WEYL COMPATIBLE TENSORS

CARLO ALBERTO MANTICA AND LUCA GUIDO MOLINARI

Abstract. We introduce the new algebraic property of Weyl compatibility for symmetric tensors and vectors. It is strictly related to Riemann compatibility, which generalizes the Codazzi condition while preserving much of its geometric implications. In particular it is shown that the existence of a Weyl compatible vector implies the Weyl tensor to be algebraically special, and it is a necessary and sufficient condition for the magnetic part to vanish. Some theorems (Derdziński and Shen, Hall) are extended to the broader hypothesis of Weyl or Riemann compatibility; Weyl compatibility includes conditions that were investigated in the literature of general relativity (as McIntosh et al.) Hypersurfaces of pseudo Euclidean spaces provide a simple example of Weyl compatible Ricci tensor.

1. Introduction

The geometry of Riemannian or pseudo-Riemannian manifolds of dimension \( n \geq 3 \) is intrinsically described by \( \mathcal{N} = \frac{1}{12} n(n-1)(n-2)(n+3) \) algebraically independent scalar fields, constructed with the Riemann and the metric tensors. The same counting is provided by the Weyl, the Ricci and the metric tensors. The Weyl tensor bears the symmetries of the Riemann tensor, with the extra property of being traceless:

\[
C_{jklm} = R_{jklm} + \frac{1}{n-2} \left( \delta_{[j} R_{k]lm} + R_{[j}^m g_{k]l} \right) - \frac{1}{(n-1)(n-2)} R \delta_{[j}^m g_{k]l}.
\]

The trace condition \( C_{j}^{j} = 0 \) reduces the parameters of the Riemann tensor by a number \( \frac{1}{2} n(n+1) \) that is accounted for by considering the Ricci tensor as algebraically independent. The two tensors are linked by functional relations as the following one:

\[
-\nabla_{m} C_{abc}^{m} = \frac{n-3}{n-2} \left[ \nabla_{[a} R_{b]c} - \frac{1}{2(n-1)} \nabla_{[a} g_{b]c} R \right].
\]

In the coordinate frame that locally diagonalizes the Ricci and the metric tensors (the latter with diagonal elements \( \pm 1 \)), the parameters that survive are the components of the Weyl tensor and the \( n \) eigenvalues of the Ricci tensor, whose number is precisely \( \mathcal{N} \). This choice of fundamental tensors offers advantages, as in the classification of manifolds and in general relativity.

Date: 18 Jan 2013.

2010 Mathematics Subject Classification. 53B20, 53C50 (Primary), 83C20 (Secondary).

Key words and phrases. Weyl tensor, Riemann compatibility, Petrov types.

1 Conventions: \( X_{[ab]} := X_{ab} - X_{ba}, R_{ab} = R_{amb}^m, u^2 = u^a u_a \).
Debever and Penrose [26] proved that in four-dimensional space-time manifolds the equation
\[ k\{b,C\}_a|_{rs}(k^r)k^s = 0 \]
always admits four null solutions (principal null directions). When two or more coincide, the Weyl tensor is named \textit{algebraically special}, and the condition for degeneracy is
\[ k\{b,C\}_a|_{rsq}k^r k^s = 0 \]
The degeneracies classify space-time manifolds in classes that coincide with the Petrov types, which are determined by the degeneracies of the eigenvalues of the self-dual part of the Weyl tensor [27].

For \( n > 4 \), Milson et al. showed in 2005 that eq. (3) may not have solution at all [24]. They introduced the notion of \textit{Weyl aligned null directions} (WAND): a null vector \( k \) is a WAND if there is a null frame including it, such that the Weyl scalars of maximal boost weight vanish. This is true if \( C_{b0|0} = 0 \). For any \( n \) the condition is equivalent to \( k \) being a solution of eq. (3) [24] Prop. IV.5). The order of alignment provides the backbone of a classification of Lorentzian manifolds [24 Table III, and [25]).

The Einstein equations of general relativity link the energy momentum tensor \( T_{ij} \) to the Ricci tensor and the curvature scalar, but not to the Weyl tensor:
\[ R_{ij} - \frac{R}{2} g_{ij} = 8\pi T_{ij}. \]
In \( n = 4 \) the Weyl tensor may be replaced by two symmetric tensors, the electric and magnetic components, and the identity [2] for the Weyl tensor translates into Maxwell-like equations for the components [30, 2]. The construction was extended to \( n > 4 \) [15].

In the study of Derdziński and Shen’s theorem [9, 3] on the restrictions imposed by a Codazzi tensor on the structure of the Riemann tensor, we introduced the new algebraic notion of \textit{Riemann compatible tensors} [18]. This enabled us to extend the theorem in two directions: the replacement of the Codazzi condition with the milder hypothesis of Riemann compatibility together with a drastic simplification of the proof, the restatement of the theorem for curvature tensors other than Riemann’s. Riemann compatible tensors were investigated in [19]. Most of the statements valid for (pseudo) Riemannian manifolds equipped with a nontrivial Codazzi tensor, such as the vanishing of Pontryagin forms, were shown to persist in presence of a non-trivial Riemann compatible tensor. The application to geodesic mappings was then discussed.

This paper is mainly about Weyl compatibility of symmetric tensors, a property which is broader than Riemann compatibility. The restriction on the structure of the Weyl tensor has consequences on the Petrov type of the manifold and the electric and magnetic components of the Weyl tensor. Since classifications of Lorentzian manifolds are mainly based on vectors, our discussion of Weyl compatibility of vectors crosses in several points definitions and properties proven by other authors, but having a different origin.

Definitions and main properties of Riemann and Weyl compatible symmetric tensors are reviewed in Sect. 2, with the subcase of Riemann and Weyl-\textit{permutable}
WEYL COMPATIBLE TENSORS 3

tensors. Tensors $u_iu_j$ naturally define Riemann and Weyl compatibility for vectors, which is discussed in Sect. 3 with various new results, as the extension of Derdziński and Shen’s theorem [9] and of Hall’s theorem [12], and a sufficient condition for the vanishing of Pontryagin forms. Riemann (Weyl) permutable vectors are considered in Sect. 4, where results by McIntosh and others [22, 13, 21] for the special case $R_{ijkl}u^l = 0$ are reobtained and extended.

In Sect. 5 it is shown that the existence of a Weyl compatible vector is a sufficient condition for the Weyl tensor to be special, with results regarding the Penrose-Debever classification of spacetimes. The electric and magnetic components of the Weyl tensor are considered in Sect. 6, with the statements that existence of a Weyl compatible vector is necessary and sufficient for the Weyl tensor to be purely electric, and that Weyl permutability implies a conformally flat space-time.

Sect. 7 is devoted to hypersurfaces; the Gauss and Codazzi equations specify the induced Riemann tensor of the hypersurface as a quadratic expression of a Codazzi tensor. It is shown that the corresponding Ricci tensor is Riemann and Weyl compatible.

