UPPER BOUNDS FOR ROPELENGTH AS A FUNCTION OF CROSSING NUMBER

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ABSTRACT. The paper provides bounds for the ropelength of a link in terms of the crossing numbers of its prime components. As in earlier papers, the bounds grow with the square of the crossing number; however, the constant involved is a substantial improvement on previous results. The proof depends essentially on writing links in terms of their arc-presentations, and has as a key ingredient Bae and Park’s theorem that an $n$-crossing link has an arc-presentation with less than or equal to $n + 2$ arcs.

1. INTRODUCTION

The ropelength of a space curve is defined to be the quotient of its length by its thickness, where thickness is the radius of the largest embedded tubular neighborhood around the curve. For a knot or link type $L$, we define the ropelength $\text{Rop}(L)$ to be the minimum ropelength of all curves with the given link type. This minimum ropelength is a link invariant which measures the topological complexity of the link, much like crossing number, or bridge number, in classical knot theory.

It has been shown that every link type contains at least one $C^{1,1}$ tight representative which achieves this minimum ropelength [3, 7]. Much effort has been invested in the project of finding lower bounds for the ropelength of various link types in terms of classical topological invariants, such as the crossing number [2, 3, 10].

In this paper, we are interested in a converse problem: given a link type $L$ of crossing number $c(L)$, can we guarantee the existence of a representative curve with ropelength less than some function of $c(L)$? That is, can we find upper bounds on ropelength in terms of crossing number? Our main theorem states the following:

Theorem 1. If $L$ is a non-split link, then

$$\text{Rop}(L) \leq 1.64 \ c(L)^2 + 7.69 \ c(L) + 6.74.$$ 

In particular, this bound holds for prime links.

Our Theorem 2 gives similar bounds for composite links.

Other groups ([3, 9]) have attacked this problem by finding upper bounds on the number of edges required to embed a given link $L$ in the unit lattice (the lattice number $k(L)$ of the link), and then observing that $\text{Rop}(L) < 2k(L)$ [3]. Both proofs rely on laying out a diagram of the knot as a graph in a planar grid and then adding bridges to form overcrossings. In this context, it has been observed that constructing a particular diagram of a link with crossing number $c(L)$ may require ropelength $O(c(L)^2)$ [9]. These authors have obtained the weaker bounds $\text{Rop}(L) < 24 \ c(L)^2$ [3], and $\text{Rop}(L) < 25 \ c(L)^2$ [9]. Johnston’s algorithm, like ours, produces an explicit realization of the knot in space, while the approach of [3] is less constructive.

By contrast, our methods are more three-dimensional and are not based on grid or lattice embeddings. Instead of using a planar diagram of a knot, we base our construction on Peter Cromwell’s

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idea of arc-presentations [4]. It is curious that our methods, too, seem to be essentially of order $c(L)^2$. While we believe that bounds with a slower order of growth must be attainable, it is becoming clear that the problem of constructing such bounds is likely to be challenging.

2. **THE DEFINITION OF ROPELENGTH**

The ropelength of a curve is defined to be the quotient of length by the radius of the largest embedded tubular neighborhood around the curve. This radius is called the thickness of the curve. For $C^2$ curves, this radius is locally controlled by curvature and globally controlled by distances of self-approach between various regions of the curve. Formally, we write

**Definition 1.** The thickness of a $C^2$ curve $c$ is given by

$$\tau[c] := \min \left\{ \min_s \frac{1}{\kappa(s)}, \frac{\text{dcsd}(c)}{2} \right\},$$

where $\kappa(s)$ is the curvature of $c$ at $s$, and $\text{dcsd}(c)$ is the shortest doubly-critical self-distance of $c$; that is, the length of the shortest chord of $c$ which is perpendicular to the tangent vector $c'$ at both endpoints.

