High Temperature Phase Transition in Two-Scalar Theories

S. Bornholdt, a N. Tetradis b and C. Wetterich a

a) Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16, 69120 Heidelberg, Germany
b) Theoretical Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP, U.K.

Abstract

Two-scalar theories at high temperature exhibit a rich spectrum of possible critical behaviour, with a second or first order phase transition. In the vicinity of the critical temperature one can observe critical exponents, tricritical points and crossover behaviour. None of these phenomena are visible to high temperature perturbation theory.
1. Introduction

Scalar field theories have been the prototype for investigations concerning the question of symmetry restoration at high temperature. Following the original argument of Kirzhnits and Linde [1], the $O(N)$-symmetric scalar theory was considered in subsequent studies of the problem [2, 3, 4]. The framework in which these studies were carried out is the perturbative evaluation of the effective potential [5] and its generalization for non-zero temperature. Even though the restoration of the spontaneously broken symmetry was qualitatively demonstrated, the investigation of the details of the phase transition was not possible, due to infrared divergences rendering the perturbative approach unreliable near the critical temperature [3, 4]. These divergences originate in the absence of an infrared cutoff in higher loop contributions when the temperature dependent mass of the scalar fluctuations approaches zero near the critical temperature. An amelioration of the situation was achieved through the summation of an infinite subclass of perturbative contributions (the “daisy” graphs) [2]. Indeed, these contributions become dominant for large $N$ and a quantitative description of the phase transition can be obtained in this limit. However, the physical picture remained unclear for small, physically relevant values of $N$, for which even the order of the transition was not established. The question was resolved [3] through the method of the effective average action [6] - [10], which relies on the renormalization group approach. The phase transition was shown to be second order for all values of $N$. The quantitative behaviour near the critical temperature was studied in detail and the critical system was found to have an effectively three-dimensional character. Its behaviour can be characterized by critical exponents [3, 10], in agreement with known results from three-dimensional field theory. The picture was verified through an independent analysis in the large $N$ limit [11], with use of other non-perturbative methods, such as the solution of the Schwinger-Dyson equations. A summary of the results can be found in ref. [12].

In a cosmological context first order phase transitions are more spectacular than second order transitions due to the departure from thermal equilibrium. One would like to have a prototype model for a first order transition, for which the methods of high temperature field theory can be tested, similarly to the $O(N)$ scalar model for second order transitions. In statistical physics it is well known that scalar models with more than one field and discrete symmetries instead of maximal $O(N)$ symmetry exhibit a rich spectrum of critical behaviour, including first and second order transitions and tricritical behaviour in between. Since high temperature field theories are in close correspondence to (three-dimensional) statistical models, it seems natural to investigate such models also as prototypes for first order transitions in high temperature field theory. In this paper we apply the method of the effective average action to the study of the high temperature phase transitions in theories with two real scalar fields. The symmetry is not $O(2)$, but rather a discrete symmetry. This model can serve as a prototype for a first order phase transition in field theories. It can be easily generalized to models in which each scalar field is an $N$-component vector.

We are interested in the phenomenon of spontaneous symmetry breaking and sym-
metry restoration at high temperature. For sufficiently low temperature our two scalar theory models the Higgs mechanism in gauge theories, through which the expectation value of a scalar field results in a mass term for gauge fields. Perturbative arguments predict a first order phase transition for this case. However, the reliability of such predictions is questionable when the transition becomes weakly first order, due to infrared divergences similar to the ones plaguing the study of the $O(N)$-symmetric scalar theory. The approximate vanishing of some mass near the critical temperature results in the absence of an infrared cutoff in higher loop contributions of perturbation theory. In two-scalar theories this is connected with the fact that one of the fields gets its mass (or part of it) through the expectation value of the other.

The present work obtains control over these infrared problems. Depending on the couplings of the model we find that the phase transition is either first or second order. For a sufficiently strongly first order transition high temperature perturbation theory may give realistic results for the two-scalar model. We concentrate here on the more problematic regions of a second order transition, a weakly first order transition and the tricritical behaviour at the separation of the two regimes. For the corresponding values of the couplings high temperature perturbation theory fails near the critical temperature. As a byproduct, our results can be used in order to establish in which region of parameter space perturbation theory gives a reasonable approximation for the description of the phase transition.

Our results are directly relevant for two specific classes of scenarios in the cosmological context: The first class concerns multi-Higgs-scalar extensions of the standard model at non-zero temperature. The prediction of perturbation theory for a first order electroweak phase transition, combined with the existence of baryon number violating processes at non-zero temperature within the standard model, has generated much interest in the possibility of creating the baryon asymmetry of the universe during the electroweak phase transition. It is not clear, however, if the phase transition in the pure standard model is sufficiently strongly first order and if there is sufficient $CP$ violation in order to create an asymmetry of reasonable size. This has led to the study of multi-Higgs-scalar extensions of the standard model, in which the additional scalar fields can be used to make the phase transition more strongly first order or to enhance the sources of $CP$ violation in the model. Also supersymmetric extensions of the standard model contain two scalar doublets. It is not clear whether the perturbative methods used in ref. for the calculation of the scalar field contributions to the effective potential are reliable for such two-scalar models, if we take into account the “warning” from the study of the $O(N)$-symmetric scalar theory.

For sufficiently small gauge coupling the present investigation and its generalization to the case where each scalar field has $N$ components gives a reasonable approximation to the gauged models even in the vicinity of the critical temperature. However, the determination of the meaning of “sufficiently small” needs a detailed investigation of high temperature gauge theories.

We should point out that, for a non-abelian Higgs model, the situation is much more involved than the perturbative results indicate, due to the presence of a confining regime in the symmetric phase of the model. As this work deals only with scalar fields, such a complication does not arise. For a discussion of gauge theories in the context of the effective average action approach see refs. 15, 16, 17.
symmetric theory. Our work gives a reliable estimate of the effect of these contributions on the nature of the transition.

The second class of scenario concerns multi-scalar models of inflation [21]. In most such studies some classical potential is employed, which may bear no resemblance to the effective potential. Thermal effects are often ignored except for the temperature dependence of the mass term. If inflation is initiated by a high temperature phase transition our formalism sets the framework for the proper study of the problem.

We consider a theory of two real scalar fields $\chi_a (a = 1, 2)$, invariant under the discrete symmetries ($\chi_1 \leftrightarrow -\chi_1$, $\chi_2 \leftrightarrow -\chi_2$, $\chi_1 \leftrightarrow \chi_2$), which we denote by (1 $\leftrightarrow$ $-1$, 2 $\leftrightarrow$ $-2$, 1 $\leftrightarrow$ 2) for brevity. The symmetry group is $Z_4 \times Z_2$, consisting of 90$^\circ$ rotations in the $(\chi_1, \chi_2)$ plane and a reflection on one of the axes. The classical potential can be written as

$$V(\chi_1, \chi_2) = \frac{1}{2} \bar{m}^2 (\chi_1^2 + \chi_2^2) + \frac{1}{8} \bar{\lambda} (\chi_1^4 + \chi_2^4) + \frac{1}{4} \bar{g} \chi_1^2 \chi_2^2$$

$$= \frac{1}{2} \bar{m}^2 (\chi_1^2 + \chi_2^2) + \frac{1}{8} \bar{\lambda} (\chi_1^2 + \chi_2^2)^2 + \frac{1}{4} x \bar{\lambda} \chi_1^2 \chi_2^2, \quad (1.1)$$

with

$$x = \frac{\bar{g}}{\bar{\lambda}} - 1. \quad (1.2)$$

For $V$ to be bounded from below we require $\bar{\lambda} > 0$, $x > -2$.

For $\bar{m}^2 > 0$ the classical theory is in the symmetric regime (which we denote by S) with the minimum of the classical potential at the origin. For $\bar{m}^2 < 0$ the theory is in the spontaneously broken regime and we distinguish two possibilities consistent with the symmetry:

I) For $x < 0$ four degenerate minima of the potential are located between the two axes at

$$\chi_{10} = \pm \chi_{20} = \pm \sqrt{-\frac{2\bar{m}^2}{\bar{\lambda} + \bar{g}}}. \quad (1.3)$$

We denote this regime by M.

II) For $x > 0$ the four minima of the potential are located on the axes at

$$\chi_{10} = \pm \sqrt{-\frac{2\bar{m}^2}{\bar{\lambda}}}, \quad \chi_{20} = 0, \quad (1.4)$$

or similarly with $\chi_{10}$ and $\chi_{20}$ interchanged. We denote this regime by AX.

The regimes M and AX are closely related. A redefinition of the fields according to

$$\tilde{\chi}_1 = \frac{1}{\sqrt{2}} (\chi_1 + \chi_2)$$

$$\tilde{\chi}_2 = \frac{1}{\sqrt{2}} (\chi_1 - \chi_2) \quad (1.5)$$

results in a rotation of the axes by 45$^\circ$, thus transforming the AX into the M regime. The couplings of the redefined theory are related to the old ones according to

$$\tilde{\lambda} = \bar{\lambda} \left(1 + \frac{x}{2}\right)$$
\[ \tilde{x} = -\frac{x}{1 + \frac{x}{2}}. \] 

There are three characteristic values of \( x \):

a) For \( x = 0 \) the symmetry of the theory is increased to \( O(2) \).

b) For \( x = -1 \) the theory decomposes into two disconnected \( Z_2 \)-symmetric models for \( \chi_1 \) and \( \chi_2 \) separately.

c) Similarly, for \( x = 2 \) (and therefore \( \tilde{x} = -1 \), according to eqs. (1.6)) the theory decomposes into two disconnected \( Z_2 \)-symmetric models for \( \tilde{\chi}_1 \) and \( \tilde{\chi}_2 \).

The above symmetries are expected to be preserved after the quantum or thermal corrections have been taken into account. This means that any renormalization group flow of the couplings that starts on the surfaces \( x = 0, -1, 2 \) in parameter space cannot take the system out of them. As a result the parameter space of the theory is divided into the four regions: \( x > 2, 2 > x > 0, 0 > x > -1, x < -1 \), which are not connected by the renormalization group flow of the couplings. The phase transitions for the theories which correspond to \( x = 0, -1, 2 \) have been discussed in detail in refs. [6, 10]. They are second order transitions governed by effectively three-dimensional fixed points. In our model these fixed points exist on surfaces separating the parameter space into disconnected regions. We shall demonstrate all the above points in the following sections.

We should point out that this model was discussed in ref. [22] through use of finite temperature perturbation theory. No part of the rich structure of critical behaviour that we shall describe in the following sections was observed. The universal, effectively three-dimensional behaviour of the system near the critical temperature is common for statistical systems and three-dimensional field theories which belong to the same universality class. (The statistical systems are characterized as two-component spin systems with cubic anisotropy.) As a consequence, various aspects of the problem have been investigated in refs. [23] - [26] (and references therein) through other methods. Our results are in very good agreement with all these studies. Similar models have been considered in ref. [27].

The outline of our procedure is the following: We make use of the effective average action \( \Gamma_k \), which results from the effective integration of quantum and thermal fluctuations with characteristic momenta \( q^2 > k^2 \). It contains all the information on the generalized couplings of the theory and their dependence on the scale \( k \). For \( k \) of the order of some ultraviolet cutoff \( \Lambda \) the effective average action is equal to the classical (bare) action (no integration of fluctuations takes place). For \( k = 0 \), \( \Gamma_k \) is equal to the effective action (all fluctuations are integrated). The dependence of \( \Gamma_k \) on the scale \( k \) is given by an exact non-perturbative renormalization group equation, which can be expressed as evolution equations for the running couplings of the theory. These equations can be solved within some appropriate approximation scheme, with the classical couplings as initial conditions for \( k = \Lambda \). The renormalized couplings of the theory are obtained for \( k = 0 \). The calculation can be performed for zero and non-zero temperature. The gradual incorporation of the effects of quantum and thermal fluctuations into the running couplings is the essential element which resolves the problem of infrared divergences that invalidates perturbative schemes. The basic formalism of the effective average action is summarized in section 2. We expect that the running of the couplings for \( k \) much smaller
than the temperature has an effectively three-dimensional character. The reason is that the effective dimensionality is reduced, when the characteristic length scale $1/k$ of the “coarse grained” system is much larger than the periodicity $1/T$ in the imaginary time direction set by the temperature. This is expected to be important near the critical temperature, where no infrared cutoffs (such as masses) other than $k$ exist. As a result, the fixed point structure of the three-dimensional theory determines the behaviour of the critical system. For this reason we present in section 3 a qualitative study of the three-dimensional theory and its fixed points, based on a crude approximation scheme for the solution of the exact renormalization group equation. In section 4, we develop more elaborate (and, therefore, more accurate) approximation schemes. They are generalized for non-zero temperature in section 5. These tools are put into work in sections 6 and 7: The evolution of the running couplings is calculated, starting with the classical theory at scales $k = \Lambda \gg T$ and finishing at $k = 0$, where the renormalized theory is obtained. In section 7 we explicitly demonstrate how the evolution of the running couplings becomes effectively three-dimensional for $k \ll T$. In section 8 we calculate the critical temperature for the phase transition. In sections 9-11 we discuss the details of this transition. We observe a rich spectrum of critical behaviour with critical exponents, crossover phenomena, tricritical points etc. None of these are visible within perturbation theory. Our conclusions are given in section 12.

