leq \rho(c(T^n x - Tx)) + \rho(\alpha l(Tx - x))$$
$$\leq k\rho(l(T^{n-1} x - x)) + r$$

By induction, we have:

$$\rho(l(T^n x - x)) \leq k^{n-1}\rho(l(Tx - x)) + k^{n-2}r + \ldots + r$$

As $\alpha > 1$, we have $\rho(l(Tx - x)) \leq r$. Then

$$\rho(l(T^n x - x)) \leq \frac{1 - k^n}{1 - k}r$$

Therefore, $\rho(c(T^{n+m}x - T^m x)) \leq k^n \frac{1-k^n}{1-k}r \to 0$ as $n, m \to +\infty$.

$X_\rho$ is $\rho$-complete and $B$ is $\rho$-closed hence, $\exists z \in B$ such that $\rho(c(T^n x - z)) \to 0$ as $n \to +\infty$.

We prove that $z$ is a fixed point of $T$. Indeed,

$$\rho\left(\frac{c}{2}(Tz - z)\right) = \rho\left(\frac{c}{2}(Tz - T^{n+1}x) + \frac{c}{2}(T^{n+1}x - z)\right)$$
$$\leq k\rho(l(z - T^{n}x)) + \rho(c(T^{n+1}x - z))$$
$$\leq \rho(c(z - T^n x)) + \rho(c(T^{n+1}x - z))$$

since $\rho(c(z - T^n x)) + \rho(c(T^{n+1}x - z)) \to 0$ as $n \to +\infty$ then $T(\frac{c}{2}(Tz - z)) = 0$ and $Tz = z$.

Remark 1.1
It results from this proof that: \( \exists x \in B \) such that \( \rho(c(T^{n+m}x - T^mx)) \leq k^m \frac{1-k^n}{1-k^n} r \). If \( \rho \) has the Fatou property, then, the fixed point \( z \) is such that:

\[
\rho(c(z - T^mx)) \leq \lim inf \rho(c(T^{n+m}x - T^mx)) \leq \frac{k^m}{1-k^n}
\]

This estimate allows an approximation to this fixed point.

Remark 1.2

If \( \rho \) is a \( s \)-convex modular, we have the same theorem 1.1. But, because of the \( s \)-convex combination \( \left( \frac{1}{s} \right)^s + \frac{1}{s} = 1 \), some technical modifications are necessary in the theorem 1.1’s proof.

The comparison between the theorem 1.1 and the theorem I.1 in [?] gives the following result.

**Corollary 2.1**

1.1. Let \( X_\rho \) be a \( \rho \)-complete modular space, where \( \rho \) is \( s \)-convex. \( B \subseteq X_\rho \) is a \( \rho \)-closed subset of \( X_\rho \). \( T : B \rightarrow B \) is a mapping such that:

\[
\exists c, k, l \in \mathbb{R}^+ \text{ with } c > Max(l, kl) \text{ and } \rho(c(Tx - Ty)) \leq k^s \rho(l(x - y)), \forall x, y \in B \quad (**) \]

Then \( T \) has a fixed point.

This result brings substantial ameliorations to theorem I.1 [?]: The insertion of the constant \( l \), the deletion of the hypothesis, the \( \Delta_2 \)-condition and the Fatou property.

**Proof of the corollary 1.1:**

Let \( l_0 \) be one constant such that \( c > l_0 > Max(l, kl) \); We have:

\[
\rho(c(Tx - Ty)) \leq k^s \rho(l(x - y)) = k^s \rho(l_0(x - y)) \leq \left( \frac{l}{l_0} \right)^s \rho(l_0(x - y))
\]

Then, \( c > l_0 \) and \( \left( \frac{l}{l_0} \right)^s < 1 \). By the theorem 1.1, \( T \) has a fixed point.

Remark 1.3

It results from the above proof that, if \( \rho \) is \( s \)-convex, the two formulations of the strong contraction of

\( T \) (\( * \) in theorem 1.1. and \( ** \) in corollary 1.1.) are equivalent.

