Long-range correlations in a locally driven exclusion process

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We show that the presence of a driven bond in an otherwise diffusive lattice gas with simple exclusion interaction results in long-range density-density correlation in its stationary state. In dimensions \(d > 1\) we show that in the thermodynamic limit this correlation decays as \(C(r,s) \sim (r^2 + s^2)^{-d}\) at large distances \(r\) and \(s\) away from the drive with \(|r - s| \gg 1\). This is derived using an electrostatic analogy whereby \(C(r,s)\) is expressed as the potential due to a configuration of electrostatic charges distributed in 2d-dimension. At bulk density \(\rho = 1/2\) we show that the potential is that of a localized quadrupolar charge. At other densities the same is correct in leading order in the strength of the drive and is argued numerically to be valid at higher orders.

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Unlike systems in equilibrium, non-equilibrium stationary states often exhibit generic scale invariance, i.e., spatial correlation of local thermodynamic variables has power-law tail \([1-6]\). This intriguing feature has been demonstrated in numerous models of non-equilibrium systems where an external drive prevents the system from achieving thermodynamic equilibrium. In particular, for a diffusive system coupled to two reservoirs at unequal densities the correlation of density fluctuations has been found to decay as \(1/r^d\) \([4]\). On the other hand in the presence of uniform bulk drive the correlation decays as \(1/r^d\) \([5]\). The role played by the drive-induced anisotropy in generating power-law correlation in systems with conserving dynamics has been elucidated in \([6]\). In some cases, long-range correlations can be related to the non-local nature of the large-deviation function of the density profile \([7,8]\).

A natural question to ask is what happens when the drive is spatially localized in the bulk. Such a drive can be thought of as a local perturbation that breaks detailed balance. Moreover, the drive breaks translational symmetry. This is a crucial difference from most of the models studied earlier which has translational symmetry \([4,5]\).

In an earlier work we studied the change in density profile induced by a driving bond in Symmetric Simple Exclusion Process (SSEP) and in a system of Non-Interacting (NI) particles \([9]\). In both cases, we found that the density in \(d\)-dimension approaches its bulk constant value algebraically as \(1/r^{d-1}\) with the distance \(r\) away from the position of the drive. It would be of great interest to compare the steady states of both cases in more detail by considering the density-density correlation function. In the case of NI particles the correlation vanishes in the thermodynamic limit. The correlation in SSEP has recently been studied in \(d = 1\) dimension \([10]\). It has been shown that the correlation scales as \(1/L\) for a system of length \(L\), and thus it vanishes in the thermodynamic limit. This is related to the vanishing of the current in the stationary state of the \(L \to \infty\) limit. The density-density correlation of models of local drive with interacting particles in higher dimensions has not been considered. As we show below, the interaction drastically modifies the steady state of the system, generating long-range correlations which persist in the thermodynamic limit in \(d \geq 2\) dimensions.

In this Letter, we study the density-density correlation of SSEP with single driven bond in \(d\)-dimensions. We show that due to the fact that the local current does not vanish in the thermodynamic limit for \(d > 1\), density-density correlations do not vanish and they are long ranged. In particular we show that at large distances \(r\) and \(s\) away from the drive with \(|r - s| \gg 1\), the density-density correlation function corresponds to a quadrupolar electrostatic potential in \(2d\)-dimension which decays with a power-law tail as

\[
C(r,s) \sim \frac{f(\hat{r},\hat{s})}{(r^2 + s^2)^d},
\]

with \(f\) being an anisotropic function of the unit directions \(\hat{r}\) and \(\hat{s}\). This is in contrast with the case of boundary-drive \([4]\) where the correlation vanishes in the thermodynamic limit, in any dimension.

The model is defined on a lattice \([-L,L]^d\) in
$d$-dimension with periodic boundary condition. The lattice sites are denoted by the Cartesian coordinates $r = x_1 e_1 + \cdots + x_d e_d$, where $e_k$ is the unit vector in the $k$-th direction. The particles interact with symmetric exclusion i.e. any site is occupied by at most one particle at a time. The particles hop across bonds with rate 1 which is symmetric for all bonds except the one between sites $r = 0 \equiv \{0, \ldots, 0\}$ and $r = -e_1 \equiv \{L - 1, 0, \ldots, 0\}$. Across this bond the hopping rate is asymmetric with rate $(1 - \epsilon)$ in the $e_1$ direction and $(1 + \epsilon)$ in the reverse direction (see Fig. 1). The asymmetric rate drives a circulating particle current around the bond, which does not vanish in the $L \to \infty$ limit.

