New nonlinear programming techniques for multiobjective optimization with applications to portfolio selection

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October 24, 2016

Abstract

We propose a new nonlinear programming approach for optimization problems with multiple criteria. In the context of no-preference methods, we introduce a new nonlinear nonconvex reformulation of the original multiobjective problem that shows distinctive significant features: computing a (global) solution of our reformulation does not depend on the objectives’ scales and provides us with a (global) Pareto optimum of the original problem that “dominates the most” with respect to the objectives space. Furthermore, as for the solvability issue, the problem of finding such Pareto solution, while nonconvex, nonetheless enjoys generalized convexity properties: in turn, this fact allows us to resort to standard nonlinear programming methods in order to calculate a global optimum of our nonconvex reformulation. We show the viability of our ideas by considering portfolio selection: numerical tests prove the effectiveness of our approach.

Keywords: Nonlinear Programming, Multiobjective Optimization, Generalized Convexity, Portfolio Selection

1 Introduction

Multiobjective optimization problems arise naturally when the decision process involves several (potentially) conflicting goals that must be taken into account simultaneously. Solving multiobjective programs usually consists in computing the Pareto optimal solution that best suites the decision maker; this is in general carried out by scalarization. Just with respect to the latter approach, it is hardly possible here to even summarize the huge amount of solution methods that have been proposed for multiobjective optimization. We only mention that solution methods are commonly grouped into four main categories: no-preference, a posteriori, a priori and interactive methods. We refer the interested reader to the fundamental \cite{7} for both theoretical bases and review of the literature on multiobjective optimization.
In this work, we propose a new scalarization technique resulting in a no-preference approach that consists in solving a nonlinear nonconvex reformulation of the original multiobjective problem. To be more specific, let us consider, for the sake of simplicity, an optimization problem with two objectives, \( f_1 \) and \( f_2 \). Each feasible solution \( x \) identifies, in the objectives space, a rectangle defined by the worst values of the objectives on the feasible region and point \((f_1(x), f_2(x))\). Clearly, \( x \) dominates all feasible points whose objectives values belong to the rectangle. In the light of this consideration, we aim at computing a Pareto optimal solution that maximizes the area of the corresponding rectangle. While this simple idea draws inspiration from the *hypervolume* paradigm (see, e.g., \([1, 5] \) and \([8] \)) and shares some conceptual similarities with other approaches (e.g., GUESS \([3] \) and the method of the global criterion, see \([7] \)), it has distinctive features: namely, computing a (global) solution of our reformulation is not sensitive to the scaling of the objective functions and provides us with a (global) Pareto optimum of the original problem that “dominates the most” (in the sense of the area of the induced rectangle) with respect to the objectives space. As usual, passing from the original multiple objectives to a conceptually simpler single objective optimization comes at a price: in general, one can not expect to solve globally the resulting single objective program in order to recover global Pareto optimal solutions. But, luckily, this is not the case for our method: under mild standard assumptions, the nonlinear nonconvex scalarization that we propose turns out to be practically (globally) solvable, since it is proven to enjoy generalized convexity properties. In particular, since any stationary point of our reformulation is shown to be global optimal, one can compute a Pareto (global) solution by simply relying on any nonlinear optimization solver.

Finally, we illustrate the application of our method to portfolio selection with two and three objectives: numerical results show the significance of the computed Pareto optima also with respect to the solutions provided by other classical approaches.

Sections 2-4 deal with the two objectives case. In Section 2 we introduce the core idea of our approach; there we prove that solving our reformulation gives us a global Pareto optimum of the original multiobjective problem. Section 3 is devoted to the theoretical results that form the basis of our method; the main result (Corollary 3.5) shows that any stationary point of our scalarized reformulating problem is also global optimal. The application of our approach to portfolio (mean-risk) selection is provided in Section 4. Finally, in Section 5 all previous results are extended to the case of three objectives.

