NON-TRAPPING MAGNETIC FIELDS AND MORREY-CAMPANATO ESTIMATES FOR SCHRÖDINGER OPERATORS

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Abstract. We prove some uniform in $\epsilon$ a priori estimates for solutions of the equation

$$\left(\nabla - iA\right)^2 u - V(x)u + (\lambda \pm i\epsilon)u = f, \quad \lambda \geq 0, \quad \epsilon \neq 0.$$  

The estimates are obtained in terms of Morrey-Campanato norms, and can be used to prove absence of zero-resonances, in a suitable sense, for electromagnetic Hamiltonians. Precise conditions on the size of the trapping component of the magnetic field and the non repulsive component of the electric field are given.

1. Introduction

In space dimension $n \geq 3$, let us consider the electromagnetic Schrödinger operator

$$H = -(\nabla - iA(x))^2 + V(x); \quad (1.1)$$

here $A = (A^1, \cdots, A^n) : \mathbb{R}^n \to \mathbb{R}^n$ is the magnetic potential, and $V : \mathbb{R}^n \to \mathbb{R}$ is the electric potential. We denote by

$$\nabla A = \nabla - iA, \quad \Delta A = \Delta^2 A.$$ 

In the theory of electromagnetic fields, a deep literature has been produced on the study of electromagnetic Schrödinger Hamiltonians (1.1). There are indeed a lot of interesting problems related to the properties of solutions of stationary and evolutive equations described by these operators. The magnetic potential $A$ is a mathematical construction which describes the interaction of particles with an external magnetic field. The vector field $A$ is standardly associated to a 1-form, whose differential $B := dA$ is the magnetic field, which is a physical object. We can define analytically $B$ as the $n \times n$ anti-symmetric matrix

$$B = DA - (DA)^t, \quad (DA)_{ij} = \frac{\partial A^i}{\partial x^j}, \quad (DA)^t_{ij} = (DA)_{ji}.$$ 

In dimension $n = 3$, the magnetic field $B$ is identified as $B = \text{curl} A$, due to the isomorphism between 1-forms and 2-forms; this fact has to be interpreted in terms of the action

$$Bv = \text{curl} A \times v, \quad \text{for all } v \in \mathbb{R}^3.$$
where the cross is the vectorial product on $\mathbb{R}^3$. We will always consider smooth potentials $A, V \in C^1$; actually, it is possible to study the validity of the results of this paper for rough potentials, but it is not in our aims. Moreover, in what follows, we always assume:

**Assumption 1.1.** the Hamiltonian $H$ is self-adjoint on $L^2(\mathbb{R}^n)$, with form-domain

$$D(H) = \{ f \in L^2(\mathbb{R}^n) : \int |\nabla A f|^2 + \int |V| \cdot |f|^2 < \infty \}.$$

Assumption 1.1 has several consequences: the spectrum $\sigma(H)$ is real, and via Spectral Theorem we can perform the functional calculus $g(H)$, for any Borel-measurable function $g$. In particular, by the powers of the operator $H$ we can define the distorted Sobolev norms

$$\|f\|_{H^s} = \|H^s f\|_{L^2}.$$

The validity of Assumption 1.1 requires local integrability conditions on $A, V$, and the literature about it is complete. For details, see the Leinfelder-Simader result in [14] and the book [6].

The aim of this paper is to prove uniform (in $\epsilon$) a priori estimates for solutions of the resolvent equation

$$-Hu(x) + (\lambda \pm i\epsilon)u(x) = f(x), \quad \lambda \geq 0, \quad \epsilon \neq 0 \quad (1.2)$$

by direct techniques based on integration by parts. In the purely electric case $A \equiv 0$, we shall mention [17] as inspirator of this multipliers technique (actually the subject there is the Helmholtz equation, and the role of $V$ is played by the rarefraction index $n(x)$). Since $\lambda \pm i\epsilon \notin \mathbb{R}$, for any $f$ in $L^2$ there exists a unique $u \in L^2$ solution of (1.2).

The integration by parts gives very precise informations about the relevant quantities (related to the electromagnetic field) which play a role in the spectral properties of $H$. It is of particular interest the part concerning the magnetic potential $A$. Let us give the following definition.

**Definition 1.2 (non trapping magnetic fields).** Let us define by $B_\tau : \mathbb{R}^n \to \mathbb{R}^n$ the tangential component of the magnetic field $B$, given by

$$B_\tau(x) := \frac{x}{|x|} B.$$

Observe that in dimension $n = 3$ it coincides with

$$B_\tau(x) := \frac{x}{|x|} \times \text{curl}A(x).$$

We say that $B$ is non-trapping if $B_\tau = 0$.

The quantity $B_\tau$ was introduced in [9], in which it is proved that weak-dispersion for the magnetic Schrödinger and wave equation holds, for example, for non-trapping potentials. Indeed, a smallness condition on $B_\tau$ is sufficient there to prove that some aspects of the free dynamics are preserved in presence of this kind of fields. This is also what happens in the stationary case, as we prove later in our main theorems. We give some examples of non-trapping fields (see also [9]), in dimension $n = 3$. 

Example 1.3. Let us take
\[ A = \frac{1}{x^2 + y^2 + z^2}(-y, x, 0) = \frac{1}{x^2 + y^2 + z^2}(x, y, z) \times (0, 0, 1). \] (1.3)
One can easily check that
\[ \nabla \cdot A = 0, \quad B = -2z(x^2 + y^2 + z^2)^{-\frac{3}{2}}(x, y, z), \quad B_\tau = 0. \]
Another (more singular) example is the following:
\[ A = \left( \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0 \right) = \frac{1}{x^2 + y^2}(x, y, z) \times (0, 0, 1). \] (1.4)
Here we have \( B = (0, 0, \delta) \), with \( \delta \) denoting Dirac’s delta function. Again we have \( B_\tau = 0 \).

Example 1.4. A natural generalization of the previous examples is the following one. Assume that \( B = \text{curl} A : \mathbb{R}^3 \to \mathbb{R}^3 \) is known; if we fix the Coulomb gauge \( \text{div} A = 0 \), then \( A \) can be obtained by the Biot-Savart formula
\[ A(x) = \frac{1}{4\pi} \int \frac{x - y}{|x - y|^3} \times B(y) \, dy. \] (1.5)
Let us assume \( B_\tau = 0 \), namely \( x \times B(x) = 0 \); by (1.5) we have
\[ A(x) = \frac{x}{4\pi} \times \int \frac{B(y)}{|x - y|^3} \, dy. \] (1.6)
Consequently, for the condition \( B_\tau = 0 \) it is necessary \( B(y) = g(y)\frac{y}{|y|} \), for some \( g : \mathbb{R}^3 \to \mathbb{R} \). Since we want \( A \neq 0 \), \( g \) has not to be radial. For example we consider
\[ g(y) = h \left( \frac{y}{|y|} \cdot \omega \right) |y|^{-\alpha}, \]
for some fixed \( \omega \in S^2 \), where \( h \) is homogeneous of degree 0 and \( \alpha \in \mathbb{R} \); as a consequence, the vector field \( B \) is homogeneous of degree \( -\alpha \). By (1.6) we have
\[ A(x) = \frac{x}{4\pi} \times \int \frac{h \left( \frac{y}{|y|} \cdot \omega \right)}{|x - y|^3} |y|^{-\alpha} \, dy. \] (1.7)
The potential \( A \) is homogenous of degree \( 1 - \alpha \), and by symmetry we have that \( A(\omega) = 0 \). These examples can be easily extended to higher dimensions.

