Hawking radiation of scalar particles and fermions from squashed Kaluza–Klein black holes based on a generalized uncertainty principle

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Abstract
We study the Hawking radiation from the five-dimensional charged static squashed Kaluza–Klein black hole by the tunneling of charged scalar particles and charged fermions. In contrast to the previous studies of Hawking radiation from squashed Kaluza–Klein black holes, we consider the phenomenological quantum gravity effects predicted by the generalized uncertainty principle with the minimal measurable length. We derive corrections of the Hawking temperature to general relativity, which are related to the energy of the emitted particle, the size of the compact extra dimension, the charge of the black hole and the existence of the minimal length in the squashed Kaluza–Klein geometry. We obtain some known Hawking temperatures in five and four-dimensional black hole spacetimes by taking limits in the modified temperature. We show that the generalized uncertainty principle may slow down the increase of the Hawking temperature due to the radiation, which may lead to the thermodynamic stable remnant of the order of the Planck mass after the evaporation of the squashed Kaluza–Klein black hole. We also find that the sparsity of the Hawking radiation modified by the generalized uncertainty principle may become infinite when the mass of the squashed Kaluza–Klein black hole approaches its remnant mass.

Keywords: Hawking radiation, quantum gravity, generalized uncertainty principle, tunneling of particles, higher-dimensional black holes, black hole evaporation, black hole remnants
1. Introduction

Hawking radiation is one of the interesting phenomena where both of general relativity and quantum theory play a role. While a black hole does not emit radiation from a classical point of view, it is shown that, using the semiclassical method of quantum theory in curved spacetimes, a black hole can radiate from an event horizon like a blackbody at the Hawking temperature which is proportional to the surface gravity of the black hole [1]. Hawking radiation and its black hole temperature are derived by several approaches. Parikh and Wilczek proposed an elegant derivation of Hawking radiation on the basis of the tunneling method [2]. The essential idea of the tunneling mechanism is that particle–antiparticle pairs are formed close to the horizon inside a black hole. While the ingoing particles moving toward the center of the black hole are trapped inside the horizon, a part of the outgoing particles escapes outside the horizon by the quantum tunneling effect. If the particle which comes out to our Universe has positive energy, we can regard such a particle outside the horizon as the radiation from the black hole. The Hawking temperature is derived by comparing the tunneling probability of an outgoing particle with the Boltzmann factor in thermodynamics. The tunneling method has also been used to obtain the Hawking radiation of scalar particles and fermions across an event horizon in some black hole geometries [3–6]. Then the derivation of Hawking radiation on the basis of the tunneling mechanism has been actively discussed in the literature (see the reference [7] as a review).

A generalized uncertainty principle with a minimal measurable length is attracted a lot of interest to consider phenomenological quantum gravity effects. Though the debates on the quantum nature of black holes are crucial and long standing, there is no systematic study by quantum gravity so far to lay a solid theoretical foundation for the arguments. Then some different models of quantum gravity are effective ways to understand gravity behaviors at a sufficiently small scale. The combination of relativistic and quantum effects implies that the conventional notion of distance would break down the latest at the Planck scale. The basic argument is that the resolution of small distances requires test particles of short wavelengths and thus of high energies. At such small scales, the gravitational effects by the high energies of test particles would significantly disturb the spacetime structure which was tried. Then some quantum gravity theories suggest that there would exist a finite limit to the possible resolution of distances, which would be of order the Planck length. From several studies in string theory, such a minimal measurable length would be obtained by a generalized uncertainty principle, which is a quantum gravity inspired modification to the conventional Heisenberg uncertainty principle. Then various types of generalized uncertainty principles have been heuristically derived from thought experiments and are often taken as phenomenological models that would accommodate a minimal length [8–10], though there exist attempts to make its formulations more mathematically rigorous [11]. Generalized uncertainty principles have been applied to some different systems and played an important role to consider its corrections by supposed quantum gravity theories [12–17]. Motivated by these discussions of generalized uncertainty principles, we are interested in performing in-depth studies of the quantum features of higher-dimensional black hole solutions. Quantum gravity effects predicted by generalized uncertainty principles on the Hawking radiation have been studied by the tunneling of various particles in a variety of spacetimes including vacuum, electrovacuum and with a vast array of scalar fields or effective fluids [18–24]. In this paper, we focus on the quantum tunneling radiation of charged particles coming from higher-dimensional black holes based on a generalized uncertainty principle in the spacetime with compact extra dimensions.
Higher-dimensional black hole solutions are actively discussed in the context of string theories and braneworld models. Since our observable world is effectively four dimensional, we can regard higher-dimensional black hole solutions with compactified extra dimensions as candidates of realistic models. We call these Kaluza–Klein black holes. The four-dimensional Schwarzschild metric uniquely describes the general relativistic gravitational field in vacuum with spherical symmetry. However, even if we impose asymptotic flatness to the four-dimensional part of the higher-dimensional spacetime model with Kaluza–Klein structure, the metric is not uniquely determined. A family of five-dimensional squashed Kaluza–Klein black hole solutions [25–29] represent fully five-dimensional black holes near the squashed $S^3$ horizons and asymptote to effective four-dimensional spacetimes with a twisted $S^1$ as an extra dimension at infinity. Then we can regard a series of squashed Kaluza–Klein black hole solutions with a twisted compactified extra dimension as one of realistic higher-dimensional black hole models. Several aspects of squashed Kaluza–Klein black holes have been discussed, for example, thermodynamics [30–32], Hawking radiation [33–36], stabilities [37, 38], gyroscope precession [39, 40], thin accretion disk [41], x-ray reflection spectroscopy [42], light deflection [43], strong gravitational lensing [44–48] and black hole shadow [49, 50].

In this paper, we investigate the Hawking radiation by the tunneling of charged particles and its quantum gravity effects based on the generalized uncertainty principle in the five-dimensional charged static squashed Kaluza–Klein black hole spacetime. To the best our knowledge, Hawking radiation as a tunneling process in the presence of a generalized uncertainty principle has not been discussed in asymptotically Kaluza–Klein spacetimes. Then, in contrast to the previous studies of Hawking radiation from squashed Kaluza–Klein black holes [33–36], we extend the derivation of Hawking radiation on the basis of the tunneling mechanism in four-dimensional black hole spacetimes to the case of the five-dimensional squashed Kaluza–Klein black hole, including the phenomenological quantum gravity effects predicted by the generalized uncertainty principle with the minimal measurable length.

