Research Article

Strict Deformation Quantization via Geometric Quantization in the Bieliavsky Plane

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1. Introduction

Let $M$ be a symplectic symmetric space, $TM$ its tangent bundle, and let $M \times M$ be the symplectic pair groupoid. Because $M$ is a symplectic symmetric space, the pushforward of the vertical fiberation of $TM$ under the map

$$\Phi : TM \to M \times M, \quad (m, ν) \mapsto (\exp_m(-ν), \exp_m(ν)), \quad (1)$$

determines a foliation $\mathcal{F}_V$ on $M \times M$ which, if regular, defines a real polarization on the symplectic pair groupoid (cf. [1–3] for instance). The regularity condition fails if $M$ is compact but is satisfied if $M$ is noncompact with no compact factors. This short paper considers only the simplest possible case: $M = \mathbb{R}^{2n}$ (actually we here fix $n = 1$ to make matters simpler without any significant loss of generality). Similarly, $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ can be identified with the cotangent bundle $T^*\left(\mathbb{R}^{2n}\right)$, via the map

$$Y : (x, y) \mapsto \left(\frac{1}{2}(x + y), \omega(x - y)\right), \quad (2)$$

where $\omega$ is the standard symplectic form on $\mathbb{R}^{2n}$ and the pullback of the vertical fiberation of $T^*\left(\mathbb{R}^{2n}\right)$ determine a foliation $\mathcal{F}_{V^*}$ on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$.

The integral version of the Weyl-Moyal product of functions on $\mathbb{R}^{2n}$, also known as the Groenewold-von Neumann product, has been obtained and reobtained in various ways since the original work of von Neumann [4]. But in [5], Gracia-Bondía and Varilly rederived this product via geometric quantization (see [6] for example), using the pairing of two nontransversal real polarizations on the pair groupoid $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ one being the polarization $\mathcal{F}_V$ described above.

Now, Pierre Bieliavsky gets more recently (see [7, 8]), as a contribution to the area of Strict Quantization, the integral product of functions on $\mathbb{R}^{2n}$ ($\mathbb{R}^{2n}$ is $\mathbb{R}^{2n}$ as a (nonmetric) symplectic symmetric space) given explicitly by

$$\left(g_1 \ast g_2\right)(p, q) = \frac{1}{\mathcal{R}} \int_{a_1, a_2} \left(\cosh(2(p_1 - p)) \cdot \cosh(2(p_1 - p)) \cdot \cosh(2(p_1 - p))\right)^{1/2}, \quad (3)$$

$$\left(g_1 \ast g_2\right)(p_1, q_1, p_2, q_2) e^{\frac{i}{\hbar}(\sinh(2(p - p_1) q) - \sinh(2(p - p_1) q) - \sinh(2(p - p_1) q) - \sinh(2(p - p_1) q))} \eta dp_1 dp_2 dq_1 dq_2 dp_1 dp_2, \quad (4)$$

This type of product was initially considered for Weinstein and Zakrzewski in the so-called WKB-quantization program (see [9]). Here, we will rederive this product, again via geometric quantization and again using pairing of polarizations,
but now pairing the real polarization \( \mathcal{F}_V \). is determined by a map of \( \mathbb{R}^{2n} \times \mathbb{R}^{2n} \) to \( T^* (\mathbb{R}^{2n}) \) given for

\[
Y_b : (x_1, y_1 ; x_2, y_2) \mapsto (m[(x_1, y_1), (x_2, y_2)], \hat{\omega}((x_2, y_2) - (x_1, y_1)));
\]

where \( m[(x_1, y_1), (x_2, y_2)] \) is the middle point function on \( \mathbb{R}^{2n} \) (see [10]). Although our derivation presented below could be considered a simple exercise in geometric quantization, we have not yet found it explicitly done in detail, in the literature (the method originally developed in [11] is totally different, using a map to the Weyl product) and it also allows us to obtain the associativity of this product as a direct corollary of our construction. In fact, the main idea for this derivation is already found in the aforementioned paper by Gracia-Bondia and Varilly ([5], for Euclidian case \( \mathbb{R}^{2n} \)). On the other hand, appropriate generalizations of this technique to other noncompact hermitian symmetric spaces can in principle be helpful (for instance, if \( M = \mathbb{H}^2 \) is the hyperbolic plane). This fact shall be thoroughly explored in subsequent papers and constitutes the main motivation for our working out this technique in detail for the case of \( \mathbb{R}^2_b \) (Bielavsky plane), in this present note. As we shall see below in detail, the geometric quantization pairing of \( \mathcal{F}_V \) and a standard real polarization on \( \mathbb{R}^2 \times \mathbb{R}^2 \) defines a integral transform from functions on \( \mathbb{R}^2_b \) to \( L^2(\mathbb{R}^2_b) \).

