Characteristics of Eigenvalues Realized by Path-Connected Sets of Matrices

Alex Kokot, Charles Johnson

1 Introduction

It is a well known fact [4] that a continuous path in the space of $n \times n$ complex matrices $A(x) : \mathbb{R} \rightarrow M_n$ induces continuous eigenvalue paths in the parameter $x$. In particular, we can choose $n$ of these paths so that each of them corresponds to an eigenvalue of $A(x)$.

Figure 1: The eigenvalues realized by three different matrix paths.

These paths will be our primary tool to study the eigen-surface associated with a set of matrices $S$, denoted $ES(S) \subseteq \mathbb{C}^{k+1}$. If $S$ is contained in the span of the matrices $A_1, A_2, \ldots, A_k$, then we can define

$$ES(S) := \{ (\alpha_1, \alpha_2, \ldots, \alpha_k, \lambda) \mid A \in S, \lambda \in \sigma(A), A = \alpha_1 A + \alpha_2 A_2 + \cdots + \alpha_k A_k \}.$$  

We define $\sigma^*(A) := (\alpha_1, \ldots, \alpha_k) \times \sigma(A)$ to distinguish elements in this higher dimensional space, and we will refer to its elements as $\lambda(A) \in \sigma^*(A)$. For eigenpaths we write $\lambda(A(x))$ indicating that for all $x$, $\lambda(A(x)) \in \sigma^*(A(x))$.

Figure 2: The eigen-surface of the convex-hull of three matrices projected into $\mathbb{C}$ (left) and represented in 3 dimensions (right).

We suppress our choice of spanning matrices in this notation as any choice will generate an eigensurface that is equivalent up to homeomorphism. We will think of this set as the collection of all eigenpaths realized by $S$. This object has particular significance in physics as it can be thought of as the potential energy surface realized by a particular atomic configuration space [2]. However, this application is typically restricted to the convex hull of Hermitian matrices, whereas here we will loosen this restriction and consider arbitrary matrices. Working in this more general setting will allow us to gain additional insight on this problem, as well as apply this research to other important questions such as the eigenvalues realized by $DS_n$, the doubly-stochastic matrices. These matrices play an important role in probability and modeling, and they relate to this work as they can be represented as the convex hull of the permutation matrices [3].

\[1\] Here we use $\sigma(A)$ to refer to the set of eigenvalues of $A \in M_n$. When discussing an eigenvalue of a particular matrix we will often shorten the notation to $\lambda(A)$. 

\[\]
2 The Eigen-Surface

Let $S \subseteq M_n$ be path-connected. We call the path-components of $ES(S)$ $k$-components, $\mathcal{K}$.

**Lemma 2.1.** Let $\mathcal{K}$ be a path-component of $ES(S)$. If $A \in S$ is such that there exists eigenvalues $\lambda_1(A), \ldots, \lambda_k(A)$ where $\lambda_i(A) \in \mathcal{K}$ for all $i \in [k]$, then there are $k$ such eigenvalues in $\mathcal{K}$ for any matrix $B \in S$.

**Proof.** For arbitrary $A \in S$, and any path in $S$ from $A$, there must be an eigenpath corresponding to each $\lambda_i(A) \in \mathcal{K}$, thus every matrix $B \in S$ must have at least as many eigenvalues as $A$ in $\mathcal{K}$. Similarly, as $S$ is path connected, for any other matrix $B \in S$ we can take a path from $B$ to $A$, thus $A$ have at least as many eigenvalues in $\mathcal{K}$ as $B$ does. Thus every matrix in $S$ must have the same number of eigenvalues, up to algebraic multiplicity, in $\mathcal{K}$. □

**Corollary 2.2.** $ES(S)$ can be partitioned into distinct path-components $\mathcal{K}_1, \ldots, \mathcal{K}_\ell$ where $\sum_{k=1}^\ell k_i = n$.

For $A, B \in S$, $\lambda(A) \sim \lambda(B)$ if $\lambda(A), \lambda(B) \in \mathcal{K}$ for some $k$-component $\mathcal{K} \subseteq ES(S)$. We say that these eigenvalues are paired. In particular, this means that there is a path $A(x) \in S$ for all $x$, $A(0) = A$, $A(1) = B$ and an eigenpath $\lambda(A(x))$ where $\lambda(A(0)) = \lambda(A)$, $\lambda(A(1)) = \lambda(B)$. We now prove a result which will be critical for the primary result of this paper.

