ELEVENTH-ORDER CALCULATION OF GREEN’S FUNCTIONS IN THE ISING LIMIT FOR ARBITRARY SPACE-TIME DIMENSION D

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ABSTRACT

This paper extends an earlier high-temperature lattice calculation of the renormalized Green’s functions of a $D$-dimensional Euclidean scalar quantum field theory in the Ising limit. The previous calculation included all graphs through sixth order. Here, we present the results of an eleventh-order calculation. The extrapolation to the continuum limit in the previous calculation was rather clumsy and did not appear to converge when $D > 2$. Here, we present an improved extrapolation which gives uniformly good results for all real values of the dimension between $D = 0$ and $D = 4$. We find that the four-point Green’s function has the value $0.62 \pm 0.007$ when $D = 2$ and $0.98 \pm 0.01$ when $D = 3$ and that the six-point Green’s function has the value $0.96 \pm 0.03$ when $D = 2$ and $1.2 \pm 0.2$ when $D = 3$.  

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There have been many attempts to calculate the coefficients in the effective potential of a Euclidean scalar quantum field theory in the Ising limit. These coefficients are just the dimensionless renormalized $2n$-point Green’s functions evaluated at zero momentum. Techniques that have been used to determine these Green’s functions include high-temperature lattice expansions, Monte Carlo methods, and epsilon expansions. For the case $D = 3$ a complete list of references is given in a recent paper by Tsypin.\(^1\)

In a series of papers\(^2,3\) high-temperature lattice techniques were used to obtain the dependence of the Green’s functions upon the Euclidean space-time dimension $D$ for $D$ ranging continuously between 0 and 4. In this work a strong-coupling calculation to sixth order was performed analytically on a hypercubic lattice in $D$ dimensions and Padé extrapolation techniques were invented to obtain the continuum limit.\(^4\) The current paper is an addendum to the work in Refs. 2 and 3. Here, we extend the strong-coupling calculation to eleventh order. Furthermore, we use an improved Padé extrapolation method that relies on information taken from the results of a large-$N$ calculation and our recent studies of dimensional expansions for quantum field theory.\(^5,6,7\)

Our strong-coupling lattice calculations are identical to those described in Ref. 2 except that the graphs were generated using a FORTRAN program and evaluated analytically using MACSYMA. The eleventh-order calculation involves several hundred times as many graphs as the sixth-order calculation. We have verified the accuracy of our expansions for the specific cases of $D = 2$, $D = 3$, and $D = 4$ dimensions by comparing them with previous calculations\(^8\).

The lattice series for the renormalized four-point and six-point Green’s functions are as follows:
\[ \gamma_4 = \frac{y^{-D/2}}{12} \left[ 1 + 4D y + (4D^2 - 10D) y^2 + 16D y^3 + (30D - 80D^2) y^4 \\
+ (256D^3 + 104D^2 - 192D) y^5 \\
+ (-704D^4 - 1736D^3 + 2508D^2 - 656D) y^6 \\
+ (1792D^5 + 10432D^4 - 11232D^3 - 3872D^2 + 4992D) y^7 \\
+ (-4352D^6 - 45600D^5 + 11456D^4 + 123672D^3 \\
- 128440D^2 + 35542D) y^8 \\
+ (10240D^7 + 168320D^6 + 181248D^5 - 1052576D^4 \\
+ 2615584/3D^3 + 76664D^2 - 681472/3D) y^9 \\
+ (-23552D^8 - 558208D^7 - 1630272D^6 + 5391904D^5 \\
- 1011536/3D^4 - 10102936D^3 + 29622092/3D^2 - 2720752D) y^{10} \\
+ (53248D^9 + 1718272D^8 + 9081856D^7 - 18274816D^6 \\
- 113682176/3D^5 + 367432576/3D^4 - 292976128/3D^3 \\
+ 18425408/3D^2 + 14757984D) y^{11} + \ldots \right] \]

