Canonical Left Cells and the Lowest Two-sided Cell in an Affine Weyl Group

Nanhua Xi*

ABSTRACT. In this paper we give some discussion to the relations between canonical left cells and the lowest two-sided cell of an affine Weyl group have closed relations. In particular, we use the relations to construct some one dimensional representations of affine Hecke algebras.

Canonical left cells of an affine Weyl group are interesting in understanding cells in affine Weyl group and have nice relations with structure and representations of algebraic groups. However, it is not easy to describe canonical left cells. In this paper we give some discussion to the relations between canonical left cells and the lowest two-sided cell of an affine Weyl group. In particular, we use the relations to construct some one dimensional representations of affine Hecke algebras. For convenience we work with an extend affine Weyl group. This work was partially motivated by [AB].

1. Canonical left cells

1.1. Let $R$ be an irreducible root system and $P$ the corresponding weight lattice. The Weyl group $W_0$ acts on $X$ naturally and the semi-direct product $W = W_0 \ltimes P$ is an extended affine Weyl group, which contains the affine Weyl group $W_a = W_0 \ltimes \mathbb{Z}R$. Let $S$ be the set of simple reflections of $W_a$. The partial order $\leq$ and the length function $l$ on $W$ are well defined.

Let $s_0$ be the unique simple reflection of $W_a$ out of $W_0$. Define $Y_0 = \{w \in W \mid ws_0 \leq w \text{ or } w = e\}$, where $e$ is the neutral element of
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$W$. Then $Y_0 \cap \Omega$ is a left cell for any two-sided cell $\Omega$ of $W$, called a canonical left cell.

In general it is not easy to describe a canonical left cell. However, it is easy to describe the set $Y_0$. Let $w_0$ be the longest element of $W_0$. The set of anti-dominant weights in $P$ is defined to be $P^- = \{x \in P \mid l(xw_0) = l(w_0) + l(x)\}$ and the set of dominant weights is $P^+ = \{x \in P \mid l(w_0x) = l(w_0) + l(x)\}$. For $w \in W$, set $L(w) = \{s \in S \mid sw \leq w\}$ and $R(w) = \{s \in S \mid ws \leq w\}$.

**Proposition 1.2.** $Y_0 = \{wx \mid w \in W_0, x \in P^-, R(w) \subseteq L(x)\}$.

Proof. Let $u \in W$. Then there exist unique $w, v \in W_0$ and $x \in P^-$ such that $R(w) \subseteq L(x)$ and $u = wxv$. Moreover, we have $l(u) = l(x) + l(v) - l(w)$. The proposition follows.

1.3. It would be interesting to see when two elements in $Y_0$ are in a left cell. Let $\rho$ be the product of all fundamental dominant weights. Then the set $\{wx\rho \mid w \in W_0, x \in P^+, R(w) \subseteq L(x)\}$ is the canonical left cell in the lowest two-sided cell $c_0$ of $W$. In general, for any $x \in P^+$ there exists a positive integer $a$ (depending on $x$) such that $x^b$ and $x^a$ are in a left cell if $b \geq a$ (see [X1, Lemma 3.2]). It seems that the number $a$ is not big, in many cases, it is among 1,2,3.

Let $S_0 = S \cap W_0$ and denote by $\Gamma_0$ the left cell $\{w \in W \mid R(w) = S_0\}$, which is in the lowest two-sided cell $c_0$ of $W$. For $x \in P$, denote by $n_x$ (resp. $m_x$) the unique shortest element in the coset $xW_0$ (resp. the double coset $W_0xW_0$). The map $x \to n_x$ defines a one-to-one correspondence from $P$ to $Y_0$, and the map $n_x \to n_xw_0$ defines a one-to-one correspondence from $Y_0$ to $\Gamma_0$. Also the map $x \to m_x$ defines a one-to-one correspondence between $P^+$ and $Y_0 \cap Y_0^{-1}$. The sets $P, Y_0$ and $\Gamma_0$ produce naturally three modules of an affine Hecke algebras of $(W, S)$. In next section we will see that the three modules are essentially the same.

2. Cell modules of affine Hecke algebras

2.1. Let $H$ be the Hecke algebra of $(W, S)$ over a field $k$ with parameter $q$. Assume that $k$ contains square roots of $q$. Let $\{T_w\}_{w \in W}$ be its standard basis. Let For any $w$ in $W$, let $C_w = q^{l(w)} \sum_{y \leq w}(-1)^{l(w) - l(y)} p_{y,w}(q^{-1}) T_y$. 
and $C'_w = q^{\frac{\ell(w)}{2}} \sum_{y \leq w} P_{y,w}(q) T_y$, where $P_{y,w}$ are the Kazhdan-Lusztig polynomials. Then the elements $C_w$, $w \in W$ form a basis of $H$, and the elements $C'_w$, $w \in W$ form a basis of $H$ as well, see [KL1].