**Theorem 2.3.** Let $S \subseteq M_n$ be path-connected and compact. There exists a neighborhood $S_\delta$ of $S$ such that $ES(S_\delta)$ has the same maximal $k$-components as $S$, that is, for $A, B \in S$, if $\lambda(A) \not\sim \lambda(B)$ in $S$ then $\lambda(A) \not\sim \lambda(B)$ in $S_\delta$.

**Proof.** First we will show that for $i \neq j$, there exists $\varepsilon_i, \varepsilon_j > 0$ such that for neighborhoods $\mathcal{K}_{\varepsilon_i}, \mathcal{K}_{\varepsilon_j}$ of $\mathcal{K}_i, \mathcal{K}_j$ respectively,

$$\mathcal{K}_{\varepsilon_i} \cap \mathcal{K}_{\varepsilon_j} = \emptyset.$$  

We argue by contradiction. Suppose not, then there must exist a sequence $(\lambda(A_n), \lambda(B_n)) \in \mathcal{K}_i \times \mathcal{K}_j$ where

$$\lim_{n \to \infty} (\lambda(A_n), \lambda(B_n)) = (\lambda(A), \lambda(A)).$$

By compactness of $S$, we see that there must be a matrix $A \in S$ for which this holds. Now, for each $k$-component, we can define the continuous function

$$f(A) : M_n \to \mathbb{R}^k \times M_n,$$

$$f(A) := \left( \max_{i \in [k]} |\lambda_i(A)|, \ldots, \min_{i \in [k]} |\lambda_i(A)|, A \right)$$

where $\lambda_1(A), \ldots, \lambda_k(A)$ are the eigenvalues in the $k$-component. We see that $f$ is continuous and as $S$ is compact, so $f(S)$ is also compact. We see then that $\lambda(A) \in \mathcal{K}_i, \mathcal{K}_j$, as the matrix $A$ is a limit point for $f(S)$. However, then $\mathcal{K}_i = \mathcal{K}_j$, contradicting that they are distinct path components. We see then that there exists such a neighborhood separating each pair of disjoint $k$-components, and as there are only finitely many, the intersection of these neighborhoods is open, completing our proof. □

**Corollary 2.4.** If $S$ is path-connected and compact then the path-components of $ES(S)$ are compact and components, that is, all distinct path-components are separable.

3 Transitivity

We say $S$ is transitive if $\lambda_i(A) \sim \lambda_j(A) \implies \lambda_i(A) = \lambda_j(A)$. Further, if $S$ has the property for any $A, B \in S$ and $\lambda \in \sigma(A)$ with algebraic multiplicity $k$, there exists $\mu \in \sigma(B)$ where $\mu$ has algebraic multiplicity $k$, we say that $S$ has unifrom multiplicity.

**Lemma 3.1.** For a matrix $A \in M_n$, there exist a neighborhood $N \subseteq M_n$ of $A$ and neighborhoods $D_i \subseteq \mathbb{C}$ of the eigenvalues $\lambda_i(A)$ such that for all matrix paths $P(x)$ contained fully within $N$, any eigenpath of $P(x)$ is contained entirely within one $D_i$. 

Proof. If $A$ has only one distinct eigenvalue, then take any $\varepsilon > 0$ and choose $\delta > 0$ such that $\|A - B\| < \delta$ gives $|\sigma(A) - \sigma(B)| < \varepsilon$\footnote{We use this notation to represent $\sup_{\lambda \in \sigma(A)} \min_{\mu \in \sigma(B)} |\lambda - \mu|$.} If $A$ has more than one distinct eigenvalue, define

$$R := \min_{\lambda_i(A) \neq \lambda_j(A)} |\lambda_i(A) - \lambda_j(A)|/2$$

and choose $\delta > 0$ such that $\|A - B\| < \delta \implies |\sigma(A) - \sigma(B)| < R$. For $i = 1, \ldots, n$, let $D_i = B_R(\lambda_i(A))$\footnote{Here we use $B_\delta(A)$ to refer to the set $\{B \in M_n \mid \|B - A\| < \delta\}$, where $\|\cdot\|$ denotes the Euclidean norm on $M_n$.} Choose any matrix path $P(x)$ contained within $B_\delta(A)$ and corresponding eigenpath $\lambda(P(x))$, and note that $\lambda(P(0))$ is in $D_k$ for some $k$. In fact, $\lambda(P(x))$ is contained in $D_k$ for all $x \in [0, 1]$ since all $D_i$ corresponding to distinct eigenvalues are disjoint by choice of $R$, so that if $\lambda(P(x))$ leaves $D_k$, by continuity it must at some point be at least distance $R$ from all eigenvalues of $A$, contradicting the choice of $\delta$. \hfill \Box

**Corollary 3.2.** Let $S \subseteq M_n$ have uniform multiplicity. If $A \in S$, then there exists $\delta > 0$ so that $B_\delta(A) \cap S$ is transitive.

