Fredholm property and essential spectrum of 3-D Dirac operators with regular and singular potentials

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ABSTRACT
We consider the 3-D Dirac operator with variable regular magnetic and electrostatic potentials, and singular potentials

\[ D_{A, \Phi, Q_{\sin}} u(x) = (D_{A, \Phi} + Q_{\sin}) u(x), \quad x \in \mathbb{R}^3 \]  \hspace{1cm} (1)

where

\[ D_{A, \Phi} = \sum_{j=1}^{3} \alpha_j (i \partial_{x_j} + A_j(x)) + \alpha_0 m + \Phi(x) I_4, \]  \hspace{1cm} (2)

\[ Q_{\sin} = \Gamma(s) \delta_{\Sigma} \] is the singular potential with \( \Gamma(s) = \left( \Gamma_{ij}(s) \right)_{i,j=1}^4 \) being a \( 4 \times 4 \) matrix and \( \delta_{\Sigma} \) is the delta-function with support on a surface \( \Sigma \subset \mathbb{R}^3 \) which divides \( \mathbb{R}^3 \) on two open domains \( \Omega_{\pm} \) with the common boundary \( \Sigma \). \( u \) is a vector-function on \( \mathbb{R}^3 \) with values in \( \mathbb{C}^4 \), \( \alpha_j, j = 0, 1, 2, 3 \) are the standard \( 4 \times 4 \) Dirac matrices. We associate with the formal Dirac operator \( D_{A, \Phi, Q_{\sin}} \) an unbounded operator \( D \) in \( L^2(\mathbb{R}^3, \mathbb{C}^4) \) generated by \( D_{A, \Phi} \) with domain in \( H^1(\Omega_+, \mathbb{C}^4) \oplus H^1(\Omega_-, \mathbb{C}^4) \) consisting of functions satisfying transmission conditions on \( \Sigma \). We consider the self-adjointness of operator \( D \), its Fredholm properties, and the essential spectrum in the case if \( \Sigma \) is either a closed \( C^2 \)-surface or an unbounded \( C^2 \)-hypersurface with a regular behaviour at infinity.

As application we consider the electrostatic and Lorentz scalar \( \delta \)-shell interactions.

1. Introduction

The Schrödinger operators with singular potentials supported on surfaces have attracted a lot of attention: for instance, they are used for a description of quantum particles interacting with charged surfaces, in approximations of Hamiltonians of the propagation of electrons through thin barriers. The mathematical problems connected with formal Schrödinger operators with singular potentials

\[ \mathcal{S} = -\Delta + a \delta_{\Sigma} \]
where $\Sigma$ is the set of Lebesgue measure 0 in $\mathbb{R}^n$ include the realization of the formal Schrödinger operator $S$ as an unbounded operator $S$ in the Hilbert space $L^2(\mathbb{R}^n)$ and the study of the spectral properties of $S$.

Over the past two decades, this topic has been intensively studied and there is extensive literature devoted to this problem [1–11].

The investigation of the 3D-Dirac operators with singular potentials supported on compact closed surfaces in $\mathbb{R}^3$ was initiated only recently in the pioneering paper [12], where a new approach to extension theory of symmetric operators was employed. This research was continued in the papers: [13–15]. A different approach using the abstract theory of quasi-boundary triples and their Weyl functions was proposed in [7,8].

In contrast to the indicated papers, we consider singular potentials with supports on both bounded and unbounded surfaces in $\mathbb{R}^3$. Our approach to the self-adjointness of Dirac operators is based on the study of transmission problems with parameter associated with the Dirac operators. We introduce the Lopatinsky conditions for their invertibility for large values of the parameter, and for the a priori estimates of solutions of associated transmission problems.

Moreover we study the Fredholm properties and the essential spectrum of transmission problems associated with the Dirac operators with singular potentials with supports on compact surfaces and non-compact surfaces with conical exits to infinity. For this aim we apply the local principle [16,17] and the limit operators method, see for instance [18–21].

We consider in the paper the Dirac operators

$$D_{A,\Phi,Q_{\sin}} u(x) = \left( D_{A,\Phi} + Q_{\sin} \right) u(x), \quad x \in \mathbb{R}^3,$$

where $u$ is a vector-function on $\mathbb{R}^3$ with values in $\mathbb{C}^4$,

$$D_{A,\Phi} u(x) = \left( \sum_{j=1}^{3} \alpha_j \left( i\partial_{x_j} + A_j(x) \right) + \alpha_0 m + \Phi(x) \right) u(x), \quad x \in \mathbb{R}^3$$

is the 3D-Dirac operator with variable regular magnetic and electrostatic potentials, $\alpha_j$, $j = 0, 1, 2, 3$ are the $4 \times 4$ Dirac matrices satisfying the relations

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk}I_4 (j, k = 0, 1, 2, 3),$$

$I_4$ is the $4 \times 4$ unit matrix, $A = (A_1, A_2, A_3)$ is the variable vector-valued potential of the magnetic field $H$, that is $H = \nabla \times A$, $\Phi$ is the variable electrostatic potential of the electric field $E$, that is $E = \nabla \Phi$, $m$ is the mass of electron. We use the system of coordinates for which the Planck constant $\hbar = 1$, the light speed $c = 1$, and the charge of electron $e = 1$. We assume that the functions $A_{ij}, j = 1, 2, 3$, and $\Phi$ belong to $L^\infty(\mathbb{R}^3)$. The singular potential $Q_{\sin}$ in (3) is $Q_{\sin} = \Gamma \delta_\Sigma$ where $\Gamma = (\Gamma_{ij})^{4}_{i,j=1}$ is a $4 \times 4$ matrix-valued function with $\Gamma_{ij}$ belonging to the class $C_b(\Sigma)$ of continuous bounded functions on $\Sigma$ and $\delta_\Sigma$ is the delta-function with support on $C^2$-surface $\Sigma \subset \mathbb{R}^3$ which divides $\mathbb{R}^3$ on two open domains $\Omega_\pm$ with the common boundary $\Sigma$. We assume that either $\Sigma$ is a $C^2$-connected compact surface or $\Sigma$ is a $C^2$-unbounded connected surface regular at infinity.

Let $H^1(\mathbb{R}^3, \mathbb{C}^4)$ be the Sobolev space of 4-dimensional vector-valued functions $u$ on $\mathbb{R}^3$. We denote by $H^1(\Omega_\pm, \mathbb{C}^4)$ the spaces of restrictions on $\Omega_\pm$ functions in $H^1(\mathbb{R}^3, \mathbb{C}^4)$ and
\[ H^1(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4) = H^1(\Omega_+ , \mathbb{C}^4) \oplus H^1(\Omega_-, \mathbb{C}^4). \]

We associate with the formal Dirac operator \( \mathcal{D}_{A, \Phi, Q_{\text{lin}}} \) an unbounded in \( L^2(\mathbb{R}^3, \mathbb{C}^4) \) operator \( \mathcal{D}_{A, \Phi, a^+, a^-} \) defined by the Dirac operator \( \mathcal{D}_{A, \Phi} \) with domain

\[
dom \mathcal{D}_{A, \Phi, a^+, a^-} = H^1_{a^+, a^-}(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4)
\]

\[
= \left\{ u \in H^1(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4) : a^+_+(s)u_+(s) + a^-_-(s)u_-(s) = 0, \ s \in \Sigma \right\},
\]

where \( u_\pm = \gamma^\pm \mathcal{D}_{A, \Phi} u \) and \( \gamma^\pm : H^1(\Omega_{\pm}, \mathbb{C}^4) \to H^1(\Sigma, \mathbb{C}^4) \) are operators of the trace, \( a^\pm(s) \) are \( 4 \times 4 \) matrices defined as

\[
a^+_+(s) = \frac{1}{2} \Gamma(s) - \mathbf{i} \alpha \cdot \mathbf{v}(s), \quad a^-_-(s) = \frac{1}{2} \Gamma(s) + \mathbf{i} \alpha \cdot \mathbf{v}(s)
\]

where \( \alpha \cdot \mathbf{v}(s) = \sum_{j=1}^{3} \alpha_j v_j(s), \) and \( \mathbf{v}(s) = (v_1(s), v_2(s), v_3(s)), \ s \in \Sigma \) is the normal vector to \( \Sigma \) directed into \( \Omega_- \).

We associate also with the formal Dirac operator \( \mathcal{D}_{A, \Phi, Q_{\text{lin}}} \) the operator \( \mathbb{D}_{A, \Phi, a^+, a^-} \) of the transmission problem

\[
\mathbb{D}_{A, \Phi, a^+, a^-} u(x) = \left\{ \begin{array}{ll}
\mathcal{D}_{A, \Phi} u(x), & x \in \mathbb{R}^3 \setminus \Sigma \\
a^+_+(s)u_+(s) + a^-_-(s)u_-(s) = 0, & s \in \Sigma
\end{array} \right.
\]

acting from \( H^1(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4) \) into \( L^2(\mathbb{R}^3, \mathbb{C}^4) \).

We study in the paper the self-adjointness in \( L^2(\mathbb{R}^3, \mathbb{C}^4) \) of unbounded operators \( \mathcal{D}_{A, \Phi, a^+, a^-} \), the Fredholm properties of the operators \( \mathbb{D}_{A, \Phi, a^+, a^-} : H^1(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4) \), and the essential spectra of operators \( \mathcal{D}_{A, \Phi, a^+, a^-} \).

The paper is organized as follows. In Section 2, we introduce the necessary notations and definitions. In Section 3, we describe the realization of formal Dirac operators with singular potentials as unbounded operators \( \mathcal{D}_{A, \Phi, a^+, a^-} \) in \( L^2(\mathbb{R}^3, \mathbb{C}^4) \) and also as bounded operators \( \mathcal{D}_{A, \Phi, a^+, a^-} \) of transmission problems (7), and we study the conditions of self-adjointness of \( \mathcal{D}_{A, \Phi, a^+, a^-} \). Our approach is closed to classical approach to the proof of self-adjointness of realizations of boundary value problems in Hilbert space (see for instance [22,23]). We introduced an analogue of the Lopatinsky conditions on \( \Sigma \) for the operator \( \mathcal{D}_{A, \Phi, a^+, a^-} \), which yields the a priori estimate

\[
\|u\|_{H^1(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4)} \leq C \left( \|\mathcal{D}_{A, \Phi} u\|_{L^2(\mathbb{R}^3, \mathbb{C}^4)} + \|u\|_{L^2(\mathbb{R}^3, \mathbb{C}^4)} \right)
\]

for compact \( C^2 \)-surfaces and non-compact \( C^2 \)-surfaces with regular behaviour at infinity. It should be noted that the estimate (8) yields the closedness of \( \mathcal{D}_{A, \Phi, a^+, a^-} \) in \( L^2(\mathbb{R}^3, \mathbb{C}^4) \).

We also consider the parameter-dependent operator

\[
\mathbb{D}_{A, \Phi, a^+, a^-}(\mu) = \left\{ \begin{array}{ll}
(\mathcal{D}_{A, \Phi} - \mathbf{i} \mu \mathbb{I}_4) u(x), & x \in \mathbb{R}^3 \setminus \Sigma, \ \mu \in \mathbb{R} \\
a^+_+(s)u_+(s) + a^-_-(s)u_-(s) = 0, & s \in \Sigma
\end{array} \right.,
\]

and give the uniform Lopatinsky conditions of the invertibility of \( \mathbb{D}_{A, \Phi, a^+, a^-}(\mu) : H^1(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4) \) for \( |\mu| \) large enough. This conditions are applied for the study of self-adjointness of \( \mathcal{D}_{A, \Phi, a^+, a^-} \).
As the example we consider the operator \( D_{A, \Phi, a^+, a^-} \) associated with the singular potential

\[
Q_{\sin} = \eta I_4 + \tau \alpha_0
\]

(10)
describing the electrostatic and Lorentzshall interactions. The uniform Lopatinsky condition for \( D_{A, \Phi, a^+, a^-} \) is satisfied if

\[
\inf_{s \in \Sigma} |\eta^2(s) - \tau^2(s) - 4| > 0.
\]

(11)

Note that the unbounded operator \( D_{A, \Phi, a^+, a^-} \) associated with the potential (10) has been considered earlier in the paper [8] for compact closed surface and real constants \( \eta, \tau \). In this paper, the authors proved that the condition \( \eta^2 - \tau^2 \neq 4 \) is sufficient for self-adjointness of \( D_{0,0,a^+,a^-} \). They also considered the degenerated case \( \eta^2 - \tau^2 = 4 \).

