Abstract. We deal with two natural examples of almost-elementary classes: the class of all Banach spaces (over $\mathbb{R}$ or $\mathbb{C}$) and the class of all groups. We show both of these classes do not have the strict order property, and find the exact place of each one of them in Shelah’s $SOP_n$ (strong order property of order $n$) hierarchy. Remembering the connection between this hierarchy and the existence of universal models, we conclude, for example, that there are “few” universal Banach spaces (under isometry) of regular cardinalities.

1. Introduction and preliminaries

In this paper we deal with two very natural abstract elementary classes. An abstract elementary class (AEC) is a class of models for some dictionary (language) with a binary relation (order) defined on them, satisfying natural axioms saying that the order has a similar behavior to the “being an elementary submodel” relation in the first order case. One can see [Sh88] for definitions. AECs capture many examples of nonelementary classes, most of which being much more complicated to understand than the first order case, for example $L_{\kappa,\omega}$ for $\kappa > \omega$. But our classes are not such good examples for seeing how general an AEC can be, as they are very similar to the first order case, and hardly even can be called “abstract”. They have almost all the important properties that a regular elementary class has, and can be treated in a very similar way.

The first class discussed in this paper is the class of all Banach spaces (real or complex), where “being a submodel” means just being a subspace - a linear subspace with the induced norm. As we have already mentioned, this class has very similar properties to an elementary class - it has amalgamation and the disjoint union property, locality of types (a type of an infinite sequence is determined by its finite subsequences) and it has compactness in a certain logic (positive strongly bounded formulae). For a special case of Banach spaces, positive strongly bounded formulae allow saying that the norm of a variable (or a constant) is in some compact set of the reals, and they are closed under conjunction, disjunction and the existential quantifier. No negation or universal quantifiers are allowed. Itay Ben-Yaacov suggested in [BenYa] to call the classes having the above three properties CATs (compact abstract classes).

Logic and model theory of Banach spaces was studied in detail by Henson and developed more by Henson and Iovino. The basic definitions can be found in [Iov], as well as a very good survey of the fundamentals of this theory, including Henson’s compactness theorem for Banach spaces. Henson’s logic allows more formulae (the “admissible” formulae are called “positive bounded”) than the theory of CATs does, but there is also a price - the compactness theorem is “local”, i.e. it is true inside
every ball, but not in general (the norms of the variables have to be bounded by some uniform bound).

The second class is even simpler to describe - it is the class of all groups, where being a submodel is just being a subgroup. Here we get all the properties of a CAT without even having to restrict our formulae. Compactness theorem holds trivially for groups in the regular first order language, as this is just a class of models of a universal first order theory. In fact, if we restrict ourselves to existentially closed groups, we will get a simple example of a Robinson theory (see [Hr]).

One of main properties of a CAT, which makes the work really similar to the first order model theory, is the existence of a “monster” model, i.e. a model which is $\kappa^*$-universal and $\kappa^*$-homogeneous for $\kappa^*$ much bigger than every cardinal mentioned in this paper (except $\kappa^*$ itself, of course, which is also mentioned in the paper). Such models that are also very saturated and strongly homogeneous were called universal domains by Hrushovski in [Hr], as they are the “playground” where all the work can be done. Every AEC with amalgamation and the disjoint union property has a “monster”. For having a universal domain it is necessary to have locality of types and compactness, but this is also sufficient (see [Hr] or [BenYac]), so our classes have in fact very “good” monster models.

So, what is the purpose of this paper? We’ve come to the questions asked and answered here from two directions that seem very different, though both deal with classification theory of models, and the connection between them was noticed by the first author in [Sh 500]. The original question arised from the project started by the first author of classifying elementary and non-elementary classes using the following “test” question:

Question 1.1. In which regular $\lambda$ can the theory/class in question have universal models, i.e. models that embed any other model of the same cardinality (and less)?

This question was naturally asked about the class of Banach spaces. So the original question we were interested in is:

Question 1.2. In which regular $\lambda$ can exist a universal Banach space?

This question is certainly interesting for model theorists, as it has to do with classification theory of classes of models and asks how complicated a certain class is, but it is also of some interest to analysts researching Banach spaces themselves and their properties. Unlike saturated, big, compact models, etc, the concept and the importance of universal objects are well understood outside logic as well. Universal Banach spaces, for example, were studied by Banach himself in [Ban], and Szlenk in [Sz] showed there is no universal reflexive separable Banach space. Logicians and analysts do not always agree, though, on the question what a universal Banach space is. As the reader could have pointed out, an “embedding” of one model into another in our case is an isomorphism (in the usual sense in logic) of the first model onto a submodel (subspace) of the second one, i.e. a linear embedding which is also an isometry. Analysts, on the other hand, usually allow more kinds of embeddings than just an isometry, and therefore right now the results presented here may not be of highest interest to them. Still, we will present a full answer to 1.2.

The second question discussed in this paper arised from the joint interest of both authors in the SOP$_n$ (strong order property of order $n$) hierarchy for theories/classes defined by the first author in [Sh500]. We will recall the definitions:
Definition 1.3.  (1) We say that a formula $\varphi(\bar{x}, \bar{y})$ exemplifies the strict order property (SOP) in the model $M$ if it defines a partial order on $M$ with infinite indiscernible chains.

(2) We say that a formula $\varphi(\bar{x}, \bar{y})$ exemplifies the SOP$_n$ (for $n \geq 3$) if it defines on $M$ a graph with infinite indiscernible chains and no cycles of size $n$.

(3) We say that a formula $\varphi(\bar{x}, \bar{y})$ exemplifies the SOP$_{\leq n}$ (for $n \geq 3$) if it defines on $M$ a graph with infinite indiscernible chains and no cycles of size smaller or equal to $n$.

(4) We say an abstract elementary class $\mathfrak{K}$ has SOP/SOP$_n$/SOP$_{\leq n}$ in a logic $L$ if: there exists a formula $\varphi(\bar{x}, \bar{y})$ in $L$, such that for any infinite totally ordered set $I$ there is a model $M$ in $\mathfrak{K}$ in which SOP/SOP$_n$/SOP$_{\leq n}$ is exemplified by $\varphi(\bar{x}, \bar{y})$ with indiscernible chains of order type $I$.

Remark 1.4.  (1) The idea of 1.3 (4) is that without the compactness theorem, we have to demand indiscernible sequences of any length exemplifying an order property (which is not equivalent to just an infinite sequence). But we will see that in our cases this demand (and even the demand of indiscernibility) is unnecessary.

(2) Sometimes we will replace a formula in the above definitions by a type.

(3) One may view the SOP$_n$ hierarchy as finite approximations of the strict order property.

The SOP$_n$ hierarchy is connected in the following way to the well-known classes of stable and simple theories: any simple (and therefore any stable) theory/class does not have SOP$_3$, which is the lowest property in the hierarchy.

