Minimal distances for certain quantum product codes and tensor products of chain complexes

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We use a map to quantum error-correcting codes and a subspace projection to get lower bounds for minimal homological distances in a tensor product of two chain complexes of vector spaces over a finite field. Homology groups of such a complex are described by the Künneth theorem. We give an explicit expression for the distances when one of the complexes is a linear map between two spaces. The codes in the construction, subsystem product codes and their gauge-fixed variants, generalize several known families of quantum error-correcting codes.

I. INTRODUCTION

Central idealization in topology is the focus on continuity while sizes are ignored. Topologically speaking, an opening through a straw is no different than a pinhole in a piece of paper, or a missing pixel in an image. Yet a missing pixel could be just an artifact of the noisy data. No wonder that in practical applications the sizes and distances are important, and are incorporated into computational algorithms in a variety of ways.[1–3].

Quantum stabilizer and, more generally, subsystem codes offer an excellent example of a problem where such a distance is extremely relevant.[4–6, 7]. Namely, a qubit quantum stabilizer code is isomorphic to a chain complex $C$ with three finite-dimensional binary spaces, where logical operators correspond to elements of the first homology group $H_1(C)$. In the case of a Calderbank-Shor-Steane (CSS) code, the rank of this group gives the number $k$ of encoded qubits, the code length $n$ is the dimension of the corresponding space $C_1$, while the distance $d$ of the quantum error correcting code (QECC), the minimum weight of a non-trivial element in $H_1(C)$ (or the corresponding co-homology group), has to be sufficiently large for the code to offer a protection against environmental errors.

In fact, topological QECCs, generalizations of the toric code[6, 10–12] invented by Kitaev[16], are presently at the crux of research in quantum error correction (QEC). Such a code can be constructed from any tessellation of an arbitrary surface or a higher-dimensional manifold. The essential advantage of topological codes is locality: stabilizer generators, operators to be measured frequently, involve only qubits in the immediate vicinity of each other; this is what makes planar surface codes so attractive and practical. However, locality also limits the parameters of topological codes.[17–21]. In particular, for a code of length $n$ with generators local in two dimensions, the number of encoded qubits $k$ and the minimal distance $d$ satisfy the inequality $kd^2 \leq O(n)$. This implies asymptotically zero rate, $R = k/n \to 0$, whenever $d$ diverges with $n$.

More general quantum low-density parity-check (LDPC) codes have stabilizer generators of bounded weight but no locality constraint. This is the only class of codes known so far to combine finite rates with non-zero fault-tolerant (FT) thresholds[21, 22], to allow scalable quantum computation with a finite multiplicative overhead[22]. However, unlike in the classical case, where capacity-approaching codes can be constructed from random sparse matrices[24–27], matrices suitable for constructing quantum LDPC codes are highly atypical in the corresponding ensembles. Thus, an algebraic ansatz is required to construct large-distance quantum LDPC codes. Precious few examples of algebraic constructions are known that give finite rate codes and also satisfy sufficient conditions[22] for fault-tolerance: bounded weight of stabilizer generators and minimum distance that scales logarithmically or faster with the block length $n$. Such constructions include hyperbolic codes on two- and higher-dimensional manifolds[28–32], and quantum hypergraph-product (QHP) & related codes.[33–36]. Further, some constructions, e.g., in Refs. 4, 37–40, have finite rates and relatively high distances, with the stabilizer generator weights that grow with $n$ logarithmically. It is not known whether these codes have non-zero FT thresholds. However, such codes can be modified into those with provable FT thresholds with the help of weight reduction.[41, 42].

The original QHP ansatz[33] by Tillich and Zémor can be seen as a tensor product of two chain complexes $A$ and $B$, each involving just two finite-dimensional binary spaces with chosen bases, so that the corresponding boundary operators are just binary matrices without any additional constraints. The resulting chain complex has three spaces; elements of the first homology group of dimension $k = \text{rank } H_1(A \times B)$ form a half of the quantum code $\mathbb{Q}_1(A \times B)$ encoding $k$ qubits (the other half comes from the corresponding co-homology group). This dimension can be immediately recovered from the Künneth formula.[43, 44]. The main result by Tillich and Zémor is the expression for the minimal distance. This was generalized by the present authors to homology groups in a tensor product of a general chain complex over binary spaces with that involving just two spaces[39].

In this work, we offer a generalization of the distance result in Ref. 39 to a tensor product of two chain complexes of vector spaces over any finite field $F$, with one of the complexes still required to be a linear map be-
between a pair of spaces. While the original proof would still work with a general field, here we give a simpler proof for the lower bound on the minimum distance, formulated in terms of a projected product complex with the level-$j$ subspace projected onto just one subspace $A_i \otimes B_{-i} \subset (A \times B)_{j}$. As a result of the projection, the quantum code $Q(A \times B)$ associated with the $j$-th homology group of the product complex is replaced by an $F$-linear quantum subsystem code; its distance gives a lower bound on the distance associated with the homology group $H_{j}(A \times B)$ of the original product complex. When one of the complexes has length two, the minimum distance of the subsystem code can be computed and, as in the binary case, the result saturates the upper bound.

While the construction also works for a product of chain complexes of arbitrary length, we failed to find a tight lower bound on the distance of the corresponding projected codes. Further, we have found a class of examples, a generalization of the homological product of Steane code with itself, where the distance in the projected complex is strictly smaller than the upper bound. However, through extensive numerics for $q \in \{2, 3, 2^2, 5, 7, 2^3, 3^2, 11\}$, we could not find a single case where the homological distance in the full product complex would fail to saturate the upper bound. We conjecture that in a product of general chain complexes, the upper bound on the homological distance is saturated.

**Potential applications:** In theory of QEC, in addition to defining new classes of quantum LDPC codes with parameters known explicitly, our construction may be useful for (i) optimizing repeated measurements in the problem of FT quantum error correction, (ii) related problem of single-shot error correction, (iii) analysis of transformations between different QECCs, like the distance-balancing trick by Hastings, and (iv) construction of asymmetric quantum CSS codes optimized for operation where error rates for $X$ and $Z$ channels may differ strongly.

More generally, Künneth formula is one of the most important and widely known results in algebraic topology, see, e.g., Ref. 53. Its well known consequence is the relation between the Betti numbers of two manifolds and their product, which can be written in terms of a product of the corresponding generating functions, the Poincare polynomials $P_k(x) = b_0 + b_1 x + b_2 x^2 + \ldots$. Generally, $b_k$ is the rank of the $k$-th homology group. For manifolds in three dimensions, the zeroth Betti number, $b_0$, gives the number of connected components, the first, $b_1$, the number of one-dimensional holes (incontractible cycles), and $b_2$ the number of closed surfaces that cut out internal cavities. In particular, for a torus, $P(x) = 1 + 2x + x^2$, which can be written as $(1 + x)^2$, the square of the Poincare polynomial for a circle.

Our results can be seen as equipping Künneth formula with a distance. For example, consider a torus defined via periodic boundary conditions on a plane, e.g., with periods $L_x$ and $L_y$ along the $x$ and $y$ directions. Then, the systola (girth in the case of a graph) is $\min(L_x, L_y)$, while the surface area (number of plaquettes) is $L_xL_y$. More generally, for a tensor product of a circle with perimeter $L$ and an arbitrary manifold with systola $L_3'\in\{L_3\}$, minimum surface area $L_3''$, etc., the corresponding dimensions are given by $\min(L, L_3'), \min(LL_3', L_3''), \ldots$.

The outline of the rest of the paper is as follows. In Sec. II we go over the necessary background facts from theory of classical and quantum error-correcting codes, as well as chain complexes of vector spaces over a finite field $F$. We also establish the relation between (co)homology groups in such a complex and $F$-linear quantum codes. In Sec. III we describe the construction and derive upper and lower bounds for minimal distances of several related families of “product” codes constructed in terms of Kronecker products of matrices associated with a pair of quantum codes whose parameters are known. In Sec. IV we formulate main results in application to chain complexes, give detailed proofs, and discuss their use in fault-tolerant quantum error correction. Finally, in Sec. V we discuss some extensions of present results.

## II. Background

### A. Classical $q$-ary codes

A classical $q$-ary code $C \subset F^n$ with parameters $(n, K, d)_q$ is a collection of $K$ strings (codewords) of length $n$ over an alphabet with $q$ symbols. The code distance $d$ is the minimum number of positions where two strings in the code differ. A linear $q$-ary code, where $q$ is a power of a prime, is a $k$-dimensional subspace of the $n$-dimensional vector space $F^n$ over the field $F \equiv \mathbb{F}_q$. Such a code contains $K = q^k$ strings. A linear code $C \subset \mathbb{C}$ with parameters $[n, k, d]_q$ can be defined in terms of a generator matrix $G$ whose rows are the chosen basis vectors; the dimension $k$ of the code $C_0$ is $k = \text{rank} G$. For a linear code, the distance $d$ is the minimum Hamming weight of a non-zero vector in the code.

A linear subspace in $F^n$ can be also specified in terms of its orthogonal subspace. To this end, one has to choose the inner product to be used. The simplest choice is the usual Euclidean scalar product, $a \cdot b \equiv a^T b$, where $a, b \in F^n$ are considered as length-$n$ row vectors, and $b^T$ is the transposed vector. Respectively, the dual $C^\perp$ of a linear code $C$ is a collection of $q$-ary row vectors orthogonal to any vector in $C$,

\[
C^\perp = \{b \in F^n | c^T b = 0, \forall c \in C\}. \tag{1}
\]

For a linear code of size $|C| = q^k$ and dimension $k$, the dual code has size $|C^\perp| = q^{n-k}$. Generator matrix $H$ of the dual code, $C_H \equiv C^\perp$, is called a parity check matrix of the original code. More generally, a pair of $n$-column matrices $G$ and $H$ with elements in $F$ are called mutually dual if

\[
GHT = 0, \quad \text{rank } G + \text{rank } H = n. \tag{2}
\]
Given a string \( c \in F^n \), denote \( V \equiv \{1, 2, \ldots, n\} \) the set indexing the individual characters. For any index set \( I \subseteq V \) of length \(|I| = r \), let \( c[I] \in F^r \) be a substring of \( c \) with the characters in all positions \( i \notin I \) dropped. Similarly, for an \( n \)-column matrix \( G \) with rows \( g_j[I] \) is formed by the rows \( g_j[I] \). If \( C = C_2 \) is an \( F \)-linear code with the generating matrix \( G \), then the code of length \(|I|\) with the generating matrix \( G[I] \) is the code punctured outside \( I \), \( C_2[I] \equiv \{c[I] \mid c \in C\} \).

The shortened code \( C_3[I] \) is formed similarly, except only from the codewords supported inside \( I \), \( C_3[I] = \{c[I] \mid c = (c_1, c_2, \ldots, c_n) \in C \) and \( c_j = 0 \) for each \( i \notin I \). The dual of a punctured code \( C_2[I] \) is the shortened dual code, \( (C_2[I])^\perp = (C_2[I])_s[I] \). To express this relation in terms of matrices, consider a pair of mutually dual matrices, \((C_2[I])\) and also a Euclidean dual of the matrix \((C_2[I])^\perp \) has generating matrix \( G[I] \) and \( H[I] \) are also mutually dual \(58\),

\[
H[I] \cdot G[I] = 0, \quad \text{rank} \, G[I] + \text{rank} \, H[I] = |I|. \tag{3}
\]

Similarly, \( H[I] \) is a dual of the punctured matrix \( G[I] \).

In relation to quantum codes, we also consider \( q \)-ary linear space \( F^{2n} \) of length-2n vectors in the form \( e = (a, b) \), where both \( a \) and \( b \) are row vectors of length \( n \). The symplectic product of two such vectors is defined as

\[
e' \cdot e \equiv a' \cdot b - b' \cdot a = e' \Sigma e^T.
\]

where \( \Sigma \) is an \( n \times n \) identity matrix. For a row vector \( e \in F^{2n} \), the (symplectic) conjugate is \( \tilde{e} = e \Sigma e^T = -e \Sigma \), so that the symplectic product can be also written as \( e' \cdot e = e' \Sigma e^T \). The code orthogonal with respect to the symplectic product to a given \( q \)-ary code \( C \subseteq F^{2n} \) is denoted \( C^\perp \). A code \( C_3[G^*] \) orthogonal to \( C_2 \) has generator matrix \( G^* \), a (symplectic) parity check matrix of the original code \( C_2 \) and also a Euclidean dual of the matrix \( \tilde{G} = -G \Sigma \), see Eq. (2), except that the code length here is \( 2n \). Explicitly, for a generator matrix in the block form \( G = (A|B) \), where each block has \( n \) columns, rows of \( G^* \) are orthogonal to the rows of \( \tilde{G} = (B|-A) \), \( \tilde{G}(G^*)^T = 0 \).

