A Modification of the Generalized Kudryashov Method for the System of Some Nonlinear Evolution Equations

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Abstract:

In this study, a comparatively new technique named the generalized Kudryashov method (gKM) has been effectively implemented to explore the exact traveling wave solutions to some nonlinear evolution equations (NLEEs) in the field of nonlinear science and engineering. The effectiveness of the new functional method has been demonstrated by investigating single as well as coupled equations with arbitrary parameters explicitly the coupled Higgs field equation, the Benney-Luke equation, and the Drinfel'd-Sokolov-Wilson (DSW) equation. As a matter of fact, the solution attained in this article thrust into the abundant wave solutions which includes kink, singular kink, periodic and solitary wave solutions. Moreover, the characteristics of these analytic solutions are interpreted depicting some 2D and 3D graph by using computer symbolic programming Wolfram Mathematica. The computational work ascertained that the employed method is sturdy, simple, precise, and wider applicable. Also, the prominent competence of this current method ensures that practically capable to reducing the size of the computational task and can be solved several nonlinear types of new complex higher order partial differential equations that originating in applied mathematics, computational physics and engineering.

Keywords: The generalized Kudryashov method; Coupled Higgs field equation; Benney-Luke equation; DSW equation; Traveling wave solution; Solitary wave solution; Exact solution.
I. Introduction

In modern era, the mechanism of most of the real-world tangible phenomena can be recounted by a class of NLEEs. Therefore, the study of exact traveling and solitary wave solution of NLEEs is of enormous importance forasmuch as their expected effectuation in various branches of scientific research investigation and physical sciences, such as plasma physics, solid state physics, fluid dynamics, electromagnetic theory, optical fibers, chemical kinematics, geochemistry, mathematical biology, acoustics and mathematical finance etc. all are materially administered by nonlinear equations. In the last decades, the swift advancement of different symbolic computer programs many mathematicians and physical scientist have been bewitched and is actually presented various efficient and masterful technique to seek a more exact solution of nonlinear equations. Furthermore, it is mentionable to notice that there is no unique method to examine all kinds of NLEEs. Therefore, several methods have been introduced and extended, such as, the modified simple equation method [IV, XXI, XXVI], the exp-function method [I, XXII, XVI], the extended tanh method [V, X], the Cantor-type cylindrical co-ordinate method [XXXXIII], the Jacobi elliptic function method [XI, XXVII], the ansatz method [XII, XXXIV], the Homotopy perturbation method (HPM) [IX, XIX, XIV], the parameter expansion method [XXIII, XXXIII], the Darboux and Backlund transformation [XV], the exponential rational function method [VII, XXXI], the Adomian decomposition method [XVIII] and so on. Recently, Kumar et al. [VI] have flourished a direct truncated expansion method to search exact solutions of NLEEs called the modified Kudryashov method. By using similar method Kabir et al. [XXIX] developed the exact solitary wave solutions for some third order nonlinear equations. More recently, Mahmud et al. [XIII] implemented the GKM to obtain exact traveling wave solution to the PHI-four and Fishers equation. The analogous method was used by Koparan et al. [XXVIII] to attain the exact solutions of some NLEEs. In our term, after this general discussion, we constrict ourselves to find the new study relating to the well-known GKM for finding traveling and other exact solution to the Coupled Higgs field, Benney-Luke and DSW equations to exhibit the congeniality and straightforwardness of the method.

First, we take into account the couple Higgs field equation which occurs in the field of quantum mechanics to interpret the mechanism of mass for gauge bosons [XXXIX]. The general form of the coupled Higgs field equation is of the form [XXXII, XXV]:

\[
\begin{align*}
    u_{tt} - u_{xx} - auu + \beta |u|^2 u - 2uv &= 0, \\
    v_{tt} + v_{xx} - \beta (|u|^2)_{xx} &= 0.
\end{align*}
\]  

(1)
Here, in this case, $v$, $u$ are real constants and indicates a complex scalar nucleon field and real scalar meson field. When $\alpha > 0$, $\beta > 0$, Eq. (1) represent coupled Higgs field equation and if $\alpha < 0$, $\beta < 0$, the model convert into the coupled nonlinear Klein-Gordon equation. Eq. (1) has already been investigated using the first integral method [XXXII], generalized exp-function method [XXV], Lie classical approach and $(G'/G)^n$-expansion method [XXXV], Backlund transformation method [III], where some exact, solitary and new soliton wave solutions are obtained.

Secondly, we consider the Benney-Luke equation of the form [XXXVII, XXX]:

$$u_{tt} - u_{xx} + au_{xxxx} - bu_{xxtt} + u_tu_x + 2u_xu_{xt} = 0$$

(2)

where $a$, $b$ are real nonzero constants. The Benney-Luke equation is used to presumptive the full water wave equations and precisely compatible for describing two-way water wave propagation in company with surface tension. Equation (2) has been investigated implementing the $(G'/G^2)$-expansion method [XXXVII], F-expansion method with Riccati equation [XXX], Ansatz method [XXVII], improved $(G'/G)^n$-expansion method [XL] etc. and attained some exact traveling and solitary wave solutions.

