Towards the graviton from spinfoams:
the complete perturbative expansion of the 3d toy model

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We consider an exact expression for the 6j-symbol for the isosceles tetrahedron, involving SU(2) group integrals, and use it to write the two-point function of 3d gravity on a single tetrahedron as a group integral. The perturbative expansion of this expression can then be performed with respect to the geometry of the boundary using a simple saddle-point analysis. We derive the complete expansion in inverse powers of the length scale and evaluate explicitly the quantum corrections up to second order. Finally, we use the same method to provide the complete expansion of the isosceles 6j-symbol with the explicit phases at all orders and the next-to-leading correction to the Ponzano-Regge asymptotics.

Introduction

A wide-spread expectation from a full theory of quantum gravity is the possibility to fix the coefficients appearing in the conventional non-renormalizable perturbative expansion seen as an effective field theory (EFT). To address this question, a necessary tool is to control the perturbative expansion of the full theory. In this paper, we investigate this issue in the spinfoam formalism, using the 3d toy model with a single dynamical variable introduced in [1] and developed in [2].

Pursuing a matching with the EFT, while right at the root of many approaches to quantum gravity, most notably string theory and the asymptotic safety scenario, has long been obstructed in the spinfoam formalism. This is due to the difficulty in consistently inserting a background metric to perform the perturbative expansion. The key idea is to relate the n-point functions to the field propagation kernel, via the introduction of a suitable boundary state [3]. The boundary state can then be taken to be a coherent state[5] peaked on a classical geometry [5]. We then expect the boundary geometry to effectively induce a semi-classical background structure in the bulk, which allows to define the graviton propagator from background-independent correlation functions.

The structure of this framework is particularly clear in 3d. Considering for simplicity the Riemannian case, the spinfoam amplitude for a single tetrahedron is the 6j-symbol of the Ponzano-Regge model. Its large spin asymptotics is dominated by exponentials of the Regge action for 3d general relativity. This is a key result, since the quantization of the Regge action is known to reproduce the correct free graviton propagator around flat spacetime [6]. The role of the boundary state is to induce the flat background and to gauge-fix the propagator [7]. Thus the framework provides a clear bridge to Regge calculus as an effective description of spinfoam gravity. However, there is more to it. Indeed, if one works with quantum Regge calculus alone, there are technical problems to go beyond the free theory approximation. These are related to the lack of a unique measure for the path integral compatible with the triangle inequalities conditions ensuring that the metric is positive definite. The issue is solved in the spinfoam formalism, where the triangle conditions are automatically imposed on the 6j-symbol by the recoupling theory of SU(2) and the measure is selected by the topological symmetry of the system. Thus the spinfoam approach does reduce to quantum Regge calculus at leading order but improves it beyond [2].

1 Similar ideas on the use of coherent states lie also behind the study the semiclassical limit in the canonical loop gravity framework [4].

2 The situation is more complicated in 4d. Developments of this idea have led to the remarkable result that the Barrett-Crane model in 4d Riemannian spacetime does reproduce at large scales the scaling behavior of the free graviton propagator (or 2-point function) [8, 9, 10, 11, 12]. This is crucial evidence towards the correctness of the semiclassical limit of LQG. However the same developments also pointed out [8, 12] that the Barrett-Crane model does not reproduce the right tensorial structure of the propagator, thus the model fails to reproduce General Relativity in the large scale limit. These results have confirmed the validity of the method, and spurred new efforts towards a better understanding of the spinfoam dynamics [13, 14]. This better behaved models should have a semiclassical limit.
In this paper we consider the simplest possible setting given by the 3d toy model introduced in [1, 2] and study analytically the full perturbative expansion of the 3d graviton. Our results are based on a reformulation of the 6j-symbol and the graviton propagator as group integrals and the saddle point analysis of these integrals. We compute explicitly the leading order then both next-to-leading and next-to-next analytically and we support these results with numerical data. Moreover, it was shown in [2] that deviations of the 6j-symbol from the leading order Ponzano-Regge asymptotics do not contribute to the next-to-leading order of the graviton, but that they enter the next-to-next order corrections. Here, the exact representation of the graviton propagator as a group integral naturally incorporates these deviations. Finally, an interesting side-product of our calculations is a formula for the next-to-leading order of the famous Ponzano-Regge asymptotics of the 6j-symbol in the special isosceles configuration.

In spite of the simplicity of the model, the framework we develop here has rather generic features useful for computing graviton correlation functions in non-perturbative quantum gravity from spinfoam amplitudes, although it does not allow us to tackle the more general issue of the existence of a relevant boundary state and of the resulting EFT-like expansion of the correlation functions for a generic spinfoam triangulation. We nevertheless show that the full perturbative expansion of the two-point function in the spinfoam quantization of 3d gravity is computable. We hope to apply these same methods and tools to 4d spinfoam models and allow a more thorough study of the full non-perturbative spinfoam graviton propagator and correlations in 4d quantum gravity.

I. THE KERNEL AND THE PROPAGATOR AS GROUP INTEGRALS

A. The boundary states and the kernel

Let us consider a triangulation consisting of a single tetrahedron. To define transition amplitudes in a background independent context for a certain region of spacetime, the main idea is to perform a perturbative expansion with respect to the geometry of the boundary. This classical geometry acts as a background for the perturbative expansion. To do so we have to specify the values of the intrinsic and extrinsic curvatures of such a boundary, that is the edge lengths and the dihedral angles for a single tetrahedron in spinfoam variables. Following the framework set in [1], we restrict attention to a situation in which the lengths of four edges have been measured, so that their values are fixed, say to a unique value \( j_t + \frac{1}{2} \). These constitute the time-like boundary and we are then interested in the correlations of length fluctuations between the two remaining and opposite edges which are the initial and final spatial slices (see figure 1). This setting is referred to as the time-gauge setting. The two opposite edges \( e_1 \) and \( e_2 \) have respectively lengths \( j_1 + \frac{1}{2} \) and \( j_2 + \frac{1}{2} \). In the spinfoam formalism, and in agreement with 3d LQG, lengths are quantized so that \( j_t, j_1 \) and \( j_2 \) are half-integers.

![Figure 1](1.png)

**FIG. 1**: Physical setting to compute the 2-point function. The two edges whose correlations of length fluctuations will be computed are in fat lines, and have length \( j_1 + \frac{1}{2} \) and \( j_2 + \frac{1}{2} \). These data are encoded in the boundary state of the tetrahedron. In the time-gauge setting, the four bulk edges have imposed lengths \( j_t + \frac{1}{2} \) interpreted as the proper time of a particle propagating along one of these edges. Equivalently, the time between two planes containing \( e_1 \) and \( e_2 \) has been measured to be \( T = (j_t + \frac{1}{2})/\sqrt{2} \).

The lengths and the dihedral angles are conjugated variables with regards to the boundary geometry, and have to satisfy the classical equations of motion. Here, it simply means that they must have admissible values to form a genuine flat tetrahedron. Note that the dimension of the SU(2)-representation of spin \( j \), \( d_j \equiv 2j + 1 \) is twice the edge

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given by a modified Regge calculus where the fundamental variables are area and angles, as the one investigated in [15].
length. Setting \( k_e = \frac{d_e}{2d_j} \), for \( e = e_1, e_2 \), the dihedral angles \( \vartheta_1, \vartheta_2 \) and \( \vartheta_t \) can be expressed in terms of the lengths:

\[
\vartheta_{1,2} = 2 \arccos \left( \frac{k_{2,1}}{\sqrt{1 - k_{1,2}^2}} \right) \quad \text{and} \quad \vartheta_t = \arccos \left( \frac{-k_1k_2}{\sqrt{1 - k_1^2} \sqrt{1 - k_2^2}} \right)
\]

provided \( k_e < 1 \), a condition ensured by the triangle inequalities. Notice the relation: \( \cos \vartheta_i = -\cos \left( \frac{\vartheta_{1,2}}{2} \right) \cos \left( \frac{\vartheta_{1,2}}{2} \right) \).

