Weighted $L^2$-norms of Gegenbauer polynomials

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Abstract. We study integrals of the form

$$\int_{-1}^{1} (C_n^{(\lambda)}(x))^2 (1-x)^\alpha (1+x)^\beta \, dx,$$

where $C_n^{(\lambda)}$ denotes the Gegenbauer-polynomial of index $\lambda > 0$ and $\alpha, \beta > -1$. We give exact formulas for the integrals and their generating functions, and obtain asymptotic formulas as $n \to \infty$.

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1. Introduction

Let $(p_n)_{n \in \mathbb{N}_0}$ be a sequence of orthogonal polynomials with respect to some weight function $\tilde{w}$ on the interval $I$ (see [29]). Integrals of the form

$$\int_I p_n^2(x) \, w(x) \, dx,$$

where $w$ is again a weight function on $I$, have occurred in different contexts. Of course, the case when $w \neq \tilde{w}$ is the interesting one.

Such integrals for Legendre, associated Legendre and Gegenbauer polynomials occur in explicit computations of angular momentum in classical as well as quantum mechanics (see [12]). Based on this interest in these computations there exists an extensive literature in a physics context (see for instance [17,22,24,30]).

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More general integrals of this form involving Jacobi polynomials occur in
the context of generalisations of Stolarsky’s invariance principle (see [28]) to
projective spaces as in [26,27]. In these cases the constellation of parameters is
so that the integrals can be expressed in closed form involving gamma functions
and factorials.

Determinantal point processes (see [15]) have been introduced also with a
strong motivation from physics; they are used to model Fermionic particles.
Since then they have become the object of mathematical research from various
perspectives. One aspect that makes these processes interesting is their
built-in repulsion between different point, which results in better distribution
properties of the sample points as compared to i.i.d. points. Also, as a special
feature of these processes the computation of expectations of discrete energy
expressions (for a comprehensive introduction and collection of recent results
see [5])

$$\sum_{i \neq j} f(\|x_i - x_j\|)$$

is computationally feasible. Here \(f\) is some potential depending only on the
distance of two points. In many cases these computations lead to integrals of
the form (1) (see [1,3,4,6]).

A further probabilistic model that leads to the study of integrals of the
form (1) was investigated in [7]. Here the Gaussian random field on the sphere
\(S^2\) given by

$$f_\ell(x) = \sqrt{\frac{4\pi}{2\ell + 1}} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x)$$

is studied. Here \((Y_{\ell m})_{m=-\ell}^{\ell}\) is an orthonormal base of the space of spherical
harmonics of degree \(\ell\) (see [20]) and \((a_{\ell m})_{m=-\ell}^{\ell}\) are independent Gaussian
random variables with mean 0 and variance 1. Then the asymptotic study
of the distribution of the Euler-Poincaré characteristic of the random field \(f_\ell\)
involves inter alia integrals of the form (1).

We took this as a motivation to provide a general study of such integrals,
where \((p_n)_n\) are Gegenbauer polynomials, and \(w(x)\) are Gegenbauer or Jacobi
weights. A special case has been studied in [13].

Outline of the paper In Sect. 2, we provide notations and collect frequently
used facts. In Sect. 3, we define the integral and present explicit formulas
in the most general case of Jacobi weights and give the generating function
relation. Section 4 gives a brief introduction into the method of singularity
analysis and Sect. 5 provides the Mellin-Barnes integral representations of the
generating functions for Jacobi and Gegenbauer weights. Section 6 discusses
the generic case for the Jacobi weight. Our main results are the asymptotic
series relation (11) with explicit coefficients and Theorem 4 concerning the
asymptotic leading term. Section 7 discusses the generic case for Gegenbauer
weights. Section 8 provides connection formulas for the integrals and selected non-generic cases.

2. Preliminaries

Throughout this paper we use the Gegenbauer polynomials with their standard normalisation (see [19]) given by

\[
\sum_{n=0}^{\infty} C^{(\lambda)}_n(x) z^n = \frac{1}{(1-2xz+z^2)^{\lambda}}.
\]

These polynomials are orthogonal with respect to the weight function \((1-x^2)^{\lambda-\frac{1}{2}}\) on the interval \([-1, 1]\) and normalised so that (see for instance [2])

\[
C^{(\lambda)}_n(1) = \frac{(2\lambda)_n}{n!} = \frac{1}{\Gamma(2\lambda)} \frac{\Gamma(n+2\lambda)}{\Gamma(n+1)} \sim \frac{1}{\Gamma(2\lambda)} n^{2\lambda-1} \quad \text{as } n \to \infty. \tag{2}
\]

Furthermore, the relation

\[
\int_{-1}^{1} \left( C^{(\lambda)}_n(x) \right)^2 (1-x^2)^{\lambda-\frac{1}{2}} \, dx = \frac{\sqrt{\pi}}{\Gamma(\lambda + \frac{1}{2})} \frac{\lambda}{n+\lambda} \frac{(2\lambda)_n}{n!} \tag{3}
\]

holds. We make frequent use of the Pochhammer symbol

\[
(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(n+a)}{\Gamma(a)},
\]

and the formulas

\[
(a)_{2k} = 2^{2k} \left( \frac{a}{2} \right)_k \left( \frac{a+1}{2} \right)_k, \quad (a)_{-k} = \frac{\Gamma(a-k)}{\Gamma(a)} = \frac{(-1)^k}{(1-a)_k} \frac{(-n)_k}{k!} = (-1)^k \binom{n}{k}. \tag{4}
\]

We also use the digamma function

\[
\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+x} \right), \tag{5}
\]

where \(\gamma\) is the Euler-Mascheroni constant.

The classical hypergeometric functions are given by

\[
_{p}F_{q} \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right| z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n n!} z^n
\]

for \(a_1, \ldots, a_p, b_1, \ldots, b_q \in \mathbb{C}\) and \(p \leq q + 1\). These power series allow for an analytic continuation to the slit complex plane \(\mathbb{C} \setminus [1, \infty)\). For further properties of these functions we refer to [2,18].
We will state some of our results in terms of asymptotic series (see [11]). We write
\[ f(x) \sim \sum_{k=0}^{\infty} \phi_k(x) \text{ as } x \to \infty, \]
if for all \( k \geq 0 \)
\[ \lim_{x \to \infty} \frac{\phi_{k+1}(x)}{\phi_k(x)} = 0 \]
and
\[ f(x) - \sum_{\ell=0}^{k} \phi_{\ell}(x) = O(\phi_{k+1}(x)). \]

In the statements of our results we will have sums of two and three asymptotic series, which we understand in the following way
\[
f(x) \sim \sum_{k=0}^{\infty} \phi_k(x) + \sum_{k=0}^{\infty} \chi_k(x) \\
= \phi_0(x) + \cdots + \phi_{k_1}(x) + \chi_0(x) + \cdots + \chi_{\ell_1}(x) + \phi_{k_1+1}(x) + \cdots \\
+ \phi_{k_2}(x) + \chi_{\ell_1+1}(x) + \cdots,
\]
where
\[
0 = \lim_{x \to \infty} \frac{\phi_{k_1}(x)}{\chi_0(x)} = \lim_{x \to \infty} \frac{\chi_{\ell_1}(x)}{\phi_{k_1+1}(x)} = \cdots;
\]
this means that we interlace the terms of the two series to obtain a new asymptotic series. In the situation where we use this notation the two constituting series will be such that this notation is well defined.

