$N = 1$ DUAL STRING PAIRS AND THEIR MASSLESS SPECTRA

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We construct two chains of fourdimensional $F$-theory/heterotic dual string pairs with $N = 1$ supersymmetry. On the $F$-theory side as well as on the heterotic side the geometry of the involved manifolds relies on del Pezzo surfaces. We match the massless spectra by using, for one chain of models, an index formula to count the heterotic bundle moduli and determine the dual $F$-theory spectra from the Hodge numbers of the fourfolds $X^4$ and of the type IIB base spaces.

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1 Introduction

Various type of duality symmetries in compactified string vacua were established during recent years; moreover several transitions among different vacua were explored. The unification of possibly all superstring theories is expected to be realized by a common underlying framework, often called $M$-theory. For example, strong/weak coupling duality symmetries among type II compactifications on Calabi-Yau threefolds and heterotic vacua on $K3 \times T^2$ with $N = 2$ supersymmetry in four dimensions \[\] were successfully tested in a broad class of models. Using these $N = 2$ string duality symmetries results about nonperturbative effects in string theory and field theory can be derived, like the computation of the nonperturbative heterotic $N = 2$ prepotential or nonperturbative effects in $N = 2$ field theory like Seiberg/Witten \[\]. Of even greater phenomenological interest is the investigation of string duality symmetries in fourdimensional string vacua with $N = 1$ space-time supersymmetry. One hopes in this way to get important nonperturbative information, like the computation of nonperturbative $N = 1$ superpotentials \[\] and supersymmetry breaking, or the stringy reformulation of many effects in $N = 1$ field theory. In addition, it is a very important question, how transitions among $N = 1$ string vacua, possibly with a different number of chiral multiplets, take place.

String dual pairs with $N = 1$ supersymmetry in four dimensions \[\], \[\], \[\], \[\] are provided by comparing heterotic string vacua on Calabi-Yau threefolds $Z$, supplemented by the specification of a particular choice of the heterotic gauge bundle $V \rightarrow Z$, with particular fourdimensional type IIB superstring vacua, which can be formulated as $F$ theory compactifications \[\], \[\] on elliptic Calabi-Yau fourfolds $X^4$. To be more specific, the elliptic fibration of the Calabi-Yau fourfold has a (complex) three dimensional base manifold $B^3$, i.e. $X^4 \rightarrow B^3$, which is the (non Calabi-Yau) type IIB compactification space from ten to four dimensions. Moreover one supposes that $B^3$ is rationally ruled, i.e. there exists a fibration $B^3 \rightarrow S$ with $P^1$ fibers, because one has to assume the fourfold $X^4$ to be a $K3$ fibration over the two dimensional surface $S$, i.e. $Z \rightarrow S$. Then, using the eightdimensional duality among $F$-theory on an elliptic $K3$ and the heterotic string on $T^2$, one derives by an adiabatic argument that the heterotic Calabi-Yau is an elliptic fibration over the same surface $S$. In this paper we will explicitly construct $N = 1$ dual $F$-theory/heterotic string pairs with $S$ being the socalled del Pezzo surfaces $dP_k$, the $P^2$’s blown up in $k$ points. Then, in our class of models the three-dimensional bases, $B^3_{n,k}$, are characterized by two parameters $k$ and $n$, where $n$ encodes the fibration structure $B^3_{n,k} \rightarrow dP_k$. Specifically we will discuss the cases $n = 0$, $k = 0, \ldots , 6$, i.e. $B^3_{6,k} = P^1 \times dP_k$, and also the cases $n = 0, 2, 4, 6, 12, k = 9$, where the 9 points are the intersection of two cubics and the $B_{n,9}$ are a fibre product $dP_9 \times_{P^1} F_{n/2}$ with common base $P^1$ and $F_{n/2}$ the rational ruled Hirzebruch surfaces. We will show that the massless spectra match on both sides.

The considered class of models is the direct generalization of the $F$-theory/heterotic $N = 1$ dual string pair with $n = 0$, $k = 9$ (again the 9 points in the mentioned special position), which we investigated in \[\], which was a $\mathbb{Z}_2$ modding of a $N = 2$ model described as $F$-theory on $T^2 \times CY^3$ (also considered there was the $\mathbb{Z}_2$ modding of $K3 \times K3$). In this model we also considered a duality check for the superpotential.

The first chain of models we will discuss here consists then in heterotic Calabi-Yau’s elliptic over $dP_k$ ($k = 0, \ldots , 3$) with a general $E_8 \times E_8$ vector bundle resp. $F$-theory
We consider F-theory on a smooth elliptically fibered Calabi-Yau fourfold \(X^4_T\) elliptic over \(dP_k \times P^1\). The heterotic Calabi-Yau of \(\chi \neq 0\) varies with \(k\) and one can also see the transition induced from the blow-ups in \(P^2\) among the fourfolds on the \(F\)-theory side (for transitions among \(N = 1\) vacua cf. also [11]).

Then we go on to construct a second chain of fourdimensional \(F\)-theory/heterotic dual string pairs with \(N = 1\) supersymmetry by \(\mathbb{Z}_2\)-modding of corresponding dual pairs with \(N = 2\) supersymmetry. The resulting Calabi-Yau four-folds \(X^4_T\) are \(K3\)-fibrations over the del Pezzo surface \(dP_3\) (with points in special position). On the heterotic side the dual models are obtained by compactification on a known Voisin-Borcea Calabi-Yau three-fold with Hodge numbers \(h^{(1,1)} = h^{(2,1)} = 19\), where, similarly as in the underlying \(N = 2\) models, the heterotic gauge bundles over this space are characterized by turning on \(\left(6 + \frac{n}{2}, 6 - \frac{n}{2}\right)\) instantons of \(E_8 \times E_8\). We work out the Higgsing chains of the gauge groups together with their massless matter content (for example the numbers of chiral multiplets in the \(27+2\overline{7}\) representation of \(E_6\)) for each model and show that the heterotic spectra of our models match the dual \(F\)-theory spectra, as computed from the Hodge numbers of the four-folds \(X^4_T\) and of the type IIB base spaces.

Note that in contrast to this second class of models the first chain (varying \(k, n = 0\)) consists of genuine \(N = 1\) models. Moreover, within the first chain the heterotic Calabi-Yau spaces have non-vanishing Euler numbers, potentially leading to theories with chiral spectra with respect to non-Abelian gauge groups which show up at certain points in the moduli spaces. Hence the transitions in \(k\) might connect \(N = 1\) models with different numbers of chiral fermions.

In section 2 we consider \(F\)-theory on smooth fourfolds \(X^4_k\) elliptic over \(dP_k \times P^1\) and propose an identification of the spectrum from the Hodge numbers of a general \(X^4\) as well as of its threefold base \(B^3\).

In section 3 we study a class of Calabi-Yau threefolds elliptic over \(dP_k\) and derive an index formula for the number of moduli of a general \(E_8 \times E_8\) bundle.

In section 4 we consider the \(\mathbb{Z}_2\) modding of the \(N = 2\) models described by \(F\)-theory on \(T^2 \times X^3_n\), where \(X^3_n\) are the well known threefold Calabi-Yau over \(F_n\).

In section 5 we make the corresponding modding in the dual heterotic \(N = 2\) model on \(T^2 \times K3\) with instanton embedding \((12 + n, 12 - n)\).

## 2 \(F\)-theory on \(X^4_k \rightarrow \mathbb{P}^1 \times dP_k\)

We consider \(F\)-theory on a smooth elliptically fibered Calabi-Yau fourfold \(X^4\) with base \(B^3_k = \mathbb{P}^1 \times dP_k\) which can be represented by a smooth Weierstrass model over \(B^3_k\) if the anti-canonical line bundle \(-K_B\) over \(B^3_k\) is very ample [12]. The Weierstrass model is described by the homogeneous equation \(y^2z = x^3 + g_2xz^2 + g_3z^3\) in a \(\mathbb{P}^2\) bundle \(W \rightarrow B^3_k\). The \(\mathbb{P}^2\) bundle is the projectivization of a vector bundle \(K_B^{-2} \oplus K_B^{-3} \oplus O\) over \(B^3_k\). Furthermore we can think of \(x, y\) and \(z\) as homogeneous coordinates on the \(\mathbb{P}^2\) fibres, i.e. they are sections of \(O(1) \otimes K_B^{-2}, O(1) \otimes K_B^{-3}\) and \(O(1)\) over \(W\) respectively; \(g_2\) and \(g_3\) are sections of \(H^0(B^3_k, K_B^{-2})\) and \(H^0(B^3_k, K_B^{-6})\) [3].

