Integrable bi-Hamiltonian systems by Jacobi structure on real three-dimensional Lie groups

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Abstract

By Poissonization of Jacobi structures on real three-dimensional Lie groups $\mathbf{G}$ and using the realizations of their Lie algebras, we obtain integrable bi-Hamiltonian systems on $\mathbf{G} \otimes \mathbb{R}$.

keywords: Poissonization, Jacobi manifold, Completely integrable Hamiltonian system.

1 Introduction

Let $(M^{2n}, \omega)$ be a symplectic manifold, and $\dot{x} = X_H(x)$ be a Hamiltonian system on it, where $H$ is a smooth function on $M^{2n}$ which is called the Hamiltonian and vector field $X_H(x) = \omega^{-1}(dH(x))$ is the corresponding Hamiltonian vector field. A completely Liouville-integrable Hamiltonian system is a Hamiltonian system with $n$ functionally independent first integrals in involution. In other words, two smooth functions $f$ and $g$ on $(M^{2n}, \omega)$ are in involution if their Poisson bracket equals zero \cite{1}. A symmetry of a Hamiltonian system $(M^{2n}, \omega, H)$ is a transformation $S : M^{2n} \rightarrow M^{2n}$ such that $S^\ast \omega = \omega$ and $S^\ast H = H$ where $S^\ast$ is the pullback of the symplectic form $\omega$ or $H$ by $S$. The set of all symmetries forms a group, which is called the symmetry group and can be a Lie group \cite{2}. One can construct a dynamical system for which the Lie group plays the role of the symmetry Lie group and the symplectic manifold plays the role of the phase space \cite{3}.

Jacobi manifolds which are a generalization of Poisson manifolds have various applications in physics and classical mechanics. Poisson manifolds play an important role as phase spaces of classical mechanics. In \cite{4}, we have classified all Jacobi structures on real three-dimensional Lie groups. In this work, using the Poissonization of the Jacobi structure $(\Lambda, E)$ on $M$ \cite{4}, we convert the Jacobi structure $(\Lambda, E)$ on $M$ into the Poisson structure on $M \times \mathbb{R}$ and we consider those Poisson structures that are non-degenerate, and so define symplectic structures. Then, applying Darboux’s theorem \cite{4} and using realizations \cite{6} of real three-dimensional Lie algebras $\mathfrak{g}$ (related Lie group $\mathbf{G}$), we construct integrable Hamiltonian systems for which the Lie group $\mathbf{G}$ plays the role of the symmetry Lie group and the symplectic manifold $\mathbf{G} \otimes \mathbb{R}$ plays the role of the phase space.

In mathematical physics and mechanics, many integrable dynamical systems admit the bi-Hamiltonian structure, that is Hamiltonian with respect to two compatible Poisson structures $P_1$ and $P_2$ \cite{7}. Here, we will calculate all equivalence-classes of Jacobi structures on real three-dimensional Lie groups $\mathbf{G}$ for which after Poissonization we have non-degenerate Poisson brackets on $M = \mathbf{G} \otimes \mathbb{R}$ and then we will study the existence of a bi-Hamiltonian structure for a completely integrable Hamiltonian system.

The outline of the paper is as follows: In Sec. 2, we briefly recall the Jacobi structures on real low-dimensional Lie groups and also the construction of the Liouville-integrable Hamiltonian system. In Sec. 3, we find integrable Hamiltonian systems such that their phase spaces are obtained by using Poissonization of the Jacobi structures on some real three-dimensional Lie groups. In Sec. 4, we study the existence of the bi-Hamiltonian structure for a completely integrable Hamiltonian system obtained in Sec. 3.
2 A review of the necessary constructions

For self-containing of the paper, we review the essential results about Jacobi structures on real low-dimensional Lie groups [1] and Integrable Hamiltonian systems \[2,3\].

2.1 Jacobi structures on real low-dimensional Lie groups

The study of the Jacobi manifolds was introduced by Lichnerowicz and Kirillov [4,5]. A Jacobi manifold \((M, \Lambda, E)\) is a manifold \(M\) admitting a bivector field \(\Lambda\) and a Reeb vector field \(E\) such that \([\Lambda, \Lambda] = 2E \wedge \Lambda, \ L_E \Lambda = [E, \Lambda] = 0\), where \([\ldots]\) stands for the Schouten-Nijenhuis bracket [6]. If \((M, \Lambda, E)\) is a Jacobi manifold, then the space \((C^\infty(M, \mathbb{R}), \{\ldots\}_\Lambda, E)\) becomes a local Lie algebra in the sense of Kirillov [7] with the following Jacobi bracket

\[
\{f, g\}_\Lambda, E = \Lambda(df, dg) + f \mathbf{E}g - g \mathbf{E}f, \quad \forall f, g \in C^\infty(M).
\]

This Lie bracket is a Poisson bracket if and only if the vector field \(E\) identically vanishes.

As shown by Lichnerowicz [4], to any Jacobi manifold \((M, \Lambda, E)\) one can associate a Poisson manifold \((M \otimes \mathbb{R}, P)\) with the Poisson bivector \(P\) as:

\[
P = e^{-s}(\Lambda + \partial_s \wedge E)
\]

where \(s\) is the coordinate on \(\mathbb{R}\). The Poisson manifold \((M \otimes \mathbb{R}, P)\) is said to be the Poissonization of the Jacobi manifold \((M, \Lambda, E)\).

Let \(x^\mu (\mu = 1, \ldots, \dim M)\) be the local coordinates chart of a Jacobi manifold \(M\), then the tensor field \(\Lambda\), the vector field \(E\) and the Jacobi bracket on \(M\) can be written as follows:

\[
\Lambda = \frac{\Lambda}{2}\Lambda^{\mu \nu} \partial_\mu \wedge \partial_\nu, \quad (3)
\]

\[
E = \mathbf{E}^\mu \partial_\mu, \quad (4)
\]

\[
\{f, g\}_{\Lambda, E} = \Lambda^{\mu \nu} \partial_\mu f \partial_\nu g + f \mathbf{E}^\mu \partial_\mu g - g \mathbf{E}^\mu \partial_\mu f, \quad \forall f, g \in C^\infty(M).
\]

Furthermore, by substituting the Jacobi bracket (5) in the Jacobi identity, one can obtain the following relations

\[
\Lambda^{\mu \nu} \partial_\mu \Lambda^{\lambda \nu} + \Lambda^{\mu \nu} \partial_\nu \Lambda^{\lambda \mu} + \Lambda^{\lambda \mu} \partial_\mu \Lambda^{\nu \mu} + \mathbf{E}^\lambda \Lambda^{\nu \lambda} + \mathbf{E}^\mu \Lambda^{\lambda \mu} = 0,
\]

\[
\mathbf{E}^{\mu} \partial_\mu \Lambda^{\nu \mu} - \Lambda^{\nu \mu} \partial_\mu \mathbf{E}^{\nu} + \mathbf{E}^{\rho} \partial_\rho \Lambda^{\mu \nu} = 0.
\]

The Eqs. (6) and (7) are called the Jacobi equations. The general solution for the Jacobi equations yields the general form of the Jacobi structures on a manifold \(M\) [10]. We have obtained the Jacobi structures on real three-dimensional Lie groups as a smooth manifold in [4]. To find these Jacobi structures, one must determine the vielbein \(e_a^\mu\) for the Lie groups, and for this, one must find the left-invariant one-form on the Lie group:

\[
g^{-1} dg = e_a^\mu X_\mu dx^\mu, \quad \forall g \in \mathbf{G}
\]

where \(\{X_\mu\}\) are generators of the Lie group. All left-invariant one-forms on real three-dimensional Lie groups were previously obtained in [11]. Therefore, one can compute the inverse of the vielbein \(e_a^\mu\) (i.e., \(e_a^\mu\) with \(e_\mu e_\nu = \delta_\nu^\mu, e_\mu e_\nu = \delta_\nu^\mu\) for Lie groups using left-invariant one-forms. Note that the elements of real three-dimensional Lie group \(\mathbf{G}\) are given by \(g = e^{x_1} e^{y_2} e^{z_3}\) for all \(g \in \mathbf{G}\), where \((x, y, z)\) is the local coordinate system on the Lie group \(\mathbf{G}\). The Jacobi structure \((\mathbf{G}, \Lambda, E)\) on the Lie group \(\mathbf{G}\) is written in terms of the non-coordinate basis [3] as

\[
\Lambda^{\mu \nu} = e_a^\nu e_b^\mu \Lambda^{ab},
\]

where \(\{X_a\}\) are generators of the Lie group.
\[ E^\mu = e_\mu^a E^a, \]  

where \( \Lambda^{ab} \) and \( E^a \) are Jacobi structures on Lie algebra \( g \) and we have assumed that these are independent of the coordinate of the Lie group, and the indices \( \mu, \nu, \cdots \) and \( a, b, \cdots \) are respectively related to the Lie group coordinates and the Lie algebra basis.

