LOWER BOUNDS ON THE CHROMATIC NUMBER OF RANDOM GRAPHS

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ABSTRACT. We prove that a formula predicted on the basis of non-rigorous physics arguments (Zdeborová and Krzakala: Phys. Rev. E (2007)) provides a lower bound on the chromatic number of sparse random graphs. The proof is based on the interpolation method from mathematical physics. In the case of random regular graphs the lower bound can be expressed algebraically, while in the case of the binomial random we obtain a variational formula. As an application we calculate improved explicit lower bounds on the chromatic number of random graphs for small (average) degrees. Additionally, we show how asymptotic formulas for large degrees that were previously obtained by lengthy and complicated combinatorial arguments can be re-derived easily from these new results.

MSC: 05C80

1. INTRODUCTION

1.1. Motivation and background. A most fascinating feature of combinatorics is how easy-to-state problems sometimes lead to deep and difficult mathematical challenges. The random graph colouring problem is a case in point. First mentioned in the seminal paper of Erdős and Rényi that started the theory of random graphs [23], the problem of finding the chromatic number of the binomial random graph $G(n,d/n)$ with a fixed average degree $d$ remains open to this day. It is, in fact, the single open problem posed in that seminal paper that still awaits a complete solution. Nor is the chromatic number of the random $d$-regular graph, a conceptually simpler object, known for all values of $d$. Nevertheless, the quest for the chromatic number has contributed tremendously to the development of new techniques, some of which now count among the standard tools of probabilistic combinatorics [39].

A series of important papers contributed ever tighter bounds on the chromatic number of random graphs. Straightforward first moment calculations show that for any $q \geq 3$ and for $G$ either the binomial or the random regular graph $G(n,d/n)$ or the random regular graph $G(n,d)$,

$$\chi(G) > q \quad \text{w.h.p.} \quad \text{if} \quad \log q + \frac{d}{2} \log(1 - 1/q) < 0. \quad (1.1)$$

To be precise, (1.1) is obtained by computing the expected number of $q$-colourings $\chi(G)$, which tends to zero as $n \to \infty$ if $\log q + d \log(1 - 1/q)/2 < 0$. A celebrated contribution of Achlioptas and Naor [4] shows that for $G = G(n,d/n)$,

$$\chi(G) \leq q \quad \text{w.h.p.} \quad \text{if} \quad d < 2(q-1) \log(q-1). \quad (1.2)$$

The proof hinges on the computation of the second moment of the number of $q$-colourings, which involves a delicate analytical optimisation task. Following up on work of Achlioptas and Moore [5], Kemkes, Pérez-Giménez and Wormald [28] showed that (1.2) holds for the random regular graph $G = G(n,d)$ as well. Expanding (1.1)–(1.2) asymptotically for large $q$, we find $\chi(G) > q$ if $d > (2q-1) \log q + o_q(1)$, while $\chi(G) \leq q$ if $d \leq (2q-2) \log q - 2 + o_q(1)$, with $o_q(1)$ vanishing as $q \to \infty$. A series of papers [9,10,16] improved these asymptotic bounds to

$$\chi(G) \begin{cases} 
q & \text{if} \quad d \leq (2q-1) \log q - 2 \log 2 + o_q(1), \\
q & \text{if} \quad d > (2q-1) \log q - 1 + o_q(1)
\end{cases} \quad \text{w.h.p.} \quad (1.3)$$

for both the binomial and the random regular graph. But in the absence of explicit estimates of the $o_q(1)$ error term [13] fails to render improved bounds for any specific value of $q$. Finally, several articles have been dedicated

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1 Research of the first author supported in part by DFG CO 646. Research of the third author supported by the Australian Research Council Discovery Project DP190108977.
2 Sometimes $G(n,d/n)$ is referred to as the Erdős-Rényi model. This is a slight misnomer as Erdős and Rényi [24] actually worked with a uniformly random graph with a given number of edges.
3 Throughout the paper the term $q$-colouring or just colouring refers to proper vertex colourings. That is, colours are assigned to the vertices of a graph such that no two adjacent vertices receive the same colour.
to the special case $q = 3$. For the binomial random graph the best bounds read \[22\]

\[\chi(G(n,d/n)) \begin{cases} 
\leq 3 & \text{if } d \leq 4.03 \\
> 3 & \text{if } d > 4.94 
\end{cases} \text{ w.h.p.} \tag{1.4}\]

For the random regular graph Diaz, Kaporis, Kemkes, Kirousis, Pérez and Wormold \[10\] showed that $\chi(G(n,5)) = 3$ w.h.p. if a certain optimisation problem attains its maximum at a specific point, for which they provided numerical evidence. Moreover, Shi and Wormold \[40,41\] proved analytically that $\chi(G(n,4)) = 3$, while \[14\] implies that $\chi(G(n,6)) > 3$. The proofs of all of the above lower bounds rely upon the first moment method, in some cases applied to cleverly designed random variables \[9,10,22\]. Similarly, all of the upper bounds derive from second moment arguments, with the exception of the upper bound from \[14\] and \[40,41\], which are algorithmic.

Additionally, physicists brought to bear a canny but non-rigorous technique called the ‘1RSB cavity method’ on the random graph colouring problem \[44\]. In the case of the random regular graph, the physics calculations predict an elegant formula. Let

$$
\Sigma_{d,q}(a) = \log \sum_{i=0}^{q-1} \left( \frac{1}{i+1} \right) \left( 1 - (i+1)(1-a)/q \right)^d - \frac{d}{2} \log(1 - (1-a)^2/q) \quad (a \in [0,1]) \tag{1.5}
$$

and let $a^* \in [0,1]$ be the solution to the algebraic equation

$$
\sum_{i=0}^{q-1} \left( \frac{1}{i+1} \right) \left( 1 - (i+1)(1-a)/q \right)^d - \frac{d}{2} \log(1 - (1-a)^2/q) = 0 \tag{1.6}
$$

that minimises $\Sigma_{d,q}(\cdot)$. (If there is more than one such value $a^*$, choose one arbitrarily.) Then \[44\] predicts that

$$
\chi(G(n,d)) \begin{cases} 
\leq q & \text{if } \Sigma_{d,q}(a^*) \geq 0 \\
> q & \text{if } \Sigma_{d,q}(a^*) < 0 
\end{cases} \text{ w.h.p.} \tag{1.7}
$$

There is a similarly precise, albeit more complicated, prediction as to the chromatic number of the binomial random graph. Section 1.3 contains references to related work. An outline of the proof strategy follows in Section 2.

1.2. The random regular graph. Given $d, q \geq 3$ and with $\Sigma_{d,q}(a)$ from \[15\] define

$$
\Sigma_{d,q} = \min_{0 \leq a \leq 1} \Sigma_{d,q}(a), \quad d_q = \min \{ d \geq 3 : \Sigma_{d,q} < 0 \}. \tag{1.8}
$$

Then we have the following lower bound on the chromatic number of the random regular graph.

**Theorem 1.1.** If $q \geq 3$ and $d \geq d_q$ then $\chi(G(n,d)) > q$ w.h.p.

The function $a \mapsto \Sigma_{d,q}(a)$ is differentiable and $\Sigma_{d,q}(1) = 0$. Furthermore, the calculations performed towards \[44\] eq. (35) show that $\Sigma'_{d,q}(0) < 0$. Hence, whenever the minimum value $\Sigma_{d,q}$ is negative, the minimiser $a$ must be a zero of $\Sigma'_{d,q}(\cdot)$. It follows, after some algebra, that the minimiser is a solution to \[16\]. Thus, Theorem 1.1 verifies the lower bound from \[17\], which \[44\] conjectures to be tight for all $q \geq 3$.

Of course, we can evaluate $\Sigma_{d,q}$ numerically and calculate $d_q$ for any given $q$. The first few values are displayed in Table 1. For those $q$ where $d_q$ is displayed in boldface, the new bound strictly improves over the first moment.
If \( d \) could try atoms but such a calculation seems to require computer assistance. In principle, the 4.697 bound could be sharpened by optimising over distributions with a (small) finite support, \( 1.3 \).

The binomial random graph. Locally the random regular graph is as ‘deterministic’ as it gets: for all but a bounded number of exceptional vertices, any bounded-depth neighbourhood is just a \( d \)-regular tree w.h.p. By contrast, in the binomial random graph \( G(n, d/n) \) the neighbourhoods are random, distributed as the trees generated by a Galton-Watson process with offspring distribution \( \text{Po}(d) \). The value of the chromatic number predicted by the cavity method mirrors this local non-uniformity. Indeed, while in the case of random regular graphs we obtained the scalar optimisation problem \( 1.8 \), in the binomial case we face an optimisation problem over a probability measure on the unit interval. To be precise, let \( a \) be a probability distribution on \([0, 1]\). Moreover, let \( (\alpha_i)_{i \geq 1} \) be a family of independent random variables with distribution \( a \). Additionally, let \( D \sim \text{Po}(d) \) be independent of the \( \alpha_i \). Then we define

\[
\Sigma^*_{q,d}(a) = E \left[ \log \left( \sum_{i=0}^{q-1} (-1)^i \binom{q}{i+1} \left( \prod_{h=1}^D 1 - (i+1)(1 - \alpha_h) / q \right) \right) - \frac{d}{2} E \left[ \log \left( 1 - (1 - \alpha_1)(1 - \alpha_2) / q \right) \right] \right].
\] (1.9)

Setting

\[
\Sigma^*_{q,d} = \inf_a \Sigma^*_{q,d}(a), \quad d^*_q = \inf \left\{ d > 0 : \Sigma^*_{q,d} < 0 \right\},
\] (1.10)

we obtain the following lower bound on the chromatic number.

**Theorem 1.3.** If \( q \geq 3 \) and \( d > d^*_q \) then \( \chi(G(n, d/n)) > q \) w.h.p.

Zdeborová and Krzakala predict that this bound is tight for all \( q \geq 3 \) \( 44 \).

Due to the optimisation over distributions \( a \), the value \( d^*_q \) may be hard to evaluate. The physics literature relies upon a numerical heuristic called population dynamics \( 34 \) to tackle such optimisation problems, but of course there is no general guarantee that the true optimiser will be found. Yet fortunately Theorem \( 1.3 \) shows that any distribution \( a \) yields an upper bound on \( d^*_q \), and thus a lower bound on the chromatic number. In particular, we could try atoms \( a = \delta_\alpha \) with \( \alpha \in [0, 1] \). For instance, we find that \( \Sigma_{4.697,3}(\delta_{0.25}) < 0 \), whence \( d^*_q \approx 4.697 \) obtained via population dynamics \( 44 \). In principle, the 4.697 bound could be sharpened by optimising over distributions with a (small) finite support, but such a calculation seems to require computer assistance.

Similarly, substituting a suitable atom \( a = \delta_q \) into \( 1.10 \) suffices to rederive the large-\( q \) asymptotic lower bound on the chromatic number from \( 1.3 \), originally established in \( 9 \) via a complicated first moment argument. As in the regular case we do not improve over \( 1.3 \) asymptotically for large \( q \); again, \( 1.3 \) is conjectured to be optimal up to the precise value of the term hidden in the \( o_q(1) \).

**Corollary 1.4.** If \( d > (2q - 1)\log q - 1 + o_q(1) \) then \( \chi(G(n, d/n)) > q \) w.h.p.
1.4. Related work. The history of the random graph colouring problem is long and distinguished. Improving a prior result of Matula [32], Bollobás [8] determined the chromatic number of the dense binomial random $G(n, p)$ for fixed $p \in (0, 1)$, up to a multiplicative error of $1+o(1)$. Kučera and Matula obtained the same result via a different proof [33]. Łuczak [31] extended the approach from [32, 33] to sparse random graphs. His main result shows that w.h.p. for $p = o(1)$,

$$\chi(G(n, p)) = (1 + O\left(\frac{\log^2 np}{\log np}\right)) \frac{\log np}{2np}. \quad (1.11)$$

Particularly for small edge probabilities $p$ the bound (1.11) is not quite satisfactory, as a result of Alon and Krivelevich [5] shows that the chromatic number of $G(n, p)$ is concentrated on two consecutive integers if $p \leq n^{-1/2-\Omega(1)}$.

Seizing upon techniques from [4, 5], Coja-Oghlan, Panagiotou and Steger [14] determined a set of three consecutive integers on which the chromatic number concentrates for $p \leq n^{-3/4-\Omega(1)}$. Furthermore, the aforementioned result of Achlioptas and Naor [4] determines the two integers on which $\chi(G(n, d/n))$ concentrates, when $d$ is fixed. Y et in this case it is widely conjectured that there exists a sharp threshold for $q$-colourability, i.e., that for each $q \geq 3$ there exists $d_q^* > 0$ such that $\chi(G(n, d/n)) \leq q$ if $d < d_q^*$ while $\chi(G(n, d/n)) > q$ if $d > d_q^*$. Clearly, if such a $d_q^*$ exists then the chromatic number would actually concentrate on a single integer for almost all $d \in (0, \infty)$.

Towards the sharp threshold conjecture, Achlioptas and Friedgut [1] established the existence of a non-uniform threshold sequence for every $q \geq 3$. Physics predictions [44] assert that the $q$-colourability threshold $d_q^*$ coincides with $d_q^*$ from (1.10) for all $q \geq 3$.

