The aim of this article is to provide a rigorous-but-simple steps to prove the hermiticity of the volume operator of Rovelli-Smolin and Ashtekar-Lewandowski using the angular momentum approach, as well as pointing out some subtleties which have not been given a lot of attention previously. Besides of being hermitian, we also prove that both volume operators are real, symmetric, and positive semi-definite, with respect to the inner product defined on the Hilbert space over SU(2). Other special properties follows from this fact, such as the possibility to obtain real orthonormal eigenvectors. Moreover, the matrix representation of the volume operators are degenerate, such that the real positive eigenvalues always come in pairs for even dimension, with an additional zero if the dimension is odd. As a consequence, one has a freedom in choosing the orthonormal eigenvectors for each 2-dimensional eigensubspaces. Furthermore, we provide a formal procedure to obtain the spectrum and matrix representation of the volume operators. In order to compare our procedure with the earlier ones existing in the literature, we give explicit computational examples for the case of monochromatic quantum tetrahedron, where the eigenvalues agrees with the standard earlier procedure.

I. INTRODUCTION

Loop Quantum Gravity (LQG) has been a fruitful field of research after these three decades. The birth of LQG was initiated by the discovery of the ‘new variables’ by Ashtekar [1], which reveal a new path on the canonical quantization of gravity, and as an additional advantage, the Hamiltonian constraint (given a special condition) could be written in polynomial form [2]. As a consequence arising from the quantization, space are discrete in the Planck scale, which is reflected by the discrete spectrum of area and volume operator [3, 4]. The origin of the discreteness could be traced from the fact that the Hilbert space of states is constructed over the space of SU(2) connection, where the dependence is reflected by the discrete spectrum of area and volume operator [3, 4]. The origin of the discreteness could be traced from the fact that the Hilbert space of states is constructed over the space of SU(2) connection, where the dependence on the connection is inserted through the holonomy of SU(2), i.e., the cylindrical functions. With this proposal in hand, one could obtain the candidate of Hilbert space of quantum gravity equipped with the Ashtekar-Lewandowski measure, as proven in [3, 7]. Due to the Peter-Weyl theorem, one could construct the basis on the Hilbert space \( \mathcal{H}_L = L^2[SU(2), d\mu_{\text{Haar}}] \) of a graph \( \Gamma \), defined by a collection of intersecting loops, from the irreducible representation of SU(2) in \( (2j + 1) \)-dimension [3, 9]. For the kinematical regime, where the Gauss constraint is taken into account, one has a spin-network state: a gauge invariant state of a graph labeled by the spin representation of SU(2).

There exists two versions of volume operators which differ by the regularization procedure. The first is due to Rovelli and Smolin [3] and the later is due to Ashtekar and Lewandowski [10, 11]. These operators are constructed from a triple surface integral over the triads, which, using the terms in [12], is called as the fluxization of the volume. This cause the dimension of the operators to be in the order of \( L^5 \), and to correctly describe the volume of a 3-dimensional region of space, one needs to take their square root. Both of the operator are well-defined in the sense that they converge to the classical volume formula in the continuum limit [11, 12], and analysis on their properties could be found in considerable amount on the literatures. However, there exists several subtleties which become a concern in this article, as some of them are already pointed out in [13, 14]. Another new volume operator had been introduced by Yang and Ma [15], where the regularized volume is constructed from a triple line integral over cotriads, instead of surfaces integral as their previous predecessors. This cause the dimension of the new alternative volume operator to be in the order of \( L^5 \), hence one does not need to take the square root to obtain the correct operator of volume. The consistency and spectrum of the new alternative operator has not been studied in great detail, since the operator was introduced so recently. Its properties are an interesting subject to pursue, but remains in the outer scope of this article.

As shown in [16], the expectation value of Ashtekar-Lewandowski (and hence the Rovelli-Smolin) volume operator coincides with the classical volume for the coherent states only for 6-valent vertices, i.e., the graph defines only cubical topology. For \( n \neq 6 \), the quantum states do not admit a correct semi-classical limit, given specific coherent states in [17] and [18]. To solve the criticism, a more general polyhedral volume operator was introduced by Bianchi, Dona’, and Speziale in [19]. The polyhedral volume operator is constructed following the procedure in [20, 21], such that in the semi-classical limit, it gives rise to any \( n \)-faces polyhedra, which is independent from the definition of the coherent states (the construction in [19] particularly, use the Livine-Speziale coherent states [22]). However, the complete
The importance of the volume operator in LQG varies from the needs of studying the properties of quanta of space, to other related operators such as the length operator [12, 23], Hamiltonian constraint operator [24, 26], and Master constraint operator [27]. It is also used in the procedure to couple matters with gravitational field [28, 29]. In the LQG literatures, it was already mentioned that the volume operator is hermitian [12, 31, 33, 34], even some articles had proven its hermiticity [30, 34]. However, the proof was not done in a formal and rigorous mathematical manner, which in our opinion is not satisfying and transparent enough to give a clear first sight, particularly for researchers and students new to LQG. Therefore, the aim of this article is to provide a rigorous-but-simple steps to prove the hermiticity of the volume operator of Rovelli-Smolin and Ashtekar-Lewandowski, as well as pointing out some subtleties which has not been given a lot of attention previously. Interestingly, we could also show that both the volume operators are real, symmetric, and positive semi-definite, with respect to the inner product defined on the Hilbert space over \( SU(2) \). In fact, the symmetricity of the Rovelli-Smolin volume operator, based on the graphical method of the spin-network [8, 30, 35], had been proven in [30]. Our work in this article could be interpreted as a complement to the result obtained in [30], in the sense that the symmetricity of the volume operators is proven from the angular momentum approach. The equivalency between these two approaches is supported by the result.

The article is organized as follows. In Section II, we briefly review the two well-known volume operators in LQG; these operators are rewritten using a common notation in order to discover easily their similarity and differences. Section III consists the proof of the hermiticity of the volume operators; this section begins with the introduction of several important mathematical definitions, followed by some claims useful for the proof. We also highlight several subtleties concerning the operators. In addition to the hermiticity condition, we show that the volume operators are real, symmetric, and positive semi-definite, with respect to the inner product defined on \( \mathcal{H}_\Gamma = L^2[SU(2), d\mu_{\text{Haar}}] \). Other special properties follows from this fact, such as the possibility to obtain real orthonormal eigenvectors, the occurrence of degenerates eigenvalues, and the freedom to choose the eigenvectors for the eigensubspaces due to the degeneracies in the spectrum of volume operators. In Section IV, we provide explicit calculations of the matrix representation of volume operators and their spectrum for a quantum tetrahedron, both for the ground state monochromatic (GSM) and first excited monochromatic (FEM) case. Finally we give a conclusion and remarks on this subject.

II. THE VOLUME OPERATOR

In this section, we will review the well-known results on the volume operators in LQG. There are two types of volume operator, the one derived by Rovelli-Smolin, which are labeled by \( \hat{V}_{RS} \) [3, 12], and the one by Ashtekar-Lewandowski, labeled as \( \hat{V}_{AL} \) [10, 11]. The main differences between these two operators are the following: (1) the constants in front of the operators, (2) the way the operator sums up the variables for each link, and (3) the absence of the sign factor of \( \epsilon \) in the connection representation: \( V^\alpha_i \). These differences are a consequence of different regularization schemes applied to the volume operator, but however they start from the same classical definition of volume as follows:

\[
V_R(x) = \int_\mathcal{R} \sqrt{\det q(x)} = \int_\mathcal{R} \sqrt{|\epsilon_{abc}\hat{E}_a^i(x)\hat{E}_b^j(x)\hat{E}_c^k(x)|},
\]

with \( q(x) \) is the determinant of metric \( q \) on 3D foliation \( \Sigma \), and \( \mathcal{R} \) is a 3D region on \( \Sigma \). Writing \( q \) in terms of densitized triads \( E \), \( \mathcal{R} \) could be directly quantized by promoting the triads to an operator \( E_{i\alpha}^a \to \hat{E}_{i\alpha}^a \), giving the (continuous) volume operator in the connection representation:

\[
\hat{V}_R(x) = \int_\mathcal{R} \sqrt{\frac{1}{3!}\epsilon^{ijk}\epsilon_{abc}\hat{E}_i^a(x)\hat{E}_j^b(x)\hat{E}_k^c(x)}.
\]

A regularization is needed to obtain a volume operator free from singularities [8, 8].

