GLUING CURVES OF GENUS 1 AND 2 ALONG THEIR 2-TORSION

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Abstract. Let $X$ (resp. $Y$) be a curve of genus 1 (resp. 2) over a base field $k$ whose characteristic does not equal 2. We give criteria for the existence of a curve $Z$ over $k$ whose Jacobian is up to twist $(2,2,2)$-isogenous to the products of the Jacobians of $X$ and $Y$. Moreover, we give algorithms to construct the curve $Z$ once equations for $X$ and $Y$ are given. The first of these involves the use of hyperplane sections of the Kummer variety of $Y$ whose desingularization is isomorphic to $X$, whereas the second is based on interpolation methods involving numerical results over $\mathbb{C}$ that are proved to be correct over general fields a posteriori. As an application, we find a twist of a Jacobian over $\mathbb{Q}$ that admits a rational 70-torsion point.

Introduction

One of the most fundamental properties of abelian varieties is their unique decomposition up to isogeny, also called Poincaré’s Complete Reducibility Theorem [3, 5.3.7]: An abelian variety $A$ over a field $k$ is isogenous to a product

$$A \sim B_1^{e_1} \times \cdots \times B_n^{e_n} \quad (0.1)$$

where the abelian varieties $B_i$ are simple and pairwise non-isogenous over $k$, and this decomposition is unique in the sense that up to reordering the isogeny classes of the abelian varieties $B_i$ and the corresponding exponents $e_i$ are uniquely determined.

When $A = \text{Jac}(Z)$ is the Jacobian of a curve of small genus, then there exist algorithms after [8] to calculate the decomposition (0.1) over the base field $k$ in terms of the Jacobians of curves over small extensions of $k$ whenever possible. The decomposition of the Jacobian of a curve of genus 2 is also discussed in [22]. Similarly, when $A = \text{Jac}(Z)$ is the Jacobian of a curve of genus 3 that admits a degree-2 map $Z \to X$ to a genus-1 curve $X$, then the results in [31] furnish a simple description of the complementary part $B$ in the decomposition $A \sim \text{Jac}(X) \times B$ in terms of the Jacobian of a genus-2 curve $Y$.

This article aims to develop algorithms for the converse construction, that is, to produce an abelian variety $A$ given factors $B_i$ as in (0.1). When $A = \text{Jac}(Z)$ and $B_i = \text{Jac}(X_i)$, we also call the curve $Z$ a gluing of the curves $X_i$.

Previous work. Gluing elliptic curves $E_1$ and $E_2$ to a genus-2 curve $Z$ was first studied in the seminal article [13] by Frey and Kani, where explicit criteria for the existence of $Z$ given $E_1$ and $E_2$ are given. In fact, [13] proves a more precise criterion, in that they also fix a degree $n$ and realize $Z$ as a degree-$n$ cover of both $E_1$ and $E_2$. Similarly, Howe–Leprévost–Poonen [21] use plane quartic curves with defining equation in the Ciani standard form

$$Z : ax^4 + by^4 + cz^4 + dx^2y^2 + ey^2z^2 + f z^2x^2 = 0 \quad (0.2)$$

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to realize three given elliptic curves $E_1, E_2, E_3$ over $k$ as factors of the Jacobian of $Z$. Finally, [7] realizes the Jacobian of a given genus-2 curve over $\mathbb{C}$ as part of the Jacobian of a smooth plane quartic, using modular methods.

**Results in this paper.** This paper considers the problem of gluing two curves of genus 1 and 2 to a curve of genus 3 along their 2-torsion over a given base field. More precisely, our main theorem, proved in Section 2, is as follows.

**Main Theorem.** Let $k$ be a field whose characteristic does not equal 2, and let $X$ and $Y$ be curves of genus 1 and 2 over $k$. Then Algorithm 2.46 returns all isomorphism classes of pairs $(Z, \mu)$ over $k$, where $Z$ is a smooth plane quartic curve over $k$ and where $\mu \in k^*/(k^*)^2$ with the property that there exists a $k$-rational quotient map

$$\text{Jac}(X) \times \text{Jac}(Y) \to \mu \ast \text{Jac}(Z)$$

(0.3)

by a symplectic subgroup of $(\text{Jac}(X) \times \text{Jac}(Y))[2]$. In particular, we have an isogeny $\text{Jac}(X) \times \text{Jac}(Y) \sim \mu \ast \text{Jac}(Z)$ over $k$. Here $\mu \ast \text{Jac}(Z)$ denotes the twist of $\text{Jac}(Z)$ by $-1$ with respect to the quadratic extension $k(\sqrt{\mu})$ of $k$.

The alternative Algorithm 3.67 in Section 3 also determines equations for curves $Z$ gluing $X$ and $Y$, but these may require a further base extension.

Our results function over any base field $k$ of characteristic not equal to 2, not necessarily algebraically closed, which makes them relevant in the broader arithmetic-geometric context. Moreover, they allow one to specify both curves $X$ and $Y$, whereas for the previous results [7] over the special base field $k = \mathbb{C}$ only $Y$ could be specified. We mention here that there is also a short and simple construction of $Z$ over a general base field $k$ once only $Y$ is specified: It is given in [17, §2.2] and involves the parametrization of a certain conic admitting a $k$-rational point. Similarly, the case where the glued curve $Z$ is hyperelliptic will be dealt with in [18]. This article restricts its consideration to the more involved case where both $X$ and $Y$ are specified and where $Z$ is smooth plane quartic curve.

**Applications.** Being able to work with decompositions (0.1) over arbitrary base fields, which our Main Theorem contributes to, is of arithmetic importance, for example when describing $L$-functions: If an abelian surface over $\mathbb{Q}$ splits up to isogeny as $A \sim E_1 \times E_2$ over $\mathbb{Q}$, then we have $L(A, s) = L(E_1, s)L(E_2, s)$, and the modular properties of $A$ can be reduced to those of $E_1$ and $E_2$.

This description extends to situations where the decomposition requires an extension of the base field. For a systematic exploration this topic for abelian surfaces, we refer to [4].

Another application is in the context of Sato–Tate groups, and more specifically around the recent classification [12] in genus 3. While many Sato–Tate groups in *loc. cit.* can be realized in a trivial way by abelian threefolds of the form $\text{Jac}(X) \times \text{Jac}(Y)$, it remains a challenge to see if the same can be achieved when considering Jacobians $\text{Jac}(Z)$ of curves $Z$ of genus 3. The current work allows one to approach the latter problem by starting from suitable products $\text{Jac}(X) \times \text{Jac}(Y)$ and attempting to find a corresponding gluing as described in the Main Theorem (as well as in the next subsection).

Finally, over $k = \mathbb{C}$, the type of decomposition that we reconstruct is important in the context of certain integrable systems, see [11].

**Outline.** We now give a more precise description of our methods, as well as of some important intermediate results on the way to the Main Theorem. We will specify our curves $X$ and $Y$ of genus 1 and 2 by equations

$$X : y^2 = p_X(x)$$

(0.4)

and

$$Y : y^2 = p_Y(x)$$

(0.5)
over $k$. While general curves over $k$ of genus 1 do not allow such a defining equation, we may reduce to this case, since our constructions only involve the Jacobian of $X$.

For an isogeny (0.3) to exist, we need a criterion for the existence of a maximal isotropic subgroup $G$ of $(\text{Jac}(X) \times \text{Jac}(Y))[2]$ that is indecomposable (in the sense of not being a product of subgroups of $\text{Jac}(X)$ and $\text{Jac}(Y)$). This is furnished by Theorem 1.53, which is the following:

**Subgroup Existence Criterion.** Let $X$ and $Y$ be as above. There exists an indecomposable maximal isotropic subgroup $G$ of $(\text{Jac}(X) \times \text{Jac}(Y))[2]$ that is defined over $k$ if and only if

1. $p_Y$ admits a quadratic factor $q_Y$ over $k$;
2. For the complementary factor $r_Y = p_Y/q_Y$ we have that the cubic resolvents $\varrho(p_X)$ and $\varrho(r_Y)$ have isomorphic splitting fields over $k$.

Note that this criterion allows one to work with simpler splitting fields than those defined by the factors $p_X$ and $r_Y$ themselves.

The existence of such a subgroup $G$ does not always guarantee that the corresponding quotient $(\text{Jac}(X) \times \text{Jac}(Y))/G$ is the twist of a plane quartic by $-1$. Indeed, the polarization on said quotient may be decomposable, or give rise to a hyperelliptic curve. Generically, however, such a quotient is isomorphic as a principally polarized abelian variety to a twist $\mu^\star \text{Jac}(Z)$ of a Jacobian of a plane quartic curve. Taking a twist is indispensable, as over an arithmetic base fields most principally polarized abelian threefolds are not Jacobians — for a geometric description of this so-called Serre obstruction, see for example the main result in [2].

In Section 2, we proceed to find an expression for the curve $Z$ in terms of the data in the Subgroup Existence Criterion whenever possible. The resulting curve $Z$ will admit a homogeneous ternary quartic equation of the form

\[ Z : G(x^2, y, z) = 0 \quad (0.6) \]

from which $\text{Jac}(X)$ can be recovered as the Jacobian of the quotient by the involution $(x, y, z) \mapsto (-x, y, z)$. Note that there is no direct map $Z \to Y$ in general, even over the algebraic closure $\overline{k}$. In fact, Proposition 3.4 shows that this can only happen when $Z$ is hyperelliptic, a case that we excluded from consideration. Therefore we cannot use constructions that only involve covers of curves.

Section 2 takes the following indirect route to constructing $Z$. We start by interpolating results over the complex numbers. When appropriately normalized, these yield formulae that can be verified a posteriori to remain valid over any field of characteristic not equal to 2. Note that evaluating these formulae, as is done in Algorithm 2.46 corresponding to the Main Theorem, once again only requires passing to the common splitting field of the aforementioned cubic resolvents, and is therefore feasible in practice also when the coefficients of the defining equations of $X$ and $Y$ are large. The formulae also yield the twisting scalar $\mu$ mentioned in the Main Theorem. Moreover, since they are obtained in a highly normalized way, applying them in concrete cases such as Example 2.53 yields small defining coefficients for $Z$ without any further simplification being required.

An alternative and more geometric version of the Main Theorem is given in Section 3. It is essentially a geometric inversion of the results in the seminal paper [31] by Ritzenthaler and Romagny, and constructs $Z$ as a double cover of $X$ obtained by realizing $X$ birationally as a hyperplane section of the Kummer variety of $Y$. This is the content of Theorem 3.9, which we state here as follows.

**Geometric Main Theorem.** Let $Z$ be a gluing of $X$ and $Y$ as in the Main Theorem. Let $\text{Kum}(Y) = \text{Jac}(Y)/\langle -1 \rangle \subset \mathbb{P}^4_k$ be the Kummer surface associated to $\text{Jac}(Y)$. Then over $\overline{k}$ there
exists a commutative diagram

\[
\begin{array}{ccc}
Z & \overset{i_Z}{\longrightarrow} & \text{Jac}(Y) \\
p \downarrow & & \downarrow \pi \\
X & \overset{i_X}{\longrightarrow} & \text{Kum}(Y).
\end{array}
\]  

(0.7)

where \( p : Z \rightarrow X \) is a degree-2 cover, where \( \pi : \text{Jac}(Y) \rightarrow \text{Kum}(Y) \) is the quotient map, and where \( i_Z \) and \( i_X \) are rational maps such that \( i_X(X) = H \cap \text{Kum}(Y) \) for a plane \( H \subset \mathbb{P}^3_k \) that passes through two singular points of \( \text{Kum}(Y) \).

Moreover, by Theorem 3.64, this construction indeed recovers all the gluings from the Main Theorem over \( \overline{\mathbb{F}} \).

Explicitly, an element whose square root gives rise to the double cover \( Z \rightarrow X \) can be obtained by restricting a Kummer generator of the extension of function fields \( k(\text{Kum}(Y)) \subset k(\text{Jac}(Y)) \) (which is described in [27]) to the hyperplane section \( X \). Various tricks are used to make this calculation feasible in practice, especially over finite fields, and many interesting phenomena in this geometric Ansatz remain to be explained and generalized.

We give examples of the aforementioned constructions, both over \( \mathbb{Q} \) and over finite fields. Moreover, in Section 2.6 we use our results to obtain a Jacobian of a plane quartic curve over \( \mathbb{Q} \) whose twist by \(-1\) with respect to the extension \( \mathbb{Q}(\sqrt{5}) \) of \( \mathbb{Q} \) admits a rational 70-torsion point. A full implementation of the results in this article is openly available via a full MAGMA implementation and example suite at [19].

Notations and conventions. Throughout the article, \( k \) denotes a fixed base field, whose characteristic we suppose not to equal 2. Its absolute Galois group is denoted by \( \Gamma_k \).

A curve over \( k \) is a separated and geometrically integral scheme of dimension 1 over \( k \). Given an affine equation for a curve, we will identify it with the smooth projective curve that has the same function field. The Jacobian of a curve \( X \) is denoted by \( \text{Jac}(X) \), and its principal polarization, which we consider as an algebraic equivalence class of line bundles on \( X \), is denoted by \( \lambda_X \).

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1. Criteria for a gluing to exist

Let \( X \) (resp. \( Y \)) be a smooth curve of genus 1 (resp. 2) over the base field \( k \). Let \( \pi_X : \text{Jac}(X) \times \text{Jac}(Y) \to \text{Jac}(X) \) and \( \pi_Y : \text{Jac}(X) \times \text{Jac}(Y) \to \text{Jac}(Y) \) be the two canonical projections.

**Definition 1.1.** A \((2,2)\)-gluing (or simply gluing) of the curves \( X \) and \( Y \) (over \( k \)) is a pair \((Z, \varphi)\), where \( Z \) is a smooth curve over \( k \) and where

\[
\varphi : \text{Jac}(X) \times \text{Jac}(Y) \to \text{Jac}(Z)
\]

(1.2)
is an isogeny with the property that \( \varphi^*(\lambda_Z) \) is algebraically equivalent to the 2-fold \( 2(\pi_X^*(\lambda_X) \otimes \pi_Y^*(\lambda_Y)) \) of the product polarization on \( \text{Jac}(X) \times \text{Jac}(Y) \).

**Remark 1.3.** A fuller theory of gluings will be developed in [16]. In the present article we restrict ourselves to gluing curves of genus 1 and 2 along their 2-torsion.

Let \( T = (\text{Jac}(X) \times \text{Jac}(Y))[2] = \text{Jac}(X)[2] \times \text{Jac}(Y)[2] \).

Consider a maximal isotropic subgroup \( G \) of \( T(\kappa) \). Then over \( \kappa \) we can form the fppf quotient

\[
Q = (\text{Jac}(X) \times \text{Jac}(Y))/G.
\]

(1.5)

Let

\[
\pi_Q : \text{Jac}(X) \times \text{Jac}(Y) \to Q
\]

(1.6)
be the quotient morphism. By [34, Proposition 11.25], there exists a unique principal polarization \( \lambda_Q \) on \( Q \) whose pullback under \( \pi_Q \) is algebraically equivalent to \( 2(\pi_X^*(\lambda_X) \otimes \pi_Y^*(\lambda_Y)) \). Since we have imposed that \( \text{char}(k) \neq 2 \) and the quotient morphism \( Q \) is defined over \( k \) if and only if the subgroup \( G \) is, we obtain the following.

**Lemma 1.7.** Giving a gluing \((Z, \varphi)\) of \( X \) and \( Y \) over \( k \) is the same as giving a maximal isotropic subgroup \( G \) of \( T(\kappa) \) with the following properties.

(i) \( G \) is stable under the action of the absolute Galois group \( \Gamma_k \).

(ii) There exists a curve \( Z \) over \( k \) such that \( (\text{Jac}(Z), \lambda_Z) \cong (Q, \lambda_Q) \).

**Remark 1.8.** For Condition (ii) in Lemma 1.7 to hold, the group \( G \) cannot be decomposable, that is, a product \( G_X \times G_Y \) of maximal isotropic subgroups of \( \text{Jac}(X)[2] \) and \( \text{Jac}(Y)[2] \). Indeed, in this case we have

\[
Q \cong \text{Jac}(X)/G_X \times \text{Jac}(Y)/G_Y
\]

(1.9)
and \( \lambda_Q \) is the corresponding product polarization. This precludes the existence of an isomorphism \( (\text{Jac}(Z), \lambda_Z) \cong (Q, \lambda_Q) \), since principal polarizations of Jacobians are indecomposable.

