Quasi-exactly solvable quartic potentials with centrifugal and Coulombic terms

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Abstract

$D$–dimensional central and complex potentials of a Coulomb plus quartic-polynomial form are considered in a $\mathcal{PT}$–symmetrized radial Schrödinger equation. Arbitrarily large finite multiplets of bound states are shown obtainable in an elementary form. Relations between their energies and couplings are determined by a finite-dimensional secular equation. The Bender’s and Boettcher’s one-dimensional quasi-exact oscillators re-emerge here as the simplest chargeless solutions.
1 Introduction

Recently, Bender and Boettcher [1] studied the three-parametric family of Hamiltonians

\[ H^{(BB)} = -\partial_x^2 - x^4 + 2iax^3 + cx^2 + i(a^3 - ac - 2J)x, \quad a, c \in \mathbb{R}, \quad J = 1, 2, \ldots \]  

(1)
defined on a certain complex curve \( x = x(t) \in \mathbb{C}, \ t \in (-\infty, \infty) \). Their main result was an explicit construction of a \( J \)-plet of bound states in an elementary and closed form

\[ \psi_j(x) = e^{-ix^3/3 - ax^2/2 - i bx} P_{j,J-1}(x), \quad b = (a^2 - c)/2, \quad j = 1, 2, \ldots, J. \]

Here, \( P_{j,J-1}(x) \) denotes certain polynomials of the \((J-1)\)-st degree. The normalizability of the multiplet is manifestly guaranteed via pre-selection of the asymptotics of \( x(t) \),

\[ x(t) \sim (\cos \varphi \pm i \sin \varphi) t, \quad t \to \mp \infty, \quad \varphi \in (0, \pi/3). \]

It is easy to check that we may even stay on the straight line \( x(t) = t - i \varepsilon \), provided only that we choose it in such a way that \( \text{Re}(2\varepsilon + a) > 0 \).

An older paper by Buslaev and Grechii [2] may be recalled for the rigorous proof that the Hamiltonian (1) has the real and discrete spectrum. The paper shows that after a constant shift \( \delta \sim b^2 + Ja/2 \) the spectrum will coincide with the energy levels of the \( D \)-dimensional anharmonic oscillator \( H^{(AHO)} = -\Delta + \frac{1}{4} r^2 + \frac{1}{4} \|r^2\|^2 \) in its \( \ell \)-th partial-wave mode such that \( 2J = D + 2\ell - 2 \). This assignment (or “BG transformation”) combines a change of variables with Fourier transformation. Its application to \( H^{(BB)} \) at an integer \( J \) specifies either our choice of the AHO partial wave \( \ell = 0, 1, \ldots \) or of the (even) dimension \( D \). In the opposite direction it throws new light on the non-Hermitian Hamiltonians \( H^{(BB)} \) which exhibit the puzzling [3] \( \mathcal{PT} \)-symmetry

\[ \mathcal{PT} H^{(BB)} = H^{(BB)} \mathcal{PT}, \quad \mathcal{P}\psi(x) = \psi(-x), \quad \mathcal{T}\psi(x) = \psi^*(x). \]

The \( J \)-plets of their exact states become transformed in the unusual anharmonic oscillator states obtainable in terms of certain elementary Fourier-type integrals (cf. also ref. [4] in this context). This adds a new and strong reason why the models of the type (1) deserve a thorough attention.

We may only feel dissatisfied by a certain incompleteness of the whole picture: Why the BG transformation prefers the even dimensions? What could be done at the odd \( D = 1, 3, \ldots \)? The free variability of the integer \( D \) would be highly welcome also in some phenomenological \( D \gg 1 \) models in nuclear physics [5], quantum chemistry [6] and atomic physics [7].