A. Rovelli-Smolin Volume Operator

The regularized Rovelli-Smolin volume is constructed by considering a quantity such that in the continuum limit it converges to the classical version of (2). The derivation here is based on [12]. A region \( \mathcal{R} \) is approximated by a set of cubic-cell \( \mathcal{R}_\alpha \) with the edge length \( \Delta x \), such that the region \( \mathcal{R} \subset \bigcup_\alpha \mathcal{R}_\alpha \). The volume of region \( \mathcal{R} \) can be calculated as:

\[
V_R(x) = \int_\mathcal{R} \sqrt{\frac{1}{3!}\epsilon^{ijk}\epsilon_{abc}\hat{E}_i^a(x)\hat{E}_j^b(x)\hat{E}_k^c(x)}.
\]
approximated by the Riemann sum of the cubic-cells volumes, namely, $\sum_{\alpha} \text{vol} (R_{\alpha})$. One considers the following triple surface integral of the cubes as follows:

\[
W_{\Delta x} (x_\alpha) = \frac{1}{8 \times 3!} \frac{1}{(\Delta x)^6} \int_{\partial R_{\alpha}} d^{2} \sigma \int_{\partial R_{\alpha}} d^{2} \sigma' \int_{\partial R_{\alpha}} d^{2} \sigma'' \times \ldots
\]

\[
.. \times |T_{x_{\alpha}}^{ijk} (\sigma, \sigma', \sigma'') E_{i}^{n} (\sigma) n_{a} (\sigma) E_{j}^{b} (\sigma') n_{b} (\sigma') E_{k}^{c} (\sigma'') n_{c} (\sigma'')|.
\]  

(3)

$\sigma$ is the local coordinate on $\Sigma$, and $n$ is the normal to surface $\partial R_{\alpha}$, written in coordinate $x^a = X^a (\sigma)$ as:

\[
n_{a} (\sigma) = \varepsilon_{abc} \frac{\partial X^{b}}{\partial \sigma^{i}} \frac{\partial X^{c}}{\partial \sigma^{j}}.
\]

$T_{x_{\alpha}}^{ijk} (\sigma, \sigma', \sigma'')$ is a function that guarantees the three fluxes to satisfies SU(2) gauge invariance, namely:

\[
T_{x_{\alpha}}^{ijk} (\sigma, \sigma', \sigma'') = \varepsilon^{i'j'k'} D^{1} \left( h_{\gamma_{x_{\alpha}} [A]} \right) \varepsilon_{i'j'k'} D^{1} \left( h_{\gamma_{x_{a}} \sigma} [A] \right) D^{1} \left( h_{\gamma_{x_{a}} \sigma''} [A] \right) k^k.
\]

(4)

$D^{1} (h_{\gamma} [A])$ is the adjoint representation of SU(2) holonomy along the loop $\gamma = \gamma_{x_{a}} \sigma$, starting at $x_{\alpha}$ in $R_{\alpha}$ and ends at the boundary $\partial R_{\alpha}$. Taking the limit $\Delta x \rightarrow 0$, (3) becomes:

\[
\lim_{\Delta x \rightarrow 0} W_{\Delta x} (x_{\alpha}) = \frac{1}{3!} |\varepsilon^{ijk} \varepsilon_{abc} E_{i}^{n} (x_{\alpha}) E_{j}^{b} (x_{\alpha}) E_{k}^{c} (x_{\alpha})|.
\]

(5)

Therefore, one has:

\[
V_R = \lim_{\Delta x \rightarrow 0} \sum_{\alpha} (\Delta x)^3 \sqrt{W_{\Delta x} (x_{\alpha})},
\]

(6)

and the regularized Rovelli-Smolin volume is:

\[
V_{RS} = \sum_{\alpha} (\Delta x)^3 \sqrt{W_{\Delta x} (x_{\alpha})}.
\]

The next step is to define a partition of surface $\partial R_{\alpha}$ into square plaquette $S^l$ such that $\partial R_{\alpha} = \bigcup_{l} S^l_{\alpha}$. Using this partition, one can write $W_{\Delta x} (x_{\alpha})$ as a Riemann summation of three fluxes [12 30]:

\[
W_{\Delta x} (x_{\alpha}) = \frac{1}{8 \times 3!} \frac{1}{(\Delta x)^6} \sum_{I,J,K} T_{x_{\alpha}}^{ijk} \left( S^l_{\alpha} \right) E_{i} \left( S^l_{\alpha} \right) E_{j} \left( S^l_{\alpha} \right) E_{k} \left( S^l_{\alpha} \right)|.
\]

(7)

Promoting the fluxes into hermitian operators: $E_i \left( S^l \right) \rightarrow \hat{E}_i \left( S^l \right) = iX^i_l$, defined as the (self-adjoint) right invariant vector field on SU(2), one immediately obtains the following relation:

\[
(\Delta x)^3 \sqrt{\hat{W}_{\Delta x} (x_{\alpha})} = \sqrt{\frac{1}{8 \times 3!} \sum_{I,J,K} |i T_{x_{\alpha}}^{ijk} X^i_{l,I} X^j_{j,O} X^K_{K,O}|}.
\]

(8)

$X^i_l$ is antihermitian, satisfying $[X^j_l, X^i_l] = -2\varepsilon^{ijk} X^k_l$. They are related to self-adjoint Pauli matrices by $\tau^i := -\frac{i}{2} X^i$, satisfying $[\tau^i, \tau^j] = i\varepsilon^{ijk} \tau^K$. Using the representation of angular momentum in $(2j + 1)$-dimension as follows: $\tau_i \mapsto \rho_j \mapsto \rho_{j_i} (\tau_i) = J_i$, the argument inside the absolute value of [8] could be written as:

\[
iT_{x_{\alpha}}^{ijk} X^i_l X^j_j X^K_{K,K} = 8 \varepsilon^{ijk} J^I_{I} J^J_{J} J^K_{K},
\]

(9)

with the adjoint representation of the holonomy on [13] is gauge-fixed to be trivial (this is possible, providing the geometrical picture of the quanta of space which is flat in the interior). The term in the RHS of [9], following [34], is written as:

\[
\hat{q}_{IJK} := \frac{4}{i} \varepsilon^{ijk} J^I_{I} J^J_{J} J^K_{K} = \frac{4}{i} J_I \cdot (J_J \times J_K).
\]
\(\hat{q}_{IJK}\) is the \textit{three-hand} operator \([12, 30]\). Therefore, the Rovelli-Smolin volume operator is:

\[
\hat{V}_{\text{RS}} = \sum_{\alpha} (\hat{v}_{\text{RS}})_{\alpha},
\]

\[
(\hat{v}_{\text{RS}})_{\alpha}^{(1)} = \frac{1}{4} \sum_{I < J < K} |i(\hat{q}_{\alpha})_{IJK}|,
\]

\[
(\hat{q}_{\alpha})_{IJK} = [(J_{IJ})^2, (J_{JK})^2]_{\alpha},
\]

(11)

The last equality comes from relation (10), together with the fact that the summation in (11) is only over distinct indices \(I, J, K\). This will be clear later in Section III.

