Maintaining maximal matching with lookahead✩

Kitti Gelle¹, Szabolcs Iván¹

ÔUniversity of Szeged, Hungary

Abstract

In this paper we study the problem of fully dynamic maximal matching with lookahead. In a fully dynamic n-vertex graph setting, we have to handle updates (insertions and removals of edges), and answer queries regarding the current graph, preferably with a better time bound than that when running the trivial deterministic algorithm with worst-case time of \( O(m) \) (where \( m \) is the all-time maximum number of the edges) and recompute the matching from scratch each time a query arrives. We show that a maximal matching can be maintained in an (undirected) general graph with a deterministic amortized update cost of \( O(\log m) \), provided that a lookahead of length \( m \) is available, i.e. we can “take a peek” at the next \( m \) update operations in advance.

1. Introduction and notation

Graph algorithms are fundamental in computer science. In most cases, graphs have been studied as static objects, however, in many real life examples (e.g. social networks, AI) they are changing in size. In the last few decades, there has been a growing interest in developing algorithms and data structures for such dynamic graphs. In this setting, graphs are subject to updates — in our case, additions and removals of an edge at a time. The aim of a so-called fully dynamic algorithm (here “fully” means that both addition and removal are supported) is to maintain the result of the algorithm after each and every update of the graph, in a time bound significantly better than recomputing it from scratch each time.

In [5], a systematic investigation of dynamic graph problems in the presence of a so-called lookahead was initiated: although the stream of update operations can be arbitrarily large and possibly builds up during the computation time, in actual real-time systems it is indeed possible to have some form of lookahead available. That is, the algorithm is provided with some prefix of the update sequence of some length (for example, in [5] an assembly planning problem is studied in which the algorithm can access the prefix of the sequence of future operations to be handled of length \( \Theta(\sqrt{m/n}\log n) \)), where \( m \) and \( n \) are the number of edges and nodes, respectively. Similarly to the results of [5] (where the authors devised dynamic algorithms using lookahead for the problems of strongly connectedness and transitive closure), we will execute the tasks in batches: by looking ahead at \( O(m) \) future update operations, we treat them as a single batch, preprocess our current graph based on the information we get from the complete batch, then we run all the updates, one at a time, on the appropriately preprocessed graph. This way, we achieve an amortized update cost of \( O(\log m) \) for maintaining a maximal matching.

We view a graph \( G \) as a set (or list) of edges, with \( |G| \) standing for its cardinality. This way notions like \( G \cup H \) for two graphs \( G \) and \( H \) (sharing the common set \( V(G) = V(H) \) of vertices) are well-defined.

In the studied problem, a matching of a (undirected) graph \( G \) is a subset \( M \subseteq G \) of edges having pairwise disjoint sets of endpoints. A matching \( M \) is maximal if there is no matching \( M' \supseteq M \) of \( G \). Given a matching \( M \), for each vertex \( v \) of \( G \) let \( \text{mate}(v) \) denote the unique vertex \( u \) such that \( (u, v) \in M \) if such a vertex exists, otherwise \( \text{mate}(v) = \text{null} \).

In the fully dynamic version of the maximal matching problem, the update operations are edge additions \(+ (u, v)\), edge deletions \(- (u, v)\) and the queries have the form \( \text{mate}(u) \).
So a fully dynamic algorithm for maximal matching problem supports the following operations on an undirected graph $G = (V, E)$:

- **INSERT**(u, v): inserts an edge between u and v
- **DELETE**(u, v): deletes an edge between u and v
- **MATE**(u): answers v if $(u, v) \in M$, where $M$ is the current maximal matching of $G$ and NULL otherwise

**Related work.** There is an interest in computing a maximum (i.e. maximum cardinality) or maximal (i.e. non-expandable) matching in the fully dynamic setting. There is no “best-so-far” algorithm, since the settings differ: Baswana, Gupta and Sen [1] presented a randomized algorithm for maximal matching, having an $O(\log n)$ expected amortized time per update. Based on this algorithm Solomon [11] gave a randomized algorithm with constant amortized update time. (Note that algorithms for maximal matching automatically provide 2-approximations for maximum matching and also vertex cover.) For the deterministic variant, Ivković and Lloyd [3] defined an algorithm with an $O((n + m)^0.7072)$ amortized update time, which was improved to an amortized $O(\sqrt{m})$ update cost by Neiman and Solomon [7]. For maximum matching, Onak and Rubinfeld [8] developed a randomized algorithm that achieves a $c$-approximation for some constant $c$, with an $O(\log^2 n)$ expected amortized update time. To maintain an exact maximum cardinality matching, Micali and Vazirani [6] gave an algorithm with a worst-case update time of $O(\sqrt{n} \cdot m)$. Allowing randomization, an update cost of $O(n^{1.495})$ is achievable due to Sankowski [9].