Taking into account that \( \hat{e}_a = e_\mu^a \partial_\mu \), then we have

\[ [\hat{e}_a, \hat{e}_b] = f^c_{ab} \hat{e}_c, \]

where \( f^c_{ab} \) (i.e., the structure constants of the Lie algebra \( g \)) are related to the vielbein \( e_\mu^a \) by the Maurer-Cartan relation

\[ \hat{e}_a = e_\mu^a (e_\nu^b \partial_\nu b - e_\nu^b \partial_\mu a) \].

Inserting Eqs. (9) and (10) into Eqs. (6) and (7) and using the Maurer-Cartan equation, one can obtain

\[ f_{bc} \Lambda^{hi} \Lambda^{jk} + f_{bd} \Lambda^{hi} \Lambda^{jk} + f_{da} \Lambda^{hi} \Lambda^{jk} + E^f \Lambda^{hi} E^g \Lambda^{jk} = 0, \]

or

\[ f_{ab} dE^a \Lambda^{ji} + f_{ab} cE^a \Lambda^{kd} = 0. \]

It is quite difficult to get results working with the tensor form of Eqs. (13) and (14); thus we propose using the adjoint representations of Lie algebras

\[ f^c_{ab} = -(\chi_a)^c_b, \quad f^c_{ac} = -(\chi_a)^c_b, \]

then the Eqs. (13) and (14) in the matrix form can be rewritten respectively as follows

\[ -(\chi_a)^c_b + \Lambda^c_j \Lambda^d_i = 0, \]

\[ \left( \Lambda^c_j \chi^d_i \right) E^a + E^f \Lambda^{ji} \Lambda^{kd} = 0. \]

The general solution of Eqs. (16) and (17) yields the general form of the Jacobi structures. In order to find general solutions of these equations, one can use the Maple program. Applying Eqs. (9) and (10), one can obtain the Jacobi structures \( \Lambda \) and \( E \) on the Lie group. In [1], we have obtained all Jacobi structures on three-dimensional Lie algebras and their Lie groups. Here, we will consider those structures such that after Poissonization of them (see relation (2)) the resulting Poisson structures are nondegenerate. The results are given in Table 1. We will see that only the Lie groups \( II \otimes \mathbb{R}, III \otimes \mathbb{R}, IV \otimes \mathbb{R}, VI_0 \otimes \mathbb{R}, VII_0 \otimes \mathbb{R} \) have nondegenerate Poisson structures (see Table 1).

### 2.2 Liouville-integrable Hamiltonian systems with symmetry Lie groups

Let \( (M^{2n}, \omega_{ij}) \) be a symplectic manifold, and let \( (x_1, \cdots, x_{2n}) \) be the local coordinates system on \( M^{2n} \) as a phase space. The relationship between the Poisson bracket on the space of smooth functions on \( M^{2n} \) and the symplectic form \( \omega_{ij} \) is given by \( \{f, g\} = P_{ij} \partial f/\partial x_i \partial g/\partial x_j \) where \( P_{ij} \) is the inverse of 2-form \( \omega_{ij} \).

**Theorem 2.1 (G.Darboux)** [1] For any point of a symplectic manifold \( (M^{2n}, \omega) \), there exists an open neighborhood possessing canonical coordinate \( (q_1, \cdots, q_n, p_1, \cdots, p_n) \) in which the symplectic structure \( \omega \) admits the canonical form \( \omega = \sum_{i=1}^{n} dq_i \wedge dp_i \). In other words, the canonicity condition for the symplectic structure can be rewritten in terms of the Poisson bracket as follows:

\[ \{p_i, p_j\} = 0, \quad \{q_i, q_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}, \quad i, j = 1, \cdots, n. \]

Given a dynamical system for which Lie group \( G \) plays the role of symmetry group and symplectic manifold \( M^{2n} \) plays the role of the phase space, one can construct independent dynamical functions \( S_i = S_i(q_1, \cdots, q_n, p_1, \cdots, p_n) \) on the phase space \( M^{2n} \) satisfying

\[ \{S_i, S_j\} = \sum_{n=1}^{2} \left( \frac{\partial S_i}{\partial q_n} \frac{\partial S_j}{\partial p_n} - \frac{\partial S_i}{\partial p_n} \frac{\partial S_j}{\partial q_n} \right) = f_{ij}^k S_k, \]

where \( f_{ij}^k \) are constants of the motion.
where \( f_{ij} \) stand for the structure constants of the Lie algebra \( g \) associated with the symmetry Lie group \( G \).

Using the relation (18), one can find the number of integrals \( S_i \) which commute with respect to the Poisson bracket related to the symplectic form. In other words, \( \{ S_i, S_j \} = 0, i, j = 1, \cdots, n; \) such that one of the functions \( S_i \) can be considered as a Hamiltonian of the integrable system [8].

In the following section, we will use Jacobi structures associated with real three-dimensional Lie groups \( G \) to construct the Poisson structure on \( G \otimes \mathbb{R} \) (Poissonization) and then will obtain related integrable Hamiltonian systems. We will perform those using of the differential realization of real three-dimensional Lie groups [6].

3 Integrable Hamiltonian systems by Jacobi structures on real three-dimensional Lie groups

In this section, we shall consider the different representations of one equivalence-class of Jacobi structures on real three-dimensional Lie groups \( G \) for which after Poissonization we have a non-degenerate Poisson bracket on \( G \otimes \mathbb{R} \) for each representation of equivalence-classes [8]. To simplify the presentation, we will discuss integrable Hamiltonian systems only for one representation of equivalence classes.

Example 3.1 Lie group \( \Pi \)

Considering the Lie group \( \Pi \) related to the Lie algebra \( \Pi \) with non-zero commutators \( [X_2, X_3] = X_1 \), it admits the Jacobi structure as follows (see Table 1):

\[
A_1 = -z\partial_x \wedge \partial_z + \partial_y \wedge \partial_z, \quad E_1 = -\partial_x
\]

where \((x, y, z)\) is the local coordinate system on the Lie group \( \Pi \). Applying the Poissonization (2) of the Jacobi manifold \((\Pi, A_1, E_1)\), it leads to the Poisson manifold \((\Pi \otimes \mathbb{R}, P_1)\) with the local coordinate system \((x, y, z, s)\) and the non-degenerate Poisson structure

\[
P_1 : \quad \{ x, z \} = -ze^{-s}, \quad \{ x, s \} = e^{-s}, \quad \{ y, z \} = e^{-s},
\]

Now one can find the following Darboux coordinates:

\[
q_1 = x, \quad q_2 = y, \quad p_1 = e^s, \quad p_2 = ze^s,
\]

such that they satisfy in the following canonical Poisson brackets:

\[
\{ q_1, p_1 \} = 1, \quad \{ q_2, p_2 \} = 1.
\]

We now consider that the Lie algebra \( \Pi \) is realized by means of smooth transformations on the phase space \( \mathbb{R}^4 \) with the canonical coordinate \((q_1, q_2, p_1, p_2)\)

\[
S_i = X_i(q_1, q_2, p_1, p_2),
\]

where the differential operator of \( p_1 = -\frac{\partial}{\partial q_1} \) and \( p_2 = -\frac{\partial}{\partial q_2} \) (quantum mechanical realization) are the conjugate momentums to \( q_1 \) and \( q_2 \), respectively. Using the results of [8] (see Table 2) one can get the \( S_i \) as follows:

\[
S_1 = -p_1 = -e^s, \quad S_2 = -p_2 = -ze^s, \quad S_3 = -q_2p_1 = -ye^s.
\]

In this way, now applying relation (18), one can show that they satisfy the following Poisson brackets

\[
\{ S_2, S_3 \} = S_1,
\]

i.e. we have the integrable Hamiltonian system with symmetry Lie group \( \Pi \) such that one can consider its Hamiltonian as:

\[
H = S_3 = -ye^s
\]

and the invariants of the system are \((H, S_1)\).