Conformal maps obviously preserve Weyl compatibility, geodesic maps preserve Riemann compatibility [19] but not necessarily Weyl compatibility; a sufficient condition is presented in Sect. 8.

The manifolds considered here are Hausdorff connected with non degenerate metric of arbitrary signature, i.e. $n$-dimensional pseudo-Riemannian manifolds. Where necessary, we specialize to the metric signature $n - 2$, i.e. to $n$-dimensional Lorentzian manifolds (space-times). We always assume a Levi-Civita connection ($\nabla_i g_{jk} = 0$).

2. RIEMANN AND WEYL COMPATIBLE TENSORS

We briefly review the concept of compatibility for symmetric tensors, first introduced in [18], and investigated in [19]. Permutable tensors are then defined, as a special class.

Definition 2.1. A symmetric tensor $b_{ij}$ is Riemann compatible if:

$$b_{am}R_{bcl}^m + b_{bm}R_{cal}^m + b_{cm}R_{abl}^m = 0. \tag{6}$$

The metric tensor is trivially Riemann compatible because of the first Bianchi identity.

Remark 2.2. The definition has a natural origin. Consider the vector-valued 1-form $B_i = b_{kl}dx^k$, where $b$ is a symmetric tensor. A covariant exterior derivative gives

$$DB_i = \frac{1}{2} C_{jkl} dx^j \wedge dx^k,$$

where $C_{ijkl} = = \nabla_i b_{jk} - \nabla_j b_{ik}$ is the “Codazzi deviation tensor”, defined in [19]. As it is well known, $DB_i = 0$ if and only if $b_{kl}$ is a Codazzi tensor [4, 20]. If $DB_i \neq 0$, another derivative gives

$$D^2 B_i = \frac{1}{3!} (\nabla_i C_{jkl} + \nabla_j C_{kil} + \nabla_k C_{ijl}) dx^j \wedge dx^k \wedge dx^k.$$

The following identity links the Codazzi deviation to Riemann compatibility [19]:

$$\nabla_i C_{jkl} + \nabla_j C_{kil} + \nabla_k C_{ijl} = b_{im}R_{jkl}^m + b_{jm}R_{kil}^m + b_{km}R_{ijl}^m \tag{7}.$$
Therefore, \( D^2 B_i = 0 \) (i.e. \( DB_i \) is closed) if and only if \( b \) is Riemann compatible. It follows that Codazzi tensors, \( \nabla_i b_{jk} = \nabla_j b_{ik} \), are Riemann compatible.

As an example, consider the Ricci tensor. Its Codazzi deviation is \( \mathcal{C}_{abc} := \nabla_a R_{bcd} = -\nabla_b R_{acd} + \nabla_c R_{abd} \), by the contracted Bianchi identity. The identity (7) turns out to be Lovelock’s identity [16]:

\[
-(\nabla_a \nabla_m R_{bcd} + \nabla_b \nabla_m R_{cad} + \nabla_c \nabla_m R_{abd}) = R_{am} R_{bcd} + R_{bm} R_{cad} + R_{cm} R_{abd}.
\]

Compatibility was extended to generalized curvature tensors \( K_{abc} \), i.e. tensors having the symmetries of the Riemann tensor in the exchange of indices, and the first Bianchi property. The Weyl tensor (1) is the most notable example. A symmetric tensor \( b_{ij} \) is Weyl compatible if:

\[
b_{am} C_{jkl}^{m} + b_{bm} C_{kl}^{m} + b_{cm} C_{ijl}^{m} = 0.
\]

In [17] (Prop. 2.4) we showed an invariance property of Lovelock’s identity (8). In particular, it remains valid if the Riemann tensor in the left hand side is replaced by the Weyl tensor. If Einstein’s equations are then used, one gets a differential condition for the Weyl tensor involving the stress-energy tensor \( T_{ij} \):

\[
\nabla_i \nabla_{m} C_{jkl}^{m} + \nabla_j \nabla_{m} C_{klm} + \nabla_k \nabla_{m} C_{ijl}^{m} = -8\pi n - \frac{3}{n-2} (T_{im} C_{jkl}^{m} + T_{jm} C_{kilm} + T_{km} C_{ijl}^{m})
\]

Since the left hand side is the exterior covariant differential of the vector valued 1-form \( \Pi_l = \nabla_m C_{jkl}^{m} dx^j \wedge dx^k \), the following theorem holds:

**Theorem 2.3.** On a \( n \)-dimensional pseudo-Riemannian manifold, the Ricci tensor and the energy-stress tensor are Weyl compatible if and only if \( D\Pi_l = 0 \).

The condition \( D\Pi_l = 0 \) is satisfied in \( n \)-dimensional Lorentzian manifolds that are conformally symmetric (\( \nabla_i C_{jkl}^{m} = 0 \)) or conformally recurrent (\( \nabla_i C_{jkl}^{m} = \alpha_i C_{jkl}^{m} \)). On these manifolds the Ricci and the stress energy tensors are Weyl compatible [17, 20].

A Weyl compatible tensor poses strong restrictions on the Weyl tensor. In [18] we proved a broad generalization of Derdzinski and Shen’s theorem that holds both in Riemannian and pseudo-Riemannian manifolds. For the Weyl tensor it reads:

**Proposition 2.4.** On a pseudo-Riemannian manifold with a Weyl compatible tensor \( b \), if \( X, Y \) and \( Z \) are eigenvectors of \( b \) with eigenvalues \( \lambda, \mu, \nu \) \( (b_{ij}X^j = \lambda X^i, \text{ etc.}) \) then:

\[
C_{abcd} X^a Y^b Z^c = 0, \quad \nu \neq \lambda, \mu.
\]

The following algebraic identity relates a symmetric tensor \( b_{ij} \) to the Weyl, the Riemann and the Ricci tensors [19]:

\[
b_{im} C_{jkl}^{m} + b_{jm} C_{kil}^{m} + b_{km} C_{ijl}^{m} = b_{im} R_{jkl}^{m} + b_{jm} R_{kil}^{m} + b_{km} R_{ijl}^{m} + \frac{1}{n-2} [g_{kl}(b_{im} R_{j}^{m} - b_{jm} R_{i}^{m}) + g_{ij}(b_{jm} R_{k}^{m} - b_{km} R_{j}^{m}) + g_{jl}(b_{km} R_{i}^{m} - b_{im} R_{k}^{m})].
\]

Any contraction with the metric tensor gives zero; the identity is trivial if \( b \) is the metric tensor. An immediate consequence is:
Theorem 2.5. A symmetric tensor is Riemann compatible if and only if it is Weyl compatible and it commutes with the Ricci tensor.

Proof. If \( b \) is Riemann compatible, contraction of (6) with \( g^{cl} \) gives
\[
R_{am} R_{bm} - b_{bm} R_{am} = 0,
\]
i.e. \( b \) commutes with the Ricci tensor. Then \( b \) is Weyl compatible by identity (11). The converse is obvious, by the same identity. □

In particular, Riemann and Weyl compatibility are equivalent for the Ricci tensor, or any symmetric tensor that commutes with it.