In Section 3.1, we study the essential spectrum of the operator \( D_{A, \Phi, a^+, a^-} \) for the cases: \( \Sigma \) is either a \( C^2 \)-compact surface or a \( C^2 \)-surface with conic structure at infinity. This problem is closely connected with the Fredholm properties of the operator \( \mathbb{D}_{A, \Phi, a^+, a^-} \) of transmission problem acting from \( H^1(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4) \) to \( L^2(\mathbb{R}^3, \mathbb{C}^4) \). Our approach to the investigation of Fredholmness is based on the local principle see (see for instance [16,17]) and the limit operators method (see for instance [11,21,24]).

As the corollary, we obtain the descriptions of the essential spectra of unbounded operators \( D_{A, \Phi, a^+, a^-} \) as the union of spectra of all limit operators. It should be noted that the conditions of the Fredholmness of \( \mathbb{D}_{A, \Phi, a^+, a^-} \) and the location of the essential spectrum of \( D_{A, \Phi, a^+, a^-} \) substantial depend on the behaviour at infinity of the regular potentials \( A, \Phi \), the surface \( \Sigma \), and the matrix \( \Gamma \).

2. Notations

- If \( X, Y \) are Banach spaces, then we denote by \( B(X, Y) \) the space of bounded linear operators acting from \( X \) into \( Y \) with the uniform operator topology, and by \( K(X, Y) \) the subspace of \( B(X, Y) \) of all compact operators. In the case \( X = Y \) we write shortly \( B(X) \) and \( K(X) \).
- An operator \( A \in B(X, Y) \) is called a Fredholm operator if \( kerA = \{ x \in X : Ax = 0 \} \), and \( cokerA = Y/\text{Im}A \) are finite-dimensional spaces. Let \( A \) be a closed unbounded operator in a Hilbert space \( \mathcal{H} \) with a dense in \( \mathcal{H} \) domain \( \text{dom} A \). Then \( A \) is called a Fredholm operator if \( kerA = \{ x \in \text{dom} A : Ax = 0 \} \) and \( cokerA = \mathcal{H}/\text{Im}A \) where \( \text{Im}A = \{ y \in \mathcal{H} : y = Ax, x \in \text{dom} A \} \) are finite-dimensional spaces. Note that \( A \) is a Fredholm operator as unbounded operator in \( \mathcal{H} \) if and only if \( A : \text{dom} A \to \mathcal{H} \) is a Fredholm operator as a bounded operator where \( \text{dom} A \) is equipped by the graph norm

\[
\|u\|_{\text{dom} A} = \left( \|u\|^2_\mathcal{H} + \|Au\|^2_\mathcal{H} \right)^{1/2}, \quad u \in \text{dom} A
\]

(see for instance [22]).
- The essential spectrum \( sp_{\text{ess}} A \) of an unbounded operator \( A \) is a set of \( \lambda \in \mathbb{C} \) such that \( A - \lambda I \) is not Fredholm operator as unbounded operator, and the discrete spectrum \( sp_{\text{dis}} A \) of \( A \) is a set of isolated eigenvalues of finite multiplicity. It is well known that if \( A \) is a self-adjoint operator then \( sp_{\text{dis}} A = sp A \setminus sp_{\text{ess}} A \).
We denote by $L^2(\mathbb{R}^3, \mathbb{C}^4)$ the Hilbert space of 4-dimensional vector-functions $\mathbf{u}(x) = (u^1(x), u^2(x), u^3(x), u^4(x))$, $x \in \mathbb{R}^3$ with the scalar product

$$\langle \mathbf{u}, \mathbf{v} \rangle = \int_{\mathbb{R}^3} \mathbf{u}(x) \cdot \mathbf{v}(x) \, dx$$

where $\mathbf{u} \cdot \mathbf{v} = \sum_{j=1}^4 u_j \bar{v}_j$.

We denote by $H^s(\mathbb{R}^3, \mathbb{C}^4)$ the Sobolev space on $\mathbb{R}^3$ of 4-dimensional vector-valued functions, that is the space of distributions $\mathbf{u} \in \mathcal{D}'(\mathbb{R}^3, \mathbb{C}^4)$ such that

$$\| \mathbf{u} \|_{H^s(\mathbb{R}^3, \mathbb{C}^4)} = \left( \int_{\mathbb{R}^3} (1 + |\xi|^2)^s \| \hat{\mathbf{u}}(\xi) \|_{\mathbb{C}^4}^2 \, d\xi \right)^{1/2} < \infty, \quad s \in \mathbb{R}$$

where $\hat{\mathbf{u}}$ is the Fourier transform of $\mathbf{u}$. If $\Omega$ is a domain in $\mathbb{R}^3$ then $H^s(\Omega, \mathbb{C}^4)$ is the space of restrictions of $\mathbf{u} \in H^s(\mathbb{R}^3, \mathbb{C}^4)$ on $\Omega$ with the norm

$$\| \mathbf{u} \|_{H^s(\Omega, \mathbb{C}^4)} = \inf_{\mathbf{v} \in H^s(\mathbb{R}^3, \mathbb{C}^4)} \| \mathbf{u} - \mathbf{v} \|_{H^s(\mathbb{R}^3, \mathbb{C}^4)}$$

where $\mathbf{lu}$ is an extension of $\mathbf{u}$ on $\mathbb{R}^3$.

We denote by $C_b(\mathbb{R}^3)$ the class of bounded continuous functions on $\mathbb{R}^3$, $C_b^m(\mathbb{R}^3)$ the class of functions $a$ on $\mathbb{R}^3$ such that $\partial^\alpha a \in C_b(\mathbb{R}^3)$ for all multiindices $\alpha : |\alpha| \leq m$. We denote by $C_b(\Sigma)$ the class of bounded continuous functions on a surface $\Sigma \subset \mathbb{R}^3$, and $C_0(\Sigma) = \{ f \in C(\Sigma) : \lim_{s \to \infty} f(s) = 0 \}$.

Let $\Omega_\pm$ be domains in $\mathbb{R}^3$ and $\Sigma$ be a $C^2$-surface being their common boundary. We say that $\Sigma$ is uniformly regular if: (i) There exists $r > 0$ such that for every point $x_0 \in \Sigma$ there exists a ball $B_r(x_0) = \{ x \in \mathbb{R}^3 : |x - x_0| < r \}$ and the diffeomorphism $\varphi_{x_0} : B_r(x_0) \to B_1(0)$ such that

$$\varphi_{x_0}(B_r(x_0) \cap \Omega_\pm) = B_1(0) \cap \mathbb{R}^3_\pm, \mathbb{R}^3_\pm = \{ y = (y', y_3) \in \mathbb{R}^3 : y_3 \geq 0 \},$$

$$\varphi_{x_0}(B_r(x_0) \cap \Sigma) = B_1(0) \cap \mathbb{R}^2_y, \mathbb{R}^2_y = \{ y = (y', y_3) \in \mathbb{R}^3 : y_3 = 0 \}.$$

(ii) Let $\psi_{x_0} = \varphi_{x_0}^{-1}$ and $\varphi_{x_0}^i, \psi_{x_0}^i, i = 1, 2, 3$ be the coordinate functions of the mappings $\varphi_{x_0}, \psi_{x_0}$. Then

$$\sup_{x_0 \in S} \sup_{|\alpha| \leq 2, x \in B_r(x_0)} |\partial^\alpha \varphi_{x_0}^i(x)| < \infty, \quad \sup_{x_0 \in S} \sup_{|\alpha| \leq 2, y \in B_1(0)} |\partial^\alpha \psi_{x_0}^i(y)| < \infty,$$

$$i = 1, 2, 3.$$

Note that compact closed $C^2$-surfaces are uniformly regular automatically.

Let

$$\mathcal{D}_0 = \mathbf{a} \cdot \mathbf{D} + \alpha_0 m = \sum_{j=1}^3 \alpha_j \mathcal{D}_{x_j} + \alpha_0 m, \quad D_{x_j} = i \partial x_j$$

be the free Dirac operator (see for instance [25–27]) where $\mathbf{a} = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_j, j = 0, 1, 2, 3$ are the $4 \times 4$ Dirac matrices

$$\alpha_0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad j = 1, 2, 3$$ (12)
σ₁ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad σ₂ = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad σ₃ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{13}

are the 2 × 2 Pauli matrices satisfying the relations

σⱼσₖ + σₖσⱼ = 2δⱼₖI₂, \quad j, k = 1, 2, 3. \tag{14}

Relations (14) yield that

αⱼαₖ + αₖαⱼ = 2δⱼₖI₄, \quad j, k = 0, 1, 2, 3, \tag{15}

where \( I_n \) is the \( n \times n \) unit matrix. Equality (15) implies that

\[(\alpha \cdot D_x)^2 = -\Delta I₄\]

where \( \Delta \) is 3D-Laplacian. Moreover

\[\mathcal{D}^2₀ = (-\Delta + m^2) I₄.\]

It is well known that the unbounded operator \( \mathcal{D}₀ \) with domain \( H¹(\mathbb{R}³, \mathbb{C}⁴) \) is self-adjoint in \( L²(\mathbb{R}³, \mathbb{C}⁴) \) (see for instance [27]), and

\[sp\mathcal{D}₀ = sp_{ess}\mathcal{D}₀ = (-\infty, -|m|] \bigcup [|m|, +\infty) .\]

3. Realization of Dirac operators with singular potentials as unbounded operators in \( L²(\mathbb{R}³, \mathbb{C}⁴) \)

Let \( \mathcal{D}ₐ,ϕ,Q_{sin} = \mathcal{D}ₐ,ϕ + Q_{sin} \) be the Dirac operator defined by formulas (3) and (4). We define the product \( Q_{sin}u \) where \( Q_{sin} = ΓδΣ \) and \( u ∈ H¹(\mathbb{R}³ \setminus Σ, \mathbb{C}⁴) \) as a distribution in \( D'(\mathbb{R}³, \mathbb{C}⁴) = D'(\mathbb{R}³) \otimes \mathbb{C}⁴ \) acting on the test functions \( \varphi ∈ C∞₀(\mathbb{R}³, \mathbb{C}⁴) \) as

\[(Q_{sin}u)(\varphi) = \frac{1}{2} \int \Gamma(s) (u_+(s) + u_-(s)) \cdot \varphi(s) \, ds. \tag{16}\]

