THE EXPONENTIAL DECAY RATE OF GENERIC TREE OF 1-D WAVE EQUATIONS WITH BOUNDARY FEEDBACK CONTROLS

YARU XIE AND GENQI XU

Department of Mathematics, Tianjin University
Haihe Education Park, Tianjin
Tianjin, MO 300000, China

(Communicated by Benedetto Piccoli)

Abstract. In this paper, we study the exponential decay rate of generic tree of 1-d wave equations with boundary feedback controls. For the networks, there are some results on the exponential stability, but no result on estimate of the decay rate. The present work mainly estimates the decay rate for these systems, including signal wave equation, serially connected wave equations, and generic tree of 1-d wave equations. By defining the weighted energy functional of the system, and choosing suitable weighted functions, we obtain the estimation value of decay rate of the systems.

1. Introduction. In this paper, our aim is to estimate decay rate of some concrete 1-d wave network systems. Before going on, we introduce some notation. Let $\mathbb{X}$ be a Banach space and $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ be a closed and densely defined linear operator. Let us consider the abstract differential equation in $\mathbb{X}$:

$$\begin{cases}
  \dot{x}(t) = Ax(t), & t > 0, \\
  x(0) = x_0.
\end{cases} \quad (1)$$

Suppose that $A$ generates a $C_0$ semigroup $T(t)$, then the solution to (1) is given by $x(t) = T(t)x_0$, and there exist constants $M > 0$ and $\omega \in \mathbb{R}$ such that

$$\|x(t)\| \leq Me^{\omega t}\|x_0\|,$$

where $\omega \geq \omega_0(A)$ and the scalar defined by

$$\omega_0(A) = \lim_{t \to \infty} \ln \frac{\|T(t)\|}{t}$$

is called the growth order of the semigroup $T(t)$.

If $\omega_0(A) = -\beta_0 < 0$ with $\beta_0 > 0$, then the system (1) is said to be the exponentially stable, and $\beta_0$ is called the exponential decay rate of the system (1).

It is well known that for given $A$, to determine $\omega_0(A)$ has been a difficult topic in mathematical system theory. Since $A$ is known, we can calculate the spectrum of $A$, and determine the scalar

$$s(A) = \sup \{\Re \lambda, \lambda \in \sigma(A)\}.$$

2010 Mathematics Subject Classification. Primary: 35L05, 37L45; Secondary: 37L15, 90B10.

Key words and phrases. Wave networks, weighted energy functional, boundary feedback control, exponential decay rate.

This research was supported by Natural Science Foundation of China(grant NSFC-61174080, 61503275, 61573252).
In general, it holds that $s(A) \leq \omega_0(A)$. If it holds equality, i.e., $s(A) = \omega_0(A)$, then the system (1) is said to satisfy the spectrum determined growth assumption. When the system (1) satisfies the spectrum determined growth assumption, we can obtain $\omega_0(A)$ by $s(A)$. For example, if $T(t)$ is eventually norm continuous semigroup, differentiable semigroup or analytic semigroup, it holds that $s(A) = \omega_0(A)$ (see, [37]). In particular, if $A$ is resolvent compact, and its eigenvectors system forms a uncondition basis for $X$, then system (1) satisfies the spectrum determined growth assumption (see, [11]).

However, in practice, even if $s(A) = \omega_0(A)$, we cannot calculate the exact value of $\omega_0(A)$, this is because we cannot calculate the exact values of $\sigma(A)$. In most case, we only obtain the asymptotical values of $\sigma(A)$. So, to obtain the approximation of $\omega_0(A)$ has been an interesting topic in practice. In this paper we concern with the estimate problem of $\omega_0(A)$ for the 1-d wave networks with boundary controls.

Let us recall briefly research development of 1-d wave networks. The study of control problem on single 1-d wave equation or string started early in 1970’s [39]. Russell [38], Loins [30] obtained the stabilization result for it. In [41], Shubov studied the spectral property of the damping string system. Cox and Zuazua [12] gave the decay rate of the energy functional of a damped string. Xu et al. in [45] studied the stability of a string with feedback time delay, and in [46] they studied the general linear feedback on the boundary and calculated all spectrum of the closed loop system. Krstic et al. in [27] studied the output feedback of an unstable wave equation, and obtained the exponential stability. Ammari in [5] studied the large time behavior of the solutions and optimal location of a homogenous string equation.

Although single string is a simple system, under the boundary velocity feedback, the closed loop system is a more complex system. We can prove the system is exponentially stable, only a few systems can be determined the explicit decay rate.

For multi-link system, Liu et al. in [31] studied the stabilization problem of a serially connected strings and proved that the closed loop system is exponentially stable. Since then, the modelling and control for the wave equations, for instance, see [42, 32, 33, 29, 28], became gradually a hot topic in the world. Dager and Zuazua [15, 14, 13] studied the controllability of star-shaped and tree-shaped networks of strings; Ammari and Jellouli [1, 2] studied the stabilization of star-shaped tree and generic tree of strings. Recently Ammari in [3] studied a chain of serially connected strings using a frequency domain method and a special analysis for the resolvent. In [4] the authors analysed the spectrum of the dissipative Schrodinger operator on binary tree-shaped networks, and proved the Riesz basis property of the system. Hence the system satisfies the spectrum determined growth assumption.

Jellouli [26] analyzed the spectrum of a degenerate tree and by the spectral decomposition. He proved the best decay rate identifying with the spectral abscissa of the system. However, they do not give the decay rate of the systems due to difficulty of spectrum exact calculation.

To study the decay rate, Xu, Guo et al in [20, 47] studied the Riesz basis property of the closed loop system. Under certain conditions, they proved that the closed loop systems have Riesz basis property, and hence the systems satisfy the spectrum determine growth assumption, i.e., $s(A) = \omega_0(A)$. Since then, there are many papers studying the Riesz basis property of the closed loop system for different 1-d wave networks, for example, see, [34, 50, 48, 17, 25, 24, 23]. These results show that the closed loop systems satisfy $s(A) = \omega_0(A)$. However, due to the difficulty
of spectrum calculation of the wave networks, they do not give the estimate of the decay rate of the systems.

Moreover, Nicaise and Valein in [36] studied the stabilization of the 1-d wave networks with a delay in the feedbacks. Under certain conditions, they proved that the networks system is exponentially stable, but have no estimate of the decay rate. More recent results on stabilization and supper-stability of the 1-d wave networks, we refer to [53, 54, 52], [19, 18]. About research development for the general 1-d wave network, we refer to literatures [51] and [49].

Stabilization is one of the most important problems in the research of 1-d wave networks. Under suitable feedback control laws, we can use the different approach, such as multiplier method [51], spectral analysis method [46], as well as resolvent estimate method [22], to prove the exponential stability of the closed loop systems. But there is no result on the estimate of the decay rate.

The estimate problem of the decay rate of the system appears not only in the 1-d wave networks, but also in the other networks, for instance, the first hyperbolic systems, [16, 10, 9, 8] for 1-d linear hyperbolic systems, [43] for thermo-elastic networks, [21] for gas networks and others [6, 35]. Based on the reasons above, in this paper, we concentrate our attention on the estimate problem of the 1-d wave networks. Our approach is inspired by the works [16, 10]. The most important thing is that we find out the conditions which make the inequalities hold and hence get the decay rate estimate.

