The WMAP results on the scalar spectral index $n$ and its running with scale, though preliminary, open a very interesting window to physics at very high energies. We address the problem of finding simple inflaton potentials well motivated by particle physics which can accommodate WMAP data.

Inflation can naturally explain the flatness, homogeneity and isotropy plus the huge entropy of our universe through a period of superluminal expansion driven by a scalar field $\phi$ with a nearly constant potential energy $V$. The time evolution of the inflaton $\phi$ and the Hubble expansion rate, $H \equiv \dot{a}/a$, where $a$ is the FRW scale factor, are given by

$$\ddot{\phi} + 3H \dot{\phi} = -\frac{\partial V}{\partial \phi} \equiv -V', \quad H^2 = \frac{1}{3M_p^2} \left( V + \frac{1}{2} \dot{\phi}^2 \right),$$  \hspace{1cm} (1)

with $8\pi M_p^2 = 1/G_{\text{Newton}}$. When the parameters

$$\epsilon \equiv \frac{1}{2} M_p^2 \left( \frac{V'}{V} \right)^2, \quad \eta \equiv M_p^2 \frac{V''}{V},$$  \hspace{1cm} (2)

are much smaller than 1 (slow-roll regime) the acceleration term in (1) and the kinetic contribution to $H^2$ can be neglected and the scale factor grows exponentially during inflation as $a(t) \rightarrow e^{Ht}a_0$. The number of $e$-folds of expansion is $N_e = \int_{t_i}^{t_f} H dt$ where $t_i$ ($t_f$) is the time when inflation begins (ends) and solving the problems mentioned above requires $N_e = 50 - 60$.

A simple way of constructing a very flat inflaton potential is to make use of the flat directions ubiquitous in supersymmetric models. Consider as a very simple example the so-called $D$-term hybrid-inflation model. It includes a gauged $U(1)$ and three chiral multiplets, $\phi(0), H_+(-1)$
and $H_{-}(-1)$, with the $U(1)$ charges as indicated. The superpotential is $W = \lambda \dot{\phi} H_{+} H_{-} - \mu^2 \dot{\phi}$ and the potential is $V = V_F + V_D$:

$$V = \left|\lambda H_{+} H_{-} - \mu^2\right|^2 + \lambda^2 \left(|H_{-}|^2 + |H_{+}|^2\right) |\Phi|^2 + \frac{g^2}{2} \left(|H_{+}|^2 - |H_{-}|^2 + \xi_D\right)^2,$$

(3)

where we have included a Fayet-Iliopoulos term, $\xi_D$. This potential has a valley along $\phi$ for $H_{\pm} = 0$ which is exactly flat at tree-level and has $V \neq 0$ with $V \approx \mu^4 + g^2 \xi_D^2/2$. A small slope is induced radiatively by $H_{\pm}$ loops giving

$$V = V_0 + \beta \ln(\phi/Q),$$

(4)

where $\beta = (g^4 \xi_D^2 + \lambda^2 \mu^4)/(8\pi^2)$ and $Q$ is the renormalization scale. Along this valley one can achieve easily inflation, with $\epsilon, |\eta| \ll 1$ and $H^2 \approx V_0/(3M_p^2)$. The "waterfall" fields $H_{\pm}$ guarantee the exit of inflation: at small enough $\phi$ they roll towards their minima stopping inflation. A virtue of this model is that $\eta$ remains small even after embedding the model in Supergravity (no $\eta$-problem).

In addition to the generic virtues already mentioned, inflation also provides a very simple explanation for the formation of the structures we observe in the universe today. It does so through the stretching of quantum fluctuations of the inflaton field, $\Phi(x,t) = \phi(t) + \varphi(x,t)$. These fluctuations are frozen at superhorizon scales with amplitudes determined by $H$ and a flat spectrum $\langle \varphi^2 \rangle = H^2/(2\pi)^2 \int dk/k$, where $k$ is the comoving momentum (the scale goes as $1/k$). Different regions of the universe have different values of $\varphi$ which leads to different times for the end of inflation (with delays given by $\delta t = \varphi/\dot{\phi}$) and this in turn causes primordial inhomogeneities $\delta\rho/\rho \sim H \delta t \sim H^2/\dot{\phi}$. In terms of field derivatives of the potential one gets

$$\frac{\delta\rho}{\rho} \sim \frac{V^{3/2}}{M_p^3 \sqrt{\eta}},$$

(5)

evaluated for each $k$-mode at the time of its horizon crossing. This results in a nearly $k$-independent spectrum of scalar perturbations.