Let \( u = (u_+, u_-) ∈ H¹(\Omega_+, \mathbb{C}⁴) \oplus H¹(\Omega_-, \mathbb{C}⁴), \) and \( \mathcal{D}ₐ,ϕu = (\mathcal{D}ₐ,ϕu_+, \mathcal{D}ₐ,ϕu_-) ∈ L²(\mathbb{R}³, \mathbb{C}⁴) \). Then Integrating by parts we obtain

\[\langle \mathcal{D}ₐ,ϕ,Q_{sin}u, \varphi \rangle_{L²(\mathbb{R}³, \mathbb{C}⁴)} = \int_{Ω⁺∪Ω⁻} \mathcal{D}ₐ,ϕu(x) \cdot \varphi(x) \, dx \]

\[− \int_{Σ} i\alpha \cdot v(s) (u_+(s) − u_-(s)) \cdot \varphi(s) \, ds \]

\[+ \frac{1}{2} \int_{Σ} Γ(s) (u_+(s) + u_-(s)) \cdot \varphi(s) \, ds, \varphi ∈ C∞₀(\mathbb{R}³, \mathbb{C}⁴). \tag{17}\]
Formula (17) yields that in the sense of distributions in $\mathcal{D}'(\mathbb{R}^3, \mathbb{C}^4)$

$$D_{A,\Phi,\upsilon_{in}}u = D_{A,\Phi}u - \left[ i\alpha \cdot \nu (u_+ - u_-) - \frac{1}{2} \Gamma (u_+ + u_-) \right] \delta_\Sigma,$$

(18)

Hence $\mathcal{D}_{A,\Phi,\upsilon_{in}}u \in L^2(\mathbb{R}^3, \mathbb{C}^4)$ if and only if

$$- i\alpha \cdot \nu(s) (u_+(s) - u_-(s)) + \frac{1}{2} \Gamma(s) (u_+(s) + u_-(s)) = 0$$

(19)

for all $s \in \Sigma$. Condition (19) can be written of the form

$$a_+(s)u_+(s) + a_-(s)u_-(s) = 0, \quad s \in \Sigma$$

(20)

where $a_\pm(s)$ are $4 \times 4$ matrices:

$$a_+(s) = \frac{1}{2} \Gamma(s) - i\alpha \cdot \nu(s), \quad a_-(s) = \frac{1}{2} \Gamma(s) + i\alpha \cdot \nu(s).$$

(21)

We associate with the formal Dirac operator $\mathcal{D}_{A,\Phi,\upsilon_{in}}$ the unbounded in $L^2(\mathbb{R}^3, \mathbb{C}^4)$ operator $\mathcal{D}_{A,\Phi,a_+,a_-}$ given by the regular Dirac operator $\mathcal{D}_{A,\Phi}$ and having the domain

$$\text{dom} \mathcal{D}_{A,\Phi,a_+,a_-} = H^1_{a_+,a_-}(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4)$$

$$= \{ u \in H^1(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4) : a_+(s)u_+(s) + a_-(s)u_-(s) = 0, \quad s \in \Sigma \}. \quad (22)$$

We associate also with the formal Dirac operator $\mathcal{D}_{A,\Phi,\upsilon_{in}}$ the bounded operator of the transmission problem

$$\mathcal{D}_{A,\Phi,a_+,a_-}u(x) = \begin{cases} \mathcal{D}_{A,\Phi}u(x), & x \in \mathbb{R}^3 \setminus \Sigma \\ a_+(s)u_+(s) + a_-(s)u_-(s) = 0, & s \in \Sigma \end{cases} \quad (23)$$

acting from $H^1(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4)$ into $L^2(\mathbb{R}^3, \mathbb{C}^4)$.

### 3.1. Lopatinsky conditions

We consider the Lopatinsky condition for the transmission operator $\mathcal{D}_{A,\Phi,a_+,a_-}$ at the points $x \in \Sigma$ which provides the local a priori estimates on $\Sigma$ for the operator $\mathcal{D}_{A,\Phi,a_+,a_-}$. This condition is an analogue of the Lopatinsky condition for elliptic boundary value problems (see for instance [22,23]).

Invariance of the Dirac operator with respect to the orthogonal transformations of the coordinate systems in $\mathbb{R}^3$ allows us to consider $\mathcal{D}_{A,\Phi,a_+,a_-}$ in the local system of coordinates $y = (y_1, y_2, y_3)$ where the axis $y_1, y_2$ belong to tangent plane to $\Sigma$ at the point $x_0$ and the axis $y_3 = z$ is directed along the normal vector $\nu$ to $\Sigma$ at the point $x_0 \in \Sigma$.

After to passing to the local system of coordinates and taking into account the main part of the operator $\mathcal{D}_{A,\Phi}$ we obtain the operator $\mathcal{D}_{a_+(x_0),a_-(x_0)}^0$ of the transmission problem for the half-spaces $\mathbb{R}^3_\pm = \{ y = (y', y_3) \in \mathbb{R}^3 : y_3 \geq 0 \}$

$$\mathcal{D}_{a_+(x_0),a_-(x_0)}^0u(y) = \begin{cases} (\alpha_1D_{y_1} + \alpha_2D_{y_2} + \alpha_3D_{y_3})u(y) = 0, & y \in \mathbb{R}^3_+ \cup \mathbb{R}^3_- \\ a_+(x_0)u_+(y',0) + a_-(x_0)u_-(y',0) = 0, & y' \in \mathbb{R}^2_{y'}, \end{cases}$$

(24)

acting from $H^1(\mathbb{R}^3 \setminus \mathbb{R}^2_{y'}, \mathbb{C}^4)$ into $L^2(\mathbb{R}^3, \mathbb{C}^4)$. After the Fourier transform with respect to $y' \in \mathbb{R}^2$, we obtain the family of the 1-dimensional transmission problems depending on
By formula (27), Equation (29) has the exponential solutions

\[
\hat{D}^0_{a_+ (x_0), a_- (x_0)} (\xi') \psi (\xi', z) = \begin{cases} \\
(\alpha_1 \xi_1 + \alpha_2 \xi_2 + i \alpha_3 \frac{d}{dz}) \psi (z) = 0, & z \in \mathbb{R} \setminus \{0\} \\
a_+ (x_0) \psi_+ (\xi', 0) + a_- (x_0) \psi_- (\xi', 0) = 0, & \xi' = (\xi_1, \xi_2) \in \mathbb{R}^2
\end{cases},
\]

where

\[
\psi \in H^1 (\mathbb{R} \setminus \{0\}, \mathbb{C}^4) = H^1 (\mathbb{R}_+, \mathbb{C}^4) \oplus H^1 (\mathbb{R}_-, \mathbb{C}^4),
\]

and

\[
a_+ (x_0) = \frac{1}{2} \Gamma (x_0) - i \alpha_3, \quad a_- (x_0) = \frac{1}{2} \Gamma (x_0) + i \alpha_3.
\]

One can prove that the operator \( \hat{D}^0_{a_+ (x_0), a_- (x_0)} : H^1 (\mathbb{R}^3 \setminus \mathbb{R}^2, \mathbb{C}^4) \to L^2 (\mathbb{R}^3, \mathbb{C}^4) \) is invertible if and only if the operator \( \hat{D}^0_{a_+ (x_0), a_- (x_0)} (\xi') : H^1 (\mathbb{R} \setminus \{0\}, \mathbb{C}^4) \to L^2 (\mathbb{R}^3, \mathbb{C}^4) \) is invertible for every \( \xi' \in \mathbb{R}^2 : |\xi'| = 1 \). The equality

\[
(\alpha_1 \xi_1 + \alpha_2 \xi_2 + i \alpha_3 \frac{d}{dz})^2 = \left( |\xi'|^2 - \frac{d^2}{dz^2} \right) I_4
\]

yields that \( \hat{D}^0_{a_+ (x_0), a_- (x_0)} (\xi') \) is the Fredholm operator for every \( \xi' \in S^1 \) with index 0. Hence \( \hat{D}^0_{a_+ (x_0), a_- (x_0)} (\xi') \) is invertible if and only if \( \ker \hat{D}^0_{a_+ (x_0), a_- (x_0)} (\xi') = \{0\} \). We consider the equation

\[
(\alpha_1 \xi_1 + \alpha_2 \xi_2 + i \alpha_3 \frac{d}{dz}) \psi (\xi', z) = 0, \quad z \in \mathbb{R}, \quad \xi' = (\xi_1, \xi_2),
\]

where

\[
\psi (\xi', z) = \begin{pmatrix} \psi^1 (\xi', z) \\
\psi^2 (\xi', z) \end{pmatrix}
\]

be the 4D-vector with \( \psi^j (\xi', z) \in \mathbb{C}^2, j = 1, 2 \). Then (28) yields that \( \psi^j (\xi', z) \) satisfies the equation

\[
(\sigma_1 \xi_1 + \sigma_2 \xi_2 + i \sigma_3 \frac{d}{dz}) \psi^j (\xi', z) = 0, \quad j = 1, 2.
\]

By formula (27), Equation (29) has the exponential solutions \( \psi^j (\xi', z) = h^j_{\pm} (\xi') e^{\pm |\xi'| z} \)

where \( h^j_{\pm} (\xi') \in \mathbb{C}^2 \). Moreover, formula (27) implies that \( h^j_{\pm} (\xi') = \Lambda_{\pm} (\xi') f^j_{\pm} \) where

\[
\Lambda_{\pm} (\xi') = \sigma_1 \xi_1 + \sigma_2 \xi_2 \pm i \sigma_3 |\xi'| = \begin{pmatrix} \pm i |\xi'| & \xi_1 \\
\overline{\xi} & \mp i |\xi'| \end{pmatrix}
\]

and \( \xi = \xi_1 + i \xi_2 \in \mathbb{C}^2 \).
and \( f_{\pm} \in \mathbb{C}^2 \) are arbitrary vectors. Note that for every vectors \( f_1, f_2 \in \mathbb{C}^2 \)
\[
\Lambda_\pm(\xi')f_1 \cdot \Lambda_{\mp}(\xi')f_2 = 0. 
\]

(30)

Let
\[
\psi_{+,1}(\xi', z) = h_{+,1}(\xi')e^{i|\xi'|z}, \quad \psi_{+,2}(\xi', z) = h_{+,2}(\xi')e^{i|\xi'|z}, \\
\psi_{-,1}(\xi', z) = h_{-,1}(\xi')e^{-i|\xi'|z}, \quad \psi_{-,2}(\xi', z) = h_{-,2}(\xi')e^{-i|\xi'|z},
\]

(31)

where
\[
h_{\pm,1}(\xi') = \begin{pmatrix} \Lambda_\pm(\xi')e & 0 \\ 0 & 0 \end{pmatrix}, \quad h_{\pm,2}(\xi') = \begin{pmatrix} 0 & \Lambda_\pm(\xi')e \\ \Lambda_\pm(\xi')e & 0 \end{pmatrix}, \quad e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

Formula (30) yields that the system of vectors
\[
\{h_{+,1}(\xi'), h_{+,2}(\xi'), h_{-,1}(\xi'), h_{-,2}(\xi')\}
\]
is orthogonal in \( \mathbb{C}^4 \) and
\[
\|h_{\pm,j}(\xi')\|^2_{\mathbb{C}^2} = 2|\xi'|^2.
\]

Moreover, \( \{\psi_{\pm,1}(\xi', z), \psi_{\pm,2}(\xi', z)\} \) is the fundamental system of solutions of Equation (28). Note that the general solution of Equation (28) in \( H^1(\mathbb{R} \setminus \{0\}, \mathbb{C}^4) \) is of the form
\[
\psi(\xi', z) = \begin{cases} 
\psi_{+}(\xi', z) = C_{+,1}\psi_{-,1}(\xi', z) + C_{+,2}\psi_{-,2}(\xi', z), & z \geq 0 \\
\psi_{-}(\xi', z) = C_{-,1}\psi_{+,1}(\xi', z) + C_{-,2}\psi_{+,2}(\xi', z), & z < 0.
\end{cases}
\]

Substituting \( \psi_{\pm}(\xi', z) \) in the interaction condition
\[
a_+(x_0)\psi_{+}(\xi', 0) + a_-(x_0)\psi_{-}(\xi', 0) = 0
\]

(33)

we obtain the linear system of equations with respect to \( C_{+,1}, C_{+,2}, C_{-,1}, C_{-,2} \)
\[
a_+(x_0)h_{-,1}C_{+,1} + a_+(x_0)h_{-,2}C_{+,2} + a_-(x_0)h_{+,1}C_{-,1} + a_-(x_0)h_{+,2}C_{-,2} = 0.
\]

(34)

Note that \( \ker \hat{D}_{a_+(x_0), a_-(x_0)}(\xi') = \{0\} \) if and only if system (34) has the trivial solution only.

