HADAMARD SEMIDIFFERENTIAL, ORIENTED DISTANCE FUNCTION, AND SOME APPLICATIONS

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Abstract. The Hadamard semidifferential calculus preserves all the operations of the classical differential calculus including the chain rule for a large family of non-differentiable functions including the continuous convex functions. It naturally extends from the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) to subsets of topological vector spaces. This includes most function spaces used in Optimization and the Calculus of Variations, the metric groups used in Shape and Topological Optimization, and functions defined on submanifolds.

Certain set-parametrized functions such as the characteristic function \( \chi_A \) of a set \( A \), the distance function \( d_A \) to \( A \), and the oriented (signed) distance function \( b_A = d_A - d_{\mathbb{R}^n \setminus A} \) can be used to identify a space of subsets of \( \mathbb{R}^n \) with a metric space of set-parametrized functions. Many geometrical properties of domains (convexity, outward unit normal, curvatures, tangent space, smoothness of boundaries) can be expressed in terms of the analytical properties of \( b_A \) and a simple intrinsic differential calculus is available for functions defined on hypersurfaces without appealing to local bases or Christoffel symbols.

The object of this paper is to extend the use of the Hadamard semidifferential and of the oriented distance function from finite to infinite dimensional spaces with some selected illustrative applications from shapes and geometries, plasma physics, and optimization.

1. Introduction. This paper is motivated by problems where the modeling, design, optimization, or control variable is a geometry, that is, a subset of the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \). For such problems metric spaces of subsets have been constructed along with a semidifferential calculus that preserves all the operations of the classical differential calculus including the chain rule. This semidifferential calculus arises from the notion of differential introduced by Hadamard [35] in 1923, which readily extends to embedded submanifolds of \( \mathbb{R}^n \) (see, [27]).

For shape optimization, metric groups of \( C^k \) diffeomorphisms with the composition as the group operation have been constructed by Micheletti [39] in 1972 and a notion of shape derivative by Zolésio [55] in 1979 that turned out to be a differential
with respect to directions in the vector space of $C^k$ transformations of $\mathbb{R}^n$, which is the tangent space at each point of the group.

For topological optimization, the metric group is the set of all characteristic functions $\chi_A$ of Lebesgue measurable subsets $A$ of $\mathbb{R}^n$ with the symmetric difference as the group operation. The topological derivative, rigorously introduced by Sokolowski-Zochowski [50] in 1999, turns out to be a semidifferential with respect to the tangent space, which is a cone that contains bounded measures generated by dilatation of points, curves, surfaces, and, more generally, $d$-rectifiable subsets, $0 \leq d < n$, of $\mathbb{R}^n$ (see, [17, 18, 19, 20]).

Other set-parametrized functions such as the distance function $d_A$ to a set $A$ or the oriented (algebraic, signed) distance function $b_A = d_A - d_{\mathbb{R}^n \setminus A}$ have been used to identify a space of subsets of $\mathbb{R}^n$ with a metric space of set-parametrized functions (see, [16]). It has been known that many geometrical features and properties of smooth domains can be expressed in terms of the analytical properties of $b_A$ (see, for instance, [33, 23]). Its gradient is the unit outward normal at each boundary point and the eigenvalues of its Hessian matrix are 0 and the $(n - 1)$ principal curvatures of the boundary $\partial A$. This provides an intrinsic tangential calculus for functions defined on hypersurfaces (boundary of smooth sets) without local bases and Christoffel symbols (see [15]). Also, the convexity of a closed non-empty subset $A$ of $\mathbb{R}^n$ is equivalent to the convexity of $d_A$, while the convexity of a closed subset $A$ of $\mathbb{R}^n$ with non-empty boundary $\partial A$ is equivalent to the convexity of $b_A$.

The last motivation is to extend to function spaces, theorems of the Karush, John, and Kuhn-Tucker type for objective and inequality constraint functions with Hadamard semidifferentials that are sublinear (see [21]). Such functions include non-differentiable continuous convex functions and differentiable functions.

The objective of this paper is to extend some of the above constructions and results from finite to infinite dimensional spaces via the Hadamard semidifferential to see whether it can be a useful and efficient tool in some old, revisited, or current applications. This paper has important intersections with the work of B. S. Mordukhovich (see, for instance, [41] and his earlier books) and of the late J. Borwein using subdifferentials in Banach spaces (cf. for instance, Borwein-Fitzpatrick [5]). As explained in section 2.7, they start from lower (liminf) or upper (limsup) semidifferentials relaxing the notion of strict differentiability (see [11]). As a result the classical differential calculus is lost and is restored in the form of a subdifferential calculus. Both families of semidifferentials contain convex continuous functions, but they are not contain in one another. It would have been nice to make systematic comparisons, but this would have required additional material, which is not realistic within a limited number of pages of a journal paper.

The paper is organized as follows. Section 2 summarizes the definitions and results from the recent paper [22] that generalized the notion of Hadamard semidifferential to topological vector spaces (TVS). This includes most function spaces used in the metric groups of Shape and Topological Optimization, the Calculus of Variations, and functions defined on submanifolds.

Section 3 recalls the variational principle for a local minimum with the illustrative example of a non-differentiable convex functional from Plasma Physics that yields a non-local partial differential equation.

Section 4 generalizes distance and oriented distance functions to a subset $A$ of $\mathbb{R}^n$ (see, for instance, [27, Chapters 6 and 7]) to normed vector spaces and study their Hadamard semidifferentiability versus the geometrical properties of $A$. For
instance, in addition to the equivalence between the convexity of a closed set and the convexity of $d_A$ in a reflexive Banach space, we prove, for a closed set $A$ with non-empty boundary, the equivalence of the convexity of $A$ and the convexity of $b_A$ avoiding the issue of the existence of a projection onto the complement (see, [27, Thm. 10.1, Chap. 7, pp. 375–376] for $\mathbb{R}^n$). We extend the notions of skeleton and set of cracks, and set of positive reach in a way that also avoids the issue of the existence of projections, which is problematic in Banach spaces. We characterize the properties of the boundary and expand the notion of crack-free sets which seems to be related with the fact that the contingent and the adjacent cones are equal.

Section 5 revisits the Karush, John, and Kuhn-Tucker type theorems from $\mathbb{R}^n$ (see, for instance, [21, Chapter 5]) to reflexive Banach spaces for the local minimum of an objective function subject to a finite number of inequality constraints specified by functions that have a sublinear Hadamard semidifferential. Then this is specialized to constrained optimization problems with respect to a closed subset $A$ by using the single inequality constraint $b_A(x) \leq 0$ on the oriented distance function $b_A$ since $x \in A$ if and only if $b_A(x) \leq 0$.

Section 6 extends the theorem of Bertsekas [3, 4] from $\mathbb{R}^n$ to a locally convex topological vector space for the semidifferential of the parametrized supremum of Danskin [14] in the convex case.

2. Hadamard semidifferentialiability and properties. The family of Hadamard semidifferentiable functions contains classically differentiable functions, all continuous (non-differentiable) convex functions, and functions defined on embedded submanifold and even on arbitrary subsets of a topological vector space. It is probably the largest family of non-differentiable functions for which all the rules of the classical differential calculus including the chain rule remain available. In this section we recall the definition and properties of such functions from the recent paper [22].

2.1. Some definitions and notation. Recall that in a topological vector space (TVS) over $\mathbb{R}$ there is a fundamental system $\mathcal{R}$ of neighborhoods of the origin for which ([36, Dfn. pp. 79–80, Thm. 1, p. 81])

(i) every $V$ in $\mathcal{R}$ is absorbing and balanced, and
(ii) for every $V \in \mathcal{R}$, there exists $U \in \mathcal{R}$ such that $U + U \subset V$.

In this paper we assume that the neighborhoods of the origin are the elements of $\mathcal{R}$.

A set $A$ is bounded if, for all $V \in \mathcal{R}$, there exists $\alpha > 0$ such that $A \subset \lambda V$ for all $\lambda \geq \alpha$ ([36, Dfn. 1, p. 108]). A complete, metrizable, locally convex topological space is called a Fréchet space ([36, Dfn. 4, p. 136]).

Definition 2.1. Let $X$ and $Y$ be topological vector spaces over $\mathbb{R}$.

(i) $f : X \to Y$ is positively homogeneous if, for all $\alpha \geq 0$, $f(\alpha x) = \alpha f(x)$.
(ii) $f : X \to \mathbb{R}$ is subadditive if, for all $x, y \in X$, $f(x + y) \leq f(x) + f(y)$; it is sublinear if it is positively homogeneous and subadditive; it is supadditive (resp. suplinear) if $-f$ is subadditive (resp. sublinear); and it is linear if for all $\alpha, \beta \in \mathbb{R}$ and $x, y \in X$, $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$.
(iii) Given a convex subset $U$ of $X$, a function $f : U \to \mathbb{R}$ is convex if

$$\forall x, y \in U, \forall \lambda \in (0, 1), \quad f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y). \quad (2.1)$$

A function $f : U \to \mathbb{R}$ is concave if $-f$ is convex.
2.2. Semidifferentiability in a topological vector space.

**Definition 2.2.** Let $X$ and $Y$ be topological vector spaces and $f : X \to Y$.

(i) $f$ is **Gateaux semidifferentiable at** $x \in X$ in the direction $v \in X$ if

$$df(x; v) \overset{\text{def}}{=} \lim_{t \downarrow 0} \frac{f(x + tv) - f(x)}{t} \text{ exists in } Y. \quad (2.2)$$

(ii) $f$ is **Gateaux semidifferentiable at** $x \in X$ if it is Gateaux semidifferentiable at $x \in X$ in all directions $v \in X$.

(iii) $f$ is **Gateaux differentiable at** $x \in X$ if $f$ is Gateaux semidifferentiable at $x \in X$ and $v \mapsto Df(x)v \overset{\text{def}}{=} df(x; v) : X \to Y$ is linear. \qed

**Definition 2.3.** An **admissible semitrajectory** at $x$ in a topological vector space $X$ is a function $h : [0, \tau) \to X$, for some $\tau > 0$, such that

$$h(0) = x \quad \text{and} \quad h'(0^+) \overset{\text{def}}{=} \lim_{t \downarrow 0} \frac{h(t) - h(0)}{t} \text{ exists in } X, \quad (2.3)$$

where $h'(0^+)$ is the semitangent to the trajectory $h$ at $h(0) = x$.

**Definition 2.4.** Let $X$ and $Y$ be topological vector spaces and $f : X \to Y$.

(i) $f$ is **Hadamard semidifferentiable at** $x \in X$ in the direction $v \in X$ if there exists $d_H f(x; v) \in Y$ such that for all admissible semitrajectories $h$ in $X$ at $x$ such that $h'(0^+) = v$, we have

$$
(f \circ h)'(0^+) \overset{\text{def}}{=} \lim_{t \downarrow 0} \frac{(f \circ h)(t) - (f \circ h)(0)}{t} = d_H f(x; v). \quad (2.4)
$$

(ii) $f$ is **Hadamard semidifferentiable at** $x \in X$ if there exists a function $v \mapsto d_H f(x; v) : X \to Y$ such that for each admissible semitrajectory $h$ in $X$ at $x$, 

$$(f \circ h)'(0^+) \text{ exists and } (f \circ h)'(0^+) = d_H f(x; h'(0^+)).$$

(iii) $f$ is **Hadamard differentiable at** $x \in X$ if $f$ is Hadamard semidifferentiable at $x$ and the function $v \mapsto Df(x)v \overset{\text{def}}{=} d_H f(x; v) : X \to Y$ is linear. \qed

**Definition 2.5** (Penot [47, p. 250], 1978). Let $X$ and $Y$ be topological vector spaces and $f : X \to Y$ a function.

(i) $f$ is **M-semidifferentiable at** $x \in X$ in the direction $v \in X$ if

$$d_M f(x; v) \overset{\text{def}}{=} \lim_{t \downarrow 0} \frac{f(x + tv) - f(x)}{t} \text{ exists in } Y. \quad (2.5)$$

(ii) $f$ is **M-semidifferentiable at** $x \in X$ if it is M-semidifferentiable at $x \in X$ in all directions $v \in X$.

(iii) $f$ is **M-differentiable at** $x \in X$ if $f$ is M-semidifferentiable at $x \in X$ and the function $v \mapsto Df(x)v \overset{\text{def}}{=} d_M f(x; v) : X \to Y$ is linear. \qed

M-semidifferentiability is stronger than Hadamard semidifferentiability, which is equivalent to following weaker sequential notion.

**Definition 2.6.** Let $X$ and $Y$ be topological vector spaces, $f : X \to Y$, and $x \in X$.

(i) $f$ is **MS-semidifferentiable at** $x \in X$ in the direction $v \in X$ if there exists $d^*_M f(x; v) \in Y$ such that for each sequence $\{v_n\}$ converging to $v$,

$$
\lim_{\substack{n \to \tau \nu
\downarrow 0}} \frac{f(x + tv_n) - f(x)}{t} = d^*_M f(x; v) \quad (2.6)
$$
Theorem 2.7. Let $X$ and $Y$ be topological vector spaces, $f: X \to Y$, and $x \in X$.

(i) The function $f$ is MS-semidifferentiable at $x$ if and only if it is Hadamard semidifferentiable at $x$.

(ii) The function $f$ is MS-semidifferentiable at $x$ if and only if $f$ is MS-semidifferentiable at $x$.

(iii) Let $f$ be MS-semidifferentiable at $x \in X$ and the function $v \mapsto Df(x)v \overset{\text{def}}{=} d_M^x f(x; v) : X \to Y$ is linear.

\begin{proof}
\end{proof}

Theorem 2.8. Let $X$ and $Y$ be topological vector spaces and $f : X \to Y$ a function.

(i) If $f$ is Hadamard semidifferentiable at $x$, then $v \mapsto d_H f(x; v) : X \to Y$ is positively homogeneous and sequentially continuous.

(ii) If $f_1$ and $f_2$ are Hadamard semidifferentiable at $x \in X$ in the direction $v \in X$, then for all $\alpha$ and $\beta$ in $\mathbb{R}$,

$$d_H(\alpha f_1 + \beta f_2)(x; v) = \alpha d_H f_1(x; v) + \beta d_H f_2(x; v). \quad (2.7)$$

(iii) (Chain rule) Let $X, Y, Z$ be topological vector spaces, $g: X \to Y$ and $f: Y \to Z$ be functions such that $g$ is Hadamard semidifferentiable at $x$ in the direction $v \in X$ and $f$ is Hadamard semidifferentiable at $g(x)$ in the direction $d_H g(x; v)$. Then $f \circ g$ is Hadamard semidifferentiable at $x$ in the direction $v \in X$ and

$$d_H(f \circ g)(x; v) = d_H f(g(x); d_H g(x; v)). \quad (2.8)$$

The next question is the continuity of a Hadamard semidifferentiable function.

Theorem 2.9. Let $X$ and $Y$ be topological vector spaces, $f : X \to Y$ a function. Assume that $f$ is Hadamard semidifferentiable at $x \in X$.

(i) If there exists a bounded neighborhood $U(0) \in \mathcal{R}$ in $X$, then $f$ is sequentially continuous at $x$.

(ii) If $X$ is a Fréchet space, then $\forall v \mapsto d_H f(x; v) : X \to Y$ is positively homogeneous and continuous. If $X$ and $Y$ are Fréchet spaces, then $f$ is continuous at $x$.

2.3. Lipschitz functions and Gateaux semidifferentiability. Lipschitz continuous functions enjoy the nice property that, if they are Gateaux semidifferentiable, they are M-semidifferentiable, and a posteriori Hadamard semidifferentiable.