**Proof.** We first note that by the continuity of the spectrum and our assumption of uniform multiplicity, we can choose $\delta > 0$ so that each $D_i$ corresponding to $B_\delta(A)$ has eigenvalues of strictly 1 algebraic multiplicity, and thus each matrix has a unique eigenvalue in this neighborhood. We further construct these neighborhoods to be disjoint by lemma 3.1 and so we see that $\lambda(A) \sim \lambda(B)$ and this is the only such pairing, thus it is transitive. \hfill \Box

**Theorem 3.3.** Let $A(x), B(x)$ be closed paths in $M_n$, and let $A(x), y, x, y \in [0, 1], A(x) = A(x, 0), B(x) = A(x, 1)$ be a continuous deformation between these paths where $A(x, y)$ is a closed path for all $y$. We say that $A := A(0) = A(1), B := B(0) = B(1)$, and we take induced eigenpaths $\lambda_i(A(0, y))$ considering the path from $A$ to $B$. If there exists a pairing $\lambda_i(A(0, 1)) \sim \lambda_j(A(0, 1))$ in the image of $B(x)$, where $\lambda_i(A(0, 0)) \not\sim \lambda_j(A(0, 0))$ in the image of $A(x)$ then the image of $A(x, y)$ does not have uniform multiplicity. Further, any continuous deformation of a path over a space of uniform multiplicity will preserve pairings.

**Proof.** We define the set

$$\mathcal{T} := \{y \in [0, 1] \mid \lambda_i(A(x, y)) \not\sim \lambda_j(A(x, y))\}$$

and $L := \sup \mathcal{T}$ where $L \geq 0$ by hypothesis. This pairing either occurs in $A(x, L)$ or there must be a sequence $(\gamma_n) \rightarrow L$, with $\gamma_n > L$ for all $n$, where this pairing occurs in $A(x, \gamma_n)$ (by definition of $L$). We can then apply theorem 2.3 again to see that this pairing must occur in $C(x) := A(x, L)$, and so this is the only case to consider. Now, we restrict ourselves to $x \in [\alpha, 1]$ for some $\alpha > 0$. By the above, we see that $\lambda_i(C(\alpha)) \sim \lambda_j(C(0))$. By definition of $L$, we see that there is a sequence $(\phi_n) \rightarrow L$, $\phi_n < L$ for all $n$ where a different pairing occurs, that is, there exists $k \neq j$ where $\lambda_i(A(\alpha, \phi_n)) \sim \lambda_k(A(0, \phi_n))$. Applying theorem 2.3 again, we get that $\lambda_i(C(\alpha)) \sim \lambda_j(C(0)), \lambda_k(C(0))$, which can only happen if two distinct eigenpaths (where the initial and terminal ends are not both equal) must intersect in $C(x)$, and so this path cannot have uniform multiplicity. \hfill \Box
Corollary 3.4. Let $S$ be a simply connected subset of $M_n$, and let $A(x)$ be a closed path in $S$ ($A(0) = A(1)$). If the image of $A(x)$ is not transitive, then not all matrices in $S$ can have the same multiplicity lists.

**Proof.** This follows by taking the contraction from $A(x)$ to the initial point $A(0)$. As $A(0)$ is trivially transitive, a different pairing must occur in $A(x)$, and so by theorem 3.3, $A(x,y)$ cannot have uniform multiplicity, and thus $S$ cannot have uniform multiplicity. 

Corollary 3.5. A simply connected set $S$ is transitive if and only if it has uniform multiplicity.

**Proof.** Clearly, transitivity implies uniform multiplicity. Further, by corollary 3.4, uniform multiplicity implies that all paths in $S$ must have the same pairings, and because the trivial constant paths are transitive, $S$ must be transitive.

Corollary 3.6. If $S$ is simply connected, and $K$ is one of its $k$-components, $k > 1$, then there must be an eigenvalue with algebraic multiplicity $> 1$ in $K$.