Most of the 84 known Fano threefolds have a very ample \(-K_B\) [12]. 18 of them are toric threefolds \(\mathcal{F}_n\) \((1 \leq n \leq 18)\) and are completely classified [4, 11, 16, 17]. They were recently studied in the context of Calabi-Yau fourfold compactifications over Fano
threefolds \([18, 13]\). In particular in \([12]\) it was shown that for \(k = 0, \ldots, 6\) over \(B^3_k\) there exists a smooth Weierstrass model (having a section); \(B^3_k\) with \(k = 0, 1, 2, 3\) correspond to the toric Fano threefolds \(F_n\) with \(n = 2, 9, 13, 17\). The corresponding fourfolds \(X^4\) have a K3 fibration over \(dP_k\) cf. \([13, 20]\). First we can determine the Hodge numbers of \(B^3_k\). The number of Kähler and complex structure parameters of \(B^3_k\) are \(h^{11}(B^3_k) = 2 + k\) (where the 2 is coming from the line in \(\mathbb{P}^2\) and the class of the \(\mathbb{P}^1\)) and \(h^{31}(B^3_k) = 0\). For \(k = 0, \ldots, 6\) we can compute the Euler number of \(X\) in terms of topological data of the base according to \([21]\)

\[
\chi = 12 \int_{B^3_k} c_1 c_2 + 360 \int_{B^3_k} c_1^3
\]  

where the \(c_i\) refer to \(B^3_k\) (the 360 is related to the Coxeter number of \(E_8\) which is associated with the elliptic fiber type \([18]\)).

One finds \(\int_{B^3_k} c_1 c_2 = 24\) and \(\int_{B^3_k} c_1^3 = 3c_1(\mathbb{P}^1)c_1(dP_k)^2 = 6(9 - k)\) which leads to

\[
\chi = 288 + 2160(9 - k).
\]  

In the following we restrict to \(k = 0, 1, 2, 3\) where one has \(h^{21} = 0, \ [18, 19], \ h^{11}(X^4_k) = 3+k, \ h^{31}(X^4_k) = \frac{1}{6} - 8 - (3 + k) = 28 + 361(9 - k) \ [18, 19, 21]\).

Next let us compute for a general model from the Hodge numbers of \(X^4\) the spectrum of massless \(N = 1\) superfields in the \(F\)-theory compactification. Just as in the six-dimensional case (see Ch. 4.1), also the Hodge numbers of the type IIB bases \(B^3\), i.e. the details of the elliptic fibrations, will enter the numbers of massless fields. So consider first the compactification of the type IIB string from ten to four dimensions on the spaces \(B^3\). Abelian \(U(1)\) \(N = 1\) vector multiplets arise from the dimensional reduction of the four-form antisymmetric Ramond-Ramond tensor field \(A_{MNPQ}\) in ten dimensions; therefore we expect that the rank of the four-dimensional gauge group, \(r(V)\), gets contributions from the \((2, 1)\)-forms of \(B^3\) such that \(h^{(2,1)}(B^3)\) contributes to \(r(V)\). Chiral (respectively anti-chiral) \(N = 1\) multiplets, which are uncharged under the gauge group, arise from \(A_{MNPQ}\) with two internal Lorentz indices as well from the two two-form fields \(A_{MN}^{12}\) (Ramond-Ramond and NS-NS) with zero or two internal Lorentz indices. Therefore we expect that the number of singlet chiral fields, \(C\), receives contributions from \(h^{(1,1)}(B^3)\). On the other hand we can study the \(F\)-theory spectrum in three dimensions upon further compactification on a circle \(S^1\). This is equivalent to the compactification of 11-dimensional supergravity on the same \(X^4\). (Equivalently we could also consider the compactifications of the IIA superstring on \(X^4\) to two dimensions.) So in three dimensions the 11-dimensional three-form field \(A_{MNP}\) contributes \(h^{(1,1)}(X^4)\) to \(r(V)\) and \(h^{(2,1)}\) to \(C\). In addition, the complex structure deformations of the 11-dimensional metric contributes \(h^{(3,1)}(X^4)\) chiral fields. (These fields arise in analogy to the \(h^{(2,1)}\) complex scalars which describe the complex structure of the metric when compactifying on a Calabi-Yau threefold from ten to four dimensions.) Since, however, vector and chiral fields are equivalent in three dimensions by Poincare duality, this implies that the sum \(r(V) + C\) must be independent from the Hodge numbers of the type IIB bases \(B^3\). Therefore, just in
analogy to the six-dimensional $F$-theory compactifications, the following formulas for the spectrum of the four-dimensional $F$-theory models on Calabi-Yau four-folds are expected [10],[19]: namely for the rank of the $N=1$ gauge group we derive
\[ r(V) = h^{(1,1)}(X^4) - h^{(1,1)}(B^3) - 1 + h^{(2,1)}(B^3), \]  
and for the number $C$ of $N=1$ neutral chiral (resp. anti-chiral) multiplets we get
\[ C = h^{(1,1)}(B^3) - 1 + h^{(2,1)}(X^4) - h^{(2,1)}(B^3) + h^{(3,1)}(X^4) \]
\[ = h^{(1,1)}(X^4) - 2 + h^{(2,1)}(X^4) + h^{(3,1)}(X^4) - r(V) \]
\[ = \frac{\chi}{6} - 10 + 2h^{(2,1)}(X^4) - r(V). \]  

Note that in this formula we did not count the chiral field which corresponds to the dual heterotic dilaton.

Now let us apply these formulae to the fourfolds $X^4_k$. For the rank of the $N=1$ gauge group we derive
\[ r(V) = h^{11}(X^4_k) - h^{11}(B^3_k) - 1 + h^{21}(B^3_k) \]
\[ = 0 \]  
and for the number of $N=1$ neutral chiral (resp. anti-chiral) multiplets $C_F$ we get
\[ C_F = h^{11}(B^3_k) - 1 + h^{21}(X^4_k) - h^{21}(B^3_k) + h^{31}(X^4_k) \]
\[ = 38 + 360(9 - k) \]

| $k$ | $\chi$ | $h^{31}$ | $h^{22}$ | $h^{11}$ | $C_F$ |
|-----|-------|---------|--------|--------|------|
| 0   | 19728 | 3277    | 13164  | 3      | 3278 |
| 1   | 17568 | 2916    | 11724  | 4      | 2918 |
| 2   | 15408 | 2555    | 10284  | 5      | 2558 |
| 3   | 13248 | 2194    | 8844   | 6      | 2198 |
| 4   | 11088 | 1833    | 7404   | 7      | 1838 |
| 5   | 8928  | 1472    | 5964   | 8      | 1478 |
| 6   | 6768  | 1111    | 4524   | 9      | 1118 |

In this table we have made for the cases $k = 4, 5, 6$ the assumption $h^{21} = 0$. With this assumption the matching we will present in this paper goes through also in these cases.

