Mediated gates between spin qubits

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(Dated: May 2, 2014)

In a typical quantum circuit, nonlocal quantum gates are applied to nonproximal qubits. If the underlying physical interactions are short-range (e.g., exchange interactions between spins), intermediate swap operations must be introduced, thus increasing the circuit depth. Here we develop a class of “mediated” gates for spin qubits, which act on nonproximal spins via intermediate ancilla qubits. At the end of the operation, the ancillae return to their initial states. We show how these mediated gates can be used (1) to generate arbitrary quantum states and (2) to construct arbitrary quantum gates. We provide some explicit examples of circuits that generate common states [e.g., Bell, Greenberger-Horne-Zeilinger (GHZ), W, and cluster states] and gates (e.g., √SWAP, swap, CNOT, and Toffoli gates). We show that the depths of these circuits are often shorter than those of conventional swap-based circuits. We also provide an explicit experimental proposal for implementing a mediated gate in a triple-quantum-dot system.

I. INTRODUCTION

Quantum dot spin qubits are promising candidates for quantum computing because of their long decoherence times and their potential to leverage existing semiconductor technologies [1]. The exchange coupling is a desirable tool for mediating interactions between spin qubits, because it can be controlled electrostatically and it is typically very fast [3]. In combination with arbitrary single-qubit operations, the exchange coupling enables universal quantum computation [4]. When logical qubits consist of two [5] or three [6] physical qubits in a decoherence-free subsystem, the exchange coupling alone is universal for quantum computation. On the other hand, the intrinsic short-range nature of the exchange coupling (typically tens of nanometers) imposes strong constraints on the physical architecture of the spin qubits. These constraints present a significant challenge to scalability during quantum error correction, particularly for linear qubit architectures, which are typical for quantum dot spin qubits [7]. Indeed, large-scale quantum computing is challenging in any qubit implementation, and the complexity of a given quantum circuit could, to a large extent, determine its success.

In a number of qubit systems, such as nuclear magnetic resonance (NMR), the physical interactions may be constant or “always on.” This is not necessarily a disadvantage. For example, it has been shown that simultaneous, multiqubit couplings can be used to enable quantum state transfer [8, 9], and other rudimentary quantum gates [10]. Similar considerations apply to quantum dot spin systems with Heisenberg couplings [11]. Quantum dots provide unique opportunities for controlling the nature of the interactions. For example, simultaneous, multiqubit couplings could provide a potential route for enhancing the effective range of the coupling, in analogy with the Ruderman-Kittel-Kasuya-Yosida (RKKY) interaction [12]. When these couplings are arranged into nontrivial topologies, such as rings, a rich spectrum of quantum gates emerges [13–15]. However, even simple topologies, like those considered here, can produce entangling gates that differ from the existing two-qubit gates in spin qubits [16–18].

In this paper, we show how to control such simultaneous, multiqubit couplings. The result is a class of “mediated” quantum gates. We focus primarily on the three-qubit geometry shown in Fig. 1, due to recent experimental progress on triple quantum dots [19–21]. In this arrangement, the mediated gate acts on the nonproximal qubits 1 and 2, leaving the ancilla or central qubit c unaffected, at the end of the operation. We characterize this well-defined gate operation, U2, and show how arbitrary two-qubit states and gates can be generated using U2 as the sole entangling resource. We also compare the circuit depth of these mediated gate protocols to more conventional swap-based protocols. Finally, we explain how long-range mediated gates can be attained by replacing qubit c with a spin bus.

![FIG. 1. (Color online) A linear triple-quantum-dot geometry with three electrons. Mediated gates can be achieved between qubit 1 and qubit 2 by applying simultaneous exchange couplings, with J1 = J2. At the end of the operation, the ancilla qubit c is restored to its initial state.](arXiv:1207.6063v4 [quant-ph] 3 Jan 2013)
II. TWO-QUBIT MEDIATED GATE, \( U_2 \)

A. Mediated gate, \( U_2 \)

We begin by characterizing the mediated gate \( U_2 \). The effective spin Hamiltonian for the quantum dot geometry in Fig. 1 derives from the exchange interaction and takes the form of nearest-neighbor Heisenberg couplings \( \mathcal{H}_J \), given by

\[
H = J_1 \mathbf{s}_1 \cdot \mathbf{s}_c + J_2 \mathbf{s}_2 \cdot \mathbf{s}_c, \tag{1}
\]

where \( \mathbf{s}_j \) are spin operators. In typical experiments, the coupling constants \( J_1 \) and \( J_2 \) are controlled by detuning the local electrostatic potentials in a given dot \cite{3}. \( J_1 \) and \( J_2 \) can usually be varied independently as a function of time \cite{19} \cite{20}. For mediated gates, however, we assume that both couplings are turned on and off simultaneously.