3. Explicit formulas and generating functions

Let \( \lambda > 0 \) and \( \alpha, \beta > -1 \). We define for non-negative integers \( n \),
\[
I_n^{(\lambda; \alpha, \beta)} := \int_{-1}^{1} \left( C_n^{(\lambda)}(x) \right)^2 (1 - x)^{\alpha} (1 + x)^{\beta} \, dx.
\]

Theorem 1. Let \( I_n^{(\lambda; \alpha, \beta)} \) be given by (6). Then we have
\[
I_n^{(\lambda; \alpha, \beta)} = 2^{\alpha+\beta+1} \frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)} \\
\times \left( \frac{2\lambda}{n!} \right)^{2} \binom{-n, n + 2\lambda, \lambda, \alpha + 1, \beta + 1}{2\lambda, \lambda + \frac{1}{2}, \frac{\alpha+\beta+2}{2}, \frac{\alpha+\beta+3}{2}} 1).
\]
Remark 1. For $\beta = \alpha = \mu - \frac{1}{2}$, the 1-balanced $5F_4$-hypergeometric polynomial reduces to a 1-balanced $4F_3$-hypergeometric polynomial

$$4F_3\left(\begin{array}{c} -n, n + 2\lambda, \lambda, \mu + \frac{1}{2} \\ 2\lambda, \lambda + \frac{1}{2}, \mu + 1 \end{array} \middle| 1 \right).$$

To simplify notation, we set

$$J_n^{(\lambda; \mu)} := I_n^{(\lambda; \mu - \frac{1}{2}, \mu - \frac{1}{2})}, \quad \lambda > 0, \mu > -\frac{1}{2}.$$ 

For $\mu = \lambda$, $4F_3$ becomes

$$3F_2\left(\begin{array}{c} -n, n + 2\lambda, \lambda \\ 2\lambda, \lambda + 1 \end{array} \middle| 1 \right),$$

hence can be computed by the Pfaff-Saalschütz theorem as

$$\frac{(\lambda)_n(-n)_n}{(2\lambda)_n(-n - \lambda)_n} = \frac{\lambda}{n + \lambda} \frac{n!}{(2\lambda)_n},$$

which, of course, reproduces the well known formula (3) for the $L^2$-norm of $C_n^{(\lambda)}$ for weight $(1 - x^2)^{\lambda - \frac{1}{2}}$ (see [2]).

Proof of Theorem 1. The result follows from [25, Eq. (16)], i.e.

$$\left(C_n^{(\lambda)}(x)\right)^2 = \left(\frac{(2\lambda)_n}{n!}\right)^2 3F_2\left(\begin{array}{c} -n, n + 2\lambda, \lambda \\ 2\lambda, \lambda + 1 \end{array} \middle| 1 - x^2 \right),$$

the relation

$$\int_{-1}^{1} (1 - x^2)^k (1 - x)^\alpha (1 + x)^\beta \, dx = 2^{2k+\alpha+\beta+1} \frac{\Gamma(k + \alpha + 1) \Gamma(k + \beta + 1)}{\Gamma(2k + \alpha + \beta + 2)},$$

and the duplication formula in (4) in order to rewrite the Pochhammer symbol $(\alpha + \beta + 2)_2k$. \qquad \square

Theorem 2. The integrals $I_n^{(\lambda; \alpha, \beta)}$ satisfy the following generating function relation

$$I_n^{(\lambda; \alpha, \beta)}(z) := \sum_{n=0}^{\infty} \frac{n!}{(2\lambda)_n} I_n^{(\lambda; \alpha, \beta)} z^n = 2^{\alpha+\beta+1} \frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 2)} \frac{1}{(1 - z)^{2\lambda}} 4F_3\left(\begin{array}{c} \lambda, \lambda, \alpha + 1, \beta + 1 \\ 2\lambda, \alpha+\beta+2, \alpha+\beta+3 \end{array} \middle| \frac{4z}{(1 - z)^2} \right).$$

Remark 2. For $\alpha = \beta = \mu - \frac{1}{2}$, we get

$$J_n^{(\lambda; \mu)}(z) := \sum_{n=0}^{\infty} \frac{n!}{(2\lambda)_n} J_n^{(\lambda; \mu)} z^n = \sqrt{\pi} \frac{\Gamma(\mu + \frac{1}{2})}{\Gamma(\mu + 1)} \frac{1}{(1 - z)^{2\lambda}} 3F_2\left(\begin{array}{c} \lambda, \lambda, \mu + \frac{1}{2} \\ 2\lambda, \mu + 1 \end{array} \middle| \frac{4z}{(1 - z)^2} \right).$$
The same right-hand side is obtained for \( \alpha = \mu + \frac{1}{2} \) and \( \beta = \mu - \frac{1}{2} \). This is obvious by the fact that

\[
\int_{-1}^{1} \left( C_n^{(\lambda)}(x) \right)^2 (1 - x)(1 - x^2)^{\mu - \frac{1}{2}} \, dx = \int_{-1}^{1} \left( C_n^{(\lambda)}(x) \right)^2 (1 - x^2)^{\mu - \frac{1}{2}} \, dx.
\]

Proof of Theorem 2. By Theorem 1,

\[
\sum_{n=0}^{\infty} \frac{n!}{(2\lambda)^n} f_{n}^{(\lambda;\alpha,\beta)} z^n = A \sum_{\ell=0}^{\infty} c_{\ell} \sum_{n=\ell}^{\infty} \frac{(2\lambda)^{n+\ell}}{(n-\ell)!} \frac{(\lambda)_{\ell}(\alpha + 1)_{\ell}(\beta + 1)_{\ell}}{(2\lambda)_{\ell}(\lambda + \frac{1}{2})_{\ell}} \frac{(\alpha+\beta+3)_{\ell}}{\ell!} z^n.
\]

Interchanging the order of summation

\[
\sum_{n=0}^{\infty} \frac{n!}{(2\lambda)^n} f_{n}^{(\lambda;\alpha,\beta)} z^n = A \sum_{\ell=0}^{\infty} \sum_{n=\ell}^{\infty} c_{\ell} \frac{(2\lambda)^{n+\ell}}{(n-\ell)!} z^n
\]

and taking into account that

\[
\sum_{n=\ell}^{\infty} \frac{(2\lambda)^{n+\ell}}{(n-\ell)!} z^n = \frac{1}{(1 - z)^{2\lambda}} (\lambda)_{\ell} \left( \lambda + \frac{1}{2} \right)_{\ell} \left( \frac{4z}{(1 - z)^2} \right)_{\ell},
\]

we arrive at the series expansion of the desired hypergeometric function. \( \square \)

4. Singularity analysis

In the last section we have found generating functions for the quantities \( f_{n}^{(\lambda;\alpha,\beta)} \) and \( J_{n}^{(\lambda;\mu)} \). In order to retrieve asymptotic information about these quantities from analytic information about the generating function at its singularity, we briefly discuss the method of singularity analysis introduced in [14]. The main advantage of this method over the classical method of Darboux (see [9,10]) is that this method can also provide asymptotic expressions for the coefficients of the generating functions in the case when the coefficients tend to 0. This difference comes from the fact that Darboux’s method uses a local approximation of the generating function inside the circle of convergence and uses the Riemann-Lebesgue-lemma to obtain an error term. Singularity analysis needs information on the behaviour of the analytic continuation to a region of the form

\[
\Delta_{\varepsilon,\phi} = \{ z \in \mathbb{C} \mid |z| < 1 + \varepsilon, |\arg(1 - z)| < \phi \}.
\]
for some $\pi > \phi > \frac{\pi}{2}$ (assuming that the radius of convergence is 1). Since in our case the generating functions have an analytic continuation to the complex plane with a branch cut connecting 1 and $\infty$, the method is readily applicable.