The second reason why the class of models (1) looks so inspiring is related to the second version of the BG transformation, presented in paper [2] as “the second main result”. It starts from a less usual, volcano-shaped model \( V(\vec{r}) \sim \omega^2 r^2 - [r^2]^2 \). The current partial-wave decomposition of its
$D$-dimensional wave functions

$$\Psi(\vec{r}) = \sum_{\ell=0}^{\infty} r^{(1-D)/2} \psi(r) \times \text{angular part}, \quad r = |\vec{r}| \in (0, \infty)$$

reduces its Schrödinger equation to the ordinary equation on the half-axis. Using a complex shift of the coordinates again, Buslaev and Grecchi arrive at a new, properly $\mathcal{PT}$-symmetrized two-parametric operator

$$H^{(BG)} = -\partial_t^2 + \frac{L(L+1)}{x^2} + \omega^2 x^2 - x^4, \quad x = x(t) = t - i \varepsilon, \quad L = \ell + (D-3)/2,$$

extended to the whole real line $t \in \mathbb{R}$ and acting in the current Hilbert space $L^2(\mathbb{R})$. They prove its isospectrality with a certain one-dimensional (and Hermitian) double-well oscillator. In the resulting quadruple scheme

| $D$-dimensional and Hermitian | 1-dimensional and Hermitian |
|-------------------------------|-----------------------------|
| $-x^4$ BB model (1) $\leftrightarrow$ $x \rightarrow i x$ | $+x^4$ double well $\leftrightarrow$ $x \rightarrow i(t - i \varepsilon)$ |
| $\uparrow$ BG transformation | $\uparrow$ BG transformation |

the two columns are related by the elementary changes of variables. The upper-left-corner item is characterized by its partial solvability. In what follows we intend to show that the lower-right-corner-like singular Hamiltonians may equally well exhibit the same (or at least very similar) multiplet or “quasi-exact” [8] solvability.

## 2 $\mathcal{PT}$-regularized singular oscillators

The problem we shall solve here is the Schrödinger equation

$$\left[ -\frac{d^2}{dx^2} + \frac{L(L+1)}{x^2} + i \frac{d}{x} + i c x + b x^2 + i a x^3 - x^4 \right] \psi(x) = E \psi(x)$$

(3)

defined on curves $x(t) \neq 0$ to be specified later. It contains the bound state problems (1) and (2) after an appropriate choice of its parameters. As long as certain characteristic features of the similar $\mathcal{PT}$-symmetric systems will be of an immediate relevance here, let us first review some of them briefly.
### 2.1 \( PT\)-symmetry

Many complex models with \( PT\) symmetry (and, let us assume, real spectra, \( \text{Im} \ E = 0 \)) are defined on real line [10]. Their potentials comprise a spatially symmetric real well plus its purely imaginary antisymmetric complement. An interpretation of these models parallels the usual real and symmetric bound state problems in one dimension since one may try to switch off the imaginary force smoothly. Typically, the spectrum splits in the convergent and divergent parts in this limit.

Recalling eq. (2) for illustration and definiteness, we identify the convergent part with the standard physical solutions \( \psi_{(qo)}(r) \sim r^{\ell+(D-1)/2} \) of a “quasi-odd” or “regular” type. The divergent part represents the “quasi-even” components \( \psi_{(qe)}(r) \sim r^{(3-D)/2-\ell} \). For a further explicit illustration one may recall the harmonic oscillator [11] or a few other solvable examples [12].

The less trivial set of the models which “weaken” their Hermiticity to the mere \( PT\) symmetry may be defined off the real line [13]. In the usual mathematical notation one speaks about sectors where the \( L^2(\mathbb{R}) \) boundary conditions are imposed. Thus, returning to our examples (1) – (3) we define sectors \( S_k = \{ x \in \mathbb{C}; x \neq 0, |\text{arg}(x) - \frac{1}{2}(2k-1)\pi| < \frac{1}{4}\pi \} \) and choose \( S_4 \) and \( S_6 \) in accord with the recommendation of ref. [1], or \( S_1 \) and \( S_3 \) as accepted in ref. [2] (cf. Figure 1).

