The completely coupled region of supercritical contact processes*

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Summary

We show the existence of an unbounded space-time cone in any dimension within which the entire descendancy of a contact process from full occupancy may with strictly positive probability be identical to that of the process started from the single site at its apex, depending merely on the existence of a successful coupling among the two processes. The proof relies upon refining a previously known coupling argument [T11] and, in particular, neither goes through block construction arguments, nor invokes large deviation estimates associated to the shape theorem. We observe that the result leads to extensions of the one dimensional so-called "formation of the descendancy barriers" of Andjel et.al. [AMPV10] in the symmetric case, in addition to the corresponding consequence of the multidimensional refinements of the shape theorem of Garet and Marchand [GM14] in the symmetric nearest neighbors case.

The contact process is an extensively studied spatial Markov process introduced by Harris [H74] that can be interpreted as a simple model for spatial growth, or the spread of an infection in a spatially structured population. The class of translation-invariant and finite range contact processes \( \xi_t \) we consider here may be briefly described as follows. Let \( Z_M \) be the simple graphs with vertices the integers \( \mathbb{Z} \), and edges \( E_M \) placed amongst pairs of sites at distance not greater than a finite integer \( M \geq 1 \). Regarding sites in \( \xi_t \) as occupied by a particle and others as vacant, the process with parameter vector \( (\mu_i; \ i = -M, \ldots, M, i \neq 0) \) evolves according to the following local prescription: (i) particles die at rate 1, (ii) a particle at site \( x \) gives

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birth to new ones at rate $\mu_{y-x}$ to each site $y$ such that $xy \in E_M$, and (iii) there is at most one particle per site, that is, births at occupied sites are disallowed. Note that, equivalently, the rate at which births occur can be any linear combination of the parameter rates, and that deaths may occur at any constant rate. Here we have opted to work out the details in dimension one for simplicity of exposition, whereas extensions of results and arguments in the proofs to higher dimensions, owing to being readily analogous, are dealt with in the corresponding remarks paragraph below. For background and standard terminology regarding the process we refer the reader to the corresponding chapters in the books \[L85, D95, L99\]. In the following we say that $\xi_t$ is supercritical when the parameter values are sufficiently large to insure that there is a positive probability for the process started from a finite set to produce infinite descendancy; note that, although this is not the classical definition of supercriticality, which is, for instance in the uniform, symmetric interaction case, i.e. $\mu := \mu_i$ for all $i$, that $\mu > \mu_c := \inf\{\mu : P(\xi_t \neq \emptyset \forall t) > 0\}$, the two definitions are equivalent for it is known that the process dies out when the parameters are on the critical surface, cf. \[BG90, BG94\]. Let us now state the result.

**Theorem 1.** Let $\xi^O_t$ and $\xi^Z_t$ be a supercritical contact process started from $\{0\}$ and $\mathbb{Z}$ respectively; further let $r_t = \sup \xi^O_t$ and $l_t = \inf \xi^O_t$ be the rightmost and leftmost sites of $\xi^O_t$ respectively. Let $I_t = [(\beta + \epsilon)t, (\alpha - \epsilon)t]$, $t \in \mathbb{R}_+$, where $\alpha$ (resp. $\beta$) is the asymptotic velocity of $r_t$ (resp. $l_t$) on survival, which are assumed to be strictly positive (resp. negative) and where $\epsilon > 0$ is sufficiently small such that $\beta + \epsilon < 0$ and $\alpha - \epsilon > 0$. We then have that

$$\xi^Z_t \cap I_t = \xi^O_t \cap I_t, \text{ for all } t \text{ has positive probability.}$$

The first formulation of a version of the above statement appears to be a key result in Andjel et.al \[AMPV10\], where it may be found, along with its proof, within section 2.2 there, and where the proviso on asymptotic velocities given here is dispensed as equivalent to supercriticality in the symmetric interaction case treated there; cf. also with Valesin \[V10\] for other applications this result founds. Regarding other available related resources in the literature, we also refer to Theorem 1.1 of Garet and Marchand \[GM14\] which offers large deviations estimates associated with the notable extension of the shape theorem for the process in a special case of a

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It may be useful for the sequel to reconceptualize this last rule and regard that those births take place, but result in merging and coalescence with the already extant, at the site that birth is given, particle.
random environment treated earlier on by Andjel [A92] which is accomplished by the authors in [GM12] by means of building upon various large deviations estimates of the shape theorem. Regarding the proof of Theorem 1 given here, we mention that the technique goes through a refinement of the argument employed in showing that there is a positive chance for two contact processes started from all sites and from any finite set to agree on that set for all times, pointed out in [T11], Chpt. 4, Remark 2] for filling in some lines in the proofs of the central limit theorem for the endmost particles of the one-dimensional non-nearest neighbors contact processes.