According to the discussion of the Hawking radiation in general relativity, primordial mini black holes in the very early Universe would have been completely evaporated [51]. However, during the final stages of the Hawking evaporation, the semiclassical approach would be expected to break down due to the dominance of quantum gravity effects. If the black holes would not evaporate completely but turn into the remnants at the end of the evaporation process, such remnants might be a constituent of dark matter, similar to other candidates including weakly interacting massive particles, the axion, the axino, the neutralino and the gravitino [52–58]. Then it is interesting to investigate how higher-dimensional black holes with compact extra dimensions evolve under a Hawking evaporation process modified by a generalized uncertainty principle. In this paper, we study the evaporation of the squashed Kaluza–Klein black hole by the tunneling of particles in the framework of the generalized uncertainty principle as one of the quantum gravity effects in the Hawking radiation.

This paper is organized as follows. In the section 2, we review the properties of five-dimensional charged static Kaluza–Klein black hole solutions with squashed horizons. In the section 3, we consider the tunneling radiation of charged scalar particles from the squashed Kaluza–Klein black hole in the context of the generalized uncertainty principle and derive the modified Hawking temperature with the quantum gravity effects. We obtain some known Hawking temperatures in five and four-dimensional black hole spacetimes by taking limits in the modified temperature. Then we investigate the evaporation process of the black hole and the sparsity of the radiation by the tunneling of scalar particles in the squashed Kaluza–Klein geometry. In the section 4, we study the Hawking radiation by the tunneling of charged fermions from the squashed Kaluza–Klein black hole based on the generalized uncertainty principle. The section 5 is devoted to summary and discussion.
2. Squashed Kaluza–Klein black holes

We consider the charged static Kaluza–Klein black holes with squashed $S^3$ horizons in the five-dimensional Einstein–Maxwell theory [27]. The metric and the Maxwell field are respectively given by

$$ds^2 = -F dt^2 + \frac{K^2}{F} dr^2 + r^2 K^2 (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{r^2}{4K^2} (d\psi + \cos \theta d\phi)^2,$$

\[ F = \frac{(\rho - \rho_+)(\rho - \rho_-)}{\rho^2}, \quad K^2 = \frac{\rho + \rho_0}{\rho}, \tag{1} \]

with

$$A_\mu dx^\mu = \pm \frac{\sqrt{3}\rho - \rho_+}{2\rho} dt, \tag{2}$$

where the parameters $\rho_+, \rho_-$ and $\rho_0$ denote the outer and the inner horizons, and the typical scale of transition from five dimensions to effective four dimensions, respectively [39].

The parameter $r_\infty = 2\sqrt{\left(\rho_+ + \rho_0\right)(\rho_- + \rho_0)}$ gives the size of the compactified extra dimension $\psi$ at infinity. The coordinates run the ranges of $-\infty < t < \infty, 0 < \rho < \infty, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi$ and $0 \leq \psi \leq 4\pi$. The squashed Kaluza–Klein black hole solution is asymptotically locally flat, i.e. the metric asymptotes to a twisted constant $S^1$ fiber bundle over the four-dimensional Minkowski spacetime. In this paper, to avoid the existence of naked singularities on and outside the horizon, we restrict ourselves to the ranges of parameters such that $\rho_+ \geq \rho_- \geq 0, \rho_- + \rho_0 > 0$. The Komar mass and the charge of the black hole are given by

$$M = \frac{\pi r_\infty}{G_5} (\rho_+ + \rho_-) = \frac{\rho_+ + \rho_-}{2G_4}, \quad |Q| = \frac{2\pi r_\infty}{G_5} \sqrt{\rho_+ \rho_-} = \frac{\sqrt{\rho_+ \rho_-}}{G_4}, \tag{4}$$

respectively, where the five-dimensional gravitational constant $G_5$ and the four-dimensional one $G_4$ are related as $G_5 = 2\pi r_\infty G_4$ [39]. In terms of the mass and the charge, the outer and the inner horizons are expressed as $\rho_\pm = G_4 \left( M \pm \sqrt{M^2 - Q^2} \right)$. The Hawking temperature and the entropy of the black hole associated with the surface gravity of the outer horizon $\kappa$ and the area of the outer horizon $A$ are obtained as

$$T_{KK} = \frac{\kappa}{2\pi} = \frac{\rho_+ - \rho_-}{4\pi \rho_+ \sqrt{\rho_+ (\rho_+ + \rho_0)}}, \tag{5}$$

$$S_{KK} = \frac{\sqrt{\rho_+ (\rho_+ + \rho_0)}}{4G_5}.$$

\[ S_{KK} = \frac{\sqrt{\rho_+ (\rho_+ + \rho_0)}}{4G_5}, \tag{6} \]

respectively, which satisfy the Smarr-type formula $M - 2QA_+ / \sqrt{3} = 2T_{KK}S_{KK}$, where $|A_+| = \sqrt{3\rho_- / \rho_+} / 2$ is the gauge potential on the outer horizon [30–32]. When $\rho_- = \rho_+$, the Hawking temperature $T_{KK}$ vanishes and the metric (1) represents the extremal charged Kaluza–Klein black hole with the mass $M = |Q|$ [27]. In the limit $\rho_0 \rightarrow 0$, we obtain the metric (1) with $K = 1$ which represents the four-dimensional Reissner–Nordström black hole with a twisted constant $S^1$ fiber. We expect the appearance of the higher-dimensional corrections,
which are related to the parameter $\rho_0$, to the tunneling radiation of particles across the black hole horizon and its Hawking temperature of four-dimensional relativity.

### 3. Tunneling of scalar particles with quantum gravity effects

#### 3.1. Derivation of modified Hawking temperature

We consider the Hawking radiation by the tunneling of charged particles from the five-dimensional charged static squashed Kaluza–Klein black hole (1), including the quantum gravity effects predicted by a generalized uncertainty principle. A number of generalized uncertainty principles are proposed as phenomenological models that would describe quantum gravity inspired uncertainty in the position of a particle in the vicinity of the black hole horizon [8, 9, 11]. In this paper, we focus on the quadratic generalized uncertainty principle with the minimal measurable length in the form [8, 9, 15]

$$\Delta x \Delta p \geq \frac{1}{2} \left[ 1 + \beta (\Delta p)^2 \right], \quad (7)$$