Another version of Rovelli-Smolin volume operator exists, according to \([13, 14]\), where the difference is on the location of the summation, which is inside the square root:

\[
\hat{V}_{\text{RS}} = \hat{R} d^3 p \hat{V}(p)\gamma,
\]

(13)

\[
\hat{V}(p)\gamma = \sum_{v \in V(\gamma)} \delta^{(3)}(v, p) (\hat{v}_{\text{RS}})_{v,\gamma},
\]

(14)

\[
(\hat{v}_{\text{RS}})_{v,\gamma}^{(2)} = \sum_{I,J,K} \sqrt{C_{\text{reg}}} \frac{1}{8} |i(\hat{q}_{\alpha})_{IJK}|.
\]

(15)

Writing (15) in terms of the angular momentum representation and using (10), one could write the second version of Rovelli-Smolin operator using a common notation with the previous one:

\[
(\hat{v}_{\text{RS}})_{v,\gamma}^{(2)} = \sum_{I,J,K} \sqrt{C_{\text{reg}}} \frac{1}{4} |i(\hat{q}_{\alpha})_{IJK}|,
\]

(16)

with \(C_{\text{reg}}\) is a constant to be specified by the regularization procedure. Notice that the differences with (12) is in the location of the summation and the constant.

\section*{B. The Ashtekar-Lewandowski Volume Operator}

On the other hand, the Ashtekar-Lewandowski volume operator \(\hat{V}_{\text{AL}}\) is obtained from different regularization scheme. The derivation here is based on \([33]\). Let us consider a cubic-cell on \(\Sigma\) centered at point \(p\), with edge length \(2\Delta_i\). Let the unit vectors \(\hat{n}_i\) be the normals to the faces- \(i\) of the cube, then the volume of the cube is:

\[
\text{vol} (\Delta) = 2^3 \Delta_1 \Delta_2 \Delta_3 \det (\hat{n}_1, \hat{n}_2, \hat{n}_3).
\]

The coordinate of the cube is defined by \(x\), such that the characteristic function of the cube is written as follows:

\[
\lambda_\Delta (x, p) = \prod_{i=1}^{3} \Theta (\Delta_i - |(\hat{n}_i, x - p)|).
\]

(17)

\(\Theta (z)\) is the Heaviside step function, with the condition \(\Theta (z) = 0\) if \(z < 0\), \(\Theta (z) = \frac{1}{2}\) if \(z = 0\), and \(\Theta (z) = 1\) if \(z > 0\). Taking the limit \(\Delta \to 0\) by setting \(\Delta_i \to 0\), (17) becomes:

\[
\lim_{\Delta \to 0} \frac{1}{\text{vol} (\Delta)} \lambda_\Delta (x, p) = \delta^{(3)} (x, p),
\]

with \(\delta^{(3)}\) is the three dimensional Dirac-delta function. One considers the triple volume integral as follows:

\[
E (p, \Delta, \Delta', \Delta'' \rangle = \frac{1}{3!} \frac{1}{\text{vol} (\Delta) \text{vol} (\Delta') \text{vol} (\Delta'')} \int_{R} d^3x \int_{R} d^3x' \int_{R} d^3x'' \times \ldots
\]

\[
\times \lambda_\Delta (x, p) \lambda_{\Delta'} \left(\frac{x + x'}{2}, p\right) \lambda_{\Delta''} \left(\frac{x + x' + x''}{2}, p\right) \varepsilon_{ijk} \varepsilon_{abc} E_i^a(x) E_j^b(x') E_k^c(x''),
\]

(18)
such that the volume of region $\mathcal{R}$ could be obtained by taking the limit of $\triangle \to 0$:

$$V_{\mathcal{R}} = \lim_{\triangle \to 0} \lim_{\triangle' \to 0} \lim_{\triangle'' \to 0} \int_{\mathcal{R}} d^3 p \sqrt{E(p, \triangle, \triangle', \triangle'')}.$$  \hfill (19)

Hence, the Ashtekar-Lewandowski regularization of volume is defined as:

$$V_{\text{AL}} = \int d^3 p \sqrt{E(p, \triangle, \triangle', \triangle'')}.$$  

Promoting the fluxes as operators, acting on the holonomy, and using the proposal completely explained in 33, one has the Ashtekar-Lewandowski volume operator as follows:

$$\hat{V}_{\text{AL}} = \hat{R} d^3 p \hat{E}(p, \gamma).$$  \hfill (20)

$$\hat{E}(p, \gamma) = \sum_{v \in V(\gamma)} \delta^{(3)}(v, p) (\hat{v}_{\text{AL}})_{v, \gamma}.$$  \hfill (21)

$$\hat{v}_{\text{AL}} = \frac{1}{8 \times 3!} \sum_{(e_I, e_J, e_K) \in E(\gamma)} \epsilon(e_I, e_J, e_K) \epsilon_{ijk} X^I_I X^J_J X^K_K.$$  \hfill (22)

Rewriting the right invariant vector fields in terms of angular momentum representation as in the previous subsections, (22) becomes:

$$\hat{v}_{\text{AL}} = \frac{1}{4} \sum_{I < J < K} \epsilon(e_I, e_J, e_K) \hat{q}_{IJK}.$$  \hfill (23)

with $\hat{q}_{IJK}$ is the three-hand operator satisfying (12). Notice that one of the main difference between (23) and (11)-(16) is the existence of the sign factor $\epsilon(e_I, e_J, e_K)$, where its value is determined by the cross product sign of the tangents vectors of the link of the spin-network.

III. THE HERMITICITY OF THE VOLUME OPERATOR

A. Basic Definitions and Facts of Hermitian Matrix

The confusion we encounter regarding the volume operators is on the definition of the absolute value of an operator, in particular, a matrix. It is not clear in the LQG literatures (at least, we could not find any article which clearly discussed this matter) how the absolute value acts on the arguments in (11), (16), and (23). However, in the mathematical literatures, such definition exists on the subject of bounded operators 31, 32. We apply a similar definition for matrices as follows:

Definition 1. Given an arbitrary (not necessarily regular) complex matrix $M$, the absolute value of $M$, namely $|M|$, is defined as follows:

$$|M| := +\sqrt{M^\dagger M},$$  \hfill (24)

with $M^\dagger$ is the complex-conjugate of $M$.

To understand the properties of $|M|$, one considers a hermitian matrix, which is special case of complex matrices satisfying $H^\dagger = H$. One could show that a hermitian matrix is unitarily diagonalizable. It is a well-known fact that all the eigenvalues of hermitian matrices are real, and vice-versa: the reality of the eigenvalues guarantees a matrix to be hermitian. It needs to be kept in mind that the hermiticity of an operator (or a matrix) is defined with respect to a choice of inner product. A hermitian matrix with respect to inner product $\langle \ldots \rangle_1$, in general will not be hermitian with respect to a distinct inner product $\langle \ldots \rangle_2$. Another definition concerning a hermitian matrix which is useful for the discussion is the following:
Definition 2. A hermitian matrix $H$ is positive semi-definite (or non-negative) if:

$$x^\dagger H x \geq 0, \quad \forall x \in \mathbb{C}^n, \, x \neq 0.$$  

and negative semi-definite (or non-positive) if:

$$x^\dagger H x \leq 0, \quad \forall x \in \mathbb{C}^n, \, x \neq 0.$$  

Notice that it is not relevant to define the notion of non-negativity (or non-positivity) for an arbitrary complex matrix $M$ since the quantity $x^\dagger M x$ in general is not restricted to real numbers. As a consequence to Definition 1 and 2, one could prove these following facts:

**Claim 1.** $M^\dagger M$ is hermitian and positive semi-definite. **Proof:** (1) $(M^\dagger M)^\dagger = M^\dagger M^\dagger = M^\dagger M$, thus $M^\dagger M$ is hermitian. (2) $x^\dagger M^\dagger M x = (Mx)^\dagger M x \geq 0$, with respect to a Riemannian inner product in complex space. Therefore $M^\dagger M$ is positive semi-definite.

