HURWITZ NUMBERS FROM FEYNMAN DIAGRAMS

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To obtain a generating function of the most general form for Hurwitz numbers with an arbitrary base surface and arbitrary ramification profiles, we consider a matrix model constructed according to a graph on an oriented connected surface \(\Sigma\) with no boundary. The vertices of this graph, called stars, are small discs, and the graph itself is a clean dessin d’enfants. We insert source matrices in boundary segments of each disc. Their product determines the monodromy matrix for a given star, whose spectrum is called the star spectrum. The surface \(\Sigma\) consists of glued maps, and each map corresponds to the product of random matrices and source matrices. Wick pairing corresponds to gluing the set of maps into the surface, and an additional insertion of a special tau function in the integration measure corresponds to gluing in Möbius strips. We calculate the matrix integral as a Feynman power series in which the star spectral data play the role of coupling constants, and the coefficients of this power series are just Hurwitz numbers. They determine the number of coverings of \(\Sigma\) (or its extensions to a Klein surface obtained by inserting Möbius strips) for any given set of ramification profiles at the vertices of the graph. We focus on a combinatorial description of the matrix integral. The Hurwitz number is equal to the number of Feynman diagrams of a certain type divided by the order of the automorphism group of the graph.

Keywords: Hurwitz number, random matrix, Klein surface, Schur polynomial, Wick law, tau function, BKP hierarchy, two-dimensional Yang–Mills theory

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1. Introduction

This paper is an extended version of a talk by one of us (A. Yu. Orlov) at the International Conference “Classical and Quantum Integrable Systems” (CQIS-2019). The following results were briefly mentioned in that talk.

1. We introduced matrix integrals constructed according to a graph with source matrices at its vertices. The source matrices play the role of coupling constants in a matrix model. We expressed these integrals as power series in Schur functions. This power series is a generating function of Hurwitz numbers of general type, i.e., in the case where the covered (base) surface is an orientable (or possibly nonorientable) connected surface without a boundary and with any Euler characteristic and any given set of ramification profiles.

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2. We calculated the obtained integral by the method of character decomposition.
3. We used a tau function as the integrand.
4. We showed a connection between the problem of finding Hurwitz numbers and the two-dimensional Yang–Mills theory.
5. We analyzed the integral using Feynman diagrams.
6. We considered combinatorial aspects of these problems.

The development of points 2–4 was published in [1] and [2]. Here, we reproduce those papers in part but with a somewhat different emphasis. Moreover, we study point 6 in more detail.

We note that an extremely rich literature has been devoted to the connection between matrix integrals and Hurwitz numbers. But various particular cases of generating functions were studied in all the papers, moreover, not for the Hurwitz numbers themselves but for some linear combinations of them. Our approach is universal and allows generating Hurwitz numbers of a general type.

We list some aspects of the studied problem and corresponding papers related to our topic (we beg pardons from the authors of important papers not mentioned here; this list is surely incomplete):

- Hurwitz numbers [3]–[14],
- Hurwitz numbers in a string theory context (Dijkgraaf obtained the key result [15], [16]; [17]–[22] and many other papers subsequently appeared),
- combinatorial aspects related to the covering problem [23]–[26],
- Klein surfaces in the context of our problems [27], [28],
- matrix models and orientable surfaces (’t Hooft made the key observation [29]; numerous important applications and development of the matrix integral topic itself were presented in [2], [30]–[38],
- Ginibre ensembles and independent Ginibre ensembles [39]–[45],
- matrix models and Hurwitz numbers [26], [45]–[58],
- integrable systems and Hurwitz numbers ( [17], [18], and [24] are fundamental; further serious advances were made in [1], [13], [21], [51], [59]–[65]; we note reviews [66] and [67]), and
- topological theories and Hurwitz numbers [10]–[12], [15], [16], [68]–[70].

Our main goal here is to describe integral (2.7) combinatorially. We obtain formulas (3.16) and (3.15) describing the Feynman diagrams for integral (2.8). We decompose integral (2.8) in the insertions of source matrices, more precisely, in the spectral functions of its products (the so-called star spectrum). The lowest-order Feynman graph is our basic matrix model. The combinatorial meaning of this graph is given by relation (3.5) in the symmetry group $S_{2n}$, where $n$ is the number of fat graph edges (moreover, $2n$ is the number of matrices in the multimatrix integral). This relation is well known as a combinatorial description of charts (see the remarkable book [71]). Subsequent orders of the perturbation theory describe the covering of the lowest Feynman graph and give a generating function for the Hurwitz numbers.

We briefly recall some other topics (integration over tau functions and nonorientable coverings). In Sec. 4, we develop the results in [13] and propose a beautiful generalization of the “cut-or-join” formula (Mironov–Morozov–Natanzon)
This formula describes the merger of a pair of branch points in the covering problem. Here, \( \mu = (\mu_1, \mu_2, \ldots) \), \( \Delta = (\Delta_1, \ldots, \Delta_\ell) \) are Young diagrams (\( \Delta \) is the branch profile of one of the points; for simplicity, we consider the case \( |\mu| = |\Delta| \)), and \( s_{\mu} \) are Schur functions written as functions of power sums. The differential operators \( \mathcal{W}^\Delta(p) \) generalize the operators of “additional symmetries” [72] in soliton theory and commute for different \( \Delta \). It was noted in [14] that if we write them in Miwa variables, i.e., in terms of eigenvalues of a matrix \( X \) such that \( p_m = \text{tr}(X^m) \), then the generalized “cut-or-join” formula is written very compactly:

\[
\mathcal{W}^\Delta s_{\mu}(X) = \varphi_{\mu}(\Delta) s_{\mu}(X),
\]

where \( \mathcal{W}^\Delta \) is the differential operator,

\[
\mathcal{W}^\Delta = \frac{1}{z_\Delta} \text{tr} D^\Delta_1 \cdots \text{tr} D^\Delta_\ell,
\]

the factor \( z_\Delta \) is given by formula (A.6) in the appendix, and \( D \) is the Euler vector field

\[
D_{a,b} = \sum_{c=1}^{N} X_{a,c} \frac{\partial}{\partial X_{b,c}}.
\]

As Olshanski pointed out to us, such formulas describing the action of a Casimir operator in a given representation appeared in papers by Perelomov and Popov [73]–[75] (also see Sec. 9 in [76]).

We propose a very simple proof and generalization of formula (1.2) using clean dessins d’enfants (or charts in the terminology in textbook [71]). In fact, we consider a modification of dessins d’enfants in which vertices are replaced with small disc-stars. We will discuss differential operators connected with dessins d’enfants in more detail in a subsequent paper. Here, we restrict ourself to only referring to the important and beautiful works [77]–[82].

In the appendix, we present some review material concerning Hurwitz numbers taken from the scientific literature and from our previous works.

2. Feynman integrals for a complex matrix model constructed for a ribbon graph with inflated vertices

We consider a connected ribbon (or fat) graph \( \Gamma \) on an orientable surface \( \Sigma \) without a boundary and with the Euler characteristic \( E \) with all faces homeomorphic to discs. Then \( E = V - n + F \), where \( V \) is the number of vertices, \( n \) is the number of edges (ribbons), and \( F \) is the number of faces. We construct a graph \( \tilde{\Gamma} \) obtained from \( \Gamma \) by replacing the vertices with small discs (in what follows, we also call them inflated vertices and stars). Then \( \tilde{\Gamma} \) has \( F + V \) faces: \( V \) small discs and \( F \) initial faces, which we call basic faces to distinguish them from small discs. The graph \( \tilde{\Gamma} \) has \( 2n \) trivalent vertices (ribbon end points): one ribbon and two segments of small disc boundaries come from each vertex. Moreover, there are \( 3n \) edges: \( n \) ribbons and \( 2n \) boundary segments of small discs.

We assign a positive orientation (counterclockwise arrow) to the edges of a main face. Then each of two ribbon edges is represented by an arrow, and the ribbon itself is represented by a pair of oppositely directed arrows. Each ribbon arrow is continued by a dotted arrow corresponding to a segment of the small disc boundary. The boundary of each small disc thus consists of a sequence of dotted arrows, each with a negative direction (clockwise arrow) if we go around the small disc center. The boundary of a main face of the graph \( \tilde{\Gamma} \) consists of solid and dotted arrows that alternate and are positively directed if we go around the “capital” of the main face. We call the chosen point inside a main face of \( \tilde{\Gamma} \) a capital.

We now assign each trivalent vertex of \( \tilde{\Gamma} \) a number from 1 to \( N \) and assign each arrow a pair of such numbers corresponding to its end points. We number the edges (ribbons) from 1 to \( n \) and number the
arrows corresponding to a given ribbon such that the arrows for ribbons with the number \( i \) have opposite signs and are numbered as \( i \) and \(-i\) (the choice of which arrow is numbered \( i \) or \(-i\) is irrelevant). We assign each dotted arrow the same number as the solid arrow that enters the starting vertex of the dotted arrow.

We thus assign each arrow three integers: the number \( i \) of the arrow, the number \( a \) of the start of the arrow, and the number \( b \) of the end of the arrow. Finally, we associate a complex number \((Z_i)_{a,b}\) with a solid arrow with such data and a complex number \((C_i)_{a,b}\) \( (i = \pm 1, \ldots, \pm n, a = 1, \ldots, N, b = 1, \ldots, N)\) with a dotted arrow with such data. We regard these numbers as elements of the \( \mathbb{C} \) of this disc in the order given by the sequence of the arrows. For each star (small disc) in the graph we number small discs from 1 to \( n \) if \( Z_i = Z_i^\dagger \) for all \( i = 1, \ldots, n \). We call \( \{C_i\} \) source matrices, and they play the role of coupling constants in the matrix model considered below. We call a graph \( \tilde{\Gamma} \) with numbered arrows and matrices associated with the arrows an equipped graph \( \tilde{\Gamma}(\{Z_i, C_i\}) \).

We number small discs from 1 to \( V \). We introduce the monodromy matrix (or, briefly, monodromy) of a disc as the product of matrices in the set \( \{C_i\} \) corresponding to the dotted arrows along the boundary of this disc in the order given by the sequence of the arrows. For each star (small disc) in the graph \( \tilde{\Gamma}(\{Z_i, C_i\}) \),

\[
W_i^* = C_{i_1} \cdots C_{i_k}, \quad i = 1, \ldots, V,
\]

where the matrices \( C_{i_1}, \ldots, C_{i_k} \) correspond to \( k \) dotted arrows successively adjacent to each other clockwise along the boundary of the star (inflated vertex) with the number \( i \) from which \( k \) ribbons exit. The monodromy is defined up to a cyclic permutation of the matrix factors.

**Remark 2.1.** We describe the geometric picture related to what was said above. If we represent matrix elements by arrows, then we can interpret the matrix product as a chain of adjacent arrows with summation of all numbers in the interval \([1, N]\) associated with the vertices. The trace of the matrix product is a closed chain of arrows, i.e., a polygon. Hence, the monodromy trace corresponds to a polygon, a piece of a two-dimensional plane homeomorphic to a disc. It is known that polygons can be glued into a surface. Interrelation of surfaces glued from polygons and matrix integrals was first discovered by ’t Hooft \[29\] and was actively used in works on two-dimensional quantum gravity \[31\], \[32\].

In fact, we need only the spectrum of the star monodromy, which, for brevity, we call the **star spectrum** in what follows:

\[
\text{Spectr } W_i^* = (w_{i,1}^*, \ldots, w_{i,N}^*), \quad i = 1, \ldots, V.
\]

We now number the main faces from 1 to \( F \) and define the monodromy of the main face as the product of the matrices corresponding to the arrows going around the boundary:

\[
M_i = Z_{i_1} C_{i_1} \cdots Z_{i_m} C_{i_m}, \quad i = 1, \ldots, F,
\]

where \( Z_{i_1}, \ldots, Z_{i_m}, C_{i_m} \) correspond to \( m \) pairs of solid and dotted arrows following each other in going around the face capital along the face boundary along \( 2m \) arrows in the positive direction. This monodromy is also defined up to a cyclic permutation, and we need only the spectrum of this matrix:

\[
\text{Spectr } M_i = (m_{i,1}, \ldots, m_{i,N}), \quad i = 1, \ldots, F.
\]

In Sec. A.6 in the appendix, we present a purely algebraic way to obtain the set \( \{W_i^*\} \) from a set \( \{M_i\} \) in \( n \) steps, and the inverse.