The main ingredient of the method is the following theorem. We adopt the notation 
\[ [z^n] f(z) = f_n \]
used in [14] for the $n$-th coefficient in the Taylor series expansion of $f(z)$
\[ f(z) = \sum_{n=0}^{\infty} f_n z^n. \]

**Theorem 3.** (Big-$O$-theorem, see [14, Theorem 1]) Assume that, with the sole exception of the singularity $z = 1$, $f(z)$ is analytic in $\Delta_{\varepsilon,\phi}$ for some $\varepsilon > 0$ and $\phi > \frac{\pi}{2}$. Assume further that as $z$ tends to 1 in $\Delta_{\varepsilon,\phi}$,
\[ f(z) = O(|1 - z|^\alpha) \]
for some real number $\alpha$. Then the $n$-th Taylor coefficient of $f(z)$ satisfies
\[ [z^n] f(z) = O(n^{-\alpha - 1}). \]

As a consequence of this theorem a local expansion around $z = 1$ of the generating function
\[ f(z) = \sum_{k=0}^{K} a_k (1 - z)^{\alpha_k} + O(|1 - z|^\beta) \]
for $\alpha_0 < \alpha_1 < \cdots < \alpha_K < \beta$ translates into an asymptotic relation for the coefficients
\[ f_n = \sum_{k=0}^{K} a_k \binom{n - \alpha_k - 1}{n} + O(n^{-\beta - 1}). \]

Each of the binomial coefficients has an asymptotic expansion in terms of powers of $n$:
\[ \binom{n - \alpha_k - 1}{n} = \frac{1}{\Gamma(-\alpha_k)} \frac{\Gamma(n - \alpha_k)}{\Gamma(n + 1)} = \frac{n^{-\alpha_k - 1}}{\Gamma(-\alpha_k)} \left\{ 1 + O\left(\frac{1}{n}\right) \right\} \text{ as } n \to \infty. \]

(7)

The paper [14] also contains more general theorems of this type suitable for more complicated asymptotic behaviour of $f(z)$ for $z \to 1$, like logarithmic singularities.
5. Mellin-Barnes formulas

In order to write the generating function $T^{(\lambda; \alpha, \beta)}(z)$ in a more tractable form, we recall the Mellin-Barnes formula for hypergeometric functions (see [2, 21]). This gives

$$
T^{(\lambda; \alpha, \beta)}(z) = \frac{2^{\alpha+\beta+1} \Gamma(2\lambda)}{\Gamma(\lambda)^2 (1-z)^{2\lambda}} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(s+\lambda) \Gamma(s+\alpha+1) \Gamma(s+\beta+1) \Gamma(-s)}{\Gamma(s+2\lambda) \Gamma(2s+\alpha+\beta+2)} \left( \frac{16z}{(1-z)^2} \right)^s ds, \quad (8)
$$

where the contour of integration is taken along the imaginary axis encircling $s = 0$ in the left half-plane so that the poles at $s = -\lambda$, $s = -\alpha - 1$, and $s = -\beta - 1$ are to the left of the contour. Similarly, we obtain

$$
J^{(\lambda; \mu)}(z) = \frac{\sqrt{\pi} \Gamma(2\lambda)}{\Gamma(\lambda)^2 (1-z)^{2\lambda}} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(s+\lambda)^2 \Gamma(s+\mu+\frac{1}{2}) \Gamma(-s)}{\Gamma(s+2\lambda) \Gamma(s+\mu+1)} \left( \frac{4z}{(1-z)^2} \right)^s ds, \quad (9)
$$

where the contour is chosen as before, this time leaving $s = -\lambda$ and $s = -\mu - \frac{1}{2}$ to the left of the contour.

6. Jacobi weights, generic case

We use the formula (8) to derive an asymptotic expansion of $T^{(\lambda; \alpha, \beta)}(z)$ around $z = 1$. This expansion is then translated into a full asymptotic expansion of $I_n^{(\lambda; \alpha, \beta)}$ in the generic case. In this case the integrand in (8) has simple poles at $-\alpha - 1 - \ell$, $-\beta - 1 - \ell$ and double poles at $-\lambda - \ell$ for $\ell \in \mathbb{N}_0$. There is no pole cancellation or pole multiplication; i.e., $\alpha - \lambda$, $\beta - \lambda$, $\alpha - \beta$, $\alpha - 2\lambda$, $\beta - 2\lambda$, $\alpha + \beta - 2\lambda$, and $\lambda$ are not integers.

Moving the contour in (8) to the left and collecting the residues at the double poles at $-\lambda - \ell$, we have

$$
\frac{\Gamma(\lambda + \frac{1}{2}) \Gamma(1+\alpha-\lambda) \Gamma(1+\beta-\lambda)}{2^{2\lambda-1-\alpha-\beta} \sqrt{\pi} \Gamma(\lambda) \Gamma(2+\alpha+\beta-2\lambda)} \times z^{-\lambda} _4F_3\left( 1-\lambda, \lambda, \frac{2\lambda-\alpha-\beta-1}{2}, \frac{2\lambda-\alpha-\beta}{2}, 1, \lambda-\alpha, \lambda-\beta \right) \left( \frac{(1-z)^2}{4z} \right) \log \frac{1}{1-z} + \text{power series in (1-z)};
$$
collecting the residues at the simple poles at $-\alpha - 1 - \ell$, we get
\[
\frac{\Gamma(\lambda + \frac{1}{2}) \Gamma(\alpha + 1) \Gamma(\lambda - \alpha - 1)^2}{2^{3\alpha+4-\beta-2\lambda} \sqrt{\pi} \ \Gamma(\lambda) \Gamma(2\lambda - \alpha - 1)} \times (1 - z)^{2+2\alpha-2\lambda} z^{-1-\alpha} \quad 4F_3 \left( \begin{array}{c} \alpha + 1, \alpha + 2 - 2\lambda, \frac{1+\alpha-\beta}{2}, \frac{2+\alpha-\beta}{2} \\ 1 + \alpha - \beta, 2 + \alpha - \lambda, 2 + \alpha - \lambda \end{array} \right) \frac{-(1-z)^2}{4z};
\]
and collecting the residues at the simple poles at $-\beta - 1 - \ell$, we obtain
\[
\frac{\Gamma(\lambda + \frac{1}{2}) \Gamma(\beta + 1) \Gamma(\lambda - \beta - 1)^2}{2^{3\beta+4-\alpha-2\lambda} \sqrt{\pi} \ \Gamma(\lambda) \Gamma(2\lambda - \beta - 1)} \times (1 - z)^{2+2\beta-2\lambda} z^{-1-\beta} \quad 4F_3 \left( \begin{array}{c} \beta + 1, \beta + 2 - 2\lambda, \frac{1+\beta-\alpha}{2}, \frac{2+\beta-\alpha}{2} \\ 1 + \beta - \alpha, 2 + \beta - \lambda, 2 + \beta - \lambda \end{array} \right) \frac{-(1-z)^2}{4z}.
\]
Thus, we arrive at
\begin{equation}
I^{(\lambda; \alpha, \beta)}(z) = \sum_{m=0}^{\infty} A_m (1 - z)^{m+2+2\alpha-2\lambda} + \sum_{m=0}^{\infty} B_m (1 - z)^{m+2+2\beta-2\lambda} + \sum_{m=0}^{\infty} D_m (1 - z)^m \log \frac{1}{1 - z} + \text{power series in } (1 - z),
\end{equation}
where
\begin{align*}
A_m := & \frac{\Gamma(\lambda + \frac{1}{2}) \Gamma(\alpha + 1) \Gamma(\lambda - \alpha - 1)^2}{2^{3\alpha+4-\beta-2\lambda} \sqrt{\pi} \ \Gamma(\lambda) \Gamma(2\lambda - \alpha - 1)} \sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} (\alpha + 2 - 2\lambda)_\ell (1 + \alpha - \beta)_\ell (\alpha + 1)_{m-\ell} (-1)^\ell (1 + \alpha - \beta)_\ell (2 + \alpha - \lambda)_\ell 2^\ell (m - 2\ell)! \quad 16^\ell, \\
B_m := & \frac{\Gamma(\lambda + \frac{1}{2}) \Gamma(\beta + 1) \Gamma(\lambda - \beta - 1)^2}{2^{3\beta+4-\alpha-2\lambda} \sqrt{\pi} \ \Gamma(\lambda) \Gamma(2\lambda - \beta - 1)} \sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} (\beta + 2 - 2\lambda)_\ell (1 + \beta - \alpha)_\ell (\beta + 1)_{m-\ell} (-1)^\ell (1 + \beta - \alpha)_\ell (2 + \beta - \lambda)_\ell 2^\ell (m - 2\ell)! \quad 16^\ell, \\
D_m := & \frac{\Gamma(\lambda + \frac{1}{2}) \Gamma(\alpha + 1 - \lambda) \Gamma(\beta + 1 - \lambda)}{2^{2\lambda-\alpha-\beta-1} \sqrt{\pi} \ \Gamma(\lambda) \Gamma(\alpha + \beta + 2 - 2\lambda)} \sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} (1 - \lambda)_\ell (2\lambda - \alpha - \beta - 1)_\ell (\lambda)_{m-\ell} (-1)^\ell (1 - \lambda)_\ell (\lambda - \beta)_\ell 2^\ell (m - 2\ell)! \quad 16^\ell.
\end{align*}

The relation (10) holds as an asymptotic relation at first. Since $I^{(\lambda; \alpha, \beta)}(z)$ has an analytic continuation to $\mathbb{C} \setminus [1, \infty)$ and satisfies a fourth order differential equation with regular singular points $0, 1, \infty$ (as a consequence of the representation in terms of hypergeometric functions), it has a power series
representation of the form (10) with radius of convergence $\geq 1$ by the Frobenius method. The asymptotic expansion has to coincide with this power series representation.

