PROPERLY DISCONTINUOUS GROUP ACTIONS ON AFFINE HOMOGENEOUS SPACES

George Tomanov

1. INTRODUCTION

Let $G$ be a real algebraic group, $H$ an algebraic subgroup of $G$, and $\Gamma$ a closed subgroup of $G$ acting on the homogeneous space $G/H$ by left translations. Given $x \in G/H$, $\Gamma_x$ is the stabilizer of $x$ in $\Gamma$. Recall that the action of $\Gamma$ is properly discontinuous (respectively, free) if for any compact $K \subset G/H$ the set \{ $g \in \Gamma \mid gK \cap K \neq \emptyset$ \} is finite (respectively, $\Gamma_x$ is trivial for all $x \in G/H$). If $\Gamma$ acts properly discontinuously and freely on $G/H$ then the manifold of double cosets $\Gamma \backslash G/H$ is called Clifford-Klein form. The following question is natural and well-known: Which homogeneous manifolds $G/H$ admit nontrivial (respectively, compact) Clifford-Klein forms $\Gamma \backslash G/H$? The question has been studied when the homogeneous spaces $G/H$ is of reductive type, that is, when both $G$ and $H$ are reductive groups (cf. [Be1-2] and [K1-3]). In the present paper we discuss some cases when $G/H$ is never of reductive type. First of all recall the notable Auslander conjecture (cf. [Au]):

Conjecture 1. Let $\Gamma$ be a subgroup of the group $\text{Aff}(\mathbb{A}^n)$ of all affine linear transformations of the $n$-dimensional real affine space $\mathbb{A}^n$. Assume that $\Gamma$ acts properly discontinuously on $\mathbb{A}^n$ and the quotient $\Gamma \backslash \mathbb{A}^n$ is compact. Then $\Gamma$ is a virtually solvable group, i.e. $\Gamma$ contains a solvable subgroup of finite index.

In other terms, the Auslander conjecture says that if $\Gamma$ acts properly discontinuously and co-compactly on $\mathbb{A}^n$ then the Levi subgroup of its Zariski closure in $\text{Aff}(\mathbb{A}^n)$ is trivial. (Recall that the maximal connected semisimple subgroups of $G$ are usually called Levi subgroups of $G$ and they are all conjugated.) As proved by G.A.Margulis in [Mar1] and [Mar2], the compactness of $\Gamma \backslash \mathbb{A}^n$ in the formulation of the conjecture is essential. (See Theorem 4.2(b) below.)

The continuous analog of the Auslander conjecture is the following result of T.Kobayashi and R.Lipsman:

Theorem 1.1. ([KI], [L]) Suppose that $H$ contains a Levi subgroup of $G$, $\Gamma$ is a connected algebraic subgroup of $G$ and $\Gamma_x$ is compact for all $x \in G/H$. Then $\Gamma$ is a compact extension of a unipotent group.

In the light of the above discussion the following generalization of Auslander’s conjecture is natural:
Conjecture 2. Suppose that $H$ contains a maximal reductive subgroup of $G$, $\Gamma$ acts properly discontinuously on $G/H$ and $\Gamma \backslash G/H$ is compact. Then $\Gamma$ is virtually solvable.

It is easy to see that Conjecture 2 implies Conjecture 1 (cf. Remark 1 in 2.1). Also note that $G/H$ is isomorphic (as a real algebraic variety) to $\mathbb{A}^n$ and $G$ acts on $\mathbb{A}^n$ by regular (polynomial) automorphisms of degree $\geq 1$. Conjecture 1 is exactly the case when this action is linear. Some known results about Auslander’s conjecture could be extended to Conjecture 2. In §2 and §3 of the present paper we prove the following

Theorem 1.2. Conjecture 2 is true if the Levi subgroup of $G$ is a product of simple real algebraic groups of ranks $\leq 1$.

The arguments used in the proof of Theorem 1.2 generalize our arguments in [To1] where the analogous result is proved for Auslander’s conjecture. Independently, K.Dekimpe and N.Petrosyan proved in a recent paper a similar to Theorem 1.2 result [D-P, Theorem A] and formulated relevant to Conjecture 2 questions [D-P, Questions 1 and 2]. At present, we can prove Conjecture 2 for $\dim G/H \leq 4$. (The proof will appear elsewhere.)

As to the Auslander conjecture, its proof is easy for $n = 2$ and due to D.Fried and W.Goldman [F-G] for $n = 3$. The proof of the conjecture for $n \leq 5$ was announced in [To2] with detailed sketch of the proof for $n = 4$. After the present paper was finished we became aware of the preprint [A-M-S5] where the conjecture is proved for $n \leq 6$. In §4 we give a full proof of Auslander’s conjecture for $n \leq 5$ (Theorem 4.5) along the lines in [To2]. Our proof for $n \leq 5$ is different and simpler than the proof in [A-M-S5], in particular, it uses less input. All one needs is the result of Margulis preprint [Mar3], published as part of [A-M-S3] (see Theorem 4.2(a) below), and [To1].

1.1. Notation and terminology. By an algebraic group (resp., algebraic variety) we will mean a real linear algebraic group (resp., real algebraic variety), that is, the set of all $\mathbb{R}$-rational points of a linear algebraic group (resp., algebraic variety) defined over $\mathbb{R}$. On every algebraic variety we have Hausdorff topology (induced by the topology on $\mathbb{R}$) and also Zariski topology. In order to distinguish the two topologies the topological notions connected with the Zariski topology will be usually used with the prefix ”Zariski”. (We say: Zariski closed, Zariski closure, Zariski connected, etc.) If $M$ is a subset of an algebraic variety $X$ then $\bar{M}$ denotes the Zariski closure of $M$ in $X$ and $\overline{M}$ denotes the closure of $M$ in $X$ for the usual (Hausdorff) topology. We denote by $G^0$ the connected component of $G$ for the Hausdorff topology and by $R(G)$ (respectively, $R_u(G)$) the radical (respectively, unipotent radical) of $G$. If $G$ acts on a set $X$ and $x \in X$, $G_x$ is the stabilizer of $x$ in $G$. Given $g \in G$, $g = g_s g_u$ is the Jordan
decomposition of $g$ where $g_s$ (resp. $g_u$) is the semi-simple (resp. unipotent) part of $g$. We let $<g>$ be the subgroup generated by $g$. Also, we will denote by Lie($G$) the Lie algebra of $G$. By rank of $G$ we mean the common dimension of the maximal $\mathbb{R}$-diagonalizable tori of $G$. Also, $D^iG$ is the $i$-th derived subgroup of $G$, that is, $D^0G = G$ and $D^{i+1}G = [D^iG, G]$ for all $i \geq 0$.

1.2. Basic affine geometry. A real affine space $\mathbb{A}^n$ is obtained from a real $n$-dimensional vector space $V$, called the direction of $\mathbb{A}^n$, by "forgetting" the origin. Most often $V = \mathbb{R}^n$. If $x$ and $y \in \mathbb{A}^n$ we denote by $x\ y$ the unique vector in $V$ such that $y = x + x\ y$. An affine automorphism $\gamma \in \text{Aff}(\mathbb{A}^n)$ determines a $\lambda(\gamma) \in \text{GL}(V)$, called the linear part of $\gamma$, such that for any pair $x, y \in \mathbb{A}^n$ we have $\gamma(x)\gamma(y) = \lambda(\gamma)(x\ y)$. The map $\lambda : \text{Aff}(\mathbb{A}^n) \to \text{GL}(V), \gamma \mapsto \lambda(\gamma)$, is a surjective group homomorphism. Fix an origin $p \in \mathbb{A}^n$. If $\vec{v} \in V$ then $\gamma(p + \vec{v}) = \gamma(p) + \lambda(\gamma)(\vec{v})$. So, every $\gamma \in \text{Aff}(\mathbb{A}^n)$ can be decomposed as the linear "vector" transformation $\mathbb{A}^n \to \mathbb{A}^n, p + \vec{v} \mapsto p + \lambda(\gamma)(\vec{v})$, followed by the translation by the vector $p\gamma(p)$. Consider the semidirect product of algebraic groups $V \rtimes \text{GL}(V)$ where the action of $\text{GL}(V)$ on $V$ is the natural one. The group $\text{Aff}(\mathbb{A}^n)$ is identified with $V \rtimes \text{GL}(V)$ via the group isomorphism $\text{Aff}(\mathbb{A}^n) \to V \rtimes \text{GL}(V), \gamma \mapsto (p\gamma(p), \lambda(\gamma))$. The structure of algebraic group on $\text{Aff}(\mathbb{A}^n)$ obtained in this way does not depend on the choice of $p$. All stabilizers $\text{Aff}(\mathbb{A}^n)_x, x \in \mathbb{A}^n$, are maximal reductive subgroups of $\text{Aff}(\mathbb{A}^n)$ isomorphic to $\text{GL}(V)$ and pairwise vector translation conjugate. Hence every reductive subgroup of $\text{Aff}(\mathbb{A}^n)$ (and, therefore, every semi-simple element in $\text{Aff}(\mathbb{A}^n)$) admits a fixed point. Further on, we will tacitely use this observation.