Before stating the main theorems, we need to introduce some notations. For \( f : \mathbb{R}^n \to \mathbb{C} \) we define the Morrey-Campanato norm as
\[ |||f|||^2 = \sup_{R > 0} \frac{1}{R} \int_{|x| \leq R} |f|^2 \, dx. \]
Moreover, we denote by \( C(j) = \{ x \in \mathbb{R}^n : 2^j \leq |x| \leq 2^{j+1} \} \),
\[ N(f) = \sum_{j \in \mathbb{Z}} \left( 2^{j+1} \int_{C(j)} |f|^2 \, dx \right)^{\frac{1}{2}}, \]
and we easily notice the duality relation
\[ \int fg \, dx \leq |||g||| \cdot N(f). \]
For any $p \geq 1$, we also define
\[ \|f\|_{L^p_r L^\infty(S_r)} = \left( \int_0^{+\infty} \sup_{|x|=r} |f(x)|^p \, dr \right)^{\frac{1}{p}}. \]

We are now ready to state our main results.

**Theorem 1.5 (3D-Morrey-Campanato estimates).** Let $n = 3$; let us assume that
\begin{align*}
\| |x|^{\frac{3}{2}} B_r \|_{L^p_r L^\infty(S_r)} &= C_1 < \infty \quad (1.8) \\
\| |x|^2 (\partial_r V)_+ \|_{L^p_r L^\infty(S_r)} &= C_2 < \infty \quad (1.9) \\
\| \langle x \rangle^{-1} |x|^2 V_+ \|_{L^p_r L^\infty(S_r)} &= C_3 < \infty, \quad (1.10)
\end{align*}
and moreover there exists $M \geq 0$ such that
\begin{equation}
\left( \frac{M + \frac{1}{2}}{M} \right)^2 C_1^2 + 2 \left( M + \frac{1}{2} \right) C_2 < 1. \quad (1.11)
\end{equation}
Assume, moreover, that $V$ satisfies the Hardy-type condition
\begin{equation}
\int |V| \cdot |u|^2 \, dx \leq C \int |\nabla_A u|^2 \, dx, \quad (1.12)
\end{equation}
for some $C > 0$. Then, any solution $u \in H^1$ of equation (1.2) satisfies the following a priori estimates:
\begin{equation}
\| \nabla_A u \|^2 + |u(0)|^2 + \frac{M}{2} \int (\partial_r V)_- |u|^2 \\
+ \delta \left( \int \langle x \rangle^{-1} V_- |u|^2 + \lambda \int \frac{|u|^2}{\langle x \rangle} + \int \frac{|\nabla_A u|^2}{|x|} + \sup_{R>0} \frac{1}{R^2} \int_{|x|=R} |u|^2 \, d\sigma \right) \\
\leq C \left[ N(f)^2 + (|\epsilon| + \lambda) \left( N \left( \frac{f}{|\lambda|^{1/2}} \right) \right)^2 \right],
\end{equation}
for some $C > 0$ and some small $\delta > 0$ depending on $C_1, C_2, C_3, M$.

**Theorem 1.6 (Higher-dimensional Morrey-Campanato estimates).** Let $n \geq 4$ and; let us assume that
\begin{align*}
\| |x|^2 B_r \|_{L^\infty} &= C_1 < \infty \quad (1.14) \\
\| |x|^3 (\partial_r V)_+ \|_{L^\infty} &= C_2 < \infty \quad (1.15) \\
\| \langle x \rangle^{-1} |x|^3 V_+ \|_{L^\infty} &= C_3 < \infty, \quad (1.16)
\end{align*}
and moreover
\begin{equation}
C_1^2 + 2C_2 < (n - 1)(n - 3). \quad (1.17)
\end{equation}
Assume, moreover, that $V$ satisfies the Hardy-type condition
\begin{equation}
\int |V| \cdot |u|^2 \, dx \leq C \int |\nabla_A u|^2 \, dx, \quad (1.18)
\end{equation}
for some $C > 0$. Then, any solution $u \in \mathcal{H}^1$ of equation (1.2) satisfies the following a priori estimates:

$$|||\nabla_A u|||^2 + \sup_{R > 0} \left( \frac{1}{R^2} \int_{|x|=R} |u|^2 d\sigma \right) + \int (\partial_r V)_- |u|^2$$

$$+ \delta \left( \int \langle x \rangle^{-1} V_- |u|^2 + \lambda \int \frac{|u|^2}{\langle x \rangle} + \int \frac{|\nabla_A u|^2}{|x|} + \int \frac{|u|^2}{|x|^3} dx \right)$$

$$\leq C \left[ N(f)^2 + (|\epsilon| + \lambda) \left( N \left( \frac{f}{|\lambda|^{1/2}} \right) \right)^2 \right],$$

for some $C > 0$ and some small $\delta > 0$ depending on $C_1, C_2, C_3, M$.

Let us make some remarks about the statements of Theorems 1.5, 1.6 and their possible applications.

**Remark 1.1.** Estimates (1.13) and (1.19) recover the uniform (with respect to $\epsilon$) estimate in the main Theorem of [17], in the purely electric case $A \equiv 0$ (actually the refraction index $n(x)$ there plays the role of our electric potential $V$). In fact, here we have some gain in the term involving $\lambda$ at the left-hand side (analogous to the term $|||n^{1/2}u|||^2$ in the main Theorem by [17]), which is due to an appropriate choice of the symmetric multiplier $\varphi$ (see Section 3 in the following).

**Remark 1.2 (Assumptions on the electromagnetic field).** Let us give an interpretation of assumptions (1.11), (1.17). Observe the difference on the decay and singularity informations about $A, V$, between the 3D case and the higher dimensional case. Indeed, in dimension $n = 3$, potentials behaving like $|A| = C/|x|$, $|V| = C/|x|^2$ are not allowed, while assumptions (1.8), (1.9), (1.10) are satisfied by potentials with these behaviors

$$|A| \leq \frac{C}{|x|^{1-\epsilon} + |x|^{1+\epsilon}}, \quad |V| \leq \frac{C}{|x|^{2-\epsilon} + |x|^{2+\epsilon}},$$

with $C > 0$, and according with the smallness of $B_\tau$ and $(\partial_r V)_+$ required by (1.11). In fact, potentials with critical decay and singularity are permitted by the higher dimensional assumptions (1.14), (1.15), (1.16) and (1.17).