So a natural question can be:

**Question 1.5.** Find natural examples of classes without the strict order property that are not trivial from the point of view of the SOP$_n$ hierarchy, i.e. are not simple.

Here we show that both Banach spaces and groups provide good examples. Banach spaces have SOP$_n$ for every natural number $n$, and still do not have the SOP. Groups have the SOP$_3$, but not the SOP$_4$, which is even more surprising for such a complicated class.

So what is the connection between the two questions? The answer was given by the first author in [Sh500]. Note that every AEC with amalgamation has a universal model in every regular $\lambda$ satisfying $\lambda = \lambda^{<\lambda}$ or $\lambda = \mu^+$ and $2^{<\mu} \leq \lambda$. So it is definitely consistent that such a class has a universal model in every regular $\lambda$-take $V = L$. Therefore, can be asking only one thing - is it consistent that there is a universal Banach space of a regular cardinality $\lambda$ that does not satisfy any of the above equalities? In [Sh500] Shelah proves that the answer is negative for any class with SOP$_4$. So finding the right location of the class of Banach spaces in the SOP$_n$ hierarchy gives automatically a full answer to the first question.

An important question that has to be dealt with before we begin our discussion is - what should be the definition of the SOP and SOP$_n$ for our abstract elementary classes? What do we mean by a “formula” that exemplifies an order property, what language do we allow? And is there a difference between demanding infinite chains, infinite indiscernible chains, or infinite indiscernible chains of any length?

Considering the question of language - the case of groups is easy, here we deal with the regular first order formulae. What about Banach spaces? In fact, here
we show the strongest possible results in each direction. We prove that \( \text{SOP}_n \) is exemplified by using the most poor language - the language of CATs, i.e. positive strongly bounded formulae, even quantifier free. In the other direction, we show that the strict order property can not be exemplified in any “locally” compact language, i.e. can not be exemplified by any formula or type which is preserved under taking ultraproducts (where infinite elements are thrown away, and infinitesimal elements are divided out). In particular, Banach spaces do not have the strict order property in the rich Henson’s language of positive bounded formulae, and even positive bounded types.

What about the infinite chains that exemplify the order properties, do we have to demand explicitly existence of chains of any length, as this is usually done for an AEC, and is there a difference between regular and indiscernible chains? It turns out that in our cases, the compactness theorem solves all the problems. Suppose there exists in a “monster” model \( M \) an infinite sequence \( \langle \bar{a}_i : i < \omega \rangle \) satisfying \( i < j \implies M \models \varphi(\bar{a}_i, \bar{a}_j) \). Then by compactness, there is such a sequence of any length (smaller than \( \kappa^* \)) in \( M \). Therefore, using the Erdos - Rado theorem, without loss of generality the original \( \omega \)-sequence \( \langle \bar{a}_i : i < \omega \rangle \) is also indiscernible (this trick is well-known for first order theories, and the proof works just the same for CATs - see [BenYac]). Now, again by compactness, there is an indiscernible chain as required of any length and order type (just a chain of the same type as \( \langle \bar{a}_i : i < \omega \rangle \)). Note that one has to be very careful, as in the case of Banach spaces negation does not exist in the language, so not every technique can be used for finding indiscernible sequences (Ramsey’s theorem won’t help), but the process described above shows there is in fact no problem. So we can summarize:

**Fact 1.6.** Suppose there exists an infinite sequence \( \langle \bar{a}_i : i \in I \rangle \) (\( I - \) some infinite ordered set) in the universal domain \( M \) of some compact abstract theory. Then for any infinite ordered set \( J \), there exists an indiscernible sequence \( \langle \bar{b}_i : i \in J \rangle \) such that for all \( n \) there exist \( i_1 < \ldots < i_n \) in \( I \) satisfying \( \text{tp}(\bar{b}_{i_0}, \ldots, \bar{b}_{i_n-1}) = \text{tp}(\bar{a}_{i_1}, \ldots, \bar{a}_{i_n}) \). In particular, if \( \varphi(\bar{a}_i, \bar{a}_j) \) for all \( i < j \) in \( I \), the same thing holds for \( \langle \bar{b}_i : i \in J \rangle \).

**Proof.** Use Erdos-Rado and compactness, exactly like the case of \( M \) a big model of a first order theory. For more details, see [BenYac]. \( \square \)

We will use the following immediate corollary:

**Corollary 1.7.**

1. If \( M \) is the universal domain of a compact abstract theory and we are interested in order properties exemplified in it, indiscernibility can be omitted from all the items of the definition.

2. If \( \mathcal{R} \) is a compact abstract theory with the universal domain \( M \) (or just an abstract elementary class satisfying the compactness theorem for a logic \( \mathcal{L} \) with the monster model \( M \)), \( \mathcal{R} \) has \( \text{SOP}/\text{SOP}_n/\text{SOP}_{\kappa^*} \) in \( \mathcal{L} \) \( \iff \) it is exemplified in \( M \) by some formula in \( \mathcal{L} \) with indiscernible infinite chains of any order type (smaller than \( \kappa^* \)).

2. **Banach spaces**

Let \( F \) be either \( \mathbb{R} \) or \( \mathbb{C} \).
Notation 2.1. We denote the “monster” Banach space (the universal domain of the compact abstract theory of Banach spaces) by \( \mathcal{B} \).

Theorem 2.2. \( \mathcal{B} \) has SOP\(_n\) for all \( n \geq 3 \). Moreover, there is a positive strongly bounded quantifier free formula \( \varphi_{n+2}(\vec{x}, \vec{y}) \) exemplifying SOP\(_{\leq n}\) in \( \mathcal{B} \) with \( \text{len} \vec{x} = \text{len} \vec{y} = 2 \), such that \( \varphi_{n+2}(\vec{x}, \vec{y}) \vdash \varphi_{n}(\vec{x}, \vec{y}) \).

Proof: Choose \( n > 2 \).

First we define a seminormed space \( B_0 \). As a vector space over \( F \), its basis is \( \{a_\alpha : \alpha < \omega\} \cup \{b_\alpha : \alpha < \omega\} \). The seminorm is defined by \( s(v) = \sup_{\gamma<\omega} \{ |f_\gamma(v)| \} \), where \( f_\gamma \) is a functional defined on the basis as follows:

\[
f_\gamma(a_\alpha) = 1 \text{ if } \alpha < \gamma, \quad f_\gamma(a_\alpha) = 0 \text{ if } \alpha \geq \gamma
\]

\[
f_\gamma(b_\alpha) = 0 \text{ if } \alpha < \gamma, \quad f_\gamma(b_\alpha) = 1 \text{ if } \alpha \geq \gamma
\]

and extended to every \( v \in B_0 \) in the only possible way. Note that in fact \( s(v) = \max_{\gamma<\omega} \{|f_\gamma(v)|\} \) (i.e. the seminorm is finite). It is not a norm: it’s easy to see that, for example, \( s(a_0 - a_1 - b_1 + b_0) = 0 \). So we define \( B_1 \) as the normed space \( B_1/\{v : s(v) = 0\} \). Note that \( \{a_\alpha : \alpha < \omega\} \cup \{b_\alpha : \alpha < \omega\} \) is no more a basis for \( B_1 \), though it certainly still is a set that generates the vector space. The following easy fact will be important for us:

\( \mathcal{B} . \) 2.2.1. \( \{a_\alpha : \alpha < \omega\} \cup \{b_\alpha : \alpha < \omega\} \) is a sequence of distinct non-zero elements in \( B_1 \).

