EQUILIBRIUM STATES FOR NON-UNIFORMLY HYPERBOLIC SYSTEMS: STATISTICAL PROPERTIES AND ANALYTICITY

Suzete Maria Afonso
Universidade Estadual Paulista (UNESP)
Instituto de Geociências e Ciências Exatas, Câmpus de Rio Claro
Avenida 24-A, 1515, Bela Vista, Rio Claro, São Paulo, 13506-900, Brazil

Vanessa Ramos and Jaqueline Siqueira*
Centro de Ciências Exatas e Tecnologia - UFMA
Av. dos Portugueses, 1966, Bacanga,65080-805, São Luís, Brazil
*Instituto de Matemática - Universidade Federal do Rio de Janeiro
Av. Athos da Silveira Ramos 149, Cidade Universitária
Ilha do Fundão, Rio de Janeiro 21945-909, Brazil

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Abstract. We consider a wide family of non-uniformly expanding maps and hyperbolic Hölder continuous potentials. We prove that the unique equilibrium state associated to each element of this family is given by the eigenfunction of the transfer operator and the eigenmeasure of the dual operator (both having the spectral radius as eigenvalue). We show that the transfer operator has the spectral gap property in some space of Hölder continuous observables and from this we obtain an exponential decay of correlations and a central limit theorem for the equilibrium state. Moreover, we establish the analyticity with respect to the potential of the equilibrium state as well as that of other thermodynamic quantities. Furthermore, we derive similar results for the equilibrium state associated to a family of non-uniformly hyperbolic skew products and hyperbolic Hölder continuous potentials.

1. Introduction. The field of ergodic theory has been developed with the goal of understanding the statistical behavior of a dynamical system via measures which remain invariant under its action. In general, a dynamical system admits more than one invariant measure which makes it necessary to choose a suitable one to analyze the system. In order to do it, one may select those maximizing the free energy of the system, called equilibrium states.

Formally, given a continuous map $f : M \to M$ defined on a compact metric space $M$ and a continuous potential $\phi : M \to \mathbb{R}$, we say that an $f$-invariant probability
measure $\mu$ is an equilibrium state for $(f, \phi)$ if it satisfies the following variational principle:

$$h_\mu(f) + \int \phi \, d\mu = \sup_{\eta \in \mathcal{P}_f(M)} \left\{ h_\eta(f) + \int \phi \, d\eta \right\},$$

where $\mathcal{P}_f(M)$ denotes the set of $f$-invariant probability measures on the Borel sets of $M$ endowed with the weak* topology.

The study of equilibrium states was initiated by Sinai, Ruelle and Bowen in the seventies through the application of techniques and results from statistical physics to smooth dynamical systems. Sinai, in his pioneering work [40], studied the problem of existence and finiteness of equilibrium state for Anosov diffeomorphisms and Hölder continuous potentials. This strategy was carried out by Ruelle in [34], [35] and Bowen in [9] to extend the theory to uniformly hyperbolic (Axiom A) dynamical systems. Since then, the study of this problem has been greatly extended to include classes beyond uniform hyperbolic systems among many others contributions we cite the recent works [11], [13], [14] and [36].

Once the existence and finiteness of equilibrium states have been established, it is natural to ask which type of information one can obtain regarding the system via the equilibrium state. For instance, how fast is the memory of the past lost as time evolves? In other words, is it possible to specify the rate of decay of correlations? Also, can one characterize weak correlations via a central limit theorem? Moreover, what type of regularity can one obtain for the equilibrium state when the potential varies?

In the context of one dimensional piecewise expanding maps, Liverani [26] proved an exponential decay of correlations and Keller [21] obtained a central limit theorem. Such statistical properties were also obtained for the unique equilibrium state associated to potentials with small variation for a class of non-uniformly expanding maps by Castro and Varandas [12] and for a class of partially hyperbolic horseshoes by Ramos and Siqueira [31].

Bruin and Todd [10] considered a class of smooth interval maps and proved analyticity of the topological pressure for a one-parameter family of potentials with bounded range: the variation is smaller than the topological entropy. For smooth deformations of generic nonuniformly hyperbolic unimodal maps, Baladi and Smania [5] proved the differentiability of the absolutely continuous invariant measure. For a family of real multimodal maps, Iommi and Todd [19] gave a characterization of the first order phase transitions of the topological pressure for the geometric potentials. In the context of countable Markov shifts, Sarig [37] studied the analyticity of the topological pressure for some one-parameter family of potentials. For a class of non-uniformly expanding maps, Bonfim, Castro, and Varandas [8] obtained linear response formulas for its equilibrium states.

In this paper we study a wide class of non-uniformly hyperbolic maps associated to hyperbolic potentials with small variation. The uniqueness of equilibrium states for this class was established in [32]; here we derive that such equilibrium state admits strong statistical properties. Namely, the correlations decay exponentially and a central limit theorem holds. We use the approach of projective metrics to prove that the transfer operator has the spectral gap property in a suitable space of Hölder continuous observables. The core of the paper is to establish this spectral gap property. From this several nice properties can be obtained applying classical analytical methods.
This technique has been successfully implemented by several authors, such as Hofbauer and Keller [18] for studying piecewise monotonic transformations, Baladi [4] and Liverani [25] for studying hyperbolic maps, and Young [43] for a class of non-uniformly hyperbolic maps. Melbourne and Nicol [28] developed an approach in an abstract setting to obtain statistical properties via the quasi-compactness of the transfer operator. In addition, Giulietti, Kloeckner, Lopes and Marcon [16] used a differential geometric approach to obtain consequences of the spectral gap property. In this paper we follow the approach of [31], where a class of partially hyperbolic horseshoes were considered.

In the context of this work, from the spectral gap property, besides deriving statistical properties, it allows us to study the behavior of the system under small perturbations of the potential. Namely, we prove that the equilibrium state, as well as others thermodynamical quantities, such as the topological pressure, vary analytically. In [2], it was proved that the equilibrium state is jointly continuous with respect to the map and the potential while here we establish analyticity but only as a function of the potential.

The main idea for establishing the spectral gap property in the space of Hölder continuous observables is proving that the transfer operator contracts a suitable cone. This requires a contractive behavior in the pre-images of points that are close enough. The strong topological mixing property guarantees that each point has at least one pre-image in the non-uniformly expanding set, giving us a good control of these pre-images. The other pre-images are then controlled by a domination condition.

This paper is organized as follows. In Section 2 we describe the setting and state the main results. In Section 3 we introduce some definitions and results used throughout the paper.

In Section 4 we prove that the transfer operator admits the spectral gap property. In Section 5 we use the spectral gap property to obtain statistical properties for the equilibrium state. We also describe the thermodynamical formalism, that is, we prove that the unique equilibrium state is given by an eigenfunction of the transfer operator and an eigenmeasure of its dual both having the spectral radius as an eigenvalue. In Section 6 we prove that the equilibrium state, as well as other thermodynamical quantities vary analytically with the potential.

In Section 7 we extend our results to a wider class of non-uniformly hyperbolic maps. Finally, in Section 8 we present several examples for which our results hold.

2. Definitions and main results. Let $M$ be a compact Riemannian manifold and let $\mathcal{F}$ be a family of $C^1$ local diffeomorphisms $f : M \to M$. Given $\sigma \in (0, 1)$, define $\Sigma_\sigma(f)$ as the set of points $x \in M$ where $f$ is non-uniformly expanding, that is,

$$
\limsup_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df^i(x)^{-1}\| \leq \log \sigma.
$$

We say that a continuous potential $\phi : M \to \mathbb{R}$ is $\sigma$-hyperbolic for $f$ if the topological pressure of $\phi$ (with respect to $f$) is equal to the relative pressure of $\phi$ on the set $\Sigma_\sigma(f)$; we recall the definition of topological pressure relative to a set in Section 3, Subsection 3.1.

Given $\alpha > 0$, consider $C^\alpha(M)$ the space of Hölder continuous functions $\varphi : M \to$
endowed with the seminorm
\[ |\varphi|_\alpha = \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{d(x, y)^\alpha} \]
and the norm
\[ \|\varphi\|_\alpha = \|\varphi\|_0 + |\varphi|_\alpha, \]
where \( \|\|_0 \) stands for the sup norm in \( C^0(M) \). We shall always consider \( \mathcal{F} \times C^\alpha(M) \) endowed with the product topology.

Given \((f, \phi)\), with \( \phi \) a \( \sigma \)-hyperbolic potential, under the assumption that \( \{f^{-n}(x)\}_{n \geq 0} \) is dense in \( M \) for all \( x \in M \), the existence of a unique equilibrium state \( \mu_{f, \phi} \) was established in [Theorem 2, [32]]. Clearly, the condition above holds whenever \( f \) is strongly topologically mixing, i.e. for every open set \( U \subset M \) there is \( N \in \mathbb{N} \) such that \( f^N(U) = M \).

Consider \( Q \) a cover by injectivity domains of \( f \). Since \( M \) is compact and \( f \) is a local diffeomorphism, we can take \( Q = \{U_1, \ldots, U_m\} \) for some \( m \in \mathbb{N} \). Defining
\[ \vartheta = \max_{1 \leq j \leq m} \left\{ \sup_{x \in U_j} \|Df^{-1}(x)\| \right\}, \]
we note that \( \vartheta \) does not depend on the choice of the cover \( Q \).

We fix \( \sigma \in (0, 1) \) satisfying
\[ \vartheta \cdot \sigma < 1. \quad (*) \]

Consider the family \( \mathcal{H}_\sigma \), consisting of pairs \((f, \phi) \in \mathcal{F} \times C^\alpha(M) \) such that \( f \) is strongly topologically mixing, satisfies condition \( (*) \) and \( \phi \) is \( \sigma \)-hyperbolic for \( f \) satisfying condition \( (**) \) that will be properly stated in Section 4.

We establish that for each \((f, \phi) \in \mathcal{H}_\sigma \) its unique equilibrium state \( \mu_{f, \phi} \) has an exponential decay of correlations for Hölder continuous observables.

**Theorem I.** There exists a constant \( 0 < \tau < 1 \) such that for all \( \varphi \in L^1(\mu_{f, \phi}), \psi \in C^\alpha(M) \) there exists \( K = K(\varphi, \psi) > 0 \) satisfying
\[ \left| \int (\varphi \circ f^n) \cdot \psi \, d\mu_{f, \phi} - \int \varphi \, d\mu_{f, \phi} \int \psi \, d\mu_{f, \phi} \right| \leq K \cdot \tau^n \quad \text{for every } n \geq 1. \]

We also derive a central limit theorem for the equilibrium state \( \mu_{f, \phi} \).

**Theorem II.** Let \( \varphi \) be an \( \alpha \)-Hölder continuous function and let \( \tilde{\sigma} > 0 \) be defined by
\[ \tilde{\sigma}^2 = \int \psi^2 \, d\mu_{f, \phi} + 2 \sum_{n=1}^{\infty} \int \psi(\psi \circ f^n) \, d\mu_{f, \phi}, \quad \text{where} \quad \psi = \varphi - \int \varphi \, d\mu_{f, \phi}. \]

Then \( \tilde{\sigma} \) is finite and \( \tilde{\sigma} = 0 \) if and only if \( \varphi = u \circ f - u \) for some \( u \in L^2(\mu_{f, \phi}) \). On the other hand, if \( \tilde{\sigma} > 0 \) then given any interval \( A \subset \mathbb{R} \),
\[ \mu_{f, \phi} \left( x \in M : \frac{1}{n} \sum_{j=0}^{n-1} (\varphi(f^j(x)) - \int \varphi \, d\mu_{f, \phi}) \in A \right) \to \frac{1}{\tilde{\sigma} \sqrt{2\pi}} \int_A e^{-\frac{t^2}{2\tilde{\sigma}^2}} \, dt, \]
as \( n \) goes to infinity.
Transfer operator and its spectrum. Given $\sigma \in (0,1)$ let $(f, \phi) \in \mathcal{H}_\sigma$. Denote by $C^0(M)$ the set of real continuous functions on $M$ endowed with the sup norm.

We define the operator $\mathcal{L}_{f,\phi} : C^0(M) \rightarrow C^0(M)$ called the Ruelle-Perron-Frobenius operator or simply the transfer operator, which associates to each $\psi \in C^0(M)$ a continuous function $\mathcal{L}_{f,\phi}(\psi) : M \rightarrow \mathbb{R}$ by

$$\mathcal{L}_{f,\phi}(\psi)(x) = \sum_{y \in f^{-1}(x)} e^{\phi(y)} \psi(y).$$

Note that $\mathcal{L}_{f,\phi}$ is a positive and bounded linear operator. Consider the resolvent set

$$Res(\mathcal{L}_{f,\phi}) := \{ z \in \mathbb{C} : (zI - \mathcal{L}_{f,\phi}) \text{ has a bounded inverse} \},$$

and its complement set, called the spectrum set,

$$Spec(\mathcal{L}_{f,\phi}) := \{ z \in \mathbb{C} : (zI - \mathcal{L}_{f,\phi}) \text{ has no bounded inverse} \}.$$

We say that the transfer operator $\mathcal{L}_{f,\phi}$ satisfies the spectral gap property if its spectrum set $Spec(\mathcal{L}_{f,\phi}) \subset \mathbb{C}$ admits a decomposition as follows: $Spec(\mathcal{L}_{f,\phi}) = \{ \lambda \} \cup \Sigma$ where $\lambda \in \mathbb{R}$ is an eigenvalue for $\mathcal{L}_{f,\phi}$ associated to a one-dimensional eigenspace and $\Sigma$ is strictly contained in a ball centered at zero and of radius strictly less than $\lambda$.

