Distorted Torsion Tensor, Teleparallelism and Spin 2 Field Equations

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Abstract

A notion of distorted torsion tensor was introduced by Okubo, in the establishment of the Nijenhuis-Bianchi identity and of BRST-like operators. These quantities are constructed with the help of the Nijenhuis tensor, which in turn is defined in terms of a (1,1) tensor $S^{\lambda}_{\mu}$. This tensor enters the construction of the distorted torsion tensor. We use this tensor to extend the teleparallel equivalent of general relativity (TEGR) into a theory defined by the tetrad fields and by the tensor $S^{\lambda}_{\mu}$. The ordinary TEGR is recovered if $S^{\lambda}_{\mu} = \delta^{\lambda}_{\mu}$. We consider the flat space-time formulation of the theory, in terms of $S^{\lambda}_{\mu}$ only, and show that this tensor satisfies the wave equation for massless spin 2 fields.

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1 Introduction

The tools of differential geometry have been applied to the study of dynamical
integrable systems \cite{1,2}. In this context, a quantity that plays a major role
is the Nijenhuis tensor \cite{3,4}, which has been considered in connection to the
formulation of integrable models \cite{5}. The vanishing of the Nijenhuis tensor
yields interesting properties in manifolds with dual symplectic structures.
One of these properties is the emergence of a number of conserved quantities
in involution, that are necessary for the complete integrability of certain
dynamical systems \cite{1,2}. The Nijenhuis tensor is defined in any differentiable
manifold $M$ of dimension $D$. In components, it reads \cite{4}

$$N^\lambda_{\mu\nu} = S^\alpha_\mu \partial_\alpha S^\lambda_\nu - S^\alpha_\nu \partial_\alpha S^\lambda_\mu - S^\lambda_\alpha \left( \partial_\mu S^\alpha_\nu - \partial_\nu S^\alpha_\mu \right), \quad (1)$$

where $S^\lambda_\mu$ is a mixed $(1,1)$ tensor on $M$ that is usually related to the dual
symplectic structures on the manifold. The tensor $N^\lambda_{\mu\nu}$ is entirely independent of any connection. In the context of complex manifolds, the vanishing of the
Nijenhuis tensor is related to the existence of integrable almost complex
structures \cite{4}. We are not aware of the applicability of this tensor in the
framework of gravity theories.

One motivation for considering the Nijenhuis tensor in a geometrical
framework is the construction of BRST-like operators. BRST operators \cite{6}
are the building blocks of the called BRST quantization, which is specially
applied to gauge theories. In particular, this quantization approach yields a
rigorous canonical quantization of Yang-Mills theories. Okubo \cite{7,8} investigated the existence of BRST-like operators in certain manifolds. He found
that the existence of these operators depends (i) on the vanishing of both
the Nijenhuis and Riemann tensors, and (ii) on a geometrical identity which
he calls the Nijenhuis-Bianchi identity. The construction of this identity is
based on an extension of the standard torsion tensor. Okubo defines dis-
torted torsion tensors of the first and second kinds. Here, we will consider
the distorted torsion tensor of the first kind only. It is established with the help of the $S^\lambda_\mu$ tensor, and reads \cite{7}

$$T^\lambda_{\mu\nu} = \partial_\mu S^\lambda_\nu - \partial_\nu S^\lambda_\mu + \Gamma^\lambda_{\mu\sigma} S^\sigma_\nu - \Gamma^\lambda_{\nu\sigma} S^\sigma_\mu
= \nabla_\mu S^\lambda_\nu - \nabla_\nu S^\lambda_\mu + S^\lambda\sigma T^\sigma_{\mu\nu}, \quad (2)$$

where $T^\sigma_{\mu\nu} = \Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\mu}$, and $\Gamma^\sigma_{\mu\nu} = \epsilon^{\alpha\sigma} \partial_\mu e_{\alpha\nu}$ is the Weitzenböck connection.
The covariant derivative $\nabla_\mu$ is constructed out of $\Gamma^\sigma_{\mu\nu}$. Note that $T^\lambda_{\mu\nu}$ reduces to $T^\lambda_{\mu\nu}$ if $S^\lambda_{\mu} = \delta^\lambda_{\mu}$.