Now denote the completion of \( B_1 \) by \( B \) (of which we can think as of a subspace of the “monster” \( \mathcal{B} \)).

Now we define a term in the language of Banach spaces (a positive bounded term) \( \tau_{n,\ell}(x, y) \) by

\[
\tau_{n,\ell}(x, y) = (n - 2\ell)x + (n - 2\ell + 1)y
\]

Now define \( c_{n,\ell,\alpha} = \tau_{n,\ell}(a_\alpha, b_\alpha) \), i.e.

\[
c_{n,\ell,\alpha} = (n - 2\ell)a_\alpha + (n - 2\ell + 1)b_\alpha
\]

It is clear from the definitions that

\[
|f_\gamma(c_{n,\ell,\alpha})| = n - 2\ell, \text{ if } \alpha \leq \gamma
\]

and

\[
|f_\gamma(c_{n,\ell,\alpha})| = n - 2\ell + 1, \text{ if } \alpha > \gamma
\]

. Therefore (check the calculation),

\( \mathcal{B} . \) 2.2.2.

\[
\bigwedge_{\alpha < \beta \leq \omega} \bigwedge_{\ell \leq n} \|c_{n,\ell+1,\beta} - c_{n,\ell,\alpha}\| = 2
\]

and

\( \mathcal{B} . \) 2.2.3.

\[
\bigwedge_{\alpha < \beta \leq \omega} \bigwedge_{m \leq n} \|c_{n,m,\alpha} - c_{n,0,\beta}\| = 2m + 1
\]
Now we define \( \varphi_n = \varphi_n(x_1x_2, y_1y_2) \):

\[
\varphi_n = \bigwedge_{\ell < n} (\|\tau_{n,\ell}(x_2, y_2) - \tau_{n,\ell}(x_1, y_1)\| \leq 2) \land \\
\bigwedge_{m \leq n} (\|\tau_{n,m}(x_1, y_1) - \tau_{n,0}(x_2, y_2)\| \geq 2m + 1) \land \\
\bigwedge_{m \leq n} (\|\tau_{n,m}(x_1, y_1) - \tau_{n,0}(x_2, y_2)\| \leq 2m + 2)
\]

Remark 2.2.4. The last demand is not needed, as the reader will see in the proof, its only purpose is to make the formula strongly bounded. Readers who are interested only in Henson and Iovino’s logic, can just omit it.

Now we shall show that \( \varphi_n \) exemplifies \( SOP_{\leq n} \) in \( \mathcal{B} \). First, by 2.2.2 and of course 2.2.1, the sequence \( \langle a_\alpha b_\alpha : \alpha < \omega \rangle \) verifies the first part of the definition (in \( B \), of which we think as of a subspace of \( \mathcal{B} \)), i.e. it is an infinite chain of the graph defined by \( \varphi_n \) on \( \mathcal{B} \) (by 1.7 we don’t need to prove indiscernibility). The only thing that is left to verify is that there are no cycles of length \( \leq n \) in this graph, and this is an immediate consequence of the triangle inequality (well hidden under the cover of long formulae):

Suppose \( 2 < m \leq n \), and suppose there are \( \langle c_i, d_i : i \leq m \rangle \) in \( \mathcal{B} \) such that \( \mathcal{B} \models \varphi_n(c_i d_i, c_{i+1} d_{i+1}) \) for \( i < m \) and \( \mathcal{B} \models \varphi_n(c_m d_m, c_0 d_0) \). Then in particular, from \( \mathcal{B} \models \varphi_n(c_i d_i, c_{i+1} d_{i+1}) \) for \( i < m \) follows (taking only the “first component” of \( \varphi_n \)):

\[
\bigwedge_{i < m} (\|\tau_{n,i+1}(c_{i+1}, d_{i+1}) - \tau_{n,i}(c_i, d_i)\| \leq 2)
\]

On the other hand, \( \mathcal{B} \models \varphi_n(c_m d_m, c_0 d_0) \) implies (taking only the “second component” of \( \varphi_n \)):

\[
\|\tau_{n,m}(c_m, d_m) - \tau_{n,0}(c_0, d_0)\| \geq 2m + 1
\]

But from 2.2.5 follows that \( \|\tau_{n,m}(c_m, d_m) - \tau_{n,0}(c_0, d_0)\| \leq \|\tau_{n,m}(c_m, d_m) - \tau_{n,m-1}(c_{m-1}, d_{m-1})\| + \ldots + \|\tau_{n,1}(c_1, d_1) - \tau_{n,0}(c_0, d_0)\| \leq 2m \), which contradicts 2.2.6.

Remark 2.2.7. Careful readers have probably pointed out that we actually showed that \( \varphi_n \) exemplifies \( SOP_{\leq n+1} \) in \( \mathcal{B} \), but it doesn’t matter for our discussion.

We know now that \( \varphi_n \) exemplifies \( SOP_{\leq n} \). In order to complete the proof of the theorem, we need to show that \( \varphi_{n+2} \vdash \varphi_n \) for all \( n \geq 3 \). For this, just note that \( \tau_{n,\ell}(x, y) = \tau_{n,\ell+1}(x, y) \), and the rest follows immediately from the definition of \( \varphi_n \). Q.E.D.

The following corollary can be summarized as “universal Banach spaces in regular cardinals exist only if they have to”, i.e. there are “few” universal Banach spaces (under isometry).

Corollary 2.3. Suppose there exists a universal Banach space (under isometry) in \( \lambda = cf(\lambda) \). Then either \( \lambda = \lambda^{<\lambda} \) or \( \lambda = \mu^+ \) and \( 2^{<\mu} \leq \lambda \).
Proof. $SOP_4$ is enough for this result - see [Sh500], Theorem 2.13. □

Remark 2.4. Note that the other direction of the last corollary is obvious - any abstract elementary class with amalgamation has a universal model in every $\lambda$ satisfying one of the above demands.

Corollary 2.5. There exists a positive strongly bounded quantifier free type type $p(\bar{x}, \bar{y})$ with $\text{len}(\bar{x}) = \text{len}(\bar{y}) = 2$, defining on $\mathcal{B}$ a graph with infinite (indiscernible) chains and no cycles at all.