We then need to assign a quantum state to the boundary, peaked on the classical geometry of the tetrahedron. Since \( j_i \) is fixed, we only need such a state for \( e_1 \), peaked on the length \( j_1 + \frac{1}{2} \), and for \( e_2 \), peaked on \( j_2 + \frac{1}{2} \). The previous works used a Gaussian ansatz for such states. However, it is more convenient to choose states which admit a well-defined Fourier transform on SU(2). In this perspective, the dihedral angles of the tetrahedron are interpreted as the class angles of SU(2) elements. As proposed in [2], the Gaussian ansatz can be replaced for the edges \( e_1 \) and \( e_2 \) by the following Bessel state:

\[
\Psi_e(j) = \frac{e^{-\gamma_e/2}}{N} \left[ I_{|j-j_i|} (\frac{\gamma_e}{2}) - I_{j+j_i+1} (\frac{\gamma_e}{2}) \right] \cos(d_j \alpha_e)
\]

with \( \gamma_e = d_{j_i} (1 - k_e^2) \).

where \( N \) is a normalization coefficient depending on \( \gamma_e \). The functions \( I_n(z) \) are modified Bessel functions of the first kind, defined by: \( I_n(z) = \frac{1}{\sqrt{\pi} \gamma_n^2} \int_0^\infty d\phi \ e^{\phi \sin(\gamma_n \phi)} \), and \( \alpha_e = \vartheta_e / 2 \) is half the dihedral angle. The asymptotics reproduce the Gaussian behavior peaked around \( j_e \), with \( \gamma_e \) as the squared width:

\[
\Psi_e(j) = \frac{1}{N} \sqrt{\frac{4}{\pi \gamma_e}} \ e^{-\frac{(j-j_i)^2}{\gamma_e}} \cos(d_j \alpha_e).
\]

The role of the cosine in (2) is to peak the variable dual to \( j \), i.e. the dihedral angle, on the value \( \alpha_e \). Then the boundary state admits a well-defined Fourier transform, which is a Gaussian on the group SU(2). We parameterize SU(2) group elements as

\[
g(\phi, \tilde{n}) = \cos \phi \ \mathbb{I} + i \sin \phi \ \hat{n} \cdot \hat{\sigma}, \quad \phi \in [0, 2\pi], \quad \hat{n} \in S^2,
\]

where \( \sigma_i \) are the Pauli matrices, satisfying \( \sigma_i^2 = \mathbb{I} \), and \( \phi \) is the class angle of \( g \). Since the group element \( g(\phi, \tilde{n}) \) is identified to \( g(\phi + \pi, -\hat{n}) \), we can restrict \( \phi \) to live in \([0, \pi]\). The Fourier transform of (2) is then given by:

\[
\widehat{\Psi}_e(\phi) = \frac{1}{2} \sum_{\eta = \pm 1} \widehat{\Psi}^{(\eta)}(\phi)
\]

with \( \widehat{\Psi}^{(\eta)} = \frac{1}{N \sin(\phi)} \sin \left( d_{j_e} (\phi + \eta \alpha_e) \right) e^{-\gamma_e \sin^2(\phi + \eta \alpha_e)} \).

This state is a class function on SU(2), but \( \widehat{\Psi}^{(\eta)} \) alone is not (due to the \( \phi \leftrightarrow -\phi \) symmetry reflecting that a SU(2) group element and its inverse are simply related by conjugation). The semiclassical analysis is crystal-clear: it is peaked around the angle \( \alpha_e \) or \( \pi - \alpha_e \), according to the sign of \( \eta \). The sine shifts the mean length to \( j_e + \frac{1}{2} \).

These states carry the information about the boundary geometry necessary to induce a perturbative expansion around it. More precisely, we are interested in the following correlator,

\[
W_{1122} = \frac{1}{N} \sum_{j_1' j_2'} \left\{ \begin{array}{ccc} j_1' & j_t & j_t \\ j_2' & j_t & j_t \end{array} \right\} O_{j_1}(j_1') \Psi_{e_1}(j_1') O_{j_2}(j_2') \Psi_{e_2}(j_2')
\]

with \( O_{j_e}(j_e') = \frac{1}{d_{j_e}} \left( d_{j_e}^2 - d_{j_e'}^2 \right) \).

where the normalisation factor \( N \) is given by the same sum, without the observable insertions \( O_{j_e}(j_e') \). \( W_{1122} \) measures the correlations between length fluctuations for the edges \( e_1 \) and \( e_2 \) of the tetrahedron, and it can be interpreted as the 2-point function for gravity \( \mathbb{I} \), contracted along the directions of \( e_1 \) and \( e_2 \).

The 6j-symbol, as it enters [2], emerges from the usual spinfoam models for 3d gravity as the amplitude for a single tetrahedron. In the previous work [2] we studied the perturbative expansion using its well-known (leading order) asymptotics in term of the discrete Regge action (for the tetrahedron). Here instead we use the fact that the
6j-symbol for the isosceles configuration admits an exact expression as group integrals:\(^3\)

\[
\begin{pmatrix} j_1 & j_2 & j_3 \\ j_1 & j_2 & j_3 \end{pmatrix} = \int_{SU(2)^2} dg_1 dg_2 \chi_{j_3}(g_1 g_2) \chi_{j_3}(g_1 g_2^{-1}) \chi_{j_3}(g_1) \chi_{j_3}(g_2)
\]

(9)

where \(\chi_j(g) = \frac{\sin(j \phi)}{\sin \phi}\) is the SU(2) character. Then, selecting a specific boundary state as described below, we are able to rewrite also \(\chi_j\) as an integral over SU(2). This allows us to study the perturbative expansion as the saddle point (or stationary phase) approximation of the integral for large lengths, \(d_s \gg 1\). With respect to \(^2\), this procedure has the advantage of including the higher order corrections coming from both the Regge action and the corrections to the \(\{6j\}\) asymptotics. We will come back to this point below.

Let us begin by looking at the saddle points of the isosceles 6j-symbol, as the computation of the propagator will have a similar structure. We first need the angle of the group elements \(g_1 g_2\) and \(g_1 g_2^{-1}\):

\[
\phi_{12}^+ = \arccos \left( \cos \phi_1 \cos \phi_2 \mp u \sin \phi_1 \sin \phi_2 \right)
\]

(10)

where we used the notation \(u = \vec{n}_1 \cdot \vec{n}_2\). Then, expanding the rapidly oscillatory phases in exponential form yields:

\[
\begin{pmatrix} j_1 & j_2 & j_3 \\ j_1 & j_2 & j_3 \end{pmatrix} = \frac{1}{8\pi^2} \sum_{\epsilon_1, \epsilon_2, \epsilon_{12}, \epsilon_{12}^\pm = \pm 1} \epsilon_1 \epsilon_2 \epsilon_{12}^+ \epsilon_{12}^- \int d\phi_1 d\phi_2 du \ f(\phi_1, \phi_2, u) e^{i\epsilon_1 \Psi_1 + \epsilon_2 \Psi_2}
\]

(11)

with

\[
f(\phi_1, \phi_2, u) = \frac{\sin(\phi_1) \sin(\phi_2)}{\sin(\phi_{12}) \sin(\phi_{12}^+)}
\]

(12)

\[
\Psi_{\epsilon}(\phi_1, \phi_2, u) = (\epsilon_{12} \phi_{12}^+ + \epsilon_{12}^\pm \phi_{12}^-) + 2k_1 \epsilon_1 \phi_1 + 2k_2 \epsilon_2 \phi_2
\]

(13)

Let us proceed to the search for the stationary points of the phase \(\Psi_{\epsilon}\). The variable \(u\) only enters \(\phi_{12}^+\), and the related equation, \(\epsilon_{12}^+ \partial_u \phi_{12}^+ + \epsilon_{12}^- \partial_u \phi_{12}^- = 0\), is solved by \(u = \vec{n}_1 \cdot \vec{n}_2 = 0\) and \(\epsilon_{12}^+ = \epsilon_{12}^- = \epsilon_{12}\). The variational equations with respect to \(\phi_1\) and \(\phi_2\) are:

\[
d_j (\epsilon_{12} \partial_{\phi_1} \phi_{12}^+ + \epsilon_{12} \partial_{\phi_1} \phi_{12}^-) + d_j \epsilon_e \phi_e = 0 \quad e = 1, 2
\]

(14)

and are solved in \([0, \pi]\) by:

\[
\begin{align*}
\tilde{\phi}_1 &= -\epsilon_{12} \epsilon_e \arccos \left( \frac{k_2}{\sqrt{1 - k_2^2}} \right) + (1 + \epsilon_{12} \epsilon_e) \frac{\pi}{2} \\
\tilde{\phi}_2 &= -\epsilon_{12} \epsilon_e \arccos \left( \frac{k_1}{\sqrt{1 - k_1^2}} \right) + (1 + \epsilon_{12} \epsilon_e) \frac{\pi}{2}
\end{align*}
\]

(15)

Notice that this result allows us to give a geometrical interpretation to the class angles entering \((11)\), which is similar to the one for the usual integral formula of the squared 6j-symbol: the isosceles 6j-symbol is peaked on half the internal, or external, dihedral angles of the classical geometry. Indeed, for example, when \(\epsilon_{12} \epsilon_2 = -1\), the stationary angle \(\phi_1\) is \(\phi_1 = \alpha_1 = \alpha_1 / 2\), while for \(\epsilon_{12} \epsilon_2 = 1\), we have \(\phi_1 = \pi \alpha_1\). \(^4\)

We perform the complete expansion of the isosceles \(\{6j\}\), using this stationary phase analysis, below in section \(\text{IV}\).

Now we turn to the graviton propagator.

### B. The propagator as group integrals

If we insert the expression \((9)\) into \((7)\), the sums over \(j_1\) and \(j_2\) give the SU(2) Fourier transform of the boundary states. Let us first look at the normalization \(\mathcal{N}\):

\[
\mathcal{N} = \int dg_1 dg_2 \chi_{j_1}(g_1 g_2^{-1}) \chi_{j_1}(g_1 g_2) \left[ \sum_{j_1} \Psi_{\epsilon_1}(j_1) \chi_{j_1}(g_1) \right] \left[ \sum_{j_2} \Psi_{\epsilon_2}(j_2) \chi_{j_2}(g_2) \right]
\]

(16)

\[
= \int dg_1 dg_2 \chi_{j_1}(g_1 g_2^{-1}) \chi_{j_1}(g_1 g_2) \left( \Psi_{\epsilon_1}(g_1) \tilde{\Psi}_{\epsilon_2}(g_2) \right)
\]

(17)

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\(^3\) For a general configuration, one has to consider the squared 6j-symbol to have an integral expression.