The latter ambiguity of the curve \( x(t) \) may be characterized by a “signature” \( \sigma = \pm 1 \) which easily distinguishes between the two \( T\)-conjugate curves \( x(\sigma)(t) = x(\sigma)(t) \) with the asymptotics bounded by the \( \varphi \to \pi/3 \) Stokes lines,

\[
x^{(\sigma)}(t) \sim (\cos \varphi \pm i \sigma \sin \varphi) t, \quad t \to \mp \infty, \quad \varphi \in (0, \pi/3).
\]

Even in the admissible straight-line or \( \varphi \to 0 \) extreme \( x^{(\sigma)}(t) = t \pm i \sigma \varepsilon \), the possible branch-point singularity in the origin must be avoided properly, e.g., in the trivial \( T\)-conjugation manner which leaves the (real) spectrum unchanged. For this reason the condition which determines \( \varepsilon \) is now more restrictive and reads \( 2\varepsilon > |a| \).

We may summarize that our curves of integration \( x(t) = x(\sigma)(t) \) may be straight lines or hyperbolic shapes open downwards (\( \sigma = +1 \), [1]) or upwards (\( \sigma = -1 \), [2]). For the fully regular forces their non-asymptotic form may be deformed almost arbitrarily. For all the other models with singularities these curves must avoid all the cuts. In particular, eqs. (2) and (3) admit the presence of an essential singularity in the origin. After we cut the plane upwards (for \( \sigma = +1 \)) or downwards (for \( \sigma = -1 \)), both our curves \( x = x^{(\sigma)}(t) \) may be kept smooth and, say, symmetric with respect to the \( T\) reflections.

As long as we are relaxing the usual Hermiticity, \( H^{(BB)} \neq [H^{(BB)}]^+ \), both the two independent wave function solutions remain equally admissible in the vicinity of the origin. This is the main formal reason why the Bender’s and Boettcher’s complex oscillator remained solvable: One only has to combine the quasi-even behaviour \( \psi_{(qe)}(r) \sim O(r^0) \) with its quasi-odd parallel \( \psi_{(qo)}(r) \sim O(r^1) \) in the single ansatz. Here we intend to proceed in the same manner.
2.2 Recurrences

Schrödinger equation (3) with the integer or half-integer values of \( L = \ell + (D - 3)/2 = K/2 \) remains perfectly regular along both our non-self-intersecting integration paths. After a change of variables \( x^{(\sigma)}(t) = -iy(t) \) we get the new differential equation for \( \psi(x) = \chi(i x) \) which contains, formally, no imaginary units,

\[
\left[ \frac{d^2}{dy^2} - \frac{L(L+1)}{y^2} - \frac{d}{y} + cy - by^2 - ay^3 - y^4 \right] \chi(y) = E\chi(y). \tag{5}
\]

We shall search for its solutions by a power-series ansatz, assuming that such a series terminates. Demanding also the manifest normalizability in a way depending on the signature \( \sigma = \pm 1 \) we arrive at a virtually unique formula

\[
\chi(y) = \exp \left( \sigma \frac{1}{3} y^3 + \frac{1}{2} Tyy^2 + Sy \right) \sum_{n=0}^{N} h_n y^{-L}. \tag{6}
\]

It combines both the respective \( y^{-L} \) and \( y^{L+1} \) quasi-even and quasi-odd components, and equation (5) determines the asymptotically correct values of our auxiliary parameters as well as the unique termination-compatible coupling \( c \),

\[
T = a/2\sigma, \quad S = (b - T^2)/2\sigma, \quad c \equiv c(N) = -2TS - \sigma \cdot (2N - 2L + 2).
\]

The resulting ansatz (6) represents the desired normalizable solutions if and only if its coefficients \( h_n \) are compatible with the recurrences

\[
h_{n+1}A_n + h_nB_n + h_{n-1}C_n + h_{n-2}D_n = 0, \quad n = 0, 1, \ldots, N+1. \tag{7}
\]

Their coefficients

\[
A_n = (n+1)(n-2L), \quad B_n = S(2n-2L)-d, \quad C_n = S^2+T(2n-2L-1)-E, \quad D_n = 2\sigma(n-N-2)
\]

are all elementary.