Theorem 1 will be derived as a consequence of Proposition 2 given below. Prior to this, we give some introductory, explanatory remarks regarding this statement in the special nearest-neighbors interaction one-dimensional case first. Under this simplifying assumptions, the process started from the site at the origin and that started from all sites may be coupled to agree amongst the endmost sites of the former provided its descendancy is not empty; this is a simple consequence of two-dimensional path intersection properties in a Harris type graphical representation [H78]. Hence, in the supercritical regime, the almost sure existence of a strictly positive asymptotic velocity of the endmost sites (a consequence of Ligget’s [L85] version of Kingman’s subadditive theorem [K68], originally derived in Durrett [D80]) provides that, on the set of survival, the two corresponding descendant processes are identical within any linearly growing from the origin at rate smaller than the asymptotic velocity limits interval for all sufficiently large times, that is, the lower inclusion of the asymptotic shape theorem (the upper inclusion counterpart being that the descendants of the former process on survival are contained within any such interval growing at rate larger than the asymptotic velocity for all sufficiently large times). In view of these comments, we may phrase the contents of Theorem 1 as showing that the coupling event associated with the lower inclusion of the asymptotic shape theorem for any supercritical contact processes may commence instantly with positive probability, provided that the corresponding random coupling time is almost surely finite.

In [DSS7, §5 and §6], Durrett and Schonmann derive the following result by means of the renormalized construction associated to their extension of the nearest-neighbors rescaling result of Durrett and Griffeath [DG83] to translation invariant, finite range, discrete time one-dimensional contact processes. (Another outline of their arguments may also be found below).

Note that we shall use coordinatewise notation as well when convenient, that is
we write $\xi(\cdot) = 1(\cdot \in \xi)$, where $1(\cdot)$ denotes the indicator function.

**Proposition 2** ((2), §6 in [DSS7]). Grant the assumptions in Theorem 1 and let $R_t = \sup_{s \leq t} r_s$ and $L_t = \inf_{s \leq t} l_s$. We have that

$$\{ x : \xi_t^0(x) = \xi_t^2(x) \text{ and } L_t \leq x \leq R_t \} \supseteq I_t \cap \mathbb{Z}, \text{ for all } t \text{ sufficiently large,}$$

almost surely on $\{ \xi_t^0 \neq \emptyset, \text{ for all } t \}$.

**Proof of Theorem 1.** Familiarity with the construction of $(\xi^A_t : A \subset \mathbb{Z})$ by a graphical representation, a realization of which will be denoted typically by $\omega$, and standard associated terminology is assumed, see for example the introductory sections in [D95, L99]. We will use the notation: for all standard associated terminology is assumed, see for example the introductory sections in [D95, L99]. We will use the notation: for all $\omega \in E_1$, $\omega \in E_2$ a.e. to denote that $\mathbb{P}(\{ \omega : \omega \in E_1, \omega \notin E_2 \}) = 0$ (where a.e. stands for almost everywhere (on $E_1$)).

Let $A_n = \{ \xi^n_t \cap I_t = \xi^O_t \cap I_t, \text{ for all } t \geq n \}$, for integer $n \geq 0$. Proposition 2 implies that for all $\omega \in \{ \xi^O_t \neq \emptyset, \text{ for all } t \}$, $\omega \in \{ \xi^n_t \cap I_t = \xi^O_t \cap I_t, \text{ for all } t \geq t_0 \}$, for some $t_0$, a.e.. Hence $\mathbb{P}(\cup_{n \geq 0} A_n)$ equals $\mathbb{P}(\xi^O_t \neq \emptyset, \text{ for all } t)$ which is positive as the process is supercritical, which implies, say, by contradiction, that there is $n_0$ finite such that $\mathbb{P}(A_{n_0}) > 0$.

We show that this last conclusion implies that, indeed, $\mathbb{P}(A_0) > 0$. Let $A'_{n_0}$ be such that $\omega' \in A'_{n_0}$ if and only if there exists $\omega \in A_{n_0}$ such that $\omega$ and $\omega'$ are identical realizations except perhaps from any $\delta$-symbols (death marks) in $B_{n_0} = V \times [0, n_0]$, where $V = \{ -v_1, \ldots, v_r \}$ and $v_r, v_l$ are the smallest integer greater than $(\alpha - \epsilon)n_0$ and than $(\beta + \epsilon)n_0$ respectively. Let further $\hat{A}_n = \{ \xi^n_t \cap I_t = \xi^V_t \cap I_t, \text{ for all } t \geq n \}$, and let $F$ be the event that no $\delta$-symbols appear in $B_{n_0}$. We have that