Proof. Choose

$$p(\bar{x}, \bar{y}) = \bigwedge_{n \in \omega} \varphi_{2n+3}(\bar{x}, \bar{y})$$

$p$ is consistent by compactness, as $\varphi_{n+2}(\bar{x}, \bar{y})$ implies $\varphi_n(\bar{x}, \bar{y})$. Now, as $\varphi_n(\bar{x}, \bar{y})$ exemplifies $SOP_{\leq n}$ in $\mathcal{B}$, and $(\bar{a}_\alpha b_\alpha : \alpha < \omega)$ from the proof of 2.2 is an infinite sequence ordered by $\varphi_n$ for every $n$, the result is clear. □

Discussion 2.6. A natural question after we showed 2.5 is: does $\mathcal{B}$ have the strict order property? Or, a more general question: does having a (type-definable) graph as in 2.5 imply the strict order property (maybe also type-definable)? Suppose we gave up compactness and allowed ourselves $L_{\omega_1, \omega}$ formulae, i.e. infinite disjunctions as well as infinite conjunctions. Then the answer to the second question is certainly positive, as one can define the transitive closure of a relation using an infinite disjunction, and the transitive closure of $p(\bar{x}, \bar{y})$ is easily seen to be a partial order on $\mathcal{B}$. But in our case the implication is not clear, and in fact turns out to be false - we will give a negative answer to the first question (and therefore to the second one). So the compact abstract theory of Banach spaces turns out to be an interesting example of a theory having a “uniform” definition of $SOP_n$, but yet without the $SOP$.

As nonstructure results for the class of Banach spaces are more likely, the following one is rather surprising (and nice):

Theorem 2.7. $\mathcal{B}$ does not have the strict order property exemplified by a positive bounded type (in particular, $\mathcal{B}$ doesn’t have the SOP exemplified by a p.b. formula).

Proof. Suppose towards a contradiction that $q(\bar{x}, \bar{y})$ is a “compact” type which exemplifies $SOP$ in $\mathcal{B}$. So for every linear order $I$, there is an indiscernible sequence $\langle \bar{a}_i : i \in I \rangle$ which is linearly ordered by $q(\bar{x}, \bar{y})$. We will choose $I = \mathbb{Z}$.

We denote $\text{len}(\bar{x}) = \text{len}(\bar{y})$ in $q(\bar{x}, \bar{y})$ by $n$ and assume wlog that there exists $n^* < n$ such that $\bigwedge_{\ell < n^*} (a_{i,\ell} = a_{i,\ell}^*)$ for all $i \in I$ and $\langle \bar{a}_{i,\ell} : n^* \leq \ell < n, i \in I \rangle$ is a linearly independent sequence. In other words, we assume

Assumption 2.7.1. $\langle a_{i,\ell}^* : \ell < n^* \rangle \cup \langle \bar{a}_{i,\ell} : n^* \leq \ell < n, i \in I \rangle$ is a basis for $\langle \bar{a}_i : i \in I \rangle_{\mathcal{B}}$

Define for $k < \omega$, $B_k = \langle \bar{a}_k, \bar{a}_{k+1} \rangle_{\mathcal{B}}$. Denote for any $k_2 > k_1 + 1$, $B_{k_1} \cap B_{k_2}$ by $V^-$ (generated by $\langle a_{i,\ell}^* : \ell < n^* \rangle$).

Pick $m < \omega$ and define $V_m$ as a vector subspace (over $\mathbb{F}$) generated by $\langle \bar{a}_0, \bar{a}_1, \ldots, \bar{a}_m \rangle$ in $\mathcal{B}$. Note that by 2.7.1 $V_m$ (as a vector space) is just a free amalgamation of $B_0', \ldots, B_{m-1}'$ over $\langle \bar{a}_1, \ldots, \bar{a}_m \rangle$ and $V^-$. We shall define three different norms on $V_m$. In order not to get confused between the original indiscernible sequence
and the new normed space that we are going to define, we’ll write \( \langle \bar{b}_i : i \leq m \rangle \) instead of \( \langle \bar{a}_i : i \leq m \rangle \). Let \( h_1 : V_m \to B \) and \( h_{-1} : V_m \to B \) be natural embeddings (isometries respecting the linear structure) such that for \( 0 \leq k \leq m \), \( h_1(b_k) = a_k \) and \( h_{-1}(b_k) = a_{-k} \). Let \( g_{\ell,k} : \langle \bar{b}_i \rangle \to \langle \bar{b}_i \rangle \) be the natural isomorphism mapping \( \bar{b}_k \) onto \( b_k \) and \( g_k = g_{0,k} \). Let \( h_k' : \langle \bar{b}_k \rangle \to \langle \bar{a}_{k} \rangle_B \) be the natural isomorphism mapping \( \bar{b}_k \) onto \( \bar{a}_k \), i.e. \( h_k' = h_1 | (\bar{b}_k') \circ g_k \).

Now we define three different norms on \( B_k' \) (for \( k < m \)). \( \| \cdot \|_1 \) is a norm induced by \( h_1 \) (which is in fact the identity), \( \| \cdot \|_1 \) is induced by \( h_{-1} \), \( \| \cdot \|_0 \) is defined by \( \max \{ \| \cdot \|_1, \| \cdot \|_1 \} \). Now we expand these definitions to \( V_m \): define for \( t \in V_m \), \( i \in \{0, 1, -1\} \), \( \| t \|_i = \inf \{ \sum_{k < m} \| t_k \| : t_k \in B_k', \sum_{k < m} t_k = t \} \).

In fact, eventually we’ll be interested only in \( \| \cdot \|_1 \). Our goal is to show that taking free amalgamations of \( \langle a_0, a_1 \rangle_B \), \ldots , \( \langle a_{m-1}, a_m \rangle_B \) leads (in the limit - and here is where the compactness will be used) to a symmetric type. Two other norms are useful for showing the limit is symmetric, and their role will become clear in 2.7.3.

Let \( r', r'' \in \langle \bar{b}_0 \rangle \). Define for \( 0 < k \leq m \), \( r_k = r' + g_k(r'') \). We will be interested in \( \| r_k \|, \| r_k \|_i, \) for \( i \in \{0, 1, -1\} \). Note that for \( i \in \{1, -1\} \), by the definition of the norm \( \| \cdot \|_i \), for each \( \epsilon > 0 \), there are \( t_p \in B_p' \) for \( p < k \) such that \( r_k = \sum_{p < k} t_p \) and \( \| r_k \|_i + \epsilon \geq \sum_{p < k} \| t_p \|_i \geq \| r_k \|_i \). In the following claim we will assume that in fact one can find \( t_p \) as above such that \( \| r_k \|_i = \sum_{p < k} \| t_p \|_i \).

Claim 2.7.2. Suppose \( i \in \{1, -1, 0\} \), \( \| r_k \|_i = \sum_{p < k} \| t_p \|_i \) where \( t_p \in B_p' \) and \( r_k = \sum_{p < k} t_p \). Then there exist \( r_p' \in \langle \bar{b}_p \rangle \) and \( s_p \in V^- \) for \( 0 < p \leq k \) such that \( t_p = r_p' + r_p + s_p \) and \( \| r_p \|_i + \epsilon \geq \sum_{p < k} \| t_p \|_i \geq \| r_k \|_i \). Moreover, we may assume \( r_0' = r' \), \( r_k' = g_k(r'') \), therefore \( \sum_{p < k} s_p = 0 \).