**B. Quantum stabilizer codes over qudits**

A single qudit is an isolated quantum-mechanical system whose pure states are described by vectors \( |\psi\rangle \) in a \( q \)-dimensional Hilbert space \( \mathcal{H}_q \). Pure states of \( n \) qudits are described by vectors in the Hilbert space \( \mathcal{H}_q^\otimes n \), the tensor product of \( n \) single-qudit spaces. The corresponding physical observables are described by Hermitian operators acting in \( \mathcal{H}_q^\otimes n \). An \( n \)-qudit quantum error-correcting code \( \mathcal{Q} \) with parameters \( ((n, K))_q \) is a \( K \)-dimensional subspace of \( \mathcal{H}_q^\otimes n \).

When \( q = p^m \) is a power of a prime, there is a particularly nice basis for single-qudit operators acting in \( \mathcal{H}_q \). Following Ref. 60, choose \( q \) orthonormal basis vectors \( |z\rangle \in \mathcal{H}_q \), \( z \in F \), enumerated by elements of the finite field \( F = \mathbb{F}_q \). Two kinds of unitary operators, \( \hat{X}(a) \) and \( \hat{Z}(a) \), \( a \in F \), also enumerated by elements of the field, are defined in terms of their action on the basis vectors,

\[
\hat{X}(a) |z\rangle = |z + a\rangle, \quad \hat{Z}(b) |z\rangle = \omega^{\text{tr}(b)} |z\rangle,
\]

where, with \( q = p^m \) a prime power,

\[
\text{tr}(x) \equiv \text{tr}_{\mathbb{F}_p}(x) = x + x^p + \ldots + x^{p^{m-1}}
\]

is the trace function from the extension field \( F = \mathbb{F}_q \) to the prime field \( \mathbb{F}_p \), and \( \omega = e^{2\pi i/p} \) is a primitive \( p \)-th root of unity. The basis of interest is formed by the \( q^2 \) operators \( \hat{X}(a)\hat{Z}(b), a, b \in F \).

The same operators can be used to construct a basis of operators acting in an \( n \)-qudit Hilbert space \( \mathcal{H}_q^\otimes n \). Namely, given a \( q \)-ary vector \( a \in F^n \), define the \( n \)-qudit operators \( \hat{X}(a) \) and \( \hat{Z}(a) \) as tensor products over components, e.g., \( \hat{X}(a) = \hat{X}(a_1) \otimes \hat{X}(a_2) \otimes \ldots \otimes \hat{X}(a_n) \). These operators generate the \( n \)-qudit Pauli group

\[
\mathcal{P}_n = \left\{ \omega^s \hat{X}(a)\hat{Z}(b)|c \in F_p, a, b \in F^n \right\}.
\]

The weight \( \text{wt}(\hat{U}) \) of an operator \( \hat{U} \in \mathcal{P}_n \) is defined as the number of qudits that \( \hat{U} \) acts upon non-trivially. Up to a phase, a Pauli operator \( \hat{U}(a, b, c) = \omega^s \hat{X}(a)\hat{Z}(b) \) can be specified by the vector \( c \equiv (a, b) \in F_q^2 \). The commutation relation between two such operators (with inessential phase factors suppressed) reads

\[
\hat{U}(a, b)\hat{U}(a', b') = \omega^{\text{tr}(a' \cdot b' - a \cdot b)} \hat{U}(a', b')\hat{U}(a, b).
\]

In particular, the two operators commute if and only if the trace symplectic form \( \text{tr}(a \cdot b' - a' \cdot b) \) vanishes.

An \( n \)-qudit stabilizer code is a common \(+1\) eigenspace of all operators in a stabilizer group \( S \),

\[
\mathcal{Q} \equiv \mathcal{Q}_S = \left\{ |\psi\rangle \in \mathcal{H}_q^\otimes n \mid \hat{U}|\psi\rangle = |\psi\rangle, \forall \hat{U} \in \mathcal{S} \right\},
\]

where \( \mathcal{S} \) is an abelian subgroup of \( \mathcal{P}_n \) whose only zero-weight member is the identity operator. It is easy to see that any Pauli operator \( \hat{E} \) which does not commute with an element of the stabilizer throws the code \( \mathcal{Q}_S \) into an orthogonal space \( E\mathcal{Q}_S \); such operators are called detectable errors. Undetectable errors commute with all elements of \( \mathcal{S} \). In particular, all elements of \( \mathcal{S} \) are undetectable. However, since these operators act trivially in the code, such errors can be ignored. Only undetectable errors outside of \( \mathcal{S} \) (up to a phase) are relevant for error correction. Such errors act non-trivially in the code and correspond to logical operators. The distance \( d \) of a stabilizer code is defined as the minimum weight of an undetectable Pauli operator not equal (up to a phase) to an
element of $\mathcal{S}$. Similarly, errors $\hat{E} \in \mathcal{P}_n$ and $\hat{E}' = \omega^c \hat{S} \hat{E}$ that differ by an element $\hat{S} \in \mathcal{S}$ of the stabilizer group (again, up to a phase) are called mutually degenerate; for all practical purposes such errors are equivalent.

Up to the choice of the phases of its generators, a stabilizer group can be also represented as a length-2 additive code over $\mathbb{F}_q$, isomorphic to a length-2nm linear code over the prime field $\mathbb{F}_p$, where $q = p^m$. The commutation condition gives an additional requirement that the rows of the generator matrix be mutually orthogonal with respect to the symplectic trace product. In general, any element $x \in \mathbb{F}_q$ of an extension of a field of prime degree $p$ is $p$-periodic with respect to addition, $px = 0$. Respectively, the size of a stabilizer group is a power of the prime $p$. This gives the code dimension $K = q^n / |S| = p^k$, which is not necessarily an integral power of $q$. Thus, excluding the case of a prime field analyzed in Ref. [62], a stabilizer code does not necessarily encode an integer number of qudits. The latter condition is satisfied under an additional constraint, $s \equiv \omega^c m \mod m = 0$.

C. F-linear quantum codes

In this work we focus on the special case of F-linear length-2n codes formed by vectors of the form $e = (a|b)$, $a, b \in F^n$, and duality implemented in terms of the Euclidean symplectic product [11]. Unlike in Eq. [12], there is no field trace in this expression. Thus, $e + e' = 0$ gives a sufficient but not a necessary condition for the Pauli operators $\hat{U}(e)$ and $\hat{U}(e')$ to commute, unless $q$ is a prime. Such an approach follows the definition of CSS codes in Ref. [62]. Alternatively, many of the same results can be obtained by classifying generators in terms of a lifted Pauli group as suggested by Gottesman [63].

Degeneracy is the key difference of quantum codes from their classical counterparts. Two vectors $e$ and $e'$ in $F^{2n}$ are called degenerate with respect to elements of the F-linear code $\mathcal{C}_G$ generated by an $r \times 2n$ matrix $G$ if there exist an $\alpha \in F^+$ such that $e' = e + \alpha G$. Degeneracy with respect to $\mathcal{C}_G$ is denoted $e' \preceq_G e$, where the generating matrix may be omitted if the meaning is clear from context.

In the simplest case rows of the generator matrix $H = (A|B)$ (here and below denoted as $H$ to indicate that orthogonality is expected) are mutually orthogonal with respect to the symplectic product,

$$H \tilde{H}^T \equiv H \Sigma H^T = AB^T - BA^T = 0,$$

which is equivalent to $\mathcal{C}_H \subseteq \mathcal{C}_H^\perp$. The space $\mathcal{C}_H$ is readily seen as the symplectic map of a stabilizer group acting in $\mathcal{H}_q^{2n}$. The corresponding dual code $\mathcal{C}_H^\perp$, with any pair of vectors degenerate with respect to $\mathcal{C}_H$ identified, is called an F-linear stabilizer code. The same object is also known as the quotient space $\mathcal{C}_H^\perp / \mathcal{C}_H$.

Given any set of $(m \text{ rank } H)$ additively independent basis vectors of $\mathcal{C}_H$, a stabilizer group $\mathcal{S} \subseteq \mathcal{P}_n$ can be constructed by assigning each generator a phase $c \in \mathbb{F}_p$. With this map, vectors in $\mathcal{C}_H^\perp$ correspond (up to a phase) to undetectable Pauli errors, i.e., operators acting in the space $Q_\mathcal{S} \subseteq \mathcal{H}_q^{2n}$ stabilized by $\mathcal{S}$. Stabilizer group being abelian, it is a subgroup of the group $\mathcal{L}_\mathcal{S}$ of all undetectable Pauli errors acting in $\mathcal{H}_q^{2n}$. Thus, mutually non-degenerate logical operators are classified by elements of the quotient group $\mathcal{L}_\mathcal{S} / \mathcal{S}$. If we ignore the phases, then this group is isomorphic to the F-linear stabilizer code $\mathcal{C}_H^\perp / \mathcal{C}_H$. Notice that the subspace $Q_\mathcal{S} \subseteq \mathcal{H}_q^{2n}$ is also called a stabilizer code, but this should not cause a confusion as we will exclusively use the former meaning.

For an F-linear stabilizer code based on the generator matrix $H$, any codeword $c$ satisfies $Hc^T = 0$, see Eq. [11]; equivalent codewords are mutually degenerate, $c' \equiv c$. Using orthogonalization, we can construct $k = n - \text{rank } H$ pairs of canonically conjugated codewords $c_i, c'_i$ such that $c_i \ast c'_j = \delta_{ij}$, $i, j \leq k$. Equivalently, we can construct a logical generator matrix $L$ whose rows are orthogonal to those of $H$, $HL^T = 0$, are linearly independent from rows of $H$, and, in addition,

$$L \Sigma_n L^T = \Sigma_k.$$  

(12)

More generally, with $\tilde{G}G^T$ not necessarily zero, $\mathcal{C}_G$-degeneracy classes of different vectors in $\mathcal{C}_G^\perp$ correspond to an F-linear subsystem code, a generalization of qubit subsystem codes [64, 65]. Elements of $\mathcal{C}_G$ form a symplectic map of subsystem code’s gauge group, while vectors $c \in \mathcal{C}_G^\perp$ correspond to bare logical operators. Multiplication of a bare logical operator $\hat{U}(c)$ by an element of the gauge group gives a dressed logical operator; with the symplectic map this corresponds to adding a linear combination of the rows of $G$. Nonequivalent logical operators in $\mathcal{P}_n$ map to vectors in $F^{2n}$ which are not degenerate with respect to $\mathcal{C}_G$, $c' \not\equiv c$.

A subsystem code can also be defined in terms of a stabilizer code whose stabilizer group maps to the space $\mathcal{C}_H = \mathcal{C}_G \cap \mathcal{C}_G^\perp$ of dimension $r = \text{rank } G - 2k$, where $2k = \text{rank}(G \tilde{G}G^T)$. The space $\mathcal{C}_H$ is generated by code’s stabilizer generator matrix $H$ whose rows are linear combinations of the rows of $G$, and also $\tilde{G}H^T = 0$. The corresponding orthogonal space $\mathcal{C}_H^\perp$ contains $k + k = n - r$ canonically conjugated vector pairs, including $k$ such pairs in $\mathcal{C}_G$ (these correspond to gauge qudits) and $k$ pairs in $\mathcal{C}_G^\perp \setminus \mathcal{C}_G$ corresponding to logical operators of the data qudits.