Moreover, notice that the size of $C_3$ is not relevant, both in (1.11) and (1.17); indeed, no smallness assumption on $V$ is needed in order to obtain estimates (1.13), (1.19). In the 3D case, assume that $C_1 = 0$, i.e. the field $B$ is non-trapping, according to Definition 1.2; hence, since $\min_{M \geq 0} 2(M + 1/2) = 1$, condition (1.11) simply reads

$$C_2 < 1.$$

On the other hand, if we assume $C_2 = 0$, in other words $V$ is repulsive, since $\min_{M \geq 0} (M + 1/2)^2/M = 2$, the condition on $C_1$ is

$$C_1^2 < \frac{1}{2}.$$

We claim that (1.11) is in fact sharp; it would be interesting to find counterexamples to estimate (1.13), with potentials satisfying (1.8), (1.9) and (1.10), but not satisfying (1.11).
Observe also that no assumptions on $A$ (except for the self-adjointness) are in the statement of Theorems 1.5, 1.6; hence the gauge invariance of these results is preserved.

Remark 1.3 (Hardy conditions on $V$). The Hardy-type conditions (1.12), (1.18) have to be interpreted by means of the magnetic Hardy inequality
\[ \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^2} \, dx \leq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx, \] (1.20)
which holds in dimension $n \geq 3$ on any function $u \in H^1$ (see [9] for a simple proof of (1.20) by integration by parts).

Remark 1.4 (Absence of resonances). One of the possible applications of Theorems 1.5 and 1.6 is to prove absence of zero-energy resonances for the Hamiltonian $H$. Actually the right definition of resonances is not completely clear, see e.g. [1], [2], [10], [11], [12], [13], [15], [19]. In fact, in the study of dispersive equations related to $H$, as the magnetic Schrödinger equation
\[ iu_t + Hu = 0, \]
or the magnetic wave equation
\[ u_{tt} + Hu = 0, \]
a typical abstract assumption of absence of zero-energy resonances is needed, in order to preserve the free dynamics (see e.g. the recent papers [4], [5], [7], [8], [20]). By the statement of Theorems 1.5 and 1.6 it is natural to consider the following definition of zero-energy resonances, introduced in [3]:

Definition 1.7. A function $u$ is a zero-resonance if
\[ u \notin L^2, \quad u \in H^1_{\text{loc}}, \quad |V|^\frac{1}{2}u \in L^2, \]
\[ \sup_{R>1} \frac{1}{R} \int_{|x| \leq R} \left( |V| + \langle x \rangle^{-2} \right) |u|^2 < \infty, \]
\[ \liminf_{R \to \infty} \frac{1}{R} \int_{|x| \leq R} \left( |V| + \langle x \rangle^{-2} \right) |u|^2 = 0, \]
and $u$ satisfies the equation
\[ -Hu = 0. \]

It is possible to see that Theorems 1.5 and 1.6 imply the absence of zero-energy resonances, according to the previous Definition. Indeed, one should repeat the proof of Lemma 2.1 (see Section 2) by performing the integration by parts on compact balls of $\mathbb{R}^n$, taking into account the boundary terms, which in fact turn out with correct signs. We omit here further details (for completeness we remand to [3], final section). Actually, we also remark that in [9] it is proved that, under assumptions of type (1.11), (1.17), weakly dispersive estimates and Strichartz estimates are true for the magnetic Schrödinger and wave equation; in that paper, no abstract assumptions on the Hamiltonian (namely absence of zero-resonances) are needed. Observe that our assumptions on the term $B_\tau$ does not appear in [10], in which First order perturbations of $-\Delta$ are also treated.
The rest of the paper is devoted to the proofs of the main theorems. These are based on the Morawetz-type Lemma 2.1, which is proved in the next section; then a suitable choice of the multipliers (last section) completes the proofs.

2. Integration by parts

In this Section we state and prove Lemma 2.1, which is our fundamental tool for the proof of the main theorems. It is based on the standard technique of Morawetz multipliers, introduced in [15] for the Klein-Gordon equation and then used in several other contests (dispersive equations, kinetic equations, Helmholtz equations ecc...). We should mention here [17], as a seminal work about the relation between Morawetz methods and Morrey-Campanato estimates for the Helmholtz equation. Later, in [3], [9] it was shown as these techniques can be adopted to prove some weak-dispersive estimates for Schrödinger and wave equations with electric and electromagnetic potentials.

We prove the following Lemma, which will be used to prove the main theorems.

Lemma 2.1. Let \( \phi(|x|), \psi(|x|) \) be two radial, real-valued multipliers and let \( u \in H^1 \) be a solution of equation (1.2). Then, the following identity holds:

\[
\int \nabla A u D^2 \phi \nabla A u dx - \int \varphi |\nabla A u|^2 dx - \int \left( \frac{1}{4} \Delta^2 \phi - \frac{1}{2} \Delta \varphi \right) |u|^2 dx \tag{2.1}
\]

\[
- \int \left[ \frac{1}{2} \varphi' \left( \partial_r V \right) + \varphi V \right] |u|^2 dx + \Im \int \varphi' u B \cdot \nabla A u dx + \Re \int \varphi |u|^2 dx
\]

\[
= \Re \int f \varphi dx + \Re \int f \phi dx \pm \epsilon \Im \int u \nabla \phi \cdot \nabla A u dx,
\]

where \( D^2 \phi, \Delta^2 \phi \) denote, respectively, the Hessian and the bi-Laplacian of \( \phi \), while \( B_r \) is as in Definition 1.2.

Proof. We divide the proof into two parts, acting on equation (1.2) with a symmetric multiplier first, and then with an anti-symmetric one.

