MEASURE DIFFERENTIAL EQUATION WITH A NONLINEAR GROWTH/DECAY TERM

CHRISTIAN DÜLL, PIOTR GWIAZDA, ANNA MARCINIAK-CZOCHRA, AND JAKUB SKRZECZKOWSKI

Abstract. We obtain an existence result for a measure differential equation with a nonlinear growth/decay term that may change the sign. The proof requires a modification of the approximating schemes proposed by Piccoli and Rossi. The new scheme combines model discretization with an exponential solution of the nonlinear growth/decay, and hence, preserves nonnegativity of the measure. Furthermore, we formulate a new analytic condition on the measure vector field, which substantially simplifies the previous proof of continuity of solutions with respect to initial data and generalizes the former condition formulated by Piccoli and Rossi.

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1. Introduction

We consider a measure differential equation (MDE) of the form
\begin{equation}
\dot{\mu}_t = V(\mu_t) \oplus c(\cdot, \mu_t) \mu_t \oplus s[\mu_t],
\end{equation}

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where $V : \mathcal{M}^+(\mathbb{R}^d) \to \mathcal{M}^+(\mathbb{R}^d \times \mathbb{R}^d)$ is the so-called measure vector field (MVF), $s : \mathcal{M}^+(\mathbb{R}^d) \to \mathcal{M}^+(\mathbb{R}^d)$ is the source term, $c : \mathbb{R}^d \to \mathbb{R}$ is a nonlinear source/decay function, and $\mathcal{M}^+(\mathbb{R}^d)$, $\mathcal{M}^+(\mathbb{R}^d \times \mathbb{R}^d)$ are spaces of finite, nonnegative Radon measures on $\mathbb{R}^d$ and $\mathbb{R}^d \times \mathbb{R}^d$, respectively.

The notion using $\oplus$ is applied to depict a summation of various effects, in this case transport, source and growth/decay processes. The rigorous meaning of (1.1) is given in Definition 2.1.

Equation (1.1) is an extension of the MDE setting for a transport of measures that was recently introduced by Piccoli in terms of a conservative transport equation [26],

$$\dot{\mu}_t = V[\mu_t],$$

and extended by Piccoli and Rossi to an equation with a nonnegative source term [28]

$$\dot{\mu}_t = V[\mu_t] \oplus s[\mu_t].$$

The main contribution of this paper is an existence result for (1.1) which includes a distribution dependent growth/decay term $c(\cdot,\mu_t)$, without any assumption on the sign of function $c$. The new MDE is motivated by applications in life sciences accounting for birth and death of individuals or cell divisions and transitions, which cannot be described solely by source terms valued in $\mathcal{M}^+(\mathbb{R}^d)$. Analysis of the resulting model requires modification of the original approach that was based on an approximating scheme using a proof of nonnegativity of solutions. We propose a new discretization that preserves nonnegativity of measures also in case of negative $c$. Furthermore, continuity of the resulting Lipschitz semigroup with respect to initial data and uniqueness in an appropriate class of solutions require an additional continuity condition on the measure vector field $V$ (MVF continuity condition). Exploring convergence of the approximating scheme, we formulate a new MVF continuity condition, see (1.4). The proposed setting significantly simplifies the original reasoning that was based on optimal transport, [26,28]. Moreover, in Appendix A we prove that the new MVF continuity condition generalizes the one exploited in [26,28].

The proposed model (1.1) extends the MDE framework from applications to pedestrian flows to population dynamics. Analysis of transport and growth phenomena in the context of population dynamics has been originally performed using so-called structured population models in $L^1$-setting [25,33,36] and has then been extended to the space of measures [9,16,17,21,23,34]. The established theory has allowed rigorous convergence analysis of numerical algorithms such as particle methods and EBT [4,10,11,13,20,24]. Recently, it has also been applied to show stability of a posteriori distributions obtained in Bayesian Inverse Problems [32]. The space of measures is a convenient setting for the analysis of transport phenomena on complex domains such as graphs [5,17,18] or manifolds [2,29]. Finally, there are some promising results concerning sensitivity analysis and optimal control problems that may be further combined with particle methods [1,19,30]. Admitting a measure-valued velocity field, the proposed MDE model (1.1) is an extension of the structured population models.
in Radon measures and may provide a new tool for model-based analysis of the evolution of heterogeneous cell populations.

2. Problem formulation and main results

MDEs provide a generalization of the concept of ordinary differential equations to spaces of measures. In this approach, evolution of measure $\mu$ is governed by a measure vector field $V[\mu]$ which is a measure on $\mathbb{R}^d \times \mathbb{R}^d$. Its first coordinate represents spatial position $x$ and the second denotes admissible values of velocity $v$. The measure $V[\mu]$ has marginal $\mu$ on the first coordinate, i.e. it satisfies $\pi_1^# V[\mu] = \mu$ for all $\mu \in \mathcal{M}(\mathbb{R}^d)$. Here $\pi_1$ denotes the projection to the first (spatial) coordinate and the superscript $#$ denotes the push-forward operator

$$\pi_1^# \mu(A) = \mu (\pi_1^1(A)) \quad \forall A \in B(\mathbb{R}^d).$$

Roughly speaking, if $(x,v)$ belongs to the support of $V[\mu]$, $\mu$ at position $x$ evolves with velocity $v$. We refer to Section 6 and [26, Section 7.1] for examples of measure vector fields and to [6] for recent work on numerical schemes for MDE.

The measure solution to the MDE (1.1) is based on the weak formulation of the problem.

**Definition 2.1.** We say that a continuous curve $\mu_\bullet : [0,T] \to (\mathcal{M}(\mathbb{R}^d), \| \cdot \|_{\text{BL}})$ is a solution to (1.1) with initial condition $\mu_0 \in \mathcal{M}(\mathbb{R}^d)$ if for all $f \in C^\infty_c(\mathbb{R}^d)$ and for all $t \in [0,T]$, it holds

$$\int_{\mathbb{R}^d} f(x) \, d\mu_t(x) - \int_{\mathbb{R}^d} f(x) \, d\mu_0(x) = \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla f(x) \cdot v \, dV[\mu_r](x,v) \, dr + \int_0^t \int_{\mathbb{R}^d} f(x) \, c(x,\mu_r) \, d\mu_r(x) \, dr + \int_0^t \int_{\mathbb{R}^d} f(x) \, ds[\mu_r](x) \, dr.$$

The bullet in the subscript of the solution above denotes the time argument.

**Remark 2.2.** In most cases the dynamics of MDEs simplifies to a transport equation in the spaces of measures as in [14]. To see this, we note that by disintegration theorem, see e.g. [15, Theorem 1.45], there exists a family of probability measures $\{\nu_{x,t}\}_{x \in \mathbb{R}^d, t \in [0,T]}$ such that

$$\int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla f(x) \cdot v \, dV[\mu_r](x,v) \, dr = \int_0^t \int_{\mathbb{R}^d} \nabla f(x) \cdot \left[ \int_{\mathbb{R}^d} v \, d\nu_{x,r}(v) \right] \, dx \, dr$$

so that Definition 2.1 boils down to measure solutions for transport equation with velocity

$$V(t,x) := \left[ \int_{\mathbb{R}^d} v \, d\nu_{x,t}(v) \right].$$

Of course, this point of view does not bring new insights as $[\int_{\mathbb{R}^d} v \, d\nu_{x,t}(v)]$ cannot be computed explicitly in general.

Now, we formulate assumptions on model functions. They are expressed in $\| \cdot \|_{\text{BL}}$ and $\| \cdot \|_{\text{BL}^*}$ norm, see (2.2) and (3.1) respectively.
Assumption 2.3.  *(V)* The measure vector field \( V : \mathcal{M}^+(\mathbb{R}^d) \to \mathcal{M}^+(\mathbb{R}^d \times \mathbb{R}^d) \) satisfies \( \pi^+_1 V[\mu] = \mu \) for all \( \mu \in \mathcal{M}^+(\mathbb{R}^d) \). Moreover, it holds:

- **Control of support in velocity:** There is a constant \( C_S > 0 \) such that
  \[
  \sup_{(x,v) \in \text{supp}(V[\mu])} |v| \leq C_S \left( 1 + \sup_{(x,v) \in \text{supp}(V[\mu])} |x| \right).
  \]

- **Lipschitz continuity with respect to the flat metric:** For all \( R > 0 \), there is a constant \( C_F(R) \) such that if \( \mu, \nu \) are supported in \( B(0,R) \), we have
  \[
  \| V[\mu] - V[\nu] \|_{BL^*} \leq C_F(R) \| \mu - \nu \|_{BL^*}.
  \]

(S) The source \( s : \mathcal{M}^+(\mathbb{R}^d) \to \mathcal{M}^+(\mathbb{R}^d) \) satisfies

- **Lipschitz continuity:** There exists \( L \) such that for all \( \mu, \nu \in \mathcal{M}^+(\mathbb{R}^d) \)
  \[
  \| s[\mu] - s[\nu] \|_{BL^*} \leq L \| \mu - \nu \|_{BL^*}.
  \]

- **Uniform boundedness of the support:** There exists \( R \) such that for all \( \mu \in \mathcal{M}^+(\mathbb{R}^d) \) it holds
  \[
  \text{supp}(s[\mu]) \subseteq B(0, R).
  \]

(C) The source/decay function \( c : \mathbb{R}^d \times \mathcal{M}^+(\mathbb{R}^d) \to \mathbb{R} \) satisfies

- **Boundedness:** There exists a constant \( C_b > 0 \) such that
  \[
  |c(x, \mu)| \leq C_b \quad \forall x \in \mathbb{R}^d, \mu \in \mathcal{M}^+(\mathbb{R}^d).
  \]

- **Lipschitz continuity:** There exists \( C_L > 0 \) such that
  \[
  |c(x, \mu) - c(y, \nu)| \leq C_L \| x - y \| + \| \mu - \nu \|_{BL^*}.
  \]

Remark 2.4. In view of the Lemmas 4.3 and Corollary 4.5 the constructed solution \( \mu^N_t \) and the \( \text{MVF} V[\mu^N_t] \) are both uniformly compactly supported so that the radius \( R \) in assumption \( (V_2) \) can be chosen uniformly. Therefore, in what follows we will write \( C_F \) instead \( C_F(R) \).

The first main result of this paper reads:

**Theorem 2.5** (Existence of solutions). Suppose that the measure vector field \( V \) and model functions \( s, c \) satisfy Assumption 2.3. Moreover, assume that \( \mu_0 \in \mathcal{M}^+(\mathbb{R}^d) \) is compactly supported. Then, there exists a measure solution \( \mu_* \) to \( (1.1) \) in the sense of Definition 2.1. This solution is Lipschitz continuous with respect to time.

To prove Theorem 2.5 we consider an approximating sequence and establish uniform bounds that enable application of the compactness argument. The main novelty of the presented analysis is the construction of approximating scheme \( (4.1) \) that preserves nonnegativity of the measure even if \( c \) is negative. The proof is presented in Section 5.
Unfortunately, a uniqueness result of solutions to problem (1.1) seems to be out of reach with Assumption 2.3 alone. Thus, we introduce an additional assumption on the measure vector field $V$.

**Assumption 2.6 (MVF continuity condition).** For all $R > 0$, there is a constant $C_H(R)$ such that if $\mu, \nu$ are supported in $B(0, R)$, we have

$$(V_3) \quad \sup_{\|\psi\|_{BL^2}} \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x + \tau v) d(V[\mu] - V[\nu])(x, v) \leq (1 + C_H(R) \tau) \|\mu - \nu\|_{BL^*}.$$

**Remark 2.7.** Note that assumption $(V_2)$, which is necessary for the existence result Theorem 2.5, only implies $(V_3)$ if $C_F \leq 1$. So in general, both $(V_2)$ and $(V_3)$ have to be assumed to establish existence and uniqueness of solutions to model (1.1). Furthermore, the same reasoning as in Remark 2.4 applies, so that we can drop the radius $R$ appearing in the constant $C_H(R)$ in $(V_3)$ and simply write $C_H$ instead.

Under this additional MVF continuity assumption, the corresponding semigroup of solutions proves to be continuous with respect to initial conditions. We remark that the alternative Lipschitz continuity condition with respect to the operator $W^\theta$ (see Definition A.1) was formulated by Piccoli in [28] and applied to obtain continuity with respect to initial conditions in [26, 28]. We show that this approach is a special case of our reasoning in Appendix A.

**Theorem 2.8.** Suppose that Assumptions 2.3 and 2.6 are satisfied. Let $\mu_0, \nu_0 \in \mathcal{M}_c^+(\mathbb{R}^d)$ be two initial data with corresponding solutions $\mu_t, \nu_t$ to problem (1.1). Then it holds that

$$(2.1) \quad \|\mu_t - \nu_t\|_{BL^*} \leq e^{Ct}\|\mu_0 - \nu_0\|_{BL^*}.$$

Uniqueness of the measure solution $\mu_\bullet$ in appropriate class can then be established as in [26, 28] and is formulated in Theorem 7.6.