\(^4\)Note that previously in [1] we have obtained all equivalence classes of Jacobi structure on a three-dimensional Lie group. Here we use only one of those equivalence classes, because on other classes after Poissonization the obtained Poisson brackets are degenerate or singular [13].
Example 3.2 Lie group III

Now consider the Lie group III related to the Lie algebra III with non-zero commutators

\[ [X_1, X_2] = -(X_2 + X_3), \quad [X_1, X_3] = -(X_2 + X_3), \]

it admits the Jacobi structure as follows (see Table 1):

\[ \mathbf{A}_1 = \partial_x \wedge \partial_y + (y + z) \partial_y \wedge \partial_z \quad \mathbf{E}_1 = \partial_y - \partial_z \]

where \((x, y, z)\) is the local coordinate system on the Lie group III. Applying the Poissonization of the Jacobi manifold \((\text{III}, \mathbf{A}_1, \mathbf{E}_1)\), it leads to the Poisson manifold \((\text{III} \otimes \mathbb{R}, P_1)\) with the local coordinate system \((x, y, z, s)\) and the non-degenerate Poisson structure

\[
P_1: \quad \{x, z\} = e^{-s}, \quad \{y, z\} = e^{-s} (y + z), \quad \{y, s\} = -e^{-s}, \quad \{z, s\} = e^{-s}.
\]

Now one can find the following Darboux coordinates:

\[
q_1 = - e^s y, \quad q_2 = x, \quad p_1 = s, \quad p_2 = ze^s
\]
such that they satisfy in the following canonical Poisson brackets:

\[
\{q_1, p_1\} = 1, \quad \{q_2, p_2\} = 1.
\]

We now consider that the Lie algebra III is realized by means of smooth transformations on the phase space \(\mathbb{R}^4\) with the canonical coordinate \((q_1, q_2, p_1, p_2)\)

\[ S_i = X_i(q_1, q_2, p_1, p_2), \]

where the differential operator of \(p_1 = -\frac{\partial}{\partial q_1}\) and \(p_2 = -\frac{\partial}{\partial q_2}\) (quantum mechanical realization) are the conjugate momentums to \(q_1\) and \(q_2\), respectively. Using the results of Table 2 one can get the \(S_i\) as follows:

\[
S_1 = -(q_1 + q_2)(p_1 + p_2) = (e^s y - x)(s + z e^s), \quad S_2 = -p_1 = -s, \quad S_3 = -p_2 = -z e^s.
\]

In this way, now applying relation (18), one can show that they satisfy the following Poisson brackets

\[
\{S_1, S_2\} = -(S_2 + S_3), \quad \{S_1, S_3\} = -(S_2 + S_3),
\]

i.e. we have the integrable Hamiltonian system with symmetry Lie group III such that one can consider its Hamiltonian as:

\[ H = S_2 = -s \]

and the invariants of the system are \((H, S_3)\).

Example 3.3 Lie group IV

Considering the Lie group IV related to the Lie algebra IV with non-zero commutators

\[ [X_1, X_2] = -(X_2 - X_3), \quad [X_1, X_3] = -X_3, \]

it admits the Jacobi structure as follows (see Table 1):

\[ \mathbf{A}_1 = \partial_x \wedge \partial_y + (y - z) \partial_y \wedge \partial_z, \quad \mathbf{E}_1 = -\partial_y - \partial_z \]

where \((x, y, z)\) is the local coordinate system on the Lie group IV. Applying the Poissonization of the Jacobi manifold \((\text{IV}, \mathbf{A}_1, \mathbf{E}_1)\), it leads to the Poisson manifold \((\text{IV} \otimes \mathbb{R}, P_1)\) with the local coordinate system \((x, y, z, s)\) and the non-degenerate Poisson structure

\[
P_1: \quad \{x, y\} = e^{-s}, \quad \{y, z\} = (y - z)e^{-s}, \quad \{y, s\} = e^{-s}, \quad \{z, s\} = e^{-s}.
\]

Now one can find the following Darboux coordinates:

\[
q_1 = x, \quad q_2 = y, \quad p_1 = (y - z)e^s, \quad p_2 = e^s
\]
such that they satisfy in the following canonical Poisson brackets:

\[ \{q_1, p_1\} = 1, \quad \{q_2, p_2\} = 1. \]

We now consider that the Lie algebra \( VI \) is realized by means of smooth transformations on the phase space \( \mathbb{R}^4 \) with the canonical coordinate \( (q_1, q_2, p_1, p_2) \)

\[ S_i = X_i(q_1, q_2, p_1, p_2), \]

where the differential operator of \( p_1 = -\frac{\partial}{\partial q_1} \) and \( p_2 = -\frac{\partial}{\partial q_2} \) (quantum mechanical realization) are the conjugate momentums to \( q_1 \) and \( q_2 \), respectively. Using the results of [14] (see Table 2) one can get the \( S_i \) as follows:

\[ S_1 = q_1(q_2 - 1)p_1 + q_2^2 p_2 = x(y - 1)(y - z) e^s + y^2 e^s, \]

\[ S_2 = -p_1 = (-y + z) e^s, \quad S_3 = -q_2 p_1 = y (-y + z) e^s. \]

In this way, now applying relation [18], one can show that they satisfy the following Poisson brackets

\[ \{S_1, S_2\} = -(S_2 - S_3), \quad \{S_1, S_3\} = -S_3, \]
i.e. we have the integrable Hamiltonian system with symmetry Lie group \( IV \) such that one can consider its Hamiltonian as:

\[ H = S_2 = (-y + z) e^s \]

and the invariants of the system are \( (H, S_3) \).

**Example 3.4 Lie group \( VI_0 \)**

Considering the Lie group \( VI_0 \) related to the Lie algebra \( VI_0 \) with non-zero commutators

\[ [X_1, X_3] = X_2, \quad [X_2, X_3] = X_1, \]

it admits the Jacobi structure as follows (see Table 1):

\[ A_2 = -\sinh (z) \partial_z \wedge \partial_z + \cosh (z) \partial_y \wedge \partial_z, \quad E_2 = -\cosh (z) \partial_x + \sinh (z) \partial_y, \]

where \((x, y, z)\) is the local coordinate system on the Lie group \( VI_0 \). Applying the Poissonization [2] of the Jacobi manifold \((VI_0, A_2, E_2)\), it leads to the Poisson manifold \((VI_0 \otimes \mathbb{R}, P_2)\) with the local coordinate system \((x, y, z, s)\) and the non-degenerate Poisson structure

\[ P_2 : \quad \{x, z\} = -e^{-s} \sinh (z), \quad \{x, s\} = e^{-s} \cosh (z), \quad \{y, z\} = e^{-s} \cosh (z), \quad \{y, s\} = -e^{-s} \sinh (z). \]

Now one can find the following Darboux coordinates:

\[ q_1 = x, \quad q_2 = y, \quad p_1 = \cosh (z) e^s, \quad p_2 = \sinh (z) e^s \]
such that they satisfy in the following canonical Poisson brackets:

\[ \{q_1, p_1\} = 1, \quad \{q_2, p_2\} = 1. \]

We now consider that the Lie algebra \( VI_0 \) is realized by means of smooth transformations on the phase space \( \mathbb{R}^4 \) with the canonical coordinate \( (q_1, q_2, p_1, p_2) \)

\[ S_i = X_i(q_1, q_2, p_1, p_2), \]

where the differential operator of \( p_1 = -\frac{\partial}{\partial q_1} \) and \( p_2 = -\frac{\partial}{\partial q_2} \) (quantum mechanical realization) are the conjugate momentums to \( q_1 \) and \( q_2 \), respectively. Using the results of [14] (see Table 2) one can get the \( S_i \) as follows:

\[ S_1 = -p_1 = -\cosh (z) e^s, \quad S_2 = -p_2 = -\sinh (z) e^s \]
\[ S_3 = -q_2p_1 - q_1p_2 = -y \cosh (z) e^s - x \sinh (z) e^s. \]

In this way, now applying relation (18), one can show that they satisfy the following Poisson brackets

\[ \{ S_1, S_3 \} = S_2, \quad \{ S_2, S_3 \} = S_1, \]

i.e. we have the integrable Hamiltonian system with symmetry Lie group \( \text{VI}_0 \) such that one can consider its Hamiltonian as:

\[ H = S_1 = -\cosh (z) e^s \]

and the invariants of the system are \((H, S_2)\).