\(^4\) This geometric interpretation of the saddle point is not surprising since the 6j-symbol is indeed the unique physical quantum state for a trivial topology and a triangulation made of a single tetrahedron \(\text{[17]}\). It satisfies the quantum flatness constraint and can serve as the boundary state in the general boundary framework.
For large spins, we are interested in evaluating $\mathcal{N}$ with a saddle point approximation. To that end, let us expand the previous expression in exponential form:

$$\mathcal{N} = \frac{1}{32\pi^2} \sum_{\epsilon_1, \epsilon_2, \epsilon_1', \epsilon_2', j_1, j_2 = \pm 1} \epsilon_1 \epsilon_2 \epsilon_{12}^+ \epsilon_{12}' \int d\phi_1 d\phi_2 d\phi f(\phi_1, \phi_2, u) e^{d_{j_1}, S_{(\epsilon, \eta)}(\phi_1, \phi_2, u)} \tag{18}$$

with $S_{(\epsilon, \eta)}(\phi_1, \phi_2, u) = i(\epsilon_1 \epsilon_2 \phi_{12}^+ + \epsilon_1' \epsilon_2' \phi_{12}') + \sum_{e=1,2} 2i k_e \epsilon_e (\phi_e + \eta e \alpha_e) - (1 - k_e^2) \sin^2(\phi_e + \eta e \alpha_e) \tag{19}$

where the label $\{\epsilon, \eta\}$ refers to the dependence on the sign variables, and $f$ is given by (12). The crucial point is that the phase of (18) (the imaginary part of $S_{(\epsilon, \eta)}$) is precisely $\Phi_{(\epsilon)}$ in (13), up to constant $\alpha_{1,2}$ terms which play no role in the stationary phase approximation. This means in particular that the imaginary part of $S_{(\epsilon, \eta)}$ has the same saddle points of the isosceles 6j-symbol.

The same analysis can be performed for the numerator of $W_{1122}$. To take into account the observables $O_{j1}$, notice that the SU(2) character is an eigenfunction for the Laplacian on the sphere $S^3$ with the Casimir as eigenvalue:

$$\Delta_{S^3} \chi_j(\phi) = \frac{1}{\sin^2 \phi} \partial_{\phi} (\sin^2 \phi \partial_{\phi} \chi_j) = -(d_j^2 - 1) \chi_j(\phi). \tag{20}$$

This allows to perform the sums over $j_1'$ and $j_2'$ in (21), introducing the Fourier transforms $\hat{\Psi}_c$. Again expanding the result of these operations into exponential form, one ends up with:

$$W_{1122} = \frac{1}{32\pi^2} \mathcal{N} \frac{k_1 k_2}{4 \cos^2 \theta} \sum_{\epsilon_1, \epsilon_2, \epsilon_1', \epsilon_2', j_1, j_2 = \pm 1} \epsilon_1 \epsilon_2 \epsilon_{12}^+ \epsilon_{12}' \int d\phi_1 d\phi_2 d\phi f(\phi_1, \phi_2, u) e^{d_{j_1}, S_{(\epsilon, \eta)}(\phi_1, \phi_2, u)} \times \left( a_{(\epsilon, \eta)}(\phi_1, \phi_2) + \frac{b_{(\epsilon, \eta)}(\phi_1, \phi_2)}{d_{j_1}} + \frac{c_{(\epsilon, \eta)}(\phi_1, \phi_2)}{d_{j_1}^2} \right) \tag{21}$$

The functions $a_{(\epsilon, \eta)}$, $b_{(\epsilon, \eta)}$, and $c_{(\epsilon, \eta)}$ stand for the observable insertions, and are given by:

$$a_{(\epsilon, \eta)}(\phi_1, \phi_2) = \prod_{e=1,2} \left( \frac{1 - k_e^2}{2k_e} \sin^2 2(\phi_e + \eta e \alpha_e) - 2i \epsilon_e \sin 2(\phi_e + \eta e \alpha_e) \right) \tag{22}$$

$$b_{(\epsilon, \eta)}(\phi_1, \phi_2) = -\frac{1}{k_2} \cos 2(\phi_2 + \eta_2 \alpha_2) \left( \frac{1 - k_1^2}{2k_2} \sin^2 2(\phi_1 + \eta_1 \alpha_1) - 2i \epsilon_1 \sin 2(\phi_1 + \eta_1 \alpha_1) \right) + (\epsilon_1 \leftrightarrow \epsilon_2) \tag{23}$$

$$c_{(\epsilon, \eta)}(\phi_1, \phi_2) = \frac{1}{k_1 k_2} \cos 2(\phi_1 + \eta_1 \alpha_1) \cos 2(\phi_2 + \eta_2 \alpha_2) \tag{24}$$

We are now ready to study the large spin expansion of (21). A common choice in the literature is to do so using a power series in $1/j_1$ (keeping $j_1/j_1$ and $j_2/j_1$ constant). However it is more convenient to take as parameter of the expansion the dimension $d_{j_1}$ (again keeping $k_1$, $k_2$ fixed). This is the natural choice, as we compute correlations with respect to the background geometry with lengths defined by the half-dimensions $d_{j_1}/2$, $d_{j_2}/2$ and $d_{j_2}/2$. Furthermore, as we show below (see (19) and (51)), these are the values of the lengths emerging in the asymptotics of the 6j-symbol.

As written in (21), $W_{1122}$ corresponds to the mean value of the function $a_{(\epsilon, \eta)} + b_{(\epsilon, \eta)} d_{j_2} + c_{(\epsilon, \eta)} d_{j_1}^2$ for the non-linear theory defined by the action $S_{(\epsilon, \eta)}$ and the integration measure $f$. The strategy is thus clear: we will compute separately the normalisation $\mathcal{N}$ and the numerator, perturbatively, with an expansion around the saddle points of the action $S_{(\epsilon, \eta)}$.

As stated above, the imaginary part of $S_{(\epsilon, \eta)}$ has the same saddle points of the isosceles 6j-symbol, namely $u = 0$ and $\tilde{\phi}_e$ given in (15) (independently of $\eta_1$ and $\eta_2$). The extremization with respect to the real part of $S_{(\epsilon, \eta)}$, on the other hand, constrains the $\eta_1$ and $\eta_2$ signs. Indeed, for a given solution $(\tilde{\phi}_1, \tilde{\phi}_2)$ from (15), characterized by $\epsilon_{12}$, $\epsilon_1$ and $\epsilon_2$, the signs $\eta_1$ and $\eta_2$ have to satisfy:

$$\sin 2(\tilde{\phi}_e + \eta_e \alpha_e) = 0, \quad \text{for } e = 1, 2 \tag{25}$$

These equations are solved by taking $\eta_1 = \epsilon_2 \epsilon_{12}$ and $\eta_2 = \epsilon_1 \epsilon_{12}$. This leads to four possibilities, summarized in the
following table,

| $\eta_2$ | $\phi_1 = \alpha_1$, and $\phi_2 = \alpha_2$, | $\phi_1 = \pi - \alpha_1$, and $\phi_2 = \alpha_2$, |
|---------|---------------------------------|---------------------------------|
| $\eta_1 = -1$ | $\epsilon_1 = \epsilon_2 = -\epsilon_{12}$ | $\epsilon_1 = -\epsilon_2 = \epsilon_{12}$ |
| $\eta_2 = 1$ | $\phi_1 = \alpha_1$, and $\phi_2 = \pi - \alpha_2$, | $\phi_1 = \pi - \alpha_1$, and $\phi_2 = \pi - \alpha_2$, |
| $\epsilon_1 = \epsilon_2 = -\epsilon_{12}$ | $\epsilon_1 = \epsilon_2 = \epsilon_{12}$ |

The condition $\epsilon_{12} = \epsilon_{12} = \epsilon_{12}$ and $\eta_1 = \epsilon_2 \epsilon_{12}$ and $\eta_2 = \epsilon_1 \epsilon_{12}$ allows us to perform three sums in (21), the configurations for which there is no saddle point being exponentially suppressed:

$$W_{1122} = \frac{1}{32\pi^2 N} \frac{k_1 k_2}{4 \cos^2 d_t} \prod_{\epsilon_1, \epsilon_2, \epsilon_{12} = \pm 1} \epsilon_1 \epsilon_2 \int d\phi_1 d\phi_2 du f(\phi_1, \phi_2, u) e^{d_t S(\epsilon_1)(\phi_1, \phi_2, u)}$$

$$\times \left( a(\epsilon_1) \phi_1 + b(\epsilon_1) \phi_2 + c(\epsilon_1) \phi_2^2 \right)$$

and the same for $N$ without the insertion of $\frac{k_1 k_2}{4 \cos^2 d_t} (a(\epsilon_1) + b(\epsilon_1)/d_j + c(\epsilon_1)/d_j^2)$.

Here the label $\{\epsilon\}$ simply indicates the dependence of the functions on the signs $\epsilon_1, \epsilon_2$ and $\epsilon_{12}$.