and

\[ \gamma_6 = \frac{y^{-D}}{30} \left[ 1 + 6D y + (12D^2 - 6D) y^2 + (8D^3 - 12D^2 - 20D) y^3 \\
+ (48D^2 + 48D) y^4 + (-96D^3 - 816D^2 + 528D) y^5 \\
+ (192D^4 + 4640D^3 - 2736D^2 - 560D) y^6 \\
+ (-384D^5 - 18432D^4 - 10800D^3 + 46512D^2 - 23040D) y^7 \\
+ (768D^6 + 61440D^5 + 188352D^4 - 510816D^3 + 357324D^2 - 72492D) y^8 \\
+ (-1536D^7 - 184576D^6 - 1274880D^5 + 2653440D^4 \\
- 77496D^3 - 2911496D^2 + 1698240D) y^9 \\
+ (3072D^8 + 517632D^7 + 6280704D^6 - 6584832D^5 - 27745840D^4 \\
+ 65401176D^3 - 49332608D^2 + 11853912D) y^{10} \\
+ (-6144D^9 - 1382400D^8 - 25928448D^7 - 13343232D^6 + 286690784D^5 \\
- 516057392D^4 + 211594432D^3 + 210150872D^2 - 153291336D) y^{11} + \ldots \right]. \]
where \( y = (Ma)^{-2} \), \( a \) is the lattice spacing, and \( M \) is the renormalized mass, which is obtained from the two-point function as explained in Ref. 2.

The quantity \( \sqrt{y} \) is the dimensionless correlation length. The continuum limit \( a \to 0 \) corresponds to infinite correlation length. To obtain the continuum Green’s functions it is necessary to extrapolate the formulas in (1) and (2) to their values at \( y = \infty \). Direct extrapolation to the continuum limit of either series in (1) or (2) leads to a sequence of extrapolants that becomes badly behaved when \( D \) increases beyond 2; we find that extrapolations as functions of \( D \) do not converge to a limiting curve (see Figs. 1 and 2). However, for \( D \) near 0 these extrapolants are well behaved and converge rapidly to the known exact values\(^2 \) \( \gamma_4 = 1/4 \) and \( \gamma_6 = 1/4 \) at \( D = 1 \).

To improve our extrapolation we make the following observation. We consider a scalar field theory having an \( O(N) \) symmetry. The model we have studied above corresponds to the case \( N = 1 \). In the limit \( N \to \infty \) one can solve for the Green’s functions exactly. We obtain the following lattice results:

\[
N\gamma_4^{(N=\infty)} = \frac{y^{-D/2}}{4 \int_0^\infty dt \ e^{-t} \left[e^{-2ty} I_0(2ty)\right]^D}
\]  

and

\[
N^2\gamma_6^{(N=\infty)} = \frac{y^{-D} \int_0^\infty dt \ t^2 e^{-t} \left[e^{-2ty} I_0(2ty)\right]^D}{12 \left(\int_0^\infty dt \ e^{-t} \left[e^{-2ty} I_0(2ty)\right]^D\right)^3},
\]

where we have summed over the external indices. In the continuum limit \( y \to \infty \) we have

\[
N\gamma_4^{(N=\infty)} = \frac{(4\pi)^{D/2}}{4 \Gamma(2 - D/2)}
\]

and

\[
N^2\gamma_6^{(N=\infty)} = \frac{(4\pi)^D \Gamma(3 - D/2)}{12 \left[\Gamma(2 - D/2)\right]^3},
\]
where $D$ lies in the range $0 \leq D \leq 4$. Each of these functions rises from its value at $D = 0$, attains a maximum, and falls to 0 at $D = 4$.

Under the assumption that the Green’s functions for $N = 1$ vanish at $D = 4$ like those in (5) and (6) we can extract such a behavior from the series (1) and (2) by performing a Borel summation as follows. Consider the lattice series in (1) for $\gamma_4$. This series has the general form

$$\gamma_4 = \frac{y^{D/2}}{12} \sum_{k=0}^{\infty} P_k(D)y^k,$$

where $P_k(D)$ are polynomials of maximum degree $k$. One can read off the first eleven polynomials $P_k(D)$ from (1). We can rewrite (7) as

$$\frac{1}{12\gamma_4} = y^{D/2} \sum_{k=0}^{\infty} Q_k(D)y^k,$$

where $Q_k(D)$ is another polynomial in $D$. Next, we insert the identity

$$1 = \frac{1}{(k+1)!} \int_0^\infty dt \, t^{k+1} e^{-t}$$

for each term in the sum. This converts (8) to the form

$$\frac{1}{12\gamma_4} = \int_0^\infty dt \, t^{1-D/2} e^{-t} f(yt),$$

where we define

$$f(x) = x^{D/2} \sum_{k=0}^{\infty} \frac{Q_k(D)x^k}{(k+1)!} = \left( x \sum_{k=0}^{\infty} R_k(D)x^k \right)^{D/2},$$

where again $R_k(D)$ are polynomials in $D$.

