The role of Cooperons in the disordered electron problem: logarithmic corrections to scaling near the metal-insulator transition

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Abstract

The effect of Cooperons on metal-insulator transitions (MIT) in disordered interacting electronic systems is studied. It is argued that previous results which concluded that Cooperons are qualitatively unimportant near the MIT might not be correct, and that the problem is much more complicated than had previously been realized. Although we do not completely solve the Cooperon problem, we propose a new approach that is at least internally consistent. Within this approach we find that in all universality classes where Cooperons are present, i.e. in the absence of magnetic impurities and magnetic fields, there are logarithmic corrections to scaling at the MIT. This result is used for a possible resolution of the so-called exponent puzzle for the conductivity near the MIT. A discussion of the relationship between theory and experiment is given. We also make a number of predictions concerning crossover effects which occur when a magnetic field is applied to a system with Cooperons.

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I. INTRODUCTION

The current theoretical description of the metal-insulator transition (MIT) problem asserts that the presence or absence of the particle-particle or Cooper channel does not qualitatively modify the MIT. At first sight this may seem surprising, as much of the early work in the modern (post-1979) theory of the localization problem concentrated on the Cooper channel and the backscattering or weak localization effects it produces. Also, numerous experiments confirmed the presence of these weak localization effects in weakly disordered metallic systems in the absence of magnetic impurities and magnetic fields. However, the assertion appears less surprising if one recalls that electron-electron interaction effects in the presence of disorder lead to many of the same effects as Cooperons. If this is so in the weak disorder regime, and if one acknowledges that electron-electron interactions are in general relevant for the MIT, then it is conceivable that Cooperons do not lead to any additional effects at the MIT over and above those produced by the interplay of interactions and disorder alone. This was in fact the conclusion reached by Finkel’stein and others who have argued that the Cooper channel is irrelevant, in the sense of the renormalization group (RG), for the MIT, that interaction effects effectively replace Cooperon effects near the MIT, and that MIT with or without Cooperons are qualitatively the same.

In this paper we argue that the latter conclusion is probably not correct. We present a consistent description within which the Cooper propagator, or the effective Cooper interaction amplitude, $\Gamma_c$, is a marginal operator rather than an irrelevant one. This marginal operator leads to logarithmic corrections to scaling that are characteristic for those universality classes where Cooperons are present.

Technically, we first show that the MIT problem in the presence of Cooperons is substantially more complicated than had previously been realized. In particular, the renormalization procedures used in existing treatments of the problem do not lead to a finite renormalized field theory. We further argue that even if this problem is ignored, the RG fixed point structure obtained in previous treatments is not stable with respect to the consideration of higher order terms that were neglected. We then present a partial solution to this problem. We first point out that a principal unanswered question is that of how many renormalization constant are needed to make the theory finite. This is closely related to the physical question of whether $\Gamma_c$ is a simple scaling variable near the MIT or whether it consists of several scaling parts. Since at present we do not have a firm answer to this question, we first derive RG flow equations for all other coupling constants which appear in the theory. These flow equations can be expressed in terms of $\Gamma_c$ whose scaling behavior is a priori unknown. We then derive an Eliashberg-type integral equation for $\Gamma_c$. General arguments lead us to the conclusion that $\Gamma_c$ consists of several scaling parts, and that it approaches its asymptotic value at the MIT, $\Gamma_c^*$, logarithmically slowly.

The most important conclusion from our considerations is that in systems without magnetic impurities or magnetic fields there are logarithmic corrections to scaling at the MIT. This in turn implies that it is virtually impossible to experimentally reach the asymptotic critical scaling regime for these universality classes, and that the existing experiments measure effective exponents, rather than asymptotic critical exponents. We show that these logarithmic corrections to scaling provide a possible resolution of a long-standing problem. The critical exponents for the conductivity in Si:P and some other systems, all of which
are believed to be in universality classes that contain Cooperons, is experimentally observed to be smaller than 2/3, in apparent violation of a rigorous bound that requires \( s \geq 2/3 \) in three-dimensional (3-D) systems.\footnote{We will see that logarithmic corrections to scaling can easily account for an apparent or effective exponent \( s_{\text{eff}} \approx 0.5 \) even if the true asymptotic critical exponent obeys \( s \geq 2/3 \). The same conclusion was reached in a previous short account of part of the work presented in this paper.} 

The plan of this paper is as follows. In Sec. II we use Finkel’sstein’s effective field theory for the MIT to calculate the perturbative corrections to the coupling constants which appear in the theory to one-loop order. In Sec. III we use a normalization point RG approach to renormalize the field theory and to derive RG flow equations for the coupling constants in terms of the Cooper propagator \( \Gamma_c \). We also show how previous results can be obtained from certain assumptions and approximations for \( \Gamma_c \). We find that these assumptions do not lead to a finite renormalized theory, and that terms neglected in the previous treatments invalidate this approach in any case. In Sec. IV we classify and discuss the possible solutions of the integral equation for \( \Gamma_c \). We argue that generic solutions of the integral equation yield a \( \Gamma_c \) that approaches its fixed point value logarithmically slowly, which leads to logarithmic corrections to scaling. In Sec. V we discuss the experimental consequences of this result. We compare the theory with existing experiments and suggest a number of new measurements to further test the theory. In particular we discuss crossover effects due to an external magnetic field which changes the universality class and eliminates the logarithmic corrections to scaling. We conclude in Sec. VI with a general discussion of our results as well as the current status of the MIT problem.

II. THE FIELD THEORY, AND THE LOOP EXPANSION

In the first part of this section we recall the basic field theoretic description of the disordered electron problem and derive the Gaussian propagators of the field theory. We then explain how to obtain perturbative corrections to the coupling constants by considering the generating functional to one-loop order.

A. The Model

The existing theoretical description of the MIT is based on the usual assumption of the theory of continuous phase transitions, viz. that the physics near the transition is dominated by the low-lying excitations of the system. As for the description of other phase transitions, the problem is then reduced to obtaining a solution for an effective field theory for these slow modes. Once the field theory has been identified, the technical apparatus used to obtain a solution is the RG. The initial problem is to obtain the field theory, which requires a physical identification of the relevant slow modes.

In the field theory which we will use, the slow modes in the metallic phase are assumed to be the diffusion of mass, spin, and energy density. The effective field theory for long-wavelength and low-frequency excitations that describe how these diffusive processes change across an MIT is defined by an action,
\[
S[Q] = -\frac{1}{2G} \int dx \, \text{tr} \left( \nabla Q(x) \right)^2 + 2H \int dx \, \text{tr} \left( \Omega Q(x) \right) \\
- \frac{\pi T}{4} \sum_{u=s,t,c} K_u \int dx \, [Q(x) \gamma^{(u)} Q(x)].
\]

(2.1)

Here the field variable \( Q \) is an infinite matrix whose matrix elements, \( Q_{nm}^{\alpha \beta} \), are complex 4-by-4 matrices (spin-quaternions) which comprise the spin and particle-hole degrees of freedom. The labels \( \alpha, \beta = 1, 2, \cdots, N \) denote replica labels. In deriving Eq. (2.1), quenched disorder has been integrated out by means of the replica trick and the limit \( N \to \infty \) is implied at the end of all calculations. \( n, m = -\infty, \cdots, +\infty \) are Matsubara frequency labels. \( \Omega = \omega_n \) with \( I \) the identity matrix, \( \omega_n = 2\pi T(n + 1/2) \), is a fermionic frequency matrix, and \( \text{tr} \) denotes a trace over all discrete degrees of freedom. \([Q\gamma^{(u)} Q]\) in Eq. (2.1) is defined as,

\[
[Q\gamma^{(s)} Q] = \sum_{n_1n_2n_3n_4} \sum_{r=0,3} (-1)^r \delta_{n_1+n_3,n_2+n_4} \text{tr} \left( (\tau_r \otimes s_0) Q_{n_1n_2}^{\alpha \alpha} \right) \text{tr} \left( (\tau_r \otimes s_0) Q_{n_3n_4}^{\alpha \alpha} \right),
\]

(2.2a)

\[
[Q\gamma^{(t)} Q] = - \sum_{n_1n_2n_3n_4} \sum_{r=0,3} (-1)^r \delta_{n_1+n_3,n_2+n_4} \sum_{\alpha} \text{tr} \left( (\tau_r \otimes s_0) Q_{n_1n_2}^{\alpha \alpha} \right) \text{tr} \left( (\tau_r \otimes s_0) Q_{n_3n_4}^{\alpha \alpha} \right),
\]

(2.2b)

\[
[Q\gamma^{(c)} Q] = - \sum_{n_1n_2n_3n_4} \sum_{r=1,2} \delta_{n_1+n_2,n_3+n_4} \sum_{\alpha} \text{tr} \left( (\tau_r \otimes s_0) Q_{n_1n_2}^{\alpha \alpha} \right) \text{tr} \left( (\tau_r \otimes s_0) Q_{n_3n_4}^{\alpha \alpha} \right),
\]

(2.2c)

with \( tr = tr_s tr_t \) where \( tr_t \) acts only on the \( \tau \)'s and \( tr_s \) acts only on the \( s \)'s. In Eqs. (2.2), \( \tau_0 = s_0 = \sigma_0 \) and \( \tau_j = -s_j = -i\sigma_j \) \((j = 1, 2, 3)\), with \( \sigma_j \) the Pauli matrices. The theory contains five coupling constants. \( G = 8/\pi\sigma_B \) with \( \sigma_B \) the bare or self-consistent Born conductivity is a measure of the disorder, and \( H = \pi N_F/4 \) plays the role of a frequency coupling parameter with \( N_F \) the bare density of states (DOS) at the Fermi level. \( K_s \) and \( K_t \) are singlet and triplet particle-hole interaction constants, respectively, and \( K_c \) is the singlet particle-particle or Cooper channel interaction constant. At zero frequency the triplet coupling constant in the particle-particle channel vanishes due to the Pauli principle. A disorder generated, frequency dependent triplet particle-particle interaction constant has been discussed elsewhere. For simplicity we formulate the theory with a short-range model interaction, i.e. the \( K_{s,t,c} \) are simply numbers. For the more realistic case of a Coulomb interaction \( K_s \) is \( x \)-dependent and must be kept under the integral in Eq. (2.1). Most results for this case are easily obtained after all calculations have been performed by essentially putting \( K_s = -H \) but occasionally subtle complications arise. Since these are purely technical in nature, and decoupled from the Cooper channel induced problems which are the subject of this paper we will not discuss them. We will, however, give results for the Coulomb interaction case in Sec. [I].

