AN EXTENSION OF ONE DIRECTION IN MARTY’S NORMALITY CRITERION

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Abstract. We prove the following extension of one direction in Marty’s theorem: If $k$ is a natural number, $\alpha > 1$ and $F$ is a family of functions meromorphic on a domain $D$ all of whose poles have multiplicity at least $\frac{k}{\alpha-1}$, then the normality of $F$ implies that the family

$$\left\{ \frac{|f^{(k)}|}{1 + |f|^\alpha} : f \in F \right\}$$

is locally uniformly bounded.

1. Introduction and main results

Our point of departure is the following famous normality criterion of F. Marty [9].

Theorem A. (Marty’s Theorem) A family $F$ of meromorphic functions on a domain $D \subseteq \mathbb{C}$ is normal if and only if the family $\{ f^\# : f \in F \}$ of the corresponding spherical derivatives $f^\# = \frac{|f'|}{1 + |f|^\alpha}$ is locally uniformly bounded.

In the present paper we investigate the question how normality can be characterized in terms of the quantity

$$\frac{|f^{(k)}|}{1 + |f|^\alpha}$$

where $k \in \mathbb{N}$, $\alpha > 0$

rather than the spherical derivative $f^\#$. A more or less complete answer is already known for the direction “$\Leftarrow$” in Marty’s theorem (locally uniform boundedness implies normality), but not for the opposite direction. Hence, we focus our attention on the latter one.

But first we summarize the known results concerning direction “$\Leftarrow$”. A substantial (and best possible) improvement of this direction in Marty’s theorem is due to A. Hinkkanen [3]: A family of meromorphic (resp. holomorphic) functions is already normal if the corresponding spherical derivatives are bounded on the preimages of a set consisting of five (resp. three) elements. (An analogous result for normal functions was earlier proved by P. Lappan [4].)

As to generalizations of Marty’s theorem to higher derivatives, S.Y. Li and H. Xie [6] obtained the following result.

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Theorem B. Let \( k \) be a natural number and \( \mathcal{F} \) a family of functions meromorphic on a domain \( D \) all of whose zeros have multiplicity at least \( k \). Then \( \mathcal{F} \) is normal in \( D \) if and only if \( \left\{ \frac{|f^{(k)}|}{1 + |f|^{k+1}} : f \in \mathcal{F} \right\} \) is locally uniformly bounded in \( D \). The direction \( \implies \) holds without the assumption on the multiplicities.

In [10] a new proof of Theorem B was given which avoids the use of Nevanlinna theory.

Finally, Y. Xu [11] proved the following extension of Hinkkanen’s normality result to higher derivatives.

Theorem C. Let \( k \) be a natural number and \( \mathcal{F} \) a family of functions meromorphic on a domain \( D \). Assume that there is a value \( w^* \in \mathbb{C} \) and a constant \( M < \infty \) such that for each \( f \in \mathcal{F} \) we have \( |f'(z)| + \cdots + |f^{(k-1)}(z)| \leq M \) whenever \( f(z) = w^* \) and that there exists a set \( E \subset \mathbb{C} \) consisting of \( k + 4 \) elements such that for all \( f \in \mathcal{F} \) and all \( z \in D \) we have

\[
|f(z)| \in E \implies \frac{|f^{(k)}|}{1 + |f|^{k+1}}(z) \leq M. \tag{1}
\]

Then \( \mathcal{F} \) is a normal family. If all functions in \( \mathcal{F} \) are holomorphic, this also holds if one merely assumes that \( E \) has at least 3 elements.

Here and in the following, terms like \( \frac{|f^{(k)}|}{1 + |f|^{k+1}} \) are understood to be continuously extended into the poles of \( f \). Of course, the use of \( \frac{|f^{(k)}|}{1 + |f|^{k+1}} \) instead of \( |f^{(k)}| \) in Theorem C is only due to the possibility that \( E \) might contain the point \( \infty \); if \( \infty \notin E \), condition (1) can be replaced by \( |f^{(k)}(z)| \leq M' \) whenever \( f(z) \in E \) with a suitable constant \( M' > M \).

In “\( \Leftarrow \)” of Theorem B and in Theorem C, the condition on the multiplicities of the functions in \( \mathcal{F} \) resp. the (slightly weaker) condition on the existence of the value \( w^* \) is essential as the non-normal family of polynomials of degree at most \( k - 1 \) demonstrates.