We now take the continuum limit of the expression (10). Assuming that $f(\infty)$ exists in the limit $y \to \infty$ so that (10) separates into a product of two terms, we have

$$\gamma_4 = \frac{1}{12 \Gamma(2-D/2) f(\infty)},$$

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Thus, we have forced the continuum limit of the four-point function to take the form of
the large-$N$ result in (5) apart from $f(\infty)$, which is a smoothly varying function of $D$.

There is an immediate indication that the Borel summation leading to (12) has a
significant impact. We find that the polynomials $R_k(D)$ in (11) are significantly simpler
than the original polynomials $P_k(D)$ in (7); the polynomials $R_k(D)$ have maximum degree
$[k/2] - 1$, about half the degree of the polynomials $P_k(D)$. The polynomials $R_k(D)$, $k = 0, 1, 2, \ldots, 11$, are as follows:

$$R_0(D) = 1,$$
$$R_1(D) = -2,$$
$$R_2(D) = 11/3,$$
$$R_3(D) = -16/3,$$
$$R_4(D) = -\frac{7}{45} D + \frac{233}{36},$$
$$R_5(D) = \frac{2}{3} D - \frac{125}{18},$$
$$R_6(D) = \frac{73}{2835} D^2 - \frac{1346}{945} D + \frac{76691}{11340},$$
$$R_7(D) = -\frac{221}{1134} D^2 + \frac{2713}{1260} D - \frac{16939}{2835},$$
$$R_8(D) = -\frac{13}{8100} D^3 + \frac{98323}{170100} D^2 - \frac{8837}{3240} D + \frac{2627137}{544320},$$
$$R_9(D) = \frac{163}{3150} D^3 - \frac{14699}{14175} D^2 + \frac{17111}{5670} D - \frac{235681}{64800},$$
$$R_{10}(D) = -\frac{457}{267300} D^4 - \frac{1338257}{5613300} D^3 + \frac{2015579}{1403325} D^2 - \frac{9257497}{3207600} D + \frac{230357209}{89812800},$$
$$R_{11}(D) = -\frac{611}{69300} D^4 + \frac{6066953}{11226600} D^3 - \frac{3923681}{2245320} D^2 + \frac{110092603}{44906400} D - \frac{75132389}{44906400}.$$

The resummation of the lattice series as performed above reduces the problem of
extracting the continuum limit to finding the value of $f(\infty)$. This is done using the same
Padé techniques as were used in Ref. 2. If we perform this numerical calculation we obtain a sequence of approximants, one for each new order in perturbation theory. The first eleven such approximants for $\gamma_4$ in (12) are plotted in Fig. 3. Each approximant is a continuous function of $D$ for $0 \leq D \leq 4$. Note that the approximants are smooth and well behaved; the sequence is monotone increasing and appears to converge uniformly to a limiting curve. The dotted line on Fig. 3 is an extrapolation of these approximants to this limiting curve obtained using Richardson extrapolation.\(^9\)

There are a number of ways to assess the accuracy of the limiting curve. First, one can Taylor expand this limiting curve about $D = 0$ as a series in powers of $D$. This Taylor series has the form

$$\gamma_4^{\text{limiting curve}}(D) = \frac{1}{12} (1 + 1.18D + 0.64D^2 + 0.19D^3 + 0.03D^4 + \ldots). \quad (14)$$

We may then compare this Taylor series with that recently obtained\(^7\) using dimensional expansion methods:

$$\gamma_4^{\text{dimensional expansion}}(D) = \frac{1}{12} (1 + 1.18D + 0.62D^2 + 0.18D^3 + 0.03D^4 + \ldots). \quad (15)$$

Note that the coefficients of these two series are almost identical. Second, we can examine the limiting curve at $D = 1$, for which the exact value $\gamma_4 = 1/4$ is known. At this value of $D$ the limiting curve has the value 0.2526 so it is slightly high by about 1%.