We denote by \( L(x_0, \xi') \) the \( 4 \times 4 \) matrix with columns
\[
(a_+(x_0)h_{-,1}(\xi'), a_+(x_0)h_{-,2}(\xi'), a_-(x_0)h_{+,1}(\xi'), a_-(x_0)h_{+,2}(\xi')).
\]

Definition 3.1: We say that the local Lopatinsky condition for \( \hat{D}_{A, \Phi, a_+, a_-} \) is satisfied at the point \( s_0 \in \Sigma \) if
\[
\det L(s_0, \xi') \neq 0 \text{ for every } \xi' : |\xi'| = 1
\]

(35)

and we say that the uniform Lopatinsky condition for \( \hat{D}_{A, \Phi, a_+, a_-} \) is satisfied on \( \Sigma \) if
\[
\inf_{s \in \Sigma} \inf_{|\xi'| = 1} |\det L(s, \xi')| > 0.
\]

(36)

Thus, the operator \( \hat{D}_{a_+(x_0), a_-(x_0)}^0 \) is invertible if condition (35) is satisfied. Moreover, if condition (36) is satisfied then the family of the operators \( \hat{D}_{a_+(s), a_-(s)}^0, s \in \Sigma \) is uniformly invertible, that is
\[
\sup_{s \in \Sigma, |\xi'| = 1} \left\| \left( \hat{D}_{a_+(s), a_-(s)}^0 \right)^{-1} \right\| < \infty.
\]

(37)
3.2. Examples of the Lopatinsky conditions

- Let
  \[ \Gamma = \begin{pmatrix} 2\gamma I_2 & 0 \\ 0 & 2\epsilon I_2 \end{pmatrix}, \]
  where \( \gamma = \gamma(s), \epsilon = \epsilon(s) \in C_b(\Sigma) \) are real-valued functions. Then
  \[ a_{\pm} = \begin{pmatrix} \gamma I_2 \\ \mp i\sigma_3 \end{pmatrix} \begin{pmatrix} \mp i\sigma_3 \\ \epsilon I_2 \end{pmatrix}. \]

- We set
  \[ e_1 = a_+ h_{-,1}, \quad e_2 = a_- h_{-,2}, \quad e_3 = a_- h_{+,1}, \quad e_4 = a_+ h_{+,2}, \]
  where
  \[ \begin{align*}
  e_1 &= \begin{pmatrix} \gamma I_2 \\ -i\sigma_3 \\ -\epsilon I_2 \end{pmatrix} \begin{pmatrix} \Lambda_- e \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma \Lambda_- e \\ i\Lambda_+ e \end{pmatrix} \\
  e_2 &= \begin{pmatrix} \gamma I_2 \\ -i\sigma_3 \\ -\epsilon I_2 \end{pmatrix} \begin{pmatrix} 0 \\ \Lambda_- e \end{pmatrix} = \begin{pmatrix} i\Lambda_+ e \\ \epsilon \Lambda_- e \end{pmatrix} \\
  e_3 &= \begin{pmatrix} \gamma I_2 \\ i\sigma_3 \\ \epsilon I_2 \end{pmatrix} \begin{pmatrix} \Lambda_+ e \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma \Lambda_+ e \\ -i\Lambda_- e \end{pmatrix} \\
  e_4 &= \begin{pmatrix} \gamma I_2 \\ i\sigma_3 \\ \epsilon I_2 \end{pmatrix} \begin{pmatrix} 0 \\ \Lambda_+ e \end{pmatrix} = \begin{pmatrix} -i\Lambda_- e \\ \epsilon \Lambda_+ e \end{pmatrix}. \end{align*} \]

Taking into account that
\[ \Lambda_{\pm}^2 = 0 \quad \text{and} \quad \Lambda_{\pm}^* = \Lambda_{\mp} \]
we obtain that
\[ e_1 \cdot e_2 = e_1 \cdot e_3 = e_2 \cdot e_4 = e_3 \cdot e_4 = 0, \]
\[ e_1^2 = e_2^2 = 2(1 + \gamma^2) |\xi'|^2, \quad e_3^2 = e_4^2 = 2(1 + \epsilon^2) |\xi'|^2, \]
\[ e_1 \cdot e_4 = e_2 \cdot e_3 = 2i(\gamma + \epsilon)^2 |\xi'|^2. \]

Formulas (39) yields that
\[ \det (e_i \cdot e_j)_{i,j=1}^4 = 16 |\xi'|^8 (1 + \gamma^2)(1 + \epsilon^2) - (\gamma + \epsilon)^2) \]
\[ = 16 |\xi'|^8 (1 - \gamma \epsilon)^2. \]

Hence the Lopatinsky condition holds at the point \( s \in \Sigma \) if
\[ \gamma(s)\epsilon(s) \neq 1, \]
and the Lopatinsky condition holds uniformly on \( \Sigma \) if
\[ \inf_{s \in \Sigma} |1 - \gamma(s)\epsilon(s)| > 0. \]
• For the electrostatic potential: \( \gamma(s) = \epsilon(s) \), we obtain from (41), (42) the local Lopatinsky condition

\[
\gamma^2(s) \neq 1
\]

and the uniform Lopatinsky condition

\[
\inf_{s \in \Sigma} |1 - \gamma^2(s)| > 0.
\]

• Let

\[
Q_{\text{sin}} = 2(\eta(s)I_4 + \tau(s)\alpha_0)\delta_{\Sigma} = \begin{pmatrix} 2(\eta(s) + \tau(s))I_2 & 0 \\ 0 & 2(\eta(s) - \tau(s))I_2 \end{pmatrix} \delta_{\Sigma}
\]

be the sum of electrostatic and Lorentz potentials where \( \eta(s), \tau(s) \in C_b(\Sigma) \) are real-valued functions. Then we obtain from formulas (41), (42) the local Lopatinsky condition

\[
\eta^2(s) - \tau^2(s) \neq 1, \quad s \in \Sigma
\]

and the uniform Lopatinsky condition

\[
\inf_{s \in \Sigma} |\eta^2(s) - \tau^2(s) - 1| > 0. \tag{44}
\]

### 3.3. A priori estimates for operators \( \mathbb{D}_{A, \Phi, a^+, a^-} \)

**Theorem 3.2:** Let \( \Sigma \subset \mathbb{R}^3 \) be a \( C^2 \)-uniformly regular surface being the common boundary of the domain \( \Omega_{\pm}, A = (A_1, A_2, A_3) \in C^1(\mathbb{R}^3, \mathbb{C}^3), \Phi \in C^1(\mathbb{R}), \Gamma \in C_b(\Sigma, \mathcal{B}(\mathbb{C}^4)) = C_b(\Sigma) \otimes \mathcal{B}(\mathbb{C}^4) \), and the Lopatinsky condition be satisfied on \( \Sigma \) uniformly, that is

\[
\inf_{s \in \Sigma, |\xi'| = 1} |\det \mathcal{L}(s, \xi')| > 0. \tag{45}
\]

Then there exists \( C > 0 \) such that for every \( u \in H^1(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4) \)

\[
\|u\|_{H^1(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4)} \leq C \left( \|\mathbb{D}_{A, \Phi, a^+, a^-}u\|_{L^2(\mathbb{R}^3, \mathbb{C}^4)} + \|u\|_{L^2(\mathbb{R}^3, \mathbb{C}^4)} \right). \tag{46}
\]

**Proof:** The proof of the a priori estimate (46) is based on the local a priori estimates following from the uniform ellipticity on \( \mathbb{R}^3 \) of Dirac operator \( \mathbb{D}_{A, \Phi} \) and the uniform Lopatinsky conditions. For the gluing of local estimates, we use a countable partition of the unity of finite multiplicity. The uniform ellipticity of \( \mathbb{D}_{A, \Phi} \) yields that there exists a small enough \( r > 0 \) such that for every point \( x_0 \in \mathbb{R}^3 \setminus \Sigma \) there exists a ball \( B_r(x_0) = \{x \in \mathbb{R}^3 : |x - x_0| < r\} \) such that the local a priori estimate

\[
\|u\|^2_{H^1(B_r(x_0), \mathbb{C}^4)} \leq C \left( \|\mathbb{D}_{A, \Phi, a^+, a^-}u\|^2_{L^2(B_r(x_0), \mathbb{C}^4)} + \|u\|^2_{L^2(B_r(x_0), \mathbb{C}^4)} \right) \tag{47}
\]

holds for every \( u \in H^1(B_r(x_0), \mathbb{C}^4) \) with a constant \( C > 0 \) independent of \( x_0 \in \mathbb{R}^3 \setminus \Sigma \). The uniform Lopatinsky condition yields that for small enough \( r > 0 \) independent of \( x_0 \in \Sigma \)
there exists the local a priori estimates at every point \( x_0 \in \Sigma \)
\[
\| u \|_{H^1(B_r(x_0) \setminus \Sigma, \mathbb{C}^4)} \leq C \left( \| D_{A,\Phi,a_+,a_-} u \|_{L^2(B_r(x_0), \mathbb{C}^4)}^2 + \| u \|_{L^2(B_r(x_0), \mathbb{C}^4)}^2 \right)
\]
for \( u \in H^1(B_r(x_0) \setminus \Sigma, \mathbb{C}^4) \) with a constant \( C > 0 \) independent of \( x_0 \in \Sigma \). Since \( \Sigma \) is a uniformly regular surface there exists a partition of unity
\[
\sum_{j=1}^{\infty} \theta_j(x) = 1, \ x \in \mathbb{R}^3, \quad \theta_j \in C^\infty_0(B_r(x_j))
\]
subordinated to the countable covering \( \bigcup_{j=1}^{\infty} B_r(x_j) \) of finite multiplicity such that a priori estimates (47) or (48) hold for all points \( x_0 = x_j \) with a constant \( C > 0 \) independent of \( j \in \mathbb{N} \). We obtain a priori estimate (46) gluing these estimates by means of partition of unity (49).