So Theorem C gives a (more or less) complete answer to the question how direction “\( \Leftarrow \)” in Marty’s theorem can be extended in terms of \( \frac{|f^{(k)}|}{1 + |f|^{k+1}} \) rather than \( f^\# \). In particular, for arbitrary \( \alpha > 0 \) the locally uniform boundedness of \( \left\{ \frac{|f^{(k)}|}{1 + |f|^{\alpha}} : f \in \mathcal{F} \right\} \) implies normality of \( \mathcal{F} \) provided that the zeros of the functions in \( \mathcal{F} \) appear only with multiplicities at least \( k \) – though the whole truth is much stronger since it suffices to investigate the preimages of “few” values.

As to the opposite direction in Marty’s theorem, we prove the following result which generalizes “\( \implies \)” in Theorem B.

Theorem 1. Let \( k \) be a natural number, \( \alpha > 1 \) be a real number and let \( \mathcal{F} \) be a family of functions meromorphic on a domain \( D \) all of whose poles have multiplicity at least \( \frac{1}{\alpha - 1} \). Then the normality of \( \mathcal{F} \) implies that

\[
\mathcal{F}_{k,\alpha} := \left\{ \frac{|f^{(k)}|}{1 + |f|^{\alpha}} : f \in \mathcal{F} \right\}
\]
is locally uniformly bounded.

We explicitly point out two special (and, in some sense, extremal) cases:

(S1) If \( \alpha \geq k + 1 \) and if \( \mathcal{F} \) is normal, then the conclusion that \( \mathcal{F}_{k,\alpha} \) is locally uniformly bounded holds without any further assumptions on the multiplicities of the poles. This is just the direction “\( \Rightarrow \)” in Theorem B. (More precisely, Theorem B settles the case \( \alpha = k+1 \). But if the locally uniform boundedness of \( \mathcal{F}_{k,\alpha} \) is proved for \( \alpha = k+1 \), it trivially also holds for \( \alpha > k+1 \) since \( x \mapsto \frac{1+x^{k+1}}{1+x^\alpha} \) is bounded on \([0, \infty)\) whenever \( \alpha > k+1 \).)

(S2) If all functions in \( \mathcal{F} \) are holomorphic, then the normality of \( \mathcal{F} \) implies that \( \mathcal{F}_{k,\alpha} \) is locally uniformly bounded for any \( \alpha > 1 \). For \( k = 1 \), this was already proven in [8, Theorem 3].

In the case \( 1 < \alpha < k + 1 \) the lower bound \( \frac{k}{\alpha-1} \) for the multiplicities in Theorem 1 is best possible. This can be seen by considering the single function \( f(z) = \frac{1}{z^p} \) with \( p < \frac{k}{\alpha-1} \) near its pole: Here, for a certain \( C > 0 \) we have

\[
\frac{|f^{(k)}|}{1 + |f|^{\alpha}}(z) \sim C \cdot |z|^{(\alpha-1)p-k} \quad \text{for} \quad z \to 0
\]

since \( (\alpha-1)p-k < 0 \). Since \( f \) is even zero-free, this example also shows that there is no analogue of Theorem 1 where the condition on the multiplicities of the poles is replaced by a condition on the multiplicities of the zeros.

Even for holomorphic functions the condition \( \alpha > 1 \) cannot be weakened any further. This is shown by the family of the functions \( f_n(z) := (z-3)^n \) which is normal in the unit disk \( \mathbb{D} \) and satisfies

\[
\frac{|f_n^{(k)}(z)|}{1 + |f_n(z)|^{\alpha}} = n(n-1) \cdots (n-k+1) \cdot \frac{|z-3|^{n-k}}{1 + |z-3|^{\alpha n}} \\
\geq \frac{1}{2} \cdot (n-k)^k \cdot |z-3|^{n(1-\alpha)-k} \longrightarrow \infty \quad (n \to \infty)
\]

for all \( z \in \mathbb{D} \) and all \( \alpha \) with \( 0 < \alpha \leq 1 \).

Theorem 1 is a consequence of the following result which we hope to be of interest for itself. To simplify its statement, we write “\( f_n \xrightarrow{n\to\infty} f \) on \( \mathcal{D} \)” to indicate that the sequence \( \{f_n\}_n \) converges to \( f \) uniformly w.r.t. the spherical metric on compact subsets of \( \mathcal{D} \) and “\( f_n \xrightarrow{\text{uniform}} f \) on \( \mathcal{D} \)” if the convergence is in the Euclidean metric.