3 Finite-dimensional Schrödinger equation

In the light of their strict postulated termination, our recurrences (7) represent just a finite set of \( N + 2 \) linear algebraic equations for \( N + 1 \) unknown coefficients \( h_n \). Such a set is, obviously, over-determined. Its \( (N+2) \times (N+1) \)-dimensional non-square matrix has in effect two main diagonals,

\[
B_n = S(2n-K) - d, \quad C_n = S^2 + T(2n-K-1) - E, \quad K = 2L.
\]
Both the quantities $d$ and $E$ can play a role of an eigenvalue simultaneously [14]. In the other words, we may interpret the whole linear set of equations as the two coupled square-matrix problems indicated, say, by the single and double line in the whole non-square system

\[
\begin{pmatrix}
B_0 & A_0 \\
C_1 & B_1 & A_1 \\
D_2 & C_2 & \ddots & \ddots \\
\vdots & \ddots & B_{N-1} & A_{N-1} \\
D_N & C_N & B_N \\
D_{N+1} & C_{N+1}
\end{pmatrix}
\begin{pmatrix}
h_0 \\
h_1 \\
h_2 \\
\vdots \\
h_N
\end{pmatrix} = 0.
\tag{8}
\]

In both cases (i.e., omitting the last or first line, respectively) we get a routine secular determinant with an eigenvalue on its main diagonal. Both these two independent secular or termination conditions have to be satisfied simultaneously.

### 3.1 Auxiliary constraint

As long as the upper diagonal $A_n = (n + 1)(n - K)$ vanishes at $n = K$, there emerges an important asymmetry between our two eigenvalues $d$ and $E$. Whenever one keeps just a few lowest partial waves, we may say that the integer $K = 2L$ remains “very small”, at least in comparison with $N$ which can/should be “large” or at least “arbitrary” in principle. This is our present fundamental observation. As a consequence of the related disappearance of $A_K = 0$ we shall achieve a thorough simplification of the construction of bound states.

At the lowest possible $K = 0$ we return immediately to the Bender’s and Boettcher’s proposal. Their choice of $d = 0$ in eq. (1) (with $J = N + 1$) gives us the highly welcome possibility of omitting the whole first row from eq. (8). The rest of this equation is the usual square-matrix diagonalization. All the exceptional quasi-exact eigenvalues $E_j$, $j = 1, 2, \ldots, N + 1$ may be determined as roots of a polynomial of the $(N + 1)$–st degree [1].

A transition to the nonzero integers $K$ is easy. In place of using the trivial $d = 0$ we have to satisfy the $(K + 1)$–dimensional sub-equation

\[
\det
\begin{pmatrix}
B_0 & A_0 \\
C_1 & B_1 & A_1 \\
D_2 & C_2 & \ddots & \ddots \\
\vdots & \ddots & B_{K-1} & A_{K-1} \\
D_K & C_K & B_K
\end{pmatrix} = 0.
\tag{9}
\]

This can fix, say, the eligible electric charges $d$ as functions of the other parameters. These functions can be used as a starting point of a facilitated solution of our original problem (8).
3.2 Closed formulae for the energies

In the simplest test of our new and general recipe let us return, once more, to \( K = 0 \) in eq. (9) and derive

\[
d = 0, \quad K = 0.
\]

This reproduces the model \( H^{(BB)} \) of ref. [1]. In the next (and the first really innovative) \( K = 1 \) step we get the double root \( d = d_{(\pm)}(E) = \pm \sqrt{E} \). Each choice of the sign specifies a different relation between the charge and the energy. Both these signs remain admitted by the inverted, unique recipe

\[
E = E_K(d) = d^2, \quad K = 1.
\]

It defines the energy as a function of the charge. The numbering of the separate elements of our new, \( K = 1 \) multiplets of bound states is in this way transferred directly to the admissible charges \( d = d_j \). Formally this means that the energy \( E \) may be eliminated from all our algebraic equations. The values of the charge \( d \) remain the only unknown quantities.

