STARLIKE FUNCTIONS ASSOCIATED WITH A NON-MA-MINDA FUNCTION

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ABSTRACT. In this work, we introduce a newly defined class of analytic functions associated with elliptical and strip domains, given by

$$\mathcal{F}[A, B] := \left\{ f \in \mathcal{A} : \left( \frac{zf'(z)}{f(z)} - 1 \right) \prec \psi_{A, B}(z) := \frac{1}{A - B} \log \frac{1 + Az}{1 + Bz} \right\},$$

where either $A = -B = \alpha$ or $A = \alpha e^{i\gamma}$ and $B = \alpha e^{-i\gamma}$ ($\alpha \in (0, 1]$ and $\gamma \in (0, \pi/2)$). Here, we study various characteristic properties of $\psi_{A, B}(z)$ as well as functions in $\mathcal{F}[A, B]$. We also obtain the sharp radius of starlikeness of order $\delta$ and univalence for the functions in $\mathcal{F}[A, B]$. Further, we find the sharp radii for functions in $\mathcal{B}(\alpha) := \{ f \in \mathcal{A} : zf'(z)/f(z) - 1 \prec z/(1 - \alpha z^2) \}$, $\mathcal{S}_{\alpha}(\alpha) := \{ f \in \mathcal{A} : zf'(z)/f(z) - 1 \prec z/(1 - \alpha z^2) \}$, and others to be in the class $\mathcal{F}[A, B]$. We also obtain $\mathcal{S}^\ast(\Phi((z))$ as well as functions in $\mathcal{F}[A, B]$. We also obtain

$$\mathcal{S}^\ast(\Phi) := \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec \Phi(z) \right\},$$

(1.1)

where $\Phi$ is univalent with $\Phi'(0) > 0$ satisfying

A. $\Phi(z) \prec (1 + z)/(1 - z)$
B. $\Phi(\mathbb{D})$ is symmetric about real axis and starlike with respect to $\Phi(0) = 1$.

Further, Kumar and Banga [4], called such $\Phi$ as Ma-Minda function and the class of all Ma-Minda functions is denoted by $\mathcal{M}$. In [3], authors extensively studied Ma-Minda functions and classified them as non-Ma-Minda function of type-A whenever condition A doesn’t hold, the class of all such functions is denoted by $\tilde{\mathcal{M}}_A$. In past many authors studied the class $\mathcal{F}[1, 1]$ for various choices of $\Phi \in \mathcal{M}$, such as $\mathcal{S}^\ast(2/(1 + e^{-z})) =: \mathcal{S}_{SG}^\ast[11]$, $\mathcal{S}^\ast(e^z) =: \mathcal{S}_{p}^\ast[9]$, $\mathcal{S}^\ast(z + \sqrt{1 + z^2})$ [10], $\mathcal{S}^\ast(\sqrt{1 + z}) =: \mathcal{S}_{L}^\ast[12]$, etc. Also the well known classes $\mathcal{S}^\ast(A, B)$ and $\mathcal{S}^\ast(\alpha)$ are also obtained by taking $\Phi(z) \in \mathcal{M}$ as $(1 + Az)/(1 + Bz)$ ($-1 \leq B < A \leq 1$ and $(1 + z)/(1 - z)^{\alpha}$ ($\alpha \in (0, 1]$) respectively. In particular, $\mathcal{S}^\ast(1, -1) =: \mathcal{S}_s^\ast$, $\mathcal{S}^\ast(1 - 2\delta, -1) =: \mathcal{S}^\ast(\delta)$ ($\delta \in [0, 1]$), the class of starlike functions of order $\delta$. Robertson [11] introduced and investigated the class of starlike functions of order $\delta$, denoted by $\mathcal{S}_s^\ast$ for $\delta \leq 1$. Note that if $\delta < 0$, then the functions in $\mathcal{S}_s^\ast$ may not be univalent, i.e. if $\delta < 0$, then $\Phi(z) = (1 + (1 - 2\delta)z)/(1 - z) \in \tilde{\mathcal{M}}_A$ and $\mathcal{S}_s^\ast \not\in \mathcal{S}^\ast$. Later

2010 Mathematics Subject Classification. 30C45, 30C50, 30C80.

Key words and phrases. Analytic, Subordination, Logarithmic function, Starlike function, Non-univalent function.
many authors considered the classes associated with non-Ma-Minda functions of type-A such as the class by Uralegaddi [14]

\[ \mathcal{M}(\beta) = \left\{ f \in A : \frac{zf'(z)}{f(z)} < \frac{1 + (2\beta - 1)z}{1 - z}, \beta > 1 \right\}, \]

the classes associated with the Booth Lemniscate and Cissoid of Diocles

\[ BS^*(\alpha) := \left\{ f \in A : \left( \frac{zf'(z)}{f(z)} - 1 \right) < \frac{z}{1 - \alpha z^2}, \alpha \in [0, 1) \right\}, \]

\[ S_{cs}^*(\alpha) := \left\{ f \in A : \left( \frac{zf'(z)}{f(z)} - 1 \right) < \frac{z}{(1 - z)(1 + \alpha z)}, \alpha \in [0, 1) \right\}, \]

which were introduced by [2] and [7] respectively. Similarly, authors in [8] studied the class

\[ ST_L^*(s) := \{ f \in A : zf'(z)/f(z) \prec L_A(z) \} \]

associated with the Limaçon of Pascal \( L_A(z) = (1 + sz)^2, s \in [-1, 1]\backslash\{0\} \). Note that \( L_A(z) \in \mathcal{M}_A \), whenever \(|s| > 1/\sqrt{2} \). Recently, Kumar and Gangania [5] studied the following class:

\[ F(\psi) := \left\{ f \in A : \frac{zf'(z)}{f(z)} - 1 \prec \psi(z) \right\}, \tag{1.2} \]

where \( \psi(z) \in S^* \). If \( 1 + \psi(z) \prec (1 + z)/(1 - z) \) and \( 1 + \psi(\mathbb{D}) \) is symmetric about the real axis with \( \psi'(0) > 0 \) then \( F(\psi) \) reduces to the class \( S^*(1 + \psi) \), a Ma-Minda class and if \( 1 + \psi(z) \prec (1 + z)/(1 - z) \) then the functions in \( F(\psi) \) may not be univalent, thus \( F(\psi) \not\subseteq S^* \). Note that all above mentioned classes associated with non-Ma-Minda functions of type-A are the special cases of \( F(\psi) \) such as

\[ F((1 + (2\beta - 1)z)/(1 - z)) =: \mathcal{M}(\beta), F(z/(1 - \alpha z^2)) =: BS^*(\alpha), F(z/((1 - z)(1 + \alpha z))) =: S_{cs}^*(\alpha) \]

and \( F((1 + sz^2)) =: L_A(z) \).

