Exact Ground States of Frustrated Spin-1 Ising-Heisenberg and Heisenberg Ladders in a Magnetic Field

Jozef Strečka, Frédéric Michaud, and Frédéric Mila

1 Institute of Physics, Faculty of Science, P. J. Šafárik University, Park Angelinum 9, 040 01, Košice, Slovakia
2 Institute of Theoretical Physics, Ecole Polytechnique Fédérale de Lausanne (EPFL), CH-1015 Lausanne, Switzerland

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Ground states of the frustrated spin-1 Ising-Heisenberg two-leg ladder with Heisenberg intra-rung coupling and only Ising interaction along legs and diagonals are rigorously found by taking advantage of local conservation of the total spin on each rung. The constructed ground-state phase diagram of the frustrated spin-1 Ising-Heisenberg ladder is then compared with the analogous phase diagram of the fully quantum spin-1 Heisenberg two-leg ladder obtained by density matrix renormalization group (DMRG) calculations. It is demonstrated that both investigated spin models exhibit quite similar magnetization scenarios, which involve intermediate plateaux at one-quarter, one-half and three-quarters of the saturation magnetization.

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I. INTRODUCTION

Over the last few decades, quantum spin ladders have been actively studied mainly in connection with spin-liquid behaviour, quantum critical points and superconductivity under hole doping of some cuprates (see Ref. [1] for a review). In particular, the frustrated spin-1/2 Heisenberg two-leg ladder exhibits a striking dimerized ground state [2] and a low-temperature magnetization process with an intermediate plateau and magnetization jumps [3].

Another challenging topic of current research interest consists of the theoretical investigation of related models such as the quantum spin-1 Heisenberg two-leg ladder [4,5]. The main goal of the present work is to find the exact ground states of a simpler spin-1 Ising-Heisenberg ladder and to contrast them with the respective ground states of the pure quantum spin-1 Heisenberg ladder. Note that the former model is analytically tractable using the procedure developed in Refs. [4,5] and it brings insight into the relevant behaviour of the latter not fully integrable model.

II. FRUSTRATED ISING-HEISENBERG LADDER

Consider first the frustrated spin-1 Ising-Heisenberg ladder with the Heisenberg intra-rung interaction and the unique Ising interaction along the legs and diagonals. The total Hamiltonian of the investigated model is given by

$$\hat{H} = \sum_{i=1}^{N} \left[ J \hat{S}_{1,i} \cdot \hat{S}_{2,i} + J_1 (\hat{S}_{1,i}^z + \hat{S}_{2,i}^z) (\hat{S}_{1,i+1}^z + \hat{S}_{2,i+1}^z) - h(\hat{S}_{1,i}^z + \hat{S}_{2,i}^z) \right],$$

where $\hat{S}_{\alpha,i} \equiv (\hat{S}_{\alpha,i}^x, \hat{S}_{\alpha,i}^y, \hat{S}_{\alpha,i}^z)$ denotes spatial components of the spin-1 operator, the former suffix $\alpha = 1$ or 2 enumerates the leg and the latter suffix specifies a lattice position within a given leg. The coupling constant $J$ denotes the isotropic Heisenberg intra-rung interaction, the parameter $J_1$ determines the Ising interaction along the legs and diagonals, $h$ is an external magnetic field.

For further convenience, let us introduce the spin operator $\hat{T}_i = \hat{S}_{1,i} + \hat{S}_{2,i}$, which corresponds to the total spin angular momentum of the $i$th rung. It can be easily proved that the operators $\hat{T}_i^z$ and $\hat{T}_i^z$ commute with the Hamiltonian [1], i.e. $[\hat{H}, \hat{T}_i^z] = [\hat{H}, \hat{T}_i^z] = 0$, which means that the total spin of a rung and its $z$ component represent conserved quantities with well defined quantum numbers. The complete energy spectrum of the frustrated spin-1 Ising-Heisenberg ladder then readily follows from the relation

$$E = -2NJ + \frac{J_1}{2} \sum_{i=1}^{N} T_i (T_i + 1) + J_1 \sum_{i=1}^{N} T_i^z T_{i+1}^z - h \sum_{i=1}^{N} T_i^z,$$

which depends just on the quantum numbers $T_i = 0, 1, 2$ and $T_i^z = -T_i, T_i + 1, \ldots, T_i$ determining the eigenvalues of the total spin of the $i$th rung and its $z$th spatial projection, respectively. Using this procedure, the spin-1 Ising-Heisenberg two-leg ladder has been rigorously mapped to some classical chain of composite spins and accordingly, we can readily find all available ground states by looking for the lowest-energy state of Eq. (2).