It is well known that geometric quantization can be used to construct the integral transforms, such as Laplace transform, Fourier transform, Segal-Bargmann transform (cf. e.g., [12–14]), and the generalized Segal-Bargmann transform for Lie groups of compact type can also be developed using geometric quantization (cf. [15, 16]).

In this short note, again via geometric quantization, we shall obtain the 2-d integral transforms given by

\[
T_b g(x_1, x_2) = C_b [\cosh ((x_1 - x_2))]^{1/2} \int_{\mathbb{R}} g\left(\frac{x_1 + x_2}{2}, q\right) e^{i(h)\sinh (x - x_2)} dq,
\]

\[
S_b f(p, q) = C_b \int_{\mathbb{R}} f\left(\frac{q^*}{2} + p, p - \frac{q^*}{2}\right) e^{i(h) \sinh \left( q^* \right) \left( \cosh \left( q^* \right) \right)^{1/2}} dq^*,
\]

where \( T_b \) is an integral transform defined on the support compact function in \( \mathbb{R}^2_b \); \( (x_1, x_2), (p, q) \in \mathbb{R}^2_b \), and Planck’s constant \( h \in \mathbb{R}^+ \) can also be considered a free positive parameter whenever this is convenient. Moreover, for an appropriate choice of constant \( C_b \), \( T_b (S_b f) = S_b (T_b f) \) for, \( f \), a support compact function.

Thus, in Section 2, we present our detailed derivation of this transform (cf. (35), (36)), which immediately generalizes to all even dimensional cases. Finally, in Section 3, combining this transform with a natural deformation of the usual convolution of functions on the pair groupoid, we obtain the integral formulation of the Bielavsky product, which is given by (4).

2. The Integral Transform Generated by the Geometric Quantization

Let \( \mathbb{R}^2_b \) the Bielavsky plane [8], this is (\( \mathbb{R}^2, o, s \), where \( o \) is the canonical symplectic form on the euclidian plan and \( s \) a symmetric on \( \mathbb{R}^2 \), such that, if \( (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2 \), then a symmetric is given by the expression:

\[
s_{(x_1, y_1)}(x_2, y_2) = 2(x_1 - x_2, 2y_1 \cosh (2(x_1 - x_2)) - y_2).
\]

Thus, we have that the middle point function is given by

\[
m[(x_1, y_1), (x_2, y_2)] = \left[ \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right] \operatorname{sech} \left( \frac{(x_1 + x_2)}{2} \right),
\]

defined by the relation \( s_m[(x_1, y_1), (x_2, y_2)](x_1, y_1) = (x_2, y_2) \).

In this case, the map \( Y_b : \mathbb{R}^2_b \times \mathbb{R}^2_b \rightarrow T^* \mathbb{R}^2_b \), above, cf. equation (5), is given explicitly by

\[
Y_b(x_1, y_1; x_2, y_2) = \left\{ \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right\} \operatorname{sech} \left( \frac{(x_1 + x_2)}{2} \right),
\]

\[
y_1 - y_2, x_2 - x_1 = (p, q : p^*, q^*).
\]

Denote by \( \mathbb{R}^2_b \times \mathbb{R}^2_b \) the symplectic manifold, such that if \( \mathbb{R}^2_b \times \mathbb{R}^2_b \) has coordinates \( (x_1, y_1; x_2, y_2) \), then the symplectic form is given by

\[
\Omega = dx_1 \wedge dy_1 - dx_2 \wedge dy_2.
\]

Moreover, if \( T^* \mathbb{R}^2_b \) has coordinates \( (p, q, p^*, q^*) \) as above and since \( Y_b \) is a diffeomorphism with inverse \( Y_b^{-1} \) given by

\[
x_1 = p - \frac{q^*}{2},
\]

\[
x_2 = p + \frac{q^*}{2},
\]

\[
y_2 = q \cosh (q^*) - \frac{p^*}{2},
\]

\[
y_1 = q \cosh (q^*) + \frac{p^*}{2},
\]

the symplectic form \( (Y_b^{-1})^* (\Omega) = \Pi \) on \( T^* \mathbb{R}^2_b \) is given by

\[
\Pi = - \cosh (q^*) dq^* \wedge dq + dp \wedge dp^*.
\]