Motivated by the above, we define a family of functions \( \psi_{A,B}(z) \) as

\[ \psi_{A,B}(z) := \frac{1}{A - B} \log \left( \frac{1 + Az}{1 + Bz} \right), \tag{1.3} \]

where either \( A = -B = \alpha \) or \( A = \alpha e^{i\gamma} \) and \( B = \alpha e^{-i\gamma} \), for \( 0 < \alpha \leq 1 \) and \( 0 < \gamma \leq \pi/2 \). The function defined by [13] is analytic everywhere except the branchcuts \( \{-\infty < \text{Re} Az \leq -1, \text{Im} Az = 0\} \cup \{-\infty < \text{Re} Bz \leq -1, \text{Im} Bz = 0\} \) with \( \psi_{A,B}(0) = \psi'_{A,B}(0) - 1 = 0 \), i.e. \( \psi_{A,B}(z) \in \mathcal{A} \). Now we introduce the following classes associated with \( \psi_{A,B}(z) \):

**Definition 1.1.** Let \( p \in S \). Then \( p \in \mathcal{L}[A,B] \) if and only if

\[ p(z) \prec \psi_{A,B}(z). \]

**Definition 1.2.** Let \( p \in A \). Then \( p \in \mathcal{F}[A,B] \) if and only if

\[ \frac{zf'(z)}{f(z)} - 1 \prec \psi_{A,B}(z). \tag{1.4} \]

This paper aims to investigate various characteristic properties of functions in the classes \( \mathcal{L}[A,B] \) and \( \mathcal{F}[A,B] \). We also obtain the sharp radius of starlikeness of order \( \delta \) and univalence for the functions in \( \mathcal{F}[A,B] \). Further, we find the sharp radii for functions in \( BS^*(\alpha), S_{cs}^*(\alpha) \), etc to be in the class \( \mathcal{F}[A,B] \).

2. **Characteristics of \( \mathcal{L}[A,B] \) and \( \mathcal{F}[A,B] \)**

We now proceed to find various geometric properties of \( \psi_{A,B}(z) \) and obtain certain bounds and inclusion relations for functions in the class \( \mathcal{L}[A,B] \). Further we deduce the extremal function and
derive the growth and covering theorems for the class $F[A, B]$. Note that
\[
\psi_{A,B}(z) = \sum_{n=1}^{\infty} (-1)^{n+1} C_n z^n,
\]
where
\[
C_n = \frac{A^n - B^n}{n(A - B)} = \frac{1}{n} \sum_{k=0}^{n-1} A^k B^{n-1-k} \quad (n = 1, 2, \ldots).
\]

Remark 2.1. From (2.1) we have

(i) Since $C_n$ is real valued for all $n$, $\psi_{A,B}(D)$ is symmetric with respect to real axis.

(ii) $|C_n| = \left|\sum_{k=0}^{n-1} A^k B^{n-1-k} \right| \leq n^{-1} \leq 1$.

**Theorem 2.2.** The function $\psi_{A,B}(z)$ is convex and univalent on $D$.

**Proof.** Let $H(z) := 1 + z\psi'_{A,B}(z)/\psi_{A,B}(z)$. By [12, Corollary 3], it is enough to show that $\text{Re} \, H(z) > 0$. Since $|A| = |B| = \alpha \leq 1$, using basic calculation with [4, Theorem 1], we obtain
\[
\text{Re} \, H(z) = \text{Re} \left( -1 + \frac{1}{1 + Az} + \frac{1}{1 + Bz} \right) > -1 + \frac{2}{1 - \alpha} > 0.
\]
Thus the result holds.

**Theorem 2.3.** Let $p \in L[A, B]$, Then in the disk $D_r = \{ z \in \mathbb{C} : |z| \leq r < 1 \}$,

(i) For $A = -B = \alpha$, we have
\[
|\text{Re} \, p(z)| \leq \frac{1}{2\alpha} \log \left( \frac{1 + \alpha r}{1 - \alpha r} \right)
\]
\[
|\text{Im} \, p(z)| \leq \frac{1}{2\alpha} \sin^{-1} \left( \frac{2\alpha r}{1 + \alpha^2 r^2} \right).
\]

(ii) For $A = \alpha e^{i\gamma}$ and $B = \alpha e^{-i\gamma}$, we have
\[
-\frac{1}{2\alpha \sin \gamma} (\eta - \tau) \leq \text{Re} \, p(z) \leq \frac{1}{2\alpha \sin \gamma} (\eta + \tau).
\]
\[
\frac{1}{2\alpha \sin \gamma} \log T_2 \leq \text{Im} \, p(z) \leq \frac{1}{2\alpha \sin \gamma} \log T_1.
\]
where $\eta := \sin^{-1} \left( \frac{2\alpha \sin \gamma}{\sqrt{1 + \alpha^4 r^4 - 2\alpha^2 r^2 \cos 2\gamma}} \right)$, $\tau := \tan^{-1} \left( \frac{-\alpha^2 r^2 \sin 2\gamma}{1 - \alpha^2 r^2 \cos 2\gamma} \right)$ and $T_j := \left( \frac{\sqrt{1 + \alpha^4 r^4 - 2\alpha^2 r^2 \cos 2\gamma} + (-1)^j 2\alpha \sin \gamma}{1 - \alpha^2 r^2} \right)^{-1} (j = 1, 2)$.