In the following, we will be mostly interested in CSS codes [64], a special class of F-linear subsystem (or stabilizer) codes whose generator matrices can be chosen in a block-diagonal form, $G = \text{diag}(G_X, G_Z)$, with each block containing $n$ columns. The corresponding stabilizer generator matrix also has a block form, $H = \text{diag}(H_X, H_Z)$; the symplectic orthogonality is equivalent to $G_X H_Z^T = 0$ and $G_Z H_X^T = 0$. Such a code, denoted CSS$(G_X, G_Z)$, is a direct sum of an $X$- and a $Z$-like codes,

$$\text{CSS}(G_X, G_Z) = \mathcal{C}_X \oplus \mathcal{C}_Z = \mathcal{C}_H^{X} / \mathcal{C}_G X \oplus \mathcal{C}_H^{Z} / \mathcal{C}_G Z,$$  

(13)
where each term in the right-hand side (r.h.s.) is a quotient of two linear spaces. Clearly, the spaces $C_X$ and $C_Z$ are identical to those in gauge-fixed stabilizer codes with generator matrices $H_1 = \text{diag}(G_X, H_Z)$ and $H_2 = \text{diag}(H_X, G_Z)$, respectively. Gauge generator matrix contains $k$ conjugate vector pairs not in $C_H$, thus rank $G_X = \text{rank} H_X + k$ and rank $G_Z = \text{rank} H_Z + k$. As a result, both codes in the r.h.s. of Eq. (19) contain $k = n - \text{rank} H_X - \text{rank} G_Z$ inequivalent vectors. The distances of the two codes are

$$d_X = \min_{x \in C_{i \in H}} \text{wgt}(x), \quad d_Z = \min_{x \in C_{j \in G}} \text{wgt}(x). \quad (14)$$

Any $k$ inequivalent codewords from $C_X$ can be chosen to form the rows of a logical generator matrix $L_X$; in general $L_X H^T_Z = 0$. However, it is convenient to choose bare codewords for the basis, so that also $L_X G^T_Z = 0$. Using bare codewords for the basis of the logical generator matrix of the other code, $L_Z$, this matrix will satisfy $L_Z G^T_X = 0$. In addition, choosing conjugate vector pairs for the two bases, we can also ensure

$$L_X L^T_Z = I_k; \quad (15)$$

with the full-code logical generator matrix in the block-diagonal form, $L = \text{diag}(L_X, L_Z)$. This is the CSS form of Eq. (12). Parameters of such a CSS code are denoted as $[n, k, (d_X, d_Z)]$, where the usual code distance is given by the minimum, $d = \min(d_X, d_Z)$.

**D. Chain complex of $F$-linear spaces.**

Generally, a chain complex is a sequence of abelian groups and a sequence of homomorphisms (boundary operators) between pairs of consecutive groups such that the image of each homomorphism be included in the kernel of the next. We will be concerned with the special case of chain complexes of finite-dimensional vector spaces $\ldots, A_{j-1}, A_j, \ldots$ over a finite field $F = \mathbb{F}_q$, where $q = p^m$ is a power of a prime $p$. In this case the boundary operators are linear transformations $\partial_j : A_{j-1} \to A_j$ that map between each pair of neighboring spaces, with the requirement $\partial_j \partial_{j+1} = 0$, $j \in \mathbb{Z}$. We define an $\ell$-complex $A \equiv K(A_1, \ldots, A_{\ell})$, a bounded chain complex which only contains $\ell+1$ non-trivial spaces with fixed bases, in terms of $n_{j-1} \times n_j$ matrices $A_j$, with elements from $F$ serving as the boundary operators, $j \in \{1, \ldots, \ell\}$:

$$A : \ldots \mapsto \{0\} \xleftarrow{\partial_j} A_0 \xrightarrow{\partial_1} A_1 \xrightarrow{\partial_2} \cdots A_{\ell-1} \xrightarrow{\partial_\ell} \{0\} \ldots (16)$$

Here the neighboring matrices must be mutually orthogonal, $A_{j-1} A_j = 0$, $j \in \{2, \ldots, \ell\}$. In addition to boundary operators given by the matrices $A_j$, implicit are the trivial operators $\partial_0 : \{0\} \to A_0$ and $\partial_{\ell+1} : A_\ell \to \{0\}$ (with the image being the zero vector in $A_0$) treated formally as rank-zero $0 \times n_0$ and $n_{\ell} \times 0$ matrices.

Elements of the subspace $\text{im}(\partial_{j+1}) \subseteq A_j$ are called boundaries; in our case these are linear combinations of columns of $A_{j+1}$ and, therefore, form a binary linear code with the generator matrix $A_{j+1}^T$, $\text{im}(A_{j+1}) = A_{j+1}^T$. In the singular case $j = \ell$, $\text{im}(\partial_{\ell+1}) = \{0\}$, a trivial vector space. Elements of $\ker(\partial_j) \subseteq A_j$ are called cycles; in our case these are vectors in a binary linear code with the parity check matrix $A_j$, $\ker(A_j) = A^\perp_j$. In the singular case $j = 0$, $\ker(\partial_0) = A_0$, the entire space.

Because of the orthogonality $\partial_j \partial_{j+1} = 0$, all boundaries are necessarily cycles, $\text{im}(\partial_{j+1}) \subseteq \ker(\partial_j) \subseteq A_j$. The structure of the cycles in $A_j$ that are not boundaries is described by the $j$th homology group,

$$H_j(A) \equiv H(A_j, A_{j+1}) = \ker(A_j)/\text{im}(A_{j+1}). \quad (17)$$

Group quotient here means that two cycles $\text{[elements of } \ker(A_j)]$ that differ by a boundary $\text{[element of } \text{im}(A_{j+1})\text{]}$ are considered equivalent; non-zero elements of $H_j(A)$ are equivalence classes of homologically non-trivial cycles. Explicitly, the equivalence of $x$ and $y$ in $A_j$ implies that for some $\alpha \in A_{j+1}$, $y = x + \alpha A^T_{j+1}$. The rank of $j$th homology group is the dimension of the corresponding vector space; one has

$$k_j = \text{rank } H_j(A) = n_j - \text{rank } A_j - \text{rank } A_{j+1}. \quad (18)$$

The homological distance $d_j$ is the minimum Hamming weight of a non-trivial element (any representative) in the homology group $H_j(A) \equiv H(A_j, A_{j+1})$,

$$d_j = \min_{0 \neq x \in H_j(A)} \text{wgt } x = \min_{x \in \ker(A_j)/\text{im}(A_{j+1})} \text{wgt } x. \quad (19)$$

By this definition, $d_j \geq 1$. To address singular cases, throughout this work we define the minimum of an empty set as infinity; $k_j = 0$ is always equivalent to $d_j = \infty$. In particular, the distance of the homology group $H_0(A)$ is $d_0 = 1$, unless $A_1$ has full row rank, giving $k_0 = 0$, in which case we get $d_0 = \infty$. In the case of the homology group $H(A)$, the distance $d_\ell$ is that of the F-linear code $C^\perp_{A_\ell}$. Again, we get $d_\ell = \infty$ if $k_\ell = 0$, which happens when $A_\ell$ has full column rank.

In addition to the homology group $H(A_j, A_{j+1})$, there is also a co-homology group $H^j(A) = H(A^T_{j+1}, A^T_j)$ of the same rank [13]; this is associated with the co-chain complex $A$ formed from the transposed matrices $A^T_j$ taken in the opposite order. A quantum CSS code with generator matrices $G_X = A_j$ and $G_Z = A^T_{j+1}$ is isomorphic with the direct sum of the groups $H_j$ and $H^j$, cf. Eq. [13],

$$\text{CSS}(A_j, A^T_{j+1}) \cong H(A_j, A_{j+1}) \oplus H(A^T_{j+1}, A^T_j). \quad (20)$$

The two terms correspond to $Z$ and $X$ logical operators, respectively. This gives for the homological distances in the chain complex and in the co-chain complex, respectively, $d_j = d_Z$ and $d_\ell = d_X$.

The tensor product $A \times B$ of two chain complexes $A$ and $B$ is defined as the chain complex formed by linear spaces decomposed as direct sums of Kronecker products,

$$(A \times B)_j = \bigoplus_{i \in \mathbb{Z}} A_i \otimes \mathcal{B}_{j-i}. \quad (21)$$
with the action of the boundary operators
\[ \partial^n(a \otimes b) \equiv \partial' a \otimes b + (-1)^i a \otimes \partial^n b, \] (22)
where \( a \in A_i, b \in B_{j-i}, \) and the boundary operators \( \partial', \partial'', \) and \( \partial^m \) act in complexes \( A, B, \) and \( A \times B, \) respectively. Notice that the two terms in Eq. (22) are supported in different subspaces of the expansion. Notice that the two terms in Eq. (22) are supported in different subspaces of the expansion. Notice that the two terms in Eq. (22) are supported in different subspaces of the expansion.  

The homology groups of the product \( C = A \times B \) are isomorphic to a simple expansion in terms of those of \( A \) and \( B \) which is given by the Künneth formula,
\[ H_j(C) \equiv \bigoplus_i H_i(A) \otimes H_{j-i}(B). \] (24)
One immediate consequence is that the rank \( k_j(C) \) of the \( j \) th homology group \( H_j(C) \) is
\[ k_j(C) = \sum_i k_i(A) k_{j-i}(B). \] (25)
Such a convolution can be also written as a product of the Poincare polynomials \( p_A(x) = \sum_j k_j(A) x^j \) corresponding to the two complexes, \( p_C(x) = p_A(x) p_B(x). \)

III. MINIMAL DISTANCES OF CERTAIN F-LINEAR CSS CODES

A. Subsystem product codes and their gauge-fixed versions

Our main tool is the map (20) between a CSS code and the homology groups of associated chain and co-chain complexes. In this section we derive the minimum distances of several classes of CSS codes which are relevant for the analysis of the homological distances in the tensor products of chain complexes. Although the derivations are not technically hard, these results may be of independent value.

The distance bounds are constructed using the following two Lemmas which, in turn, follow from Eq. (8) and the fact that for any CSS stabilizer code \( CSS(H_X, H_Z) \) with logical generator matrix \( L = \text{diag}(L_X, L_Z) \), the dual code \( C^\perp_{H_X} \) coincides with the space generated by the combined rows of \( H_Z \) and \( L_Z \), while \( C^\perp_{H_Z} \) coincides with the space generated by rows of \( H_X \) and \( L_X \) combined.

Lemma 1 (Z-puncturing bound). Consider a stabilizer code \( Q = CSS(H_X, H_Z) \) with parame- ters \([n, k, (d_X, d_Z)]\) \( q \) and a qudit index set \( V = \{1, 2, \ldots, n\} \). Given a partition into complementary sets \( I \subset V \) and \( J = V \setminus I \), suppose a logical generator matrix \( L_X \) can be chosen so that none of its \( k \) rows is supported both in \( I \) and in \( J \). Let \( Q' = CSS((H_X)_I, H_Z[I]) \) and \( Q'' = CSS((H_X)_J, H_Z[J]) \) be the codes whose \( X \) generator matrices are shortened and \( Z \) generator matrices punctured to \( I \) and \( J \), respectively. Then, the \( Z \)-distances of the three codes satisfy the inequality \( d_Z \geq \min(d'_Z, d''_Z) \).

Proof. The case \( k = 0 \) is trivial since it gives infinite \( d_Z \); assume \( k > 0 \). The distance \( d_Z \) of the code is the minimum weight in the set \( Q = C^\perp_{H_X} \setminus C^\perp_{H_Z} \) of all non-trivial \( Z \)-like codewords and their equivalent vectors. For any \( c \in Q_Z \), the punctured vectors \( c[I] \) and \( c[J] \) are orthogonal to the rows of \((H_X)_I \) and \((H_X)_J \), respectively; the corresponding Pauli errors are undetectable. Further, since \( L_X c^T \neq 0 \), it is impossible that \( c[I] \) be orthogonal to the rows of \((L_X)_I \) and \((L_X)_J \) and at the same time \( c[J] \) be orthogonal to the rows of \((L_X)_J \). Therefore, at most one of the vectors \( c[I] \) and \( c[J] \) can be trivial in the corresponding code.

