Spatial distribution of the electric field of laser pulses in optical fibers having a gradient profile of the refractive index. Linear case

I Bozhikoliev¹, V Slavchev¹, V¹, A Dakova¹, D Dakova², K Kovachev¹ and L Kovachev¹

¹Institute of Electronics, Bulgarian Academy of Sciences, 72 Tsarigradsko Chaussee, 1784 Sofia, Bulgaria,
²Faculty of Physics and Technology, Paisii Hilendarski University of Plovdiv, 24 Tsar Asen str., 4000 Plovdiv, Bulgaria,
³Department of Medical Physics and Biophysics, Faculty of Pharmacy, Medical University, Plovdiv, 15-A Vasil Aprilov blvd., 4002 Plovdiv, Bulgaria

E-mail: valerislavchev@yahoo.com

Abstract. The aim of the present research is to investigate the spatiotemporal regime of propagation of three-dimensional optical pulses in fibers with spatial dependence of the refractive index. An exact analytical solution of the linearized equation is found. Numerical simulations are conducted of the solutions obtained.

1. Introduction
In recent years, the evolution of three-dimensional optical pulses in isotropic dispersive linear and nonlinear media has been actively studied. Such pulses are used in telecommunications, in optical methods for encoding and recording information, in modern medical laser systems for precision cutting and ablation of tissues. Advances in waveguide technologies due to the modernization of equipment and the development of the scientific knowledge in the field of fiber optics provoked our interest in the theoretical study of the evolution of 3D+1 optical pulses in fibers with spatial dependence of the refractive index in a linear regime of propagation. In the present work, we investigated the dynamic of laser pulses in optical fibers with quadratic dependence of the linear refractive index. The evolution of these pulses in a media with dispersion of the group velocity shows a specific behavior. Gradient-index fibers are used in optical communication systems for improving their parameters. The refractive index of such kind of waveguides can be presented by the expression [1-6]:

\[ n = n_0(\omega) + S_g(x^2 + y^2) + n_2|\vec{U}|^2, \]  

(1)

where \( n_0(\omega) \) and \( n_2 \) correspond to the linear and nonlinear refractive indices of the medium; \( \vec{U} \) is the vector amplitude function describing the pulse’s envelope; \( S_g \) is a constant connected with the refractive
index of the fiber and \( S_g(x^2 + y^2) \) characterizes the spatial profile of the refractive index. Depending on the sign of the constant \( S_g \), the gradient-index waveguides can be divided in two different groups [7]:

1. \( S_g < 0 \) – when the constant is negative, we have convex gradient fibers. The linear refractive index has a maximal value on the fiber axis and decreases smoothly to the periphery of the waveguide.
2. \( S_g > 0 \) – when the constant is positive, then we have concave gradient fibers. The refractive index rises smoothly towards the periphery of the waveguide.

These kinds of optical waveguides and photonic crystals have a number of applications in different devices controlling and manipulating laser light. They can be used in modern communication systems and sensors, and as parts of optical computers, mirrors and lenses in thin-film optics [8-13].

2. Basic equations

To investigate the evolution of ultra-short laser pulses, it is necessary to use the more general nonlinear amplitude equation. The equation, which describes the dynamics of linearly polarized amplitude of the electric field \( \vec{U} = (U_x, U_y, 0) \) in local time coordinates, has the following form [7]:

\[
\begin{align*}
  i \frac{\partial \vec{U}}{\partial z} + \frac{1}{2} \left( \Delta \vec{U} - S_d \frac{\partial^2 \vec{U}}{\partial t^2} \right) + S_g \left( x^2 + y^2 \right) \vec{U} + \gamma |\vec{U}|^2 \vec{U} &= 0.
\end{align*}
\]

(2)

The above equation is applied to both narrow-band pulses and broad-band ones. The third term characterizes the spatial dependence of the refractive index and \( S_g \) is the dispersion parameter of the medium. The authors in [7] found an approximate solution of equation (2) in a scalar form for standard gradient fibers with \( S_g < 0 \).

In our paper we search for an analytical solution of the vector amplitude equation (2) in a linear case (\( \gamma = 0 \)) describing the propagation of laser pulses in concave gradient optical fibers (\( S_g > 0 \)), so that we neglect the last term in equation (2) (\( \gamma |\vec{U}|^2 \vec{U} = 0 \)) which corresponds to the nonlinearity of the medium.