**Proof.** Let $w$ be the Schwarz function such that $w(0) = 0$ and $|w(z)| \leq |z| = r < 1$ for all $z \in D$. Define the function $F_w : D \to \mathbb{C}$ such that $p(z) = \psi_{A,B}(w(z)) = \frac{1}{A - B} \log (F_w(z))$. We have
\[
F_w(z) := \frac{1 + Aw(z)}{1 + Bw(z)},
\]
which maps unit disk into the disk
\[ F_w(D) \subset \left\{ \zeta \in \mathbb{C}; \left| \zeta - \frac{1 - AB}{1 - |B|^2} \right| < \frac{|A - B|}{1 - |B|^2} \right\}. \]

As \(|A| = \alpha \in (0, 1]|, therefore origin \(O \notin F_w(D)\) (for more details see [4]). Now using [4, Theorem 1] with \(|z| = r < 1\), we have
\[
\frac{|1 - AB|^2 - |A - B|^2}{1 - |B|^2 \alpha r^2} \leq |F_w(z)| \leq \frac{|1 - AB|^2 + |A - B|^2}{1 - |B|^2 \alpha r^2},
\]
\[
\left| \arg F_w(z) - \tan^{-1} \frac{\text{Im}(AB) r^2}{\text{Re}(AB) r^2 - 1} \right| < \sin^{-1} \frac{|A - B|^2}{1 - |1 - AB|^2}.
\]

Since \(\log f(z) = \log|f(z)| + i \arg f(z)\) and \(p(z) = \frac{1}{A - B} \log(F_w(z))\), thus for \(A = -B = \alpha\) we obtain (2.2) and (2.3) and for \(A = \alpha e^{i\gamma}\) and \(B = \alpha e^{-i\gamma}\), we have (2.4) and (2.5).

For the sake of computational convenience, we shall make the following assumptions:
\[
h_1 := \frac{1}{2\alpha} \log \left( \frac{1 + \alpha}{1 - \alpha} \right), \quad h_2 := \frac{1}{2\alpha} \sin^{-1} \left( \frac{2\alpha}{1 + \alpha^2} \right),
\]
\[
k_1 := \frac{1}{2\alpha \sin \gamma} \sin^{-1} \left( \frac{2\alpha \sin \gamma}{\sqrt{1 + \alpha^4 - 2\alpha^2 \cos 2\gamma}} \right), \quad k_2 := \frac{1}{2\alpha \sin \gamma} \log \left( \frac{\sqrt{1 + \alpha^4 - 2\alpha^2 \cos 2\gamma} + 2\alpha \sin \gamma}{1 - \alpha^2} \right).
\]

Now we shall discuss the image domain of \(\psi_{A,B}\) for different cases of \(\alpha\):
For \(0 < \alpha < 1\), \(\psi_{A,B}(D) = \{u + iv \in \mathbb{C} : (u, v) \in \Omega_1\}\) where
\[
\Omega_1 := \left\{ \begin{array}{ll}
\frac{u^2}{h_1^2} + \frac{v^2}{k_1^2} < 1, & \text{when } A = -B = \alpha \\
\frac{(u - k)^2}{h_2^2} + \frac{v^2}{k_2^2} < 1, & \text{when } A = \alpha e^{i\gamma}, B = \alpha e^{-i\gamma} \text{ for } 0 < \gamma \leq \pi/2
\end{array} \right\}. \tag{2.8}
\]

Moreover when \(\alpha = 1\), the major axis \(h_1\) and \(k_2\) respectively, tends to infinity, hence \(\psi_{A,B}(D) = \{u + iv \in \mathbb{C} : (u, v) \in \Omega_2\}\) where \(\Omega_2\) is the following strip domain
\[
\Omega_2 := \left\{ \begin{array}{ll}
-\frac{\pi}{4} < v < \frac{\pi}{4}, & \text{when } A = -B = 1 \\
\frac{\gamma - \pi}{2 \sin \gamma} < u < \frac{\gamma}{2 \sin \gamma}, & \text{when } A = e^{i\gamma}, B = e^{-i\gamma} \text{ for } 0 < \gamma \leq \pi/2
\end{array} \right\}. \tag{2.9}
\]

Remark 2.4. 1. \(\max_{|z|=1} \text{Re } \psi_{A,B}(z) = \psi_{A,B}(r)\) and \(\min_{|z|=1} \text{Re } \psi_{A,B}(z) = \psi_{A,B}(-r)\).
2. If \(\alpha < 1\), then \(p < \psi_{A,B}(z)\) if and only if \(p(D) \subseteq \Omega_1\) and if \(\alpha = 1\), then \(p < \psi_{A,B}(z)\) if and only if \(p(D) \subseteq \Omega_2\), where \(\Omega_1\) and \(\Omega_2\) are as defined in (2.8) and (2.9) respectively.
3. \(1 + \psi_{A,B} \in \mathcal{M}_A\).

Let \(w = 1 + \psi_{A,B}(z)\), then we have
\[
|w - 1| = |\psi_{A,B}(z)| = \frac{1}{|A - B|} \log \left( \frac{1 + Az}{1 + Bz} \right).
\]

It can be easily seen that for \(\alpha \in (0, 1), A = -B = \alpha\)
\[
\min_{|z|=1} |\psi_{A,B}(z)| = h_2 \quad \text{and} \quad \max_{|z|=1} |\psi_{A,B}(z)| = h_1.
\]
and for $A = \alpha e^{i\gamma}$ and $B = \alpha e^{-i\gamma}$
\[
\min_{|z|=1} |\psi_{A,B}(z)| = k_1 + k \quad \text{and} \quad \max_{|z|=1} |\psi_{A,B}(z)| = k_2.
\]

**Remark 2.5.** For $\alpha = 1$, $h_1$ and $k_2$ tends to $\infty$, $h_2 = \pi/4$ and $k_1 + k = \gamma/(2 \sin \gamma)$.

