DISCRETE APPROXIMATION OF STATIONARY MEAN FIELD GAMES

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Abstract. In this paper, we focus on stationary (ergodic) mean-field games (MFGs). These games arise in the study of the long-time behavior of finite-horizon MFGs. Motivated by a prior scheme for Hamilton–Jacobi equations introduced in Aubry–Mather’s theory, we introduce a discrete approximation to stationary MFGs. Relying on Kakutani’s fixed-point theorem, we prove the existence and uniqueness (up to additive constant) of solutions to the discrete problem. Moreover, we show that the solutions to the discrete problem converge, uniformly in the nonlocal case and weakly in the local case, to the classical solutions of the stationary problem.

1. Introduction

The mean-field games (MFG) theory aims to model and analyze systems with many competing rational agents. These agents have preferences encoded in a cost functional, which they seek to optimize. This functional depends both on the agent’s and on the other agent’s states. Individually, agents have little effect on the entire system. Accordingly, their cost depends only on their state and on aggregate or statistical quantities. MFGs theory was introduced in the mathematics community by J-M. Lasry and P-L. Lions in [37, 38] and independently in the engineering community by M. Huang, P. Caines, and R. Malhamé in [34, 33]. This theory has expanded tremendously and has found applications in population dynamics [36, 5, 29], economics [4, 30, 28], finance [16, 18], engineering [21, 35], to name just a few.

An important model in the theory of MFG is the stationary (ergodic) MFG. This problem arises in the study of the long-time behavior of finite-horizon MFGs. The stationary (ergodic) MFG is determined by the following system of PDEs

\[
\begin{aligned}
\Delta v + H(x, Dv) &= \rho + F(x, m), \\
\Delta m - \text{div}(m \, D_p H(x, Dv)) &= 0, \quad m > 0, \quad \int_{\mathbb{T}^d} m \, dx = 1.
\end{aligned}
\]

(1)

It is usual to study the preceding system under periodic boundary conditions. Hence, the spatial variable $x$ takes values on the $d$-dimensional torus $\mathbb{T}^d$. The data of this problem are a smooth Tonelli Hamiltonian, $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$, and a continuous coupling,
\( F: \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}; \) the unknowns are the distribution of players, \( m \in \mathcal{P}(\mathbb{T}^d), \) the value function, \( v: \mathbb{T}^d \to \mathbb{R}, \) and the ergodic constant, \( \rho \in \mathbb{R}. \) Because of the periodicity, \( v \) can only be determined up to additive constants.

Much progress has been achieved in understanding stationary MFGs since the first results in \([37, 39]\) on the existence and uniqueness of weak solutions to (1). For example, in \([32]\), the existence of classical solutions for (1) in the local coupling case \((F(\cdot, m) = F(m(\cdot)))\) was proven; these results were later improved in \([31, 42]\). The nonlocal case with quadratic and sub-quadratic Hamiltonian was addressed in \([11]\). In parallel, the theory of Sobolev solutions was developed in \([12]\). In \([41]\), stationary MFGs with density constraints were addressed. Finally, general existence results using monotone operator methods were developed in \([23]\).

In contrast, the theory for the convergence of discrete schemes to (1) is less well developed. The first systematic approaches to numerical methods for MFGs were discussed in \([2\text{ and } 3]\). See also the survey \([1]\) and the subsequent paper \([6]\). A new class of estimates for MFGs with congestion and the rigorous convergence of corresponding numerical schemes was discussed in \([7, 8]\). The ergodic problem was examined from a numerical perspective in several works; for example, \([14]\) studied the ergodic problem both analytically and numerically, semi-discrete approximations were considered in \([15]\), and semi-Lagrangian methods were investigated in \([17]\). An alternative approach using proximal methods was developed in \([13]\). In the last section of this paper, we will use the ideas first developed in \([9]\) for justifying convergence of numerical schemes for monotone MFGs.

Let \( L: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) represent the Legendre transform of the Hamiltonian \( H, \)

\[
L(x, v) = \sup_{p \in \mathbb{R}^d} pv - H(x, p).
\]

Here, we introduce a discrete approximation (2) of the system (1). This scheme is the MFG analog of the approximation scheme for Hamilton-Jacobi equations and Mather measures considered in \([19\text{ and } 20]\).

**Problem 1.** Let \( \tau > 0, \) \( L: \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R} \) be a smooth Tonelli Lagrangian, and \( F: \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d) \to \mathbb{R} \) be continuous. Find \((\rho, u, \tilde{\mu}) \in \mathbb{R} \times C(\mathbb{T}^d) \times \mathcal{C}_\tau\) satisfying

\[
\begin{aligned}
\tau \rho &= \mathcal{L}_\tau u(x) - u(x) - \tau F(x, \mu), \\
-\rho &= \int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, q) + F(x, \mu) \, d\tilde{\mu}(x, q), \quad \mu = \Pr_{1\#}\tilde{\mu},
\end{aligned}
\]

where the set of holonomic measures \( \mathcal{C}_\tau \) is introduced in Definition 2, and the Lax operator \( \mathcal{L}_\tau \) is defined by (8).

Note that our discrete system depends on the parameter \( \tau. \) To stress that, we call it \( \tau \)-discrete MFG system.

Our primary goal is to establish the existence of solutions to the preceding problem and, then, prove the convergence, uniform in the nonlocal case and weak in the local
case, to classical solutions of (1). For that, in Section 2, we review prior results on the approximation of stationary Hamilton–Jacobi equations and the connection with Aubry–Mather theory. In Section 3.1, we detail our discrete MFG problem and present its relation to other discrete models. Then, in Section 4, we present our main assumptions.

The existence of solutions is addressed in Section 5. There, relying on Kakutani’s fixed-point Theorem, we prove the following theorem.

**Theorem 1.** Suppose that $L$ satisfies Assumption (A1). In the nonlocal case, further assume that $F$ satisfies (B1) and (B2); in the local case, suppose that $F(\cdot, m) = F(m)$ satisfies (B4) and (B5). Then, there exists a solution $(\rho_\tau, u_\tau, \bar{\mu}_\tau)$ of the $\tau$–discrete MFG, (2). In addition, $\rho_\tau$ and $\mu_\tau = \Pr_{1#}\bar{\mu}_\tau$, are unique and $u_\tau$ is unique up to additive constants.

Moreover, in the local case when $F(m) = \log(m)$, we have

\[
\rho_\tau = \min_{\mu \in C_\tau} \int_{\mathbb{T}^d \times \mathbb{R}^d} L + \log \mu \, d\bar{\mu} \\
= -\log \min_{\phi \in C(\mathbb{T}^d)} \int_{\mathbb{T}^d} e^{(L_\tau \phi(x) - \phi(x))/\tau} \, dx. \tag{3}
\]

Next, in Section 6, we establish our main results; that is, we prove the convergence of the solutions of the $\tau$–discrete MFG system, Problem 1, to the classical solution of (1), when $\tau \to 0^+$. First, we prove helpful properties of the operator $\mathcal{L}_\tau$ (see Subsection 6.1). In Subsection 6.2, we examine the nonlocal coupling case and, relying on semi-convexity estimates, prove

**Theorem 2.** Suppose that $L$ satisfies (A1) and (A2) and $F$ satisfies (B1), (B2), and (B3). Let $x_0 \in \mathbb{T}^d$ and let $(\rho_\tau, u_\tau, \bar{\mu}_\tau)$ be a solution of the discrete MFG (2) with $u_\tau(x_0) = 0$ and $\mu_\tau = \Pr_{1#}\bar{\mu}$. Let $(\rho, u, m)$ solves the ergodic MFG (1) with $u_0(x_0) = 0$. Then,

1. $\lim_{\tau \to 0^+} \rho_\tau = \rho$,
2. $\lim_{\tau \to 0^+} u_\tau = u$ uniformly,
3. $\mu_\tau \rightharpoonup m$.

**Remark 1.** Under the assumptions of Theorem 2, Theorem 2.1 in [11] ensures the existence of classical solutions to (1).

The particular case of (2) with the local coupling $F(m) = m^a$ is examined in Subsection 6.3. There, we use the hypercontractivity of the Hamilton–Jacobi equation to get uniform bounds for the solution of the $\tau$–discrete MFG (2). Using these bounds and using the monotonicity (see [22, 23, 24]), we define weak solutions to the $\tau$–discrete MFG system and to the ergodic MFG system (see Definition 3). Finally, in Section 6.4, using Minty’s method, we prove that normalized solutions to the discrete MFG system (2) weakly converge to a classical solution to (1).
Theorem 3. Suppose that $L$ satisfies (A3) and $F$ satisfies (B6). Let $(\rho_\tau, u_\tau, \tilde{\mu}_\tau)$ with $\mu_\tau = \Pr_{1 \#} \tilde{\mu}_\tau$ be the solution to the discrete MFG (2) such that $\max \eta^\tau * u_\tau = 0$. Then, as $\tau \to 0$, $(\rho_\tau, \mu_\tau)$ converges in $\mathbb{R} \times \mathcal{P}_\ell$ to $(r, m)$, $u_\tau$ converges weakly in $L^{1+1/a}$ to $u$, and $(r, u, m)$ is a classical solution to (1).

Remark 2. Under the assumptions of Theorem 3, Theorem 1 in [42] ensures the existence of classical solutions to (1) for this local case.

2. Background

Let $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a smooth Tonelli Lagrangian, $\mathbb{Z}^d$–periodic in the space variable $x$, or equivalently, let $L$ be defined on $\mathbb{T}^d \times \mathbb{R}^d$. Let $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, the Hamiltonian, be the Legendre transform of $L$.

Our discretized model is motivated by our previous study in [19] of a discrete approximation of the viscous Hamilton–Jacobi equation,

$$\Delta u + H(x, Du) = \alpha_0,$$

and by the discrete approximation of the stochastic Mather measures introduced in [26], whose definition we recall next.

Let $\mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ be the set of Borel probability measures on $\mathbb{T}^d \times \mathbb{R}^d$. Let $C^0_\ell$ be the set of continuous functions $f : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ having at most linear growth; that is,

$$\|f\|_\ell := \sup_{(x,q) \in \mathbb{T}^d \times \mathbb{R}^d} \frac{|f(x,q)|}{1 + \|q\|} < +\infty.$$

Let $\mathcal{P}_\ell$ be the set of measures in $\mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ such that

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} \|q\| \, d\mu < +\infty.$$

We endow $\mathcal{P}_\ell$ with the topology corresponding to the following convergence: for $\mu_n \in \mathcal{P}_\ell$, $\lim_{n \to \infty} \mu_n = \mu$ if and only if

$$\lim_{n \to \infty} \int f \, d\mu_n = \int f \, d\mu,$$

for all $f \in C^0_\ell$.

Let $(C^0_\ell)'$ be the dual of $C^0_\ell$. Then, $\mathcal{P}_\ell$ is naturally embedded in $(C^0_\ell)'$, and its topology is the weak*- induced topology on $(C^0_\ell)'$. This topology is metrizable. To see this, let $\{f_n\}$ be a sequence of functions with compact support on $C^0_\ell$, which is dense on $C^0_\ell$ in the topology of uniform convergence on compact sets of $\mathbb{T}^d \times \mathbb{R}^d$. The metric $d$ on $\mathcal{P}_\ell$ defined by

$$d(\mu_1, \mu_2) = \left| \int |q| \, d\mu_1 - \int |q| \, d\mu_2 \right| + \sum_n \frac{1}{2^n \|f_n\|_\infty} \left| \int f_n \, d\mu_1 - \int f_n \, d\mu_2 \right|$$

gives the topology of $\mathcal{P}_\ell$. 
Remark 3. According to [40], for \( c \in \mathbb{R} \), the set

\[
A(c) := \left\{ \tilde{\mu} \in \mathcal{P}_\ell : \int_{\mathbb{T}^d \times \mathbb{R}^d} L \, d\tilde{\mu} \leq c \right\}
\]

is compact in \( \mathcal{P}_\ell \).