The group $\mathbb{R}^n \rtimes \text{GL}_n(\mathbb{R})$ is identified with its image in $\text{GL}_{n+1}(\mathbb{R})$ under the embedding $(\vec{v}, m) \mapsto \begin{pmatrix} m & \vec{v} \\ 0 & 1 \end{pmatrix}$, where the elements from $\mathbb{R}^n$ are vector columns. If $\vec{v}_1, \cdots, \vec{v}_n$ is a basis of $V$, then $\mathcal{F} = \{ p; \vec{v}_1, \cdots, \vec{v}_n \}$ is a frame of $\mathbb{A}^n$. We have an isomorphism $\text{Aff}(\mathbb{A}^n) \to \mathbb{R}^n \rtimes \text{GL}_n(\mathbb{R}), \gamma \mapsto \begin{pmatrix} l(\gamma) & \vec{v}(\gamma) \\ 0 & 1 \end{pmatrix}$, where $l(\gamma)$ is the matrix of $\lambda(\gamma)$ in the basis $\vec{v}_1, \cdots, \vec{v}_n$ and $\vec{v}(\gamma)$ is the vector-column of the coordinates of $p\gamma(p)$ in this basis, called the matrix representation of $\text{Aff}(\mathbb{A}^n)$ in the frame $\mathcal{F}$. Note that $l(\gamma)$ is the same in any translated frame $\mathcal{F} + \vec{v} = \{ p + \vec{v}; \vec{v}_1, \cdots, \vec{v}_n \}, \vec{v} \in V$.

2. Rational actions of $\Gamma$ on $\mathbb{A}^n$

2.1. Let $G$ and $\Gamma$ be as in the formulation of Conjecture 2. Replacing $\Gamma$ by a subgroup of finite index, we suppose from now on that $G$ is Zariski connected. Our main goal is to reduce the proof of Conjecture 2 to the case when $\Gamma \cap R(G) = \{ e \}$. 
Proposition 2.1. Assume that G is acting rationally on $\mathbb{A}^n$, the restriction of this action to $\Gamma$ is properly discontinuous, the quotient $\Gamma \backslash \mathbb{A}^n$ is compact, and there exists $x_o \in \mathbb{A}^n$ such that $Sx_o = x_o$ for a maximal reductive subgroup $S$ of $G$. Then $R_u(G)$ acts transitively on $\mathbb{A}^n$.

Proof. Note that $Gx_o = R_u(G)x_o$ and $R_u(G)x_o$ is closed and isomorphic as a real algebraic variety to an affine space $\mathbb{A}^k$ (see [Bi] and [Ro]). The group $\Gamma$ acts properly discontinuously and with compact quotient on $R_u(G)x_o$. In view of [Se],

$$vcd(\Gamma) = \dim R_u(G)x_o = \dim \mathbb{A}^n,$$

where $vcd(\Gamma)$ denotes the virtual cohomological dimension of $\Gamma$. So, $R_u(G)x_o = \mathbb{A}^n$, completing the proof. □

Remarks: 1. Let $\Gamma$ be as in the formulation of the Auslander conjecture, $G$ be the Zariski closure of $\Gamma$ in Aff($\mathbb{A}^n$) and $S$ be a maximal reductive subgroup of $G$. Let $x_o \in \mathbb{A}^n$ be fixed by $S$. Put $H = G_{x_o}$. In view of Proposition 2.1, $G$ acts transitively on $\mathbb{A}^n$. So, $\mathbb{A}^n$ can be identified with $G/H$ and, therefore, Conjecture 2 implies Conjecture 1.

2. The argument used in the proof of Proposition 2.1 is identical to that used in the proof of [To1] Lemma 1.1. As indicated to the author by the referee of [To1], in a different way, the result was first proved by W.Goldman and M.W.Hirsch [G-H Theorem 2.6]. [D-P] Lemma 2.5] corresponds to [To1 Lemma 1.1].

The following assertion is implicitly contained in the proof of [To1 Proposition 1.4].

Lemma 2.2. Let $\Delta \subset GL_n(\mathbb{R})$ be a discrete solvable subgroup. Then there exists a connected (for the Hausdorff topology on $GL_n(\mathbb{R})$) solvable subgroup $R \subset GL_n(\mathbb{R})$ such that $R \cap \Delta$ is a normal subgroup of finite index in $\Delta$ and $R/R \cap \Delta$ is compact.

Proof. The group $\Delta \cap \Delta^o$ is a normal subgroup of finite index in $\Delta$. Replacing $\Delta$ by $\Delta \cap \Delta^o$ it is enough to prove the existence of a connected subgroup $R$ such that $\Delta \subset R \subset \Delta^o$ and $R/\Delta$ is compact. Note that $D\Delta$ is a Zariski dense discrete subgroup in the connected unipotent group $D\Delta = D\Delta^o$. Therefore $\Delta \cap D\Delta$ is a co-compact lattice in $D\Delta$. This implies that $\Delta \cap D\Delta/D\Delta$ is a discrete subgroup of $\Delta^o/D\Delta$ (cf. [Rag Theorem 1.13]). Hence, it is enough to prove the lemma when $\Delta$ is abelian and $\Delta \subset \Delta^o$. The Lie group $\Delta^o$ is isomorphic to $K \times \mathbb{R}^m$ where $K$ is a compact torus. Let $\pi : \Delta^o \to \mathbb{R}^m$ be the natural projection and $R'$ be the linear span of $\pi(\Delta)$. Then $\Delta$ is co-compact in $R = \pi^{-1}(R')$. □
2.2. Let $G, H$ and $\Gamma$ be as in the formulation of Conjecture 2. Using Proposition 2.1, we see that the unipotent radical of the Zariski closure of $\Gamma$ in $G$ is acting transitively on $G/H$. So, replacing $G$ by the Zariski closure of $\Gamma$ we may (as we will) assume that $\Gamma$ is Zariski dense in $G$. Put $\Delta = \Gamma \cap R(G)$. Then $\tilde{\Delta}$ is a normal subgroup of $G$. Denote by $G_1$ the Zariski closure of $G/\tilde{\Delta}$, by $H_1$ the Zariski closure of $H \tilde{\Delta}/\tilde{\Delta}$ in $G_1$, and by $\Gamma_1$ the natural imbedding of $\Gamma/\Delta$ into $G_1$. Clearly, $\Gamma_1 \cap R(G_1) = \{e\}$.

The next proposition, which is the central one, allows to “eliminate” the solvable radical when dealing with the Auslander conjecture or with some of its generalizations. Actually, it coincides with [To1, Proposition 1.4(a)]. For reader’s convenience we provide a somewhat more detailed than in [Tol] proof of the proposition.

**Proposition 2.3.** With the above notation and assumptions, $\Gamma_1$ acts properly discontinuously on $G_1/H_1$ and $\Gamma_1 \backslash G_1/H_1$ is compact.