Thus we have the following relation:

**Lemma 2.6.** Let $\alpha \in (0, 1)$ and $\gamma \in (0, \pi/2]$, then we have

(i) for $A = -B = \alpha$,
\[
\{ w \in \mathbb{C} : |w - 1| < h_2 \} \subset 1 + \psi_{A,B}(\mathbb{D}) \subset \{ w \in \mathbb{C} : |w - 1| < h_1 \}.
\]

(ii) for $A = \alpha e^{i\gamma}$ and $B = \alpha e^{-i\gamma}$,
\[
\{ w \in \mathbb{C} : |w - 1| < k_1 + k \} \subset 1 + \psi_{A,B}(\mathbb{D}) \subset \{ w \in \mathbb{C} : |w - 1| < k_2 \}.
\]

Further for $\alpha = 1$, we have

(iii) for $A = -B = 1$,
\[
\{ w \in \mathbb{C} : |w - 1| < \frac{\pi}{4} \} \subset 1 + \psi_{A,B}(\mathbb{D}).
\]

(iv) for $A = e^{i\gamma}$ and $B = e^{-i\gamma}$,
\[
\{ w \in \mathbb{C} : |w - 1| < \frac{\gamma}{2 \sin \gamma} \} \subset 1 + \psi_{A,B}(\mathbb{D}).
\]

In general, if we consider a disc $\mathcal{D}(a, r) := \{ w \in \mathbb{C} : |w - a| < r \}$ where $a \in \mathbb{R}$ and $1 \in \mathcal{D}(a, r)$, then we observe that $\mathcal{D}(a, r) \subset 1 + \psi_{A,B}(\mathbb{D})$ if and only if
\[
\mathcal{D}(a, r) \subset \begin{cases} 
\mathcal{D}(h_2, 1), & \text{for } A = -B = \alpha, \\
\mathcal{D}(k_1 + k, 1), & \text{for } A = \alpha e^{i\gamma}, B = \alpha e^{-i\gamma},
\end{cases} \tag{2.10}
\]
i.e. if and only if
\[
|a - 1| \leq \begin{cases} 
h_2 - r, & \text{for } A = -B = \alpha, \\
k_1 + k - r, & \text{for } A = \alpha e^{i\gamma}, B = \alpha e^{-i\gamma}. \tag{2.11}
\end{cases}
\]

Thus we have the following theorem:

**Theorem 2.7.** Let $-1 < D < C \leq 1$ and $p(z) := (1 + Cz)/(1 + Dz)$. Then $p(z) < \mathcal{L}[A, B]$ if and only if

(i) for $A = -B = \alpha$
\[
C \leq \begin{cases} 
h_2 + (1 - h_2)D, & \text{when } (1 - CD)/(1 - D^2) \leq 1, \\
h_2 + (1 + h_2)D, & \text{when } (1 - CD)/(1 - D^2) \geq 1. \tag{2.12}
\end{cases}
\]

(ii) for $A = \alpha e^{i\gamma}$ and $B = \alpha e^{-i\gamma}$,
\[
C \leq \begin{cases} 
k_1 + k + (1 - k_1 - k)D, & \text{when } (1 - CD)/(1 - D^2) \leq 1, \\
k_1 + k + (1 + k_1 + k)D, & \text{when } (1 - CD)/(1 - D^2) \geq 1. \tag{2.13}
\end{cases}
\]

**Proof.** Since $p(z) = (1 + Cz)/(1 + Dz)$ maps $\mathbb{D}$ onto $\mathcal{D}(a, r)$, where
\[
a = \frac{1 - CD}{1 - D^2} \quad \text{and} \quad r = \frac{C - D}{1 - D^2},
\]
the result follows at once from (2.10) and (2.11).

From Theorem 2.7 we deduce the following corollary, which ensures that $\mathcal{F}[A, B]$ is nonempty.
Corollary 2.8. Let \(-1 < D < C \leq 1\) and \(f \in A\) be such that \(zf'(z)/f(z) = (1 + Cz)/(1 + Dz)\), then \(f \in F[A, B]\) if and only if, the condition (2.12) or (2.13) holds.

In particular, we illustrate below that the class \(F[A, B]\) is non-empty.

Example 1. 1. If \(f(z) = z \exp(Cz)\), then \(f(z) \in F[A, B]\) if and only if
\[
C \leq \begin{cases} h_2, & \text{for } A = -B = \alpha, \\ k_1 + k, & \text{for } A = \alpha e^{i\gamma} \& B = \alpha e^{-i\gamma}. \end{cases}
\]

2. If \(f(z) = z/(1 + Cz)\), then \(f(z) \in F[A, B]\) if and only if
\[
C \geq \begin{cases} -h_2/(1 + h_2), & \text{for } A = -B = \alpha, \\ -(k_1 + k)/(1 + k_1 + k), & \text{for } A = \alpha e^{i\gamma} \& B = \alpha e^{-i\gamma}. \end{cases}
\]

3. If \(f(z) = z/(1 - Cz)^2\), then \(f(z) \in F[A, B]\) if and only if
\[
C \leq \begin{cases} h_2/(2 + h_2), & \text{for } A = -B = \alpha, \\ (k_1 + k)/(2 + k_1 + k), & \text{for } A = \alpha e^{i\gamma} \& B = \alpha e^{-i\gamma}. \end{cases}
\]

Now from (1.4), we have \(f \in F[A, B]\) if and only if there exists \(p(z) \prec \psi_{A,B}(z)\) such that
\[
f(z) = z \exp\left( \int_0^z \frac{p(t)}{t} \, dt \right).
\]

If we take \(p(z) = \psi_{A,B}(z)\), then we obtain from (2.14)
\[
f_{A,B}(z) := z \exp\left( \frac{Li_2(-Bz) - Li_2(-Az)}{A - B} \right),
\]

where \(Li_2(x) = \sum_{n=1}^{\infty} x^n/n^2\), denotes the Spence’s (or dilogarithm) function. The function \(f_{A,B}(z)\)
is nonunivalent but have the extremal properties for many problems for the class \(F[A, B]\) (see Fig[1]). Kumar and Gangania [5] obtained various geometric properties such as the growth and covering theorem for the case when \(1 + \psi(z) \neq (1 + z)/(1 - z)\). As a consequence of the same and Remark 2.4, we obtain the following result:

Theorem 2.9. Let \(f \in F[A, B]\) and \(f_{A,B}\) be defined as in (2.15), then for \(|z| = r\)

(i) Growth Theorem: \(-f_{A,B}(-r) \leq |f(z)| \leq f_{A,B}(r)\).