An example of Riemann tensor with a Riemann compatible symmetric tensor can be constructed by the Kulkarni-Nomizu product of two symmetric tensors [3, 4, 9]:

Proposition 2.6. Suppose that the Riemann tensor has the form:
\[
R_{jklm} = b_l \left[ j a_k \right] m + b_m \left[ k a_j \right] l
\]
with symmetric tensor fields \( a_{ij} \) and \( b_{ij} \). If they commute, \( a_i^m b_{mj} = a_j^m b_{mi} \), then they are both Riemann compatible.

Proof. Evaluate:
\[
b^i_m R_{jklm} = b_{lj} \left( b a_i \right)_{ik} - b_{lk} \left( b a_i \right)_{ij} + b^2_{ij} a_{jl} - b^2_{ij} a_{kl}.
\]
The sum on cyclic permutations of \( ijk \) cancels the r.h.s. i.e. \( b \) is Riemann compatible. Because of the symmetry of (12) in the exchange of \( a \) and \( b \), also \( a \) is Riemann compatible. □

The commuting tensors \( a, b \) are also Weyl compatible for the Weyl tensor computed from (12). The same Kulkarni-Nomizu product can be used to construct another Weyl tensor:

Proposition 2.7. Let \( a \) and \( b \) be commuting symmetric tensor fields, such that
\[
b^m_m a_{kl} + a^m_m b_{kl} - 2b_{km} a^m_l = 0,
\]
then \( C_{jklm} = b_{lj} \left[ j a_k \right] m + b_m \left[ k a_j \right] l \) has the symmetries of the Weyl tensor, and \( a \) and \( b \) are Weyl compatible.

The additional equation (13) is required to enforce tracelessness, and can be solved to obtain the “potential” \( a \) that produces \( b \).

Suppose that a symmetric tensor has the property \( b_{im} R_{jklm} = \omega b_{im} R_{jklm} \), where \( \omega \) is a scalar. Then either \( \omega = \pm 1 \) or \( b_{im} R_{jklm} = 0 \). The three cases define interesting classes of tensors that will be shown to be Riemann compatible. The same conditions are found with the Weyl tensor.

The class \( \omega = -1 \) was studied by McIntosh and others [21, 13] and is presented in Sect. 4. The class \( b_{im} R_{jklm} = 0 \) was studied in [22] for \( b_{ij} = u_i u_j \), i.e. \( R_{jklm} u_m = 0 \). Now we consider the class \( \omega = 1 \):

Definition 2.8. A symmetric tensor \( b_{ij} \) is Riemann permutable if:
\[
b_{im} R_{jklm} = b_{lm} R_{jklm}
\]
It is Weyl permutable if:
\[
b_{im} C_{jklm} = b_{lm} C_{jklm}
\]

Proposition 2.9. If a symmetric tensor is Riemann (Weyl) permutable then it is Riemann (Weyl) compatible.
Proof. In the relation for Riemann compatibility use (14) for each term: 
\[ b_{im}R_{jklm} + b_{jm}R_{iklm} + b_{km}R_{ijlm} = b_{lm}(R_{jki} + R_{kij} + R_{ijk}) = 0 \]
by the first Bianchi identity. An analogous proof holds for Weyl permutable tensors. □

Note that Riemann permutability does not imply Weyl permutability. A Riemann permutable tensor (being Riemann compatible) commutes with the Ricci tensor.

Derdziński and Shen’s theorem for the Riemann tensor and theorem 2.4 for the Weyl tensor, become more stringent for permutable tensors:

**Proposition 2.10.** If \( b \) is a symmetric tensor and \( X, Y \) are two eigenvectors, \( b_{ij}X_j = \lambda X_i \) and \( b_{ij}Y_j = \mu Y_i \) with \( \lambda + \mu \neq 0 \), then

1) if \( b \) is Riemann permutable it is: 
\[ R_{jklm}X_lY_m = 0 \]
2) if \( b \) is Weyl permutable it is: 
\[ C_{jklm}X_lY_m = 0 \]

**Proof.** Contraction of (14) with \( X_iY_l \) gives 
\[ \lambda R_{jklm}X_mY_l = \mu R_{jki}X_iY_m , \]
then 
\[ 0 = (\lambda + \mu)R_{jklm}X_lY_m \]. The proof for the Weyl tensor is analogous. □

3. Riemann and Weyl compatible vectors

The notion of \( K \)-compatible symmetric tensor includes vectors \( u_i \) in a natural way, through the symmetric tensor \( u_iu_j \):

**Definition 3.1.** A vector field \( u_i \) is \( K \)-compatible (where \( K \) is the Riemann, the Weyl or a generalized tensor) if:

\[ (u_iK_{jkl} + u_jK_{kli} + u_kK_{ijl})u_m = 0 \]

On a Lorentzian manifold, let \( K \) be the Riemann (Weyl) tensor. If \( u \) is a timelike vector, the definition corresponds to the statement that \( u_iu_j \) is a purely electric Riemann (Weyl) tensor ([15] Prop. 3.5 and Prop. 4.3). If \( u \) is a null vector, it is a double WAND and the manifold is type II(d) ([25], Table 1).

**Theorem 3.2.** A vector field \( u \) with \( u^2 \neq 0 \) is \( K \)-compatible if and only if there is a symmetric tensor \( D_{ij} \) such that:

\[ K_{abcm}u^m = D_{ac}u_b - D_{bc}u_a \]

**Proof.** Multiplication by \( u_d \) and cyclic summation on \( abd \) makes the r.h.s. vanish and \( K \)-compatibility is obtained.

If \( u \) is \( K \)-compatible then multiplication of (16) by \( u^i \) gives

\[ (u^2K_{jkl} + u_ju^iK_{kli} + u_ku^iK_{ijl})u_m = 0 \]

where we read \( D_{jl} = K_{ijlm}u^i u^m / u^2 \).

It can be easily shown that \( u \) is an eigenvector of the symmetric tensor \( D \). For the Weyl tensor, \( D \) will be identified with its electric component (see Sect. 5). On a Lorentzian manifold, the theorem with \( K = C \) (Weyl tensor) follows from eq. 20 in [15].

