Mutual Visibility by Luminous Robots
Without Collisions

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Abstract

Consider a finite set of identical computational entities that can move freely in the Euclidean plane operating in Look-Compute-Move cycles. Let $p(t)$ denote the location of entity $p$ at time $t$; entity $p$ can see entity $q$ at time $t$ if at that time no other entity lies in the line segment $p(t)q(t)$. We consider the basic problem called Mutual Visibility: starting from arbitrary distinct locations, within finite time the entities must reach, without collisions, a configuration where they all see each other. This problem must be solved by each entity autonomously executing the same algorithm. We study this problem in the luminous robots model; in this generalization of the standard model of oblivious robots, each entity, called robot, has an externally visible persistent light which can assume colors from a fixed set. The case where the number of colors is $c \leq 1$ corresponds to the classical model without lights.

The extensive literature on computability in such a model, mostly for $c \leq 1$ and recently for $c > 1$, has never considered this problem because it has always assumed that three collinear robots are mutually visible.

In this paper we remove this assumption, and investigate under what conditions luminous robots can solve Mutual Visibility without collisions and at what cost (i.e., with how many colors). We establish a spectrum of results, depending on the power of the adversary, on the number $c$ of colors, and on the a-priori knowledge the robots have about the system. Among such results, we prove that Mutual Visibility can always be solved without collisions in SSynch with $c = 2$ colors and in ASynch with $c = 3$ colors. If an adversary...
can interrupt and stop a robot moving to its computed destination, Mutual Visibility is still always solvable without collisions in SSYNCH with \( c = 3 \) colors, and, if the robots agree on the direction of one axis, also in ASYNCH. All the results are obtained constructively by means of novel protocols.

As a byproduct of our solutions, we provide the first obstructed-visibility solutions to two classical problems for oblivious robots: collision-less convergence to a point and circle formation.

1. Introduction

1.1. Computational Framework

Consider a distributed system composed of a team of mobile computational entities, called robots, moving and operating in the Euclidean plane \( \mathbb{R}^2 \), initially each at a distinct point. Each robot can move freely in space, and operates in Look-Compute-Move cycles. During a cycle, a robot determines the position (in its own coordinate system) of the other robots (Look); it executes a protocol, the same for all robots, to determine a destination point (Compute); and moves towards the computed destination (Move); after each cycle, a robot may be inactive for an arbitrary but finite amount of time. The robots are anonymous, without a central control, and oblivious (i.e., at the beginning of a cycle, a robot has no memory of any observation or computation of its previous cycles). What is computable by such entities has been the object of extensive research within distributed computing; e.g., see [2, 6, 7, 12, 19, 23, 25, 27, 32, 33, 35]; for a recent review see [21].

Vision and mobility provide the robots with stigmergy, enabling the robots to communicate and coordinate their actions by moving and sensing their relative positions; they are otherwise assumed to be without any means of explicit direct communication. This restriction could enable deployment in extremely harsh environments where communication is impossible or can be jammed. Nevertheless, in many other situations it is possible to assume the availability of some sort of direct communication. The theoretical interest is obviously for weak communication capabilities.

A model employing a weak explicit communication mechanism is that of robots with lights, or luminous robots, initially suggested by Peleg [31]. In this model, each robot is provided with a local externally visible light, which can assume colors from a fixed set; the robots explicitly communicate with each other using these lights; the lights are persistent (i.e., the color is not erased at the end of a cycle), but otherwise the robots are oblivious [10, 11].
Notice that a light with only one possible color is the same as no light; hence the luminous robots model generalizes the classical one.

Both in the classical model and in that with lights, depending on the assumptions on the activation schedule and the duration of the cycles, different settings are identified. In the synchronous setting, the robots operate in rounds, and all robots activated in a round perform their cycle in perfect synchrony; the system is fully synchronous if all robots are activated at all rounds, semi-synchronous otherwise. In the asynchronous setting, there is no common notion of time, and no assumption is made on timing of activation, other than fairness, nor on the duration of each computation and movement, other than that it is finite.

The choice of when a robot is activated (in the semi-synchronous and asynchronous settings) and the duration of an activity within a cycle (in the asynchronous setting) is made under the control of an adversary, or scheduler. Similarly, the choices of the initial location of each robot and of its private coordinate system are made under adversarial conditions.

A crucial distinction is whether or not the adversary has also the power to stop a moving robot before it reaches its destination. If so, the moves are said to be non-rigid, and the only constraint is that, if interrupted before reaching its destination, the robot moves at least a minimum distance $\delta > 0$ (otherwise, no destination can ever be reached). If the adversary does not have such a power, the moves are said to be rigid.

1.2. Obstructed Visibility

The classical model and the more recent model of robots with lights share a common assumption, that three or more collinear robots are mutually visible. It can be easily argued against such an assumption, and for the importance of investigating computability when visibility is obstructed by the presence of other robots; that is, if two robots $r$ and $s$ are located at $r(t)$ and $s(t)$ at time $t$, they can see each other if and only if no other robot lies in the segment $r(t)s(t)$ at that time.

Very little is known on computing with obstructed visibility. In fact, the few studies on obstructed visibility have been carried out in other models: the model of robots in the one-dimensional space $\mathbb{R}$ [8]; and the so-called fat robots model, where robots are not geometric points but occupy unit disks, and collisions are allowed and can be used as an explicit computational tool (e.g., [1, 4, 9]). In our model, collisions can create unbreakable symmetries: since robots are oblivious and anonymous and execute the same protocol, if
\[ r(t) = s(t) \] (a collision), then the activation adversary can force \( r(t') = s(t') \) for all \( t' > t \) if the two robots do not have lights or their lights have the same color. Thus, unless this is the intended outcome, collision avoidance is always a requirement for all algorithms in the model considered here.

In this paper we focus on luminous robots in presence of obstructed visibility, and investigate computing in such a setting. Clearly, obstructed visibility increases the difficulty of solving problems without the use of additional assumptions. For example, with unobstructed visibility, every active robot can determine the total number \( n \) of robots at each activity cycle. With obstructed visibility, unless a robot has a-priori knowledge of \( n \) and this knowledge is persistently stored, the robot might be unable to decide if it sees all the robots; hence it might be unable to determine the value \( n \).

The main problem we investigate, called Mutual Visibility, is perhaps the most basic in a situation of obstructed visibility: starting from arbitrary distinct positions in the plane, within finite time the robots must reach a configuration in which they are in distinct locations, can all see each other, and no longer move. This problem is clearly at the basis of any subsequent task requiring complete visibility. Notice that this problem does not exist under unobstructed visibility, and has never been investigated before.

Among the configurations that achieve mutual visibility, a special class is that where all robots are in a strictly convex position; within that class, of particular interest are those where the robots are on the perimeter of a circle, possibly equally spaced. The problems of forming such configurations have been extensively studied both directly (e.g., [13, 15, 16, 22]) and as part of the more general Pattern Formation problem (e.g., [23, 25, 32, 33, 35]). Unfortunately, none of these investigations consider obstructed visibility, and those algorithms do not work in the setting considered here.

1.3. Main Contributions

In this paper we investigate under what conditions luminous robots can solve Mutual Visibility and at what cost (i.e., with how many colors).

We establish a spectrum of results, depending on the power of the adversary, on the number \( c \) of colors, and on the a-priori knowledge the robots have about the systems.

We first consider the case when the adversary can choose the activation schedule and (in the case of asynchronous setting) the duration of the operation, but cannot interrupt the movements of the robots; that is, movements are rigid. In this case, we show the following.
Theorem 1.1. Mutual Visibility is solvable without collisions by Rigid robots 

(a) with no colors in SSYNCH, if the robots know their number, $n$;  
(b) with 2 colors in SSYNCH, always;  
(c) with 3 colors in ASYNCH, always. 

We then consider the case when the adversary has also the power to interrupt the movements of the robots; that is, movements are non-rigid. The only restriction is that there exists a constant absolute length $\delta > 0$ such that, even if a robot’s move is interrupted before it reaches the destination, it travels at least a length $\delta$ towards it (otherwise it may never be able to reach any destination). In the case of non-rigid movements, we prove the following.

Theorem 1.2. Mutual Visibility is solvable without collisions by Non-Rigid robots 

(a) with no colors in SSYNCH, if the robots know $\delta$ and their number, $n$;  
(b) with 2 colors in SSYNCH, if the robots know $\delta$;  
(c) with 3 colors in SSYNCH, always;  
(d) with 3 colors in ASYNCH, if the robots agree on the direction of one coordinate axis. 

All these results are established constructively. We present and analyze two protocols, Algorithm 1 (Shrink) and Algorithm 2 (Contain), whose goal is to allow the robots to position themselves at the vertices of a convex polygon, solving Convex Formation, and thus Mutual Visibility. These two algorithms are based on different strategies, and are tailored for different situations. Protocol Shrink uses two colors and requires rigid movements, while protocol Contain uses more colors but operates also with non-rigid movements. We prove their correctness for SSYNCH robots (Sections 3 and 4). We then show how, directly or with simple expansions and modifications, all the claimed results follow (Sections 5 and 6). Finally, we propose some open problems (Section 7).

Let us point out that, to prove the correctness of Shrink, we solve a seemingly unrelated problem, Communicating Vessels, which is interesting in its own right.
As a byproduct of our solutions, we provide the first obstructed-visibility solution to a classical problem for oblivious robots: collision-less convergence to a point (Near-Gathering) (see [21, 30]). In fact, if the robots continue to follow algorithm Shrink once they reach full visibility, the convex hull of their positions converges to a point, and the robots approach it without colliding; thus solving Near-Gathering (Section 6.3). This algorithm has an interesting fault tolerance property: if a single robot is faulty and becomes unable to move, the robots will still solve Near-Gathering, converging to the faulty robot’s location (Section 6.6).

Additionally, both protocols can be modified so that the robots can position themselves on the perimeter of a circle, thus providing an obstructed-visibility solution to the classical problem of Circle Formation. The problem can be solved with 2 colors in Rigid SSynch, with 3 colors in Non-Rigid SSynch, and with 4 colors in Rigid ASynch, and Non-Rigid ASynch with agreement on one axis (Section 6.2).

2. Model and Definitions

2.1. Modeling Robots

We mostly follow the terminology and definitions of the standard model of oblivious mobile robots (e.g., see [21]).

By $\mathcal{R} = \{r_1, r_2, \ldots, r_n\}$ we denote a set of oblivious mobile computational entities, called robots, operating in the Euclidean plane, and initially placed at distinct points. Each robot is provided with its own local coordinate system centered in itself, and its own unit of distance. There is no agreement among different robots on the orientation of the coordinate system, on its handedness, or on the unit of distance. We denote by $r(t) \in \mathbb{R}^2$ the position occupied by robot $r \in \mathcal{R}$ at time $t$; these positions are expressed here in a global coordinate system, which is used for description purposes, but is unknown to the robots. Two robots $r$ and $s$ are said to collide at time $t$ if $r(t) = s(t)$. A robot $r$ can see another robot $s$ (equivalently, $s$ is visible to $r$) at time $t$ if and only if no other robot lies in the segment $r(t)s(t)$ at that time.

The robots are luminous: each robot $r$ has a persistent state variable, called light, which may assume any value in a finite set $\mathcal{C}$ of colors. The color of $r$ at time $t$ can be seen by all robots visible by $r$ at that time.