**Theorem 2.** Let \( \mathcal{D} \) be a domain in \( \mathbb{C} \) and let \( k, m, p \) be natural numbers.

(a) Let \( \{g_n\}_{n=1}^\infty \) be a sequence of holomorphic functions \( g_n \neq 0 \) on \( \mathcal{D} \) all of whose zeros have multiplicity at least \( m \). If \( g_n \xrightarrow{n\to\infty} 0 \), then

\[
\frac{\left(g_n^{(k)}\right)^m}{g_n^{m-k}} \xrightarrow{n\to\infty} 0.
\]
Let \( \{f_n\}_{n=1}^\infty \) be a sequence of meromorphic functions on \( D \) all of whose poles have multiplicity at least \( p \). If \( f_n \xrightarrow{n \to \infty} \infty \), then

\[
\left( \frac{f_n^{(k)}}{f_n^{p+k}} \right) \to 0.
\]

For \( p = 1 \), a proof of (b) was given in [10], essentially based on Weierstraß’s theorem and induction. For holomorphic functions (where \( p \) can be chosen arbitrarily large) (b) has been proved by H. Chen and X. Hua [1]. Their reasoning (as the proof of (S2) for \( k = 1 \) in [8]) uses just Harnack’s inequality (applied to the harmonic functions \( \log |f_n| \)) and Cauchy’s formula. In the general case considered here, a much more careful analysis is required.

In the case \( m \leq k \), (a) almost trivially follows from Weierstraß’s theorem. So the interesting case in (a) is the case \( k < m \).

We note that the exponents in \( \left( \frac{g_n^{(k)}}{g_n^{m-k}} \right) \) and \( \left( \frac{f_n^{(k)}}{f_n^{p+k}} \right) \) are chosen in such a way that (under the respective assumptions in (a) and (b)) these functions are holomorphic and that (for \( k < m \)) the assumptions on the multiplicities cannot be weakened without losing the holomorphy. (Explicit counterexamples are provided by the sequences of the functions \( g_n(z) := z^{m-1}/n \) and \( f_n(z) := n/z^{p-1} \) in the unit disk.) In this sense, Theorem 2 is best possible.

2. Proofs

First let us define some notations. For \( z_0 \in \mathbb{C} \) and \( r > 0 \), we set \( \Delta(z_0, r) := \{ z \in \mathbb{C} : |z - z_0| < r \} \) and \( \Delta'(z_0, r) := \Delta(z_0, r) \setminus \{ z_0 \} \). Furthermore, we denote the open unit disk by \( \mathbb{D} := \Delta(0, 1) \).

The proof of Theorem 2 is inspired by several ideas used in the proof of the lemma on the logarithmic derivative (see [7], VI. §3). At some (crucial) point it makes use of the general form of Poisson-Jensen-Nevanlinna’s formula which as a special case yields the First Fundamental Theorem of Nevanlinna theory. To simplify notations, we state this application of Poisson-Jensen-Nevinlinna’s formula in terms of a modification of Nevanlinna theory that was discussed in [2]. If \( f \) is a function meromorphic on a disk \( \Delta(0, R_0) \) and if \( \alpha \in \Delta(0, R_0) \) is not a pole of \( f \), for \( |\alpha| < r < R_0 \) we define

\[
m_\alpha(r, f) := \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{it})| \cdot \Re \frac{re^{it} + \alpha}{re^{it} - \alpha} dt,
\]

\[
N_\alpha(r, f) := \sum_{|b_k| < r} \log \left| \frac{r^2 - \overline{b_k} \alpha}{r(\alpha - b_k)} \right|
\]

and

\[
T_\alpha(r, f) := m_\alpha(r, f) + N_\alpha(r, f),
\]

where the \( b_k \) are the poles of \( f \), each taken into account according to its multiplicity. We call \( m_\alpha(r, f) \), \( N_\alpha(r, f) \) and \( T_\alpha(r, f) \) the modified proximity function, counting
function and characteristic of $f$ with respect to $\alpha$. Using these quantities, Poisson-Jensen-Nevanlinna’s formula takes the following form [2, Theorem 1]

$$T_\alpha \left( r, \frac{1}{f} \right) = T_\alpha(r, f) + \log \frac{1}{|f(\alpha)|} \quad \text{for} \ |\alpha| < r < R_0$$

(2)

provided that $\alpha$ is not a zero or pole of $f$.

