In this note we prove that:

**Theorem 1.** for \(2 \leq s < \frac{n}{2}\) or \(1 \leq s < \frac{2n}{n+1}\) or \(1 \leq s < \frac{n}{2}\) but \(n\) is even,

\[
(-\Delta)^s(u) = |u|^{q-2}u, \quad q = \frac{2n}{n-2s}
\]

has infinitely many sign changing solutions or equivalently we can say that there exist solutions \(u_k\) such that \(\int u_k(-\Delta)^s(u_k)dx \to \infty\) as \(k \to \infty\)

A brief history of this problem: In 1979, Gidas, Ni and Nirenberg [4] classified all the positive solutions when \(s = 1\). In 1999, Wei and Xu [5] classified the positive solutions when \(s\) is integer. In 2004 and 2006, Li [9] and Chen, Li and Ou [6] classified the positive solutions for \(0 < s < \frac{n}{2}\). In 1986, Ding [1] proved the above theorem for case \(s = 1\), In 2004 Bartsch, Schneider and Weth [3] proved the cases when \(s\) is integer. The ideas used are same, both pull back the energy functional to sphere and then use symmetric criticality to obtain the critical points.

Proof of the theorem:

In the following, \(dx\) always denotes the volume form of \(\mathbb{R}^n\), \(d\xi\) denotes the volume form of sphere \(S^n\) with standard metric, \(c\) denotes some constant.

First notice that the solutions of the equation in the theorem are exactly the critical points of the functional \(J(u) = \frac{1}{2} \int u(-\Delta)^s u dx - \frac{1}{q} \int |u|^q dx\)

Step 1:

Use stereographic projection to lift \((-\Delta)^s\) to sphere

\[
\pi : \mathbb{R}^n \to S^n, \quad \pi(x) = \left(\frac{2x}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2}\right)
\]

By direct computation we have that the Jacobian is given:

\[
J_\pi = (\frac{2}{1+|x|^2})^n = cU^{\frac{2n}{n-2s}} = cU^q.
\]

Then the intertwining operator \(A_s\) on \(S^n\) ([8]) is defined:

\[
A_s(w) \circ \pi = cJ_\pi^{\frac{n+2s}{2n}} (-\Delta)^s(J_\pi^{\frac{n-2s}{2n}} (w \circ \pi)) = cU^{1-q}(-\Delta)^s(U(w \circ \pi))
\]

The eigenvalues of \(A_s\) are given by \(\lambda_l = \frac{\Gamma\left(\frac{n}{2}+l-s\right)}{\Gamma\left(\frac{n}{2}+l+s\right)}\), the corresponding eigenspaces are spanned by orthonormal spherical harmonics (same with the standard laplacian on \(S^n\)).
Consider the map:
\[ \Theta : H^s(R^n) \to H^s(S^n), \]
\[ \Theta(v) = (U^{-1}v) \circ \pi^{-1} \]
where \( H^s(R^n) \) is the completion of compact supported functions under norm \( \sqrt{((-\Delta)^s(\cdot), \cdot)} \) and \( H^s(S^n) \) is the completion of smooth functions under norm \( \sqrt{(A_s(\cdot), \cdot)} \)

Direct computations show that \( \Theta \) is an isometry and it also preserves the \( L^q \) norm so we have:
\[ \tilde{J}(\Theta(v)) := \frac{1}{2} \int \Theta(v)A_s(\Theta(v))d\xi - \frac{1}{q} \int |\Theta(v)|^q d\xi = J(v) = \frac{1}{2} \int v(-\Delta)^svdx - \frac{1}{q} \int |v|^q dx \]
Which means that the critical points of \( \tilde{J} \) and \( J \) are in 1-1 correspondence.

Step 2:

Now we study the critical points of \( \tilde{J} \).

Since \( \tilde{J} \) is invariant under conformal transform of \( S^n \), which means that it does not satisfy the Palais-Smale condition. However, applying as in [1] or [3] the symmetric mountain pass theorem [7] and the principle of symmetric criticality [2] we have the following.

**Lemma 1.** Let \( G \) be a compact subgroup of \( O(n+1) \) acting linearly and isometrically on \( H^s(S^n) \) such that

1. \( \tilde{J} \) is \( G \)-invariant;
2. the embedding \( H^s_G(S^n) \hookrightarrow L^q(S^n) \) is compact;
3. \( H^s_G(S^n) \) has infinite dimension.

Then \( \tilde{J} \) has a sequence of critical points \( w_k \), such that \( \int w_kA_s(w_k)d\xi \to \infty \) as \( k \to \infty \).

Here we denote by \( H^s_G(S^n) \) the subspace of \( H^s(S^n) \) consisting of \( G \)-invariant functions: \( H^s_G(S^n) = \{ w \in H^s(S^n) | w(g\xi) = w(\xi), \text{ every } g \in G \text{ and a.e. } \xi \in S^n \} \)

(1) and (3) in Lemma 1 are obvious, in the next step we will check (2) for some \( G \).
Step 3:

Now let $G = O(\lfloor \frac{n}{2} \rfloor) \times O(\lfloor \frac{n+1}{2} \rfloor)$ (here $\lfloor \frac{n}{2} \rfloor$ means the greatest integer less or equal than $\frac{n}{2}$), let $m < s < m + 1$, $m$ is an integer. It is easy to see that the minimum dimension of the orbit of $G$ is $d_G := \lfloor \frac{n}{2} \rfloor$.

Elementary computations show that $q = \frac{2n}{n-2s} < \frac{2(n-d_G)}{n-d_G-2m}$, apply Lemma 3.2 in [3] we see that the embedding $H^m_G(S^n) \hookrightarrow L^q(S^n)$ is compact.

Then from the formula of eigenvalues of $A_s$ it is easy to see that $\sqrt{(A_s(\cdot), \cdot)}$ is increasing as $s$ increases, which means that we have a continuous embedding $H^s_G(S^n) \hookrightarrow H^m_G(S^n)$.

So the embedding $H^s_G(S^n) \hookrightarrow L^q(S^n)$ is compact and it satisfies (3) in Lemma 1.

Finally, by applying Lemma 1 we prove Theorem 1.

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