Note: $\mu, \nu, ...$ are space-time indices and run from 0 to 3, $\mu = (0, i)$, where $i = 1, 2, 3$; $a, b, c, ...$ are Lorentz (tangent space) indices and also run from 0 to 3, $a = ((0), (i))$.

In this article we will investigate the consequences of the distorted torsion tensor (2) in the framework of the teleparallel equivalent of general relativity (TEGR). For this purpose, we will abandon the usual relation of the tensor $S^\lambda_{\mu}$ with the dual symplectic structures of integrable models (if $f_{\mu\nu} = -f_{\nu\mu}$ and $F_{\mu\nu} = -F_{\nu\mu}$ are two symplectic forms on a manifold, then one may identify $S^\lambda_{\mu} = f_{\mu\rho}F^{\rho\lambda}$ in the context of integrable dynamical systems [1]). We will treat $S^\lambda_{\mu}$ as a regular $(1, 1)$ tensor. Quantities similar to eq. (2) may possibly generalize the TEGR to theories that incorporate spin 1 or spin 2 fields with non-trivial couplings with the tetrad fields.

In section 2 we establish the first Lagrangian formulation of the extended TEGR, based on the distorted torsion tensor (2), and obtain the field equations for both the tetrad fields and the tensor $S^\lambda_{\mu}$. In order to probe the nature of the tensor $S^\lambda_{\mu}$ in the present gravitational context, we consider a TEGR-type theory with only the $S^\lambda_{\mu}$ tensor, i.e., in flat space-time. It turns out that this tensor obeys the standard wave equation, without assuming any linearisation of the field variables. Thus, in this simplified formulation, the tensor $S^\lambda_{\mu}$ describes propagating massless spin 2 fields. We discuss some features and limitations of this model, regarding the flat space-time limit, and then in sections 3 and 4 we present the second and third models where the flat space-time limit is obtained in a more conventional way. Finally, in section 5 we present the final comments.

2 The extended TEGR - the first model

Teleparallelism is a geometrical framework where the notion of distant parallelism is well defined. In order to understand this notion, let us consider a vector field $V^\mu(x)$ in space-time. The projection of this vector on a certain frame, in the tangent space at the position $x^\alpha$, is given by $V^\alpha(x^\alpha) = e^a_\mu(x^\alpha) V^\mu(x^\alpha)$. At the position $x^\alpha + dx^\alpha$, we have $V^\alpha(x^\alpha + dx^\alpha) = e^a_\mu(x^\alpha + dx^\alpha) V^\mu(x^\alpha + dx^\alpha)$. It is easy to show that $V^\alpha(x^\alpha)$ and $V^\alpha(x^\alpha + dx^\alpha)$ are parallel, i.e., $V^\alpha(x^\alpha) = V^\alpha(x^\alpha + dx^\alpha)$, if the covariant derivative $\nabla_\mu V^\lambda$ vanishes. This covariant
derivative is constructed out of the Weitzenböck connection. Therefore, the equation \( \nabla_\mu V^\lambda = 0 \) establishes a condition of distant parallelism in space-time, and the Weitzenböck connection plays a crucial role in this concept. It is easy to verify that the tetrad fields \( e^a_\mu \), and a quadratic combination of the torsion tensor of the Weitzenböck connection, \( T^a_\mu \nu \),

\[
\nabla_\mu V^\lambda = 0 \text{ establishes a condition of distant parallelism in space-time, and the Weitzenböck connection plays a crucial role in this concept. It is easy to verify that the tetrad fields } e^a_\mu , \text{ and a quadratic combination of the torsion tensor of the Weitzenböck connection, } T^a_\mu \nu , \text{,}
\]