We are not aware of any results on allowing lookahead for any of the matching problems, but the notion has been applied to several problems in this field: following the seminal work of Khanna, Motwani and Wilson [5], where lookahead was investigated for the problems of maintaining the transitive closure and the strongly connectedness of a directed graph, Sankowski and Mucha [10] also considered the transitive closure with lookahead via the dynamic matrix inverse problem, devising a randomized algorithm, and Kavitha [4] studied the dynamic matrix rank problem.

### 2. Maximal matching with lookahead

In this section we present an algorithm that maintains a maximal matching in a dynamic graph $G$ with constant query and $O(\log m)$ update time (note that $O(\log m)$ is also $O(\log n)$ as $m = O(n^2)$), provided that a lookahead of length $m$ is available in the sequence of (update and query) operations. This is an improvement over the currently best-known deterministic algorithm [7] that has an update cost of $O(\sqrt{m})$ without lookahead.

The following is clear:

**Proposition 1.** Suppose $G$ is a graph in which $M$ is a maximal matching. Then a maximal matching in the graph $G + (u, v)$ is

- $M \cup \{(u, v)\}$, if MATE(u) = MATE(v) = NULL,
- $M$, otherwise.

This proposition gives the base algorithm GREEDY for computing a maximal matching in a graph:

Note that if one initializes the MATE array in the above code so that it contains some non-NULL entries, then the result of the algorithm represents a maximal matching within the subgraph of $G$ spanned by the vertices having NULL mates initially. Also, with $M$ represented by a linked list, the above algorithm runs in $O(m)$ total time using no lookahead. Hence, by calling this algorithm on each update operation (after inserting or removing the edge in question), we get a dynamic graph algorithm with no lookahead (hence it uses a lookahead of at most $m$ operations), a constant query cost (as it stores the MATE array explicitly) and an $O(m)$ update cost. Using this algorithm $A_1$, we build up a sequence $A_k$ of algorithms, each having a smaller update cost than the previous ones. (In a practical implementation there would be a single
algorithm $A$ taking $k$ as a parameter along with the graph $G$ and the update sequence, but for proving the time complexity it is more convenient to denote the algorithms in question by $A_1$, $A_2$, and so on.)

In our algorithm descriptions the input is the current graph $G$ (which is $\emptyset$ the first time we start running the program) and a sequence $(q_1, \ldots, q_t)$ of operations. Of course as the sequence can be arbitrarily long, we do not require an explicit representation, just the access of the first $m$ elements (that is, we have a lookahead of length $m$).

To formalize our main lemma in a more concise way, we first define the invariant property, which we call $h(m)$-ensuring, of these algorithms:

**Definition 1.** We say that an algorithm $A$ is an $h(m)$-ensuring algorithm for maximal matching, if $A$ is a fully dynamic algorithm maintaining a maximal matching in a graph such that if it gets as input a graph $G$, as an edge list, having $m_0$ edges initially, and a (possibly infinite) stream $(q_1, q_2, \ldots, q_t)$ of updates with $t \geq m_0$, then $A$ can process these queries with an amortized update cost of $h(m)$ using a lookahead of length $m$, such that between handling of these updates, queries of the form $\text{Mate}(u)$, asking for the mate of vertex $u$ in the current maximal matching, can be answered in a constant time.

In the definition above, $m$ stands for the maximum number of edges in $G$ during its life cycle, formally, $m := \max \{|Gq_1q_2\ldots q_i|: 0 \leq i \leq t\}$.

