Research article

A closed-form expansion for the conditional expectations of the extended CIR process

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A R T I C L E   I N F O

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A B S T R A C T

This paper derives a closed-form expansion for the conditional expectation of a continuous-time stochastic process, given by $V_{t,T} := e^{-\int_0^T \theta(s) \, ds} f(v_T)$ for $0 \leq t \leq T$, where $v_t$ evolves according to the extended Cox-Ingersoll-Ross process, for any $C^\infty$ functions $f$ and $g$. We apply the Feynman-Kac theorem to state a Cauchy problem associated with $V_{t,T}$ and solve the problem by using the reduction method. Furthermore, we extend our method to any piecewise $C^\infty$ function $f$, demonstrating our method can be applied to price options in financial derivative markets. In numerical study, we employ Monte Carlo simulations to demonstrate the performance of the current method.

1. Introduction

In this paper, we consider the Cox-Ingersoll-Ross (CIR) process [1] having a form of

$$d v_t = \kappa (\theta - v_t) \, dt + \sigma \sqrt{v_t} \, dW_t$$

(1.1)

for $t \in [0, T]$, $T > 0$ and $v_0 = \nu > 0$, where $v_t$ is an instantaneous variance, $\theta$, $\kappa$, and $\sigma$ are parameters, and $W_t$ is a standard Brownian motion under a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t \geq 0}$. For more general class of the CIR process, we focus on the extended Cox-Ingersoll-Ross (ECIR) process [9],

$$d v_t = \kappa (\theta(t) - v_t) \, dt + \sigma(t) \sqrt{v_t} \, dW_t,$$

(1.2)

for $t \in [0, T]$, $T > 0$ and $v_0 = \nu > 0$, where all parameters in (1.2) depend on time.

Considering option pricing when the underlying process is described by the ECIR process (1.2), we define a continuous-time stochastic process, namely, the evaluation process of a contingent claim $(f, g)$ by

$$V_{t,T} := e^{-\int_t^T \theta(s) \, ds} f(v_T)$$

(1.3)

for real-valued functions $f$ and $g$. In this situation, the function and process $f(v_t)$ and $g(v_t)$ for $t \in [0, T]$ interpret a terminal payoff and an interest rate process, respectively.

In the past several decades, the CIR process (1.1) is one of the most well-known processes used to model the dynamics of interest rates [8] when parameters are assumed to be constant. Hence, the CIR process (1.1) has been extended to the ECIR process (1.2), in which the parameters depend on time for various applications. Therefore, calculation of conditional expectations of the process (1.3) is a challenging topic for researchers in financial mathematics.

Let us focus on the process (1.3). Applying the theorem for option pricing proposed in [7] gives us the fair price of the contingent claim $(f, g)$ at a current time $t$ can be stated as

$$E^P[V_{t,T} | \mathcal{F}_t] = E^P[V_{t,T} | v_t = \nu]$$

(1.4)

for $0 \leq t \leq T$ and $\nu > 0$, where $P$ denotes the risk-neutral probability measure and $\mathcal{F}_t$ is the current $\sigma$-field.

In terms of computation, the Monte Carlo (MC) method can be directly adopted to obtain approximate values for (1.4), but it takes a lot of effort and time since $V_{t,T}$ is a path dependence process, depending on the underlying process $v_t$. To avoid this problem, an analytical approach should be developed in order to compute the conditional expectation (1.4) instead of using MC simulations.

Some special cases of (1.4) have closed-form formulas. For example, Dufresne [2] derived an exact formula for the case $f(v) = v^\gamma$ when $\gamma > \frac{\nu}{2\sigma^2}$ and $g = 0$ in which $v_t$ evolves according to the CIR process (1.1). Rujivan [10] developed a different method from the one proposed in [2] to obtain an exact formula for the ECIR processes (1.2).

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for any \( r \in \mathbb{R} \). Suthimat et al. [13, 14] applied Rujivan’s [10] method to path-dependent product of polynomial and exponential functions. Thamrongrat and Rujivan [16] derived an exact formula for the case \( f(v) = v^r \) for any \( r \in \mathbb{R} \) to compute a conditional expectation of the form: 
\[ V_{x,r} := e^{-\int_0^t c(s) U(v_s) ds} \]
for any integrable function \( r \).

