Spin Glasses: a Perspective

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Abstract

A brief personal perspective is given of issues, questions, formulations, methods, some answers and selected extensions posed by the spin glass problem, showing how considerations of an apparently insignificant and practically unimportant group of metallic alloys stimulated an explosion of new insights and opportunities in the general area of complex many-body systems and still is doing so.

1 Introduction

What are spin glasses? The answer to this apparently innocuous question has evolved from an initially obscure, if interesting, small special class of metallic alloys to ones concerned with the globally pervasive issue of the understanding of emergent complex behaviour in many-body systems, the development of new mathematical, simulational and conceptual tools, new experimental protocols, new algorithms and even a new class of mathematical probability problems. In this article I shall review some of this history and try to expose some of the key issues, challenges, solutions and opportunities of the topic.

2 Random magnetic alloys

The story starts with magnetic aspects of metallic alloys. In the early 1960’s there was much interest in the solid state physics community in the behaviour of isolated impurities in metals, first with the formation of local magnetic moments on magnetic metal impurities in non-magnetic hosts (Anderson 1961) and then with the strong coupling of a localized moment to the conduction electrons at low temperatures and its consequences for the electrical resistivity (Kondo 1964). The later 60’s and early 70’s saw the emergence of interest in the effects of inter-impurity correlations through spin glasses (see e.g. Coles 1983 and Mézard et al 1987) and the Kondo lattice (Doniach 1977).
The appellation “spin glass” is due to Bryan Coles in the late 60’s to label the low temperature state of a class of substitutional magnetic alloys, typified by CuMn or AuFe, with finite concentrations of the magnetic ions Mn or Fe in the non-magnetic hosts Cu and Au. The reason for the name is two-fold, first that in the state the magnetic moments (traditionally called “spins”) on the magnetic ions seem to freeze in orientation but without any periodic ordering (so conceptually reminiscent of the amorphous freezing of the locations of atoms in a conventional (structural) glass), and secondly that the low temperature specific heat is linear in $T$, again a feature of conventional glasses. Experiments at that time indicated a non-sharp onset of the state as the temperature was reduced from the paramagnetic one, suggesting a rapid onset of sluggishness but not a phase transition, again as believed to be characteristic of glasses. There were attempts to understand the behaviour in the 60’s but nothing very extensive.

But then more accurate experiments in the early 1970’s exposed a new source of theoretical interest, an apparently sharp phase transition signalled by a cusp in the magnetic susceptibility when external magnetic fields were kept very small (Cannella and Mydosh 1972). This had to be a new type of phase transition and therefore worthy of extra notice. But still theoretical work was minimal until Edwards and Anderson (1975) produced a paper that at one fell swoop recognised the importance of the combination of frustration and quenched disorder as fundamental ingredients, introduced a more convenient model, a new and novel method of analysis, new types of order parameters, a new mean field theory, new approximation techniques and the prediction of a new type of phase transition apparently explaining the observed susceptibility cusp. This paper was a watershed.

Edwards and Anderson’s new approach was beautifully minimal, fascinating and attractive but also their analysis was highly novel and sophisticated, involving radically new concepts and methods but also unusual and unproven ansätze, as well as several different approaches. So it seemed sensible to look for an exactly soluble model for which their techniques could be verified. Such a model was suggested by Sherrington and Kirkpatrick (1975). It extends the Edwards-Anderson model, in which exchange interactions are range-dependent and effectively short-range, to one with interactions between all spins, chosen randomly and independently from an intensive distribution (and so ‘infinite-ranged’ but not uniform). It offered the possibility of exact solution in the thermodynamic limit and an exact mean field theory, in analogy but subtle extension of the infinite-range ferromagnet for which naïve mean field theory is correct. Study of the SK model, or the mean field theory of the EA model that it defines, has proven highly non-trivial and instructive, and opened many new conceptual doors. It has also proven to be an entry point to many applications and extensions, which are still ongoing.
2.1 More details

2.1.1 Experimental spin glasses

Let me be more explicit. The original experimental spin glasses can be characterised by Hamiltonians

\[ H = -\frac{1}{2} \sum_{i,j} J(R_i - R_j) \mathbf{S}_i \cdot \mathbf{S}_j, \tag{1} \]

where the \( i, j \) label magnetic ions with Heisenberg spins \( \mathbf{S}_i \) and locations \( R_i \) and \( J(R) \) is an exchange interaction which oscillates in sign as a function of the spin separation. In metallic systems the origin of \( J(R) \) is the Ruderman-Kittel-Kasuya-Yoshida (RKKY) interaction. In the original alloys the disorder is substitutional on a lattice.

2.1.2 Edwards-Anderson

What Edwards and Anderson (correctly) surmised was that the important aspect of (1) is the combination of frustration, corresponding to the fact that the spins receive conflicting relative ordering instructions (as a consequence of the oscillation of the exchange with separation), and the quenched disorder in the location of the spins. For theoretical convenience they effectively replaced the Hamiltonian by one that can be written as

\[ H = -\frac{1}{2} \sum_{i,j} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j, \tag{2} \]

with spins on all the sites of a lattice but the \( J_{ij} \) between neighbouring spins and chosen randomly from a distribution having weight of either sign.\(^1\) They further chose the distribution to be Gaussian of zero mean, thereby both eliminating the possibility of any conventional order (with spatially uniform or periodic magnetization) and also having a minimal one-parameter characterization.\(^2\) This necessitated the introduction of a new form of order parameter to describe magnetic freezing without periodicity. In fact EA gave two versions: one based explicitly on temporal freezing,

\[ q = \lim_{t \to \infty, \tau \to \infty} q(t, t + \tau); \quad q(t, t + \tau) \equiv N^{-1} \sum_i \langle \mathbf{S}_i(t) \cdot \mathbf{S}_i(t + \tau) \rangle, \tag{3} \]

where \( \langle \cdot \rangle \) refers to a dynamical average, and the other based on ensemble-averaging,

\[ q = N^{-1} \sum_i |\langle \mathbf{S}_i \rangle|^2, \tag{4} \]

\(^1\)In fact the EA Hamiltonian was first written explicitly by Sherrington and Southern (1975).