As an example, the following algorithm $A_0$ that runs Greedy after each update, is a $c \cdot m$-ensuring algorithm for maximal matching, for some universal constant $c$:

1. Initialize a global array $\text{Mate}$ of vertices, set $\text{Mate}(u) := \text{NULL}$ for each vertex $u$.
2. Upon receiving an update sequence $(q_1, \ldots, q_t)$, the algorithm does the following:
   (a) Let $M$ be an empty list of edges.
   (b) For processing $q_i$, we
      i. first modify $G$ accordingly, $G := G \cdot q_i$,
      ii. then we iterate through the current matching $M$ and set $\text{Mate}(u) = \text{Mate}(v) = \text{NULL}$ for each $(u, v) \in M$, emptying $M$ during the process,
      iii. we set $M := \text{Greedy}(G, \text{Mate})$.
   (c) Having processed $q_i$, we now can answer queries of the form $\text{Mate}(u)$ in a constant time, by accessing the global array $\text{Mate}$.

Step 1 has a setup cost of $O(n)$. When we receive the update sequence, the local initialization of $M$ takes a constant time. Note that for processing $q_i$ we do not use any lookahead which is fine. Modifying the current graph $G$ in Step 2.b.i. takes $O(m)$ time, since adding/removing an entry to a list of unique entries takes a time proportional to the size of the list, which is by definition of $m$, at most $m$ at any given time point $i$. Then, as the matching $M$ is also a list of at most $m$ edges, iterating through it takes $O(m)$ iterations, setting the $\text{Mate}$ array for a constant time each, so Step 2.b.ii. also takes $O(m)$ time. Finally, Step 2.b.iii. also takes $O(m)$ time, and after that, we clearly have a maximal matching for $G_i := Gq_1 \ldots q_i$, stored in the $\text{Mate}$ array. The total cost for handling a single update is thus $c \cdot m$ for some universal constant $c$.

Note after in each step we erase our “local” matching $M$ from the $\text{Mate}$ array for a total cost of $O(m)$ since we do not want to rely on the number $n$ of nodes: this is crucial since at the end, we’ll apply the above algorithm for very small graphs with $m = o(n)$ edges.

So starting from the above algorithm $A_0$, we can build up a sequence $A_k$ of algorithms, each having a better update cost till $k = \log m$ by the following lemma:

**Lemma 1.** There is a universal constant $C$ such that if there exists an $(f(k) + g(k) \cdot m)$-ensuring algorithm $A_k$ for maximal matching, with a setup cost of $h(k, m, n)$, then there also exists an $(f(k) + C + \frac{m^2}{2}) \cdot m$-ensuring algorithm $A_{k+1}$ for maximal matching as well, with a setup cost of $h(k, m, n) + O(n^2)$.

Before proving the above lemma, we derive the main result of the section. As $A_0$ is an $c \cdot m$-ensuring algorithm, that is, $f(k) = 0$ and $g(k) = c \cdot m$, by induction we get the existence of an algorithm a $(k \cdot C + \frac{m}{k}) \cdot m$-ensuring algorithm for maximal matching. Now setting $k = \log m$ we get that $A_{\log m}$ maintains a maximal matching with an amortized update cost of $C \cdot \log m + \frac{m}{2} \cdot m = C \cdot \log m + 2 = O(\log m)$, thus we get:
Theorem 1. There exists a fully dynamic graph algorithm for maintaining a maximal matching with an $O(\log n)$ amortized update cost and constant query cost, using a lookahead of length $m$, with a setup cost of $O(n^2 \cdot \log m)$.

Now we prove Lemma 1 by defining the algorithm $A_{k+1}$ below.

- The algorithm $A_{k+1}$ works in phases and returns a graph $G$ (as an edge list) and a matching $M$ (also as an edge list).
- The algorithm accesses the global mate array in which the current maximal matching of the whole graph is stored. ($A_{k+1}$ might get only a subgraph of the whole actual graph as input.)
- The algorithm manages a boolean array $T_{k+1}$ of size $n \times n$, initialized to be all-zero in the start of the program (hence the plus setup cost of $n^2$).
- As input, $A_{k+1}$ gets a graph $G$ and the update sequence $(q_1, \ldots, q_t)$, with a promise of $t \geq m_0$, where $m_0$ is the number of edges in $G$.
- The algorithm $A_{k+1}$ maintains a local matching $M$ as a list of edges (similarly to $A_0$), which is set to the empty list when calling $A_{k+1}$.
- In one phase, $A_{k+1}$ either handles a block $\vec{q} = (q_1, \ldots, q_t)$ of $t$ operations for some $\frac{m_0}{4} \leq t' \leq \frac{m_0}{2}$, or a single operation.
- If $|G|$ is smaller than our favorite constant 42, then the phase handles only the next update by explicitly modifying $G$, afterwards recomputing a maximal matching from scratch, in $O(42)$ (constant) time. That is,
  1. We iterate through all the edges $(u, v) \in M$, and set $\text{mate}[u]$ and $\text{mate}[v]$ to NULL (in effect, we remove the “local part” $M$ of the global matching);
  2. We apply the next update operation on $G$;
  3. We set $M := \text{Greedy}(G, \text{mate})$.
- Otherwise the phase handles $t'$ operations as follows. First, if there are at most $m_0$ unprocessed queries remaining (that can be checked by a lookahead of length $m_0 \leq m$), then we finish the processing of the sequence in exactly two phases, each having $t' = \frac{t}{2}$ updates. Otherwise, we set $t' = \frac{m_0}{4}$, and handle the next $t'$ updates in a single phase.