**Claim 2A.** A hermitian and positive semi-definite matrix $T$ has non-negative eigenvalues. **Proof:** Consider the eigenvalue problem $Tn = \lambda_n n$, with $(\lambda_n, n)$ is the spectrum of $T$, where the eigenvalues $\lambda_n \in \mathbb{R}$. Acting on the left by $n^\dagger$, one has $n^\dagger Tn = \lambda_n n^\dagger n$. Both $n^\dagger Tn \geq 0$ and $n^\dagger n \geq 0$, therefore $\lambda_n \geq 0$.

**Claim 2B.** If all the eigenvalues of a diagonalizable matrix $T$ are non-negative, then $T$ is a hermitian, positive semi-definite matrix. **Proof:** (1) Since the eigenvalues are real, $T$ is clearly hermitian which we are not going to prove. (2) Since a hermitian matrix is unitarily diagonalizable, one has $\Lambda^{-1} T \Lambda = \lambda_n P_n$, with $(P_n)_{ij} = \delta_{ni}\delta_j^*$ is the spectral projector of $T$. Acting a non-zero vector $x \in \mathbb{C}^n$ on the left and the right of $T$ one has:

$$x^\dagger T x = x^\dagger \Lambda \lambda_n P_n \Lambda^\dagger x,$$

using the fact that $\Lambda^{-1} = \Lambda^\dagger$. With $\bar{x} = \Lambda^\dagger x$, one has $x^\dagger T x = \bar{x}^\dagger \Lambda \lambda_n P_n \bar{x}$. Moreover,

$$x^\dagger T x = \sum_n \bar{\lambda}_n \lambda_n P_n \bar{x}_n = \sum_n \lambda_n |\bar{x}_n|^2.$$

If $\lambda_n \geq 0$, then $T$ is positive semi-definite, since $\bar{x}$ is not a zero vector.

**Claim 3.** A hermitian and positive semi-definite matrix $T$ admits a unique hermitian and positive semi-definite square root. **Proof:** Consider the diagonalization $\Lambda^{-1} T \Lambda = \lambda_n P_n$. One could write the diagonalization as $\Lambda^{-1} \sqrt{T} \Lambda = \sqrt{\lambda_n} P_n$, or moreover:

$$\Lambda^{-1} \sqrt{T} \Lambda = \left(\Lambda^{-1} \sqrt{T} \Lambda^\dagger\right)^2 = \left(\sqrt{\lambda_n} P_n\right)^2,$$

using the fact that $P_n^2 = P_n$. Finally one has $\Lambda^{-1} \sqrt{T} \Lambda = \pm \sqrt{\lambda_n} P_n$. The following is the positive solution to the square root of $T$, which is unique:

$$\sqrt{T} = \Lambda \sqrt{\lambda_n} P_n \Lambda^{-1}.$$

Since $\lambda_n \in \mathbb{R}$ and $\lambda_n \geq 0$, then from Claim 2B, $\sqrt{T}$ is hermitian and positive semi-definite.

**Claim 4.** The absolute value of an arbitrary matrix defined as in [23], namely $|M|$, is hermitian and positive semi-definite. **Proof:** $M^\dagger M$ is hermitian, and therefore diagonalizable. It is also positive semi-definite, and hence from Claim 3, admits a unique, hermitian, positive semi-definite square root, namely $+\sqrt{M^\dagger M}$ which by definition [23] is $|M|$.

As direct consequence of Claim 3 and 4, we have as follows, the main important fact which will be used to prove the hermiticity of the volume operators:

**Claim 5.** The absolute value of an arbitrary matrix, namely $|M|$, admits a unique hermitian and positive semi-definite square root, namely $\sqrt{|M|}$. The proof for this is clear.

### B. Subleties on the Volume Operators

Let us return to the volume operators, which are listed as follows (the indices $\alpha$, and $v, \gamma$ are neglected for simplicity, and the constants are written in $Z$’s):

$$v^{(1)}_{RS} = \sqrt{Z_1 \sum_{I,J,K} |\hat{q}_{IJJK}|}, \quad (25)$$
Using this results, we could show that the Rovelli-Smolin volume operator (11) and (16), could we written as:

\[
\hat{v}_{RS}^{(2)} = \sum_{I,J,K} \sqrt{Z_2} |\hat{q}_{IJK}|,
\]

\[
\hat{v}_{AL} = \sqrt{Z} \sum_{I,J,K} \epsilon (e_I,e_J,e_K) |\hat{q}_{IJK}|.
\]

(26) (27)

In this subsection, we will study the properties of the three-hand operator \(\hat{q}_{IJK}\) defined in (10); this will be followed by the subleties neglected in the literatures of volume operators. Firstly, from the definition in (10), \(\hat{q}_{IJK}\) could classified into three cases, where: all the three indices are equal, two of them are equal, and none of them are equal. The first and second case, namely \(\hat{q}_{IIJ}\) and \(\hat{q}_{IIJ}\), are interestingly, not zero. By a direct calculation from (10), one could obtain:

\[
\hat{q}_{III} = 4 |J_I|^2, \\
\hat{q}_{IIJ} = 4 J_I \cdot J_J,
\]

where both operators are clearly hermitian (real, symmetric) since the angular momentum \(J_I\) is hermitian, with respect to the inner product defined on the Hilbert space. The second case, particularly has the following symmetries:

\[
\hat{q}_{III} = -\hat{q}_{IIJ} = \hat{q}_{IIJ} = \hat{q}_{III}.
\]

The last case is the condition where the indices \(I, J, K\) are distinct. With this condition, (10) could be written as (12), which is an antihermitian (real, antisymmetric) matrix, since the commutator of distinct components of angular momentum is antihermitian. It has the following symmetries:

\[
\hat{q}_{IJK} = \hat{q}_{IJK} = -\hat{q}_{KIJ} = -\hat{q}_{IKJ} = -\hat{q}_{IKJ}.
\]

We found that the hermiticity of three-hand operator (10) depends on the indices \(I, J, K\). Using the symmetry properties of these three case, one could prove the following relation:

\[
\sum_{I,J,K=1}^{N} \hat{q}_{IJK} = \sum_{I=1}^{N} 4 |J_I|^2.
\]

(28)

It should be kept in mind that since \(\hat{q}_{III}\) and \(\hat{q}_{IIJ}\) are symmetric, the imaginary factor \(i\) of the volume operator alter them to their antihermitian counterparts, while for antisymmetric \(\hat{q}_{IJK}\) as shown in [30, 33, 34], it becomes hermitian. Thanks to the existence of the absolute value in the operator, all the \(i\hat{q}_{III}\) and \(i\hat{q}_{IIJ}\) terms becomes hermitian, which is the consequence from Claim 4. Moreover they satisfy:

\[
|\hat{q}_{III}| = |\hat{q}_{IIJ}|, \\
|\hat{q}_{IIJ}| = |\hat{q}_{III}|.
\]

Using this results, we could show that the Rovelli-Smolin volume operator (11) and (16), could we written as:

\[
\hat{v}_{RS}^{(1)} = \sqrt{Z_1} \left( \sum_{I=1}^{N} |\hat{q}_{III}| + 3! \sum_{I<J}^{N} |\hat{q}_{IIJ}| + 3! \sum_{K<L<M}^{N} |\hat{q}_{KLM}| \right),
\]

\[
\hat{v}_{RS}^{(2)} = \sum_{I=1}^{N} \sqrt{Z_2} |\hat{q}_{III}| + 3! \sum_{I<J}^{N} \sqrt{Z_2} |\hat{q}_{IIJ}| + 3! \sum_{K<L<M}^{N} \sqrt{Z_2} |\hat{q}_{KLM}|.
\]

(29) (30)

However, following the argument derived in [30], as the three-hand operator acts on the nodes of a spin-network \(\Gamma\) [12, 30], the linearly dependent terms \(i\hat{q}_{III}\) and \(i\hat{q}_{IIJ}\) gives zero contributions to the sum, since, in the graphical representation of spin-network, only terms in which each hand of the operator ‘grasps’ a distinct links give non-vanishing contributions. Another argument which strengthen the removal of the linearly dependent terms by hand is the fact that the volume operator acting on gauge-invariant trivalent graph must be zero since it is a planar graph [37]. It is clear that in the angular-momentum representation of spin-network, the action of (29) and (30) on a planar graph is not zero by the existence of the linearly dependent terms, thus giving different results with the graphical representation. Therefore, by consciously removing the linearly dependent terms due to the reason explained previously, the Rovelli-Smolin operator are exactly written as (11) and (16). Nevertheless, the ambiguities on the regularization procedure of the Rovelli-Smolin operator is dicussed in detailed in [13, 14].