The terms power series in $(1 - z)$ correspond to a function holomorphic around $z = 1$. Since this function does not contribute to the asymptotic expansion we are aiming for, we do not work out these terms, which are slightly more elaborate than the remaining terms.

By Theorem 3 the local expansion (10) around $z = 1$ translates into an asymptotic series for the coefficients

$$I_n^{(\lambda; \alpha, \beta)} \sim (2\lambda)_n \left( \sum_{m=0}^{\infty} D_m (-1)^m \frac{m!}{n(n-1)\cdots(n-m)} + \sum_{m=0}^{\infty} A_m \frac{n + 2\lambda - 2\alpha - 3 - m}{n} + \sum_{m=0}^{\infty} B_m \frac{n + 2\lambda - 2\beta - 3 - m}{n} \right).$$

(11)

This formula provides the full asymptotic expansion for $I_n^{(\lambda; \alpha, \beta)}$ in the generic case in terms of binomial coefficients. The asymptotic expansion

$$\frac{\Gamma(n + x)}{n!} \sim n^{x-1} \left( 1 + \sum_{k=1}^{\infty} \frac{g_k(x)}{n^k} \right) \text{ as } n \to \infty$$

(12)

(in principle) allows us to rewrite (11) in terms of powers of $n$; expressions for $g_k$ in terms of Bernoulli-polynomials can be found at [8, Eqn. 5.11.13]. For instance, this would allow us to give a full asymptotic expansion of the expectation of Riesz energies of samples of determinantal processes as studied in [4] and [3].

In order to make the results more transparent and applicable, we state the asymptotic main terms as a theorem.

**Theorem 4.** Let $-1 < \alpha < \beta$ and $\lambda > 0$ be real numbers. Then

$$I_n^{(\lambda; \alpha, \beta)} = \begin{cases} 
\frac{2^{\alpha+\beta+2-4\lambda}(\alpha + 1 - \lambda)\Gamma(\alpha + 1 - \lambda)}{(\lambda - \alpha - 1)!} n^{2\lambda - 2} + O(n^\eta) & \alpha > \lambda - 1, \\
\frac{2^{\beta+2-3\lambda}}{\Gamma(\lambda)^2} n^{2\lambda - 2} \left( \log n - A(\lambda, \beta) \right) + O(n^{2\lambda - 3} \log n) + O(n^{4\lambda - 2\beta - 5}) & \alpha = \lambda - 1, \\
\frac{2^{\beta-3\alpha-3} \Gamma(\alpha + 1) \Gamma(\lambda - \alpha - 1)^2}{\Gamma(\lambda)^2 (2\lambda - \alpha - 1)!} n^{4\lambda - 2\alpha - 4} + O(n^\eta) & \alpha < \lambda - 1,
\end{cases}$$

(13)

where

$$\eta = \begin{cases} 
\max(2\lambda - 3, 4\lambda - 2\alpha - 4) & \alpha > \lambda - 1, \\
\max(2\lambda - 2, 4\lambda - 2\alpha - 5, 4\lambda - 2\beta - 4) & \alpha < \lambda - 1,
\end{cases}$$

and

$$A(\lambda, \beta) = \frac{1}{2} \left( \gamma - 4\log 2 + \psi(\beta + 1 - \lambda) + 2\psi(\lambda) \right),$$

where $\psi$ is given by (5).
Proof. The asymptotic main term in (11) is given by the first term of the first series, if $\alpha > \lambda - 1$; it is given by the first term of the second series, if $\alpha < \lambda - 1$. Except when $\alpha = \lambda - 1$, this also holds in the non-generic case, since only higher order asymptotic terms would be affected.

It remains to discuss the case $\alpha = \lambda - 1$. In this case, when taking into account a triple pole of the integrand in (8) at $-\lambda$, the generating function has the local expansion

$$I^{(\lambda; \lambda - 1, \beta)}(\lambda) = 2^{\beta - \lambda} \frac{\Gamma(\lambda + \frac{1}{2})}{\sqrt{\pi} \Gamma(\lambda)} \left( \left( \log \frac{1}{1-z} \right)^2 - \left( 3\gamma - 4 \log 2 + \psi(1 + \beta - \lambda) + 2\psi(\lambda) \right) \log \frac{1}{1-z} \right) + C + O \left( (1-z)(\log(1-z))^2 \right) + O \left( (1-z)^{(2(\beta+1)-\lambda)} \right),$$

where $C$ is a constant that will play no role in the singularity analysis. This relation translates into the asymptotic expansion

$$I^{(\lambda; \lambda - 1, \beta)}_n = \frac{(2\lambda)_n}{n!} 2^{\beta - \lambda} \frac{\Gamma(\lambda + \frac{1}{2})}{\sqrt{\pi} \Gamma(\lambda)} \left( \frac{2}{n} \sum_{k=1}^{n-1} \frac{1}{k} - \left( 3\gamma - 4 \log 2 + \psi(1 + \beta - \lambda) + 2\psi(\lambda) \right) \frac{1}{n} \right) + O \left( \frac{\log n}{n^2} \right) + O \left( \frac{1}{n^{2(\beta+1)-\lambda+1}} \right).$$

Using the asymptotic expansion

$$\sum_{k=1}^{n-1} \frac{1}{k} = \psi(n) + \gamma = \log n + \gamma + O \left( \frac{1}{n} \right),$$

we obtain the stated expression.

We remark that the leading asymptotic term in the case $\alpha = \lambda - 1$ could be obtained by taking the limit as $a \to \lambda - 1$ in

$$\frac{(2\lambda)_n}{n!} \left( D_0 + A_0 \left( \frac{n + 2\lambda - 2\alpha - 3}{n} \right) \right)$$

derived from the general asymptotics (11).
7. Gegenbauer weights, generic case

The Gegenbauer weights are a special case of Jacobi weights in the non-generic setting. We present the asymptotic evaluation of the integrals

\[ J_n^{(\lambda;\mu)} := \int_{-1}^{1} \binom{C_n^{(\lambda)}(x)}{\lambda}^2 (1 - x^2)^{\mu - \frac{1}{2}} \, dx = I_n^{(\lambda;\mu - \frac{1}{2}, \mu - \frac{1}{2})} \]  

(14)

using the more appropriate specialisation \( \alpha = \beta = \mu - \frac{1}{2} \), so that for \( \mu = \lambda \) we get back the well-known result (3).

Because of the symmetry in the Gegenbauer weight, there is a second obvious approach to a formula for \( J_n^{(\lambda;\mu)} \) based on connection formulas between \( C_n^{(\lambda)} \) and \( C_n^{(\mu)} \).