Definition 1. We denote by \( C \) the set of holonomic measures; that is, the closed convex subset of measures in \( \mathcal{P}_\ell \) that satisfy

\[
\int_{\mathbb{T}^d \times \mathbb{R}^d} (\Delta \varphi(x) + \langle D\varphi(x), q \rangle) \, d\tilde{\mu}(x, q) = 0
\]

for all \( \varphi \in C^2(\mathbb{T}^d) \).

The set \( C \) is non-empty as the measure \( \tilde{\nu} \) given by

\[
\int f \, d\tilde{\nu} = \int f(x, 0) dx, \quad f \in C_0^0 \tag{5}
\]

is holonomic.

More generally, let \( V : \mathbb{T}^d \to \mathbb{R}^d \) be a \( C^1 \) vector field and let \( \mu \in \mathcal{P}(\mathbb{T}^d) \) be the unique weak solution of the following Fokker–Planck equation.

\[
\Delta \mu - \text{div} (V(x) \mu) = 0 \quad \text{in} \ \mathbb{T}^d. \tag{6}
\]

For \( G_V(x) = (x, V(x)) \), we have that \( \tilde{\mu} := G_V \# \mu \) is a holonomic measure.

According to [26], the ergodic constant in (4) is determined both by the minimization problem

\[
\alpha_0 = \min_{\varphi \in C^2(\mathbb{T}^d)} \max_x \Delta \varphi(x) + H(x, D\varphi(x))
\]

and by the dual problem

\[
-\alpha_0 := \min_{\tilde{\mu} \in \mathcal{P}} \int_{\mathbb{T}^d \times \mathbb{R}^d} L \, d\tilde{\mu}. \tag{7}
\]

Holonomic measures achieving the minimum in (7) are called stochastic Mather measures.

Theorem 4 ([26]). A measure \( \tilde{\mu} \) is a stochastic Mather measure if and only if \( \tilde{\mu} := G_V \# \mu \) where \( \mu \in \mathcal{P}(\mathbb{T}^d) \) is the solution of (6) with \( V(x) := D_\mu H(x, Du(x)) \) and \( u \) is any solution to (4).

Because solutions to (4) are unique, up to additive constants, there is only one stochastic Mather measure for each Lagrangian.

3. The discrete MFG problem

Now, we examine in detail Problem 1 and explore related discrete MFG models.
3.1. **Statement of the problem.** For the discrete problem, the source of randomness corresponds to the heat kernel \( \eta^\tau \) on \( T^d = \mathbb{R}^d / \mathbb{Z}^d \); that is,

\[
\eta^\tau(z) = \frac{1}{(4\pi \tau)^{\frac{d}{2}}} \sum_{k \in \mathbb{Z}^d} e^{-\frac{|z + k|^2}{4\tau}} = \sum_{k \in \mathbb{Z}^d} e^{-\tau|2\pi k|^2 + 2\pi ik \cdot z}.
\]

Accordingly, for \( u \in C(\mathbb{R}^d), \mathbb{Z}^d \)-periodic, we have

\[
(\eta^\tau * u)(y) = \int_{T^d} \eta^\tau(y - z) u(z) \, dz = \int_{\mathbb{R}^d} e^{-\frac{|y - z|^2}{4\tau}} u(z) \, dz.
\]

Next, we define the operator \( \mathcal{L}_\tau : C(T^d) \to C(T^d) \) as

\[
\mathcal{L}_\tau u(x) := \max_{q \in \mathbb{R}^d} \left( (\eta^\tau * u)(x + \tau q) - \tau L(x, q) \right) \text{ for } x \in \mathbb{R}^d.
\]  

As discussed in [19], the discrete version \( \mathcal{L}_\tau u - \tau \alpha_\tau \) arises from the discretization of the stochastic Lax formula.

We recall the following results concerning \( \mathcal{L}_\tau \) that were established in [19].

(i) There is a unique value \( \alpha_\tau \in \mathbb{R} \) for which \( \mathcal{L}_\tau \) admits solutions. This value is given by

\[
\alpha_\tau = \min_{\varphi \in C(T^d)} \max_{x,q} \frac{\eta^\tau * \varphi(x + \tau q) - \varphi(x)}{\tau} - L(x, q)
\]

and satisfies

\[
\min_{T^d \times \mathbb{R}^d} L \leq -\alpha_\tau \leq \max \min_{x \in T^d, q \in \mathbb{R}^d} L(x, q).
\]

(ii) A solution \( u \in C(T^d) \) to \( \mathcal{L}_\tau \) is unique up to additive constants.

For the continuous second-order Hamilton-Jacobi equation, the preceding results were established in [26].

Now, we fix \( u \in C(T^d) \) and let

\[
V_u(x) := \arg \max \left[ (\eta^\tau * u)(x + \tau q) - \tau L(x, q) \right].
\]

Then, we have the following two additional results.

(iii) If \( q \in V_u(x) \) then

\[
D(\eta^\tau * u)(x + \tau q) = D_q L(x, q)
\]

or, equivalently,

\[
q = D_p H(x, D(\eta^\tau * u)(x + \tau q)).
\]

(iv) There is a Borel measurable map \( V : T^d \to \mathbb{R}^d \) that is optimal for \( u \); that is, \( V(x) \in V_u(x) \) for all \( x \in T^d \). Moreover, there is \( \tau_u > 0 \) such that for all \( \tau < \tau_u \), \( V_u(x) \) is a singleton and

\[
D_{qq}^2 \left[ \frac{\eta^\tau * u(x + \tau q) - u(x)}{\tau} - L(x, q) \right] = \tau D^2(\eta^\tau * u)(x + \tau q) - D_{qq}^2 L(x, q)
\]
is negative definite for $q = \nabla u(x)$. 

(v) If $u \in C(\mathbb{T}^d)$ and $V : \mathbb{T}^d \to \mathbb{R}^d$ Borel measurable satisfies 

$$(\eta^\tau * u)(x + \tau V(x)) - u(x) - \tau L(x, V(x)) = \tau \alpha_{\tau},$$

then $u$ solves (9) and $V$ is optimal for $u$.

We follow the generalized scheme in [27] to describe the dual of the minimization problem (10).

**Definition 2.** We denote by $C_{\tau}$ the set of $\tau$–holonomic measures; that is, the closed convex subset of measures $\tilde{\mu} \in \mathcal{P}$ such that the projected measure $\mu = \Pr_1 \# \tilde{\mu}$ satisfies 

$$\mu(A) = \int_A \int_{\mathbb{T}^d \times \mathbb{R}^d} \eta^\tau(x + \tau q - z) \, d\tilde{\mu}(x, q) \, dz$$

for any Borel set $A \subset \mathbb{T}^d$.

We observe that 

$$m_{\tilde{\mu}}(z) = \int_{\mathbb{T}^d \times \mathbb{R}^d} \eta^\tau(x + \tau q - z) \, d\tilde{\mu}(x, q)$$

defines a smooth function. Thus, when $\tilde{\mu}$ is $\tau$–holonomic, its projected measure has a smooth density that we identify with this projected measure. Note that we have the following upper bound

$$m_{\tilde{\mu}} \leq (4 \pi \tau)^{-d/2}.$$  

(12)

The dual problem to (10) is

$$-\alpha_{\tau} = \min \left\{ \int_{\mathbb{T}^d \times \mathbb{R}^d} L \, d\tilde{\mu} : \tilde{\mu} \in C_{\tau} \right\}.$$  

(13)

The measure given in (5) is $\tau$-holonomic. More generally,

**Proposition 1.** [20] Any Borel measurable map $V : \mathbb{T}^d \to \mathbb{R}^d$, defines a $\tau$–holonomic measure $\tilde{\mu}^V$ such that

(a) If $u \in C(\mathbb{T}^d)$ and $V$ satisfy (11), then

$$-\alpha_{\tau} = \int_{\mathbb{T}^d \times \mathbb{R}^d} L \, d\tilde{\mu}^V.$$ 

(b) If $V$ is continuous there is $\mu \in \mathcal{P}(\mathbb{T}^d)$ such that $\tilde{\mu}^V = G_V \# \mu$, and in particular, for any $f \in C(\mathbb{T}^d)$ we have

$$\int_{\mathbb{T}^d} \eta^\tau * f(x + \tau V(x)) - f(x) \, d\mu(x) = 0.$$ 

The previous discussion motivates the $\tau$-discrete MFG model (2), for $(\rho, u, \tilde{\mu}) \in \mathbb{R} \times C(\mathbb{T}^d) \times C_{\tau}$,

$$
\begin{aligned}
\tau \rho &= \mathcal{L}_{\tau} u(x) - u(x) - \tau F(x, \mu) \\
-\rho &= \int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, q) + F(x, \mu) \, d\tilde{\mu}(x, q), \quad \mu = \Pr_1 \# \tilde{\mu},
\end{aligned}$$

where $\mathcal{L}_{\tau}$ is a suitable operator.
that was presented in Problem 1.

3.2. Relation with other discrete MFG models. The work [43] examined a model for discrete average cost MFG. The model is determined by a quadruple, \((X, A, p, c)\), where \(X\) and \(A\) stand for the state and control spaces, \(p\) is a stochastic kernel, \(p: X \times A \times \mathcal{P}(X) \to \mathcal{P}(X)\), that gives the transition probability law, and \(c: X \times A \times \mathcal{P}(X) \to [0, \infty)\) is the one-stage cost. A policy is a stochastic kernel on \(A\) given \(X\), \(\pi: X \to \mathcal{P}(A)\). Each policy, together with an initial distribution \(\mu_0 \in \mathcal{P}(X)\) and the transition probability \(p\), defines a measure \(P^\pi\) on \((X \times A)^\infty\). We denote the expectation with respect to \(P^\pi\) by \(E^{\pi}\).

Given a policy \(\pi\), the corresponding average cost is
\[
J_{\mu}(\pi) = \limsup_{n \to \infty} \frac{1}{n} E^{\pi} \left[ \sum_{i=0}^{n-1} c(x_i, a_i, \mu) \right].
\]
The average cost problem consists of finding \(\pi^*\) that is optimal for \(\mu\); that is, such that
\[
J_{\mu}(\pi^*) = \inf_{\pi} J_{\mu}(\pi).
\]

For any set \(A\) let \(2^A\) be the set of all subsets of \(A\). Consider the map \(\Psi: \mathcal{P}(X) \to 2^\Pi\) given by
\[
\Psi(\mu) = \{\pi \in \Pi: \pi \text{ is optimal for } \mu\}.
\]
Furthermore, define the map \(\Lambda: \Pi \to 2^{\mathcal{P}(X)}\) as follows: given \(\pi \in \Pi\), \(\mu \in \Lambda(\pi)\) if
\[
\mu(B) = \int_{X \times A} p(B \mid x, a, \mu) \pi(da \mid x) \mu(dx).
\]
Under the assumption discussed in [43], \(\Lambda(\pi)\) has a unique element for all \(\pi\). A pair \((\pi, \mu) \in \Pi \times \mathcal{P}(X)\) is a mean-field equilibrium if \(\pi \in \Psi(\mu)\) and \(\mu \in \Lambda(\pi)\).

For us, the state and control spaces are the torus \(T^d\) and \(\mathbb{R}^d\). The transition law \(p: T^d \times \mathbb{R}^d \to \mathcal{P}(T^d)\) is given by the dynamics
\[
x_{i+1} = x_i + \tau q_i + \nu_i,
\]
where \((\nu_i)\) is a sequence of Gaussian random variables, i.i.d. with common law
\[
\eta^\tau(s) = \frac{1}{(4\pi \tau)^{\frac{d}{2}}} e^{-|s|^2/4\tau}.
\]
Thus,
\[
p(B \mid x, q) = \int_{T^d} \chi_B(x + \tau q + s) \eta^\tau(s) ds = \int_B \eta^\tau(z - x - \tau q) dz.
\]
The one-stage cost function is \(c_\tau(x, q, \mu) = \tau(L(x, q) + F(\mu))\).

