Open Archive Toulouse Archive Ouverte (OATAO)

OATAO is an open access repository that collects the work of some Toulouse researchers and makes it freely available over the web where possible.

This is a publisher's version published in: https://oatao.univ-toulouse.fr/27761

Official URL: https://doi.org/10.1016/j.ifacol.2020.12.1311

To cite this version:

Vergara-Hermosilla, Gastón and Matignon, Denis and Tucsnak, Marius Well-Posedness and input-output stability for a system modelling rigid structures floating in a viscous fluid. (2020) In: IFAC - World Congress 2020, 11 July 2020 - 17 July 2020 (Virtual event, Germany).

Any correspondence concerning this service should be sent to the repository administrator:

tech-oatao@listes-diff.inp-toulouse.fr
Open Archive Toulouse Archive Ouverte (OATAO)

OATAO is an open access repository that collects the work of some Toulouse researchers and makes it freely available over the web where possible.

This is an author's version published in: https://oatao.univ-toulouse.fr/27761

Official URL: https://doi.org/10.1016/j.ifacol.2020.12.1311

To cite this version:

Vergara-Hermosilla, Gastón and Matignon, Denis and Tucsnak, Marius Well-Posedness and input-output stability for a system modelling rigid structures floating in a viscous fluid. (2020) In: IFAC - World Congress 2020, 11 July 2020 - 17 July 2020 (Virtual event, Germany).

Any correspondence concerning this service should be sent to the repository administrator:
tech-oatao@listes-diff.inp-toulouse.fr
Well-Posedness and input-output stability  
for a system modelling rigid structures  
floating in a viscous fluid  

Gastón Vergara-Hermosilla  
Denis Matignon  
Marius Tucsnak  

* Institut de Mathématiques de Bordeaux, Université de Bordeaux, 351,  
cours de la Libération - 33405 Talence, France (e-mail:  
coibungo@gmail.com, marius.tucsnak@u-bordeaux.fr)  
** ISAE-SUPAERO, Université de Toulouse, 31055 Toulouse Cedex 4,  
France (e-mail: denis.matignon@isae-sup Aero.fr)  

Abstract:  We study a PDE based linearized model for the vertical motion of a solid floating at  
the free surface of a shallow viscous fluid. The solid is controlled by a vertical force exerted  
via an actuator. This force is the input of the system, whereas the output is the distance from the  
solid to the bottom. The first novelty we bring in is that we prove that the governing equations  
define a well-posed linear system. This is done by considering adequate function spaces and  
convenient operators between them. Another contribution of this work is establishing that the  
system is input-output stable. To this aim, we give an explicit form of the transfer function and  
we show that it lies in the Hardy space \( H^\infty \) of the right-half plane.  

Copyright © 2020 The Authors. This is an open access article under the CC BY-NC-ND license  
(http://creativecommons.org/licenses/by-nc-nd/4.0)  

Keywords:  Well-posed systems, input-output stability, transfer function, infinite dimensional  
systems.  

1. INTRODUCTION  

In this work we consider an infinite dimensional system describing the vertical motion of a solid floating at  
the free surface of a viscous fluid with finite depth and flat bottom. This system is motivated by the growing interest  
of wave energy extractors that float on the sea and extract energy by activating a hydraulic pump, which in turn  
drives a hydraulic motor connected to a generator. In such an arrangement, the torque on the generator can  
be controlled, leading to a controllable vertical force on the floating object, see for instance Korde and Ringwood  
(2016) or Pecher and Kofoid (2017). The input of the considered system is the force acting on the solid by an  
actuator, whereas the output is the distance from the solid to the sea bottom. The novelty brought by this work  
is twofold:  

- The viscous effects are taken in consideration from the beginning of the modelling process, by adapting  
a method describing viscous free boundary value flows which has been introduced in Maity et al. (2018).  
- We give an explicit form of the transfer function, allowing, in particular, to establish the input-output  
stability of the system. In a future work we aim at using this explicit form to implement simple feedback  
laws.  