**Symmetric multiplier.** Let us multiply equation (1.2) by \( \varphi u \) in the \( L^2 \)-sense; taking the resulting real parts, and observing that

\[
- \Re (Hu, \varphi u)_{L^2} = - \int \varphi |\nabla A u|^2 dx + \frac{1}{2} \int \Delta \varphi |u|^2 dx - \int \varphi V |u|^2 dx,
\]

it gives the identity

\[
- \int \varphi |\nabla A u|^2 dx + \frac{1}{2} \int \Delta \varphi |u|^2 dx - \int \varphi V |u|^2 dx + \lambda \int \varphi |u|^2 dx \tag{2.2}
\]

\[
= \Re \int f \varphi dx.
\]

On the other hand, the imaginary parts give

\[
\pm \epsilon \int \varphi |u|^2 dx = \Im \int f \phi dx. \tag{2.3}
\]
Anti-symmetric multiplier. Let us multiply equation (1.2) by
\[ \frac{1}{2}[H, \phi]u = \nabla \phi \cdot \nabla_A u + \frac{1}{2}(\Delta \phi)u, \]
in the sense of \( L^2 \). It gives
\[
-\frac{1}{2}(Hu, [H, \phi]u)_{L^2} + \frac{\lambda}{2}(u, [H, \phi]u)_{L^2} \pm \frac{i}{2}(u, [H, \phi]u)_{L^2} = (f, [H, \phi]u)_{L^2}.
\]
(2.4)

Now we take the real part of identity (2.4). First observe that, since the commutator \([H, \phi]\) is anti-symmetric, we have
\[
\Re(u, [H, \phi]u)_{L^2} = 0,
\]
(2.5)

\[
\pm \Re \frac{\epsilon}{2}(u, [H, \phi]u)_{L^2} = \mp \epsilon \Im \int u \nabla \phi \cdot \nabla_A u dx.
\]
(2.6)

For the same reason, we see immediately that
\[
-\frac{1}{4} \Re(Hu, [H, \phi]u)_{L^2} = -\frac{1}{4}([H, [H, \phi]]u, u)_{L^2}.
\]
(2.7)

The explicit computation of the second commutator \([H, [H, \phi]]\) has been already performed in [9]; this is the point in which the trapping component \(B_\tau\) appears. By see formulas (1.13) and (2.3) in [9] we obtain
\[
-\frac{1}{4}([H, [H, \phi]]u, u)_{L^2} = \int \nabla_A u D^2 \phi \nabla_A u dx - \frac{1}{4} \int |u|^2 \Delta^2 \phi dx
\]
\[
- \frac{1}{2} \int \phi' \partial_r V |u|^2 dx + \Im \int \phi' u B_\tau \cdot \nabla_A u dx.
\]
(2.8)

We remark that the idea of the computation (2.8) in [9] is to use the Leibnitz formula for \( \nabla_A \) in the form \( \nabla_A (fg) = (\nabla_A f)g + (\nabla g)f \); hence we can put all the distorted derivatives on the solution and the straight derivatives on the multiplier.

Finally, by (2.4), (2.5), (2.6), (2.7) and (2.8) we obtain the following identity:
\[
\int \left( \nabla_A u D^2 \phi \nabla_A u - \frac{1}{4}|u|^2 \Delta^2 \phi - \frac{1}{2} \phi' |\partial_r V||u|^2 \right) dx + \Im \int \phi' u B_\tau \cdot \nabla_A u dx
\]
\[
= \Re \int f \left( \nabla \phi \cdot \nabla_A u + \frac{1}{2}(\Delta \phi)u \right) dx \pm \epsilon \Im \int u \nabla \phi \cdot \nabla_A u dx.
\]
(2.10)

Now identity (2.1) follows by summing up (2.2) with (2.10). The following regularity remark completes the proof.

Remark 2.1. We must notice that the term requiring more regularity on \( u \), in order to justify the integration by parts, is the one involving second commutator \([H, [H, \phi]]\). In principle, it requires \( u \in D(H^2) \) to make sense; actually, the integration by parts on it shows that a term of the form \( \int \Delta u \nabla \phi \cdot \nabla_A u \) needs to be a priori bounded, and \( u \in H^3 \) is sufficient. The proof of identity (2.1) for \( H^1 \)-solutions follows by approximation on \( f \). Indeed, if \( f \in D(H^s) \), \( s \geq 0 \), and \( \epsilon \neq 0 \), there exists a unique solution \( u \in D(H^{s}) \) of (1.2); now the density of \( C_0^\infty \) in \( D(H^s) \) completes the argument.

\[ \square \]
3. Proof of the main Theorems 1.5, 1.6

We pass now the the proofs of our main theorems. These are based on identity (2.1), by suitable choices of the multipliers $\phi, \varphi$. Our choice of the multipliers follows an idea introduced in [3], and then used in [9] with explicit definitions. The multipliers are analogous, in dimensions $n = 3, n \geq 4$, but give different results and conditions on the potentials (see Remark 1.2).

3.1. Proof of Theorem 1.5

We denote by $r = |x|$; following [9], we define $\phi_0$ as

$$
\phi_0(x) = \int_0^x \phi_0'(s) \, ds,
$$

where

$$
\phi_0'(r) = \begin{cases} 
M + \frac{1}{3} R, & r \leq 1 \\
M + \frac{1}{2} \frac{1}{6r^2}, & r > 1,
\end{cases}
$$

and $M$ is given by assumption (1.11). We have

$$
\phi_0''(r) = \begin{cases} 
\frac{1}{3}, & r \leq 1 \\
\frac{1}{3}, & r > 1
\end{cases}
$$

and the bilaplacian is given by

$$
\Delta^2 \phi_0(r) = -4\pi \delta_{x=0} - \delta_{|x|=1},
$$

in the distributional sense. By scaling, for any $R > 0$ we define

$$
\phi_R(r) = R \phi_0\left(\frac{r}{R}\right),
$$

hence

$$
\phi_R'(r) = \begin{cases} 
M + \frac{r}{3R}, & r \leq R \\
M + \frac{1}{2} \frac{R^2}{6r^2}, & r > R
\end{cases}
$$

(3.1)

$$
\phi_R''(r) = \begin{cases} 
\frac{1}{3R}, & r \leq R \\
\frac{1}{R}, & r > R
\end{cases}
$$

(3.2)

$$
\Delta \phi_R(r) = \begin{cases} 
\frac{1}{R} + \frac{2M}{r}, & r \leq R \\
\frac{1}{R} + \frac{2M}{r}, & r > R
\end{cases}
$$

(3.3)

$$
\Delta^2 \phi_R(r) = -4\pi \delta_{x=0} - \frac{1}{R^2} \delta_{|x|=R}.
$$

(3.4)

Observe that $\phi_R', \phi_R'', \Delta \phi_R \geq 0$ and moreover

$$
\sup_{r \geq 0} \phi_R'(r) \leq M + \frac{1}{2}, \quad \sup_{r \geq 0} \phi_R''(r) \leq \frac{1}{3R}, \quad \sup_{r \geq 0} \Delta \phi_R \leq \frac{1 + 2M}{r},
$$

(3.5)

$$
\inf_{r \geq 0} \phi_R'(r) \geq M.
$$

(3.6)

In fact, this choice of $\phi_R$ had been made in the reverse way; we started from the bi-laplacian, which contains the term $\delta_{r=R}$ and seems to optimize the size condition (1.11), as we see in the following.