In the outcomes presented in [11, 12] and Rujivan and Rakwongwan [12] show that the method proposed in [10] can be used for pricing variance swaps and volatility swaps. Thamrongrat and Rujivan [17] used the outcomes proposed in [13] to the pricing of interest rate swaps in terms of bond prices when the interest rate process follows the ECIR model (1.2).

For a different framework from the ones proposed in [2, 10, 13, 14, 15, 16, 18], this paper develops a method for obtaining a closed-form expansion of (1.4) for any \( C^\infty \) real-valued functions \( f \) and \( g \). Utilizing the Feynman-Kac theorem [7], we can state an initial value problem, i.e. a Cauchy problem, \(^1\) written in terms of a partial differential equation (PDE) in parabolic type \(^2\) with an initial condition, associated with the valuation process (1.3).

Our closed-form expansion can be obtained by using the reduction method. Generally, we assume that the Cauchy problem can be written as a power series of the time-step size defined by \( \tau = T-t \), in which the coefficient functions can be computed from the \( n \)th derivatives of \( f \) and \( g \) for \( n = 1, 2, \ldots \), recursively. Furthermore, we apply our method for obtaining a closed-form expansion for (1.4) when \( f \) is a piecewise \( C^\infty \) function; demonstrating our method can be used to price options in derivative financial markets.

The rest of the paper is organized as follows. Section 2 provides the theorems used to derive a closed-form expansion based on the CIR process (1.1) and ECIR process (1.2), respectively. Section 3 demonstrates the accuracy of our current method by comparing our numerical results with MC simulations. Moreover, we apply our current approach for the case that \( f \) is a piecewise \( C^\infty \) function and also show that our method can be adopted to price European options in the section. We conclude the paper in Section 4.

2. Materials and methods

This section starts by imposing the following assumptions.

**Assumption 1.** The parameter functions in (1.2) are continuous on [0, \( T \)] and strictly positive. In addition, \( \delta(t) := \frac{\delta(t)}{\sigma^2(t)} \), is assumed to be bounded on [0, \( T \)].

**Assumption 2.** \( \frac{1}{2} \delta(t) \geq 1 \) for all \( t \in [0, T] \).

**Assumption 3.** The functions \( f \) and \( g \) are \( C^\infty \) real-valued functions and satisfy the polynomial growth condition \( |f(v)| + |g(v)| \leq Cv^N \) for some constant \( C > 0 \), positive integer \( N \), and for all \( v > 0 \).

2.1. Our method for the CIR process

Next, we present the first theorem of the paper.

**Theorem 2.1.** Let \( v_t \) be described by the CIR process (1.1). Under Assumptions 1-3, we write
\[
U_C(v, \tau) := E^P[V_{x,r} | v_t = f] \quad \text{for } v > 0 \text{ and } \tau = T-t \geq 0.
\]

Then,
\[
U_C(v, \tau) = \sum_{k=0}^{\infty} A_k(v) \frac{\tau^k}{k!}
\]
for all \((v, \tau) \in D_C\) where \( D_C \) is a subset of \((0, \infty) \times (0, \infty)\),

\[
A_0(v) = f(v)
\]
\[
A_k(v) = \frac{\sigma^2}{2} A_k''(v) + \left\{ a_2(T) - a_1(T) v \right\} B_k'(v) - g(v) B_k(v)
\]
for \( k = 1, 2, \ldots \) and the derivatives are computed with respect to \( v \).

**Proof.** Using the Feynman-Kac theorem [7], \( U_C \) solves the parabolic PDE:
\[
-\frac{dU_C}{dv} + \frac{1}{2} \sigma^2 v^2 \frac{d^2 U_C}{dv^2} + \kappa v - \theta \frac{dU_C}{dv} - g(v) U_C(v, \tau) = 0
\]
and satisfies
\[
U_C(v, 0) = f(v)
\]
for all \( v > 0 \).

It should be noted that the initial value problem (2.5)-(2.6) is known as a Cauchy problem [4]. Furthermore, the existence and uniqueness of the solution for the Cauchy problem (2.5)-(2.6) are ensured when Assumptions 1-3 hold (see Friedman [5]).

Next, we shall show that the solution form of \( U_C \) as expressed on the RHS of (2.1) satisfies the PDE (2.5) and initial condition (2.6) when the coefficients \( A_k(v), k = 0, 1, \ldots \), satisfy (2.3)-(2.4), respectively.