2. The evolution equation for the effective average potential

We consider a theory of two real scalar fields $\chi_a (a = 1, 2)$, in $d$-dimensional Euclidean space, with an action $S[\chi]$ invariant under the $(1 \leftrightarrow -1, 2 \leftrightarrow -2, 1 \leftrightarrow 2)$ symmetry. We specify the action together with some ultraviolet cutoff $\Lambda$, so that the theory is properly regulated. We add to the kinetic term an infrared regulating piece

$$\Delta S = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} R_k(q) \chi_a^*(q) \chi^a(q), \quad (2.1)$$

where $\chi_a(q)$ are the Fourier modes of the scalar fields. The function $R_k$ is employed in order to prevent the propagation of modes with characteristic momenta $q^2 < k^2$. This can be achieved, for example, by the choice

$$R_k(q) = \frac{q^2 f_k^2(q)}{1 - f_k^2(q)}, \quad (2.2)$$

with

$$f_k^2(q) = \exp \left( -\frac{q^2}{k^2} \right). \quad (2.3)$$

We point out that there are many alternative choices for $R_k(q)$, some of which were used in refs. [7] - [12]. The physical results which are obtained when the cutoff is removed
are scheme independent. The choice of eqs. (2.2), (2.3) is the most natural one and convenient for numerical calculations. For a massless field the inverse propagator derived from the action \( S + \Delta S \) has a minimum \( \sim k^2 \). The modes with \( q^2 \gg k^2 \) are unaffected by the infrared cutoff, while the low frequency modes with \( q^2 \ll k^2 \) are cut off, as \( R_k \) acts like a mass term
\[
\lim_{q^2 \to 0} R_k(q) = k^2.
\] (2.4)

We subsequently introduce sources and define the generating functional for the connected Green functions for the action \( S + \Delta S \). Through a Legendre transformation we obtain the generating functional for the 1PI Green functions \( \tilde{\Gamma}_k[\phi^a] \), where \( \phi^a \) is the expectation value of the field \( \chi^a \) in the presence of sources. The use of the modified propagator for the calculation of \( \tilde{\Gamma}_k \) results in the effective integration of only the fluctuations with \( q^2 > k^2 \).

Finally, the effective average action is obtained by removing the infrared cutoff
\[
\Gamma_k[\phi^a] = \tilde{\Gamma}_k[\phi^a] - \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} R_k(q) \phi^*_a(q) \phi^a(q).
\] (2.5)

For \( k \) equal to the ultraviolet cutoff \( \Lambda \), \( \Gamma_k \) becomes equal to the classical action (no effective integration of modes takes place), while for \( k \to 0 \) it tends towards the effective action \( \Gamma \) (all the modes are included) which is the generating functional of the 1PI Green functions computed from \( S \) (without infrared cutoff). For intermediate values of \( k \) the effective average action realizes the concept of a coarse grained effective action in the sense of ref. [28].

The interpolation of \( \Gamma_k \) between the classical and the effective action makes it a very useful field theoretical tool. The means for practical calculations is provided by an exact flow equation \( \frac{\partial}{\partial t} \Gamma_k[\phi^a] = \frac{1}{2} \Tr \left\{ (\Gamma_k^{(2)}[\phi^a] + R_k)^{-1} \frac{\partial}{\partial t} R_k \right\} \) which describes the response of the effective average action to variations of the infrared cutoff \( t = \ln(k/\Lambda) \).

\[
\frac{\partial}{\partial t} \Gamma_k[\phi^a] = \frac{1}{2} \Tr \left\{ (\Gamma_k^{(2)}[\phi^a] + R_k)^{-1} \frac{\partial}{\partial t} R_k \right\}.
\] (2.6)

Here \( \Gamma_k^{(2)} \) is the second functional derivative of the effective average action with respect to \( \phi^a \). For real fields it reads in momentum space
\[
(\Gamma_k^{(2)})^a_{\;\,b}(q, q') = \frac{\delta^2 \Gamma_k}{\delta \phi^a(q) \delta \phi^b(q')}.
\] (2.7)

with
\[
\phi^a(-q) = \phi^*_a(q).
\] (2.8)

The non-perturbative flow equation has the form of an one-loop expression involving the exact inverse propagator \( \Gamma_k^{(2)} \) together with an infrared cutoff provided by \( R_k \). No contributions from higher loops appear in this exact equation.

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4 See ref. [29] for other versions of exact renormalization group equations.
For the solution of eq. (2.6) one has to develop an efficient truncation scheme. The form of the effective average action is constrained by the $(1 \leftrightarrow -1, 2 \leftrightarrow -2, 1 \leftrightarrow 2)$ symmetry. However, there is still an infinite number of invariants to be considered. Throughout this paper we shall work with an approximation which neglects the effects of wave function renormalization. We shall, therefore, keep only a classical kinetic term in the effective average action

$$\Gamma_k = \int d^d x \left\{ U_k(\rho_1, \rho_2) + \frac{1}{2} \partial^\mu \phi_a \partial_\mu \phi^a \right\}, \quad (2.9)$$

and neglect all invariants which involve more derivatives of the fields. We have used the definition $\rho_1 = \frac{1}{2} \phi_1^2$ and similarly for $\rho_2$. The justification for our approximation lies in the smallness of the anomalous dimension, which is expected to be $\eta \simeq 0.03 - 0.04$ for the three-dimensional theory. We estimate the corrections arising from the proper inclusion of wave function renormalization effects to be of the same order as $\eta$ (a few %). An improved treatment will be given elsewhere [34].

In order to obtain an evolution equation for $U_k$ from eq. (2.6), we have to expand around a constant field configuration (so that the derivative terms in the parametrization (2.9) do not contribute to the l.h.s. of eq. (2.6)). Eq. (2.6) then gives [8] - [10]

$$\frac{\partial}{\partial t} U_k(\rho_1, \rho_2) = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \left( \frac{1}{P(q^2) + M_1^2} + \frac{1}{P(q^2) + M_2^2} \right) \frac{\partial}{\partial t} R_k(q). \quad (2.10)$$

$P(q^2)$ results from the combination of the classical kinetic contribution $q^2$ and the regulating term $R_k$ into an effective inverse propagator (for massless fields)

$$P(q^2) = q^2 + R_k = \frac{q^2}{1 - f_k^2(q)}, \quad (2.11)$$

with $f_k^2(q)$ given by eq. (2.3). For $q^2 \gg k^2$ the inverse “average” propagator $P(q)$ approaches the standard inverse propagator $q^2$ exponentially fast, whereas for $q^2 \ll k^2$ the infrared cutoff prevents the propagation. $M_{1,2}^2$ are the eigenvalues of the mass matrix at the point $(\rho_1, \rho_2)$

$$M_{1,2}^2(\rho_1, \rho_2) = \frac{1}{2} \left\{ U_1 + U_2 + 2U_{11}\rho_1 + 2U_{22}\rho_2 \right. \pm \left[ (U_1 - U_2 + 2U_{11}\rho_1 - 2U_{22}\rho_2)^2 + 16U_{12}^2\rho_1\rho_2 \right]^{\frac{1}{2}} \right\}. \quad (2.12)$$

and we have introduced the notation $U_1 = \frac{\partial U_k}{\partial \rho_1}, U_{12} = \frac{\partial^2 U_k}{\partial \rho_1 \partial \rho_2}$ etc.

Eq. (2.10) is the master equation for our investigation. It is a non-linear partial differential equation for three independent variables ($t, \rho_1, \rho_2$). Since it is difficult to solve it exactly we again resort to some approximation scheme. We first introduce a Taylor expansion of $U_k(\rho_1, \rho_2)$ around its minimum. This turns eq. (2.10) into an infinite system of ordinary (coupled) differential equations for the $k$-dependence of the minimum and the derivatives of the effective average potential, with independent variable $t = \ln(k/\Lambda)$. We
solve this system approximately by truncating at a finite number of derivatives. This approach has been used in the past for the study of the $O(N)$-symmetric scalar theory. It has provided a full, detailed picture of the high temperature phase transition for this theory \[3, 10 - 12\], with accurate determination (at the few % level) of such non-trivial quantities as the critical exponents \[14\]. An estimate \[30\] of the residual errors for high level truncations indicates that they are smaller than the uncertainties introduced by the imprecise treatment of the wave function renormalization effects. For this work we shall use the lowest level truncation which keeps only the second derivatives of the potential $U_{11}, U_{22}, U_{12}$. This will be sufficient for a reliable determination of the phase diagram and a crude estimate of universal quantities such as critical exponents and crossover curves. For an improved treatment see ref. \[33\], and for a discussion which takes into account the next level in the truncation for $U_k$ and the first corrections arising from wave function renormalization see ref. \[34\].

3. The phase structure of the three-dimensional theory

Before performing a more detailed analysis we would like to gain some understanding of the phase structure of the theory. As we have already mentioned in the introduction, the behaviour of the four-dimensional theory near a high temperature second order phase transition is expected to have a three-dimensional character. The reason for this is the divergence of the correlation length for the fluctuations of the system (the mass of some field goes to zero). As a result, the characteristic length scale for the critical system is much larger than the periodicity in the imaginary time direction due to temperature (for details see the following sections). \[5\] This leads to dimensional reduction and the critical system has effectively three-dimensional behaviour. For this reason we are interested in the phase structure of the three-dimensional theory. More specifically we want to investigate the possible existence of fixed points which govern the dynamics of second order phase transitions.

For the purpose of this section we parametrize the potential by its derivatives at the origin (S regime)

$$\begin{align*}
m^2(k) &= U_1(0) = U_2(0) \\
\bar{\lambda}(k) &= U_{11}(0) = U_{22}(0) \\
\bar{g}(k) &= U_{12}(0) \\
x(k) &= \bar{g}(k) - 1.
\end{align*}$$

The equality of $U_1, U_2$ and $U_{11}, U_{22}$ is imposed by the $(1 \leftrightarrow -1, 2 \leftrightarrow -2, 1 \leftrightarrow 2)$ symmetry of the theory. For the potential to be bounded we also require $x > -2$. For

\begin{footnote}{5If the phase transition is strongly first order this need not be true, because the mass of the fluctuations does not go to zero and the correlation length does not diverge near the transition.}
a rough estimate the three-dimensional couplings are related to the effective couplings of the four-dimensional theory at high temperature by \( \tilde{\lambda}(2\pi T) = \lambda_4 T \), \( \tilde{g}(2\pi T) = g_4 T \), \( \tilde{m}^2(2\pi T) = m_4^2 + cT^2 \), with appropriate \( c \) (for details see sections 6 and 7). Evolution equations for the above parameters are obtained by taking derivatives of eq. (2.10) with respect to \( \rho_{1,2} \). The truncated flow equations read for \( d = 3 \)

\[
\begin{align*}
\frac{d\tilde{m}^2}{dt} &= v_3 k(4 + x)\tilde{\lambda}L^3_1(\tilde{m}^2) \\
\frac{d\tilde{\lambda}}{dt} &= -v_3 k^{-1}(10 + 2x + x^2)\tilde{\lambda}^2L^3_2(\tilde{m}^2) \\
\frac{dx}{dt} &= v_3 k^{-1}(x + 1)x(x - 2)\tilde{\lambda}L^3_2(\tilde{m}^2),
\end{align*}
\]

with \( v_3 = 1/8\pi^2 \). The threshold functions \( L^3_n(w) \) suppress the contributions of massive modes to the evolution equations. They are studied in detail in the following sections and in appendix A. It is convenient to define the dimensionless couplings

\[
\begin{align*}
m^2(k) &= \frac{\tilde{m}^2(k)}{k^2} \\
\lambda(k) &= \frac{\tilde{\lambda}(k)}{k} \\
g(k) &= \frac{\tilde{g}(k)}{k}.
\end{align*}
\]

In terms of these quantities the evolution equations have a scale invariant form, in the sense that the r.h.s. does not explicitly involve a dependence on \( k \)

\[
\begin{align*}
\frac{dm^2}{dt} &= -2m^2 + v_3(4 + x)\lambda L^3_1(m^2) \\
\frac{d\lambda}{dt} &= -\lambda - v_3(10 + 2x + x^2)\lambda^2L^3_2(m^2) \\
\frac{dx}{dt} &= v_3(x + 1)x(x - 2)\lambda L^3_2(m^2).
\end{align*}
\]

We are interested in the fixed points of the last set of equations. For any \( x \), there is an ultraviolet attractive Gaussian fixed point with \( m^2 = \lambda = 0 \). There are also three fixed points with at least one infrared attractive direction \([23] - [26] \). They all appear for \( m^2 < 0, \lambda > 0 \). (The exact values are not important since the discussion in this section is only qualitatively correct.) For their identification we use their standard names in statistical physics \([23], [25] \).

a) The Heisenberg fixed point has \( x = 0 \) and corresponds to a theory with symmetry increased to \( O(2) \), as we have discussed in the introduction.

b) The Ising fixed point has \( x = -1 \) and corresponds to two disconnected \( Z_2 \)-symmetric theories.
c) The Cubic fixed point has \( x = 2 \) and corresponds to two disconnected theories, if the fields are redefined according to eqs. (I.5).