To obtain a mean-field equilibrium \((\pi, \mu)\) from a solution of the MFG system (2), we disintegrate \(\bar{\mu}\) as \(\bar{\mu}(dx, dq) = \pi(dq \mid x) \mu(dx)\). This approach was used in [43]. The setting in that reference is substantially different from ours: there, the control space \(A\) is compact and the one-stage cost \(c\) is bounded. In particular, the results in [43] do not imply our results directly.
4. Assumptions

Here, we describe the main assumptions used in the sequel.

(1) The Lagrangian $L : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is smooth. Moreover, we will need some of the following assumptions for our results.

(A1) There is $c_0 > 0$ such that $D^2_{qq}L \geq c_0 I$.

(A2) There are $k_1, k_2$, such that for any $x, y, q, v \in \mathbb{R}^d$

$$L(x + h, q + v) - 2L(x, q) + L(x - h, q - v) \leq k_1|h|^2 + k_2|v|^2.$$ 

(A3) The Lagrangian $L$ is separable; that is, $L(x, v) = K(v) - U(x)$. Furthermore, $K(v) \geq c|v|^q$, $q = 2$ for $d = 1$ and $q > d$ for $d > 2$; $K(0) = 0$.

Assumption (A1) implies that there are $c_1, c_2 > 0$ such that for any $x \in \mathbb{T}^d$, $q \in \mathbb{R}^d$

$$|q|^2 \leq c_1L(x, q) + c_2$$ 

(14)

For the convergence, in the nonlocal case, we work under Assumption (A2); for the local case, instead, we use (A3). Observe that for $d = 1$, a Lagrangian that satisfies Assumption (A3) also satisfies Assumption (A2).

(2) The coupling $F : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is either nonlocal or local. For our various results, we will work with the following assumptions.

(a) **Nonlocal.** We assume that

(B1) $F$ is continuous.

(B2) $F$ strictly monotone; that is,

$$\int_{\mathbb{T}^d}(F(x, m) - F(x, m'))d(m - m')(x) > 0 \quad \text{for any } m \neq m'.$$

(B3) There is $k_0 > 0$ such that for any $x, h \in \mathbb{R}^d$, $m \in \mathcal{P}(\mathbb{T}^d)$

$$F(x + h, m) - 2F(x, m) + F(x - h, m) \leq k_0|h|^2.$$ 

The monotonicity assumption is used to establish uniqueness, as it is standard in MFG theory. The last assumption provides uniform semiconcavity bounds that are crucial for the convergence results.

(b) **Local.** We assume $F(x, m) = F(m(x))$ with $F : \mathbb{R}^+ \to \mathbb{R}$ smooth and, we will work with the following additional properties.

(B4) $F$ concave and strictly increasing

(B5) $F$ bounded below or $F(m) = \log m$.

(B6) $F$ has the following form $F(m) = m^\alpha$, $0 < \alpha < 1$

Assumptions (B6) is used in the analysis of the convergence. Of course, (B5) implies (B6) since $m^\alpha$ is bounded by below.

In the sequel, we will use the following constants

(1) In the nonlocal case, $a_F = \min F$, $b_F = \max F$.
(2) In the local case, $a_F = \min F$ when $F$ is bounded below, $a_F = 0$ when $F(m) = \log m$, and $b_F = F(1)$.

5. Solving the discrete MFG

Fix $\tilde{\mu} \in C_T$ and let $\mu = \text{Pr}_{1\#}\tilde{\mu}$. According to properties (i) and (ii) concerning (9) that were discussed in subsection 3.1, there is a unique $\rho^\mu \in \mathbb{R}$ and $u_\mu \in C(\mathbb{T}^d)$, unique up to additive constants, solving the first equation of (2). Thus, it follows from [20] that the set

$$
\Psi_{\tau}(\tilde{\mu}) := \{ \tilde{\nu} \in C_T : \int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, q) + F(x, \mu)\, d\tilde{\nu}(x, q) = -\rho^\mu \}
$$

is non-empty. Furthermore, $\Psi_{\tau}(\tilde{\mu})$ is also convex.

**Lemma 1.** Assume $L$ satisfies (A1) and $F$ satisfies (B2) or (B4). Let $(\rho_1, u_1, \tilde{\mu}_1)$, $(\rho_2, u_2, \tilde{\mu}_2)$ solve the $\tau$–discrete MFG in (2). Then $\text{Pr}_{1\#}\tilde{\mu}_1 = \text{Pr}_{1\#}\tilde{\mu}_2$, $\rho_1 = \rho_2$ and $u_1 - u_2$ is constant.

**Proof.** Let $\mu^i = \text{Pr}_{1\#}\tilde{\mu}_i$, $i = 1, 2$. Then,

$$
\int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, q) + F(x, \mu^1)\, d\tilde{\mu}_1(x, q) = -\rho_1 \leq \int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, q) + F(x, \mu^1)\, d\tilde{\mu}_2(x, q),
$$

and

$$
\int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, q) + F(x, \mu^2)\, d\tilde{\mu}_2(x, q) = -\rho_2 \leq \int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, q) + F(x, \mu^2)\, d\tilde{\mu}_1(x, q).
$$

Adding these inequalities, we have

$$
\int L d\tilde{\mu}_1 + \int L d\tilde{\mu}_2 + \int F(x, \mu^1)\, d\mu_1(x) + \int F(x, \mu^2)\, d\mu_2(x) \\
\leq \int L d\tilde{\mu}_1 + \int L d\tilde{\mu}_2 + \int F(x, \mu^1)\, d\mu_2(x) + \int F(x, \mu^2)\, d\mu_1(x).
$$

Therefore,

$$
\int_{\mathbb{T}^d} (F(x, \mu^1) - F(x, \mu^2))\, d(\mu^1 - \mu^2)(x) \leq 0.
$$

By the strict monotonicity of $F$, in the nonlocal case, or the strict monotonicity of $F$, in the local case, if we had $\mu^1 \neq \mu^2$, then we would have

$$
\int_{\mathbb{T}^d} (F(x, \mu^1) - F(x, \mu^2))\, d(\mu^1 - \mu^2)(x) > 0.
$$

Thus, $\mu^1 = \mu^2$, so $\rho_1 = \rho_2$. Furthermore, by the fact (ii) about equation (9), we have that $u_1 - u_2$ is constant. \qed

To solve the discrete MFG, we must show that the map $\Psi_{\tau} : C_T \rightarrow 2^{C_T}$ has a fixed point. This is achieved using Kakutani’s fixed-point theorem.
Proposition 2. Suppose that \( L \) satisfies (A1) and that either \( F \) is nonlocal satisfying (B1) or that \( F \) is local with \( F \) satisfying (B4) and (B5). Moreover, if \( F \) is local and \( F(m) = \log m \) suppose that \( 0 < \tau < 2/c_1 \).

Then, there is a constant, \( C_\tau > 0 \), such that for

\[
A_\tau = \left\{ \tilde{\mu} \in C_\tau : \int_{\mathbb{T}^d \times \mathbb{R}^d} L \, d\tilde{\mu} \leq C_\tau \right\},
\]

\( F|_{\mathbb{T}^d \times A_\tau} \) is bounded and \( \Psi_\tau(A_\tau) \subset 2^{A_\tau} \).

Proof. Consider the problem in (13). In the nonlocal case, (B1) and the compactness of \( \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \) imply that \( F \) is bounded. In the local case, from (B4) and the upper bound (12), we have \( F(\mu) \leq F((4\pi\tau)^{-d/2}) \), for \( \mu = \text{Pr}_\# \tilde{\mu}, \tilde{\mu} \in C_\tau \).

We examine first the following case: \( F \) is nonlocal satisfying (B1) or \( F \) is local with \( F \) satisfying (B4) and bounded below. We address the logarithmic case separately.

By the definition of \( \Psi_\tau(\tilde{\mu}) \) in (15), and applying Proposition 1 using the measure defined in (5), we get

\[
- \rho^n \leq \int_{\mathbb{T}^d} L(x, 0) + F(x, \mu) \, dx \leq \max_{x \in \mathbb{T}^d} [L(x, 0) + b_F - F].
\]

where, in the local case, we use the concavity of \( F \). Hence, in the case (1) for \( \nu \in \Psi_\tau(\tilde{\mu}) \), we have

\[
\int_{\mathbb{T}^d \times \mathbb{R}^d} L \, d\nu + a_F \leq - \rho^n \leq \max_{x \in \mathbb{T}^d} [L(x, 0) + b_F - F].
\]

Thus, it suffices to take \( C_\tau = \max_{x \in \mathbb{T}^d} L(x, 0) + b_F - a_F \).

Now, we consider the case where \( F(m) = \log m \). From the convexity of the exponential and the fact that \( \mu \in C_\tau \), we have

\[
\mu(z) = (4\pi\tau)^{-d/2} \sum_{k \in \mathbb{Z}^d} \int_{\mathbb{T}^d \times \mathbb{R}^d} \exp \left( -\frac{|x + \tau q - z + k|^2}{4\tau} \right) \, d\tilde{\mu}(x, q)
\]

\[
\geq (4\pi\tau)^{-d/2} \sum_{k \in \mathbb{Z}^d} \exp \left( \int_{\mathbb{T}^d \times \mathbb{R}^d} -\frac{|x + \tau q - z + k|^2}{4\tau} \, d\tilde{\mu}(x, q) \right)
\]

\[
\geq (4\pi\tau)^{-d/2} \sum_{k \in \mathbb{Z}^d} \exp \left( \int_{\mathbb{T}^d \times \mathbb{R}^d} \frac{-(|k|^2 + 1)^2}{2\tau} - \frac{\tau}{2} |q|^2 \, d\tilde{\mu}(x, q) \right)
\]

\[
= \exp \left( -\int_{\mathbb{T}^d \times \mathbb{R}^d} \frac{\tau}{2} |q|^2 \, d\tilde{\mu}(x, q) \right) B_\tau,
\]

where

\[
B_\tau := (4\pi\tau)^{-d/2} \sum_{k \in \mathbb{Z}^d} e^{-\frac{(|k|^2 + 1)^2}{4\tau}}.
\]
From (14), we have
\[ \log \mu(z) \geq -\frac{\tau}{2} \left( \int_{T^d \times \mathbb{R}^d} c_1 L \, d\tilde{\mu} + c_2 \right) + \log B_{\tau}. \] (17)

Therefore, by (16) for \( \tilde{\nu} \in \Psi_{\tau}(\tilde{\mu}) \), we get
\[ \int_{T^d \times \mathbb{R}^d} L \, d\tilde{\nu} + \log B_{\tau} - \frac{\tau}{2} \left( \int_{T^d \times \mathbb{R}^d} c_1 L \, d\tilde{\mu} + c_2 \right) \leq -\rho^\mu \leq \max_{x \in T^d} L(x, 0). \]

Consequently,
\[ \int_{T^d \times \mathbb{R}^d} L \, d\tilde{\nu} \leq \max_{x \in T^d} L(x, 0) - \log B_{\tau} + \frac{\tau}{2} \left( c_1 \int_{T^d \times \mathbb{R}^d} L \, d\tilde{\mu} + c_2 \right). \]

For \( \tau < 2/c_1 \), let
\[ C_{\tau} = \left( \max_{x \in T^d} L(x, 0) - \log B_{\tau} + \frac{\tau}{2} c_2 \right)/(1 - \frac{\tau}{2} c_1). \]

It follows from (17) that \( F|_{T^d \times A_{\tau}} \) is bounded from below. If
\[ \int_{T^d \times \mathbb{R}^d} L \, d\tilde{\mu} \leq C_{\tau}, \]
then
\[ \int_{T^d \times \mathbb{R}^d} L \, d\tilde{\nu} \leq C_{\tau} (1 - \frac{\tau}{2} c_1) + \frac{\tau}{2} c_1 C_{\tau} = C_{\tau} \]
and so \( \Psi_{\tau}(A_{\tau}) \subset 2^{A_{\tau}} \). \qed

**Lemma 2.** Assume the hypothesis of Proposition 2. Then, the map \( \Psi_{\tau} \) has a fixed point.

**Proof.** Let \( C_{\tau} > 0 \) and \( A_{\tau} \subset C_{\tau} \) be given by Proposition 2. Consider the map \( \Psi_{\tau} : A_{\tau} \to 2^{A_{\tau}} \). The set \( A_{\tau} \) is convex. Moreover, according to Remark 3, \( A_{\tau} \) is also compact. Thus, to apply Kakutani’s theorem, it suffices to check that \( \Psi_{\tau} \) has a closed graph. For this, let \( \tilde{\mu}_n, \tilde{\nu}_n \) be sequences in \( A_{\tau} \) converging to \( \tilde{\mu} \) and \( \tilde{\nu} \) respectively such that \( \tilde{\nu}_n \in \Psi(\tilde{\mu}_n) \).