An equipped graph \( \tilde{\Gamma}(\{Z_i, C_i\}) \) is a least-order Feynman graph for the matrix model

\[
\int \left( \prod_{i=1}^F e^{\text{tr } M_i} \right) d\Omega(Z_1, \ldots, Z_n),
\]

(2.5)
where
\[ d\Omega(Z_1, \ldots, Z_n) = e_n^N \prod_{i=1}^{n} \prod_{a,b=1}^{N} e^{-N|(Z_i)_{a,b}|^2} d^2(Z_i)_{a,b} \]
with the normalization
\[ \int d\Omega(Z_1, \ldots, Z_n) = 1. \]
The set \( \{Z_i\} \) of matrices together with the measure \( d\Omega(Z_1, \ldots, Z_n) \) is known as \( n \) independent complex Ginibre ensembles.\(^1\)

In our problem, we assume that the number \( n = 1, 2, \ldots \) is given, and instead of \( d\Omega(Z_1, \ldots, Z_n) \), we write \( d\Omega \) everywhere. Hence, \( \tilde{\Gamma} \) is the Feynman graph for the integral
\[ \int \left( \prod_{i=1}^{F} \text{tr} M_i \right) d\Omega = N^{-n} \prod_{i=1}^{V} \text{tr} W_i^* . \] (2.6)
We discuss this equality below and now write the answer for all orders of integral decomposition (2.5). For the parameter set \( d_1, \ldots, d_F \) (where \( d_1 = d \)), we have
\[ \int \prod_{i=1}^{F} (\text{tr} M_i)^{d_i} d\Omega = \delta_{d,d_1,\ldots,d_F} N^{-nd} \sum_{\Delta^1,\ldots,\Delta^V; |\Delta^1|=\cdots=|\Delta^V|=d} H\Sigma(\Delta^1, \ldots, \Delta^V) C(\Delta^1, \ldots, \Delta^V), \] (2.7)
where \( \delta_{d,d_1,\ldots,d_F} = 1 \) if \( d_1 = \cdots = d_F \) and \( \delta_{d,d_1,\ldots,d_F} = 0 \) otherwise. An important point is that because the source matrices are arbitrary, this is not one relation but a family of relations. In Sec. 3 below, we use this fact to construct differential operators related to \( \tilde{\Gamma} \). We note that the source matrices are in different combinations in the left- and right-hand sides of the equality.

As a result, we obtain the model
\[ \int \left( \prod_{i=1}^{F} e^{\frac{1}{\hbar} \text{tr} M_i} \right) d\Omega = \sum_{d=0}^{\infty} (N\hbar)^{-nd} \sum_{\Delta^1,\ldots,\Delta^V; |\Delta^1|=\cdots=|\Delta^V|=d} H\Sigma(\Delta^1, \ldots, \Delta^V) C(\Delta^1, \ldots, \Delta^V), \] (2.8)
which should be regarded as a formal power series in the parameter \( \hbar^{-1} \). Here,
\[ C(\Delta^1, \ldots, \Delta^V) := \prod_{i=1}^{V} \prod_{k=1}^{\infty} \text{tr}(W_i^*)^{\Delta^i_k} \] (2.9)
is a quantity depending only on the star spectra and \( H\Sigma(\Delta^1, \ldots, \Delta^V) \) is the Hurwitz number counting the degree-\( d \) coverings of the orientable connected surface \( \Sigma \) without a boundary and with ramification profiles of the form \( \Delta^1, \ldots, \Delta^V \) at \( V \) points (see the appendix for an exact definition of Hurwitz numbers). A ramification profile \( \Delta^i \) is a Young diagram \( \Delta^i = (\Delta^i_1, \Delta^i_2, \ldots) \) that encodes ways to merge the sheets of the covering of \( \Sigma \). For more visuality, we can regard the branch points as star centers although it is known that Hurwitz numbers for surfaces without a boundary are independent of locations of the ramification points.

In the case \( d = 1 \) (first order of the perturbation theory), we obtain relation (2.6) because the Hurwitz number \( H((1), \ldots, (1)) \) describing the covering of \( \Sigma \) by itself is equal to unity.

Formula (2.8) was proved geometrically in [2] based on the fact that the Hurwitz number is defined as the weighted number of ways to glue polygons into a covering surface.

\(^1\)In particular, such ensembles are used in the theories of quantum chaos and quantum information transmission. In [1], we showed that they can also be used to describe the two-dimensional Yang–Mills theory, which is close to the approach in [83], [84].
Remark 2.2. We can generalize the indicated integral by replacing the integrand with a tau function or, in a more general case, with a product of tau functions.\textsuperscript{2} We consider the first variant. The tau function of the multicomponent KP equation \cite{85} has the form \cite{86}

\[ \tau_g(M_1, \ldots, M_F) = \sum_{\lambda^1, \ldots, \lambda^F} g(\lambda^1, \ldots, \lambda^F) s_{\lambda^1}(M_1) \cdots s_{\lambda^F}(M_F), \]

(2.10)

where each set \( \lambda^i = (\lambda^i_1, \lambda^i_2, \ldots) \) is some partition and \( s_{\lambda^i}(M_i) \) is the Schur function determined by the expression \cite{87}

\[ s_{\lambda^i}(M_i) = \frac{\det[m_{i,k}^i]}{\det[m_{i,k}]} \]

(see formula (2.4)). Moreover, the function \( g(\lambda^1, \ldots, \lambda^F) \) satisfies some concrete equation, which we omit here. We merely note that for matrix model (2.5), we have

\[ g(\lambda^1, \ldots, \lambda^F) = \prod_{i=1}^F \frac{\dim \lambda^i}{d!}. \]

If the number \( F \) is even, then we obtain \cite{1}

\[ \int \tau_g(M_1, \ldots, M_F) d\Omega = \sum_{\lambda: \ell(\lambda) \leq N} N^{-n|\lambda|} \left( \frac{\dim \lambda}{d!} \right)^{-n} g(\lambda, \ldots, \lambda) s_{\lambda}(W_1^*) \cdots s_{\lambda}(W_N^*) = \]

\[ = \sum_{d=0}^{\infty} N^{-nd} \sum_{|\Delta^1| = \cdots = |\Delta^V| = d} H_{\Sigma'}(g|\Delta^1, \ldots, \Delta^V) C(\Delta^1, \ldots, \Delta^V), \]  

(2.11)

where

\[ s_{\lambda^i}(W_i^*) = \frac{\det[w_{i,k}^i]}{\det[w_{i,k}]} \]

(2.12)

and

\[ H_{\Sigma'}(g|\Delta^1, \ldots, \Delta^V) = \sum_{\lambda: \ell(\lambda) \leq N} \left( \frac{\dim \lambda}{d!} \right)^{E-F} g(\lambda, \ldots, \lambda) \varphi_\lambda(\Delta^1) \cdots \varphi_\lambda(\Delta^V) \]

(2.13)

is an analogue of the Hurwitz number. An example of a tau function for which

\[ g(\lambda, \ldots, \lambda) = \prod_i \exp \left\{ \sum_{k>0} t_k (\lambda_i - i)^k \right\} \]

was considered in \cite{1}. In this case, number (2.13) is a direct analogue of the Hurwitz number for replenished cycles introduced in \cite{18}, but it describes coverings of a surface \( \Sigma' \) obtained from \( \Sigma \) by gluing \( F/2 \) handles and not a covering of a sphere, as was done in \cite{18}.

The nonorientable case corresponds to the matrix model

\[ \int \prod_{i=1}^{F_1} e^{(1/h) \tr M_i} \prod_{i=F_1+1}^F \mathfrak{M}(M_i) d\Omega = \sum_{d=0}^{\infty} (Nh)^{-nd} \sum_{|\Delta^1| = \cdots = |\Delta^V| = d} H_{\Sigma}(\Delta^1, \ldots, \Delta^V) C(\Delta^1, \ldots, \Delta^V), \]  

(2.14)

\[ \text{for instance, computing the integral over a product of some particular tau functions, we can obtain the correlation functions known from [83], [84] of a two-dimensional gauge theory [2].} \]
where $F_1$ is a given number, $0 < F_1 < F$, and

$$\mathcal{M}(M_i) = \det \frac{(1 + (1/\hbar)M_i)^{1/2}(1 - (1/\hbar)M_i)^{-1/2}}{(I_N \otimes I_N - (1/\hbar^2)M_i \otimes M_i)^{1/2}}. \quad (2.15)$$

We are interested only in surface topological structures. Therefore, the nonorientable surface $\tilde{\Sigma}$ with the Euler characteristic $\tilde{E} = F_1 - n + V$ can be interpreted as a surface $\Sigma$ with $F - F_1$ Möbius strips glued in. We can formally derive Eq. (2.14) from (2.7), formula (A.18) in Sec. A.4, and relation (A.16) in Sec. A.3. But this equality also has a geometric meaning: it defines a function on the so-called orientating double cover of the real projective plane with a hole, i.e., on a sphere with an involution and two holes that covers the projective plane $\mathbb{RP}^2$ with a hole (also see Sec. A.4).

Because we glue in Möbius strips, we can also glue handles in $\Sigma$. We let $\tilde{\Sigma}$ denote the obtained surface. Let $h$ be the number of additional handles, $m$ be the number of Möbius strips, and $m = F_2 - F_1$, $0 < F_2 < F_1$, $F = F_1 + m + 2h$. The matrix model describing the covering $\tilde{\Sigma}$ of the surface $\Sigma$ has the form

$$\int \prod_{i=1}^{F_1} e^{(1/\hbar)\text{tr} M_i} \prod_{i=F_1+1}^{F_2} \mathcal{M}(M_i) \prod_{i=F_2+2,F=4,...}^{F} \mathcal{S}(M_{i-1}, M_i) \, d\Omega =$$

$$= \sum_{d=0}^{\infty} (N\hbar)^{-nd} \sum_{|\Delta^1| = \cdots = |\Delta^V| = d} H^\Sigma_{\Sigma}(\Delta^1, \ldots, \Delta^V)C(\Delta^1, \ldots, \Delta^V), \quad (2.16)$$

where the factors $\mathcal{M}(M_i)$ given by (2.15) correspond to the insertions of Möbius strips and the factors

$$\mathcal{S}(M_i, M_{i+1}) = \det \left( I_N \otimes I_N - \frac{1}{\hbar^2} M_i \otimes M_{i+1} \right)^{-1} \quad (2.17)$$

correspond to the insertions of handles (also see Sec. A.4 for the geometric meaning of this factor).

**Remark 2.3.** We can interpret factor (2.17) as a sphere with two holes with the matrices $M_i$ and $M_{i+1}$ on the boundaries. Moreover, we can interpret factor (2.15) as an orienting cover of the projective sphere with a hole with $M_i$ on the boundary (i.e., as a sphere with two holes and an involution). We note that expression (2.17) was previously used without a geometric interpretation in [88] to describe a matrix model comprising a chain of (Hermitian) matrices $M_1, \ldots, M_n$. More concretely, expressions of the form $\mathcal{S}(M_1, M_2)\mathcal{S}(M_2, M_3)\ldots$ were integrated in that model.

**Remark 2.4.** The right-hand sides of equalities (2.8), (2.14), and (2.16) contain the same factor $C(\Delta^1, \ldots, \Delta^V)$, which depends only on the star spectrum. Moreover, the Hurwitz numbers in these expressions have the same set of ramification profiles corresponding to these stars, and the distinction between them is only in the respective covered surfaces $\Sigma$, $\tilde{\Sigma}$, and $\tilde{\Sigma}$.

**Remark 2.5.** The right-hand side of Eq. (2.16) is independent of the distribution of matrices in the set $\{M_i\}$ among the three factors in the integrand in (2.16): they can be permuted. It is only important how many matrices are used for Möbius strips and how many are used for handles. What happens if we replace the three factors in integral (2.16) with tau functions will be a topic in a subsequent paper.

**Remark 2.6.** Any isospectral deformations of set $V$ of monodromies (2.1) do not change the values of the considered integrals.
Remark 2.7. The case where all star monodromies $W^*_i (i = 1, \ldots, \nu - 1)$ are degenerate matrices is interesting because we have an integral over rectangular matrices in this case. If $\Sigma = \mathbb{S}^2$ and also the monodromies of all stars except one or two (let them be $W^*_i, i = 1, 2$) have spectra $\text{Spectr} W^*_i = (1, \ldots, 1, 0, 0, \ldots)$, rank $W^*_i = n_i$. This means that for any $F$ (under the condition $F - n + \nu = 2$), integral (2.8) is equal to

$$
\sum_{d \geq 0} (N\hbar)^{-nd} \sum_{\lambda: |\lambda| = d} s_\lambda(W^*_1) s_\lambda(W^*_2) \prod_{i=3}^\nu (n_i)_\lambda,
$$

where $s_\lambda(W^*_i) = (n_i)_\lambda \dim \lambda/d!$ and $(x)_\lambda = (x)_\lambda (x-1)_{\lambda_2} \cdots (x-\ell+1)_{\lambda_\ell}$ is the Pochhammer symbol. This sum is an example of the hypergeometric tau function [89], [90] of the KP hierarchy [86], [91] and the Toda chain [92]. The spectrum of the stars $W^*_{1,2}$ is called the set of Miwa variables in this case.