1.1. Structure of maximally isotropic subgroups. Ideas similar to those in this section were explored in [28]. Consider a maximally isotropic subgroup \( G \) of \( T(\kappa) \), where \( T = \text{Jac}(X)[2] \times \text{Jac}(Y)[2] \). Let \( V_X = \text{Jac}(X)[2](\kappa) \) and \( V_Y = \text{Jac}(Y)[2](\kappa) \), so that

\[
T(\kappa) = V_X \times V_Y
\]

(1.10)
and let

\[
E = E_X \times E_Y : V \times V \to \mathbb{F}_2
\]

(1.11)
be the product of the Weil pairings \( E_X \) and \( E_Y \) on \( V_X \) and \( V_Y \). Finally, let \( \pi_X : G \to V_X \) and \( \pi_Y : G \to V_Y \) be the canonical projections.
Proposition 1.12. If \( G \) is indecomposable, then \( \pi_X \) is surjective and we have
\[
\dim(\ker(\pi_X)) = \dim(G \cap (0 \times V_Y)) = 1. \tag{1.13}
\]

Proof. We have \( \dim(G) = 3 \). If \( \dim(\text{im}(\pi_X)) = 0 \), then \( \ker(\pi_X) \) can be identified with an isotropic subgroup of \( 0 \times V_Y \) of dimension 3. No such subgroups exist since \( V_Y \) has dimension 4. If \( \dim(\text{im}(\pi_X)) = 1 \), then \( \text{im}(\pi_X) \) is a symplectic subgroup of \( V_X \), so that for all \((x_1, y_1)\) and \((x_2, y_2)\) in \( G \subset V_X \times V_Y \) we have
\[
0 = E(((x_1, y_1), (x_2, y_2))) = E_X((x_1, x_2)) + E_Y((y_1, y_2)) = E_Y((y_1, y_2)). \tag{1.14}
\]
This implies that \( \text{im}(\pi_Y) \) is an isotropic subgroup of \( V_Y \). Since \( G \subset \text{im}(\pi_X) \times \text{im}(\pi_Y) \) this forces \( \dim(\text{im}(\pi_Y)) = 2 \) and \( V = \ker(\pi_X) \times \ker(\pi_Y) \). This is a contradiction with the indecomposability of \( G \). The second statement of the proposition then follows from the dimension formula. \( \square \)

Now fix an indecomposable maximal isotropic subgroup \( G \subset V_X \to V_Y \). Following ideas as in [28] (described in more detail and generality in [16]) one shows the following. Let \( H \subset V_Y \) be the 1-dimensional subgroup defined by
\[
H = \pi_Y(\ker(\pi_X)) = \pi_Y(G \cap (0 \times V_Y)) \subset V_Y. \tag{1.15}
\]
In other words, \( H \) is the subgroup of \( V_Y \) generated by the second components of the vectors of the form \((0, w)\) in \( G \). The orthogonal complement \( H^\perp \subset V_Y \) is of dimension 3, so that we have
\[
\dim(H^\perp/H) = 2. \tag{1.16}
\]
The symplectic pairing \( E_Y \) on \( V_Y \) induces one on \( H^\perp/H \), which we will denote by \( E_\perp \). There is a multivalued map
\[
V_X \to H^\perp/H \tag{1.17}
\]
that sends \( x \in V_X \) to \( \pi_Y(\pi_X^{-1}(x)) \). Since \( H \times 0 \subset G \) and \( G \) is isotropic, the elements of \( \pi_Y(\pi_X^{-1}(x)) \) are in \( H^\perp \). The map (1.17) factors to a single-valued linear map
\[
\ell : V_X \to H^\perp/H. \tag{1.18}
\]
Now for all \((x_1, y_1)\) and \((x_2, y_2)\) in \( G \) we have
\[
0 = E(((x_1, y_1), (x_2, y_2))) = E_X((x_1, x_2)) + E_Y((y_1, y_2)), \tag{1.19}
\]
By construction of \( \ell \) we have \( y_i + H = \ell(x_i) + H \). Therefore the map \( \ell \) is antisymplectic (or for that matter symplectic, since \( V \) is a vector space over the finite field \( \mathbb{F}_2 \)). In particular, \( \ell \) is an isomorphism.

Thus an indecomposable maximal isotropic subgroup \( G \) gives rise to a subgroup \( H \subset V_Y \) and an anti-symplectic linear isomorphism \( \ell : V_X \to H^\perp/H \). As is shown more generally in [16], there is a converse to this result:

Proposition 1.20. Let \((H, \ell)\) be a pair with \( H \subset V_Y \) of dimension 1 and with \( \ell : V_X \to H^\perp/H \) an anti-symplectic isomorphism. Define
\[
G = \{(x, y) \in V_X \times V_Y : \ell(x) = y + H\}. \tag{1.21}
\]
Then \( G \subset V_X \times V_Y \) is indecomposable and maximal isotropic.

This construction of \( G \) from \((H, \ell)\) is inverse to that of \((H, \ell)\) from \( G \) above and yields a bijective correspondence between indecomposable maximal isotropic subgroups \( G \subset V_X \times V_Y \) on the one hand and the pairs \((H, \ell)\) under consideration on the other.

Proof. The first part follows by the same methods as above. Note in particular that \( G \) is indecomposable because its intersection with \( 0 \times V_Y \) is of dimension 1. The remainder of the statement follows by direct verification. \( \square \)

Corollary 1.22. There exist exactly 90 indecomposable maximal isotropic subgroups of \( V_X \times V_Y \).
Proof. By Proposition 1.20, giving such an indecomposable maximal isotropic subgroup is the same as giving a pair \((H, \ell)\). Since \(V_Y\) has dimension 4, there are \(2^4 - 1 = 15\) possible ways to choose \(H\). Given \(H\), there are 6 choices for the (anti-)symplectic isomorphism \(\ell\). Indeed, since \(V_X\) and \(H^\perp / H\) are both of dimension 2, any linear isomorphism between them is symplectic, and the group \(\text{GL}(V_X)\) has cardinality 6. \(\square\)

Remark 1.23. The total number of maximal isotropic subgroups (not necessarily indecomposable) of \(V_X \times V_Y\) equals 135.

1.2. Interpretation in terms of roots. The symplectic vector spaces \(V_X\) and \(V_Y\) have the following concrete descriptions. Choose quadratic defining equations

\[
X : y^2 = p_X(x) \quad (1.24)
\]

and

\[
Y : y^2 = p_Y(x) \quad (1.25)
\]

over \(\overline{k}\), as one may since \(\text{char}(k) \neq 2\). For now, suppose that \(p_X\) (resp. \(p_Y\)) is of degree 4 (resp. 6). Let

\[
\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \overline{k} \quad (1.26)
\]

be the roots of \(p_X\), and let

\[
\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6 \in \overline{k} \quad (1.27)
\]

be the roots of \(p_Y\). Consider the corresponding sets

\[
\mathcal{P} = \left\{ P_i = (\alpha_i, 0) \in X(\overline{k}) : i \in \{1, \ldots, 4\} \right\}, \quad \mathcal{Q} = \left\{ Q_j = (\beta_j, 0) \in Y(\overline{k}) : j \in \{1, \ldots, 6\} \right\}. \quad (1.28)
\]

Remark 1.29. If either one of the degrees of \(p_X\) and \(p_Y\) is odd, then we can formally consider \(\infty\) as an element of \(\mathcal{P}\) or \(\mathcal{Q}\) (or both). To simplify the exposition, we ignore these cases.

Given a set \(T\) of even cardinality, we can define a symplectic \(\mathbb{F}_2\)-vector space \(\mathcal{G}(T)\) as follows: The elements of \(\mathcal{G}(T)\) are the subsets \(S\) of \(T\) of even cardinality up to the equivalence \(S \sim S^c\). The symmetric difference operation

\[
(S_1, S_2) \mapsto S_1 \oplus S_2 = (S_1 \cup S_2) \setminus (S_1 \cap S_2). \quad (1.30)
\]

descends to a group structure on \(\mathcal{G}(T)\), for which the empty subgroup corresponds to the identity element. Finally, we equip \(\mathcal{G}(T)\) with the symplectic pairing induced by

\[
(S_1, S_2) \mapsto (-1)^{|S_1 \cap S_2|}. \quad (1.31)
\]

Now [29] shows the following.

Proposition 1.32. The symplectic \(\mathbb{F}_2\)-vector space \(V_X\) (resp. \(V_Y\)) can be identified with the \(\mathcal{G}(\mathcal{P})\) (resp. \(\mathcal{G}(\mathcal{Q})\)). To the equivalence classes \(\overline{S}\) of a subgroup \(S = \{P_1, P_2\}\) of cardinality 2 there corresponds the 2-torsion point \([P_1] - [P_2]\).

Remark 1.33. For the genus-2 curve \(Y\), the subsets of \(\mathcal{Q}\) of cardinality 2 are in bijective correspondence with the non-zero elements of \(V_Y\); since the subsets \(S\) of \(\mathcal{P}\) of even cardinality that do not give rise to \(0 \in V_Y\) are of cardinality 2 or 4, so that exactly one of \(S\) and \(S^c\) is of cardinality 2.

By contrast, for the genus-1 curve \(X\) the non-zero elements of \(V_X\) are no longer in bijective correspondence with the subsets \(S\) of \(\mathcal{P}\) cardinality 2: this needs the identification of a set \(S\) with its complement \(S^c\).

Consider a pair \((H, \ell)\) as in Proposition 1.20. In terms of \(\mathcal{P}\) and \(\mathcal{Q}\), giving \(H\) is nothing but giving a subset \(\mathcal{T}\) of \(\mathcal{Q}\) of cardinality 2. Let \(\mathcal{R} = \mathcal{Q} \setminus \mathcal{T}\) be the complement of \(\mathcal{T}\).
Proposition 1.34. The inclusion \( \iota : \mathcal{R} \hookrightarrow \mathcal{Q} \) induces a canonical isomorphism of symplectic \( \mathbb{F}_2 \)-vector spaces

\[
i_R: \mathcal{G}(\mathcal{R}) \to H^\perp/H. \tag{1.35}\]

Proof. Both vector spaces involved are of dimension 2. The injection \( \iota \) gives rise to a well-defined map on equivalence classes \( \mathfrak{R} \) since for \( S \subset \mathcal{R} \) we have

\[
\iota(S^c) = \iota(\mathcal{R} \setminus S) = \mathcal{Q} \setminus (\mathcal{T} \cup S) \sim \mathcal{T} \cup S. \tag{1.36}
\]

Taking the symmetric difference with the non-trivial element \( \mathcal{T} \) in \( H \), we obtain the class

\[
(\mathcal{T} \cup S) \oplus \mathcal{T} = S = \iota(S). \tag{1.37}
\]

This shows that the images of \( \iota(S^c) \) and \( \iota(S) \) in \( H^\perp/H \) indeed coincide.

The map \( i_R \) is linear (and hence symplectic) since

\[
\iota(S_1 \oplus S_2) = \iota((S_1 \cup S_2) \setminus (S_1 \cap S_2)) = (S_1 \cup S_2) \setminus (S_1 \cap S_2) = \iota(S_1) \oplus \iota(S_2). \tag{1.38}
\]

Finally, \( i_R \) is injective since the image of the equivalence class of a subset \( S \subset R \) of cardinality 2 remains an equivalence class of a subset of cardinality 2 and is therefore non-trivial. We conclude that \( i_R \) is indeed a symplectic isomorphism.

Combining the above results, we get the following.

Corollary 1.39. Giving an indecomposable maximal isotropic subgroup \( G \subset V_X \times V_Y \) is the same as giving a subset \( \mathcal{T} \) of \( \mathcal{Q} \) of cardinality 2 along with a symplectic isomorphism

\[
\ell: \mathcal{G}(\mathcal{P}) \to \mathcal{G}(\mathcal{R}), \tag{1.40}
\]

where \( \mathcal{R} = \mathcal{Q} \setminus \mathcal{T} \).

1.3. Rationality criteria. We now consider criteria for a gluing to exist over the given base field \( k \). First, note that we may assume \( X \) and \( Y \) to admit defining equations (1.24) and (1.25) over \( k \). For \( Y \) this follows from the fact that every genus-2 curve over \( k \) admits such an equation, whereas for \( X \) we may make this assumption because only the Jacobian \( \text{Jac}(X) \) intervenes in our constructions, and this Jacobian is an elliptic curve, which therefore admits an equation (1.24) since \( k \) is not of characteristic 2. We can say more:

Proposition 1.41. Let \((Z, \varphi)\) be a gluing of \( X \) and \( Y \) over \( k \). Then \( \text{Jac}(Y) \) has a rational 2-torsion point.

Proof. By Remark 1.8, the group \( G \subset V_X \times V_Y \) that corresponds to \((Z, \varphi)\) by Lemma 1.7 is indecomposable. Since \( G \) and \( V_Y \) are both stable under the action of \( \Gamma_k \) the same is true for the intersection \( G \cap (0 \times V_Y) \), which has dimension 1 by Proposition 1.12. The result follows.

We proceed to give concrete criteria for the Galois stability in Part (i) of Lemma 1.7 in terms of the equivalent interpretations of the maximal isotropic subgroup \( G \) that we developed.

Proposition 1.42. Let \((H, \ell)\) be a pair with \( H \subset V_Y \) of dimension 1 and with \( \ell: V_X \to H^\perp/H \) an anti-symplectic isomorphism, and let \( G \) be the corresponding subgroup, as described in Proposition 1.20. Then \( G \) is Galois stable if and only if:

(i) \( H \) is Galois stable;
(ii) \( \ell \) : \( V_X \to H^\perp/H \) is Galois equivariant.

Proof. The first condition is necessary because of Proposition 1.41. As for the second, if \( G \) is Galois stable, then if \((x, y) \in G\), the same holds for \((\sigma(x), \sigma(y))\). We conclude that \( \ell(\sigma(x)) = \sigma(y) + H \) in \( H^\perp/H \), so indeed \( \ell \) is Galois equivariant, as

\[
\sigma(\ell(x)) = \sigma(y) + H = \ell(\sigma(x)). \tag{1.43}
\]
Now suppose conversely that \((H, \ell)\) fulfills the conditions of the proposition. Let \((x, y)\) be an element of the corresponding group \(G\). We have \(\ell(x) = y + H\). Since \(\ell\) is Galois equivariant, we have
\[
\ell(\sigma(x)) = \sigma(\ell(x)) = \sigma(y) + \sigma(H) = \sigma(y) + H, \tag{1.44}
\]
which implies \((\sigma(x), \sigma(y)) \in G\). Therefore \(G\) is Galois stable. \(\square\)

Now choose defining equations for \(X : y^2 = p_X\) and \(Y : y^2 = p_Y\) as in Subsection 1.2. The rationality of the subgroup \(H\) yields a quadratic factor \(q_Y\) of \(p_X\) over \(k\), which corresponds to the roots in the set \(T\). Let \(r_Y = p_Y/q_Y\) be the complementary factor corresponding to the roots in \(R = Q - T\).

Recall that given a quartic polynomial
\[
p = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \in k[x], \tag{1.45}
\]
its cubic resolvent \(g(p)\) is defined by
\[
g(p) = x^3 - a_2x^2 + (a_1a_3 + 4a_0)x + (4a_0a_2 - a_1^2 - 4a_0a_3^2) \in k[x] \tag{1.46}
\]
For simplicity of exposition, we define the cubic resolvent of a general quartic polynomial as the cubic resolvent of the polynomial obtained by dividing it by its leading coefficient. If \(\alpha_1, \ldots, \alpha_4\) are the roots of \(p\), then the roots of \(g(p)\) are given by
\[
\gamma_1 = \alpha_1\alpha_2 + \alpha_3\alpha_4, \gamma_2 = \alpha_1\alpha_3 + \alpha_2\alpha_4, \gamma_3 = \alpha_1\alpha_4 + \alpha_2\alpha_3. \tag{1.47}
\]
Considering the symmetries of its roots, the cubic resolvent \(g(p)\) cuts out the extension of \(k\) that corresponds to the normal subgroup \(V_4 \cap \text{Gal}(p)\) of the Galois group \(\text{Gal}(p) \subset S_4\) of \(p\). The resulting conjugation action
\[
\text{Gal}(p) \to \text{Sym}(V_4) \tag{1.48}
\]
coincides with the Galois module structure on the symplectic group \(G\) defined by the 4 roots of the given polynomial \(p\). Indeed, the non-trivial elements \((12)(34), (13)(24), (14)(23)\) of \(V_4\) correspond to the non-trivial elements of \(G\), as well as to the roots \(\gamma_1, \gamma_2, \gamma_3\) of \(g(p)\). This allows us to prove the following result:

**Proposition 1.49.** The Galois module structure of the group \(G(P)\) (resp. \(G(R)\)) in Corollary 1.39 is determined by the cubic resolvent \(g(p_X)\) (resp. \(g(r_Y)\)). Moreover, there is a Galois equivariant isomorphism \(\ell : G(P) \to G(R)\) if and only if the splitting fields of \(g(p_X)\) and \(g(r_Y)\) are isomorphic.

**Proof.** Since the trivial element of \(G(P)\) is fixed by \(\Gamma_k\), it suffices to consider the action on the non-trivial elements. By the above, this action corresponds with that on the roots of \(g(p_X)\). The same argument applies to \(G(R)\) and \(g(r_Y)\).

The Galois module structure on \(G(P) \setminus \{0\}\) coincides with the \(\Gamma_k\)-set defined by \(g(p_X)\) in the sense of [25], and similarly for \(G(R) \setminus \{0\}\). These two \(\Gamma_k\)-sets give rise to representations \(g_X\) and \(g_Y\) of \(\Gamma_k\) with image in \(S_3\). The former are isomorphic as \(\Gamma_k\)-sets exactly if \(g_X\) and \(g_Y\) are \(S_3\)-conjugate. In this case the kernels of \(g_X\) and \(g_Y\) coincide, which means that the splitting fields of \(p_X\) and \(r_Y\) are isomorphic.

Conversely, if these splitting fields are isomorphic, then \(g_X\) and \(g_Y\) have a common kernel \(N\). These representations are therefore conjugate by the following lemma. \(\square\)

**Lemma 1.50.** Let \(\Gamma\) be a group, and let
\[
g_1, g_2 : \Gamma \to S_3 \tag{1.51}
\]
be two representations. Then \(g_1\) and \(g_2\) are \(S_3\)-conjugate if and only if their kernels coincide and their images in \(S_3\) are isomorphic.