where $\Delta x$ and $\Delta p$ respectively represent the uncertainties in the position and the momentum of a particle, $\beta = \beta_0 \rho_0^2 = \beta_0 / m_0^2$ is the parameter encoding quantum gravity effects on the particle dynamics, $\beta_0$ is a dimensionless positive deformation parameter, $l_0 = \sqrt{\hbar G_4}$ is the Planck length and $m_0 = \sqrt{\hbar / G_4}$ is the Planck mass. When $\Delta x \gg l_0$, one recovers the Heisenberg uncertainty principle as $\Delta x \Delta p \geq 1/2$. In standard quantum mechanics, $\Delta x$ can be made arbitrarily small by letting $\Delta p$ grow correspondingly. However, this is no longer the case if the generalized uncertainty relation (7) holds. When $\Delta p$ increases with decreasing $\Delta x$, the term $\beta (\Delta p)^2$ on the right-hand side of the uncertainty relation (7) will eventually grow faster than the left-hand side. Then $\Delta x$ can no longer be made arbitrarily small and there exists a minimal measurable length $\Delta x_0 = l_0 \sqrt{\beta_0}$. The generalized uncertainty relation (7) implies a small correction term to the commutation relation in the associative Heisenberg algebra for mirror-symmetric states, i.e. $[x_i, p_j] = i \left[ 1 + \beta g^{ij} p_k p_l \right] \delta_{ij}$, where $x_i = x_{i0}$ is the position operator, $p_i = p_{i0} \left[ 1 + \beta g^{ij} p_j p_k \right]$ is the momentum operator, and $x_{i0}$ and $p_{i0}$ satisfy the canonical commutation relation $[x_{i0}, p_{j0}] = i \hbar \delta_{ij}$ [8, 9]. As the consequences of the generalized uncertainty principle (7), using the energy mass shell condition $E^2 = g^{ij} p_i p_j + m^2$, the modified energy operator takes the form $\tilde{E} = E \left( 1 - \beta m^2 - \beta g^{ij} p_i p_j \right)$, where $E = i (\partial / \partial t)$ is the energy operator in standard quantum mechanics and $m$ is the mass of a particle [13].

The Hawking radiation emitted by black holes may contain several kinds of particles. The choice of emitted particles is based on the prediction that the Hawking radiation would favor lower spin, lighter particles and it would be expected to be dominated by scalar particles [59–61]. Then, in this section, we consider the tunneling of charged scalar particles across the horizon of the squashed Kaluza–Klein black hole with the quantum gravity effect. Using the momentum operator $p_i$ and the energy operator $\tilde{E}$, we obtain the modified Klein–Gordon equation with the Maxwell field up to the first order in the parameter $\beta$ [22] as

$$\left[ g^{\mu \nu} (\hbar \partial_\nu + ieA_\nu) \left( \hbar \partial_\mu + ieA_\mu \right) + g^{ij} (\hbar \partial_i + ieA_i) \left( \hbar \partial_j + ieA_j \right) - m^2 \right] \left( 1 - 2 \beta m^2 + 2 \beta \hbar^2 g^{ij} g^{kl} \partial_i \partial_l \right) \Psi = 0, \quad (8)$$

where $\partial_\mu = \partial / \partial x^\mu$, $e$ is the charge of the particle, $\Psi$ is the modified scalar field. When $A_\mu = 0$, the equation (8) describes the modified Klein–Gordon equation without the Maxwell field [21].
We assume that the wave function of the Klein–Gordon equation (8) takes the form

$$\Psi = \exp \left( \frac{i}{\hbar} I(\chi) \right),$$

(9)

where $I$ is the action of the emitted scalar particle. Substituting the metric (1), the Maxwell field (2) and the ansatz (9) into the Klein–Gordon equation (8), the Wentzel–Kramers–Brillouin approximation to the leading order in $\hbar$ yields the equation of motion

$$\frac{1}{F} (\partial_t I + eA_\nu)^2 + \sigma (2\beta \sigma - 1) = 0,$$

(10)

with

$$\sigma = m^2 + \frac{F}{K^2} (\partial_\mu I)^2 + \frac{1}{\rho^2 K^2} (\partial_\rho I)^2 + \frac{\left( \partial_\rho I - \cos \theta \partial_\phi I \right)^2}{\rho^2 K^2 \sin^2 \theta} + \frac{4K^2}{r^2} (\partial_\phi I)^2,$$

(11)

where we relax the possible restriction on the parameter $\beta_0$ and regard the parameter $\beta$ as an independent variable. According to the Killing vector fields in the squashed Kaluza–Klein spacetime, $\partial/\partial t, \partial/\partial \phi$ and $\partial/\partial \psi$, we consider the action $I$ in the form

$$I = -\omega t + W(\rho, \theta) + J\phi + L\psi,$$

(12)

where $\omega, J$ and $L$ are the emitted particle’s energy and the angular momenta in the $\phi$ and the $\psi$ directions, respectively. Substituting the action (12) into the equation (10), we obtain

$$m^2 \left( 1 - 2\beta m^2 \right) - \frac{1}{F} (\omega - eA_\nu)^2 + \frac{1 - 4\beta m^2}{K^2} F \left( \frac{\partial W}{\partial \rho} \right)^2$$

$$- \frac{2\beta F^2}{K^4} \left( \frac{\partial W}{\partial \rho} \right)^4 = 0,$$

(13)

where we restrict ourselves to the s-wave particles with $\theta = \text{const}$, $J = 0, L = 0$, since the tunneling effect is a quantum one arising within the Planck length near the horizon region [2–6]. From the equation (13), we see that the function $W(\rho, \theta)$ can be written as

$$W(\rho, \theta) = R(\rho) + \Theta(\theta).$$

Then, solving the equation (13) with the relation $\partial W/\partial \rho = dR/d\rho$ on the black hole horizon $\rho = \rho_+$ yields the imaginary part of the action,

$$\text{Im } R_{\text{out}} = -\text{Im } R_{\text{in}} = \frac{\pi \rho_+ \left( \rho_+ + \rho_0 \right) (\omega - eA_+) \left( \rho_+ - \rho_- \right)}{\rho_+ - \rho_-} \left( 1 + \beta \Xi_s \right) + O \left( \beta^2 \right),$$

(14)

with

$$\Xi_s = \frac{m^2}{2} + \frac{\rho_+ \left( \omega - eA_+ \right)}{2 \left( \rho_+ + \rho_0 \right) \left( \rho_+ - \rho_- \right)^2} \left[ \omega \left( 4\rho_+ \left( \rho_+ - 2\rho_- \right) + \rho_0 \left( 3\rho_+ - 7\rho_- \right) \right) \right.$$

$$\left. + eA_+ \left( 2\rho_+ \left( \rho_+ + \rho_- \right) + \rho_0 \left( 3\rho_+ + \rho_- \right) \right) \right],$$