Concerning the random regular graph $G(n, d)$, Frieze and Łuczak [25] obtained an asymptotic bound akin to (1.11) for $d = o(n^{1/3})$, which Cooper, Frieze, Reed and Riordan [17] extended to $d = n^{1-\Omega(1)}$. Further, Krivelevich, Sudakov, Vu and Wormald [29] obtained an asymptotic formula akin to Bollobás’ result [8] for degrees $n^{6/7+\Omega(1)} \leq d \leq 0.9n$. The best prior bounds on $\chi(G(n, d))$ with fixed $d$ were stated in Section 1.1.

The physicists’ cavity method has inspired a great deal of rigorous work. Perhaps the most prominent example is the proof of the $k$-SAT threshold conjecture for large $k$ by Ding, Sly and Sun [21]. The proof of the lower bound on the $k$-SAT threshold is based on an impressive second moment argument, while the proof of the upper bound relies on the interpolation method. The way we use the interpolation method here is reminiscent of its application in [21]. Further problems in which the 1RSB cavity method has been vindicated include the independent set problem on random regular graphs [20], the regular $k$-NAESAT problem [19] and the regular $k$-SAT problem [13].

As for the history of the interpolation method itself, Guerra [27] invented the technique in order to study the free energy of the Sherrington-Kirkpatrick spin glass model. The interpolation method went on to become a mainstay of the mathematical physics of spin glasses (see, e.g., [36]). Franz and Leone [24] pioneered the use of the interpolation method for combinatorial problems. The approach was further elaborated and generalised by Panchenko and Talagrand [38], and their version of the interpolation method was applied to the $k$-SAT problem in [21]. We will use (and adapt) the Panchenko–Talagrand version as well. Moreover, an important contribution
Giants defined by Gular graph, which plays a prominent role in the present paper as well. In particular, for the random regular graph the partition function for a variety of models. These models include the Potts antiferromagnet on the random regular graph. [42, Theorem E.3] shows that the formula provided by the 1RSB cavity method yields an upper bound on the entropy of Bayati, Gamarnik and Tetali [7] applied a different variant of the interpolation method to prove, e.g., the existence of the limit limn→∞ a(G)/n of the normalised independence number of the random graph G = G(n, d) or G = G(n, d/n). This version of the interpolation method does not provide estimates of the value of such limits. Sly, Sun and Zhang [42] combined the combinatorial interpolation scheme from [7] with the interpolation arguments from [24, 38] to derive bounds on the partition functions of random regular (and uniform) hypergraphs. For instance, [24, 38] to derive bounds on the partition functions of random regular graphs in [15, 42] and for binomial random graphs in [38] are rather brisk, and since, strictly speaking, [38] does not cover the Potts model, we present the interpolation method from [38] to derive bounds on the partition functions of random regular graphs in [15, 42] and for binomial random graphs in [38, 42].

1.5. Preliminaries and notation. In order to avoid repetitions and case distinctions, throughout the paper we use the shorthand G to denote either the random regular graph G(n, d) or the binomial random graph G(n, d/n). Most of the statements and arguments in the following sections are generic and apply to either model. There are just a few steps where we will need to treat the two models separately. If G = G(n, d/n) is the binomial random graph then we let D ~ Po(d) be a Poisson variable, while in the case of the random regular graph we let D = d deterministically. In either case we let (D1)i=1 be independent copies of D.

As per common practice, we use the O(·)-notation to refer to the limit n → ∞. In our calculations we tacitly assume that n is sufficiently large for the various estimates to be valid. In addition, in Section 5 we use Oq(·)-notation to refer to the limit of large q as in Corollaries 1.2 and 1.3.

For a finite set Ω ≠ ∅ we denote by P(Ω) the set of probability distributions on Ω. We identify P(Ω) with the standard simplex in RΩ. Accordingly, P(Ω) inherits its topology from RΩ. Further, we write P2(Ω) for the space of probability measures on P(Ω). We endow P2(Ω) with the weak topology, thus obtaining a Polish space. Additionally, P2(Ω) denotes the space of probability measures on P2(Ω).

For a probability measure µ on a discrete probability space X we denote by σ1, σ2, . . . ∈ X independent samples drawn from µ. Where the reference to µ is apparent we omit µ from the superscripts and just write σ, σ1, etc.

Finally, we need the following version of a Markov random field. A factor graph consists of

- a finite set V of variable nodes,
- a finite set C of constraint nodes,
- a finite or countable range Ωv for each v ∈ V,
- a subset ∂a ⊆ V for each a ∈ C,
- a weight function ψa : Πv∈∂a Ωv → [0, ∞) for each a ∈ C.

A factor graph can be represented by a bipartite graph with vertex sets V and C where the neighbourhood of a ∈ C is just ∂a. We further define the function ψd : Πv∈V Ωv → [0, ∞) by

\[ \sigma \mapsto \prod_{a \in C} \psi_a(\sigma_{\partial a}) \]  

for all σ = (σv)v∈V ∈ Ω, where σ_{\partial a} denotes the restriction of σ to ∂a. Finally, the partition function Z(\mathcal{G}) of G is defined by

\[ Z(\mathcal{G}) = \sum_{\sigma \in \prod_{v \in V} \Omega_v} \psi_d(\sigma). \]
If $0 < Z(\mathcal{G}) < \infty$ then $\mathcal{G}$ gives rise to a probability distribution

$$\mu_{\mathcal{G}}(\sigma) = \psi_{\mathcal{G}}(\sigma) / Z(\mathcal{G})$$

for $\sigma \in \prod_{v \in V} \Omega_v$ (1.15)

that is called the Boltzmann distribution of $\mathcal{G}$.

2. Outline

We proceed to survey the proofs of the main results, deferring most technical details to the following sections.

2.1. The Potts antiferromagnet. The goal is to derive a lower bound on the chromatic number of the random graph $G = G(n, d)$ or $G = G(n, d/n)$. We tackle this problem indirectly by way of a weighted version of the $q$-colourability problem. To be precise, the $q$-spin Potts antiferromagnet at inverse temperature $\beta > 0$ on a multigraph $G = (V, E)$ is the probability distribution $\mu_{G, \beta}$ on $[q]^V$ defined by

$$\mu_{G, \beta}(\sigma) = \frac{1}{Z(\beta)} \prod_{v \in E(G)} 1 - (1 - e^{-\beta}) \mathbf{1}[\sigma(v) = \sigma(w)],$$

$$Z(\beta) = \sum_{\sigma \in [q]^V} \prod_{v \in E(G)} 1 - (1 - e^{-\beta}) \mathbf{1}[\sigma(v) = \sigma(w)].$$

Here it is understood that each edge of $G$ contributes to the products in (2.1) and (2.2) according to its multiplicity. The strictly positive quantity $Z(\beta)$, known as the partition function, ensures that $\mu_{G, \beta}$ is a probability measure. Moreover, we observe that the probability mass $\mu_{G, \beta}(\sigma)$ is governed by the number of edges that $\sigma$ renders monochromatic. Indeed, the product in (2.1) imposes an $\exp(-\beta)$ ‘penalty factor’ for every monochromatic edge. Thus, larger values of $\beta$ deliver higher penalties to monochromatic edges. In particular, if $\sigma$ is a $q$-colouring of $G$ then the product evaluates to one. Therefore, the partition function is lower-bounded by the total number of $q$-colourings of $G$ and $\lim_{\beta \to \infty} Z(\beta) = \#q\text{-colourings of } G$. Hence, $\chi(G) > q$ if there exists $\beta > 0$ such that $Z(\beta) < 1$.

Thus, our approach is to show that there exists $\beta > 0$ such that if $d$ exceeds the thresholds stated in Theorems 1.1 and 1.3 then w.h.p. $\log Z(\beta) < 0$. To facilitate the analysis of $Z(\beta)$ we will work with slightly modified and (for our purposes) more amenable random graph models. Specifically, fixing $\epsilon > 0$, we let

$$m \sim \text{Po}\left((1 - \epsilon)dn/2\right)$$

be a Poisson variable conditioned on not exceeding $dn/2$. Define $G(n, d/n)$ as the random multigraph on the vertex set $V_n = \{v_1, \ldots, v_n\}$ obtained by inserting $m$ independent random edges $e_1, \ldots, e_m$ chosen uniformly out of all $\binom{n^2}{2}$ possible edges. Similarly, let $G(n, d)$ be the random multigraph obtained from the following version of the configuration model: choose a matching $\Gamma$ of size $m$ of the complete graph on $V_n \times \{d\}$ uniformly at random. Then obtain $G(n, d)$ by inserting one $v w$-edge for every matching edge $\{(v, i), (w, j)\} \in \Gamma$. In order to avoid case distinctions, we use the symbol $G$ to denote either $G(n, d/n)$ or $G(n, d)$.

Working with the Potts antiferromagnet rather than directly with the graph colouring problem offers two advantages. First, the partition function $Z(\beta)$ is always positive and $\log Z(\beta)$ enjoys a Lipschitz property with respect to edge additions/deletions. Indeed, adding or deleting a single edge can change $\log Z(\beta)$ by an additive term of at most $\beta$ in absolute value. (See Section 3.1 below.) Second, as a consequence of this Lipschitz property it is easy to prove that $\log Z(\beta)$ is tightly concentrated about its expectation. Although similar statements already appear in the literature (e.g., [6, 15]), we include the proof for completeness.

**Proposition 2.1.** For any $\epsilon, \delta, \beta > 0$ there is $\xi > 0$ such that for sufficiently large $n$ we have

$$\mathbb{P} \left[ |\log Z(\beta) - \mathbb{E}[\log Z(\beta)]| > \delta n \right] \leq \exp(-\xi n), \quad \mathbb{P} \left[ |\log Z(\beta) - \mathbb{E}[\log Z(\beta)]| > \delta n \right] \leq \exp(-\xi n).$$

**Proposition 2.1** implies that the partition functions of $G$ and $G$ do not differ too much.

**Corollary 2.2.** For any $\beta > 0$ we have

$$\lim_{n \to \infty} \limsup_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \left| \mathbb{E}[\log Z(\beta)] - \mathbb{E} \left[ \log Z(\beta) \right] \right| = 0.$$

Finally, thanks to the following corollary it suffices to bound $\mathbb{E}[\log Z(\beta)]$ to show that $G$ fails to be $q$-chromatic.

**Corollary 2.3.** If there is $\beta > 0$ such that $\limsup_{n \to \infty} \frac{1}{n} \mathbb{E} \left[ \log Z(\beta) \right] < 0$, then $\chi(G) > q$ w.h.p.
Proof. If \( \chi(G) \leq q \) then \( Z_\beta(G) \geq 1 \) for all \( \beta > 0 \). Hence,
\[
\limsup_{n \to \infty} P[\chi(G) \leq q] > 0 \Rightarrow \forall \beta > 0 : \limsup_{n \to \infty} P[\log Z_\beta(G) \geq 0] > 0.
\]
(2.5)
Now, assume that \( \limsup_{n \to \infty} \frac{1}{n} \mathbb{E}[\log Z_\beta(G)] < -\delta < 0 \) for some \( \beta > 0 \). Then Proposition 2.1 implies that \( \log Z_\beta(G) \leq -\delta n/2 \) w.h.p., and thus \( \limsup_{n \to \infty} P[\log Z_\beta(G) \geq 0] = 0 \). Thus, (2.5) shows that \( \chi(G) > q \) w.h.p. \( \square \)
The proofs of Proposition 2.1 and Corollary 2.2 can be found in Section 3. At the end of Section 2 we show how these results are used to prove our main theorems.

2.2. The interpolation scheme. The study of the partition function \( Z_\beta(G) \) is closely intertwined with the study of the probability distribution \( \mu_{\beta,G} \) from (2.1). What turns the latter task into a challenge is the possible presence of extensive stochastic dependencies amongst the colours that \( \sigma \in \{ q \}^V \), drawn from \( \mu_{\beta,G} \), assigns to the different vertices. While there are short range dependencies between the colour of a vertex \( v \) and the colours of vertices in its proximity, the expansion properties of \( G \) are apt to cause long-range dependencies as well.

To cope with this issue, we are going to compare \( G \) with another random graph model \( G_1 \) in which the dependencies between the vertices are more manageable. Specifically, we will upper-bound \( \mathbb{E}[\log Z_\beta(G)] \) in terms of \( \mathbb{E}[\log Z_\beta(G_1)] \). To this end we will construct an interpolating family of random graphs \( (G_t)_{t \in [0,1]} \) such that \( G_0 \) essentially coincides with the random graph \( G \) from Section 2.1. To compare \( G_0 \) and \( G_1 \) we will show that \( \mathbb{E}[\log Z_\beta(G_1)] \) is non-negative. This general proof strategy is known as the interpolation method. The specific interpolation scheme \( (G_t)_{t \in [0,1]} \) that we use is an adaptation of the construction that Panchenko and Talagrand [38] used to study binary problems on binomial random hypergraphs (e.g., random \( k \)-SAT formulas). In the case of random regular graphs, the present construction can actually be viewed as a special case of the interpolation scheme from [15]. But since we need to perform the analysis for the binomial random graph anyway, a unified treatment of both models incurs little overhead.