Finally, we consider the well-known equation which introduced as essential wave model in physics is Drinfel’d-Sokolov-Wilson (DSW) equation takes the form [XXIV]:

$$\begin{cases} u_t + pv_x = 0, \\ v_t + qv_{xx} + ruv_x + su_xv = 0. \end{cases}$$

(3)

herein $p$, $q$, $r$, $s$ are real parameters. Equation (3) has been studied by applying the improved F-expansion method [XXV], first integral method [VII], Backlund transformation of Riccati equation and trial function approach [II] and gained some exact, solitary and traveling wave solution.

The structure of this paper is systematized as follows: In section II, we will illustrate the generalized Kudryashov method. The implementation of the introduced method to the above-mentioned equations is carried out in section III. In section IV, we presented results and discussion and lastly, in section V, we have drawn conclusions of this study.

**II. The generalized Kudryashov method**

In this section, we illustrate the novel GKM for investigating traveling wave solutions to NLEEs. Presume that a nonlinear equation for a function $v$, say in two independent variables $x$ and $t$ take the form:

$$H(v, v_t, v_x, v_y, v_{tt}, v_{xt}, v_{yt}, v_{xx}, v_{yx}, v_{yy}, \ldots \ldots) = 0.$$  (4)
\( H \) is a polynomial of unknown function in \( v = v(x, y, t) \) and its several partial derivatives and nonlinear terms are employed. The algorithm of the generalized Kudryashov method is described step by step as follows [XX]:

Step 1: Taking into account the traveling wave transformation, we assume

\[ v(x, y, t) = V(\eta), \eta = x + y \pm \omega t, \]  

where \( \omega \) is the speed of traveling wave.

Using this wave transformation (5) authorizes us reducing Eq. (4) to an ordinary differential equation (ODE) of the form for \( V = v(\eta) \) is:

\[ L(V, V', V'', V''', \ldots) = 0, \]  

herein ' indicates the derivative with respect to \( \eta \).

Step 2: The aforementioned technique suggests that the exact solution of (6) can be revealed in the following rational form:

\[ V(\eta) = \frac{\sum_{i=0}^{N} a_i Q^i(\eta)}{\sum_{j=0}^{M} b_j Q^j(\eta)} = \frac{a[Q(\eta)]}{b[Q(\eta)]}, \]  

where \( a_i (i = 0, 1, 2, \ldots, N), \ b_j (j = 0, 1, 2, \ldots, M) \) are constants to be determined later. Also \( a_N \neq 0, b_M \neq 0 \) and we note that \( Q(\eta) \) is the solution of the auxiliary ODE

\[ \frac{dQ}{d\eta} = Q(\eta)(Q(\eta) - 1). \]  

It is obvious that Eq. (8) has solution of the form:

\[ Q(\eta) = \frac{1}{1 + Be^{\eta}}, \]  

wherein \( B \) being an integrating constant.

Step 3: The positive integers \( N \) and \( M \) appearing in Eq. (7) can be evaluated by using the homogeneous balance principle. As for instance, we balance the highest order derivative with the highest order nonlinear term appears in Eq. (7).

Step 4: Plugging Eq. (8) into Eq. (7) along with Eq. (9), we achieve a polynomial \( S(Q(\eta)) \) of \( Q(\eta) \). Installing all of the coefficients of like powers of \( S(Q(\eta)) \) to zero, we obtain a set of algebraic equations. For solving these sets of equations, we use computer symbolic program Mathematica and acquiring the unknown parameters \( a_i, b_j \) and \( \omega \). In this process, ultimately we embellish the exact solutions of the reduced Eq. (7).
III. Applications of the method

In this section, the GKM has been attributed to seeking more exact and traveling wave solutions of the Coupled Higgs field equation, Benney-Luke equation and DSW equation.

III a. Coupled Higgs field equation

In this subsection, we will make use of the GKM to constitute the exact traveling wave solutions of the Coupled Higgs field equation (1). Take into account the following transformation

\[ u(x, t) = e^{i\theta(x,t)} U(\eta), \quad v(x, t) = V(\eta), \]

wherein \( \theta = kx + \omega t, \eta = x + ct \), and a relation \( k = \omega c \), hence the system of Eq. (1) transform to the following system of ODEs

\[ (\omega^2(c^2 - 1) - \alpha) U(\eta) + (c^2 - 1) U'(\eta) + \beta U^3(\eta) - 2U(\eta)V(\eta) = 0, \]  
\[ (c^2 + 1)V''(\eta) - \beta (U^2(\eta))'' = 0. \]

Integrating Eq. (12) two times with respect to \( \eta \), we obtain

\[ V(\eta) = \frac{R + \beta U^2(\eta)}{c^2 + 1}, \]

here \( R \) is the constant of second integration and letting the first integrating constant to be zero.