(15)
where the $\rho$-integral is performed by deforming the contour around the pole at the horizon which lies along the line of integration and gives $\pi i$ times the residue, and $R_{\text{out}}$ and $R_{\text{in}}$ correspond to the outgoing and the ingoing solutions, respectively. Then the tunneling probability amplitude of the charged scalar particle takes the form

$$
\Gamma \simeq \exp\left(-\frac{4\pi \rho_+ \sqrt{\rho_+ (\rho_+ + \rho_0) (1 + \beta \Xi_s)}}{\rho_+ - \rho_-} (\omega - eA_+) \right).
$$

Thus, by comparing the probability amplitude (16) to the first order in the energy with the Boltzmann factor $\Gamma = \exp\left(-\frac{(\omega - eA_+)}{T}\right)$ in a thermal equilibrium state at the temperature $T$, we obtain the modified Hawking temperature of the squashed Kaluza–Klein black hole (1) as

$$
T = T_{\text{KK}} (1 - \beta \Xi_s) + O (\beta^2),
$$

where the temperature $T_{\text{KK}}$ and the correction $\Xi_s$ are given by the equations (5) and (15), respectively. We see that the modified Hawking temperature (17) depends on the energy $\omega$, the mass $m$ and the charge $e$ of the emitted scalar particle, and is modified by the squashed Kaluza–Klein geometry, the Maxwell field and the generalized uncertainty principle through the parameters $\rho_0$, $\rho_-$ and $\beta$, respectively.

By taking some limits in the equation (17), we obtain the Hawking temperatures of some five and four-dimensional black holes. First, when $\beta = 0$, the equation (17) coincides with the Hawking temperature of the five-dimensional squashed Kaluza–Klein black hole [30–32]. Second, when $\rho_- = 0$, introducing the new parameters $\rho_+ = r_+^2 / \left(2 \sqrt{r_\infty^2 - r_+^2}\right)$ and $\rho_0 = \sqrt{r_\infty^2 - r_+^2} / 2$, then taking the limit $r_\infty \to \infty$, the equation (17) represents the modified Hawking temperature of the five-dimensional Schwarzschild–Tangherlini black hole obtained by the uncharged scalar particle tunneling [22]. Lastly, when $\rho_0 = 0, \rho_- = 0$, the equation (17) represents the modified Hawking temperature of the four-dimensional Schwarzschild black hole by the uncharged scalar particle tunneling [21].

### 3.2. Evaporation of black holes

We consider the evaporation process of the squashed Kaluza–Klein black hole by the tunneling of scalar particles, which is affected by the correction $\Xi_s$ in the modified Hawking temperature $T$, as one of the quantum gravity effects in the Hawking radiation. Since all particles emitted by the Hawking radiation near the horizon region are effectively massless [2–6], the mass of the emitted scalar particle is not taken into account in the following discussion, i.e. $m = 0$. The mass of the squashed Kaluza–Klein black hole would decrease due to the radiation. When the black hole mass approaches the order of the Planck mass, the quantum gravity effect could be considered and we may discuss the value of the correction $\Xi_s$. First, when $\Xi_s = 0$, the effects related to the energy of the scalar particle, the asymptotically Kaluza–Klein structure and the Maxwell field are canceled. Then the Hawking temperature without the quantum gravity correction $T_{\text{KK}}$ appears and results in the complete evaporation. Second, when $\Xi_s < 0$, the temperature $T$ is higher than the temperature $T_{\text{KK}}$. Then the black hole accelerates the evaporation and there is no remnant left. Lastly, when $\Xi_s > 0$, the temperature $T$ is lower than the temperature $T_{\text{KK}}$. This implies that the combination of the effects related to the
energy of the scalar particle, the compact extra dimension, the Maxwell field and the generalized uncertainty principle slows down the increase of the temperature due to the radiation. Then the evaporation may cease at the particular mass of the black hole and the black hole may be in a stable balanced state, leading to the remnant mass. We show the parameter regions of $\Xi_s$ in the figure 1. From the equation (15) and the figure 1, we see that, for fixed $\rho_0/\rho_+$ and $e/\omega$, the value of $\Xi_s$ decreases and changes from positive to negative as $\rho_-/\rho_+$ increases. Then the black holes with the large $Q$ would favor negative $\Xi_s$ and evaporate completely. We also find that the positive value of $\Xi_s$ and its parameter region decrease with increasing $\rho_0/\rho_+$ for fixed $e/\omega$. Then the effectively four-dimensional black holes with the small $\rho_0$ and the small $Q$ would favor positive $\Xi_s$ and turn into the remnants at the final stage of the evaporation.

Here, we restrict ourselves to the uncharged scalar particle radiation from the uncharged Kaluza–Klein black hole (1), i.e. $\rho_- = 0$. In this case, the correction $\Xi_s$ is positive. Then, using the lower bound on the energy of the emitted particle $\omega \geq 1/\Delta x$ [14] and the uncertainty in the position $x$ for the events near the black hole horizon $\Delta x \simeq 2\rho_+$ [14], we have the Hawking temperature (17) in terms of $M, r_{\infty}, \beta_0, m_p$ and $l_p$ as

$$T = \frac{m_p}{4\pi} \sqrt{\mu \left(2\mu + \sqrt{4\mu^2 + \nu^2}\right)} \left(1 - \beta_0 \frac{10\mu + 3\sqrt{4\mu^2 + \nu^2}}{32\mu^2 \left(2\mu + \sqrt{4\mu^2 + \nu^2}\right)}\right) + O\left(\beta_0^2\right),$$

(18)

where $\mu = M/m_p$ and $\nu = r_{\infty}/l_p$. We find that, for the large black hole masses, since the quantum gravity effect is negligible at that scale, the modified temperature (18) asymptotes to the
temperature $T_{\text{KK}}$ with $\rho_+ = 0$, which monotonically increases with decreasing $M/m_p$ for fixed $r_\infty/l_p$ and is higher than the temperature (18) with $\beta_0 \neq 0$. Using the temperature (18) and the Smarr-type formula, we have the entropy with the quantum gravity effect as

$$S = 2\pi \mu \sqrt{\mu \left(2\mu + \sqrt{4\mu^2 + \nu^2}\right)} \left(1 + \beta_0 \frac{10\mu + 3\sqrt{4\mu^2 + \nu^2}}{32\mu^2 \left(2\mu + \sqrt{4\mu^2 + \nu^2}\right)}\right) + O(\beta_0^2).$$

(19)