Consider the following respective polarization on \( T^* \mathbb{R}^2_b \) and \( \mathbb{R}^2_b \times \mathbb{R}^2_b \),

\[
\tilde{P} = \left< dp^*, dq^* \right>, F = \left< \partial y_1, \partial y_2 \right>.
\]

Thus, from (12), the symplectic potential adapted to the polarizations \( \tilde{P} \) is given by
where
\[ \Theta_p = -\sinh(q^*)dq - p^*dp, \]

while from (10), the symplectic potential adapted to the polarization $F$ is given by
\[ \Theta_F = y_2dx_2 - y_1dx_1. \]

For $\tilde{F} = (Y_b)_*(F)$, we have from (11) that
\[
\Theta_{\tilde{F}} = \left( q \cos(q^*) - \frac{p^*}{2} \right) \left( dp + \frac{1}{2} dq^* \right) \\
- \left( \frac{p^*}{2} + q \cos(q^*) \right) \left( dp - \frac{1}{2} dq^* \right) - q \cos(q^*)dp \\
+ \frac{p^*}{4} dq^* - \frac{p^*}{2} dp = -p^*dp + q \cos(q^*)dq^*,
\]

Therefore,
\[ \Theta_p - \Theta_{\tilde{F}} = d\bar{\Psi}, \]

where
\[ \bar{\Psi}(p, q; p^*, q^*) = -\frac{i}{\hbar} q \sin(q^*), \]

which in terms of the coordinates on $\mathbb{R}^2_+ \times \mathbb{R}^2_-$ can be written as
\[ \bar{\Psi} \circ Y_b = \Psi(x_1, y_1; x_2, y_2) \\
= -\frac{i}{\hbar} \left( \frac{y_1 + y_2}{2} \right) \text{sech}(x_1 + x_2) \sinh(x_2 - x_1). \]

Now, recall that a connection on a hermitian line bundle $L$ associated to the prequantum principal $S^1$-bundle over a symplectic manifold $M$ is given locally by
\[ \nabla_X = X - (i/\hbar)\Theta(X), \quad X \in \mathfrak{X}(M), \]

where $\Theta$ is a symplectic potential. Then, consider the polarized section $\tilde{s}_0$ of $L$ over $\mathbb{R}^2_+ \times \mathbb{R}^2_-$ adapted to the symplectic potential $\Theta_p$ and its push-forward $\tilde{s}_0$ adapted to $\Theta_p$, as well as the polarized section $\tilde{t}_0$ of $\tilde{L}$ over $T^* \mathbb{R}^2_0$ adapted to the symplectic potential $\Theta_{\tilde{F}}$ and its push-forward $\tilde{t}_0$ adapted to $\Theta_{\tilde{F}}$, where $P = (Y_b)^{-1}$, satisfying
\[
\nabla_X \tilde{t}_0 = -(i/\hbar)\Theta_p \tilde{t}_0; \quad (\tilde{t}_0, \tilde{t}_0) = 1, \\
\nabla_X \tilde{s}_0 = -(i/\hbar)\Theta_{\tilde{F}} \tilde{s}_0; \quad (\tilde{s}_0, \tilde{s}_0) = 1,
\]