\[
\text{Neglecting the total divergence, the Lagrangian density for the TEGR in asymptotically flat space-times is given by } [9, 10]
\]

\[
L(e) = -k e \, \left( \frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - T^a_{\mu} T^\mu_a \right) - \frac{1}{c} L_M
\]

\[
\equiv -k e \Sigma^{abc} T_{abc} - \frac{1}{c} L_M,
\]

where \( k = c^3/(16\pi G) \), \( L_M \) represents the Lagrangian density for the matter fields, and \( \Sigma^{abc} \) is defined by [9]

\[
\Sigma^{abc} = \frac{1}{4} \left( T^{abc} + T^{bac} - T^{cab} \right) + \frac{1}{2} \left( \eta^{ac} T^b - \eta^{ab} T^c \right).
\]

The extended, generalized form of the TEGR is constructed by replacing \( T^a_\mu \nu \) with \( T^a_\mu \nu \), the latter given by eq. (2). Thus, the vacuum formulation of the theory is determined by the Lagrangian density constructed out of \( e^a_\mu \) and of the tensor \( S^\lambda_\mu \), and reads

\[
L(e^a_\mu, S^\lambda_\mu) = -k \Sigma^{abc} T_{abc},
\]

where \( \Sigma^{abc} \) is now given by

\[
\Sigma^{abc} = \frac{1}{4} \left( T^{abc} + T^{bac} - T^{cab} \right) + \frac{1}{2} \left( \eta^{ac} T^b - \eta^{ab} T^c \right).
\]
\[ e_{b\mu} \partial_\sigma (e \Sigma^{b\nu} S_{\nu}^\mu e_{a\rho}) - \frac{1}{2} (e \Sigma^\sigma_{\mu} e_{a\lambda} T^{\lambda}_{\sigma\nu}) + e \Sigma_{\lambda a}^\nu T^{\lambda}_{\mu\nu} - \frac{1}{4} e e_{a\mu} \Sigma^{bcd} T_{bcd} = 0, \] (8)

\[ \nabla_\sigma (e \Sigma^\lambda_{\sigma\mu}) = \partial_\sigma (e \Sigma^\lambda_{\sigma\mu}) - \Gamma^\rho_{\sigma\lambda} (e \Sigma^\rho_{\sigma\mu}) = 0. \] (9)

We note that if we enforce \( S^\lambda_{\mu} = \delta^\lambda_{\mu} \) in eq. (8), we obtain

\[ e^\nu e_{b\mu} \partial_\sigma (e \Sigma^{b\nu} S_{\nu}^\mu e_{a\rho}) - e (\Sigma_{\lambda a}^\nu T^{\lambda}_{\mu\nu} - \frac{1}{4} e e_{a\mu} \Sigma^{bcd} T_{bcd}) = 0, \] (10)

which are the vacuum space-time field equations for the tetrad field in the TEGR, and which are equivalent to Einstein’s equations in vacuum [10]. However, this enforcement \( (S^\lambda_{\mu} = \delta^\lambda_{\mu}) \) is not so trivial, as we will see below.

In order to probe the nature of the field \( S^\lambda_{\mu} \) in the equations above, let us assume \( e^a_{\mu}(t, x, y, z) = \delta^a_{\mu} \) in the field equation (9). We remark that this is not a flat space-time limit of the theory, but just a flat space-time formulation of a TEGR-type theory, where eq. (2) reduces to \( T^{\lambda}_{\mu\nu} = \partial_\mu S^\lambda_{\nu} - \partial_\nu S^\lambda_{\mu} \), and eqs. (6) and (7) remain unchanged. Thus, this is a new theory defined in flat space-time. In this theory, the field equations for \( S^\lambda_{\nu} \) are

\[ \nabla_\rho (e \Sigma^\lambda_{\rho\nu}) = \partial_\rho (e \Sigma^\lambda_{\rho\nu}) = 0. \] (11)

