An Axially Symmetric Transitioning models with Observational Constraints

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Abstract

In this study, we have demonstrated the expansion history of an axially symmetric Bianchi type-I model of the universe. Our model as of now presents an accelerating universe, which had been in the decelerating phase in the past. Roles of the two crucial Hubble $H(z)$ and deceleration $q(z)$ parameters are examined. The energy parameters of the universe are estimated with the help of the latest observational Hubble data (46-data points) and Pantheon data (the latest compilation of SNIa with 40 binned in the redshift range $0.014 \leq z \leq 1.62$). We also discuss the stability analysis of the model by state finder diagnosis. The analysis reveals that in late time, the model is a quintessence type and points towards the ΛCDM model. Our developed model agrees with observational findings in a proper way. We have discussed some of the physical aspects of the model.

Keywords: LRS Bianchi-I, ΛCDM Model, Statefinder diagnosis.
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1 Introduction

Type Ia Supernova observations, announced in 1998, indicate that the physical universe is accelerating which had been in the decelerating phase in the past[1, 2]. The background behind these observations are the studies of supernova exploration and explosions. SN Ia supernovae are standard candles (populations II and III stars). At shorter redshift, nearby SN Ia help in the determination of Hubble constant $H$ whereas large red-shifted long distant objects provide an estimation of deceleration constant $q$[3, 4, 5]. The Supernova Cosmology Project was launched
in 1988, in which magnitude-redshift relations were used to estimate the cosmological parameters. In this project, cosmological parameters were estimated with the help of more than 75 SN Ia (0.18 ≤ z ≤ 0.86) magnitude-redshift relation. Authors\cite{6} discussed that the cosmological constant-based dark energy is dominant over baryon matter-energy. Authors \cite{7} worked with the 33 additional high-red shifted supernovae and developed a confidence region of an accelerating universe. They have also searched for a possibility of a low-mass Λ = 0 cosmology. In this context, Riess et al. \cite{8} had also done an independent study of 10 high-redshift supernovae and found that the universe is accelerating. In this direction, CMB anisotropy \cite{9}, SN Ia magnitude-redshift relation \cite{10,11}, baryonic acoustic oscillation (BAO), peak length scale \cite{12,13,14,15}, and Hubble parameter versus redshift measurements \cite{16,17,18,19} point to the accelerating universe.

Most recent, Hubble parameters for various red-shifts have been determined with the help of cosmic chronometric (CC) and BAO techniques \cite{16,19,20}. They provided cosmic “deceleration-acceleration transition” when the universe is thought to have the presence of dark energy in abundance along with non-relativistic baryon matter. This redshift transition were discussed in \cite{19,21,22}. Hubble H(z) data sets are also used to derive energy parameters Ω_m and Ω_de \cite{23,24,25,26,27,28,29,30} and are comparable with the results with SNIa data, BAO and CMB observations. They estimated the present value of Hubble constant $H_0$ \cite{31,32}. The derived values of $H_0$ are compatible with Huchra’s $H_0$ compilation statistical analysis \cite{17}. All of the above findings were discovered by a simple analysis of “55 supernova data points”.

46 Hubble parameters H(z) data set in the redshifts range 0 ≤ z ≤ 2.36 were used to estimate present value of Hubble constant along with baryon and dark energy parameters \cite{33,34,35,36}. The authors of these references have also analyzed the cumulative effect of H(z) data, SN Ia pantheon compilation data, and BAO data in the estimation of model parameters. They have used the χ² minimization technique. SN Ia pantheon compilation data set is given in Scolnic et al. \cite{37}.