In Figs. 4 and 5 we demonstrate how we obtain the limiting curve for the cases $D = 2$ and $D = 3$. We have plotted the $n$th-order Richardson extrapolants for the approximants in Fig. 3 for $n = 1, 2, \ldots, 5$ versus the inverse order of the approximants. We then determine where each of these extrapolants crosses the vertical axis (each intersection is indicated by a horizontal bar). Finally, we extrapolate to the limiting value of these intersection points.
This procedure gives the value $\gamma_4 = 0.620 \pm 0.007$ at $D = 2$, indicated in Fig. 4 by a fancy square. This result is to be compared with $\gamma_4 = 0.6108 \pm 0.0025$ obtained by Baker and Kincaid$^{10}$. Similarly, in Fig. 5 we find that the limiting curve gives $\gamma_4 = 0.986 \pm 0.010$ at $D = 3$. This value compares reasonably well with previous results, as tabulated in Ref. 1. For example, Baker and Kincaid$^{11}$ obtain 0.98, Monte Carlo studies give results between 0.9 and 1, and renormalization group studies give results around 0.98. Note that the limiting curve in Fig. 3 has a maximum extremely close to $D = 3$; numerically, the maximum occurs at $D = 3.03$.

The same procedure that was used to extrapolate (1) to the continuum and thereby to obtain a plot of $\gamma_4$ as a function of $D$ can be applied to (2). We perform a Borel summation of the series in (2) as follows. The lattice series in (2) for $\gamma_6$ has the general form

$$\gamma_6 = \frac{y^{-D}}{30} \sum_{k=0}^{\infty} S_k(D)y^k, \quad (16)$$

where $S_k(D)$ are polynomials of maximum degree $k$. One can read off the first eleven polynomials $S_k(D)$ from (2). From the structure of $\gamma_6^{(N=\infty)}$ in (4) we are motivated to rewrite (16) in the form

$$y^{D/2} \int_0^{\infty} dt \frac{t^2 e^{-t} [e^{-2tyI_0(2ty)}]^D}{\frac{y}{30} \gamma_6} = \left[y^{D/2} \sum_{k=0}^{\infty} T_k(D)y^k\right]^3, \quad (17)$$

where $T_k(D)$ is another polynomial in $D$. Again, we insert the identity (9) for each term in the sum in (17). This converts (17) to the form

$$y^{D/2} \int_0^{\infty} dt \frac{t^2 e^{-t} [e^{-2tyI_0(2ty)}]^D}{\frac{y}{30} \gamma_6} = \left[\int_0^{\infty} dt \frac{t^{1-D/2} e^{-t} g(ty)}{\gamma_6}\right]^3, \quad (18)$$
where we define
\[ g(x) = x^{D/2} \sum_{k=0}^{\infty} \frac{T_k(D) x^k}{(k + 1)!} \]
\[ = \left( x \sum_{k=0}^{\infty} U_k(D) x^k \right)^{D/2}, \] (19)

where \( U_k(D) \) are polynomials in \( D \) of degree \( [k/2] - 1 \) similar in structure to those in (13).

Next, we take the continuum limit of the expression (18). Assuming that \( g(\infty) \) exists in the limit \( y \to \infty \) we find that (18) separates into a product of several terms and we have
\[ \gamma_6 = \frac{(4\pi)^{-D/2} \Gamma(3 - D/2)}{30 \Gamma(2 - D/2) g(\infty)^3}. \] (20)

Thus, we have forced the continuum limit of the six-point function to take the form of the large-\( N \) result given in (6) apart from \( g(\infty) \), which is a function of \( D \).

Again, the resummation of the lattice series reduces the problem of extracting the continuum limit to finding the value of \( g(\infty) \). This is done using the same Padé techniques as were used in Ref. 2. We perform this numerical calculation and obtain a sequence of approximants, one for each new order in perturbation theory. The first eleven such approximants for \( \gamma_6 \) in (20) are plotted in Fig. 6. Each approximant is a continuous function of \( D \) for \( 0 \leq D \leq 4 \). As in Fig. 3 the approximants are smooth and well behaved; the sequence is monotone increasing and appears to converge uniformly to a limiting curve indicated in Fig. 6 by a dotted line. This limiting curve is again obtained using fifth-order Richardson extrapolation. The limiting curve at \( D = 1 \) passes through the value \( \gamma_6 = 0.240 \) which differs from the exact value \( \gamma_6 = 1/4 \) by about 4%.