The Hamiltonian \( H \) induces three-spin dynamics according to the time evolution operator \( U(t) = e^{-iHt} \), where we set \( \hbar = 1 \). However, the mediated gate we seek has the special form \( U = \mathbb{U}_2 \otimes I \), where \( \mathbb{U}_2 \) acts on qubits 1 and 2, and the identity operator \( I \) acts on the ancilla qubit \( c \). In Appendix A, we prove that only one nontrivial mediated gate exists for the geometry in Fig. 1, corresponding to the unique parameter combination \( J_1 = J_2 = J \) and the special evolution period \( T_g = 4\pi/3J \) (with periodic recurrences \cite{22}). The gate is robust against control errors, similarly to conventional two-qubit exchange gates. For example, if \( J_2 = J_1(1 + \delta) \) results in the gate \( U(\delta) \), where \( U(0) \) is the desired gate, and if the fidelity is defined as \( F = |\text{Tr}[U(\delta)U(0)]/|\text{Tr}[U(0)U(0)]| \), then we obtain a quadratic error in the fidelity: \( 1 - F \approx 0.97\delta^2 \) when \( \delta \leq 0.4 \).

Any unitary two-qubit operator \( U_2 \in \text{SU}(4) \), including \( \mathbb{U}_2 \), can be expressed in the form of a Cartan decomposition, given by \cite{24} \cite{25}

\[
U_2 \overset{\text{U.}}{=} e^{\frac{i}{2} \sum \sigma_i \otimes \sigma_i \delta + c_{12} \sigma_x \otimes \sigma_y + c_3 \sigma_z \otimes \sigma_z}, \tag{2}
\]

where \( \sigma_x, \sigma_y, \sigma_z \) are the Pauli matrices, and \( s = \sigma/2 \) in spinor notation. Here, the relation \( \overset{\text{U.}}{=} \) means “equal, up to local unitary gates,” where the latter may be applied before and/or after the nonlocal operator. The decomposition is unique when the parameters \( (c_1, c_2, c_3) \) are restricted to the tetrahedron \( c_1 - c_2 \geq c_1 \geq c_2 \geq c_3 \geq 0 \), known as the Weyl chamber. (Note that special considerations apply to the base of the tetrahedron \cite{21}.)

There is a one-to-one mapping between the Weyl chamber and the Makhlin invariants \cite{26}, which provides an alternative representation of the nonlocal properties of \( U_2 \in \text{SU}(4) \) (except on the bottom surface of the chamber). The Cartan decomposition for our two-qubit mediated gate is given by \( (c_1, c_2, c_3) = (2, 1, 1)(\pi/3) \) and it has the explicit form (see Appendix A for details)

\[
\mathbb{U}_2 = -\begin{pmatrix}
\frac{1}{2}(1 + i\sqrt{3}) & 0 & 0 & 0 \\
0 & \frac{1}{2}(-1 + i\sqrt{3}) & \frac{1}{2}(3 + i\sqrt{3}) & 0 \\
0 & \frac{1}{2}(3 + i\sqrt{3}) & \frac{1}{2}(-1 + i\sqrt{3}) & 0 \\
0 & 0 & 0 & \frac{1}{2}(1 + i\sqrt{3})
\end{pmatrix}. \tag{3}
\]

The position of \( \mathbb{U}_2 \) in the Weyl chamber is shown in Fig. 2 along with several other common two-qubit gates.

The gating capabilities of \( U_2 \in \text{SU}(4) \) derive from its entangling properties, which can be characterized in part by its position in the Weyl chamber. The operators known as “perfect entanglers” lie inside a polyhedron, which fills half of the chamber \cite{24}, as shown in Fig. 2. Combined with local unitaries, a perfect entangler can generate a maximally entangled state from a separable state. For example, if we quantify two-qubit entanglement in terms of the “concurrence” measure \( C \) \cite{27} \cite{28}, then a separable state exhibits no entanglement, with \( C = 0 \), while a highly nonlocal state like the singlet Bell state \( |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \) exhibits maximal entanglement, with \( C = 1 \). Thus, for some initial two-qubit state with \( C = 0 \), one application of a perfect entangler produces a state with \( C = 1 \). The standard \textsc{cnot} gate is known to be a perfect entangler \cite{24}, as indicated in Fig. 2. However, \textsc{cnot} does not arise naturally from the exchange interaction between spin qubits; it must be constructed from more basic gates \cite{4}. In contrast, the mediated gate \( \mathbb{U}_2 \) does arise naturally in many-body spin systems as we have shown; however, its location in the Weyl chamber indicates that it is not a perfect entangler. Using the methods of \cite{25}, we find that \( \mathbb{U}_2 \) achieves a maximum concurrence of \( C_{\text{max}} = \sqrt{3}/2 < 1 \), when acting on a separable state.