Integral (2.14) with the insertion of one factor (2.15), i.e., in the case where $\Sigma = \mathbb{R} \mathbb{P}^2$, can also be a tau function of the form

$$
\sum_{d \geq 0} (N\hbar)^{-nd} \sum_{\lambda: |\lambda| = d} s_\lambda(W^*_1) \prod_{i=2}^\nu (n_i)_\lambda.
$$

This is a hypergeometric tau function [93], only not for the KP hierarchy but for the hierarchy introduced in [94]. Both tau functions are regarded as formal power series in $\hbar$.

3. Combinatorial meaning of the matrix integral

We can associate the permutation group $S_{2n}$, where $n$ is the number of edges (or $2n$ is the number of half-edges) to a graph all of whose faces are homeomorphic to a disc. For this, we should label all edges with positive numbers from 1 to $n$ and the sides of edge $i$ ($i = 1, \ldots, n$) with the numbers $i$ and $-i$. In this case, it is unimportant which edge side has the positive number and which has the negative number, but this choice must be fixed. We let $J = [-n, n] \setminus \{0\}$ denote set of side numbers. From the permutation group, we then choose cycles corresponding to faces: these cycles are formed by the set of numbers of the edge sides passed in the positive direction around the capital of a chosen face. We number the faces and associate the face numbered by $m$ ($m = 1, \ldots, F$) with the cycle $f_m$.

Let the cycle for a face with $k_1$ sides comprise the numbers $i_1, i_2, \ldots, i_{k_1} \in J$. We let $(i_1, \ldots, i_{k_1})$ denote the corresponding cycle in the group $S_{2n}$. The face monodromy $Z_{i_1} C_{i_1} \cdots Z_{i_{k_1}} C_{i_{k_1}}$ is constructed exactly according to this cycle. We call such a matrix product constructed according to a given cycle the dressing of the cycle by matrices. The cycles corresponding to all the faces are denoted by

$$
f_1 = (i_1, \ldots, i_{k_1}), \quad f_2 = (i_{k_1+1}, \ldots, i_{k_1+k_2}), \quad \ldots, \quad f_F = (i_{k_1+\ldots+1}, \ldots, i_{2n}),
$$

where the pairwise distinct numbers $i_1, \ldots, i_{2n}$ comprise the set $J$. These cycles are in correspondence with the traces of the face monodromies

$$
\begin{align*}
\text{tr } M_1 &= \text{tr} (Z_{i_1} C_{i_1} \cdots Z_{i_{k_1}} C_{i_{k_1}}) := D_Z[f_1], \\
\text{tr } M_2 &= \text{tr} (Z_{i_{k_1+1}} C_{i_{k_1+1}} \cdots Z_{i_{k_1+k_2}} C_{i_{k_1+k_2}}) := D_Z[f_2], \\
&\vdots \\
\text{tr } M_F &= \text{tr} (Z_{i_{k_1+\ldots+1}} C_{i_{k_1+\ldots+1}} \cdots Z_{i_{2n}} C_{i_{2n}}) := D_Z[f_F].
\end{align*}
$$

The operation $D_Z[\text{cycle}]$ (dressing a cycle by the matrices $\{Z_x C_x\}$) denotes replacing the cycle with the trace of the product of the matrices $Z_x C_x$, where $x$ is equal to the numbers contained in the cycle.
We also introduce cycles related to the vertices: with each vertex, we associate the set of side numbers of
the edges that are encountered immediately after the intersection of the edge when the vertex is gone around
in the negative direction (clockwise). We number the vertices. The vertex with the number
\( s = 1, \ldots, V \) corresponds to the cycle \( \sigma_s \) and the trace of the monodromy related to this vertex. We have
\[
\sigma_1 = (j_1, \ldots, j_{s_1}), \quad \sigma_2 = (j_{s_1+1}, \ldots, j_{s_1+s_2}), \quad \ldots, \quad \sigma_V = (j_{s_1+\ldots+1}, \ldots, j_{2n}),
\] (3.3)
where the pairwise distinct numbers \( j_1, \ldots, j_{2n} \) comprise the set \( J \). The traces of the vertex (star) mon-
odromies corresponding to these cycles are
\[
\text{tr } W_1^* = \text{tr} (C_{j_1} \cdots C_{j_{s_1}}) =: D[\sigma_1],
\]
\[
\text{tr } W_2^* = \text{tr} (C_{j_{s_1+1}} \cdots C_{j_{s_1+s_2}}) =: D[\sigma_2],
\]
\[
\vdots
\]
\[
\text{tr } W_V^* = \text{tr} (C_{j_{s_1+\ldots+1}} \cdots C_{j_{2n}}) =: D[\sigma_V].
\] (3.4)
The operation \( D[\text{cycle}] \) (dressing a cycle by the matrices \( \{C_x\} \)) denotes replacing the cycle with the trace
of the product of the matrices \( C_x \), where \( x \) is equal to the numbers contained in the cycle.

As we already noted, the edge (ribbon) with the number \( i \) (\( i = 1, \ldots, n \)) corresponds to numbers \( i \) and \(-i\) located on different sides of the edge. We associate the transposition \( \alpha_i \) that permutes these two
numbers with each edge \( i \).

We have a remarkable relation (analyzed in detail in [71]) for any connected graph \( \Gamma \) represented on a
surface:
\[
\prod_{i=1}^{n} \alpha_i \prod_{i=1}^{F} f_i = \prod_{i=1}^{V} \sigma_i.
\] (3.5)
We can say that the involution \( \prod_{i=1}^{n} \alpha_i \) maps cycles of faces of a graph to cycles of faces of the dual graph.
We call graphs represented on a covered surface in correspondence with (3.5) “clean dessins d’enfants.”

**Remark 3.1.** We place the indices \( i \) and \(-i\) associated with the half-edge arrows not only on the side
of the edge \( i \) where the arrow ends but also at the start of the arrow. The transposition \( \alpha_i \) \( (i > 0) \) then
corresponds to transposition of the half-edges.

We consider integral (2.6) and write it in the form
\[
\int DZ \left[ \prod_{i=1}^{F} f_i \right] d\Omega(Z) = D \left[ \prod_{i=1}^{V} \sigma_i \right].
\] (3.6)
The integrand is the sum of a large number of monomials consisting of the product of elements of the \( Z \)
and \( C \) matrices. Because of Gauss integration, only terms containing \( (Z_i)_{a,b}(Z_{-i})_{b,a}(C_i)_{b,x}(C_{-i})_{a,y} \) are
important. Moreover, as a result of integration, only the product \( (1/N)(C_i)_{b,x}(C_{-i})_{a,y} \) remains from this
expression:
\[
(Z_i)_{a,b}(Z_{-i})_{b,a}(C_i)_{b,x}(C_{-i})_{a,y} \longrightarrow \frac{1}{N} (C_i)_{b,x}(C_{-i})_{a,y}.
\] (3.7)
We recall that a matrix \( Z_i \) \( (i = 1, \ldots, n) \) is represented by an arrow that corresponds to side \(|i|\) of the
ribbon and \( Z_{-i} \) corresponds to the other side of this ribbon, which is represented by a reverse arrow. The
arrow directions correspond to the positive orientation of faces.
Each solid arrow on a graph $\Gamma$ is continued by a dotted arrow with the same number. Therefore, as follows from (3.7), after integration, dotted arrows around a small disc correspond to the product of the source matrices consecutively located at the sectors of this disc boundary, and closure of the arrows corresponds to the trace of the star monodromy. As a result of integrating over all matrices $Z_i$, we obtain the product of the traces of all star monodromies, which proves formula (2.6). The factor $N^{-nd}$ in (2.7) and other formulas for a $d$-sheet covering comes from the factor $N^{-1}$ in formula (3.7). We can represent this visually as follows. We erase all ribbons, and only small discs (stars) remain. With the cycles, we associate the traces of the matrix products.

We consider the integrand in formula (2.7). We want to number all products of matrices $Z_iC_i$ with some fixed $i$ and do this as follows. We count these matrices from left to right and set the number as a superscript in parentheses: if a matrix occurs for the first time, then we write $\prod_j Z_i^{(1)}C_i^{(1)}$, and so on, going from left to right along the whole integrand in (2.7). We do the same for the right-hand side of (2.7). In this case, we fix $i$ and number the matrices $C_i$ as $C_i^{(1)}, C_i^{(2)}, \ldots$, moving from left to right along the whole right-hand side of (2.7). There is an arbitrariness in this procedure because it is possible to interchange matrices in the trace and to permute the traces themselves. We fix this arbitrariness “by hand.”

Further, we can associate a set of cycles in the group $S_{2nd}$ with the integrand in the left-hand side of (2.7). Analogically, we can associate a set of cycles in $S_{2nd}$ with the right-hand side of (2.8). We can call the inverse procedure dressing a cycle. This procedure for dressing cycles in the left-hand side of (2.7) is denoted by $D_Z[\text{cycle}]$ and for dressing cycles in the right-hand side of (2.7) is denoted by $D[\text{cycle}]$. We replace each index $i^{(a)}$ in the left-hand side with $Z_i^{(a)}C_i^{(a)}$ and replace each index $j^{(a)}$ in the right-hand side with $C_j^{(a)}$. We call products of dressed cycles the procedure of dressing cycle products. We set $D_Z[fg] = D_Z[f]D_Z[g]$. As a result of integrating over all matrices $Z_i$, we obtain the product of the traces of all star monodromies, which proves formula (2.6). The factor $N^{-nd}$ in (2.7) and other formulas for a $d$-sheet covering comes from the factor $N^{-1}$ in formula (3.7).

We set

$$f_1^{(1)} = (i_1^{(1)}, \ldots, i_k^{(1)}),$$
$$f_2^{(1)} = (i_{k+1}^{(1)}, \ldots, i_{k+k_2}^{(1)}) \cdot (i_k^{(d)}, \ldots, i_{k+k_2}^{(d)}),$$
$$\vdots$$
$$f_k^{(1)} = D_Z[(i_1^{(k+1)}, \ldots, i_2n^{(1)}, \ldots, i_{k+1}^{(d)}, \ldots, i_{2n}^{(d)})],$$

(3.8)
where $(1^d)$ denotes the partition $(1, \ldots, 1)$ containing only units. The reason it is convenient to mark cycles with Young diagrams becomes clear later. The right-hand sides of these equalities contain all possible numbers $i_j^{(a)}$ ($j = 1, \ldots, 2n, a = 1, \ldots, d$) with the total number $2nd$. We then have

$$\text{(tr} M_i)^d = D_Z[f_i^{(1)}], \quad i = 1, \ldots, F$$

(3.9)
(cf. expressions (3.2)). The cycles in the right-hand side can be regarded as an unramified $d$-sheet covering of the cycles in the right-hand side of (3.1): each index $i_j^{(a)}$ with its corresponding neighborhood inside of each of the cycles (cycles are labeled by the index $a = 1, \ldots, d$) is projected on $i_j$.

With cycles, we can also associate polygons with numbered sides. The polygons on the base surface $\Sigma$ cut by the edges of the graph $\Gamma$ then correspond to the cycles in (3.1). With the cycles in the right-hand side of (3.8), we associate $d$ copies of the polygons covering them (we have a “kebab” of skewered polygons with the skewer positioned vertically). Integral (2.7) glues polygons from the “kebab” into a surface that covers $\Sigma$ in the same way as integral (2.6) glues polygons (3.1) into the base surface. But we now have a choice of which polygons to glue to which because each matrix $Z_i$ occurs $d$ times: we label these $d$ matrices $Z_i^{(1)}, \ldots, Z_i^{(d)}$. The Wick theorem is responsible for the pairing, and as a result, we obtain the sum in the right-hand side of (2.7).
Each pairing corresponds to a transposition indicating which matrices are glued to which. We let $\alpha^{(a,b)}_i$ denote these transpositions, where the indices mean that $Z_i^{(a)}$ is paired with $Z_i^{(b)}$. Gauss integral (2.7) describes the complete pairing of all $\nu d$ matrices $Z_i^{(a)}$ ($a = 1, \ldots, d, i = 1, \ldots, 2n$). This leads to replacing the product $\prod_{i=1}^n \alpha_i$ that describes gluing polygons into the base surface with the sum (an element of the group algebra of $S_{\nu d}$)

$$
\sum_{w_1, \ldots, w_n \in S_d} \hat{\alpha}(w_1, \ldots, w_n),
$$

where

$$
\hat{\alpha}(w_1, \ldots, w_n) := \prod_{a=1}^{d} \prod_{i=1}^{n} \alpha^{(a,w_i(a))}_i
$$

and the permutation $w_i$ corresponds to the Wick law: these permutations correspond to all possible pairings. We call element (3.11) of the group $S_{\nu d}$ a gluing element. Formula (3.10) describes the sum over all gluing elements. The involution $\prod_{i=1}^n \alpha_i$ in formula (3.5) corresponding to the edges of the graph $\Gamma$ is a particular case of (3.10) obtained at $d = 1$.