## II. THE COMPLETE PERTURBATIVE EXPANSION

The perturbative expansion of the two-point function $W$ is formulated as an asymptotic power series expansion in $1/d_j$, of the type:

$$W = \frac{1}{d_j} \left[ w_0 + \frac{1}{d_j} w_1 + \frac{1}{d_j^2} w_2 + \ldots \right].$$

Let us remind the reader that the dimension $d_j$ defines the length scale of the tetrahedron $L = d_j L_P/2$, with the Planck length $L_P = \hbar G$. Such an expansion thus matches the typical expansion of quantum field theory correlations with quantum corrections ordered in increasing powers of $\hbar$ (and of the coupling constant $G$) and with $L$ corresponding to the renormalization scale. We can thus call the coefficients $w_1, w_2, \ldots$ the one-loop and two-loop (and so on) corrections.

This perturbative expansion is obtained studying the power series expansion in $1/d_j$ around each of the four saddle points of both the denominator $\int f \exp S_{\{\epsilon\}}$ and the numerator $\int (a(\epsilon_1) + b(\epsilon_1)/d_j + c(\epsilon_1)/d_j^2) \exp S_{\{\epsilon\}}$. More precisely, we expand the action $S_{\{\epsilon\}}$ around its saddle point; the evaluation of $S$ at the stationary point gives a numerical factor, there is no linear term obviously, the quadratic term defines the Hessian matrix $A_{\{\epsilon\}}$ and finally all the remaining higher order terms (cubic onwards) are kept together to define the potential $\Omega_{\{\epsilon\}}$. This potential thus contains all higher order corrections to the quadratic approximation to the action $S$. As such, it does not enter the leading order of the two-point function but largely enters its NLO, NNLO and so on, (the loop corrections) as in quantum field theory. Then each term in the power series is evaluated as the Gaussian moment with respect to the Hessian matrix $A_{\{\epsilon\}}$ of terms coming from the expansion of $f \exp d_j \Omega_{\{\epsilon\}}$ in powers of $d_j$. In general many terms actually contribute to the same overall order in $1/d_j$. This intricate structure comes about precisely as in $\Phi^4$ because the expansion of $\exp d_j \Omega_{\{\epsilon\}}$ gives increasing powers of $d_j$ while the Gaussian moments have increasing powers in $1/d_j$.

On the other hand, a simplification of our calculations comes from the fact that each saddle point gives the same contribution. This is a consequence of the symmetry properties of the functions involved under the transformation of $\phi_c$ into $\pi - \phi_c$. Further, for a given saddle point, the two possible configurations of signs are simply related by complex conjugation. The actual sum then ensures the reality of the result. Without loss of generality, we can thus restrict the computation to the saddle point ($\alpha_1, \alpha_2, 0$) with $\epsilon_{12} = 1 = -\epsilon_1 = -\epsilon_2$. There we have:

$$a(\phi_1, \phi_2) = \prod_{\epsilon = 1, 2} \left( 1 - \frac{k_1^2}{2k_c} \sin^2 \phi_c - \alpha_c + 2i \sin \phi_c - \alpha_c \right)$$

$$b(\phi_1, \phi_2) = -\frac{1}{k_2} \cos 2\phi_2 - \alpha_2 \left( \frac{1 - k_1^2}{2k_1} \sin^2 \phi_1 - \alpha_1 + 2i \sin \phi_1 - \alpha_1 \right) \pm (\epsilon_1 \leftrightarrow \epsilon_2)$$

$$c(\phi_1, \phi_2) = \frac{1}{k_1 k_2} \cos 2\phi_1 - \alpha_1 \cos 2\phi_2 - \alpha_2$$

(28) (29) (30)
and the potential $\Omega$ is extracted from the derivatives of $S$ greater than three, with $S$ given by (19) with the chosen signs,

$$S(\phi_1, \phi_2, u) = i(\phi_{12}^+ + \phi_{12}^-) - \sum_{e=1,2} (1 - k_e^2) \sin^2(\phi_e - \alpha_e) + \text{linear terms} \quad (31)$$

Expanding around the background, the inverse of the Hessian matrix is (see the Appendix A for details):

$$A^{-1} = \frac{1}{4} \left( \begin{array}{ccc} \frac{1}{\cos \phi_1 k_1 k_2} & \frac{1}{\cos \phi_2 k_1 k_2} & e^{i\phi_1} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{2i \tan \phi_2}{1 - (k_1^2 + k_2^2)} \end{array} \right). \quad (32)$$

Introducing the shorthand notation

$$A^{-1} = \sum_{\text{all possible pairings}} A^{-1}_{\beta_1 \beta_2} \cdots A^{-1}_{\beta_{2n-1} \beta_{2n}} \quad (33)$$

for $\bar{\beta} \in \{1, 2, 3\}^{2n}$, the complete perturbative expansion of the propagator can be written as

$$W_{122} = \frac{k_1 k_2}{4 \cos^2 \vartheta_t} \frac{\sqrt{1 - k_1^2} \sqrt{1 - k_2^2}}{2d_{j_1}} \sum_{i,j=1,2} \partial_{j_t}^2 a \ A_{ij}^{-1} + \sum_{p \geq 2} \frac{W_p}{d_{j_t}^p}. \quad (34)$$

In the numerator, the first term gives the leading order contribution in $1/d_{j_t}$ (see next section). It comes entirely from the $a$ term in (28). In fact, $a$ and $b$ vanish at the saddle point, and so does the gradient of $a$, so the expansion of $a$, $b$, and $c$ is dominated by the quadratic term of $a$.

All the higher order corrections have been collected in the summations. The coefficients $N_p$ and $W_p$ correspond to finite sums:

$$N_p = \sum_{n=0}^{2p} \sum_{\bar{\beta} \in \{1, 2, 3\}^{2(P+n)}} \frac{1}{(2(P+n))!} \mathbb{R}\left( \int e^{-i(2d_{j_1} - \frac{1}{2}) \vartheta_t} \partial_{\bar{\beta}}^2 (P+n) (f \Omega^n) A_{\bar{\beta}}^{-1} \right) |_{\phi_1 = \alpha_1, \phi_2 = \alpha_2, u=0} \quad (35)$$

and

$$W_p = \sum_{n \geq 0} \frac{1}{n!} \sum_{\bar{\beta} \in \{1, 2, 3\}^{2(P+n)}} \frac{1}{(2(P+n))!} \mathbb{R}\left( i e^{-i(2d_{j_1} - \frac{1}{2}) \vartheta_t} \partial_{\bar{\beta}}^2 (P+n) (a f \Omega^n) A_{\bar{\beta}}^{-1} \right)$$

$$+ \sum_{\bar{\beta} \in \{1, 2, 3\}^{2(P+n-1)}} \frac{1}{(2(P+n-1))!} \mathbb{R}\left( i e^{-i(2d_{j_1} - \frac{1}{2}) \vartheta_t} \partial_{\bar{\beta}}^2 (P+n-1) (b f \Omega^n) A_{\bar{\beta}}^{-1} \right)$$

$$+ \sum_{\bar{\beta} \in \{1, 2, 3\}^{2(P+n-2)}} \frac{1}{(2(P+n-2))!} \mathbb{R}\left( i e^{-i(2d_{j_1} - \frac{1}{2}) \vartheta_t} \partial_{\bar{\beta}}^2 (P+n-2) (c f \Omega^n) A_{\bar{\beta}}^{-1} \right) |_{\phi_1 = \alpha_1, \phi_2 = \alpha_2, u=0} \quad (35)$$

The three lines of (36) are the separate contributions of the insertions $a$, $b$ and $c$. The sum over $n$ defining $W_p$ is finite for each of these contributions: $n$ is bounded by $2P - 2, 2P - 3$ and $2P - 4$ for $a$, $b$ and $c$ respectively. The derivatives of highest order of $\Omega$ involved in $W_p$ are respectively the $2P$-th derivatives, the $(2P - 1)$-th ones and the $(2P - 2)$-th ones, and for $N_p$, the $(2P + 1)$-th derivatives, all corresponding to $n = 1$.

The intricacy of the formulas was anticipated at the beginning of the section. However the reader should be reassured that they are simple, if tedious, algebraic expressions.

The real part $\mathbb{R}$ in (35) is consistent with the reality of the initial expression (21), and arises from the summation over the $\epsilon$ sign.

### III. THE LEADING ORDER, ONE-LOOP AND TWO-LOOP CORRECTIONS

We now use (35) and (36) to obtain explicitly the first orders of the expansion. The leading order (LO) and the next to leading order (NLO) have already been obtained in a quite different way in (2). We here recover them quickly.
The computation of the next to next to leading order (NNLO) is then completely new. It is shown in [2] that the NNLO needs the corrections to the asymptotics of the 6j-symbol, i.e., to the Ponzano-Regge formula. The success of our method resides in the fact that such corrections are naturally contained in the exact group integral expression 9 of the kernel.

