Phase Transitions of the Bilayered Spin-$S$ Heisenberg Model and Its Extension to Fractional Dimensions

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Abstract

We study the ground state and the phase transitions of the bilayered spin-$S$ antiferromagnetic Heisenberg model using the Schwinger boson mean field theory. The interplane coupling initially stabilizes but eventually destroys the long-range antiferromagnetic order. The transition to the disordered state is continuous for small $S$, and first order for large $S$. The latter is consistent with an argument based on the spin wave theory. The phase diagram and phase transitions in corresponding model in fractional dimensions are also discussed.

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I. INTRODUCTION

Recently there has been considerable interest in quantum spin liquids, which are magnetic systems without LRO at low temperature. While in general, the ground state of quantum spin systems lack true LRO in 1D, the ground state of the 2D Heisenberg antiferromagnet exhibits Neel ordering even for $S = 1/2$, albeit with an sublattice magnetization that is considerably decreased from its classical value. Since spin is quantized, the spin value cannot be decreased beyond 1/2, hence the model does not have a spin liquid ground state. On the other hand, when two planes of antiferromagnetic spins are coupled together [3–11], and if the interplane coupling is strong enough, the ground state is easily seen to be one of valence bond solid of interplane singlets (IVBS). Thus, there should be a transition from the LRO Neel state to a spin liquid state as the interplane coupling is increased. It has been suggested that the unusual magnetic properties of YBCO, with its basic unit of a pair of coupled CuO planes, may be due to its lying close to this quantum transition [3].

It is of interest to study the nature of this quantum transition. Within a non-linear sigma-model (NLSM) description, Haldane [12] has pointed out that for a single plane of spins, topological Berry phase terms exist which differ between half-integer, odd integer, and even integer spins. One way to understand this is to consider the degeneracy of the valence bond solid states which maximize the number of resonating plaquettes in each case (4-fold, 2-fold, and non-degenerate respectively). On the other hand, the mapping of the two-plane system to the NLSM does not yield a topological term, which is consistent with the valence bond solid state for two planes with large interplane vs. intraplane coupling being zero-dimensional like and non-degenerate. Since the 2+1 D NLSM has only one phase transition which is second order, this suggests the same for the 2D quantum Heisenberg antiferromagnet at $T = 0$. However, the NLSM mapping assumes slow variation on the scale of lattice spacing, and so additional disordered phases and/or first order transition cannot be ruled out conclusively.

In this paper we investigate the ground state of the 2D bilayered Heisenberg AF for gen-
eral $S$ using the Schwinger boson mean field theory [1,2] with no additional approximation. Our calculation complements previous calculations for $S = 1/2$ only and/or using additional approximations, as well as a calculation using the related Takahasi bosons approach [4,5].

Our results show that a first order transition is favored by decreasing quantum fluctuations, i.e., increasing $S$. In particular, in agreement with those previous works, the transition for $S = 1/2$ is first order. The critical $S$ separating first from second order transition equals 0.35. While increasing interplane coupling $J_\perp$ eventually destroys LRO, ordering is stabilized by small $J_\perp$, and the critical $S$ for no LRO is shifted from the single plane value of 0.2 to 0.13. A simple argument using spin wave theory helps to explain why first order transition occurs for large $S$. Since quantum fluctuations increases both with decreasing $S$ and decreasing $d$, we also study the dimensionality dependence of the ”bilayer” hypercubic system. For $S = 1/2$, we find that for $d < 1.86$, the first order transition is replaced by second order one. En route, we also calculate the $S$ vs. $d$ phase diagram for $J_\perp = 0$, i.e. the hypercubic Heisenberg AF. In addition to contradicting the NLSM description by having the possibility of a first order phase transition, whenever the transition is continuous, the Schwinger boson MFT gives an additional phase transition between two disordered phases for all $S$, corresponding to a jump in the ratio of short-ranged intraplane to interplane correlations. However, since the Schwinger boson order parameter is not related directly to any physical symmetry breaking, it is likely this does not constitute a real phase transition but a sharp cross-over in behavior.

