Stellar scintillation on large and extremely large telescopes

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Accepted 2012 July 4. Received 2012 July 3; in original form 2012 June 13

ABSTRACT

The accuracy of ground-based astronomical photometry is limited by two factors: photon statistics and stellar scintillation arising when starlight passes through the Earth’s atmosphere. This paper examines the theoretical role of the outer scale $L_O$ of the optical turbulence (OT), which suppresses the low-frequency component of the scintillation. It is shown that for typical values of $L_O \sim 25–50$ m, this effect becomes noticeable for telescopes of diameter about 4 m. On extremely large, 30–40 m, telescopes with exposures longer than a few seconds, the inclusion of the outer scale in the calculation reduces the scintillation power by more than a factor of 10 relative to conventional estimates. The details of this phenomenon are discussed for various models of non-Kolmogorov turbulence. In addition, a quantitative description of the influence of the telescope central obscuration on the measured scintillation noise is introduced and combined with the effect of the outer scale. Evaluation of the scintillation noise on future TMT and E-ELT telescopes predicts an amplitude of approximately 10 $\mu$mag for a 60-s exposure.

Key words: turbulence – atmospheric effects – techniques: photometric.

1 INTRODUCTION

Stellar scintillation is the random fluctuation of the radiation flux entering the aperture of a telescope caused by amplitude distortions of light waves passing through the turbulent terrestrial atmosphere (Tatarskii 1967; Roddier 1981). The photometric error arising from scintillation, the scintillation noise, has often been studied (see e.g. Young 1969; Dravins et al. 1998), as it often determines the fundamental limit of the accuracy of ground-based photometry.

The basic dependences required to calculate scintillation noise have been known by the astronomical community for some time (Heasley et al. 1996; Gilliland et al. 1993). However, these relationships were obtained for ideal cases and cannot always be used for the accurate prediction of the scintillation noise. This became especially noticeable when accurate measurements of the intensity of the optical turbulence (OT) in the atmosphere at altitudes responsible for the occurrence of scintillation on large telescopes began to be possible (Kenyon et al. 2006).

This paper comprises a theoretical study of two factors affecting the power of stellar scintillation on large and extremely large telescopes: the influence of the OT outer scale and the telescope central obscuration (CO).

Section 2 recalls the basic description of the phenomenon and its relationship with several parameters of the OT in the atmosphere. In Section 3, we assess the influence of the outer scale in the case of various simplified non-Kolmogorov models. The effect of the CO is considered in Section 4, first for the Kolmogorov model, and then for the general case. In the final section, the conclusions are formulated and a prediction of the scintillation noise for future extremely large telescopes is given.

2 THEORY

2.1 Scintillation noise in photometry

Measurements of the brightness of astronomical objects are always burdened by several types of noise. Depending on the origin, these noises are included in the measured signal in different ways. Noise caused by the stellar scintillation enters in multiplicative way; that is, for it, the signal-to-noise ratio is independent of brightness. Stellar scintillation is usually characterized by the index $s^2$, representing the variance of the relative fluctuations of the intensity $I$ of the radiation passing through the receiving aperture:

$$s^2 = \langle (I - \langle I \rangle)^2 \rangle / \langle I \rangle^2. \quad (1)$$

Although in this definition an averaging over all possible states (over the ensemble) is performed, in practice, using the ergodic property for turbulent phenomena (Tatarskii 1967), it is replaced by averaging over time.

For astronomical applications, the telescope entrance pupil $D$ is considered as a receiving aperture. The scintillation noise can be expressed in magnitudes $\sigma_{\text{sc}}^2$. It then becomes the additive noise, associated with the scintillation index: $\sigma_{\text{sc}}^2 = 1.179 s^2$.

Under nighttime conditions, suitable for astronomical observations, the OT is usually such that the phase and amplitude
distortions at the pupil plane are well described using the approximation of weak perturbations (Tatarskii 1967). It is believed that for large apertures (geometric optics regime), this is even more the case. In this approximation, the turbulence of each layer is independent of the previous layers, and the scintillation index observed on the surface is linearly related to the distribution of the structural coefficient $C_n^2(z)$ on the line of sight throughout the whole atmosphere (see, e.g. Roddier 1981) by the following integral:

$$s^2 = \int_D A_n^2(z) Q(z) \, dz,$$

where $z$ is the distance to the turbulent layer, which in the case of measurements at zenith coincides with the altitude $h$ above the observatory. The weighting function $Q(z)$ relates the output from the layer to its corresponding effect at the surface.

### 2.2 Basic relations

The calculation of the weighting function $Q(z)$ is described in detail in Tokovinin (2002, 2003). It involves the integration of the 2D spatial spectrum of the amplitude perturbation over all frequencies. In the case of axial symmetry of integrand functions, it is easier to perform the integration in polar coordinates, in which, after averaging over the polar angle, the functions depend only on the modulus $f$ of the vector of spatial frequencies. For an isotropic and locally homogeneous OT with a spatial spectrum of the wavefront phase fluctuations $\Phi(f)$, normalized to the value of $C_n^2(z)$,

$$Q(z) = 9.61 \int_0^\infty \Phi(f) S(z, f) A(f) \, df,$$  \hspace{1cm} (3)

where the Fresnel filter $S(z, f)$ describes the evolution of the amplitude perturbations in the propagation of the light wave. In the case of monochromatic radiation with wavelength $\lambda$, the filter $S(z, f) = \sin^2(\pi f z / \lambda^2)$, and its intrinsic spatial scale is defined by the Fresnel radius $r_F = (\lambda z)^{1/2}$.