**Remark 3.3.** Suppose that on a pseudo-Riemannian manifold there is a concircular vector, \( \nabla_k u_l = A g_{kl} + B u_k u_l \), with constant \( A \) and \( B \). The condition implies \( R_{jkl}u_m = AB(u_j g_{kl} - u_k g_{jl}) \), which has the form (17). Therefore, a concircular vector is both Riemann and Weyl compatible.
The general statements valid for compatible tensors [19] become stronger for compatible vectors, and new facts arise. For example, the generalized Derdziński and Shen’s theorem has now a surprisingly simple proof, with no need of auxiliary $K$–tensors:

**Theorem 3.4.** Let $K$ be a generalized curvature tensor, and $u$ be a $K$–compatible vector.

1) If $u^2 \neq 0$ and $v$, $w$ are vectors orthogonal to $u$ ($u_a v^a = 0$, $u_a w^a = 0$), then:

$$K_{abcd} v^a w^b v^c = 0.$$  

(18)

2) If $u^2 = 0$ and $v$ is orthogonal to $u$ ($u_a v^a = 0$) then:

$$K_{abcd} u^a v^b v^c = 0.$$  

(19)

**Proof.** 1) The $K$–compatibility condition, $(u_a K_{bcde} + u_b K_{caed} + u_c K_{abde}) u^e = 0$, is contracted with $u^a v^b w^c$:

$$(u^a u_b) v^b w^c K_{bcde} u^e + u^a (u_b v^b) w^c K_{caed} u^e + u^a v^b (w^c u_c) K_{abde} u^e = 0.$$  

The last two terms cancel because of orthogonality. 2) The $K$–compatibility condition is contracted with $u^a v^b$ and equation $u_a K_{abde} u^a v^b u^c = 0$ is obtained. The result follows if $u$ is non zero.

**Remark 3.5.** For the Riemann tensor, $R_{abcd} v^a w^b u^c$ is the vector obtained through parallel transport of $u$ along a parallelogram with infinitesimal vectors $v$ and $w$. It is known that, if $R_{abcd} v^a w^b u^c = 0$ for any $v$ and $w$, then it is $R_{abcd} u^c = 0$. If $u$ is Riemann compatible, then it has zero variations along infinitesimal parallelograms with directions orthogonal to it.

**Theorem 3.6.** Let $X(1), . . . , X(n)$ be an orthonormal basis of a $n$–dimensional pseudo-Riemannian manifold, $X(a)_k X(b)^k = \pm \delta_{ab}$. If $X(3) . . . X(n)$ are Riemann compatible, then all Pontryagin forms vanish.

**Proof.** Among three vectors, one is certainly Riemann compatible; therefore it is always $R_{ij}^{kl} X(a)^i \wedge X(b)^j X(c)^k = 0$, by Theorem [31]. This means that the column vectors of the matrix $R_{ij}^{kl} X(a)^i \wedge X(b)^j$ are orthogonal to all vectors $X(c)$ with $c \neq a, b$, i.e. they belong to the subspace spanned by $X(a)$ and $X(b)$. Because of the antisymmetry in $k, l$, it is necessarily

$$R_{ij}^{kl} X(a)^i \wedge X(b)^j = \lambda_{ab} X(a)^k \wedge X(b)^l.$$  

This condition of pureness of the Riemann tensor implies the vanishing of all Pontryagin forms [19] Theor. 5.2.

The identity [11] relating Riemann and Weyl compatibility, is rewritten for vectors:

$$(u_a C_{bclm} + u_b C_{caim} + u_c C_{ablm}) u^m = (u_a R_{bclm} + u_b R_{caim} + u_c R_{ablm}) u^m + \frac{1}{n-2} [g_{cl} u[a R_{b|m} + g_{at} u[b R_{c|m} + g_{bt} u[c R_{a|m}]] u^m].$$  

(20)

A first consequence is the restatement of theorem [25] for vectors:

**Proposition 3.7.** A vector field $u$ is Riemann compatible if and only if it is Weyl compatible and $u[a R_{b|m} u^m = 0$.  

On a Lorentzian manifold the proposition is equivalent to Prop. 4.2 (time-like vectors) and Prop. A13 (null vectors) in Ref. [15].

A second consequence is the extension of a theorem by Hall, which he proved for null vectors in $n = 4$ space-times [12]. It is valid in any dimension and metric signature, and for vectors not necessarily null:

**Theorem 3.8.** Consider the following conditions on a vector field $u$:

- **A)** $u_{[a}R_{b]cimd}u^c u^m + u^2 R_{ablm}u^m = 0$,
- **B)** $u_{[a}C_{b]cimd}u^c u^m + u^2 C_{ablm}u^m = 0$,
- **C)** $u_{[a}R_{b]m}u^m = 0$.

Two conditions imply the third one. In particular, if $u^2 \neq 0$ the stronger statement holds: $A$ is true if and only if $B$ and $C$ are true.

**Proof.** Eq. (20) is contracted with $u^c$,  

$$u_{[a}C_{b]cimd}u^c u^m + u^2 C_{ablm}u^m = u_{[a}R_{b]cimd}u^c u^m + u^2 R_{ablm}u^m + \frac{1}{2} \left[ u u_{[a}R_{b]m}u^m + (g_{al}u_b - g_{bl}u_a)u^c u^m R_{cm} - u^2 (g_{al}R_{bm} - g_{bl}R_{am})u^m \right].$$

If condition $C$ is true, its contraction with $u^b$ gives $(u_a u^b R_{bml} - R_{bml})u^m = 0$ and (21) becomes $u_{[a}C_{b]cimd}u^c u^m + u^2 C_{ablm}k^m = u_{[a}R_{b]cimd}u^c u^m + u^2 R_{ablm}u^m$. Therefore $B$ and $C$ imply $A$, or $A$ and $C$ imply $B$.

Suppose now that $A$ is true; contraction of condition $A$ by $g^{al}$ gives $u^2 R_{bml}u^m - u_b (u^c u^m R_{cm}) = 0$, and (21) becomes:

$$u_{[a}C_{b]cimd}u^c u^m + u^2 C_{ablm}u^m = \frac{1}{n - 2} u u_{[a}R_{b]m}u^m$$

Validity of $A$ and $B$ imply that $uu_{[a}R_{b]m}u^m = 0$ i.e. $C$ is true. A stronger result holds if $u^2 \neq 0$. Contraction of (22) by $u^l$ makes the left-hand-side vanish and condition $C$ is true. Then, the same equation (22) states that also $B$ is true, i.e. $A$ implies $B$ and $C$. $\square$

**Remark 3.9.** Condition $C$ is met in Einstein spaces, defined by $R_{ab} - \frac{1}{n} R g_{ab} = 0$.

**Remark 3.10.** Condition $B$ plays a special role in the classification of manifolds. Some cases where it holds are: 1) $u^m R_{abcd} = 0$ (22) [30]: 2) $k$ is a recurrent null vector, $\nabla_a k_b = \lambda_a k_b$, with $\nabla_{[a} \lambda_{b]} = 0$ (31) page 69: 3) Manifolds with constant curvature (31) page 101:

$$R_{acdl} = \frac{R}{n(n - 1)}(g_{a}g_{cm} - g_{cl}g_{bm})$$

In cases 1,2 the vector is Riemann compatible.

**Proof.** 1) The relation implies $R_{am}u^m = 0$. Then the whole r.h.s. of (20) is zero and $(u_a C_{b]cimd} + u_b C_{aclm} + u_c C_{ablm})u^m = 0$. Multiply by $u^c$ and obtain $A$.

2) $[\nabla_a, \nabla_b] u_c = R_{abc} u_m$; because of recurrency and closedness, the l.h.s. is $\nabla_a (\lambda_b u_c) - \nabla_b (\lambda_a u_c) = (\lambda_b \nabla_a - \lambda_a \nabla_b) u_c = 0$. Then case 1) is obtained.

