Irreducibility of the Punctual Quotient Scheme of a Surface

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Let $X$ be a smooth projective surface, $E$ a locally free sheaf of rank $r \geq 1$ on $X$, and let $\ell \geq 1$ be an integer. $\text{Quot}(E, \ell)$ denotes Grothendieck’s quotient scheme that parametrises all surjections $E \to T$, where $T$ is a zero-dimensional sheaf of length $\ell$, modulo automorphisms of $T$. Sending a quotient $E \to T$ to the point $\sum_{x \in X} \ell(T_x)x$ in the symmetric product $S^\ell(X)$ defines a morphism $\pi : \text{Quot}(E, \ell) \to S^\ell(X)$. It is the purpose of this note to prove the following theorem:

**Theorem 1** — $\text{Quot}(E, \ell)$ is an irreducible scheme of dimension $\ell(r + 1)$. The fibre of the morphism $\pi : \text{Quot}(E, \ell) \to S^\ell(X)$ over a point $\sum x \ell_x x$ is irreducible of dimension $\sum x(r\ell_x - 1)$.

If $r = 1$, i.e. if $E$ is a line bundle, then $\text{Quot}(E, \ell)$ is isomorphic to the Hilbert scheme $\text{Hilb}^\ell(X)$. For this case, the first assertion of the theorem is due to Fogarty [4], whereas the second assertion was proved by Briançon [2]. For general $r \geq 2$, the first assertion of the theorem is a result due to J. Li and D. Gieseker [8],[6]. We give a different proof with a more geometric flavour, generalising a technique from Ellingsrud and Størme [4]. The second assertion is a new result for $r \geq 2$.

1 Elementary Modifications

Let $X$ be a smooth projective surface and $x \in X$. If $N$ is a coherent $\mathcal{O}_X$-sheaf, $e(N_x) = \text{hom}_X(N, k(x))$ denotes the dimension of the fibre $N(x)$, which by Nakayama’s Lemma is the same as the minimal number of generators of the stalk $N_x$. If $T$ is a coherent sheaf with zero-dimensional support, we denote
by \( i(T_x) = \text{hom}_X(k(x), T) \) the dimension of the socle of \( T_x \), i.e. the submodule 
\( \text{Soc}(T_x) \subset T_x \) of all elements that are annihilated by the maximal ideal in 
\( \mathcal{O}_{X,x} \).

Lemma 2 — Let \([g : E \to T] \in \text{Quot}(E, \ell)\) be a closed point and let \( N \) be
the kernel of \( q \). Then the socle dimension of \( T \) and the number of generators 
of \( N \) at \( x \) are related as follows:

\[ e(N_x) = i(T_x) + r. \]

Proof. Write \( e(N_x) = r + i \) for some integer \( i \geq 0 \). Then there is a minimal 
free resolution 
\[ 0 \to \mathcal{O}_{X,x}^r \to \mathcal{O}_{X,x}^i \to N_x \to 0, \]
where all coefficients of 
the homomorphism \( \alpha \) are contained in the maximal ideal of \( \mathcal{O}_{X,x} \). We have 
\( \text{Hom}(k(x), T_x) \cong \text{Ext}^1_X(k(x), N_x) \) and applying the functor \( \text{Hom}(k(x), .) \) one
finds an exact sequence

\[ 0 \to \text{Ext}^1_X(k(x), N_x) \to \text{Ext}^2_X(k(x), \mathcal{O}_{X,x}^i) \to \text{Ext}^2_X(k(x), \mathcal{O}_{X,x}^{r+i}). \]

But as \( \alpha \) has coefficients in the maximal ideal, the homomorphism \( \alpha' \) is zero. 
Thus \( \text{Hom}(k(x), T) \cong \text{Ext}^2_X(k(x), \mathcal{O}_{X,x}^i) \cong k(x)^i \). \[ \square \]