Thus, the equation (2) takes this form:

\[
\begin{align*}
  i \frac{\partial \vec{U}}{\partial z} + \frac{1}{2} \left( \Delta \vec{U} - S_d \frac{\partial^2 \vec{U}}{\partial t^2} \right) + S_g \left( x^2 + y^2 \right) \vec{U} &= 0,
\end{align*}
\]

(3)

where \( \Delta \) is the Laplace operator. We are interested in the two components of the laser radiation \( \vec{U} = (U_x, U_y, 0) \). Equation (3) can be transformed in a linear set of two partial differential equations, describing the components \( U_x \) and \( U_y \) of the vector amplitude function. In a scalar form, the system of equation is:

\[
\begin{align*}
  i \frac{\partial U_x}{\partial z} + \frac{1}{2} \left( \frac{\partial^2 U_x}{\partial x^2} + \frac{\partial^2 U_x}{\partial y^2} \right) + S_d \frac{\partial^2 U_x}{\partial t^2} + S_g \left( x^2 + y^2 \right) U_x &= 0, \quad (4) \\
  i \frac{\partial U_y}{\partial z} + \frac{1}{2} \left( \frac{\partial^2 U_y}{\partial x^2} + \frac{\partial^2 U_y}{\partial y^2} \right) + S_d \frac{\partial^2 U_y}{\partial t^2} + S_g \left( x^2 + y^2 \right) U_y &= 0. \quad (5)
\end{align*}
\]

The equations for \( U_x \) and \( U_y \) are of the same type. For simplicity, we will present the solution for only one of them and we will give the result for the second one. We rewrite equation (4) in spherical coordinates \( U_x = U (r, \theta, z, t) \). After the substitution
\[ U_x = V(z, t) P(r, \theta), \]  
\[ \text{(6)} \]
equation (4) can be presented as follows:
\[ iP \frac{\partial V}{\partial z} - \frac{1}{2} S_d \frac{\partial^2 V}{\partial t^2} + \frac{1}{2} \left( \frac{1}{r} \frac{\partial P}{\partial r} + \frac{\partial^2 P}{\partial r^2} V + \frac{1}{r^2} \frac{\partial^2 P}{\partial \theta^2} \right) + S g r^2 PV = 0, \]
\[ \text{(7)} \]
where \( V \) and \( P \) are new unknown functions. The following step requires separation of the variables on both sides of the equation. In order to keep the equality, each of the sides of equation (7) must be equal to a same constant \( a \) of division of the variables. Thus, we obtain the following two equations:
\[ i \frac{\partial V}{\partial z} - \frac{1}{2} S_d \frac{\partial^2 V}{\partial t^2} = aV, \]
\[ \text{(8)} \]
\[ \frac{1}{r} \frac{\partial P}{\partial r} + \frac{\partial^2 P}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 P}{\partial \theta^2} + 2S g r^2 P = 2aP. \]
\[ \text{(9)} \]
Let us first consider equation (8). It is a partial differential equation of parabolic type with respect to the unknown function \( V \). To solve it we use a similar method as that above. For this purpose, we make the following substitution:
\[ V = Z(z) \Phi(t). \]
\[ \text{(10)} \]
Thus, equation (8) takes the form:
\[ i \frac{Z'}{Z} - \frac{1}{2} S_d \frac{\Phi''}{\Phi} = a. \]
\[ \text{(11)} \]
After some transformations and assumptions, we find a partial solution of equation (11):
\[ V = e^{i\sqrt{\frac{2a+c}{S_d}}t} e^{icz}, \]
\[ \text{(12)} \]
where \( c \) is a complex division constant.

Let us now return to equation (9), which is a partial differential equation of the second order. To solve it we make another substitution:
\[ P = R(r) T(\theta). \]
\[ \text{(13)} \]
Thus, equation (9) takes the form:
\[ \frac{1}{r} TR' + TR'' + \frac{1}{r^2} RT'' + 2S g r^2 RT + 2aRT = 0. \]
\[ \text{(14)} \]
Its general solution is a linear combination of a Bessel function \( (J_n(\eta)) \) and a Neumann function \( (N_n(\eta)) \) with constant \( \lambda = n/2 \), where \( n = 1, 2, 3, 4, \ldots \) and \( r^2 \sqrt{\frac{S g}{2}} = \eta \):
\[ R = C_1 J_n(\eta) + C_2 N_n(\eta), \]
\[ \text{(15)} \]
where \( C_1 \) and \( C_2 \) are constants.
Taking into account all the assumptions and substitutions above, we find the following class of analytical solutions for the components of the vector amplitude function $\mathbf{U}$ described by the linear system of equations (4) and (5):

$$U_x = C_1 \frac{J_n}{2} \left( \sqrt{\frac{S}{2}} r^2 \right) \cos(n\theta) e^{i\theta \frac{2\pi S}{\sqrt{Cd}}},$$

$$U_y = C_1 \frac{J_n}{2} \left( \sqrt{\frac{S}{2}} r^2 \right) \sin(n\theta) e^{i\theta \frac{2\pi S}{\sqrt{Cd}}}. \tag{16}$$

The expressions above are derived under the boundary conditions $\eta \to \infty$, $J_{\frac{n}{2}}(\eta) \to 0$, $N_{\frac{n}{2}}(\eta) \to \infty$ and $C_2 = 0$, as we need the solutions to be descending.