## 2 Preliminaries

We start, for the sake of comprehension, considering the case of an optimization problem with two objectives

\[
\begin{align*}
\text{minimize} & \quad (f_1(x), f_2(x))^T \\
\text{s.t.} & \quad x \in X,
\end{align*}
\]

where \( f_1, f_2 : \mathbb{R}^n \to \mathbb{R} \) are smooth and \( X \subseteq \mathbb{R}^n \) is nonempty and compact.

Any point \( x \in X \) identifies in the objectives space a rectangle \( R_x \) with base \((f_1^{\text{max}} - f_1(x))\) and height \((f_2^{\text{max}} - f_2(x))\), where \( f_1^{\text{max}} \triangleq \max_{x \in X} f_1(x) \) and \( f_2^{\text{max}} \triangleq \max_{x \in X} f_2(x) \). Note that \( x \in X \) strictly dominates, in a Pareto sense, all points \( y \in X \) such that \((f_1(y), f_2(y))\) belongs to the rectangle \( R_x \).

By conceptually referring to the *hypervolume* paradigm, we aim at finding a feasible \( x^* \) that maximizes the area of the corresponding rectangle. Thus, we address the following
problem:
\[
\begin{align*}
\text{minimize} & \quad -\left(f_1^\text{max} - f_1(x)\right)\left(f_2^\text{max} - f_2(x)\right) \\
\text{s.t.} & \quad x \in X.
\end{align*}
\]  

Dealing with problem (1) by solving reformulation (2) has the following prominent advantages and distinctive features:

(i) any global solution of (2) is a global Pareto optimum of (1) (see Proposition 2.1);

(ii) among all Pareto optima of (1), solution \(x^*\), maximizing the area of the rectangle, “dominates the most” with respect to the objectives space;

(iii) finding a solution of (2) does not depend on the objectives’ scales, thus allowing to “robustly” deal with non homogeneous objectives.

With the following proposition we prove the claim in (i) establishing the fundamental link between (2) and the original multiobjective problem (1).

**Proposition 2.1** Any global solution \(x^*\) of problem (2) such that \((f_1^\text{max} - f_1(x^*))\left(f_2^\text{max} - f_2(x^*)\right) > 0\) is a global Pareto optimum of (1).

**Proof.** Suppose by contradiction that there exists \(\hat{x} \in X\) such that, without loss of generality, \(f_1(\hat{x}) < f_1(x^*), f_2(\hat{x}) \leq f_2(x^*)\).

Therefore, we have
\[
(f_1^\text{max} - f_1(\hat{x}))\left(f_2^\text{max} - f_2(\hat{x})\right) > (f_1^\text{max} - f_1(x^*))\left(f_2^\text{max} - f_2(x^*)\right)
\]
that is an absurdum since \(x^*\) is optimal for (2). \(\square\)

We remark that assumption \((f_1^\text{max} - f_1(x^*))\left(f_2^\text{max} - f_2(x^*)\right) > 0\) in Proposition 2.1 is not demanding: indeed, in the convex case, if this condition is not verified, then at least an objective reaches its worst value for every feasible point, and, thus, can be dropped (see Proposition 3.1).

Here we recall some generalized convexity properties.

**Definition 2.2** A function \(f : \mathbb{R}^n \to \mathbb{R}\) is quasiconvex on an open convex set \(C \subseteq \mathbb{R}^n\) if all its sublevel sets 
\[
\mathcal{L}_f^b \triangleq \{x \in C \mid f(x) \leq b\}
\]
are convex.

**Definition 2.3** A function \(f : \mathbb{R}^n \to \mathbb{R}\) is pseudoconvex on an open convex set \(C \subseteq \mathbb{R}^n\) if, for all \(x, y \in C\),
\[
\nabla f(x)^T (y - x) \geq 0 \Rightarrow f(y) \geq f(x).
\]

Finally, we denote with \(\mathbb{M}_n\) the space of symmetric \(n \times n\) matrices. Given a matrix \(A\), we indicate with \(A_{ij}\) its \(i\)th row and \(j\)th column element. Finally, \(\| \cdot \|\) is the Euclidean norm.