The aperture filter $A(f)$ takes into account the averaging of the wave amplitude by a receiver. For a circular entrance aperture $D$ it is the Airy function $[2J_1(\pi f D)/\pi f D]^2$ if there are no other factors for averaging. In general, however, stellar scintillations are registered as temporary fluctuations in light intensity averaged over a measurement (exposure). The dependence of the scintillation on the exposure has been investigated repeatedly (e.g. Young 1967; Dravins et al. 1998) with the help of time-averaging of the signal.

On the basis of the ‘frozen turbulence’ hypothesis (Taylor 1938), the temporal averaging can be replaced with a spatial filtering of the scintillation spectrum (see e.g. Martin 1987; Tokovinin 2002; Kornilov 2011), extending the concept of aperture filtering. The scintillation power in the case of a finite exposure is still described by the expression (3) if instead of $A(f)$ we substitute the product $A(f)A_n(f)$, where $A_n(f)$ is the wind shear filter (Tokovinin 2002; Kornilov 2011). The real ‘freeze’ of the OT is not required, as the invariability of the spatial spectrum is enough.

In cases where measurements are performed with an exposure $\tau$ so short that the wind $w$ in the atmosphere does not shift the OT by a significant distance, namely $\tau w \ll D$, the additional averaging can be neglected (the filter $A_n(f)$ is much wider than the filter $A(f)$ and multiplication does not change the integrand). Hereafter, this situation is called the short- (zero-) exposure (SE) regime.

In real astronomical observations, the opposite situation is much more common: during the exposure, the wind shifts the OT by distances exceeding the aperture of the telescope, namely $\tau w \gg D$, and temporal averaging becomes the dominant effect (the filter $A_n(f)$ is narrower than $A(f)$ and it defines the integrand). This case is denoted below as the long-exposure (LE) regime.

### 2.3 Large-aperture approximation

Expression (3) is valid for arbitrary parameters, but in a general form it can be integrated only numerically. To study the dependence of $Q(z)$ on distance $z$ and other parameters, it is necessary to perform some simplifications.

To describe the scintillation on a typical telescope, the approximation of large aperture $D \gg r_F$ can be used. This approximation is well satisfied for telescopes with diameters of 1 m or more, as for any reasonable atmospheric condition $r_F \lesssim 0.1$ m in the optical and near-IR spectral range.

In this case, one can use a well-known feature of the spectral filters included in the integral (3): the filters $A(f)$ and $S(f)$ overlap a little. After replacing $\sin(\pi f z / \lambda^2)$ by $(\pi f z / \lambda^2)$, the integrand is still largely unchanged (Roddier 1981). As a consequence, the dependence on $\lambda$ disappears not only for the monochromatic case, but also in the case of a wide spectral band of the detector, so the more accurate description of this filter from Tokovinin (2003) is not necessary.

Moving to the dimensionless frequency $q = fD$, we can write, for a circular aperture and the SE regime,

$$Q_s(z) = 38.44 D^{-7/3} \int_0^\infty \Phi(q) q^3 (J_1(\pi q))^2 \, dq.$$  \hspace{1cm} (4)

In the LE regime, the approximation technique is particularly suitable because the wind smoothing additionally suppresses the high frequencies. We use the asymptotic behaviour for the corresponding filter of the wind shear $A_n = D/r_F w q$ from Kornilov (2011). Multiplying the integrand by it, we obtain

$$Q_L(z) = 12.24 D^{-4/3} \int_0^\infty \Phi(q) q^2 (J_1(\pi q))^2 \, dq.$$  \hspace{1cm} (5)

It should be noted again that the conventional dependences on telescope diameter $D$ and propagation distance $z$ are the result of the Fresnel filter approximation under the condition $D \gg r_F$ and are not associated with a particular form of the OT spectrum.

For the Kolmogorov model of a normalized spatial spectrum of phase perturbations $\Phi(q) = q^{-11/3}$, the integrals $I_0$ and $I_1$ on the right-hand sides of these expressions are easily calculated and are equal to 0.4508 and 0.8699, respectively (see equations A1 and A2). As a result, the previous expressions reduce to the known dependences $Q_s(z) = 17.34 D^{-7/3} z^2$ for the SE regime and $Q_L(z) = 10.66 D^{-4/3} z^2 (\tau w)^{-1}$ for the LE regime. After integration over the whole atmosphere, they lead to the following formulae of observed scintillation noise (Young 1967; Gilliland et al. 1993; Kenyon et al. 2006):

$$s_s^2 = 17.34 D^{-7/3} \int_A C_n^2(z) z^2 \, dz,$$  \hspace{1cm} (6)

and

$$s_L^2 = 10.66 D^{-4/3} \tau^{-1} \int_A C_n^2(z) w(z)^2 \, dz.$$  \hspace{1cm} (7)

