A note on continuity of strongly singular Calderón-Zygmund operators in Hardy-Morrey spaces

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Abstract In this note we address the continuity of strongly singular Calderón-Zygmund operators on Hardy-Morrey spaces $HM^p(\mathbb{R}^n)$, assuming weaker integral conditions on the associated kernel. Important examples that falls into this scope are pseudodifferential operators on the Hörmander classes $Op(S^m_{\sigma, \mu}(\mathbb{R}^n))$ with $0 < \sigma \leq 1$, $0 \leq \mu < 1$, $\mu \leq \sigma$ and $m \leq -n(1 - \sigma)/2$.

1 Introduction

J. Álvarez and M. Milman [1] introduced a new class of Calderón-Zygmund operators, called strongly singular Calderón-Zygmund operator and established continuity of those operators in real Hardy space $H^q(\mathbb{R}^n)$. More precisely, a continuous function $K \in C(\mathbb{R}^{2n} \setminus \Delta)$, where $\Delta = \{ (x, x) : x \in \mathbb{R}^n \}$ is a $\delta$-kernel of type $\sigma$, if there exists some $0 < \delta \leq 1$ and $0 < \sigma \leq 1$ such that

$$|K(x, y) - K(x, z)| + |K(y, x) - K(z, x)| \leq C \frac{|y - z|^\delta}{|x - z|^{n+\frac{\delta}{\sigma}}}$$

for all $|x - z| \geq 2|y - z|^\sigma$. A bounded linear operator $T : S(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n)$ is called a strongly singular Calderón-Zygmund operator, if it is associated to a $\delta$-kernel of

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type $\sigma$ in the sense $\langle T f, g \rangle = \int \int K(x, y) f(y) g(x) dy dx$, for all $f, g \in S(\mathbb{R}^n)$ with disjoint supports; it has bounded extension from $L^2(\mathbb{R}^n)$ to itself and in addition $T$ and $T^*$ extend to a continuous operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where

$$\frac{1}{p} = \frac{1}{2} + \frac{\beta}{n}$$

for some $\left(1 - \sigma \right)^2 \leq \beta < \frac{n}{2}$. When $\sigma = 1$ and $\beta = 0$ we recover the standard non-convolution Calderón-Zygmund operators (see [4]).

The authors in [1, Theorem 2.2] established the continuity of those classes of operators in real Hardy spaces $H^p(\mathbb{R}^n)$ as follows: under the condition $T^*(1) = 0$, strongly singular Calderón-Zygmund operators associated to a kernel satisfying (1) are bounded from $H^p(\mathbb{R}^n)$ to itself for every $0 < q < 1$ where

$$\frac{1}{q_0} := \frac{1}{2} + \frac{\beta \left( \frac{\delta}{2} + \frac{\beta}{4} \right)}{n \left( \frac{\delta}{2} - \delta + \beta \right)}.$$  

(2)

The case $q = q_0$ is still open, however the conclusion continues to hold replacing the target space by $L^p(\mathbb{R}^n)$ (see [2, Theorem 3.9]).

In this note, we establish results on continuity of strongly singular Calderón-Zygmund operators on Hardy-Morrey spaces $\mathcal{HM}^{\alpha,\beta}(\mathbb{R}^n)$ assuming weaker integral conditions on the kernel, introduced by the second and third authors in [12]. Let $0 < \sigma \leq 1$, $r \geq 1$ and $\delta > 0$. We say that $K(x, y)$ associated to $T$ is a $D_{\delta,r}$ kernel of type $\sigma$ if

$$\left( \int_{C_j(z, \ell)} |K(x, y) - K(z, z)|^r + |K(y, x) - K(z, x)|^r \, dx \right)^{\frac{1}{r}} \lesssim |C_j(z, \ell)|^{\frac{1}{r} - 1} 2^{-j \delta}$$

(3)

for $\ell \geq 1$ and

$$\left( \int_{C_j(z, \ell')} |K(x, y) - K(z, z)|^r + |K(y, x) - K(z, x)|^r \, dx \right)^{\frac{1}{r}} \lesssim |C_j(z, \ell')|^{\frac{1}{r} - 1 + \frac{\delta}{r} \left( \frac{1}{2} - \frac{\delta}{r} \right)} 2^{-j \delta}$$

(4)

for $\ell < 1$, where $z \in \mathbb{R}^n$, $|y - z| < \ell$, $0 < \rho \leq \sigma$ and $C_j(z, \eta) := \{x \in \mathbb{R}^n : 2^j \eta < |x - z| \leq 2^{j+1} \eta\}$. Those conditions also covers the standard case $\sigma = 1$, by choosing $\rho = \sigma$ in (4), and in that case both conditions are the same. It is easy to check that $D_{\delta,r_1}$ condition is stronger than $D_{\delta,r_2}$ for $r_1 > r_2$ and any $\delta > 0$ and $0 < \sigma \leq 1$. Moreover, $\delta$-kernels of type $\sigma$ satisfying (1) also satisfies $D_{\delta,r}$ condition for all $r \geq 1$. It has also been shown in [12, Proposition 5.3] that pseudodifferential operators associated to symbols in the Hörmander classes $S_{\mu,m}^\sigma(\mathbb{R}^n)$ with $0 < \sigma \leq 1$, $0 \leq \mu < 1$, $\mu \leq \sigma$ and $m \leq -n(1 - \sigma)/2$, satisfies the $D_{1,r}$ condition for $1 \leq r \leq 2$. We refer to [12] for more details. In particular, the continuity of operators associated to symbols given by $e^{i|\xi|^\mu} |\xi|^{-m}$ away from the origin are also examples of this type of operators and have been extensively studied, for instance in [5, 6, 8, 13].

Our main result is the following:
Theorem 1 Let $T$ be a strongly singular Calderón-Zygmund operator associated to a $D_{\delta, r}$ kernel of type $\sigma$ for some $1 \leq r \leq 2$. Under the assumptions that $T^\ast (x^\alpha) = 0$ for every $|\alpha| \leq \lfloor \delta \rfloor$, $T$ can be extended to a bounded operator from $\mathcal{HM}^1_q (\mathbb{R}^n)$ to itself for any $0 < q < \lambda < r$ and $q_0 < q \leq 1$, where $q_0$ is given by (2).