where $(\cdot, \cdot)$ is the hermitean product of the line bundle $\tilde{L}$ and $\tilde{X} \in \mathfrak{X}(T^* \mathbb{R}^2_0)$, with similar expressions for $\tilde{s}_0$ and $\tilde{t}_0$ in terms of $P, F, (\cdot, \cdot)$ on $L$, and $X \in \mathfrak{X}(\mathbb{R}^2_+ \times \mathbb{R}^2_-)$. The polarized sections $\tilde{t} \in \Gamma_p \tilde{L}$ are given by $\tilde{t} = \tilde{g} \tilde{t}_0$, with $\tilde{g} \in \mathcal{C}^{\infty}(T^* \mathbb{R}^2_0)$ satisfying $X \tilde{g} = 0$, for $X \in \mathfrak{X}(T^* \mathbb{R}^2_0, \tilde{P})$; thus, it follows that $\tilde{g}$ depends only on the variables $(x, y) \in \mathbb{R}^2_0$ seen as the zero section of $T^* \mathbb{R}^2_0$. Similarly, the polarized sections $s \in \Gamma_p L$ of the form $s = f \tilde{s}_0$, where $f \in \mathcal{C}^{\infty}(\mathbb{R}^2_+ \times \mathbb{R}^2_-)$ and depends only on the variables $(x_1, x_2)$ for $(x_1, y_1; x_2, y_2) \in \mathbb{R}^2_+ \times \mathbb{R}^2_-$. Furthermore, as the prequantum line bundle is a linear bundle, we have that $\tilde{s}_0 = \tilde{\phi} \tilde{t}_0$ for a nonvanishing function $\tilde{\phi} \in \mathcal{C}^{\infty}(T^* \mathbb{R}^2_0)$. Therefore,
\[ \nabla_X \tilde{t}_0 = \nabla_X \tilde{\phi} \tilde{t}_0 = (\tilde{X} \tilde{\phi}) \tilde{t}_0 + \tilde{\phi} \nabla_X \tilde{t}_0, \]

whence, we get
\[ \frac{d\tilde{\phi}}{\tilde{\phi}} = i \left( \Theta_p - \Theta_{\tilde{F}} \right) = d\bar{\Psi}, \]

cf. (17), (18), and (19), thus $\tilde{\phi} = Ce^{i\phi/\hbar}$. Hence, we have the following.

**Lemma 1.** For $t \in \Gamma_p L$, $\tilde{t} \in \Gamma_p \tilde{L}$, $s \in \Gamma_p s$, in the hermitian products of these polarized sections are given modulo multiplicative constants by the formulas:
\[ (s, \tilde{t}) = g(p, q)f(x_1, p, q; p^*, q^*), x_2(p, q; p^*, q^*)e^{i\psi(p, q; p^*, q^*, x_2)/\hbar}, \]
\[ (t, s) = g(p, q)f(x_1, y_1; x_2, y_2), q(x_1, y_1; x_2, y_2)e^{i\psi(p, q; p^*, q^*, x_2)/\hbar}, \]
with $p(x_1, y_1; x_2, y_2), q(x_1, y_1; x_2, y_2)$ and $x_1(p, q; p^*, q^*)$, $x_2(p, q; p^*, q^*)$ obtained from (9) and (11), with $\Psi(p, q; p^*, q^*)$ and $\Psi(x_1, y_1; x_2, y_2)$ given by (18) and (19).

Now, as the polarization $\tilde{P}$ is the natural polarization of cotangent bundle, given $q \in \mathbb{R}^2_+$ and $m \in \pi^{-1}(q)$, with $\pi$ the canonical projection, $T_q(T^* \mathbb{R}^2_0)/\tilde{P}_m = T\mathbb{R}^2_0$, thus
\[ \Delta_{1/2}(\tilde{P}_m) = \Delta_{1/2}(T_m(T^* \mathbb{R}^2_0)) \otimes \Delta_{1/2}(T(p,q)(\mathbb{R}^2_0)). \]

and so the volume form $\epsilon_{1/2}$ of $T^* \mathbb{R}^2_0$ and $\epsilon_m$ of $\mathbb{R}^2_0$ determine the natural section of $\Delta_{1/2}(\tilde{P}_m)$ given by
\[ \tilde{\nu} = |\epsilon_{1/2}|^{1/2}, |\epsilon_m|^{1/2}. \]

Analogously, for the polarization $F$ in the pair groupoid $\mathbb{R}^2_+ \times \mathbb{R}^2_-$, we have
\[ T(\mathbb{R}^2_+ \times \mathbb{R}^2_-)/F = T\mathbb{R}^2_0, \]
thus, a natural section of $\Delta_{1/2}(F_m)$ is given by
\[ \nu = |\epsilon_{1/2}|^{1/2}, |\epsilon_m|^{1/2}, \]
where $\epsilon_{1/2}$ is the volume form in $\mathbb{R}^2_+ \times \mathbb{R}^2_-$. Since $(Y_b)_*$ is an isomorphism, for $P = (Y_b)^{-1}_*(\tilde{P})$ and $\tilde{F} = (Y_b)_*(F)$, the natural half density in $\Delta_{1/2}(\tilde{F})$ are given, respectively, by

\[
\Theta_{\tilde{F}} = y_2dx_2 - y_1dx_1. \]
\[ \psi' = |Y_\phi(\epsilon \Pi)|^{-1/2} \cdot |Y_\phi^*(\epsilon \omega)|^{1/2} \psi' = |E(Y_{\phi}^*)|^{1/2} \cdot |F(Y_{\phi}^*)|^{-1/2}. \]  

(29)

On the other hand, we can see that the polarizations \( \tilde{P}, F \), above, satisfy the following property.