At the subsequent integer \( K = 2 \) we get, with a bit of luck, a highly compact energy formula again,

\[
E = E_K(d) = \frac{d^2}{4} + 2 \frac{ST + \sigma N}{d}, \quad K = 2.
\]

The further step gives the two rules or roots \( E_3(d) = F \) of the quadratic equation

\[
9 F^2 - 10 d^2 F + d^4 + 48 dST + 48 d\sigma N - 24 d - 72 S - 36 T^2 - 0, \quad K = 3.
\]

Hence, all the \( K \geq 3 \) cases require the so called Gröbner elimination which should (and easily can) be performed by a computer. We only quote a similar calculation [6] and omit the further details.

4 Secular determinant and its roots

In the usual quasi-exact manner [8] the multiplets of the (real) values of \( d_j \) have to follow from our termination postulate at any positive integer \( N \). Let us now assume that the value of \( K \) is fixed. After the insertion of its energy formula \( E = E_K(d) \) in the recurrences or equation (8) we are left with the square-matrix secular equation of the dimension \( (N + 1) \times (N + 1) \),

\[
\begin{vmatrix}
C_1(d) & B_1(d) & A_1 \\
D_2 & C_2(d) & \ddots & \ddots \\
& \ddots & \ddots & B_{N-1}(d) & A_{N-1} \\
D_N & C_N(d) & B_N(d) \\
D_{N+1} & C_{N+1}(d)
\end{vmatrix} = 0.
\]

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This equation has to determine the unknown eigencharges. In its analysis we may skip the already known Bender’s and Boettcher’s $K = 0$ case [1] and pay attention to the next few $K = 1, 2, \ldots$.

### 4.1 The first nontrivial option: $K = 1$

Insertion of the available formulae gives us the $K = 1$ secular determinants with the only $N$– and $\sigma$–dependent elements $D_j(N) = 2\sigma(n - N - 2),$

\[
\begin{vmatrix}
S^2 - d^2 & S - d & 0 \\
D_2(N) & S^2 + 2T - d^2 & 3S - d & 3 \\
D_3(N) & S^2 + 4T - d^2 & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots \\
D_{N+1}(N) & S^2 + 2NT - d^2 & \end{vmatrix} = 0. \tag{10}
\]

At any $N$ this equation gives an “exceptional” explicit root $d = S$. Although it looks like an artifact of our present approach, its further inspection reveals its acceptability. For example, at $N = 1$ this root gives a solution provided that we have $ST = -\sigma/2$.

All the other roots of eq. (10) remain manifestly $N$–dependent. They also exhibit certain symmetries. For example, the change of the signature $\sigma \to -\sigma$ may be compensated by the simultaneous sign-change of $S \to -S$ and $d \to -d$.

In a more constructive setting it makes sense to pick up a specific $N$. Starting from the smallest $N = 1$ we arrive at the cubic equation

\[-d^3 - S d^2 + (S^2 + 2T) d + 2 \sigma + S (S^2 + 2T) = 0, \quad N = 1.\]

We may determine the $S$– and $T$–dependence of its three roots $d_1$, $d_2$ and $d_3$ via Cardano formulae. If needed, the boundaries of the domain of their reality may be determined numerically, in a complete parallel to the $K = 0$ study [1]. In order to prove that this domain is not empty at $K = 1$, we may recall a sample triplet, say, of $d_{1,2,3}(S,T) = (-5.303953910, -3.103253421, 5.407207331)$ at $(S,T) = (3,10)$.