Firstly, we derive the partial derivatives of \( U_C \) contained in (2.5) by utilizing the solution form (2.1). Substituting the results obtained into (2.5) gives
\[
\sum_{k=0}^{\infty} \frac{A_k(v) \tau^k}{k!} = -a_2(v) \frac{\tau^k}{k!} + \frac{1}{2} \sigma^2 v^2 \sum_{k=0}^{\infty} \frac{A_k''(v) \tau^k}{k!} + \kappa v - \theta \sum_{k=0}^{\infty} \frac{A_k''(v) \tau^k}{k!} - g(v) \sum_{k=0}^{\infty} \frac{A_k'(v) \tau^k}{k!} = 0
\]
for all \((v, \tau) \in D_C\) where \( D_C \) is a subset of \((0, \infty) \times (0, \infty)\), that shall be determined in our numerical study.

Secondly, we collect the coefficients of \( \tau^k \) in (2.7) for \( k = 0, 1, \ldots \). Using the initial condition (2.6), we have that \( A_0(v) \) must satisfy (2.3).

Since the RHS of (2.7) is zero, the coefficients of \( \tau^k, k = 1, \ldots \), must be zero which implies \( A_k(v), k = 1, \ldots \), satisfy (2.4), recursively. It should be noted from (2.3)-(2.4) that \( A_k(v), k = 0, 1, \ldots \), are \( C^\infty \) real-valued functions due to \( f \) and \( g \) are \( C^\infty \) real-valued functions.

2.2. Our method for the ECIR process

The second theorem of the paper needs the additional assumption as follows.

**Assumption 4.** The parameter functions in (1.2) are \( C^\infty \) real-valued functions.

**Theorem 2.2.** Let \( v_t \) be described by the ECIR process (1.2). Under Assumptions 1-4, we write
\[
U_E(v, \tau) := E^P[V_{x,r} | v_t = v]
\]
for \( v > 0 \) and \( \tau = T-t \geq 0 \). We set \( a_1(t) = \sigma^2(t), a_2(t) = \kappa(t) \theta(t), \) and \( a_3(t) = \kappa(t) \) for \( t \geq 0 \). Then, \( U_E(v, \tau) \) can be written as
\[
U_E(v, \tau) = \sum_{k=0}^{\infty} B_k(v) \frac{\tau^k}{k!}
\]
for all \((v, \tau) \in D_E\) where \( D_E \) is a subset of \((0, \infty) \times (0, \infty)\),

\[
B_0(v) = f(v)
\]
\[
B_k(v) = \frac{1}{2} a_1(T) v B_k''(v) + \left\{ a_2(T) - a_1(T) v \right\} B_k'(v) - g(v) B_k(v)
\]
for all \((v, \tau) \in D_E\) where \( D_E \) is a subset of \((0, \infty) \times (0, \infty)\),

1 We refer to the textbook written by Evans [4].
2 We refer to the textbook written by Friedman [5].
\[ B_k(v) = \sum_{j=0}^{k-1} \left\{ \frac{1}{j!} \left( k-j-1 \right) v B_j''(v) + \left[ a_2^{(k-j-1)}(T) - a_3^{(k-j-1)}(T) v \right] B_j'(v) \right\} - g(v)B_{k-1}(v) \]

(2.12)

where \( \phi_j = \alpha^{(k-j-1)}(\alpha-1) \) for \( k = 2, 3, \ldots \). The \( (k-j-1) \)th derivatives of \( a_i(t), i = 1, 2, 3, \) are computed with respect to \( t \) while the derivatives \( B_j''(v) \) and \( B_j'(v) \) are computed with respect to \( v \).

\[ \frac{\partial U_E^k}{\partial t} + \frac{1}{2} a_1(t) \frac{\partial^2 U_E^k}{\partial v^2} + (a_2(t) - a_3(t)v) \frac{\partial U_E^k}{\partial v} - g(v)U_E^k(v, t) = 0 \]

(2.13)

and satisfies

\[ U_E(v, 0) = f(v) \]

(2.14)

for all \( v > 0 \).

Note that, the existence and uniqueness of the solution for the Cauchy problem (2.13)-(2.14) are guaranteed when Assumptions 1-3 are fulfilled (see Friedman [3]).

Following the method as presented in Theorem 2.1 gives

\[ a_i(t) = \sum_{k=0}^{\infty} a_i^{(k)}(T) \frac{(-1)^k t^k}{k!} \]

(2.16)

for \( (v, t) \in D_E \), where \( D_E \) is a subset of \((0, \infty) \times [0, \infty)\), that shall be determined in our numerical method.