The rest of this paper is organized as follows: In section 2, we discuss the decay rate of a serially connected 1-d wave equations. At first we discuss a single 1-d wave equation, and from it we will obtain some information about the decay rate of the weighted energy functional and the spectrum of the system. Using this information we can assert the decay rate of the serially connected wave system. In section 3, we discuss the decay rate of the generic tree of 1-d wave networks. Herein we will extend the approach used in section 2 to the generic tree of 1-d wave networks. At first, we discuss the simple tree of 1-d wave network. From this simple model, we will find out some rule of the parameter choices. After then we use this rule to get the estimate of the decay rate of the generic tree of 1-d wave networks. Finally, in section 4, we conclude this paper.

2. The decay rate of serially connected wave equations. In this section we estimate the decay rate of serially connected 1-d wave equations. At first, we study a control problem of 1-d wave equation. By defining a weighted energy functional of the system, we get a feedback control law. Furthermore, we study the decay rate of the closed loop system. For a single wave equation, Loins [30] proved the exponential stability and Xu [46] proved the system has the Riesz basis by the spectral analysis. Next, we study the serially connected strings, and determine its decay rate. This model was studied early in 1989 by Liu et al in [31], they proved the exponential stability, but they had not given the decay rate of the system.

Although some results of this section are known, we hope to find a general approach which can be apply to more complex system.

2.1. The decay rate of single wave equation. In this subsection, we study the decay rate of signal 1-d wave equation. Although it is a simple model and has been studied in [46] by the spectral analysis method, we hope one can find a general approach from it.

We begin with recalling a control problem of 1-d wave equation:
Using (2), we can find out
\[ u(x,t) \in \gamma_c \]
where \( \gamma, p \) are positive constants, they are determined later. Obviously,
\[ \beta > 0 \]

For (2), we introduce new functions
\[ w(x,t) = w(x) + cw_x(x,t), \]
\[ \eta(x,t) = w(x) - cw_x(x,t). \] (3)

Using (2), we can find out
\[ \xi_t(x,t) = c\xi_t(x,t), \]
\[ \eta_t(x,t) = -c\eta_t(x,t). \]

Now we define a weighted energy functional of (2) by
\[ V(t) = \int_0^1 [p(x)\xi^2(x,t) + q(x)\eta^2(x,t)]dx, \] (4)
where the weighting functions \( p(x) \) and \( q(x) \) is defined as follows:
\[ p(x) = p_1e^{\gamma x}, \quad q(x) = q_1e^{-\gamma x}, \] (5)
where \( \gamma, p_1 \) and \( q_1 \) are positive constants, they are determined later. Obviously,
\[ p'(x) = \gamma p(x), \quad q'(x) = -\gamma q(x). \]

Differentiating \( V(t) \) leads to
\[ \dot{V}(t) = 2\int_0^1 [p(x)\xi(x,t)\xi_t(x,t) + q(x)\eta(x,t)\eta_t(x,t)]dx \]
\[ = 2c\int_0^1 [p(x)\xi(x,t)\xi_x(x,t) - q(x)\eta(x,t)\eta_x(x,t)]dx \]
\[ = c\int_0^1 [p(x)\partial_x\xi^2(x,t) - q(x)\partial_x\eta^2(x,t)]dx. \]

Integration by parts, we obtain
\[ \dot{V}(t) = cp_1e^{\gamma}(w_t + cw_x)^2(1, t) - cq_1e^{-\gamma}(w_t - cw_x)^2(1, t) \]
\[ -cp_1(w_t + cw_x)^2(0, t) + cq_1(w_t - cw_x)^2(0, t) - \gamma cV(t) \]
\[ = cp_1e^{\gamma}(w_t(1,t) - \alpha u(t))^2 - cq_1e^{-\gamma}(w_t(1,t) + \alpha u(t))^2 \]
\[ +c^3(q_1 - p_1)w_x^2(0, t) - \beta V(t) \]
\[ = -\beta V(t) + B(t), \]
where \( \beta = \gamma c. \)
Theorem 2.1. \( V(t) \) and \( \beta \) are located in the line \( \mathbb{R} \).

By the spectral analysis, we know that all eigenvalues of the system \( \text{Remark 1.} \)

rate is at least \( \beta \) to (2) is

\[
B(t) = cp_1e^{-\gamma (w_1(1, t) - \alpha u(t))^2} - c q_1 e^{-\gamma (w_1(1, t) + \alpha u(t))^2} \\
+ c^3 (q_1 - p_1) u_x^2(0, t) \\
\leq 0,
\]

then the system (2) can be exponentially stable.

For example, we take \( u(t) = w_1(1, t) \), \( p_1 = q_1 \), then the boundary parts have the form

\[
B(t) = cp_1 w_1^2(1, t) [e^{-\gamma (1 - \alpha)^2} - e^{-\gamma (1 + \alpha)^2}] \\
= cp_1 e^{-\gamma (1 - \alpha)^2} w_1^2(1, t) [e^{2\alpha} - \frac{(1 + \alpha)^2}{(1 - \alpha)^2}] .
\]

Therefore, the \( \gamma \) satisfying \( 0 < \gamma \leq \ln \left| \frac{1 + \alpha}{1 - \alpha} \right| \) is desired. In this case, we have

\[
\dot{V}(t) \leq -\beta V(t) \quad \text{and hence}
\]

\[
V(t) \leq V(0) e^{-\beta t}.
\]

The decay rate of the weighted functional of the system (2) is at least \( \beta = \gamma c \).

Note that according to above choice of \( u(t) \), the closed loop system corresponding to (2) is

\[
\left\{ \\
\begin{array}{l}
w_{tt}(x, t) = c^2 w_{xx}(x, t), \ x \in (0, 1), \ t > 0, \\
w_x(0, t) = 0, \\
c w_x(1, t) = -\alpha w_1(1, t), \\
w(x, 0) = w_0(x), \ w_1(x, 0) = w_1(x).
\end{array}
\right.
\]

(6)

Note that the weighted energy functional satisfies the following inequality

\[
2 e^{-\gamma p_1} \int_0^1 [ |c w_x(x, t)|^2 + |w_1(x, t)|^2 ] dx \\
\leq V(t) \leq 2 e^{-\gamma p_1} \int_0^1 [ |c w_x(x, t)|^2 + |w_1(x, t)|^2 ] dx.
\]

So the decay rate of the system (6) is \( \gamma \).

Summarizing above discussion, we have proved the following result.

**Theorem 2.1.** The closed loop system (6) is exponentially stable, and the decay rate is at least \( \frac{\alpha}{2} = \frac{\gamma c}{2} \) where \( 0 < \gamma \leq \ln \left| \frac{1 + \alpha}{1 - \alpha} \right| \).

**Remark 1.** By the spectral analysis, we know that all eigenvalues of the system (6) are located in the line \( \Re \lambda = -\frac{c}{2} \ln \left| \frac{1 + \alpha}{1 - \alpha} \right| \), obviously,

\[
\frac{c}{2} \ln \left| \frac{1 + \alpha}{1 - \alpha} \right| = \frac{c}{2} \max \left\{ \gamma; 0 < \gamma \leq \ln \left| \frac{1 + \alpha}{1 - \alpha} \right| \right\} .
\]

This means that \( \gamma = \ln \left| \frac{1 + \alpha}{1 - \alpha} \right| \) is the optimal value, and the maximal decay rate of (6) is \( \frac{\alpha}{2} = \frac{\gamma c}{2} \ln \left| \frac{1 + \alpha}{1 - \alpha} \right| .
\]

Note that the result of Theorem 2.1 is obtained under the condition \( \alpha \neq 1 \). If \( \alpha = 1 \), we have a better result.