\(^2\)Actually, the Gaussian choice for the single parameter description was also useful for the further analytic methods employed. The alternative simple single-parameter symmetric distribution having two delta functions of equal weight at \( \pm J \) has often been employed in (later) computer simulations.
with $\langle \cdot \rangle$ now referring to an ensemble-average restricted to one symmetry-breaking macro-state. The phase transition is signalled by $q$ becoming non-zero.

EA did not attempt a full solution but used several new variants of mean field theory, all requiring novel treatment beyond those conventional for a simple ferromagnet. The most sophisticated of them introduced and employed the so-called ‘replica trick’ which replaces the average of the logarithm of the partition function\(^3\), the physical generating function\(^4\), by the limiting behaviour of a partition function of an effective periodic system of higher dimensional spins:

$$
\ln Z = \lim_{n \to 0} \frac{\partial}{\partial n}(Z^n) = \lim_{n \to 0} \frac{\partial}{\partial n}(\prod_{\alpha=1,...,n} Z(\alpha)) = \lim_{n \to 0} \frac{\partial Z_{\text{eff}}(n)}{\partial n}, \quad (5)
$$

where the overbar refers to the average over the distribution of the $J$, $Z$ is the usual partition function $Z = \text{Tr}\{\sigma\} \exp\{-\beta H\}$, $Z(\alpha)$ is the partition function for spins with dummy labels $\alpha$ and $Z_{\text{eff}}(n)$ is the partition function of a periodic Hamiltonian $H_{\text{eff}}(\{\sigma^\alpha_i\})$ of effectively higher dimensional pseudo-spins with extra replica labels $\alpha = 1, \ldots, n$ and higher order interactions now between spins with different replica, as well as site, labels. Within this new description EA devised a new mean field theory with a new order parameter measuring inter-replica overlap

$$
q^{\alpha\beta} = N^{-1} \sum_i S_i^\alpha \cdot S_i^\beta; \quad \alpha \neq \beta. \quad (6)
$$

To go further, however, they employed several assumptions and approximations whose validity was difficult to assess, although they do yield results with some qualitative similarity to several experimental features.

### 2.1.3 Sherrington-Kirkpatrick

In view of the many uncertainties of the EA analysis and the fact that the model was surely not soluble with current techniques, it seemed sensible to look for a model in which a mean-field theory might be exact. Since the conventional ferromagnet is soluble in the thermodynamic limit provided that all spins interact equally with one another and correspondingly the exchange interaction scales inversely with the number of spins, it seemed reasonable to look for an analogue in the spin glass problem. This led to the formulation of the Sherrington-Kirkpatrick (SK) model whose Hamiltonian is similar to

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\(^3\)The argument for studying the average of the logarithm of the partition function is that the physical quantities it generates should be self-averaging, independent in the thermodynamic limit of the specific instance of choice of the disorder.

\(^4\)Certain physical observables can already be expressed as derivatives of $\ln Z$ with $Z$ defined as above with the bare $H$. Others, in principle, require the addition to $H$ of terms involving appropriate conjugate fields so that desired observables follow from derivatives of $\ln Z$ with respect to these fields.
that of EA but with interactions between all spins, chosen randomly and independently from a distribution whose mean and variance scale inversely with the number of spins. Simplifying to Ising spins and allowing for a ferromagnetic bias and an external field, the SK model is characterised by

\[ H = - \sum_{(ij)} J_{ij} \sigma_i \sigma_j - h \sum_i \sigma_i; \quad \sigma = \pm 1; \quad J_{ij} \text{i.i.d.}; \quad J_{ij} = J_0/N, \quad J_{ij}^2 = J^2/N. \]  

(7)

Despite its apparent simplicity, this model has turned out to expose many subtleties; for statistical physics, for mathematical physics and for probability theory; as well as having much wider application relevance. Extensions to other related models with extensive and super-extensive constraints independently drawn from identical (intensive) distributions have led to further novelties and applications. In this introductory perspective I shall restrict discussion to outlines at the level of conceptual theoretical statistical physics, leaving mathematical rigour to other authors.

2.1.4 Replica theory

Within the replica theory of EA but applied to the SK model the averaged free energy can be expressed in a form

\[ \mathcal{F} = -T \ln Z = -T \lim_{n \to 0} \frac{\partial}{\partial n} \left( \frac{\text{Tr}}{n} \exp \left\{ f \left( \sum_{i} \sigma_i^\alpha, \sum_{i} \sigma_i^\alpha \sigma_i^\beta \right) \right\} \right), \]  

(8)

in which \( f \) involves the spin variables only in the form of sums over all sites and those sums only up to quadratic order. Hence, by introducing auxiliary (macroscopic) variables to linearize these sums, the trace over the spins may be taken to yield

\[ \mathcal{F} = -T \ln Z = -T \lim_{n \to 0} \frac{\partial}{\partial n} \int \prod_{\alpha = 1, \ldots, n} dm^\alpha \prod_{(\alpha \beta)} dq^{\alpha \beta} \exp \left\{ -N \Phi \left( \{m^\alpha; q^{\alpha \beta}\} \right) \right\}, \]  

(9)

with \( \Phi \) intensive. Thus, provided the limit \( n \to 0 \) and the thermodynamic limit \( N \to \infty \) can be inverted, the method of steepest descents in principle yields a solution determined by an extremum of \( \Phi \). However, to take the limit \( n \to 0 \) an appropriate analytic form continuable to non-integer \( n \) is needed and the correct way to achieve this is not obvious.