**Proof.** This is a direct consequence of the fact that two subgroups of \(S_3\) are conjugate if and only if they are isomorphic. \(\square\)
Remark 1.52. It is not much more difficult to calculate the number of Galois equivariant isomorphisms \( \ell : \mathcal{G}(\mathcal{P}) \to \mathcal{G}(\mathcal{R}) \). Indeed, if the given \( \Gamma_\ell \)-sets are conjugate, then the number of isomorphisms is nothing but their common number of automorphisms, which in turn is the number of elements of the centralizer of either of their images in \( S_3 \).

Theorem 1.53. Let \( X \) (resp. \( Y \)) be a curve of genus 1 (resp. 2) admitting a defining equation \( X : y^2 = p_X \) (resp. \( Y : y^2 = p_Y \)). There exists a Galois stable indecomposable maximal isotropic subgroup \( G \subset V_X \times V_Y \) if and only if

(i) \( p_Y \) admits a quadratic factor \( q_Y \) over \( k \);

(ii) For the complementary factor \( r_Y = p_Y/q_Y \) we have that the cubic resolvents \( g(p_X) \) and \( g(r_Y) \) have isomorphic splitting fields over \( k \).

Proof. This follows by combining Proposition 1.41 and Proposition 1.49. \( \square \)

Remark 1.54. As mentioned at the start of Section 1.2, some changes take place when either \( p_X \) of \( p_Y \) is of odd degree. If \( p_Y \) has degree 5, then we should also consider the case where \( p_Y \) has a linear factor over \( k \) in Part (i) of Theorem 1.53. Similarly, if \( p_X \) or \( r_Y \) is of odd degree, then we should use this polynomial directly in Part (ii) instead of taking a Galois resolvent.

1.4. Intervening twists. The previous section has given a concrete characterization of Part (i) of Lemma 1.17. For Part (ii), we restrict ourselves in this article to the case where the quotient \( (\text{Jac}(X) \times \text{Jac}(Y))/G \) is a Jacobian over \( \bar{k} \). When \( \bar{k} \subset \mathbb{C} \), it is possible to characterize the case when this occurs by numerical complex-analytic methods, which are further discussed in Section 2): namely, none of the even theta null values of the complex torus corresponding to \( (\text{Jac}(X) \times \text{Jac}(Y))/G \) should be 0.

Remark 1.55. We note in passing and without detail that it is not difficult to characterize when there exists a hyperelliptic gluing \((Z, \varphi)\) over \( \bar{k} \). This is the case if and only if one of the cross ratios of the roots of the polynomial \( p_X \) that defines \( X \) coincides with one of the cross ratios of four of the roots of the polynomial \( p_Y \) that defines \( Y \), as one observes by noting that the Prym variety of the obvious morphism from

\[
Z : y^2 = x^8 + a_3 x^6 + a_2 x^4 + a_1 x^2 + a_0
\]

(1.56) to

\[
X : y^2 = x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0
\]

(1.57) is given by

\[
Y : y^2 = x(x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0).
\]

(1.58)

However, issues of rationality remain to be explored thoroughly. Note in particular that it is possible for the glued curve \( Z \) to be hyperelliptic over \( \bar{k} \) but not over \( k \), which happens when \( Z \) is a degree-2 cover of a non-trivial conic. The upcoming work [18] will explore these themes in detail.

For non-algebraically closed fields, there is an additional complication: It is possible that \((Z, \lambda_Z)\) is a Jacobian over \( \bar{k} \), but not over \( k \). More precisely, we have the following by [2]:

Theorem 1.59. Let \((Q, \lambda_Q)\) be a principally polarized abelian threefold over \( k \) that is not a product of abelian varieties of smaller dimension over \( \bar{k} \). Then there exists a curve \( Z \) over \( k \) and a field extension \( \ell | k \) with \(|\ell : k| \leq 2 \) such that \((Q, \lambda_Q)\) is isomorphic to the quadratic twist of \((\text{Jac}(Z), \lambda_Z)\) with respect to the automorphism \(-1\) and the extension \( \ell | k \). Moreover, if \( Z \) is hyperelliptic, then \( \ell = k \).

Definition 1.60. Let \( Q = (\text{Jac}(X) \times \text{Jac}(Y))/G \) and \( Z \) be as in the preceding theorem, and let \( \mu \in k^* \). If \((Q, \lambda_Q)\) is isomorphic to the quadratic twist of \((Z, \lambda_Z)\) corresponding to the class of \( \mu \in k^*/(k^*)^2 \), then we call \( Z \) a twisted gluing of \( X \) and \( Y \) (by the twisting scalar \( \mu \)).
Given generic $X$ and $Y$, the twisting scalar is usually non-trivial. In the plane quartic case that we will consider in what follows, we will calculate it explicitly in terms of the polynomials $p_X$ and $p_Y$ that define the curves $X$ and $Y$ and their cubic resolvents.

2. Gluing via interpolation

2.1. Numerical algorithms over $\mathbb{C}$. Consider the base field $k = \mathbb{C}$. In this section, we consider gluings from an analytic point of view. As in Section 1.2, we choose defining equations

$$X : y^2 = p_X(x)$$

(2.1)

and

$$Y : y^2 = p_Y(x).$$

(2.2)

Consider the sets of roots $\mathcal{P} = \{\alpha_1, \ldots, \alpha_4\}$ of $p_X$ and $\mathcal{Q} = \{\beta_1, \ldots, \beta_6\}$ of $p_Y$. Let $\mathcal{T} = \{\beta_5, \beta_6\}$ and $\mathcal{R} = \mathcal{Q} - \mathcal{T}$. Via the correspondence in Corollary 1.39, we consider the maximal isotropic subgroup $G$ defined by the pair $(\mathcal{T}, \ell)$, where $\mathcal{T} = \{\beta_5, \beta_6\}$ and where $\ell$ is defined as

$$\ell : \mathcal{G}(\mathcal{P}) \rightarrow \mathcal{G}(\mathcal{R})$$

$$\{\alpha_1, \alpha_1\} \mapsto \{\beta_1, \beta_1\}. \quad (2.3)$$

for $i \in \{2, 3, 4\}$. In other words, we fix a root pairing determined by our choice of ordering of the roots of $p_X$ and $p_Y$. We intend to find a corresponding genus-3 curve defined by a homogeneous ternary quartic equation

$$Z : F(x, y, z) = 0 \subset \mathbb{P}^2$$

(2.4)

provided that such an equation exists.

Definition 2.5. Let $X$ be a curve over $\mathbb{C}$, and let $\mathcal{B} = \{\omega_1, \ldots, \omega_g\}$ be a basis of the $\mathbb{C}$-vector space of global differentials $H^0(X, \omega_X)$. The period lattice $\Lambda_{X, \mathcal{B}}$ of $X$ with respect to $\mathcal{B}$ is the lattice in $\mathbb{C}^g$ defined by

$$\Lambda_{X, \mathcal{B}} = \left\{ \left( \int_{\gamma} \omega_i \right)_{i=1,\ldots,g} : \gamma \in H_1(X, \mathbb{Z}) \right\}. \quad (2.6)$$

A period lattice in $\mathbb{C}^g$ is a lattice $\Lambda \subset \mathbb{C}^g$ that is of the form $\Lambda_{X, \mathcal{B}}$ for some choice of $X$ and $\mathcal{B}$.

The defining equation (2.1) picks out a distinguished basis of $H^0(X, \omega_X)$, namely

$$\mathcal{B}_X = \{dx/y\} \quad (2.7)$$

Similarly, the defining equations (2.2) and (2.4) pick out the bases

$$\mathcal{B}_Y = \{xdx/y, dx/y\} \quad (2.8)$$

and

$$\mathcal{B}_Z = \{xdx/(\partial f/\partial y), ydx/(\partial f/\partial y), dx/(\partial f/\partial y)\}, \quad (2.9)$$

where $f(x, y) = F(x, y, 1)$. We need the following result, which is a more down-to-earth version of considerations in [2].

Proposition 2.10. Giving a period lattice $\Lambda$ for the genus-2 curve $Y$ (resp. for the plane quartic curve $Z$) is the same as giving a defining polynomial $p_Y$ as in (2.2) (resp. a defining ternary quartic $F$ as in (2.4) up to sign).

Proof. Given a period lattice $\Lambda$, there always exists a corresponding equation. Indeed, consider an initial equation with associated period lattice $\Lambda_0$. Choose $T \in GL_2(\mathbb{C})$ such that $T\Lambda_0 = \Lambda$. Applying the corresponding fractional linear transformation in $x$ (resp. projective linear map $\mathbb{P}^2 \rightarrow \mathbb{P}^2$) to $p_Y$ (resp. $F$) we obtain a new equation whose associated period lattice equals $\Lambda$ up to a scalar. Since moreover scaling $p_Y$ and $F$ induces a non-trivial scaling of the basis of differentials, we can indeed find an equation corresponding to the specified matrix $\Lambda$. 

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As for uniqueness, first consider the genus-2 case. Once more considering the above induced map shows that for two defining equations $y^2 = p_1(x)$ and $y^2 = p_2(x)$ giving rise to $\Lambda$, there exists an isomorphism between them that maps $x$ to $x$. Since the hyperelliptic involution does not affect these defining equations, we conclude that they coincide.

For plane quartics, we can similarly conclude the existence of an isomorphism with trivial tangent representation between the curves defined by two defining equations $F_1 = 0$ and $F_2 = 0$ giving rise to $\Lambda$. However, in this case the ternary quartic is determined by the curve up to a scalar only. As multiplying a ternary quartic $F$ by $\mu$ scales the lattice by $\mu^{-1}$, we conclude that there remains the ambiguity of scaling $F$ by $\mu = -1$. \qed

Remark 2.11. For curves of genus 2, an explicit algorithm to recover $Y$ from a given period lattice with a symplectic pairing is given in [15].

Remark 2.12. The statement in Proposition 2.10 does not hold for the genus-1 curve $X$, because translations by elements of its Jacobian induce trivial maps on differentials.

Let $\Lambda_X$ and $\Lambda_Y$ be the period lattices resulting from the choices of basis (2.7) and (2.8). These matrices can be calculated especially fast by using new algorithms by Molin–Neurohr [26]. This functionality also includes the calculation of the Abel–Jacobi map, which means that we can construct elements of the group $G$, as follows.

Given a hyperelliptic curve $X$ with Weierstrass points $P_1$ and $P_2$, the corresponding element of $H^0(X, \omega_X)^*/H_1(X, \mathbb{Z})$ under the Abel–Jacobi map is

$$\omega \mapsto \int_{P_1}^{P_2} \omega,$$

where the integral can be taken along any path between $P_1$ and $P_2$. Now the period lattice corresponding to $\text{Jac}(X) \times \text{Jac}(Y)$ is

$$\Lambda = \Lambda_X \times \Lambda_Y \subset \mathbb{C}^1 \times \mathbb{C}^2 = \mathbb{C}^3.$$  \hspace{1cm} (2.14)

Because of the construction in 1.20 and the above remark, a basis for $G \cong (\mathbb{Z}/2\mathbb{Z})^3$ is given by the elements $\{v_1, v_2, v_3\}$ of

$$\left\{ \left(0, \int_{\beta_5}^{\beta_6} \frac{dx}{y}, \int_{\beta_6}^{\beta_5} \frac{dx}{y}\right), \left(\int_{\beta_1}^{\beta_2} \frac{dx}{y}, \int_{\beta_2}^{\beta_1} \frac{dx}{y}, \int_{\beta_1}^{\beta_2} \frac{dx}{y}\right), \left(\int_{\beta_1}^{\beta_3} \frac{dx}{y}, \int_{\beta_3}^{\beta_1} \frac{dx}{y}, \int_{\beta_1}^{\beta_3} \frac{dx}{y}\right) \right\}$$

of $\mathbb{C}^3/\Lambda$. Choosing a basis $\{e_1, \ldots, e_6\}$ of the $\mathbb{Z}$-module $\Lambda$, we can use numerical calculation to find $q_{i,j} \in (1/2)\mathbb{Z}$ such that

$$v_i = q_{i,1}e_1 + \cdots + q_{i,6}e_6.$$  \hspace{1cm} (2.16)

Using linear algebra over $\mathbb{Z}$, we can find a basis for the lattice $\Lambda_Z$ obtained by adjoining the elements of $G$. Moreover, the principal polarization on $\Lambda$ (which is returned by the algorithms in [26]) extends to a polarization on $\Lambda_Z$, which by construction of $G$ is a 2-fold of a principal polarization $E$.

Our task is to find a plane quartic equation (2.4) corresponding to $(\Lambda_Z, E)$, if it exists. This is accomplished by the following algorithm. It essentially requires the generalized algorithms in [30].

Algorithm 2.17. This algorithm (numerically) reconstructs a ternary quartic from a period lattice.

**Input:** A period lattice $\Lambda$ with a principal polarization $E$.

**Output:** A ternary plane quartic $F$ whose corresponding period lattice with respect to (2.9) equals $\Lambda$, if it exists.

**Steps:**

(i) Choose a matrix $P = (P_1 \ P_2) \in M_{3,6}(\mathbb{C})$ with respect to a symplectic basis for $E$, and let $\tau = P_2^{-1}P_1$ be a corresponding small period matrix.
(ii) As in [23], check whether \( \tau \) has associated vanishing even theta-null values. If so, terminate the algorithm.

(iii) As in [23], construct a Weber model for \( \tau \), determine the corresponding invariants, and reconstruct a corresponding ternary quartic \( F \).

(iv) Calculate the period lattice \( \Lambda_F \) associated to \( F \).

(v) Using the methods from [5, §4.1], find a matrix \( T \) such that \( T \Lambda_F = \Lambda R \) for some \( R \in \text{GL}_6(\mathbb{Z}) \). Let \( F_0 = T \cdot F \) be the transform of \( F \) by \( T \).

(vi) Calculate the period lattice \( \Lambda_0 \) for \( F_0 \) and (again using [5]) find \( \mu \) be such that \( \mu \Lambda = \Lambda_0 \). Return the ternary quartic \( \mu F_0 \).

The correctness of Algorithm 2.17 follows from the sources cited therein and Proposition 2.10.

Remark 2.18. The calculation in Step (vi) is in fact superfluous, as the effect of applying \( T \) to \( G \) on the resulting period lattices can be described in terms of a power of its determinant. For ease of exposition, we have used the description above.

2.2. Interpolation. The results so far are purely numerical and specific to the base field \( \mathbb{C} \). We now interpolate them to obtain explicit formulae. This process leads to very large formulae that we cannot display in this article, and in fact all of the considerations in this section will be descriptive rather than explicit. However, the process of obtaining the relevant formulae is documented in the ZIP file available at [20]. We sketch the main ideas.

We start with formal monic defining equations

\[
X : y^2 = (x - \alpha_1) \cdots (x - \alpha_4)
\]  

(2.19)

and

\[
Y : y^2 = (x - \beta_1) \cdots (x - \beta_4)(x^2 + ax + b).
\]  

(2.20)

We can then consider the gluing for the group \( G \) specified in the previous section, with \( \beta_5, \beta_6 \) the roots of the symmetrized polynomial \( x^2 + ax + b \). This gives rise to a ternary quartic form \( F(x, y, z) \) that defines a curve \( Z \). While Proposition 2.10 shows that this equation is only determined up to a minus sign, the corresponding curve is still canonically determined as a subvariety of \( \mathbb{P}^2_\mathbb{C} \). More concretely, we can obtain a unique normalized defining equation for \( Z \) by dividing by the coefficient of \( x^4 \) in \( F \). We want to determine the dependence of this equation on the \( \alpha_i \) and \( \beta_j \), and will achieve this by a suitable interpolation process that we will prove correct a posteriori in Section 2.3.

It is far too ambitious to start working with all \( \alpha_i \) and \( \beta_j \) simultaneously. Instead, we have gradually worked our way up. We sketch our procedure.

(i) First we work with as few moduli parameters as possible and consider equations of the form

\[
X : y^2 = x(x - 1)(x - \alpha)
\]  

(2.21)

and

\[
Y : y^2 = x(x - 1)(x - \beta)(x^2 + ax + b)
\]  

(2.22)

That is, we take \( \alpha_1 = \beta_1 = \infty, \alpha_2 = \beta_2 = 0, \alpha_3 = \beta_3 = 1, \alpha_4 = \alpha, \beta_4 = \beta \), while \( \beta_5, \beta_6 \) are the roots of \( x^2 + ax + b \) as before.