A universal quantum processor must be able to generate arbitrary entangled states or implement arbitrary quantum circuits. For example, \textsc{cnot} gates combined with single-qubit unitaries are known to be universal \cite{29} \cite{31}. Any two-qubit entangling gate \( U_2 \) can replace \textsc{cnot} in this scheme \cite{32}, although the entangling capabilities of the gate will affect the overall circuit depth. In the remainder of this section, we explore methods for generating arbitrary entangled states and entangling gates using the mediated gate \( \mathbb{U}_2 \) and we determine the circuit depth of such protocols.
B. Generation of arbitrary states

Our goal here is to construct arbitrary, two-qubit, entangled pure states between qubit 1 and qubit 2 in the geometry shown in Fig. 1 using the mediated gate $U_2$ as our nonlocal entangling resource. Allowing for local operations and classical communication (LOCC), it is possible to transform maximally entangled states, such as Bell states, into arbitrary pure states \[2\]. We therefore focus on using $U_2$ to generate Bell states. For simplicity, we ignore global phase factors throughout this paper.

The strategy we adopt is to apply $U_2$ repeatedly, assisted by single-qubit unitary rotations $U_1$, as needed:

$$|\psi\rangle = (U_1 \otimes U_1)(U_2(U_1 \otimes U_1))^n|00\rangle.$$  \[4\]

Note that each application of $U_1$ here represents an arbitrary rotation, and that, in general, the rotations can all be different.

We would like to be able to compare the speed or efficiency of disparate gating protocols, particularly between mediated and conventional gates. The most convenient measure of this efficiency is the “circuit depth,” which we define here as the total number of exchange gates. For example, in Eq. \[4\], the circuit depth is equal to $n$. For conventional quantum dot circuits, there may be cases where it is possible to implement gates between different pairs of qubits simultaneously, due to physical separation. We define the circuit depth of such parallel gates to be 1, since they occur simultaneously. On the other hand, conventional circuits typically require intermediate SWAP gates to be applied sequentially when the qubits are non-proximal, causing the circuit depth to increase by 1 with each SWAP application. This notion of circuit depth plays an important role in the gate times and fidelities of quantum circuits, and we speak of circuit depth throughout the following discussion. To conclude, we note that since $U_2$ is not a perfect entangler, the value of $n$ in Eq. \[4\] must be greater than 1 when we generate a Bell state.

We have solved Eq. \[4\] numerically, obtaining several two-qubit states of interest. Our procedure involves maximizing the state fidelity, $f = |\langle \psi_{\text{des}} | \psi_{\text{actual}} \rangle|^2$, where $|\psi_{\text{actual}}\rangle$ is the outcome of Eq. \[4\] and $|\psi_{\text{des}}\rangle$ is the desired outcome. The two-qubit mediated gate used in the simulations is given by Eq. \[3\], and the individual single-qubit rotations $U_1$ are determined using global optimization methods, as described in Appendix B. In principle, we could also allow the circuit depth $n$ to vary. However, we find that maximally entangled Bell states can already be obtained when $n = 2$. The resulting circuit for the singlet Bell state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ is shown in Fig. 3(a). Other Bell states can be generated in a similar fashion.

We can compare our mediated gate protocol to the conventional Bell-state protocol based on nearest-neighbor gates, as shown in Fig. 3(b). In the latter case, CNOT is used to generate the Bell state, while the SWAP gates are used to make the qubits proximal. The minimal circuit depth needed to construct CNOT is 2. Comparing Figs. 3(a) and 3(b), we obtain an exchange gate circuit depth of $n = 2$ for the mediated gate protocol and $n = 4$ for the SWAP-based protocol. The mediated gate therefore offers distinct advantages for generating arbitrary states.
C. Experimental proposal for a triple quantum dot

Triple quantum dots have been investigated in several laboratories [20, 21]. Here, we suggest a specific protocol for generating a Bell state in a triple quantum dot, using the mediated gate protocol in Fig. 3a). Our proposal includes the supporting initialization and verification steps, and it is based on existing experimental methods. We note that Bell states can also be produced via standard, conventional (i.e., nonmediated) techniques [3]. The purpose of this section is simply to outline a proof-of-principle experiment that employs mediated gates.

To generate a Bell state using mediated gates, we must first initialize the triple dot into the separable state...
The verification step is performed most conveniently via spin-to-charge conversion, using a singlet projection procedure. We first perform a swap operation between qubit 1 and qubit 2, so that the two spins in the singlet state become proximal: $|\Psi^-\rangle_{12} \rightarrow |\Psi^-\rangle_{1c} \langle c|\rangle_{2}$. This result (together with (3)), constitutes a formal proof that $n \leq 10$ for a mediated gate with optimal circuit depth, as described in Eq. (5). It forms a constructive protocol for generating an arbitrary two-qubit gate using 10 $U_{2}$ gates. However, we note that the bound $n \leq 10$ does not appear to be tight, since none of the gates we have solved requires more than five applications of $U_{2}$.