(ii) Covering Theorem: Either \(f\) is a rotation of \(f_{A,B}\) or \(\{ w \in \mathbb{C} : |w| \leq -f_{A,B}(-1) \} \subset f(\mathbb{D})\).

For \(z = re^{i\theta}\), where \(\theta\) is fixed but arbitrary, as a consequence of growth theorem and \(\psi_{A,B}(-r) \leq \Re \psi_{A,B}(re^{i\theta}) \leq \psi_{A,B}(r)\), we obtain
\[
\log \frac{f(z)}{z} = \int_0^r \frac{p(te^{i\theta})}{t} \, dt,
\]

where \(p(z) := \psi_{A,B}(w(z))\) and \(w\) is a Schwarz function. Further
\[
\frac{f(z)}{z} = \exp \int_0^r \frac{p(te^{i\theta})}{t} \, dt = \exp \left( \int_0^r \Re \frac{p(te^{i\theta})}{t} \, dt + i \int_0^r \Im \frac{p(te^{i\theta})}{t} \, dt \right),
\]
\[
\exp \int_0^r \frac{\psi_{A,B}(-t)}{t} \, dt \leq \left| \frac{f(z)}{z} \right| \leq \exp \int_0^r \frac{\psi_{A,B}(t)}{t} \, dt.
\]
Corollary 2.10. Let \( f \in \mathcal{F}[A, B] \) and \( M(r) = \exp \int_0^r \frac{T_A(t)}{t} \, dt \), then for \( |z| = r \), we have

(i) \( M(-r) \leq \left| \frac{f(z)}{z} \right| \leq M(r) \).

(ii) \( (1 + \max_{|z| \leq r} |\psi_{A,B}(z)|)M(-r) \leq |f'(z)| \leq (1 + \max_{|z| \leq r} |\psi_{A,B}(z)|)M(r) \).

(iii) \( 2\pi r (1 + \max_{|z| \leq r} |\psi_{A,B}(z)|)M(-r) \leq L(f, r) \leq 2\pi r (1 + \max_{|z| \leq r} |\psi_{A,B}(z)|)M(r) \).

(iv) \( f(z)/z < f_{A,B}(z)/z \).

3. Radius Estimates

In this section, we obtain the sharp radius of univalence and radius of starlikeness of order \( \delta \) for functions in \( \mathcal{F}[A, B] \). Further, we find the sharp radii for functions in \( \mathcal{B}S(\alpha) \), \( S_{cs}(\alpha) \) and others to be in the class \( \mathcal{F}[A, B] \). From Theorem 2.3 we conclude that \( f \in \mathcal{S}[A, B] \) is starlike of order \( 1 + \frac{1}{|A - B|} (\varphi - \vartheta) < 0 \), hence \( f \) may not be univalent in \( \mathbb{D} \), thus \( \mathcal{F}[A, B] \not\subseteq \mathbb{S}^* \). Therefore, in the following result, we find the radius of starlikeness of order \( \delta \), where \( \delta \in [0, 1) \) for functions in the class \( \mathcal{F}[A, B] \).

Theorem 3.1. Let \( \alpha \in (0, 1], \gamma \in (0, \gamma_0) \), where \( \gamma_0 \simeq 1.2461 \ldots \) and \( \delta \in [0, 1) \) be given numbers. If \( f \in \mathcal{F}[A, B] \), then \( f \) is starlike of order \( \delta \) in the disc \( |z| < r(\delta) \), where

(i) If \( A = -B = \alpha \),

\[
r(\delta) = \frac{\exp(2\alpha(1 - \delta)) - 1}{\alpha(\exp(2\alpha(1 - \delta)) + 1)}.
\]

(ii) If \( A = \alpha e^{i\gamma} \) and \( B = \alpha e^{-i\gamma} \), \( r(\delta) \) is the smallest positive root of

\[
\tan(2\alpha \sin \gamma (\delta - 1)) - \frac{\alpha^4 r^4 \sin 2\gamma + 2\alpha^3 r^3 \sin \gamma \cos 2\gamma - \alpha^2 r^2 \sin 2\gamma - 2\alpha r \sin \gamma}{\alpha^4 r^4 \cos 2\gamma - 2\alpha^3 r^3 \sin \gamma \sin 2\gamma - \alpha^2 r^2 \cos 2\gamma - \alpha^2 r^2 + 1} = 0.
\]

The result is sharp.

Proof. Let \( f \in \mathcal{F}[A, B] \), then using Theorem 2.3 for \( |z| < r \), we have

Case 1. when \( A = -B = \alpha \),

\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > 1 - \frac{1}{2\alpha} \log \left( \frac{1 + \alpha r}{1 - \alpha r} \right) = h_1(r),
\]
Now as $h_1'(r) < 0$ for all choices of $\alpha$, thus $h_1(r)$ is strictly decreasing function from 1 to $1 - 1/(2\alpha) \log((1 + \alpha)/(1 - \alpha)) < 0$. Hence the root $r(\delta)$, given in (3.1), of the equation $h_1(r) = \delta$, is the radius of starlikeness of order $\delta$ of $F[A, B]$.