Let $n(r, f)$ denote the number of poles of $f$ and $n(r, c, f)$ for $c \in \mathbb{C}$ the number of poles of $\frac{1}{f-c}$ in the closed disk $\Delta(0, r)$, counted according to their multiplicities. Then for $|\alpha| < r < R < R_0$ we have the estimate [2, Lemma 3]

$$n(r, f) \cdot \frac{(R-r)(R-|\alpha|)}{R^2 + r|\alpha|} \leq N_\alpha(R, f) - N_\alpha(r, f)$$

(3)

which is also required in the proof of Theorem 2.

Proof of Theorem 2. Without loss of generality we may assume that $D = \mathbb{D}$ is the unit disk and that the convergence of $\{g_n\}_n$ and $\{f_n\}_n$ is uniform in $\mathbb{D}$.

I. First we show that

$$g^k_n \cdot \left[ \left( \frac{g'_n}{g_n} \right)^{(k-1)} \right]^m \to 0.$$  

(4)

For this purpose we fix $r < R < 1$ and set $s := \frac{1}{2}(r+R)$. There exists some $x_0 \in (0, 1/e]$ such that for all $x \in (0, x_0]$ the function $y \mapsto H(x, y)$ where

$$H(x, y) := \left( \frac{x}{y} \right)^k \cdot \left( m + \frac{1}{(s-r)^2} \log \frac{y}{x} \right)^2m$$

is monotonically decreasing on the interval $[1, \infty)$.

We consider some fixed function $g \not\equiv 0$ holomorphic on $\mathbb{D}$ all of whose zeros have multiplicity at least $m$ and which satisfies $|g(z)| \leq x_0$ for all $z \in \mathbb{D}$.

We define

$$G_a(z) := \frac{s^2 - \pi z}{s(z-a)} \quad \text{and} \quad B := \prod_{|a_j| < s} G_{m_j}^{a_j}$$

where the $a_j$ are the distinct zeros of $g$ and $m_j (\geq m)$ their respective multiplicities. Then

$$h := g \cdot B$$

is holomorphic on $\mathbb{D}$ and non-vanishing in $\Delta(0, s)$, and we have

$$g^k \cdot \left[ \left( \frac{g'}{g} \right)^{(k-1)} \right]^m = \frac{h^k}{B^k} \cdot \left[ \left( \frac{h'}{h} \right)^{(k-1)} - \left( \frac{B'}{B} \right)^{(k-1)} \right]^m.$$  

(5)

From Poisson’s formula one easily gets (cf. [5, Satz 9.2]) for $|z| < s$

$$\frac{h'}{h}(z) = \frac{1}{2\pi} \int_0^{2\pi} \log |h(se^{it})| \cdot \frac{2se^{it}}{(se^{it} - z)^2} dt$$

and

$$\left( \frac{h'}{h} \right)^{(k-1)}(z) = \frac{k!}{2\pi} \int_0^{2\pi} \log |h(se^{it})| \cdot \frac{2se^{it}}{(se^{it} - z)^{k+1}} dt.$$
Here, in view of $|B(\zeta)| = 1$ for $|\zeta| = s$ we have

$$|h(\zeta)| = |g(\zeta)| \leq x_0 \quad \text{for } |\zeta| = s,$$

hence by the maximum principle

$$|h(z)| \leq x_0 \quad \text{for } |z| \leq s.$$ In particular, $\log |h(\zeta)| < 0$ for $|\zeta| = s$. Therefore, we obtain for $|z| \leq r$

$$\left| \left( \frac{h'}{h} \right)^{(k-1)} (z) \right| \leq - \frac{2 \cdot k!}{2\pi (s - r)^{k+1}} \int_0^{2\pi} \log |h(se^{it})| \, dt
= - \frac{2 \cdot k!}{(s - r)^{k+1}} \cdot \log |h(0)|. \quad (6)$$