The robots are autonomous (i.e., without any external control), anonymous (i.e., without internal identifiers), indistinguishable (i.e., without ex-
ternal markings), without any direct means of communication, other than their lights. At any time, robots can be active or inactive, and initially they are all inactive.

When activated, a robot performs a Look-Compute-Move sequence of operations: it first obtains a snapshot of the positions, expressed in its local coordinate system, of all visible robots, along with their respective colors (Look phase); using the last obtained snapshot as an input, the robot executes an algorithm, which is the same for all robots, to compute a destination point \( x \in \mathbb{R}^2 \) and a color \( c \in \mathcal{C} \), and it sets its light to \( c \) (Compute phase); finally, it moves towards \( x \) (Move phase). It then stays inactive until the next activation.

The robots are oblivious in the sense that, when a robot becomes inactive, all its local memory, except for the light, is reset. In other words, upon becoming active again, a robot has no memory of past computations and snapshots.

With regards to the activation and timing of the robots, there are two basic settings: semi-synchronous (SSynch) and asynchronous (ASynch). In SSynch, the time is discrete; at each time instant \( t \) (called a round or a turn) a subset of the robots is activated and performs its operations atomically, ending at time \( t+1 \). At any given round, any subset of robots may be activated, from the empty set to all of \( \mathcal{R} \). In particular, if all robots are activated at every round, the setting is called fully synchronous (FSynch). In ASynch, there is no common notion of time; each robot is activated independently, the Look operation is instantaneous, but the Compute and Move operation can take an unpredictable (but bounded) amount of time, unknown to the robot.

The choice of the activations is done by an adversary, which however activates each robot infinitely often. In ASynch, the adversary chooses also the (finite) duration of each operation. The adversary also determines the initial position of the robots and their local coordinate system; in particular, the coordinate system of a robot might not be preserved over time and might be modified by the adversary when the robot is inactive.

The adversary might or might not have the power to interrupt the movement of a robot before it reaches its destination in the Move operation. If it does, the system is said to be NON-RIGID; the only constraint on the adversary is that there exists a constant \( \delta > 0 \) such that, if interrupted before reaching its destination, a robot moves at least a distance \( \delta \), not known to the robot itself. Notice that, otherwise, the adversary would be able to pre-
vent a robot from reaching any given destination in a finite number of turns. If movements are not under the control of the adversary, and every robot reaches its destination at every turn, the system is said to be **Rigid**.

2.2. **Mutual Visibility and Related Problems**

The **Mutual Visibility** problem requires $n$ robots to form a configuration where they are in $n$ distinct locations, no three of them are collinear, and the will no longer move. A protocol $P$ is a solution of **Mutual Visibility** if it allows the robots to solve **Mutual Visibility** starting from any initial configuration in which their positions are all distinct, and regardless of the decisions of the adversary.

Let us stress that, since robots are oblivious and anonymous and execute the same protocol, if $r(t) = s(t)$ (a collision), then the adversary can force $r(t') = s(t')$ for all $t' > t$ if the two robots do not have lights or their lights have the same color. Hence the two robots will never again occupy distinct locations, and will no longer be able to solve **Mutual Visibility**. Thus, collision avoidance of robots with the same color is a requirement for any solution protocol.

Among the configurations that achieve mutual visibility, a special class is that in which all robots are in a strictly convex position; within that class, of particular interest are those where the robots are on the perimeter of a circle; among those, of special interest are the configurations where the robots are at the vertices of a regular polygon. The problems of forming such configurations are called **Convex Formation**, **Circle Formation**, and **Uniform Circle Formation**, respectively.

2.3. **Geometric Notions and Observations**

Let $\mathcal{H}(t)$ denote the convex hull of $\{r_1(t), r_2(t), \ldots, r_n(t)\}$ at time $t$. The robots lying on its boundary are called **external robots** at time $t$, while the ones lying in its interior are the **internal robots** at time $t$.

Observe that a robot may not know where the convex hull’s vertices are located, because its view may be obstructed by other robots. However, it can easily determine whether it is an external or an internal robot. In fact, a robot $r$ is external at time $t$ if and only if there is a half-plane bounded by a straight line through $r(t)$ whose interior contains no robots at time $t$. In other words, $r$ is external if and only if it lies on the boundary of the convex hull of the robots that it can currently see. If, in addition, $r$ lies at a non-degenerate vertex of the (visible) convex hull, it is said to be a **vertex**
robot. Note also that the neighbors of an external robot on its visible convex hull are indeed its neighbors on the actual convex hull.

Finally, observe that a robot is able to tell if \( H \) is a line segment, i.e., if all the robots are collinear. In particular, if a robot can see only one other robot, it understands that it is an endpoint robot. Conversely, non-endpoint robots can always see more than one other robot.

3. Solving Mutual Visibility for Rigid SSynch Robots

In this section we consider the Mutual Visibility problem in the Rigid SSynch setting. We present and analyze a protocol, Algorithm \( \text{[Shrink]} \), and we prove it solves Mutual Visibility in such a setting using only two colors.

3.1. Description of Algorithm \( \text{[Shrink]} \)

The main idea of Algorithm \( \text{[Shrink]} \) is to make only the external robots move, so to shrink the convex hull. When a former internal robot becomes external, it starts moving as well. Eventually, all the robots reach a strictly convex configuration, and at this point they all see each other and they can terminate (by not moving).

If an active robot \( r_i \), located at \( p \), realizes that it is not a vertex robot, it does not move. Otherwise, it locates its clockwise and counterclockwise neighbors (in its own coordinate system) on the convex hull’s boundary, say located at \( a \) and \( b \), which are necessarily visible. Then, \( r_i \) attempts to move somewhere in the triangle \( pab \), in such a way to shrink the convex hull, and possibly make one more robot become a vertex robot. To avoid collisions with other robots that may be moving at the same time, \( r_i \)’s movements are restricted to a smaller triangle, shaded in gray in Figure 1. Moreover, to avoid becoming a non-vertex robot, \( r_i \) does not cross any line parallel to \( ab \) that passes through another robot, and it carefully positions itself on the closest of such lines, as shown in Figure 1(a). In particular, if no such line intersects the gray area, \( r_i \) makes a default move, and it moves halfway toward the midpoint of the segment \( ab \), as indicated in Figure 1(b).

In order to recognize that the Mutual Visibility problem has been solved, and to correctly terminate, the robots carry visible lights of two possible colors: namely, \( \mathcal{C} = \{ \text{Off}, \text{Vertex} \} \). All robots’ lights are initially set to Off. If an active robot realizes that it is a vertex of the convex hull, it sets its light to the other value, Vertex. Hence, when a robot sees only robots whose
Algorithm 1: Solving the Mutual Visibility problem for RIGID SSYNCH robots with 2-colored lights

**Input:** \( \mathcal{V} \): set of robots visible to me (myself included) whose positions are expressed in a coordinate system centered at my location.

1. \( r^* \leftarrow \text{myself} \)
2. \( \mathcal{P} \leftarrow \{r.\text{position} \mid r \in \mathcal{V}\} \)
3. \( \mathcal{H} \leftarrow \text{convex hull of } \mathcal{P} \)
4. **if** \( |\mathcal{V}| = 3 \) \text{ and } \mathcal{H} \text{ is a line segment} **then**
   5. Move orthogonal to \( \mathcal{H} \) by any positive amount
5. **else**
   6. **if** \( r^*.\text{position} \) is a vertex of \( \mathcal{H} \) **then**
      7. \( r^*.\text{light} \leftarrow \text{Vertex} \)
      8. **if** \( \forall r \in \mathcal{V}, r.\text{light} = \text{Vertex} \) **then**
         Terminate **else** **if** \( |\mathcal{V}| > 2 \) **then**
         \( a \leftarrow \text{position of my ccw neighbor on the boundary of } \mathcal{H} \)
         \( b \leftarrow \text{position of my cw neighbor on the boundary of } \mathcal{H} \)
         \( u \leftarrow a/2 \)
         \( \gamma \leftarrow 1/2 \)
         **foreach** \( r \in \mathcal{V} \setminus \{r^*\} \) **do**
         \( \text{Let } \alpha, \beta \text{ be such that } r.\text{position} = \alpha \cdot a + \beta \cdot b \)
         **if** \( \alpha + \beta < \gamma \) **then**
         \( u \leftarrow r.\text{position} \)
         \( \gamma \leftarrow \alpha + \beta \)
         **else if** \( \alpha + \beta = \gamma \) \text{ and } \( r.\text{position} \) is closer to \( b \) than \( u \)
         **then** \( u \leftarrow r.\text{position} \)
         \( v \leftarrow \gamma \cdot b \)
         Move to \( (u + v)/2 \)
      9. **else if** \( \forall r \in \mathcal{V} \setminus \{r^*\}, r.\text{light} = \text{Vertex} \) \text{ and } \( r^*.\text{position} \) lies in the interior of \( \mathcal{H} \) **then**
         Move to the midpoint of any edge of \( \mathcal{H} \)
(a) Making \( c \) become a vertex robot, without moving past it

(b) Default move

Figure 1: Move of an external robot, in two different cases (robots’ locations are indicated by small circles)

lights are set to \textit{Vertex}, it knows it can see all the robots in the swarm, and hence it terminates.

The above rules are sufficient to solve the \textbf{Mutual Visibility} problem in most cases, but there are some exceptions. It is easy to see that there are configurations in which \textbf{Mutual Visibility} is never solved until an internal robot moves, regardless of the algorithm employed. For instance, suppose that the configuration is centrally symmetric, with one robot lying at the center. Let the local coordinate systems of any two symmetric robots be oriented symmetrically and have the same unit distance, and assume that the scheduler chooses to activate all robots at every turn. Then, every two symmetric robots have symmetric views, and therefore they move symmetrically. If the central robot—which is an internal robot—never moves, then the configuration remains centrally symmetric, and the central robot always obstructs all pairs of symmetric robots. Hence \textbf{Mutual Visibility} is never solved, no matter
what algorithm is executed.

It turns out that our rules can be fixed in a simple way to resolve also this special case: whenever an internal robot sees only robots whose lights are set to Vertex (except its own light), it moves to the midpoint of any edge of the convex hull.

Finally, the configurations in which all the robots are collinear need special handling. In this case it is impossible to solve Mutual Visibility unless some robots leave the current convex hull. Suppose that a robot \( r \) realizes that all robots lie on a line, and that it is not an endpoint (i.e., \( r \) can see only two other robots, which are collinear with it). Then, \( r \) moves by any positive amount, orthogonal to the line formed by the other two visible robots. When this is done, the previous rules apply.

### 3.2. Correctness of Algorithm 1

#### 3.2.1. Invariants

In the following we discuss some basic invariants, which will serve to prove the correctness of Algorithm 1.

Suppose that, for some \( t \in \mathbb{N} \), \( \mathcal{H}(t) \) is not a line segment: the situation is illustrated in Figure 2. If a vertex robot is activated, it is bound to remain in the corresponding gray triangle, called move region of the robot. More precisely, the move region consists of the interior of the gray triangle, plus the vertex where the robot currently is, plus the interior of the edge that is opposite to the robot. Hence all move regions are disjoint. Moreover, if there is only one internal robot and it sees only robots whose light is set to Vertex, it moves to the midpoint of an edge of \( \mathcal{H}(t) \), which does not lie in any move region. It follows that, no matter which robots are activated at time \( t \), they will not collide at time \( t + 1 \). Also, \( \mathcal{H}(t + 1) \subseteq \mathcal{H}(t) \).