For any $x \in P$ there is a well defined element $\theta_x = q^{-\frac{\ell(y)}{2}} T_y q^{\frac{\ell(y)}{2}} T_z - 1$ where $y, z \in P^+$ such that $x = yz^{-1}$. Then $\theta_x \theta_y = \theta_y \theta_x$ for any $x, y \in P$ and the elements $w \theta_x$ (resp. $\theta_x w$), $w \in W_0$, $x \in P$, form a basis of $H$.

The group algebra $k[P]$ is isomorphic to the subalgebra $\Theta$ of $H$ generated by all $\theta_x$, $x \in P$. Lusztig defined several $H$-module structures on $k[P]$, see [L2, Section 7]. They are actually isomorphic to the modules provided by the cell $\Gamma_0$. Let $M$ (resp. $M'$) be the subspace of $H$ spanned by all $C_w$, $w \in \Gamma_0$ (resp. $C'_w$, $w \in \Gamma_0$). Then $M_0$ and $M'_0$ are left ideals of $H$ and generated by $C = C_{w_0}$ and $C' = C'_{w_0}$ respectively. The elements $\theta_x C$, $x \in P$, form a basis of $M$ and the elements $\theta_x C'$, $x \in P$, form a basis of $M'$.

Let $I$ (resp. $I'$) be the subspace of $H$ spanned by all $C_w$, $w \in W - Y_0$ (resp. $C'_w$, $w \in W - Y_0$). Then $J$ and $J'$ are left ideals of $H$. Let $N = H/J$ and $N' = H/J'$. Essentially the following result is due to Arkhipov and Bezrukavnikov (see [AB, 1.1.1]).

**Lemma 2.2.** As $H$-modules $N$ is isomorphic to $M'$, and $N'$ is isomorphic to $M$.

**Proof.** Consider the surjective homomorphism $H \to M'$, $h \to hC'$. It is easy to check that the kernel is $I$. So $N$ is isomorphic to $M'$. Similarly the surjective homomorphism $H \to M$, $h \to hC$ induces an isomorphism $N' \to M$ of $H$-module. The lemma is proved.

2.3. The geometric explanation of the isomorphism is that Thom isomorphism for a certain equivariant $K$-group of the cotangent bundle of flag variety is compatible with certain actions of the affine Hecke $H$, see [L2, Section 7].

Lemma 2.2 seems helpful in understanding the structure of $H$-modules $M$ and $M'$, and may be useful to understand canonical left cells. A natural question is to consider the submodule of $M'$ (resp. $M$) generated by all $C_w C'$ (resp. $C'_w C$), $w \in c_0 \cap Y_0$. Modulo a central character of $H$, we can get a finite dimensional quotient algebra of $H$. 
In next section we will give some discussion to the images in such quotient algebras of the modules. It seems possible that the images are either irreducible modules of $H$ or 0 when $k$ is algebraically closed and $\text{char} k = 0$.

3. A realization of one dimensional representations

3.1. Assume that $k$ is algebraically closed. The center $Z(H)$ of $H$ is in the subalgebra $\Theta$ of $H$ and is isomorphic to $k \otimes \mathbb{Z} R_G$, where $G$ is a simply connected simple algebraic group over $k$ with root system $R$ and $R_G$ is the representation ring of $G$. Thus the set of $k$-algebra homomorphism from $Z(H)$ to $k$ is in one-to-one correspondence to the set of semisimple classes of $G$. For each semisimple class $\tilde{t}$ in $G$, let $\phi_t : Z(H) \to k$ be the corresponding homomorphism. Let $T$ be a maximal torus of $G$ and identify $P$ with the character group $\text{Hom}(T, k^*)$ of $T$. For each semisimple element $t$ in $T$, let $I_t$ be the two-sided ideal of $H$ generated by all $z - \phi_t(z)$, $z \in Z(H)$. Define $H_t = H/I_t$. Then $\dim H_t = |W_0|^2$.

For each simple $H$-module $L$, there exist some $t$ in $T$ such that $Z(H)$ acts on $L$ through the homomorphism $\phi_t$. So to study simple modules of $H$ it is enough to study simple modules of the quotient algebras $H_t$ for $t \in T$. We shall use the same notations $C_w, C'_w, C, C', \theta_x, ...$ for their images in $H_t$.

**Theorem 3.2.** Let $t \in T$. The following statements are equivalent.

(a) $CH_t C = 0$. (Recall that $C = C_{w_0}$ and $C' = C'_{w_0}$.)
(b) $CH_t C' = 0$.
(c) $C'H_t C = 0$.
(d) $C'H_t C' = 0$.
(e) For any simple $H_{t^{-1}}$-module $L$ we have $CL = 0$.
(f) For any simple $H_t$-module $L$ we have $C'L = 0$.