There are several well-known techniques for constructing arbitrary two-qubit gates, which can be adapted for mediated gates. The most efficient method involves the so-called B gate, which is defined as the only two-qubit gate that can generate generic two-qubit operations from two successive applications. Figure 4(g) shows our globally optimized circuit for a B gate, which employs five $U_{2}$ gates. This result (together with (35)), constitutes a formal proof that $n \leq 10$ for a mediated gate with optimal circuit depth, as described in Eq. (5). It forms a constructive protocol for generating an arbitrary two-qubit gate using 10 $U_{2}$ gates. However, we note that the bound $n \leq 10$ does not appear to be tight, since none of the gates we have solved requires more than five applications of $U_{2}$.
III. THREE-QUBIT MEDIATED GATE, $U_3$

A. Mediated gate, $U_3$

We now consider the mediated gate geometry shown in Fig. 5(a), with three qubits coupled through a single mediating spin, $c$. The system Hamiltonian is given by

$$H = J_1 c s_1 \cdot s_c + J_2 c s_2 \cdot s_c + J_3 c s_3 \cdot s_c,$$

and the time evolution operator is given by $U(t) = e^{-iHt}$. If qubits 1–3 are arranged in a linear geometry rather than the “star” geometry shown in Fig. 5 then an effective star geometry can still be achieved by introducing a spin bus architecture [22], where $c$ is an odd-size bus.

For larger geometries, the group theoretical methods described in Appendix A become cumbersome. However, for the special case of equal couplings, $J = J_1 c = J_2 c = J_3 c$, we can still obtain mediated gates analytically. We do this by computing $U(t)$ in the angular momentum basis, where it is diagonal [22]. We then transform it to the computational basis and identify the gate periods $t = T_g$ for which the special decomposition $U = U_3 \otimes I$ is satisfied. Here, $U_3$ is the mediated gate acting on qubits 1–3, while $I$ is the single-qubit identity operator acting on spin $c$. This procedure produces four different mediated gates [22]. The first gate is the trivial identity operator, obtained at the gate periods $T_g = (8m)\pi/J$ ($m$ is an integer). The second gate occurs at the gate periods $T_g = (8m + 2)\pi/J$, and takes the form

$$U_3 = i \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/3 & 2/3 & 0 & 2/3 & 0 & 0 & 0 \\ 0 & 2/3 & -1/3 & 0 & 2/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/3 & 0 & 2/3 & 2/3 & 0 \\ 0 & 2/3 & 2/3 & 0 & -1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2/3 & 0 & -1/3 & 2/3 & 0 \\ 0 & 0 & 0 & 2/3 & 0 & 2/3 & -1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$  

The third gate occurs at the gate periods $T_g = (8m + 4)\pi/J$, and is given by $U_3^3 = -I$. The fourth gate occurs at the gate periods $T_g = (8m + 6)\pi/J$, and is given by $U_3^4 = -U_3$.

B. Generation of arbitrary states

The methods used to generate two-qubit states and gates can also be extended to three-qubit problems. However, three-qubit protocols are slightly more complicated because they can involve two-qubit gates, three-qubit gates, or both. The most general scheme for generating a three-qubit state is shown in Fig. 6(a). We note that higher order gates such as the three-qubit mediated gate $U_3$ can potentially achieve shorter circuit depths, because they are more parallel than two-qubit gates. The global optimization techniques used to solve Eq. (4) can also be applied to Fig. 6(a).

There are known to be two nonfungible forms of entanglement for three qubits [37]: the $W$-state family, characterized by the symmetric form

$$|W_3\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle),$$  

and the Greenberger-Horne-Zeilinger (GHZ)-state family

$$|\text{GHZ}_3\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$  

The GHZ state is understood to be maximally entangled for three qubits.

We have applied global optimization methods to obtain $|\text{GHZ}_3\rangle$ and $|W_3\rangle$, obtaining the results shown in

FIG. 5. (Color online) Multiqubit “star” geometries for implementing mediated gates. Here, the qubits are labeled with numbers and the ancilla spins labeled $c$ mediate the gates. (a) The $U_3$ gate acts on qubits 1–3 when the three qubit couplings $J$ are equal. (b) To generate the four-qubit cluster state $|C_4\rangle$, we implement three-qubit mediated gates $U_3$ by turning on the couplings to three of the qubits at a time. (c) To generate a $W_N$ state, we consider $N$ qubits connected simultaneously to the ancilla spin $c$, with equal couplings $J$. In each of these geometries, the ancilla spin can be replaced with an odd-size spin bus. In this case, $c$ represents the pseudospin of the bus ground state [22].
Figs. 6(b)–6(d). For \(|W_3\rangle\), we provide two different strategies. One uses a combination of \(U_2\) and \(U_3\); the other uses \(U_2\) only. They both have the same circuit depth, \(n = 2\). Remarkably, we find that the GHZ state can be attained using \(U_2\) as the only entangling resource with just a single application:

\[|\text{GHZ}_3\rangle = (U_1 \otimes U_1 \otimes U_1)U_3(U_1 \otimes U_1 \otimes U_1)|000\rangle. \quad (10)\]

The circuit is optimal \((n = 1)\), indicating that \(U_3\) is a perfect entangler for the three-qubit GHZ state family. We can compare this result to the conventional, pairwise gating circuit for \(|\text{GHZ}_3\rangle\), which uses two \textsc{cnot} gates. In a quantum dot quantum computer, this would require at least four exchange gate operations, or \(n = 4\). It is interesting to note that \(U_3\) is locally equivalent to the time evolution operator describing the three-qubit triangular geometry (evaluated at a special time) \([10]\). The latter gate is also capable of generating \(|\text{GHZ}_3\rangle\) in a single time step.