Using Assumption 4, we can write \( a_i(t) \) in terms of a Taylor expansion centered at \( T \) as follows:

\[ a_i(t) = \sum_{k=0}^{\infty} a_i^{(k)}(T) \frac{(-1)^k (t-T)^k}{k!} \]

(2.21)

for \( i = 1, 2, 3 \) for all \( t \in (T - \epsilon, T) \) for some \( \epsilon > 0 \).

Replacing the Taylor expansion (2.21) into (2.15) and then matching the coefficients of \( t^k \) for \( k = 0, 1, \ldots \), the coefficient function \( B_k(t) \) must satisfy (2.10) derived by using the initial condition (2.14). Consequently, the remaining coefficient functions \( B_k(t), k = 1, \ldots, \) must follow (2.21)-(2.12), recursively. The proof is now complete.\(^3\)

3. Main results and discussion

This section illustrates the accuracy and efficiency of (2.9) comparing with MC simulations. We shall focus on the ECIR process, presented by Egorov et al. [3] as follows:

\[ dV_t = \left( \frac{\sigma^2 \sigma^2 dW_t}{4} - \theta \right) dt + \sigma dW_t \]

(3.1)

for \( t \in [0, T] \), \( T > 0 \), and \( V_0 = v > 0 \), where \( k(t) = k, \theta(t) = \frac{1}{4k} \sigma^2 dW_t \), and \( \sigma(t) = \sigma_0 e^{\kappa t}, \) with \( k, \sigma, \kappa, \) and \( \sigma_0 \) are positive constants.

It is easy to show that \( \delta(t) = d \) and this implies the parameters functions \( \theta(t), k(t), \) and \( \sigma(t) \) satisfy Assumptions 1-2 when \( d \geq 2 \).

For a unique situation, when \( g = 0 \), we can compute \( U_E(v, r) \) by using the formula:

\[ U_E(v, r) = \mathbb{E}^P[f(\psi_v)[v, t = v]] = \int_0^\infty f(w) p_r(w, t + r, t) dw \]

(3.2)

where \( p_r(w, t + r, t) \) for all \( w > 0 \) is the transition density of \( \psi_v \).

It should be noticed that the integral on the RHS of (3.2) is always finite due to \( f \) satisfies the polynomial growth condition in Assumption 3. Nevertheless, as discussed in [3], the integral (3.2) has not an explicit form and hence a numerical method shall be used to approximate \( U_E(v, r) \). To ease this problem, one can use MC simulations for approximating the value of \( U_E(v, r) \).

3.1. Absolute relative errors in MC simulations

In this section, \( U_E^{(k)}(K, \tau) \) denotes a closed-form expansion of \( U_E \) in the form of (2.9), in which the partial sum of the infinite series (2.9) is used up to order \( K \), with the Taylor expansion (2.16) up to order \( K \).

Additionally, \( U_E^{(k, N)}(K, \tau) \) represents an approximation of \( U_E \), obtained by MC simulations with a number of sample paths is \( N \).

In the case that \( U_E \) has not been found in closed-form, for a fixed \( N \), we consider a sequence of absolute relative errors as

\[ e_k(v, \tau) = \left| \frac{U_E^{(k+1, N)}(K, \tau) - U_E^{(k)}(K, \tau)}{U_E^{(k)}(K, \tau)} \right| \]

(3.3)

for any \((v, \tau) \in (0, \infty) \times (0, \infty) \), and \( k = 1, 2, \ldots \) providing that \( U_E^{(k+1)}(K, \tau) \neq 0 \). In addition, if \( e_k(v, \tau) < 0.5 \times 10^{-n} \) then we shall conclude that the approximate \( U_E^{(k, N)}(K, \tau) \) of \( U_E(v, \tau) \) matches at least \( n \) significant digits.