Putting Eq. (13) into Eq. (11) yields

\[ \left( \omega^2(c^2 - 1) - \alpha - \frac{2R}{c^2 + 1} \right) U(\eta) + (c^2 - 1) U'(\eta) + \beta \left( 1 - \frac{2}{c^2 + 1} \right) U^3(\eta) = 0. \]

Again, the second order ODE can be written as

\[ U''(\eta) + \frac{\beta}{c^2 + 1} U^3(\eta) + \left( \omega^2 - \frac{\alpha}{c^2 - 1} - \frac{2R}{c^4 - 1} \right) U(\eta) = 0. \]

Utilizing homogeneous balance principle, balancing the highest order derivative \( U'' \) with the highest order non-linear term \( U^3 \) in Eq. (15), we get

\[ N = M + 1. \]

Selecting \( M = 1 \), then \( N = 2 \), so that the exact solution of Eq. (7) derived as follows

\[ U(\eta) = \frac{a_0 + a_1 Q + a_2 Q^2}{b_0 + b_1 Q}, \]

wherein \( a_0, a_1, a_2, b_0 \) and \( b_1 \) are constants to be discerned later and \( Q = Q(\eta) \) conciliate Eq. (9). Setting Eq. (17) into Eq. (15) and cleaning the denominator, we
We obtain a polynomial in $Q(\eta)$. Equating all the coefficients like powers of $Q(\eta)^k$ to zero, we attain the following system of algebraic equations

\begin{align*}
Q^0: & \quad -\beta a_0^3 + c^2 \beta a_0^3 - 2 Ra_0 b_0^2 - a a_0 b_0^2 - c^2 a a_0 b_0^2 - \omega^2 a_0 b_0^2 + c^4 \omega^2 a_0 b_0^2 = 0, \\
Q^1: & \quad -3\beta a_0^2 a_1 + 3c^2 \beta a_0^2 a_1 - a_1 b_0^2 + c^2 a_0 b_0^2 - 2 Ra_1 b_0^2 - a a_1 b_0^2 - c^2 a a_1 b_0^2 - \omega^2 a_1 b_0^2 + c^4 \omega a_1 b_0^2 + a_0 b_0 b_1 - c^4 a_0 b_0 b_1 - 4 Ra_0 b_0 b_1 - 2 a a_0 b_0 b_1 - 2c^2 a a_0 b_0 b_1 - 2\omega^2 a_0 b_0 b_1 + 2c^4 \omega^2 a_0 b_0 b_1 - a_0 b_1^2 + c^4 a_0 b_1^2 - 2 Ra_0 b_1^2 - a a_0 b_1^2 - c^2 a a_0 b_1^2 - \omega^2 a_0 b_1^2 + c^4 \omega^2 a_0 b_1^2 = 0, \\
Q^2: & \quad -3\beta a_0 a_1^2 + 3c^2 \beta a_0 a_1^2 - 3\beta a_0^2 a_2 + 3c^2 \beta a_0^2 a_2 + 3a_1 b_0^2 - 3c^4 a_1 b_0^2 - 4a_2 b_0^2 + 4c^2 a_2 b_0^2 - 2Ra_2 b_0^2 - a a_2 b_0^2 - c^2 a a_2 b_0^2 - \omega^2 a_2 b_0^2 + c^4 \omega a_2 b_0^2 - a_0 b_0 b_1 + a_1 b_0 b_1 - c^4 a_1 b_0 b_1 - 4 Ra_1 b_0 b_1 - 2 a a_1 b_0 b_1 - 2c^2 a a_1 b_0 b_1 - 2\omega^2 a_1 b_0 b_1 + 2c^4 \omega^2 a_1 b_0 b_1 - a_0 b_1^2 + c^4 a_0 b_1^2 - 2 Ra_0 b_1^2 - a a_0 b_1^2 - c^2 a a_0 b_1^2 - \omega^2 a_0 b_1^2 + c^4 \omega^2 a_0 b_1^2 = 0, \\
Q^3: & \quad -\beta a_1^3 + c^2 \beta a_1^3 - 6\beta a_0 a_1 a_2 + 6c^2 \beta a_0 a_1 a_2 - 2a_1 b_0^2 + 2c^4 a_1 b_0^2 + 10 a_2 b_0^2 - 10c^2 a_2 b_0^2 + 2a_0 b_0 b_1 - 2c^4 a_0 b_0 b_1 - a_1 b_0 b_1 + c^4 a_1 b_0 b_1 - 3a_2 b_0 b_1 + 3c^4 a_2 b_0 b_1 - 4Ra_2 b_0 b_1 - 2 a a_2 b_0 b_1 - 2c^2 a a_2 b_0 b_1 - 2\omega^2 a_2 b_0 b_1 + 2c^4 \omega^2 a_2 b_0 b_1 + a_0 b_1^2 - c^4 a_0 b_1^2 - 2 Ra_0 b_1^2 - a a_0 b_1^2 - c^2 a a_0 b_1^2 - \omega^2 a_0 b_1^2 + c^4 \omega^2 a_0 b_1^2 = 0, \\
Q^4: & \quad -3\beta a_1^2 a_2 + 3c^2 \beta a_1^2 a_2 - 3\beta a_0 a_2^2 + 3c^2 \beta a_0 a_2^2 - 6 a_2 b_0^2 + 6c^4 a_2 b_0^2 + 9 a_2 b_0 b_1 - 9c^2 a_2 b_0 b_1 - a_2 b_1^2 + c^4 a_2 b_1^2 - 2 Ra_2 b_1^2 - a a_2 b_1^2 - c^2 a a_2 b_1^2 - \omega^2 a_2 b_1^2 + c^4 \omega^2 a_2 b_1^2 = 0, \\
Q^5: & \quad -3\beta a_1 a_2^3 + 3c^2 \beta a_1 a_2^3 - 6a_2 b_0 b_1 + 6c^4 a_2 b_0 b_1 + 3a_2 b_1^2 - 3c^4 a_2 b_1^2 = 0, \\
Q^6: & \quad -\beta a_1^2 b_2^2 - 2a_2 b_1^2 + 2c^4 a_2 b_1^2 = 0.
\end{align*}