**Proof.** Since $\tilde{\Delta}$ is a normal subgroup in $G$, the action of $G$ on $G/H$ permutes the $\tilde{\Delta}$-orbits on $G/H$. So, we can identify the space of $\tilde{\Delta}$-orbits on $G/H$ with $G/H'$ where $H' = \tilde{\Delta}H$. Let $\tilde{\Delta}_u$ be the unipotent radical of $\tilde{\Delta}$ and $T$ be a maximal reductive subgroup of $\tilde{\Delta}$. Then $\tilde{\Delta}$ is equal to the semidirect product $\tilde{\Delta}_u \rtimes T$. Remark that $T$ is conjugated to a subgroup of $H$ and $\tilde{\Delta}_u$ is normal in $G$. Hence $H' = \tilde{\Delta}_uH$ and $H'$ is an algebraic subgroup of $G$.

Let $\phi : \tilde{\Delta} \to \tilde{\Delta}_u$ be the natural projection and $R$ be a connected subgroup of $\tilde{\Delta}$ such that $\Delta \cap R$ is a normal subgroup of finite index in $\Delta$ and $R/\Delta \cap R$ is compact (see Lemma 2.2). We will prove that $\phi(R) = \tilde{\Delta}_u$. Denote by $\tilde{\Delta}^*$ the Zariski connected component of $\tilde{\Delta}$. Then $\tilde{\Delta}_u$ is the unipotent radical of $\tilde{\Delta}^*$. Suppose that $\tilde{\Delta}^*$ is abelian. In this case the restriction of $\phi$ to $\tilde{\Delta}^*$ is a homomorphism of algebraic groups and $\phi(R)$ is connected and, therefore, algebraic subgroup of $\tilde{\Delta}_u$. Since $\phi(R)$ is Zariski dense in $\tilde{\Delta}_u$ we get that $\phi(R) = \tilde{\Delta}_u$. Now, let $\tilde{\Delta}^*$ be arbitrary. Since $R$ is connected, the commutator $D(R)$ is unipotent and $\phi(R)$ contains $D(R)$. It is enough to prove that $D(R) = D(\tilde{\Delta}^*)$. Indeed, if so, we may factorize by $D(R)$ and reduce the proof to the case when $\tilde{\Delta}^*$ is abelian. Let us prove that $D(R)$ contains $D(\tilde{\Delta}^*)$. (The inclusion $D(R) \subset D(\tilde{\Delta}^*)$ is obvious.) Since $R$ is Zariski dense in $\tilde{\Delta}^*$ and $D(R)$ is an algebraic subgroup of $\tilde{\Delta}^*$ we have that $D(R)$ is normal in $\tilde{\Delta}^*$. But $R/D(R)$ is Zariski dense in $\tilde{\Delta}^*/D(R)$. Therefore $\tilde{\Delta}^*/D(R)$ is abelian which implies that $D(R)$ contains $D(\tilde{\Delta}^*)$, as required.

In view of the above, if $x \in G/H$ then $\tilde{\Delta}x = \tilde{\Delta}_u x = Rx$ is closed and the quotient $\Delta \backslash \tilde{\Delta}x$ is compact. Since $\Delta$ acts trivially on $G/H'$, the natural action of $\Gamma$ on $G/H'$ induces an action of $\Gamma_1$ on $G/H'$. Let us prove that $\Gamma_1$ acts properly discontinuously on $G/H'$ and that $\Gamma_1 \backslash G/H'$ is compact. Indeed, let $\psi : G/H \to G/H'$ be the natural map, $K_o \subset G/H$ be a compact subset and
$K = \psi(K)$. Since $\psi$ is $\Gamma$-equivariant we have that $\Gamma_1 K = G/H'$ if $\Gamma K_o = G/H$, proving that $\Gamma_1 \backslash G/H'$ is compact. Let $\{\gamma_i' \mid i \in I\}$ be the set of all elements in $\Gamma_1$ such that $\gamma_i' K \cap K \neq \emptyset$. For each $i$ we fix a $\gamma_i \in \Gamma$ such that $\gamma_i' = \gamma_i \Delta$. Every fiber of $\psi$ is a $\Delta$-orbit and, by the above, an $L$-orbit. Therefore for every $i \in I$ there exist $a_i, b_i \in K_o$ and $l_i \in R$ such that $\gamma_i a_i = l_i b_i$. Fix a compact $C \subset R$ such that $R = \Delta C$ and write $l_i = \delta_i c_i$, where $\delta_i \in \Delta$ and $c_i \in C$. Then $(\delta_i^{-1} \gamma_i) a_i = c_i b_i$. But $\Gamma$ acts properly discontinuously on $G/H$. Therefore $\{\delta_i^{-1} \gamma_i \mid i \in I\}$ is finite, which implies that $\Gamma_1$ acts properly discontinuously on $G/H'$.

In order to complete the proof of the proposition it remains to notice that $G/H'$ and $G_1/H_1$ are both canonically homeomorphic to the affine variety $\text{R}_u(G)/\tilde{\Delta}_u \cdot \text{R}_u(H)$. □

We will use Proposition 2.3 together with the following:

**Proposition 2.4.** With the notation and assumptions of Conjecture 2, additionally assume that $\Gamma \cap R(G) = \{e\}$ and $\Gamma$ is Zariski dense in $G$. Denote by $L$ a Levi subgroup of $G$ and by $K$ a maximal compact subgroup of $L$. Then

$$\dim(G/H) \leq \dim(L/K).$$

**Proof.** Since $\Gamma$ acts properly discontinuously and with compact quotient on the affine space $G/H$, we have

$$\dim(G/H) = \text{vcd}(\Gamma).$$

On the other hand, the projection of $\Gamma$ into $G/R(G)$ is injective and the connected component of its closure in $G/R(G)$ is solvable by a result of Auslander (cf. [Rag, Theorem 8.24]). Therefore the projection of $\Gamma$ into $G/R(G)$ is discrete. This implies that $\Gamma$ acts properly discontinuously on the symmetric space of $L$. Therefore,

$$\text{vcd}(\Gamma) \leq L/K,$$

completing the proof. □

The following proposition is useful.

**Proposition 2.5.** With $G$ and $H$ as in the formulation of Conjecture 2, let $g \in G$. Let $U = \langle g_u \rangle$ where $g_u$ is the unipotent part of $g$. Then there exists $p \in G/H$ with the following properties:

(i) The orbit $U p$ is closed and $g$-invariant;
(ii) $g_s$ fixes $U p$ element-wise;
(iii) $\dim U p = 1$ if $g_u \neq e$ and $gp = p$ if $g_u = e$.

**Proof.** Since $H$ contains a maximal reductive subgroup of $G$ there exists a $\sigma \in G$ such that $g_s \in \sigma H \sigma^{-1}$. Let $p = \sigma H$. It follows from $g_s g_u = g_u g_s$ that $g_s$ fixes $U p$ element-wise and that $U p$ if $g$-invariant. It is well known (and easy to prove) that $\dim U = 1$ if $g_u \neq e$. Finally, $U p$ is closed as a unipotent orbit on an affine algebraic variety (cf. [Bi]). □
Remark: If \( g \in \text{Aff}(\mathbb{A}^n) \) and all eigenvalues of \( \lambda(g) \) are different from 1 and pairwise different then \( g = g_0 \) and according to (iii) (applied to \( G = \text{Aff}(\mathbb{A}^n) \) and \( H = \text{GL}_n(\mathbb{R}) \)) there exists a \( p \in \mathbb{A}^n \) such that \( gp = p \). This assertion is well-known and is easy to prove directly. Using it one proves easily that if \( \Gamma \) acts properly discontinuously on \( \mathbb{A}^2 \) then \( \Gamma \) is virtually solvable, in particular, the Auslander conjecture holds when \( n = 2 \).

Finally, let us also mention:

**Proposition 2.6.** Let \( G \) and \( H \) be as in the formulation of Conjecture 2 and \( S \) be a maximal reductive subgroup of \( G \) contained in \( H \). Then the action of \( S \) on \( G/H \) by left translations is linearizable.