**Remark 1.4**

If \( X_\rho \) is equipped with the following convergence: \( x_n \overset{\rho}{\rightarrow} x \iff \rho(x_n - x) \rightarrow 0 \) as \( n \rightarrow +\infty \).

Then the theorem 1.1 takes the following form:

**Theorem 2.2**

1.2. Let \( X_\rho \) be a \( \rho \)-complete modular space, and \( B \subseteq X_\rho \) a \( \rho \)-closed subset of \( X_\rho \).

Let \( T : B \rightarrow B \) be a mapping such that:

\[
\exists c, k, l \in IR^+ \text{ with } c > l, \quad k \in ]0, 1[ \text{ and } \rho(c(Tx - Ty)) \leq k\rho(l(x - y)); \forall x, y \in B.
\]

Then \( T \) has a fixed point if one of the following assumptions is satisfied:

i) \( 1 \leq c \)

ii) \( 0 < c < 1 \) and \( \rho \) satisfies the \( \Delta_2 \)-condition.
Proof.
It results from the theorem 1.1.'s proof that \( \rho(c(T^{n+m}x - T^nx)) \to 0 \) as \( n, m \to +\infty \). \( X_\rho \) is \( \rho \)-complete, hence, \( \exists z \in X_\rho \) such that \( \rho(cT^n x - z) \to 0 \) as \( n \to +\infty \). Then \( \frac{z}{c} \in B \). Indeed, \( \rho(cT^n x - z) = \rho(c(T^n x - \frac{z}{c})) \).

If \( 1 \leq c \), then \( \rho(T^n x - \frac{z}{c}) \leq \rho(c(T^n x - \frac{z}{c})) \to 0 \) as \( n \to +\infty \), and \( \frac{z}{c} \in B \). If \( 0 < c < 1 \) and \( \rho \) satisfies the \( \Delta_2 \)-condition, then \( \rho(c(T^n x - \frac{z}{c})) \to 0 \) as \( n \to +\infty \Rightarrow \rho(T^n x - \frac{z}{c}) \to 0 \) as \( n \to +\infty \), and \( \frac{z}{c} \in B \).

We prove that \( \frac{z}{c} \) is a fixed point of \( T \). Indeed,

\[
\rho\left(\frac{c}{2}(T\frac{z}{c} - \frac{z}{c})\right) = \rho\left(\frac{c}{2}(T\frac{z}{c} - T^{n+1}x) + \frac{c}{2}(T^{n+1}x - \frac{z}{c})\right) \\
\leq \rho(c(T\frac{z}{c} - T^{n+1}x)) + \rho(cT^{n+1}x - z) \\
\leq k\rho\left(\frac{1}{c}(z - cT^nx)\right) + \rho(c(T^{n+1}x - z)) \\
\leq k\rho(z - cT^nx) + \rho(cT^{n+1}x - z)
\]

Since \( k\rho(z - cT^nx) + \rho(cT^{n+1}x - z) \to 0 \) as \( n \to +\infty \), then \( \rho\left(\frac{c}{2}(T\frac{z}{c} - \frac{z}{c})\right) = 0 \) and \( T\frac{z}{c} = \frac{z}{c} \).

Remark 4
If \( B \) is a subspace of the \( X_\rho \) in theorem 1.2, then the constraints on \( c \) (\( 1 \leq c \) or \( 0 < c < 1 \) and \( \rho \) satisfies the \( \Delta_2 \)-condition ) are useless.

3 II-Strict \( \rho \)-contraction mappings

Let us note that if \( c = l \) or \( c = l = 1 \) , the adopted method in the theorem 1.1.'s proof is not valid. The following result can be considered as the second approach of the Banach’s fixed point theorem in the normed spaces.

