1. **Introduction and preliminaries**

Enumerative problems arising in various fields of mathematics, from combinatorics and representation theory to algebraic geometry and low-dimensional topology, often bear much in common. More precisely, in many cases the generating functions associated with these problems enjoy similar properties – e.g., they may satisfy

- Virasoro constraints,
- an evolution equations of the “cut-and-join” type,
- an integrable hierarchy (such as Toda, Kadomtsev-Petviashvili (KP) or Korteweg-DeVries (KdV) equations),
- a topological recursion (also known as Eynard-Orantin recursion).

Simple Hurwitz numbers provide one of the best studied examples of such an enumerative problem – indeed, their generating function satisfies the celebrated cut-and-join equation [10], the Virasoro constraints (via the ELSV formula [6] and the famous Mumford’s Grothendieck-Riemann-Roch formula [18] it reduces to the...
Witten-Kontsevich potential), KP hierarchy \cite{20}, \cite{14} or \cite{13}, and the topological recursion \cite{7}. Other examples include the Witten-Kontsevich theory, Mirzakhani’s Weil-Petersson volumes, Gromov-Witten invariants of the complex projective line, invariants of knots, etc. (see \cite{8}, \cite{9} for a review).

These remarkable integrability properties of generating functions usually result from matrix model reformulations of the corresponding counting problems. However, in this paper we show that for the enumeration of Grothendieck’s dessins d’enfants all these properties follow from pure combinatorics in a rather straightforward way.

The origin of Grothendieck’s theory of dessins d’enfants \cite{12} lies in the famous result by Belyi:

**Theorem 1.** (Belyi, \cite{4}) A smooth complex algebraic curve \(C\) is defined over the field of algebraic numbers \(\mathbb{Q}\) if and only if there exist a non-constant meromorphic function \(f\) on \(C\) (or a holomorphic branched cover \(f : C \to \mathbb{C}P^1\)) that is ramified only over the points \(0, 1, \infty \in \mathbb{C}P^1\).

We call \((C, f)\), where \(C\) is a smooth complex algebraic curve and \(f\) is a meromorphic function on \(C\) unramified over \(\mathbb{C}P^1 \setminus \{0, 1, \infty\}\), a Belyi pair. For a Belyi pair \((C, f)\) denote by \(g\) the genus of \(C\) and by \(d\) the degree of \(f\). Consider the inverse image \(f^{-1}([0, 1]) \subset C\) of the real line segment \([0, 1] \subset \mathbb{C}P^1\). This is a connected bicolored graph with \(d\) edges, whose vertices of two colors are the preimages of 0 and 1 respectively, and the ribbon graph structure is induced by the embedding \(f^{-1}([0, 1]) \hookrightarrow C\). (Recall that a ribbon graph structure is given by prescribing a cyclic order of half-edges at each vertex of the graph.) The following is straightforward (cf. also \cite{15}):

**Lemma 1.** (Grothendieck, \cite{12}) There is a one-to-one correspondence between the isomorphism classes of Belyi pairs and connected bicolored ribbon graphs.

**Definition 1.** A connected bicolored ribbon graph representing a Belyi pair is called Grothendieck’s dessin d’enfant\footnote{An important observation of Grothendieck that, by Belyi’s theorem, the absolute Galois group \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\) naturally acts on dessins, lies beyond the scope of this paper; we refer the reader to \cite{15} for details.}.

Let \((C, f)\) be a Belyi pair of genus \(g\) and degree \(d\), and let \(\Gamma = f^{-1}([0, 1]) \hookrightarrow C\) be the corresponding dessin. Put \(k = |f^{-1}(0)|\), \(l = |f^{-1}(1)|\) and \(m = |f^{-1}(\infty)|\), then we have \(2g - 2 = d - (k + l + m)\). We assume that the poles of \(f\) are labeled and denote the set of their orders by \(\mu = (\mu_1, \ldots, \mu_m)\), so that \(d = \sum_{i \geq 1} \mu_i\). The triple \((k, l, \mu)\) will be called here the type of the dessin \(\Gamma\), and the set of all dessins of type \((k, l, \mu)\) will be denoted by \(D_{k, l; \mu}\).

Actually, instead of the dessin \(\Gamma = f^{-1}([0, 1])\) corresponding to a Belyi pair \((C, f)\) it will be more convenient to consider its dual graph \(\Gamma^* = f^{-1}(1/2 + \sqrt{-1}\mathbb{R})\), where the bar denotes the closure in \(C\), see Fig. \(\#\). The graph \(\Gamma^*\) is connected, has \(m\) ordered vertices of even degrees \(2\mu_1, \ldots, 2\mu_m\) at the poles of \(f\) and inherits a natural ribbon graph structure. Moreover, the boundary components (faces) of \(\Gamma^*\) are naturally colored: a face is colored in white (resp. in gray) if it contains a preimage of 0 (resp. 1), and every edge of \(\Gamma^*\) belongs to precisely two boundary components of different color.
In this paper we are interested in the weighted count of labeled dessins d’enfants of a given type. Namely, define
\[ N_{k,l}(\mu) = N_{k,l}(\mu_1, \ldots, \mu_m) = \sum_{\Gamma \in D_{k,l,\mu}} \frac{1}{|\text{Aut}_b \Gamma|}, \]
where $\text{Aut}_b \Gamma$ denotes the group of automorphisms of $\Gamma$ that preserve the boundary componentwise.

Consider the total generating function
\[ F(s, u, v, p_1, p_2, \ldots) = \sum_{k,l,m \geq 1} \frac{1}{m!} \sum_{\mu \in \mathbb{Z}^+} N_{k,l}(\mu)s^d u^{k} v^{l} p_{\mu_1} \cdots p_{\mu_m}, \tag{1} \]
where the second sum is taken over all ordered sets $\mu = (\mu_1, \ldots, \mu_m)$ of positive integers, and $d = \sum_{i=1}^{m} \mu_i$.

The objective of this paper is to show that the generating function $F$ satisfies all four integrability properties listed at the beginning of this section – namely, Virasoro constraints, an evolution equation, the KP (Kadomtsev-Petviashvili) hierarchy and a topological recursion. We prove the Virasoro constraints by a bijective combinatorial argument and derive from them all other properties of $F$. As a result, we obtain a simpler version of the topological recursion in terms of homogeneous components of $F$. We also revisit the problem of enumeration of the ribbon graphs with a prescribed boundary type. Topological recursion for this problem was first established in [5]. In this paper we give a different, more streamlined proof of it based on the Virasoro constraints and show that the corresponding generating function satisfies an evolution equation and the KP hierarchy as well. These (and

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**Figure 1.** Decomposition of $\mathbb{C}P^1$ into two 1-gons.

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2Equivalently, we can put $N_{k,l}(\mu) = \sum_{\Gamma \in D_{k,l,\mu}} \frac{1}{|\text{Aut}_v \Gamma^*|}$, where $\text{Aut}_v \Gamma^*$ is the group of automorphisms of the dual graph $\Gamma^*$ preserving each vertex pointwise. A closely related problem of the weighted count of unlabeled dessins $\Gamma$ with weights $\frac{1}{|\text{Aut}_v \Gamma^*|}$ is equivalent to the above one. If one treats $\mu$ as the unordered partition $[1^{m_1} 2^{m_2} \ldots]$, where $m_j = \# \{ \mu_i = j \}$, then the corresponding number of dessins of type $(k, l, \mu)$ is equal to $\frac{1}{|\text{Aut}_v \mu|} N_{k,l}(\mu)$ with $|\text{Aut}_v \mu| = m_1! m_2! \cdots$.

3While this paper was in preparation, similar results were independently obtained by matrix integration methods in [3] and generalized further in [2].
other) examples convincingly demonstrate that Virasoro constrains imply topological recursion and are in fact equivalent to it.

2. Virasoro constraints

For any integer \( n \geq 0 \) consider the differential operator

\[
L_n = \frac{n + 1}{s} \frac{\partial}{\partial p_{n+1}} + (u + v)n \frac{\partial}{\partial p_n} + \sum_{j=1}^{\infty} p_j(n + j) \frac{\partial}{\partial p_{n+j}} + \sum_{i+j=n} ij \frac{\partial^2}{\partial p_i \partial p_j} + \delta_{0,n} uv .
\]  

(2)

A straightforward check shows that for any integer \( m, n \geq 0 \)

\[ [L_m, L_n] = (m - n)L_{m+n} . \]

In other words, the operators \( L_n \) form (a half of) a representation of the Virasoro (or, rather, Witt) algebra.