We consider the monomials in \( \alpha, \beta, a, b \) of degree at most 4. There are 70 of these. We therefore generate 200 quartics over \( \mathbb{Q} \) by taking random integer values for \( \alpha, \beta, a, b \) between \(-10\) and \(10\), and apply the LLL algorithm to the result of Algorithm 2.17 in order to obtain good rational approximations numerical values for the coefficients of the normalized defining equation of \( Z \). (Note that we know the resulting equation to be defined over \( \mathbb{Q} \), since by our choice of \( \alpha, \beta, a, b \) the curves \( X \) and \( Y \) as well as the gluing datum are defined over that field.)
We then try to find rational expressions in $\alpha, \beta, a, b$ for the coefficients of $Z$ that interpolate these equations. It turns out that all instances are interpolated by the ternary quartic
\begin{equation}
(a^2 \beta^2 - a^2 \beta - a \beta^2 + a \beta) z^4 + (a \alpha^2 - a \alpha \beta^2 - a \alpha \beta + a \beta^2 + a b \beta - 2 b \alpha \beta + b \beta^2 + a \alpha^2 - a \alpha^2) x \beta^2 y^2 + (-2 a \alpha \beta^2 + 4 a \alpha^2 - 2 a \beta^2 + 2 a \alpha^2 - 2 a \alpha \beta - 2 b \beta^2 + 2 a \alpha^2 + 2 a \beta^2 - 2 b \beta^2) x y z + (-a a \beta^2 + a b \beta^3 - b a \beta^2 + 2 a \beta^3 - 2 b a \beta^2 + 2 b a \beta + a \beta^3 - 2 a \beta^3 - 2 b \beta^3) y^2 z^2 + (2 a a \alpha^2 - 2 a a \beta^2 + 2 a \alpha^2 + 2 a \beta^2 + 2 b \alpha \beta + 2 a \alpha \beta^2 + a b \beta^2 - 2 a \alpha \beta - a b \beta^2 + a b \beta - 2 b \alpha \beta + 2 b \beta^2 + 2 b a \alpha \beta + 2 b \beta^3 - 2 b \beta^3) y^2 z^2 + (2 a a \alpha^2 - 2 a a \beta^2 - 2 a \alpha^2 - 2 b \alpha \beta + 2 b \alpha \beta + 2 b \beta^2 - 2 b \beta^3 - 2 b \beta^3 - 2 b \beta^3 - 2 b \beta^3) y^3 z^3 + (a \alpha \beta^3 - a \alpha \beta^2 - a \alpha^2 + a \beta^2 + b a \beta^3 - a b \beta - a b \beta^2 - a b \beta + a a \beta^2 - a a \beta^2 - a a \beta + a a \beta - a b \beta - a b \beta^2 - a b \beta - a b \beta + a a \beta^2 - a a \beta^2 - a a \beta + a a \beta) y^4 + (2 a a a \alpha^2 - 2 a a a \beta^2 + 2 a a \alpha^2 + 2 a a \beta^2 + 2 b a \alpha \beta + 2 a a \beta^2 + 2 b a \alpha \beta + 2 b \alpha \beta + 2 a a \beta^2 + 2 b a \alpha \beta + 2 b \alpha \beta + 2 b \beta^2 + 2 b \beta^2 + 2 b \beta^2 + 2 b a \alpha \beta + 2 b \beta^3 - 2 b \beta^3 - 2 b \beta^3 - 2 b \beta^3) y^3 z^4
\end{equation}
which is still of somewhat acceptable size — at least after simplifying by changing the factor in front of $x^4$ to $a^2 \beta^2 - a^2 \beta - a \beta^2 + a \beta$. Testing this result on a few thousand quartics more confirms it, completing the first step of our approach. We also observe that the new factor in front of $z$ that clears denominators is nothing but a product of the discriminants of the polynomials $x(x - 1)(x - \alpha)$ and $x(x - 1)(x - \beta)$.

(ii) The result in (i) already speeds up further considerations, since it obviates all but the final two steps in Algorithm 2.17, saving considerable calculation time. We now explore further by keeping one of (2.21) and (2.22) fixed and considering the general expression (2.19) and (2.20) for the other factor. At this point, we suspect that the resulting expressions are polynomial once the normalized defining equation for $Z$ is multiplied with the product of the discriminants of $(x - x_1) \cdots (x - x_4)$ and $(x - y_1) \cdots (x - y_4)$. This turns out to be the case: the corresponding interpolation needs a few thousand curves but finds corresponding results with very small coefficients, though involving many monomials. We jot down the homogeneity degrees of these monomials in the $a_i$ and $\beta_j$ for later use.

(iii) We now consider the equations (2.19) and (2.20) simultaneously, multiplying the normalized equation for $Z$ with the same product of discriminants as before. Knowing what degrees of homogeneity in the $a_i$ and $\beta_j$ to expect for every defining coefficient cuts the number of candidate monomials down enormously, although it often remains considerable, in the order of several thousand at worst. However, finding a corresponding number of curves for interpolation is no problem, and with enough patience, the corresponding linear-algebraic calculations terminate. They again yield formulae with very modest coefficients (typically small powers of 2), but with a very large number of monomials.

(iv) Having found these interpolated formulae, we can stress-test them further on several thousand more curves, until we are convinced that everything checks out.

To summarize the heuristic results so far: Starting with $X$, $Y$ and $G$ at the beginning of this section, we have obtained a formula for a plane quartic

\begin{equation}
Z : a_{400} x^4 + a_{229} x^2 y^2 + a_{211} x^2 y z + a_{202} x^2 z^2 + a_{040} y^4 + a_{031} y^3 z + a_{022} y^2 z^2 + a_{013} y z^3 + a_{004} z^4,
\end{equation}

with the $a_{ijk}$ polynomials in the $a_i$, $\beta_j$, $a$ and $b$ such that (conjecturally!) for generic values of the parameters, the resulting substitution yields a plane quartic whose Jacobian is isomorphic over $\mathbb{C}$ to the quotient $(\text{Jac}(X) \times \text{Jac}(Y))/G$ with its induced principal polarization. Note the pleasant form of (2.23), which is a consequence of its canonicity.

Remark 2.24. We will not give a precise analysis of the degeneracy locus of the formulae obtained above. Since the discriminant of the resulting ternary quartic can be verified to be non-zero, it is generically well-defined. Its exact locus of definition is best studied in the context of a more
detailed theoretical approach, perhaps by using the theory of algebraic theta functions, than the more ad hoc methods of this article.

**Remark 2.25.** While we have considered the formal equations (2.19) and (2.20) above, one also has to consider the cases where the defining polynomial of X and/or Y has odd degree. The resulting interpolation procedures are, however, completely similar, so we do not consider them further here or in what follows.

2.3. **Rationality considerations and verification.** Because we have determined our equation for a quotient by a subgroup of the 2-torsion in a canonical way, it stands to reason to expect that the resulting construction remains valid over base fields whose characteristic does not equal 2. This turns out to be the case. First, however, we will discuss how to prove when the heuristically interpolated equation (2.23) is actually correct, that is, when the Jacobian of the curve Z in (2.23) actually splits as a product of the Jacobians of the given curves X and Y. We first consider these questions over \( k \) and discuss the base field \( k \) in Section 2.4, where we will also generalize to the case where \( p_X \) and \( p_Y \) are not necessarily monic.

2.3.1. **The genus-1 factor.** The curve Z has an obvious involution \( \iota : (x, y, z) \mapsto (-x, y, z) \). We claim that the Jacobian of the corresponding quotient is generically indeed \( k \)-isomorphic to \( \text{Jac}(X) \).

**Algorithm 2.26.** This algorithm gives a method to verify the existence of a \( k \)-isomorphism \( \text{Jac}(Z/\iota) \cong \text{Jac}(X) \).

**Input:** A curve X defined by an equation (2.1) and a curve Z defined by an equation (2.23).

**Output:** A boolean that indicates whether there exists a \( k \)-isomorphism \( \text{Jac}(Z/\iota) \cong \text{Jac}(X) \).

**Steps:**

(i) Write (2.23) in the form \( Ax^4 + Bx^2 + C \), with \( A, B, C \in \overline{k}[y, z] \).

(ii) Let \( p = B^2 - 4AC \), and let \( p_0 \) be the homogenization of \( p_X \). Define \( X' \) to be the curve defined by \( p \).

(iii) Let \( I, J \) (resp. \( I_0, J_0 \)) be the binary quartic invariants of \( p \) (resp. \( p_0 \)), as defined in [9].

(iv) Check whether we have \( (I : J) = (\mu^2 I_0, \mu^3 J_0) \) for some \( \mu \in \overline{k} \). If so, return true, and return false otherwise.

The correctness of the algorithm follows from the observation that the quotient \( Z/\iota \) is isomorphic to \( X' \) and the fact that given a binary quartic \( p \) with invariants \( I, J \) as in [9], the Jacobian of the curve corresponding to \( p \) is defined by \( y^2 = x^3 - 27Ix - 27J \), see the first footnote in loc. cit.

**Remark 2.27.** The astute reader will note that Algorithm 2.26 does not work in characteristic 3. Over finite fields, which is a usual case of interest, we can still circumvent this problem. Indeed, the Hasse–Weil bound shows that the relevant curves of genus 1 always admit a rational point. Putting these at infinity, we are reduced to testing for isomorphism of elliptic curves, which this time becomes a calculation of equivalence in a weighted \((1, 2, 3)\)-space.

Algorithm 2.26 is sufficiently simple to be run for the generic expression (2.23) (considered as a ternary quartic over a rational function field). It yields a positive response. The implementation at [19] performs the corresponding check for every gluing that it constructs.

2.3.2. **The genus-2 factor.** It remains to check whether a complementary factor of the Jacobian of the curve Z in (2.23) is given by the specified curve Y. For this, we use the result [31] by Ritzenthaler and Romagny. We summarize their result in the following way:

**Theorem 2.28 ([31]).** Consider a smooth plane quartic curve

\[
Z : x^4 + h(y, z)x^2 + f(y, z)g(y, z) = 0
\]  

(2.29)
over $k$, where $h \in k[y, z]$ and $f, g \in \overline{k}[y, z]$ are binary quadratic forms. Let $\iota$ be the involution $\iota : (x, y, z) \mapsto (-x, y, z)$. Then there exists a polynomial $p(h, f, g) \in k[x]$ whose coefficients are polynomial expressions in those of $h, f, g$ such that the Jacobian of the genus-2 hyperelliptic curve

$$Y' : y^2 = p(h, f, g)$$

(2.30)
defines a degree-2 cover of the Prym variety of $Z \to Z/\iota$.

Remark 2.31. As mentioned in [31], the formulae from Theorem 2.28 only apply under certain genericity assumptions. We quietly pass over these in what follows, as we do with most results in [31]. The corresponding calculations are also checked at [19].

Lemma 2.32. Denoting the substitution action of $A \in \text{GL}_2(k)$ on a binary quadratic $q$ by $q.A$, we have the following.

(i) $p(h, f, g) = p(h, g, f)$.
(ii) $p(h, f, g) = p(h, \mu^{-1}f, \mu g)$ for all $\mu \in \overline{k}^\times$.
(iii) $p(h.A, f.A, g.A) = p(h, f, g).A'$, where $A' = UAU^{-1}$ for $U = (-1 0 \ 0 1)$.

Proposition 2.33. In the situation of Theorem 2.28, there exists a $k$-rational surjective map $\text{Jac}(Z) \to \text{Jac}(Y')$ if and only if the polynomial $p$ has coefficients in $k$.

Proof. The condition is certainly necessary, since $k$-rationality is not a meaningful notion otherwise. Suppose from now on that $p$ has coefficients in $k$. One way to conclude is to invoke the canonicity of the constructions in [31], which ensure rationality over the base field even when meaningfully possible. However, there is also a more direct proof.

First consider the special case where $f = f_0 := yz$. Then $Z$ admits the $k$-rational point $P_0 = (0 : 0 : 1)$. To construct a $k$-rational map $\text{Jac}(Z) \to \text{Jac}(Y')$, it suffices to construct such a map for divisors of the form $[P] - [P_0] \in \text{Jac}(Z)$. Moreover, by smoothness of the Jacobian, it in turn suffices to indicate the image of a generic point $P$. Given our equation for $Y'$, we let $\infty$ be the degree-2 divisor of $Y'$ at infinity. We can then specify an element $D = \infty$ of $\text{Jac}(Y')$ using the Mumford representation

$$0 = t^2 + a_1 x + a_2$$
$$y = b_1 x + b_2$$

(2.34)
of a degree 2 divisor $D$. Since $P$ is a generic point and the map involved is non-constant, the coefficients $a_1, a_2, b_1, b_2$ are in the function field of $Z$.

The results at [19] contain an equation for a non-trivial divisor (2.34) that was obtained via another interpolation. It is defined over the base field. Moreover, it is verified that in terms of the bases (2.9) and (2.8) the corresponding pullback of differentials is represented by the transpose of the matrix $T_0 = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

The latter property proves that the map $\text{Jac}(Z) \to \text{Jac}(Y')$ is surjective and that the corresponding factor of the Jacobian is complementary to the quotient by $\iota$. Moreover, it is what we need to extend our results to general $Z$ (on which there may not be a $k$-rational point, so that describing a map $\text{Jac}(Z) \to \text{Jac}(Y')$ becomes problematic).

Indeed, now let $F = x^4 + h(y, z)x^2 + f(y, z)g(y, z) \in k[x, y, z]$ with $f \in \overline{k}[y, z]$ be general. Then there exists $A \in \text{GL}_2(\overline{k})$ such that $f = f_0.A$. Let $F_0 = F.A^{-1} \in \overline{k}[x, y, z]$, where $A = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$. Construct the polynomial $p$ (resp. $p_0$) and the curve $Y'$ (resp. $Y'_0$) corresponding to $F$ (resp. $F_0$). The preceding special case shows that there is a middle map in

$$\text{Jac}(Z) \to \text{Jac}(Z_0) \to \text{Jac}(Y'_0) \to \text{Jac}(Y').$$

(2.35)
The map $\text{Jac}(Z) \to \text{Jac}(Z_0)$ is induced by the map of curves defined by $\tilde{A}$, and by Lemma 2.32(iii) there is an isomorphism $\text{Jac}(Y'_0) \to \text{Jac}(Y')$ induced by $UA^{-1}U^{-1}$. The tangent representation of the composition (2.35) is therefore

$$
UA^{-1}U \cdot T_0 \cdot \tilde{A} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} A^{-1} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}
$$

$$
= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} A^{-1} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}
$$

$$
= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} A^{-1} 2A
$$

$$
= \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} = T_0.
$$

We conclude that the composition (2.35) still has tangent representation $T_0$. Since the tangent representation uniquely determines the corresponding morphism, the latter is defined over $k$ as well.

Given an equation $Z$ of the form

$$
Z : x^4 + h(y, z)x^2 + q(y, z) = 0
$$

there are 3 choices for the factorization in (2.29) over $\overline{k}$ up to reordering. By Lemma 2.32(i) and (ii), the resulting equation for $Y'$ depends only on the chosen such factorization, or in other words, of a choice of partition of the roots of $q$ into two pairs.

**Proposition 2.38.** Given an equation (2.37) for a plane quartic curve $Z$, there exists a choice of factorization (2.29) that gives rise to a curve $Y'$ defined over $k$ under the construction in 2.28 if and only if the cubic resolvent $\varrho(q)$ of $q$ admits a root over $k$.

**Proof.** This follows because the choice of a pair partition of roots of $q$ uniquely determines $Z$. The resulting defining polynomial $p$ of $Y'$ is therefore defined over $k$, that is, Galois invariant, if and only if the given partition is. This is the case if and only if $\varrho(q)$ admits a root. Indeed, classical Galois theory shows that the splitting field of $\varrho(q)$ generates the subfield corresponding to the kernel of the conjugation action $\text{Gal}(q) \to \text{Aut}(V_4)$ on pairs of roots, and from the description of the roots in (1.47) we then see that the stabilizers of an individual root of $\varrho(q)$ is nothing but the stabilizer of a given pair partition.

Now consider the genus-2 curve $Y$ with which we started. The matrix $T_0$ involved in Proposition 2.33 shows that we should expect the defining polynomial of $p_Y$ of $Y$ to coincide with the transformation $p(-x)$ of the polynomial defining the recovered factor $Y'$. This turns out to be the case, but only up to a constant. In other words, we have

$$
p_Y(x) = cp(-x)
$$

for some non-trivial $c \in k$. The reason for this phenomenon is the same as our reason for only considering isomorphisms over $\overline{k}$ in Section 2.3.1, namely the presence of twists, a problem to which we turn in the next section. For now, we record our result over $\overline{k}$:

**Algorithm 2.40.** If the result of this algorithm is positive, then the Jacobian of $Y$ is $\overline{k}$-isogenous to the complement of the image of $\text{Jac}(Z/\iota)$ in $\text{Jac}(Z)$.

**Input:** A curve $Y$ defined by an equation (2.2) and a curve $Z$ defined by an equation (2.23).

**Output:** A boolean that, if true, shows that there is a $\overline{k}$-isogeny between $\text{Jac}(Y)$ and the complement of the image of $\text{Jac}(Z/\iota)$ in $\text{Jac}(Z)$.

**Steps:**
(i) Rescale \((2.23)\) to the form \((2.29)\) and consider the three possible factorizations up to scalars 
\[ q = fg \]
into binary quadratic factors.

(ii) For each factorization pair, calculate a defining polynomial \(p\). If for one of these polynomials
we have \(p_Y(x) = cp(-x)\) for some \(c \in \overline{\mathbb{F}}\), then return true. Otherwise return false.

In contrast to Algorithm \(2.26\), it costs large amounts of time and memory to run Algorithm \(2.40\)
on the generic expression \((2.23)\). Still, the implementation at [19] performs the corresponding check
for every concrete gluing that it constructs before returning its results.