Proof. There is a unique involutive automorphism $h \to h^*$ of the $k$-algebra $H$ such that $T_r^* = -qT_r^{-1} = q - 1 - T_r$ ($r \in S_0$), $\theta_{x^*} = \theta_{x^{-1}}$ ($x \in P$).
Noting that $C^* = (-1)^{l(w_0)} C'$, we see that (a) and (d) are equivalent, (e) and (f) are equivalent.

There is a unique involutive anti-automorphism $h \rightarrow \tilde{h}$ of the $k$-algebra $H$ such that $\tilde{T}_r = T_r \ (r \in S_0)$, $\tilde{\theta}_x = \theta_x \ (x \in P)$ [KL2, 2.13(c)]. Noting that $\tilde{C} = C$ and $\tilde{C}' = C'$, we see that (b) and (c) are equivalent.

Since the two-sided ideal $H_{c_0}$ of $H$ spanned by all $C'w, \ w \in c_0$ is generated by $C'$, using [X3, 7.7] we know that (d) and (f) are equivalent.

Now we show that (d) and (b) are equivalent. Since $T_w C' = C'T_w = q^{l(w)}$ if $w \in W$, we see that $C'H'C'$ is spanned by $C\theta_x C'$. The Weyl group $W_0$ acts on $\Theta$ by $w(\theta_x) = \theta_{w(x)}$. In $H$ we have the Macdonald formula [NR, Theorem 2.22]

$$C'\theta_x C' = q^{l(w_0)} \sum_{w \in W_0} w(\theta_x \prod_{\alpha \in R^+} \frac{1 - q\theta_{\alpha}}{1 - \theta_{\alpha}})C'.$$

So we have

1. The condition $C'H'C' = 0$ is equivalent to

$$\phi_t(\sum_{w \in W_0} w(\theta_x \prod_{\alpha \in R^+} \frac{1 - q\theta_{\alpha}}{1 - \theta_{\alpha}})) = 0, \ \text{for all } x \in P.$$

Let $\Delta$ be the set of simple roots of $R$ and denote $x_\alpha$ the fundamental dominant weight corresponding to a simple root $\alpha$. Similar to [X2, Lemma 2.10], we see that $HC'$ is spanned by all $T_w z \theta_I C'$, $w \in W_0, \ z \in Z(H), \ I \subseteq \Delta$ and $\theta_I = \prod_{\alpha \in I} \theta_{x_\alpha}$. Thus we get

2. The condition $C'H'C' = 0$ is equivalent to

$$\phi_t(\sum_{w \in W_0} w(\theta_I \prod_{\alpha \in R^+} \frac{1 - q\theta_{\alpha}}{1 - \theta_{\alpha}})) = 0, \ \text{for all } I \subseteq \Delta.$$

Since $CT_w = T_w C = (-1)^{l(w)}$ if $w \in W_0$ and $C\theta_IC' = 0$ if $I \neq \Delta$, as an $Z(H)$-module, $CHC'$ is generated by $C\theta_\rho C'$, where $\rho = x_\Delta$ is the product of all fundamental dominant weights. According to [L1, p.222, Lemma 7.4 (iii)], we have

$$C\theta_\rho C' = (-1)^\nu q^{\frac{\nu}{2}} C' \sum_{I \subseteq R^+} (-q)^{|I|} \theta_{\rho}^{-1} \theta_{\alpha_I},$$

here $\nu = l(w_0) = |R^+|$, $\alpha_I$ is the sum of all roots in $I$ and $|I|$ is the cardinality of $I$. 

For \( w \in W_0 \) define
\[
e_w = w \left( \prod_{\alpha \in \Delta} x_\alpha \right).
\]
(Recall that here \( x_\alpha \) is the fundamental dominant weight corresponding to \( \alpha \in \Delta \).) Then \( \Theta \) is a free \( Z(H) \)-module with a basis \( \theta_{e_w}, \ w \in W_0 \).

For \( \theta, \theta' \in \Theta \), define
\[
(\theta, \theta') = \theta \rho \prod_{\alpha \in R^+} (1 - \theta_{\alpha})^{-1} \sum_{w \in W_0} (-1)^{l(w)} w(\theta \theta' \rho) \in Z(H).
\]
By [KL2, p.163] there exist \( \theta' u \in \Theta (u \in W_0) \) such that \( (\theta_{e_w}, \theta' u) = \delta_{w,u} \) and the elements \( \theta' u \) form a \( Z(H) \)-basis of \( \Theta \).

Let \( A = \sum_{I \subseteq R^+} (-q)^{|I|} \rho^{-1} \theta_{\alpha_I} \). Note that \( A = \theta^{-1} \prod_{\alpha \in R^+} (1 - q \theta_{\alpha}) \).