The main technical statement of this section is the following

**Theorem 2.** The partition function \( e^F = e^{F(s,u,v,p_1,p_2,...)} \) satisfies the infinite system of non-linear differential equations (Virasoro constraints)

\[ L_n e^F = 0 . \]  

(3)

The equations (3) determine the partition function \( e^F \) uniquely.

**Proof.** The Virasoro constraints (3) can be re-written as follows:

\[
\frac{n + 1}{s} \frac{\partial F}{\partial p_{n+1}} = \sum_{j=1}^{\infty} p_j(n + j) \frac{\partial F}{\partial p_{n+j}} + (u + v)n \frac{\partial F}{\partial p_n} + \sum_{i+j=n} ij \left( \frac{\partial^2 F}{\partial p_i \partial p_j} + \frac{\partial F}{\partial p_i} \frac{\partial F}{\partial p_j} \right) + \delta_{0,n} uv .
\]  

(4)

Eq. (4) for \( n + 1 = \mu_1 \) can be further re-written as a recursion relation for the coefficients \( N_{k,l}(\mu) \) of \( F \):

\[
\mu_1 N_{k,l}(\mu_1, \ldots, \mu_m) = \sum_{j=2}^{m} (\mu_1 + \mu_j - 1) N_{k,l}(\mu_1 + \mu_j - 1, v_2, \ldots, \tilde{\mu_j}, \ldots, \mu_m) 
+ (\mu_1 - 1)(N_{k-1,l}(\mu_1 - 1, \mu_2, \ldots, \mu_m) + N_{k,l-1}(\mu_1 - 1, \mu_2, \ldots, \mu_m))
+ \sum_{i+j=\mu_1-1} ij \left( N_{k,l}(i, j, \mu_2, \ldots, \mu_m) 
+ \sum_{k_1+k_2=k} \sum_{l_1+l_2=l} N_{k_1,l_1}(i, \mu_1) N_{k_2,l_2}(j, \mu_j) \right) ,
\]  

(5)

where \( \mu_1 = \mu_i, \ldots, \mu_{i_k}, I = \{i_1, \ldots, i_k\} \), and the hat means that the corresponding term is omitted. This recursion is valid for \( \sum_{i=1}^{m} \mu_i > 1 \) and expresses the numbers \( N_{k,l}(\mu) \) recursively in terms of \( N_{1,1}(1) = 1 \).

We prove this recursion similar to [19] (cf. also [1], [5], [21]) by establishing a direct bijection between dessins counted in the left and right hand sides of (5).

Here it is more convenient to deal with the dual graphs instead. Let \( \Gamma^* \) be the
ribbon graph dual to a dessin $\Gamma$ of type $(k,l,\mu)$. There are $2\mu_1$ ways to pick a half-edge incident to the first vertex of $\Gamma^*$ (following [3] we label this half-edge with an arrow). When $\Gamma$ varies over the set $\mathcal{D}_{k,l,\mu}$, this gives twice the number in the l.h.s. of (5).

Let us now express the same number in terms of dessins with one edge less. This can be done by contracting (or expanding) the labeled edges in the dual graphs in a way that preserves the proper coloring of faces. The following possibilities can occur:

(i) The labeled edge connects the first vertex with the $j$-th vertex, $j \neq 1$. Contracting this edge we get a ribbon graph with properly bicolored faces of type $(k,l,\mu_1 + \mu_j - 1,\mu_2,\ldots,\hat{\mu}_j,\ldots,\mu_m)$, see Fig. 2. Conversely, given a graph of type $(k,l,\mu_1 + \mu_j,\mu_2,\ldots,\hat{\mu}_j,\ldots,\mu_m)$, there are $2(\mu_1 + \mu_j - 1)$ ways to split its first vertex into two ones of degrees $2\mu_1$ and $2\mu_j$. Since $j$ can vary from 2 to $m$, this gives twice the first sum in the r.h.s. of (5).

(ii) The labeled edge forms a loop that bounds a white 1-gon, see Fig. 3. Contracting such a loop we reduce both $k$ and $\mu_1$ by 1, leaving $l$ and $\mu_j$, $j = 2,\ldots,m$, unchanged. Conversely, if we have a graph of type $(k,l,\mu_1 - 1,\mu_2,\ldots,\mu_m)$, we can insert a loop into any of the $\mu_1 - 1$ gray sectors at the first vertex in order to get a graph of type $(k,l,\mu_1,\ldots,\mu_m)$, and 2 ways to label one of its half-edges. The case of a loop bounding a gray 1-gon can be treated verbatim, giving twice the second term in the r.h.s. of (5).

(iii) The labeled edge forms a loop whose half-edges are not adjacent relative to the cyclic order of half-edges at the first vertex. Contracting such a loop we split the first vertex into two ones, say, of degrees $2i$ and $2j$, where $i + j = \mu_1 - 1$,
see Fig. [3] Under this operation the graph may remain connected, or may split into two connected components. In the former case we get a graph of type 
\((k, l, i, j, \mu_2, \ldots, \mu_m)\). Reversing this operation, we join the first two vertices and add a loop. We can place the labeled half-edge of the loop in any of the

\(2i\) sectors at the first vertex, but its other half-edged can be placed only in one of \(j\) sectors of different color at the second edge (otherwise it will not be compatible with the face coloring). This gives us twice the third term in the r.h.s. of (5). The latter case when the graph becomes disconnected can be treated similarly.

The operations (i)–(iii) are reversible and compatible with the face coloring, thus establishing a required bijection. This proves the Virasoro constraints (4). It is also not hard to see that the Virasoro constraints determine the partition function \(e^F\) uniquely, since they are equivalent to the recursion (5).

\[\Lambda_1 = \sum_{i=2}^{\infty} (i-1)p_i \frac{\partial}{\partial p_{i-1}},\]
\[M_1 = \sum_{i=2}^{\infty} \sum_{j=1}^{i-1} \left( (i-1)p_j p_{i-j} \frac{\partial}{\partial p_{i-1}} + j(i-j)p_{i+1} \frac{\partial^2}{\partial p_j \partial p_{i-j}} \right).\]

Then the partition function \(e^F\) satisfies the evolution equation

\[\frac{\partial e^F}{\partial s} = ((u + v)\Lambda_1 + M_1 + uv p_1)e^F,\]  

(6)

and is uniquely determined by the initial condition \(F|_{s=0} = 0\). In other words, \(e^F\) is explicitly given by the formula

\[e^F = e^{s((u+v)L_1+M_1+uv p_1)} 1\]

were “1” stands for the constant function identically equal to 1.
Proof. Multiply the both sides of (2) by $p_{n+1}$ and sum over $n$. We get
\[
\sum_{n=0}^{\infty} p_{n+1} L_n = \sum_{n=0}^{\infty} p_{n+1} \left( -\frac{n+1}{s} \frac{\partial}{\partial p_{n+1}} + (u+v)n \frac{\partial}{\partial p_n} 
+ \sum_{j=1}^{\infty} p_j (n+j) \frac{\partial}{\partial p_{n+j}} + \sum_{i+j=n} ij \frac{\partial^2}{\partial p_i \partial p_j} \right) + uwp_1 
\]
\[
= -\frac{\partial}{\partial s} + (u+v)\Lambda_1 + M_1 + uwp_1 ,
\]
and the required statement immediately follows from (3).
\[\square\]

Remark 1. A different proof of Corollary 1 has recently appeared in [21].

Remark 2. Consider the principal specialization of the partition function $e^F$:
\[
\psi = \psi(s,u,v,t) = e^{F(s,u,v,p_1,p_2,\ldots)} \big|_{p_i=t^i} ,
\]
where $t$ is a new formal variable. Then, with the help of the obvious identity $s \frac{\partial \psi}{\partial s} = t \frac{\partial \psi}{\partial t}$, the evolution equation (6) translates into the Schrödinger equation for the wave function $\psi$:
\[
t^2 \frac{d^2 \psi}{dt^2} + \left( (u+v+1)t - \frac{1}{s} \right) \frac{d\psi}{dt} + uv\psi = 0 .
\]
Eq. (7) is often referred to as the quantum curve equation in the literature on topological recursions.