It should be mentioned that besides of the spectral gap property having its intrinsic interest, this property is the core of this paper since it will be used to obtain our main results.

**Theorem III.** For every $(f, \phi) \in \mathcal{H}_\sigma$ the transfer operator, $\mathcal{L}_{f,\phi}$, has the spectral gap property in the space of Hölder continuous observables.

According to Riesz–Markov Theorem we can consider the dual operator $\mathcal{L}_{f,\phi}^* : P(M) \rightarrow P(M)$ as the operator that satisfies

$$\int \psi \, d\mathcal{L}_{f,\phi}^* \eta = \int \mathcal{L}_{f,\phi}(\psi) \, d\eta,$$

for every $\psi \in C^0(M)$ and every $\eta \in P(M)$, where $P(M)$ denotes the space of probability measures on the Borel sets of $M$.

Now consider the spectral radius $\lambda_{f,\phi}$ of the transfer operator defined by $\lambda_{f,\phi} := \sup \{ |z| : z \in Spec(\mathcal{L}_{f,\phi}) \}$. Since $\mathcal{L}_{f,\phi}$ is positive, the spectral radius can be computed by the following formula

$$\lambda_{f,\phi} = \lim_{n \rightarrow \infty} \sqrt[n]{\| \mathcal{L}_{f,\phi}^n \|}.$$

Using that $\| \mathcal{L}_{f,\phi}^n \| = \| \mathcal{L}_{f,\phi}^n \|_{1}$ for any $n \geq 1$ we deduce that

$$\deg(f)e^{\inf \phi} \leq \lambda_{f,\phi} \leq \deg(f)e^{\sup \phi}. \quad (2)$$

As a byproduct of the spectral gap property of the transfer operator we can describe the Thermodynamical Formalism for a pair $(f, \phi) \in \mathcal{H}_\sigma$.

**Theorem IV.** Let $\lambda_{f,\phi}$ be the spectral radius of the transfer operator $\mathcal{L}_{f,\phi}$. There exist a probability measure $\nu_{f,\phi} \in P(M)$ and a Hölder continuous function $h_{f,\phi} : M \rightarrow \mathbb{R}$ bounded away from zero and infinity which satisfies

$$\mathcal{L}_{f,\phi}^* \nu_{f,\phi} = \lambda_{f,\phi} \nu_{f,\phi} \quad \text{and} \quad \mathcal{L}_{f,\phi} h_{f,\phi} = \lambda_{f,\phi} h_{f,\phi}.$$

Moreover, the invariant measure $\mu_{f,\phi}$ given by $\mu_{f,\phi} = h_{f,\phi} \nu_{f,\phi}$ is the unique equilibrium state associated to $(f, \phi) \in \mathcal{H}_\sigma$. 
We also prove that some thermodynamical quantities vary analytically with the potential, in particular, the equilibrium state as well. We describe the setting as follows: consider \( f \in \mathcal{F} \) strongly topologically mixing satisfying condition (\(*)\). Denote by \( \mathcal{P}_\sigma \) the set of \( \alpha \)-Hölder continuous potentials which are \( \sigma \)-hyperbolic for \( f \) and satisfy condition (\(*\)).

**Theorem V.** The following functions defined in \( \mathcal{P}_\sigma \) are analytic:

(i) The topological pressure function \( \phi \mapsto P_f(\phi) \in \mathbb{R} \);
(ii) The invariant density function \( \phi \mapsto h_{f,\phi} \in C^\alpha(M) \);
(iii) The conformal measure function \( \phi \mapsto \nu_{f,\phi} \in (C^\alpha(M))^* \).
(iv) The equilibrium state function \( \phi \mapsto \mu_{f,\phi} = h_{f,\phi} \nu_{f,\phi} \in (C^\alpha(M))^* \).

The meaning of analyticity in this context is the one given in Section 6. We point out that according [2] the family of hyperbolic potentials is an open class in the \( C^0 \)-topology. Since (\(*\)) is an open condition on the potential we derive that \( \mathcal{P}_\sigma \) is an open subset of \( C^\alpha(M) \).

In Section 8 we present some examples for which our results hold.

3. **Preliminaries.** In this section we present some basic definitions and results that will be useful in the following sections. We begin with the definition of pressure relative to a set, not necessarily compact. This will allow us to precise the concept of hyperbolic potentials stated in the last section.

3.1. **Topological pressure.** Let \( M \) be a compact metric space and consider \( T: M \to M \) and \( \phi: M \to \mathbb{R} \) both continuous. Given \( \delta > 0 \), \( n \in \mathbb{N} \) and \( x \in M \), define the dynamical ball by

\[
B_\delta(x,n) = \{ y \in M : d(T^j(x), T^j(y)) < \delta, \text{ for } 0 \leq j \leq n \}.
\]

For each \( N \in \mathbb{N} \) consider \( \mathcal{F}_N \) the following collection of dynamical balls

\[
\mathcal{F}_N = \{ B_\delta(x,n) : x \in M \text{ and } n \geq N \}.
\]

Given \( \Lambda \subset M \), not necessarily compact, denote by \( \mathcal{F}_N(\Lambda) \) the finite or countable families of elements in \( \mathcal{F}_N \) which cover \( \Lambda \). For \( n \in \mathbb{N} \), let

\[
R_{n,\delta}(x) = \sup_{y \in B_\delta(x,n)} S_n\phi(y),
\]

where \( S_n\phi(y) := \sum_{j=0}^{n-1} \phi(f^j(y)) \) is the Birkhoff’s sum.

Assume \( \Lambda \subset M \) is invariant under \( T \) and define, for each \( \gamma > 0 \),

\[
m_T(\phi, \Lambda, \delta, N, \gamma) = \inf_{U \subset \mathcal{F}_N(\Lambda)} \left\{ \sum_{B_\delta(x,n) \in U} e^{-\gamma n + R_{n,\delta}(x)} \right\}.
\]

Define

\[
m_T(\phi, \Lambda, \delta, \gamma) = \lim_{N \to +\infty} m_T(\phi, \Lambda, \delta, N, \gamma),
\]

and

\[
P_T(\phi, \Lambda, \delta, \gamma) = \inf \{ \gamma > 0 : m_T(\phi, \Lambda, \delta, \gamma) = 0 \}.
\]

The **relative pressure** of \( \phi \) on \( \Lambda \) is defined by

\[
P_T(\phi, \Lambda) = \lim_{\delta \to 0} P_T(\phi, \Lambda, \delta).
\]

The **topological pressure** of \( \phi \) is by definition \( P_T(\phi, M) \), and it satisfies

\[
P_T(\phi) = \sup \{ P_T(\phi, \Lambda), P_f(\phi, \Lambda^c) \},
\]

(3)
where $\Lambda^c$ is the complement of the set $\Lambda$ on $M$. We refer the reader to [Section 11, [30]] for the proof of (3) and for additional properties of the pressure.

We precise the definition of hyperbolic potentials, first we consider potentials with respect to diffeomorphisms and after with respect to skew products.

Let $M$ be a compact Riemannian manifold and let $f: M \to M$ be a local $C^1$ diffeomorphism. Considering the set $\Sigma_{\sigma}(f)$ as in Section 2, we point out that it is an invariant set although not necessarily compact. A real continuous function $\phi: M \to \mathbb{R}$ is said to be a hyperbolic potential for $f$ if

$$P_f(\phi, (\Sigma_{\sigma}(f))^{c}) < P_f(\phi, \Sigma_{\sigma}(f)) = P_f(\phi).$$

The class of hyperbolic potentials for a map consists in an open class in the $C^0$-topology [2].

3.2. Hyperbolic pre-balls. Here we present the concept of hyperbolic times that will help us to deal with the lack of hyperbolicity in the family of maps that we consider. We say that $n$ is a hyperbolic time for $x \in M$ if

$$n - 1 \prod_{j=n-k}^{n-1} \|Df(f^j(x))^{-1}\| \leq \sigma k/2$$

for all $1 \leq k < n$.

Since we consider only maps with no critical or singular sets, the definition of hyperbolic times given in [1, Definition 5.1] reduces to the one that we have presented.

Condition (1) of non-uniform expansion is enough to guarantee the existence of infinitely many hyperbolic times for points in $\Sigma_{\sigma}(f)$. See [Lemma 5.4, [1]].

The next result guarantees that points associated to a hyperbolic time admit a neighborhood for which all orbits behave as uniformly expanded ones. We refer the reader to [Lemma 5.2, [1]] for its proof.

**Proposition 1.** There exists $\delta > 0$ such that if $n$ is a hyperbolic time for $x \in M$, then there exists a neighborhood of $x$, $V_n(x)$, satisfying

(i) $f^n$ maps $V_n(x)$ diffeomorphically onto the ball centered on $f^n(x)$ and of radius $\delta$;

(ii) for all $1 \leq k < n$ and $y, x \in V_n(x)$,

$$d(f^{n-k}(y), f^{n-k}(z)) \leq \sigma^{k/2} d(f^n(y), f^n(z)).$$

Moreover, for every $y \in V_n(x)$ we have $\|Df^n(y)^{-1}\| \leq \sigma^{n/2}$.

We will refer to the sets $V_n$ as hyperbolic pre-balls. Note that the $n$-th iterate of a hyperbolic pre-ball, $f^n(V_n)$, is actually a topological ball of radius $\delta > 0$.

Given $f \in F$, fix $\delta > 0$ provided by Proposition 1. Note that, $\delta > 0$ depends only on $f$ and $\sigma$. Moreover, $\delta$ can be taken uniformly in a neighborhood of $f$, see [Remark 3.5, [2]]. From now on we fix $\delta > 0$ as above.

**Lemma 3.1.** There exists $\tilde{N}$ such that

$$f^{\tilde{N}}(B(x, \delta)) = M \text{ for all } x \in M.$$

Moreover, for any $n \in \mathbb{N}$ and any hyperbolic pre-ball $V_n$ we have

$$f^{\tilde{N}+n}(V_n) = M.$$
Proof. Consider a cover of $M$ by balls of radius $0 < \varepsilon < \frac{\delta}{4}$. Since $M$ is a compact manifold, we can extract a finite subcover of $M$, namely $B = \{B_1, \ldots, B_l\}$. From the property of topologically mixing of $f$, we can consider $N_1, \ldots, N_l \in \mathbb{N}$ such that

$$f^{N_1}(B_1) = \ldots, f^{N_l}(B_l) = M. \tag{4}$$

Let $\tilde{N} = N_1 + \cdots + N_l$. It is straightforward to see that

$$f^{\tilde{N}}(B_j) = M, \quad 1 \leq j \leq l. \tag{5}$$

Given a ball $B(x, \delta)$ for some $x \in M$, there exists $B_{\tilde{l}} \in B$, $1 \leq \tilde{l} \leq k$, such that $B_{\tilde{l}} \subset B(x, \delta)$. Therefore,

$$f^{\tilde{N}}(B(x, \delta)) = M. \tag{6}$$

Now, for the second part of the lemma, consider the hyperbolic pre-ball $V_n(x)$ associated to some $x \in M$. From Proposition 1 we have

$$f^n(V_n) = B(f^n(x), \delta). \tag{4}$$

On the other hand, from what we have proved

$$f^{\tilde{N}}(B(f^n(x), \delta)) = M. \tag{5}$$

Thus, by (4) and (5) we conclude that

$$f^{\tilde{N}+n}(V_n) = M. \tag{6}$$

Fix $\tilde{N} \in \mathbb{N}$ given by the first part of the lemma. Consider

$$\bar{n} = \inf_{n \in \mathbb{N}} \{n \text{ hyperbolic time for some } x \in M \mid n \geq 2\tilde{N}\}.$$

Let $V_{\bar{n}}(z)$ be the hyperbolic pre-ball associated to some $z \in M$. We define

$$N := \tilde{N} + \bar{n} \tag{6}$$

Note that $N > 3\tilde{N}$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{hyperbolic_pre-balls.png}
\caption{Hyperbolic pre-balls.}
\end{figure}
By the second part of Proposition 1, given $x, y \in M$, there exist $x_N, y_N \in \mathbb{V}_N(z)$ such that

$$f^N(x_N) = x \quad \text{and} \quad f^N(y_N) = y.$$  

Moreover, if $d(x, y) < \delta$, thus

$$d(x_N, y_N) \leq \sigma^{\bar{n}/2} \cdot d(f^\bar{n}(x_N), f^\bar{n}(y_N))$$

$$\leq \sigma^{\bar{n}/2} \cdot \varphi^{\bar{N}} \cdot d(x, y)$$

$$\leq \sigma^{\bar{n}/2 - \bar{N}} \cdot (\sigma \cdot \varphi)^\bar{N} \cdot d(x, y)$$

$$= \gamma \cdot d(x, y), \quad (7)$$

where we denote $\gamma = \sigma^{\bar{n}/2 - \bar{N}} \cdot (\sigma \cdot \varphi)^\bar{N}$. Note that $\gamma \ll 1$ since $\sigma < 1$, $\bar{n} > 2\bar{N}$ and by hypothesis $(\ast)$ we have $\sigma \cdot \varphi < 1$.