The LO is obtained evaluating the normalization at the saddle point, $f_0 = f(\alpha_1, \alpha_2, 0)$, and the numerator at the non-zero second derivatives of $a$:

$$W_{1122}^{LO} = \frac{-k_1 k_2}{4 \cos^2 \vartheta_t} \frac{\Re \left( i e^{-i(2d_j t - \frac{1}{2}) \vartheta_t} \partial_{ \vartheta_t}^2 a A_{12}^{-1} \right)}{f_0 \Re \left( i e^{-i(2d_j t - \frac{1}{2}) \vartheta_t} \right)}$$

This reproduces the expected 1/d$_j$ scaling behavior of the LO. The difference in the coefficient with $[1, 2]$ comes from the different boundary state used. In particular notice that while $\vartheta_t (k_1, k_2)$ is a constant, the dependence upon $d_j$, of the second fraction produces spurious oscillations. These can be reabsorbed in the boundary state, replacing $\sin d_j (\phi + \eta \alpha_e)$ in $\Psi_e$ with $\sin (d_j (\phi + \eta \alpha_e) + \eta d_j \vartheta_t)$. The Fourier transform is then:

$$\Psi_e (j) = \frac{e^{-\gamma j/2}}{N} \left[ I_{|j-j_0|} (\frac{\gamma e}{2}) \cos (d_j \alpha_e + d_j \vartheta_t) - I_{j+j_0+1} (\frac{\gamma e}{2}) \cos (d_j \alpha_e - d_j \vartheta_t) \right]$$

This does not affect the asymptotic behavior of $\Psi_e (j)$. With this replacement, we obtain the same result of [2] (cf. equation (37)) for the isosceles case,

$$W_{1122}^{LO} = \frac{-1}{d_j \cos \vartheta_t} \frac{\sin \frac{3}{2} \vartheta_t}{\sin \frac{5}{2} \vartheta_t}$$

Even if the LO now matches the previous results presented in [2], the higher orders will differ because of the different boundary state used.

For the sake of a simpler presentation, we will report the NLO and the NNLO for the equilateral tetrahedron, $k_1 = k_2 = 1/2$ and $\vartheta_t = \arccos -\frac{1}{2}$. The general expressions in terms of $k_1$ and $k_2$ are indeed quite cumbersome. This choice will also facilitate the comparison with numerical simulations of [7].

The NLO is then obtained from the coefficients $N_1$ and $W_2$. To keep compact expressions, we adopt the following symbolic notation for the contractions of derivatives with Gaussian moments: for functions $f$ and $h$ of $\phi_1$, $\phi_2$ and $u$, define:

$$f_n h_m A_n^{-1} = \sum_{i_1 \ldots i_n} \sum_{j_1 \ldots j_m} f_{i_1 \ldots i_n} h_{j_1 \ldots j_m} \partial_{i_1 \ldots i_n} A^{-1}_{j_1 \ldots j_m}$$

evaluated at the saddle point $(\alpha_1, \alpha_2, 0)$ with $\epsilon_{12} = -\epsilon_1 = -\epsilon_2 = 1$. For example, the LO of (43) can be written $\frac{1}{d_j} a_2^2 A_1^{-1}$. In $N_1$, three powers of $\Omega$ appear, $\Omega^0$, $\Omega$ and $\Omega^2$. Using the boundary state [39], we have:

$$N_1 = \Re \left( i e^{\vartheta_t} \left[ f_2 A_2^{-1} + (f_1 S_3 + f_0 S_4) A_4^{-1} + \frac{f_0}{2} S_3 S_3 A_6^{-1} \right] \right)$$

We proceed in the same way for the three contributions to $W_2$:

$$W_2 = \Re \left( i e^{\vartheta_t} \left[ (a_2 f_2 + a_3 f_1 + f_0 a_4) A_4^{-1} + (a_2 f_1 S_3 + f_0 (a_3 S_3 + a_2 S_4)) A_6^{-1} + \frac{f_0}{2} a_2 S_3 S_3 A_8^{-1} + (f_0 b_2 + b_1 f_1) A_2^{-1} + f_0 b_1 S_3 A_4^{-1} + f_0 c_0 \right] \right)$$

After straightforward algebra we obtain the NLO, of order 1/d$_j^2$:

$$W_{1122}^{NLO} = \frac{1}{d_j} - \frac{511}{432} \frac{d_j^2}{d_j^2}$$

These results for the LO and NLO are well-confirmed by numerical simulations, as one can see from figure [2]. An agreement with 0.58% of error for the LO, and with 1.7% error for the NLO is reached between the coefficients of these orders for $d_j = 201$ (i.e., the representation of spin $j_z = 100$).

All orders of the expansion can be computed using the above recipe. From this point of view, the NNLO is of no particular specificity. We need the expansion of the action (or equivalently $\Omega$) until the sixth order. The highest order
correlator $A^{-1}_{ββ}$ is of order 12 for the normalisation, and respectively 14, 10 and 6 for the insertion of $a$, $b$ and $c$.

$$N_2 = \Re \left( i e^{2iθ_1} \left[ f_4 A_4^{-1} + (f_3 S_3 + f_2 S_4 + f_1 S_5 + f_0 S_6)A_6^{-1} + \frac{1}{2} (f_2 S_3 S_3 + 2f_1 S_4 S_4 + f_0 S_4 S_4 + 2f_0 S_3 S_5)A_8^{-1} + \frac{1}{3!} (f_1 S_3 S_3 S_3 + 3f_0 S_3 S_3 S_4)A_{10}^{-1} + \frac{f_0}{4!} S_3 S_3 S_3 A_{12}^{-1} \right] \right)$$

(44)

We also write $W_3 = W_3^{(a)} + W_3^{(b)} + W_3^{(c)}$, with:

$$W_3^{(a)} = \Re \left( i e^{2iθ_1} \left[ (f_0 a_6 + f_1 a_5 + f_2 a_4 + f_3 a_3 + f_4 a_2)A_6^{-1} + ((f_0 a_5 + f_1 a_4 + f_2 a_3 + f_3 a_2)S_3 + (f_0 a_4 + f_1 a_3 + f_2 a_2)S_4 + (f_0 a_3 + f_1 a_2)S_5 + (f_0 a_2)S_6)A_8^{-1} + \frac{1}{2} ((f_0 a_4 + f_1 a_3 + f_2 a_2)S_3 S_3 + (f_0 a_3 + f_1 a_2)S_4 S_4 + (f_0 a_2 S_5)S_5 + (f_0 a_1)S_6)A_{10}^{-1} + \frac{1}{3!} ((f_0 a_3 + f_1 a_2)S_3 S_3 S_3 + 3f_0 a_2 S_3 S_3 S_4)A_{12}^{-1} + \frac{f_0}{4!} a_2 S_3 S_3 S_3 A_{14}^{-1} \right] \right)$$

(45)

$$W_3^{(b)} = \Re \left( i e^{2iθ_1} \left[ (f_0 b_4 + f_1 b_3 + f_2 b_2 + f_3 b_1)A_4^{-1} + ((f_0 b_3 + f_1 b_2 + f_2 b_1)S_3 + (f_0 b_2 + f_1 b_1)S_4 + (f_0 b_1)S_5)A_6^{-1} + \frac{1}{2} ((f_0 b_2 + f_1 b_1)S_3 S_3 + 2f_0 b_1 S_3 S_4)A_8^{-1} + \frac{f_0}{3!} b_1 S_3 S_3 S_3 A_{10}^{-1} \right] \right)$$

(46)

$$W_3^{(c)} = \Re \left( i e^{2iθ_1} \left[ (f_0 c_2 + f_1 c_1 + f_2 c_0)A_2^{-1} + ((f_0 c_1 + f_1 c_0)S_3 + (f_0 c_0)S_4)A_4^{-1} + \frac{f_0}{2} c_0 S_3 S_3 A_6^{-1} \right] \right)$$

(47)

The NNLO is thus computed to be:

$$W_{1122}^{NNLO} = \frac{1}{d_j} - \frac{511}{432 d_j^2} + \frac{520507}{157464 d_j^3}$$

(48)

This result is again supported by numerical simulations, see figure 2. An agreement with 11.3% of error is obtained for the coefficient of $1/d_j^3$ at $d_j = 201$. The error can be reduced by pushing the simulations to higher values of $d_j$.

**IV. PERTURBATIVE EXPANSION OF THE ISOSCELES 6J-SYMBOL**

The procedure described above can be applied directly to the isosceles 6j-symbol (9), obtaining the known Ponzano-Regge formula and its corrections. This is interesting for a number of reasons. As discussed in [2], the corrections to the Ponzano-Regge formula are a key difference between the spin foam perturbative expansion studied here, and the one that would arise from quantum Regge calculus. The 6j-symbol is also the physical boundary state of 3d gravity for a trivial topology and a one-tetrahedron triangulation. In 4d, it appears as a building block for the spin foams amplitudes, such as the 15j-symbol. Thus, with regards to many aspects of spin foams in 3d and 4d, in particular for the quantum corrections to the semiclassical limits, it would be good to have a better understanding of this object.
of the asymptotics of the 6j-symbol, we are here scaling the lengths of the tetrahedron (or equivalently the truncated Taylor expansion of \( \Phi \)).

The expansion of this isosceles 6j-symbol is then (see appendix B for more details):

\[
\epsilon^{id_{ij}} \Phi_{(e)}(\hat{\theta}_1, \hat{\phi}_2, 0) = \epsilon_1 \epsilon_2 e^{-\epsilon_1 \epsilon_2 \epsilon_1 \epsilon_2 S_R}
\]

\[
S_R = d_{ji} (2 \tilde{\partial}_t + 2k_1 \alpha_1 + 2k_2 \alpha_2)
\]

We then proceed exactly as for the propagator, knowing that for each configuration of signs, the \( \{6j\} \) is peaked on the classical geometry of the tetrahedron. The perturbative expansion with respect to this flat geometry is thus given by the Gaussian moments of the Hessian matrix \( H_{(e)} \) of \( \Phi_{(e)} \). Let us stress that, in contrast with the previous studies of the asymptotics of the 6j-symbol, we are here scaling the lengths of the tetrahedron (or equivalently \( d_{ji} \)), keeping the length ratios \( k_1 \) and \( k_2 \) fixed, instead of scaling \( j_t \).