Before we come to the heterotic side let us make some comments. A consistent $F$-theory compactification on a Calabi-Yau fourfold requires the presence of $\frac{1}{3}$ threebranes in the vacuum [21]. Under duality with the heterotic string, the threebranes turn into fivebranes wrapping the elliptic fibers and in [1] it was shown that one can understand the appearance of the fivebranes by using the non-perturbative anomaly cancellation
condition with fivebranes. The non-perturbative anomaly cancellation condition is given by\[^6\]

\[\lambda(V_1) + \lambda(V_2) + [W] = c_2(TZ)\] (2.7)

here \(TZ\) is the tangent bundle of the Calabi-Yau threefold \(Z\) on which the heterotic string is compactified, \(V_1 \times V_2\) is an \(E_8 \times E_8\) bundle with \(\lambda(V)\) denoting its fundamental characteristic class and \([W]\) is the cohomology class of the fivebranes where \([W] = h[F] = \pi^*(p)\) and \([F]\) is the class in \(H^4\) of the fiber of the elliptic fibration and \(h\) is the number of fivebranes on the heterotic string side. In order to show that \(h = \chi_{24}\) one has to express the number of threebranes in terms of topological data defined on \(B^2\). These was done by Friedman, Morgan and Witten\[^3\]. They considered the \(F\)-theory base \(B^3\) as a \(\mathbb{P}^1\) bundle over \(B^2\) being the projectivization of a vector bundle \(O \oplus T\) with \(O\) and \(T\) being line bundles over \(B^2\) and \(r = c_1(O(1)), t = c_1(T)\) (note that \(t\) plays the role of the \(n\) of \(F_n\) in the N=2 string dualities in six dimensions between the heterotic string on \(K3\) with \((12 + n, 12 - n)\) instantons embedded in each \(E_8\) and \(F\) theory on elliptic fibered Calabi-Yau threefold over \(F_n\)). In our case we have \(t = 0\). For the number of threebranes one has\[^3\] :

\[
\frac{\chi}{24} = \int_{B^2} (c_2 + 91c_1^2 + 30t^2).
\] (2.8)

which simplifies in our case using \(t = 0\). Finally for our base \(B^2\) we can write:

\[
\frac{\chi}{24} = 822 - 90k.
\] (2.9)

From the last expression we learn that between each blow up there is a threebrane difference of 90. Note that Sethi, Vafa and Witten\[^21\] had a brane difference of 120 as they blow up in a threefold whereas we do this in \(B^2\).

### 3 Heterotic string on \(Z \rightarrow dP_k\)

To compactify the heterotic string on elliptically fibered Calabi Yau threefold we have in addition to specify a vector bundle \(V\) with fixed second Chern class. Since we had zero for the rank \(r(V)\) of the N=1 gauge group on the \(F\) theory side we have to switch on a \(E_8 \times E_8\) bundle that breaks the gauge group completely.

Now let \(Z\) be a nonsingular elliptically fibered Calabi-Yau threefold over \(dP_k\) with a section. Recall the Picard group of \(dP_k\) is \(\text{Pic } dP_k = \mathbb{Z}\ell \oplus \mathbb{Z}\ell_i\) where \(\ell\) denotes the class of the line in \(\mathbb{P}^2\) and \(\ell_i, i = 1, \ldots, k\) are the classes of the blown up points. The intersection form is defined by\[^22\]

\[
\ell^2 = 1, \quad \ell_i \cdot \ell = 0, \quad \ell_i \cdot \ell_j = -\delta_{ij},
\] (3.10)
and the canonical class of $dP_k$ is

$$K_B = -3\ell + \sum_{i=1}^{k} \ell_i. \quad (3.11)$$

Assuming again that we are in the case of a general smooth Weierstrass model one has $h^{11}(Z) = 2+k$. So the fourth homology of $Z$ is generated by the following divisor classes: $D_0 = \pi^*\ell$, $D_i = \pi^*\ell_i$ and $S = \sigma(B_k)$. The intersection form on $Z$ is then given by

$$S^3 = 9 - k, \quad D_0^2S = 1, \quad D_1^2S = -1, \quad D_0S^2 = -3, \quad (3.12)$$

all other triple intersections are equal to zero.

The canonical bundle for $Z \to B$ is given by

$$K_Z = \pi^*(K_B + \sum a_i[\Sigma_i]) \quad (3.13)$$

where $a_i$ are determined by the type of singular fiber and $\Sigma_i$ is the component of the locus within the base on which the elliptic curve degenerates. In order to get $K_Z$ trivial one requires $K_B = -\sum a_i[\Sigma_i]$.

Since we are interested in a smooth elliptic fibration we have to check that the elliptic fibration does not degenerate more worse than with $I_1$ singular fiber over codimension one in the base which then admits a smooth Weierstrass model. That this indeed happens in our case of del Pezzo base can be proved by similar methods as for the $F_n$ case in [9].

As $a_i = -1$ we have a smooth elliptic fibration.

The Euler number of $Z$ is given by (assuming $E_8$ elliptic fiber type, cf. [18])

$$\chi(Z) = -60 \int c_1^2(B) \quad (3.14)$$

and

$$h^{11}(Z) = 2 + k = 11 - (9 - k) \quad (3.15)$$

$$h^{21}(Z) = (272 - 29k) = 11 + 29(9 - k). \quad (3.16)$$

In order to compare the spectrum of the heterotic side with the $F$ theory side we have still to determine the number of moduli coming from the $E_8 \times E_8$ bundle over $Z$. Therefore we use the $\mathbb{Z}_2$ character valued index theorem [3] using the fact that a elliptic manifold

2i.e. having only one section [13], so a typical counterexample would be the $CY^{19,19} = B_9 \times_{\mu_3} B_9$ with the 9 points being the intersection of two cubics

3Note that the transition by 29 in $h^{21}(Z)$ in going from $k$ to $k+1$ has a well known interpretation if one uses these Calabi-Yau threefolds as $F$-theory compactification spaces cf. [24]
with a section has a \( \mathbb{Z}_2 \) symmetry generated by an involution \( \tau \) that leaves the section invariant and acts as \(-1\) on each fiber or in terms of the Weierstrass model as \( y \to -y \). As \( h^{21}(X^4) = 0 \) the index gives the number of bundle moduli \( \mathbb{g} \) (we assume no 4-flux has been turned on (which is not a free decision in general \( \mathbb{P}^3 \))).

\[
I = rk - \sum_i \int_{U_i} \lambda(V)
\]  
(3.17)

The \( U_i \) are the components of the fixed point set. \( U_1 \) is given by \( x = z = 0, y = 1 \) and is the section of \( Z \to B \); the second component, \( U_2 \), is given by \( y = 0 \) and is a triple cover of \( B \) given by the trisection \( 0 = 4x^3 - g_2xz^2 - g_3z^3 \) which is defined in a \( \mathbb{P}^1 \) bundle \( W' \) over \( B \) where \( x, z \) are sections of \( \mathcal{O}(1) \otimes K_B^{-2} \) and \( \mathcal{O}(1) \) respectively with \( c_1(\mathcal{O}(1)) = r \) and \( c_1(\mathcal{O}(1) \otimes K_B^{-2}) = r + 2c_1 \). Furthermore \( \lambda(V) \) is the fundamental characteristic class of the \( E_8 \times E_8 \) bundle.

Now one has the identity \( \int_{U_1} \lambda(V) = \int_B \lambda(V) |_{U_1} + 3 \int_B \lambda(V) |_{U_2} \) where the 3 appears since \( U_2 \) is a triple cover of \( B \). One has

\[
\lambda(V_1) |_{U_1} + \lambda(V_2) |_{U_1} + [W] |_{U_1} = c_2(TZ |_{U_1})
\]  
(3.18)

Now the restriction of the tangent bundle to \( U_i \) can be derived by the exact Euler sequence \( 0 \to \mathcal{T}U_i \to TZ |_{U_i} \to N_Z U_i \to 0 \) where \( N_Z U_i \) is the normal bundle to \( U_i \) in \( Z \). As \( U_1 \) is the section of \( Z \to dP_k \) we can identify \( U_1 \) with the base and express the total Chern class of \( TZ |_{U_1} \) in terms of \( c_i = \pi^*(c_i(dP_k)) \). We find

\[
c_1(TZ |_{U_1}) = c_1 + c_1(N_Z U_1)
\]  
(3.19)

\[
c_2(TZ |_{U_1}) = c_2 + c_1(N_Z U_1)c_1(dP_k)
\]  
(3.20)

and using the Calabi-Yau condition \( c_1 = -c_1(N_Z U_1) \) we find for the restricted second Chern class:

\[
c_2(TZ |_{U_1}) = c_2 - c_1^2.
\]  
(3.21)