Then
\[
(A, \theta_{e_w}) = (-1)^{|I|} \sum_{w \in W_0} w(\theta_{\rho w} \prod_{\alpha \in R^+} \frac{1 - q \theta_{\alpha}}{1 - \theta_{\alpha}}).
\]
Since \( A = \sum_{w \in W_0} (A, \theta_{e_w}) \theta' w \), we obtain
4 The condition \( CH_tC' = 0 \) is equivalent to
\[
\phi_t \left( \sum_{w \in W_0} w(\theta_{\rho w} \prod_{\alpha \in R^+} \frac{1 - q \theta_{\alpha}}{1 - \theta_{\alpha}}) = 0 \right) \ 	ext{for all} \ w \in W_0.
\]

Using (1) and (4) we see that (d) implies (b). For any \( I \) there exists \( w \in W_0 \) such that \( \rho w = \theta_I \). Using (2) and (4) we see that (b) implies (d). The theorem is proved.

Theorem 3.3. Let \( t \in T \) be such that \( \alpha(t) = q \) for all simple roots \( \alpha \) of \( R \). Then
(a) \( CH_tC' \) (resp. \( C'H_tC \)) is a two-sided ideal of \( H_t \) with dimension 1 if \( \sum_{w \in W_0} q^{l(w)} \neq 0 \).
(b) \( CH_tC' = 0 \) if \( \sum_{w \in W_0} q^{l(w)} = 0 \).

Proof. We have seen that \( CH_tC' \) is spanned by the image in \( H_t \) of \( C\theta_C' \). To see it is a two-sided ideal of \( H_t \) it suffices to prove that the images in \( H_t \) of \( C\theta_C' \theta_x \) and \( \theta_x C\theta_C' \) for all \( x \in \Theta \) are scalar multiples of the image in \( H_t \) of \( C\theta_C' \).

1 If \( w \) is not the neutral element of \( W_0 \), then there exists a positive root \( \beta \) such that \( w(\beta) = \alpha^{-1} \) for some simple root \( \alpha \). Thus \( w(1 - q \beta)(t) = 0 \).
Since

\[(A_{\theta x}, \theta_{w}) = (-1)^{\nu} \sum_{w \in W_0} w(\theta_{w}) \prod_{\alpha \in R^+} \frac{1 - q^{1+\langle \rho, \alpha \rangle}}{1 - q^{\langle \rho, \alpha \rangle}},\]

using (1) we get

\[(A_{\theta x}, \theta_{w})(t) = x(t)w(t) \prod_{\alpha \in R^+} \frac{1 - q^{1+\langle \rho, \alpha \rangle}}{1 - q^{\langle \rho, \alpha \rangle}},\]

if \(1 - q^{\langle \rho, \alpha \rangle} \neq 0\) for all positive roots \(\alpha\). Then [NR, Corollary 2.17]

\[\prod_{\alpha \in R^+} \frac{1 - q^{1+\langle \rho, \alpha \rangle}}{1 - q^{\langle \rho, \alpha \rangle}} = \sum_{w \in W_0} q^{l(w)},\]

if \(1 - q^{\langle \rho, \alpha \rangle} \neq 0\) for all positive roots \(\alpha\). Now \((A_{\theta x}, \theta_{w})\) is in \(Z(H)\), so \((A_{\theta x}, \theta_{w})(t)\) is a regular function in \(q \in k^*\). Thus we have \((A_{\theta x}, \theta_{w})(t) = x(t)w(t) \sum_{w \in W_0} q^{l(w)}\) for all \(q \in k^*\). So the images in \(H_t\) of \(C\theta_{\rho}C^\theta\) for all \(x \in \Theta\) are scalar multiples of the image in \(H_t\) of \(C\theta_{\rho}C^\theta\), and \(CH_tC^\theta\) is a right ideal of \(H_t\). Using the composition of involutions \(h \rightarrow h^*\) and \(h \rightarrow \tilde{h}\) of \(H\) we see that \(CH_tC^\theta\) is a right ideal of \(H_t\) implies that it is also a left ideal of \(H_t\). The theorem is proved.

It is easy to check that \(T_sCH_tC^\theta = -CH_tC^\theta\) and \(CH_tC^\theta T_s = qCH_tC^\theta\) for all simple reflections \(s\) if \(\alpha(t) = q\) for all simple roots \(\alpha\). So the ideals \(CH_tC^\theta\) and \(C^\theta H_tC\) give natural realizations of some one dimensional representations of \(H_q\).

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* HUA LOO-KENG KEY LABORATORY OF MATHEMATICS AND INSTITUTE OF MATHEMATICS, CHINESE ACADEMY OF SCIENCES, BEIJING, 100190, CHINA
E-mail address: nanhua@math.ac.cn