The transformations between $N = 2$ supersymmetric KdV and HD equations

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The $N = 2$ supercomformal transformations are employed to study supersymmetric integrable systems. It is proved that two known $N = 2$ supersymmetric Harry Dym equations are transformed into two $N = 2$ supersymmetric modified Kortweg-de Vries equations, thus are connected with two $N = 2$ supersymmetric Kortweg-de Vries equations.
I. INTRODUCTION

The reciprocal transformation, also known as hodograph transformation, plays an important role when we investigate relations among some nonlinear evolution equations. For instance, the Harry Dym (HD) equation (or hierarchy)\(^4\), which is invariant under a kind of reciprocal transformation, is also reciprocally linked to the Korteweg-de Vries (KdV) equation (or hierarchy). The Camassa-Holm equation is shown to be linked to the first negative flow of the KdV hierarchy\(^6\). The Kawamoto equation is transformed to the common modification of the Sawada-Kotera and Kaup-Kupershmidt equations by reciprocal transformation\(^7\) and many other examples exist. Apart from its own interest, the reciprocal transformation is also a powerful tool to investigate integrable properties of nonlinear evolution equations. Indeed, recursion operators, bi-Hamiltonian structures and solutions of one given equation could be constructed from the corresponding properties of the other equation through the associated reciprocal transformation. With the help of the reciprocal transformation, the abundant symmetry structures of Kawamoto-type equations and Harry Dym equation were revealed\(^5\).

Very recently, the reciprocal transformation was generalized to \(N = 1\) supersymmetric equations\(^12\), where a general procedure to construct supersymmetric reciprocal transformation was presented. As applications, one of the supersymmetric HD equations was shown to be reciprocally linked to the supersymmetric modified KdV equation, and the supersymmetric Kawamoto equation, as a fifth order analog of HD equation, was transformed to the supersymmetric modified Sawada-Kotera equation. As in the classical case, the supersymmetric reciprocal transformation could be employed to explore integrable properties of supersymmetric equations, which was illustrated by constructing the recursion operators and bi-Hamiltonian structures of the supersymmetric HD and Kawamoto equations. More examples of supersymmetric reciprocal transformation can be found in Ref.\(^13\).

Besides \(N = 1\) supersymmetric generalizations, the integrable systems also admit \(N = 2\) extended supersymmetric generalizations. The idea could almost be traced back to the usage of the supersymmetry in the quantum field theory\(^8\). As a striking feature, \(N = 2\) extended case distinguishes itself from \(N = 1\) non-extended case by the possibility to supply new classical integrable systems. The \(N = 2\) supersymmetric KdV equations were proposed more than twenty years ago\(^9,10\) and have been studied extensively since then. Various results
for these equations have been obtained, including Lax representations\textsuperscript{10,16}, bi-Hamiltonian structures\textsuperscript{15}, bilinear formulism\textsuperscript{18} and so on. In addition to the KdV equation, some other integrable equations, such as HD equation, Camassa-Holm equation were also generalized to the $N = 2$ super space\textsuperscript{3,17} (see also Refs. 1 and 11).

With the success of establishing the $N = 1$ supersymmetric reciprocal transformation, it would be important to extend the results to the $N = 2$ supersymmetric case and to study the possible links between $N = 2$ supersymmetric HD and KdV equations. In this paper, we will show that such extension is indeed possible. We will show that the superconformal transformations serve our need. The paper is organized as follows. In the next section, we recall the $N = 2$ supersymmetric KdV and HD equations. In the section three, the $N = 2$ superconformal transformations and the relevant results will be reviewed and elaborated. Then in the section four, the transformations between the two $N = 2$ supersymmetric HD equations and supersymmetric KdV equations are given. Final section contains some comments and open questions.