Recall that a robot \( r \in \mathcal{R} \) is a vertex robot if an only if it lies at the vertex of a reflex angle whose interior does not contain any robots. Now, referring to Figure 1, it is clear that a vertex robot will remain a vertex robot after a move. Additionally, if no new vertex robots are acquired between time \( t \) and \( t + 1 \), then the ordering of the vertex robots around the convex hull is preserved from time \( t \) to time \( t + 1 \). This easily follows from the fact that every robot remains in its own move region (cf. Figure 2).

#### 3.2.2. Convergence

We seek to prove that Algorithm 1 makes every robot eventually become a vertex robot. As it will be apparent in the proof of Theorem 3.1 the
crux of the problem is the situation in which only default moves are made (cf. Figure 1(b)). In Lemma 3.3 we will prove that, if all external robots are vertex robots, and no new robots ever become vertex robots or terminate, then all the vertex robots converge to the same limit point. We reduce this sub-problem to the Communicating Vessels problem, as detailed next.

Since we are assuming that only the vertex robots move, and that their movements depend only on the positions of other visible vertex robots, we may as well assume that all robots are vertex robots, and that their indices follow their order around the convex hull. Indeed, by the invariants observed in Section 3.2.1, all robots will remain vertex robots throughout the execution, and their ordering around the convex hull will remain the same. So, let \( r_{i-1}, r_i, r_{i+1} \) be three vertex robots, which appear on the boundary of \( H(t) \) consecutively in this order. Let \( r_i \) perform a default move at time \( t \). Then, the new position of \( r_i \) is a convex combination of the current positions of these three robots, and precisely

\[
    r_i(t+1) = \frac{r_{i-1}(t)}{4} + \frac{r_i(t)}{2} + \frac{r_{i+1}(t)}{4}. \tag{1}
\]

In general, as different sets of vertex robots are activated in several rounds, and nothing but default moves are made, the new location of each robot is always a convex combination of the \textit{original} positions of all the robots, obtained by applying (1) to the set of active robots, at every round. In
formulas,
\[ r_i(t_0 + t) = \sum_{j=1}^{n} \alpha_{i,j,t} \cdot r_j(t_0), \]
with \( \alpha_{i,j,t} \geq 0 \) and \( \sum_{j=1}^{n} \alpha_{i,j,t} = 1 \), assuming that the robots start making only default moves at time \( t_0 \). Let \( I = \{1, 2, \ldots, n\} \). We fix \( j \in I \), and we let \( w_{i,t} = \alpha_{i,j,t} - \alpha_{i-1,j,t} \), where indices are taken modulo \( n \). We claim that
\[ \lim_{t \to \infty} w_{1,t} = \lim_{t \to \infty} w_{2,t} = \cdots = \lim_{t \to \infty} w_{n,t} = 0. \] (2)

If such a claim is true (for all \( j \in I \)), it implies that the robots get arbitrarily close to each other, as \( t \) grows. This, paired with the fact that \( \mathcal{H}(t_0 + t + 1) \subseteq \mathcal{H}(t_0 + t) \) for every \( t \), as observed in Section 3.2.1, allows us to conclude that the robots converge to the same limit point.

Our claim can be generalized and reformulated in the following terms.

**Communicating Vessels.** Suppose that \( n \) vessels containing water are arranged in a circle, and there is a pipe between each pair of adjacent vessels, regulated by a valve. At every second, some of the valves are opened and others are closed, in such a way that each of the \( n \) valves stays open for infinitely many seconds, in total. If a valve between two adjacent vessels stays open between seconds \( t \) and \( t + 1 \), then \( 1/4 \) of the surplus of water, measured at second \( t \), flows from the fuller vessel to the emptier one. Our claim is that the amount of water converges to the same limit in all vessels, no matter how the valves are opened and closed. We call this problem **Communicating Vessels**.

In this formulation, the amount of water in the \( i \)-th vessel at time \( t \in \mathbb{N} \) would be our previous \( w_{i,t} \). However, here we somewhat abstract from the **Mutual Visibility** problem, and we consider a slightly more general initial configuration, in which the \( w_{i,0} \)'s are arbitrary real numbers. We set \( v_{i,t} = 1 \) if the valve between the \( i \)-th and the \( (i + 1) \)-th vessel is open between time \( t \) and \( t + 1 \) (indices are taken modulo \( n \)), and \( v_{i,t} = 0 \) otherwise. It is easy to verify that activating robot \( r_i \) at time \( t \) in our previous discussion corresponds to setting \( v_{i,t} = 1 \) in the **Communicating Vessels** formulation.

Let us denote by \( w_t \) the vector whose \( i \)-th entry is \( w_{i,t} \), and let \( q_{i,t} = w_{i+1,t} - w_{i,t} \). We first prove an inequality on the Euclidean norms of the vectors \( w_t \). Note that the inequality holds regardless of what assumptions are made on the opening pattern of the valves.
Lemma 3.1. For every $t \in \mathbb{N}$,

$$\|w_t\|^2 - \|w_{t+1}\|^2 \geq \frac{1}{4} \sum_{i=1}^{n} v_{i,t} \cdot q_{i,t}^2. \quad (3)$$

Proof. For brevity, let $a = w_{i-1,t}$, $b = w_{i,t}$, $c = w_{i+1,t}$; hence, $q_{i-1,t} = b - a$ and $q_{i,t} = c - b$.

Suppose first that $v_{i-1,t} = v_{i,t} = 1$, i.e., both valves connecting the $i$-th vessel with its neighbors are open. Then, $w_{i,t+1} = (a + 2b + c)/4$. We have

$$\frac{w_{i-1,t}^2}{4} + \frac{w_{i,t}^2}{2} + \frac{w_{i+1,t}^2}{4} - \frac{w_{i,t+1}^2}{4} \geq \frac{q_{i-1,t}^2}{8} + \frac{q_{i,t}^2}{8}, \quad (4)$$

which can be obtained by dropping the term $(a - c)^2/16$ from the algebraic identity

$$\frac{a^2}{4} + \frac{b^2}{2} + \frac{c^2}{4} - \frac{(a + 2b + c)^2}{16} = \frac{(a - b)^2}{8} + \frac{(b - c)^2}{8} + \frac{(a - c)^2}{16}.$$

Now, suppose instead that $v_{i-1,t} = 1$ and $v_{i,t} = 0$. Then we have $w_{i,t+1} = (a + 3b)/4$, and

$$\frac{w_{i-1,t}^2}{4} + \frac{3w_{i,t}^2}{4} - \frac{w_{i,t+1}^2}{4} = \frac{3q_{i-1,t}^2}{16} \geq \frac{q_{i-1,t}^2}{8}, \quad (5)$$

where the first equality comes from the identity

$$\frac{a^2}{4} + \frac{3b^2}{4} - \frac{(a + 3b)^2}{16} = \frac{3(a - b)^2}{16}.$$

If $v_{i-1,t} = 0$ and $v_{i,t} = 1$, an analogous argument gives

$$\frac{3w_{i,t}^2}{4} + \frac{w_{i+1,t}^2}{4} - \frac{w_{i,t+1}^2}{4} \geq \frac{q_{i,t}^2}{8}. \quad (6)$$

Finally, if $v_{i-1,t} = v_{i,t} = 0$, $w_{i,t+1} = w_{i,t}$, and trivially

$$w_{i,t}^2 - w_{i,t+1}^2 = 0. \quad (7)$$

We sum for each $i \in I$ the relevant inequality among (4), (5), (6), (7), depending on the value of $v_{i-1,t}$ and $v_{i,t}$. Each of the terms $q_{i,t}^2/8$ appears twice if and only if $v_{i,t} = 1$, and the coefficients of the terms in $w_{i,t}^2$ sum to 1 for every $i$, hence we get (3). \qed
From the previous lemma, it immediately follows that the sequence \((\|w_t\|)_{t \geq 0}\) is non-increasing. Since it is also bounded below by 0, it converges to a limit, which we call \(\ell\). Let \(M_t = \max_{i \in I} \{w_{i,t}\}\) and \(m_t = \min_{i \in I} \{w_{i,t}\}\). Observe that each entry of \(w_{t+1}\) is a convex combination of entries of \(w_t\), hence \((M_t)_{t \geq 0}\) is non-increasing and \((m_t)_{t \geq 0}\) is non-decreasing. Therefore they both converge, and we let \(M = \lim_{t \to \infty} M_t\) and \(m = \lim_{t \to \infty} m_t\).

**Corollary 3.1.**

\[ m \leq \frac{\ell}{\sqrt{n}} \leq M. \]

**Proof.** For every \(t \in \mathbb{N}\), we have

\[ nM_t^2 \geq \sum_{i=1}^{n} w_{i,t}^2 = \|w_t\|^2 \geq \ell^2, \]

which proves the second inequality. As for the first inequality, for every \(\varepsilon > 0\) and large-enough \(t\), we have \(nm_t^2 \leq \|w_t\|^2 \leq \ell^2 + \varepsilon\).

For the next lemma, we let \(V_i = \{t \in \mathbb{N} \mid v_{i,t} = 1\}\).

**Lemma 3.2.** Suppose that \(|V_i| = \infty\) for at least \(n-1\) distinct values of \(i \in I\). Then,

\[
M = m = \frac{\ell}{\sqrt{n}}.
\]

**Proof.** Due to Corollary 3.1, it is enough to prove that \(M - m = 0\). By contradiction, assume \(M - m > 0\), and let \(\delta = (M - m)/(n + 1) > 0\). We have

\[
\lim_{t \to \infty} \left( \|w_t\|^2 - \|w_{t+1}\|^2 \right) = \ell^2 - \ell^2 = 0,
\]

hence there exists \(T \in \mathbb{N}\) such that \(\|w_t\|^2 - \|w_{t+1}\|^2 < \delta^2/4\) for every \(t \geq T\).

By Lemma 3.1,

\[
\frac{q_{i,t}^2}{4} \leq \|w_t\|^2 - \|w_{t+1}\|^2 < \frac{\delta^2}{4}
\]

for every \(t \geq T\) and every \(i\) such that \(v_{i,t} = 1\). This implies \(|q_{i,t}| < \delta\), that is, a necessary condition for the valve between the \(i\)-th and the \((i+1)\)-th vessel to be open at time \(t \geq T\) is that \(|w_{i+1,t} - w_{i,t}| < \delta\). Consider now the \(n+1\) open intervals

\[(m, m + \delta), (m + \delta, m + 2\delta), \ldots, (m + n\delta, M),\]
each of width $\delta$. Since $M_T \geq M$ and $m_T \leq m$, there are $w_{i,T}$'s above and below all these intervals. Moreover, by the pigeonhole principle, at least one of the intervals contains no $w_{i,T}$'s, for any $i \in I$. In other words, we can find a partition $I_1 \cup I_2 = I$, with $I_1$ and $I_2$ both non-empty, and a threshold value $\lambda$ such that $w_{i,T} \leq \lambda$ for every $i \in I_1$, and $w_{i,T} \geq \lambda + \delta$ for every $i \in I_2$. Hence, at time $T$, only valves between entries of $w_i$ whose indices belong to the same $I_k$ can be open. It is now easy to prove by induction on $t \geq T$ the following facts:

- $\max_{i \in I_1} \{w_{i,t}\} \leq \lambda$,
- $\min_{i \in I_2} \{w_{i,t}\} \geq \lambda + \delta$,
- $v_{i,t} = 0$ whenever $i$ and $i + 1$ belong to two different classes of the partition.