Proof. As \( t_p \in B_p' \), we can write for every \( p < k \), \( t_p = r_{p+1} - \bar{r}_p + \bar{s}_p \) for \( \bar{r}_p' \in \langle \bar{b}_p \rangle \setminus V^- \), \( \bar{r}_p \in \langle \bar{b}_p \rangle \setminus V^- \) and \( s_p \in V^- \). So we get \( r_k = r' + g_k(r'') - \bar{r}_0 + \sum_{0 < p < k} (\bar{r}_p' - \bar{r}_p) + \bar{r}_k + \sum_{p < k} s_p \). By 2.7.1 and the definition of \( V_m \), \( \langle \bar{b}_i : \ell < n^- \rangle \cup \langle \bar{h}_i : n^- \leq \ell < n, i \leq m \rangle \) is a basis of \( V_m \). As \( \bar{r}_p' \) and \( \bar{r}_p \) are both elements of \( \langle \bar{b}_p \rangle \setminus V^- \), remembering the fact that \( r_k = r' + g_k(r'') \), where \( r' \in \langle \bar{b}_0 \rangle \) and \( r'' \in \langle \bar{b}_k \rangle \), we get that necessarily \( \bar{r}_p = \bar{r}_p' \). For \( 0 < p < k \), this is going to be \( r_k' \). As the claim does not demand \( r_0', r_k' \not\in V^- \), and we know that \( r' + \bar{r}_0 \in V^- \), as well as \( g_k(r'') - \bar{r}_0 \), by changing \( s_0 \) and \( s_{k-1} \), we may assume \( \bar{r}_0 = -r'' \) and \( \bar{r}_k = g_k(r'') \). As \( r_k = -r_0' + r_k' + \sum_{p < k} s_p \), we get \( \sum_{p < k} s_p = 0 \). Q.E.D.

Now we shall show

Claim 2.7.3. 

(1) For each \( i \in \{0, 1, -1\} \), \( \| r_k \|_i \) is an ascending uniformly bounded sequence (the bound does not depend on \( m \)).

(2) For each \( j > 1, m = j^2 \), for each \( i \in \{0, 1, -1\} \), \( \| r_m \|_0 \geq \| r_m \|_i \geq (1 + \frac{2}{j})^{-1} \cdot \| r_j \|_0 \)

Proof:

(1) First we show the boundedness. \( \| r' + g_k(r'') \|_i \leq \| r' \|_i + \| g_k(r'') \|_i = \| h_1(r') \|_B + \| h_1(g_k(r'')) \|_B = \| h_1(r') \|_B + \| h_1(g_k(r'')) \|_B \). So as we see, the bound does not depend on \( m \).

Now suppose \( k < \ell \). We aim to show that \( \| r_k \|_i \leq \| r_{\ell} \|_i \). First we’ll prove this for \( i = 1 \).
As proving that for every $\epsilon > 0$, $\| r_k \|_1 \leq \| r_\ell \|_1 + \epsilon$ is enough, we may assume there exist $t_p \in B^*_p$ such that $r_\ell = \sum_{p<\ell} t_p$ and $\| r_\ell \|_1 = \sum_{p<\ell} \| t_p \|_1$. Let $r'_p \in (\bar{B}_p)$ and $s_p \in V^-$ for $0 \leq p \leq \ell$ be as in (2).

Then

$$r_k = r' + g_k(r'') = -r'_0 + g_k(r'')$$

Therefore, by the definition of $\| \cdot \|_1$ and $h_1$ being a linear function,

$$\| r_k \|_1 = \| -r'_0 + g_k(r'') \|_1 = \| h_1(-r'_0 + g_k(r'')) \|_2 = \| h_1(-r'_0) + h_1(g_k(r'')) \|_2$$

But by indiscernibility of $\bar{a}_i$ in $B$ and the definitions of $g_k, g_\ell, h_1$,

$$\| h_1(-r'_0) + h_1(g_k(r'')) \|_2 = \| h_1(-r'_0 + h_1(g_k(r'')) \|_2$$

So we get

$$\| r_k \|_1 \leq \| h_1(-r'_0 + h_1(g_k(r''))) \|_2 = \| h_1(-r'_0 + r'_\ell) \|_2 = \| -r'_0 + r'_\ell \|_1 = \| -r'_0 + r'_1 + s_0 - r'_1 + r'_2 + s_1 - \ldots - r'_{\ell-1} + s_{\ell-1} = \sum_{p<\ell} s_p \|_1 = \| \sum_{p<\ell} t_p - \sum_{p<\ell} s_p \|_1$$

Remembering that $\sum_{p<\ell} s_p = 0$ (see 2.7.2), we conclude

$$\| r_k \|_1 \leq \sum_{p<\ell} \| t_p \|_1 = \| r_\ell \|_1$$

finishing the proof for $i = 1$. The same argument is used for $i = -1$, and the case $i = 0$ follows.

(2) Define (just for the proof) $B_{i,j} = \langle \bar{a}_i, \bar{a}_j \rangle_{\bar{B}}$. Just as in case of $\langle \bar{b}_i, \bar{b}_j \rangle$, we can define three norms on $B_{i,j}$: one is induced from the original norm on $B$ (an analog of $\| \cdot \|_1$), the second one is induced from the norm on $B_{j,i}$, using the isomorphism from $B_{j,i}$ onto $B_{i,j}$ taking $\bar{a}_i$ onto $\bar{a}_j$ and vice versa (an analog of $\| \cdot \|_{-1}$). The third norm on $B_{i,j}$ (the one we will actually interested in) will be denoted by $\| \cdot \|_{\bar{B}_{max}}$, and it is naturally an analog of $\| \cdot \|_0$, i.e. the maximum of the first two norms.

So we start the proof with the following

**Main Claim 2.7.4.** Suppose $m > k + 1$ and $r = c_m - c_k \in (\bar{b}_k, \bar{b}_m)$, then $\| r \|_1 \geq (1 + \frac{2}{m-k})^{-1} \cdot \| h_1(r) \|_{\bar{B}_{max}}$.