All these points are infrared unstable in the \( m^2 \) direction and are located on a critical surface \( m^2_{cr} = m^2_{cr}(\lambda, x) < 0 \). Solutions of the evolution equations which start above the critical surface, with \( m^2 > m^2_{cr} \), flow towards the region of positive \( m^2 \) for \( t \to -\infty \), and correspond to theories in the symmetric phase. Solutions with \( m^2 < m^2_{cr} \) flow deep into the region of negative \( m^2 \) and correspond to theories in the phase with spontaneous symmetry breaking.

The relative stability of the fixed points on the critical surface determines which one governs the dynamics of the phase transition very close to the critical temperature. For a first simple investigation of the relative stability in the \((\lambda, x)\) directions we fix \( m^2 \) to an arbitrary value (we choose \( m^2 = 0 \) for convenience) and solve eqs. (3.7), (3.8) numerically. The results are presented in fig. 1. We observe that all three fixed points are attractive in the \( \lambda \) direction. However, the Ising and Cubic fixed points are repulsive in the \( x \) direction, while the Heisenberg fixed point is totally attractive. We observe four disconnected regions:

a) \( 2 > x > 0 \) : The trajectories flow away from the Cubic and towards the Heisenberg fixed point.
b) \( 0 > x > -1 \) : The trajectories flow away from the Ising and towards the Heisenberg fixed point.
c) \( x > 2 \) : The trajectories flow away from the Cubic fixed point and into a region of large \( x \) and small \( \lambda \). Eventually \( \lambda \) turns negative at a finite value of \( k \). (This can be verified through the explicit solution of eqs. (3.7), (3.8) in this region.) At this point an instability arises, as the potential seems not to be bounded from below. Our treatment is not sufficient for a detailed investigation of the nature of this instability, since our truncation scheme is very crude. A detailed discussion is given in ref. [33], where improved truncations are employed. It is shown that the instability is not real since the higher derivatives of the potential remain positive. The change of sign for \( \lambda \) corresponds to the disappearance of a false vacuum of the theory and results in a first order phase transition. We shall return to this point in the following sections.
d) \( x < -1 \) : The trajectories flow away from the Ising fixed point and cross the line \( x = -2 \) at a finite \( k \). This again implies the presence of an instability whose true nature is related to the disappearance of a false vacuum. The model exhibits a first order transition also for \( x < -1 \).

Flows that start on the lines \( x = 0, -1, 2 \) remain on these lines. No trajectories exist which connect the four regions \( x > 2, 2 > x > 0, 0 > x > -1, x < -1 \). All this is in agreement with the discussion at the end of the introduction.

The diagram of fig. 1 determines the phase structure of the theory when the behaviour of the system becomes effectively three-dimensional (i.e. close to the critical temperature). For parameters in the regions \( 2 > x > 0, 0 > x > -1 \) we expect second order phase transitions, with critical dynamics governed by the three fixed points. These two regions can be mapped onto each other through a redefinition of the fields according to eqs. (I.5).
This indicates that the Ising and Cubic fixed point should lead to identical universal behaviour (and therefore to identical universal quantities, such as critical exponents). Very close to the critical temperature we expect the Heisenberg fixed point to dominate the dynamics, but the other two can be relevant if the initial values of the running parameters are sufficiently close to them. In the parameter regions \( x > 2, x < -1 \) we expect first order phase transitions. In the following sections we shall verify the above conclusions with improved quantitative accuracy.

4. Truncations of the evolution equation

We proceed now to a more detailed study of the evolution equation and its truncations. As we have discussed at the end of section 2, we parametrize the effective average potential by its minimum and its derivatives at the minimum. For this work we shall use a truncation that preserves up to second derivatives of the potential.

In the symmetric regime (which we denote by S) the minimum of the potential is at \( \rho_{10}(k) = \rho_{20}(k) = 0 \) and we use the definitions of eqs. (3.1). The evolution equations in arbitrary dimension \( d \) preserve automatically the symmetry. They read

\[
\frac{d\bar{m}^2}{dt} = v_d k^{d-2}(4 + x)\bar{\lambda} L^d_1(\bar{m}^2) \tag{4.1}
\]

\[
\frac{d\bar{\lambda}}{dt} = -v_d k^{d-4}(10 + 2x + x^2)\bar{\lambda}^2 L^d_2(\bar{m}^2) \tag{4.2}
\]

\[
\frac{dx}{dt} = v_d k^{d-4}(x + 1)x(x - 2)\bar{\lambda} L^d_2(\bar{m}^2), \tag{4.3}
\]

with the dimensionless integrals \( L^d_n(w) \) given by

\[
L^d_n(w) = -n k^{2n-d} \pi^{-\frac{d}{2}} \Gamma \left( \frac{d}{2} \right) \int dq q^n \frac{\partial P}{\partial t} (P + w)^{-(n+1)}
\]

\[
= -n k^{2n-d} \int_0^{\infty} dx x^{4-1} \frac{\partial P}{\partial t} (P + w)^{-(n+1)}. \tag{4.4}
\]

Here \( P \) is given by eq. (2.11), and

\[
v^{-1}_d = 2^{d+1} \pi^{\frac{d}{2}} \Gamma \left( \frac{d}{2} \right). \tag{4.5}
\]

In the spontaneously broken regime there are two possibilities consistent with the symmetry:

1) In the M regime the minimum of the potential is located symmetrically between the \( \rho \)-axes at \( \rho_{10}(k) = \rho_{20}(k) = \frac{1}{2} \rho_0(k) \). We define the couplings

\[
\bar{\lambda}(k) = U_{11}(\rho_0) = U_{22}(\rho_0) > 0
\]

\[
\bar{g}(k) = U_{12}(\rho_0)
\]

\[
x(k) = \frac{\bar{g}(k)}{\bar{\lambda}(k)} - 1. \tag{4.6}
\]
The requirement that the point \((\frac{1}{2}\rho_0, \frac{1}{2}\rho_0)\) is the minimum of the potential imposes \(x < 0\), while the potential is bounded at infinity for \(x > -2\). For \(x = 0\) the symmetry of the theory is increased to \(O(2)\) and the potential develops a series of degenerate minima along the circle \(\rho_{10} + \rho_{20} = \rho_0\). For \(x = -1\) the theory decomposes into two disconnected \(Z_2(\phi_{1,2} \leftrightarrow -\phi_{1,2})\)-symmetric models. The mass eigenvalues are given by \(M_1^2 = (2 + x)\lambda\rho_0\), \(M_2^2 = -x\lambda\rho_0\). The evolution equation for the minimum \(\rho_0(k)\) is obtained by considering the total \(t\)-derivative of the conditions \(\frac{\partial {\mathcal L}}{\partial \rho_0} \big|_{\rho_0 = \rho_0} = \frac{\partial {\mathcal L}}{\partial \rho_2} \big|_{\rho_0 = \rho_0} = 0\). Again, the truncated evolution equations automatically preserve the symmetry and read

\[
\frac{d\rho_0}{dt} = -v_d k^{d-2} \left\{ 3L_1^d((2 + x)\lambda\rho_0) + \frac{2 - x}{2 + x} L_1^d(-x\lambda\rho_0) \right\} \tag{4.7}
\]

\[
\frac{d\lambda}{dt} = -v_d k^{d-4} \left\{ 3x\lambda 1 + \frac{3}{4} L_1^d((2 + x)\lambda\rho_0) - L_1^d(-x\lambda\rho_0) \right\}
\]

\[
- v_d k^{d-4} \lambda^2 \left\{ \left( 1 + \frac{x}{2} \right)^2 L_2^d((2 + x)\lambda\rho_0) + \left( 1 - \frac{x}{2} \right)^2 L_2^d(-x\lambda\rho_0) \right\} \tag{4.8}
\]

\[
\frac{dx}{dt} = v_d k^{d-2} \left\{ \frac{2 + x}{\rho_0} 1 + \frac{x}{4} \left\{ L_1^d((2 + x)\lambda\rho_0) - L_1^d(-x\lambda\rho_0) \right\} 
\]

\[
+ v_d k^{d-4} x\lambda \left\{ \left( 1 + \frac{x}{2} \right)^2 L_2^d((2 + x)\lambda\rho_0) + \left( 1 - \frac{x}{2} \right)^2 L_2^d(-x\lambda\rho_0) \right\}. \tag{4.9}
\]

For \(x = 0\) the above evolution equations reproduce the equations of the \(O(2)\)-symmetric theory, while for \(x = -1\) those of the \(Z_2\)-symmetric one (compare with refs. \[8, 10\]).

II) In the regime which we denote by AX, two degenerate minima of the potential exist on each one of the \(\rho\)-axes. Without loss of generality we concentrate on the minimum at \(\rho_{10}(k) = \rho_0(k), \rho_{20}(k) = 0\). At the level of truncations that we are considering, the remaining parameters of the theory are conveniently defined as

\[
\tilde{\lambda}(k) = U_{11}(\rho_0)
\]

\[
\tilde{g}(k) = U_{12}(\rho_0)
\]

\[
x(k) = \frac{\tilde{g}(k)}{\tilde{\lambda}(k)} - 1
\]

\[
\tilde{m}^2_2(k) = U_2(\rho_0). \tag{4.10}
\]

The symmetry demands that for the truncated potential

\[
\tilde{m}^2_2(k) = x(k)\tilde{\lambda}(k)\rho_0(k). \tag{4.11}
\]

The requirement that the point \((\rho_0, 0)\) is the minimum of the potential imposes \(x > 0\). As before, for \(x = 0\) the symmetry of the theory is increased to \(O(2)\). The mass eigenvalues are given by \(M_1^2 = 2\tilde{\lambda}\rho_0\), \(M_2^2 = x\tilde{\lambda}\rho_0\). At this point we encounter a difficulty. The derivation of truncated evolution equations is hindered by the fact that the parametrization around a minimum located on one of the axes is asymmetric between the two fields. As
a result the symmetry ($\phi_1 \leftrightarrow \phi_2$), is not maintained by the evolution equations at each level of the truncations. More specifically, the flow equations for the couplings $U_{11}(\rho_0)$, $U_{22}(\rho_0)$ are different. Also eq. (4.11) is not preserved by the evolution equation. This is not surprising, since these relations are not expected to hold for the exact potential without truncation. It is easy to see that they are altered as soon as third derivatives of the potential are included. This is in constrast with what happens in the M regime, where the formulation is symmetric between the two fields. For example, in the M regime the couplings $U_{11}, U_{22}$ are expected to remain equal at every level of truncations, and indeed this is guaranteed by the evolution equations. The above remarks indicate a natural method of preserving the (1 $\leftrightarrow$ 2, 2 $\leftrightarrow$ 1, 1 $\leftrightarrow$ 2) symmetry in the AX regime. A redefinition of the fields according to
\begin{align*}
\tilde{\phi}_1 &= \frac{1}{\sqrt{2}}(\phi_1 + \phi_2) \\
\tilde{\phi}_2 &= \frac{1}{\sqrt{2}}(\phi_1 - \phi_2)
\end{align*} (4.12)
results in a rotation of the axes by 45°, thus transforming the AX into the M regime. The couplings of the redefined theory are related to the old ones according to
\begin{align*}
\tilde{\rho}_0 &= \rho_0 \\
\tilde{\lambda} &= \tilde{\lambda} \left(1 + \frac{x}{2}\right) \\
\tilde{x} &= -\frac{x}{1 + \frac{x}{2}}. (4.13)
\end{align*}
The evolution equations for the new quantities have already been worked out and are given by eqs. (4.7)-(4.9). By simply rewriting them in terms of the old quantities defined in eqs. (4.11) we obtain
\begin{align*}
\frac{d\rho_0}{dt} &= -v_d k^{d-2} \left\{3L_1^d(2\tilde{\lambda}\rho_0) + (1 + x)L_1^d(x\tilde{\lambda}\rho_0)\right\} \quad (4.14) \\
\frac{d\tilde{\lambda}}{dt} &= -v_d k^{d-4}\tilde{\lambda}^2 \left\{9L_2^d(2\tilde{\lambda}\rho_0) + (1 + x)^2L_2^d(x\tilde{\lambda}\rho_0)\right\} \quad (4.15) \\
\frac{dx}{dt} &= v_d k^{d-2} \frac{6}{\rho_0} \left(\frac{x + \frac{x^2}{2}}{1 + \frac{x}{2}} \left\{L_1^d(2\tilde{\lambda}\rho_0) - L_1^d(x\tilde{\lambda}\rho_0)\right\} \right. \\
&\quad \left. + v_d k^{d-4}x\tilde{\lambda} \left\{9L_2^d(2\tilde{\lambda}\rho_0) + (1 + x)^2L_2^d(x\tilde{\lambda}\rho_0)\right\}\right. \quad (4.16).
\end{align*}
For $x = 0$ the above evolution equations reproduce the ones of the $O(2)$-symmetric theory. Another special point is $x = 2$ for which the theory, when expressed in terms of the redefined fields $\tilde{\phi}_1, \tilde{\phi}_2$, decomposes into two disconnected $Z_2(\tilde{\phi}_{1,2} \leftrightarrow -\tilde{\phi}_{1,2})$-symmetric models. We should point out that eqs. (4.14)-(4.16) could have been obtained by defining $\tilde{\lambda} = U_{11}(\rho_0)$ and $x = U_2(\rho_0)/U_{11}(\rho_0)\rho_0$ (in agreement with eq. (4.11)) and inserting eqs.
(4.10) on the r.h.s. of the flow equations. The advantage of the redefinition (4.12) is that it makes transparent how this apparently arbitrary choice of parameters preserves the original symmetry at this truncation level.