**Theorem 5.** Let \( \mu > -\frac{1}{2} \) and \( \lambda > 0 \). Let \( J_n^{(\lambda;\mu)} \) be given by (14). Then we have

\[ J_n^{(\lambda;\mu)} = \sqrt{\pi} \frac{\Gamma(\mu + \frac{1}{2})}{\Gamma(\mu + 1)} \frac{\left( (2\lambda)_n \right)^2}{n!} \, _4F_3 \left( \begin{array}{c} -n, n + 2\lambda, \lambda, \lambda + \frac{1}{2} \\ 2\lambda, \lambda + \frac{1}{2}, \mu + 1 \end{array} \right) 1; \]

alternatively, we have

\[ J_n^{(\lambda;\mu)} = \sqrt{\pi} \frac{\Gamma(\mu + \frac{1}{2})}{\mu \Gamma(\mu + 1)} \left( \frac{2\lambda}{n} \right)^2 \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(\mu + 1)_{n-k}(\lambda - \mu)^2_k}{\mu n + 2k (n + 2k)} \frac{(2\mu)_{n-2k}}{(n - 2k)!}. \]

(15)

**Proof.** The first equation is an immediate consequence of Theorem 1 after specialising \( \alpha = \beta = \mu - \frac{1}{2} \). The second expression can be obtained using the connection formula

\[ C_n^{(\lambda)}(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(\lambda)_{n-k}(\lambda - \mu)_k}{(\mu + 1)_{n-k} k!} \frac{n + \mu - 2k}{\mu} C_n^{(\mu)}(x) \]

(see [2, (7.1.11)]) and relation (3).

**Remark 3.** The two formulas for \( J_n^{(\lambda;\mu)} \) give the identity

\[ \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(\lambda)_{n-k}(\lambda - \mu)_k^2}{(\mu + 1)_{n-k}^2 k!} \frac{(2\mu)_{n-2k}}{(n - 2k)!} \]

\[ = \mu \left( \frac{(2\lambda)_n}{n!} \right)^2 \, _4F_3 \left( \begin{array}{c} -n, n + 2\lambda, \lambda, \lambda + \frac{1}{2} \\ 2\lambda, \lambda + \frac{1}{2}, \mu + 1 \end{array} \right) 1. \]

Notice that the sum on the left-hand side has only positive terms, whereas the (implicit) sum on the right-hand side is alternating. Alternatively, this identity could be proved using Zeilberger’s algorithm (see [16]). This algorithm is based on polynomial algebra and finds recurrence relations for sums involving factorials and Pochhammer symbols, if such exist. The algorithm becomes quite elaborate in our case, so we relied on the help of Mathematica and used...
the implementation [23] of this algorithm. We found that both expressions satisfy the linear recurrence relation

\[(n + 2\lambda)^2 (n + 2\lambda - \mu) J_n^{(\lambda; \mu)} - 2(n + \lambda + 1)(n^2 + 2(\lambda + 1)n + 3\lambda + 1) J_{n+1}^{(\lambda; \mu)} + (n + 2)^2 (n + \mu + 2) J_{n+2}^{(\lambda; \mu)} = 0.\]

This relation could be used to give an independent proof of the above identity.

The generating function

\[J(\lambda; \mu)(z) = \sum_{n=0}^{\infty} \frac{n!}{(2\lambda)_n} J_n^{(\lambda; \mu)} z^n\]

satisfies the differential equation

\[z^2 (z - 1)^2 y''' + (z - 1) z ((4(\lambda + 1) - \mu) z - 2(\lambda + 1) - \mu) y'' + 2 \left( (\lambda + 1) (2(\lambda + 1) - \mu) - 1 \right) z^2 - \left( 2(\lambda + 1)^2 - 1 \right) z + \lambda (\mu + 1) y' + 2\lambda ((2\lambda - \mu) z - \lambda) y = 0.\]

This is a third order differential equation with regular singular points at 0, 1, \(\infty\). The indicial equation at \(z = 1\) reads as

\[x^2 (x + 2\lambda - 2\mu - 1) = 0.\]

Thus, the fundamental solutions about 1 will take the form of a power series in \((1 - z)\), \(\log \frac{1}{1 - z}\) times a power series in \((1 - z)\), and \((1 - z)^{1+2\mu-2\lambda}\) times a power series in \((1 - z)\).

From Remark 2 we have that

\[J_n^{(\lambda; \mu)}(z) = \frac{\sqrt{\pi} \Gamma(\mu + \frac{1}{2})}{\Gamma(\mu + 1)} \frac{1}{(1 - z)^{2\lambda}} \binom{\lambda, \lambda, \mu + \frac{1}{2} \mid _{\lambda + \mu + 1}}{3F_2 \left( \frac{-4z}{(1 - z)^2} \right). (16)\]

The asymptotic behaviour of \(J_n^{(\lambda; \mu)}\) is encoded in the behaviour of the function \(J(\lambda; \mu)(z)\) around \(z = 1\) (see [14]). To obtain this local expansion we proceed as before by using (9) and shifting the line of integration to the left.

We consider the generic case for \(\lambda \notin \mathbb{Z}, \mu + 1 - \lambda \notin \mathbb{Z}, \mu + \frac{1}{2} - 2\lambda \notin \mathbb{Z},\) and \(\mu + \frac{1}{2} - \lambda \notin \mathbb{Z}.

Collecting residues at the double poles \(-\lambda - \ell, \ell \in \mathbb{N}_0\), we get

\[\frac{\Gamma(\lambda + \frac{1}{2}) \Gamma(\frac{1}{2} + \mu - \lambda)}{\Gamma(\lambda) \Gamma(1 + \mu - \lambda)} \left( z^{-\alpha} \binom{1 - \lambda, \lambda, \lambda - \mu \mid _{1 + \frac{1}{2} + \lambda - \mu} - (1 - z)^2}{4z} \right) \log \frac{1}{1 - z} + \text{power series in } (1 - z).\]
Collecting residues at the simple poles $-\mu - \frac{1}{2} - \ell$, $\ell \in \mathbb{N}_0$, we obtain
\[
\frac{\Gamma(\lambda + \frac{1}{2}) \Gamma(\lambda - \mu - \frac{1}{2})^2 \Gamma(\mu + \frac{1}{2})}{4^{1+\mu-\lambda} \sqrt{\pi} \Gamma(\lambda) \Gamma(2\lambda - \mu - \frac{1}{2})} \frac{(1-z)^{1+2\mu-2\lambda}}{z^{\mu+\frac{1}{2}}} \left(\frac{1}{3} \mu + \frac{1}{2}, \mu - \lambda, \frac{3}{2} + \mu - \lambda \right) - \frac{(1-z)^2}{4z} \right).
\]

Thus, we arrive at
\[
\mathcal{J}(\lambda; \mu)(z) = \sum_{m=0}^{\infty} A_m (1-z)^m \log \frac{1}{1-z} + \sum_{m=0}^{\infty} B_m (1-z)^{1+2\mu-2\lambda+m} + \text{power series in } (1-z),
\]

where
\[
A_m = \frac{\Gamma(\lambda + \frac{1}{2}) \Gamma(\frac{1}{2} + \mu - \lambda)}{\Gamma(\lambda) \Gamma(1+\mu-\lambda)} \sum_{\ell=0}^{\lfloor \frac{m+1}{2} \rfloor} \frac{(1-\lambda)_\ell (\lambda - \mu)_\ell (\lambda)_{m-\ell}}{(\frac{1}{2} + \lambda - \mu)_\ell \ell! (m-2\ell)!} \frac{(-1)^\ell}{4^\ell},
\]
\[
B_m = \frac{\Gamma(\lambda + \frac{1}{2}) \Gamma(\lambda - \mu - \frac{1}{2})^2 \Gamma(\mu + \frac{1}{2})}{4^{1+\mu-\lambda} \sqrt{\pi} \Gamma(\lambda) \Gamma(2\lambda - \mu - \frac{1}{2})} \sum_{\ell=0}^{\lfloor \frac{m+1}{2} \rfloor} \frac{(\frac{1}{2})_\ell (\frac{3}{2} + \mu - 2\lambda)_\ell (\mu + \frac{1}{2})_{m-\ell}}{(\frac{3}{2} + \mu - \lambda)_\ell (\frac{1}{2} + \mu - \lambda)_\ell \ell! (m-2\ell)!} \frac{(-1)^\ell}{4^\ell}.
\]

Using singularity analysis, this translates into the following asymptotic series:
\[
J_n^{(\lambda; \mu)} = \frac{(2\lambda)_n}{n!} \left( \sum_{m=0}^{\infty} (-1)^m \frac{m!}{n(n-1) \cdots (n-m)} A_m + \sum_{m=0}^{\infty} \frac{(n+2\lambda - 2\mu - 2 - m)}{n} B_m \right).
\]

(17)

Regarding the leading term, we have the following result.