**Case 2.** when $A = \alpha e^{i\gamma}$ and $B = \alpha e^{-i\gamma}$

$$\text{Re} \frac{zf'(z)}{f(z)} > 1 - \frac{1}{2\alpha \sin \gamma}(\eta - \tau) =: h_2(r),$$

where $\eta$ and $\tau$ are given in Theorem 2.3. Since $h_2'(r) = \delta$ can be simplified to the equation (3.2) by a simple computation, thus it is enough to find the smallest positive root of (3.2). Further $h_2'(r) = -\frac{1 - \alpha^4 r^4 + 2\alpha r \cos \gamma (1 - \alpha^2 r^2)}{(1 - \alpha^2 r^2)\sqrt{1 + \alpha^4 r^2 - 2\alpha^2 r^2 \cos 2\gamma}} < 0$

for all choices of $\alpha$ and $\gamma$, $h_2(r) = \delta$ is a strictly decreasing function from $1$ to $g(\alpha, \gamma)$, where

$$g(\alpha, \gamma) := 1 + \frac{1}{2\alpha \sin \gamma} \left( \tan^{-1} \left( \frac{\alpha^2 \sin 2\gamma}{\alpha^2 \cos 2\gamma - 1} \right) - \sin^{-1} \left( \frac{2\alpha \sin \gamma}{\sqrt{1 + \alpha^4 - 2\alpha^2 \cos 2\gamma}} \right) \right).$$

Note that $g(\alpha, \gamma)$ is real valued, non-constant analytic function on a bounded domain $R := (0, 1) \times (0, \pi/2)$ and is bounded above on $R$. Thus by Maximum Modulus Theorem, $g(\alpha, \gamma)$ attains its maximum value on the boundary of $R$ and the four boundaries are enlisted below:

1. $g(0, \gamma) = 0$.
2. $g(1, \gamma) = 1 + (\gamma - \pi)/(2 \sin \gamma)$. Now as $g'(1, \gamma) = (1 + (\pi - \gamma) \cot \gamma)/(2 \sin \gamma) > 0$, therefore $g(1, \gamma)$ is strictly increasing from $-\infty$ to $1 - \pi/4 \simeq 0.2146 \ldots$ and $\gamma_0$ is the zero of it.
3. $g(\alpha, 0) = 1 - 1/(1 - \alpha)$. As $g'(\alpha, 0) = \log(1 - \alpha) < 0$, therefore $g(\alpha, 0)$ is strictly decreasing from 0 to $-\infty$.
4. $g(\alpha, \pi/2) = 1 - \left( \sin^{-1}(2\alpha/(1 + \alpha^2)) \right)/(2\alpha)$. As $g'(\alpha, \pi/2) = \left( (2\alpha/(1 + \alpha^2)) + \sin^{-1}(2\alpha/(1 + \alpha^2)) \right)/(2\alpha^2) > 0$, therefore $g(\alpha, \pi/2)$ is also strictly increasing from 0 to $1 - \pi/4 \simeq 0.2146$.

From Figure 2 and all four boundaries, we observe that $g(\alpha, \gamma) < 0$ for all choices of $\alpha \in (0, 1]$, $\gamma \in (0, \gamma_0)$, where $\gamma_0$ is the root of the equation $g(1, \gamma) = 0$. Hence $r(\delta)$ is the radius of starlikeness of order $\delta$ of $F[A, B]$.

![Figure 2. Figure of $g(\alpha, \gamma)$](image)
Theorem 3.2. Let $\alpha \in (0, 1]$ and $\delta \in [0, 1)$. If $f \in \mathcal{F}[A, B]$, for the case when $A = \alpha e^{i\gamma}$ and $B = \alpha e^{-i\gamma}$, then there is an unique $\gamma' \in (\gamma_0, \pi/2)$ such that

(i) when $\gamma \in (\gamma_0, \gamma']$, $f$ is starlike of order $\delta$ in $D_{\gamma}(\delta)$, where $r(\delta)$ the root of the equation (3.2).

(ii) when $\gamma \in (\gamma', \pi/2)$, $f$ is starlike of order $\delta \in [0, g(\alpha, \pi/2))$ in $D$ and starlike of order $\delta \in [g(\alpha, \pi/2, 1)$ in $D_{\gamma}(\delta)$.

Proof. By observing the Figure 2 and the nature of all boundary curves discussed in Theorem 3.1, we arrive at $g(\alpha, \gamma_0) < 0$ and $g(\alpha, \pi/2) > 0$ for all $\alpha \in (0, 1)$. Thus by IVP for any choice of $\alpha \in (0, 1)$, there is an unique $\gamma' \in (\gamma_0, \pi/2)$ such that $g(\alpha, \gamma') = 0$, $g(\alpha, \gamma) < 0$ for all $\gamma \in (\gamma_0, \gamma')$ and $g(\alpha, \gamma) > 0$ for all $\gamma \in (\gamma', \pi/2)$. Hence the result follows.

Taking $\delta = 0$ in Theorems 3.1 and 3.2, we obtain the following result:

Corollary 3.3. Let $\alpha \in (0, 1]$ and $\gamma \in (0, \pi/2]$. If $f \in \mathcal{F}[A, B]$ then

(i) For $A = -B = \alpha$, $f$ is starlike univalent in the disc $|z| < r_0$, where

$$ r_0 = \frac{\exp(2\alpha) - 1}{\alpha(\exp(2\alpha) + 1)}. $$

(ii) For $A = \alpha e^{i\gamma}$ and $B = \alpha e^{-i\gamma}$, $f \in S^*$, whenever $\gamma \in (\gamma', \pi/2)$, and $f$ is starlike univalent in the disc $|z| < r_0$, whenever $\gamma \in (0, \gamma) \cup (\gamma_0, \gamma']$, where $\gamma_0, \gamma'$ are given in Theorems 3.1 and 3.2, respectively and $r_0$ is the smallest positive root of

$$ \tan(2\alpha \sin \gamma) + \frac{\alpha^4 r^4 \sin 2\gamma + 2\alpha^3 r^3 \sin \gamma \cos 2\gamma - \alpha^2 r^2 \sin 2\gamma - 2\alpha r \sin \gamma}{\alpha^4 r^4 \cos 2\gamma - 2\alpha^3 r^3 \sin \gamma \sin 2\gamma - \alpha^2 r^2 \cos 2\gamma - \alpha^2 r^2 + 1} = 0. $$

The result is sharp.