The integrals in these formulae (the atmospheric moments) are determined by the particular state of the atmosphere and are the figures of merit when monitoring and/or forecasting conditions for photometry (Kenyon et al. 2006; Kornilov 2011). The aim of this paper is to consider the effects that modify the coefficients before the atmospheric moments.
3 THE IMPACT OF THE OUTER SCALE OF TURBULENCE

The immediate perception of scintillation from observations with the naked eye, together with numerous experiments performed with small telescopes have created the enduring impression that scintillation is a sufficiently high-frequency process. The problem of the contribution of high and low frequencies in the scintillation has been discussed for a long time (see e.g. Young 1967). For example, in one of the first papers devoted to stellar scintillations (Reiger 1963), the author claimed that the turbulence outer scale has no effect on the scintillation intensity, in contrast to the inner scale. The fact that the situation changes radically when the measurements are made on large telescopes usually goes unnoticed.

Nevertheless, we decided to re-evaluate their relative contribution. Assuming a Kolmogorov spectrum, the fraction of the scintillation power in the frequency interval between 0 and some dimensionless frequency \( q \) was calculated. The calculation shows that most of the scintillation power (\( \sim 70 \) per cent) passes through the first (main) peak of the filter \( A(q) \). In the LE regime, the fraction of the power for this peak is even greater (\( \sim 95 \) per cent). The main peak of the aperture filter is located at \( q < 1.22 \), which for large telescopes corresponds to the metre scale. It is logical to expect that the distinction of the real OT from the usual Kolmogorov model at such scales may significantly affect the measured power.

3.1 Non-Kolmogorov models with outer scale

Skipping the discussion of the physical meaning of the outer scale of turbulence \( L_o \) (Tatarskii 1967), which plays an important role in the generation of turbulence per se, we will consider it simply as an additional parameter in the mathematical description of the spectrum of perturbations in the refractive index. The main purpose of this parameter is to limit the infinite spectral density at frequency \( f = 0 \) inherent in the Kolmogorov spectrum, or to overcome the divergence in the relationships derived from this spectrum.

In astronomical applications, the outer scale of the OT was studied for a long time in connection with long-baseline interferometry (Davis et al. 1995; Avila et al. 1997; Maire et al. 2006). The development of adaptive optics on large telescopes also requires the correct description of the spectrum of phase perturbations in the low-frequency region (Conan et al. 2003; Tokovinin, Sarazin & Smette 2007; Martinez et al. 2010).

The most generally accepted OT model that includes the outer scale is the Von Karman (VK) model (von Karman 1948; Tatarskii 1967). In this model, the normalized spectrum of phase perturbation can be written as

\[
\Phi(f, L_o) = \left(f^2 + L_o^2\right)^{-11/6}. \tag{8}
\]

The main feature of the VK model is a saturation of the power density in the range \( f \lesssim L_o^{-1/3} \) at the level of \( L_o^{11/3} \).

An alternative model without saturation but with a change in the exponent of the power law was proposed by Greenwood and Tarazano (GT) (Greenwood & Tarazano 1974):

\[
\Phi(f, L_o) = (f^2 + f L_o^{-1})^{-11/6}. \tag{9}
\]

It is easy to see that at low frequencies the spectrum has an asymptote \( \sim L_o^{-11/6} f^{-11/6} \). The intermediate range is very wide, so in this model the divergence from the Kolmogorov power spectrum becomes apparent at relatively high frequencies.

Figure 1. Spectral power density of the phase perturbations as a function of the dimensionless frequency \( q \) for various models of non-Kolmogorov turbulence and various values of the dimensionless outer scale \( v \). Solid lines depict the VK model; short-dashed, the GT model; long-dashed, the EM model; dash–dotted, the ME model. See abbreviations in the text. (Online version in colour.)

The exponential model (EM) behaves similarly; it was considered in Lukin & Pokasov (1981) as follows:

\[
\Phi(f, L_o) = f^{-11/3} \left[1 - \exp \left(-f^2 L_o^2\right)\right]. \tag{10}
\]

This spectrum is asymptotic to \( \sim L_o^{-11/3} f^{-5/3} \) at low frequencies; that is, it has a slightly smaller slope than the GT models. The intermediate range of the spectrum is quite narrow.

We modified the EM model so that the spectral density at the origin is finite and coincides with \( L_o^{11/3} \), as predicted by the VK model. This modified exponential model (ME) is described by the dependence

\[
\Phi(f, L_o) = f^{-11/3} \left[1 - \exp \left(-f^{11/6} L_o^{11/6}\right)\right]^2. \tag{11}
\]

The only difference between this spectrum and the VK model is a faster transition from one asymptotic branch to another.

The behaviours of the spectral densities for these four models are shown in Fig. 1 as a function of dimensionless frequency \( q = fD \) for various dimensionless outer-scale frequencies \( v = D/L_o \). The vertical lines indicate the frequency domain where the main peak of the aperture filter is located. Note that for any model, the spectral density can be described as a product of the Kolmogorov spectrum with some spectral filter suppressing low frequencies.

The question of which model best describes the OT in the atmosphere is still open, despite many experimental attempts to solve it (e.g. Maire et al. 2008; Wheelon, Short & Townes 2007). For this reason, in further calculations we use all of the above models with a variety of behaviours.