2.4. Twists. Over an algebraically closed field \(k\), positive results to the checks in Algorithms \(2.26\)
and \(2.40\) previous sections suffice to demonstrate that \(\text{Jac}(Z)\) is indeed isogenous to \(\text{Jac}(X) \times \text{Jac}(Y)\).
We are now interested in doing likewise for general base field \(k\), for which twists play a subtle role,
as mentioned in Theorem \(1.59\). The main problems in passing to the base field \(k\) are the following:

(i) The isomorphism \(\text{Jac}(Z/\iota) \rightarrow \text{Jac}(X)\) in Algorithm \(2.26\) may be defined over a proper
extension of \(k\);

(ii) The isomorphism \(\text{Jac}(Y) \rightarrow \text{Jac}(Y')\) (or equivalently the isomorphism \(Y \rightarrow Y')\) in Algo-
rithm \(2.40\) may be defined over a proper extension of \(k\).

In other words, when the necessary hypothesis of Proposition \(2.33\) is satisfied, we will have

\[ \text{Jac}(Z) \sim \text{Jac}(X') \times \text{Jac}(Y') \]  
(2.41)

for suitable curves \(X'\) and \(Y'\) of genus 1 and 2 over \(k\), but \(X'\) and \(Y'\) need not be isomorphic to
the specified curves \(X\) and \(Y\) over the base field \(k\) itself.

We are in Case (i) when \((I : J)\) and \((I_0 : J_0)\) are equivalent in weighted \((2, 3)\)-space over \(k\), but
not in weighted \((4, 6)\), whereas Case (ii) occurs when the scalar \(c\) in \((2.39)\) is not a square in \(k\). In
either case, the issue is the presence of quadratic twists.

We can resolve this problem by appropriately twisting the Jacobian \(\text{Jac}(Z)\). Choose a defining
equation \((2.29)\) for \(Z\), and let \(\nu \in k\). Then we can take a quadratic twist by \(\nu\) with respect to the
involution \(\iota : (x, y, z) \mapsto (-x, y, z)\) to obtain the curve

\[ Z_\nu : \nu^2 x^4 + h(y, z)\nu x^2 + f(y, z)g(y, z) = 0. \]  
(2.42)

Similarly, given a curve \(X : y^2 = p_X\) of gonality 2, we write \(X_\nu : y^2 = \nu^4 p_X\) for its quadratic twist
by \(\nu\).

Moreover, \(\text{Jac}(Z)\) has the automorphism \(-1\) that does not come from an automorphism of the
plane quartic curve \(Z\). We write \(\mu \ast \text{Jac}(Z)\) to denote the quadratic twist of \(\text{Jac}(Z)\) with respect
to this automorphism by a given element \(\mu \in k\).

Lemma 2.43. Let \(Z\) be a genus-3 curve defined by an equation \((2.37)\). Suppose that \(X'\) and \(Y'\)
are such that \(\text{Jac}(Z) \sim \text{Jac}(X') \times \text{Jac}(Y')\) over \(k\). We have:

(i) \(\mu \ast \text{Jac}(Z) \sim \text{Jac}(X'_\mu) \times \text{Jac}(Y'_\mu)\);

(ii) \(\text{Jac}(Z_\nu) \sim \text{Jac}(X'_\nu) \times \text{Jac}(Y'_\nu)\).

Proof. Part (i) follows because the projections \(\text{Jac}(Z) \rightarrow \text{Jac}(X')\) and \(\text{Jac}(Z) \rightarrow \text{Jac}(Y')\) commute
with the automorphism \(-1\), which implies that over \(k\) the twist \(\mu \ast \text{Jac}(Z)\) admits the factor
\(\mu \ast \text{Jac}(X) = \text{Jac}(X_\mu)\).

Part (ii) can be verified by explicit calculation. Indeed, the quotient

\[ Z/\iota : x^2 + h(y, z)x + q(y, z) = 0 \]  
(2.44)

is clearly \(k\)-isomorphic to

\[ Z_\nu/\iota : \nu^2 x^2 + h(y, z)\nu x + q(y, z) = 0. \]  
(2.45)

As for the new \(Y'_\mu\), a direct calculation that its defining polynomial is \(\nu^{-2}\) that of \(Y'\), which implies
the claim. \(\square\)
Since combining the actions in Lemma 2.43 allows us to twist both separate factors $X'$ and $Y'$ in any way desired, we can find $\mu$ and $\nu$ such that $\mu \ast \text{Jac}(Z_\nu)$ has the requested factorization $\text{Jac}(X) \times \text{Jac}(Y)$ up to isogeny over the base field $k$.

This observation allows us equally well to deal with general defining equations (2.1) and (2.2) rather than merely their monic versions (2.19) and (2.20) that we used in our interpolation algorithms up until now. Moreover, the verification algorithms 2.26 and 2.26 function equally well over $k$: In Algorithm 2.26, it suffices to check for equivalence of $(I : J)$ and $(I_0 : J_0)$ in weighted projective $(4,6)$-space over $k$ instead of in weighted projective $(2,3)$-space over $\overline{k}$, and in Algorithm 2.40, it suffices to demand that $c$ be a square in $k$.

Summarizing all that went before in this section, we have therefore obtained the following main algorithm, and with it, the Main Theorem:

**Algorithm 2.46.** This algorithm finds gluings of genus-1 and genus-2 curves along their torsion over the base field.

**Input:** Equations $X : y^2 = p_X$ and $Y : y^2 = p_Y$ that define curves of genus 1 and 2 over $k$.

**Output:** A (possibly empty) list $L$ of pairs $(Z, \mu)$, where $Z$ is a smooth plane quartic and where $\mu \in k^*$ is a constant such that $\mu \ast \text{Jac}(Z) \sim \text{Jac}(X) \times \text{Jac}(Y)$.

**Steps:**

(i) Initialize the empty list $L$. For all quadratic factors $q_Y$ of $p_Y$, let $r_Y = p_Y/q_Y$ and perform all next steps but the final one.

(ii) Check if the splitting fields of $p_X$ and $q_Y$ are isomorphic. If so, consider labelings of roots of $p_X$ and $q_Y$ such that the Galois actions on the corresponding roots (1.47) of the quadratic resolvents are compatible. For all such labelings, perform all next steps but the final one.

(iii) Construct the interpolated curve $Z$ in (2.23). Check that the coefficients of $Z$ belong to $k$.

Construct $X' = Z/I$ as in Algorithm 2.26 and check that $X'$ is a quadratic twist of $X$ by $\mu$ say. Check that the cubic resolvent of $q = a_{040}y^4 + a_{031}y^3z + a_{022}y^2z^2 + a_{013}yz^3 + a_{004}z^4$ admits at least one root over $k$, and that for one of the roots of this resolvent, we have $p_Y(x) = c\mu(x)$ for some $c \in k$, where $p'$ is the defining polynomial for the curve $Y'$ corresponding to the chosen root, as in Algorithm 2.40.

(iv) Let $\nu = c\mu$, so that $\mu \ast \text{Jac}(Z_\nu) \sim \text{Jac}(X_\nu') \times \text{Jac}(Y_\nu') \cong \text{Jac}(X_\mu') \times \text{Jac}(Y_\mu') \cong \text{Jac}(X) \times \text{Jac}(Y)$ by Lemma 2.43. Append $(Z, \nu)$ to $L$.

(v) Return the list $L$.

**Remark 2.47.** The algorithms at [19] apply a more precise version of the above results, which considers the effect of twists on defining equations (2.1) (and hence on bases of differentials (2.7)) rather than on isomorphism classes. We omit the calculations, which only slightly refine the twisting scalars involved, and describe the result. Recall from Section 2.1 that given defining equations for the curves $X$, $Y$, and $Z$, we can consider the bases of $\text{Jac}(Z)$ (resp. $\text{Jac}(X) \times \text{Jac}(Y)$) that are duals of the bases (2.9) and the union of the pullbacks of (2.7) and (2.8).

Now let defining polynomials $p_X$ and $p_Y$ for the input curves $X$ and $Y$ be given. Then the formulas at [19] give a ternary quartic equation $F$ and a constant $\mu \in k$ such that there exists a map $\text{Jac}(Z) \to \text{Jac}(X) \times \text{Jac}(Y)$ with tangent representation $4\sqrt{\mu}$ with respect to the bases corresponding to $p_X$, $p_Y$, and $F$. Thus $\mu \ast \text{Jac}(Z)$ is isogenous over $k$ to the $\text{Jac}(X) \times \text{Jac}(Y)$. The factor 4 is inserted because it makes both $F$ and $\mu$ smaller without losing integrality.

In practice, choosing this completely canonical approach gives rise to very agreeable expressions for $Z$ and $\mu$, especially when $p_X$ and $p_Y$ are in reduced minimized form. For example, the simple equations in (2.58), were found using this method, without any further optimization or reduction being needed.

### 2.5. A crucial symmetrization

A symmetrization of the formulae obtained above that is important in practice is the following. It is unpleasant, when starting with polynomials $p_X$ and $p_Y$
defining $X$ and $Y$, to have to determine their roots $\alpha_i$ and $\beta_j$, as is currently required to apply the first formula (2.23). This leads to the determination of a compositum of the splitting field of two quartic polynomials, which already over $\mathbb{Q}$ involves difficult field arithmetic that cannot be circumvented by reduction algorithms like PARI’s polredabs since the extensions involved may have degree larger than 20.

Instead, we use the more symmetric presentations

$$p_X = (x - \alpha_1) \cdots (x - \alpha_4) = x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$$

and

$$p_Y = (x - \beta_1) \cdots (x - \beta_4)(x^2 + ax + b) = (x^4 + b_1x^3 + b_2x^2 + b_3x + b_4)(x^2 + ax + b).$$

The considerations from Section 1 show that our chosen gluing only depends on a pairing of the roots $\gamma_1 = \alpha_1 \alpha_2 + \alpha_3 \alpha_4$, $\gamma_2 = \alpha_1 \alpha_3 + \alpha_2 \alpha_4$, $\gamma_3 = \alpha_1 \alpha_4 + \alpha_2 \alpha_3$ of the cubic resolvent of $p_X$ with the roots $\delta_1, \delta_2, \delta_3$ of the cubic resolvent of $x^4 + b_1x^3 + b_2x^2 + b_3x + b_4$. We therefore expect that the coefficients $a_{ijk}$ in (2.23) are invariant under the Klein Vierergruppen of permutations of $\alpha_i$ and $\beta_j$ that stabilize all of the $\gamma_i$ and $\delta_i$. This indeed turns out to be the case.

Invariant theory shows that the polynomial expressions in $\alpha_1, \ldots, \alpha_4$ that are invariant under the distinguished Vierergruppe are polynomial expressions in the invariants $\gamma_1, \gamma_2, \gamma_3$ of weight 2 and the coefficients $a_1$ and $a_3$ of $p_X$, which are of weight 1 and 3, respectively. A similar result of course holds for $\beta_1, \ldots, \beta_4$. We obtain the following result.

**Proposition 2.50.** Starting with equations (2.19) and (2.20), the coefficients of the interpolated polynomial $Z$ in (2.23) can be expressed as polynomials in the compositum of the splitting fields of the cubic resolvents of $p_X$ and $p_Y$. In particular, if these splitting fields coincide (a necessary and sufficient condition for a gluing over the base field to exist by Theorem 1.53), then explicitly determining the coefficients of $Z$ only requires intermediate calculations in this common splitting field of a cubic polynomial.

**Remark 2.51.** Formally writing down the invariant expressions Proposition 2.50 already cuts down their length by a factor almost 20.

In practice, the formulae obtained in this way determine gluings over finite base fields (not necessarily prime fields) in a fraction of a second, whereas gluing curves over the rationals whose defining coefficients have about 100 decimal digits needs a bit over a minute (and results in defining coefficients with about 1600 decimal digits). Corresponding test suites are available at [19].

**Remark 2.52.** When considering the case where the curve $Y$ admits a defining equation of the form $Y : y^2 = p(x^2)$, our algorithms directly recover the Ciani form (0.2). When Jac($Y$) is itself a 2-gluing of elliptic curves $E_1$ and $E_2$, embedding $X$ into its Jacobian allows use to recover maps $X \to E_1$ and $X \to E_2$ of degree 4 by functoriality.

### 2.6. Examples.

**Example 2.53.** Consider the genus-1 curve defined by the equation

$$X : y^2 = 4x^3 + 5x^2 - 98x + 157 = p_X.$$  \hspace{1cm} (2.54)

It is isomorphic to the elliptic curve with label 118.\texttt{c1} in the LMFDB [33]. Similarly, let

$$Y : y^2 = x^6 + 2x^3 - 4x^2 + 1 = p_Y.$$  \hspace{1cm} (2.55)

be the genus-2 curve isomorphic to the curve in the LMFDB with label 295.\texttt{a.295.1}.

First we consider these curves in light of Section 1. The defining polynomial $p_Y$ of $Y$ factors as

$$(x - 1)(x^2 + x - 1)(x^3 + 2x + 1).$$  \hspace{1cm} (2.56)
We see that there is a unique quadratic factor \( q_Y = x^2 + x - 1 \), with complement \( r_Y = x^4 - x^3 + 2x^2 - x - 1 \), and hence a unique subgroup \( H \subset V_Y \) of dimension 1 that is Galois-stable. The cubic resolvent of \( r_Y \) is given by

\[
\varrho(r_Y) = x^3 - 2x^2 + 5x - 8.
\]  

(2.57)

This defines the same splitting field as \( p_X \); in fact both polynomials already define a common number field, which is isomorphic to that defined by \( x^3 + 2x - 1 \). Note that we have not taken a resolvent of \( p_X \), in line with Remark 1.54.

The common splitting field of \( p_X \) and \( \varrho(r_Y) \) has Galois group \( S_3 \). Remark 1.52 shows that there is a single Galois equivariant isomorphism \( \ell : G(\mathcal{P}) \to G(\mathcal{R}) \) for \( H \). Since \( H \) itself was also unique, we conclude that \( V_X \times V_Y \) has a single Galois stable maximal isotropic subgroup \( G \).

The algorithms of Section 2 show that the quotient \( (Q, \lambda_Q) \) is a twist by 5 of the Jacobian \( (\text{Jac}(Z), \lambda_Z) \) of the plane quartic curve

\[
Z : 32x^4 + 3x^2y^2 - 132x^2yz + 37x^2z^2 + 3y^4 - 14y^3z + 7y^2z^2 - 6yz^3 - 2z^4 = 0.
\]

(2.58)

(More precisely, the twisting scalar \( \mu \) from Remark 2.47 is given by \( 5^3 \).)

The LMFDB tells us that the Jacobian \( \text{Jac}(X) \) has a rational 5-torsion point, and that \( \text{Jac}(Y) \) has a 14-torsion point. As the isogeny defined by \( G \) has degree that is a power of 2, we can conclude that \( (Q, \lambda_Q) \) has a rational 70-torsion point if we show that the Galois module

\[
W = (\text{Jac}(X)[2](\overline{k}) \times \text{Jac}(Y)[2](\overline{k}))/G
\]

(2.59)

has a Galois stable subspace of dimension 1. For this, we use our knowledge of the subgroup \( G \). The splitting field \( L \) of \( p_X p_Y \) is of degree 12 over the base field \( k = \mathbb{Q} \). We can label the roots \( \alpha_1 = \infty, \alpha_2, \ldots, \alpha_4 \) of \( p_X \) and \( \beta_1, \ldots, \beta_6 \) of \( p_Y \) in such a way that (i) \( \beta_1 = 1 \), (ii) \( \beta_5 \) and \( \beta_6 \) correspond to the quadratic factor \( q_Y \), and (iii) the Galois action on the 2-torsion points \( [\alpha_2] - [\alpha_1], [\alpha_3] - [\alpha_1], [\alpha_4] - [\alpha_1] \) in \( G(\mathcal{P}) \) coincides with that on the elements \( [\beta_2] - [\beta_1], [\beta_3] - [\beta_1], [\beta_4] - [\beta_1] \) of \( G(\mathcal{R}) \) defined by \( r_Y \).

Now consider the bases \( \{ [\alpha_2] - [\alpha_1], [\alpha_3] - [\alpha_1] \} \) of \( V_X \) and \( \{ [\beta_2] - [\beta_1], [\beta_3] - [\beta_1], [\beta_4] - [\beta_1] \} \) of \( V_Y \). A calculation (performed in [19]) shows that the right action of two generators of \( \text{Gal}(L/k) \) is given by the matrices

\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix} \quad \quad \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}.
\]

(2.60)

Because of our ordering of the roots, the subgroup \( G \) corresponds to the subspace \( U \) given by

\[
U = \langle (1 \ 0 \ 1 \ 0 \ 0 \ 0), (0 \ 1 \ 0 \ 1 \ 0 \ 0), (0 \ 0 \ 0 \ 0 \ 1 \ 0) \rangle.
\]

(2.61)

This subspace is indeed stable under the action of the matrices above. If we take the images of the standard basis vectors \( e_2, e_3, e_6 \) as a basis for the corresponding quotient \( W = V/U \), then the induced Galois action on \( W \) is described by the matrices

\[
\begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}, \quad \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}.
\]

(2.62)

There is a single non-trivial vector fixed under this action, which is the image of \( e_6 = [\beta_5] - [\beta_1] \). Indeed the Galois action sends \( [\beta_3] - [\beta_1] \) either to itself or to \( [\beta_6] - [\beta_1] \), which is equivalent to \( [\beta_5] - [\beta_1] \) modulo \( G \), since the latter group contains the generator \( [\beta_5] - [\beta_1] \) of \( H \).
We have therefore shown that the twist of the Jacobian of the curve (2.58) by 5 indeed contains a rational 70-torsion point.