**3 Theoretical results**

In this section we lay down the theoretical foundation of our approach: specifically, we show the viability of our idea by proving that problem (2) enjoys useful generalized convexity properties. From now on, we make the following blanket assumptions:
\begin{itemize}
\item $X$ is convex;
\item $f_1, f_2$ are convex on an open convex set containing $X$.
\end{itemize}

Preliminarily, we prove that the optimal value of reformulation (2) reaches zero only in pathological cases.

**Proposition 3.1** Let $x^*$ be any global solution of problem (2). Then, $(f_1^{\text{max}} - f_1(x^*)) (f_2^{\text{max}} - f_2(x^*)) = 0$ if and only if $i \in \{1, 2\}$ exists such that $f_i(x) = f_i^{\text{max}}$ for all $x \in X$.

**Proof.** First, let $(f_1^{\text{max}} - f_1(x^*)) (f_2^{\text{max}} - f_2(x^*)) = 0$ and suppose by contradiction that $x^1, x^2 \in X$ exist such that $f_1(x^1) < f_1^{\text{max}}$ and $f_2(x^2) < f_2^{\text{max}}$. Clearly, we have $(f_1^{\text{max}} - f_1(x))(f_2^{\text{max}} - f_2(x)) = 0$ for every $x \in X$, and, in turn, $f_1(x^1) = f_1^{\text{max}}$ and $f_2(x^1) = f_2^{\text{max}}$. Let $\tilde{x} = \frac{1}{2} x^1 + \frac{1}{2} x^2 \in X$. We obtain $f_1(\tilde{x}) \leq \frac{1}{2} f_1(x^1) + \frac{1}{2} f_1^{\text{max}} < f_1^{\text{max}}$ and $f_2(\tilde{x}) \leq \frac{1}{2} f_2^{\text{max}} + \frac{1}{2} f_2(x^2) < f_2^{\text{max}}$. Therefore, $(f_1^{\text{max}} - f_1(\tilde{x}))(f_2^{\text{max}} - f_2(\tilde{x})) > 0$, in contradiction with the fact that the optimal value $(f_1^{\text{max}} - f_1(x^*))(f_2^{\text{max}} - f_2(x^*))$ is zero.

The other sense is trivial. \hfill $\square$

We observe that, even if $f_1$ and $f_2$ are convex on $\mathbb{R}^n$, the objective function in (2) is in general nonconvex on $\mathbb{R}^n$. We resort to the following slightly perturbed version of (2):

\[
\begin{align*}
\text{minimize} & \quad -(f_1^{\text{max}} + \varepsilon_1 - f_1(x))(f_2^{\text{max}} + \varepsilon_2 - f_2(x)) \\
\text{s.t.} & \quad x \in X,
\end{align*}
\]

where $\varepsilon_1, \varepsilon_2$ are positive constants. Clearly, with $\varepsilon_1 = \varepsilon_2 = 0$, problem (3) reduces to problem (2).

Lemmas 3.2 and 3.3 are instrumental to prove in Theorem 3.4 that the objective in (3), while nonconvex, turns out to be pseudoconvex.

**Lemma 3.2** Let functions $p_1, p_2 : \mathbb{R}^q \to \mathbb{R}$ be concave and positive on an open and convex set $S \subseteq \mathbb{R}^q$. Then, function $h \hat{=} -p_1 p_2$ is quasiconvex on $S$.

**Proof.** Let us consider the level set $\mathcal{L}_h^c \hat{=} \{ u \in S : h(u) \leq c \}$. We need to show that $\mathcal{L}_h^c$ is a convex set for every choice of $c \leq 0$.