3.2 The effect of the outer scale on scintillation power

The impact of the outer scale can be calculated by substituting the appropriate model in formulae (4) and (5) expressed as a function of dimensionless frequency \( q = fD \) and outer scale \( v = D/L_o \). It is obvious that after such replacement, as in the case of the Kolmogorov spectrum, the dependence on the parameters \( z \) and \( D \) can be taken out of the integral completely. In the case of the SE

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regime we obtain
\[ \hat{Q}_s(z, \nu) = 38.44 D^{-7/3} z^2 \int_0^\infty \Phi(q, \nu) q^3 (J_1(\pi q))^2 dq. \]  
(12)

A similar expression can be written for \( \hat{Q}_l(z, \nu) \) in the case of the LE regime. The value of the integral in the formula depends explicitly only on the parameter \( \nu \). However, if the outer scale \( L_o \) varies with altitude, then there is an implicit dependence of the integral on the altitude and the diameter of the telescope via the parameter \( \nu \).

The approximation for a large aperture (4, 5) can be used if not only \( D > r_L \) but also \( L_o \gg r_L \). Otherwise, the integrals diverge and a formula with the exact expression for the Fresnel filter should be used. In that case the integrand maximum is located near \( \sim 1/r_L \) and the situation becomes similar to the scintillation in a small aperture.

Multiplying and dividing the right-hand side of (12) by the value of the integral \( I_s \) we find that \( \hat{Q}(z, \nu) = Q(z)G(\nu) \), where the function \( G(\nu) \) describes the impact of the outer scale, and in the SE regime is equal to
\[ G_\nu(\nu) = I_s^{-1} \int_0^\infty \Phi(q, \nu) q^3 (J_1(\pi q))^2 dq. \]  
(13)

The smaller this term is, the greater the effect. The factorization of the weighting functions leads to the fact that the scintillation index \( \tilde{s}^2 \) for non-Kolmogorov OT is related to the index \( s^2 \), defined by formulae (6) or (7), by the simple relationship \( \tilde{s}^2 = s^2 \tilde{G}(\nu) \).

In principle, for exponential models the integral (13) can be calculated analytically, but the resulting expression is so cumbersome that it is meaningless. For VK and GT models we failed to obtain an analytic solution, and the use of expansions of the type in Maire et al. (2007) leads to the appearance of improper integrals.

We therefore investigated the overall behaviour of \( G(\nu) \) using a piecewise power-law dependence with a break point at frequency \( \nu \). The necessary integrals are calculated analytically within the two segments \( \{0, \nu \} \) and \( \{\nu, \infty \} \). We refer to the integral with the spectrum of \( q^{-11/3} \) in the range \( \{\nu, \infty \} \) as \( G^- (\nu) \) after its normalization with \( I \). The integral with the spectrum \( \nu^{-11/3} \) (the model with saturation) or \( \nu^{-2} q^{-5/3} \) (the model without saturation) in the range \( \{0, \nu \} \) is referred to as \( G^+ (\nu) \). Those constants provide continuity to the spectrum at \( q = \nu \). We specify the SE or LE regime, using the appropriate subscript. The integration results of these functions are given in Appendix A2.

The functions \( G^- (\nu) \) and \( G^+ (\nu) \) are shown in Fig. 2 for both regimes. It is evident that the decrease in scintillation power caused by the outer scale arises from a sharp fall in the function \( G^+ (\nu) \), which is not compensated by an increase in \( G^- (\nu) \). The figure also shows their sum, namely the dependence of \( G(\nu) \). At the point \( \nu \approx 0.5 \) the power falls by \( \approx 20 \) per cent. As expected, the effect in the case of the saturated spectrum is slightly greater than for the non-saturated one.

The asymptotes of the \( G_\nu(\nu) \) for small \( \nu \) follow from the evaluated integrals in Appendix A2 and properly describe the functions while \( \nu \lesssim 0.4 \):
\[ G_\nu(\nu) = 1 - 1.433 \nu^{7/3} + 1.428 \nu^{13/3} + \cdots, \quad \Phi = \nu^{-11/3} \]
\[ G_\nu(\nu) = 1 - 1.083 \nu^{7/3} + 0.984 \nu^{13/3} + \cdots, \quad \Phi = \nu^{-2} q^{-5/3}. \]  
(14)

For large \( \nu \), these functions approach the asymptotes \( 0.412 \nu^{-2/3} \) and \( 0.508 \nu^{-2/3} \), respectively.

In the LE regime, the effect of the outer scale is much stronger: at \( \nu \approx 0.5 \) the power is reduced by almost half and at \( \nu \approx 1 \) by about 5 times. The approximations of the functions \( G_\nu(\nu) \) look like:
\[ G_L(\nu) = 1 - 1.560 \nu^{4/3} + 1.099 \nu^{10/3} + \cdots, \quad \Phi = \nu^{-11/3} \]
\[ G_L(\nu) = 1 - 1.276 \nu^{4/3} + 0.787 \nu^{10/3} + \cdots, \quad \Phi = \nu^{-2} q^{-5/3}. \]  
(15)

The difference between the behaviours of the functions evaluated for the spectra with and without saturation is not fundamental and can hardly be detected in actual measurements. As will be seen from calculations using the models described in Section 3.1, a much greater effect occurs because of the length of the intermediate region in these models.