We start by considering the steady state density profile. Let $n(r)$ denote the occupation number of site $r$ which takes values 1 or 0, depending on whether or not the site is occupied. The density in the stationary state is defined by $\phi(r) = \langle n(r) \rangle$, where the angular brackets denote ensemble average. It is straightforward to show that in the stationary state $\phi(r)$ satisfies

$$\Delta \phi(r) = c(Q) (\delta_{r,0} - \delta_{r+e_1,0}),$$

(2)

where

$$Q = n(-e_1) [1 - n(0)] + n(0) [1 - n(-e_1)],$$

(3)

and $\Delta$ denotes Laplacian operator on the lattice. In an electrostatic analogy $\phi(r)$ corresponds to the potential due to a dipole across the driven bond, and at large distances it decays to its bulk constant value as $1/r^{d-1}$ with the distance $r$ away from the position of the drive.

We use similar electrostatic analogy to determine the spatial correlation of the density fluctuations defined as $C(r,s) = \langle n(r)n(s) \rangle - \phi(r)\phi(s)$. Using the identity $n(r)^2 = n(r)$ the correlation at $r = s$ can be expressed in terms of the density as $C(r,r) = \phi(r) [1 - \phi(r)]$. It is thus convenient to subtract this value and define

$$c(r,s) = C(r,s) - \phi(r) [1 - \phi(r)] \delta_{r,s}.$$ 

(4)

Thus $c(r,r) = 0$ for any $r$ and $c(r,s)$ vanishes at large distances, $r$ and $s$.

From the dynamics of the model it can be shown (see Supplemental Material [11]) that in the stationary state, $c(r,s)$ follows a Poisson equation on a 2$d$-dimensional lattice made of $(r,s)$ vectors.

$$\Delta c(r,s) = \sigma(r,s),$$

(5)

with $\sigma(r,s)$ given below in (6)-(9). In an electrostatic analogy, $c(r,s)$ is the potential due to the charge $\sigma(r,s)$ on the 2$d$-dimensional lattice. The large distance profile of $c(r,s)$ is determined by the lowest non-vanishing multi-pole moment of the charge $\sigma(r,s)$ which we show to be an effectively localized quadrupole near the origin in the thermodynamic limit.

For convenience the charge density is expressed below as a combination of three charges

$$\sigma(r,s) = \sigma_1(r,s) + \sigma_2(r,s) + \sigma_3(r,s),$$

(6)

where

$$\sigma_1(r,s) = \sum_{\nu=1}^d (\delta_{r+e_{\nu},s} + \delta_{r,s}) [c(r + e_{\nu},s) - 2c(r,s) + c(r,s - e_{\nu})],$$

(7)

$$\sigma_2(r,s) = \sum_{\nu=1}^d \left[ (\phi(r + e_{\nu}) - \phi(r))^2 - \delta_{\nu,1} \delta_{r,-e_1} \frac{c^2(Q)}{d} \right] \delta_{s,r+e_{\nu}} + (r \leftrightarrow s),$$

(8)

$$\sigma_3(r,s) = c(Q\hat{n}(s)) (\delta_{r,0} - \delta_{r,-e_1}) (1 - \delta_{s,0} - \delta_{s,-e_1}) + (r \leftrightarrow s).$$

(9)

Here $\hat{n}(r) \equiv n(r) - \phi(r)$ and $(r \leftrightarrow s)$ denotes the term obtained by interchanging $r$ and $s$ in the preceding expression. The latter is related to the symmetry of $c(r,s)$ under exchange of $r$ and $s$.

The charge $\sigma_1(r,s)$ is a linear function of the the correlation itself and is introduced to include the $r = s$ sites in the Poisson equation (see Supplemental Material [11]). The other two charge densities result from the asymmetric hopping rates across the driven bond. Both vanish for $\epsilon = 0$ which corresponds to equilibrium state where $c(r,s) = 0$ is the solution in the thermodynamic limit.

