Robust self-testing of multipartite GHZ-state measurements in quantum networks

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Self-testing is a device-independent examination of quantum devices based on correlations of observed statistics. Motivated by elegant progresses on self-testing strategies for measurements [Phys. Rev. Lett. \textbf{121}, 250507 (2018)] and for states [New J. Phys. \textbf{20}, 083041 (2018)], we develop a general self-testing procedure for multipartite generalized GHZ-state measurements. The key step is self-testing all measurement eigenstates for a general $N$-qubit multipartite GHZ-state measurement. Following our procedure, one only needs to perform local measurements on $N-2$ chosen parties of the $N$-partite eigenstate and maintain to achieve the maximal violation of tilted Clauser-Horne-Shimony-Holt (CHSH) Bell inequality for remaining two parties. Moreover, this approach is physically operational from an experimental point of view. It turns out that the existing result for three-qubit GHZ-state measurement is recovered as a special case. Meanwhile, we develop the self-testing method to be robust against certain white noise.

1 Introduction

The rapid development of quantum communication in recent years creates an exigent requirement for devising certification methods to guarantee correctness of quantum information tasks. To rule out any potential attacks by malicious third party, such certification methods must be device-independent. As the first device-independent tool, the Bell nonlocality has been extensively studied in recent decades [1]. It has brought great breakthroughs in quantum physics. Recently, as the strongest form of device-independent certification, self-testing has been developed, which is also based on Bell nonlocality. Such certification method can characterize the target objects (quantum states, measurements) fully, only up to local isometries, in a device-independent manner.

Self-testing, acting as a device-independent certification method, has attracted lots of attention since the pioneer works of Mayers and Yao [2]. It can be used to certify entangled pure states and measurements [3–22]. Up to now, a wide range of entangled
quantum states are proved to be self-testable, such as the elegant results for all pure bipartite entangled states [23], three-qubit W states [24], and graph states [25]. It has also been shown that all pure multipartite GHZ states and Dicke states can be self-tested [26]. Recently, the self-testing method for quantum channels has also been developed [27]. Moreover, there have been many applications about self-testing, such as quantum key distribution [28], randomness expansion [29], detection for entanglement [30], certification of genuinely entangled subspaces [31, 32], coarse-grained self-testing of a many-body singlet [33], as well as verification of quantum computations [34, 35].

In this work, we will focus on self-testing entangled measurements in quantum networks. Self-testing entangled quantum measurements is of great potential to develop practical quantum networks, which has been preliminarily studied [36, 37]. For a star-network as shown in Fig. 1, where $N$ observers share entangled states with central node, respectively. A self-testable entangled measurement can guarantee the success of quantum information tasks, which are based on distributing entangled states between remote parties in such a network. Meanwhile, the entanglement between observers and central node can also be certified. In Ref. [36], the authors presented a self-testing method for the Bell-state measurement (BSM) and three-qubit GHZ-state measurement (GSM). Furthermore, a more robust self-testing scheme for BSM has also been proposed in Ref. [37]. However, there have not been a detailed characterization for self-testing multipartite ($N > 3$) entangled measurements directly.

By generalizing the idea of Ref. [36], we present herewith a self-testing method for $N$-qubit tilted GSM, whose eigenstates are partially entangled $N$-qubit GHZ states (tilted GHZ states). That is to say, one of the eigenstates of $N$-qubit tilted GSM can be written as $|\text{GHZ}^\theta_N\rangle = \cos \theta |0\rangle^\otimes N + \sin \theta |1\rangle^\otimes N$, $\theta \in (0, \pi/4]$. In the $N + 1$-partite star-network shown in Fig. 1, after performing measurement in central node (Roy), the remaining particles shared with Alice 1, Alice 2, …, and Alice $N$ will be projected into $|\text{GHZ}^\theta_N\rangle$ for all $r$, the measurement performed by Roy is equivalent to the tilted GSM.

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further a general method for self-testing multipartite tilted GSM in a star-network, and
the method is operational from an experimental point of view. We also show that one can
self-test more entangled measurements by our developed method straightforwardly.

The paper is organized as following. In Sec. 2, we provide a preparation review of
tilted CHSH scenario which constitute important ingredient of our self-testing method. In
Sec. 3, self-testing method of multipartite tilted GSM is presented. In Sec. 4, a noise-robust
self-testing scheme of three-qubit GSM is presented, with the help of semidefinite program
(SDP) method. Finally, we conclude our results and make a discussion on potential future
works in Sec. 5.

2 Preliminaries

To self-test tilted multipartite GSM, the tilted CHSH inequality is necessary [39]. Let us
consider a task: Alice and Bob share a two-qubit state and they want to know whether the
shared state is partially entangled or not. They perform local measurements (dichotomic
observables) respectively. The tilted CHSH inequality is given by

$$\alpha \langle A_0 \rangle + \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \leq 2 + \alpha,$$

where the maximal value of violation is $\sqrt{8 + 2\alpha^2}$, $\alpha \in [0, 2)$, $A_i$ and $B_i$ being observables
with outcomes $\{-1, +1\}$ measured locally by Alice and Bob. Here, we omit the notation
“$\otimes$” between systems $A$ and $B$ and write $A_0 \otimes I$ as $A_0$ for short. After performing local
measurements, if Alice and Bob obtain the maximal violation of tilted CHSH inequality,
the state shared by them is a certain partially entangled two-qubit state (tilted Bell state).

For detailed case, the four tilted Bell states $|Bell^{b}_{\theta}\rangle$, $b = 0, 1, 2, 3$ are given by

$$|Bell^{0}_{\theta}\rangle = c_\theta|00\rangle + s_\theta|11\rangle, \quad |Bell^{1}_{\theta}\rangle = s_\theta|00\rangle - c_\theta|11\rangle,$$

$$|Bell^{2}_{\theta}\rangle = c_\theta|01\rangle + s_\theta|10\rangle, \quad |Bell^{3}_{\theta}\rangle = s_\theta|01\rangle - c_\theta|10\rangle,$$

where $c_\theta = \cos \theta, s_\theta = \sin \theta, \theta \in [0, \frac{\pi}{4}]$. Let $\mu$ satisfy $\tan \mu = \sin 2\theta$ and $\sigma_Z, \sigma_X$ be Pauli
matrices. If one fixes the measurement settings of Alice and Bob as $A_0 = \sigma_Z, A_1 = \sigma_X, B_0 = \cos \mu \sigma_Z + \sin \mu \sigma_X, B_1 = \cos \mu \sigma_Z - \sin \mu \sigma_X$, the output statistics obtained by
these measurements will maximally violate some tilted CHSH inequalities. The maximal
violation is $CHSH^{\theta}_b = \langle Bell^{b}_{\theta}|W^{\alpha}_b|Bell^{b}_{\theta}\rangle = \sqrt{8 + 2\alpha^2}$ with $\alpha = 2 \cos 2\theta/\sqrt{1 + \sin^2 2\theta}$, where

$$W^{\alpha}_0 = \alpha A_0 + A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1,$$