Observe that by this method, the value of $t'$ is always between $\frac{m_0}{4}$ and $\frac{m_0}{2}$.

1. Using lookahead (observe that $t' < m$) we collect all the edges involved in $\vec{q}$ (either by an insert or a remove operation) into a graph $G'$.
2. We iterate through all the edges $(u, v) \in M$, and set $\text{mate}[u] := \text{null}, \text{mate}[v] := \text{null}$.
3. Iterating over all the edges $(u, v) \in G'$, we set $T_{k+1}(u, v)$ and $T_{k+1}(v, u)$ to 1.
4. Using $T_{k+1}$ containing the adjacency matrix of $G'$ now, we split the list $G$ into the lists $G - G'$ and $G \cap G'$ by iterating through $G$ and putting $(u, v)$ to either $G - G'$ (if $T_{k+1}(u, v)$ is zero) or to $G \cap G'$ (otherwise).
5. We reset $T_{k+1}$ to be an all-zero matrix by iterating over $G'$ again and resetting the corresponding entries.
6. We run $M := \text{Greedy}(G - G', \text{mate})$.
7. We call $A_k(G \cap G', (q_1, \ldots, q_t))$. Let $G^*$ and $M^*$ be the graph and matching returned by $A_k$.
8. We set $G := (G - G') \cup G^*$ and $M := M \cup M^*$.

In order to give the reader a better insight, we give an example before analyzing the time complexity. To make the example more manageable, we adjust the constants as follows: we shall use the constant 1 instead of 42 (that is, if $G$ contains at most one edge, we do not make a recursive call but recompute the matching) and also, the block size $A_2$ handles in one phase will be set to 1 while $A_3$, which we call at the topmost level, will handle 3 operations in one phase.
Example 1. Let us assume that we call the algorithm $A_3$ on the graph
$G = [(a, b), (b, g), (a, f), (g, e), (e, c), (d, e)]$ of Figure 1 (a). As the graph contains 6 edges, which is more than our threshold $1$, a block of update operations of length $6/2 = 3$ will be handled in a phase, using lookahead.

Now assume the next three update operations are $+(f, g), -(a, f)$ and $+(d, c)$. Thus $G' = [(f, g), (a, f), (d, c)]$ is the list of edges involved, that’s for Step 1. In Steps 2 and 3, we construct the graph $G'' = G - G'$ and run the greedy matching algorithm on it, the (possible) result is shown in Figure 1 (b). (Note that the actual result depends on the order in which the edges are present in $G$.)

In the Figure, thick circles denote those vertices having a non-null mate at this point (that is, $\text{mate} [a] = b, \text{mate} [b] = a$, and so on, $c, d$ and $f$ having a NULL mate). Now, $A_2$ is called on $G \cap G'$ (depicted in Figure 2 (a)), and the whole block of three updates is passed to $A_2$ as well.

Now as the input graph of $A_2$ has only one edge, $A_2$ just handles the next update $+(f, g)$; that is, it inserts the edge $(f, g)$ into its input of Figure 2 (a) and runs GREEDY on this, resulting in the graph of Figure 2 (b).

Observe that at this point $\text{mate}[a] = b$ and $\text{mate}[g] = e$, so neither of these two edges is added to the maximal matching managed by $A_2$. That is due to the fact that the $\text{mate}$ array is a global variable. This is vital: this way one can ensure that the union of the matchings of different recursion levels is still a matching, and also ensures a constant-time query cost.