Now we have to determine \( c_2(TZ |_{U_2}) \). Therefore recall that \( U_2 \) was given by \( y = 0 \) defining a triple cover of \( dP_k \) in a \( \mathbb{P}^1 \) bundle \( W' \to dP_k \). To determine \( c_2(TZ |_{U_2}) \) we use techniques from [21]. The cohomology ring of \( W' \) is generated over the cohomology ring of \( dP_k \) by the element \( r = c_1(\mathcal{O}(1)) \) with the relation \( r(r + 2c_1) = 0 \) which expresses the fact that \( x \) and \( z \) have no common zero. Since \( U_2 \) is defined by the vanishing of a section of \( \mathcal{O}(1)^3 \otimes K_{dP_k}^{-6} \) which is a line bundle over \( W' \) with \( c_1(\mathcal{O}(1)^3 \otimes K_{dP_k}^{-6}) = 3r + 6c_1 \). Any cohomology class on \( U_2 \) that can be extended over \( W' \) can be integrated over \( U_2 \) by multiplying it with the cohomology class of \( U_2 \) in \( H^2(Z) \) which is \( 3r + 2c_1 \) cf. [8], i.e. multiplication by \( 3r + 2c_1 \) can be understood as restriction from \( W' \) to \( U_2 \). The relation \( r(r + 2c_1) \) can then be simplified to \( r = 0 \). Now we are able to compute the
total Chern class of the projective space bundle $W'$ given by the adjunction formula $c(W') = c(dP_k)(1 + r)(1 + r + 2c_1)$. Thus the total Chern class of the tangent bundle of $U_2$ is obtained by dividing by $(1 + 3r + 6c_1)$ and using $r = 0$

$$c(TU_2) = c(dP_k)\frac{(1 + 2c_1)}{(1 + 6c_1)}.$$ \hspace{1cm} (3.22)

From the total Chern class we derive

$$c_1(TZ | U_2) = -3c_1 + c_1(N_Z U_2)$$ \hspace{1cm} (3.23)
$$c_2(TZ | U_2) = 20c_1^2 + c_2 - 3c_1 + c_1(N_Z U_2)$$ \hspace{1cm} (3.24)

and with the Calabi-Yau condition $3c_1 = c_1(N_Z U_2)$ we get

$$c_2(TZ | U_2) = c_2 + 11c_1^2.$$ \hspace{1cm} (3.25)

Taking into account the restricted five brane correction term $h = \int_B c_2 + 91c_1^2 + 30t^2$ we find for the index a formula first derived by E. Witten [25]

$$I = 16 + 332 \int_B c_1^2(B) + 120 \int_B t^2$$ \hspace{1cm} (3.26)

Remember that in our case the last term vanishes and we find for the number of bundle moduli

$$I = 16 + 332(9 - k)$$ \hspace{1cm} (3.27)

For the number of $N = 1$ neutral chiral (resp. antichiral) multiplets $C_{\text{het}}$ we find

$$C_{\text{het}} = h^{21}(Z) + h^{11}(Z) + I$$
$$= 38 + 360(9 - k)$$ \hspace{1cm} (3.28)

which agrees with the number of chiral multiplets $C_F$ eq.(2.6) on the $F$-theory side.

| $k$ | $\chi$ | $h^{21}$ | $h^{11}$ | $I$ |
|-----|-------|--------|--------|-----|
| 0   | -540  | 272    | 2      | 3004|
| 1   | -480  | 243    | 3      | 2672|
| 2   | -420  | 214    | 4      | 2340|
| 3   | -360  | 185    | 5      | 2008|
| 4   | -300  | 156    | 6      | 1676|
| 5   | -240  | 127    | 7      | 1344|
| 6   | -180  | 98     | 8      | 1012|

\footnote{We thank S. Kachru for pointing out this reference to us.}
4 F-theory on the $K3$-fibred four-folds $X_n^4$

4.1 The $N=2$ models: F-theory on $X_n^3(\times T^2)$

We will start constructing the four-folds $X_n^4$ by first considering $F$-theory compactified to six dimensions on an elliptic Calabi-Yau threefold $X^3$, and then further to four dimensions on a two torus $T^2$, i.e. the total space is given by $X^3 \times T^2$. This leads to $N=2$ supersymmetry in four dimensions. As explained in [8], this four-dimensional $F$-theory compactification is equivalent to the type IIA string compactified on the same Calabi-Yau $X^3$.

To be more specific let us discuss the cases where the Calabi-Yau threefolds, which we call $X_n^3$, are elliptic fibrations over the rational ruled Hirzebruch surfaces $F_n$ [9]; the Hirzebruch surfaces $F_n$ with complex coordinates $z_1$ and $z_2$ are $P^1$ fibrations over $P^1_{z_1}$. The corresponding type IIB base spaces are given by the Hirzebruch surfaces $F_n$ in six dimensions. In the following it will become very useful to describe the elliptically fibred Calabi-Yau spaces $X_n^3$ in the Weierstrass form [9]:

$$X_n^3 : \quad y^2 = x^3 + \sum_{k=-4}^{4} f_{8-nk}(z_1) z_2^{4-k} x + \sum_{k=-6}^{6} g_{12-nk}(z_1) z_2^{6-k}. \quad (4.1)$$

Here $f_{8-nk}(z_1)$, $g_{12-nk}(z_1)$ are polynomials of degree $8-nk$, $12-nk$ respectively, where the polynomials with negative degrees are identically set to zero. From this equation we see that the Calabi-Yau threefolds $X_n^3$ are $K3$ fibrations over $P^1_{z_1}$, with coordinate $z_1$; the $K3$ fibres themselves are elliptic fibrations over the $P^1_{z_2}$ with coordinate $z_2$.

The Hodge numbers $h^{(2,1)}(X_n^3)$, which count the number of complex structure deformations of $X_n^3$, are the given by the the number of parameters of the curve (4.1) minus the number of possible reparametrizations, which are given by 7 for $n = 0, 2$ and by $n + 6$ for $n > 2$. On the other hand, the Hodge numbers $h^{(1,1)}(X_n^3)$, which count the number of Kähler parameters of $X_n^3$, are determined by the Picard number $\rho$ of the $K3$-fibre of $X_n^3$ as

$$h^{(1,1)}(X_n^3) = 1 + \rho. \quad (4.2)$$

Let us list the Hodge numbers of the $X_n^3$ for those cases which are relevant for our following discussion:

| $n$ | $h^{(1,1)}(X_n^3)$ | $h^{(2,1)}(X_n^3)$ |
|-----|-------------------|-------------------|
| 0   | 3                 | 243               |
| 2   | 3                 | 243               |
| 4   | 7                 | 271               |
| 6   | 9                 | 321               |
| 12  | 11                | 491               |

(4.3)

Let us also recall [8, 9] how the Hodge numbers of $X_n^3$ determine the spectrum of the $F$-theory compactifications. In six dimensions the number of tensor multiplets $T$ is given
by the number of Kähler deformations of the (complex) two-dimensional type IIB base $B^2$ except for the overall volume of $B^2$:

$$T = h^{(1,1)}(B^2) - 1.$$  \hspace{1cm} (4.4)

These tensor fields become Abelian $N = 2$ vector fields upon further $T^2$ compactification to four dimensions. Since the four-dimensional $F$-theory is equivalent to the type IIA string on $X_n^3$, it follows that the number of four-dimensional Abelian vector fields in the Coulomb phase is given by $T + r(V) + 2 = h^{(1,1)}(X_n^3)$, where $r(V)$ is the rank of the six-dimensional gauge group, and the additional two vector fields arise from the toroidal compactification. This then leads to the following equation for $r(V)$:

$$r(V) = h^{(1,1)}(X_n^3) - h^{(1,1)}(B^2) - 1.$$  \hspace{1cm} (4.5)

Finally, the number of hypermultiplets $H$, which are neutral under the Abelian gauge group, is given in four as well as in six dimensions by the number of complex deformations of $X_n^3$ plus the freedom in varying the the size of the base $B^2$:

$$H = h^{(2,1)}(X_n^3) + 1.$$  \hspace{1cm} (4.6)

For the cases we are interested in, namely $B^2 = F_n$, $h^{(1,1)}(F_n)$ is universally given by $h^{(1,1)}(F_n) = 2$. Therefore one immediately gets that $T = 1$, which corresponds to the universal dilaton tensor multiplet in six dimensions, and

$$r(V) = \rho - 2 = h^{(1,1)}(X_n^3) - 3.$$  \hspace{1cm} (4.7)

At special loci in the moduli spaces of the hypermultiplets one obtains enhanced non-Abelian gauge symmetries. These loci are determined by the singularities of the curve \([4.1]\) and were analyzed in detail in \([26]\). These $F$-theory singularities correspond to the perturbative gauge symmetry enhancement in the dual heterotic models.