EA and SK both used the natural ‘replica-symmetric’ ansatze,

\[ m^\alpha = m, \text{ all } \alpha; \quad q^{\alpha \beta} = q, \text{ all } \alpha \neq \beta. \]  

(10)

\[ \text{5We use notation } (ij) \text{ to denote a pair of unequal sites.} \]
This ansatz also yields the identifications

\[ m = \langle \sigma_i \rangle \text{ and } q = |\langle \sigma_i \rangle|^2. \] (11)

Already it gives many features qualitatively similar to ones found experimentally. In fact, though, it does not in general give a stable solution (de Almeida & Thouless 1978) and a much more subtle replica-symmetry-breaking ansatz for \( q \) is needed to yield stability in all regions of control-parameter space. The Parisi ansatz (Parisi 1980) has satisfied this need and passed all subsequent stability tests.

Let me first describe Parisi’s ansatz in terms of its original replica theory formulation and only turn later to its physical interpretation. \( q^{\alpha\beta} \) can be viewed as an \( n \times n \) matrix with zeros on its diagonal elements\(^7\). The Parisi ansatz may be viewed as the result of a sequence of operations in which (i) \( n(\equiv m_0) \) is initially considered as an integer which is subdivided sequentially into an integral number of smaller intervals; first into \( n/m_1 \) blocks of size \( m_1 \), then each of the \( m_1 \) blocks into \( m_1/m_2 \) blocks of size \( m_2 \) and so on sequentially, with all the \( m_i \) integers and the ratios \( m_i/m_{i+1} \) also integers, until \( m_{k+1} = 1 \) (ii) \( q^{\alpha\beta} \) is taken to have the value \( q_i \) if \( I(\alpha/m_i) = I(\beta/m_i) \), \( I(\alpha/m_i) \neq I(\beta/m_{i+1}) \) where \( I(x) \) is equal to the smallest integer greater than or equal to \( x \), as illustrated below for the sequence \( n = m_0 = 12; \; m_1 = 4; \; m_2 = 2; \; m_3 = 1 \)

\[
q^{\alpha\beta} = \begin{pmatrix}
q_0 & q_0 & q_0 \\
q_0 & q_0 & q_0 \\
q_0 & q_0 & q_0 \\
q_0 & q_0 & q_0 \\
q_0 & q_0 & q_0 \\
q_0 & q_0 & q_0 \\
q_0 & q_0 & q_0 \\
q_0 & q_0 & q_0 \\
q_0 & q_0 & q_0 \\
q_0 & q_0 & q_0 \\
q_0 & q_0 & q_0 \\
q_0 & q_0 & q_0 \\
\end{pmatrix}
\]

(iii) in the limit \( n \to 0 \) the \( m \) are continued to real values with \( 0 \leq m_1 \leq m_2 \leq \cdots \leq m_k \leq m_{k+1} \equiv 1 \) and \( q \) is replaced by a function \( q(x) \) given by \( q(x) = q_i; m_i < x < m_{i+1} \) \((i = 1, \ldots, k)\) with \( x \) in \([0,1]\), (iv) the limit \( k \to \infty \) is taken. Insertion into eqn. \( \text{[9]} \) yields a functional integral which in the limit \( N \to \infty \) is extremally dominated and yields

\(^6\)One also requires that the extremum of \( \Phi \) be a minimum for the single-replica order parameter \( m^\alpha \) but a maximum for the two-replica order parameter \( q^{\alpha\beta} \).

\(^7\)Some authors take \( q^{\alpha\alpha} \) as unity (c.f. an extension of eqn. \( \text{[6]} \)) but here I am assuming the \( \alpha\alpha \) term is so taken explicitly.
self-consistency equations for the dominating $q(x)$; hereafter $q(x)$ is taken to refer to this extremal function, which is the the mean-field order function for the problem. For different regions of the $(J, J_0, h, T)$ parameter space the stable solutions are of one of two forms:

(i) $q(x) = q = \text{constant}; \text{ replica-symmetric (RS)}$

(ii) $q(x) = q_0$ for $0 \leq x \leq x_1$, monotonically increasing smoothly between $x_1$ and $x_2$, and $q(x) = q_1$ for $x_2 \leq x \leq 1$; full replica symmetry breaking (FRSB) \(^8\).

They are separated by a plane in $(J, J_0, h, T)$ which marks a continuous transition, with RSB on the higher-$J$ side.

Replica symmetry breaking signals the existence of many non-equivalent macrostates. $q(x)$ provides a measure of the extent of similarity between these states. It follows from consideration of the concept of overlaps (Parisi 1983). The overlap between two macrostates $S, S'$ is defined by $q^{SS'} = N^{-1} \sum_i \langle \sigma_i \rangle^S \langle \sigma_i \rangle^{S'}$, where $\langle \cdot \rangle^S$ refers to a thermodynamic average over macrostate $S$, and the distribution of overlaps is given by $P(q) = \sum_{S,S'} W_S W_{S'} \delta(q - q^{SS'})$ where $W_S$ is the probabilistic weight of state $S$, given in equilibrium by $W_S = \exp(-\beta F_S)/\sum_{S'} \exp(-\beta F_{S'})$ where $F_S$ is the free energy of macrostate $S$. The relation to $q(x)$ is $P(q) = \int dx \delta(q - q(x)) = dq/dx$. Consequently it follows that an RS system has a single macrostate (aside from trivial global inversion or rotation), whereas FRSB implies a hierarchy of non-equivalent relevant macrostates at the temperature of interest\(^9\). This in turn implies that the macroscopic dynamics will be slow and glassy and that practical equilibration will be very difficult to achieve. Already, however, the existence of RSB predicts different kinds of response functions; for the susceptibility one may experience either single-macrostate response $\chi_{SS} \equiv \chi_{EA} = T^{-1}(1 - q(1))$ or the full Gibbs average $\chi_G = T^{-1}(1 - \int dx q(x))$. These in turn can be identified with the experimental zero-field-cooled and field-cooled susceptibilities and used to explain their difference in the spin glass phase (see e.g. Nagata et al. 1979); this non-ergodicity was already observed before EA in the difference between thermoremanent and isothermal remanent magnetisations (e.g. Tholence and Tournier 1974). The Parisi replica analysis also demonstrates a number of other interesting properties (Mézard et al. 1984), such as

\(^8\)For the intermediate approximations mentioned above one would have a $k$-step replica symmetry-breaking with $k + 1$ sections of constant $q(x)$ separated by $k$ discontinuities, but it is believed that the only stable situations for the SK model are $k = 0$ (RS) and $k = \infty$ (FRSB). There are stable 1 step RSB solutions for several other problems (see section 5).

