Large Two-Loop Contributions to $g - 2$
from a Generic Pseudoscalar Boson

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Abstract

We calculate the dominant contributions to the muon $g - 2$ at the two-loop level
due to a generic pseudoscalar boson that may exist in any exotic Higgs sector in
most extensions of the standard model. The leading effect comes from diagrams
of the Barr-Zee type. A sufficiently light pseudoscalar Higgs boson can give rise
to contribution as large as the electroweak contribution which is measurable in the
next round of $g - 2$ experiment. Through the contribution we calculated here, the
anticipated improved data in the recent future on the muon $g - 2$ can put the best
limit on the possible existence of a light pseudoscalar boson in physics beyond the
standard model.

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Precision measurement of the anomalous magnetic moment of the muon, \( a_\mu \equiv \frac{1}{2} (g_\mu - 2) \), can provide not only a sensitive test of quantum loop effects in the electroweak standard model (SM), but can also probe the effects of some potential “new physics”. The experimental average in 1998 Particle Data Book gives \( a_\mu^{\exp} = 11659230(84) \times 10^{-10} (\pm 7.2 \text{ ppm}) \). Recent measurements by E821 experiment at Brookhaven give \( a_\mu^{\exp} = 11659250(150) \times 10^{-10} (\pm 13 \text{ ppm}) \) (1997 data) and \( a_\mu^{\exp} = 11659191(59) \times 10^{-10} (\pm 5 \text{ ppm}) \) (1998 data).

The E821 experiment has just announced the most accurate result from the 1999 data sample. The weighted mean measurement is

\[
a_\mu^{\exp} = 116592023(151) \times 10^{-11} (\pm 1.3 \text{ ppm}) .
\]

(1)

Its high precision poses a direct challenge to the theoretical prediction of \( a_\mu \), which are usually divided into four sources,

\[
a_\mu = a_\mu^{\text{QED}} + a_\mu^{\text{Hadronic}} + a_\mu^{\text{EW}} + \Delta a_\mu ,
\]

(2)

representing QED, hadronic, electroweak and the exotic (beyond the standard model) contributions respectively. The QED loop contributions have been computed to very high order \( a_\mu^{\text{QED}} = \frac{\alpha}{2\pi} + 0.765857376(27) \left( \frac{\alpha}{\pi} \right)^2 + 24.05050898(44) \left( \frac{\alpha}{\pi} \right)^3 \\
+ 126.07(41) \left( \frac{\alpha}{\pi} \right)^4 + 930(170) \left( \frac{\alpha}{\pi} \right)^5 .
\]

(3)

The most precise value for the fine structure constant \( \alpha = 1/137.03599958(52) \) can be obtained by inverting the similar formula for the electron \( g_e - 2 \) from the data. This gives

\[
a_\mu^{\text{QED}} = 116584705.7(2.9) \times 10^{-11} ,
\]

(4)

with precision much higher than the expected experimental reach. The SM electroweak contribution up to two-loop level gives \( a_\mu^{\text{EW}} = 152(4) \times 10^{-11} \) for \( \sin^2 \theta_W = 0.224 \) and \( M_H = 250 \text{ GeV} \) (In comparison, the one-loop SM electroweak contribution is \( 195 \times 10^{-11} \)). The hadronic contribution due to the hadronic vacuum polarization diagram has the largest uncertainty from the strong interaction, \( a_\mu^{\text{Hadronic}} = 6739(67) \times 10^{-11} \). However,
such uncertainty is still smaller than the experimental error in $a_{\mu}$, even before results from future planned experiments which intend to measure the hadronic vacuum polarization directly. The total value in standard model is\[9\],

$$a_{\mu}^{\text{SM}} = a_{\mu}^{\text{QED}} + a_{\mu}^{\text{Hadronic}} + a_{\mu}^{\text{EW}} = 116591597(67) \times 10^{-11} (\pm 0.57 \text{ ppm}).$$ \hspace{1cm} (5)

The current measurement is well above the standard model prediction by a $2.6\sigma$ effect, $\Delta a_{\mu} = (42.6 \pm 16.5) \times 10^{-10}$. Given that $a_{\mu}^{\text{Hadronic}}$ and $a_{\mu}^{\text{EW}}$ are both positive, one can conclude that the current data have already probed these contributions, but they are not enough to fit the data. Extra contributions from new physics is needed. At 90\%CL, $\Delta a_{\mu}$ lies\[9\] between $(+21.5 - 63.7) \times 10^{-10}$. Note that the negative $a_{\mu}^{\text{EW}}$ (two-loop) plays an important role in claiming this discrepancy.

