On the Cosmological Constant in a Conformally Transformed Einstein Equation

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We explore the stress-energy tensor arising from the interaction of U(1) symmetric quantum and gravitational fields. Using scalar-tensor theories of gravity, a conformal factor $\Omega^2$ is defined as the rest mass corrected by the quantum potential. The quantum potential, derived from the Klein-Gordon equation, allows for matter’s intrinsic interaction with spacetime. A Lagrange multiplier $\lambda$ is used as a constraint to properly couple matter with gravity. The Heisenberg uncertainty principle appears as a natural artifact of $\lambda$. Unlike the classical limit, $\lambda$ in the quantum regime strongly influence the stress-energy tensor and it is therefore suggested that it is characteristic of the quantum vacuum. Additionally, the cosmological constant $\Lambda$, defined from the modified Einstein’s equation, is formulated for any particle of mass $m$. The mysterious variation in $\Lambda$ is properly evaluated from its cosmological value to that of an electron, from which we obtained a 77 order difference.

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Extended versions of Einstein’s theory of gravity are getting more attractive in recent years [1, 2]. There are many different extension of the general theory of relativity [3–9]. Weyl conformal gravity is especially interesting due to the fact it is a higher derivative theory of gravity with many advantages over Einstein’s theory [10]. Weyl conformal theory shines light at many theoretical issues in General Relativity, but many conceptual problems have yet to be tackled. The main problem facing conformal gravity is that it is non-unitary, making it incompatible with our current understanding of quantum mechanics. As a resolution, Manheim and Bender proposed that the Hermitian nature of quantum mechanics can be generalized to account for space-time reflection, or PT-Symmetry [11, 12]. Assuming PT-Symmetry is a proper extension of Hermiticity, conformal gravity can play a fundamental role in unifying quantum and gravitational fields.

Many of the difficulties in quantum-gravity can be circumvented if one were to accept the duality between the two theories [13, 14]. The classical field theories of Quantum Mechanics and General Relativity are governed by different fundamental assumptions. Quantum theory defines a set of field variables over a flat Minkowski space-time. General Relativity, on the other hand, displaces the gravitational forces onto a manifold structure using a space-time metric $g_{\mu\nu}$. Given the presumed flat space-time geometry of quantum theory, properly incorporating gravity can be a difficult task. Additionally, unlike the deterministic interpretation of General Relativity, quantum mechanics is inherently defined as a probabilistic theory with complex quantities. All of these difficulties can be circumvented by reformulating the U(1) symmetry of quantum mechanics into its geometrical form [15].

In this letter, we show that one could define the required conformal frames necessary to allow matter and spacetime to coexist. Geometrically, this can be achieved by identifying the conformal factor with quantum potential emerging from the quantum mechanical scalar matter field (e.g. Klein-Gordon field). Initially, we define the stress energy tensors related to matter, quantum, and vacuum contributions within the framework of a scalar-tensor theory. In our framework, the gravitational scalar field appearing in the theory is just the Klein-Gordon quantum potential. An interpretation is given for the less obvious vacuum contribution arising from coupling of matter with gravity via the Lagrange multiplier $\lambda$. As a special case, $\lambda$ is studied for a static Gaussian distribution to validate our general approach. $\lambda$ is shown to decay from the quantum to classical regime, as might be expected from the quantum vacuum. The standard deviation $s$ associated to the Gaussian distribution is confined to the Planck length in order to conform with cosmological observables. Finally, an analytical expression is defined for the cosmological constant $\Lambda$ and the alleged 60-120 order difference in $\Lambda$ is exemplified for an arbitrary mass $m$ ranging from the universe to a single electron.

Interpreting classical trajectories using the de-Broglie-Bohm picture of quantum-gravity is one possible way for integrating the classical and quantum theories consistently. Classically, one must ensure the particle trajectories contain the fluctuations arising from quantum mechanics. One possible way of incorporating such fluctuations is through a conformal factor. Nalikar and Padmanabhan studied a quantized version of conformal fluctuations associated to the spacetime geometry [16, 17]. Later on, Santamato derived a modified Schrödinger equation by considering the scale-invariant Weyl theory [18]. Thereafter, Sidharth attempted to provide a geometrical interpretation of quantum mechanics [19]. Then Shojai et al. [20] inspired by their work, defined a conformally transformed action along with a Lagrange multiplier. Here we define a more generalized form of the action to fully incorporate the general exponential constraint condition using the
Lagrange multiplier $\lambda$, taking $c = 1$
\[
A[\{\mu, \Omega, S, \rho, \lambda\}] = 
\frac{1}{2\kappa} \int d^4x \sqrt{-g} \left( R \Omega^2 - 6 \nabla_\mu \Omega \nabla^\mu \Omega \right)
+ \int d^4x \sqrt{-g} \left( \frac{\rho}{m} \Omega^2 \nabla_\mu S \nabla^\mu S - m^2 \Omega^2 \right)
+ \int d^4x \sqrt{-g} \lambda \left[ \ln \Omega^2 - \left( \frac{\hbar^2}{m^2} \nabla_\mu \nabla^\mu \sqrt{\rho} \right) \right] 
\] (1)