II. $N = 2$ SUPERSYMMETRIC KDV AND HD EQUATIONS

The $N = 2$ supersymmetric KdV equation is given by the one-parameter system

$$
\Phi_x = \frac{1}{4} \left( \Phi_{3y} - 3\Phi(D_1 D_2 \Phi) |_{y} - \frac{a - 1}{2} [D_1 D_2 \Phi^2] |_{y} - 3a \Phi^2 \Phi_y \right),
$$

(1)

which is denoted by SKdV$_a$ equation in literature. Some conventional criteria have been adopted to study its integrability and the system is known to be integrable only for certain values of the parameter $a$. In fact, the existence of infinitely many conservation laws implies that $a$ only takes three values, $-2$, $1$ and $4$\textsuperscript{10}. The singularity analysis also leads to the same conclusion\textsuperscript{2}. For these three cases, Eq. (1) was shown to admit Lax representations\textsuperscript{10,16}

\begin{align*}
a = 4 : & \quad L_4 = -(D_1 D_2 + \Phi)^2, \quad \frac{\partial}{\partial r_4} L_4 = \{[L_4^{\frac{1}{2}} L_4] \geq 0, L_4\}, \\
a = -2 : & \quad L_{-2} = \partial^2_y + D_1 D_2 - D_2 D_1, \quad \frac{\partial}{\partial r_{-2}} L_{-2} = \{[L_{-2}] \geq 0, L_{-2}\}, \\
a = 1 : & \quad L_1 = -\partial^2_y D_1 D_2 (D_1 D_2 + \Phi), \quad \frac{\partial}{\partial r_1} L_1 = \{[L_1^{3/2}] \geq 1, L_1\},
\end{align*}

where we are using the standard notations and the subscripts $\geq_k$ denote the corresponding projections. It is remarked that that the operator $L_4$ possesses two different square roots,
namely
\[ \mathbb{L}_4 = i(D_1 D_2 + \Phi), \quad \text{and} \quad \mathbb{L}_4^\perp = \partial_y + \cdots. \]

The non-uniqueness of roots of Lax operator results in the SKdV\(_4\) equation which admits twice as many conserved quantities as those of the other two systems\(^{13}\). It is easy to see, from the Lax representation, that the SKdV\(_4\) equation is not the first non-trivial flow in its hierarchy, while the first one is
\[ \Phi_{\tau_1} = \frac{1}{2}(D_1 D_2 \Phi_y) + 2\Phi \Phi_y. \quad (2) \]

Let us now turn to the Harry Dym case. By considering the most general \(N = 2\) super Lax operator which consists of differential operators only, two different \(N = 2\) supersymmetric Harry Dym hierarchies were presented\(^3\). One is formulated as
\[ \frac{\partial}{\partial t_n} L_1 = \left[ (\hat{L}_1 L_1^n)_{\geq 2}, L_1 \right], \quad n = 0, 1, 2, \ldots \quad (3) \]
where the Lax operator is
\[ L_1 = -(W D_1 D_2)^2, \quad (4) \]
whose two different square roots are given by
\[ \hat{L}_1 = i W D_1 D_2, \]
\[ L_1^\perp = W \partial_x + \frac{1}{2} \left[ (D_1 W) D_1 + (D_2 W) D_2 - W_x \right] \]
\[ + \frac{1}{4} \left[ -2(D_1 D_2 W) - (D_2 W)(D_1 W) W^{-1} \right] D_1 D_2 \partial_x^{-1} + \cdots, \]
respectively. Based on the Lax equation \(^3\), it is easy to write down the first two non-trivial flows in this hierarchy explicitly, namely,
\[ W_{t_1} = \frac{i}{2} \left( D_1 D_2 W_x \right) W^2, \quad (5) \]
\[ W_{t_2} = \frac{1}{8} \left( 2 W^3 - 6(D_1 D_2 W_x)(D_1 D_2 W^2 - 3(D_2 W_{2x})(D_2 W) W^2 \right. \]
\[ - 3(D_1 W_{2x})(D_1 W) W^2 \right). \quad (6) \]

The other \(N = 2\) supersymmetric Harry Dym hierarchy is defined as
\[ \frac{\partial}{\partial t_n} L_2 = \left[ (L_2^n)_{\geq 2}, L_2 \right] \quad (7) \]
with the Lax operator
\[ L_2 = \frac{1}{2} (D_1 W^2 D_1 + D_2 W^2 D_2) \partial_x. \quad (8) \]
The first non-trivial flow of this hierarchy reads as

\[
W_{t_3} = \frac{1}{8} \left( 2W_{3x}W^3 - 3(D_2 W_{2x})(D_2 W)W^2 - 3(D_1 W_{2x})(D_1 W)W^2 \\
+ 3(D_2 W)(D_1 W)(D_1 D_2 W_x)W \right).
\]  

(9)

In both cases, \( W = W(x, \theta_1, \theta_2, t) \) is a bosonic super field.