The structure of the paper is as follows. In Section 3 we introduce the flat norm and some results on compactness of measures, which will be used in the proof of Theorem 2.5. Section 4 is devoted to the explicit construction of the lattice approximate solution $\mu_t^{N}$. Furthermore, we establish useful bounds and estimates for the supports. In Section 5 we prove the existence result (Theorem 2.5). We continue in Section 6 with our second main result on continuity of solutions with respect to initial data (Theorem 2.8). For the proof we need to upgrade assumption $(V_2)$ with the additional regularity hypothesis $(V_3)$. In Section 7 we summarise the theory introduced in [26, 28] to show uniqueness of the resulting semigroup based on the concept of Dirac germs (see Definition 7.4 and Theorem 7.6). Additionally, in Appendix A we prove that our MVF continuity condition $(V_3)$ generalizes the one exploited formerly in [26, 28].

### 3. Flat norm on the space of measures

In this section, we present our functional analytic setting. Let $\mathcal{M}(\mathbb{R}^d)$ be the space of bounded real-valued signed Borel measures on $\mathbb{R}^d$ and let $\mathcal{M}^+(\mathbb{R}^d)$ be the cone consisting of nonnegative measures
The space of compactly supported nonnegative measures is denoted by $\mathcal{M}_c^+(\mathbb{R}^d)$. We can define a partial ordering on $\mathcal{M}^+(\mathbb{R}^d)$ via

$$\mu \leq \nu :\iff \mu(A) \leq \nu(A) \quad \text{for all } A \in \mathcal{B}(\mathbb{R}^d).$$

We recall Hahn-Jordan decomposition: If $\mu \in \mathcal{M}(\mathbb{R}^d)$ is a signed measure, there are two (uniquely determined) nonnegative measures $\mu^+, \mu^- \in \mathcal{M}^+(\mathbb{R}^d)$ with disjoint supports such that

$$\mu = \mu^+ - \mu^-.$$

To perform analysis on $\mathcal{M}(\mathbb{R}^d)$, we need a notion of distance. The standard one is given by the total variation norm:

$$\|\mu\|_{TV} := \mu^+(\mathbb{R}^d) + \mu^-(\mathbb{R}^d).$$

Unfortunately, total variation generates a topology which is too strong for applications [23, Examples 1.1,1.2], so in this paper we will work in spaces equipped with the flat norm (or bounded Lipschitz distance, Fortet-Mourier distance) defined as

$$\|\mu\|_{BL^*} := \sup \left\{ \int_{\mathbb{R}^d} \psi \, d\mu \mid \psi \in BL(\mathbb{R}^d), \|\psi\|_{BL} \leq 1 \right\}.$$

The space of bounded Lipschitz functions $BL(\mathbb{R}^d)$ is given by

$$BL(\mathbb{R}^d) = \left\{ f : \mathbb{R}^d \to \mathbb{R} \text{ is continuous and } \|f\|_{\infty} < \infty, |f|_{Lip} < \infty \right\},$$

where

$$\|f\|_{\infty} = \sup_{x \in \mathbb{R}^d} |f(x)|, \quad |f|_{Lip} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

Equipped with the norm

$$\|f\|_{BL} = \max (\|f\|_{\infty}, |f|_{Lip}) \leq \|f\|_{\infty} + |f|_{Lip},$$

the space $(\mathcal{M}^+(\mathbb{R}^d), \|\cdot\|_{BL^*})$ is a separable and complete metric space [13 Corollary 1.38, Theorem 1.61] or [22 Theorem 2.7 (ii)]. We also remark that if $\mu \in \mathcal{M}^+(\mathbb{R}^d)$, then $\|\mu\|_{TV} = \|\mu\|_{BL^*}$.

**Remark 3.1.** In [27 Theorem 13], the following alternative characterization for the flat norm of two measures $\mu, \nu \in \mathcal{M}^+(\mathbb{R}^d)$ is proven

$$\|\mu - \nu\|_{BL^*} = W_{1,1}^1(\mu, \nu) := \inf_{\tilde{\mu} \leq \mu, \tilde{\nu} \leq \nu} \|\mu - \tilde{\mu}\|_{TV} + \|\nu - \tilde{\nu}\|_{TV} + W_1(\tilde{\mu}, \tilde{\nu}).$$

Here, $W_1$ denotes the classical $1-$Wasserstein distance with respect to the cost function $c(x, y) = |x - y|$. The decomposition into terms with total variation and the term with Wasserstein distance admits a heuristic interpretation: any share $\delta \mu$ of the mass of $\mu$ can either be transported from $\mu$ to $\nu$ at cost $W_1(\delta \mu, \delta \nu)$ or removed at cost $\|\delta \mu\|_{TV}$. As such, the minimal "sub-measures" $\tilde{\mu}, \tilde{\nu}$ achieve an optimal compromise between mass transportation and cancellation.

In this paper we will use Arzelà-Ascoli theorem [31 Theorem 9.4.13] in the space of measures. We briefly discuss the technical details below.
Theorem 3.1 (Arzelà-Ascoli). Let $X$ be a separable metric space and $Y$ be a complete metric space. Let $F \subset C(X,Y)$ be a family of continuous functions such that

- $F$ is equibounded,
- $F$ is equicontinuous,
- for each $x \in X$, the set $\{ f(x) \mid f \in F \}$ is relatively compact in $Y$.

Then, each sequence of functions $(f_n)_{n \in \mathbb{N}} \subset F$ has a subsequence converging uniformly on compact subsets of $X$.

In our case $X = [0, T]$ is the time interval while $Y = (\mathcal{M}^+(\mathbb{R}^d), \| \cdot \|_{BL^*})$ is the space of nonnegative measures equipped with the flat metric. To verify pointwise compactness, we will use the following result.

Lemma 3.2. Suppose that $(\mu_n)_{n \in \mathbb{N}}$ is a sequence of nonnegative measures supported on some compact set $K \subseteq \mathbb{R}^d$ such that $\| \mu_n \|_{TV} \leq C$ for all $n \in \mathbb{N}$. Then, there exists a subsequence $(\mu_{nk})_{k \in \mathbb{N}}$ converging to $\mu$ in $\| \cdot \|_{BL^*}$ norm.

Proof. First, as the sequence of measures $(\mu_n)_{n \in \mathbb{N}}$ is supported on the compact set $K \subseteq \mathbb{R}^d$, it holds that $\mu_n(\mathbb{R}^d \setminus K) = 0$ for all $n \in \mathbb{N}$ so that $(\mu_n)_{n \in \mathbb{N}}$ is tight. Therefore the theorem of Prokhorov [12, Theorem 2.3] implies that there is a subsequence converging narrowly to a measure $\mu \in \mathcal{M}^+(\mathbb{R}^d)$, i.e. for all continuous and bounded functions $\psi : \mathbb{R}^d \to \mathbb{R}$ we have

$$\int_{\mathbb{R}^d} \psi(x) \mathrm{d}\mu_{nk}(x) \to \int_{\mathbb{R}^d} \psi(x) \mathrm{d}\mu(x).$$

But then $\|\mu_{nk} - \mu\|_{BL^*} \to 0$ as $k \to \infty$ according to [23 Theorem 2.10 (ii)] or [14, Theorem 1.57].

□

4. Construction of the solution

To construct a solution to (1.1), we will define an approximating scheme which is based on discretization of time, space and velocity. We use the notation of Piccoli for meshes and $\Delta_N$ for a mesh step. In particular, for $N \in \mathbb{N}$ the time step size is given by $\Delta_N = 1/N$, the velocity step size by $\Delta_N^v = 1/N$ and the space step size by $\Delta_N^s = 1/N^2$. Furthermore, we use the same equispaced space mesh $(\mathbb{Z}^d/(N^3)) \cap [-N,N]^d$ with discretization points $x_i, i = 1, \ldots, I = I(N) := (2N^3 + 1)^d$ and the same equispaced velocity mesh $(\mathbb{Z}^d/N) \cap [-N,N]^d$ with discretization points $v_j, j = 1, \ldots, J = J(N) := (2N^2 + 1)^d$. The time interval $[0,T]$ is divided into $M + 1$ subintervals (with $M = M(N) := \lfloor T \rfloor N + [(T - \lfloor T \rfloor)/N] - 1$) of length at most $\Delta_N$, where the intervals are of the form $[t_l, t_{l+1})$ with $t_l = l/N$, for $l = 0, \ldots, M - 1$, and the last one is given by $[t_M, t_{M+1}]$ with $t_{M+1} = T$. 
Using the above mesh, we can introduce the following discretization operators in the space and in the velocity variable

\[ A^e_N : \mathcal{M}^+ (\mathbb{R}^d) \to \mathcal{M}^+ (\mathbb{R}^d) : \quad A^e_N (\mu) = \sum_{i=1}^I m^e_i (\mu) \delta_{x_i}, \]

\[ A^v_N : \mathcal{M}^+ ((\mathbb{R}^d)^2) \to \mathcal{M}^+ ((\mathbb{R}^d)^2) : \quad A^v_N (V) = \sum_{i=1}^I \sum_{j=1}^J m^v_{ij} (V) \delta_{(x_i, v_j)}, \]

where we used the following measure dependent weights

\[ m^e_i (\mu) = \mu (x_i + [0, \Delta^e_N)^d) := \mu (x_i + Q) := \mu (Q_i) \quad \text{and} \]

\[ m^v_{ij} (V) = V ((x_i + [0, \Delta^e_N)^d) \times (v_j + [0, \Delta^e_N)^d)) := V ((x_i + Q) \times (v_j + Q')) := V (Q_{i,j}), \]

Now, starting with an initial measure \( \mu_0 \in \mathcal{M}^+_e (\mathbb{R}^d) \), we can define the lattice approximate solution \( \mu^N_t \): At \( t = 0 \) we set

\[ \mu^N_0 = A^e_N (\mu_0) \]

and for a time mesh point \( t_i \) and \( \tau \in [0, \Delta^e_N] \) we set via recursion

\[ \mu^N_{t_i + \tau} = \tau \sum_{i=1}^I m^e_i (s[\mu^N_{t_i}]) \delta_{x_i} + \sum_{i=1}^I \sum_{j=1}^J m^v_{ij} (V[\mu^N_{t_i}]) \delta_{x_i + \tau v_j} e^{c(x_i, \mu^N_{t_i}) \tau}. \]

Note that \( \mu^N_{t_i + \tau} \) is a nonnegative measure, independent of the sign of \( c \).

We adapt [28, Proposition 19] to our setting.

**Proposition 4.1.** Let \( \mu \in \mathcal{M}^+_e (\mathbb{R}^d) \). Then for \( N \) sufficiently large it holds

\[ \| \mu - A^e_N (\mu) \|_{BL^*} \leq \sqrt{d} \Delta^e_N \| \mu \|_{BL^*}. \]

Similarly,

\[ \| V[\mu] - A^v_N (V[\mu]) \|_{BL^*} \leq 2 \sqrt{d} \Delta^e_N \| \mu \|_{BL^*}. \]

**Proof.** Let \( \psi \in BL (\mathbb{R}^d) \) with \( \| \psi \|_{BL} \leq 1 \) and \( N \in \mathbb{N} \) so large that \( \text{supp} (\mu) \subseteq [-N, N]^d \). Then

\[ \int_{\mathbb{R}^d} \psi (x) \left( \mu - \sum_{i=1}^I m^e_i \delta_{x_i} \right) = \sum_{i=1}^I \int_{Q_i} \psi (x) \left( \mu - \sum_{i=1}^I m^e_i \delta_{x_i} \right) \]

\[ = \sum_{i=1}^I \int_{Q_i} \psi (x) - \psi (x_i) \, d\mu \leq \| \psi \|_{BL} \sum_{i=1}^I \int_{Q_i} \| x - x_i \| \, d\mu \leq \sqrt{d} \Delta^e_N \| \mu \|_{BL^*}. \]

The other statement follows similarly. Just note that

\[ \| (x, v) - (x_i, v_j) \| \leq \sqrt{d \frac{1}{N^4} + d \frac{1}{N^2}} \leq 2 \sqrt{d} \Delta^e_N. \]

\( \square \)
The following lemma is a simple consequence of the push-forward condition in the Definition of the measure vector fields.

**Lemma 4.2.** Let $\mu \in \mathcal{M}^{+}(\mathbb{R}^{d})$. Then, $\|V[\mu]\|_{TV} = \|\mu\|_{TV}$. Moreover, if for some $\{x_{i}\}_{i=1,\ldots,l}$, $\{v_{j}\}_{j=1,\ldots,J} \subset \mathbb{R}^{d}$ we have

$$V[\mu] = \sum_{i=1}^{l} \sum_{j=1}^{J} m_{i,j} \delta_{(x_{i},v_{j})},$$

then $\mu = \sum_{i=1}^{l} m_{i} \mu_{x_{i}}$ and $m_{i} = \sum_{j=1}^{J} m_{i,j}$.