Here, $\lambda$ is employed to constrain the conformal factor $\Omega^2$ to the quantum nature of the Klein-Gordon field. The constraint effectively bypasses the fine-tuning problem articulated by Weinberg (see Appendix A on how Lagrange multiplier bypasses Weinberg’s no-go theorem). The proposed action (Eq. 1) reduces to its classical form in the $\hbar \to 0$ limit (e.g. $\Omega^2 \to 1$). By minimizing the action with respect to $S, \rho, \Omega, g_{\mu\nu}$ and $\lambda$, here we derive the equations of motion for a relativistic matter field. In taking the variation of $\lambda$ with respect to $\rho$, the equation of motion for the particle (e.g. matter field) is obtained
\[
(\nabla_\mu S \nabla^\mu S - m^2 \Omega^2) \Omega^2 \sqrt{\rho} + f(\lambda, \rho) = 0
\] (2)

Here, $f(\lambda, \rho) = \frac{\hbar^2}{m^2} \left[ \nabla_\mu \left( \frac{\lambda}{\sqrt{\rho}} \right) - \left( \frac{\lambda}{\sqrt{\rho}} \right) \right]$ is the coupling contribution arising from the Lagrange multiplier. The equation of motion is fully defined in terms of the density $\rho$, Hamilton’s principal function $S$, and the Lagrange multiplier $\lambda$. In taking the variation with respect to the classical action $S$, one can similarly arrive at the corresponding continuity equation
\[
\nabla_\nu (\rho \Omega^2 \nabla^\nu S) = 0.
\] (3)

The quantum mechanical behavior of the particle can therefore fully be described by two real fields $\rho$ and $S$, along with the yet to be defined coupling contribution. For $f(\lambda, \rho) = 0$ one gets the usual Klein-Gordon equation. It is therefore apparent that the equations of motion arising from the conformal factor are more general, with an additional contribution given by $f(\lambda, \rho)$. Unfortunately, Shojai assumed $\lambda$ to be zero giving her no coupling contributions. The physical meaning of $\lambda$ is worthwhile to explore in the quantum mechanical context. In this article we interpret $\lambda$ as a quantity associated to the energy density of the quantum vacuum. There are many reasons to suspect $\lambda$ is a vacuum contribution (as will be discussed in later sections). Even in a flat spacetime, the $\lambda$ contribution seems to generalize the governing quantum mechanical equations of motion. It can be seen that, in the absence of matter’s interaction with gravity, Eq. 2 yields $f(\lambda, \rho) = 0$, which results in a wave equation for $\lambda$. In addition to the equation of motion (Eq. 2) and the continuity equation (Eq. 3), the equation associated to the scalar curvature $R$ can be determined by varying the action with respect to $\Omega$
\[
R \Omega + 6 \Box \Omega + \frac{2\kappa}{m^2} \rho \Omega (\nabla_\mu S \nabla^\mu S - 2m^2 \Omega^2) + \frac{2\kappa \lambda}{\Omega} = 0
\] (4)

Here, the conformal factor $\Omega^2$ is defined as exp ($Q$), where $Q$ is a quantum mechanical quantity known as the quantum potential $[15, 21, 22]$. Here $Q$ is contained within the constraint equation, which can be obtained by similarly varying the action (Eq.1) with respect to $\lambda$
\[
\Omega^2 = \exp \left( \frac{\hbar^2}{m^2} \nabla_\mu \nabla^\mu \sqrt{\rho} \right). \] (5)

This constraint equation is particularly interesting since we are identifying a purely geometrical quantity $\Omega$ with a quantum mechanical descriptor. Similarly, the variation of the action with respect to the metric tensor $g_{\mu\nu}$ generates the modified Einstein equation
\[
G_{\mu\nu} = T_{\mu\nu}^{\text{matter}}(S, \rho) + T_{\mu\nu}^{\text{qm}}(\Omega) + T_{\mu\nu}^{\text{vac}}(\lambda, \rho). \] (6)