$$W^{\alpha}_1 = -\alpha A_0 + A_0 B_0 + A_0 B_1 - A_1 B_0 - A_1 B_1,$$

$$W^{\alpha}_2 = -W^{\alpha}_1, W^{\alpha}_3 = -W^{\alpha}_0.$$

Here the $W^\alpha_b$ is Bell operator acting on the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ of Alice and Bob.
It is easy to show that the eigenvalue $\sqrt{8 + 2\alpha^2}$ of the Bell operator $W^\alpha_b$ is nondegenerate
with associated eigenvector $|Bell^{b}_{\theta}\rangle$. Hence, if the maximal violation of $CHSH^{\theta}_b$ is
$\sqrt{8 + 2\alpha^2}$, the shared state will be $|Bell^{b}_{\theta}\rangle$. One can discriminate the four tilted Bell states
by the maximal violations of four tilted Bell inequality with fix measurement settings.
Furthermore, other tilted Bell states that are local-unitary (constructed by $\sigma_Z, \sigma_X$) equivalent
to the above four tilted Bell states can also be discriminated. For example, the state
$|\Phi\rangle = c_\theta|00\rangle - s_\theta|11\rangle = \sigma_Z A|Bell^{0}_{\theta}\rangle$. It can maximally violate tilted CHSH inequality with
$CHSH^{\theta}_b = \langle \sigma_Z A W^0_b \sigma^\dagger_Z A \rangle = \alpha \langle A_0 \rangle + \langle A_0 B_0 + A_0 B_1 - A_1 B_0 + A_1 B_1 \rangle$ and fixed measurements
given above.
In the entanglement swapping scenario [40] shown in Fig. 2, let Charlie perform tilted BSM whose measurement eigenstates are tilted Bell states with outcomes \( b \). Then, the remaining state will be projected into one of the four tilted Bell states \( |\text{Bell}_b^{\theta}\rangle \) conditioned on the outcomes \( b \). Conversely, if one finds that Alice and Bob share tilted Bell states \( |\text{Bell}_b^{\theta}\rangle \) for \( b \in \{0, 1, 2, 3\} \), the performed measurement of Charlie is a tilted BSM. Motivated by this idea, we will develop a procedure for preforming self-testing of tilted multipartite GSM.

**Figure 2:** An entanglement swapping scenario: Charlie shares a maximally entangled two-qubit state with each of the other two observers (Alice and Bob). If Charlie performs tilted BSM and obtains outcome \( b \), then Alice and Bob will be projected into \( |\text{Bell}_b^{\theta}\rangle \), i.e., Alice and Bob can observe the maximal violation of the specific tilted CHSH inequality with \( CHSH_b^{\alpha} = \sqrt{8 + 2\alpha^2} \).

### 3 Self-test tilted multipartite GSM

As shown in Ref. [41], any completely positive and trace preserving (CPTP) map can be implemented, by tracing out degrees of freedom that does not involve effective information after applying a local isometry. Therefore, one can adopt the approach presented in [15, 36, 38] to present the definition for self-testing multipartite measurements via simulation: denote an ideal \( d \)-outcome measurement for Roy acting on \( H_{R_1} \otimes H_{R_2} \otimes \cdots \otimes H_{R_N} \) as \( P' = \{ P'_{R_1 R_2 \ldots R_N} \}_{r=1}^d \), and a real measurement acting on \( H_{R_1} \otimes H_{R_2} \otimes \cdots \otimes H_{R_N} \) as \( P = \{ P_{R_1 R_2 \ldots R_N} \}_{r=1}^d \). If there exist completely positive and unital maps \( \Lambda_{R_j} : L(H_{R_j}) \to L(H_{A'_j}) \), for \( j \in \{1, 2, \ldots, N\} \), such that

\[
\Lambda_{R_1} \otimes \Lambda_{R_2} \otimes \cdots \otimes \Lambda_{R_N} (P_{R_1 R_2 \ldots R_N}) = P'_{R_1 R_2 \ldots R_N},
\]

for all \( r \), we say \( P \) is capable of simulating \( P' \). In the above definition, we adopt the assumption that the different physical sources are independent in a quantum network. The construction of a quantum network as shown in Fig. 1 guarantees the well-defined \( N \)-partition for Roy’s measurement device, i.e., \( H_R = H_{R_1} \otimes H_{R_2} \otimes \cdots \otimes H_{R_N} \).

The idea of our self-testing method relies on the task of entanglement swapping as shown in Fig. 1. There are \( N \) initially uncorrelated parties Alice 1 (A1), Alice 2 (A2), …, Alice N (AN). They are independently entangled with an additional party, Roy. Specifically, the \( A_i \) and Roy share a Bell state \( |\text{Bell}_{\pi/4}^{A_i R_i}\rangle \) \( A_i R_i \in H_{A_i} \otimes H_{R_i} \), \( i \in \{1, 2, \ldots, N\} \). To distribute entanglement between \( A_1, A_2, \ldots, A_N \) in such a quantum network, Roy performs the tilted GSM and obtains outcomes \( r \in \{k_1 k_2 \ldots k_N\} \) with \( k_1, k_2, \ldots, k_N \in \{0, 1\} \). For simplicity, we denote the outcomes as \( r \in \{0, 1, \ldots, 2^N - 1\} \). Then, the states shared by \( A_1, A_2, \ldots, A_N \) are projected to one of the \( 2^N \) tilted GHZ states \( |\text{GHZ}_r^{\theta}\rangle \) based on the outcome \( r \). The
tilted GHZ states are measurement eigenstates of tilted GSM given by

\[ |GHZ_\theta^\alpha\rangle = (-1)^r |k_1 k_2 \ldots k_N\rangle \cos \theta |\tilde{k}_1 \tilde{k}_2 \ldots \tilde{k}_N\rangle, \]

where \( \tilde{k}_i = 1 - k_i \) and \( k_i \in \{0, 1\}, \ i \in \{1, \ldots, N\} \). The tilted GSM can be denoted as
\( GSM_\theta = \{GHZ_\theta^\alpha\}_r, \) with \( GHZ_\theta = |GHZ_\theta^\alpha\rangle \langle GHZ_\theta^\alpha| \). If \( A_1, A_2, \ldots, A_N \) obtain the outcomes \( r \) of Roy, they can apply a special local unitary operation on their qubits, so that they share a special tilted GHZ state. With the above operations, we have implemented distribution of entanglement between \( N \) remote parties.

The self-testing procedure is similar to the task of entanglement swapping, without assumptions on the dimensions, initial states and operators. From now on, let us adopt labels on the same letter to make a distinction between two Hilbert spaces, e.g., \( Q \) and \( Q' \). Specially, the \( Q' \) is in a two-dimensional Hilbert space and the dimension of \( Q \) is unknown. Let us start with presenting a self-testing method for \( N \)-partite tilted GHZ states given in Ref. [26].