3.2. Comparison with MC simulations

Next, we employ MC simulations to demonstrate the accuracy and efficiency of our current method for some cases of contingent claims in which the corresponding conditional expectations, written in the form of (1.4), have not been found in close form.\(^3\)

Employing a numerical method called European-Maruyama for (1.2), we then obtain

\[ v_i(o) = v_{i-1}(o) + \kappa(t_{i-1}) (\theta(t_{i-1}) - v_{i-1}(o)) \Delta t + \sigma(t_{i-1}) \sqrt{v_{i-1}(o)} \sqrt{\Delta W_{t_i}(o)} \]

(3.4)

for any sample path \( o \in \Omega \). We produce sample paths of \( v_i \) on \([0, T]\) for \( r > 0 \) with setting \( \kappa = 0.05, \sigma = 0.01, \) and \( \sigma_r = 0.02 \). Our calculation employs Mathematica 11 to generate the sample paths.

Example 1. Let \( f(v) = \cos(v) \) and \( g(v) = \frac{v}{v^2 + 1} \) for \( v \in \mathbb{R} \). It should be noted that \( f \) and \( g \) satisfy Assumptions 3-4. Hence, we can apply Theorem 2.2 to derive a closed-form expansion for \( U_E(v, r) \). Utilizing (2.9) with the contingent claim \( (f, g) \) gives a closed-form expansion for \( K_1 = K_2 = 1 \) as

\[ U_E^{(1)}(v, r) = \cos(v) + \left( \frac{1}{2} e^{2\kappa t} \sigma_0^2 \cos(v) - \frac{\cos(v)}{v^2 + 1} - \sin(v) \left( \frac{1}{4} e^{2\kappa t} \sigma_0^2 - \text{tk} \right) \right) r \]

(3.5)

where \( r = T - t \).

One can see from Fig. 4.1 that our numerical results obtained by using our closed-form expansion (3.5) are very close to the results from the MC simulations with \( N_s = 10,000 \) at \( v \in [0.1, 0.2, \ldots, 2] \) and \( r = 0.1 \).

Due to the convergence of the MC simulations to the exact solution when \( N_s \) approaches infinity, this confirms that our closed-form

\(^3\) It is easy to show that (2.11)-(2.12) reduce to (2.4) when the parameter functions in (1.2) are constants.\(^4\)

To illustrate the accuracy of our current method in the case that the conditional expectation (1.4) has a closed-form formula as found in [10, 16], one can use our method to compute a closed-form expansion. However, we omit showing our results here for brevity reasons.

\(^5\) Note that one can use the symbolic computing package to obtain \( U_E^{(k, N)}(K, \tau) \) for \( k = 2, 3, 4, \ldots \).
expansion (3.5) produces accurate approximates for the exact solution when \( v \in (0.1, 0.2, \ldots, 2) \) and \( r = 0.1 \). Very interestingly, our method consumes a very much shorter time than the MC simulations.

It should be pointed here that the exact solution \( U_E \) has not been discovered in closed-form under this setting of contingent claim \((f,g)\). Therefore, we shall study the accuracy of (3.5) by considering the error defined in (3.3). Hence, we calculate the absolute relative errors for \( r = 0.01, 0.1 \) and \( k = 1, 2, 3 \), in (3.3), where \( v \) varies from 0.0001 to 2.0.

By fixing \( r \) and \( v \), Table 4.1 shows that the accuracy of (3.5) increases when \( k \) increases. Furthermore, we can conclude from our numerical results shown in Table 4.1 that using \( k = 1 \) is sufficient for obtaining an accurate approximation for the exact solution when \( r \) is small.

It should be remarked before leaving this example as follows. As shown by Maghsoudi [9], the process \( v_t \) driven by the SDE (1.2) never hits zero a.s. \( P \) for all \( t \in [0,T] \) providing that Assumptions 1-2 hold. Therefore, the accuracy of (1.4) when \( v_t \) approaches its boundary, i.e. \( v \to 0^+ \), will be discussed. It can be clearly seen from Table 4.1 that the absolute relative error \( e^{(v)}(v,t) \) decreases when \( v \) approaches zero for all \( r = 0.01, 0.1 \) and \( k = 1, 2, 3 \); ensuring that our method produces accurate results for the conditional expectation (1.4) near the boundary of the ECIR process (1.2).

3.3. An extension of our approach to piecewise \( C^\infty \) functions

The following example applies Theorem 2.2 to obtain a closed-form expansion for (1.4) when Assumption 3 is violated.

**Example 2.** We choose \( f(v) = |v-1| \) and \( g(v) = \cos(v) \) for \( v \in \mathbb{R} \). It should be noticed that \( f \) is not differentiable at \( v = 1 \). Nevertheless, the existence and uniqueness of the solution of the PDE (2.13) subject to an initial condition, which is a piecewise smooth function, can be obtained by using the results presented in [5].