The late-time acceleration of the universe is described in General relativity (GR) by dark energy density along with matter density in Einstein’s field equation \cite{38,39,40,41}. It is suggested that dark energy of repulsive anti gravitating origin present in abundance in our universe, which is responsible for the change in the mode of the universe. In some modified theories of gravity accelerated expansion of the universe has been explained without applying the DE component in a different way of thinking. A variety of gravity theories such as f(R) \cite{42,43,44}, f(T) \cite{45}, and f(G) \cite{46}, f(Q,T) \cite{47,48}, f(Q) \cite{49,50,51}, f(R,G) \cite{52}, f(T,B) gravity \cite{53}, and Einstein-Gauss-Bonnet theory \cite{54,55,56,57,58,59} were proposed. In the same direction, Harko et al. \cite{60} developed f(R,T) gravity theory. In these modified theories, non-linear curvature scalar and energy traces describe acceleration in the universe. Their origin lies in the corrections towards curvature and energy trace.

Off late considerable number of Bianchi type-I cosmological models that fit well with the latest observations and exhibit acceleration in the universe were surfaced in the literature \cite{61,62,63,64,65,66}. It is a well-known fact that neutrino viscosity develops anisotropy in the universe and was in abundance during the primordial fireball(P.F.) \cite{67,68}. As of now the contribution of neutrinos and CMBR(Cosmic Microwave Background Radiations) in the total content of the universe are 0.1% – 0.5% \cite{69}. They are leftover remains of the P.F.
just after the investigation of the CMBR phenomenon, Spatially homogeneous and anisotropic cosmological models attracted the scientific community [70, 71]. It is 1962, when Bianchi type-I cosmological models were introduced in the book “An Introduction to Current Research” [70]. Sometimes they were called Heckmann and Schucking models in the author’s name. The Bianchi-type models are the best and simplest anisotropic models, which completely describe the anisotropic effects. The advantages of these anisotropic models are that they play an important role in describing the history of the early Universe and that they contribute to the formation of more generalized cosmological models than isotropic FRW models [72, 73, 74].

Locally rotational symmetric (LRS) spacetimes are a sub-class of Bianchi type-I models in the sense that they are spatially homogeneous and describe symmetry along a particular direction (axis). They may also be called axially symmetric spacetimes. A considerable number of present days accelerating universe models on LRS spacetimes have appeared in the literatures [75, 76, 77, 78, 79, 80, 81].

In this study, we have demonstrated the expansion history of an axially symmetric Bianchi type-I model of the universe. Our model as of now presents an accelerating universe, which had been in the decelerating phase in the past. Roles of the two crucial Hubble \( H(z) \) and deceleration \( q(z) \) parameters are examined. The energy parameters of the universe are estimated with the help of the latest observational Hubble data (46-data points) and Pantheon data (the latest compilation of SNIa with 40 binned in the redshift range \( 0.014 \leq z \leq 1.62 \)). We also discuss the stability analysis of the model by state finder diagnosis. The analysis reveals that in late time, the model is a quintessence type and points towards the \( \Lambda \) CDM model. Our developed model agrees with observational findings in a proper way. We have discussed some of the physical aspects of the model.

The paper is structured as follows: The present cosmological scenario is discussed in section I. In section 2 we present a model and field equation. In section 3, we constrain the model parameters using Hubble data set \( H(z) \) and Pantheon data set. We have also discussed the Luminosity Distance, Apparent Magnitude, and Distance Modulus of the model. Section 4 contains the age of the universe. Section 5 shows the behavior of the deceleration parameter. Evolutionary trajectories are discussed in section 6. Conclusions are mentioned in section 7.