Here the stress-energy tensors are given by,
\[
T_{\mu\nu}^{\text{matter}}(S, \rho) = -\frac{2\kappa}{m^2} \rho \nabla_\mu S \nabla^\nu S + \frac{\kappa}{m^2} \rho g_{\mu\nu} \nabla_S S \nabla^\sigma S - \kappa m^2 \Omega^2 g_{\mu\nu},
\] (7)

and
\[
T_{\mu\nu}^{\text{qm}}(\Omega) = \frac{g_{\mu\nu} \Box \Omega^2 - \nabla_\mu \nabla_\nu \Omega^2}{\Omega^2} + 6 \frac{\nabla_\mu \Omega \nabla^\nu \Omega}{\Omega^2} - 3 g_{\mu\nu} \frac{\nabla_\sigma \nabla^\sigma \Omega}{\Omega^2}. \] (8)

The remaining components associated to $\lambda$ are defined as the vacuum energy contributions for reasons to later be clarified
\[
T_{\mu\nu}^{\text{vac}}(\lambda, \rho) = -\frac{\kappa \hbar^2}{m^2 \Omega^2} \left[ \nabla_\mu \sqrt{\rho} \nabla_\nu \left( \frac{\lambda}{\sqrt{\rho}} \right) + \nabla_\nu \sqrt{\rho} \nabla_\mu \left( \frac{\lambda}{\sqrt{\rho}} \right) \right]
+ \frac{\kappa \hbar^2}{m^2 \Omega^2} g_{\mu\nu} \nabla_\sigma \left( \frac{\lambda}{\sqrt{\rho}} \right). \] (9)

The modified equation contains a stress-energy tensor related to the matter $T_{\mu\nu}^{\text{matter}}(S, \rho)$, quantum $T_{\mu\nu}^{\text{qm}}(\Omega)$ and coupling $T_{\mu\nu}^{\text{vac}}(\lambda, \rho)$ contributions. The coupling contribution $T_{\mu\nu}^{\text{vac}}(\lambda, \rho)$ is usually ignored by applying perturbative schemes to the Lagrangian multiplier $\lambda$ (typically assumed to be a parameter). It is worthwhile to explore the physical meaning of $\lambda$ as a density-dependent field, arising from the coupling of quantum matter with gravity. The physical interpretation of the coupling contribution and its relevance to the Planck scale is of important physical consequence. As will be seen, an expression of $\lambda$ for a single-boson can be easily defined for a static density, otherwise $\lambda$ must be interpreted dynamically. In Eq. 6 the Lagrange multiplier $\lambda$ appears to mediate the interaction between Klein-Gordon and Gravitational fields within the stress-energy tensor $T_{\mu\nu}^{\text{vac}}(\lambda, \rho)$. We therefore seek to introduce a new physical meaning to the Lagrange multiplier $\lambda$ appearing in the theory.

Additionally, it is apparent that $T_{\mu\nu}^{\text{vac}}(\lambda, \rho)$ brings about negative energies within the modified Einstein equation. General Relativists are particularly interested in negative energies because of their correspondence to expansion behavior $[23]$. On the other hand, to particle physicists, negative energies are simply a consequence of the particle’s interaction with the quantum vacuum. Negative energies are well known to
contribute to spacetime expansion. In the Brans-Dicke theory, scalar fields contribute to the negative energies of gravitational fields \[24\]. Similarly, \(f(R)\) theories also predict a negative energy contribution \[25\–27\]. Hence, it is fair to assume that the source of negative energy appearing in the currently proposed theory is simply a geometrical manifestation of the quantum vacuum. A connection to the quantum vacuum is difficult to interpret geometrically since there is no ‘geometrodynamic’ theory of quantum mechanics, rather just a probabilistic one. Nonetheless, in the Bohmian framework, one can more easily conceive the physical meaning of the coupling of matter to gravity via the conformal factor \(\Omega^2\). Instead of assuming \(\lambda\) to be spatially and temporally uniform, one can assume a spacetime dependence. As will be speculated later, the reason for the spacetime dependence can be attributed to the nontrivial nature of the vacuum energy. Assuming a near-zero velocity and mass, the scalar curvature \(R\) of the trace equation (Eq. 4) contains an intimate relationship to \(\lambda\)

\[
R \approx -2\kappa \frac{\lambda}{\Omega^2} \tag{10}
\]