II. BILAYERED ANTIFERROMAGNET IN 2D

We begin with a quick review of Schwinger boson mean field theory [3] as applied to the translationally invariant nearest neighbor Heisenberg antiferromagnet on a bipartite lattice. The Hamiltonian is

$$H = \sum_{\langle ij \rangle} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j; \quad J_{ij} > 0, \quad \langle ij \rangle = n.n.$$
In the Schwinger boson representation, spin operators in each lattice site are replaced by spin 1/2 bosons as follows:

\[ S^+_i = b^\dagger_i b^\downarrow_i, \quad S^-_i = b^\dagger_i b^\uparrow_i \]

\[ S^z_i = \frac{1}{2}(b^\dagger_i b^\uparrow_i - b^\dagger_i b^\downarrow_i) \]

The number of bosons at each lattice site is subject to the constraint:

\[ \sum_{\sigma} b^\dagger_{i\sigma} b_{i\sigma} = 2S, \]

which can be implemented by introducing a Lagrange multiplier on each site. The Hamiltonian can now be written as

\[ H = -2 \sum_{\langle ij \rangle} J_{ij} \tilde{A}_{ij}^\dagger \tilde{A}_{ij} + \frac{1}{2} N z J S^2 + \sum_i \lambda_i (\tilde{b}^\dagger_i \tilde{b}^\downarrow_i - 2S), \]

where \( \tilde{A}_{ij}^\dagger = \frac{1}{2} \sum_{\sigma} \tilde{b}^\dagger_{i\sigma} \tilde{b}^\dagger_{j\sigma} \) and \( \tilde{b}^\dagger_{i\uparrow} = b^\downarrow_{i\downarrow}, \quad \tilde{b}^\dagger_{i\downarrow} = -b^\uparrow_{i\uparrow} \) for sites on one sublattice and \( \tilde{b}^\dagger_{i\sigma} = b^\sigma_{i\sigma} \) for sites on the other sublattice. Physically, the product \( \tilde{A}_{ij}^\dagger \tilde{A}_{ij} \) acts as the valence bond (singlet) number operator for sites \((i,j)\). In the mean field approximation, this product is decoupled by the Hartree-Fock decomposition. In addition, the exact local constraint is relaxed to one for the average:

\[ \langle \sum_{\sigma} b^\dagger_{i\sigma} b_{i\sigma} \rangle = 2S, \]

leading to the mean field Hamiltonian

\[ H_{MF} = E_0 + \lambda \sum_{i\sigma} \tilde{b}^\dagger_{i\sigma} \tilde{b}_{i\sigma} - 2 \sum_{\langle ij \rangle} J_{ij} A_{ij} (\tilde{A}_{ij}^\dagger + \tilde{A}_{ij}), \]

where we have taken \( A_{ij} = \langle \tilde{A}_{ij} \rangle \) to be real.

First consider the case that all the bonds are identical by symmetry, and assuming no spontaneous dimerization, then all \( A_{ij} \) must be the same \( A_{ij} = A \). In this case \( E_0 = \frac{1}{2} N z J S^2 - 2\lambda N S + JA^2 N z \), where \( z \) is the coordination number. \( H_{MF} \) can be diagonalized by going to momentum space and performing the Bogoliubov transformation:
\[ H_{MF} = E_0 - \lambda N + \sum_k \omega_k (a_k^\dagger a_k + \beta_k^\dagger \beta_k + 1), \]

where \( \omega_k = [\lambda^2 - (J\tilde{A}z\gamma_k)^2]^{1/2}, \gamma_k = \frac{1}{z} \sum \delta e^{ik\delta} = \sum_{i=1}^d \cos k_i/d. \) At \( T = 0, \) the energy should be minimized with respect to \( \lambda \) and \( A, \) yielding the set of self-consistent equations:

\[
S + \frac{1}{2} = \frac{1}{2N} \sum_k \frac{\mu}{(\mu^2 - \gamma_k^2)^{1/2}},
\]

\[
\tilde{A} = \frac{1}{2N} \sum_k \frac{\gamma_k^2}{(\mu^2 - \gamma_k^2)^{1/2}},
\]