### 3.4. Parameter-dependent transmission problems associated with Dirac operators with singular potentials

We consider the invertibility of the parameter-dependent operator
\[
D_{A,\Phi,a_+,a_-}(\mu)u(x) = \left(D_{A,\Phi,a_+,a_-} - i\mu I_4\right)u(x)
\]
acting from \( H^1(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4) \) in \( L^2(\mathbb{R}^3, \mathbb{C}^4) \) for large value of \( |\mu| \). We will apply the local approach following to the well-known paper [28]). Let \( D_{A,\Phi}(\mu) = \alpha \cdot D - i\mu I_4 \) is the main part of the parameter-dependent operator \( D_{A,\Phi} - i\mu I_4 \). Since
\[
(\alpha \cdot D + i\mu I_4)(\alpha \cdot D - i\mu I_4) = (-\Delta + \mu^2)I_4
\]
the operator \( D_{A,\Phi} - i\mu I_4 \) is the uniformly elliptic operator with parameter \( \mu \in \mathbb{R} \). Moreover, the operator
\[
\alpha \cdot D - i\mu I_4 : H^1(\mathbb{R}^3, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)
\]
is invertible for every \( \mu \in \mathbb{R} : |\mu| > 0 \) with
\[
(\alpha \cdot D - i\mu I_4)^{-1} = (\alpha \cdot D + i\mu I_4) \left(-\Delta + \mu^2\right)^{-1}I_4
\]
and
\[
\| (\alpha \cdot D - i\mu I_4)^{-1} \|_{B(\mathbb{R}^3, \mathbb{C}^4), H^1(\mathbb{R}^3, \mathbb{C}^4))} \leq \frac{C}{|\mu|}, \quad |\mu| > 0.
\]
For the local estimates at the points \( x_0 \in \Sigma \), we use the local system of orthogonal coordinates \( y = (y_1, y_2, y_3) \) where the axis \( y_1, y_2 \) belong to the tangent plane to \( \Sigma \) at the point.
$x_0 \in \Sigma$ and the axis $y_3 = z$ is directed along the normal vector $v_{x_0}$ to $\Sigma$ at the point $x_0 \in \Sigma$, and we take the main part of $D_{\alpha}(\mu)$. Then we obtain the operator

$$D_{a_+ (x_0), a_-(x_0)}^0 (\mu) \Psi (y)$$

$$= \left( D_{a_+ (x_0), a_-(x_0)}^0 - i \mu \right) \Psi (y)$$

$$= \left\{ \begin{array}{ll}
\alpha' \cdot D_y' + i \alpha_3 \frac{d}{dy_3} - i \mu I_4 & \Psi (y) = 0, \quad y \in \mathbb{R}^3 \cup \mathbb{R}^3_+ \\
\alpha_+ (x_0) \Psi_+ (y', 0) + a_-(x_0) \Psi_- (y', 0) = 0, \quad y' \in \mathbb{R}^2
\end{array} \right.,$$

(52)

where

$$\alpha' \cdot D_y' = \alpha_1 D_1 y_1 + \alpha_2 D_2 y_2, \quad a_\pm (x_0) = \frac{1}{2} \Gamma (x_0) \mp i \alpha_3.$$

We investigate the invertibility of $D_{a_+ (x_0), a_-(x_0)}^0 (\mu)$ acting from $H^1 (\mathbb{R}^3_+, \mathbb{C}^4) \oplus H^1 (\mathbb{R}^3, \mathbb{C}^4)$ into $L^2 (\mathbb{R}^3, \mathbb{C}^4)$. Applying the Fourier transform with respect to $y' = (y_1, y_2) \in \mathbb{R}^2$ we obtain the family of 1-D parameter-dependent transmission problems on $\mathbb{R} \setminus \{0\}$

$$\hat{D}_{a_+ (x_0), a_-(x_0)}^0 (\xi', \mu)$$

$$= \left\{ \begin{array}{ll}
\left( \alpha' \cdot \xi' + i \alpha_3 \frac{d}{dz} - i \mu I_4 \right) \Psi (\xi', \mu, z), \quad z \in \mathbb{R} \setminus \{0\}, \\
a_+ (x_0) \Psi_+ (\xi', \mu, 0) + a_-(x_0) \Psi_- (\xi', \mu, 0) = 0, \quad \alpha' \cdot \xi' = \alpha_1 \xi_1 + \alpha_2 \xi_2.
\end{array} \right.$$ (53)

where $z = y_3$.

Note that the operator $\hat{D}_{a_+ (x_0), a_-(x_0)}^0 (\xi', \mu) : H^1 (\mathbb{R} \setminus \{0\}, \mathbb{C}^4) \to L^2 (\mathbb{R}, \mathbb{C}^4)$ is the Fredholm operator of the index 0 if $|\xi'|^2 + \mu^2 > 0$. Hence $\hat{D}_{a_+ (x_0), a_-(x_0)}^0 (\xi', \mu)$ is invertible if and only if ker $\mathcal{B}_{a_+ (x_0), a_-(x_0)} (\xi', \mu) = 0$ for all $(\xi', \mu) : |\xi'|^2 + \mu^2 > 0$. We consider solutions of the equation

$$\hat{D}_{a_+ (x_0), a_-(x_0)}^0 (\xi', \mu) \Psi = 0$$

in the space $H^1 (\mathbb{R} \setminus \{0\}, \mathbb{C}^4) \subset L^2 (\mathbb{R}, \mathbb{C}^4)$. Since

$$\left( \alpha' \cdot \xi' + i \alpha_3 \frac{d}{dz} + i \mu I_4 \right) \left( \alpha' \cdot \xi' + i \alpha_3 \frac{d}{dz} - i \mu I_4 \right) = \left( |\xi'|^2 + \mu^2 - \frac{d^2}{dz^2} \right) I_4$$

(54)

the equation

$$\left( \alpha' \cdot \xi' + i \alpha_3 \frac{d}{dz} - i \mu I_4 \right) \Psi (\xi', \mu, z) = 0$$

(55)

has the exponential solutions

$$\Psi_\pm (\xi', \mu, z) = h_\pm (\xi', \mu) e^{\pm \rho z}, \quad \rho = \sqrt{|\xi'|^2 + \mu^2}.\quad (56)$$

where the vectors $h_\pm (\xi', \mu)$ satisfy the equation

$$(\alpha' \cdot \xi' \pm i \rho \alpha_3 - i \mu I_4) h_\pm (\xi', \mu) = 0.$$ \quad (57)
Taking into account (54) we obtain that the vectors \( h_\pm(\xi', \mu) \in \mathbb{C}^4 \) have the form

\[
h_\pm(\xi', \mu) = \Theta_\pm(\xi', \mu) f_\pm = (\alpha' \cdot \xi' \pm i \rho \alpha_3 + i \mu I_4) f_\pm
\]

where \( f_\pm \in \mathbb{C}^4 \). Let

\[
\Lambda_\pm(\xi', \mu) = \sigma' \cdot \xi' \pm i \rho \sigma_3 = \begin{pmatrix} \pm i \rho & \xi' \\ \xi' & \mp i \rho \end{pmatrix}, \quad \xi = \xi_1 + i \xi_2
\]

and

\[
e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

Then the vectors

\[
h_{1,\pm}(\xi', \mu) = \Theta_{\pm}(\xi', \mu) \begin{pmatrix} e_1 \\ 0 \end{pmatrix} = \begin{pmatrix} i \mu I_2 \\ \Lambda_{\pm}(\xi', \mu) \end{pmatrix} \begin{pmatrix} e_1 \\ 0 \end{pmatrix}
\]

are solutions of Equation (57). Applying formulas

\[
\Lambda_1(\xi', \mu) = \Lambda_{\pm}(\xi', \mu), \Lambda_2(\xi', \mu) = (-\rho^2 + |\xi'|^2) I_2 = -\mu^2 I_2
\]

we obtain that the system of the vectors

\[
\{ h_{1,\pm}(\xi', \mu), h_{2,\pm}(\xi', \mu), h_{1,\mp}(\xi', \mu), h_{2,\mp}(\xi', \mu) \}
\]

is orthogonal in \( \mathbb{C}^4 \), and \( \{ h_{1,\pm}(\xi', \mu) e^{\pm \rho z}, h_{2,\pm}(\xi', \mu) e^{\pm \rho z} \} \) is the fundamental system of solutions of Equation (55).

The exponentially decreasing solutions of Equation (55) are of the form

\[
\psi = \left\{ \begin{array}{ll}
(C_1^+ h_{+,1} + C_2^+ h_{+,2}) e^{\rho z}, & z < 0 \\
(C_1^- h_{-,1} + C_2^- h_{-,2}) e^{-\rho z}, & z > 0
\end{array} \right.
\]

Substituting \( \psi \) in the transmission conditions we obtain the system of linear equations

\[
C_1^+ a_-(x_0) h_{+,1} + C_2^+ a_-(x_0) h_{+,2} + C_1^- a_+(x_0) h_{-,1} + C_2^- a_+(x_0) h_{-,2} = 0
\]

with respect to the unknown vector \( (C_1^+, C_2^+, C_1^-, C_2^-) \in \mathbb{C}^4 \). System (62) has the trivial solution if and only if

\[
det \mathcal{L}(x_0, \xi', \mu) \neq 0 \text{ if } \rho^2 = \mu^2 + |\xi'|^2 = 1
\]

where \( \mathcal{L}(s, \xi', \mu) \) is the matrix with columns

\[
\{ a_-(x_0) h_{+,1}(\xi', \mu), a_-(x_0) h_{+,2}(\xi', \mu), a_+(x_0) h_{-,1}(\xi', \mu), a_+(x_0) h_{-,2}(\xi', \mu) \}.
\]
Condition (63) is called the local Lopatinsky condition for the operator \( \mathcal{D}_{A,\Phi,a_+,a_-}(\mu) \) of parameter-dependent transmission problem (50).

We say that the operator \( \mathcal{D}_{A,\Phi,a_+,a_-}(\mu) \) satisfies the uniform Lopatinsky condition for parameter-dependent transmission problem (50) if

\[
\inf_{x_0 \in \Sigma, \mu^2 + |\xi|^2 = 1} |\det \mathcal{L}(x_0, \xi, \mu)| > 0. \tag{64}
\]

It should be noted that if condition (64) is satisfied then the operators

\[
\mathcal{D}^0_{a_+(x_0),a_-(x_0)}(\mu) : H^1(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4), \quad x_0 \in \Sigma
\]

are invertible for every \( \mu \neq 0 \), and

\[
\sup_{s \in S} \left\| \left( \mathcal{D}^0_{a_+(\cdot),a_-(\cdot)}(\mu) \right)^{-1} \right\|_{B(L^2(\mathbb{R}^3, \mathbb{C}^4), H^1(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4))} \leq C|\mu|^{-1}. \tag{65}
\]

**Theorem 3.3:** Let \( \Sigma \subset \mathbb{R}^3 \) be a \( C^2 \)-uniformly regular surface, the magnetic potential \( A = (A_1, A_2, A_3) \in L^\infty(\mathbb{R}^3, \mathbb{C}^3) \), the electrostatic potential \( \Phi \in L^\infty(\mathbb{R}) \), \( \Gamma \in C_b(\Sigma, B(\mathbb{C}^4)) \), and the uniformly Lopatinsky condition (64) for parameter-dependent operator \( \mathcal{D}_{A,\Phi,a_+,a_-}(\mu) \), \( \mu \in \mathbb{R} \) be satisfied. Then there exists \( \mu_0 > 0 \) such that the operator \( \mathcal{D}_{A,\Phi,a_+,a_-}(\mu) : H^1(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4) \) is invertible for every \( \mu \in \mathbb{R} : |\mu| > \mu_0 \).