Before we proceed, let us establish the following convention:

\[ S^\lambda_{\mu} \longrightarrow S^\lambda_{-\mu}. \] (12)

As a consequence, we have \( S^{\lambda\mu} = \eta^{\mu\rho} S^\lambda_{\rho} \), and \( S_{\alpha\mu} = \eta_{\alpha\lambda} S^\lambda_{\mu} \), where \( \eta_{\mu\nu} = (-1, +1, +1, +1) \) is the flat Minkowski metric tensor. With the help of the notation above, and in view of eq. (7), the field equations (11) read

\begin{align*}
\partial_\rho \Sigma^{\lambda\rho\nu} &= \frac{1}{4} (\partial_\rho \partial^\rho) (S^{\nu\lambda} + S^{\lambda\nu}) \\
&\quad - \frac{1}{4} (\partial_\rho \partial^\nu S^{\lambda\rho} + \partial_\rho \partial^\lambda S^{\nu\rho} + \partial_\rho \partial^\nu S^{\rho\lambda} + \partial_\rho \partial^\nu S^{\nu\rho}) \\
&\quad + \frac{1}{2} (\eta^{\lambda\nu} \partial_\rho \partial_\sigma S^{\rho\sigma} - \eta^{\lambda\nu} (\partial_\rho \partial^\nu) S^{\sigma\rho} + \partial^\lambda \partial^\nu S^{\rho\rho}) \\
&= 0,
\end{align*}
(13)
where $\partial^\lambda = \eta^{\lambda\mu} \partial_\mu$.

We note that the left hand side of the equation above is naturally symmetric in the indices $(\lambda\nu)$. Therefore, the antisymmetric part of the tensor $S^{\lambda\nu}$ is completely undetermined from the field equations. For the time being, and without loss of generality, we will assume in the present context that the tensor $S^{\lambda\nu}$ is symmetric, i.e., $S^{\lambda\nu} = S^{\nu\lambda}$.

Equation (13) may be simplified by first implementing the symmetry condition $S^{\lambda\nu} = S^{\nu\lambda}$, and then by contracting it with the metric tensor $\eta_{\lambda\nu}$. This last operation yields

$$\left(\partial_\rho \partial^\rho\right) S^{\rho\lambda} = \partial_\rho \partial_\lambda S^{\rho\lambda}.$$ \hspace{1cm} (14)

As a consequence, eq. (13) simplifies to

$$\left(\partial_\rho \partial^\rho\right) S^{\lambda\nu} - \partial_\rho \partial_\nu S^{\rho\lambda} - \partial_\lambda S^{\rho\rho} + \partial^\lambda \partial^\rho S^{\rho\rho} = 0.$$ \hspace{1cm} (15)

It is interesting to note that the left hand side of the equation above is similar to the linearised Ricci tensor for $h_{\mu\nu}$, which is considered in the investigation of linearised gravitational waves, where $g_{\mu\nu} \simeq \eta_{\mu\nu} + h_{\mu\nu}$ (see eq. (20.10) of Ref. [11]). In contrast, eq. (15) is exact, as it does not follow from any linearisation procedure.

It is very easy to verify that eq. (15) is invariant under the gauge transformation

$$S'^{\lambda\mu} = S^{\lambda\mu} + \partial^\lambda V^\mu + \partial^\mu V^\lambda,$$ \hspace{1cm} (16)

where $V^\lambda(x)$ is an arbitrary vector field. This vector field may be used to fix the de Donder, or Einstein, or Hilbert, or Fock gauge [11],

$$\partial_\rho S'^{\rho\lambda} - \frac{1}{2} \partial^\lambda S'^{\rho\rho} = 0,$$ \hspace{1cm} (17)

which is obtained by requiring the vector field $V^\lambda$ to satisfy

$$\left(\partial_\rho \partial^\rho\right) V^\lambda = -\partial_\rho S^{\lambda\rho} + \frac{1}{2} \partial^\lambda S^{\rho\rho}.$$ \hspace{1cm} (18)