**Theorem 6.** Let $\mu > -\frac{1}{2}$ and $\lambda > 0$. Then
\[
J_n^{(\lambda; \mu)} = \begin{cases} 
\sqrt{\pi} \Gamma(\mu + \frac{1}{2} - \lambda) n^{2\lambda-2} + O(n^\eta) & \mu > \lambda - \frac{1}{2}, \\
\frac{1}{2} \sqrt{2^{2\lambda-1} \Gamma(\lambda)^2} (\log n + 2 \log 2 - \psi(\lambda)) + O\left(n^{2\lambda-3} \log n\right) & \mu = \lambda - \frac{1}{2}, \\
\frac{1}{2} \sqrt{2^{2\lambda-2} \Gamma(\lambda)^2} n^{2\lambda-2} & \mu < \lambda - \frac{1}{2}, \\
\frac{1}{2} \sqrt{2^{2\lambda-1} \Gamma(\lambda)^2} (\log n + 2 \log 2 - \psi(\lambda)) + O(n^\eta) & \mu = \lambda - \frac{1}{2}, \\
\sqrt{\pi} \Gamma(\lambda - \mu - \frac{1}{2}) \Gamma(\mu + \frac{1}{2}) n^{4\lambda-2\mu-3} + O(n^\eta) & \mu < \lambda - \frac{1}{2}, \\
\end{cases}
\]

where
\[
\eta = \begin{cases} 
\max(2\lambda - 3, 4\lambda - 2\mu - 3) & \mu > \lambda - \frac{1}{2}, \\
\max(2\lambda - 2, 4\lambda - 2\mu - 4) & \mu < \lambda - \frac{1}{2}.
\end{cases}
\]

(18)
Proof. In the generic case, the asymptotic terms can be read off from the general asymptotics (17) (setting \( m = 0 \)) and using (2) and (7). The leading term (and the remainder term) is still valid in the non-generic case, except for \( \mu = \lambda - \frac{1}{2} \), when the first two terms need to be combined to allow for a cancellation of the poles of \( \Gamma(\mu + \frac{1}{2} - \lambda) \) and \( \Gamma(\lambda - \mu - \frac{1}{2}) \) as \( \mu \to \lambda - \frac{1}{2} \).

It only remains to study the case \( \mu = \lambda - \frac{1}{2} \). Then the generating function becomes

\[
\mathcal{J}^{(\lambda;\lambda-\frac{1}{2})}(z) = \frac{\sqrt{\pi} \Gamma(\lambda)}{\Gamma(\lambda+\frac{1}{2})} \frac{1}{(1-z)^{2\lambda}} \, 3F_2\left( \begin{array}{c} \lambda, \lambda, \lambda \\ 2\lambda, \lambda+\frac{1}{2} \end{array} \middle| -\frac{4z}{(1-z)^2} \right).
\]

In this case the Mellin-Barnes formula reads as

\[
\mathcal{J}^{(\lambda;\lambda-\frac{1}{2})}(z) = \frac{\sqrt{\pi} \Gamma(2\lambda)}{\Gamma(\lambda^2 (1-z)^{2\lambda})} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(s+\lambda) \Gamma(-s)}{\Gamma(s+2\lambda) \Gamma(s+\lambda+\frac{1}{2})} \left( \frac{4z}{(1-z)^2} \right)^s \, ds;
\]

where the integrand has triple poles at \( s+\lambda \in -\mathbb{N}_0 \). Shifting the line of integration to \( \Re(s) = -\lambda - 1 \) gives

\[
\mathcal{J}^{(\lambda;\lambda-\frac{1}{2})}(z) = \frac{\Gamma(\lambda+\frac{1}{2})}{\sqrt{\pi} \Gamma(\lambda)} \, z^{-\lambda} \left( \gamma - \log 2 + \psi(\lambda) - \frac{1}{2} \log \frac{4z}{(1-z)^2} \right)^2
\]

\[
+ \frac{\sqrt{\pi} \Gamma(2\lambda)}{\Gamma(\lambda^2 (1-z)^{2\lambda})} \frac{1}{2\pi i} \int_{-\lambda-1-i\infty}^{-\lambda-1+i\infty} \frac{\Gamma(s+\lambda) \Gamma(-s)}{\Gamma(s+2\lambda) \Gamma(s+\lambda+\frac{1}{2})} \left( \frac{4z}{(1-z)^2} \right)^s \, ds.
\]

The contour of integration is chosen along the vertical line \( \Re(s) = -\lambda - 1 \) with a small circular arc encircling \( -\lambda - 1 \) on the right.

The integral is \( \mathcal{O}\left((1-z)^2 \log((1-z)^2)\right) \), thus the first term is the asymptotic main term. We note that in principle a full asymptotic expansion of \( \mathcal{J}^{(\lambda;\lambda-\frac{1}{2})}(z) \) around \( z = 1 \) could be developed by shifting the line of integration further to the left. The terms originating from the residues of the triple poles at \( s = -\lambda - \ell, \ell \in \mathbb{N}_0 \), become more complicated. Thus we confine ourselves to the first term.

This gives

\[
\mathcal{J}^{(\lambda;\lambda-\frac{1}{2})}(z) = \frac{\Gamma(\lambda+\frac{1}{2})}{\sqrt{\pi} \Gamma(\lambda)} \left( \log \frac{1}{1-z} \right)^2 - 2 \left( \gamma - 2 \log 2 + \psi(\lambda) \right) \log \frac{1}{1-z} + C
\]

\[
+ \mathcal{O}\left((1-z) \log((1-z)^2)\right),
\]

where \( C \) is an explicit constant that does not influence the later results.

The method of singularity analysis (see [14]), explained in Sect. 4 and applied in the proof of Theorem 4, allows us to translate (21) into an asymptotic expressions for \( J_n^{(\lambda;\lambda-\frac{1}{2})} \) as given in the result. \( \square \)
8. Special cases

In this section we consider connection formulas and relations between the parameters $\lambda$, $\alpha$, $\beta$, and $\mu$ not covered by the generic case.

8.1. General connection formulas

A direct corollary of [25, Theorem 3] is the following result connecting $I_n^{(\lambda;\alpha,\beta)}$ and $I_k^{(\rho;\alpha,\beta)}$.

**Proposition 7.** Let $\alpha, \beta > -1$ and $\lambda, \rho > 0$. Then

$$I_n^{(\lambda;\alpha,\beta)} = \frac{(2\lambda)_n}{(n!)^2} \sum_{k=0}^{n} \binom{n}{k} (k!)^2 (k + 2\lambda) (\lambda)_k (\rho + \frac{1}{2})_k$$

$$\times 5F_4 \left( \begin{array}{c} k - n, k + n + 2\lambda, k + \lambda, k + 2\rho, k + \rho + \frac{1}{2} \\ 2k + 2\rho + 1, k + \rho, k + 2\lambda, k + \lambda + \frac{1}{2} \end{array} \right) _1 I_k^{(\rho;\alpha,\beta)}.$$

For the case $\beta - \alpha \in \mathbb{N}$, we have the following connection formula.