Since $I_1$ and $I_2$ are non-empty, there must be at least two distinct indices $i' \in I_1$ and $i'' \in I_2$ such that $i' + 1 \in I_2$ and $i'' + 1 \in I_1$ (where indices are taken modulo $n$). It follows that the $i'$-th and $i''$-th valve are never open for $t \geq T$, and this contradicts the hypothesis that $|V_i| < \infty$ for at most one choice of $i \in I$.

This solves the Communicating Vessels problem.

**Corollary 3.2.** Under the hypotheses of Lemma 3.2, for every $i \in I$,

$$\lim_{t \to \infty} w_{i,t} = \ell \frac{\sqrt{n}}{n} = \frac{\sum_{j=1}^{n} w_{j,0}}{n}.$$ 

**Proof.** By Lemma 3.2, since $m_t \leq w_{i,t} \leq M_t$, all the limits coincide. Moreover, the sum of the $w_{i,t}$'s does not depend on $t$; hence their average, taken at any time, must be equal to the joint limit.

Let us return to the Mutual Visibility problem, to prove our final lemma.

**Lemma 3.3.** If, at every round, each robot makes a default move (cf. Figure 1(b)) or stays still, all external robots have their lights set to Vertex, and no new robots become vertex robots or terminate, then all robots' locations converge to the same limit point.

**Proof.** As discussed at the beginning of Section 3.2.2, this is implied by (2). Recall that $w_{i,0} = \alpha_{i,j,0} - \alpha_{i-1,j,0}$, and hence $\sum_{i=1}^{n} w_{i,0} = 0$. Then, (2) follows immediately from Corollary 3.2.
We are now ready to prove our main theorem.

**Theorem 3.1.** Algorithm 1 solves Mutual Visibility for Rigid SSYNCH robots with 2-colored lights.

*Proof.* If the initial convex hull is a line segment, it becomes a non-degenerate polygon as soon as one or more of the non-vertex robots are activated. It is also easy to observe (cf. Figure 2) that, from this configuration, the convex hull may never become a line segment. So the invariants discussed in Section 3.2.1 apply, possibly after a few initial rounds: no two robots will ever collide, and a vertex robot will never become a non-vertex robot.

Assume by contradiction that the execution never terminates. Note that a robot terminates if and only if all robots terminate. Indeed, if there are any non-vertex robots (whose lights are still set to Off), then each vertex robot can see at least one of them. Hence we are assuming that all robots execute the algorithm forever.

At some point, the set of vertex robots reaches a maximum \( M \subseteq \mathcal{R} \), and as soon as all of these robots have been activated, they permanently set their lights to Vertex. Let \( T \in \mathbb{N} \) be a time at which all the robots in \( M \) have their lights set to Vertex. Suppose that there are external robots that are not vertex robots after time \( T \), and let \( r \) be one such robot that is adjacent to a vertex robot \( r' \). Then, after \( r' \) is activated and moves, \( r \) becomes a vertex robot as well, contradicting the maximality of \( M \). Hence the external robots are exactly the robots in \( M \), and no other robot may become external after time \( T \).

If there is only one internal robot at time \( t \geq T \), it becomes external as soon as it is activated, due to line 23 of the algorithm, which is impossible, as argued in the previous paragraph. Therefore there are at least two internal robots at every time \( t \geq T \). On the other hand, if a vertex robot makes a non-default move at any time \( t \geq T \), a new robot becomes external at time \( t + 1 \). Indeed, referring to Figure 1(a), the line \( uv \) passes through \( p(t+1) \) and \( c(t+1) \), and no robot lies above this line at time \( t + 1 \). Hence \( c \) becomes a new external robot, which again is impossible.

As a consequence, only default moves are made after time \( T \). Moreover, no robot becomes external or becomes a vertex robot after time \( T \), and no robot ever terminates. Therefore Lemma 3.3 applies, and the robots converge to the same limit point. But since there are at least two internal robots, this means that at least one of them has to move, implying that it becomes a
vertex robot at some point (by the above assumptions, only vertex robots can move), a contradiction.

Hence the execution terminates, meaning that at some point one of the robots sees only vertex robots. This implies that there are no non-vertex robots, hence the configuration is strictly convex, all robots can see each other, and they all terminate without moving as soon as they are activated, thus solving the Mutual Visibility problem.

4. Solving Mutual Visibility for Non-Rigid SSynch Robots

Here we give a protocol, Algorithm 2 (Contain), for the Mutual Visibility problem that works for Non-Rigid robots and the SSynch scheduler. Recall that, in the Non-Rigid model, the robots make unreliable moves, that is, the scheduler can stop them before they reach their destination point, but not before they have moved by at least a constant $\delta > 0$. Since these robots are weaker than the ones considered in Section 3, they will require lights of three possible colors, as opposed to two.

4.1. Description of Algorithm 2

Algorithm 2 consists of two phases: an interior depletion phase and a vertex adjustments phase, to be executed in succession. In the first phase, the internal robots move toward the boundary of the convex hull, and in the second phase the robots (who are now all external) make small movements to finally reach a strictly convex configuration. Three colors are used: $\mathcal{C} = \{\text{Off}, \text{External}, \text{Adjusting}\}$.

Initially, all the robots’ lights are set to Off, and as soon as an external robot is activated, it sets its own light to External and does not move as long as it can still see robots whose light is Off. We denote by $\mathcal{H'}(t)$ the convex hull of the positions of the internal robots at time $t \in \mathbb{N}$. Note that a robot $r$ that occupies a vertex of $\mathcal{H'}$ eventually becomes aware of it, by looking at the convex hull of the visible robots whose lights are Off. These may not all be internal robots, because perhaps not all external robots have been activated yet, but eventually $r$ gets to see a good-enough approximation of a “neighborhood” of $\mathcal{H'}$, and it realizes it occupies one of its vertices.

So, when a robot understands that it lies on a vertex of $\mathcal{H'}$, it moves toward the boundary of $\mathcal{H}$, part of which is also identifiable by $r$. We distinguish three cases. If $r$ realizes it is the only internal robot, it moves toward the midpoint of an edge of the convex hull. To avoid bouncing back and
Algorithm 2: Solving the Mutual Visibility problem for NON-RIGID SSYNCH robots and RIGID ASYNCH robots with 3-colored lights

Input: \( V \): set of robots visible to me (myself included) whose positions are expressed in a coordinate system centered at my location.

1. \( r^* \leftarrow \) myself
2. \( P \leftarrow \{\text{r.position} \mid r \in V\} \)
3. \( H \leftarrow \) convex hull of \( P \)
4. \( \partial H \leftarrow \) boundary of \( H \)
5. If \( |V| = 1 \) then Terminate else if \( |V| = 2 \) then
   6. If \( r^*.light = \text{Adjusting} \) then
      7. \( r^*.light \leftarrow \text{External} \)
      8. Terminate
   else
      9. \( r^*.light \leftarrow \text{Adjusting} \)
      10. Move orthogonal to \( H \) by the length of \( H \)
   else if \( H \) is a line segment then
      11. If \( \forall r \in V \setminus \{r^*\}, r\.light = \text{External} \) then
         12. \( r^*.light \leftarrow \text{Adjusting} \)
         13. Move orthogonal to \( H \) by any positive amount
   else if \( r^*.position \in \partial H \) then
      14. If \( r^*.light = \text{Adjusting} \) then
         15. If \( \forall r \in V, r\.light \neq \text{Off} \) or \( \exists r \in V, r\.light = \text{External} \) then
            16. \( r^*.light \leftarrow \text{External} \)
            17. Terminate
         else if \( r^*.position \) is a vertex of \( H \) and \( \forall r \in V, r\.light = \text{External} \) then
            18. \( a \leftarrow \) position of my ccw neighbor on \( \partial H \)
            19. \( b \leftarrow \) position of my cw neighbor on \( \partial H \)
            20. \( r^*.light \leftarrow \text{Adjusting} \)
            21. Move to \( (a + b)/4 \)
      else if (the internal angle of \( H \) at \( r^*.position \) is acute and \( |V| = 3 \) or \( |\{r \in V \mid r\.light = \text{Adjusting}\}| \neq 1 \) then
         22. \( r^*.light \leftarrow \text{External} \)
else if $\forall r \in V, r.light \neq \text{Adjusting}$ then

$P' \leftarrow \{r.position \mid r \in V \land r.light = \text{Off}\}$

$H' \leftarrow$ convex hull of $P'$

$\partial H' \leftarrow$ boundary of $H'$

if $|P'| = 1$ then

Move to a closest midpoint of a connected component of $\partial H \setminus P$

else if $|P'| = 2$ then

$\ell \leftarrow$ line orthogonal to $H'$

$A \leftarrow$ half-plane bounded by $\ell$ such that $A \cap H' = \{r^*.position\}$

Move to any point of $(A \cap \partial H) \setminus P$

else if $r^*.position$ is a vertex of $H'$ then

$A \leftarrow$ internal angle of $H'$ whose vertex is $r^*.position$

$\alpha \leftarrow$ measure of $A$

$\ell \leftarrow$ axis of symmetry of $A$

if $\alpha \leq \pi/2$ then

$\alpha' \leftarrow \alpha$ else $\alpha' \leftarrow \pi - \alpha$

$A' \leftarrow$ angle of measure $\alpha'$ with axis of symmetry $\ell$ such that $A' \cap H' = \{r^*.position\}$

Move to any point of $(A' \cap \partial H) \setminus P$
forth at different turns, it always chooses the closest of such midpoints. If 
 andrealsizes that $H'$ is a line segment and it occupies one endpoint of it, it 
moves like in Figure 3(a). That is, it moves to the boundary of $H$, while 
remaining in the half-plane bounded by the line orthogonal to $H'$. Finally, 
if $r$ “believes” that $H'$ is a non-degenerate polygon and that it lies on one 
of its vertices, it moves as in Figure 3(b). Remember that $r$ may believe so 
even if $H'$ is actually degenerate, because some external robots may still be 
Off. However, $r$ gets an approximation of $H'$, which we call $H'_r$, and it knows 
it lies on a vertex of $H'_r$, implying that it also lies on a vertex of the “real” 
$H'$. Now, if the internal angle of $H'_r$ at $r(t)$ is acute, $r$ moves as the robot 
in $p$ in Figure 3(b) it moves to the boundary of $H$ while remaining between 
the extensions of its two incident edges of $H'_r$. Otherwise, if the angle is not 
acute, $r$ moves as the robot in $q$ in Figure 3(b) it moves to the boundary of 
$H$ while staying between the two perpendiculars to its incident edges of $H'_r$.