Hence, for any \( \tilde{\eta} \in C_{\tau} \), we have
\[ \int_{T^d \times \mathbb{R}^d} L(x, q) + F(x, \mu_n) \, d\tilde{\nu}_n(x, q) \leq \int_{T^d \times \mathbb{R}^d} L(x, q) + F(x, \mu_n) \, d\tilde{\eta}(x, q). \]

Note that the sequences of smooth functions \( \mu_n = \text{Pr}_{1\#} \tilde{\mu}_n, \nu_n = \text{Pr}_{1\#} \tilde{\nu}_n \) converge pointwise to \( \mu = \text{Pr}_{1\#} \tilde{\mu} \) and \( \nu = \text{Pr}_{1\#} \tilde{\nu} \), respectively.

Since \( F|_{T^d \times A_{\tau}} \) is bounded and \( 0 \leq \nu_n \leq (4\pi \tau)^{-\frac{d}{2}} \), by dominated convergence,
\[ \lim_{T^d} \int F(x, \mu_n) d\nu_n(x) = \int_{T^d} F(x, \mu) d\nu(x), \quad \lim_{T^d} \int F(x, \mu_n) d\eta = \int_{T^d} F(x, \mu) d\eta. \]

Therefore, recalling that \( L \) is bounded from below, we deduce
\[ \int_{T^d \times \mathbb{R}^d} L + F(x, \mu) \, d\tilde{\nu} \leq \liminf \int_{T^d \times \mathbb{R}^d} L + F(x, \mu_n) \, d\tilde{\nu}_n \leq \int_{T^d \times \mathbb{R}^d} L + F(x, \mu) \, d\tilde{\eta}, \]
showing that \( \tilde{\nu} \in \Psi_{\tau}(\mu) \). Thus, \( \Psi_{\tau} \) has a closed graph. Hence, by Kakutani fixed-point Theorem, it follows that there is \( \tilde{\mu} \in A_{\tau} \) such that \( \tilde{\mu} \in \Psi(\tilde{\mu}) \). \qed
Now, we are ready to prove Theorem 1.

Proof of Theorem 1. The existence and uniqueness of solutions to the $\tau$–discrete MFG, (2), follows from Lemmas 1 and 2. To complete the proof, it remains to prove (3) for the local case when $F(m) = \log(m)$. First, we claim that for any $\tilde{\mu} \in \mathcal{C}_\tau$ with $\mu = \Pr_{1#\tilde{\mu}}$ and $\phi \in C(\mathbb{T}^d)$, the following inequality holds

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} L + \log(\mu) \, d\tilde{\mu} \geq -\log \int_{\mathbb{T}^d} e^{(\mathcal{L}_\tau \phi(x) - \phi(x))/\tau} \, dx. \quad (18)$$

To verify (18), let $\tilde{\mu} \in \mathcal{C}_\tau$ with $\mu = \Pr_{1#\tilde{\mu}}$ and $\phi \in C(\mathbb{T}^d)$. We have

$$\int L(x, v) d\tilde{\mu} = \int \left( L(x, q) - \frac{\phi(x) - \eta^* \phi(x + \tau q)}{\tau} \right) d\mu \geq -\int \frac{\phi(x) - \mathcal{L}_\tau \phi(x)}{\tau} d\mu.$$

Define

$$a(m, \phi) = \int \left( \frac{\mathcal{L}_\tau \phi(x) - \phi(x)}{\tau} - \log(m) \right) \, dm,$$

for any absolutely continuous Borel probability measure $m$ in $\mathbb{T}^d$. Then

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} L + \log(\mu) \, d\tilde{\mu} \geq -a(\mu, \phi).$$

Let

$$\lambda_\phi = \log \int e^{(\mathcal{L}_\tau \phi(x) - \phi(x))/\tau} \, dx,$$

and

$$m_\phi(x) = e^{(\mathcal{L}_\tau \phi(x) - \phi(x))/\tau - \lambda_\phi}. \quad (19)$$

Note that $a(m_\phi, \phi) = \lambda_\phi$. Recall that $t - t \log t \leq 1$ for all $t > 0$. Therefore, for any probability measure $m$ in $\mathbb{T}^d$, absolutely continuous with respect to the Lebesgue, we have

$$m \log(m) - m_\phi \log(m_\phi) \geq (1 + \log(m_\phi))(m - m_\phi).$$

Consequently, we obtain

$$a(m, \phi) \leq a(m_\phi, \phi) + \int \left( \frac{\mathcal{L}_\tau \phi(x) - \phi(x)}{\tau} - \log(m_\phi) - 1 \right)(m(x) - m_\phi(x)) \, dx.$$

Recalling that $m$ and $m_\phi$ are probability measures (19) implies that the second term on the right-hand side vanishes. Thus,

$$\lambda_\phi = \sup_m a(m, \phi).$$

This implies (18). Therefore,

$$\inf_{\mu \in \mathcal{C}_\tau} \int_{\mathbb{T}^d \times \mathbb{R}^d} L + \log(\mu) \, d\tilde{\mu} \geq -\log \inf_{\phi \in C(\mathbb{T}^d)} \int_{\mathbb{T}^d} e^{(\mathcal{L}_\tau \phi(x) - \phi(x))/\tau} \, dx. \quad (20)$$
From the first equation in (2), for \( \mu = \Pr_{1 \# \tilde{\mu}} \), we have
\[
\exp\left( \frac{\mathcal{L}_\tau u_\tau(x) - u_\tau(x)}{\tau} \right) = e^{\rho_\tau \mu_\tau}.
\]
Thus, from (20), we get
\[
\inf_{\mu \in \mathcal{C}_{\tau}} \int_{T^d \times \mathbb{R}^d} L + \log(\mu) \, d\tilde{\mu} \geq -\log \inf_{\phi \in \mathcal{C}(T^d)} \int_{T^d} e^{(\mathcal{L}_\tau \phi(x) - \phi(x))/\tau} \geq -\log \int_{T^d} e^{(\mathcal{L}_\tau u_\tau(x) - u_\tau(x))/\tau} = -\log \int_{T^d} e^{\rho_\tau \mu_\tau} = -\rho_\tau = \int_{T^d \times \mathbb{R}^d} L + \log(\mu_\tau) \, d\tilde{\mu}_\tau.
\]

**Remark 4.** Let \((\rho, u, \tilde{\mu})\) with \(\mu = \pi_{1 \# \tilde{\mu}}\) solve the \(\tau\)-discrete MFG, (2), and let \(\mathbb{V}_u(x) := \arg \max[(\eta^\tau * u)(x + \tau q) - \tau L(x, q)]\). Since
\[
0 = \int (\eta^\tau * u)(x + \tau q) - u(x) \, d\tilde{\mu} \leq \int \tau (L(x, q) + F(\mu(x)) + \rho) \, d\tilde{\mu} = 0,
\]
we have that \(\tilde{\mu}\) is supported in the set \(\{(x, q) : q \in \mathbb{V}_u(x)\}\).

**Remark 5.** For a solution \((\rho_\tau, u_\tau, \tilde{\mu}_\tau)\) of the \(\tau\)-discrete MFG, (2), we have
\[
\min_{T^d \times \mathbb{R}^d} L + a_F \leq -\rho_\tau \leq \max_{x \in T^d} L(x, 0) + b_F
\]
and
\[
\min_{T^d \times \mathbb{R}^d} L \leq \int_{T^d \times \mathbb{R}^d} L \, d\tilde{\mu} \leq \max_{x \in T^d} L(x, 0) + b_F - a_F.
\]

Thus, by Remark 3, we have that
\[
\{ (\rho_\tau, \tilde{\mu}_\tau) : (\rho_\tau, u_\tau, \tilde{\mu}_\tau) \text{ solves (2)} \}
\]
is precompact in \(\mathbb{R} \times \mathcal{P}_T\).

6. Convergence

Now, we study the convergence of solutions of the discrete MFG problem, (2) to solutions to the MFG system, (1). We begin with proving some preliminary results, which are needed to prove Theorems 2 and 3. We treat nonlocal and local cases separately. First, in Subsection 6.2, we examine the nonlocal case. We begin by proving the uniformly semi-convexity of solutions of the discrete problem. This estimate leads to the proof of Theorem 2. In Subsection 6.3, we address the local case. There, using the hypercontractivity of the Hamilton-Jacobi equation, we establish uniform bounds for the solutions of the discrete MFG problem in the local case. Finally, relying on these bounds and using Minty’s method, in Subsection 6.4, we obtain Theorem 3.
6.1. Preliminary Computations. In this section, we prove several results, which are needed to get convergence of the solutions of discrete MFG (2) to solutions of the ergodic problem (1).

We begin by recalling the following Lemma from [19].

**Lemma 3.** [19] Let $\varphi \in C^2(\mathbb{T}^d)$. For every $R > 0$, there exists a non-decreasing continuous function $\omega : [0, +\infty) \to [0, +\infty)$ vanishing at 0, only depending on $R$ and $\varphi$, such that

$$\left| \frac{(\eta^* \ast \varphi)(x + \tau q) - \varphi(x)}{\tau} - (D\varphi(x), q) - \Delta \varphi(x) \right| \leq \omega(\tau),$$

for all $(x, q) \in \mathbb{T}^d \times B_R$ and $\tau > 0$.

**Proposition 3.** Assume that $L$ satisfies (A1). Let $\varphi \in C^2(\mathbb{T}^d)$. Then, there exist $t_\varphi > 0$ and $V_{\varphi} \in C^2_1([0, t_\varphi) \times \mathbb{T}^d, \mathbb{R}^d)$ such that $V_{\varphi}(x, 0) = D_p H(x, D\varphi(x))$ and

$$\mathcal{L}_\tau \varphi(x) = (\eta^* \ast \varphi)(x + \tau V_{\varphi}(\tau, x)) - \tau L(x, V_{\varphi}(\tau, x)) \text{ for } \tau < t_\varphi, x \in \mathbb{T}^d.$$

**Proof.** We aim to prove that for $\tau$ small, $V_{\varphi}(x) = \arg \max (\eta^* \ast \varphi)(x + \tau q) - \tau L(x, q)$ is a singleton that defines a $C^2_1$ map. To do so, we define the map $\Phi : [0, +\infty) \times \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}^d$ by

$$\Phi(\tau, x, q) = D(\eta^* \ast \varphi)(x + \tau q) - D_q L(x, q).$$

Then,

$$\Phi(0, x, D_p H(x, D\varphi(x))) = 0,$$

$$\Phi_q(\tau, x, q) = \tau D^2(\eta^* \ast \varphi)(x + \tau q) - D_{qq}^2 L(x, q).$$