In the following sections we shall use the evolution equations (4.1)-(4.3), (4.7)-(4.9), (4.14)-(4.16), for the S, M, AX regimes respectively, in order to obtain the renormalized theory in its various phases.

5. The integrals \( L_n^d \) for zero and non-zero temperature

The integrals \( L_n^d(w) \), defined in eq. (4.4), have been discussed extensively in refs. [8, 10, 31] (for various shapes of the infrared regulating function \( R_k(q) \), for which eqs. (2.2), (2.3) are the most natural choice [9]). For completeness, we give in appendix A a summary of the properties of \( L_n^d(w) \). Also, as an example, we plot in fig. 2 the integrals \( L_3^1(w), L_3^2(w) \). Their most interesting property, for our discussion, is that they fall off for large values of \( w/k^2 \), following a power law. As a result they introduce threshold behaviour for the contributions of massive modes to the evolution equations. It is obvious from eqs. (4.1)-(4.3), (4.7)-(4.9), (4.14)-(4.16), for the S, M, AX regimes respectively, that the various contributions to the evolution equations involve \( L_n^d \) integrals with the mass eigenvalues as their arguments. When the running squared mass of a massive mode becomes much larger than the scale \( k^2 \) (at which the system is probed), these contributions vanish and the massive modes decouple. We evaluate the integrals \( L_n^d(w) \) numerically and use numerical fits for the solution of the evolution equations.

In order to extend the formalism of the previous section to non-zero temperature we only need to recall that, in Euclidean formalism, non-zero temperature \( T \) results in periodic boundary conditions in the time direction (for bosonic fields), with periodicity \( 1/T \). This leads to a discrete spectrum for the zero component of the momentum \( q_0 \)

\[
q_0 \rightarrow 2\pi m T, \quad m = 0, \pm 1, \pm 2, \ldots
\]  

As a consequence the integration over \( q_0 \) is replaced by a summation over the discrete spectrum

\[
\int \frac{d^d q}{(2\pi)^d} \rightarrow T \sum_m \int \frac{d^{d-1} \tilde{q}}{(2\pi)^{d-1}}.
\]

With the above remarks in mind we can easily generalize our master equation (2.10) in order to take into account the temperature effects. For the temperature dependent effective average potential \( U_k(\rho_1, \rho_2, T) \) we obtain

\[
\frac{\partial}{\partial t} U_k(\rho_1, \rho_2, T) = \frac{1}{2} (2\pi)^{-(d-1)} T \sum_m \int d^{d-1} \tilde{q} \left( \frac{1}{P + M_1^2} + \frac{1}{P + M_2^2} \right) \frac{\partial}{\partial t} R_k,
\]

with the implicit replacement

\[
q^2 \rightarrow \tilde{q}^2 + 4\pi^2 m^2 T^2.
\]
in eqs. (2.2), (2.3) and (2.11) for $R_k$ and $P$. Again, the usual temperature dependent effective potential [3]-[4] is obtained from $U_k(\rho_1, \rho_2, T)$ in the limit $k \to 0$. As before, we can parametrize $U_k(\rho_1, \rho_2, T)$ in terms of its minimum and its derivatives at the minimum. The evolution equations are given by (4.1)-(4.3), (4.7)-(4.9), (4.14)-(4.16), with the obvious generalizations

$$\rho_0(k) \to \rho_0(k, T)$$
$$\bar{\lambda}(k) \to \bar{\lambda}(k, T)$$
$$x(k) \to x(k, T)$$
$$\bar{m}^2(k) \to \bar{m}^2(k, T).$$

(5.5)

The $L_n^d$ integrals for non-vanishing temperature read

$$L_n^d(w, T) = -2nk^{2n-d} \pi^{-\frac{d}{2}+1} \Gamma \left( \frac{d}{2} \right) T \sum_m \int d^{d-1}q \frac{\partial P}{\partial t} (P + w)^{-(n+1)},$$

(5.6)

where the implicit replacement (5.4) is assumed in $P$. Their basic properties can be established analytically. For $T \ll k$ the summation over discrete values of $m$ in the expression (5.6) is equal to the integration over a continuous range of $q_0$ up to exponentially small corrections. Therefore

$$L_n^d(w, T) = L_n^d(w) \quad \text{for} \quad T \ll k.$$  

(5.7)

In the opposite limit $T \gg k$ the summation over $m$ is dominated by the $m = 0$ contribution. Terms with non-zero values of $m$ are suppressed by $\sim \exp \left(- (mT/k)^2 \right)$. The leading contribution gives the the simple expression

$$L_n^d(w, T) = \frac{v_d}{v_d} \frac{T}{k} L_n^{d-1}(w) \quad \text{for} \quad T \gg k,$$

(5.8)

with $v_d$ defined in (4.3). The two regions of $T/k$ in which $L_n^d(w, T)$ is given by the equations (5.7), (5.8) are connected by a small interval, in which the exponential corrections result in a more complicated dependence on $w$ and $T$.

The above conclusions are verified by a numerical calculation of $L_1(w, T)$. In fig. 3 we plot $L_1(w, T)/L_1(0)$ as a function of $T/k$, for various values of $w/k^2$. We distinguish three regions:

a) $T/k \leq \theta_1$: This is the **low temperature region** where $L_{1,2}(w, T)$ are very well approximated by their zero temperature value. We take $\theta_1 = 0.15$ and use $L_{1,2}^4(w, 0)$ in the evolution equations for $k \geq T/\theta_1$.

b) $\theta_1 < T/k < \theta_2$: In the **threshold region** we perform a numerical fit of the curve corresponding to $w = 0$ which we use for all values of $w$. This is a good approximation since

\[ \text{Comparison with results presented in ref. [6] shows that the form of these functions depends on the details of the infrared regulating function $R_k(q)$. However, the physical results, which are obtained when the cutoff is removed, are independent of the shape of the cutoff. This will be apparent in the next sections and is a verification of the scheme independence of our conclusions.} \]
the relevant $w/k^2$ turns out to be small in this region (see next sections).

c) $T/k \geq \theta_2$: We take $\theta_2 = 0.4$. For the high temperature region we use for the numerical solution of the evolution equations

$$L^4_{1,2}(w, T) = 4 \frac{T}{k} L^3_{1,2}(w).$$

(5.9)

The three dimensional character of the effective theory for modes with $q^2 \ll T^2$ manifests itself in the appearance of the three dimensional momentum integrals. It acquires here a precise quantitative meaning.

We have now developed the necessary formalism for the study of the four-dimensional zero and non-zero temperature theory. In the following two sections we study the evolution of the running parameters of the theory, which leads to the determination of the renormalized theory at zero and non-zero temperature.

6. The running in the low temperature and threshold regions

In section 4 we derived the zero temperature evolution equations for the parameters (masses, vacuum expectation values and couplings) of the truncated theory as a function of the scale $k$ in the various regimes (S, M, AX). In section 5 we generalized the formalism in order to take into account non-zero temperature effects. The evolution equations can be solved for a given set of initial conditions, specified as the values of the running parameters at a scale equal to the ultraviolet cutoff of the theory ($k = \Lambda$). As we pointed out in section 2, at this scale the effective average action is equal to the classical action. Therefore, the initial values for the parameters correspond to their classical (or bare) values. Also, the discussion in section 5 has shown that in the low temperature region ($k \geq T/\theta_1$) there is no difference between the zero and non-zero temperature theory. As a result, we can define the theory in terms of the classical values of its parameters at $k = \Lambda \gg T$, independently of the temperature. The integration of the evolution equations gives the running couplings at lower scales. No temperature effects are observed in the evolution inside the low temperature region ($k \geq T/\theta_1$). We can, therefore, use the values of the running couplings at $k = T/\theta_1$ for the definition of the theory, since they are in one-to-one correspondence with the classical couplings, independently of the temperature. This turns out to be the most convenient choice and we shall use it for the rest of the paper. The temperature starts to become important when the evolution enters the threshold region ($T/\theta_1 > k > T/\theta_2$). In the high temperature region ($k \leq T/\theta_2$) the evolution is effectively three-dimensional, as we discussed in section 5. Finally in the limit $k \to 0$ the effective average action becomes the effective action, and the integration of the evolution equations gives the renormalized values for the couplings at various temperatures. All the information on the various phases of the theory is contained in these renormalized couplings and their temperature dependence.
We have seen in the introduction and in section 4 that the AX regime \((x > 0)\) and the M regime \((x < 0)\) can be mapped onto each other through a simple redefinition of the fields (see eqs. (4.12), (4.13)). For this reason, the physical behaviour in the two regimes is the same. For example, the Cubic and Ising fixed points generate the same universal behaviour (characteristic of a \(Z_2\)-symmetric scalar theory). For this reason, we shall concentrate on the region \(x > 0\) only. All the results can be easily extended to the region \(x < 0\), through the transformations of eqs. (4.12), (4.13).

Since we are interested in symmetry restoration at non-zero temperature, we first consider the theory in the spontaneously broken regime. The evolution equations in the AX regime (which is the relevant one for \(x > 0\)) in four dimensions and non-zero temperature can be easily derived from eqs. (4.14)-(4.16) and read

\[
\begin{align*}
\frac{d\rho}{dt} &= -v_4 k^2 \left\{ 3L_1^4(2\lambda\rho_0)t_1(2\lambda\rho_0, T) + (1 + x)L_1^4(x\lambda\rho_0)t_1(x\lambda\rho_0, T) \right\} \quad (6.1) \\
\frac{d\lambda}{dt} &= -v_4 \lambda^2 \left\{ 9L_2^4(2\lambda\rho_0)t_2(2\lambda\rho_0, T) + (1 + x)^2L_2^4(2\lambda\rho_0)t_2(2\lambda\rho_0, T) \right\} \quad (6.2) \\
\frac{dx}{dt} &= v_4 \frac{6 x + x^2}{\rho_0} \left\{ L_1^4(2\lambda\rho_0)t_1(2\lambda\rho_0, T) - L_1^4(x\lambda\rho_0)t_1(x\lambda\rho_0, T) \right\} \\
&\quad + v_4 x \lambda \left\{ 9L_2^4(2\lambda\rho_0)t_2(2\lambda\rho_0, T) + (1 + x)^2L_2^4(2\lambda\rho_0)t_2(2\lambda\rho_0, T) \right\}. \quad (6.3)
\end{align*}
\]

where \(v_4 = 1/32\pi^2\). We have not indicated explicitly the \(k\) and \(T\) dependence of the running parameters \(\rho_0(k, T), \lambda(k, T), x(k, T)\). They are defined at zero temperature according to eqs. (4.10), and generalized for non-zero temperature according to eqs. (5.5). The functions \(t_{1,2}\) are defined as

\[
t_n(w, T) = \frac{L_n^4(w, T)}{L_n^4(w)}, \quad (6.4)
\]

with \(t_1(w, T)\) plotted in fig. 3.