As for the Ashtekar-Lewandowski volume operator, the existence of the sign term \(\epsilon (e_I,e_J,e_K)\) is crucial for three reasons: (1) Without the existence of \(\epsilon (e_I,e_J,e_K)\), the Ashtekar-Lewandowski volume operator (22) will also encounter the similar problem with the Rovelli-Smolin operator, namely, the non-vanishing of linearly dependent terms, due to condition (28). (2) The existence of \(\epsilon (e_I,e_J,e_K)\) removes the linearly dependent terms from the summation since parallel \(e_I\)’s will give zero, and this cause the possibility to write (22) as (23). (3) Its existence also guarantees that \(\hat{v}_{AL}\) gives the same result either in the angular momentum or the graphical representation of spin-network.
C. Arguments for the Hermiticity of the Volume Operators

From the previous section, we have found that the hermiticity of $\hat{q}_{IJK}$ depends on the indices $I, J, K$. For distinct $I, J, K$, $\hat{q}_{IJK}$ is antisymmetric, therefore, it is clear that the arguments inside the absolute value in the volume operators (25), (26) and (27) are hermitian (for the case of $\hat{v}_{AL}$, the quantity inside the absolute value is only a linear combination of hermitian matrices, which is still hermitian). Let us call these hermitian quantities as $M \sim i\eta$. Regardless of the hermiticity of $M$, the quantity $|M|$ is guaranteed to be hermitian and positive semi-definite, by Claim 4. Thus one could conclude further that the arguments inside the square root in (25), (26) and (27) are hermitian and positive semi-definite (the case of $\hat{v}_{RS}^{(1)}$, the hermiticity and non-negativity are closed under addition) since the constant $Z$’s are positive. Finally, using Claim 3, it is clear that the volume operators (25), (26) and (27) are hermitian, and positive semi-definite (where the closure of hermiticity and positive semi-definiteness under addition is again used for the case of $\hat{v}_{RS}^{(2)}$). This ends the proof of the hermiticity of the volume operators in LQG.

It needs to be kept in mind that the hermiticity (and the non-negativity as well) of the operators is defined with respect to the inner product defined on the Hilbert space of the graph, namely $H_L = L_2[\text{SU}(2), d\mu_{\text{Haar}}]$, which could be extended to the Ashtekar-Lewandowski Hilbert space, $H_{AL} = L_2[A, d\mu_{\text{AL}}]$, $A$ is the functional space of $\text{su}(2)$-valued 1-form and $d\mu_{\text{AL}}$ is the Ashtekar-Lewandowski measure $[10, 11]$.

Furthermore, from definition (24), one could show that:
\[
x^I |\hat{q}_{IJK}| x = x^I \sqrt{-\hat{q}_{IJK}^* \hat{q}_{IJK}} = x^I \sqrt{-\hat{q}_{IJK} \hat{q}_{IJK}^*} = x^I \sqrt{\hat{q}_{IJK}^\dagger \hat{q}_{IJK}} \geq 0,
\]
where the last equality comes from the antisymmetry of $\hat{q}_{IJK}$. Therefore the volume operators (25), (26) and (27) could be written as:
\[
\hat{v}_{RS}^{(1)} = Z_1^{1/2} \sqrt{\sum_{I,J,K} \hat{q}_{IJK}^* \hat{q}_{IJK}},
\]
\[
\hat{v}_{RS}^{(2)} = \sum_{I,J,K} Z_2^{1/2} \sqrt{\hat{q}_{IJK}^T \hat{q}_{IJK}},
\]
\[
\hat{v}_{AL} = Z_3^{1/2} \sqrt{\sum_{i<j<k} \epsilon(e_i, e_j, e_k) \hat{q}_{IJK}}.
\]

Since for distinct $I, J, K$, $\hat{q}_{IJK}$ is real and antisymmetric as proven in $[30, 33, 34]$, $\hat{q}_{IJK}^T \hat{q}_{IJK}$ is therefore real, symmetric and positive semi-definite (in fact, this statement could be proven using a similar procedure used to prove Claim 1, by replacing $\mathbb{C}^n \rightarrow \mathbb{R}^n$ and $\dagger \rightarrow T$). Moreover, one could convince her/himself that $\hat{q}_{IJK}^T \hat{q}_{IJK}$ admits a unique real, symmetric, and positive semi-definite square root (in fact, this statement is the real version of Claim 4, which could be proven by a similar manner), namely $\sqrt{\hat{q}_{IJK}^T \hat{q}_{IJK}}$, equivalent to $|\hat{q}_{IJK}|$ by relation (31). This proves that $|\hat{q}_{IJK}|$ is real, symmetric and positive semi-definite. Finally, realizing that symmetricity and non-negativity are closed under addition, we could conclude that, in the angular-momentum representation of spin-network, the Rovelli-Smolin and the Ashtekar-Lewandowski volume operator are real, symmetric, and positive semi-definite with respect to the inner product defined on $H_L = L_2[\text{SU}(2), d\mu_{\text{Haar}}]$.

As mentioned in the Introduction, the symmetry of the Rovelli-Smolin volume operator had been proven in $[30]$, based on the graphical representation of the spin-network. In this approach $[30, 35]$, one could obtain the matrix representation of an operator without fixing a notion of inner product. Therefore, the reality and symmetricity of the area and volume operator obtained from the graphical approach, could be used to fix an inner product on the representation space of the graph, such that it matches the inner product induced by the Haar measure on $\text{SU}(2)$ $[30]$. Our work in this article could be interpreted as a complement to the result obtained in $[30]$; in which given a Hilbert space $H_L = L_2[\text{SU}(2), d\mu_{\text{Haar}}]$, we prove, based on the angular-momentum approach, that volume operators acting on $H_L$ are real, symmetric, and positive semi-definite, with respect to the inner product induced by the Haar measure. The result gives a positive argument on the equivalency between the graphical and the angular momentum approach of spin-network.

The reality, symmetricity, and positive semi-definiteness of an operator are already some advantages, hence for the volume operators case, we could reveal another interesting property. One notice that the symmetric matrix comes from a product of a purely imaginary antisymmetric matrix with its complex-conjugate as in $[31]$. The eigenvalues of this type of matrix always come in pairs, namely $\pm \lambda$ for even dimension, with an additional zero for the odd dimension. Therefore the resulting symmetric matrix always have positive degenerate eigenvalues coming in pairs $\lambda^2_i$, with an additional zero if the dimension is odd. One could prove that for a real symmetric matrix, the eigenspaces of
distinct eigenvalues are orthogonal to each other. For repeated eigenvalues with multiplicity \(m\), one could obtain \(m\) orthonormal eigenvectors which spans the eigensubspace. These orthonormal eigenvectors can be chosen arbitrarily in the sense that they are invariant up to an orthogonal transformation in \(m\)-dimension. Moreover, the eigenvectors of real, symmetric matrices can always be chosen to be real. We would like to stress that there exists a freedom in choosing the orthonormal eigenvectors for each 2-dimensional eigensubspaces as a consequence of the degeneracies in the eigenvalues. The eigenvectors do not necessarily need to be the ones which correspond to the non-degenerate eigenvalues pair \(\pm \lambda\) of \(\hat{q}_{IJK}\), as suggested in the following literatures \[12, 33, 34, 36\].