We can extend this definition to $C^{1,1}$ curves by adjusting our idea of the radius of curvature as follows (c.f. [3]):

**Definition 2.** Let $s$ be a point on a $C^{1,1}$ curve. Consider a decreasing sequence of open neighborhoods $U_n$ of $s$. The infimal radius of curvature at $s$ is given by

$$\inf_{U_n} \left\{ \inf_{t \in U_n} \frac{1}{\kappa(t)} \right\},$$

where the inner infimum is restricted to $t$ in $U_n$ such that $\kappa(t)$ exists.

Figure 1 shows examples of curves where thickness is controlled by curvature and by the doubly-critical self-distance.

![Figure 1](image)

**Figure 1.** These are two curves of unit thickness in the plane with their largest embedded tubular neighborhoods. In the left curve, thickness is controlled by curvature while in the right curve, thickness is controlled by the length of the doubly-critical chord shown.

Gonzalez and Maddocks have given another definition of thickness which looks somewhat less natural, but is often more useful. (See [8] for details). Another useful way to look at thickness comes from Federer’s notion of reach, which agrees with the thickness for curves [6].

**Definition 3.** The reach of a set $S$ inside $\mathbb{R}^n$ is the greatest non-negative $r$ so that each point within distance $r$ of $S$ has a unique nearest neighbor in $S$. 
3. ARC-PRESENTATIONS

We start with a definition:

**Definition 4.** An arc-presentation of a link $L$ is an embedding of $L$ in a finite collection of $\alpha$ open half-planes arrayed around a common axis, or binding, so that the intersection of $L$ with each half-plane is a single simple arc. The number of half-planes $\alpha$ is called the arc-index of the arc-presentation. The minimal arc-index over all arc-presentations of a link $L$ is an invariant of the link type.

By isotopy, we can arrange that $L$ intersects the axis only at the points $1, \ldots, \alpha$. We call these the *levels* of the arc-presentation. Such an arc-presentation is then specified by combinatorial data: a collection of $\alpha$ triples in the form $(x_i, y_i, \theta_i)$, where each denotes an arc from level $x_i$ to level $y_i$ on the half-plane at angle $\theta_i$ around the axis.

Figure 2 shows an arc-presentation for the trefoil and the corresponding set of triples.

![Combinatorial Data](image)

**Figure 2.** This figure shows an arc-presentation for a trefoil knot. The presentation has arc-index 5. To the right we see the combinatorial data which describes this arc-presentation: 5 triples in the form $(x_i, y_i, \theta_i)$, each indicating an arc from level $x_i$ to level $y_i$ on page $\theta_i$ of the “5-page book” shown on the left.

We will assemble our ropelength bounds from two ingredients. First, we define a notion of the total distance travelled by the arcs in an arc-presentation:

**Definition 5.** The total skip of an arc-presentation $A$, denoted $\text{Skip}(A)$, is

\[
\text{Skip}(A) = \sum_{i=1}^{\alpha} |x_i - y_i|.
\]

For a given arc-presentation we can construct a realization of the knot in space with ropelength bounded in terms of $\text{Skip}(A)$ and $\alpha$: 
Proposition 1. An arc-presentation $A$ composed of $\alpha$ half-planes can be realized with ropelength smaller than

\begin{equation}
\frac{2\alpha}{\tan(\pi/\alpha)} + (\pi - 2)\alpha + 2 \text{Skip}(A).
\end{equation}

For the arc-presentation of the trefoil in Figure 2, we have $\alpha = 5$ and $\text{Skip}(A) = 12$; so Proposition 1 yields an upper bound on the ropelength of the trefoil of about 43.47. Numerical experiments estimate the ropelength of the tight trefoil to be about 32.66 [11], so the slack in our estimate is about 33% of the total value. Figure 3 shows the tubular neighborhoods of this trefoil knot and an arc-presentation of the knot $7_1$ as realized by the algorithm of Proposition 1.

Figure 3. Here we see a trefoil knot (top left) and a $7_1$ knot (bottom left) together with the tubular neighborhoods around them constructed by Proposition 1. Our trefoil knot appears much tighter: its ropelength (43.47) is proportionally closer to the minimum ropelength for its knot type (32.66) than the ropelength of our $7_1$ knot (97.05) is to the minimum for its knot type (61.40 [11]).

