POSITIVE SCALAR CURVATURE AND CONNECTED SUMS

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ABSTRACT. Let $N$ be a closed enlargeable manifold in the sense of Gromov-Lawson and $M$ a closed spin manifold of equal dimension, a famous theorem of Gromov-Lawson states that the connected sum $M \# N$ admits no metric of positive scalar curvature. We present a potential generalization of this result to the case where $M$ is nonspin. We use index theory for Dirac operators to prove our result.

0. Introduction

It has been an important subject in differential geometry to study when a smooth manifold carries a Riemannian metric of positive scalar curvature (cf. [6, Chap. IV]). A famous theorem of Gromov and Lawson [4], [5] states that an enlargeable manifold (in the sense of [5]) does not carry a metric of positive scalar curvature.

Definition 0.1. (Gromov-Lawson [5, Definition 5.5]) One calls a closed manifold $N$ (carrying a metric $g_{TN}$) an enlargeable manifold if for any $\epsilon > 0$, there is a covering manifold $\hat{N}_\epsilon \to N$, with $\hat{N}_\epsilon$ being spin, and a smooth map $f : \hat{N}_\epsilon \to S^{\dim N}(1)$ (the standard unit sphere), which is constant near infinity and has non-zero degree, such that for any $X \in \Gamma(T\hat{N}_\epsilon)$, $|f_*(X)| \leq \epsilon |X|$.

It is clear that the enlargeability does not depend on the metric $g_{TN}$.

We assume from now on that $N$ is a closed enlargeable manifold. Let $M$ be a closed manifold such that there is a closed codimension two submanifold $W \subset M$ such that $M \setminus W$ is spin. Without loss of generality, we assume that $\dim M = \dim N = n$ is even. Let $h^{TN}$ be a metric on $TN$.

We fix a point $p \in N$. For any $r \geq 0$, let $B^N_p(r) = \{ y \in N : d(p,y) \leq r \}$. Let $a_0 > 0$ be a fixed sufficiently small number. Then the connected sum $M \# N$ can be constructed so that the hypersurface $\partial B^N_p(a_0)$, which is the boundary of $B^N_p(a_0)$, cuts $M \# N$ into two parts: the part $N \setminus B^N_p(a_0)$ and the rest part coming from $M$ (by attaching the boundary of a ball in $M \setminus W$ to $\partial B^N_p(a_0)$).

Let $\varphi : M \# N \to [0,1]$ be an arbitrary smooth function such that $\varphi \equiv 1$ on $N \setminus B^N_p(a_0)$ and $\text{Supp}(\varphi) \subseteq M \# N \setminus W$. The main result of this paper can be stated as follows.

Theorem 0.2. There is no metric $g^{TM \# N}$ on $T(M \# N)$ such that the associated scalar curvature $k^{TM \# N}$ verifies the following inequality on $\text{Supp}(\varphi)$,

$$c - 6|d\varphi|^2_{g^{TM \# N}} \geq \max \left\{ 0, \frac{3c}{2} - \frac{k^{TM \# N}}{4} \right\}$$

for some constant $c > 0$. 
When $M$ is spin, one can take $W = \emptyset$, $\varphi \equiv 1$ on $M \# N$ and $c > 0$ small enough to recover the theorem of Gromov-Lawson \cite{4, 5} mentioned at the beginning.

Our proof of Theorem 0.2 is index theoretic and is inspired by \cite{8}, where a new proof of the above mentioned Gromov-Lawson theorem is given without using index theorems on noncompact manifolds. The details will be carried out in Section 1.

1. Proof of Theorem 0.2

Assume there is a metric $g^{T(M \# N)}$ on $T(M \# N)$ such that (0.1) holds for $c = \alpha^2 > 0$.

For any $\epsilon > 0$, let $\pi : \hat{N}_\epsilon \to N$ be a covering manifold verifying Definition 0.1, carrying lifted geometric data from that of $N$. Let $a_0 > 0$ be small enough so that for any $p', q' \in \pi^{-1}(p)$ with $p' \neq q'$, $B_{p'}^N(4a_0) \cap B_{q'}^N(4a_0) = \emptyset$. It is clear that one can choose $a_0 > 0$ not depending on $\epsilon$.