Since $D^2(\eta^* \ast \varphi)(x) = (\eta^* \ast D^2 \varphi)(x)$ and $L(x, q)$ satisfies (A1), there is $s_\varphi > 0$ such that $\Phi_q(\tau, x, q)$ is strictly negative definite for all $\tau < s_\varphi$ and $(x, q) \in \mathbb{T}^d \times \mathbb{R}^d$. Therefore, the map $q \mapsto \eta^* \ast \varphi(x + \tau q) - \tau L(x, q)$ is concave for any $\tau < s_\varphi$, $x \in \mathbb{T}^d$. Thus, for any $\tau < s_\varphi$, $x \in \mathbb{T}^d$, the solution $q = V_{\varphi}(\tau, x)$ to $\Phi(\tau, x, q) = 0$ is the maximizer of this map. The compactness of $\mathbb{T}^d$ and the implicit function theorem imply that there is $0 < t_\varphi \leq s_\varphi$ such that $V_{\varphi} \in C^2_1([0, t_\varphi) \times \mathbb{T}^d, \mathbb{R}^d)$ and $V_{\varphi}(0, x) = D_p H(x, D\varphi(x))$. Furthermore, from the definition of the Legendre transform, we have $V_{\varphi}(\tau, x) = D_p H(x, D(\eta^* \ast \varphi)(x + \tau V_{\varphi}(\tau, x))).$

**Corollary 1.** Assume $L$ satisfies (A1). Let $\varphi \in C^2(\mathbb{T}^d)$, then

$$\lim_{\tau \to 0} \| \frac{\mathcal{L}_\tau \varphi(x) - \varphi(x)}{\tau} - \Delta \varphi(x) - H(x, D\varphi) \|_{C^0} = 0.$$

**Proof.** Let $V \in C^2([0, t_\varphi) \times \mathbb{T}^d, \mathbb{R}^d)$ be the map given in Proposition 3 and $R = \max |V_{\varphi}|$. Let $\omega : [0, +\infty) \to [0, +\infty)$ be the function given by Lemma 3. Then

$$\left| \frac{\mathcal{L}_\tau \varphi(x) - \varphi(x)}{\tau} - \Delta \varphi(x) - (D\varphi, V_{\varphi}(\tau, x)) + L(x, V_{\varphi}(\tau, x)) \right| \leq \omega(\tau). \quad (22)$$
Since $V_\varphi(0, x) = D_\rho H(x, D\varphi(x))$

$$
\langle D\varphi, V_\varphi(\tau, x) \rangle - L(x, V_\varphi(\tau, x)) - H(x, D\varphi(x)) = \\
\langle D\varphi(x), V_\varphi(\tau, x) - V_\varphi(0, x) \rangle - L(x, V_\varphi(\tau, x)) + L(x, V_\varphi(0, x)).
$$

(23)

From (22), (23) the Corollary follows. $\square$

**Lemma 4.** Let $u \in C(\mathbb{T}^d)$ and suppose that $L$ satisfies (A1) and (A2). Then, $L_\tau u$ is semi-convex.

**Proof.** Fix $x \in \mathbb{T}^d$ and let $q \in \mathbb{R}^d$ be such that $L_\tau u(x) = -\tau L(x, q) + (\eta^\tau \ast u)(x + \tau q))$. Using (A2), for $h \in \mathbb{R}^d$, we have

$$
L_\tau u(x + h) - 2L_\tau u(x) + L_\tau u(x + h)
\geq -\tau L(x + h, q - \frac{h}{\tau}) + 2\tau L(x, q) - \tau L\left(x - h, q + \frac{h}{\tau}\right)
\geq -\tau k_1|h|^2 - k_2 \frac{|h|^2}{\tau} = -(\tau k_1 + k_2) |h|^2.
$$

(24)

6.2. The nonlocal case. Here, we examined the discrete MFG (2) with a nonlocal $F$ and prove Theorem 2. In the following Lemma, we establish a fundamental estimate, the uniform semiconvexity of the approximations.

**Lemma 5.** Under the hypothesis of Proposition 2 and assuming that $L$ satisfies (A2) and that $F$ satisfies (B3), let $(\rho_\tau, u_\tau, \bar{\mu}_\tau)$ solve (2). Then, $(u_\tau)_{0<\tau<1}$ is uniformly semi-convex; that is, there is a constant, $C > 0$, independent of $\tau$, such that

$$
u_\tau(x + h) - 2u_\tau(x) + u_\tau(x - h) \geq -C|h|^2,
$$

(24)

for any $x, h \in \mathbb{R}^d$, $0 < \tau < 1$.

**Proof.** Fix $0 < \tau < 1$, and to simplify the notation, we denote the solution to (2) by $(\rho, u, \bar{\mu})$. Since $u = L_\tau u - \tau F(x, \mu) - \tau \rho$, Lemma 4 and (B3) imply that $u$ is semi-convex. To find the constant $C$ in (24), we define inductively a sequence $\{\Lambda_n\}_{n=0}^\infty$ of semi-convexity moduli of $u$, taking the modulus $\Lambda_0$ such that $\Lambda_0^2 > (k_2 + \Lambda_0)(k_1 + k_0)$. Assuming we have chosen $\Lambda_n$ such that

$$u(x + h) - 2u(x) + u(x - h) \geq -\Lambda_n|h|^2,
$$

we have

$$
(\eta^\tau \ast u)(x + h) - (\eta^\tau \ast u)(x) + (\eta^\tau \ast u)(x - h)
= \int_{\mathbb{T}^d} \eta^\tau (y) (u(x + h - y) - 2u(x - y) + u(x - h - y)) dy
\geq -\int_{\mathbb{T}^d} \eta^\tau (y) \Lambda_n|h|^2 dy = -\Lambda_n|h|^2.
$$

(25)
For $x \in \mathbb{T}^d$, let $q \in \mathbb{R}^d$ be such that $u(x) = -\tau L(x, q) - \tau F(x, \mu) + (\eta^\tau * u)(x + \tau q) - \tau \rho$. Using (25) and (B2), for $h \in \mathbb{R}^d$ and $0 \leq \theta \leq 1$, we get

$$u(x + h) - 2u(x) + u(x - h) \geq -\tau \left[ L(x + h, q - \theta \frac{h}{\tau}) - 2L(x, q) + L(x - h, q + \theta \frac{h}{\tau}) \right] - \tau k_0|h|^2 + (\eta^\tau * u)(x + \tau q + (1 - \theta)h) - 2(\eta^\tau * u)(x + \tau q) + (\eta^\tau * u)(x + \tau q - (1 - \theta)h) \geq -\tau (k_1 + k_0)|h|^2 - k_2 \frac{\theta^2|h|^2}{\tau} - \Lambda_n(1 - \theta)^2|h|^2.$$ Optimizing over $\theta$, we obtain $\Lambda_{n+1}$ as

$$\Lambda_{n+1} = \tau (k_1 + k_0) + \frac{k_2 \Lambda_n}{k_2 + \Lambda_n \tau}.$$ Recalling that $\tau < 1$ and $\Lambda_0$ satisfies $\Lambda_0^2 > (k_2 + \Lambda_0)(k_1 + k_0)$, we have

$$\Lambda_1 = \frac{\tau (k_1 + k_0)(k_2 + \Lambda_0 \tau) + k_2 \Lambda_0}{k_2 + \Lambda_0 \tau} < \frac{\tau \Lambda_0^2 + k_2 \Lambda_0}{k_2 + \Lambda_0 \tau} = \Lambda_0.$$ Since

$$\Lambda_{n+1} - \Lambda_n = \frac{k_2 \Lambda_n}{k_2 + \Lambda_n \tau} - \frac{k_2 \Lambda_{n-1}}{k_2 + \Lambda_{n-1} \tau} = \frac{k_2^2(\Lambda_n - \Lambda_{n-1})}{(k_2 + \Lambda_n \tau)(k_2 + \Lambda_{n-1} \tau)},$$ we get by induction that $\{\Lambda_n\}$ is a decreasing sequence. In the limit, this procedure gives a universal modulus

$$\Lambda(\tau) = \frac{(k_1 + k_0)\tau + \sqrt{(k_1 + k_0)^2 \tau^2 + 4k_1k_2^2}}{2}.$$ Finally, we take $C = \Lambda(1)$. $\square$

A uniformly semi-convex family of functions on $\mathbb{T}^d$ is uniformly Lipschitz. Thus, by the preceding result, we have that $(u_\tau)_\tau$ is a uniformly Lipschitz family of functions.

**Proof of Theorem 2.** From Proposition 1, we know that the pair $(\rho_\tau, \mu_\tau)$ is unique. Let $\tau_n \to 0^+$. From Remark 5, by passing to a sub-sequence, we can assume that $(\rho_{\tau_n}, \mu_{\tau_n})$ converges to $(\rho, \mu) \in \mathbb{R} \times \mathcal{P}_\ell$, and from Proposition 4.2 in [20], $\mu \in \mathcal{C}$. Then $\mu_{\tau_n} \rightharpoonup \mu = \text{Pr}_{1\#} \tilde{\mu}$. Since $F$ is uniformly continuous, $F(x, \mu_{\tau_n})$ converges uniformly to $F(x, \mu)$ and so

$$\lim_{n \to \infty} \int_{\mathbb{T}^d} F(x, \mu_{\tau_n}) \, d\mu_{\tau_n}(x) = \int_{\mathbb{T}^d} F(x, \mu) \, d\mu(x).$$

Thus,

$$-\rho = \lim_{n \to +\infty} -\rho_{\tau_n} = \lim_{n \to +\infty} \int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, q) + F(x, \mu_{\tau_n}) \, d\mu_{\tau_n}(x, q)$$

$$= \int_{\mathbb{T}^d \times \mathbb{R}^d} L(x, q) + F(x, \mu) \, d\mu(x, q). \quad (26)$$
Because \((u_\tau)_\tau\) is uniformly Lipschitz, we can extract a subsequence if necessary, and therefore assume that \((u_{\tau_n})_n\) converges uniformly to some function \(u \in C(\mathbb{T}^d)\). We claim that \(u\) solves

\[
\Delta v + H(x, Dv(x)) = \rho + F(x, \mu) \tag{27}
\]

in the viscosity sense. This will be established by proving that \(u\) is both a viscosity sub-solution and a viscosity super-solution.

First, we prove that \(u\) is a sub-solution of \((27)\). For that, let \(\varphi \in C^2(\mathbb{T}^d)\) and suppose that \(x_0 \in \text{argmax}(u - \varphi)\) is a strict maximum point. Accordingly, we can find a sequence \((x_n)_n\) of points of maximum of \(u_{\tau_n} - \varphi\) that converges to \(x_0\) in \(\mathbb{T}^d\). Let \(\varepsilon_n := \max(u_{\tau_n} - \varphi)\) and define \(\varphi_n := \varphi + \varepsilon_n\). By construction, we have

\[
u_{\tau_n} \leq \varphi_n \text{ in } \mathbb{T}^d
\]

and, in addition, \(u_{\tau_n}(x_n) = \varphi_n(x_n)\).

Because \(\mathcal{L}_\tau u \geq \mathcal{L}_\tau v\) if \(u \geq v\), we conclude that

\[
\varphi_n(x_n) = u_{\tau_n}(x_n) = \mathcal{L}_{\tau_n} u_{\tau_n}(x_n) - \tau_n \rho_{\tau_n} - \tau_n F(x, \mu_n)
\]

\[
\leq \mathcal{L}_{\tau_n} \varphi_n(x_n) - \tau_n \rho_{\tau_n} - \tau_n F(x, \mu_n).
\]

Therefore, from the identity \(\varphi_n = \varphi + \varepsilon_n\), we obtain

\[
\frac{\mathcal{L}_{\tau_n} \varphi_n(x_n) - \varphi_n(x_n)}{\tau_n} \geq \rho_{\tau_n} + F(x, \mu_n).
\]

For each \(n \in \mathbb{N}\), we select \(q_n \in \mathbb{R}^d\) such that \(\mathcal{L}_{\tau_n} = (\eta^{\tau_n} * \varphi)(x_n + \tau_n q_n) - \tau_n L(x_n, q_n)\). Accordingly, the preceding inequality becomes

\[
\frac{(\eta^{\tau_n} * \varphi)(x_n + \tau_n q_n) - \varphi(x_n)}{\tau_n} - L(x_n, q_n) \geq \rho_{\tau_n} + F(x, \mu_n).
\]

Combining the estimate \(|D(\eta^{\tau_n} * \varphi)(x_n)| \leq \|D\varphi\|_\infty\) with fact (iii) in subsection 3.1, we conclude that there exists \(R > 0\) such that \(|q_n| \leq R\) for every \(n \in \mathbb{N}\). Thus, by extracting a further subsequence if necessary, we have that \(q_n\) converges to some \(q \in \mathbb{R}^d\). By considering the limit \(n \to +\infty\) in \((28)\), we obtain

\[
\Delta \varphi(x_0) + \langle D\varphi(x_0), q \rangle - L(x_0, q) \geq \rho + F(x_0, \mu).
\]

Thus,

\[
\Delta \varphi(x_0) + H(x_0, D\varphi(x_0)) \geq \rho + F(x_0, \mu),
\]

Therefore, \(u\) is a viscosity sub-solution to \((27)\).