Figs 3 and 4 show the functions \( G(\nu) \) as calculated by numerical integration for various non-Kolmogorov models. It is seen that the

![Figure 2](https://academic.oup.com/mnras/article-abstract/426/1/647/1010313/fig2)

**Figure 2.** Dependences of the integrals \( G^- (\nu) \) (thin solid and dashed lines) and \( G^+ (\nu) \) (dot-dashed line) and the sum \( G(\nu) \) (thick lines) on the dimensionless frequency \( \nu \) for the piecewise model. Solid lines depict the saturated model; the dashed, the non-saturated model. Left, SE regime; right, LE regime.

![Figure 3](https://academic.oup.com/mnras/article-abstract/426/1/647/1010313/fig3)

**Figure 3.** Dependence of \( G(\nu) \) on the dimensionless outer frequency \( \nu \) for various models of non-Kolmogorov turbulence for the SE regime. Solid line depicts the VK model; short-dashed, the GT model; long-dashed, the EM model; dash–dotted, the ME model.
of the filter, which should replace the \((J_1(\pi \epsilon q))^2\) in the integrands:
\[ a(q, \epsilon) = \Upsilon (J_1(\pi \epsilon q) - \epsilon J_1(\pi \epsilon q))^2, \]
where the accessory parameter \(\Upsilon = (1 - \epsilon^2)^{-2}\). While there is virtually no difference within the main spectral peak from the case of the aperture filter of a circular aperture, the filter has a significantly greater transmission within the second and subsequent peaks (e.g. for \(\epsilon = 0.3\), by about 2.5 times). As a result, the scintillation spectrum at high frequencies rises and the total scintillation power increases as well.

Substituting expression (16) in (4), we obtain the weighting function \(Q_3(\epsilon, \epsilon)\), which can be represented in the form \(Q_3(\epsilon, \epsilon) = Q_2(\epsilon)C_3(\epsilon)\), where the integral \(C_3(\epsilon)\) is expressed in terms of hypergeometric functions as follows:
\[
C_3(\epsilon) = \Upsilon \left(1 - \frac{\sqrt{3} \pi \Gamma \left(\frac{5}{6}\right)}{\sqrt{3} \pi \Gamma \left(\frac{5}{6}\right)} \right) \left(\frac{1}{\epsilon} + \frac{2}{3} \epsilon^2 + e^{5/3}\right).
\]

There is a simple and good approximation for this function. Four terms are enough to ensure an accuracy better than 1 per cent for \(\epsilon < 0.6\):
\[
C_3(\epsilon) \approx \Upsilon (1 + e^{5/3} - 1.5682 e^2 - 0.1525 e^4).
\]

The function \(C_3(\epsilon)\) is shown in Fig. 5. As can be seen, in the SE regime, the CO effect leads to a twofold increase in the power of the telescope with \(\epsilon \approx 0.6\). With a widespread CO of 0.3–0.4, the effect is much smaller, although still 20–40 per cent.

In the LE regime, the CO effect is calculated similarly using formula (5). The dependence of the scintillation on the parameter \(\epsilon\) is given by the formula
\[
C_L(\epsilon) = \Upsilon \left(1 - \frac{8}{27} \pi^{3/2} \sqrt{3} \Gamma \left(\frac{3}{2}\right) F_1 \left(\frac{1}{2}, \frac{3}{2}; \epsilon^2 + e^{5/3}\right) \right).
\]

The simple approximation provides accuracy better than 0.005 in the range \(0 < \epsilon < 0.6\) and can be written as
\[
C_L(\epsilon) \approx \Upsilon (1 - 2.355 \epsilon^2 + e^{5/3} + 0.2617 e^4).
\]

The curve in Fig. 5 shows that in this regime the CO does not introduce any significant effect for most telescopes. Even with

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**Figure 4.** Dependence of \(G(\nu)\) on the dimensionless outer frequency \(\nu\) for various models of non-Kolmogorov turbulence for the LE regime. See designations in Fig. 3.

**Figure 5.** Relative change in the scintillation power \(z^2\) for the SE regime (thick solid line, left scale) and for the LE regime (dashed line, right scale) on the parameter \(\epsilon\). The corresponding thin lines depict the approximations by formulae (18) and (20).
$\epsilon = 0.6$ the power increases only by 8 per cent. The smallness of the effect is explained by the fact that for the Kolmogorov spectrum, the power decrease in the main peak is almost balanced by the increase in the second peak of the spectrum.

Because the weighting function $Q_s(z, \epsilon) = Q(z)C_0(\epsilon)$, all asymptotic dependences on the altitude and diameter are retained. Accordingly, the effect on the measured scintillation index $\nu^2$ is also described by the function $C(\epsilon)$. It is worth noting that if the inner diameter of the aperture $\epsilon D$ becomes comparable with $r_p$ (the telescope diameter is less than 0.5 m), then the functions presented above are approximate.

4.2 The combined effect of the outer scale and the central obscuration

The formulae obtained in the previous section are valid if the outer-scale effect is negligible, that is, if the dimensionless parameter $\nu \ll 1$. Otherwise, the fraction of the high-frequency scintillation spectrum grows and the CO effect increases as well.