Now to the vertex adjustments phase. When a robot lies at a vertex of $H$ 
and it sees only robots whose light is set to External, it makes the “default 
move” of Figure 1(b), where $a$ and $b$ are the locations of its two neighbors on 
$H$. Moreover, while doing so it also sets its light to the third value, Adjusting, 
as a “self-reminder”. So, when it is activated again, it knows it has already 
adjusted its position, and it terminates, after reverting its light to External. 
This way we make sure that each vertex robot adjusts its position exactly 
onece, and we ensure termination. When the adjustment is done, the robots 
at $a$ and $b$ are guaranteed to occupy vertices of $H$, instead of lying in the 
middle of an edge. So, each external robot becomes a vertex robot at some 
point, then it adjusts its position while remaining a vertex, possibly making 
its adjacent robots become vertices as well, and it terminates. When all 
robots have terminated, the configuration is strictly convex, and therefore 
Mutual Visibility is solved.

There is one more special case to discuss, that is, when all robots are 
initially collinear. Let robots $r$ and $s$ be the two endpoints of $H$: as soon as 
one of them is activated (possibly both), it sets its lights to Adjusting, moves 
 orthogonal to $H$ by a small-enough amount, and then waits. Meanwhile, the 
other robots do not do anything until some conditions are met (see line 27). 
If only $r$ moves, $s$ realizes and sets its light to External (and vice versa). If 
both $r$ and $s$ move together, some other robot realizes that it is external and 
that it can see both $r$ and $s$ set to Adjusting: in this case, it sets itself to 
External. When $r$ or $s$ sees some robots set to External, it finally sets itself to 
External too, and it terminates. After this is done, the execution transitions
seamlessly into one of the general cases.

If $n \leq 3$ this is not sufficient. Suppose first that $n = 3$. Then, because $r$ and $s$ may move in such a way that the configuration remains centrally symmetric, with the middle robot $q$ obstructing $r$ and $s$. However, after moving once, $r$ and $s$ become External and terminate (possibly after after $q$ has in turn become External), and finally $q$ moves orthogonal to $\mathcal{H}$ (line 16), thus solving Mutual Visibility also in this special case.

If $n = 2$, each robot moves once, and then it detects a situation in which it can safely terminate terminate (line 9).
4.2. Correctness of Algorithm 2

4.2.1. Interior Depletion Phase

We first prove that no collisions occur during the interior depletion phase, and then that the phase itself eventually terminates, with all the robots becoming external.

It is easy to observe that, during this phase, all external robots keep seeing (internal) robots whose lights are set to \textit{Off}, and therefore none of them moves. On the other hand, no internal robot moves outside of the convex hull.

\textbf{Observation 4.1.} \textit{If there are internal robots at time }t\textit{, no external robot moves, and }\mathcal{H}(t) = \mathcal{H}(t + 1)\textit{.}

\textbf{Lemma 4.1.} \textit{If }r\textit{ and }s\textit{ are two internal robots at time }t\textit{, then}

\[(r(t + 1) - r(t)) \cdot (s(t) - r(t)) \leq 0.\]

\textit{Proof.} If \(r\) is not activated at time \(t\), or it is activated but it does not move, then the left-hand side is zero, and therefore the inequality holds. Suppose now that \(r\) moves by a positive amount, so \(r(t + 1) - r(t)\) is not the null vector. Let \(\ell\) be the line through \(r(t)\) that is orthogonal to the segment \(r(t)r(t + 1)\). By construction, \(r\) moves in such a way that \(r(t + 1)\) lies in the open half-plane bounded by \(\ell\) that does not contain \(\mathcal{H}'(t)\) (note that this holds \textit{a fortiori} also if some external robots have not set their lights to \textit{External} yet, and therefore the \(\mathcal{H}'\) computed by \(r\) is larger than the real one). Since \(s(t) \in \mathcal{H}'(t)\), \(s(t)\) lies on \(\ell\) or in the half-plane bounded by \(\ell\) that does not contain \(r(t + 1)\). This is equivalent to saying that the dot product between \(r(t + 1) - r(t)\) and \(s(t) - r(t)\) is not positive. \(\blacksquare\)

\textbf{Lemma 4.2.} \textit{As long as there are internal robots, no collisions occur.}

\textit{Proof.} If there are internal robots, every external robot sees robots whose light is set to \textit{Off}, and hence it does not move. By construction, the internal robots avoid moving on top of external robots, and therefore there can be no collision involving external robots.

Suppose by contradiction that two robots \(r\) and \(s\) that are internal at time \(t\) collide for the first time at \(t + 1\), and therefore \(r(t + 1) = s(t + 1) = p\). By \textbf{Lemma 4.1} applied to \(r\) and \(s\), we have

\[(p - r(t)) \cdot (s(t) - r(t)) \leq 0.\] (8)
Applying Lemma 4.1 again with \( r \) and \( s \) inverted, we also have
\[
(p - s(t)) \cdot (r(t) - s(t)) \leq 0.
\] (9)

Adding 8 and 9 together and doing some algebraic manipulations, we obtain
\[
(p - r(t)) \cdot (s(t) - r(t)) + (p - s(t)) \cdot (r(t) - s(t)) \leq 0,
\]
\[
(s(t) - r(t)) \cdot ((p - r(t)) - (p - s(t))) \leq 0,
\]
\[
(s(t) - r(t)) \cdot (s(t) - r(t)) \leq 0.
\]
The latter is equivalent to \( \| s(t) - r(t) \| \leq 0 \), implying that \( r(t) = s(t) \). This contradicts the fact that \( r \) and \( s \) collide for the first time at \( t + 1 \).

We still have to prove that the interior depletion phase terminates, that is, eventually all robots become external. Due to Observation 4.1, when a robot becomes external, it stops moving and remains external, at least as long as there are other internal robots. Thus, if by contradiction this phase does not terminate, the set of internal robots reaches a non-empty minimum, and from that time on no new robot becomes external. After possibly some more turns, say at time \( T \in \mathbb{N} \), all external robots have been activated and have set their lights to \textit{External}, and hence no robot changes its light any more.

In the following lemmas, we will show that these assumptions on \( T \) yield a contradiction. We will prove that, if the area of \( \mathcal{H}'(T) \) is positive, then it grows unboundedly, and therefore at some point \( \mathcal{H}' \) cannot be a subset of \( \mathcal{H} \) any more. (On the other hand, the analysis when \( \mathcal{H}'(T) \) has null area is easy.)

By inspecting Algorithm 2 and referring to Figure 4 it is easy to observe the following.

**Observation 4.2.** Let robots \( r \) and \( s \) lie at adjacent vertices of \( \mathcal{H}'(t) \) at time \( t \geq T \). Then, \( r(t+1) \) and \( s(t+1) \) lie on the same side of the line through \( r(t) \) and \( s(t) \) (or possibly on the line itself).

**Lemma 4.3.** If \( t \geq T \) and the area of \( \mathcal{H}'(t) \) is positive, then \( \mathcal{H}'(t) \subseteq \mathcal{H}'(t+1) \).

**Proof.** Let \( \mathcal{R}' \subset \mathcal{R} \) be the set of robots that lie at vertices of \( \mathcal{H}'(t) \), at time \( t \geq T \). Let \( P \) be the polygon (illustrated in Figure 4 as a thick dashed polygon) whose vertices are the locations at time \( t + 1 \) of the robots of \( \mathcal{R}' \),
taken in the same order as they appear around the boundary of $\mathcal{H}'$. Note that, since the robots are NON-RIGID and not all of them are necessarily activated at time $t$, $\mathcal{P}$ is not necessarily a convex polygon.

Because the property stated in Observation 4.2 holds for all the edges of $\mathcal{H}'(t)$ and $\mathcal{P}$, we have that $\mathcal{H}'(t) \subseteq \mathcal{P}$. But, by definition of $T$, none of the robots of $\mathcal{R}'$ ever becomes external, and hence $\mathcal{H}'(t+1)$ is the convex hull of $\mathcal{P}$. We conclude that $\mathcal{H}'(t) \subseteq \mathcal{P} \subseteq \mathcal{H}'(t+1)$. 

![Figure 4: Combined motion of internal robots (external robots are not shown)](image)

**Corollary 4.1.** If $r$ is an internal robot at time $t \geq T$, and $r(t)$ is not a vertex of $\mathcal{H}'(t)$, then $r$ is internal at any time $t' \geq t$, and $r(t')$ is not a vertex of $\mathcal{H}'(t')$.

**Proof.** By Lemma 4.3, if $t' \geq t$, then $\mathcal{H}'(t) \subseteq \mathcal{H}'(t')$. Moreover, according to Algorithm 2, an internal robot that does not lie at a vertex of $\mathcal{H}'$ does not move. It follows that, after time $t$, robot $r$ will not move, hence it will remain internal, and it will never lie at a vertex of $\mathcal{H}'$. 

As a consequence of the previous corollary, no new robot becomes a vertex of $\mathcal{H}'$ after time $T$, but some robots may indeed cease to be vertices of $\mathcal{H}'$. 

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and stop moving forever. Hence, at some time \( T' \geq T \), the set of robots that lie at vertices of \( \mathcal{H}' \) reaches a minimum \( \mathcal{M} \subset \mathcal{R} \). By Lemma 4.3, the area of \( \mathcal{H}' \) is positive at every time \( t \geq T' \), and hence we have the following.

**Observation 4.3.** \(|\mathcal{M}| \geq 3\).

In the next lemma, we denote by \( \alpha(r,t) \), where \( r \in \mathcal{M} \) and \( t \geq T' \), the measure of the internal angle of \( \mathcal{H}'(t) \) at vertex \( r(t) \). Also, similarly to Section 3, we call the gray regions in Figures 3 and 4 the *move regions* of the respective robots.

**Lemma 4.4.** After time \( T' \), the following statements hold.

a. *The cyclic order of the robots of \( \mathcal{M} \) around \( \mathcal{H}' \) is preserved.*

b. *The length of the shortest edge of \( \mathcal{H}'(t) \) does not decrease as \( t \) increases.*

c. *If \( r \in \mathcal{M} \) is not activated at time \( t \), then \( \alpha(r,t) \leq \alpha(r,t+1) \).*

**Proof.** Let \( t \geq T' \), and consider the polygon \( \mathcal{P} \) as defined in the proof of Lemma 4.3. By our assumptions on \( \mathcal{M} \), \( \mathcal{P} \) is a convex polygon, or else some robot would cease to occupy a vertex of \( \mathcal{H}' \). Hence \( \mathcal{P} = \mathcal{H}'(t+1) \) and, by Lemma 4.3, \( \mathcal{H}'(t) \subseteq \mathcal{H}'(t+1) \). Let \( r,s \in \mathcal{M} \) occupy two adjacent edges of \( \mathcal{H}' \) at time \( t \), and let \( \ell \) be the axis of the segment \( r(t)s(t) \). It follows from Algorithm 2 that \( \ell \) separates the move regions of \( r \) and \( s \) at time \( t \) (cf. Figure 4). This, paired with the fact that the move region of a robot of \( \mathcal{M} \) at time \( t \) does not intersect the interior of \( \mathcal{H}' \), implies (a).

Now, to prove (b), it is sufficient to note that, with the previous paragraph’s notation, the distance between the move regions of \( r \) and \( s \) at time \( t \) is precisely the distance between \( r(t) \) and \( s(t) \) (see Figure 4). Therefore, the segment \( r(t+1)s(t+1) \) is not shorter than \( r(t)s(t) \), implying that the length of the shortest edge of \( \mathcal{H}' \) cannot decrease.