Before analyzing the charge distribution in detail it is instructive to examine its symmetry. The model is invariant under two symmetry operations. The first is a combination of reversal of the drive ($\epsilon$) and space inversion,

$$c_{-\epsilon,\rho}(-r - e_1, -s - e_1) = c_{\epsilon,\rho}(r,s).$$

(10)

The second is a combination of space inversion and particle-hole exchange

$$c_{\epsilon,1-\rho}(-r - e_1, -s - e_1) = c_{\epsilon,\rho}(r,s).$$

(11)

The symmetry of the model enables one to extract the large distance profile of the correlation function for $\rho = 1/2$ without actually solving the Poisson equation. At this density the second symmetry (11) implies that $c(r,s)$ is even under space inversion. Then the large distance profile is determined by the lowest non-vanishing “even” multi-pole moment which could be monopole, quadrupole etc. Since, as shown below, the net charge vanishes at all densities, and the quadrupolar component is shown to be non-vanishing and effectively localized near the origin, the correlation function for $\rho = 1/2$ decays at large distance as the electrostatic potential of a quadrupole moment in $2d$ dimension, namely, $c(r,s) \sim (r^2 + s^2)^{-d}$. 


The fact that the monopole moment of the charge distribution vanishes in the case of periodic boundary conditions is evident from integrating the Poisson equation (5). It can be shown, by using the dipolar form of the density $\phi(r)$, that the same is true for an infinite lattice with boundary condition of vanishing correlation at infinity.

In order to analyze the correlation function at densities $\rho \neq 1/2$ we examine in more detail the expressions for the charge density. While the charges $\sigma_1$ and $\sigma_2$ are functions of the density $\phi$ and the correlation function $c(r,s)$ itself, the $\sigma_3$ charge is a function of higher order correlations. To see this let us examine the expression for $\langle Q\delta(r) \rangle$ in $\sigma_3$ which using (3) yields

$$\langle Q\delta(r) \rangle = \left(1 - 2\phi(0)\right)c(r, -e_1) + (1 - 2\phi(-e_1))c(r, 0) - 2c(r, 0, -e_1),$$

for $r \neq 0, -e_1$. Here $c(r, 0, -e_1)$ is the three point correlation function which itself depends on higher order correlation functions. This hierachical dependence of correlation makes the solution difficult. However, the hierarchy can be handled by a perturbative expansion in $\epsilon$, and in principle one could solve the problem order by order in $\epsilon$. This becomes possible due to the pre-factor $\epsilon$ in the expression of $\sigma_3(r,s)$ in (6).

Consider an expansion around the zero drive state as

$$c(r,s) = c_0(r,s) + \epsilon c_1(r,s) + \epsilon^2 c_2(r,s) + \epsilon^3 c_3(r,s) + \ldots$$

(13)

The first term, $c_0(r,s)$ corresponds to the correlation in equilibrium which is zero in the thermodynamic limit. Also, $\langle Q\delta(r) \rangle$ in (12) vanishes in equilibrium. Using these observations in equations (6)-(9) it is evident that $c_1(r,s) = 0$ in the thermodynamic limit. Then, the leading non-vanishing term in (13) is of order $\epsilon^2$.

A similar argument can be used to show that the three point correlation $c(r,s,t)$ is of order $\epsilon^3$ or higher. An immediate consequence of the latter is that, in the thermodynamic limit, the terms up to order $\epsilon^3$ in (13) do not depend on the higher order correlations, and can in principle be determined using equations (6)-(9).

We proceed by noting that the first symmetry relation in (10) when applied to (13) implies that all the even terms in $\epsilon$ are invariant under space inversion, whereas the odd terms change sign. Thus,

$$c_{2k}(r - e_1, s - e_1) = c_{2k}(r,s)$$

(14)

$$c_{2k+1}(r - e_1, s - e_1) = -c_{2k+1}(r,s),$$

(15)

with integer $k \geq 0$. Thus, all the even terms in $\epsilon$ are generated by charges of “even” multi-pole moments like monopole, quadrupole etc. As argued earlier, the monopole vanishes at all densities. Thus, to leading order in $\epsilon$ the large distance correlation function is expected to be determined by the quadrupole component of the charge at any density.