**Lemma 1.** (Please Refer to Ref. [26]). Suppose an \( N \)-partite state \( |\psi\rangle \), and a pair of binary observables \( A_{0,i}, \ A_{1,i} \) for the \( i \)-th party, for \( i = 1, \ldots, N \). For an observable \( D \), let \( P_D^a = [I + (-1)^a D]/2, \ a \in \{0, 1\} \). Let \( \mu \) satisfy \( \tan \mu = \sin 2 \theta, \ Z_i = A_{0,i}, \ X_i = A_{1,i}, \) for \( i = 1, \ldots, N - 1 \). Then, let \( Z_N \) be \( (A_{0,N} + A_{1,N})/(2 \cos \mu) \) with zero eigenvalues replaced by 1 and \( X_N' \) be \( (A_{0,N} - A_{1,N})/(2 \sin \mu) \) with zero eigenvalues replaced by 1. Define

\[
Z_N = Z_N' |Z_N'|^{-1} \quad \text{and} \quad X_N = X_N' |X_N'|^{-1}.
\]

If the following relations are satisfied:

\[
\langle \psi | P_{A_{0,i}}^0 |\psi\rangle = \langle \psi | P_{A_{0,j}}^0 P_{A_{1,i}}^1 |\psi\rangle = c_\theta^2, \quad \forall i, j \in \{1, \ldots, N - 1\},
\]

\[
\langle \psi | \prod_{i=1}^{N-2} P_{A_{i,j}}^{a_i} |\psi\rangle = \frac{1}{2^{N-2}}, \quad \forall a_i \in \{0, 1\},
\]

\[
\langle \psi | \prod_{i=1}^{N-2} P_{A_{1,i}}^{a_i} |\psi\rangle (\alpha A_{0,N-1} \otimes I_N + A_{0,N-1} A_{0,N} + A_{0,N-1} A_{1,N} +
\]

\[
(-1)^{\sum_{i=1}^{N-2} a_i} (A_{1,N-1} A_{0,N} - A_{1,N-1} A_{1,N}) |\psi\rangle = \frac{\sqrt{8 + 2 \alpha^2}}{2^{N-2}},
\]

\[
\forall a_i \in \{0, 1\},
\]

where \( \alpha = 2 \cos 2 \theta / \sqrt{1 + \sin^2 2 \theta} \) and \( c_\theta = \cos \theta, \ \theta \in (0, \pi/4] \), there exists a local isometry \( \Phi \) such that

\[
\Phi(|\psi\rangle) = |\text{junk}\rangle |GHZ_\theta^0\rangle,
\]

for some junk state \( |\text{junk}\rangle \). Hence, these relations for correlations self-test the state \( |GHZ_\theta^0\rangle = \cos \theta |0\rangle \otimes |\psi\rangle + \sin \theta |1\rangle \otimes |\psi\rangle \).

The junk state in the Lemma 1 can be any state and can be removed by tracing out the \( A_1 A_2 \ldots A_N \) space. It should be noted that this self-testing method is also suitable for a general \( \rho \) [26]. Without loss of generality, let the \( N \)-partite state be a pure state. Here, the \( Z_N \) and \( X_N \) act on \( |\psi\rangle \) in the same way as \( (A_{0,N} + A_{1,N})/(2 \cos \mu) \) and \( (A_{0,N} - A_{1,N})/(2 \sin \mu) \), respectively [38]. For details, the ideal measurements achieving these correlations in the Lemma 1 are: \( A_{0,i}' = \sigma_Z, \ A_{1,i}' = \sigma_X, \) for \( i = 1, \ldots, N - 1 \), and \( A_{0,N}' = \cos \mu \sigma_S + \sin \mu \sigma_X, \ A_{1,N}' = \cos \mu \sigma_Z - \sin \mu \sigma_X \).

From the Lemma 1, all partially entangled \( N \)-partite GHZ states can be self-tested by checking whether the projected state of the remaining two parties \( (A_{N-1} \text{ and } A_N) \) maximally violates the tilted CHSH inequality. The remaining two parties are the parties
The detailed form of a swap gate is shown in Fig. 3. From the Lemma 1 in Ref. [36], one relations are satisfied:

after performing local measurements on the other \( N - 2 \) parties. Moreover, for different \( r \in \{0, 1, \ldots, 2^N - 1\} \), the \( |GHZ_\theta^0\rangle \) can all be self-tested by correlations in the Lemma 1 with different measurement settings up to local isometries. In other words, one can obtain a local isometry, such that \( \Phi'(|\psi^r\rangle) = |\text{junk}\rangle|GHZ_\theta^0\rangle \), for each \( r \). As the isometry can always be constructed by local operations which does not depend on \( r \), one can always construct a single isometry, such that \( \Phi(|\psi^r\rangle) = |\text{junk}\rangle|GHZ_\theta^0\rangle \). The detailed description will be shown in the next lemma.

Now, let us firstly introduce some notations. For an observable \( O' \) acting on Hilbert space \( \mathcal{H}' = \otimes_{i=1}^N \mathcal{H}'_i \), let \( \tilde{O}' = U^{tr}O'U^{tr} \), where \( U^{tr} = \otimes_{i=1}^N U^{tr}_i \) acting on \( \mathcal{H}_i \). The \( \mathcal{H}'_i, i \in \{1, 2, \ldots, N\} \), are two dimensional Hilbert spaces. The unitary operator \( U^{tr} \) satisfies the equation \( U^{tr}|GHZ_\theta^0\rangle = |GHZ_\theta^0\rangle \) and is constructed by the product of identity matrix \( I_i \), and Pauli matrices \( X_i, Z_i \). By the above special unitary transformation, one can obtain following Lemma 2.

Lemma 2. Let \( |\psi\rangle \) be an \( N \)-partite state, and let \( A_{0,i}, A_{1,i} \) be a pair of binary observables for the \( i \)-th party, for \( i = 1, \ldots, N \). Suppose that, for all \( r \in \{0, 1, \ldots, 2^N - 1\} \), the following relations are satisfied:

where \( \bar{a} \equiv a_1 \ldots a_{N-2} \) and

The detailed forms for \( P^{tr}_{A_{0,i}} \), \( P^{tr}_{A_{1,i}} \) are easy to calculate and the details for \( W^{tr}_{\bar{a}} \) as an example are provided in the Appendix B. The measurements here are the same as shown in the Lemma 1. Then, there exists a single local isometry such that \( \Phi(|\psi^r\rangle) = |\text{junk}\rangle|GHZ_\theta^0\rangle \), for all \( r \).