**Theorem 2.2.** If \( \alpha = 1 \), the closed loop system (6) is super-stable. That is , there exist a positive constant \( \tau \), when \( t \geq \tau \), \( (w(x, t), w_1(x, t)) \equiv (0, 0) \).
Proof. If $\alpha = 1$, we can take $p_1 > q_1$. In this case, the boundary parts always satisfy
$$-4cq_1w_1^2(1, t) - c^3(p_1 - q_1)w_2^2(0, t) \leq 0.$$ 
Thus
$$\dot{V}(t) + \beta V(t) = -4cq_1w_1^2(1, t) - c^3(p_1 - q_1)w_2^2(0, t).$$ 
From above we get
$$e^{\beta t}V(t) = V(0) - 4cq_1 \int_0^t e^{\beta s}w_2^2(1, s)ds + c^3(p_1 - q_1) \int_0^t e^{\beta s}w_2^2(0, s)ds,$$
that holds for all $\beta = \gamma c$. Therefore, it must exist a $\tau$ such that $V(t) = 0$, $\forall t \geq \tau$. Hence when $t \geq \tau$, we have $(w(x, t), w_t(x, t)) \equiv (0, 0)$. \hfill \Box

**Remark 2.** We can prove that $\tau = \frac{c}{\gamma}$.

**Remark 3.** The super-stability problem of a system was studied by [7]. The similar questions for the 1-d wave network were studied in [46, 40, 54].

### 2.2. The decay rate of serially connected 1-d wave equations.

Let us recall the model studied in [31]. Under suitable change, the model can be written as follows:

$$\begin{align*}
&\begin{cases}
    w_{i,t,t}(x, t) = c_i^2w_{i,x,x}(x, t), x \in (0, 1), t > 0, i = 1, 2, \cdots, n, \\
    w_i(0, t) = 0, \\
    w_i(1, t) = w_{i+1}(0, t), i = 1, 2, \cdots, n - 1, \\
    c_iw_{i,x}(1, t) = c_{i+1}w_{i+1,x}(0, t), i = 1, \cdots, n - 1, \\
    c_nw_{n,x}(1, t) = -\alpha w_{n,t}(1, t), \\
    w_i(x, 0) = w_{i,0}(x), \\
    w_{i,t}(x, 0) = w_{i,1}(x), x \in (0, 1), i = 1, 2, \cdots, n.
\end{cases} \tag{7}
\end{align*}$$

Similarly, we take transform of the variable:

$$\begin{align*}
\xi_i(x, t) &= w_{i,t}(x, t) + c_iw_{i,x}(x, t), \\
\eta_i(x, t) &= w_{i,t}(x, t) - c_iw_{i,x}(x, t).
\end{align*}$$

Under the transform the equations become

$$\begin{align*}
&\begin{cases}
    \xi_{i,t}(x, t) = c_i\xi_{i,x}(x, t), \\
    \eta_{i,t}(x, t) = -c_i\eta_{i,x}(x, t), i = 1, 2, \cdots, n, \\
    \xi_1(0, t) = -\eta_1(0, t), \\
    \xi_i(1, t) = \xi_{i+1}(0, t), i = 1, 2, \cdots, n - 1, \\
    (1 + \alpha)\xi_n(1, t) = (1 - \alpha)\eta_n(1, t), \\
    \xi(x, 0) = \xi_0(x), \quad \eta(x, 0) = \eta_0(x).
\end{cases} \tag{8}
\end{align*}$$

We define the weighted energy functional of the system (7) as

$$V(t) = \sum_{i=1}^{n} V_i(t)$$

where

$$V_i(t) = \int_0^1 [p_i(x)\xi_i^2(x, t) + q_i(x)\eta_i^2(x, t)]dx,$$ 
and

$$p_i(x) = p_i e^{\gamma_i x}, \quad q_i(x) = q_i e^{-\gamma_i x},$$
with $p_i, q_i, \gamma_i > 0$. 

[532] YARU XIE AND GENQI XU
Proof. If \( p_1 = q_1 = p \), and \( c_{i+1}q_{i+1} = c_ie^{-\gamma_i}, \ c_{i+1}p_{i+1} = c_ip_i e^{\gamma_i}, \ i = 1, 2, \cdots, n-1 \), then we have recursion relationship
\[
p_{i+1} = \frac{c_i}{c_{i+1}} e^{\gamma_i} p_{i}, \quad q_{i+1} = \frac{c_i}{c_{i+1}} e^{-\gamma_i} q_{i}.
\]
Therefore,

\[ p_n = \frac{c_1}{c_n} \sum_{i=1}^{n-1} \gamma_i, \quad q_n = \frac{c_1}{c_n} \sum_{i=1}^{n-1} \gamma_i. \]

In this case, we have

\[
B(t) = c_n [p_n e^{\gamma_n} - q_n e^{-\gamma_n} (\frac{1 + \alpha}{1 - \alpha})^{2}] \xi_n(1, t)
= c_1 p \left[ \sum_{i=1}^{n} \gamma_i - e - \sum_{i=1}^{n} \gamma_i \left( \frac{1 + \alpha}{1 - \alpha} \right)^2 \right] \xi_n(1, t).
\]

Taking

\[ e^{2 \sum_{i=1}^{n} \gamma_i} \leq \left( \frac{1 + \alpha}{1 - \alpha} \right)^{2}, \]

i.e.,

\[ \sum_{i=1}^{n} \gamma_i \leq \ln \left| \frac{1 + \alpha}{1 - \alpha} \right|, \]

we get \( B(t) \leq 0 \). So

\[ \dot{V}(t) \leq - \sum_{i=1}^{n} \beta_i V_i(t) \leq - \beta V(t), \]

and

\[ V(t) \leq V(0) e^{-\beta t}. \]

In particular, when \( \gamma_i = \frac{1}{\sum_{j=1}^{n} \frac{1}{c_j}} \ln \left| \frac{1 + \alpha}{1 - \alpha} \right| \), we have

\[ \sum_{i=1}^{n} \gamma_i = \ln \left| \frac{1 + \alpha}{1 - \alpha} \right|, \]

and

\[ \beta = \min_{i=1, 2, \ldots, n} \beta_i = \min_{i=1, 2, \ldots, n} \gamma_i c_i = \frac{1}{\sum_{i=1}^{n} \frac{1}{c_i}} \ln \left| \frac{1 + \alpha}{1 - \alpha} \right|. \]

The desired result follows.