Another observation is that the generating function $F = F(s,u,v,p_1,p_2,\ldots)$ satisfies an infinite system of non-linear partial differential equations called the KP (Kadomtsev-Petviashvili) hierarchy (this means that the numbers $N_{k,l}(\mu)$ additionally obey an infinite system of recursions). The KP hierarchy is one of the best studied completely integrable systems in mathematical physics. Below are the several first equations of the hierarchy:
\[
F_{22} = -\frac{1}{2} F_{11}^2 + F_{31} - \frac{1}{12} F_{1111} ,
\]
\[
F_{32} = -F_{11} F_{21} + F_{41} - \frac{1}{6} F_{2111} ,
\]
\[
F_{42} = -\frac{1}{2} F_{21}^2 - F_{11} F_{31} + F_{51} + \frac{1}{8} F_{1111} + \frac{1}{12} F_{11} F_{1111} - \frac{1}{4} F_{3111} + \frac{1}{120} F_{1111111} ,
\]
\[
F_{33} = \frac{1}{3} F_{11}^3 - F_{21}^2 - F_{11} F_{31} + F_{51} + \frac{1}{4} F_{1111} + \frac{1}{3} F_{11} F_{1111} - \frac{1}{3} F_{3111} + \frac{1}{45} F_{1111111} ,
\]
where the subscript $i$ stands for the partial derivative with respect to $p_i$.

The exponential $Z = e^F$ of any solution is called a tau function of the hierarchy. The space of solutions (or the space of tau functions) has a nice geometric interpretation as an infinite-dimensional Grassmannian (called the Sato Grassmannian), see [16] or [13] for details. In particular, the space of solutions is homogeneous: there is a Lie algebra $\mathfrak{gl}(\infty)$ that acts infinitesimally on the space of solutions, and the action of the corresponding Lie group is transitive.

Corollary 2. The generating function $F = F(s,u,v,p_1,p_2,\ldots)$ satisfies the infinite system of KP equations (8) with respect to $p_1,p_2,\ldots$ for any parameters $s,u,v.$
Equivalently, the partition function $Z = e^F$ is a 3-parameter family of KP tau functions.

Proof. To begin with, we notice that 1 is obviously a KP tau function. Then, since $p_1, L_1, M_1 \in \mathfrak{gl}(\infty)$ (cf. [13]), the linear combination $s((u + v)L_1 + M_1 + uvp_1)$ also belongs to $\mathfrak{gl}(\infty)$ for any $s, u, v$. Therefore, the exponential $e^{s((u + v)L_1 + M_1 + uvp_1)}$ preserves the Sato Grassmannian and maps KP tau functions to KP tau functions. Thus, $e^F$ is a KP tau function as well, and $F$ is a solution to KP hierarchy. □

Remark 3. Corollary 2 was earlier proven in [11] by a different method. However, [11] contains no analogs of the Virasoro constraints or the evolution equation.

A closely related, but somewhat different enumerative problem was considered in [5]. Recall that a dessin d’enfant $f^{-1}([0, 1])$ is a bicolored connected ribbon graph with vertices “colored” by either 0 or 1 depending on whether $f$ maps the vertex to 0 or 1 in $\mathbb{C}P^1$. One can similarly try to enumerate all (not necessarily bicolored) connected ribbon graphs, and this is the problem that was addressed in [5]. To make it precise, let us label the boundary components of a ribbon graph $\Gamma$ (or, equivalently, the vertices of the dual graph $\Gamma^*$) by integers from 1 to $m$, and let $\mu_1, \ldots, \mu_m$ be the lengths of the boundary components of $\Gamma$ (or the degrees of vertices of $\Gamma^*$).

Ribbon graphs can naturally be represented by dessins of a special type called clean dessins in [5]. Namely, color each vertex of a ribbon graph in white and place black vertices at the midpoints of its edges. Such a dessin corresponds to a covering of $\mathbb{C}P^1$ of even degree $d$ with ramification of type $[2d/2]$ over 0 and arbitrary ramification over 0 and $\infty$. As before, we put $k = |f^{-1}(0)|$ (the number of vertices of the ribbon graph $\Gamma$), $l = |f^{-1}(1)| = d/2$ (the number of edges of $\Gamma$), and $m = |f^{-1}(\infty)|$ (the number of boundary components of $\Gamma$). Clearly, we have $k - d/2 + m = 2 - 2g$.

Following [5], denote by $D_{g,m}(\mu) = D_{g,m}(\mu_1, \ldots, \mu_m)$ the number of genus $g$ ribbon graphs with $m$ labeled vertices of degrees $\mu_1, \ldots, \mu_m$ counted with weights $\frac{1}{|\text{Aut}_\Gamma|}$, where the automorphisms preserve each vertex of $\Gamma$ pointwise. Apparently, the same numbers enumerate pure dessins with $m$ labeled boundary components of lengths $(2\mu_1, \ldots, 2\mu_m)$. The following recursion for $D_{g,m}(\mu)$ was derived in [5]:

\[
\mu_1 D_{g,m}(\mu_1, \ldots, \mu_m) = \sum_{j=2}^{m} (\mu_1 + \mu_j - 2)D_{g,m-1}(\mu_1 + \mu_j - 2, \mu_2, \ldots, \widehat{\mu_j}, \ldots, \mu_m) \\
+ 2(\mu_1 - 2)D_{g,m}(\mu_1 - 2, \mu_2, \ldots, \mu_m) \\
+ \sum_{i+j=\mu_1-2} ij \left( D_{g-1,m+1}(i, j, \mu_2, \ldots, \mu_m) \\
+ \sum_{g_1 + g_2 = g} \sum_{I \cup J = \{2, \ldots, m\}} D_{g_1, |I|+1}(i, \mu_I)D_{g_2, |J|+1}(j, \mu_J) \right).
\] (9)

\footnote{Note that the second term in the r.h.s. of (9), corresponding to a loop bounding a 1-gon, was omitted in the original formula (3.15) of [5]. This required some “modification” of the numbers $D_{g,m}(\mu_1, \ldots, \mu_m)$ in [5].}
Recursion (9) is valid for all $g \geq 0$, $m \geq 1$, and $\mu$ such that $d = \sum_{i=1}^{m} \mu_i > 2$, whereas for $d = 2$ the only nonzero numbers are $D_{0,1}(2) = 1/2$, $D_{0,2}(1,1) = 1$. Below we give a convenient interpretation of this recursion in terms of PDEs.

Similar to (1), introduce the generating function

$$\tilde{F}(s,u,p_1,p_2,\ldots) = \sum_{g=0}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{\mu \in \mathbb{Z}_m^+} D_{g,m}(\mu)s^d u^k p_{\mu_1} \cdots p_{\mu_m},$$

where $d = \sum_{i=1}^{m} \mu_i$, $k = 2 - 2g - m + d/2$, and $\mu = (\mu_1, \ldots, \mu_m)$ (compared to (1), we omit here the trivial factor $v^{d/2}$ that carries no additional information in this case).

**Theorem 3.** The generating function $\tilde{F}$ enjoys the following integrability properties:

(i) Let

$$\tilde{L}_n = -\frac{n+2}{s^2} \frac{\partial}{\partial p_{n+2}} + 2u n \frac{\partial}{\partial p_n} + \sum_{j=1}^{\infty} p_j (n+j) \frac{\partial}{\partial p_{n+j}} + \sum_{i+j=n} i j \frac{\partial^2}{\partial p_i \partial p_j} + \delta_{-1,n} u p_1 + \delta_{0,n} u^2,$$

where $n \geq -1$. Then the partition function $e^\tilde{F}$ satisfies the infinite system of PDE’s (“Virasoro constraints”)

$$\tilde{L}_n e^\tilde{F} = 0$$

that determine $\tilde{F}$ uniquely.

(ii) Put

$$\Lambda_2 = \sum_{i=3}^{\infty} (i-2) p_i \frac{\partial}{\partial p_{i-2}} + \frac{1}{2} p_1^2,$$

$$M_2 = \sum_{i=2}^{\infty} \sum_{j=1}^{i-1} \left( (i-2) p_j p_{i-j} \frac{\partial}{\partial p_{i-2}} + j (i-j) p_{i+2} \frac{\partial^2}{\partial p_j \partial p_{i-j}} \right).$$

Then $e^\tilde{F}$ satisfies the evolution equation

$$\frac{1}{s} \frac{\partial e^\tilde{F}}{\partial s} = (2u \Lambda_2 + M_2 + u^2 p_2) e^\tilde{F},$$

that, together with the initial condition $\tilde{F}|_{s=0} = 0$, determines $\tilde{F}$ uniquely. In other words, $e^\tilde{F}$ is explicitly given by the formula

$$e^\tilde{F} = e^{\frac{1}{2}(2u \Lambda_2 + M_2 + u^2 p_2)}.$$

(iii) The partition function $e^\tilde{F}$ is a tau function of the KP hierarchy, i.e. its logarithm $\bar{F}(s,u,p_1,p_2,\ldots)$ satisfies (8) for any $s$ and $u$.