The proof relies on showing that $T$ maps atoms into molecules and a molecular decomposition in $\mathcal{HM}^1_q (\mathbb{R}^n)$ for $0 < q \leq 1$ and $q \leq \lambda < \infty$ under restriction $\lambda < r$ (see Theorem 3 and the Remark 2). As an immediate consequence of previous theorem, we also obtain the continuity of standard non-convolution Calderón-Zygmund operators ($\sigma = 1$) associated to kernels satisfying integral conditions. The corresponding result in the convolution setting for kernels satisfying derivative conditions can be found in [9, Section 2.2].

Corollary 1 Under the same hypothesis of the previous theorem, if $T$ is a standard Calderón-Zygmund operator, then it is bounded from $\mathcal{HM}^1_q (\mathbb{R}^n)$ to itself provided that $n/(n + \delta) < q \leq 1$.

The organization of the paper is as follows. In Section 2 we recall some basic definitions and a general atomic and molecular decomposition of Hardy-Morrey spaces. In particular, in Section 2.1 we present an atomic decomposition in terms of $\mathcal{F}$ atoms by showing the equivalence with classical $\mathcal{F}_\infty$ atomic space and in Section 2.2 we show an appropriated molecular decomposition of Hardy-Morrey spaces. Finally, in Section 3 we present the proof of Theorem 1 showing that $T$ maps atoms into molecules.

Notation: throughout this work, the symbol $f \lesssim g$ means that there exist a constant $C > 0$, not depending on $f$ nor $g$, such that $f \leq C g$. By a dyadic cube we mean cubes on $\mathbb{R}^n$, open on the right whose vertices are adjacent points of the lattice $(2^{-k} \mathbb{Z})^n$ for some $k \in \mathbb{Z}$. Given a set $A \subset \mathbb{R}^n$ we denote by $|A|$ its Lebesgue measure. Given a cube $Q$ (dyadic or not), we will always denote its center and side-length by $x_Q$ and $\ell_Q$ respectively. By $Q^\ast$ we mean the cube with same center as $Q$ and side-length $2\ell_Q$. We also denote by $\int_Q f(x)dx := \frac{1}{|Q|} \int_Q f(x)dx$.

2 Hardy-Morrey spaces $\mathcal{HM}^1_q (\mathbb{R}^n)$

In this section, we recall and present some properties of Hardy-Morrey spaces. For $0 < q \leq \lambda < \infty$, the Morrey spaces, denoted by $\mathcal{M}^1_q (\mathbb{R}^n)$, are defined to be the set of measurable functions $f \in L^q_{\text{loc}} (\mathbb{R}^n)$ such that

$$
\|f\|_{\mathcal{M}^1_q} := \sup_J \|J^\frac{1}{q - 1} \left( \int_J |f(y)|^q dy \right)^\frac{1}{q} \| < \infty,
$$

where the supremum is taken over all cubes $J \subset \mathbb{R}^n$. 

For any tempered distribution \( f \in S'(\mathbb{R}^n) \) and any fixed \( \varphi \in S(\mathbb{R}^n) \) with \( \int \varphi \neq 0 \), consider the smooth maximal function \( M_\varphi f(x) = \sup_{\varepsilon > 0} |(\varphi_\varepsilon \ast f)(x)| \), where \( \varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(x/\varepsilon) \). For any \( 0 < q \leq \lambda < \infty \), we say that \( f \in S'(\mathbb{R}^n) \) belongs to Hardy-Morrey space \( H_{\mathcal{M}^q_\lambda}(\mathbb{R}^n) \) if the smooth maximal function \( M_\varphi f \in \mathcal{M}^q_\lambda(\mathbb{R}^n) \). The functional \( \| f \|_{H_{\mathcal{M}^q_\lambda}} := \| M_\varphi f \|_{\mathcal{M}^q_\lambda} \) defines a quasi-norm as \( 0 < q < 1 \) and is a norm if \( q \geq 1 \).

In the same way as Hardy spaces, the Hardy-Morrey spaces have also equivalent maximal characterizations (see \([10, \text{Section 2}]\)). Clearly, Hardy-Morrey spaces cover the classical Hardy spaces \( H^p(\mathbb{R}^n) \) when \( \lambda = q \) and Morrey spaces \( \mathcal{M}^q_\lambda(\mathbb{R}^n) \) if \( 1 < q \leq \lambda < \infty \).

### 2.1 Atomic decomposition in Hardy-Morrey spaces

**Definition 1** \([9, \text{Definiton 2.2}]\). Let \( 0 < q \leq 1 \leq r \leq \infty \) with \( q < r \) and \( q \leq \lambda < \infty \). A measurable function \( a_Q \) is called a \((q, \lambda, r)\)-atom if it is supported on a cube \( Q \subset \mathbb{R}^n \) and satisfies: (i) \( \| a_Q \|_{L_r} \leq |Q|^{1/r} \) and (ii) \( \int_{\mathbb{R}^n} x_\alpha a_Q(x) dx = 0 \) for all \( \alpha \in \mathbb{N}_0^n \) such that \( |\alpha| \leq N_q := \lfloor (1/q - 1) \rfloor \), where \( \lfloor \cdot \rfloor \) denotes the floor function.

The following lemma is an extension of \([3, \text{Proposition 2.5}]\) and the proof will be presented for completeness.

**Proposition 1** Let \( 0 < q \leq 1 \leq r \leq \infty \) with \( q < r \) and \( q \leq \lambda < \infty \) with \( \lambda \leq r \). If \( f \) is a compactly supported function in \( L^r(\mathbb{R}^n) \) satisfying the moment condition

\[
\int_{\mathbb{R}^n} x_\alpha f(x) dx = 0 \quad \text{for all } |\alpha| \leq N_q,
\]

then it belongs to \( H_{\mathcal{M}^q_\lambda}(\mathbb{R}^n) \) and moreover \( \| f \|_{H_{\mathcal{M}^q_\lambda}} \leq \| f \|_{L^r} |Q|^{1/r - 1/r} \) for all cube \( Q \supset \text{supp} \, (f) \). In particular, if \( f = a_Q \), then \( \| a_Q \|_{H_{\mathcal{M}^q_\lambda}} \leq 1 \) uniformly.