Theorem 3.1 2.1.Let \( X_\rho \) be a \( \rho \)-complete modular space where \( \rho \) satisfies the \( \Delta_2 \)-condition. \( B \subseteq X_\rho \) a \( \rho \)-closed subset of \( X_\rho \). Let \( T : B \to B \) be a strict \( \rho \)-contraction mapping, i.e., \( \exists c, k \in \mathbb{R}^+ \) with \( k \in ]0,1[ \) and \( \rho(c(Tx - Ty)) \leq k\rho(c(x - y)) \), \( \forall x, y \in B \).

We suppose that \( \rho(c(x - y)) < +\infty \), \( \forall x, y \in B \). Then \( T \) has a unique fixed point.

Remarks.
This result, As the theorem 1.1, has permitted to delete the boundedness conditions concerning the domain of \( T \) in [2]-[3]-[5].
But the \( \Delta_2 \)-condition, in the following proof, is essential, that is , the iterative techniques are to happen locally.

Proof of the theorem 2.1

1st step
If \( \rho \) satisfies the \( \Delta_2 \)-condition, then, \( \exists \delta, \ L, \ M \in \mathbb{R}^+ \) such that:
\[
\rho(x) \leq \delta \Rightarrow \rho(2x) \leq L\rho(x) + M \quad (\Delta_2)
\]
Otherwise, for \( \delta = \frac{1}{n} \) and \( L = M = 1 \), we have \( \rho(x_n) \to 0 \) as \( n \to +\infty \) and \( \rho(2x_n) > \rho(x_n) + 1 \geq 1 \). Absurd.

2\textsuperscript{nd} step

\( \exists x_0 \in B \) such that \( r = \rho(2c(Tx_0 - x_0)) \) is arbitrary small, because:
\[
\rho(c(Tx - x)) < \infty, \rho(c(T^{n+1}x - T^nx)) \leq k^n\rho(c(Tx - x)) \to 0 \text{ as } n \to +\infty,
\]
and, by the \( \Delta_2 \)-condition, \( \rho(2c(T^{n+1} - T^nx)) \to 0 \) as \( n \to +\infty \).

We suppose that the constant \( k \) is such that:
\[
0 < k \leq \frac{\delta}{M + r + L\delta} \quad (1)
\]
Let us note that \( (1) \Rightarrow Lk < 1 \). We prove that \( \{T^nx_0\}_{n \in \mathbb{N}} \) is \( \rho \)-Cauchy in \( X_\rho \). Indeed, by induction we have:
\[
\rho(c(T^{n+m}x_0 - T^mx_0)) \leq k^n\rho(c(T^nx_0 - x_0))
\]
We show that
\[
\rho(c(T^nx_0 - x_0)) \leq \frac{1-(Lk)^n}{1-Lk}(M + r) \quad (2)
\]
Indeed, for \( n = 1 \), \( \rho(c(Tx_0 - x_0)) \leq r \leq M + r \). We suppose that \( (2) \) is satisfied. Then
\[
\rho(c(T^{n+1}x_0 - x_0)) \leq \rho(2c(T^{n+1}x_0 - Tx_0)) + r
\]
Or
\[
\rho(c(T^{n+1}x_0 - Tx_0)) \leq kp(c(T^nx_0 - x_0)) \\
\leq k \frac{1-(Lk)^n}{1-Lk}(M + r) \\
\leq \frac{k}{1-Lk}(M + r)
\]
By \( (1) \), \( \rho(c(T^{n+1}x_0 - Tx_0)) \leq \delta \). Therefore
\[
\rho(c(T^{n+1}x_0 - x_0)) \leq Lk \frac{1-(Lk)^n}{1-Lk}(M + r) + M + r \\
\leq \frac{1-(Lk)^n}{1-Lk}(M + r)
\]
Finally, we have \( \rho(c(T^{n+m}x_0 - T^mx_0)) \leq k^m\frac{1-(Lk)^n}{1-Lk}(M + r)) \to 0 \) as \( n, m \to +\infty \).