Definition 2.10. Let $X$ and $Y$ be normed spaces. A function $f : X \to Y$ is Lipschitz continuous at $x \in X$ if there exists a constant $c(x) > 0$ and a ball $B_r(x)$ of radius $r > 0$ such that

$$\forall y, z \in B_r(x), \quad \|f(y) - f(z)\|_Y \leq c(x) \|y - z\|_X. \quad (2.9)$$

A function $f : X \to Y$ is Lipschitz continuous in a subset $U$ of $X$ if there exists a constant $c(U) > 0$ such that

$$\forall y, z \in U, \quad \|f(y) - f(z)\|_Y \leq c(U) \|y - z\|_X. \quad (2.10)$$
Theorem 2.11. Let $X$ and $Y$ be normed spaces, $f : X \to Y$ be a function which is Lipschitz continuous at $x \in X$. If $f$ is Gateaux semidifferentiable at $x$ in the direction $v$ (that is, $df(x; v)$ exists), then $f$ is M-semidifferentiable and hence Hadamard semidifferentiable at $x$ in the direction $v$, and $d_H f(x; v) = d_M f(x; v) = df(x; v)$.

2.4. Convex functions. In a locally convex topological vector space, all convex functions continuous (resp. sequentially continuous) at $x$ are M- (resp. Hadamard) semidifferentiable at $x$. In particular, the norm is M-semidifferentiable.

Theorem 2.12. Let $X$ be a locally convex topological vector space and $U$ a convex open subset of $X$. A function $f : U \to \mathbb{R}$ is convex if and only if

(i) for all $y \in U$, $f$ is Gateaux semidifferentiable at $y$, that is, $df(y; v)$ exists in all directions $v \in X$ at all points $y \in U$,

$$\forall y \in U, \forall v \in X, \quad df(y; v) + df(y; -v) \geq 0,$$

(2.11)

$$\forall x, y \in U, \quad f(y) \geq f(x) + df(x; y - x),$$

(2.12)

(ii) and for each $y \in U$, the function

$$v \mapsto df(y; v) : X \to \mathbb{R}$$

(2.13)

is positively homogeneous, convex, and subadditive, that is,

$$\forall v, w \in X, \quad df(y; v + w) \leq df(y; v) + df(y; w).$$

(2.14)

Corollary 1. In a normed space, the norm is M-semidifferentiable.

Proof. The norm is convex and Lipschitz continuous. By Theorem 2.12, it is Gateaux semidifferentiable and, by Theorem 2.11, it is M-semidifferentiable. \qed

The next theorem connects continuity and semidifferentiability.

Theorem 2.13 ([22, Thm. 3.14 (i), p. 1055]). Let $X$ be a locally convex topological vector space, $x \in X$, and $f : V(x) \to \mathbb{R}$ a convex function in a convex neighborhood $V(x)$ of $x \in X$.

(i) If $f$ is continuous\(^1\) at $x$, then $f$ is M-semidifferentiable at $x$. Hence, $v \mapsto d_M f(x; v) : X \to \mathbb{R}$ is continuous, convex, subadditive, and sublinear.

(ii) If $f$ is sequentially continuous at $x$, then $f$ is Hadamard semidifferentiable at $x$. Hence, $v \mapsto d_H f(x; v) : X \to \mathbb{R}$ is sequentially continuous, convex, subadditive, and sublinear.

(iii) If $X$ is a Fréchet space, then $f$ is continuous at $x$ if and only if $f$ is Hadamard semidifferentiable at $x$.

Remark 1. If, in addition, $X$ has a bounded neighborhood of 0, $f$ is continuous at $x$ if and only if $f$ is M-semidifferentiable at $x$ in (i) and $f$ is sequentially continuous at $x$ if and only if $f$ is Hadamard semidifferentiable at $x$ in (ii). Recall that if $X$ is a locally convex TVS in which there exists a bounded neighborhood of 0, the topology of $X$ can be defined with a single semi-norm ([36, Prop. 1, p. 109]).

Proof. To complete the proof of [22, Thm. 3.14 (i) and (ii), p. 1055], we prove that $v \mapsto d_M f(x; v) : X \to \mathbb{R}$ is convex, subadditive, an hence sublinear.

\(^1\)It is sufficient to assume that $f$ is upper semicontinuous at $x$. For $\mu$ such that $f(x) < \mu$ there exists a convex open neighborhood $V(x)$ such that $f(y) < \mu$ for all $y \in V(x)$ and $f$ is bounded above in $V(x)$. Therefore, it is continuous from [29, Prop. 2.1, p. 11].
Choose an open convex neighborhood $U$ of $x$ such that $U \subset V(x)$. We want to show that for all $\alpha, 0 \leq \alpha \leq 1$, and $v, w \in H$,
\[
df(x; \alpha v + (1 - \alpha)w) \leq \alpha df(x; v) + (1 - \alpha)df(x; w).
\]
Since $x \in U$ and $U$ is open and convex,
\[
\exists \theta_0, 0 < \theta_0 < 1, \text{ such that } \forall \theta, 0 < \theta < \theta_0, \quad x + \theta v \in U \text{ and } x + \theta w \in U
\]
\[
\Rightarrow \forall 0 \leq \alpha \leq 1, \quad x + \alpha(\theta v + (1 - \alpha)w) = \alpha(x + \theta v) + (1 - \alpha)(x + \theta w) \in U,
\]
and by convexity of $f$,
\[
f(x + \theta(\alpha v + (1 - \alpha)w)) = f(\alpha [x + \theta v] + (1 - \alpha)[x + \theta w])
\leq \alpha f(x + \theta v) + (1 - \alpha) f(x + \theta w)
\Rightarrow [f(x + \theta(\alpha v + (1 - \alpha)w)) - f(x)]
\leq \alpha [f(x + \theta v) - f(x)] + (1 - \alpha) [f(x + \theta w) - f(x)].
\]
Dividing by $\theta$ and going to the limit as $\theta$ goes to 0, we get
\[
df(x; \alpha v + (1 - \alpha)w) \leq \alpha df(x; v) + (1 - \alpha)df(x; w).
\]
Combining the positive homogeneity and the convexity,
\[
df(x; v + w) = df(x; \frac{1}{2}v + \frac{1}{2}w) \leq \frac{1}{2} df(x; 2v) + \frac{1}{2} df(x; 2w) = df(x; v) + df(x; w),
\]
we get the subadditivity. \qed

2.5. Sublinear and suplinear functions.

**Theorem 2.14.** Let $f : X \to \mathbb{R}$ be sublinear in a topological vector space $X$.

(i) The functions $f$ and $|f|$ are convex and $f(0) = 0$.

(ii) If, in addition, $X$ is a normed vector space and $f$ is continuous at 0, then
$f$ is M-semidifferentiable and $x \mapsto f(x)$ and $v \mapsto df(x; v)$ are Lipschitz continuous in $X$: there exists $c(f) > 0$ such that
\[
\forall x, y \in X, \quad |f(y) - f(x)| \leq c(f) \|y - x\|,
\]
\[
\forall v_1, v_2 \in X, \quad |d_H f(x; v_2) - d_H f(x, v_1)| \leq c(f) \|v_2 - v_1\|.
\]

**Proof.** (i) By definition, a sublinear function $f$ and its absolute value $|f|$ are convex.
Since $f(0)$ is finite, for $\lambda > 0$, $f(\lambda 0) = \lambda f(0)$, and $f(0) = 0$.

(ii) By continuity at 0, for $\varepsilon > 0$, there exists $\delta > 0$ such that
\[
\forall y, \|y\| < \delta, \quad |f(y)| = |f(y) - f(0)| < \varepsilon.
\]
Since $f$ is sublinear
\[
\forall x, y, x \neq y, \quad f(y) - f(x) \leq f(y - x) \Rightarrow \frac{f(y) - f(x)}{\|y - x\|} \leq \frac{2f\left(\frac{\delta}{2}\frac{y - x}{\|y - x\|}\right)}{\|y - x\|} < \frac{2\varepsilon}{\delta},
\]
\[
\Rightarrow \forall x, y, x \neq y, \quad \frac{|f(y) - f(x)|}{\|y - x\|} \leq c(f) \overset{\text{def}}{=} \frac{2\varepsilon}{\delta}
\]
\[
\Rightarrow \forall x, y \in X, \quad |f(y) - f(x)| \leq c(f)\|y - x\|.
\]
From Theorem 2.13 (i), $d_H f(x; v)$ exists for all $v \in X$. For all $v_1, v_2 \in X$
\[
\left|\frac{f(x + tv_2) - f(x)}{t} - \frac{f(x + tv_1) - f(x)}{t}\right| \leq c(f) \left|\frac{tv_2 - tv_1}{t}\right| = c(f) \|v_2 - v_1\|
\]
\[
\Rightarrow \forall v_1, v_2 \in X, \quad |df(x; v_2) - df(x; v_1)| \leq c(f)\|v_2 - v_1\|.
\]
Since $f$ is Lipschitzian this inequality holds with $d_H f(x; v)$ in place of $df(x; v)$ \qed
2.6. Semidifferentials of functions defined on unstructured sets. For functions on a smooth embedded submanifold of \( \mathbb{R}^n \) of dimension \( d < n \) or an unstructured subset \( A \) of a TVS \( X \), the Hadamard semidifferential is the natural choice over the M-semidifferential since it uses semitrajectories that do not require some algebraic structure on \( A \). For a subset \( A \) of \( X \), the tangent space at interior points of \( A \) is \( X \), but, at the boundary \( \partial A \), the tangent space will generally be only a cone. For instance, for a smooth embedded submanifold of dimension \( d < n \), \( \overline{A} = \partial A \) and all points of \( A \) are boundary points where the tangent space is \( \mathbb{R}^d \).

Definition 2.15. Let \( A \) be a non-empty subset of a topological vector space \( X \). An admissible semitrajectory at \( x \in \overline{A} \) in \( A \) is a function \( h : [0, \tau) \to A \) such that

\[
h(0) = x \quad \text{and} \quad h'(0^+) \overset{\text{def}}{=} \lim_{t \downarrow 0} \frac{h(t) - h(0)}{t}
\]

exists in \( X \), where \( h'(0^+) \) is the semitangent to the trajectory \( h \) in \( A \) at \( h(0) = x \).

Given a subset \( A \neq \emptyset \) of a normed vector space \( X \),

\[
d_A(x) \overset{\text{def}}{=} \inf_{a \in A} \|x - a\|
\]

is the distance function from a point \( x \) to \( A \). It is readily seen that \( d_{\overline{A}} = d_A \) and

\[
\forall x, y \in X, \quad |d_A(y) - d_A(x)| \leq \|y - x\|.
\]

Definition 2.16. Let \( A \) be a non-empty subset of a topological vector space \( X \).

(i) The Bouligand contingent cone\(^2\) to \( A \) at \( x \in \overline{A} \) is defined as

\[
T_A(x) \overset{\text{def}}{=} \left\{ v \in X : \exists \{t_n \downarrow 0\}, \exists\{x_n\} \subset A \text{ such that } \lim_{n \to \infty} \frac{x_n - x}{t_n} = v \right\}.
\]

(ii) The adjacent or intermediary tangent cone\(^3\) to \( A \) at \( x \in \overline{A} \) is defined as

\[
T^b_A(x) \overset{\text{def}}{=} \left\{ v \in X : \forall\{t_n \downarrow 0\}, \exists\{x_n\} \subset A \text{ such that } \lim_{n \to \infty} \frac{x_n - x}{t_n} = v \right\}.
\]

(iii) A set \( A \) is derivable at \( x \in \overline{A} \) if \( T_A(x) = T^b_A(x) \).

If \( A \) is convex in a normed vector space, \( T_A(x) = T^b_A(x) = \{ \lambda(A - x) : \lambda \geq 0 \} \).\(^4\) \( T^b_A(x) \) is directly related to the notion of admissible semitrajectories at \( x \) in \( A \).

Theorem 2.17. Let \( A \) be a subset of a topological vector space \( X \). For \( x \in \overline{A} \),

\[
T^b_A(x) = \{ h'(0^+) : h \text{ an admissible semitrajectory in } A \text{ at } x \}.
\]

Theorem 2.18. Let \( A \subset X, X \) a normed vector space, and \( x \in \overline{A} \), then

\[
T^b_A(x) \overset{\text{def}}{=} \left\{ v \in X : \lim_{t \downarrow 0} \frac{d_A(x + tv)}{t} = 0 \right\} = \left\{ v \in X : d_H d_A(x; v) = 0 \right\}
\]

and \( T^b_A(x) \) is a closed cone at 0.

\(^2\)Bouligand [10], 1930

\(^3\)In the terminology of Aubin and Frankowska [2, Definition 4.1.5, pp. 126–129, Definition 4.1.5, pp. 126–127]. See [2, Figure 4.4, p. 161] for an example in dimension two where \( T_A(x) \neq T^b_A(x) \).

\(^4\)Aubin and Frankowska [2, Thm. 4.2.1, p. 138].
Proof. In a normed vector space

\[ T^0_A(x) = \left\{ v \in X : \lim_{t \downarrow 0} \frac{d_A(x + tv)}{t} = 0 \right\}. \]

This means that the limit \((d_A(x + tv) - d_A(x))/t\) exists and is zero. Since \(d_A\) is Lipschitzian, \(d_Hd_A(x; v)\) exists by Theorem 2.11.

We now have all the elements to extend the definition of the Hadamard semidifferent to a subset of a TVS.

**Definition 2.19.** Let \(X\) and \(Y\) be TVS, \(A, \emptyset \neq A \subset X\), and \(f : A \to Y\).

(i) The function \(f\) is Hadamard semidifferentiable at \(x \in A\) in the direction \(v \in T^0_A(x)\) if there exists \(g(x, v) \in Y\) such that, for all admissible semitrajectories \(h\) in \(A\) at \(x\) such that \(h'(0^+) = v\),

\[ (f \circ h)'(0^+) \overset{\text{def}}{=} \lim_{t \downarrow 0} \frac{f(h(t)) - f(h(0))}{t} = g(x, v). \]  \hspace{1cm} (2.18)

The element \(g(x, v)\) will be denoted \(d_Hf(x; v)\).

(ii) \(f\) is Hadamard semidifferentiable at \(x \in A\) if \(f\) is Hadamard semidifferentiable at \(x\) in all directions \(v \in T^0_A(x)\).

(iii) \(f\) is Hadamard differentiable at \(x \in A\) if \(T^0_A(x)\) is a linear subspace, \(f\) is Hadamard semidifferentiable at \(x \in A\), and the function \(v \mapsto d_Hf(x; v) : T^0_A(x) \to Y\) is linear in which case it will be denoted \(Df(x)\).

The Hadamard semidifferentiability enjoys all the nice properties of the classical finite dimensional differential calculus including the chain rule.

**Theorem 2.20.** Let \(X\) and \(Y\) be topological vector spaces and \(A, \emptyset \neq A \subset X\).

(i) If \(f : A \to Y\) is Hadamard semidifferentiable at \(x \in A\) in the direction \(v \in T^0_A(x)\), then for all admissible semitrajectory \(h\) in \(A\) such that \(h'(0^+) = v\), \(f \circ h\) is an admissible trajectory in \(f(A)\) such that \((f \circ h)'(0^+) = d_Hf(x; v) \in T^0_{f(A)}(f(x))\). The positively homogeneous mapping

\[ v \mapsto d_Hf(x; v) : T^0_A(x) \to T^0_{f(A)}(f(x)) \subset Y \]  \hspace{1cm} (2.19)

is sequentially continuous for the induced topologies.

(ii) If \(f_1 : A \to Y\) and \(f_2 : A \to Y\) are Hadamard semidifferentiable at \(x \in A\) in the direction \(v \in T^0_A(x)\), then for all \(\alpha\) and \(\beta\) in \(\mathbb{R}\),

\[ d_H(\alpha f_1 + \beta f_2)(x; v) = \alpha d_Hf_1(x; v) + (1 - \alpha) d_Hf_2(x; v), \]  \hspace{1cm} (2.20)

and \(\alpha f_1 + \beta f_2\) is Hadamard semidifferentiable at \(x\) in the direction \(v\).

(iii) (Chain rule) Let \(X, Y, Z\) be topological vector spaces, \(A \subset X\), \(g : A \to Y\), and \(f : g(A) \to Z\) be functions such as \(g\) is Hadamard semidifferentiable at \(x\) in the direction \(v \in T^0_A(x)\) and \(f\) is Hadamard semidifferentiable at \(g(x)\) in \(g(A)\) in the direction \(d_Hg(x; v) \in T^0_{g(A)}(x)\). Then \(d_Hg(x; v) \in T^0_{g(A)}(x)\), \(f \circ g\) is Hadamard semidifferentiable at \(x\) in the direction \(v \in T^0_A(x)\), and

\[ d_H(f \circ g)(x; v) = d_Hf(g(x); d_Hg(x; v)). \]  \hspace{1cm} (2.21)

The next question is the continuity of a semidifferentiable function.