3) Contraction with $g^{cm}$ shows that the manifold is Einstein, then condition $C$ holds. If $u$ is a vector, obtain $u^2 R_{bclm}u^c u^m = \frac{R}{n(n - 1)} u_a (g_{a}u^2 - u_b u_l)$; then $u_{[a}R_{b]cimd}u^m = u^2 \frac{R}{n(n - 1)} (u_a g_{b} - u_b g_{a}) = -u^2 R_{ablm}u^m$ i.e. condition $B$ is true, and $B$ and $C$ imply $A$. $\square$
4. PERMUTABLE VECTORS

In the same way that compatibility is defined for vectors, permutability of a vector is defined by permutability of the tensor $u_iu_j$:

**Definition 4.1.** A vector is Riemann (Weyl) permutable if $R_{kl[m}u_{nj]u_m} = 0$ ($C_{kl[m}u_{nj]u_m} = 0$).

On Lorentzian manifolds and for null vectors, the definition is equivalent to the Bel-Debever condition for Weyl type II(abd) ([25] Table 1).

**Remark 4.2.** If $u$ is Riemann (Weyl) permutable and $u^2 \neq 0$, then $R_{kljm}u_m = 0$ ($C_{kljm}u_m = 0$).

A special class of Riemann permutable vectors is:

$$R_{abcm}u_m = 0.$$ (23)

Null vectors of this sort describe gravitational waves in Einstein’s linearized theory (see [30] page 244). A complete classification of space times that satisfy (23) is given in Theorem 1.1 of ref. [22].

Eq. (23) arises as the integrability condition for the equation

$$\nabla_a u_b + \nabla_b u_a = 2\lambda g_{ab}$$

with constant $\lambda$ and the constraint $\nabla_a u_b - \nabla_b u_a = 0$ ($u$ is a homothetic vector, see [31] pp. 69, 564). Vectors that fullfill (23) also arise in the symmetric solution of the equation

$$R_{abcd} x_{dm} + R_{abcd} x_{cm} = 0$$

which, by the Ricci identity, is equivalent to $[\nabla_a, \nabla_b] x_{cd} = 0$. It has a trivial solution $x_{ab} = \phi g_{ab}$, where $\phi$ is a scalar. McIntosh and Halford [21] investigated spacetimes whose Riemann tensor admits a non trivial solution, such as Einstein spaces, Gödel metric, Bertotti-Robinson metric. McIntosh and Hall proved [13] that the only nontrivial solution is $x_{ab} = \alpha u_a u_b$, where $u$ has the property (23) and $\alpha$ is a scalar field.

Besides the uniqueness stated above in $n = 4$, we prove in general:

**Proposition 4.3.** Let $x_{ab}$ be a symmetric tensor that fulfills (24).
1) $x_{ab}$ is Riemann (and Weyl) compatible;
2) If $X$ and $Y$ are two eigenvectors, $X^m x_{cm} = \lambda X_c$ and $Y^m x_{cm} = \mu Y_c$ with $\lambda \neq \mu$, then $R_{abcm} X^c Y^m = 0$.

**Proof.** Summation on cyclic permutations of indices $abc$ in (24) gives a vanishing term (first Bianchi identity) and Riemann compatibility. Property 2 is proven exactly as in Prop. 2.10. □

5. PETROV TYPES AND WEYL COMPATIBLE VECTORS

In 1954 Petrov classified $n = 4$ space-times according to the degeneracy of the eigenvalues of the self-dual part of the Weyl tensor. The eigenvalues solve an equation of degree four [27]. In type I spaces they are distinct, in type II spaces two are coincident and two are distinct, in type D spaces they are pairwise coincident, type III spaces have three equal eigenvalues, and finally in type N spaces all eigenvalues coincide [30]. Type O spaces are conformally flat. The same types arise in the
classification by Bel and Debever [1, 7], which is based on null vectors that solve increasingly restricted equations:

\[(25)\] type I \( k \) 
\[k_{[\alpha}C_{\beta\rho]k^\alpha k^\rho = 0\]

\[(26)\] type II, D \( k \) 
\[k_{[\alpha}C_{\beta\rho]k^\alpha k^\rho = 0\]

\[(27)\] type III \( k \) 
\[k_{[\alpha}C_{\beta\rho]k^\alpha k^\rho = 0\]

\[(28)\] type N \( C \) 
\[C_{\alpha\beta\rho\sigma} = 0\]

\[(29)\] type O \( C \) 
\[C_{\alpha\beta\rho\sigma} = 0\]

When at least two vectors \( k \) are degenerate, i.e. \( k \) meets condition (26), the Weyl tensor is named algebraically special [28, 30]. The classification was generalized to \( n > 4 \) and includes the above relations [5, 6, 15, 25].

Let’s consider the above classification in the perspective of Weyl compatibility. According to the general definition (16), a vector is Weyl compatible if

\[(30)\] \((u_i C_{jklm} + u_j C_{kilm} + u_k C_{ijlm})u^m = 0\).

**Theorem 5.1.** On a Lorentzian manifold, if a null vector \( k \) is Weyl compatible (or Riemann compatible), then the Weyl tensor is algebraically special.

**Proof.** Multiply (30) by \( k^c \) and use the antisymmetry of Weyl’s tensor:
\[0 = (k_a C_{bcdm} + k_b C_{cadm}) k^c k^d = k_{[a} C_{b]cd} k^c k^d = -k_{[a} C_{b]cmd} k^c k^d.\] \(\square\)

In ref. [25] (Table 1), the condition (30) is the statement that the Weyl tensor is type II(d).

If a space-time admits a null concircular vector, \( \nabla_k u_l = A_{kl} + B_{kl} u_l \), then the Weyl tensor is algebraically special (see Remark 3.3).

For a null-dust \( n \)-dimensional space-time, \( T_{ab} = \Phi^2 k_a k_b \) (eq. 5.8 in [31]), the condition \( D k^m_m = 0 \) is verified if and only if the Weyl tensor is type II(d) with respect to \( k \) (see theorem 2.3). The theorem extends Theorem 1.1 in [22], which holds for null vectors such that \( R_{ijk}^m k^m = 0 \).

Space-times with a null Weyl-compatible vector are Petrov type II or D. Are they more special than II or D? In general the answer is no. In a type III space-time there are three coincident principal directions, i.e. there is a null vector such that \( k_{[\alpha}C_{\beta\rho]k^\alpha k^\rho = 0\). This means that the null \( k \) is Weyl-permutable, a property that implies Weyl compatibility (see def. 4.1):

**Proposition 5.2.** A null vector \( k \) solves (27), which corresponds to \( n = 4 \) space-times of Petrov type III, if and only if it is Weyl-permutable.

**Proposition 5.3.** If \( u_k u_l \) is a Codazzi tensor and \( u_j \) is a closed 1-form, then:
1) if \( u^2 = 0 \) the Weyl tensor is algebraically special;
2) if \( u^2 \neq 0 \) the integral curves of \( u \) are geodesic lines.