Further, if we can bound $\text{Skip}(A)$ in general, we will be able to draw conclusions about the ropelength of an arbitrary link. A combinatorial argument yields:
Proposition 2. If an arc-presentation $A$ has arc-index $\alpha$, then

$$\text{Skip}(A) \leq \begin{cases} \frac{\alpha^2-1}{2} & \text{if } \alpha \text{ is odd,} \\ \frac{\alpha^2}{2} & \text{if } \alpha \text{ is even.} \end{cases}$$

This bound is sharp.

It is shown in [1] that any non-split link $L$ admits an arc-presentation with $\alpha \leq c(L) + 2$. This result, when coupled with the previous two propositions, gives Theorem 1. We obtain an even stronger statement for composite links:

Theorem 2. If $L$ is a non-split composite link with prime components $L_1, L_2, \ldots, L_n$, then

$$\text{Rop}(L) \leq 1.64 \sum_{i=1}^{n} c(L_i)^2 + 7.69 \sum_{i=1}^{n} c(L_i) + 6.74n.$$
and the total length of the fin is \( \pi - 2 + 2|x_i - y_i| \). Summing over \( i = 1, \ldots, \alpha \) and using Definition 5 proves the claim. \( \square \)

4.2. **The Binding Prism.** We denote the sections of the curve inside each floor of the binding prism by \( B_1, \ldots, B_\alpha \). Each \( B_i \) is a circular arc joining the midpoints of two edges of the regular polygon which is the cross-section of the prism as shown in Figure 4.

![Figure 4](image-url)

**Figure 4.** The sections of our curve \( B_i \) within the binding prism are circular arcs joining the midpoints of edges of the cross-section of the prism. The plane of this picture is located in the center of a floor of the prism.

Because the sides of the polygon have length 2, each of these is an arc of a circle of radius at least one; so each arc has curvature bounded above by one. Further, since each floor has height 2 and only one \( B_i \) lies in each floor, the tubes around each of the \( B_i \) are disjoint. Thus these \( B_i \) can be constructed with a tube of unit thickness.

**Claim 2.** If \( \text{Rop}(B_i) \) denotes the length of the segment of the curve \( B_i \), then

\[
\sum_{i=1}^{\alpha} \text{Rop}(B_i) \leq \frac{2\alpha}{\tan(\pi/\alpha)}.
\]

**Proof.** Each of these circular arcs is contained in a sector of the circle inscribed within the polygonal cross-section of the prism as shown in Figure 5. Since each arc is convex, its length is bounded above by the diameter of the inscribed circle. This diameter is exactly \( 2 \cot(\pi/\alpha) \). Summing over \( i = 1, \ldots, \alpha \) proves the claim. \( \square \)

Combining Claims 1 and 2 yields the statement of Proposition 1. \( \square \)

**Proof of Proposition 2.** Our job is to find an upper bound for \( \text{Skip}(A) = \sum_{i=1}^{\alpha} |x_i - y_i| \). We first observe that the difference \( |x_i - y_i| \) is one unit larger than the number of levels skipped over. For example, jumping from level 3 to level 6, a difference of 3 levels, skips the fourth and fifth levels. Thus, we can rewrite the sum

\[
\text{Skip}(A) = \alpha + \sum_{i=1}^{\alpha} \{\text{number of levels skipped by the arc } (x_i, y_i, \theta_i)\}.
\]
Figure 5. Each of the paths $B_i$ through the floors of the binding prism is a circular arc connecting two sides of the polygon which is that prism’s cross-section. Here we see that each of these arcs is contained within a sector of the circle inscribed within that polygon. Since each arc is a convex curve, this means that its length is bounded by the length of the two radii which bound the sector. That is, it is bounded by the diameter of the inscribed circle.