The matrix \( Q \) is subject to the nonlinear constraints,
\begin{align}
Q^2 &= 1, \quad (2.3a) \\
tr Q &= 0, \quad (2.3b)
\end{align}

and satisfies,
\begin{align}
Q^\dagger &= C^T Q^T C = Q, \quad (2.3c)
\end{align}

with,
\begin{align}
C &= i (\tau_1 \otimes s_2). \quad (2.3d)
\end{align}

The \( \tau_i \) are the quaternion basis and span the particle-hole and particle-particle space, while the \( s_i \) serve as our basis in spin space. For convenience we expand \( Q_{\alpha\beta}^{nm} \) in this basis,
\begin{align}
Q_{\alpha\beta}^{nm} &= \sum_{r=0}^{3} \sum_{i=0}^{3} i \rho_{nm}^{\alpha\beta} (\tau_r \otimes s_i) \quad (2.4)
\end{align}

From the first equality in Eq. (2.3c) it follows that the elements of \( Q \) describing the particle-hole degrees of freedom are real, while those describing the particle-particle degrees of freedom are purely imaginary,
\begin{align}
i \rho_{nm}^{\alpha\beta} &= i \rho_{nm}^{\alpha\beta}, \quad (r = 0, 3), \quad (2.5a) \\
i \rho_{nm}^{\alpha\beta} &= -i \rho_{nm}^{\alpha\beta}, \quad (r = 1, 2), \quad (2.5b)
\end{align}

In addition, from the hermiticity condition one obtains,
\begin{align}
0 \rho_{nm}^{\alpha\beta} &= (-1)^r \rho_{nm}^{\alpha\beta}, \quad (r = 0, 3), \quad (2.6a) \\
i \rho_{nm}^{\alpha\beta} &= (-1)^{r+1} i \rho_{nm}^{\alpha\beta}, \quad (r = 0, 3; i = 1, 2, 3), \quad (2.6b) \\
0 \rho_{nm}^{\alpha\beta} &= 0 \rho_{nm}^{\alpha\beta}, \quad (r = 1, 2), \quad (2.6c) \\
i \rho_{nm}^{\alpha\beta} &= -i \rho_{nm}^{\alpha\beta}, \quad (r = 1, 2; i = 1, 2, 3). \quad (2.6d)
\end{align}

The constraints given by Eqs. (2.3a,2.3b) and the hermiticity requirement, Eq. (2.3c), can be eliminated by parametrizing the matrix \( Q \) by,
\begin{align}
Q &= \begin{cases}
(1 - qq^\dagger)^{1/2} - 1 & \text{for } n \geq 0, \ m \geq 0 \\
q & \text{for } n \geq 0, \ m < 0 \\
q^\dagger & \text{for } n < 0, \ m \geq 0 \\
-(1 - q^\dagger q)^{1/2} - 1 & \text{for } n < 0, \ m < 0
\end{cases} \quad (2.7a)
\end{align}

Here the \( q \) are matrices with spin-quaternion valued elements \( q_{nm}^{\alpha\beta}; \ n = 0, 1, \cdots; \ m = -1, -2, \cdots \). Like the matrix \( Q \), they can be expanded as
\[ q_{nm}^{\alpha\beta} = \sum_{r=0}^{3} \sum_{i=0}^{3} i q_{nm}^{\alpha\beta} (\tau_r \otimes s_i) \]  

Note that the q do not satisfy the symmetry relations given by Eqs. (2.6) for the Q.

We have so far given the theory for the so-called generic (G) universality class which is realized by systems without magnetic fields, magnetic impurities, or spin-orbit scattering. Apart from class G, the second universality class with Cooperons is the one with strong spin-orbit scattering (class SO). For class SO the action is shown as above, except that the particle-hole spin triplet channel is absent, i.e. the sum over the spin index \( i \) in Eqs. (2.4) and (2.7b) is restricted to \( i = 0 \). In what follows we will give results for both class G and class SO.

With the help of Eqs. (2.7) one can expand the action in powers of q,

\[ S[Q] = \sum_{n=2}^{\infty} S_n[q] \]  

where \( S_n[q] \sim q^n \). We first concentrate on the Gaussian part of the action,

\[ S_2[q] = -4 \int_\mathbb{P} \sum_{r,i} q_{12}(\mathbf{p}) \epsilon_i M_{12,34} \epsilon_i q_{34}(\mathbf{-p}) \]  

where \( \mathbb{P} \equiv \int d\mathbf{p} / (2\pi)^D \), and \( 1 \equiv (n_1, \alpha_1) \), etc. The matrix M is given by,

\[ i_{0,3} M_{12,34}(p) = \frac{\delta_{1-2,3-4}}{G} \{ \delta_{13}[p^2 + GH(\omega_{n_1} - \omega_{n_2})] + \delta_{\alpha_1\alpha_2} \delta_{\alpha_1\alpha_2} 2\pi TGK \} \]  

where \( \nu_0 = s, \nu_{1,2,3} = t \), and

\[ i_{1,2} M_{12,34}(p) = -\frac{\delta_{1+2,3+4}}{G} \{ \delta_{13}[p^2 + GH(\omega_{n_1} - \omega_{n_2})] + \delta_{\alpha_1\alpha_2} \delta_{\alpha_1\alpha_2} \delta_{i,9} 2\pi TGK \} \]

The Gaussian propagators are given by,

\[ \langle \epsilon_i q_{12}(\mathbf{p}_1) \epsilon_j q_{34}(\mathbf{p}_2) \rangle^{(2)} = \int D[q] \epsilon_i q_{12}(\mathbf{p}_1) \epsilon_j q_{34}(\mathbf{p}_2) e^{S_2[q]} / \int D[q] e^{S_2[q]} \]

\[ = \frac{1}{8} \delta_{ij} (2\pi)^D \delta(\mathbf{p}_1 + \mathbf{p}_2) \epsilon_i M_{12,34}^{-1}(p_1) \]

with,

\[ i_{0,3} M_{12,34}^{-1}(p) = \delta_{1-2,3-4} G \left( \delta_{13} D_{n_1-n_2}(p) + \frac{\delta_{\alpha_1\alpha_2}}{n_1 - n_2} \Delta D_{n_1-n_2}(p) \right) \]

\[ i_{1,2} M_{12,34}^{-1}(p) = -\delta_{1+2,3+4} G \left( \delta_{13} D_{n_1-n_2}(p) - \delta_{\alpha_1\alpha_2} \delta_{i9} D_{n_1-n_2}(p) D_{n_3-n_4}(p) \right) \times \frac{G2\pi TK}{1 + G2\pi TK c^\delta n_{1+n_2}(p)} \]
\[ f_n(p) = \sum_{n_1 \geq 0, n_2 < 0} \delta_{n,n_1+n_2} D_{n_1-n_2}(p). \]  

(2.10d)

Here we have introduced the propagators,

\[ D_n(p) = [p^2 + GH \Omega_n]^{-1}, \]  

(2.10e)

\[ D^{s,t}_n(p) = [p^2 + G(H + K_{s,t})\Omega_n]^{-1}, \]  

(2.10f)

\[ \Delta D^{s,t}_n(p) = D^{s,t}_n(p) - D_n(p), \]  

(2.10g)

with \( \Omega_n = 2\pi T_n \) a bosonic Matsubara frequency. Physically, \( D_n, D_s^{n}, D_t^{n} \) are the energy, mass, and spin diffusion propagators. Equations (2.10b,2.10c) can be put into a more standard form by summing over, e.g., \( n_3 \) and \( n_4 \),

\[ \sum_{n_3 \geq 0, n_4 < 0} i_{0,3} M_{12,34}^{-1}(p)/G = (1 - \delta_{\alpha_1 \alpha_2}) D_{n_1-n_2}(p) + \delta_{\alpha_1 \alpha_2} D^{\nu_1}_{n_1-n_2}(p), \]  

(2.11a)

\[ \sum_{n_3 \geq 0, n_4 < 0} i_{1,2} M_{12,34}^{-1}(p)/G = -(1 - \delta_{\alpha_1 \alpha_2}) D_{n_1-n_2}(p) - \delta_{\alpha_1 \alpha_2}(1 - \delta_{i0}) D_{n_1-n_2}(p) \]  

\[ \quad - \delta_{i0} \delta_{\alpha_1 \alpha_2} \frac{D_{n_1-n_2}(p)}{1 + G2\pi TK_c f_{n_1+n_2}(p)}. \]  

(2.11b)

Examining the various terms in Eqs. (2.11) we see that all of them have a standard propagator structure except for the last contribution in Eq. (2.11b). In interpreting these propagators as having a standard structure, the Matsubara frequencies in Eqs. (2.10e)-(2.10g) are taken to be analogous to a magnetic field at a magnetic phase transition, i.e., the MIT occurs at \( \Omega_n \to 0 \) and \( \Omega_n \) or the temperature is a relevant perturbation in the RG sense. Using Eq. (2.10a), changing the sum to an integral, and placing an ultraviolet cutoff, \( \Omega_0 \), on the result shows that \( 2\pi T f_{n_1+n_2}(p) \) in Eq. (2.11b) diverges logarithmically in the long wavelength, low temperature limit. With \( K_c > 0 \), we see that the last term in Eq. (2.11b) is logarithmically small compared to the other terms in Eqs. (2.11).

We conclude this subsection with two remarks. Firstly, the logarithm discussed above that appears in the particle-particle density correlation function is just the usual BCS logarithm. However, since we consider a system with a repulsive Cooper channel interaction, \( K_c > 0 \), which is not superconducting in the clean limit, this does not lead to a Cooper instability. Rather, the last term in Eq. (2.11b) vanishes logarithmically in the limit \( p,T \to 0 \). If the structure of this term persists for disorder values up to the MIT, and if it couples to the physical quantities like, e.g., the conductivity, then it will lead to logarithmic corrections to scaling. Secondly, considering the Gaussian theory one can already anticipate a fundamental problem with any RG treatment of the field theory. To see this, note that at the Gaussian level the two-point vertex functions are given by \( S_2[q] \). Examining the Eqs. (2.9) we see that at this order no singularities, neither in the ultraviolet nor in the infrared, are present in the vertex functions, and there is no explicit cutoff dependence. This should be contrasted with the corresponding two-point Gaussian propagators given by Eqs. (2.10). Because of the last term in Eq. (2.10d), both an infrared singularity and a dependence on an ultraviolet cutoff
appear. In the usual RG approach such cutoff dependences are eliminated from the field theory by the introduction of suitable renormalization constants. Here, unusual features are that vertex functions and propagators behave differently with respect to their cutoff dependence, and that the cutoff dependent term is logarithmically small rather than large. In previous RG treatments of this problem, the procedure used was effectively to renormalize $K_c$ in Eqs. (2.9c) and (2.10c) such that the renormalized two-point propagators were finite as $\Omega_0 \to \infty$. It is easy to see that such a procedure leads to renormalized two-point vertex functions that contain a singularity in this limit, cf. Eq. (3.10) below. In Sec. III we will further discuss this problem, and we will propose an alternative renormalization procedure.