**Lemma 2** (see [17]). Let \( \tilde{P} = (Y_\phi)_x(F) \), we have that \( \tilde{P} \) and \( F \) are always not transverse, specifically \( \tilde{P} \cap F = \langle -\partial p^* \rangle \).

Now, if \( F \cap \tilde{P} = D = \langle -\partial p^* \rangle \), \( D^0 = \langle -\partial q, \partial p^*, \partial q^* \rangle \) and \( R = T^*\mathbb{R}^2_{\phi}/D \), for \( x \in R \) and \( m \in \pi^{-1}(x) \), with \( \pi \) the canonical projection on \( T^*\mathbb{R}^2_{\phi} \), we have

\[ T_x R = \frac{T_{xR}M}{D_m} e V_m = \frac{D^0}{D_m}. \]  

(30)

Then,

\[ \Delta_1(T_x R) = \Delta_{-1/2}(\tilde{F}_m) \otimes \Delta_{-1/2}(\tilde{F}_m) \otimes \Delta_{1/2}(V_m) \otimes \Delta_1(T_{xM}), \]  

(31)

and so \( T^*(\mathbb{R}^2_{\phi})/D \equiv \mathbb{R}^3 \), for \( x_1 = (q^*/2) + p \); \( x_2 = p - (q^*/2) \), we obtain

\[ <\tilde{t} \otimes \tilde{v}, \tilde{t} \otimes \tilde{v}'>_{pr} = \int_{\mathbb{R}^3} f \left( \frac{q^*}{2} + p, p - \frac{q^*}{2} \right) g(p, q) e^{-i(h)q \sinh(q')} \times [\cosh (q')]^{1/2} dpdq. \]  

(32)

\[ = \langle S_h f \rangle, g \rangle_{L^2(\mathbb{R}^3_{\phi})}. \]

and

\[ S_h f(p, q) = C \int_{\mathbb{R}} \int \left( \frac{q^*}{2} + p, p - \frac{q^*}{2} \right) e^{i(h)q \sinh(q')} [\cosh (q')]^{1/2} dq'. \]  

(36)

Now, we have the following result for the integral transform, \( T_h \) and \( S_h \). This result is a direct consequence of the approximation unit theorem and associativity guarantees that define the product to follow.

**Proposition 3.** For the support compact function on \( \mathbb{R}^2_{\phi} \), the integral transform \( T_h \) above ((35)) has inverse given by \( S_h \) above ((36)), this is, \( S_h(T_h f)(p, q) = f(p, q) \). Inversely, \( T_h(S_h f)(p, q) = f(p, q) \).

The proof of this result is given in the appendix.

### 3. Rederiving the Bieliavsky Product

Starting from the usual convolution of functions on the symplectic pair groupoid \((\mathbb{R}^3_{\phi} \times \mathbb{R}^3_{\phi}, \mathbb{A}_{\mathbb{R}^3_{\phi}})\)

\[ (f \circ g)(x_1, y_1; x_2, y_2) = \int_{\mathbb{R}^3} f(x_1, x_2; y_2) g(x_2, y_2; x_3, y_3) dx_2 dy_2. \]  

(37)

is possible to construct a deformed convolution of the functions on \((\mathbb{R}^3_{\phi} \times \mathbb{R}^3_{\phi})/F \) (see appendix) as follows:

\[ (f \circ g)(x_1, x_3) = \int_{\mathbb{R}} f(x_1, x_2) g(x_2, x_3) dx_2, \]  

(38)

which can be straightforwardly checked to satisfy the following.

**Lemma 4.** The deformed convolution defined by ((38)) above is associative.

From this, we define a new product on \( L^2(\mathbb{R}^3_{\phi}) \) as follows:

\[ *_{\mathbb{R}^3_{\phi}} : L^2(\mathbb{R}^3_{\phi}) \times L^2(\mathbb{R}^3_{\phi}) \rightarrow L^2(\mathbb{R}^3_{\phi}), \]  

(39)

\[ (g_1, g_2) \rightarrow g_1 *_{\mathbb{R}^3_{\phi}} g_2, \]

where

\[ g_1 *_{\mathbb{R}^3_{\phi}} g_2 = S_h[T_h g_1 \otimes T_h g_2], \]  

(40)

which from Proposition 3 and Lemma 4 satisfies the following.