The leading assumptions on the fluid are that it is one dimensional and unbounded in the horizontal direction,  
that the flow can be described within the shallow water approximation (this mean that the horizontal length scale  
of motion \( L \) is much greater than the perpendicular fluid depth \( D \), i.e. \( D/L \ll 1 \)) and that the viscosity effects  
cannot be neglected. Concerning the solid, we assume that it has vertical walls, that it can move only vertically and  
that it is subject to a vertical control force. The output signal is the distance from the bottom of the solid to the  
sea bottom. More precisely, we consider the model introduced in Maity et al. (2018), with the particularity that  
the fluid is supposed to be infinite in the horizontal direction, denoting \( I := [a, b] \) the projection on the fluid  
bottom of the solid domain and setting \( \mathcal{E} := \mathbb{R} \setminus [a, b] \). The floating solid is supposed, without loss of generality,  
to have mass \( M = 1 \) and it is constrained to move only in the vertical direction. Given \( t > 0 \), we denote by \( h(t, x) \) the  
height of the free surface of the fluid, by \( g(t, x) \) the flux of viscous fluid in the direction \( x \) and by \( h_S(t) \) the distance  
from the bottom of the rigid body to the bottom of the fluid, supposed to be horizontal, as described in Fig. 1. We  
denote by \( \hat{h} \) and \( \hat{h}_S \) the equilibrium height for the fluid and the solid, respectively. Then, following Maity et al. (2018), we have  

\[
\hat{h} = \hat{h}_S + \frac{1}{b - a},
\]

and for simplicity, we assume that  

\[
\hat{h} = 1, \quad g = 1, \quad \bar{p} = \frac{1}{b - a}.
\]

Hence, Linearizing around the trajectory \((h_S, h, q, p) = (\bar{h}_S, \bar{h}, 0, \bar{p})\) we we obtain the equations.
Theorem 1. Equations (1)-(8) can be recast as \[ p \text{ where } \] is the input function whereas the output is \( X \) system with state space (2014), or also the book Tucsnak and Weiss (2009)), the Using the classical definition of well-posed linear systems Remark 3. Our first main result is the following reformulation of the \[ A \] and \( b \), \( h \), \( t, a \), \( h \), \( \cdot \), \( t, \cdot \), \( q(\cdot, a) \) and \( q(t, b) \), \( B \) is in \( \mathcal{L}(\mathbb{C}, X) \), \( C \) is in \( \mathcal{L}(X, \mathbb{C}) \) and \( A \) is the generator of an analytic semigroup on \( X \).

Using the classical definition of well-posed linear systems (see for instance the survey paper Tucsnak and Weiss (2014), or also the book Tucsnak and Weiss (2009)), the above theorem implies the following result: Corollary 2. Equations (1)-(8) define a well-posed linear system with state space \( X \) defined in (9) and input and output spaces \( U = Y = \mathbb{C} \).

Remark 3. From the above results it follows, in particular, that for every and every \( u \in L^2(0, \infty) \), the initial value problem formed of (1)-(8) and the initial condition \( z(0) = z_0 \) admits a unique solution \[ z(t, x) = \begin{bmatrix} h_S(t) \\ h(t, x) \\ q(t, x) \\ q(t, a) \\ q(t, b) \end{bmatrix}, \] in \( C([0, \infty); X) \). Moreover, it is easily checked that \( \tilde{z} \) defined by \[ \tilde{z}(t, x) = \begin{bmatrix} h(t, a + b - x) \\ -q(t, a + b - x) \\ -q(t, b) \end{bmatrix}, \] satisfies (1)-(8). Moreover, if we assume that \( q_0(x) = -q_0(a + b - x) \), \( h_0(x) = h_0(a + b - x) \) \( (x \in \mathcal{E}) \), then \( \tilde{z}(0, \cdot) = z_0 \), thus \( \tilde{z} \) satisfies the same initial value problem as \( z \). Using the uniqueness of solutions of this initial value problem we deduce that \( \tilde{z} = z \). This means, in particular, that for initial data satisfying (11) we have \( q(t, a) = -q(t, b) \), \( h(t, a) = h(t, b) \) \( (t \geq 0) \).