Now we define $\varphi_R$ as follows:

$$
\varphi_R(r) = \begin{cases} 
\frac{\beta}{R}, & r \leq R \\
\frac{\beta}{R}, & r > R
\end{cases}
$$

(3.7)
for some $\beta < \frac{1}{3}$ to be chosen later. The reason of the bound $1/3$ for $\beta$ will be clear in Section 3.1.2. Observe that

$$C_0(x)^{-1} \varphi_R(r) \leq C(x)^{-1}, \quad (3.8)$$

for some $C = C(\beta) > 0$, and $C_0 = C_0(\beta) > 0$ such that $C, C_0 \to 0$ as $\beta \to 0$. Here $\langle x \rangle = (1 + |x|^2)^{1/2}$. By a direct computation we obtain

$$\Delta \varphi_R = -\frac{\beta}{R^2} \delta_{|x|=R}, \quad (3.9)$$

which is true in the distributional sense.

Let us now put the multipliers $\varphi_R, \varphi_R$ in identity (2.1) and begin to estimate. We start with the estimate of the right-hand side.

3.1.1. **Estimate of the RHS in (2.1).** By (3.5) and Cauchy-Schwartz, we have

$$\left| \int f \nabla \varphi \cdot \nabla_A u \right| \leq \left( M + \frac{1}{2} \right) \sum_{j \in \mathbb{Z}} \int_{C(j)} |f| \cdot |\nabla_A u| \quad (3.10)$$

$$\leq \left( M + \frac{1}{2} \right) \sum_{j \in \mathbb{Z}} \left( 2^{-j} \int_{C(j)} |\nabla_A u|^2 \right)^{\frac{1}{2}} \left( 2^{j+1} \int_{C(j)} |f|^2 \right)^{\frac{1}{2}}$$

$$\leq \left( M + \frac{1}{2} \right) \left( \sup_{R > 0} \frac{1}{R} \int_{|x| \leq R} |\nabla_A u|^2 \right)^{\frac{1}{2}} \sum_{j \in \mathbb{Z}} \left( 2^{j+1} \int_{C(j)} |f|^2 \right)^{\frac{1}{2}}$$

$$\leq \alpha \||\nabla_A u||^2 + C(\alpha)N(f)^2,$$

with $\alpha, C(\alpha) > 0$. Analogously, by (3.5) and (3.7),

$$\left| \int f \left( \frac{1}{2} \Delta \varphi + \varphi \right) u \right| \leq \left( \frac{1}{2} + M + \beta \right) \sum_{j \in \mathbb{Z}} \int_{C(j)} |f| \cdot \frac{|u|}{|x|} \quad (3.11)$$

$$\leq \left( \frac{1}{2} + M + \beta \right) \sum_{j \in \mathbb{Z}} \left( 2^{-j} \int_{C(j)} \frac{|u|^2}{|x|^2} \right)^{\frac{1}{2}} \left( 2^{j+1} \int_{C(j)} |f|^2 \right)^{\frac{1}{2}}$$

$$\leq \left( \frac{1}{2} + M + \beta \right) \left( \sup_{R > 0} \frac{1}{R^2} \int_{|x| = R} |u|^2 d\sigma \right)^{\frac{1}{2}} \sum_{j \in \mathbb{Z}} \left( 2^{j+1} \int_{C(j)} |f|^2 \right)^{\frac{1}{2}}$$

$$\leq \alpha \sup_{R > 0} \frac{1}{R^2} \int_{|x| = R} |u|^2 d\sigma + C(\alpha)N(f)^2.$$

It remains now to estimate the last term at the RHS of (2.1). Observe that, multiplying (1.2) by $u$ in the $L^2$-sense and taking the resulting imaginary parts, we get (see identity (2.3))

$$\epsilon \int |u|^2 dx \leq \int |fu| dx. \quad (3.12)$$

On the other hand, taking the real parts we obtain (see identity (2.2))

$$\int |\nabla_A u|^2 = -\int V|u|^2 + \lambda \int |u|^2 - \Re \int fu.$$
Hence by assumption (1.12) we have
\[ \int |\nabla_A u|^2 \leq C \left( \lambda \int |u|^2 + \int |fu| \right), \quad (3.13) \]
As a consequence of (3.12) and (3.13), by (3.5) we can estimate
\[ \left| \epsilon \int u \nabla \phi \cdot \nabla_A u \right| \leq C|\epsilon|^{1/2} \left( |\lambda| \int |u|^2 + \int |fu| \right)^{\frac{1}{2}} \left( \int |fu| \right)^{\frac{1}{2}} \quad (3.14) \]
\[ \leq C|\epsilon|^{1/2} \int |fu| + C \left( |\epsilon\lambda| \int |fu| \right)^{\frac{1}{2}} \leq C(|\epsilon| + |\lambda|)^{\frac{1}{2}} \int |fu| \]
\[ \leq C(|\epsilon| + |\lambda|)^{\frac{1}{2}} \cdot ||u||_1 ||\lambda||_1 \cdot N \left( \frac{f}{|\lambda|^{1/2}} \right) \]
\[ \leq \alpha ||u||_1 ||\lambda||_1^2 + C(\alpha)(|\epsilon| + |\lambda|) \left( N \left( \frac{f}{|\lambda|^{1/2}} \right) \right)^2, \]
for \( \alpha, C(\alpha) > 0 \). In conclusion, by (3.10), (3.11) and (3.14), for the right-hand side of (2.1) we have
\[ |\Re \int f \left( \nabla \phi \cdot \nabla_A u + \frac{1}{2}(\Delta \phi) u \right) dx + \Re \int f \phi u dx \pm \epsilon \Im \int u \nabla \phi \cdot \nabla_A u dx | \]
\[ \leq \alpha \left( ||\nabla_A u||_2^2 + ||u||_1 ||\lambda||_1^2 \right)^2 + \sup_{R > 0} \frac{1}{R^2} \int_{|x|=R} |u|^2 d\sigma \]
\[ + C(\alpha) \left[ N(f)^2 + (|\epsilon| + |\lambda|) \left( N \left( \frac{f}{|\lambda|^{1/2}} \right) \right)^2 \right], \]
for arbitrary \( \alpha > 0 \).

Our next step is to prove the positivity of the left-hand side of (2.1).