Proof. For \( r = 0 \), the correlations in the Lemma 2 are same as the Lemma 1. Hence these correlations self-test state \( |GHZ_\theta^0\rangle \). Denote \( |\psi\rangle \) in the self-testing procedure as \( |\psi^0\rangle \). From the Lemma 1, there exists a local isometry \( \Phi \) such that \( \Phi(|\psi^0\rangle) = |\text{junk}\rangle|GHZ_\theta^0\rangle \). Meanwhile, \( X_f^2 = Z_f^2 = I \) and \( X_f, Z_f \) anti-commute over the support of the state \( |\psi^0\rangle \), for all \( f \in \{A_1, A_2, \ldots, A_N\} \) [26]. Then, one can construct this isometry by ancillary qubits \( |0\rangle^\otimes N \) and swap gates \( \{S_{X_f, Z_f}\} \) as

The detailed form of a swap gate is shown in Fig. 3. From the Lemma 1 in Ref. [36], one knows that \( S_{X_f, Z_f, X} |0\rangle |\xi_f\rangle = X^f S_{X_f, Z_f} |0\rangle |\xi_f\rangle \) and \( S_{X_f, Z_f, Z} |0\rangle |\xi_f\rangle = Z^f S_{X_f, Z_f} |0\rangle |\xi_f\rangle \).
Let $S_{A_1,A_2,...,A_N} = (\otimes_{i=1}^{N} S_{X_{A_i},Z_{A_i}})$. As the $U^r$ is constructed by $I$, $X$, $Z$, one has
\[
\Phi(U^r|\psi^0) = S_{A_1,A_2,...,A_N}|0^\otimes NU^r|\psi^0 = U^r S_{A_1,A_2,...,A_N}|0^\otimes N|\psi^0 = U^r|\psi^0 \rangle = |junk\rangle \otimes |GHZ^\theta_0\rangle.
\]
Here $U^r|\psi^0 = |\psi\rangle$. One has $\Phi(|\psi\rangle) = |junk\rangle|GHZ^\theta_0\rangle$. Therefore, the relations for correlations in the Lemma 2 self-test state $|GHZ^\theta_0\rangle$. The $|\psi\rangle$ can be denoted as $|\psi^r\rangle$. Thus, one has $\Phi(|\psi^r\rangle) = |junk\rangle|GHZ^\theta_0\rangle$.

\[
|0\rangle \quad |\xi_f\rangle
\]
\begin{center}
\begin{figure}
\centering
\begin{circuitikz}
\draw[blue,fill=blue!20] (-2.5,0) rectangle (2.5,2.5);
\node (A) at (0,0) [shape=rectangle,draw,fill=white] {$S_{X_f, Z_f}$};
\node (B) at (0,1.5) [shape=rectangle,draw,fill=white] {$H'$};
\node (C) at (1.5,1.5) [shape=rectangle,draw,fill=white] {$H'$};
\node (D) at (-1.5,1.5) [shape=rectangle,draw,fill=white] {$H'$};
\node (E) at (0,-1) [shape=rectangle,draw,fill=white] {$X$};
\node (F) at (-1.5,-1) [shape=rectangle,draw,fill=white] {$X$};
\node (G) at (1.5,-1) [shape=rectangle,draw,fill=white] {$X$};
\node (H) at (0,2.5) [shape=rectangle,draw,fill=white] {$S_{X_f, Z_f}$};
\node (I) at (-2.5,0) [shape=rectangle,draw,fill=white] {$S_{X_f, Z_f}$};
\node (J) at (2.5,0) [shape=rectangle,draw,fill=white] {$S_{X_f, Z_f}$};
\draw (A) -- (B) -- (C) -- (D) -- (A);
\draw (E) -- (F) -- (G) -- (H) -- (I) -- (J);
\end{circuitikz}
\caption{Swap gate is constructed by unitary $X$, $Z$ and $H'$, where $H'$ is the Hadamard gate, and $X$, $Z$ are anti-commute over the support of the state $|\xi_f\rangle \in \mathcal{H}$. The $|0\rangle$ is in the qubit Hilbert space $\mathcal{H}'$.}
\end{figure}
\end{center}

From the Lemma 2, self-testing method with fixed measurements can be used to distinguish special entangled pure states. Here, let $\{|GHZ^\theta_0^r\rangle\}_{r=0}^{2^N-1}$ be reference states and $|GHZ^\theta_0^0\rangle$ be a standard reference state. For example, there is a set of states $\{|\psi^s\rangle\}_{s=0}^{2^{N-1}}$ shared by $A_1, A_2, \ldots, A_N$. If one shared state $|\psi^s\rangle$ satisfies the correlations in the Lemma 2 with $r = 0$, one can specify the shared state $|\psi^s\rangle$ as state $|\psi^0\rangle$ according to standard reference state $|GHZ^\theta_0^0\rangle$. Then, for another shared state $|\psi^s_{\text{2}}\rangle$ with $s_2 \in \{0, 1, 2, \ldots, s_1 - 1, s_1 + 1, \ldots, 2^N - 1\}$, if it satisfies correlations in the Lemma 2 for one $r$ with $r \in \{1, 2, \ldots, 2^N - 1\}$, e.g., $r = 3$, then, one resets the $s_2$ as $s_2 = 3$. In other words, the state $|\psi^s_{\text{2}}\rangle$ can be rewritten as $|\psi^3\rangle$ and these correlations have self-tested the $|GHZ^\theta_0^3\rangle$. Therefore, the states $|\psi^s\rangle$ and $|\psi^s_{\text{2}}\rangle$ are actually different. Now, the main result of the paper is following.

**Theorem 1.** Let $A_1, A_2, \ldots, A_N$ share respectively a pair of quantum state with Roy as $\tau_{A_1R_1A_2R_2\ldots A_NR_N} = \tau_{A_1R_1} \otimes \tau_{A_2R_2} \otimes \ldots \otimes \tau_{A_NR_N}$ and let $\mathcal{R} = \{R_{R_1R_2\ldots R_N}^r\}_{r=0}^{2^N-1}$ be a $2^N$-outcome measurement acting on $\mathcal{H}_{R_1} \otimes \mathcal{H}_{R_2} \otimes \ldots \otimes \mathcal{H}_{R_N}$. For the $A_1, A_2, \ldots, A_N$, if there exist measurements such that the observed correlations conditioned on outcome $r$ of Roy’s measurement satisfy the relations in the Lemma 2, then there exist completely positive and unital maps $\Lambda_{R_i} : \mathcal{L}(\mathcal{H}_{R_i}) \to \mathcal{L}(\mathcal{H}'_{A_i})$, $i \in \{1, 2, \ldots, N\}$, for $\dim(\mathcal{H}'_{A_i}) = 2$ such that
\[
\Lambda_{R_1} \otimes \Lambda_{R_2} \otimes \ldots \otimes \Lambda_{R_N}(R_{R_1R_2\ldots R_N}^r) = GHZ^\theta_0^r
\]
for $r \in \{0, 1, 2, \ldots, 2^N - 1\}$.