### III. THE SUPERCONFORMAL TRANSFORMATION OF \( N = 2 \) SUPER SPACE

A super diffeomorphism between two super spaces is named as a superconformal transformation providing the super derivatives are transformed covariantly (see Ref. 14 and the references there). Let us consider the super diffeomorphism between the super spaces \((y, \varrho_1, \varrho_2)\) and \((x, \theta_1, \theta_2)\)

\[
y \rightarrow x = x(y, \varrho_1, \varrho_2),  \\
\varrho_1 \rightarrow \theta_1 = \theta_1(y, \varrho_1, \varrho_2),  \\
\varrho_2 \rightarrow \theta_2 = \theta_2(y, \varrho_1, \varrho_2).
\]  

(10)

The associated super derivatives for each super space are respectively denoted by

\[
\mathbb{D}_k = \partial_{\varrho_k} + \varrho_k \partial_y, \quad \mathcal{D}_k = \partial_{\theta_k} + \theta_k \partial_x, \quad (k = 1, 2).
\]

For the super diffeomorphism (10), we have the following relations

\[
\mathbb{D}_1 = \left( (\mathbb{D}_1 x) - \theta_1 (\mathbb{D}_1 \theta_1) - \theta_2 (\mathbb{D}_1 \theta_2) \right) \frac{\partial}{\partial x} + (\mathbb{D}_1 \theta_1) \mathbb{D}_1 + (\mathbb{D}_1 \theta_2) \mathbb{D}_2,  \\
\mathbb{D}_2 = \left( (\mathbb{D}_2 x) - \theta_1 (\mathbb{D}_2 \theta_1) - \theta_2 (\mathbb{D}_2 \theta_2) \right) \frac{\partial}{\partial x} + (\mathbb{D}_2 \theta_1) \mathbb{D}_1 + (\mathbb{D}_2 \theta_2) \mathbb{D}_2,
\]

(11a)

\[
\mathcal{D}_1 = (\mathcal{D}_1 \theta_1) \mathcal{D}_1 + (\mathcal{D}_1 \theta_2) \mathcal{D}_2,  \quad \mathcal{D}_2 = (\mathcal{D}_2 \theta_1) \mathcal{D}_1 + (\mathcal{D}_2 \theta_2) \mathcal{D}_2,
\]

(11b)

hence, to ensure the super derivatives transforming covariantly, i.e.

\[
\mathbb{D}_1 = (\mathbb{D}_1 \theta_1) \mathbb{D}_1 + (\mathbb{D}_1 \theta_2) \mathbb{D}_2,  \quad \mathbb{D}_2 = (\mathbb{D}_2 \theta_1) \mathbb{D}_1 + (\mathbb{D}_2 \theta_2) \mathbb{D}_2,
\]

(12)

the super diffeomorphism (10) must be constrained by

\[
(\mathbb{D}_1 x) = \theta_1 (\mathbb{D}_1 \theta_1) + \theta_2 (\mathbb{D}_1 \theta_2),  \quad (\mathbb{D}_2 x) = \theta_1 (\mathbb{D}_2 \theta_1) + \theta_2 (\mathbb{D}_2 \theta_2).
\]

(13)

However, the constraints (13) are not sufficient to ensure the super diffeomorphism to be superconformal. Some further constraints are resulted from the relation

\[
\mathcal{D}_1^2 = \mathcal{D}_2^2 = \partial_x.
\]

5
Through cumbersome, otherwise straightforward calculation, one obtains

\[(\mathcal{D}_1 \theta_2) = -(\mathcal{D}_2 \theta_1), \quad (\mathcal{D}_2 \theta_2) = (\mathcal{D}_1 \theta_1).\] (14)

Summing up, we emphasize that under the constraints (13) and (14), the super diffeomorphism (10) is a superconformal transformation, which can be formulated by

\[\mathcal{D}_1 = K^{-1}\left((\mathcal{D}_1 \theta_1)\mathcal{D}_1 + (\mathcal{D}_2 \theta_1)\mathcal{D}_2\right),\] (15a)

\[\mathcal{D}_2 = K^{-1}\left(- (\mathcal{D}_2 \theta_1)\mathcal{D}_1 + (\mathcal{D}_1 \theta_1)\mathcal{D}_2\right),\] (15b)

where \(K = (\mathcal{D}_1 \theta_1)^2 + (\mathcal{D}_2 \theta_1)^2\).