In order to demonstrate that the correlation function indeed decays at large distances as $1/(r^2 + s^2)^{d}$ we analyze the charge in more detail arguing that its quadrupolar component does not vanish and that this component is effectively localized near the origin. A schematic representation of the charge distribution in the $(r_1e_1, s_1e_1)$ plane is shown in Fig. 2. The charges are distributed on $d$-dimensional planes whose intersection with the $(r_1e_1, s_1e_1)$ plane are straight lines.

The simplest charge to analyze is $\sigma_1$ in (7) which using $c(r,r) = 0$ yields

$$\sigma_1(r,s) = \sum_{\nu} \left[ 2c(r + e_\nu, r) + 2c(r - e_\nu, r) \right] \delta_{s,r} \left[ -2c(r,s) \left( \delta_{s,r} + \delta_{s.r-e_\nu} \right) \right],$$

(16)

From this expression it is clear that the charge is a collection of quadrupoles along $r = s$ plane, each made of four charges $2c(r + e_\nu, r)$ of alternating signs placed on the corners of the square plaquettes shown in Fig. 2. The filled and empty circles denote charges of opposite sign.

The $\sigma_2$ charge, shown in Fig 2(b), is made of positive charges (○) on $s = r \pm e_\nu$ planes and a localized negative charge (●) at $(-e_1,0)$ and $(0,-e_1)$. Using the density profile $\phi(r)$ it is straightforward to show that the quadrupolar moment is non-zero.

The $\sigma_3$ charge is shown in Fig 2(c) where (□) denotes a charge $\langle Q\delta(r) \rangle$, while (■) denotes a charge $-\langle Q\delta(r) \rangle$. At $\rho = 1/2$ one has $\langle Q\delta(-r - e_1) \rangle = \langle Q\delta(r) \rangle$. This can be shown using (12) and noting that the three point correlation is odd under space inversion for this density. Then clearly the quadrupole moment is non-zero. At other densities, using continuity with variation of $\rho$, the quadrupole moment remains non-zero.

In order for the large distance behavior of the correlation function to be that of a quadrupole, one has to demonstrate that the quadrupole moment of the charges $\sigma_1, \sigma_2$ and $\sigma_3$ is effectively localized around the origin. By this we mean that the quadrupolar moment density decays sufficiently fast from the origin such that the large distance decay of the potential behaves as that of a localized quadrupole. This can be argued using the following observation: consider a potential in $2d$-dimension due to distributed quadrupoles on a $d$-dimensional plane with moment density $\gamma(\mathbf{R})$ at $\mathbf{R}$. For $\gamma(\mathbf{R})$ which de-
cays at large distances faster than $R^{-d}$, the generated potential can be shown to have the same power-law tail as the potential due to a localized quadrupole at origin (see Supplemental Material [11]). Indeed one can show that the quadrupole density of each of the three charge densities decays at large $R$ faster than $R^{-d}$. The charge density $\sigma_1(R)$ is proportional to the reduced correlation $c(R)$. Therefore a quadrupolar profile of $c(R)$ implies that $\sigma_1(R)$ decays as $1/R^{d}$ which is faster than required for effective localization, indicating self-consistency of the solution. Similar argument also applies for $\sigma_3(r, s)$ whose strength $(Q\delta(r))$ decays as $c(r, s)$. The charge $\sigma_2(R)$ is composed of two parts: a distributed positive charge $|\nabla_0(R)|^2$ which decays as $R^{-d}$, and a localized negative charge at the origin. These two charges effectively generate localized quadrupole moment. We thus conclude that the three charges generate correlation function $c(r, s)$ which at large distance decays as the potential generated by a localized quadrupole in $2d$ dimensions.

We now consider the effect of the odd terms in the expansion [13]. At density $\rho = 1/2$ all odd terms in the expansion vanish due to inversion symmetry and the decay of correlation is expected to be that of quadrupole charge for any $\epsilon$. However, for densities $\rho \neq 1/2$ the odd terms do not vanish and they can support odd multi-pole moments such as dipole, hexapole etc. We have shown above that in the thermodynamic limit $c_1(r, s)$ vanishes. Thus to leading order in $\epsilon$ (namely $\mathcal{O}(\epsilon^2)$) the density-density correlation function decays as the potential of a localized quadrupole. Our numerical studies in $d = 2$ dimensions, presented below, show that the decay of correlation indeed corresponds to that of a quadrupolar charge in $d = 4$ dimensions even at finite $\epsilon$. This suggests that the dipole contribution which can come from higher order terms, such as $c_3(r, s)$, vanish in thermodynamic limit.