Continuing with our results, we remember that a well-posed linear system of the form (10) is said input-output stable if equations (10) define, for \( z(0) = 0 \) a bounded map from \( L^2([0, \infty); U) \) to \( L^2([0, \infty); Y) \). Considering this our second main result can be stated as: Theorem 4. The system described by (1)-(8) is input-output stable.

The remaining part of this paper is organized as follows. In Section 2 we prove Theorem 1. Finally, Section 3 is devoted to the proof of our second main result, asserting the input-output stability of the considered system.

2. PROOF OF THEOREM 1

For \( t \geq 0 \), we set \( q_a(t) := q(t, a) \) and \( q_b(t) := q(t, b) \). Since (5) implies that \( q \) is a linear function of \( x \) on \( \mathcal{I} \), for every \( t \geq 0 \) and \( x \in \mathcal{I} \), \[ \dot{h}_S(t) = \frac{\partial q}{\partial x}(t, a) - \frac{\partial q}{\partial x}(t, b) \] \[ q(t, x) = q_a(t) \frac{x - b}{a - b} + q_b(t) \frac{x - a}{b - a}, \] \[ \frac{\partial q}{\partial x}(t, x) = \frac{q(t, x) - q(t, a)}{b - a} \] We differentiate (6) with respect to \( x \) and use (3)-(5) to arrive at \[ \frac{\partial^2 p}{\partial x^2}(t, x) = \dot{h}_S(t) \] \[ p(t, a^+ = p_a(t), p(t, b^-) = p_b(t), \] where \[ p_a(t) := h(t, a^-) - \frac{\partial q}{\partial x}(t, a^-) - h_{sol}(t) - \mu h_{sol}(t), \] \[ p_b(t) := h(t, b^-) - \frac{\partial q}{\partial x}(t, b^-) - h_{sol}(t) - \mu h_{sol}(t). \]
Moreover, the first equation in (15) implies that, for every \( t \geq 0 \), \( p(t, x) \) is a second order polynomial in \( x \) so that
\[
\int_a^b p(t, x) \, dx = p(t, a) l - q_a(t) l^2 \frac{1}{3} - \dot{q}_b(t) l^2 \frac{1}{6} = p(t, b) l + \dot{q}_a(t) l^2 \frac{1}{3} + \dot{q}_b(t) l^2 \frac{1}{6},
\]
where we set \( l := b - a \). Combining this with (7) and (12) we deduce that
\[
\begin{aligned}
1 + \frac{l^3}{3} \dot{q}_a(t) - \left[ 1 - \frac{l^3}{6} \right] \dot{q}_b(t) = p(t, a) l^2 + l u(t), \\
- \left[ 1 - \frac{l^3}{6} \right] \dot{q}_a(t) + \left[ 1 + \frac{l^3}{3} \right] \dot{q}_b(t) = -p(t, b) l^2 - l u(t).
\end{aligned}
\]
Inverting the above linear system, we get
\[
\begin{bmatrix} \dot{q}_a(t) \\ \dot{q}_b(t) \end{bmatrix} = M \begin{bmatrix} p(t, a) \\ -p(t, b) \end{bmatrix} + \frac{1}{l} M \begin{bmatrix} u(t) \end{bmatrix},
\]
where \( M \) is the matrix given by
\[