Now, consider the identity \( wgt c[I] + wgt c[J] = wgt c > 0 \). The punctured pieces \( c[I] \) and \( c[J] \) contribute to the distances \( d'_Z \) and \( d''_Z \), respectively only if the corresponding vectors are non-trivial. Let \( d(c) \) equal infinity if \( c \) is trivial in \( Q \), and \( wgt c \geq 1 \) otherwise, and define similar functions \( d'(c) \) and \( d''(c) \) for vectors corresponding to undetectable errors in \( Q' \) and \( Q'' \), respectively. Then, \( d_Z' \leq \min_{c \in Q_Z} d'(c[I]) \) and \( d''_Z \leq \min_{c \in Q_Z} d''(c[J]). \) The stated result is obtained by minimizing the inequality \( \min(d'(c[I]), d''(c[J])) \leq d(c) \) over all \( c \in Q_Z \).

Lemma 2 (Z-shortening bound). Consider a stabilizer code \( Q = CSS(H_X, H_Z) \) with the set \( V \) indexing its variable nodes. For any index set \( I \subset V \), let \( Q' = CSS(H_X[I], (H_Z)_I) \) be the code whose \( X \) generator matrix is punctured and \( Z \) generator matrices shortened to \( I \). Then (i) the \( Z \)-distances of the original code does not exceed that of \( Q' \), \( d_Z \leq d'_Z \). (ii) This inequality is saturated if the support of a minimum-weight codeword in \( Q_Z \) is contained in \( I \).

Proof. This follows from the facts that (a) any codeword in \( Q_Z \) is also in \( Q_Z \), and (b) that any codeword in \( Q_Z \) which is supported on \( I \) is also in \( Q_Z \).

We now consider several “product” codes related to the subsystem code \( Q^{\text{obs}} = CSS(G_X, G_Z) \) with the gauge generator matrices
\[ G_X = \left( \begin{array}{cc} H_X \otimes I(n_B) \\ I(n_A) \otimes H_Z^B \end{array} \right), \quad G_Z = \left( \begin{array}{cc} H_X^A \otimes I(n_B) \\ I(n_A) \otimes H_Z^B \end{array} \right), \] (26)
constructed in terms of generator matrices of a pair of stabilizer codes \( Q_A = CSS(H_X^K, H_Z^K) \) and \( Q_B = CSS(H_X^K, H_Z^K) \) with parameters \([n_A, k_a, (d_X^K, d_Z^K)] \) and \([n_B, k_B, (d_X^K, d_Z^K)] \), respectively.

Lemma 3 (Subsystem product code). Denote \( L_X^A, L_Z^A \) and \( L_X^B, L_Z^B \) the logical generator matrices of the CSS stabilizer codes \( CSS(H_X^K, H_Z^K) \) and \( CSS(H_X^K, H_Z^K) \), respectively, chosen so that
\[ L_X^A (L_Z^A)^T = I(k_A), \quad L_X^B (L_Z^B)^T = I(k_B). \] (27)
Then the subsystem product code with CSS gauge generator matrices \((20)\) has logical generator matrices
\[
L_X = L_X^A \otimes L_X^B, \quad L_Z = L_Z^A \otimes L_Z^B.
\]
and stabilizer generator matrices
\[
H_X = \begin{pmatrix} H_X^A \otimes H_X^B \\ H_X^A \otimes L_X^B \\ L_X^A \otimes H_X^B \end{pmatrix}, \quad H_Z = \begin{pmatrix} H_Z^B \otimes H_Z^A \\ H_Z^B \otimes L_Z^A \\ L_Z^B \otimes H_Z^A \end{pmatrix}.
\]
Proof. Matrices
\[
P_A = \begin{pmatrix} L_X^A \\ H_X^A \end{pmatrix}, \quad P_B = \begin{pmatrix} L_Z^B \\ H_Z^B \end{pmatrix}
\]
are the parity check matrices for the classical \(F\)-linear codes with generator matrices \(H_X^A \) and \(H_Z^B\), respectively. Thus, a classical code with generator matrix \(G_X\) in Eq. \((20)\) has a parity check matrix \(P_A \otimes P_B\). Out of the four row blocks of the \(n\times n\) matrix, only rows of \(L_X = L_X^A \otimes L_X^B\) are linearly independent from the rows of \(G_Z\), as can be verified by taking scalar products with the rows of \(L_X\). The remaining row blocks can be readily seen as linear combinations of the rows of \(G_Z\); they form the matrix \(H_Z\). The proof for \(L_X\) and \(H_X\) is similar.

Theorem 4 (Concatenated-stabilizer CSS code). Let \(Q_A\) and \(Q_B\) be two \(F\)-linear CSS stabilizer codes used to define matrices \((20)\), with logical generator matrices \((27)\). Use \(n_A\) copies of the code \(Q_A\), with logical operators used as qudits for the outer code, to form a concatenated-stabilizer code \(Q\) with CSS generator matrices
\[
\overline{H}_X = \begin{pmatrix} H_X^A \otimes I(n_B) \\ L_X^A \otimes H_X^B \end{pmatrix}, \quad \overline{H}_Z = \begin{pmatrix} H_Z^B \otimes I(n_B) \\ L_Z^B \otimes H_Z^A \end{pmatrix}.
\]
The logical generator matrices of this constructed code are given by Eq. \((28)\), and the parameters are given by the corresponding products \([n_A n_B, k, k_{AB}, (d_A d_B)(d_A d_B)]\).

Proof. It is easy to check that \(\overline{H}_X \overline{H}_X^T = 0\); this is a stabilizer code. Similarly, we get \(\overline{H}_X L_X^T = 0, \overline{H}_Z L_Z^T = 0\), and the matrix ranks
\[
\text{rank}(\overline{H}_X) = \text{rank}(H_X^A n_B + k_A \text{rank}(H_X^B), \quad \text{rank}(\overline{H}_Z) = \text{rank}(H_Z^B n_B + k_B \text{rank}(H_Z^A), \quad \text{rank}(L_X) = \text{rank}(L_Z) = k_{AB}.
\]
these expressions add up to the code length \(n_A n_B\). This verifies the CSS construction and the number of encoded qudits \(k = k_{AB}\). The case \(k = 0\) is trivial; in the following, assume \(k > 0\). To construct the upper distance bounds, e.g., \(d_Z \leq d_Z^A d_Z^B\), consider pairs of conjugated codewords \(a, a'\) and \(b, b'\) in \(Q_A\) and \(Q_B\), respectively, where \(a\) and \(b\) are \(Z\)-like with \(\text{wgt}(a) = d_Z^A, \text{wgt}(b) = d_Z^B\), and \(a' b = b' a = 1\). Then the vector \(c = a \otimes b\) of weight \(d_Z^A d_Z^B\) satisfies \(\overline{H}_X c^T = 0\). Further, its dual \(a' \otimes b'\) is orthogonal to the rows of \(\overline{H}_Z\), which implies that \(c\) cannot be a linear combination of the rows of \(\overline{H}_Z\). Taken together, this proves \(c \in \overline{Q}_Z\), thus its weight gives a valid upper bound on \(d_Z\).

To construct a matching lower distance bound, assume there is a non-trivial codeword \(c \in \overline{Q}_Z\) such that \(\text{wgt}(c) < d_Z^A d_Z^B\). This implies \(\overline{H}_X c^T = 0\), and also that \(c\) must be linearly independent from the rows of \(\overline{H}_Z\).

Let \(e_j \in F^{n_A}\), \(j \in \{1, \ldots, n_B\}\) be vectors with all zero components except a one at position \(j\). Consider a decomposition
\[
c = \sum_j a_j \otimes e_j, \quad \text{where } a_j \in F^{n_A}.
\]
From the upper row blocks of the generators \((31)\), each non-zero \(a_j\) must either be a non-trivial \(Z\)-like vector in the code \(Q_A\), or a linear combination of the rows of \(H_Z^A\). This implies that any non-zero \(a_j\) such that \(\text{wgt}(a_j) < d_Z^A\) can be removed from \(c\) (set to zero) without any other changes; the resulting vector \(c'\) should remain in the code as the two vectors are degenerate with respect to \(C_{\overline{Q}_Z}\).

This vector has weight \(\text{wgt}(c') \leq \text{wgt}(c) < d_Z^A d_Z^B\), and any non-zero component \(a_j\) in its expansion \((35)\) has weight \(d_Z^A\) or larger. Let \(J = \{1, 2, \ldots, n_B\}\) be the set of positions \(j\) corresponding to non-zero \(a_j\) in the expansion of \(c'\). By this logic,
\[
d_Z^A d_Z^B > \text{wgt}(c') = \sum_{j \in J} \text{wgt}(a_j) \geq d_Z^A |J|;
\]
the total number of positions in \(J\) satisfies \(|J| < d_Z^B\). Denote \(V_A = \{1, 2, \ldots, n_A\}\) and \(J = V_A \cup J\); the punctured vector \(c'[J]\) preserves all non-zero positions in \(c'\). Thus, \(c'[J]\) should be in the code \(Q' = \text{CSS}(\overline{H}_X[J], (\overline{H}_Z)[J])\), see Lemma 2. By construction, the matrices \(\overline{H}_X[J]\) and \(\overline{H}_Z[J]\) have the same structure \((31)\), except the code \(Q_B\) is replaced with \(Q_B' = \text{CSS}(H_X[J], (H_Z)[J])\) of length \(|J|\). This latter code also satisfies Lemma 2. We expect the corresponding distance to serve as an upper bound to \(d_Z^B\). However, since its length \(|J| < d_Z^B\), the only possibility is for the code \(Q_B'\) to encode no qudits, \(k_B' = 0\). Necessarily, the code \(Q'\) also has \(k' = k_{AB}\), which makes the initial assumption about the existence of the codeword \(c\) invalid; this proves \(d_Z = d_Z^A d_Z^B\).

**B. Bounds on the minimal distance**

Notice that rows of \(H_Z\) in Eq. \((29)\) are linear combinations of rows of \(\overline{H}_Z\) in Eq. \((31)\), whose rows are, in turn, linear combinations of rows of \(G_Z\) in Eq. \((28)\). Similar relation exists between the corresponding \(X\) matrices. As a result, there is a sequence of inclusions
\[
C_{G_Z} \setminus C_{H_Z} \subseteq C_{\overline{H}_X} \setminus C_{\overline{H}_Z} \subseteq C_{H_X} \setminus C_{G_Z},
\]
which implies a sequence of inequalities for the three related codes:
\[
d_Z(G_X, H_Z) \geq d_Z(\overline{H}_X, \overline{H}_Z) \geq d_Z(H_X, G_Z),
\]
where, e.g., $d_Z(G_X, H_Z)$ is the $Z$-distance in the code $\text{CSS}(G_X, H_Z)$.

On the other hand, from linear relations between the rows of matrices involved, Lemma \ref{lem:linear_relations} and Theorem \ref{thm:linear_relations} it follows that all of the three codes in Eq. (37) are gauge-fixed versions of the subsystem code with the generators \cite{20}. They share the logical generator matrices \cite{28}, which implies a common upper bound $d_2 \leq d_2^B$, the proof is similar to that in Statement \ref{thm:linear_relations}.

We get:

\begin{align}
  d_2(G_X, G_Z) &= d_2(H_X, G_Z) \leq d_2^A d_2^B, \\
  d_2(G_X, H_Z) &= d_Z(H_X, H_Z) = d_2^A d_2^B. 
\end{align}

Unfortunately, we are not able to get the exact values for the $Z$-distances in the l.h.s. of Eq. (39). It is clear that the upper bound \cite{39} is sharp. In particular, the upper bound is saturated whenever one of the codes has distance one. This follows from the following two lower bounds which we adapted from Ref. \cite{16}.

Statement 5 (Lower distance bound I). Consider an $F$-linear code $\text{CSS}(H_X, G_Z)$ with stabilizer generator matrices $H_X$ and $G_Z$ given by Eqs. \cite{29} and \cite{29}, respectively. (a) The corresponding $Z$-distance satisfies the inequality

\begin{align}
  d_Z(H_X, G_Z) &\geq \max(d_2^A, d_2^B). 
\end{align}

(b) In addition, assume that $d_2^A > 1$. Then, with $F = \mathbb{F}_q$,

\begin{align}
  d_Z(H_X, G_Z) &\geq \frac{q}{q-1} d_2^B. 
\end{align}

The proof is based on the following Lemma from Ref. \cite{16}.