Notice that any level $j$ contributes to the above sum exactly when it is skipped over. We can rewrite our sum in terms of $j$ as

$$\text{Skip}(A) = \alpha + \sum_{j=1}^{\lfloor \alpha/2 \rfloor} \{ \text{number of times level } j \text{ is skipped} \}$$

$$= \alpha + \sum_{j=1}^{\lfloor \alpha/2 \rfloor} \{ \text{number of times level } j \text{ is skipped} \}$$

$$+ \sum_{j=0}^{\alpha - \lfloor \alpha/2 \rfloor - 1} \{ \text{number of times level } \alpha - j \text{ is skipped} \}, \quad (11)$$

where in the final equality we have split the second half of the sum off and let $j \mapsto \alpha - j$.

Now we bound the number of times level $j$ is skipped over. The only way to hop over $j$ from a higher level is to land on a lower level. There are $j - 1$ levels below the $j$th on which such a jump can land. Further, each of these levels can act as a launch pad for a jump back up which crosses the $j$th level again. This gives at most $2(j - 1)$ skips over level $j$. Similarly, the number of times we can skip over the $\alpha - j$th level is twice the number of levels above it, or $2j$.

For even $\alpha$, these estimates are sharp (as we will see below). However, when level $\alpha - j$ is the central level of an arc-presentation with $2k + 1$ levels ($j = k = \frac{\alpha - 1}{2}$), the situation is slightly different. Here all of the $j$ levels above the middle cannot be initial and terminal levels of arcs which skip level $\alpha - j$. For if so, then no arcs land on level $\alpha - j$, and we could have eliminated level $\alpha - j$ from the original arc-presentation. Thus level $\alpha - j$ is skipped at most $2j - 1 = \alpha - 2$ times.

Inserting these bounds into Equation (11), we apply the sum formulae for arithmatic progressions. When $\alpha$ is odd, we get

$$\text{Skip}(A) \leq \alpha + \sum_{j=1}^{\frac{\alpha - 1}{2}} 2(j - 1) + \sum_{j=0}^{\frac{\alpha - 3}{2}} 2j + (\alpha - 2) = \frac{\alpha^2 - 1}{2}. \quad (12)$$

If $\alpha$ is even, the proof is similar.
We now construct arc-presentations which show that these results are sharp. Consider the arc-presentation with even arc-index $\alpha = 2k$ described by the data
\[
(\alpha, \alpha/2, \theta_1), (\alpha/2, \alpha-1, \theta_2), (\alpha - 1, \alpha/2 - 1, \theta_3), (\alpha/2 - 1, \alpha - 2, \theta_4), \\
\ldots, (\alpha/2 + 1, 1, \theta_{2k-1}), (1, \alpha, \theta_{2k}).
\]
If we add up the lengths of the jumps, we get
\[
\text{Skip}(A) = \alpha^2/2.
\]
The same approach yields a realization of $A$ so that $\text{Skip}(A) = \frac{\alpha^2-1}{2}$ for odd $\alpha$.

**Proof of Theorem 1.** Taylor’s theorem gives the approximation $\frac{1}{\tan(x)} \leq 1/x - x/3$ for $x > 0$. Via Propositions 1 and 2 we gather that
\[
\text{Rop}(L) \leq \frac{2\alpha}{\tan(\pi/\alpha)} + (\pi - 2)\alpha + \alpha^2 \\
\leq (2/\pi + 1)\alpha^2 + (\pi - 2)\alpha - 2\pi/3.
\]
By Bae and Park [1], for any non-split link $L$ there exists an arc-presentation with $\alpha \leq c(L) + 2$. Inserting this into the above bound for ropelength yields
\[
\text{Rop}(L) \leq (2/\pi + 1) c(L)^2 + (8/\pi + 2 + \pi) c(L) + (8/\pi + 4\pi/3),
\]
and each of these constants evaluates to something smaller than the approximations given in the statement of the theorem. To gain the final remark in the theorem, we note that any prime link $L$ is non-split (otherwise it would consist of split components $L_1$ and $L_2$ and would admit the nontrivial factors $L_1$ and $L_2$ union a split unknot).