Obviously, after replacing the aperture filter in the expressions (12) for $Q(z)\epsilon$ or $Q(z), \nu$ on aperture filter (16), the integral $G(\nu)$ becomes dependent on the parameter $\epsilon$. The separation of it into two multiplicands, each depending on its own parameters, cannot be performed. So, the combined impact of these effects can be written either as $\hat{Q}(\nu, \epsilon, \nu) = Q(z, \epsilon)C(\epsilon)$ or as $\hat{Q}(\nu, \epsilon, \nu) = Q(z)G(\nu)C(\epsilon, \nu)$.

We chose the latter form, because for extremely large telescopes the effect of the outer scale is dominant. Accordingly, the function $C(\epsilon, \nu)$ depends on the adopted model and not only on the parameter $\nu$. The normalization used in these calculations means that $C(\epsilon, \nu) = C(\epsilon)$ when $\nu = 0$.

The curves $C(\epsilon, \nu)$, calculated for the VK and GT models, are shown in Fig. 6. For both models the CO effect increases monotonically with $\nu$. It is seen that for the VK model, which experiences saturation at low spatial frequencies, the effect is more significant and at $\nu \approx 1$ almost reaches its maximum. This indicates that the scintillation within the main peak of the aperture filter is suppressed almost completely.

In the LE regime, the behaviour of $C(\epsilon, \nu)$ is also very interesting. For small $\nu$ the CO effect is negligible on real telescopes, but at $\nu \sim 0.5$ it becomes comparable to the effect in the SE regime and is growing further. Nevertheless, this does not compensate for the scintillation reduction caused by the outer-scale effect (Fig. 4). The VK and GT models differ more than in the SE regime.

5 DISCUSSION

5.1 Scintillation noise on large and extremely large telescopes

The value of the outer scale is important for many applications, so it is often measured by a number of methods and its estimates and altitude profiles are available for several astronomical observatories (Abahamid et al. 2004; Maire et al. 2007; Dali Ali et al. 2010; Floyd, Thomas-Osip & Prieto 2010). These data show that at altitudes greater than 8 km the typical $L_0$ is 20–25 m, although values from 10 to 100 m are sometimes observed. We will use these figures for further numerical evaluation.

Figs 7 and 8 show $G(\nu)$ as a function of the telescope diameter for $L_0 = 25$ m and $L_0 = 50$ m. Here the CO effect is not considered, so the curves are applicable only to telescopes with $\epsilon \leq 0.2$. We see that on telescopes of the 10-m class, even in the SE regime, the scintillation power is $\approx 0.6$ from the power predicted by the standard relation (6), assuming the actual spectrum of perturbations close to the VK or GT models. Exponential models lead to a smaller effect.

For the designed extremely large telescopes TMT and E-ELT, $G$ decreases to $\sim 0.25$, which certainly needs to be taken into account in the estimates of the error budget in high-precision photometric observations. Note that the CO for the TMT ($\epsilon = 0.11$) and for the E-ELT ($\epsilon = 0.15$) is small, so that accounting for it increases corrective functions by no more than 10 per cent.

For the LE regime, the effect is more significant: the measured scintillation power is reduced by a factor of 10–20. Using formula (7) and typical estimates of atmospheric moments from Kenyon et al. (2006), we can predict that for a 60-s exposure the scintillation noise becomes $\sim 10$ μmag instead of the $\sim 40$ μmag predicted by the classical formula. For comparison, the similar estimate for a 4-m telescope is $\sim 200$ μmag.

![Figure 6](https://北大數科數理學會-my.sharepoint.com/:f:/g/sbridges_eCE_RCQ1uASvNbrwA7R2gy5XJ8DQs82Z_m84riog?e=2z0C3W)

Figure 6. $C(\epsilon, \nu)$ for $\epsilon = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$ as a function of $\nu$ for the SE (left) and LE (right) regimes. The solid lines depict the VK model; the dashed lines, the GT model.

![Figure 7](https://北大數科數理學會-my.sharepoint.com/:f:/g/sbridges_eCE_RCQ1uASvNbrwA7R2gy5XJ8DQs82Z_m84riog?e=2z0C3W)

Figure 7. Effect of the outer scale as a function of the diameter of the telescope for two values of $L_0$ for the SE regime. The curve for the VK model is shown with the solid line; for the GT model, with the dashed; for the EM model, with the long-dashed; and for the ME model, with the dash–dotted line.
A function of the outer scale $L$ for different telescopes is illustrated in Fig. 9, where the effect is represented as $(VLT$ and LSST) and two extremely large (TMT and E-ELT) telescopes.

$\sim 7.0$ mag. should provide illumination of about 25 photons cm$^{-2}$ s$^{-1}$, which corresponds to a star with $V \approx 11.5$ mag. Of course, for such photon fluxes, special techniques should be used to avoid strong non-linearity effects.

Evaluation of the scintillation impact in the case of fast photometry (on time-scales of 0.001–0.01 s) for the same telescope leads to a noise of 100–150 $\mu$mag, which is about 20 times less than for 4-m telescopes. The brightness of objects available for such measurements depends on the exposure, and for $\tau = 0.01$ s is equal to $\sim 7.0$ mag.