**Proof.** For all test functions $\psi \in BL(\mathbb{R}^{d})$ we have

$$\int_{\mathbb{R}^{d}} \psi(x) \, d\mu(x) = \int_{\mathbb{R}^{d} \times \mathbb{R}^{d}} \psi(\pi_{1}(x,v)) \, dV[\mu](x,v).$$

The first part of the lemma follows from taking $\psi = 1$. For the second part we observe that

$$\int_{\mathbb{R}^{d}} \psi(x) \, d\mu(x) = \int_{\mathbb{R}^{d} \times \mathbb{R}^{d}} \psi(\pi_{1}(x,v)) \, dV[\mu](x,v) = \sum_{i=1}^{l} \sum_{j=1}^{J} m_{i,j} \psi(x_{i}).$$

Considering $\psi$ which vanishes at $\{x_{i}\}_{i=1,\ldots,l}$ we obtain that $\mu$ may be supported only at points $\{x_{i}\}_{i=1,\ldots,l}$. Hence, taking $\psi$ which is one at $x_{i}$ and vanishes at $x_{k}$ for $k \neq i$ we conclude the proof. \qed

**Lemma 4.3.** Let $\tilde{R} = \max\{R, R_{0}\}$, where $R$ is the maximal radius corresponding to the support of $s$ and $R_{0}$ is chosen such that $\text{supp}(\mu_{0}) \subseteq B(0,R_{0})$. Then for all $l = 0, \ldots, M$, $\tau \in [0, \Delta_{N}]$ and $N \in \mathbb{N}$ big enough

$$\text{supp}(\mu_{t_{i}+\tau}^{N}) \subseteq B\left(0, e^{C_{S}T}(\tilde{R} + 2) - 1\right).$$

In particular, $\mu_{t_{i}+\tau}^{N}$ has a compact support $K$ which is independent of $N \in \mathbb{N}$, $l \in \{0, \ldots, M\}$ and $\tau \in [0, \Delta_{N}]$.

**Proof.** We first note the following auxiliary statement: If $\mu \in \mathcal{M}_{e}^{+}(\mathbb{R}^{d})$ with $\text{supp}(\mu) \subseteq B(0,r)$, then (4.2)

$$\text{supp} (A_{N}^{x}(\mu)) \subseteq B(0,r+1).$$

Indeed, $m_{i}^{\tau}(\mu) = \mu(Q_{i}) = 0$ if $Q_{i} \not\subseteq B(0,r)$ or equivalently if $x_{i} \not\in B(0,r + \sqrt{d}/N^{2}) \subseteq B(0,r+1)$ for $N$ big enough, which was to show.

From (4.2) and the definition of $\mu_{0}^{N}$ it follows directly that

$$\text{supp}(\mu_{0}^{N}) \subseteq \text{supp}(0, \tilde{R} + 1).$$

Now let $l \in \{1, \ldots, M\}$ and $\tau \in [0, \Delta_{N}]$. Suppose $\text{supp}(\mu_{t_{i}^{N}}) \subseteq B(0,R_{i}^{N})$ with $R_{i}^{N} \geq \tilde{R} + 1$. Then we claim that

(4.3) $$\text{supp}(\mu_{t_{i}+\tau}^{N}) \subseteq B(0,R_{i}^{N} + \Delta_{N} C_{S}(1 + R_{i}^{N})).$$

We consider the summands in (4.1) separately. For the first term, we note that by (4.2) sup $\text{supp}(s|\mu_{t_{i}^{N}}|) \subseteq B(0,\tilde{R})$ and thus sup $\left(\tau \sum_{i=1}^{l} m_{i}^{\tau}(s|\mu_{t_{i}^{N}}|)\right) \subseteq B(0,\tilde{R}+1) \subseteq B(0,\tilde{R})$ by (4.2). For the second term,
we invoke (V1) and see that if \((x_i, v_j) \in \text{supp} V[\mu_i^N]\) then the assumption implies \(|v_j| \leq C_S(1 + R_i^N)\) and consequently \(\text{supp}(\delta_{x_i,\tau v_j}) \subseteq B(0, R_i^N + \Delta_N C_S(1 + R_i^N))\). Now, (4.3) follows directly. By induction over \(l\) it can be shown that for all \(k = 0, \ldots, l\)

\[
(4.4) \quad R_i^N \leq R_{i-k}^N (1 + \Delta_N C_S)^k + (1 + \Delta_N C_S)^k - 1,
\]

where the induction base follows from (4.3) choosing \(\tau = \Delta_N\).

We conclude the proof by applying (4.4) with \(k = l\) and using that \(R_0^N = R_0 \leq \tilde{R} + 1\)

\[
R_i^N \leq R_0^N (1 + \Delta_N C_S)^l + (1 + \Delta_N C_S)^l - 1 \leq (\tilde{R} + 1)e^{\Delta_N C_S l} + e\Delta_N C_S l - 1 \leq e^{T C_S} (\tilde{R} + 2) - 1,
\]

since \(l \Delta_N \leq T\). □

**Lemma 4.4.** Let \(t \in [0, T]\) and let \(\mu_t^N\) be defined by (4.1). If \((x, v) \in \text{supp} (V[\mu_t^N])\) for some \(t \in [0, T]\), then

\[
|v| \leq C_S e^{C_S T} (\tilde{R} + 2),
\]

where \(\tilde{R}\) has been defined in Lemma 4.3.

**Proof.** Before we prove the statement, we note the following: If \(\mu \in \mathcal{M}_+^c(\mathbb{R}^d)\), then

\[
\text{supp} (V[\mu]) \subseteq \text{supp} (\mu) \times \mathbb{R}^d.
\]

Indeed, if \(A \not\subseteq \text{supp} (\mu), A \subseteq \mathbb{R}^d,\) then

\[
0 = \mu(A) = V[\mu] \circ \pi_1^{-1}(A) = V[\mu](A \times \mathbb{R}^d)
\]

and the claim follows by contraposition.

Now we prove the Lemma. According to (V1), (4.5) and Lemma 4.3

\[
|v_j| \leq C_S \left( 1 + \sup_{(x, v) \in \text{supp} (V[\mu_i])} |x| \right) \leq C_S \left( 1 + \sup_{x \in \text{supp} (\mu_i)} |x| \right) \leq C_S \left( 1 + e^{C_S T} (\tilde{R} + 2) - 1 \right) = C_S e^{C_S T} (\tilde{R} + 2).
\]

Combining the results of Lemma 4.3 and 4.4 with equation (4.5) we note the following:

**Corollary 4.5.** \(V[\mu_t^N]\) is compactly supported in some set \(K_V\) which is independent of \(l\) and \(N\).

**Lemma 4.6** (Lipschitz continuity of \(t \mapsto \mu_t^N\) and further estimates). There is a constant \(C_d\) (independent of \(N\) and \(t \in [0, T]\)) such that

\[
(4.6) \quad \|\mu_t^N - \mu_s^N\|_{B_L^r} \leq C_d |t - s|.
\]
Moreover, we have estimates

\begin{equation}
\|V[\mu^N]\|_{BL^*} = \|\mu^N\|_{BL^*} \leq C_d, \quad \sum_{i=1}^l \sum_{j=1}^j m_{i,j}^v(V[\mu^N_i]) (1 + |v_j| + |v_j|^2) \leq C_d.
\end{equation}

**Proof.** First, consider \( l \in \{0, \ldots, M\} \) and \( \tau_1, \tau_2 \in [0, \Delta_N] \). Let \( \psi \in BL(\mathbb{R}^d) \) with \( \|\psi\|_{BL} \leq 1 \). Using the representation (4.4) we obtain

\[ \int_{\mathbb{R}^d} \psi(x) d(\mu^N_{l_i+\tau_1} - \mu^N_{l_i+\tau_2}) (x) = (\tau_1 - \tau_2) \sum_{i=1}^l m_i^v(s[\mu^N_i]) \psi(x_i) + \sum_{i=1}^l \sum_{j=1}^j m_{i,j}^v(V[\mu^N_i]) \left[ \psi(x_i + \tau_1 v_j) e^{c(x_i, \mu^N_i)\tau_1} - \psi(x_i + \tau_2 v_j) e^{c(x_i, \mu^N_i)\tau_2} \right]. \]

Now, the second term above can be further rewritten as

\[ \sum_{i=1}^l \sum_{j=1}^j m_{i,j}^v(V[\mu^N_i]) \left[ \psi(x_i + \tau_1 v_j) - \psi(x_i + \tau_2 v_j) \right] e^{c(x_i, \mu^N_i)\tau_1} + \sum_{i=1}^l \sum_{j=1}^j m_{i,j}^v(V[\mu^N_i]) \left[ e^{c(x_i, \mu^N_i)\tau_1} - e^{c(x_i, \mu^N_i)\tau_2} \right]. \]

Using Lipschitz continuity of \( \psi \) and of the exponential function (on the bounded interval \([0, \Delta_N]\) with constant \( ||c||_{\infty} ||c||_{\infty} \Delta_N \)) we obtain

\begin{equation}
\left| \int_{\mathbb{R}^d} \psi(x) d(\mu^N_{l_i+\tau_1} - \mu^N_{l_i+\tau_2}) (x) \right| \leq |\tau_1 - \tau_2| \left[ B_{1}^l + B_{2}^l + B_{3}^l \right],
\end{equation}

where

\[ B_{1}^l := \sum_{i=1}^l m_i^v(s[\mu^N_i]) \psi(x_i), \quad B_{2}^l := \sum_{i=1}^l \sum_{j=1}^j m_{i,j}^v(V[\mu^N_i]) \|\psi\|_{Lip} |v_j| e^{\|c\|_{\infty} \Delta_N}, \]

\[ B_{3}^l = \sum_{i=1}^l \sum_{j=1}^j m_{i,j}^v(V[\mu^N_i]) \|\psi\|_{\infty} e^{\|c\|_{\infty} \Delta_N} \|c\|_{\infty} \|
\]

We want to estimate the terms \( B_{1}^l, B_{2}^l \) and \( B_{3}^l \) and start with \( B_{2}^l \) and \( B_{3}^l \). According to Lemma 4.3 \( |v_j| \) is uniformly bounded and by Lemma 4.2

\[ \sum_{i=1}^l \sum_{j=1}^j m_{i,j}^v(V[\mu^N_i]) \leq \|V[\mu^N_i]\|_{BL^*} = \|\mu^N_i\|_{TV} = \|\mu^N_i\|_{BL^*}, \]

so the sum can be controlled by \( \|\mu^N_i\|_{BL^*} \). Thus, we obtain a constant \( C \) (independent of \( \psi \)) such that for all \( l \leq M \) and \( \tau_1, \tau_2 \in [0, \Delta_N] \)

\begin{equation}
|B_{2}^l + B_{3}^l| \leq C \|\mu^N_i\|_{BL^*}.
\end{equation}
Next, we try to bound $B^l_1$

\begin{equation}
\sum_{i=1}^{l} m_i^N(s[\mu^N_{ti}])\psi(x_i) \leq \|\psi\|_{BL} \left| \sum_{i=1}^{l} m_i^N(s[\mu^N_{ti}]) \right| \leq \|s[\mu^N_{ti}]\|_{BL^*} \leq [\|s[\mu^N_{ti}]\| - \|s[\mu_0]\|_{BL^*} + \|s[\mu_0]\|_{BL^*}] \leq [L\|\mu^N_{ti} - \mu_0\|_{BL^*} + \|s[\mu_0]\|_{BL^*}].
\end{equation}

Note that we used (4.1) for $\psi$ by induction over $l$ proving (4.12). Hence, as (4.13) and by a series of triangle inequalities we also get for arbitrary $\tau$ that

\begin{equation}
\|\mu^N_{ti+\tau_1} - \mu^N_{ti+\tau_2}\|_{BL^*} \leq |\tau_1 - \tau_2| [C\|\mu^N_{ti}\|_{BL^*} + L\|\mu^N_{ti} - \mu_0\|_{BL^*} + \|s[\mu_0]\|_{BL^*}] \leq |\tau_1 - \tau_2| [C + C\|\mu^N_{ti} - \mu_0\|_{BL^*}].
\end{equation}

In order to bound the right-hand side of (4.11) uniformly, we still have to show that the term $\|\mu^N_{ti} - \mu_0\|_{BL^*}$ in (4.11) is bounded independent of $N$ and $l$. To see this, note that by Proposition 4.1 for $N$ large enough $\|\mu^N_0 - \mu_0\|_{BL^*} \leq \|\mu_0\|_{BL^*}\Delta_N$. Thus, there exists a constant $C$ such that for all $N \in \mathbb{N} \|\mu^N_0 - \mu_0\|_{BL^*} \leq C$. Similarly to [28], we prove the following estimate

\begin{equation}
\|\mu^N_{ti} - \mu_0\|_{BL^*} \leq (1 + C\Delta_N)^l C (1 + C\Delta_N)^l - 1
\end{equation}

by induction over $l$. As the case for $l = 0$ is already proven, we assume that (4.12) holds for some $l \in \mathbb{N}_0$. Applying (4.11) with $\tau_1 = 0$ and $\tau_2 = \Delta_N$ yields

$$\|\mu^N_{ti+\tau_1} - \mu_0\|_{BL^*} \leq \|\mu^N_{ti+1} - \mu_0\|_{BL^*} + \|\mu^N_{ti} - \mu_0\|_{BL^*} \leq \Delta_N [C + C\|\mu^N_{ti} - \mu_0\|_{BL^*}] + \|\mu^N_{ti} - \mu_0\|_{BL^*} = (1 + C\Delta_N)\|\mu^N_{ti} - \mu_0\|_{BL^*} + C\Delta_N \leq (1 + C\Delta_N) [(1 + C\Delta_N)^l C (1 + C\Delta_N)^l - 1] + C\Delta_N = (1 + C\Delta_N)^{l+1} C (1 + C\Delta_N)^{l+1} - 1,$$

proving (4.12). Hence, as $l\Delta_N \leq T$, we have

\begin{equation}
\|\mu^N_{ti} - \mu_0\|_{BL^*} \leq Ce^{CT} e^{CT} - 1 < \infty.
\end{equation}