**Proof.** Put \( U = \text{R}_u(G) \). Then \( U_1 = H \cap U \) is the unipotent radical of \( H \) and \( G/H \) is rationally isomorphic to \( U/U_1 \). Since \( U \) and \( U_1 \) are \( \text{Int}(S) \)-invariant there exists \( \text{Ad}(S) \)-invariant vector subspace \( W \subset \text{Lie}(U) \) such that \( \text{Lie}(U) = W \oplus \text{Lie}(U_1) \) and \( W = \exp W \) is a regular cross-section for \( U/U_1 \) (i.e. the map \( W \times U_1 \to U \), \( (x,y) \to xy \), is a regular isomorphism of real algebraic varieties), cf. [Bo-Spr, 9.13]. Since \( \exp \circ \text{Ad}(x) = \text{Int}(x) \circ \exp \) for any \( x \in S \), we have that the map \( W \to G/H \), \( w \to (\exp w)H \), is \( S \)-equivariant isomorphism of algebraic varieties. Therefore, the action of \( S \) on \( G/H \) is linearizable. \( \square \)

**Corollary 2.7.** Suppose that the action of \( S \) on \( G/H \) is irreducible. Then the action of \( G \) on \( G/H \) is linearizable, that is, there exists an isomorphism \( \varphi : G/H \to \mathbb{A}^n \) such that if \( g \in G \) and \( l_g : G/H \to G/H, xH \mapsto gxH \), then \( \varphi \circ l_g \circ \varphi^{-1} \in \text{Aff}(\mathbb{A}^n) \) for all \( g \).

**Proof.** We use the notation \( U \) and \( U_1 \) as in the proof of Proposition 2.6. Let \( \mathcal{N}_U(U_1) \) be the normalizer of \( U_1 \) in \( U \). We suppose that \( \dim G/H > 0 \). Then \( \mathcal{N}_U(U_1) \supsetneq U_1 \). Since \( \mathcal{N}_U(U_1) \) is \( S \)-invariant and the action of \( S \) on \( G/H \) is irreducible, \( U_1 \) is a normal subgroup of \( U \) and, therefore, of \( G \) too. Factorizing \( G \) and \( H \) by \( U_1 \) we reduce the proof to the case when \( U_1 = \{e\} \), i.e., when \( H = S \). Since \( \mathcal{D}(U) \cdot S \) is a proper subgroup of \( G \) and the action of \( S \) on \( G/H \) is irreducible, \( \mathcal{D}(U) = \{e\} \). Hence \( G \) is a semidirect product of \( S \) and the vector group \( U \) on which \( S \) acts linearly, implying the corollary. \( \square \)

Corollary 2.7 shows that Conjectures 1 and 2 coincide for irreducible actions of \( S \) on \( G/H \).

3. Proof of Theorem 1.2

3.1. Some representation theory. Let \( G, H, \Gamma \) be as in the formulation of Theorem 1.2. Let \( L \) be a Levi subgroup of \( H \). Then \( L \) is an almost direct product of simple algebraic groups \( L_1, \cdots, L_r \) each of rank \( \leq 1 \).

Using Proposition 2.3 we reduce the proof of the theorem to the case when \( \Gamma \cap R(G) \) is trivial and \( \Gamma \) is Zariski dense in \( G \). Moreover, by a theorem of Selberg (see [S]) \( \Gamma \) contains a torsion free subgroup of finite index. Hence, we
may (and will) suppose that $\Gamma$ is torsion free. In view of Proposition 2.4 the relation (1) holds. We will denote by $V$ the tangent space of $G/H$ at the origin and by $\rho$ the representation of $L$ on $V$ induced by the action of $L$ on $G/H$ by left translations (Proposition 2.6). Since the kernel of the action of $G$ on $G/H$ is a normal algebraic subgroup $N$ of $G$ contained in $H$, factorizing $G$ and $H$ by $N$ we may (and will) suppose that $G$ acts faithfully on $G/H$. In this case the representation $\rho$ is also faithful.

The following proposition is an improved version of [To1, 2.5].

**Proposition 3.1.** With the above notation and assumptions, $L = S_1 \times S_2 \times \ldots \times S_m$, where $S_i = SL_2(\mathbb{R})$ or $S_i = SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$, and $V = V_1 \bigoplus V_2 \bigoplus \ldots \bigoplus V_m$ where each $V_i$ is an $L$-module such that each $S_j, j \neq i$, acts trivially on $V_i$, $V_i$ is the standard representation of $SL_2(\mathbb{R})$ if $S_i = SL_2(\mathbb{R})$, and $V_i$ is the tensor product of two standard representations of $SL_2(\mathbb{R})$ if $S_i = SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$.

The next lemma is derived from [H, Table 5, p.518].

**Lemma 3.2.** Let $Q$ be a simple real algebraic group and $\text{rank}_Q Q \leq 1$. Let $d$ be the dimension of the minimal nontrivial representation of $Q$ and $s$ be the dimension of the symmetric space of $Q$. Then $d \geq s$ and $d = s$ if and only if $Q$ is isomorphic to $SL_2(\mathbb{R})$ and $d = s = 2$.

**Proof of Proposition 3.1.** Let $V = V_1 \bigoplus V_2 \bigoplus \ldots \bigoplus V_m$ be a direct sum of irreducible $L$-submodules. Each $V_i$ is a tensor product of irreducible nontrivial $L_{ij}$-modules $V_{ij}$ (i.e. $V_i = \bigotimes_{1 \leq j \leq r_i} V_{ij}, r_i \in \mathbb{N}$), where $L_{ij} \in \{L_1, L_2, \ldots, L_r\}$. Put $n = \dim V$ and $n_{ij} = \dim V_{ij}$. Then

$$n = \sum_{1 \leq i \leq m} \left( \prod_{1 \leq j \leq r_i} n_{ij} \right). \quad (2)$$

For every $L_i$, we let $d_i$ (respectively, $s_i$) be the dimension of the minimal nontrivial real representation of $L_i$ (respectively, the dimension of the symmetric space of $L_i$). Remark that $s_1 + s_2 + \ldots + s_r$ is the dimension of the symmetric space of $L$. Using (1), (2) and the faithfulness of $\rho$, we get

$$d_1 + d_2 + \ldots + d_r \leq n \leq s_1 + s_2 + \ldots + s_r. \quad (3)$$

According to Lemma 3.2 $d_i \geq s_i$ for all $i$. It follows from (3) that $d_i = s_i = 2$ and $L_i = SL_2(\mathbb{R})$ for all $i$. In particular,

$$n = 2r. \quad (4)$$

Since $\sum_{1 \leq i \leq m} r_i \geq r$ and $n_{ij} \geq 2$ for all $i$ and $j$, it follows from (2) and (4) that $\sum_{1 \leq i \leq m} r_i = r, 1 \leq r_i \leq 2$ and all $n_{ij} = 2$ (i.e. each $V_{ij}$ is a standard $SL_2(\mathbb{R})$-module). Moreover, we see that $V$ is a faithful representation of $L_1 \times \cdots \times L_r$. This implies that $L = S_1 \times S_2 \times \ldots \times S_m$ as in the formulation of the proposition. \[\square\]
3.2. End of the proof. Let $\rho$ be as in the formulation of Proposition 3.1. The isomorphism $DG/R_u(DG) \cong L$ gives a natural surjective homomorphism $\pi : DG \to L$. Put $\phi = \rho \circ \pi$. Let $\gamma \in DG$. By Proposition 2.5 $\gamma_s$ fixes element-wise a smooth curve on $G/H$. There exists $g \in DG$ such that $g\gamma_s g^{-1} \in L$. Hence $g\gamma_s g^{-1}$ fixes element-wise a smooth curve on $G/H$ passing through the origin. So, 1 is an eigenvalue of $\phi(g\gamma_s g^{-1})$ and, therefore, of $\phi(\gamma_s)$ and $\phi(\gamma)$ too. But $\phi(\Gamma)$ is Zariski dense in $L$. Therefore 1 is an eigenvalue of $\rho(s)$ for every $s \in L$. In view of Proposition 3.1 $L$ is trivial, that is, $\Gamma$ is solvable. □

4. On Auslander’s conjecture

4.1. Some known results. First we formulate a general result which is often useful in tackling Auslander’s conjecture. So, let $S$ be a real, connected, non-compact, and semi-simple algebraic group. An element $g \in S$ is said to be $\mathbb{R}$-regular if the number of eigenvalues having modulus 1 (counted with multiplicity) of $Ad(g)$ is minimal possible. Note that every $\mathbb{R}$-regular element in a semi-simple (or reductive) group is semi-simple. It is known (and can be checked by direct computation) that if $S = SL_n(\mathbb{R})$ or $Sp_{2n}(\mathbb{R}), n \geq 2$, (the cases arising in section 4.2) then $g \in S$ is $\mathbb{R}$-regular if and only if all its eigenvalues are real and their moduli are distinct. The following theorem is proved by different methods in [Be-L], [P] and [A-M-S1]. (Concerning to its second part, we refer to [P, Remark, p.545].)