The growth of the dimension $N + 1$ makes the practical determination of the roots $d$ of eq. (10) more complicated. This is well illustrated by its next two $K = \sigma = 1$ explicit polynomial representants,

\[-d^5 + S d^4 - (2 S^2 + 6T) d^3 - (6 + 2 S^3 + 6 ST) d^2 + (6 S^2 T + 4 S + 8 T^2 + S^4) d + 10 S^2 + 16 T + 6 S^3 T + 8 ST^2 + S^5 = 0, \quad N = 2.\]
\[-d^7 - Sd^6 + (12T + 3S^2)d^5 + (12 + 12ST + 3S^3)d^4 - (16S + 44T^2 + 3S^4 + 24S^2T)d^3
- (40S^2 + 88T + 44ST^2 + 3S^5 + 24S^3T)d^2
+ (12 + 16S^3 + S^6 + 44S^2T^2 + 64ST + 48T^3 + 12S^4T)d
+ 152S^2T + 28S^4 + 84S + 144T^2 + S^7 + 44S^3T^2 + 48ST^3 + 12S^5T = 0, \quad N = 3.\]

To this list with the respective 16 and 31 terms one could add the next \(N = 4\) item containing 53 terms, etc. This would complement the similar \(K = 0\) formulae displayed in detail in ref. [1].

### 4.2 Numerical illustration

An overall insight in the structure of the spectra may be mediated by their simplest \(S = T = 0\) sample. *A posteriori*, the main merit of such a choice may be seen in a drastic formal reduction of the underlying secular polynomials. One may factorize many of them by the purely non-numerical means.

#### 4.2.1 \(K = 0\)

In the paper [1] the domain of existence of the full quasi-exact \(N\)–plets with \(K = 0\) has been determined numerically. This analysis excluded the point \(S = T = 0\) where, empirically, the \(\mathcal{PT}\) symmetry becomes spontaneously broken. Computationally, this means that some of the energies coalesce and move in pairs off the real line. Hence, our understanding of the \(K = S = T = 0\) spectra is not complete yet.

Direct computation at \(N = 0,\ N = 1\) and \(N = 2\) gives just the trivial \(E = 0\). In the next two cases we get the \(\sigma\)–independent equations

\[
E^4 - 96E = 0, \quad N = 3
\]
\[
-E^5 + 336E^2 = 0, \quad N = 4
\]

with the single nonzero real root \(E \sim 4.48\) at \(N = 3\) and \(E \sim 6.95\) at \(N = 4\). For the general \(N\) the secular equation reads

\[
\begin{vmatrix}
-E & 0 & 1 \cdot 2 \\
2N\sigma & -E & 0 & 2 \cdot 3 \\
& \ddots & \ddots & \ddots & \ddots \\
6\sigma & -E & 0 & (N-1)N \\
4\sigma & -E & 0 \\
2\sigma & -E
\end{vmatrix}
= 0.
\]

Its \(\sigma\)–independence and polynomiality in \(x = E^3 \neq 0\) can be proved in an easy exercise. As a consequence, its roots remain non-numerical up to \(N = 13\). In the latter extreme we get the real quadruplet of nonzero energies \(E \sim 9.381, 17.768, 26.487\) and 35.535.
4.2.2 $K = 1$

After one moves to the negative signature $\sigma = -1$, only certain signs change in the secular polynomials of sect. 4.1. At both $\sigma = \pm 1$ we may reduce the first few cases to their pertaining $S = T = 0$ forms. They remain non-numerical and exactly solvable up to the dimension $N = 6$. Their first few samples are

\[
\begin{align*}
-d^3 + 2\sigma &= 0, \\
-d^5 - 6\sigma d^2 &= 0, \\
-d^7 + 12\sigma d^4 + 12d &= 0, \\
-d^9 - 20\sigma d^6 - 76d^3 + 512\sigma &= 0,
\end{align*}
\]

In detail, the single real root $d \approx 1.26 \cdot \sigma$ is obtained at $N = 1$, and one double zero and one real nonzero root $d \approx 1.71 \cdot \sigma$ appear at $N = 2$. One simple zero and one positive and one negative root $(d_1 \approx 2.35 \cdot \sigma, \; d_2 \approx -0.975 \cdot \sigma)$ follow at $N = 3$ while, finally, three nonzero real roots $\sigma \cdot d_{1,2,3} \approx (2.82, 1.55, -1.83)$ emerge from our last displayed polynomial at $N = 4$.