Plugging (4.13) into (4.11) yields

$$\|\mu^N_{ti+\tau_1} - \mu^N_{ti+\tau_2}\|_{BL^*} \leq C|\tau_1 - \tau_2|,$$

and by a series of triangle inequalities we also get for arbitrary $s, t \in [0, T]$

\begin{equation}
\|\mu^N_{ti} - \mu^N_s\|_{BL^*} \leq C|t - s|.
\end{equation}

In particular, $(\mu^N_{ti})_{N \in \mathbb{N}}$ is uniformly Lipschitz continuous with respect to $t$ as the Lipschitz constant is independent of $N$. Furthermore, we see that for all $N \in \mathbb{N}$ and all $t \in [0, T]$

\begin{equation}
\|\mu^N_{ti}\|_{BL^*} \leq \|\mu^N_{ti} - \mu_0\|_{BL^*} + \|\mu_0\|_{BL^*} \leq CT + \|\mu_0\|_{BL^*},
\end{equation}

where
i.e. \((\mu_t^N)_{N \in \mathbb{N}}\) is uniformly bounded. Now, the first part of (4.7) follows from Lemma 4.2 and the second by a combination of Lemma 4.4 and (4.15)

\[
\sum_{i=1}^{I} \sum_{j=1}^{J} m_{i,j}^n (V[\mu_t^N]) (1 + |v_j| + |v_j|^2) \leq C \|V[\mu_t^N]\|_{BL^*} = C \|\mu_t^N\|_{BL^*} \leq C_d.
\]

\(\square\)

**Corollary 4.7.** The term \(s[\mu_t^N]\) is bounded in the flat norm by a constant which is independent of \(t\) and \(N\).

**Proof.** From (5.1) and estimate (4.13) we see

\[
\|s[\mu_t^N]\|_{BL^*} \leq \|s[\mu_t^N] - s[\mu_0]\|_{BL^*} + \|s[\mu_0]\|_{BL^*} \leq L \|\mu_t^N - \mu_0\|_{BL^*} + C \leq C.
\]

\(\square\)

5. **Proof of the existence result**

In this section we focus on the existence result formulated in Theorem 2.5

**Proof of Theorem 2.5.** Let \(\mu_t^N\) be the lattice approximate solution constructed in (4.1). We start by extracting a converging subsequence with the theorem of Arzelà-Ascoli. Combining the results on Lipschitz continuity and uniform boundedness of Lemma 4.6 with the considerations on pointwise compactness in Lemma 3.2, all requirements of Theorem 3.1 (Arzelà-Ascoli) are fulfilled. Consequently, the sequence \((\mu_t^N)_{N \in \mathbb{N}}\) has a subsequence (still denoted by \((\mu_t^N)_{N \in \mathbb{N}}\)) converging in the space

\[
E := C([0, T]; (\mathcal{M}^+(\mathbb{R}^d), \| \cdot \|_{BL^*}))
\]

with limit measure \(\mu_t\). We claim that \(\mu_t\) solves (1.1), i.e. it satisfies Definition 2.1. To prove this, we fix \(t \in [0, T]\) and \(f \in C_c^\infty(\mathbb{R}^d)\). Introducing the notation \(s_1 \wedge s_2 := \min(s_1, s_2)\) we write

\[
\int_{\mathbb{R}^d} f(x) \, d(\mu_t^N - \mu_0^N)(x) = \sum_{i=0}^{M} \int_{\mathbb{R}^d} f(x) \, d(\mu_{t+1 \wedge t} - \mu_{i+1 \wedge t})(x)
\]
and we want to study each summand separately. Let $\tau_t = (t_{i+1} \land t) - (t_i \land t) \in [0, \Delta N]$, then representation (4.1) implies

$$
\int_{\mathbb{R}^d} f(x) \, d(\mu_{t_{i+1}}^N - \mu_{t_i}^N)(x)
$$

$$
= \tau_t \sum_{i=1}^{l} m_i^v(s[\mu_i^N]) f(x_i) + \sum_{i=1}^{l} \sum_{j=1}^{J} m_{i,j}^v(V[\mu_i^N]) f(x_i + \tau_j v_j) \left[ e^{c(x_i, \mu_i^N) \tau_t} - f(x_i) \right]
$$

(5.1)

$$
= \tau_t \sum_{i=1}^{l} m_i^v(s[\mu_i^N]) f(x_i) + \sum_{i=1}^{l} \sum_{j=1}^{J} m_{i,j}^v(V[\mu_i^N]) f(x_i + \tau_j v_j) \left[ e^{c(x_i, \mu_i^N) \tau_t} - 1 \right]
$$

$$
+ \sum_{i=1}^{l} \sum_{j=1}^{J} m_{i,j}^v(V[\mu_i^N]) \left[ f(x_i + \tau_j v_j) - f(x_i) \right]
$$

$$
=: \tau_t \sum_{i=1}^{l} m_i^v(s[\mu_i^N]) f(x_i) + \sum_{i=1}^{l} \sum_{j=1}^{J} A_{i,j,l} + \sum_{i=1}^{l} \sum_{j=1}^{J} B_{i,j,l}.
$$

Terms $B_{i,j,l}$. We claim that

$$
\sum_{l=0}^{M} \sum_{i=1}^{l} \sum_{j=1}^{J} B_{i,j,l} \rightarrow \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla f(x) \cdot v \, dV[\mu_i](x,v) \, dt \quad (N \rightarrow \infty).
$$

(5.2)

Indeed, as $f \in C_c^\infty(\mathbb{R}^d)$, Taylor’s expansion implies that

$$
|f(x_i + \tau_j v_j) - f(x_i) - \tau_j \nabla f(x_i) \cdot v_j| \leq C(f) \tau_j^2 |v_j|^2 \leq C(f) \Delta_N^2 |v_j|^2.
$$

Therefore, we can replace $B_{i,j,l}$ with

$$
B'_{i,j,l} := m_{i,j}^v(V[\mu_i^N]) \tau_t \nabla f(x_i) \cdot v_j
$$

and the error is controlled by

$$
\sum_{l=0}^{M} \sum_{i=1}^{l} \sum_{j=1}^{J} |B_{i,j,l} - B'_{i,j,l}| \leq \sum_{l=0}^{M} \sum_{i=1}^{l} \sum_{j=1}^{J} m_{i,j}^v(V[\mu_i^N]) C(f) \Delta_N^2 |v_j|^2 \leq C_d C(f) \Delta_N T
$$

thanks to (4.7) and $\sum_{l=0}^{M} \Delta_N \leq T$. Now, for $B'_{i,j,l}$ we have

$$
\sum_{l=0}^{M} \sum_{i=1}^{l} \sum_{j=1}^{J} B'_{i,j,l} = \sum_{l=0}^{M} \int_{t_i \land t}^{t_{i+1} \land t} \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla f(x) \cdot v \, dA^v_N(V[\mu_i^N])(x,v) \, dr.
$$

(5.3)

Note that the map $(x,v) \mapsto \nabla f(x) \cdot v$ may be assumed to be in $BL(\mathbb{R}^d \times \mathbb{R}^d)$ thanks to [4.1] and Corollary 4.3. Hence, according to Proposition 4.1 we can replace the right-hand side of (5.3) by

$$
\sum_{l=0}^{M} \int_{t_i \land t}^{t_{i+1} \land t} \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla f(x) \cdot v \, dV[\mu_i^N](x,v) \, dr
$$

(5.4)
and the error is controlled by
\[
M \sum_{l=0}^{M} \int_{t_l}^{t_{l+1}} \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla f(x) \cdot v \, d \left[ A_M^\tau (V[\mu^N_r]) - V[\mu^N_r] \right] (x, v) \, dr \\
\leq M \sum_{l=0}^{M} \tau_l \| \nabla f(x) \cdot v \|_{BL} \| A_M^\tau (V[\mu^N_l]) - V[\mu^N_l] \|_{BL^*} \leq CT \Delta_N \| \nabla f(x) \cdot v \|_{BL} \| \mu^N_l \|_{BL^*} \to 0
\]
as \( N \to \infty \) because \( \| \mu^N_r \|_{BL^*} \) is uniformly bounded by (4.7).

Lastly, we want to replace the measure \( V[\mu^N_r] \) in [5.4] by \( V[\mu^N_r] \) for an arbitrary time point \( r \in [t_l \wedge t, t_{l+1} \wedge t] \). Therefore, using (5.2) and Lipschitz continuity from (4.11) we obtain
\[
M \sum_{l=0}^{M} \int_{t_l}^{t_{l+1}} \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla f(x) \cdot v \, d (V[\mu^N_l] - V[\mu^N_r]) (x, v) \, dr \\
\leq \| \nabla f(x) \cdot v \|_{BL} M \sum_{l=0}^{M} \int_{t_l}^{t_{l+1}} \| V[\mu^N_l] - V[\mu^N_r] \|_{BL^*} \, dr \\
\leq \| \nabla f(x) \cdot v \|_{BL} C_F M \sum_{l=0}^{M} \int_{t_l}^{t_{l+1}} \| \mu^N_l - \mu^N_r \|_{BL^*} \, dr \\
\leq \| \nabla f(x) \cdot v \|_{BL} C M \sum_{l=0}^{M} \int_{t_l}^{t_{l+1}} | r - t_l | \, dr \\
\leq \Delta_N \| \nabla f(x) \cdot v \|_{BL} C T,
\]
where we used the following inequality in the last line
\begin{equation}
(5.5) \quad M \sum_{l=0}^{M} \int_{t_l}^{t_{l+1}} | r - t_l | \, dr \leq \Delta_N T.
\end{equation}

Indeed, this is true as we note that
\begin{equation}
(5.6) \quad | r - t_l | \leq \Delta_N \quad \forall r \in [t_l \wedge t, t_{l+1} \wedge t].
\end{equation}

Then claim (5.5) follows by combining inequality (5.6) with the estimates \((t_{l+1} \wedge t) - (t_l \wedge t) \leq \Delta_N \) and \( \sum_{l=1}^{M} \Delta_N \leq T \). To sum up, up to an error of order \( \Delta_N \), we have
\[
M \sum_{l=0}^{M} \sum_{i=1}^{M} \sum_{j=1}^{M} B_{i,j,l} \approx M \sum_{l=0}^{M} \int_{t_l}^{t_{l+1}} \int_{\mathbb{R}^d} \nabla f(x) \cdot v \, dV[\mu^N_r](x, v) \, dr \quad (N \to \infty).
\]

We conclude by seeing
\[
M \sum_{l=0}^{M} \int_{t_l}^{t_{l+1}} \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla f(x) \cdot v \, dV[\mu^N_r](x, v) \, dr = \int_{0}^{t} \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla f(x) \cdot v \, dV[\mu^N_r](x, v) \, dr
\]
and use \( \mu^N_r \to \mu_r \) in \( E \) together with assumption (5.2) to deduce (5.2).
Terms $A_{i,j,l}$. We claim that

\[
\sum_{l=0}^{M} \sum_{i=1}^{I} \sum_{j=1}^{J} A_{i,j,l} \to \int_{0}^{t} \int_{\mathbb{R}^d} f(x) c(x, \mu_{\tau}) \, d\mu_{\tau}(x) \, dr \quad (N \to \infty).
\]

Note simple Taylor’s estimates

\[
\left| e^{c(x, \mu_{\tau})\tau_l} - 1 \right| \leq e^{\|c\|_{\infty} \Delta_N} \|c\|_{\infty} \tau_l, \quad \left| e^{c(x, \mu_{\tau}^N)\tau_l} - 1 - c(x, \mu_{\tau}^N)\tau_l \right| \leq e^{\|c\|_{\infty} \Delta_N} \|c\|_{\infty}^2 \tau_l^2.
\]

Thanks to the first estimate, the term $A_{i,j,l}$ can be replaced by

\[
A'_{i,j,l} := m_{i,j}^v \left( V[\mu_{\tau}^N] \right) f(x_i) \left[ e^{c(x, \mu_{\tau}^N)\tau_l} - 1 \right]
\]

and the total error is controlled by

\[
\sum_{l=0}^{M} \sum_{i=1}^{I} \sum_{j=1}^{J} \left| A_{i,j,l} - A'_{i,j,l} \right| \leq \sum_{l=0}^{M} \sum_{i=1}^{I} \sum_{j=1}^{J} m_{i,j}^v \left( V[\mu_{\tau}^N] \right) |f|_{\text{Lip}} |v_j| e^{\|c\|_{\infty} \Delta_N} \|c\|_{\infty} \tau_l
\]

\[
\leq C(c, f) \sum_{l=0}^{M} \sum_{i=1}^{I} \sum_{j=1}^{J} m_{i,j}^v \left( V[\mu_{\tau}^N] \right) |v_j| (\Delta_N)^2 \leq C(c, f) C_d T \Delta_N
\]

due to (4.7) and $\sum_{l=0}^{M} \Delta_N \leq T$. Similarly, term $A''_{i,j,l}$ can now be replaced by

\[
A''_{i,j,l} := m_{i,j}^v \left( V[\mu_{\tau}^N] \right) f(x_i) c(x_i, \mu_{\tau}^N) \tau_l
\]

and this time we use the second estimate in (3.8) for controlling the error. From the definition of $\mu_{\tau}^N$ we obtain

\[
\sum_{i=1}^{I} \sum_{j=1}^{J} A''_{i,j,l} = \tau_l \int_{\mathbb{R}^d} f(x) c(x, \mu_{\tau}^N) \, d\mu_{\tau}^N(x) = \int_{t_{l+1}^{\Lambda t}}^{t_{l+1}^{\Lambda t}} \int_{\mathbb{R}^d} f(x) c(x, \mu_{\tau}^N) \, d\mu_{\tau}^N(x) \, dr.
\]