### 4.2 The $N = 1$ models: $F$-theory on $X_n^4 = (X_n^3 \times T^2)/Z_2$

Now we will construct from the $N = 2$ $F$-theories on $X_n^3 \times T^2$ the corresponding $N = 1$ models on Calabi-Yau four-folds $X_n^4$ by a $Z_2$ modding procedure, i.e.

$$X_n^4 = \frac{X_n^3 \times T^2}{Z_2}.$$  \hspace{1cm} (4.8)

First, the $Z_2$ modding acts as quadratic redefinition on the coordinate $z_1$, the coordinate of the base $P_{z_1}^1$ of the $K3$-fibred space $X_n^3$, i.e. the operation is $z_1 \to -z_1$. This means that the modding is induced from the quadratic base map $z_1 \to \tilde{z}_1 := z_1^2$ with the two branch points 0 and $\infty$. So the degrees of the corresponding polynomials $f(z_1)$ and $g(z_1)$ in eq.\([4.1]\) are reduced by half (i.e. the moddable cases are the ones where only even degrees occur). So instead of the Calabi-Yau threefolds $X_n^3$ we are now dealing with the non-Calabi-Yau threefolds $B_n^3 = X_n^3/Z_2$ which can be written in Weierstrass form as follows:

$$B_n^3: \quad y^2 = x^3 + 4 \sum_{k=-4}^4 f_{4-k}(z_1)z_2^{4-k}x + 6 \sum_{k=-6}^6 g_{6-k}(z_1)z_2^{6-k}.$$  \hspace{1cm} (4.9)
The $B_3^3$ are now elliptic fibrations over $F_{n/2}$ and still $K3$ fibrations over $P_{z_1}^1$. Note that the unmodded 3-folds $X_3^3$ and the modded spaces $B_3^3$ have still the same $K3$-fibres with Picard number $\rho$. The Euler numbers of $B_3^3$ can be computed from the Euler numbers of $X_3^3$ from the ramified covering as

$$\chi(X_3^3) = 2\chi(B_3^3) - 2 \cdot 24. \quad (4.10)$$

Using $\chi(X_3^3) = 2(1 + \rho - h^{(2,1)}(X_3^3))$ and $\chi(B_3^3) = 2 + 2(1 + \rho - h^{(2,1)}(B_3^3))$ we derive the following relation between $h^{(2,1)}(X_3^3)$ and $h^{(2,1)}(B_3^3)$:

$$h^{(2,1)}(B_3^3) = \frac{1}{2}(\rho - 2 + h^{(2,1)}(X_3^3) - 19). \quad (4.11)$$

Second, $X_4^3$ are of course no more products $B_3^3 \times T^2$ but the torus $T^2_{z_4}$ now is a second elliptic fibre which varies over $P_{z_1}^1$. More precisely, this elliptic fibration describes just the emergence of the del Pezzo surface $dP_9$, which is given in Weierstrass form as

$$dP_9: \quad y^2 = x^3 + f_4(z_1)x + g_6(z_1). \quad (4.12)$$

Therefore the spaces $X_4^3$ have the form of being the following fibre products:

$$X_4^3 = dP_9 \times P_{z_1}^1 B_3^3. \quad (4.13)$$

All $X_4^3$ are $K3$ fibrations over the mentioned $dP_9$ surface. The Euler numbers of all $X_4^3$’s are given by the value

$$\chi = 12 \cdot 24 = 288. \quad (4.14)$$

The corresponding (complex) three-dimensional IIB base manifolds $B_3^3$ have the following fibre product structure

$$B_3^3 = dP_9 \times P_{z_1}^1 F_{n/2}. \quad (4.15)$$

For the case already studied in [10] with $n = 0$, $B_3^3$ is just the product space $dP_9 \times P_{z_1}^1$.

The fibration structure of $X_4^3$ provides all necessary informations to compute the Hodge numbers of $X_4^3$ from the number of complex deformations of $B_3^3$, which we call $N_{B_3^3}$. These can be calculated from eq. (4.14) and are summarized in table (4.22). Note that in the cases $n > 2$ we have to subtract in the $N = 2$ setup $7 + n - 1 = 6 + n$ reparametrizations, whereas in the $N = 1$ setup only $6 + n/2$ (for $n = 0, 2$ we have to subtract 7 reparametrizations both for $N = 1$ and $N = 2$).

Knowing that the number of complex deformations of $dP_9$ is eight, as it can be easily be read off from eq. (4.12), we obtain for the number of complex structure deformations of $X_4^3$ the following result:

$$h^{(3,1)}(X_4^3) = 8 + 3 + N_{B_3^3} = 11 + N_{B_3^3}. \quad (4.16)$$

Next compute the number of Kähler parameters, $h^{(1,1)}(X_4^3)$, of $X_4^3$. Since $h^{(1,1)}(dP_9) = 10$ we obtain the formula

$$h^{(1,1)}(X_4^3) = 10 + \rho, \quad (4.17)$$
where $\rho$ is the Picard number of the $K3$ fibre of $X^4_n$.

Finally for the computation of $h^{(2,1)}(X^4_n)$ of $X^4_n$ we can use the condition \[ \text{tadpole cancellation}, \] which tells us that $h^{(1,1)}(X^4_n) - h^{(2,1)}(X^4_n) + h^{(3,1)}(X^4_n) = \frac{8}{6} - 8$. Hence we get for $h^{(2,1)}(X^4_n)$

$$h^{(2,1)}(X^4_n) = \rho + N_{B^3} - 19. \quad (4.18)$$

Using eqs.\(\ref{eq:4.16},\ref{eq:4.17},\ref{eq:4.18}\) we have summarized the spectrum of Hodge numbers of $X^4_n$ in table \(\ref{tab:4.22}. \) The computation of these 4-fold Hodge numbers, which was based on the counting of complex deformations of the Weierstrass form eq.\(\ref{eq:4.9},\) can be checked in a rather independent way, by noting that $h^{(1,1)}(X^4_n) - h^{(2,1)}(X^4_n) + h^{(3,1)}(X^4_n) = \chi$. Hence we get for $h^{(2,1)}(X^4_n)$

$$h^{(2,1)}(X^4_n) = \rho + N_{B^3} - 19. \quad (4.18)$$

Using eqs.\(\ref{eq:4.16},\ref{eq:4.17},\ref{eq:4.18}\) we have summarized the spectrum of Hodge numbers of $X^4_n$ in table \(\ref{tab:4.22}. \) The computation of these 4-fold Hodge numbers, which was based on the counting of complex deformations of the Weierstrass form eq.\(\ref{eq:4.9},\) can be checked in a rather independent way, by noting that $h^{(1,1)}(X^4_n) - h^{(2,1)}(X^4_n) + h^{(3,1)}(X^4_n) = \chi$. Hence we get for $h^{(2,1)}(X^4_n)$

$$h^{(2,1)}(X^4_n) = \rho + N_{B^3} - 19. \quad (4.18)$$

Let us compute the spectrum for our chain of models using eqs. \(\ref{eq:2.3},\ref{eq:2.4}\). The Hodge numbers of $B^3_n \quad \text{eq.}\(\ref{eq:4.15}\) are universally given as $h^{(1,1)}(B^3_n) = 11$, $h^{(2,1)}(B^3_n) = 0$. Thus we obtain using eq.\(\ref{eq:4.17}\) that

$$r(V) = h^{(1,1)}(X^3_n) - 12 = \rho - 2; \quad (4.19)$$

observe that the rank of the $N = 1$ four-dimensional gauge groups agrees with the rank of the six-dimensional gauge groups of the corresponding $N = 2$ parent models (see eq.\(\ref{eq:4.7}\)). Second, using eqs.\(\ref{eq:4.16},\ref{eq:4.18}\) we derive that

$$C = 38 - r(V) + 2h^{(2,1)}(X^4_n) = 2 + \rho + 2N_{B^3}. \quad (4.20)$$

Using eq.\(\ref{eq:4.11}\), $C$ can be expressed by the number of hypermultiplets of the $N = 2$ parent models as follows:

$$C = 38 + H - 20. \quad (4.21)$$

This relation will become clear when considering in the next chapter the dual heterotic models. The explicit results for $N_{B^3}$, $h^{(1,1)}(X^4_n)$, $h^{(2,1)}(X^4_n)$, $h^{(3,1)}(X^4_n)$, $r(V)$ and $C$ are contained in the following table:

| $n$ | $N_{B^3}$ | $h^{(1,1)}(X^4_n)$ | $h^{(2,1)}(X^4_n)$ | $h^{(3,1)}(X^4_n)$ | $r(V)$ | $C$ |
|-----|-----------|-------------------|-------------------|-------------------|--------|-----|
| 0   | 129       | 12                | 112               | 140               | 0      | 262 |
| 2   | 129       | 12                | 112               | 140               | 0      | 262 |
| 4   | 141       | 16                | 128               | 152               | 4      | 290 |
| 6   | 165       | 18                | 154               | 176               | 6      | 340 |
| 12  | 249       | 20                | 240               | 260               | 8      | 510 |