In order to test our result we calculated the correlation function in $d = 2$ dimensions numerically, and show that at large distance it indeed decays as an electrostatic potential generated by a quadrupole at origin in $d = 4$ dimensions. The difficulty in determining $c(r, s)$ has to do with the charge $\sigma_3$ in [9] which depends on the three point correlation. Ignoring the three point correlation, which is justified for small $\epsilon$, we determine $c(r, s)$ by numerically solving the Poisson equation [5].

The solution is obtained by iterating the equation

$$\partial_t c(r, s) = \Delta c(r, s) - \sigma(r, s), \quad (17)$$

on a four dimensional integer lattice $[-40 : 40]^4$. The long time profile $c(r, s)$ is the solution of the [5] and corresponds to the correlation for a two-dimensional system. For a fast convergence we have conditioned $c(r, s) = 0$ at the boundary. As the charges are themselves a function of $c(r, s)$, at every time step they are updated with the existing profile. The profile after $10^6$ iterations varies by $\mathcal{O}(10^{-10})$ and considered as the asymptotic solution.

FIG. 3. Contour plot on $(r_1e_1, s_1e_1)$ plane of the numerical solution of the Poisson equation [5] on a 4-dimensional lattice. (color online)

FIG. 4. Log-Log plot of the $c(r, s)$ along three different directions. The dashed lines denote a power-law fit by $1/x^4$. The (▲) data is shifted by one unit in log scale for better presentation.

The resulting profile on the $(r_1, 0, s_1, 0)$ plane for $\epsilon = 1/2$ and $\rho = 1/4$ is shown in Fig.3. The different colors denote the value of $c(r, s)$ as indicated in the color bar. The profile has a slightly broken quadrupolar symmetry at large distances. The small asymmetry is likely due to finite system size and higher odd multi-pole moments. In the thermodynamic limit the asymmetry is expected to vanish at large distances.

The power-law tail of the $c(r, s)$ along three different directions away from the origin is shown in Fig.4. The result is consistent with a quadrupolar decay $1/(r^2 + s^2)^2$. The exponential decay near the tail is due to finite-size of the lattice.

To conclude, we have shown that in the presence of a single driven bond in a bulk SSEP in $d$-dimension, the density-density correlation is long-ranged. Using symmetry considerations we showed that, for $\rho = 1/2$, the correlation at large distances is expressed in terms of the potential due to a localized quadrupole at origin. At other densities, we use a perturbative analysis to argue
that the same is true in the small drive ($\epsilon$) limit. In particular, we showed that, in the thermodynamic limit, the leading non-vanishing term of $c(r,s)$ is of order $\epsilon^2$ and is generated by a localized quadrupole. For general $\epsilon$, the evidence for the quadrupolar profile of $c(r,s)$ comes from numerical solution of the stationary equation [5] for $d = 2$ which shows that $c(r,s) \sim (r^2 + s^2)^{-d}$ for large $r$ and $s$. It would be interesting to see how this quadrupolar decay in correlations may change in presence of more than one driving bonds.

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[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. 59, 381 (1987)
[2] J. R. Dorfman, T. R. Kirkpatrick, and J. V. Sengers, Ann. Rev. Phys. Chem. 45, 213 (1994)
[3] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim (2007), arXiv:0705.2996
[4] H. Spohn, J. Phys. A 16, 4275 (1983)
[5] P. L. Garrido, J. L. Lebowitz, C. Maes, and H. Spohn, Phys. Rev. A 42, 1954 (1990)
[6] G. Grinstein, D.-H. Lee, and S. Sachdev, Phys. Rev. Lett. 64, 1927 (1990)
[7] B. Derrida, J. Stat. Mech. 2007, P07023 (2007)
[8] B. Derrida, J. Lebowitz, and E. Speer, J. Stat. Phys. 107, 599 (2002)
[9] T. Sadhu, S. N. Majumdar, and D. Mukamel, Phys. Rev. E 84, 051136 (2011)
[10] T. Bodineau, B. Derrida, and J. Lebowitz, J. Stat. Phys. 140, 648 (2010)
[11] See Supplemental Material for an outline of the derivation of the stationary equation for correlation, and an argument for the effective localization of the quadrupolar moments.