IV. THE MATRIX REPRESENTATION OF THE VOLUME OPERATOR

On a series of remarkable articles, Thiemann and Brunemann gave an explicit closed formula for the matrix \(\hat{q}_{IJK}\), which are given as follows. Let \(\ket{\hat{a}(12)}\) be a state of \(N\)-coupled spins, written in the recoupling scheme basis \[34\]:

\[
|J\rangle = \left| g(J), j, j = 0, m = 0 \right>,
\]

which diagonalize the following complete sets of commuting observables:

\[
|J|^2 = \sum_{I=1}^{N} J_I^2, \quad J^3, \quad |J_1|^2, I = 1, ..., N,
\]

and the modulus of the following \(N-2\) operators (let us suppose for the moment \(I, J \neq 1, 2\)), with their corresponding spins eigenvalues are:

\[
\begin{align*}
G_1 & := J_1, \quad g_1 = j_1, \\
G_2 & := G_1 + J_2, \quad g_2 = g_2 (g_1, j_2), \\
G_3 & := G_2 + J_1, \quad g_3 = g_3 (g_2, j_1), \\
G_4 & := G_3 + J_2, \quad g_4 = g_4 (g_3, j_2), \\
& \vdots \\
G_I & := G_{I-1} + J_{I-2}, \quad g_I = g_I (g_{I-1}, j_{I-2}), \\
G_{I+1} & := G_I + J_{I-1}, \quad g_{I+1} = g_{I+1} (g_I, j_{I-1}), \\
G_{I+2} & := G_{I+1} + J_{I+1}, \quad g_{I+2} = g_{I+2} (g_{I+1}, j_{I+1}), \\
G_{I+3} & := G_{I+2} + J_{I+2}, \quad g_{I+3} = g_{I+3} (g_{I+2}, j_{I+2}), \\
& \vdots \\
G_J & := G_{J-1} + J_{J-1}, \quad g_J = g_J (g_{J-1}, j_{J-1}).
\end{align*}
\]

The matrix representation of \(\hat{q}_{IJK}\) in this basis is:

\[
\langle \hat{a}(12) | \hat{q}_{IJK} | \hat{a}^\dagger(12) \rangle = \frac{1}{4} (-1)^{j_I + j_K + a_{I-1} + a_K} (-1)^{a_I-1} (-1)^{\sum_{n=1}^{j_I} j_n} (-1)^{\sum_{n=1}^{j_K} j_n} \prod_{n=2}^{I-1} \delta_{a_n a_n} \prod_{n=K}^{N} \delta_{a_n a_n} \times \\
\times X (j_I, j_K) \sqrt{2(a_I + 1)} (2a_I' + 1) \sqrt{2(a_K + 1)} (2a_K' + 1) \times \\
\times \left\{ \prod_{n=I+1}^{J-I+1} \delta_{a_n a_n} \right\} \left\{ \prod_{n=J+1}^{K-J+1} \delta_{a_n a_n} \right\} \times \\
\times \left\{ \prod_{n=1}^{I+1} \delta_{a_n a_n} \right\} \left\{ \prod_{n=1}^{K+1} \delta_{a_n a_n} \right\} \times \\
\times (a_I - a_I') \left\{ a_{I-1} j_I a_I' j_I' \right\} \left\{ a_{I+1} j_I a_I' j_I' \right\} \left\{ a_{I+1} j_I a_I' j_I' \right\} - .. \\
\times (a_K - a_K') \left\{ a_{K-1} j_K a_K' j_K' \right\} \left\{ a_{K+1} j_K a_K' j_K' \right\} \left\{ a_{K+1} j_K a_K' j_K' \right\} - ..
\]

with \(X (j_I, j_K) = 2j_I (2j_I + 1) (2j_K + 1) (2j_K + 2) 2j_J (2j_J + 1) (2j_J + 2)\) and \(|\hat{a}(12)\rangle = |\hat{a}^\dagger(12), j, j = 0, m = 0 \rangle\). In this step, one use a specific inner product \(\langle ..., \rangle\) to obtain the matrix components. Using this formula, one can obtain the matrix representation of the volume operator, which is need to be done with an extra care. Nevertheless, it needs
to be kept in mind that it is not possible to obtain a closed analytical formula for the matrix representation of the volume operator, due to the complicated nature of (35).

The gauge invariance is implemented by setting the total angular momentum operator to be zero \[34\]:

\[ J |\psi\rangle = \sum_{i=1}^{N} J_i |\psi\rangle = 0. \]  

(36)

Projecting the operator to the kinematical Hilbert space \( K_{\Gamma} \subset H_{\Gamma} \) defined by \[36\], they satisfy the following condition:

\[ J = \sum_{i=1}^{N} J_i = G_N = G_{N-1} + J_N \approx 0. \]  

(37)

This cause a recursive constraints to occur on each steps of the spins addition:

\[ G_{N-1} = G_{N-2} + J_{N-1} \approx -J_N \]

\[ G_{N-2} = G_{N-3} + J_{N-2} \approx -J_N - J_{N-1}. \]  

(38)

The equivalence notation \( \approx \) is used to denoted that these relations are valid only on the constrained Hilbert space. Therefore, the spin quantum number of spin-network state are also restricted:

\[ \max ((|J_{N-2} - g_{N-3}|, |J_N - J_{N-1}|)) \leq g_{N-2} \leq \min ((|J_{N-2} + g_{N-3}|, |J_N + J_{N-1}|)). \]  

(39)

An important thing we need to stress is that relation (37)-(38) are only valid on the kinematical Hilbert space. Acting on the full, non-gauge invariant Hilbert space \( H_{\Gamma} \), the quantum gauge invariance is not restricting the spin operators to satisfy (37)-(38), but to select the set of invariant states such that they satisfies the Gauss constraint, and thus defines \( K_{\Gamma} \).

The formal steps to obtain the matrix representation of volume operators in the angular momentum representation of the spin-network could be summarize as follows:

1. Starting from (32), (33), and/or (34), one finds the \( \hat{q}^{T}_{IJK} \hat{q}^{T}_{IJK} \) for \( \hat{v}_{RS} \) and/or the terms under the 4-root for \( \hat{v}_{AL} \).

2. The next step is to diagonalize the terms under the root to obtain the spectrum, particularly the pair \( (\Lambda, \lambda_n P_n) \), with \( \lambda_n P_n \) is the diagonal matrix of \( \hat{q}^{T}_{IJK} \hat{q}^{T}_{IJK} \), and \( \Lambda \) is the corresponding orthogonal transformation. In this step one should check that all the eigenvalues are positive.

3. Following similar procedure as in Claim 3, one takes the 4-root of \( \lambda_n P_n \) for \( \hat{v}_{RS}^{(2)} \) and \( \hat{v}_{AL} \) (or the square root for \( \hat{v}_{RS}^{(1)} \)) by simply 4-rooting (or square-rooting for \( \hat{v}_{RS}^{(1)} \)) the eigenvalues \( \lambda_n \) to obtain \( +\sqrt{\lambda_n P_n} \) (or \( +\sqrt{\lambda_n P_n} \) for \( \hat{v}_{RS}^{(1)} \)).

4. The next step is to de-diagonalize the matrix resulting from Step 3: \( \Lambda^{-1} \sqrt{\lambda_n P_n} \Lambda \) to obtain \( \sqrt{\hat{q}^{T}_{IJK} \hat{q}^{T}_{IJK}} \) for \( \hat{v}_{RS}^{(2)} \) (and the same procedure for \( \hat{v}_{AL} \)), while for \( \hat{v}_{RS}^{(1)} \), one needs to de-diagonalize the following matrix \( \Lambda^{-1} \sqrt{\lambda_n P_n} \Lambda \) to obtain \( \sqrt{\hat{q}^{T}_{IJK} \hat{q}^{T}_{IJK}} \), then to sum over distinct \( I, J, K \), and finally to repeat once again Step 4.

5. In this step, one already obtain the matrix representation of volume operators (32), (33), and/or (34); they are ready to be diagonalized to obtain the spectrum.