Let $u, v$ belong to $\mathcal{L}_h^c$ and $w = \lambda u + (1 - \lambda)v$, with $\lambda \in [0, 1]$. We have

\[
\begin{align*}
h(w) = & \quad -p_1(\lambda u + (1 - \lambda)v) p_2(\lambda u + (1 - \lambda)v) \\
\overset{(a)}{\leq} & \quad [-p_1(u) + (1 - \lambda)p_1(v)] p_2(\lambda u + (1 - \lambda)v) \\
\overset{(b)}{\leq} & \quad [-p_1(u) + (1 - \lambda)p_1(v)][\lambda p_2(u) + (1 - \lambda)p_2(v)] \\
= & \quad \lambda^2 h(u) - \lambda(1 - \lambda)[p_1(u)p_2(v) + p_1(v)p_2(u)] + (1 - \lambda)^2 h(v) \\
\leq & \quad \lambda^2 c + \lambda(1 - \lambda)c \left\{ \frac{p_1(u)}{p_1(v)} + \frac{p_1(v)}{p_1(u)} \right\} + (1 - \lambda)^2 c \\
\leq & \quad \lambda^2 c + \lambda(1 - \lambda)c \left\{ \frac{p_1(u)^2 + p_1(v)^2}{p_1(u)p_1(v)} \right\} + (1 - \lambda)^2 c \\
\overset{(c)}{\leq} & \quad \lambda^2 c + 2\lambda(1 - \lambda)c + (1 - \lambda)^2 c \\
= & \quad c,
\end{align*}
\]
where (a) and (b) are due to the concavity of \( p_1 \) and \( p_2 \), respectively; (c) holds since 
\[
p_1(u)^2 + p_1(v)^2 \geq 2p_1(u)p_1(v).
\]

\[\Box\]

**Lemma 3.3** Let \( h : \mathbb{R}^q \to \mathbb{R} \) be differentiable and quasiconvex on an open and convex set 
\( S \subseteq \mathbb{R}^q \). If \( \nabla h(u) \neq 0 \) for every \( u \in S \), then \( h \) is pseudoconvex on \( S \).

**Proof.** See [4, Lemma 2.1]. \[\Box\]

**Theorem 3.4** The objective function in (3) is pseudoconvex on an open convex set containing \( X \).

**Proof.** Let \( p_1(x, y) \triangleq f_1^{\max} + \epsilon_1 - f_1(x) \) and \( p_2(x, y) \triangleq f_2^{\max} + \epsilon_2 - f_2(x) - y \). We observe that there exists an open convex set \( S \) containing \( X \times \{0\} \), such that \( p_1 \) and \( p_2 \) are concave and positive on \( S \). Therefore, by Lemma 3.2, \( h \triangleq -p_1 p_2 \), is quasiconvex on \( S \). We note that \( \nabla_y h(x, y) = p_1(x, y) > 0 \) for every \( (x, y) \in S \), where \( \nabla_y h(x, y) \) denotes the derivative of \( h \) with respect to the second argument. Hence, \( \nabla h(x, y) \neq 0 \) in \( S \) and, by Lemma 3.3, \( h \) is pseudoconvex on \( S \). In turn, the objective function in (3) is pseudoconvex on an open convex set containing \( X \) and the thesis holds. \[\Box\]

Finally, since the objective function in (3) is pseudoconvex, one can resort, by means of Corollary 3.5, to standard nonlinear optimization techniques in order to find a global optimum of (3).

**Corollary 3.5** Any stationary solution is global for problem (3).

**Proof.** The proof is a straightforward consequence of the very definition of pseudoconvexity. \[\Box\]

**Remark 3.6** One may choose a reference point \((f_1^{\text{ref}}, f_2^{\text{ref}})\) different from \((f_1^{\max}, f_2^{\max})\), for example whenever values \( f_1^{\max}, f_2^{\max} \) are not readily at hand. In our framework, this can be easily done by adding to the original feasible set \( X \) convex constraints \( f_1(x) \leq f_1^{\text{ref}} \) and \( f_2(x) \leq f_2^{\text{ref}} \).

4 An application to portfolio selection

We consider mean-risk models for portfolio selection. In particular, referring to Markowitz’s seminal work [6], we employ variance as risk measure. Thus, having \( n \) available assets, let \( m^T x \) and \( x^T V x \) denote mean (expected value) and variance of the return associated with portfolio \( x \in \mathbb{R}^n \) in terms of mean \( m \in \mathbb{R}^n \) and variance \( \mathbb{M}_n \succ V \succeq 0 \) of the return of the \( n \) assets, respectively.