\(^9\)There are even more macrostates of relevance at different temperatures.
ultrametricity (Mézard and Virasoro 1985)\textsuperscript{10} and non-self-averaging of certain non-trivial overlap measures, but these will not be dwelt upon further here\textsuperscript{11}.

### 2.1.5 Short-range spin glasses

‘Real’ experimental spin glasses have short-range or spatially decaying exchange interactions, whereas the replica theory above is exact only for infinite-range problems. Many of the predictions of mean field theory are mimicked qualitatively in the experiments; some are thought to be real, but others are still subjects of controversy in true Gibbs equilibrium although often apparent as non-equilibrium experimental features. The Edwards-Anderson model with nearest neighbour interactions is considered representative of such real spin glasses but remains without full exact solution.

### 2.1.6 Spin glasses on dilute random networks

A class of model spin glasses with finite inter-spin connectivities, as is the case for EA, but range-free and offering the possibility of exact solution, was introduced by Viana & Bray (1985) and characterised by an analogue of SK with

\[ H = - \sum_{(ij)} c_i c_j J_{ij} \sigma_i \sigma_j; \text{ random quenched } c \text{ and } J_{ij}; c_i = 0, 1; \overline{J}_{ij} = J_0; \overline{J}_{ij}^2 = J^2, \tag{12} \]

where the annealed spins \( \sigma \) are located on the quenched vertices of a finite-connectivity Erdős-Rényi\textsuperscript{12} graph, with \( c_i = 1 \) denoting a vertex, but without the need for inverse \( N \)-scaling of the exchange distribution.\textsuperscript{13} This problem, which is often considered a ‘half-way house’ between SK and EA, requires more order parameters \( m^\alpha, q^{\alpha\beta}, q^{\alpha\beta\gamma}, q^{\alpha\beta\gamma\delta}, \ldots \) and, although soluble in RS approximation via a mapping \( q^{\alpha\beta\ldots r} = \int P(h) \{ \tanh(\beta h) \}^r \), also poses greater challenges than SK for FRSB (see e.g. Wong and Sherrington 1988).

\textsuperscript{10}This corresponds to the hierarchical clustering of overlaps as illustrated by the branching cartoon

\[
\begin{align*}
q^{(15)} &= q^{(16)} = q^{(17)} = \ldots = q_0; \\
q^{(13)} &= q^{(14)} = q^{(23)} = \ldots = q_1; \\
q^{(12)} &= q^{(34)} = q^{(56)} = \ldots = q_2;
\end{align*}
\]

\textsuperscript{11}For a recent review of the topic of overlaps and their interpretation see (Parisi 2004).

\textsuperscript{12}In an Erdős-Rényi graph of degree \( p \) any vertex is connected to any other with a probability \( p/N \).

\textsuperscript{13}A simple extension utilises as underlying network a random graph with fixed degree at each vertex (Banavar et. al. 1987).
2.1.7 Itinerant spin glasses

Thus far we have discussed only systems with magnetic moments even in the absence of interaction. However, it is well known that ferromagnetism in periodic systems can occur not only through the orientation of effectively pre-existing localized moments, as typified by Curie-Weiss mean field theory and found in insulating magnets and in some rare earth metals, but also through the spontaneous cooperative ordering of metallic itinerant electrons, as in Stoner-Wohlfarth ferromagnetism in transition metals. Similarly, one can readily envisage itinerant spin glass behaviour (Sherrington and Mihill 1974) and indeed it is found in alloys such as \( \text{RhCo} \) (Coles et al. 1974). A simple model is given by the Hamiltonian

\[
H = \sum_{ij\sigma} t_{ij} a_{i\sigma}^{\dagger} a_{j\sigma} + \sum_i V_i a_{i\uparrow}^{\dagger} a_{i\downarrow} + \sum_i U_i n_{i\uparrow} n_{i\downarrow},
\]

(13)

where the \( a_i, a_i^{\dagger} \) are Wannier electron creation and annihilation operators, \( n_{i\sigma} \equiv a_{i\sigma}^{\dagger} a_{i\sigma} \) are number operators, and the parameters \( t_{ij}, V_i, U_i \) depend upon the types of atom at sites \( i, j \). The simplest instance takes randomly quenched alloys with two atomic types (A, B) but with the \( t_{ij} \) independent of the atom types and considers only magnetic fluctuations

\[
H = \sum_{ij\sigma} t_{ij} a_{i\sigma}^{\dagger} a_{j\sigma} - \frac{1}{2} \sum_i U_i (n_{i\uparrow} - n_{i\downarrow})^2,
\]

(14)

in which the \( U_i \) take one value \( U_A \) at the sites associated with atom type A and take the another value \( U_B \) at the sites associated with atom type B. Of particular interest is the itinerant case in which (i) A is not spontaneously magnetically ordered, i.e. \( 1 - U_A \chi^0(q) > 0 \) where \( \chi^0(q) \) is the wave-vector dependent susceptibility associated with the bare band structure, (ii) the pure B system is spontaneously itinerantly ferromagnetic (so \( 1 - U_B \chi^0(0) < 0 \)), but also (iii) there is no magnetic moment associated with an isolated B atom in an A matrix, even in the mean field sense of Anderson (1961). Analogy with the phenomenon of Anderson localization (Anderson 1958) leads to the expectation of statistical fluctuation nucleation of cluster moments within the conceptual framework of Anderson local moment formation, while further cluster interaction can stabilise cluster glass behaviour beyond a critical B concentration (Sherrington and Mihill 1974), as well as ferromagnetism at a higher concentration. However, in fact isolated paramagnetic cluster moments are not necessary precursors for spontaneous spin glass order (as emphasised by Hertz 1979), as neither there are local moments in pure itinerant ferromagnets nor well-defined bosons in BCS superconductivity above their respective onset temperatures.