To estimate the contribution of $\frac{B'}{B}$ to (5), we use

$$\frac{G'_a}{G_a}(z) = \frac{-\overline{a}}{s^2 - \overline{a}z} - \frac{1}{z - a}$$

and

$$\left( \frac{G'_a}{G_a} \right)^{(k-1)} (z) = (k-1)! \cdot \left( \frac{-\overline{a}^k}{(s^2 - \overline{a}z)^k} + \frac{(-1)^k}{(z - a)^k} \right)$$

and note that for $|z| = r$ and $|a_j| < s$ we have $|s^2 - \overline{a}z| \geq s \cdot (s - r)$, hence

$$\left| \left( \frac{G'_{a_j}}{G_{a_j}} \right)^{(k-1)} (z) \right| \leq (k-1)! \cdot \left( \frac{1}{(s - r)^k} + \frac{1}{|z - a_j|^k} \right).$$

So we obtain for $|z| \leq r$

$$\left| \left( \frac{B'}{B} \right)^{(k-1)} (z) \right| \leq \sum_{|a_j| < s} m_j \cdot \left| \left( \frac{G'_{a_j}}{G_{a_j}} \right)^{(k-1)} (z) \right|
\leq (k-1)! \cdot \left( n(s, 0, g) \frac{1}{(s - r)^k} + \sum_{|a_j| < s} \frac{m_j}{|z - a_j|^k} \right) \quad (7)$$

Let’s assume that $g$ has at least one zero in $\Delta(0, s)$, i.e. that $n(s, 0, g) \geq 1$. Then from (6), (7) and the (trivial) estimate

$$a + b + c \leq 3abc \quad \text{for all } a, b, c \geq 1$$
we obtain for $|z| \leq r$

$$\left| \left( \frac{h'}{h} \right)^{(k-1)} - \left( \frac{B'}{B} \right)^{(k-1)} \right|(z)$$

$$\leq \frac{2 \cdot k!}{(s-r)^{k+1}} \cdot \left( \log \frac{1}{|h(0)|} + n(s, 0, g) + 2^k \cdot \sum_{|a_j| < s} \frac{m_j}{|z - a_j|^k} \right)$$

$$\leq \frac{6 \cdot k! \cdot 2^k}{(s-r)^{k+1}} \cdot \log \frac{1}{|h(0)|} \cdot n(s, 0, g) \cdot \sum_{|a_j| < s} \frac{m_j}{|z - a_j|^k}. \quad (8)$$

(Here we have used that $-\log |h(0)| \geq -\log x_0 \geq 1$ and $2^k \cdot \sum_{|a_j| < s} \frac{m_j}{|z - a_j|^k} \geq 1$ since $|z - a_j| \leq 2$.)

We fix some $z_0 \in \Delta(0, r)$ such that $g(z_0) \neq 0$. Then there is some $j_* = j_*(z_0)$ such that $|z_0 - a_{j_*}| = \min_{|a_j| < s} |z_0 - a_j|$. We define

$$B_* := G_{a_{j_*}}^{m_{j_*} - m} \cdot \prod_{|a_j| < s, j \neq j_*} G_{a_j}^{m_j} \quad \text{and} \quad g_* := \frac{h}{B_*}. \quad (9)$$

Then $g_*$ is holomorphic on $\Delta(0, s)$, and we have

$$B = B_* \cdot G_{a_{j_*}}^{m} \quad \text{and} \quad g_* = g \cdot G_{a_{j_*}}^{m}.$$

Using $|G_{a_{j_*}}(z)| \leq 1$ for $|z| \geq s$, $|g(z)| = |g_*(z)|$ for $|z| = s$ and the maximum principle we deduce

$$|g_*(z)| \leq \begin{cases} |g(z)| & \leq x_0 \quad \text{for} \ 0 \leq |z| \leq R, \\ \max_{|z| = s} |g(\zeta)| & \leq x_0 \quad \text{for} \ |z| < s, \end{cases}$$

i.e. $|g_*(z)| \leq x_0$ for $|z| \leq R$, and we obtain

$$\frac{1}{|B(z_0)|^k} \cdot \left( \sum_{|a_j| < s} \frac{m_j}{|z_0 - a_j|^k} \right)^m$$

$$\leq \frac{1}{|B_*(z_0)|^k} \cdot \left( \frac{s \cdot |z_0 - a_{j_*}|}{s^2 - a_{j_*}^2} \right)^{km} \cdot (n(s, 0, g))^m \cdot \frac{1}{|z_0 - a_{j_*}|^km}$$

$$\leq \frac{1}{|B_*(z_0)|^k} \cdot \frac{1}{(s - r)^{km}} \cdot (n(s, 0, g))^m.$$
We combine this estimate with (5) and (8) and arrive at