When the deformation parameter of the generalized uncertainty principle vanishes, i.e. $\beta_0 = 0$, the equation (19) coincides with the entropy $S_{\text{KK}}$ with $\rho_+ = 0$. Using the temperature (18) and the entropy (19), we obtain the modified heat capacity $C = T \left(\partial S/\partial T\right)$ as

$$C = \pi \left(2048\mu^5 + 448\mu^3 \nu^2 + 2\beta_0 \mu \nu^2 + (1024\mu^4 + 96\mu^2 \nu^2 - 3\beta_0 \nu^2) \sqrt{4\mu^2 + \nu^2}\right)$$

$$\times \left(64\mu^3 - 10\beta_0 \mu + (32\mu^2 - 3\beta_0) \sqrt{4\mu^2 + \nu^2}\right) \sqrt{\mu \left(2\mu + \sqrt{4\mu^2 + \nu^2}\right)}$$

$$\times \left[48\beta_0 \mu \left(512\mu^4 + 136\mu^2 \nu^2 + 5\nu^4 + (256\mu^3 + 36\mu^2 \nu^2) \sqrt{4\mu^2 + \nu^2}\right)\right]^{-1},$$

(20)

which gives the information on the thermodynamic stability of the present system. We find that the heat capacity (20) with $\beta_0 = 0$ monotonically increases with decreasing $M/m_p$ for fixed $r_\infty/l_p$ and is negative for $M > 0$. We show the behaviors of the Hawking temperature $T/m_p$ and the heat capacity $C$ versus $M/m_p$ in the figures 2 and 3. From the left panels of the figures 2 and 3, we see that, as the black hole mass decreases, the modified temperature (18) with $\beta_0 \neq 0$ reaches the local maximum value at the critical mass

$$M_{\text{cr}} = \frac{m_p}{4} \sqrt[4]{\frac{6\beta_0 (3\beta_0 - 5\nu^2) + 3\beta_0 \sqrt{6\beta_0 (6\beta_0 + 5\nu^2)}}{2 (3\beta_0 - 2\nu^2)}}.$$

(21)

and then decreases to zero at the minimum value of the mass

$$M_{\text{m}} = \frac{m_p}{4} \sqrt[4]{\frac{2\beta_0 (\beta_0 + 3\nu^2) + \beta_0 \sqrt{2\beta_0 (2\beta_0 + 3\nu^2)}}{2 (\beta_0 + 2\nu^2)}}.$$

(22)

The existence of local maximum and vanishing values of the Hawking temperature are related to a phase transition of the black hole and an evaporation remnant. From the right panels of the figures 2 and 3, we find that there exist three regimes in the heat capacity (20) with $\beta_0 \neq 0$, i.e. $C < 0$ for $M > M_{\text{cr}}$, $C \geq 0$ for $M_{\text{m}} \leq M < M_{\text{cr}}$, and $C \to \infty$ for $M \to M_{\text{cr}}$. The discontinuity between negative and positive values of the heat capacity indicates a phase transition from a thermodynamic unstable phase to a stable one. Then we see that, at the local maximum temperature specified by the mass $M_{\text{cr}}$, the system undergoes a transition from an unstable negative heat capacity phase to a stable positive heat capacity cooling down toward a cold extremal configuration with the mass $M_{\text{m}}$ due to the radiation. At the minimum mass $M_{\text{m}}$, the black hole no longer radiates since the Hawking temperature (18) vanishes. At the same time, the heat capacity (20) also vanishes, which implies that the black hole may
not exchange its energy with the surrounding environment. Then the generalized uncertainty principle prevents the squashed Kaluza–Klein black hole to completely evaporate and results in the thermodynamic stable remnant, similar to the evaporation of the black holes in the noncommutative model and the asymptotically safe gravity [62–64]. From the left panels of the figures 2 and 3, we see that the mass of the black hole remnant $M_{\text{rm}}$ increases with increasing $\beta_0$ for fixed $r_\infty/l_p$, while it decreases with increasing $r_\infty/l_p$ for fixed $\beta_0$. When $r_\infty = 0$, the equation (22) represents the mass of the four-dimensional Schwarzschild black hole remnant [22]. If the deformation parameter is $\beta_0 \simeq 1$ [22] and the size of the extra dimension is $r_\infty \simeq 0.1 \text{ mm}$ [39], we can estimate that the mass of the squashed Kaluza–Klein black hole remnant is $M_{\text{rm}} \simeq 10^{-8} \text{ kg}$, which is of order the Planck mass like the black hole remnants in the minimally geometric deformation model, the quadratic gravity and the asymptotically safe gravity [23, 24, 62].

We consider the impact of the quantum gravity effect onto the sparsity of the Hawking radiation during the evaporation process of the squashed Kaluza–Klein black hole. Sparsity is defined by the average time gap between two successive emissions of the particle over the characteristic timescale of the emission of the individual particle [65, 66]. It has been shown that the Hawking radiation is sparse throughout the whole Hawking evaporation process of
the four-dimensional black hole [66]. However, it has also been shown that the sparsity of the Hawking radiation would be modified by some quantum gravity effects in some higher-dimensional black hole spacetimes [64, 67–74]. Further, once the wave effect of the radiation is taken into account, the sparsity becomes a crucial feature that extends the black hole lifetime. Then we consider the sparsity of the Hawking radiation in the form $\eta = \lambda^2 / A_{\text{eff}}$, where $\lambda = 2\pi / T$ is the thermal wavelength of the Hawking particle, $A_{\text{eff}} = 27 \tilde{A} / 4$ is the effective area that corresponds to the universal cross section at high frequencies and $\tilde{A}$ is the area of the black hole horizon [70–73]. If the sparsity is much less than 1, the Hawking radiation is a typical blackbody radiation where its thermal wavelength is much shorter than the size of the emitting body. On the other hand, if the sparsity is much greater than 1, the Hawking radiation is not a continuous emission of particles but a sparse radiation, i.e. most particles are randomly emitted in a discrete manner with pauses in between. Using the temperature (18) and the area of the horizon $\tilde{A} = 4\pi \mu \ell_p^2 \left(2\mu + \sqrt{4\mu^2 + \nu^2}\right)$, we obtain the modified sparsity of the Hawking radiation as

$$\eta = \frac{65536\pi^3 \mu^4 \left(2\mu + \sqrt{4\mu^2 + \nu^2}\right)^2}{27 \left(64\mu^3 - 10\beta_0 \mu + \left(32\mu^2 - 3\beta_0\right) \sqrt{4\mu^2 + \nu^2}\right)^2}.$$ (23)