The efficient solutions in a mean-risk model are Pareto efficient solutions of a multi-objective problem in which the expected return is maximized and the risk is minimized:

\[
\begin{align*}
\text{minimize} & \quad ( - m^T x, x^T V x )^T \\
\text{s.t.} & \quad e^T x = 1 \\
& \quad x \geq 0,
\end{align*}
\]

(4)
Clearly, since function \(x^T V x\) is convex, the results in Section 3 hold for problem (5). Thanks to Corollary 3.5, we can employ any standard nonlinear programming technique in order to compute a global solution of (5): in fact, solving KKT conditions for (5) leads to a global minimum of the nonconvex problem.

We studied the behavior of our selection strategy referring, as benchmark, to the datasets presented in [2]: in particular, data consist in weekly returns time series for assets and indexes belonging to several major stock markets (see summary Table 1). For further details we refer the reader to [2]. In Table 1 denoting by \(m_i\) and \(\%\)\(\min\) belonging to several major stock markets (see summary Table 1). For further details we presented in [2]: in particular, data consist in weekly returns time series for assets and indexes belonging to several major stock markets (see summary Table 1). For further details we refer the reader to [2].

All experiments have been carried out on an Intel Core i7-4702MQ CPU @ 2.20GHz x 8 with Ubuntu 14.04 LTS 64-bit and by using Matlab 7.14.0.739 (R2012a). When addressing nonconvex problem (5), we resorted to \texttt{fmincon} with \texttt{sqp} algorithm and allowing the computation of derivatives both of the objective function and the constraints. On the other hand, we solved the quadratic problems obtained by means of the \(\varepsilon\)-constraints method by using \texttt{quadprog} with \texttt{interior-point-convex} algorithm. All \(\varepsilon_i\) have been set to 1e\(-10\).
Table 1: Datasets

| Index       | #assets | #weeks | From-To          | \(\%m^{\text{min}}\) | \(\%m^{\text{max}}\) | \(\%V^{\text{min}}\) | \(\%V^{\text{max}}\) |
|-------------|---------|--------|------------------|-----------------------|-----------------------|-----------------------|-----------------------|
| DowJones    | 28      | 1363   | 2/1990 - 4/2016  | 0.128                 | 0.605                 | 0.040                 | 0.385                 |
| NASDAQ100   | 82      | 596    | 11/2004 - 4/2016 | 0.012                 | 1.030                 | 0.039                 | 0.705                 |
| FTSE100     | 83      | 717    | 7/2002 - 4/2016  | -0.061                | 0.802                 | 0.030                 | 0.774                 |
| SP500       | 442     | 595    | 11/2004 - 4/2016 | -0.063                | 1.032                 | 0.022                 | 1.417                 |
| NASDAQComp  | 1203    | 685    | 2/2003 - 4/2016  | -0.306                | 1.174                 | 0.005                 | 2.788                 |
| FF49Industries | 49     | 2325   | 7/1969 - 7/2015  | 0.275                 | 0.544                 | 0.029                 | 0.303                 |

Figure 1: Plot of results for DowJones data set.

5 Extension to the three objectives case

In this section we extend all previous results to the more general and challenging case of a three objectives optimization:

\[
\begin{align*}
\text{minimize} & \quad (f_1(x), f_2(x), f_3(x))^T \\
\text{subject to} & \quad x \in X,
\end{align*}
\]

where \(X \subseteq \mathbb{R}^n\) is a nonempty compact convex set and \(f_1, f_2, f_3 : \mathbb{R}^n \to \mathbb{R}\) are smooth and convex on an open convex set containing \(X\).

