THE RENORMALIZED EULER CHARACTERISTIC AND L-SPACE SURGERIES

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ABSTRACT. Using the equivalence between the renormalized Euler characteristic of Ozsváth and Szabó, and the Turaev torsion normalized by the Casson-Walker invariant, we make calculations for \( S_p^3 \) \((K)\). An alternative proof of a theorem by Ozsváth and Szabó on L-space surgery obstructions is provided.

1. Introduction

In the beautiful paper [2], Ozsváth and Szabó consider the correction terms of \( Y = S_{2n-1/2}^3(K) \), where \( K \) is a knot in \( S^3 \), and prove that if \( Y \) is an L-space, then there is a symmetry among these correction terms. Given an L-space whose correction terms are known, one can check whether this symmetry is satisfied, and if not, it would follow that the manifold at hand cannot be obtained as a \( \frac{2n-1}{2} \) surgery on a knot in \( S^3 \). This obstruction is then used to calculate some new unknotting numbers. The symmetry was later generalized to the case of \( p/q \) surgery in [4].

We remind that for a rational homology sphere \( Y \) and a Spin\(^c\) structure \( t \) on it, the renormalized Euler characteristic \( \hat{\chi}(Y, t) \) is defined as

\[
\hat{\chi}(Y, t) = \chi(HF_{\text{red}}(Y, t)) - \frac{1}{2}d(Y, t),
\]

where \( d(Y, t) \) denotes the correction terms. By definition, \( Y \) is an L-spaces if and only if \( HF_{\text{red}}^{+}(Y, t) \cong 0 \). One immediately sees that for L-spaces any symmetry of correction terms is equivalent to the symmetry of \( \hat{\chi} \)'s. This points out that calculating the renormalized Euler characteristic for \( S^3_{p/q}(K) \) could be interesting.

Given a knot \( K \), let

\[
\Delta_K(T) = a_0 + \sum_{j > 0} a_j \left( T^j + T^{-j} \right)
\]

be the symmetrized Alexander polynomial of \( K \), and define

\[
t_i(K) = \sum_{j \geq 1} ja_{|i|+j}.
\]

**Theorem 1.1.** Let \( p \neq 0 \) and \( q \) be relatively prime integers. For each \( t \), the renormalized Euler characteristic \( \hat{\chi}(S^3_{p/q}(K), t) \) can be expressed in terms of the coefficients of the Alexander polynomial of \( K \).
This theorem follows from the identification \( \hat{\chi} = -\tau + \lambda \), where \( \tau \) is the Turaev torsion, and \( \lambda \) is the Casson-Walker invariant, see [5]. In fact, we give a precise formula for the renormalized Euler characteristic in Proposition 3.1. The next theorem is proved using this formula.

**Theorem 1.2.** Let \( p \) and \( q \) be relatively prime integers with \( p/q > 1 \). Suppose that the Alexander polynomial of knot \( K \) satisfies \( a_j = 0 \) for \( j > \frac{p}{2q} + 1 \). Let \( n \) be any integer, and denote \( r = \lceil \frac{np}{q} - 1 \rceil \in \mathbb{Z}/p\mathbb{Z} \), then for any \( |i| \leq p/2q \) we have

\[
\hat{\chi}(S^3_{p/q}(K), r + i) - \hat{\chi}(S^3_{p/q}(U), r + i) = t_i,
\]

where \( U \) is the unknot, and we have used the affine identification \( \text{Spin}^c(S^3_{p/q}(K)) \cong \mathbb{Z}/p\mathbb{Z} \) explained in Section 2; also see Remark 3.2.

Note that if \( p/q \leq 1 \), then the condition of the theorem will force \( a_j = 0 \) for \( j > 1 \). In this case, it follows from the calculations that

\[
\hat{\chi}(S^3_{p/q}(K), -1) - \hat{\chi}(S^3_{p/q}(U), -1) = \lceil q/p \rceil a_1,
\]

and the difference is zero for all remaining \( \text{Spin}^c \) structures.

As a corollary of this theorem we obtain an alternative proof for the Theorem 1.2 of [4].

**Corollary 1.3.** Let \( K \) be a knot which admits an L-space surgery for some \( \frac{p}{q} > 1 \). Then, for all integers \( i \) with \( |i| \leq \frac{p}{2q} \) we have that

\[
d(S^3_{p/q}(K), i - 1) - d(S^3_{p/q}(U), i - 1) = -2t_i,
\]

while for all \( |i| > \frac{p}{2q} \), we have \( t_i(K) = 0 \).