The main technique for proving the theorem will be induction on the
length of \( T \). Let \( N \) be the kernel of a surjection \( E \to T \), let \( x \in X \) be a
closed point, and let \( \lambda : N \to k(x) \) be any surjection. Define a quotient 
\( E \to T' \) by means of the following push-out diagram:

\[
\begin{array}{ccccccc}
0 & 0 & & & & & \\
\uparrow & \uparrow & & & & & \\
0 & \to & k(x) & \xrightarrow{\mu} & T' & \to & T & \to & 0 \\
\uparrow & \uparrow & \uparrow & \parallel & & & & \\
0 & \to & N & \to & E & \to & T & \to & 0 \\
\uparrow & \uparrow & \uparrow & \parallel & & & & \\
N' & = & N' & & & & & \\
\uparrow & \uparrow & \uparrow & & & & & \\
0 & 0 & & & & & 
\end{array}
\]

In this way every element \( \langle \lambda \rangle \in \mathbb{P}(N(x)) \) determines a quotient \( E \to T' \)
together with an element \( \langle \mu \rangle \in \mathbb{P}(\text{Soc}(T_x')) \). (Here \( W^\vee := \text{Hom}_k(W, k) \)
denotes the vector space dual to \( W \).) Conversely, if \( E \to T' \) is given, any
such \( \langle \mu \rangle \) determines \( E \to T \) and a point \( \langle \lambda \rangle \). We will refer to this situation
by saying that \( T' \) is obtained from \( T \) by an elementary modification.
We need to compare the invariants for $T$ and $T'$. Obviously, $\ell(T') = \ell(T) + 1$. Applying the functor $\text{Hom}(k(x), \cdot)$ to the upper row in the diagram we get an exact sequence

$$0 \rightarrow k(x) \rightarrow \text{Soc}(T'_x) \rightarrow \text{Soc}(T_x) \rightarrow \text{Ext}^1_X(k(x), k(x)) \cong k(x)^2,$$

and therefore $|i(T_x) - i(T'_x)| \leq 1$. Moreover, we have $0 \leq e(T'_x) - e(T_x) \leq 1.$ Two cases deserve more attention:

**Lemma 3** — Consider the natural homomorphisms $g : N(x) \rightarrow E(x)$ and $f : \text{Soc}(T'_x) \rightarrow T' \rightarrow T'(x)$. The following assertions are equivalent

1. $e(T') = e(T) + 1$
2. $\langle \mu \rangle \not\in \mathbb{P}(\ker(f)^\vee)$
3. $\langle \lambda \rangle \in \mathbb{P}(\text{im}(g))$.

Moreover, if these conditions are satisfied, then $T' \cong T \oplus k(x)$ and $i(T'_x) = i(T_x) + 1$.

**Proof.** Clearly, $e(T') = e(T) + 1$ if and only if $\mu(1)$ represents a non-trivial element in $T'(x)$ if and only if $\mu$ has a left inverse if and only if $\lambda$ factors through $E$. \(\square\)

**Lemma 4** — Still keeping the notations above, let $E \rightarrow T'_x$ be the modification of $E \rightarrow T$ determined by the point $\langle \lambda \rangle \in \mathbb{P}(N(x))$. Similarly, for $\langle \mu' \rangle \in \mathbb{P}(\text{Soc}(T_x)^\vee)$ let $T^\vee_{\mu'} = T/\mu'(k(x))$. If $i(T^\vee_{\lambda, x}) = i(T_x) + 1$ for all $\langle \lambda \rangle \in \mathbb{P}(N(x))$, then $i(T_x) = i(T^\vee_{\mu', x}) - 1$ for all $\langle \mu' \rangle \in \mathbb{P}(\text{Soc}(T_x)^\vee)$ as well.