**Remark 4.** By the spectral analysis, we can show that the asymptote of the spectrum is

\[ \frac{1}{\sum_{i=1}^{n} \frac{1}{c_i}} \ln \left| \frac{1 + \alpha}{1 - \alpha} \right|. \]

### 3. The decay rate of the generic tree of 1-d wave equations.

In this section we will extend the approach used in section 2 to more complex models. Here we mainly study the decay rate of the generic tree of 1-d wave equations. It is well known that if a network is of tree-shaped, and all boundary vertices (but one) are acted on control, then the system is exactly controllable, and under the feedback control laws, then the closed loop system is exponentially stable, see [13, 15, 1, 2, 50]. But there is no the estimate of decay rate. In this section we will estimate the decay rate of the generic tree wave networks.
3.1. A simple tree-shaped network of 1-d wave equations. In this subsection we consider a simple tree-shaped network of wave equations, which is governed by the following partial differential equations:

\[
\begin{align*}
    w_{i,t,t}(x,t) &= c_i^2 w_{i,x,x}(x,t), \quad x \in (0,1), \quad t \geq 0, \quad i = 1, 2, 3, \\
    w_1(0,t) &= 0, \\
    w_1(1,t) &= w_2(0,t) = w_3(0,t), \\
    c_1 w_{1,x}(1,t) &= c_2 w_{2,x}(0,t) + c_3 w_{3,x}(0,t), \\
    c_2 w_{2,x}(1,t) &= -\alpha_2 u_2(t), \\
    c_3 w_{3,x}(1,t) &= -\alpha_3 u_3(t), \\
    w_i(x,0) &= w_{i,0}(x), \quad w_{i,t}(x,0) = w_{i,1}(x).
\end{align*}
\]  

where \( c_i > 0, \alpha_i > 0, \ i = 1, 2, 3. \) The functions \( u_i(t), \ i = 2, 3, \) are controls.

For the equations (11), we let

\[
\begin{align*}
    \xi_i(x,t) &= w_{i,t}(x,t) + c_i w_{i,x}(x,t), \\
    \eta_i(x,t) &= w_{i,t}(x,t) - c_i w_{i,x}(x,t).
\end{align*}
\]

Using (11) we can find out

\[
\begin{align*}
    \xi_{i,t}(x,t) &= c_i \xi_{i,x}(x,t), \\
    \eta_{i,t}(x,t) &= -c_i \eta_{i,x}(x,t), \quad i = 1, 2, 3.
\end{align*}
\]

Similarly, we define the weighted energy functional by

\[
V(t) = \sum_{i=1}^{3} V_i(t)
\]

where

\[
V_i(t) = \int_0^1 (p_i(x)\xi_i^2(x,t) + q_i(x)\eta_i^2(x,t)) dx,
\]

the weight functions \( p_i(x) \) and \( q_i(x) \) are defined as follows:

\[
p_i(x) = p_i e^{\gamma_i x}, \quad q_i(x) = q_i e^{-\gamma_i x},
\]

with \( p_i > 0, q_i > 0, \gamma_i > 0, \ i = 1, 2, 3. \)

A direct calculation gives

\[
\begin{align*}
    \dot{V}_i(t) &= 2 \int_0^1 p_i(x)\xi_i(x,t)\xi_{i,t}(x,t) dx + 2 \int_0^1 q_i(x)\eta_i(x,t)\eta_{i,t}(x,t) dx \\
    &= 2c_i \int_0^1 p_i(x)\xi_i(x,t)\xi_{i,x}(x,t) dx - 2c_i \int_0^1 q_i(x)\eta_i(x,t)\eta_{i,x}(x,t) dx \\
    &= c_i [p_i(1)\xi_i^2(1,t) - q_i(1)\eta_i^2(1,t)] - c_i [p_i(0)\xi_i^2(0,t) - q_i(0)\eta_i^2(0,t)] - \beta_i V_i(t),
\end{align*}
\]

where \( \beta_i = \gamma_i c_i. \) Thus, it holds that

\[
\dot{V}(t) = -\sum_{i=1}^{3} \beta_i V_i(t) + \sum_{i=1}^{3} c_i [p_i(1)\xi_i^2(1,t) - q_i(1)\eta_i^2(1,t)] \\
- \sum_{i=1}^{3} c_i [p_i(0)\xi_i^2(0,t) - q_i(0)\eta_i^2(0,t)]
\]
For simplicity, we denote the boundary parts by

\[ B \]

that

\[ p \]

\[ \sum_{i=1}^{3} \beta_i V_i(t) + \sum_{i=1}^{3} c_i p_i e^{\gamma_i} (w_{i,t}(1, t) + c_i w_{i,x}(1, t))^2 \]

\[ - \sum_{i=1}^{3} c_i q_i e^{-\gamma_i} (w_{i,t}(1, t) - c_i w_{i,x}(1, t))^2 - \sum_{i=1}^{3} c_i p_i (w_{i,t}(0, t) + c_i w_{i,x}(0, t))^2 \]

\[ + \sum_{i=1}^{3} c_i q_i (w_{i,t}(0, t) - c_i w_{i,x}(0, t))^2. \]

For simplicity, we denote the boundary parts by

\[ B(t) = \sum_{i=1}^{3} c_i p_i e^{\gamma_i} (w_{i,t}(1, t) + c_i w_{i,x}(1, t))^2 - \sum_{i=1}^{3} c_i q_i e^{-\gamma_i} (w_{i,t}(1, t) - c_i w_{i,x}(1, t))^2 \]

\[ - \sum_{i=1}^{3} c_i p_i (w_{i,t}(0, t) + c_i w_{i,x}(0, t))^2 + \sum_{i=1}^{3} c_i q_i (w_{i,t}(0, t) - c_i w_{i,x}(0, t))^2. \]

Using the boundary conditions in (11), we have

\[ B(t) = \sum_{i=2}^{3} c_i p_i e^{\gamma_i} (w_{i,t}(1, t) - \alpha_i u_i(t))^2 - \sum_{i=2}^{3} c_i q_i e^{-\gamma_i} (w_{i,t}(1, t) + \alpha_i u_i(t))^2 \]

\[ + c_1 p_1 e^{\gamma_1} (w_{1,t}(1, t) + c_1 w_{1,x}(1, t))^2 - c_1 q_1 e^{-\gamma_1} (w_{1,t}(1, t) - c_1 w_{1,x}(1, t))^2 \]

\[ - \sum_{i=2}^{3} c_i p_i (w_{i,t}(0, t) + c_i w_{i,x}(0, t))^2 + \sum_{i=2}^{3} c_i q_i (w_{i,t}(0, t) - c_i w_{i,x}(0, t))^2 \]

\[ + c_1^2 (q_1 - p_1) w_{1,t}^2(0, t). \]

Our aim is to prove that we can select parameters \( p_i, q_i \) and \( \gamma_i \) and control \( u_i(t) \) such that \( B(t) \leq 0 \) for all \( t \geq 0 \).

The simple selections for \( u_i(t), i = 2, 3 \) are

\[ u_i(t) = w_{i,t}(1, t), \quad i = 2, 3. \]

Hence the closed loop system associated with (11) is

\[ \begin{align*}
    w_{i,t}(x, t) &= c_i^2 w_{i,xx}(x, t), \quad x \in (0, 1), \ t > 0, \ i = 1, 2, 3, \\
    w_1(0, t) &= 0, \\
    w_1(1, t) &= w_2(0, t) = w_3(0, t), \\
    c_1 w_{1,t}(1, t) &= c_2 w_{2,t}(0, t) + c_3 w_{3,t}(0, t), \\
    c_2 w_{2,t}(1, t) &= -\alpha_2 w_{2,t}(1, t), \\
    c_3 w_{3,t}(1, t) &= -\alpha_3 w_{3,t}(1, t), \\
    w_i(x, 0) &= w_{i,0}(x), \ w_{i,t}(x, 0) = w_{i,1}(x). 
\end{align*} \]

The following theorem gives the selection condition of the parameters.