M := \frac{1}{l(1 + \frac{l^3}{17})} \begin{bmatrix} 1 + \frac{l^3}{3} & 1 - \frac{l^3}{6} \\ 1 - \frac{l^3}{6} & 1 + \frac{l^3}{3} \end{bmatrix}.
\]
Considering equation (3)-(4) together with (14) we deduce that
\[
p(t, a) = h(t, a^-) - \mu \frac{\partial q}{\partial x} (t, a^-) - h_s(t) + \mu \frac{\partial q}{\partial b} - a, \tag{20}
\]
and
\[
p(t, b) = h(t, b^+) - \mu \frac{\partial q}{\partial x} (t, b^+) - h_s(t) + \mu \frac{\partial q}{\partial b} - a. \tag{21}
\]
Finally, the system (1)-(7) writes in the equivalent form
\[
\dot{h}_S(t) = -\frac{q_b(t) - q_a(t)}{b - a} \quad (t \geq 0),
\]
\[
\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = 0 \quad (x \in \mathcal{E}),
\]
\[
\frac{\partial q}{\partial t} + \frac{\partial h}{\partial x} + \mu \frac{\partial^2 q}{\partial x^2} = 0 \quad (x \in \mathcal{E}),
\]
\[
\begin{bmatrix} \dot{q}_a(t) \\ \dot{q}_b(t) \end{bmatrix} = M \begin{bmatrix} u(t) \\ -u(t) \end{bmatrix} + \begin{bmatrix} h(t, a^-) - \mu \frac{\partial q}{\partial x} (t, a^-) - h_s(t) + \mu \frac{\partial q}{\partial b} - a \\ -h(t, b^+) + \mu \frac{\partial q}{\partial x} (t, b^+) + h_s(t) - \mu \frac{\partial q}{\partial b} - a \end{bmatrix}.
\]
Let \( X \) be defined by (9), set
\[
W := \mathbb{C} \times H^2(\mathcal{E}) \times H^2(\mathcal{E}) \times \mathbb{C} \times \mathbb{C},
\]
and denote by \( z := [h_S, h, q_a, q_b]^T \) a generic element of \( X \). Consider the operator \( A : D(A) \rightarrow X \) defined by
\[
D(A) := \{ z \in W \mid q(a) = q_a, q(b) = q_b \},
\]
where
\[
Az := \begin{bmatrix} -\frac{q(b) - q(a)}{l} \\ -\frac{q(b) - q(a)}{l} \\ -\frac{\dot{h}}{R_1 z} \\ -\frac{\dot{h}}{R_2 z} \end{bmatrix}.
\]
In the situation when \( \mathcal{E} \) is supposed to be bounded (which means that the fluid is contained in a container), it has been proved in (Maity et al., 2018, Section 6) that the corresponding version of \( A \) defined in (26)-(27) generates an analytic semigroup. This proof can be transposed with obvious modifications to our case so that the operator \( A \) generates an analytic semigroup on \( X \). We set
\[
Bu := [0, 0, \frac{lu}{2 (1 + \frac{l^3}{17})}, -\frac{lu}{2 (1 + \frac{l^3}{17})}]^T \quad \text{and} \quad Cz := h_s,
\]
and we observe that \( B \in \mathcal{L}(\mathbb{C}, X) \) and \( C \in \mathcal{L}(X, \mathbb{C}) \). Hence the proof of Theorem 1 is completed.