Now, we prove that \(u\) is a viscosity super-solution of \((27)\). For that, fix \(\varphi \in C^2(\mathbb{T}^d)\) and suppose that \(x_0 \in \text{argmin}(u - \varphi)\) is a strict minimum point. Because \(x_0\) is a strict minimum of \(u - \varphi\), we can find a sequence \((x_n)_n\) converging to \(x_0\) in \(\mathbb{T}^d\) such that \(x_n \in \text{argmin}(u_{\tau_n} - \varphi)\). Let \(\varepsilon_n := \min(u_{\tau_n} - \varphi)\) and set \(\varphi_n := \varphi + \varepsilon_n\). Arguing as in the first part, we end up with

\[
\frac{\mathcal{L}_{\tau_n} \varphi_n(x_n) - \varphi_n(x_n)}{\tau_n} \leq \rho_{\tau_n} + F(x, \mu_n).
\]
By the definition of $L_{\tau_n}$, for every fixed $q \in \mathbb{R}^d$, we have
\[
\frac{(\eta^{\tau_n} * \varphi)(x_n + \tau_n q) - \varphi(x_n)}{\tau_n} - L(x_n, q) \leq \rho_{\tau_n} + F(x, \mu_{\tau_n}).
\]
Thus, considering the limit $n \to +\infty$ and using Proposition 3, we conclude that
\[
\Delta \varphi(x_0) + \langle D\varphi(x_0), q \rangle - L(x_0, q) \leq \rho + F(x_0, \mu).
\]
Finally, taking the supremum with respect to $q \in \mathbb{R}^d$ in the prior inequality, we get
\[
\Delta \varphi(x_0) + H(x_0, D\varphi(x_0)) \leq \rho + F(x_0, \mu),
\]
using the definition of the Legendre transform. Consequently, $u$ is a viscosity supersolution to (27).

Applying Theorem 4 to the Lagrangian $L(x, q) + F(x, \mu)$, we conclude that (26) implies that
\[
\tilde{\mu} = G_{V\# \nu},
\]
where $\nu \in \mathcal{P}(T^d)$ solves (6) with $V(x) := D_p H(x, Du(x))$. Therefore, $\mu = \nu$, and so $(\rho, u, \mu)$ solves the ergodic MFG (1).

6.3. Bounds for the local case. In this part, we assume that $L$ satisfies (A3) and $F$ satisfies (B6). Let $(\rho_{\tau}, u_{\tau}, \tilde{\mu}_{\tau})$ solve the $\tau$-discrete MFG and set $\mu_{\tau} = \Pr_{1\#} \tilde{\mu}_{\tau}$. According to Remark 5, we have that
\[
\int_{T^d} F(\mu_{\tau}) \, d\mu_{\tau} = -\rho_{\tau} - \int_{T^d \times \mathbb{R}^d} L \, d\tilde{\mu}_{\tau},
\]
is uniformly bounded in $\tau$. Therefore, for $F(m) = m^a$, $0 < a < 1$, we have $F(m)m = F(m)^{1+1/a}$. Consequently, the $L^{1+1/a}$ norm of $F(\mu_{\tau})$ is uniformly bounded in $\tau$.

Define the operators
\[
\Gamma_{\tau}[u] := \eta^{\tau} * u,
\]
\[
\mathcal{N}_{\tau}[u](x) := \max_{v \in \mathbb{R}^d} \left[ u(x + \tau v) - \tau K(v) \right].
\]
Then, $\mathcal{L}_{\tau} u = \mathcal{N}_{\tau} \circ \Gamma_{\tau}[u] + \tau U(x)$. First, we observe that for any $\tau > 0$, $x \in T^d$
\[
u(x) \leq \mathcal{N}_{\tau}[u](x) \leq \max u.
\]
Moreover, if $u(x_0) = \max u$, we have $\mathcal{N}_{\tau}[u](x_0) \geq u(x_0)$ and so
\[
\mathcal{N}_{\tau}[u](x_0) = \max u.
\]
Because $\eta^{\tau}$ is a probability density and $s \to |s|^r$ is convex for $r > 1$, we have
\[
|\eta^{\tau} * u(x)|^r = \left| \int_{T^d} \eta^{\tau}(y - x)u(y) \, dy \right|^r \leq \int_{T^d} \eta^{\tau}(y - x)|u(y)|^r \, dy.
\]
Consequently, integrating the foregoing expression, we have
\[ \int_{\mathbb{T}^d} | \eta^r * u(x)|^r \, dx \leq \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} | \eta^r(y - x)|u(y)|^r \, dy \, dx \]
\[ = \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} | \eta^r(y - x)|u(y)|^r \, dx \, dy = \int_{\mathbb{T}^d} |u(y)|^r \, dy. \]
Thus, for any \( r \geq 1 \), we have
\[ \| \Gamma_\tau [u] \|_{L^r} \leq \| u \|_{L^r}. \] (29)
Define
\[ f^r := \mathcal{N}_\tau \circ \Gamma_\tau[u] - u. \] (30)
Then, we have \( f_\tau = \rho_\tau + F(\mu_\tau) - U \). Thus, \( \| f_\tau \|_{L^{1+1/a}} \) is bounded uniformly in \( \tau \). Note that (30) does not depend on the choice of the solution \( u_\tau \) as solutions are unique up to the addition of constants. Let \( v_\tau = u_\tau - \max \Gamma_\tau[u_\tau] \), then \( \max \Gamma_\tau[v_\tau] = 0 \). Thus, we have
\[ v_\tau = -\tau f^r + \mathcal{N}_\tau \circ \Gamma_\tau[v_\tau] \] (31)
Observe that \( w(t, x) = \mathcal{N}_t[u](x) \) solves the initial value problem
\[ \begin{cases} w_t - K(Dw) = 0 \\ w(0, x) = u(x) \end{cases}. \]
We have that \( \max \Gamma_\tau[v_\tau] = 0 \). Additionally, by performing a translation, we can assume that \( \Gamma_\tau[v_\tau](0) = 0 \). Letting \( W(t, x) = -\mathcal{N}_t \circ \Gamma_\tau[v_\tau](x) \), we have \( W(t, x) \geq 0 = W(t, 0) \). We deduce
\[ \frac{d}{dt} \int_{\mathbb{T}^d} W^r = -r \int_{\mathbb{T}^d} W^{r-1}K(-DW) \leq -cr \int_{\mathbb{T}^d} W^{r-1}|DW|^q \]
\[ = -cr \left( \frac{q}{r + q - 1} \right)^q \int_{\mathbb{T}^d} |DW|^{\frac{r+q-1}{q}} \] (32)
Proposition 4. Let \( q > d \), \( f \in W^{1,q}(\mathbb{T}^d) \) and suppose that \( f(0) = 0 \). Then, there exists a positive constant, \( C \), such that
\[ \int_{\mathbb{T}^d} |f|^q \, dx \leq C \int_{\mathbb{T}^d} |Df|^q \, dx. \] (33)
Proof. Suppose that the inequality in (33) is not true. Then, there exists a sequence \( \{ f^n \}_{n=1}^\infty \subset W^{1,q}(\mathbb{T}^d) \) satisfying \( f^n(0) = 0 \), \( \| f^n \|_{L^q(\mathbb{T}^d)} = 1 \) and
\[ \| Df^n \|_{L^q(\mathbb{T}^d)} \leq \frac{1}{n}. \] (34)
By Morrey theorem, we have that \( W^{1,q}(\mathbb{T}^d) \subset C^{0,\gamma}(\mathbb{T}^d) \), for some \( 0 < \gamma < 1 \); that is,
\[ \| \varphi \|_{C^{0,\gamma}(\mathbb{T}^d)} \leq C \| \varphi \|_{W^{1,q}(\mathbb{T}^d)}, \quad \varphi \in W^{1,q}(\mathbb{T}^d). \]
Therefore, \( \{f^n\}_{n=1}^{\infty} \subset C^{0,\gamma}(\mathbb{T}^d) \). The sequence \( \{f^n\} \) is equicontinuous and \( f^n(0) = 0 \). Hence, by the Arzela-Ascoli theorem, there is a subsequence \( f^{n_k} \) that converges uniformly to a function \( \bar{f} \) with \( \bar{f}(0) = 0 \). The sequence \( |f^{n_k}|^2 \) converges uniformly to \( |\bar{f}|^2 \). Thus,

\[
\int_{\mathbb{T}^d} |\bar{f}|^2 = 1. \tag{35}
\]

On the other hand, from (34), we have that \( D\bar{f} = 0 \). Hence, \( \bar{f} \) is constant and, thus, identically 0. This contradicts (35).

\[\square\]

From (32), Proposition 4, and \( \|w\|_{L^r} \leq \|w\|_{L^{r+q-1}} \), we obtain a constant \( C_r > 0 \) such that

\[
\frac{d}{dt} \int_{\mathbb{T}^d} W^r \, dx \leq -rC_r \int_{\mathbb{T}^d} W^{r+q-1} \, dx \leq -\frac{rC_r}{q-1} \left( \int_{\mathbb{T}^d} W^r \, dx \right)^{\frac{r+q-1}{q}}.
\]

Denoting \( h(t) = \int_{\mathbb{T}^d} W^r \, dx \), from the preceding estimate, we have

\[
\frac{h'(t)}{h^{\frac{q+1}{q-1}}(t)} \leq -\frac{rC_r}{q-1}.
\]

Integrating the previous inequality over \([0, \tau]\), we obtain

\[
-\frac{q-1}{r} \left( h^{\frac{q+1}{r}}(\tau) - h^{\frac{q+1}{r}}(0) \right) \leq -\frac{rC_r}{q-1}. \tag{36}
\]

Taking into account the definitions of \( h \) and \( W \), from (36), we obtain the estimate

\[
\|N_\tau \circ \Gamma_\tau[v_\tau]\|_{L^r}^{q-1} \leq \frac{\|\Gamma_\tau[v_\tau]\|_{L^r}^{q-1}}{1 + C_r \tau \|\Gamma_\tau[v_\tau]\|_{L^r}^{q-1}}. \tag{37}
\]

Because the function \( s \mapsto s/(1 + \tau C_r s) \) is increasing, from (29) we have

\[
\frac{\|\Gamma_\tau[v_\tau]\|_{L^r}^{q-1}}{1 + C_r \tau \|\Gamma_\tau[v_\tau]\|_{L^r}^{q-1}} \leq \frac{\|v_\tau\|_{L^r(\mathbb{T}^d)}^{q-1}}{1 + \tau C_r \|v_\tau\|_{L^r(\mathbb{T}^d)}^{q-1}}.
\]