**Proof of the main claim.** First of all, wlog $k = 0$. As in the previous proof, we assume the existence of $t_p \in B^*_p$ for $p < m$ such that $r = \Sigma t_p$ and $\| r \|_1 = \Sigma \| t_p \|_1$. Therefore, by 2.7.2 there are $c_p \in (\bar{b}_p) \setminus V^-$ for $p < m$ and $s_p \in V^-$ for $p < m$ such that for all $p$, $t_p = c_{p+1} - c_p + s_p$. Denote $\| t_p \|_1 = \| h_1(t_p) \|_{\bar{B}}$ by $\varrho_p$. So $\| r \|_1 = \Sigma_{p<m} \varrho_p$ and we aim to show

$$\| h_1(r) \|_{\bar{B}_{max}} \leq (1 + \frac{2}{m}) \cdot \Sigma \varrho_p.$$

Trivially (the triangle inequality) $\| h_1(r) \|_{\bar{B}} \leq \Sigma \varrho_p \leq (1 + \frac{2}{m}) \cdot \Sigma \varrho_p$. Therefore it’s left to show that

$$\| h_1(r) \|_{\bar{B}} \leq (1 + \frac{2}{m}) \cdot \Sigma \varrho_p$$

Denote for $p < m$ and $\alpha \in I$, $c^\alpha_p = h^\alpha_p(c_p)$. By the indiscernibility of $\bar{a}_\alpha$, for all $\alpha < \beta \in I$,

$$\varrho_p = \| c^\beta_{p+1} - c^\alpha_p \|_{\bar{B}}$$
Also, denote for some/all $\alpha < \beta$
\[ g^* = \| e_0^\beta - e_m^\alpha \|_B \]

For every $\alpha < \beta \in I$ there is a functional $f_{\alpha, \beta} : B \to F$, such that
\[ \| f_{\alpha, \beta} \| = 1, f_{\alpha, \beta}(e_0^\beta - e_m^\alpha) = g^* \]

Choose $\ell$ such that $g_\ell$ is minimal. In particular,
\[ \Box \quad 2.7.5. \]

Choose $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ in $I$.

\[ g^* = \| e_0^{\alpha_3} - e_m^{\alpha_4} \| = |f_{\alpha_1, \alpha_3}(e_0^{\alpha_3} - e_m^{\alpha_4})| + \sum_{p=1}^{\ell-1} f_{\alpha_1, \alpha_3}(e_p^{\alpha_3+p} - e_{p+1}^{\alpha_4+p}) + f_{\alpha_1, \alpha_3}(e_\ell^{\alpha_3+\ell-1} - e_{\ell+1}^{\alpha_4+\ell-1}) + \sum_{p=\ell+1}^{m-2} f_{\alpha_1, \alpha_3}(e_p^{\alpha_3-m+p} - e_{p+1}^{\alpha_4-m+p}) + f_{\alpha_1, \alpha_3}(e_{m-1}^{\alpha_3-m+\ell+1} - e_m^{\alpha_4-m+\ell+1}) \]

\[ \leq g_0 + \ldots + g_{\ell-1} + |f_{\alpha_1, \alpha_3}(e_\ell^{\alpha_3+\ell-1} - e_{\ell+1}^{\alpha_4+\ell-1})| + g_{\ell+1} + \ldots + g_m \]

The last inequality is true as $\| f_{\alpha_1, \alpha_3} \| = 1$.

Denote $\beta_1 = \alpha_4 + \ell - 1$, $\beta_2 = \alpha_0 - m + \ell + 1$. Find $\beta_0 < \beta_1 < \beta_2 < \beta_3$ in $I$.

Now note that
\[ c_\ell^{\beta_2} - c_{\ell+1}^{\beta_1} = (c_\ell^{\beta_2} - c_\ell^{\beta_1}) - (c_\ell^{\beta_0} - c_\ell^{\beta_2}) - (c_\ell^{\beta_0} - c_\ell^{\beta_1}) \]

Therefore, $\| c_\ell^{\beta_2} - c_{\ell+1}^{\beta_1} \| \leq 3g_\ell$. But (as $\| f_{\alpha_1, \alpha_3} \| = 1$),
\[ |f_{\alpha_1, \alpha_3}(e_\ell^{\alpha_3+\ell-1} - e_{\ell+1}^{\alpha_4+\ell-1})| \leq \| c_\ell^{\beta_2} - c_{\ell+1}^{\beta_1} \| \leq 3g_\ell \]

Putting all the inequalities together (including \[ \Box \quad 2.7.5. \]), we conclude:
\[ g^* \leq g_0 + \ldots + g_{\ell-1} + |f_{\alpha_1, \alpha_3}(e_\ell^{\alpha_3+\ell-1} - e_{\ell+1}^{\alpha_4+\ell-1})| + g_{\ell+1} + \ldots + g_m \leq (\sum_{p=0}^{m-1} g_p) + 3g_\ell = \sum_{p=0}^{m-1} g_p + 2g_\ell \leq \sum_{p=0}^{m-1} g_p + 2 - \frac{1}{m} \sum_{p=0}^{m-1} g_p = (1 + \frac{2}{m}) \sum_{p=0}^{m-1} g_p \]

which finishes the proof of the main claim.

Now assume $m = j^2 > 1$. We aim to show $\| r_j \|_0 \leq \| r_m \|_1 \cdot (1 + \frac{2}{m})$.

As usual, we assume $\| r_m \|_1 = \sum_{p<m} \| t_p \|_1$ for some $t_p \in B_p'$ satisfying $r = \sum_{p<m} t_p$, where $t_p = -r_p' + r_{p+1}' + s_p$ as in \[ \Box \quad 2.7.2. \]. Denote for $\ell \leq j$,
\[ \hat{r}_\ell = g_{t_\ell,j}(r_{t_\ell,j}) \]

i.e. $\hat{r}_\ell$ is a copy of $r_{t_\ell,j}$ in $\{b_\ell\}$.

So by the definition of $r_j = r' + g_j(r''')$, we have
\[ \| r_j \|_0 = \| r' + \hat{r}_1 - \hat{r}_1 + \hat{r}_2 - \hat{r}_2 + \ldots + \hat{r}_{j-1} - \hat{r}_{j-1} + g_j(r''') \|_0 \]

Now note that $g_j(r''') = \hat{r}_j$: $g_m(r''') = r_m''$ (by \[ \Box \quad 2.7.2. \]), therefore $g_j(r''') = g_{m,j}(r_m(r''')) = g_{m,j}(r_m)$. Remembering that $m = j^2$ and the definition of $\hat{r}_j$, we get the desired.

Also remember that $r' = -r_0'$ and $\sum_{p<m} s_p = 0$ (see \[ \Box \quad 2.7.2. \]). We get:
Remark on $\omega$.

Let $V$ be a Banach space with the norm $\| \cdot \|_p$. We will use several times the analog of Łoś’s theorem for positive bounded formulae, claiming $V \models \varphi(\langle \xi_i : i < \omega \rangle)$ if and only if $V_i \models \varphi(\xi_i)$ for “almost all” $i$.

By 2.7.3 (1), each one of the three sequences $\langle \| r_m \|_i : m < \omega \rangle$ converges. By 2.7.3 (2), all of them converge to the same limit. Let us denote this limit by $\rho(r', r'') \in \mathbb{R}$.