**Proof.** Part (i) of the theorem is just a reformulation of the recursions (9) for $\mu_1 = n + 2$. Note that the operators $\tilde{L}_n$ obey the commutation relations $[\tilde{L}_m, \tilde{L}_n] = (m-n) \tilde{L}_{m+n}$ for $m, n \geq -1$. 

\[\]
To prove (ii) we multiply \( \tilde{L}_n \) by \( p_{n+2} \) and sum over \( n \):

\[
\sum_{n=1}^\infty p_{n+2} \tilde{L}_n = \sum_{n=1}^\infty p_{n+2} \left( -\frac{n+2}{s} \frac{\partial}{\partial p_{n+2}} + 2u n \frac{\partial}{\partial p_n} \right. \\
\left. + \sum_{j=1}^\infty p_j (n+j) \frac{\partial}{\partial p_{n+j}} + \sum_{i+j=n}^\infty i j \frac{\partial^2}{\partial p_i \partial p_j} \right) + u p_1^2 + u^2 p_2 \\
= -\frac{1}{s} \frac{\partial}{\partial s} + 2u \Lambda_2 + M_2 + u^2 p_2.
\]

(11)

Part (iii) follows from the fact that \( \Lambda_2, M_2 \) and \( p_2 \) belong to \( \widehat{gl}(\infty) \), cf. Corollary 2.

**Remark 4.** Similar to Eq. (7) we can write the quantum curve equation for the wave function 

\[
\tilde{\psi} = \tilde{\psi}(s,u,t) = e^{\tilde{F}(s,u,p_1,p_2,\ldots)} \mid_{p_i=t_i},
\]

that reads in this case as follows:

\[
t^2 \frac{d^2 \tilde{\psi}}{dt^2} + \left( 2(u+1) t - \frac{1}{s^2} t \right) \frac{d \tilde{\psi}}{dt} + (u + u^2) \tilde{\psi} = 0.
\]

Note that this equation differs from the one obtained in [17].

3. **Topological recursion**

The generating function \( F = \sum_{g,m} F_{g,m} \) of \( \{1\} \) enumerating Belyi pairs \((C,f)\) (or Grothendieck’s dessins) can be naturally decomposed into components with fixed \( g \) and \( m \), where \( g \) is the genus of \( C \) and \( m \) is the number of poles of \( f \):

\[
F_{g,m}(s,u,v,p_1,p_2,\ldots) = \frac{1}{m!} \sum_{\mu_1,\ldots,\mu_m} \sum_{k+l=d-m+2-2g} N_{k,l}(\mu)s^d u^k v^l p_{\mu_1} \ldots p_{\mu_m}
\]

(12)

(here, as usual, \( d = \sum \mu_i \)). Another way to collect these numbers into a generating series is to use the \( m \)-point correlation functions

\[
W_{g,m}(x_1,\ldots,x_m) = \sum_{\mu_1,\ldots,\mu_m} \sum_{k+l=d-m+2-2g} N_{k,l}(\mu) x_1^{\mu_1} \ldots x_m^{\mu_m}.
\]

**Topological recursion** (cf. [9]) is an “ansatz” that allows to reconstruct the coefficients of certain generating series recursively in \( g \) and \( m \). Traditionally, it is formulated in terms of correlation functions or, rather, differentials

\[
w_{g,m}(x_1,\ldots,x_m) = \frac{\partial^m W_{g,m}}{\partial x_1 \ldots \partial x_m} dx_1 \ldots dx_m
\]

\[
= \sum_{\mu} \sum_{k+l=d-m+2-2g} N_{k,l}(\mu) \prod_{i=1}^m \mu_i x_i^{\mu_i-1} dx_i.
\]

We will present the topological recursion in terms of components \( F_{g,m} \) of the generating function \( F \). The advantage of this approach is that we need only one set of variables \( p_i \) for all \( g \) and \( n \). The two approaches being equivalent, it proves out, however, that many properties of the recursion become more clear in terms
of \( p \)-variables. The main feature of the topological recursion is the polynomiality of the components \( F_{g,m} \) relative to a special set of variables that are linear in \( p_i \).

Introduce the formal variables \( x \) and \( z \) related by

\[
z(x) = \sqrt{\frac{1 - \beta sx}{1 - \alpha sx}}, \quad x(z) = \frac{z^2 - 1}{s(\alpha z^2 - \beta)},
\]

where

\[
\alpha = (\sqrt{u} - \sqrt{v})^2, \quad \beta = (\sqrt{u} + \sqrt{v})^2.
\]

We consider (13) as a formal change of coordinates on the complex projective line \( \mathbb{C}P^1 \) near the point \( x = 0 \) (resp., \( z = 1 \)) depending on the parameters \( \alpha, \beta, s \) (or \( u, v, s \)).

Put

\[
T_j(p) = \sum_{i=1}^{\infty} c_j^{(i)} p_i,
\]

where the coefficients \( c_j^{(i)} \) are defined by the relation

\[
\sum_{i=1}^{\infty} c_j^{(i)} x^i = (z(x))^j - 1.
\]

**Theorem 4.** Let \( F_{g,m} \) be the infinite series defined by (12). Then

(i) For each \( g, m \) with \( 2g - m > 0 \) there exists a polynomial \( G_{g,m} \) of the variables \( t_j, j \in \mathbb{Z}_{odd} \), such that

\[ F_{g,m}(p) = G_{g,m}(t) \big|_{t_j = T_j(p)} \]

(i.e. each \( F_{g,m} \) is a polynomial in the linear functions \( T_{\pm 1}, T_{\pm 3}, T_{\pm 5}, \ldots \)).

(ii) The polynomials \( G_{g,m} \) can be recursively computed starting from \( G_{0,3} \) and \( G_{1,1} \), cf. Eqs. (18)-(22) below.

Let us now formulate the recursion for the polynomials \( G_{g,m} \) precisely. This can be done in terms of the so-called spectral curve. In our case the spectral curve is the projective line \( \mathbb{C}P^1 \) equipped with the globally defined holomorphic involution \( z \mapsto -z \) with respect to some affine coordinate \( z \).

By a **Laurent form** we understand here a globally defined meromorphic 1-form on \( \mathbb{C}P^1 \) with poles only at 0 and \( \infty \). Denote by \( L \) the space of odd Laurent forms relative to the involution \( z \mapsto -z \). The forms \( d(z^j) = j z^{j-1} dz, j \in \mathbb{Z}_{odd} \), provide a convenient basis in \( L \). Let \( P_L \) denote the projector to the space \( L \) in the space of all Laurent forms. For a Laurent form \( \phi \) its projection \( \psi = P_L(\phi) \) to \( L \) is uniquely determined by the requirement that the form \( \psi - \phi_{odd} \) is regular at both 0 and \( \infty \), where \( \phi_{odd}(z) = \frac{1}{2}(\phi(z) - \phi(-z)) \) is the odd part of \( \phi \). More explicitly, the action of \( P_L \) is given by the formula

\[
(P_L \phi)(z) = \sum_{i=0}^{\infty} \text{Res}_{w=0}(\phi(w)w^{2i+1}) z^{-2i-2} dz - \sum_{i=0}^{\infty} \text{Res}_{w=\infty}(\phi(w)w^{-2i-1}) z^{2i} dz
\]

\[= \text{Res}_{w=0} \left( \phi(w) \frac{w dz}{z^2 - w^2} \right) + \text{Res}_{w=\infty} \left( \phi(w) \frac{w dz}{z^2 - w^2} \right). \tag{16}
\]

Note that for the validity of this definition it will suffice to assume that \( \phi \) is defined in a neighborhood of the points 0 and \( \infty \), or even that \( \phi \) is a formal Laurent
series at these points. On the other hand, the form $P_L \phi \in L$ is always globally defined on $\mathbb{C}P^1$.