Even without the recent experimental improvement, $g - 2$ data has already provided non-trivial constraints\[10\] on physics beyond the standard model. For example, the constraint on the minimal supersymmetric standard model (MSSM) due to its one-loop contribution to $g - 2$, via smuon-smuon-neutralino and chargino-chargino-sneutrino loops, is well known\[11\]. The resulting constraint depends on the masses of supersymmetric particles and $\tan \beta$.

In theories beyond the standard model, there are usually additional scalar or pseudoscalar bosons. In particular, some of the pseudoscalar bosons can potentially be light because of its pseudo-Goldstone nature, accidentally or otherwise. However, in collider search, it is known that searching for the pseudoscalar neutral boson is much harder than the scalar neutral or charged one. Therefore it is particularly interesting to see if one can constrain the pseudoscalar boson using low energy precision experiments. In this paper we wish to report that if the extended theory has a light enough pseudoscalar boson, its two-loop contribution to muon $g - 2$ can be as large as the one-loop electroweak effect. As a result the muon $g - 2$ can provide a very strong probe on a large class of theories beyond the standard model.

The one-loop contributions to $g - 2$ from a scalar or pseudoscalar boson have been presented many times in the literature\[12\]. Besides two powers of $m_{\mu}$ that is required by kinematics and definitions, the one-loop contribution is further suppressed by another two powers of $m_{\mu}/M_a$. However, the result is enhanced by a logarithmic loop factor, $\ln m_{\mu}/M_a$, coming from the diagram in which the photon is emitted by the internal muon. Therefore
for a light enough Higgs mass, some limit can be derived from \( g - 2 \) data just based on one-loop result. Nevertheless, as we shall see later, the two loop contribution is typically larger than the one-loop one by a factor of 2–10 for Higgs mass from 10 – 100 GeV. In addition, the one-loop and two-loop contributions have different signs for both scalar and pseudoscalar contributions. Therefore the one-loop contribution actually partially cancels the larger the two-loop contribution.

The two-loop contribution of a scalar boson has been calculated in Ref.\[\text{[6, 7]}\] in the context of the standard model. The contribution of any scalar boson beyond the standard model can in principle be extracted from that calculation and we shall not dwell on this here except to note that the scalar boson gives negative contribution while the pseudoscalar gives positive contribution to \( \Delta a_\mu \). Also, we have parameterized our input Lagrangian as model independent as possible in order to make our gauge invariant result widely applicable to a large class of models.

For Higgs mass larger than roughly 3 GeV, the dominant Higgs related contribution to the muon anomalous magnetic moment is through the two-loop Barr-Zee type diagram\[\text{[13]},\] as in Fig. 1. Compared with the one-loop graph, the Yukawa coupling of the heavy fermion \( f \) in the inner loop together with the mass insertion of the heavy fermion in the two-loop graph will give rise to \( (m_f/m_\mu)^2 \) enhancement which can overcome the extra loop suppression factor of \( \alpha/16\pi^2 \). The internal gauge boson can be a photon or a \( Z^0 \). The \( Z^0 \) contribution is typically smaller by two order of magnitude. It is included in this manuscript just for completeness. Note that unless CP violation occurs in the Higgs potential, there is no two-loop Barr-Zee type contribution to \( g_\mu - 2 \) associated with pseudoscalar boson and an inner gauge boson loop.
The dominant two-loop graph involving a pseudoscalar boson that contributes to $g_\ell - 2$. The cross location denotes a possible mass insertion.

The form of the gauge invariant vertex function $\Gamma^{\mu\nu}$ of a pseudoscalar boson $a^0$ of momentum $(p)$ turning into two photons ($-k, \nu$), $(q, \nu)$ due to the internal fermion or gauge boson loop is

$$\Gamma^{\mu\nu} = P(q^2)\epsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta.$$  \hspace{1cm} (6)

In general, the heavy fermion generation dominates in the loop. The Yukawa coupling is parameterized in a model independent expression,

$$L = ig_A f_m f^{\mu\nu} \bar{f} \gamma_5 f a^0.$$  \hspace{1cm} (7)