A classical mean field theory follows from (14) by formulating the partition function as a functional integral over a Grassmann representation of the electron field (Sherrington 1971), linearizing the interaction term involving the \( U \) over auxiliary Hubbard-Stratonovich fields, integrating out the electron fields and taking the static approximation.
This yields an effective classical field theory with

$$Z = \int D\mathbf{m} \exp(-\beta F(\mathbf{m})),$$

(15)

where

$$F(\mathbf{m}) = \sum_i U_i m_i^2 - \sum_{ij} U_i U_j m_i m_j \chi_{ij}^0 - \sum_{ijkl} U_i U_j U_k U_l m_i m_j m_k m_l \Lambda_{ijkl}^0 + \ldots,$$

(16)

where \(m_i = \langle n_{i\uparrow} - n_{i\downarrow} \rangle\) and \(\chi_{ij}^0, \Lambda_{ijkl}^0, \ldots\) are two-, four- and ..-point correlation functions of the bare band structure (in real space). Taking the extremum yields a set of self-consistent mean field equations which are the analogue of the Thouless-Anderson-Palmer (1977) (TAP) equations for the SK equation. The analogy with Anderson localization follows from writing these equations as

$$U_i^{-1} M_i - \sum_j \chi_{ij}^0 M_j - \sum_{jkl} \Lambda_{ijkl}^0 M_j M_k M_l + \ldots = 0; M_i = U_i m_i$$

(17)

and comparing with the Anderson wave-function localization equation

$$\epsilon_i \phi_i + \sum_j t_{ij} \phi_j - E \phi_i = 0,$$

(18)

with disorder in the \(\epsilon_i (= U_i^{-1})\); naively, local moment clusters of (17) are related to negative energy states of (18) and long range magnetic order is related to the mobility edge. But in fact there are more subtle effects, both bootstrap effects as mentioned earlier (contained in the non-linear terms of (17)) and effects differentiating spin glass and ferromagnetic cooperative order.

A simple conceptual model of itinerant spin glass ordering, further simplified in the EA spirit, is given by an effective field theory with

$$Z = \int \prod_i d\phi_i \exp(-F(\phi_i)); F(\phi) = r \sum_i \phi_i^2 + u \sum_i \phi_i^4 - \sum_{ij} J_{ij} \phi_i \phi_j; u > 0,$$

(19)

with the \(J_{ij}\) random as in EA or SK; this model encompasses local moment spin glasses for \(r < 0\) and itinerant spin glasses for \(r > 0\).

2.1.8 Other induced moment models

There are other classical models allowing the bootstrapping of magnetic order. One such is the spin glass analogue of the induction of ferromagnetism due to exchange interaction
lifting of singlet ground state preference of isolated atoms. A simple example is the spin-1 Ising model

\[ H = -D \sum_i S_i^2 - \frac{1}{2} \sum_{ij} J_{ij} S_i S_j; \quad S_i = 0, \pm 1. \]  

(20)

If \( D > 0 \) then the system behaves analogously to the usual spin 1/2 Ising model, but if \( D < 0 \) then in the absence of \( J \) the ground state preference is for non-magnetic \( S_i = 0 \). However even if \( D < 0 \) a sufficient exchange can self-consistently lift the preference to the magnetically ordered state via a first-order transition. If the \( J_{ij} \) are quenched random as in the SK model, this system is known as the Ghatak-Sherrington (GS) model (1977) and has induced spin glass behaviour; it has been analysed extensively in FRSB by Crisanti and Leuzzi (2002). The Fermionic Ising Spin Glass (FISG) model (Rosenow and Oppermann 1996) is closely related (Pérez Castillo and Sherrington 2005).

### 2.1.9 Vector spin glasses

Magnetic alloy spin glasses are not restricted to Ising systems. Indeed Heisenberg magnets are more common experimentally. It is straightforward to extend the exactly soluble models to encompass vector spins (see e.g. Sherrington 1983). In the absence of a magnetic field or a ferromagnetic component there is little change of note beyond the extension of random spin glass freezing to the full spin dimensionality. Within the infinite-range/mean field model, an axial symmetry-breaking due to an applied field or to ferromagnetism still permits a spin glass freezing in the orthogonal directions (Gabay and Toulouse 1981) with strong onset of transverse non-ergodicity and induced weaker longitudinal non-ergodicity, crossing over to strong RSB in all directions at a lower temperature (Cragg et al. 1982, Elderfield and Sherrington 1982, 1984). Anisotropy effects can also be included (Cragg and Sherrington 1982b).

### 3 Discontinuous transitions

For the case of conventional 2-spin interactions, as employed in both the SK and EA models and believed to be appropriate for conventional experimental magnetic alloy spin glasses, mean field theory yields full replica symmetry breaking once the spin glass state occurs. However, in extensions which lack reflection and definiteness symmetries, such as \( p \)-spin models for \( p > 2 \) (Crisanti and Sommers 1992) or Potts or quadrupolar spin...
glasses beyond critical Potts or vector dimensions (Gross et al 1985, Goldbart and Sherrington 1985) one finds that the spin glass transition is discontinuous to one step of replica symmetry breaking with finite overlap magnitude (D1RSB)\textsuperscript{16}. This behaviour is thought to be characteristic also of (even short-range interaction) structural glasses, in which crystallization is dynamically avoided in favour of self-consistent glassiness.

4 Beyond magnetic alloys

4.1 Complex many body problems

The formalism and concepts developed for model magnetic alloys have found significant application more generally; in particular for a large class of problems that can be characterised by control functions of the form

$$H = H(\{J_{ij...k}\}, \{S_{ij...l}\}, \{X\}),$$

where the $i, j$ are microscopic identification labels; the $\{J_{ij...k}\}$ symbolise a set of quenched parameters depending on one or more of the identification labels and in general different for different labels; the $\{S_{ij...l}\}$ symbolise the (annealed) microscopic variables again depending on one or more identification variables; and the $\{X\}$ are macroscopic intensive control variables. The specific identifications of the $\{J, S, X\}$ can however be quite different, as also the manner of operation of the control function. In the spirit of statistical physics and probability theory one often concerns oneself with problems in which the $\{J, X\}$ are drawn from intensive distributions independent of the specific labels.