Let $h : N \to N$ be a smooth map such that $h = \text{Id}$ on $N \backslash B_3^N(3a_0)$, while $h(B_2^N(2a_0)) = \{p\}$. It lifts to a map $\hat{h} : \hat{N}_\epsilon \to \hat{N}_\epsilon$ verifying that $\hat{h} = \text{Id}$ on $\hat{N}_\epsilon \setminus \bigcup_{p' \in \pi^{-1}(p)} B_{p'}^N(3a_0)$, while for any $p' \in \pi^{-1}(p)$, $\hat{h}(B_{p'}^N(2a_0)) = \{p'\}.$

Let $f : \hat{N}_\epsilon \to S^n(1)$ be as in Definition 0.1. Set $\hat{f} = f \circ \hat{h} : \hat{N}_\epsilon \to S^n(1)$. Then $\deg(\hat{f}) = \deg(f)$ and there is a constant $c' > 0$ such that for any $X \in \Gamma(T\hat{N}_\epsilon)$, one has

\begin{equation}
|\hat{f}_* (X)| \leq c' |X|.
\end{equation}

To simplify the presentation, we assume first that each $\hat{N}_\epsilon$ is compact, i.e., $N$ is a compactly enlargeable manifold. Since $M \setminus W$ is spin, one can construct a compact spin manifold with boundary $M_W \subset M \setminus W$ such that $\partial M_W \subset M \setminus \text{Supp}(\varphi)$. Let $M'_W$ be another copy of $M_W$. Then one gets a closed spin manifold by gluing $M_W$ and $M'_W$ along the boundary. We denote the resulting double by $\tilde{M}_W$. Then one can extend the connected sum $M_W \# N$ to $\tilde{M}_W \# N$ obviously. It lifts naturally to $\tilde{N}_\epsilon$ where near each $p' \in \pi^{-1}(p)$, we do the lifted connected sum. We denote the resulting manifold by $\tilde{M}_W \# \tilde{N}_\epsilon$. Clearly, the metric $g^{T(M \# N)}$ induces a metric $g^{T(M_W \# N)}$ such that $g^{T(M_W \# N)|_{M_W \# N}} = g^{T(M \# N)|_{M_W \# N}}$. Let $k^{T(M_W \# N)}$ be the associated scalar curvature. They determine the corresponding metric $g^{T(M_W \# N)}$ and scalar curvature $k^{T(M_W \# N)}$ on $\tilde{M}_W \# \tilde{N}_\epsilon$.

The cut-off function $\varphi$ extends to $\tilde{M}_W \# N$ by setting $\varphi(M'_W) = 0$. It lifts to $\tilde{M}_W \# \tilde{N}_\epsilon$ obviously and we still denote the lifting by $\varphi$.

We extend $\hat{f} : \hat{N}_\epsilon \to S^n(1)$ to $\tilde{f} : \tilde{M}_W \# \tilde{N}_\epsilon \to S^n(1)$ by setting $\tilde{f}(\tilde{M}_W \# B_{p'}(4a_0)) = f(p')$ for any $p' \in \pi^{-1}(p)$.

Following \cite{4, 5}, let $(E_0, g^{E_0})$ be a Hermitian vector bundle on $S^n(1)$ carrying a Hermitian connection $\nabla^{E_0}$ such that

\begin{equation}
\langle \text{ch} (E_0) , [S^n(1)] \rangle \neq 0.
\end{equation}

Let $(E_1 = C_k|_{S^n(1)}, g^{E_1}, \nabla^{E_1})$, with $k = \text{rk}(E_0)$, be the canonical Hermitian trivial vector bundle on $S^n(1)$.