Hence \(\lambda\) gives negative scalar curvature \(R\) in the near-zero velocity classical regime (when \(\Omega \approx 1\)). This is just a heuristic argument concerning the relationship between \(\lambda\) and scalar curvature \(R\). A complete mathematical relationship can be obtained by combining Eq. 4 and Eq. 4

\[
R = +2\kappa \rho m \Omega^2 - 6\frac{\lambda}{\Omega^2} - \frac{\kappa}{m^2 \Omega^2} \left( \frac{\lambda}{\sqrt{\rho}} - \frac{\lambda}{\rho} \right) \tag{11}
\]

It is to be noted that the conformal factor \(\Omega^2\) gives a negative contribution to the scalar curvature \(R\) which is independent of the gravitational constant \(\kappa\). The last three terms in Eq. 11 give the quantum mechanical correction to the scalar curvature due to \(\lambda\). There is a positive contribution to \(R\) from rest mass \(m\) of the quantum particle while the contribution from \(\lambda\) can be negative. Spatial and temporal dynamics of \(\lambda\) play an important role in the limit of small mass. As \(m \to 0\) the vacuum coupling contributions within the scalar curvature dominate and therefore become critical in the quantum regime. Additionally, the vacuum coupling contribution vanishes in the classical limit as \(m \to \infty\). It is then easy to conceive that the rest mass, characterizing a positive curvature, can be accompanied with negative curvature as a consequence of quantum mechanical effects. Given negative curvature plays a central role in the quantum regime, we claim \(\lambda\) to be a vacuum contribution. In order to determine the quantum vacuum contributions, the first task is to find a way to express \(\lambda\) analytically. Once \(\lambda\) is known, it is possible to compute all the observable quantities appearing in the theory, which can be compared to experimental observations.

The spacetime behavior of \(\lambda\) can be obtained by substituting the scalar curvature Eq. 11 into the contracted Einstein equation (Eq. 6)

\[
2\nabla_\alpha \left( \frac{\nabla^\alpha \sqrt{\rho}}{\sqrt{\rho}} \right) - \nabla_\mu \nabla^\mu \left( \frac{\lambda}{\sqrt{\rho}} \right) = \frac{\lambda m^2}{h^2}. \tag{12}
\]

Simplifying the L.H.S using Eq. 12 followed by rearranging the terms involving \(\lambda\) and \(\rho\), one can arrive at a complete equation for \(\lambda\)

\[
\nabla_\mu \sqrt{\rho} \nabla^\mu \frac{\lambda}{\sqrt{\rho}} = \left( \frac{m^2 (1 - Q)}{h^2} - \frac{\nabla_\mu \sqrt{\rho} \nabla^\mu \sqrt{\rho}}{\rho} \right) \tag{13}
\]

Assuming a spherically symmetric form of the density \(\rho\) and ignoring variations in \(\theta\) and \(\phi\), one arrives at a more appealing form of the alleged vacuum energy equation

\[
\nabla_\mu \sqrt{\rho} \nabla^\mu \frac{\lambda}{\sqrt{\rho}} = \left( \frac{m^2 (1 - Q)}{h^2} - \frac{\nabla_\mu \sqrt{\rho} \nabla^\mu \sqrt{\rho}}{\rho} \right) \tag{14}
\]

Here, \(\lambda\) needs to be solved dynamically. In the static case, with only a radial contribution, \(\lambda\) can be analytically represented in a much simpler exponential form

\[
\lambda(r) = \exp \left( -\int dr \beta(r) + C_\beta \right) \tag{15}
\]

\[
\beta(r) = \frac{\sqrt{\rho}}{r} \left( \frac{m^2 (1 - Q)}{h^2} + \frac{\nabla_\mu \sqrt{\rho} \nabla^\mu \sqrt{\rho}}{\rho} \right) \tag{16}
\]

The subscript \(r\) denotes differentiation in the radial component and \(\lambda_0 = \exp(C_\beta)\) is the resulting integration constant. As of now, the constraint for \(\lambda_0\) is unknown and open for debate. It is speculated that, like the vacuum energy, \(\lambda_0\) could be defined by a group theory characterizing the particles allowed in nature. In the case of a separable density, it can also be shown that \(\lambda\) is of the form

\[
\lambda = \lambda_0 \exp \left( -\int dt \alpha_1(\rho(t)) + \int dr \beta_2(\rho(r)) \right). \tag{17}
\]

Integrating \(\beta(r)\) within the exponential and computing \(C_\beta\) results in the expression of \(\lambda\). Once \(\lambda\) is known, interpreting the negative energies arising from the quantum mechanical nature of matter can become a trivial task. Therefore, finding an analytical expression for \(\lambda\) is critical to understanding the physical implications of the coupling contribution.