3.3. Projective metrics. In this section we will state some definitions and results regarding projective metrics associated to convex cones. The notion of projective metric associated to a convex cone in a vector space was introduced by Garrett Birkhoff [7] and provides a nice way to explicit spectral properties of the transfer operator (see [4], [25] and [42], for instance). In particular, we will derive the spectral gap property for the transfer operator through this notion.

Let $E$ be a Banach space. A subset $\mathcal{C}$ of $E\setminus\{0\}$ is called a cone in $E$ if $\mathcal{C}\cap(-\mathcal{C}) = \{0\}$ and it satisfies:

$$v \in \mathcal{C} \quad \text{and} \quad \lambda > 0 \quad \Rightarrow \quad \lambda \cdot v \in \mathcal{C}.$$  

A cone $\mathcal{C}$ is called convex if

$$v, w \in \mathcal{C} \quad \text{and} \quad \lambda, \eta > 0 \quad \Rightarrow \quad \lambda \cdot v + \eta \cdot w \in \mathcal{C}.$$  

The closure of $\mathcal{C}$, denoted by $\bar{\mathcal{C}}$, is defined by

$$\bar{\mathcal{C}} := \{w \in E : \text{ there are } v \in \mathcal{C} \text{ and } \lambda_n \to 0 \text{ such that } (w + \lambda_nv) \in \mathcal{C} \text{ for all } n \geq 1\}.$$  

A cone $\mathcal{C}$ is called closed if $\bar{\mathcal{C}} = \mathcal{C} \cup \{0\}$.

Let $\mathcal{C}$ be a closed convex cone and given $v, w \in \mathcal{C}$ we define

$$A(v, w) = \sup \{t > 0 : w - tv \in \mathcal{C}\} \quad \text{and} \quad B(v, w) = \inf \{s > 0 : sv - w \in \mathcal{C}\},$$

with the convention $\sup \emptyset = 0$ and $\inf \emptyset = +\infty$, where $\emptyset$ denotes the empty set.

It is easy to verify that $A(v, w)$ is finite, $B(v, w)$ is positive and $A(v, w) \leq B(v, w)$ for all $v, w \in \mathcal{C}$ (see [42]). We set

$$\Theta(v, w) = \log \left( \frac{B(v, w)}{A(v, w)} \right),$$

with $\Theta$ possibly infinity in the case $A = 0$ or $B = +\infty$.

By virtue of properties of $A$ and $B$, we derive that $\Theta(v, w)$ is well-defined and takes values in $[0, +\infty]$. Since $\Theta(v, w) = 0$ if and only if $v = tw$ for some $t > 0$, we derive $\Theta$ defines a pseudo-metric on $\mathcal{C}$. In this way, $\Theta$ induces a metric on a projective quotient space of $\mathcal{C}$ called the projective metric of $\mathcal{C}$.

We point out that the projective metric depends in a monotone way on the cone:

if $\mathcal{C}_1 \subset \mathcal{C}_2$ are two convex cones in $E$, then $\Theta_2(v, w) \leq \Theta_1(v, w)$ for any $v, w \in \mathcal{C}_1$, where $\Theta_1$ and $\Theta_2$ are the projective metrics in $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively.

Furthermore, note that if $E_1, E_2$ are Banach space, $L : E_1 \to E_2$ is a linear operator, and $\mathcal{C}_1, \mathcal{C}_2$ are convex cones in $E_1, E_2$, respectively, such that $L(\mathcal{C}_1) \subset$
continuous observables defined on $C_2$, then $\Theta_2(L(v), L(w)) \leq \Theta_1(v, w)$ for any $v, w \in C_1$, where $\Theta_1$ and $\Theta_2$ are the projective metrics in $C_1$ and $C_2$, respectively.

In general, $L$ need not be a strict contraction, that will be the case for instance if $L(C_1)$ had finite diameter in $C_2$. The next result will be a key tool to establish the spectral gap for the Ruelle–Perron–Frobenius operator. Its proof can be found in [[42], Proposition 2.3].

**Proposition 3.** Let $C_1$ and $C_2$ be closed convex cones in the Banach spaces $E_1$ and $E_2$, respectively. If $L: E_1 \to E_2$ is a linear operator such that $L(C_1) \subset C_2$ and $\Delta = \text{diam}_{\Theta_2}(L(C_1)) < \infty$, then

$$\Theta_2(L(\varphi), L(\psi)) \leq (1 - e^{-\Delta}) \cdot \Theta_1(\varphi, \psi) \quad \text{for all } \varphi, \psi \in C_1.$$

Our goal is to apply the last result to the transfer operator acting in a special class of cones. More precisely those of locally Hölder continuous functions that we define as follows.

Given $\delta > 0$ a function $\varphi$ is said to be $(C, \alpha)$-Hölder continuous in balls of radius $\delta$ if for some constant $C > 0$ we have

$$|\varphi(x) - \varphi(y)| \leq C d(x, y)^\alpha \quad \text{for all } y \in B(x, \delta).$$

Denote by $|\varphi|_{\alpha, \delta}$ the smallest Hölder constant of $\varphi$ in balls of radius $\delta > 0$.

We fix $\delta > 0$ and consider for each $k > 0$ the convex cone of locally Hölder continuous observables defined on $M$:

$$C_{k, \delta} = \left\{ \varphi: \varphi > 0 \text{ and } \frac{|\varphi|_{\alpha, \delta}}{\inf \varphi} \leq k \right\}. \quad (8)$$

Considering the classes of cones of locally Hölder continuous observables, it is possible to give a more explicit expression to the projective metric, which we will be denoted by $\Theta_k$. The use of this expression allow to prove that the diameter of a cone, $C_{k, \delta}$ is finite, if $k$ is large enough. We state these results below and refer the reader to [31] for their proofs.

**Lemma 3.2 ([31], Lemma 4.3).** The metric $\Theta_k$ in the cone $C_{k, \delta}$ is given by

$$\Theta_k(\varphi, \psi) = \log \left( \frac{B_k(\varphi, \psi)}{A_k(\varphi, \psi)} \right),$$

where

$$A_k(\varphi, \psi) := \inf_{d(x, y) < \delta, z \in M} \frac{k|x - y|^\alpha \varphi(z) - (\psi(x) - \psi(y))}{k|x - y|^\alpha \varphi(z) - (\varphi(x) - \varphi(y))}$$

and

$$B_k(\varphi, \psi) := \sup_{d(x, y) < \delta, z \in M} \frac{k|x - y|^\alpha \varphi(z) - (\psi(x) - \psi(y))}{k|x - y|^\alpha \varphi(z) - (\varphi(x) - \varphi(y))}.$$

In particular, we have

$$A_k(\varphi, \psi) \leq \inf_{x \in M} \left\{ \frac{\varphi(x)}{\psi(x)} \right\} \quad \text{and} \quad B_k(\varphi, \psi) \geq \sup_{x \in M} \left\{ \frac{\varphi(x)}{\psi(x)} \right\}.$$

**Proposition 3 ([31], Proposition 5.3).** The cone $C_{\lambda k, \delta}$ has finite diameter for $k > 0$ sufficiently large.

We end this section with some technical results regarding Hölder continuous functions that will be used in the sequel.

**Lemma 3.3.** If $\varphi: M \to \mathbb{R}$ is a $(C, \alpha)$-Hölder continuous function in balls of radius $\delta > 0$, then there exists $m = m(\delta) > 0$ such that $\varphi$ is $(m \cdot C, \alpha)$-Hölder continuous.
Proof. By the compactness of \( M \), there exists \( N \in \mathbb{N} \) which depends only on \( \delta \) such that given \( x, y \in M \) there are \( z_0 = x, z_1, \ldots, z_{N+1} = y \) with \( d(z_i, z_{i+1}) \leq \delta \) for all \( i = 0, \ldots, N \) and \( d(z_i, z_{i+1}) \leq d(x, y) \).

Since \( \varphi \) is \((C, \alpha)\)-Hölder continuous in balls of radius \( \delta \) it follows that

\[
|\varphi(x) - \varphi(y)| \leq \sum_{i=0}^{N} |\varphi(z_i) - \varphi(z_{i+1})| \leq \sum_{i=0}^{N} Cd(z_i, z_{i+1})^\alpha \leq C(N+1)d(x, y)^\alpha.
\]

Therefore, \( \varphi \) is \((m \cdot C, \alpha)\)-Hölder continuous where \( m = N+1 \).

Remark 1. Note that considering balls of radius \((\vartheta^N \cdot \delta)\), by the proof of Lemma 3.3, with \( \vartheta \) satisfying \((*)\), we can conclude that \( \varphi \) is \(((\vartheta^N) + 1) \cdot C, \alpha)\)-Hölder continuous, where \( [\vartheta^N] \) denotes the greatest integer less than or equal to \( \vartheta^N \).

Lemma 3.4. For each \( \varphi \in C_{k, \delta} \),

\[
\sup \varphi \leq \inf \varphi \cdot (2 \cdot m \cdot d^\alpha) \cdot k,
\]

where \( d \) denotes the diameter of \( M \).

Proof. Let \( \varphi \in C_{k, \delta} \). It follows from Lemma 3.3 that

\[
\sup \varphi - \inf \varphi \leq m \cdot |\varphi|_{\alpha, \delta} \cdot d^\alpha \leq m \cdot \inf \varphi \cdot k \cdot d^\alpha.
\]

Therefore, \( \sup \varphi \leq \inf \varphi \cdot (2 \cdot m \cdot d^\alpha) \cdot k \).

3.4. Eigenprojections. In this section we state some classical results of spectral theory of bounded operators. Let \((X, ||\cdot||)\) be a complex Banach vector space. Let \( \mathcal{B}(X) \) be the space of all the linear operators \( T : X \to X \) that are bounded.

Although the next result being classical we present it here since its proof will be useful to obtain our results in the next section.

Lemma 3.5. Let \( X \) be a Banach space and let \( T \in \mathcal{B}(X) \) be a bounded linear operator. If \( T \) is invertible and \( \|T - S\| < \|T^{-1}\|^{-1} \), then \( S \) is invertible. In particular, the set of invertible operators is open on \( \mathcal{B}(X) \).

Proof. Since \( T \) is invertible and \( \|(T - S)T^{-1}\| < 1 \) then \( I - (T - S)T^{-1} \) has a bounded inverse given by

\[
\sum_{n=0}^{\infty} [(T - S)T^{-1}]^n.
\]

Moreover

\[
T^{-1} \sum_{n=0}^{\infty} [(T - S)T^{-1}]^n = T^{-1}[I - (T - S)T^{-1}]^{-1}
\]

\[
= T^{-1}[I - T(T - S)^{-1}]
\]

\[
= T^{-1} - (T - S)^{-1}
\]

\[
= [T - (T - S)]^{-1} = S^{-1},
\]

thus \( S \) is invertible and \( T^{-1} \sum_{n=0}^{\infty} [(T - S)T^{-1}]^n \) is the inverse of \( S \).

We point out that what we present next remains valid even if the Banach space considered is a real one and not complex. This is because we can consider the complexification of the space and the operator acting on it.
A bounded linear operator $E : X \to X$ is called a projection if it satisfies $E^2 = E$ in which case we can write the following direct sum decomposition

$$X = \text{Im}(E) \oplus \text{Ker}(E).$$

Note that this decomposition is such that $x = E(x) + (I - E)(x)$ for all $x \in X$, where $I$ is the identity map.

**Theorem 3.6 (Separation of the spectrum).** Let $X$ be a Banach space and $T \in B(X)$. Suppose that the spectrum of $T$ has the following decomposition $\text{Spec}(T) = \sigma_1 \cup \sigma_2$ where $\sigma_1$ and $\sigma_2$ are disjoint compact sets. If $\gamma$ is a closed smooth simple curve which does not intersect $\text{Spec}(T)$ and which contains $\sigma_1$ in its interior and $\sigma_2$ in its exterior then the operator defined by

$$E := \frac{1}{2\pi i} \int_{\gamma} (zI - T)^{-1} \, dz$$

is a projection and it satisfies:

(i) $ET = TE$ and $\text{Ker}(E), \text{Im}(E)$ are $T$-invariant;

(ii) $\text{Spec}(T|_{\text{Im}(E)}) = \sigma_1$ and $\text{Spec}(T|_{\text{Ker}(E)}) = \sigma_2$.