**Proof** Let \( J \subset \mathbb{R}^n \) be an arbitrary cube and \( Q \) a cube such that \( \text{supp} \, (f) \subset Q \). Split the integral over \( J \) into \( J \cap Q^* \) and \( J \setminus Q^* \). Since the maximal function \( M_\varphi \) is bounded from \( L^r(\mathbb{R}^n) \) to itself for every \( 1 < r \leq \infty \), it follows that

\[
\int_{J \cap Q^*} |M_\varphi f(x)|^q dx \leq \| M_\varphi f \|_{L^r}^q |J \cap Q^*|^{1 - \frac{q}{r}} \leq \| f \|_{L^r}^q |J \cap Q^*|^{1 - \frac{q}{r}}.
\]

For \( r = 1 \) and \( 0 < q < 1 \), setting \( R = \| f \|_{L^1}|J \cap Q^*|^{-1} \) and using that \( M_\varphi \) satisfies weak \((1, 1)\) inequality we get the analogous inequality:

\[
\int_{J \cap Q^*} |M_\varphi f(x)|^q dx = \int_0^\infty \omega^{q-1} \left| \{ x \in J \cap Q^* : |M_\varphi f(x)| > \omega \} \right| d\omega \\
\leq |J \cap Q^*| \int_0^R \omega^{q-1} d\omega + \| f \|_{L^1} \int_R^\infty \omega^{q-2} d\omega \leq \| f \|_{L^1}^q |J \cap Q^*|^{1 - q}.
\]

(6)
If $|Q| < |J|$, since $q/\lambda - 1 \leq 0$ and $1 - q/r > 0$ for all $1 \leq r < \infty$, one has $|J|^{q/\lambda - 1} |J \cap Q|^{1-q/r} \leq |Q|^{1-q/r}$. On the other hand, if $|J| < |Q|$, using that $\lambda \leq r$ it follows $|J|^{\frac{q}{\lambda} - \frac{1}{r}} |J \cap Q|^{1 - \frac{q}{r}} = |J|^{\frac{q - 1}{r}} \left( \frac{|J \cap Q|}{|J|} \right)^{\frac{1}{r}} \leq |Q|^{\frac{q - 1}{r}}$. Hence $|J|^{\frac{q}{r} - 1} \int_{J \cap Q^c} |M_{\varphi} f(x)|^q dx \leq \|f\|_{L^r}^q |Q|^{1 - \frac{q}{r}}$.

To estimate the integral on $J \setminus Q^c$, using the moment condition (5) we write $\varphi_t * f(x) = \int f(y) (\varphi_t(x-y) - P_{\varphi_t}(y)) dy$, where $P_{\varphi_t}(y) = \sum_{|\alpha| \leq N_q} C_{\alpha} \partial^\alpha \varphi_t(x) (-y)^\alpha$ denotes the Taylor polynomial of degree $N_q$ of the function $y \mapsto \varphi_t(x-y)$. The standard estimate of the remainder term (see [11, p. 106]) yields $|\varphi_t(x-y) - P_{\varphi_t}(y)| \leq |y - x_Q|^{N_q+1} |x - x_Q|^{-n+\varepsilon}$ and since supp $(f) \subseteq Q$, we have the pointwise control

$$|M_{\varphi} f(x)| \leq \frac{\ell^{N_q+1}}{|x - x_Q|^n + N_q+1} \int_{Q} |f(y)| dy \leq \frac{\ell^{N_q+1}}{|x - x_Q|^n + N_q+1} \|f\|_{L^r} |Q|^{1 - \frac{1}{r}}.$$ 

If $|Q| < |J|$, since $N_q + 1 > n (1/q - 1)$, we estimate $|J|^{\frac{q}{r} - 1} \int_{J \setminus Q^c} |M_{\varphi} f(x)|^q dx$ by

$$\|f\|_{L^r}^q |Q|^{\frac{q}{r} - 1} \int_{Q^c \setminus J} |x - x_Q|^{-q(n+\varepsilon)} dx \leq \|f\|_{L^r}^q |Q|^{\frac{q}{r} - \frac{1}{r}}.$$ 

Finally, if $|J| < |Q|$ we use

$$|J|^{\frac{q}{r} - 1} \int_{J \setminus Q^c} |M_{\varphi} f(x)|^q dx \leq \|f\|_{L^r}^q |J|^{\frac{q}{r} - 1} |Q|^{\frac{q}{r} - \frac{1}{r}} \ell^{nq} |J \setminus Q^c| \leq \|f\|_{L^r}^q |Q|^{\frac{q}{r} - \frac{1}{r}},$$

which concludes the proof. \(\square\)

Given $1 \leq r \leq \infty$, we denote the atomic space $\mathfrak{atHM}^{1,r}_q(\mathbb{R}^n)$ by the collection of $f \in S'(\mathbb{R}^n)$ such that $f = \sum_{Q \in \mathcal{D}} s_Q a_Q$ in $S'(\mathbb{R}^n)$, where $\{a_Q\}$ are $(q, \lambda, r)$-atoms and $\{s_Q\}$ is a sequence of complex scalars satisfying

$$\|s_Q\|_{i,q} := \sup_f \left( \left| \left( \sum_{Q \subset J} \left| \frac{1}{|Q|^{1/\lambda}} |s_Q| \right|^q \right)^{\frac{1}{q}} \right| < \infty.$$ 

The functional $\|f\|_{\mathfrak{atHM}^{1,r}_q} := \inf \{ \|s_Q\|_{i,q} : f = \sum_{Q \in \mathcal{D}} s_Q a_Q \}$, where the infimum is taken over all such atomic representations, defines a quasi-norm in $\mathfrak{atHM}^{1,r}_q(\mathbb{R}^n)$. Clearly, if $1 \leq r_1 < r_2 \leq \infty$ then $\mathfrak{atHM}^{1,r_1}_q(\mathbb{R}^n)$ is continuously embedded in $\mathfrak{atHM}^{1,r_2}_q(\mathbb{R}^n)$. The converse of this simple embedding is the content of the next result.