**Theorem 3.1.** Let the boundary control laws \( u_2, u_3 \) be defined by (15). Suppose that \( p_i, q_i, i = 1, 2, 3 \) satisfy the following conditions:

\[ \begin{align*}
    p_1 &= q_1, \\
    p_2 c_2 &= p_3 c_3 = \frac{3}{2} p_1 c_1, \\
    q_2 c_2 &= q_3 c_3 = \frac{5}{2} p_1 c_1, \\
    \gamma_1 &\leq \ln \frac{5}{4}, \\
    \gamma_2 &\leq \frac{1}{2} \ln \left(1 + \alpha_2 \right)^2, \\
    \gamma_3 &\leq \frac{1}{2} \ln \left(1 + \alpha_3 \right)^2. 
\end{align*} \]
then $B(t) \leq 0$, and hence for $\beta_i = \gamma_i \alpha_i$, and

$$\beta = \min_{i = 1, 2, 3} \{\beta_i\}, \quad (18)$$

it holds that $\dot{V}(t) \leq -\beta V(t)$, i.e., the exponential decay rate of the closed loop systems (16) is at least $\frac{\beta}{2}$.

Proof. Note that

$$\dot{V}(t) = -\sum_{i = 1}^{3} \beta_i V_i(t) + B(t) \leq -\min_{i = 1, 2, 3} \{\beta_i\} V(t) + B(t) = -\beta V(t) + B(t).$$

So, we only need to prove that under the conditions (17), it holds $B(t) \leq 0$.

Suppose that (15) and (17) hold, then we have

$$e^{2\gamma_i} - \frac{(1 + \alpha_i)^2}{3(1 - \alpha_i)^2} \leq 0, \quad i = 2, 3,$$

and

$$\begin{align*}
B(t) &= \frac{3}{2} p_1 c_1 \sum_{i = 2}^{3} e^{\gamma_i} (1 - \alpha_i)^2 w_{i,i}(1, t) - \frac{1}{2} p_1 c_1 \sum_{i = 2}^{3} e^{-\gamma_i} (1 + \alpha_i)^2 w_{i,i}(1, t) \\
&\quad + c_1 p_1 e^{\gamma_i} (w_{1,i}(1, t) + c_1 w_{1,x}(1, t))^2 - c_1 p_1 e^{-\gamma_i} (w_{1,i}(1, t) - c_1 w_{1,x}(1, t))^2 \\
&\quad - \frac{3}{2} p_1 c_1 \sum_{i = 2}^{3} (w_{1,i}(1, t) + c_1 w_{1,x}(0, t))^2 + \frac{1}{2} p_1 c_1 \sum_{i = 2}^{3} (w_{1,i}(1, t) - c_1 w_{1,x}(0, t))^2 \\
&\leq c_1 p_1 e^{\gamma_i} (w_{1,i}(1, t) + c_1 w_{1,x}(1, t))^2 - c_1 p_1 e^{-\gamma_i} (w_{1,i}(1, t) - c_1 w_{1,x}(1, t))^2 \\
&\quad - \frac{3}{2} p_1 c_1 \sum_{i = 2}^{3} (w_{1,i}(1, t) + c_1 w_{1,x}(0, t))^2 + \frac{1}{2} p_1 c_1 \sum_{i = 2}^{3} (w_{1,i}(1, t) - c_1 w_{1,x}(0, t))^2 \\
&=: c_1 p_1 B_1(t).
\end{align*}$$

We expand above last terms as follows:

$$\begin{align*}
B_1(t) &= e^{\gamma_i} (w_{1,i}(1, t) + c_1 w_{1,x}(1, t))^2 - e^{-\gamma_i} (w_{1,i}(1, t) - c_1 w_{1,x}(1, t))^2 \\
&\quad - \frac{3}{2} \sum_{i = 2}^{3} (w_{1,i}(1, t) + c_1 w_{1,x}(0, t))^2 + \frac{1}{2} \sum_{i = 2}^{3} (w_{1,i}(1, t) - c_1 w_{1,x}(0, t))^2 \\
&= e^{\gamma_i} \left[w_{1,i}^2(1, t) + 2c_1 w_{1,i}(1, t)w_{1,x}(1, t) + c_1^2 w_{1,x}^2(1, t)\right] \\
&\quad - e^{-\gamma_i} \left[w_{1,i}^2(1, t) - 2c_1 w_{1,i}(1, t)w_{1,x}(1, t) + c_1^2 w_{1,x}^2(1, t)\right] \\
&\quad - \frac{3}{2} \sum_{i = 2}^{3} \left[w_{1,i}^2(1, t) + 2c_1 w_{1,i}(1, t)w_{1,x}(0, t) + c_1^2 w_{1,x}^2(0, t)\right] \\
&\quad + \frac{1}{2} \sum_{i = 2}^{3} \left[w_{1,i}^2(1, t) - 2c_1 w_{1,i}(1, t)w_{1,x}(0, t) + c_1^2 w_{1,x}^2(0, t)\right] \\
&= w_{1,i}^2(1, t) [(e^{\gamma_i} - e^{-\gamma_i}) - 2] + 2w_{1,i}(1, t) [(e^{\gamma_i} + e^{-\gamma_i}) c_1 w_{1,x}(1, t) \\
&\quad - 2 \sum_{i = 2}^{3} c_1 w_{1,x}(0, t)] + (e^{\gamma_i} - e^{-\gamma_i}) c_1^2 w_{1,x}^2(1, t) - \sum_{i = 2}^{3} c_1^2 w_{1,x}^2(0, t)
\end{align*}$$
we can choose networks which are governed by the following partial differential equations: section, we extend the previous model to more extensive case. We consider 1-d wave

\[ 2w_{1,t}(1,t)[\sinh \gamma_1 - 1] + 4w_{1,x}(1,t)c_1w_{1,x}(1,t)[\cosh \gamma_1 - 1] + 2 \sinh \gamma_1c^2_1w^2_{1,x}(1,t) - \sum_{i=2}^{3} c_i^2 w^2_{i,x}(0,t) \]

\[ \leq 2w_{1,t}(1,t)[\sinh \gamma_1 - 1] + 2[\cosh \gamma_1 - 1][w^2_{1,t}(1,t) + c_1^2 w^2_{1,x}(1,t)] + 2 \sinh \gamma_1c^2_1w^2_{1,x}(1,t) - \sum_{i=2}^{3} c_i^2 w^2_{i,x}(0,t) \]

\[ = 2w_{1,t}(1,t)(e^{\gamma_1} - 2) + 2(e^{\gamma_1} - 1)c^2_1w^2_{1,x}(1,t) - \sum_{i=2}^{3} c_i^2 w^2_{i,x}(0,t). \]

Using the condition

\[ c_1w_{1,x}(1,t) = c_2w_{2,x}(0,t) + c_3w_{3,x}(0,t), \]

\[ c^2_1w^2_{1,x}(1,t) = [c_2w_{2,x}(0,t) + c_3w_{3,x}(0,t)]^2 \leq 2 \sum_{i=2}^{3} c_i^2 w^2_{i,x}(0,t). \]

so \( B(t) \leq c_1p_1B_1(t) \) and

\[ B_1(t) \leq 2w^2_{1,t}(1,t)(e^{\gamma_1} - 2) + (4e^{\gamma_1} - 5) \sum_{i=2}^{3} c_i^2 w^2_{i,x}(0,t). \]

Obviously, when \( e^{\gamma_1} \leq \frac{5}{4} \), i.e., \( 0 \leq \gamma_1 \leq \ln \frac{5}{4} \), we have \( B(t) \leq 0 \). The desired result follows.