For the proof of the last result the reader can consult, for instance, [[20], Theorem 6.17]. We are interested in a particular case of this result when the spectrum admits an isolated point. That is, there exist an eigenvalue $\lambda$ and a closed smooth simple curve $\gamma$ such that $\lambda$ is the unique element of the spectrum in the interior of $\gamma$. In this way we state the next corollary.

**Corollary 1.** Let $\lambda \in \text{Spec}(T)$ be an isolated eigenvalue and let $\gamma$ be a closed smooth simple curve that separates $\lambda$ from the rest of the spectrum. Then

$$E := \frac{1}{2\pi i} \int_{\gamma} (zI - T)^{-1} \, dz$$

is a projection. Moreover, $E$ is the eigenprojection of $\lambda$, $\text{Im}(E)$ is the eigenspace of $\lambda$ and $\text{dim} (\text{Im}(E))$ is the geometric multiplicity of $\lambda$.

4. **Spectral gap of the transfer operator.** In this section we prove that the transfer operator admits the spectral gap property when restrict to the space Hölder continuous observables. As previously mentioned, this property is the core for proving the main results of this paper. We closely follow the ideas of [31] where it is considered a model of a non-uniformly expanding map where expansion rates are explicit. In the present work, we consider a more general class of non-uniformly expanding maps which contains their example. Still some of their results remain true without major alterations, in which case, we will refer the reader to their proof.

Let $f \in \mathcal{F}$. As seen before, each $x \in \Sigma_\sigma$ admits infinitely many hyperbolic times. Recall that we fixed $\delta > 0$ in Subsection 3.2 depending only on $f$ and $\sigma$. We also take $N$ given by equation (6).

We assume that $\phi : M \to \mathbb{R}$ is a Hölder continuous potential, hyperbolic for $f$ and satisfying

$$\left( e^{N \text{Var} \phi} \left[ \vartheta^N + 1 \right] + \frac{2m^{\alpha} e^{N \phi}}{e^{N \inf \phi}} \right) \left( \frac{[(\deg(f))^N - 1] \vartheta^{N\alpha}}{[\deg(f)]^N} + \frac{\gamma^\alpha}{[\deg(f)]^N} \right) < 1 \quad (\ast \ast)$$

for some $0 < \alpha < 1$.

The role of this assumption will be transparent in the proof of Proposition 4. However, $(\ast \ast)$ can be weaken as described in Remark 2.
Note that for each $\varphi \in C^0(M)$ and each $x \in M$ we have
\[
\mathcal{L}^N(\varphi)(x) = \sum_{y \in f^{-N}(x)} e^{S_N\phi(y)}\varphi(y) \geq [\deg(f)]^N e^{N\inf \phi} \inf \varphi. \tag{9}
\]

We recall that the convex cone of locally Hölder continuous observables is defined by:
\[
C_{k,\delta} = \left\{ \varphi : \varphi > 0 \text{ and } \frac{\|\varphi\|_{\alpha,\delta}}{\inf \varphi} \leq k \right\}.
\]

Next we prove that for $k$ large enough, the cone $C_{k,\delta}$ is invariant under the $N$-th iterate of the transfer operator.

**Proposition 4.** There exists $0 < \lambda < 1$ such that
\[
\mathcal{L}^N(C_{k,\delta}) \subset C_{k,\delta}, \quad \text{for } k \text{ large enough.}
\]

**Proof.** Let $\varphi \in C_{k,\delta}$. Thus $\varphi > 0$ and, by definition, we have $\mathcal{L}^N(\varphi) > 0$. Since $\mathcal{L}^N$ is a bounded operator we derive that $\mathcal{L}^N(\varphi)$ is continuous.

In order to prove that $\mathcal{L}(\varphi) \subset C_{k,\delta}$ we must show that
\[
\frac{|\mathcal{L}^N(\varphi)|_{\alpha,\delta}}{\inf \mathcal{L}^N(\varphi)} \leq \lambda k \quad \text{for some } 0 < \lambda < 1.
\]

Set $N > 2\bar{N}$, where $\bar{N}$ is given by Lemma 3.1. Given $x, y \in M$ satisfying $d(x,y) < \delta$, we denote by $x_j, y_j$, $1 \leq j \leq [\deg(f)]^N$, the pre-images of $x$ and $y$ under $f^N$, respectively.

By the definition of the operator $\mathcal{L}^N$ and the constant $|\mathcal{L}^N(\varphi)|_{\alpha,\delta}$ we obtain, by using triangle inequality, the following
\[
\frac{|\mathcal{L}^N(\varphi)|_{\alpha,\delta}}{\inf \mathcal{L}^N(\varphi)} \leq \sup_{d(x,y) < \delta} \frac{|\mathcal{L}^N(\varphi(x)) - \mathcal{L}^N(\varphi(y))|}{\inf \mathcal{L}^N(\varphi) d(x,y)^\alpha} \\
\leq \frac{\sum_{j=1}^{[\deg(f)]^N} e^{S_N\phi(x_j)}|\varphi(x_j) - e^{S_N\phi(y_j)}\varphi(y_j)|}{\inf \mathcal{L}^N(\varphi) d(x,y)^\alpha} \\
\leq \frac{\sum_{j=1}^{[\deg(f)]^N} e^{S_N\phi(x_j)}|\varphi(x_j) - \varphi(y_j)|}{\inf \mathcal{L}^N(\varphi) d(x,y)^\alpha} + \frac{\sum_{j=1}^{[\deg(f)]^N} \|\varphi(y_j)||e^{S_N\phi(x_j)} - e^{S_N\phi(y_j)}|}{\inf \mathcal{L}^N(\varphi) d(x,y)^\alpha}. \tag{10}
\]

From the definition of the supremum of functions, it follows that the last sums are bounded from above by
\[
e^{N\sup \varphi} \sum_{j=1}^{[\deg(f)]^N} |\varphi(x_j) - \varphi(y_j)| + \sup \varphi \frac{e^{N\phi(x_j)} - e^{N\phi(y_j)}}{\inf \mathcal{L}^N(\varphi) d(x,y)^\alpha}. \tag{11}
\]

On the other hand, by Remark 1 and equation (9), the previous sums can be bounded from above by
\[
e^{N\sup \varphi} [\varphi + 1]|\varphi|_{\alpha,\delta} \sum_{j=1}^{[\deg(f)]^N} d(x_j, y_j)^\alpha + \sup \varphi e^{N\phi}|\alpha| \sum_{j=1}^{[\deg(f)]^N} d(x_j, y_j)^\alpha. \tag{11}
\]
Now, by Lemma 3.4 we conclude that the expression 11 is less or equal than
\[
\frac{e^N \sup_{\varphi} |\varphi^N + 1||\varphi|_{\alpha, \delta} \sum_{j=1}^{[\deg(f)]^N} d(x_j, y_j)^\alpha}{[\deg(f)]^N e^N \inf_{\varphi} \varphi^N d(x, y)^\alpha} \leq 2md^\alpha k \inf_{\varphi} e^N |\varphi|_{\alpha} \sum_{j=1}^{[\deg(f)]^N} d(x_j, y_j)^\alpha + \frac{2md^\alpha k e^N |\varphi|_{\alpha} \sum_{j=1}^{[\deg(f)]^N} d(x_j, y_j)^\alpha}{[\deg(f)]^N e^N \inf_{\varphi} \varphi^N d(x, y)^\alpha}.
\]

Let \( \text{Var} \phi = \sup_{\varphi} \phi - \inf_{\varphi} \phi \). Since \( \varphi \in \mathcal{C}_{k, \delta} \), we can rewrite the expression above as
\[
\frac{e^N \text{Var} \phi |\varphi^N + 1|k \sum_{j=1}^{[\deg(f)]^N} d(x_j, y_j)^\alpha}{[\deg(f)]^N d(x, y)^\alpha} \leq 2md^\alpha k e^N |\varphi|_{\alpha} \sum_{j=1}^{[\deg(f)]^N} d(x_j, y_j)^\alpha + \frac{2md^\alpha k e^N |\varphi|_{\alpha} \sum_{j=1}^{[\deg(f)]^N} d(x_j, y_j)^\alpha}{[\deg(f)]^N e^N \inf_{\varphi} \varphi^N d(x, y)^\alpha}.
\]

Rearranging the indexes, if necessary, we can suppose that
\[
d(x_j, y_j) \leq \varphi^N \cdot d(x, y) \quad \text{for} \quad 1 \leq j \leq [\deg(f)]^N.
\]

Moreover, according to (7), there exists at least one index, let us call it \( M \), such that
\[
d(x_M, y_M) \leq \gamma \cdot d(x, y).
\]

Hence, the sums \( 12 \) are bounded from above by
\[
\frac{e^N \text{Var} \phi |\varphi^N + 1|k \sum_{j=1}^{[\deg(f)]^N} d(x_j, y_j)^\alpha}{[\deg(f)]^N d(x, y)^\alpha} \leq 2md^\alpha k e^N |\varphi|_{\alpha} \sum_{j=1}^{[\deg(f)]^N} d(x_j, y_j)^\alpha + \frac{2md^\alpha k e^N |\varphi|_{\alpha} \sum_{j=1}^{[\deg(f)]^N} d(x_j, y_j)^\alpha}{[\deg(f)]^N e^N \inf_{\varphi} \varphi^N d(x, y)^\alpha}.
\]

From (10), (13) and (14), it follows that
\[
\frac{|\mathcal{L}^N(\varphi)|_{\alpha, \delta}}{\inf \mathcal{L}^N(\varphi)} \leq \left( \frac{e^N \text{Var} \phi |\varphi^N + 1| \left[ (\deg(f))^N - 1 \right] \varphi^N + \gamma^\alpha}{[\deg(f)]^N} + \frac{2md^\alpha k e^N |\varphi|_{\alpha} \left[ (\deg(f))^N - 1 \right] \varphi^N + \gamma^\alpha}{[\deg(f)]^N e^N \inf_{\varphi} \varphi^N d(x, y)^\alpha} \right) k.
\]

Therefore, we conclude that
\[
\frac{|\mathcal{L}^N(\varphi)|_{\alpha, \delta}}{\inf \mathcal{L}^N(\varphi)} \leq \hat{\lambda} k,
\]
where
\[
\hat{\lambda} = \left( e^N \text{Var} \phi |\varphi^N + 1| + \frac{2md^\alpha k e^N |\varphi|_{\alpha} |\varphi^N - 1|}{[\deg(f)]^N + \gamma^\alpha} \right) k.
\]

By condition (**), we have \( \hat{\lambda} < 1 \), which completes the proof.

\textbf{Remark 2.} If we assume that \( \varphi^N < 2 \), then estimate (11) can be improved. Indeed, this extra assumption allows us to replace
\[
|\varphi(x_j) - \varphi(y_j)| \leq |\varphi^N + 1||\varphi|_{\alpha, \delta} d(x_j, y_j)^\alpha
\]
by the following
\[
|\varphi(x_j) - \varphi(y_j)| \leq |\varphi^N - 1| + 1 |\varphi|_{\alpha, \delta} d(x_j, y_j)^\alpha.
\]

Now replacing \( |\varphi^N + 1| \) by \( |\varphi^N - 1| + 1 \) in the assumption (**), we enlarge the class of potentials that satisfy this assumption.
We point out that the extra assumption $\vartheta^N < 2$ is in fact quite general. For instance, it is satisfied by the examples presented in Section 8 (as can be seen in [8] for Example 8.1 and in [31] for Example 8.3).

Moreover, note that condition $(**)$ could be improved if we controlled the visits of the orbits to the complement of the non-uniformly expanding set.

The last Proposition shows that the cone $C_{k,\delta}$ is invariant under $L^N_{f,\phi}$, moreover, Proposition 3 ensures that $C_{k,\delta}$ has finite diameter. Therefore Proposition 2 implies the next result.

**Proposition 5.** The operator $L^N_{f,\phi}$ is a contraction in the cone $C_{k,\delta}$ for the constant
\[ \Delta = \text{diam}(C_{k,\delta}) > 0 \] we have
\[ \Theta_k(L^N_{f,\phi}(\varphi), L^N_{f,\phi}(\psi)) \leq (1 - e^{-\Delta}) \cdot \Theta_k(\varphi, \psi) \quad \text{for all } \varphi, \psi \in C_{k,\delta}. \]

Let $\lambda_{f,\phi}$ be the spectral radius of the transfer operator $L_{f,\phi}$. The existence of a probability measure $\nu_{f,\phi}$ satisfying $L^N_{f,\phi} \nu_{f,\phi} = \lambda_{f,\phi} \nu_{f,\phi}$ and $\nu_{f,\phi} (\Sigma_\sigma) = 1$ was proven in [32]. Moreover, [32] also guarantees that $\log \lambda_{f,\phi} = P_f(\phi)$. From the last proposition we will obtain the existence of an eigenfunction $h_{f,\phi}$ of $L_{f,\phi}$ associated to the spectral radius.