Similarly to what done in Section 3 in order to cope with problem (6), we solve the following program:

\[
\begin{align*}
\text{minimize} & \quad -\left(f_1^{\text{max}} + \varepsilon_1 - f_1(x)\right)\left(f_2^{\text{max}} + \varepsilon_2 - f_2(x)\right)\left(f_3^{\text{max}} + \varepsilon_3 - f_3(x)\right) \\
\text{subject to} & \quad x \in X,
\end{align*}
\]

with \(f_1^{\text{max}} \triangleq \max_x \{f_1(x) | x \in X\}, f_2^{\text{max}} \triangleq \max_x \{f_2(x) | x \in X\}, f_3^{\text{max}} \triangleq \max_x \{f_3(x) | x \in X\}\) and where \(\varepsilon_1, \varepsilon_2, \varepsilon_3\) are positive constants.
| Method            | %return | %risk | Area   | $\beta_1$ | $\beta_2$ | $\|\beta\|$ | #ptf assets |
|-------------------|---------|-------|--------|-----------|-----------|-------------|-------------|
| DowJones          | hyper2  | 0.538 | 0.127  | 0.106     | 0.140     | 0.252       | 0.289       | 6           |
|                   | %return ≥ 0.133 | 0.214 | 0.040  | 0.030     | 0.820     | 0.000       | 0.820       | 13          |
|                   | %return ≥ 0.247 | 0.247 | 0.041  | 0.041     | 0.750     | 0.003       | 0.750       | 16          |
|                   | %return ≥ 0.367 | 0.367 | 0.058  | 0.078     | 0.500     | 0.052       | 0.502       | 13          |
|                   | %return ≥ 0.486 | 0.486 | 0.100  | 0.102     | 0.250     | 0.174       | 0.304       | 9           |
|                   | %return ≥ 0.601 | 0.601 | 0.240  | 0.069     | 0.010     | 0.580       | 0.500       | 2           |
| NASDAQ100         | hyper2  | 0.897 | 0.159  | 0.483     | 0.131     | 0.180       | 0.223       | 7           |
|                   | %return ≥ 0.022 | 0.242 | 0.039  | 0.154     | 0.774     | 0.000       | 0.774       | 12          |
|                   | %return ≥ 0.266 | 0.266 | 0.039  | 0.170     | 0.750     | 0.000       | 0.750       | 11          |
|                   | %return ≥ 0.521 | 0.521 | 0.058  | 0.330     | 0.500     | 0.028       | 0.501       | 18          |
|                   | %return ≥ 0.775 | 0.775 | 0.109  | 0.456     | 0.250     | 0.105       | 0.272       | 11          |
|                   | %return ≥ 1.020 | 1.020 | 0.552  | 0.155     | 0.010     | 0.770       | 0.770       | 2           |
| FTSE100           | hyper2  | 0.943 | 0.194  | 1.230     | 0.081     | 0.123       | 0.148       | 5           |
|                   | %return ≥ -0.052 | 0.189 | 0.022  | 0.352     | 0.770     | 0.000       | 0.770       | 25          |
|                   | %return ≥ 0.155 | 0.254 | 0.030  | 0.234     | 0.635     | 0.000       | 0.635       | 23          |
|                   | %return ≥ 0.370 | 0.370 | 0.035  | 0.319     | 0.500     | 0.007       | 0.501       | 17          |
|                   | %return ≥ 0.586 | 0.586 | 0.091  | 0.442     | 0.250     | 0.082       | 0.263       | 9           |
|                   | %return ≥ 0.793 | 0.793 | 0.567  | 0.177     | 0.010     | 0.722       | 0.722       | 2           |
| SP500             | hyper2  | 1.125 | 0.215  | 3.684     | 0.033     | 0.075       | 0.082       | 9           |
|                   | %return ≥ -0.291 | 0.149 | 0.005  | 1.266     | 0.693     | 0.000       | 0.693       | 352         |
|                   | %return ≥ 0.064 | 0.149 | 0.005  | 1.266     | 0.693     | 0.000       | 0.693       | 352         |
|                   | %return ≥ 0.434 | 0.434 | 0.013  | 2.055     | 0.500     | 0.003       | 0.500       | 95          |
|                   | %return ≥ 0.804 | 0.804 | 0.058  | 3.032     | 0.250     | 0.019       | 0.251       | 59          |
|                   | %return ≥ 1.160 | 1.160 | 0.307  | 3.637     | 0.010     | 0.108       | 0.109       | 4           |
| NASDAQComp        | hyper2  | 0.535 | 0.083  | 0.057     | 0.033     | 0.197       | 0.200       | 4           |
|                   | %return ≥ 0.278 | 0.325 | 0.029  | 0.014     | 0.814     | 0.000       | 0.814       | 6           |
|                   | %return ≥ 0.343 | 0.343 | 0.030  | 0.018     | 0.750     | 0.004       | 0.750       | 7           |
|                   | %return ≥ 0.410 | 0.410 | 0.037  | 0.036     | 0.500     | 0.029       | 0.500       | 9           |
|                   | %return ≥ 0.477 | 0.477 | 0.055  | 0.050     | 0.250     | 0.095       | 0.266       | 7           |
|                   | %return ≥ 0.541 | 0.541 | 0.089  | 0.057     | 0.011     | 0.219       | 0.219       | 3           |