A. The 4-Vertex Case

The volume operator acts on the nodes of a spin-network \( \Gamma \), as already discussed in (12). Let us consider a 4-vertex node, where the direction of the four links labelled by \( j_i, I = 1, 2, 3, 4 \), is chosen to be pointing outward from the node. The possible three-hand operator \( \hat{q}_{IJK} \) for the 4-vertex volume operators are the ones with distinct indices, where \( I, J, K \) runs from 1 to 4. As already shown in (34), restricting to the kinematical Hilbert space, the possible three-hands operator for 4-vertex case satisfies the following relations:

\[ \hat{q}_{123} \approx -\hat{q}_{124} \approx \hat{q}_{134} \approx -\hat{q}_{234}. \]
Inserting this condition to (32), (33), and (35) gives:

\[ \hat{\psi}_{RS}^{(1)} = 2Z_1^{1/2} \sqrt{\mu_{123}}, \]

\[ \hat{\psi}_{RS}^{(2)} = 4Z_2^{1/2} \sqrt{\mu_{123}} = 2\sqrt{\frac{Z_2}{Z_1}} \hat{\psi}_{RS}^{(1)}, \]

\[ \hat{\psi}_{AL} = 2Z_2^{1/2} \sqrt{\mu_{123}} = \frac{Z_3}{Z_1} \hat{\psi}_{RS}^{(1)}. \]

The three volume operators give the same expression and only differ by the constants.

For a 4-vertex case, (35) greatly simplifies into:

\[ \langle j_{12} | \hat{q}_{123} | j_{12} - 1 \rangle = \frac{1}{((2j_{12} - 1)(2j_{12} + 1))^{1/2}} ((j_1 + j_2 + j_1 + 1)(j_1 - j_2 + j_1 + j_2)(j_1 + j_2 - j_{12} + 1) \times \ldots \]

\[ \times (j_3 + j_4 + j_1 + 1)(j_3 - j_4 + j_3 + j_4)(j_3 + j_4 - j_{12} + 1))^{1/2}, \]

where we write \(|\hat{a}(12)| = |a_2 = j_{12}|\), or simply \(|j_{12}|\). Moreover, \(|a'_2 = j_{12}'| = |j_{12} - 1|\), where the detailed explanation is discussed in [34]. The possible value of \(j_{12}\) needs to satisfy the following restrictions coming from [35]:

\[ \max (|j_1 - j_2|, |j_3 - j_4|) \leq j_{12} \leq \min (|j_1 + j_2|, |j_3 + j_4|). \]

Therefore, the dimension \(k\) of the intertwiner space (or kinematical space \(K_T\)), which is the dimension of the matrix representation of \(\langle j_{12} | \hat{q}_{123} | j_{12} - 1 \rangle\) is \(k = j_{12}^{\min} - j_{12}^{\min} + 1\). Sorting the possible value of \(j_{12}\) from the smallest one, namely \(j_{12}^{\min}\), and using the following notation: \(a_k = \langle j_{12}^{\min} + k | \hat{q}_{123} | j_{12}^{\min} + k - 1 \rangle\), one could construct the matrix representation of \(\hat{q}_{123}\), which is a type of Jacobi matrix with banded structure [34]:

\[ \hat{q}_{123} = \begin{bmatrix} 0 & -a_1 & 0 & \cdots & 0 \\ a_1 & 0 & -a_2 & \vdots & \\ 0 & a_2 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -a_{n-1} \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}. \]

We found that for 4-vertex case, the product of \(\hat{q}_{123}\) with its transpose could always be written as the following matrix:

\[ \hat{q}_{123}' \hat{q}_{123} = \begin{bmatrix} a_1^2 & 0 & -a_1a_2 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & a_1^2 + a_2^2 & 0 & -a_2a_3 & 0 & 0 & \cdots & \\ -a_1a_2 & 0 & a_2^2 + a_3^2 & 0 & -a_3a_4 & 0 & \cdots & \vdots \\ 0 & -a_2a_3 & 0 & a_3^2 + a_4^2 & 0 & -a_4a_5 & 0 & \cdots \\ 0 & 0 & -a_3a_4 & 0 & \ddots & 0 & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & -a_{n-3}a_{n-2} & 0 \\ 0 & \cdots & \cdots & 0 & a_{n-3}^2 + a_{n-2}^2 & 0 & -a_{n-2}a_{n-1} & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & -a_{n-2}a_{n-1} & 0 & a_{n-2}^2 + a_{n-1}^2 \end{bmatrix}, \]

which is real, symmetric, and positive semi-definite, with respect to the inner product defined in (35). With these simplification, one could proceed to the formal steps to obtain the matrix representation of volume operators and their spectrum.

**B. Monochromatic Quantum Tetrahedron**

A 4-vertex node is geometrically interpreted as a quantum tetrahedron. Here we give, as examples, the explicit calculation of the matrix representation of volume operators and their spectrum for 4-vertex in two cases: (1) Ground State Monochromatic (GSM) case, where the spin \(j_1 = j_2 = j_3 = j_4 = \frac{1}{2}\), and the First Excited Monochromatic (FEM) case with \(j_1 = j_2 = j_3 = j_4 = 1\). Since for 4-vertex case the three volume operators only differ by the constants, we will only calculate the \(\hat{\psi}_{RS}^{(1)}\) for brevity.
Using (31), (40) becomes:

\[ \hat{v}_{RS}^{(1)} = 2Z_1^{1/2} \sqrt{q_{123}^T q_{123}}. \]

and from (44), we obtain, respectively:

\[ \hat{q}_{123} = \begin{bmatrix} 0 & \sqrt{3} \\ -\sqrt{3} & 0 \end{bmatrix}, \quad \hat{q}_{123}^T \hat{q}_{123} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \quad |\hat{q}_{123}| = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{3} \end{bmatrix}. \]

Since the matrix is already diagonal (in 2-dimension, the degrees of freedom for an antisymmetric matrix is one, therefore, \( \hat{q}_{123}^T \hat{q}_{123} \) is already diagonal in this step. This, in general, is not the case for larger dimensional matrix), then we can directly obtain the unique positive square root:

\[ \hat{v}_{RS}^{(1)} = 2Z_1^{1/2} \begin{bmatrix} \sqrt{3} \\ 0 \end{bmatrix}, \sqrt{3} \]

so that the eigenvalues are \( \lambda_{1,2} = 2Z_1^{1/2} \sqrt{3} \), giving exactly the same result with the procedure describe in [12, 34, 36].

The eigenvalues are degenerate, therefore one could freely choose two orthonormal eigenvectors by the Gramm-Schmidt procedure for the volume operator:

\[ |n_1\rangle = \cos \theta |a_{12}(0)\rangle - \sin \theta |a_{12}(1)\rangle, \quad |n_2\rangle = \sin \theta |a_{12}(0)\rangle + \cos \theta |a_{12}(1)\rangle. \]  

(44)

The freedom is described by a parameter \( \theta \), which correspond to the angle of rotation in the 2-dimensional eigenspace.