As for the propagator, the four saddle points give the same contribution, and the two sign configurations of a given saddle point are related by complex conjugation. This can be done in a quite explicit way. Introduce \( \omega \) to be the truncated Taylor expansion of \( \Phi_{(e)} \), starting at order three onwards, around the saddle point \((\alpha_1, \alpha_2, 0)\) with \( \epsilon_{12} = -\epsilon_1 = -\epsilon_2 = 1 \). Let \( H^{-1} \) be the corresponding inverse of \( H_{(e)} \):

\[
H^{-1} = \frac{1}{2} \begin{pmatrix}
\frac{1}{1-k_1^2} \cot \theta_t & -\frac{1}{\sqrt{1-k_1^2} \sqrt{1-k_2^2} \sin \theta_t} & 0 \\
-\frac{1}{\sqrt{1-k_1^2} \sqrt{1-k_2^2} \sin \theta_t} & \frac{1}{k_2^2} \cot \theta_t & 0 \\
0 & 0 & \frac{\tan \theta_t}{1-(k_1^2+k_2^2)}
\end{pmatrix}
\]

We also introduce the volume of the tetrahedron, which enters the Gaussian integrals of \( H \):

\[
V_t = \frac{d_3^4}{12} k_1 k_2 \sqrt{1-(k_1^2+k_2^2)}.
\]

The expansion of this isosceles 6j-symbol is then (see appendix B for more details):

\[
\begin{pmatrix}
j_1 & j_t & j_t \\
j_2 & j_t & j_t
\end{pmatrix} = \frac{1}{\sqrt{1-k_1^2} \sqrt{1-k_2^2} 12 \pi V_t} \sum_{p \geq 0} (-1)^p (C_{2p \sigma}^{2p}) \cos(S_R + \frac{\pi}{4}) + C_{2p+1 \sigma}^{2p+1} \sin(S_R + \frac{\pi}{4})
\]

where the coefficients \( C_P \), for \( P = 2p, 2p + 1 \), are given by finite sums:

\[
C_P = \sum_{n=0}^{P} \frac{(-1)^n}{[2(P+n)]!n!} \sum_{\beta \in \{1,2,3\}^{2(P+n)}} \hat{\partial}^{2(P+n)}_{\beta} \left( f_{\omega}^n \right) |_{(\alpha_1, \alpha_2, 0)} H^{-1}_{(e)}
\]

Thus, all even orders are in phase with the leading order asymptotics, given by the original Ponzano-Regge formula in \( \cos(S_R + \pi/4) \). This leading order is easily recovered by computing the coefficient \( C_0 \), with \( f(\alpha_1, \alpha_2, 0) = \sqrt{1-k_1^2} \sqrt{1-k_2^2} \):

\[
\begin{pmatrix}
j_1 & j_t & j_t \\
j_2 & j_t & j_t
\end{pmatrix} \sim \frac{1}{\sqrt{12 \pi V_t}} \cos \left( S_R + \frac{\pi}{4} \right).
\]

On the other hand, all odd orders are out of phase (or in quadrature of phase) with this leading order. If we were scaling the spin \( j_t \) instead of the length \( d_{ji} / 2 \), the result would not have had such a simple structure with sines and cosines being mixed up at all orders (but leading).

This asymptotic series formula for the isosceles tetrahedron shows that only the Regge action is relevant and no other frequency appears in the 6j-symbol. We believe this feature to generalize to the generic 6j-symbol since its asymptotics can also be extracted using saddle point techniques [10].

The coefficient of a given order is simply given by the contractions of the derivatives of \( f_{\omega}^n \) with the Gaussian moments. For a given order \( P \), the highest derivatives of \( \omega \) involved correspond to \( n = 1 \) in [53] and equals \( 2(2P+1) \).

For instance, the NLO is obtained by setting \( P = 1 \). With the notations of the previous section, we have:

\[
C_1 = f_2 H_{2}^{-1} - (f_1 \omega_3 + f_0 \omega_4) H_{4}^{-1} + \frac{f_0}{2} \omega_3 \omega_3 H_{6}^{-1}
\]

\[\text{above for the propagator.} \]
and introducing the reduced volume \( v = V / d_{ji}^3 \):

\[
\left\{ j_1, j_t, j_t \right\} \sim \frac{1}{\sqrt{12\pi V_t}} \cos \left( S_R + \frac{\pi}{4} \right) - \frac{\cos^2 \vartheta_t}{d_{ji} \sqrt{12\pi V_t}} P_1(k_1, k_2) \frac{P_1(k_1, k_2)}{48(12\pi)^3} \sin \left( S_R + \frac{\pi}{4} \right),
\]

where \( P(k_1, k_2) \) is a symmetric polynomial in \( k_1^2 \) and \( k_2^2 \):

\[
P_1(k_1, k_2) = 3(1 - k_1^2)^2(1 - 2k_1^2) + 3(1 - k_2^2)^2(1 - 2k_2^2) - 3 + 46k_1^2k_2^2 + 25k_1^4k_2^2 - 44(k_1^2k_2^2 + k_1^2k_2^2) + 10(k_1^6k_2^2 + k_1^2k_2^6).
\]

This polynomial is not simply related to the volume and we haven’t succeeded in providing it with a geometric interpretation. It would nevertheless be very interesting to understand its geometrical origin in order to interpret physically the higher order corrections to the graviton propagator.

For extremal values of \( k_1 \), this polynomial simplifies. We get \( P_1(0, k) = 3(1 - k^2)^2(1 - 2k^2) \) for \( k_1 = 0 \). At the other end at \( k_1 = 1 \), we obtain \( P_1(1, k) = -4k^4(1 - k^2) \) with obvious roots 0 and 1. Let us point out that \( k_{1,2} \) actually never physically reaches these extreme values 0 and 1, but its bounds depend on the representation \( j_t \) (due to the SU(2) recoupling theory):

\[
\frac{1}{2d_{ji}} \leq k_e \leq 1 - \frac{1}{2d_{ji}}.
\]

When \( k_1 \) reaches these extreme values, the coefficients of \( P_1 \) are polynomials in \( 1/d_{ji}^2 \).  

The result \([50]\) is confirmed by numerical simulations, see figure 3. These plots represent numerical simulations of the \{6j\} minus the analytical formula \([56]\), for three pairs \((k_1, k_2)\). We have used in these simulations the particular case \( k_1 = k_2 = k \), for which

\[
P_1(k, k) = (1 - k^2) \left( 3 - 21k^2 + 55k^4 - 45k^6 \right)
\]

whose only root in \([0, 1]\) is \( k = \frac{1}{\sqrt{15}} \sqrt{15((10(81\sqrt{3}10 + 1450))^{1/2} + (10(81\sqrt{3}100 - 1450))^{1/2} + 55)} \approx 0.8248 \) thus inducing the vanishing of the NLO. To enhance the comparison, we have multiplied by \( d_{ji}^{5/2} \) to see how the coefficient of the NLO is approached, and suppressed the oscillations by dividing by those of the NNLO, \( \cos(S_R + \frac{\pi}{4}) \). The numerics support both the coefficient and the phase.

Notice that in the equilateral situation, \( k_1 = k_2 = 1/2 \) represent (half) the length ratios but also (half) the spin

---

5 For instance, when the edge \( e_1 \) is at minimal length, \( j_1 = 0 \) or \( k_1 = \frac{1}{2d_{ji}} \), the coefficients of \( P_1 \) read:

\[
P_1\left( \frac{1}{2d_{ji}}, k_2 \right) = 3(1 - \frac{1}{d_{ji}^2} + \frac{5}{16d_{ji}^4} - \frac{1}{32d_{ji}^6}) + \left( -12 + \frac{23}{2d_{ji}^2} - \frac{11}{4d_{ji}^4} + \frac{5}{32d_{ji}^6} \right) k_2^2 + \left( 15 - \frac{11}{d_{ji}^2} + \frac{25}{16d_{ji}^4} \right) k_2^4 + \left( -6 + \frac{5}{2d_{ji}^2} \right) k_2^6.
\]
ratios. We can thus switch easily to the usual $1/j_t$ expansion:

$$\langle 6j \rangle^{NLO} = \frac{25/4}{\sqrt{\pi d_j}} \cos \left( S_R + \frac{\pi}{4} \right) - \frac{31}{72 \cdot 2^{1/4} \sqrt{\pi d_j}} \sin \left( S_R + \frac{\pi}{4} \right)$$

This point of view shows that it is much more natural to study the asymptotics of the 6j-symbol in terms of the inverse length $1/d_j$ instead of the inverse spin label $1/j$. For instance, the leading order coefficient is given in terms of the volume $V$ of the tetrahedron with edge lengths given by the $d_j$'s and not the $j$'s.

Finally, we point out that the asymptotics given above in terms of the cosine and sine of the Regge action holds for mid-range values of $k_1, k_2$ and it breaks down for $k_1, k_2$ close to their extremal values 0 and 1. Indeed when $k_2 = 0$ the asymptotics are given in terms of Airy functions while when $k_2 = 1$ they are given by the (non-oscillatory) exponential of the Regge action. The interested reader can find details and references in [17].