Formally, there exists a solution to the equation above for $V^\lambda$, and therefore the gauge (17) is consistent. With the imposition of this gauge, the field equations (15) reduce to

$$\left(\partial_\rho \partial^\rho\right) S^{\lambda\nu} = 0.$$ \hspace{1cm} (19)
where we have dropped the prime. This is the wave equation for a massless spin 2 field, but not for a pure spin 2 field \[12\], as the trace $S^{\rho}_{\rho}$ is non-vanishing. Since $S^\lambda_{\mu} = \delta^\lambda_{\mu}$ is a condition that leads eq. (8) to eq. (10), solutions of eq. (19) may have the form

$$S_{\lambda\mu} = \eta_{\lambda\mu} + \text{wave solution}. \quad (20)$$

The formalism presented above allows to make a clear distinction between the propagation of non-linear gravitational waves, that arise from eq. (10), and of spin 2 fields (gravitons, in a quantum theory), an issue that is not satisfactorily addressed in the standard formulation of general relativity. In the framework of the coupled field equations (8) and (9), a source of the field $S^\lambda_{\mu}$ is the dynamical geometry of the space-time, represented by time dependent tetrad fields $e^a_{\mu}$.

It is certainly not easy to obtain a solution for $S^\lambda_{\mu}$ of the coupled equations (8) and (9). It is necessary to make some simplifications and approximations to handle these equations. Equation (9) is a wave equation for $S^\lambda_{\mu}$ in the presence of a non-trivial set of tetrad fields, i.e., in a non-flat space-time. In the context of the coupled equations (8) and (9), $S^\lambda_{\mu}$ should be part of the geometry, as we conclude from eq. (20). In particular, the tensor

$$S_{ab} = e^a_{\lambda} e^b_{\mu} S_{\lambda\mu} , \quad (21)$$

could be an effective metric tensor for the tangent space, i.e., an extension of the flat Minkowski (tangent space) metric tensor $\eta_{ab} = (-1, +1, +1, +1)$, which includes oscillations (fluctuations) of the background geometry. This interpretation is physically possible at least in the context of weak gravitational fields. Such background oscillations could be similar to (continuous) gravitational waves that permeate the universe, and even to the background noises that are significant in the detection of gravitational waves, but otherwise very weak to be characterized in ordinary circumstances.

We note that $e^a_{\mu} = \delta^a_{\mu}$ and $S^\lambda_{\mu} = \delta^\lambda_{\mu}$ are, together, solutions of the field equations (8) and (9). However, it seems that neither $e^a_{\mu} = \delta^a_{\mu}$ alone, nor $S^\lambda_{\mu} = \delta^\lambda_{\mu}$ alone, are solutions of the field equations. Thus, the flat space-time (the vacuum) is also determined by the condition $S^\lambda_{\mu} = \delta^\lambda_{\mu}$. By enforcing $S^\lambda_{\mu} = \delta^\lambda_{\mu}$, without requiring $e^a_{\mu} = \delta^a_{\mu}$, eq. (9) imposes further, additional restrictions on the tetrad fields, that may lead to inconsistencies. It is not impossible that the framework described above is physically correct, but certainly it is not sufficiently convincing, at least at present. Thus, in the
next section we will address the issue of how to obtain the limit $S^\lambda_\mu \to \delta^\lambda_\mu$, independently of the flat space-time limit, by formulating a second, more conventional model.

3 The second model

The aim of this section is to speculate on another extended version of the TEGR where the limit $S^\lambda_\mu \to \delta^\lambda_\mu$ may be obtained independently of the flat space-time limit $e^a_\mu \to \delta^a_\mu$. The converse does not hold, since a non-vanishing spin 2 field, or any other form of matter fields, is always a source to the gravitational field equations. The model to be discussed below is a more conservative description of the physical system.