The equations \(\ref{eq:4.19}\) and \(\ref{eq:4.20}\) count the numbers of Abelian vector fields and the number of neutral chiral moduli fields of the four-dimensional $F$-theory compactification. Let us now discuss the emergence of $N = 1$ non-Abelian gauge groups together with their

\footnote{The Abelian vector fields which arise in the $N = 2$ situation from the $T^2$ compactification from six to four dimensions do not appear in the modded $N = 1$ spectrum – see the discussion in the next chapter.}
matter contents. Namely, non-Abelian gauge groups arise by constructing 7-branes over which the elliptic fibration has an ADE singularity \([4]\). Specifically, one has to consider a (complex) two-dimensional space, which is a codimension one subspace of the type IIB base \(B^3\), over which the elliptic fibration has a singularity. (In order to avoid adjoint matter the space \(S\) must satisfy \(h^{(2,0)}(S) = h^{(1,0)}(S) = 0\).) The world volume of the 7-branes is then given by \(R^4 \times S\); if \(n\) parallel 7-branes coincide one gets for example an \(SU(n)\) gauge symmetry, i.e. the elliptic fibration acquires an \(A_{n-1}\) singularity. \(N_F\) chiral massless matter fields in the fundamental representation of the non-Abelian gauge group can be geometrically engineered by bringing \(N_F\) 3-branes near the 7-branes, i.e close to \(S\) \([5]\). The Higgs branches of these gauge theories should then be identified with the moduli spaces of the gauge instantons on \(S\).

In our class of models, the space \(S\) is just given by the \(dP_9\) surface which is the base of the \(K3\) fibration of \(X^4_n\). The singularities of the elliptic four-fold fibrations are given by the singularities of the Weierstrass curve for \(B^3_n\), given in eq.(4.9). So the non-Abelian gauge groups arise at the degeneration loci of eq.(4.9). However with this observation we are in the same situation as in the \(N = 2\) parent models, since the singularities of the modded elliptic curve \(B^3_n\) precisely agree with the singularities of the elliptic 3-folds \(X^3_n\) in eq.(4.1). In other words, the non-Abelian gauge groups in the \(N = 2\) and \(N = 1\) models are identical. This observation can be explained from the fact that the gauge group enhancement already occurs in eight dimensions at the degeneration loci of the elliptic \(K3\) surfaces as \(F\)-theory backgrounds. However the underlying eight-dimensional \(K3\) singularities are not affected by the \(Z_2\) operation on the coordinate \(z_1\), but are the same in eqs.(1.1) and (4.9).

In the following section about the heterotic dual models we will explicitly determine the non-Abelian gauge groups and the possible Higgsing chains. We will show that after maximal Higgsing of the gauge groups the dimensions of the instanton moduli spaces, being identical the the dimensions of the Higgsing moduli spaces, precisely agree with \(2h^{(2,1)}(X^4_n) - \rho + 2\) on the \(F\)-theory side; in addition we also verify that the ranks of the unbroken gauge groups after the complete Higgsing precisely match the ranks of the \(F\)-theory gauge groups, as given by \(r(V)\) in table eq.(4.22).

5 Heterotic String on the \(CY^{19,19}\)

5.1 The \(N = 2\) models: the heterotic string on \(K3(\times T^2)\)

In this section we will construct the heterotic string compactifications dual to \(X^4_n\) with \(N = 1\) supersymmetry by \(Z_2\) modding of \(N = 2\) heterotic string compactifications which are the duals of the \(F\)-theory models on \(X^3_n \times T^2\). Heterotic string models with \(N = 2\) supersymmetry in four dimensions are obtained by compactification on \(K3 \times T^2\) plus the specification of an \(E_8 \times E_8\) gauge bundle over \(K3\). So in the heterotic context, we have to specify how the \(Z_2\) modding acts both on the compactification space \(K3 \times T^2\) as well as on the heterotic gauge bundle.

The \(N = 2\) heterotic models, that are dual to the \(F\)-theory compactifications on \(X^3_n \times T^2\), are characterized by turning on \((n_1, n_2) = (12 + n, 12 - n)\) \((n \geq 0)\) instantons of the heterotic gauge group \(E_8^{(I)} \times E_8^{(II)}\). Let us recall briefly the resulting spectrum, first in
six dimensions on $K3$. For this class of models there is one tensor multiplet which contains the heterotic dilaton field. Next consider the massless vector and hypermultiplets in six dimensions. While the first $E_8^{(I)}$ is generically completely broken by the gauge instantons, the second $E_8^{(II)}$ is only completely broken for the cases $n = 0, 1, 2$; for bigger values of $n$ there is a terminating gauge group $G^{(II)}$ of rank $r(V)$ which cannot be further broken. The quaternionic dimensions of the instanton moduli space of $n$ instantons of a gauge group $H$, living on $K3$, is in general given by

$$\dim_Q(M_{\text{inst}}(H, k)) = c_2(H)n - \dim H,$$

where $c_2(H)$ is the dual Coxeter number of $H$. Then, in the examples we are discussion, we derive the following formula for the quaternionic dimension of the instanton moduli space:

$$\dim_Q M_{\text{inst}}(E_8^{(I)} \times H^{(II)}, n)) = 112 + 30n + (12 - n)c_2(H^{(II)}) - \dim H^{(II)};$$

here (for $n \neq 12$) $H^{(II)}$ is the commutant of the unbroken gauge group $G^{(II)}$ in $E_8^{(II)}$. Specifically, the following gauge groups $G^{(II)}$ and dimensions of instanton moduli spaces are derived:

| $n$ | $G^{(II)}$ | $\dim_Q M_{\text{inst}}$ |
|-----|-----------|----------------------|
| 0   | 1         | 224                  |
| 2   | 1         | 224                  |
| 4   | $SO(8)$  | 252                  |
| 6   | $E_6$     | 302                  |
| 12  | $E_8$     | 472                  |

(5.25)

The number of massless gauge singlet hypermultiplets is then simply given by

$$H = 20 + \dim_Q M_{\text{inst}},$$

where the 20 corresponds to the complex deformations of $K3$. It is well known that comparing the spectra of $F$-theory on the 3-folds $X_n^3$ (see eqs.(4.14.7) and table (1.3)) with the spectra of the heterotic string on $K3$ with instanton numbers $(12 + n, 12 - n)$ (see eq.(5.26) and table (5.27)) one finds perfect agreement. Note that on the heterotic side there is an perturbative gauge symmetry enhancement at special loci in the hypermultiplet moduli spaces. Specifically, by embedding the $SU(2)$ holonomy group of $K3$, namely the $SU(2)$ bundles with instanton numbers $(12 + n, 12 - n)$ in $E_8^{(I)} \times E_8^{(II)}$, the six-dimensional gauge group is broken to $E_7^{(I)} \times E_7^{(II)}$ (or $E_7^{(I)} \times E_8^{(II)}$ for $n = 12$); in addition one gets charged hyper multiplet fields, which can be used to Higgs the gauge group via several intermediate gauge groups down to the terminating groups. The dimensions of the Higgs moduli space, i.e. the number of gauge neutral hypermultiplets, agrees with the dimensions of the instanton moduli spaces eq.(5.24).