4 Classifying Eigenvalues

We now restrict ourselves to the convex-hull of matrices, that is, for matrices $A_1, \ldots, A_\ell \in M_n$,

$$Co(A_1, \ldots, A_k) := \{ A \in M_n \mid A = \sum_{i=1}^{k} \alpha_i A_i, \sum_{i=1}^{k} \alpha_i = 1, \forall i \alpha_i \in [0,1] \},$$

which is simply-connected and compact. If $A, B \in M_n$, then we say that $A \triangleright B$ if the multiplicity list of $A$ is greater than or equal to the multiplicity list of $B$ in the lexicographic topology, and we omit the bar for strict inequality. Here, we take the multiplicity list of a matrix $A \in M_n$ to be an ordered $n$-tuple $(a_n, \ldots, a_1)$ where $a_i$ is the number of $i$ algebraic multiplicity eigenvalues of $A$, and our imposed ordering $\triangleright$ is such that these $a_i$ are compared in descending order with respect to $i$, for example,

$$(3,4,2) \triangleright (2,5,3).$$

**Lemma 4.1.** There is a set $C \subseteq Co(A_1, \ldots, A_k)$ that is dense with uniform multiplicity. Additionally, it is open in the subspace topology. Further, if $B \in \mathcal{U} := Co(A_1, \ldots, A_k) \setminus C$ then $B \triangleright C$ for any $C \in C$. We call this the core of the set.

**Proof.** As there are finitely many multiplicity lists for matrices in $M_n$, we can take the minimum that occurs over $S$. We first argue that it is dense. It is sufficient to show that for arbitrary $A \in S$, for all $\varepsilon > 0$, there exists $C \in C \cap B_\varepsilon(A)$. We argue by contradiction. Suppose not, then there exists $B_\varepsilon(A)$ such that for all $B \in B_\varepsilon(A)$, $B \triangleright C$. If we take $C \in C$ and the line segment $A(x)$ from $A$ to $C$, we see that $A(x) \triangleright B$ for all $x \in [0,1]$. This is a contradiction. The fact that this is open in the subspace topology follows from the continuity of the spectrum, as for any $C \in C$ there must be some $\varepsilon > 0$ where for any $B \in B_\varepsilon(C)$, $C \triangleright B$. 

Figure 4: A 3-dimensional representation of an eigen-surface that is not transitive. The “saddle point” in the center of the figure is indicative of a higher multiplicity eigenvalue.
Corollary 4.2. \( ES(C) \) is topologically a manifold with boundary.

Proof. By corollary 3.2 \( C \) is locally transitive, and thus the eigenvalues can locally be parameterized as continuous (smooth) functions, thus this result is clear by the construction of \( ES(C) \). \( \Box \)

We say that, for a matrix \( A \in U \), \( A \) is locally non-transitive if, for any \( \varepsilon > 0 \), \( B_\varepsilon(A) \cap C \) is not transitive, and further, for all eigenvalues \( \lambda(A) \in \sigma^*(A) \) and any matrix \( B \in B_\varepsilon(A) \cap \mathcal{C} \), if \( \lambda_i(B) \sim \lambda(B) \to \lambda_j(B) \) in \( B_\varepsilon(A) \cap \mathcal{C} \). What this definition signifies is that there is locally a path in \( \mathcal{C} \) which connects all of the eigenvalues which are path-connected through \( A \). We say these eigenvalues collide at \( A \).

![Figure 5: An example of a \( Co(A_1,A_2,A_3) \) where \( U \) is a single matrix that is locally non-transitive.](image)

Lemma 4.3. Let us describe the matrices in \( S \) by their coefficients \( (\alpha_1,\ldots,\alpha_k) \). There is an analytic functions of \( k \) variables such that for any \( A \in U \), the coefficients corresponding to \( A \) are such that \( f(A) = 0 \). Further, for any \( C \in \mathcal{C} \), \( f(C) \neq 0 \).

Proof. When we consider the characteristic polynomials of the matrices in \( \mathcal{C} \) as a function of these coefficients, we see that if all of the eigenvalues are simple then this forms an irreducible function, and if not, it can be reduced to the product of irreducible functions, all of which have distinct roots [5]. The discriminant of these irreducible analytic functions is zero precisely when the algebraic multiplicity of the eigenvalues increases, thus the claim is proven, as either we only need to consider the discriminant, or we can consider the product of the discriminants of each reduced function. \( \Box \)

Lemma 4.4. \( Co(A_1,\ldots,A_k) \) is almost everywhere locally transitive. Further, we can partition all but an arbitrarily small subset of \( Co(A_1,\ldots,A_k) \) into finitely many transitive, simply connected sets.