Although \(U_3\) acts on just three qubits at a time, it is interesting to note that it can also be used as an entangling resource for larger systems. For example, we can consider cluster states, which represent an important entanglement family used for one-way quantum computing \([41, 42]\). The four-qubit cluster state \(|C_4\rangle\) is defined as

\[|C_4\rangle \equiv \frac{1}{2}(|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle). \quad (11)\]

We have solved \(|C_4\rangle\) numerically, for the geometry shown in Fig. 4(b). Here, the ancilla spin \(c\) can be connected to each of the four ancillas. However, we assume that only three of the couplings are turned on at a time. For example, \(U_3(1, 2, 3)\) indicates that the couplings between \(c\) and qubits 1–3 are turned on, thus implementing the gate \(U_3\) between those three qubits. Hence, we obtain a numerical solution for \(|C_4\rangle\) of the form

\[|C_4\rangle = U_1^{\otimes 4}U_3(1, 2, 3)U_1^{\otimes 4}U_3(1, 2, 4)U_1^{\otimes 4}U_3(1, 2, 3) \times U_1^{\otimes 4}U_3(2, 3, 4)U_1^{\otimes 4}U_3(1, 2, 4)|0000\rangle. \quad (12)\]

Here, \(U_1^{\otimes 4}\) represents arbitrary single-qubit rotations acting on each of the four qubits. According to our definition of circuit depth, this protocol corresponds to \(n = 4\).

We can compare our mediated gate solution to a conventional sequence for generating \(|C_4\rangle\), based on nearest-neighbor pairwise gates. The conventional scheme involves three sequential applications of the phase gate diag(1, 1, 1, −1), in addition to single-qubit rotations \([11, 12]\). Since the phase gate is locally equivalent to \textsc{cnot}, it can be decomposed into two exchange gates plus single-qubit rotations. The resulting circuit depth for the conventional protocol is therefore \(n = 6\). Thus, again, we find that mediated gates offer a considerable improvement in terms of circuit depth.

C. Construction of arbitrary gates

We now turn to the construction of three-qubit quantum gates using \(U_3\). As an example, we determine an explicit gate sequence for generating the Toffoli gate, defined as

\[
U_T = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}.
\quad (13)
\]

Our strategy is analogous to the state-generating circuit in Fig. 6(a), where we interspersed \(U_2\) or \(U_3\) gates with arbitrary single-qubit rotations. Our best result for constructing the Toffoli gate by this method is a gate sequence containing five \(U_2\) gates and seven \(U_3\) gates, giving a total exchange-gate circuit depth of \(n = 12\).
We can compare our mediated gate solution to a conventional Toffoli gate construction. A Toffoli circuit using CNOT gates as the entangling resource has been presented in [30] and [43]; it consists of six sequential CNOT gates. We can decompose this into a sequence of nearest-neighbor exchange gates, including intermediate SWAP gates when necessary. After identifying the exchange gates that can be performed in parallel, this procedure gives a circuit depth of \( n = 16 \). Alternatively, if we allow other two-qubit gates in this procedure, in addition to CNOT, it can be shown that five sequential two-qubit gates are necessary and sufficient for implementing a Toffoli gate [44]. However, some of these gates are decomposed into exchange gate sequences with \( n > 2 \). Based on such considerations, it appears that the mediated gate circuit with a circuit depth of \( n = 12 \) for constructing a Toffoli gate is always more efficient than a conventional gate circuit.

IV. MEDIATED GATES, \( U_{2N+1} \) (\( N > 1 \))

The previous approach to state generation and gate construction using mediated gates can be extended to systems with more than three qubits. There are many qubit architectures of interest. Here, we consider the “star” geometry shown in Fig. 4(c). In cases where it is experimentally challenging to fabricate a star geometry, due to physical constraints, it may be convenient to replace the central spin \( c \) with an odd-size spin bus [22]. In this case, nontrivial mediated gates can be obtained when an odd number of qubits is simultaneously coupled to the bus. These multiqubit mediated gates, \( U_{2N+1} \), are highly parallel and potentially very efficient.

Here, we demonstrate that multiqubit \( W \) states can be generated using mediated gates, with very small circuit depths. The \( N \)-qubit \( W \) state is defined as

\[
|W_N\rangle = \frac{1}{\sqrt{N}}(|00\ldots01\rangle + |00\ldots10\rangle + \ldots + |10\ldots00\rangle).
\]

In Figs. 6(c) and 6(d) we indicate two methods for generating \(|W_3\rangle\). An alternative method is shown in Fig. 7.