Let us first establish the notation. In this section, we will use the following definitions:

$$\Sigma^{abc} = \frac{1}{4} \left( T^{abc} + T^{bac} - T^{cab} \right) + \frac{1}{2} \left( \eta^{ac} T^{b} - \eta^{ab} T^{c} \right),$$

(22)

$$\Phi^{abc} = \frac{1}{4} \left( T^{abc} + T^{bac} - T^{cab} \right) + \frac{1}{2} \left( \eta^{ac} T^{b} - \eta^{ab} T^{c} \right),$$

(23)

$$T^a_\mu_\nu = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu,$$

(24)

$$T^\lambda_\mu_\nu = \nabla_\mu S^\lambda_\nu - \nabla_\nu S^\lambda_\mu.$$  

(25)

Equation (25) replaces eq. (2). As before, the tetrad fields convert space-time into Lorentz indices, and vice-versa. The covariant derivatives are constructed out of the Weitzenböck connection, $\Gamma^\lambda_\mu_\sigma = e^a_\mu \partial_\mu e^a_\nu$. Thus, we have

$$\nabla_\mu S^\lambda_\nu = \partial_\mu S^\lambda_\nu + \Gamma^\lambda_\mu_\sigma S^\sigma_\nu - \Gamma^\sigma_\mu_\nu S^\lambda_\sigma.$$  

(26)

Note that if we make $S^\lambda_\mu = \delta^\lambda_\mu$, the covariant derivative vanishes. This feature is important for ensuring that in the limit $S^\lambda_\mu \to \delta^\lambda_\mu$ we obtain the standard TEGR in vacuum, in the equations below.

The theory to be considered in this section is defined by the Lagrangian density

$$L = -ke\Sigma^{abc} T_{abc} - ke\Phi^{abc} T_{abc}.$$  

(27)

Variations of $L$ with respect to $S^\lambda_\mu$ and $e^{a_\mu}$ yield the field equations
\[ \nabla_\mu (e \Phi_\lambda ^{\mu \nu}) = 0 , \tag{28} \]

where
\[ \nabla_\mu (e \Phi_\lambda ^{\mu \nu}) = \partial_\mu (e \Phi_\lambda ^{\mu \nu}) - e \Gamma^\sigma_{\mu \lambda} \Phi_\sigma ^{\mu \nu} + \Gamma^\nu_{\mu \sigma} \Phi_\lambda ^{\mu \sigma} , \tag{29} \]

and
\[ e_{a \nu} e_{b \mu} \partial_\sigma (e \Sigma^\lambda_{\sigma \nu}) - e (\Sigma_{\lambda a} ^{\nu} T^\lambda _{\mu \nu} - \frac{1}{4} e_{a \mu} e_{b \nu} T_{bcd}) = - \frac{1}{2} e_{a \lambda} \Phi_\mu ^{\rho \nu} T^\lambda _{\rho \nu} + e \Phi_\lambda ^{\rho \nu} a T_{\lambda \mu \nu} - \frac{1}{4} e_{a \mu} e_{b \nu} T_{bcd} \]
\[ + \nabla_\rho [e_{a \sigma} (\Phi_\mu ^{\rho \sigma} S^\lambda _{\sigma \nu} - \Phi_\nu ^{\rho \sigma} S^\mu _{\nu \lambda})] . \tag{30} \]

The left hand side of the eq. (28) is equivalent to Einstein’s equations [10]. In view of eq. (26), we see that the right hand side of this equation vanishes in the limit \( S_{\lambda \mu} \rightarrow \delta_{\lambda \mu} \), and thus we obtain the expected equations for the tetrad field in vacuum. According to eq. (30), the spin 2 field is a source to the gravitational field, as any other kind of matter fields.