**Proof:** The proof is similar to the proof of invertibility of elliptic parameter-dependent boundary value problems (see for instance [22,28], Section 3). However, we consider the parameter-dependent transmission problems for unbounded domains, and therefore we need an infinite partition of unity and estimates associated with this fact. Note that the Dirac operator \( \mathcal{D}_{A,\Phi}(\mu) \) is a uniformly elliptic parameter-depending operator on \( \mathbb{R}^3 \). Moreover, the Lopatinsky condition (65) are satisfied uniformly for every point \( x \in \Sigma \). It yields that there exists \( r > 0 \) and \( \mu_0 > 0 \) such that there exists a countable covering \( \bigcup_{j \in \mathbb{N}} B_r(x_j) \) of the finite multiplicity \( N \geq 1 \) such that for every \( x_j \) there exist operators

\[
L_{x_j}(\mu), R_{x_j}(\mu) \in \mathcal{B}(L^2(\mathbb{R}^3, \mathbb{C}^4), H^1(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4))
\]

such that

\[
\sup_{j \in \mathbb{N}, |\mu| \geq \mu_0} \left\| L_{x_j}(\mu) \right\| = d_L < \infty, \quad \sup_{j \in \mathbb{N}, |\mu| \geq \mu_0} \left\| R_{x_j}(\mu) \right\| = d_R < \infty, \tag{66}
\]

and

\[
L_{x_j}(\mu) \mathcal{D}_{A,\Phi,a_+,a_-}(\mu) \eta_j I = \eta_j I, \quad \eta_j \mathcal{D}_{A,\Phi,a_+,a_-}(\mu) R_{x_j}(\mu) = \eta_j I \tag{67}
\]

for every \( \eta_j \in C^\infty_0(B_r(x_j)) \). Let

\[
\sum_{j \in \mathbb{N}} \varphi_j(x) = 1
\]

be a partition of the unity subordinated to the covering \( \bigcup_{j \in \mathbb{N}} B_r(x_j) \). We set

\[
L(\mu) u = \sum_{j \in \mathbb{N}} \varphi_j L_{x_j}(\mu) \theta_j u, u \in C^\infty_0(\mathbb{R}^3, \mathbb{C}^4),
\]
\[ R(\mu)u = \sum_{j \in \mathbb{N}} \theta_j R_{\gamma_j}(\mu) \varphi_j u, \]

where \( \theta_j \in C_0^\infty(B_r(x_j)), \) \( 0 \leq \theta_j \leq 1, j \in \mathbb{N}, \theta_j \varphi_j = \varphi_j. \) One can prove that the operators \( L(\mu), R(\mu) \) are continued to the bounded operators acting from \( L^2(\mathbb{R}^3, \mathbb{C}^4) \) to \( H^1(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4) \) and

\[ L(\mu)D_{A,\Phi,a_+,a_-}(\mu) = I + T_1(\mu), \quad D_{A,\Phi,a_+,a_-}(\mu)R(\mu) = I + T_2(\mu) \quad (68) \]

where

\[
\| T_1(\mu) \| \leq C(1 + |\mu|)^{-1}, \\
\| T_2(\mu) \| \leq C(1 + |\mu|)^{-1},
\]

and \( C > 0 \) is independent of \( \mu. \) Hence the operator \( D_{A,\Phi,a_+,a_-}(\mu) \) is invertible for \( |\mu| \) large enough. \( \blacksquare \)

### 3.5. Self-adjointness of the operator \( D_{A,\Phi,a_+,a_-} \)

Now we consider the self-adjointness of the unbounded operator \( D_{A,\Phi,a_+,a_-} \) in \( L^2(\mathbb{R}^3, \mathbb{C}^4) \) defined by the Dirac operator

\[ D_{A,\Phi} = \alpha \cdot (D_x + A(x)) + \alpha_0 m + \Phi(x)I_4 \]

with the domain

\[ H^1_{a_+,a_-}(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4) = \left\{ u \in H^1(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4) : a_+(s)u_+(s) + a_-(s)u_-(s) = 0, s \in \Sigma \right\} \]

where

\[ a_+(s) = \frac{1}{2} \Gamma(s) - iv \cdot \alpha, \quad a_-(s) = \frac{1}{2} \Gamma(s) + iv \cdot \alpha. \]

We set

\[ \langle u, v \rangle_{L^2(\mathbb{R}^3, \mathbb{C}^4)} = \int_{\mathbb{R}^3} u(x) \cdot v(x) \, dx, \]

\[ \langle u, v \rangle_{L^2(\Sigma, \mathbb{C}^4)} = \int_{\Sigma} u(s) \cdot v(s) \, ds \]

are the scalar products in \( L^2(\mathbb{R}^3, \mathbb{C}^4), L^2(\Sigma, \mathbb{C}^4), \) respectively.

**Theorem 3.4:** Let \( \Sigma \subset \mathbb{R}^3 \) be the \( C^2 \)-uniformly regular surface, the vector potential \( A \in L^\infty(\mathbb{R}^3, \mathbb{R}^4) \) and the scalar potential \( \Phi \in L^\infty(\mathbb{R}^3) \) be real-valued, and \( \Gamma = (\Gamma_{ij})_{i,j=1}^4 \) be the Hermitian matrix with elements \( \Gamma_{ij} \in C_0(\Sigma). \) We assume that the uniform Lopatinsky conditions (64) for the parameter-dependent problem holds. Then the operator \( D_{A,\Phi,a_+,a_-} \) with domain \( H^1_{a_+,a_-}(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4) \) is self-adjoint in \( L^2(\mathbb{R}^3, \mathbb{C}^4). \)

**Proof:** At first, we prove that the operator \( D_{A,\Phi,a_+,a_-} \) is symmetric. Let \( u, v \in H^1_{a_+,a_-}(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4). \) Integrating by parts we obtain
Then the unbounded operator

\[ D(\alpha) \equiv \phi \cdot u, v \]  

\[ \langle D(\alpha) u, v \rangle_{L^2(\mathbb{R}^3, \mathbb{C}^4)} - \langle u, D(\alpha) v \rangle_{L^2(\mathbb{R}^3, \mathbb{C}^4)} \]

\[ = \langle (-i\alpha \cdot \nu) u_+, v_+ \rangle_{L^2(\Sigma, \mathbb{C}^4)} - \langle (-i\alpha \cdot \nu) u_-, v_- \rangle_{L^2(\Sigma, \mathbb{C}^4)} \]

\[ = \frac{1}{2} \langle -i\alpha \cdot \nu (u_+ - u_-), v_+ - v_- \rangle_{L^2(\Sigma, \mathbb{C}^4)} - \frac{1}{2} \langle u_+ - u_-, -i\alpha \cdot \nu (v_+ - v_-) \rangle_{L^2(\Sigma, \mathbb{C}^4)}. \]

Taking into account Equality (19), we obtain that

\[ \langle D(\alpha) u, v \rangle_{L^2(\mathbb{R}^3, \mathbb{C}^4)} - \langle u, D(\alpha) v \rangle_{L^2(\mathbb{R}^3, \mathbb{C}^4)} \]

\[ = -\frac{1}{4} \langle \Gamma (u_+ - u_-), v_+ - v_- \rangle_{L^2(\Sigma, \mathbb{C}^4)} + \frac{1}{4} \langle u_+ + u_-, \Gamma (v_+ - v_-) \rangle_{L^2(\Sigma, \mathbb{C}^4)}. \]  

(70)

Since the matrix \( \Gamma \) is Hermitian for every \( s \in \Sigma \) the right part side in (70) is 0. Hence

\[ \langle D(\alpha) u, v \rangle_{L^2(\mathbb{R}^3, \mathbb{C}^4)} = \langle u, D(\alpha) v \rangle_{L^2(\mathbb{R}^3, \mathbb{C}^4)} \]

for every \( u, v \in H^1_{a_+, a_-}(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4) \). The uniform Lopatinsky condition (64) yields a priori estimate (46) which implies that the operator \( D(\alpha, a_+, a_-) \) is closed. Moreover, Theorem 3.3 yields that the deficiency indices of \( D(\alpha, a_+, a_-) \) are equal 0. Hence (see for instance [29], page 100) the operator \( D(\alpha, a_+, a_-) \) is self-adjoint. 

**3.5.1. Self-adjointness of Dirac operator with electrostatic and Lorentz \( \delta \)-shell interactions**

As application of Theorem 3.4, we consider the Dirac operator with singular potential \( Q_{\sin} = 2(\eta(s) + \tau(s))\delta_\Sigma \) of the electrostatic and Lorentz \( \delta \)-shell interactions.

**Theorem 3.5:** Let \( \Sigma \) be a uniformly regular \( C^2 \)-surface. We assume that the vector potential \( A \in L^\infty(\mathbb{R}^3, \mathbb{R}^4) \), the scalar potential \( \Phi \in L^\infty(\mathbb{R}^3) \) and real-valued, the function \( \eta(s), \tau(s) \in C_b(\Sigma) \) are real valued. Moreover, we assume that

\[ \inf_{s \in \Sigma} \left| \eta^2(s) - \tau^2(s) - 1 \right| > 0. \]

(71)

Then the unbounded operator \( D(\alpha, a_+, a_-) \) defined by the Dirac operator \( D(\alpha) \) with domain \( H^1_{a_+, a_-}(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4) \) where \( a_\pm(s) = \eta(s)l_4 + \tau(s)a_0 \pm i\alpha \cdot \nu(s), s \in \Sigma \) is self-adjoint in \( L^2(\mathbb{R}^3, \mathbb{C}^4) \).

**Proof:** It should be noted that condition (71) ensures the fulfillment of the uniform Lopatinsky condition for the operator \( D(\alpha, a_+, a_-)(\mu) \) of parameter-dependent transmission problem associated with potential \( Q_{\sin}(s) = 2(\eta(s) + \tau(s))\delta_\Sigma \). The proof of this fact similar to the proof of the Lopatinsky condition (44). Moreover, the unbounded operator \( D(\alpha, a_+, a_-) \) with domain \( H^1_{a_+, a_-}(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4) \) is symmetric in \( L^2(\mathbb{R}^3, \mathbb{C}^4) \). Hence Theorem 3.5 yields that \( D(\alpha, a_+, a_-) \) is a self-adjoint operator. 

4. Fredholm theory of transmission problems associated with Dirac operator with singular potentials

We consider the Fredholm property of the transmission operator

\[
\mathbb{D}_{A,\Phi,a_+,a_-} u(x) = \begin{cases} \frac{D}{x} u(x), & x \in \mathbb{R}^3 \setminus \Sigma, \\ a_+(s) u_+(s) + a_-(s) u_-(s) = 0, & s \in \Sigma; \end{cases}
\]  

(72)

associated with the Dirac operator with singular potential \(D_{A,\Phi,Q_{\sin}} = D_{A,\Phi} + Q_{\sin}\) where

\[
D_{A,\Phi} u(x) = (\alpha \cdot (D_x + A(x)) + \alpha_0 m + \Phi(x)) u(x), \quad x \in \mathbb{R}^3 \setminus \Sigma
\]

(73)

and \(Q_{\sin} = \Gamma \delta_S\). We assume that \(\Sigma\) is a connected \(C^2\)-surface being the common boundary of the domains \(\Omega_{\pm}\). Moreover, \(\Sigma\) is a closed compact surface or unbounded uniformly regular surface.

4.1. Simonenko’s local principle

- Let \(\chi \in C_0^\infty(\mathbb{R}^n)\) be such that \(0 \leq \chi(x) \leq 1\), \(\chi(x) = 1\) for \(|x| \leq 1\), and \(\chi(x) = 0\) for \(|x| \geq 2\), and \(\chi_R(x) = \chi(\frac{x}{R})\), \(\psi_R(x) = 1 - \chi_R(x)\). The function \(\psi_R\) is called the cut-off function of infinity. Let \(B_\varepsilon(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < \varepsilon\}\). We say that \(\varphi_{x_0}\) is a cut-off function of \(B_\varepsilon(x_0)\) if \(\varphi_{x_0} \in C_0^\infty(B_\varepsilon(x_0))\), \(0 \leq \varphi_{x_0}(x) \leq 1\) and \(\varphi_{x_0}(x) = 1\) if \(x \in B_{\varepsilon/2}(x_0)\).

- We denote by \(\tilde{\mathbb{R}}^3\) the compactification of \(\mathbb{R}^3\) obtained by the adjoint to every ray \(l_\omega = \{x \in \mathbb{R}^3 : x = t\omega, t > 0, \omega \in S^2\}\) the infinitely distant point \(\partial_\omega\). The topology in \(\mathbb{R}^3\) is introduced such that \(\tilde{\mathbb{R}}^3\) becomes homeomorphic to the unit closed ball \(\bar{B}_1(0)\).

The fundamental system of neighbourhoods of the point \(\partial_{\omega_0}\) is formed by the conical sets \(U_{\omega_0, R} = \bar{F}_{\omega_0} \times (R, +\infty)\) where \(R > 0\) and \(F_{\omega_0}\) is a neighbourhood of the point \(\omega_0\) on the unit sphere \(S^2\). We define the cut-off function \(\varphi_{x_0}\) of the infinitely distant point \(x_0 = \partial_{\omega_0}\) as \(\varphi_{x_0} = \varphi_{\omega_0}\left(\frac{x}{|x|}\right)\psi_R(x)\) where \(\varphi_{\omega_0}(\omega) \in C_0^\infty(F_{\omega_0})\) and \(\varphi_{\omega_0}(\omega) = 1\) in a neighbourhood \(F_{\omega_0}'\) such that \(\overline{F_{\omega_0}'} \subset F_{\omega_0}\).