This circuit requires a maximally entangled Bell state, \(|\Psi^-\rangle\), as input. The total circuit depth for this solution (\( n = 3 \)) is larger than in Figs. 6(c) and 6(d) because the circuit depth for generating \(|\Psi^-\rangle\) is \( n = 2 \). However, the scheme has the advantage that it may be scalable for odd-size \( W \) states.

For the cases \( N = 1-3 \), we have numerically verified the result that

\[
|W_{2N+1}\rangle = (U_1 \otimes U_1)U_{2N+1}(U_1 \otimes U_1)|\Psi^-\rangle|0\rangle^{\otimes(2N-1)},
\]

which includes the result in Fig. 7 for the case \( N = 1 \). For all cases, we note that the single-qubit rotations are applied only to qubits 1 and 2 (the qubits in the Bell state). For the cases of \( N = 2, 3 \), the generating circuits are similar to Fig. 7 but with different angles \( \theta_1 \) and \( \theta_2 \). For each of these cases, the circuit depth is given by \( n = 3 \).

A related, probabilistic scheme can be used to generate the even-size \( W \) states. We first generate the odd-size \( W \) state, as described above. This state can be expressed as

\[
|W_{2N+1}\rangle = \frac{1}{\sqrt{2N+1}}|0\rangle^{\otimes 2N}|1\rangle + \sqrt{\frac{2N}{2N+1}}|W_2\rangle|0\rangle.
\]

Hence, if one of the qubits is measured in the \( z \) basis, with outcome 0, then the state of the remaining qubits will collapse to \(|W_2\rangle\). When \( N \) is large, this protocol is successful with a high probability, \( P = 2N/(2N+1) \).

V. SUMMARY AND CONCLUSIONS

| State or gate | No. mediated gates | No. pairwise gates |
|---------------|-------------------|-------------------|
| \(|\Psi^-\rangle\) | 2 | 4 |
| \(|W_3\rangle\) | 2 | 2 |
| \(|W_N\rangle\) | 3 | \(N - 1\) |
| \(|GHZ_3\rangle\) | 1 | 4 |
| \(|C_4\rangle\) | 4 | 6 |
| \(|\text{SWAP}\rangle\) | 4 | 3 |
| \(|\text{SWAP}\rangle\) | 5 | 3 |
| \(|B\rangle\) | 5 | 5 |
| Toffoli | 12 | 16 |

In this paper, we developed the concept of a mediated gate between nonproximal qubits. This gate is implemented by coupling the qubits simultaneously through a central, ancilla qubit, which is restored to its initial state at the end of the operation. We have focused on two and three-qubit gates, although higher dimensional gates can be obtained in similar fashion. We investigated protocols,
based on global optimization techniques, for generating arbitrary states and gates, using mediated gates as the sole entangling resource.

Several promising results were obtained using mediated gates, as summarized in Table I. We showed that a maximally entangled Bell state can be achieved with just two applications of a mediated gate $U_2$, and we proposed an experimental protocol for implementing this procedure in a triple quantum dot. We showed that several important two-qubit quantum gates can be obtained using five or fewer mediated gates, and we proved that ten exchange gates is the maximum needed for generating an arbitrary two-qubit gate. We showed how the central ancilla qubit can be replaced with a spin bus, leading to significant improvements in scaling properties, for both the total gate time and the circuit depth. We also considered the mediated gates $U_N$ with $N \geq 3$, and showed how mediated gate methods might be generalized to higher dimensions.

We find that mediated gates compare favorably with conventional, pairwise gating schemes, which make use of SWAP gates when qubits are not proximal. For each of the results reported in Table I, we compare the circuit depths based on mediated gates to those involving conventional pairwise gates.

We thank Jun Zhang and Andrei Galaitoudin for helpful discussions. This work was supported by the DARPA QuEST program through a grant from AFSOR, by NSA/LPS through grants from ARO (W911NF-09-1-0393, W911NF-08-1-0482, and W911NF-12-1-0607), by NSF-PIF (PHY-1104672), and by the United States Department of Defense. The US government requires publication of the following disclaimer: the views and conclusions contained in this document are those of the authors and should not be interpreted as representing the official policies, either expressly or implied, of the US Government.

**Appendix A: EXISTENCE PROOF FOR $U_2$**

Here, we prove that the gate $U_2$, presented in Eq. [3] of the main text (and its family), represents the only solutions to the mediated gate problem for two qubits. For convenience, we adopt slightly different notation than in the main text, as indicated in Fig. 8. Spins 1 and 3 are the two nonproximal qubits, while spin 2 is the central ancilla qubit.

We consider the following Hamiltonian for a linear three-qubit array:

$$H = J_a s_1 \cdot s_2 + J_b s_2 \cdot s_3,$$

(A1)

where $s_j$ is the spin operator for qubit $j$. In principle, $J_a$ and $J_b$ may take any value. However, we limit our search to the case where the couplings are turned on and off simultaneously. $J_a$ and $J_b$ are therefore constant throughout the gate operation. The goal of this Appendix is to identify specific relations between $J_a$ and $J_b$ that lead to mediated gates.