Using the preceding inequality in (37), from (31), we deduce that

\[
\|v_\tau\|_{L^r(\mathbb{T})} \leq \tau \|f^\tau\|_{L^r(\mathbb{T})} + \frac{\|v_\tau\|_{L^r(\mathbb{T})}}{1 + \tau C_r \|v_\tau\|_{L^r(\mathbb{T})}},
\]

for \( d = 1 \) and \( q = 2 \). Therefore,

\[
\|v_\tau\|_{L^r(\mathbb{T})} \leq \frac{\tau C_r \|f^\tau\|_{L^r(\mathbb{T})} + \sqrt{(\tau C_r \|f^\tau\|_{L^r(\mathbb{T})})^2 + 4C_r \|f^\tau\|_{L^r(\mathbb{T})}}}{2C_r}.
\]
For $d > 2$, defining $p$ by $\frac{1}{q} + \frac{1}{p} = 1$, from (31), Minkowsky and Hölder inequalities, we have

$$\|v_\tau\|_{L^r(T^d)}^{q-1} \leq (\tau \|f^\tau\|_{L^r(T^d)} + \|N_\tau \circ \Gamma[v_\tau]\|_{L^r(T^d)})^{q-1}$$

$$\leq (\tau \|f^\tau\|_{L^r(T^d)} + \|N_\tau \circ \Gamma[v_\tau]\|_{L^r(T^d)})^{q-1} (\tau + 1)^{\frac{q-1}{p}}$$

which implies

$$\|v_\tau\|_{L^r(T^d)}^{q-1} \leq \frac{\tau}{2} (\tau + 1)^{\frac{q-1}{p}} \|f^\tau\|_{L^r(T^d)}^{q-1} + \frac{(\tau + 1)^{\frac{q-1}{p}} - 1}{2C_r \tau}$$

$$+ \sqrt{\left(\frac{\tau}{2} (\tau + 1)^{\frac{q-1}{p}} \|f^\tau\|_{L^r(T^d)}^{q-1} + \frac{(\tau + 1)^{\frac{q-1}{p}} - 1}{2C_r \tau}\right)^2 + \frac{(\tau + 1)^{\frac{q-1}{p}} \|f^\tau\|_{L^r(T^d)}^{q-1}}{C_r}}$$

(38)

Thus, recalling that $\|f_\tau\|_{L^{1+1/a}}$ is uniformly bounded in $\tau$, we deduce that for $r = 1+1/a$, $\|v_\tau\|_{L^r(T^d)}$ is bounded uniformly in $\tau \in (0, 1)$. Using (21), max $N_\tau \circ \Gamma[v_\tau] = 0$ and $F(m) \geq 0$, we also have

$$v_\tau = \tau(U - \rho_r - F \circ \mu_\tau) + N_\tau \circ \Gamma[v_\tau] \leq \tau(\max U - \min U + F(1)).$$

6.4. The weak solutions for local coupling. In this subsection, using the monotonicity properties of the MFG, we define weak solutions (see Definition 5) for the ergodic MFG system (1) and for the $\tau$-discrete MFG problem (2) in the local case. Then, we verify that solutions of the $\tau$-discrete MFG problem (2) are also weak solutions. Next, relying on the a priori estimate of the previous subsection and using Minty’s method (see [22]), we conclude that our approximations (up to a normalization) converge weakly to a weak solution to (1). Finally, using an unpublished result due to Vardan Voskanyan, we prove that any weak solution of (1) is a classical solution. Therefore, normalized solutions of the $\tau$-discrete MFG system (2) weakly converge to a classical solution of the MFG system (1).

Let $H^\tau : C(T^d) \to C(T^d)$ be

$$H^\tau u(x) = (L_{\tau} u(x) - u(x))/\tau.$$

Next, we prove several properties of $H^\tau$, which are crucial for our study of the weak solutions to the MFG system (1) in the sense of Definition 5.

**Proposition 5.** Fix $x \in T^d$. Then, the map $u \mapsto H^\tau u(x)$ is convex. Moreover, let $\partial_u H^\tau(x)$ be the subdifferential of $H^\tau$ at $u$. Consider the set

$$V_u(x) := \arg \max[\eta^\tau * u](x + \tau q) - \tau L(x, q)].$$
Then, for \( w \in C(\mathbb{T}^d) \) and \( v \in \mathbb{V}_u(x) \), the functional \( w \mapsto ((\eta^\tau * w)(x + \tau v) - w(x))/\tau \) belongs to \( \partial_u \mathcal{H}_\tau(x) \).

Proof. The convexity of \( \mathcal{H}_\tau \) follows from the inequality

\[
\tau \mathcal{H}_\tau(\lambda u_1 + (1 - \lambda) u_2)(x) = \max_q \lambda (\eta^\tau * u_1(x + \tau q) - u_1(x) - \tau L(x, q)) + (1 - \lambda) (\eta^\tau * u_2(x + \tau q) - u_2(x) - \tau L(x, q)) \leq \lambda \tau \mathcal{H}_\tau(u_1) + (1 - \lambda) \tau \mathcal{H}_\tau(u_2),
\]

for any \( u_1, u_2 \in C(\mathbb{T}^d) \) and \( 0 \leq \lambda \leq 1 \). Note that for any \( w \in C(\mathbb{T}^d) \), we have

\[
\mathcal{L}_\tau(u + w)(x) \geq (\eta^\tau * (u + w))(x + \tau v) - L(x, v) = \mathcal{L}_\tau u(x) + (\eta^\tau * w)(x + \tau v),
\]

because \( v \in \mathbb{V}_u(x) \). Therefore,

\[
\tau \mathcal{H}_\tau(u + w)(x) \geq \tau \mathcal{H}_\tau(u)(x) + (\eta^\tau * w)(x + \tau v) - w(x).
\]

Thus, the linear map \( w \mapsto ((\eta^\tau * w)(x + \tau v) - w(x))/\tau \) belongs to \( \partial_u \mathcal{H}_\tau(x) \). \( \square \)

For \( u \in C(\mathbb{T}^d) \) consider a Borel measurable map \( V : \mathbb{T}^d \to \mathbb{R}^d \) such that \( V(x) \in \mathbb{V}_u(x) \) for all \( x \in \mathbb{T}^d \). For each \( x \in \mathbb{T}^d \), the linear map \( \zeta_V(x) : C(\mathbb{T}^d) \to \mathbb{R} \), defined by \( \zeta_V(x)w = ((\eta^\tau * w)(x + \tau V(x)) - w(x))/\tau \) belongs to \( \partial_u \mathcal{H}_\tau(x) \). For \( m \in \mathcal{P}(\mathbb{T}^d) \), we define \( \zeta^*_V m \in C(\mathbb{T}^d)^* \) by

\[
\langle \zeta^*_V m, w \rangle = \int_{\mathbb{T}^d} \zeta_V(x)w \, dm(x),
\]

and let

\[
\partial_u \mathcal{H}_\tau^* m = \{ \zeta^*_V m \mid V : \mathbb{T}^d \to \mathbb{R}^d \text{ Borel measurable}, \forall x \in \mathbb{T}^d, \, V(x) \in \mathbb{V}_u(x) \}.
\]

Next, we define the multivalued operator \( A^\tau : D(A^\tau) \subset \mathbb{R} \times L^1(\mathbb{T}^d) \times L^1(\mathbb{T}^d) \to \mathbb{R} \times L^1(\mathbb{T}^d) \times L^1(\mathbb{T}^d) \)

\[
A^\tau \begin{bmatrix} \rho \\ u \\ m \end{bmatrix} = \begin{bmatrix} 1 - \int_{\mathbb{T}^d} dm \\ \mathcal{H}_\tau^* m \\ -\mathcal{H}_\tau u + F(m) + \rho \end{bmatrix}, \quad (\rho, u, m) \in D(A^\tau), \tag{41}
\]

where \( D(A^\tau) = \mathbb{R} \times C(\mathbb{T}^d) \times \{ m \in L^1(\mathbb{T}^d) : m \geq 0 \} \).

Relying on Proposition 5, we prove the monotonicity of \( A^\tau \), which is crucial for defining weak solutions.

**Proposition 6.** Suppose that Assumption (B4) holds and that \((\rho_1, u_1, m_1), (\rho_2, u_2, m_2) \in D(A^\tau)\). Let \((\sigma_1, \nu_1, v_1) \in A^\tau(\rho_1, u_1, m_1)\) and \((\sigma_2, \nu_2, v_2) \in A^\tau(\rho_2, u_2, m_2)\). Then,

\[
\left\langle \begin{bmatrix} \sigma_1 - \sigma_2 \\ \nu_1 - \nu_2 \end{bmatrix}, \begin{bmatrix} \rho_1 - \rho_2 \\ u_1 - u_2 \\ m_1 - m_2 \end{bmatrix} \right\rangle_{((\mathbb{R} \times L^1(\mathbb{T}^d)) \times L^1(\mathbb{T}^d))^* \times D(A^\tau)}} \geq 0. \tag{42}
\]
Proof. The proof results from the following computations. For \( i = 1, 2 \), we have that \((\sigma_i, \nu_i, v_i) \in A^\tau(\rho_i, u_i, m_i) \), \( \nu_i = \zeta_i^* m_i \), \( V_i : T^d \to \mathbb{R}^d \) Borel measurable, \( V_i(x) \in \mathbb{V}_u(x) \). Therefore, recalling that the map \( u \mapsto H_\tau u(x) \) is convex (see Proposition 5) and taking into account that \( F \) is increasing (see Assumption (B4)), we deduce

\[
\left\langle \begin{bmatrix} \sigma_1 - \sigma_2 \\ \nu_1 - \nu_2 \\ v_1 - v_2 \end{bmatrix}, \begin{bmatrix} \rho_1 - \rho_2 \\ u_1 - u_2 \\ m_1 - m_2 \end{bmatrix} \right\rangle = \int_{T^d} (H_\tau(u_2) - H_\tau(u_1)) \, d(m_1 - m_2)
+ \int_{T^d} (F(m_1) - F(m_2) + \rho_1 - \rho_2) \, d(m_1 - m_2)
+ \int_{T^d} (u_1 - u_2) \, d(\nu_1 - \nu_2) + (\rho_1 - \rho_2) \int_{T^d} d(m_2 - m_1)
= \int_{T^d} (H_\tau(u_2)(x) - H_\tau(u_1)(x) + \zeta_{V_i}(x)(u_1 - u_2)) \, dm_1
+ \int_{T^d} (H_\tau(u_1)(x) - H_\tau(u_2)(x) + \zeta_{V_2}(x)(u_2 - u_1)) \, dm_2
+ \int_{T^d} (F(m_1) - F(m_2)) \, d(m_1 - m_2) \geq 0. \quad \Box
\]

Note that by the monotonicity of the operator \( A^\tau \), we mean that \( A^\tau \) satisfies (42). Setting

\[
A^0 \begin{bmatrix} \rho \\ u \\ m \end{bmatrix} = \begin{bmatrix} 1 - \int_{T^d} dm \\ \Delta m - \text{div}(m D_y H(x, Du)) \\ -\Delta u - H(x, Du) + F(m) + \rho \end{bmatrix},
\]

and using (41), we can write the MFG system (1) and the \( \tau \)-discrete MFG system (2) as

\[
A^\tau \begin{bmatrix} \rho \\ u \\ m \end{bmatrix} = 0, \quad \tau \geq 0. \quad (43)
\]

Now, using the concept of the weak solutions for the monotone MFGs developed in the series of papers [23, 24, 25, 10], we define the weak solutions to (43).

**Definition 3.** We say that \((\rho, u, \mu) \in \mathbb{R} \times L^1(T^d) \times L^1(T^d) \) with \( \mu \geq 0 \) is a weak solution to (43) if there is \( k > 1 \) such that \( u \in L^k(T^d) \) and for any \( m, \varphi \in C^2(T^d) \), \( \lambda \in \mathbb{R} \) with \( m \geq 0 \), and any \((\sigma, \nu, v) \in A^\tau(\lambda, \varphi, m) \), we have

\[
\left\langle \begin{bmatrix} \sigma \\ \nu \\ v \end{bmatrix}, \begin{bmatrix} \lambda - \rho \\ \varphi - u \\ m - \mu \end{bmatrix} \right\rangle \geq 0. \quad ((\mathbb{R} \times L^k(T^d) \times L^1(T^d))^*, \mathbb{R} \times L^k(T^d) \times L^1(T^d))
\]

The following proposition verifies that solutions of \( \tau \)-discrete MFG system (2) are weak solutions to (43) for \( \tau > 0 \).
Proposition 7. Suppose that \( L \) satisfies (A1) and \( F \) satisfies (B4) and (B5). Then, a solution \((\rho_\tau, u_\tau, \mu_\tau) \in \mathbb{R} \times C(\mathbb{T}^d) \times \mathcal{C}_\tau\) to (2) with \( \tau > 0 \), is a weak solution to (43).