Then, as the current graph has two edges (which is larger than the threshold), $A_2$ handles a complete block of operations in a phase. (Now the length of the block happens to be $2 = 1$ so this does not make that much of a difference.) Thus, using a lookahead of length 1, the only operation to be handled is $-(a, f)$. So we compute the difference graph and run GREEDY on it (Figure 3 (a)), compute the intersection graph and

![Figure 1: Executing Steps 1 – 3 of $A_3$ on $G$, looking ahead the operations $+(f, g), -(a, f), +(d, c)$.](image)

(a) The original graph $G$. (b) $G - G'$ with a maximal matching.

![Figure 2: Handling the first recursive call.](image)

(a) The graph $G \cap G'$ (b) $A_2$ adds $(f, g)$ directly
call $A_1$ on this along with the update sequence consisting of the single operation $-(a,f)$ (Figure 3 (b)).

As the input of $A_1$ is now a graph consisting of a single edge, it gets removed (as the edge in question is not involved in the matching, which can be seen e.g. from the MATE array, the global matching is not changed), resulting in an empty graph on which GREEDY gives an empty matching as well. Then, $A_1$ returns, as it handled the only operation it received. Now $A_2$ takes control. Concluding the second phase, it constructs the union of its intersection graph and the empty graph returned by $A_1$, so its current graph $G$ becomes the graph on Figure 3 (a). As now the graph has only one edge, the next update $+(d,c)$ is handled directly: the edge $(c,d)$ is inserted and GREEDY is run (Figure 4 (a)).

Now as $A_2$ has handled its whole input block, it returns its current graph: $A_3$ takes control and glues together its difference graph from Figure 1 (b) and the returned graph 4 (a), resulting in the graph in Figure 4 (b) which would be the starting graph of further updates.

Having completed this example, we will now show its correctness. That is, we claim that each $A_k$ maintains a maximal matching among those vertices having a NULL MATE when the algorithm is called. This is true for the greedy algorithm $A_0$. Now assuming $A_k$ satisfies our claim, let us check $A_{k+1}$. When the graph is small, the algorithm throws away its locally stored matching $M$, resetting the MATE array to its original value in the process (in fact, this is the only reason why we store the local matching at each recursion level: the global matching state can be queried by accessing the MATE array alone). Then we handle the update and run GREEDY, which is known to compute a maximal matching on the subgraph of $G$ spanned by the vertices having a NULL mate. So this case is clear.

For the second case, if a block of $t'$ operations involving the edges of the edge list $G'$ is handled, then we split the graph into two, namely into a difference graph $G - G'$ and an intersection graph $G'' := G \cap G'$. By construction, when handling the block, the edges belonging to $G - G'$ do not get touched and they are present in the graph during the whole phase.

Hence, at any time point, a maximal matching of $G$ can be computed by starting from a maximal matching of $G - G'$ and then extending the matching by a maximal matching in the subgraph of $G''$ not covered by the matching of $G'$. Thus, if we compute a maximal matching $M'$ in the subgraph of $G - G'$ spanned by the vertices having a NULL MATE, updating the MATE array accordingly (that is, calling GREEDY...
on $G'$), and maintaining a maximal matching $M''$ over the vertices of $G''$ having a null mate after that point (which is done by $A_k$, by the induction hypothesis), we get that at any time $M' \cup M''$ is a maximal matching of $G$. Hence, the algorithm is correct.

Now we analyze the time complexity of $A_{k+1}$. Upon calling $A_{k+1}$, we set the local matching $M$ to be the empty list, in constant time. Then, a phase either handles a single operation (if $|G|$ is bounded by a constant threshold), or a batch of $t'$ operations.

If $|G|$ is below the threshold 42, then so is $|M|$, thus running Greedy also takes a constant time.

Assume the phase handles $t'$ operations for some $t'$ between $\frac{m}{4}$ and $\frac{m}{2}$. Then, collecting the first $t'$ updates into a list $G'$ of edges (containing possibly duplicates) takes $O(m_0)$ time. Now constructing the intersection and the difference graphs maintaining an $O(m_0)$ time can be done by using the global boolean matrix $T_k$ of size $n \times n$, which is initialized to an all-zero matrix in the very beginning of the program (hence, an initialization cost of $n^2$ is needed to do that), then, the algorithm $A_k$ sets those entries $T_k(u, v)$ and $T_k(v, u)$ for which $(u, v) \in G'$ to one. Using $T_k$, the list $G$ can be split into $G - G'$ and $G \cap G'$ using $O(m_0)$ time. Then, as $G - G'$ also has at most $m_0$ edges, Greedy runs in $O(m_0)$ steps on it as well.