The elements \( G_t \) of the interpolation scheme will not be plain random graphs but random factor graphs. To construct the interpolating family, fix a probability measure \( \tau \in \mathcal{P}^3(\{ q \}) \) as well as parameters \( \epsilon, \beta > 0 \) and a probability distribution \( \gamma \) on \( N \). Let \( (r_i, r_{i,j}, r'_{i,j}, r''_{i,j}, r_i), i,j \geq 1 \) be mutually independent random variables with distribution \( \tau \); thus, \( r_i, r_{i,j}, r'_{i,j}, r''_{i,j}, r_i \sim \sigma \in \mathcal{P}^3(\{ q \}) \). Next, given \( (r_i, r_{i,j}, r'_{i,j}, r''_{i,j}, r_i), i,j \geq 1 \) let
\[
\{ \rho_{h,i}, \rho_{h,i,j}, \rho'_{h,i}, \rho''_{h,i}, \hat{\rho}_h \mid i, j, h \geq 1 \}
\]
(2.6)
be a set of mutually independent random variables such that the \( \rho_{h,i} \) have distribution \( r_i \), the \( \rho_{h,i,j} \) have distribution \( r_{i,j} \), the \( \rho'_{h,i} \) have distribution \( r'_{i,j} \), the \( \rho''_{h,i} \) have distribution \( r''_{i,j} \) and the \( \hat{\rho}_h \) have distribution \( \hat{\rho} \). Thus, all random variables in \( \{ \rho_{h,i}, \rho_{h,i,j}, \rho'_{h,i}, \rho''_{h,i}, \hat{\rho}_h \mid i, j, h \geq 1 \} \) are mutually independent given \( (r_i, r_{i,j}, r'_{i,j}, r''_{i,j}, r_i), i,j \geq 1 \).

Additionally, let
\[
M_t \sim \text{Po}((1-\epsilon)(1-t)dn/2), \quad M'_t \sim \text{Po}((1-\epsilon)tdn), \quad M''_t \sim \text{Po}((1-\epsilon)(1-t)dn/2)
\]
(2.7)
be mutually independent and independent of everything else. Define the event
\[
\mathcal{M} = \{2M_t + M'_t \leq dn, M_t + M'_t + M''_t \leq dn \}
\]
(2.8)
and write \( (m_t, m'_t, m''_t) \) for \( (M_t, M'_t, M''_t) \) given \( \mathcal{M} \).

Remark 2.4. Although the above description of the random variables is complete and correct, now seems to be a propitious moment to dwell on the measure-theoretic basis of the construction. It can be implemented on a standard Borel space. To this end we identify the space \( \mathcal{P}(\{ q \}) \) with the standard simplex in \( \mathbb{R}^q \). Thus, \( \mathcal{P}(\{ q \}) \) inherits the Euclidean topology and the corresponding Borel algebra. Let
\[
\mathcal{R} : [0,1]^2 \to \mathcal{P}(\{ q \}), \quad (x, s) \to \mathcal{R}_{x,s}
\]
be a measurable function and let
\[
(x_{i,j}, x'_{i,j}, x''_{i,j}, x, y_{h,i,j}, y'_{h,i,j}, y''_{h,i,j}, y_{h,i,j}, y'_{h,i,j}, y''_{h,i,j}, y_{h,i,j} h, i, j \geq 1)
\]
be mutually independent random variables that are uniformly distributed on the unit interval \([0,1]\), all defined on a common standard Borel space. Then \( \mathcal{R} \) induces a distribution \( \tau \in \mathcal{P}^3(\{ q \}) \) as for a given \( x \in [0,1] \) we naturally obtain a distribution \( \mathcal{R}_x \in \mathcal{P}^2(\{ q \}) \), namely the distribution of the \( \mathcal{P}(\{ q \}) \)-valued random variable \( \mathcal{R}_{x,y_{h,i,j}} \). Consequently, the distribution \( \tau \) of the \( \mathcal{P}^3(\{ q \}) \)-valued random variable \( \mathcal{R}_x \) belongs to the space \( \mathcal{P}^3(\{ q \}) \). Indeed, since \( \mathcal{P}(\{ q \}) \) is a complete separable metric space, any distribution \( \tau \in \mathcal{P}^3(\{ q \}) \) can be represented by a map \( \mathcal{R} \) in
this manner. Now, the above \( r_i \) can be identified with the \( \mathcal{P}^2((q)) \)-valued random variables \( \mathcal{R}_X \), and similarly for \( r_{i,j}, r_{i,j}', \tilde{r}_i \). Moreover, the \( \rho_{h,i}, \rho_{h,i,j}, \tilde{\rho}_h \) can be identified with \( \mathcal{R}_{X,Y_{h,i}}, \mathcal{R}_{X,Y_{h,i,j}}, \mathcal{R}_X,Y_{h,i}', \mathcal{R}_X,Y_{h,i}'' \).

All the factor graphs \( G_i \) have variable nodes

\[
\{s, v_1, \ldots, v_n, e_1, \ldots, e_{m_1}, a_1, \ldots, b_{m_1}, b_{m_1}' \}, \nonumber \]

with \( s \) ranging over \( \mathbb{N} \) (that is, \( \Omega = \mathbb{N} \)), and \( v_1, \ldots, v_n \) ranging over \( |q| \). The constraint nodes are

\[
e_1, \ldots, e_{m_1}, a_1, \ldots, b_{m_1}, b_{m_1}' \].

How constraint and variable nodes are connected depends on whether \( G \) is the binomial or the regular random graph.

**Definition 2.5** (binomial case). The connections between the constraint and variable are as follows.

- Each \( e_i \), \( i \in \{m_1\} \), is adjacent to a random pair of two distinct variable nodes from \( V_n \); these pairs are drawn uniformly and independently of everything else.
- Each \( a_i \), \( i \in \{m_1\} \), is adjacent to \( s \) and one random variable node from \( V_n \) drawn uniformly and independently of everything else.
- The constraint nodes \( e, b_1, \ldots, b_{m_1}' \) are adjacent to the variable node \( s \) only.

The construction in the random regular case resembles the ‘configuration model’.

**Definition 2.6** (regular case). Let \( \Gamma_1 \) be a uniformly random maximal matching of the complete bipartite graph with vertex classes

\[
\left( \bigcup_{i=1}^{m_1} \{e_i\} \times \{1,2\} \right) \cup \bigcup_{i=1}^{m_1'} \{a_i\} \quad \text{and} \quad \bigcup_{i=1}^{n} \{v_i\} \times \{d\};
\]

this matching covers the left vertex set completely because \( 2m_1 + m_1' \leq d \).

- Each constraint node \( e_i \) is adjacent to the variable nodes \( v, w \) for which \( \Gamma_1 \) contains edges between \( (e_i, 1) \) and \( \{v\} \times \{d\} \) and \( (e_i, 2) \) and \( \{w\} \times \{d\} \).
- Each \( a_i \) is adjacent to \( s \) and to the variable node \( w \) for which \( \Gamma_1 \) contains an edge between \( a_i \) and \( \{w\} \times \{d\} \).
- The constraints \( e, b_1, \ldots, b_{m_1}' \) are adjacent to \( s \) only.

Finally, we need to define the weight functions of the constraint nodes: let

\[
\psi_G(s) = \gamma(s) \quad (s \in \mathbb{N}),
\]

\[
\psi_{e_i}(s, w) = 1 - (1 - e^{-\beta}) \mathbf{1}[s = \sigma] \quad (\partial e_i = \{v, w\}, \sigma_v, \sigma_w \in \{q\}),
\]

\[
\psi_{a_i}(s, v) = 1 - (1 - e^{-\beta}) \rho_{a_i}(s, v) \quad (\partial a_i = \{s, v\}, s \in \mathbb{N}, \sigma_v \in \{q\}),
\]

\[
\psi_{b_i}(s) = 1 - (1 - e^{-\beta}) \sum_{\tau \in \{q\}} \rho_{b_i}(s) \rho_{b_i}''(\tau) \quad (s \in \mathbb{N}).
\]

Thus, \( \psi_G \) simply weights the value \( s \) according to the given probability distribution \( \gamma \). Moreover, the constraint nodes \( e_i \) simulate the effect of the edges of the original graph \( G \) as in the definition (2.1) of the Potts model. Indeed, if the variable nodes adjacent to \( e_i \) are coloured the same then the weight is \( \exp(-\beta) \); otherwise it is one. Moreover, \( \psi_{a_i} \) weighs the colour \( s \) of the adjacent variable node from \( V_n \) according to \( \rho_{a_i} \). Further, \( \psi_{b_i} \) is determined by the probability that two colours chosen independently from \( \rho_{b_i}', \rho_{b_i}'' \in \mathcal{P}([q]) \) coincide. The total weight \( \psi_G \) and partition function \( Z(G) \), and the Boltzmann distribution \( \mu_G \), are defined by the general formulas (1.13)–(1.15). In the physics literature the \( a_i \) are called external fields [34]. A similar construction involving an extra \( \mathbb{N} \)-valued variable node \( s \) was used in [42].

At ‘time’ \( t = 1 \) (2.2) ensures that \( m_1 = m_1' = 0 \). Thus, the only constraints present are the \( a_i \). Each of them is connected to the variable node \( s \) and to one other variable node. Hence, the factor graph is star-shaped with constraint node \( s \) at the centre. In effect, the variable nodes \( v_1, \ldots, v_n \) are dependent only through \( s \).

By contrast, at \( t = 0 \) (2.7) yields \( m_1' = 0 \). Thus, the factor graph contains only constraints of type \( e_i \) and of type \( b_j \). In effect, \( G_0 \) decomposes into two parts. The connected component of \( s \) contains all the constraint nodes \( b_j \), none of which is connected with \( v_1, \ldots, v_n \). Thus, once more there is a star structure with \( s \) at the centre, and it is not too difficult to write out the partition function of this component. Furthermore, the factor graph induced on \( v_1, \ldots, v_n \) and \( e_1, \ldots, e_{m_1} \) is essentially identical to the original graph \( G \). More specifically, the Boltzmann distribution \( \mu_G \) mimics that of the Potts antiferromagnet \( \mu_{G,\beta} \) from (2.1). The only, for our purposes negligible, difference is that
$G_0$ typically has slightly fewer than $dn/2$ constraint nodes of the type $e_i$. Thus, we can relate the partition functions $Z_{\beta}(G)$ and $Z(G_0)$; see Figure 2 for an illustration.

**Figure 2.** The factor graphs $G_0$ (left) and $G_1$ (right).

We observe that the distribution of the degrees of $v_1, \ldots, v_n$ remains essentially the same for $0 \leq t \leq 1$. Specifically, in the regular case most variables have degree exactly $d$ throughout the interpolation, and in the binomial case the degrees are approximately $\text{Po}(1 - \varepsilon)d$ distributed. Additionally, the total number of constraints remains (essentially) constant throughout the interpolation as well. Indeed, at $t = 0$ there are about $(1 - \varepsilon)dn/2$ constraints of type $e_i$ and about the same number of constraints $b_i$, while at $t = 1$ we have about $(1 - \varepsilon)dn/2$ constraints of type $a_i$.

As mentioned above, the idea behind the construction is to compare $\mathbb{E}[\log Z_{\beta}(G)]$ with the partition function of a simpler model where correlations amongst $v_1, \ldots, v_n$ are amenable to a precise analysis. The following two propositions spell out this relationship precisely.

**Proposition 2.7.** Let

$$Y' = \sum_{\sigma} \gamma(\sigma) \prod_{1 \leq i \leq \lfloor d/2 \rfloor} \left(1 - (1 - e^{-\beta}) \sum_{i=1}^{q} \rho_{i,j}(\tau) \rho_{i,j}(\tau)\right).$$

Then for any $\delta, \beta > 0$ there exists $\varepsilon_0(d, \delta, \beta) > 0$ such that for all $0 < \varepsilon < \varepsilon_0(d, \delta, \beta)$ and for all $n > 1/\varepsilon_0(d, \delta, \beta)$ we have $\mathbb{E}[\log Z(G_0)] = \mathbb{E}[\log Z_{\beta}(G)] + \mathbb{E}[\log Y'] - \delta n$.

Furthermore, the following proposition shows that $Z(G_1)$ dominates $Z(G_0)$. The proof is based on estimating the derivative $\frac{d}{d\beta}\mathbb{E}[\log Z(G_1)]$.

**Proposition 2.8.** We have $\mathbb{E}[\log Z(G_0)] \leq \mathbb{E}[\log Z(G_1)] + o(n)$.

Finally, we introduce a convenient proxy for the partition function of $G_1$: let

$$Y = \sum_{\sigma} \gamma(\sigma) \prod_{i=1}^{q} \prod_{\sigma_{v_i}=1}^{q} \sum_{j=1}^{D_i} \left(1 - (1 - e^{-\beta}) \rho_{\sigma_{v_i},\sigma_{v_j}}(\sigma_{v_j})\right).$$

(2.9)

**Corollary 2.9.** For any $\beta > 0$ we have $\mathbb{E}[\log Z_{\beta}(G)] \leq \mathbb{E}[\log Y] - \mathbb{E}[\log Y'] + o(n)$.