The following formulas, which can be checked directly,

\[\mathcal{D}_1 \mathcal{D}_2 = K^{-1}\left[\mathcal{D}_1 \mathcal{D}_2 + \frac{1}{2}(\mathcal{D}_2 \log K)\mathcal{D}_1 - \frac{1}{2}(\mathcal{D}_1 \log K)\mathcal{D}_2\right],\] (16)

\[\partial_2 = K^{-1}\left[\partial_y - \frac{1}{2}(\mathcal{D}_2 \log K)\mathcal{D}_2 - \frac{1}{2}(\mathcal{D}_1 \log K)\mathcal{D}_1\right],\] (17)

are very useful when we study the link between the \(N = 2\) supersymmetric HD hierarchies to supersymmetric MKdV type hierarchies.

**IV. RECIPROCAL TRANSFORMATIONS**

In this section, we show that both \(N = 2\) supersymmetric HD hierarchies mentioned in the section two are linked with supersymmetric MKdV type hierarchies via superconformal transformations. First, let us consider the behavior of the Lax operators \(L_1\) and \(L_2\) under certain superconformal transformations.

For the Lax operator \(L_1 = -(WD_1D_2)^2\), if we assume \(W = K\), then according to the formula (16), we have the new operator in the super space \((y, \varrho_1, \varrho_2)\)

\[L_{1m} = -\left[\mathcal{D}_1 \mathcal{D}_2 + \frac{1}{2}((\mathcal{D}_2 \log K)\mathcal{D}_1 - \frac{1}{2}(\mathcal{D}_1 \log K)\mathcal{D}_2\right]^2.\] (18)

Let \(U = 1/2 \log K\), then

\[L_{1m} = -[\mathcal{D}_1 \mathcal{D}_2 + (\mathcal{D}_2 U)\mathcal{D}_1 - (\mathcal{D}_1 U)\mathcal{D}_2]^2.\] (19)

Applying the gauge transformation on \(L_{1m}\), we have

\[e^{-U}L_{1m}e^U = -[\mathcal{D}_1 \mathcal{D}_2 + (\mathcal{D}_1 \mathcal{D}_2 U) + (\mathcal{D}_2 U)(\mathcal{D}_1 U)]^2 \equiv -[\mathcal{D}_1 \mathcal{D}_2 + \Phi]^2\]
which is nothing but the Lax operator $\mathbb{L}_4$ of SKdV$_4$ equation. This implies that $L_{1m}$ should be a Lax operator for the modification of SKdV$_4$ equation. In fact, we have

**Proposition 1** The Lax equation

$$\frac{\partial}{\partial \tau_n} L_{1m} = \left[ (\hat{L}_{1m} L_{1m}^{\frac{1}{2}})_{\geq 1}, L_{1m} \right], \quad n = 0, 1, 2, \ldots \tag{20}$$

defines a $N = 2$ supersymmetric hierarchy, where

$$\hat{L}_{1m} = i [\mathbb{D}_1 \mathbb{D}_2 + (\mathbb{D}_2 U) \mathbb{D}_1 - (\mathbb{D}_1 U) \mathbb{D}_2], \quad \text{and} \quad L_{1m}^{\frac{1}{2}} = \partial_y + \cdots .$$

The first two non-trivial flows in this hierarchy are explicitly given by

$$U_{\tau_1} = \frac{i}{2} \left[ (\mathbb{D}_1 \mathbb{D}_2 U_y) + 2(\mathbb{D}_1 \mathbb{D}_2 U) U_y + 2(\mathbb{D}_2 U_y)(\mathbb{D}_1 U) + 2(\mathbb{D}_2 U)(\mathbb{D}_1 U) \right], \tag{21}$$

$$U_{\tau_2} = \frac{1}{4} \left[ U_{3y} - 2U_y^3 - 6(\mathbb{D}_1 \mathbb{D}_2 U_y)(\mathbb{D}_1 \mathbb{D}_2 U) - 6(\mathbb{D}_1 \mathbb{D}_2 U)^2 U_y - 3(\mathbb{D}_2 U_y)(\mathbb{D}_2 U)U_y - 9(\mathbb{D}_2 U_y)(\mathbb{D}_1 U)(\mathbb{D}_1 \mathbb{D}_2 U) - 9(\mathbb{D}_2 U)(\mathbb{D}_1 U_y)(\mathbb{D}_1 \mathbb{D}_2 U) - 6(\mathbb{D}_2 U)(\mathbb{D}_1 U)(\mathbb{D}_1 \mathbb{D}_2 U_y) - 3(\mathbb{D}_1 U_y)(\mathbb{D}_1 U)U_y \right]. \tag{22}$$