**Example 3.5 Lie group \( \text{VII}_0 \)**

Considering the Lie group \( \text{VII}_0 \) related to the Lie algebra \( \text{VII}_0 \) with non-zero commutators

\[ [X_1, X_3] = -X_2, [X_2, X_3] = X_1, \]

it admits the Jacobi structure as follows (see Table 1):

\[ \Lambda_1 = -\sin (z) \partial_x \wedge \partial_z + \cos (z) \partial_y \wedge \partial_z, \quad E_1 = -\cos (z) \partial_x - \sin (z) \partial_y, \]

where \((x, y, z)\) is the local coordinate system on the Lie group \( \text{VII}_0 \). Applying the Poissonization (2) of the Jacobi manifold \( \text{VII}_0 \cdot \Lambda_1, E_1 \), it leads to the Poisson manifold \( \text{VII}_0 \oplus \mathbb{R}, P_1 \) with the local coordinate system \((x, y, z, s)\) and the non-degenerate Poisson structure

\[ P_1 : \quad \{ x, z \} = -e^{-s} \sin (z), \quad \{ y, z \} = e^{-s} \cos (z), \quad \{ x, s \} = e^{-s} \cos (z), \quad \{ y, s \} = e^{-s} \sin (z). \]

Now one can find the following Darboux coordinates:

\[ q_1 = x, \quad q_2 = y, \quad p_1 = \cos (z) e^s, \quad p_2 = e^s \sin (z) \]

such that they satisfy in the following canonical Poisson brackets:

\[ \{ q_1, p_1 \} = 1, \quad \{ q_2, p_2 \} = 1. \]

We now consider that the Lie algebra \( \text{VII}_0 \) is realized by means of smooth transformations on the phase space \( \mathbb{R}^4 \) with the canonical coordinate \((q_1, q_2, p_1, p_2)\)

\[ S_i = X_i(q_1, q_2, p_1, p_2), \]

where the differential operator of \( p_1 = -\frac{\partial}{\partial q_1} \) and \( p_2 = -\frac{\partial}{\partial q_2} \) (quantum mechanical realization) are the conjugate momentums to \( q_1 \) and \( q_2 \), respectively. Using the result results of (8) (see Table 2) one can get the \( S_i \) as follows:

\[ S_1 = -p_1 = -\cos (z) e^s, \quad S_2 = -p_2 = -e^s \sin (z) \]

\[ S_3 = -q_2p_1 + q_1p_2 = -y \cos (z) e^s + xe^s \sin (z). \]

In this way, now applying relation (18), one can show that they satisfy the following Poisson brackets

\[ \{ S_1, S_3 \} = -S_2, \quad \{ S_2, S_3 \} = S_1, \]

i.e. we have the integrable Hamiltonian system with symmetry Lie group \( \text{VII}_0 \) such that one can consider its Hamiltonian as:

\[ H = S_1 = -\cosh (z) e^s \]

and the invariants of the system are \((H, S_2)\).
4 Bi-Hamiltonian systems by Jacobi structures on real three-dimensional Lie groups

The study of bi-Hamiltonian systems started with the pioneering work by Franco Magri.  

**Definition 4.1** A pair \((P_1, P_2)\) of Poisson structures on \(M\) is said to be compatible if \(^4\)

\[
[P_1, P_1] = [P_2, P_2] = [P_1, P_2] = 0, \tag{19}
\]

where \([\cdot, \cdot]\) is the Schouten–Nijenhuis bracket and the resulting bracket is the three vectors such that their components \([P_1, P_2]^{BCD}\) have the following forms: \(^5\)

\[
[P_1, P_2]^{BCD} = P_i^{AB} \partial_A P_j^{CD} + P_i^{AD} \partial_A P_j^{BC} + P_i^{AC} \partial_A P_j^{DB}. \tag{20}
\]

**Definition 4.2** The manifold \(M\) equipped with compatible Poisson structures \(P_1\) and \(P_2\) is called the bi-Hamiltonian manifold.

**Definition 4.3** A bi-Hamiltonian system is a dynamical system possessing two compatible Hamiltonian formulations.

**Theorem 4.4** Let \((\Lambda', E')\) and \((\Lambda, E)\) be two Jacobi structures. If there exists an automorphism \(A\) of the Lie algebra \(a\) such that

\[
\Lambda' = A^t \Lambda A, \tag{21}
\]

and

\[
E'^\rho = E^b A^\rho_b, \tag{22}
\]

then the Jacobi structures \((\Lambda', E')\) and \((\Lambda, E)\) are equivalent.

**Proof.** The proof is given in \([1]\). \(\blacksquare\)

**Lemma 4.5** Suppose that \((M \times \mathbb{R}, P_1)\) is the Poissonization of the Jacobi manifold \((M, \Lambda, E)\) and \((M \times \mathbb{R}, P_2)\) is the Poissonization of the Jacobi manifold \((M, \Lambda', E')\), and Jacobi structures \((\Lambda, E)\) and \((\Lambda', E')\) are equivalent. In the general case, structures \(P_1\) and \(P_2\) are not compatible Poisson structures.

**Proof.** By the definition of the Poissonization of the Jacobi manifold \((M, \Lambda, E)\), we have

\[
P_1 = e^{-s}(\Lambda + \partial_s \wedge E) = e^{-s} \Lambda^{\rho} \partial_\rho \wedge \partial_\lambda + e^{-s} E^\lambda \partial_s \wedge \partial_\lambda,
\]

such that \(P_1^{\rho\lambda} = e^{-s} A^{\rho\lambda} = e^{-s} e_i^\rho e_j^\lambda A^{ij}\) and \(P_1^{\lambda\lambda} = e^{-s} E^\lambda = e^{-s} e_i^\lambda E^k\). Moreover, by definition of the Poissonization of the Jacobi manifold \((M, \Lambda', E')\), and using \((21), (22)\), we have

\[
P_2 = e^{-s}(\Lambda' + \partial_s \wedge E') = e^{-s}(A'^t \Lambda A + \partial_s \wedge EA) = e^{-s}(A'^t a_b \Lambda^{bc} A^d_e e^\mu e^r_e\partial_\mu \wedge \partial_r + e^{-s} e^\mu e^k A^f_k \partial_\lambda \wedge \partial_\mu).
\]

Here we employ also the notation \(P_2^{\mu\nu} = e^{-s}(A'^t a_b \Lambda^{bc} A^d_e e^\mu e^r_e\partial_\mu \wedge \partial_r)\) and \(P_2^{\nu\mu} = e^{-s} e^\mu e^k A^f_k\). One can show that \([P_1, P_2]^{BCD} \neq 0\), for \(A = (s, \rho), B = \lambda, C = s, D = \mu\). \(\blacksquare\)

However, in the following, we will find examples where structures \(P_1\) and \(P_2\) are compatible. Note that these Poisson structures are obtained from the Poissonization of the Jacobi structures. Investigation of general Poisson structures on real four-dimensional Lie groups is previously studied in \([11]\).

**Example 4.6** Lie group \(\Pi\)

In Example 3.1, applying the Poissonization of the Jacobi manifold \((\Pi, A_1, E_1)\), we show that it leads to the Poisson manifold \((\Pi \otimes \mathbb{R}, P_1)\) with the local coordinate system \((x, y, z, s)\) and the non-degenerate Poisson structure:

\[
P_1: \quad \{x, z\}_1 = -ze^{-s}, \quad \{x, s\}_1 = e^{-s}, \quad \{y, z\}_1 = e^{-s}.
\]

\(^4\)Here we use Einstein’s summation convention.