Solving above system of algebraic equations by using Mathematica, we achieve the following several types of solutions:

**Type 1:**

\[
\begin{align*}
a_0 &= \frac{a_2 b_0}{2 b_1}, \quad a_1 = \frac{a_2 (-2b_0 + b_1)}{2 b_1}, \quad R = \frac{\beta a_2^2 (\beta a_2^3 + 2 \beta \omega^2 a_2^2 - 4 b_1^2 + 4 \omega^2 b_1^2)}{16 b_1^2}, \\
c &= \frac{\sqrt{-\beta a_1^2 - 2 b_1^2}}{\sqrt{2b_1}}, \quad \frac{\sqrt{-\beta a_1^2 - 2 b_1^2}}{\sqrt{2b_1}}.
\end{align*}
\]
herein $a_2$, $b_0$ and $b_1$ are free parameters.

Type 2:

$$a_0 = 0, a_1 = -a_2, b_0 = -\frac{b_1}{2}, R = \frac{\beta a_2^2(\beta a_2^2 + \beta \omega^2 a_2^2 + 4b_1^2 + 2ab_1^2 + 4\omega^2 b_1^2)}{8b_1^4},$$

$$c = \left(\frac{-\beta a_2^2 - 2b_1^2}{\sqrt{2b_1}}, -\frac{-\beta a_2^2 - 2b_1^2}{\sqrt{2b_1}}\right),$$

where $a_2$ and $b_1$ are free parameters.

Type 3:

$$a_0 = \frac{a_2}{2}, a_1 = -a_2, b_0 = -\frac{b_1}{2}, R = \frac{\beta a_2^2(-2\beta a_2^2 + \beta \omega^2 a_2^2 - 8b_1^2 + 2ab_1^2 + 4\omega^2 b_1^2)}{8b_1^4},$$

$$c = \left(\frac{-\beta a_2^2 - 2b_1^2}{\sqrt{2b_1}}, -\frac{-\beta a_2^2 - 2b_1^2}{\sqrt{2b_1}}\right),$$

whither $a_2$ and $b_1$ be the free parameters.

Inserting these results into equations (9) and (10) into Eq. (17), we attain exactly traveling wave solutions step by step as follows

Type 1 corresponds to the exact and traveling wave solutions of Eq. (1) takes the form

$$U_{1,1}(\eta) = \frac{(1 - B(Cosh(\eta) + Sinh(\eta)))a_2}{2(1 + B(Cosh(\eta) + Sinh(\eta)))b_1},$$

where $\eta = x - \frac{-\beta a_2^2 - 2b_1^2}{\sqrt{2b_1}}t$.

If we set $a_2 = 1, b_1 = \frac{1}{2}$ and $B = 1$ in Eq. (18), yields

$$u_{1,1}(x, t) =$$

$$\left(Csch\left(x - \frac{-\beta a_2^2 - 2b_1^2}{\sqrt{2b_1}}t\right) - Coth\left(x - \frac{-\beta a_2^2 - 2b_1^2}{\sqrt{2b_1}}t\right)\right)e^{i(kx + \omega t)}. (19)$$
\[ v_{1,1}(x, t) = \frac{1}{2} \left( \beta - \alpha - 2\omega^2 - 2\beta \omega^2 - 2\text{Csch}^2 \left( x - \sqrt{\frac{-\beta a_2^2 - 2b_1^2}{\sqrt{2b_1}}} \right) t \right) + 2\text{Coth}x \left( -\beta a_2 - 2b_122b_1t \right) \text{Csch}x - \beta a_2 - 2b_122b_1t. \]  

(20)

Type 2 converts the exact and traveling wave solution of Eq. (1) is in the form

\[ U_{1,2}(\eta) = \frac{2B(\cos h(\eta) + \sin h(\eta))a_2}{(-1 + \left( B(\cos h(\eta) + \sin h(\eta)) \right)^2)b_1}. \]  

(21)

here \( \eta = x - \sqrt{\frac{-\beta a_2^2 - 2b_1^2}{\sqrt{2b_1}}} t. \)

Choose \( a_2 = 1, b_2 = -\frac{1}{2} \) and \( B = 1 \), Eq. (21) transmutes

\[ u_{1,2}(x, t) = \left( -2\text{Csch} \left( x - \sqrt{\frac{-\beta a_2^2 - 2b_1^2}{\sqrt{2b_1}}} \right) t \right) e^{i(kx + \omega t)}. \]  