**Theorem 2.21.** Let \(X\) and \(Y\) be topological vector spaces, \(\emptyset \neq A \subset X\), and \(f : A \to Y\). Assume that \(f\) is Hadamard semidifferentiable at \(x \in A\).
(i) If there exists a bounded neighborhood $U(0) \subset \mathcal{R}$ in $X$, then $f$ is sequentially continuous$^5$ at $x$ in $A$ for the induced topology on $A$.

(ii) If $X$ is a Fréchet space, then $v \mapsto d_H f(x; v) : T^*_A(x) \rightarrow T^*_{f(A)}(f(x))$ is positively homogeneous and continuous for the induced topologies. If $X$ and $Y$ are Fréchet spaces, then $f$ is continuous at $x$.

2.7. **Strict differentiability and upper and lower semidifferentiability.** To complete this section, we compare the Hadamard semidifferential with some other notions for real valued functions available in the literature.

The following notion which is strictly stronger than the M-differentiability (Hadamard, Fréchet) was introduced by the school of Bourbaki in the fifties.

**Definition 2.22** (Borwein-Lewis [7, p. 132]). A Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is strictly differentiable at $x$ if there exists $Df(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ linear such that

$$
\forall v \in \mathbb{R}^n, \lim_{t \searrow 0} \frac{f(y + tv) - f(y)}{t} = Df(x)v,
$$

(2.23)

For Lipschitz functions $f : V(x) \rightarrow \mathbb{R}$ in a neighborhood $V(x)$ of $x$, lower and upper notions of Gateaux, M-, and strict differentiability can be introduced by replacing the limit by the liminf or the limsup (Cannarsa and Sinestrari [11]):

- **Lower Gateaux semidifferential** at $x$ in the direction $v$:
  \[ d_f(x; v) \overset{\text{def}}{=} \liminf_{t \searrow 0} \frac{f(x + tv) - f(x)}{t} \]

- **Upper Gateaux semidifferential** at $x$ in the direction $v$:
  \[ d_f(x; v) \overset{\text{def}}{=} \limsup_{t \searrow 0} \frac{f(x + tv) - f(x)}{t} \]

- **Lower M-semidifferential** at $x$ in the direction $v$:
  \[ d_M f(x; v) \overset{\text{def}}{=} \liminf_{t \searrow 0} \frac{f(x + tw) - f(x)}{t} \]

- **Upper M-semidifferential** at $x$ in the direction $v$:
  \[ d_M f(x; v) \overset{\text{def}}{=} \limsup_{t \searrow 0} \frac{f(x + tw) - f(x)}{t} \]

- **Clarke lower semidifferential** at $x$ in the direction $v$:
  \[ d_C f(x; v) \overset{\text{def}}{=} \liminf_{t \searrow 0} \frac{f(y + tv) - f(y)}{t} \]

- **Clarke upper semidifferential** at $x$ in the direction $v$:
  \[ d_C f(x; v) \overset{\text{def}}{=} \limsup_{t \searrow 0} \frac{f(y + tv) - f(y)}{t} \]

The upper notion of strict differentiability corresponds to the upper semidifferential developed by Clarke$^6$ in 1973 under the name generalized directional derivative.

---

$^5$Note the following natural equivalence for the semicontinuity in terms of semitrajectories. Let $X$ and $Y$ be topological spaces and $A$ a subset of $X$. A function $f : A \rightarrow Y$ is sequentially continuous at $a \in A$ if and only if for all semitrajectories $h : [0, \tau) \rightarrow A$

\[
\lim_{t \searrow 0} h(t) = a \Rightarrow \lim_{t \searrow 0} f(h(t)) = f(a),
\]

(2.22)

where $A$ is endowed with the topology induced by $X$.

$^6$See Clarke [12, 13], Cannarsa and Sinestrari [11].
Upper and lower semidifferentials are more general, but the basic operations of the differential calculus are lost and one has to introduce the notion of subdifferential to restore some form of calculus. This is a disadvantage over the simpler Hadamard semidifferential, which readily handles non-differentiable convex continuous functions without going to upper or lower notions.

3. Variational principle for a local minimum and an example from plasma physics. We begin with the standard first order necessary condition.

Theorem 3.1. Let $X$ be a topological vector space, $A \neq \emptyset$ be a closed subset of $X$, and $f : A \to \mathbb{R}$ be Hadamard semidifferentiable at $x \in A$.

(i) If $x \in A$ is a local minimizer of $f$ with respect to $A$, then
\[ d_H f(x; v) \geq 0 \text{ for all } v \in T^\flat_A(x), \] (3.1)
where $T^\flat_A(x)$ is the adjacent tangent cone of Definition 2.16.

(ii) If $A$ is convex and $x \in A$ is a minimizer of $f$ with respect to $A$, then
\[ d_H f(x; y - x) \geq 0 \text{ for all } y \in A. \] (3.2)
If, in addition, $f$ is convex, condition (3.2) is necessary and sufficient.

Proof. (i) By definition, there exists a neighborhood $V$ of the origin such that
\[ f(y) \geq f(x) \text{ for all } y \in (x + V) \cap A. \]
Given $v \in T^\flat_A(x)$, there exists an admissible trajectory $h$ in $A$ at $x$ such that $h(0) = x$ and $h'(0^+) = v$. So, there exists $\delta > 0$ such that $h(t) \in A \cap (x + V)$ for $t < \delta$ and
\[ \forall t, 0 < t < \delta, \quad f(h(t)) - f(x) \geq 0 \Rightarrow \frac{f(h(t)) - f(x)}{t} \geq 0. \]
Since $f$ is Hadamard semidifferentiable at $x$, for all $v \in T^\flat_A(x)$, $d_H f(x; v) \geq 0$.

(ii) and (iii) By standard arguments. \hfill \Box

The use of the Hadamard semidifferential in the modeling of physical phenomena provides a first order approximation that is not a linearization. We illustrate Theorem 3.1 (ii) with the Grad-Mercer equation of Plasma Physics [34, 38].

Let $\Omega$ be a bounded open domain with locally Lipschitz boundary $\Gamma$ and finite volume $|\Omega| = \text{meas} (\Omega)$. Consider the problem of finding a function $u \in H^1_0(\Omega) \cap H^2(\Omega)$ solution of the equation
\[ -\Delta u + \beta(u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma, \] (3.3)
where
\[ \beta(u)(x) \overset{\text{def}}{=} \text{meas} \left( \{ y \in \Omega : u(x) \geq u(y) \} \right) \] (3.4)
is a non-local function. This equation has been studied and put on solid mathematical ground by Temam [51, 52] and Mossino [43, eq. (**)]. It is related to the Grad-Mercier equations for the normalized flux of electrons in the adiabatic evolution of plasma equilibrium [34, 38].

Mossino and Zolésio [44] in 1977 and Zolésio [54, 55] in 1979 considered the infimum of the following non-differentiable convex continuous functional on $H^1_0(\Omega)$
\[ f(v) \overset{\text{def}}{=} \int_\Omega (||\nabla v(x)||^2 + |\Omega| v(x)) \, dx + \int_\Omega \int_\Omega [v(x) - v(y)]^+ \, dx \, dy, \] (3.5)
where \( [y]^+ = \max\{y, 0\} \). The introduction of the non-differentiable term provided a simpler way to obtain equation (3.3) than direct approaches and it also revealed for the first time that one of its solutions arises from a variational problem.

**Theorem 3.2** (Mossino-Zolésio [44] and Zolésio [54, 55]). Assume that \( \Omega \) is a bounded open domain with locally Lipschitzian boundary \( \Gamma \) and that \( f \) is given by (3.5). There exists a unique \( u \in H^1_0(\Omega) \) that minimizes \( f \) over \( H^1_0(\Omega) \).

The necessary and sufficient condition for a minimizer \( u \in H^1_0(\Omega) \) is

\[
\forall v \in H^1_0(\Omega), \quad df(u; v - u) \geq 0,
\]

\[
df(u; v) = 2\int_\Omega \nabla u(x) \cdot \nabla v(x) dx + |\Omega| \int_\Omega v(x) dx + \int_\Omega \left( \int_{u(y)<u(x)} [v(x) - v(y)]^+ \right) dy
\]

\[
+ \int_\Omega \left( \int_{u(y)=u(x)} [v(x) - v(y)]^+ \right) dy.
\]

**Theorem 3.3** (Mossino-Zolésio [44] and Zolésio [54, 55]). Let \( \Omega \) be a bounded open domain with locally Lipschitz boundary \( \Gamma \) and finite volume \( |\Omega| = \text{mes}(\Omega) \) and \( f \) be the functional (3.5). The unique minimizer \( u \) of \( f \) over \( H^1_0(\Omega) \) of Theorem 3.2 is the solution in \( H^1_0(\Omega) \cap H^2(\Omega) \) of the following system

\[
-\Delta u + \beta_-(u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma,
\]

\[
\text{mes} \{ y \in \Omega : u(x) = u(y) \} = 0,
\]

where

\[
\beta_-(u)(x) = \text{mes} \{ y \in \Omega : u(x) > u(y) \}.
\]

This theorem says that the variational solution \( u \) is not constant on any subset of \( \Omega \) of positive measure and is the unique solution of (3.6) with that property.

4. **Distance and oriented distance functions.** In this section, we work in a normed vector space where M-semidifferentiability is equivalent to Hadamard semidifferentiability and continuity is equivalent to sequential continuity.

4.1. **Distance function.** For a non-empty subset \( A \) of a normed vector space \( X \), the distance function from a point \( x \) to \( A \) is

\[
d_A(x) \overset{\text{def}}{=} \inf_{a \in A} \|x - a\|\tag{4.1}
\]

and the set of projections of \( x \) onto \( A \) is

\[
\Pi_A(x) \overset{\text{def}}{=} \{ p \in A : \|x - p\| = d_A(x) \}.
\]

The multivalued mapping \( x \mapsto \Pi_A(x) : X \to 2^A \) is called the metric projection. A subset \( A \) is called a proximinal set if for all \( x \in X \), \( \Pi_A(x) \neq \emptyset \); it is called a Chebyshev set if \( \Pi_A(x) \) is a singleton for all \( x \in X \). If \( X \) is finite dimensional, \( \Pi_A(x) \neq \emptyset \) and, for \( x \in X \setminus A \), \( \Pi_A(x) = \partial A \cap \{ y \in X : \|y - x\| = d_A(x) \} \). Quoting

If \( X \) is reflexive then, as is well known, any weakly closed set is proximinal; hence every closed convex set is proximinal. Similarly, if \( X \) is a conjugate Banach space then any weak*-closed set is proximinal. On the other hand any nonreflexive Banach space contains a closed subspace which fails to be proximinal. (Edelstein [28]).

**Theorem 4.1.** Let \( A \neq \emptyset \) be a subset of a normed vector space \( X \).
(i) For all \( a \in A \) and \( x, y \in X \),
\[
|d_A(y) - d_A(x)| \leq \|y - x\|.
\]
(ii) For all \( x \in X \), \( d_A(x) = d_{\overline{A}}(x) \) and \( \Pi_\mathbb{H}(x) = \Pi_A(x) \).
(iii) The norm \( x \mapsto n(x) = \|x\| \) and the function
\[
f_A(x) \overset{\text{def}}{=} \frac{1}{2} \left( \|x\|^2 - d_A(x)^2 \right)
\]
are convex, continuous, Hadamard semidifferentiable at every \( x \in X \), and \( v \mapsto d_H n(x; v) \) and \( v \mapsto d_H f_A(x; v) \) are sublinear and continuous. Moreover,
\[
\forall v \in X, \quad d_H d_A^2(x; v) = n(x) d_H n(x; v) - d_H f_A(x; v),
\]
where \( d_A \) is Hadamard semidifferentiable, \( v \mapsto d_H d_A^2(x; v) \) is the difference of two sublinear functions, and, for all \( x \in \overline{A} \), \( d_A \) is Hadamard differentiable and \( d_H d_A^2(x; v) = 0 \) for all \( v \in X \).

(iv) \( d_A \) is Hadamard semidifferentiable in \( X \setminus \partial \overline{A} \), \( \partial \overline{A} \) is nowhere dense,\(^7\) and
\[
\forall v \in X, \quad |d_H d_A(x; v)| \leq \|v\|.
\]

Remark 2. In finite dimension, \( \Pi_A(x) \) is not empty and the gradient \( \nabla d_A(x) \) exists almost everywhere but not at points where the projection is not unique. Yet, the Hadamard semidifferential exists in \( X \setminus \partial \overline{A} \) even at points where the projection onto \( \overline{A} \) is not unique. So, we need not worry about the \textit{almost everywhere}.

Proof. (i) For \( a \in A \) and \( x, y \in X \), by the triangle inequality,
\[
\|x - a\| \leq \|x - y\| + \|y - a\| \quad \Rightarrow \quad d_A(x) \leq \|x - y\| + d_A(y).
\]
Changing the order of \( x \) and \( y \), \( |d_A(y) - d_A(x)| \leq \|y - x\| \).
(ii) In view of the continuity of \( x \mapsto \|x - a\| \), \( A \) can be replaced by \( \overline{A} \):
\[
d_A(x) = \inf_{a \in A} \|x - a\| = \inf_{a \in \overline{A}} \|x - a\| = d_{\overline{A}}(x).
\]
(iii) For \( a \in A \), the function
\[
x \mapsto \ell_A(x) \overset{\text{def}}{=} \frac{1}{2} \left( \|x\|^2 - \|x - a\|^2 \right) = a \cdot x - \frac{1}{2} \|a\|^2
\]
is affine (hence convex) in \( x \). So, its upper enveloppe
\[
x \mapsto \sup_{a \in A} \ell_A(x) = \frac{1}{2} \left( \|x\|^2 - d_A(x)^2 \right)
\]
is convex and continuous since \( d_A \) is continuous on \( X \). From Theorem 2.13 (i), \( f_A \) is Hadamard semidifferentiable and \( v \mapsto d_H f_A(x; v) \) is sublinear. Since the norm \( x \mapsto n(x) = \|x\| \) is Hadamard semidifferentiable and sublinear by Corollary 1 to Theorem 2.12, then
\[
d_H n^2(x; v) = 2 n(x) d_H n(x; v) \quad \text{and} \quad d_H d_A^2(x; v) = n(x) d_H n(x; v) - d_H f_A(x; v).
\]
So, \( d_A^2 \) is Hadamard semidifferentiable and \( v \mapsto d_H d_A^2(x; v) \) is the difference of two sublinear functions.

(iv) But \( d_H d_A(x; v) = 0 \) in \( \text{int} \overline{A} \), \( d_H d_A(x; v) = d_H d_A^2(x; v)/(2d_A(x)) \) in \( X \setminus \overline{A} \), and \( d_A \) is Hadamard semidifferentiable in \( X \setminus \partial \overline{A} \).
\[\square\]

\(^7\)See [27, Ex. 6.2, Chapter 5, pp. 249-250] for an example of a set \( A \) such that \( \partial A \neq \emptyset \). Recall that \( \overline{B} = \text{int} (\overline{B} \cup \partial B) = \text{int} B \cup \partial B \). A set \( B \) is \textit{nowhere dense} if \( \text{int} \overline{B} = \emptyset \), that is \( \overline{B} = \partial \overline{B} \).

The boundary of a closed or an open set \( A \) is nowhere dense, that is, \( \partial A = \partial (\partial A) = \partial A \cup \partial \overline{A} \) (see footnote 10 on page 17). Every closed nowhere dense set is the boundary of an open set.
For simplicity, the following existence theorem is restricted to reflexive Banach spaces. However, the existence of a projection onto a closed convex subset of a Banach space is not restricted to reflexive Banach spaces. For instance, a closed convex bounded subset of \( L^\infty(\Omega) \) is compact in the weak* topology and, since the space of continuous functions is closed in \( L^\infty(\Omega) \), a closed convex bounded subset of continuous functions is also weak* compact. See also [28, 5] for examples of almost proximinal sets for which \( \Pi_A(x) \neq \emptyset \) at \( x \) in a dense \( G_\delta \) subset of \( X \setminus A \).