The proofs of Propositions 2.7 and 2.8 and Corollary 2.9 can be found in Section 4. We are thus left to study $Y, Y'$, which are approximations to the partition function of the factor graph $G_1$ and the partition function of the $s$-component of $G_0$, respectively.

Let us wrap up by dwelling on the intended combinatorial semantics of construction. The nodes $v_1, \ldots, v_n$ and $e_1, \ldots, e_m$, clearly mimic the original Potts antiferromagnet. But as we move along we replace more and more $e_i$ by external fields $a_j$. These are meant to capture the physics intuition as to the nature of the interactions between variable nodes in the random graph $G$. Corollary 2.9 corroborates the physics picture to the extent that it yields an upper bound on the partition function. Specifically, the impact that an actual edge $e = vw$ of $G$ has on an incident vertex $v$ is thought to be governed by the local graph structure around the other vertex $w$ in the graph $G - e$ obtained by removing $e$ [34]. Since short cycles are scarce, the local graph structure will likely be a tree. Indeed, it
will just be a \((d - 1)\)-ary tree in the random regular graph, and a Po\((d)\) Galton-Watson tree in the binomial case. In the binomial case, the specific tree structure is apt to impact the influence that \(w\) exerts on \(v\). For example, if the Galton-Watson tree dies out quickly, then it will be easy to colour the entire tree properly regardless of the colour of \(v\). Thus, the edge \(e = uv\) will be of little consequence. By contrast, in the event of a relatively dense tree, choosing a specific colour for \(v\) might have repercussions on a large number of other vertices. The random variables \(r_i\) are meant to capture the randomness of the tree structure pending on vertex \(w\). But for the sake of simplicity, we do not incorporate an actual distribution on trees into our construction. Instead, we make do with the distribution \(r_i \in \mathcal{P}(\{q\})\) that is meant to just capture the ensuing impact that \(w\) has on \(v\).

Furthermore, the variable node \(s\) is intended to represent the conjectured structure of the Boltzmann distribution \(\mu_{G, \beta}\). To elaborate: according to physics intuition, the distribution \(\mu_{G, \beta}\) partitions the phase space \([q]^V\) into an unbounded number of ‘clusters’ \((S_1)\) for \(d\) close to the \(q\)-colourability threshold and \(\beta\) large \([35, 36]\). Inside a cluster, i.e., under the conditional distribution \(\mu_{G, \beta}(\cdot | S_1)\), most vertices \(w\) are strongly polarised towards a particular colour. In other words, the conditional marginals \(\mu_{G, \beta}(\{\sigma_w = c\} | S_1)\) for \(c \in \{q\}\) are typically either fairly close to zero or to one, while of course overall the marginal of the colour of each vertex is just uniform. The variable node \(s\) is intended to represent the choice of the cluster \(S_1\). Thus, the distribution \(r_i \in \mathcal{P}(\{q\})\), which mimics the local graph structure, determines how the marginal of \(w\) is distributed given a cluster index, and then the sample \(p_{\sigma_i, j}(c)\) represents the actual realisation of the distribution of the colour inside cluster number \(\sigma_s\). Finally, \(y(s)\) models the distribution of the relative cluster volumes \(\mu_{G, \beta}(S_1)\).

2.3. Poisson-Dirichlet weights. While the expression \(E[\log Y] - E[\log Y']\) from Corollary \([2,9]\) already bears a certain resemblance to \([1,9]\), an important difference remains. Namely, the expressions \(Y, Y'\) inside the logarithm still contain \(n\), the number of vertices. If the probability distribution \(y\) is an atom, that is, \(y(h) = 1\) for some \(h \in \mathbb{N}\), then we can produce the same joint distribution on \(\sigma_{v_1}, \ldots, \sigma_{v_n}\) by deleting \(s\) and \(g\) from the factor graphs \(G_0\) and \(G_1\) and replacing \(\sigma_s\) by \(h\) in the expressions for \(Y\) and \(Y'\). This causes \(Y\) and \(Y'\) to factorise:

\[
E[\log Y] = \sum_{i=1}^{n} \log \left[ \prod_{r=1}^{q} \left( 1 - e^{-\beta} \right) \rho_{h,i,j}(r) \right], \quad E[\log Y'] = \sum_{i \leq d n/2} \log \left( 1 - \left( 1 - e^{-\beta} \right) \sum_{r=1}^{q} \rho'_{h,i}(r) \rho''_{h,i}(r) \right).
\]

In particular, long-range correlations are completely absent in the target \(G_1\) of the interpolation. (The modified \(G_1\) with \(s\) and \(g\) deleted consists of \(n\) connected components, each containing exactly one \(v_i\).) In physics jargon the bound on \(E[\log Z_{\beta}(G)]\) that can be obtained from \([2,10]\) is called the replica symmetric bound. While the replica symmetric bound easily implies the first moment bound \([1,1]\), it does not appear sufficient to prove Theorems \([1,1]\) and \([13]\) for any \(q \geq 3\).

Fortunately there is another choice of the distribution \(y\) that leads to a simple formula. Recall that the Poisson-Dirichlet distribution with parameter \(\gamma > 0\) is defined as follows. Let \(P \subset (0, \infty)\) be the countable point set generated by a Poisson point process on \((0, \infty)\) with density \(\lambda x^{-\gamma - 1} dx\), independent of all other sources of randomness that have been introduced thus far. Further, let \((p_h)_{h \geq 1}\) be the sequence that comprises the points from \(P\) in decreasing order, i.e., \(p_h \geq p_{h+1}\) for all \(h\). Since \(y > 0\), we have \(\sum_{h=1}^{\infty} p_h < \infty\) almost surely. Therefore,

\[
y(h) = p_h / \sum_{\ell=1}^{\infty} p_{\ell}
\]

defines a probability measure on \(\mathbb{N}\), the Poisson-Dirichlet law. To be precise, \(y\) is a random probability measure which depends on \(P\). This distribution is used in the following lemma, which enables us to simplify \(E[\log Y]\), \(E[\log Y']\).

**Lemma 2.10** \([38\text{ Proposition 1}] \) and \([43\text{ Proposition 6.5.15}]\). Suppose that \(0 < \gamma < 1\) and that \((X_h)_{h \geq 1}\) are positive identically distributed random variables with bounded second moments, mutually independent and independent of \(y\). Then

\[
E[\log \sum_{h=1}^{\infty} y(h) X_h] = \frac{1}{\gamma} \log \mathbb{E}[X_1^\gamma].
\]

In the physics literature, the Poisson-Dirichlet distribution has been postulated as the correct distribution of the relative cluster sizes \([34, 35]\). Moreover, Panchenko and Talagrand \([38]\) used Lemma 2.10 to bound the partition function of the random \(k\)-SAT model. We apply Lemma 2.10 in a similar manner to upper bound \(E[\log Z_{\beta}(G)]\).
Specifically, let $\mathcal{R}$ be the $\sigma$-algebra generated by $(r_{i, r_{i,j}, r_{i,j}', r_{i,j}''}, D_i)_{i,j=1}$. Thanks to Lemma 2.10, we can simplify the bound from Corollary 2.9 as follows.

**Corollary 2.11.** For any $y, \beta > 0$ and $t \in \mathcal{P}^3([q])$ we have

$$\limsup_{n \to \infty} \frac{1}{n} \mathbb{E}[\log Z_\beta(G)] \leq \phi_{\beta, y}(t)/y,$$

where

$$\phi_{\beta, y}(t) = \mathbb{E} \left[ \log \mathbb{E} \left[ \left( \sum_{r_{i,j} \in \mathbb{P}} \prod_{j=1}^q 1 - (1 - e^{-\beta}) \rho_{1,1,j}(r_{i,j}) \right)^y \bigg| \mathcal{R} \right] \right] - \frac{d}{2} \mathbb{E} \left[ \log \mathbb{E} \left[ \left( 1 - (1 - e^{-\beta}) \sum_{r_{i,j} \in \mathbb{P}} \rho_{1,1,j}(r_{i,j}) \right)^y \bigg| \mathcal{R} \right] \right]. \quad (2.12)$$

**Proof.** Choose $\epsilon > 0$ small enough and assume that $n$ is sufficiently large. Moreover, for all $h \in \mathbb{N}$ let

$$X_h = \prod_{i=1}^n \sum_{a_{i,j} \in \mathbb{P}} \prod_{j=1}^q (1 - (1 - e^{-\beta}) \rho_{h,i,j}(\sigma_{i,j})), \quad X'_h = \prod_{1 \leq i \leq \rho(n/2)} (1 - (1 - e^{-\beta}) \sum_{r_{i,j} \in \mathbb{P}} \rho_{h,i,j}(r_{i,j})).$$

Applying Corollary 2.9 to the random distribution $\gamma$, we obtain

$$\mathbb{E}[\log Z_\beta(G)] \leq \mathbb{E} \left[ \log \sum_{h=1}^\infty \gamma(h) X_h \right] - \mathbb{E} \left[ \log \sum_{h=1}^\infty \gamma(h) X'_h \right] + o(n).$$

Hence, Lemma 2.10 yields

$$y \mathbb{E}[\log Z_\beta(G)] \leq \mathbb{E} \left[ \log \mathbb{E} \left[ X'^y \bigg| \mathcal{R} \right] - \log \mathbb{E} \left[ X'^y \bigg| \mathcal{R} \right] \right] + o(ny); \quad (2.13)$$

clearly, since $X, X'_h$ do not depend on $P$, the outer $\mathbb{E} [\cdot]$ in (2.13) is on $(r_{i, r_{i,j}, r_{i,j}', r_{i,j}''}, D_i)_{i,j=1}$ only. Further, because the $\rho_{h,i,j}, \rho_{h,i}', \rho_{h,i}''$ are mutually independent given $\mathcal{R}$, we obtain

$$\mathbb{E} \left[ \log \mathbb{E} \left[ X'^y \bigg| \mathcal{R} \right] \right] = n \mathbb{E} \left[ \log \mathbb{E} \left[ \left( \sum_{r_{i,j} \in \mathbb{P}} \prod_{j=1}^q 1 - (1 - \exp(-\beta)) \rho_{1,1,j}(r_{i,j}) \right)^y \bigg| \mathcal{R} \right] \right], \quad (2.14)$$

$$\mathbb{E} \left[ \log \mathbb{E} \left[ X'^y \bigg| \mathcal{R} \right] \right] = \frac{dn}{2} \mathbb{E} \left[ \log \mathbb{E} \left[ \left( 1 - (1 - \exp(-\beta)) \sum_{r_{i,j} \in \mathbb{P}} \rho_{1,1,j}'(r_{i,j}) \right)^y \bigg| \mathcal{R} \right] \right]. \quad (2.15)$$

The assertion follows from (2.13) – (2.15).

**2.4. The zero temperature limit.** To actually deduce a bound on the chromatic number from Proposition 2.11, we need to fix the three remaining parameters $\beta, y, t$. Since the Potts model approaches the graph colouring problem in the limit of large $\beta$, it seems natural to take the limit $\beta \to \infty$. In physics jargon, we take the ‘temperature’ $1/\beta$ to zero. Moreover, physics intuition suggests sending the ‘Parisi parameter’ $y$ to zero as well. Ding et al. [21] took similar limits to derive the upper bound on the $k$-SAT threshold from the formula for the $k$-SAT partition function from [30].

With respect to $t$, we make two different choices, depending on whether $G$ is regular or binomial. Let us begin with the regular case. For $i \in [q]$, let $\eta_i \in \mathcal{P}([q])$ be the atom on colour $i$. Moreover, let $\eta_0 = q^{-1} 1 \in \mathcal{P}([q])$ be the uniform distribution on the $q$ colours. Then for a given $\alpha \in [0, 1]$ we define

$$r_a = \alpha \delta_{\eta_0} + \frac{1 - \alpha}{q} \sum_{i=1}^q \delta_{\eta_i} \in \mathcal{P}^3([q]). \quad (2.16)$$

Geometrically, we can think of $r_a$ as a discrete distribution on the standard simplex $\mathcal{P}([q]) \subset \mathbb{R}^q$ that places mass $\alpha$ on the centre and distributes the remaining mass $1 - \alpha$ equally amongst the $q$ vertices of the simplex. Let

$$t_a = \delta_{r_a} \in \mathcal{P}^3([q]) \quad (2.17)$$

be the atom on $r_a$. Further, the expression (1.10) for the binomial random graph involves a probability distribution $a$ on $[0,1]$. Given any $a \in \mathcal{P}([0,1])$, we define

$$t_a = \int_0^1 t_a \, da(a). \quad (2.18)$$

Observe that the integrand is the distribution $t_a \in \mathcal{P}^3([q])$ from (2.17), and thus $t_a \in \mathcal{P}^3([q])$. Plugging $t_a$ or $t_a$ into Proposition 2.11, we finally obtain the expressions from (1.8) and (1.10).
Proposition 2.12. If \( G = G(n,d) \) is the random regular graph then
\[
\lim_{y \to 0} \lim_{\beta \to \infty} \phi_{\beta,y}(t_a) = \Sigma_{d,q}(\alpha)
\]
for all \( \alpha \in [0,1] \).