In fact, the recursion applies not to the polynomials $G_{g,m}$ themselves, but to certain 1-forms $U_{g,m}$. For the set of variables $z$ and $t = (t_{\pm 1}, t_{\pm 3}, \ldots)$ introduce the differential operator

$$\delta_{z,t} = \sum_{j \in \mathbb{Z}_{\text{odd}}} j z^{j-1} \frac{\partial}{\partial t_j}.$$  

For $2g - 2 + m > 0$ put

$$U_{g,m}(z) = \delta_{z,t} G_{g,m}(z) = \sum_{j \in \mathbb{Z}_{\text{odd}}} j z^{j-1} \frac{\partial G_{g,m}}{\partial t_j}.$$  

As we will see later, $U_{g,m} = U_{g,m}(z)dz$ is an odd Laurent form on $\mathbb{C}P^1$ that is polynomial in $t$.

Remark 5. The operator $\delta_{z,t}$ written in terms of $x$ and $p$-variables becomes

$$\delta_{x,p} = \sum_{i=1}^{\infty} i x^{i-1} \frac{\partial}{\partial p_i},$$

where $x$ is related to $z$ by (13) and $t_j = T_j(p)$, see (15). (For brevity we will omit the subscripts ‘$p$’ and ‘$t$’ by $\delta$, always associating $p$- and $t$-variables with $x$ and $z$ respectively.) More precisely, assume that a function $f$ of $p$-variables can be expressed as a composition $f = h \circ T$, where $h$ is a function of $t$-variables and $T$ is the linear change (15). Then we have $\delta_x f dx = (\delta_x h)_{t_k = T_k(p)} dz$. In particular, if $g$ is a polynomial in $t$-variables, then $\delta_x f dx$ is a Laurent form in $z$ (with coefficients depending on $p_i$'s). Indeed, the operators on both sides satisfy the Leibnitz rule, and therefore it is sufficient to prove the equality for the case $h = t_k$, that is,

$$\delta_x T_k(p)dz = k z^{k-1} dz, \quad z = z(x),$$

which is essentially the definition of the linear functions $T_k(p)$, cf. (15).

In the unstable cases (i.e. when $2g - 2 + m \leq 0$) the definition of $U_{g,m}$ should be modified. Namely, we set $U_{0,1} = 0$ and define $U_{0,2}$ by the following formal expansions

$$U_{0,2}(z)dz = -\sum_{i=0}^{\infty} t_{-2i-1} z^{2i} dz = -(t_{-1} + t_{-3} z^2 + \ldots) dz, \quad z \to 0,$$

$$U_{0,2}(z)dz = -\sum_{i=0}^{\infty} t_{2i+1} z^{2i} dz = (t_1 z^{-2} + t_3 z^{-4} + \ldots) dz, \quad z \to \infty,$$

In general, the homogeneous degree $m$ polynomial $G_{g,m}$ can be recovered form the form $U_{g,m}$ by the Euler formula

$$G_{g,m} = \frac{1}{m} \sum_{k} t_k \frac{\partial G_{g,m}}{\partial t_k} = \frac{1}{m} \Omega(U_{g,m}, U_{0,2}),$$

where for odd forms $\phi$ and $\psi$ we set

$$\Omega(\phi, \psi) = -\Omega(\psi, \phi) = \text{Res}_{z=0} \left( \phi \int \psi \right) + \text{Res}_{z=\infty} \left( \phi \int \psi \right).$$
The last ingredient needed to write down the topological recursion is the form
\[
\eta = \eta(z) dz = \frac{\sigma z^2 dz}{(z^2 - 1)^2 (\alpha z^2 - \beta)}, \quad \sigma = (\alpha - \beta)^2 = 16 uv. \tag{19}
\]
This form is odd and has the property that the dual vector field
\[
\frac{1}{\eta} \frac{d}{dz} = \frac{1}{\eta(z)} d = \frac{\alpha z^4 - (2 \alpha + \beta) z^2 + \alpha + 2 \beta - 2 \sigma}{\sigma} \frac{d}{dz}
\]
is meromorphic with poles of order 2 at \( z = 0 \) and \( z = \infty \) and regular elsewhere in \( \mathbb{C}P^1 \).

The main recursive relation of this paper is
\[
U_{g,m} = P_L \left( \frac{1}{2\eta} \left( \delta U_{g-1,m+1} + \sum_{g_1 + g_2 = g, m_1 + m_2 = m+1} U_{g_1,m_1} U_{g_2,m_2} \right) \right) \tag{20}
\]
(here and below we tacitly assume that \( \delta f = \delta_z f dz \)). Note that almost all terms in the sum on the right hand side of (20) belong to \( L \), so that \( P_L \) is identical on these terms. Therefore, (20) can equivalently be rewritten as
\[
U_{g,m} = \frac{1}{2\eta} \left( \delta U_{g-1,m+1} + \sum_{g_1 + g_2 = g, m_1 + m_2 = m+1}^{*} U_{g_1,m_1} U_{g_2,m_2} \right) + P_L \left( \frac{1}{\eta} U_{g,m-1} U_{0,2} \right),
\]
where the star \( ^{*} \) by the summation sign means that the terms involving \( U_{0,2} \) are excluded (recall that \( U_{0,1} = 0 \) by assumption). This recursion relation is valid for all \( g \) and \( n \) with \( 2g - 2 + n > 1 \). Moreover, it applies for \((g,n) = (0,3)\) as well:
\[
U_{0,3} = P_L \left( \frac{(U_{0,2})^2}{2\eta} \right). \tag{21}
\]
In the case \((g,n) = (1,1)\) the formula is not applicable since \( \delta U_{0,2} \) is not defined. This is why we set by definition
\[
U_{1,1} = U_{1,1}(z) dz = P_L \left( \frac{1}{2\eta} \left( \frac{dz}{2z} \right)^2 \right) = \frac{1}{2\eta} \left( \frac{dz}{2z} \right)^2 = \frac{1}{8z^2 \eta(z)} dz. \tag{22}
\]

Eqs. (18–22) concretize the second statement of Theorem 4.
Below we list the polynomials $U_{g,m}$ and $G_{g,m}$ for small $g$ and $m$:

$$
\sigma U_{0,3}(z) = \frac{1}{2} t_2 z^2 - \frac{3}{2} t_2 z^{-2},
$$

$$
\sigma G_{0,3} = \frac{1}{6} t_2^2 + \frac{2}{2} t_2 z^2 - \frac{3}{2} z^{-2} - \frac{5}{2} z^{-4},
$$

$$
\sigma U_{1,1}(z) = \frac{1}{8} z^2 - \frac{2\alpha + \beta}{8} + \frac{\alpha + 2\beta}{8} z^2 - \frac{4\beta}{8} z^{-2} - \frac{5}{4} z^{-4},
$$

$$
\sigma G_{1,1} = \frac{1}{24} t_3 - \frac{2\alpha + \beta}{8} t_1 - \frac{\alpha + 2\beta}{8} t_{-1} + \frac{2\beta}{24} t_{-3},
$$

$$
\sigma^2 U_{0,4}(z) = \frac{1}{2} t_2^3 z^2 + \left(\frac{2\alpha + \beta}{2} t_3 t_2^2 - \frac{2\alpha + \beta}{2} t_1^2 - \frac{\alpha \beta}{2} t_2 t_1\right) z^2 - \frac{2\beta^2}{2} t_2 z^{-2} - \frac{2\beta^2}{2} t_2 z^{-4},
$$

$$
\sigma^2 G_{0,4} = \frac{1}{6} t_3 t_2^2 - \frac{2\alpha + \beta}{2} t_1^2 - \frac{\alpha \beta}{2} t_2 t_1 - \frac{\alpha \beta}{2} t_{-1} - \frac{\alpha \beta}{2} t_{-3},
$$

$$
\sigma^2 U_{1,2}(z) = \frac{5}{8} t_2^3 z^2 + \left(\frac{2\alpha + \beta}{2} t_3 t_2^2 - \frac{2\alpha + \beta}{2} t_1^2 - \frac{\alpha \beta}{2} t_2 t_1\right) z^2 - \frac{2\beta^2}{2} t_2 z^{-2} - \frac{2\beta^2}{2} t_2 z^{-4},
$$

$$
\sigma^2 G_{1,2} = \frac{1}{8} t_3 t_2^2 + \frac{2\alpha + \beta}{2} t_1^2 - \frac{2\alpha + \beta}{2} t_1 t_{-1} + \frac{\alpha \beta}{2} t_{-1} - \frac{\alpha \beta}{2} t_{-3},
$$

where $\alpha, \beta, \sigma$ are given by \[14\], \[19\]. This list can be continued further on.