3.1.2. Positivity of the LHS in (2.1). Let us consider the first term. Since \( \phi_R \) is radial, we can exploit the formula
\[ \nabla_A u D^2 \phi_R \nabla_A u = \phi''_R |\nabla_A u|^2 + \phi'_R |\nabla_A u|^2, \quad (3.16) \]
where \( \nabla_A u = \nabla_A u \cdot x/|x| \) denotes the radial component of the distorted gradient and \( |\nabla_A u| \) the modulus of the tangential component, i.e.
\[ \nabla_A^r u \cdot \nabla_A u = 0, \quad |\nabla_A^r u|^2 = |\nabla_A u|^2 - |\nabla_A^r u|^2. \]
By (3.16), (3.5) and (3.7), since \( \beta < 1/3 \) we estimate
\[ \int \nabla_A u D^2 \phi_R \nabla_A u - \int \phi_R |\nabla_A u|^2 \geq M \int \frac{|\nabla_A^r u|^2}{|x|} + \frac{1 - 3\beta}{3} \frac{1}{R} \int_{|x| \leq R} |\nabla_A u|^2. \quad (3.17) \]
For the third term, by (3.15) and (3.9) we have
\[ \int \left( -\frac{1}{4} \Delta^2 \phi_R + \frac{1}{2} \Delta \phi_R \right) |u|^2 dx = \pi |u(0)|^2 + \frac{1 - 2\beta}{4R^2} \int_{|x| = R} |u|^2 d\sigma, \quad (3.18) \]
and again this is a positive term. Now we pass to the terms containing $\partial_\tau V$ and $B_\tau$. First observe that, by splitting $\partial_\tau V = (\partial_\tau V)_+ - (\partial_\tau V)_-$ and using (3.5), (3.6), we obtain

$$-\frac{1}{2} \int \phi'_R(\partial_\tau V)|u|^2 \geq \frac{M}{2} \int (\partial_\tau V)_-|u|^2 - \frac{2M + 1}{4} \int (\partial_\tau V)_+|u|^2$$

(3.19)

$$\geq \frac{M}{2} \int (\partial_\tau V)_-|u|^2 - \frac{2M + 1}{4} \int_0^\infty dp \int_{|x| = \rho} (\partial_\tau V)_+|u|^2 d\sigma$$

$$\geq \frac{M}{2} \int (\partial_\tau V)_-|u|^2 - \frac{2M + 1}{4} \sup_{R > 0} \left( \frac{1}{R^2} \int_{|x| = R} |u|^2 d\sigma \right) \| |x|^{-1}(\partial_\tau V)_+ \|_{L^1 L^\infty(S_\tau)}.$$  

Analogously, by (3.8) we have

$$-\int \varphi_R V|u|^2 \geq C_0(\beta) \int \langle x \rangle^{-1}V_-|u|^2 - C(\beta) \int \langle x \rangle^{-1}V_+|u|^2$$

(3.20)

$$\geq C_0(\beta) \int \langle x \rangle^{-1}V_-|u|^2 - C(\beta) \sup_{R > 0} \left( \frac{1}{R^2} \int_{|x| = R} |u|^2 d\sigma \right) \| \langle x \rangle^{-1}|x|^2V_+ \|_{L^1 L^\infty(S_\tau)}.$$  

The term containing $B_\tau$ does not have sign in principle; hence, noticing that

$$|B_\tau \cdot \nabla_A u| = |B_\tau| \cdot |\nabla_A u|,$$

since $B_\tau$ is a tangential vector, we estimate

$$\Im \int_{\mathbb{R}^n} u \phi'_R B_\tau \cdot \nabla_A u \, dx \geq -\frac{2M + 1}{2} \int_{\mathbb{R}^n} |u| \cdot |B_\tau| \cdot |\nabla_A u| \, dx$$

(3.21)

$$\geq -\frac{2M + 1}{2} \left( \int \frac{|\nabla_A u|^2}{|x|} \right)^{\frac{1}{2}} \left( \int_0^{+\infty} dp \int_{|x| = \rho} |x| \cdot |u|^2 \cdot |B_\tau|^2 d\sigma \right)^{\frac{1}{2}}$$

$$\geq -\frac{2M + 1}{2} \left( \int \frac{|\nabla_A u|^2}{|x|} \right)^{\frac{1}{2}} \left( \sup_{R > 0} \frac{1}{R^2} \int_{|x| = R} |u|^2 d\sigma \right)^{\frac{1}{2}} \| |x|^{-\frac{3}{2}}B_\tau \|_{L^2 L^\infty(S_\tau)}.$$  

We are ready now to sum (3.17), (3.18), (3.19), (3.20), and (3.21). Due to the freedom on the choice of $R$ we can take the supremum over $R$ in (3.17), (3.18). In order to simplify the reading, let us introduce the following notations:

$$a := \left( \int \frac{|\nabla_A u|^2}{|x|} \right)^{\frac{1}{2}}; \quad b := \left( \sup_{R > 0} \frac{1}{R^2} \int_{|x| = R} |u|^2 d\sigma \right)^{\frac{1}{2}}.$$  

Moreover, according to assumption (1.11), we denote

$$C_1 := \| |x|^{-\frac{3}{2}}B_\tau \|_{L^2 L^\infty(S_\tau)};$$

$$C_2 := \| |x|^2(\partial_\tau V)_+ \|_{L^1 L^\infty(S_\tau)};$$

$$C_3 := \| \langle x \rangle^{-1}|x|^2V_+ \|_{L^1 L^\infty(S_\tau)}.$$  

Hence we have obtained
\[
\begin{align*}
\int \nabla_A u D^2 \phi_R \nabla_A u - \int \phi_R |\nabla_A u|^2 + \int \left( -\frac{1}{4} \Delta^2 \phi_R + \frac{1}{2} \Delta \phi_R \right) |u|^2 \\
- \int \left[ \frac{1}{2} \phi_R' (\partial_r V) + \varphi_R V \right] |u|^2 + 3 \int_{\mathbb{R}^n} u \phi_R' B_r \cdot \nabla_A u
\end{align*}
\]
\[
\geq \frac{1 - 3\beta}{3} \sup_{R > 0} \left( \frac{1}{R} \int_{|x| \leq R} |\nabla_A u|^2 \right) + \pi |u(0)|^2
\]
\[
+ \frac{M}{2} \int (\partial_r V)_{-} |u|^2 + C_0(\beta) \int \langle x \rangle^{-1} V_{-} |u|^2
\]
\[
+ Ma^2 - \frac{2M + 1}{2} C_1 a b + \frac{1}{4} [1 - 2\beta - (2M + 1)C_2 - 4C(\beta)C_3] b^2.
\]
Then we need to prove that
\[
+Ma^2 - \frac{2M + 1}{2} C_1 a b + \frac{1}{4} [1 - 2\beta - (2M + 1)C_2 - 4C(\beta)C_3] b^2 > 0,
\]
for any \(a, b\). By homogeneity, it is sufficient to prove that
\[
+Ma^2 - \frac{2M + 1}{2} C_1 a b + \frac{1}{4} [1 - 2\beta - (2M + 1)C_2 - 4C(\beta)C_3] > 0,
\]
for any \(a\). Since \(\beta\) is arbitrary in the definition (3.7) of \(\varphi\), we can choose \(\beta \in (-\gamma, \gamma)\), for \(\gamma > 0\) arbitrarily small. As a consequence also the constant \(C(\beta)\) is arbitrarily small (see (3.8)). Hence we can neglect the terms containing \(\beta\), \(C(\beta)\), and (3.23) is satisfied if
\[
\frac{(M + \frac{1}{2})^2}{M} C_1^2 + 2 \left( M + \frac{1}{2} \right) C_2 < 1,
\]
which in fact coincides with (1.11). In conclusion, we have proved that, under assumption (1.11),
\[
\begin{align*}
\int \nabla_A u D^2 \phi_R \nabla_A u - \int \phi_R |\nabla_A u|^2 + \int \left( -\frac{1}{4} \Delta^2 \phi_R + \frac{1}{2} \Delta \phi_R \right) |u|^2 \\
- \int \left[ \frac{1}{2} \phi_R' (\partial_r V) + \varphi_R V \right] |u|^2 + 3 \int_{\mathbb{R}^n} u \phi_R' B_r \cdot \nabla_A u + \lambda \int \varphi_R |u|^2
\end{align*}
\]
\[
\geq \frac{1 - 3\beta}{3} \sup_{R > 0} \left( \frac{1}{R} \int_{|x| \leq R} |\nabla_A u|^2 \right) + \pi |u(0)|^2
\]
\[
+ \frac{M}{2} \int (\partial_r V)_{-} |u|^2 + C_0(\beta) \int \langle x \rangle^{-1} V_{-} |u|^2
\]
\[
+ \delta \left( \int \frac{|\nabla_A u|^2}{|x|} \right) + \sup_{R > 0} \frac{1}{R^2} \int_{|x| = R} |u|^2 d\sigma + C_0(\beta) \lambda \int \frac{|u|^2}{\langle x \rangle} \geq 0,
\]
if \(\lambda \geq 0\), for a sufficiently small \(\delta > 0\) depending on \(B_r, (\partial_r V)_+\).