**Proof.** 1) Codazzi tensors are Riemann compatible, and thus Weyl compatible. If moreover \( u^2 = 0 \), prop. 5.1 applies. 2) The Codazzi condition \( \nabla_a (u_b u_c) = \nabla_b (u_a u_c) \)
and closedness $\nabla_a u_b = \nabla_b u_a$ give $u_b \nabla_a u_c - u_a \nabla_b u_c = 0$. Exchange $a$ with $c$ and subtract to obtain: $u_c \nabla_b u_a - u_a \nabla_b u_c = 0$. Multiply by $u^a u^b$:

$$(u^b \nabla_b) u_a = \left[ \frac{u^a u^b \nabla_b u_c}{u^2} \right] u_a$$

i.e. the integral curves of $u$ are geodesics (see [8], eqs. 2.9.4 and 2.9.5). □

6. Electric and magnetic tensors

In an $n = 4$ space-time the Weyl tensor has 10 independent components that can be accounted for by two symmetric tensors. Given a vector $u$ with $u^a u_a = -1$, the electric and magnetic components of the Weyl tensor are [2]:

$$E_{ab} = u^j u^m C_{j a b m}, \quad H_{ab} = u^j u^m \tilde{C}_{j a b m}$$

(31)

where $\tilde{C}_{abcd} = \frac{1}{2} \epsilon_{a b r s} C^{r s c d}$ is the dual tensor. The two tensors are symmetric, traceless, and satisfy $E_{ab} u^b = 0$, $H_{ab} u^b = 0$. Then they each have 5 independent components, and completely describe the Weyl tensor.

If the electric and magnetic components are proportional, $\nu E = \mu H$ for some scalar fields $\mu$ and $\nu$ (including the case when one of them is zero), the space is type I, D or O [31] (page 73). The following theorem was partly proven in [19] and is stated in [25] for any $n$:

**Theorem 6.1.** On an $n = 4$ space-time, a vector $u$ is Weyl-compatible if and only if $H = 0$.

**Proof.** Consider the following chain of identities:

$$H_{ab} = u_j u^m \tilde{C}_{j a b m} = \frac{1}{2} u_j u^m C_{b s r m} \epsilon^{j a b s r}$$

$$= \frac{1}{6} [u_j C_{r s b m} \epsilon^{j a b s r} + u_r C_{s j b m} \epsilon^{a j s r} + u_s C_{j r b m} \epsilon^{a j s r}] u^m$$

$$= \frac{1}{6} [u_j C_{r s b m} + u_r C_{s j b m} + u_s C_{j r b m}] u^m \epsilon^{j a b s r}$$

(32)

The equality shows that $H = 0$ is equivalent to Weyl compatibility. □

It follows that a $n = 4$ space-time with a Weyl compatible time-like vector is type I, D or O. This extends Theorem 1.1 in [22]. If a spacetime admits a time-like concircular vector $\nabla_k u_l = A g_{k l} + B u_k u_l$, with constant $A$ and $B$, then the magnetic part vanishes.

**Theorem 6.2.** A $n = 4$ space-time with a non-null Weyl permutable vector is conformally flat, $C_{j k l m} = 0$ (type O).

**Proof.** Let $E$ and $H$ be the electric and magnetic components evaluated with $u$. If $u$ is Weyl permutable, then it is Weyl compatible and $H = 0$. Let’s show that also $E$ is zero. Multiply the relation (30) for Weyl compatibility by $u^j$: $u^j C_{k l m} u^m = u_k E_{d l} - u_l E_{d k}$. Because $u$ is Weyl permutable, it is $C_{k l m} u^m = 0$ (see remark [22]); then $0 = u_k E_{d l} - u_l E_{d k}$. Multiply by $u^k$ and use $u^k E_{k l} = 0$ to obtain $E_{d l} = 0$. □

The definitions of electric and magnetic components of the Weyl tensor can be generalized by replacing the symmetric tensor $u^i u^j$ by a symmetric tensor $T^{ij}$:

$$E_{ab} = T^{jm} C_{j a b m}, \quad H_{ab} = T^{jm} \tilde{C}_{j a b m}.$$
Proposition 6.3. \( E \) and \( H \) are symmetric and traceless.

Proof. The first statement follows from the symmetry of \( C_{ijkl} \) or \( \tilde{C}_{ijkl} \) in the exchange of \( ij \) with \( kl \), and symmetry of \( T \). The second follows from tracelessness of the Weyl tensor and its dual. \( \square \)

Proposition 6.4.

1) If \( T \) is Weyl-compatible then \( E \) commutes with \( T \);
2) \( H = 0 \) if and only if \( T \) is Weyl compatible.

Proof. The proof is based on the following identities:

\[
E_{ab}T^b_{bc} - T^b_{ab}E_{bc} = [C_{jabm}T^b_{cm} - C_{jcbm}T^b_{am}]T^{jm}
\]

\[
H^a_{b} = \frac{1}{6}[T^m_{j}C_{rsbm} + T^m_{r}C_{sjbm} + T^m_{s}C_{jrbm}]\epsilon^{jars}
\]

The second identity is proven along the same line as (32). The first equation is proven here:

\[
E_{ab}T^b_{c} - T^b_{a}E_{bc} = [C_{jabm}T^b_{cm} - C_{jcbm}T^b_{am}]T^{jm}
\]

\[
= - [T^b_{c}C_{jab} + T^b_{a}C_{cjmb} + T^b_{j}C_{acmb}]T^{jm}
\]

\[
= - [T^b_{c}C_{jab} + T^b_{a}C_{cjmb} + T^b_{j}C_{acmb}]T^{jm}
\]

where the last term added in the second line is identically zero. \( \square \)

7. Hypersurfaces

Let \( \mathcal{M}_n \) be a hypersurface in a pseudo Riemannian manifold \( (\mathbb{V}^n_{n+1}, \tilde{g}) \). The metric tensor (first fundamental form) is \( g_{ij} = \tilde{g}(B_i, B_j) \), where \( B_1 \ldots B_n \) are the tangent vectors. If \( N \) is the vector normal to the hypersurface it is \( \tilde{g}(B_i, N) = 0 \). The Riemann tensor is given by the Gauss equation [16]:

\[
R_{jklm} = \bar{R}_{\mu
u\rho\sigma}B^\mu_j B^\nu_k B^\rho_l B^\sigma_m \pm (\Omega_j \Omega_k - \Omega_i \Omega_l)
\]

with a symmetric tensor \( \Omega_{ij} \) (second fundamental form) constrained by the Codazzi equation:

\[
\nabla_k \Omega_{jl} - \nabla_j \Omega_{kl} = N^{\mu} \bar{R}_{\mu
u\rho\sigma}B^\nu_j B^\rho_k B^\sigma_l
\]