**Proposition 6.** There exists a Hölder continuous function $h_{f,\phi} : M \to \mathbb{R}$ bounded away from zero and infinity which satisfies $L_{f,\phi} h_{f,\phi} = \lambda_{f,\phi} h_{f,\phi}$.

**Proof.** Define $L = \lambda^{-N}_{f,\phi} L^N_{f,\phi}$ and consider the sequence $\{L^n(1)\}_{n \in \mathbb{N}}$. Since $\nu_{f,\phi}$ is an eigenmeasure associated to $\lambda_{f,\phi}$, we have for every $n \geq 1$
\[ \int L^n(1) \, d\nu_{f,\phi} = \int \lambda^{-nN} L^N_{f,\phi}(1) \, d\nu_{f,\phi} = \lambda^{-nN} \int 1 \, d(L^N_{f,\phi})^n \nu_{f,\phi} = \int 1 \, d\nu_{f,\phi} = 1. \]

Thus each term of the sequence satisfies $\sup L^n(1) \geq 1$ and $\inf L^n(1) \leq 1$. Note that $1 \in C_{k,\delta}$ and by Proposition 4, the cone $C_{k,\delta}$ is invariant under $L$, then $\{L^n(1)\}$ is a sequence in $C_{k,\delta}$. By Lemma 3.4 every $\varphi \in C_{k,\delta}$ satisfies $\sup \varphi \leq \inf \varphi \cdot (2 \cdot m \cdot d^\alpha) \cdot k$, therefore we conclude that $\{L^n(1)\}$ is uniformly bounded away from zero and infinity by
\[ (2 \cdot m \cdot d^\alpha \cdot k)^{-1} \leq \inf L^n(1) \leq 1 \leq \sup L^n(1) \leq 2 \cdot m \cdot d^\alpha \cdot k. \]

Moreover, since $L^n(1)$ is $\alpha$-Hölder continuous in balls of radius $\delta$ for all $n \geq 1$, Lemma 3.3 implies that $L^n(1)$ is an $\alpha m$-Hölder continuous function.

Now we prove that $\{L^n(1)\}$ is a Cauchy sequence in the sup norm. Let $\Delta = \text{diam}(C_{k,\delta})$ and $\tau = 1 - e^{-\Delta}$. Proposition 5 implies that for every $j, l \geq n$ the projective metric satisfies
\[ \Theta_k(L^j(1), L^l(1)) \leq \Delta^{\tau^n}. \]

According to Lemma 3.2 we can write
\[ \Theta_k(L^j(1), L^l(1)) = \log \left( \frac{B_k(L^j(1), L^l(1))}{A_k(L^j(1), L^l(1))} \right), \]
and combining with the last inequality we obtain
\[ e^{-\Delta^{\tau^n}} \leq A_k(L^j(1), L^l(1)) \leq \inf \frac{L^j(1)}{L^l(1)}. \]
that for every Proposition 7.

5.5] and therefore it will be omitted here.

Note that second and fifth inequalities follow from the second part of Lemma 3.2.

Then for all \( j, l \geq n \), we have:

\[
\|L^{j}(1) - L^{l}(1)\|_{0} \leq \|L^{j}(1)\|_{0} \left(\frac{\|L^{l}(1)\|_{0} - 1}{\|L^{j}(1)\|_{0}}\right) \leq \|L^{l}(1)\|_{0} \left(e^{\Delta \tau^{n}} - 1\right) \leq \hat{R} \Delta \tau^{n},
\]

which proves that \( \{L^{n}(1)\} \) is a Cauchy sequence. Therefore \( \{L^{n}(1)\} \) converges uniformly to a function \( h_{f,\phi} : M \to \mathbb{R} \) in the cone \( C_{\delta} \) and consequently \( \alpha \mu \)-Hölder continuous and bounded away from zero and infinity.

It remains to check that \( L_{f,\phi} h_{f,\phi} = \lambda_{f,\phi} h_{f,\phi} \). Notice that if we replace in the definition of the sequence the function \( 1 \) by \( \lambda_{f,\phi}^{-1} L_{f,\phi}(1) \), a similar argument shows that the sequence \( \{L^{n}(\lambda_{f,\phi}^{-1} L_{f,\phi}(1))\} \) converges to \( h_{f,\phi} \). From the continuity of the transfer operator we conclude that

\[
L_{f,\phi}(h_{f,\phi}) = L_{f,\phi} \left(\lim_{n \to \infty} L^{n}(1)\right) = L_{f,\phi} \left(\lim_{n \to \infty} \lambda_{f,\phi}^{-n} L_{f,\phi}^{n}(1)\right)
\]

\[= L_{f,\phi} \left(\lambda_{f,\phi}^{-n} L_{f,\phi}^{n}(1)\right) = \lambda_{f,\phi} L_{f,\phi} \left(\lambda_{f,\phi}^{-n-1} L_{f,\phi}^{n+1}(1)\right)\]

\[= \lambda_{f,\phi} h_{f,\phi} \phi.
\]

Proposition 5 implies the next result. Its proof is analogous to [31], Proposition 5.5 and therefore it will be omitted here.

**Proposition 7.** Let \( (f, \phi) \in \mathcal{H}_{\sigma} \). There exist a constant \( R > 0 \) and \( 0 < \tau < 1 \) such that for every \( \varphi \in C_{\delta} \) satisfying \( \int \varphi \, dv_{f,\phi} = 1 \) we have

\[
\left\|\lambda_{f,\phi}^{-n} L_{f,\phi}^{n}(\varphi) - h_{f,\phi}\phi\right\|_{\alpha} \leq R \tau^{n} \quad \forall n \geq 1.
\]

We finish this section by proving the spectral gap property of the transfer operator.

**Theorem 4.1.** For \( (f, \phi) \in \mathcal{H}_{\sigma} \), the spectrum of the operator \( L_{f,\phi} \), acting on the space \( C^{\alpha}(M) \), has a decomposition: there exists \( 0 < r < \lambda_{f,\phi} \) such that \( \text{Spec}(L_{f,\phi}) = \{\lambda_{f,\phi}\} \cup \Sigma \) with \( \Sigma \) contained in a ball \( B(0, r) \) centered at zero and of radius \( r \).

**Proof.** Let \( \mathcal{L} = \lambda_{f,\phi}^{-1} L_{f,\phi} \) be the normalized operator. Consider the space \( E_{0} = \{\psi \in C^{\alpha}(M) : \int \psi \, dv_{f,\phi} = 0\} \) and let \( E_{1} \) be the eigenspace of dimension 1 of \( \mathcal{L} \) associated to the eigenvalue 1. We point out that \( \dim E_{1} = 1 \) since \( h_{f,\phi} \phi \) is the unique eigenfunction associated to the spectral radius. Notice that it is possible to decompose \( C^{\alpha}(M) \) as a direct sum of \( E_{0} \) and \( E_{1} \) by writing any \( \varphi \in C^{\alpha}(M) \) as follows

\[
\varphi = \left[\varphi - \left(\int \varphi \, dv_{f,\phi}\right) \cdot h_{f,\phi}\right] + \left[\left(\int \varphi \, dv_{f,\phi}\right) \cdot h_{f,\phi}\right] = \varphi_{0} + \varphi_{1}
\]

with \( \varphi_{0} \in E_{0} \) and \( \varphi_{1} \in E_{1} \).
In fact, since \( \int h_{f,\phi} \, d\nu_{f,\phi} = 1 \), we derive that \( \varphi_0 := \left[ \varphi - \int \varphi \, d\nu_{f,\phi} \cdot h_{f,\phi} \right] \) satisfies
\[
\int \varphi_0 \, d\nu_{f,\phi} = \int \left( \varphi - \int \varphi \, d\nu_{f,\phi} \cdot h_{f,\phi} \right) \, d\nu_{f,\phi} = \int \varphi \, d\nu_{f,\phi} - \int \varphi \, d\nu_{f,\phi} \cdot \int h_{f,\phi} \, d\nu_{f,\phi} = 0
\]
and so it is an element of \( E_0 \). Moreover \( \varphi_1 = \left[ (\int \varphi \, d\nu_{f,\phi}) \cdot h_{f,\phi} \right] \) belongs to \( E_1 \) because
\[
\mathcal{L}(\varphi_1) = \lambda_{f,\phi}^{-1} \mathcal{L}_{f,\phi}(\varphi_1) = \lambda_{f,\phi}^{-1} \mathcal{L}_{f,\phi} \left( \int \varphi \, d\nu_{f,\phi} \cdot h_{f,\phi} \right)
\]
\[
= \lambda_{f,\phi}^{-1} \int \varphi \, d\nu_{f,\phi} \cdot \lambda_{f,\phi} h_{f,\phi} = \varphi_1.
\]
Now it is enough to show that \( \mathcal{L}^n \) is a contraction in \( E_0 \) for \( n \) sufficiently large.

Fix \( k > 0 \) large enough. Given \( \varphi \in E_0 \) with \( |\varphi|_{\alpha,\delta} \leq 1 \) notice that \( \varphi \) does not necessarily belong to the cone \( C_{k,\delta} \) but for example \( (\varphi + 2) \in C_{k,\delta} \) since
\[
\frac{|\varphi + 2|_{\alpha,\delta}}{\inf (\varphi + 2)} = \frac{|\varphi|_{\alpha,\delta}}{\inf (\varphi + 2)} \leq \frac{1}{\inf (\varphi + 2)} \leq k \quad \text{for } k \text{ large}.
\]
Therefore applying Proposition 7 we derive that
\[
||\mathcal{L}^n(\varphi)||_\alpha = ||\mathcal{L}^n(\varphi + 2) - \mathcal{L}^n(2)||_\alpha
\]
\[
\leq ||\mathcal{L}^n(\varphi + 2) - 2h_{f,\phi}||_\alpha + ||\mathcal{L}^n(2) - 2h_{f,\phi}||_\alpha
\]
\[
\leq \left\| \left( \int \varphi + 2 \, d\nu_{f,\phi} \right) \mathcal{L}^n \left( \frac{\varphi + 2}{\int \varphi + 2 \, d\nu_{f,\phi}} \right) - 2h_{f,\phi} \right\|_\alpha
\]
\[
+ ||\mathcal{L}^n(2) - 2h_{f,\phi}||_\alpha
\]
\[
\leq 2 \left\| \mathcal{L}^n \left( \frac{\varphi + 2}{\int \varphi + 2 \, d\nu_{f,\phi}} \right) - h_{f,\phi} \right\|_\alpha + 2 ||\mathcal{L}^n(1) - h_{f,\phi}||_\alpha
\]
\[
\leq 2L^n \tau^n + 2L^n \tau^n = 4L^n \tau^n.
\]

This contraction shows that the spectrum of \( \mathcal{L} \) admits a decomposition \( \text{Spec}(\mathcal{L}) = \{1\} \cup \Sigma_0 \) where \( \Sigma_0 \) is contained in a ball centered at zero and radius strictly less than one. To conclude the proof just observe that we obtain the spectrum of \( \mathcal{L}_{f,\phi} \) by multiplying the spectrum of \( \mathcal{L} \) by \( \lambda_{f,\phi} \). □

The spectral gap property established in the previous result is the key tool we will use to derive our main results. In the following sections we prove these results employing probabilistic and analytic arguments. In [16] the authors use a differential geometrical approach to obtain consequences of the spectral gap property for a rather general framework. Some of our results could also be obtained using [16]. More specifically, Theorem II follows from [16, Theorem 5.6], while Theorem V follows from [16, Corollary B].
5. Statistical behavior of the equilibrium state. In this section we conclude the proof of Theorem IV and derive statistical properties of the equilibrium state, namely Theorem I and Theorem II. We show that a classical proof of the exponential decay of correlations holds in this context, for the sake of completeness. We end the section recalling that a central limit theorem can be obtained by applying the well known Gordin Theorem.

Let \( \mu_{f,\phi} := h_{f,\phi} \nu_{f,\phi} \), where \( \mathcal{L}_{f,\phi} h_{f,\phi} = \lambda_{f,\phi} h_{f,\phi} \) and \( \mathcal{L}_{f,\phi}^{\tau} \nu_{f,\phi} = \lambda_{f,\phi} \nu_{f,\phi} \). It is straightforward to check that \( \mu_{f,\phi} \) is invariant under \( f \). Moreover, since \( \nu_{f,\phi}(\Sigma_\alpha) = 1 \), we also have that \( \mu_{f,\phi}(\Sigma_\alpha) = 1 \).