**Remark 6.** Here we compare our form of the topological recursion \[20\] with the one that can be found in the literature, see, e.g. \[5\], \[9\].

(i) Traditionally, the spectral curve comes with an embedding to (a compactification of) $\mathbb{C}^2$ by means of certain meromorphic functions $X, Y$ on $\mathbb{C}P^1$. These functions are chosen so that $X$ is even with respect to the involution $z \mapsto -z$, and $\eta = Y dX$. In our case we could have set, for example,

$$
X = \frac{1}{x} = s \frac{\alpha z^2 - \beta}{z^2 - 1}, \quad Y = \frac{z}{2s} \frac{\alpha z^2 - \beta}{z^2 - 1}.
$$

The formulas of the topological recursion, however, involve the coordinates $X$ and $Y$ only in the combination $\eta = Y dX$.

(ii) The topological recursion is usually formulated in terms of $m$-point correlators. They are related to the homogeneous components $F_{g,m}$ of the generating
function \( F \) by the formulas

\[
w_{g,m}(x_1, \ldots, x_m) = \delta_{x_1} \ldots \delta_{x_m} F_{g,m} \, dx_1 \ldots dx_m
= \delta_{x_2} \ldots \delta_{x_m} U_{g,m}(x_1) \, dx_1 \ldots dx_m,
\]

where \( \delta_{x_j} = \sum_{i=1}^{\infty} i x_j^{i-1} \frac{\partial}{\partial p_i} \). Via the change of variables (13), \( w_{g,m} \) can be viewed as a meromorphic \( m \)-differential on \((\mathbb{C}P^1)^n\) that is a Laurent form with respect to each of its arguments provided \( 2g - 2 + n > 0 \).

(iii) A version of (23) holds also for \((g, m) = (2, 0)\). Namely, \( w_{0,2}(z_1, z_2) = \delta_{z_2} U_{0,2} \, dz_2 \) is the odd part of the Bergman kernel \( B(z_1, z_2) = \frac{dz_1 \, dz_2}{(z_1 - z_2)^2} \):

\[
w_{0,2}(z_1, z_2) = \delta_{z_2} U_{0,2} \, dz_2 = \frac{1}{2} \left( B(z_1, z_2) - B(z_1, -z_2) \right).
\]

This can be interpreted as an equality of asymptotic expansions of the left and right hand sides at \( z_1 \to 0 \) and \( z_1 \to \infty \), cf. (17):

\[
w_{0,2}(z_1, z_2) = \begin{cases} - \sum_{i=0}^{\infty} d(z_2^{-2i-1}) z_1^{2i} \, dz_1, & z_1 \to 0, \\ - \sum_{i=0}^{\infty} d(z_2^{2i+1}) z_1^{-2i} \, dz_1^{-1}, & z_1 \to \infty. \end{cases}
\]

(iv) The projector \( P_L \), see (16), is given by the contour integral

\[
(P_L \phi)(z) \, dz = \frac{1}{2\pi \sqrt{-1}} \left( \frac{w \, dw}{z^2 - w^2} + \frac{w \, dw}{|w| = 1/\epsilon} \right) \, dz,
\]

for small \( \epsilon > 0 \), where

\[
\frac{w \, dz}{z^2 - w^2} = \frac{1}{2} \int_{-w}^{w} B(z, \cdot) \, dz.
\]

This explains the appearance of the Bergman kernel in the majority of expositions of the topological recursion.

(v) The above items (i)–(iv) demonstrate that the traditional form of the topological recursion in terms of the correlators \( w_{g,n} \) is obtainable from (20) by applying \( \delta_{z_2} \ldots \delta_{z_n} \) to the both sides of it.

4. Proof of the topological recursion

4.1. Master Virasoro equation. As we will see below, the topological recursion relations are just the equivalently reformulated Virasoro constraints. To begin with, let us collect the Virasoro constraints (4) into a single equation by multiplying the \( n \)th equation by \( x_n \) (where \( x \) is a formal variable) and summing them up:

\[
\sum_{n=0}^{\infty} x_n \left( -\frac{1}{8} (n + 1) \frac{\partial F}{\partial p_{n+1}} + (u + v) \frac{\partial F}{\partial p_n} + \sum_{j=1}^{\infty} p_j (n + j) \frac{\partial F}{\partial p_{n+j}} 
+ \sum_{i+j=n} ij \left( \frac{\partial^2 F}{\partial p_i \partial p_j} + \frac{\partial F}{\partial p_i} \frac{\partial F}{\partial p_j} \right) \right) + uv = 0.
\]
This equation can be simplified. Notice that
\[
\sum_{n=0}^{\infty} x^n \left( \frac{-1}{s} (n+1) \frac{\partial F}{\partial p_{n+1}} + (u+v)n \frac{\partial F}{\partial p_n} \right) = \left( \frac{-1}{s} + (u+v)x \right) \delta_x F,
\]
\[
\sum_{n=0}^{\infty} \sum_{i+j=n} ij \left( \frac{\partial^2 F}{\partial p_i \partial p_j} + \frac{\partial F}{\partial p_i} \frac{\partial F}{\partial p_j} \right) = x^2 \left( \delta_x^2 F + (\delta_x F)^2 \right),
\]
where, as in the previous section, \( \delta_x = \sum_{n=1}^{\infty} nx^{n-1} \frac{\partial}{\partial p_n} \) (cf. Remark 5). As for the third term in the sum, we use the identity \( p_j = \delta^{-1}_y (j y^{j-1}) = \delta^{-1}_y d_y (y^j) \), where \( y \) is a new independent formal variable and \( d_y = \frac{\partial}{\partial y} \), to re-write it as follows:
\[
\sum_{n=0}^{\infty} x^n \sum_{j=1}^{\infty} p_j (n+j) \frac{\partial F}{\partial p_{n+j}} = \sum_{n=0}^{\infty} \sum_{i+j=n} x^i p_j n \frac{\partial F}{\partial p_n} = \delta^{-1}_y d_y \left( \sum_{n=0}^{\infty} \sum_{i+j=n} x^i y^j n \frac{\partial F}{\partial p_n} \right) = \delta^{-1}_y d_y \left( \frac{x^2 \delta_x F - y^2 \delta_y F}{x-y} \right).
\]
This yields the following master Virasoro equation that unifies all Eqs. (4):
\[
\left( \frac{-1}{s} + (u+v)x \right) \delta_x F + x^2 \left( \delta_x^2 F + (\delta_x F)^2 \right) + \delta^{-1}_y d_y \left( \frac{x^2 \delta_x F - y^2 \delta_y F}{x-y} \right) + uv = 0.
\]

4.2. Unstable terms and the spectral curve. Our immediate goal is to extract the homogeneous terms in \( \delta_x F \) contributing to \( \delta_x F_{g,n} \) for fixed \( g \) and \( n \). We start with the unstable cases. For \( g = 0 \) and \( n = 1 \) we get
\[
x^2 (\delta_x F_{0,1})^2 + \left( \frac{-1}{s} + (u+v)x \right) \delta_x F_{0,1} + uv = 0.
\]
Solving this equation for \( x \delta_x F_{0,1} \), choosing the proper root and expanding it into the Taylor series at \( x = 0 \) we get
\[
x \delta_x F_{0,1} = \frac{1}{2} \left( \frac{1}{sx} - u - v - \sqrt{\left( \frac{1}{sx} - u - v \right)^2 - 4uv} \right)
= uv s x + uv (u+v) s^2 x^2 + uv (u^2 + 3uv + v^2) s^3 x^3 + \ldots
\]
and
\[
F_{0,1} = uv s p_1 + \frac{1}{2} uv (u+v) s^2 p_2 + \frac{1}{3} uv (u^2 + 3uv + v^2) s^3 p_3 + \ldots
\]

**Definition 2.** The spectral curve is the Riemann surface of the algebraic function \( x \delta_x F_{0,1} \) in the \( x \)-variable.