Lemma 6 (Lower distance bound II). Consider an $F$-linear stabilizer code $Q = \text{CSS}(H_X, G_Z)$ with generator matrices $H_X$ and $G_Z$ in Eqs. \cite{29} and \cite{29}, respectively. Given $a \in Q_A$, consider a set $\Omega_A(a) = \{x_1, x_2, \ldots, x_N\}$ of vectors degenerate with $a$ with respect to $C_{H_X}$, such that each $i \in \{1, 2, \ldots, n_A\}$ is the support of no more than $K$ of these vectors. Then, for any $Z$-like codeword $c \in Q_Z$ such that $[a \otimes I(n_B)] c^T \neq 0$, we have

\begin{align}
  \text{wtg}(c) \geq \left\lceil \frac{N}{K} \frac{d_B}{d_Z} \right\rceil. 
\end{align}

Proof. Given $c$ in Eq. (43), consider an expansion

\begin{align}
  c = \sum_{j=1}^{n_A} f_j \otimes b_j, \quad b_j \in F^{n_B},
\end{align}

where components of $f_j \in F^{n_A}$ are all zero except for $f_j[j] = 1$, $j \in \{1, \ldots, n_A\}$. By assumption, the dot-product $a_i \otimes I(n_B)$ with $c$ is non-zero; for any $a_i \in \Omega_A(a)$, we have

\begin{align}
  x_i^T \equiv (a_i \otimes I(n_B)) c^T = \sum_j a_i[j] b_j^T.
\end{align}

It is easy to check that the resulting vector $x_i \in F^{n_A}$ satisfies $H_X x_i^T = 0$, while $L_X x_i^T \neq 0$. That is, $x_i$ is in $Q_Z^B$, so that $\text{wtg}(x_i) \geq d_2^B$. Let us now sum the weights of vectors $x_i$ corresponding to all elements of $\Omega_A(a)$,

\begin{align}
  N d_2^B \leq \sum_{i=1}^{n_A} \text{wtg}(x_i) \leq \sum_{j=1}^{n_A} \sum_{i=1}^{N} \text{wtg}(a_i[j] b_j^T) \\
  \leq K \sum_{j=1}^{n_A} \text{wtg}(b_j) = K \text{wtg}(c),
\end{align}

which gives Eq. (43) since $\text{wtg}(c)$ is an integer. \hfill \square

Proof of Statement 5. Both (a) and (b) are trivial if $k_A k_B = 0$; assume otherwise below. (a) The construction is symmetric with respect to constituent codes $Q_A$ and $Q_B$; without limiting generality assume $d_2^A \geq d_2^B$. Use the set $\Omega_A(a) = \{a\}$ in Lemma \ref{lem:linear_relations} with $N = K = 1$, which proves $d_2(H_X, G_Z) \geq d_2^B$. (b) The condition $d_2^A > 1$ implies that any all-zero column in $H_X^T$ (say, at position $i \leq n_A$) must be matched by a row (or a linear combination of rows) of $H_Z^T$ with the only non-zero element at $i$. This guarantees that any $X$-like codeword $a$ has no support at such position(s). For any $a \in Q_A$, consider the set $\Omega = \Omega_A(a)$ of size $N = q^{\text{rank}H_X^T}$ which contains all vectors degenerate with $a$. For any $i \leq n_A$, the set of characters $\Omega[i] = \{x[i] : \forall x \in \Omega_A(a)\}$ either contains all zeros, or contains equal numbers of all elements of $F$—this can be seen by considering a generating matrix with all except one row not supported on $i$. For such a set, $K = (q-1)q^{\text{rank}H_X^T-1}$, which proves Eq. (42). \hfill \square

Another application of Lemma \ref{lem:linear_relations} is demonstrated by the following

Example 7. Let $Q_A = \text{CSS}(H_A^X, H_A^Z)$ be a single-qubit encoding (consta)cyclic CSS code with parameters $[n_A, 1, (d_A^X, d_A^Z)]$. Then, for any $Q_B = \text{CSS}(H_B^X, H_B^Z)$, the $Z$-distance of the product code $\text{CSS}(H_X, G_Z)$ with stabilizer generator matrices \cite{29} and \cite{29} satisfies

\begin{align}
  d_Z(H_X, G_Z) \geq [n_A d_B^Z/d_X^A].
\end{align}

Proof. Use Lemma \ref{lem:linear_relations} with $\Omega_A(a)$ a set of size $N = n_A$ constructed by shifting an $X$-like minimum-weight codeword $a \in Q_A$ by $0, 1, \ldots, n_A-1$ positions. The resulting vectors $x_i \in \Omega_A(a)$ cannot be linear combinations of rows $H_A^X$, or else the original vector $a$ would be too, thus they must be in the code. Since $k_A = 1$, they must be degenerate with $a$. The lower bound \cite{44} is obtained if we notice that for this set, $K = d_A^A$. \hfill \square

The discussed lower distance bounds are pretty far from the generic upper bound \cite{50}. On the other hand, at least in the binary case, it is not easy to construct an example of a subsystem product code with the distance strictly below the upper bound. Discovering such examples is dramatically simplified with the help of the ansatz in the following Theorem \ref{thm:ansatz} a generalization of the construction based on the homological product of Steane’s $[[7, 1, 3]]$ code with itself \cite{55} (see Example \ref{ex:ansatz} below)
Theorem 8 (X–Z-symmetric product codes). Consider codes \( Q_A = \text{CSS}(H_X^A, H_Z^A) \) and \( Q_B = \text{CSS}(H_X^B, H_Z^B) \) with \( X \) and \( Z \) generator matrices interchanged. The distances of the corresponding subsystem product code \( \text{CSS}(G_X, G_Z) \) with generators \( \{ G_A, G_B \} \) satisfy

\[
d_X(G_X, G_Z) \leq n_A, \quad d_Z(G_X, G_Z) \leq n_A.
\]  

The inequality \( \{2\} \) becomes strict if \( n_A < d_A^X d_B^Z \equiv d_Z^X d_A^Z \).

Proof. The construction is symmetric with respect to \( X \) and \( Z \) parts of \( Q_A \); it is sufficient to prove the bound for \( d_Z = d_Z(G_X, G_Z) = d_Z(H_X, G_Z) \) with \( H_X \) in Eq. \( \{3\} \). We have \( n_A = n_B \); consider a vector \( c = \sum_{j=1}^{n_A} e_j \otimes e_j \) of weight \( n_A \), where \( e_j \) are weight-one vectors as in Eq. \( \{5\} \). Using Eq. \( \{24\} \) and the orthogonality between the rows of remaining \( X \) and \( Z \) generator matrices, verify that \( H_X c^T = 0 \) while \( L_X c^T \neq 0 \). Thus, \( c \) is a valid \( Z \)-like codeword in \( \text{CSS}(H_X, G_Z) \) and \( d_Z \leq n_A \). \( \square \)

It is known that long CSS codes with distances scaling linearly with the code length \( n \) exist \( \{3\} \). For a pair of such codes, the generic upper bound \( \{39\} \) has asymptotic scaling \( d \sim O(n) \), linear in the length of the product code. On the other hand, the upper bound for the corresponding \( X–Z \)-symmetric product codes, see Theorem \( \{8\} \) gives \( d \leq n_A \), a square root of the length of the product code. Thus, we can not expect the generic upper bound to be saturated. The following explicit Examples demonstrate that such a saturation does not happen for any finite field \( F \).

Example 9. For any field \( F = \mathbb{F}_q \) with \( q = 2t + 1 \) odd, consider a \( \{3,1,2,2\}\)|\( q \) code with \( G \) generators \( H_X = (1,1,1), H_Z = (t,t,1) \). The corresponding \( X–Z \)-symmetric product code in Theorem \( \{8\} \) has distances \( d_X = d_Z = 3 \), smaller than the upper bound \( \{39\} \). For \( q = 3 \), this saturates the lower bound \( \{32\} \).

Example 10. For any \( q = 2^m \) with \( m \) even, so that \( r \equiv (q - 1)/3 \) is an integer, consider a stabilizer code \( \{3,1,2,2\}\)|\( q \) with cyclic \( H_X^A \) and constacyclic \( H_Z^B \) generators,

\[
H_X^A = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}, \quad H_Z^B = \begin{pmatrix} 1 & x^r & x^{2r} \end{pmatrix},
\]

where \( x \in \mathbb{F}_q \) is a primitive element, \( i.e., x^{q-1} = 1 \). Construct an \( X–Z \)-symmetric product code as in Theorem \( \{8\} \). Combining Eq. \( \{42\} \) with the lower bound \( \{44\} \) again gives \( d_Z(G_X, G_Z) = 3 \), smaller than \( d_A^X d_B^Z = d_Z^X d_A^Z = 4 \).

Example 11 (Square of Steane’s code \( \{138\} \)). For any \( q = 2^m \), \( m \in \mathbb{N} \), consider a pair of identical cyclic codes \( \{7,1,3,3\}\)|\( q \) with stabilizer generator polynomials \( h_X^A(x) = h_Z^B(x) = 1 + x^2 + x^3 + x^4 \). Combination of the \( X–Z \) symmetric product construction from Theorem \( \{8\} \) and the lower bound \( \{47\} \) gives \( d_Z(G_X, G_Z) = 7 \), smaller than \( d_A^X d_B^Z = 9 \).

C. Previously known constructions

In the remainder of this Section, we discuss several existing code families which can be described as special cases of the subsystem product code construction in Lemma \( \{3\} \) or as gauge-fixed versions of such codes.

The first such family, homological product codes from Refs. \( \{4,38\} \), is based on square nilpotent matrices such that \( \delta^2 = 0 \), with elements from a field \( F = \mathbb{F}_q \) with \( q = 2^m, m \in \mathbb{N} \). Such a matrix \( \delta \) and its transposed \( \delta^T \) can be used to construct the stabilizer code \( \text{CSS}(\delta, \delta^T) \) and its symmetric \( \text{CSS}(\delta^T, \delta) \). Alternatively, stabilizer generators of a CSS code with rank \( H_X = \text{rank} H_Z \) can be used to form such a nilpotent matrix, \( \delta = H_X^T M H_Z \), where \( M \) is a matrix of appropriate dimensions chosen to preserve the rank of the product.

Example 12 (Homological product codes). For \( q = 2^m, m \in \mathbb{N} \), consider a pair of \( F \)-linear stabilizer codes \( Q' = \text{CSS}(\delta, \delta^T) \) with parameters \( [n_\mu, k_\mu, (d_A^\mu, d_B^\mu)]_q \) based on nilpotent matrices \( \delta_\mu \), where \( \mu \in \{A, B\} \). Then the matrix \( \delta_C = I(n_A) \otimes \delta_B + \delta_A \otimes I(n_B) \) is also nilpotent. The corresponding code \( \text{CSS}(\delta_C, \delta_C^T) \) has logical generator matrices given by Eq. \( \{20\} \), and parameters \( [n_{A\mu B}, k_{A\mu B}, (d_A^\mu, d_B^\mu)]_q \), where, e.g.,

\[
d_Z(G_X, G_Z) \leq d_Z^C \leq d_A^X d_B^Z. \]  

Proof. It is easy to check that the logical generator matrices are given by Eq. \( \{28\} \); the upper bound on the distance follows. On the other hand, rows of \( \delta_C \) and \( \delta_C^T \), respectively, are linear combinations of the rows of \( G_X \) and \( G_Z \), see Eq. \( \{29\} \). This implies that the stabilizer code defined by this matrix is a gauge-fixed version of the subsystem product code \( \text{CSS}(G_X, G_Z) \), which gives the lower bound. \( \square \)

As before, the upper distance bound is sharp, but it is not necessarily saturated. In particular, an example \( \{35\} \) can be constructed along the lines of Example \( \{11\} \) as a homological product code combining two Steane’s codes with identical symmetric nilpotent matrices \( \delta \). Such a code has distance \( d = 7 \), while the the upper bound in Eq. \( \{47\} \) gives \( d \leq 9 \).