Then, we call $A_k(G \cap G', (q_1, ..., q_{t'}))$. Observe that since $G'$ is the graph constructed from the $t'$ queries, it has at most $t'$ edges, hence $|G' \cap G'| \leq t'$. Now by assumption, $A_k$ guarantees in this case that the queries can be processed in an amortized time of $f(k) + g(k) t'$, since $t'$ is an upper bound for the size of this dynamic graph during its whole lifecycle. As $\frac{m}{4} \leq t' \frac{m}{2}$, this gives an amortized cost at most $f(k) + \frac{g(k)}{2} m_0$ per update.

Finally, at the end of the phase we have to concatenate the two lists containing the graphs $G - G'$ and $G^*$ returned by $A_k$, and the two matchings $M$ and $M^*$, and clear the entries $(u, v)$ of $T_k$ for which $(u, v)$ is present in $G'$ (this is needed to ensure that at the beginning of each phase, $T_k$ is an all-zero helper matrix). This can be done in $O(m_0)$ steps as well, by simply iterating through $G'$.

Overall, to process the $t'$ updates, the algorithm takes an amortized cost of $f(k) + \frac{g(k)}{2} m_0$ per update, plus a total cost of $O(m_0)$ for some universal constant $C$, which makes the amortized cost to be $C + f(k) + \frac{g(k)}{2} m_0$ for some universal constant $C$, since the number $t'$ of updates is at least $\frac{m_0}{4}$.

Since in each phase $m_0 \leq m$ (as $m_0$ is the size of the graph $G$ in a specific time point, while $m$ is the maximum of those values over time), we proved Lemma 1 and thus Theorem 1.

2.1. Implementation details and improving the setup cost

The careful reader might observe the fact that the algorithms $A_k$ never use the value $m$ to make decisions, neither in the length of the lookahead it uses, nor for setting the length of $t'$. Hence the amortized update cost is guaranteed to be an actual $O(\log m)$. Also, the sequence of these algorithms can be constructed as a single algorithm $A$, taking as argument a graph $G$, the sequence $\hat{q}$ of updates, and the recursion depth $k$ as an integer – but this latter value is used only in order to determine which helper table $T_k$ can be used when constructing the graphs $G - G'$ and $G \cap G'$. However, the construction of these graphs happens during substeps 3 – 5 in which no recursive call is made, and after which $T_k$ is again guaranteed to be an all-zero matrix – hence, the algorithm $A$ can use the very same helper array $T$ on each recursion level. This already improves the setup cost to be $O(n^2)$ instead of $O(n \cdot \log m)$ as there is only one adjacency matrix $n \times n$ we have to handle globally, which we initialize by zeros.

However, we can do even better: during the construction of $G - G'$ and $G \cap G'$, we only check those entries of $T$ which correspond to edges already present in $G$. Thus, we can postpone the initialization: it suffices to set $T(u, v)$ and $T(v, u)$ to zero for those entries for which $(u, v)$ is in $G$, which can be handled during the processing of an insert operation for a constant increase in the amortized run-time.

The $\text{MATE}$ array has to be initialized to an all-zero vector in the beginning, though, requiring an $O(n)$ setup cost.

So the final form of our main result is the following:

**Theorem 2.** There exists a fully dynamic graph algorithm for maintaining a maximal matching with an $O(\log m)$ amortized update cost and constant query cost, using a lookahead of length $n$, with either an $O(n)$ setup cost (if the memory model allows getting an uninitialized memory chunk of size $n \times n$ in constant time) or a setup cost of $O(n^2)$ (if in the memory model we have to pay $n^2$ even when the memory is uninitialized).
Note that in the latter case if $O(n^2)$ is too much of a cost for either in storage space, or as a setup cost, then the set operations required for splitting the graph $G$ can be implemented by using balanced binary trees for the set operations. That way, splitting the list $G$ of size $m_0$, we can make a searchable set from $G'$ in $O(t' \cdot \log t')$ time, then doing a search operation for each element of the list $G$ takes an additional time of $O(m_0 \cdot \log t')$. As $t' \leq m_0$, that’s a total time of $O(m_0 \log m_0)$ which yields and additional $\log m_0$ amortized cost per update on each recursion level. This way, an amortized update cost of $O(\log^2 m)$ can be gained, for a setup cost of $O(n)$, that only uses a single global mate array of size $n$, and lists/sets having in total $O(m)$ elements, with a setup cost of $O(n)$, which might be a more memory-efficient solution for graphs which are guaranteed to be sparse at any given time.