**Theorem 8.** Let $\alpha > -1$ and $\lambda > 0$. For $k \in \mathbb{N}$ with $k = 2m + \eta$ and $\eta \in \{0, 1\}$,

$$I_n^{(\lambda;\alpha,\alpha+k)} = \sum_{\ell=0}^{m} (-1)^\ell b_\ell J_n^{(\lambda;\ell+\alpha+\frac{1}{2})},$$

where

$$b_\ell = \sum_{\mu=\ell}^{m} \binom{2m + \eta}{2\mu} \binom{\mu}{\ell} = \binom{m}{\ell} \frac{(m + \eta)_{m-\ell}}{(\frac{1}{2} + \eta)_{m-\ell}} \times \begin{cases} 1, & \eta = 0 \\ 2m + 1, & \eta = 1. \end{cases}$$

**Remark 4.** By Theorem 8, the asymptotic expansion of $I_n^{(\lambda;\alpha,\beta)}$ when $\beta - \alpha$ is a positive integer can be derived from the asymptotic expansions of $J_n^{(\lambda;\ell+\alpha+\frac{1}{2})}$, $\ell = 0, 1, \ldots, \lfloor \frac{\beta - \alpha}{2} \rfloor$. For $\beta - \alpha$ is a negative integer, the roles of $\alpha$ and $\beta$ are interchanged.

**Proof of Theorem 8.** Using the definition of $I_n^{(\lambda;\alpha,\beta)}$, we write

$$I_n^{(\lambda;\alpha,\alpha+k)} = \int_{-1}^{1} (1 - x)^k (1 - x^2)^\alpha \left( C_n^{(\lambda)}(x) \right)^2 \, dx$$

$$= \sum_{\ell=0}^{k} \binom{k}{\ell} \int_{-1}^{1} x^\ell (1 - x^2)^\alpha \left( C_n^{(\lambda)}(x) \right)^2 \, dx.$$

The result follows by observing that for odd integers $\ell$ the integral vanishes and

$$(x^2)^\mu = (1 - (1 - x^2))^\mu = \sum_{\ell=0}^{\mu} \binom{\mu}{\ell} (-1)^\ell (1 - x^2)^\ell.$$
8.2. The case $\lambda - \mu \in \mathbb{N}$

**Theorem 9.** Let $\lambda > 0$ and $k \in \mathbb{N}$ with $k \leq \lambda$. Then $J^{(\lambda; \lambda-k)}(z)$ is a rational function with denominator $(1 - z)^{2k-1}$. As a consequence

$$J_n^{(\lambda; \lambda-k)} = \frac{(2\lambda)_n}{n!} q_k(n),$$

where $q_k$ is a polynomial of degree $2k - 2$. The generating function and the explicit formula for the coefficients are given in (26) and (27). The main term of the asymptotics is given in (28).

**Proof.** We first show that for $k \geq 1$ we obtain a rational generating function

$$J^{(\lambda; \lambda-k)}(z) = \frac{p_k(z)}{(1 - z)^{2k-1}},$$

where $p_k$ is a polynomial of degree $2k - 2$.

This can be seen by considering the differential equation satisfied by

$$z^2(z - 1)^2 y''' + z(z - 1)((3\lambda - 5k + 7)z - 3\lambda + k - 2)y''$$

$$+ 2 \left( (1 + \lambda - k)(\lambda - 4k + 5)z^2 - \left( (1 + \lambda - k)(2\lambda - 4k + 5) - 2(k - 1)^2 \right)z$$

$$+ \lambda(1 + \lambda - k) \right) y' - 2(k - 1)(2\lambda - 2k + 1)((1 + \lambda - k)z - \lambda)y = 0. \quad (22)$$

It translates into the three-term recurrence relation

$$(2 - 2k + n)(1 + 2\lambda - 2k + n)(1 + \lambda - k + n)p_n$$

$$- \left( (2 - 2k + n) \left( (2 - 2k + n) (4 + 3\lambda - k + 2n) + \lambda + k + 4\lambda k \right) + 2\lambda k \right) p_{n+1}$$

$$+ (3 - 2k + n)^2(2 + n)p_{n+2} = 0 \quad (23)$$

for the coefficients of

$$p(z) = \sum_{n=0}^{\infty} p_n (1 - z)^n.$$

Specialising $n = 2k - 2$ and $n = 2k - 3$ gives

$$p_{2k} = \lambda p_{2k-1},$$

$$(k - 1) p_{2k-2} = (\lambda + k - 2) p_{2k-3}$$

which shows that $p_{2k-1}$ can be chosen as 0, without influencing the coefficients $p_0, \ldots, p_{2k-2}$. Then $p_\ell = 0$ for $\ell \geq 2k - 1$. Thus (22) has a polynomial solution, which has to coincide with the only solution holomorphic around $z = 0$. 
In order to compute the coefficients in the partial fraction decomposition

\[ \mathcal{J}^{(\lambda;\lambda-k)}(z) = \sum_{\ell=1}^{2k-1} \frac{c_\ell}{(1-z)^\ell} \]

we use (9) and shift the line of integration to the left to line \( \Re(s) = -\lambda \); this time we choose the contour to encircle \( s = -\lambda \) on the right. Collecting the residues gives

\[
\mathcal{J}^{\lambda;\lambda-k}(z) = \frac{\sqrt{\pi} \Gamma(2\lambda)}{\Gamma(\lambda)^2 (1-z)^{2\lambda}} \frac{1}{2\pi i} \int_{-\lambda-i\infty}^{-\lambda+i\infty} \frac{\Gamma(s+\lambda)^2 \Gamma(s+\lambda-k+\frac{1}{2}) \Gamma(-s)}{\Gamma(s+2\lambda) \Gamma(s+\lambda-k+1)} \left( \frac{4z}{(1-z)^2} \right)^s ds
\]

+ \frac{\sqrt{\pi} \Gamma(2\lambda)}{\Gamma(\lambda)^2 (1-z)^{2\lambda}} \sum_{\ell=0}^{k-1} \frac{(-1)^{\ell}}{\ell!} \frac{\Gamma(k-\ell-\frac{1}{2})^2 \Gamma(\lambda-k+\ell+\frac{1}{2})}{\Gamma(\lambda+k-\ell-\frac{1}{2}) \Gamma(-\ell+\frac{1}{2})} \left( \frac{4z}{(1-z)^2} \right)^{-\lambda+k-\ell-\frac{1}{2}}.
\]

The sum of residues simplifies to

\[
\frac{\Gamma(2\lambda)}{\Gamma(\lambda)^2} \sum_{\ell=0}^{k-1} \frac{(-1)^{\ell}}{\ell!} \frac{\Gamma(k-\ell-\frac{1}{2})^2 \Gamma(\lambda-k+\ell+\frac{1}{2})}{\Gamma(\lambda+k-\ell-\frac{1}{2})} (4z)^{-\lambda+k-\ell-\frac{1}{2}} (1-z)^{2\ell-2k+1}.
\]

We already know that \( \mathcal{J}^{(\lambda;\lambda-k)}(z) \) is a rational function with denominator \((1-z)^{2k-1}\). Furthermore, the integral in (24) behaves like \( O(1) \) for \( z \to 1 \). Thus, the rational function and the coefficients in its partial fraction decomposition can be obtained from the corresponding asymptotic terms of the sum (25) for \( z \to 1 \). For this purpose we rewrite the powers of 4z in terms of the binomial series to obtain

\[
\mathcal{J}^{(\lambda;\lambda-k)}(z) = \frac{4^{k-1} \sqrt{\pi} \Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)(1-z)^{2k-1}} \sum_{r=0}^{2k-2} (1-z)^r \sum_{\ell=0}^{\lfloor \frac{r}{2} \rfloor} \frac{\left( \frac{1}{2} \right)_{k-\ell-1} \left( \frac{1}{2} \right)_{\ell}}{4^\ell (r-2\ell)! \ell! (\lambda-k-\ell+r+\frac{1}{2})_{2k-r-1}}.
\]