Conclusions

We have shown it is possible to compute analytically the two-point function -- the graviton propagator -- at all orders in the Planck length for the 3d toy model (the Ponzano-Regge model for a single isosceles tetrahedron) introduced in [1]. This builds on the previous work [2] where the leading order and first quantum corrections were computed using the asymptotics of the 6j-symbol in terms of the Regge action. Here, we introduced a representation of the relevant 6j-symbol and of the full graviton propagator as group integrals over SU(2). Then one obtains the expansion of the two-point function as a power series in the inverse spin label (or equivalently in the Planck length) by expanding these group integrals around their saddle points. We computed explicitly the first and second order corrections to the leading order behavior and matched them successfully against numerical simulations.

A side-product of these calculations is the corrections to the Ponzano-Regge asymptotic formula for the 6j-symbol for an isosceles tetrahedron (when four representations are taken equal). We obtain a series alternating cosines and sines of the Regge action for the tetrahedron (shifted by $\pi$). We computed explicitly the next-to-leading order correction and checked it numerically. An open issue is the geometrical interpretation of the polynomial coefficient $P_1(k_1, k_2)$ in front of this first order correction.

To conclude, we have shown how to carry out the calculations of the spinfoam graviton propagator at all orders at least in this simple setting. We hope to apply the present methods and tools to more refined 3d triangulations [18] and to compute spinfoam correlations for 4d quantum gravity along the lines of [3, 11, 12, 13, 19].

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APPENDIX A: DETAILS FOR THE PROPAGATOR EXPANSION

The key object containing the quadratic fluctuations and their corrections is the generating function $Z(J)$, which is the Gaussian integral of the Hessian matrix $A$ with a source $J$, evaluated at each saddle point:

$$Z(J) = \int dX e^{-\frac{dJ}{2} X A_{(s)} X + J X}$$

with $X = (\phi_1, \phi_2, u)$. The Hessian matrix is given, for all configurations of signs, by:

$$A_{(s)} = \begin{pmatrix}
\frac{2(1-k^2)}{\sin \vartheta_1} e^{i \epsilon_1 \epsilon_2 \epsilon_{12}} (\phi_2 - \phi_1) & 2 \frac{k_1 k_2}{\sin \vartheta_2 \cos \vartheta_1} e^{-i \epsilon_1} \frac{\pi}{2} & 0 \\
2 \frac{k_1 k_2}{\sin \vartheta_1 \cos \vartheta_2} e^{-i \epsilon_1} \frac{\pi}{2} & 0 & 0 \\
0 & 2 \frac{k_1 k_2}{\sin \vartheta_2 \cos \vartheta_1} e^{i \epsilon_1 \epsilon_2 \epsilon_{12}} (\phi_2 - \phi_1) & 0 \\
-2i \epsilon_1 \epsilon_2 \epsilon_{12} \left( 1 - (k_1^2 + k_2^2) \right) \frac{\cos \vartheta_l}{\sin \vartheta_1} & 0 & 0
\end{pmatrix}$$

(A2)
Due to the initial symmetry between the group elements $g_1$ and $g_2$, clearly expressed in (17), $A_{\epsilon}$ is invariant under the reversing of $\epsilon_1$ and $\epsilon_2$. $A_{\epsilon}$ has also the property of transforming into its complex conjugated matrix when inverting all signs, an operation which does not change the saddle point $(\phi_1, \phi_2, 0)$ considered, and equivalently under the flipping of $\epsilon_{12}$. A straightforward calculation yields:

$$Z_{\epsilon}(J) = Z_{\epsilon} e^{-\frac{1}{\hbar} \sum_{j} A^{-1}_{\epsilon}(J)}$$  \hspace{1cm} (A3)

$$Z_{\epsilon} = e\left(\frac{\pi}{d_j}\right)^{3/2} \frac{\epsilon_{1} \epsilon_{2} \epsilon_{12}}{(1-k_1^2)(1-k_2^2)\sqrt{2}\cos \partial_t} e^{\epsilon_{1} \epsilon_{2} \epsilon_{12} \partial_t},$$  \hspace{1cm} (A4)

$$A^{-1}_{\epsilon} = \frac{1}{4} \left( \begin{array}{cc}
\epsilon_{1} \epsilon_{2} \cos \frac{\partial_t}{k_1 k_2} e^{\epsilon_{1} \epsilon_{2} \epsilon_{12} \partial_t} & 0 \\
0 & 1 - k_2^2
\end{array}\right).$$  \hspace{1cm} (A5)

$A^{-1}_{\epsilon}$ and $Z_{\epsilon}$ benefit from the previously mentioned symmetries of $A_{\epsilon}$. The symmetry flipping $\epsilon_1$ and $\epsilon_2$ means that the saddle points $(\alpha_1, \alpha_2, 0)$ and $(\pi - \alpha_1, \pi - \alpha_2, 0)$ have the same Hessian matrices, for a fixed $\epsilon_{12}$. Moreover, flipping $\epsilon_{12}$ while going to the saddle points $(\alpha_1, \pi - \alpha_2, 0)$ and $(\pi - \alpha_1, \alpha_2, 0)$ does not change $A_{\epsilon}$ and $Z_{\epsilon}$, up to a change of sign for the non-diagonal coefficients of $A_{\epsilon}$.

Let us focus on the normalization $N$. The Gaussian moments are generated by successive derivations of $Z(J)$ with respect to the source, and they are contracted with the derivatives of $f \exp d_j \Omega_{\epsilon}$, which we expand into powers of $d_j$. We thus have:

$$N = \frac{1}{32 \pi^2} \sum_{\epsilon_1, \epsilon_2, \epsilon_{12}} \epsilon_1 \epsilon_2 Z_{\epsilon} e^{d_j \epsilon_1 \epsilon_2 \epsilon_{12}} \sum_{N \in \mathbb{N}} \sum_{n \geq 0} \frac{1}{(2N)!} d_j^{2N} (f \Omega^n_{\epsilon})_{\phi_{1}, \phi_2, \beta} \epsilon^{-1}_{\epsilon} \beta$$  \hspace{1cm} (A6)

where the correlators $A^{-1}_{\epsilon, \beta}$ are defined according to Wick’s theorem:

$$A^{-1}_{\epsilon, \beta} = \sum_{\text{all possible pairings}} A^{-1}_{\epsilon, \beta_{1}, \beta_{2}} \cdots A^{-1}_{\epsilon, \beta_{2N-1}, \beta_{2N}}$$  \hspace{1cm} (A7)

As $\Omega$ is a Taylor expansion into powers of $(\phi_1 - \bar{\phi}_1)$, $(\phi_2 - \bar{\phi}_2)$ and $u$, whose minimal order is $3$, the power $n$ of $\Omega$ in (A6) is bounded from above by $N$: $3n \leq 2N$, and the sum over $n$ is thus finite for each $N$. The power of $1/d_j$ receives two contributions: one, positive, from the Gaussian moments, and the other, negative, from the expansion of $\exp d_j \Omega_{\epsilon}$. We can identify the coefficients of a given order by the simple change of variables $P = N - n$. Introducing the explicit expressions of $Z_{\epsilon}$ and $S_{\epsilon}(\tilde{\phi}_1, \tilde{\phi}_2, 0)$:

$$N = \frac{-1}{32(1-k_1^2)(1-k_2^2)\sqrt{2\pi \cos \partial_t}} \epsilon_1 \epsilon_2 \epsilon_{12} e^{-\epsilon_1 \epsilon_2 \epsilon_{12} (2d_j - 1/2) \partial_t} \frac{1}{(2P+n)!} d_j^{2P+n} (f \Omega^n_{\epsilon})_{\phi_{1}, \phi_2, \beta} A^{-1}_{\epsilon, \beta}$$  \hspace{1cm} (A8)

with $u_P = \sum_{n=0}^{2P} \sum_{\beta \in \{1,2,3\}^{2(P+n)}} \sum_{\epsilon_1 \epsilon_2 \epsilon_{12}} \epsilon_1 \epsilon_2 \epsilon_{12} e^{-\epsilon_1 \epsilon_2 \epsilon_{12} (2d_j - 1/2) \partial_t} \frac{1}{(2P+n)!} d_j^{2P+n} (f \Omega^n_{\epsilon})_{\phi_{1}, \phi_2, \beta} A^{-1}_{\epsilon, \beta}$  \hspace{1cm} (A9)

Let us further simplify the coefficients $u_P$ by performing the sums over $\epsilon_1$, $\epsilon_2$ and $\epsilon_{12}$. First, notice that the sign of the imaginary part of $\Omega_{\epsilon}$ is $\epsilon_1 \epsilon_2 \epsilon_{12}$. Since $f$ is real, and considering the symmetry properties of $A^{-1}_{\epsilon}$ given in (A5), it is clear that when the signs $\epsilon_1$, $\epsilon_2$ and $\epsilon_{12}$ are all flipped, the derivatives are evaluated at the same saddle point and $u_P$ is transformed into its complex conjugate. Thus, let us work with a fixed value of $\epsilon_1 \epsilon_2 \epsilon_{12}$, say 1, and consider the basic properties of the functions $f$ and $S_{\epsilon}$ minus the linear parts in $\phi_1$ and $\phi_2$ (its derivatives greater than three are those of $\Omega$). More precisely, we are interested in how these functions and their derivatives, evaluated at a given saddle point, transform when the saddle point is changed. Let us see for instance the differences when going between the saddle points $(\phi_1 = \alpha_1, \phi_2 = \alpha_2)$ and $(\phi_1 = \pi - \alpha_1, \phi_2 = \alpha_2)$.