Table 2: Numerical results for the two objectives case
Preliminarily, we observe that Propositions 2.1 and 3.1 are still valid in the three objectives case. Furthermore, the following theoretical results generalize the ones in Section 3.

**Lemma 5.1** Let functions $p_1, p_2, p_3: \mathbb{R}^q \to \mathbb{R}$ be concave and positive on an open and convex set $S \subseteq \mathbb{R}^q$. Then, function $h \triangleq -p_1 p_2 p_3$ is quasiconvex on $S$.

**Proof.** Let us consider the level set $L^c_h \triangleq \{ u \in S : h(u) \leq c \}$. We need to show that $L^c_h$ is a convex set for every choice of $c \leq 0$.

Let $u, v$ belong to $L^c_h$ and $w = \lambda u + (1 - \lambda)v$, with $\lambda \in [0, 1]$. We have

$$h(w) = -p_1(\lambda u + (1 - \lambda)v)p_2(\lambda u + (1 - \lambda)v)p_3(\lambda u + (1 - \lambda)v) \leq (a) - [\lambda p_1(u) + (1 - \lambda)p_1(v)] [\lambda p_2(u) + (1 - \lambda)p_2(v)] [\lambda p_3(u) + (1 - \lambda)p_3(v)]$$

$$= \lambda^3 h(u) - \lambda^2 (1 - \lambda) \left[ p_1(u)p_2(u)p_3(v) + p_1(u)p_2(v)p_3(u) + p_1(v)p_2(u)p_3(u) \right]$$

$$\leq \lambda^3 h(u) - \lambda^2 (1 - \lambda) \left[ p_1(u)p_2(u)p_3(v) + p_1(u)p_2(v)p_3(u) + p_1(v)p_2(u)p_3(u) \right] + (1 - \lambda)^3 h(v)$$

$$\leq \lambda^3 c + \lambda^2 (1 - \lambda) c \left[ p_1(u)p_2(u)p_3(v) + p_1(u)p_2(v)p_3(u) + p_1(v)p_2(u)p_3(u) \right] + (1 - \lambda)^3 c$$

where (a) is due to the concavity of $p_1, p_2$ and $p_3$; (b) holds since, for any $r, s > 0$, denoting $t = \sqrt{rs} > 0$, we obtain

$$rs + \frac{1}{r} + \frac{1}{s} = rs + \frac{r + s}{rs} = t^2 + \frac{r + s}{t^2} \geq t^2 + \frac{2t}{t^2} - 3 + 3 = \frac{(t - 1)^2(t + 2)}{t} + 3 \geq 3.$$  

\[\square\]