(22)

\[ v_{1,2}(x, t) = -\frac{a}{2} - (1 + \beta)(1 + \omega^2) - 2\text{Csch}^2 \left( x - \sqrt{\frac{-\beta a_2^2 - 2b_1^2}{\sqrt{2b_1}}} \right) t. \]  

(23)

Type 3 performs the closed form and traveling wave solution of Eq. (1) is

\[ U_{1,3}(\eta) = \frac{(1 + \left( B(\cosh(\eta) + \sinh(\eta)) \right)^2)a_2}{(1 - \left( B(\cosh(\eta) + \sinh(\eta)) \right)^2)b_1}, \]  

(24)

herein \( \eta = x + \sqrt{\frac{-\beta a_2^2 - 2b_1^2}{\sqrt{2b_1}}} t. \)

If we select \( a_2 = \frac{3}{2}, b_1 = 1 \) and \( B = 1 \), Eq. (24) derives

\[ u_{1,3}(x, t) = \left( -\frac{3}{2 \text{tanh} \left( x + \sqrt{\frac{-\beta a_2^2 - 2b_1^2}{\sqrt{2b_1}}} \right) } \right) e^{i(kx + \omega t)}. \]  

(25)

\[ v_{1,3}(x, t) = -\frac{a}{2} - \frac{1}{16} (16 + 9\beta)(-2 + \omega^2) - 2\text{Coth}^2 \left( x + \sqrt{\frac{-\beta a_2^2 - 2b_1^2}{\sqrt{2b_1}}} \right) t. \]  

(26)
III b. Benney-Luke equation

In this subsection, GKM is treating to enucleate the exact and then traveling wave solutions of the Benney-Luke equation (2). Consider the traveling wave transformation equation

\[ u(x,t) = U(\eta), \quad \eta = x - ct. \]  (27)

Here \( c \) is the speed of the traveling wave. With the help of Eq. (27), Eq. (2) reduces into the following ODEs

\[ (c^2 - 1)U'' + (a - c^2 b)U^{(4)} - 3cU' U''' = 0. \]  (28)

Integrating Eq. (28) once with respect to \( \epsilon \) and assuming the constant of integration to zero, we get

\[ (c^2 - 1)U' + (a - c^2 b)U''' - \frac{3}{2} c(U')^2 = 0. \]  (29)

Balancing \( U''' \) with \((U')^2\) in Eq. (29) becomes \( N = M + 1 \). Now, by implementing GKM, we attain new exact solution of Eq. (29). That solution is the similar solution of Eq. (15) and is in the form of Eq. (17).

Plugging Eq. (17) in Eq. (29) and putting off the denominator, we gain a system of algebraic equations by solving it, different types of results derived as follows

Type 1:

\[ a_0 = 0, \quad a_2 = -\frac{4(a-b)b_1}{\sqrt{a-1}\sqrt{b-1}}, \quad b_0 = 0, \quad c = \frac{\sqrt{a-1}}{\sqrt{b-1}}, \]

where \( b_1 \) is a free parameter.

Type 2:

\[ a_1 = -2a_0, \quad a_2 = \frac{8\left((1-\sqrt{a}b_0, \sqrt{1-a}b_0)\right)}{4\sqrt{a-1}}, \quad b_1 = -2b_0, \quad c = \frac{\sqrt{1-4a}}{\sqrt{1-4b}}, \]

here \( a_0 \) and \( b_0 \) are free parameters.

Type 3:

\[ a_1 = \frac{4\sqrt{a-1}b_0}{\sqrt{b-1}} - \frac{4\sqrt{a-1}b_0^2}{\sqrt{b-1}}a_0 b_1 + a_0 b_1, \quad a_2 = \frac{4\left((1-\sqrt{a}b_1, \sqrt{1-a}b_1)\right)}{a-1}, \quad c = -\frac{\sqrt{a-1}}{\sqrt{b-1}}, \]

herein \( a_0, b_0 \) and \( b_1 \) are free parameters.

Type 1 transforms into the following exact traveling wave solution of Eq. (2) will be in the form

\[ U_{2,1}(\eta) = \frac{(1+B(Cosh(\eta)+Sinh(\eta)))a_1 + \frac{4(b-a)b_1}{\sqrt{a-1}\sqrt{b-1}}}{(1+B(Cosh(\eta)+Sinh(\eta)))b_1}, \]  (30)
where $\eta = x - \frac{a-1}{\sqrt{b-1}} t$.

If we input $a_1 = 1, b_1 = 1$ and $B = 1$, Eq. (30) turn out

$$u_{2,1}(x,t) = 1 - \frac{2(b-a)}{\sqrt{a-1} \sqrt{b-1}} \left( \text{Coth} \left( x - \frac{\sqrt{a-1}}{\sqrt{b-1}} t \right) - \text{Csch} \left( x - \frac{\sqrt{a-1}}{\sqrt{b-1}} t \right) - 1 \right).$$

Type 2 performs the new exact traveling wave solution of Eq. (2) takes the form

$$U_{2,2}(\eta) = \frac{(4a-1)\sqrt{1-4b} \left( B(\text{Cosh}(\eta) + \text{Sinh}(\eta)) \right)^2 - 1 \right) a_0 + 8\sqrt{1-4a} (a-b) b_0}{(4a-1)\sqrt{1-4b} \left( B(\text{Cosh}(\eta) + \text{Sinh}(\eta)) \right)^2 - 1 \right) b_0},$$

herein $\eta = x - \frac{\sqrt{1-4a}}{\sqrt{1-4b}} t$.