One can parametrize departures from exact scale independence with the scalar spectral index $n$, writing

$$\left(\frac{\delta\rho}{\rho}\right)^2 \propto k^{n-1}.$$

(6)

The value $n = 1$ corresponds to a scale independent spectrum while $n > 1$ ($n < 1$) corresponds to a blue (red) spectrum with more power at small (large) scales. In terms of slow-roll parameters one gets

$$n \approx 1 + 2\eta - 6\epsilon,$$

(7)

so that inflation typically predicts $n \approx 1$. In our previous example one easily gets $n$ as a function of the scale analytically

$$n = 1 - \frac{1}{N_e - \ln(k/k_*)},$$

(8)

where $N_e$ is the number of e-folds and $k_*$ corresponds to the largest scales in the observable Universe ($1/k_* \sim 10^4$ Mpc). The model therefore predicts a red spectrum and a very small dependence of $n$ with scale, given by

$$-\frac{dn}{d\ln k} = (n - 1)^2 \ll 1.$$

(9)

The values of $n$ and $dn/d\ln k$ measured by WMAP with the 3-year data sample were presented right at the time of the Moriond conference and are

$$n \approx 1.06 \pm 0.06, \quad \frac{dn}{d\ln k} \approx -0.055 \pm 0.025,$$

(10)
evaluated at \( k = 0.002 \text{ Mpc}^{-1} \) (I have rounded off errors and assumed negligible tensor perturbations. At the time of writing the revised analysis of the errors in \(^2\) has not appeared yet). With this amount of running for the index, \( n \) gets red for larger values of \( k \) (eventually the running should become smaller at smaller scales, with \( n \) stabilized at some red value. Otherwise it would be difficult to achieve a sufficient number of e-folds.). In fact, fitting the data with a constant \( n \) gives the red value
\[
n = 0.951^{+0.015}_{-0.015}.
\] (11)

Although the fit with a running index is better than without it, the improvement is marginal and the evidence for a non-zero running remains inconclusive. However, a running index is a challenge for inflation model building and if confirmed it would have important implications.

The main goal of the work \(^3\) reported in this talk was to explore whether simple models of inflation, with a good motivation from the particle physics point of view, can give such a large value of \( dn/d\ln k \) (see e.g. \(^4\), \(^5\), \(^6\), \(^7\), \(^8\) for related work). Before proceeding, let me remind you that, as the inflaton field rolls down its potential towards smaller values (like in the example discussed before), it is scanning smaller and smaller energy scales. As the number of e-folds builds up, fluctuations of increasing \( k \) are leaving the horizon so that one is also scanning different scales \( 1/k \) in the observable universe. In this way, earlier values of the field correspond to higher energies and larger structure scales. In this sense, observations of structure at the highest observable scales offer a window to very high energies.

Our strategy to construct an inflaton potential able to accommodate (10) was to start with the simple \( D \)-term model discussed before and modify it at high energy (in a well motivated way) to reproduce the running of \( n \), which is associated to high scales. In slow-roll inflation one gets
\[
\frac{dn}{d\ln k} = 16\epsilon\eta - 24\epsilon^2 - 2\xi,
\] (12)
where \( \xi \equiv M_p^4V'V''/V^2 \). We see that having a sizable value of \( dn/d\ln k \) will require sizable \( \xi \) (\( \epsilon \) and \( \eta \) are necessarily small) and therefore a sizable \( V'' \).

Ref.\(^3\) describes several attempts at constructing such potential. First, one can try to modify the radiative corrections to \( V \), which control the slope, by taking into account the renormalization group evolution of the couplings. In a regime in which some of the couplings get strong in the ultraviolet one might hope to have a large effect on the shape of the potential eventually inducing a large running of \( n \) (a similar idea with the couplings getting strong in the infrared was exploited in ref.\(^9\)). Although this can be achieved, the price to pay is a small number of e-folds. In fact, one obtains a no-go relation of the form \( (N_e/50)^2|(dn/d\ln k)/0.055| \ll 1 \), so that if \( N_e \) is large enough the running is necessarily small and vice versa.

One can also try to change the shape of the potential assuming that the inflaton field crosses in its evolution a physical threshold so that above it the potential is affected by radiative contributions from some additional heavy degrees of freedom. Again, a large \( |dn/d\ln k| \) requires a small \( N_e \). In both of the cases just described the failure can be traced back to the fact that the modifications of the shape of the potential are not sharp enough but rather gradual and smooth, see \(^3\) for details. Although having a small \( N_e \) might not be a problem if the model is supplemented by additional stages of inflation,\(^10\) we would like to achieve both goals (large \( N_e \) and sizable \( dn/d\ln k \)) using a single potential.