At zero temperature one has \(t_n(w, 0) = 1\) and the evolution equations have only one infrared attractive fixed point, the Gaussian one. In the limit of small \(\lambda\) we shall neglect the slow logarithmic running of \(\lambda\), which is eventually stopped by the mass terms in the threshold functions \(L_{1,2}^4\). Similarly the running of \(x\) can also be neglected since it is suppressed by \(\lambda/32\pi^2\). (For small \(\lambda\) the difference of the two \(L_1^4\) functions in the first line of eq. (6.3) gives a contribution \(\propto \lambda L_2^4(0)\).) For large \(\lambda\) the evolution equations can be integrated numerically and the small resulting corrections can be reliably computed. This has been done in ref. [3] for the \(O(N)\)-symmetric scalar theory. In this paper we concentrate on small couplings for which analytical expressions can be obtained. Eq. (6.1) can be integrated easily for small \(\lambda\) and we obtain \((L_1^4(0) = -2)\)

\[
\rho_0(k, 0) = \rho_0 \left( \frac{T}{\theta_1} \right) + \frac{1}{32\pi^2}(x + 4) \left[ k^2 - \left( \frac{T}{\theta_1} \right)^2 \right], \quad (6.5)
\]
where we used the point \( k = T/\theta_1 \) instead of \( k = \Lambda \) to start the evolution, as we have explained in the first paragraph of this section. We define the renormalized couplings of the theory in the limit \( k \to 0 \) as

\[
\begin{align*}
\rho_0 &= \rho_0(0, 0) \\
\lambda_R &= \tilde{\lambda}(0, 0) \\
x_R &= x(0, 0).
\end{align*}
\tag{6.6}
\]

and conclude that

\[
\rho_0 \left( \frac{T}{\theta_1} \right) = \rho_0 \left( \frac{T}{\theta_1} \right) + \frac{1}{16\pi^2} (x_R + 4) \left( \frac{T}{\theta_1} \right)^2.
\tag{6.7}
\]

At non-zero temperature, the evolution in the low temperature region \( (k \geq T/\theta_1) \) is identical to the zero temperature case. In the threshold region \( (T/\theta_1 < k < T/\theta_2) \), the form of \( t_{1,2}(w, T) \) is not given by a simple analytical expression. For small \( \tilde{\lambda} \) we neglect the running of \( \tilde{\lambda}, x \) in this region and find

\[
\begin{align*}
\rho_0 \left( \frac{T}{\theta_2}, T \right) &= \rho_0 \left( \frac{T}{\theta_1} \right) - \frac{1}{16\pi^2} (x_R + 4) T^2 I \\
&= \rho_0 + \frac{1}{16\pi^2} (x_R + 4) T^2 \left( \frac{1}{2\theta_1^2} - I \right) \\
\tilde{\lambda} \left( \frac{T}{\theta_2}, T \right) &= \tilde{\lambda} \left( \frac{T}{\theta_1} \right) = \lambda_R \\
x \left( \frac{T}{\theta_2}, T \right) &= x \left( \frac{T}{\theta_1} \right) = x_R,
\end{align*}
\tag{6.8}
\]

where

\[
I = \int_{1/\theta_2}^{1/\theta_1} dy \int_{0}^{1} t_1 \left( 0, \frac{1}{y} \right)
\tag{6.9}
\]

and we have made use of the fact that \( t_1 \left( w, T \right) \) depends on \( T \) only through the combination \( T/k \). The integral \( I \) can be evaluated numerically. For \( \theta_1 = 0.15 \), \( \theta_2 = 0.4 \) we find \( I = 19.97 \). Eqs. \ref{6.8} set the initial values for the evolution in the high temperature region.

### 7. The running in the high temperature region

In the high temperature region \( (k \leq T/\theta_2) \) the functions \( L_{1,2}^A(w, T) \) are given by the simplified expression \ref{5.3}. We can rewrite eqs. \ref{6.1}-\ref{6.3} in terms of effective three-dimensional couplings

\[
\rho'_0(k, T) = \frac{\rho_0(k, T)}{T}
\]

\(^7\)In the case that Goldstone modes are present (as for \( x = 0 \)) the couplings are defined at some appropriate non-zero \( k \). The same applies for non-zero temperature. This does not affect our results for small \( \tilde{\lambda} \). For a detailed discussion see ref. \cite{10}.
\bar{\lambda}'(k, T) = \bar{\lambda}(k, T) T
\bar{g}'(k, T) = \bar{g}(k, T) T
x(k, T) = \frac{\bar{g}'(k, T)}{\bar{\lambda}'(k, T)} - 1. \quad (7.1)

The flow equations then read
\[
\frac{d\rho'_0}{dt} = -v_3 k \left\{ 3L_1^3(2\bar{\lambda}'\rho'_0) + (1 + x) L_1^3(x\bar{\lambda}'\rho'_0) \right\} \quad (7.2)
\]
\[
\frac{d\bar{\lambda}}{dt} = -v_3 k^{-1} [\bar{\lambda}']^2 \left\{ 9L_2^3(2\bar{\lambda}'\rho'_0) + (1 + x)^2 L_2^3(x\bar{\lambda}'\rho'_0) \right\} \quad (7.3)
\]
\[
\frac{dx}{dt} = v_3 k^{-1} \frac{6 x + x^2}{\rho'_0 - \frac{1}{2}} \left\{ L_1^3(2\bar{\lambda}'\rho'_0) - L_1^3(x\bar{\lambda}'\rho'_0) \right\} \\
+ v_3 k^{-1} x \bar{\lambda}' \left\{ 9L_2^3(2\bar{\lambda}'\rho'_0) + (1 + x)^2 L_2^3(x\bar{\lambda}'\rho'_0) \right\}, \quad (7.4)
\]

with \( v_3 = 1/8\pi^2 \). Comparison with eqs. (4.14)-(4.16) shows that the above equations are exactly the ones of the three-dimensional theory at zero temperature. In order to make their fixed point structure more transparent we define the dimensionless parameters

\[
\kappa(k, T) = \frac{\rho'_0(k, T)}{k} = \frac{\rho_0(k, T)}{kT}
\]
\[
\lambda(k, T) = \frac{\bar{\lambda}'(k, T)}{k} = \frac{\bar{\lambda}(k, T) T}{k}
\]
\[
g(k, T) = \frac{\bar{g}'(k, T)}{k} = \frac{\bar{g}(k, T) T}{k}. \quad (7.5)
\]

In terms of these we obtain the scale invariant form of the evolution equations
\[
\frac{d\kappa}{dt} = -\kappa - v_3 \left\{ 3L_1^3(2\lambda\kappa) + (1 + x) L_1^3(x\lambda\kappa) \right\} \quad (7.6)
\]
\[
\frac{d\lambda}{dt} = -\lambda - v_3 \lambda^2 \left\{ 9L_2^3(2\lambda\kappa) + (1 + x)^2 L_2^3(x\lambda\kappa) \right\} \quad (7.7)
\]
\[
\frac{dx}{dt} = v_3 \frac{6 x + x^2}{\kappa \left( 1 - \frac{1}{2} \right)} \left\{ L_1^3(2\lambda\kappa) - L_1^3(x\lambda\kappa) \right\} \\
+ v_3 x \lambda \left\{ 9L_2^3(2\lambda\kappa) + (1 + x)^2 L_2^3(x\lambda\kappa) \right\}. \quad (7.8)
\]

No explicit dependence on the scale \( k \) appears on the r.h.s.

The first of the above equations defines a critical surface \( \kappa_{cr} = \kappa_{cr}(\lambda, x) \). It consists of the points \( \kappa_{cr} \) for which the solution of eqs. (7.6)-(7.8) approaches, for large negative \( t \), a scaling solution with \( \kappa, \lambda \) and \( x \) independent of \( t \). (For a weakly first order transition the scaling holds only approximately.) Every point on the critical surface is unstable in the \( \kappa \) direction. Trajectories which start at \( \kappa > \kappa_{cr} \) continue towards the region of large
\(\kappa\), in such a way that \(\rho_0(k,T) = \kappa(k,T)Tk\) reaches asymptotically a constant value for \(k \to 0\). As a result the renormalized theory settles down in the phase with spontaneous symmetry breaking. If the evolution starts at \(\kappa < \kappa_{cr}\), the flows cross the surface \(\kappa = 0\) at some finite \(k_s\). From this point on the system is in the symmetric regime. In order to continue the evolution, we define appropriate parameters according to eqs. (3.1). The evolution equations read

\[
\frac{d\bar{m}^2}{dt} = v_3k(4 + x)\bar{\lambda}'L_1^3(m^2) \tag{7.9}
\]

\[
\frac{d\bar{\lambda}'}{dt} = -v_3k^{-1}(10 + 2x + x^2)[\bar{\lambda}]^2L_2^3(\bar{m}^2) \tag{7.10}
\]

\[
\frac{dx}{dt} = v_3k^{-1}(x + 1)x(x - 2)\bar{\lambda}'L_2^3(\bar{m}^2), \tag{7.11}
\]

with \(\bar{\lambda}'\) related to \(\bar{\lambda}\) by eqs. (7.1). Comparison with eqs. (4.1) - (4.3) shows that the above equations are the ones for the three-dimensional theory in the symmetric regime. We define the dimensionless couplings

\[
m^2(k,T) = \frac{\bar{m}^2(k,T)}{k^2}, \quad \lambda(k,T) = \frac{\bar{\lambda}'(k,T)}{k} = \frac{\bar{\lambda}(k,T)T}{k}, \quad g(k,T) = \frac{\bar{g}'(k,T)}{k} = \frac{\bar{g}(k,T)T}{k}. \tag{7.12}
\]

In terms of these quantities the evolution equations in the symmetric regime read

\[
\frac{dm^2}{dt} = -2m^2 + v_3(4 + x)\lambda L_1^3(m^2) \tag{7.13}
\]

\[
\frac{d\lambda}{dt} = -\lambda - v_3(10 + 2x + x^2)\lambda^2L_2^3(m^2) \tag{7.14}
\]

\[
\frac{dx}{dt} = v_3(x + 1)x(x - 2)\lambda L_2^3(m^2). \tag{7.15}
\]

We start the evolution in this regime at \(k = k_s\) with \(m^2(k_s,T) = 0\) and \(\lambda(k_s,T), x(k_s,T)\) taking their values at the end of the running in the spontaneously broken regime. For \(k \to 0\) the evolution is stopped by the mass terms in the threshold functions \(L_{1,2}\) and the theory settles down in the symmetric phase. Obviously the critical temperature \(T_{cr}\) is related to \(\kappa_{cr}\) (for given \(\lambda(T_{cr}/\theta_2), x(T_{cr}/\theta_2)\)) by

\[
\kappa\left(\frac{T_{cr}}{\theta_2}\right) = \kappa_{cr}. \tag{7.16}
\]

On the critical surface there are two fixed points with at least one attractive direction for the flow towards the infrared \((k \to 0)\):

a) the Cubic fixed point located at

\[
\kappa_C = 5.674 \times 10^{-2} \quad \lambda_C = 8.747 \quad x_C = 2, \tag{7.17}
\]
b) and the Heisenberg fixed point located at

\[ \kappa_H = 4.486 \times 10^{-2} \quad \lambda_H = 15.265 \quad x_H = 0. \tag{7.18} \]

Both fixed points are attractive in the \( \lambda \) direction, but the Cubic fixed point is unstable in the \( x \) direction while the Heisenberg one is stable. For fixed \( x \) there is also the infrared unstable Gaussian fixed point. It is located at

\[ \kappa_G = -v_3(x + 4)L_1^3(0) = \frac{1}{8\pi^{3/2}}(x + 4) \quad \lambda_G = 0. \tag{7.19} \]

These fixed points are the same as the ones observed in section 3. The only difference lies in the parametrization. In section 3 we used an expansion around the origin of the potential and the evolution equations in the symmetric regime. For this reason the fixed points appeared for negative values of the mass parameter (the curvature at the origin), indicating a minimum away from the origin. In this section we rely on a parametrization around the minimum of the potential, which results in increased quantitative accuracy for the truncation.

The flows on the critical surface are qualitatively similar to those in fig. 1 for \( x > 0 \). There are two disconnected regions:

a) \( 2 > x > 0 \): The trajectories flow away from the Cubic fixed point and towards the Heisenberg fixed point. This region corresponds to a second order phase transition.

b) \( x > 2 \): The trajectories flow away from the Cubic fixed point and into a region of small \( \lambda \) and large \( x \). Similarly to our discussion in section 3, we expect \( \lambda(k, T) \) to become negative at some finite \( k \). This indicates that the minimum of the potential becomes unstable (it turns into a maximum). Our crude truncation is not sufficient for the investigation of this situation, since the higher derivatives of the potential are important. A detailed study is presented in ref. [33] for the four-dimensional theory at zero temperature. The dimensionality of the theory is not crucial for the qualitative behaviour in this region of parameter space. The first term in the r.h.s of eq. (7.7) (which is present only for the effectively three-dimensional theory) is not important for large \( x \). The second term, which drives the dynamics, is the same as for the four-dimensional theory (with the replacement of \( v_3L_1^3 \) by \( v_4L_1^4 \) generating only quantitative corrections.) In ref. [33] higher derivatives of the potential are taken into account. Also a parametrization is used which simultaneously follows the evolution of the potential at its minimum at non-zero \( \rho_0(k, T) \) and at the origin. This permits the study of the global properties of the potential. It is found that during the evolution in the region \( x > 2 \) a second minimum appears at the origin which subsequently becomes the absolute minimum of the potential. This results in a discontinuity in the order parameter and a first order phase transition. At some point in the evolution, \( \lambda \) turns negative and the minimum at non-zero \( \rho_0 \) disappears. From this point on the deeper minimum at zero is the only minimum. During the whole evolution the higher derivatives stay positive guaranteeing that the potential remains bounded. We cannot reproduce the above picture within the crude truncation of a quartic polynomial for the potential. Instead we shall give in section 11 an approximate solution of the evolution equation for the potential in this region, which will demonstrate the existence of the first order transition.
8. The critical temperature

A quantity which can be easily calculated from the discussion in the last two sections is the critical temperature for the phase transitions. From eqs. (6.8) and (7.5) we obtain

\[
\kappa\left(\frac{T}{\theta_2}, T\right) = \theta_2 \rho_0 + \frac{\theta_2}{16\pi^2} \left(\frac{1}{2\theta_1^2} - I\right) (x_R + 4)
\]

\[
\lambda\left(\frac{T}{\theta_2}, T\right) = \theta_2 \lambda_R
\]

\[
x\left(\frac{T}{\theta_2}, T\right) = x_R.
\] (8.1)