If \( \mathbb{V}^n_{n+1} \) is a constant curvature manifold, the Gauss and Codazzi equations simplify:

\[
R_{jklm} = \frac{\bar{R}}{n(n+1)}(g_{jl}g_{km} - g_{jm}g_{kl}) \pm (\Omega_j \Omega_k - \Omega_i \Omega_l),
\]

\[
\nabla_k \Omega_{jl} - \nabla_j \Omega_{kl} = 0,
\]

If \( \mathbb{V}^n_{n+1} \) is (pseudo)-Euclidean, the terms proportional to the scalar curvature \( \bar{R} \) vanish [31]. For this case Stephani proved that eqs. (38) and (39) are sufficient conditions for a manifold \( \mathcal{M}_4 \) to have an embedding in \( \mathbb{V}^5_{5} \) [31] (page 587). A general theorem by Goenner, restricted to pseudo-Euclidean manifolds \( \mathbb{V}^n_{n+1} \), states that if \( \Omega_{kl} \) is invertible, then it is a Codazzi tensor [31] (page 587). A simple proof of the same fact is here given, for the case of constant curvature \( \mathbb{V}^n_{n+1} \):

**Theorem 7.1.** If \( R_{jklm} \) has the form (38) and \( \Omega \) is invertible, then \( \Omega \) is a Codazzi tensor.
Proof. The second Bianchi identity for the Riemann tensor is
\[ \Omega_{mk}(\nabla_i \Omega_{jl} - \nabla_j \Omega_{il}) + \Omega_{ml}(\nabla_j \Omega_{kl} - \nabla_k \Omega_{lj}) + \Omega_{mj}(\nabla_k \Omega_{il} - \nabla_l \Omega_{ik}) + + \Omega_{jl}(\nabla_i \Omega_{mk} - \nabla_k \Omega_{jm}) + \Omega_{il}(\nabla_k \Omega_{mj} - \nabla_j \Omega_{km}) = 0 \]
Multiplication by \((\Omega^{-1})^{km}\) gives:
\[ (n - 3)(\nabla_i \Omega_{jl} - \nabla_j \Omega_{il}) = - (\Omega^{-1})^{km}[\Omega_{jl}(\nabla_i \Omega_{mk} - \nabla_k \Omega_{jm}) + \Omega_{il}(\nabla_k \Omega_{mj} - \nabla_j \Omega_{km})]. \]
Multiplication by \((\Omega^{-1})^{ij}\) gives: \[2(n - 2)(\Omega^{-1})^{ij}(\nabla_i \Omega_{jl} - \nabla_j \Omega_{il}) = 0.\] This result is used to simplify the previous equation: \((n - 3)(\nabla_i \Omega_{jl} - \nabla_j \Omega_{il}) = 0,\) which for \(n > 3\) is the Codazzi property.

Theorem 7.2. Let \( \mathcal{M}_n \) be a hypersurface isometrically embedded in a pseudo-Riemannian space \( \mathbb{V}_{n+1} \) with constant curvature. Then:
1) \( \Omega \) is Weyl compatible;
2) the eigenvectors of \( \Omega \) are Weyl compatible;
3) the Ricci tensor is Weyl compatible.

Proof. 1) For a hypersurface that is isometrically embedded in a constant curvature space, \( \Omega_{ij} \) is a Codazzi tensor, and then it is both Riemann and Weyl compatible.
2) Given the form (33) of the Riemann tensor, if \( \Omega_{km}u^m = \lambda u_k \) then:
\[ u_iu^m R_{jklm} = ku_i(u_k g_{jl} - u_j g_{kl}) \pm \lambda u_i(\Omega_{jl}u_k - \Omega_{kj}u_l) \]
where, for shortness, \( k = \bar{R}/(n + 1). \) Summation on cyclic permutations of \( i, j, k \) cancels all terms in the right-hand-side, and one is left with Riemann compatibility:
\[ u_iu^m R_{jklm} + u_ju^m R_{iklm} + u_ku^m R_{ijlm} = 0. \]
3) The Ricci tensor for a hypersurface isometrically embedded in a constant curvature space is \( R_{kl} = \pm (\Omega^2_{kl} - \Omega_p \Omega^p_{kl}) + k(n - 1)g_{kl}. \) Let us first show that \( \Omega^2 \) is Riemann compatible. Evaluate the expression \((\Omega^2)^{im} R_{jklm} + (\Omega^2)^{jm} R_{iklm} + (\Omega^2)^{km} R_{ijlm} \) with the Riemann tensor (33). The first term is:
\[ \Omega^2_{im} R_{jklm} = k \left( g_{jl} \Omega^2_{ik} - g_{il} \Omega^2_{jk} \right) + \left( \Omega_{jl} \Omega^3_{ik} - \Omega_{ik} \Omega^3_{jl} \right) \]
While summing on cyclic permutations of \( ijk \), all terms in the r.h.s cancel. Therefore the tensor \( \Omega^2 \) is Riemann compatible and thus Weyl compatible. Since the Ricci tensor is the sum of Riemann - compatible terms, it is itself Riemann compatible, and thus Weyl compatible.

By considering Einstein’s equation (3) one also has:

Corollary 7.3. On a space-time that is isometrically embedded as a hypersurface is a pseudo Riemannian space \( \mathbb{V}_{n+1} \) with constant curvature, the energy momentum tensor is Weyl compatible.
If the energy momentum tensor has the form \( T_{kl} = au_ku_l + bg_{kl} \) with \( u^i u_i = -1 \), then the Weyl tensor is purely electric.

8. Geodesic maps

Let \((\mathcal{M}, g)\) be a pseudo-Riemannian manifold. A geodesic map \( \mathcal{M} \to \mathcal{M} \) induces a pseudo-Riemannian structure \((\mathcal{M}, \tilde{g})\) with Christoffel symbols \( \tilde{\Gamma}^{k}_{ij} = \Gamma^{k}_{ij} + \delta^{k}_{j} X_{i} + \delta^{k}_{i} X_{j} \), where \( X \) is a closed 1-form (11). Accordingly, the new Riemann tensor is \( \tilde{R}_{jklm} = R_{jklm} + \delta^{m}_{j} P_{kl} - \delta^{m}_{k} P_{jl}, \) with deformation tensor \( P_{kl} = \nabla_{k} X_{l} - X_{k} X_{l}. \)
Since $X$ is closed, the deformation tensor is symmetric. The new Ricci tensor is $\tilde{R}_{kl} = R_{kl} - (n-1) P_{kl}$.