B. One-loop perturbation theory

We will not be able to present a final solution of the Cooperon renormalization problem. we will therefore consider several renormalization procedures, and will discuss which ones are at least internally consistent and which ones are not. One of these procedures is based on perturbation theory for the generating functional for the vertex functions, $\Gamma[q]$, which is closely related to the thermodynamic potential. The perturbation expansion for $\Gamma[q]$ can be generated by standard techniques. First, a source term with an external potential $J_{\alpha\beta}^{nm}(x)$ is added to the action so that the average of $q_{\alpha\beta}^{nm}$, which we denote by $\bar{q}_{\alpha\beta}^{nm}$, is nonzero. We can then obtain a loop or disorder expansion for $\Gamma[\bar{q}]$. It is convenient to expand $\Gamma[\bar{q}]$ in powers of $\bar{q}$,

$$\Gamma[\bar{q}] = \sum_{N=1}^{\infty} \frac{1}{N!} \sum_{1,2,\ldots,2N} \int dx_1 \cdots dx_N \Gamma^{(N)}_{1,2,3,4,\ldots,2N-1,2N}(x_1, \cdots, x_N) \bar{q}_{12}(x_1) \cdots \bar{q}_{2N-1,2N}(x_N),$$

(2.12)

where $\Gamma^{(N)}$ is the N-point vertex function.

To zero-loop order, one has

$$\Gamma_0[\bar{q}] = -S[\bar{q}].$$

(2.13)

At higher-loop order the various coefficient in $S$ acquire perturbative corrections, and in addition structurally new terms are generated. For simplicity we restrict our considerations to the two-point vertex function $\Gamma^{(2)}$, and to the one-point vertex function $\Gamma^{(1)}$ which is related to the one-point propagator $P^{(1)} = \langle 0 | Q^{\alpha\alpha}_{nm}(x) | 0 \rangle$. At one-loop order $\Gamma^{(2)}$ is given by the second derivative with respect to $\bar{q}$ of the right-hand side of Eq. (2.9a) with $q \to \bar{q}$, and with the matrix $M$ replaced by

$$i_{0,3} M'_{12,34}(p) = \delta_{1-2,3-4} \left\{ \delta_{13} \left[ \frac{p^2}{G_{nn}} + H_{nn}(\omega_n - \omega_n) \right] + \delta_{\alpha_1 \alpha_2} \delta_{\alpha_3 \alpha_4} 2\pi TK_{n1n2}^{\nu_1} \right\},$$

(2.14a)

and,

$$i_{1,2} M'_{12,34}(p) = -\delta_{1+2,3+4} \left\{ \delta_{13} \left[ \frac{p^2}{G_{nn}} + H_{nn}(\omega_n - \omega_n) \right] + \delta_{\alpha_1 \alpha_2} \delta_{\alpha_1 \alpha_4} 2\pi TK_{n1n2}^{\nu_1} \right\},$$

(2.14b)
where $G_{n_1n_2}$, $H_{n_1n_2}$, and $K_{s,t,c}^{n_1n_2n_3n_4}$ are given by $G$, $H$, and $K_{s,t,c}$, respectively, plus frequency dependent one-loop perturbative corrections. In our notation we have suppressed the fact that these corrections are in general also momentum dependent. In the absence of Cooperons these corrections have been discussed in detail elsewhere. In general the momentum and frequency dependence of the corrections is quite complicated. Here we just give the results to leading order in $1/\epsilon$, with $\epsilon = D - 2$. With $\Lambda$ an ultraviolet momentum cutoff, and $\bar{G} = G_{SD}/(2\pi)^D$ with $S_D$ the surface of the $D$-dimensional unit sphere, we obtain,

$$G_{n_1n_2} = G + G\frac{\bar{G}}{4\epsilon} \lbrack \lambda^\epsilon - (GH\Omega_{n_1-n_2})^{\lambda/2} \rbrack - G^2 \int_p I_1^s(p, \Omega_{n_1-n_2}) + 3I_1^t(p, \Omega_{n_1-n_2})$$

$$+ I_1^c(p, \Omega_{n_1-n_2}) + G^2 \int_p I_2^s(p, \Omega_{n_1-n_2}) + 3I_2^t(p, \Omega_{n_1-n_2}) \rbrack \quad , \quad (2.15a)$$

$$H_{n_1n_2} = H + GH \int_p [I_1^s(p, \Omega_{n_1-n_2}) + 3I_1^t(p, \Omega_{n_1-n_2}) + \frac{1}{2} I_1^c(p, \Omega_{n_1}) + \frac{1}{2} I_1^c(p, \Omega_{n_2})]$$

$$- GH \int_p [I_2^s(p, \Omega_{n_1-n_2}) + 3J_2^t(p, \Omega_{n_1-n_2})] \quad , \quad (2.15b)$$

$$K_{n_1n_2n_3n_4} = K_s + H - H_{n_1n_2} \quad , \quad (2.15c)$$

$$K_{s,t,c}^{n_1n_2n_3n_4} = K_t + G(K_t - K_s) \int_p J_1(p, \Omega_{n_1-n_2}) - \frac{GK_t}{2} \int_p J_3(p, \Omega_{n_1-n_2})$$

$$- G^2K_t \int_p I_1^s(p, \Omega_{n_1-n_2}) - \frac{G}{2}(H + K_t)^2 \int_p J_2^c(p, \Omega_{n_1-n_2}) \quad , \quad (2.15d)$$

$$K_{s,t,c}^{n_1n_2n_3n_4} = K_c - \frac{G}{8\epsilon}[\lambda^\epsilon - (GH\Omega_{n_3-n_2})^{\lambda/2}](K_s - 3K_t) + G \int_p J_1^c(p, \Omega_{n_3-n_2}) \quad , \quad (2.15e)$$

Here, $\Omega_0 = O(\Lambda^2/GH)$ is a cutoff frequency, and,

$$I_1^{s,t}(p, \Omega_n) = \frac{1}{8} \sum_{m=n}^{\Omega_0/2\pi T} \frac{1}{m} \Delta D_m^{s,t}(p) \quad , \quad (2.16a)$$

$$I_1^c(p, \Omega_n) = - \frac{1}{8} G^2 \pi T \sum_{m=n}^{\Omega_0/2\pi T} \frac{K_c}{1 + G^2\pi TK_c f_{-m}(p)} (D_m(p))^2 \quad , \quad (2.16b)$$

$$I_2^{s,t}(p, \Omega_n) = - \frac{G^2}{8}(K_{s,t})^2 2\pi T \sum_{m=n}^{\Omega_0/2\pi T} \Omega_m D_m^{s,t}(p)(D_{n+m}(p))^2 \times [1 - 2p^2 D_{n+m}(p)] \quad , \quad (2.16c)$$

$$J_2^{s,t}(p, \Omega_n) = \frac{1}{8} (K_{s,t}/H) \pi T \sum_{m=n}^{\Omega_0/2\pi T} \left[ \Delta D_m^{s,t}(p) + GK_{s,t}\Omega_m D_m^{s,t}(p)D_{m+n}(p) \right] \quad , \quad (2.16d)$$
\[ J_1(p, \Omega_n) = \frac{2\pi T}{8\Omega_n} \sum_{m=1}^{n-1} \left[ (1 - \frac{m}{n})D_{m+n}(p) + \frac{m}{n}D_m(p) \right] , \quad (2.16e) \]

\[ J_3(p, \Omega_n) = -\sum_{m,n=1}^{\Omega_0/2\pi T} \frac{1}{m} \Delta D^t_m(p) - GK_t \sum_{m=1}^{\Omega_0/2\pi T} D_m(p)D_{m+n}(p) + \frac{G}{2} K_s \sum_{m=1}^{\Omega_0/2\pi T} D^s_m(p)D_{m+n}(p) + \frac{G}{4} K_t \sum_{m=1}^{\Omega_0/2\pi T} n m D^s_m(p)D_{m+n}(p) + \frac{G}{4} K_t \sum_{m=1}^{\Omega_0/2\pi T} \left( 2 - \frac{m}{n} \right) D^t_m(p)D_{m+n}(p) , \quad (2.16f) \]

\[ J_2^c(p, \Omega_n) = 2\pi T \sum_{m=1}^{\Omega_0/2\pi T} \Omega_m \left( D_m(p) \right)^3 \frac{K_c}{1 + K_c G^2 \pi T f_{m+1}(p)} , \quad (2.16g) \]

\[ J_1^c(p, \Omega_n) = \frac{1}{2} G^2 \pi T \sum_{m,n=1}^{\Omega_0/2\pi T} \frac{1}{m} \Delta D^s_m(p) \frac{K_c}{1 + G^2 \pi T K_c f_{m+1}(p)} , \quad (2.16h) \]

For the one-point vertex function one finds,

\[ \Gamma^{(1)}(\Omega_n) = 1 + G \int_p [I^s_1(p, \Omega_n) + 3I^t_1(p, \Omega_n) + I^c_1(p, \Omega_n)] . \quad (2.17) \]

In giving Eqs. (2.13)-(2.17) we have neglected terms that are finite in \( D = 2 \) as \( \Lambda, \Omega_0 \rightarrow \infty \), and a delta function constraint (cf. Eqs. (2.14)) is understood in Eqs. (2.15c,2.15d,2.15e). Some of these terms depend on \( n_1 \) and \( n_2 \) separately, not just on the difference \( n_1 - n_2 \). In Eq. (2.15d) we have written a separate dependence on \( n_1 \) and \( n_2 \) explicitly for later reference. Also, the complete frequency dependence of \( K^t_{n_1n_2,n_3n_4} \) and \( K^c_{n_1n_2,n_3n_4} \) is more complicated than the one shown. However, for most of our purposes it is sufficient to treat all 'external' frequencies as equal, and we do not have to deal with the (substantial) complications that arise from the full perturbation theory.

Finally, let us discuss one important point. To one-loop order the two-point propagators are given by the right-hand side of Eq. (2.10a) with the inverse matrix \( M^{-1} \) replaced by the matrix \( M^* \) which contains the perturbative corrections. For the particle-hole degrees of freedom one can show in general that, except for irrelevant terms, the matrix \( M^{-1} \) has the same form as the matrix \( M^{-1} \), with the only difference being that the corrected coupling constants appear in \( M^* \). The underlying reason for this feature is the conservation laws for mass, spin, and energy density. For the particle-particle degrees of freedom the situation is different. There are no conservation laws which guarantee that the form of the last term in Eq. (2.10c) will not change at higher orders in the loop expansion. In general, the matrix \( 1/(M^*)^{-1} \) is given by Eq. (2.10c) with the replacement,

\[ \frac{K_c}{1 + G^2 \pi T K_c f_{n_1+n_2}(p)} \rightarrow \Gamma^c_{n_1n_2,n_3n_4}(p) , \quad (2.18a) \]

where \( \Gamma^c \) satisfies an Eliashberg-type integral equation,

\[ \Gamma^c_{n_1n_2,n_3n_4} + 2\pi T \sum_{n_1',n_2' > 0} \Gamma^c_{n_1n_2,n_1'n_2'} \frac{K^c_{n_1'n_2'n_3n_4} \delta_{n_3+n_4,n_1'+n_2'}}{p^2/G_{n_1'n_2'} + H_{n_1'n_2'}(\omega_{n_1'} - \omega_{n_2'})} = K^c_{n_1n_2,n_3n_4} . \quad (2.18b) \]
Here we have again suppressed the momentum dependence of $\Gamma_c$. Note that if we make the substitutions $K_{n_1n_2,n_3n_4}^c \rightarrow K_c$, $H_{n_1n_2}^c \rightarrow H$, $G_{n_1n_2}^c \rightarrow G$, replace the remaining sum by an integral, let $p, T \rightarrow 0$, and use an ultraviolet cutoff $\Omega_0$ on the frequency integral, then $\Gamma_c$ has the standard BCS form,

$$\Gamma_c(p, \omega_{n_1+n_2}) = \frac{K_c}{1 + \frac{\gamma_c^{(0)}}{2} \ln \left( \frac{\Omega_0}{D^{(0)} p^2 + |\omega_{n_1+n_2}|} \right)} \ ,$$

(2.18c)

with $\gamma_c^{(0)} = K_c/H$, and $D^{(0)} = 1/GH$.