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Now one can consider other representations of the same equivalence class of Jacobi structures on Lie group II (see Table 1) for which after Poissonization one can obtain the following non-degenerate Poisson structures \((P_2, P_3, P_4)\) on \(\mathbb{R}^2\):

\[
P_2 : \quad \{x, z\}_2 = (1 - z)e^{-s}, \quad \{x, s\}_2 = e^{-s}, \quad \{y, z\}_2 = e^{-s}, \\
P_3 : \quad \{x, y\}_3 = e^{-s}, \quad \{x, z\}_3 = -ze^{-s}, \quad \{x, s\}_3 = e^{-s}, \quad \{y, z\}_3 = e^{-s}, \\
P_4 : \quad \{x, y\}_4 = e^{-s}, \quad \{x, z\}_4 = (1 - z)e^{-s}, \quad \{x, s\}_4 = e^{-s}, \quad \{y, z\}_4 = e^{-s},
\]

such that the above structures are compatible with each other:

\[
[P_i, P_j] = 0, \quad i, j = 1, 2, 3, 4.
\]

Note that there are other non-degenerate Poisson structures on Lie group II \(\otimes \mathbb{R}\). These structures can be calculated from relation (16) with \(E = 0\). After the simple calculations, one can obtain the following representation (see Appendix and Table 3)

\[
P'_1 = \partial_x \wedge \partial_s + (1 - z)\partial_e \wedge \partial_s + \partial_y \wedge \partial_s + \partial_z \wedge \partial_s
\]

all other compatible Poisson structures on II \(\otimes \mathbb{R}\) (i.e., \(P'_2, \ldots, P'_{12}\)) which are not compatible with \((P_1, P_2, P_3, P_4)\) are given in Table 3 of the Appendix.

**Example 4.7 Lie group III**

In Example 3.3, applying the Poissonization of the Jacobi manifold \((\text{III}, \mathbb{A}_1, \mathbb{E}_1)\), we show that it leads to the Poisson manifold \((\text{III} \otimes \mathbb{R}, P_1)\) with the local coordinate system \((x, y, z, s)\) and the non-degenerate Poisson structure:

\[
P_1 : \quad \{x, z\}_1 = e^{-s}, \quad \{y, z\}_1 = (y + z)e^{-s}, \quad \{y, s\}_1 = e^{-s}, \quad \{z, s\}_1 = e^{-s}.
\]

Now one can consider other representations of the same equivalence class of Jacobi structures on Lie group III (see Table 1) for which after Poissonization can obtain the following non-degenerate Poisson structures \((P_2, P_3, P_4)\) on III \(\otimes \mathbb{R}\):

\[
P_2 : \quad \{x, y\}_2 = e^{-s}, \quad \{y, z\}_2 = -(y + z)e^{-s}, \quad \{y, s\}_2 = e^{-s}, \quad \{z, s\}_2 = -e^{-s}, \\
P_3 : \quad \{x, z\}_3 = e^{-s}, \quad \{y, z\}_3 = (y + z + 1)e^{-s}, \quad \{y, s\}_3 = e^{-s}, \quad \{z, s\}_3 = e^{-s}, \\
P_4 : \quad \{x, y\}_4 = e^{-s}, \quad \{y, z\}_4 = -(y + z - 1)e^{-s}, \quad \{y, s\}_4 = e^{-s}, \quad \{z, s\}_4 = -e^{-s},
\]

such that the above structures are compatible with each other:

\[
[P_i, P_j] = 0, \quad i, j = 1, 2, 3, 4.
\]

**Example 4.8 Lie group IV**

In Example 3.3, applying the Poissonization of the Jacobi manifold \((\text{IV}, \mathbb{A}_1, \mathbb{E}_1)\), we show that it leads to the Poisson manifold \((\text{IV} \otimes \mathbb{R}, P_1)\) with the local coordinate system \((x, y, z, s)\) and the non-degenerate Poisson structure:

\[
P_1 : \quad \{x, y\}_1 = e^{-s}, \quad \{y, z\}_1 = (y - z)e^{-s}, \quad \{y, s\}_1 = e^{-s}, \quad \{z, s\}_1 = e^{-s}.
\]

Now one can consider other representations of the same equivalence class of Jacobi structures on Lie group IV (see Table 1) for which after Poissonization one can obtain the following non-degenerate Poisson structures \((P_2, P_3, P_4)\) on IV \(\otimes \mathbb{R}\):

\[
P_2 : \quad \{x, y\}_2 = e^{-s}, \quad \{y, z\}_2 = e^{-s}, \quad \{y, s\}_2 = e^{-s}, \quad \{z, s\}_2 = e^{-s}, \\
P_3 : \quad \{x, y\}_3 = e^{-s}, \quad \{x, z\}_3 = e^{-s}, \quad \{y, z\}_3 = (2y - z)e^{-s}, \quad \{y, s\}_3 = e^{-s}, \quad \{z, s\}_3 = 2e^{-s}, \\
P_4 : \quad \{x, y\}_4 = e^{-s}, \quad \{x, z\}_4 = e^{-s}, \quad \{y, z\}_4 = (2y - z + 1)e^{-s}, \quad \{y, s\}_4 = e^{-s}, \quad \{z, s\}_4 = 2e^{-s},
\]

such that the above structures are compatible with each other:

\[
[P_i, P_j] = 0, \quad i, j = 1, 2, 3, 4.
\]
Example 4.9 Lie group $\mathbf{VI}_0$

In Example 3.4, applying the Poissonization of the Jacobi manifold $(\mathbf{VI}_0, \mathbf{A}_2, \mathbf{E}_2)$, we show that it leads to the Poisson manifold $(\mathbf{VI}_0 \otimes \mathbb{R}, P_2)$ with the local coordinate system $(x, y, z, s)$ and the non-degenerate Poisson structure:

$P_2$ : $\{x, z\}_2 = -e^{-s}\sinh(z)$, $\{y, z\}_2 = -e^{-s}\cosh(z)$, $\{x, s\}_2 = e^{-s}\cosh(z)$, $\{y, s\}_2 = -e^{-s}\sinh(z)$.

Now one can consider other representations of the same equivalence class of Jacobi structures on Lie group $\mathbf{VI}_0$ (see Table 1) for which after Poissonization one can obtain the following non-degenerate Poisson structures $(P_1, P_3, P_4)$ on $\mathbf{VI}_0 \otimes \mathbb{R}$:

$P_1$ : $\{x, z\}_1 = e^{-s}\cosh(z)$, $\{y, z\}_1 = -e^{-s}\sinh(z)$, $\{x, s\}_1 = -e^{-s}\sinh(z)$, $\{y, s\}_1 = e^{-s}\cosh(z)$,

$P_3$ : $\{x, y\}_3 = -e^{-s}$, $\{x, z\}_3 = -e^{-s}\cosh(z)$, $\{y, z\}_3 = -e^{-s}\sinh(z)$, $\{y, s\}_3 = e^{-s}\cosh(z)$,

$P_4$ : $\{x, y\}_4 = e^{-s}$, $\{x, z\}_4 = -e^{-s}\sinh(z)$, $\{y, z\}_4 = e^{-s}\cosh(z)$, $\{y, s\}_4 = -e^{-s}\sinh(z)$,

such that the above structures are compatible with each other:

$[P_i, P_j] = 0, \quad i, j = 1, 2, 3, 4.$

Example 4.10 Lie group $\mathbf{VII}_0$

In Example 3.5, applying the Poissonization of the Jacobi manifold $(\mathbf{VII}_0, \mathbf{A}_1, \mathbf{E}_1)$, we show that it leads to the Poisson manifold $(\mathbf{VII}_0 \otimes \mathbb{R}, P_1)$ with the local coordinate system $(x, y, z, s)$ and the non-degenerate Poisson structure:

$P_1$ : $\{x, z\}_1 = e^{-s}\sin(z)$, $\{y, z\}_1 = e^{-s}\cos(z)$, $\{x, s\}_1 = e^{-s}\cos(z)$, $\{y, s\}_1 = e^{-s}\sin(z)$.