Moreover, if \( G = G(n,d/n) \) is the binomial model then
\[
\lim_{y \to 0} \lim_{\beta \to \infty} \phi_{\beta,y}(t_a) = \Sigma_{d,q}(\alpha)
\]
for all \( \alpha \in \mathcal{P}([0,1]) \).

The proof of Proposition 2.12 can be found in Section 4.4.

Now we have all the pieces in place to complete the proofs of the main theorems.

Proof of Theorem 1.7. Fix \( q \geq 3 \) and assume that \( \Sigma_{d,q} < 0 \) for some \( d \geq 3 \). (This holds when \( d = d_q \), for example.) Then Proposition 2.12 yields \( y, \beta > 0 \) and \( \alpha \in [0,1] \) such that \( \phi_{\beta,y}(t_\alpha) < 0 \). In particular we can take \( \alpha \) to be the value which minimises \( \Sigma_{d,q}(\cdot) \). Consequently, Corollary 2.11 implies that \( \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[\log Z_\beta(G)] < 0 \). Therefore, Corollary 2.3 implies that
\[
\chi(G(n,d)) > q \quad \text{w.h.p.}
\]
(2.19)
We are left to prove that \( G(n,d') \) also fails to be \( q \)-chromatic w.h.p. for all \( d' > d \). To see this, we observe that the property of being \( q \)-colourable is decreasing; that is, if a graph \( G \) is \( q \)-colourable then so is every subgraph \( G' \) of \( G \). Now, (2.8) Theorem 9.36 shows that if \( d' > d \) and if a decreasing property \( \mathcal{A} \) is satisfied for \( G(n,d') \) w.h.p. then \( G(n,d) \) enjoys \( \mathcal{A} \) w.h.p. Thus, (2.19) implies that \( \chi(G(n,d')) > q \) for all \( d' > d \).

Proof of Theorem 1.3. Once more we fix \( q \geq 3 \) and suppose that \( \Sigma_{d,q} < 0 \) for some \( d > 0 \). (This holds when \( d = d_q^* \), for example.) Then by Proposition 2.12 there exist \( y, \beta > 0 \) and \( \alpha \in \mathcal{P}([0,1]) \) such that \( \phi_{\beta,y}(t_\alpha) < 0 \) and thus \( \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[\log Z_\beta(G)] < 0 \) by Corollary 2.11. Hence, Corollary 2.3 yields
\[
\chi(G(n,d/n)) > q \quad \text{w.h.p.}
\]
(2.20)
Finally, due to monotonicity, (2.20) implies that \( \chi(G(n,d'/n)) > q \) w.h.p. for all \( d' > d \).

Given Theorems 1.1 and 1.3 the asymptotic formulas detailed in Corollary 1.2 and Corollary 1.4 follow from routine calculations, which we defer to Section 5.

3. Concentration

After proving Proposition 2.1 in Section 4.1, we prove Corollary 2.2 in Section 5.2.

3.1. Proof of Proposition 2.1. The proof is based on Azuma’s inequality and the Lipschitz property of the random variable \( \log Z_\beta(\cdot) \). Indeed, (2.2) shows that if a multigraph \( G' \) is obtained from \( G \) by adding one single edge then
\[
e^{-\beta} \leq Z_\beta(G')/Z_\beta(G) \leq 1,
\]
and hence
\[
|\log Z_\beta(G') - \log Z_\beta(G)| \leq \beta.
\]
(3.1)
We pick a small enough \( \zeta = \zeta(\epsilon, \delta, \beta) > 0 \) and a smaller \( \xi = \zeta(\epsilon, \delta, \beta, \xi) > 0 \). We treat the binomial random graph and the random regular graph separately, tacitly assuming in either case that \( n \) is sufficiently large.

3.1.1. The binomial random graph. Writing \( M \sim \text{Bin}(\binom{n}{2}, d/n) \) for the number of edges of \( G \) and invoking the Chernoff bound, we obtain
\[
P\left[ |M - dn/2| < \zeta n \right] \geq 1 - \exp(-4\zeta n).
\]
(3.2)
Further, let \( G_{n,m} \) be the random multigraph on \( n \) vertices comprising \( m \) edges chosen uniformly and independently out of all \( \binom{n}{2} \) possible edges. Let \( \mathcal{F} \) be the event that \( G_{n,m} \) is simple. It is well known that
\[
P\left[ G_{n,m} \in \mathcal{F} \right] = \Omega(1)
\]
uniformly for all \( m \leq dn/2 + \zeta n \).
(3.3)
Moreover, providing \( \xi \) is chosen small enough, Azuma’s inequality and (3.1) imply that
\[
P\left[ |\log Z_\beta(G_{n,m}) - \mathbb{E} [\log Z_\beta(G_{n,m})]| > \zeta n \right] \leq \exp(-6\xi n)
\]
for all \( m \leq dn/2 + \zeta n \).
(3.4)
The estimates (3.3)–(3.4) imply that for all \( m \leq dn/2 + \zeta n \),
\[
P\left[ |\log Z_\beta(G_{n,m}) - \mathbb{E} [\log Z_\beta(G_{n,m})]| > \zeta n \mid \mathcal{F} \right] \leq \exp(-5\xi n).
\]
(3.5)
Since
\[
|\log Z_\beta(G_{n,m})| \leq n \log q + m \beta = O(n + m),
\] (3.6)
the bound 3.5 shows that for all \(m \leq dn/2 + \zeta n\),
\[
\mathbb{E}[|\log Z_\beta(G_{n,m})| - \mathbb{E}[\log Z_\beta(G_{n,m})]|] \leq 2\zeta n.
\] (3.7)
Further, because \(G\) \((M = m)\) and \(G_{n,m}\) \(\mathcal{F}\) are identically distributed, 3.4 and 3.7 show that
\[
\mathbb{P}
\left|\log Z_\beta(G) - \mathbb{E}[\log Z_\beta(G) | M = m] \right| > 3\zeta n \right| M = m = \mathbb{P}
\left|\log Z_\beta(G_{n,m}) - \mathbb{E}[\log Z_\beta(G_{n,m}) | M = m = m] \right| > 3\zeta n
\]
\[
\leq \mathbb{P}
\left[|\log Z_\beta(G_{n,m}) - \mathbb{E}[\log Z_\beta(G_{n,m})] > \zeta n | \mathcal{F} \right] \leq \exp(-5\zeta n)
\] for all \(m \leq dn/2 + \zeta n\).

Moreover, combining 3.2, 3.6 and 3.8, we obtain 2.4.

To prove the second assertion, we recall that \(m \sim \mathcal{P}_{0 \leq d n/2}((1 - \varepsilon)dn/2)\). We thus obtain the tail bound
\[
\mathbb{P}
\left|m - (1 - \varepsilon)dn/2 > \zeta n \right| \leq \exp(-2\zeta n)
\] (3.9)
for sufficiently small \(\zeta\). Since \(G\) \((m = m)\) and \(G_{n,m}\) are identically distributed, 3.4 yields
\[
\mathbb{P}
\left[|\log Z_\beta(G) - \mathbb{E}[\log Z_\beta(G) | m = m] | > \zeta n | m = m \right] \leq \exp(-3\zeta n).
\] (3.10)
Finally, providing that \(\zeta\) is chosen small enough, 3.1 and 3.9 imply that
\[
|\mathbb{E}[\log Z_\beta(G) | m = m] - \mathbb{E}[\log Z_\beta(G)]| \leq \delta n/2
\] for all \((1 - \varepsilon)dn/2 - \zeta n \leq m \leq (1 - \varepsilon)dn/2 + \zeta n\).
Therefore, the second part of 2.4 follows from 3.9 and 3.10.

3.1.2. The random regular graph. We recall that the random regular graph \(G\) can be constructed via the configuration model by drawing a perfect matching \(e_1, \ldots, e_{dn/2}\) of the complete graph on the vertex set \(V_n \times [d]\) uniformly at random. To be precise, the sequence \(e_1, \ldots, e_{dn/2}\) is constructed by successively drawing a uniformly random edge \(e_{i+1}\) that connects two distinct vertices of the complete graph on \(V_n \times [d]\) that are not incident with \(e_1, \ldots, e_i\). Let \(G'\) be the random multigraph on \(|n|\) obtained by inserting for each matching edge \(e_i = ((v, h), (w, j))\) an edge between \(v\) and \(w\) and let \(\mathcal{F}\) denote the event that \(G'\) is simple. It is well known that
\[
\mathbb{P}[\mathcal{F}] = \Omega(1);
\] (3.11)
see, e.g., [26, Corollary 9.7]. Moreover, \(G\) is distributed as \(G'\) given \(\mathcal{F}\).

To prove the first inequality we consider the filtration \((\mathcal{F}_t)_{t \in \{dn/2\}}\) with \(\mathcal{F}_t\) generated by \(e_1, \ldots, e_t\). Then the sequence \((\mathbb{E}[\log Z_\beta(G') | \mathcal{F}_t])_{t \in \{dn/2\}}\) is a Doob martingale. Moreover, 3.1 implies that
\[
|\mathbb{E}[\log Z_\beta(G') | \mathcal{F}_1] - \mathbb{E}[\log Z_\beta(G') | \mathcal{F}_{t+1}]| \leq \beta.
\] (3.12)
Therefore, Azuma’s inequality yields
\[
\mathbb{P}
\left[|\log Z_\beta(G') - \mathbb{E}[\log Z_\beta(G')]| > \delta n/8 \right] \leq \exp(-2\zeta n).
\] (3.13)
The first assertion thus follows from 3.11 and 3.13. Further, we can think of \(G\) as the multigraph obtained by inserting the edges induced by \(e_1, \ldots, e_m\) only. Hence, arguing as for 3.13 but stopping after \(m\) steps gives
\[
\mathbb{P}
\left[|\log Z_\beta(G) - \mathbb{E}[\log Z_\beta(G) | m] | > \delta n/8 | m \right] \leq \exp(-2\zeta n).
\] (3.14)
Finally, the second assertion follows from 3.11, 3.9 and 3.14.

3.2. Proof of Corollary 2.2. Given \(\delta, \beta > 0\) we choose small enough \(\varepsilon = \varepsilon(\delta, \beta) > 0, \zeta = \zeta(\delta, \beta, \varepsilon), \xi = \xi(\delta, \beta, \varepsilon, \zeta)\) and assume that \(n\) is sufficiently large. Once more we treat the binomial and the regular models separately.
3.2.1. The binomial random graph. We continue to denote the total number of edges of the binomial graph \( G = \binom{G(n, d/n)}{M} \) and by \( G_{n, M} \) the random multigraph obtained by including \( M \) uniformly and independently chosen edges. Due to \((3.2)\) and \((3.3)\), with probability \( 1 - \exp(-2d/n) \), we can obtain \( G_{n, M} \) from \( G \) by adding or removing no more than \( 2c \) new edges. Hence, provided \( c \) is small enough, \((3.1)\) ensures that
\[
\left| E\left[ \log Z_{\beta}(G) \right] - E\left[ \log Z_{\beta}(G_{n, M}) \right] \right| \leq 2c\beta d n (1 - e^{-\Theta(n)}) + O(n^2) e^{-\Theta(n)} \leq \delta n/3. \tag{3.15}
\]
Furthermore, with \( \mathcal{S} \) the event that \( G_{n, M} \) is simple, \( G \) is distributed as \( G_{n, M} \) given \( \mathcal{S} \). Therefore, \((3.3)\) and \((3.4)\) imply that
\[
\left| E\left[ \log Z_{\beta}(G) \right] - E\left[ \log Z_{\beta}(G_{n, M}) \right] \mid \mathcal{S} \right| \leq \delta n/3. \tag{3.16}
\]
Finally, the assertion follows from \((3.15)\) and \((3.16)\).

3.2.2. The random regular graph. As in Section \(3.1.2\) we denote by \( G' \) the random multigraph with \( dn/2 \) edges drawn from the configuration model. By the principle of deferred decisions we can think of \( G' \) as being obtained from \( G \) by adding the missing \( dn/2 - m \) edges. Hence, provided that \( \varepsilon \) is sufficiently small, \((3.1)\) implies that
\[
\left| E\left[ \log Z_{\beta}(G) \right] - E\left[ \log Z_{\beta}(G') \right] \right| \leq \delta n/3. \tag{3.17}
\]
Furthermore, as \( G \) is distributed as \( G' \) given the event \( \mathcal{S} \), \((3.11)\) and \((3.12)\) yield
\[
\left| E\left[ \log Z_{\beta}(G) \right] - E\left[ \log Z_{\beta}(G') \right] \right| \leq \delta n/3. \tag{3.18}
\]
The assertion follows from \((3.17)\) and \((3.18)\).

4. Interpolation

In this section we carry out the technical details of the interpolation argument. Section \(4.1\) contains the proof of Proposition \(2.7\) while Section \(4.2\) deals with the proof of Proposition \(2.8\). Subsequently, Section \(4.3\) contains the proof of Proposition \(2.9\) and finally, in Section \(4.4\) we prove Proposition \(2.12\).