**Theorem 4.2.** Let \( A \neq \emptyset \) be a subset of a reflexive Banach space \( X \).

(i) If \( A \neq \emptyset \) is a convex subset of \( X \), the projection of \( x \) onto \( \overline{A} \) exists, is unique, and is denoted by \( p_A(x) \), that is, \( \Pi_A(x) = \{ p_A(x) \} \).

(ii) \( \overline{A} \) is convex if and only if \( d_A \) is convex.

(iii) If \( \overline{A} \) is convex, \( d_A \) is Hadamard semidifferentiable and \( v \mapsto d_Hd_A(x; v) \) is positively homogeneous, convex, sublinear and continuous.

**Proof.** (i) The function \( y \mapsto \|y - x\| : X \to \mathbb{R} \) is strictly convex, continuous, has the growth property, that is, \( \|y - x\| \to \infty \) as \( \|y\| \) goes to infinity, and a bounded closed convex subset of a reflexive Banach space \( X \) is weakly compact.

(ii) From Theorem 4.1 (iii), if \( \overline{A} \) is convex, for \( x \) and \( y \) in \( X \), there exist \( \overline{x} \) and \( \overline{y} \) in \( \overline{A} \) such that \( d_A(x) = \|x - \overline{x}\| \) and \( d_A(y) = \|y - \overline{y}\| \). By convexity of \( \overline{A} \), for all \( \lambda \), \( 0 \leq \lambda \leq 1 \), \( \lambda \overline{x} + (1 - \lambda) \overline{y} \in \overline{A} \) and

\[
d_A(\lambda x + (1 - \lambda) y) \leq \|\lambda x + (1 - \lambda)y - (\lambda \overline{x} + (1 - \lambda) \overline{y})\| \leq \lambda \|x - \overline{x}\| + (1 - \lambda) \|y - \overline{y}\| = \lambda d_A(x) + (1 - \lambda) d_A(y)
\]

and \( d_A \) is convex in \( X \). Conversely, if \( d_A \) is convex, then

\[
\forall \lambda \in [0,1], \forall x, y \in \overline{A}, \quad d_A(\lambda x + (1 - \lambda)y) \leq \lambda d_A(x) + (1 - \lambda) d_A(y).
\]

But \( x \) and \( y \) in \( \overline{A} \) imply that \( d_A(x) = d_A(y) = 0 \) and hence

\[
\forall \lambda \in [0,1], \quad d_A(\lambda x + (1 - \lambda)y) = 0.
\]

Thus \( \lambda x + (1 - \lambda)y \in \overline{A} \) and \( \overline{A} \) is convex.

(iii) From part (v), for \( \overline{A} \) convex, \( d_A \) is convex and Lipschitz continuous, and the other properties follow from Theorem 2.13 (i). \( \square \)

**Theorem 4.3.** Let \( A \neq \emptyset \) be a subset of a Hilbert space \( H \) with inner product \( x \cdot y \)

(i) The function \( d_A^2 \) is Hadamard semidifferentiable,

\[
\forall x \in H, \forall v \in H, \quad d_Hd_A^2(x; v) = 2 \left[ x \cdot v - d_Hf_A(x; v) \right], \quad (4.5)
\]

and \( v \mapsto d_Hd_A^2(x; v) \) is suplinear. For \( x \in \text{int} A, \) \( d_Hd_A(x; v) = 0; \) for \( x \in H \setminus \overline{A}, \) \( d_Hd_A(x; v) \) exists and \( v \mapsto d_Hd_A(x; v) \) is suplinear; and for \( x \in \partial \overline{A}, \) \( d_Hd_A(x; v) \) exists if and only if \( ^9 \)

\[
\lim_{t \searrow 0} \frac{d_A(x + tv) - d_A(x)}{t} = \lim_{t \searrow 0} \frac{d_A(x + tv)}{t} \text{ exists.}
\]

For all \( v \in T_A^0(x), \) \( d_Hd_A(x; v) \) exists and \( d_Hd_A(x; v) = 0. \)

(ii) If \( \overline{A} \) is convex, for \( x \in H \setminus \partial \overline{A}, \) \( v \mapsto d_Hd_A(x; v) \) is linear, and, for \( x \in \partial \overline{A}, \)

\( v \mapsto d_Hd_A(x; v) \) is sublinear and \( d_Hd_A(x; v) = 0 \) for all \( v \in T_A^0(x). \)

---

8It extends [27, Thm. 8.1, p. 318] from \( \mathbb{R}^n \) to a reflexive Banach space \( X \).

9Several notions are available. \( A \) is cracked if, \( \forall x \in A, \exists d, \|d\| = 1, \) such that \( \lim inf_{t \searrow 0} d_A(x + td)/t > 0 \) (see [26], [27, sec. 15, chap. 7, pp. 394–395] for application in image segmentation).
Theorem 4.4. Let $A \neq \emptyset$ be a closed subset of a Hilbert space $H$. 

(iii) If $\overline{A}$ is convex, for all $x \in H, \Pi_A(x) = \{p_A(x)\}$ is a singleton, 

$$
\text{d}_H d_A^2(x;v) = 2(x - p_A(x)) \cdot v, \quad \text{d}_H f_A(x;v) = p_A(x) \cdot v, \quad (4.6)
$$

and $d_A^2$ and $f_A$ are Hadamard differentiable. The projection $p_A(x) \in \overline{A}$ is solution of the variational inequality 

$$
\forall a \in \overline{A}, \quad (p_A(x) - x) \cdot (a - p_A(x)) \geq 0 \Rightarrow \forall x, y \in H, \quad \|p_A(y) - p_A(x)\| \leq \|y - x\|.
$$

Proof. It is sufficient to give the proof for $A$ closed. (i) In a Hilbert space, $x \mapsto \|x\|^2$ is Hadamard differentiable and $v \mapsto d_H d_A^2(x;v)$ is sublinear. For $x \in \text{int} \overline{A}$, $d_A(x) = 0$ and $d_H d_A(x;v) = 0$. For $x \in H \setminus \overline{A}$, $d_A(x) > 0$ and 

$$
v \mapsto d_H d_A(x;v) = \frac{1}{2d_A(x)} d_H d_A^2(x;v) = \frac{1}{2d_A(x)} [2x \cdot v - d_H f_A(x;v)]
$$

is sublinear. For $x \in \partial \overline{A}, d_H d_A(x;v)$ exists if and only if 

$$
\lim_{t \downarrow 0} \frac{d_A(x + tv) - d_A(x)}{t} = \lim_{t \downarrow 0} \frac{d_A(x + tv)}{t} \quad \text{exists}.
$$

(ii) Since, from Theorem 4.2 (iii), for $\overline{A}$ is convex, $v \mapsto d_H d_A(x;v)$ is sublinear for all $x$, then, from part (i), it is linear for $x \in H \setminus \overline{A} \cup \text{int} \overline{A} = H \setminus \partial \overline{A}$ and $d_H d_A(x;v)$ exists and is sublinear for $x \in \partial \overline{A}$.

(iii) From Theorem 4.2 (i), since $H$ is a Hilbert space, there exists a unique minimizer $p_A(x) \in \overline{A}$ to the minimization problem 

$$
\|p_A(x) - x\|^2 = \inf_{a \in \overline{A}} \|a - x\|^2,
$$

which is solution of the variational equation 

$$
\forall a \in \overline{A}, \quad (p_A(x) - x) \cdot (a - p_A(x)) \geq 0.
$$

For $x, y \in H$, 

$$
(p_A(x) - x) \cdot (p_A(y) - p_A(x)) \geq 0 \quad \text{and} \quad (p_A(y) - y) \cdot (p_A(x) - p_A(y)) \geq 0
\Rightarrow (p_A(x) - p_A(y)) \cdot (p_A(y) - p_A(x)) - (y - x) \geq 0
$$

$$
\|p_A(x) - p_A(y)\|^2 \leq (p_A(y) - p_A(x)) \cdot (y - x) \leq \|p_A(x) - p_A(y)\| \|y - x\|
$$

and $\|p_A(y) - p_A(x)\| \leq \|y - x\|$. \hfill \square

For a convex subset $A$ of a Hilbert space $H$ we have the existence of a unique projection $p_A(x)$ of $x$ onto $\overline{A}$ for all $x \in H$. This property is equivalently characterized by the Hadamard differentiability of $d_A^2$.

For non-convex subsets $A$ in $\mathbb{R}^n$, Federer [31] introduced the notion of set of positive reach, which is equivalent to the existence of a unique projection but only in an $h$-tubular neighborhood of $A$ defined as 

$$
U_h(A) \overset{\text{def}}{=} \{x \in H : d_A(x) < h\}, \quad h > 0, \quad (4.7)
$$

that is, in the vicinity of $A$ away from its skeleton. The projection onto $A$ exists and is unique for all $x \in U_h(A)$ if and only if $d_A^2$ is Hadamard differentiable in $U_h(A)$. Moreover, the projection $p_A$ is Lipschitz in $U_h(A)$. However, the results are not as optimal in infinite dimension.

**Theorem 4.4.**
(i) If the projection onto $A$ exists and is unique for an open ball $B_r(x)$, $r > 0$, $x \in H$, and the projection $p_A$ is weakly continuous at $x$, then $d^2_A$ is Hadamard differentiable at $x$, $\nabla d^2_A(x) = 2(x-p_A(x))$, and $\|\nabla d^2_A(x)\| = 2d_A(x)$.

(ii) If $d^2_A$ is Hadamard differentiable at $x \in H$ and $\|\nabla d^2_A(x)\| = 2d_A(x)$, then $f_A$ is Hadamard differentiable at $x$ and

$$\nabla f_A(x) = x - \frac{1}{2} \nabla d^2_A(x) \in \Pi_A(x). \quad (4.8)$$

(iii) The projection onto $A$ exists, is unique, and is weakly continuous for all $x \in U_h(A)$ if and only if $d^2_A$ is Hadamard differentiable and $\|\nabla d^2_A(x)\| = 2d_A(x)$ for all $x \in U_h(A)$.

Remark 3. In finite dimension, for an integer $k \geq 2$, the condition $d^2_A \in C^k$ characterizes a submanifold of class $C^k$ (see, for instance, [48] or [27, Thm. 6.5]).

Proof. (i) For $0 \neq v \in H$, there exists $\hat{t} > 0$, for all $t$, $0 < |t| < \hat{t}$, $x + tv \in B_r(x)$. Since $d^2_A$ is Lipschitz continuous in $B_r(x)$, it is sufficient to check that the limit

$$\lim_{t \searrow 0} \frac{d_A(x + tv)^2 - d_A(x)^2}{t} \text{ exists.}$$

By assumption, $\Pi_A(x + tv) = \{p_A(x + tv)\}$. In one direction,

$$\frac{d_A(x + tv)^2 - d_A(x)^2}{t} \leq \frac{||p_A(x) - (x + tv)||^2 - ||p_A(x) - x||^2}{t} = -2\langle p_A(x) - x - tv, v \rangle \Rightarrow \limsup_{t \searrow 0} \frac{d_A(x + tv)^2 - d_A(x)^2}{t} \leq 2(x - p_A(x)) \cdot v.$$

In the other direction,

$$\frac{d_A(x + tv)^2 - d_A(x)^2}{t} \geq \frac{||p_A(x + tv) - (x + tv)||^2 - ||p_A(x + tv) - x||^2}{t} = -2\langle p_A(x + tv) - p_A(x), v \rangle - 2(x - p_A(x)) \cdot v - t\|v\|^2$$

and, by weak continuity of $p_A$ at $x$,

$$\liminf_{t \searrow 0} \frac{d_A(x + tv)^2 - d_A(x)^2}{t} \geq 2(x - p_A(x)) \cdot v \Rightarrow \liminf_{t \searrow 0} \frac{d_A(x + tv)^2 - d_A(x)^2}{t} \geq 2(x - p_A(x)) \cdot v$$

and $d_A^2$ is Hadamard differentiable at $x$.

(ii) If $d_A^2$ is Hadamard differentiable at $x$, there exists $p \in H$ such that

$$d_H f_A(x; v) = x \cdot v - \frac{1}{2} \nabla d^2_A(x) \cdot v = p \cdot v,$$

$$p = x - \frac{1}{2} \nabla d^2_A(x), \quad x - p = \frac{1}{2} \nabla d^2_A(x), \quad \|p - x\| = d_A(x).$$

By convexity of $f_A$,

$$f_A(p) \geq f_A(x) + df_A(x; p - x)$$

$$\frac{1}{2} \left[\|p\|^2 - d_A(p)^2\right] \geq \frac{1}{2} \left[\|x\|^2 - d_A(x)^2\right] + x \cdot (p - x) - \frac{1}{2} \nabla d^2_A(x) \cdot (p - x)$$

$$d_A(p)^2 \leq \|p\|^2 - \|x\|^2 + 2x \cdot (p - x) - \nabla d^2_A(x) \cdot (p - x) - d_A(x)^2$$

$$= \|p\|^2 - \|x\|^2 - 2x \cdot (p - x) - 2\|p - x\|^2 + \|p - x\|^2 = 0$$

$d_A(p) = 0$, $p \in A$, $\|p - x\| = d_A(x)$, and $p \in \Pi_A(x)$. 

4.2. Oriented distance function. For a subset $A$ of a normed vector space $X$ such that $\partial A \neq \emptyset$, the oriented distance function from a point $x$ to $A$ is defined as

$$b_A(x) \overset{\text{def}}{=} d_A(x) - d_{\overline{A}}(x), \quad \mathcal{C} A \overset{\text{def}}{=} X \setminus A.$$  \hfill (4.9)

It provides a level set description of the set $A$: $b_A(x) < 0$ for $x \in \text{int } A$, $b_A(x) = 0$ on $\partial A$, and $b_A(x) > 0$ in $\overline{C A}$. We use the term “oriented” rather than “algebraic” or “signed” to emphasize that $\nabla b_A(x)$ is the outward unit normal at $x \in \partial A$ for an open domain with smooth boundary. The eigenvalues of the Hessian matrix $D^2 b_A(x)$ are 0 and the $(n - 1)$ principal curvatures of the boundary and $\Delta b_A(x)/(n - 1)$ is the mean curvature. It is easy to verify the following properties

$$|b_A(x)| = \max \{d_A(x), d_{\overline{A}}(x)\} = d_{\partial A}(x)$$

$$\Pi_{\partial A}(x) = \{p \in \partial A : \|x - p\| = |b_A(x)|\}.$$  \hfill (4.10, 4.11)

Theorem 4.5. Let $A$ be a subset of a normed vector space $X$ such that $\partial A \neq \emptyset$.

(i) $b_A$ is well-defined, Lipschitz continuous, and

$$\forall x, y \in X, \quad |b_A(y) - b_A(x)| \leq \|y - x\|.$$  \hfill (4.12)

(ii) For $x \in \partial A$, $T_{\partial A}^b(x)$ is a closed cone at 0 and

$$T_{\partial A}^b(x) = \left\{ v \in X : \lim_{t \searrow 0} \frac{d_{\partial A}(x + tv)}{t} = 0 \right\} = \left\{ v \in X : d_H d_{\partial A}(x; v) = 0 \right\}$$

$$= \left\{ v \in X : \lim_{t \searrow 0} \frac{b_A(x + tv)}{t} = 0 \right\} = \left\{ v \in X : d_H b_A(x; v) = 0 \right\}.$$  \hfill (4.13)

(iii) The norm $x \mapsto n(x) = \|x\|$ and the function

$$f_{\partial A}(x) \overset{\text{def}}{=} \frac{1}{2} \left[ \|x\|^2 - b_A(x)^2 \right]$$

are convex, continuous, Hadamard semidifferentiable at every $x \in X$, and $v \mapsto d_H n(x; v)$ and $v \mapsto d_H f_{\partial A}(x; v)$ are sublinear. Moreover, for all $v \in X$

$$d_H b_A^2(x; v) = n(x)d_H n(x; v) - d_H f_{\partial A}(x; v),$$

$b_A^2$ is Hadamard semidifferentiable, $v \mapsto d_H b_A^2(x; v)$ is the difference of two sublinear functions, and, for all $x \in \partial A$, $b_A(x)$ is Hadamard differentiable and $d_H b_A^2(x; v) = 0$ for all $v \in X$.