Furthermore, the $\tau_2$ flow (22) is the modification of SKdV$_4$ equation with the Miura transformation

$$\Phi = (\mathbb{D}_1 \mathbb{D}_2 U) + (\mathbb{D}_2 U)(\mathbb{D}_1 U).$$

**Proof:** Direct calculations.

The superconformal transformation provides us a link between the spatial variables. To have a complete picture, we need to find the counterpart for the temporal variables. For the couple of Lax hierarchies (3) and (20), the relations are given by

$$\frac{\partial}{\partial t_n} - (\hat{L}_{1} L_{1}^{\frac{1}{2}})_{\geq 2} = \frac{\partial}{\partial \tau_n} - (\hat{L}_{1m} L_{1m}^{\frac{1}{2}})_{\geq 1}. \tag{23}$$

We now calculate the explicit transformations for the first and second flows. In the simplest case, namely $n = 1$, since

$$(\hat{L}_1 L_1^{\frac{1}{2}})_{\geq 2} = iW^2 \partial_x \mathbb{D}_1 \mathbb{D}_2 + \frac{i}{2}(\mathbb{D}_1 W)W \partial_x \mathbb{D}_2 - \frac{i}{2}(\mathbb{D}_2 W)W \partial_x \mathbb{D}_1 + \frac{i}{2} W \mathbb{D}_1 \mathbb{D}_2$$

$$= i \partial_y \mathbb{D}_1 \mathbb{D}_2 - i(\mathbb{D}_1 U) \partial_y \mathbb{D}_2 + i(\mathbb{D}_2 U) \partial_y \mathbb{D}_1 - iU_y \mathbb{D}_1 \mathbb{D}_2$$

$$+ \left( - i(\mathbb{D}_1 U_y) + i(\mathbb{D}_1 U)U_y \right) \mathbb{D}_2 + \left( i(\mathbb{D}_2 U_y) - i(\mathbb{D}_2 U)U_y \right) \mathbb{D}_1,$$

$$(\hat{L}_{1m} L_{1m}^{\frac{1}{2}})_{\geq 1} = i \partial_y \mathbb{D}_1 \mathbb{D}_2 - i(\mathbb{D}_1 U) \partial_y \mathbb{D}_2 + i(\mathbb{D}_2 U) \partial_y \mathbb{D}_1 + i \left( (\mathbb{D}_1 \mathbb{D}_2 U) + (\mathbb{D}_2 U)(\mathbb{D}_1 U) \right) \partial_y - iU_y \mathbb{D}_1 \mathbb{D}_2.$$
By direct calculation, we have

\[
\frac{\partial}{\partial t_1} = \frac{\partial}{\partial r_1} - i \left( (D_1 D_2 U) + (D_2 U)(D_1 U) \right) \frac{\partial y}{\partial t_1} + \frac{i}{2} \left( (D_1 U_y) - (D_1 U)U_y - (D_2 U)(D_1 D_2 U) \right) D_2 + \frac{i}{2} \left( - (D_2 U_y) + (D_2 U)U_y - (D_1 U)(D_1 D_2 U) \right) D_1.
\]

Hence, we have

\[
\frac{\partial}{\partial t_2} = \frac{\partial}{\partial r_2} + \left( \hat{L}_1^{1,7} \right)_{\gtrless 2} - \left( \hat{L}_1^{1,7} \right)_{\gtrless 1}
\]

\[
= \frac{\partial}{\partial r_2} + \frac{1}{2} \left( 3(D_1 D_2 U)^2 - U_{2y} + U_y^2 + (D_2 U_y)(D_2 U) + 4(D_2 U)(D_1 U)(D_1 D_2 U) \\
+ (D_1 U_y)(D_1 U) \right) \frac{\partial y}{\partial t_2}
+ \frac{1}{4} \left( - (D_2 U_{2y}) + (D_2 U_y)U_y + 4(D_2 U)(D_1 D_2 U)^2 + (D_2 U)U_{2y} + 4(D_2 U)(D_1 U_y)(D_1 U) \\
- 5(D_1 U_y)(D_1 D_2 U) - (D_1 U)(D_1 D_2 U_y) + 4(D_1 U)(D_1 D_2 U)U_y \right) D_2
+ \frac{1}{4} \left( - (D_1 U_{2y}) + (D_1 U_y)U_y + 4(D_1 U)(D_1 D_2 U)^2 + (D_1 U)U_{2y} + 4(D_2 U_y)(D_2 U)(D_1 U) \\
+ 5(D_2 U_y)(D_1 D_2 U) + (D_2 U)(D_1 D_2 U_y) - 4(D_2 U)(D_1 D_2 U)U_y \right) D_1.
\]