4.1. Proof of Proposition \(2.7\). A glimpse at \((2.7)\) reveals that the random factor graph \( G_0 \) consists of constraint nodes \( e_1, \ldots, e_{m_0} \) and \( b_1, \ldots, b_{m_0} \) only. (See also the left side of Figure \(2\)). The constraints \( e_1, \ldots, e_{m_0} \) are adjacent to the variables \( V_n \) but not to \( s \), while \( b_1, \ldots, b_{m_0} \) are adjacent to \( s \) but not to \( V_n \). Consequently, the partition function factorises:
\[
Z(G_0) = \mathcal{Y} \cdot \mathcal{Z}, \quad \text{where} \quad \mathcal{Y} = \sum_{\sigma = 1}^\infty \gamma(\sigma) \prod_{i=1}^{m_0} \psi_{b_i}(\sigma_i), \quad \mathcal{Z} = \sum_{\sigma \in \{\pm 1\}^n} \prod_{i=1}^{m_0} \psi_{e_i}(\sigma_{b_i}).
\]
Hence
\[
E\left[ \log Z(G_0) \right] = E\left[ \log \mathcal{Z} \right] + E\left[ \log \mathcal{Y} \right] \quad \tag{4.1}
\]
and by construction we have
\[
E\left[ \log \mathcal{Z} \right] = E\left[ \log Z_{\beta}(G) \right] + O(m_0 - dn/2) + o(n). \quad \tag{4.2}
\]
Additionally, \( Y' \) is distributed as \( \mathcal{Y} \) given \( m_0' = \lfloor dn/2 \rfloor \). Hence, since \( \mathbb{P}[m_0' > dn/2] = e^{-\Theta(n)} \), we can couple \( \mathcal{Y} \) and \( Y' \) such that
\[
E[\log Y' - \log \mathcal{Y}] = E\left[ \begin{cases} \mathcal{Y} & \text{if } m_0' \leq \lfloor dn/2 \rfloor \\ \sum_{\sigma = 1}^\infty \gamma(\sigma) \prod_{1 \leq i \leq \lfloor dn/2 - m_0' \rfloor} \psi_{b_i}(\sigma_i) \end{cases} \right] + e^{-\Theta(n)}. \tag{4.3}
\]
Since for any \( s \in \mathbb{N} \) we have \( \exp(-\beta) \leq \gamma(\sigma) \leq 1 \) for all \( i \), by \((2.7)\) and \((4.3)\) and applying Poisson tail bounds gives
\[
E[\log Y'] - E[\log \mathcal{Y}] \leq \varepsilon \beta d n/2 + o(n). \tag{4.4}
\]
Combining \((4.1)\), \((4.2)\) and \((4.4)\), we obtain
\[
E[\log Z(G_0)] \geq E[\log Z_{\beta}(G)] + E[\log Y'] - \varepsilon \beta d n/2 + o(n). \tag{4.5}
\]
Finally, the assertion follows from \((4.5)\) and Corollary \(2.2\).
functions of the constraint nodes

In the binomial case \((G)\) is the binomial random graph \('c\) let \(c\) be the uniform distribution on \(c\). In the regular case \((G)\) is the random regular graph let \(c\) be the set of all vertices \(v \in V\) of degree \(d_{G_i}(v)\) strictly less than \(d\) in \(G\), and providing that \(c\) we define, for all \(v \in c\),

\[
P_t(v) = \frac{d - d_{G_i}(v)}{\sum_{w \in c_i} (d - d_{G_i}(w))}.
\]

In both the binomial and the regular case we refer to the elements of \(c\) as cavities. Assuming that \(c \neq \emptyset\), we define \(c\) by letting \(c_1, c_2, c_3, \ldots \in c\) cavities drawn independently from \(P_t\). Note that \(\mathbb{P}(c_i = \emptyset) = e^{-\Omega(n)}\).

The proof of Proposition 2.8 relies on coupling arguments. Specifically, we will couple \(G\) with three random factor graphs obtained by adding one more constraint of each of the three types of constraints:

- assuming that \(2m_i + m'_i \leq dn - 2\), we obtain \(G_i\) from \(G\) by adding one more constraint \(e_{m_i+1}\) as per Definition 2.5 or 2.6 respectively; if \(2m_i + m'_i > dn - 2\) then we let \(G_i = G\).
- assuming that \(2m_i + m'_i < dn\), we obtain \(G_i\) from \(G\) by adding one more constraint \(a_{m_i+1}\) in accordance with Definition 2.5 or 2.6 respectively; if \(2m_i + m'_i = dn\) then we let \(G_i = G\).
- finally, obtain \(G_i\) from \(G\) by adding one more constraint \(b_{m_i+1}\).

The following lemma expresses the derivative of \(E[\log Z_p(G_i)]\) in terms of these three enhanced factor graphs. Let us observe for future reference that

\[
|\log Z_p(G_i)| \leq n \log q + \beta(m_i + m'_i + m''_i) = O(n),
\]

which follows from the fact that \(G_i\) has at most \(1 + m_i + m'_i + m''_i \leq dn + 1\) constraint nodes, and that the weight functions of the constraint nodes \(e_i, a_i, b_i\) satisfy \(\exp(-\beta) \leq \psi_{e_i}, \psi_{a_i}, \psi_{b_i} \leq 1\).

**Lemma 4.1.** We have

\[
\frac{2}{(1 - \epsilon)dn} \frac{\partial}{\partial t} \mathbb{E}[\log Z_p(G_i)] = -\mathbb{E}[\log Z_p(G_i)] + 2\mathbb{E}[\log Z_p(G_i^t)] - \mathbb{E}[\log Z_p(G_i^r)] + o(1).
\]

**Proof.** Recalling that \(m_i, m'_i, m''_i\) are distributed as the independent Poisson variables \(M_i, M'_i, M''_i\) from (2.7) given the event \(\mathcal{A} = [\sum m_i + M'_i \leq dn, M_i + M'_i + M''_i \leq dn]\) from (2.8), we see that

\[
\mathbb{E}[\log Z_p(G_i)] = \sum_{(m_i, m'_i, m''_i) \in \mathcal{A}} \mathbb{P}[M_i = m, M'_i = m', M''_i = m'' | \mathcal{A}] \mathbb{E}[\log Z_p(G_i) | M_i = m, M'_i = m', M''_i = m''].
\]

The conditional expectation on the right hand side is independent of \(t\). But the means of \(M_i, M'_i, M''_i\) are governed by \(t\). Hence, we need to differentiate \(\mathbb{P}[M_i = m, M'_i = m', M''_i = m'' | \mathcal{A}]\). For \((m_i, m'_i, m''_i) \in \mathcal{A}\) we obtain

\[
\frac{\partial}{\partial t} \mathbb{P}[M_i = m, M'_i = m', M''_i = m'' | \mathcal{A}] = \frac{\partial}{\partial t} \mathbb{P}[M_i = m] \mathbb{P}[M'_i = m'] \mathbb{P}[M''_i = m'']
\]

The product rule yields

\[
\frac{\partial}{\partial t} \mathbb{P}[M_i = m] \mathbb{P}[M'_i = m'] \mathbb{P}[M''_i = m'']
\]

\[= \left( \frac{\partial}{\partial t} \mathbb{P}[M_i = m] \right) \mathbb{P}[M'_i = m'] \mathbb{P}[M''_i = m''] + \mathbb{P}[M_i = m] \left( \frac{\partial}{\partial t} \mathbb{P}[M'_i = m'] \right) \mathbb{P}[M''_i = m'']
\]

\[+ \mathbb{P}[M_i = m] \mathbb{P}[M'_i = m'] \left( \frac{\partial}{\partial t} \mathbb{P}[M''_i = m''] \right)
\]

\[= -\left( \frac{1 - \epsilon}{2}dn \mathbb{P}[M_i = m - 1] - \mathbb{P}[M_i = m] \right) \mathbb{P}[M'_i = m'] \mathbb{P}[M''_i = m'']
\]

\[+ \left( 1 - \epsilon \right)dn \mathbb{P}[M_i = m] \left( \mathbb{P}[M'_i = m' - 1] - \mathbb{P}[M'_i = m'] \right) \mathbb{P}[M''_i = m'']
\]

\[= \left( \frac{1 - \epsilon}{2}dn \mathbb{P}[M_i = m] \mathbb{P}[M'_i = m'] \left( \mathbb{P}[M''_i = m' - 1] - \mathbb{P}[M''_i = m'] \right),
\]

(4.10)
which simplifies to

\[ \frac{\partial}{\partial t} P[M_t = m]P[M'_t = m']P[M''_t = m''] = - \frac{(1-\varepsilon) dn}{2} \left[ P[M_t = m - 1]P[M'_t = m']P[M''_t = m''] - 2P[M_t = m]P[M'_t = m']P[M''_t = m'' - 1] \right]. \]  

(4.11)

Moreover, differentiating \(-P[\mathcal{\Upsilon}] = P[\mathcal{\Upsilon}'] - 1\) gives

\[ \frac{\partial}{\partial t} P[\mathcal{\Upsilon}] = \sum_{(m',m'')} P[M_t = m]P[ M'_t = m' ]P[ M''_t = m''] = \exp(-\Omega(n)). \]  

(4.12)

Combining (4.12) and using \(P[\mathcal{\Upsilon}] = 1 - \exp(-\Omega(n))\), we obtain

\[ - \frac{2}{(1-\varepsilon)dn} \frac{\partial}{\partial t} E[\log Z_\beta(G_t)] = o(1) + \]

\[ \sum_{(m',m'')} E[\log Z_\beta(G_t) | M_t = m, M'_t = m', M''_t = m''] \left[ P[M_t = m - 1]P[M'_t = m']P[M''_t = m''] + P[M_t = m]P[M'_t = m']P[M''_t = m'' - 1] \right]. \]  

(4.13)

By the principle of deferred decisions, if \((m',m'',m''')\in\mathcal{\Upsilon}\) then we can think of \(G_t\) given \(m_t = m, m'_t = m', m''_t = m''\) as resulting from \(G_t\) given \(m_t = m - 1, m'_t = m', m''_t = m''\) via the insertion of one more constraint \(e_{m_t}\). Therefore,

\[ \sum_{(m',m'')} E[\log Z_\beta(G_t) | M_t = m, M'_t = m', M''_t = m''] \left[ P[M_t = m - 1]P[M'_t = m']P[M''_t = m''] \right] = E[1(\{m_t + 1, m'_t, m''_t\} \in \mathcal{\Upsilon})] \log Z_\beta(G'_t)]. \]  

(4.14)

The definition (2.7) of the Poisson variables ensures that \(P\{\{m_t + 1, m'_t, m''_t\} \in \mathcal{\Upsilon}\} = 1 - \exp(-\Omega(n))\). Hence, (4.6) and (4.14) yield

\[ \sum_{(m',m'')} E[\log Z_\beta(G_t) | M_t = m, M'_t = m', M''_t = m''] \left[ P[M_t = m - 1]P[M'_t = m']P[M''_t = m''] \right] = E[\log Z_\beta(G'_t)] + o(1). \]  

(4.15)

Similarly,

\[ \sum_{(m',m'')} E[\log Z_\beta(G_t) | M_t = m, M'_t = m', M''_t = m''] \left[ P[M_t = m]P[m'_t = m''] \right] = E[\log Z_\beta(G'_t)] + o(1), \]  

(4.16)

\[ \sum_{(m',m'')} E[\log Z_\beta(G_t) | M_t = m, M'_t = m', M''_t = m''] \left[ P[M_t = m]P[m'_t = m']P[m''_t = m'' - 1] \right] = E[\log Z_\beta(G''_t)] + o(1). \]  

(4.17)

Thus, the assertion follows from (4.13), (4.15), (4.16) and (4.17).

Let \(\mathcal{C}\) be the event that \(|\mathcal{C}| \geq n^{2/3}\). The choice of the parameters (2.7) ensures that

\[ P[\mathcal{C}] = 1 - \exp(-\Omega(n)). \]  

(4.18)

We proceed to calculate the three expressions on the r.h.s. of (4.17). Recall the function \(\psi_{G_t}\) and the Boltzmann distribution \(\mu_{G_t}\) which correspond to \(G_t\), defined as in (1.13) and (1.15) with \(\gamma\) from (2.11). Also recall the bracket notation from (1.12).