**Lemma** Let $0 < q < 1 \leq r$ with $q < r$ and $q \leq \lambda < \infty$. Then $\mathfrak{atHM}^{1,r}_q(\mathbb{R}^n) = \mathfrak{atHM}^{1,\infty}_q(\mathbb{R}^n)$ with comparable quasi-norms.
The proof is based on the corresponding theorem for Hardy spaces (see [7, Theorem 4.10]). Let \( a_Q \) be a \((q, \lambda, r)\)-atom and we show that \( a_Q = \sum_j s_{Q_j}a_{Q_j} \), where \( \{s_{Q_j}\}_j \) are \((q, \lambda, \infty)\)-atoms and \( \|s_{Q_j}\|_{L^{q,\lambda}} \leq C \) independently. Consider \( b_Q = |Q|^{1/4} a_Q \) and since \( \int_Q |b_Q(x)|^r \, dx \leq |Q| \), from Calderón-Zygmund decomposition applied for \( |b_Q|^r \in L^1(Q) \) at level \( \alpha^r > 0 \), there exists a sequence \( \{Q_j\}_j \) of disjoint dyadic cubes (subcubes of \( Q \)) such that \( |b_Q(x)| \leq \alpha, \ \forall x \notin \bigcup_j Q_j \), \( \alpha^r \leq \int_{Q_j} |b_Q(x)|^r \, dx \leq 2^n \alpha^r \) and \( \bigcup_j |Q_j| \leq \alpha^{-r} \int_Q |b_Q(x)|^r \, dx \leq |Q| \alpha^{-r} \). Let \( P_{N_q} \) to be the space of polynomials in \( \mathbb{R}^n \) with degree at most \( N_q \) and \( P_{N_q, j} \) its restriction to \( Q_j \). Since \( P_{N_q, j} \) is a subspace of the Hilbert space \( L^2(Q_j) \), let \( P_{Q_j} b P_{N_q, j} \) to be the unique polynomial such that \( \int_{Q_j} [b_Q(x) - P_{Q_j}(b)(x)]^2 \, dx = 0 \) for all \( |\beta| \leq N_q \).

Now we write \( b_Q = g_0 + \sum_j h_j \), where \( h_j(x) = [b_Q(x) - P_{Q_j}(b)(x)] \mathbb{I}_{Q_j}(x) \) and \( g_0(x) = b_Q(x) \) if \( x \notin \bigcup_j Q_j \) and \( g_0(x) = P_{Q_j}(b)(x) \) if \( x \in Q_j \). Clearly \( \int h_j(x) \xi^2 \, dx = 0 \) and since \( |g_0(x)| \leq c\alpha \) almost everywhere (see [11, Remark 2.1.4 p. 104]), this implies

\[
\left( \int_{Q_j} |h_j(x)|^r \, dx \right)^{1/r} \leq \left( \int_{Q_j} |b_Q(x)|^r \, dx \right)^{1/r} + \left( \int_{Q_j} |g_0(x)|^r \, dx \right)^{1/r} \leq c\alpha.
\]

For each \( j_0 \in \mathbb{N} \), let \( b_{j_0}(x) := (c\alpha)^{-1} h_{j_0}(x) \) and write \( b_Q(x) = g_0(x) + (c\alpha) \sum_{j_0} b_{j_0}(x) \), where \( \int_{Q_{j_0}} |b_{j_0}(x)|^r \, dx \leq |Q_{j_0}| \). Applying the previous argument for each \( b_{j_0} \) we obtain the identity

\[
b_Q = g_0 + (c\alpha) \sum_{j_0} b_{j_0} = g_0 + c\alpha \sum_{j_0} g_{j_0} + (c\alpha)^2 \sum_{j_0,j_1} b_{j_0,j_1},
\]

where \( \sum_{j_0,j_1} |b_{j_0,j_1}(x)|^r \, dx \leq |Q_{j_0,j_1}| \) and \( \{Q_{j_0,j_1}\}_j \) is a sequence of disjoint dyadic cubes (subcubes of \( Q_{j_0} \)) such that \( |g_{j_0}(x)| \leq c\alpha \) a.e., \( \alpha^r \leq \int_{Q_{j_0,j_1}} |b_{j_0,j_1}(x)|^r \, dx \leq 2^n \alpha^r \) and \( |\bigcup_{j_1} Q_{j_0,j_1}| \leq c\alpha^{-r} \int_{Q_{j_0}} |b_{j_0}(x)|^r \, dx \leq |Q_{j_0}| \alpha^{-r} \). Employing an induction argument, we can find a family \( \{Q_{k_{-1}, \ldots, k_{-1}}\}_j := \{Q_{j_0, \ldots, j_{k-1}}\}_j \) of disjoint dyadic sub-cubes of \( Q_{k_{-1}} := \bigcup_{j_0, j_1, \ldots, j_{k-1}} \) for \( k = 1, 2, \ldots \) with \( i_{k-1} = \{j_0, j_1, \ldots, j_{k-1}\} \) such that

\[
b_Q = g_0 + c\alpha \sum_{i_1} g_{i_1} + (c\alpha)^2 \sum_{i_2} g_{i_2} + \cdots + (c\alpha)^{k-1} \sum_{i_{k-1}} g_{i_{k-1}} + (c\alpha)^k \sum_{i_k} h_{i_k}, \quad (7)
\]

in which \( g_{i_{k-1}} \) and \( h_{i_k} \), for every \( i_k = (j_0, j_1, \ldots, j_{k-1}, f) \), satisfies \( |g_{i_k}(x)| \leq c\alpha \) a.e. \( x \in \mathbb{R}^n \), \( \alpha^r \leq \int_{Q_{i_{k-1}, f}} |h_{i_k}(x)|^r \, dx \leq 2^n \alpha^r \) and \( |\bigcup_{i_{k-1}} Q_{i_{k-1}, f}| \leq c|Q_{i_{k-1}}| \alpha^{-r} \). The sum at (7) is interpreted as \( \sum_{i_{k-1}} g_{i_{k-1}} := \sum_{j_0 \in \mathbb{N}} \cdots \sum_{j_{k-1} \in \mathbb{N}} g_{j_0, \ldots, j_{k-1}} \) (analogously
to \( \sum_{i} h_{i} \). We claim that the reminder term \((ca)^k \sum_{i} h_{i} \) in (7) goes to zero in \( L^{1}(\mathbb{R}^n) \) as \( k \to \infty \). Indeed, writing \( Q_k := Q_{i_{k-1},j} \) for some fixed \( j \) we have

\[
\int_{\mathbb{R}^n} |h_{i_k}(x)| \, dx = \int_{Q_{i_k}} |h_{i_k}(x)| \, dx \leq \left( \int_{Q_{i_k}} |h_{i_k}(x)|^r \, dx \right)^{\frac{1}{r}} |Q_{i_k}|^{1-\frac{1}{r}} \leq c a |Q_{i_k}|
\]

and iterating \((k + 1)\)-times the previous argument one has

\[
\sum_{i_k} |Q_{i_k}| \leq \left( \frac{c}{a^r} \right)^{k+1} |Q|.
\]