Let $V$ be an ultrapower of all the $V_m$ modulo some nonprincipal ultrafilter $\mathcal{D}$ on $\omega$ (where $V_m$ is a normed space with the norm $\| \cdot \|_1$):

$$V = \prod_{m < \omega} V_m / \mathcal{D}$$

Remark 2.7.6. (1) Certainly, this is where the compactness becomes important. We will use several times the analog of Loś’s theorem for positive bounded formulae, claiming $V \models \varphi(\langle \xi_i : i < \omega \rangle)$ if and only if $V_i \models \varphi(\xi_i)$ for “almost all” $i$. 

(2) Instead of looking at $V_m$ and $V$, we should have looked at their completions, which are Banach spaces, and not just normed spaces, but it doesn’t matter.

(3) We will think of $V$ as embedded into our “monster” $\mathcal{B}$.

(4) Note that there is a natural embedding $i_m$ of $V_m$ into $V$:

$$i_m(r) = (0, \ldots, 0, r, \ldots)$$

i.e. $i_m(r) = g : \omega \to \bigcup V_m$ s.t. $g(k) = 0$ for $k < m$ and $g(k) = r$ for $k \geq m$. Moreover, for $k < m$ we get $i_m | k = i_k$.

So we will not distinguish between elements of $V_m$ for some $m$ (in fact, for all $k \geq m$) and the appropriate elements of $V$.

The following discussion will be done inside $V$ (and therefore inside $\mathcal{B}$). Let $\bar{b}_\omega \in V$ be the “limit” of the sequence $(\bar{b}_m : m \in \omega)$, i.e. $\bar{b}_\omega = (\bar{b}_m : m \in \omega)/\mathcal{D}$. Let $g_\omega$ be the “limit” of $(g_m : m \in \omega)$ taking $\bar{b}_0$ onto $\bar{b}_\omega$.

**Claim 2.7.7.** Let $r', r'' \in (\bar{b}_0)$, define $r_1 = r' + g_\omega(r'')$, $r_{-1} = r'' + g_\omega(r')$. Let $\rho = \rho(r', r'')$. Then $\| r_1 \|_V = \| r_{-1} \|_V = \rho$.

**Proof.** Denote $r_m = r' + g_m(r'')$, $r_{-m} = r'' + g_m(r')$. Then

$$\| r_m \|_V = \| r_{-m} \|_V = \| r_m \|_{-1}$$

Remember that by 2.7.3 both $\| r_m \|_1$ and $\| r_m \|_{-1}$ are ascending sequences converging to $\rho$. So on one hand, $\| r_m \|_1 \leq \rho$ and $\| r_m \|_{-1} \leq \rho$ for all $m$, and therefore

2.7.8.

$$\| r_1 \|_V \leq \rho, \quad \| r_{-1} \|_V \leq \rho$$

On the other hand, for every real $\varepsilon > 0$, for almost all $m$,

$$\| r_m \|_1 \geq \rho - \varepsilon, \quad \| r_m \|_{-1} \geq \rho - \varepsilon$$

Therefore, for all real $\varepsilon > 0$,

2.7.9.

$$\| r_1 \|_V \geq \rho - \varepsilon, \quad \| r_{-1} \|_V \geq \rho - \varepsilon$$

Combining 2.7.8 with 2.7.9 we get the desired equalities. □

The following claim can be viewed as the heart of the proof we’ve been working hard for:

**Claim 2.7.10.** $\text{tp}(\bar{b}_0, \bar{b}_\omega)$ is symmetric, i.e. $\text{tp}(\bar{b}_0, \bar{b}_\omega) = \text{tp}(\bar{b}_\omega, \bar{b}_0)$

**Proof.** Define the obvious mapping $\Phi$ from $(\bar{b}_0, \bar{b}_\omega)$ onto itself, extending $g_\omega \cup g_\omega^{-1}$ (“exchanging” $\bar{b}_0$ and $\bar{b}_\omega$ and respecting the linear structure). It is obviously an isomorphism of vector spaces, so we just have to show it is also an isometry. Take $r \in (\bar{b}_0, \bar{b}_\omega)$, then for some $r', r'' \in (\bar{b}_0)$, $r = r' + g_\omega(r'')$. Therefore $\Phi(r) = r'' + g_\omega(r')$. Now, by 2.7.7 $\| r \|_V = \| \Phi(r) \|_V = \rho(r', r'')$, q.e.d. □

Now we have obviously reached a contradiction. Why? First of all, note that as $\text{tp}(\bar{b}_i, \bar{b}_{i+1}) = \text{tp}(\bar{a}_i, \bar{a}_{i+1})$ for all $i \in \omega$, we get $q(\bar{b}_i, \bar{b}_{i+1})$ for all $i$ (remember: $q(\bar{x}, \bar{y})$ defines a partial order on $\mathcal{B}$, $\langle \bar{a}_i : i < \omega \rangle$ is ordered by $q$). As $q(\bar{x}, \bar{y})$ is a partial order, it is in particular transitive, so $q(\bar{b}_0, \bar{b}_m)$ holds for all $m > 0$, and therefore $\mathcal{B} \models q(\bar{b}_0, \bar{b}_\omega)$ (compactness + the fact that $q$ is a positive bounded type). But by 2.7.10 $\mathcal{B} \models q(\bar{b}_\omega, \bar{b}_0)$, a contradiction to $q$ being a partial order!

□
Note that the only property of positive bounded formulae we used in the proof is that they satisfy the compactness theorem, therefore in fact we proved:

**Theorem 2.8.** Let $\mathcal{L}$ be a logic satisfying the compactness theorem for the AEC of Banach spaces. Then the class of Banach spaces does not have the SOP in $\mathcal{L}$.

\[ \square \]

3. Groups

Let $\mathcal{G}$ be the “monster” group (the universal domain). Our first theorem in this section is a non-structure result that once again doesn’t seem to be surprising as after seeing the undecidability of the word problem, we feel that any “bad” syntactic property can be somehow found in the class of groups.

**Proposition 3.1.** $\mathcal{G}$ has SOP\(_3\)

**Proof.** Consider the formula $\varphi(x, y)$ defined by $“(xyx^{-1} = y^2) \land (x \neq y)”$.

* First, we have to show that there is a sequence $\langle a_i : i < \omega \rangle$ such that $i < j \Rightarrow \varphi(a_i, a_j)$. But this is trivial by using HNN extensions and compactness.