4.2.3 $K = 2$

In a move to the higher $K > 1$ one has to notice that the first few choices of $N \leq K$ are too formal and do not make an explicit use of the present “advantage” $A_K = 0$ at all. In this sense, the $K = 2$ illustration has to start from the first nontrivial $N = 3$. After we fix $\sigma = +1$ for brevity, we get the secular polynomial of the deterring twelfth degree. Fortunately, in the same spirit as observed above the abbreviation $x = d^3 \neq 0$ reduces it again to a (still solvable) quartic equation,

\[
x^4 + 331776 - 96x^3 + 384x^2 + 18432x = 0, \quad \sigma = 1, \quad K = 2, \quad N = 3.
\]

This equation possesses the two real and positive roots, namely, $x_1 = 24$ and

\[
x_2 = 24 + 16\sqrt[9]{9 + \sqrt{17}} + \frac{64}{\sqrt[9]{9 + \sqrt{17}}} \approx 88.87294116.
\]

These roots lead to the real and positive charges $d_{1,2} \approx (2.88, 4.46)$ and to the $K = 2$ energies $E_2(d_j)$ as prescribed above. The parallel problem with $\sigma = -1$ leads to the full quadruplet of the negative real roots

\[
x_{1,2,3,4} \approx (-199.78, -72.00, -14.65, -1.57), \quad \sigma = -1
\]

and to the eigencharges with the same signs, $d_{1,2,3,4} \approx (-5.84, -4.16, -2.45, -1.16)$.

We may conclude that at a fixed $K$ (i.e., for a specific partial wave $\ell$) we get in general a multiplet of states connected by some broken lines in the two-dimensional charge-energy plane. Only in the simplest $K = 0$ special case this “guide for eye” becomes the Bender-Boettcher straight line $d = const = 0$. 

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5 Discussion

We have shown that the $N$-plets of exact bound-state solutions of our general quartic problem (3) can be constructed in closed form at any integer degree $N$ and dimension $D$. In the other words, under certain relationship between the couplings and energies, arbitrarily large multiplets of solutions (with real energies!) proved obtainable from a single and finite-dimensional secular equation. The Bender’s and Boettcher’s quartic example re-appears here as the simplest one-dimensional special case with the unique (and, incidentally, vanishing) Coulombic charge $d$.

In many applications of practical interest the closed quasi-exact solutions are of the similar Sturmian type. Their bound states are numbered by one of the couplings but still share, mostly, the same value of the energy. Many Hermitian models belong to this category, be it one of the most popular non-polynomial examples [15] or one of the historically first models of the quasi-exact type [16]. Only the most popular partially solvable sextic model [17] owes for its popularity to its most standard variable-energy character.

We have presented the explicit construction of a few multiplets which lie along the more general curves (e.g., parabolas) in the energy-coupling space. In combination with their overall $\mathcal{PT}$ symmetric quantum mechanical framework these multiplets offer several new challenges, e.g., in their completely missing abstract interpretation, say, from the modern Lie-algebraic point of view [18].

In our present approach the recent unexpected construction of Bender and Boettcher [1] finds one of its simplest and quite natural explanations. We have found that its solvable status has been mediated first of all by its complex, non-Hermitian background. In a way which clarifies the whole paradox the exceptional features of the Bender’s and Boettcher’s oscillator were related to its special chargeless form.

In another approach to the same problem one may just speak about the underlying system of recurrences. In this language, it remained unnoticed in the current literature on anharmonic oscillators that the coefficient $A_K$ can vanish at the integer $K = 2L = 2\ell + D - 3$. Still, in one dimension, this was just the core of feasibility of the Bender’s and Boettcher’s surprising construction. This immediately implies that the Bender’s and Boettcher’s one-dimensional potential is just a very special case of the general class of quasi-exactly solvable quartic models. Their description has been offered here.