If we consider $a_0 = 1, b_0 = -1$ and $B = 1$, Eq. (32) promotes into

$$u_{2,2}(x,t) = -1 + 4\sqrt{1-4a} (a-b) \left( \text{Coth} \left( x - \frac{\sqrt{1-4a}}{\sqrt{1-4b}} t \right) - 1 \right).$$

Type 3 represents the exact and singular soliton solution of Eq. (2) will be in the form

$$U_{2,3}(\eta) = \frac{(a-1)^{\frac{1}{2}} \left( 1+B(\text{Cosh}(\eta) + \text{Sinh}(\eta)) \right) a_0 + 4\sqrt{a-1} (a-b) b_0}{(a-1)\sqrt{b-1} \left( 1+B(\text{Cosh}(\eta) + \text{Sinh}(\eta)) \right) b_0},$$

here $\eta = x + \frac{a-1}{\sqrt{b-1}} t$.

If we select $a_0 = 1, b_0 = \frac{1}{2}$ and $B = -1$, Eq. (34) derives

$$u_{2,3}(x,t) = 2 - \frac{a-b}{\sqrt{a-1} \sqrt{b-1}} \left( \text{Coth} \left( x + \frac{\sqrt{a-1}}{\sqrt{b-1}} t \right) + \text{Csch} \left( x + \frac{a-1}{\sqrt{b-1}} t \right) - 1 \right).$$

**III c. Drinfel’d-Sokolov-Wilson equation**

In this subsection, GKM will be employed to establish the exact traveling wave solution of DSW equation (3). First of all, we take into consideration the following traveling wave transformation,

$$u(x,t) = U(\eta), v(x,t) = V(\eta), \eta = x + \lambda t,$$

here $\lambda$ is wave speed. Substituting this transformation into Eq. (3), this transforms into the ODE is of the forms

$$\lambda U' + p V V' = 0.$$  \hspace{1cm} (37)

$$\lambda V' + q V'' + r U V + s U' V = 0.$$  \hspace{1cm} (38)

Integrating Eq. (37) once and ignoring the constant of integration, we get
\[ U = -\frac{pV^2}{2\lambda}. \]  

(39)

Setting the values of \( U \) and \( U' \) from Eq. (39) and (37) into Eq. (38), then

\[ 2\lambda qV'' + 2\lambda^2 V' - pV^2V'(r + 2s) = 0. \]  

(40)

Integrating Eq. (40) once and assuming zero be the constant of integration, we attain

\[ 2\lambda qV' + 2\lambda^2 V - \frac{pV^3(r + 2s)}{3} = 0. \]  

(41)

Balancing highest order derivative with highest order nonlinear term in Eq. (41) becomes \( N = M + 1 \). By introducing GKM, the solution of Eq. (41) is similar to the solution of Eq. (15) and the form is in Eq. (17). Inserting Eq. (17) into Eq. (41) and reducing the denominator, we achieve a system of algebraic equations by solving it, several types of results represent as follows

Type 1:

\[
\begin{align*}
    a_0 &= \frac{\sqrt{2} qa_0}{\sqrt{p(r + 2s)}}, \\
    a_1 &= 0, \\
    a_2 &= -\frac{2\sqrt{6} qa_0}{\sqrt{p(r + 2s)}}, \\
    b_1 &= 2b_0, \\
    \lambda &= \frac{q}{2},
\end{align*}
\]

therein \( b_0 \) is a free parameter.

Type 2:

\[
\begin{align*}
    a_0 &= 0, \\
    a_2 &= -2a_1, \\
    b_0 &= 0, \\
    b_1 &= -\frac{\sqrt{2} qa_1}{q}, \\
    \lambda &= \frac{q}{2},
\end{align*}
\]

where \( a_1 \) is free parameter.

Type 3:

\[
\begin{align*}
    a_0 &= 2 \left( -\frac{20\sqrt{6} pqr b_0}{(p(r + 2s))^2} - \frac{40\sqrt{6} pq^2 b_0}{(p(r + 2s))^2} + \frac{21\sqrt{6} q b_0}{\sqrt{p(r + 2s)}} \right), \\
    a_1 &= -\frac{4\sqrt{6} qa_0}{\sqrt{p(r + 2s)}}, \\
    a_2 &= \frac{24 q b_0}{\sqrt{p(r + 2s)}^2 \sqrt{p(r + 2s)}}, \\
    b_1 &= -2b_0, \\
    \lambda &= 2q,
\end{align*}
\]

here \( b_0 \) is free parameter.