**Proof of Theorem 2.** The strategy for this proof is to arrange the prime components of our composite link so that we can make use of the bounds given by Theorem 1. So suppose that we have found arc-presentations with minimal arc-index for these components and embedded them as unit-thickness curves $L_1, \ldots, L_n$ according to the algorithm of Proposition 1.

We will now prove that for any links $L_1$ and $L_2$, constructed by the algorithm of Proposition 1, we can construct a curve $L_1 \# L_2$ with ropelength less than or equal to $\text{Rop}(L_1) + \text{Rop}(L_2)$. This is all that is required to complete the proof of our Theorem since the bound in the statement is just the sum of the bounds obtained for the $L_i$ by Theorem 1.

We begin by preparing $L_1$ and $L_2$. The top floor of $L_1$ contains only a single horizontal circular arc joining the centers of two sides of the binding prism. Since no fins jump over this level, we may rotate these quarter-circles to face one another and replace the horizontal circular arc with a horizontal line segment of shorter length without changing thickness or knot type. We do the same for the bottom floor of $L_2$. This procedure is shown in Figure 6.

We now arrange $L_1$ and $L_2$ in space so that the horizontal segments are colinear and share an endpoint. If we keep each oriented so that its floors are horizontal, the only overlap between the tubes surrounding each curve occurs on the shared floor. At the shared endpoint, we may delete two quarter-circles and replace them with a vertical line segment of length 2. We could keep track of this savings and get a slightly better constant term in the statement of Theorem 2. For each prime component we add, we save $\pi - 2$ in length.
Handling the other endpoints of the curve will prove to be a little more work. We may assume that both line segments lie along the x-axis with the shared endpoint at the origin. Suppose $L_2$’s segment has length $\ell_2$, while $L_1$’s segment has the smaller length $\ell_1$.

We now rotate the remaining vertical quarter-circle of $L_1$ to face the corresponding quarter-circle of $L_2$. If $\ell_1 \leq \ell_2 - 2$, we may replace both horizontal line segments with a single, shorter horizontal line segment joining the ends of these vertical quarter circles to obtain the desired curve. See Figure 7.

If $\ell_1 > \ell_2 - 2$, we cannot simply connect the endpoints of the quarter-circles after rotating the lower quarter-circle to face right. The resulting curve would have cusps on both ends. We solve this problem by finding a line tangent to both circles and following the composite path shown in Figure 8.

It is less obvious that these changes reduce length. To see that they do, we consider the diagonal line tangent to both circles shown in Figure 8. Since both circles are also tangent to a horizontal line, by symmetry this horizontal line cuts the diagonal line in half. Consider Figure 9. We need only show that half of the diagonal line (labelled $x$ in the Figure) is shorter than the portion of the quarter-circle it replaces (twice the angle $\theta$).

Since the lower quarter-circle has unit radius, this amounts to proving that $\tan \theta \leq 2\theta$ for $0 \leq \theta \leq \pi/4$. This is shown by a simple computation.
Figure 8. This figure shows the two extreme arcs of the components in the case where $\ell_1 > \ell_2 - 2$. When we rotate the lower quarter-circle to face right, it cannot be joined by a horizontal straight to the upper quarter-circle to create a $C^{1,1}$ curve; instead we find the diagonal line tangent to both quarter-circles and follow the composite path shown.

Figure 9. This detailed Figure enlarges the right-hand side of Figure 8. Consider the triangle with the following vertices: the point of tangency of the diagonal segment with the lower circle, the center of the lower circle, and the midpoint of the diagonal segment. The portion of the lower quarter-circle replaced by this half of the line segment has length $2\theta$ (again by symmetry). The length of this portion of the line segment is given by $x$.

Since the resulting curve remains $C^{1,1}$, is still of unit thickness, and has less length than the total length of the initial curves, this completes the proof.

An example of this construction is shown in Figure 10.

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Figure 10. Here we see the results of the construction of Theorem 2. Two mirror-image trefoil knots, generated by the method of Proposition 1 from the arc-presentation given in Figure 2, have been joined by the methods of Theorem 2 to obtain the composite knot $3_1 \# 3_1$.

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