$$\left| g^k \cdot \left[ \left( \frac{g'}{g} \right)^{(k-1)} \right]^m \right| (z_0)$$

$$\leq \left| \frac{h}{B(z_0)} \right|^k \cdot \left( \frac{6 \cdot k! \cdot 2^k}{(s-r)^{2k+1}} \cdot \log \frac{1}{|h(0)|} \right) \cdot (m + \frac{1}{(s-r)^2} \cdot \log \frac{|B_*(z_0)|}{|h(z_0)|})$$

$$\leq \left| \frac{h}{B_*(z_0)} \right|^k \cdot \left( \frac{6 \cdot k! \cdot 2^k}{(s-r)^{2k+1}} \cdot \log \frac{1}{|h(0)|} \right) \cdot (m + \frac{1}{(s-r)^2} \cdot \log \frac{|B_*(z_0)|}{|h(z_0)|})$$

(9)

Here, applying the estimate (3) and the First Fundamental Theorem (2) to $g_*$ and observing that $|g_*(z)| \leq 1$ for $|z| \leq R$ implies $T_{s_0}(R, g_*) = 0$, we obtain

$$n(s, 0, g) = m + n(s, 0, g_*)$$

$$\leq m + \frac{R^2 + s|z_0|}{(R - s)(R - |z_0|)} \cdot N_{z_0} \left( R, \frac{1}{g_*) \right)$$

$$\leq m + \frac{2R^2}{(R - s)(R - r)} \cdot \left( T_{s_0}(R, g_*) + \log \frac{1}{|g_*(z_0)|} \right)$$

$$\leq m + \frac{1}{(s-r)^2} \cdot \log \frac{|B_*(z_0)|}{|h(z_0)|}.$$

Inserting this into (9) and keeping in mind $|B_*(z_0)| \geq 1$, $|h(z_0)| \leq x_0$ and the definition of $x_0$ we conclude that

$$\left| g^k \cdot \left[ \left( \frac{g'}{g} \right)^{(k-1)} \right]^m \right| (z_0)$$

$$\leq \left| \frac{h}{B_*(z_0)} \right|^k \cdot \left( \frac{6 \cdot k! \cdot 2^k}{(s-r)^{2k+1}} \cdot \log \frac{1}{|h(0)|} \right) \cdot (m + \frac{1}{(s-r)^2} \cdot \log \frac{|B_*(z_0)|}{|h(z_0)|})$$

$$\leq \left| h(z_0) \right|^k \cdot \left( \frac{6 \cdot k! \cdot 2^k}{(s-r)^{2k+1}} \cdot \log \frac{1}{|h(0)|} \right) \cdot (m + \frac{1}{(s-r)^2} \cdot \log \frac{1}{|h(z_0)|})$$

(10)

i.e. we have eliminated the function $B_*$ which depended on $z_0$. We have shown this estimate for all $z_0 \in \Delta(0, r)$ with $g(z_0) \neq 0$. By continuity, it even holds for all $z_0 \in \Delta(0, r)$. (Note that the function $g^k \cdot \left[ \left( \frac{g'}{g} \right)^{(k-1)} \right]^m$ is holomorphic on $\mathbb{D}$ by our assumption on the multiplicities of the zeros of $g$.)

These considerations were subject to the assumption $n(s, 0, g) \geq 1$. However, if $n(s, 0, g) = 0$, then $B \equiv 1$ and $g = h$, and from (5) and (6) we immediately obtain

$$\left| g^k \cdot \left[ \left( \frac{g'}{g} \right)^{(k-1)} \right]^m \right| (z) \leq |h(z)|^k \cdot \left( \frac{2 \cdot k!}{(s-r)^{k+1}} \cdot \log \frac{1}{|h(0)|} \right)^m$$

for $|z| \leq r$.