We see that the sparsity (23) depends on the black hole mass $M$, the extra dimension size $r_\infty$ and the deformation parameter $\beta_0$. When $\beta_0 = 0$, the equation (23) coincides with the sparsity of the Hawking radiation from the four-dimensional Schwarzschild black hole, which keeps a constant value $64\pi^3 / 27$ during the whole evaporation process [70]. We show the behaviors of the sparsity $\eta$ versus $M/m_p$ in the figure 4. From the figure 4, we see that the sparsity (23) with $\beta_0 \neq 0$ increases with decreasing $M/m_p$ for fixed $r_\infty/l_p$ and $\beta_0$, and then diverges at $M = M_{\text{rm}}$. This implies that the average time between the emission of the successive Hawking quanta becomes much larger than the timescales set by the energies of the emitted quanta, so the radiation is very sparse. Then the sparsity of the Hawking radiation is enhanced due to the quantum gravity effect. Thus, similar to the evaporation of the black holes in the non-commutative model and the asymptotically safe gravity [62, 64], the squashed Kaluza–Klein black hole with the generalized uncertainty principle would take an infinite amount of time to radiate a particle at the final stage of the evaporation, and then turn into the remnant when the black hole mass $M$ approaches the mass $M_{\text{rm}}$. Since the squashed Kaluza–Klein black hole remnant would not radiate and its gravitational interaction would be very weak, it would be difficult to observe the remnants in our Universe directly. However, it would be expected that one possible indirect signature of the black hole remnant might be associated with the cosmic gravitational wave background [53, 54].

4. Tunneling of fermions with quantum gravity effects

We consider the Hawking radiation by the tunneling of charged fermions in the five-dimensional charged static squashed Kaluza–Klein black hole spacetime (1) with the quantum gravity effects. As the consequences of the generalized uncertainty principle (7), using the momentum operator $p$, and the energy operator $\tilde{E}$, which leads effectively to a replacement of third order time derivatives by spatial derivatives, we obtain the modified Dirac equation with the Maxwell field up to the first order in the parameter $\beta$ [13, 19] which describes the behaviors
of spinning particles in curved spacetimes as

\[
[i\hbar\gamma^\mu \partial_\mu + (i\hbar (\gamma^i \partial_i + \gamma^\mu \Omega_\mu) - e_\mu^a \gamma^a + m) \left(1 - \beta m^2 + \beta \hbar^2 \eta^{jk} \partial_j \partial_k \right)] \tilde{\Psi} = 0,
\]

(24)

where \(\tilde{\Psi}\) is the modified Dirac field, \(\Omega_\mu = i \omega^{ab}_\mu \Sigma_{ab} / 2\), \(\omega^{ab}_\mu = e_\mu^a e_\mu^b \Gamma^a_{bc} \), \(e_\mu^a\) is the spin connection defined by the Christoffel symbol \(\Gamma^\lambda_{\mu\nu}\) and the vielbein \(e_\mu^a\), and \(\Sigma^{ab} = i \left[\gamma^a, \gamma^b\right] / 4\) is the Lorentz spinor generator. The vielbeins \(e_\mu^a\) and the gamma matrices \(\gamma^\mu = e_\mu^a \gamma^a\) satisfy the conditions \(g^{\mu\nu} e^a_\mu e^b_\nu = \eta^{ab} = \text{diag}(-1, 1, 1, 1, 1)\) and \(\{\gamma^a, \gamma^b\} = 2g^{\mu\nu}\), respectively. When \(A_\mu = 0\), the equation (24) describes the modified Dirac equation without the Maxwell field [18].

For a spin-1/2 fermion, there are a spin-up and a spin-down states. In this paper, we only consider the tunneling radiation of the fermion with the spin-up state, since the discussion of the spin-down state is fully analogous. For the spin-up state, we employ the following ansatz for the spinor field describing the fermion,

\[
\tilde{\Psi} = \begin{pmatrix}
U(x^\mu) \\
0 \\
V(x^\mu)
\end{pmatrix} \exp \left(\frac{i}{\hbar} \tilde{I}(x^\mu)\right),
\]

(25)

with the action \(\tilde{I}\) and the functions \(U\) and \(V\). The corresponding gamma matrices of the squashed Kaluza–Klein spacetime (1) are
\[
\gamma^t = \frac{1}{\sqrt{F}} \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}, \quad \gamma^\rho = \frac{1}{\sqrt{F}} \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}, \quad \gamma^\theta = \frac{1}{\rho K} \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\gamma^\phi = \frac{1}{\rho K \sin \theta} \begin{pmatrix}
0 & 0 & -i \tilde{r} & 0 \\
0 & i \tilde{r} & 0 & 0 \\
i \tilde{r} & 0 & 0 & 0 \\
i \tilde{r} & 0 & 0 & 0
\end{pmatrix}, \quad \gamma^\psi = \begin{pmatrix}
-2K/r_\infty & 0 & 0 & i\tilde{K} \\
0 & -2K/r_\infty & -i\tilde{K} & 0 \\
i\tilde{K} & 0 & 2K/r_\infty & 0 \\
i\tilde{K} & 0 & 0 & 2K/r_\infty
\end{pmatrix},
\]

(26)

where \( \tilde{K} = \cos \theta / (\rho K \sin \theta) \) [35, 36]. Substituting the metric (1), the Maxwell field (2), the ansatz (25) and the matrices (26) into the Dirac equation (24), the Wentzel–Kramers–Brillouin approximation to the leading order in \( \hbar \) yields the equations of motion

\[
U \left( \frac{1}{\sqrt{F}} \left( \partial_\tilde{t} \tilde{I} + eA_\tilde{t} \tilde{\sigma} \right) - \tilde{\sigma} \frac{\sqrt{F}}{K} \partial_\tilde{r} \tilde{I} \right) - \tilde{\sigma} V \left( \frac{2K}{r_\infty} \partial_\tilde{r} \tilde{I} - m \right) = 0,
\]

(27)

\[
V \left( \frac{1}{\sqrt{F}} \left( \partial_\tilde{t} \tilde{I} + eA_\tilde{t} \tilde{\sigma} \right) + \tilde{\sigma} \frac{\sqrt{F}}{K} \partial_\tilde{r} \tilde{I} \right) - \tilde{\sigma} U \left( \frac{2K}{r_\infty} \partial_\tilde{r} \tilde{I} + m \right) = 0,
\]

(28)

\[
\tilde{\sigma} \left( \partial_\tilde{t} \tilde{I} + \frac{i}{\sin \theta} \left( \partial_\tilde{r} \tilde{I} - \cos \theta \partial_\tilde{\theta} \tilde{I} \right) \right) = 0,
\]