Proof. The first statement follows from the zero set of a multi-variable analytic function being negligible, so that \( Co(A_1,\ldots,A_k) = C \) almost everywhere. For the second statement, by lemma 4.1 for each matrix \( C \in \mathcal{C} \), we take \( \varepsilon > 0 \) such that \( B_\varepsilon(C) \cap U = \emptyset \). Then, for arbitrary \( \delta > 0 \), we can take the \( \delta \)-neighborhood \( U_\delta \) of \( U \). These open sets form a cover of \( Co(A_1,\ldots,A_k) \), and because it is compact, we can take a finite sub-cover. All of the sets except for \( U_\delta \) are transitive by corollary 3.5, and it is clear that we can form a partition of the convex hull by taking unions and intersections of these sets as necessary. \( \Box \)

Lemma 4.5. If \( A \) is locally non-transitive then there exists \( \varepsilon > 0 \) such that

\[
ES(B_\varepsilon(A) \cap Co(A_1,\ldots,A_k))
\]

is topologically a manifold.

Proof. It suffices to show that we can parameterize these eigenvalues as a single continuous function. By hypothesis, we can take a closed path \( B(x) \in B_\varepsilon(A) \cap \mathcal{C} \) for all \( x \), where for \( B := B(0) \) there is an eigenpath \( \lambda(B(x)) \) that achieves all eigenvalues which collide at \( \lambda(A) \). By lemma 4.3 we see that for sufficiently small \( \varepsilon > 0 \), \( C \cap B_\varepsilon(A) \) forms the same homotopy equivalence class, thus by theorem 3.3 we can continuously parameterize all eigenvalues realized by this set in this way. We can extend this function continuously to \( \lambda(A) \) and the other matrices in \( U \cap B_\varepsilon(A) \). \( \Box \)
We observe here that this methodology leads to a natural symmetry with analytic perturbations of a single variable. If we take the line segment from a matrix $C \in \mathcal{C}$ to $A \in \mathcal{U}$ where $A$ is locally non-transitive, $C(1-x)+xA$, we can interpret this as the analytic perturbation $C+x(A-C)$. If we consider this function locally about $x=1$, we see that the eigenvalues must also form a cycle about this algebraic singularity as this path is in the same homotopy class as the one described above, and so we can describe these eigenvalues as Puiseux series [4]. Thus this non-transitive behaviour generally can be thought of as continuous deformations of the branches of an algebraic function.

**Theorem 4.6.** If $A \in \mathcal{U}$ is not locally non-transitive, then there is no $\varepsilon > 0$ such that

$$ES(B_\varepsilon(A) \cap \text{Co}(A_1,\ldots,A_k))$$

is topologically a manifold.

**Proof.** It suffices to show that there is no local chart to Euclidean space at $A$. By lemma [4.3], $\mathcal{U}$ can be represented as a manifold of at most $k-2$ real dimensions, while $\text{Co}(A_1,\ldots,A_k)$ is a $k-1$ dimensional manifold. We suppose there is such a local chart $f$. If $A \notin \partial\mathcal{C}$, then it follows that we can take a sufficiently small open neighborhood $V$ about $A$ that $V \cap \text{Co}(A_1,\ldots,A_k) = V \cap \mathcal{U}$. Thus $V \cap \mathcal{U}$ is locally homeomorphic to $\mathbb{R}^{k-1}$ by $f$, but it is also locally homeomorphic to $\mathbb{R}^{k-2}$, which is a contradiction. If $A \in \partial\mathcal{C}$, then we see that removing $\mathcal{U}$ from $V \cap \text{Co}(A_1,\ldots,A_k)$ forms a separation of this set into at least 4 connected sets, which is not possible when removing a $\mathbb{R}^{k-2}$ manifold from a $\mathbb{R}^{k-1}$ manifold, contradicting $\text{Co}(A_1,\ldots,A_k)$ is locally homeomorphic to $\mathbb{R}^{k-1}$ at $A$. $\square$

**Corollary 4.7.** If $A \in \mathcal{U}$ has higher dimensional commutator space than all matrices in $\mathcal{C}$, then there is no $\varepsilon > 0$ such that

$$ES(B_\varepsilon(A) \cap \text{Co}(A_1,\ldots,A_k))$$

is topologically a manifold.