i.e. an estimate even better than (10). So in both cases $n(s, 0, g) \geq 1$ and $n(s, 0, g) = 0$, (10) holds for all $z$ with $|z| \leq r$. 
Now, by Harnack’s inequality we have
\[
\log \frac{1}{|h(0)|} \leq \frac{s + r}{s - r} \cdot \log \frac{1}{|h(z)|} \quad \text{for } |z| \leq r
\]
and we finally conclude that for $|z| \leq r$
\[
\left| g^k \cdot \left[ \frac{g'}{g} \right]^{(k-1)m} \right| (z)
\]
\[
\leq |h(z)|^k \cdot \left( \frac{12s \cdot k! \cdot 2^k}{(s - r)^{2k+2}} \cdot \log \frac{1}{|h(z)|} \right)^m \cdot \left( m + \frac{1}{(s - r)^2} \cdot \log \frac{1}{|h(z)|} \right)^{2m}.
\]  
(11)

This estimate holds for all holomorphic functions $g$ on $\mathbb{D}$ all of whose zeros have multiplicity at least $m$ and which satisfy $|g(z)| \leq x_0$ for all $z \in \mathbb{D}$.

We apply this estimate to the sequence $\{g_n\}_n$. To each $g_n$, as in (4) we construct a function $h_n$ holomorphic on $\mathbb{D}$ and non-vanishing in $\Delta(0, s)$ such that $|h_n(z)| = |g_n(z)|$ for $|z| = s$, hence
\[
\max_{|z| \leq s} |h_n(z)| = \max_{|z| \leq s} |g_n(z)| \longrightarrow 0 \quad (n \to \infty)
\]
by the maximum principle. Now from (11) we immediately obtain that the sequence
\[
\left\{ g^k \cdot \left[ \frac{g'}{g} \right]^{(k-1)m} \right\}_{n}
\]
converges to 0 uniformly in $\Delta(0, r)$. Since this holds for any $r < 1$, our assertion in I. is proved.

II. Now we prove (a) by induction. The case $k = 1$ follows immediately from I.

Let some $k \geq 2$ be given and assume that
\[
\left( \frac{g^{(j)}}{g^{m-j}} \right)^m \Longrightarrow 0 \quad \text{for } j = 1, \ldots, k - 1
\]
has already been proved. Now by induction there are certain universal constants $c_{k,l;j_1,\ldots,j_l}$ such that
\[
\frac{g^{(k)}}{g} = \left( \frac{g'}{g} \right)^{(k-1)} + \sum_{l=2}^{k} \sum_{j_1+\ldots+j_l=k}^{j_l \geq 1} c_{k,l;j_1,\ldots,j_l} \cdot \prod_{\mu=1}^{l} \frac{g^{(j_{\mu})}}{g}
\]
for all functions $g \not\equiv 0$ holomorphic on $\mathbb{D}$. Setting
\[
S_{n,k} := \sum_{l=2}^{k} \sum_{j_1+\ldots+j_l=k}^{j_l \geq 1} c_{k,l;j_1,\ldots,j_l} \cdot \prod_{\mu=1}^{l} \frac{g^{(j_{\mu})}}{g}
\]
we obtain
\[
\left| g^k \cdot \left[ \frac{g_n}{g_n} \right]^{(k-1)m} \right| \leq \left| g^k \cdot \left[ \frac{g}{g_n} \right]^{(k-1)m} \right| + \sum_{\sigma=0}^{m-1} \left( \frac{m}{\sigma} \right) \cdot \left( \frac{g'}{g_n} \right)^{(k-1)\sigma} \cdot |g_n|^{k\sigma/m} \cdot |S_{n,k}|^{m-\sigma} \cdot |g_n|^{k(m-\sigma)/m}.
\]  
(12)
Here, from I. we know
\[ \left| \left( \frac{g_n'}{g_n} \right)^{(k-1)} \right|^\sigma \cdot |g_n|^{k\sigma/m} \Rightarrow 0 \quad (n \to \infty) \quad \text{for } \sigma = 1, \ldots, m - 1, \]
and from
\[ |S_{n,k}| \cdot |g_n|^{k/m} \leq \sum_{l=2}^{k} \sum_{j_{1}+\cdots+j_{l}=k} |c_{k; j_{1}, \ldots, j_{l}}| \prod_{\mu=1}^{l} \left| \frac{\left( g_{j_{\mu}} \right)^{m}}{g_n^m} \right|^{1/m} \]
and the induction hypothesis we deduce that
\[ |S_{n,k}| \cdot |g_n|^{k/m} \Rightarrow 0 \quad (n \to \infty). \]
Inserting this into (12) and observing I. once more yields
\[ g_n^k \cdot \left( \frac{g_n^{(k)}}{g_n} \right)^{m} \Rightarrow 0 \quad (n \to \infty), \]
as asserted.