We now recall a fact about the composition of a transposition and the product of nonintersecting cycles. A transposition exchanges some two elements. There are two possible cases: the elements belong to the same or different cycles. In the first case, the product of the transposition and the cycle gives two vertices of the graph $\tilde{\Gamma}$ (stars of the graph $\tilde{S}$). We note that the obtained cycles can be interpreted as covering initial set (3.3) because each element $\alpha^{(a,b)}_i$ acts on its corresponding edge $i$ of the graph $\Gamma$.

We consider a vertex with the number 1, for example, with $s_1$ outgoing edges (see (3.3)). Let the covering cycles have the lengths $s_1\Delta^1_1 \geq \cdots \geq s_1\Delta^1_k$ (the sum of all lengths must be equal to $\nu d$). We let $\sigma^{(1)}_1, \sigma^{(2)}_1$ denote the corresponding cycles, and the cycle $\sigma^{(\Delta^1_k)}_1$ covers cycle (3.3) with $\Delta^1_k$ sheets.

The first two cycles have the forms

$$
\sigma^{(1)}_1 = (j^{(1)}_1, \ldots, j^{(1)}_{s_1}), (j^{(2)}_1, \ldots, j^{(2)}_{s_1}), \ldots, (j^{(s_1)}_1, \ldots, j^{(s_1)}_{s_1}),
$$

$$
\sigma^{(\Delta^1_1)}_1 = (j^{(\Delta^1_1)}_1, \ldots, j^{(\Delta^1_1)}_{s_1}), (j^{(\Delta^1_1+1)}_1, \ldots, j^{(\Delta^1_1+1)}_{s_1}), \ldots, (j^{(\Delta^1_1+s_1-1)}_1, \ldots, j^{(\Delta^1_1+s_1-1)}_{s_1}).
$$

We set

$$
\sigma^\Delta := \prod_{m=1}^{s} \sigma^\Delta_m.
$$

The set $\{ \Delta^s = (\Delta^s_1, \Delta^s_2, \ldots), s = 1, \ldots, \nu \}$ is a set of Young diagrams that we assign to the corresponding vertices of the graph $\Gamma$ (stars of the graph $\tilde{\Gamma}$). Each trace of matrix powers corresponds to some cycle of coverings. We have

$$
\sigma^{\Delta^1_i} \cdots \sigma^{\Delta^1_{k_1}} \longrightarrow \text{tr}(W^{\Delta^1_i}_1) \cdots \text{tr}(W^{\Delta^1_{k_1}}_1) =: \mathcal{D}[\sigma^{\Delta^1_i} \cdots \sigma^{\Delta^1_{k_1}}].
$$

**Remark 3.2.** For a Feynman graph $\tilde{\Gamma}$, instead of a vertex, we should consider the boundary of a small disc, regarding it as a polygon with sides consisting of $s_1$ dotted boundary segments (a segment connects neighboring outgoing ribbon edges), and speak about the covering of this polygon by a system of polygons with $s_1\Delta^1_1, \ldots, s_1\Delta^1_k$ sides.
We choose a set of permutations \( w_1, \ldots, w_n \) and consider the equation

\[
\hat{\alpha}(w_1, \ldots, w_n) \prod_{i=1}^{F} f_i^{(1^{i_{d_i}})} = \prod_{i=1}^{V} \sigma_i^{\Delta_i}
\]

for unknown \( \Delta^1, \ldots, \Delta^V \), where the right-hand side describes the result of composing involution (3.11) and the product of the cycles \( f_i^{(1^{i_{d_i}})}, \ldots, f_i^{(d_i)} \) (interpreted as preimages of the cycles \( f_1, \ldots, f_F \)). It is easy to understand that the cycles in the set \( \{ \sigma_i^{\Delta_m}, i = 1, \ldots, n, m = 1, \ldots, \ell(\Delta_i) \} \) are pairwise nonintersecting, which follows from the properties of the transposition \( \alpha_{i(a,b)} \).

Equation (3.14) corresponds to a given Wick pairing determined by the set \( w_1, \ldots, w_n \) and a given Feynman graph. It plays the role of Eq. (3.5) but for another graph, namely, for the graph \( \hat{\Gamma} \) covering \( \Gamma \). A graph \( \hat{\Gamma} \) with punctured vertices has \( d \) sheets and no ramifications and covers the graph \( \Gamma \) with punctured vertices. The vertices of \( \Gamma \) are branch points with the profiles \( \Delta^1, \ldots, \Delta^V \).

We now consider a given set of Young diagrams \( \Delta^1, \ldots, \Delta^V \) of the same weight \( d \) and regard (3.14) as an equation for unknown \( w_1, \ldots, w_n \). We divide the number of solutions of this equation by \((dl)^N\). As a result, we obtain exactly the Hurwitz number \( H_{\Sigma}(\Delta^1, \ldots, \Delta^V) \). It describes the number of ways we can glue polygons into a covering of the surface \( \Sigma \) if all ramification profiles are given. Dividing by \((dl)^N\) is natural because the covering of each cycle \( f_i \) \( (i = 1, \ldots, n) \) contains \( d \) identical cycles (polygons) in the construction. This corresponds to the geometric definition of the Hurwitz number as the number of nonequivalent coverings for a given set of ramification profiles (see Sec. A.1).

We write matrix integral (2.7) as

\[
\frac{N^{nd}}{(dl)^N} \int \mathcal{D}Z \left[ \prod_{i=1}^{F} f_i^{(1^{i_{d_i}})} \right] d\Omega = \sum_{\Delta^1, \ldots, \Delta^V} H_{\Sigma}(\Delta^1, \ldots, \Delta^V) \mathcal{D} \left[ \prod_{i=1}^{V} \sigma_i^{\Delta_i} \right].
\]

This expression corresponds to the equality

\[
\sum_{w_1, \ldots, w_n \in S_d} \hat{\alpha}(w_1, \ldots, w_n) \prod_{i=1}^{F} f_i^{(1^{i_{d_i}})} = \sum_{\Delta^1, \ldots, \Delta^V} H_{\Sigma}(\Delta^1, \ldots, \Delta^V) \prod_{i=1}^{V} \sigma_i^{\Delta_i}, \tag{3.16}
\]

which describes how higher-order Feynman graphs of integral (2.8) cover a lower-order graph \( \hat{\Gamma} \). The types of preimages of the small discs of the graph \( \hat{\Gamma} \) are given by the set \( \Delta^1, \ldots, \Delta^V \).

If we take \( f_i^{\tilde{\Delta_i}} \) instead of the cycles \( f_i^{(1^{i_{d_i}})} \), then instead of (3.16) or (2.7), we should write the equality

\[
\sum_{w_1, \ldots, w_n \in S_d} \hat{\alpha}(w_1, \ldots, w_n) \prod_{i=1}^{F} f_i^{\tilde{\Delta_i}} = \sum_{\Delta^1, \ldots, \Delta^V} H_{\Sigma}(\tilde{\Delta}^1, \ldots, \tilde{\Delta}^F, \Delta^1, \ldots, \Delta^V) \prod_{i=1}^{V} \sigma_i^{\Delta_i},
\]

where \( z_{\Delta} \) is defined by formula (A.6) in the appendix and we have the equality

\[
\sum_{\Delta^1, \ldots, \Delta^V} H_{\Sigma}(\tilde{\Delta}^1, \ldots, \tilde{\Delta}^F, \Delta^1, \ldots, \Delta^V) \mathcal{D} \left[ \prod_{i=1}^{V} \sigma_i^{\Delta_i} \right].
\]

The Hurwitz numbers in the right-hand sides of these relations contain two sets of profiles: one set corresponds to preimages of the faces of the graph \( \Gamma \) (preimages of the basic loops of the Feynman graph \( \hat{\Gamma} \)), and the other set corresponds to preimages of the vertices of \( \Gamma \) (or, equivalently, preimages of the small discs of \( \hat{\Gamma} \)) under a higher-order Feynman graph covering of a lower-order one. The locations of branch points
does not affect the Hurwitz numbers, but for visuality, we can assume that the branch points are located at the face capitals and the vertices of $\Gamma$ or, equivalently, at the vertices of the graph dual to the Feynman graph $\hat{\Gamma}$.

We write a combinatorial equation for the diagrams corresponding to integral (2.16). Instead of (3.16), we obtain

$$
\sum_{w_1, \ldots, w_F \in S_d} \hat{\alpha}(w_1, \ldots, w_n) \prod_{i=1}^{F_1} \int_{(1, \ldots, 1)} \frac{d^4l}{d^4l} \prod_{i=F_1+1}^{F} \left( \sum_{\mu_i \in \mathcal{P}} f^{(\mu_i)} D(\mu_i) \right) =
$$

$$
\sum_{\Delta^1, \ldots, \Delta^V} H_{\Sigma}^{(\Delta^1, \ldots, \Delta^V)} \prod_{i=1}^{V} \sigma_i^{\Delta^i},
$$

(3.18)

where $\mathcal{P}$ is the set of all partitions and $D(\mu)$ is given in Sec. A.3. We have

$$
\int \sum_{w_1, \ldots, w_F \in S_d} \hat{\alpha}(w_1, \ldots, w_n) \prod_{i=1}^{F_1} \frac{f^{(1, \ldots, 1)}}{d^4l} \prod_{i=F_1+1}^{F} \frac{m(f_i)}{d^4l} \prod_{i=F_2+2}^{F} \frac{b(f_{i-1}, f_i)}{d^4l} =
$$

$$
\sum_{\Delta^1, \ldots, \Delta^V} H_{\Sigma}^{(\Delta^1, \ldots, \Delta^V)} C(\Delta^1, \ldots, \Delta^V),
$$

(3.19)

where the factors

$$
m(f_i) = \sum_{\mu_i \in \mathcal{P}} f^{(\mu_i)}(\mu_i),
$$

$$
b(f_i, f_{i+1}) = \sum_{\mu_i \in \mathcal{P}} f^{(\mu_i)}(\mu_i) \frac{1}{2\mu_i},
$$

(3.20)

respectively correspond to insertions of Möbius strips and handles.

4. Discussion

4.1. Differential operators. We can interpret the Gauss integral as an integral over $n$-component two-dimensional charged boson fields $Z_i$ and $Z_i^\dagger$:

$$
\int (Z_i^\dagger)_{a,b}(Z_j)_{b',a'} d\Omega = \left((Z_i^\dagger)_{a,b}(Z_j)_{b',a'}\right) = \frac{1}{N} \delta_{a,a'} \delta_{b,b'} \delta_{i,j}, \quad i, j = 1, \ldots, n, \quad a, b = 1, \ldots, N.
$$

The Fock space of these fields is formed by all possible polynomials in elements of the matrices $Z_1, \ldots, Z_n$, and the polynomial identically equal to unity is the vacuum vector. We can regard the $(Z_i)_{ab}$ as creation operators and the $(Z_i^\dagger)_{ba} = (1/N)\partial/\partial Z_{ab}$ as annihilation operators acting in this space. We should regard the integrand in (2.5) as antidered, i.e., regard all creation operators as moved to the left and all annihilation operators (all derivatives) as moved to the right while the sequence of matrices inside these groups is preserved. We let $:: A ::$ denote antidering of a polynomial $A$ in the elements of $Z_1, Z_1^\dagger, \ldots, Z_n, Z_n^\dagger$. From this standpoint, we place the canonical conjugate coordinates $(Z_i)_{a,b}$ and momenta $\partial/\partial\partial (Z_i)_{a,b}$ at different sides of the $i$th ribbon of the graph $\hat{\Gamma}$. Then, for example, we can write relation (2.11) as the action of the differential operator on unity:

$$
:: \tau_g(M_1, \ldots, M_F) :: \cdot 1 = \sum_{d=0}^{\infty} \frac{1}{N^{nd}} \sum_{\Delta^1, \ldots, \Delta^V, \sigma_i^{\Delta_i} \vdash \delta_{\Delta^1, \ldots, \Delta^V}} H_{\Sigma}^{(g, \Delta^1, \ldots, \Delta^V)} C(\Delta^1, \ldots, \Delta^V),
$$

This expression looks more compact than (2.11). We give one more relation:

$$
N^{nd} \left( :: \prod_{i=1}^{F} s_{\lambda_i}(M_i) :: \right) \cdot 1 = \delta_{\lambda^1, \ldots, \lambda^F} \left( \frac{\dim \lambda}{d!} \right)^{-n} \prod_{i=1}^{V} s_{\lambda_i}(W_i^*),
$$

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our partitions have the weight d. We obtain expressions are useful from the application standpoint. In relation to this, it would be interesting to establish a connection with the approach to Hurwitz numbers developed in [13], [14].