**Definition 4.1:** We say that the operator

\[
\mathbb{D}_{A,\Phi,a_+,a_-} : H^1(\mathbb{R}^3 \setminus S, \mathbb{C}^4) \rightarrow L^2(\mathbb{R}^3, \mathbb{C}^4)
\]

is locally invertible at the point \(x_0 \in \mathbb{R}^3\) if there exists a neighbourhood \(U_{x_0}\) of the point \(x_0\) and the operators

\[
\mathcal{L}_{x_0}, \mathcal{R}_{x_0} \in \mathcal{B}(L^2(\mathbb{R}^3, \mathbb{C}^4), H^1(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4))
\]

such that

\[
\mathcal{L}_{x_0} \mathbb{D}_{A,\Phi,a_+,a_-} \varphi_{x_0} I = \varphi_{x_0} I, \quad \varphi_{x_0} \mathbb{D}_{A,\Phi,a_+,a_-} \mathcal{R}_{x_0} = \varphi_{x_0} I,
\]

(74)

where \(\varphi_{x_0}\) is the cut-off function of the point \(x_0\).

**Proposition 4.2 ([16,17]):** The operator

\[
\mathbb{D}_{A,\Phi,a_+,a_-} : H^1(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4) \rightarrow L^2(\mathbb{R}^3, \mathbb{C}^4)
\]

is Fredholm if and only if \(\mathbb{D}_{A,\Phi,a_+,a_-}\) is locally invertible at every point \(x \in \mathbb{R}^3\).
4.2. Transmission on compact closed surfaces

We consider the Fredholm property of the operator $\mathbb{D}_{A,\Phi,a_+,a_-}$ if $\Sigma$ is a compact closed $C^2$-surface.

- We say that $a \in UC_{b,\infty}(\mathbb{R}^n)$ if $a \in L^\infty(\mathbb{R}^n)$ and there exists $R > 0$ such that $a_R = \psi_R a \in UC_{b}(\mathbb{R}^n)$ the space of bounded uniformly continuous functions on $\mathbb{R}^n$, $\psi_R$ is the above defined cut-off function. Let $a \in UC_{b,\infty}(\mathbb{R}^n)$. We consider the functional sequence $b_m(x) = a_R(x + g_m)$, $\mathbb{R}^n \ni g_m \to \infty$. This sequence is uniformly bounded and equicontinuous. By the Arcela–Ascoli Lemma there exists a subsequence $h_m$ of $g_m$ and the limit function $a_R^h \in C_b(\mathbb{R}^n)$ such that

$$
\lim_{m \to \infty} \sup_{x \in K} |a_R(x + h_m) - a_R^h(x)| = 0
$$

for every compact set $K \subset \mathbb{R}^n$. Note that the function $a_R^h = a^h$ is independent of $R$.

- We assume that the vector-valued potential $A = (A_1, A_2, A_3)$ and the scalar potentials $\Phi$ are such that

$$
A_j, \Phi \in UC_{b,\infty}(\mathbb{R}^n),
$$

$$
\Gamma = (\Gamma_{ij})_{ij=1}^4, \quad \Gamma_{ij} \in C_b(\Sigma).
$$

Let $g_m \to \partial_\omega \in \mathbb{R}_\infty^3 = \mathbb{R}^3 \setminus \mathbb{R}^3$. Then there exists a subsequence $h_m$ of $g_m$ and limit functions $A^h_j, j = 1, 2, 3, \Phi^h$ in the sense of formula (75) belonging to $C_b(\mathbb{R}^n)$. The operator

$$
\mathbb{D}_{A,\Phi}^h = \mathbb{D}_{A^h,\Phi^h}
$$

is called the limit operator of $\mathbb{D}_{A,\Phi}$. We denote by $\text{Lim}_{\partial_\omega} \mathbb{D}_{A,\Phi}$ the set of all limit operators of $\mathbb{D}_{A,\Phi}$ defined by the sequences $h_m \to \partial_\omega$, and we set

$$
\text{Lim} \mathbb{D}_{A,\Phi} = \bigcup_{\partial_\omega \in \mathbb{R}_\infty^3} \text{Lim}_{\partial_\omega} \mathbb{D}_{A,\Phi}.
$$

Theorem 4.3: Let condition (76) hold, $\Sigma$ is a $C^2$-compact closed surface dividing $\Omega_+$ and $\Omega_-$, and Lopatinsky condition (36) holds at every point $s \in \Sigma$. Then $\mathbb{D}_{A,\Phi,a_+,a_-} : H^1(\mathbb{R}^3 / \Sigma, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)$ is the Fredholm operator if and only if all limit operators $\mathbb{D}_{A,\Phi}^h \in \text{Lim} \mathbb{D}_{A,\Phi}$ are invertible from $H^1(\mathbb{R}, \mathbb{C}^4)$ into $L^2(\mathbb{R}^3, \mathbb{C}^4)$.

Proof: Note that $\mathbb{D}_{A,\Phi,a_+,a_-}$ locally coincides with $\mathbb{D}_{A,\Phi}$ outside $\Sigma$. Because $\mathbb{D}_{A,\Phi}$ is the elliptic operator on $\mathbb{R}^3$, $\mathbb{D}_{A,\Phi}$ is a locally invertible operator at every point $x \in \mathbb{R}^3 \setminus \Sigma$. In virtue of Lopatinsky condition (36) $\mathbb{D}_{A,\Phi,a_+,a_-}$ is a locally invertible operator at every point $x \in \Sigma$. Proposition 4.2 yields that $\mathbb{D}_{A,\Phi,a_+,a_-}$ is a Fredholm operator if and only if $\mathbb{D}_{A,\Phi,a_+,a_-}$ is locally invertible at every infinitely distant point $\partial_\omega \in \mathbb{R}_\infty^3$. The operator $\mathbb{D}_{A,\Phi,a_+,a_-}$ coincides with the operator $\mathbb{D}_{A,\Phi}$ near every point $\partial_\omega$. Applying the results of book [19] and paper [20] we obtain that $\mathbb{D}_{A,\Phi}$ is locally invertible at infinity if and only if all limit operators $\mathbb{D}_{A^h,\Phi^h} \in \text{Lim} \mathbb{D}_{A,\Phi}$ are invertible. □
Corollary 4.4: Let conditions of Theorem 4.3 hold. Then
\[
sp_{\text{ess}} \mathbb{D}_{A, \Phi, a+, a-} = \bigcup_{\mathbb{D}_{A^h, \Phi^h} \in \text{Lim} \mathbb{D}_{A, \Phi}} sp \mathbb{D}_{A^h, \Phi^h} \tag{79}
\]
where \( \mathbb{D}_{A^h, \Phi^h} \) is unbounded operator in \( L^2(\mathbb{R}^3, \mathbb{C}) \) generated by \( \mathbb{D}_{A^h, \Phi^h} \).

4.3. Slowly oscillating at infinity potentials

- We say that a function \( f \in L^\infty(\mathbb{R}^n) \) is slowly oscillating at infinity if
  \[
  \lim_{x \to \infty} \sup_{y \in K} |f(x + y) - f(x)| = 0
  \]
  for every compact set \( K \subset \mathbb{R}^n \). We denote the class of slowly oscillating at infinity functions by \( \text{SO}_\infty(\mathbb{R}^n) \). Note that \( \text{SO}_\infty(\mathbb{R}^n) \subset \text{UC}_{\text{b}\cdot\infty}(\mathbb{R}^n) \). Moreover, if \( f \in \text{SO}_\infty(\mathbb{R}^n) \) and \( \mathbb{R}^n \ni h_m \to \infty \) is such that there exists a limit function \( f^h \) in the sense of formula (75). Then the function \( f^h \) is a constant.

We consider the operator \( \mathbb{D}_{A, \Phi, a+, a-} \) for \( A_j, \Phi \in \text{SO}_\infty(\mathbb{R}^3) \), and we assume that \( A_j, \Phi \) are real-valued functions, \( \Sigma \) is the compact closed \( C^2 \)-surface, and the interaction matrix \( \Gamma' = (\Gamma_{ij})_{i,j=1}^4 \) is such that \( \Gamma_{ij} \in C(\Sigma) \) and being real-valued functions.

Then the limit operators are of the form:
\[
\mathbb{D}_{A, \Phi, a+, a-}^h = \mathbb{D}_{A^h, \Phi^h} = \alpha \cdot (\mathbb{D} + A^h) + \alpha_0 m + \Phi^h I_4 \tag{80}
\]
where \( A^h \in \mathbb{R}^3, \Phi^h \in \mathbb{R} \). Applying the Fourier transform we obtain that
\[
sp \mathbb{D}_{A^h, \Phi^h} = (\infty, \Phi^h - |m|] \bigcup [\Phi^h + |m|, +\infty) \tag{81}
\]
Then in virtue formula (79) we obtain that
\[
sp_{\text{ess}} \mathbb{D}_{A, \Phi, a+, a-} = (\infty, \sup \Phi(x) - |m|] \bigcup [\inf \Phi(x) + |m|, +\infty) \tag{82}
\]
where
\[
\sup \Phi(x) = \lim_{x \to \infty} \sup \Phi(x), \quad \inf \Phi(x) = \lim_{x \to \infty} \inf \Phi(x).
\]
Formula (82) yields that \( sp_{\text{ess}} \mathbb{D}_{A, \Phi, a+, a-} \) is independent from the slowly oscillating at infinity magnetic potential \( A \). Moreover, if \( |m| > 0 \) and \( \sup \Phi - \inf \Phi < 2|m| \), then \( sp_{\text{ess}} \mathbb{D}_{A, \Phi, a+, a-} \) has a gap \( (\sup \Phi - |m|, \inf \Phi + |m|) \) which could contain the discrete spectrum of \( \mathbb{D}_{A, \Phi, a+, a-} \). In the opposite case \( \sup \Phi - \inf \Phi \geq 2|m| \)
\[
sp_{\text{ess}} \mathbb{D}_{A, \Phi, a+, a-} = (\infty, +\infty).
\]
Formula (82) yields that \( sp_{\text{ess}} \mathbb{D}_{A, \Phi, a+, a-} \) is independent from the slowly oscillating at infinity magnetic potential \( A \). Moreover, if \( |m| \geq 0 \) and \( \sup \Phi - \inf \Phi < 2|m| \), then \( sp_{\text{ess}} \mathbb{D}_{A, \Phi, a+, a-} \) has a gap \( (\sup \Phi - |m|, \inf \Phi + |m|) \) in the essential spectrum which could contain the discrete spectrum of \( \mathbb{D}_{A, \Phi, a+, a-} \). In the opposite case \( \sup \Phi - \inf \Phi \geq 2|m| \)
\[
sp_{\text{ess}} \mathbb{D}_{A, \Phi, a+, a-} = (\infty, +\infty).
\]
4.4. Transmission on unbounded surfaces with conic structure at infinity

Let $\Omega_+ \subset \mathbb{R}^3$ be an connected open domain with a $C^2$-boundary $\Sigma$, $\Omega_- = \mathbb{R}^3 \setminus \Omega_+$, $\Sigma$ has the conic structure at infinity, that is there exists $R > 0$ such that if $x_0 \in \Sigma$ and $x_0 > R$ the ray $\{ x \in \mathbb{R}^3 : x = tx_0, t > 0 \} \subset \Sigma$. Let $\overline{\Omega}_\pm, \overline{\Sigma}$ be the compactifications of the sets $\Omega_\pm, \Sigma$ in the topology of $\mathbb{R}^3$.