Now one can consider other representations of the same equivalence class of Jacobi structures on Lie group $\mathbf{VII}_0$ (see Table 1) for which after Poissonization one can obtain the following non-degenerate Poisson structures $(P_2, P_3, P_4, P_5, P_6)$ on $\mathbf{VII}_0 \otimes \mathbb{R}$:

$P_2$ : $\{x, z\}_2 = e^{-s}\cos(z)$, $\{y, z\}_2 = e^{-s}\sin(z)$, $\{x, s\}_2 = -e^{-s}\sin(z)$, $\{y, s\}_2 = -e^{-s}\cos(z)$,

$P_3$ : $\{x, z\}_3 = e^{-s}(\cos(z) - \sin(z))$, $\{y, z\}_3 = e^{-s}(\cos(z) + \sin(z))$, $\{x, s\}_3 = -e^{-s}(\cos(z) + \sin(z))$, $\{y, s\}_3 = -e^{-s}(\cos(z) - \sin(z))$,

$P_4$ : $\{x, y\}_4 = -e^{-s}$, $\{x, z\}_4 = -e^{-s}\sin(z)$, $\{y, z\}_4 = e^{-s}\cos(z)$, $\{x, s\}_4 = e^{-s}\cos(z)$, $\{y, s\}_4 = e^{-s}\sin(z)$,

$P_5$ : $\{x, y\}_5 = e^{s}$, $\{x, z\}_5 = e^{s}\cos(z)$, $\{y, z\}_5 = -e^{s}\sin(z)$, $\{x, s\}_5 = -e^{s}\sin(z)$, $\{y, s\}_5 = -e^{s}\cos(z)$,

$P_6$ : $\{x, y\}_6 = e^{s}$, $\{x, z\}_6 = -e^{s}(\cos(z) - \sin(z))$, $\{y, z\}_6 = -e^{s}(\cos(z) + \sin(z))$, $\{x, s\}_6 = e^{s}(\cos(z) + \sin(z))$, $\{y, s\}_6 = -e^{s}(\cos(z) - \sin(z))$,

such that the above structures are compatible with each other:

$[P_i, P_j] = 0, \quad i, j = 1, 2, 3, 4, 5, 6.$
Table 1: Jacobi structures on real three-dimensional Lie algebras and Lie groups for which after Poissonization we have a non-degenerate Poisson brackets on $M = G \otimes \mathbb{R}$. 

| Jacobi structures on Lie algebra II | Representation of one equivalence class on Lie algebra II | Representation of one equivalence class on Lie group II | compatible Poisson structures on $M = \mathbb{II} \otimes \mathbb{R}$ |
|----------------------------------|----------------------------------|----------------------------------|----------------------------------|
| $\Lambda = \lambda_{12} \partial_x \wedge \partial_y + \lambda_{13} \partial_x \wedge \partial_z + \lambda_{23} \partial_y \wedge \partial_z$ | $\Lambda_1 = \partial_y \wedge \partial_z$ | $\Lambda_1 = -z \partial_z \wedge \partial_y + \partial_x \wedge \partial_y$ | $P_1 = -e^{-s} \partial_x \wedge \partial_z + e^{-s} \partial_y \wedge \partial_x$ |
| $E = -\lambda_{23} \partial_x$ | $E_1 = -\partial_x$ | $E_1 = -e^{-s} \partial_x \wedge \partial_z$ | $+ e^{-s} \partial_y \wedge \partial_z$ |
| Comment: $\lambda_{23} \neq 0$ | | | |

| Jacobi structures on Lie algebra III | Representation of one equivalence class on Lie algebra III | Representation of one equivalence class on Lie group III | compatible Poisson structures on $M = \mathbb{III} \otimes \mathbb{R}$ |
|----------------------------------|----------------------------------|----------------------------------|----------------------------------|
| $\Lambda = \lambda_{12} \partial_x \wedge \partial_y + \lambda_{13} \partial_x \wedge \partial_z + \lambda_{23} \partial_y \wedge \partial_z$ | $\Lambda_1 = \partial_x \wedge \partial_y$ | $\Lambda_1 = \partial_x \wedge \partial_z + (y + z) \partial_y \wedge \partial_z$ | $P_1 = e^{-s} \partial_x \wedge \partial_z + (y + z) e^{-s} \partial_y \wedge \partial_z$ |
| $E = (\lambda_{13} - \lambda_{12}) \partial_y - (\lambda_{13} - \lambda_{12}) \partial_z$ | $E_1 = \partial_y - \partial_z$ | $E_1 = \partial_y - \partial_z$ | $P_2 = e^{-s} \partial_x \wedge \partial_y - (y + z) e^{-s} \partial_y \wedge \partial_z$ |
| Comment: $\lambda_{12} \neq \pm \lambda_{13}$ | | | |
| Table 1: continue. | Representation of one equivalence class on Lie algebra IV | Representation of one equivalence class on Lie group IV | compatible Poisson structures on $M = \text{IV} \otimes \mathbb{R}$ |
|---------------------|------------------------------------------------------|----------------------------------------------------|--------------------------------------------------|
| Jacobi structures on Lie algebra IV | | | |
| $\Lambda = \lambda_1 \partial_x \wedge \partial_y + \lambda_2 \partial_x \wedge \partial_z + \lambda_3 \partial_y \wedge \partial_z$ | $\Lambda_1 = \partial_x \wedge \partial_y$ | $\Lambda_1 = \partial_x \wedge \partial_y + (y - z) \partial_y \wedge \partial_z$ | $P_1 = e^{-s} \partial_x \wedge \partial_y + (y - z) e^{-s} \partial_y \wedge \partial_z$ |
| $E = -\lambda_1 \partial_y - (\lambda_2 + \lambda_3) \partial_z$ | $E_1 = -\partial_y - \partial_z$ | $E_1 = -\partial_y - \partial_z$ | $e^{-s} \partial_y \wedge \partial_z + e^{-s} \partial_z \wedge \partial_z$ |
| Comment: $\lambda_{12} \neq 0$ | | | |
| $\Lambda_2 = \partial_x \wedge \partial_y + \partial_y \wedge \partial_z$ | | | |
| $E_2 = -\partial_y - \partial_z$ | | | |
| $\Lambda_3 = \partial_x \wedge \partial_y + \partial_y \wedge \partial_z$ | | | |
| $E_3 = -\partial_y - 2 \partial_z$ | | | |
| $\Lambda_4 = \partial_x \wedge \partial_y + \partial_y \wedge \partial_z + \partial_y \wedge \partial_z$ | | | |
| $E_4 = -\partial_y - 2 \partial_z$ | | | |
| Jacobi structures on Lie algebra VI | Representation of one equivalence class on Lie algebra VI0 | Representation of one equivalence class on Lie group VI0 | compatible Poisson structures on $M = \text{VI0} \otimes \mathbb{R}$ |
| | | | |
| $\Lambda = \lambda_1 \partial_x \wedge \partial_y + \lambda_2 \partial_x \wedge \partial_z + \lambda_3 \partial_y \wedge \partial_z$ | $\Lambda_1 = \partial_x \wedge \partial_z$ | $\Lambda_1 = \cosh(z) \partial_x \wedge \partial_z$ | $P_1 = e^{-s} \cosh(z) \partial_x \wedge \partial_z - e^{-s} \sinh(z) \partial_y \wedge \partial_z$ |
| $E = -\lambda_2 \partial_x - \lambda_3 \partial_y$ | $E_1 = -\partial_y$ | $\cosh(z) \partial_x \wedge \partial_z$ | $e^{-s} \sinh(z) \partial_x \wedge \partial_z + e^{-s} \cosh(z) \partial_y \wedge \partial_z$ |
| Comment: $\lambda_{13} \neq \pm \lambda_{23}$ | | | |
| $\Lambda_2 = \partial_y \wedge \partial_z$ | | | |
| $E_2 = -\partial_z$ | | | |
| $\Lambda_3 = \partial_x \wedge \partial_y + \partial_x \wedge \partial_z$ | | | |
| $E_3 = -\partial_y$ | | | |
| $\Lambda_4 = \partial_x \wedge \partial_y - \sinh(z) \partial_x \wedge \partial_z$ | | | |
| $E_4 = -\partial_x$ | | | |
Table 1: continue.