Our goal is to prove that \( \mu_{f,\phi} \) is the unique equilibrium state of \( (f, \phi) \), which finishes the proof of Theorem IV. However, we first establish that the decay of correlations is exponential for the probability measure \( \mu_{f,\phi} \). For this we can borrow some ideas from [31] since we have already proved that the transfer operator has a spectral gap property.

**Proposition 8.** For every \( (f, \phi) \in \mathcal{H}_\alpha \) the invariant measure \( \mu_{f,\phi} \) has exponential decay of correlations for Hölder continuous observables: there exists \( 0 < \tau < 1 \) such that for all \( \varphi \in L^1(\mu_{f,\phi}) \) and \( \psi \in C^\alpha(M) \) there exists a positive constant \( K(\varphi, \psi) \) satisfying:

\[
\left| \int (\varphi \circ f^n) \cdot \psi \, d\mu_{f,\phi} - \int \varphi \, d\mu_{f,\phi} \int \psi \, d\mu_{f,\phi} \right| \leq K(\varphi, \psi) \tau^n \quad \text{for all } n \geq 1.
\]

**Proof.** Let \( \varphi, \psi \in C^\alpha(M) \) and note that the transfer operator satisfy the following for all \( n \in \mathbb{N} \)

\[
\mathcal{L}^n((\varphi \circ f^n) \cdot \psi) = \varphi \cdot \mathcal{L}^n(\psi).
\]

Recall that by Proposition 6 the eigenfunction of the transfer operator \( h_{f,\phi} \) is bounded away from zero and infinity. We first assume that \( \psi \cdot h \in C_{k,\delta} \) for \( k \) large enough and without loss of generality, we can consider \( \int \psi \, d\mu_{f,\phi} = 1 \).

\[
\left| \int (\varphi \circ f^n) \cdot \psi \, d\mu_{f,\phi} - \int \varphi \, d\mu_{f,\phi} \int \psi \, d\mu_{f,\phi} \right| = \left| \int \varphi \cdot \lambda_{f,\phi}^{-n} \mathcal{L}_{f,\phi}^{n} \psi \, d\nu_{f,\phi} - \int \varphi \, d\mu_{f,\phi} \right|
\]

\[
= \int \varphi \cdot \left[ \frac{\lambda_{f,\phi}^{-n} \mathcal{L}_{f,\phi}^{n} (\psi \cdot h_{f,\phi})}{h_{f,\phi}} - 1 \right] \, d\mu_{f,\phi}
\]

\[
\leq \left\| \varphi \right\| \left\| \frac{\lambda_{f,\phi}^{-n} \mathcal{L}_{f,\phi}^{n} (\psi \cdot h_{f,\phi})}{h_{f,\phi}} - 1 \right\|_0.
\]

Therefore, applying Proposition 7, there exists some positive constant \( L_1 \) such that

\[
\left\| \frac{\lambda_{f,\phi}^{-n} \mathcal{L}_{f,\phi}^{n} (\psi \cdot h_{f,\phi})}{h_{f,\phi}} - 1 \right\|_0 \leq \left\| h_{f,\phi} \right\|_0 \left\| \lambda_{f,\phi}^{-n} \mathcal{L}_{f,\phi}^{n} (\psi \cdot h_{f,\phi}) - h_{f,\phi} \right\|_0 \leq L_1 \tau^n.
\]

Now if \( \psi \cdot h \notin C_{k,\delta} \) we fix \( B = k^{-1} \| \psi \cdot h \|_{k,\delta} \) and consider \( \xi := \psi \cdot h \) where

\[
\xi = \xi^+_B - \xi^-_B \quad \text{and} \quad \xi^\pm_B = \frac{1}{2} (|\xi| \pm \xi) + B.
\]

Thus \( \xi^\pm_B \in C_{k,\delta} \) and then we apply the previous estimates to \( \xi^\pm_B \). The result follows by linearity. \( \square \)
We recall that a measure that is invariant under a map $f$ is called exact if it satisfies \( \lim_{n \to +\infty} \mu(f^n(A)) = 1 \) for all measurable set $A$ such that $\mu(A) > 0$. In particular, an exact probability measure is ergodic.

The exponential decay of correlations implies the exactness property. The reader can see a proof of this result in [[31], Corollary 6.2].

**Corollary 2.** The invariant measure $\mu_{f,\phi}$ is exact.

Now we can prove that $\mu_{f,\phi} := h_{f,\phi}\nu_{f,\phi}$ is the equilibrium state associated to $(f,\phi) \in \mathcal{H}_{\sigma}$ and complete the proof of Theorem IV.

In [32] it was proved that $\nu_{f,\phi}$ satisfies a type of Gibbs property at hyperbolic times: for $\varepsilon \leq \delta$ there exists $C = C(\varepsilon) > 0$ such that if $n$ is a hyperbolic time for $x \in M$ then

\[
C^{-1} \leq \frac{\nu_{f,\phi}(B_\varepsilon(x,n))}{\exp(S_n\phi(y) - n \log \lambda_{f,\phi})} \leq C.
\]

for all $y \in B_\varepsilon(x,n)$. Recalling that the density $h_{f,\phi}$ is bounded away from zero, it follows that $\mu_{f,\phi}$ is equivalent to $\nu_{f,\phi}$ and, thus, $\mu_{f,\phi}$ also satisfies the Gibbs property at hyperbolic times:

\[
\tilde{C}^{-1} \leq \frac{\mu_{f,\phi}(B_\varepsilon(x,n))}{\exp(S_n\phi(y) - n \log \lambda_{f,\phi})} \leq \tilde{C}.
\]

Rewriting the inequalities above we have for $\mu_{f,\phi}$-almost every point $x \in M$

\[
\log \lambda_{f,\phi} - \lim_{n \to \infty} S_n\phi(y) \leq \limsup_{n \to \infty} \frac{1}{n} \log \mu_{f,\phi}(B_\varepsilon(x,n)) \leq \log \lambda_{f,\phi} - \lim_{n \to \infty} S_n\phi(y),
\]

where the limit was considered when $n$ goes to infinity since $\mu_{f,\phi}$-almost every point $x \in M$ admit infinitely many hyperbolic times. From Birkhoff’s Ergodic Theorem we obtain that

\[
\log \lambda_{f,\phi} - \int \phi \, d\mu_{f,\phi} \leq \limsup_{n \to \infty} \frac{1}{n} \log \mu_{f,\phi}(B_\varepsilon(x,n)) \leq \log \lambda_{f,\phi} - \int \phi \, d\mu_{f,\phi}.
\]

Taking the limit when $\varepsilon$ goes to zero the Brin-Katok entropy formula implies

\[
h_{\mu_{f,\phi}}(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu_{f,\phi}(B_\varepsilon(x,n)) = \log \lambda_{f,\phi} - \int \phi \, d\mu_{f,\phi}.
\]

Recalling that the topological pressure $P_f(\phi)$ is equal to $\log \lambda_{f,\phi}$ we have proved that $\mu_{f,\phi}$ is an equilibrium state for $(f,\phi) \in \mathcal{H}_{\sigma}$. By the uniqueness established in [32] we conclude the proof of Theorem IV.

A central limit theorem for $\mu_{f,\phi}$ can be obtained from the exponential decay of correlations, as stated in Theorem II. Its proof is obtained by applying a non-invertible case of an abstract central limit theorem due to Gordin, which can be found in [[42], Theorem 2.11]. The reader can verify the steps of a similar proof in [[31], Theorem E].
6. Analyticity of thermodynamical quantities with respect to the potential. In this section we treat the analyticity of the thermodynamical quantities as the potential varies. Since these quantities are intrinsically related with the spectrum of the transfer operator, we will analyze its behavior under analytic perturbations.

More specifically, the spectral gap property of the transfer operator allows us to consider the projection operator defined in Subsection 3.4. Using the perturbation theory due to Kato in [20], we prove the analyticity of this operator. Finally, we can describe the thermodynamical quantities in terms of the projection operator and thus, the analytical dependence will follow.

Since we are fixing the underlying dynamics and varying only the potential we simplify the notation by omitting the dynamics as follows \( \mathcal{L}_{f, \phi} = \mathcal{L}_\phi, \mu_{f, \phi} = \mu_\phi, \nu_{f, \phi} = \nu_\phi \) and \( h_{f, \phi} = h_\phi \).

We begin defining analyticity for operators on Banach spaces. Most properties of the classical analytic functions setting remain true in this context.

Let \( X, Y \) be Banach vector spaces. Denote by \( L^k_k(X, Y) \) the space of symmetric \( k \)-linear maps from the \( k \)-fold product \( X^k := X \times \cdots \times X \) into \( Y \). Given \( T_k \in L^k_k(X, Y) \) and \( (H, \ldots, H) \in X^k \) we write \( T_k(H^k) := T_k(H, \ldots, H) \).

We say that a mapping \( \Gamma: U \subset X \to Y \) defined on an open subset \( U \subset X \) is analytic if for all \( x \in U \) there exist \( \varepsilon > 0 \) and \( T_k := T_k(x) \in L^k_k(X, Y) \), for every \( k \geq 1 \), depending only on \( x \) such that
\[
\Gamma(x + H) = \Gamma(x) + \sum_{k=1}^{+\infty} \frac{1}{k!} T_k(H^k)
\]
for all \( H \) in an \( \varepsilon \)-neighborhood of zero and the series is uniformly convergent.

Given a potential \( \phi \in C^\alpha(M) \) it easily follows that \( \mathcal{L}_\phi(\psi) \in C^\alpha(M) \) for any \( \psi \in C^\alpha(M) \). In [8] it was proved that the application which associates \( \phi \in C^\alpha(M) \) to the transfer operator \( \mathcal{L}_\phi: C^\alpha(M) \to C^\alpha(M) \) is analytic.

In the next theorem we prove the analyticity on the potential of the projection mapping. We follow closely the ideas of Sarig [39, Theorem 5.6].

**Theorem 6.1.** Given \( \phi_0 \in \mathcal{P}_\sigma \), let \( \lambda_{\phi_0} \) be the spectral radius of \( \mathcal{L}_{\phi_0} \) and let \( \gamma \) be a closed smooth simple curve which separates \( \lambda_{\phi_0} \) from the rest of the spectrum. Then the projection mapping
\[
E(\phi) := \frac{1}{2\pi i} \int_{\gamma} (zI - \mathcal{L}_\phi)^{-1} \, dz
\]
is analytic in a neighborhood of \( \phi_0 \) contained in \( \mathcal{P}_\sigma \).

**Proof.** Consider the set
\[
\Omega := \{(z, \phi) \in \mathbb{C} \times \mathcal{P}_\sigma : (zI - \mathcal{L}_\phi) \text{ has a bounded inverse}\}.
\]
Note that \( \Omega \) is an open set in the product topology. In fact, given \( (z, \phi) \in \Omega \) since \( \mathcal{P}_\sigma \) is open and the map \( \phi \in \mathcal{P}_\sigma \mapsto \mathcal{L}_\phi \) is analytic (and therefore continuous) we can obtain \( \delta > 0 \) sufficiently small such that for all \( \psi \in B(\phi, \delta) \) its transfer operator \( \mathcal{L}_\psi \) is close to \( \mathcal{L}_\phi \). Moreover we have \( (zI - \mathcal{L}_\psi) \) is close to \( (zI - \mathcal{L}_{\phi_0}) \) for all \( z \in \mathbb{C}(\tau, \delta) \) and for all \( \psi \in B(\phi, \delta) \). Lemma 3.5 assures that \( (zI - \mathcal{L}_\psi) \) has a bounded inverse for all \( z \in \mathbb{B}(\tau, \delta) \) and for all \( \psi \in B(\phi, \delta) \) and, thus, \( \Omega \) is open.

Let \( \phi_0 \in \mathcal{P}_\sigma \) and let \( \gamma \) be a closed smooth simple curve that separates \( \lambda_{\phi_0} \) and \( \text{Spec}(\mathcal{L}_{\phi_0}) \setminus \{\lambda_{\phi_0}\} \). We can assume that \( \gamma \) contains \( \lambda_{\phi_0} \) in its interior and
Spec(\mathcal{L}_{\phi_0}) \setminus \{\lambda_{\phi_0}\} \text{ in its exterior. In this way, each } z \in \gamma \text{ belongs to the set } Res(\mathcal{L}_{\phi_0}) \text{ and thus } (zI - \mathcal{L}_{\phi_0}) \text{ has a bounded inverse. Furthermore, since } \gamma \text{ is compact and the existence of a bounded inverse is an open property (see Lemma 3.5), there exists } 
\varepsilon > 0 \text{ small enough such that } B(\phi_0, \varepsilon) \subset P_\sigma \text{ and for all } z \in \gamma \text{ and } \phi \in B(\phi_0, \varepsilon) \text{ we have } (z, \phi) \in \Omega. \text{ In particular, } \Omega \text{ contains the set } \gamma \times B(\phi_0, \varepsilon).