The detailed proof is shown in Appendix A. Here, we present a brief description. Let the $\tau_{A_1A_2\ldots A_N}^r = |\psi^r\rangle \langle \psi^r| \otimes |\psi^r\rangle \langle \psi^r|$ acting on $\otimes_{i=1}^{N} \mathcal{H}_i$ be the state shared by $A_1, A_2, \ldots, A_N$ conditioned on outcome $r$. From the Lemma 2, there exists a single isometry such that $\Phi(|\psi^r\rangle) = |junk\rangle|GHZ^\theta_0\rangle$. By tracing out the subsystems $\mathcal{H}_{1}, \ldots, \mathcal{H}_{N}$, one can construct a single pair of swap channels $\Gamma_{A_i} : \mathcal{L}(\mathcal{H}_{A_i}) \to \mathcal{L}(\mathcal{H}'_{A_i})$, $i \in \{1, 2, \ldots, N\}$, such that
\[
(\otimes_{i=1}^{N} \Gamma_{A_i}(\tau_{A_1A_2\ldots A_N}^r)) = |GHZ^\theta_0^r\rangle \langle GHZ^\theta_0^r|,
\]
for all $r$. With the help of Choi-Jamiolkowski map [36], one can construct completely positive and unital maps which are associated with above swap channels, such that

$$(\otimes_{i=1}^N A_{R_i})(R_{R_1R_2...R_N}) = (\otimes_{i=1}^N \Gamma_{A_i})(\tau^f_{A_1A_2...A_N}) = GHZ^f_r.$$  

The $2^N$ equations given by Eq. (7) imply that a real measurement $\mathcal{R} = \{R_{R_1...R_N}\}_{r=0}^{2^N-1}$ is capable of simulating ideal tilted GSM, $\{GHZ^f_r\}_{r=0}^{2^N-1}$, i.e., the Theorem 1 self-tests the tilted GSM. The method presents a unified form of the theorem for multipartite case without resorting to different Bell inequalities. Furthermore, one can also self-test multipartite GSM, if $\alpha = 0, \theta = \pi/4$. Moreover, if $N = 3$, one can recover the case of three-qubit GSM [36].

Remarkably, for any self-testing method of tilted GHZ-states, if the ideal measurements in the self-testing procedure are constructed by Pauli matrices, it can be adopted to self-test tilted GSM. Such a property can be a rule to construct the self-testing method for tilted GSM.

4 Robust self-testing of the GSM

The ideal self-testing method is an excellent tool to device-independently certify quantum information tasks. However, due to the imperfection of quantum devices, the accurate correlations in the above theorem may not be satisfied. Hence, a robust version of self-testing is necessary from an experimental point of view. For convenience, we will study here a robust self-testing scheme of three-qubit GSM, where $N = 3, \alpha = 0, \theta = \pi/4$. The method for studying robustness of other cases is similar.

Before presenting the robustness of GSM, let us firstly study the robust self-testing of the GHZ state with semi-definite programs (SDP) method. One can rewrite $A_1, A_2, A_3$ as $A, B, C$ and let $A_{i,1} = A_i, A_{i,2} = B_i, A_{i,3} = C_i, i \in \{0, 1\}$. Let the state shared by $A, B$ and $C$ with outcome $r = 0$ be $\tau^0_{ABC} = |\psi^0\rangle\langle\psi^0|$. In a general way, one can adopt the fidelity $F = \langle GHZ|\sigma_{A'B'C'}^0|GHZ \rangle$ to capture the distance of the unknown state to the target state [42], where $|GHZ \rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}$ and $\sigma_{A'B'C'}^0 = \Gamma_A \otimes \Gamma_B \otimes \Gamma_C (\tau_{ABC}^0)$. The maps $\Gamma_f, f \in \{A, B, C\}$ are defined from Fig. 3 as $\Gamma_f(\xi) = Tr_{H_f}(S_{X_fZ_f}|0\rangle\langle 0| \otimes \xi)\langle \xi|S^\dagger_{X_fZ_f})$ with $f \in \{A, B, C\}$. Here, the assumption that $X, Z$ are anti-commutative in the definition of $\Gamma$ has been removed. The state $\sigma_{A'B'C'}^0$ can be written as

$$\sigma_{A'B'C'}^0 = Tr_{ABC}(S_{ABC}|000\rangle\langle 000| \otimes \tau^0_{ABC} S^\dagger_{ABC}).$$  

From the definition of fidelity, one has

$$F = \langle GHZ|\sigma_{A'B'C'}^0|GHZ \rangle$$

$$= \frac{1}{128} Tr_{ABC} \{8(1 + Z_A)(1 + Z_B)(1 + Z_C)\tau^0_{ABC}$$

$$+ 8(1 - Z_A)(1 - Z_B)(1 - Z_C)\tau^0_{ABC}$$

$$+ (\Pi_{f \in \{A,B,C\}}(1 + Z_f)X_f(1 - Z_f))\tau^0_{ABC}$$

$$+ (\Pi_{f \in \{A,B,C\}}(1 - Z_f)X_f(1 + Z_f))\tau^0_{ABC} \},$$  

where the fidelity can be expressed as a linear function of the expectation values. Suppose the channel suffers with white noise (weight $\epsilon$), one can transform the problem of robustness
into the problem that finding a lower bound on the fidelity. It can be solved by SDP [24, 42–44]:
\[
\begin{align*}
\min \ F &= \langle GHZ|\sigma^0_{ABC'}|GHZ\rangle, \\
\text{s.t.} \ M &\geq 0, \\
\langle \psi|P^0_{A0}|\psi\rangle &= \langle \psi|P^0_{B0}|\psi\rangle = \frac{1}{2}, \\
\langle \psi|P^0_{A0}P^0_{B0}|\psi\rangle &= \frac{1-\epsilon}{2} + \frac{\epsilon}{4}, \\
\langle \psi|P^a_{A1}|\psi\rangle &= \frac{1}{2}, \quad \text{for} \ a \in \{0, 1\}, \\
\langle \psi|P^a_{A1}(\alpha B_0 + B_0 C_0 + B_0 C_1 + (-1)^a(B_1 C_0 - B_1 C_1))|\psi\rangle &= \sqrt{2}(1-\epsilon),
\end{align*}
\]

where \( M \) is a moment matrix defined by \( M_{ij} = \text{Tr}(\tau^0_{ABC}D_i^\dagger D_j) \) with set \( \{D_1 = I, D_2 = Z_A, D_3 = X_A, \ldots\} \) [45]. For an ideal case, the fidelity is 1 when error \( \epsilon = 0 \). For other \( \epsilon \) up to 0.1225, the relations between minimal fidelity and error are shown in Fig. 4. Thus, the Fig. 4 gives a lower bound of fidelity for different \( \epsilon \). Without loss of generality, one can define the relation between minimal fidelity and \( \epsilon \) as a function \( G(\epsilon_0) \), which will be used to study the robustness of GSM. Here, the \( \epsilon \) has been rewritten as \( \epsilon_0 \).

\[\text{Figure 4: The lower bound on the fidelity } F \text{ between GHZ state and unknown state } \sigma^0_{ABC'}. \text{ When the fidelity is above the nontrivial bound of 0.5 (i.e., } \epsilon \leq 12.25\%, \text{) the unknown state is close to a GHZ state.}\]

For defining quality of real measurement \( \mathcal{R} \) as a simulation of ideal GSM \( \mathcal{P} \), where \( \mathcal{R} = \{R_{R_1R_2R_3}^R\}_0^7 \) and \( \mathcal{P} = \{GHZ^r\}_0^7 \), we directly extend the definition in Ref. [36] to three parties as
\[
Q(\mathcal{R}, \mathcal{P}) = \frac{1}{8} \times \max_{\Lambda_{R_1}, \Lambda_{R_2}, \Lambda_{R_3}} \sum_{i=0}^7 \langle (\Lambda_{R_1} \otimes \Lambda_{R_2} \otimes \Lambda_{R_3})(R_{R_1R_2R_3}^R), GHZ^r \rangle.
\]

Here, we omit the subscript of \( GHZ^r_\epsilon \) as \( GHZ^r \) and \( \Lambda_{R_1}, \Lambda_{R_2}, \Lambda_{R_3} \) are unital CPTP maps with \( \Lambda_{R_1} : \mathcal{L}(\mathcal{H}_{R_1}) \to \mathcal{L}(\mathcal{H}_A), \Lambda_{R_2} : \mathcal{L}(\mathcal{H}_{R_2}) \to \mathcal{L}(\mathcal{H}_{B'}), \Lambda_{R_3} : \mathcal{L}(\mathcal{H}_{R_3}) \to \mathcal{L}(\mathcal{H}_{C'}) \). The symbol \( \langle , , \rangle \) for two matrices \( L_1 \) and \( L_2 \) implies
\[
\langle L_1, L_2 \rangle = \text{Tr}(L_1 L^\dagger_2).
\]
Now, the robust version of self-testing method is presented as following.