**Theorem 5.2** Any stationary solution is global for problem (7).

**Proof.** Let $p_1(x, y) \triangleq f_1^{\max} + \varepsilon_1 - f_1(x), p_2(x, y) \triangleq f_2^{\max} + \varepsilon_2 - f_2(x)$ and $p_3(x, y) \triangleq f_3^{\max} + \varepsilon_3 - f_3(x) - y$. We observe that there exists an open convex set $S$ containing $X \times \{0\}$, such that $p_1, p_2$ and $p_3$ are concave and positive on $S$. Therefore, by Lemma 5.1, $h \triangleq -p_1 p_2 p_3$, is quasiconvex on $S$. We note that $\nabla_h h(x, y) = p_1(x, y)p_2(x, y) > 0$ for every $(x, y) \in S$. Hence, $\nabla h(x, y) \neq 0$ in $S$ and, by Lemma 5.3, $h$ is pseudoconvex on $S$. In turn, the objective function in (7) is pseudoconvex on an open convex set containing $X$ and the thesis holds.

\[\square\]

In portfolio optimization, one may consider a further objective in order to diversify the selection. In that light, here, in addition to expected return and variance (see [43]), we consider a third objective, namely $f_3 \triangleq \left\| x - \frac{\sigma^2}{\varepsilon} \right\|^2$. The introduction of the latter objective is intended
to avoid “too sparse” solutions by minimizing the distance, in terms of Euclidean norm, with respect to an equally-weighted portfolio.

\[
\begin{align*}
\text{minimize} & \quad \left( -m^T x, x^T V x, \left\| x - \frac{e^T x}{n} e \right\|^2 \right)^T \\
\text{s.t.} & \quad e^T x = 1 \\
& \quad x \geq 0.
\end{align*}
\] (8)

Notice that

\[
\left\| x - \frac{e^T x}{n} e \right\|^2 = x^T x - \frac{(e^T x)^2}{n} = x^T x - \frac{1}{n},
\]

where the second equality follows from the budget constraint.

In the same spirit of the previous analysis, we cope with problem (8) by solving the following program:

\[
\begin{align*}
\text{minimize} & \quad - \left( -m^{\text{min}} + \varepsilon_1 + m^T x \right) \left( V^{\text{max}} + \varepsilon_2 - x^T V x \right) \left( 1 + \varepsilon_3 - x^T x \right) \\
\text{s.t.} & \quad e^T x = 1 \\
& \quad x \geq 0.
\end{align*}
\] (9)

The results of our tests are summarized in Table 3. For each index we compared the behavior of two objectives approach hyper2 with that of three objective one hyper3 in terms of \%return, \%risk, DivIdx \( \triangleq x^T x - \frac{1}{\# \text{assets}} \), Volume \( \triangleq 10000 \left( -m^{\text{min}} + m^T x \right) \left( V^{\text{max}} - x^T V x \right) \left( 1 - x^T x \right) \) and \#ptf assets. The performed experiments show that, in all the cases that have been considered, the introduction of the third objective, resulting in an increased diversification of the portfolio, reduces the overall risk at the cost of slightly worse returns.

| Method     | %return | %risk | DivIdx | Volume | #ptf assets |
|------------|---------|-------|--------|--------|-------------|
| DowJones   | hyper3  | 0.485 | 0.102  | 0.107  | 0.087       | 13          |
|            | hyper2  | 0.538 | 0.127  | 0.197  | 0.081       | 6           |
| NASDAQ100  | hyper3  | 0.826 | 0.134  | 0.087  | 0.420       | 16          |
|            | hyper2  | 0.897 | 0.159  | 0.166  | 0.397       | 7           |
| FTSE100    | hyper3  | 0.565 | 0.091  | 0.103  | 0.378       | 18          |
|            | hyper2  | 0.660 | 0.136  | 0.272  | 0.329       | 4           |
| SP500      | hyper3  | 0.856 | 0.149  | 0.112  | 1.032       | 15          |
|            | hyper2  | 0.943 | 0.194  | 0.228  | 0.948       | 5           |
| NASDAQComp | hyper3  | 1.021 | 0.144  | 0.053  | 3.318       | 29          |
|            | hyper2  | 1.125 | 0.215  | 0.273  | 2.675       | 9           |
| FF49Industries | hyper3 | 0.493 | 0.068  | 0.087  | 0.046       | 14          |
|            | hyper2  | 0.535 | 0.083  | 0.670  | 0.018       | 4           |

Table 3: Comparison between the two and the three objectives models
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