In other words, the spectral curve is an algebraic curve such that \( x \) and \( x \delta_x F_{0,1} \) are globally defined univalued meromorphic functions on it. In our case the spectral curve is given by \( \{25\} \). It is rational (admits a rational parametrization). Let \( z \) be an affine coordinate on \( \mathbb{C}P^1 \). Its choice is not important, but, for convenience, we choose it in such a way that the two critical points of the function \( x \) on \( \mathbb{C}P^1 \)
are $z = 0$ (with the critical value $x = \frac{1}{2g}$) and $z = \infty$ (with the critical value $x = \infty$). The corresponding rational parametrization has the following form:

$$x = \frac{z^2 - 1}{s(\alpha z^2 - \beta)}, \quad z = \sqrt{\frac{1 - \beta x}{1 - \alpha x}},$$

$$x \delta_x F_{0,1} = \sqrt{u} \frac{1 - z}{1 + z}, \quad \delta_x F_{0,1} \frac{dx}{dz} = \frac{8 u v z}{(z + 1)^2 (\alpha z^2 - \beta)},$$

where $\alpha, \beta$ are related to $u, v$ by (14). All functions entering these equalities can be regarded either as rational functions in $z$-variable or as formal power expansions of these functions at $z = 1$.

We continue with the terms with $g = 0$ and $n = 2$ in (24). We have

$$\left(-\frac{1}{s} + (u + v)x\right) \delta_x F_{0,2} + 2 x^2 \delta_x F_{0,1} \delta_x F_{0,2} + \delta_y^{-1} dy \left(\frac{x^2 \delta_x F_{0,1} - y^2 \delta_y F_{0,1}}{x - y}\right) = 0,$$

from where we get

$$\delta_y \delta_x F_{0,2} = \frac{dy \left(\frac{x^2 \delta_x F_{0,1} - y^2 \delta_y F_{0,1}}{x - y}\right)}{\frac{1}{s} - (u + v)x - 2 x^2 \delta_x F_{0,1}}.$$

This equality uniquely determines $\delta_y \delta_x F_{0,2} dx dy$ as a meromorphic bidifferential on $\mathbb{C}P^1 \times \mathbb{C}P^1$. Substituting the obtained above expressions for $x$, $dx/dz$ and $\delta_x F_{0,1}$ into the last formula, after some miraculous cancellations we finally get

$$\delta_y \delta_x F_{0,2} dx dy = \frac{dz dw}{(z + w)^2} = -B(z, -w), \quad (26)$$

where $B(z, w) = \frac{dz dw}{(z - w)^2}$ is the Bergman kernel, and $w$ is related to $y$ by the same formulas (13) that relate $z$ to $x$.

4.3. **Rational recursion formula.** Now we look at the homogeneous terms of degrees $g$ and $n$ with $2g - 2 + n > 1$ in (24). To begin with, let us extract the unstable terms from the expression $(\delta F)^2$ in (24) in order to re-group them with the other summands. Multiplying (24) by $x^{-2}$ we get

$$2 \left(-\frac{1}{2 s x^2} + \frac{u + v}{2 x} + \delta_x F_{0,1}\right) \delta_x (F - F_{0,1} - F_{0,2}) + \delta_x^2 F + (\delta_x (F - F_{0,1} - F_{0,2}))^2$$

$$+ x^{-2} \delta_y^{-1} dy \left(\frac{x^2 \delta_x (F - F_{0,1}) - y^2 \delta_y (F - F_{0,1})}{x - y}\right) + 2 \delta_x (F - F_{0,1}) \delta_x F_{0,2} - (\delta_x F_{0,2})^2 = 0.$$

Let us rewrite the coefficients of this equation in the $z$-coordinate. It is convenient to put

$$\eta = \eta(x) dx = -\left(-\frac{1}{2 s x^2} + \frac{u + v}{2 x} + \delta_x F_{0,1}\right) dx,$$

so that in terms of the coordinate $z$

$$\eta = \eta(z) dz = \frac{(\alpha - \beta)^2 z^2}{(z^2 - 1)^2 (\alpha z^2 - \beta)} dz.$$
Then, using the already known expressions for $x, \frac{dx}{dz}, \delta_x F_{0,1}$, and $\delta_y \delta_x F_{0,2}$, we find that
\[
\delta_x F_{0,2} = -\delta_y^{-1} d_y \left( \frac{1}{z+w} \right).
\] (27)

From the above identities we obtain the following equation for $\delta_x F$:
\[
\delta_x (F - F_{0,1} - F_{0,2}) = \frac{1}{2 \eta(x)} \left( \delta_x^2 F + \left( \delta_x (F - F_{0,1} - F_{0,2}) \right)^2 - (\delta_x F_{0,2})^2 \right)
+ \delta_y^{-1} d_y \left( \frac{w}{z^2 - w^2} \left( \frac{\delta_x (F - F_{0,1})}{\eta(x)} - \frac{\delta_y (F - F_{0,1})}{\eta(y)} \right) \right) \frac{dz}{dx}.
\] (28)

In particular, for the homogeneous components $U_{g,m} = \delta_x F_{g,m}$ with $2g - 2 + m > 1$, this equation reads
\[
U_{g,m}(z) = \frac{1}{2 \eta(z)} \left( \delta_z U_{g-1,m+1}(z) + \sum_{g_1+g_2=g, m_1+m_2=m+1}^\ast U_{g_1,m_1}(z) U_{g_2,m_2}(z) \right)
+ \delta_w^{-1} d_w \left( \frac{w}{z^2 - w^2} \left( \frac{U_{g,m-1}(z)}{\eta(z)} - \frac{U_{g,m-1}(w)}{\eta(w)} \right) \right),
\] (29)
where the star $\ast$ by the summation sign means that the unstable terms with $2g - 2 + m \leq 0$ are omitted. We refer to this equation as the rational recursion formula for the forms $U_{g,m} = U_{g,m}(z) dz$.

This equation allows one to prove the polynomiality property for the forms $U_{g,m}$ by induction in $g$ and $m$. Indeed, assume that $U_{g',m'}$ is a Laurent form in $z$ with coefficients polynomially depending on $t_k = T_k(p)$, $k \in Z_{odd}$, for all $(g', m')$ with $0 < 2g' - 2 + m' < 2g - 2 + m$. Then the first summand in (29) obviously also has this form. Let us examine the second summand. Note that the operator $\delta_w^{-1} d_w$ is well defined on the space of odd Laurent polynomials in variable $w$, so let us check that this condition always holds. Indeed, the function $U_{g,m-1}(z) / \eta(z)$ is an even Laurent polynomial in $z$, therefore, it can be represented as a Laurent polynomial in $z^2$. Therefore, $\frac{1}{z^2 - w^2} \left( \frac{U_{g,m-1}(z)}{\eta(z)} - \frac{U_{g,m-1}(w)}{\eta(w)} \right)$ is a Laurent polynomial in $z^2$ and $w^2$ regular at $z = \pm w$. Multiplying it by $w$ and applying $d_w$, we obtain an odd Laurent form in $w$. The polynomiality property for the forms $U_{g,m}$ follows now from Remark 5.

Utilising Remark 5 once again, we obtain the polynomiality property for $F_{g,m}$ with $2g - 2 + m > 0$ as well. This proves the main assertion of Theorem 4 (under the assumption that it is valid for the initial terms $F_{0,3}$ and $F_{1,1}$; this is checked below).

4.4. The residual formalism. Since the both sides of (29) belong to $L$, it is convenient to equate the projections of the terms entering this equality by applying $P_L$ to each of them. The terms of the first summand on the right hand side already belong to $L$, so $P_L$ is identical on them. Compute the image of the second summand on the right hand side under the projection $P_L$. The key observation is that $P_L$ can be applied to the two terms of this summand separately. In particular, the form $\frac{dz}{z^2 - w^2}$ is regular both at $z = 0$ and at $z = \infty$ so that it does not contribute
This proves the equality (20) of the topological recursion. It remains to check it for the initial terms of the recursion, that is, for \( P_L \) and \( z = 0 \) using the explicit formula (26) for \( P_L \) to the image. It remains to compute the term
\[
P_L \left( \delta_w^{-1} d_w \left( \frac{w}{z^2 - w^2} U_{g,m-1}(z) \right) dz \right) = P_L \left( \frac{U_{g,m-1}(z)}{\eta(z)} \delta_w^{-1} \left( \frac{z^2 + w^2}{(z^2 - w^2)^2} dz \right) dw \right).
\]

The form \( \frac{(z^2 + w^2)}{(z^2 - w^2)^2} \) \( dz \) \( dw \) is not Laurent so that \( \delta_w^{-1} \) is not applicable to it directly. However, what we actually need in order to apply \( P_L \) is the expansion of this form at \( z = 0 \) and \( z = \infty \). The coefficients of these expansions are Laurent with respect to \( w \):
\[
\frac{(z^2 + w^2)}{(z^2 - w^2)^2} \frac{dz}{dw} = -\sum_{i=0}^{\infty} d(w^{-2i-1}) z^{2i} dz, \quad z \to 0, \quad z \to \infty.
\]

In other words, the form \( \delta_w^{-1} \left( \frac{z^2 + w^2}{(z^2 - w^2)^2} \right) \) is well defined in some neighborhoods of the points \( z = 0 \) and \( z = \infty \) and coincides with the form \( U_{0,2} \) defined by (17). This proves the equality (20) of the topological recursion.