Proof. For simplicity, we omit the subscript \( \tau \) for \((\rho_\tau, u_\tau, \mu_\tau)\). Let \( \varphi, m \in C^2(\mathbb{T}^d), \lambda \in \mathbb{R} \). Since \( \mu \) is supported in \( \{(x, q) : q \in \mathcal{V}_u(x)\} \), we have from (39) and (40) that

\[
\int_{\mathbb{T}^d} \mathcal{H}_\tau(\varphi) \, d\mu \geq \int_{\mathbb{T}^d} \mathcal{H}_\tau(u) \, d\mu + \int_{\mathbb{T}^d} \frac{(\eta^\tau_\varphi - u)(x + \tau q) - (\varphi - u)(x)}{\tau} \, d\mu = \int_{\mathbb{T}^d} \mathcal{H}_\tau(u) \, d\mu.
\]

For \((\sigma, \nu, v) \in A^\tau(\lambda, \varphi, m), \nu = \zeta^*_\nu m, V : \mathbb{T}^d \to \mathbb{R}^d\) Borel measurable, \( V(x) \in \mathcal{V}_\varphi(x) \), we have

\[
\left\langle \begin{bmatrix} \sigma \\ \nu \\ \nu \end{bmatrix}, \begin{bmatrix} \lambda - \rho \\ \varphi - u \\ m - \mu \end{bmatrix} \right\rangle = \int_{\mathbb{T}^d} (-\mathcal{H}_\tau \varphi + F(m) + \lambda) \, d(m - \mu) + \int_{\mathbb{T}^d} \zeta_V(\varphi - u) \, dm + (\lambda - \rho) \int_{\mathbb{T}^d} d(\mu - m)
= \int_{\mathbb{T}^d} (\zeta_V(\varphi - u) - \mathcal{H}_\tau \varphi) \, dm + \int_{\mathbb{T}^d} \mathcal{H}_\tau \varphi \, d\mu + \int_{\mathbb{T}^d} F(m) \, d(m - \mu) - \rho \int_{\mathbb{T}^d} d(\mu - m)
\geq \int_{\mathbb{T}^d} (-\mathcal{H}_\tau u + F(m) + \rho) \, d(m - \mu)
= \int_{\mathbb{T}^d} (F(m) - F(\mu)) \, d(m - \mu) \geq 0. \qedhere
\]

We aim to prove the weak convergence of solutions to (2). For that purpose, we need the following proposition.

Proposition 8. Suppose that \( L \) satisfies (A1) and let \( \varphi, m \in C^2(\mathbb{T}^d), \rho \in \mathbb{R} \). Then,

\[
\lim_{\tau \to 0} A^\tau \begin{bmatrix} \rho \\ \varphi \\ m \end{bmatrix} = A^0 \begin{bmatrix} \rho \\ \varphi \\ m \end{bmatrix}.
\]

Proof. According to Corollary 1, we have

\[
\lim_{\tau \to 0} -\mathcal{H}_\tau \varphi + F(m) + \rho = -\Delta \varphi - H(x, D\varphi) + F(m) + \rho,
\]

where the convergence is uniform in \( x \). Let \( t_\varphi \) and \( V_\varphi \) be given by Proposition 3 and \( R = \max |V_\varphi| \).

Next, we compute explicitly \( \partial_\varphi \mathcal{H}^*_\tau \). For \((\tau, x) \in (0, t_\varphi) \times \mathbb{T}^d\), we have

\[
\partial_\varphi \mathcal{H}^*_\tau m = \zeta^*_\varphi m.
\]
\[
\langle \partial_\varphi \mathcal{H}^*_\tau m, w \rangle = \int_{\mathbb{T}^d} \zeta_{V_\varphi}(x) w \, dm(x) = \int_{\mathbb{T}^d} \eta^\tau * w(x + \tau V_\varphi(\tau, x)) - w(x) \, m(x) \, dx.
\]

Note that
\[
\mathcal{I} := \int_{\mathbb{T}^d} \eta^\tau * w(x + \tau V_\varphi(\tau, x)) m(x) \, dx = \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \eta^\tau(y) w(x + \tau V_\varphi(\tau, x) - y) \, dy \, m(x) \, dx
\]
\[
= \int_{\mathbb{T}^d} \eta^\tau(y) \int_{\mathbb{T}^d} w(x + \tau V_\varphi(\tau, x) - y) m(x) \, dx \, dy.
\]

Let \( \Phi_\tau = \Phi(\tau, \cdot) \) be the inverse map of \( I + \tau V_\varphi(\tau, \cdot) \). The implicit function theorem and the compactness of \( \mathbb{T}^d \) imply that there is \( \tau_0 \) such that \( \Phi_\tau \) is well defined for \( \tau < \tau_0 \) and \( \Phi \) is smooth in \( \tau \). Therefore, we can change variables in \( \mathcal{I} \) such that \( z = x + \tau V_\varphi(\tau, x) - y \) and \( x = \Phi_\tau(y + z) \). Hence,
\[
\mathcal{I} = \int_{\mathbb{T}^d} \eta^\tau(y) \int_{\mathbb{T}^d} w(z) m(\Phi_\tau(y + z)) \det D \Phi_\tau(y + z) \, dz \, dy
\]
\[
= \int_{\mathbb{T}^d} w(z) \int_{\mathbb{T}^d} \eta^\tau(y) m(\Phi_\tau(y + z)) \det D \Phi_\tau(y + z) \, dy \, dz
\]
\[
= \int_{\mathbb{T}^d} w(z) \eta^\tau * (m \circ \Phi_\tau \det D \Phi_\tau)(z) \, dz.
\]

Setting \( \Psi(\tau, z) = \eta^\tau * (m \circ \Phi_\tau \det D \Phi_\tau)(z) \), we notice that \( \Psi(0, z) = m(z) \) and
\[
\partial_\varphi \mathcal{H}^*_\tau m(z) = \frac{\Psi(\tau, z) - \Psi(0, z)}{\tau} = \int_0^1 D_1 \Psi(s\tau, z) \, ds.
\]

Consequently,
\[
\lim_{\tau \to 0} \partial_\varphi \mathcal{H}^*_\tau m(z) = \frac{\partial \Psi}{\partial \tau}(0, z).
\]

On the other hand, taking \( W_\tau \) such that \( \Phi_\tau = I - \tau W_\tau \), we observe that
\[
x + \tau (V_\varphi(\tau, x) - W_\tau(x + \tau V_\varphi(\tau, x))) = x.
\]

Therefore,
\[
W_\tau(x + \tau V_\varphi(\tau, x)) = V_\varphi(\tau, x) = D_\varphi H(x, D(\eta^\tau * \varphi)(x + \tau V_\varphi(\tau, x))),
\]
which implies
\[
\frac{\partial \Psi}{\partial \tau} = \Delta \Psi + \eta^\tau * \left( Dm \circ \Phi_\tau \cdot \frac{d \Phi_\tau}{d \tau} \det D \Phi_\tau + m \circ \Phi_\tau \frac{d \det D \Phi_\tau}{d \tau} \right),
\]
and
\[
\frac{\partial \Psi}{\partial \tau}(0, z) = \Delta m - Dm \cdot W_0 - m \div W_0 = \Delta m - \div (m D_\varphi H(x, \varphi)). \qedhere
\]
The next result of this subsection is the following weak-strong uniqueness result. If there exists a classical solution to (43) with \( \tau = 0 \), then any weak solution agrees with this solution. This result was first proven by Vardan Voskanyan but was never published. We present here the proof for completeness.

**Lemma 6.** Suppose that \( L \) satisfies (A1) and \( F \) is strictly increasing. Let \( k > 1 \) and \( (r,u,m) \in \mathbb{R} \times L^k(\mathbb{T}^d) \times L^1(\mathbb{T}^d) \) be a weak solution (in the sense of Definition 3) of (43) with \( \tau = 0 \), and let \((\rho,\varphi,\mu) \in \mathbb{R} \times C^2(\mathbb{T}^d) \) be a classical solution. Then, \((r,m) = (\rho,\mu)\) and \( u - \varphi \) is constant.

**Proof.** For any \( \nu,\psi \in C^2(\mathbb{T}^d) \) and \( \lambda \in \mathbb{R} \), the function

\[
    h(\varepsilon) = \left< A^0 \begin{bmatrix} \rho + \varepsilon \lambda \\ \varphi + \varepsilon \psi \\ \mu + \varepsilon \nu \\ \mu + \varepsilon \nu - m \end{bmatrix}, \begin{bmatrix} \rho + \varepsilon \lambda - r \\ \varphi + \varepsilon \psi - u \\ \mu + \varepsilon \nu - m \end{bmatrix} \right>
\]

has a minimum at \( \varepsilon = 0 \), so \( h'(0) = 0 \). Thus, taking into account that \((\rho,\varphi,\mu)\) is a classical solution of the equation involving \( A^0 \), we obtain

\[
\begin{aligned}
\left< DA^0 \begin{bmatrix} \rho \\ \varphi \\ \mu \\ \nu \end{bmatrix}, \begin{bmatrix} \lambda \\ \varphi - u \\ \mu - m \end{bmatrix} \right> &= \int_{\mathbb{T}^d} (\Delta \varphi - D_p H(x, D\varphi) D\psi + F'(\mu)\nu + \lambda) d(\mu - m) \\
&\quad + \int_{\mathbb{T}^d} (\Delta \nu - \text{div}(\nu D_p H(x, D\varphi)) d(\mu - m) - (\rho - r) \int_{\mathbb{T}^d} d\nu \\
&\quad + \int_{\mathbb{T}^d} \mu D^2_{pp} H(x, D\varphi) D\psi (\varphi - u) \, dx = 0.
\end{aligned}
\]

(44)

Letting \( \lambda = \rho - r \), and using a density argument, we take \( \psi = \varphi - u \), \( \nu = \mu - m \) such that

\[
\int_{\mathbb{T}^d} F'(\mu)(\mu - m)^2 + \mu \langle D^2_{pp} H(x, D\varphi) D(\varphi - u), D(\varphi - u) \rangle = 0.
\]

Hence, relying on the facts that \( \mu, F'(\mu) > 0 \) and \( D^2_{pp} H \) is positive definite, we get \( m = \mu \) and \( Du = D\varphi \). Using these in (44), we obtain \( r = \rho \).

Now, using the previous results, we prove Theorem 3.

**Proof of Theorem 3.** By Proposition 7, \( (\rho_\tau, u_\tau, \bar{\mu}_\tau) \in \mathbb{R} \times C(\mathbb{T}^d) \times C_{r+1} \) is a weak solution to (43) with \( \tau > 0 \). From (38), we have that \( \|u_\tau\|_{L^{1+1/a}(\mathbb{T}^d)} \) is uniformly bounded in \( \tau \in (0,1) \). Let \( \tau_n > 0 \) be a sequence converging to zero. By Lemma 8, there is a subsequence, still denoted \( \tau_n \), such that \( u_{\tau_n} \) converges weakly in \( L^{1+1/a}(\mathbb{T}^d) \) to \( u \). By Remark 5, there is a further subsequence, still denoted \( \tau_n \), such that \((\rho_{\tau_n}, \mu_{\tau_n})\) converges in \( \mathbb{R} \times \mathcal{P}_{\ell} \) to \((r,m)\). Thus, \((r,u,m)\) is a weak solution to (43) with \( \tau = 0 \). Therefore, by Lemma 6, we conclude the proof. 

\[ \square \]
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