where we define \( \mu \equiv \lambda/(J\tilde{A}z). \) An essential point of the theory is that a non-zero mean field amplitude \( A, \) which gives rise to boson hopping, indicates short-ranged antiferromagnetic order. Long-ranged order is achieved if the hopping amplitude is sufficiently large to give Bose condensation. This occurs when these eqs. cannot be satisfied by having \( \mu > 1, \) in which case \( \mu = 1, \) and the \( k = 0 \) term gives a finite contribution when converting the momentum sums into integrals:

\[
S + \frac{1}{2} = m_s + \frac{1}{2} \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \frac{1}{(1 - \gamma_k^2)^{1/2}},
\]

\[
A = m_s + \frac{1}{2} \int_{-\pi}^{\pi} \frac{d^d k}{(2\pi)^d} \frac{\gamma_k^2}{(1 - \gamma_k^2)^{1/2}},
\]

\[ (1) \]

It has been shown that the condensate density \( m_s \) is also the sublattice magnetization. For the Heisenberg antiferromagnet on a square lattice, it was found that Bose condensation occurs for all \( S > S_c, \) where \( S_c = 0.2, \) with a gapless linear excitation spectrum characteristic of spin waves. For \( S < S_c, \mu > 1, \) and there is an energy gap for excitations. Thus, for all physical values of \( S, \) there is AFLRO.

On the other hand, if two such planes are coupled together antiferromagnetically, and the interplane coupling is very large compared to intraplane coupling, the ground state is obviously a valence bond solid of interplane singlets, and the intraplane correlation length is zero. Thus, there must be at least one phase transition as the interplane coupling is increased. We now analyze this for general \( S \) using Schwinger boson MFT.

The Hamiltonian in this case is
\[ H = J \sum_{(ij)} S_i S_j + J_\perp \sum_{(ij)_z} S_i S_j, \]

where \( \sum_{(ij)} \) sums over n.n. on the same plane and \( \sum_{(ij)_z} \) sums over n.n. on different planes. Since there is still translational invariance, the mean field Lagrange multiplier will be the same on all sites. However, the lack of symmetry between intraplane and interplane bonds means two mean field amplitudes must be introduced for the bond decoupling. Letting these be \( A \) and \( B \) respectively, and taking them to be both real, the mean field Hamiltonian is now

\[ H_{MF} = E_0 + \lambda \sum_{i\sigma} (\tilde{b}_i^\dagger \tilde{b}_i) - 2JA \sum_{(ij)} (\tilde{A}_i^\dagger + \tilde{A}_{ij}) - 2J_\perp B \sum_{(ij)_z} (\tilde{A}_{ij} + \tilde{A}_{ij}), \]

where \( E_0 = 2NJS^2 - 2\lambda NS + 4JA^2N + NJ_\perp S^2/2 + J_\perp B^2N. \) As before, we diagonalize \( H_{MF} \) through the Bogoliubov transformation, giving

\[ H_{MF} = E_0 - \lambda N + \sum_{k\sigma} \omega_{k\sigma} (\alpha_{k\sigma}^\dagger \alpha_{k\sigma} + \beta_{k\sigma}^\dagger \beta_{k\sigma} + 1), \]

The excitation energies are given by

\[ \omega_{k\sigma} = [\lambda^2 - (2J_A \sum_{i=1}^d \cos k_i + J_\perp B\sigma)^2]^{\frac{1}{2}}, \]

where \( \sigma = \pm 1. \) Minimizing \( H_{MF} \) with respect to \( \lambda, A, \) and \( B \) gives the self-consistent equations:

\[ S + \frac{1}{2} = \frac{1}{2N} \sum_{k,\alpha} \frac{\mu}{(\mu^2 - \Gamma_{k,\alpha}^2)^{\frac{1}{2}}}, \]

\[ A = \frac{1}{2N} \sum_{k,\alpha} \frac{\Gamma_{k,\alpha}}{(\mu^2 - \Gamma_{k,\alpha}^2)^{\frac{1}{2}}} \left( \sum_{i=1}^d \cos k_i \right), \]