Integrating the fermion loop momentum, we obtain the form factor

$$P(q^2) = N_c^f g_A f^2 q^2 m_f^2 \int_0^1 \frac{dz}{m_f^2 - z(1-z)q^2},$$  \hspace{1cm} (8)

where $m_f$ and $q_f$ are the mass and the charge of the internal fermion in the loop. The color trace gives $N_c^b = N_c^l = 3$, $N_c^7 = 1$. The above vertex is further connected to the lepton propagator to produce anomalous magnetic dipole moment $a^{a^0}_\ell$ for the lepton $\ell$,

$$a^{a^0}_\ell = \frac{\alpha^2}{8\pi^2 \sin^2 \theta_W} \frac{m_f^2 A_f}{M_W^2} \sum_{f=t, b, \tau} N_c^f q_f^2 A_f \frac{m_f^2}{M_a^2} \mathcal{F} \left( \frac{m_f^2}{M_a^2} \right),$$  \hspace{1cm} (9)

$$\mathcal{F}(x) = \int_0^1 \frac{\ln \frac{x}{z(1-z)}}{x - z(1-z)}. \hspace{1cm} (10)$$

$\mathcal{F}(1) = \frac{4}{\sqrt{3}} \text{Cl}_2(\frac{\pi}{3})$, with the Clausen’s function $\text{Cl}_2(\theta) = -\int_0^\theta \ln \left(2 \sin \frac{x}{2}\right) d\theta$. As $x \gg 1$, $x \mathcal{F}(x)$ has the asymptotic form $2 + \ln x$. On the other extreme limit $x \ll 1$, $\mathcal{F}(x)$ approaches to $\frac{x^2}{3} + \ln^2 x$. Our result is consistent with that from an unphysical Higgs boson in SM\cite{7].

For the graph with the inner photon replaced by $Z^0$ boson, its contribution to $a_\mu$ can be calculated in a similar fashion,

$$a^{a^0}_\ell = \frac{\alpha^2 m_f^2 A_f g_f^\ell}{8\pi^2 \sin^4 \theta_W \cos^4 \theta_W M_Z^2} \sum_{f=t, b, \tau} N_c^f A_f q_f g_f^\ell m_f^2 \left[ \mathcal{F} \left( \frac{m_f^2}{M_Z^2} \right) - \mathcal{F} \left( \frac{m_f^2}{M_a^2} \right) \right],$$  \hspace{1cm} (11)

with $g_f^\ell = -\frac{1}{2} T_3(f_L) - q_f \sin^2 \theta_W$. Note that, for both pseudoscalar and scalar boson contributions, only the vector coupling of $Z^0$ to heavy fermion contributes to the effective vertex due to Furry theorem. Numerically, this $Z^0$ mediated contribution turns out to
be about two order of magnitude smaller than that of the photon mediated one. One suppression factor comes from the massive $Z^0$ propagator and the other one comes from the smallness of the leptonic vector coupling of $Z^0$ boson, which is proportional to $(-\frac{1}{4} + \sin^2 \theta_W) \sim -0.02$.

Taking the pattern of Yukawa couplings in MSSM as an example, we set $A_f$ as $\cot \beta$ ($\tan \beta$) for the $u$ (or $d$)-type fermion. The contributions due to top quark $t$, bottom quark $b$ and tau lepton $\tau$ in the loop respectively as well as the total are displayed in Fig. 2 for both $\tan \beta = 30$ and 50. In this MSSM pattern the $t$ contribution is insensitive to $\tan \beta$. In addition, both the $b$ and the $\tau$ contributions, which are roughly the same order of magnitude, dominate over that of the top quark one for large enough $\tan \beta$ and light enough pseudoscalar mass $M_a$. For $M_a \lesssim 15$ GeV, the $\tau$ contribution is larger than the $b$-quark contribution. The total two-loop photonic contribution from the pseudoscalar boson, $a^\gamma a^0$, can be as large as $10^{-8}$ for a large $\tan \beta$ when $M_a \leq 10$ GeV as shown in Fig. 2. For example, for $M_a = 10$ GeV and $\tan \beta = 50$, $a^\gamma a^0 (2\text{-loop}) = 1.2 \times 10^{-8}$, which is above the upper limit allowed by the current experiment bound. Generically, for $M_a \sim 80$–100 GeV, $\tan \beta \sim 50$, $a_\mu$ ranges in $(7 - 9) \times 10^{-10}$ which is close to the electroweak contribution. Note that the pseudoscalar contribution has the same sign as the hadronic or electroweak contributions. To derive constraint from the data one must combine the one- and two-loop contributions. The well-known one-loop contribution due to the pseudoscalar $a^0$ is

$$a^\ell_\mu (1\text{-loop}) = -\frac{m_\ell^2}{8\pi^2M_a^2} \left( \frac{gA_\ell m_\ell}{2M_W} \right)^2 H \left( \frac{m_\ell^2}{M_a^2} \right), \quad \text{with} \quad H(y) = \int_0^1 \frac{x^3dx}{1 - x + x^2y}. \quad (12)$$