The actual Cooper propagator to all orders in perturbation theory would have the simple form given by Eq. (2.18c) only if the coupling constants $K_c$ and $H$ were constant to all orders. In general, the structure of the Cooper propagator will be more complicated and to obtain it one has to solve the integral equation, Eq. (2.18b). For later reference we symmetrize the equation by defining

$$\gamma_{n_1n_2,n_3n_4} = \frac{K_{n_1n_2,n_3n_4}^c}{[H_{n_1n_2} H_{n_3n_4}^{1/2}]} \ ,$$

(2.19a)

and,

$$\tilde{\Gamma}_{n_1n_2,n_3n_4} = \frac{\Gamma_{n_1n_2,n_3n_4}^c}{[H_{n_1n_2} H_{n_3n_4}^{1/2}]} \ .$$

(2.19b)

For $T \rightarrow 0$ the integral equation for $\tilde{\Gamma}$ then reads,

$$\tilde{\Gamma}(\omega, \Omega, \omega'') + \int_0^{\Omega_0} d\omega' \frac{\gamma(\omega, \Omega, \omega') \tilde{\Gamma}(\omega', \Omega, \omega'')}{D(\omega', -\Omega - \omega') p^2 + 2\omega' + \Omega} = \gamma(\omega, \Omega, \omega'') \ ,$$

(2.20a)

with,

$$D(\omega', -\Omega - \omega') = [G(\omega', -\Omega - \omega') H(\omega', -\Omega - \omega')]^{-1} \ ,$$

(2.20b)

Since we consider the zero temperature limit we have made the replacements,

$$2\pi T(n_1 + n_2) = 2\pi T(n_3 + n_4) \rightarrow -\Omega \ ,$$

$$2\pi T n_3 \rightarrow \omega \ ,$$

$$2\pi T n_1' \rightarrow \omega' \ ,$$

$$2\pi T n_1 \rightarrow \omega'' \ ,$$

$$\tilde{\Gamma}_{n_1n_2,n_3n_4} \rightarrow \tilde{\Gamma}(\omega, \Omega, \omega'') \ ,$$

(2.20c)

e tc., and for definiteness we have assumed $\Omega \geq 0$. We stress again that Eq. (2.20a) has the form of an Eliashberg equation with a repulsive kernel.

### III. THE RENORMALIZATION GROUP FLOW EQUATIONS

In the first part of this section we review the general procedure of the field theoretic RG as applied to a nonlinear sigma-model. We then use a normalization point RG procedure to obtain RG flow equations for all of the parameters that appear in the field theory except for the Cooper propagator $\tilde{\Gamma}$. Finally, we discuss the reasons why previous attempts to derive a flow equation for $\tilde{\Gamma}$ are problematic.
A. The Nonlinear Sigma-Model and the Renormalization Group

The basic philosophy behind the field theoretic RG approach is to eliminate all singular ultraviolet cutoff dependences, i.e. singularities of the theory as \((\Lambda, \Omega_0) \to \infty\), by introducing renormalization constants \(Z_i, \ (i = 1, 2, \ldots)\). RG flow equations are then obtained by examining how the \(Z_i\) depend on the cutoff. In principle this approach is equivalent to the Wilsonian RG, which examines how the theory changes when the ultraviolet cutoff is changed from, e.g., \(\Lambda\) to \(\Lambda/b\) with \(b\) a RG rescaling factor. However, only a limited amount of work has been done on the formal relationship between these two formulations of the RG.

A point of fundamental importance in any RG approach to a field theory is that one has to determine how many independent scaling operators there are. In the field theoretic RG approach, one needs to know how many renormalization constants are needed to make the theory finite. The field theory defined by Eqs. (2.1)-(2.3) has the form of a nonlinear sigma-model with perturbing operators. Let us first consider the pure sigma-model part, i.e. the first term on the r.h.s. of Eq. (2.1) with the constraints given by Eqs. (2.3). This term is invariant under the symplectic symmetry group \(Sp(8nN)\), with \(N\) the number of replicas and \(2n\) the number of Matsubara frequencies. This model is well known to be renormalizable with two renormalization constants, one for the coupling constant \(G\) (the disorder) and one for the renormalization of the \(Q\)-field. The second term on the r.h.s. of Eq. (2.1) breaks the symmetry to

\[
\underbrace{Sp(4N) \otimes Sp(4N) \otimes \cdots \otimes Sp(4N)}_{\text{2n times}}.
\]

This term does not require any additional renormalization constants because the coupling constant \(H\) in Eq. (2.1) just multiplies the basic \(Q\)-field, and therefore the renormalization of \(H\) is determined by the field renormalization constant. This model represents the non-interacting localization problem. From a physical point of view the model has a rather restrictive property: The only interaction taken into account is the elastic electron-impurity scattering, and consequently the different Matsubara frequencies in Eq. (2.1) are decoupled. Effectively, \(n\) is held fixed, and this is crucial for the simple renormalization properties of the noninteracting model. The situation remains relatively simple if further terms are added which respect the non-mixing of the frequencies. In this case, one needs one additional renormalization constant for each operator that represents a different irreducible representation of \(Sp(8nN)\).

The situation changes fundamentally with the addition of the last term in Eq. (2.1). Physically, this term describes the electron-electron interaction and hence the exchange of energy between electrons. Technically, this leads to a coupling between the Matsubara frequencies, and an examination of the perturbation theory shows that this term introduces new infrared and ultraviolet singularities as \(n \to \infty\). In the absence of interactions, singularities arise only from momentum integrations and the symmetry group is fixed to be \(Sp(8nN)\). With interactions there are singularities due to both momentum and frequency integrations, and the symmetry properties of the model change continuously during the RG procedure. As a consequence, the results mentioned above concerning the renormalizability of nonlinear sigma-models with perturbing operators, which apply to models with a fixed
symmetry, are inapplicable. No general results concerning the number of renormalization constants needed are available, and it is unclear how to renormalize the model given by the full Eq. (2.1). The symmetry arguments quoted above can provide only a lower bound on the number of Z’s needed to renormalize the theory, and it is not known whether the model is renormalizable with a finite number of renormalization constants.

All RG treatments of the field theory defined by Eqs. (2.1)-(2.3) so far have ignored this general renormalizability problem. They have assumed, explicitly or implicitly, that the full model is still renormalizable with one extra renormalization constant for each interaction coupling constant which is added. In addition, H acquires a renormalization constant of its own once interactions are present. In the absence of Cooperons, i.e. in a theory which contains only K_s and K_t, there is empirical evidence based on perturbation theory for this assumption being correct. In the presence of Cooperons things are more complicated as we will see. For pedagogical reasons, and to make contact with previous work, we nevertheless proceed for a while using this assumption. It is most convenient to use a normalization point RG. The renormalized disorder, frequency coupling, interaction constants, and q-fields are denoted by g, h, k_s, k_t, k_c, and q_R, respectively. They are defined by,

\[ \tilde{G} = \mu^{-\varepsilon} Z_g g \quad , \quad (3.1a) \]
\[ H = Z_h h \quad , \quad (3.1b) \]
\[ K_{s,t,c} = Z_{s,t,c} k_{s,t,c} \quad , \quad (3.1c) \]
\[ q = Z^{1/2} q_R \quad , \quad (3.1d) \]

where µ is an arbitrary momentum scale that is introduced in Eq. (3.1a) to make g dimensionless. The renormalized N-point vertex functions are related to the bare ones by,

\[ \Gamma^{(N)}_R (p, \Omega_n; g, h, k_s, k_t, k_c; \mu, \Lambda) = Z^{N/2} \Gamma^{(N)} (p, \Omega_n; G, H, K_s, K_t, K_c; \Lambda) \quad . \quad (3.2) \]

The theory is called renormalizable with these renormalization constants if all of the \( \Gamma^{(N)}_R \) are finite as \( \Lambda \to \infty \) for fixed renormalized coupling constants. The Z’s in Eqs. (3.1) are not unique. We fix them by the following normalization conditions for the two-point vertex functions,

\[ \frac{1}{8} \frac{\partial}{\partial p^2} i \left( \Gamma^{(2)}_R \right)_{n_1 n_2, n_1 n_2} (p) \big|_{p=0, \omega_{n_1} - \omega_{n_2} = \mu^D / gh, \omega_{n_1} + \omega_{n_2} = 0} = \frac{\mu^D}{g} S_D / (2\pi)^D \quad , \quad (3.3a) \]
\[ \frac{1}{8} \frac{\partial}{\partial (\omega_{n_1} - \omega_{n_2})} i \left( \Gamma^{(2)}_R \right)_{n_1 n_2, n_1 n_2} (p) \big|_{\alpha \neq \beta, p=0, \omega_{n_1} - \omega_{n_2} = \mu^D / gh, \omega_{n_1} + \omega_{n_2} = 0} = h \quad , \quad (3.3b) \]
\[ \frac{1}{8} \left[ i \left( \Gamma^{(2)}_R \right)_{n_1 n_2, n_1 n_2} (p = 0) - i \left( \Gamma^{(2)}_R \right)_{n_1 n_2, n_1 n_2} (p = 0) \right]_{\alpha \neq \beta, \omega_{n_1} - \omega_{n_2} = \mu^D / gh, \omega_{n_1} + \omega_{n_2} = 0} = 2\pi T k_{\nu_i} \quad , \quad (3.3c) \]
\[
\frac{1}{8} \left[ \frac{1}{2} (\Gamma_R^{(2)})_{\alpha\alpha,\alpha\alpha} (p = 0) - 0 \right] \left[ (\Gamma_R^{(2)})_{\alpha\beta,\alpha\beta} (p = 0) \right]_{\omega_1 + \omega_2 = -\mu^D/gh, \omega_1 - \omega_2 = 0} = -2\pi T k_c ,
\]

(3.3d)

These conditions determine the renormalization constants \( Z_{g,h,s,t,c} \). The wavefunction or field renormalization constant \( Z \) we fix by a normalization condition for the one-point vertex function \( \Gamma^{(1)} \). We require,

\[
0(\Gamma^{(1)})_{\alpha\alpha} |_{\Omega_\nu = \mu^D/gh} = 1 .
\]

(3.3e)

Note that the normalization conditions given by Eqs. (3.3) are the usual ones: At scale \( \mu \) the renormalized vertex functions are taken to have their tree level structure.