It should be noted here that the exact solution under this case of contingent claims has not been found in closed-form. Hence, we employ MC simulations to produce an approximate, denoted by \( U^{(M,N)}_{E,f}(v,t) \), for the exact solution \( U_{E,f}(v,t) \) for any \( v > 0 \) and \( t > 0 \).

Note that the \( m^\delta \)th derivative of \( f \) at \( v = 1 \) does not exist for all \( n = 1, 2, \ldots \). Hence, our goal is that the closed-form expansion (2.9) is applicable to obtain good approximates for the exact solution \( U_{E,f}(v,t) \) for any \( v > 0 \) and \( t > 0 \).

To achieve our aim, we set \( f^-(v) := 1 - v \) and \( f^+(v) := v - 1 \) for \( v > 0 \). Note that \( f^- \) and \( f^+ \) satisfy Assumption 3. Thus, we can apply Theorem 2.2 to derive closed-form expansions for \( U_{E,f}(v,t) \) with respect to the contingent claims \((f^-,-g)\) and \((f^+,g)\).

Utilizing the symbolic computing package in Mathematica, one can use the closed-form expansion (2.9) with the contingent claims \((f^-,-g)\) and \((f^+,g)\) to derive \( U_{E,f,K}^{(M,N)}(v,t) \) and \( U_{E,f,K}^{(K,K)}(v,t) \), respectively, for any \( K = 1, 2, \ldots \).

Nevertheless, for a fixed \( r \) and sufficient large value of \( K \), we shall justify that \( U_{E,f}(v,t) \approx U_{E,f,K}^{(M,N)}(v,t) \) if \( v < v_t \) and \( U_{E,f}(v,t) \approx U_{E,f,K}^{(K,K)}(v,t) \) if \( v > v_t \) for some \( v_t > 0 \) depending on \( r \).

In our numerical test, we first compute \( U_{E,f,K}^{(2,2)}(v,t) \) and \( U_{E,f,K}^{(2,2)}(v,t) \) for \( \tau = t = 0.1 \) and 100 values of \( v \), sampled uniformly in a neighborhood of the non-differentiable point \( v = 1 \). Next, we compare our numerical results with \( U_{E,f,K}^{(M,N)}(v,t) \) for all sampled points \( v \) where we set \( N_p = 10,000 \) in MC simulations. Alternatively, we claim that \( U_{E,f}(v,t) \approx U_{E,f,K}^{(M,N)}(v,t) \) for any \( v > 0 \) and \( t > 0 \).

Fig. 4.2 illustrates that \( U_{E,f,K}^{(2,2)}(v,t) \) for \( v < v_t \) and \( U_{E,f,K}^{(2,2)}(v,t) \) for \( v > v_t \) are very close to \( U_{E,f,K}^{(M,N)}(v,t) \) in the neighborhood where \( v_t \approx 1.003 \) is determined by the intersection point between the two curves. When we set \( \tau = r = 1 \), however, \( U_{E,f,K}^{(2,2)}(v,t) \) and \( U_{E,f,K}^{(2,2)}(v,t) \) are away from \( U_{E,f,K}^{(M,N)}(v,t) \), as shown in Fig. 4.3.

To obtain better approximates for \( U_{E,f}(v,	au_2) \), we increase the value of \( K \) from 2 to 4 and compute \( U_{E,f,K}^{(4,4)}(v,t) \) and \( U_{E,f,K}^{(4,4)}(v,t) \) for all sampled points \( v \). As shown in Fig. 4.4, \( U_{E,f,K}^{(4,4)}(v_,t) \) for \( v < v_t \) and \( U_{E,f,K}^{(4,4)}(v,t) \) for \( v > v_t \) are very close to \( U_{E,f,K}^{(M,N)}(v,t) \) in the neighborhood where \( v_t \approx 1.0309 \). Therefore, we can conclude from our numerical results that \( U_{E,f,K}^{(K,K)}(v,t) \) and \( U_{E,f,K}^{(K,K)}(v,t) \) converge to the exact solution \( U_{E,f}(v,t) \) when \( r \) approaches zero. Nonetheless, as illustrated in Figs. 4.3-4.4, we should increase the value of \( K \) in order to get better approximates when \( r \) increases.