For part (c), let \( q,r,s \in \mathcal{M} \) occupy three consecutive vertices of \( \mathcal{H}' \) at time \( t \geq T' \) (in this order, as in Figure 4), and suppose that \( r \) is not activated, that is, \( r(t) = r(t+1) \). By definition of \( \mathcal{M} \), and due to part (a), \( q,r,s \) occupy three consecutive vertices of \( \mathcal{H}' \) also at time \( t+1 \), and they appear in the same order. Moreover, by Lemma 4.3, \( \mathcal{H}'(t) \subseteq \mathcal{H}'(t+1) \), implying that the internal angle of \( \mathcal{H}' \) at \( r \)'s location at time \( t \) must be contained in the corresponding angle at time \( t+1 \), and in particular \( \alpha(r,t) \leq \alpha(r,t+1) \).
We are finally ready to prove the correctness of the interior depletion phase.

**Lemma 4.5.** If Algorithm 2 is executed from a non-collinear configuration, after finitely many turns all robots become external and none of them collide.

**Proof.** The non-collision part has already been proven in Lemma 4.2, so we need to prove that eventually all robots become external.

By assumption $\mathcal{H}$ has positive area, and we have to show that all robots become external in finitely many turns. If there is just one internal robot, it keeps moving somewhere within $\mathcal{H}$, until it either becomes external, or all external robots have been activated. When all external robots have their lights set to *External*, if there is still a single internal robot, it keeps moving toward the same edge of $\mathcal{H}$, until it finally reaches it.

Suppose that there is more than one internal robot, but $\mathcal{H}'$ has null area, that is, all the internal robots are collinear. Then, according to Algorithm 2, only the two endpoint robots of $\mathcal{H}'$ are allowed to move, and they move in roughly opposite directions, as Figure 3(a) shows. If they move in such a way that the internal robots keep remaining collinear, eventually one of them reaches the boundary of $\mathcal{H}$, and there is one less internal robot. Otherwise, they reach a situation in which $\mathcal{H}'$ has strictly positive area.

Therefore we may assume that $\mathcal{H}'$ has positive area, and we suppose for a contradiction that some internal robots never become external. By the previous lemmas and observations, we know that at some time $T'$ the situation becomes “stable”, in that $\mathcal{H}$ never changes, the robots that are vertices of $\mathcal{H}'$ keep remaining the same, and the area of $\mathcal{H}'$ does not decrease. Let $a(t)$ be the length of the shortest edge of $\mathcal{H}'(t)$. By Lemma 4.4.b, we know that $a(t)$ is a weakly increasing function of $t \geq T'$. Moreover, by Observation 4.3, $\mathcal{H}'$ has at least three vertices after time $T'$. Let $\tilde{r} \in \mathcal{M}$ be such that $\alpha(\tilde{r}, T')$ is maximum. If $|\mathcal{M}| = k \geq 3$, then

$$\sum_{r \in \mathcal{M}} \alpha(r, T') = \pi(k - 2),$$

implying that

$$\alpha(\tilde{r}, T') \geq \frac{\pi(k - 2)}{k} \geq \frac{\pi}{3}.$$

Let $T'' \geq T'$ be the first time (after $T'$) at which $\tilde{r}$ is activated. By Lemma 4.4.c, $\alpha(\tilde{r}, T'') \geq \alpha(\tilde{r}, T') \geq \pi/3$. Let $q, s \in \mathcal{M}$ occupy the two
vertices of $H'(T'')$ adjacent to vertex $\tilde{r}(T'')$, such that

$$\frac{\pi}{2} < \angle q(T'')\tilde{r}(T'')\tilde{r}(T''+1) \leq \angle\tilde{r}(T''+1)\tilde{r}(T'')s(T'') \leq \frac{2\pi - \alpha(\tilde{r}, T'')}{2} \leq \frac{5}{6}\pi,$$

as Figure 5 shows. Then, recalling that $H'(T'') \subseteq H'(T''+1)$, we have that the area of $H'$ increases at least by the area of the triangle $q(T'')\tilde{r}(T'')\tilde{r}(T''+1)$, which in turn is at least

$$\|q(T'') - r(T'')\| \cdot \|r(T'' + 1) - r(T'')\| \cdot \sin(\angle q(T'')\tilde{r}(T'')\tilde{r}(T'' + 1)) \geq a(T'') \cdot \delta \cdot \sin(5\pi/6) \geq a(T') \cdot \delta/4.$$

Concluding, the area of $H'$ increases by at least $a(T') \cdot \delta/4$ between times $T'$ and $T''$. By repeating the same argument, there is a time $T''' > T''$ by which the area of $H'$ has increased by at least another $a(T') \cdot \delta/4$, and so on. Since $a(T') \cdot \delta/4$ is a positive constant, it follows that the area of $H'$ grows unboundedly, contradicting the fact that $H'(t) \subseteq H(t)$, and $H(t)$ is independent of $t$. □

4.2.2. Vertex Adjustments Phase

Due to Lemma 4.5, all robots become external at some point, and it remains to show that they finally reach a strictly convex configuration and correctly terminate. This turns out to be a significantly easier task.
Lemma 4.6. If Algorithm 2 is executed from a configuration in which all robots are external, then after finitely many turns all robots have terminated, none have collided, and the configuration is strictly convex.

Proof. In the vertex adjustments phase, each robot eventually sets its own light to 
*External* (if not already done in the interior depletion phase). Then, when a vertex robot sees only robots whose lights are set to *External* as well, it sets its light to *Adjusting* and makes a default move, as in Figure 1(b). Recall that robots are *NON-RIGID*, hence a vertex robot may be stopped before reaching its destination, but not before having moved by at least $\delta > 0$. When the robot is activated again, it sees itself in the *Adjusting* state, and it finally terminates, after setting its light back to *External*, thus allowing other robots to move.

As observed in Section 3, where a similar procedure was used to reduce the size of the convex hull while increasing the set of vertex robots, when a robot occupying a vertex of the convex hull moves, it becomes a vertex of the new convex hull. On the other hand, if a robot $r$ lies in the interior of an edge of the convex hull and one of its two neighbors $s$ lies on a vertex, then, as soon as $s$ moves, $r$ becomes a vertex of the new convex hull. Hence, eventually all robots become vertices of the convex hull. After that, whenever a robot is activated, it permanently sets its light to *External* and terminates. It follows that eventually all robots terminate in a strictly convex configuration.

Moreover, by the observations already made in Section 3.2.1 and referring to Figure 2, it is clear that no collisions can occur in this phase.

4.2.3. Collinear Case

From Lemmas 4.5 and 4.6, the correctness of Algorithm 2 immediately follows, provided that the robots are not initially collinear. This case is considered in the following.

Theorem 4.1. Algorithm 2 solves Mutual Visibility for *NON-RIGID* SSYNCH robots with 3-colored lights.

Proof. Due to Lemmas 4.5 and 4.6, we only have to show that the case in which the robots are initially collinear correctly evolves into one of the other cases. If $n \leq 3$, this is easy to verify through a case analysis. So, let $n \geq 4$, and let robots $r$ and $s$ initially occupy the vertices of the line segment $H(0)$. Nothing happens until $r$ or $s$ is activated; then, at least one of them becomes *Adjusting* and moves orthogonal to $H$ by a small-enough amount (line 12),
i.e., at most the (initial) distance to its neighboring robot. This movement
length is chosen in such a way that, even if $r$ and $s$ move simultaneously
in opposite directions, both neighbors of $r$ and $s$ become vertex robots, as
Figure 6(b) suggests. After $r$ or $s$ has moved, some robots eventually become
External (line 28). Indeed, if both $r$ and $s$ move, some external robot sees two
Adjusting robots, and therefore it becomes External. If only $r$ moves, then
$s$ sees only three robots (including itself), and its corresponding convex hull
angle is acute, hence it becomes External (and vice versa). In the latter case,
only $s$ is allowed to become External, while the other robots wait. When a
robot is External, any robot that was Adjusting can see it, and hence becomes
External as well, and terminates.

(a) $r$ and $s$ move in the same direction

(b) $r$ and $s$ move in opposite directions

Figure 6: Evolutions of a collinear configuration
At this point the execution proceeds normally, except that there are one of two vertex robots that have already terminated, and we have to show that this does not prevent the others from forming a strictly convex configuration. Obviously the interior depletion phase causes no trouble and terminates correctly, but the vertex adjustments phase could “get stuck”. We will prove that this is not the case. Clearly, if only $r$ or only $s$ has terminated, all robots except one are able to move in the vertex adjustments phase, and therefore they all become vertices. If $r$ and $s$ initially move in the same direction, they become neighboring vertices, and all the other robots become consecutive external robots (Figure 6(a)). It is easy to see that in this case the external robots are still able to make adjusting movements in cascade, and become vertices. Finally, suppose that $r$ and $s$ initially move in opposite directions. As already noted, their two neighbors $r'$ and $s'$ become vertex robots, and hence all the robots between them become internal. Note that $r$ and $s$ do not lie at adjacent vertices of $H$, due to the presence $r'$ and $s'$ (Figure 6(b)). After the interior depletion phase, $r'$ and $s'$ are able to adjust, thus enabling all other external robots to become vertices, in cascade.

5. Solving Mutual Visibility for ASynch Robots

In this section we briefly touch on the ASynch model. In the Rigid case, we show that Algorithm 2 solves the Mutual Visibility problem. In the Non-Rigid case, we show how to solve Mutual Visibility assuming that the robots agree on the direction of one coordinate axis.

5.1. Rigid ASynch Robots

Algorithm 2 turns out to solve the Mutual Visibility problem for Rigid ASynch robots, as well. For the interior depletion phase, the collision avoidance proof gets slightly more complex, but termination is easier to prove. On the other hand, the vertex adjustments phase and the collinear case work basically in the same way.

First we state an equivalent of Lemma 4.1. The only difference is that, instead of a generic time $t \in \mathbb{N}$, now we have to consider a specific time $t \in \mathbb{R}$ at which a robot $r$ performs a Look. Also, instead of considering the position of $r$ at time $t + 1$, we consider the destination point computed after such a Look. After these changes, the proof of Lemma 4.1 works in the ASynch case as well, and therefore we have the following.
Lemma 5.1. Let Rigid ASynch robots execute Algorithm 2, and let $r$ and $s$ be two internal robots at time $t \in \mathbb{R}$. If $r$ executes a Look phase at time $t$, and the next destination point of $r$ is $p$, then 

$$(p - r(t)) \cdot (s(t) - r(t)) \leq 0.$$ 

The previous lemma can be used to prove that no collisions occur during the interior depletion phase.

Lemma 5.2. If Rigid ASynch robots execute Algorithm 2 from a non-collinear configuration, no collisions occur as long as there are internal robots.

Proof. Suppose for a contradiction that the internal robot $r$ performs a Look at time $t$, then robot $s$ performs a Look at time $t' \geq t$, and they collide at time $t'' \geq t'$, in $r(t'') = s(t'')$. We may further assume that this is the first collision between the two robots, and therefore $r(t) \neq s(t)$. Because the model is Rigid and each internal robot’s destination point is on the convex hull, it follows that each internal robot makes exactly one move and then becomes external. Therefore, $r(t)$, $r(t')$, $r(t'')$, and the destination point of $r$ are all collinear, and the same holds for $s$. Additionally, we have $s(t) = s(t')$ (see Figure 7).