## 2 Metric and the Field Equations

We consider the LRS Bianchi type I metric of the form

\[
ds^2 = -A(dx)^2 - B^2(dy^2 + dz^2) + dt^2
\]  

(1)

Here, “A and B” are the time-dependent metric functions. In general relativity, Einstein’s field equation with the cosmological constant is:

\[
\Lambda g_{ij} + R_{ij} - \frac{1}{2}g_{ij}R = -T_{ij}
\]  

(2)

We consider the energy–momentum tensor in the form \( T_{ij} = (p_m + \rho_m)u_iu_j + p_mg_{ij} \), where \( p_m \) and \( \rho_m \) “represents the matter pressure and matter energy density”. For the LRS Bianchi type-I
universe, the field equations are as follows:

\[ 2 \frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} = -p_m + \Lambda \]  
(3)

\[ \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{A} \dot{B}}{AB} = -p_m + \Lambda \]  
(4)

\[ 2 \frac{\dot{A} \dot{B}}{AB} + \frac{\dot{B}^2}{B^2} = \rho_m + \Lambda \]  
(5)

The volume for the model is given by \( V = AB^2 \), the scale factor is assumed as \( a = (AB^2)^{1/3} \) and the Hubble’s parameter can be written as \( H = \frac{1}{3} \left( \frac{\dot{A}}{A} + 2 \frac{\dot{B}}{B} \right) = \frac{\dot{a}}{a} \) in order to obtained the solutions of the field equations.

From Eq. (3)-Eq. (4), we have

\[ \frac{d}{dt} \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) + \left( \frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) \left( \frac{\dot{A}}{A} + 2 \frac{\dot{B}}{B} \right) = 0 \]  
(6)

On integration, we get

\[ \frac{\dot{A}}{A} - \frac{\dot{B}}{B} = \frac{c_1}{a^3} \]  
(7)

Also, we have

\[ \frac{\dot{A}}{A} + 2 \frac{\dot{B}}{B} = 3 \frac{\dot{a}}{a} \]  
(8)

Solving Eq. (7) and Eq. (8), we get

\[ \frac{\dot{A}}{A} = \frac{\dot{a}}{a} + \frac{2 c_1}{3 a^3} \]  
(9)

\[ \frac{\dot{B}}{B} = \frac{\dot{a}}{a} - \frac{1 c_1}{3 a^3} \]  
(10)

Using Eq. (9), Eq. (10) in Eq. (5), we get

\[ H^2 = \frac{1}{3} \left( \rho_m + \Lambda + \frac{1}{3 a^6} \right) \]  
(11)

For barotropic matter, the energy conservation law holds i.e., \( \rho_m + 3H(p_m + \rho_m) = 0 \). The present universe is “dust filled for which \( p_m = 0 \)” and \( \rho_m \propto a^{-3} \). Using the relation \( \frac{\dot{a}}{a} = 1 + z \) between scale factor \( a \) and redshift \( z \), we get \( \rho_m = \rho_{m0}(1 + z)^3 \). The term \( \frac{1}{3} \frac{c_1^2}{a^6} = \rho_{\sigma0}(1 + z)^6 = \rho_{\sigma} \) represents the anisotropy energy density. For the dust filled universe \( (p_m = 0) \), the density parameters are defined as \( \Omega_m = \frac{\rho_m}{\rho_c} \) and \( \Omega_\sigma = \frac{\rho_\sigma}{\rho_c} \), where \( \rho_c = \frac{3H^2}{8\pi G} \approx 8\pi G \approx 1 \). Thus, Eq. (11) can also be written as

\[ H^2 = H_0^2 \left[ (1 + z)^3 \Omega_{m0} + \Omega_{\Lambda0} + (1 + z)^6 \Omega_{\sigma0} \right] \]  
(12)

For \( z = 0 \), the relationship between energy parameters is obtained from Eq. (12) as:

\[ 1 = \Omega_{m0} + \Omega_{\Lambda0} + \Omega_{\sigma0} \]  
(13)
The deceleration parameter (DP) for the model can be expressed as:

\[ q = \frac{(1 + z)^3\Omega_m - 2\Omega_\Lambda + 4(1 + z)^6\Omega_\sigma}{2[(1 + z)^3\Omega_m + \Omega_\Lambda + (1 + z)^6\Omega_\sigma]} \] (14)

The Hubble parameter \( H \) and deceleration parameter \( q \) is the important physical quantities for describing the evolution of the universe. For explaining the expansion of the Universe, \( H \) plays a vital role and is also very useful in the estimation of the age of the universe. On the other hand, the deceleration parameter describes the phase transition (acceleration or deceleration) during the evolution of the universe.