$$\mathbb{P}(A'_{n_0} \cap F) = \mathbb{P}(A'_{n_0}) \mathbb{P}(F) \geq \mathbb{P}(A_{n_0}) e^{-(2V+1)n_0} > 0$$

and that $\hat{A}_0 \supseteq A'_{n_0} \cap F$, where to see that the last claim is true note that if $\omega$ and $\omega'$ are identical except that $\omega'$ does not contain any $\delta$-symbols in $B_{n_0}$ that possibly exist in $\omega$, then $\omega \in A_{n_0}$ implies that $\hat{\omega}' \in \hat{A}_{n_0}$, and indeed $\omega' \in \hat{A}_0$. Hence $\mathbb{P}(\hat{A}_0) > 0$.

\footnote{To infer this, consider sampling from $A'_{n_0} \cap F$ and note that adjusting the death marks by mapping on $F$ results in enlarging $\xi^O_t$ and further that, as paths outside of the box $B_{n_0}$ are left intact by such adjustments, particles of $\xi^V_t \times t$ cannot go through $\xi^O_t \cap I_t \times t$, for all $t \in [n_0, \infty)$, via trajectories not intersecting with the box (property inherited from $A_{n_0}$). Further, as $B_{n_0} \subset (\xi^V_t \times t : t \leq n_0)$, those particles cannot go through $\xi^V_t \cap I_t \times t$, for all $t \in [n_0, \infty)$, via paths intersecting with the box, because they are assimilated (by merging and coalescence) to $\xi^V_t$. Finally, they cannot go through $\xi^V_t \cap I_t \times t$, for $t \in [0, n_0)$, simply because by definition of $V$, $I_t \times t \subset B_{n_0}$.}
and from that the result follows by the Markov property for $\xi_t^O$ at an appropriately chosen sufficiently small time.

Higher dimensions. The method developed in [DS87], §5, 6 relies on a simplified version of the imbedding arguments in [DG82], §3 which allows giving a direct proof of Proposition 2 via a sharpened version of hypothesis (7) in DG82, (1) below. The extension of the approach from [DS87] in higher dimensions is carried out in [§5, Durrett D91], see also [§5, Bezuidenhout and Grimmett BG90], from which the analog of Proposition 2 is similarly also derived directly by employing the rescaling results of [BG90] in the symmetric (translation invariant) finite-range interaction case, and that in [BG91] for the case that the interaction is not necessarily symmetric. To see that in the extension of these arguments to higher dimensions symmetry is not necessitated, note that only its consequence that $\lim\inf_{t \to \infty} S_t(i)/t > 0$, for all $i$, where $S_t(i) = \max\{x \cdot e_i : x \in \xi_t^O\}$ and $e_i$ has one coordinate $= \pm 1$, and all others $= 0$, is actually used there. The extension of the arguments in the proof of Theorem 1 are omitted as directly analogous.

Outline of proof of Proposition 2. We follow arguments from §5, 6 in [DS87] that regard the discrete-time version of the process, giving some additional remarks and referring to the original work for more details whenever necessary. Consider the graphical representation for a supercritical contact process which satisfies the assumptions given in Theorem 1. We will argue that for any $\epsilon > 0$ there exist $C, \gamma \in (0, \infty)$ such that

$$\mathbb{P}(\xi_t^O(x) \neq \xi_{2t}^O(x), \xi_t^O \neq \emptyset) \leq C e^{-\gamma t}$$

(1)

for all $x \in I_{2t}$ and $t \geq 0$. To see that this suffices, note that the result for integer times then follows from (1) and the 1st Borel-Cantelli lemma immediately, since

$$\sum_{n \geq 1} \mathbb{P}\left( \bigcup_{x \in I_{2n}} \xi_{2n}^O(x) \neq \xi_{2n}^O(x) \mid \xi_t^O \neq \emptyset, \text{ for all } t \right) < \infty,$$

where we first used that $\mathbb{P}(\xi_t^O \neq \emptyset \cap \{\xi_t^O \neq \emptyset, \text{ for all } t\})$ is exponentially bounded in $t$, a result proved by emulating a standard version of a restart argument, see Theorem 2.30 (a) in [L99]. Whereas, obtaining the result for all $t$ then follows elementarily by using a ”filling in” argument, emulating for instance (4.2.2) in Chpt. 4 in [T11].