We make use of the identity

$$4 s_1 \cdot s_j = 2p^{ij} - I,$$

(A2)

where $p^{ij}$ is the SWAP (i.e., transposition) operator between spin $i$ and spin $j$, and $I$ is the two-qubit identity operator. Hamiltonian (A1) can then be rewritten as

$$H = \frac{1}{2}(J_a p^{12} + J_b p^{23}) - \frac{1}{4}(J_a + J_b).$$

(A3)

The time evolution operator is given by

$$U(t) = e^{-iHt} = e^{(J_a + J_b)t/4} e^{-iQJ_4t/2} = e^{(J_a + J_b)t/4} \sum_{n=0}^{\infty} \frac{(-iJ_4t/2)^n}{n!} Q^n,$$

(A4)

where $\hbar = 1$ and we have defined

$$Q \equiv p^{23} + Jp^{12},$$

(A5)

with $J = J_a/J_b$. Since $p^{23}$ and $p^{12}$ are generators of the symmetric group $S_3$, we may expand $Q^n$ in terms of the $S_3$ group elements:

$$Q^n = a_n p^{231} + b_n p^{312} + c_n p^{123} + d_n p^{13} + e_n p^{23} + f_n I.$$  

(A6)

Here, $p^{ijk}$ is the tripartite, cyclic permutation operator.

**TABLE II.** (Color online) Cayley table for the symmetric group $S_3$.

|   | $I$ | $p^{12}$ | $p^{13}$ | $p^{23}$ | $p^{312}$ | $p^{313}$ |
|---|---|---|---|---|---|---|
| $I$ | $I$ | $p^{12}$ | $p^{13}$ | $p^{23}$ | $p^{312}$ | $p^{313}$ |
| $p^{12}$ | $p^{12}$ | $I$ | $p^{23}$ | $p^{13}$ | $p^{313}$ | $p^{312}$ |
| $p^{13}$ | $p^{13}$ | $p^{312}$ | $I$ | $p^{23}$ | $p^{313}$ | $p^{312}$ |
| $p^{23}$ | $p^{23}$ | $p^{312}$ | $p^{313}$ | $I$ | $p^{12}$ | $p^{13}$ |
| $p^{312}$ | $p^{312}$ | $p^{313}$ | $p^{13}$ | $p^{23}$ | $I$ | $p^{23}$ |

The full set of $S_3$ group operations is listed in Table II. We then deduce the recursion relations for $Q^n+1 = Q^n$:

$$a_{n+1} = c_n + Jd_n,$$

(A7)

$$b_{n+1} = d_n + Jc_n,$$

(A8)

$$c_{n+1} = a_n + Jf_n,$$

(A9)

$$d_{n+1} = b_n + Jd_n,$$

(A10)

$$e_{n+1} = f_n + Jb_n,$$

(A11)

$$f_{n+1} = e_n + Jc_n.$$  

(A12)
These relations can be expressed compactly as

\[ v_{n+1} = T v_n, \]  
(A13)

where

\[ v_n = [f_n \ c_n \ c_n \ a_n \ b_n \ d_n]^T, \]  
(A14)

and

\[
T = \begin{pmatrix}
0 & J & 1 & 0 & 0 & 0 \\
J & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & J & 0 \\
0 & 1 & 0 & 0 & 0 & J \\
0 & 0 & J & 0 & 0 & 1 \\
0 & 0 & 0 & J & 1 & 0
\end{pmatrix}. \]  
(A15)

We now solve the recursion problem analytically. The \( n = 0 \) term of the summation in Eq. (A4) corresponds to the initial condition \( v_0 = [1 \ 0 \ 0 \ 0 \ 0 \ 0]^T \). Equation (A13) then leads to

\[
a_n = \frac{1}{3}[(1 + J)^n - (1 - J + J^2)^{\frac{n}{2}}], \]  
(A16)

\[
b_n = \frac{1}{3}[(1 + J)^n - (1 - J + J^2)^{\frac{n}{2}}], \]  
(A17)

\[
f_n = \frac{1}{3}[(1 + J)^n + 2(1 - J + J^2)^{\frac{n}{2}}], \]  
(A18)

\[
c_n = d_n = e_n = 0, \]  
(A19)

when \( n \) is even, and

\[
c_n = \frac{1}{3}[(1 + J)^n + (2J - 1)(1 - J + J^2)^{\frac{n-1}{2}}], \]  
(A20)

\[
d_n = \frac{1}{3}[(1 + J)^n - (1 + J)(1 - J + J^2)^{\frac{n-1}{2}}], \]  
(A21)

\[
e_n = \frac{1}{3}[(1 + J)^n + (2 - J)(1 - J + J^2)^{\frac{n-1}{2}}], \]  
(A22)

\[
a_n = b_n = f_n = 0, \]  
(A23)

when \( n \) is odd. Performing the sum over \( n \), the time evolution operator can finally be written as