3.4. An application of our approach to the pricing European options

As introduced by Hull [6], the problem of calculation the conditional expectation (1.4) becomes the pricing European call options under the ECIR process (1.2) when we set \( f(v) = \max(v-K,0) \) and \( g(v) = r \) for some strike price of a call option \( K > 0 \) and risk-free interest rate \( r > 0 \).

Therefore, the fair price of a call option at initial time \( t = 0 \) having maturity time \( T > 0 \) can be written as

\[
U_{E,f}^{(K)}(v,T) = e^{-rT} E^F \left[ \max(v_T - K,0) | v_t = v \right].
\]

(3.6)

It should be noted that \( f \) is a \( C^\infty \) function on \( \mathbb{R} - \{K\} \). To obtain a closed-form expansion for \( U_{E,f}^{(K)}(v,T) \) given in (3.6), we can apply the method presented in Example 2 by setting \( f^-(v) := 0 \) and \( f^+(v) := v - K \) for \( v > 0 \).

A closed-form expansion for the fair price of a put option at initial time \( t = 0 \) having maturity time \( T > 0 \) defined as

\[
U_{E,f}^{(K)}(v,T) = e^{-rT} E^F \left[ \max(K - v_T,0) | v_t = v \right].
\]

(3.7)

can be obtained by setting \( f^-(v) := K - v \) and \( f^+(v) := 0 \) for \( v > 0 \) in Example 2.

4. Conclusions

In this work, we have applied the Feynman-Kac theorem to state a Cauchy problem associated with a continuous-time stochastic process \( V_{t,T} := \int_0^T e^{(\delta_1 t - \delta_2 v_t)} f(v_T) \) for \( 0 \leq t \leq T \), where \( v_t \) evolves according to the ECIR process (1.2), for any \( C^\infty \) real-valued functions \( f \) and \( g \). The Cauchy problem has been solved by using the reduction method to yield a closed-form expansion for the conditional expectation of \( V_{t,T} \). Our numerical study has confirmed the accuracy and efficiency of the current method comparing with the MC results. Finally, we have extended our

![Fig. 4.1. Comparison between the approximates for the exact solution \( U_{E,f}(v,0.1) \) obtained by using the closed-form expansion (3.5) and MC simulations where the computational times (seconds) are written in the parentheses.](image-url)
Table 4.1. Sequences of absolute relative errors with different time-step sizes $\tau = 0.01$ and $\tau = 0.10$ obtained by using the one-step closed-form expansion (3.5) where the contingent claim is set in Example 1.

| $k$ | $c^{13}(v,v,\tau = 0.01)$ | $c^{13}(v,v,\tau = 0.10)$ |
|-----|--------------------------|--------------------------|
|     | $v = 0.0001$ | $v = 0.001$ | $v = 0.1$ | $v = 0.5$ | $v = 1.0$ | $v = 1.5$ | $v = 2.0$ |
| 1   | 0.14 E-09 | 0.15 E-08 | 0.20 E-07 | 0.63 E-06 | 0.10 E-04 | 0.20 E-05 | 0.13 E-04 |
| 2   | 0.12 E-13 | 0.15 E-12 | 0.30 E-11 | 0.32 E-09 | 0.14 E-07 | 0.56 E-07 | 0.19 E-07 |
| 3   | 0.00 0   | 0.33 E-15 | 0.15 E-12 | 0.15 E-10 | 0.91 E-10 | 0.20 E-10 | 0.13 E-10 |

Fig. 4.2. Comparison between the approximates for the exact solution $U_{E,f}(v,0.1)$ obtained by using the contingent claims $(f^{-}, g)$, $(f^{+}, g)$ with $K = 2$ in Example 2 and MC simulations.

Fig. 4.3. Comparison between the approximates for the exact solution $U_{E,f}(v,1)$ obtained by using the contingent claims $(f^{-}, g)$, $(f^{+}, g)$ with $K = 2$ in Example 2 and MC simulations.

method to the case that $f$ is a piecewise $C^\infty$ function and shown its application to the pricing European options. Of course, how to resolve this problem when $f$ and $g$ are both piecewise $C^\infty$ functions may be a potential topic in the future.

Declarations

Author contribution statement

Sanae Rujivan: Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

Nopporn Thamrongrat: Conceived and designed the experiments; Performed the experiments; Wrote the paper.

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Data availability statement

No data was used for the research described in the article.
**Declaration of interests statement**

The authors declare no conflict of interest.

**Additional information**

No additional information is available for this paper.

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