| Jacobi structures on Lie algebra $\mathfrak{vi}_0$ | Representation of one equivalence class on Lie algebra $\mathfrak{vi}_0$ | Representation of one equivalence class on Lie group $\mathfrak{vi}_0$ | compatible Poisson structures on $M = \mathfrak{vi}_0 \otimes \mathbb{R}$ |
|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|
| $\Lambda = \lambda_1 \partial_x \wedge \partial_y + \lambda_2 \partial_x \wedge \partial_z + \lambda_3 \partial_y \wedge \partial_z$ | $\Lambda_1 = \partial_y \wedge \partial_z$ | $\Lambda_1 = -\sin(z)\partial_x \wedge \partial_z$ | $P_1 = -e^{-s}\sin(z)\partial_x \wedge \partial_z + e^{-s}\cos(z)\partial_y \wedge \partial_z$ |
| $E = -\lambda_3 \partial_x + \lambda_3 \partial_y$ | $E_1 = \partial_x$ | $+\cos(z)\partial_y \wedge \partial_z$ | $+e^{-s}\cos(z)\partial_x \wedge \partial_z + e^{-s}\sin(z)\partial_y \wedge \partial_z$ |
| Comment: $\lambda_1^2 + \lambda_2^2 \neq 0$ | | $E_1 = -\cos(z)\partial_x - \sin(z)\partial_y$ | |
| $\Lambda_2 = \partial_x \wedge \partial_z$ | $E_2 = \partial_y$ | $\Lambda_2 = \cos(z)\partial_x \wedge \partial_y$ | $P_2 = +e^{-s}\cos(z)\partial_x \wedge \partial_z + e^{-s}\sin(z)\partial_y \wedge \partial_z$ |
| $E_3 = -\partial_x + \partial_y$ | $\Lambda_3 = (\cos(z) - \sin(z))\partial_x \wedge \partial_y$ | $+\sin(z)\partial_y \wedge \partial_z$ | $+e^{-s}\sin(z)\partial_x \wedge \partial_z - e^{-s}\cos(z)\partial_y \wedge \partial_z$ |
| | $\Lambda_3 = \cos(z)\partial_x \wedge \partial_y + (\sin(z) + \cos(z))\partial_y \wedge \partial_z$ | | |
| $\Lambda_4 = \partial_x \wedge \partial_y + \partial_y \wedge \partial_z$ | $E_4 = -\partial_x$ | $\Lambda_4 = \partial_x \wedge \partial_y - \sin(z)\partial_x \wedge \partial_z$ | $P_3 = +e^{-s}(\cos(z) - \sin(z))\partial_x \wedge \partial_z$ |
| $E_4 = \partial_x$ | $+\cos(z)\partial_y \wedge \partial_z$ | $+\sin(z)\partial_y \wedge \partial_z$ | $+e^{-s}(\sin(z) + \cos(z))\partial_y \wedge \partial_z$ |
| | $E_4 = -\cos(z)\partial_x - \sin(z)\partial_y$ | | $-e^{-s}(\cos(z) - \sin(z))\partial_y \wedge \partial_z$ |
| $\Lambda_5 = \partial_x \wedge \partial_y + \partial_z \wedge \partial_z$ | $\Lambda_5 = \partial_x \wedge \partial_y + \cos(z)\partial_x \wedge \partial_z$ | $P_4 = +e^{-s}\partial_x \wedge \partial_y - e^{-s}\sin(z)\partial_x \wedge \partial_z$ | $+e^{-s}(\cos(z) \partial_y \wedge \partial_z$ |
| $E_5 = \partial_y$ | $+\sin(z)\partial_y \wedge \partial_z$ | $+e^{-s}(\cos(z) \partial_x \wedge \partial_z$ | $+e^{-s}(\sin(z) + \cos(z))\partial_y \wedge \partial_z$ |
| $\Lambda_6 = \partial_x \wedge \partial_y + \partial_x \wedge \partial_z$ | $\Lambda_6 = \partial_x \wedge \partial_y + (\cos(z) - \sin(z))\partial_x \wedge \partial_z$ | $P_5 = +e^{-s}\partial_x \wedge \partial_y + e^{-s}\cos(z)\partial_x \wedge \partial_z$ | $+e^{-s}(\sin(z) + \cos(z))\partial_y \wedge \partial_z + e^{-s}(\sin(z) + \cos(z))\partial_y \wedge \partial_z$ |
| $\Lambda_6 = \partial_x \wedge \partial_y + \partial_y \wedge \partial_z$ | $E_6 = \partial_x \wedge \partial_y$ | | $-e^{-s}(\cos(z) + \sin(z))\partial_x \wedge \partial_z$ |
| $\Lambda_6 = \partial_x \wedge \partial_y + \partial_x \wedge \partial_z$ | $E_6 = -\partial_x + \partial_y$ | $P_6 = +e^{-s}\partial_x \wedge \partial_y$ | $+e^{-s}(\cos(z) - \sin(z))\partial_x \wedge \partial_z$ |
| $\Lambda_6 = \partial_x \wedge \partial_y$ | | $+e^{-s}(\cos(z) \partial_y \wedge \partial_z$ | $+e^{-s}(\sin(z) + \cos(z))\partial_y \wedge \partial_z$ |
| | $+\sin(z)\partial_y \wedge \partial_z$ | | $+e^{-s}(\sin(z) + \cos(z))\partial_y \wedge \partial_z$ |
| | $\Lambda_6 = (\sin(z) - \cos(z))\partial_x \wedge \partial_y$ | | $-e^{-s}(\cos(z) - \sin(z))\partial_y \wedge \partial_z$ |
Table 2: Realizations of some three-dimensional Lie algebras on $\mathbb{R}^2$.

| Lie algebra with non-zero commutation relations | Realization on $\mathbb{R}^2$ with coordinates $(q_1, q_2)$ |
|------------------------------------------------|--------------------------------------------------|
| $II$ $[X_2, X_3] = X_1$                           | $X_1 = \partial_1, X_2 = \partial_2, X_3 = q_2 \partial_1$ |
| $III$ $[X_1, X_2] = -(X_2 + X_3), [X_1, X_3] = -(X_2 + X_3)$ | $X_1 = (q_1 + q_2) \partial_1 + (q_1 + q_2) \partial_2, X_2 = \partial_1, X_3 = \partial_2$ |
| $IV$ $[X_1, X_2] = -(X_2 - X_3), [X_1, X_3] = -X_3$ | $X_1 = -q_1(q_2 - 1) \partial_1 - q_2^2 \partial_2, X_2 = \partial_1, X_3 = q_2 \partial_1 + q_1 \partial_2$ |
| $V I_0$ $[X_1, X_3] = X_2, [X_2, X_3] = X_1$        | $X_1 = \partial_1, X_2 = \partial_2, X_3 = q_2 \partial_1 + q_1 \partial_2$ |
| $V II_0$ $[X_1, X_3] = -X_2, [X_2, X_3] = X_1$       | $X_1 = \partial_1, X_2 = \partial_2, X_3 = q_2 \partial_1 - q_1 \partial_2$ |

Appendix: Other non-degenerate compatible Poisson structures on the real four-dimensional Lie group $II \otimes \mathbb{R}$

In this appendix, we describe the details for obtaining Poisson structures on real four-dimensional Lie algebra $II \oplus \mathbb{R}$. We also obtain the non-degenerate compatible Poisson structures on the related Lie group. Note that in the classification of these Poisson structures, some of the structures are equivalent, and therefore define an equivalence relation and apply the following theorem:

**Theorem 4.11** Two Poisson structures $P$ and $P'$ are equivalent if there exists $A \in \text{Aut}(g)$, (i.e., automorphism group of the Lie algebra $g$) such that

$$P' = A^t P A,$$

(23)

**Proof.** The proof is given in [4].

**Lie algebra** $II \oplus \mathbb{R} \cong A_{3,1} \oplus A_1$

We first assume the matrix form of the Poisson structure on real four-dimensional Lie algebra $II \oplus \mathbb{R}$ as follows:

$$P = \begin{pmatrix} 0 & p_{12} & p_{13} & p_{14} \\ -p_{12} & 0 & p_{23} & p_{24} \\ -p_{13} & -p_{23} & 0 & p_{34} \\ -p_{14} & -p_{24} & -p_{34} & 0 \end{pmatrix},$$

(24)

where $p_{ij}$ are arbitrary real constants. Using (15), we obtain adjoint representations $\chi_i$ of the Lie algebra $II \oplus \mathbb{R}$:

$$\chi_1 = \chi_4 = 0, \chi_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \chi_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

(25)
and antisymmetric matrices $\mathcal{Y}_i$:

$$
\mathcal{Y}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{Y}_2 = \mathcal{Y}_3 = \mathcal{Y}_4 = 0. 
$$

(26)

Substituting $E = 0$ in (13), we have:

$$
P^{cc}(\chi^t, P) + P\mathcal{Y}^t P + (P\chi_b)P^{bc} = 0.
$$

(27)

Now inserting (24)–(26) in (27), one can obtain the Poisson structures for the Lie algebra $II \oplus \mathbb{R}$ as follows:

$$
P = \begin{pmatrix} 0 & p_{12} & p_{13} & p_{14} \\ -p_{12} & 0 & 0 & p_{24} \\ -p_{13} & 0 & 0 & p_{34} \\ -p_{14} & -p_{24} & -p_{34} & 0 \end{pmatrix}.
$$

(28)

Then applying the automorphism group of the Lie algebra $II \oplus \mathbb{R}$, we have:

$$
a \in \mathbb{R}.
$$

where $a_{ij} \in \mathbb{R}$.