**Theorem 4.1.** Any Zariski dense sub-semigroup $\Delta$ of $S$ contains an $\mathbb{R}$-regular element. Moreover, the set of $\mathbb{R}$-regular elements in $\Gamma$ is dense in $G$ in the Zariski topology.

Note that if $S = SL_n(\mathbb{R})$ or $Sp_{2n}(\mathbb{R}), n \geq 2$, then the set of elements in $S$ with all eigenvalues different from 1 is Zariski open and non-empty which implies that the set of $\mathbb{R}$-regular elements in $\Delta$ with all eigenvalues different from 1 is Zariski dense in $G$. We will use this assertion in the course of our proof in 4.2 of the Auslander conjecture for $n \leq 5$.

Further on, we denote by $SO_{p,q}(\mathbb{R})$ the special orthogonal group of a quadratic form on $\mathbb{R}^n$ of signature $(p,q)$, $n = p + q$, and by $Sp_{2n}(\mathbb{R})$ the symplectic subgroup of $SL_{2n}(\mathbb{R})$. If $n = p$ we use the standard notation $SO_n(\mathbb{R})$ instead of $SO_{n,0}(\mathbb{R})$.

Now, let $\lambda : Aff(A^n) \to GL_n(\mathbb{R})$ be the natural projection (see 1.2) and $H$ be an algebraic subgroup of $GL_n(\mathbb{R})$. A subgroup $\Gamma \subset Aff(A^n)$ is called $H$-linear if $\lambda(\Gamma) \subset H$. If $H = SO_n(\mathbb{R})$, i.e. if $\Gamma$ consists of Euclidean transformations of $A^n$, the Auslander conjecture follows from the classical Bieberbach theorem. Goldman and Kamishima (see [G-K]) proved the conjecture for Lorentz space.

---

Note that Theorem 4.1 is not indispensable for the proof of Auslander’s conjecture for $n \leq 5$. It could be replaced by a weaker claim in the spirit of [Ti, Lemmas 2.1-2.5] but this would make the proof less natural and somewhat more complicate.
forms, i.e. for $H = \text{SO}_{n-1,1}(\mathbb{R})$, and Grunewald and Margulis proved it when $H$ is a reductive group of real rank $\leq 1$ (see [Gr-Mar]).

Recall the following results of Abels, Margulis and Soifer.

**Theorem 4.2.** ([A-M-S3 Theorems A and B]) Suppose that $n = 2k + 1 \geq 3$. Then the following holds:

(a) if $k$ is even there is no $\Gamma$ acting properly discontinuously on $\mathbb{A}^n$ with $\lambda(\Gamma)$ Zariski dense in $\text{SO}_{k+1}(\mathbb{R})$;
(b) if $k$ is odd there are free groups $\Gamma$ acting properly discontinuously on $\mathbb{A}^n$ with $\lambda(\Gamma)$ Zariski dense in $\text{SO}_{k+1}(\mathbb{R})$.

Theorem 4.2(a) was proved in the 1991 Margulis’ preprint [Mar3]. Theorem 4.2(b) is a generalization of Margulis’ results [Mar1] and [Mar2] where Theorem 4.2(b) is proved for $\text{SO}_{2,1}(\mathbb{R})$ disproving a conjecture of J.Milnor [Mi]. Different aspects of [Mar1] and [Mar2] were developed in [Dr1], [Dr2] and [Dr-G].

The results from [A-M-S3] are sharpened by the following:

**Theorem 4.3.** ([A-M-S3 Theorems A,B and C]) Suppose that $\lambda(\Gamma) \subset \text{SO}_{p,q}(\mathbb{R})$, $n = p + q$. Denote by $H$ the Zariski closure of $\lambda(\Gamma)$ in $\text{GL}_n(\mathbb{R})$. Then:

(a) $\Gamma$ can not act properly discontinuously on $\mathbb{A}^n$ if $p - q \geq 2$ and $H \supset \text{SO}_{p,q}(\mathbb{R})$;
(b) $\Gamma$ can not act properly discontinuously on $\mathbb{A}^n$ if $q$ is even and the homogeneous space $\text{SO}_{p,q}(\mathbb{R})/H$ is compact;
(c) $\Gamma$ is virtually solvable if $q = 2$ and $\Gamma \backslash \mathbb{A}^n$ is compact.

4.2. **Proof of Auslander’s conjecture for** $n \leq 5$. From now on $\Gamma$ is a subgroup of $\text{Aff}(\mathbb{A}^n)$, $n \leq 5$, and $G$ is its Zariski closure in $\text{Aff}(\mathbb{A}^n)$. We will prove the following:

**Proposition 4.4.** If $G$ contains a simple algebraic group of rank $\geq 2$ then $\Gamma$ does not act properly discontinuously on $\mathbb{A}^n$.

In view of Theorem 1.2 (or [To1]), Proposition 4.4 implies immediately:

**Theorem 4.5.** The Auslander conjecture is true for $n \leq 5$.

4.2.1. We will use the notation and the terminology of section 1.2. In order to prove Proposition 4.4 we need a particular case of the following general

**Proposition 4.6.** (cf. [To2 Lemma 4.2]) Let $H$ be a Zariski connected algebraic subgroup of $\text{Aff}(\mathbb{A}^m)$, $S$ be a maximal reductive subgroup of $H$ and $L$ be the Levi subgroup of $S$. Then there exists a decomposition $V = W_1 \bigoplus \cdots \bigoplus W_k$, where $W_i$ are irreducible $\lambda(S)$-modules, such that for every $i \geq 1$ the sum $W_1 \bigoplus \cdots \bigoplus W_i$ is $\lambda(H)$-invariant. Hence, for every $p \in \mathbb{A}^m$ there exists a
frame with origin \( p \) in which the matrix representation of \( H \) is of the form:

\[
\begin{pmatrix}
\rho_1 & * & * \\
\rho_2 & & * \\
& \ddots & \ddots \\
0 & \ddots & \ddots \\
& & \rho_k \\
0 & 0 & \cdots & 0 & 1
\end{pmatrix},
\]

where \( \rho_i, \, i = 1, \ldots, k, \) are irreducible matrix representations of both \( \lambda(H) \) and \( \lambda(S) \). Moreover, if some restriction \( \rho_i|_{\lambda(L)} \) is not irreducible then \( \rho_i|_{\lambda(L)} = \sigma_i \oplus \sigma_i \) where \( \sigma_i \) is an irreducible representation of \( \lambda(L) \).

**Proof.** The existence of the decomposition \( V = W_1 \oplus \cdots \oplus W_k \) as in the formulation of the proposition will be proved by induction on \( \text{dim} \, V \). Denote by \( U \) the unipotent radical of \( \lambda(H) \). Let \( W_1 \) be an irreducible \( \lambda(H) \)-submodule of \( V \). Since \( U \) is a unipotent group there exists a \( U \)-invariant vector \( \overrightarrow{v} \in W_1 \setminus \{0\} \).