For small \(\lambda\) the critical surface, which separates the symmetric phase from the phase with spontaneous symmetry breaking, goes through the Gaussian fixed point given by eq. (7.19). The critical temperature can be computed as the temperature for which \(\kappa\left(\frac{T}{\theta_2}, T\right)\) coincides with the Gaussian fixed point \(\kappa_G\). This gives

\[
\frac{T^2_{cr}}{\rho_0} = \frac{C}{x_R + 4}
\]

\[
C^{-1} = \frac{1}{8\pi^2} \left[\frac{\sqrt{\pi}}{\theta_2} - \frac{1}{2} \left(\frac{1}{2\theta_1^2} - I\right)\right].
\] (8.2)

independently of \(\lambda_R\). Substitution of the values \(\theta_1 = 0.15, \theta_2 = 0.4, I = 19.97\), which we computed in part I of this section, gives

\[
C = 23.89.
\] (8.3)

We should point out that the above value for the critical temperature is not strictly valid for \(x_R > 2\). In this region the first order phase transition occurs for \(\kappa\left(\frac{T}{\theta_2}, T\right)\) slightly above the critical surface. As a result the transition takes place at a temperature slightly lower than the one given by eq. (8.2). In the language of the effective three-dimensional theory, the distance from the phase transition can be parametrized, for small \(\lambda_R\), in terms of \(\delta\kappa_{cr} = \kappa\left(\frac{T}{\theta_2}, T\right) - \kappa_G\). We establish the connection between this quantity and the temperature as

\[
\delta\kappa_{cr} = \kappa\left(\frac{T}{\theta_2}, T\right) - \kappa_G = \theta_2 \rho_0 \left(\frac{1}{T^2} - \frac{1}{T^2_{cr}}\right).
\] (8.4)

The critical temperature can also be calculated in high temperature perturbation theory through the perturbative expansion of the effective potential [3] and its generalization to non-zero temperature [2]-[4]. The calculation is straightforward and we do not present the details here. When the leading term in the high temperature expansion of the one-loop contribution to the effective potential is retained, the critical temperature is found to be

\[
\frac{T^2_{cr}}{\rho_0} = \frac{24}{x_R + 4}.
\] (8.5)
This value is in excellent agreement with our result. The slight discrepancy is due to small deviations of the form of $L^4(w, T)$ that we have used from the exact expression, and could probably be removed by using lower $\theta_1$ and larger $\theta_2$. It is well known that the perturbative expansion of the effective potential breaks down near the critical temperature for a second or weakly first order phase transition, due to infrared divergences $\mathcal{O}$. The surprising accuracy of the perturbative estimate for the critical temperature is due to the fact that the infrared divergences appear at temperatures $|T - T_{cr}|/T_{cr} = \mathcal{O}(\lambda_R)$. For sufficiently small $\lambda_R$ the location of the transition can be accurately computed. However, naive perturbative predictions for the details of the transition (even its order) can be misleading. This is the case for our model, for which the next terms in the naive high temperature expansion of the perturbative result fail to reproduce even the correct qualitative picture $\mathcal{P}$ (apart from a small region in parameter space which will be discussed at the end of the next section). We emphasize that our approach may also be used for large values of $\lambda_R$ not so easily accessible to perturbation theory. In this case $\kappa_G$ should be replaced by the relevant exact point on the critical surface $\kappa_{cr}$.

9. Second order phase transition and the critical behaviour

Let us briefly summarize the main results of the previous sections. We have considered theories which at zero temperature are in the phase with spontaneous symmetry breaking corresponding to the AX regime. They are defined in terms of the classical (bare) parameters at the ultraviolet cutoff $\Lambda \gg T$. The renormalized parameters are obtained by solving the evolution equations from $k = \Lambda$ to $k = 0$. In the low temperature region ($\Lambda \geq k \geq T/\theta_1$) there is no difference between the zero and non-zero temperature case for the evolution of the parameters of the theory. The first temperature effects are observed in the threshold region ($T/\theta_1 > k > T/\theta_2$). For small $\lambda_R$ the values of the running parameters at the beginning of the evolution in the high temperature region ($k = T/\theta_2$) can be expressed in terms of the renormalized parameters of the zero temperature theory. The relation is given by eqs. (6.8). In the high temperature region ($k \leq T/\theta_2$) the character of the evolution is effectively three-dimensional and is governed by the fixed points of the three-dimensional theory. These are most transparent in terms of the dimensionless parameters defined in eqs. (7.3). The evolution equations are given by eqs. (7.4)-(7.8). These equations define a critical surface $\kappa_{cr} = \kappa_{cr}(\lambda, x)$, which is unstable in the $\kappa$ direction and separates the phase with spontaneous symmetry breaking from the symmetric one. The system ends up in either phase depending on the values of the running parameters at $k = T/\theta_2$. For small $\lambda$ the critical surface goes through the Gaussian fixed point given by eq. (7.19). As a result, for small $\lambda_R$ the crucial quantity is the distance from the

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*Perturbation theory for gap equations $\mathcal{P}$ may lead to more reliable results concerning the order of the transition but will fail for critical exponents unless the effectively three-dimensional running of the quartic scalar coupling is properly included.*
critical surface \( \delta \kappa_{cr} = \kappa \left( \frac{T}{\rho_0}, T \right) - \kappa_G \). For \( \delta \kappa_{cr} > 0 \) the theory ends up in the phase with spontaneous symmetry breaking. For \( \delta \kappa_{cr} < 0 \) it ends up in the symmetric one. There is a direct connection between \( \delta \kappa_{cr} \) and the distance from the critical temperature, which is expressed in eq. \( (8.4) \).

In this section we discuss the region in parameter space \( 0 < x < 2 \) where the phase transition is second order. Sufficiently near to the critical temperature the evolution is governed by the Heisenberg fixed point. In fig. 4 we plot the numerical solution of eqs. \( (7.6) \)-\( (7.8) \) for the evolution in the high temperature region, for a theory with zero temperature renormalized parameters \( \lambda_R = 0.01, x_R = 1 \) and critical temperature \( T_{cr}^2/\rho_0 = 4.78 \). We display two trajectories, which start slightly above and below the critical surface (and therefore correspond to temperatures slightly below and above the critical one). We observe that the system flows towards the Heisenberg fixed point which is attractive in both the \( \lambda \) and \( x \) directions. It stays around this fixed point for several orders of magnitude in \( t \) and then deviates towards either the phase with spontaneous symmetry breaking or the symmetric one. During the “time” \( t = \ln(k/\Lambda) \) that the system stays close the fixed point it loses memory of the initial conditions of the evolution. Its dynamics is fixed solely by the fixed point, which has a purely three-dimensional character (as we demonstrated in section 7). As a result we expect that the behaviour of the theory near the critical temperature is independent of the details of the zero temperature theory. It must display universal critical behaviour characteristic of systems with Heisenberg fixed point. As long as \( \kappa(k, T) \) stays almost constant around its fixed point value \( \kappa_H \) and \( k \to 0 \) as \( t \to -\infty \), the minimum of the effective average potential evolves towards zero according to

\[
\rho_0(k, T) = \kappa_H kT. \tag{9.1}
\]

If the temperature is equal to the critical one, the system never leaves the fixed point and \( \rho_0(0, T_{cr}) = 0 \). If the temperature is slightly below \( T_{cr} \), \( \kappa(k, T) \) eventually runs away from the fixed point and diverges, so that \( \rho_0(k, T) = \kappa(k, T)kT \) reaches a constant non-zero value as \( k \to 0 \). This value corresponds to the renormalized minimum of the effective potential at non-zero temperature and we denote it by

\[
\rho_0(T) = \rho_0(0, T). \tag{9.2}
\]

For a temperature slightly above \( T_{cr} \), \( \kappa(k, T) \) (and therefore \( \rho_0(k, T) \)) runs to zero at a finite \( k_s \). From this point on the system is in the symmetric regime and the appropriate evolution equations are given by eqs. \( (7.13) \)-\( (7.15) \). We start the evolution in this regime at \( k = k_s \) with \( m^2(k_s, T) = 0 \) and \( \lambda(k_s, T), x(k_s, T) \) taking their values at the end of the running in the spontaneously broken regime. For \( k \to 0 \) the evolution is stopped by the mass terms in the threshold functions \( L_{1,2}^3 \) and the theory settles down in the symmetric phase. We define the renormalized mass in the symmetric phase as

\[
m_R^2(T) = \bar{m}^2(0, T). \tag{9.3}
\]

We also define the renormalized couplings in both phases as

\[
\lambda_R(T) = \bar{\lambda}(0, T)
\]
\[ x_R(T) = x(0,T). \quad (9.4) \]

It is important to point out that, while the system is staying close to the fixed point, the coupling \( \lambda(k, T) \) evolves towards zero according to
\[
\bar{\lambda}(k, T) = \lambda_{H} \frac{k}{T}. \quad (9.5)
\]

As a result \( \lambda_R(T) \) goes to zero as the critical temperature is approached. Its strong renormalization near \( T_{cr} \) provides the resolution of the problem of infrared divergences. The ratio \( \lambda_R(T)T/m_R(T) \) does not diverge near the critical temperature, in contrast to \( \lambda_R T/m_R(T) \). (Here \( \lambda_R \) is the renormalized coupling of the zero-temperature theory, which is approximately equal to the bare one for small couplings.) We shall not elaborate on this point, but we refer the reader to ref. [6] for an extensive discussion. We also mention that as the temperature deviates from the critical one the system spends less “time” \( t = \ln(k/\Lambda) \) near the critical point. Its flow deviates from the those depicted in fig. 4 at earlier stages. As a result the universal behaviour ceases to dominate.

The behaviour of the renormalized theory at various temperatures is shown in fig. 5 for zero temperature parameters \( \lambda_R = 0.01, x_R = 1 \) and critical temperature \( T_{cr}^2/\rho_0 = 4.78 \). We observe that \( \rho_0(T) \) moves continuously to zero, indicating a second order phase transition. The mass \( m_{R}^2(T) \) is zero at \( T_{cr} \) and positive for larger temperatures. The quartic coupling \( \lambda_R(T) \) stays close to its zero temperature value for most temperatures, but is strongly renormalized towards zero near \( T_{cr} \). The ratio of couplings \( x_R(T) \) again takes its zero temperature value, unless the temperature is sufficiently close to \( T_{cr} \) for the flow to reach the Heisenberg fixed point. The universal behaviour near \( T_{cr} \) can be parametrized by critical exponents, which we define similarly to ref. [6] as
\[
\begin{align*}
\rho_0(T) &\propto (T_{cr}^2 - T^2)^{2\beta} \\
m_R^2(T) &\propto (T^2 - T_{cr}^2)^{2\nu} \\
\lambda_R(T) &\propto (T^2 - T_{cr}^2)^{\zeta} \\
x_R(T) &\propto (T^2 - T_{cr}^2)^{\mu}. \quad (9.6)
\end{align*}
\]

More precisely, \( 2\beta \) is given by the derivative of \( \ln \rho_0(T) \) with respect to \( \ln(T_{cr}^2 - T^2) \), and similarly for the other parameters. The definition of \( \zeta \) and \( \mu \) applies only to the symmetric phase. The exponent \( \beta(T) \) is plotted in fig. 6 along with \( x_R(T) \) for temperatures approaching \( T_{cr} \), for a theory with \( \lambda_R = 0.2, x_R = 1 \). It is apparent from the temperature dependence of \( x_R \) that near \( T_{cr} \) the Heisenberg fixed point becomes important. During its evolution the system stays long enough on the critical surface for this fixed point to generate universal critical behaviour. The exponent \( \beta(T) \) approaches a temperature independent value which is independent of \( \lambda_R \) and \( x_R \) (as long as \( x_R < 2 \)) and characteristic of systems with Heisenberg critical behaviour. This value is
\[
\beta_H = 0.32 \quad (9.7)
\]

\[ \text{25} \]
in agreement with ref. [4]. Two other exponents are fixed by the scaling laws and the finite value of the ratio $\lambda_R(T)/m_R(T)$. They are $\nu_H = \zeta_H = 2\beta_H$. The above values for the exponents are in rough agreement with known values from three-dimensional field theory [24, 36]. The agreement improves dramatically when less restrictive truncations are used for the study of the evolution equation for the potential, and wave function renormalization effects are taken into account [10]. We have performed this more accurate calculation and obtained results which agree with the known values at the 4-5 % level. This work will be described in ref. [34]. Finally, the exponent $\mu$ also approaches asymptotically a constant value (c.f. eq. (7.8))

$$\mu_H = \frac{\nu_H}{8\pi^2} \left\{ \frac{6}{\kappa_H} \left[ L_3^3(2\lambda_H\kappa_H) - L_1^3(0) \right] + \lambda_H \left[ 9L_2^3(2\lambda_H\kappa_H) + L_2^3(0) \right] \right\}$$

$$= 0.64.$$  

(9.8)