3. PROOF OF THEOREM 4

It has been shown in Maity et al. (2018) that in the case of a bounded container, the linearized system describing the motion of the floating body is exponentially stable. It is not difficult to check that in our case we have that 0 lies in the spectrum of \( A \), thus the system is no longer exponentially stable. However, we have the following result where \( \mathbb{C}_0 \) denotes the open right-half plane
\[
\mathbb{C}_0 := \{ s \in \mathbb{C} : \text{Re } s > 0 \}. \tag{29}
\]
Proposition 5. The resolvent set \( \rho(A) \) contains \( \mathbb{C}_0 \).

Proof. Let \( \lambda \in \mathbb{C}_0 \) and \( F = [f_1, f_2, f_3, f_4, f_5]^T \) in \( X \). The equation \((\lambda I - A)z = f\) for \( z \in D(A) \) reads
\[
\lambda h_S + \frac{q(b) - q(a)}{b - a} = f_1, \tag{30}
\]
\[
\lambda h(x) + \frac{\partial q}{\partial x} = f_2(x) \quad (x \in \mathcal{E}), \tag{31}
\]
\[
\lambda q(x) + \frac{\partial h}{\partial x} - \mu \frac{\partial^2 q}{\partial x^2} = f_3(x) \quad (x \in \mathcal{E}), \tag{32}
\]
\[
M \begin{bmatrix} h(a^-) - \mu \frac{\partial q}{\partial x} (a^-) - h_s + \mu \frac{q(b) - q(a)}{b - a} \\ -h(b^+) + \mu \frac{q(b)}{x} (b^+) + h_s + \mu \frac{q(b) - q(a)}{b - a} \end{bmatrix} = \begin{bmatrix} f_4 \\ f_5 \end{bmatrix} - \lambda \begin{bmatrix} q_a \\ q_b \end{bmatrix}, \tag{33}
\]
\[
\lim_{x \rightarrow -\infty} q(x) = \lim_{x \rightarrow +\infty} q(x) = 0, \tag{34}
\]
\[
q(a) = q_a, \quad q(b) = q_b. \tag{35}
\]
From (33) and (25), it follows that
\[
\mu \frac{q(b) - q(a)}{l} + h(a^-) - h_s + \mu \frac{\partial q}{\partial x} (a^-) = \begin{bmatrix} 1 + \frac{l^3}{3} \cdot (\lambda q_a - f_4) + \frac{1}{l^2} \cdot (f_5 - \lambda q_b) \end{bmatrix}, \tag{36}
\]
\[
-\mu \frac{q(b) - q(a)}{l} - h(b^+) + h_s + \mu \frac{\partial q}{\partial x} (b^+) = \begin{bmatrix} 1 - \frac{l^3}{6} \cdot (\lambda q_b - f_5) + \frac{1}{l^2} \cdot (\lambda q_a - f_4) \end{bmatrix}. \tag{37}
\]
We next transform (30)-(35) into a boundary value problem for \( q \) by eliminating \( h, h_s, q_a, q_b \) from the above mentioned equations. First, from (31) and (32), we deduce

\[
\lambda q - \left( \mu + \frac{1}{\lambda} \right) \frac{d^2q}{dx^2} = \phi_1 \quad (x \in \mathcal{E}),
\]  

(38)

where

\[
\phi_1 := f_3 - \frac{1}{\lambda} \frac{df_2}{dx} \in L^2(\mathcal{E}).
\]

Next, using (30), (31) and (35) in (36) and (37) it follows that

\[
\begin{align*}
\left( \mu + \frac{1}{\lambda} \right) \frac{dq}{dx} (a^-) &= \left( \mu + \frac{1}{\lambda} + \frac{\lambda}{T} \right) \frac{q(b) - q(a)}{l} - \frac{\lambda}{6} \left( 2q(a) + q(b) \right) + \phi_2 \\
\left( \mu + \frac{1}{\lambda} \right) \frac{dq}{dx} (b^+) &= \left( \mu + \frac{1}{\lambda} + \frac{\lambda}{T} \right) \frac{q(b) - q(a)}{l} + \frac{\lambda}{6} \left( q(a) + 2q(b) \right) + \phi_3
\end{align*}
\]

(39)

with

\[
\begin{align*}
\phi_2 := \left( \frac{f_2(a^-)}{\lambda} - \frac{f_1}{\lambda} \right) + \left[ 1 + \frac{\lambda}{3} \right] \frac{f_4}{T^2} - \left[ 1 - \frac{\lambda}{6} \right] \frac{f_5}{T^2}, \\
\phi_3 := \left( \frac{f_2(b^+)}{\lambda} - \frac{f_1}{\lambda} \right) + \left[ 1 - \frac{\lambda}{6} \right] \frac{f_4}{T^2} + \left[ 1 + \frac{\lambda}{3} \right] \frac{f_5}{T^2}
\end{align*}
\]

(40)