From this we can read off an exact formula for the coefficients

\[
J_n^{\lambda;\lambda-k} = \frac{(2\lambda)_n}{n!} \frac{4^{k-1} \sqrt{\pi} \Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)} \sum_{r=0}^{2k-2} \binom{n+2k-2-r}{n} \sum_{\ell=0}^{\lfloor \frac{r}{2} \rfloor} \frac{\left( \frac{1}{2} \right)_{k-\ell-1} \left( \frac{1}{2} \right)_{\ell}}{4^\ell (r-2\ell)! \ell! (\lambda-k-\ell+r+\frac{1}{2})_{2k-r-1}}.
\]
The asymptotic main term is given by

\[
J_n^{(\lambda; \lambda - k)} = \frac{\sqrt{\pi} \Gamma((1 - k) + \lambda)}{2^{2 \lambda - 1} \Gamma(\lambda)^2 (k - 1)! \Gamma(k + \lambda - \frac{1}{2})} n^{2 \lambda + 2k - 3} + \mathcal{O}(n^{2 \lambda + 2k - 4}). \tag{28}
\]

\[\square\]

8.3. The case \( \mu - \lambda \in \mathbb{N} \)

In this case we observe that (15) has at most \( k := \mu - \lambda \) terms and there occur some extra cancellations in the Pochhammer-symbols. We get in particular

\[
J_n^{(\lambda; \lambda)} = \frac{(2\lambda)_n}{n!} \sqrt{\pi} \Gamma(\lambda + \frac{1}{2}) \frac{1}{\Gamma(\lambda)} \frac{1}{n + \lambda}
\]

and for \( k \in \mathbb{N} \):

\[
J_n^{(\lambda; \lambda + k)} = \frac{2\pi}{4^{\lambda + k} \Gamma(\lambda)^2} \sum_{\ell = 0}^{\min(k, \lfloor \frac{n}{2} \rfloor)} \binom{k}{\ell}^2 \left( \frac{\Gamma(n - \ell + \lambda)}{\Gamma(n + k - \ell + 1 + \lambda)} \right)^2 
\times (n - 2\ell + k + \lambda) \frac{\Gamma(n + 2k - 2\ell + 2\lambda)}{\Gamma(n - 2\ell + 1)}.
\]

All terms in the sum have the same asymptotic order as \( n \to \infty \). We use

\[
\sum_{\ell = 0}^{k} \binom{k}{\ell}^2 = \binom{2k}{k} = \frac{2^{2k} \Gamma(k + \frac{1}{2})}{\sqrt{\pi} \Gamma(k + 1)}
\]

to obtain the following asymptotic formula.

**Theorem 10.** Let \( \lambda > 0 \) and \( k \in \mathbb{N} \). Then

\[
J_n^{(\lambda; \lambda + k)} = \frac{2\pi}{4^{\lambda + k} \Gamma(\lambda)^2} \binom{2k}{k} n^{2\lambda - 2} + \mathcal{O}(n^{2\lambda - 3}) \quad \text{as } n \to \infty.
\]

Of course, a full asymptotic expansion could be given with more effort.

8.4. The case \( \lambda \in \mathbb{N} \) and \( \mu, 2\mu \notin \mathbb{Z} \)

Let \( \lambda = k, k \in \mathbb{N} \). The case \( \mu \in \mathbb{Z} \) is covered above. The case when \( \mu \) is a half-integer is more involved and not done here. The setting is a non-generic case for the Gegenbauer weight. To obtain the local expansion around \( z = 1 \), we proceed as before by using (9) and shifting the line of integration to the left. Collecting the residues at the double poles in \(-k - \ell, 0 \leq \ell \leq k - 1\), we obtain

\[
\frac{\sqrt{\pi} \Gamma(\mu + \frac{1}{2})}{\Gamma(\mu + 1)} \frac{(1/2)_k (-\mu)_k}{\Gamma(k)(1/2 - \mu)_k} z^{-k} F_2 \left( \frac{1 - k, k - \mu}{1, k + 1/2 - \mu} \mid -\frac{(1 - z)^2}{4z} \right) \log \frac{1}{1 - z}
\]

+ power series in \((1 - z)\).
Due to cancellation, the integrand in (9) has simple poles in \(-k - \ell\) for \(\ell \geq k\). Collecting those residues, we obtain

\[
\frac{\sqrt{\pi} \Gamma(\mu + \frac{1}{2})}{4 \Gamma(\mu + 1)} \frac{(-\mu)_k}{k!} \frac{1}{(\frac{1}{2} - \mu)_k} (1 - z)^{2k} z^{2k} F_3 \left( \begin{array}{c} 1, 1, 2k, 2k - \mu \\ k + 1, k + 1, 2k + \frac{1}{2} - \mu \end{array} \right) \frac{(1 - z)^2}{4z}
\]

which is holomorphic about \(z = 1\) and as a power series in \((1 - z)\) will, thus, not play a role in the singularity analysis. Collecting the residues at the simple poles in \(-\mu - \frac{1}{2} - \ell, \ell \in \mathbb{N}_0\), we have

\[
2^{2k-2\mu-2} \Gamma(-\frac{1}{2} - \mu) \Gamma(\mu + \frac{1}{2}) \frac{1}{(\frac{1}{2} - \mu)_k} \frac{1}{(\frac{1}{2} - \mu)_k} \sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(1 - k)_\ell (k - \mu)_\ell (k)_m}{(k + \frac{1}{2} - \mu)\ell!} \frac{(-1)^\ell}{4^\ell}
\]

\[
\times (1 - z)^{1+2\mu-2k} z^{-\frac{1}{2} - \mu} \sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(\frac{3}{2} + \mu - 2k)_\ell (\frac{3}{2} + \mu - k)_\ell (\frac{3}{2} + \mu - k)_\ell}{(\frac{3}{2} + \mu - k)_\ell!} \frac{(-1)^\ell}{4^\ell}.
\]

Thus, we arrive at

\[
J^{(k; \mu)}(z) = \sum_{m=0}^{\infty} A_m (1 - z)^m \log \frac{1}{1 - z} + \sum_{m=0}^{\infty} B_m (1 - z)^{1+2\mu-2k+m} + \text{power series in } (1 - z),
\]

where

\[
A_m = \frac{\sqrt{\pi} \Gamma(\mu + \frac{1}{2})}{\Gamma(\mu + 1)} \frac{(-\mu)_k}{k!} \frac{1}{(\frac{1}{2} - \mu)_k} \sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(1 - k)_\ell (k - \mu)_\ell (k)_m}{(k + \frac{1}{2} - \mu)\ell!} \frac{(-1)^\ell}{4^\ell},
\]

\[
B_m = 2^{2k-2\mu-2} \Gamma(-\frac{1}{2} - \mu) \Gamma(\mu + \frac{1}{2}) \frac{1}{(\frac{1}{2} - \mu)_k} \frac{1}{(\frac{1}{2} - \mu)_k} \sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(\frac{3}{2} + \mu - 2k)_\ell (\frac{3}{2} + \mu - k)_\ell (\frac{3}{2} + \mu - k)_\ell}{(\frac{3}{2} + \mu - k)_\ell!} \frac{(-1)^\ell}{4^\ell}.
\]

Observe, that the sum in the expression for \(A_m\) has at most \(k - 1\) terms. Using singularity analysis, this translates into the following asymptotic series:

\[
J^{(k; \mu)}_n = \frac{(2k)_n}{n!} \left( \sum_{m=0}^{\infty} \frac{(-1)^m}{m! (n-m)!} A_m + \sum_{m=0}^{\infty} \frac{(n+2k-2\mu-2-m)_n}{n} B_m \right).
\]

### 8.5. Other non-generic cases

In principle, our method can be used to get the full asymptotic expansion. Due to pole cancellation and pole multiplication in the integrands of (8) and (9) depending on assumptions on interrelations between the parameters \(\lambda, \alpha, \beta,\) and \(\mu,\) and the position of the line of integration as it is moved to the left, computations are rather involved. For example, in the case when \(\lambda \in \mathbb{N}\) and \(\mu\) is a positive half-integer such that \(0 < m = \mu + \frac{1}{2} < \lambda = k,\) the
integrand in (9) has in $-m - \ell$: simple poles for $0 \leq \ell < k - m$, triple poles for $k - m \leq \ell < 2k - m$, and double poles for $\ell \geq 2k - m$.

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