Type 1 accomplishes the exact traveling wave solution of Eq. (3), which represent as

\[ V_{3.1}(\eta) = \frac{\sqrt{2}^{-1} (-1 + B(Cosh(\eta) + Sinh(\eta)))}{\sqrt{p(r + 2s)}})^q, \]  

(42)

wherein \( \eta = x + \frac{q}{2} t \).
If we select \( B = 1 \) in Eq. (42), converts into
\[
v_{3,1}(x, t) = \frac{\sqrt{2} q}{\sqrt{p(r+2s)}} \left( \text{Csch}(x + \frac{q}{2}t) + \text{Coth}(x + \frac{q}{2}t) \right),
\]
and
\[
u_{3,1}(x, t) = -\frac{3 q}{2(r+2s)} \left( \text{Csch}(x + \frac{q}{2}t) + \text{Coth}(x + \frac{q}{2}t) \right)^2.
\]

Type 2 executes the following closed form solution of Eq. (3), which represent as
\[
V_{3,2}(\eta) = -\frac{\sqrt{2}}{\sqrt{p(r+2s)} \sqrt{2} \left( \beta \text{Cosh}(\eta) + \text{Sinh}(\eta) \right)^2 q}.
\]

where \( \eta = x + \frac{q}{2} t \).

If we input \( B = -1 \) in Eq. (45), transforms
\[
v_{3,2}(x, t) = -\frac{\sqrt{2} q}{\sqrt{p(r+2s)}} \left( \text{Csch}(x + \frac{q}{2}t) + \text{Coth}(x + \frac{q}{2}t) \right),
\]
and
\[
u_{3,2}(x, t) = -\frac{3 q}{2(r+2s)} \left( \text{Csch}(x + \frac{q}{2}t) + \text{Coth}(x + \frac{q}{2}t) \right)^2.
\]

Type 3 ascertain the exact traveling wave solution of eq. (3), takes the form
\[
V_{3,3}(\eta) = \frac{2\sqrt{2}}{\sqrt{p(r+2s)} \sqrt{2} \left( \beta \text{Cosh}(\eta) + \text{Sinh}(\eta) \right)^2 q}.
\]

herein \( \eta = x + 2qt \).

If we choose \( B = 1 \), Eq. (48) takes the form
\[
v_{3,3}(x, t) = \frac{2\sqrt{2} q}{\sqrt{p(r+2s)}} \text{Coth}(x + 2qt).
\]
and
\[
u_{3,3}(x, t) = -\frac{6q}{(r+2s)} \text{Coth}^2(x + 2qt).
\]

Remark: All of the above solutions have been tested by substituting back into the main equation via the symbolic computer program Mathematica and produced them appropriately.

IV. Results and Discussion

The fundamental keystone of the GKM is to punctuate the new exact solutions of the above-mentioned Equations (1-3). Many researchers also solved these equations by using various techniques and obtained different types of exact, solitary and traveling wave solutions. In our term, we attain several types of new exact, closed-
form and traveling wave solutions, which represents as hyperbolic (tan, cot, and csc) form. Moreover, we compare our obtained solutions with Taghizadeh and Mirzazadeh [XXXII], Islam et al. [XXX], Lu et al. [VII] results for equation (1), (2), (3) respectively. Wherein Taghizadeh and Mirzazadeh [XXIX] solved Coupled Higgs field equation by using the first integral method, Islam et al. [XXI] solved the Benney-Luke equation by implementing improved F-expansion method combined with Riccati equation and Lu et al. [VII] solved DSW equation by applying first integral method. The comparison is represented as follows:

### Coupled Higgs field equation

| Taghizadeh and Mirzazadeh [XXXII] solution | Obtained solution |
|--------------------------------------------|-------------------|
| For $B_0 = 0$, $A_1 = \frac{1}{\beta}\left[-\alpha(c^2 + 1) - 2\beta c e + 1\right]$ and $R = \frac{1}{2}\left[-\alpha(c^2 + 1) - 2\beta c e + 1\right]$, then the solution of Eq. (38) arrives, | If $B = 1$, $a_2 = 1$, $b_1 = \frac{1}{2}$ and $c = \frac{-\beta a_2^2 - 2b_1^2}{\sqrt{2b_1}}$, the traveling wave solution is |
| $u(x,t) = -\frac{A_0}{\beta}\sqrt{-2\beta(c^2 + 1)} e^{i(\omega x + \omega t)}$ | $u_{1,1}(x,t) = \left(Csch\left(x - \frac{-\beta a_2^2 - 2b_1^2}{\sqrt{2b_1}} t\right) - Ccosh - \beta a_2 - 2b_1 t + i\omega x + \omega t\right)$ |
| $\times \tan\left(\frac{A_0}{\beta}\sqrt{-2\beta(c^2 + 1)} (x + ct + \xi_0)\right)$, and | |
| $v(x,t) = \frac{1}{2}\left[-\alpha + \omega^2(c^2 - 1) + \sqrt{-2\beta(c^2 + 1)} A_0(c^2 - 3\tan 2\alpha 2^2 - 2\beta c e + 1)x + ct + \xi_0\right]$ | $v_{1,1}(x,t) = \frac{1}{2}\left(\beta - \alpha - 2\omega^2(1 + \beta) - 2Csch2\omega - \beta a_2 - 2b_1 t + Ccosh - \beta a_2 - 2b_1 t + i\omega x + \omega t\right)$ |