Thus, \( \int (ca)^k \sum_{i_k} h_{i_k}(x) \, dx \leq (ca)^{k+1} \sum_{i_k} |Q_{i_k}| \leq (c^2a^{1-r})^{(k+1)} |Q| \). That means, \((ca)^k \sum_{i_k} h_{i_k}(x) \) goes to 0 in \( L^{1}(\mathbb{R}^n) \) as \( k \to \infty \), provided that \( c^2a^{1-r} < 1 \). Therefore,

\[
b_Q = g_{i_0} + ca \sum_{i_1} g_{i_1} + (ca)^2 \sum_{i_2} g_{i_2} + \cdots + (ca)^{k-1} \sum_{i_{k-1}} g_{i_{k-1}} + (ca)^k \sum_{i_k} g_{i_k} + \cdots
\]

in \( L^{1}(\mathbb{R}^n) \), where \( |g_{i_k}(x)| \leq ca \) a.e. and for all \( |\beta| \leq N_q \) we have \( \int x^\beta g_{i_k}(x) \, dx = \int x^\beta b_{i_k}(x) \, dx + \sum_{j} \int_{Q_{i_{k-1},j}} x^\beta P_{Q_{i_{k-1},j}} b(x) \, dx = \int x^\beta b_{i_k}(x) \, dx = 0 \). From the above considerations it is clear that \( a_{i_0} := (ca)^{-1} |Q|^{-1/4} g_{i_0} \) and \( a_{i_k} := (ca)^{-1} |Q_{i_k}|^{-1/4} g_{i_k} \) are \((q, \lambda, \infty)\)–atoms, for all \( k = 1, 2, \ldots \). Moreover, we can write

\[
a_Q = a_{i_0} a_{i_0} + \sum_{i_1} s_{i_1} a_{i_1} + \sum_{i_2} s_{i_2} a_{i_2} + \cdots + \sum_{i_k} s_{i_k} a_{i_k} + \cdots
\]

where each coefficient \( \{s_{i_k}\} \) is defined by \( s_{i_k} = (ca)^{k+1} |Q_{i_k}|^{-1/4} |Q_{i_k}|^{1/4} \). It remains to show that \( \|\{s_{i_k}\}_k\|_{\lambda,q} \leq C \), uniformly. Fixed \( J \subset \mathbb{R}^n \) a dyadic cube, we may estimate

\[
|J|^q \sum_{k=0}^{\infty} \sum_{Q_{i_k} \subset J} |s_{i_k}|^q |Q_{i_k}|^{1-\frac{q}{2}} = |J|^q \sum_{k=0}^{\infty} (ca)^{q(k+1)} \left( \sum_{Q_{i_k} \subset J} |Q_{i_k}| \right) \leq |J|^q \sum_{k=0}^{\infty} (ca)^{q(k+1)} \left( \frac{c}{a^r} \right)^{k+1} \leq C
\]

provided \( c^{q+1}a^{q-r} < 1 \) (weaker than the previous one) and \( q \leq \lambda \). Note that here we have used a refinement of (8) given by \( \sum_{i_k} : Q_{i_k} \subset J \) \( |Q_{i_k}| \leq \left( \frac{c}{a^r} \right)^{k+1} |J \cap Q| \) and the uniform control \( |J|^{q/2} |Q|^{-q/2} |J \cap Q| \leq 1. \)

The previous lemma allow us to study Hardy-Morrey spaces \( \mathcal{HM}^q_{\lambda}(\mathbb{R}^n) \) with any of the atomic spaces \( \mathcal{A} \mathcal{HM}^q_{\lambda,r}(\mathbb{R}^n) \) for \( 1 \leq r \leq \infty \) provided \( q < r \). In addition, we announce an atomic decomposition in terms of \((q, \lambda, r)\)–atoms, which is a direct consequence of the one proved in [10, p. 100] for \((q, \lambda, \infty)\)–atoms and the Lemma 1, since they are in particular \((q, \lambda, r)\)–atoms.

**Theorem 2** Let \( 0 < q \leq 1 \leq r \leq \infty \) with \( q < r \) and \( q \leq \lambda < \infty \). Then, \( f \in \mathcal{HM}^q_{\lambda}(\mathbb{R}^n) \) if and only if there exist a collection of \((q, \lambda, r)\)–atoms \( \{a_Q\}_Q \) and
a sequence of complex numbers \( \{s_Q\}_Q \) such that \( f = \sum_Q s_Q a_Q \) in \( S'(\mathbb{R}^n) \) and \( \|f\|_{\mathcal{A}^{\lambda}_q} \approx \|f\|_{\mathcal{M}_q^\lambda}. \)

### 2.2 Molecular decomposition in Hardy-Morrey spaces

**Definition 2** Let \( 0 < q \leq 1 \leq r < \infty \) with \( q < r, \, q \leq \lambda < \infty, \) and \( s > n (r/q - 1). \) A function \( m(x) \) is called a \((q, \lambda, s, r)\)-molecule in \( \mathcal{M}_q^\lambda(\mathbb{R}^n), \) or simply an \( L^r \) molecule, if there exist a cube \( Q \) such that

\[
(M_1) \quad \int_{\mathbb{R}^n} |m(x)|^r \, dx \leq \ell_Q^{-n(1-\frac{s}{r})} \\
(M_2) \quad \int_{\mathbb{R}^n} |m(x)|^r |x - x_Q|^s \, dx \leq \ell_Q^{1+n(1-\frac{s}{r})}
\]

and also the cancellation condition \((M_3)\)

\[
\int_{\mathbb{R}^n} m(x) x^\alpha \, dx = 0 \text{ for all } |\alpha| \leq N_q.
\]

**Remark 1** Equivalently, we can replace the previous global estimates by \((M_1)\) on \( 2Q \) and \((M_2)\) on \( 2Q^c. \)

**Lemma 2** Let \( m(x) \) to be an \( L^r \) molecule. Then \( m = \sum_Q a_Q a_Q + \sum_Q b_Q b_Q \) in \( L^r(\mathbb{R}^n), \) where each \( \{a_Q\}_Q \) are \((q, \lambda, r)\)-atoms and \( \{b_Q\}_Q \) are \((q, \lambda, \infty)\)-atoms, for a suitable sequence of scalars \( \{d_Q\}_Q \) and \( \{t_Q\}_Q. \)