It is probably worth re-emphasizing that the vanishing of the coefficient $A_K = 0$ does occur for the $s$-waves in three dimensions as well as for $p$-waves in one dimension. This is our reinterpretation of the known $K = 0$ results. In the same sense, the present new $K = 1$ case covers the $s$-waves in four dimensions and $p$-waves in two dimensions. Similarly, we have $K = 2$ for $s$-waves in five dimensions and for $p$-waves in three dimensions, and we encounter the triple possibility of $(\ell, D) = (0, 6), (1, 4)$ and $(2, 2)$ at $K = 3$. Etc. The first problem formulated in
our introductory section is satisfactorily settled: In principle, the multiplet solvability occurs in all dimensions.

Our second initial motivation concerned the possible tractability of strong singularities of the centrifugal and Coulombic type. We have preserved their certain constrained variability within the Buslaev’s and Grecchi’s \( \mathcal{PT} \) regularization scheme. In this way we achieved a satisfactory balance in the picture given at the end of section 1.

Further lessons from our construction are not quite clear. In the future, we intend to pay more attention to the underlying Fourier-mediated \( p \leftrightarrow x \) symmetries, trying to move beyond their known harmonic-oscillator, BG-transformation [2] or quasi-harmonic-oscillator [19] exemplifications. In such a context, the role of the centrifugal-like singularities does not seem to have said its last word yet.

**Acknowledgements**

My thanks belong to Rajkumar Roychoudhury (ISI Calcutta) and to Francesco Cannata (INFN Bologna) who insisted that the Bender-Boettcher model deserves a deeper study. Partial support by the GA AS CR grant Nr. A 1048004 is acknowledged.
References

[1] Bender C M and Boettcher S 1998 J. Phys. A: Math. Gen. 31 L273
[2] Buslaev V and Grechi V 1993 J. Phys. A: Math. Gen. 26 5541
[3] Bender C M and Boettcher S 1998 Phys. Rev. Lett. 24 5243
[4] Flessas G P and Watt A 1981 J. Phys. A: Math. Gen. 14 L315
[5] Sotona M and Žofka J 1974 Phys. Rev. C 10 2646
[6] Znojil M 1999 J. Math. Chem. 26 157
[7] Panja M, Dutt R and Varshni Y P 1990 Phys. Rev. A 42 106
[8] Ushveridze A G 1994 Quasi-exactly solvable models in quantum mechanics (Bristol: IOPP)
[9] Bender C M, Boettcher S and Meisinger P N 1999 J. Math. Phys. 40 2201
[10] Fernández F M, Guardiola R, Ros J and Znojil M 1999 J. Phys. A: Math. Gen. 32 3105;
  Bagchi B and Roychoudhury R 2000 J. Phys. A: Math. Gen. 33 L1;
  Znojil M 2000 J. Phys. A: Math. Gen. 33 L61
[11] Znojil M 1999 Phys. Lett. A 259 220
[12] Znojil M 1999 J. Phys. A: Math. Gen. 32 4563;
  Znojil M 1999 arXiv quant-ph/9912027 and arXiv quant-ph/9912079
[13] Alvarez G 1995 J. Phys. A: Math. Gen. 27 4589;
  Cannata F, Junker G and Trost J 1998 Phys. Lett. A 246 219;
  Delabaere F and Pham F 1998 Phys. Lett. A 250 25
[14] Znojil 1994 J. Phys. A: Math. Gen. 27 4945
[15] Flessas G P 1981 Phys. Lett. A 83 121
[16] Hautot A 1972 Phys. Lett. A 38 305
[17] Singh V, Biswas S N and Data K 1978 Phys. Rev. D 18 1901
[18] Turbiner A V 1988 Comm. Math. Phys. 118 467
[19] Znojil M 1981 Phys. Rev. D 24 903.
Figure 1. Integration paths for eq. (3)