For every \( g \in \lambda(S), \, g \overrightarrow{v} \) is also \( U \)-invariant. Since \( \lambda(H) = \lambda(S) \rtimes U \), \( W_1 \) consists of \( U \)-invariant vectors and, therefore, is an irreducible \( \lambda(S) \)-module. Suppose that \( W_1, \ldots, W_i \) are irreducible nontrivial \( \lambda(S) \)-submodules of \( V \) such that \( W_1 + \cdots + W_i = W_1 \oplus \cdots \oplus W_i \subseteq V \) and \( W_1 + \cdots + W_j \) is \( \lambda(H) \)-invariant for every \( 1 \leq j \leq i \). Let \( W_{i+1}' \) be a \( \lambda(H) \)-submodule of \( V \) containing \( W_1 + \cdots + W_i \) and such that \( W_{i+1}'/W_1 + \cdots + W_i \) is a non-trivial irreducible \( \lambda(H) \)-submodule of \( V/W_1 + \cdots + W_i \). By the complete reducibility of the action of \( \lambda(S) \) on \( W_{i+1}' \) there exists an irreducible \( \lambda(S) \)-submodule \( W_{i+1} \) of \( W_{i+1}' \) such that \( W_{i+1} = W_1 \oplus \cdots \oplus W_{i+1} \), completing the proof of the existence of the decomposition.

Now, suppose that \( \rho : S \to \text{GL}(W) \) is an irreducible representation but \( \rho|_{L} \) is not. Let \( Z \) be the center of \( S \). If \( W' \) is an irreducible \( Z \)-module then \( \text{dim} \, W' = 2 \) and \( W \) is a direct sum of translations of \( W' \) by elements from \( S \).

This implies that if \( W'' \) is an irreducible \( L \)-module then there exists a \( c \in Z \) such that \( W = W'' \oplus cW'' \), proving the last assertion of the proposition. \( \square \)

**Remark.** It is worth mentioning that \( \lambda(S) \) is isomorphic to \( S \) and the representations \( \{ \rho_i|_{\lambda(S)} : i = 1, \ldots, k \} \), do not depend (up to isomorphism) on the choice of \( S \).

4.2.2. **Proof of Proposition 4.4.** We divide the proof in two steps: the first being mostly routine and the second containing the main ingredients of the proof.

**Step 1.** Replacing \( \Gamma \) by a subgroup of finite index we may (and will) assume that the algebraic group \( G \) is Zariski connected. We will apply Proposition 4.6 with \( H = G \). We fix a frame \( \{ \rho_i, \overrightarrow{v}_i, \cdots, \overrightarrow{v}_n \} \) of \( \mathbb{A}^n, \, n \leq 5 \), such that the matrix representation of \( G \) in this frame is as given by Proposition 4.6 and we keep the notation \( S, \, L \) and \( \rho_i, \, i = 1, \cdots, k \), from its formulation. Since \( \lambda|_S \) is injective, when this does not lead to confusion, we also write \( \rho_i \) instead of \( \rho_i \circ \lambda \).
There exists l such that $D^lG = D^{l+1}G$. Since $D^l\Gamma$ is Zariski dense in $D^lG$ and $D^lG$ contains a simple algebraic group of rank $\geq 2$ (because $G$ does), replacing $\Gamma$ by $D^l\Gamma$, we reduce the proof of Proposition 4.4 to the case when $G = D^lG$. In this case $L = S$. In view of the classification in [Bou, Table 2] of the dimensions of the irreducible representations of the simple algebraic groups, if $H$ is a simple algebraic group of rank $d > 0$ and $\rho : H \to \text{SL}(W)$ is its non-trivial representation then $\dim W \geq d + 1$ and $\dim W = d + 1$ if and only if $\rho$ is an isomorphism. It follows from Proposition 4.6 and the assumption that $L$ contains a simple group of rank $\geq 2$ that: (a) if $n = 3$ then $L = \text{SL}_3(\mathbb{R})$, and (b) if $L$ is not simple then $n = 5$ and $L = \text{SL}_3(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$. Using Theorem 4.4 we get that in both cases (a) and (b) there exists an element $\gamma \in \Gamma$ of infinite order such that all eigenvalues of $\lambda(\gamma)$ are different from 1 and pairwise different. According to Proposition 2.5(iii), $\gamma$ admits a fixed point proving that $\Gamma$ does not act properly discontinuously on $\mathbb{A}^n$.

So, it remains to consider the case when $4 \leq n \leq 5$ and $L$ is a simple algebraic group of rank $\geq 2$. It is enough to prove that if Proposition 4.4 is valid for $n = 1$ it is also valid for $n$. Assume to the contrary that $\Gamma$ acts properly discontinuously on $\mathbb{A}^n$ and Proposition 4.4 is true for $n - 1$. If one of the representations $\rho_i, 1 \leq i \leq k$, as in the formulation of Proposition 4.6, is not trivial then $\dim \rho_i \geq 3$ and, since $n \leq 5$, all remaining representations $\rho_j, j \neq i$, are trivial. Also, if $\rho_k$ is trivial it follows from $G = D^lG$ that $\Gamma$ acts properly discontinuously of the hyperplane $p + \mathbb{R}v_1 + \cdots + \mathbb{R}v_{n-1}$ contradicting the assumption that Proposition 4.4 is true for $n - 1$. Hence $\rho_k$ is a non-trivial representation, $3 \leq \dim \rho_k \leq 5$, and all $\rho_i, 1 \leq i \leq k - 1$, are trivial. Moreover, it follows from the remark after Proposition 2.5 that $\lambda(L)$ should not contain an element with pairwise different eigenvalues different from 1. Now, let $k = 1$. In view of [Bou, Table 2], $L = \text{SO}_{3,2}(\mathbb{R})$, $n = 5$ and $\rho_{1|L}$ is the standard representation. By Margulis’ Theorem 4.2(a), in this case the action of $\Gamma$ on $\mathbb{A}^5$ is not properly discontinuous\footnote{Formally, [Bou, Table 2] concerns the representations of the algebraic groups over $\mathbb{C}$ but since the absolute rank (over $\mathbb{C}$) of an algebraic group over $\mathbb{R}$ is greater than or equal to its real rank the assertion remains true for real algebraic groups as in the context of this paper.}.

Therefore $k > 1$ and $3 \leq \dim \rho_k \leq 4$. Since $\text{rank}(L) \geq 2$, [Bou, Table 2] implies that $\rho_k$ is one of the standard representations of $\text{SL}_3(\mathbb{R})$, $\text{SL}_4(\mathbb{R})$ or $\text{Sp}_4(\mathbb{R})$. So, in order to complete the proof it remains to consider the possibilities (i), (ii), (iii) and (iv) below. (In the formulations of (i) – (iv) we use the notation: $M_{i,j}(\mathbb{R})$ is the set of all real matrices with $i$ rows and $j$ columns, $0_{i,j}$ is the zero in $M_{i,j}(\mathbb{R})$ and $I_i$ is the unit matrix in $M_{i,i}(\mathbb{R})$.)

(i) $\lambda(G) \subseteq \left\{ \begin{pmatrix} I_1 & m_{1,4} \\ 0_{4,1} & g \end{pmatrix} \middle| g \in L, m_{1,4} \in M_{1,4}(\mathbb{R}) \right\}$ and $L = \text{SL}_4(\mathbb{R})$;

(ii) $\lambda(G) \subseteq \left\{ \begin{pmatrix} I_1 & m_{1,4} \\ 0_{4,1} & g \end{pmatrix} \middle| g \in L, m_{1,4} \in M_{1,4}(\mathbb{R}) \right\}$ and $L = \text{Sp}_4(\mathbb{R})$;

\footnote{The treatment of this case in [To2] contains an error.}
Choose \( L \) for any choice of \( L \) its eigenvalues are different from 1. (Note that this property of \( \gamma \) remains valid for any choice of \( L \) in Step 1.) Let \( \gamma = \gamma_s \gamma_u \) be the Jordan decomposition of \( \gamma \). Choose \( L \) such that \( \gamma_s \in L \). Fix a \( p_0 \in \mathbb{A}^n \) with \( L p_0 = p_0 \).