We derive a beautiful formula in [14] (see Theorem 5.1 there and also [77]-[82]). We recall that all our partitions have the weight d. For the partition \( \Delta = (\Delta_1, \ldots, \Delta_\ell) \), we introduce a notation for the monodromies of the faces \( M_i \) and of the stars \( W_i^* \):

\[
M_i^{\Delta_i} = \text{tr}(M_i)^{\lambda_1^{\Delta_i}} \cdots \text{tr}(M_i)^{\lambda_F^{\Delta_i}} \quad \text{and} \quad C_i^{\Delta_i} = \text{tr}(W_i^*)^{\lambda_1^{\Delta_i}} \cdots \text{tr}(W_i^*)^{\lambda_F^{\Delta_i}}.
\]

Then (3.17) becomes

\[
N^{nd} \int \prod_{i=1}^F \frac{M_i^{\tilde{\Delta}_i}}{z^{\tilde{\Delta}_i}} \, d\Omega = \sum_{\Delta_1, \ldots, \Delta_\ell} H_\Sigma(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_F, \Delta_1, \ldots, \Delta_\ell) \mathcal{D} \left[ \prod_{i=1}^V c_i^{\Delta_i} \right].
\]

We write the most general generating function for Hurwitz numbers [2]:

\[
N^{nd} \int \prod_{i=1}^F \frac{M_i^{\tilde{\Delta}_i}}{z^{\tilde{\Delta}_i}} \prod_{i=F_1+1}^{F_2} \mathcal{M}(M_i) \prod_{i=F_2+2}^{F_4} \mathcal{S}(M_{i-1}, M_i) d\Omega = \sum_{|\Delta_1|=\ldots=|\Delta_\ell|=d} H_\Sigma(\tilde{\Delta}_1^{F_1}, \Delta_1, \ldots, \Delta_\ell) C(\Delta_1, \ldots, \Delta_\ell),
\]

where

\[
H_\Sigma(\tilde{\Delta}_1^{F_1}, \Delta_1, \ldots, \Delta_\ell) = \sum_{\mu \in \mathcal{P}} \left( \frac{\dim \mu}{d!} \right)^{F+V-2h-n-m} \varphi_\mu(\tilde{\Delta}_1) \cdots \varphi_\mu(\Delta_1) \cdots \varphi_\mu(\Delta_\ell). \tag{4.4}
\]

Here, \( h = (F - F_2)/2 \) is the number of handles, and \( m = F_2 - F_1 \) is the number of Möbius strips glued into the surface \( \Sigma \) on which the graph \( \tilde{\Gamma} \) (a modified “dessin d’enfants”) is represented. The Euler characteristic of the surface \( \Sigma \) is

\[
F - n + V - 2h - m = F_1 - n + V.
\]

Hurwitz number (4.4) counts the coverings of \( \Sigma \) with the ramification profiles \( \tilde{\Delta}_1, \ldots, \tilde{\Delta}_F, \Delta_1, \ldots, \Delta_\ell \).

We multiply both sides of (4.3) by

\[
\prod_{i=k+1}^{F_1} \frac{\dim \mu_i^{\tilde{\Delta}_i}}{d!} \varphi_{\mu_i}(\tilde{\Delta}_i) z^{\tilde{\Delta}_i}, \quad k \leq F,
\]

and then sum (2.16) over the partitions \( \tilde{\Delta}_i \) \((i = k+1, \ldots, F_1)\) taking the formulas

\[
s_\mu(X) = \frac{\dim \mu}{d!} \sum_{\Delta} \varphi_\mu(\Delta) \chi^{\Delta}, \quad \chi^{\Delta} = \text{tr}(X^{\Delta}) \cdots \text{tr}(X^{\Delta_\ell}),
\]

into account while calculating the left-hand side of (4.1) and orthogonality relation (A.5) into account while calculating the right-hand side. We obtain

\[
N^{nd} \int \prod_{i=1}^F \frac{M_i^{\tilde{\Delta}_i}}{z^{\tilde{\Delta}_i}} \prod_{i=F_1+1}^{F_2} \mathcal{M}(M_i) \prod_{i=F_2+2}^{F_4} \mathcal{S}(M_{i-1}, M_i) \prod_{i=k+1}^{F_1} s_{\mu_i}(M_i) d\Omega = \delta_{\mu_1, \ldots, \mu_{F_1}} \left( \frac{\dim \mu}{d!} \right)^{F_1 - V - h - n} \varphi_\mu(\tilde{\Delta}_1) \cdots \varphi_\mu(\tilde{\Delta}_F) \prod_{i=1}^V s_\lambda(W_i^*). \tag{4.5}
\]
We recall that we can interpret \( m \) as the number of M"obius strips and \( h \) as the number of handles glued into the surface \( \Sigma \) with the represented graph \( \Gamma \) and that \( F_1 \) is a number of Young diagrams \( \Delta^i \) in the right-hand side of the equality.

**Remark 4.1.** We assume that the faces of a graph \( \Gamma \) can be colored black and white like a chessboard such that a face of one color borders only faces of the other color. We can then assign the sides of edges of white-face matrices in the set \( \{ Z_i \} \) (i.e., differential operators) and sides of black-face matrices in the set \( \{ Z_i \} \). In this case, monodromies of white faces are differential operators acting on monodromies of black faces.

The most natural and simple case is the following “polarization”: we assume that monodromies of \( M_i \) \((i = k + 1, \ldots, F_1)\) correspond to black faces and the remaining monodromies correspond to white faces (see remark 4.1). Let \( m = k = 0 \). We take dessin d’enfants of a sunflower with \( n \) petals as the graph \( \Gamma \). We have one inflated vertex of \( \Gamma \) and obtain a small disc, the flower center. We have \( n+1 \) faces of \( \Gamma \): \( n \) petals and a large face containing all petals and the point at infinity. We place all momenta inside petals:

\[
M_i = C_i Z_i^i, \quad i = 1, \ldots, n.
\]

All “coordinates” (the set \( \{ (Z_i)_{a,b} \} \) of elements) then turn out to be located on the (outer) other side of a ribbon; they are located along the boundary of the black face containing the petals:

\[
M_{n+1} = Z_1 C_1 \cdots Z_n C_n.
\]

We remove the tilde from the notation for Young diagrams. Then

\[
N^{n_d} \left[ \prod_{i=1}^{n} \frac{M_{\Delta^i}}{z_{\Delta^i}} \right] s_{\mu}(Z_1 C_1 \cdots Z_n C_n) d\Omega = \phi_{\mu}(\Delta^1) \cdots \phi_{\mu}(\Delta^1) s_{\mu}(C_{-1} C_1 \cdots C_{-n} C_n). \tag{4.6}
\]

This equality is equivalent to

\[
W_{\Delta^1}^{i} \cdots W_{\Delta^n}^{n} \cdot s_{\mu}(Z_1 C_1 \cdots Z_n C_n) = \phi_{\mu}(\Delta^1) \cdots \phi_{\mu}(\Delta^1) s_{\mu}(C_{-1} C_1 \cdots C_{-n} Z_n C_n), \tag{4.7}
\]

where each \( N \times N \) matrix \( C_{-i} \) can now, for example, depend polynomially on \( Z_i \) \((i = 1, \ldots, n)\) and

\[
W_{\Delta^i}^{i} = : \text{tr}((C_{-i} Z_i \partial_i)^{\Delta^i}) \cdots \text{tr}((C_{-i} Z_i \partial_i)^{\Delta^i}): \tag{4.8}
\]

Here, \( C_{-i} \partial_i \) represents a matrix whose elements are differential operators, more precisely, vector fields

\[
(C_{-i} \partial_i)_{a,b} := \sum_{c=1}^{N} (C_{-i})_{a,c} \frac{\partial}{\partial(Z_i)_{b,c}}, \tag{4.9}
\]

where (starting with formula (4.7)) we neglect the dependence on \( N \) because it cancels anyway. The normal ordering symbol in (4.8) denoted by a colon is the same here as in [14], i.e., it preserves the matrix structure: the derivative operators do not act on \( C_{-i} = C_{-i}(Z_i) \). It is important that the normal ordering procedure is necessary for (4.7) to be equivalent to (4.6)!

If we now consider the case \( n = 1 \) (one petal) and in addition set \( C_{-1} = Z_i \), then we obtain the sought formula (1.2). If we take \( C_{-i} = I_1 \) \((i = 1, \ldots, n)\), then we obtain the rule for multiplying the classes \( C_{\Delta^i} \) by an idempotent \( \mathfrak{F}_\lambda \) given by (A.8) in the center of the group algebra \( S_d \) (see Sec. A.3).
We consider one more example with the same graph \( \tilde{\Gamma} \) and the same monodromies but with \( F_1 = 0 \) in formula (4.5). In this case, we can rewrite integral (4.6) as

\[
\widehat{\mathfrak{M}}_1 \cdots \widehat{\mathfrak{M}}_m \widehat{\mathfrak{N}}_1 \cdots \widehat{\mathfrak{N}}_h \cdot s_\mu(Z_1 C_1 \cdots Z_n C_n) = \left( \frac{\dim \mu}{|\mu|!} \right)^{-2h-n} s_\mu(C_{-1}(Z_1) \cdots C_{-n}(Z_n) C_n),
\]

(4.10)

where

\[
\widehat{\mathfrak{M}}_i = \exp \left\{ \frac{1}{2} \sum_{j>0} \frac{1}{j} \left( \text{tr}(C_{-j} \partial_j) \right)^2 + \sum_{\text{odd } j>0} \frac{1}{j} \text{tr}(C_{-j} \partial_j) \right\}, \quad i = 1, \ldots, m,
\]

\[
\widehat{\mathfrak{N}}_i = \exp \left\{ \sum_{j>0} \frac{1}{j} \text{tr}(C_{-j} \partial_j) \right\} \text{tr}(C_{-j} \partial_j) \}, \quad i = 1, \ldots, h.
\]

**Remark 4.2.** If \( C_{-i} = Z_i \) (\( i = 1, \ldots, n \)), then Eqs. (4.7) and (4.10) describe eigenvalue problems for the corresponding Hamiltonians in two-dimensional boson theory. A comparison with the case analyzed by Dubrovin [62] might be relevant here. In this case, the operators \( W^{(n)} \) \( (n = 1, 2, \ldots) \) are Hamiltonians of the dispersion-free KdV equations.

**Remark 4.3.** The case where \( C_{-i} \) is independent of the random matrices \( Z_i \) is interesting if the matrices of the star monodromies are degenerate. In this case, the integration is over rectangular matrices. For example, we can choose \( \text{diag}(1, \ldots, 1, 0, \ldots, 0) \) as the matrix \( C_{-1}C_1 \cdots C_{-n}C_n \) in (4.6). We then obtain the Pochhammer symbol in the right-hand side, which allows associating the whole integral with the hypergeometric tau function [89]. We will discuss this in a future paper, where we plan to relate our research to some research in [77]–[82].