### 3 Observational Constraints

#### 3.1 Observational Hubble Data of 46 Data Set of \( H(z) \)

In this segment, we present the observational data as well as the statistical methodological analysis that was used to constrain the model parameters of the derived Universe see in (Fig.1, 2). Here we applied 46 \( H(z) \) observational data points in the ranges \( 0 \leq z \leq 2.36 \), which were obtained by using the “cosmic chronometric approach (CCA)” [18, 21].

To determine the best-fitting values and limits for a fitted model, we use the \( \chi^2 \) statistic as

\[ \chi^2(p) = \sum_{i=1}^{46} \frac{(H_{th}(i) - H_{ob}(i))^2}{\sigma(i)^2}. \] (15)

Our estimated values for various cosmological parameters for minimum \( \chi^2 \) have been computed as \( H_0 = 69.48, \Omega_m = 0.2548, \Omega_\Lambda = 0.7411, \Omega_\sigma = 0.0005, \chi^2 = 29.1136 \).

Here \( p \) is the set of model parameters, where \( (p = H_0, \Omega_m) \). The \( \chi^2 \) expression in Eq. (15) holds for the \( H(z) \) measurements listed in Table 1. This table contains a data set of the observed values of the Hubble parameters \( H(z) \) versus redshift \( z \) with a possible error obtained using the different age approach by various cosmologists.
Table 1: “The behaviour of Hubble parameter \( H(z) \) with redshift

| S.No | Z  | \( H(\text{Obs}) \) | \( \sigma_i \) | References | S.No | Z  | \( H(\text{Obs}) \) | \( \sigma_i \) | References |
|------|----|----------------|-----------|---------|------|----|----------------|-----------|---------|
| 1    | 0  | 67.77          | 1.30      | [82]   | 24   | 0.4783        | 0.009     | [22]   |
| 2    | 0.07 | 69          | 19.6      | [83]   | 25   | 0.48          | 0.01      | [22]   |
| 3    | 0.09 | 69          | 12        | [84]   | 26   | 0.51          | 0.01      | [22]   |
| 4    | 0.01 | 69          | 12        | [85]   | 27   | 0.57          | 0.01      | [22]   |
| 5    | 0.12 | 68.6        | 26.2      | [86]   | 28   | 0.593         | 0.01      | [22]   |
| 6    | 0.17 | 83          | 8         | [87]   | 29   | 0.60          | 0.01      | [22]   |
| 7    | 0.179 | 75          | 4         | [88]   | 30   | 0.61          | 0.01      | [22]   |
| 8    | 0.1993 | 75         | 5         | [89]   | 31   | 0.68          | 0.01      | [22]   |
| 9    | 0.2  | 72.9        | 29.6      | [90]   | 32   | 0.73          | 0.01      | [22]   |
| 10   | 0.24 | 79.7        | 2.7       | [91]   | 33   | 0.781         | 0.01      | [22]   |
| 11   | 0.27 | 77          | 14        | [92]   | 34   | 0.875         | 0.01      | [22]   |
| 12   | 0.28 | 88.8        | 36.6      | [93]   | 35   | 0.88          | 0.01      | [22]   |
| 13   | 0.35 | 82.7        | 8.4       | [94]   | 36   | 0.9           | 0.01      | [22]   |
| 14   | 0.352 | 83         | 14        | [95]   | 37   | 1.037         | 0.01      | [22]   |
| 15   | 0.38 | 81.5        | 1.9       | [96]   | 38   | 1.3           | 0.01      | [22]   |
| 16   | 0.3802 | 83       | 13.5      | [97]   | 39   | 1.363         | 0.01      | [22]   |
| 17   | 0.4  | 95          | 17        | [98]   | 40   | 1.43          | 0.01      | [22]   |
| 18   | 0.4004 | 77        | 10.2      | [99]   | 41   | 1.53          | 0.01      | [22]   |
| 19   | 0.4247 | 87.1       | 11.2      | [100]  | 42   | 1.75          | 0.01      | [22]   |
| 20   | 0.43 | 86.5        | 3.7       | [101]  | 43   | 1.965         | 0.01      | [22]   |
| 21   | 0.44 | 82.6        | 7.8       | [102]  | 44   | 2.3           | 0.01      | [22]   |
| 22   | 0.4497 | 92.8       | 12.9      | [103]  | 45   | 2.34          | 0.01      | [22]   |
| 23   | 0.47 | 89          | 40.6      | [104]  | 46   | 2.36          | 0.01      | [22]   |