To better understand this, we define the density \(\sqrt{\rho}\) of a single quantum mechanical particle as a Gaussian wavepacket in spherical coordinates

\[
\sqrt{\rho}(r, s) = \left( \frac{1}{\pi s^2} \right)^{3/4} \exp \left( -\frac{r^2}{2s^2} \right). \tag{17}
\]

Here, \(N(s) = \left( \frac{1}{\pi s^2} \right)^{3/4}\) is the normalization constant and \(s = \sigma + \sqrt{2} \ell_p\) is the spatial variation of the particle. We assume that an external potential, necessary to allow for a Gaussian density, is added to the equation of motion (Eq. 4). Although \(\sigma\) can be chosen freely, a fundamental Planck length limit \(\ell_p\) is considered so as to conform with cosmological observables. As will be seen, the Gaussian standard deviation \(s\)
cannot arbitrarily decrease to zero, rather must obey the minimum allowable standard deviation $\sqrt{2/\hbar^2}$. Here, $\sqrt{2}$ is taken to eliminate the singularity which will later be shown to appear in the cosmological constant expression. The partial differential equation of $\lambda$ (Eq. 13) can be simplified for a general quantum mechanical density $\rho$

$$\lambda = \frac{1}{(1 - Q)} \frac{\hbar^2}{m^2} \nabla_\sigma (\sqrt{\lambda} \frac{\nabla^\sigma \sqrt{\rho}}{\sqrt{\rho}}).$$

(18)

By ansatz, one can easily verify that this simple function obeys Eq. 13. Similarly, the total vacuum energy density in the static case also simplifies to $T_{00}^{\text{vac}} = \kappa \lambda / \Omega^2$. Using Eq. 13 and Eq. 16 along with the predefined Gaussian density, $\lambda$ appears to take a fascinating form

$$\lambda = \lambda_0 \exp \left( -\frac{r^2}{2s^2} \right) r^{(s/l_c)^2}.$$  

(19)

As can be seen, a natural expansion and decay behavior is characterized by the polynomial and Gaussian, respectively. The form of Eq. 13 for a linear constraint

$$\lambda_{\text{lin}} = \frac{\hbar^2}{m^2} \nabla_\sigma (\lambda_{\text{lin}} \frac{\nabla^\sigma \sqrt{\rho}}{\sqrt{\rho}}).$$

(20)

Unlike Eq. 13, it can be seen that the expansion behavior of $\lambda_{\text{lin}}$ ceases to exist

$$\lambda_{\text{lin}} = \lambda_0 \exp \left( -\frac{(D-1)s}{l_c} \right).$$

(21)

Here, $D$ is representative of the number of dimensions and $l_c = \hbar/m$ is the Compton wavelength of a particle of mass $m$. It can be seen that $\lambda_{\text{lin}}$ contains a singularity when $s = l_c$. Therefore, $s \geq \sqrt{(D - 1)} l_c$ must be satisfied to avoid the singularity. For a 1D-Gaussian ($1 + 1$ spacetime ($D = 2$)), it can be shown that, once the free parameter $s$ is identified as $s = \sqrt{2\Delta x}$ and the maximum uncertainty in momentum as $\Delta p \propto m \Rightarrow l_c = \hbar / \sqrt{2\Delta p}$ (where $c = 1$), avoiding the singularity in Eq. 21 implies a new but familiar relation

$$\Delta x \Delta p \geq \hbar / 2.$$  

(22)

This is just the uncertainty principle in $1 + 1$-dimension. Here, the uncertainty principle emerges from a more fundamental condition; that is avoiding the singularity of the quantum vacuum. This singularity can be eliminated by taking into account a minimal length $\ell_p$ requirement, enforcing a fundamental threshold. Note that, the singularity in $\lambda_{\text{lin}}$ appears for even $r \geq 0$ in the linear order theory ($\Omega^2 = 1 + Q$). Since we go beyond linear order theory, this problem doesn’t appear for $r > 0$ and $s = 0$ in Eq. 19. But $\lambda$ in Eq. 19 is undefined when $r = 0$ and $s = 0$, again this can be solved by taking into account a minimal length $\ell_p$ requirement.