We consider another example. Let the graph \( \Gamma \) have two vertices connected by four edges. We obtain

\[
: p^{(1)}(\partial_1 C_{-2} \partial_2 C_{-3}) p^{(2)}(\partial_2 C_{-3} \partial_4 C_{-4} : s_\lambda(Z_1 C_1 Z_4 C_4) s_\mu(Z_2 C_2 Z_3 C_3) =
\]

\[
= \delta_{\lambda, \mu} \left( \frac{\dim \mu}{d!} \right)^{-2} \varphi_\mu(\Delta_1) \varphi_\mu(\Delta_2) s_\mu(C_{-2} C_{-3} C_{-2}) s_\mu(C_{-1} C_{-2} C_{-3} C_{-2}) .
\]

In particular, if \( C_3 = C_4 = C \), \( C_{-1} = C_{-1}(Z_2) \), \( C_{-2} = C_{-2}(Z_1) \), \( C_{-3} = C_{-3}(Z_3) \), and \( C_{-4} = C_{-4}(Z_4) \), then

\[
: p^{(1)}(C_{-2}(Z_1) \partial_1 C_{-1}(Z_2) \partial_2 p^{(2)}(\partial_2 C_{-3}(Z_3) \partial_4 C_{-4}(Z_4)) : s_\lambda(Z_1 C_1 Z_4 C_4) s_\mu(Z_2 C_2 Z_3 C_3) =
\]

\[
= \delta_{\lambda, \mu} \left( \frac{\dim \mu}{d!} \right)^{-2} \varphi_\mu(\Delta_1) \varphi_\mu(\Delta_2) s_\mu(C_{-2} Z_1 C_{-1} C_{-4}(Z_4) C_{-1}(Z_2) C_{-3} C_{-2}) s_\mu(C_{-1} C_{-2} C_{-3} C_{-2}) .
\]

If \( C_3 = C_4 = C \), \( C_{-1} = Z_2 \), \( C_{-2} = Z_1 \), \( C_{-3} = Z_3 \), and \( C_{-4} = Z_4 \) (Euler fields), then we obtain the eigenvalue problem

\[
: p^{(1)}(Z_1 \partial_1 Z_2 \partial_2 p^{(2)}(\partial_2 Z_3 \partial_4 Z_4) : (s_\lambda(Z_1 C_1 Z_4 C_4) s_\mu(Z_2 Z_2 Z_3 C_3)) =
\]

\[
= \delta_{\lambda, \mu} \left( \frac{\dim \mu}{d!} \right)^{-2} \varphi_\mu(\Delta_1) \varphi_\mu(\Delta_2) s_\mu(Z_1 C_1 Z_4 C_4) s_\mu(Z_2 Z_2 Z_3 C_3) .
\]

If \( C_{-1} = C_{-2} = C_{-3} = C_{-4} = I_N \), then

\[
: p^{(1)}(\partial_1 \partial_2) p^{(2)}(\partial_3 \partial_4) : (s_\lambda(Z_1 C_1 Z_4 C_4) s_\mu(Z_2 Z_2 Z_3 C_3)) =
\]

\[
= \delta_{\lambda, \mu} \left( \frac{\dim \mu}{d!} \right)^{-2} \varphi_\mu(\Delta_1) \varphi_\mu(\Delta_2) s_\mu(C_1 C_3) s_\mu(C_2 C_4) .
\]
Now let \( n = 4 \) and \( \Gamma \) be the simplest graph with two edges and one vertex represented on a torus. We double both edges of this graph and obtain a graph with one vertex, four edges, and three faces. We then have

\[
\begin{align*}
\phi_{\Delta}(\partial_1 C_1 \partial_2 C_3, \partial_4 C_4) & \cdot \phi_{\Delta}(\partial_2 C_2 \partial_3 C_4, \partial_2 C_4) = s_\lambda(Z_1 C_1 Z_2 C_2 Z_3 C_3 Z_4 C_4) = \\
& = \left(\frac{\dim \lambda}{d!}\right)^{-2} s_\lambda(C_1 C_2 C_3 C_4 C_4 C_3 C_2) = \left(\frac{\dim \lambda}{d!}\right)^{-2} s_\lambda(C_1 C_2 C_3 C_4 C_4) \cdot \phi_{\Delta}(\Delta^1) \cdot \phi_{\Delta}(\Delta^2) \cdot s_\lambda(C_1 C_2 C_3 C_4 C_4)
\end{align*}
\]

Let \( C_1 = C_1(Z_3), C_2 = C_2(Z_2), C_3 = C_3(Z_1), C_4 = C_4(Z_4) \), and \( C_2 = C_4 = C \). Then we obtain

\[
\phi_{\Delta}(\partial_1 C_1 \partial_3 C_3, \partial_3 C_4) \cdot \phi_{\Delta}(\partial_2 C_2 \partial_4 C_4, \partial_2 C_4) = s_\lambda(Z_1 C_1 Z_2 C_2 Z_3 C_3 Z_4 C) = \\
= \left(\frac{\dim \lambda}{d!}\right)^{-2} s_\lambda(C_1 C_2 C_3 C_4 C_3 C_2) = \left(\frac{\dim \lambda}{d!}\right)^{-2} s_\lambda(C_1 C_2 C_3 C_4 C_4) \cdot \phi_{\Delta}(\Delta^1) \cdot \phi_{\Delta}(\Delta^2) \cdot s_\lambda(C_1 C_2 C_3 C_4 C_4)
\]

We note that if we take the graph dual to the sunflower graph with \( n = 1 \) (the dual graph related to one petal, which is merely an interval), then we have one face and two vertices, and we obtain an equality of the type of a Capelli relation. In this case, it makes sense to compare such relations with the beautiful results in [79]–[82].

There are several hints of the existence of interesting structures related to quantum integrability. First, as noted in [2], this is suggested by the appearance of the two-dimensional Yang–Mills theory [84] (also see [95]). Moreover, we note the appearance of Yangians in [77], [78], which we hope can be related to our topic. And we finally mention [62].

There is a direct analogy between integrals over complex matrices and over unitary matrices, but we do not consider this topic here.

**4.2. Comparison with the Hermitian matrix model.** For comparison, we write the famous one-matrix model [31], [32] with an additional source matrix:

\[
Z_N(p, C) = \int \exp \left\{ N \sum_{m > 0} \frac{N}{m} \text{tr}(CX) \right\} e^{-N \text{tr}(X^2)} \prod_{a=1}^{N} dX_{a,a} \prod_{a > b}^{N} d\text{Re} X_{a,b} d\text{Im} X_{a,b},
\]

where \( C \) and \( X \) are \( N \times N \) matrices and \( X \) is Hermitian. Such a model was considered in [33]. This model can also be regarded as the generating function for coverings of a graph \( \Gamma \) on \( S^2 \), which we describe without details. The graph \( \Gamma \) has only one edge connecting images of the middle of each edge of a Feynman graph with images of all vertices; the Feynman graphs are represented not on the base surface but on its even-sheeted coverings. We note that the edge of \( \Gamma \) is not a ribbon. Because we inserted a source matrix, inflated vertices (stars) should be considered instead of vertices.

We let \( \tilde{\Gamma} \) denote such a graph on the base surface. In contrast to the complex matrix model, this graph is not a Feynman diagram. The first branch point corresponds to the pairing of half-edges (i.e., all coverings of degree \( 2d \) have ramifications of type \( (2d) \)). The second branch point corresponds to vertices, and the ramification profiles over this point are given by a Young diagram in the factor \( \text{tr} C^{\Delta_1} \cdots \text{tr} C^{\Delta_v} =: C^\Delta \), where \( \Delta \) is a ramification profile. The third ramification point corresponds to edges and is defined by a Young diagram that appears in

\[
\text{tr}(MC)^{\Delta_1} \cdots \text{tr}(MC)^{\Delta_f} =: X^\Delta.
\]

The combinatorial equation for Feynman graphs of order \( 2d \)

\[
a \prod_{i=1}^{E} f_i = \prod_{i=1}^{V} \sigma_i
\]
is an equation in the group \( S_{2d} \), where \( \alpha \) has the cycle type \((2^d)\), each element \( f_i \) is a cycle of length \( \Delta_i \), and each element \( \sigma_i \) is a cycle of length \( \Delta_i \). The number of nonisomorphic coverings of the sphere \( S^2 \) with profiles of type \((2^d)\), \( \Delta, \bar{\Delta} \) is the Hurwitz number \( H_{S^2}((2^d), \Delta, \bar{\Delta}) \). It is natural to assume that
\[
Z_N(p, C) = \sum_{d>0} N^{-2nd} \sum_{\Delta, \bar{\Delta}} H_{S^2}((2^d), \Delta, \bar{\Delta}) C^d p_\bar{\Delta}.
\]
We will consider this relation in a future paper.

Appendix A: Definitions and review of known results

Presenting common facts, we generally follow [2] in this appendix.

A.1. Hurwitz numbers. A Hurwitz number is the weighted number of ramified covers of a surface with a given topological type of critical values. Hurwitz numbers for oriented surfaces without a boundary were introduced by Hurwitz at the end of the 19th century. They subsequently turned out to be related to moduli spaces of Riemann surfaces [19], integrable systems [17], modern models in mathematical physics (matrix models), and closed topological field theories [15]. Here, we consider only Hurwitz numbers for compact surfaces without a boundary. The definition and important properties of Hurwitz numbers for arbitrary compact surfaces (with and without a boundary) are given in [96].

We refine the given definition. We consider a ramified covering \( \varphi: \tilde{\Sigma} \to \Sigma \) of degree \( d \) over a compact surface without a boundary. The map \( \varphi \) in a neighborhood of each point \( z \in \tilde{\Sigma} \) is topologically equivalent to the complex map \( u \to u^p \) in a neighborhood of the point \( u = 0 \) on the complex plane \( \mathbb{C} \). The number \( p = p(z) \) is called the degree of the covering \( \varphi \) at the point \( z \). A point \( z \in \tilde{\Sigma} \) is called a branch point or critical point if \( p(z) \neq 1 \). The number of critical points is finite. The images \( \phi(z) \) of a critical point are called critical values.

We associate all points \( z_1, \ldots, z_\ell \in \tilde{\Sigma} \) for which \( \varphi(z_i) = s \) with the point \( s \in \Sigma \). Let \( p_1, \ldots, p_\ell \) be the degree of the covering \( \varphi \) at these points. The sum \( d = p_1 + \cdots + p_\ell \) is equal to the degree of the map \( \varphi \). Hence, the partition \( d = p_1 + \cdots + p_\ell \) of \( d \) corresponds to each point \( s \in \Sigma \). Ordering the degrees of the covering as \( p_1 \geq \cdots \geq p_\ell > 0 \) at each point \( s \in \Sigma \), we can introduce the Young diagram \( \Delta^s = [p_1, \ldots, p_\ell] \) of degree \( d \) with \( \ell = \ell(\Delta^s) \) rows of lengths \( p_1, \ldots, p_\ell \). This Young diagram \( \Delta^s \) is called the topological type of the point \( s \). A point \( s \) is critical if not all \( p_i \) are equal to unity.

We note that the Euler characteristics \( E(\tilde{\Sigma}) \) and \( E(\Sigma) \) of surfaces \( \tilde{\Sigma} \) and \( \Sigma \) are related by the Riemann–Hurwitz relation
\[
E(\tilde{\Sigma}) = E(\Sigma)d + \sum_{z \in \tilde{\Sigma}} (p(z) - 1) = E(\Sigma)d + \sum_{i=1}^n (\ell(\Delta^s_i) - d),
\]
where \( s_1, \ldots, s_n \) are critical values.

Two coverings \( \varphi_1: \tilde{\Sigma}_1 \to \Sigma \) and \( \varphi_2: \tilde{\Sigma}_2 \to \Sigma \) are considered equivalent if there exists a homeomorphism \( F: \tilde{\Sigma}_1 \to \tilde{\Sigma}_2 \) such that \( \varphi_1 = \varphi_2 F \). The equivalence of a covering to itself is called an automorphism of the covering. The automorphisms of a covering \( \varphi \) form a finite-order automorphism group \(|Aut(\varphi)|\). Equivalent coverings have isomorphic automorphism groups, and the orders of these groups coincide.

We fix points \( s_1, \ldots, s_n \in \Sigma \) and a Young diagram \( \Delta^1, \ldots, \Delta^n \) of degree \( d \). We consider a set \( \Phi \) of equivalent covering classes for which the given points are critical values and the Young diagrams are topological classes of these critical values. Everywhere, we assume that the surface \( \Sigma \) is connected unless otherwise specified.

By definition, a Hurwitz number of degree \( d \) is the number
\[
H^d(\Delta^1, \ldots, \Delta^n) = \sum_{\varphi \in \Phi} 1 / |Aut(\varphi)|.
\]
It is easy to prove that the Hurwitz number is finite and depends only on the Euler characteristic \(E = E(\Sigma)\) and the Young diagrams \(\Delta^1, \ldots, \Delta^n\) of the surface \(\Sigma\) and is independent of the location of the points \(s_1, \ldots, s_n \in \Sigma\). Therefore, we write \(H_E\) instead of \(H_\Sigma\) everywhere below, in particular, \(H_2 = H_{\mathbb{Z}^2}\) and \(H_1 = H_{\mathbb{R} \mathbb{Z}^2}\).

**A.2. Hurwitz numbers and the symmetric group.** We describe the Hurwitz number

\[
H^n_E(\Delta^1, \ldots, \Delta^n)
\]

in terms of the center \(Z(C(S_d))\) of the group algebra \(C(S_d)\) of the symmetric group \(S_d\) of permutations. It is essential for the description that Young diagrams of degree \(d\) are in one-to-one correspondence with the conjugacy classes of the elements of \(S_d\). Moreover, a Young diagram \(\Delta\) with rows of lengths \(d_1, \ldots, d_k\) corresponds to the conjugacy class comprising permutations of cycles of type \(d_1, \ldots, d_k\), i.e., permutations \(\sigma \in S_d\) generating a group isomorphic to \(\mathbb{Z}/d_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_k \mathbb{Z}\). We let the same symbol \(\Delta\) denote this conjugacy class and \(|\Delta|\) denote the number of elements in \(\Delta\). The sum \(\hat{\Delta}_i\) of elements of \(\Delta^i\) belongs to the center of the algebra \(Z(C(S_d))\). Moreover, the sums \(\hat{\Delta}_i\) generate the algebra \(Z(C(S_d))\).