(29)

Using (23) with two equivalent Poisson structure $P$ and $P'$ from (28), we get $\det A = \frac{p'_{12}p'_{34} - p'_{13}p'_{24}}{p_{12}p_{34} - p_{13}p_{24}}$. Since we must have $\det A \neq 0$, it follows that $p'_{12}p'_{34} - p'_{13}p'_{24} \neq 0$. Moreover, $\det A$ does not depend on parameters $p'_{14}$; thus these parameters can take any value.

By using Theorem 4.11 we show that the Poisson structure $P$ consists of the following structures in one equivalence class:

(1) If $p'_{12} = 1, p'_{34} = 1, p'_{13} = 0, p'_{24} = 1, p'_{14} = 1$, then

$$
A = \begin{pmatrix} a_{22}a_{33} - a_{23}a_{32} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & 0 & 0 & a_{44} \end{pmatrix},
$$

(29)

where

$$
a_{34} = \frac{\left((p_{12}p_{34} - p_{13}p_{24}) (a_{21}p_{24} + a_{31}p_{34}) a_{32}^2 - a_{32}p_{14}p_{24} - p_{24}^2\right) a_{32}}{p_{24}^2}
$$

with $\det A = \frac{1}{p_{12}p_{34} - p_{13}p_{24}}$ and the Poisson structure $P$ is equivalent to

$$
P'_1 = \partial_x \wedge \partial_y + \partial_z \wedge \partial_s + \partial_y \wedge \partial_x + \partial_z \wedge \partial_s,
$$

(30)

(2) If $p'_{12} = 1, p'_{34} = 1, p'_{13} = 1, p'_{24} = 0, p'_{14} = 0$, then the Poisson structure $P$ is equivalent to

$$
P'_2 = \partial_x \wedge \partial_y + \partial_z \wedge \partial_s + \partial_x \wedge \partial_z,
$$

(30)
In view of the above relation, we get

Then one can obtain the inverse of the vielbein $e^\alpha_\mu$, that is $e_{a\mu}$, for the Lie group $\Pi \otimes \mathbb{R}$ as follows:

$$g^{-1}dg = e^a_\mu X_a dx^\mu = dxX_1 + dy(X_2 + zX_1) + dzX_3 + dsX_4.$$ 

In view of the above relation, we get

$$e^a_\mu = \begin{pmatrix} 1 & z & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

Then one can obtain the inverse of the vielbein $e^a_\mu$, that is $e_{a\mu}$, for the Lie group $\Pi \otimes \mathbb{R}$

$$e_{a\mu} = \begin{pmatrix} 1 & -z & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ (31)
Hence, substituting (31) and (30) in (9), one can calculate the Poisson structure $P_1'$ on the Lie group $\Pi \otimes \mathbb{R}$:

$$P_1' = \begin{pmatrix} 0 & 1 & 0 & 1-z \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ -1+z & -1 & -1 & 0 \end{pmatrix}$$

In the same way, one can obtain the other structures $P_2', \ldots, P_{12}'$ on the Lie group $\Pi \otimes \mathbb{R}$ (see Table 3). Note that we consider non-degenerate and compatible Poisson structures on four-dimensional Lie groups $\Pi \otimes \mathbb{R}$ (see Table 3). In the same way, we have obtained the Poisson structures on Lie groups $\Pi_3 \otimes \mathbb{R}$, $\Pi_4 \otimes \mathbb{R}$ and $\Pi_6 \otimes \mathbb{R}$ but unfortunately all of them are degenerate. Also, for the Lie group $\Pi_7 \otimes \mathbb{R}$ we have not found any solution.

**Table 3:** Other non-degenerate compatible Poisson structures on the four-dimensional Lie algebra $\Pi \otimes \mathbb{R}$ and its Lie group $\Pi \otimes \mathbb{R}$.

| Poisson structures on Lie algebra $\Pi \otimes \mathbb{R}$ | Representation of one equivalence class on Lie algebra $\Pi \otimes \mathbb{R}$ | Representation of one equivalence class on Lie groups $\Pi \otimes \mathbb{R}$ |
|----------------------------------------------------------|--------------------------------------------------------------------------|--------------------------------------------------------------------------|
| $P = p_{12} \partial_x \wedge \partial_y + p_{13} \partial_x \wedge \partial_z + p_{14} \partial_y \wedge \partial_s + p_{24} \partial_y \wedge \partial_s + p_{34} \partial_z \wedge \partial_s$ | $P_1' = \partial_x \wedge \partial_y + \partial_z \wedge \partial_s + \partial_y \wedge \partial_s + \partial_z \wedge \partial_s$ | $P_1' = \partial_x \wedge \partial_y + (1-z) \partial_x \wedge \partial_s + \partial_y \wedge \partial_s + \partial_z \wedge \partial_s$ |
| Comment: $p_{12} p_{14} - p_{13} p_{24} \neq 0$ | $P_2' = \partial_y \wedge \partial_s + \partial_x \wedge \partial_z$ | $P_2' = \partial_x \wedge \partial_y + \partial_z \wedge \partial_s + \partial_x \wedge \partial_s$ |
| | $P_3' = \partial_x \wedge \partial_y + \partial_z \wedge \partial_s + \partial_y \wedge \partial_s$ | $P_3' = \partial_x \wedge \partial_y + \partial_z \wedge \partial_s + \partial_y \wedge \partial_s - \partial_z \wedge \partial_s$ |
| | $P_4' = \partial_x \wedge \partial_y + \partial_z \wedge \partial_s + \partial_x \wedge \partial_s$ | $P_4' = \partial_x \wedge \partial_y + \partial_z \wedge \partial_s + \partial_x \wedge \partial_s$ |
| | $P_5' = \partial_x \wedge \partial_y + \partial_z \wedge \partial_s + \partial_y \wedge \partial_s$ | $P_5' = \partial_x \wedge \partial_y + \partial_z \wedge \partial_s + \partial_x \wedge \partial_s$ |
| | $P_6' = \partial_x \wedge \partial_y + \partial_z \wedge \partial_s$ | $P_6' = \partial_x \wedge \partial_y + \partial_z \wedge \partial_s$ |
| | $P_7' = \partial_x \wedge \partial_y + \partial_z \wedge \partial_s$ | $P_7' = \partial_x \wedge \partial_y + \partial_z \wedge \partial_s - \partial_z \wedge \partial_s$ |
| | $P_8' = \partial_x \wedge \partial_y + \partial_z \wedge \partial_s + \partial_y \wedge \partial_s - \partial_z \wedge \partial_s$ | $P_8' = \partial_x \wedge \partial_y + \partial_z \wedge \partial_s + \partial_y \wedge \partial_s - \partial_z \wedge \partial_s$ |
| | $P_9' = \partial_x \wedge \partial_y + \partial_z \wedge \partial_s + \partial_y \wedge \partial_s - \partial_z \wedge \partial_s$ | $P_9' = \partial_x \wedge \partial_y + \partial_z \wedge \partial_s + \partial_y \wedge \partial_s - \partial_z \wedge \partial_s$ |
| | $P_{10}' = \partial_x \wedge \partial_y + \partial_z \wedge \partial_s + \partial_x \wedge \partial_s$ | $P_{10}' = \partial_x \wedge \partial_y + \partial_z \wedge \partial_s + \partial_x \wedge \partial_s$ |
| | $P_{11}' = \partial_x \wedge \partial_y + \partial_z \wedge \partial_s + \partial_y \wedge \partial_s + \partial_x \wedge \partial_s$ | $P_{11}' = \partial_x \wedge \partial_y + \partial_z \wedge \partial_s + \partial_x \wedge \partial_s + (1-z) \partial_x \wedge \partial_s$ |
| | $P_{12}' = \partial_x \wedge \partial_y + \partial_z \wedge \partial_s + \partial_y \wedge \partial_s + \partial_x \wedge \partial_s$ | $P_{12}' = \partial_x \wedge \partial_y + \partial_z \wedge \partial_s + \partial_y \wedge \partial_s + (1-z) \partial_x \wedge \partial_s$ |
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