\textsuperscript{1}Here and in what follows, the involved balls are determined by $h^{TN}$. 
For any \( p' \in \pi^{-1}(p) \), let \( v_{f(p')} : \Gamma(E_0|_{f(p')}) \rightarrow \Gamma(E_1|_{f(p')}) \) be an isometry. Let \( v^{*}_{f(p')} : \Gamma(E_1|_{f(p')}) \rightarrow \Gamma(E_0|_{f(p')}) \) be the adjoint of \( v_{f(p')} \) with respect to \( g^{E_0|_{f(p')}} \) and \( g^{E_1|_{f(p')}} \). Set
\[
V_{f(p')} = v_{f(p')} + v^{*}_{f(p')}. \tag{1.3}
\]

Let \((\xi, g^\xi, \nabla^\xi) = (\xi_0 \oplus \xi_1, g^{\xi_0} \oplus g^{\xi_1}, \nabla^{\xi_0} \oplus \nabla^{\xi_1}) = (\hat{f}^*E_0 \oplus \hat{f}^*E_1, \hat{f}^*g^{E_0} \oplus \hat{f}^*g^{E_1}, \hat{f}^*\nabla^{E_0} \oplus \hat{f}^*\nabla^{E_1}) \) be the \( \mathbb{Z}_2 \)-graded Hermitian vector bundle with Hermitian connection over \( \hat{M}_W \# \hat{N}_e \). Let \( R^\xi = (\nabla^\xi)^2 \) be the curvature of \( \nabla^\xi \).

Let \( D^\xi : \Gamma(S(T(\hat{M}_W \# \hat{N}_e)) \otimes \xi) \rightarrow \Gamma(S(T(\hat{M}_W \# \hat{N}_e)) \otimes \xi) \) be the canonical (twisted) Dirac operator (cf. [6]) associated to \( (T(\hat{M}_W \# \hat{N}_e), g^{T(\hat{M}_W \# \hat{N}_e)}) \) and \((\xi, g^\xi, \nabla^\xi) \). Let \( D^\xi_+ : \Gamma((S(T(\hat{M}_W \# \hat{N}_e)) \otimes \xi)_+) \rightarrow \Gamma((S(T(\hat{M}_W \# \hat{N}_e)) \otimes \xi)_+) \) be the obvious restrictions, where \((S(T(\hat{M}_W \# \hat{N}_e)) \otimes \xi)_+ = S_+(T(\hat{M}_W \# \hat{N}_e)) \otimes \xi_0 \oplus S_-(T(\hat{M}_W \# \hat{N}_e)) \otimes \xi_1 \), while \((S(T(\hat{M}_W \# \hat{N}_e)) \otimes \xi)_- = S_-(T(\hat{M}_W \# \hat{N}_e)) \otimes \xi_0 \oplus S_+(T(\hat{M}_W \# \hat{N}_e)) \otimes \xi_1 \). By the Atiyah-Singer index theorem [11] (cf. [3] and [5]), one has
\[
\text{ind} \left( D^\xi_+ \right) = \left\langle A \left( T \left( \hat{M}_W \# \hat{N}_e \right) \right) \left( \text{ch} \left( \xi_0 \right) - \text{ch} \left( \xi_1 \right) \right), \left[ \hat{M}_W \# \hat{N}_e \right] \right\rangle = \left( \text{deg}(f) \right) \left( \text{ch} \left( E_0 \right), [S^\alpha(1)] \right). \tag{1.4}
\]

Following [22] p. 115, let \( \varphi_1, \varphi_2 : \hat{M}_W \# \hat{N}_e \rightarrow [0, 1] \) be defined by
\[
\varphi_1 = \frac{\varphi}{(\varphi^2 + (1 - \varphi)^2)^{\frac{1}{2}}}, \quad \varphi_2 = \frac{1 - \varphi}{(\varphi^2 + (1 - \varphi)^2)^{\frac{1}{2}}}. \tag{1.5}
\]
Then \( \varphi_1^2 + \varphi_2^2 = 1 \).

Recall that \( \alpha^2 = c > 0 \). Let \( D^\xi_\alpha : \Gamma(S(T(\hat{M}_W \# \hat{N}_e)) \otimes \xi) \rightarrow \Gamma(S(T(\hat{M}_W \# \hat{N}_e)) \otimes \xi) \) be the deformed operator defined by
\[
D^\xi_\alpha = D^\xi + \alpha \sum_{p' \in \pi^{-1}(p)} \varphi_2 V_{p'}, \tag{1.6}
\]
where \( V_{p'} = \hat{f}^{-1}(V_{f(p')}) \) lives on the lift of \( M_W \# B_p(4a_0) \) near \( p' \).