We show how the Hurwitz number \(H^2_2(\Delta^1, \ldots, \Delta^n)\) for the sphere \(\Sigma = \mathbb{S}^2\) is described in terms of \(Z(C(S_d))\). Let the points \(p_1, \ldots, p_n \in \mathbb{S}^2\) be pairwise distinct and \(p \in \mathbb{S}^2 \setminus \{p_1, \ldots, p_n\}\). We consider the standard generators of the fundamental group \(\pi_1(\mathbb{S}^2 \setminus \{p_1, \ldots, p_n\}, p)\). They are represented by simple closed pairwise nonintersecting contours \(\gamma_1, \ldots, \gamma_n\) starting and ending in \(p\) and sequentially going around the points \(p_1, \ldots, p_n\). Their product \(\gamma_1 \cdots \gamma_n = 1\).

We now consider a cover \(\varphi: \hat{\Sigma} \to \mathbb{S}^2\) of type \((\Delta^1, \ldots, \Delta^n)\) with the critical points \(p_1, \ldots, p_n\). The complete preimage \(\varphi^{-1}(p)\) comprises \(d\) points \(q_1, \ldots, q_d\). Going around a contour \(\gamma_i\) determines a permutation \(\sigma_i \in S_d\) of these points. The conjugacy class of \(\sigma_i\) is described by the Young diagram \(\Delta^i\) and the product \(\sigma_1 \cdots \sigma_n\) gives the identical permutation. Hence, the covering \((\Delta^1, \ldots, \Delta^n)\) of the sphere generates an element of the set

\[
M(\Delta^1, \ldots, \Delta^n) = \{(\sigma_1, \ldots, \sigma_n) \in (S_d)^n \mid \sigma_i \in \Delta^i \ (i = 1, \ldots, n), \ \sigma_1 \cdots \sigma_n = 1\}.
\]

Equivalent coverings correspond to sets \((\sigma_1, \ldots, \sigma_n)\) obtained from each other by conjugation using some permutation.

We construct the reverse correspondence; in other words, we associate an equivalence class of coverings \(\varphi: \hat{\Sigma} \to \mathbb{S}^2\) of type \((\Delta^1, \ldots, \Delta^n)\) with critical values \(p_1, \ldots, p_n\) with each point in the conjugacy class \(M(\Delta^1, \ldots, \Delta^n)\). To construct such a covering, we must make cuts \(r_i\) between the points \(p\) and \(p_i\) in the contour \(\gamma_i\), consider \(d\) numbered examples of such cut spheres, and glue the boundaries of cuts together according to the permutations \(\sigma_1, \ldots, \sigma_n\). Consequently, the number of equivalence classes of coverings of type \((\Delta^1, \ldots, \Delta^n)\) coincides with the number of conjugacy classes of sets \((\sigma_1, \ldots, \sigma_n)\) in \(M = M(\Delta^1, \ldots, \Delta^n)\). This number is

\[
\frac{1}{d!} \sum_{(\sigma_1, \ldots, \sigma_n) \in M} |\text{Aut}(\sigma_1, \ldots, \sigma_n)|, \quad (A.1)
\]

where the subgroup \(\text{Aut}(\sigma_1, \ldots, \sigma_n) \subset S_n\) comprises permutations such that conjugation using them preserves the set \((\sigma_1, \ldots, \sigma_n)\). Under the established associations, automorphisms of coverings of type \((\Delta^1, \ldots, \Delta^n)\) are in a one-to-one correspondence with the permutations in \(\text{Aut}(\sigma_1, \ldots, \sigma_n)\). Therefore,

\[
H^2_2(\Delta^1, \ldots, \Delta^n) = \frac{1}{d!} |M(\Delta^1, \ldots, \Delta^n)|.
\]
We now describe the number $|M(\Delta_1, \ldots, \Delta_n)|$ in terms of the algebra $Z(C(S_d))$. We consider the linear functional $l: Z(C(S_d)) \rightarrow \mathbb{R}$ taking the value $1/d!$ on the Young diagram corresponding to identity and the value zero on the other Young diagrams. Then

$$\frac{1}{d!}|M(\Delta_1, \ldots, \Delta_n)| = l(\Delta^1 \Delta \cdots \Delta^n),$$

and consequently

$$H_2^d(\Delta^1, \ldots, \Delta^n) = l(\Delta^1 \Delta^2 \cdots \Delta^n).$$

This argument with some modifications can be applied to other surfaces $\Sigma$. We consider the case of the projective plane $\Sigma = \mathbb{P}^2$. The relation $\sigma_1 \cdots \sigma_n = 1$ in the definition of the set $M(\Delta_1, \ldots, \Delta_n)$ is then replaced with $\sigma_1 \cdots \sigma_n = \sigma^2$, where $\sigma$ is an arbitrary permutation. Therefore, in the calculation of the sets of permutations giving the needed covering, the number $l(\Delta^1 \Delta^2 \cdots \Delta^n)$ is replaced with $l(\Delta^1 \Delta^2 \cdots \Delta^n U)$, where $U = \sum_{\sigma \in S_d} \sigma^2$. The final formula has the form

$$H_1^d(\Delta_1, \ldots, \Delta^n) = l(\Delta^1 \Delta^2 \cdots \Delta^n U).$$

In accordance with [95], the Hurwitz number for a surface $\Sigma$ with the Euler characteristic $E$ in the general case is equal to

$$H_E^d(\Delta_1, \ldots, \Delta^n) = l(\Delta^1 \Delta^2 \cdots \Delta^n U^{2-E}).$$

A.3. Hurwitz numbers and representation theory. Hurwitz numbers can be described in terms of characters of symmetric groups. The corresponding formula has the form

$$H_E^d(\Delta_1, \ldots, \Delta^n) = (n!)^{1-E}[\Delta^1] \cdots [\Delta^n] \sum_{\lambda} \frac{\chi(\Delta^1) \cdots \chi(\Delta^n)}{\chi(1)^{n-E}}, \quad (A.2)$$

where the summation is over all characters of an irreducible representation of the group $S_d$ and $|\langle \Delta \rangle|$ is the cardinality of the class $\Delta$ of cycles of $S_d$. Frobenius and Schur proved this formula for covers without ramifications over the sphere and projective plane. They sought a formula for the cardinality of set (A.1) without its geometric interpretation. The proof in the general case differs little from the proof in the simplest case but was obtained significantly later in connection with an increased interest in Hurwitz numbers (see [6]–[8], [15]).

A partition $\lambda$ of weight $d$ generates an irreducible representation of the group $S_d$ of dimension $\dim \lambda$. Let $\chi(\lambda)$ be the character of this representation. Then $\dim \lambda = \chi(\lambda(e_{[1, \ldots, 1]}))$. For an arbitrary diagram $\Delta$ and $\lambda$, we define the normalized character

$$\varphi_\lambda(\Delta) := |\langle \Delta \rangle| \frac{\chi_\lambda(\Delta)}{\dim \lambda}. \quad (A.3)$$

We have the known orthogonality relations [87]

$$\sum_{\lambda} \left( \frac{\dim \lambda}{d!} \right)^2 \varphi_\lambda(\mu) \varphi_\lambda(\Delta) = \frac{\delta_{\Delta, \mu}}{z_\Delta}, \quad (A.4)$$

$$\left( \frac{\dim \lambda}{d!} \right)^2 \sum_\Delta z_\Delta \varphi_\lambda(\Delta) \varphi_\mu(\Delta) = \delta_{\lambda, \mu}, \quad (A.5)$$

where $d = |\Delta| = |\lambda|$ and

$$z_\Delta = \prod_i m_i i^{m_i} = \frac{d!}{|\langle \Delta \rangle|} \quad (A.6)$$
is the order of the automorphism group of the Young diagram $\Delta$ (in this formula, $m_i$ is a number of rows of length $i$ in $\Delta$).

The elements

$$\mathfrak{g}_\lambda = \left( \frac{\dim \lambda}{d!} \right)^2 \sum_\Delta z_\Delta \varphi_\lambda(\Delta) \mathfrak{c}_\Delta$$

(A.7)

form a basis of idempotents in $\mathbb{Z}[\mathbb{C}[S_d]]$, i.e., $\mathfrak{g}_\lambda^2 = \mathfrak{g}_\lambda$ and $\mathfrak{g}_\lambda \mathfrak{g}_\mu = 0$ for $\lambda \neq \mu$. Further,

$$\mathfrak{c}_\Delta = \sum_\lambda \varphi_\lambda(\Delta) \mathfrak{g}_\lambda,$$

(A.8)

and therefore

$$\mathfrak{c}_\Delta \mathfrak{c}_\Delta = \sum_\lambda \varphi_\lambda(\Delta^1) \varphi_\lambda(\Delta^2) \mathfrak{g}_\lambda = \sum_\Delta H_2(\Delta^1, \Delta^2, \Delta) z_\Delta \mathfrak{c}_\Delta.$$  

(A.9)

Moreover,

$$\langle \mathfrak{c}_\Delta, \cdots \mathfrak{c}_{\Delta^E} U^{2-E} \rangle = \sum_\lambda \varphi_\lambda(\Delta) \cdots \varphi_\lambda(\Delta^E) \langle \mathfrak{g}_\lambda U^{2-E} \rangle$$

(A.10)

and

$$\langle \mathfrak{g}_\lambda U^{2-E} \rangle = \left( \frac{\dim \lambda}{|\lambda|!} \right)^E.$$  

(A.11)

Hence, we have the relation

$$H_{E(\Sigma)}(\Delta^1, \ldots, \Delta^F) = \langle \mathfrak{c}_\Delta, \cdots \mathfrak{c}_{\Delta^F} \rangle_\Sigma = \langle \mathfrak{c}_\Delta, \cdots \mathfrak{c}_{\Delta^F} U^{2-E(\Sigma)} \rangle =$$

$$= \sum_\lambda \varphi_\lambda(\Delta^1) \cdots \varphi_\lambda(\Delta^F) \left( \frac{\dim \lambda}{|\lambda|!} \right)^E,$$

(A.12)

which is equivalent to (A.2).

From (A.12) and (A.4), we obtain

$$H_2(\Delta^1, \Delta) = \frac{\delta_{\Delta^1, \Delta}}{z_\Delta}. $$

For the number $D(\Delta)$, which in topological theory [96] describes the so-called Möbius cut, we have

$$D(\Delta) = z_\Delta H_1(\Delta),$$

(A.13)

where $H_1(\Delta)$ are the Hurwitz numbers counting covers of the real projective plane $\mathbb{R}P^2$ with the profile $\Delta$ at a single critical point. Formulas (A.12) and (A.5) give an independent proof of the axioms of the topological theory of Hurwitz numbers.

**Proposition A.1.** Let the numbers $H_{E(\Sigma)}(\Delta^1, \ldots, \Delta^F)$ be given by expression (A.12) for Young diagrams $\Delta^i$ ($i = 1, \ldots, F_1 + F_2$) of the same weight $d$. Then we have the equalities, called handle pinching
relations,
\[
H_{E-2}(\Delta^1, \ldots, \Delta^F) = \sum_{\Delta: |\Delta| = d} H_E(\Delta^1, \ldots, \Delta^F, \Delta, z_\Delta) =
\]
\[
= \sum_{\Delta: |\Delta| = d} \frac{H_E(\Delta^1, \ldots, \Delta^F, \Delta)}{H_2(\Delta, \Delta)}, \tag{A.14}
\]
\[
H_{E_1+E_2-2}(\Delta^1, \ldots, \Delta^{F_1+F_2}) = \sum_{\Delta: |\Delta| = d} H_{E_1}(\Delta^1, \ldots, \Delta^{F_1}, \Delta) z_\Delta H_{E_2}(\Delta, \Delta^{F_1+1}, \ldots, \Delta^{F_1+F_2}) =
\]
\[
= \sum_{\Delta: |\Delta| = d} \frac{H_{E_1}(\Delta^1, \ldots, \Delta^{F_1}, \Delta) H_{E_2}(\Delta, \Delta^{F_1+1}, \ldots, \Delta^{F_1+F_2})}{H_2(\Delta, \Delta)}, \tag{A.15}
\]
\[
H_{E-1}(\Delta^1, \ldots, \Delta^F) = \sum_{\Delta} H_E(\Delta^1, \ldots, \Delta^F, \Delta) D(\Delta) =
\]
\[
= \sum_{\Delta} \frac{H_E(\Delta^1, \ldots, \Delta^F, \Delta) H_1(\Delta)}{H_2(\Delta, \Delta)}, \tag{A.16}
\]
where
\[
\frac{H_1(\Delta)}{H_2(\Delta, \Delta)} = D(\Delta) = z_\Delta H_1(\Delta) = \sum_{\lambda: |\lambda| = |\Delta|} \chi_\lambda(\mathcal{C}_\Delta) \tag{A.17}
\]
are rational numbers (see (A.2)).