As \( X_\rho \) is \( \rho \)-complete and \( B \) is \( \rho \)-closed, \( \exists z \in B \) such that \( \rho(c(T^nx_0 - z)) \to 0 \) as \( n \to +\infty \). Then \( z \) is a fixed point of \( T \). Indeed,
\[
\rho(\frac{cTz - z}{2}) = \rho(\frac{c}{2}(Tz - T^{n+1}x_0 + T^{n+1}x_0 - z)) \\
\leq kp(c(z - T^nx_0)) + \rho(c(T^{n+1}x_0 - z))
\]
So \( kp(c(z - T^nx_0)) + \rho(c(T^{n+1}x_0 - z)) \to 0 \) as \( n \to +\infty \). Hence, \( \rho(\frac{cTz - z}{2}) = 0 \) and \( Tz = z \).

Since \( \rho(c(x - y)) < \infty \), \( \forall x, y \in B \), then \( z \) is a unique

3\textsuperscript{rd} step

As \( k^n \to 0 \) as \( n \to +\infty \), then, \( \exists p_0 \in \mathbb{N} \) such that
\[
k^{p_0} \leq \frac{\delta}{M + r + L\delta}
\]
We take $S = T^{p_0}$ and $k_0 = k^{p_0}$ we have:

$$\rho(c(Sx - Sy)) \leq k_0 \rho(c(x - y)), \forall x, y \in B$$

By the same approaches as in the $2^{nd}$ step, we verify that $S$ has a unique fixed point $z$. Therefore $z$ is also a unique fixed point of $T$.

**Remark**

Let $(\Omega, \Sigma, \mu)$ be a measure space. $(L^\varphi, \rho)$ is the Musielak-Orlicz space where $\mu$ is $\sigma$-finite and atomless, and $\varphi$ is locally integrable. If $\rho$ satisfies the $\Delta_2$-condition, then, by ([6], theorem 8.14), $\exists L, M \in \mathbb{R}^+$ such that $\rho(2x) \leq L\rho(x) + M, \forall x \in L\varphi$.

In this case, the constant $\delta$, in the above proof, is arbitrary; hence, this proof is valid with the constraint: $\exists p_0 \in \mathbb{N}$ such that $k^{p_0}L < 1$

## 4 III-$\rho$-nonexpansive mappings

In this paragraph, we consider the modular space $X_\rho$ equipped with the convergence:

$x_n \xrightarrow{\rho} x \iff \rho(x_n - x) \to 0$ as $n \to +\infty$

**Definition 4.1.** The modular $\rho$ satisfies the regular growth condition if $W_\rho(t) < 1$ for all $t \in [0, 1[, \text{ where } W_\rho(t) = \sup \{ \frac{\rho(tx)}{\rho(x)} \mid x \in X_\rho, \ 0 < \rho(x) < \infty \}$.

All $s$-convex function modulars satisfy the regular growth condition. For other examples see [2].

The set $B$ is said to be star-shaped if there exists $z \in B$ such that $\alpha z + \beta x \in B$, $\forall x \in B$, whenever $\alpha, \beta \in \mathbb{IR}$ with $\alpha + \beta = 1$. Such a point $z$ is called a center of $B$.

A subset $B$ of $X_\rho$ is said to be $\rho$-bounded in the sense of topological vector spaces ($\tau_\rho$-bounded) if: For every sequence $\{x_n\} \subset B$ and any sequence of numbers $\epsilon_n \to 0$, there holds $\rho(\epsilon_n x_n) \to 0$ as $n \to +\infty$.

The following result can be considered as of the Schauder’s type.

**Theorem 4.1** 3.1. Let $X_\rho$ be a $\rho$-complete modular space where $\rho$ satisfies the regular growth condition. $B \subseteq X_\rho$ a $\rho$-closed and star-shaped subset of $X_\rho$.

$T : B \to B$ be a $\rho$-nonexpansive mapping, i.e., $\rho(Tx - Ty) \leq \rho(x - y), \forall x, y \in B$.

If $T(B)^\rho$ is $\rho$-compact, then $T$ has a fixed point.