10. Tricritical point and crossover

In the above discussion the Heisenberg fixed point was the only one which played any role. This was expected since the Cubic fixed point is repulsive in the $x$ direction. Any flow which starts sufficiently far from it is further repelled and the system never feels its effect. However, it is possible that the values of the running parameters at the beginning of the evolution in the high temperature region are within the region of influence of the Cubic fixed point. An example is given in fig. 7, for a theory with $\lambda_R = 0.01$, $x_R$ slightly smaller than 2, and $T_{cr}^2/\rho_0 = 3.98$. As we have discussed in the introduction and section 3, flows that start on the surface $x = 2$ in parameter space never move out of it. The flows depicted in fig. 7 start with $x(k, T) = 2 - \delta x$ and $\delta x \ll 1$. For this reason, their deviation from the surface $x = 2$ is very slow. We display two trajectories which start a small distance $\delta \kappa_{cr}$ above and below the critical surface (and therefore correspond to temperatures slightly below and above the critical one, according to eq. (8.4)). For $|\delta \kappa_{cr}| \ll \delta x \ll 1$ the flows stay on the critical surface and close to $x = 2$ for a large initial part of the evolution. During this “time” they approach the Cubic fixed point and stay near it. Finally $x(k, T)$ starts growing and the system moves away from the repulsive (in the $x$ direction) Cubic fixed point and towards the Heisenberg one. After it approaches this attractive (in the $x$ direction) fixed point the evolution is similar to the one depicted in fig. 4. Systems which start with larger values of $|\delta \kappa_{cr}|$ behave similarly to fig. 7, but deviate from the critical surface at earlier stages of the evolution. As a result, they can feel the influence of both the Cubic and Heisenberg fixed point, or only the Cubic one, or they can deviate from the critical surface too soon for any universal behaviour to be induced. We calculate the renormalized parameters of the theory (at various temperatures) similarly to the previous subsection. Their behaviour as a function of temperature is analogous to that in fig. 5. The main difference concerns the small region around $T_{cr}$. In this region the temperature dependence should reflect the influence of the two fixed points during the evolution. We first concentrate on values of $\delta \kappa_{cr}$ for
which the critical behaviour is dominated by the cubic fixed point. For this region we plot in fig. 8 the critical exponents corresponding to \( \rho_0(T) \) and \( x_R(T) \), which are defined according to

\[
\rho_0(T) \propto (T^2_{cr} - T^2)^{2\beta},
\]

\[
2 - x_R(T) \propto (T^2_{cr} - T^2)^{-\varphi}.
\]

(10.1)

We observe that they reach constant values as the critical temperature is approached. The value for \( \beta \) should be characteristic of the Cubic fixed point. We find

\[
\beta_C = 0.25
\]

(10.2)

and \( \nu_C = \zeta_C = 2\beta_C \), in agreement with the scaling laws and the finite value of the ratio \( \lambda_R(T)/T/m_R(T) \). We expect the Cubic fixed point to generate the universal behaviour characteristic of an Ising system. This is due to the fact that the theory decomposes into two disconnected \( Z_2 \)-symmetric theories for \( x = 2 \) (see introduction and sections 3 and 4). Indeed, the critical exponents that we have calculated are in exact agreement with the results of ref. [6] for \( N = 1 \), which were obtained at the same level of the truncation scheme. Improved truncations result in values for the exponents which are in agreement with three-dimensional field theory [23, 30] at the few percent level [34]. The exponent \( \varphi = 0.16 \) is a typical example of a crossover exponent [23, 24]. It is related to the growth of the unstable coupling at the Cubic fixed point, and therefore to the negative eigenvalue of the matrix which governs the evolution of small perturbations around the fixed point value of the parameters. We postpone a more detailed discussion of the crossover behaviour for a future publication [34].

The behaviour shown in fig. 8 changes if the critical temperature is further approached (extension of the graph to the right). Eventually the system moves away from the Cubic fixed point and the exponents \( \beta, \nu, \zeta \) take values different from those typical of an Ising system. Also the temperature dependence of \( 2 - x_R(T) \) cannot be described by a crossover exponent anymore and \( x_R(T) \) will rather follow eqs. (9.6), (9.8). We display this behaviour in fig. 9. The values of \( x_R(T) \) give an indication of which fixed point influences the system. It is clear that the Heisenberg fixed point takes over from the Cubic one very close to \( T_{cr} \). The temperature dependence of the exponent \( \beta \) is a characteristic example of a crossover curve. It demonstrates how the critical dynamics changes from Ising-like (for \( \ln \left( \frac{T^2 - T^2_{cr}}{T^2_{cr}} \right) \approx -20 \) ) to Heisenberg-like (for \( \ln \left( \frac{T^2 - T^2_{cr}}{T^2_{cr}} \right) \approx -50 \)). A detailed discussion of this behaviour within more accurate truncation schemes will be given in ref. 34].

### 11. First order phase transition

We turn now to the region \( x > 2 \) where we expect a first order phase transition, as we have explained in sections 3 and 7. Our truncation scheme is too crude to describe the behaviour of the potential in this region. We have approximated \( U_k(\rho_1, \rho_2, T) \) by a
second order polynomial in \( \rho_{1,2} \). This permits the discussion of potentials with only one minimum. The study of first order transitions requires the use of improved truncations, where higher \( \rho \)-derivatives of \( U_k \) are taken into account and the possibility of two distinct minima is permitted. This has been done in ref. 33 for the zero temperature theory, and the existence of a first order transition has been established. We shall not repeat this calculation here. Instead we shall derive an explicit solution of the evolution equation in the region of large \( x \), which will demonstrate the existence of first order transitions for the high temperature theory.

In fig. 10 we plot the numerical solution of eqs. (7.6)-(7.8) in the high temperature region, for zero temperature renormalized parameters \( \lambda_R = 0.01, x_R = 2.01, 3, 5 \). The temperature is very close to the critical one. We notice that for all three sets of parameters the evolution leads to a region of large \( x \). In fig. 10 the curves for \( \kappa \) and \( \lambda \) are terminated when \( x = 30 \). We observe that the running parameters tend towards the same area of parameter space. More specifically, for \( x = 30 \) we find (very roughly) \( \lambda \sim 3, \kappa \sim 0.08 \). This convergence of flows was already apparent in fig. 1. The difference in the evolution lies in the “time” \( t = \ln(k/\Lambda) \) that it takes for the various flows to reach the same region. The flows (a) and (b) are fast, while the trajectory (c) starts very close to the surface \( x = 2 \), is first attracted towards the Cubic fixed point, and finally deviates towards the region of large \( x \). The Cubic fixed point separates the region \( x \leq 2 \), where we have observed second order phase transitions, from the region \( x > 2 \), for which we expect first order transitions. For this reason it is characterized as a tricritical point. (The Ising fixed point exhibits similar behaviour.)

In the regions of large \( x \) we have \( \bar{g} \gg \bar{\lambda} \). As a result, the contribution of the \( \phi_1 \) fluctuations to the evolution of \( U_k(\rho_1, 0, T) \) is suppressed as compared to the contribution of \( \phi_2 \). Moreover, the increase of \( x \) in this region is mainly due to the fast decrease of \( \bar{\lambda} = \lambda k/T \). In contrast, the coupling \( \bar{g} \) evolves only slowly. The \( \rho_1 \)-dependent mass term for the \( \phi_2 \) field is approximately given by \( \bar{g}\rho_1 \) (for \( \rho_2 = 0 \) and apart from a very small region around the origin) and has again a mild \( k \)-dependence. Let us assume that for a given scale \( k_0 \) the solution of the truncated evolution equations depicted in fig. 10 gives a good approximation to the exact solution for the potential. (This means, in particular, that a two-minimum structure has not appeared yet at this scale for the true potential.) We denote the parameters of the theory at the scale \( k_0 \) by \( \kappa_0 = \kappa(k_0, T), \bar{\lambda}_0 = \bar{\lambda}(k_0, T), \bar{g}_0 = \bar{g}(k_0, T), x_0 = x(k_0, T) \), and the mass term for the \( \phi_2 \) field by \( \bar{g}_0 \rho_1 \). Based on the remarks at the beginning of this paragraph we can obtain in the high temperature region an approximate solution of the evolution equation (5.3) for the potential on the \( \rho_1 \)-axis (\( \rho_2 = 0 \)). By neglecting the first term in the r.h.s. of eq. (5.3) and the \( k \)-dependence of \( U_2 = \frac{\partial U_k}{\partial \rho_2} \), the differential equation (5.3) is easily integrated. We obtain in the limit \( k \to 0 \) (up to an irrelevant constant)

\[
U(\rho_1, 0, T) = U_{k=0}(\rho_1, 0, T) = \frac{1}{2} \bar{\lambda}_0(\rho_1 - \kappa_0 k_0 T)^2 - \frac{T}{8\pi^2} \int_0^\infty dx \sqrt{x} \ln \left[ \frac{P_{k_0}(x) + \bar{g}_0 \rho_1}{x + \bar{g}_0 \rho_1} \right]. \tag{11.1}
\]
The effective inverse propagator $P(x)$ is given by eq. (2.11) and we have indicated that it must be evaluated for $k = k_0$. Together with the numerical solution of the flow equations near the critical surface for $k > k_0$, which provides the “integration constants” $\tilde{\lambda}_0$, $\tilde{g}_0$ and $\kappa_0k_0T$, we expect the effective potential of eq. (11.1) to be a very good approximation. (For a sufficiently small ratio of couplings $\lambda_R/g_R$ we may identify $k_0$ with $T/\theta_2$. This essentially reproduces the results of high temperature perturbation theory.)

The effective potential of eq. (11.1) describes indeed a first order phase transition. This can be most easily visualized if we approximate for the purpose of demonstration

$$P_{k_0} = x \quad \text{for} \quad x > k_0^2$$
$$P_{k_0} = k_0^2 \quad \text{for} \quad x < k_0^2. \quad (11.2)$$

One finds for the $\rho_1$-derivative

$$U_1(\rho_1, 0, T) = \frac{\partial U(\rho_1, 0, T)}{\partial \rho_1}$$
$$= - \kappa_0 \tilde{\lambda}_0 k_0 T + \tilde{\lambda}_0 \rho_1 + \frac{\tilde{g}_0}{8\pi^2} T \int_0^{k_0^2} dx \sqrt{x} \left[ \frac{1}{x + \tilde{g}_0 \rho_1} - \frac{1}{k_0^2 + \tilde{g}_0 \rho_1} \right]. \quad (11.3)$$

Using a rescaled field variable

$$\bar{\rho} = \frac{\tilde{g}_0 \rho_1}{k_0^2} \quad (11.4)$$

this yields (with $\lambda_0 = \tilde{\lambda}_0 T/k_0$)

$$U_1(\bar{\rho}) = \frac{\tilde{g}_0 k_0 T}{4\pi^2} \left\{ \frac{2}{3} - \frac{4\pi^2 \kappa_0}{1 + x_0} - \sqrt{\bar{\rho}} \arctan \left( \frac{1}{\sqrt{\bar{\rho}}} \right) + \frac{4\pi^2}{\lambda_0 (1 + x_0)^2} \bar{\rho} + \frac{1}{3} \frac{\bar{\rho}}{1 + \bar{\rho}} \right\}. \quad (11.5)$$

For $\kappa_0 < \kappa_A = \frac{1 + x_0}{6\pi^2}$ the potential $U(\rho_1, 0, T)$ develops a minimum at the origin ($\rho_1 = 0$). For $\kappa_0$ only slightly below $\kappa_A$ the origin is only a local minimum whereas the global minimum occurs at $\rho_1 \neq 0$ and the model is in the phase with spontaneous symmetry breaking. For sufficiently small $\kappa_0/\kappa_A$, however, the absolute minimum is at the origin and the model is in the symmetric phase. (Note that $\frac{2}{3} - \sqrt{\bar{\rho}} \arctan \left( \frac{1}{\sqrt{\bar{\rho}}} \right) + \frac{1}{3} \frac{\bar{\rho}}{1 + \bar{\rho}}$ is a positive function for all $\bar{\rho}$.) There is a critical ratio $\kappa_0/\kappa_A$ (depending on the value of $\lambda_0(1 + x_0)^2$) for which the minima at $\rho_1 = 0$ and $\rho_1 \neq 0$ are degenerate in depth, but they are still well separated from each other. Changing $\kappa_0$ (which is a function of $T$) through this critical value leads to a first order phase transition with a jump in the order parameter.

The necessity of a first order phase transition can also be seen by considering the $\rho_1$-dependent quartic coupling $U_{11}(\rho_1) = \frac{\partial^2 U(\rho_1, 0, T)}{\partial \rho_1^2}$ which obeys

$$U_{11}(\rho_1) = \tilde{\lambda}_0 - \frac{\tilde{g}_0^2}{8\pi^2} T \int_0^{\infty} dx \sqrt{x} \left[ \frac{1}{(x + \tilde{g}_0 \rho_1)^2} - \frac{1}{(P_{k_0} + \tilde{g}_0 \rho_1)^2} \right]. \quad (11.6)$$
By keeping only the most singular behaviour of the integral for \( \rho_1 \to 0 \) we obtain

\[
U_{11}(\rho_1) = \bar{\lambda}_0 + \frac{\bar{g}_0^2}{3\pi^2 k_0} \frac{T}{\sqrt{\rho_1}} - \frac{\bar{g}_0^{3/2}}{16\pi} T \frac{1}{\sqrt{\rho_1}}.
\] (11.7)

We have recovered the leading perturbative result for the behaviour of the quartic coupling near a first order phase transition. If the minimum of the potential \( \rho_{10}(k) \) is sufficiently close to zero at the scale \( k_0 \), the remaining evolution of \( \rho_{10}(k) \) from \( k_0 \) to \( k = 0 \) causes \( U_{11} \) to vanish at some scale \( k \) between 0 and \( k_0 \). As a consequence, the minimum at \( \rho_{10} \neq 0 \) becomes a saddlepoint and disappears subsequently. Already before, a new minimum has been generated at the origin, which remains the only minimum in the subsequent evolution to \( k = 0 \). Since \( U_{11}(\rho_1) \) is always negative for sufficiently small \( \rho_1 \), the phase transition can never be second order and all values \( x > 2 \) must lead to a first order phase transition.