We first prove the existence of a weak solution to (38)-(40) with the condition (34) at infinity. We thus set \( V := H^1(\mathcal{E}) \). The weak formulation of (38)-(40) with the condition (34) is to find \( q \in V \) such that

\[
B_\lambda(q, \psi) := \int_\mathcal{E} \left[ \lambda q \psi + \left( \mu + \frac{1}{\lambda} \right) \frac{dq}{dx} \frac{d\psi}{dx} \right] dx
\]

(41)

where

\[
B_\lambda(q, \psi) := \int_\mathcal{E} \left[ \lambda q \bar{\psi} + \left( \mu + \frac{1}{\lambda} \right) \frac{dq}{dx} \frac{d\bar{\psi}}{dx} \right] dx
\]

(42)

\[
\begin{align*}
&- \left[ \left( \frac{1}{\lambda} + \mu + \frac{\lambda}{T} \right) \frac{q(b) - q(a)}{l} - \frac{\lambda}{6} \left( 2q(a) + q(b) \right) \right] \bar{\psi}(a) \\
&+ \left[ \left( \frac{1}{\lambda} + \mu + \frac{\lambda}{T} \right) \frac{q(b) - q(a)}{l} + \frac{\lambda}{6} \left( q(a) + 2q(b) \right) \right] \bar{\psi}(b).
\end{align*}
\]

For \( q, \psi \in V \),

\[
B_\lambda(q, \psi) := \int_\mathcal{E} \left[ \lambda q \bar{\psi} + \left( \mu + \frac{1}{\lambda} \right) \frac{dq}{dx} \frac{d\bar{\psi}}{dx} \right] dx
\]

(43)

\[
\begin{align*}
&+ \left( \frac{1}{\lambda} + \mu + \frac{\lambda}{T} \right) \frac{q(b) - q(a)}{l} \left( \bar{\psi}(b) - \bar{\psi}(a) \right) + \\
&\frac{\lambda}{6} \left[ q(a) \psi(a) + q(b) \psi(b) + (q(b) + q(a)) \left( \psi(b) + \psi(a) \right) \right].
\end{align*}
\]

From the above formula, it follows that for any \( \lambda \in \mathbb{C}_0 \) we can consider positive constants \( C = C(\lambda), \alpha = \min \{ \Re \lambda, \Re 1/\lambda \} \) such that

\[
|B_\lambda(q, \psi)| \leq C ||q||_{V} ||\psi||_{V}, \quad \text{Re} \ B_\lambda(q, q) \geq \alpha ||q||_{V}^2.
\]

(44)

where \( q, \psi \in V \), thus \( B_\lambda \) is a bounded and coercive form on \( V \). Moreover, the right hand side of (41) clearly defines a bounded linear functional on \( V \). Thus, the conclusion follows by the complex version of the Lax-Milgram Lemma (see, for instance, Lemma 5.4 on Arendt et al. (2014)).

3.1 Transfer function

From Proposition 5 it follows that the transfer function

\[
G(s) = C(sI - A)^{-1},
\]

of the system (1)-(8) is defined for every \( s \in \mathbb{C}_0 \). In this subsection we compute this transfer function and we show that it lies in the Hardy space \( H^\infty(\mathbb{C}_0) \) and thus obtain the main ingredient of the proof of Theorem 4. In other terms, we compute the Laplace transform of the solution of (1)-(8) with zero initial data. More precisely, we have:

Proposition 6. The transfer function of the system (1)-(8) is given by

\[
G(s) := \frac{1}{(1 + \frac{l}{12}) s^2 + \frac{l^2}{2} s \sqrt{1 + s \mu} + s \mu + l \mu s + l} \quad (s \in \mathbb{C}_0).
\]

(45)

Proof. We first express \( h(t, a^-) - \frac{\partial h}{\partial t}(t, a^-) \) in terms of \( h_s \) and \( \hat{h}_S \). To this end, for \( x \in \mathcal{I} \), we first note that

\[
q(t, b) - q(t, a) = -\hat{h}_S(t).
\]

(46)

Moreover, using Remark 3 we obtain

\[
h(t, a^-) = h(t, b^+), \quad -q(t, a^-) = q(t, b^+),
\]