**Remark 3.1**

This result is presented under supplementary conditions in [2]-[3], where $X_\rho = L_\rho$; in particular, $B$ is $\rho$-bounded ($\delta_\rho(B) = \sup_{x,y \in B} \rho(x - y) < \infty$) and $\rho$-compact. We note that if $B$ is $\rho$-compact, then $B$ is $\rho$-closed and $T(B)$ is $\rho$-compact.

**Proof of the theorem 3.1**

$1^{st}$ step

**Lemma 4.1** 3.1. $X_\rho$ and $B$ are as in theorem 3.1. $T : B \to B$ is $\rho$-nonexpansive. Then

a) The equation $x = \alpha z + \beta Tx$ ($z$ a center of $B$, $x \in B$, $(\alpha, \beta) \in \mathbb{IR}^+ \times \mathbb{IR}^+$ with $\alpha + \beta = 1$) has one solution.

b) If moreover, $TB$ is $\tau_\rho$-bounded, then $T$ has an approximating fixed point.
Proof
a) For \( \beta \in [0, 1[ \), let \( \lambda \in ]1, \frac{1}{\beta}[ \). We consider \( Sx = \alpha z + \beta Tx \). Then \( S : B \to B \) and

\[
\rho(\lambda(Sx - Sy)) = \rho(\lambda\beta(Tx - Ty)) \\
\leq W_\rho(\lambda\beta)\rho(Tx - Ty) \\
\leq W_\rho(\lambda\beta)\rho(x - y)
\]

By the theorem 1.2, \( S \) has a fixed point

b) Let \( k_n \in ]0, 1[ \) with \( k_n \not\to 1 \). By a), we have \( x_n = (1 - k_n)z + k_nTx_n \). Hence

\[
\rho(Tx_n - x_n) \leq \rho(2(1 - k_n)Tx_n) + \rho(2(1 - k_n)z)
\]

By the definition of \( X_\rho \), \( \rho(2(1 - k_n)z) \to 0 \) as \( n \to +\infty \). As \( TB \) is \( \tau_\rho \)-bounded, then,

\[
\rho(2(1 - k_n)Tx_n) \to 0 \text{ as } n \to +\infty.
\]

Therefore \( \rho(Tx_n - x_n) \to 0 \) as \( n \to +\infty \) i.e. \( T \) has an approximating fixed point.

2\textsuperscript{nd} step

As \( \overline{T(B)}^\rho \) is \( \rho \)-compact, then \( \overline{T(B)}^\rho \) is \( \tau_\rho \)-bounded. [jm]. Hence, there exists \( x_n \in B \) such that \( \rho(Tx_n - x_n) \to 0 \) as \( n \to +\infty \). Also, there exists \( \{T_{x_{n'}}\} \) a subsequence of \( \{T_{x_n}\} \) that is \( \rho \)-convergent to \( y \in B \). We prove that \( y \) is a fixed point of \( T \). Indeed, we have:

\[
\rho\left(\frac{Ty - y}{3}\right) = \rho\left(\frac{(Ty - T^2x_{n'}) + (T^2x_{n'} - Tx_{n'}) + (Tx_{n'} - y)}{3}\right) \leq 2\rho(Tx_{n'} - y) + \rho(Tx_{n'} - x_{n'})
\]

Since \( 2\rho(Tx_{n'} - y) + \rho(Tx_{n'} - x_{n'}) \to 0 \) as \( n' \to +\infty \). Therefore \( \rho\left(\frac{Ty - y}{3}\right) = 0 \) and \( Ty = y \).

Remark 3.3

Recall that \( B \) is \( \rho \)-bounded is, in general, not equivalent to \( B \) is \( \tau_\rho \)-bounded, see [5].

Finally, we present one result, for the strict \( \rho \)-contraction mappings, using the lemma 3.2.

**Proposition 4.1** 3.1. Let \( X_\rho \) be a \( \rho \)-complete modular space where \( \rho \) is convex subset of \( X_\rho \) with \( 0 \in B \).