**Proof** The proof follows the corresponding result for Hardy spaces [7, Theorem 7.16]. Let \( m \) to be a \((q, \lambda, s, r)\)-molecule centered in the cube \( Q. \) For each \( j \in \mathbb{N}, \) let \( Q_j := Q(x_j, \ell_j) \) in which \( \ell_j = 2^j \ell_Q. \) Consider the collection of annulus \( \{E_j\}_{j \in \mathbb{N}_0} \) given by \( E_0 = Q \) and \( E_j = Q \setminus Q_{j-1} \) for \( j \geq 1, \) and let \( m_j(x) := m(x) 1_{E_j}(x). \) By the same arguments presented in the proof of Lemma 1, there exist polynomials \( \{\phi_j^\gamma(x)\}_{|\gamma| \leq N_q} \) uniquely determined in \( E_j \) such that

\[
2^j \ell_Q^{-|\gamma|} |\phi_j^\gamma(x)| \leq 1 \quad \text{and} \quad \frac{1}{|E_j|} \int_{E_j} \phi_j^\gamma(x) x^\beta \, dx = \begin{cases} 1, & \gamma = \beta \\ 0, & \gamma \neq \beta \end{cases}
\]

where the implicit constant is uniformly on \( E_j. \) Let \( m_j^\gamma = \int_{E_j} m_j(x) x^\gamma \, dx \) and consider \( P_j(x) = \sum_{|\gamma| \leq N_q} m_j^\gamma \phi_j^\gamma(x). \) Splitting \( m = \sum_{j=0}^{\infty} (m_j - P_j) + \sum_{j=0}^{\infty} P_j, \) with convergence in \( L^r(\mathbb{R}^n), \) we claim that for each \( j, \) \( m_j - P_j \) is multiple of a \((q, \lambda, r)\)-atom and \( P_j \) is a finite linear combination of \((q, \lambda, \infty)\)-atoms.

For the first sum, since \( m_j \) and \( P_j \) are supported on \( E_j, \) so is \( m_j - P_j \) and by definition one has the desired vanish moments up to the order \( N_q. \) It remains show that \( m_j - P_j \) satisfies the size estimate. Indeed, from conditions \((M_1)\) and \((M_2)\) it follows that for every \( j \in \mathbb{N}_0 \)

\[
|m_j|_{L^s} \leq |E_j|^{\frac{1}{r} - \frac{1}{s}} (2^j)^{-n(\frac{s}{r} - 1)}.
\]

(11)
Also, from (10) it follows $|P_j(x)| \leq \left( \sum_{|\beta| \leq N_q} |\phi^\beta_p(x)|2^{j|\beta|} \right) \int_{E_j} |m_j(x)| dx \lesssim |E_j|^{-\frac{1}{2}}|m_j|_{L^r}$, where the implicit constants are independent of $j$. Hence, if we write $(m_j - P_j)(x) = d_j a_{Q_j}(x)$ for $d_j = \|m_j - P_j\|_{L^r} |Q_j|^{\frac{1}{r} - \frac{1}{2}}$ and $a_{Q_j} = \frac{m_j - P_j}{\|m_j - P_j\|_{L^r}} |Q_j|^{\frac{1}{r} - \frac{1}{2}}$, for each $j \in \mathbb{N}_0$, it is clear that $\{a_{Q_j}\}_j$ is a sequence of $(q, \lambda, r)$-atoms supported on $Q_j$. Moreover, from (11) we have $\|m_j - P_j\|_{L^r} \lesssim \|m_j\|_{L^r} \lesssim |Q_j|^{\frac{1}{r} - \frac{1}{2}} (2^j)^{-\frac{rn(\frac{1}{r} - \frac{1}{2})}{2}}$. Hence, since $s > n(r/q - 1)$

$$\sum_{j=0}^\infty |d_j|^q |Q_j|^{1 - \frac{q}{2}} \leq |Q|^{1 - \frac{q}{2}} \sum_{j=0}^\infty (2^j)^q \left(2^j)^{-\frac{rn(\frac{1}{r} - \frac{1}{2})}{2}}\right) \leq |Q|^{1 - \frac{q}{2}}.$$ 

For the second sum, let $\psi^j_\gamma(x) := N^{j+1}_\gamma \left| |E_{j+1}|^{-\frac{1}{2}} \phi^{j+1}_\gamma(x) - |E_j|^{-\frac{1}{2}} \phi^j_\gamma(x) \right|$, where $N^j_\gamma = \sum_{E_k \in E_j} m_\gamma \mathcal{L}_k = \sum_{E_k \in E_j} \mathcal{L}_k \mathcal{M}_Q(x) \chi_{E_j} dx$. Then, we can represent $P_j$ (using the vanish moments $(M_j)$) as $\sum_{j=0}^\infty P_j(x) = \sum_{j=0}^\infty \sum_{|\gamma| \leq N_q} \psi^j_\gamma(x)$. The function $\psi^j_\gamma$ is supported on $E_{j+1}$ and by construction also satisfies vanish moments conditions up to the order $N_q$. It remains to check the size condition. Since $|\gamma| \leq n(1/\lambda - 1)$ and $s > n(r/q - 1)$ we have $|N^{j+1}_\gamma| \leq |Q_j|^{-1/4} (2^j)^{1/(1/2 - 1)}$. The previous estimate and $(2^j \ell_Q)^{|\gamma|} |\phi^j_\gamma(x)| \leq C \text{ yields for all } x \in E_j$

$$|N^{j+1}_\gamma| |E_j|^{-\frac{1}{2}} \phi^j_\gamma(x) \leq C |Q_j|^{\frac{1}{r} - \frac{1}{2}} (2^j)^{1/(1/2 - 1/2)}.$$ 

Let $\psi^j_\gamma = t_j b^j_\gamma$, where $t_j = (2^j)^{-\frac{1}{2}}$ and $b^j_\gamma(x) = (2^j)^{1/(1/2 - 1)} |\phi^j_\gamma(x)|$. Hence, we can write $\sum_{j=0}^\infty P_j(x) = \sum_{j=0}^\infty \sum_{|\gamma| \leq N_q} t_j b^j_\gamma(x)$, and for each $j \in \mathbb{N}$ the function $b^j_\gamma(x)$ is a $(q, \lambda, \omega)$-atom, since it is supported on $E_{j+1}$ and satisfies $|b^j_\gamma(x)| \lesssim |Q_j|^{\frac{1}{r} - \frac{1}{2}}$, as desired. Moreover from $s > n(r/q - 1)$ one has

$$\sum_{j=0}^\infty |t_j|^q |Q_j|^{1 - \frac{q}{2}} = |Q|^{1 - \frac{q}{2}} \sum_{j=0}^\infty (2^j)^q \left(2^j)^{-\frac{rn(\frac{1}{r} - \frac{1}{2})}{2}}\right) \leq |Q|^{1 - \frac{q}{2}}.$$ 

Now we ready to announce a molecular decomposition in Hardy-Morrey spaces.