**Proof.** Let $\Phi : \text{Hom}_X(N, k(x)) \rightarrow \text{Hom}_k(\text{Ext}^1_X(k(x), N), \text{Ext}^1_X(k(x), k(x)))$ be the homomorphism which is adjoint to the natural pairing

$$\text{Hom}_X(N, k(x)) \otimes \text{Ext}^1_X(k(x), N) \rightarrow \text{Ext}^1_X(k(x), k(x)).$$

Identifying $\text{Soc}(T_x) \cong \text{Ext}^1_X(k(x), N)$, we see that $i(T^\vee_{\lambda, x}) = 1 + i(T_x) - \text{rank}(\Phi(\lambda))$. The action of $\Phi(\lambda)$ on a socle element $\mu' : k(x) \rightarrow T$ can be described by the following diagram of pull-backs and push-forwards

$$
\begin{array}{ccccccc}
0 & \rightarrow & N & \rightarrow & E & \rightarrow & T & \rightarrow & 0 \\
\| & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & N & \rightarrow & N^\vee_{\mu'} & \rightarrow & k(x) & \rightarrow & 0 \\
\lambda & \downarrow & & \downarrow & & & \| & & \\
0 & \rightarrow & k(x) & \rightarrow & \xi & \rightarrow & k(x) & \rightarrow & 0
\end{array}
$$
The assumption that $i(T'_{\lambda,x}) = 1 + i(T_x)$ for all $\lambda$, is equivalent to $\Phi = 0$. This implies that for every $\mu'$ and every $\lambda$ the extension in the third row splits, which in turn means that every $\lambda$ factors through $N_{\mu'}^-$, i.e. that $N(x)$ embeds into $N_{\mu'}^-$. Hence, for $T^-_{\mu'} = E/N_{\mu'}^- = \text{coker}(\mu)$ we get $i(T^-_{\mu',x}) = e(N_{\mu',x}) - r = e(N_x) + 1 - r = i(T_x) + 1$. □

2 The Global Case

Let $Y_\ell = \text{Quot}(E, \ell) \times X$, and consider the universal exact sequence of sheaves on $Y_\ell$:

$$0 \to N \to \mathcal{O}_{\text{Quot}} \otimes E \to \mathcal{T} \to 0.$$

The function $y = (s, x) \mapsto i(T_{s,x})$ is upper semi-continuous. Let $Y_{\ell,i}$ denote the locally closed subset \{\(y = (s, x) \in Y_\ell | i(T_{s,x}) = i\}\} with the reduced subscheme structure.

**Proposition 5** — $Y_\ell$ is irreducible of dimension $(r + 1)\ell + 2$. For each $i \geq 0$ one has $\text{codim}(Y_{\ell,i}, Y_\ell) \geq 2i$.

Clearly, the first assertion of the theorem follows from this.

**Proof.** The proposition will be proved by induction on $\ell$, the case $\ell = 1$ being trivial: $Y_1 = \mathbb{P}(E) \times X$, the stratum $Y_{1,1}$ is the graph of the projection $\mathbb{P}(E) \to X$ and $Y_{1,i} = \emptyset$ for $i \geq 2$. Hence suppose the proposition has been proved for some $\ell \geq 1$.

We describe the ‘global’ version of the elementary modification discussed above. Let $Z = \mathbb{P}(N)$ be the projectivization of the family $\mathcal{N}$ and let $\varphi = (\varphi_1, \varphi_2) : Z \to Y_\ell = \text{Quot}(E, \ell) \times X$ denote the natural projection morphism. On $Z \times X$ there is canonical epimorphism

$$\Lambda : (\varphi_1 \times \text{id}_X)^*\mathcal{N} \to (\text{id}_Z, \varphi_2)^*\mathcal{N} \to (\text{id}_Z, \varphi_2)_*\mathcal{O}_Z(1) =: \mathcal{K}.$$ 

As before we define a family $\mathcal{T}'$ of quotients of length $\ell + 1$ by means of $\Lambda$:

$$0 \to \mathcal{K} \overset{\Lambda}{\to} \mathcal{T}' \overset{(\varphi_1, \text{id}_X)^*\mathcal{T}}{\to} 0$$

$$0 \to (\varphi_1, \text{id}_X)^*\mathcal{N} \overset{\text{id}_Z \otimes E}{\to} (\varphi_1, \text{id}_X)^*\mathcal{T} \to 0$$

Let $\psi_1 : Z \to \text{Quot}(E, \ell + 1)$ be the classifying morphism for the family $\mathcal{T}'$, and define $\psi := (\psi_1, \psi_2 := \varphi_2) : Z \to Y_{\ell+1}$. The scheme $Z$ together with the
morphisms \( \varphi : Z \to Y \) and \( \psi : Z \to Y_{\ell+1} \) allows us to relate the strata \( Y_{\ell,i} \) and \( Y_{\ell+1,j} \). Note that \( \psi(Z) = \bigcup_{j \geq 1} Y_{\ell+1,j} \).