From Eqs. (2.14), (2.15), (3.1), (3.2), and (3.3) the \( Z \)'s can be determined. We obtain,

\[
Z = \left( 1/\Gamma^{(1)}(\Omega = \mu^D/gh) \right)^2 ,
\]

(3.4a)

\[
Z_g = Z - \frac{\mu^\varepsilon S_D}{g (2\pi)^D} [G_{n_1 - n_2} - G]_{\omega_1 - \omega_2 = \mu^D/gh}
\] + \( O(g^2) \),

(3.4b)

\[
Z_h = Z^{-1} - \frac{1}{h} [H_{n_1 - n_2} - H]_{\omega_1 - \omega_2 = \mu^D/gh} + O(g^2) ,
\]

(3.4c)

\[
Z_{s,t,c} = Z^{-1} - \frac{1}{k_{s,t,c}} [K_{n_1 n_2 n_3 n_4} - K_{s,t,c}]_{\omega_1 - \omega_2 = \mu^D/gh}
\] + \( O(g^2) \),

(3.4d)

The fastest way to derive the RG flow equations is to switch from our cutoff regularized theory to a minimal subtraction scheme. We can do so by putting \( \varepsilon < 0 \) in Eqs. (3.4), letting \( \Lambda \to \infty \), and then analytically continuing to \( \varepsilon > 0 \). We find,

\[
Z = 1 - \frac{g}{4\varepsilon} \left[ \frac{-2}{\varepsilon} + 3l_t + \tilde{\Gamma} \right] ,
\]

(3.5a)

\[
Z_g = 1 + \frac{g}{4\varepsilon} \left[ 5 - 3(1 + 1/\gamma_t)l_t - \frac{\tilde{\Gamma}}{2} \right] + O(g^2) ,
\]

(3.5b)

\[
Z_h = 1 + \frac{g}{8\varepsilon} \left[ -1 + 3/\gamma_t + \tilde{\Gamma} \right] ,
\]

(3.5c)

\[
Z_s = Z_h ,
\]

(3.5d)

\[
Z_t = 1 + \frac{g}{2\varepsilon} \left[ \frac{1}{4\gamma_t} + \frac{1}{4} + \gamma_t + \tilde{\Gamma} \left( \frac{1}{4\gamma_t} + 1 + \frac{1}{2} \gamma_t \right) \right] ,
\]

(3.5e)
\[ Z_c = 1 + \frac{g}{4\epsilon} \left[ \frac{-2}{\epsilon} + 3l + \tilde{\Gamma} \right] + \frac{g}{4\epsilon} (1 + 3\gamma_t) / \gamma_c \]
\[ + \frac{g}{\epsilon^2} \tilde{\Gamma} / \gamma_c , \quad (3.5f) \]

with \( l_t = \ln(1 + \gamma_t) \), \( \gamma_{t,c} = k_{t,c} / \hbar \). In giving Eqs. (3.3), and (3.4) below, we have for simplicity neglected terms of order \( g\tilde{\Gamma}^n \) with \( n \geq 2 \). The omission of these terms does not affect our conclusions.

The one-loop RG flow equations follow from Eqs. (3.1) and (3.5) in the usual way. With \( b \sim \mu^{-1} \) and \( x = \ln b \) one obtains,
\[ \frac{dg}{dx} = -\epsilon g + \frac{g^2}{4} \left\{ 5 - 3(1 + 1/\gamma_t)l_t - \frac{\tilde{\Gamma}}{2} \right\} + O(g^3) , \quad (3.6a) \]
\[ \frac{dh}{dx} = \frac{gh}{8} [3\gamma_t - 1 + \tilde{\Gamma}] + O(g^2) , \quad (3.6b) \]
\[ \frac{d\gamma_t}{dx} = \frac{g}{8} (1 + \gamma_t)^2 + \frac{g}{8} (1 + \gamma_t)(1 + 2\gamma_t) \tilde{\Gamma} + O(g^2) , \quad (3.6c) \]
\[ \frac{d\gamma_c}{dx} = \frac{g}{4} (3\gamma_t + 1) + \frac{g\gamma_c}{8} \left\{- \frac{4}{\epsilon} + 6l_t + 1 - 3\gamma_t + \tilde{\Gamma} \right\} + O(g^2) . \quad (3.6d) \]

In Eqs. (3.3), (3.6) \( \tilde{\Gamma} \) is the Cooper propagator at scale \( \mu \):
\[ \tilde{\Gamma}(\mu) = \tilde{\Gamma}(\omega, \Omega, \omega')|_{\omega = \omega' = \mu^{1/2} / 2g = \Omega / 2} . \quad (3.7a) \]

To lowest order in the disorder one has,
\[ \tilde{\Gamma}(\mu) = \frac{\gamma_c}{1 + \gamma_c x} + O(g) . \quad (3.7b) \]

In giving Eqs. (3.3), (3.6) we have specialized to the case of a Coulomb interaction between the electrons. In this case a compressibility sum rule enforces \( \gamma_s = k_s / \hbar = -1 \) and the terms \( \ln(1 + \gamma_s) \) that appear in doing the integrals become \( -2 / \epsilon \) in Eqs. (3.6). The presence of these terms in the RG flow equation for \( \gamma_c \) reflects the well-known \( (\ln)^2 \) singularity that exists in the disorder expansion of the single-particle DOS in \( D=2 \). It has sometimes been claimed that these DOS effects are absent in all flow equations for the interactions constants.\(^2\) We stress that this statement depends on what quantity exactly one tries to derive a flow equation for. We find that it is true for \( \gamma_s \) and \( \gamma_t \), but not for the Cooperon amplitude \( \gamma_c \). It is also important to note that the presence of these \( 1 / \epsilon \)-terms in the flow equations per se does not create any problems. They do appear, e.g., in the renormalization of the single-particle DOS, for which a careful application of the RG\(^3\) leads to results that are consistent both internally and with those obtained by other methods.\(^4\)
The Eqs. (3.6) are valid for the universality class G. For the spin-orbit class SO the analogous results are,

\[
\frac{dg}{dx} = -\epsilon g + \frac{g^2}{8}[1 - \tilde{\Gamma}] + O(g^3) \quad ,
\]

\[
\frac{dh}{dx} = -\frac{gh}{8}[1 - \tilde{\Gamma}] + O(g^2) \quad ,
\]

\[
\frac{d\gamma_c}{dx} = \frac{g}{4} + \frac{g\gamma_c}{8}\left\{ -\frac{4}{\epsilon} + 1 + \frac{8}{\epsilon} \frac{\tilde{\Gamma}}{\tilde{\Gamma}} \right\} + O(g^2) \quad .
\]

Note that the Eqs. (3.8) for the universality class SO have a fixed point describing a MIT, while the Eqs. (3.6) for the generic universality class do not. The description of the MIT in the class G is more complicated. For our present purposes we do not need a detailed description of the MIT and the Eqs. (3.6) are sufficient.

**B. A Scaling equation for \( \tilde{\Gamma} \)**

The RG flow equations given by Eqs. (3.6) are not closed because they contain the Cooper propagator \( \tilde{\Gamma} \). Ref. 10 just used Eq. (3.7b) in Eqs. (3.6). In general this is not satisfactory, since for a consistent one-loop RG description one needs \( d\tilde{\Gamma}/dx \) to one-loop order. In Ref. 10 we argued that this was justified near the MIT since we found that \( \gamma_c \) approaches a finite FP value at the transition. Here we try to improve on this point. In principle, \( \tilde{\Gamma} \) can be determined by solving the integral equation given by Eqs. (2.20) together with Eqs. (3.6) and (3.7). This structure is very different from the one usually encountered in RG approaches where the flow equations are an autonomous set of coupled differential equations. In this subsection we discuss attempts to reduce the integral equation for \( \tilde{\Gamma} \) to a single differential equation. We note that for generic integral equations this can not be done. What has effectively been assumed in the previous literature is that the reduction is possible in this case. As we will see, this reduction leads to severe structural problems which went unnoticed in the previous treatments quoted since the techniques used were not sensitive to them. In the next section we will therefore turn to a different method, which determines the behavior of \( \tilde{\Gamma} \) directly from Eqs. (2.21). This will qualitatively yield the same result as inserting Eq. (3.7b) into Eqs. (3.6). Here we investigate possible avenues for deriving a flow equation for \( \tilde{\Gamma} \) strictly in order to make contact with previous work.

Within the normalization point RG approach we can formally obtain a differential equation for \( \tilde{\Gamma} \) by renormalizing the Cooper propagator rather than the Cooper vertex function. We impose a normalization condition,

\[
\tilde{\Gamma}(\mu) = \gamma_c \quad ,
\]

and define a renormalization constant \( \bar{Z}_c \) by

\[
\gamma_c^{(0)} = \bar{Z}_c \gamma_c \quad .
\]
Notice that this approach assumes that $\tilde{\Gamma}$ is a simple scaling quantity. To zeroth order in the disorder Eqs. (3.7) and (3.9) give,

$$\bar{Z}_c = (1 - \gamma_c x)^{-1}, \quad (3.10a)$$

$$\frac{d\gamma_c}{dx} = -\gamma_c^2. \quad (3.10b)$$

A few remarks are in order in the context of Eq. (3.10b): (1) It has the standard form of a RG flow equation for a marginal operator. (2) In this approach the RG is used to obtain the BCS logarithm. This is in contrast to, e.g., Eq. (2.11b) where we obtained the BCS logarithm at Gaussian order by inverting the vertex function. (3) In the present subsection the physical meaning of $\gamma_c$ is different from the rest of this paper. Due to Eq. (3.9a) it plays the role of the Cooper propagator rather than that of the Cooper interaction amplitude that appeared in the previous subsection. (4) A crucial question is what the structure of the higher loop corrections to Eq. (3.10b) will be. In a particular approximation that has been made in the literature and which we will discuss below, $\gamma_c$ flows to a nonzero fixed point value, $\gamma_c \to \gamma_c^*$. In this case Eq. (3.10b) naively implies that $\bar{Z}_c$ has a Cooper-type singularity at a finite scale $x = \log b = 1/\gamma_c^*$ which does not correspond to any physical phase transition. Of course, this conclusion is in general not necessarily correct, since terms of higher order in the disorder could change the behavior of $\bar{Z}_c$. Nevertheless, we will see that the appearance of such an unphysical singularity represents a serious problem. We will discuss this point in connection with Eq. (3.14) below, and in the Conclusion.