**Example 2.63.** The complex-analytic reconstruction techniques in the previous section also allow one to construct gluings along 3-torsion. While it is more difficult to find examples of such gluings over the base field, one can still inspect which quotients by overlattices have invariants that are numerically in the base field. For example, consider the case \( k = \mathbb{Q} \) and the elliptic curve with LMFDB label 675.d2

\[
X : y^2 = 4x^3 + 25
\]

(2.64)

together with the genus-2 curve

\[
Y : y^2 = x^5 + 20x^3 + 36x
\]

(2.65)

which is a twist of the curve in the LMFDB with label 2916.b.11664.1. Over \( \mathbb{Q} \), the curves \( X \) and \( Y \) admit gluings along 3-torsion whose invariants are in \( \mathbb{Q} \), as is shown in the example files at [19]. Two such gluings are given by the base extensions of

\[
6x^4 - 27x^2y^2 + 42x^2yz + 13x^2z^2 - 18y^4 - 30y^3z + 12y^2z^2 + 24yz^3 + 16z^4 = 0
\]

(2.66)

and

\[
14x^3z + 3x^2y^2 - 210xyz^2 + 10y^3z + 1225z^4 = 0.
\]

(2.67)

The former of these curves has its full endomorphism ring defined over a number field of degree 12, whereas the latter requires an extension of degree 18. Exact verification of these numerical results above is possible via [8]. In fact, these also show that the curve \( Y \) is itself isogenous to a product of elliptic curves.

The algorithms at [19] also allow for the direct gluing of threefold products of elliptic curves along 3-torsion (or 2-torsion), as shown in the example files at [19].

**Example 2.68.** A final example is given by considering the curves

\[
X : y^2 = x^4 + 2x^3 + x + 1
\]

(2.69)

and

\[
Y : y^2 = 2x^6 + x^4 + x^3 + x^2 + 2x + 1
\]

(2.70)

over \( \mathbb{F}_3 \). Our algorithms give two rise to two gluings, defined by the equations

\[
Z_1 : x^4 + x^2yz + 2x^2z^2 + 2y^3z + y^2z^2 + z^4 = 0
\]

(2.71)

and

\[
Z_2 : x^4 + 2x^2yz + x^2z^2 + 2y^3z + y^2z^2 + z^4 = 0.
\]

(2.72)

For the former, the twisting scalar is trivial, whereas the second requires a quadratic twist by the non-square \(-1 \in \mathbb{F}_3\) to recover the relevant abelian quotient variety.

### 3. Gluing via the Kummer Variety

In this section we describe a geometric algorithm that allows us to construct gluings of the curves \( X \) and \( Y \) over any algebraically closed base field \( \mathbb{F} \). The algorithm we describe reverses the construction of Ritzenthaler and Romagny in [31], mentioned above in Theorem 2.28.

We begin by recalling some basic facts about Kummer surfaces. Throughout this section let \( X, Y, \) and \( Z \) be curves over \( k \) of genera 1, 2, and 3, respectively.

**Definition 3.1.** Let \( Y \) be a curve of genus 2 over \( k \). The Kummer surface of \( Y \) is the quotient of \( \text{Jac}(Y) \) by the negation map, i.e., \( \text{Kum}(Y) = \text{Jac}(Y)/(\langle -1 \rangle) \).

**Proposition 3.2 ([14, Proposition 2.16]).** Let \( X \) be a genus 2 curve and \( K = \text{Kum}(X) \). Then \( K \) has 16 singular points, each one a node, and there exist 16 planes such that these planes and nodes form a nondegenerate \((16, 6)\)-configuration.
3.1. **Plane sections of a Kummer variety.** We first give a condition under which a gluing produces a hyperelliptic curve. For this, we need the following Lemma.

**Lemma 3.3.** Let $(Z, \varphi)$ be a gluing of $X$ and $Y$ over $k$. Then there is a degree-2 map $Z \to X$ over $\overline{k}$.

**Proof.** Dualizing the gluing map $\varphi$ gives a map $\varphi' : \text{Jac}(Z) \to \text{Jac}(X) \times \text{Jac}(Y)$. Choosing base points on $Z$ and $X$, we obtain an inclusion $Z \to \text{Jac}(Z)$ and an identification $\text{Jac}(X) \cong X$. We can compose to obtain a non-constant map $f : Z \to X$. The corresponding pullback map $f^* : \text{Jac}(X) \to \text{Jac}(Z)$ is obtained by composing the canonical inclusion of $\text{Jac}(X)$ into $\text{Jac}(X) \times \text{Jac}(Y)$ with $\varphi$. Because of the defining property of $\varphi$, the principal polarization $\lambda_Z$ satisfies $(f^*)^*(\lambda_Z) \equiv 2\lambda_X$, which implies that $f$ is of degree 2 by [3, Lemma 12.3.1].

**Proposition 3.4.** Suppose that $Z$ is a gluing of $X$ and $Y$. If there exists a degree 2 morphism $\pi_2 : Z \to Y$, then $Z$ is hyperelliptic.

**Proof.** There exists a degree 2 morphism $\pi_1 : Z \to X$ by Proposition 3.3. Both $\pi_1$ and $\pi_2$ induce involutions of $Z$, which we denote $i_1$ and $i_2$, respectively. By considering the action of $i_1$ and $i_2$ on $\text{End}^0(\text{Jac}(Z))$, we see that $i_1 \circ i_2$ induces the negation map $-1$ on $\text{Jac}(Z)$. Then $Z$ is hyperelliptic by [24, Appendice, Théorème 3].

**Proposition 3.5.** With notation as in Theorem 2.28, the curve $Z$ is a gluing of the curves $Y$ and $X$.

**Proof.** Consider the degree-2 cover $p : Z \to X$ given above. The map $p$ induces an inclusion $p^* : (\text{Jac}(X), \mathcal{P}_X) \to (\text{Jac}(Z), \mathcal{P}_Z)$ of polarized abelian varieties, and by [3, Lemma 12.3.1] we find that we get

$$(p^*)^* \lambda_Z = 2\lambda_X,$$

the pullback of $p^*$.

Now $\text{Jac}(Y)$ is isomorphic to the Prym of an allowable singular cover $\tilde{p} : \tilde{Z} \to \tilde{X}$ whose normalization is equal to $p : Z \to X$ as is shown in the proof of Theorem 1.1 in [31].

By [1, Theorem 3.7], the principal polarization on the generalized Jacobian $\text{Jac}(\tilde{Z})$ restricts to $2\lambda_Y$ on $\text{Pr}(\tilde{Z} / \tilde{X})$ where $\lambda_Y$ is the principal polarization on $\text{Pr}(\tilde{Z} / \tilde{X}) \cong \text{Jac}(Y)$. From [10, Lemma 1] we get a commutative diagram of polarized abelian varieties:

$$\begin{array}{ccc}
\text{Pr}(\tilde{Z} / \tilde{X}) & \xrightarrow{\nu} & \text{Pr}(Z / X) \\
\downarrow & & \downarrow i \\
\tilde{Z} & \xrightarrow{i} & Z
\end{array}$$

(3.7)

where $\nu$ is induced by the morphism $\tilde{Z} \to Z$. This implies that

$$(i \circ \nu)^*(\lambda_Z) = 2\lambda_Y.$$  

(3.8)

Now consider the map $j : \text{Jac}(X) \times \text{Jac}(Y) \to \text{Jac}(Z)$ defined by $(x, y) \mapsto p^*(x) + (i \circ \nu)(y)$. As $j = (i \circ \nu) \circ \pi_Y$ on the restriction to $\{0\} \times X$, we get that

$$j^*(\lambda_Z)|_{\{0\} \times X} = (\pi_Y^* \circ (i \circ \nu)^*(\lambda_Z))|_{\{0\} \times X} = \pi_Y^*(2\lambda_Y)|_{\{0\} \times X}.$$

An analogous argument shows that

$$j^*(\lambda_Z)|_{X \times \{0\}} = (\pi_X^* \circ (p^*)^*(\lambda_Z))|_{X \times \{0\}} = \pi_X^*(2\lambda_X)|_{X \times \{0\}}.$$

Using that our construction is generic, and that generically $\text{Hom}(A, B) = 0$ so that $\text{NS}(A) \times \text{NS}(B) \cong \text{NS}(A \times B)$, we conclude that $j^*(\lambda_Z)$ is algebraically equivalent to $\pi_X^*(2\lambda_X) \otimes 2\pi_Y^*(\lambda_Y)$, so $Z$ is a gluing of $\text{Jac}(X)$ and $\text{Jac}(Y)$.
A more concrete approach in the case $k = \mathbb{C}$, which generalizes to arbitrary fields by the use of étale cohomology, is the following. The map $p^*$ gives us an inclusion of $L_X = H_1(X, \mathbb{Z})$ into $L_Z = H_1(Z, \mathbb{Z})$ and the equality $(p^*)^*\lambda_Z = 2\lambda_X$ shows that the restriction of $\lambda_Z$ to $L_X$ gives us $2\lambda_X$. Now the kernel of the map $p_*$ gives us the Prym variety $P(Z/X)$, which, using Lemma 1 of [10], comes equipped with a $(2,1)$-polarization that is the restriction of $\lambda_Z$ to $P(Z/X)$. Let $L_P$ be the sublattice of $L_Z$ that corresponds to $P(Z/X)$. Then $p^*(L_X) \oplus L_P \subset L_Z$. The construction in [31] shows that the curve $Y$ corresponds to a sublattice $q^*(L_Y)$ of index 2 in $L_P$ on which the $(2,1)$-polarization restricts to a $(2,2)$-polarization. As a consequence, the restriction of $\lambda_Z$ to $p^*(L_X) \oplus q^*(L_Y)$ is 2 times the product polarization, which ensures that the induced map $j : \text{Jac}(X) \times \text{Jac}(Y) \to \text{Jac}(Z)$ has the property that $j^*(\lambda_Z) = 2\pi_X^*(\lambda_X) \otimes 2\pi_Y^*(\lambda_Y)$ on $\text{Jac}(Z)$, which is what we wanted to show.

The main theorem of this section is the following:

**Theorem 3.9.** Let $(Z, \varphi)$ be a gluing of $X$ and $Y$ over $k$ and assume that $Z$ is a non-hyperelliptic curve. Let $Kum(Y) = \text{Jac}(Y)/(-1) \subset \mathbb{P}^3_k$ be the Kummer surface associated to $\text{Jac}(Y)$. Then over $\overline{k}$ there exists a commutative diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{i_Z} & \text{Jac}(Y) \\
p & & \downarrow \pi \\
X & \xrightarrow{i_X} & \text{Kum}(Y)
\end{array}
$$

(3.10)

where $p : Z \to X$ is the degree 2 cover from Proposition 3.3, $\pi : \text{Jac}(Y) \to \text{Kum}(Y)$ is the quotient map and $i_Z$ and $i_X$ are rational maps such that $i_X(X) = H \cap \text{Kum}(Y)$ for a plane $H \subset \mathbb{P}^3_k$ that passes through two singular points of $\text{Kum}(Y)$.

**Proof.** According to Proposition 3.3 the gluing $Z$ gives rise to a cover as in Theorem 2.28. The code in [32] contains an explicit rational map $Z \dashrightarrow \text{Jac}(Y)$. After a change of coordinates we may assume that $Z$ has an affine open $V$ of the form

$$v^4 + v^2g(u) + uh(u) = 0$$

(3.11)

where $g(u) = g_2u^2 + g_1u + g_0$ and $h(u) = h_2u^2 + h_1u + h_0$. We calculate an equation for $Y$ using 2.28 and use this equation to construct the affine open $U \subset \overline{k}[a_1, a_2, b_1, b_2]$ of the Jacobian $\text{Jac}(Y)$ given by the equations in Proposition 3.40.

Let

$$
\begin{align*}
\alpha(u, v) &= (g_2h_0 - g_0h_2)u^2 + (g_2^2h_0 - g_2g_0h_2)u^2 + (g_2g_1h_0 - g_2g_0h_1 - h_2h_0)u, \\
\beta(u, v) &= g_2^2h_0 - g_2g_0h_2u^2 + (g_2^2h_0 - g_2g_0h_2)u^2 \\
&\quad + ((g_2^3h_1 - g_2g_1g_0h_2 - g_2h_0h_2 + g_0h_2^2)u + g_2^3g_0h_0 - g_2g_0h_2)v \\
N(u, v) &= (g_2^2h_1 - g_2g_1h_2 + h_2^2)u + g_2^2h_0 - g_2g_0h_2.
\end{align*}
$$

(3.12)

Then the map $i_Z : V \to U$ is explicitly given by

$$
(u, v) \mapsto (\alpha(u, v)/N(u, v), 0, \beta(u, v)/(N(u, v)), \beta(u, v)/(uN(u, v))).
$$

(3.13)

In the code it is shown that the image of $i_Z$ is generically contained in $U$. A proof for the generic injectivity of the map $i_Z : Z \to \text{Jac}(Y)$ due to D. Lombardo is the following: Assume that $i_Z$ is not injective. If $i_Z(Z)$ is of genus 2, then $Z$ would be hyperelliptic by Proposition 3.4, which gives us a contradiction, so $i_Z(Z)$ is either of genus 1 or of genus 0. It is impossible for $i_Z(Z)$ to be of genus 0, as then by the theory of abelian varieties the map $i_Z$ would be constant. On the other hand, if $i_Z(Z)$ is a curve of genus 1 then $\text{Jac}(Y)$ would be isogenous to the product of two elliptic curves, which cannot be true generically.
Let $\pi : U \to \text{Kum}(Y)$ be the map given in Corollary 3.52. As $i_Z(Z)$ is contained in the plane given by $a_2 = 0$, it follows that $\pi(i_Z(Z))$ is a curve contained in the plane $H$ defined by $\kappa_3 = 0$. This means we have a rational map $Z \dashrightarrow \pi(i_Z(Z))$ of degree 2. We claim that the curve $\pi(i_Z(Z))$ is of genus 1. Indeed, if $\pi(i_Z(Z))$ is not of genus 1 then it will either be of genus 2 or of genus 0. But in both of these cases $Z$ can be shown to be a hyperelliptic curve, which is a contradiction as $Z$ is non-hyperelliptic by assumption. Indeed, if $\pi(i_Z(Z))$ has genus 2 then Proposition 3.4 tells us that $Z$ is a hyperelliptic curve. In the second case we have a degree 2 cover from $Z$ to a genus 0 curve, so the statement follows by definition. We conclude that $\pi(i_Z(Z))$ is of genus 1. As any plane section of a quartic surface in $\mathbb{P}^3$ has arithmetic genus 3 this means that the plane $H$ has to intersect $\text{Kum}(Y)$ in two singular points. Finally, it remains to be shown that the above diagram commutes. Let $i : Z \to Z$ be the involution $(u, v) \mapsto (u, -v)$ that corresponds to the degree 2 cover $Z \to X$. Then the explicit form of the maps at [32] shows that

$$i_Z(i(u, v)) = i_Z((u, -v)) = (\alpha(u, -v)/(N(u, -v)), 0, \beta(u, -v)/(N(u, -v)), \beta(u, v)/(-vN(u, -v)))$$

$$= (\alpha(u, v)/N(u, v), 0, -\beta(u, v)/(N(u, v)), -\beta(u, v)/(uN(u, v))).$$

Now the map $(a_1, a_2, b_1, b_2) \mapsto (a_1, a_2, -b_1, -b_2)$ sends the divisor $P + Q$ corresponding to the equations $x^2 + a_1x + a_2$ and $y = b_1x + b_2$ to the divisor $P' + Q'$ corresponding to the equations $x^2 + a_1x + a_2$ and $y = -b_1x - b_2$. We see that $P' = -P$ and $Q' = -Q$ on $\text{Jac}(Y)$. So $i_Z \circ (i)$ is multiplication by $-1$ on $\text{Jac}(Y)$ and we conclude that we have found a commutative diagram as in (3.10).

**Remark 3.15.** Theorem 3.9 tells us that any gluing of $X$ and $Y$ occurs as the pullback of a plane section of $\text{Kum}(Y)$ that passes through two singular points. In paragraph 3.2 we will describe this construction explicitly.

**Lemma 3.16.** Let $K$ be a Kummer surface in $\mathbb{P}^3_K$ and let $P_1, P_2$ be a pair of singular points on $K$. Consider the 1-dimensional family $H(\lambda)$ of planes passing through $P_1$ and $P_2$. Then the family of curves $H(\lambda) \cap K$ consists generically of genus 1 curves that have exactly two nodes.

**Proof.** A general member $C$ of the pencil $H(\lambda) \cap K$ is irreducible by Bertini’s first theorem, and is nonsingular outside of $\{P_1, P_2\}$ by Bertini’s second theorem. Since $C$ is an irreducible quartic curve with 2 nodes, it has arithmetic genus 3 and geometric genus 1 by the genus-degree formula. □

**Lemma 3.17.** With notation as above, let $P_1, P_2, P_3$ be three distinct nodes on $K$, and let $H$ be a plane passing through $P_2$ and $P_3$. Then there is a linear automorphism $\sigma$ such that $\sigma(H)$ passes through $P_1$ and a second singular point.

**Proof.** By the properties of (16,6)-configurations described in Paragraph 1 of [14] such a morphism exists if $K$ is of the form described in [14, Proposition 1.6]. But [14, Theorem 1.45] tells us this is always possible up to an automorphism of $\mathbb{P}^3_K$. Thus the automorphism is linear. □

**Lemma 3.18.** Let $\pi : \text{Jac}(Y) \to \text{Kum}(Y)$ be the quotient morphism and let $T_1, T_2 \in \text{Jac}(Y)[2]$. Let $P_1 = \pi(T_1)$ (so $P_1$ is a singular point on $\text{Kum}(Y)$.) Then the automorphism of $\text{Jac}(Y)$ given by $x \mapsto x + T_1 - T_2$ induces a linear automorphism of $\text{Kum}(Y)$ that maps $P_2$ to $P_1$.