To better understand the coupling contributions defined in Eq. 9 one must determine whether observables are properly reproduced in the quantum and universal domains. One such observable is the cosmological constant $\Lambda$. The alleged 60-120 order difference in the transition of $\Lambda$ from the classical to quantum regime is a long-lived problem yet to be solved. The difficulty arises due to the lack of: 1) the proper characterization of quantum mechanical matter within the framework of General Relativity; 2) the lack of geometrical interpretation in quantum mechanics, via conformal frame. The cosmological constant can be identified as the negative term of the stress-energy tensor containing $g_{\mu\nu}$

$$\Lambda_{v+q+g} = 3 \frac{\nabla_\sigma \Omega \nabla^\sigma \Omega}{\Omega^2} - \frac{\Box \Omega^2}{\Omega^2} - \frac{\kappa \lambda (1 - Q)}{\Omega^2} + \frac{\kappa\hbar^2}{2m^2\Omega^2} \left( \frac{\lambda}{\sqrt{\rho}} - \frac{\nabla^2 \sqrt{\rho}}{\rho} \right).$$

(23)

Substituting the equation of motion (Eq. 2) into Eq. 23, one gets a quantity completely analogous to the vacuum and quantum contributions

$$\Lambda_{v+q+g} = 3 \frac{\nabla_\sigma \Omega \nabla^\sigma \Omega}{\Omega^2} - \frac{\Box \Omega^2}{\Omega^2} - \frac{\kappa \lambda (1 - Q)}{\Omega^2} + \frac{\kappa^2 \hbar^2}{2m^2\Omega^2} \left( \frac{\lambda}{\sqrt{\rho}} - \frac{\nabla^2 \sqrt{\rho}}{\rho} \right).$$

(24)

For quantum mechanical particles ($m \rightarrow 0$), the vacuum contributions in Eq. 24 can play a significant role in characterizing $\Lambda$. Terms containing $\Omega$ will dominate in Eq. 24 deeming the conformal factor, arising from the quantum potential $Q$, an important contribution. Once $m$ is of the order of the Planck mass $\sqrt{\frac{\hbar c}{\kappa}}$, the gravitational contribution suppresses the vacuum in the short distance. For smaller particles (i.e. mass of an electron), the vacuum tends to dominate the gravitational contribution. There is also an important point to take into consideration: the cosmological constant $\Lambda_{v+q+g}$ in Eq. 24 contains a spatial dependence, via $r$. For the more generalized vacuum contribution, a temporal dependence also naturally arises. By evaluating $\Lambda_{v+q+g}$ in Eq. 24 for the presumed density (Eq. 17), using vacuum contribution $\Lambda$ (Eq. 25), and defined conformal factor $\Omega^2$ (Eq. 5), one arrives at an expression for the cosmological constant

$$\Lambda_{v+q+g} = -6 \left( \frac{l_c^2}{s^4} \right) - \left( \frac{l_c^2}{s^4} \right)^2 r^2 - \kappa r^2 \frac{l_c^2}{s^4} \exp \left( -\frac{r^2}{2s^2} \right) \left( 1 + \frac{l_c^2}{s^2}(3s^2 - r^2) \right) \times \left[ s^6(l_c^2 + s^2) - l_c^6(r^4 - 3r^2 s^2) \right].$$

(25)

Equation 25 contains the standard deviation of the Gaussian density $s$, allowing for a smooth quantum to classical transition. By considering the dominant contributions in Eq. 25, one arrives at a simplified expression

$$\Lambda = 6 \left( \frac{l_c^2}{s^4} \right) + \left( \frac{l_c^2}{s^4} \right)^2 r^2$$

(26)
Here, \( r \) defines the scale of observation and can vary from the Planck length to the observable universe. To properly conform to cosmology, one must fix \( s = \sigma + \sqrt{2} \rho \). Here, \( s = \sigma + \sqrt{2} \rho \) comes from the condition \( s \geq \sqrt{2} l \) to avoid \( r^{-2} \) singularity in the third term of the expression \( \Lambda_{c+q+g} \) (See Eq. 25). The minimum-length element is chosen to avoid illogical mathematical scenario \((\frac{\rho}{\sigma})^0 \) appearing in \( \Lambda_{c+q+g} \) (See Eq. 25) when \( r = 0 \) and \( s = 0 \). Fixing the minimum length \( l_p \) transforms \((\frac{\rho}{\sigma})^0 \rightarrow (\frac{\rho}{\sigma})^{r^2} \). Interestingly enough, the number 6 in the conformal transformation is a result of the dimensionality of our spacetime \((D = 4) \Rightarrow (D - 1)(D - 2) = 6 \). In the cosmological scale \((s \approx \sqrt{2} l_p l_c \Rightarrow l_c \approx 6 \times 10^{34} \text{m}) \), Eq. 26 can be written entirely in terms of the Schwartzschild radius \( r_s = 2GM/c^2 \).