**Theorem 2.** Let $A, B, C$ share a pair of quantum state with Roy respectively as $\tau_{AR_1} \otimes \tau_{BR_2} \otimes \tau_{CR_3}$ and let $\mathcal{R} = \{R_{R_1R_2R_3}\}^{r=0}_7$ be a 8-outcome measurement acting on $\mathcal{H}_{R_1} \otimes \mathcal{H}_{R_2} \otimes \mathcal{H}_{R_3}$. Let $p_r$ be the probability of Roy observing the outcome $r$. Define the function $G(\epsilon)$ as the lower bound on the fidelity between $\Gamma_A \otimes \Gamma_B \otimes \Gamma_C(\tau_{ABC}^{r})$ and $\text{GHZ}^r$ under noise $\epsilon$. For $A, B$ and $C$, suppose there exist measurements, such that the observed correlations conditioned on outcomes $r$ satisfy the relations in the Lemma 2 with error $\epsilon$ and $G(\epsilon) > 0.5$. Define $q = \sum_r p_r G(\epsilon)$, then one has

$$Q(\mathcal{R}, \mathcal{P}) \geq \frac{1}{2(1 + 2\sqrt{q(1-q)})^2} \cdot \min_{u \in [0, 2\sqrt{q(1-q)}]} \left( \frac{2q - 1}{\sqrt{(1-u^2)}} + \frac{1}{(1+u)} \right).$$

(12)

The detailed proof is given in Appendix C. One can always let every $\epsilon$ be $\max \{\epsilon\}^{r=0}_7$ and denote it as $\epsilon$. Then, one has $q = G(\epsilon)$, which can be obtained by numerical method of SDP problem. The relation between quality of unknown real measurement and the noise $\epsilon = \max \{\epsilon\}^{r=0}_7$ is shown in Fig. 5. Thus, we have shown the robust self-testing scheme of the GSM with the noise tolerance up to 0.28%. From the definition of quality $Q(\mathcal{R}, \mathcal{P})$ (11), it should go through all possible unital CPTP maps $\Lambda_{R_1}, \Lambda_{R_2}, \Lambda_{R_3}$ and then choose the maximal value. However, our result is currently based on only one choice of these maps. Hence, if one optimizes this question and finds the maximum result, a better robustness can be expected. With the help of SDP method, one can straightforwardly obtain the robust version of our self-testing method for multipartite tilted GSM, similar to the robust self-testing method done for three-qubit GSM here.

![Figure 5: The lower bound on the quality of the unknown real measurement is numerically estimated as a function about the weight of white noise $\epsilon$. When the weight of white noise $\epsilon \leq 0.28\%$ (i.e., the quality is above the nontrivial bound of 0.5), the presented procedure guarantees the unknown measurement is close to a three-qubit GHZ-state measurement.](image)

5 Conclusion

In quantum network, it is extremely vital to certify multipartite entangled measurements. Here, we have presented the first self-testing method for the important class of general GHZ-state measurements. The procedure is operational for arbitrary number of parties.
from experimental point of views, and does not resort to the common method of verifying \( N \)-partite Bell inequalities. Meanwhile, the approach can recover the case of three-qubit GHZ-state measurement directly. In addition, we have provided robustness of the self-testing procedure with the help of semi-definite program. The noise tolerance is up to 0.28% when certifying a three-qubit GHZ-state measurement.

For future works, it is interesting to develop more robust method to open the possibility to estimate the robustness of arbitrary multipartite entangled measurements, and enable experiments about self-testing quantum networks. It is expected that our approach can also be extended to high dimensional case, as the self-testing method done for high dimensional entangled states [26].

6 Acknowledgements

We sincerely thank Xinhui Li for insightful discussions about the technology of semi-definite programs. This work has been supported by the National Natural Science Foundation of China (Grants No. 62031024, 11874346), the National Key R&D Program of China (2019YFA0308700) and the Anhui Initiative in Quantum Information Technologies (AHY060200).

Appendix A: Proof of the Theorem 1

As shown in the Theorem 1, if the observed correlations conditioned on outcome of Roy’s measurement satisfy the relations in the Lemma 2, the measurement performed by Roy is a tilted GHZ-state measurement. Now, let us present the detailed proof of it.

**Proof.** Let \( p_r \) be the probability of Roy observing the outcome \( r \), and \( \tau^f_{A_1A_2...A_N} = |\psi^r\rangle_{A_1A_2...A_N} \langle \psi^r| \) be the state shared between \( A_1...A_N \) conditioned on outcome \( r \in \{0,...,2^N-1\} \), i.e., \( p_r \tau^f_{A_1A_2...A_N} = Tr_{R_1R_2...R_N}[(I_{A_1A_2...A_N} \otimes R^r_{R_1R_2...R_N}) (\otimes_{i=1}^N \tau^f_{A_iR_i})] \). One can always choose \( p_r = \frac{1}{2^N} \). By the definition of swap gate in Fig. 3, one can construct swap channels as

\[
\Gamma_f(|\xi\rangle_f\langle \xi|) = Tr_{H_f}(S_{X_f, Z_f}|0\rangle\langle 0| \otimes |\xi\rangle_f\langle \xi| S_{X_f, Z_f}^{\dagger}),
\]

where \( f \in \{A_1, A_2, \ldots A_N\} \). Define

\[
\sigma_{A_iR_i}^{f} \equiv \Gamma_{\{i\}}(\tau_{A_iR_i}^f), \quad i \in \{1, 2, \ldots, N\},
\]

\[
\sigma_{A_1'...A_N'}^{f} \equiv (\otimes_{i=1}^N \Gamma_{\{i\}}(\tau_{A_iA_i'}^f)
\]

\[
= (\frac{1}{p_r}) Tr_{R_1R_2...R_N}(R^r_{R_1R_2...R_N} (\otimes_{i=1}^N \sigma_{A_iR_i}^f))
\]

\[
= (2^N) Tr_{R_1R_2...R_N}(R^r_{R_1R_2...R_N} (\otimes_{i=1}^N \sigma_{A_iR_i}^f)).
\]