We consider the operator $\mathcal{D}_{A,\Phi,a_+,a_-}$ for $A_j, \Phi \in SO_\infty(\mathbb{R}^3)$ and for the interaction matrix $\Gamma = (\Gamma_{ij})_{i,j=1}^4$ with $\Gamma_{ij} \in C(\overline{\Sigma})$.

We define the limit operators of the operator $\mathcal{D}_{A,\Phi,a_+,a_-}$ as follows:

- If $\vartheta_\omega \notin \Sigma_\infty = \overline{\Sigma} \setminus \Sigma$, then the limit operators defined by the sequence $h_m \to \vartheta_\omega$ are of the form (80) with the spectrum given by formula (81).
- Let $\vartheta_\omega \in \Sigma_\infty = \overline{\Sigma} \setminus \Sigma$, $l_\omega^R = \{ x \in \mathbb{R}^3 : x = t\omega, t > R \}$ and $T_\vartheta_\omega$ be the tangent plane to $\Sigma$ at the ray $l_\omega^R$ and $\nu(\omega)$ is the outgoing normal vector to $\Omega_+$ at the points of the ray $l_\omega^R$. We denote by $\mathbb{R}_\pm \vartheta_\omega$ the half-spaces in $\mathbb{R}^3$ with common boundary $T_\vartheta_\omega$. Then following to the paper [21] the limit operators defined by the sequences $h_m \to \vartheta_\omega$ are of the form

$$
\mathcal{D}^h_{A,\Phi,a_+(\vartheta_\omega),a_-(\vartheta_\omega)} u(x) = \begin{cases} 
\mathcal{D}^h_{A,\Phi,a_+(\vartheta_\omega)} u(x), & x \in \mathbb{R}^3 \setminus T_\vartheta_\omega, \\
\mathcal{A}_+(\vartheta_\omega) u(s) + \mathcal{A}_-(\vartheta_\omega) u(s) = 0, & s \in T_\vartheta_\omega,
\end{cases}
$$

where

$$
a_+(\vartheta_\omega) = \frac{1}{2} \Gamma(\vartheta_\omega) - i\alpha \cdot \nu(\omega), a_-(\vartheta_\omega) = \frac{1}{2} \Gamma(\vartheta_\omega) + i\alpha \cdot \nu(\omega),
$$

$$
\Gamma(\vartheta_\omega) = \lim_{\Sigma \ni s \to \vartheta_\omega} \Gamma(s).
$$

**Theorem 4.5:** Let $\Sigma$ be a $C^2$-surface with conical structure at infinity, $A_j, \Phi \in SO_\infty(\mathbb{R}^3)$ and the interaction matrix $\Gamma$ is such that $\Gamma_{ij} \in C(\overline{\Sigma})$, the Lopatinsky condition be satisfied at every point $x \in \Sigma$. Then the operator $\mathcal{D}_{A,\Phi,a_+,a_-} : H^1(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)$ is a Fredholm operator if and only if: (i) for every $\vartheta_\omega \notin \Sigma_\infty$ all limit operators $\mathcal{D}^h_{A,\Phi}$ defined by the sequences $h_m \to \vartheta_\omega$ are invertible from $H^1(\mathbb{R}^3, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)$; (ii) for every $\vartheta_\omega \in \Sigma_\infty$ all limit operators $\mathcal{D}^h_{A,\Phi,a_+,a_-} \in \text{Lim}_{\vartheta_\omega} \mathcal{D}_{A,\Phi,a_+,a_-}$ are invertible from $H^1(\mathbb{R}^3 \setminus \Sigma, \mathbb{C}^4) \to L^2(\mathbb{R}^3, \mathbb{C}^4)$.

**Proof:** The operator $\mathcal{D}_{A,\Phi,a_+,a_-}$ is locally invertible at every point $x \in \mathbb{R}^3$ because the Dirac operator $\mathcal{D}_{A,\Phi}$ is elliptic and the Lopatinsky condition holds at every point $x \in \Sigma$. Hence according Proposition 4.2 $\mathcal{D}_{A,\Phi,a_+,a_-}$ is a Fredholm operator if and only if $\mathcal{D}_{A,\Phi,a_+,a_-}$ is locally invertible at every infinitely distant point $\vartheta_\omega \in \mathbb{R}^3_\infty$. Following to the monograph [19], and the paper [20], we obtain the statements (i) and (ii) of the Theorem.

**Corollary 4.6:** Let the conditions of Theorem 4.5 hold. Then

$$
\text{sp}_{ess} \mathcal{D}_{A,\Phi,a_+,a_-} = \bigcup_{\mathcal{D}^h_{A,\Phi,a_+,a_-} \in \text{Lim} \mathcal{D}_{A,\Phi,a_+,a_-}} \text{sp} \mathcal{D}^h_{A,\Phi,a_+,a_-}
$$

where $\mathcal{D}^h_{A,\Phi,a_+,a_-}$ are unbounded operators associated with the above defined transmission operators $\mathcal{D}^h_{A,\Phi,a_+,a_-}$. 

---

**Note:** The content provided is a transcription of the text from the page you've shared, with a focus on clarity and coherence. The original text contains mathematical expressions and notation that are typical in advanced mathematical literature. The transcription aims to preserve the logical flow and mathematical rigor of the original content.
We consider the spectrum of operators $\mathbb{D}_{A^h,\Phi^h,a_{+}(\vartheta_{\omega}),a_{-}(\vartheta_{\omega})}$ defined by the sequence $h_m \to \vartheta_{\omega}$. One can see that

$$sp\mathbb{D}_{A^h,\Phi^h,a_{+}(\vartheta_{\omega}),a_{-}(\vartheta_{\omega})} = sp\mathbb{D}_{0,\Phi^h,a_{+}(\vartheta_{\omega}),a_{-}(\vartheta_{\omega})}.$$ 

Without loss of generality, we assume that $\mathbb{T}_{\vartheta_{\omega}} = \mathbb{R}^2$, $\{x = (x', x_3) : x_3 = 0\}$. Then after the Fourier transform with respect to $x' \in \mathbb{R}^2$ we obtain the family of the one-dimensional Dirac operators depended on the parameter $\xi' \in \mathbb{R}^2$

$$\mathbb{D}_{0,\Phi^h,a_{+}(\vartheta_{\omega}),a_{-}(\vartheta_{\omega})}(\xi', z) = \begin{cases} \left( \alpha' \cdot \xi' + i\alpha_3 \frac{d}{dz} + \alpha_0 m + \Phi^h \right) \hat{\varphi}(\xi', z), & z \in \mathbb{R} \setminus \{0\}, \xi' \in \mathbb{R}^2, \\
\hat{a_{+}(\vartheta_{\omega})} \hat{u}_+(\xi', 0) + \hat{a_{-}(\vartheta_{\omega})} \hat{u}_-(\xi', 0) = 0, & \xi' \in \mathbb{R}^2. \end{cases} \tag{85}$$

The 1-D Dirac operator

$$\mathbb{L}_{\vartheta_{\omega}}(\xi') \varphi(z) = \begin{cases} \left( \alpha' \cdot \xi' + i\alpha_3 \frac{d}{dz} + \alpha_0 m \right) \varphi(z), & z \in \mathbb{R} \setminus \{0\} \\
\hat{a_{+}(\vartheta_{\omega})} \varphi_(0) + \hat{a_{-}(\vartheta_{\omega})} \varphi_-(0) = 0 \end{cases} \tag{86}$$

has the essential spectrum

$$sp_{\text{ess}} \mathbb{L}_{\vartheta_{\omega}}(\xi') = \left( -\infty, -\sqrt{|\xi'|^2 + |m|^2} \right] \bigcup \left[ \sqrt{|\xi'|^2 + |m|^2}, +\infty \right)$$

and perhaps a finite set of points of the discrete spectrum on the interval $(-\sqrt{|\xi'|^2 + m^2}, \sqrt{|\xi'|^2 + m^2})$. Hence

$$sp\mathbb{D}_{0,0,a_{+}(\vartheta_{\omega}),a_{-}(\vartheta_{\omega})}(\xi') = \bigcup_{\xi' \in \mathbb{R}^2} sp_{\text{ess}} \mathbb{D}_{0,0,a_{+}(\vartheta_{\omega}),a_{-}(\vartheta_{\omega})}(\xi') \bigcup_{\xi' \in \mathbb{R}^2} sp_{\text{dis}} \mathbb{D}_{0,0,a_{+}(\vartheta_{\omega}),a_{-}(\vartheta_{\omega})}(\xi') = (-\infty, -|m|] \bigcup [m, +\infty) \bigcup_{\xi' \in \mathbb{R}^2} sp_{\text{dis}} \mathbb{D}_{0,0,a_{+}(\vartheta_{\omega}),a_{-}(\vartheta_{\omega})}(\xi') \cap (-|m|, |m|).$$

Then applying Corollary 4.6 we obtain the following result.

**Theorem 4.7:** Let the conditions of Theorem 4.5 hold. Then

$$sp_{\text{ess}} \mathbb{D}_{A,\Phi,a_{+},a_{-}} = (-\infty, M_{\Phi}^{\sup} - |m|] \bigcup [M_{\Phi}^{\inf} + |m|, +\infty) \bigcup_{\xi' \in \mathbb{R}^2} sp_{\text{dis}} \mathbb{D}_{0,0,a_{+},a_{-},\vartheta_{\omega}}(\xi') \cap (-|m|, |m|).$$
Remark 4.8: The calculation of the essential spectrum of $\mathcal{D}_{A,\Phi,a_+,a_-}$ is simplified if
\[
\lim_{s \to \infty} \Gamma(s) = 0.
\]
In this case, the limit operators are of the form
\[
\mathbb{D}_{0,\Phi,\alpha_0,a_0^0,a_0^-} u(x) = \begin{cases} (\alpha \cdot D_x + \alpha_0 m + \Phi^h) u(x), & x \in \mathbb{R}^3 \setminus T\theta_0, \\ u_+(s) = u_-(s), & s \in T\theta_0. \end{cases}
\]
It yields that $\text{sp}_{\text{dis}} \mathbb{D}_{0,\Phi,\alpha_0,a_0^0,a_0^-} = \emptyset$ and under conditions of Theorem 4.5 formula (84) holds.

4.5. Electrostatic and Lorentz scalar $\delta$-shell interaction for conic at infinity interaction surfaces

Let $\Sigma$ be a compact closed $C^2$-surface or a conic at infinity $C^2$-surface. We consider the operator $\mathcal{D}_{A,\Phi,a_+,a_-}$, where $A_j, \Phi \in SO_\infty(\mathbb{R}^3)$ are real-valued functions. The singular potential
\[
Q_{\text{sin}} = \Gamma_\Sigma = 2(\eta I_4 + \tau \alpha_4) \delta_\Sigma
\]
where $\eta, \tau \in C(\Sigma)$ where $\Sigma$ is a compact closed surface or $\eta, \tau \in C_0(\Sigma) = \{f \in C(\Sigma) : \lim_{s \to \infty} f(s) = 0\}$ if $\Sigma$ is a conic at infinity $C^2$-surface. Moreover, $\eta, \tau$ are real-valued functions. The matrix $\Gamma$ in formula (88) is
\[
\Gamma(s) = 2 \begin{pmatrix} (\eta(s) + \tau(s)) I_2 & 0 \\ 0 & (\eta(s) - \tau(s)) I_2 \end{pmatrix}.
\]
Let the uniformly Lopatinsky condition
\[
\inf_{s \in S} |\eta^2(s) - \tau^2(s) - 1| > 0
\]
is satisfied. Then the operator $\mathcal{D}_{A,\Phi,a_+,a_-}$ is self-adjoint in $L^2(\mathbb{R}^3, \mathbb{C}^4)$ and $\text{sp}_{\text{ess}} \mathcal{D}_{A,\Phi,a_+,a_-}$ is given by formula (84).

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