The fibre of \( \varphi \) over a point \((s, x) \in Y_{\ell,i}\) is given by \( \mathbb{P}(\mathcal{N}_s(x)) \cong \mathbb{P}^{r-1+i} \), since \( \dim(\mathcal{N}_s(x)) = r + i(T_{s,x}) = r + i \) by Lemma 3. Similarly, the fibre of \( \psi \) over a point \((s', x) \in Y_{\ell+1,j}\) is given by \( \mathbb{P}(\text{Soc}(T'_{s',x})) \cong \mathbb{P}^{j-1} \). If \( T' \) is obtained from \( T \) by an elementary modification, then \(|i(T') - i(T)| \leq 1\) as shown above. This can be stated in terms of \( \varphi \) and \( \psi \) as follows: For each \( j \geq 1 \) one has:

\[
\psi^{-1}(Y_{\ell+1,j}) \subset \bigcup_{|i-j| \leq 1} \varphi^{-1}(Y_{\ell,i}).
\]

Using the induction hypothesis on the dimension of \( Y_{\ell,i} \) and the computation of the fibre dimension of \( \varphi \) and \( \psi \), we get

\[
\dim(Y_{\ell+1,j}) + (j - 1) \leq \max_{|i-j| \leq 1} \{(r + 1)\ell + 2 - 2i + (r + 1 - i)\}
\]

and

\[
\dim(Y_{\ell+1,j}) \leq (r + 1)(\ell + 1) + 2 - 2j - \min_{|i-j| \leq 1} \{i - j + 1\}.
\]

As \( \min_{|i-j| \leq 1} \{i - j + 1\} \geq 0 \), this proves the dimension estimates of the proposition.

It suffices to show that \( Z \) is irreducible. Then \( \text{Quot}(E, \ell + 1) = \psi_1(Z) \) and \( Y_{\ell+1} \) are irreducible as well.

Since \( X \) is a smooth surface, the epimorphism \( \mathcal{O}_{\text{Quot}} \otimes E \to T \) can be completed to a finite resolution

\[
0 \to A \to B \to \mathcal{O}_{\text{Quot}} \otimes E \to T \to 0
\]

with locally free sheaves \( A \) and \( B \) on \( Y_\ell \) of rank \( n \) and \( n + r \), respectively, for some positive integer \( n \). It follows that \( Z = \mathbb{P}(\mathcal{N}) \subset \mathbb{P}(\mathcal{B}) \) is the vanishing locus of the composite homomorphism \( \varphi^*A \to \varphi^*B \to \mathcal{O}_{\mathbb{P}(\mathcal{B})}(1) \). In particular, assuming by induction that \( Y_\ell \) is irreducible, \( Z \) is locally cut out from an irreducible variety of dimension \( (r + 1)\ell + 2 + (r + n - 1) \) by \( n \) equations. Hence every irreducible component of \( Z \) has dimension at least \( (r + 1)(\ell + 1) \).