The combined effect of the outer scale and the CO for two large (VLT and LSST) and two extremely large (TMT and E-ELT) telescopes is illustrated in Fig. 9, where the effect is represented as a function of the outer scale $L_0$. Clearly, the result for the LSST ($D = 8.36$ m) with a large $\epsilon = 0.61$ stands out. In the SE regime, the scintillation is larger than the estimate given by formula (6), as the CO effect dominates over the outer-scale effect. In the LE regime and for the most probable value of $L_0 = 25$ m, however, the scintillation power is half that of the estimated value given by formula (6).

The complex rarefied configuration of the aperture of the GMT has no evident diameter, so we cannot obtain an estimate of the scintillation power in the usual way. Instead, we use the fact that the scintillation is almost uncorrelated at distances greater than the diameter of a telescope (Kornilov 2012). In the LE regime, the correlation can be as small as $\sim 0.2$, but, owing to the outer-scale effect, it should be further reduced.

Ignoring the CO, although for the central GMT mirror it is large enough, we find that the scintillation power on the GMT should be 4–7 times smaller than for a single mirror with diameter 8.36 m. Therefore, the curves for the VLT, shown in Fig. 9, can be used as an estimation.

5.2 General questions

All extremely large telescopes are designed with a segmented primary mirror. The gaps between the individual segments and secondary mirror spiders have a characteristic size of about 0.2–2 cm, and increase the power of high-frequency scintillation. The maximum effect falls within the second–third transmission peak of the aperture filter, as in the case of the CO. In the first approximation, the effect is proportional to the obscured relative area. The area of gaps between segments amounts to less than 1 per cent (for the TMT it is 0.3 per cent), so that, even in the SE regime, we do not expect their influence to be noticeable.

In assessing the observed effect of the outer scale in Section 3.2, we assumed that $L_0$ is constant on the line of sight. Otherwise, the function $\mathcal{G}(v)$ cannot be factored out of integrals (6) or (7), and this effect in the scintillation index should be evaluated for each layer and then integrated over the whole atmosphere. Evidently, it is necessary to know the vertical distribution of the OT and $L_0$ for this.

However, the calculated effect is model-dependent, and the imprecision introduced by the uncertainty of the model is quite large (see Figs 3 and 4), so a variation of $L_0$ with altitude on the order of ±50 per cent can be neglected. Furthermore, as has already been mentioned, the effect, in large, is defined by the $L_0$ in a bounded region of altitude (10–15 km), which generates the main part of the scintillation for large telescopes.

The curves in Fig. 9 also demonstrate that the temporal variations of the measured scintillation power occur not only because of changes in OT intensity in the upper atmosphere, but also because of variations in the outer scale. The net effect should depend strongly on how these two parameters are connected. This issue is virtually unexplored, but it is clear that if an increase in the intensity corresponds to a decrease of the $L_0$, then the scintillation power will be a quite stable characteristic, and vice versa.

The functions shown in Figs 7 and 8 can be used for verification of the type of turbulence model. For this, the initial part of the dependences (diameters of 2–8 m) for the SE regime is best suited. Here, simultaneous measurements with apertures of different diameters are required. One can apply the known methods of integration of exit pupil image, selecting the desired subapertures during processing. In order to minimize the impact of the CO, these subapertures should be similar. For example, on the VLT one can distinguish a number of subapertures with diameters of 8.2, 3.5 m and less with the same $\epsilon = 0.15$. 

Figure 8. Effect of the outer scale as a function of the diameter of the telescope for two values of $L_0$ for the LE regime. See designations in Fig. 7.

Naturally, the scintillation noise at the level of 10 $\mu$mag requires an appropriate photon noise. It is easy to calculate that $10^{10}$ photons are to be accumulated at least, or, in the case of TMT, the source should provide illumination of about 25 photons cm$^{-2}$ s$^{-1}$, which corresponds to a star with $V \approx 11.5$ mag. Of course, for such photon fluxes, special techniques should be used to avoid strong non-linearity effects.

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Figure 8. Effect of the outer scale as a function of the diameter of the telescope for two values of $L_0$ for the LE regime. See designations in Fig. 7.
The reduction of low-frequency components of the scintillation for non-Kolmogorov OT leads to yet another consequence: the effectiveness of the pupil apodization, proposed in (Young 1967), should greatly increase. In the case of extremely large telescopes, it is expected that optimal apodization should lead to a gain of more than 2 in the scintillation noise.

6 CONCLUSION

In this paper we considered two factors that modify the scintillation intensity in observations on large (4–10 m) and extremely large (20–50 m) telescopes. The first effect is caused by the deviation of the real OT from the Kolmogorov model at scales of the order of 10 m. In the models that describe the real turbulence using the outer scale \( L_o \), the scintillation power at low spatial frequencies is much lower than for a pure Kolmogorov spectrum.

The effect of the outer scale can be described by an additional function, which depends on the ratio of \( D/L_o \). This function is equal to 1 when the telescope aperture is small and decreases with increasing diameter. We considered its general behaviour by the example of a piecewise power-law spectrum with the salient point at spatial frequency \( L_o^{-1} \). Particular features were investigated by numerical integration for four models: the Von Karman and Greenwood–Tarazano models, and two exponential models.