We have \( V = W_o \oplus W \) where \( W_o = \{ \overrightarrow{v} \in V | \gamma_s(\overrightarrow{v}) = \overrightarrow{v} \} \) \( \{ \overrightarrow{v} \in V | \gamma_s(\overrightarrow{v}) = \overrightarrow{v} \} \) \( \{ \overrightarrow{v} \in V | \gamma_s(\overrightarrow{v}) = \overrightarrow{v} \} \) \( \{ \overrightarrow{v} \in V | \gamma_s(\overrightarrow{v}) = \overrightarrow{v} \} \) \( \{ \overrightarrow{v} \in V | \gamma_s(\overrightarrow{v}) = \overrightarrow{v} \} \) \( \{ \overrightarrow{v} \in V | \gamma_s(\overrightarrow{v}) = \overrightarrow{v} \} \) \( \{ \overrightarrow{v} \in V | \gamma_s(\overrightarrow{v}) = \overrightarrow{v} \} \) and \( W \) is \( \lambda(\gamma_s) \)-invariant. Using \( \lambda(\gamma_s) \lambda(\gamma_u) = \lambda(\gamma_u) \lambda(\gamma_s) \) and the choice of \( \gamma \), one proves easily that \( \gamma_u \) is a translation belonging to the center of \( G \). Indeed, let \( \lambda(\gamma) = \begin{pmatrix} I_i & m_{i,j} \\ 0_{j,i} & g \end{pmatrix} \) and \( g = g_s g_u \) be the Jordan decomposition of \( g \). By the choice of \( L \) and Proposition 4.6, we have \( \lambda(\gamma_s) = \begin{pmatrix} I_i & 0_{j,i} \\ 0_{j,i} & g_s \end{pmatrix} \) and \( \lambda(\gamma_u) = \begin{pmatrix} I_i & m_{i,j} \\ 0_{j,i} & g_u \end{pmatrix} \). Since the eigenvalues of \( g_s \) are pairwise different we get that \( g_u = I_j \) and since they are all different from 1 we get that \( m_{i,j} = 0 \). Therefore, \( \lambda(\gamma_u) = I_d \), equivalently, \( \gamma_u \) is a translation by a vector \( \overrightarrow{v}_\gamma \). We have \( \gamma_s \gamma_u p_o = p_o + \lambda(\gamma_s)(\overrightarrow{v}_\gamma) = \gamma_u \gamma_s p_o = p_o + \overrightarrow{v}_\gamma \).

Hence \( \overrightarrow{v}_\gamma \in W_o \). Let \( h \in G \). Then \( \lambda(h) \overrightarrow{v}_\gamma = \overrightarrow{w}_\gamma \). So, if \( x \in \mathbb{A}^n \) then \( h \gamma_u(x) = h(x + \overrightarrow{v}_\gamma) = h(x) + \overrightarrow{w}_\gamma = \gamma_u h(x) \), proving that \( \gamma_u \) belongs to the center of \( G \).

Put \( E^o(\gamma) = \{ p \in \mathbb{A}^n | \gamma_s p = p \} \).

Then \( E^o(\gamma) = p_o + W_o \). Denote \( V^+(\gamma) = \{ \overrightarrow{v} \in V | \lim_{n \to -\infty} \gamma_s^n(p_o + \overrightarrow{v}) = p_o \} \) \( V^+(\gamma) = \{ \overrightarrow{v} \in V | \lim_{n \to -\infty} \gamma_s^n(p_o + \overrightarrow{v}) = p_o \} \) \( V^+(\gamma) = \{ \overrightarrow{v} \in V | \lim_{n \to -\infty} \gamma_s^n(p_o + \overrightarrow{v}) = p_o \} \) \( V^+(\gamma) = \{ \overrightarrow{v} \in V | \lim_{n \to -\infty} \gamma_s^n(p_o + \overrightarrow{v}) = p_o \} \) \( V^+(\gamma) = \{ \overrightarrow{v} \in V | \lim_{n \to -\infty} \gamma_s^n(p_o + \overrightarrow{v}) = p_o \} \) and \( V^-(\gamma) = \{ \overrightarrow{v} \in V | \lim_{n \to +\infty} \gamma_s^n(p_o + \overrightarrow{v}) = p_o \} \).

Let \( E^+(\gamma) = E^o(\gamma) + V^+(\gamma) \) and \( E^-(\gamma) = E^o(\gamma) + V^-(\gamma) \). Since \( E^\pm(\gamma) = E^\pm(\gamma^{-1}) \), we may (and will) assume that \( \dim E^+(\gamma) \geq \dim E^-(\gamma) \).

Let \( \delta \in \Gamma \). Since \( \gamma_u \) is central, we have \( \gamma_u = (\delta \gamma \delta^{-1}) \). Also, \( E^o(\delta \gamma \delta^{-1}) = \delta E^o(\gamma) \) and \( E^\pm(\delta \gamma \delta^{-1}) = \delta E^\pm(\gamma) \).
Remark that $E^o(\gamma) \cap E^o(\delta \gamma \delta^{-1})$ are parallel subspaces of $\mathbb{A}^n$ directed by $W_\gamma$.

Suppose that $E^o(\gamma) = \delta E^o(\gamma)$ for all $\delta \in \Gamma$. Recall that every subgroup of Aff($\mathbb{A}^m$), $m \leq 2$, acting properly discontinuously on $\mathbb{A}^m$ is virtually solvable. Since $G$ is connected, $G = DG$ and $\dim E^o(\gamma) \leq 2$, we get that the action of $\Gamma$ on $E^o(\gamma)$ is trivial which contradicts the assumption that $\Gamma$ acts properly discontinuously on $\mathbb{A}^n$. Therefore there exists $\delta \in \Gamma$ such that

(5) \[ E^o(\gamma) \cap E^o(\delta \gamma \delta^{-1}) = \emptyset. \]

With such a $\delta$, $E^+(\gamma) \cap E^+(\delta \gamma \delta^{-1})$ contains a line $l = q + \mathbb{R} \overrightarrow{v_\gamma}$. We can write $q = q_1 + \overrightarrow{w_1}$, where $q_1 \in E^o(\gamma)$ and $\overrightarrow{w_1} \in V^+(\gamma) \setminus \{0\}$, and $q = q_2 + \overrightarrow{w_2}$, where $q_2 \in E^o(\delta \gamma \delta^{-1})$ and $\overrightarrow{w_2} \in V^+(\delta \gamma \delta^{-1}) \setminus \{0\}$.

Put $p_m = q + mw_\gamma$, $m \in \mathbb{N}$. Then

\[ \lim_{m \to +\infty} \gamma^{-m}(p_m) = q_1 \quad \text{and} \quad \lim_{m \to +\infty} (\delta \gamma \delta^{-1})^{-m}(p_m) = q_2. \]

Note that the set $X = \{\gamma^{-m}(p_m) \mid m \in \mathbb{N}\} \cup \{(\delta \gamma \delta^{-1})^{-m}(p_m) \mid m \in \mathbb{N}\}$ is relatively compact and $(\delta \gamma \delta^{-1} \gamma^m)X \cap X \neq \emptyset$ for all $m$. It follows from (3) that $\delta \gamma^{-m_1} \delta \gamma \delta^{-1} \gamma^{m_1} \neq \delta \gamma^{-m_2} \delta \gamma \delta^{-1} \gamma^{m_2}$ if $m_1 \neq m_2$ which contradicts the assumption that $\Gamma$ acts properly discontinuously on $\mathbb{A}^n$. This completes our proof. \(\square\)

**Remark:** As noted in [To2], the element $\delta$ can be chosen in such a way that the subgroup spanned by $\gamma^m$ and $\delta \gamma^m \delta^{-1}$ is free if $m$ is sufficiently large. This can be achieved by a well-known argument of Tits [T3]. Indeed, let $\varphi : G \to L$ be the natural projection and $L$ be identified with its image in SL($W$). Let $P(W)$ be the projective space of $W$. If $g \in G$ and $\varphi(g)$ has pairwise different positive eigenvalues $\alpha_1 > \cdots > \alpha_t > 0$ with respective eigenvectors $\overrightarrow{w_1}, \ldots, \overrightarrow{w_t}$, we denote by $A(g)$ (resp. $A'(g)$) the projectivization of the vector space $\mathbb{R} \overrightarrow{w_1} + \cdots + \mathbb{R} \overrightarrow{w_t}$. Since $L$ acts irreducibly on $W$, we can choose $\delta$ in such a way that $A(\gamma) \cup A(\gamma^{-1}) \subset P(W)$ \(\setminus (A'(\delta \gamma \delta^{-1}) \cup A'(\delta \gamma^{-1} \delta^{-1}))\) and $A(\delta \gamma \delta^{-1}) \cup A(\delta \gamma^{-1} \delta^{-1}) \subset P(W) \setminus (A'(\gamma) \cup A'(\gamma^{-1})$. Now, it follows from the ”ping-pong lemma” that the subgroup spanned by $\gamma^m$ and $\delta \gamma^m \delta^{-1}$ is free if $m$ is sufficiently large.