**Corollary 5.** The product \( *_{\mathbb{R}^3_{\phi}} \) defined by ((40)) above is associative.

Finally, by a straightforward computation, one can easily check the following.
**Theorem 6.** The formula for the product $*^h_k$ defined by ((40)) via ((35)), ((36)), and ((38)) coincides in an obvious with formula ((4)) for the Bieliavsky product.

## Appendix

### A. Symplectic Groupoids

A general example of the symplectic groupoid is the fundamental groupoid $\pi(M)\tilde{\mathcal{M}}\tilde{\mathcal{M}}$ of a symplectic manifold $(M, \omega)$. Its elements are homotopy classes of smooth paths $\sigma : [0, 1] \to M$, with the usual concatenation product of paths whose endpoints match; reserving the path gives the involution. Here $\alpha([\sigma]) = \sigma(0)$, $\beta([\sigma]) = \sigma(1)$ are the endpoint assignment maps. The manifold $M$ embeds en $\pi(M)$ as the submanifold of constant paths, which is Lagrangian with respect to the symplectic structure $\Omega = \alpha^*\omega - \beta^*\omega$ on $\pi(M)$. When $M$ is simply connected, $[\sigma]$ is determined by its endpoints, and the fundamental groupoid may be reexpressed as

$$M \times \tilde{\mathcal{M}}\tilde{\mathcal{M}}.$$  

We can then write $\alpha(q, p) = q, \beta(q, p) = p,$ and identify $M$ with the diagonal submanifold $\{(q, q) : q \in M\}$. The multiplication and involution are given by

$$(q, p) \cdot (p, r) = (q, r),$$  

$$(q, p)^* = (p, q).$$  

When $M$ is simply connected, $[\sigma]$ is determined by its endpoints, and the fundamental groupoid may be reexpressed as

$$M \times \tilde{\mathcal{M}}\tilde{\mathcal{M}}.$$  

where $\alpha(q, p) = q, \beta(q, p) = p$, the product and involution is given by

$$\alpha(p, q) = p, \beta(p, q) = q, (p, q)\iota(q, r) = (p, r) \text{ and } (p, q) = (q, p).$$

We call this groupoid of symplectic pair groupoid.

Now, if $(M, \omega)$ a symplectic manifold simply connected and $F$ a polarization of manifold $M$. As $M$ is simply connected, we can consider the symplectic groupoid above.

On the other hand, we have $F \times F$ the polarization of the symplectic manifold $M \times M$. Without loss of generality, we assume that $M$ has dimension 2, if $M \times M$ has coordinates $(p_1, q_1; p_2, q_2)$, consider the polarization

$$F := \langle \partial q_1 \rangle.$$  

Thus, the polarization $\mathfrak{F} = F \times F$ is generated by $\{\partial q_1, \partial q_2\}$; then, the foliation space $M \times M\mathfrak{F}$ has coordinates $(p_1, p_2)$, as $M$ is identified with the diagonal of $M \times M$ if $(p_1, q_1; p_1, q_1) \in \text{Diag}(M \times M)$, then $\text{Diag}(M \times M)/\mathfrak{F}$ has coordinates $(p_1, p_1) = M/F$. Suppose that $\{M \times M\tilde{\mathcal{M}}\tilde{\mathcal{M}}/\mathfrak{F}, M/F$ are simply connected, we will show that $M \times M\tilde{\mathcal{M}}\tilde{\mathcal{M}}$ is a groupoid on $M/F$. In effect, we considered the following graph:

$$M \times \tilde{\mathcal{M}}[d]^\pi[r]^\pi(M \times \tilde{\mathcal{M}})/\mathfrak{F}[d]^\pi[M][r]^\pi[M/F]$$

where $\pi$ and $\pi$ are the canonical projections.