**Lemma 4.2.** We have

\[ E[\log Z_\beta(G'_t)] - E[\log Z_\beta(G_t)] = o(1) - \sum_{\ell=1}^{\infty} \frac{(1-e^{-\beta})^\ell}{\ell} E \left[ 1_{\mathcal{C}} \cdot \langle \sigma_{e_1}, \mu_{G_t} \rangle^\ell \right]. \]
Moreover, conditioned on the event \( C \), the factor graph \( G' \) results from \( G_t \) via the addition of a single constraint \( e_{m_t+1} \) joins, we obtain

\[
\log Z_{\beta}(G') - \log Z_{\beta}(G_t) = \log \sum_{\sigma \in \{0,1\}^n} \psi_{e_{m_t+1}}(\sigma, \sigma) \psi_{G_t,\beta}(\sigma) Z_{\beta}(G_t) = \log \left( \psi_{e_{m_t+1}, \mu_{G_t}} \right). \tag{4.19}
\]

(Here the sum is over all \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \{0,1\}^n \), recalling that \( V_n = \{v_1, \ldots, v_n\} \). In particular, since \( \exp(-\beta) \leq \psi_{e_{m_t+1}}(\sigma) \leq 1 \), we have \( -\beta \leq \log Z_{\beta}(G') - \log Z_{\beta}(G_t) \leq 0 \). Further, conditioned on \( C \), the probability that two cavities \( c_1, c_2 \) chosen independently with distribution \( P_t \) coincide is \( o(1) \). Hence, recalling the construction of the probability distribution \( P_t \) on the set \( \mathcal{C}_t \) of cavities, we notice that the distribution of the pair \( (u, v) \) and the distribution of the pair \( (c_1, c_2) \) have total variation distance \( o(1) \). Consequently, (4.20) yields

\[
E \left[ 1 \cdot \log Z_{\beta}(G') \right] - E \left[ 1 \cdot \log Z_{\beta}(G_t) \right] = o(1) - \sum_{\ell = 1}^{\infty} \frac{(1-e^{-\beta})^\ell}{\ell} E \left[ 1 \cdot \langle \rho_{\sigma, m_t+1}^\ell(\sigma_{c_1}), \mu_{G_t}^\ell \rangle \right]. \tag{4.21}
\]

The assertion follows from (4.19) and (4.21). \( \square \)

**Lemma 4.3.** We have

\[
E \left[ \log Z_{\beta}(G'_{n_t}) \right] - E \left[ \log Z_{\beta}(G_t) \right] = o(1) - \sum_{\ell = 1}^{\infty} \frac{(1-e^{-\beta})^\ell}{\ell} E \left[ 1 \cdot \langle \rho_{\sigma, m_t+1}^\ell(\sigma_{c_1}), \mu_{G_t}^\ell \rangle \right]. \tag{4.22}
\]

**Proof.** Just as in the proof of Lemma 4.2 we have

\[
E \left[ \log Z_{\beta}(G'_{n_t}) \right] - E \left[ \log Z_{\beta}(G_t) \right] = E \left[ 1 \cdot \log Z_{\beta}(G'_{n_t}) \right] - E \left[ 1 \cdot \log Z_{\beta}(G_t) \right] + o(1). \tag{4.23}
\]

Denote by \( v \in V_n \) the variable node adjacent to the new constraint \( a_{m_t+1} \) of \( G_t \). Then conditioned on \( C \) we have

\[
\log Z_{\beta}(G'_{n_t}) - \log Z_{\beta}(G_t) = \log \sum_{\sigma \in \{0,1\}^n} \psi_{a_{m_t+1}}(\sigma, \sigma) \psi_{G_t,\beta}(\sigma) Z_{\beta}(G_t) = \log \left( \psi_{a_{m_t+1}, \mu_{G_t}} \right).
\]

By construction, the variable node \( v \) is distributed according to \( P_t \), the law of \( c_1 \). Hence,

\[
E \left[ 1 \cdot \log Z_{\beta}(G'_{n_t}) \right] - E \left[ 1 \cdot \log Z_{\beta}(G_t) \right] = E \left[ 1 \cdot \log \left( 1 - (1-e^{-\beta})^\ell \right) \langle \rho_{\sigma, m_t+1}^\ell(\sigma_{c_1}), \mu_{G_t}^\ell \rangle \right] = - \sum_{\ell = 1}^{\infty} \frac{(1-e^{-\beta})^\ell}{\ell} E \left[ 1 \cdot \langle \rho_{\sigma, m_t+1}^\ell(\sigma_{c_1}), \mu_{G_t}^\ell \rangle \right]. \tag{4.24}
\]

Combining (4.22) and (4.23) completes the proof. \( \square \)

**Lemma 4.4.** We have

\[
E \left[ \log Z_{\beta}(G'_{n_t}) \right] - E \left[ \log Z_{\beta}(G_t) \right] = o(1) - \sum_{\ell = 1}^{\infty} \frac{(1-e^{-\beta})^\ell}{\ell} E \left[ 1 \cdot \sum_{t=1}^q \rho^\ell_{\sigma, m_t+1}(\tau) \rho^\ell_{\sigma, m_t+1}(\tau), \mu_{G_t}^\ell \right]. \tag{4.25}
\]

**Proof.** This follows from similar manipulations as in the proofs of Lemmas 4.2 and 4.3. \( \square \)

**Proof of Proposition 4.3** Let

\[
\Delta \ell = E \left[ 1 \cdot \langle 1 \cdot (\sigma_{c_1} = \sigma_{c_1}), \mu_{G_t} \rangle - 2 \langle \rho_{\sigma, m_t+1}^\ell(\sigma_{c_1}), \mu_{G_t}^\ell \rangle - \sum_{t=1}^q \rho^\ell_{\sigma, m_t+1}(\tau) \rho^\ell_{\sigma, m_t+1}(\tau), \mu_{G_t}^\ell \rangle \right]. \tag{4.26}
\]

Combining Lemmas 4.1, 4.4 we see that

\[
\frac{2}{(1-e)\, \partial n} \frac{\partial}{\partial t} E \left[ \log Z_{\beta}(G_t) \right] = o(1) + \sum_{\ell = 1}^{\infty} \frac{(1-e^{-\beta})^\ell}{\ell} \Delta \ell. \tag{4.27}
\]
We are going to show that $\Delta_\ell \geq 0$ for all $\ell \geq 1$; then the assertion follows from (4.24).

Thus, fix $\ell \geq 1$ and let $\sigma^{(1)}, \sigma^{(2)}, \ldots, \sigma^{(\ell)}$ denote independent samples from $\mu_{G_1}$. Since the expectation of the product of random variables equals the product of their expectations, we can rewrite $\Delta_\ell$ as

$$
\Delta_\ell = \mathbb{E} \left[ 1 \mathbf{1}_c \cdot \left( \prod_{h=1}^{\ell} \mathbb{1}_c^{(h)} = \mathbf{e}_2^{(h)} \right) - 2 \left( \prod_{h=1}^{\ell} \rho_{\sigma^{(h)}_s, m_{i+1}^{(h)} + 1}^{(h)}(\mathbf{e}_2^{(h)}) \cdot \mu_{G_1} \right) \right] 
$$

$$
= \sum_{c \in c_{\ell}} \mathbb{E} \left[ 1 \mathbf{1}_c \cdot \left( \prod_{h=1}^{\ell} \mathbb{1}_c^{(h)} = \mathbf{e}_2^{(h)} \right) - 2 \left( \prod_{h=1}^{\ell} \rho_{\sigma^{(h)}_s, m_{i+1}^{(h)} + 1}^{(h)}(\mathbf{e}_2^{(h)}) \cdot \mu_{G_1} \right) \right].
$$

(4.25)

To simplify the last expression, we introduce for $\tau \in \{q\}^\ell$.

$$
X_\tau = \sum_{c \in c_{\ell}} P_{\ell}(c) \prod_{h=1}^{\ell} 1 \mathbf{1}_c^{(h)} = \tau_h.
$$

Further, recall the family $(\hat{\rho}_\ell)_{\ell \geq 1}$ of distributions $\hat{\rho}_\ell \in \mathcal{P}(\{q\})$ drawn from $\hat{r} \in \mathcal{P}^2(\{q\})$ from (2.6). Writing $\mathbb{E}'$ for the expectation over $(\hat{\rho}_\ell)_{\ell \geq 1}$ only, let

$$
Y_\tau = \mathbb{E}' \left[ \prod_{h=1}^{\ell} \hat{\rho}_{\sigma^{(h)}_s(\tau_h)} \right].
$$

Since $c_1, c_2$ and $(\rho_{s, m_{i+1}^{(1)}, \rho_{s, m_{i+1}^{(1)}}})_{\ell \geq 1}$ in (4.25) are mutually independent, we can interchange the order in which expectations are taken and rewrite (4.25) as

$$
\Delta_\ell = \sum_{c \in c_{\ell}} \mathbb{E} \left[ 1 \mathbf{1}_c \cdot \left( X_\tau^2 - 2X_\tau Y_\tau + Y_\tau^2, \mu_{G_1} \right) \right] = \sum_{c \in c_{\ell}} \mathbb{E} \left[ 1 \mathbf{1}_c \cdot (X_\tau - Y_\tau)^2, \mu_{G_1} \right] \geq 0.
$$

(4.26)

Finally, the assertion follows from (4.24) and (4.26).

4.3. Proof of Corollary (2.9) We begin by estimating the partition function of $G_1$.

**Lemma 4.5.** For any $\delta > 0$ there is $\varepsilon > 0$ such that for all large enough $n$ we have $\mathbb{E}[\log Z(G_1)] \leq \mathbb{E}[\log Y] + \delta n$.

**Proof.** Let $d_1, d_2, \ldots, d_n$ denote the degrees of the variable nodes $v_1, \ldots, v_n$ in $G_1$. Each of the constraints $d_j$ is adjacent to only one of the variable nodes from $V_1$. For each $v_1$, suppose the constraints $a_{i_1}, \ldots, a_{i_d}$ are adjacent to $v_1$ and let $\rho_{a_{i_1, h}}$ denote the distribution associated with $a_{i_1}$, for $h \in \{d_j\}$. (In the definition of the interpolation scheme, this distribution is denoted $\rho_{a_{i_1, h}}$.) Then we can write

$$
\mathbb{E}[\log Z(G_1)] = \mathbb{E} \log \sum_{\sigma_1} \gamma(\sigma_1) \sum_{(\sigma),(\gamma) \in \{q\}^{\ell}} \prod_{i=1}^{m} \psi_{a_j}(\sigma_1, \sigma_2, \ldots, \sigma_n) 
$$

$$
= \mathbb{E} \log \sum_{\sigma_1} \gamma(\sigma_1) \prod_{i=1}^{n} \sum_{a_{i_1}} \prod_{j=1}^{m} \left( 1 - (1 - \exp(-\beta)) \rho_{a_{i_1, h}}(\sigma_1, \ldots, \sigma_n) \right).
$$

(4.27)

Suppose first that $G$ is the binomial random graph and let $d'_1, d'_2, \ldots, d'_n \sim \text{Po}(1 - \varepsilon) d$ be independent random variables. The construction of $G_1$ ensures that $(d'_1, \ldots, d'_n)$ is distributed as $(d'_1, \ldots, d'_n)$ given $d'_1 + \cdots + d'_n \leq d n$. Since this event occurs with probability $1 - \exp(-\Omega(n))$, we conclude that $d_{Y_1}((d'_1, \ldots, d'_n)) = \exp(-\Omega(n))$. Therefore, (4.27) yields

$$
\mathbb{E}[\log Z(G_1)] = \mathbb{E} \log \sum_{\sigma_1} \gamma(\sigma_1) \prod_{i=1}^{n} \sum_{a_{i_1}} \prod_{j=1}^{m} \left( 1 - (1 - \exp(-\beta)) \rho_{a_{i_1, h}}(\sigma_1, \ldots, \sigma_n) \right) + o(n).
$$

(4.28)

To compare this last expression with $Y$ from (2.9), let $\Delta_\ell \sim \text{Po}(\varepsilon d)$ be independent random variables for $\ell \in [n]$. Then we can couple the $D_\ell$ from (2.9) and the $d'_\ell$ from (4.28) by letting $D_\ell = d'_\ell + \Delta_\ell$. Thus, since each factor in (4.28) lies in the interval $[\exp(-\beta), 1]$, we obtain the estimate

$$
\mathbb{E}[\log Z(G_1)] \leq \mathbb{E}[\log Y] + \beta \mathbb{E} \sum_{i=1}^{n} \Delta_\ell + o(n) \leq \mathbb{E}[\log Y] + \beta \varepsilon d n + o(n),
$$

(4.29)
whence the assertion follows. Second, if $G$ is the random regular graph then $D_i = d$ deterministically. Hence, letting $\Delta_i = d - D_i$, we obtain (4.29) in this case as well.

Proof. Let Lemma 4.7. \hfill \Box

Proof of Corollary 2.9. The corollary follows from Proposition 2.7, Proposition 2.8 and Lemma 4.5 by taking $\delta$ to zero. \hfill \Box

4.4. Proof of Proposition 2.12. We will calculate the limits of the two terms appearing in (2.12) separately. To facilitate a unified treatment, let $a \in \mathcal{P}([0,1])$ be the given probability distribution in the binomial case and let $a = \delta$ for $a \in [0,1]$ in the case of the random regular graph. Also let $(\alpha_i)_{i=1}^\infty$ be independent samples from $a$.