\[ \square \]

Remark 5. Since the decay rate is determined by

\[ \beta = \min_{j=1,2,3} \beta_j = \min_{j=1,2,3} \gamma_j c_j, \]

we can choose

\[ \gamma_1 = \ln \frac{5}{4}, \gamma_j = \frac{1}{2} \ln \left| \frac{1 + \alpha_j}{1 - \alpha_j} \right|, j = 2, 3, \]

thus the decay rate estimate is given by

\[ \frac{\beta}{2} = \min \left\{ \frac{c_1}{2} \ln \frac{5}{4}, \frac{c_2}{4} \ln \left| \frac{1 + \alpha_2}{1 - \alpha_2} \right|, \frac{c_3}{4} \ln \left| \frac{1 + \alpha_3}{1 - \alpha_3} \right| \right\}. \]

From above we see that even if \( \alpha_2 = \alpha_3 = 1 \), the decay rate of the system is \( \frac{\beta}{2} = \frac{\alpha_1}{2} \ln \frac{5}{4} \). So the system is not super-stable.

3.2. The decay rate of the generic tree of 1-d wave equation. In this subsection, we extend the previous model to more extensive case. We consider 1-d wave networks which are governed by the following partial differential equations:

\[
\begin{cases}
  w_{i,t}(x,t) = c^2_i w_{i,xx}(x,t), i = 1, 2, \ldots, m, x \in (0,1), t > 0, \\
  w_1(0,t) = 0, \\
  w_1(1,t) = w_j(0,t), j = 2, 3, \ldots, m, \\
  c_1 w_{1,x}(1,t) = \sum_{j=2}^{m} c_j w_{j,x}(0,t), \\
  c_j w_{j,x}(1,t) = -\alpha_j w_{j,t}(1,t), j = 2, 3, \ldots, m, \\
  w_1(x,0) = w_{i,0}(x), w_{i,t}(x,0) = w_{i,1}(x).
\end{cases}
\]  

(19)

where \( c_i > 0, \alpha_i > 0, i = 2, \ldots, m. \)
As before, we set
\[
\begin{align*}
\xi_i(x,t) &= w_{i,t}(x,t) + c_i w_{i,x}(x,t), \\
\eta_i(x,t) &= w_{i,t}(x,t) - c_i w_{i,x}(x,t).
\end{align*}
\]
then
\[
\begin{align*}
\xi_{i,t}(x,t) &= c_i \xi_{i,x}(x,t), \\
\eta_{i,t}(x,t) &= -c_i \eta_{i,x}(x,t), \quad i = 1, 2, \ldots, m.
\end{align*}
\]
We define the weighted energy functional by
\[
V_i(t) = \int_0^1 (p_i(x)\xi_i^2(x,t) + q_i(x)\eta_i^2(x,t))dx,
\]
the weight functions \(p_i(x)\) and \(q_i(x)\) are defined as follows:
\[
p_i(x) = p_i e^{\gamma_i x}, \quad q_i(x) = q_i e^{-\gamma_i x},
\]
with \(p_i > 0, q_i > 0, i = 1, 2, \ldots, m.
\]
A direct calculation gives
\[
\dot{V}(t) = -\sum_{j=1}^{m} \beta_j V_j(t) + c_1 q_1 [w_{1,t}(0,t) - c_1 w_{1,x}(0,t)]^2
\]
\[
- c_1 p_1 [w_{1,t}(0,t) + c_1 w_{1,x}(0,t)]^2 + c_1 p_1 e^{\gamma_1} [w_{1,t}(1,t) + c_1 w_{1,x}(1,t)]^2
\]
\[
- c_1 q_1 e^{-\gamma_1} [w_{1,t}(1,t) - c_1 w_{1,x}(1,t)]^2 - \sum_{j=2}^{m} c_j p_j [w_{j,t}(0,t) + c_j w_{j,x}(0,t)]^2
\]
\[
+ \sum_{j=2}^{m} c_j q_j [w_{j,t}(0,t) - c_j w_{j,x}(0,t)]^2 + \sum_{j=2}^{m} c_j p_j e^{\gamma_j} [w_{j,t}(1,t) + c_j w_{j,x}(1,t)]^2
\]
\[
- \sum_{j=2}^{m} c_j q_j e^{-\gamma_j} [w_{j,t}(1,t) - c_j w_{j,x}(1,t)]^2,
\]
where \(\beta_i = \gamma_i c_i.
\]

For simplicity, we denote the connection parts by
\[
N(t) = c_1 q_1 [w_{1,t}(1,t) + c_1 w_{1,x}(1,t)]^2 - c_1 q_1 e^{-\gamma_1} [w_{1,t}(1,t) - c_1 w_{1,x}(1,t)]^2
\]
\[
- \sum_{j=2}^{m} c_j p_j [w_{j,t}(0,t) + c_j w_{j,x}(0,t)]^2 + \sum_{j=2}^{m} c_j q_j [w_{j,t}(0,t) - c_j w_{j,x}(0,t)]^2
\]
\[
(20)
\]
and the boundary parts by
\[
B(t) = c_1 q_1 [w_{1,t}(0,t) - c_1 w_{1,x}(0,t)]^2 - c_1 p_1 [w_{1,t}(0,t) + c_1 w_{1,x}(0,t)]^2
\]
\[
+ \sum_{j=2}^{m} c_j p_j e^{\gamma_j} [w_{j,t}(1,t) + c_j w_{j,x}(1,t)]^2 - \sum_{j=2}^{m} c_j q_j e^{-\gamma_j} [w_{j,t}(1,t) - c_j w_{j,x}(1,t)]^2
\]
\[
(21)
\]
Using the boundary conditions in (19), taking \(p_1 = q_1, c_i p_i = \mu_p\) and \(c_i q_i = \mu_q\),
\[
B(t) = \sum_{j=2}^{m} w_{j,t}^2(1,t) [\mu_p e^{\gamma_j} (1 - \alpha_j)^2 - \mu_q e^{-\gamma_j} (1 + \alpha_j)^2].
\]
Using the connection conditions in (19), we get
\[
N(t) = [2c_1 p_1 \sinh \gamma_1 + (m-1)(\mu_q - \mu_p)]c_1^2 w_{1,x}^2(1,t) + 2c_1 p_1 \sinh \gamma_1 c_1^2 w_{1,x}^2(1,t) + 2w_{1,x}(1,t)c_1 w_{1,x}(1,t) [2c_1 p_1 \cosh \gamma_1 - (\mu_p + \mu_q)] + (\mu_q - \mu_p) \sum_{j=2}^{m} c_j^2 w_{j,x}^2(0,t).
\]

When \(2c_1 p_1 \cosh \gamma_1 - (\mu_p + \mu_q) \geq 0\), it holds that
\[
N(t) \leq w_{1,x}^2(1,t) [2c_1 p_1 e^{\gamma_1} + (m-1)(\mu_q - \mu_p) - (\mu_p + \mu_q)] + [2c_1 p_1 e^{\gamma_1} - (\mu_p + \mu_q)] c_1^2 w_{1,x}^2(1,t) + (\mu_q - \mu_p) \sum_{j=2}^{m} c_j^2 w_{j,x}^2(0,t)
\]
\[
\leq w_{1,x}^2(1,t) [2c_1 p_1 e^{\gamma_1} + (m-1)(\mu_q - \mu_p) - (\mu_p + \mu_q)] + (m-1) \sum_{j=2}^{m} c_j^2 w_{j,x}^2(0,t) [2c_1 p_1 e^{\gamma_1} - (\mu_p + \mu_q) + (\mu_q - \mu_p)].
\]