Corollary 3.4. Let $f \in \mathcal{F}[A, B]$. Then for the disc $|z| < r \leq r_0$, where $r_0$ is given in Corollary 3.3, we have

(i) For $A = -B = \alpha$,

$$ \left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \tan^{-1} \left( \frac{\sin^{-1} \left( \frac{2\alpha r}{1 + \alpha^2 r^2} \right)}{2\alpha - \log \left( \frac{1 + \alpha r}{1 - \alpha r} \right)} \right), $$

(ii) For $A = \alpha e^{i\gamma}$ and $B = \alpha e^{-i\gamma}$,

$$ \left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \tan^{-1} \left( \frac{\log T_2}{2\alpha \sin \gamma - \eta + \tau} \right), $$

where $T_2, \eta$ and $\tau$ are given in Theorem 2.3.

Corollary 3.5. Let $f \in \mathcal{F}[A, B]$. Then for the disc $|z| < r_s \leq r_0$, where $r_0$ is given in Corollary 3.3, $f \in SS^*(\beta)$, where $r_s \in (0, r_0]$ is the smallest positive root of the following equation:

(i) For $A = -B = \alpha$,

$$ \sin^{-1} \left( \frac{2\alpha r}{1 + \alpha^2 r^2} \right) = \tan \left( \frac{\beta\pi}{2} \right) \left( 2\alpha - \log \left( \frac{1 + \alpha r}{1 - \alpha r} \right) \right) $$

(ii) For $A = \alpha e^{i\gamma}$ and $B = \alpha e^{-i\gamma}$,

$$ \frac{\log T_2}{2\alpha \sin \gamma - \eta + \tau} = \tan \left( \frac{\beta\pi}{2} \right), $$

where $T_2, \eta$ and $\tau$ are given in Theorem 2.3.
We now discuss the radius estimates for the classes $\mathcal{BS}^*(\alpha)$ and $\mathcal{S}^*_C$, which are not contained in $S^*$. Further we derive radii for functions in $\mathcal{BS}^*(\alpha)$ and $\mathcal{S}^*_C$ to be in $\mathcal{F}[A,B]$.

**Theorem 3.6.** Let $\alpha \in (0,1)$, $r_0 = (-1 + \sqrt{1 + 4\alpha h_1^2})/(2ah_1)$ and $\alpha_0 \in (0,1)$ is the smallest solution of

$$1 - \frac{r_0^2 \cos^2 \theta (1 - \alpha r_0^2)^2}{(1 + \alpha^2 r_0^4 - 2\alpha r_0^2 \cos 2\theta)^2 h_1^2} - \frac{r_0^2 \sin^2 \theta (1 + \alpha r_0^2)^2}{(1 + \alpha^2 r_0^4 - 2\alpha r_0^2 \cos 2\theta)^2 h_2^2} = 0,$$

where $\theta \in (0, \pi/2)$. If $f \in \mathcal{BS}^*(\alpha)$ then for the disc $|z| \leq r_b < 1$, $f \in \mathcal{F}[A,B]$, where

$$r_b = \begin{cases} r_0, & \text{for } A = B = \alpha \leq \alpha_0 \\ r_1 := -1 + \sqrt{1 + 4\alpha h_2^2}, & \text{for } A = -B = \alpha > \alpha_0 \\ r_2 := \frac{-1 + \sqrt{1 + 4\alpha(1+k)^2}}{2\alpha(1+k)}, & \text{for } A = \alpha e^{i\gamma} \text{ and } B = \alpha e^{-i\gamma}. \end{cases}$$

Moreover the radius $r_0$ and $r_2$ are sharp.

**Proof.** Let $f \in \mathcal{BS}^*(\alpha)$, then we have

**Case 1.** When $A = -B = \alpha$: Since $\max_{|z|=1} \text{Re } \psi_{A,B}(z) = h_1$, to find such $r < 1$ for which the image of $zf'(z)/f(z) - 1$ under the disc $|z| < r$ lies inside $\psi_{A,B}(\mathbb{D})$, it is necessary that

$$\max_{|z|=r<1} \text{Re } \left( \frac{z}{1 - \alpha z^2} \right) = \frac{r}{1 - \alpha r^2} \leq h_1 \quad (3.3)$$

should hold. Clearly for $|z| \leq r_0$, the inequality $[3.3]$ holds. Now to see that for $\alpha \leq \alpha_0$, the radius $r_0$ is also sufficient for $zf'(z)/f(z) - 1 < z/(1 - \alpha z^2)$ to be in $\psi_{A,B}(\mathbb{D})$ in the disc $|z| \leq r_0$. For $\zeta = re^{i\theta}$ ($\theta \in [0, 2\pi]$), we have

$$B_r(\theta) := \frac{\zeta}{1 - \alpha \zeta^2} = \frac{r \cos \theta (1 - \alpha r^2)}{1 + \alpha^2 r^4 - 2\alpha r^2 \cos 2\theta} + i \frac{r \sin \theta (1 + \alpha r^2)}{1 + \alpha^2 r^4 - 2\alpha r^2 \cos 2\theta}.$$

Since $\text{Re } B_r(\theta) = \text{Re } B_r(-\theta)$, $\text{Re } B_r(\theta) = -\text{Re } B_r(\pi - \theta)$ and $\text{Im } B_r(\theta) = \text{Im } B_r(\pi - \theta)$, therefore the curve $B_r(\theta)$ is symmetric about real and imaginary axis thus it is sufficient to consider for $\theta \in [0, \pi/2]$. Now for $r = r_0$, the square of the distance from the origin to the points of $B_{r_0}(\theta)$ is given by

$$\text{Dist}(0; B_{r_0}(\theta)) := \frac{r_0^2}{1 + \alpha^2 r_0^4 - 2\alpha r_0^2 \cos 2\theta}.$$

Since $\text{Dist}(0; B_{r_0}(\theta))' < 0$, thus $\text{Dist}(0; B_{r_0}(\theta))$ is a decreasing function of $\theta$. Hence the farthest point of $B_{r_0}(\theta)$ from origin is $(r_0/(1 - \alpha r_0^2), 0)$, which lies on the boundary of $\Omega_1$. Now for $A = -B = \alpha > \alpha_0$ and $A = \alpha e^{i\gamma}$, $B = \alpha e^{-i\gamma}$ with $|z| = r$, we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{r}{1 - \alpha r^2}.$$

Therefore, by Lemma 2.6, we see that $\mathcal{F}[A,B]$—radius for the class $\mathcal{BS}^*(\alpha)$ are the smallest positive roots $r_1$ and $r_2$ of the equations $r/(1 - \alpha r^2) = h_2$ and $r/(1 - \alpha r^2) = k_1 + k$, respectively.