Let \( T : B \to B \) be a mapping such that \( \exists k \in ]0, 1[ \) with \( \rho(Tx - Ty) \leq k\rho(x - y) \); \( \forall x, y \in B \).

Let \( A = \{x \in B : x = \lambda Tx, \ \lambda \in ]0, 1[\} \). If \( \sup_{x \in A} \rho(x) < \infty \), then \( T \) has a fixed point.

Proof.

By the lemma 3.2, \( A \neq \emptyset \). Let \( \lambda_n \in ]0, 1[ \), with \( \lambda_n \not\to 1 \), and \( x_n = \lambda_nTx_n \). We show that \( \{x_n\} \) is \( \rho \)-Cauchy. Indeed, for \( m > n \), we have

\[
\rho(x_m - x_n) = \rho(\lambda_mTx_m - \lambda_nTx_n) \\
= \rho(\lambda_n(Tx_m - Tx_n) + (\lambda_m - \lambda_n)Tx_m) \\
= \rho(\lambda_n(Tx_m - Tx_n) + \frac{\lambda_m - \lambda_n}{\lambda_m}x_m)
\]

As \( \lambda_n + \frac{\lambda_m - \lambda_n}{\lambda_m} \leq 1 \), then

\[
\rho(x_m - x_n) \leq \lambda_nk\rho(x_m - x_n) + \frac{\lambda_m - \lambda_n}{\lambda_m}\rho(x_m)
\]
So $\rho(x_m - x_n) \leq \frac{\lambda_m - \lambda_n}{\lambda_m(1-k)} \sup_m \rho(x_m) \to 0$ as $m, n \to +\infty$.

$X_\rho$ is a $\rho$-complete space and $B$ is a $\rho$-closed, hence, $\exists x \in B$ such that $\rho(x_n - x) \to 0$ as $n \to +\infty$.

We show that $x$ is a fixed point of $T$. Indeed, we have

$$\rho\left(\frac{T x - x}{3}\right) = \rho\left(\frac{T x - Tx_n + Tx_n - x_n + x_n - x}{3}\right) \leq (k + 1)\rho(x - x_n) + \rho(T x_n - x_n)$$

Or $T x_n - x_n = x_n\left(1 - \frac{1}{\lambda_n}\right)$ and for $n$ very large, we have $0 < 1 - \frac{1}{\lambda_n} < 1$; hence

$\rho(T x_n - x_n) \leq (\frac{1}{\lambda_n} - 1) \sup_n \rho(x_n) \to 0$ as $n \to +\infty$. Therefore $\rho\left(\frac{T x - x}{3}\right) = 0$ and $T x = x$.

**References**

[1] Ait Taleb, A. - Hanebaly, E. A fixed point theorem and its application to integral equations in modular function spaces. Proc. Amer. Math. Soc. 127, no 8, 2335-2342 (1999) 128, no. 2, 419-426 (2000).

[2] Khamsi, M. A.-Kozlowski, W. M.-Reich, S. Fixed point theory in modular function spaces. Nonlinear Analysis, theory, methods and applications, Vol. 14, N° 11 (1990). 935-953.

[3] Khamsi, M. A. A convexity property in Modular Function Spaces. These d’etat. Departement de Mathematique, Rabat (1994).

[4] Hajji, A.-Hanebaly, E.1) Perturbed integral equations in modular function spaces. E.J. Qualitative Theory of Diff.Equ, No.20. (2003), pp. 1-7.2) A fixed point theorem and its application to perturbed integral equations in modular function spaces. EJDE.(2005) no 105 pp 1-11

[5] Lami Dozo, E-P.Turpin. Nonexpansive maps in generalized Orlicz spaces. Studia Math. 86,155-188 (1987).

[6] Musielak, J. Orlicz spaces and modular spaces. Lecture notes in mathematics, vol. 1034, S.V (1983)

Adress:Boulevard Mohammed Elyazidi.S12.C6.Hay Riad Rabat (Morocco)

E-mail: hanebaly@hotmail.com .