\[
\Lambda \approx 6 \left( \frac{1}{r_s^2} \right) + \left( \frac{1}{r_s^4} \right) r^2 \quad (27)
\]

For \( r \ll r_s \), the static term dominates and can be shown to play a fundamental role in defining its astronomical value

\[
\Lambda_{\text{astr}} \approx 6 \left( \frac{1}{r_s^2} \right) \quad (28)
\]

The cosmological constant can be determined theoretically once we know the analytical expression for the mass of the universe. It is well known that the Hoyle-Carvalho relation \([28]\) gives a theoretical estimation for the mass of the universe, where they had shown that the mass of the universe can be determined using only microscopic quantities. In addition, D. V. Valev pointed out that, using dimensionality arguments only, the Hoyle-Carvalho relation can be derived \([29]\)

\[
M_u \propto \frac{e^3}{G H_0} \quad (29)
\]

Given the recently measured Hubble constant \((H_0 = 73.52 \pm 1.62 \text{ km s}^{-1} \text{ Mpc}^{-1}) \) \([31, 32]\), the mass of the universe can be estimated \(M_u \approx 1.6 \times 10^{53} \text{ kg} \). Using Eq. 28 and Eq. 29, the cosmological constant can be written in terms of Hubble’s constant \(H_0 \).

\[
\Lambda_{\text{astr}} \approx 3 \left( \frac{H_0^2}{c^2} \right) \quad (30)
\]

This is just like the standard result of the cosmological constant arising in the Friedmann equation for a flat universe when the gravitational mass density contribution is ignored \([32]\). Note that, we started with a pure quantum mechanical problem and arrived at a standard result in Einstein’s gravity differing only by a factor of two. Approximately taking the mass of the universe \(M \approx 1.6 \times 10^{53} \text{ kg} \), and using Eq. 28 one gets a static contribution of \(\Lambda_{\text{astr}} = 1.06 \times 10^{-52} \text{ m}^{-2} \). This is close to the value measured in a recent experiment \([33]\). The radially dependent term in Eq. 27 plays a fundamental role in characterizing the expansion of our universe. At \( r = r_s \), the rate of expansion naturally increases \( \Lambda = 1.24 \times 10^{-52} \text{ m}^{-2} \). The quadratic trend in \( r \) is the result of considering the exponential form of the conformal factor \( \Omega^2 = e^q \) to ensure non-tachyonic behavior (e.g. beyond the linear form of \( \Omega^2 \)). In the quantum regime, similar conclusions can be made for the standard deviation in Eq. 26 when \( s \approx l_c \gg \sqrt{2} l_p \).

\[
\Lambda_{\text{qm}} \approx 6 \left( \frac{1}{l_c^2} \right) + \left( \frac{1}{l_c^2} \right) r^2 \quad (31)
\]

Similarly, ignoring the spatially dependent contribution, one arrives at the static value of the cosmological constant for an electron mass \(\Lambda_{\text{cl}} = 4.02363 \times 10^{29} \text{ m}^{-2} \). At the Compton wavelength of an electron \( r = l_c \), the spatial dependence of the cosmological constant once again begins to dominate \(\Lambda_{\text{cl}} = 4.69424 \times 10^{25} \text{ m}^{-2} \). The large discrepancy from the astronomical value has been pointed out by physicists for decades \([34, 35]\). Here, a natural variation in the value of \( \lambda \) is present simply by the consideration of the conformal factor. For a more accurate expression of the cosmological constant, particularly in the quantum regime (where the quantum vacuum plays a more fundamental role), one can use the generalized expression in Eq. 25.