Then, one has

\[
(\otimes_{i=1}^N \sigma_{A_i}^f)(\tau_{A_1A_2...A_N}^r)
\]

\[
= Tr_{A_1A_2...A_N}(S_{A_1A_2...A_N}|0\rangle_{A_1}^{N} \otimes \tau_{A_1A_2...A_N} \otimes_{i=1}^N \sigma_{A_i^{f}})
\]

\[
= Tr_{A_1A_2...A_N}(S_{A_1A_2...A_N}|0\rangle_{A_1}^{N} \otimes \tau_{A_1A_2...A_N} \otimes_{i=1}^N \sigma_{A_i^{f}})
\]

\[
= Tr_{A_1A_2...A_N}(|\text{junk}\rangle_{A_1A_2...A_N} \otimes |GHZ_\theta^{f}\rangle \langle GHZ_\theta^{f}|)
\]

\[
= |GHZ_\theta^{f}\rangle \langle GHZ_\theta^{f}|.
\]
The third equality is from the Lemma 2. From the definition of the state $\sigma_{A_1' A_2' \cdots A_N'}$, one has
\[
\sigma_{A_1' A_2' \cdots A_N'} = GHZ_\theta^r,
\]
for all $r \in \{0, 1, \ldots, 2^N - 1\}$. Let us firstly present the definition of Choi-Jamiołkowski map \cite{36}. If $\rho_{AB}$ acts on $\mathcal{H}_A \otimes \mathcal{H}_B$, the Choi-Jamiołkowski map ($\Lambda_B : \mathcal{H}_B \rightarrow \mathcal{H}_A$) associated to it is defined by $\Lambda_B(\sigma_B) = TR_B[(I_A \otimes \sigma_B^T)\rho_{AB}]$ for $\forall \sigma_B$. Here, $\rho_{AB}$ is the Choi state and can be unnormalized. Now, let $\Lambda_{R_1} : \mathcal{L}(\mathcal{H}_{R_1}) \rightarrow \mathcal{L}(\mathcal{H}_{A'})$, be respectively the Choi-Jamiołkowski maps associated to the operators $2\sigma_{A_1' i} \in \{1, 2, \ldots, N\}$. By decomposing the operator $R_{R_1 R_2 \cdots R_N}$ as $R_{R_1 R_2 \cdots R_N} = \sum_k \omega_k \Phi_k$, where $\omega_k$ is the operator of $\mathcal{H}_{R_k}$, one has $\Lambda_{R_1} \otimes \Lambda_{R_2} \otimes \cdots \otimes \Lambda_{R_N} (R_{R_1 R_2 \cdots R_N} = (2^N)\sum_{i=1}^{N} \sigma_{A_i'}(R_i)) = \sigma_{A_1' A_2' \cdots A_N'} = GHZ_\theta^r$. Moreover, we will prove that these Choi maps $\Lambda_{R_i}, i \in \{1, 2, \ldots, N\}$, are unital maps. First, let us consider $\Lambda_{R_1}$, the other cases being similar. By the definition of Choi-Jamiołkowski map, one has
\[
\Lambda_{R_1}(I_{R_1}) = TR_{R_1}(2\sigma_{A_1' R_1})
= 2TR_{R_1 R_2 \cdots R_N A_2' \cdots A_N'} (\sum_{i=1}^{N} \sigma_{A_i'}(R_i))
= 2\sum_{r=0}^{2^N-1} TR_{R_1 R_2 \cdots R_N A_2' \cdots A_N'} (R_{R_1 R_2 \cdots R_N} (\sum_{i=1}^{N} \sigma_{A_i'}(R_i)))
= \frac{1}{2^N - 1} \sum_{r=0}^{2^N-1} TR_{A_2' \cdots A_N'} (\sigma_{A_1' A_2' \cdots A_N'}(I_{A_1'})) = I_{A_1'},
\]
where we have used the fact that $\sum_{r=0}^{2^N-1} R_{R_1 R_2 \cdots R_N} = I$ and $\sigma_{A_1' A_2' \cdots A_N'} = GHZ_\theta^r$.

Therefore, we have proven that the joint measurement performed by central node Roy is actually a tilted GHZ-state measurement under the conditions in the Lemma 2.

Appendix B: The detailed form of $\tilde{W}_a^r$ in the Lemma 2

In the Lemma 2, a new form of self-testing statement has been presented. The notation “$\sim$” in the $\tilde{O}^r$ means that local unitary transformations are performed on the observable $O$. Here, we will provide the details of $\tilde{W}_a^r$, where $a = a_1 \cdots a_{N-2}$. For convenience, let $N = 3$. Rewrite $A_1$, $A_2$, $A_3$ as $A$, $B$, $C$ and let $A_{i,1} = A_i$, $A_{i,2} = B_i$, $A_{i,3} = C_i$, $i \in \{0, 1\}$. The $W_0^a$ and $W_1^a$ can be obtained from the Lemma 2 as $W_0^a = \alpha B_0 \otimes I + B_0 C_0 + B_0 C_0 + B_1 C_0 - B_1 C_0 + B_1 C_0$. From the Lemma 1, one knows that $B_0 = Z$, $B_1 = X$, and $C_0 = \cos \mu Z + \sin \mu X, C_1 = \cos \mu Z - \sin \mu X$ with $\mu = \sin \theta$. The local unitary transformation performed on $W_0^a$ is $U^r = U_B^r \otimes U_C^r$. As the $U^r$ is the local unitary transformation between tilted GHZ states, one can always choose $U_B^r C^r C_1 \in \{X' \otimes X', I' \otimes X', X' \otimes I', I' \otimes I'\}$. For $r = 7$ case, one has $X' \otimes X' \otimes X' (GHZ_{\theta^7}' = [GHZ_{\theta^7}'_a$, where $[GHZ_{\theta^7}'_a = \sin \theta(000) - \cos \theta(111)$ and $[GHZ_{\theta^7}'_a = \cos \theta(000) + \sin \theta(111)$. Here, the $U^r = X' \otimes X' \otimes X'$. Thus, $\tilde{W}_a^7 = U_B^r W_0^a U_C^r = -\alpha B_0 + W_0^a$ and $\tilde{W}_a^7 = U_B^r W_1^a U_C^r = -\alpha B_0 + W_0^a$. After calculating $\tilde{W}_a^7$ for all $r \in \{0, 1, \ldots, 7\}$ and $a \in \{0, 1\}$, the detailed formulas of $\tilde{W}_a^r$ can be obtained. By replacing the symbols $I', B_i, C_i, i \in \{0, 1\}$ in $\tilde{W}_a^r$ with $I, B_i, C_i, i \in \{0, 1\}$, one can obtain the detailed form of $\tilde{W}_a^r$. In short, the $\tilde{W}_a^r$ is acquired by deleting the superscript prime of $\tilde{W}_a^r$. The $\tilde{W}_a^r$ is obtained by performing local unitary transformations on $W_a^r$. The local unitary transform-
mation depends on the transformation between states $|GHZ^0_{\theta}\rangle$ and $|GHZ^0_{\theta}\rangle$. Therefore, one can easily write the detailed form of $W^\alpha_{\bar{a}}$ in the Lemma 2.