**Proof.** According to [14, Proposition 4.15] an automorphism of $\text{Jac}(Y)$ given by translation with an element of $\text{Jac}(Y)[2]$ induces an automorphism of the Kummer surface as described in [14, §1]. In the same paragraph these automorphisms are explicitly written down as linear maps in the special case that the 16 singular points of the Kummer surface are of the form $(\pm a, \pm b, \pm c, \pm d)$ for $a, b, c, d \in \bar{K}$. But [14, Theorem 1.45] tells us that every Kummer surface in $\mathbb{P}^3_K$ can be written in such a form after a linear transformation of $\mathbb{P}^3_K$. □
Remark 3.19. Let $Z \to X$ be a cover and let $H$ be a plane that passes through two singular points $P_2$ and $P_3$ such that $i_X(X) = H \cap \text{Kum}(Y)$ as in Theorem 3.9. Then the above lemma implies that, after choosing suitable automorphisms of $\text{Jac}(Y)$ and $\text{Kum}(Y)$, we may assume that $H$ passes through $P_1$.

Lemma 3.20. Let $k$ be a field, and let $K \subset \mathbb{P}^3$ be a Kummer surface with singular points $P_1, \ldots, P_{16}$. Let $H_{i,j}(\lambda)$ be the family of planes going through $P_i$ and $P_j$ with $i \neq j$. Fix $\lambda_0 \in k$ and let $U \cong \mathbb{A}^2_k$ be an affine open of $H_{i,j}(\lambda_0)$ containing both $P_i$ and $P_j$. Let $\widetilde{C}_{\lambda_0} = K \cap U$. Let $(x_i, y_i)$ be the coordinates of $P_i$ in $U$. Define the function $\tilde{g} : \widetilde{C}_{\lambda_0} \to \mathbb{P}^1$ through $\tilde{g}(x, y) = \frac{y - y_i}{x - x_i}$. (3.21)

Then $\tilde{g}$ extends to a function $g : C_{\lambda_0} \to \mathbb{P}^1$ of degree 2 where $C_{\lambda_0}$ is the normalization of $\widetilde{C}_{\lambda_0}$.

Proof. Given a point $Q_1 = (x, y) \in \widetilde{C}_{\lambda_0}$, then $\tilde{g}(x, y)$ is exactly the slope of the line $\ell$ through $P_1$ and $Q_1$. As $K$ is a quartic surface and $U$ is a plane, then $K \cap U$ is a quartic plane curve. Then $\ell$ and $\widetilde{C}_{\lambda_0}$ have intersection number 4 by Bézout’s theorem. Since $P_i$ is a node it contributes 2 to the intersection number, so $\ell$ generically intersects $\widetilde{C}_{\lambda_0}$ is third point $Q_2$. Thus $\tilde{g}$ is generically 2-to-1, hence has degree 2. □

Proposition 3.22. Let $H(\lambda)$ be the family of planes going through $P_1$ and $P_2$. Then the $j$-invariant of the family $H(\lambda) \cap K$ is a rational function $j(H(\lambda)) \in \mathbb{k}(\lambda)$ of degree at most 12.

Proof. By [14, Proposition 2.20] we may assume that $K$ is given by the homogeneous polynomial

$$
\kappa(x, y, z, t) = x^4 + y^4 + z^4 + t^4 + 2Dxyzt + A(x^2t^2 + y^2z^2) + B(y^2t^2 + x^2z^2) + C(z^2t^2 + x^2y^2)
$$

in $\mathbb{P}^3_k$ with singular points $P_1 = (d, -c, b, -a)$ and $P_2 = (d, c, b - a)$. In this case the family of planes going through $P_1$ and $P_2$ is given by

$$
H(\lambda) = ax + by + cz + dt + \lambda(ax - by - cz + dt).
$$

Without loss of generality we assume that $b, d \neq 0$. Let $U$ be the affine open subset of $H_{1,2}(\lambda)$ that we get by setting $z = 1$ to get a plane that contains both $P_1$ and $P_2$. Let $\widetilde{C}_{\lambda_0} = U \cap K$. It follows that we can describe $\widetilde{C}_{\lambda_0}$ as a curve in $\mathbb{A}^2_k$ given by the equation $F(x, y) = 0$ where

$$
F(x, y) = \kappa \left( x, y, 1, \frac{(1 + \lambda)ax + (1 - \lambda)(by + c)}{d(-1 - \lambda)} \right)
$$

and define an isomorphism $\varphi : \widetilde{C}_{\lambda_0} \to K \cap U$ by

$$
\varphi(x, y) = \left( x, y, 1, \frac{(1 + \lambda)ax + (1 - \lambda)(by + c)}{d(-1 - \lambda)} \right).
$$

Using this isomorphism we get $\varphi^{-1}(d, -c, b, -a) = (d/b, -c/b)$. Let

$$
g : (U \cap K) \setminus \{(d/b, -c/b)\} \to k
$$

be the function defined by mapping a point $P$ to the slope of the line passing through $(d/b, -c/b)$ and $P$ as in Lemma 3.20. We will find the ramification points of $g$ to calculate the $j$-invariant of the family.

A line in $U$ with slope $\mu$ passing through $(d/b, -c/b)$ satisfies the equation

$$
y = \mu x - c/b - \mu d/b.
$$
Consider the polynomial

\[ F(x, \mu x - c/b - \mu d/b) \]  

in \( k(\lambda, \mu)[x] \).

Let \( D(\mu) \in k(\lambda) \) be the discriminant of \( F/(x - d/b)^2 \) with respect to \( x \). Solving \( D(\mu) = 0 \) gives us the values of \( \mu \) for which the intersection number of \( L \) with \( \overline{C}_{\lambda_0} \) is greater than 2. We divide by \( (x - d/b)^2 \) to exclude the case where \( L \) intersects \( P_1 \).

A calculation shows that the zeroes of \( D(\mu) \) are:

\[ 0, \]

\[ x_1(\lambda) = ((ab\lambda + ab + cd\lambda + cd)/(b^2\lambda - b^2 - d^2\lambda - d^2)), \]

\[ x_2(\lambda) = ((ab\lambda + ab - cd\lambda - cd)/(b^2\lambda - b^2 + d^2\lambda + d^2)), \]

\[ x_3(\lambda) = ((-ac\lambda - ac - bd\lambda - bd)/(ad\lambda + ad - bc\lambda + bc)), \]

\[ x_4(\lambda) = ((-ac\lambda - ac + bd\lambda + bd)/(ad\lambda + ad - bc\lambda + bc)). \]

The 0 coincides with the horizontal line that passes through \( P \).

We can use these coefficients as coordinates on an open affine subset of \( \text{Jac}(C) \).

**Proposition 3.39.**

With notation as in Proposition 3.38, let \( \sigma \) be the linear automorphism of \( K \) interchanging \( P_1 \) and \( P_2 \) (cf., Lemma 3.18). Let \( \lambda, \mu \in K \) such that \( \sigma \) carries \( H(\lambda) \) isomorphically onto \( H(\mu) \), and let \( j_0 \) denote the common value \( j(\lambda) = j(\mu) \). Then \( \{\lambda, \mu\} \) is a solution pair for \( j_0 \).

**Lemma 3.38.** With notation as in Proposition 3.22, we have \( j(H(\lambda)) = j(H(1/\lambda)) \).

**Proof.** One can show that automorphism \( \sigma \) that swaps \( P_1 \) and \( P_2 \) maps the plane \( H(\lambda) \) to \( H(1/\lambda) \). This induces an isomorphism between the curves \( H(\lambda) \cap K \) and \( H(1/\lambda) \cap K \).

### 3.2. Making the construction explicit

Using the work of Mumford [29] and Cantor [6] on Jacobians of hyperelliptic curves, as well as the work of Müller [27] on Kummer surfaces, we will give explicit descriptions of the objects and maps used in Theorem 3.9.

We first recall how a divisor of degree 2 on a genus 2 curve can be represented in Mumford coordinates as a pair of polynomials.

**Proposition 3.39.** Let \( Y \) be a smooth curve of genus 2 over \( k \) given by a Weierstrass equation \( y^2 = f(x) \) in \( \mathbb{P}^2_k \). Then there exists a bijection between the sets

\[ \mathcal{S} := \{(x_1, y_1), (x_2, y_2)\} \in \text{Sym}^2(Y) \mid x_1 \neq x_2 \]

and

\[ \mathcal{P} := \{(a(x), b(x)) \in k[x] \times k[x] \mid \deg(a) = 2, a \text{ is monic, } \deg(b) \leq 1\} \]

**Proof.** See [29, Proposition 1.2] or [6, §2].

Given a pair \( (a(x), b(x)) \in \mathcal{P} \), then \( a(x) = x^2 + a_1 x + a_2, b(x) = b_1 x + b_2 \) for some \( a_1, a_2, b_1, b_2 \in k \). We can use these coefficients as coordinates on an open affine subset of \( \text{Jac}(Y) \) as described in the following proposition.
Proposition 3.40. Let $Y$ be given by the equation $y^2 = f(x)$ in $\mathbb{P}^2_k$. Let $g_1$ and $g_2$ be polynomials in $k[a_1, a_2, b_1, b_2]$ such that
\[
g_1(a_1, a_2, b_1, b_2)x + g_0(a_1, a_2, b_1, b_2) \equiv b(x)^2 - f(x) \mod a(x).
\] (3.41)

Then the system of equations
\[
g_1(a_1, a_2, b_1, b_2) = 0, \quad g_2(a_1, a_2, b_1, b_2) = 0
\] (3.42) (3.43)
describes an affine open subset $U$ of $\text{Jac}(Y)$ in $\mathbb{A}^4_k$.

Proof. See Proposition 1.3 and Chapter IIIa, §2 of [29]. \hfill \Box

Proposition 3.44. Let $Y$ be a curve of genus 2 over a field $k$ given by the equation
\[
y^2 = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + f_5 x^5 + f_6 x^6
\] (3.45)
in $\mathbb{A}^2_k$. Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are two points on $Y$ and let $P + Q \in U \subset \text{Jac}(Y)$ where $U$ is as in Proposition 3.40. Let
\[
k_1 = 1,
k_2 = x_1 + x_2,
k_3 = x_1 x_2,
k_4 = \frac{F_0(x_1, x_2) - 2y_1 y_2}{(x_1 - x_2)^2},
\] (3.46)
where
\[
F_0(x, y) = 2f_0 + f_1(x + y) + 2f_2(xy) + f_3(x + y)xy + 2f_4(xy)^2 + f_5(x + y)(xy)^2 + 2f_6(xy)^3.
\] (3.47)

Then we can define a map $\pi : U \to \text{Kum}(Y)$ given by $(P, Q) \mapsto (k_1 : k_2 : k_3 : k_4)$ such that $\pi$ is equal to the quotient morphism $\text{Jac}(Y) \to \text{Kum}(Y)$ restricted to $U$.

The functions $k_1, k_2, k_3, k_4$ satisfy the quartic equation
\[
K(k_1, k_2, k_3, k_4) = K_2(k_1, k_2, k_3)k_4^2 + K_1(k_1, k_2, k_3)k_4 + K_0(k_1, k_2, k_3) = 0
\] (3.48)
and this equation gives us a projective embedding of $\text{Kum}(Y)$ in $\mathbb{P}^3_k$.

Here
\[
K_2(k_1, k_2, k_3) = k_2^2 - 4k_1 k_3
\] (3.49)
\[
K_1(k_1, k_2, k_3) = -4k_1^3 f_0 - 2k_1^2 k_2 f_1 - 4k_1^2 k_3 f_2 - 2k_1 k_2 k_3 f_3
\] (3.50)
\[
- 4k_1 k_2^2 f_4 - 2k_2 k_3^2 f_5 - 4k_3^3 f_6
\]
\[
K_0(k_1, k_2, k_3) = -4k_1^4 f_0 f_2 + k_1^4 f_2^2 - 4k_1^3 k_2 f_0 f_3 - 2k_1^3 k_3 f_1 f_3
\] (3.51)
\[
- 4k_1^2 k_2^2 f_0 f_4 + 4k_1^2 k_2 k_3 f_0 f_5 - 4k_1 k_2 k_3 f_1 f_4 - 4k_1^2 k_3^2 f_2 f_6
\]
\[
+ 2k_1 k_2^2 f_1 f_5 - 4k_1^2 k_3^2 f_2 f_4 + k_1^2 k_3^2 f_3^2 - 4k_1 k_2^2 f_3 f_5
\]
\[
+ 8k_1 k_2 k_3^2 f_0 f_6 - 4k_1 k_2 k_3 f_1 f_5 + 4k_1 k_2^2 k_3 f_1 f_6
\]
\[
- 4k_1 k_2 k_3^2 f_2 f_5 - 2k_1 k_3^3 f_3 f_5 - 4k_1^4 f_0 f_6 - 4k_1^2 k_3^2 f_1 f_6
\]
\[
- 4k_1 k_2^2 k_3 f_2 f_6 - 4k_2 k_3^2 f_3 f_6 - 4k_3^4 f_4 f_6 + k_3^4 f_2
\]

Proof. See [27, §2]. \hfill \Box

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Corollary 3.52. Let $U$ be an affine open subset of $\text{Jac}(Y)$ in $\mathbb{A}^4 = k[a_1, a_2, b_1, b_2]$ given by the system of equations $g_1 = 0, g_2 = 0$ as in Proposition 3.40. Then the map $U \to \text{Kum}(Y)$ described in Proposition 3.44 can be explicitly described as

$$(a_1, a_2, b_1, b_2) \mapsto \left(1 : -a_1 : a_2 : \frac{F_0(-a_1, a_2) - 2(b_1^2 a_2 - b_1 b_2 a_1 + b_2^2)}{a_1^2 - 4a_2}\right) \quad (3.53)$$

where

$$F_0(x, y) = 2f_0 + f_1x + 2f_2y + f_3xy + 2f_4y^2 + f_5xy^2 + 2f_6y^3. \quad (3.54)$$

Proof. The correspondence described in Proposition 3.39 sends a pair of points $\{P_1, P_2\} \in S$ with $P_i = (x_i, y_i), i = 1, 2$, to the pair of polynomials $(a(x), b(x))$ with

$$a(x) = (x - x_1)(x - x_2) \quad \text{and} \quad b(x) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1. \quad (3.53)$$

The result now follows from equating coefficients and then substituting these expressions into the formula for $F_0$ given in Proposition 3.44. \qed

Remark 3.55. The point $(0 : 0 : 0 : 1)$ is always a singular point on the projective embedding of $\text{Kum}(Y)$ in $\mathbb{P}_k^3$ given by equation (3.48), as can be verified by a straightforward computation of partial derivatives.

Lemma 3.56. Let $\varphi : k(\text{Kum}(Y)) \to k(\text{Jac}(Y))$ be the inclusion of function fields induced by the morphism $\pi : \text{Jac}(Y) \to \text{Kum}(Y)$. Then

(i) There exist $\alpha_i, \beta_j \in k(a_1, a_2)$ such that

$$b_1 b_2 = \alpha_1(a_1, a_2) + \alpha_2(a_1, a_2)b_2^2, \quad (3.57)$$

$$b_2^2 = \beta_1(a_1, a_2) + \beta_2(a_1, a_2)b_1^2. \quad (3.58)$$

(ii) Let

$$h := \frac{(k_2^2 - 4k_3)\kappa_4 - F_0(\kappa_2, \kappa_3) + 2\kappa_2\kappa_1(\kappa_2, \kappa_3) + 2\beta_1(\kappa_2, \kappa_3)}{2\kappa_3 - 2\kappa_2\kappa_1(\kappa_2, \kappa_3) - 2\beta_2(\kappa_2, \kappa_3)}. \quad (3.59)$$

Then $\varphi(h) = b_1^2$.

Proof. Note that the polynomials $g_1$ and $g_2$ of Proposition 3.40 can be computed by dividing $b(x)^2 - f(x) = (b_1 x + b_2)^2 - f(x)$ by $a(x)$ using polynomial long division, hence are contained in the subring $k[a_1, a_2][b_1^2, b_2^2, b_1 b_2]$. Considering the system of equations $g_1 = g_2 = 0$ as linear equations in $b_1^2, b_1 b_2, b_2^2$ over the field $k(a_1, a_2)$, we can solve for $b_1 b_2$ and $b_2^2$ in terms of $b_1^2$.