By enforcing \( e_{a \mu} \rightarrow \delta^a_\mu \) in eq. (28) (i.e., assuming that the theory is formulated in flat space-time in terms of \( S_{\lambda \mu} \) only, as in section 2), we obtain from (28) the equation
\[ \partial_\mu \Phi_\lambda ^{\mu \nu} = 0 , \tag{32} \]
which reminds Maxwell’s equations in covariant form, and which is strictly equivalent to eq. (11). Therefore, in the present context, we also obtain eqs. (13), (15), and ultimately eq. (19),
\[ (\partial_\rho \partial^\rho) S^{\lambda \nu} = \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) S^{\lambda \nu} = 0 . \tag{33} \]

Thus, the field \( S^{\lambda \mu} \) again satisfies the wave equation in the flat space-time formulation. In addition, the left hand side of eq. (28) vanishes in the limit
\[ S^\lambda_{\mu} \to \delta^\lambda_{\mu}. \] This is the feature that characterizes the theory defined by eq. \( (27) \).

By contracting eq. \( (30) \) with \( e^\alpha_{\, \beta} \), the left hand side of the equation becomes symmetric in the indices \( \beta \mu \). This fact can be verified by transforming the resulting tensor into Einstein’s tensor, which is symmetric. However, it is not immediately clear that the right hand side of \( (30) \) is also symmetric. If it turns out that the right hand side of \( (30) \) is not symmetric, the antisymmetric part imposes conditions on 6 of the 16 components of the tetrad fields. This issue might be a novel feature, and will be investigated elsewhere.

### 4 The third model

The field equations resulting from the variation of the Lagrangian with respect to the tetrad fields are symmetric (in the \( \beta \mu \) indices, according to the discussion at the end of section 3) if we replace the Weitzenböck connection in eq. \( (25) \) by the Christoffel symbols \( \Gamma^\lambda_{\mu\nu} \). Therefore, in this section we also assume eqs. \( (22), (24), (24) \) and \( (25) \) of section 3, as well as the Lagrangian density \( (27) \), but eq. \( (25) \) is now constructed out of the Christoffel symbols. This change characterizes the third model. The field equations of this model, resulting from variations of the Lagrangian with respect to \( S^\lambda_{\mu} \) and \( e^a_{\, \mu} \), are given by

\[ \nabla_\mu (e \Phi^\lambda_{\mu\nu}) = 0, \quad (34) \]

where

\[ \nabla_\mu (e \Phi^\lambda_{\mu\nu}) = \partial_\mu (e \Phi^\lambda_{\mu\nu}) - e \Gamma^{\lambda}_{\nu} (e \Phi^\sigma_{\mu\nu}), \quad (35) \]

and

\[
e_{ab} e_{b\nu} \partial_\sigma (e \Sigma_{\nu}^{\lambda\sigma}) - e (\Sigma_{\lambda a} T^\lambda_{\mu\nu} - 1/4 e_{ab} \Sigma^{bcd} T_{bcd}) \]
\[= -1/4 e_{ab} \Phi^{\lambda\rho\sigma} T_{\lambda\rho\sigma} - 1/2 e_a^\lambda E_{(\lambda\mu)}, \quad (36)\]

where

\[
E_{\lambda\mu} = e (\Phi^{\nu}_{\lambda} T_{\mu\nu} - 2 \Phi^{\rho\sigma}_{\mu} T_{\rho\sigma\lambda})
\]
\[ -2 g_{\lambda\alpha} g_{\mu\sigma} (\Phi^{\alpha\nu\beta} S_{\nu\beta} + \Phi^{\alpha\nu\beta} S_{\nu\beta}) \partial^\sigma_{\mu\nu} \]
\[ -2 g_{\lambda\alpha} g_{\mu\sigma} \left[ \partial_{\nu} \left( e \left( \Phi^{\alpha\nu\beta} S_{\sigma\beta} + \Phi^{\alpha\nu\beta} S_{\sigma\beta} \right) \right) \right] \]
\[ - \frac{1}{2} \partial_{\nu} \left( e \left( \Phi^{\nu\alpha\beta} S_{\sigma\beta} + \Phi^{\nu\sigma\beta} S_{\alpha\beta} \right) \right) \], \quad (37)

In the last quantity on the right hand side of eq. (36), the symbol \((\lambda\mu)\) means symmetrization in the indices \(\lambda\) and \(\mu\).