* Secondly we have to make sure there is no “triangle”, but this is actually a well-known example in geometric group theory (see [Grp]) of a triangle $X = \langle a, b : aba^{-1} = b^2 \rangle, Y = \langle b, c : cbc^{-1} = c^2 \rangle, Z = \langle a, c : cac^{-1} = a^2 \rangle$ that generates a trivial group when put together. Therefore, $\mathcal{G} \models (\forall x, y, z)(xyx^{-1} = y^2 \land zyz^{-1} = z^2 \land yxy^{-1} = y \rightarrow x = y = z = e)$ (where $e$ is the group identity). Therefore $\mathcal{G} \models \neg(\exists x, y, z)(\varphi(x, y) \land \varphi(y, z) \land \varphi(z, x))$, as required.

\[ \square \]

The proof uses the fact that there can not be a triangle of a certain kind. A natural question now is - what about quadrangles? In particular, is the group $H = \langle a, b, c, d : aba^{-1} = b^2, cbc^{-1} = c^2, dcd^{-1} = d^2, dad^{-1} = a^2 \rangle$ also trivial? Once again, it’s a well-known fact that it is actually infinite, and the proof is even more interesting than the fact itself, as it seems very general - it doesn’t speak at all about the relations between the generators. In fact, the proof suggests a generalization that roughly speaking says that it is impossible to “collapse” a group with four generators by forcing relations between only adjacent pairs. Model theoretically, it leads to the following surprising structure result, showing that unlike what people might have thought, there is a hope for some model-theoretic structure theory for the class of all groups.

**Theorem 3.2.** $\mathcal{G}$ does not have SOP\(_4\)

**Proof.** Suppose towards a contradiction that $\varphi(\bar{x}, \bar{y})$ exemplifies SOP\(_3\) in $\mathcal{G}$. In particular, there exists an indiscernible sequence $\langle \bar{a}_i : i < \omega \rangle$ such that $i < j \Rightarrow \varphi(\bar{a}_i, \bar{a}_j)$. Define for all $i \in \omega$, $H_i = \langle \bar{a}_i \rangle_{\mathcal{G}}$.

We denote $\text{len}(\bar{x}) = \text{len}(\bar{y})$ in $\varphi(\bar{x}, \bar{y})$ by $n$ and assume wlog (by indiscernibility) that there exists $n^* < n$ such that $\bigwedge_{\ell < n^*}(a_i, \bar{a}_i = a^*_\ell)$ for all $i < \omega$ and $\langle a_i, \bar{a}_i : n^* \leq \ell < n, i < \omega \rangle$ is a sequence of distinct elements. Define $H^* = \langle a^*_\ell : \ell < n^* \rangle$, i.e. $H^* = H_i \cap H_j$ for all $i < j < \omega$.

By the indiscernibility, there exists for $i \neq j \in \omega$, an isomorphism $f_{i,j} : H_i \rightarrow H_j$ mapping $\bar{a}_i$ onto $\bar{a}_j$. Define for all $i < j < \omega$, $H_{i,j} = \langle \bar{a}_i, \bar{a}_j \rangle_{\mathcal{G}}$. For $j < i < \omega$ we define $H_{i,j}$ by “relabeling”, changing the roles of $\bar{a}_i$ and $\bar{a}_j$, i.e. as a set $H_{i,j}$ equals
and the group action is defined on it such that there exists \( f_{j,i}^{i,j} : H_{j,i} \to H_{j,j} \) an isomorphism extending \( f_{j,j} \cup f_{j,i} \). So for \( j < i \), \( H_{j,j} \) does not have to be a subgroup of \( \mathbb{G} \) (but we can embed it into \( \mathbb{G} \), as \( \mathbb{G} \) is universal).

Given two groups \( G_1 \) and \( G_2 \) and a subgroup of both, \( G_0 \), we shall denote the free amalgamation of the two over \( G_0 \) by \( G_1 *_{G_0} G_2 \). Now let us concentrate on \( H_0, H_1, H_2, H_3 \). Define \( K_0 = H_0 *_{H^-} H_2, K_1 = H_{0,1} *_{H^+} H_{1,2}, K_2 = H_{2,3} *_{H^-} H_{3,0} \). Once again, those groups do not have to be subgroups of \( \mathbb{G} \). It is obvious, though, that \( K_0 \) is a subgroup of both \( K_1 \) and \( K_2 \) (by definition of free product and amalgamation of groups). So we define \( K = K_1 *_{K_0} K_2 \).

\( \mathbb{G} \) is universal, so we can embed \( K \) into \( \mathbb{G} \). Denote the image of \( \bar{a}_i \) under this embedding by \( \bar{b}_i \in \mathbb{G} \). Now we note

**Claim 3.2.1.** \( \text{tp}(\bar{b}_0 \bar{b}_1, \mathbb{G} ) = \text{tp}(\bar{b}_1 \bar{b}_2, \mathbb{G} ) = \text{tp}(\bar{b}_2 \bar{b}_3, \mathbb{G} ) = \text{tp}(\bar{b}_3 \bar{b}_0, \mathbb{G} ) = \text{tp}(\bar{a}_0 \bar{a}_1, \mathbb{G} ) \).

**Proof.** \( \text{tp}(\bar{b}_0 \bar{b}_1, \mathbb{G} ) = \text{tp}(\bar{a}_0 \bar{a}_1, K) = \text{tp}(\bar{a}_0 \bar{a}_1, K_1) = \text{tp}(\bar{a}_0 \bar{a}_1, H_{0,1}) = \text{tp}(\bar{a}_0 \bar{a}_1, \mathbb{G} ) \). The first equality is true because types are preserved under group isomorphisms ("embeddings"), and the rest - just the definitions of the groups. Using the same arguments for \( \text{tp}(\bar{b}_1 \bar{b}_2, \mathbb{G} ) \), we get \( \text{tp}(\bar{b}_1 \bar{b}_2, \mathbb{G} ) = \text{tp}(\bar{a}_1 \bar{a}_2, \mathbb{G} ) \), but the latter equals \( \text{tp}(\bar{a}_0 \bar{a}_1, \mathbb{G} ) \) by indiscernibility. The same argument (replacing \( K_1 \) by \( K_2 \) shows \( \text{tp}(\bar{b}_2 \bar{b}_3, \mathbb{G} ) = \text{tp}(\bar{a}_0 \bar{a}_1, \mathbb{G} ) \). Now \( \text{tp}(\bar{b}_2 \bar{b}_3, \mathbb{G} ) = \text{tp}(\bar{a}_3 \bar{a}_0, K) = \text{tp}(\bar{a}_3 \bar{a}_0, K_2) = \text{tp}(\bar{a}_3 \bar{a}_0, H_{3,0}) \), but the latter equals \( \text{tp}(\bar{f}_3, \mathbb{G} ) = \text{tp}(\bar{a}_0 \bar{a}_3, \mathbb{G} ) \) by the definition of \( H_{3,0} \) and \( \bar{f}_3 \), and by indiscernibility we’re done. \( \square \)

Now we obviously get a contradiction, as by \( \text{tp}(\bar{b}_0 \bar{b}_1, \mathbb{G} ) = \text{tp}(\bar{b}_1 \bar{b}_2, \mathbb{G} ) = \text{tp}(\bar{b}_2 \bar{b}_3, \mathbb{G} ) \) \( \varphi(\bar{x}, \bar{y}) \) exemplifies \( \text{SOP}_4 \) in \( \mathbb{G} \). \( \square \)

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