(iv) $b_A$ is Hadamard semidifferentiable in $X \setminus \partial(\partial A)$,\footnote{Recall Remark 2 on page 13. In addition,}

$$\forall v \in X, \quad |d_H b_A(x; v)| \leq \|v\|.$$  \hfill (4.16)

Then $\partial A$ is nowhere dense, that is, $\partial A = \partial(\partial A)$, if and only if $\text{int } \overline{A} \cap \text{int } \overline{\partial A} = \emptyset$, which is true if $A$ is open or close.
Remark 4. In $X = \mathbb{R}^n$, $\Pi_{\partial A}(x) \neq \emptyset$ for all $x \in X$, and, by Rademacher’s theorem, $b_A$ is Hadamard (Fréchet) differentiable almost everywhere. The gradient $\nabla b_A$ does not exist at points of the skeleton and the set of cracks:

$$\text{Sk}_b(A) \overset{\text{def}}{=} \{ x \in X : \Pi_{\partial A}(x) \text{ is not a singleton} \} \subset X \setminus \partial(\partial A)$$

(4.17)

$$\text{Ck}_b(A) \overset{\text{def}}{=} \{ x \in \partial(\partial A) : \nabla b_A(x) \text{ does not exist} \} \subset \partial(\partial A).$$

(4.18)

However, the Hadamard semidifferential exists in $X \setminus \partial(\partial A)$ even at points where the projection onto $\partial A$ is not unique or does not exist. So, we need not worry about making sense of the differentiability almost everywhere in a normed space as, for instance, in Mignot [40, Thm. 1.2, p. 133] and Aronszajn [1, Chapter II]: any Lipschitz continuous function $f : H_1 \to H_2$ between a separable Hilbert space $H_1$ and a Hilbert space $H_2$ is Gateaux differentiable (and hence Hadamard differentiable by Theorem 2.11) in a dense set.

Proof. (i) If $\partial A \neq \emptyset$, both $A$ and $\overline{\partial A}$ are non-empty, $d_A(x)$ and $d_{\overline{\partial A}}(x)$ are finite, and $b_A(x)$ is well-defined. Therefore,

$$\forall x, y \in \overline{A}, \quad |b_A(y) - b_A(x)| = | - d_{\overline{\partial A}}(y) + d_{\overline{\partial A}}(x)| \leq \|y - x\|,$$

$$\forall x, y \in \overline{\partial A}, \quad |b_A(y) - b_A(x)| = |d_A(y) - d_A(x)| \leq \|y - x\|.$$

If $x \in A$ and $y \in \overline{\partial A} \setminus \partial A = \overline{\partial A}$, then $d_A(y) > 0$ and

$$b_A(y) - b_A(x) = b_A(y) + d_{\overline{\partial A}}(x) > 0 \quad \Rightarrow \quad |b_A(y) - b_A(x)| = d_A(y) + d_{\overline{\partial A}}(x) > 0.$$

By assumption on $y \in \overline{\partial A}$, $B_{d_{\partial A}(y)}(y) \subset \overline{\partial A} = \text{int } \overline{\partial A}$ Define the point

$$\overline{y} = y + \frac{d_A(y)}{\|x - y\|}(x - y) \in B_{d_{\partial A}(y)}(y) \subset \overline{\partial A}$$

$$\Rightarrow \quad d_{\overline{\partial A}}(x) \leq \|x - \overline{y}\| = \left\| \left(1 - \frac{d_A(y)}{\|x - y\|}\right)(y - x) \right\| = \|y - x\| - d_A(y)$$

$$\Rightarrow \quad |b_A(y) - b_A(x)| = d_A(y) + d_{\overline{\partial A}}(x) \leq \|y - x\|.$$

The argument is similar for $x \in \text{int } A = \overline{\partial A} \setminus \partial A$ and $y \in \overline{\partial A}$:

$$|b_A(y) - b_A(x)| = \| - d_{\overline{\partial A}}(x) - d_A(y)| = d_A(y) + d_{\overline{\partial A}}(x).$$

Since $\overline{\partial A}$ is open, the closed ball $\overline{B}_{d_{\partial A}(x)}(x)$ is contained in $\overline{A}$ and

$$\overline{x} = x + \frac{d_{\overline{\partial A}}(x)}{\|y - x\|}(x - y) \in \overline{B}_{d_{\partial A}(x)}(x) \subset \overline{A}$$

$$\Rightarrow \quad d_A(y) \leq \|y - \overline{x}\| = \|y - x\| - d_{\overline{\partial A}}(x)$$

$$|b_A(y) - b_A(x)| = d_A(y) + d_{\overline{\partial A}}(x) \leq \|y - x\|.$$

(ii) From Theorem 2.18. Since $d_{\partial A}(x) = |b_A(x)|$, at $x_0 \in \partial A$

$$\frac{b_A(x + tv) - b_A(x)}{t} \to 0 \quad \Rightarrow \quad \frac{d_{\partial A}(x + tv)}{t} = \frac{|b_A(x + tv)|}{t} \to 0$$

and conversely.

(iii) For $a \in \partial A$, the function

$$x \mapsto \ell_a(x) = \frac{1}{2} \left[ \|x\|^2 - \|x - a\|^2 \right] = a \cdot x - \frac{1}{2} \|a\|^2$$
is affine in \( x \). So, its upper enveloppe
\[
\sup_{a \in \partial A} \ell_a(x) = \frac{1}{2} \left[ \|x\|^2 - d_{\partial A}(x)^2 \right] = \frac{1}{2} \left[ \|x\|^2 - b_A(x)^2 \right]
\]
is convex and continuous by continuity of \( d_{\partial A} \). The remainder of the proof of (iii) and (iv) is the same as the one of Theorem 4.1 (iii) and (iv).

Since \( b_A^2 \) is Hadamard differentiable at all \( x \in \partial A \), we can characterize the points where \( b_A \) is not Hadamard differentiable by modifying the definitions of \( \text{Sk} (\partial A) \) and \( \text{Ck} (\partial A) \) in such a way that they do not use \( \Pi_A(x) \).

**Definition 4.6.** Let \( A \) be a subset of a Banach space \( X \) such that \( \partial A \neq \emptyset \). The \textit{skeleton} and the \textit{set of cracks} are defined as follows:

\[
\text{Sk}_b(A) \overset{\text{def}}{=} \{ x \in X : b_A^2 \text{ is not Hadamard differentiable at } x \} \subset X \setminus \partial(\partial A) \tag{4.19}
\]

\[
\text{Ck}_b(A) \overset{\text{def}}{=} \left\{ x \in \partial(\partial A) : \begin{array}{l}
 b_A^2 \text{ is Hadamard differentiable at } x \\
 b_A \text{ is not Hadamard differentiable at } x
\end{array} \right\} \subset \partial(\partial A). \tag{4.20}
\]

In view of Theorem 4.5 (iv), \( b_A \) is Hadamard semidifferentiable except possibly in \( \partial(\partial A) \), which means that the properties of the set are characterized by the properties of \( b_A \) at or in the vicinity of its boundary as in Differential Geometry.

**Definition 4.7.** Let \( A \) be a subset of a Banach space \( X \) such that \( \partial A \neq \emptyset \).

(i) The boundary \( \partial A \) is \textit{semi-regular} if, at each \( x \in \partial A \), \( T_{\partial A}(x) = T_{\partial A}^b(x) \) and \( b_A \) is Hadamard semidifferentiable at \( x \).

(ii) The boundary \( \partial A \) is \textit{regular} if, at each \( x \in \partial A \), \( T_{\partial A}(x) = T_{\partial A}^b(x) \) and \( b_A \) is Hadamard differentiable at \( x \).

(iii) A set \( A \) such that \( \text{int} A \neq \emptyset \) is \( C^k \), \( k \geq 0 \), if, at each \( x \in \partial A \), \( T_{\partial A}(x) = T_{\partial A}^b(x) \) and \( b_A \) is \( C^k \) in some neighborhood of \( x \).

For a regular boundary, we have at each boundary point \( x \in \partial A \)
\[
T_{\partial A}^b(x) = \{ v \in X : d_H b_A(x; v) = 0 \} \tag{4.21}
\]
and the tangent space \( T_{\partial A}^b(x) \) is a closed linear subspace of \( X \) since the semidifferential is linear and continuous. In finite dimension this is a property of smooth open domains, but here we are not making an a priori smoothness assumption in a neighborhood of each boundary point.

**Definition 4.8.** Let \( A \) be a subset of a Banach space \( X \) such that \( \overline{A} = \partial A \neq \emptyset \).

(i) \( A \) is \textit{regular} if, at each \( x \in A \), \( T_{\partial A}(x) = T_{\partial A}^b(x) \) and \( b_A^2 \) is Hadamard differentiable in a neighborhood of \( x \).

(ii) A set \( A \) is \( C^k \), \( k \geq 0 \) an integer, if, at each \( x \in \partial A \), \( T_{\partial A}(x) = T_{\partial A}^b(x) \) and \( b_A^2 \) is \( C^k \) in some neighborhood \( x \).\footnote{In \( \mathbb{R}^n \) for \( k \geq 2 \) such a set is locally a \( C^k \) submanifold of dimension equal to the dimension of the image of \( D_{\partial A}(x) \) (see, for instance [48], [27, Thms. 6.4 and 6.5, pp. 311]).}

In \( \mathbb{R}^n \) the regularity is the analogue of a the notion of positive reach; as for the second notion it is related to embedded \( C^k \) submanifolds.

The assumption \( T_{\partial A}^b(x) = T_{\partial A}(x) \) is not innocent. It eliminates many pathological subsets and seems to be related to the following notion.

**Definition 4.9.** Let \( A \neq \emptyset \) be a subset of a topological space \( X \).
(i) $A$ is crack-free at $x \in \overline{A}$ if there exists a neighborhood $V(x)$ of $x$ such that $V(x) \cap \text{int} \overline{A} \cap \partial A = \emptyset$.

(ii) $A$ is crack-free if int $\overline{A} \cap \partial A = \emptyset$.

A closed set is crack-free (and $b_A = b_\partial$ in a normed space). An open set $\Omega \neq \emptyset$ is crack free $\iff \partial \Omega = \partial \Omega \iff \Omega = \text{int} \Omega \iff b_\Omega = b_{\Omega \overline{\Omega}}$ in a normed space.

**Definition 4.10.** Let $A$ be a subset of a topological space $X$ such that $\partial A \neq \emptyset$.

(i) $\partial A$ is crack-free at $x \in \partial A$ if there exists a neighborhood $V(x)$ of $x$ such that $V(x) \cap \text{int} \overline{A} \cap \partial A = \emptyset$ and $V(x) \cap \overline{A} \cap \partial A = \emptyset$.

(ii) $\partial A$ is crack-free if int $\overline{A} \cap \partial A = \emptyset$ and int $\overline{A} \cap \partial A = \emptyset$.

If $\partial A$ is crack-free, $\partial A = \overline{A} = \partial \overline{A} = \partial \partial A = (\partial \overline{A})$, int $A = \text{int} \overline{A}$, int $\overline{A} = \text{int} \partial A$, $\partial \partial A = \partial A$, and in a Banach space, $b_{\text{int} A} = b_A = b_{\overline{A}}$.

**Conjecture 1.** If $\partial A$ is crack-free at $x_0 \in \partial A$, then $T_{\partial A}^b(x_0) = T_{\partial A}(x_0)$.

We have seen that the convexity of $d_A$ is equivalent to the convexity of $\overline{A}$. This characterization remains true with $b_A$ in place of $d_A$. We now extend [27, Thm 10.1 (iii), p. 375–376] from $\mathbb{R}^n$ to a reflexive Banach space by arguments avoiding the issue of the existence of projections onto $\partial A$.

**Theorem 4.11.** Let $A$ be a subset of a reflexive Banach space $X$ such that $\partial A \neq \emptyset$.

(i) $b_A$ is convex if and only if $\overline{A}$ is convex.

(ii) If $\overline{A}$ is convex, then $b_A$ is Hadamard semidifferentiable and $v \mapsto d_Hb_A(x;v)$ is sublinear. In particular, $b_A$, $d_{\partial A}$, and $d_{\partial A} = d_A + d_{\partial A}$ are Hadamard semidifferentiable, and $A$ is semi-regular. If int $A \neq \emptyset$, int $\overline{A} = \overline{A}$ and for each $x \in \partial A$, there exists $v \in X$ such that $d_Hb_A(x;v) < 0$.

**Proof.** (i) Let $x_\lambda = \lambda x + (1 - \lambda)y$ for $x$ and $y$ in $\overline{A}$ and $\lambda \in [0,1]$. If $b_A$ is convex, $b_A(x_\lambda) \leq \lambda b_A(x) + (1 - \lambda)b_A(y) = -[\lambda d_{\partial A}(x) + (1 - \lambda)d_{\partial A}(y)] \leq 0$, $d_A(x_\lambda) = [b_A(x_\lambda)]^{\dagger} = 0$, $x_\lambda \in \overline{A}$, and $\overline{A}$ is convex.

Conversely, assume that $\overline{A}$ is convex. If int $A = \emptyset$, then $\overline{A} = X$, $d_{\partial A} = 0$ and $b_A = d_A$ is convex by Theorem 4.2 (ii). For int $A \neq \emptyset$, consider three cases: a) $x,y \in \overline{A}$; b) $x,y \in \overline{A}$; c) $x \in \overline{A}$, $y \in \partial A$.

a) If $x,y \in \overline{A}$, then $x,y \in \overline{A}$, $d_{\partial A}(x) = 0 = d_{\partial A}(y)$ and, since $d_A$ is convex, $b_A(\lambda x + (1 - \lambda)y) \leq d_A(x) + (1 - \lambda)y) \leq \lambda d_A(x) + (1 - \lambda)d_A(y) = \lambda b_A(x) + (1 - \lambda)b_A(y)$.

b) Associate with $x$ and $y$ in $\overline{A}$ the radii $r_x = d_{\partial A}(x)$, $r_y = d_{\partial A}(y)$, and $r_\lambda = \lambda r_x + (1 - \lambda)r_y$ and the open (possibly empty) balls $B_x$ of center $x$ and radius $r_x$, $B_y$ of center $y$ and radius $r_y$, and $B_\lambda$ of center $x_\lambda$ and radius $r_\lambda$. By definition, $B_x \subset \text{int} A$ and $B_y \subset \text{int} A$.

If $r_x = 0$ and $r_y = 0$, then $x,y \in \partial A$, $x,y \in \overline{A} \cap \overline{A} = \partial A$, and $x_\lambda \in \overline{A}$. Therefore, $b_A(x) = b_A(y) = 0$, $d_A(x_\lambda) = 0$, and $b_A(x_\lambda) = -d_{\partial A}(x_\lambda) \leq 0 = \lambda b_A(x) + (1 - \lambda)b_A(y)$.