As expected, the field equations (34) are equivalent to eqs. (11) and (28). Thus, eq. (34) reduces to eqs. (13), (15) and (19) in the flat space-time formulation of the theory, and ultimately eq. (34) describes the propagation of massless spin 2 fields. Similarly to the model in section 3, the left hand side of eq. (36) is equivalent to Einstein’s tensor (except for a factor \(1/2\)). Equation (36) reduces to Einstein’s equation in vacuum in the limit \(S_{\lambda\mu} \rightarrow \delta_{\lambda\mu}\). In spite of being considerably more intricate than eq. (30), equation (36) is covariant under local Lorentz transformations, and strictly invariant if we convert the free Lorentz index \(a\) into a space-time index. Finally, as asserted at the beginning of this section, the field eq. (36) is symmetric if the free Lorentz index is converted into a space-time index.

5 Discussion

We have speculated on possible generalizations of the TEGR. The model described by eqs. (6), (8) and (9) is no longer equivalent to the standard general relativity. In addition to the tetrad fields, the theory contains the tensor \(S_{\lambda\mu}\) that, in the flat space-time formulation, obeys the ordinary wave equation in vacuum for a spin 2 field. This flat space-time formulation is determined by eqs. (6), (7) and \(T_{\lambda\mu\nu} = \partial_{\mu} S_{\lambda\nu} - \partial_{\nu} S_{\lambda\mu}\), and yields the wave equation (19), which may describe the propagation of gravitons in a quantum formulation of the theory. This theory is a by-product of the more general theory determined by eqs. (2), (6) and (7).

The formalism presented here allows to make a clear distinction between the propagation of gravitational waves and of spin 2 fields, an issue that is not properly and satisfactorily addressed in the ordinary formulation of general relativity. In the framework of the coupled field equations (8) and (9), the source of the field \(S_{\lambda\mu}\) is the dynamical geometry of the space-time, represented by time dependent tetrad fields \(e_\mu^a\).
In the realm of the theory defined by eqs. (6), (8) and (9), if we take the limit $S^{\lambda \mu} \rightarrow \delta^{\lambda \mu}$, we must necessarily make $e^{a \mu} \rightarrow \delta^{a \mu}$ simultaneously, otherwise the tetrad fields would be over-determined. This difficulty (or in fact, this feature) does not take place in the models addressed in sections 3 and 4.

In the framework of eqs. (27), (28) and (30) (second model), or (34) and (36) (third model), by taking the limit $S^{\lambda \mu} \rightarrow \delta^{\lambda \mu}$ we arrive at the standard form of the vacuum Einstein’s equations in the TEGR. In contrast, the model established in section 2 may be thought as a significant change of paradigm regarding the ordinary formulation of general relativity, since the establishment of the flat space-time is more subtle and unconventional. Of course, as a possible physical realization, we may have a nearly flat space-time described by the tetrad fields, with non-vanishing background weak oscillating fields $S^{\lambda \mu}$.

An issue that certainly deserves a careful investigation is the interaction of both $e^{a \mu}$ and $S^{\lambda \mu}$ with matter fields. The analysis of this issue could ensure or not the viability of the models presented in sections 2, 3 and 4.

Quantization in the teleparallel geometry has already been carried out in references [13, 14]. It is possible that the techniques developed in the latter references could be applied to $S^{\lambda \mu}$ in a simplified framework where $e_{a \mu}$ represents a weak gravitational field.

Finally, an interesting issue to be addressed is the establishment of a possible dynamics for the anti-symmetric part of the tensor $S^{\lambda \mu}$, since this quantity is not fixed by the field equations (13).

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