**Theorem 3** Let $\{m_\gamma\}_Q be a collection of $L^r$–molecules and $\{s_\gamma\}_Q$ be a sequence of complex numbers such that $\|\{s_\gamma\}_Q\|_{L,q} < \infty$. If the series $f = \sum_{\gamma} s_\gamma m_\gamma$ converges in $S'(\mathbb{R}^n)$ and $\lambda < r$, then $f \in \mathcal{M}^\lambda_\gamma(\mathbb{R}^n)$ and moreover, $\|f\|_{\mathcal{M}^\lambda_\gamma} \lesssim \|\{s_\gamma\}_Q\|_{L,q}$ with implicit constant independent of $f$.

**Proof** Suppose $f = \sum_{\gamma} s_\gamma m_\gamma$ in $S'(\mathbb{R}^n)$ and $\|\{s_\gamma\}_Q\|_{L,q} < \infty$. Since $0 < q \leq 1$, for a fixed dyadic cube $J \subset \mathbb{R}^n$ we may estimate $\int_J |M_{|\cdot|} f(x)|^q dx$ by
Estimate of $I_1$. From Lemma 2, write $m_Q = \sum_{j=0}^{\infty} d_j a_{Q_j}$ (convergence in $L^r$) where \( \{a_{Q_j}\} \) are $\langle q, \lambda, r \rangle$-atoms and moreover $\sum_{j=0}^{\infty} |d_j| q |Q_j|^{1-\frac{r}{q}} \leq |Q|^{1-\frac{r}{q}}$. It follows from analogous estimates of Proposition 1 that

\[ I_1 \leq \sum_{Q \subseteq J} |s_Q|^q \int_{\mathbb{R}^n} |M_{\varphi} m_Q(x)|^q dx + \sum_{J \subseteq Q} |s_Q|^q \int_{\mathbb{R}^n} |M_{\varphi} m_Q(x)|^q dx : = I_1 + I_2. \]

Estimate of $I_2$. Since $1 < r < \infty$ and $M_{\varphi}$ is bounded on $L^r(\mathbb{R}^n)$, it follows

\[ |J|^{q-1} \int_{\mathbb{R}^n} |M_{\varphi} m_Q(x)|^q dx \leq |J|^q \left( \sum_{j=0}^{\infty} (2^j \ell_Q)^{-r} \int_{E_j} |m_Q(x)|^r |x-x_Q|^s dx \right)^{\frac{q}{r}} \leq |J|^q |Q| q (\frac{1}{q}) \left( \sum_{j=0}^{\infty} 2^{-js} \right)^{\frac{q}{r}} \leq \left( \frac{|J|}{|Q|} \right)^{q (\frac{1}{q})}. \]

If $r = 1$ and $0 < q < 1$, we proceed like in (6) and then

\[ |J|^{q-1} \left[ |J| \int_0^{\infty} \omega^{q-1} d\omega + |Q|^{q-1} \omega^{q-2} d\omega \right] \leq \left( \frac{|J|}{|Q|} \right)^{q (\frac{1}{q})}. \]

Fixed a dyadic cube $J$, we point out there exists a subset $N \subseteq \mathbb{N}$ such that each cube $J \subseteq Q$ is uniquely determined by a dyadic cube $Q_{k,J} \in \{Q \text{ dyadic : } J \subseteq Q \text{ and } \ell_Q = 2^k \ell_J \}$. Hence, we can write

\[ \sum_{Q \subseteq J} |s_Q|^q \left| \frac{|J|}{|Q|} \right|^q = \sum_{k \in N} |s_{Q_{k,J}}|^q 2^{-kq\gamma} \]

with $\gamma := 1/\lambda - 1/r > 0$. Then,

\[ |J|^{q-1} \sum_{Q \subseteq J} |s_Q|^q \|M_{\varphi}(a_Q)\|_{L^q(J)}^q \leq \sum_{k \in N} \left| \frac{Q_{k,J}}{Q} \right|^{1-\frac{q}{2}} |Q_{k,J}|^{\frac{q}{2}-1} 2^{-kq\gamma} \leq \sum_{k \in N} \left( \sum_{Q \subseteq Q_{k,J}} |s_Q|^q |Q|^{1-\frac{q}{2}} \right) |Q_{k,J}|^{\frac{q}{2}-1} 2^{-kq\gamma} \leq \| \{s_Q\}_Q \|_{L^q(J)} \sum_{k \in N} 2^{-kq \gamma} \leq \| \{s_Q\}_Q \|_{L^q}. \]

\[ \square \]
Remark 2 The Theorem 3 covers [9, Theorem 2.6] when \( r = 2 \) where the natural restriction \( \lambda < 2 \) was omitted.