If $r_x > 0$ and $r_y > 0$, associate with each $z \in B_\lambda$ the points $z_x \overset{\text{def}}{=} x + r_x \overline{A} - x_\lambda \Rightarrow z_x \in B_x$, and $z_y \overset{\text{def}}{=} y + r_y \overline{A} - x_\lambda \Rightarrow z_y \in B_y$.
\[ \Rightarrow \lambda x + (1-\lambda)z = x\lambda + \frac{\lambda r_y + (1-\lambda)r_x}{r_x} (z - x\lambda) = z. \]

Therefore, \( B_\lambda \subset \lambda B_x + (1-\lambda)B_y \subset \text{int } A = \overline{\mathbb{C}A} \), since \( \text{int } A \) is convex. Then \( \overline{\mathbb{C}A} \subset \mathbb{C}B_\lambda \) and

\[ d_{\mathbb{C}A}(x\lambda) \geq d_{\mathbb{C}B_\lambda}(x\lambda) \geq r_\lambda = \lambda d_{\mathbb{C}A}(x) + (1-\lambda)d_{\mathbb{C}A}(y). \]

But \( d_A(x) = d_A(y) = d_A(x\lambda) = 0 \) and

\[ b_{\mathbb{C}A}(x\lambda) \leq \lambda b_{\mathbb{C}A}(x) + (1-\lambda)b_{\mathbb{C}A}(y). \]

If \( r_x = 0 \) and \( r_y > 0 \), then \( r_\lambda = (1-\lambda)r_y, \ x \in \partial A, \) and \( y \in \mathbb{C}A \). For \( \lambda = 1, \ x = x, \ b_A(x) = 0, \ r_\lambda = 0, \) and

\[ b_A(x\lambda) = b_A(x) = 0 = \lambda b_A(x) + (1-\lambda)b_A(y). \]

For \( 0 \leq \lambda < 1, \ b_A(x) = 0, \ r_\lambda = (1-\lambda)r_y > 0, \ b_A(y) = -d_{\mathbb{C}A}(y) < 0, \ x \in \partial A \) and \( y \in \text{int } A \). Hence \( \{ \lambda x + (1-\lambda)y : 0 \leq \lambda < 1 \} \subset \text{int } A \) and

\[ \forall \lambda, \ 0 \leq \lambda < 1, \ \lambda x + (1-\lambda)B_y \subset \text{int } A. \]

For \( z \in B_\lambda \)

\[ \|z - x\lambda\| < r_\lambda = (1-\lambda)r_y \Rightarrow \frac{z - x\lambda}{1-\lambda} < r_y \Rightarrow y + \frac{z - x\lambda}{1-\lambda} \in B_y \]

\[ z = (1-\lambda)\frac{z - x\lambda}{1-\lambda} + x\lambda = \lambda x + (1-\lambda) \left[ y + \frac{z - x\lambda}{1-\lambda} \right] \in \lambda x + (1-\lambda)B_y. \]

\[ \Rightarrow B_\lambda \subset \lambda x + (1-\lambda)B_y \subset \text{int } A \Rightarrow \overline{\mathbb{C}A} \subset \mathbb{C}B_\lambda. \]

Conversely, for \( z \in B_y \)

\[ \|\lambda x + (1-\lambda)z - x\lambda\| = (1-\lambda)\|z - y\| < (1-\lambda)r_y = r_\lambda \Rightarrow \lambda x + (1-\lambda)z \in B_\lambda, \]

\( \lambda x + (1-\lambda)B_y \subset B_\lambda, \) and \( \lambda x + (1-\lambda)B_y = B_\lambda \). Finally,

\[ d_{\mathbb{C}A}(x\lambda) = d_{\mathbb{C}B_\lambda}(x\lambda) \geq r_\lambda = \lambda d_{\mathbb{C}A}(x) + (1-\lambda)d_{\mathbb{C}A}(y). \]

But \( b_A(x) = d_A(x) = 0, \ d_A(x\lambda) = 0, \ d_A(y) = 0, \) and

\[ b_A(x\lambda) \leq \lambda b_A(x) + (1-\lambda)b_A(y). \]

(ii) The Hadamard semidifferentiability of \( b_A \) and the sublinearity of \( v \mapsto d_H b_A(x; v) \) follow from Theorem 2.13 (i). Since \( d_A \) is also Hadamard semidifferentiable by Theorem 4.2 (iii) \( d_{\mathbb{C}A} = d_A - b_A \) and \( d_{\mathbb{C}A} = d_A + d_{\mathbb{C}A} \) are also Hadamard semidifferentiable. If \( x \in \partial A \), choose a point \( y \in \text{int } A \) and by convexity of \( b_A \)

\[ 0 > -d_{\mathbb{C}A}(y) = b_A(y) = b_A(y) - b_A(x) \geq d_H b_A(x; y - x) \]

and choose \( v = y - x \).

\[ \square \]

**Theorem 4.12.** Let \( A \) be a subset of a Hilbert space \( H \) such that \( \partial A \neq \emptyset \).

(i) The function \( b_A^2 \) is Hadamard semidifferentiable,

\[ \forall x \in H, \forall v \in H, \quad d_H b_A^2(x; v) = 2 \ |x - v - d_H f_{\partial A}(x)|, \quad (4.22) \]

and \( v \mapsto d_H b_A^2(x; v) \) is suplinear.

\[ \underline{12} \text{See, for instance, [29, sec. 1.2, p. 4],} \]
(ii) For \( x \in H \setminus \partial A \) and \( v \in H \), \( d_{H} b_{A}(x; v) \) exists. For \( x \in \text{int } A \), \( v \mapsto d_{H} b_{A}(x; v) \) is sublinear and for \( x \in \overline{A} \), \( v \mapsto d_{H} b_{A}(x; v) \) is sublinear. As for \( x \in \partial A \), \( d_{H} b_{A}(x; v) \) exists if and only if
\[
\lim_{t \searrow 0} \frac{b_{A}(x + tv) - b_{A}(x)}{t} = \lim_{t \searrow 0} \frac{b_{A}(x + tv)}{t} \quad \text{exists.}
\]

**Proof.** (i) From Theorem 2.13 (i), \( f_{\partial A} \) is Hadamard semidifferentiable in \( H \) and \( v \mapsto d_{H} f_{\partial A}(x; v) \) is sublinear. Since, in a Hilbert space, \( \|x\|^2 \) is Hadamard differentiable, \( v \mapsto d_{H} b_{\partial A}(x; v) \) is sublinear.

(ii) For \( x \notin \partial A \), \( b_{A}(x) \neq 0 \) and
\[
v \mapsto d_{H} b_{A}(x; v) = \frac{1}{2b_{A}(x)} d_{H} b_{\partial A}^{2}(x; v) = \frac{1}{2b_{A}(x)} \left[ 2x \cdot v - d_{H} f_{\partial A}(x; v) \right]
\]
For \( x \in \text{int } A \), \( v \mapsto d_{H} b_{A}(x; v) \) is sublinear since \( b_{A}(x) < 0 \) and, for \( x \in \overline{A} \), it is sublinear since \( b_{A}(x) > 0 \). As, for \( x \in \partial A \), \( d_{H} b_{A}(x; v) \) exists if and only if
\[
\lim_{t \searrow 0} \frac{b_{A}(x + tv) - b_{A}(x)}{t} = \lim_{t \searrow 0} \frac{b_{A}(x + tv)}{t} \quad \text{exists.}
\]
For closed subsets of a normed vector space without interior, that is, \( A = \partial A \) and \( b_{A} = d_{A} = d_{\partial A} \), we are back to the distance function of section 4.1.

5. Minimization problem for semidifferentiable objective and constraint functions. To illustrate some advantages of the oriented distance function in non-differentiable optimization, we extend the finite dimensional constructions and theorems of [21, Chapter 5].

5.1. Karush, John, Kuhn-Tucker for a finite number of constraints. In this section, we provide a theorem of the Karush, John, Kuhn-Tucker type for objective and constraint functions \( f \) that are Hadamard semidifferentiable and \( v \mapsto d_{M} f(x_{0}; v) \) is sublinear. This includes both functions that are Hadamard differentiable and continuous convex functions.

5.1.1. Saddle point and auxiliary lemma.

**Theorem 5.1** (Ekeland-Temam [29, Prop. 2.1, p. 171]). Let \( X \) and \( Y \) be two reflexive Banach spaces. Let \( G : X \times Y \to \mathbb{R} \) and \( A \subset X \) and \( B \subset Y \) two bounded closed convex nonempty subsets with the following assumptions:

(a) for each \( y \in B \), \( x \mapsto G(x, y) : A \to \mathbb{R} \) is convex and lower semicontinuous;
(b) for each \( x \in A \), \( y \mapsto G(x, y) : B \to \mathbb{R} \) is concave and upper semicontinuous.

Then \( G \) has a saddle point in \((\hat{a}, \hat{b}) \in A \times B\), that is,
\[
\forall a \in A, \forall b \in B, \quad G(\hat{a}, b) \leq G(\hat{a}, \hat{b}) \leq G(a, \hat{b}). \quad (5.1)
\]

**Lemma 5.2.** Let \( p \geq 1 \) be an integer, \( M \) a closed linear subspace of a reflexive Banach space \( X \), and \( s_{i} : M \to \mathbb{R}, 1 \leq i \leq p \), lower semicontinuous sublinear functions such that
\[
\forall x \in M, \quad \max\{s_{1}(x), \ldots, s_{p}(x)\} \geq 0. \quad (5.2)
\]
Then, there exists
\[
\alpha = (\alpha_{1}, \ldots, \alpha_{p}) \in \mathbb{S} \overset{\text{def}}{=} \left\{ (\alpha_{1}, \ldots, \alpha_{p}) : \alpha_{i} \geq 0, \sum_{i=1}^{p} \alpha_{i} = 1 \right\}. \quad (5.3)
\]
such that

$$\forall x \in M, \sum_{i=1}^{p} \alpha_i s_i(x) \geq 0. \quad (5.4)$$

**Proof.** Introduce the function

$$(\alpha, x) \mapsto \ell(\alpha, x) \overset{\text{def}}{=} \sum_{i=1}^{p} \alpha_i s_i(x) : \mathbb{R}^p \times M \to \mathbb{R}.$$ 

Since the $s_i(x)$ are positively homogeneous, it is sufficient to prove the result for the weakly compact convex set $B = \{ x \in M : \|x\| \leq 1 \}$. For each $\alpha$ the mapping $x \mapsto \ell(\alpha, x)$ is convex and continuous. For each $x$, the mapping $\alpha \mapsto \ell(\alpha, x)$ is linear and continuous. The set $S$ is convex and compact and $B$ is convex and weakly compact. By Theorem 5.1, there exists a saddle point $(\hat{\alpha}, \hat{x}) \in S \times B$:

$$\forall x \in B, \forall \alpha \in S, \quad \ell(\alpha, \hat{x}) \leq \ell(\hat{\alpha}, \hat{x}) \leq \ell(\alpha, x).$$

In view of the special form of $\ell(\alpha, \hat{x})$ and assumption (5.2),

$$\sup_{\alpha \in S} \ell(\alpha, \hat{x}) = \max_{1 \leq i \leq p} s_i(\hat{x}) \geq 0.$$ 

This yields the result

$$\forall x \in B, \quad \sum_{i=1}^{p} \hat{\alpha}_i s_i(x) = \ell(\hat{\alpha}, x) \geq \ell(\hat{\alpha}, \hat{x}) \geq \sup_{\alpha \in S} \ell(\alpha, \hat{x}) \geq 0$$

and hence inequality (5.4).

5.1.2. **Fundamental Lemma.** Let $X$ be a topological vector space and $g_i : X \to \mathbb{R}$, $1 \leq i \leq m$, be the functions defining the set of constraints

$$U \overset{\text{def}}{=} \{ x \in X : g_i(x) \leq 0, 1 = 1, \ldots, m \}$$

and let $x_0 \in U$. Assume that the functions $g_i$ are Hadamard semidifferentiable and continuous\textsuperscript{13} at $x_0$ and that their semidifferential is sublinear. The $m$ constraint functions $g_i$ can be lumped into a single one

$$U = \{ x \in X : g(x) \leq 0 \}, \quad g(x) \overset{\text{def}}{=} \max_{1 \leq i \leq m} g_i(x),$$

$$d_H g(x_0; v) = \max_{i \in I(x)} d_H g_i(x_0; v), \quad I(x) \overset{\text{def}}{=} \{ i : g_i(x_0) = g(x_0) \},$$

$$\{ v \in X : d_H g(x_0; v) \leq 0 \} = \{ v \in X : \forall i \in I(x_0) : d_H g_i(x_0; v) \leq 0 \},$$

$v \mapsto d_H g(x_0; v)$ is sublinear and $g$ is continuous at $x_0$.

**Lemma 5.3.** Let $X$ be a TVS, $g : X \to \mathbb{R}$, the set of constraint

$$\overset{\text{def}}{=} \{ x \in X : g(x) \leq 0 \}. \quad (5.5)$$

Assume that $g$ is continuous and Hadamard semidifferentiable at $x_0 \in U$ and that its semidifferential is sublinear.\textsuperscript{14} For $x_0$ such that $g(x_0) = 0$ define the cone

$$C \overset{\text{def}}{=} \{ v \in X : d_H g(x_0; v) < 0 \}. \quad (5.6)$$

\textsuperscript{13}For simplicity we add the continuity to avoid adding the existence of a bounded neighborhood of the origin of Theorem 2.9).

\textsuperscript{14}Since $g$ is continuous, int $U = \{ x \in X : g(x) < 0 \}$ and $\partial U = \{ x \in X : g(x) = 0 \}$.
Proof of Corollary 2. If, in addition to the assumptions of Lemma 5.3, \( g \) is convex, then \( C \neq \emptyset \) if there exists \( x^* \in X \) such that \( g(x^*) < 0 \) or, equivalently, if \( \text{int} \, U \neq \emptyset \).

Proof. (i) Since \( g \) is Hadamard semidifferentiable at \( x_0 \), \( g \) is continuous at \( x_0 \). If \( g(x_0) < 0 \), there exists \( r > 0 \) such that \( B_r(x_0) \subset \{ x \in X: g(x) < 0 \} \) and \( x_0 \in \text{int} \, U \), which means that \( C = T^0_U(x_0) = X \).

If \( g(x_0) = 0 \), then \( U = \{ x \in X: g(x) \leq g(x_0) \} \). By definition, for all \( v \in T^0_U(x_0) \),

\[
\forall \{ t_n > 0 \}, \, t_n \searrow 0, \quad \exists \{ x_n \}, \, g(x_n) \leq g(x_0), \quad \lim_{n \to \infty} \frac{x_n - x_0}{t_n} = v
\]

\[\Rightarrow \frac{g(x_n) - g(x_0)}{t_n} \leq 0 \Rightarrow d_H g(x_0; v) \leq 0 \Rightarrow T^0_U(x_0) \subset \{ v \in X: d_H g(x_0; v) \leq 0 \}.\]

Therefore, \( T^0_U(x_0) \subset \{ v \in X: d_H g(x_0; v) \leq 0 \} \).

If \( C = \emptyset \), \( C \subset T^0_U(x_0) \subset \{ h \in X: d_H g(x_0; v) \leq 0 \} \). If \( v \in C \), then \( d_H g(x_0; v) < 0 \) and for any semitrajectory \( h \) such that \( h'(0^+) = v \), there exists \( \delta > 0 \) such that

\[
\forall 0 < t < \delta, \quad \frac{g(h(t))}{t} = \frac{g(h(t)) - g(x_0)}{t} \leq \frac{1}{2} d_H g(x_0; v) < 0
\]

\[\Rightarrow \forall 0 < t < \delta, \quad g(h(t)) \leq \frac{t}{2} d_H g(x_0; v) < 0 \text{ and } h(t) \in U \Rightarrow v \in T^0_U(x_0)
\]

and \( C \subset T^0_U(x_0) \subset \{ v \in X: d_H g(x_0; v) \leq 0 \} \).

(II) Let \( v \) such that \( d_H g(x_0; v) \leq 0 \). Since \( C \neq \emptyset \), there exists \( v^* \) such that \( d_H g(x_0; v^*) < 0 \). Associate with \( v \), the sequence

\[v_n = v + \frac{1}{n} v^* \to h, \quad n \geq 1.\]

By sublinearity

\[d_H g(x_0; v_n) \leq d_H g(x_0; v) + \frac{1}{n} d_H g(x_0; v^*) < d_H g(x_0; v) \leq 0
\]

\[\Rightarrow d_H g(x_0; v_n) < 0 \Rightarrow v_n \in C \Rightarrow v_n \to v \in C\]

\[C \subset T^0_U(x_0) \subset \{ h \in X: d_H g(x_0; h) \leq 0 \} \subset C\]

(III) By definition of \( C \) for \( g(x_0) = 0 \), \( C = \emptyset \) if and only if \( d_H g(x_0; v) \geq 0 \) for all \( v \in X \). For \( g(x_0) = 0 \) and any semitrajectory \( h \) in \( U \) at \( x_0 \) such that \( h'(0^+) = v \)

\[
\frac{g(h(t)) - g(x_0)}{t} \leq 0 \Rightarrow 0 \leq d_H g(x_0; v) \leq 0 \Rightarrow d_H g(x_0; v) = 0.
\]

and \( T^0_U(x_0) \subset \{ v \in X: d_H g(x_0; v) = 0 \} \).

\[\Box\]

Proof of Corollary 2. By convexity of \( g \) for \( x_0 \) such that \( g(x_0) = 0 \)

\[g(x^*) \geq g(x_0) + d_H g(x_0; x^* - x_0)
\]

\[0 > g(x^*) = g(x^*) - g(x_0) \geq d_H g(x_0; x^* - x_0).
\]
Then, the direction $x^* - x_0 \in C$ and hence $C \neq \emptyset$.