In [19] we showed that for geodesic maps the following identity holds for any symmetric tensor:

\[
\tilde{b}_{im} \tilde{R}_{jkl} \tilde{m} + \tilde{b}_{jm} \tilde{R}_{kilm} \tilde{m} + \tilde{b}_{km} \tilde{R}_{i}j\tilde{l} \tilde{m} = \tilde{b}_{im} R_{jkl} \tilde{m} + \tilde{b}_{jm} R_{kilm} \tilde{m} + \tilde{b}_{km} R_{i}j\tilde{l} \tilde{m}.
\]

as a consequence, the property of Riemann compatibility is conserved. What about Weyl compatibility? For general symmetric tensors the answer is difficult by the fact that the expression (1) for the new Weyl tensor contains $\tilde{g}$, which is not simply related to $g$. This is a sufficient condition:

**Proposition 8.1.** If a symmetric tensor $b$ commutes with the Ricci and the deformation tensors, then

\[
b_{im} \tilde{C}_{jkl} \tilde{m} + b_{jm} \tilde{C}_{kilm} \tilde{m} + b_{km} \tilde{C}_{ijl} \tilde{m} = b_{im} C_{jkl} \tilde{m} + b_{jm} C_{kilm} \tilde{m} + b_{km} C_{ijl} \tilde{m}.
\]

Proof. If $b$ commutes with the Ricci and the deformation tensors, then it commutes with $\tilde{R}_{ij}$. With these conditions, (11) implies that

\[
b_{im} C_{jkl} \tilde{m} + b_{jm} C_{kilm} \tilde{m} + b_{km} C_{ijl} \tilde{m} = b_{im} R_{jkl} \tilde{m} + b_{jm} R_{kilm} \tilde{m} + b_{km} R_{ijl} \tilde{m}
\]

and the same relation with tensors $\tilde{C}_{jkl} \tilde{m}$ and $\tilde{R}_{jkl} \tilde{m}$. Since (10) holds for geodesic maps, (11) follows. $\square$

A simplification occurs for special geodesic maps, defined by the property $P_{kl} = \gamma g_{kl}$, meaning that $X$ is a concircular vector: $\nabla_k X_l - X_k X_l = \gamma g_{kl}$ [10].

**References**

[1] L. Bel, *Radiation states and the problem of energy in General Relativity* (reprint from (1962) Cah. Phys. 16 p.59), Gen. Rel. Grav. 32 n. 10 (2000) 2047-2078.
[2] E. Bertschinger and A. J. S. Hamilton, *Lagrangian evolution of the Weyl tensor*, Astroph. J. 435 (1994) 1-7.
[3] A. L. Besse, *Einstein Manifolds*, Springer (1987).
[4] J. P. Bourguignon, *Les variétés de dimension 4 à signature non nulle dont la courbure est harmonique sont d’Einstein*, Invent. Math. 63 n. 2 (1981) 263-286.
[5] A. Coley, *Classification of the Weyl tensor in higher dimensions and applications*, Class. Quantum Grav. 25 (2008) 033001.
[6] A. Coley, R. Milson, V. Pravda and A. Pravdova, *Classification of the Weyl tensor in higher dimensions*, Class. Quantum Grav. 21 n. 7 (2004) L35-41.
[7] R. Debever, *Tensor de super-énergie, tenseur de Riemann: cas singuliers*, C. R. Acad. Sci. (Paris) 249 (1959) 1744-1746.
[8] F. De Felice and C. J. S. Clarke, *Relativity on curved manifolds*, Cambridge University Press (2001).
[9] A. Derdziński and C. L. Shen, *Codazzi tensor fields, curvature and Pontryagin forms*, Proc. London Math. Soc. 47 n. 3 (1983) 15-26.
[10] R. Deszcz and M. Hotlos, *Notes on pseudo-symmetric manifolds admitting special geodesic mappings*, Soochow J. Math. 15 n. 1 (1989) 19-27.
[11] S. Formella, *On some class of nearly conformally symmetric manifolds*, Colloq. Math. 68 (1995) 149-164.
[12] G. S. Hall, *On the Petrov classification of gravitational fields*, J. Phys. A: Math. Nucl. Gen. 6 n. 5 (1973) 619-623.
[13] G. S. Hall and C. B. G. McIntosh, *The algebraic determination of the metric from the curvature in General Relativity*, Int. J. Theor. Phys. 22 (1983) 469-476.
[14] S. W. Hawking and G. F. R. Ellis, *The large scale structure of space time*, Cambridge University Press (1973).
[15] S. Hervik, M. Ortaggio, and L. Wylleman, *Minimal tensors and purely electric and magnetic spacetimes of arbitrary dimensions*, arXiv:1203.1060 [gr-qc].
[16] D. Lovelock and H. Rund, Tensors, differential forms and variational principles, reprint Dover Ed. (1988).
[17] C. A. Mantica and L. G. Molinari, A second order identity for the Riemann tensor and applications, Colloq. Math. 122 n. 1 (2011) 69-82.
[18] C. A. Mantica and L. G. Molinari, Extended Derdziński-Shen theorem for curvature tensors, Colloq. Math. 128 n. 1 (2012) 1-6.
[19] C. A. Mantica and L. G. Molinari, Riemann compatible tensors, Colloq. Math. 128 n. 2 (2012) 197-210.
[20] C. A. Mantica and Y. J. Suh, The closedness of some generalized 2-forms on a Riemannian manifold I, Publ. Math. Debrecen 81 n. 3-4 (2012) 313-326.
[21] C. B. G. McIntosh and W. D. Halford, The Riemann tensor, the metric tensor, and curvature collineations in general relativity, J. Math. Phys. 23 (1982) 436-441.
[22] C. B. G. McIntosh and E. H. van Leeuwen, Spacetimes admitting a vector field whose inner product with the Riemann tensor is zero, J. Math. Phys. 23 (1982) 1149-1152.
[23] J. Mikes, Geodesic mappings of affine-connected and Riemannian spaces, J. Math. Sci. 78 n. 3 (1996) 311-334.
[24] R. Milson, A. Coley, V. Pravda and A. Pravdova, Alignment and algebraically special tensors in Lorentzian geometry, Int. J. Geom. Meth. Mod. Phys. 2 (2005) 41-61.
[25] M. Ortaggio, BelDebever criteria for the classification of the Weyl tensor in higher dimensions, Class. Quantum Grav. 26 (2009) 195015 (8pp).
[26] R. Penrose, A spinor approach to General Relativity, Ann. Phys. 10 n. 2 (1960) 171-201.
[27] A. Z. Petrov, The classification of spaces defining gravitational fields (a reprint), Gen. Rel. Grav. 32 (2000) 1665-1685.
[28] R. Sachs, Gravitational waves in General Relativity. VI. The outgoing radiation condition, Proc. Roy. Soc. 264 (1961) 309-338.
[29] N. S. Sinyukov, Geodesic mapping of Riemannian spaces, Nauka, Moscow (1979).
[30] H. Stephani, General Relativity, Cambridge University Press, 3rd ed. (2004).
[31] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers and E. Hertl, Exact Solutions of Einstein’s Field Equations, Cambridge University Press, 2nd ed. (2003).
[32] S. Weinberg, Gravitation and Cosmology, Wiley (1972).

C. A. Mantica: I.I.S. Lagrange, Via L. Modignani 65, 20161, Milano, Italy – L. G. Molinari (corresponding author): Physics Department, Università degli Studi di Milano and I.N.F.N. sez. Milano, Via Celoria 16, 20133 Milano, Italy.
E-mail address: carloalberto.mantica@libero.it, luca.molinari@mi.infn.it