![Figure 7: Two colliding internal robots](image-url)
By Lemma 5.1 applied to $s$ at time $t'$, and because $s(t'')$ lies between $s(t')$ and the destination point of $s$, we have

$$(s(t'') - s(t')) \cdot (r(t') - s(t')) \leq 0.$$ 

On the other hand, $\|s(t'') - s(t')\| \geq 0$, implying that

$$(s(t'') - s(t')) \cdot (s(t') - s(t'')) \leq 0.$$ 

By adding the two inequalities together, we obtain

$$(s(t'') - s(t')) \cdot (r(t') - s(t'')) \leq 0.$$ 

Recall that $s(t'') = r(t'')$ and that $r(t')$ lies between $r(t)$ and $r(t'')$, and therefore the last inequality implies

$$(s(t'') - s(t')) \cdot (r(t) - r(t')) \leq 0,$$

hence

$$(s(t'') - s(t')) \cdot (r(t) - s(t') + s(t') - r(t')) \leq 0,$$

and

$$(s(t'') - s(t')) \cdot (r(t) - s(t')) \leq (s(t'') - s(t')) \cdot (r(t') - s(t')).$$

But we already know that the right-hand side is not positive, hence so is the left-hand side:

$$(s(t'') - s(t')) \cdot (r(t) - s(t')) \leq 0.$$ 

Now, by Lemma 5.1 applied to $r$ at time $t$, and recalling that $r(t'')$ lies between $r(t)$ and the destination point of $r$, we have

$$(r(t'') - r(t)) \cdot (s(t) - r(t)) \leq 0.$$ 

If we add together the last two inequalities and we recall that $s(t') = s(t)$, we get

$$(r(t) - s(t)) \cdot (r(t) - r(t'') + s(t'') - s(t')) \leq 0.$$ 

Because $r(t'') = s(t'')$ and $s(t') = s(t)$, we finally obtain

$$(r(t) - s(t)) \cdot (r(t) - s(t)) \leq 0,$$

which is equivalent to $\|r(t) - s(t)\| \leq 0$, implying that $r(t) = s(t)$, a contradiction. $\square$
We can now prove that Algorithm 2 works also with Rigid ASynch robots.

**Theorem 5.1.** Algorithm 2 solves Mutual Visibility for Rigid ASynch robots with 3-colored lights.

**Proof.** In the interior depletion phase there can be no collisions, due to Lemma 5.1. Also, whenever an internal robot on the boundary of \( H' \) performs a Look phase, it then computes a destination point on the convex hull’s perimeter, and it eventually reaches it in one move. Therefore, in finite time all robots become external, and the interior depletion phase terminates.

When all robots are external, none of them moves unless it sees only robots set to External. This means that, in the vertex adjustments phase, a robot waits until it is sure that no robot is in the middle of a move (note that this holds also for robots that it cannot see). Hence the robots synchronize themselves, and we may pretend them to be SSynch, as opposed to ASynch. Then, the proof proceeds exactly as in Lemma 4.6.

In the case in which the robots are initially collinear, the proof follows the lines of Theorem 4.1, with a few minor complications. Indeed, note that, despite being ASynch, the robots manage to wait for each other and synchronize their actions. Suppose that one endpoint robot \( r \) becomes Adjusting and starts moving to its destination. Then, every robot is bound to wait for the other endpoint robot, \( s \), to take action. So, \( s \) could either become Adjusting as well and start moving (if it performed its Look before \( r \) started moving), or it could notice \( r \) and become External. If \( r \) and \( s \) are both Adjusting and moving asynchronously, some other robots eventually become External, but do not move yet. In particular, referring to Figure 6, at least robots \( r' \) and \( s' \) can become external in this phase. Notice that, if \( r \) and \( s \) move asynchronously in opposite directions (Figure 6(b)), \( r' \) and \( s' \) may switch between internal and external several times. However, as soon as they set their light to External, they do not set it back to Off even if they become internal again. Moreover, they will indeed be external when \( r \) and \( s \) reach their destination, due to line 12 of Algorithm 2 which prevents \( r \) and \( s \) from moving “too much”. Therefore the colors of \( r' \) and \( s' \) are eventually consistent, despite asynchrony. So, every robot waits for both \( r \) and \( s \) to see some External robots and thus become External themselves. Only then do other robots start moving (cf. line 22). As a consequence, we may once again pretend that the robots in this phase are SSynch, and the proof is completed as in Theorem 4.1.
5.2. Non-Rigid ASynch Robots with Agreement on One Axis

Unfortunately, for Non-Rigid ASynch robots, our correctness proof of the interior depletion phase of Algorithm 2 fails. Indeed, to prove collision avoidance, we used in a crucial way the fact that, if two internal robots are moving at the same time, then at most one of them saw the other robot in the middle of a move. This is true under the Non-Rigid SSynch model (obviously) and under the Rigid ASynch model (because each internal robot becomes external after only one move), but not under the Non-Rigid ASynch model. However, the vertex adjustments phase of Algorithm 2 works under all models, therefore we only need to replace the interior depletion phase and the collinear case.

With the additional requirement that all robots agree on one axis, there is an easy way to fix the interior depletion phase, which is illustrated in Figure 8. Say that the robots agree on the y axis, i.e., they agree on the “North” direction, but they may disagree on “East” and “West”. Then, if an internal robot sees that every robot that lies to the North is set to External (i.e., if its own y coordinate is maximum among the internal robots), it is eligible to move. If there is a row of several internal robots that are all eligible to move (as in Figure 8), then only the two endpoints are allowed to move, and the others wait. The left endpoint moves to the upper-left quadrant, and the right endpoint moves to the upper-right quadrant, and their destination points are on the convex hull, but not on locations already occupied by external robots. To guarantee termination, we make sure that each robot always moves to the same edge of the convex hull (for instance by always moving to the closest edge among those that it can reach), or we can make it always move straight to the North, as soon as there are no external robots in the way.

Also the protocol for the collinear case needs some modifications: indeed, referring to Figure 6(b), in which r and s move in opposite directions, it is no longer true that r' and s' will eventually be external robots when r and s stop (recall that robots are Non-Rigid now). Unfortunately though, r' and s' may become temporarily external while and r and s move, and thus they may (permanently) set themselves to External, which could lead to inconsistent behaviors. Once again, we can fix the protocol if the robots agree on the y axis: now, in the collinear case, an endpoint robot moves according to Algorithm 2 only if it has the maximum y coordinate. This makes only one endpoint move in most cases, which eliminates the aforementioned issue. Moreover, if both endpoints have the same y coordinate, they will both move...
North, thus forming a configuration like in Figure 6(a), which causes to trouble.

**Theorem 5.2.** The Mutual Visibility problem is solvable by Non-Rigid ASynch robots carrying 3-colored lights, provided that they agree on one axis.

**Proof.** We show that the above algorithm is correct. In the interior depletion phase, there can be no collisions, and each internal robot eventually reaches the convex hull. Indeed, suppose that initially there is a unique internal robot \( r \) with largest \( y \) coordinate. As soon as enough external robots have set themselves to External, \( r \) starts moving to the North, and no other robot moves. Eventually \( r \) becomes external without colliding with any robot.

If several internal robots have the largest \( y \) coordinate, as in Figure 8, the argument is similar. At most two robots can move at the same time, and they cannot collide because the difference of their \( x \) coordinates cannot decrease. After enough cycles, either they have reached the convex hull, or one of them has been “left behind” and is no longer eligible to move. Either way, at least one internal robot eventually becomes external.

Once an internal robot has become external, the same argument repeats for all other internal robots. Note that these “sub-phases” do not interfere with each other, because a new robot becomes eligible to move only after the previous eligible robots have stopped on the convex hull.
The moment the last internal robot becomes external, no robot is moving, and therefore the whole swarm correctly transitions to the vertex adjustments phase, which works exactly as described in Theorem 5.1 and Lemma 4.6.

If the robots are initially collinear, they correctly transition to a non-collinear configuration, as in Theorem 5.1. Indeed, note that the two endpoint robots cannot move in opposite directions (as in Figure 6(b)), and hence it does not matter if they are RIGID or NON-RIGID, since in this case it is not harmful if they move by smaller amounts than those indicated by Algorithm 2 (cf. Figure 6(a)). The same clearly holds if \( n = 3 \) and the middle robot executes line 12.

\[ \square \]

6. Related Problems and Alternative Models

Here we discuss some applications of the previous Mutual Visibility algorithms to other problems, and we also discuss different robot models.

6.1. Forming a Convex Configuration

As already observed, Algorithm 1 also solves the Convex Formation problem, where the robots have to terminate in a strictly convex position. Moreover, no robot ever crosses the perimeter of the initial convex hull unless, of course, all the robots are initially collinear. This works for RIGID SSYNCH robots carrying 2-colored lights.

For NON-RIGID SSYNCH robots carrying 3-colored lights, Algorithm 2 also solves the Convex Formation problem, but it has an additional property: during the interior depletion phase, the convex hull of the robots remains unaltered (unless all robots are collinear), and in the vertex adjustments phase it shrinks a little, due to the small movements of the vertices. We can actually make these movements as small as we want, by changing line 26 of Algorithm 2 into

\[
\text{Move to } (a + b) \cdot \frac{\varepsilon}{\|a + b\|},
\]

where \( \varepsilon \) is an arbitrarily chosen positive constant. Similarly, in lines 12 and 16 we can make the robot move orthogonal to \( \mathcal{H} \) by \( \varepsilon \) or less. As a result, we can guarantee that the robots will terminate in a (strictly convex) configuration whose vertices are contained in an \( \varepsilon \)-wide band around the initial convex hull’s boundary.

Similar observations hold for the algorithms and models discussed in Section 5.
6.2. Forming a Circle

As a followup to Algorithms 1 and 2, the robots can even solve the Circle Formation problem, in which they have to become concircular and then terminate. Moreover, if they are Rigid SSynch (respectively, Non-Rigid SSynch), they can do so with the same 2-colored (respectively, 3-colored) lights that they used to solve Mutual Visibility.

First, it is necessary to slightly modify the termination condition of the algorithms: in Algorithm 1 when a robot sees only robots set to Vertex, it does not terminate, but it starts executing a circle formation phase. Similarly, in Algorithm 2, we remove lines 19–21, thus preventing vertex robots from reverting their color to External and terminating after they have adjusted their position. Instead, they wait until they only see robots set to Adjusting. Accordingly, in lines 22 and 33 we remove the conditions that prevent robots from moving if they see other robots set to Adjusting. Since we are assuming that robots are SSynch, it is straightforward to see that the correctness proof of Section 4 goes through even after these modifications to the protocol, and eventually all robots are set to Adjusting. At this point, the circle formation phase starts.

Since all robots are set to Adjusting, each robot knows that all of them occupy non-degenerate vertices of the convex hull, and that there are no other robots in the swarm. Hence the phase starts in a strictly convex configuration, and all the robots see each other. In particular, the Smallest Enclosing Circle (SEC) computed by each active robot is the same. In the circle formation phase, the robots move toward the perimeter of the SEC in a precise order, as illustrated in Figure 8. If a robot lies in p, which is not on the SEC, and one of its neighbors lies in s, which is on the SEC, then the robot in p moves along the extension of the edge of the convex hull that is incident to p and not to s. If both neighbors of the robot lie on the SEC (as with the robot in q in Figure 8), it chooses one of its two incident edges, and moves along the extension of that edge.