3.2 Pantheon Data

In this investigation, we use “Pantheon data (the latest compilation of SNIa with 40 binned in the redshift range \( 0.014 \leq z \leq 1.62 \))” [37]. In this scenario, \( \chi^2 \) for Pantheon data was calculated by using the following formula.

\[
\chi^2(H_0, \Omega_m) = \sum_{i=1} \frac{\left( \mu_{\text{th}}(z(i), H_0, \Omega_m) - \mu_{\text{obs}}(i) \right)^2}{\sigma(i)^2},
\]

(16)

where \( \mu_{\text{th}} \) denoted the theoretical distance modulus and \( \mu_{\text{obs}} \) represents the observed “distance modulus” with the standard error \( \sigma(i)^2 \). In earlier work [38, 40] researchers presented a technique for estimating the present values of the energy parameters \( \Omega_m, \Omega_{\Lambda}, \) and \( \Omega_{\sigma} \) by “comparing the theoretical and observed results” with the help of the \( \chi^2 \) statistic.

Our estimated values for various cosmological parameters for minimum \( \chi^2 \) have been computed as \( H_0 = 70.02, \Omega_m = 0.2728, \Omega_{\Lambda} = 0.7262, \Omega_{\sigma} = 0.001, \chi^2 = 562.242. \)

We have formed a new data set by joining OHD and Pantheon datas and estimated an other set of values for these parameters with minimum \( \chi^2 \). In this case we obtaine \( H_0 = 70.17, \)
\( \Omega_{m0} = 0.2542, \Omega_{\Lambda0} = 0.745731, \Omega_{s0} = 0.000069, \chi^2 = 593.356. \)

Figure 1: Two-dimensional contours for (OHD, Pantheon, OHD+Pantheon data set) at 1\( \sigma \), 2\( \sigma \) and 3\( \sigma \) confidence regions, bounded with latest 46 OHD and Pantheon data.

Figure 2: Hubble parameter versus redshift error bar plot. The plot of Hubble rate \( H(z)/(1+z) \) versus \( z \).

The Hubble parameter is clearly not a constant as demonstrated in the figures. Over the redshift, it varies slowly. Using the differential age technique and galaxy clustering method, many physicists have computed the values of the Hubble constant at various redshifts. They have described a variety of observed Hubble constant \( H_{ob} \) values as well as the corrections in the range \( 0 \leq z \leq 2.36 \). The observed and theoretical values are found to agree quite well, indicating
that our model is stable. We can see in Fig 2a that H grows as the redshift increases. The
dots in the figures represents 46 observed Hubble constant $H_0$ values with corrections, whereas
the linear curve represents the theoretical graphs of the Hubble constant. Fig2b represents, the
error bar plot of Hubble rate $H(z)/(1 + z)$ versus the redshift $z$. It is clear that using a joint
dataset gives raise to a better fit of the data. The dots with bars indicate the experimental
observations of H(z) data, $H_0$ is the present value of the Hubble constant.