\[ B = \frac{1}{2N} \sum_{k,\alpha} \frac{\Gamma_{k,\alpha}}{(\mu^2 - \Gamma_{k,\alpha}^2)^{\frac{1}{2}}}, \]

where \( \Gamma_{k,\alpha} = (\sum_{i=1}^d \cos k_i + Q\alpha)/d \) and \( Q = J_\perp B/(2JA). \) Note that \( \mu, \) the excitation gap, must be greater than or equal \( 1 + Q/d. \) In particular, in the case of bose condensation, the value of \( \mu \) is fixed to \( 1 + Q/d \) and hence the summations in Eq. (2) turn out to be a function
of the parameter $Q$ only. The magnetization $m_s$ is calculated by solving the self-consistent equations with the summations converted into integrals:

\[
S + \frac{1}{2} = m_s + \frac{1}{4} \int \frac{d^d k}{(2\pi)^d} \sum_{\alpha} \frac{\mu_0}{(\mu_0^2 - \Gamma_{k,\alpha}^2)^{\frac{d}{2}}} \\
\tilde{A} = m_s + \frac{1}{4} \int \frac{d^d k}{(2\pi)^d} \sum_{\alpha} \frac{\Gamma_{k,\alpha}}{(\mu_0^2 - \Gamma_{k,\alpha}^2)^{\frac{d}{2}}} \left( \sum_{i=1}^{d} \cos k_i \right) \\
\tilde{B} = m_s + \frac{1}{4} \int \frac{d^d k}{(2\pi)^d} \sum_{\alpha} \frac{\Gamma_{k,\alpha} \alpha}{(\mu_0^2 - \Gamma_{k,\alpha}^2)^{\frac{d}{2}}},
\]

where $\mu_0 = 1 + Q/d$. These equations \[3\] hold so long as they give $m_s > 0$, i.e., Bose condensation, otherwise Eqs. (3) should be used with $\mu$ also as an unknown parameter. In principle, we can solve for $Q$ and $m_s$ or $\mu$. In practice, the form of these equations allow us to avoid this by plugging in an arbitrary values of $Q$ into the equations to find out $m_s$ or $\mu$, and then $A$ and $B$. The self-consistency is then reduced to using the values of $A$, $B$, and $Q$ to determine the value of $\beta \equiv J_\perp/J$.

The behavior of $Q$ as a function of $\beta$ for $S = 1/2$ is shown in Fig. 1, and is representative of all $S$. Discounting the trivial solutions $Q = 0$ and $Q = \infty$, corresponding to independent planes and IVBS respectively, there are $Q \neq 0$ solutions indicating both intraplane and interplane correlations. For small $\beta$, there is only one solution, with $Q$ increasing from 0 with $\beta$. For $\beta > 4(S + 1/2)^2$, a second branch of solution, beginning at infinity appears. The two solutions merge at some larger value of $\beta$, and beyond that, only the trivial solutions remain. The significance of these solutions can be understood if we consider the energy $E(Q)$ obtained by minimizing the energy with respect to all other parameters except $Q$. Then, the non-trivial solutions are extrema of $E(Q)$. Thus, for $\beta < \beta_0 = 4(S + 1/2)^2$, the solution of $Q$ corresponds to a global minimum in $E(Q)$, and describes the ground state. For $\beta > \beta_0$, the upper branch corresponds to a local maximum while the lower branch remains a local minimum. The local maximum begins at $Q = \infty$ at $\beta_0$, and moves towards the local minimum with increasing $\beta$. Eventually, the two extrema merge into a saddle point at $\beta_2$. Beyond that, $E(Q)$ is strictly decreasing with $Q$. By continuity, this means somewhere between $\beta_0$ and $\beta_2$, $E(\infty)$ must cross from being greater than $E(Q_1)$ to less than it, where
$Q_1$ is the lower branch solution. Thus, at this value $\beta_1$, the ground state jumps from the 2D correlated state described by $Q_1$ to the interplane VBS state.