A.4. Inserting a Möbius strip and a handle. We have the equalities
\[
\frac{\det^{1/2}(1 + X)(1 - X)^{-1}}{\det^{1/2}(I_N \otimes I_N - X \otimes X)} = \sum_{\lambda} s_\lambda(X) = \exp \left\{ \frac{1}{2} \sum_{m > 0} \frac{1}{m} (\text{tr } X^m)^2 + \sum_{\text{odd } m > 0} \frac{1}{m} \text{tr } X^m \right\} = \sum_{\Delta \in \mathcal{P}} p_\Delta D(\Delta), \tag{A.18}
\]
where \(\mathcal{P}\) is the set of all partitions and we set \(p_m = \text{tr } X^m\) and \(p_\Delta = p_{\Delta_1} p_{\Delta_2} \ldots\). Function (A.18) written in the variables \(\{p_m\}\) was used in [53] as the generating function for one-point Hurwitz numbers in the case of covering the projective plane \(\mathbb{R}P^2\). This function was given in [93] as the simplest nontrivial example of a hypergeometric tau function \(\tau_B^1\) of the B-type BKP hierarchy.

We write a generating function of the one-point Hurwitz number for \(\Sigma = \mathbb{R}P^2\). We first consider the simplest case of one branch point connected with all \(r_i = 1\) and \(N = \infty\). This case is generated by the function \(\tau_B^1\), in which it is reasonable to change the variables \(p_m \to h^{-1} c^m p_m\):
\[
\exp \left\{ \frac{1}{h^2} \sum_{m > 0} \frac{1}{2m} p_m^2 c^{2m} + \frac{1}{h} \sum_{\text{odd } m > 0} \frac{1}{m} p_m c^m \right\} = \sum_{d > 0} c^d \sum_{\Delta: |\Delta| = d} h^{-\ell(\Delta)} p_\Delta H^{1,a}(d; \Delta), \tag{A.19}
\]
where \(a = 0\) if \(\Delta = (1^d)\) and \(a = 1\) otherwise. Then \(H^{1,1}(d; \Delta)\) is the Hurwitz number describing the covering with \(d\) sheets of the plane \(\mathbb{R}P^2\) with a single branch point with the profile \(\Delta = (d_1, \ldots, d_l), |\Delta| = d\) by a Klein surface (not necessarily simply connected) with the Euler characteristic \(E' = \ell(\Delta)\). For example, for \(d = 3\) and \(E' = 1\), we obtain \(H^{1,1}(3; \Delta) = \delta_{\Delta,(3)}/3\).

Further, we note that the exponent in the left-hand side can be rewritten as a generating series for connected Hurwitz numbers:
\[
\frac{1}{h^2} \sum_{d=2m} c^{2m} p_m^2 H_{\text{con}}^{1,1}(d; (m, m)) + \frac{1}{h} \sum_{d=2m-1} c^{2m-1} p_{2m-1} H_{\text{con}}^{1,1}(d; (2m - 1)),
\]

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where \( H_{\text{con}}^{1,1} \) describes a \( d \)-fold covering by either the Riemann sphere (\( d = 2m \)) or the projective plane (\( d = 2m - 1 \)). In the case of one branch point, these are the only ways to cover \( \mathbb{R}P^2 \) with a connected surface.

The geometric meaning of the exponent in (A.19) can be explained as follows. We can regard the projective plane as the unit disk with opposite points \( z \) and \( -z \) on its boundary \( |z| = 1 \) identified. If we cover the Riemann sphere with a Riemann sphere, \( z \to z^m \), then we obtain two critical points with the same profiles. But covering the projective plane \( \mathbb{R}P^2 \) with the Riemann sphere, we have the composition of the map \( z \to z^m \) on the Riemann sphere and the quotient by the antipodal involution \( z \to -1/z \). Hence, the ramification profile \((m, m)\) appears at the single critical point 0 on the plane \( \mathbb{R}P^2 \). The automorphism group is the dihedral group of order 2\( m \) comprising rotations through the angle \( \pi/m \) and the antipodal involution \( z \to -1/z \).

Therefore, we find that \( H_{\text{con}}^{1,1}(d; (m, m)) = 1/2m \), which coincides with the factor in the terms in the first sum in the exponent in (A.19).

We now consider a covering of \( \mathbb{R}P^2 \) by the projective plane \( \mathbb{R}P^2 \) via \( z \to z^d \). We have the critical point 0 for even \( d \); moreover, each point of the unit circle \( |z| = 1 \) is also critical (multiple). But from the beginning, we restrict our consideration to the case of isolated critical points. There is one critical point 0 for odd \( d = 2m - 1 \), and the automorphism group consists of rotations through the angle \( \pi/(2m - 1) \).

Therefore, in this case, we have \( H^{1,1}(d; (2m - 1)) = 1/(2m - 1) \), which coincides with the factor in the terms in the second sum in the exponent in (A.19).

**A.5. Generating function of simple Hurwitz numbers.** Important applications of Hurwitz numbers are related to generating functions for 1- and 2-Hurwitz numbers. The nonconnected simple 1-Hurwitz number \( h_{m, \Delta}^0 \) is by definition the nonconnected Hurwitz number \( H_{\text{con}}^{1,1} \) for all \( \Delta \) (\( \Gamma_i \) should not be confused with the graph notation in the main body of the paper) and \( |\Delta| = \cdots = |\Gamma_m| = d \).

We associate the monomial \( p_{\Delta} = p_{d_1}, \ldots, p_{d_k} \) with the Young diagram \( \Delta \) with rows of lengths \( d_1, \ldots, d_k \). The generating function for 1-Hurwitz numbers depends on an infinite number of formal variables \( p_1, p_2, \ldots \) and is defined as

\[
F^0(u|p_1, p_2, \ldots) = \sum_{m=0}^{\infty} \sum_{\Delta} \frac{u^m}{m!} h_{m, \Delta}^0 p_{\Delta}.
\]

This function has several remarkable properties discovered rather recently. The first is the relation between \( u \) and the \( p_i \):

\[
\frac{\partial F^0}{\partial u} = LF^0, \quad L = \frac{1}{2} \sum_{a,b=1}^{\infty} \left( (a+b)p_a p_b \frac{\partial}{\partial p_{a+b}} + ab p_{a+b} \frac{\partial^2}{\partial p_a \partial p_b} \right).
\]  

(A.20)

This relation was first found in [23] by purely combinatorial methods, It can also be obtained via vertex operators [97] and a tau function [91], [13], [89], but it also has a geometric explanation [14].

We consider a covering \( \varphi : \Sigma \to S^2 \) of type \((\Delta, \Gamma_1, \ldots, \Gamma_m)\). Let \( q, p \in S^2 \) be critical points of \( \varphi \) corresponding to the Young diagrams \( \Delta \) and \( \Gamma_m \). We join the points \( q \) and \( p \) by a line \( l \) with no self-intersections. The preimage \( \varphi^{-1}(l) \) comprises \( d-1 \) connected components of which only one, \( \tilde{l} \), contains the critical point \( \tilde{p} \) with the critical value \( p \). The endpoints of \( \tilde{l} \) are the preimages \( \tilde{q}_1 \) and \( \tilde{q}_2 \) of the point \( q \). We now move \( p \) along the line \( l \) in the direction of \( q \) by varying \( \varphi \) continuously in an appropriate way. As a result, we obtain the covering \( \varphi' \) of type \((\Delta', \Gamma_1, \ldots, \Gamma_{m-1})\). We consider what kind of Young diagram \( \Delta' \) is obtained in this case. Let \( \tilde{q}_1 = \tilde{q}_2 \) and \( c \) be the ramification order of \( \varphi \) at the point \( \tilde{q} = \tilde{q}_1 = \tilde{q}_2 \). The orders of critical points that are not \( \tilde{q} \) do not change in the process of deforming \( \varphi \) into \( \varphi' \). As a result of the deformation, \( \tilde{q} \) is split into two points with the ramification orders \( a \) and \( b \), where \( a + b = c \). Hence, the monomial \( p_{\Delta} \) is mapped into the monomial \( p_a p_b \partial p_{\Delta}/\partial p_c \).
Let the critical points $\tilde{q}_1$ and $\tilde{q}_2$ not coincide and their respective ramification orders be equal to $a$ and $b$. Then, as before, the orders of critical points other than $\tilde{q}_1$ and $\tilde{q}_2$ do not change during the process of deforming $\varphi$ into $\varphi'$. As a result of the deformation, $\tilde{q}_1$ and $\tilde{q}_2$ are mapped into one critical point of the order $c = a + b$. Hence, the monomial $p_\Delta$ is mapped into the monomial $p_\Delta \frac{\partial^2 p_\Delta}{\partial p_a \partial p_b}$.

Summation over all possible equivalence classes of coverings of all types $(\Delta, \Gamma_1, \ldots, \Gamma_m)$ and all their deformations into coverings of types $(\Delta', \Gamma_1, \ldots, \Gamma_{m-1})$ does indeed give relation (A.20).

The differential properties of the function $F^0(u|p_1, p_2, \ldots)$ were studied in [13], [14], [22], [59], [61].

**A.6. Transition from dressed cycles of edges to dressed cycles of stars.** Here, we follow [1], where the procedure for passing from monodromy set (2.1) to (dual) monodromy set (2.3) was described.

In face monodromy (3.2), we set each matrix in the set $\{\tilde{Z}_i\}$ equal to the $N \times N$ identity matrix and introduce the notation for the obtained matrices

$$W_a := M_a|_{z_i = 1, \ldots, n} = D[f_a],$$ (A.21)

where $f_a$ is a cycle of faces with the number $a$. We call such monodromies words. They are defined up to a cyclic permutation.

We consider the tensor product $W_1 \otimes W_2 \otimes \cdots \otimes W_F$ (the order of factors is not important) and the set of involutions $T_i (i = 1, \ldots, n)$ acting on this tensor product as follows. Each involution $T_i$ does not act on $W_a$ that contain neither $C_i$ nor $C_{-i}$. There are two possible cases. Let the matrices $C_i$ and $C_{-i}$ belong to one word, for example, $W_a$. Because we can cyclicly interchange matrices in a word, we transform it into the form $C_i X C_{-i} Y$, where $X$ and $Y$ are some matrices. In the other case, $C_i$ and $C_{-i}$ belong to different words. We write these words as $C_i X$ and $C_{-i} Y$. As a result, we obtain

$$T_i[\cdots \otimes C_i X C_{-i} Y \otimes \cdots] = \cdots \otimes C_i X \otimes C_i Y \otimes \cdots,$$

$$T_i[\cdots \otimes C_i X \otimes C_i Y \otimes \cdots] = \cdots \otimes C_i X C_{-i} Y \otimes \cdots.$$

It is easy to see that the involutions commute: $T_i[T_j[\cdot]] = T_j[T_i[\cdot]]$.

If we return to a graph $\Gamma$ with ribbons, then the operation $T_i$ is as follows. We must somehow widen the ribbon with the number $i$. We assume that this ribbon is transformed into a rectangle with the vertices 1, 2, 3, and 4 and the sides of the ribbon are the arrow $1 \rightarrow 2$ and the arrow paired with it $3 \rightarrow 4$. As a result of the action of $T_i$, the rectangle becomes a new ribbon, but its sides are now the paired arrows $1 \rightarrow 4$ and $3 \rightarrow 2$. The transformation

$$M_1, \ldots, M_F \rightarrow W_1^*, \ldots, W_F^*$$

proceeds purely algebraically in $n$ steps.

We have the relations

$$\prod_{i=1}^n T_i[W_1 \otimes W_2 \otimes \cdots \otimes W_F] = W_1^* \otimes \cdots \otimes W_N^*,$$

$$\prod_{i=1}^n T_i[W_1^* \otimes \cdots \otimes W_N^*] = W_1 \otimes W_2 \otimes \cdots \otimes W_F.$$

This is a version of Eq. (3.5) in the language of dressed cycles. The involution $\prod_{i=1}^n T_i$ without fixed points maps the graph $\Gamma$ to its dual graph.

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