**Proof.** If \( S^3_{p/q}(K) \) is an L-space, then it is known that \( S^3_{|p/q|}(K) \) is also an L-space. However, then \( \lceil p/q \rceil \geq 2g(K) - 1 \), where \( g(K) \) is the degree of the Alexander polynomial of \( K \). Thus, \( a_j = 0 \) for \( j > \frac{|p/q| + 1}{2} \), i.e. also for \( j > \frac{p}{2q} + 1 \). The corollary follows by taking \( n = 0 \) in the previous theorem and observing that for L-spaces

\[
\hat{\chi}(Y, t) = -\frac{1}{2}d(Y, t).
\]

The organization of this paper is as follows: the required preliminaries are presented in Section 2. Calculations needed for Theorem 1.1 are made in Section 3. We finish with the proof of Theorem 1.2 in Section 4.

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2. Preliminaries

Let \( Y \) be a rational homology sphere, \( t \) be a Spin\(^c\) structure on it. We can consider the Heegaard Floer homology group \( HF^+(Y, t) \). This is a \( \mathbb{Q} \) graded module over \( \mathbb{Z}[U] \). We can also consider a simpler version, \( HF^\infty(Y, t) \) for which one can prove

\[
HF^\infty(Y, t) \cong \mathbb{Z}[U, U^{-1}]
\]

for each \( t \). There is a natural \( \mathbb{Z}[U] \) equivariant map

\[
\pi : HF^\infty(Y, t) \to HF^+(Y, t)
\]

which is zero in sufficiently negative degrees and an isomorphism in all sufficiently positive degrees. \( HF^\text{red}^+(Y, t) \) is defined as \( HF^\text{red}^+(Y, t) = HF^+(Y, t) / \text{Im} \pi \).

Let \( d(Y, t) \) be the correction term, defined as the minimal degree of any non-torsion class of \( HF^+(Y, t) \) lying in the image of \( \pi \). As was mentioned, the renormalized Euler characteristic \( \hat{\chi}(Y, t) \) is defined by

\[
\hat{\chi}(Y, t) = \chi(HF^\text{red}(Y, t)) - \frac{1}{2} d(Y, t).
\]

The Casson-Walker invariant, \( \lambda : \{\text{rational homology spheres modulo homeomorphisms}\} \to \mathbb{Q} \), is an extension of Casson’s invariant to rational homology spheres, see [8]. It perhaps worth noting that in our normalization \( \lambda(\Sigma(2, 3, 5)) = -1 \), where \( \Sigma(2, 3, 5) \) is oriented as the boundary of the negative definite \( E_8 \) plumbing.

We will also consider the normalized Casson-Walker invariant given by \( \lambda'(Y) = |H_1(Y; \mathbb{Z})| \lambda(Y) \). Note that \( \lambda'(S^3) = 0 \). Let \( U \) be the unknot, and \( K \) be any knot in \( S^3 \). Walker’s surgery formula implies that

\[
\lambda'(S^3_{p/q}(K)) - \lambda'(S^3_{p/q}(U)) = q \cdot \sum_{j \geq 1} j^2 a_j,
\]

where \( a_j \) are the coefficients of the Alexander polynomial of \( K \).

The Turaev torsion function is another important invariant of three-manifolds. Turaev first defined it for combinatorial Euler structures, but later the connection with these and Spin\(^c\) structures emerged, providing us with a function

\[
\tau(Y, \cdot) : \text{Spin}^c(Y) \to \mathbb{Z},
\]

see [7].

The Turaev torsion can be also introduced in the case of three-manifolds with torus boundary. Indeed, consider the manifold \( M = S^3 - \text{nd}(K) \) with torus boundary. We denote the set of relative Spin\(^c\) structures on \( M \) by \( \text{Spin}^c(M) \). The Turaev torsion of this manifold is a function

\[
\tau(M, \cdot) : \text{Spin}^c(M) \to \mathbb{Z}.
\]
Actually, under a certain affine map $\text{Spin}^c(M) \cong \mathbb{Z}_{\text{odd}}$, one has
\[ \tau(M, k) = \text{sign}(k) \cdot \sum_{j \geq |k|+1} a_j, \]
for any odd integer $k$, see 9.1.4 of [7]. Here $a_i$ are again the coefficients of the Alexander polynomial of $K$. In what follows, we will use this particular identification of $\text{Spin}^c(M)$ with $\mathbb{Z}_{\text{odd}}$. Of course, given a relative $\text{Spin}^c$ structure on $M = S^3 - \text{nd}(K)$ we can look at the $\text{Spin}^c$ structure on $S^3_{p/q}(K)$ which extends it. Thus, we can get an affine identification $\text{Spin}^c(S^3_{p/q}(K)) \cong \mathbb{Z}/p\mathbb{Z}$ as follows: if $k \in \mathbb{Z}_{\text{odd}} \cong \text{Spin}^c(M)$ extends to $i \in \mathbb{Z}/p\mathbb{Z} \cong \text{Spin}^c(S^3_{p/q}(K))$, then
\[ i \equiv \frac{k - 1}{2} \mod p. \]