3.2.1 Luminosity Distance

The luminosity distance relation \[94, 95\] is a useful method for examining the evolution of the
universe. The equation for luminosity distance ($D_L$) is defined in terms of redshift, as the light
coming out of a distant luminous body gets red-shifted as the cosmos expands. The luminosity
distance is used to calculate a source’s flux. It is written as

$$D_L = a_0(1 + z)r$$  \hspace{1cm} (17)

Here $r$ is denoted as the radial co-ordinate of the source. The luminosity distance can be
expressed as \[26\].

$$D_L = (1 + z)a_0c \int_0^z \frac{dz}{H(z)}$$  \hspace{1cm} (18)

We noticed that the luminosity distance is an increasing function of redshift.

3.2.2 Apparent Magnitude and Distance Modulus in the model

The luminosity distance is related to the distance modulus by the formula below.

$$\mu = M - m_b = 5\log_{10} \frac{D_L}{Mpc} + 25$$  \hspace{1cm} (19)

where “$m_b$ and $M$ are the apparent and absolute magnitude of the source”, respectively.

We use the equation to calculate the $D_L$ for a supernova at very small redshift.

$$D_L = \frac{cz}{H_0}$$  \hspace{1cm} (20)

There are many supernovae with low red shift whose “apparent magnitudes” are known in
the literature. The common “absolute magnitude (M)” for Pantheon data is determined by
this. Using Eqs. \[19\] and \[20\] in our previous study \[96, 97\], we derived $(M)$ as follows:

$$M = 5\log_{10} \frac{H_0}{.026c} - 8.92$$  \hspace{1cm} (21)

We obtained the expression for the apparent magnitude by using Eqs. \[18\] and \[19\] as:

$$m_b = 16.08 + 5\log_{10}[\frac{1 + z}{.026c} \int_0^z \frac{dz}{h(z)}]$$  \hspace{1cm} (22)

We are numerically solving Eqs.\[12\] & \[21\] and by using Pantheon data (the latest compi-
lation of SN Ia with 40 binned in the redshift range 0.014 $\leq z \leq 1.62$. According to our model,
we derive the corresponding theoretical values. Figures 3 show the closeness of observational and theoretical results, demonstrating the validity of our model.

Figure 3 shows the progression of $\mu(z)$ for the best-fit values of model parameters. Here, $\mu(z)$ represents the distance modulus, which is the difference between the “apparent magnitude and the absolute magnitude” of the observed supernova, is given by $\mu(z) = 25 + 5\log_{10}(d_L/Mpc)$, where $d_L$ is the luminosity distance. The blue dots in this plot correspond to the Pantheon data (the latest compilation of SNIa with 40 binned in the redshift range $0.014 \leq z \leq 1.62$) for error bar plot.

4  Age of the Universe

The age of the cosmos is calculated as

$$dt = -\frac{dz}{(1+z)H(z)} \implies \int_t^{t_0} dt = -\int_z^0 \frac{1}{(1+z)H(z)} dz$$

(23)

Using Eq.(12) and Eq.(23), we get

$$t_0 - t = \int_0^z \frac{1}{H_0(z+1)\sqrt{\Omega_{\sigma0}(z+1)^6 + \Omega_{m0}(z+1)^3 + \Omega_{\Lambda0}}} \, dz$$

(24)

The present age of universe is represents as $t_0$ and it is written by

$$t_0 = \lim_{x \to \infty} \int_0^x \frac{1}{H_0(z+1)\sqrt{\Omega_{\sigma0}(z+1)^6 + \Omega_{m0}(z+1)^3 + \Omega_{\Lambda0}}} \, dz$$

(25)

Integrating Eq.(25), we get

$$H_0 \, t_0 = 0.97786$$

(26)

In this paper, we have estimated the numerical value of $H_0$ as $0.6948 \, Gyr^{-1} \sim 69.48 \, km/s/Mpc^{-1}$. Therefore, the present age of universe for the derived model is estimated as $t_0 = \frac{0.97786}{H_0} = 13.79 \, Gyr$s.
Figure 4 depicts the fluctuation of $H_0(t_0 - t)$ as a function of redshift $z$. According to WMAP data, the empirical value of the universe’s current age is $t_0 = 13.73^{+0.13}_{-0.17}$.