(29)

with

\[
\tilde{\sigma} = 1 - \beta m^2 - \frac{\beta F}{K^2} (\partial_\tilde{r} \tilde{I})^2 - \frac{\beta}{\rho^2 K^2} (\partial_\tilde{\theta} \tilde{I})^2 - \frac{\beta (\partial_\tilde{\theta} \tilde{I} - \cos \theta \partial_\tilde{r} \tilde{I})^2}{\rho^2 K^2 \sin^2 \theta} - \frac{4\beta K^2}{r_\infty^2} (\partial_\tilde{\theta} \tilde{I})^2,
\]

(30)

where we relax the possible restriction on the parameter \( \beta_0 \) and regard the parameter \( \beta \) as an independent variable. It would be difficult to solve the equations (27)–(29) directly. Then, according to the Killing vector fields in the squashed Kaluza–Klein spacetime, we consider the action \( \tilde{I} \) in the form

\[
\tilde{I} = -\tilde{\omega} t + \tilde{W}(\rho, \theta) + \tilde{J} \phi + \tilde{L} \psi,
\]

(31)

where \( \tilde{\omega} \), \( \tilde{J} \) and \( \tilde{L} \) are the emitted fermion’s energy and the angular momenta in the \( \phi \) and the \( \psi \) directions, respectively. Substituting the action (31) into the equation (29), we have

\[
\left( \frac{\partial \tilde{W}}{\partial \theta} + \frac{i}{\sin \theta} (\tilde{J} - \tilde{L} \cos \theta) \right) \times \left[ 1 - \beta m^2 - \beta F K^2 \left( \frac{\partial \tilde{W}}{\partial \rho} \right)^2 - \beta \rho^2 K^2 \left( \frac{\partial \tilde{W}}{\partial \theta} \right)^2 - \beta \left( \tilde{J} - \tilde{L} \cos \theta \right)^2 \frac{4\beta K^2}{r_\infty^2} \right] = 0.
\]

(32)
Since the parameter $\beta$ is a small quantity which represents the quantum gravity effects by the generalized uncertainty principle, the expression inside the square brackets in the equation (32) cannot vanish. Then we obtain

$$\frac{\partial \tilde{W}}{\partial \theta} + \frac{i}{\sin \theta} (J - \tilde{L} \cos \theta) = 0.$$  \hspace{1cm} (33)

Substituting the action (31) and the equation (33) into the equations (27) and (28), and canceling the functions $U$ and $V$, we have

$$\left( \frac{4\beta^2 L^2}{r_\infty^2} - \frac{e^2 A_+^2}{F} - m^2 \right) \left( \frac{4\beta^2 L^2}{r_\infty^2} + \beta m^2 - 1 \right)^2 - 2\omega e A_+ \left( \frac{4\beta^2 L^2}{r_\infty^2} + \beta m^2 - 1 \right) - \frac{\tilde{\omega}^2}{F}$$

$$+ \left( \frac{F}{K} \left( \frac{4\beta^2 L^2}{r_\infty^2} + \beta m^2 - 1 \right) \right) \left( \frac{12\beta^2 L^2}{r_\infty^2} - \frac{2\beta^2 e^2 A_+^2}{F} - \beta m^2 - 1 \right) - 2\omega e A_+ \left( \frac{4\beta^2 L^2}{r_\infty^2} \right) \left( \frac{\partial \tilde{W}}{\partial \rho} \right)^2$$

$$+ \beta^2 e^2 \left( \frac{12\beta^2 L^2}{r_\infty^2} - \beta e^2 A_+^2 \right) + \beta m^2 - 2 \right) \left( \frac{\partial \tilde{W}}{\partial \rho} \right)^4 + \frac{\beta^2 F^3}{K^2} \left( \frac{\partial \tilde{W}}{\partial \rho} \right)^6 = 0.$$  \hspace{1cm} (34)

From the equations (33) and (34), we see that the function $\tilde{W}(\rho, \theta)$ can be written as $\tilde{W}(\rho, \theta) = \tilde{R}(\rho) + \tilde{\Theta}(\theta)$. Then, from the equation (33) with the relation $\partial \tilde{W}/\partial \theta = d\tilde{\Theta}/d\theta$, we find that $\tilde{\Theta}$ must be a complex function. Neglecting the higher-order terms of $\beta$ in the equation (34) with the relation $\partial \tilde{W}/\partial \rho = d\tilde{R}/d\rho$ and solving the obtained equation on the black hole horizon $\rho = \rho_+$ for the classically forbidden trajectory yield the imaginary part of the action,

$$\text{Im} \tilde{R}_{\text{out}} = -\text{Im} \tilde{R}_{\text{in}} = \frac{\pi \rho_+ \sqrt{\rho_+ (\rho_+ + \rho_0)} \left( \tilde{\omega} - e A_+ \right)}{\rho_+ - \rho_-} (1 + \beta \Xi_t) + O (\beta^2),$$  \hspace{1cm} (35)

where

$$\Xi_t = \frac{3\tilde{\omega} m^2}{2(\tilde{\omega} - e A_+)} + \frac{\tilde{\omega} L^2}{2 \rho_+ (\rho_+ + \rho_0)(\tilde{\omega} - e A_+)}$$

$$+ \frac{\tilde{\omega}^2 \rho_+ (4\rho_+ (\rho_+ + 2\rho_-) + \rho_0 (3\rho_+ + 7\rho_-)) + e\tilde{\omega} \rho_+ A_+ (4\rho_+ \rho_+ + \rho_0 (\rho_+ + 3\rho_-))}{2 (\rho_+ + \rho_0) (\rho_+ - \rho_-)^2}.$$  \hspace{1cm} (36)

Then the tunneling probability amplitude of the charged fermions reads

$$\tilde{\Gamma} \approx \frac{\exp (-2 \text{Im} \tilde{R}_{\text{out}})}{\exp (-2 \text{Im} \tilde{R}_{\text{in}})} \approx \exp \left( -\frac{4\pi \rho_+ \sqrt{\rho_+ (\rho_+ + \rho_0)} (1 + \beta \Xi_t)}{\rho_+ - \rho_-} (\tilde{\omega} - e A_+) \right),$$  \hspace{1cm} (37)

where the contribution from the function $\tilde{\Theta}$ cancels out upon dividing the outgoing probability by the ingoing one, since the same solution for $\tilde{\Theta}$ is obtained for both the outgoing and the ingoing cases. Thus, by comparing the probability amplitude (37) to the first order in the energy with the Boltzmann factor $\tilde{\Gamma} = \exp \left( - (\tilde{\omega} - e A_+)/\tilde{T} \right)$ in a thermal equilibrium state at the
Figure 5. Parameter regions of the correction (36) with $\rho_+ \geq \rho_0 \geq 0, \rho_- + \rho_0 > 0$ for $m = 0, \tilde{L} = 0, e/\tilde{\omega} = 0.1$. The positive $\Xi_f$ region (dark region) increases with increasing $e/\tilde{\omega}$.