Within this approach the one-loop RG flow equation can be obtained by using Eqs. (2.15) and (2.19a) in Eq. (2.20a), and iterating to first order in the disorder. From Eqs. (3.1) and (3.9) one can then obtain a flow equation for $\gamma_c$. The resulting equation is quite complicated. Since we have come to the conclusion that this is not a viable approach we will not give a complete discussion, but rather illustrate only a few points. First, let us make contact with previous work\cite{1,2} by retaining only the first two terms in Eq. (2.15e), neglecting the corrections to $H$, putting $\bar{p} = 0$, and working in $D=2$. In this approximation, $\gamma$ in Eq. (2.20a) is given to leading logarithmic accuracy by,

$$\gamma(\omega, \Omega, \omega') = \gamma_c^{(0)} - a_1 \ln \left( \frac{\omega + \omega' + \Omega}{\Omega_0} \right), \quad (3.11a)$$

with,

$$a_1 = \begin{cases} (g/16)(-\gamma_s + 3\gamma_t) & \text{for class G} \\ (g/16) & \text{for class SO} \end{cases}, \quad (3.11b)$$

for the generic and spin-orbit universality classes, respectively. Using Eq. (3.11a) in the procedure discussed above, we obtain the following one-loop flow equation for $\gamma_c$,

$$\frac{d\gamma_c}{dx} = -\gamma_c^2[1 - a_1 c] - 4a_1 \gamma_c \ln 2 + 2a_1 + O(g^2, x^2 e^{-2x}) \quad (3.12a)$$

with,
\[ c = \int_{0}^{\infty} \frac{dz}{z + 1/2} \ln \left( \frac{z + 3/2}{z + 1/2} \right) - \frac{\pi^2}{12} \quad . \]  

(3.12b)

Two things should be noted: (1) The coefficients in Eq. (3.12a) are universal and, consequently, this equation has the form of a standard RG flow equation. (2) At the MIT in, e.g., the SO universality class, \( a_1 \), which is of \( O(g) \), is a constant of order \( g^* = O(\epsilon) \). If Eq. (3.12a) is valid, then this implies that \( \gamma_c \) goes to a fixed point value at the MIT which is of order \( \epsilon^{1/2} \). In a strict \( \epsilon \)-expansion the terms of \( O(g\gamma_c) \) and higher on the r.h.s of Eq. (3.12a) can be neglected, and to leading order the equation takes the form,

\[
\frac{d\gamma_c}{dx} = -\gamma_c^2 + 2a_1 \quad .
\]  

(3.12a')

This result is identical to those in Refs. 1 and 2, except for the prefactor of the \( a_1 \), which is 4 in Refs. 1,2. This difference is due to the fact that for a Coulomb interaction, which was considered in these references, an additional term appears in Eq. (2.15c). This leads to a more complicated kernel than the one in Eq. (3.11a), and to an additional factor of 2 in Eq. (3.12a'). Since this difference is not relevant for our purposes we restrict ourselves to the simpler kernel for the short-range case, Eq. (3.11a).

In the approximation discussed above one finds \( \gamma_c \to \text{const.} \) at the MIT. As noted below Eqs. (3.10), this result creates some problems. To illustrate this point we add to Eq. (3.11a), via Eq. (2.19a), the Cooper propagator contribution to \( H_{n_1n_2} \) and \( H_{n_3n_4} \), i.e. the integral over \( I_1^c \) in Eq. (2.15b). The result is a kernel in Eq. (2.20a) which is given by,

\[
\gamma(\omega, \Omega, \omega') = \gamma_c^{(0)} - a_1 \ln \left( \omega + \omega' + \Omega \right) \Omega_0 + a_2 \gamma_c^{(0)} \left[ \ln \left[ 1 - \frac{1}{2} \gamma_c^{(0)} \ln(\omega/\Omega_0) \right] + \ln \left[ 1 - \frac{1}{2} \gamma_c^{(0)} \ln(\omega'/\Omega_0) \right] \right] \quad ,
\]  

(3.13a)

with,

\[
a_2 = g/16 \quad .
\]  

(3.13b)

The last two terms in Eq. (3.13a) occur because of the dependence of \( \gamma \) on the Cooper propagator. In the last term in Eq. (3.13a) we have neglected a dependence on \( \Omega \) which is irrelevant for our purposes. Using Eq. (3.13a) in the RG procedure yields the flow equation,

\[
\frac{d\gamma_c}{dx} = -\gamma_c^2 \left[ 1 - a_1 c - 2a_2 \right] - 4a_1 \gamma_c \ln 2 + 2a_1 - 2a_2 \gamma_c^2 \int_{0}^{b^2} \frac{dz}{(z + 1/2)^2} \ln \left[ 1 - \frac{\gamma_c}{2 \ln(2z)} \right]
\]

\[ + O(g^2) \quad .
\]  

(3.14)

The crucial point is that as \( x = \ln b \to \infty \) the last term in Eq. (3.14) does not exist because of a singularity at \( b \sim \exp(1/\gamma_c) \sim \exp(1/\epsilon^{1/2}) \). Notice that this breakdown of the RG flow equation for the Cooper propagator is nonperturbative in nature. Previous attempts to derive a RG flow equation for the Cooper propagator\[\[\] were based on a frequency-momentum shell RG approach. This method is by necessity restricted to low order perturbative expansions in both \( g \) and \( \gamma_c \) and cannot be used to discuss the singularity in Eq. (3.14). Furthermore, if one expands the last term in Eq. (3.14) in powers of \( \gamma_c \) then a non-Borel
summable divergent series is obtained. The structure of this singularity resembles what is known as the renormalon problem in quantum field theory. We also note that this singularity is obviously related to the one discussed below Eq. (3.10b).

In the following section we will argue that the Cooper propagator actually consists of multiple scaling parts, and that any theoretical approach that does not acknowledge this feature is invalid. It is unclear whether or not the singularity discussed in connection with Eq. (3.14) is related to this problem, which in turn is related to the question of whether or not the theory is renormalizable. Usually it is necessary to go to higher than one-loop order in order to verify the presence of multiple scaling parts. In the present case, however, the fact that a renormalization of the propagator and the vertex function, respectively, led to different results is already an indication of their presence.

IV. THE ELIASHBERG EQUATION, AND LOGARITHMIC CORRECTIONS TO SCALING

In the first part of this section we discuss some general features of the Eliashberg equation with a repulsive kernel given by Eq. (2.20a). We then give theoretical arguments in favor of the existence of logarithmic corrections to scaling at the MIT, and discuss their experimental relevance.

A. The Eliashberg Equation

In order to complete the RG description started in Sec. III A, the scaling properties of the Cooper propagator given by Eqs. (2.19), (2.20), and (3.7) need to be determined. In Sec. III B we showed why previous attempts to reduce the integral equation for $\tilde{\Gamma}$ to a differential equation are not satisfactory. Here we pursue a different approach. We acknowledge that one has to actually solve the integral equation in order to obtain information about $\tilde{\Gamma}$. Since this is very difficult to do in general, we classify possible solutions for different behaviors of the kernel $\gamma(\omega, \Omega, \omega')$.

To simplify our considerations we work at zero momentum. We further specialize to a model with a separable kernel. After drawing some conclusions for this special case we will argue that these are actually generic. The main points can be illustrated using a kernel that is a sum of two separable parts,

$$\gamma(\omega, \Omega, \omega'') = f_1(\omega, \Omega) f_1(\omega'', \Omega) + f_2(\omega, \Omega) f_2(\omega'', \Omega) \quad .$$

Note that Eq. (eq:4.1a) satisfies the symmetry requirement,

$$\gamma(\omega, \Omega, \omega'') = \gamma(\omega'', \Omega, \omega) \quad .$$

Eq. (4.1b) is a consequence of the symmetry property $K^{\epsilon}_{n_1 n_2 n_3 n_4} = K^{\epsilon}_{n_3 n_4 n_1 n_2}$, which in turn follows from Eq. (2.2c). Inserting Eq. (4.1a) into Eq. (2.20a) leads to a separable integral equation that can be easily solved. We obtain,
\[ \bar{\Gamma}(\omega, \Omega, \omega'') = \left[ (1 + J_1)(1 + J_2) - J_3^2 \right]^{-1} \]
\[ \times \left[ f_1(\omega, \Omega) f_1(\omega'', \Omega) (1 + J_2) + f_2(\omega, \Omega) f_2(\omega'', \Omega) (1 + J_1) \right. \]
\[ \left. - [f_1(\omega, \Omega) f_2(\omega'', \Omega) + f_1(\omega'', \Omega) f_2(\omega, \Omega)] J_3 \right] \]
with,
\[ J_{1,2} = \int_{0}^{\Omega} d\omega' \frac{[f_{1,2}(\omega', \Omega)]^2}{2\omega' + \Omega} \]
and,
\[ J_3 = \int_{0}^{\Omega} d\omega' \frac{f_1(\omega', \Omega) f_2(\omega', \Omega)}{2\omega' + \Omega} \]

In the previous literature it has been suggested that \( \bar{\Gamma}(\mu) \) given by Eq. (3.7a) goes to a finite fixed point value at the MIT (cf. the discussion in the previous subsection), and that the approach to criticality is characterized by a conventional power law. To see how this type of behavior can be realized from Eqs. (4.1), (4.2), we specialize to criticality, and put \( f_2 = 0 \). If \( f_1 \) diverges like, e.g.,
\[ f_1(\omega, \Omega) \sim (\omega + \Omega)^{-\alpha} \quad (\alpha > 0) \]
then the fixed point (FP) value of \( \bar{\Gamma} \) is,
\[ \bar{\Gamma}^* = 2^{1-2\alpha} 3^{-2\alpha} \left[ \int_{0}^{\infty} \frac{dx}{(x + 1/2)(x + 1)^{2\alpha}} \right]^{-1} \]
and near the FP \( \bar{\Gamma} \) satisfies the flow equation,
\[ \frac{d\bar{\Gamma}}{d \ln \mu^2} = \bar{\Gamma}^2 2\alpha/\bar{\Gamma}^* - 2\alpha \bar{\Gamma} \]
that is, \( \bar{\Gamma} - \bar{\Gamma}^* \sim \mu^{4\alpha} \). Similarly, if \( f_2 = 0 \) and \( f_1 \) vanishes at the MIT (Eq. (4.3) with \( \alpha < 0 \)), then \( \bar{\Gamma} \) also vanishes and satisfies a flow equation with universal coefficients. The marginal, logarithmic approach to zero occurs if \( f_2 \) vanishes and \( f_1 \) approaches a constant at the MIT.

If we assume that these asymptotic results are not tied to the separable kernel, but are generic properties of the general Eliashberg equation, then we have the following situation: If \( \gamma \) diverges (vanishes) at the MIT, then \( \bar{\Gamma} \) has a finite (zero) FP value, and if \( \gamma \) goes to a constant then \( \bar{\Gamma} \) vanishes logarithmically slowly. In all of these cases \( \bar{\Gamma} \) satisfies an autonomous differential equation with universal coefficients.

However, from a more general point of view, going beyond the asymptotic behavior, one expects a more complex result. For instance, even if \( \gamma \rightarrow \gamma^* \) at the MIT one expects a correction that vanishes either as a power law or as a logarithm. In fact, the Eq. (3.6d) predicts this kind of behavior. In our model calculation this happens if \( f_2 \neq 0 \). With Eqs. (4.2), it is easy to see that \( \bar{\Gamma}(\mu) \) does not satisfy an autonomous differential equation if \( f_2 \neq 0 \).
Of course, this just reflects the obvious fact that in general an integral equation cannot be reduced to a single differential equation.