![Figure 4: Plot of $H_0(t_0 - t)$ versus $z$.](image)

Figure 4 shows, the plot of the age of the universe versus redshift in our derived model. In several cosmological research, the age of the universe is estimated as $14.46 \pm 0.8$ Gyrs [99], $14.3 \pm 0.6$ Gyrs [100] and $14.5 \pm 1.5$ Gyrs [101].

5 Deceleration Parameter

Figure 5 depicts the dynamics of the decelerating parameter ($q$) with respect to $z$ for “OHD, Pantheon, and OHD + Pantheon”. As we know the early universe was in decelerated era $q$, with the signature flipping at $z_t = 0.809$ for (OHD+Pantheon), at $z_t = 0.781$ for (OHD), at $z_t = 0.705$ for (Pantheon), due to the dominance of dark energy in the universe. As a result, the current universe evolves with a negative sign of $q$, causing the accelerated expansion of the universe. In this way, our developed model depicts a transition from the early deceleration phase to the current speeding phase. Here $q_0 = -0.42$ is the current value of the deceleration parameter. Furthermore, we find that the deceleration parameter will remain negative in the future, $z \to -1, q \to -1$.

![Figure 5: Plot of $q$ versus $z$.](image)
6 Statefinders

In this section, we focus on the diagnosis of the statefinder. The Hubble parameter $H$, which represents the universe’s expansion rate, and the deceleration parameter $q$, which represents the rate of acceleration/deceleration of the expanding cosmos, are two well-known geometrical variables that characterize the universe’s expansion history. They only depend on the scaling factor $a$. However, with the enhancing amount of cosmological models and the remarkable increase in the accuracies of cosmological observational data, these two parameters are no longer sensitive enough to discriminate between different models. As a consequence, the statefinder diagnosis was developed to distinguish between an increasing number of cosmological models containing dark energy. Since different cosmological models have different evolutionary paths in the $(r-s)$ plane, the statefinder diagnostic is likely a good way distinguish cosmological models. The remarkable property is that $\left( r - s \right) = \left( 1 \; 0 \right)$ corresponds to the $\Lambda CDM$ model seen in figure 6a.

There are so many dark energy models, for example, the quintessence, the phantom, the Chaplygin gas, the holographic dark energy models, and the interacting DE models, which have been studied in the literature [102, 103, 104, 105, 106], one can clearly identify the distance from a given cosmological model to a $\Lambda CDM$ model in the $(r-s)$ plane.

\begin{align*}
r &= \frac{\dddot{H}}{H^3} + 3\frac{\dddot{H}}{H^2} + 1 = \frac{\Omega_m(1+z)^3 + \Omega_{\Lambda 0} + 10\sigma_0(1+z)^6}{\Omega_m(1+z)^3 + \Omega_{\Lambda 0} + \Omega_{\sigma 0}(1+z)^6} \\
s &= \frac{r - 1}{3(q - \frac{1}{2})} = \frac{2\sigma_0(1+z)^6}{-\Omega_{\Lambda 0} + \Omega_{\sigma 0}(1+z)^6}
\end{align*}  

(27)  

(28)

It’s worth noting that different combinations of $r$ and $s$ indicate different DE models, as referenced in [102, 107, 108]. Figure 6(a) depicts that our model lies in quintessence region ($r < 1, s > 0$) and CG region ($r > 1, s < 0$) and also meets SCDM ($r = 1, s = 1$) point and LCDM ($r = 1, s = 0$) point.