From (1.6), it is easy to verify that
\[
(D^\xi_\alpha)^2 = (D^\xi)^2 + \alpha \sum_{p' \in \pi^{-1}(p)} \langle c(d\varphi_2) V_{p'}, s \rangle + \alpha^2 \|\varphi_2 s\|^2, \tag{1.7}
\]
Thus for any \( s \in \Gamma(S(T(\hat{M}_W \# \hat{N}_e)) \otimes \xi) \), we have
\[
\|D^\xi_\alpha s\|^2 = \|D^\xi s\|^2 + \alpha \sum_{p' \in \pi^{-1}(p)} \langle c(d\varphi_2) V_{p'}, s \rangle + \alpha^2 \|\varphi_2 s\|^2 = \|\varphi_1 D^\xi s\|^2 + \|\varphi_2 D^\xi s\|^2 + \alpha \sum_{p' \in \pi^{-1}(p)} \langle c(d\varphi_2) V_{p'}, s \rangle + \alpha^2 \|\varphi_2 s\|^2. \tag{1.8}
\]

By direct computation we have for \( i = 1, 2 \) that
\[
\|\varphi_i D^\xi s\|^2 = \|D^\xi (\varphi_i s)\|^2 - \|d\varphi_i |s\|^2 - \left\langle D^\xi s, \frac{c(d\varphi_i^2)}{2} s \right\rangle - \left\langle \frac{c(d\varphi_i^2)}{2} s, D^\xi s \right\rangle. \tag{1.9}
\]
By (1.8) and (1.9), we have

\[ (1.10) \]

\[
\| D_{\alpha}^s \|^2 = \sum_{i=1}^{2} (\| D_{\alpha}^{s_i}(\varphi_1 s) \|^2 - \| d\varphi_1 \| s \|^2) + \alpha \sum_{p'\in\pi^{-1}(p)} \langle c (d\varphi_2) V_{p'} s, s \rangle + \alpha^2 \| \varphi_2 s \|^2.
\]

By (1.5), we have

\[ (1.11) \]

\[ d\varphi_1 = \frac{\varphi_2 d\varphi}{\varphi^2 + (1-\varphi)^2}, \quad d\varphi_2 = -\frac{\varphi_1 d\varphi}{\varphi^2 + (1-\varphi)^2}. \]

Let \( e_1, \ldots, e_n \) be an orthonormal basis of \((T(\hat{M}_W \# \hat{N}_e), g^{T(\hat{M}_W \# \hat{N}_e)})\). By (1.11), (1.10), (1.11), the Lichnerowicz formula [7] (cf. [6]) and proceed as in [5], one has

\[ (1.12) \]

\[
\| D_{\alpha}^s \|^2 = -\langle \Delta (\varphi_1 s), \varphi_1 s \rangle + \left\langle \frac{k^{T(\hat{M}_W \# \hat{N}_e)}}{4} \varphi_1 s, \varphi_1 s \right\rangle + \| D_{\alpha}^{s_2}(\varphi_2 s) \|^2
\]

\[
- \left\| \frac{|d\varphi|}{\varphi^2 + (1-\varphi)^2} \right\|^2 - \alpha \sum_{p'\in\pi^{-1}(p)} \langle \varphi_1 c(d\varphi) V_{p'}^2 s, s \rangle + \alpha^2 \| \varphi_2 s \|^2
\]

\[
+ \frac{1}{2} \sum_{i,j=1}^{n} \langle c(e_i) c(e_j) R^s(e_i, e_j) s, s \rangle
\]

\[
\geq \left\langle \left( \frac{k^{T(\hat{M}_W \# \hat{N}_e)}}{4} \varphi_1^2 + \alpha^2 \varphi_2^2 \right) - \frac{|d\varphi|^2}{(\varphi^2 + (1-\varphi)^2)^2} - \alpha \sum_{p'\in\pi^{-1}(p)} \frac{\varphi_1 c(d\varphi) V_{p'}^2}{\varphi^2 + (1-\varphi)^2} \right\rangle s, s\right\rangle
\]