Let $x$ be any integer that satisfies $qx \equiv -1 \mod p$. Remember that $H_1(S^3_{p/q}(K); \mathbb{Z})$ is generated by the meridian of the knot $K$, and the homology class of $K$ is $x$ times this generator. The following formula is a consequence of 10.6.3.2 of [3]:
\[ \tau(S^3_{p/q}(K), i) - \tau(S^3_{p/q}(K), i + x) = \tau(S^3_{p/q}(U), i) - \tau(S^3_{p/q}(U), i + x) + \sum_{\{k \mid i \equiv \frac{k - 1}{2} \mod p\}} \tau(M, k), \] (3)
where $i \in \mathbb{Z}/p\mathbb{Z}$. Note that, to get this formula we have replaced the purely homological data of the original formula by terms coming from the surgery on unknot.

Let $Y$ be a rational homology sphere, $t$ be $\text{Spin}^c$ structure on it. We have the following formula for the renormalized Euler characteristic, see [5]:
\[ \hat{\chi}(Y, t) = -\tau(Y, t) + \lambda(Y). \] (4)
Note that since Turaev torsions add up to zero, we have
\[ \sum_{t \in \text{Spin}^c(Y)} \hat{\chi}(Y, t) = \chi(M). \] (5)

3. Calculations

From now on we fix the knot $K$, and two relatively prime integers $p$ and $q$. Note that calculating the renormalized Euler characteristics $\hat{\chi}(S^3_{p/q}(K), i)$ is equivalent to computing
\[ S_i = \hat{\chi}(S^3_{p/q}(K), i) - \hat{\chi}(S^3_{p/q}(U), i), \]
because the second term is already known. [3]. For $0 \leq i < p$ let
\[ T_i = \sum_{\{k \mid i \equiv \frac{k - 1}{2} \mod p\}} \tau(M, k), \]
Thus, using (2) and (5), we get

\[ S_{t+x} - S_t = T_i, \]

for any \( i \in \mathbb{Z}/p\mathbb{Z} \). Fix any \( l \in \mathbb{Z}/p\mathbb{Z} \). To get a formula for \( S_l \), we write the following equations

\[ S_{t+(j+1)x} - S_{t+jx} = T_{l+jx}, \]

for \( j = 0, 1, \ldots, p-1 \). It follows that \( S_{t+x} = S_t + T_i \), \( S_{t+2x} = S_t + T_i + T_{i+x} \) and so on. Using (2) and (5), we get

\[ pS_t + \sum_{j=0}^{p-1} (p - j - 1)T_{l+jx} = q \cdot \sum_{j \geq 1} j^2a_j. \]

Thus,

\[ S_t = \frac{1}{p} \left( q \cdot \sum_{j \geq 1} j^2a_j - \sum_{j=0}^{p-1} (p - j - 1)T_{l+jx} \right), \tag{6} \]

which establishes the following proposition.

**Proposition 3.1.** Let \( K \) be a knot in \( S^3 \), \( p > 0 \) and \( q \) relatively prime integers, and \( x = -q^{-1} \mod p \), then

\[ \hat{\chi}(S^3_{p/q}(K), l) = \hat{\chi}(S^3_{p/q}(U), l) + \frac{q}{p} \cdot \sum_{j \geq 1} j^2a_j - \frac{1}{p} \cdot \sum_{j=0}^{p-1} (p - j - 1)T_{l+jx}, \]

where \( \hat{\chi}(S^3_{p/q}(U), i) = \hat{\chi}(L(-p, q), i) = -d(L(-p, q), i)/2 \).

**Remark 3.2.** Let \( p \) be odd, so there is a unique Spin structure on \( S^3_{p/q}(K) \). In the canonical affine identification \( \text{Spin}^c(S^3_{p/q}(K)) \cong \mathbb{Z}/p\mathbb{Z} \) this Spin structure corresponds to 0, and conjugation is equivalent to multiplication by \(-1\). We denote the Spin\(^c\) structure corresponding to \( i \in \mathbb{Z}/p\mathbb{Z} \) in this identification by \( \bar{s}_i \) (thus, \( \bar{s}_0 \) is the Spin structure, \( \bar{s}_1 = s_{-i} \)), and in our identification by \( t_i \). Let us spell out the correspondence between these two identifications. Note that we always have \( T_{l+1} = 0 \), which means \( S_{l+1} = S_{l+1} \). This universal equality must be a consequence of the conjugation symmetry \( \hat{\chi}(Y, \bar{t}) = \hat{\chi}(Y, t) \). Thus, \( t_{l+1} \) and \( t_{l-1} \) are conjugate. If \( s_i = t_{l+1} \), then \( i + x = -i \mod p \), i.e. \( i = \frac{p-1}{2}x \mod p \). As a result, \( t_{(p-1)(1-x) \mod p} \) is the Spin structure of \( S^3_{p/q}(K) \).