![Figure 6: (a) Plot of $r$ vs $s$. (b) Plot of $r$ vs $q$.](image)

Figure 6(b) shows the evolutionary trajectories in the $(r-q)$ plane. In the figure the fixed point ($q = -1, r = 1$) shows the SS model (de sitter expansion) and the horizontal line represents the evolution of the trajectory corresponding to $\Lambda CDM$ ($r = 1, q = 0$). The trajectories corresponding to our model start from the fixed point $\Lambda CDM$ and passes through the SS point.
With the currently available data points, we have analyzed the Pantheon and OHD data and constrain several cosmological parameters. We would like to point out that our research does not take the effects of correlation. The goal of this study was to concentrate on a few key theoretical issues that have received insufficient attention in the literature. The 46 OHD data points and Pantheon data (the latest compilation of SNIa with 40 binned in the redshift range $0.014 \leq z \leq 1.62$) were used in this analysis. In particular, we have examined the estimated values of cosmological parameters from the OHD+Pantheon data sets. While the parameter estimation errors decreased dramatically for all the data sets.

We have summarise the main findings once more:

- Fig.1 shows the 2-dimensional joint contours (OHD+Pantheon), at 68% and 95% confidence regions, bounded with latest.

- Fig.2 represents the error bar plots for OHD and Pantheon data sets. In both graphs, the dots represent observed values with corrections, whereas red lines are the present model compared with the $\Lambda CDM$ model shown in black dashed lines.

- Figure 3 shows the progression of $\mu(z)$ for the best-fit values of model parameters

- Our model is based on latest observational findings of 46 OHD data and Pantheon data (the latest compilation of SNIa with 40 binned in the redshift range $0.014 \leq z \leq 1.62$). Figure 4 depicts the age-redshift plot of the universe in our derived model. In this paper, we have estimated the numerical value of $H_0$ as $0.6948 \text{ Gyr}^{-1} \sim 69.48 \text{ km s}^{-1} \text{ Mpc}^{-1}$. Therefore, the present age of universe for the derived model is estimated as $t_0 = \frac{0.97786}{H_0} = 13.79 \text{ Gyrs}$.

- The deceleration parameter $q$ fluctuates with $z$ from positive to negative, as seen in the graph. This shows a transition from early slowdown to the current acceleration of the universe (see fig 5).

- We also use the $(r, s)$ and $r, q$ planes parameters to diagnose the DE model geometrically, as shown in fig.6. Our derived model initially shows a Chaplygin gas (CG) type DE model, which later on evolves into a quintessence DE model at a few points. The model, later on, reverts back again in CG. Interestingly, the model deviates significantly from the point $(r, s) = (1, 0)$ and it does not coincide with $\Lambda CDM$ (see Fig. 6).

- The estimated results on the basis of $\chi^2$ statistic using OHD, Pantheon, and OHD+Pantheon datas are shown in the following Table-2:

| Datasets          | $H_0$  | $\Omega_{m0}$ | $\Omega_{\Lambda0}$ | $\Omega_{s0}$ | $\chi^2$ | $s_1$  |
|-------------------|--------|----------------|----------------------|--------------|---------|--------|
| OHD               | 69.48  | 0.2584         | 0.7411               | 0.0005       | 29.1136 | 0.781  |
| Pantheon          | 70.02  | 0.2728         | 0.7262               | 0.001        | 562.242 | 0.705  |
| OHD + Pantheon    | 70.17  | 0.2542         | 0.745731             | 0.000069     | 593.356 | 0.809  |
Finally, we can state that all of the preceding conclusions in LRS Bianchi-I are good and consistent with recent cosmological observations.

**CRediT authorship contribution statement**

Vinod Kumar Bhardwaj: Conceptualization, Ideas, Formulation, Writing – original draft. Archana Dixit: Writing-review & editing, Conceptualization. Rita Rani: Analysis of figures and review of literatures. G. K. Goswami: Formal analysis, Writing – review & editing. Anirudh Pradhan: Methodology, Supervision, Final writing & editing.

**Declaration of competing interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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