In this paper, we have explored the vacuum energy contributions resulting from the Bohmian framework of Quantum-Gravity. The conformal factor was defined by the quantum potential associated to the Klein-Gordon equation. By a geometrical means and an inherently probabilistic interpretation, we were able to couple the U(1) symmetry of quantum mechanics (beyond linear order) to arbitrary gravitational fields. Weinberg no-go theorem is bypassed using a Lagrange multiplier within the action, resulting in a vacuum density field \( \lambda \). The identified vacuum contribution plays a significant role in the quantum regime (in the form of a correction), and naturally decays in the classical regime. After rigorous analysis, the proposed theory potentially serves as a solution to the cosmological constant problem. \( \lambda \) naturally varies by 77 orders from the cosmological to quantum scale and almost perfectly conforms to the measured astronomical value. Further experiments are needed to confirm the legitimacy of the order of expansion, via the cosmological constant, identified in the quantum regime. We hope that these results will shine light at the unification of quantum and gravitational fields. We plan to further explore the defined vacuum energy contribution \( \lambda \) and its consequence to a geometrical realization of quantum mechanics.

**Appendix A - Bypassing Weinberg’s no-go**

Weinberg proves his no-go theorem based on the usual understanding of coupling scalar fields to gravity, but our action couples the scalar field, \( \Omega^2 = e^q \), in an entirely different manner. The key difference lies in our constraint field \( \lambda \), which we identify as the vacuum density. \( \lambda \) does not simply fix the conformal factor to the exponential of the quantum potential, rather allows one to overcome the no-go theorem by enforcing the scalar field to conform with the contracted stress-energy tensor. In this section, We make it apparent that Weinberg’s no-go theorem can be bypassed using a Lagrangian constraint.
Weinberg starts with the Euler Lagrange Equation (See Equation 6.2 and 6.3 in his article [24]). Looking for stationary solutions of the scalar and tensor equations, he finds that, for mathematical consistency, fine tuning is needed.

We identify that the aforementioned fine tuning problem arises within any conformally transformed Lagrangian $L$ (e.g., Brans-Dicke theories) and suggest to overcome it by considering a yet unexplored field $\lambda$. Here $\lambda$ acts as a Lagrange multiplier enforcing the metric and scalar field to conform with the particle’s background potential, $Q$. A more thorough physical interpretation of $\lambda$ is given in this article. To simplify matters, we articulate a simplified action (of little physical meaning) instead of Weinberg’s proposed action to show the inconsistency can be eliminated

$$\mathcal{L} = e^{A\phi} \sqrt{-g} \mathcal{L}_0(\sigma) + \sqrt{-g} \lambda \left( \phi - Q(\rho, \nabla_\mu \rho, \Box \rho, \ldots) \right)$$  \hspace{1cm} (A.32)

By varying the action with respect to $g_{\mu\nu}$, $\phi$, $\lambda$ we get a stress-energy tensor, scalar field equation, constraint equation, and, the yet unexplored, $\lambda$ equation

$$T_{\mu\nu} = T_{\mu\nu}^{WB} - 2 \frac{\delta}{\delta g_{\mu\nu}} \left( \lambda \left[ Q(\rho, \nabla_\mu \rho, \Box \rho, \ldots) \right] \right)$$  \hspace{1cm} (A.33)

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0 \iff (T_{\mu}^{\mu})^{WB} + \lambda = 0$$  \hspace{1cm} (A.34)

The constraint equation is given by,

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \iff \phi = Q(\rho, \nabla_\mu \rho, \Box \rho, \ldots)$$  \hspace{1cm} (A.35)

According to Weinberg, consistency of the above equations requires that the trace of the energy-momentum tensor $T_{\mu\nu}$ ($g^{\mu\nu} T_{\mu\nu} = T^{\mu}_\mu$) be equivalent to $\frac{\delta \mathcal{L}}{\delta \phi} = 0$. The added field variable $\lambda$ allows for the scalar field $\phi$ to properly conform within Einstein’s equation by implying the following constraint field equation (Obtained using Eq. A.33 and Eq. A.34)

$$\lambda = -2g^{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} \left( \lambda \left[ Q(\rho, \nabla_\mu \rho, \Box \rho, \ldots) \right] \right)$$  \hspace{1cm} (A.36)

With this field equation (See Eq. A.36) satisfied, the scalar field and Einstein’s equations are consistent. The conditions imposed by Weinberg are satisfied:

$$\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} = 0$$  \hspace{1cm} (A.37)

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0$$  \hspace{1cm} (A.38)

The Lagrange multiplier imposes a condition on the scalar field, allowing one to overcome the no-go theorem. In the simplified scenario proposed by Weinberg, $\lambda = 0$ leads to the fine tuning problem, suggesting a nonzero $\lambda$ should play an essential role in properly balancing the scalar (Eq. A.34) and contracted Einstein (Eq. A.33) equations.
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