(47)

thus

\[
q(t, a) = \frac{l}{2} \hat{h}_S, \quad q(t, b) = -\frac{l}{2} \hat{h}_S.
\]

(48)

From equations (22)-(23) it follows that for \( x \in (-\infty, a] \) we have

\[
\frac{\partial^2 q}{\partial t^2} - \frac{\partial^2 q}{\partial x^2} - \mu \frac{\partial^3 q}{\partial x \partial t^2} = 0,
\]

(49)

\[
q(t, x) \to 0 \text{ as } x \to -\infty, \quad q(t, a) = \frac{b - a}{2} \hat{h}_S(t), \quad \hat{q}(0, x) = \frac{\partial q}{\partial t}(0, x) = 0.
\]

For \( f \in L^1(0, \infty) \), we denote by \( \hat{f} \) the Laplace transform of \( f \). Applying the Laplace transform to both sides of (49), we obtain

\[
s^2 \hat{q} - (1 + s \mu) \frac{\partial^2 \hat{q}}{\partial x^2} = 0
\]

(50)

\[
\hat{q}(s, x) \to 0 \text{ as } x \to -\infty, \quad \hat{q}(s, a) = \frac{b - a}{2} \hat{h}_S(s), \quad \Re(s) > 0.
\]

Hence we can conclude that

\[
\hat{q}(s, x) = \frac{b - a}{2} e^{\sqrt{s+\mu} x} e^{\sqrt{s+\mu} a} \hat{h}_S(s)
\]

(51)

\[
\hat{h}(s, a^-) - \mu \frac{\partial \hat{q}}{\partial x}(s, a^-) = -\frac{l}{2} \left( 1 + \mu s \right) S \hat{h}_S(s) - \frac{l}{2} \left( 1 + s \mu \right) \hat{h}_S(s).
\]

(52)

In a similar way, we obtain

\[
\hat{h}(s, b^+) - \mu \frac{\partial \hat{q}}{\partial x}(s, b^+) = -\frac{l}{2} \left( 1 + \mu s \right) S \hat{h}_S(s).
\]

(53)

Moreover, applying the Laplace transform to (48) we obtain

\[
\hat{q}(s, a) = \frac{l}{2} \hat{h}_S, \quad \hat{q}(s, b) = -\frac{l}{2} \hat{h}_S.
\]

(54)

Finally, applying Laplace transform to (25), we obtain
Let us prove that \( P(z) \neq 0 \) and \( Q(z) \neq 0 \). To this aim we write \( z = x + iy \) with \( x > 0 \) (due to (57) and (58)) and \( y \in \mathbb{R} \). We note that:

\[
\text{Re}(P(z)) = \mu \text{Re}(s) + \frac{l^2 s}{4(1 + \frac{\mu^2}{12})} - y \frac{\mu \sqrt{l}}{4(1 + \frac{\mu^2}{12})} \sqrt{16 + \frac{\mu^3}{3}},
\]

\[
\text{Im}(P(z)) = y \left( 2x + \frac{l^2 s^2}{4(1 + \frac{\mu^2}{12})} + x \frac{\mu \sqrt{l}}{4(1 + \frac{\mu^2}{12})} \sqrt{16 + \frac{\mu^3}{3}} \right),
\]

\[
\text{Re}(Q(z)) = \mu \text{Re}(s) + \frac{l^2 s}{4(1 + \frac{\mu^2}{12})} + y \frac{\mu \sqrt{l}}{4(1 + \frac{\mu^2}{12})} \sqrt{16 + \frac{\mu^3}{3}},
\]

\[
\text{Im}(Q(z)) = y \left( 2x + \frac{l^2 s^2}{4(1 + \frac{\mu^2}{12})} - x \frac{\mu \sqrt{l}}{4(1 + \frac{\mu^2}{12})} \sqrt{16 + \frac{\mu^3}{3}} \right).
\]

If \( \text{Im}(P(z)) = 0 \), since \( l, \mu, x \) are positive, then \( y < 0 \). But since \( \text{Re}(z^2 - 1) \geq 0 \), this implies that \( \text{Re}(P(z)) > 0 \). Therefore \( P(z) \neq 0 \).