Our last example shows that subsystem product codes and the corresponding gauge-fixed codes from Lemma \( \{3\} \) can be seen as a generalization of subsystem hypergraph-product codes and corresponding gauge-fixed codes recently constructed by Li and Yoder \( \{64\} \) which are, in turn, a generalization of Bacon-Shor \( \{65\} \) and Shor’s \( \{67\} \) codes, respectively. The Li–Yoder construction is based on a pair of classical codes, it is similar but not identical to those in Refs. \( \{68, 69\} \). Namely, the gauge and stabilizer generator matrices can be obtained from Eqs. \( \{20\} \) and \( \{29\} \) by considering the classical codes as degenerate quantum codes with empty \( H_X^B \) and \( H_Z^A \) matrices.

Example 13 (Subsystem QPH codes \( \{66\} \)). Given a pair of \( F \)-linear classical codes with parameters \( [n_\mu, k_\mu, d_\mu]_q \),
parity check matrices $P_\mu$, and generator matrices $Q_\mu$, where $\mu \in \{A, B\}$, consider a subsystem code $CSS(G_X, G_Z)$ with gauge generator matrices
\[
G_X = (P_A \otimes I_{n_B}), \quad G_Z = (I_{n_A} \otimes P_B).
\] (48)

The corresponding stabilizer generator matrices are
\[
H_X = (P_A \otimes Q_B), \quad H_Z = (Q_A \otimes P_B).
\] (49)

Assuming $k_Ak_B > 0$, the parameters of the subsystem and both gauge-fixed codes are $[[n_An_B, k_Ak_B, (d_A, d_B)]_q$.

The parameters of the codes follow from Theorem 11 where we should use $d_A^1 = d_Z^2 = 1$. In particular, we get the original Bacon-Shor (BS) and Shor’s codes if we take repetition codes for both classical codes.

We also notice that a subsystem product code constructed from a BS code and a repetition code coincides with the 3-dimensional BS code as proposed by Napp and Preskill [74] (this construction differs from the 3D code originally suggested by Bacon [65]). Napp & Preskill construction can be seen as a three-fold subsystem product of repetition codes, and can be generalized to higher dimensions. However, it is easy to check that these single-qubit encoding codes are just rearrangements of conventional BS codes from a 2D lattice to higher dimensions. The only differences are the measurement redundancy and local connectivity of neighboring qubits, as defined by the specific sets of gauge generators used in the construction.

IV. HOMOLOGICAL DISTANCES IN TENSOR PRODUCTS OF CHAIN COMPLEXES

Example 13 may serve as a nice introduction to the subject of this section. Indeed, Bacon-Shor code can be obtained from Kitaev’s toric code by erasing qubits on all vertical (or all horizontal) bonds. The latter code corresponds exactly to a CW-complex associated with a square lattice with periodic boundary conditions—a tensor product of two cycle graphs. More general gauge generator matrices [58] can be seen as a result of erasing one of the blocks in a QHP code [53, 54] with stabilizer generator matrices
\[
H_X = (P_A \otimes I_{n_B}|I_B \otimes P^T_T), \\
H_Z = (I_{n_A} \otimes P_B| - P^T_T \otimes I_A),
\] (50)

where the dimensions of the identity matrices $I_A$ and $I_B$ match the numbers of rows in the two check matrices. The matrices $H_X$ and $H^T_Z$ correspond exactly to the boundary operator matrices in a product of the chain complexes $K(P_A)$ and $K(P^T_T)$. In this section we consider tensor products of general bounded $F$-linear chain complexes. The corresponding boundary operators, see Eq. (57) below, have row- and column-blocks with the structure of the gauge generator matrices [20].

A. Main results for $F$-linear chain complexes

Our main result is the expression for the homological distance in a tensor product of two bounded chain complexes of finite-dimensional vector spaces over a finite field $F$, where one of the complexes contains just two non-trivial spaces. Specifically, let $A$ be such a complex of any length specified in terms of boundary operators $\partial_j : A_{j-1} \rightarrow A_j$ defined explicitly as matrices, $\partial_j = A_j$ such that $A_jA_{j+1} = 0$, and $B$ a complex with just two non-trivial spaces $B_0$ and $B_1$ and a single non-trivial boundary operator (matrix) $B_1 : B_0 \rightarrow B_1$ mapping between them. Then, the homological distance $d_j(C)$ for the $j$th homology group in the tensor product $C = A \otimes B$ of the two complexes is
\[
d_j(C) = \min (d_j(A)d_0(B), d_{j-1}(A)d_1(B)).
\] (51)

This is a generalization of the identical expression for the tensor product of binary chain complexes from Ref. 50.

There is actually a stronger statement which concerns the homological distance $d_j(C_{i,j-1})$ after a projection onto a single subspace $C_{i,j-1} = A_i \otimes B_{j-1}$, where $j - i \in \{0, 1\}$. Here, a chain complex with the space $C_j$ reduced to its subspace has modified boundary operators $\partial'_i$ and $\partial'_{i+1}$. The latter is defined as a composition of a projector $P$ and the original boundary operator $\partial_i$, $\partial'_{i+1} = P\partial_{i+1}$, where $P^2 = P$ and the image of $P$ is the subspace of interest. The modified boundary operator $\partial'_i$ is defined to ensure the composition to vanish, $\partial'_i \partial'_{i+1} = 0$. For thus defined chain complex $C'_{i,j-1}$ with the space $C_j$ in the original product complex $C$ projected to its subspace $C_{i,j-1}$, the homological distance at level $j$ is given by one term only,
\[
d_j(C'_{i,j-1}) = d_i(A)d_{j-i}(B), \quad j - i \in \{0, 1\}.
\] (52)

Our third result concerns with the minimal distance in a tensor product of two arbitrary-length chain complexes of vector spaces over a finite field $F$. Here the upper bound on the homological distance reads
\[
d_j(C) \leq \min_{i \in \mathbb{Z}} d_i(A)d_{j-i}(B).
\] (53)

A lower bound for the same distance $d_j(C)$ can be constructed by projecting onto the individual product spaces $A_i \otimes B_{j-i}$, $i \in \mathbb{Z}$, whose direct sum gives the degree-$j$ space $C_j$ in the product complex. This gives $d_j(C) \geq \min_i d(C'_{i,j-1})$. The result of the projection can be seen as an $F$-linear quantum subsystem code with CSS gauge generator matrices in the product form [20],
\[
G_X = \left( I(a_i) \otimes B_{j-i} \right), \quad G_Z = \left( I(a_i) \otimes B^T_{j-i+1} \right),
\] (54)

where $I(a)$ \equiv $I_i$ is the size-$i$ identity matrix, and $a_i$ and $b_i$, respectively, are the dimensions of the degree-$i$ spaces in the chain complexes $A$ and $B$. Thus, the $Z$-distance of the subsystem code with CSS generators [54] may serve as a lower bound for $d_j(C)$, complimentary to Eq. (53).
Unfortunately, such a projection is not an ideal tool for finding the minimum distances in the product complex, as the distance may actually be reduced in some cases. The examples of such a reduction are based on Theorem 5 in the previous Section; it may happen for any finite field.

However, this reduction only concerns the minimum distances in tensor products of chain complexes after projection to one of the subspaces, it does not prevent the inequality (53) from being saturated. We conducted extensive numerical calculations finding homological distances for products of random \( \mathbb{F}_2 \)-linear chain complexes with \( q \in \{2, 3, 2^2, 5, 7, 2^3, 3^2, 11\} \) and space dimensions of up to 12, and an exhaustive enumeration of products of binary chain complexes with individual spaces of dimension up to 7. Yet we haven’t been able to find a single example of a pair of chain complexes whose product would fail to reach the upper bound (53). Combined with analytical results for multiple products of chain complexes involving just two spaces, we conjecture that in general, for any finite field \( F = \mathbb{F}_q \), the homological distances in a tensor product of a pair of bounded chain complexes of vector spaces over \( F \) satisfy the equality

\[
d_j(A \times B) = \min_{i \in \mathbb{Z}} d_i(A) d_{j-i}(B).
\]

(55)

B. Upper bound on the distance

**Statement 14.** Consider two \( F \)-linear chain complexes \( A = \mathcal{K}(A_1, \ldots, A_T) \) and \( B = \mathcal{K}(B_1, \ldots, B_{T'}) \). Then, for any \( i, j \in \mathbb{Z} \), the homological distance of the product complex \( C = A \times B \) at level \( j \) satisfies the inequality

\[
d_j(C) \leq d_i(A) d_{j-i}(B).
\]

(56)

**Proof.** By definition, the distances \( d_i(A) \) and \( d_{j-i}(B) \) are natural or infinite. Thus, if one or both homology groups are trivial, \( k_i(A) = 0 \) or \( k_{j-i}(B) = 0 \) (in which case the corresponding distance is infinite), the r.h.s. of Eq. (56) equals infinity, so that the inequality in question is trivially satisfied.

Otherwise, with both homology groups non-trivial, consider a pair of minimum-weight homologically non-trivial vectors \( a \in \mathcal{H}_i(A) \) and \( b \in \mathcal{H}_j(B) \) such that \( \text{wgt}(a) = d_i(A) \) and \( \text{wgt}(b) = d_j(B) \). Vector \( a \) is a non-trivial \( Z \)-like codeword in the stabilizer code \( \text{CSS}(A_i, A_{i+1}) \): denote \( a' \) an \( X \)-like codeword in the same code conjugate to \( a \), that is, \( a' \cdot a = 1 \). In other words, this vector is a co-cycle in \( A_i \). [In fact, \( a' \) is a member of the co-homology group \( H_i(\hat{A}) \), but this is not needed for the proof.] Similarly, denote \( b' \) an \( X \)-like codeword in the code \( \text{CSS}(B_{j-i}, B_{j-i+1}) \) conjugate to \( b \), a co-cycle in \( \hat{B}_j \). Construct \( c \in C_j \) by assigning non-zero value \( c_{i-j-i} = a \otimes b \) in the subspace \( A_i \otimes B_{j-i} \), and zero in all other subspaces at level \( j \). Clearly, \( \text{wgt}(c) = d_i(A) d_{j-i}(B) \); to prove the upper bound (56) we just need to show that \( c \cdot c' = 0 \). To this end, consider a vector \( c' \) constructed similarly to \( c \) but from vectors \( a' \) and \( b' \); it is easy to check that \( c \cdot c' = 1 \). In addition, this vector is a co-cycle in \( \hat{C}_j \), i.e., \( c' \in C_{j+1} \), where the matrix is a boundary operator in the product complex \( C \), cf. Eq. (22). Any vector equivalent to \( c \) has the form \( c + x(C_{j+1}) \), for some \( x \in C_{j+1} \). However, such a combination is never zero, as can be verified by taking a dot product with \( c' \).

The upper bound (53) immediately follows from Statement 14 by minimizing over \( i \).

C. Lower bounds on the distance

To make the map with the product codes in Sec. III evident, we start by writing out the block form of a matrix in the product complex \( C = A \otimes B \), where the spaces \( A_i \) and \( B_j \) have dimensions \( a_i \) and \( b_j \), respectively:

\[
C_j = \begin{pmatrix}
A_j \otimes I(b_0) & (-1)^j \cdot 1 \cdot I(a_{j-1}) \otimes B_1 \\
A_{j-1} \otimes I(b_1) & (-1)^{j-2} \cdot I(a_{j-2}) \otimes B_2 \\
& \ddots & \ddots & \ddots \\
& & & A_1 \otimes I(b_{j-1}) & I(a_0) \otimes B_j
\end{pmatrix}.
\]

(57)

For ease of mapping of the homology group \( H_j(C) \) to the CSS stabilizer code with generators \( H_X = C_j \) and \( H_Z = C_{j+1}^T \), we also write the latter matrix explicitly

\[
C_{j+1}^T = \begin{pmatrix}
A_{j+1}^T \otimes I(b_0) & (-1)^j I(a_j) \otimes B_1^T \\
& A_1^T \otimes I(b_1) & (-1)^{j-1} I(a_{j-1}) \otimes B_2^T \\
& & \ddots & \ddots & \ddots \\
& & & -I(a_1) \otimes B_{j-1}^T & A_1^T \otimes I(b_j) \\
& & & & I(a_0) \otimes B_{j+1}^T
\end{pmatrix}.
\]

(58)
Clearly, in general, the generator matrices \( H_X = C_j \) and \( H_Z = C_{j+1}^T \) have \( j + 1 \) column blocks, with each block row and block column incident on no more than two non-zero blocks. Our strategy is to construct bounds on the distance of these codes using Lemmas 3 and 4. Notice that for decomposition along the block boundaries, the condition in Lemma 3 can be verified by explicitly constructing bases of the product chain and product co-chain complexes, and using the K"unneth formula to make sure that no vectors are lost.