3 Proof of Theorem 1

Proof Let \( a \) be a \((q, \lambda, r)\)–atom supported in the cube \( Q \). From Theorem 3, it suffices to show that \( Ta \) is a \((q, \lambda, s, r)\)– molecule associated to \( Q \). Suppose first that \( \ell_Q \geq 1 \). Since \( T \) is bounded in \( L^2(\mathbb{R}^n) \) to itself and \( 1 \leq r \leq 2 \), condition \((M_1)\) follows by

\[
\int_{2Q} |Ta(x)|^r \, dx \leq |2Q|^{1-\frac{r}{p}} \|Ta\|_{L^p}^r \leq |Q|^{1-\frac{r}{p}} \|a\|_{L^p}^r \leq |Q|^{1-\frac{r}{p}} \ell_Q n^{1-\frac{r}{p}}. \tag{12}
\]

For \((M_2)\) using the moment condition of the atom \( a \), Minkowski inequality and (3), we estimate \( \int_{2Q^c} |Ta(x)|^r |x-x_Q|^\delta \, dx \) by

\[
\sum_{j=1}^\infty \int_{2Q^c} \left| K(x,y) - K(x,x_Q) \right| a(y) \, dy |x-x_Q|^\delta \, dx \\
\leq \sum_{j=1}^\infty \left( 2^j \ell_Q \right)^s \int_{2Q^c} |a(y)| \left( \int_{C_j(x_Q, \ell_Q)} \left| K(x,y) - K(x,x_Q) \right| \, dy \right)^\frac{r}{s} \, dx \\
\leq \sum_{j=1}^\infty \left( 2^j \ell_Q \right)^{s-n(r-1)} 2^{-jr} \ell_Q n^{1-\frac{r}{p}} \ell_Q^{r-n(1-\frac{r}{p})} \sum_{j=1}^\infty 2^{j(s-n(r-1)+r)} \leq \ell_Q^{s+n(1-\frac{r}{p})},
\]

assuming \( s < n(r-1)+r\delta \). We remark that for the case \( r = 1 \), one needs to consider \((q, \lambda, s, 1)\)–molecules and hence \( 0 < q \leq \lambda < 1 \). Suppose now that \( \ell_Q < 1 \). Since \( T \) is a bounded operator from \( L^p(\mathbb{R}^n) \) to \( L^2(\mathbb{R}^n) \) and \( 1 < r \leq 2 \), condition \((M_1)\) follows by

\[
\int_{2Q} |Ta(x)|^r \, dx \leq |2Q|^{1-\frac{r}{p}} \|Ta\|_{L^p}^r \leq |Q|^{1-\frac{r}{p}} \|a\|_{L^p}^r \leq |Q|^{1-\frac{r}{p}} \ell_Q^{r+n(1-\frac{r}{p})} \approx |Q|^{1-\frac{r}{p}}.
\]

To estimate the global \((M_2)\) condition, we consider \( 0 < \rho \leq \sigma \leq 1 \) a parameter that will be chosen conveniently later, denote by \( 2Q^\rho := Q(x_Q, 2\ell_Q^\rho) \) and split the integral over \( \mathbb{R}^n \) into \( 2Q^\rho \) and \( (2Q^\rho)^c \). For \( 2Q^\rho \) we use the boundedness from \( L^\rho(\mathbb{R}^n) \) to \( L^2(\mathbb{R}^n) \) again and obtain

\[
\int_{2Q^\rho} |Ta(x)|^r |x-x_Q|^\delta \, dx \leq \ell_Q^{\rho s+n(1-\frac{r}{p})} \|4Q^\rho |^r \|Ta\|_{L^p}^r \leq \ell_Q^{\rho s+n(1-\frac{r}{p})} \|a\|_{L^p}^r \\
\leq \ell_Q^{\rho s+n(1-\frac{r}{p})} \leq \ell_Q^{s+n(1-\frac{r}{p})}.
\]
assuming \( s \leq -n \left( 1 - \frac{r}{2} \right) + \frac{nr}{1 - \rho} \left( \frac{1}{p} - \frac{1}{2} \right) \). For \((2Q^p)^c\), we use (4) to estimate 
\[ \int (2Q^p)^c |Ta(x)|^r |x - x_Q|^s dx \] by

\[ \sum_{j=1}^{\infty} (2^j \ell_j^{Q^c})^s \left\{ \int_Q |a(y)| \left( \int_{C_j(x_Q, \ell_j^{Q^c})} |K(x, y) - K(x, x_Q)|^p dx \right)^{\frac{1}{p}} dy \right\}^r \]

\[ \leq \sum_{j=1}^{\infty} (2^j \ell_j^{Q^c})^s \left( C_j(x_Q, \ell_j^{Q^c}) \right)^{\frac{1}{p} + \frac{n}{n - \rho}} \left( 2 \frac{r}{p} \right)^r \ell_j^{Q^c} \left( \frac{n}{p} \right) \]

\[ \approx \ell_j^{Q^c} \left[ 1 + \frac{n}{n - \rho} \right] \sum_{j=1}^{\infty} 2^{-n(r - 1)} \]

\[ \leq \ell_j^{Q^c} \left[ 1 + \frac{n}{n - \rho} \right] \]

where the convergence follows assuming \( s < n(r - 1) + \frac{nr}{1 - \rho} \) and we choose \( \rho \) to be such that \( r + \frac{r\delta}{n - \rho} \left( r - 1 + \frac{r\delta}{n - \rho} \right) = \rho \left( 1 - \frac{r}{2} \right) + \frac{r}{p} \Leftrightarrow \rho := \frac{n \left( 1 - \frac{1}{p} \right) + \delta}{\frac{n}{2} + \frac{\delta}{\sigma}} \).

By the choice of \( \rho \) we have

\[ -n \left( 1 - \frac{r}{2} \right) + \frac{nr}{1 - \rho} \left( \frac{1}{p} - \frac{1}{2} \right) < n(r - 1) + r\delta < n(r - 1) + \frac{r\delta}{\sigma} \]

In particular, collecting the restrictions on \( s \) we get

\[ n \left( \frac{r}{q} - 1 \right) < s \leq -n \left( 1 - \frac{r}{2} \right) + \frac{nr}{1 - \rho} \left( \frac{1}{p} - \frac{1}{2} \right) \Rightarrow \frac{1}{q} < \frac{1}{2} + \frac{\beta (\frac{\delta}{\sigma} + \frac{1}{2})}{n \left( \frac{\delta}{\sigma} - \delta + \beta \right)} := \frac{1}{q_0} \]

We point out that when \( \sigma = 1 \), only condition \( s < n(r - 1) + r\delta \) is imposed to verify \((M_1)\) and \((M_2)\). Condition \((M_3)\), given formally by \( T''(x^\sigma) = 0 \) for all \( |x| \leq N_q \), is trivially valid, since \( n/(n + \delta) < q_0 < q \leq 1 \) implies \( N_q \leq \lfloor \delta \rfloor \).

Remark 3  The previous proof remains the same if one consider integral conditions incorporating derivatives of the kernel. For a complete discussion and the precise definition of such conditions see [12, Section 4.2].

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