If \( \text{Re}(Q(z)) = 0 \), since \( \text{Re}(z^2 - 1) \geq 0 \) and \( x > 0 \) are positive we conclude that \( y < 0 \). This implies that \( \text{Im}(Q(z)) < 0 \). Therefore \( Q(z) \neq 0 \).

Thus \( F(s) \neq 0 \) on \( \mathbb{C}_0 \), which concludes the proof of the Lemma.

Note that this result can be extended to the case of a more function defined by \( F(s) := s^2 + \alpha_1 s \sqrt{s} + \tau + \alpha_2 s + \alpha_3 \), only requiring the positivity of the coefficients \( \alpha_i \), but not relying at all on the specific values of the physical parameters and the algebraic relations between them, see the companion paper Vergara-Hermosilla et al. (2020).

3.2 Proof of Theorem 4

By Lemma 7 we know that the function \( F(s) \) defined in (56) is not vanishing on \( \mathbb{C}_0 \). Moreover, since

\[
\lim_{|s| \to \infty} |F(s)| = +\infty,
\]

we have that the map \( s \mapsto |F(s)| \) is bounded from below on \( \mathbb{C}_0 \). We conclude that the transfer function \( G \) defined in (45) is such that

\[
\sup_{s \in \mathbb{C}_0} |G(s)| < \infty.
\]

By the Paley-Wiener theorem (see, for instance, (Rudin, 1987, Section 19.2)), this implies that (1)-(8) is input-output stable, so that the proof of the theorem is complete.

4. FURTHER WORK

Other issues that have not been mentioned in this work, but that are interesting for our future research are the following ones:

- Study of the sectorial properties of the resolvent of the operator \( A \) defined in eq. (27), in order to establish results related to the existence and uniqueness of strong solutions and strong stability of the system.
- Study of the new system with vertical velocity as output, instead of the height: would it be a well-posed system in some suitable state space? Moreover, the stability properties of this new system and its applications to energy wave extractors will be investigated.
- Time-domain analysis of both these infinite-dimensional systems, together with a careful study of the asymptotic behaviour as a function of the physical parameters, following Vergara-Hermosilla et al. (2020).

ACKNOWLEDGEMENTS

The authors are very grateful to Dr. Franck Sueur for many helpful comments and suggestions, to Dr. Karim Kellay for his help in the proof of Lemma 7 and to the referees for their valuable suggestions.

REFERENCES

Arendt, W., Chill, R., Seifert, C., Vogt, H., and Voigt, J. (2014). Form methods for evolution equations, and applications. In Lecture Notes of the 18th Internet Seminar on Evolution Equations, volume 15.

Korde, U.A. and Ringwood, J. (2016). Hydrodynamic Control of Wave Energy Devices. Cambridge University Press.

Maity, D., San Martín, J., Takahashi, T., and Tucsnak, M. (2018). Analysis of a simplified model of rigid structure floating in a viscous fluid. Journal of Nonlinear Science, 1–46.

Pecher, A. and Kofoed, J.P. (2017). Handbook of Ocean Wave Energy. Springer London.

Rudin, W. (1987). Real and Complex Analysis. McGraw-Hill, Inc.

Tucsnak, M. and Weiss, G. (2009). Observation and Control for Operator Semigroups. Springer Science & Business Media.

Tucsnak, M. and Weiss, G. (2014). Well-posed systems – the LTI case and beyond. Automatica, 50(7), 1757–1779.

Vergara-Hermosilla, G., Matignon, D., and Tucsnak, M. (2020). Asymptotic behaviour of an input-output stable system modelling rigid structures floating in a viscous fluid. In Proc. 24th International Symposium on Mathematical Theory of Networks and Systems, 8 p. MTNS, Cambridge, United Kingdom. Accepted for publication.