4.1.1 Examples

We have already seen one example in the case of a magnet with the $i$ labelling the spins, the $J$ exchange interactions, the $S$ spin orientations, $X$ the temperature and $H$ the Hamiltonian determining the distribution of the $S$ through the Boltzmann measure. Other examples include:

(i) The Hopfield neural network: Here the $i$ label neurons, $\{S_i\}$ indicate the states of the neurons as firing or not firing, $\{J_{ij}\}$ label synaptic efficacies given in terms of (randomly chosen quenched) stored patterns $\{\xi_{\mu}\}$; $\mu = 1, \ldots, p = \alpha N$ by $J_{ij} = N^{-1} \sum_{\mu} \xi_{\mu}^{i} \xi_{\mu}^{j}$, $X \equiv T \equiv \beta^{-1}$ is a measure of the rounding of the sigmoidal response

\textsuperscript{16}In a Potts or quadrupolar model for a range of intermediate Potts or vector dimensions the transition to 1RSB is continuous (Elderfield and Sherrington 1983, Goldbart and Sherrington 1985, Sherrington 1986); a similar transition to C1RSB occurs in a $p > 2$-spin model in a sufficient applied field (Crisanti and Sommers 1992). Except for spherical spins, there is also a lower temperature transition from 1RSB to FRSB (Gardner 1985, Gillin et al (2001).
of a neuron to the sum of its incoming signals, $H = -\frac{1}{2} \sum_{ij} J_{ij} S_i S_j$ and $P\{\{S\}\} \sim \exp(-\beta H)$ characterises the stationary macro firing states. From the neural retrieval perspective, however, interest is not in the full Gibbs average but rather in the individual retrieval macrostates with macroscopic overlaps $m^\mu = N^{-1} \sum_i \xi^\mu_i \langle S_i \rangle$ with the patterns coded in the $\{J\}$; retrieval corresponds to a finite overlap with a single pattern and is an analogue of ferromagnetism in the examples of section 2. Spin glass states do occur due to pattern interference but are not the desired states in neural operation and their dominance indicates breakdown of retrievable memory\textsuperscript{17}.

(ii) \textit{Hard optimization:} Here the objective is to minimise a cost function $H$ as a function of variables $\{S\}$ with constraints $\{J\}$. An example is the problem of partitioning the vertices $i$ of a random graph into two groups of equal size but with the minimum number of edges of the graph between the two groups. This can be formulated as finding the ground state of a Viana-Bray-like spin glass. Consequently it can be studied by an analogue of the procedure of studying the thermodynamics of the VB spin glass. If the interest is in finding the average minimum spanning cut then replica procedure may be employed, inventing an artificial annealing temperature $T$ and taking it to zero at the end of the calculation. Of course the actual calculation involves all the subtleties of replica symmetry breaking and computer simulation involves all the corresponding issues of slow glassy dynamics\textsuperscript{18}. Another optimization problem in artificial neural network theory is to determine the maximum number of patterns which can be stored and retrieved with a specified maximum error; in this case the variables are the synaptic efficacies and the quenched parameters are the stored patterns. More recently many other optimization problems have been studied by techniques derived from spin glass studies.

(iii) \textit{Error-correcting codes:} One procedure for coding and retrieval is to code the information to be transmitted in the form of exchange interactions whose insertion into an effective magnetic Hamiltonian yields a ground state which identifies the desired message. In practice, however, transmission lines add noise and retrieval is required to best eliminate the effects of the noise. This yields yet another optimization problem, with best retrieval resulting from the introduction of an effective retrieval temperature-noise matching that on the line\textsuperscript{19}. Indeed there are several other problems in which the optimal character of noise matching can be demonstrated.

\textsuperscript{17}For further discussion see for example Sherrington (1992) or Nishimori (2001).
\textsuperscript{18}Simulation also exhibits the spin-glass features of ultrametricity and non-self-averaging (Banavar et al 1987).
\textsuperscript{19}Again see Nishimori (2001) for further details.
5 Dynamics

Thus far discussion has been about equilibrium or quasi-equilibrium. However, often one wishes to consider dynamics, including away from equilibrium;\(^\text{20}\) indeed if detailed balance is not present one cannot use usual Boltzmann equilibrium theory. As before, we are normally interested in systems characterised by simple distributions. Again one can utilise the general picture of a controlling function as in (21) but now operating in an appropriate microscopic dynamics (and without necessarily symmetries such as \(J_{ij} = J_{ji}\)). The analogue of the use of the partition function for thermodynamics is to use a dynamical generating functional (de Dominicis 1978) which can be expressed symbolically either, for random sequential updates, as

\[
Z(\Lambda) = \int DS(t) \Pi \delta \left( \text{eqn of motion} \right) \exp \left( \int dt \Lambda(t) \cdot S(t) \right), \tag{22}
\]

where the integral is over all variable paths in the full space-time, the \(\Pi \delta\) term indicates that the microscopic equations of motion are always satisfied and the \(\Lambda \cdot S\) term is a generating term, or, for parallel updates, as

\[
Z(\Lambda) = \int \Pi ds(t) \prod_t W(S(t+1)|S(t)) P_0(S) \exp \left( \sum_t \Lambda(t) \cdot S(t) \right). \tag{23}
\]

where \(P_t(S)\) indicates the ensemble distribution of \(S\) at time \(t\) and \(W(S(t+1)|S(t))\) indicates the probability of updating from \(S(t)\) to \(S(t+1)\). With suitable Jacobian normalization (not shown explicitly) \(Z(\Lambda = 0) = 1\) and one can average over the quenched disorder without need for replicas; instead of interactions between replicas one gets effective interactions between different epochs. In the case of range-free problems one can again eliminate microscopic variables in place of macroscopic ones by the artifice of introducing new variables via relations such as

\[
1 = \int dC(t, t') \delta(C(t, t') - N^{-1} \sum_i S_i(t) S_i(t'))
\]

\[
= \int d\tilde{C}(t, t') dC(t, t') \exp \left\{ i \tilde{C}(t, t') (C(t, t') - N^{-1} \sum_i S_i(t) S_i(t')) \right\} \tag{24}
\]

and similarly for response functions (involving also operators corresponding to \(\partial/\partial S_i(t)\)). One can then integrate out the microscopic variables to leave purely macroscopic measures; in the simplest cases of the form

\[
Z_{\text{eff}} \sim \int D\tilde{C}(t, t', t'', \ldots) \exp(N \Phi(\{\tilde{C}\})), \tag{25}
\]