5.2 The $N = 1$ models: the heterotic string on $Z = (K3 \times T^2)/Z_2$

Now let us construct the four-dimensional heterotic compactifications with $N = 1$ supersymmetry, which are dual to $F$-theory on $X_n^4$, by $Z_2$ modding of the heterotic string
compactifications on $K3 \times T^2$. In the first step we discuss the $Z_2$ modding of the compactification space $K3 \times T^2$ which results in a particular Calabi-Yau 3-fold $Z$: \[ Z = \frac{(K3 \times T^2)}{Z_2}. \] (5.27)

Specifically, the $Z_2$-modding reduces $K3$ to the del Pezzo surface $dP_9$. This corresponds to having on $K3$ a Nikulin involution of type $(10,8,0)$ with two fixed elliptic fibers in the $K3$ leading to

\[ K3 \rightarrow dP_9 \]
\[ \downarrow \quad \downarrow \]
\[ P_y^1 \rightarrow P_{\tilde{y}}^1 \] (5.28)

induced from the quadratic base map $y \rightarrow \tilde{y} := y^2$ with the two branch points $0$ and $\infty$ (being the identity along the fibers). In the Weierstrass representation of $K3$

\[ K3 : \ y^2 = x^3 - f_8(z)x - g_{12}(z), \] (5.29)

the mentioned quadratic redefinition translates to the representation

\[ dP_9 : \ y^2 = x^3 - f_4(z)x - g_6(z) \] (5.30)

of $dP_9$ (showing again the $8 = 5 + 7 - 3 - 1$ deformations). So the $Z_2$ reduction of $K3$ to the non Ricci-flat $dP_9$ corresponds to the reduction of the $X^3_n$ to the non Calabi-Yau space $B^3_n$ (cf. eqs. (4.1 and 4.9)). Representing $K3$ as a complete intersection in the product of projective spaces as $K3 = \left[ P^2_0 \times P^2_1 \times P^2_2 \right]$, the $Z_2$ modding reduces the degree in the $P^1$ variable by half; hence the $dP_9$ can be represented as $dP_9 = \left[ \frac{P^2_0}{P^1_0} \times P^3_1 \right]$. This makes visible on the one hand its elliptic fibration over $P^1$ via the projection onto the second factor; on the other hand the defining equation $C(x_0, x_1, x_2) y_0 + C'(x_0, x_1, x_2) y_1 = 0$ shows that the projection onto the first factor exhibits $dP_9$ as being a $P^2_x$ blown up in 9 points (of $C \cap C'$), so having as nontrivial hodge number (besides $b_0, b_4$) only $h^{1,1} = 1 + 9$. Furthermore the $dP_9$ has 8 complex structure moduli: they can be seen as the parameter input in the construction of blowing up the plane in the 9 intersection points of two cubics (the ninth of which is then always already determined as they sum up to zero in the addition law on the elliptic curve; so one ends up with $8 \times 2 - 8$ parameters).

As in the dual $F$-theory description a second $dP_9$ emerges by fibering the $T^2$ in eq.(5.27) over the $P^1$ base of $dP_9$. So in analogy to eq.(4.13) the heterotic Calabi-Yau 3-fold $X^3_{het}$, which is elliptically fibred over $dP_9$, has the following fibre product structure

\[ Z = dP_9 \times_{P^1} dP_9. \] (5.31)

The number of Kähler deformations of $Z$ is given by the sum of the deformations of the two $dP_9$'s minus one of the common $P^1$ base, i.e. $h^{1,1}(Z) = 19$. Similarly we obtain $h^{2,1}(Z) = 8 + 8 + 3 = 19$. This Calabi-Yau 3-fold is in fact well known being one of the Voisin-Borcea Calabi-Yau spaces. It can be obtained from $K3 \times T^2$ by the

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6At the orbifold point of $K3$ one can construct $Z$ as $T^6/(Z_2 \times Z_2)$, where one of the $Z_2$'s acts freely, see e.g. [10].
Voisin-Borcea involution, which consists in the ‘del Pezzo’ involution (type (10,8,0) in Nikulin’s classification) with two fixed elliptic fibers in the K3 combined with the usual “-”-involution with four fixed points in the $T^2$. Writing $K3 \times T^2$ as $K3 \times T^2 = \begin{bmatrix} p_1^2 & 1 & 3 \\ p_2^2 & 1 & 0 \\ 3 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$, the Voisin-Borcea involution changes this to $Z = \begin{bmatrix} p_1^2 & 1 & 3 \\ p_2^2 & 1 & 0 \\ 3 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} = dP \times p_1 dP$. Observe that the base of the elliptic fibration of $Z$ is given by the $dP_3$ surface which ‘emerges’ (from the trivial elliptic fibration) after the $Z_2$ modding.

After having described the $Z_2$ modding of $K3 \times T^2$ we will now discuss how this operation acts on the heterotic gauge bundle. Recall that in the $N = 2$ heterotic models on $K3 \times T^2$ the heterotic gauge group $E_8^{(I)} \times E_8^{(II)}$ is living on the four-dimensional space $K3$. We will now consider a $N = 1$ situation where after the $Z_2$ modding the heterotic gauge group still lives on a four manifold, namely on the del Pezzo surface $dP_9$, which arises from the $Z_2$ modding of the K3 surface. Then the complex dimension of the instanton moduli space of $k$ gauge instantons of a gauge group $H$, which lives on $dP$, is given by (see for example [27])

$$\dim_C \mathcal{M}_{\text{inst}}(H, k) = 2c_2(H)k - \dim H. \quad (5.32)$$

The action of the $Z_2$ modding on the gauge bundle is now defined in such a way that the gauge instanton numbers are reduced by half (think of the limit case of pointlike instantons):

$$k_{1,2} = \frac{n_{1,2}}{2}. \quad (5.33)$$

So the total number of gauge instantons in $E_8^{(I)} \times E_8^{(II)}$ will be reduced by two, i.e. $k_1 + k_2 = 12$ and we are considering $(k_1, k_2) = (6 + \frac{n}{2}, 6 - \frac{n}{2})$ instantons in $E_8^{(I)} \times E_8^{(II)}$. The reduction of the total instanton number by half from 24 to 12 can be explained from the observation that on the $F$-theory side the tad-pole anomaly can canceled either by $\gamma/24 = 12$ 3-branes or by 12 gauge instantons of the gauge group $H$, which lives over the four manifold $S = dP_9$. So, with $k_1 + k_2 = 12$ and using eq. (5.32) we can compute the complex dimensions of the instanton moduli space for the gauge group $E_8^{(I)} \times H^{(II)}$, where again $H^{(II)}$ is the commutant of the gauge group $G^{(II)}$ which cannot be further broken by the instantons:

$$\dim_C \mathcal{M}_{\text{inst}}(E_8^{(I)} \times H^{(II)}, n)) = 112 + 30n + (12 - n)c_2(H^{(II)}) - \dim H^{(II)}. \quad (5.34)$$

This result precisely agrees with the quaternionic dimensions of the instanton moduli space, eq.(5.24), in the unmodded $N = 2$ models. So we see that we obtain as gauge bundle deformation parameters of the heterotic string on $Z$ the same number of massless, gauge neutral $N = 1$ chiral multiplets as the number of massless $N = 2$ hyper multiplets of the heterotic string on $K3$. This means that the $Z_2$ modding keeps in the massless sector just one of the two chiral fields in each $N = 2$ hyper multiplet. These chiral multiplets describe the Higgs phase of the $N = 1$ heterotic string compactifications.

The gauge fields in $N = 1$ heterotic string compactifications on $Z$ are just given by those gauge fields which arise from the compactification of the heterotic string on $K3$ to six dimensions; therefore they are invariant under the $Z_2$ modding. However the complex scalar fields of the corresponding $N = 2$ vector multiplets in four dimensions do not
survive the $\mathbb{Z}_2$ modding. Therefore there is no Coulomb phase in the $N = 1$ models in contrast to the $N = 2$ parent compactifications. Also observe that the two vector fields, commonly denoted by $T$ and $U$, which arise from the compactification from six to four dimensions on $T^2$ disappear from the massless spectrum after the modding. This is expected since the Calabi-Yau space has no isometries which can lead to massless gauge bosons. Finally, the $N = 2$ dilaton vector multiplet $S$ is reduced to a chiral multiplet in the $N = 1$ context.