Let us now consider the other \( N = 2 \) supersymmetric Harry Dym equation, whose Lax operator is \( L_2 = 1/2(D_1 W^2 D_1 + D_2 W^2 D_2) \partial_x \). In this case, we also assume \( W = K \). From the formulas \( (15a) \) \( (15b) \), we have

\[
\frac{1}{2} (D_1 W^2 D_1 + D_2 W^2 D_2) = K \partial_y + \frac{1}{2} (D_2 K) D_2 + \frac{1}{2} (D_1 K) D_1.
\]

Then taking the formula \( (17) \) into account, \( L_2 \) is transformed to

\[
L_{2n} = \left[ K \partial_y + \frac{1}{2} (D_2 K) D_2 + \frac{1}{2} (D_1 K) D_1 \right] K^{-1} \left[ \partial_y - \frac{1}{2} (D_2 \log K) D_2 - \frac{1}{2} (D_1 \log K) D_1 \right]
= \partial_y^2 - U_y \partial_y - \frac{1}{2} (D_2 U)(D_1 U) D_1 D_2 + \frac{1}{4} \left( - 2(D_2 U_y) + (D_2 U)U_y - (D_1 U)(D_1 D_2 U) \right) D_2
+ \frac{1}{4} \left( - 2(D_1 U_y) + (D_1 U)U_y + (D_2 U)(D_1 D_2 U) \right) D_1.
\]

By direct calculation, we have
Proposition 2 \textit{The Lax equation}

\[
\frac{\partial}{\partial \tau_n} L_{2m} = \left[ (L_{2m}^2)^{n \geq 1}, L_{2m} \right], \quad n = 1, 3, \ldots
\]  \hspace{1cm} (25)

defines a $N = 2$ supersymmetric hierarchy, whose first non-trivial flow is

\[
U_{\tau_3} = \frac{1}{16} \left( 4U_{3y} - 2U_y^3 - 3(\mathcal{D}_2 U_y)(\mathcal{D}_2 U)U_y + 3(\mathcal{D}_2 U_y)(\mathcal{D}_1 U)(\mathcal{D}_1 \mathcal{D}_2 U) \\
+ 3(\mathcal{D}_2 U)(\mathcal{D}_1 U_y)(\mathcal{D}_1 \mathcal{D}_2 U) - 3(\mathcal{D}_1 U_y)(\mathcal{D}_1 U)U_y \right). \hspace{1cm} (26)
\]

Furthermore, the equation (26) is a modification of the SKdV$_{-2}$ equation with the Miura type transformation

\[
\Phi = \frac{1}{2} (\mathcal{D}_1 \mathcal{D}_2 U) + \frac{1}{4} (\mathcal{D}_2 U)(\mathcal{D}_1 U). \hspace{1cm} (27)
\]

\textbf{Proof:} Direct calculations.

\textbf{Remark:} As a byproduct of above results, a relation could be inferred between the bosonic limit of SKdV$_{-2}$ and that of Eq. (26). Let $\Phi = \phi_0 + \theta_2 \theta_1 \phi_1$ and $U = u_0 + \theta_2 \theta_1 u_1$, then the bosonic limits of SKdV$_{-2}$ and Eq. (26) are respectively given by

\[
\begin{align*}
\phi_{0,\tau_3} &= \frac{1}{4} \left( \phi_{0,3y} + 6\phi_{0,y}\phi_0^2 \right), \\
\phi_{1,\tau_3} &= \frac{1}{4} \left( \phi_{1,3y} - 6\phi_{1,y}\phi_1 + 6\phi_{1,y}\phi_0^2 + 12\phi_1 \phi_{0,y} \phi_0 - 6\phi_{0,2y} \phi_{0,y} \right),
\end{align*}
\]

and

\[
\begin{align*}
u_{0,\tau_3} &= \frac{1}{8} \left( 2u_{0,3y} - u_{0,y}^3 \right), \\
u_{1,\tau_3} &= \frac{1}{8} \left( 2u_{1,3y} + 3u_{1,y} u_1^2 \right).
\end{align*}
\]