**Acknowledgements:** After the initial version of this paper has been written several people contributed to its improvement. Thanks are due to Grisha Margulis and Andrei Rapinchuk for informing us about the existence of the papers [A-M-S5] and [D-P], respectively, and to Gopal Prasad for his valuable remarks and suggestions.

**References**

[A-M-S1] Abels, H., Margulis, G.A., Soifer, G.A., Semigroups containing proximal linear maps. Israel J. of Math. 91 (1995), 1-30.

[A-M-S2] Abels, H., Margulis, G.A., Soifer, G.A., Properly discontinuous groups of affine transformations with orthogonal linear part. CRAS Paris 324 (1997), 253-258.
[A-M-S3] Abels, H., Margulis, G.A., Soifer, G.A., On the Zariski closure of the Linear Part of a Properly Discontinuous Group of Affine Transformation. *J.Diff.Geom.* 60 (2002), no 2, 315-344.

[A-M-S4] Abels, H., Margulis, G.A., Soifer, G.A., The linear part of an affine group acting properly discontinuously and leaving a quadratic form invariant. *Geom.Dedicata* 153 (2011), 1-46.

[A-M-S5] Abels, H., Margulis, G.A., Soifer, G.A., The Auslander’s conjecture for dimension less than 7. [arXiv:1211.2525v4[math.GR] 2 Jun 2013], pages 1-43.

[Au] Auslander, L., The structure of compact locally affine manifolds. *Topology* 3 (1964), 131-139.

[Be1] Benoist, Y., Actions propres sur les espaces homogenes reductifs. *Annales of Math.* 144 (1996), 315-347.

[Be2] Benoist, Y., Proprietes asymptotiques des groupes lineaires. *GAFA Geom. funct. anal.* 7 (1997), 1-47.

[Be-L] Benoist, Y., Labourie, F., Sur les diffomorphismes d’Anosov affines a feuilletage stable et instable differentiables. *Invent. Math.* 111 (1993), 285-308.

[Bi] Birkes, D., Orbits of linear algebraic groups (Appendix). *Ann. of Math. (2)* 93 (1971), 459-495.

[Bo-Spr] Borel, A., Springer, T.A. Rationality properties of linear algebraic groups. *Tôhoku Math.* 20 (1968), 443-497.

[Bou] Bourbaki, N., Groupes et algebres de Lie. ch.7-8, Herman Paris (1975).

[D-P] Dekimpe, K., Petrosyan N., Cristalographic actions on contractable algebraic manifolds. *Trans. AMS* 367 (2015), 2765-2786.

[Dr1] Drumm, T., Fundamental polyhedra for Margulis space-times. *Topology* 31 (1992), 677-683.

[Dr2] Drumm, T., Linear holonomy of Margulis space-times. *J. Diff. Geometry* 38 (1993), 679-691.

[Dr-G] Drumm, T., Goldman, W., Complete flat Lorentz 3-manifold with free fundamental group. *Internat. J. Math.* 1 (1990), 149-161.

[F-G] Fried, D., Goldman, W., Three-dimensional affine crystallographic groups. *Advances in Math.* 48 (1983), 1-49.

[G-H] Goldman, W., Hirsch, W.H., Affine manifolds and orbits of algebraic groups. *Transactions of the AMS* 295, no.1, (1986), 175-198.

[G-K] Goldman, W, Kamishima, Y., The fundamental group of a compact Clifford-Klein forms of homogeneous spaces is virtually polycyclic. *J. Differential Geometry* 19 (1984), 233-240.

[Gr-Mar] Grunewald, F., Margulis, G.A., Transitive and quasitransitive actions of affine groups preserving a generalized Lorentz-structure. *J. Geom. Phys.* 5 (1988), 493-531.

[H] Helgason, S., Differential geometry, Lie groups and symmetric spaces. *Academic Press, New York* (1978).

[K1] Kobayashi, T., Discontinuous groups acting on homogeneous spaces of reductive type. *Representation Theory of Lie Groups and Lie Algebras* (Proc. Conf. at Lake Kawaguchi-Ko, Japan, 1990) World Scientific Press, Singapore (1991), 59-75.

[K2] Kobayashi, T., Discontinuous groups and Clifford-Klein forms of pseudo-riemannian homogeneous manifolds. *Algebraic and Analytic Methods in Representation Theory (edited by B. Orsted and H. Schlichtkrull)*, *Prespective in Math., Academy Press* (1997).

[K3] Kobayashi, T., A necessary condition for the existence of compact Clifford-Klein forms of homogeneous spaces of reductive type, *Duke Math. Jour.* 67 (1992), 653-664.

[L] Lipsman, R., Proper Actions and a Compactness Condition. *Journal of Lie Theory* 9 (1995), 25-39.
[Mar1] Margulis, G.A., Free completely discontinuous groups of affine transformations. *Dokl. Acad. USSR* **272** (1983), 785-788; English transl., *Soviet Math. Dokl.* **28** (1983), 435-439.

[Mar2] Margulis, G.A., Complete affine locally flat manifolds with a free fundamental group. *J. Soviet Math.* **272** (1987), 129-134.

[Mar3] Margulis, G.A., On the Zariski closure of the linear part of a properly discontinuous group of affine transformations. *Preprint, Princeton* (1991).

[Mi] Milnor, J., On fundamental groups of complete affinely flat manifold. *Advances in Math.* **25** (1977), 178-187.

[P] Prasad, G., R-regular elements in Zariski dense subgroups. *Quaterly Jour. Math.* **45** (1994), 541-545.

[Rag] Raghunathan, M.S., Discrete subgroups of Lie groups. *Ergebnisse Math. u. i. Grenzgeb.*, *Springer-Verlag Berlin Heidelberg* **68** (1972).

[Ro] Rosenlicht, M., On quotient varieties and the affine embedding of certain homogeneous spaces. *Trans. Amer. Math. Soc.* **101** (1961), 221-233.

[S] Selberg, A., On discontinuous groups of higher-dimensional symmetric spaces. *Contributions to Function Theory, Tata Inst. Fund. Res.*, *Bombay* (1960), 147-164.

[Se] Serre, J.-P., Cohomologie des groupes discrets. *Prospects in Math., Ann. of Math. Studies* **70** Princeton University Press, Princeton, NJ, (1971), 77-169.

[So1] Soifer, G.A., Affine discontinuous groups. In *L.A.Bokut’, M.Hazewinkel, Yu.Reshetnyak (eds): Proc. of the Third Siberian school Algebra and Analysis, Irkutsk State University, Irkutsk, August 30-September 4, 1989; Amer. Math. Soc. Translations (2)* **163** (1995), 165-170.

[So2] Soifer, G.A., Free subgroups of affine groups. *Preprint Tata Institute of Fundamental Research* (1990).

[Ti] Tits, J., Free subgroups in linear groups. *Journal of Algebra* **20** (1972), 250-270.

[To1] Tomanov, G., The virtual solvability of the fundamental group of a generalized Lorentz space form. *Journal of Diff. Geometry* **32** (1990), 539-547.

[To2] Tomanov, G., On a conjecture of L. Auslander. *C.R. de l’Ac. bulgare des Sciences* **43**, n.2 (1990), 9-12.

*Institut Camille Jordan, Université Claude Bernard - Lyon I, Bâtiment de Mathématiques, 43, Bld. du 11 Novembre 1918, 69622 Villeurbanne Cedex, France tomanov@math.univ-lyon1.fr*