If even maintaining the array mate is too much, then one can trade it for a global tree map in which those vertices having a mate appear as key, with their mate as value. That decision makes mate accesses to have the cost of $O(\log m)$ (as there are at most $2m$ nodes actually having a mate at each time iteration: bear in mind that $O(\log m)$ is automatically $O(\log n)$ as well, but not necessarily vice versa), allowing for a constant initialization cost and a total memory needed is only that of storing $O(m)$ nodes/edges. The query cost becomes $O(\log m)$ in that case. Managing the mate tree map in the code yields an additional total cost of $O(m_0 \times \log m)$ (erasing the local part of the matching) for a phase, which translates to an additional amortized cost of $\log m_0$ per update – which is free if we already traded the helper array $\tilde{T}$ for set-operations.

So in that case we get an algorithm with an amortized update cost of $O(\log^2 m)$, query cost of $O(\log m)$, a constant setup cost and a memory footprint proportional to storing $O(m)$ nodes. This might be the correct choice if we do not know the size of the graph in advance, or if the nodes are not numbers but strings, say, whose possible domain is not known in advance.

3. Conclusion

In this study we dealt with a problem arising in the context of fully dynamic graph algorithms. We showed that by using a lookahead of linear length, there is a deterministic algorithm achieving an $O(\log m)$ amortized update cost, without knowing the maximal size $m$ of the graph in advance. (Note that once again $O(\log m)$ is $O(\log n)$ as well, since $m \leq n^2$.)

This result shows that lookahead can help in the dynamic setting for problems other than the transitive closure (and the SCC) properties, studied in [5]: indeed, the best known deterministic algorithm for the problem using no lookahead has an update cost of $O(\sqrt{m})$.

It is an interesting question to study further the possibilities of using lookahead for different problems, and maybe factor in also randomization as well, albeit for the randomized setting, an algorithm with a (both expected and whp) constant update cost is already known without lookahead.

References

[1] Surender Baswana, Manoj Gupta, and Sandeep Sen. Fully dynamic maximal matching in $O(\log n)$ update time. SIAM Journal on Computing, 44(1):88–113, 2015.

[2] Kitti Gelle and Szabolcs Ivn. DFS is unsparsable and lookahead can help in maximal matching. Acta Cybernetica, 23(3):887–902, 2018.

[3] Zoran Ivković and Errol L. Lloyd. Fully dynamic maintenance of vertex cover, pages 99–111. Springer Berlin Heidelberg, Berlin, Heidelberg, 1994.

[4] Telikepalli Kavitha. Dynamic matrix rank with partial lookahead. Theor. Comp. Sys., 55(1):229–249, July 2014.

[5] S. Khanna, R. Motwani, and R. H. Wilson. On certificates and lookahead in dynamic graph problems. Algorithmica, 21(4):377–394, Aug 1998.

[6] S. Micali and V. V. Vazirani. An $O(\sqrt{|V| \cdot |e|})$ algorithm for finding maximum matching in general graphs. In 21st Annual Symposium on Foundations of Computer Science (sfcs 1980), pages 17–27, Oct 1980.

[7] Ofer Neiman and Shay Solomon. Simple deterministic algorithms for fully dynamic maximal matching. ACM Trans. Algorithms, 12(1):7:1–7:15, November 2015.

[8] Krzysztof Onak and Ronitt Rubinfeld. Maintaining a large matching and a small vertex cover. In Proceedings of the Forty-second ACM Symposium on Theory of Computing, STOC ’10, pages 457–464, New York, NY, USA, 2010. ACM.

[9] Piotr Sankowski. Faster dynamic matchings and vertex connectivity. In Proceedings of the Eighteenth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA ’07, pages 118–126, Philadelphia, PA, USA, 2007. Society for Industrial and Applied Mathematics.
[10] Piotr Sankowski and Marcin Mucha. Fast dynamic transitive closure with lookahead. *Algorithmica*, 56(2):180–197, February 2010.

[11] S. Solomon. Fully dynamic maximal matching in constant update time. In *2016 IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 325–334, Oct 2016.