These relations between the spectra of the $N = 1$ and $N = 2$ models can be understood from the observation that the considered $\mathbb{Z}_2$ modding corresponds to a spontaneous breaking of $N = 2$ to $N = 1$ spacetime supersymmetry [28].

In summary, turning on $(6 + \frac{n}{2}, 6 - \frac{n}{2})$ gauge instantons of $E_8^{(I)} \times E_8^{(II)}$ in our class of $N = 1$ heterotic string compactifications on $Z$, the unbroken gauge groups $G^{II}$ as well as the number of remaining massless chiral fields (not counting the geometric moduli from $Z$, see next paragraph) agree with the unbroken gauge groups and the number of massless hyper multiplets (again without the 20 moduli from $K3$) in the heterotic models on $K3$ with $(12 + n, 12 - n)$ gauge instantons. The specific gauge groups and the numbers of chiral fields are already summarized in table (5.25).

Now comparing with the $F$-theory spectra we first observe that the ranks of the gauge groups after maximally possible Higgsing perfectly match in the two dual descriptions (see tables (4.22) and (5.23)). Next compare the number of chiral $N = 1$ moduli fields in the heterotic/F-theory dual pairs. First, looking at the Hodge numbers of the dual $F$-theory fourfolds $X^4_n$, as given in table (4.22) we recognize that

$$2h^{(2,1)}(X^4_n) - (\rho - 2) = \dim_C \mathcal{M}_{\text{inst}}.$$  \hfill (5.35)

Let us give an argument for this independent of the case by case calculation. Namely, using $C_F = 38 + (2h^{(2,1)}(X^4_n) - (\rho - 2)) = H + 18$ (cfr. eqs. (4.20) and (4.21)) and $H = 20 + \dim_Q \mathcal{M}_{\text{inst}} = 20 + \dim_C \mathcal{M}_{\text{inst}}$, one gets

$$C_F = 38 + (2h^{(2,1)}(X^4_n) - (\rho - 2)) = 38 + \dim_C \mathcal{M}_{\text{inst}}.$$  \hfill (5.36)

On the heterotic side the total number $C$ of chiral moduli fields is given by the dimension of the gauge instanton moduli space plus the number of geometrical moduli $h^{(1,1)}(Z) + h^{(2,1)}(Z)$ from the underlying Calabi-Yau space $Z$, which is 38 for our class of models, i.e.

$$C_{\text{het}} = 38 + \dim_C \mathcal{M}_{\text{inst}}.$$  \hfill (5.37)

So we have shown that the massless spectra of Abelian vector multiplets and of the gauge singlet chiral plus antichiral fields agree for all considered dual pairs. The next step in the verification of the $N = 1$ string-string duality after the comparison of the massless states is to show that the interactions, i.e. the $N = 1$ effective action, agree. In particular, the construction of the superpotentials is important to find out the ground states of these theories. This was already done [28, 10] for one particular model (the model with $k = 9, n = 0$), where on the heterotic side the superpotential was entirely
generated by world sheet instantons. It would be interesting to see, whether also space
time instantons would contribute to the heterotic superpotential in some other models
and whether supersymmetry can be broken by the superpotential. In addition, it would
be also interesting to compare the holomorphic gauge kinetic functions in $N = 1$ dual
string pairs, in particular in those models which are obtained from $N = 2$ dual pairs
by $\mathbb{Z}_2$ moddings respectively by spontaneous supersymmetry breaking from $N = 2$ to $N = 1$.

5.3 Non-Abelian gauge groups and Higgsing chains

For the computation of $C$ and $r(V)$ we have considered a generic point in the moduli
space where the gauge group is broken as far as possible to the group $G^{II}$ by the vacuum
expectation values of the chiral fields. In this section we now want to determine the
non-Abelian gauge groups plus their matter content which arise in special loci of the
moduli space. Since the $N = 1$ gauge bundle is identical to the $N = 2$ bundle, which
is given by $SU(2) \times SU(2)$, the maximally unbroken gauge group is for $k_1, k_2 \geq 3$ (i.e.
$n \leq 6$) given by $E^{(I)}_7 \times E^{(II)}_7$; the $N = 1$ chiral representations follow immediately from
the $N = 2$ hypermultiplet representations and transform as

\[ E_7 \times E_7 : \quad (k_1 - 2)(\mathbf{56}, 1) + (k_2 - 2)(1, \mathbf{56}) + (4(k_1 + k_2) - 6)(1, 1). \quad (5.38) \]

Higgsing $E^{(I)}_7 \times E^{(II)}_7$ to $E^{(I)}_6 \times E^{(II)}_6$ one is left with chiral matter fields in the following
representations of the gauge group $E^{(I)}_6 \times E^{(II)}_6$:

\[ E_6 \times E_6 : \quad (k_1 - 3)((\mathbf{27}, 1) + (\overline{\mathbf{27}}, 1)) + (k_2 - 3)((1, \mathbf{27}) + (1, \overline{\mathbf{27}})) + (6(k_1 + k_2) - 16)(1, 1). \quad (5.39) \]

When $k_1 = 12$ ($n = 12$) the gauge group is $E^{(I)}_6 \times E^{(II)}_8$ with chiral matter fields

\[ E_6 \times E_8 : \quad 9[(\mathbf{27}, 1) + (\overline{\mathbf{27}}, 1)] + 64(1, 1). \quad (5.40) \]

Since $k_1 \geq 6$, the number of $(\mathbf{27}, 1) + (\overline{\mathbf{27}}, 1)$ is always big enough that the first $E^{(I)}_6$
can be completely broken. On the other hand, only for the cases $n = 0, 2$ the group $E^{(II)}_6$
can be completely Higgsed away by giving vacuum expectation values to the fields
$(1, \mathbf{27}) + (\mathbf{1}, \overline{\mathbf{27}})$. For $k_2 = 4$ ($n = 4$), $E^{(II)}_6$ can be only Higgsed to the group $G^{(II)} = SO(8)$, and for $k_2 = 3$ ($n = 6$) there are no charged fields with respect to $E^{(II)}_6$ such that
the terminating gauge group is just $E^{(II)}_6$. Clearly, the ranks of these gauge groups are
in agreement with the previous discussions, i.e. with the results for $r(V)$; in addition,
assuming maximally possible Higgsing of both gauge group factors the complex dimension
of the Higgs moduli space agrees with the dimensions of the instanton moduli spaces as
given in eq. (5.34) and in table (5.25).

Consider the Higgsing of, say, the first gauge group $E^{(I)}_6$. Namely, like in the $N = 2$ cases
\[ \text{[24]}, \] it can be Higgsed through the following chain of Non-Abelian gauge groups:

\[ E_6 \to SO(10) \to SU(5) \to SU(4) \to SU(3) \to SU(2) \to SU(1). \quad (5.41) \]
In the following we list the spectra for all gauge groups within this chain:

\[
SO(10) : \quad (k_1 - 3)(10 + \overline{10}) + (k_1 - 4)(16 + \overline{16}) + (8k_1 - 15)1, \tag{5.42}
\]
\[
SU(5) : \quad (3k_1 - 10)(5 + 5) + (k_1 - 5)(10 + \overline{10}) + (10k_1 - 24)1, \tag{5.43}
\]
\[
SU(4) : \quad (4k_1 - 16)(4 + 4) + (k_1 - 5)(6 + 6) + (16k_1 - 45)1, \tag{5.44}
\]
\[
SU(3) : \quad (6k_1 - 27)(3 + 3) + (24k_1 - 78)1, \tag{5.45}
\]
\[
SU(2) : \quad (12k_1 - 56)2 + (36k_1 - 133)1, \tag{5.46}
\]
\[
SU(1) : \quad (60k_1 - 248)1. \tag{5.47}
\]

In order to keep contact with our previous discussion, we see that the number of massless chiral fields at a generic point in the moduli space, i.e. for complete Higgsing down to $SU(1)$, is given by $60k_1 - 248 + 2k_2c_2(H^{(II)}) - \dim H^{(II)}$ which precisely agrees with $\dim_{\mathbb{C}} M_{\text{inst}}$.

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