I will now describe a successful modification of the \( D \)-term hybrid inflation model (also presented in \(^3\)) which is able to give a large running of \( n \) and a large number of e-folds\(^8\). Again one assumes some heavy physics threshold but now the associated energy scale \( M \) is above the

\(^*\)It offers an explicit counterexample to the claim of ref.\(^11\) according to which a slow-roll single field inflation model with a large \( dn/d\ln k \) (at \( k = 0.002 \text{ Mpc}^{-1} \)) like the one suggested by WMAP cannot achieve more than \( N_e = 30 \).
region probed by the inflaton. In this case the only effect of that heavy new physics on the evolution of the inflaton field is through the presence of a non-renormalizable operator (NRO) that modifies the potential as:

\[ V = V_0 + \beta \ln(\phi/Q) + \phi^4 \frac{\phi^{2N}}{M^{2N}} \]  

(13)

(For this potential to be reliable, one has to make sure that during inflation \( \phi \) is always significantly lower than \( M \).) For \( N = 9 \) the potential is given by fig. 1. The star marks \( \phi_* \), the start of inflation and the circle, \( \phi_0 \), the point at which \( n = 1 \). We have \( \rho \approx (10^{-3} \sqrt{MM_p})^4 \) and \( \beta \approx (10^{-4}M)^4 \). Choosing for instance \( M \approx 0.95M_p \), we get \( \phi_* \approx 0.15M_p \) while \( \phi_0 \approx 0.142M_p \). The amplitude of the scalar fluctuations is \( P_k \approx (2.95 \times 10^{-9}) \times (0.8) \). and the evolution of \( n \) with \( k \) is given by the lower plot in fig. 2. One can see how the index is blue at high scales and gets red for lower scales, just as WMAP3 indicates. Numerically, we have \( dn/d\ln k|_* \approx -0.03 \) and \( N_e \approx 50. \) This \( n \) is well approximated analytically by

\[ n = n_* + \frac{1}{N_e} - \frac{1}{N_e - \ln(k/k_*)} \] 

\[ - \frac{N_e}{(N+1)} \left( \frac{dn}{d\ln k}|_* + \frac{1}{N_e^2} \right) \left[ (1 - \frac{1}{N_e} \ln \frac{k}{k_*})^{N+1} - 1 \right], \]  

(14)

to be compared with (8). A careful fit of WMAP3 data with this \( n(k) \) is underway.

A number of comments is appropriate. First note that the effect of the NRO can become important because the potential is so flat to begin with. Also the effect on the third derivative is enhanced with respect to the effect on the first and second derivatives making it easy to affect \( \xi \) while keeping small \( \epsilon \) and \( \eta \) (see upper plot in fig. 2 for the evolution of the slow-roll parameters).

Then, notice that the lifting of supersymmetric flat directions by very high-order NROs, like the one we need here, is naturally expected. Examples of this are known already in the MSSM 12 and in strings 13. Moreover, it is natural to expect that, of all the flat directions of a model, the flattest is the most likely candidate to drive inflation.
Figure 2: Slow-roll parameters (upper plot) and scalar spectral index as a function of scale in Mpc$^{-1}$ (lower plot) for the inflaton potential of Figure 1.
Finally, as the model is supersymmetric one should find a superpotential suitable to give a potential of the form (13). One simple example is the following

\[ W = \lambda \hat{\Phi} \hat{H}_+ \hat{H}_- + \frac{1}{2} m \phi'^2 + \frac{1}{(P + 2)} \phi' \Phi P \hat{\Phi}^P, \]  

(15)

where \( \phi' \) is an additional field with mass \( m \ll M \) (so that \( \phi' \) really belongs in the effective theory below \( M \)). The potential along \( \phi \) is then

\[ V = V_D + \phi^4 \frac{\phi'^2 \phi^{4P}}{m^2 M^{4P}}. \]  

(16)

Other examples are of course possible. This is interesting because a rather modest \( P = 4 \) in (15) gives \( N = 9 \) in (13).

To conclude, the WMAP indication of a running of the scalar spectral index \( n \) is very interesting for Physics at high energy scales. We have shown that it is not too difficult to accommodate such large running in simple and well motivated inflaton models. Still, it is puzzling that such running occurs precisely at the largest scales observed in the universe (i.e. near \( N_e \sim 50 - 60 \)). If this experimental indication is confirmed, it would point towards a new cosmic coincidence problem.

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