Then, we get $\tilde{\pi} = \pi \circ \alpha; \text{ analogously, we define } \tilde{\beta} = \pi \circ \beta$. Thus,

$$\tilde{\beta}, \tilde{\alpha} : M \times \tilde{\mathcal{M}}\tilde{\mathcal{M}} \to M/F$$

given by $\tilde{\alpha}(p_1, p_2) = \tilde{p}_1$ and $\tilde{\beta}(p_1, p_2) = \tilde{p}_2$. Then, as $M_2 := \{(p, q; p', q') \in (M \times M) \times (M \times M) \mid (p, q) = (p', q')\}$ is the domain of the product in $M \times M\tilde{\mathcal{M}}\tilde{\mathcal{M}}$, as they were defined by applications $\tilde{\alpha}, \tilde{\beta}$ above, we can consider the following set:

$$M_2 = \{(p_1, p_2; p_1', p_2') \in (M \times M\mathfrak{F}) \times (M \times M\mathfrak{F}) \mid \tilde{\alpha}(p_1, p_2) = \tilde{\beta}(p_1', p_2')\}.$$  

Hence, the product in the groupoid $M \times \tilde{\mathcal{M}}\tilde{\mathcal{M}}$ is given by

$$\circ : M_2 \subset (M \times \tilde{\mathcal{M}}) \times (M \times \tilde{\mathcal{M}}) \to M \times \tilde{\mathcal{M}}$$

$$(p_1, q_1; p_2, q_2) \circ (p_2, q_2; p_3, q_3) = (p_1, q_1; p_2, q_3),$$

define our product in $M \times M\tilde{\mathcal{M}}\tilde{\mathcal{M}}/F$ as

$$\circ : M_2 \subset \left(M \times \frac{M}{\mathfrak{F}}\right) \times \left(M \times \frac{M}{\mathfrak{F}}\right) \to \frac{M \times M}{\mathfrak{F}},$$

$$(\tilde{p}_1, \tilde{p}_2) \circ (\tilde{p}_2, \tilde{p}_3) = (\tilde{p}_1, \tilde{p}_3),$$

where we can see that

$$\tilde{\alpha}(\tilde{p}_1, \tilde{p}_2)\iota(\tilde{p}_2, \tilde{p}_3) = \tilde{p}_1 = \tilde{\alpha}(\tilde{p}_1, \tilde{p}_2),$$

$$\tilde{\beta}(\tilde{p}_1, \tilde{p}_2)\iota(\tilde{p}_2, \tilde{p}_3) = \tilde{p}_3 = \tilde{\beta}(\tilde{p}_1, \tilde{p}_2).$$

On the other hand, know that the involution in $M \times \tilde{\mathcal{M}}\tilde{\mathcal{M}}$ is given by $i(p_1, q_1; p_2, q_2) = (p_2, q_2; p_1, q_1)$. Thus, define the involution on $M \times M\tilde{\mathcal{M}}\tilde{\mathcal{M}}/F$ by

$$(\tilde{p}_1, \tilde{p}_2)^* = (\tilde{p}_2, \tilde{p}_1),$$

where we have

$$(\tilde{p}_1, \tilde{p}_2)^* \circ ((\tilde{p}_1, \tilde{p}_2) \circ (\tilde{p}_2, \tilde{p}_3)) = (\tilde{p}_2, \tilde{p}_1) \circ (\tilde{p}_1, \tilde{p}_3) = (\tilde{p}_2, \tilde{p}_3).$$

and

$$((\tilde{p}_2, \tilde{p}_3)\iota(\tilde{p}_1, \tilde{p}_1)\iota(\tilde{p}_2, \tilde{p}_3)) = (\tilde{p}_2, \tilde{p}_1)\iota(\tilde{p}_1, \tilde{p}_3) = (\tilde{p}_2, \tilde{p}_3).$$

Therefore, using the definition above, we have that $M \times M\tilde{\mathcal{M}}\tilde{\mathcal{M}}/F$ together with $\tilde{\alpha}, \tilde{\beta}, M_2,$ and $\circ, \circ$ is a groupoid.
Lemma A.1. Let \((\mathcal{M}, \omega)\) a symplectic manifold simply connected, \((\mathcal{M} \times \mathcal{M}, (\omega, -\omega))\) the symplectic fundamental groupoid on \(\mathcal{M}\) with the operations \(\alpha, \beta, \ast, \ast\), and the polarization \(F\) as above \(((A.7))\) and \(((A.11))\). Then, \(\mathcal{M} = F \times F\) is the polarization of \(\mathcal{M} \times \mathcal{M}\) and \(\mathcal{M}, \mathcal{M}, \mathcal{M}\) give the applications given above \(((A.7))\), \(((A.11))\), and \(((A.13))\); \(\mathcal{M} \times \mathcal{M}\) with this application is a groupoid on \(\mathcal{M}/\mathcal{F}\) denoted by \(\mathcal{M} \times \mathcal{M}/\mathcal{F}\).