5.1.3. Main Theorem.

**Theorem 5.4.** Let $X$ be a reflexive Banach space, $g : X \to \mathbb{R}$,

$$U \overset{\text{def}}{=} \{x \in X : g(x) \leq 0\}, \quad (5.10)$$

and $x_0$ a local minimizer of the objective function $f : X \to \mathbb{R}$ with respect to $U$. Assume that

(a) the functions $f$ and $g$ are Hadamard semidifferentiable at $x_0$

(b) and that $v \mapsto d_H f(x_0; v)$ and $v \mapsto d_H g(x_0; v)$ are sublinear.

Then, the following conditions are verified.

(i) There exists $0 \leq \lambda \leq 1$ such that

$$\forall v \in X, \quad (1 - \lambda) d_H f(x_0; v) + \lambda d_H g(x_0; v) \geq 0. \quad (5.11)$$

(ii) If $C \neq \emptyset$ or $g(x_0) < 0$, there exists $0 \leq \lambda < 1$ such that

$$\lambda \overset{1 - \lambda}{\geq} 0, \quad g(x_0) \leq 0, \quad \frac{\lambda}{1 - \lambda} g(x_0) = 0,$$

$$\forall v \in X, \quad d_H f(x_0; v) + \frac{\lambda}{1 - \lambda} d_H g(x_0; v) \geq 0. \quad (5.12)$$

(iii) If $C = \emptyset$ and $g(x_0) = 0$, for all $\mu \geq 0$,

$$g(x_0) \leq 0, \quad \mu g(x_0) = 0,$$

$$\forall v \in X, \quad \mu d_H g(x_0; v) = 0. \quad (5.13)$$

(iv) There exist multipliers $\mu_0 \geq 0$ and $\mu_1 \geq 0$, not both zero, such that

$$g(x_0) \leq 0, \quad \mu_1 g(x_0) = 0,$$

$$\forall v \in X, \quad \mu_0 d_H f(x_0; v) + \mu_1 d_H g(x_0; v) \geq 0. \quad (5.15)$$

**Proof of Theorem 5.4.** (i) Let $x_0 \in U$ be a local minimizer of $f$ with respect to $U$. Consider the auxiliary function

$$x \mapsto F(x) \overset{\text{def}}{=} \max \{f(x) - f(x_0), g(x)\} : X \to \mathbb{R}$$

for which, for all $x \in U$, $F(x) \geq 0$. Since $x_0 \in U$ is a local minimum, there exists $r > 0$ such that

$$\inf_{x \in U \cap B_r(x_0)} F(x) = F(x_0) = \max \{0, g(x_0)\} = 0.$$

Since each function is Hadamard semidifferentiable, the max is also Hadamard semidifferentiable and is given by

$$d_H F(x_0; v) = \begin{cases} 
\max \{d_H f(x_0; v), d_H g(x_0; v)\}, & g(x_0) = 0, \\
\max \{d_H f(x_0; v), d_H g(x_0; v)\}, & g(x_0) < 0.
\end{cases}$$

The optimality condition gives for all $v \in X$

$$0 \leq d_H F(x_0; v) = \begin{cases} 
\max \{d_H f(x_0; v), d_H g(x_0; v)\}, & g(x_0) = 0, \\
\max \{d_H f(x_0; v), d_H g(x_0; v)\}, & g(x_0) < 0.
\end{cases}$$

For $g(x_0) = 0$

$$\forall v \in X, \quad \max \{d_H f(x_0; v), d_H g(x_0; v)\} \geq 0.$$
Since those semidifferentials are sublinear, the conditions of Lemma 5.2 are satisfied:

\[ \exists \lambda, 0 \leq \lambda \leq 1, \quad \forall v, \quad (1 - \lambda) d_H f(x_0; v) + \lambda d_H g(x_0; v) \geq 0. \quad (5.16) \]

If \( g(x_0) < 0 \), then

\[ \forall v \in X, \quad d_H f(x_0; v) \geq 0 \]

and we can choose \( \lambda = 0 \).

(ii) From part (i) there exists \( 0 \leq \lambda \leq 1 \) that satisfies (5.16). If \( g(x_0) < 0 \), \( x_0 \in \text{int} U \), \( T_U(x_0) = X \), \( d_H f(x_0; v) \geq 0 \) for all \( v \in X \), and we can pick \( \lambda = 0 \) and \( \lambda g(x_0) = 0 \). If \( C \neq \emptyset \) and \( g(x_0) = 0 \), assume that \( \lambda = 1 \). Then

\[ \forall v \in X, \quad d_H g(x_0; v) \geq 0 \quad \Rightarrow \quad C = \emptyset \]

and we have a contradiction. So there exists \( 0 \leq \lambda < 1 \) such that

\[ \frac{\lambda}{1 - \lambda} \geq 0, \quad g(x_0) \leq 0, \quad \frac{\lambda}{1 - \lambda} g(x_0) = 0, \quad (5.17) \]

\[ \forall v \in X, \quad d_H f(x_0; v) + \frac{\lambda}{1 - \lambda} d_H g(x_0; v) \geq 0. \]

(iii) From Lemma 5.3 (iii).

(iv) From (ii) and (iii). \( \Box \)

5.1.4. Sufficient Conditions for Constraint Qualification. We have shown in Lemma 5.3 (ii) that the condition \( C \neq \emptyset \) or \( g(x_0) < 0 \) leads to the constraint qualification. Then, according to Theorem 5.4 (ii), the constraint qualification (5.8) implies that the set of multipliers

\[ \Lambda(x_0) \overset{\text{def}}{=} \left\{ \mu \geq 0 \ \bigg| \forall v \in X, \right. \]

\[ \left. d_H f(x_0; v) + \mu d_H g(x_0; v) \geq 0 \right\} \quad (5.18) \]

is not empty. We have the equivalence with the condition of J. Gauvin [32] and the condition of R. W. Cottle.

**Theorem 5.5.** Let the assumptions of Theorem 5.4 be verified for a minimizer \( x_0 \in U \) of \( f \) such that \( g(x_0) = 0 \). Then, we have the following equivalences:

(i) \( C \neq \emptyset ; \)

(ii) \( \Lambda(x_0) \) is (convex) nonempty and bounded;

(iii) \( \mu = 0 \) is the only solution of the inequality system

\[ \forall v \in X, \forall \mu \geq 0, \quad \mu d_H g(x_0; v) \geq 0. \]

**Proof.** (i) \( \Rightarrow \) (ii). If \( C \neq \emptyset \), then from Lemma 5.3 (ii),

\[ \overline{C} = T_U(x_0) = \{ v \in X : d_H g(x_0; v) \leq 0 \} \]

and, from Theorem 5.4, there exists \( \mu \in \Lambda(x_0) \), that is,

\[ \forall v \in X, \quad d_H f(x_0; v) + \mu d_H g(x_0; v) \geq 0. \]

Since \( C \neq \emptyset \), there exists \( \bar{v} \in C \) such that \( d_H g(x_0; \bar{v}) < 0 \), and for any \( \mu \in \Lambda(x_0) \)

\[ 0 \leq -\mu d_H g(x_0; \bar{v}) \leq d_H f(x_0; \bar{v}) \quad \Rightarrow \quad 0 \leq \mu \leq -\frac{d_H f(x_0; \bar{v})}{d_H g(x_0; \bar{v})}. \]

So, the set \( \Lambda(x_0) \) is bounded.

(ii) \( \Rightarrow \) (iii). If \( \Lambda(x_0) \neq \emptyset \) is bounded, then there exists \( \overline{\mu} \geq 0 \) such that

\[ \forall v \in X, \quad d_H f(x_0; v) + \overline{\mu} d_H g(x_0; v) \geq 0. \]
If there exists $\mu > 0$ such that for all $v \in X$, $\mu d_H g(x_0; v) \geq 0$, then for all integers $n \geq 1$, $n^p + n \mu \in \Lambda(x_0)$ that contradicts the fact that $\Lambda(x_0)$ is bounded.

(iii) $\Rightarrow$ (i). From Lemma 5.3 (iii), $C \neq \emptyset$.

5.2. Constrained optimization via the oriented distance function. Let $X$ be a Banach space and $f : X \rightarrow \mathbb{R}$ an objective function. For a closed subset $A$ of $X$ with $A \neq \emptyset$, the constrained minimization problem of section 3 can be written with an equality constraint on the distance function $d_A$

$$\inf_{x \in A} f(x) = \inf_{x \in X, d_A(x) = 0} f(x). \quad (5.19)$$

Alternatively, for a closed subset $A$, $\partial A \neq \emptyset$, the constrained minimization problem can be written with an inequality constraint on the oriented distance function $b_A$

$$\inf_{x \in A} f(x) = \inf_{x \in X, b_A(x) \leq 0} f(x) \quad (5.20)$$

for which the results of section 5.1 are applicable. The two formulations are equivalent but an inequality constraint does not require some implicit function theorem.

If $x$ is a local minimizer, we have two cases: $x \in \text{int} A$ and $x \in \partial A$. In the first case, $T_A^b(x) = X$,

$$x \in \text{int} A, \quad d_H f(x; v) \geq 0 \text{ for all } v \in X, \quad b_A(x) < 0, \quad (5.21)$$

and the problem is unconstrained. In the second case, $b_A(x) = 0$ and the constraint is saturated. For a closed convex subset $A$ of a reflexive Banach space, $d_H b_A(x; v)$ exists and $v \mapsto d_H b_A(x; v)$ is sublinear. This is also true at points $x \in \text{int} A$ of an arbitrary closed set $A$ of a Hilbert space by Theorem 4.12 (ii). So, for the minimization of an objective function $f$, it is reasonable to assume that, for $x \in A$, $v \mapsto d_H f(x; v)$ and $v \mapsto d_H b_A(x; v)$ exist and are sublinear. Under those assumptions, we can use the results of section 5.1 with the single inequality constraint function $g = b_A$.

**Assumption 1.** Let $A$ be a closed subset of a Banach space $X$ such that $\partial A \neq \emptyset$. Assume that, for $x \in A$, $d_H b_A(x; v)$ exists and

$$v \mapsto d_H b_A(x; v) : X \rightarrow \mathbb{R} \text{ is sublinear.} \quad (5.22)$$

This assumption is verified for convex subsets with a non-empty boundary in a reflexive Banach space. For closed non-convex $A$ n a Hilbert space, we need to add, at each $x \in \partial A$, that $d_H b_A(x; v)$ exists and

$$v \mapsto d_H b_A(x; v) : X \rightarrow \mathbb{R} \text{ is sublinear.} \quad (5.23)$$

This is also verified if $b_A$ is Hadamard differentiable in a neighborhood of $x \in \partial A$. It generalizes the case of a smooth bounded open domain $\Omega$ in $\mathbb{R}^n$, where the boundary $\partial \Omega$ of $\Omega$ is $C^k$, $k \geq 2$, if and only if $b_A \in C^k$ in a tubular neighbourhood of $\partial \Omega$ (see [27, Thms. 8.1 and 8.2, sec. 8. Chapter 7, pp. 365–366]). There is a clear advantage of the oriented distance function over the distance function.

As for $f$, the property that $v \mapsto d_H f(x; v)$ is sublinear is verified when $f$ is continuous and convex (Theorems 2.13 (i)) and when $f$ is Hadamard differentiable.
5.2.1. **Fundamental Lemma.** The set of constraints $U$ of Lemma 5.3 becomes

$$U = \{ x \in X : b_A(x) \leq 0 \} = A. \quad (5.24)$$

If $x_0 \in \text{int} \ A$, $T_A(x_0) = X$. Recall that for $x_0 \in A$ (Theorem 2.18)

$$T_A^+(x_0) = \left\{ v \in X : \lim_{t \searrow 0} \frac{d_A(x_0 + tv)}{t} = 0 \right\}, \quad x_0 \in A, \quad (5.25)$$

$$T_{\partial A}^+(x_0) = \left\{ v \in X : \lim_{t \searrow 0} \frac{b_A(x_0 + tv)}{t} = 0 \right\}, \quad x_0 \in \partial A. \quad (5.26)$$

Under Assumption 1, for $x_0 \in \partial A$, we would like to find under which conditions

$$T_A^+(x_0) = \{ v \in X : d_H b_A(x_0; v) \leq 0 \} \quad (5.27)$$

in order to get a finer description of the adjacent tangent cone at points $x_0 \in \partial A$ that would allow the use of the results of Theorem 5.4 in the previous section.

**Lemma 5.6.** Let $A$ verify Assumption 1. For $x_0 \in \partial A$ define the cone

$$C(x_0) \overset{\text{def}}{=} \{ v \in X : d_H b_A(x_0; v) < 0 \}. \quad (5.28)$$

(i) Then, for $x_0 \in \text{int} \ A$, $T_A^+(x_0) = X$; for $x_0 \in \partial A$,

$$C(x_0) \subset T_A^+(x_0) \subset \{ v \in X : d_H b_A(x_0; v) \leq 0 \}. \quad (5.29)$$

(ii) For $x_0 \in \partial A$ and $C(x_0) \neq \emptyset$,

$$C(x_0) = T_A^+(x_0) = \{ v \in X : d_H b_A(x_0; v) \leq 0 \}. \quad (5.30)$$

(iii) For $x_0 \in \partial A$, $C(x_0) = \emptyset$ if and only if, for all $v \in X$, $d_H b_A(x_0; v) \geq 0$.

Moreover, $T_A^+(x_0) = \{ v \in X : d_H b_A(x_0; v) = 0 \} = T_{\partial A}^+(x_0)$.

(iv) If $A = \partial A$, then $b_A = d_A = d_{\partial A}$, for all $x_0 \in \partial A$ and $v \in X$, $d_H b_A(x_0; v) \geq 0$,

$$C(x_0) = \emptyset, \quad \text{and} \quad T_{\partial A}^+(x_0) = \{ v \in X : d_H b_A(x_0; v) = 0 \}.$$

**Remark 5.** From part (iv), the constraint qualification $C(x_0) \neq \emptyset$ will not be verified for a set $A$ without interior.

**Proof of Lemma 5.6.** (i) to (ii) From Lemma 5.3.

(iii) From Lemma 5.3 (iii) with $g = b_A$. For $x_0 \in \partial A$,

$$T_A^+(x_0) \subset \{ v \in X : d_H b_A(x; v) = 0 \} = T_{\partial A}^+(x_0) \subset T_A^+(x_0). \quad \Box$$

(iv) If $A = \partial A$, then $b_A = d_A = d_{\partial A}$ and, for all $x_0 \in \partial A$ and $v \in X$, $d_H b_A(x_0; v) \geq 0$,

and $T_A^+(x_0) = \{ v \in X : d_H b_A(x; v) = 0 \} = T_{\partial A}^+(x_0)$.

5.2.2. **Main Theorem.**

**Theorem 5.7.** Let $A$ be a closed subset of a reflexive Banach space $X$ such that $\partial A \neq \emptyset$ and let $x_0 \in A$ be a local minimizer of the objective function $f : X \to \mathbb{R}$ with respect to $A$. Assume that

(a) the function $f$ is Hadamard semidifferentiable at $x_0$ and that $v \mapsto d_H f(x_0; v)$ is sublinear

(b) and that Assumption 1 is verified for $A$.

Then, the following conditions are verified.

(i) There exist $0 \leq \lambda \leq 1$ such that

$$\forall v \in X, \quad (1 - \lambda) d_H f(x_0; v) + \lambda d_H b_A(x_0; v) \geq 0. \quad (5.31)$$
5.2.3. The convex case for $\overline{A} = \partial A$. For a convex subset $A$ without interior, the constraint qualification $d_H b_A(x; v) < 0$ for some $v$ is not verified.