The limiting curve predicts that \( \gamma_6 = 0.96 \pm 0.04 \) at \( D = 2 \) and \( \gamma_6 = 1.2 \pm 0.1 \) at \( D = 3 \). This value is lower than most previous results, as tabulated in Ref. 1, but it is
certainly larger than zero. By comparison, an epsilon expansion around $D = 4$ gives
\begin{equation}
\frac{\gamma_6}{(\gamma_4)^2} = 2\epsilon - \frac{20}{27}\epsilon^2 + 1.2759\epsilon^3 + \ldots .
\end{equation}

This series appears to be divergent but a direct optimal truncation of the series after one term with $\epsilon = 1$ gives the value $\gamma_6 = 1.9 \pm 0.7$. (Here, we have substituted the value $\gamma_4 = 0.986$ given above.) However, if we perform a (1, 1)-Padé summation of this series, which seems justified because of the alternating sign pattern, we obtain the smaller value $\gamma_6 = 1.66 \pm 0.28$, in better agreement with our predicted value.

Finally, we observe that the maxima of $\gamma_{2i}$ as a function of $D$ appear to follow a pattern. We observed already that $\gamma_4$ has a maximum that is close to $D = 3$. Here, we find that the limiting curve for $\gamma_6$ has a maximum at $D = 2.66$ which is very close to $8/3$. An interesting conjecture is that in general the maximum might be located at $D_{max} = \frac{2(i+1)}{i}$, the value of $D$ for which a $\phi^{2i+2}$ theory becomes free.

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FIGURE CAPTIONS

Figure 1. The first $k$ Padé approximants, $k = 1, 2, \ldots, 11$, to the continuum limit of the four-point Green’s function. These approximants are constructed from the lattice series in (1) using the Padé procedure explained in Ref. 2. Observe that the approximants are well behaved when $D$ is near 0; in particular, they converge nicely to the exact value $1/4$ at $D = 1$ (indicated by a plus sign). However, when $D$ increases beyond 2 the approximants become irregular and do not seem to converge to a limiting function of $D$. (Some of the approximants reach zero and terminate as $D$ increases because they become complex.)

Figure 2. Same as in Fig. 1 except that we have plotted the first $k$ approximants to the six-point Green’s function for $k = 1, 2, \ldots, 11$. Again, we find that the approximants converge to the exact value $1/4$ at $D = 1$ but that they are irregular for $D > 2$.

Figure 3. First eleven approximants to $\gamma_4$ plotted as functions of $D$ for $0 \leq D \leq 4$ [see (12)]. The approximants form a monotone increasing sequence of curves; the labeling indicates the order. Note that the approximants are smooth curves that seem to be tending uniformly to a limiting curve. We have obtained this limiting curve (dotted curve) by means of fifth-order Richardson extrapolation$^9$. The exact result $\gamma_4 = 1/4$ at $D = 1$ is indicated by a plus sign; the limiting curve passes within 1% of this point.

Figure 4. Plot of the $n$th-order Richardson extrapolants for the approximants in Fig. 3 for $n = 1, 2, \ldots, 5$ versus the inverse order of the approximants for the case $D = 2$. Each Richardson extrapolant is linearly extended (dash-dotted line) until it intersects the vertical axis. Each intersection is indicated by a horizontal bar. We then extrapolate to the limiting value of these intersection points, indicated by a fancy square. This procedure predicts that $\gamma_4 = 0.620$ at $D = 2$. 

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Figure 5. Same as in Fig. 4 except that $D = 3$. This extrapolation procedure predicts that $\gamma_4 = 0.986$ at $D = 3$.

Figure 6. First eleven approximants to $\gamma_6$ plotted as functions of $D$ for $0 \leq D \leq 4$ [see (20)]. The approximants form a monotone increasing sequence of curve as indicated by the labeling. As in Fig. 3 for the four-point function, the approximants are smooth curves that seem to be tending uniformly to a limiting curve. This limiting curve (dotted curve) is a fifth-order Richardson extrapolation. The exact result $\gamma_6 = 1/4$ at $D = 1$ is indicated by a plus sign; the limiting curve passes within 4% of this point.
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\[ \gamma_4 \text{ at } D=2 \]

Graph showing \( n^{th} \) Richardson Extrapolant versus \( 1/k \), with markers for different values of \( n \): n=1, n=2, n=3, n=4, n=5.
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