For small $y$, $H(y) \to -\ln y - \frac{1}{6} > 0$. Note that the one-loop contribution is always negative in contrast to the two-loop contribution. In Fig. 2, we draw the absolute value of the one-loop contribution for easy comparison. For small $M_a$ such as 10 GeV, the one-loop contribution can be as large as half of the two loop contribution and produce cancelling effect in $g_\mu - 2$. Complete cancellation occurs around 3 GeV for large $\tan \beta$. However the one-loop effect becomes smaller for larger $M_a$ due to its additional suppression factor of $(m_\mu^2/M_a^2) \ln(M_a^2/m_\mu^2)$ and basically negligible for $M_a > 50$ GeV.

The up-to-dated measurement from E821 (1999) has already indicated 2.6 $\sigma$ deviation from the standard model. The allowed contribution from new physics falls into a small interval, $\Delta a_\mu$ between $(+21.5 - +63.7) \times 10^{-10}$ at 90%CL. The positive two-loop con-
tribution is able to fit the data for large $\tan\beta \sim 50$ and $M_a \lesssim 40$ GeV, as illustrated in Fig. 2. Note that for $M_a$ lighter than roughly 3 GeV, the negative one-loop contribution dominates and gives the overall negative $\Delta a_\mu$, which is disfavored by the current E821 data.

In CP conserving MSSM, there is a lower bound\cite{14, 15} on $M_a \geq 88$ GeV, which is only based on partial scanning with certain choices of benchmarks in the MSSM. Furthermore, in more general supersymmetric models or in general two or more Higgs doublet models\cite{16}, very little can be said about the potentially light pseudoscalar Higgs boson. Experimental constraint\cite{17} on $M_a$ from LEP data is correlated to a rather light scalar Higgs boson. The model independent nature of our calculation makes it possible to derive useful information of the pseudoscalar boson sector in any theory beyond the standard model using the hard earned data on muon $g - 2$.

Note that in general multi-Higgs doublet models, the $\tan\beta$ factor in our analysis may be supplemented by additional factor of mixing matrix elements. In addition, in any specific model, there may be additional two-loop contributions, such as the ones involving the physical charged Higgs boson or the neutral scalar boson. We assume that these contributions do not accidentally cancel each other. Given that the experimental limit on the masses of the charged Higgs boson as well as the neutral scalar boson are already quite high, it is very unlikely they will cancel the contribution of a relatively light pseudoscalar boson.

In conclusion, in this letter we report a set of analytic formulas for the two-loop contributions of a generic pseudoscalar boson to lepton anomalous magnetic moment. Such pseudoscalar bosons may exist in any theory beyond the standard model and they are typically harder to constrain using collider experimental data. In this paper, we show that strong constraint on such sector can be derived from the precision data on muon anomalous magnetic moment from the going and future experiments. We hope our work add importance and urgency to these low energy precision experiments.

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Note added: While the original manuscript of this work was in the review process, the E821 experiment announced its new measured value. Based on this non-trivial result, we have updated our Fig. 2, which shows the implication on the mass of the pseudoscalar Higgs boson.

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Fig. 2: The dotted lines plot the positive two-loop contributions from the inner $t,b,\tau$ loops to $g_\mu - 2$ due to the pseudoscalar $a^0$ versus $M_a$ at $\tan\beta = 50$, while the dashed-dotted line plots the negative contribution from the one-loop diagram. The sum in solid (dashed) curve shows cancellation at low $M_a$ mass for $\tan\beta = 50$ (30). The shaded areas shows the allowed contribution to $\Delta a_\mu$ from new physics at 90%CL based on E821 measurement (1999). The region of negative $\Delta a_\mu$, mainly from the one-loop contribution for a very light $M_a < 2.9$ GeV, is not favored by data.