At this point, the proof of Theorem 1.5 is complete by (3.15) and (3.25), up to choose \(\alpha\) in (3.15) sufficiently small; actually one needs to notice that trivially
\[
|\langle \lambda^{\frac{1}{2}} u \rangle| \leq \lambda \int \frac{|u|^2}{\langle x \rangle}.
\]
3.2. Proof of Theorem 1.6. The proof in dimension \( n \geq 4 \) is completely analogous to the 3D case. We first define the following multipliers:

\[
\phi_0(x) = \int_0^x \phi_0'(s) \, ds,
\]

where

\[
\phi_0' = \phi_0'(r) = \begin{cases} 
M + \frac{1}{2n} - \frac{1}{2n^{n-1}r}, & r \leq 1 \\
M + \frac{1}{2} - \frac{1}{2n^{n-1}r}, & r > 1,
\end{cases}
\]

and \( M > 0 \) is now an arbitrary constant. Observe that \( \phi_0 \) coincides exactly with the one introduced in the 3D proof. Again, by scaling we define

\[
\phi_R(r) = R\phi_0\left(\frac{r}{R}\right),
\]

and by direct computations we obtain

\[
\phi_R'(r) = \phi_0' \left( \frac{r}{R} \right) = \begin{cases} 
M + \frac{1}{2n} - \frac{r}{R^n}, & r \leq R \\
M + \frac{1}{2} - \frac{R^{n-1}}{2n^{n-1}r}, & r > R,
\end{cases}
\]

\[\phi_R''(r) = \phi_0'' \left( \frac{r}{R} \right) = \begin{cases} 
\frac{n-1}{2n} \cdot \frac{1}{R^n}, & r \leq R \\
\frac{n-1}{2n} \cdot \frac{1}{r^n}, & r > R;
\end{cases}\]

\[\Delta \phi_R(r) = \begin{cases} 
\frac{n-1}{2n} \cdot \frac{M(n-1)}{r}, & r \leq R \\
\frac{n-1}{2n} \cdot \frac{M(n-1)}{r^3} \chi(R,\infty), & r > R;
\end{cases}\]

moreover, the bi-laplacian gives now

\[
\Delta^2 \phi_R(r) = -\frac{n-1}{2R^2} \delta_{|x|=R} - M \frac{(n-1)(n-3)}{r^3} \chi_{[0,R]} - \left( M + \frac{1}{2} \right) \frac{(n-1)(n-3)}{r^3} \chi(R,\infty),
\]

in the distributional sense, where \( \chi \) denotes the characteristic function. Observe that also here the bi-laplacian is negative; the terms involving the characteristic functions turn out to be crucial in view to improve the 3D condition (1.11) in (1.17). Moreover let us notice that, as in 3D case, \( \phi_R', \phi_R'', \Delta \phi_R \geq 0 \) and

\[
\sup_{r \geq 0} \phi_R'(r) \leq M + \frac{1}{2}, \quad \sup_{r \geq 0} \phi_R''(r) \leq \frac{n-1}{2nR}, \quad \sup_{r > 0} \Delta \phi_R(r) \leq \frac{(2M + 1)(n-1)}{2r},
\]

\[
\inf_{r \geq 0} \phi_R' \geq M.
\]

As in (3.7), we define

\[
\varphi_R(r) = \begin{cases} 
\frac{\beta}{R}, & r \leq R \\
\frac{\beta}{r}, & r > R
\end{cases}
\]

for some \( \beta < (n-1)/2n \). Obviously (3.8) is still true. Moreover we have

\[
\Delta \varphi_R(r) = -\frac{\beta}{R^2} \delta_{|x|=R} - \frac{\beta(n-3)}{r^3} \chi(R,\infty).
\]

From now on the proof is almost the same as in the 3D case.
3.2.1. Estimate of the RHS in (2.1). This stuff is identical as in subsection 3.1.1. Actually, with the same argument, by (3.30), (3.8) and assumption (1.18) we obtain (3.15), exactly as in the 3D case. We omit further details.

3.2.2. Positivity of the LHS in (2.1). Here we have a difference with respect to the 3D case. Indeed, the two terms involving the characteristic functions in (3.29) have to be exploited in order to get positivity with optimal conditions on the potentials.