Note that whereas in real physical situations the true microscopic dynamics is determined by nature, in computer simulations the dynamics is chosen by the simulator and there exists the opportunity to optimise that choice. Similarly, the control fields \(X\) are choosable.
where \( \tilde{C} \) is used to denote the generic set and the temporal dependence is two-time for full connectivity of the SK type but includes all numbers of different times for VB finite connectivity. Steepest descents then yields self-consistent coupled equations for the macroscopic correlation and response functions, although of course boundary conditions need care. This is the dynamical analogue of replica thermodynamics. In general, however, it is more difficult than replica theory and fewer cases have been solved fully. Also, in some cases the convenience of a final expression purely in terms of coupled correlation and response functions is not available, although alternative descriptions in terms of ensembles of effective single agents can often be obtained.

An alternative procedure invoking an infinite multiplicity of single-time order parameters has also been considered but will not be pursued here (see e.g. Coolen et al. 1996).

5.1 Examples

5.1.1 \( p \)-spin spherical spin glass

One example of the above procedure has been the analysis of the (infinite-range) \( p(>2) \)-spin spherical spin glass, of Hamiltonian

\[
H = - \sum_{i_1 < i_2 < \ldots < i_p} J_{i_1 i_2 \ldots i_p} S_{i_1} S_{i_2} \ldots S_{i_p}; \quad \sum_i S_i^2 = N; \quad \bar{J}_{i_1 i_2 \ldots i_p}^2 = J^2 p! / 2N^{p-1} \quad (26)
\]

and obeying Langevin dynamics, for which closed equations in terms of correlation and response functions have been obtained (Cugliandolo & Kurchan 1993). In general these equations are not restricted to stationarity. Analysis has indicated that above a critical temperature, known as the dynamical transition temperature, stationary solutions do exist, with \( \tilde{C}(t, t') \equiv \tilde{C}(t - t') \) and satisfying the normal fluctuation-dissipation theorem and mode-coupling theory, but below this temperature equilibration does not occur, the normal fluctuation-dissipation theorem \( -dR/dC = \beta \) (where \( R \) is the integrated response, \( C \) is the correlation function and \( \beta \) is the inverse temperature) is replaced by a modified relation \( -dR/dC = \beta X(C) \) where \( X(C) = x(q) \) with \( x(q) \) the inverse of the Parisi function \( q(x) \), the \( R \) and \( C \) are now two-time (and non-stationary) and the \( C \)-dependence of \( X(C) \) is instantaneous-parenthetic\(^{21} \). \(^{22} \)

These and related dynamical studies vindicate and quantify the concepts of glassy dynamics deduced from the thermodynamic existence of many non-equivalent metastable macrostates and the barriers between them.

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\(^{21}\)See for example Parisi (2004)  
\(^{22}\)In this case the onset of RSB is discontinuous and the transition temperature is given differently by simple extremization of replica theory and dynamically. The correct comparison with dynamics within replica theory is determined using marginal stability.
5.1.2 Dynamical SK-model

In the $p(>2)$-spin spherical spin glass model there is only one step of RSB in the replica equilibrium theory and similarly only 2 straight slope regions for $X(C)$. The $p = 2$ SK Ising system is more complicated with more structure, corresponding to the hierarchy of FRSB, and dynamical analogues of ultrametricity (Cugliandolo and Kurchan 1994). Other models can show regions of 1-RSB and of FRSB thermodynamics while it seems likely that dynamical vestiges of FRSB may occur in many systems, even with 1RSB thermodynamics.

5.1.3 Minority game

A rather different example is found in the so-called Minority Game in econophysics (see e.g. Challet et al. 2004, Coolen 2004), which mimics a system of speculative agents in a model market trying to gain by minority action. In the batch version of this game the system obeys microdynamics

$$p_i(t+1) = p_i(t) - h_i - \sum_j J_{ij} \text{sgn} p_j(t); \quad h_i = N^{-1} \sum_j \xi_i \cdot \omega_j; \quad J_{ij} = N^{-1} \xi_i \cdot \xi_j,$$

with the $i$ labelling agents, the $p$ unbounded variables corresponding to strategy point-weightings, and the $\xi$ and $\omega$ quenched random vectors in a $D$-dimensional strategy space. This system is soluble along the lines outlined above, utilizing large-$N$ steepest descent domination, in terms of an ensemble of independent agents obeying non-Markovian stochastic dynamics with ensemble-self-consistently determined coloured noise. On the macroscopic level it exhibits an ergodic-nongergodic transition at a critical value $d_c$ of $d = D/N$, asymptotically independent of preparation for $d > d_c$ but preparation-dependent for $d < d_c$.