To show (1) one considers the dual process $(\tilde{\xi}_s^x; 0 \leq s < t)$ defined on the same graphical representation by reversal of arrows over the time interval $(t, 2t]$. Note that this process is independent of $(\xi_s^O, 0 \leq s \leq t)$ by independence of the Poisson process.
in disjoint parts of the representation and that its distribution w.r.t. the space-
time point $x \times t$ is equal to a copy of the process w.r.t. the origin and parameters
$
\mu_t = \mu_{-t}.
$(Hence, observe that in particular, if $\tilde{t}_s = \inf\{y : y \in \tilde{\xi}^x_s\}$, $(\tilde{t}_s - x)$ is equal
in distribution to $(-r_s)$, and similarly for $\sup\{y : y \in \tilde{\xi}^x_s\}$). Then, (11) follows by
showing that, there exist $C, \gamma \in (0, \infty)$, such that
\[
\mathbb{P}(\xi^O_t \cap \tilde{\xi}^x_t = \emptyset, \xi^O_t \neq \emptyset, \tilde{\xi}^x_t \neq \emptyset) \leq C e^{-\gamma t},
\]
for all $x \in I_{2t}$, where, to see that this suffices, note that $\{\xi^O_t \cap \tilde{\xi}^x_t \neq \emptyset, \xi^O_t \neq \emptyset, \tilde{\xi}^x_t \neq \emptyset\}$
is contained in $\{\xi^O_t(x) = \xi^x_t(x) = 1, \xi^O_t \neq \emptyset\}$.

The proof of (2) then goes through imbedding the rescaled coupled with oriented
site percolation construction in order to deduce this from the corresponding result
for the last process with density arbitrarily close to 1. To show this first, let $K_n$
be independent retaining probability $p_S < 1$ oriented site percolation on the usual
lattice $\mathbb{L}^2$ with set of sites the space-time points $(y, n) \in \mathbb{Z}^2$ such that $y + n$ is even
and $n \geq 0$ obtained by adding a bond from each such point $(y, n)$ to $(y - 1, n + 1)$
and to $(y + 1, n + 1)$. Further, let $\tilde{K}_n$ be an independent copy of $K_n$ and let also
$X_n = \{x_1, \ldots, x_{[cn]}\}$ and $\tilde{X}_n = \{\tilde{x}_1, \ldots, \tilde{x}_{[cn]}\}$ be collections of points at level $n$
with spatial coordinate $y \leq an$, $a < 1$, $c > 0$. Furthermore, let $E_k$ denote the event that
both $x_k \in K_n$ and $\tilde{x}_k \in \tilde{K}_n$, and also let $(E'_k)$ denote an independent thinning of
probability $p' > 0$ of the $(E_k)$, $k = 1, \ldots, [cn]$. We can now state the following easy
consequence of Theorem 1 in [TF] we need to employ. If $p_S$ is sufficiently close to 1
then, for any $c > 0$, there are $q < 1$ and $C < \infty$ such that
\[
\mathbb{P}\left(\bigcap_{k=1}^{[cn]} E'_k, K_n \neq \emptyset, \tilde{K}_n \neq \emptyset\right) \leq C q^n,
\]
for any $X_n$, $\tilde{X}_n$, and where $E'_k$ denotes the complement of $E_k$.

Assuming that the comparison of $(\xi^O_s, 0 \leq s < t)$ with $(K_n)$ takes effect (success-
ful embedding) by time 1, then the renormalized sites spread across the entire
interval $I_{nT}$, where $n$ is such that $nT < t - 1 < (n + 1)T$ and $T$ is the correspond-
ing renormalized time constant, hence that if the comparison of $(\tilde{\xi}^x_s, 0 \leq s < t)$
with $(\tilde{K}_n)$ also takes effect after time 1, then we have that, for any $x$, there is a
$c = c(\alpha, \beta) > 0$ such that $[ct]$ renormalized sites of the two processes spatially over-
lap (to check this, take the observation following the definition $\tilde{\xi}^x_s$ into account and
note that the rightmost (resp. leftmost) renormalized sites in a successful construc-
tion advance with asymptotic velocity $\alpha - \epsilon$ (resp. $\beta + \epsilon$)). From there, we derive (2)
from (3) by simply pairing up such spatially overlapping sites and choosing $p' > 0$
there to be the probability of joining a pair by a path spatially constrained over the
span of the paired sites for providing with independence among joining of distinct
pairs. Finally, to work around the difficulty arising from an arbitrary starting point
of the successful embeddings, one employs a “restart” technique that incorporates
some basic geometrical considerations, see e.g. Proposition 2.8 in [T11]. Following
the arguments there, this difficulty may be overcome by showing that outside of
exponentially small probability in \( t \) one is deduced to a scenario for which the con-
clusion regarding the order of spatially overlapping renormalized sites for the two
processes again holds, and the last argument applies to finish the proof.

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