We further note that even if $\gamma$ diverges at the MIT, then one generically still expects a finite subleading contribution. Using this in either Eq. (1.2a) or in the actual Eliashberg equation, Eq. (2.20a), one finds that (1) $\tilde{\Gamma}(\mu)$ does not satisfy a single differential equation, and (2) $\tilde{\Gamma}$ approaches a finite FP value $\tilde{\Gamma}^*$, but logarithmically slowly so.

We conclude that for both the case where $\gamma$ diverges and where it approaches a constant at the MIT one expects a logarithmically slow approach to the FP value of $\tilde{\Gamma}$. The only other possibility is that $\gamma$ vanishes as a power law at the MIT. For this case $\tilde{\Gamma}$ also vanishes as a power law, and in a scaling description (cf. below) $\tilde{\Gamma}$ is a conventional irrelevant variable. While at present we cannot exclude this scenario, we consider it unlikely because of the first term on the r.h.s. of Eq. (3.6c) (or the second term on the r.h.s of Eq. (2.15c)), which tends to drive $\gamma_c$ and hence $\gamma$ towards larger values.

### B. Logarithmic Corrections to Scaling

In the previous subsection we have argued that in general one expects $\tilde{\Gamma}$ to approach its fixed point value logarithmically slowly. This result is consistent with Eq. (3.6c) which gives $\gamma_c \to \gamma_c^*$ at the MIT, which in turn implies that $\tilde{\Gamma}$ vanishes logarithmically slowly at the MIT. Because $\tilde{\Gamma}$ couples to all physical quantities, cf. Eqs. (3.1), we conclude that in all universality classes where Cooperons are present, logarithmic corrections to scaling will appear. Note that this conclusion is independent of the spatial dimensionality and depends only on whether or not the kernel $\gamma$ has a constant contribution at the MIT. We also note that this is a zero-temperature, quantum mechanical effect that might be relevant for other quantum phase transitions.

For a specific example of an observable quantity, let us consider the electrical conductivity $\sigma$, which is related to the disorder by,

$$\sigma = 8S_D b^{-\epsilon}/(2\pi)^D \pi g(b) .$$  \hspace{1cm} (4.5)

Note that in giving Eq. (4.5) we have used units such that $e^2/\hbar$ is unity, and we have ignored the possibility of charge renormalization. For a discussion of the latter point we refer the reader elsewhere. With $t$ the dimensionless distance from the critical point at zero temperature, and $\delta\tilde{\Gamma} = \tilde{\Gamma} - \tilde{\Gamma}^*$, the conductivity satisfies the scaling equation,

$$\sigma(t,T) = b^{-\nu} F[b^{1/\nu} t, b^z T, \delta\tilde{\Gamma}(b)] .$$  \hspace{1cm} (4.6)

Here $\nu$ is the correlation length exponent, $z$ is the dynamical scaling exponent, and of the irrelevant variables in the scaling function $F$ we have kept only the one that decays most slowly at the MIT, i.e. $\delta\tilde{\Gamma}$.

At zero temperature we let $b = t^{-\nu}$, and assume that $F[1,0,x]$ is an analytic function of $x$ since it is evaluated far from the MIT. We obtain,

$$\sigma(t \to 0, T = 0) \approx \sigma_0 t^\epsilon \left\{ 1 + \frac{a_1}{\ln(1/t)} + \frac{a_2}{\ln^2(1/t)} + \ldots \right\} ,$$  \hspace{1cm} (4.7)
with \( s = \nu(D - 2) \). Here \( \sigma_0 \) is an unknown amplitude, and the \( a_i \) are unknown expansion coefficients. Depending on what the subleading behavior of \( \delta \Gamma(b) \) is, the \( a_i \) with \( i \geq 2 \) could carry a very weak \( t \)-dependence (e.g. \( a_2 \sim \ln \ln t \)).

An interesting consequence of the logarithmically marginal operator \( \tilde{\Gamma} \) is that the dynamical scaling exponent in Eq. (4.6) is ill-defined. To see this we use that \( z \) is normally defined by,

\[
\sigma(b, T) = b^{-\epsilon} \mathcal{F}[b^{1/\nu}T, b^D h(b)T, \delta \tilde{\Gamma}(b)].
\]

At \( t = 0 \) and as \( T \to 0 \), the Eqs. (4.10) and (4.11) give,

\[
\sigma(t = 0, T \to 0) \approx \sigma_0 T^{\epsilon/z} [\ln(1/T)]^{\epsilon \alpha/z} \times \left\{ 1 - \frac{\epsilon \alpha^2 \ln \ln(1/T^{\epsilon/z})}{z \ln(1/T)} + \ldots \right\}.
\]

We conclude that for \( \sigma(t = 0, T) \) the asymptotic scaling is in general determined by logarithms.

V. EXPERIMENTAL RELEVANCE

A. Experiments in Zero Magnetic Field

It is well known that for the experimental determination of critical exponents, and for the comparison of theoretical and experimental values for these quantities, one must take into account corrections to scaling. This is so mainly because the asymptotic critical region, where corrections are negligible, is too small to be experimentally accessible. Furthermore, a reliable determination of the critical exponents usually requires experimental data that cover many decades of the control parameter. For conventional phase transitions, where the control parameter is the temperature which is relatively easy to control, these conditions can be met. In the case of the MIT the situation is much less favorable. The main reason is that changing the control parameter, i.e. the impurity concentration, usually requires the preparation of a new sample. The only known way to avoid this problem is the stress-tuning method.
Also, since the transition occurs at $T = 0$, measurements at very low temperatures and a careful extrapolation to $T = 0$ are required. The application of these techniques to Si:P has led to the most accurate determination of the critical behavior of a MIT system to date. Still, by the standards of critical phenomena experiments the data taken are relatively far from the critical point, with $t \geq 10^{-3}$, and corrections to scaling have not been considered in the data analysis. This means that the measured exponent for the conductivity cannot be identified with the asymptotic critical exponent $s$. Rather, the measured value must be taken to represent some effective exponent $s_{\text{eff}}$, which is different from $s$ due to corrections to scaling. This observation offers an interesting possibility to explain the discrepancy between the measured value $0.51$ for $s$, and the theoretical bound $s \geq 2/3$. If the measured value represents an effective exponent, then the latter is not subject to the theoretical constraint for $s$. According to this hypothesis, the discrepancy between the measured value $s_{\text{eff}} = 0.51$ and the theoretical lower bound $s \geq 0.67$ must then be due to the corrections to scaling. With the usual, power-law corrections such a large discrepancy would be hard to explain. In the present case, however, where the corrections are logarithmic, it turns out that they provide a viable explanation for the observations.

We can use Eq. (4.7) to reconcile experiments near the MIT which seem to give $s < 2/3$ with the theoretical bound $s \geq 2/3$. In Fig. 1 we show experimental data, extrapolated to zero temperature, for the conductivity of Si:P. The data points were chosen as follows. For small $t$, roundoff due to sample inhomogeneities sets a limit at $t \approx 10^{-3}$. At large $t$, at some point one leaves the region where scaling, even with corrections taken into account, is valid. We chose to include points up to $t = 10^{-1}$. Obviously, for $t \to 1$ the concept of corrections to scaling loses its meaning, and the expansion, Eq. (4.7), in powers of $1/\ln t$ breaks down. We assumed a standard deviation of a quarter of the symbol size for all points except for the one at the smallest $t$, where we assumed it to be three times as large. In order to improve the statistics with many correction terms taken into account, we augmented the thirteen data points for $10^{-3} \leq t \leq 10^{-1}$ by another twelve points obtained by linear interpolation between neighboring points. If all $a_i$ are set equal to zero then a best fit to the data yields $s = 0.51 \pm 0.05 < 2/3$. We now assume, somewhat arbitrarily, $s = 0.7$, and use Eq. (4.7) with the $a_i \neq 0$. We have used a standard $\chi^2$ fitting routine with singular value decomposition to optimize the values of the $a_i$. The dotted, dashed, and full lines, respectively, in Fig. 1 represent the best fits obtained with one, two, and three logarithmic correction terms. These fits are of significantly higher quality than a straight line fit optimizing $s$. More than three correction terms did not lead to further improvements in the fit quality. While the value of $s$ was chosen arbitrarily, this demonstrates that this experiment is certainly consistent with a lower bound $s \geq 2/3$ once the logarithmic corrections to scaling are taken into account. We have also tried to optimize $s$ by choosing different values for $s$ and comparing the quality of the resulting fits with a fixed number of coefficients $a_i$ taken into account (letting $s$ float together with the $a_i$ proved unstable). The result was a very shallow minimum in the fit quality around $s=0.7$. Relatively large ($\pm 0.15$) fluctuations in the best value of $s$ were observed when large $t$ data points were successively eliminated. From our fitting procedure for this experiment we estimate $s = 0.70^{+0.20}_{-0.03}$, where the lower bound is set by the theoretical bound rather than by the fit.
B. Experiments in a Magnetic Field

While this success in fitting the Si:P data is encouraging, one might object that the model invoked contains an infinite set of unknown parameters, namely the $a_i$ in Eq. (eq:4.7), and that therefore the quality of the fit obtained is of no significance. Also, since the value of the asymptotic exponent in 3-D is not known, no quantitative statements can be made. It is therefore important to see if the model can predict any qualitative features that are independent of the actual value of $s$, and whether or not such features are observed.

There are obvious features predicted by our model which follow from the qualitative magnetic field dependence of the Cooperon. The first one is a strong magnetic field dependence of the effective exponent $s_{eff}$. If the logarithmic corrections to scaling are caused by the Cooper propagator, which acquires a mass in a magnetic field, then any nonzero field must act to destroy the logarithms. It thus follows from the model that Si:P, or any material, in a magnetic field must show a value of $s_{eff}$ that is larger than 2/3. Any observation of an $s_{eff}$ smaller than 2/3 in a magnetic field would rule out the logarithmic corrections to scaling as the source of the anomalously small value of $s_{eff}$.

The second feature concerns the temperature dependence of the conductivity. It is well known that those materials which show an $s_{eff} < 0$ also show a change of sign of the temperature derivative $d\sigma/dT$ of the conductivity as the transition is approached. Within the RG description of the MIT this phenomenon can be explained by a change of sign of the $g^2$-term in the g-flow equation, Eq. (3.6a) or (3.8a). Since it is observed in Si:B, for which Eq. (3.8a) is the relevant flow equation, as well as in Si:P, the change of sign cannot be associated with the scaling behavior of the triplet interaction constant, but must be due to $\Gamma$. Since a magnetic field suppresses the Cooperon channel, this feature should therefore disappear in a sufficiently strong magnetic field.

Let us make these predictions somewhat more quantitative. The relevant magnetic field scale $B_x$ is given by the magnetic length $l_B = \hbar c/eB$ being equal to the correlation length $\xi \approx k_F t^{-\nu}$. Let us assume, as we did in connection with Fig. 1, that the boundary of the critical region is given by $t \approx 0.1$. With $k_F \approx 4 \times 10^6 cm^{-1}$ for Si:B or Si:P near the MIT, and with $\nu \approx 1$ we then obtain $B_x \approx 1T$. The model thus predicts that for magnetic fields exceeding about 1T the observed effective conductivity exponent should be larger than 2/3, and the change of sign of $d\sigma/dT$ observed in zero field should disappear.