First, let us construct the codes \( Q_i^{(i,j-i)} \), \( i \in \mathbb{Z} \), each projected into a single subspace \( A_i \otimes B_{j-i} \) as in Lemma 1. The corresponding lower bound on the homological distance at the level \( j \) of the product complex \( C \) reads

\[
d_j(C) \geq \min_{i \in \mathbb{Z}} d_z \left( Q_i^{(i,j-i)} \right).
\]

Denote \( I \equiv I_1^j \) the index set corresponding to the subspace \( A_i \otimes B_{j-i} \) in \( C_j \). The punctured matrix \( G_Z[I] \) is obtained by selecting the appropriate column block in the matrix \( Q_i^{(i,j-i)} \). When expressed in terms of the two small stabilizer codes \( Q_A = \text{CSS}(A_i, A_i^T) \) and \( Q_B = \text{CSS}(B_{j-i}, B_{j-i}^T) \) associated with the homology groups \( H_i(A) \) and \( H_{j-i}(B) \), respectively, the resulting matrix has exactly the form of the gauge generator matrix \( G_Z \) in Eq. (26). To construct the matching shortened matrix \( (H_X)_I \), notice that only two row blocks in \( C_j \) give non-zero contribution,

\[
C_j[I_{j+1} \cup I_i^j \cup I_{i-1}^j] = \begin{pmatrix}
(-1)^i I(a_i) \otimes B_{j-i-1} & (1)^i I(a_i) \otimes B_{j-i} & (1)^i I(a_{i-1}) \otimes B_{j-i+1} \\
A_{i+1} \otimes I(b_{j-i-1}) & A_i \otimes I(b_{j-i}) & A_{i-1} \otimes I(b_{j-i+1})
\end{pmatrix}.
\]

The shortening to the middle column block, \( I_i^j \), is achieved with the help of row operations equivalent to left multiplication of the second row block by \( A^* \otimes I(b_{j-i-1}) \) and of the third row block by \( I(a_{i-1}) \otimes B^* \), where

\[
A^* = \begin{pmatrix}
A_i \\
L_X^i
\end{pmatrix}, \quad B^* = \begin{pmatrix}
B_{j-i} \\
L_X^i
\end{pmatrix}
\]

are the largest-rank matrices with rows orthogonal to the columns of \( A_{j+1} \) and \( B_{j-i+1} \), respectively. Here and below, we denote \( L_X^i \) and \( L_Z^j \) the canonical logical generator matrices of the same stabilizer codes, \( Q_A \) and \( Q_B \). As a result of the multiplication, we obtain the shortened matrix \( (H_X)_I \) in the exact form of the stabilizer generator matrix \( H_X \) in Eq. (24), again, when expressed in terms of the matrices associated with the codes \( Q_A \) and \( Q_B \). According to Lemma 3, the corresponding stabilizer code \( \text{CSS}(H_X[I], H_Z[I]) \) has exactly the same \( Z \)-distance as the subsystem code \( (H_X)[I], H_Z[I] \) obtained by puncturing both matrices \( H_X = C_j \) and \( H_Z = C_{j+1}^T \) to the single subspace \( A_i \otimes B_{j-i} \).

With the help of the upper bound \( 39 \) and the loose lower bound \( 41 \), we obtain

**Statement 15.** The \( Z \)-distance \( d_z \equiv d_z(Q_i^{(i,j-i)}) \) of the \( F \)-linear CSS code \( Q_i^{(i,j-i)} \) obtained by \( Z \)-puncturing the CSS code corresponding to homology group \( H_j(A \otimes B) \) to the subspace \( A_i \otimes B_{j-i} \) satisfies the bounds

\[
\max \{d_i(A), d_{j-i}(B)\} \leq d_z \leq d_i(A)d_{j-i}(B).
\]

Since \( d_0(A) \) and \( d_0(B) \) are restricted to be either zero or infinity, this gives exact values for the distance in two special cases:

\[
\begin{align*}
d_z(Q_i^{(j,0)}) &= d_i(A)d_0(B), \quad \text{(62)} \\
d_z(Q_i^{(0,j)}) &= d_0(A)d_j(B). \quad \text{(63)}
\end{align*}
\]

In addition, the structure of the homologically non-trivial vectors is somewhat clarified by the following restricted result:

**Statement 16.** Consider a vector \( c \in C_j \) at level \( j \) in the product chain complex \( C = A \times B \), and assume that for some \( i \leq j \), \( c \) has a non-zero weight in \( H_i(A) \otimes H_{j-i}(B) \), while the components of \( c \) are zero in spaces \( A_i \otimes B_{j-i} \) with \( i' < i \). Then \( \text{wt}(c) \geq d_i(A)d_{j-i}(B) \).

Proof. The vector is a \( Z \)-like codeword in the CSS code with generator matrices \( 37 \) and \( 35 \). The condition can be used to construct a \( Z \)-shortened code, with all blocks to the right of the block \( A_i \otimes B_{j-i} \) removed as in Lemma 3. This amounts to dropping all column blocks of \( C_j \) and \( C_{j+1}^T \) to the right of the \( (j-i+1) \)th block-column which corresponds to the subspace \( A_i \otimes B_{j-i} \), and multiplication of the last block-row that remains non-zero in \( C_{j+1}^T \) by \( (A^*)^T \otimes I(b_{j-i}) \), where \( A^* \) is given by Eq. (21). After a subsequent application of a \( Z \)-puncture, so that all block columns to the left of the block \( A_i \otimes B_{j-i} \) are removed as in Lemma 4, we obtain exactly the concatenated-stabilizer code in Theorem 1 constructed from \( Q_A = \text{CSS}(A_i, A_i^T) \) and \( Q_B = \text{CSS}(B_{j-i}, B_{j-i}^T) \). The \( Z \)-distance of this code is

\[
d_z = d_A^2d_B^0d_z = d_i(A)d_{j-i}(B).
\]

Moreover, by assumption, vector \( c \) punctured to the space \( A_i \otimes B_{j-i} \) is non-trivial in the product code, which guarantees \( \text{wt}(c) \geq d_z \). \( \blacksquare \)

Clearly, the same lower bound also applies for vectors with zero weight in all spaces \( A_{i'} \otimes B_{j-i'} \) with \( i' < j \). In
addition, the condition of Statement [16] is automatically satisfied when $B_{j-i}$ is the last non-trivial matrix in the complex $B$, i.e., $j - i = \ell'$, see Statement [14] In this case, again, the upper bound in Eq. (61) is saturated, $d_2(Q(i,\ell')) = d_i(A)d_{\ell'}(B)$. The same is true also when $A_i$ is the last non-trivial boundary operator in the complex $A$, $i = \ell$; we have $d_2(Q(\ell,j)) = d_\ell(A)d_j(B).

The special cases in Statements [15] and [16] combine to give exact distances in the case where one of the complexes in the product contains just one non-trivial boundary operator. This gives an extension of the main result in Ref. [80] to $F$-linear chain complexes:

**Theorem 17.** Consider a tensor product $C = A \times B$ of two $F$-linear chain complexes, where one of the complexes contains just two non-trivial spaces, e.g., $A = K(A_1, \ldots, A_j)$ and $B = K(B_k)$. Then, for any $j \in \mathbb{Z}$, the homological distance at level $j$ of the product complex $C = A \times B$ is

$$d_j(C) = \min_{i \in \mathbb{Z}} d_i(A)d_{j-i}(B). \quad (64)$$

In Ref. [36] we conjectured that in the binary case, $q = 2$, the identity (64) be applicable to products of arbitrary bounded complexes. The conjecture was based on extensive numerical simulations of products of length-three binary complexes corresponding to pairs of randomly generated CSS codes.

In addition, here we have conducted numerical simulations of product chain complexes based on pairs of random $F_q$-linear stabilizer codes, with all CSS generators of full-row-rank, so that in the corresponding chain complexes only the homology groups $H_1(A)$ and $H_1(B)$ be non-trivial. For each $q \in \{2, 3, 2^2, 5, 7, 2^3, 3^2, 11\}$, we generated some $2 \times 10^4$ such code pairs of length $3 \leq a_1 \leq b_1 \leq 11$, and calculated the homological distances $d_2(C)$ and $d_2(\tilde{C})$ of the corresponding (co)chain product complexes using a version of the covering set algorithm [71, 73]. Not a single instance was found where the inequality (64) would not be saturated.

Notice that our search went over a tiny fraction of all code pairs, in particular, since the number of codes (matrices) scales exponentially with the number of entries, i.e., super-exponentially with the matrix size. To ensure that we did not miss any instances, we also enumerated all pairs of non-trivial binary CSS codes of size $n \leq 7$, and constructed tensor products of the corresponding chain complexes. Eq. (63) was satisfied for all of these.

Based on these numerical results, combined with the analytical result in Theorem 17 and the results for multiple products of 1-complexes, see Sec. [1V.D] we propose

**Conjecture 18.** The homological distances $d_j(A \times B)$ in a product of any pair of bounded chain complexes of vector spaces over a finite field is given by Eq. (64).

Of course, one should be aware that, even when highly suggestive, numerical evidence cannot substitute a proof.

A recent example is the Hedetniemi conjecture about the chromatic number in a tensor product of graphs [74, 75]. The conjecture held up for over half a century; a counterexample was only recently discovered by Yaroslav Shitov in a beautiful 2019 paper [76, 77]. Significantly, the smallest graphs known so far to provide a counterexample to Hedetniemi’s conjecture have over $10^4$ vertices [78].

### D. Applications in quantum error correction

In classical error correction it is usually safe to assume a channel model, where errors may happen during transmission but not during encoding/decoding. In comparison, when a quantum error-correcting code (QECC) is used, errors may happen at any step; to measure a syndrome one has to perform a complex set of elementary measurements. Error measurement is going to be only one homology group associated with the transposed matrix $\tilde{K}$ of $K$. In particular, when $K = K(R)$ is a 1-complex associated with a circulant check matrix of the repetition code, $K^\times D$ recovers all the $D$-dimensional toric codes.

First, consider an $r \times c$ full-row-rank $q$-ary matrix $P$ with $r < c$, and assume that the $F$-linear code $C_P$ has distance $\delta$. The 1-complex $K \equiv K(P)$ has two non-trivial dimensions $r$ and $c$; the corresponding homology groups have ranks $0, \kappa$ and the distances $\infty, \delta$. The 1-complex $\tilde{K} \equiv K(P^\top)$ generated by the transposed matrix has equivalent spaces taken in the opposite order, with the same homology group ranks, but the distances are now $1$ and $\infty$, respectively. It is easy to see that in any chain complex constructed as tensor products of $K$ and/or $\tilde{K}$, there is going to be only one homology group with a non-zero rank. Since order of the products is not important, we will write these as powers. For $(a + b)$-complex $K^{(a,b)} \equiv K^{x_a} \times K^{x_b}$, the only non-trivial homology group is $H_d(K^{(a,b)})$, acting in the space of dimension

$$n_d(K^{(a,b)}) = \sum_{i=0}^{a} 2^{2i+2a-2i} \binom{a}{i} \binom{b}{i} < (r + c)^{a+b}.$$
it has rank \(\kappa^{a+b}\) and distance \(\delta^a\). The corresponding quantum CSS code has the minimum distance \(\min(\delta^a, \delta^b)\), and its stabilizer generators have weights not exceeding \((a + b)\max(v, \omega)\).

Good weight-limited classical LDPC codes with asymptotically finite rates \(\kappa/c\) and finite relative distances \(\delta/c\) can be obtained from ensembles of large random matrices \([24, 27, 79, 80]\). Any of these can be used in the present construction. Then, for any pair \((a, b)\) of natural numbers, we can generate weight-limited quantum LDPC codes with finite rates and the distances \(d_X = \delta^a\), \(d_Z = \delta^b\) whose product scales linearly with the code length. The quantum hypergraph-product codes are a special case of this construction with \(a = b = 1\).