Theorem 5.9. Let $A$, $\partial A \neq \emptyset$, be a convex subset of a reflexive Banach space $X$.

(i) If $\overline{A} = \partial A$ then

$$\exists x \in \partial A \text{ such that } \forall v \in X, d_H b_A(x; v) \geq 0.$$  

(ii) If $\overline{A} = \partial A$, then

$$\overline{A} \subset \text{aff } A \triangleq \{ \alpha x + (1 - \alpha) y : x, y \in \overline{A}, \alpha \in \mathbb{R} \},$$  

where $\text{aff } A$ is a closed affine subspace of $X$ and $M \triangleq \text{aff } A - a$, $a \in \overline{A}$, is a closed linear subspace of $X$ which is independent of $a$.

Remark 6. From (ii), if $f$ is convex, $A$ is closed convex, and $C(x_0) \neq \emptyset$, then

$$\forall y \in \overline{A}, \quad f(y) \geq f(y) + \lambda b_A(y) \geq f(x_0) + \lambda b_A(x_0) = f(x_0)$$

and $x_0$ is a global minimizer.

We have shown in Lemma 5.6(ii) that the condition $C(x_0) \neq \emptyset$ or $x_0 \in \text{int } A$ leads to the constraint qualification. Then, according to Theorem 5.7(ii), the constraint qualification (5.30) implies that, for $x_0 \in \partial A$, the set of multipliers

$$A(x_0) \triangleq \left\{ \mu \geq 0 : \forall v \in X, d_H f(x_0; v) + \mu d_H b_A(x_0; v) \geq 0 \right\} \neq \emptyset.$$  

Theorem 5.8. Let the assumptions of Theorem 5.7 be verified for a minimizer $x_0 \in A$ of $f$. If $x_0 \in \partial A$, then we have the following equivalences:

(i) $C(x_0) \neq \emptyset$;

(ii) $A(x_0)$ is (convex) nonempty and bounded;

(iii) $\mu = 0$ is the only solution of the inequality system

$$\forall v \in X, d_H b_A(x_0; v) \geq 0, \quad \mu \geq 0.$$  

Proof. By Theorem 5.5.
Proof. (i) If \( \overline{A} = \partial A, \overline{\mathcal{A}} = X, b_A = d_A \) and for all \( x \in \partial A, v \in X \) and \( t > 0 \)
\[
\frac{b_A(x+t v) - b_A(x)}{t} = \frac{d_A(x+t v) - d_A(x)}{t} \geq 0 \quad \Rightarrow \quad d_H b_A(x; v) \geq 0.
\]
Conversely, by convexity of \( b_A \), for all \( y \in X \) and \( x \in \partial A \)
\[
b_A(y) \geq b_A(x) + d_H b_A(x; y-x) \geq 0 \quad \Rightarrow \quad \overline{\mathcal{A}} = X \quad \Rightarrow \quad \partial A = \overline{A}.
\]
(ii) Standard arguments. \( \square \)

If \( p_{\text{aff}} A(x) \) is the projection of \( x \) onto \( \text{aff} A \),
\[
\forall x \in X, \quad M = \text{aff} A - p_{\text{aff}} A(x).
\]
From this we can define distance functions in \( \text{aff} A \):
\[
d_A(x) \overset{\text{def}}{=} \inf_{a \in A} \| a - p_{\text{aff}} A(x) \|, \quad d_{\mathcal{A}}(x) \overset{\text{def}}{=} \inf_{b \in \mathcal{A}} \| p_{\text{aff}} A(b) - p_{\text{aff}} A(x) \|
\]
\[
b_A(x) \overset{\text{def}}{=} d_A(x) - d_{\mathcal{A}}(x),
\]
\[
\Rightarrow \quad b_A(x)^2 = \| x - p_{\text{aff}} A(x) \|^2 + b_A(x)^2
\]
\[
\Pi_{\mathcal{A}}(x) \overset{\text{def}}{=} \{ p \in \overline{A} : \| p - p_{\text{aff}} A(x) \| = | b_A(x) | \}
\]
\[
\Rightarrow \quad \Pi_{\mathcal{A}}(x) = \Pi_A(x) = \{ p \in \partial A : \| p - x \| = | b_A(x) | \}.
\]

With those definitions, the constraint qualification will be verified by modifying the construction of section 5.1 and Theorem 5.7 where the space \( X \) is replaced by \( \mathcal{A} = \text{aff} A \) that would play the role of an implicit affine constraint.

6. Danskin-Bertesekas Theorem for a parametrized supremum. At the beginning of section 5.1 we have lumped a finite number of inequality constraints \( g_i(x) \leq 0, 1 \leq i \leq m, \) into a single one by introducing the supremum \( g(x) = \sup_{1 \leq i \leq m} g_i(x) \leq 0 \). We now consider a set of indices \( Y \), which is a compact space (not necessarily finite), as was done by Danskin [14] and earlier by John [37].

Let \( X \) be a locally convex topological vector space. Recall that, for a family \( x \mapsto G_y(x) \overset{\text{def}}{=} G(x, y) : X \rightarrow \mathbb{R} \cup \{ +\infty \} \) of convex lower semicontinuous (lsc) functions indexed by \( y \in Y \), the upper envelope \( x \mapsto g(x) = \sup_{y \in Y} G_y(x) : X \rightarrow \mathbb{R} \cup \{ +\infty \} \) is convex lsc. Since this function can be \( +\infty \) at some points, we can only speak of a semidifferential at a point \( x \in \text{int} (\text{dom } g) \). If \( g \) is convex and upper semicontinuous in a convex neighborhood of \( x \), then \( g \) will be continuous and Hadamard semidifferentiable at each point of this neighborhood. This amounts to starting with the assumption of a family of functions \( G : U \times Y \rightarrow \mathbb{R} \) for an open convex set \( U \) in \( X \), such as, for instance, a ball \( B_\rho(x) \) of radius \( \rho > 0 \).

We follow the elements of the proof of D. Bertsekas [4, pp. 717–719] in 1971.

**Theorem 6.1.** Let \( Y \neq \emptyset \) be a compact topological space,\(^{15}\) \( U \neq \emptyset \) an open convex subset of a locally convex topological vector space \( X \), and \( G : U \times Y \rightarrow \mathbb{R} \) an upper semicontinuous function such that
\[
\forall y \in Y, \quad x \mapsto G(x, y) : U \rightarrow \mathbb{R} \quad \text{is convex.} \tag{6.1}
\]
For each \( x \in U \) consider the function
\[
g(x) \overset{\text{def}}{=} \sup_{y \in Y} G(x, y), \quad Y(x) \overset{\text{def}}{=} \{ y \in Y : G(x, y) = g(x) \}. \tag{6.2}
\]

\(^{15}\)For instance, \( Y \) could be a compact subset of \( \mathbb{R}^n \) or a bounded closed convex subset of a reflexive Banach space or \( Y = \{ 1, \ldots, m \} \) as in Theorem 5.4 of section 5.1.
Since $x \in U$, $Y(x)$ is nonempty compact, $g : U \to \mathbb{R}$ is convex, continuous, Hadamard semidifferentiable in $U$, and $v \mapsto d_H g(x; v)$ is sublinear and continuous;

(i) for each $x \in U$, there exists $y^0 \in Y(x)$ such that

$$d_H g(x; v) = \sup_{y \in Y(x)} dG(x, y; v, 0) = dG(x, y^0; v, 0). \tag{6.3}$$

Therefore, $dG(x, y; v, 0)$ is sublinear and continuous, but between the stronger Hadamard semidifferentials.

We need the following theorem of Rockafellar [49] for convex functions whose proof can be slightly modified to obtain not only an inequality between the Gateaux semidifferentials, but between the stronger Hadamard semidifferentials.

**Theorem 6.2** (Rockafellar [49, Thm. 24.5]). Let $U$ be an open convex subset of a locally convex topological vector space $X$, $F_k : U \to \mathbb{R}$, $k \geq 1$, and $F : U \to \mathbb{R}$ be convex upper semicontinuous functions on $U$ such that for all $x \in U$ and all sequences $\{x_k\}$ such that $x_k \to x$, we have $\lim_{k \to \infty} F_k(x_k) = F(x)$. Then, for all $x \in U$, $v \in X$, and sequences $\{x_k\}$ and $\{v_k\}$ such that $x_k \to x$ and $v_k \to v$,

$$\limsup_{k \to \infty} d_H F_k(x_k; v_k) \leq d_H F(x; v).$$

**Proof.** We go back to the proof of [49] replacing the simple semidifferential by the stronger Hadamard semidifferential. By Theorems 2.13 (i) and 2.12, a convex upper semicontinuous function $f : U \to \mathbb{R}$ on a convex open $U$ is continuous, Hadamard semidifferentiable in $U$, and for all $x \in U$ and $v \in X$, the differential quotient $t \mapsto (f(x + tv) - f(x))/t$, $t > 0$, is monotone increasing, and

$$d_H f(x; v) = df(x; v) = \inf_{t > 0} \frac{f(x + tv) - f(x)}{t}.$$  

Therefore, $F_k$ and $F$ are Hadamard semidifferentiable in $U$. Let $\mu$ be such that $dF(x; v) < \mu$. There exists $\bar{t} > 0$ such that

$$\forall t, 0 < t \leq \bar{t}, \quad \frac{F(x + tv) - F(x)}{t} < \mu.$$  

Since $x_k + tv_k \to x + tv$ and $x_k \to x$, we get, by assumption,

$$\frac{F_k(x_k + tv_k) - F_k(x_k)}{t} \to \frac{F(x + tv) - F(x)}{t}.$$  

Therefore, there exists $K$ such that

$$\forall k > K, \quad \frac{F_k(x_k + tv_k) - F_k(x_k)}{t} < \mu.$$  

By convexity of $F_k$,

$$F_k(x_k + tv_k) - F_k(x_k) \geq d_H F_k(x_k; tv_k) = td_H F_k(x_k; v_k) \Rightarrow \limsup_{k \to \infty} d_H F_k(x_k; v_k) \leq \limsup_{k \to \infty} \frac{F_k(x_k + tv_k) - F_k(x_k)}{t} \leq \mu.$$  

Since this inequality is true for all $\mu$ such that $d_H F(x; v) < \mu$, let $\mu$ go to $d_H F(x; v)$:

$$\limsup_{k \to \infty} d_H F_k(x_k; v_k) \leq \limsup_{k \to \infty} \frac{F_k(x_k + tv_k) - F_k(x_k)}{t} \leq d_H F(x; v). \qed$$
Proof of Theorem 6.1. (i) Since $Y$ is compact and that, for $x \in U$, $y \mapsto G(x, y) : Y \to \mathbb{R}$ is upper semicontinuous, there exists $\hat{y} \in Y$ such that
\[ g(x) = G(x, \hat{y}) \quad \Rightarrow \quad Y(x) \neq \emptyset \quad \text{and} \quad x \in \text{dom } g \quad \Rightarrow \quad U \subset \text{dom } g. \]
Since for $y \in Y$, $x \mapsto G(x, y) : Y \to \mathbb{R}$ is convex and upper semicontinuous, it is continuous and Hadamard semidifferentiable in $U$. For all $\lambda \in [0, 1]$, $x_1, x_2 \in U$,
\[ G(\lambda x_1 + (1 - \lambda)x_2, y) \leq \lambda G(x_1, y) + (1 - \lambda)G(x_2, y) \leq \lambda g(x_1) + (1 - \lambda)g(x_2) \quad \Rightarrow \quad g(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda g(x_1) + (1 - \lambda)g(x_2). \]
Therefore, the function $g : U \to \mathbb{R}$ is convex and lower semicontinuous, that is,
\[ \liminf_{x' \to x} g(x') \geq g(x), \]
but it is also upper semicontinuous. Let $x \in U$ and $\{x_n\} \subset U$ and $\{y_n\} \subset Y$ be sequences such that
\[ \lim_{n \to \infty} g(x_n) = \limsup_{x' \to x} g(x'), \quad g(x_n) = G(x_n, y_n). \]
Since $Y$ is compact, there exists $y^0 \in Y$ and a subsequence, still denoted $\{y_n\}$, such that $y_n \to y^0 \in Y$. For $y \in Y$, $G(x_n, y) \leq g(x_n) = G(x_n, y_n)$ and since $G$ is usc and $x \mapsto F(x, y)$ is continuous,
\[ G(x_n, y_n) \leq G(x, y^0) = g(x). \]
Therefore, $g$ is convex, continuous in $U$, Hadamard semidifferentiable in $U$, and its semidifferential is sublinear.

(ii) For $t > 0$, pick $y_t \in Y(x + tv)$ and $y_0 \in Y(x)$,
\[ \frac{g(x + tv) - g(x)}{t} = \frac{G(x + tv, y_t) - G(x, y_0)}{t} \geq \frac{G(x + tv, y_0) - G(x, y_0)}{t} \]
\[ \Rightarrow \quad dg(x; v) \geq dG(x, y_0; v, 0) \Rightarrow \quad dg(x; v) \geq \sup_{y_0 \in Y(x)} dG(x, y_0; v, 0). \]
In the other direction, given $\{t_n\}$, $t_n \downarrow 0$, let $y_n \in Y(x + t_nv)$, $n \geq 1$. By compactness of $Y$, there exists $y^0 \in Y$ and a subsequence, still denoted $\{y_n\}$, such that $y_n \to y^0$ as $t_n \downarrow 0$. By usc of $G$ and continuity of $x \mapsto G(x, y)$,
\[ \forall y \in Y, \quad G(x + t_nv, y) \leq G(x + t_nv, y_n) \quad \Rightarrow \quad \forall y \in Y, \quad G(x, y) \leq G(x, y^0) \]
and $y^0 \in Y(x)$. On the other hand, since $g$ is convex,
\[ dg(x; v) \leq \frac{g(x + t_nv) - g(x)}{t_n} \leq \frac{G(x + t_nv, y_n) - G(x, y^0)}{t_n}. \]
By convexity of $G$
\[ \frac{G(x, y_n) - G(x + t_nv, y_n)}{t_n} \geq dG(x + t_nv, y_n; -v, 0) \geq -dG(x + t_nv, y_n; v, 0) \]
\[ \Rightarrow \quad dg(x; v) \leq dG(x + t_nv, y_n; v, 0). \]
We now apply Theorem 6.2 to the following convex functions $F_n$ and $F$:
\[ F_n(x) \overset{\text{def}}{=} G(x + t_nv, y_n), \quad dF_n(x; v) = dG(x + t_nv, y_n; v, 0), \]
\[ F(x) \overset{\text{def}}{=} G(x, y^0), \quad dF(x; v) = dG(x, y^0; v, 0). \]
By assumption, $G$ is upper semicontinuous. Therefore, for all sequences $x_n \to x$
$$\limsup_{n \to \infty} G(x_n + t_n v, y_n) \leq G(x, y^0),$$
and, on the other hand, by continuity of $x' \mapsto G(x' y^0)$,
$$G(x_n + t_n v, y_n) \geq G(x_n + t_n v, y^0) \Rightarrow \liminf_{n \to \infty} G(x_n + t_n v, y_n)$$
$$\geq \lim_{n \to \infty} G(x_n + t_n v, y^0) = G(x, y^0).$$

Therefore, we have for all sequences $x_n \to x$
$$F_n(x_n) = G(x_n + t_n v, y_n) \to G(x, y^0) = F(x)$$
$$\Rightarrow \sup_{y \in Y(x)} G(x, y) = \limsup_{n \to \infty} \sup_{y \in Y(x)} G(x, y) \leq \limsup_{n \to \infty} dG(x; v) \leq \sup_{y \in Y(x)} dG(x, y^0; v, 0)$$
putting everything together,
$$\sup_{y \in Y(x)} dG(x, y^0; v, 0) \leq \sup_{y \in Y(x)} dG(x, y^0; v, 0) \leq \sup_{y \in Y(x)} dG(x, y^0; v, 0)$$
and we have equality everywhere. Moreover, the function $v \mapsto d_H g(x; v)$ is convex since $g$ is convex from (i) and hence sublinear.

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