But the dimension estimates for the stratum \( Y_{\ell,i} \) and the fibres of \( \varphi \) over it yield:

\[
\dim(\varphi^{-1}(Y_{\ell,i})) \leq (r + 1)\ell + 2 - 2i + (r + i - 1) = (r + 1)(\ell + 1) - i,
\]

which is strictly less than the dimension of any possible component of \( Z \), if \( i \geq 1 \). This implies that the irreducible variety \( \varphi^{-1}(Y_{\ell,0}) \) is dense in \( Z \). Moreover, since the fibre of \( \psi \) over \( Y_{\ell+1,1} \) is zero-dimensional, \( \dim(Y_{\ell+1}) = \dim(Y_{\ell+1,1}) + 2 = \dim(Z) + 2 \) has the predicted value. \( \square \)
3 The Local Case

We now concentrate on quotients $E \to T$, where $T$ has support in a single fixed closed point $x \in X$. For those quotients the structure of $E$ is of no importance, and we may assume that $E \cong O_X$. Let $Q^r_\ell$ denote the closed subset

$$\left\{ [O_X^r \to T] \in \text{Quot}(O_X^r, \ell) \mid \text{Supp}(T) = \{x\} \right\}$$

with the reduced subscheme structure. We may consider $Q^r_\ell$ as a subscheme of $Y^r_\ell$, by sending $[q]$ to $([q], x)$. Then it is easy to see that $\phi^{-1}(Q^r_\ell) = \psi^{-1}(Q^r_{\ell+1})$. Let this scheme be denoted by $Z^{'}_\ell$.

We will use a stratification of $Q^r_\ell$ both by the socle dimension $i$ and the number of generators $e$ of $T$ and denote the corresponding locally closed subset by $Q^r_{i,e}$. Moreover, let $Q^r_{i,e} = \bigcup Q^r_{i,e,i}$ and define $Q^r_{i,e}$ similarly. Of course, $Q^r_{i,e}$ is empty unless $1 \leq i \leq \ell$ and $1 \leq e \leq \min\{r, \ell\}$.

To prove the second half of the theorem it suffices to show:

**Proposition 6** — $Q^r_\ell$ is an irreducible variety of dimension $r\ell - 1$.

**Lemma 7** — $\dim(Q^r_{i,e}) \leq (r\ell - 1) - (2(i - 1) + \binom{e}{2})$.

**Proof.** By induction on $\ell$: if $\ell = 1$, then $Q^r_1 \cong \mathbb{P}^{r-1}$, and $Q^r_{1,i} = \emptyset$ if $e \geq 2$ or $i \geq 2$. Assume that the lemma has been proved for some $\ell \geq 1$.

Let $[q'] : O_X^r \to T'$ be a closed point. Suppose that the map $\mu : k(x) \to T'(x)$ represents a point in $\psi^{-1}([q']) = \mathbb{P}(\text{Soc}(T'_x)^\vee)$ and that $T_\mu = \text{coker}(\mu)$ is the corresponding modification. If $i = i(T_\mu,x)$ and $e = e(T_\mu,x)$, then, according to Section [I], the pair $(i, e)$ can take the following values:

$$\begin{align*}
(i, e) &= (j - 1, e - 1), (j - 1, e), (j, e) \text{ or } (j + 1, e),
\end{align*}$$

in other words:

$$\psi^{-1}(Q^r_{\ell+1,j}) \subset \varphi^{-1}(Q^r_{\ell,j-1}) \cup \bigcup_{|i-j|\leq 1} \varphi^{-1}(Q^r_{i,j}).$$

Subdivide $A = Q^r_{i,j}$ into four locally closed subsets $A_{i,e}$ according to the generic value of $(i, e)$ on the fibres of $\psi$. Then

$$\dim(A_{i,e}) + (j - 1) \leq \dim(Q^r_{i,j}) + d_{i,e},$$
where \( d_{i,\varepsilon} \) is the fibre dimension of the morphism

\[
\varphi : \psi^{-1}(A_{i,\varepsilon}) \cap \varphi^{-1}(Q_{\ell,i}^{r,\varepsilon}) \longrightarrow Q_{\ell,i}^{r,\varepsilon}.
\]

By the induction hypothesis we have bounds for \( \dim(Q_{\ell,i}^{r,\varepsilon}) \), and we can bound \( d_{i,\varepsilon} \) in the four cases as follows:

A) If \([q : \mathcal{O}_X \to T] \in Q_{\ell,j-1}^{r,e} \) is a closed point with \( N = \ker(q) \), then according to Lemma 3

\[
\varphi^{-1}([q]) \cap \psi^{-1}(A_{e-1,j-1}) \cong \mathbb{P}(\text{im}(g : N(x) \to k(x)^r)) \cong \mathbb{P}(\ker(k(x)^r \to T(x)) \cong \mathbb{P}^{r-e},
\]

since \( \text{im}(k(x)^r \to T(x)) \cong k^{e-1} \). Hence \( d_{j-1,e-1} = r - e \) and

\[
\dim(A_{j-1,e-1}) \leq \dim Q_{\ell,j-1}^{r,e-1} + (r - e) - (j - 1)
\]

\[
\leq \left\{ (r \ell - 1) - 2(j - 2) - \binom{e - 1}{2} \right\} + (r - e) - (j - 1)
\]

\[
= \left\{ (r \ell + 1 - 1) - 2(j - 1) - \binom{e}{2} \right\} - (j - 1).
\]

Note that this case only occurs for \( j \geq 2 \), so that \( (j-2) \) is always nonnegative.

B) In the three remaining cases

\( \varepsilon = e \) and \( i = j - 1, j, \) or \( j + 1 \)

we begin with the rough estimate \( d_{i,e} \leq r + i - 1 \) as in Section 2. This yields:

\[
\dim(A_{i,e}) \leq \left\{ (r \ell - 1) - 2(i - 1) - \binom{e}{2} \right\} + (r + i - 1) - (j - 1) \quad (2)
\]

\[
= \left\{ (r \ell + 1 - 1) - 2(j - 1) - \binom{e}{2} \right\} - (i - j). \quad (3)
\]

Thus, if \( i = j \) we get exactly the estimate asserted in the Lemma, if \( i = j + 1 \) the estimate is better than what we need by 1, but if \( i = j - 1 \), the estimate is not good enough and fails by 1. It is this latter case that we must further study: let \([q : \mathcal{O}_X \to T] \) be a point in \( Q_{\ell,j-1}^{r,e} \) with \( N = \ker(q) \). By Lemma 4 there are two possibilities:

— Either the fibre \( \varphi^{-1}([q]) \cap \psi^{-1}(A_{j-1,e}) \) is a proper closed subset of \( \mathbb{P}(N(x)) \) which improves the estimate for the dimension of the fibre \( \varphi^{-1}([q]) \) by 1,
— or this fibre equals with \( \mathbb{P}(N(x)) \), in which case we have \( i(T^-) = i(T) + 1 \) for every modification \( T^- = \text{coker}(\mu^- : k(x) \to T) \). But, as we just saw, calculation (3), applied to the contribution of \( Q_{r,e}^{r,e} \) to \( Q_{r,j}^{r,e} \), shows that the dimension estimate for the locus of such points \([q]\) in \( Q_{r,j}^{r,e} \) can be improved by 1 compared to the dimension estimate for \( Q_{r,j}^{r,e} \) as stated in the lemma.

Hence in either case we can improve estimate (3) by 1 and get

\[
\dim(A_{j-1,e}) \leq (r(\ell + 1) - 1) - 2(j - 1) - \left(\frac{e}{2}\right)
\]

as required. Thus, the lemma holds for \( \ell + 1 \). \( \square \)

**Lemma 8** — \( \psi(\varphi^{-1}(Q_{r,e}^{r,e})) \subset Q_{r,e}^{r,e+1} \).

**Proof.** Let \([q : O_X^r \to T] \in Q_{r,e}^{r,e}\) be a closed point with \( N = \ker(q) \). Then \( \varphi^{-1}([q]) = \mathbb{P}(N(x)) \cong \mathbb{P}^{r+i-1} \) and \( \varphi^{-1}([q]) \cap \psi^{-1}(Q_{r,e}^{r,e+1}) \cong \mathbb{P}(\text{im}(G)) \cong \mathbb{P}^{r-e-1} \). Since we always have \( e \geq 1, i \geq 1 \), a dense open part of \( \varphi^{-1}([q]) \) is mapped to \( Q_{r,e}^{r,e} \). \( \square \)

**Lemma 9** — If \( r \geq 2 \) and if \( Q_{r,e}^{r,e} \) is irreducible of dimension \( (r - 1)\ell - 1 \), then \( Q_{r,e}^{r,e} := \bigcup_{e < r} Q_{r,e}^{r,e} \) is an irreducible open subset of \( Q_{r,e}^{r,e} \) of dimension \( r\ell - 1 \).