**Theorem 3.7.** Let $\alpha \in (0,1)$, $r_0 = (1 + (1 - \alpha) h_1 + \sqrt{(1 + (1 - \alpha) h_1)^2 + 4ah_1^2})/(2ah_1)$ and $\alpha_0 \in (0,1)$ is the smallest solution of

$$1 - \frac{r_0^2 ((1 - \alpha) r_0 + \cos \theta (1 - \alpha^2 r_0^2))^2}{(1 + r_0^2 - 2r_0 \cos \theta)^2 (1 + \alpha^2 r_0^4 + 2\alpha r_0 \cos \theta)^2 h_1^2} - \frac{r_0^2 (1 + \alpha^2 r_0^2)^2 \sin^2 \theta}{(1 + r_0^2 - 2r_0 \cos \theta)^2 (1 + \alpha^2 r_0^4 + 2\alpha r_0 \cos \theta)^2 h_2^2} = 0,$$
where \( \theta \in (0, \pi/2) \). If \( f \in S^*_c(\alpha) \) then for the disc \( |z| \leq r_{cs} < 1 \), \( f \in F[A,B] \), where

\[
\begin{align*}
    r_{cs} = \begin{cases} 
        r_0, & \text{for } A = -B = \alpha \leq \alpha_0 \\
        r_1 := \frac{-(1 + (1 - \alpha)h_2) + \sqrt{1 + (1 - \alpha)h_2^2 + 4\alpha h_2^2}}{2\alpha h_2}, & \text{for } A = -B = \alpha > \alpha_0 \\
        r_2 := \frac{-(1 + (1 - \alpha)(k_1 + k)) + \sqrt{1 + (1 - \alpha)(k_1 + k)^2 + 4\alpha(k_1 + k)^2}}{2\alpha(k_1 + k)}, & \text{for } A = \alpha e^{i\gamma}, B = \alpha e^{-i\gamma}.
    \end{cases}
\end{align*}
\]

Moreover the radius \( r_0 \) and \( r_2 \) are sharp.

**Proof.** Let \( f \in S^*_c(\alpha) \). Since for \( A = -B = \alpha \), \( \max_{|z|=1} \text{Re} \psi_{A,B}(z) = h_1 \). Thus to find such \( r < 1 \) for which the image of \( zf'(z)/f(z) - 1 \) under the disc \( |z| < r \) lies inside \( \psi_{A,B}(D) \), it is necessary that

\[
\max_{|z|=r<1} \text{Re} \left( \frac{z}{(1-z)(1+\alpha z)} \right) = \frac{r}{(1-r)(1+\alpha r)} \leq h_1 \tag{3.4}
\]

must hold. Clearly for \( |z| \leq r_0 \), the equation \( (3.4) \) holds. Now to see that for \( \alpha \leq \alpha_0 \) radius \( r_0 \) is also sufficient for \( zf'(z)/f(z) - 1 \prec z/((1-z)(1+\alpha z)) \in \psi_{A,B}(D) \) in the disc \( |z| \leq r_0 \). For \( \zeta = re^{i\theta} \) \( (\theta \in [0,2\pi]) \), we have
\[ CS_r(\theta) := \frac{\zeta}{(1 - \zeta)(1 + \alpha \zeta)} \]
\[ = \frac{r((\alpha - 1)r + \cos \theta(1 - \alpha r^2))}{(1 + r^2 - 2r \cos \theta)(1 + \alpha^2 r^2 + 2\alpha r \cos \theta)} + i \frac{r(1 + \alpha r^2)}{(1 + r^2 - 2r \cos \theta)(1 + \alpha^2 r^2 + 2\alpha r \cos \theta)}. \]

Since \( \text{Re} CS_r(\theta) = \text{Re} CS_r(-\theta) \), therefore the curve \( CS_r(\theta) \) is symmetric about real axis thus it is sufficient to consider for \( \theta \in [0, \pi] \). Now for \( r = r_0 \), the square of the distance from the origin to the points of \( CS_{r_0}(\theta) \) is given by

\[ \text{Dist}(0; CS_{r_0}(\theta)) := \frac{r_0^2}{(1 + r^2 - 2r \cos \theta)^2(1 + \alpha^2 r^2 + 2\alpha r \cos \theta)^2}. \]

Since \( \text{Dist}(0; CS_{r_0}(\theta))' = 0 \) for \( \theta = 0, \theta_0 \) and \( \pi \) with \( \text{Dist}(0; CS_{r_0}(\theta))' < 0 \) whenever \( \theta \in (0, \theta_0) \) and \( \text{Dist}(0; CS_{r_0}(\theta))' > 0 \) whenever \( \theta \in (\theta_0, \pi) \). And \( \text{Dist}(0; CS_{r_0}(0)) - \text{Dist}(0; CS_{r_0}(\pi)) > 0 \), hence the farthest point of \( CS_{r_0}(\theta) \) from origin is equal to \( h_1 \) obtained \( \theta = 0 \). Now for \( A = -B = \alpha > \alpha_0 \) and \( A = \alpha e^{i\gamma}, B = \alpha e^{-i\gamma} \) with \( |z| = r \), we have

\[ \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{r}{(1 - r)(1 + \alpha r)}. \]

Therefore, by Lemma [2,6] we see that \( \mathcal{F}[A, B] \)-radius for the class \( S^*_c(\alpha) \) are the smallest positive roots \( r_1 \) and \( r_2 \) of the equations \( r/((1 - r)(1 + \alpha r)) = h_2 \) and \( r/((1 - r)(1 + \alpha r)) = k_1 + k \), respectively.

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