### 4.5. Initial terms of the recursion

In order to finish the proof of Theorem 1 it remains to check it for the initial terms of the recursion, that is, for \( U_{1,1} = \delta_x F_{1,1} dx \) and \( U_{0,3} = \delta_x F_{0,3} dx \).

For the case \( g = n = 1 \), equating the corresponding terms in (28) we get
\[
U_{1,1} = \frac{1}{2\eta(x)} \frac{\delta^2 F_{0,2}}{F_{0,2}} dx,
\]
and using the explicit formula (20) for \( \delta^2 F_{0,2} \), we obtain the required formula (22) for \( U_{1,1} \).

For the case \( g = 0, n = 3 \) the computations are slightly more involved. Eq. (28) uniquely determines the form \( U_{0,3} \) as
\[
U_{0,3} = \delta_{w^{-1}} \left( \frac{w}{z^2 - w^2} \right) \left( \frac{\delta_x F_{0,2}(x)}{\eta(x)} - \frac{\delta_y F_{0,2}(y)}{\eta(y)} \right) dx - \frac{(\delta_x F_{0,2})^2}{2\eta(x)} dx,
\]
where, as before, \( y \) is related to \( w \) by the same formulas (13) that relate \( x \) to \( z \).

The form \( \delta_x F_{0,2} dx \) being known, cf. (27), we directly compute
\[
U_{0,3} = \frac{1}{32w} \left( \frac{\delta_x F_{0,2}(x)}{\eta(x)} \right) = P_L \left( \frac{U_{0,2}(z)^2}{\eta(z)} \right) dz
\]
which agrees with (21). This completes the proof of Theorem 1.

### 5. Topological recursion for ribbon graphs

Here we discuss the topological recursion for the numbers \( D_{g,m} \) of genus \( g \) ribbon graphs with \( m \) labeled boundary components of lengths \( \mu_1, \ldots, \mu_m \), see Section 2. A topological recursion for these numbers was first obtained in [5]. We present a simpler proof of this recursion in terms of the homogeneous components \( \tilde{F}_{g,m} \) of the generating function (10) that follows directly from the Virasoro constraints, cf. Theorem [3] (i). In fact, the argument is quite parallel to that of the case of dessins d’enfants. This is why we skip the details paying attention only to the differences between these two enumerative problems. More specifically, we have the same spectral curve \( \mathbb{CP}^1 \) equipped with an affine coordinate \( z \) and the involution \( z \mapsto -z \).
What is different, is the choice of the local coordinate $x$ at the point $z = 1$ and the form $\eta$.

Consider the linear functions $\tilde{T}_k(p)$ given by (15) with

$$z(x) = \sqrt{\frac{1 + 2\sqrt{u} s x}{1 - 2\sqrt{u} s x}}$$  \hfill (30) 

**Theorem 5.** Let $tF_{g,m}$ be the infinite series defined by (11). Then

(i) For each $g,m$ with $2g - 2 + m > 0$ there exist a polynomial $\tilde{G}_{g,m}$ of the variables $t_j$, $j \in \mathbb{Z}_{\text{odd}}$, such that

$$\tilde{F}_{g,m}(p) = \tilde{G}_{g,m}(t) \big|_{t_j = \tilde{T}_j(p)}$$

(i.e. each $\tilde{F}_{g,m}$ is a polynomial in the linear functions $\tilde{T}_{k,1}, \tilde{T}_{k,3}, \tilde{T}_{k,5}, \ldots$).

(ii) The polynomials $\tilde{G}_{g,m}$ can be recursively computed, starting from $\tilde{G}_{0,3}$ and $\tilde{G}_{1,1}$, by the same Eqs. (20) – (22) with $\eta$ given by the formula

$$\eta = \eta(z) \, dz = -\frac{16uz^2}{(1 - z^2)^3}.$$  

**Proof.** Like in the case of dessins, we start with the master Virasoro equation that readily follows from Theorem 4 (i):

$$\left( -\frac{1}{s^2} + 2u x^2 \right) \delta_x \tilde{F} + x^3 \left( \delta_x^2 \tilde{F} + (\delta_x \tilde{F})^2 \right)$$

$$+ \delta_y^{-1} d_y \left( \frac{x^3 \delta_x \tilde{F} - y^3 d_y \tilde{F}}{x - y} \right) + u^2 x + up_1 = 0.$$  

The spectral curve in this case (an analog of (25) above) is given by the equation

$$x^2 (\delta_x \tilde{F}_{0,1})^2 - x \delta_x \tilde{F}_{0,1} \left( \frac{1}{s^2 x^2} - 2u \right) + u^2 = 0.$$  

Solving this equation for $x \delta_x \tilde{F}_{0,1}$ we get the following rational parametrization of the spectral curve

$$x(z) = \frac{z^2 - 1}{2s\sqrt{u} (z^2 + 1)},$$

$$x \delta_x \tilde{F}_{0,1} = \frac{1 - 2s^2 u x^2 - \sqrt{1 - 4s^2 u x^2}}{2s^2 x^2} = u \left( \frac{1 - z}{z + 1} \right)^2,$$

with the inverse change $z(x)$ given by (30). The genus 0 two point correlator is the same as in the case of dessins:

$$\delta_y \delta_x \tilde{F}_{0,2} \, dx \, dy = \frac{dz \, dw}{(z + w)^2},$$

and instead of the form $\eta$ we have

$$\tilde{\eta} = \tilde{\eta}(x) \, dx = \left( -\delta_x \tilde{F}_{0,1} - \frac{u}{2s^2 x^2} \right) \, dx = -\frac{16 u z^2}{(1 - z^2)^3} \, dz.$$  

Thus, the master Virasoro equation in $z$-coordinate acquires the the same form (28) and implies the topological recursion (29) with $\eta$ replaced by $\tilde{\eta}$. \hfill \Box
Below are the first few terms of the recursion:

\[ U_{0,3} = \frac{1}{32 u} \left( t_1^2 - t_{-1}^2 z^{-2} \right), \]
\[ G_{0,3} = \frac{1}{96 u} \left( t_1^3 + t_{-1}^3 \right), \]
\[ U_{1,1} = \frac{1}{128 u} \left( z^2 - 3 + 3 z^{-2} - z^{-4} \right), \]
\[ G_{1,1} = \frac{1}{384 u} \left( t_3 - 9 t_1 - 9 t_{-1} + t_{-3} \right), \]
\[ U_{0,4} = \frac{1}{512 u^2} \left( t_1^3 \left( z^2 + (t_3 t_1 - 3 t_1^3 - t_{-1}^3 t_1) \right) \right.
\[ + (t_{-1}^3 t_1 + 3 t_{-1}^3 t_{-1}^2 - t_{-3} t_{-1}^2) z^{-2} - t_{-1}^3 z^{-4} \right), \]
\[ G_{0,4} = \frac{1}{6144 u^2} \left( 4 t_1^3 t_3^2 - 9 t_1^4 - 6 t_1^2 t_{-1}^2 - 9 t_1^4 + 4 t_{-3} t_{-1}^3 \right), \]
\[ U_{1,2} = \frac{1}{2048 u^2} \left( 5 t_1 z^4 + (t_3 - 18 t_1) z^2 + (4 t_{-1} - 27 t_1 - 6 t_3 + 6 t_5) \right.
\[ + (-t_{-5} + 6 t_{-3} - 27 t_{-1} - 4 t_1) z^{-2} + (18 t_{-1} - t_{-3}) z^{-4} - 5 t_{-1} z^{-6} \right), \]
\[ G_{1,2} = \frac{1}{12288 u^2} \left( 6 t_1 t_5 + t_3^2 - 36 t_1 t_3 + 81 t_3^2 + 24 t_{-1} t_1 \right.
\[ + 81 t_{-1}^2 - 36 t_{-1} t_{-3} + t_{-2}^2 + 6 t_{-5} t_{-1} \right). \]

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