More generally, take arbitrary \(r_i \times c_i\) matrices \(P_i\), \(i = 1, 2, \ldots\) with elements from \(F \equiv \mathbb{F}_q\). Let \(F\)-linear codes with parity check matrices \(P_i\) and \(P_i^T\), respectively, have parameters \([c_i, \kappa_i, \delta_i]\) and \([r_i, \tilde{\kappa}_i, \tilde{\delta}_i]\), where the distance is assumed infinite whenever the corresponding code is trivial, \(\kappa = 0\). Then, for a product of \(m\) such 1-complexes, the space dimensions and ranks of the homology groups following from the K"unneth formula can be written in terms of the generating polynomials

\[
\begin{align*}
\nu_{i}^{(m)} (x) &:= \nu_{i}^{(0)} (x) + x \nu_{i}^{(1)} (x) + \ldots + x^{m} \nu_{i}^{(m)} (x) = \prod_{j=1}^{m} (r_{j} + x c_{j}), \\
\kappa_{i}^{(m)} (x) &:= \kappa_{i}^{(0)} (x) + x \kappa_{i}^{(1)} (x) + \ldots + x^{m} \kappa_{i}^{(m)} (x) = \prod_{j=1}^{m} (\tilde{\kappa}_{j} + x \tilde{\delta}_{j}).
\end{align*}
\]

The homological distance \(d_j^{(m)}\) can be seen as the minimum over the products of distances corresponding to those terms that give non-zero contributions to \(k_j^{(m)}\), with the substitution \(\kappa_j \rightarrow \delta_j, 0 \neq \tilde{\kappa}_j \rightarrow 1\).

It is easy to check that none of the higher-dimensional quantum hypergraph-product codes discussed here have parameters that are better than for regular QHP codes \((m = 2)\) originally constructed by Tillich and Zémor [38]. In addition, the row- and column-weights of the corresponding matrices tend to get bigger with increasing \(m\). The advantage of higher-dimension QHP codes, or, more generally, codes from \(m\)-chain complexes with \(m \geq 4\), is that the rows of matrices \(G_{X}^{(m)} = K_a, G_{Z}^{(m)} = K_{a+1}\) satisfy a large number of linear relations resulting from the orthogonality with the matrices \(K_{a-1}\) and \(K_{a+2}\), respectively. These can be used to correct syndrome measurement errors. Even though the resulting syndrome codes do not have large distances (with a finite probability some errors remain), the use of such codes in repeated measurement setting could simplify the decoding and/or improve the decoding success probability in the case of adversarial noise [51]. Such improvements with stochastic noise have been demonstrated numerically in the case of \(D = 4\) toric codes in Ref. [81].

V. EXTENSIONS

Throughout this work, we concentrated on the Hamming distance. A simple, and yet offering a range of possible applications, extension of Theorem [3] and Theorem [17] can be given by using weighted distances, defined for a vector \(c \in F^n\) in terms of the norm

\[
\text{wgt}_W(c) = \sum_{i: c[i] \neq 0} W_i,
\]

where \(W = (W_1, W_2, \ldots, W_n)\) is a vector of positive weights \(W_i > 0, i \leq n\). For the corresponding proofs to work, the only requirement is that the weights \(W^{C_{i,j}}\) in each space \(C_{i,j} ^{\bigotimes A_i \otimes B_j}\) used to form the product complex \(C = A \times B\) be related to the weights \(W^{A_i}\) and \(W^{B_j}\) in the original complexes, namely, \(W^{C_{i,j}} = W^{A_i} \otimes W^{B_j}\). Indeed, all the proofs are based either on Eq. [25], or a projection inequality as in Eq. [30]: both arguments are readily modified to account for weighted norm [65].

In particular, this implies an extension to extremal length \(L_1\) (systole) and higher-dimensional analogs \(L_j, j > 1\), representing minimal structures with non-trivial homology on a given manifold [82]. Indeed, in the simplest case, the edge \((j = 1)\), plaquette \((j = 2)\), etc. weights associated with a given tessellation can be chosen as the corresponding Euclidean length, area, etc. Then the weighted norm \(65\) gives the corresponding measure of the elements in the structure, and the homological distance—the corresponding minimum, going over to \(L_j\) in the continuum limit. We assume the manifolds be sufficiently smooth so that the corresponding limits exist [83].

Second, an extension of some of the bounds to chain complexes of \(K\)-modules, modules over a commuting ring \(K\), is possible if \(K\) is a principal ideal domain (PID). Here we only consider the ring \(K = \mathbb{Z}_q\) of modular integers, and assume torsion-free case, i.e., with all Smith normal form invariants of all matrices either zero or one. In this case one gets \([34] d_Z^2 \geq d_Z^2 d_X^2\) for the stabilizer-product code in Theorem [4]. Further, the lower bound in Theorem [5] remains intact, while Eq. [61] also becomes an inequality, \(d_j(C) \geq \min_{i \in \mathbb{Z}} d_i(A) d_j-i(B)\).

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[1] T. Kaczynski, K. Mischaikow, and M. Mrozek, Computing homology, Homology, Homotopy and Applications 5, 233 (2003).

[2] T. Kaczynski, K. Mischaikow, and M. Mrozek, Computational homology, Applied Mathematical Sciences, Vol. 157 (Springer-Verlag, New York, 2004).
A. Gonzalez-Lorenzo, A. Bac, J.-L. Mari, and P. Real, Two Measures for the Homology Groups of Binary Volumes, in 19th IAPR International Conference on Discrete Geometry for Computer Imagery, Vol. 9647, edited by Springer (Nantes, France, 2016) pp. 154–165.

B. Audoux and A. Couvreur, On tensor products of CSS codes, arXiv:1512.07081 (2015), to appear in Ann. Inst. Henri Poincaré D.

T. K. Dey, T. Li, and Y. Wang, Efficient algorithms for computing a minimal homology basis, in LATIN 2018: Theoretical Informatics, edited by M. A. Bender, M. Farach-Colton, and M. A. Mosteiro (Springer International Publishing, Cham, 2018) pp. 376–398.

H. Bombin and M. A. Martin-Delgado, Homological error correction: Classical and quantum codes, Journal of Mathematical Physics 48, 062105 (2007).

S. S. Bullock and G. K. Brennen, Qudit surface codes and gauge theory with finite cyclic groups, Journal of Physics A: Mathematical and Theoretical 40, 3481 (2007).

A. R. Calderbank and P. W. Shor, Good quantum error-correcting codes exist, Phys. Rev. A 54, 1098 (1996).

A. M. Steane, Simple quantum error-correcting codes, Phys. Rev. A 54, 4739 (1996).

S. Bravyi and A. Y. Kitaev, Quantum codes on a lattice with boundary, quant-ph/9811052 (1998), unpublished.

M. H. Freedman and D. A. Meyer, Projective plane and planar quantum codes, Foundations of Computational Mathematics 1, 325 (2001) quant-ph/9810055.

E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, Topological quantum memory, J. Math. Phys. 43, 4452 (2002).

C. Castelnovo and C. Chamon, Topological order in a three-dimensional toric code at finite temperature, Phys. Rev. B 78, 155120 (2008).

D. Mazić and A. Hamma, Topological order, entanglement, and quantum memory at finite temperature, Annals of Physics 327, 2096 (2012).

H. Bombin, R. W. Chhajlany, M. Horodecki, and M. A. Martin-Delgado, Self-correcting quantum computers, New Journal of Physics 15, 055023 (2013).

A. Y. Kitaev, Fault-tolerant quantum computation by anyons, Ann. Phys. 303, 2 (2003).

S. Bravyi and B. Terhal, A no-go theorem for a two-dimensional self-correcting quantum memory based on stabilizer codes, New Journal of Physics 11, 043029 (2009).

S. Bravyi, D. Poulin, and B. Terhal, Tradeoffs for reliable quantum information storage in 2D systems, Phys. Rev. Lett. 104, 050503 (2010) 0909.5200.

N. Delfosse, Tradeoffs for reliable quantum information storage in surface codes and color codes, in Information Theory Proceedings (ISIT), 2013 IEEE International Symposium on (IEEE, 2013) pp. 917–921.

S. T. Flammia, J. Haah, M. J. Kastoryano, and I. H. Kim, Limits on the storage of quantum information in a volume of space, Quantum 1, 4 (2017) 1610.06169.

A. A. Kovalev and L. P. Pryadko, Fault tolerance of quantum low-density parity check codes with sublinear distance scaling, Phys. Rev. A 87, 020304(R) (2013).

I. Dumer, A. A. Kovalev, and L. P. Pryadko, Thresholds for correcting errors, erasures, and faulty syndrome measurements in degenerate quantum codes, Phys. Rev. Lett. 115, 050502 (2015) 1412.0172.

D. Gottesman, Fault-tolerant quantum computation, Quantum Information and Computation 14, 1338 (2014) 1310.2984.

R. Gallager, Low-density parity-check codes, IRE Trans. Inf. Theory 8, 21 (1962).

R. G. Gallager, Low-Density Parity-Check Codes (M.I.T. Press, Cambridge, Mass., 1963).

S. Litsyn and V. Shevelev, On ensembles of low-density parity-check codes: asymptotic distance distributions, IEEE Trans. Inf. Theory 48, 887 (2002).

T. J. Richardson, M. A. Shokrollahi, and R. L. Urbanke, Design of capacity-approaching irregular low-density parity-check codes, Information Theory, IEEE Transactions on 47, 619 (2001).

G. Zémor, On Cayley graphs, surface codes, and the limits of homological coding for quantum error-correcting, in Coding and Cryptology: Second International Workshop, IWCC 2009, edited by Y. M. Chee, C. Li, S. Ling, H. Wang, and C. Xing (Springer, Berlin, Heidelberg, 2009) pp. 259–273.

N. Delfosse and G. Zémor, Quantum erasure-correcting codes and percolation on regular tilings of the hyperbolic plane, in Information Theory Workshop (ITW), 2010 IEEE (2010) pp. 1–5.

N. P. Breuckmann and B. M. Terhal, Constructions and noise threshold of hyperbolic surface codes, IEEE Trans. on Inf. Th. 62, 3731 (2016) 1506.04209.

N. P. Breuckmann, C. Vuillot, E. Campbell, A. Krishna, and B. M. Terhal, Hyperbolic and semi-hyperbolic surface codes for quantum storage, Quantum Science and Technology 2, 035007 (2017).

L. Guth and A. Lubotzky, Quantum error correcting codes and 4-dimensional arithmetic hyperbolic manifolds, Journal of Mathematical Physics 55, 082202 (2014) arXiv:1310.5555.

J.-P. Tillich and G. Zémor, Quantum LDPC codes with positive rate and minimum distance proportional to $\sqrt{n}$, in Proc. IEEE Int. Symp. Inf. Theory (ISIT) (2009) pp. 799–803.

J.-P. Tillich and G. Zémor, Quantum LDPC codes with positive rate and minimum distance proportional to the square root of the blocklength, IEEE Transactions on Information Theory 60, 1193 (2014).

A. A. Kovalev and L. P. Pryadko, Quantum Kroncker sum-product low-density-parity-check codes with finite rate, Phys. Rev. A 88, 012311 (2013).

W. Zeng and L. P. Pryadko, Higher-dimensional quantum hypergraph-product codes with finite rates, Phys. Rev. Lett. 122, 230501 (2019) 1810.01519.

A. Couvreur, N. Delfosse, and G. Zémor, A construction of quantum LDPC codes from Cayley graphs, CoRR abs/1206.2656 (2012) arXiv:1206.2656.

S. Bravyi and M. B. Hastings, Homological product codes, arXiv:1311.0885 (2013), unpublished.

B. Audoux, An application of Khovanov homology to quantum codes, Ann. Inst. Henri Poincare 1, 185–223 (2014).

M. B. Hastings, Quantum codes from high-dimensional manifolds, arXiv:1608.05089 (2016), unpublished.

M. B. Hastings, Weight reduction for quantum codes,
Quantum codes over $\mathbb{Z}_q$ and $q$-state Potts models (2020), unpublished.