**Corollary 4.8.** If $ES(\text{Co}(A_1,\ldots,A_k))$ is topologically a manifold with boundary then it has uniform commutator space dimension, except perhaps in $\mathcal{C}$ where uniform multiplicity is sufficient.

## 5 Computational Methods

Computationally, it is impossible to consider all paths in $\text{Co}(A_1,\ldots,A_k)$, thus we provide justification to study a simpler, more amenable collection of paths. We define $ES^\ell(S)$ to be the eigen-surface of a set $S$ where we only consider eigenvalue pairings by polygonal paths in $S$, that is, paths that are the union of line segments.

**Lemma 5.1.** All pairings can be achieved by polygonal paths.

**Proof.** It is sufficient to show that for arbitrary $A \in \text{Co}(A_1,\ldots,A_k)$, if $\lambda_i(A) \sim \lambda_k(A)$ in the image of an arbitrary path, then it also does by a polygonal path. By theorem [3.3], pairings are entirely determined by path homotopy class about $\mathcal{U}$ and intersections with $\mathcal{U}$. It is clear that we
can construct a polygonal path that intersects the same components of $\mathcal{U}$, and we can also construct it to be in the same homotopy class about each component $\mathcal{U}$ as $\mathcal{C}$ is open as a subspace of $Co(A_1, \ldots, A_k)$, so it follows that path connected $\iff$ polygonal path connected, so we are done.

This argument tells us that it is sufficient to consider only representative paths of each homotopy class, reducing the problem to a finite computation. For a $k$-component $\mathcal{K} \subseteq ES(S)$, we can define a graph $G(\mathcal{K})$ where $V(G(\mathcal{K})) := \{\lambda(A) \in \mathcal{K}\}$ and $E(G(\mathcal{K})) := \{((\lambda(A), \lambda(B)) \in \mathcal{K} \times \mathcal{K} | \lambda(A) \sim \lambda(B)\}$. Without restricting to a particular type of path, this graph has little meaning, as each $G(\mathcal{K})$ will be complete, however, if we consider $G(\mathcal{K})$ where only pairings induced by a line-segment correspond to an edge, we can deduce some less trivial properties.

**Lemma 5.2.** $G(\mathcal{K}^\ell)$ has finite diameter for $\mathcal{K}^\ell \subseteq ES^\ell(\mathcal{K})$

**Proof.** We argue by contradiction. We define
d($\lambda(A), \lambda(B)) : V(G(\mathcal{K}^\ell)) \times V(G(\mathcal{K}^\ell)) \to \mathbb{Z}$

to be the maximum path length between two vertices in $G(\mathcal{K}^\ell)$ and we call $S := Co(A_1, \ldots, A_k)$.

By lemma 5.1, $d(\lambda(A), \lambda(B)) = c < \infty$ for any eigenvalues in $\mathcal{K}^\ell$. If $G(\mathcal{K}^\ell)$ were to have infinite diameter, then there must be a sequence of matrices $(A_n, B_n) \in S \times S$ where

$$\lim_{n \to \infty} d(\lambda(A_n), \lambda(B_n)) = \infty.$$ 

Now, we can take a convergent subsequence of these tuples, where

$$\lim_{n_i \to \infty} (A_{n_i}, B_{n_i}) =: (A, B) \in S \times S,$$

$$\lim_{n_i \to \infty} (\lambda(A_{n_i}), \lambda(B_{n_i})) =: (\lambda(A), \lambda(B)).$$ 

By lemma 3.1 for $n_i$ sufficiently large, the line segment from $A_{n_i}$ to $A$ and from $B_{n_i}$ to $B$ will induce pairing between $\lambda(A_{n_i})$ and $\lambda(A)$, and $\lambda(B_{n_i})$ and $B$. We have then that

$$d(\lambda(A_{n_i}), \lambda(B_{n_i})) \leq d(\lambda(A), \lambda(B)) + 2$$

for all $n_i$ large enough. However, we also have that $d(\lambda(A_{n_i}), \lambda(B_{n_i}))$ is arbitrarily large for $n_i$ large enough, giving us a contradiction.

We define the principle graph of $Co(A_1, \ldots, A_k)$ to be the graph generated only by considering representative matrices in $\mathcal{U}$ and the matrices $A_1, \ldots, A_k$. This object provides information on the overall structure of its eigen-surface, highlighting regions of high transitivity, and how these regions are joined together as a partition of $Co(A_1, \ldots, A_k)$.