\[
U(t) = e^{iJ_0t(1 + J)/4} \left[ p^{231} \left\{ \cos[J_0t(1 + J)/2] - \cos(J_0t\sqrt{1 - J + J^2}/2) \right\} + p^{312} \left\{ \cos[J_0t(1 + J)/2] - \cos(J_0t\sqrt{1 - J + J^2}/2) \right\} \\
+ \frac{2J - 1}{\sqrt{1 - J + J^2}} \sin(J_0t\sqrt{1 - J + J^2}/2) \right] + \frac{1 + J}{\sqrt{1 - J + J^2}} \sin(J_0t\sqrt{1 - J + J^2}/2) \\
- ip^{13} \left\{ \sin[J_0t(1 + J)/2] - \frac{1 + J}{\sqrt{1 - J + J^2}} \sin(J_0t\sqrt{1 - J + J^2}/2) \right\} \\
- ip^{32} \left\{ \sin[J_0t(1 + J)/2] + \frac{2 - J}{\sqrt{1 - J + J^2}} \sin(J_0t\sqrt{1 - J + J^2}/2) \right\} \right]. \]  
(A24)

The mediated gates we search for can be decomposed as

\[ U = U_2 \otimes I, \]  
(A25)

where \( U_2 \) acts on qubits 1 and 3, while \( I \) is the single-qubit identity operator acting on spin 2. Condition (A25) is satisfied when the coefficients of \( p^{231}, p^{312}, p^{13}, \) and \( p^{32} \) in Eq. (A24) all vanish. The solution is given by

\[ J = \frac{J_a}{J_b} = 1, \]  
(A26)

with

\[ \cos(J_0t/2) = \cos(J_0t), \]  
(A27)

\[ \sin(J_0t/2) = -\sin(J_0t). \]  
(A28)

We then solve Eqs. (A27) and (A28) to obtain the mediated gate periods, \( t = T_g \):

\[ J_0T_g = 0, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{4\pi}{3}, \ldots \]  
(A29)

The time evolution operator obtained from Eqs. (A26)–(A28) is given by

\[ U(T_g) = e^{iJ_0T_g/2} [I \cos(J_0t) - ip^{13} \sin(J_0t)]. \]  
(A30)

In this Appendix, we outline the global optimization methods used to solve Eqs. (1) and (2), which act on two qubits. Identical methods can also be used to generate states and construct gates involving more than two qubits.
Equations (4) and (5) can be summarized as follows. An arbitrary two-qubit quantum circuit is formed of units comprised of one entangling gate, $U_2$, sandwiched between single-qubit unitary rotations. One or more of these units can be combined, sequentially, to form a circuit. The single-qubit rotations in this protocol are arbitrary. However, the entangling gate $U_2$ is fixed, with the form shown in Eq. (3).

Three scalar parameters are required, to fully specify an arbitrary single-qubit rotation, up to a global phase factor (e.g., the Euler angle construction). Here, we adopt the ZYX decomposition [30]:

$$U_1(\alpha, \beta, \gamma) = e^{-i\alpha\sigma_z/2}e^{-i\beta\sigma_y/2}e^{-i\gamma\sigma_x/2}.$$ (B1)

In the most general case, the rotations will be applied to both qubits, before and after each exchange gate. The construction can be further simplified by noting that terms such as $(U_1 \otimes U_1)(U_1 \otimes U_1)$ are redundant and can be collapsed into the form $U_1 \otimes U_1$. Thus, up to $6(n + 1)$ rotation angles are required, to specify an arbitrary gate sequence of circuit depth $n$.

Here, we employ global optimization techniques, to search through this large parameter space. We have found that multistart clustering algorithms [48, 49] are particularly effective for solving this problem. We first define an appropriate objective function to be minimized. For generating arbitrary states, as in Eq. (4), we use the infidelity $(1 - f)$ of the desired final state $|\psi_{\text{des}}\rangle$ as the objective function, where

$$f = |\langle \psi_{\text{des}} | U_{\text{actual}} \rangle|^2$$
$$= |\langle \psi_{\text{des}} | (U_1 \otimes U_1)(U_2 \otimes U_1)|00\rangle|^2.$$ (B2)

For generating arbitrary gates, as in Eq. (5), we use the operator error norm $\epsilon$ as the objective function, where

$$\epsilon = ||U_{2,\text{des}} - U_{2,\text{actual}}||.$$ (B3)

The global optimization is performed in two steps. In the first step, we use the multistart algorithm to identify potential candidate solutions. Then, we use these solutions as a first guess in a local Nelder-Mead downhill simplex search [50]. The final outcome generally provides results with very low or very high accuracy. The latter are accepted as valid solutions. We begin our searches using the minimal exchange gate sequence $(n = 1)$. If no valid solutions are obtained for a given sequence length, we increment $n$ by 1 and repeat the procedure. Once an optimal, numerical solution has been obtained, it is sometimes possible to work backwards, to determine the exact rotation angles, as in Figs. 3 [4] and 6 [6]. These identifications can then be confirmed analytically.

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