where $\xi_0$ is arbitrary constant.
The Benney-Luke equation

| Islam et al. [XXX] solution | Obtained solution |
|-----------------------------|-------------------|
| For $c = \pm \sqrt[4k+1]{4b+1}$, $\alpha_1 = 0$, $\beta_1 = \pm \sqrt[4(bk-am^2-ak+bm^2)]{(4ak+1)(4bk+1)}$ the solution of Benney-Luke equation is $u_{1,2}(\eta) = \alpha_0 \pm \frac{4(bk-am^2-ak+bm^2)}{\sqrt[4ak+1]{4b+1}{(4ak+1)(4bk+1)}{m-\sqrt{\tan h(\sqrt{-\eta})}}}$, herein $\eta = x \mp \sqrt[4ak+1]{4b+1} t$. |
| If $B = 1$, $a_1 = 1$, $b_1 = 1$ and $c = \frac{\sqrt{a-1}}{\sqrt{b-1}}$, the closed form solution is $u_{2,1}(x,t) = 1 - \frac{2(b-a)}{\sqrt{a-1}\sqrt{b-1}} \left(Coth \left(x - a-1b-1t-Cschx-a-1b-1t-1 \right) \right.$ |

| For $c = \pm \sqrt[16ak+1]{16bk+1}$, $m = 0$, $\alpha_1 = \pm \frac{\sqrt[16ak+1]{16bk+1}{(16ak+1)(16bk+1)}}{\sqrt[4(bk-am^2-ak+bm^2)]{(4ak+1)(4bk+1)}{m-\sqrt{\tan h(\sqrt{-\eta})}}}$, the solution takes the form $u_{5,6}(\eta) = \alpha_0 \mp \frac{4\sqrt{-E(a-b)}}{\sqrt[16ak+1]{16bk+1}} \left( tanh\left(\sqrt{-k}\eta\right) + Coth-k\eta \right)$, where $\eta = x \mp \sqrt[16ak+1]{16bk+1} t$. |
| If $B = 1$, $a_0 = 1$, $b_0 = -1$ and $c = \frac{\sqrt{1-4a}}{\sqrt{1-4b}}$, hence the solution $u_{2,2}(x,t) = -1 + \frac{4\sqrt{-E(a-b)}}{\sqrt[1-4b]{1-4a1-4b-t-1}} \left(Coth \left(x - 1-4a1-4b-t-1 \right) \right.$ |
The DSW equation

| Lu et al. [VII] solution | Obtained solution |
|--------------------------|-------------------|
| For $A_0 = 0$, $B = 0$, $B_0 = \frac{1}{\beta A_1} (c + \sqrt{\alpha})$, $c = -\frac{a}{3\beta A_1} (y + 2x)$, the shock wave solution of Eq. (5) transforms $v_1(x,t) = \sqrt{-\frac{2(c^2+y^2)}{c \beta A_1}} \tanh \left( A_1 \sqrt{-\frac{c^2+y^2}{2 \beta A_1}} (x - ct + \xi_{01}) \right)$, and $u_1(x,t) = \frac{\pi}{2c} \sqrt{-\frac{2(c^2+y^2)}{c \beta A_1}} \tanh \left( A_1 \sqrt{-\frac{c^2+y^2}{2 \beta A_1}} (x - ct + \xi_{012} - Ac) \right)$ | If $B = 1$, $a_0 = \frac{2b_0}{\sqrt{p(r+2s)}}$, $a_1 = 0$, $a_2 = \frac{-2\sqrt{r}b_0}{\sqrt{p(r+2s)}}$, $b_1 = 2b_0$, $\lambda = \frac{q}{2}$ then the solution derives $v_{3,1}(x,t) = \frac{\sqrt{p}}{\sqrt{p(r+2s)}} \frac{\csc \left( \sqrt{r} \left[ \frac{t}{2} + \text{Csc} \left( \frac{x+y}{2} \right) \right] \right)}{\text{Cot} \left( \frac{x+y}{2} \right)}$ and $u_{3,1}(x,t) = \frac{-3q}{2(r+2s)} \frac{\csc \left( \sqrt{r} \left[ \frac{t}{2} + \text{Csc} \left( \frac{x+y}{2} \right) \right] \right)}{\text{Cot} \left( \frac{x+y}{2} \right)}$. |

From the above manifestation, it is obvious that all of our attained results are in hyperbolic (tan, cot, cosec) form and these are totally renewed from other scholars.

V. Conclusion

The fundamental objective in this work has been effectively utilized novel GKM for searching new exact traveling wave solutions of three NLEEs namely Coupled Higgs field, Benney-Luke, and DSW equations. The obtained solutions are discerned by putting them back into the said physical model and are accomplished to be very commodious over various existing methods. Without the help of the symbolic program, we settle upon arbitrary values of free parameters and achieved different shapes of wave solutions which are quivered graphically. These solutions have an extraordinary specialty that keeps its uniformity upon interacting with others. The results out solutions emphasize that the GKM provides a powerful mathematical instrument that some new exact solutions are frequently extracted. This method is very legitimate, modest, convenient and effective from the theoretical and pragmatical point of view. Finally, it is essential to notice that this method can be more frequently applicable for the large category of physical problems in the wide range of applied nonlinear science.
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