Figure 7: The principle graphs of two eigen-surfaces, each one corresponding to the convex hull of 3 matrices. The first two images show a 2-component and a 1-component easily identified as a wheel graph and a $K_4$ (a complete graph on 4 vertices), while the second two images are much more complex, and the graph only shows the general structural properties.

### 6 Examples

#### 6.1 Hermitian Matrices

Hermitian matrices have real eigenvalues, and we can use this fact to show that all Hermitian
matrix closed paths are weakly transitive, that is, for all \( i \in [n] \), we can take paths \( \lambda_i(A(x)) \) where \( \lambda_i(A(0)) = \lambda_i(A(1)) \). This proof also holds for any other path connected set where the matrices have real eigenvalues.

**Lemma 6.1.** Let \( S \subseteq M_n \) be a path connected set. If \( S \) is such that all \( A \in S, A \) is Hermitian \( \implies S \) is weakly transitive.

**Proof.** Suppose that \( \lambda_i(A) \not\sim \lambda_j(A) \) and \( \lambda_k(A) \not\sim \lambda_l(A) \), with respect to a Hermitian matrix path. We see then that we can take continuous real functions \( \lambda_i(A(x)), \lambda_j(A(x)), \lambda_k(A(x)), \) and \( \lambda_l(A(x)) \). Where \( \lambda_i(A(0)) \leq \lambda_j(A(0)) = \lambda_k(A), \) and \( \lambda_j(A) = \lambda_i(A(1)) \leq \lambda_j(A(1)) = \lambda_l(A) \). We see then by the intermediate value theorem that these paths must intersect, and so we can construct the desired path. We can argue similarly when considering the reverse inequalities.

We further note that as the convex hull of Hermitian matrices is everywhere diagonalizable, its eigen-surface fails to be a topological manifold if it does not have uniform multiplicity, that is, all eigenvalues must be simple. These results also hold when considering the singular values of a convex hull, as these are the eigenvalues of Hermitian matrices.

### 6.2 Nonnegative Irreducible Matrices

Suppose \( A_1, \ldots, A_k \in M_n \) are primitive and irreducible matrices. Then any convex combination thereof is also primitive irreducible. Thus, by the Perron-Frobenius theory, every element of \( Co(A_1, \ldots, A_k) \) has a Perron root, a real positive eigenvalue of highest magnitude amongst all eigenvalues, that is of multiplicity 1 [1]. As the spectrum is continuous, we see then that the Perron roots of matrices in \( Co(A_1, \ldots, A_k) \) must all be contained in a single 1-component.

---

4here we consider the eigenpaths as real functions.
6.4 Tri-Diagonal Toeplitz

Suppose $A_1, \ldots, A_k$ are tri-diagonal toeplitz matrices. Then all convex combinations $\sum_i \alpha_i A_i$ are tri-diagonal toeplitz as well, with diagonal entries given by $\sum_i \alpha_i a_i$, where $a_i$ is the diagonal element of $A_i$, and so on. Likewise, if $b_i$ are the super-diagonal elements of $A_i$ and $c_i$ are the sub-diagonal elements of $A_i$, then

\[
\sum_i \alpha_i A_i = \begin{bmatrix}
\sum_i \alpha_i a_i & \sum_i \alpha_i b_i & \cdots \\
\sum_i \alpha_i c_i & \sum_i \alpha_i a_i & \cdots \\
\cdots & \cdots & \cdots \\
\sum_i \alpha_i c_i & \sum_i \alpha_i a_i & \cdots
\end{bmatrix}
\]

and, using the known formula [7] for eigenvalues of tri-diagonal toeplitz matrices, has eigenvalues

\[
\lambda_k = \sum_i \alpha_i a_i + 2 \sqrt{\left(\sum_i \alpha_i b_i\right)\left(\sum_i \alpha_i c_i\right) \cos\left(\frac{\pi k}{n+1}\right)}.
\]

6.5 Shared Eigenvectors

Suppose $A_1, \ldots, A_k$ are a family of matrices in which all eigenvectors are shared. We can take them to be circulant matrices [9], commuting matrices [3], diagonal matrices, etc. As we can simultaneously triangularize all matrices in $Co(A_1, \ldots, A_k)$, it is clear that these eigenvalues are weakly transitive as they are well-defined continuous functions of the convex coefficients. The criteria then for these sets to be transitive is to verify that these continuous eigenvalue functions never intersect.