We can solve for the value of $m_s$ at the non-trivial solutions. Initially, $m_s$ increases with increasing $Q$, but eventually will decrease, vanishing at some $Q_c$. For sufficiently large $S$, $Q_c$ will belong to the upper branch (maximum energy) solution. More importantly, in the case where $Q_c$ lies in the lower branch, its $\beta$ value changes from less than to greater than $\beta_1$ with increasing $S$. Thus, the transition from the LRO’ed state to disordered state is second order for small $S$, but becomes first order for larger $S$. In the former case, there is a subsequent transition from a disordered state with finite $Q$, hence with both interplane and intraplane short-ranged correlations, to the $Q = \infty$ state with only interplane correlations. Along with the jump in $Q$ is a discontinuous jump in the gap. It is tempting to associate this jump as a transition from some disordered state associate with a single plane to the non-degenerate IVBS. More likely this transition is probably an artifact of the Schwinger boson MFT, and indicates a relative sharp drop in the intraplane correlation length and a sharp rise in the gap. This is similar to the finite temperature MFT solution for a single plane, where $A$, hence short-ranged correlation, drops to zero above some finite temperatures $[2]$. In the latter case of first order transition in sublattice magnetization, the ground state jumps from one with LRO to the IVBS state. Since this latter state should be the correct ground state only in the $\beta$ goes to infinity limit, we interpret this as the MFT way of showing a transition into a disordered state with a very short intraplane correlation length. The behavior of $m_s$ and the gap $\Delta$ as a function of $\beta$ is shown in Fig. 2 for representative values of $S$. Fig. 2c shows an example of reentrance, where LRO first develops with increasing $\beta$, but is subsequently destroyed when $\beta$ gets too large. This occurs for $S$ smaller than approximately 0.2, the MFT value of $S$ below which the ground state has no LRO, but greater than approximately 0.13, the minimum value of $S$ for LRO at some $\beta$.

The phase diagram of $S$ vs. $\beta$ is shown in Fig. 3. For $S < 0.13$, the ground state is always disordered. For $0.13 < S < 0.2$, the system undergoes first a disorder-order and then a order-disorder continuous transition with increasing $\beta$. For $S > 0.2$, there is LRO.
for $\beta = 0$, and only the order-disorder transition remains. This transition is continuous until it terminates at a tricritical point at $S \approx 0.35, \beta \approx 2.92$, beyond which the continuous transition is preempted by a first order transition. Thus, for $S > 0.35$, there are values of $\beta$ where the LRO'ed state is not the ground state, but is nevertheless metastable. The continuous transition phase boundary remains metastable until $S \approx 0.4$, beyond which the $m_s = 0_+$ state moves into the upper branch and becomes unstable. In all cases of $S$ where a disordered ground state with finite $Q$ exists, a subsequent "first-order transition" occurs, with a discontinuous jump in $Q$ and the gap $\Delta$. As mentioned above, we interpret the jump as unphysical, and represents in reality a relatively sharp drop in the 2D correlation length.

We can understand why large $S$ favors a first order transition quite simply in terms of spin wave theory [4,5]. The Neel state energy is $E_N = S^2(2Jz + J_\perp)$ while the energy of the IVBS state is $E_V = J_\perp S(S+1)$. Equating the two implies an estimate for the first order transition at $\beta_1$ of the order of $S$ for large $S$. Within spin wave theory, the sublattice magnetization is given by $m_s$ in Eq. (3) with $B/A = 1$. For large $\beta$, the integral on the LHS scales as $\sqrt{\beta}$. If we set $m_s = 0$ as an estimate for the critical value $\beta_c$ for continuous transition, then $\beta_c$ is of order $S^2$. Thus, for large $S$, $\beta_1$ is much less than $\beta_c$.