It is clear that the combined motion of the robots does not cause collisions or obstructions, and that the SEC is always preserved. Indeed, any robot that is already on the SEC remains still, and those that are inside the SEC are allowed to move only within the SEC itself. Moreover, the direction in which each robot moves is preserved until one of them reaches the SEC. Hence, even if robots are Non-Rigid, after finitely many turns at least one of them reaches the SEC, and therefore eventually they all reach the SEC. At this point, they correctly terminate.
The same circle formation phase can also be used in conjunction with the algorithms discussed in Section 5 for ASYNCH robots. The only difference is that, instead of modifying the ASYNCH algorithms like we did with the SSYNCH ones, we simply add an extra state, called Done, to synchronize robots and make them transition correctly from the vertex adjustments phase and the circle formation phase. That is, instead of terminating, a robot sets itself to Done, and then waits until all other robots are set to Done, as well. Only then does it proceed to executing the circle formation phase described above. This works with both RIGID and NON-RIGID ASYNCH robots carrying 4-colored lights.

6.3. Converging to a Point Without Colliding

A simple modification of Algorithm 1 solves the Near-Gathering problem, which requires all the robots to converge to a point without colliding: it is sufficient to remove lines 8, 9, and 23, that is, all the operations involving colors, and the termination condition. Indeed, if there is only one internal robot, either it will become external, or the other robots will converge to its location. On the other hand, if all robots become external, the convex
hull will keep shrinking until its vertices converge to a point. This works for Rigid SSynch robots, even without the use of colored lights.

However, if the robots carry 2-colored lights, they can also terminate when they get close enough to one another. This is done by simply modifying the termination condition of line 9:

\[
\text{if } \forall r, s \in \mathcal{V}, r.\text{light} = \text{Vertex} \land \|r.\text{position} - s.\text{position}\| < \varepsilon \text{ then } \text{Terminate}
\]

where \(\varepsilon\) is any given positive constant.

6.4. Non-Rigid SSynch Robots with Knowledge of \(\delta\)

Suppose that the robots are Non-Rigid SSynch, and as such they can be stopped by the scheduler at each turn before they reach their destination point, but not before they have moved by at least \(\delta\). Recall that in this case they have an algorithm for Mutual Visibility that uses 3-colored lights, described in Section 4. However, if the robots know the exact value of \(\delta\) and they can use it in their computations, they can solve Mutual Visibility even with 2-colored lights, by executing a slightly modified version of Algorithm 1.

If all the robots are initially collinear, Algorithm 1 makes them reach a non-collinear configuration, even if they are Non-Rigid. Otherwise, the invariants discussed in Section 3.2.1 keep holding, and in particular the convex hull of the robots never grows, and vertex robots never become non-vertex robots. We have to show that a version of Lemma 3.1 can be obtained for this model, as well. Referring to Figure 1, we can make the robot in \(p\) move toward \((a + b)/2\) by a smaller amount, never passing internal robots, and never colliding with them, unless they are closer than \(\delta\). If an internal robot \(r\) is closer than \(\delta\) and it stands between \(p\) and \((a + b)/2\), the robot in \(p\) moves close enough to \(r\), on the line parallel to \(ab\), and it sets its light to the correct value (note that it knows before moving whether it will end up being a vertex robot or not). This “lateral move” cannot be stopped by the scheduler, and it is guaranteed to create a new external robot, and eventually increase by one the number of vertex robots.

On the other hand, if only “non-lateral moves” are made, the analysis in Section 3.2.2 can be generalized, because Equation 1 takes the form

\[
r_i(t + 1) = \frac{\mu}{2} \cdot r_{i-1}(t) + \mu \cdot r_i(t) + \frac{\mu}{2} \cdot r_{i+1}(t),
\]

where \(\mu \in [\mu_0, 1/2]\), and \(\mu_0\) is a constant. Indeed, if the convex hull of the robots never grows, and its initial diameter is \(d\), then each moving robot
is guaranteed to move by at least a fraction of $\mu_0 = \delta / d$ of its computed movement vector. Therefore, all the lemmas in Section 3.2.2 can be reproved by merely adjusting some coefficients in the formulas.

It remains to prove that, if only one internal robot is left, it eventually reaches the boundary of the convex hull without colliding with other robots. But since $\delta$ is known, we can make this robot stay still until it either becomes external (due to other robots’ movements), or the diameter of the convex hull becomes smaller than $\delta$. As soon as it is guaranteed to make a reliable move, it can reach the midpoint of an edge of the convex hull, and therefore become external.

When all robots are external, they eventually reach a strictly convex configuration and they correctly terminate, as detailed in the proof of Theorem 3.1.

6.5. Trading Lights with the Knowledge of $n$

Suppose that the robots do not carry visible lights and have no internal memory, but they share the knowledge of the total number of robots in the swarm, $n$. If the robots are Rigid SSynch, it is possible to slightly modify Algorithm 1 to solve Mutual Visibility in this model, as well.

Note that the information given by other robots’ visible lights is used only when a robot has to terminate (line 9), or when it is the only internal robot and it has to move to the perimeter of the convex hull (line 23). However, both these situations can be recognized locally by counting the vertices of the convex hull of the visible robots: if it has $n$ non-degenerate vertices, Mutual Visibility has been solved, and the executing robot can terminate. If the convex hull has $n - 1$ vertices and the executing robot is internal, it moves to the boundary, as in line 23 of Algorithm 1.

The same techniques can be used to modify the algorithm of Section 6.4, so that Non-Rigid SSynch robots with knowledge of $\delta$ and knowledge of $n$ can solve Mutual Visibility without the use of colored lights.

We are also able to optimize Algorithm 2 for robots with knowledge of $n$: namely, we can achieve termination detection even if the robots do not use the Adjusting color, as follows. When all robots are external and a vertex robot makes a default move, it does not change its color, but remains External. Then, when a vertex robot sees $n$ robots, it terminates. Note that making a default move allows a robot to see all other robots at its next activation, and therefore each external robot makes at most one default move before terminating. Moreover, when all robots are collinear, we apply this simple
protocol: if a robot is an endpoint of the convex hull, it moves orthogonal to it, otherwise it stays still. This way, as soon as an endpoint is activated, the configuration becomes non-collinear. The only exception to this rule is the case $n = 3$, in which the central robot has to move orthogonal to the segment, while the other two robots stay still.

This technique allows NON-RIGID SSYNCH robots with knowledge of $n$ to solve Mutual Visibility with 2 colors as opposed to 3.

6.6. Fault Tolerance

Observe that Lemma $3.2$ requires only $n - 1$ valves to be opened infinitely often, as opposed to $n$. This implies that, if RIGID SSYNCH robots execute the modification of Algorithm $1$ described in Section $6.3$, they converge to a point even if one robot is unable to move. Therefore, in the presence of one faulty robot, Near-Gathering is still solvable, even without the use of colored lights. (On the other hand, it is easy to see that, if two robots are faulty, Near-Gathering is unsolvable.) Additionally, if $n$ is known, Mutual Visibility and Convex Formation are solved in the presence of a faulty robot, provided that it is located on the boundary of the convex hull.

6.7. Sequential Scheduler

Suppose that the scheduler is sequential, i.e., it is SSYNCH and it activates exactly one robot at each turn. In this very special model there is a simple algorithm to solve Mutual Visibility with no termination detection, even with no colored lights and no knowledge of $n$, and even if the robots are NON-RIGID and two of them are faulty. (If three robots are faulty, Mutual Visibility is clearly unsolvable.) When a robot is activated, it just moves by a small amount, without crossing or landing on any line that passes through two robots that it can currently see (including itself), as illustrated in Figure $10$. Clearly, when a robot moves as described, it becomes visible to all other robots. Hence, when all robots (except possibly two of them) have moved at least once, they can all see each other.

This protocol solves Mutual Visibility with no termination detection, in the sense that, after finitely many turns, the robots will keep seeing each other. However they will never stop moving because they will never know if their task is terminated or not. If we want to achieve termination detection, we need either 2-colored lights, or the knowledge of $n$. With 2-colored lights, a robot can change its own color the first time it moves, and terminate at
the next activation. With knowledge of $n$, a robot simply terminates when it sees $n$ robots.

7. Concluding Remarks

In this paper we initiated the investigation of the computational capabilities of a swarm of anonymous mobile robots with obstructed visibility: in this model, which has never been considered in the literature, two robots cannot see each other if a third robot lies between them. We focused on the basic problem of **Mutual Visibility**, in which the robots, starting from an arbitrary configuration, have to reach a configuration in which they all see each other, and stop moving. This task is clearly impossible if the robots are completely oblivious, unable to communicate, and do not have any additional information. Therefore we employed the extended model of luminous robots, in which each robot is carrying a visible light that it can set to different colors. The goal is then to minimize the number of colors required by the robots to solve the **Mutual Visibility** problem under different settings and restrictions. Namely, we considered SSYNCH and ASYNCH robots, and RIGID and NON-RIGID movements. We also discussed how to reduce the number of used colors if some information is given to the robots, such as the size of the swarm, $n$, or a minimum distance $\delta$ that a robot is guaranteed to cover in each movement. Our main results are summarized in Theorems 1.1 and 1.2.
We then touched on more complex problems, and proposed solutions that use our Mutual Visibility protocols as a preprocessing step. Notably, we gave the first algorithms for the Near Gathering problem (with fault tolerance) and the Circle Formation problem that work under the obstructed visibility model.

We proposed two main algorithms, and several modifications and adaptations to various models. Algorithm 1 (Shrink) uses 2 colors and makes the convex hull of the robots shrink, ideally converging to a point. Algorithm 2 (Contain) uses 3 colors, and keeps the initial convex hull basically unaltered. It is therefore suited for practical situations in which the robots have to surround a large-enough area, as well as solving Mutual Visibility. Also, both algorithms keep the robots strictly within the initial convex hull, which is useful in practice, for instance in the presence of hazardous areas around the swarm.

Some interesting research problems remain unsolved. We would like to reduce the number of colors used by our algorithms in the various models, or to prove our algorithms optimal. Our solutions to Mutual Visibility in some models use only 2 colors (or no lights at all if $n$ is known), which is clearly optimal. For other models, such as NON-RIGID SSYNCH and RIGID ASYNCH, we used 3 colors, and our question is whether this can be improved. We conjecture that Algorithm 1, which uses only 2 colors and has been designed and proven correct for RIGID SSYNCH robots, can be applied with no changes also to NON-RIGID SSYNCH robots (we could prove that 2 colors are sufficient in this model under the assumption that the robots know $\delta$). In the NON-RIGID ASYNCH setting we were only able to solve Mutual Visibility (with 3 colors) assuming that the robots agree on the direction of one coordinate axis. We ask if this assumption can be dropped, perhaps if more colors are used. Another question is whether Mutual Visibility can be solved deterministically without using colored lights, and without termination detection. That is, we allow the robots to move forever, but we require them to remain mutually visible from a certain time on. We proposed a simple solution that works under the very weak sequential scheduler, and we ask if this can be generalized to SSYNCH or even ASYNCH.

We emphasize that obstructed visibility represents a broad line of research in the field of computation by mobile robots, and this paper explored just a few directions. Several classical problems are worth studying under this model, such as the general Pattern Formation problem, Flocking, Scattering, with or without bounded visibility, etc.
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