Let us start again by formula (3.16); by this, (3.26), (3.27) and (3.32) we easily see that

\[ \int \nabla A u D^2 \varphi_R \nabla A u - \int \varphi_R |\nabla A u|^2 \geq M \int \frac{\nabla^r u^2}{|x|} + \left( \frac{n-1}{2n} - \beta \right) \cdot \frac{1}{R} \int_{|x| \leq R} \nabla A u^2, \]

for any \( R > 0 \). This term is positive since \( \beta < (n-1)/2n \). By (3.29) and (3.33) we get

\[ \frac{1}{4} \Delta^2 \varphi_R - \frac{1}{2} \Delta \varphi_R = \Delta \left( \frac{1}{4} \Delta \varphi_R - \frac{1}{2} \varphi_R \right) = - \frac{n-1-4\beta}{8} \cdot \frac{1}{R^2} \delta_{|x|=R} - \frac{M(n-1)(n-3)}{4r^3} \chi_{[0,R]} \]

\[ - (2M+1)(n-1)(n-3) - 4\beta(n-3) \]

As a consequence

\[ \int \left( -\frac{1}{4} \Delta^2 \varphi_R + \frac{1}{2} \Delta \varphi_R \right) |u|^2 \geq \frac{n-1-4\beta}{8R^2} \int_{|x|=R} |u|^2 d\sigma + \left( \frac{M(n-1)(n-3)}{4} - K(\beta) \right) \int \frac{|u|^2}{|x|^3} dx, \]

with \( 0 \leq K(\beta) \to 0 \) as \( \beta \to 0 \); this term is positive, up to choose \( \beta \) small enough. As in the previous case, we now observe that, by (3.30)

\[ - \frac{1}{2} \int \phi_R' (\partial_r V) |u|^2 \geq M \int (\partial_r V)_- |u|^2 - \frac{2M+1}{4} \int (\partial_r V)_+ |u|^2 \]

\[ \geq M \int (\partial_r V)_- |u|^2 - \frac{2M+1}{4} \| |x|^3 (\partial_r V)_+ \|_{L^\infty} \int \frac{|u|^2}{|x|^3} dx, \]

\[ - \int \varphi_R V |u|^2 \geq C_0(\beta) \int \langle x \rangle^{-1} V_- |u|^2 - C(\beta) \int \langle x \rangle^{-1} V_+ |u|^2 \]

\[ \geq C_0(\beta) \int \langle x \rangle^{-1} V_- |u|^2 - C(\beta) \| \langle x \rangle^{-1} |x|^3 V_+ \|_{L^\infty} \int \frac{|u|^2}{|x|^3} dx. \]

With a similar computation, for the term involving \( B_r \) we estimate

\[ \Im \int_{\mathbb{R}^n} u \varphi_R' B_r \cdot \nabla_A u dx \geq - \frac{2M+1}{2} \int_{\mathbb{R}^n} |u| \cdot |B_r| \cdot |\nabla^r u| dx \]

\[ \geq - \frac{2M+1}{2} \left( \int \frac{|\nabla^r u|^2}{|x|} \right)^{\frac{1}{2}} \left( \int \frac{|u|^2}{|x|^3} \right)^{\frac{1}{2}} \| |x|^2 B_r \|_{L^\infty}. \]
Now we can sum (3.34), (3.36), (3.37), (3.38) and (3.39), taking the supremum over $R$; we denote by

$$
a := \left( \int \frac{|\nabla u|^2}{|x|} \right)^{\frac{1}{2}}, \quad b := \left( \int \frac{|u|^2}{|x|^3} \right)^{\frac{1}{2}},
$$

and according to assumption (1.17)

$$\| |x|^2 B_\tau \|_{L^\infty} \leq C_1,$$

$$\| |x|^3 (\partial_r V) \|_{L^\infty} \leq C_2,$$

$$\| \langle x \rangle^{-1} |x|^3 V_+ \|_{L^\infty} := C_3 < \infty.$$

We obtain

$$\int \nabla_A u D^2 \phi_R \nabla_A u - \int \varphi_R |\nabla_A u|^2 + \int \left( -\frac{1}{4} \Delta^2 \phi_R + \frac{1}{2} \Delta \varphi_R \right) |u|^2 \quad (3.40)$$

$$- \int \left[ \frac{1}{2} \phi_R' (\partial_r V) + \varphi R V \right] |u|^2 + 3 \int u \phi_R B_\tau \cdot \nabla_A u \geq \left( \frac{n-1}{2n} - \beta \right) \||\nabla_A u||^2 + \frac{n-1-4\beta}{8} \sup_{R>0} \left( \frac{1}{R^2} \int_{|x|=R} |u|^2 \, d\sigma \right)$$

$$+ \frac{M}{2} \int (\partial_r V)_{-} |u|^2 + C_0(\beta) \int \langle x \rangle^{-1} V_{-} |u|^2$$

$$+ M a^2 - \left( M + \frac{1}{2} \right) C_1 a b$$

$$+ \frac{1}{4} [M(n-1)(n-3) - (2M + 1)C_2 - 4C(\beta)C_3 - 4K(\beta)] b^2.$$

It remains to prove that

$$M a^2 - \left( M + \frac{1}{2} \right) C_1 a b$$

$$+ \frac{1}{4} [M(n-1)(n-3) - (2M + 1)C_2 - 4C(\beta)C_3 - 4K(\beta)] b^2 > 0,$$

for any $a, b$. Again, by homogeneity it is sufficient to show that

$$M a^2 - \left( M + \frac{1}{2} \right) C_1 a$$

$$+ \frac{1}{4} [M(n-1)(n-3) - (2M + 1)C_2 - 4C(\beta)C_3 - 4K(\beta)] > 0,$$

for any $a$. This is satisfied if

$$\frac{1}{(n-1)(n-3)} \left[ \left( \frac{M + \frac{1}{2}}{M} \right)^2 C_1 + 2 \left( \frac{M + \frac{1}{2}}{M} \right) C_2 \right] < 1.$$
and the infimum is reached in the limit as $M \to \infty$. Since $M$ is arbitrary in the definition of $\phi_R$ we can optimize in terms of $C_1, C_2$, and conclude that the last condition is

$$C_1^2 + 2C_2 < (n-1)(n-3), \tag{3.41}$$

which is in fact assumption (1.17). In conclusion, assumption (1.17) implies that

$$\int \nabla_A u D^2 \phi_R \nabla_A u - \int \varphi_R |\nabla_A u|^2 + \int \left( -\frac{1}{4} \Delta^2 \phi_R + \frac{1}{2} \Delta \varphi_R \right) |u|^2 \geq \int \left[ \frac{1}{2} \phi_R' (\partial_r V) + \varphi_R V \right] |u|^2 + \Im \int_{\mathbb{R}^n} u \varphi_R B_\tau \cdot \nabla_A u + \lambda \int \varphi_R |u|^2$$

$$\geq \left( \frac{n-1}{2n} - \beta \right) ||| \nabla_A u |||^2 + \sup_{R>0} \left( \frac{1}{R^2} \int_{|x|=R} |u|^2 d\sigma \right)$$

$$\int (\partial_r V)_- |u|^2 + C_0(\beta) \int \langle x \rangle^{-1} V_- |u|^2$$

$$+ \delta \left( \int \frac{|\nabla_A u|^2}{|x|} + \int \frac{|u|^2}{|x|^3} dx \right) + C_0(\beta) \lambda \int \frac{|u|^2}{\langle x \rangle} \geq 0,$$

if $\lambda \geq 0$, for a sufficiently small $\delta > 0$ depending on $B_\tau, (\partial_r V)_+$. The proof of Theorem 1.6 is complete by (3.15) and (3.42), up tho choose $\alpha > 0$ sufficiently small in (3.15).

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