2. First Excited Monochromatic (FEM) Case

From (43) we obtain, respectively:

\[ \hat{q}_{123} = \begin{bmatrix} 0 & -8\sqrt{3} \\ 8\sqrt{3} & 0 \\ 0 & 4\sqrt{3} \end{bmatrix}, \quad \hat{q}_{123}^T \hat{q}_{123} = \begin{bmatrix} 64/3 & 0 & -32\sqrt{3}/3 \\ 0 & 48 & 0 \\ -32\sqrt{3}/3 & 0 & 80/3 \end{bmatrix}. \]

In contrast with the GSM case, \( \hat{q}_{123}^T \hat{q}_{123} \) is not diagonal; one needs to perform the following diagonalization, to obtain the root:

\[ \Lambda^{-1} \hat{q}_{123}^T \Lambda = \lambda_n \hat{P}_n = \begin{begin{bmatrix} 48 & 0 & 0 \\ 0 & 48 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \sqrt[4]{\lambda_n} \hat{P}_n = \begin{bmatrix} 2\sqrt{3} & 0 & 0 \\ 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]

De-diagonalizing the root using \( \Lambda^{1/2} \lambda_n^{1/2} \Lambda^{-1} \), one obtains, respectively:

\[ \Lambda^{1/2} \hat{q}_{123}^T = \begin{bmatrix} 8\sqrt{3}/3 & 0 & -4\sqrt{3}/\sqrt{3} \\ 0 & 2\sqrt{3} & 0 \\ -4\sqrt{3}/\sqrt{3} & 0 & 10/\sqrt{3} \end{bmatrix}, \quad \hat{v}_{RS}^{(1)} = \sqrt{Z_1} \begin{bmatrix} 48/3 & 0 & -8\sqrt{3}/3 \\ 0 & 4\sqrt{3} & 0 \\ -8\sqrt{3}/3 & 0 & 2\sqrt{3} \end{bmatrix}. \]

The eigenvalues are \( \lambda_0 = 0 \) and \( \lambda_{1,2} = 4Z_1^{1/2} \sqrt{3} \). \( \lambda_{1,2} \) are the degenerate eigenvalues, and \( \lambda_0 \) is the 'accidental zero' in [12], which occurs due to the oddity of the intertwiner space. The eigenvectors are:

\[ |n_0\rangle = \frac{\sqrt{3}}{3} |a_{12}(0)\rangle + 2/3 |a_{12}(2)\rangle, \]

\[ |n_1\rangle = \cos \theta \left( \frac{\sqrt{2}}{3} |a_{12}(0)\rangle - \frac{\sqrt{10}}{6} |a_{12}(2)\rangle \right) - \sin \theta |a_{12}(1)\rangle, \]

\[ |n_2\rangle = \sin \theta \left( \frac{\sqrt{2}}{3} |a_{12}(0)\rangle - \frac{\sqrt{10}}{6} |a_{12}(2)\rangle \right) + \cos \theta |a_{12}(1)\rangle. \]
V. CONCLUSIONS.

Different regularization scheme applied to the volume of a 3D region gives two types of volume operators in LQG: the Rovelli-Smolin ($\hat{v}_{RS}$) and Ashtekar-Lewandowski ($\hat{v}_{AL}$) volume operator. In the literatures, there exists two types of $\hat{v}_{RS}$ as had been discussed in Section II, which differ only by the location of the summation with respect to the square root. Based on the angular momentum representation of the spin-network, a careful analysis on the three-hand operator $\hat{q}_{IJK}$ shows that the hermiticity of this operator depends on the indices $I, J, K$. It follows that the linearly dependent terms $\hat{q}_{III}$ and $\hat{q}_{IJJ}$ are real and symmetric (hence hermitian), and in general do not give zero contribution to the volume operators. For $\hat{v}_{RS}$, these linearly dependent term is removed by the argument stemming from the graphical representation of spin-networks, while for $\hat{v}_{AL}$, they are removed by the existence of the sign factor $\epsilon(\epsilon_1, \epsilon_2, \epsilon_3)$. The later is more natural in the sense that it gives consistent result, either in the angular momentum or graphical approach, without additional assumptions.

The formal and rigorous proof for the hermiticity and positive semi-definiteness of the volume operators had not been given explicitly in the literatures. In this article, we had provided a proof for these two conditions. The first step is to provide a clear definition on the absolute value of an operator, or particularly, for our case, the absolute value of a matrix. A mathematical definition of the absolute value of a bounded operator already existed, namely $|M| := +\sqrt{M^†M}$. Thus we apply this definition for matrices; this is done in Section III. Consequently, one could derive several claims. The first claim is the hermiticity and the positive semi-definiteness of the quantity $M^†M$, which is followed by the non-negativity of its eigenvalues, as proven in the second claim. By the third claim, $M^†M$ admits a unique hermitian and positive semi-definite square root, which, by definition, is $|M|$. Moreover, since $|M|$ is guaranteed to be hermitian and positive semi-definite, it also had a hermitian and positive semi-definite square root, namely $\sqrt{|M|}$, which is the form of matrix we expected. Therefore, considering that each volume operator contains the square root of the absolute value of a matrix, we could use the claims to prove that volume operators in LQG are hermitian and positive semi-definite. Furthermore, since the arguments inside the absolute value of the operators, namely $M$, are purely imaginary and antisymmetric (that is, $M = i\hat{q}_{IJK}$ for $\hat{v}_{RS}$ and its linear combination for $\hat{v}_{AL}$), the absolute value of $M$ is the square root of the product of an imaginary antisymmetric matrix with its complex-conjugate, namely $|M| = \sqrt{\hat{q}_{IJK}^†\hat{q}_{IJK}}$, with $\hat{q}_{IJK}$ is real antisymmetric. As a consequence to this, the volume operator $\hat{v} \sim \sqrt{|M|}$ is real, symmetric and positive semi-definite. The reality and symmetry (and hence, the hermiticity) of these operators are defined with respect to the inner product defined in the Hilbert space of the graph, $H_G = L_2[\text{SU}(2), dq_{\text{Haar}}]$. As an advantage to this fact, one could choose the corresponding eigenvectors to be real. Moreover, the diagonalization of the volume operator $\hat{v}$ always yield pairs of positive degenerate eigenvalues with an additional zero if the dimension is odd. As a result of this degeneracy, one has a freedom to choose eigenbasis for each eigensubspaces.

There exists two equivalent representations of spin-network, namely, the angular momentum and the graphical representation. In the later, one does not necessarily fix the notion of inner product, albeit it is possible to obtain the matrix representation of an operator. In particular, one could obtain the reality and symmetry of the area and volume operator solely from the rules of the graphical approach. With these conditions in hand, one could fix an inner product on the representation space of the graph, such that it matches the one defined on the Hilbert space in the angular momentum approach; this had been done in the literatures. Our work in this article could be interpreted as a complement to this result: in which given a Hilbert space $H_G = L_2[\text{SU}(2), dq_{\text{Haar}}]$, we prove, based on the angular-momentum approach, that volume operators acting on $H_G$ are real, symmetric, and positive semi-definite, with respect to the inner product induced by the Haar measure on $\text{SU}(2)$. The result supports the equivalency between the graphical and the angular momentum approach of spin-network.

In Section IV, we provide a formal procedure to obtain the matrix representation of volume operator, together with their spectrum. In order to compare our procedure with the earlier ones existing in the literatures, we give explicit computational examples for the case of monochromatic quantum tetrahedron. The spectrum is obtained directly from the diagonalization of the volume operators matrices. The results are consistent with existing results found in the literatures, although the two procedures differ in which the spectrum resulting from the standard approach is obtained from the diagonalization of $i\hat{q}_{IJK}$ rather than $\hat{v}$. This consistency gives an approval to our procedure as a formal approach in obtaining the eigenvalues of volume operator, based on the angular momentum representation of spin-network. Furthermore, our approach provides a clearer view on the symmetries of the volume operator. Since in our approach the matrices of the volume operators are obtained explicitly, it becomes clear that they contain degeneracies, particularly, the eigensubspaces corresponding to each distinct positive eigenvalues are two-folds. It might be interesting to see how this fact will affect the calculation concerning the length operator, Hamiltonian (and Master) constraint operator, and other operators related to the volume operator. However, as explained in the Introduction, the Ashtekar-Lewandowski (and Rovelli-Smolin) volume operator could be considered as a ‘special’ case of the polyhedral volume operator introduced by Bianchi, Dona’, and Speziale. The latter had a correct and
well-defined semi-classical limit for any number of vertices, in contrast with the former, which only admit a correct semi-classical limit for \( n = 6 \). Due to this reason, it is important to study the polyhedral volume operator in detail for the further works, as well as obtaining its complete spectral properties.

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