Both of these features have been observed in the recent experiments by Sarachik et al. The Cooperon induced logarithmic corrections to scaling thus provide a consistent explanation for several observed features of doped Silicon which otherwise would be mysterious.

VI. CONCLUSION

In this paper we have reconsidered the disordered electron problem in the presence of Cooperons, previous treatments of which had led to conflicting results. In particular we have analyzed the technical differences between Refs. 1,2, which renormalized the Cooper propagator, and Ref. 10, which renormalized the Cooper vertex function. Let us summarize the results of this analysis.

(1) Crucial differences exist between the electron-electron interaction in the particle-hole channel and the particle-particle or Cooper channel, respectively. In the particle-hole channel
the conservation laws for particle number and spin enforce an essentially scalar structure of the propagators and vertex functions. As a result, vertex functions are simply scalar inverses of propagators, and standard renormalization procedures applied to either object lead to identical results. In contrast, in the Cooper channel there is no conservation law which would lead to an analogous simplification, and vertex functions and propagators are related by complicated matrix inversion procedures, cf. Sec. II. This leads to fundamentally different structures of the singularities in the two objects, which poses a problem for the renormalization, cf. Sec. II A.

(2) An attempt to reduce the Cooper propagator renormalization to a single RG flow equation encounters severe structural problems which we have discussed in Sec. II B. At the Gaussian level, the renormalization constant needed displays a BCS-like singularity at a finite scale. This leads to a renormalized vertex function which is not finite, but shows the same singularity. If one ignores this problem and proceeds to derive a flow equation for the Cooper propagator, the singularity in the Gaussian renormalization constant produces imaginary terms in the flow equation at one-loop level which resemble the renormalon singularity known in quantum field theory. This problem is nonperturbative in nature with respect to the Cooper interaction constant \( \gamma_c \). This explains why it went unnoticed in Refs. 1, 2, which expanded in both \( \gamma_c \) and in the disorder \( g \). It invalidates the simple fixed point structure found in these references, according to which the Cooper channel interaction is a conventional irrelevant operator.

(3) Any renormalization scheme for the Cooper vertex function must acknowledge the fact that the object that appears in perturbation theory is the Cooper propagator \( \tilde{\Gamma} \), not just the Cooper interaction amplitude \( \gamma_c \). In fact the propagator plays the role of an effective interaction amplitude, cf. Sec. II A. Due to the inversion problem mentioned in point (1) above it is a much more complicated object than \( \gamma_c \). This point was missed in Ref. 10. The procedure there was effectively equivalent to inserting the zero-loop order result for \( \tilde{\Gamma} \) into the one-loop order flow equations for the other coupling constants. The RG flow equations of Ref. 10 are therefore not consistent in general. They are valid near the transition only if \( \gamma_c \) has a finite FP value.

(4) A possible solution of these problems is to derive flow equations for all coupling constants in terms of \( \tilde{\Gamma} \), and to determine the latter from an Eliashberg-type integral equation which deals with the inversion problem. This is the approach taken in Sec. IV. Solutions of the integral equation obtained for separable model kernels suggest that \( \tilde{\Gamma} \) is not a simple scaling variable and does not satisfy an autonomous differential equation with universal coefficients. Plausible assumptions about the kernel of the integral equation lead to the conclusion that logarithmic corrections to scaling, as predicted in Ref. 10, should exist regardless of whether \( \tilde{\Gamma} \) approaches a nonzero fixed point value, as asserted in Refs. 1, 2, or vanishes logarithmically at large scales, as assumed in Ref. 10. However, this conclusion could not be made mathematically precise since the full integral equation has not been solved.

(5) Logarithmic corrections to scaling, as discussed in Sec. IV B, can explain some otherwise mysterious properties of materials that belong to universality classes with Cooperons, cf. Sec. V. The discrepancy between the asymptotic critical exponent for the conductivity and the observable effective exponent in this case is large enough to provide an explanation for the 'exponent puzzle' in Si:P. This explanation is bolstered by two predictions which
have been verified by recent experiments. In the presence of a sufficiently strong magnetic field the observed conductivity exponent $s$ must satisfy the lower bound $s \geq 2/3$, and the characteristic change in the temperature dependence of the conductivity observed in zero field as the transition is approached should disappear.

While these results are encouraging, some serious problems remain, most notably the question of whether or not the theory is renormalizable. We recall (cf. the discussion in Sec. III A) that the pure nonlinear sigma-model is renormalizable with two renormalization constants. The proof of this makes heavy use of the symmetry properties of the model. Since the interaction terms in the action break the symmetry of the sigma-model, the proof of Ref. 21 is no longer applicable for the interacting model and renormalizability has never been proven. It is conceivable, however, that the conservation laws in the particle-hole channel are still sufficient to ensure renormalizability for the model without Cooperons. This would be consistent with the considerable body of perturbative evidence of renormalizability for this case. For the case with Cooperons the absence of a conservation law and the unusual structure encountered in attempts to apply renormalization techniques makes one much more doubtful. It would be useful to further study this question. For instance, it would be interesting to see whether the conservation laws are indeed sufficient to ensure renormalizability. It would also be useful to have an alternative approach to the subject which avoids the generalized nonlinear sigma-model with its many technical problems. One could, for instance, try to work directly with the Grassmannian action as in recent work on interacting fermions without disorder. One could also imagine an order parameter description of the MIT, using the Q-field theory, but not the sigma-model. Work in these directions is underway and will be presented in future publications.

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REFERENCES

1. A. M. Finkel’stein, Z. Phys. B 56, 189 (1984); Zh. Eksp. Teor. Fiz. 84, 168 [Sov. Phys. JETP 57, 97].
2. C. Castellani, C. DiCastro, G. Forgacs, and S. Sorella, Solid State Commun. 52, 261 (1984).
3. E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. 42, 673 (1979); for a review, see, P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. 57, 287 (1985).
4. G. Bergmann, Phys. Rep. 101, 1 (1984).
5. B. L. Altshuler and A. G. Aronov, Solid State Commun. 30, 115 (1979); B. L. Altshuler, A. G. Aronov, and P. A. Lee, Phys. Rev. Lett. 44, 1288 (1980).
6. T. F. Rosenbaum, R. F. Milligan, M. A. Paalanen, G. A. Thomas, R. N. Bhatt, and W. Lin, Phys. Rev. B 27, 7509 (1983).
7. P. Dai, Y. Zhang, and M. P. Sarachik, Phys. Rev. B 45, 3984 (1992).
8. W. N. Shafarman, D. W. Koon, and T. G. Castner, Phys. Rev. B 40, 1216 (1989).
9. J. Chayes, L. Chayes, D. S. Fisher, and T. Spencer, Phys. Rev. Lett. 57, 2999 (1986). This paper proved $\nu \geq 2/D$ with $\nu$ the correlation length exponent. For a heuristic argument to this effect, see, A. B. Harris, J. Phys. C 7, 1671 (1974). In order to conclude that $s \geq 2/D$ one needs in addition Wegner’s (Z. Phys. B 25, 327 (1976)) scaling law $s = \nu(D-2)$ which holds in all existing theories of the MIT.
10. T. R. Kirkpatrick and D. Belitz, Phys. Rev. Lett. 70, 974 (1993).
11. F. Wegner, Z. Phys. B 35, 207 (1979).
12. see, e.g., G. Grinstein, in Fundamental Problems in Statistical Mechanics VI, edited by E. G. D. Cohen (North Holland, Amsterdam 1985), p.147.
13. D. Belitz and T. R. Kirkpatrick, Phys. Rev. B 46, 8393 (1992).
14. M. Grilli and S. Sorella, Nucl. Phys. B 295 [FS 21], 422 (1988); D. Belitz and T. R. Kirkpatrick, Nucl. Phys. B 316, 509 (1989).
15. K. B. Efetov, A. I. Larkin, and D. E. Khmelnitskii, Zh. Eksp. Teor. Fiz. 79, 1120 (1980) [Sov. Phys. JETP 52, 568].
16. T. R. Kirkpatrick and D. Belitz, Phys. Rev. B 41, 11082 (1990).
17. C. Castellani, C. DiCastro, P. A. Lee, M. Ma, S. Sorella, and E. Tabet, Phys. Rev. B 30, 1596 (1986); C. Castellani and C. DiCastro, Phys. Rev. B 34, 5935 (1986); C. Castellani, C. DiCastro, G. Kotliar, P.A. Lee, and G. Strinati, Phys. Rev. Lett. 59, 477 (1987); Phys. Rev. B 37, 9046 (1988).
18. see, e.g., J. Zinn-Justin, Quantum Field Theory and Critical Phenomena (Clarendon, Oxford 1989); M. Le Bellac, Quantum and Statistical Field Theory (Clarendon, Oxford 1991).
19. K. G. Wilson and J. Kogut, Phys. Rep. 12, 75 (1974).
20. J. Polchinski, Nucl. Phys. B 231, 269 (1984).
21. E. Brézin, J. Zinn-Justin, and J. C. Le Guillou, Phys. Rev. D 14, 2615 (1976).
22. D. R. Nelson and R. A. Pelcovits, Phys. Rev. B 16, 2191 (1977).
23. C. Castellani, C. DiCastro, and G. Forgacs, Phys. Rev. B 30, 1593 (1984); C. Castellani, C. DiCastro, and S. Sorella, Phys. Rev. B 34, 1349 (1986).
24. A. M. Finkel’stein, in Anderson Localization, ed. by T. Ando and H. Fukuyama (Springer, New York 1988).
Often the RG is thought of as strictly a technique to resum logarithmic divergencies. If one adopts this interpretation one is tempted to expand Eq. 3.10a in powers of $\gamma x$ and ignore the singularity implied by the infinite series. We will show below Eq. 3.14 that such an expansion is invalid.

Very recently, the interpretation of the data in Ref. 6 has been questioned (H. Stupp, M. Hornung, M. Lakner, O. Madel, and H.v. Löhneysen, Karlsruhe preprint 1993). These authors assert that the rounding of the data ascribed to sample inhomogeneities in Ref. 6 instead signals a crossover, and that the effective exponent is 1.3. In that case, logarithmic corrections to scaling may still be present, but in the absence of a theoretical value for $s$ in $D=3$ the experiment would not be a suitable test of their presence. In what follows we show that even if the value of Ref. 6 is the correct one, the presence of logarithmic corrections to scaling can solve the exponent puzzle.
FIGURES

FIG. 1. Zero-temperature conductivity of Si:P vs. electron concentration. The data points have been redrawn from Ref. 3. The dotted, dashed, and full lines, respectively, are best fits using one, two, and three corrections terms in Eq. (4.7) with an asymptotic critical exponent $s = 0.7$. Best fit values for the coefficients in Eq. (4.7) are $\sigma_0 = 54.87; 94.82; 132.16; a_1 = -1.84; -4.42; -6.61; a_2 = 0; 6.21; 17.73; a_3 = 0; 0; -16.38$ for the three curves, respectively. See the text for more details.