Set \(\mu_p > \mu_q\) and
\[
2c_1 p_1 e^{\gamma_1} - (\mu_p + \mu_q) + \frac{(\mu_q - \mu_p)}{m-1} \leq 0,
\]
that implies
\[
2c_1 p_1 e^{\gamma_1} + (m-1)(\mu_q - \mu_p) - (\mu_p + \mu_q) \leq 0.
\]
Thus we need solve the following inequality
\[
\begin{align*}
\mu_p &> \mu_q, \quad \gamma_1 > 0, \\
2c_1 p_1 \cosh \gamma_1 - (\mu_p + \mu_q) &\geq 0, \\
2c_1 p_1 e^{\gamma_1} - (\mu_p + \mu_q) - \frac{(\mu_q - \mu_p)}{m-1} &\leq 0.
\end{align*}
\]
(22)

We can set
\[
\mu_p = (1 + (m-2)\varepsilon)c_1 p_1, \\
\mu_q = (1 - (m-2)\varepsilon)c_1 p_1,
\]
then
\[
\mu_p + \mu_q = 2c_1 p_1, \quad (\mu_p - \mu_q) = 2(m-2)\varepsilon c_1 p_1,
\]
and
\[
2c_1 p_1 e^{\gamma_1} - (\mu_p + \mu_q) = 2c_1 p_1 (e^{\gamma_1} - 1) \geq 0, \forall \gamma_1 > 0.
\]
Substituting the last inequality we get
\[
e^{\gamma_1} - 1 - \frac{m-2}{m-1} \varepsilon \leq 0.
\]
If we take
\[
\varepsilon = \frac{m-1}{m-2}(e^{\gamma_1} - 1), \quad (1 - (m-2)\varepsilon) > 0,
\]
we have
\[
\varepsilon = \frac{m-1}{m-2}(e^{\gamma_1} - 1) = \frac{1}{m-1} < \frac{1}{m-2}.
\]
From above we get
\[
\gamma_1 = \ln \frac{m^2 - m - 1}{(m-1)^2}, \quad \varepsilon = \frac{1}{m-1}.
\]
The following theorem gives the selection conditions of the parameters.
Theorem 3.2. Let the boundary control laws \( u_i, i = 2, \ldots, m \), be defined by (15). Suppose that \( p_i, q_i, i = 1, \ldots, m \), satisfy the following conditions:

\[
\begin{align*}
\begin{cases}
p_1 = q_1, \\
p_j c_j = p_1 c_1 (1 + \frac{m-2}{m-1}), \\
n_j c_j = p_1 c_1 (1 - \frac{m-2}{m-1}), \\
\gamma_1 \leq \ln \frac{m^2 - m - 1}{(m-1)^2}, \\
\gamma_j \leq \frac{1}{2} \ln \left( \frac{(1 + \alpha_j)^2}{(2m - 3)(1 - \alpha_j)^2} \right), j = 2, 3, \ldots, m.
\end{cases}
\end{align*}
\]

(23)

it holds that \( \dot{V}(t) \leq -\beta V(t) \), i.e., the exponential decay rate of the closed loop systems (19) is at least \( \frac{\beta}{2} \), and

\[
\frac{\beta}{2} = \min_{j=1, \ldots, m} \frac{\beta_i}{2} = \min_{j=1, \ldots, m} \frac{\gamma_j c_j}{2} = \min \left\{ \frac{c_j}{2} \ln \frac{m^2 - m - 1}{(m-1)^2}, \frac{1}{4} \ln \left( \frac{(1 + \alpha_j)^2}{(2m - 3)(1 - \alpha_j)^2} \right), j = 2, 3, \ldots, m \right\}
\]

4. Conclusions. In this paper, we use the weighted energy functional of the systems to estimate the decay rate of the wave networks. Under the boundary velocity feedback control laws, we calculate the decay rate \( \beta \) of the weighted energy functional, and hence the system has decay rate estimate \( \frac{\beta}{2} \). The key point of this approach is to select the parameters such that boundary parts and the connection parts are less or equal to zero. This will lead to solve an inequality group for the parameters. By a suitable choices of parameters, we can get the estimate of decay rate of the systems. Usually, if the parameters \( \gamma_j \) are selected suitable large, we can get more accurate estimate for the decay rate of the system. But according to the current method, we notice that the parameters \( \gamma_j \) are more and more small, and may tend to zero. Because we can’t get accurate decay rate for complicated tree-shaped wave networks by spectral methods, in next step, we will analyze whether the parameters \( \gamma_j \) are smaller and smaller with the increase of \( m \). This problem is worth thinking.