This can be replaced with $\int_{t_{l+1}^{\Lambda t}}^{t_{l+1}^{\Lambda t}} \int_{\mathbb{R}^d} f(x) c(x, \mu_{\tau}^N) \, d\mu_{\tau}^N(x) \, dr$ and the total error can be estimated by

\[
\sum_{l=0}^{M} \left| \int_{t_{l+1}^{\Lambda t}}^{t_{l+1}^{\Lambda t}} \int_{\mathbb{R}^d} f(x) c(x, \mu_{\tau}^N) - c(x, \mu_{\tau}^N) \right| \, d\mu_{\tau}^N(x) \, dr
\]

\[
+ \sum_{l=0}^{M} \left| \int_{t_{l+1}^{\Lambda t}}^{t_{l+1}^{\Lambda t}} \int_{\mathbb{R}^d} f(x) c(x, \mu_{\tau}^N) \, d(\mu_{\tau}^N - \mu_{\tau}^N)(x) \, dr \right|.
\]

For the first term we note that

\[
\sum_{l=0}^{M} \left| \int_{t_{l+1}^{\Lambda t}}^{t_{l+1}^{\Lambda t}} \int_{\mathbb{R}^d} f(x) c(x, \mu_{\tau}^N) - c(x, \mu_{\tau}^N) \right| \, d\mu_{\tau}^N(x) \, dr
\]

\[
\leq \|f\|_{BL} \sum_{l=0}^{M} \int_{t_{l+1}^{\Lambda t}}^{t_{l+1}^{\Lambda t}} C_L \|\mu_{\tau}^N - \mu_{\tau}^N\|_{BL} \|\mu_{\tau}^N\|_{BL} \, dr \leq C_d^2 C_L \|f\|_{BL} \Delta_N T,
\]
while the second term can be bounded by

\[
\sum_{i=0}^{M} \left| \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}^d} f(x) c(x, \mu_N^N) \, d(\mu_N^N - \mu_N^N)(x) \, dr \right|
\leq \|f\|_B L \sum_{i=0}^{M} \int_{t_i}^{t_{i+1}} \|\mu_N^N - \mu_N^N\|_{BL^*} \, dr \leq \|f\|_B C_d \Delta_N T
\]

so that the total error is controlled by \(C(c, f) \Delta_N T\). Note that we used (5.5) and Lemma 4.6 again. Noting that

\[
\sum_{i=0}^{M} \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}^d} f(x) c(x, \mu_N^N) \, d\mu_N^N(x) \, dr = \int_0^t \int_{\mathbb{R}^d} f(x) c(x, \mu_N^N) \, d\mu_N^N(x) \, dr
\]

and \(\mu_N^N \to \mu_r\) in \(E\) we prove (5.7).

Terms \(\tau_i \sum_{i=1}^l m^r_i(s[\mu_N^N])f(x_i)\). We claim that

(5.9) \[\sum_{i=0}^{M} \tau_i \sum_{i=1}^l m^r_i(s[\mu_N^N])f(x_i) \to \int_0^t \int_{\mathbb{R}^d} f(x) \, ds[\mu_r](x) \, dr \quad (N \to \infty).\]

First, observe that

\[
\sum_{i=0}^{M} \tau_i \sum_{i=1}^l m^r_i(s[\mu_N^N])f(x_i) - \int_0^t \int_{\mathbb{R}^d} f(x) \, ds[\mu_r](x) \, dr = \left( \sum_{i=0}^{M} \tau_i \sum_{i=1}^l m^r_i(s[\mu_N^N])f(x_i) - \int_0^t \int_{\mathbb{R}^d} f(x) \, ds[\mu_r^N](x) \, dr \right) + \int_0^t \int_{\mathbb{R}^d} f(x) \, ds[\mu_r^N - s[\mu_r]](x) \, dr
\]

\[
= : X + Y.
\]

The convergence of the second summand to zero as \(N \to \infty\) can be seen directly via

(5.10) \[|Y| = \left| \int_0^t \int_{\mathbb{R}^d} f(x) \, d(s[\mu_N^N] - s[\mu_r])(x) \, dr \right| \leq \int_0^T \|f\|_{BL} \|s[\mu_N^N] - s[\mu_r]\|_{BL^*} \, dr \leq T \|f\|_{BL} L \sup_{r \in [0, T]} \|\mu_N^N - \mu_r\|_{BL^*} \to 0,
\]

where we used Assumption (5.1) and that \(\mu_N^N \to \mu_r\) in \(E\) with respect to the supremum norm as \(N \to \infty\). For term \(X\), we observe that

\[
\sum_{i=0}^{M} \tau_i \sum_{i=1}^l m^r_i(s[\mu_N^N])f(x_i) = \sum_{i=0}^{M} \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}^d} f(x) \, dS^N[\mu_N^N](x) \, dr, \quad S^N[\mu_N^N] := \sum_{i=1}^l m^r_i(s[\mu_N^N]) \delta_{x_i},
\]
so that
\[
|X| \leq \sum_{l=0}^{M} \left( \int_{t_{l+1} \wedge t}^{t_{l} \wedge t} \int_{\mathbb{R}^d} f(x) \, d(x) \, dr \right) \quad \text{(5.11)}
\]
\[
\leq \|f\|_{BL} \sum_{l=0}^{M} \int_{t_{l} \wedge t}^{t_{l+1} \wedge t} \|S^N[\mu^N_{t_l}] - s[\mu^N_{t_l}]\|_{BL^*} \, dr.
\]

Using \((5.11)\), Proposition 4.1 and (4.14), we see
\[
\|S^N[\mu^N_{t_l}] - s[\mu^N_{t_l}]\|_{BL^*} \leq \|S^N[\mu^N_{t_l}] - s[\mu^0_{t_l}]\|_{BL^*} + \|s[\mu^0_{t_l}] - s[\mu^N_{t_l}]\|_{BL^*}.
\]
\[
\leq \sqrt{d} \Delta_N^2 \|s[\mu^N_{t_l}]\|_{BL^*} + L\|\mu^N_{t_l} - \mu^N_{t_l}\|_{BL^*} \leq \sqrt{d} \Delta_N^2 \|s[\mu^N_{t_l}]\|_{BL^*} + L\Delta_N^2 \|\mu^N_{t_l}\|_{BL^*}.
\]
Again using \((5.11)\), (4.14) and Proposition 4.1 we note that
\[
\|s[\mu^N_{t_l}]\|_{BL^*} \leq \|s[\mu_0]\|_{BL^*} + \|s[\mu^N_{t_l}] - s[\mu_0]\|_{BL^*}
\]
\[
\leq \|s[\mu_0]\|_{BL^*} + L\|\mu^N_{t_l} - \mu_0\|_{BL^*}.
\]
\[
\leq \|s[\mu_0]\|_{BL^*} + L\left[\|\mu^N_{t_l} - \mu_0\|_{BL^*} + \|\mu^N_{t_l} - \mu_0\|_{BL^*}\right]
\]
\[
\leq \|s[\mu_0]\|_{BL^*} + L \left[C|t_l - 0| + \Delta_N^2 \|\mu_0\|_{BL^*}\right]
\]
\[
= \|s[\mu_0]\|_{BL^*} + LCT + L\Delta_N^2 \|\mu_0\|_{BL^*}.
\]

Plugging (5.13) into (5.12) and then into (5.11) leads to
\[
|X| \leq CT \Delta_N \to 0 \quad (N \to \infty).
\]

\[\square\]

**Remark 5.1.** Let \(\mu_\star\) be a solution to problem \((1.1)\) according to Theorem 2.5. Then Lemma 4.3 implies that \(\mu_t\) has a compact support \(K\) which is independent of \(t \in [0,T]\).

6. Continuity with respect to initial data

This section is devoted to the continuity of the semigroup characterized by Theorem 2.8. Building
the proof on the new MVF continuity condition \((V_3)\), we explore its applicability in examples
proposed previously as the MDE test cases by Piccoli [26].

**Proof of Theorem 2.8** Let \(\tau \in [0,\Delta_N]\) with \(N\) big enough and consider the lattice approximate
solutions \(\mu^N_{t_{i+\tau}}, \nu^N_{t_{i+\tau}}\). In a first step, we want to estimate the difference \(\|\mu^N_{t_{i+\tau}} - \nu^N_{t_{i+\tau}}\|_{BL^*}\) by
means of \(\|\mu^N_{t_{i}} - \nu^N_{t_{i}}\|_{BL^*}\). By construction \((4.1)\) it holds
\[
\mu^N_{t_{i+\tau}} = \tau \sum_{i=1}^{I} m^r_i(s[\mu^N_{t_{i}}])\delta_x + \sum_{i=1}^{I} \sum_{j=1}^{J} m^v_{i,j} (V[\mu^N_{t_{i}}]) \delta_{x_{i} + \tau v_j} e^{c(x,\mu^N_{t_{i}})}.
\]
Now, up to an error of size $\Delta_{NL}^2$, we can replace $\mu_{t_i}^{N,i}$ with

\[
\tau \sum_{i=1}^{I} m^x_i (s[\mu_{t_i}^{N}]) \delta_{x_i} + \sum_{i=1}^{I} \sum_{j=1}^{J} m_{i,j}^v (V[\mu_{t_i}^{N}]) \delta_{x_i+\tau v_j} c(x_i, \mu_{t_i}^{N}) \tau + \sum_{i=1}^{I} \sum_{j=1}^{J} m_{i,j}^v (V[\mu_{t_i}^{N}]) \delta_{x_i+\tau v_j},
\]

in the flat norm which follows from a Taylor estimate similar to the second one in (5.3). Analogously, by paying with an error of order $\Delta_{NL}^2$ we replace $\nu_{t_i}^{N,i}$ with

\[
\tau \sum_{i=1}^{I} m^x_i (s[\nu_{t_i}^{N}]) \delta_{x_i} + \sum_{i=1}^{I} \sum_{j=1}^{J} m_{i,j}^v (V[\nu_{t_i}^{N}]) \delta_{x_i+\tau v_j} c(x_i, \nu_{t_i}^{N}) \tau + \sum_{i=1}^{I} \sum_{j=1}^{J} m_{i,j}^v (V[\nu_{t_i}^{N}]) \delta_{x_i+\tau v_j},
\]

Now, we estimate the difference between (6.1) and (6.2) in the flat norm by comparing the related terms separately.