The resolvent map

\[ R(z, \phi) := (zI - \mathcal{L}_{\phi})^{-1} \]

is well-defined and bounded on \( \Omega \).

In order to prove that the projection \( E(\phi) = \frac{1}{2\pi i} \int_\gamma (zI - \mathcal{L}_{\phi})^{-1} dz \) is analytic, we will show that the resolvent map is analytic on \( \Omega \).

Given \((z_0, \phi_0) \in \Omega\), using the proof of Lemma 3.5 we can write

\[ R(z, \phi) = (z_0I - \mathcal{L}_{\phi_0} - [(z_0 - z)I + (\mathcal{L}_{\phi} - \mathcal{L}_{\phi_0})])^{-1} = R(z_0, \phi_0)I - [(z_0 - z)I + (\mathcal{L}_{\phi} - \mathcal{L}_{\phi_0})]R(z_0, \phi_0))^{-1} = R(z_0, \phi_0) \sum_{n=0}^{\infty} [[(z_0 - z)I + (\mathcal{L}_{\phi} - \mathcal{L}_{\phi_0})]R(z_0, \phi_0)]^n, \quad (15) \]

provided that \( \|(z_0 - z)I + (\mathcal{L}_{\phi} - \mathcal{L}_{\phi_0})\| < 1 / \|R(z_0, \phi_0)\| \). Note that the inequality is satisfied in a small neighborhood \( V(z_0, \phi_0) \) of \((z_0, \phi_0)\) and the convergence is uniform on compact subsets of that neighborhood. Moreover, since the map \( \phi \in P_\sigma \mapsto \mathcal{L}_{\phi} \) is analytic at \( \phi_0 \) we can conclude that \( R(z, \phi) \) can be expressed as a double power series in \((z - z_0)\) and \((\phi - \phi_0)\) in \( V(z_0, \phi_0) \). This ensures that the resolvent map is analytic at \((z_0, \phi_0)\). Since the choice of \((z_0, \phi_0)\) was arbitrary the resolvent map is analytic on \( \Omega \).

Now to prove that the projection is analytic we write \( R(z, \phi) \) as a power series in \((\phi - \phi_0)\) with coefficients depending on \( z \). We can set \( z_0 = z \) in equation \( (15) \) obtaining

\[ R(z, \phi) = R(z, \phi_0) \sum_{n=0}^{\infty} [(\mathcal{L}_{\phi} - \mathcal{L}_{\phi_0})R(z, \phi_0)]^n. \]

Again, \( \mathcal{L}_{\phi} \) is analytic at \( \phi_0 \) and therefore \( (\mathcal{L}_{\phi} - \mathcal{L}_{\phi_0}) \) can be written as a power series in \((\phi - \phi_0)\). If we collect the terms which multiply \((\phi - \phi_0)^n\), we obtain the following series expansion for \( R(z, \phi) \) on \( V(z, \phi_0) \):

\[ R(z, \phi) = R(z, \phi_0) + \sum_{n=1}^{\infty} A_n(z)(\phi - \phi_0)^n, \]

where \( A_n(z) \) are operator valued functions which are analytic on \( \{\phi: \|\phi - \phi_0\| < \eta\} \) for some \( \eta = \eta(z, \phi_0) \).

By compactness we can cover the set \( \{(z, \phi): z \in \gamma,\|\phi - \phi_0\| \leq \eta\} \) with a finite number of neighborhoods \( V(z, \phi_0) \) as above. Therefore there exists \( \eta > 0 \) small enough such that for any \( z \in \gamma \) and \( \|\phi - \phi_0\| < \eta \) we have

\[ R(z, \phi) = R(z, \phi_0) + \sum_{n=1}^{\infty} A_n(z)(\phi - \phi_0)^n, \]

where \( A_n(z) \) are continuous on \( \gamma \) and the series converges uniformly in norm.
Integrating over $\gamma$, we conclude that $E(\phi)$ has a norm convergent power series expansion as follows

$$E(\phi) = \frac{1}{2\pi i} \int_{\gamma} R(z, \phi) \, dz$$

$$= \frac{1}{2\pi i} \int_{\gamma} \left[ R(z, \phi_0) + \sum_{n=1}^{\infty} A_n(z)(\phi - \phi_0)^n \right] \, dz$$

$$= E(\phi_0) + \frac{1}{2\pi i} \sum_{n=1}^{\infty} (\phi - \phi_0)^n \int_{\gamma} A_n(z) \, dz.$$

Then the projection is analytic in a neighborhood of $\phi_0$.

Now we prove Theorem V. We start by showing that the topological pressure function is analytic.

Given $\phi_0 \in \mathcal{P}_\sigma$ let $\lambda_{\phi_0}$ be the spectral radius of $\mathcal{L}_{\phi_0}$. Consider $\gamma$ a closed smooth simple curve which contains $\lambda_{\phi_0}$ in its interior and separates $\lambda_{\phi_0}$ from the rest of the spectrum. Recall that equation (2) gives us bounds for the spectral radius of a transfer operator depending only on the map $f$ (which is fix here) and on the potential. From this, if $\phi$ is close enough to $\phi_0$ we infer that the spectral radius $\lambda_\phi$ is also in the interior of $\gamma$. Moreover, by considering a smaller neighborhood of $\phi_0$, if needed, we can assume by Lemma 3.5 that $\gamma$ also separates $\lambda_\phi$ from the rest of the spectrum of $\mathcal{L}_\phi$. Hence, applying Corollary 1, we conclude that

$$E_\phi := E(\phi) = \frac{1}{2\pi i} \int_{\gamma} (zI - \mathcal{L}_\phi)^{-1} \, dz$$

is the eigenprojection of $\lambda_\phi$ for $\mathcal{L}_\phi$.

We claim that the spectral radius function is analytic on a neighborhood of $\phi_0$. Indeed, since the family $\{\mathcal{L}_\phi\}_{\phi \in \mathcal{P}_\sigma}$ has the spectral gap property (and by Corollary 1) it follows that $\dim(\text{Im}(E_\phi)) = 1$ for every $\phi \in \mathcal{P}_\sigma$. In particular, there is $\varphi \in C^0(M)$ such that $E_{\phi_0}(\varphi) \neq 0$ and by the well-known Hahn-Banach Theorem there exists $\eta \in (C^0(M))^*$ satisfying $\eta(E_{\phi_0}(\varphi)) \neq 0$. By continuity of $\eta(E_{\phi_0}(\varphi))$, we have $\eta(E_{\phi}(\varphi)) \neq 0$ for every $\phi$ in a small neighborhood of $\phi_0$.

Now define the mapping

$$\Psi(\phi) := \eta(\mathcal{L}_\phi(E_\phi(\varphi))) = \int \mathcal{L}_\phi \circ E_\phi(\varphi) \, d\eta.$$

By the analyticity of the transfer operator and the projection map, $\Psi$ is analytic at $\phi_0$. Corollary 1 guarantees that $\text{Im}(E_\phi)$ is the eigenspace associated to $\lambda_\phi$ then it follows

$$\eta(\mathcal{L}_\phi(E_\phi(\varphi))) = \int \mathcal{L}_\phi \circ E_\phi(\varphi) \, d\eta = \int \lambda_\phi E_\phi(\varphi) \, d\eta = \lambda_\phi \eta(E_\phi(\varphi)).$$

Since $\eta(E_\phi(\varphi)) \neq 0$ we can write

$$\lambda_\phi = \frac{\eta(\mathcal{L}_\phi(E_\phi(\varphi)))}{\eta(E_\phi(\varphi))}$$

and conclude that the map $\phi \in \mathcal{P}_\sigma \mapsto \lambda_\phi$ is analytic at $\phi_0$. Recalling that for all $\phi \in \mathcal{P}_\sigma$ the topological pressure $P_f(\phi)$ satisfies $\lambda_\phi = \exp(P_f(\phi))$ we have as an immediate consequence that the map $\phi \in \mathcal{P}_\sigma \mapsto P_f(\phi)$ is analytic. Thus we prove item (i) of Theorem V.
Let $E_\phi^0 = \{ \psi \in C^\alpha(M) : \int \psi \, d\nu_\phi = 0 \}$ and let $E_\phi^1$ be the eigenspace associated to the spectral radius $\lambda_\phi$ of $\mathcal{L}_\phi$. As in Theorem 4.1, we decompose $C^\alpha(M)$ as a direct sum of $E_\phi^0$ and $E_\phi^1$, given $\varphi \in C^\alpha(M)$ we can write

$$\varphi = \left[ \varphi - \left( \int \varphi \, d\nu_\phi \right) \cdot h_\phi \right] + \left[ \left( \int \varphi \, d\nu_\phi \right) \cdot h_\phi \right] = \varphi_0 + \varphi_1$$

with $\varphi_0 \in E_\phi^0$ and $\varphi_1 \in E_\phi^1$.

Now, considering $\varphi \equiv 1$, we derive that $\varphi_1 = h_\phi$ and this implies that $h_\phi$ is the projection of the function 1 on the space $E_\phi^1$. It follows from Corollary 1

$$h_\phi = E_\phi(1) = \frac{1}{2\pi i} \int_\gamma (zI - \mathcal{L}_\phi)^{-1} dz \, (1).$$

Applying Theorem 6.1 we conclude that $h_\phi$ varies analytically with respect to $\phi$ which proves item (ii).

Moreover, since the projection of any $\varphi \in C^\alpha(M)$ on the space $E_\phi^1$ is given by $\varphi_1 = \left[ \left( \int \varphi \, d\nu_\phi \right) \cdot h_\phi \right]$ it follows

$$E_\phi(\varphi) = \varphi_1 = \int \varphi \, d\nu_\phi \cdot E_\phi(1).$$

Therefore from the relation

$$\nu_\phi(\cdot) = \frac{E_\phi(\cdot)}{E_\phi(1)}$$

we obtain that the map $\phi \mapsto \nu_\phi \in (C^\alpha(M))^*$ is analytic, thus proving item (iii) of Theorem V. In particular, item (iv) follows from the previous results since that $\mu_\phi = h_\phi \nu_\phi$.

7. Applications: a class of non-uniformly hyperbolic skew products. In this section we extend our results to a wider class by considering a family of skew products over non-uniformly expanding maps.

Consider $N$ a compact metric space with distance $d$ and let $g : M \times N \to N$ be a continuous map uniformly contractive on $N$, i.e., there exists $0 < \lambda < 1$ such that for all $x \in M$ and all $y_1, y_2 \in N$ we have

$$d\left(g(x, y_1), g(x, y_2)\right) \leq \lambda d(y_1, y_2). \quad (16)$$

Suppose that there exists some $\bar{y} \in N$ such that $g(x, \bar{y}) = \bar{y}$ for every $x \in M$. We consider a family $\mathcal{S}$ of skew-product maps $F : M \times N \to M \times N$ such that

$$F(x, y) = (f(x), g(x, y))$$

for all $(x, y) \in M \times N$, where the base dynamics $f : M \to M$ belongs to $\mathcal{F}$ and is strongly topologically mixing and the fiber dynamics $g : M \times N \to N$ satisfies (16).

**Remark 3.** If we can write $N = N_1 \cup \cdots \cup N_n$, where $N_1, \ldots, N_n$ are pairwise disjoint compact sets then the condition above, $g(x, \bar{y}) = \bar{y}$ for all $x \in M$, can be replaced by $g_i(x, y_i) = y_i$ for all $x \in M$ and some $y_i \in N_i$, $i = 1, \ldots, n$. That is because we can define $n$ fiber dynamics $g_i : M \times N_i \to N_i$ by $g_i(x, y) = g(x, y)$ for $y \in N_i$, for each $i = 1, \ldots, n$, since $M \times N$ is a product space and $N = N_1 \cup \cdots \cup N_n$. See Example 3.
Given $\sigma \in (0,1)$ we say that a continuous potential $\Phi: M \times N \to \mathbb{R}$ is a $\sigma$-hyperbolic potential for $F \in \mathcal{S}$ if the topological pressure of the system $(F, \Phi)$ is equal to the relative pressure on the set $\Sigma_\sigma(f) \times N$. More precisely,

$$P_F(\phi, [\Sigma_\sigma(f)]^c \times N) < P_F(\phi, \Sigma_\sigma(f) \times N) = P_F(\phi).$$

We consider the family $\mathcal{G}_{\sigma}$ of pairs $(F, \Phi) \in \mathcal{S} \times C^\alpha(M \times N)$ such that $(\ast)$ holds for $f$ and $\Phi$ is hyperbolic for $F$ satisfying condition $(\ast \ast)$. The uniqueness of the equilibrium state for $(F, \Phi) \in \mathcal{G}_{\sigma}$ was proven in [2].