For the observed values of \( L_o \) and models with a wide intermediate zone, the effect becomes visible on 4-m-class telescopes. For measurements with long exposures this effect is more important than in the case of short exposures, and for the extremely large telescopes TMT and E-ELT it can reduce the scintillation power by a factor of \( \sim 10 \) compared with classical estimates.

For the Kolmogorov OT, the effect of amplification of the scintillation power, caused by the CO inherent in every large telescope, results in the multiplication of the power for a circular aperture by a function that depends only on the CO parameter. In the case of models with outer scale, the effect is described in a more complicated way, and it becomes significant for large apertures for both short and long exposures.

The significant reduction of scintillation noise, owing to the outer scale, enhances the potential of ground-based telescopes to study the variability of astronomical objects at the \( \sim 10^{-3} \) level. This accuracy is sufficient to see, for example, the transit of Earth-like planets across the disc of solar-like stars. A drastic increase in the accuracy of fast photometry (up to \( \sim 10^{-3} \)) makes it possible to study the micro-variability of many astronomical objects without the accumulation of long time series suitable for temporal spectral analysis.

ACKNOWLEDGMENTS

The author thanks his colleagues and in particular B. Safonov and A. Tokovinin for valuable comments and suggestions during discussions of this work. He is especially grateful to T. Travouillon, whose corrections significantly improved the paper.

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APPENDIX A: INTEGRALS

A1 Integrals in the case of a Kolmogorov spectrum

Here the known integrals, needed for calculating the weighting function, are listed for formula (4) in the SE regime:

\[
I_5 = \int_0^\infty q^{-11/3} q^2 (J_1(q))^2 dq = 2 \pi^{7/6} \sqrt{3} \left( \frac{\zeta}{\Gamma(\frac{5}{3})} \right)^{1/2} = 0.4508, \tag{A1}
\]

and for formula (5) in the LE regime:

\[
I_6 = \int_0^\infty q^{-11/3} q^2 (J_1(q))^2 dq = \frac{272^{2/3} \sqrt{3} \Gamma(\frac{5}{3}) \Gamma^2(\frac{5}{6})}{32 \pi^{5/3}} = 0.8699. \tag{A2}
\]

A2 Integrals in the case of finite limits

In this appendix, the results of the integration over two spectral regions delimited by the dimensionless frequency \( \nu \) are written for the cases with a different exponent of the power spectrum of the phase perturbations \( \Phi(q) \) (see Section 3.2). Expressions for the SE and LE regimes are marked with the corresponding subscript. The
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The high-frequency part \( q > \nu \) with spectrum \( \Phi(q) = q^{-11/3} \) is

\[
I_\nu^h(q) = \int_\nu^\infty q^{-11/3} q^3 (J_1(\pi q))^2 dq \\
= \mathcal{I}_S - \frac{3\pi^2}{28} \nu^{7/2} F_3 \left( \frac{3}{2}, \frac{5}{2}, \frac{13}{6}; 2, \frac{11}{6}, 3; -\pi^2 \nu^2 \right),
\]

\[
I_\nu^l(q) = \int_\nu^\infty q^{-11/3} q^2 (J_1(\pi q))^2 dq = \frac{81 \Gamma \left( \frac{5}{3} \right) \Gamma \left( \frac{11}{6} \right)}{32 \pi^{11/6}} \\
- \frac{3\pi^2}{16} \nu^{4/3} F_3 \left( \frac{3}{2}, \frac{7}{2}, \frac{5}{6}, 2, 3; -\pi^2 \nu^2 \right). \tag{A3}
\]

The low-frequency part \( q < \nu \) in the case of the spectrum with saturation \( \Phi(q) = \nu^{-2} q^{-5/3} \) is

\[
I_\nu^h(q) = \int_\nu^\infty q^{-11/3} q^3 (J_1(\pi q))^2 dq \\
= \frac{\nu^{1/3}}{6} \left( (J_2(\pi \nu))^2 + (J_4(\pi \nu))^2 \right),
\]

\[
I_\nu^l(q) = \int_\nu^\infty q^{-11/3} q^2 (J_1(\pi q))^2 dq \\
= \frac{\pi^2}{20} \nu^{4/3} F_3 \left( \frac{1}{2}, \frac{5}{2}, 2, 3, \frac{7}{12}; -\pi^2 \nu^2 \right). \tag{A4}
\]

The low-frequency part \( q < \nu \) in the case of the spectrum without saturation \( \Phi(q) = \nu^{-2} q^{-5/3} \) is

\[
I_\nu^h(q) = \int_0^\nu q^{-11/3} q^3 (J_1(\pi q))^2 dq \\
= \frac{3\pi^2}{52} \nu^{7/2} F_3 \left( \frac{3}{2}, \frac{11}{6}, \frac{19}{6}; 2, 3, \frac{7}{2}; -\pi^2 \nu^2 \right),
\]

\[
I_\nu^l(q) = \int_0^\nu q^{-11/3} q^2 (J_1(\pi q))^2 dq \\
= \frac{3\pi^2}{40} \nu^{4/3} F_3 \left( \frac{3}{2}, \frac{7}{2}, 2, 3, \frac{11}{6}; -\pi^2 \nu^2 \right). \tag{A5}
\]

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