**Term with source $s$.** Concerning the source term, we have

\[
\left\| \tau \sum_{i=1}^{I} m^x_i (s[\mu_{t_i}^{N}]) \delta_{x_i} - \tau \sum_{i=1}^{I} m^x_i (s[\nu_{t_i}^{N}]) \delta_{x_i} \right\|_{BL^*} = \tau \left\| A^x_N(s[\mu_{t_i}^{N}]) - A^x_N(s[\nu_{t_i}^{N}]) \right\|_{BL^*} \leq \tau \left\| A^x_N(s[\mu_{t_i}^{N}]) - s[\mu_{t_i}^{N}] \right\|_{BL^*} + \tau \left\| s[\mu_{t_i}^{N}] - s[\nu_{t_i}^{N}] \right\|_{BL^*} + \tau \left\| A^x_N(s[\nu_{t_i}^{N}]) - s[\nu_{t_i}^{N}] \right\|_{BL^*}.
\]

Using Proposition 4.1, Corollary 4.7 and assumption (S1), we obtain

\[
\left\| \tau \sum_{i=1}^{I} m^x_i (s[\mu_{t_i}^{N}]) \delta_{x_i} - \tau \sum_{i=1}^{I} m^x_i (s[\nu_{t_i}^{N}]) \delta_{x_i} \right\|_{BL^*} \leq C \Delta_{NL}^3 + L \Delta_{NL} \| \mu_{t_i}^{N} - \nu_{t_i}^{N} \|_{BL^*}.
\]

**Term with growth function $c$.** First, we want to transform this term to a simpler expression. Using $\| \delta_a - \delta_b \|_{BL^*} \leq |a - b|$, we observe

\[
\left\| \sum_{i=1}^{I} \sum_{j=1}^{J} m_{i,j}^v (V[\mu_{t_i}^{N}]) \delta_{x_i+\tau v_j} c(x_i, \mu_{t_i}^{N}) \tau - \sum_{i=1}^{I} \sum_{j=1}^{J} m_{i,j}^v (V[\mu_{t_i}^{N}]) \delta_{x_i} c(x_i, \mu_{t_i}^{N}) \tau \right\|_{BL^*} \leq \tau \left\| c \right\|_{\infty} \sum_{i=1}^{I} \sum_{j=1}^{J} m_{i,j}^v (V[\mu_{t_i}^{N}]) \| \delta_{x_i+\tau v_j} - \delta_{x_i} \|_{BL^*} \leq \tau^2 \left\| c \right\|_{\infty} \sum_{i=1}^{I} \sum_{j=1}^{J} m_{i,j}^v (V[\mu_{t_i}^{N}]) |v_j| \leq C \Delta_{NL}^2,
\]

where we applied Lemma 4.6 in the last step. Furthermore, as $\pi_1^* V[\mu] = \mu$, we note that

\[
\sum_{i=1}^{I} \sum_{j=1}^{J} m_{i,j}^v (V[\mu_{t_i}^{N}]) \delta_{x_i} c(x_i, \mu_{t_i}^{N}) \tau = \tau \sum_{i=1}^{I} m^v_i (\mu_{t_i}^{N}) \delta_{x_i} c(x_i, \mu_{t_i}^{N}).
\]
The latter measure is an approximation of \( \tau \mu_t^N c(\cdot, \mu_t^N) \). To see that, we consider a test function \( \psi \in BL(\mathbb{R}^d) \) with \( \| \psi \|_{BL} \leq 1 \) and compute

\[
\tau \left| \int_{\mathbb{R}^d} \psi(x) c(x, \mu_t^N) \, d\mu_t^N - \sum_{i=1}^I \psi(x_i) c(x_i, \mu_t^N) \, m_i^r(\mu_t^N) \right|
\leq \tau \sum_{i=1}^I \int_{Q_i} |\psi(x) c(x, \mu_t^N) - \psi(x_i) c(x, \mu_t^N)| \, d\mu_t^N
\leq \tau \sum_{i=1}^I \int_{Q_i} |\psi(x) - \psi(x_i)| |c(x, \mu_t^N)| + |\psi(x_i)| |c(x, \mu_t^N)| - c(x, \mu_t^N)| \, d\mu_t^N \leq C \Delta_3^N.
\]

Note that we used assumption (C2) and Lemma 4.6. We conclude that

\[
\left\| \sum_{i=1}^I \sum_{j=1}^J m_i^{v_j} (V[\mu_t^N]) \delta_{x_i, +\tau v_j} c(x_i, \mu_t^N) \tau - \tau \mu_t^N c(\cdot, \mu_t^N) \right\|_{BL^*} \leq C \Delta_3^N + C \Delta_3^N
\]

and similar estimate holds for the expression with \( \nu_t^N \) instead of \( \mu_t^N \). Therefore, to compare terms containing function \( c \), it is sufficient to estimate \( \tau \mu_t^N c(\cdot, \mu_t^N) - \tau \mu_t^N c(\cdot, \nu_t^N) \) which can be bounded by

\[
\left\| \tau \nu_t^N c(\cdot, \nu_t^N) - \tau \mu_t^N c(\cdot, \nu_t^N) \right\|_{BL^*} \leq C \tau \| \nu_t^N - \mu_t^N \|_{BL^*}
\]

using assumption (C2) and Lemma 4.6. So in total we get the estimate

\[
\left(6.4\right) \left\| \sum_{i=1}^I \sum_{j=1}^J m_i^{v_j} (V[\mu_t^N]) \delta_{x_i, +\tau v_j} c(x_i, \mu_t^N) \tau - \sum_{i=1}^I \sum_{j=1}^J m_i^{v_j} (V[\mu_t^N]) \delta_{x_i, +\tau v_j} c(x_i, \mu_t^N) \tau \right\|_{BL^*}
\leq C \Delta_3 \| \nu_t^N - \mu_t^N \|_{BL^*} + C \Delta_2^N + C \Delta_2^N.
\]

**Transport term.** This is the most difficult term to handle and the additional assumption (V3) will be needed here. We want to estimate

\[
\left(6.5\right) \gamma[\mu_t^N, \nu_t^N] := \sum_{i=1}^I \sum_{j=1}^J \left( m_{i,j}^v (V[\mu_t^N]) - m_{i,j}^v (V[\nu_t^N]) \right) \delta_{x_i, +\tau v_j}, \quad \Delta := \| \gamma[\mu_t^N, \nu_t^N] \|_{BL^*}.
\]

The most important observation is that

\[
\int_{\mathbb{R}^d} \psi(x) \, d \gamma[\mu_t^N, \nu_t^N](x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x + \tau v) \, d \left( \mathcal{A}_N(V[\mu_t^N]) - \mathcal{A}_N(V[\nu_t^N]) \right)(x, v).
\]

Now, we want to replace \( \mathcal{A}_N(V[\mu_t^N]) \) and \( \mathcal{A}_N(V[\nu_t^N]) \) by \( V[\mu_t^N] \) and \( V[\nu_t^N] \) respectively. By a reasoning similar to the one in Proposition 4.4 we have

\[
\left(6.6\right) \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x + \tau v) \, d \left( \mathcal{A}_N(V[\mu_t^N]) - V[\mu_t^N] \right)(x, v) \right| \leq C \Delta_2^N.
\]
Indeed,
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x + \tau v) \, d\left(A^N_t(V[\mu^N_t]) - V[\mu^N_t]\right) (x, v)
\]
\[
= \sum_{i=1}^I \sum_{j=1}^J \int_{Q_i \times Q_j} [\psi(x_i + \tau v_j) - \psi(x + \tau v)] \, dV[\mu^N_t](x, v).
\]

Using the Lipschitz continuity of \(\psi\), so that (6.6) follows as
\[
V = \int_{\mathbb{R}^d} \psi(x) \, d\mu^N_t(x),
\]

Let us demonstrate that standard examples of MVFs indeed satisfy both conditions (of optimal transport theory. This in turn, significantly simplifies the proof. Piccoli and Rossi (see [28]), the proof of Theorem 2.8 presented here avoids the technical application
\[
\text{Indeed,} \quad \Delta \leq (1 + C_H \tau) \|\mu^N_t - \nu^N_t\|_{BL^*} + C \Delta^2_N \leq (1 + C_H \Delta_N) \|\mu^N_t - \nu^N_t\|_{BL^*} + C \Delta^2_N.
\]

Combining the above estimates concerning the source term \(s\) (bound (6.3)), the growth term \(c\) (bound (6.4)) and the transport term (bound (6.7)), we get an estimate of the form
\[
(6.8) \|\mu^N_{t+T} - \nu^N_{t+T}\|_{BL^*} \leq (1 + K \Delta_N) \|\mu^N_t - \nu^N_t\|_{BL^*} + C \Delta^2_N
\]

which for \(t = \Delta_N\) can be iterated to get
\[
\|\mu^N_t - \nu^N_t\|_{BL^*} \leq (1 + K \Delta_N)^t \|\mu^0 - \nu^0\|_{BL^*} + C \Delta^2_N \sum_{r=0}^{t-1} (1 + K \Delta_N)^r
\]
\[
\leq e^{l \Delta_N K} \|\mu^0 - \nu^0\|_{BL^*} + C \Delta^2_N \frac{(1 + K \Delta_N)^t - 1}{1 + K \Delta_N - 1}
\]
\[
\leq e^{K t} \|\mu^0 - \nu^0\|_{BL^*} + C \Delta^2_N \frac{e^{l \Delta_N K} - 1}{K \Delta_N}
\]
\[
\leq e^{K t} \|\mu^0 - \nu^0\|_{BL^*} + C \Delta_N \frac{e^{K t} - 1}{K},
\]

as \(l \Delta_N \leq t\). Using triangle inequality with Proposition 4.11, we finally get
\[
\|\mu^N_t - \nu^N_t\|_{BL^*} \leq e^{K t} \|\mu^0 - \nu^0\|_{BL^*} + e^{K t} C \Delta^2_N + C \Delta_N \frac{e^{K t} - 1}{K}
\]

and thus (2.1) follows by letting \(N \to \infty\). \(\square\)

**Remark 6.1.** In contrast to the proof of the continuity with respect to initial data provided by Piccoli and Rossi (see [28]), the proof of Theorem 2.8 presented here avoids the technical application of optimal transport theory. This in turn, significantly simplifies the proof.

Let us demonstrate that standard examples of MVFs indeed satisfy both conditions (V2) and (V3).

**Example 6.2.** Let \(\nu(x)\) be a Lipschitz vector field and consider
\[
V_1[\mu] = \mu \otimes \delta_{\nu(x)}.
\]
Then for any $\psi$ with $\|\psi\|_{BL(\mathbb{R}^d)} \leq 1$

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x + \tau v) \, d(V_1[\mu] - V_1[\nu])(x,v) = \int_{\mathbb{R}^d} \psi(x + \tau \nu(x)) \, d(\mu - \nu)(x).$$

The map $x \mapsto \psi(x + \tau \nu(x))$ is bounded by 1 and Lipschitz continuous with constant $1 + \tau |\nu|_{\text{Lip}}$. It follows that assumption $[V_{\psi}]$ is satisfied with $C_H := |\nu|_{\text{Lip}}$. Similarly, the MVF $V_1$ also satisfies $[V_{\psi}]$ with $C_F := 1 + |\nu|_{\text{Lip}}$ as the map $\tilde{\psi} : x \mapsto \psi(x, \nu(x))$ has bound $\|\tilde{\psi}\|_{BL(\mathbb{R}^d \times \mathbb{R}^d)} \leq 1 + |\nu|_{\text{Lip}}$. So in view of Remark 2.7 existence and uniqueness of solutions to model (1.1) with $V = V_1$ can be provided.

**Example 6.3.** Suppose that $d = 1, c = 0$ and $s = 0$ so that we consider a 1D conservative MDE, i.e. the map $t \mapsto \int_{\mathbb{R}} dq_\mu$ is constant (to see this, consider a compactly supported test function which is identity on the support of the solution $\mu_t$). In such situation, we may consider the MDE in the space of probability measures as in [20]. Moreover, it is sufficient to obtain the estimates in the Wasserstein distance $W_1$ rather than flat metric so that condition $[V_{\psi}]$ can be replaced by

$$\sup_{\|\psi\|_{\text{Lip}(\mathbb{R})} \leq 1} \int_{\mathbb{R} \times \mathbb{R}} \psi(x + \tau v) \, d(V[\mu] - V[\nu])(x,v) \leq (1 + C_H \tau) W_1(\mu, \nu).$$

For $\mu \in \mathcal{P}(\mathbb{R})$ we define

$$B(\mu) := \sup \left\{ x \in \mathbb{R} : \mu(-\infty, x] \leq \frac{1}{2} \right\}, \quad b_\mu := \frac{1}{2} \frac{\mu(-\infty, B(\mu))}{\mu(B(\mu))},$$

the latter quantity is only well-defined when $\mu(B(\mu)) > 0$. Then the MVF is defined as $V_2[\mu] = \mu \otimes \gamma_x$ where

$$\gamma_x = \begin{cases} 
\delta_{-1} & \text{if } x < B(\mu), \\
\delta_1 & \text{if } x > B(\mu), \\
\mu \delta_{-1} + (1 - \mu) \delta_1 & \text{if } x = B(\mu) \text{ and } \mu(B(\mu)) > 0.
\end{cases}$$

To prove that this MVF satisfies the assumption, we will need the following decomposition device concerning Wasserstein distance. Let $\mu \in \mathcal{P}(\mathbb{R})$. We define equal decomposition of $\mu$ for left and right part

$$\mu^l := \mu 1_{(-\infty, B(\mu))} + b_\mu \mu(B(\mu)) 1_{x = B(\mu)},$$

$$\mu^r := \mu 1_{(B(\mu), +\infty)} + (1 - \mu) \mu(B(\mu)) 1_{x = B(\mu)}.$$

**Lemma 6.4** (decomposition formula for the Wasserstein distance). Consider $\mu, \nu \in \mathcal{P}(\mathbb{R})$. Then

$$W_1(\mu, \nu) = W_1(\mu^l, \nu^r) + W_1(\mu^r, \nu^r).$$

**Remark 6.5.** The formula above is another way of expressing the fundamental (and well-known) fact concerning optimal transport in 1D: The optimal maps are always monotone, i.e. they transfer the mass from left to right.
Applying formula (6.9) to \( \tilde{G} \) general inverse \( \tilde{G} \).

Moreover, let \( F \) and \( G \) be their generalized inverses

\[
F^{-1}(y) = \inf \{ x : F(x) > y \}, \quad G^{-1}(y) = \inf \{ x : G(x) > y \}.
\]

It is well-known cf. \[35, \text{Theorem 2.18, Remark 2.19 (ii)}\] that

\[
(6.9) \quad W_1(\mu, \nu) = \int_0^1 |F^{-1}(y) - G^{-1}(y)| \, dy.
\]

First, we consider the term \( W_1(\mu^l, \nu^l) \). Note that \( 2\mu^l \) and \( 2\nu^l \) are probability measures. Moreover, the CDF of \( 2\mu^l \) is given by \( \tilde{F}_l : \mathbb{R} \to [0,1] \), where

\[
\tilde{F}_l(x) = \begin{cases} 
2F(x) & x < B(\mu) \\
1 & x \geq B(\mu).
\end{cases}
\]

Thus, its generalized inverse is of the form \( \tilde{F}_l^{-1} = F^{-1}(\cdot/2) \) except possibly at \( y = 1 \) which will be negligible as a set of Lebesgue measure zero. Analogously, for \( 2\nu^l \) we get the CDF \( \tilde{G}_l \) and its generalized inverse \( \tilde{G}_l^{-1} \) by replacing \( \mu \) by \( \nu \) and \( F \) by \( G \) in the corresponding function for \( 2\mu^l \).