Recall from Corollary 3.52 that on the affine open of $\text{Jac}(Y)$ with coordinates $a_1, a_2, b_1, b_2$ we have $\varphi(\kappa_2) = -a_1, \varphi(\kappa_3) = a_2$, and

$$\varphi(\kappa_4) = \frac{F_0(-a_1, a_2) - 2(b_1^2 a_2 - b_1 b_2 a_1 + b_2^2)}{a_1^2 - 4a_2}. \quad (3.59)$$

Solving for $b_1 b_2$ and $b_2^2$ in terms of $a_1, a_2$, and $b_1^2$ as in part (i), we can express $\varphi(\kappa_4)$ as a function of $\varphi(\kappa_2), \varphi(\kappa_3)$, and $b_1^2$. A straightforward but laborious computation then shows that

$$b_1^2 = \frac{(\varphi(\kappa_2)^2 - 4\varphi(\kappa_3))\varphi(\kappa_4) - F_0(\varphi(\kappa_2), \varphi(\kappa_3)) + 2\varphi(\kappa_2)\alpha_1(\varphi(\kappa_2), \varphi(\kappa_3)) + 2\beta_1(\varphi(\kappa_2), \varphi(\kappa_3))}{-2\varphi(\kappa_3) - 2\varphi(\kappa_2)\alpha_2(\varphi(\kappa_2), \varphi(\kappa_3)) - 2\beta_2(\varphi(\kappa_2), \varphi(\kappa_3))}. \quad (3.59)$$

Thus for $h$ as defined as in the statement of the lemma, we have $\varphi(h) = b_1^2$. \qed
Corollary 3.60. Let $\pi, \varphi$ and $h$ be as in Lemma 3.56. Then we can extend $\varphi$ to a morphism $\overline{\varphi} : k(\text{Kum}(Y))[\sqrt{h}] \to k(\text{Jac}(Y))$ such that $\overline{\varphi}$ is an isomorphism. Furthermore, let $C$ be a curve on $\text{Kum}(Y)$ and let $k(C)$ be the function field of $C$. Then $k(C)[\sqrt{h}]$ is the function field of $\pi^{-1}(C)$ in $\text{Jac}(Y)$.

Proof. Define $\overline{\varphi}(f + g\sqrt{h}) = \varphi(f) + \varphi(g)b_1$. As $a_1, a_2, b_1, b_2$ are the coordinates of a dense affine open subset of $\text{Jac}(Y)$, it suffices to show that they are in the image of $\overline{\varphi}$. From Corollary 3.52 we have $a_1 = -\overline{\varphi}(\kappa_2)$, $a_2 = \overline{\varphi}(\kappa_3)$ and $\overline{\varphi}(\sqrt{h}) = b_1$, and from equation (3.57) we have

$$b_2 = \frac{a_1(a_1, a_2)}{b_1} + a_2(a_1, a_2)b_1$$

so $b_2$ is also in the image of $\overline{\varphi}$.

The final statement follows from the fact that the inclusion of function fields $k(C) \hookrightarrow k(C)[\sqrt{h}]$ corresponds to the morphism of curves $\pi^{-1}(C) \to C$. □

The following result is a generalization of [11, Theorem 1.1] over a general base field.

Proposition 3.61. Let $C$ be a genus 1 curve over an algebraically closed field $k$ with $\text{char}(k) \neq 2$, and let $P_1, P_2, P_3, P_4$ be distinct points in $C$. Then

(i) There are exactly four distinct covers $X_j \to C$ (where $j = 1, \ldots, 4$) of degree 2 that are ramified above the $P_i$ and unramified everywhere else.

(ii) If $f$ is a function such that $k(C)[\sqrt{T}] \cong k(X)$ then every $X_j$ is isomorphic to a curve with function field $k(C)[\sqrt{T}]$ with $\text{div}(f_T) = \text{div}(f) + 2T$ for some $T \in \text{Pic}(T)$.

(iii) Let $T_1, \ldots, T_4$ be the order-2 Weierstrass points on $C$. There exist non-constant functions $u \in L(T_1 + T_1)$ and $v \in L(2T_1 + T_1)$ with $\text{div}(v) = P_1 + P_2 + P_3 + P_4 - 2T_1 - 2T_1$ such that

(a) The curve $C$ has an equation of the form

$$v^2 + vh(u) + f(u) = 0$$

where $h$ is a polynomial of degree 2 and $f$ is a polynomial of degree 4.

(b) The curve $X_i$ has an equation over $\overline{k}$ of the form

$$t^4 + t^2h(s) + f(s) = 0$$

where $h$ is a polynomial of degree 2 and $f$ is a polynomial of degree 4.

(c) The cover $\pi_i : X_i \to C$ is explicitly given by $\pi_i(s, t) = (s, t^2)$.

Proof. See [17, Paragraph 4.4]. □

3.3. The algorithm. In this section we combine the previous results into an algorithm for computing all non-hyperelliptic gluings of a genus 1 and a genus 2 curve. The main result is the following.

Theorem 3.64. Let $X$ be a curve of genus 1 over $k$ and let $Y$ be a curve of genus 2 over $k$. Then:

(i) Every gluing of $X$ and $Y$ that is a non-hyperelliptic curve over $k$ can be found using Algorithm 3.67.

(ii) Generically there is a bijection between the indecomposable maximal isotropic subgroups of $\text{Jac}(X)[2] \times \text{Jac}(Y)[2]$ and tuples $(P_i, (\lambda, \mu))$ where

$$P_i \in \text{Sing}(\text{Kum}(Y)) \setminus \{1 : 0 : 0 : 0\}$$

and $(\lambda, \mu)$ is a solution pair for $j(X)$.

Proof.
(i) Let \( Z \) be a non-hyperelliptic gluing of \( X \) and \( Y \). Then \( Z \) can be embedded in \( \mathbb{P}^2 \) as a smooth quartic curve. Again by Proposition 3.3, there exists a degree 2 map \( \pi_1 : Z \to X \). By [31, Theorem 1.1], we can find a change of coordinates such that \( X, Y \), and \( Z \) can be written as in the proof of Theorem 3.9. Given a double cover \( p : Z \to X \) as in Proposition 3.3, by Theorem 3.9 there exist maps \( \pi, i_Z \), and \( i_X \) as in (3.10), such that \( \pi(i(Z)) \) is the intersection of a plane with Kum(\( Y \)), and \( p : Z \to X \) is the desingularization of the cover \( \pi_{i(Z)} : i(Z) \to \pi(i(Z)) \). Thus every gluing can be constructed from Algorithm 3.67.

(ii) By Lemma 3.17, we can apply an automorphism so that the embedding \( i_X(X) = H \cap \text{Kum}(Y) \) and the plane \( H \) passes through \( P_1 \). This yields 15 planes, passing through the nodes \( P_1 \) and \( P_i \) for \( i = 2, \ldots, 16 \). For each \( i \), let \( H_i(\lambda) \) be the pencil of planes passing through \( P_1 \) and \( P_i \). We then determine the values \( \lambda \) such that \( j(H_i(\lambda)) = j(X) \). Since \( j(H_i(\lambda)) \) generically has degree 12 by Theorem 3.22, we obtain 12 values for \( \lambda \), resulting in 6 solution pairs. Thus we obtain 90 curves, exactly corresponding to the 90 indecomposable maximal isotropic subgroups described in Corollary 1.22.

\[ \square \]

Remark 3.66. The following algorithms may require extensions of the base field in order to produce explicit equations.

Algorithm 3.67. This algorithm constructs all gluings of \( X \) and \( Y \) that produce non-hyperelliptic curves.

**Input:** Curves \( X : y^2 = p_X \) and \( Y : y^2 = p_Y \) of genera 1 and 2, respectively.

**Output:** A list \( L \) of maps of curves \( p : Z \to X \) such that \( Z \) is a gluing of \( X \) and \( Y \).

**Steps:**

(1) Initialize the empty list \( L \).

(2) Calculate an affine model for Jac(\( Y \)) as in Corollary 3.40.

(3) Compute \( j(X) \).

(4) Calculate a model for Kum(\( Y \)) and the projection map Jac(\( Y \)) \to Kum(\( Y \)) as in Proposition 3.44.

(5) Calculate a function \( h \) with the property that \( k(\text{Kum}(Y))[\sqrt{\eta}] \cong k(\text{Jac}(Y)) \) as in Lemma 3.56.

(6) For each \( P_i \in \text{Sing}(\text{Kum}(Y)) \setminus \{(1 : 0 : 0 : 0)\} \):
   
   (a) Calculate the 1-dimensional family \( H_{1,i}(\lambda) \) of planes that pass through \( P_1 := (1 : 0 : 0 : 0) \) and \( P_i \).

   (b) Calculate the set \( \Lambda(X) \) of all \( \lambda \) such that \( j(H_{1,i}(\lambda) \cap \text{Kum}(Y)) = j(X) \).

   (c) For each \( \lambda_0 \in \Lambda(X) \):

      (i) Determine the singular genus 1 curve \( \tilde{X}(\lambda_0) = H_{1,i}(\lambda_0) \cap \text{Kum}(Y) \)

      (ii) Calculate the curve \( Z \) with function field \( k(\tilde{X}(\lambda_0))[\sqrt{\eta}] \) using Algorithm 3.68; this gives us the desired gluing and a natural projection map to \( X \).

      (iii) Add the gluing to \( L \).

(7) Return the list \( L \) of all gluings \( Z \) found above.

Algorithm 3.68. This algorithm calculates an equation for the curve \( Z \) with function field \( k(\tilde{X}(\lambda_0))[\sqrt{\eta}] \).

**Input:**

- A singular genus-1 curve \( \tilde{X}(\lambda_0) \) with exactly two nodes that is given by the intersection of a plane and a quartic surface

- A function \( h \in k(\tilde{X}(\lambda_0)) \) whose divisor is of the form \( P_1 + P_2 + P_3 + P_4 - 2T \) for some divisor \( T \).

**Output:** A degree 2 cover \( p : Z \to X(\lambda_0) \) such that \( k(Z) \cong k(\tilde{X}(\lambda_0))[\sqrt{\eta}] \) where \( X(\lambda_0) \) is the normalization of \( \tilde{X}(\lambda_0) \).
Steps:

1. Use Lemma 3.20 to compute the branch points $\alpha_1, \ldots, \alpha_4$ of the map $g : \tilde{X}(\lambda_0) \to \mathbb{P}^1$ that maps a point $P$ to the slope of the line that connects $P$ with the singular point $(1 : 0 : 0 : 0)$.

2. Define the curve $C$ by the equation $y^2 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4)$ and compute a birational map $\tau : C \to \tilde{X}(\lambda_0)$.

3. Let $Q, R$ be the singular points on $\tilde{X}(\lambda_0)$ and calculate $\tau^{-1}(Q) = \{Q_1, Q_2\}$ and $\tau^{-1}(R) = \{R_1, R_2\}$.

4. Calculate the divisor $D$ of the image of $h$ in $k(C)$.

5. Find divisors $D_1, D_2, D_3, D_4$ that correspond to the four distinct degree 2 covers with ramification points $Q_1, Q_2, R_1, R_2$ as in Proposition 3.61.

6. For each $i \in \{1, \ldots, 4\}$:

   a) If there exists a principal divisor $T$ such that $D_i - D = 2T$

   i) Calculate the degree 2 cover $Z \to X$ corresponding to $D_i$ using Riemann-Roch spaces which is possible according to Proposition 3.61(iii).

   ii) Return a quartic equation of $Z$ with a degree 2 cover to $X$.

Example 3.69. Let $X$ be the curve given by

$$y^2 = x^4 + 2x^3 - x^2 - 2x$$

and let $Y$ be the curve given by

$$y^2 = x^6 - 2x^5 - 10x^4 + 20x^3 + 9x^2 - 18x$$

over $\mathbb{Q}$. An affine open of $\text{Jac}(Y)$ is given by the following system of equations in $\mathbb{Q}[a_1, a_2, a_3, a_4]$.

$$
\begin{align*}
-a_1^2a_2 - 2a_1^3a_2 + 3a_1^2a_2^2 + 10a_1^2a_2 + 4a_1a_2^2 + 20a_1a_2 - a_2 - 10a_1^2 + 2a_1b_1 - 9a_2 - b_2^2 &= 0, \\
-a_1^2 - 2a_1^3a_2 + 10a_1^2a_2 + 6a_1a_2^2 + 20a_1^2 - 3a_1a_2^2 - 20a_1a_2 + a_1b_1^2 - 9a_1 - 2a_2 - 20a_2 - 2b_1b_2 - 18 &= 0.
\end{align*}
$$

(3.72)

The equation for $\text{Kum}(Y)$ in $\mathbb{P}^3_{\mathbb{Q}}$ is

$$
32a_1^4 + 720a_1^3x_3 - 720a_1^2x_2x_3 - 144a_1x_2^2x_3 + 72a_2^2x_3^2 + 832a_1^3x_2^2 - 36a_1^2x_2^3 - 80x_1x_2x_3^2 + 80x_2x_3^3 + 44x_4^3 + 36a_1^2x_4x_3 - 36a_1^3x_2x_4 - 40x_1x_2x_3x_4 + 40x_1x_2^2x_4 + 4x_2^2x_4^2 - 4x_3x_4^2 - 4x_4^2 = 0
$$

(3.73)

and the morphism $\pi : \text{Jac}(Y) \to \text{Kum}(Y)$ is explicitly given by

$$
\pi(a_1, a_2, b_1, b_2) = \left[1: -a_1 : a_2 : \frac{2a_1a_2^2 - 20a_1a_2 + 2a_1b_1b_2 + 18a_1 + 2a_2^2 - 20a_2^2 - 20a_2b_1^2 + 18a_2 - 2b_2^2}{a_1^2 - 4a_2}\right].
$$

(3.74)

We consider the family of planes $H_{1,2}(\lambda)$ passing through the singular points $P = (0 : 0 : 0 : 1)$ and $Q = (-1/6 : 1/3 : 1/2 : 1)$. The $j$-invariant of $X$ is $35152/9$, and we seek to find the values of $\lambda$ such that $H_{1,2}(\lambda) \cap \text{Kum}(Y)$ has the same $j$-invariant. We calculate the $j$-invariant $j(\lambda)$ of $H_{1,2}(\lambda) \cap \text{Kum}(Y)$ and find that the numerator of $j(\lambda) - 35152/9$ factors as

$$
\left(\lambda - \frac{9}{25}\right) \left(\lambda - \frac{1}{11}\right) \left(\lambda^2 - \frac{38}{67}\lambda - \frac{9}{67}\right) \left(\lambda^2 - \frac{98}{195}\lambda - \frac{3}{195}\right) \left(\lambda^2 - \frac{42}{85}\lambda + \frac{1}{85}\right) \left(\lambda^2 - \frac{22}{47}\lambda + \frac{3}{47}\right) \left(\lambda^2 - \frac{2}{5}\lambda + \frac{1}{5}\right).
$$

(3.75)

We will construct the degree 2 cover above $\tilde{X}(9/23) = H_{P,Q}(9/23) \cap \text{Kum}(Y)$. (The other covers can be computed in the same way by choosing other roots of $((3.75))$.) To compute $k(\tilde{X}_1(-3/2))/(t^2 - h)$ we proceed as in Algorithm 3.68 and calculate the branch points of the degree 2 map $g : \tilde{X}(9/23) \to \mathbb{Q}$. We compute the Legendre form Weierstrass model $y^2 = x(x - 1)(x - 1/4)$ for $\tilde{X}(9/23)$, which we denote $\tilde{X}_{\text{leg}}$. Although $\tilde{X}_{\text{leg}}$ is a nontrivial twist of $\tilde{X}(9/23)$ over $\mathbb{Q}$, the curves become isomorphic upon extending scalars to $\mathbb{Q}(\alpha)$, where $\alpha$ is a root of

$$
t^2 - 156026658225043557710221401/34308279913908709968852208000.
$$
We explicitly compute the image of the function $h$ in the function field of $k(X_{\text{leg}}) \otimes \mathbb{Q}(\alpha)$ and obtain a rational function of degree 14 with rather large coefficients, which we denote $h_{\text{leg}}$.

Let $P_1, P_2$ (resp., $Q_1, Q_2$) be the two points obtained by desingularizing $P$ (resp., $Q$). We find that the divisor $P_1 + P_2 + Q_1 + Q_2$ is defined over the field $\mathbb{Q}(\beta, \gamma)$ where $\beta$ is a root of $t^2 + 3/2$ and $\gamma$ is a root of $t^2 - 327/250t + 4761/10000$. Let $T_1, \ldots, T_4$ be the 2-torsion points of $X_{\text{leg}}$. For each $i = 1, \ldots, 4$, we calculate functions $f_i$ with $\text{div}(f_i) = P_1 + P_2 + Q_1 + Q_2 - 2T_i - 2T_i$ for $i = 1, \ldots, 4$. We determine which $f_i$ corresponds to our covering $Z \rightarrow X$ by checking if $\text{div}(f_i/h_{\text{leg}})$ is divisible by 2 for each $i$. Applying Riemann-Roch as in Proposition 3.61 we compute the equation

$$u^4 - \frac{244312307247680}{12491063134299} \alpha u^3 + \left(\frac{286839015625}{36438849216} \beta \gamma - \frac{250115773625}{48585132288} \beta^{25} \right) u^2 \alpha^2 + \frac{5876}{8855} u^2$$

$$+ \left(\frac{-50500786167745625000}{1335797989660883737} \alpha \beta \gamma + \frac{1100917138568562500}{440193266220294579} \alpha \beta \gamma \right) u^2$$

$$- \frac{1045490681265625}{171408346712064} \alpha u^3 + \left(\frac{32518171875}{63767968128} \beta \gamma - \frac{45795845875}{85023981504} \beta \gamma \right) u^2 \frac{1460}{111573} = 0.$$

for the gluing over $\mathbb{Q}(\alpha, \beta, \gamma)$. A simplified equation of this curve over $\mathbb{Q}$ is

$$12x^4 - 111x^2y^2 + 478x^2yz - 577xz^2z^2 - 533y^4 + 948y^3z - 2574y^2z^2 + 2196yz^3 - 2277z^4 = 0.$$

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