6.6 Dimensional Relationships

Here we only consider pairings that occur in the line-segments between matrices.

**Lemma 6.2.** If $\lambda(T) \rightsquigarrow \lambda(A + B)$, $\lambda(A + B) \rightsquigarrow \lambda(A)$, $\lambda(B)$, then there exist continuous paths $p_A(t), p_B(t) in ES(Co(T, A, B))$, such that $p_A(0) = p_B(0) = \lambda(T)$, $p_A(1) = \lambda(A), p_B(1) = \lambda(B)$.

**Proof.** By hypothesis, $\lambda(T) \rightsquigarrow \lambda(A + B)$, and a simple algebraic computation reveals that pairings are unchanged by positive scaling, so we have that

\[
\lambda(T) \rightsquigarrow \lambda(1/2(A + B)) \in ES(Co(T, A, B)).
\]
We also have, by the same way, that \( \lambda(1/2(A + B)) \sim \lambda(A) \). Thus if we take the function

\[
p_A(t) := \begin{cases} 
\lambda(T(2t)), & t \in [0, 1/2] \\
\lambda(1/2(A + B)(2t)), & t \in [1/2, 1]
\end{cases}
\]

where \( T(x) \) is the line-segment from \( T \) to \( 1/2(A + B) \) and \( 1/2(A + B)(x) \) is the line-segment from \( 1/2(A + B) \) to \( A \), we get the desired function \( p_B(t) \) is defined similarly).

\[\square\]

We also see that this result generalizes As we can take sums of \( k \) matrices \( A_1 + \cdots + A_k \) with \( T \), and thus obtain a relationship between eigenvalues of \( T \) and \( A_1, \ldots, A_k \) subject to paths in \( Co(A_1, \ldots, A_k) \).

7 Open Problems

- Efficient methods to compute eigenvalue pairings and higher multiplicity eigenvalues in the convex hull of arbitrary matrices.

- Partitioning of the convex hull of arbitrary matrices into finitely many weakly transitive, simply connected, sets.

- Characterization of the graphs generated by the eigen-surface of the convex hull of arbitrary matrices.

- Detailed topological characterization of \( ES(DS_n) \).

- From our results, it follows that if any closed polygonal-path generating the boundary of the convex hull are not transitive, then the set must contain matrices with higher multiplicity eigenvalues. Thus a study of the probability of transitivity in these paths for random matrices, such as those drawn from the Gaussian measure, would reveal important general characterization of the eigen-surface.

Acknowledgements

This project was greatly advanced by the insightful collaboration of fellow REU participants Derek Lim, Eric Jankowski and Amit Harlev, as well as the generous support provided by NSF grant 1757603.

References

[1] Boyle, M. Notes on the Perron Frobenius Theory of Nonnegative Matrices. https://www.math.umd.edu/~mboyle/courses/475sp05/spec.pdf

[2] Herzberg, G., & Longuet-Higgins, H. C. (1963). Intersection of potential energy surfaces in polyatomic molecules. Discussions of the Faraday Society, 35, 77. doi:10.1039/df635000077

[3] Horn, R. A., & Johnson, C. R. (2017). Matrix analysis. New York, NY: Cambridge University Press.

[4] Kato Toshio. (1995). Perturbation theory for linear operators. Berlin: Springer.

[5] Knopp, Konrad (1947), Theory of Functions, Part Two: Applications and Continuation of the General Theory, New York, Dover Publications.
[6] Monov, V. (1999). On the spectrum of convex sets of matrices. *IEEE Transactions on Automatic Control, 44*(5), 1009-1012. doi:10.1109/9.763218

[7] Noschese, S., Pasquini, L., & Reichel, L. (2012). Tridiagonal Toeplitz matrices: properties and novel applications. *Numerical Linear Algebra with Applications, 20*(2), 302-326. doi:10.1002/nla.1811

[8] Stone, A. J. (1976). Spin-Orbit Coupling and the Intersection of Potential Energy Surfaces in Polyatomic Molecules. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 351*(1664), 141-150. doi:10.1098/rspa.1976.0134

[9] Tee, G. (2007). Eigenvectors of Block Circulant and Alternating Circulant Matrices. *New Zealand Journal of Mathematics, 36*, 195-211.

[10] Ding, J., & Zhou, A. (2007). Eigenvalues of rank-one updated matrices with some applications. *Applied Mathematics Letters, 20*(12), 1223-1226. doi:10.1016/j.aml.2006.11.016