Applying formula (6.9) to \( \mu^l, \nu^l \) we obtain

\[
(6.10) \quad W_1(\mu^l, \nu^l) = \frac{1}{2} W_1(2\mu^l, 2\nu^l) = \frac{1}{2} \int_0^1 |\tilde{F}_l^{-1}(y) - \tilde{G}_l^{-1}(y)| \, dy
\]

Similarly, we consider the term \( W_1(\mu^r, \nu^r) \). This time, the CDF of \( 2\mu^r \) is

\[
\tilde{F}_r(x) = \begin{cases} 
0 & x < B(\mu) \\
2(1 - b_\mu)\mu(\{B(\mu)\}) & x = B(\mu) \\
2[x, \infty) + (1 - b_\mu)\mu(\{B(\mu)\}) & x \geq B(\mu).
\end{cases}
\]

The corresponding generalized inverse is thus \( \tilde{F}_r^{-1} = F^{-1}(\cdot/2 + 1/2) \) neglecting possibly the point \( y = 0 \) as a set of Lebesgue measure zero. Analogously, \( 2\nu^r \) has CDF \( \tilde{G}_r \) with generalized inverse \( \tilde{G}_r^{-1} \). We apply (6.9) to \( 2\mu^r, 2\nu^r \) and deduce

\[
(6.11) \quad W_1(\mu^r, \nu^r) = \frac{1}{2} W_1(2\mu^r, 2\nu^r) = \frac{1}{2} \int_0^1 |\tilde{F}_r^{-1}(y) - \tilde{G}_r^{-1}(y)| \, dy
\]

Combining (6.10) and (6.11) with (6.9) yields the desired result. \( \square \)
Corollary 7.2. Under Assumption 2.3 with semigroup of solutions to problem (1.1) it is not trivial to generalize Example 6.3 to the non-conservative case. Indeed, let $\psi \in BL(\mathbb{R})$ with $\|\psi\|_{BL} \leq 1$. Then,
\[
\int_{\mathbb{R} \times \mathbb{R}} \psi(x + \tau v) \, dV_2[\mu](x, v) = \int_{x \in B(\mu)} \psi(x - \tau) \, d\mu(x) + \int_{x \notin B(\mu)} \psi(x + \tau) \, d\mu(x) + B_\mu(B(\mu)) \psi(B(\mu) - \tau) + (1 - B_\mu(B(\mu))) \psi(B(\mu) + \tau)
\]
\[
= \int_{\mathbb{R}} \psi(x - \tau) \, d\mu^l(x) + \int_{\mathbb{R}} \psi(x + \tau) \, d\mu^r(x).
\]
It follows that
\[
\int_{\mathbb{R} \times \mathbb{R}} \psi(x + \tau v) \, d(V_2[\mu] - V_2[\nu])(x, v) = \int_{\mathbb{R}} \psi(x - \tau) \, d(\mu^l - \nu^l)(x) + \int_{\mathbb{R}} \psi(x + \tau) \, d(\mu^r - \nu^r)(x).
\]
As maps $x \mapsto \psi(x - \tau), \psi(x + \tau)$ are Lipschitz continuous with constant 1, we deduce
\[
\int_{\mathbb{R} \times \mathbb{R}} \psi(x + \tau v) \, d(V_2[\mu] - V_2[\nu])(x, v) \leq W_1(\mu^l, \nu^l) + W_1(\mu^r, \nu^r) = W_1(\mu, \nu).
\]
A similar computation, replacing $\psi(x + \tau v)$ and $\psi(x + \tau)$ with $\psi(x, v)$ and $\psi(x, \pm 1)$ respectively, yields that $V_2$ also satisfies assumption (V2) with $C_F = 1$ where $\|\mu - \nu\|_{BL(\mathbb{R}^d)}$ is replaced by $W_1(\mu, \nu)$. According to [26], this is sufficient to guarantee existence in the conservative case.

Remark 6.6. It is not trivial to generalize Example 6.3 to the non-conservative case. The main issue is that Lemma 6.4 does not have a natural generalization for flat metric. One could try to prove this with the variational formula for the flat metric (cf. Remark 3.1) but the barycenter of $\mu$ does not tell too much about the barycenter of the submeasure $\tilde{\mu}$.

7. Lipschitz semigroup of solutions and uniqueness

We conclude this paper with a short consideration of the uniqueness concept introduced by Piccoli and Rossi [28]. We start with the fact that the solutions of (1.1) form a Lipschitz semigroup in the following sense.

Definition 7.1. A Lipschitz semigroup of solutions $S_t$ to problem (1.1) is a map $S : [0, T] \times \mathcal{M}^+(\mathbb{R}^d) \to \mathcal{M}^+(\mathbb{R}^d)$ satisfying

(1) $S_0 \mu_0 = \mu_0$ and $S_{t+s} \mu_0 = S_t S_s \mu_0$.
(2) The map $t \mapsto S_t \mu_0$ is a solution to (1.1) with initial condition $\mu_0$.
(3) For every $R, M > 0$ there exists $C = C(R, M) > 0$ such that if $\text{supp}(\mu_0) \cup \text{supp}(\nu_0) \subseteq B(0, R)$ and $\|\mu_0 + \nu_0\|_{TV} \leq M$, then it holds
   (a) $\text{supp}(S_t \mu_0) \subseteq B(0, e^{Ct}C)$
   (b) $\|S_t \mu_0 - S_t \nu_0\|_{BL^*} \leq e^{Ct}\|\mu_0 - \nu_0\|_{BL^*}$
   (c) $\|S_t \mu_0 - S_t \nu_0\|_{BL^*} \leq C|t - s|$.

Corollary 7.2. Under Assumption 2.3 with (V2) supplemented by (V3), there exists a Lipschitz semigroup of solutions to problem (1.1).
Proof. For $t \in [0, T]$ we define the semigroup as the limit of the lattice approximate solution, i.e.,

$$S_t \mu_0 := \mu_t = \lim_{N \to \infty} \mu_t^N.$$ 

Then properties (1), (2), (3a) and (3c) of Definition 7.1 follow directly from the proof of Theorem 2.5, and (3b) follows from Theorem 2.8. □

Piccoli and Rossi introduced a uniqueness concept of the Lipschitz semigroup based on given Dirac germs. The proof follows analogously to [28, Theorem 4]. For convenience of the reader, we present here the necessary definitions and the result.

**Definition 7.3.**

1. We define the positive linear span of Dirac deltas as

$$\mathcal{M}_D^+(\mathbb{R}^d) := \{ \mu \in \mathcal{M}^+(\mathbb{R}^d) \mid \mu = \sum_{k=1}^{l} \alpha_k \delta_{x_k} \}.$$ 

2. For constants $R, M > 0$ we set

$$\mathcal{M}_{D,R,M}^+ := \{ \mu \in \mathcal{M}_D^+(\mathbb{R}^d) \mid \text{supp}(\mu) \subseteq B(0, R), \|\mu\|_{TV} \leq M \}.$$ 

Next, we introduce the Dirac germs which will be used to obtain a uniqueness result.

**Definition 7.4.** Fix a MVF $V$.

1. A Dirac germ $\gamma$ compatible with $V$ is a map assigning to measures in $\mathcal{M}_{D,R,M}^+$ a solution to (1.1). More explicitly, for all $R > 0$, $M > 0$ and all measures $\mu \in \mathcal{M}_{D,R,M}^+(\mathbb{R}^d)$, there exists a constant $\varepsilon(R, M) > 0$ and a Lipschitz curve $\gamma_\mu : [0, \varepsilon(R, M)] \to \mathcal{M}^+(\mathbb{R}^d)$ solving (1.1).

2. Let $\gamma$ be a Dirac germ compatible with $V$. A Lipschitz semigroup of solutions to (1.1) is called compatible with $\gamma$ if for all constants $R, M > 0$ there exists a constant $C = C(R, M)$ such that

$$\forall t \in [0, \varepsilon(R, M)] : \sup_{\mu \in \mathcal{M}_{D,R,M}^+} \|S_t \mu - \gamma_\mu(t)\|_{BL^*} \leq C t^2.$$ 

**Theorem 7.5.** [28, Theorem 4] Assume (V1), (s) and (c) from Assumption 2.3 hold. Let $\gamma$ be a Dirac germ to model (1.1) compatible with $V$. Then there exists at most one Lipschitz semigroup compatible with $\gamma$.

Proof. The proof follows the same lines as the proof of Theorem 4 in [28]. □

**Appendix A. Alternative MVF Continuity Condition**

In this Appendix we prove that results of [26] and [28] are in fact special cases of our work. In these papers, Lipschitz continuity of solutions to MDE with respect to initial conditions has been established under the following assumption.
Definition A.1. Consider two measures \( V_1, V_2 \in \mathcal{M}^+(\mathbb{R}^d \times \mathbb{R}^d) \) with \( \pi_1^\# V_1 = \mu_i, \ i = 1, 2 \). For each pair \((\tilde{V}_1, \tilde{V}_2)\) with \( \tilde{V}_1 \leq V_1 \) and \( \tilde{V}_2 \leq V_2 \) set \( \tilde{\mu}_i = \pi_1^\# \tilde{V}_i \) and define the operator

\[
W^\circ(\tilde{V}_1, \tilde{V}_2) := \inf \left\{ \int_{(\mathbb{R}^d)^4} |v - w| \, dp(x, v, y, w) \left| \begin{array}{c} \tilde{V}_1 \leq V_1, \tilde{V}_2 \leq V_2, p \in \mathcal{P}(\tilde{V}_1, \tilde{V}_2), \\
\|\mu_1 - \mu_2\|_{BL^*} = \|\mu_1 - \tilde{\mu}_1\|_{TV} + \|\mu_2 - \tilde{\mu}_2\|_{TV} + W_1(\tilde{\mu}_1, \tilde{\mu}_2) \text{ and } \pi_1^\# p \in \mathcal{P}^{\text{opt}}(\tilde{\mu}_1, \tilde{\mu}_2) \right\}.
\]

Here, \( \mathcal{P}(\tilde{V}_1, \tilde{V}_2) \) denotes the set of all transference plans between measures \( \tilde{V}_1 \) and \( \tilde{V}_2 \). Analogously, \( \mathcal{P}^{\text{opt}}(\tilde{\mu}_1, \tilde{\mu}_2) \) denotes the set of all optimal transference plans between \( \tilde{\mu}_1 \) and \( \tilde{\mu}_2 \).

Assumption A.2. For all \( R > 0 \), there is a constant \( C_F(R) \) such that if \( \mu, \nu \) are supported in \( B(0, R) \), then

\[
W^g(\tilde{V}_1, \tilde{V}_2) \leq C_F(R)\|\mu - \nu\|_{BL^*}.
\]

Now, we prove that in fact condition \( (\tilde{V}_{2,3}) \) implies conditions \( (V_2) \) and \( (V_3) \).

Lemma A.3. Suppose that condition \( (\tilde{V}_{2,3}) \) in Assumption \( A.2 \) holds true. Then both \( (V_2) \) in Assumption \( 2.3 \) and \( (V_3) \) in Assumption \( 2.6 \) are satisfied.