4. **Proof of Theorem 1.2**

From now on we assume that \( q > 1 \). The case when \( q = 1 \) is only slightly different, and is left to the reader. Note that we can simplify the previous formula under certain
conditions on the Alexander polynomial. Let the knot $K$ be such that $a_j = 0$ for $j \geq p/2$, which means that for $0 \leq i < p/2$,

$$T_i = \sum_{\{k \equiv \frac{i}{p} \mod p\}} \tau(M, k) = \tau(M, 2i + 1) = \sum_{j \geq i+1} a_j,$$

and similarly, for $p/2 \leq i \leq p - 1$,

$$T_i = -\sum_{j \geq p-i} a_j.$$

With $l$ fixed, let $0 \leq u_j \leq p - 1$ satisfy

$$l + u_j x = j - 1 \mod p,$$

and $0 \leq v_j \leq p - 1$ satisfy

$$l + v_j x = -j \mod p.$$

Let

$$c_i = \sum_{j=1}^{i} (u_j - v_j),$$

and since $x = -q^{-1} \mod p$, one has

$$c_i = p \cdot \sum_{j=1}^{i} \left( \left\{ \frac{q(l + 1 - j)}{p} \right\} - \left\{ \frac{q(l + j)}{p} \right\} \right),$$

where $\{\alpha\} = \alpha - \lfloor \alpha \rfloor$ denotes the fractional value of the number $\alpha$. Now we can rewrite (6) as

$$S_l = \frac{1}{p} \sum_{i \geq 1} (q^2 + c_i) a_i. \quad (7)$$

The proof of Theorem 1.2 now becomes an exercise in arithmetic. The assumption of the theorem is that $p/q > 1$ and $q > 1$, and that the Alexander polynomial of knot $K$ satisfies $a_j = 0$ for $j > \frac{p}{q} + 1$. Given any $l \in \mathbb{Z}/p\mathbb{Z}$, let us find the coefficient of $a_1$ in $S_l$. We have find the value of

$$c_1 = p \left( \left\{ \frac{ql}{p} \right\} - \left\{ \frac{q(l+1)}{p} \right\} \right).$$

Obviously, $\frac{q(l+1)}{p} - \frac{ql}{p} = \frac{q}{p} < 1$, which means that $c_1 = -q$ unless there is an integer $n$ so that

$$\frac{ql}{p} < n \leq \frac{q(l+1)}{p},$$

i.e. $l = \lfloor \frac{qn}{p} - 1 \rfloor$, in which case we have $c_1 = p - q$. As a result, the coefficient of $a_1$ in $S_l$ is equal to 1 if $l = \lfloor \frac{qm}{p} - 1 \rfloor$, and equal to 0 otherwise.
Any $a_j$ with $j < \frac{p}{2q} + \frac{1}{2}$ can be analyzed similarly, with the result that $a_j$’s coefficient in $S_l$ is equal to zero unless $l = \left\lceil \frac{mn}{p} - 1 \right\rceil + i$, where $|i| \leq j - 1$, in which case the coefficient is equal to $j - i$.

The case of $a_j$ with $\frac{p}{2q} + \frac{1}{2} \leq j \leq \frac{p}{2q} + 1$ is a bit different, because the difference $\frac{q(l+j)}{p} - \frac{q(l+1-j)}{p}$ is not necessarily less than 1, thus there is a chance that between these two fractions two integers may appear. However, since we are interested in $S_l$ with $l = \left\lceil \frac{mn}{p} - 1 \right\rceil + i$, where $i \leq p/2q$, this does not happen. As a result, $a_j$’s coefficient in $S_l$ for $l = \left\lceil \frac{mn}{p} - 1 \right\rceil + i$, where $|i| \leq j - 1$ is equal to $j - i$. Since $a_j = 0$ for $j > \frac{p}{2q} + 1$, we have proved that given any integer $n$, for every $|i| \leq p/2q$ we have

$$\hat{\chi}(S^3_{p/q}(K), r + i) - \hat{\chi}(S^3_{p/q}(U), r + i) = t_i,$$

where $r = \left\lceil \frac{mn}{q} - 1 \right\rceil \in \mathbb{Z}/p\mathbb{Z}$.

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