III. We turn to the proof of (b). If we apply I. to the functions \( g_n := 1/f_n \) (all of whose zeros have multiplicity at least \( p \)), we obtain that under the assumptions in (b) for all \( k \geq 1 \) we have
\[ \frac{1}{f_n^k} \cdot \left[ \left( \frac{f_n'}{f_n} \right)^{(k-1)} \right]^p \Rightarrow 0 \quad (n \to \infty). \]
From this we deduce (b) almost literally as in II. we have deduced (a) from I.

Once Theorem 2 (b) is available, Theorem 1 can be proved with the same method as in the proof of Theorem B given in [10]. For completeness, we provide the details.

Proof of Theorem 1. We assume that \( \mathcal{F} \) is normal but that \( \mathcal{F}_{k,\alpha} \) is not locally uniformly bounded in \( D \). Then we find a \( z_0 \in D \), functions \( f_n \in \mathcal{F} \) and points \( z_n \in D \) such that \( \lim_{n \to \infty} z_n = z_0 \) and
\[ \frac{|f_n^{(k)}|}{1 + |f_n|^{\alpha}}(z_n) \to \infty. \quad (13) \]
Since \( \mathcal{F} \) is normal, after extracting a suitable subsequence we may assume that \( \{f_n\}_n \) converges locally uniformly to some limit function \( f \), possibly \( f \equiv \infty \). Now let us consider several cases.

Case 1. \( f(z_0) \in \mathbb{C} \).
Then there are \( r > 0 \) and \( N \in \mathbb{N} \) such that \( f \) and \( f_n \) are holomorphic on \( \Delta(z_0, r) \) for all \( n \geq N \), and by Weierstraß’s theorem we obtain
\[ \lim_{n \to \infty} \frac{f_n^{(k)}(z_n)}{1 + |f_n(z_n)|^{\alpha}} = \frac{f^{(k)}(z_0)}{1 + |f(z_0)|^{\alpha}} \not\equiv \infty, \]
a contradiction to (13).

Case 2. \( f \not\equiv \infty \), but \( f(z_0) = \infty \).
Here, we can find $r > 0$ such that $f$ is holomorphic on $\Delta'(z_0, 2r)$ and $|f(z)| \geq 1$ and $|f_n(z)| \geq 1$ for all $z \in \Delta(z_0, 2r)$ and all $n \geq N$ for a certain $N \in \mathbb{N}$.

If $p := \lceil \frac{k}{\alpha - 1} \rceil$ is the smallest integer $\geq \frac{k}{\alpha - 1}$, then, by assumption, each pole of $f_n$ has multiplicity at least $p$. Hence the functions

$$d_n := \frac{(f_n^{(k)})^p}{f_{n+k}^p}$$

are holomorphic on $\Delta(z_0, 2r)$ for $n \geq N$. Since they converge to $\frac{(f^{(k)})^p}{f_{n+k}^p}$ uniformly on $\partial \Delta(z_0, r)$, from the maximum principle we deduce that there is a constant $C < \infty$ such that

$$|d_n(z)| \leq C \quad \text{for all } z \in \Delta(z_0, r) \text{ and } n \text{ large enough.}$$

In particular, for $z = z_n$ we get for $n$ large enough

$$\left( \frac{|f_n^{(k)}(z_n)|}{1 + |f_n(z_n)|^\alpha} \right)^p \leq \frac{|f_n^{(k)}(z_n)|^p}{f_n(z_n)^{p+k}} \leq C;$$

here we have used $|f_n(z_n)| \geq 1$ and $\alpha p \geq k + p$. This is a contradiction to (13).

**Case 3.** $f \equiv \infty$.

Again, each pole of $f_n$ has multiplicity at least $p := \lceil \frac{k}{\alpha - 1} \rceil$. So from Theorem 2 (b) we obtain that the sequence $\{d_n\}$ where $d_n$ is defined as in (14) converges to 0 locally uniformly in $D$, and in view of $k + (1-\alpha)p \leq k + (1-\alpha) \cdot \frac{k}{\alpha - 1} = 0$ we deduce

$$\frac{|f_n^{(k)}|}{1 + |f_n|^\alpha} \leq (|d_n| \cdot |f_n|^{k+(1-\alpha)p})^{1/p} \Rightarrow 0$$

for $n \to \infty$, a contradiction to (13) once more.  

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