B. Inverse of the Integral Transform

In this appendix, I give a proof of Proposition 3. We consider \(T_h\) and \(S_h\) as \((35)\) and \((36)\), respectively, then

\[ S_h(T_h f)(p, q) = C \int_{\mathbb{R}} T_f \left( \frac{y}{2} + \beta y - \frac{\beta}{2} \right) \cosh^{1/2}(y) \exp \left( \frac{i}{\hbar} q \sinh(y) \right) dy, \]

\[ = C^2 \int_{\mathbb{R}} \int_{\mathbb{R}} f(p, \xi) \cosh^{1/2}(\xi) \exp \left( \frac{i}{\hbar} q \sinh(y) \right) dy d\xi. \]

(B.1)

But, for \(u = \sinh(y)\), we have that

\[ S_h(T_h f)(p, q) = C^2 \int_{\mathbb{R}} \int_{\mathbb{R}} f(p, \xi) \exp \left( \frac{i}{\hbar} q \xi \right) u d\xi, \]

(B.2)

where we note that the integral \((A.17)\) is not absolutely convergent.

Being as we can not to change the order integration in the usual sense, then introduce the Gaussian factor \(e^{-|y|^2 u^2}\) in the integral above, this is,

\[ C^2 \int_{\mathbb{R}} \int_{\mathbb{R}} f(p, \xi) \exp \left( \frac{i}{\hbar} q \xi \right) u \exp \left( -\frac{1}{\hbar^2} t |u|^2 \right) d\xi, \]

(B.3)

Notice that when \(t \to 0\), the integral \((B.3)\) above is \(S(Tf)\) \((p, q)\). Moreover, it is absolutely convergent and we can take the iterated integral in any order. Thus,

\[ C^2 \int_{\mathbb{R}} f(p, \xi) \left( \int_{\mathbb{R}} e^{i(\hbar/2) y^2} \exp \left( -\frac{1}{\hbar^2} t |y|^2 \right) dy d\xi \right) \]

where

\[ G_\xi(q - \xi) = C^2 \int_{\mathbb{R}} e^{i(\hbar/2) y^2} \exp \left( -\frac{1}{\hbar^2} t |y|^2 \right) dy. \]

(B.4)

Then,

\[ C^2 \int_{\mathbb{R}} \int_{\mathbb{R}} f(p, \xi) \exp \left( \frac{i}{\hbar} q \xi \right) u \exp \left( -\frac{1}{\hbar^2} t |u|^2 \right) d\xi, \]

\[ = \int_{\mathbb{R}} f(p, \xi) G_\xi(q - \xi) d\xi, \]

(B.5)

for all \(t > 0\). If we take the limits when \(t \to 0\), the left side in \((B.6)\) tends to \(S_h(T_h f)(p, q)\).

Now, making a change of variables in \((B.6)\), we have that

\[ G_\xi(q - \xi) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(x-q)u} e^{-\frac{1}{2\hbar} u^2} dv. \]

(B.7)

Thus, of the Fourier transform for the Gaussian function in \((B.7)\), we get

\[ G_\xi(x) = \frac{1}{2\pi \hbar} e^{-\frac{1}{2\hbar} x^2}, \]

(B.8)

which has the following properties:

(i) \[ \int_{-\infty}^{\infty} G_\xi(x) dx = 1 \] para to do \(t\).

(ii) \[ G_\xi(x) \geq 0 \]

(iii) \[ \lim_{|x| \to \infty} G_\xi(x) = 0 \] se \(x \neq 0\), uniformly in \(|x| \geq \varepsilon\) for any \(\varepsilon > 0\).

Therefore, \(G_\xi\) is an approximation unit and the inversion formula is a direct consequence of approximation unit theorem, because \(G_\xi\) is an approximation unit \(f \to f\) when \(t \to 0\), whence \(S_h(T_h f)(p, q) = f(p, q)\).

Analogously, it shows that \(T_h(S_h f)(p, q) = f(p, q)\).

As a last remark, we emphasize that the whole treatment presented in this paper generalizes in an obvious way from \(\mathbb{R}^2\) to \(\mathbb{R}^{2n}\), for every \(n \in \mathbb{N}\).

Data Availability

The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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