Proof. Note that in view of Remark 3.1 and [28, Proposition 28] we have

\[
\|V[\mu] - V[\nu]\|_{BL^*} \leq W^g(V[\mu], V[\nu]) + \|\mu - \nu\|_{BL^*},
\]

so that \( (V_2) \) is satisfied. To see \( (V_3) \), we fix \( \mu, \nu \in \mathcal{M}^+(\mathbb{R}^d) \) and observe that condition \( (\tilde{V}_{2,3}) \) implies that \( W^g(\tilde{V}_1, \tilde{V}_2) < \infty \). It is a priori not clear that the infimum of \( W^g \) is actually attained. But for sure it is almost attained and thus it follows that for all \( \varepsilon > 0 \), there exist almost optimal submeasures \( V_1^\varepsilon \leq V[\mu], V_2^\varepsilon \leq V[\nu] \) and a transference plan \( p^\varepsilon \in \mathcal{P}(V_1^\varepsilon, V_2^\varepsilon) \) such that

- \( \int_{(\mathbb{R}^d)^4} |v - w| \, dp^\varepsilon(x, v, y, w) \leq W^g(V[\mu], V[\nu]) + \varepsilon, \)
- \( \|\mu - \nu\|_{BL^*} = \|\mu - \mu^\varepsilon\|_{TV} + \|\nu - \nu^\varepsilon\|_{TV} + W_1(\mu^\varepsilon, \nu^\varepsilon), \)
- \( \pi_1^\# p^\varepsilon \in \mathcal{P}^{\text{opt}}(\mu^\varepsilon, \nu^\varepsilon). \)

Now, fix \( \tau > 0 \) and \( \psi \in BL(\mathbb{R}^d) \) with \( \|\psi\|_{BL} \leq 1 \). We compute

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x + \tau v) \, d(V[\mu] - V[\nu])(x, v) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x + \tau v) \, d(V[\mu] - V_2^\varepsilon)(x, v)
\]

\[
+ \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x + \tau v) \, d(V_1^\varepsilon - V[\nu])(x, v) + \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x + \tau v) \, d(V_1^\varepsilon - V_2^\varepsilon)(x, v).
\]
Therefore, using the definition of the transference plan we obtain
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x + \tau v) \, d(V^\varepsilon - V^\mu)(x, v)
\]
\[
= \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x + \tau v) \, dV^\varepsilon_{\mu}(x, v) - \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x + \tau v) \, dV^\mu_{\varepsilon}(x, v)
\]
\[
= \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x + \tau v) \, d\left(\pi_{1, 2}^\varepsilon p^\varepsilon\right)(x, v) - \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(y + \tau w) \, d\left(\pi_{3, 4}^\mu p^\mu\right)(y, w)
\]
\[
= \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x + \tau v) \, dp^\varepsilon(x, v, y, w) - \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(y + \tau w) \, dp^\mu(x, v, y, w)
\]
\[
= \int_{\mathbb{R}^d \times \mathbb{R}^d} \left(\psi(x + \tau v) - \psi(y + \tau w)\right) \, dp^\varepsilon(x, v, y, w).
\]

Here, \(\pi_{1, 2}\) and \(\pi_{3, 4}\) denote the projections to the first and last two coordinates, respectively. Notice carefully that we introduced the second pair of variables \((y, w)\). To handle the first term appearing in \((A.2)\) we first note that \(V[\mu] - V^\mu\) and \(\mu - \mu^\varepsilon\) are nonnegative measures by construction so that we can apply Lemma 4.2 and the linearity of the total variation norm to see
\[
\|V[\mu] - V^\mu\|_{TV} = \|V[\mu]\|_{TV} - \|V^\mu\|_{TV} = \|\mu\|_{TV} - \|\mu^\varepsilon\|_{TV} = \|\mu - \mu^\varepsilon\|_{TV}.
\]
Thus, we have for the first term
\[
\left|\int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x + \tau v) \, d(V[\mu] - V^\mu)(x, v)\right| \leq \|\mu - \mu^\varepsilon\|_{TV}
\]
because \(\|\psi\|_\infty \leq 1\). Similarly,
\[
\left|\int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(y + \tau w) \, d(V^\varepsilon - V[\mu])(y, w)\right| = \left|\int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(y + \tau w) \, d(V[\nu] - V^\varepsilon)(y, w)\right| \leq \|\nu - \nu^\varepsilon\|_{TV}.
\]
For the last term in \((A.2)\) we have
\[
|\psi(x + \tau v) - \psi(y + \tau w)| \leq |x - y| + |v - w|.
\]
Therefore, using the definition of the transference plan we obtain
\[
\left|\int_{\mathbb{R}^d \times \mathbb{R}^d} (\psi(x + \tau v) - \psi(y + \tau w)) \, dp^\varepsilon(x, v, y, w)\right| \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} (|x - y| + |v - w|) \, dp^\varepsilon(x, v, y, w)
\]
\[
\leq W_1(\mu^\varepsilon, \nu^\varepsilon) + \tau W_2(V[\mu], V[\nu]) + \tau \varepsilon \leq W_1(\mu^\varepsilon, \nu^\varepsilon) + \tau C_F \|\mu - \nu\|_{BL^*} + \tau \varepsilon,
\]
where we applied condition \((V_{2, 3})\) in the last step. It follows that
\[
\left|\int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x + \tau v) \, d(V[\mu] - V[\nu])(x, v)\right|
\]
\[
\leq \|\mu - \mu^\varepsilon\|_{TV} + \|\nu - \nu^\varepsilon\|_{TV} + W_1(\mu^\varepsilon, \nu^\varepsilon) + \tau C_F \|\mu - \nu\|_{BL^*} + \tau \varepsilon
\]
\[
= \|\mu - \nu\|_{BL^*} + \tau C_F \|\mu - \nu\|_{BL^*} + \tau \varepsilon = (1 + \tau C_F) \|\mu - \nu\|_{BL^*} + \tau \varepsilon.
\]
As \(\varepsilon > 0\) can be arbitrarily small, the conclusion follows. \(\square\)
Remark A.4. In the case of conservative problem in the space of probability measure as in [26] the setting above can be substantially simplified. First, the definition of $\mathcal{W}^g$ in (A.1) boils down to
\begin{equation}
\mathcal{W}^g(V_1, V_2) := \inf \left\{ \int_{\mathbb{R}^d} |v - w| \, dp(x, v, y, w) \big| p \in \mathcal{P}(V_1, V_2), \pi_{13}^# p \in \mathcal{P}^{opt}(\tilde{\mu}_1, \tilde{\mu}_2) \right\}.
\end{equation}

because $V_1$, $V_2$ are probability measures cf. [26, Definition 4.1]. Moreover, continuity conditions simplify to
\begin{align}
(V_{2,3,c}) & \quad \mathcal{W}^g(V[\mu], V[\nu]) \leq C_F W_1(\mu, \nu), \\
(V_{2,c}) & \quad W_1(V[\mu], V[\nu]) \leq C_F W_1(\mu, \nu), \\
(V_{3,c}) & \quad \sup_{|\psi|_{L_{\text{lip}}} \leq 1} \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x + \tau v) \, d(V[\mu] - V[\nu])(x, v) \leq (1 + C_H(R) \tau) W_1(\mu, \nu).
\end{align}

for $(V_{2,3,c})$, $(V_{2,c})$, $(V_{3,c})$ respectively. Then the same proof as in Lemma A.3 shows that $(V_{2,3,c})$ implies $(V_{2,c})$ and $(V_{3,c})$. 
References

[1] A. S. Ackleh, N. Saintier, and J. Skrzeczowski. Sensitivity equations for measure-valued solutions to transport equations. *Math. Biosci. Eng.*, 17(1):514–537, 2020.
[2] A. Aydoğdu, S. T. McQuade, and N. Pouradier Duteil. Opinion dynamics on a general compact Riemannian manifold. *Netw. Heterog. Media*, 12(3):489–523, 2017.
[3] V. I. Bogachev. *Measure theory. Vol. I*. Springer-Verlag, Berlin, 2007.
[4] Å. Brännström, L. Carlsson, and D. Simpson. On the convergence of the escalator boxcar train. *SIAM J. Numer. Anal.*, 51(6):3213–3231, 2013.
[5] S. Cacace, F. Camilli, R. De Maio, and A. Tosin. A measure theoretic approach to traffic flow optimisation on networks. *European J. Appl. Math.*, 30(6):1187–1209, 2019.
[6] F. Camilli, G. Cavagnari, R. De Maio, and B. Piccoli. Superposition principle and schemes for measure differential equations. *Kinet. Relat. Models*, 14(1):89–113, 2021.
[7] F. Camilli, R. De Maio, and A. Tosin. Transport of measures on networks. *Netw. Heterog. Media*, 12(2):191–215, 2017.
[8] F. Camilli, R. De Maio, and A. Tosin. Measure-valued solutions to nonlocal transport equations on networks. *J. Differential Equations*, 264(12):7213–7241, 2018.
[9] J. A. Carrillo, R. M. Colombo, P. Gwiazda, and A. Ulikowska. Structured populations, cell growth and measure valued balance laws. *J. Differential Equations*, 252(4):3245–3277, 2012.
[10] J. A. Carrillo, P. Gwiazda, K. Kropielnicka, and A. K. Marciniak-Czochra. The escalator boxcar train method for a system of age-structured equations in the space of measures. *SIAM J. Numer. Anal.*, 57(4):1842–1874, 2019.
[11] J. A. Carrillo, P. Gwiazda, and A. Ulikowska. Splitting-particle methods for structured population models: convergence and applications. *Math. Models Methods Appl. Sci.*, 24(11):2171–2197, 2014.
[12] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992.
[13] A. M. de Roos. Numerical methods for structured population models: the escalator boxcar train. *Numer. Methods Partial Differential Equations*, 4(3):173–195, 1988.
[14] C. Düll, P. Gwiazda, A. Marciniak-Czochra, and J. Skrzeczowski. *Spaces of measures and their applications to structured population models*, volume 36 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, Cambridge, 2022.
[15] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015.
[16] J. H. M. Evers, S. C. Hille, and A. Muntean. Mild solutions to a measure-valued mass evolution problem with flux boundary conditions. *J. Differential Equations*, 259(3):1068–1097, 2015.
[17] J. H. M. Evers, S. C. Hille, and A. Muntean. Measure-valued mass evolution problems with flux boundary conditions and solution-dependent velocities. *SIAM J. Math. Anal.*, 48(3):1929–1953, 2016.
[18] G. B. Folland. *Real analysis*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1984. Modern techniques and their applications, A Wiley-Interscience Publication.
[19] P. Gwiazda, S. C. Hille, K. Lyczek, and A. Świerczewska-Gwiazda. Differentiability in perturbation parameter of measure solutions to perturbed transport equation. *Kinet. Relat. Models*, 12(5):1093–1108, 2019.
[20] P. Gwiazda, J. Jabłoński, A. Marciniak-Czochra, and A. Ulikowska. Analysis of particle methods for structured population models with nonlocal boundary term in the framework of bounded Lipschitz distance. *Numer. Methods Partial Differential Equations*, 30(6):1797–1820, 2014.
[21] P. Gwiazda, K. Kropielnicka, and A. Marciniak-Czochra. The escalator boxcar train method for a system of age-structured equations. *Netw. Heterog. Media*, 11(1):123–143, 2016.
[22] P. Gwiazda, T. Lorenz, and A. Marciniak-Czochra. A nonlinear structured population model: Lipschitz continuity of measure-valued solutions with respect to model ingredients. *J. Differential Equations*, 248(11):2703–2735, 2010.
[23] P. Gwiazda and A. Marciniak-Czochra. Structured population equations in metric spaces. *J. Hyperbolic Differ. Equ.*, 7(4):733–773, 2010.

[24] P. Gwiazda, B. Miasojedow, J. Skrzeczkowski, and Z. Szmyńska. Convergence of the EBT method for a non-local model of cell proliferation with discontinuous interaction kernel. *arXiv preprint arXiv:2106.05115*, 2021.

[25] J. A. J. Metz and O. Diekmann. Age dependence. In *The dynamics of physiologically structured populations (Amsterdam, 1983)*, volume 68 of *Lecture Notes in Biomath.*, pages 136–184. Springer, Berlin, 1986.

[26] B. Piccoli. Measure differential equations. *Arch. Ration. Mech. Anal.*, 233(3):1289–1317, 2019.

[27] B. Piccoli and F. Rossi. Generalized Wasserstein distance and its application to transport equations with source. *Arch. Ration. Mech. Anal.*, 211(1):335–358, 2014.

[28] B. Piccoli and F. Rossi. Measure dynamics with probability vector fields and sources. *Discrete Contin. Dyn. Syst.*, 39(11):6207–6230, 2019.

[29] F. Rossi, N. P. Duteil, N. Yakoby, and B. Piccoli. Control of reaction-diffusion equations on time-evolving manifolds. In *2016 IEEE 55th Conference on Decision and Control (CDC)*, pages 1614–1619. IEEE, 2016.

[30] J. Skrzeczkowski. Measure solutions to perturbed structured population models—differentiability with respect to perturbation parameter. *J. Differential Equations*, 268(8):4119–4182, 2020.

[31] H. H. Sohrab. *Basic real analysis*. Birkhäuser/Springer, New York, second edition, 2014.

[32] Z. Szmyńska, B. Miasojedow, J. Skrzeczkowski, and P. Gwiazda. Bayesian inference of a non-local proliferation model. *arXiv preprint arXiv:2106.05955*, pages 1–29, 2021.

[33] H. R. Thieme. *Mathematics in population biology*. Princeton Series in Theoretical and Computational Biology. Princeton University Press, Princeton, NJ, 2003.

[34] A. Ulikowska. An age-structured two-sex model in the space of Radon measures: well posedness. *Kinet. Relat. Models*, 5(4):873–900, 2012.

[35] C. Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

[36] G. Webb. *Theory of nonlinear age-dependent population dynamics*. Marcel Dekker, Inc., 1985.

Christian Düll: Institute of Applied Mathematics, Heidelberg University, 69120 Heidelberg, Germany

Email address: duell@math.uni-heidelberg.de

Piotr Gwiazda: Institute of Mathematics of Polish Academy of Sciences, Jana i Jędrzeja Śniadeckich 8, 00-656 Warsaw, Poland

Email address: pgwiazda@mimuw.edu.pl

Anna Marciniak-Czochra: Institute of Applied Mathematics, Heidelberg University, 69120 Heidelberg, Germany

Email address: anna.marciniak@iwr.uni-heidelberg.de

Jakub Skrzeczkowski: Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Stefana Banacha 2, 02-097 Warsaw, Poland

Email address: jakub.skrzeczkowski@student.uw.edu.pl