**Proof.** Let \( M \) be the variety of all \( r \times (r - 1) \) matrices over \( k \) of maximal rank, and let \( 0 \to O_M^{r-1} \to O_M^r \to \mathcal{L} \to 0 \) be the corresponding tautological sequence of locally free sheaves on \( M \). Consider the open subset \( U \subset M \times Q_r^e \) of points \((A, [O^r \to T])\) such that the composite homomorphism

\[
O^r-1 \xrightarrow{A} O^r \rightarrow T
\]

is surjective. Clearly, the image of \( U \) under the projection to \( Q_r^e \) is \( Q_{r,e}^{r,e} \). On the other hand, the tautological epimorphism

\[
O_{U \times X}^{r-1} \rightarrow O_{U \times X}^r \rightarrow (O_M \otimes T)|_{U \times X}
\]

induces a classifying morphism \( g' : U \to Q_{r,e}^{r-1} \). The morphism \( g = (pr_1, g') : U \to M \times Q_{r,e}^{r-1} \) is surjective. In fact, it is an affine fibre bundle with fibre

\[
g^{-1}(g(A, [O^r-1 \to T])) \cong \text{Hom}_k(\mathcal{L}(A), T) \cong A_{k}^\ell.
\]

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Since $Q_{r}^{\ell-1}$ is irreducible of dimension $(r - 1)\ell - 1$ by assumption, $U$ is irreducible of dimension $r\ell - 1 + \dim(M)$, and $Q_{r}^{\ell,<r}$ is irreducible of dimension $r\ell - 1$. □

Proof of Proposition [4]. The irreducibility of $Q_{r}^{\ell}$ will be proved by induction over $r$ and $\ell$: the case ($\ell = 1, r$ arbitrary) is trivial; whereas ($\ell$ arbitrary, $r = 1$) is the case of the Hilbert scheme, for which there exist several proofs ([2], [4]). Assume therefore that $r \geq 2$ and that the proposition holds for ($\ell, r$) and ($\ell + 1, r - 1$). We will show that it holds for ($\ell + 1, r$) as well.

Recall that $Z' := \varphi^{-1}(Q_{r}^{\ell}) = Q_{r}^{\ell} \times_{Y_{r}} Z$. Every irreducible component of $Z'$ has dimension greater than or equal to $\dim(Q_{r}^{\ell}) + r - 1 = r(\ell + 1) - 2$ (compare Section [?]). On the other hand, $\dim(\varphi^{-1}(Q_{r}^{\ell,i})) \leq r\ell - 1 - 2(i - 1) + (r + i - 1) = r(\ell + 1) - i$. Thus an irreducible components of $Z'$ is either the closure of $\varphi^{-1}(Q_{r}^{\ell,1})$ (of dimension $r(\ell + 1) - 1$) or the closure of $\varphi^{-1}(W)$ for an irreducible component $W \subset Q_{r}^{\ell,2}$ of maximal possible dimension $r\ell - 3$. But according to Lemma [8] the image of $\varphi^{-1}(W)$ under $\psi$ will be contained in the closure of $Q_{r}^{\ell,r-1}$, unless $W$ is contained in $Q_{r}^{\ell,2}$. But Lemma [7] says that $Q_{r}^{\ell,2}$ has codimension $\geq 2 + (\ell \choose 2) \geq 3$ if $r \geq 2$, and hence cannot contain $W$ for dimension reasons. Hence any irreducible component of $Z'$ is mapped by $\psi$ into the closure of $Q_{r}^{\ell,r-1}$ which is irreducible by Lemma [8] and the induction hypothesis. This finishes the proof of the proposition. □

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