Let us finally discuss a suitable choice of the scale \( k_0 \) from which on we can replace the numerical solution of the flow equations (7.6) - (7.8) by the approximate solution given by eq. (11.7). On one hand \( x_0 \) must be sufficiently large in order to justify the neglect of the contribution of the \( \phi_1 \)-fluctuations in the approximate solution. On the other hand \( k_0 \) should be sufficiently high so that a second minimum at the origin has not yet been generated and the truncation of a polynomial around \( \rho_0 \) is still valid. This requires that trajectories near the critical trajectory not end at \( k = 0 \) too deeply in the symmetric phase. A realistic choice of \( k_0 \) should rather correspond at \( k = 0 \) to the situation where two minima exist simultaneously. For the "quasicritical" trajectories depicted in fig. 10 a reasonable compromise for \( k_0 \) seems to be given by the value for which \( x_0 \) reaches 30. (This corresponds to \( \kappa_A \simeq 0.6 \).) The trajectories (a), (b) and (c) shown in fig. 10 correspond then to potentials \( U(\rho_1,0,T) \) with two different minima, as can be seen from fig. 11. They are close to, but not equal, to the critical trajectories for which \( \kappa \) would deviate from fig. 10 towards the end of the running, thus leading to a potential with two degenerate minima.

In summary, we have established the occurrence of a first order phase transition for \( x > 2 \). Moreover, we have reproduced the perturbative prediction for the form of the potential near the origin. We should emphasize, however, that the perturbative expression applies only to the integration of fluctuations from the scale \( k_0 \) (at which \( x \gg 1 \)) to zero. The flow from the region of \( x \) near 2 to the region where the perturbative expression becomes valid can be computed only through the use of evolution equations. The different flows correspond to first order transitions of varying strength. The discontinuity in the expectation value is of the same order as \( k_0 \). Also the mass gap at the critical temperature is proportional to this scale. In consequence the discontinuities in \( \rho \) for the flows (a), (b), (c) in fig. 10 have a ratio of \( \Delta \rho_a/\Delta \rho_b/\Delta \rho_c = 1/0.016/5.1 \times 10^{-9} \). The last flow, which is remains in the vicinity of the tricritical point before deviating towards the region of large \( x \), corresponds to an extremely weakly first order transition.

Our results can easily be extended to the region \( x < 0 \). We have seen in the introduction and section 4 that the AX regime \( (x > 0) \) and the M regime \( (x < 0) \) can be mapped
onto each other through a simple redefinition of the fields (see eqs. (4.12), (4.13)). For this reason, the physical behaviour in the two regimes is the same. For example, the Cubic and Ising fixed points generate the same universal behaviour, characteristic of a \(Z_2\)-symmetric scalar theory. Similarly, a first order phase transition occurs in the region \(x < -1\). We shall not repeat our discussion for \(x < 0\). All our results can be extended to this region by the redefinition of fields and couplings according to eqs. (4.12), (4.13).

12. Conclusions

We have used the formalism of the effective average action for the study of the high temperature phase transition for a theory of two real scalar fields \(\chi_{1,2}\), with the symmetry \(\chi_1 \leftrightarrow -\chi_1, \chi_2 \leftrightarrow -\chi_2, \chi_1 \leftrightarrow \chi_2\), and quartic potential

\[
V(\chi_1, \chi_2) = \frac{1}{2} \tilde{m}^2 (\chi_1^2 + \chi_2^2) + \frac{1}{8} \tilde{\lambda} (\chi_1^2 + \chi_2^2)^2 + \frac{1}{4} x \tilde{\lambda} \chi_1^2 \chi_2^2. \tag{12.1}
\]

The phase diagram of the theory is divided into four disconnected regions: \(x > 2\), \(2 > x > 0\), \(0 > x > -1\), \(x < -1\). Three fixed points with at least one infrared stable direction exist on the surfaces separating these regions: The Heisenberg fixed point \((x = 0)\) is attractive in the \(\lambda\) and \(x\) directions, and corresponds to a theory whose symmetry is increased to \(O(2)\). The Cubic \((x = 2)\) and the Ising \((x = -1)\) fixed points are attractive in the \(\lambda\) direction and repulsive in the \(x\) direction and correspond to two disconnected \(Z_2(\chi_{1,2} \leftrightarrow -\chi_{1,2})\)-symmetric theories, which are equivalent. The model has a second or first order phase transition, with critical temperature well approximated by the perturbative expression if \(\tilde{\lambda}\) is small.

Theories with classical parameters in the regions \(2 > x > 0\), \(0 > x > -1\) have a second order phase transition. Very close to the critical temperature the behaviour of the system is universal. It is characterized by critical exponents, which are determined by the Heisenberg fixed point. For theories with classical parameters near the surfaces \(x = 2\), \(x = -1\) the influence of the Cubic or Ising fixed point can be observed near - but not too close to - the critical temperature. This leads to a crossover phenomenon, characterized by a crossover exponent and a crossover curve, for temperatures approaching \(T_{cr}\). The universal behaviour is initially determined by the Cubic or Ising fixed point for small enough \((T - T_{cr})/T_{cr}\). As the critical temperature is further approached the more attractive Heisenberg fixed point dominates. No part of this rich structure associated with the second order phase transition can be observed within perturbation theory. We should mention that for small values of \(\tilde{\lambda}\) the region in temperature where these phenomena appear is rather narrow. This changes for larger \(\tilde{\lambda}\), where the critical behaviour extends over a larger temperature domain without changing the universal results. Even though we concentrated in the present paper on small values of \(\tilde{\lambda}\) for the purpose of comparing with analytical results, our method applies equally well to large \(\tilde{\lambda}\).

A first order phase transition is observed in the regions \(x > 2\), \(x < -1\). Therefore, the Cubic and Ising fixed points are tricritical points separating regions of second and
first order transitions. The perturbative expression for the effective potential is a good approximation only for $x \gg 2$ and $x \simeq -2$. All theories near the critical temperature with classical couplings $x > 2$ or $x < -1$ correspond to renormalized theories with $x \gg 2$ or $x \simeq -2$ at scales of the order of the mass gap of the model. However, we distinguish two classes of theories:

I) For classical parameters $x \gg 2$ or $x \simeq -2$ one finds a strongly first order phase transition. Here the effects of quantum or thermal fluctuations are well approximated by the perturbative expression for the effective potential.

II) For classical parameters $x \simeq 2$ or $x \simeq -1$ we predict a very weakly first order transition. The use of the renormalization group is indispensable for the correct incorporation of the quantum or thermal effects which strongly renormalize the theory towards the regions $x \gg 2$ or $x \simeq -2$.

Our results are relevant for multi-Higgs-scalar extensions of the standard model [20] and multi-scalar models of inflation [21]. They cast doubts on the general validity of perturbative predictions for the high temperature behaviour of these models even in the case of small scalar couplings. High temperature perturbation theory was found to give a reliable estimate for the effective potential only in limited regions of the parameter space. Our non-perturbative method works for arbitrary values of the couplings in eq. (12.1). With straightforward modifications - inclusion of $\chi^6$ couplings and anomalous dimension - it gives quantitatively precise predictions for all temperatures and all regions in the phase diagram. As demonstrated earlier [6, 10] these modifications are sufficient for computing critical exponents with a few percent accuracy. The model is easily extended to the case where $\chi_1$ and $\chi_2$ are $N$-component vectors with internal $SO(N)$ symmetries. The high temperature phase transition in other two-scalar models with a different structure of the potential - as for example supersymmetric two-doublet models - can be treated in complete analogy with the present work.
Appendix A: The integrals $L_d^d$

The integrals $L_n^d(w)$ defined in eq. (4.4), with the choice of eqs. (2.2), (2.3) for the infrared regulator, read

$$L_n^d(\tilde{w}) = -2n \int_0^\infty dy y^{\frac{d}{2}+1} \frac{\exp(-y)}{[1 - \exp(-y)]^2} \left[ \frac{y}{1 - \exp(-y)} + \tilde{w} \right]^{-(n+1)}, \quad (A.1)$$

where $\tilde{w} = w/k^2$. In the text we use $L_n^d(w)$ or $L_n^d(\tilde{w})$ synonymously for the integrals (A.1), even though they depend only on the dimensionless ratio $\tilde{w}$. The integrals obey the relations

$$\frac{\partial}{\partial \tilde{w}} L_n^d(\tilde{w}) = -nL_n^{d+1}(-\tilde{w}). \quad (A.2)$$

For $\tilde{w} > -1$, the integrals $|L_n^d(\tilde{w})|$ are finite, monotonic decreasing functions of $\tilde{w}$. We define

$$L_n^d(0) = -2l_n^d \quad (A.3)$$

and give $l_n^d$ for the first three values of $n$

$$l_1^d = \Gamma \left( \frac{d}{2} \right)$$

$$l_2^d = 2 \left( 1 - 2^{1-\frac{d}{2}} \right) \Gamma \left( \frac{d}{2} - 1 \right)$$

$$l_3^d = 3 \left( 1 - 2^{3-d} + 3^{2-d} \right) \Gamma \left( \frac{d}{2} - 2 \right). \quad (A.4)$$

The asymptotic expansion of $L_n^d(\tilde{w})$ for large arguments $\tilde{w} \to \infty$ is

$$L_n^d(\tilde{w}) = -2n \zeta \left( \frac{d}{2} + 1 \right) \Gamma \left( \frac{d}{2} + 2 \right) \tilde{w}^{-(n+1)}$$

$$+ n(n+1) \left[ \zeta \left( \frac{d}{2} + 1 \right) + \zeta \left( \frac{d}{2} + 2 \right) \right] \Gamma \left( \frac{d}{2} + 3 \right) \tilde{w}^{-(n+2)} \ldots \quad (A.5)$$

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9 The pole structure of $L_n^d(\tilde{w})$ is not relevant for this work. For a discussion see ref. [3].
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Figures

Fig. 1 Flows on the $(\lambda, x)$ plane for the three-dimensional theory. The evolution is determined by eqs. (3.7), (3.8) with $m^2 = 0$.

Fig. 2 The integrals $L^3_1(w), L^3_2(w)$.

Fig. 3 $L^4_1(w, T)/L^4_1(w)$ as a function of $T/k$, for various values of $w/k^2$.

Fig. 4 The evolution of $\kappa$, $\lambda$, $x$ in the high temperature region, for temperatures slightly above and below the critical one, and $\lambda_R = 0.01$, $x_R = 1$, $T^2_{cr}/\rho_0 = 4.78$. The system approaches the Heisenberg fixed point before deviating towards the symmetric phase or the phase with spontaneous symmetry breaking.

Fig. 5 The phase transition for a theory with $\lambda_R = 0.01$, $x_R = 1$, $T^2_{cr}/\rho_0 = 4.78$.

Fig. 6 The critical exponent $\beta(T)$ and the parameter $x_R(T)$ as the phase transition is approached, for a theory with $\lambda_R = 0.2$, $x_R = 1$.

Fig. 7 The evolution of $\kappa$, $\lambda$, $x$ in the high temperature region, for temperatures slightly above and below the critical one, and $\lambda_R = 0.01$, $x_R$ slightly smaller than 2, $T^2_{cr}/\rho_0 = 3.98$. The system approaches first the Cubic and then the Heisenberg fixed point, before deviating towards the symmetric phase or the phase with spontaneous symmetry breaking.

Fig. 8 The critical exponents $\beta(T)$ and $\varphi(T)$ as the phase transition is approached, for a theory with $\lambda_R = 0.5$ and $x_R$ slightly smaller than 2.

Fig. 9 The critical exponent $\beta(T)$ and the parameter $x_R(T)$ as the phase transition is approached, for a theory with $\lambda_R = 0.2$, $x_R = 1.99$.

Fig. 10 The evolution of $\kappa$, $\lambda$, $x$ in the high temperature region for temperatures close to the critical ones. The zero temperature parameters are:
   a) $\lambda_R = 0.2$, $x_R = 2.01$,
   b) $\lambda_R = 0.2$, $x_R = 3$,
   c) $\lambda_R = 0.2$, $x_R = 5$.  

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Fig. 11 The effective potential $\bar{U} = 4\pi^2 U/k_0^3 T$ as a function of $\bar{\phi} = \sqrt{2g_0 p_1}/k_0$, resulting from the integration of eq. (11.3) with $\kappa_0 = 0.08$, $\lambda_0 = 3$ and $x_0 = 30$. 
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