III. FRUSTRATED HEISENBERG LADDER

Next, we will also consider the frustrated spin-1 Heisenberg two-leg ladder defined by the Hamiltonian

$$\hat{H} = \sum_{i=1}^{N} \left[ J \hat{S}_{1,i} \cdot \hat{S}_{2,i} + J_1 (\hat{S}_{1,i} + \hat{S}_{2,i}) (\hat{S}_{1,i+1} + \hat{S}_{2,i+1}) \right],$$
which represents the pure quantum analogue of the frustrated spin-1 Ising-Heisenberg ladder discussed previously. Taking advantage of the definition for the total spin of each rung, the Hamiltonian (3) of frustrated spin-1 Heisenberg two-leg ladder can be rewritten into the form

$$\hat{H} = -2NJ + \sum_{i=1}^{N} T_{i}^{z} + J_{i}^{z} \sum_{i=1}^{N} T_{i} \cdot T_{i+1} + h \sum_{i=1}^{N} T_{i}^{z} \cdot T_{i}^{x}.$$  \hspace{1cm} (4)

According to Eq. (4), the frustrated spin-1 Heisenberg ladder can be rigorously decomposed into the direct sum of quantum spin chains with spin 0, 1 or 2 at each site. The ground state of such a system can be shown to be either a homogeneous chain, with the same spin at all sites, or a chain with alternating spins on every other site. The energy of the different chains, obtained either analytically or using DMRG simulations, the exact ground-state phase diagram of the frustrated spin-1 Heisenberg ladder can be constructed [15].

IV. RESULTS AND DISCUSSION

The constructed ground-state phase diagrams of the frustrated spin-1 Ising-Heisenberg and Heisenberg ladders are depicted in Fig.1 and Fig.2, respectively. The ground states of the spin-1 Ising-Heisenberg ladder can be discerned according to the zth projection of the total spin on two consecutive rungs \([T_{i}^{z}, T_{i+1}^{z}]\), because \(T_{i} = |T_{i}^{z}|\) holds for all available ground states. The quantum ground states \([0,0], [0,1], [1,1], [2,1]\) and \([2,0]\) represent six different phases, whereas \(T_{1} = T_{1}^{z} = 0\) implies a formation of two singlets on the i-th rung, \(T_{i} = |T_{i}^{z}| = 1\) entails only one singlet, and \(T_{i} = |T_{i}^{z}| = 2\) denotes fully polarized rungs without singlets. Besides, two classical ground states \([2,2]\) and \([2,-2]\) are pertinent to the ferromagnetic and antiferromagnetic ordering. The magnetization normalized with respect to its saturation equals zero for \([0,0]\) and \([2,-2]\), one-quarter for \([0,1]\) and \([2,-1]\), one-half for \([1,1]\) and \([2,0]\), three-quarters for \([2,1]\) and unity for \([2,2]\). Altogether, it can be concluded that the frustrated spin-1 Ising-Heisenberg ladder always exhibits a stepwise magnetization curve, which involves intermediate plateaux at one-quarter, one-half and three-quarters of the saturation magnetization that are however of different origin.

It is quite clear from Fig.1 that the ground-state phase diagram of the pure quantum Heisenberg ladder exactly coincides with that of the Ising-Heisenberg ladder just for sufficiently weak inter-rung interactions \(J_{1}/J \leq 0.5\). A relatively good agreement between both phase diagrams is still observed in the parameter space \(0.63 \geq J_{1}/J \geq 0.5\), where the gapless phase \([2]\) with a continuously varying magnetization is present between the intermediate plateaux instead of direct magnetization jumps. The gapless phase \([2]\) corresponds to the

Luttinger-liquid phase of the effective spin-2 quantum Heisenberg chain. Finally, the gapped Haldane phase of the effective spin-2 quantum Heisenberg chain emerges for \(J_{1}/J \geq 0.63\) at sufficiently low fields.

In conclusion, we have rigorously found the ground states of the frustrated spin-1 Ising-Heisenberg and Heisenberg ladders in a magnetic field. It has been demonstrated that the Ising-Heisenberg ladder always exhibits a stepwise magnetization curve with three different intermediate plateaux, while the same quantum ground states can be identified in the pure quantum Heisenberg ladder provided the intra-rung coupling is sufficiently large.

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