The evaporation process of the squashed Kaluza–Klein black hole by the quantum tunneling radiation of fermions depends on the correction $\Xi_f$ in the modified temperature $\tilde{T}$. Since all particles emitted by the Hawking radiation near the horizon region are effectively massless...
[2–6], the mass of the emitted fermion is not taken into account in the following discussion, i.e. \( m = 0 \). Further we assume that the fermion has no momentum in the direction of the extra dimension [39], i.e. \( \tilde{L} = 0 \). We show the parameter regions of \( \Xi_f \) in the figure 5. We see that the correction (36) and its parameter regions are similar to those in the tunneling of scalar particles discussed in the section 3. Then the equation (38) with the positive \( \Xi_f \) shows that the generalized uncertainty principle slows down the increase of the Hawking temperature due to the radiation and the evaporation may cease at the particular mass of the black hole, leading to the thermodynamic stable Planck mass remnant. From the equation (36) and the figure 5, we find that, while the black holes with the large \( Q \) would be completely evaporated, the effectively four-dimensional black holes with the small \( \rho_0 \) and the small \( Q \) would turn into the remnants at the end of the evaporation process.

5. Summary and discussion

We study the Hawking radiation of charged scalar particles and charged fermions from the five-dimensional charged static squashed Kaluza–Klein black hole on the basis of the tunneling mechanism, including the quantum gravity effects predicted by the quadratic generalized uncertainty principle with the minimal measurable length. We derive the modified Hawking temperature with the corrections related to the energy of the emitted particle, the compact extra dimension, the Maxwell field and the generalized uncertainty principle in the squashed Kaluza–Klein black hole background. Some known Hawking temperatures in five and four-dimensional black hole spacetimes are obtained by taking limits in the modified temperature.

We consider the evaporation process of the five-dimensional squashed Kaluza–Klein black hole with the generalized uncertainty principle by the tunneling of particles as one of the quantum gravity effects in the Hawking radiation. We see that the generalized uncertainty principle may prevent the complete evaporation of the squashed Kaluza–Klein black hole. As the black hole mass decreases due to the radiation, the black hole with the negative heat capacity may undergo a phase transition to the one with the positive heat capacity. At the minimum mass of the black hole, both the Hawking temperature and the heat capacity may vanish. This implies that the black hole may not exchange its energy with the surrounding environment. Then the evaporation of the squashed Kaluza–Klein black hole may cease, leading to the thermodynamic stable remnant. Related to the corrections in the modified Hawking temperature, we see that, while the black holes with the large charges would evaporate completely, the effectively four-dimensional black holes with the small charges would turn into the remnants at the end of the evaporation process. Further we consider the sparsity of the Hawking radiation from the squashed Kaluza–Klein black hole in the presence of the quantum gravity effect. We find that the sparsity may increase with decreasing the black hole mass and become infinite at the final stage of the evaporation. This also indicates that the generalized uncertainty principle may stop the Hawking radiation and lead to the black hole remnant. We see that the mass of the squashed Kaluza–Klein black hole remnant increases with increasing the deformation parameter of the generalized uncertainty principle, while decreases with increasing the size of the extra dimension. If the deformation parameter is of order 1 and the extra dimension size is of order 0.1 mm, the black hole remnant mass is of order the Planck mass. Since such a Planck mass remnant would be a ground state mass of the black hole, the squashed Kaluza–Klein black hole remnant would have no hair. This would be a desirable property as a dark matter [23, 24, 52–54, 64]. If the squashed Kaluza–Klein metric would describe the geometry around a primordial black hole and the generalized uncertainty principle considered in this paper would play an important role in the quantum nature of the black hole, the squashed Kaluza–Klein
black hole remnants would be a dark matter candidate. Moreover, the variations of the deformation parameter and the extra dimension size provide specific signatures on the quantum features of the squashed Kaluza–Klein black hole solutions with the generalized uncertainty principle which would open the possibility of testing such higher-dimensional models by using future astronomical and astrophysical observations.

At present, gravity has been experimentally verified at $9 \times 10^{-5}$ kg or more [75] and the quantum control has been realized at $5 \times 10^{-11}$ kg or less [76]. Then it is expected that theoretical and experimental studies of macroscopic quantum phenomena on the Planck mass scale in gravitational fields would bring significant progress in quantum gravity researches. Since the energy of the higher-dimensional black hole remnant would decrease to a detectable range in the Large Hadron Collider, such remnants would be observed in future accelerator experiments [77–81]. If higher-dimensional black holes would be created in an accelerator and we assume that the five-dimensional squashed Kaluza–Klein black hole solutions would describe geometries around such black holes, we expect that our present work would make a contribution to the verification of the Hawking radiation and the extra dimension in asymptotically Kaluza–Klein spacetimes.

In this paper, we consider the Hawking radiation with the quantum gravity effects inspired from the modification on the commutation relation of the matter field in the fixed Kaluza–Klein background geometry. When the black hole mass approaches the order of the Planck mass due to the radiation, it would be expected that some quantum gravity effects would lead to some quantum fluctuations in the background metric. In four dimensions, such modified background geometries would be given by the quantum deformed Schwarzschild black holes [82] and the black holes in the noncommutative model [83] and the asymptotically safe gravity [62]. Since these black hole solutions are analogous to the Reissner–Nordström solution, the quantum corrections in these frameworks affect things in the same way as the black hole charge [62, 84–86]. If there would exist a correspondence between the charged Kaluza–Klein black holes with squashed horizons and some quantum-corrected black holes in five dimensions, the black hole charge in the squashed Kaluza–Klein spacetime might have something to do with the very structure of the spacetime manifold and result in some quantum fluctuations in the background geometry as a quantum gravity effect. Moreover, if we take into account the effect of the emitted particle’s self-gravitation in the Hawking evaporation process, the mass and the charge of the black hole may decrease to satisfy the energy conservation. Then the background metric may become dynamical by the backreaction effect of the quantum tunneling radiation [2, 7, 87]. We might regard such a dynamical geometry associated with the radiation as a modified background spacetime with some quantum fluctuations. Studies of these topics are currently in progress.

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Data availability statement

No new data were created or analysed in this study.
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