Given $\Phi: M \times N \to \mathbb{R}$ a Hölder continuous potential, hyperbolic for $F$ satisfying condition $(\ast \ast)$. It was proved in [[2], Section 5] that $\Phi$ induces a Hölder continuous potential $\phi: M \to \mathbb{R}$ which is hyperbolic for the base dynamics $f$ and satisfies $\operatorname{Var}(\phi) \leq \operatorname{Var}(\Phi)$. Therefore, the induced potential $\phi$ satisfies condition $(\ast \ast)$ as well.

Moreover, by [[2], Lemma 5.3], the unique equilibrium state $\mu_{F,\phi}$ associated to the system $(F, \Phi)$ is given by the push-forward $\pi_* \mu_{F,\phi} = \mu_{f,\phi}$ where $\mu_{f,\phi}$ is the unique equilibrium state of $(f, \phi)$. In other words, for every Borel set $A$ of $M \times N$ we have

$$\mu_{F,\phi}(A) = \mu_{f,\phi}(\pi(A)).$$

Note that the map $\pi$ is analytic and does not depend on the potential $\Phi$. From this and item (iv) of Theorem V, we can derive the analyticity of the equilibrium state when we fix the skew product $F$ and vary the potential $\Phi$ within the family $\mathcal{G}_{\sigma}$.

**Corollary 3.** The equilibrium state varies analytically on the potential within the family $\mathcal{G}_{\sigma}$.

In what follows, we state some statistical properties for the unique equilibrium state of $(F, \Phi) \in \mathcal{G}_{\sigma}$.

The key idea to obtain the exponential decay of correlations for the equilibrium state associated to $(F, \Phi)$ (see the statement of the next result) is to disintegrate this measure as a product of conditional measures on stable fibers by the equilibrium state of the base dynamics. Details of a similar proof of this result can be analyzed in [31]. Due to the similarity, we choose to omit such proof here.

**Corollary 4.** The equilibrium state $\mu_{F,\phi}$ has exponential decay of correlations for Hölder continuous observables: there exists $0 < \tau < 1$ such that for every $\varphi \in L^1(\mu_{F,\phi})$ and $\psi \in C^\alpha(M \times N)$ there exists $K(\varphi, \psi) > 0$ so that

$$\left| \int (\varphi \circ F^n) \psi d\mu_{F,\phi} - \int \varphi d\mu_{F,\phi} \int \psi d\mu_{F,\phi} \right| \leq K(\varphi, \psi) \tau^n, \quad \forall n \geq 1.$$

We also obtain a central limit theorem for the equilibrium state $\mu_{F,\phi}$ of the skew-product $F$ with respect to a potential $\Phi$ as considered above.

**Corollary 5.** Let $\varphi$ be an $\alpha$-Hölder continuous function and let $\tilde{\sigma} \geq 0$ be defined by

$$\tilde{\sigma}^2 = \int \psi^2 d\mu_{F,\phi} + 2 \sum_{n=1}^\infty \int \psi(\psi \circ F^n) d\mu_{F,\phi} \quad \text{where} \quad \psi = \varphi - \int \varphi d\mu_{F,\phi}.$$
Then $\tilde{\sigma}$ is finite and $\tilde{\sigma} = 0$ if and only if $\phi = u \circ F - u$ for some $u \in L^2(\mu_\Phi)$. On the other hand, if $\tilde{\sigma} > 0$ then given any interval $A \subset \mathbb{R}$,

$$
\mu_{F,\Phi}\left(x \in M \times N : \frac{1}{\sqrt{n}} \sum_{j=0}^{n} \left( \phi(F^j(x)) - \int \phi \, d\mu_{F,\Phi} \right) \in A \right) \rightarrow \frac{1}{\tilde{\sigma} \sqrt{2\pi}} \int_A e^{-t^2/2\tilde{\sigma}^2} dt,
$$

as $n$ goes to infinity.

The proof of Corollary 5 also follows from Gordin’s Theorem, which can be applied once we have established the exponential decay of correlations. Therefore, in view of Corollary 4, the proof of Corollary 5 is exactly analogous to the one of Theorem II.

8. Examples. In this section we describe some examples of systems which satisfy our results. We start by presenting a robust class of non-uniformly expanding maps introduced by Alves, Bonatti and Viana [1] and studied by Arbieto, Matheus and Oliveira [3], Oliveira and Viana [29], Varandas and Viana [41].

**Example 1.** Consider $f: M \rightarrow M$ a $C^1$ local diffeomorphism defined on a compact manifold $M$. For $\delta > 0$ small and $\sigma < 1$, consider a covering $Q = \{Q_1, \ldots, Q_q, Q_{q+1}, \ldots, Q_s\}$ of $M$ by domains of injectivity for $f$ and a region $A \subset M$ satisfying:

- (H1) $\|Df^{-1}(x)\| \leq 1 + \delta$, for every $x \in A$;
- (H2) $\|Df^{-1}(x)\| \leq \sigma$, for every $x \in M \setminus A$;
- (H3) $A$ can be covered by $q$ elements of the partition $Q$ with $q < \text{deg}(f)$.

The authors aforementioned showed that there exists a constant $\sigma \in (0, 1)$ and a set $H \subset M$ such that for every $x \in H$ we have

$$
\limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df(f^i(x))^{-1}\| \leq -\sigma.
$$

Furthermore, it was proved that for a Hölder continuous potential $\phi: M \rightarrow \mathbb{R}$ with small variation, i.e.

$$
\sup \phi - \inf \phi < \log \text{deg}(f) - \log q,
$$

it follows that the relative pressure $P(\phi, H)$ satisfies

$$
P_f(\phi, H) < P_f(\phi, H) = P_f(\phi),
$$

and thus $\phi$ is $\sigma$-hyperbolic for $f$.

Denote by $\mathcal{F}$ the class of $C^1$ local diffeomorphisms satisfying the conditions (H1)-(H3) and assume that every $f \in \mathcal{F}$ is strongly topologically mixing. Let $\mathcal{H}$ be the family of pairs $(f, \phi)$, such that $f \in \mathcal{F}$ and $\phi: M \rightarrow \mathbb{R}$ is Hölder continuous, with small variation and satisfies condition $(\ast \ast)$. Our results imply that in this family the equilibrium state has exponential decay of correlations, satisfies a central limit theorem and varies analytically with the potential.

In the next example we present a family of intermittent maps, although it is a particular case of the previous example, we present it here because the reader can check the hypotheses for the results in this paper more easily. Moreover, we want to compare our results for the equilibrium state with some of the ones existing in the literature also for the absolutely continuous invariant probability measure.
Example 2. Let $\beta \in (0, 1)$ be a positive constant and consider the local diffeomorphism defined in $S^1$ by

$$f_{\beta}(x) = \begin{cases} 
  x(1 + 2^\beta x^\beta), & \text{if } 0 \leq x \leq \frac{1}{2} \\
  2x - 1, & \text{if } \frac{1}{2} \leq x \leq 1.
\end{cases}$$

It is straightforward to check that $f_{\beta}$ is strongly topologically mixing. Consider $\mathcal{F}$ the class of $C^1$ local diffeomorphisms $\{f_{\beta}\}_{\beta \in (0,1)}$ and let $\phi : S^1 \to \mathbb{R}$ be Hölder continuous with small variation and satisfying condition $(\ast \ast)$.

Then the unique equilibrium state satisfy a central limit theorem, has exponential decay of correlations, and varies analytically with the potential.

We remark that the statistical properties mentioned above were previously obtained in [24], along with the real analyticity of the topological pressure. We note that the meaning of analyticity in this paper differs from the one in the aforementioned work.

Now, still considering the intermittent maps, we discuss a particular family of potentials of intrinsic interest in the literature.

For each $\beta \in (0,1)$ consider $\{\phi_{\beta,t}\}_{t}$ the family of potentials defined by $\phi_{\beta,t} = -t \log |f'_{\beta}|$. The variation of $\phi_{\beta,t}$ can be made small if $t$ is small:

$$|\phi_{\beta,t}(x) - \phi_{\beta,t}(y)| = |-t \log |f'_{\beta}|(x) + t \log |f'_{\beta}|(y)| = |t| \log \frac{|f'(y)|}{|f'(x)|} \leq |t| \log (2 + \beta).$$

Then for $|t| \leq t_0$ sufficiently small this family satisfies the hypotheses of our results. In particular, this implies that for $|t| \leq t_0$ the topological pressure function $t \to P_{f_{\beta}}(-t \log |f'_\beta|)$ is analytic as well as the equilibrium state.

We point out that our results can be applied for $|t|$ small, the scenario outside this range can be quite different. For example when $t = 1$, it is well-known that $\phi_{\beta} = -\log |f'_\beta|$ admits two ergodic equilibrium states: the Dirac measure centered at the fixed point and the unique invariant probability measure absolutely continuous with respect to Lebesgue (a.c.i.p.).

For the a.c.i.p. the correlations for Hölder observables decay polynomially [[27], [14]]. The polynomial decay is optimal [[38]]. Also, a central limit theorem holds for $\beta < 1/2$ [[38]] and holds for $\beta > 1/2$ considering observables vanishing in a neighborhood of the fixed point and with zero integral [[17]]. Moreover, in [6] and in [22], independently, it was proven that the a.c.i.p. is differentiable.

To end our discussion about this model we highlight that considering a one parameter family of potentials whose derivative near to zero behaves as a polynomial, Sarig [[37]] has proven a phase transition for its topological pressure. In other words, there exists a critical parameter for which the topological pressure is not analytic.

As an application of our corollaries we describe a family of partially hyperbolic horseshoes. This class of maps was defined by Díaz, Horita, Rios and Sambarino in [15] and has been studied in several works [[23], [32] and [33]]. In particular, in [33], it was shown that this family can be modeled by a skew product which base dynamics is strongly topologically mixing and non-uniformly expanding.
Example 3. Consider the cube $R = [0, 1] \times [0, 1] \times [0, 1] \subset \mathbb{R}^3$ and the parallelepipeds

$$R_0 = [0, 1] \times [0, 1] \times [0, 1/6] \quad \text{and} \quad R_1 = [0, 1] \times [0, 1] \times [5/6, 1].$$

For $(x, y, z) \in R_0$ consider a map defined as

$$F_0(x, y, z) = (\rho x, f(y), \beta z),$$

where $0 < \rho < 1/3$, $\beta > 6$ and

$$f(y) = \frac{1}{1 - \left(1 - \frac{1}{6}\right)e^{-y}}.$$

For $(x, y, z) \in R_1$ consider a map defined as

$$F_1(x, y, z) = \left(\frac{3}{4} - \rho x, \eta(1 - y), \beta_1 \left(z - \frac{5}{6}\right)\right),$$

where $0 < \eta < 1/3$ and $3 < \beta_1 < 4$. We define the horseshoe map $F$ on $R$ as

$$F|_{R_0} = F_0, \quad F|_{R_1} = F_1,$$

with $R \setminus (R_0 \cup R_1)$ being mapped injectively outside $R$.

For fixed parameters $\rho, \beta, \beta_1$ and $\eta$ satisfying conditions above, the non-wandering set of $F$ is partially hyperbolic, see [15]. Let $\Omega$ be the maximal invariant set for $F^{-1}$ on the cube $R$ and consider $\Phi: R_0 \cup R_1 \to \mathbb{R}$ a Hölder continuous potential with small variation, i.e.

$$\sup \Phi - \inf \Phi < \frac{\log \omega}{2}, \quad \text{where} \quad \omega = \frac{1 + \sqrt{5}}{2}.$$  \hfill (17)

The partially hyperbolic horseshoe admits only one equilibrium state associated to the potential $\Phi$ with small variation [see [31]]. Moreover, in [33] it was shown that small variation implies that the potential is hyperbolic. Also in [33], it was proved that the map $F^{-1}$ can be written as a skew product whose base dynamics, which they call projected map, is strongly topologically mixing and non-uniformly expanding. Moreover the fiber dynamics is a uniform contraction.

Let $\mathcal{S}$ be the family of partially hyperbolic horseshoes $F$, depending on the parameters $\rho, \beta, \beta_1$ and $\eta$ as above. Consider $\mathcal{G}$ the family of pairs $(F^{-1}, \Phi)$ such that $F \in \mathcal{S}$ and $\Phi$ satisfies (17) and condition $(\ast \ast)$. Note that each pair $(F^{-1}, \Phi) \in \mathcal{G}$ satisfies the hypotheses of our results. Since the equilibrium state, $\mu_{F^{-1}, \Phi}$, associated to $(F^{-1}, \Phi)$ coincides with the equilibrium state, $\mu_{F, \Phi}$, associated to $(F, \Phi)$ we can apply our results to $\mu_{F, \Phi}$. In other words, the correlations of the equilibrium state $\mu_{F, \Phi}$ decay exponentially, a central limit theorem holds and $\mu_{F, \Phi}$ varies analytically with respect to the potential. We note that the statistical properties were previously obtained in [31], the novelty is the analyticity.

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E-mail address: s.afonso@unesp.br
E-mail address: ramos.vanessa@ufma.br
E-mail address: jaqueline@im.ufrj.br