REFERENCES

[1] K. Ammari and M. Jellouli, Stabilization of star-shaped tree of elastic strings, Differential & Integral Equations, 17 (2004), 1395–1410.
[2] K. Ammari, M. Jellouli and M. Khenissi, Stabilization of generic trees of strings, Journal of Dynamical & Control Systems, 11 (2005), 177–193.
[3] K. Ammari and D. Mercier, Boundary feedback stabilization of a chain of serially connected strings, Evol. Equ. Control Theory, 4 (2015), 1–19.
[4] K. Ammari, D. Mercier and V. Régnier, Spectral analysis of the Schrödinger operator on binary tree-shaped networks and applications, Journal of Differential Equations, 259 (2015), 6923–6959.
[5] K. Ammari, A. Henrot and M. Tucsnak, Asymptotic behaviour of the solutions and optimal location of the actuator for the pointwise stabilization of a string, Asymptotic Analysis, 28 (2001), 215–240.
[6] K. Bartecki, A general transfer function representation for a class of hyperbolic distributed parameter systems, International Journal of Applied Mathematics & Computer Science, 23 (2013), 291–307.
[7] A. V. Balakrishnan, On superstable semigroup of operators, Dynamic Systems & Applications, 5 (1996), 371–384.
[8] J. M. Coron, B. D’Andrea-Novel and G. Bastin, A lyapunov approach to control irrigation canals modeled by the Saint Venant equations, Control Conference. IEEE, 1999.
[9] J. M. Coron, J. De Halleux and G. Bastin, On boundary control design for quasi-linear hyperbolic systems with entropies as Lyapunov functions, Proceedings of the IEEE Conference on Decision and Control, 3 (2003), 3010–3014.
[10] J. M. Coron, B. d’Andrea-Novel and G. Bastin, A strict Lyapunov function for boundary control of hyperbolic systems of conservation laws, Proceedings of the IEEE Conference on Decision & Control, 52 (2007), 2–11.
[11] R. F. Curtain and H. J. Zwart, An Introduction to Infinite-Dimensional Linear Systems Theory, Springer–Verlag, New York, Berlin, Heidelberg, 1995.
[12] S. Cox and E. Zuazua, The rate at which energy decays in a damped string, Communication Partial Differential Equations, 19 (1994), 213–243.
[13] R. Dager and E. Zuazua, Controllability of star-shaped networks of strings, C. R. Acad. Sci. Paris, Series I, 332 (2001), 621–626.
[14] R. Dager, Observation and control of vibrations in tree-shaped networks of strings, SIAM J. Control & Optim, 43 (2004), 590–623.
[15] R. Dager and E. Zuazua, Wave Propagation, Observation and Control in 1-d Flexible Multi-Structures, Springer Berlin, 2006.
[16] A. Diagne, G. Bastin and J. M. Coron, Lyapunov exponential stability of 1-D linear hyperbolic systems of balance laws, Automatica, 48 (2012), 109–114.
[17] Y. N. Guo and G. Q. Xu, Stability and Riesz basis property for general network of strings, Journal of dynamical & control systems, 15 (2009), 223–245.
[18] Y. N. Guo and G. Q. Xu, Exponential stabilization of a tree shaped network of strings with variable coefficients, Glasgow Mathematical Journal, 53 (2011), 481–499.
[19] Y. N. Guo and G. Q. Xu, Exponential stabilization of variable coefficient wave equations in a generic tree with small time-delays in the nodal feedbacks, Journal of Mathematical Analysis & Applications, 395 (2012), 727–746.
[20] B. Z. Guo and Y. Xie, A sufficient condition on Riesz basis with parentheses of non-selfadjoint operator and application to a serially connected string system under joint feedbacks, SIAM journal on control & optimization, 43 (2004), 1234–1252.
[21] M. Gugat and M. Herty, Existence of classical solutions and feedback stabilization for the flow in gas networks, ESAIM Control Optimisation & Calculus of Variations, 17 (2011), 28–51.
[22] F. L. Huang, Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces, Ann. Differential Equations, 1 (1985), 43–56.
[23] Z. J. Han and L. Wang, Riesz basis property and stability of planar networks of controlled strings, Acta Applicandae Mathematicae, 110 (2010), 511–533.
[24] Z. J. Han and G. Q. Xu, Spectrum and dynamical behavior of a kind of planar network of non-uniform strings with non-collocated feedbacks, Networks & Heterogeneous Media, 5 (2010), 315–334.
[25] Z. J. Han and G. Q. Xu, Stabilization and SDG condition of serially connected vibrating strings system with discontinuous displacement, Asian Journal of Control, 14 (2012), 95–108.
[26] M. Jellouli, Spectral analysis for a degenerate tree and applications, International Journal of Control, 88 (2015), 1647–1662.
[27] M. Krstic, B. Z. Guo and A. Balogh, Output-feedback stabilization of an unstable wave equation, Automatica, 44 (2008), 63–74.
[28] J. E. Lagnese, G. Leugering and E. J. P. G. Schmidt, Modelling, Analysis and Control of Dynamic Elastic Multi-Link Structures, Systems & Control: Foundations & Applications, 1994.
[29] J. E. Lagnese, Recent progress and open problems in control of multi-link elastic structures, Contemp. Math, 209 (1997), 161–175.
[30] J. L. Lions, Exact controllability, stabilization and perturbations for distributed parameter system, SIAM Review, 30 (1988), 1–68.
[31] K. S. Liu, F. L. Huang and G. Chen, Exponential stability analysis of a long chain of coupled vibrating strings with dissipative linkage, SIAM Journal on Applied Mathematics, 49 (1989), 1694–1707.
[32] G. Leugering and E. Zuazua, On exact controllability of generic trees, Esaim Proceedings, 8 (2000), 95–105.
[33] G. Leugering, Dynamic domain decomposition of optimal control problems for networks of strings and Timoshenko beams, SIAM Journal on Control & Optimization, 37 (1999), 1649–1675.
[34] D. Y. Liu, Y. F. Shang and G. Q. Xu, Design of controllers and compensators of a serially connected string system and its Riesz basis, Kongzhi Lilun Yu Yinyong/Control Theory & Applications, 5 (2008), 815–818.

[35] M. Najafi, G. R. Sarhangi and H. Wang, Stabilizability of coupled wave equations in parallel under various boundary conditions, Automatic Control IEEE Transactions on, 42 (1997), 1308–1312.

[36] S. Nicaise and J. Valein, Stabilization of the wave equation on 1-D networks with a delay-term in the feedbacks, Networks & Heterogeneous Media, 2 (2007), 425–479.

[37] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Berlin, Germany: Springer-Verlag, 1983.

[38] D. L. Rusell, Mathematical models for the elastic beam and their control-theoretic implications, Semigroups, Theory and Applications, 152 (1986), 177–216.

[39] S. Rolewicz, On controllability of systems of strings, Studia Math, Studia Mathematica, 36 (1970), 105–110.

[40] Y. F. Shang, D. Y. Liu and G. Q. Xu, Super-stability and the spectrum of one-dimensional wave equations on general feedback controlled networks, IMA Journal of Mathematical Control & Information, 31 (2014), 73–99.

[41] M. A. Shubov, Basis property of eigenfunctions of nonselfadjoint operator pencils generated by the equations of nonhomogeneous damped string, Integral Equations & Operator Theory, 25 (1996), 289–328.

[42] E. J. P. G. Schmidt, On the modelling and exact controllability of networks of vibrating strings, Siam Journal on Control & Optimization, 30 (1992), 229–245.

[43] L. Wang, Z. J. Han and G. Q. Xu, Exponential stability of serially connected thermoelastic system of type II with nodal damping, Applicable Analysis, 93 (2014), 1495–1514.

[44] H. Wang and G. Q. Xu, Exponential stabilization of 1-d wave equation with input delay, Wseas Transactions on Mathematics, 10 (2013), 1001–1013.

[45] G. Q. Xu, S. P. Yung and L. K. Li, Stabilization of wave systems with input delay in the boundary control, ESAIM Control Optimisation & Calculus of Variations, 12 (2006), 770–785.

[46] G. Q. Xu, Stabilization of string system with linear boundary feedback, Nonlinear Analysis Hybrid Systems, 1 (2007), 383–397.

[47] G. Q. Xu and B. Z. Guo, Riesz basis property of evolution equations in Hilbert spaces and application to a coupled string equation, SIAM Journal on Control & Optimization, 42 (2003), 966–984.

[48] G. Q. Xu, D. Y. Liu and Y. Q. Liu, Abstract second order hyperbolic system and applications to controlled network of strings, SIAM Journal on Control & Optimization, 47 (2008), 1762–1784.

[49] G. Q. Xu and N. E. Mastorakis, Differential Equations on Metric Graph, Athens: World Scientific and Engineering Academy and Society, 2010.

[50] G. Q. Xu and Y. X. Zhang, The exponential stability of complex differential networks, Journal of Systems Science & Mathematical Sciences, 29 (2009), 1399–1418.

[51] E. Zuazua, Control and stabilization of waves on 1-d networks, Modelling and Optimisation of Flows on Networks, Springer Berlin Heidelberg, 2062 (2013), 463–493.

[52] Y. X. Zhang and G. Q. Xu, Controller design for Bush-type 1-D wave networks, ESAIM Control Optimisation & Calculus of Variations, 18 (2012), 208–226.

[53] Y. X. Zhang and G. Q. Xu, A new approach for the stability analysis of wave networks, Abstract and Applied Analysis, 2014 (2014), Art. ID 724512, 10 pp.

[54] Y. X. Zhang and G. Q. Xu, Exponential and super stability of a wave network, Acta Applicandae Mathematicae An International Survey Journal on Applying Mathematics & Mathematical Applications, 124 (2013), 19–41.

Received October 2014; revised February 2016.
E-mail address: yaruxie@126.com
E-mail address: gqxu@tju.edu.cn