3. Numerical solutions

In optical communication systems, Bessel functions are used to describe the oscillations of the electric field in the core of the optical fiber. The modified Bessel functions determine the exponential attenuation of the field in the envelope of the waveguides [19].

We made numerical simulations for different values of the parameters in the exact analytical solutions (16) and (17). These expressions describe the envelope of optical pulses propagating through a concave gradient fiber. In the following figures, the intensity profiles of the optical pulses are shown for different values of the constant $n$.

**Figure 1.** Optical pulse intensity for $n = 0$.

**Figure 2.** Optical pulse intensity for (a) $n = 1$, (b) $n = 2$ and (c) $n = 4$.

Figure 2 shows the formation of the typical mode structure with a minimum at the center of the pulse and a periphery of higher intensity. For $n = 4$, the energy distribution of the laser radiation has a complex modal structure with four peaks. The theoretical and numerical results can be used for studying the
modal structure in optical fibers and the transfer of laser signals at longer distances with minimal losses [14-19].

4. Conclusions
In the present work, the evolution is studied of laser pulses in a linear dispersive medium with spatial dependence of the refractive index \( S_0 > 0 \). The gradient optical fiber with such type of characteristics has a concave profile of the refractive index. These kinds of waveguides have specific applications, which make them very attractive for scientific research. The propagation of the light pulses in such media is governed by the linear partial differential equation (3) describing the evolution of the vector amplitude function \( \vec{U}(x, y, z, t) \) of the pulses.

We found a new class of analytical solutions which describe the evolution of the components of the vector amplitude function of a laser pulse propagating in optical fibers with a concave profile of the refractive index. The solutions are in the form of Bessel functions. They characterize the modal structures in the fiber. Numerical simulations of the solutions for different values of the constant \( n \) are presented.

Acknowledgments
This paper is based upon work supported by the Air Force Office of Scientific Research under award number FA9550-19-1-7003. It is also funded by the Bulgarian National Science Fund by grant DN18/11.

References
[1] Senior J M and Jamro M Y 2009 Optical Fiber Communications: Principles and Practice 3rd edition (Prentice Hall Pearson Education Limited)
[2] Cao Y, Schenk J O and Fiddy M A 2008 Third Order Nonlinear Effect Near a Degenerate Band Edge Optics and Photonics Letters 1 (1) 1-7
[3] Joannopoulos J D, Johnson S G, Winn J N, Meade R D 2007 Photonic Crystals, Molding the Flow of Light Second Edition (Princeton University Press)
[4] Karlsson M, Anderson D and Desai M 1992 Opt. let. 17 1259
[5] Raghavan S and Agrawal G P 2000 Opt. Commun. 180 377
[6] Nye J F and Berry M V 1974 Proc. R. Soc. A 336 165–190
[7] Kivshar Y S and Agraval G P 2003 Optical solitons (Academic Press)
[8] Xu Q, Almeida V R, Panepucci R R and Lipson M 2004 Optics letters 29 1626-1628
[9] Almeida V R, Xu Q, Barrios C A and Lipson M 2004 Optics letters 29 1209-1211
[10] Jurkevičiūtė A, Armakavičius N, Virganavičius D, Šimatonis L, Tamulevičius T and Tamulevičius S 2017 J. Optoelectron. Adv. M. 19 (3-4) 119
[11] Ebadi N, Yadipour R and Baghban H 2017 J. Optoelectron. Adv. M. 19 (8–7) 454
[12] Wang H and Wang H 2017 J. Optoelectron. Adv. M. 19 (9-10) 575
[13] Hu S, Mei M, Yang H, Jiang P, Caiyang W and Qian L 2018 J. Optoelectron. Adv. M. 20 (3–4) 102
[14] Agrawal G P 2005 Lightwave Technology: Telecommunication Systems (NJ: Wiley, Hoboken)
[15] Gloge D 1971 Applied Optics 10 (10) 2252-2258
[16] Xiao L, Demokan M S, Jin W, Wang Y and Zhao C-L 2007 J. Lightwave Technol. 25 3563
[17] Ramaswami R and Sivarajan K 2002 Optical Networks: A Practical Perspective 2nd ed. (San Francisco: Morgan Kaufmann Publishers)
[18] Agrawal G P 2002 Fiber-Optic Communication Systems 3rd ed. (New York: Wiley)
[19] I P Kaminow and T Li ed 2002 Optical Fiber Telecommunications vol. 4A and 4B (Boston: Academic Press)