6 Conclusion

The spin glass problem has yielded many new concepts and techniques in both theoretical and experimental physics. These concepts and techniques have in turn inspired new insights and practical opportunities in the wider field of complex many-body problems, ranging through physics, computer science, biology and economics, with pastures still open in these and the social sciences. Most of this work has been on simple models with a single level of microscopic timescale (but many resulting macro timescales) but some work has started and much remains to do when different parameters are allowed different microdynamic time-scales (as for example in neural networks where both neurons and

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\footnote{For example the $p > 2$-spin Ising model has 1-RSB thermodynamic behaviour above a critical temperature but then FRSB behaviour below; see e.g. Gardner (1985), Gillin et.al. (2001).}
synapses evolve, the former much faster than the latter, or in biological evolution where the timescales of organism operation and species evolution and mutation are very different). Although physical systems normally obey detailed balance, others need not (e.g. biological or economic or social systems). Most of the theoretical work has been performed at a level of uncertain if physically reasonable approximation or assumption. Greater mathematical physics rigour is now needed and will be the topic of other authors in this volume. The spin glass models have introduced also new concepts in probability theory that are stimulating new mathematics. Spin glass dynamics poses challenges yet to be investigated with mathematical rigour. Much has been achieved but much remains to do.

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References

[1] de Almeida J. and Thouless, D.J., J.Phys. A11, 983 (1978)
[2] Anderson P.W., Phys.Rev. 109, 1492 (1958)
[3] Anderson P.W., Phys.Rev. 124, 41 (1961)
[4] Banavar J.R., Sherrington D. and Sourlas N., J.Phys. A20, L1 (1987)
[5] Cannella V. and Mydosh J.A., Phys. Rev. B6, 4220 (1972)
[6] Challet D., Marsili M. and Zhang Y-C., “Minority Games” (OUP 2004)
[7] Coles B.R., “The Origins and Influences of the Spin Glass Problem” in “Multicritical Phenomena” eds. R.Pynn and A. Skjeltorp (Plenum 1983)
[8] Coles B.R., Tari A. and Jamieson H.C., in “Proc. Low Temp. Phys. LT-13” 2, 414 (Plenum 1974)
[9] Coolen A.C.C., “The Mathematical Theory of Minority Games” (OUP 2004)
[10] Coolen A.C.C., Laugton S.N. and Sherrington D., Phys.Rev. **B53**, 8184 (1996)
[11] Cragg D., Sherrington D. and Gabay M., Phys.Rev.Lett. **49**, 158 (1982)
[12] Cragg D. and Sherrington D., Phys.Rev.Lett. **49**, 1190 (1982)
[13] Crisanti A. and Leuzzi L., Phys.Rev.Lett. **89**, 237204 (2002)
[14] Crisanti A. and Sommers H-J., Z.Phys. **B87**, 341 (1992)
[15] Cugliandolo L. and Kurchan J., Phys.Rev.Lett. **71**, 173 (1993)
[16] Cugliandolo L. and Kurchan J., J.Phys. **A26**, 5749 (1994)
[17] de Dominicis C., Phys.Rev. **B18**, 4913 (1978)
[18] Doniach S., Physica **B91**, 231 (1977)
[19] Edwards S.F. and Anderson P.W., J.Phys. **F5**, 965 (1975)
[20] Elderfield D. and Sherrington D., J.Phys. **C16**, L497 (1983)
[21] Elderfield D. and Sherrington D., J.Phys. **A15**, L513 (1982)
[22] Elderfield D. and Sherrington D., J.Phys. **C17**, 1923 (1984)
[23] Gabay M. and Toulouse G., Phys.Rev.Lett. **47**, 201 (1981)
[24] Gardner E., Nuc. Phys. **B257**, 747 (1985)
[25] Ghatak S.K. and Sherrington D., J.Phys. **C10**, 3149 (1977)
[26] Gillin P., Nishimori H. and Sherington D., J.Phys. **A34**, 2949 (2001)
[27] Goldbort P.M. and Sherrington D., J.Phys. **C18**, 1923 (1985)
[28] Gross D., Kanter I. and Sompolinsky H., Phys. Rev. Lett. **55**, 304 (1985)
[29] Hertz J.A., Phys.Rev. **B19**, 4796 (1979)
[30] Kondo J., J.Prog.Theor.Phys. **32**, 37 (1964)
[31] Mézard M., Parisi G. and Virasoro M.A., “Spin Glass Theory and Beyond” (World Scientific 1987)
[32] Mézard M., Parisi G., Sourlas N., Toulouse G. and Virasoro M.A., J.Physique **45**, 843 (1984)
[33] Mézard M. and Virasoro M.A., J.Physique 46, 1293 (1985)

[34] Nagata S., Keesom P.H. and Harrison H.R., Phys.Rev. B19, 1633 (1979)

[35] Nishimori H. “Statistical Physics of Spin Glasses and Information Processing ”

[36] Rosenow B and Oppermann R, Phys. Rev. Lett. 77, 1608 (1996)

[37] Parisi G., J.Phys. A13, 1101 (1980)

[38] Parisi G., Phys.Rev.Lett. 50, 1946 (1983)

[39] Parisi G., in “Stealing the Gold: a Celebration of the Pioneering Physics of Sam Edwards”, eds. P.M. Goldbart, N. Goldenfeld & D. Sherrington (OUP 2004), p 192.

[40] Pérez Castillo I and Sherrington D., B72, 104427 (2005)

[41] Sherrington D., “The infinite-ranged m-vector spin glass” in “Heidelberg Colloquium on Spin Glasses” eds. I. Morgenstern & L van Hemmen (Springer 1983) p 125

[42] Sherrington D., “Neural Networks: the spin glass approach” in “Mathematical Studies of Neural Networks” ed. J.G. Taylor (Elsevier 1992) p 261

[43] Sherrington D., Prog. Theor. Phys. Supp. 87, 180 (1986)

[44] Sherrington D. and Mihill K., J.Physique 35 Colloque C4, 199 (1972)

[45] Sherrington D. and Kirkpatrick S., Phys.Rev.Lett. 35, 1792 (1975)

[46] Sherrington D. and Southern B.W., J.Phys. F5, L49 (1975)

[47] Tholence J.L. and Tournier R., J.Physique 35, C4-229 (1974)

[48] Thouless D.J., Anderson P.W. and Palmer R.G., Phil.Mag. 35, 137 (1977)

[49] Viana L. and Bray A.J., J.Phys. C18, 3037 (1985)

[50] Wong K.Y.M. and Sherrington D., J.Phys. A21, L359 (1988)