The Miura transformation between them is

\[
\phi_0 = \frac{1}{2} u_1, \quad \phi_1 = \frac{1}{4} \left( -2u_{0,2y} + u_{0,y}^2 + u_1^2 \right).
\]

As above, we derive the transformation between the temporal variables. For the couple of hierarchies (7) and (25), the relations between vector fields of time variables are given by

\[
\frac{\partial}{\partial t_n} - (L_{2}^\frac{n}{2})^{\geq 2} = \frac{\partial}{\partial \tau_n} - (L_{2m}^\frac{n}{2})^{\geq 1}. \hspace{1cm} (28)
\]

When $n = 3$, we have

\[
\frac{\partial}{\partial t_3} = \frac{\partial}{\partial \tau_3} + (L_{2}^\frac{3}{2})^{\geq 2} - (L_{2m}^\frac{3}{2})^{\geq 1} \\
= \frac{\partial}{\partial \tau_3} + \frac{1}{8} \left( -2U_{2y} + U_y^2 + (\mathcal{D}_2 U_y)(\mathcal{D}_2 U) - (\mathcal{D}_2 U)(\mathcal{D}_1 U)(\mathcal{D}_1 \mathcal{D}_2 U) + (\mathcal{D}_1 U_y)(\mathcal{D}_1 U) \right) \partial_y
\]
\[ + \frac{1}{16} \left( -2(D_2 U_y) + (D_2 U_y) U_y - (D_2 U)(D_1 D_2 U)^2 + (D_2 U) U_{2y} - (D_2 U)(D_1 U_y)(D_1 U) \\
+ (D_1 U)(D_1 D_2 U) - (D_1 U)(D_1 D_2 U_y) - (D_1 U)(D_1 D_2 U) U_y \right) D_2 \]
\[ + \frac{1}{16} \left( -2(D_1 U_{2y}) + (D_1 U_y) U_y - (D_1 U)(D_1 D_2 U)^2 + (D_1 U) U_{2y} - (D_2 U_y)(D_2 U)(D_1 U) \\
- (D_2 U_y)(D_1 D_2 U) + (D_2 U)(D_1 D_2 U_y) + (D_2 U)(D_1 D_2 U) U_y \right) D_1. \]

V. SUMMARY AND PROBLEMS

With the help of superconformal transformations, we have established the relations between two \( N = 2 \) supersymmetric KdV equations and two supersymmetric HD equations, therefore we generalize our results of \( N = 1 \) supersymmetric reciprocal transformations to the \( N = 2 \) case. While this is interesting, there are a number of problems to be solved. We list some of them as follows

- Our construction above relies on the Lax representations, so it would be important to recover the transformations by means of other property such as conservation laws.

- Studying the implications of our transformations is another interesting problem. Indeed, more properties are known for the KdV cases than for the HD cases, for instance, bi-Hamiltonian structures have been constructed for the \( N = 2 \) supersymmetric KdV systems, but we know little about Hamiltonian structures for the \( N = 2 \) supersymmetric HD equations apart from the following observation. The systems (5) and (6) can be reformulated as

\[ W_{t_1} = \mathbb{B} \frac{\delta H_1}{\delta W}, \quad W_{t_2} = \mathbb{B} \frac{\delta H_2}{\delta W}, \]

where \( \mathbb{B} = W^2 D_1 D_2 \partial_x W^2 \) is a Hamiltonian operator and Hamiltonian functionals are given by

\[ H_1 = \frac{i}{2} \int \ln W \, dz d\theta_1 d\theta_2, \quad \text{and} \quad H_2 = \frac{1}{8} \int (D_1 W)(D_2 W) W^{-1} \, dz d\theta_1 d\theta_2. \]

- In the Harry Dym case, two known \( N = 2 \) integrable supersymmetric extensions are proved to be linked with two \( N = 2 \) supersymmetric KdV equations, but there are three rather than two such systems. Thus, this fact seems to indicate that one \( N = 2 \) integrable supersymmetric Harry Dym equation is missing.
The progress on solving these and other related problems may be reported elsewhere.

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