Within MFT, the tricritical point and even the metastable continuous transition boundary occurs below the minimum physical value of $S = 1/2$. Thus, a first order transition is predicted for all physical systems described by the model. In fact, the sublattice magnetization jump at transition for $S = 1/2$ is about 30% of that at $\beta = 0$, clearly contradicting the results of numerical work on the model for $S = 1/2$, which supports a continuous transition in the same universality class as the finite temperature transition of the 3D classical Heisenberg model. On the other hand, there is no reason to expect the Schwinger boson MFT to give the exact answer, so the true position of the tricritical point might very well be above $S = 1/2$. This is particularly so since by relaxing the local constraint to a global one, unphysical states are included in the mean field solution, and the MF energy is not even variational. Thus, using these MF energies to find the position of first order transition is necessarily suspect. Nevertheless, we believe the prediction of larger $S$ favoring a first
order transition to be correct, and the nature of phase transition in the bilayer system is non-universal. For example, the transition for $S = 1/2$ may become first order if there is a sufficiently large next nearest neighbor ferromagnetic interaction. Conversely, first order transition may become continuous if frustration is introduced. In other words, the value of $S$ at the tricritical point can be changed by enlarging the parameter space. The seeming contradiction to the fact that the 2+1 $D$ NLSM has only continuous transition is resolved by noting that the mapping of the Heisenberg model into the NLSM is legitimate only if the correlation length is long, which does not have to be the case of the disordered state close to a first order transition. Also of interest is that with sufficient frustration, the single-layered system can be disordered for $S = 1/2$ or other physical values, and the reentrance behavior for small $S$ discussed above in the bilayered system can be physically observed.

Within our MFT and according to general arguments, first order transition implies the existence of metastable states with finite sublattice magnetization. This may lead to observable dynamics characteristic of macroscopic quantum tunneling. It would also be of significance with respect to Monte Carlo type numerical calculations [6–9] due to problems of being "stuck" in the metastable minimum. For example, the first order transition may be missed if the metastability persists till the would-be continuous transition.

III. EXTENSION TO THE FRACTIONAL DIMENSIONS

We have seen that for the 2D bilayered square lattice antiferromagnet, Schwinger boson MFT shows the physically interesting case of $S = 1/2$ as undergoing a first order transition. Since the continuous transition is favored by small $S$, hence increasing quantum fluctuations, one way of getting a continuous transition in MFT for $S = 1/2$ is to go to a dimension below 2. In this section, we show that indeed this is the case. En route, we present the phase diagram of $S$ vs. $d$ (Fig. 4) for a single layer. These non-integer dimension results may be relevant to the physics of the Heisenberg antiferromagnet on percolating clusters, which have fractal dimensionality.
We first perform the Schwinger boson MFT for a single hypercubic lattice in \( d \) dimension (square lattice for \( d = 2 \)). The self-consistent equations (Eq. \( 1 \)) depend on \( d \) only through the momentum sum, which must be analytically continued to non-integer dimensions. This can be done by using the gaussian identity:

\[
\frac{1}{(1 \pm \gamma k)^2} = 2 \sqrt{\frac{d}{\pi}} \int_0^\infty dx \ e^{-x^2(d \pm \sum_{i=1}^d \cos k_i)},
\]

(4)

to rewrite Eq. \( 1 \) as

\[
\frac{2d}{\pi} \int_0^\infty dx\, dy \ e^{-(x^2+y^2)d} \left( \int_{-\pi}^\pi \frac{dk}{2\pi} e^{-\cos k(x^2-y^2)} \right)^d.
\]

(5)

In this form, the analytic continuation to arbitrary \( d \) is obvious (see Appendix). In Fig. \( 3 \), the result for \( S = 1/2 \) is shown. As \( d \) is decreased below 2, \( m_s \) decreases and vanishes at some critical dimension \( d_c = 1.46 \). Below \( d_c \), the excitation spectrum has a gap, which rises with decreasing \( d \). These behaviors are representative for all \( S \). However, it is known that for the simple Heisenberg Hamiltonian, \( 1/2 \) integer spin chains and integer spin chains are intrinsically different in that the former should be gapless, which can be understood as due to the presence of a topological term the appropriate NLSM. Thus MFT must break down for \( 1/2 \) integer spins even qualitatively as \( d \) gets sufficiently close to 1, and \( \Delta \) must decrease again.