\[
+ \frac{1}{2} \sum_{i,j=1}^{n} \langle c(e_i) c(e_j) \hat{f}_s^*(\nabla E_0) \left( \hat{f}_s(e_i), \hat{f}_s(e_j) \right) \rangle s, s \rangle
\]

\[
\geq \left\langle \left( \frac{k^{T(\hat{M}_W \# \hat{N}_e)}}{4} \varphi_1^2 + \alpha^2 \varphi_2^2 \right) - \frac{\alpha^2 |\varphi_1|^2}{2} \right\rangle_{\text{Supp}(d\varphi)} + \frac{3}{2} \left( |d\varphi|^2 \right) \right\rangle s, s \rangle
\]

\[
+ \langle O \left( \varepsilon^2 \right) s, s \rangle \right\rangle_{\hat{N}_e \cup \cup_{p'\in\pi^{-1}(p)} B^{g^s}_{\varepsilon^2}(a_0)}.
\]

For any \( x \in \text{Supp}(d\varphi) \), if \( \frac{k^{T(\hat{M}_W \# \hat{N}_e)}}{4} - \frac{3\alpha^2}{2} \geq 0 \) at \( x \), then one has at \( x \) that, in view of (1.11),

\[ (1.13) \]

\[
\frac{k^{T(\hat{M}_W \# \hat{N}_e)}}{4} \varphi_1^2 + \alpha^2 \varphi_2^2 - \frac{\alpha^2 |\varphi_1|^2}{2} \right\rangle_{\text{Supp}(d\varphi)} - 6|d\varphi|^2
\]

\[
= \left( \frac{k^{T(\hat{M}_W \# \hat{N}_e)}}{4} - \frac{3\alpha^2}{2} \right) \varphi_1^2 + \alpha^2 - 6|d\varphi|^2 \geq 0,
\]
while if \( \frac{kT(\tilde{M}_W \# \tilde{N}_e)}{4} - \frac{3a^2}{2} \leq 0 \) at \( x \), then by (0.1), one has at \( x \) that

\[
(1.14) \quad \frac{kT(\tilde{M}_W \# \tilde{N}_e)}{4} \varphi_1^2 + \alpha^2 \varphi_2^2 - \frac{\alpha^2 \varphi_1^2}{2} \bigg|_{\text{Supp}(d\varphi)} - 6|d\varphi|^2
\]

\[
= \left(3\alpha^2 - \frac{kT(\tilde{M}_W \# \tilde{N}_e)}{4}\right) \varphi_2^2 + \frac{kT(\tilde{M}_W \# \tilde{N}_e)}{4} - \frac{\alpha^2}{2} - 6|d\varphi|^2 \geq 0.
\]

On the other hand, (0.1) implies that

\[
(1.15) \quad \frac{kT(\tilde{M}_W \# \tilde{N}_e)}{4} \geq \frac{\alpha^2}{2}
\]

on \( \pi^{-1}(N \setminus B_p(a_0)) = \tilde{N}_e \setminus \cup_{\nu \in \pi^{-1}(p)} B_{\nu}^{\tilde{N}_e}(a_0) \), on which \( \varphi \equiv 1 \).

From (1.12)-(1.15), we see that if (0.1) holds for \( c = \alpha^2 \), then one has

\[
(1.16) \quad \|D_{\alpha}^s\|^2 \geq \left\langle \left(\frac{\alpha^2}{2} + O(\epsilon^2)\right) s, s \right\rangle_{\tilde{N}_e \setminus \cup_{\nu \in \pi^{-1}(p)} B_{\nu}^{\tilde{N}_e}(a_0)},
\]

which implies (when \( \epsilon > 0 \) is small enough) \( \ker(D_{\alpha}^s) = \{0\} \) (cf. [3, Theorem 8.2]), which contradicts (1.4) where the right hand side is nonzero. This completes the proof of Theorem 0.2 for the case where \( N \) is a compactly enlargeable manifold.

For the general case where \( \tilde{N}_e \) is noncompact, one can combine the above arguments with the method in [8] to complete the proof of Theorem 0.2. We leave the details to the interested reader.

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