Lemma 4.6. We have

$$\lim_{y \to 0} \lim_{\beta \to \infty} \mathbb{E} \left[ \log \left( \mathbb{E} \left[ \left( \sum_{r=1}^q \prod_{h=1}^{D_i} 1 - (1-e^{-\beta}) \rho_{1,1,h}(\tau) \right)^y \right] \right) \right] = \mathbb{E} \left[ \log \left( \sum_{i=0}^{q-1} \left( -1 \right)^i \left( \prod_{h=1}^{D_i} (1 - (i+1)(1 - \alpha_h)/q) \right) \right) \right].$$

Proof. For $c \in [q]$ let $U_c = \{ \forall h \in \{D_i\} : \rho_{1,1,h} \neq \delta_c \}$ and let $U = \bigcup_{c \in [q]} U_c$. Then

$$0 \leq \sum_{r=1}^q \prod_{h=1}^{D_i} 1 - (1-e^{-\beta}) \rho_{1,1,h}(\tau) \leq q \exp(-\beta) \quad \text{if } U \text{ does not occur},$$

$$(1 - (1-e^{-\beta})/q)^{D_i} \leq \sum_{r=1}^q \prod_{h=1}^{D_i} 1 - (1-e^{-\beta}) \rho_{1,1,h}(\tau) \leq q \quad \text{if } U \text{ occurs}.$$

(The lower bound in the first line is trivial, while the upper bound follows since $U_c$ fails for each $c \in [q]$. The upper bound in the second line is trivial, while the lower bound follows by taking the term corresponding to some colour $c$ where $U_c$ holds.) Consequently, we obtain

$$\lim_{y \to 0} \lim_{\beta \to \infty} \mathbb{E} \left[ \log \left( \mathbb{E} \left[ \left( \sum_{r=1}^q \prod_{h=1}^{D_i} 1 - (1-e^{-\beta}) \rho_{1,1,h}(\tau) \right)^y \right] \right) \right] = \mathbb{P}[U \mid \mathcal{R}] \quad (4.30)$$

pointwise. Furthermore, by inclusion/exclusion,

$$\mathbb{P}[U \mid \mathcal{R}] = \mathbb{P} \left[ \bigcup_{c \in [q]} U_c \mid \mathcal{R} \right] = \sum_{k=1}^{q} (-1)^{k-1} \sum_{Q \subset [q] : |Q| = k} \mathbb{P} \left[ \bigcap_{c \in Q} U_c \mid \mathcal{R} \right]. \quad (4.31)$$

Since $\rho_{1,1,1,\ldots,\rho_{1,1,D_i}$ are mutually independent given $\mathcal{R}$, for any set $Q \subset [q]$ of size $k$ we find that

$$\mathbb{P} \left[ \bigcap_{c \in Q} U_c \mid \mathcal{R} \right] = \prod_{h=1}^{D_i} \left( 1 - k(1 - \alpha_h)/q \right)$$

using (2.16)–(2.18). Hence, (4.31) yields

$$\mathbb{P}[U \mid \mathcal{R}] = \sum_{k=1}^{q} (-1)^{k-1} \binom{q}{k} \prod_{h=1}^{D_i} (1 - k(1 - \alpha_h)/q). \quad (4.32)$$

Finally, the assertion follows from (4.30) and (4.32). \hfill \Box

Lemma 4.7. We have

$$\lim_{y \to 0} \lim_{\beta \to \infty} \mathbb{E} \left[ \log \left( \mathbb{E} \left[ \left( 1 - (1-e^{-\beta}) \sum_{r=1}^q \rho_{1,1}(\tau) \rho_{1,1}(\tau)^\prime \right)^y \right] \right) \right] = \mathbb{E} \left[ \log(1 - (1 - \alpha_1)(1 - \alpha_2)/q) \right].$$

Proof. Let $U$ be the event that there exists $c \in [q]$ such that $\rho_{1,1} = \rho_{1,1}^\delta = \delta_c$. Then

$$0 \leq 1 - (1-e^{-\beta}) \sum_{r=1}^q \rho_{1,1}(\tau) \rho_{1,1}(\tau)^\prime = \exp(-\beta) \quad \text{if } U \text{ occurs},$$

$$1 - (1-e^{-\beta})/q \leq 1 - (1-e^{-\beta}) \sum_{r=1}^q \rho_{1,1}(\tau) \rho_{1,1}(\tau)^\prime \leq 1 \quad \text{if } U \text{ does not occur}. \hfill \Box
(The sum over $\tau$ equals 0 if $\rho_{t,1}^\prime$ and $\rho_{t,1}^\prime$ are atoms on two different colours, and equals $1/q$ otherwise.) Therefore, we have pointwise convergence

$$\lim_{y \to 0} \lim_{\beta \to -\infty} \mathbb{E} \left[ \left( 1 - (1 - e^{-\beta}) \sum_{t=1}^{q} \rho_{t,1}^\prime(\tau) \rho_{t,1}^\prime(\tau) \right)^y \right] = 1 - \mathbb{P} [ U \mid \mathcal{R} ] . \quad (4.33)$$

Since $\mathbb{P} [ U \mid \mathcal{R} ] = (1 - \alpha_1)(1 - \alpha_2)/q$, using (2.16)–(2.18), the assertion follows from (4.33). \hfill \square

**Proof of Proposition 5.1.** The proposition follows from Lemmas 4.6 and 4.7 immediately. \hfill \square

## 5. Asymptotics

We perform asymptotic expansions of $\Sigma_{d,q}(\cdot)$, $\Sigma_{d,q}^*(\cdot)$ in the limit of large $q$ to prove Corollaries 1.2 and 1.4. In this section, the notation $\tilde{O}_q(\cdot)$ suppresses polynomials in $\log q$, and both $O_q(\cdot)$ and $\tilde{O}_q(\cdot)$ refer to the limit $q \to \infty$.

### 5.1. Proof of Corollary 1.2

Write $\Sigma_{d,q}(\alpha) = S - T$, where

$$S = \log \sum_{i=0}^{q-1} (-1)^i \left( \frac{q}{i+1} \right) (1 - (i + 1) (1 - \alpha)/q)^d, \quad T = \frac{d}{2} \log \left( 1 - (1 - \alpha)^2/q \right) .$$

We will let

$$\alpha = \frac{1}{2q}, \quad d = (2q - 1) \log q - c \quad (5.1)$$

with $c = O_q(1)$, and expand $S$ and $T$ asymptotically in the limit $q \to \infty$. Substituting for $d$ in $S$ gives

$$S = \log \sum_{i=0}^{q-1} (-1)^i \left( \frac{q}{i+1} \right) \exp \left( ((2q - 1) \log q - c) \log \left( 1 - (i + 1) (1 - \alpha)/q \right) \right) . \quad (5.2)$$

Observe that

$$\log(1 - (1 - \alpha)/q) = \frac{1}{q} + O(q^{-1}) , \quad (5.3)$$

and so, for the $i = 0$ term we have the expansion

$$\exp \left( ((2q - 1) \log q - c) \log \left( 1 - (1 - \alpha)/q \right) \right) = \exp \left( -((2q - 1) \log q - c)/q + \tilde{O}_q(q^{-2}) \right) = \exp \left( -2 \log q + (\log q)/q + c/q + \tilde{O}_q(q^{-2}) \right) . \quad (5.4)$$

Moreover, for $i \geq 1$ we have

$$\exp \left( ((2q - 1) \log q - c) \log \left( 1 - (i + 1) (1 - \alpha)/q \right) \right) = q^{-(i+1)} \left( 1 + \tilde{O}_q \left( \frac{1}{q} + \frac{(i+1)^2}{q^2} \right) \right) . \quad (5.5)$$

Plugging (5.4) and (5.5) into (5.2) gives

$$S = \log \left( q \cdot \exp \left( -2 \log q + (\log q)/q + c/q + \tilde{O}_q(q^{-2}) \right) - \frac{1}{2} q^{-2} + \tilde{O}_q(q^{-2}) \right) \right)$$

$$= -\log q + \frac{\log q}{q} + \frac{c}{q} + \tilde{O}_q(q^{-2}) + \log \left( 1 - 1/(2q) + \tilde{O}_q(q^{-2}) \right)$$

$$= -\log q + \frac{\log q}{q} + \frac{2c - 1}{2q} + \tilde{O}_q(q^{-2}) .$$

Similarly, substituting for $\alpha$ and $d$ in $T$ gives

$$T = \left( q \log q - \frac{1}{2} \log q - c/2 \right) \cdot \left( -1/q + 1/(2q^2) + O_q(q^{-3}) \right) = -\log q + \frac{\log q}{q} + \frac{c}{2q} + \tilde{O}_q(q^{-2}) .$$

Hence,

$$\Sigma_{d,q}(\alpha) = S - T = \frac{c - 1}{2q} + \tilde{O}_q(q^{-2}) .$$

Consequently, if $c \leq 1 - \epsilon_q$ where $\epsilon_q \to 0$ slowly enough then $\Sigma_{d,q}(\alpha) < 0$ for large enough $q$. This completes the proof of Corollary 1.2.
5.2. Proof of Corollary 1.4 With $\alpha, d$ as in 5.1 we consider the distribution $\alpha = \delta_d$. With $D \sim \text{Po}(d)$ let

$$S = \mathbb{E} \left[ \log \sum_{i=0}^{q-1} (-1)^i \left( \frac{q}{i+1} \right) (1 - (i+1)(1 - \alpha)/q)^D \right], \quad T = \frac{d}{2} \log (1 - (1 - \alpha)^2/q). \quad (5.6)$$

First,

$$\sum_{i=0}^{q-1} (-1)^i \left( \frac{q}{i+1} \right) (1 - (i+1)(1 - \alpha)/q)^D \leq 1. \quad (5.7)$$

To see this, we interpret the sum in the middle as an inclusion/exclusion formula. Namely, choose $c_1, \ldots, c_D \in \{0, 1, \ldots, q\}$ independently such that the probability of drawing 0 equals $\alpha$ and the probability of drawing $i \in [q]$ equals $(1 - \alpha)/q$. Then the sum equals the probability of the event $|q \setminus \{c_1, \ldots, c_D\} \neq \emptyset$, which is clearly lower bounded by $\alpha^D = (2q)^{-D}$. Poisson tail bounds show that $\mathbb{P}(|D - d| \geq 10\sqrt{d}\log q) = O_q(q^{-3})$ and combining this with (5.7) gives

$$S = \mathbb{E} \left[ \log \sum_{i=0}^{q-1} (-1)^i \left( \frac{q}{i+1} \right) (1 - (i+1)(1 - \alpha)/q)^D \right] |D - d| \leq 10\sqrt{d}\log q + O_q(q^{-2}).$$

Hence, let $\Delta$ be distributed as $D - d$ given $|D - d| \leq 10\sqrt{d}\log q$. Then

$$S = \mathbb{E} \left[ \log \sum_{i=0}^{q-1} (-1)^i \left( \frac{q}{i+1} \right) (1 - (i+1)(1 - \alpha)/q)^{d + \Delta} \right] + O_q(q^{-2})$$

$$= \mathbb{E} \left[ \log \sum_{i=0}^{q-1} (-1)^i \left( \frac{q}{i+1} \right) \exp((d + \Delta) \log(1 - (i+1)(1 - \alpha)/q)) \right] + O_q(q^{-2}). \quad (5.8)$$

For the $i = 0$ term, using (5.3) and (5.3), we have the expansion

$$\exp((d + \Delta) \log(1 - (1 - \alpha)/q)) = \exp\left(-\frac{(d + \Delta)}{q} + \tilde{O}_q(q^{-2})\right)$$

$$= \exp\left(-2\log q + 2\log q/q + c/q - \Delta/q + \tilde{O}_q(q^{-2})\right)$$

$$= q^{-2} \left(1 + 2\log q/q + c/q + \tilde{O}_q(q^{-2})\right) \exp(-\Delta/q). \quad (5.9)$$

Moreover, for $i \geq 1$, using the fact that $\Delta/q = \tilde{O}(q^{-1/2})$, we obtain

$$\exp\left((d + \Delta) \log\left(1 - (i+1)(1 - \alpha)/q\right)\right) = q^{-2(i+1)} \left(1 + \tilde{O}_q\left(1 + \frac{(i+1)^2}{q^2}\right)\right). \quad (5.10)$$

Plugging (5.10) and (5.11) into (5.8), we obtain

$$S = \mathbb{E} \left[ \log \sum_{i=0}^{q-1} \left(1 + \log q/q + c/q + \tilde{O}_q(q^{-2})\right) \exp(-\Delta/q) \right] + O_q(q^{-2})$$

$$= -\log q + \log \left[1 + \log q/q + c/q + \tilde{O}_q(q^{-2})\right] - 2\log q/q + \tilde{O}_q(q^{-3/2}) - \mathbb{E} \left[\Delta/q\right] \log \left(1 + 1/(2q) + \tilde{O}_q(q^{-3/2})\right)$$

$$= -\log q + \frac{\log q}{q} + \frac{2c - 1}{q} - \mathbb{E} \left[\Delta/q\right] + \tilde{O}_q(q^{-3/2}). \quad (5.12)$$

Since $d = \mathbb{E}[D]$ and since conditioning on $|D - d| \leq 10\sqrt{d}\log q$ does not shift the mean of $D$ by more than $O_q(1/q)$, we obtain $\mathbb{E}[\Delta] = O_q(1/q)$. Using this and (5.12) yields

$$S = -\log q + \log q/q + \frac{2c - 1}{2q} + \tilde{O}_q(q^{-3/2}). \quad (5.13)$$

Combining (5.13) with the expansion (5.1) of $T$, we finally obtain

$$\Sigma^*_{d,q}(\delta) = S - T = \frac{c - 1}{2q} + \tilde{O}_q(q^{-3/2}).$$

Thus, setting $c \leq 1 - \epsilon_q$ with $\epsilon_q \to 0$ slowly, we see that $\Sigma^*_{d,q} < 0$ for large enough $q$. This completes the proof of Corollary 1.4

Acknowledgment. We thank Viktor Harangi and an anonymous reviewer for their very careful reading of our manuscript and their extremely accurate and helpful comments, which led to several improvements and corrections.
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