Next we generalize our bilayer calculation to two coupled hypercubes in \( d \) dimension. The analytic continuation of Eqs. \( 3 \) can again be done using the gaussian identity \( 4 \). As expected, a continuous transition can now be observed within MFT if \( d \) is reduced sufficiently from 2. For \( d < 1.46 \), the critical dimension for LRO for a single hypercube, the reentrance seen with increasing \( \beta \) for \( S < 0.2 \) is seen for \( S = 1/2 \). So far, we have concentrated on lowering the dimension from 2. Of course, raising it would have the opposite effect. For example, for two 3D hypercubes, the \( S = 1/2 \) transition would be strongly first order within MFT, and so even taking into account inaccuracy of MFT, strongly implies a first order transition.

Finally we discuss the critical phenomena of the continuous transition of this model. Analyzing Eqs. \( 2 \) and \( 3 \) close to the transition, we find the staggered magnetization
vanishes linearly, while the gap vanishes as \((\beta - \beta_c)^s\) with \(s = 1/(d - 1)\) for \(d < 3\), and \(s = 1/2\) for \(d > 3\) (there are logarithm corrections at \(d = 3\)). These MF exponents are the same as those for the finite temperature transition of a single hypercube with the substitution \(d \to d + 1\), reflecting the quantum nature of the present transition.

IV. APPENDIX

Notice that in Eq. (5), the integral inside the bracket is a modified Bessel function of the first kind \(I_0(x^2 - y^2)\). Therefore the first equation of Eq. (3) can be written as:

\[
S + \frac{1}{2} = m_s + \frac{2d}{\pi} \int_0^\infty dx dy \left( e^{-(x^2 + y^2)} I_0(x^2 - y^2) \right)^d.
\]

Take the transformation:

\[
u = x^2 - y^2, \quad v = x^2 + y^2
\]

with \(v \geq |u|\), for the integration variable will give us:

\[
S + \frac{1}{2} = m_s + \frac{2d}{\pi} \int_{-\infty}^\infty du (I_0(u))^d \int_{|u|}^\infty dv \frac{e^{-vd}}{\sqrt{v^2 - u^2}}
\]

\[
= m_s \frac{2d}{\pi} \int_0^\infty du (I_0(u))^d K_0(ud).
\]

It reduces the final formula into a single integral of \(I_0\) and \(K_0\), modified Bessel function of the second kind. In this form, the integral indeed converges much faster than the original form in Eq. (3) and consequently save much of the computation time. The same trick is also applied to both interplane and intraplane mean field equations.
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FIGURES

FIG. 1. The behavior of $Q$ as a function of $\beta$ for $S = 1/2$. It is representative of all $S$. $\beta_0 = 4$ and $\beta_2 \approx 4.36$. The star symbol stands for the location of 1st order transition in which $\beta \approx 4.25$.

FIG. 2. Magnetization (solid line) $m_s$ and spin gap (dashed line) $\Delta/\beta$ as a function of $\beta$. (a) shows 1st order transition with the absent of 2nd order transition for $S = 1/2$ while both transitions are observed for smaller $S$ in (b). (c), reentrance of magnetization occurs for $S < 0.2$.

FIG. 3. Phase diagram of $S$ vs $\beta$ for $d = 2$. 1st order transiton exists as long as $\beta < 2.92$ in which 1st order transition coincides with 2nd order transition (star symbol). The zero magnetization line (solid) terminates at a tricritical point, $\beta \approx 3.38$, beyond which the zero magnetization state is no longer stable.

FIG. 4. Phase diagram of $S$ vs $d$ for a single layer. The curve goes to infinity as $d$ tends to 1.

FIG. 5. Magnetization $m_s$ and spin gap $\Delta$ for $1 \leq d \leq 2$ and $S = 1/2$. The critical dimension that separates order and disorder state is $\approx 1.46$. 
Fig 2a

S = 0.5

Interplane coupling

Magnetization

Spin gap

\( \Delta/\beta \)
Fig 2b
S=0.25

Spin gap

Interplane coupling

Magnetization

Spin gap

$\Delta/\beta$
Fig 2c
S=0.15