Hoping to be able to use $\epsilon_1 \epsilon_2 \epsilon_{12}$, this change of saddle point is determined by the flips of $\epsilon_1$ and $\epsilon_{12}$. We have:

$$f(\pi - \phi_1, \phi_2, u) = f(\phi_1, \phi_2, u)$$

$$f(\phi_1 + \phi_2, u) = f(\phi_1, \phi_2, u)$$

$$f(\phi_1 + \phi_2, \pi - \phi_2, u) = 2\pi - f(\phi_1 + \phi_2, \phi_2, u)$$

The real part of $S_{\epsilon}$ is non-zero only when derivated an even number of times. Thus, $f^{2P}(\phi_1, \phi_2, u)$ equals $f^{2P}(\pi - \phi_1, \phi_2, u)$ when we flip in the same time $\epsilon_{12}$ in front of $(\phi_1 + \phi_2, \pi - \phi_2)$ in $S_{\epsilon}$. This means that each derivation with respect to $\phi_1$ flips the sign between the two saddle points considered. There is now three possibilities: (i) such a derivation is contracted with
another derivation w.r.t. \( \phi_1 \) through \( A_{11}^{-1} = \frac{1}{4(1-k_1^2)} \), then the sign is changed twice, i.e. there is no change of sign.

(ii) It is contracted with a derivation w.r.t. \( u \) via \( A_{11}^{-1} \) which is zero, so that there is in fact no contribution. (iii) It is contracted with a derivation w.r.t. \( \phi_2 \) via \( A_{12}^{-1} \) whose sign changes under the flip of \( \epsilon_1 \). Thus these two saddle points give the same contribution. The proof can be repeated between the four saddle points. Finally:

\[
N = \frac{-1}{4(1-k_1^2)(1-k_2^2)\sqrt{2\pi|\cos \vartheta_t|}} \sum_{P \in \mathbb{N}} \frac{N_P}{d_P^{1/2}}
\]

with \( N_P \) given by (35).

The same analysis can be performed for the numerator of the propagator. One has simply to take into account the fact that the insertion of \( \frac{k_1k_2}{4\cos \vartheta_t}(a_{\epsilon_1} + b_{\epsilon_1}/d_{j_1} + c_{\epsilon_1}/d_{j_1}^2) \) involves three different powers of \( d_{j_1} \). To perform the sums over the signs \( \epsilon_1, \epsilon_2 \), first notice, like for the denominator, that flipping all of them three transforms the coefficients into its complex conjugate. Then, restricting attention to a fixed value of the product \( \epsilon_1\epsilon_2\epsilon_{12} \), it is easy to check that the derivative of \( a, b \) and \( c \) w.r.t. \( \phi_1 \) evaluated at \( \phi_1 = \pi - \alpha_1 \) is equal to \((-1)^{p_1} \) times that evaluated at \( \phi_1 = \alpha_1 \), while flipping \( \epsilon_1 \) and \( \epsilon_{12} \), with \( p_1 \) being the number of derivatives w.r.t. \( \phi_1 \). The same is true for \( \phi_2 \). Thus, we can reproduce the previous argument showing that the four saddle points give the same contribution. This leads us to:

\[
W_{122} = \frac{-k_1k_2}{16(1-k_1^2)(1-k_2^2)\cos^2 \vartheta_t \sqrt{2\pi|\cos \vartheta_t|}} N \cdot \frac{1}{d_{j_1}^{3/2}} \left\{ \sum_{i,j=1,2} \frac{\partial_{P}^2 a_{i,j} A_{i,j}^{-1}}{d_{j_1}^{1/2}} + \sum_{P \geq 2} \frac{W_P}{d_{j_1}^{1/2}} \right\}
\]

with \( W_P \) given by (36).

**APPENDIX B: DETAILS FOR THE EXPANSION OF THE 6J-SYMBOL**

Let us compute the generating function:

\[
Z^{(6j)}(J) = \int dX \ e^{-\frac{\pi \cdot j}{2} X H(t) X J X} = \int dX \ e^{-\frac{\pi \cdot j}{2} X H(t) X J X}
\]

with \( H(t) = \frac{2\epsilon_{12}}{\sin \vartheta_t} \left( \begin{array}{ccc} \epsilon_1 \epsilon_2 (1-k_1^2) \cos \vartheta_t & \sqrt{1-k_1^2} \sqrt{1-k_2^2} & 0 \\ \sqrt{1-k_2^2} \sqrt{1-k_1^2} & \epsilon_1 \epsilon_2 (1-k_2^2) \cos \vartheta_t & 0 \\ 0 & 0 & -\epsilon_1 \epsilon_2 \left[ 1 - (k_1^2 + k_2^2) \right] \cos \vartheta_t \end{array} \right) \).

Taking care of the fact that \( H(t) \) has purely imaginary coefficients, one has:

\[
Z^{(6j)}(J) = Z^{(6j)}(J) e^{rac{\pi}{2} \cdot j H^{-1}(t) J}
\]

with

\[
Z^{(6j)}(J) = \frac{\pi^2}{\sqrt{1-k_1^2 \sqrt{1-k_2^2}} \sqrt{2\pi V_t}} e^{-i \epsilon_1 \epsilon_2 \epsilon_{12} \varphi_t},
\]

and

\[
H^{-1}(t) = \frac{i\epsilon_{12}}{2} \left( \begin{array}{ccc} \frac{\epsilon_1 \epsilon_2}{1-k_1^2} \cot \vartheta_t & \sqrt{1-k_1^2} \sqrt{1-k_2^2} \sin \vartheta_t & 0 \\ \sqrt{1-k_2^2} \sqrt{1-k_1^2} \sin \vartheta_t & \frac{\epsilon_1 \epsilon_2}{1-k_2^2} \cot \vartheta_t & 0 \\ 0 & 0 & \frac{\epsilon_1 \epsilon_2}{1-(k_1^2+k_2^2)} \tan \vartheta_t \end{array} \right),
\]

where the volume \( V_t \) is given by \( (51) \). Using \( (49) \), we obtain an expression similar to \( (A8) \):

\[
\left\{ \begin{array}{ccc} j_1 & j_1 & j_1 \\ j_2 & j_2 & j_2 \end{array} \right\} = \frac{1}{8 \sqrt{1-k_1^2 \sqrt{1-k_2^2}} \sqrt{2\pi V_t}} \sum_{P \geq 0} \tilde{C}_P \quad (B6)
\]

with the series coefficients in term of the Hessian:

\[
\tilde{C}_P = \sum_{n=0}^{2P} \frac{1}{(2(P+n)!)} \sum_{\epsilon_1, \epsilon_2, \epsilon_{12}} (i \epsilon_{12})^n e^{-i \epsilon_1 \epsilon_2 \epsilon_{12} \varphi_t} \sum_{\beta \in \{1,2,3\}} \partial_\beta^{(2(P+n))} (f_{\omega_{\beta}}(\varphi_t))_{\epsilon_1 \varphi_2} \quad (B7)
\]
We are now in position to repeat the arguments of the previous section. The symmetries of the functions $f$, $i\epsilon_{12}\omega$, combined with those of $\tilde{H}_{\{1\}}$ imply that the four saddle points contribute the same. Moreover, the two configurations of signs corresponding to a given saddle point, which are related by flipping $\epsilon_1$, $\epsilon_2$, and $\epsilon_{12}$, are related by complex conjugation. The coefficient $\tilde{C}_p$ is thus completely determined by the saddle point $(\alpha_1, \alpha_2, 0)$ with $\epsilon_{12} = -\epsilon_1 = -\epsilon_2 = 1$. Writing $H_{\{1\}}^{-1} = i\epsilon_{12}H_{\{1\}}^{-1}$, we have that $H_{\{1\},\beta}^{-1} = (i\epsilon_{12})^{P+n}H_{\{1\},\beta}$, and:

$$\tilde{C}_p = 8 \Im \left(i^{P-2}\frac{\pi}{2} e^{-i(S_R + \frac{\pi}{4})} \sum_{n=0}^{2P} \frac{(-1)^{n+\frac{P}{2}}}{(2P+2n)! n!} \sum_{\beta \in \{1,2,3\}^{2(P+n)}} \partial_{\beta}^2(f\omega^n)|_{(\alpha_1,\alpha_2,0)} H_{\beta}^{-1} |_{\epsilon_{12}=-\epsilon_1=-\epsilon_2=1} \right) \tag{B8}$$

It is then clear that $\Re \left(i^{P-2}\frac{\pi}{2} e^{-i(S_R + \frac{\pi}{4})} \sum_{n=0}^{2P} \frac{(-1)^{n+\frac{P}{2}}}{(2P+2n)! n!} \sum_{\beta \in \{1,2,3\}^{2(P+n)}} \partial_{\beta}^2(f\omega^n)|_{(\alpha_1,\alpha_2,0)} H_{\beta}^{-1} |_{\epsilon_{12}=-\epsilon_1=-\epsilon_2=1} \right)$ is simply $\cos (S_R + \frac{\pi}{4})$ for even $P$, and $\sin (S_R + \frac{\pi}{4})$ for odd $P$.

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