Appendix C: Proof of the Theorem 2

In this section, we give a proof of the Theorem 2 that shows the robust self-testing of three-qubit GHZ-state measurement. If the observed correlations can not perfectly satisfy the conditions in the Lemma 2, one can not adopt the ideal self-testing method presented in the Theorem 1 directly. We should bound the quality of the unknown measurement under the certain white noise, i.e., study how close the unknown measurement performed by Roy to ideal three-qubit GHZ-state measurement. Before presenting proof of the Theorem 2, we firstly generalize the result of semi-definite program in main text as following lemma.

Lemma 3. Let $A_0, A_1, B_0, B_1, C_0, C_1$, be the pairs of observables for the three parties. If the correlations in the Lemma 2 with error $\epsilon'$ ($\theta = \pi/4, \alpha = 0$) satisfy the following relations:

$$\langle \psi | \bar{P}^r_{A_0} | \psi \rangle = \langle \psi | P^r_{B_0} | \psi \rangle = \frac{1}{2}, \quad (C1)$$

$$\langle \psi | \bar{P}^r_{A_0} \bar{P}^r_{B_0} | \psi \rangle = (1 - \epsilon')^2 + \frac{\epsilon'}{4}, \quad (C2)$$

$$\langle \psi | P^r_{A_1} | \psi \rangle = \frac{1}{2}, \quad \text{for } a \in \{0, 1\}, \quad (C3)$$

$$\langle \psi | \bar{P}^r_{A_1} W^\alpha_{\bar{a}} | \psi \rangle = \sqrt{2}(1 - \epsilon'), \quad a \in \{0, 1\}, \quad (C4)$$

then there exist fixed CPTP maps $\Gamma_A, \Gamma_B, \Gamma_C$ as shown in Appendix A, such that

$$F((\Gamma_A \otimes \Gamma_B \otimes \Gamma_C)(\tau^r_{ABC}), GHZ^r_{A'B'C'}) \geq G(\epsilon'),$$

for all $r \in \{0, 1, \ldots, 7\}$. The function $G(x)$ is defined in main text as a function about lower bound of fidelity and white noise $\epsilon'$. It is a numerical solution from SDP.

Proof. For $r = 0$, we have given the detailed process of SDP to derive this result in Sec. 4. The CPTP maps are fixed for all $r \in \{0, 1, \ldots, 7\}$. For different $r$, the observables in above correlations are all equivalent to the $r = 0$ case, up to local unitary transformations. Thus, the lower bound of fidelity for different $r$ have the same form, i.e., they have a same function $G(x)$. \hfill $\Box$

Now, we start to prove the Theorem 2 that finding the lower bound on the quality of the unknown real measurement $\{R^r_{R_1R_2R_3}\}_{r=0}^7$. As $GHZ^r_{A'B'C'}$ are pure states and from Eq. (A1), one has

$$p_r F((\Gamma_A \otimes \Gamma_B \otimes \Gamma_C)(\tau^r_{ABC}), GHZ^r_{A'B'C'}) = p_r \langle (\Gamma_A \otimes \Gamma_B \otimes \Gamma_C)(\tau^r_{ABC}), GHZ^r_{A'B'C'} \rangle = \langle \sigma^r_{A'R_1} \otimes \sigma^r_{B'R_2} \otimes \sigma^r_{C'R_3}, GHZ^r_{A'B'C'} \otimes R^r_{R_1R_2R_3} \rangle.$$ 

From the Lemma 3, there is

$$\langle \sigma^r_{A'R_1} \otimes \sigma^r_{B'R_2} \otimes \sigma^r_{C'R_3}, GHZ^r_{A'B'C'} \otimes R^r_{R_1R_2R_3} \rangle \geq p_r G(\epsilon'). \quad (C5)$$

To derive the main result, one should construct unital CPTP maps $\Lambda_{R_1} : \mathcal{L}(\mathcal{H}_{R_1}) \to \mathcal{L}(\mathcal{H}_{A'})$, $\Lambda_{R_2} : \mathcal{L}(\mathcal{H}_{R_2}) \to \mathcal{L}(\mathcal{H}_{B'})$ and $\Lambda_{R_3} : \mathcal{L}(\mathcal{H}_{R_3}) \to \mathcal{L}(\mathcal{H}_{C'})$, and then find the lower
bound on \((\Lambda_{R_1} \otimes \Lambda_{R_2} \otimes \Lambda_{R_3}(R'_{R_1R_2R_3}), GHZ_{A'B'C'})\). Let \(\lambda_{A'R_1}, \lambda_{B'R_2}\) and \(\lambda_{C'R_3}\) be the Choi states of the maps \(\Lambda_{R_1}, \Lambda_{R_2}\) and \(\Lambda_{R_3}\). One has

\[
\langle \Lambda_{R_1} \otimes \Lambda_{R_2} \otimes \Lambda_{R_3}(R'_{R_1R_2R_3}), GHZ_{A'B'C'} \rangle = \langle Tr_{R_1R_2R_3}(\lambda_{A'R_1} \otimes \lambda_{B'R_2} \otimes \lambda_{C'R_3})(I_{A'B'C'} \otimes (R'_{R_1R_2R_3})^T), GHZ_{A'B'C'} \rangle = \langle \lambda_{A'R_1} \otimes \lambda_{B'R_2} \otimes \lambda_{C'R_3}, GHZ_{A'B'C'} \otimes (R'_{R_1R_2R_3})^T \rangle = \langle \lambda_{A'R_1}^T \otimes \lambda_{B'R_2}^T \otimes \lambda_{C'R_3}^T, GHZ_{A'B'C'} \otimes R'_{R_1R_2R_3} \rangle.
\]

(C6)

To utilize the relation in Eq. (C5) into above equation, the Choi states should be constructed by \(\sigma_{A'R_1}, \sigma_{B'R_2}, \sigma_{C'R_3}\), respectively. One can bound the marginals \(\sigma_{A'}, \sigma_{B'}\) and \(\sigma_{C'}\) to guarantee the marginals of the constructed Choi states are proportional to \(I\). From the Eq. (A1), we have

\[
F((\Gamma_A \otimes \Gamma_B \otimes \Gamma_C)((\tau_{A'B'C'}^r)), GHZ_{A'B'C'}^r) = F(\sigma_{A'B'C'}^r, GHZ_{A'B'C'}^r) = \langle \sigma_{A'B'C'}^r, GHZ_{A'B'C'}^r \rangle \geq G(\epsilon').
\]

(C7)

Here, we adopt the definition in main text about notation \(\{..\}^T\) and define

\[
\sigma_{A'B'C'}^r = \sum_{r=0}^{7} p_r(\sigma_{A'B'C'}^r)^\dagger = \sum_{r=0}^{7} p_r(U_{A'}^{rR} \otimes U_{B'}^{rR} \otimes U_{C'}^{rR})\sigma_{A'B'C'}^r(U_{A'}^{rR} \otimes U_{B'}^{rR} \otimes U_{C'}^{rR})^\dagger.
\]

By calculation, one has

\[
F(\sigma_{A'B'C'}^r, GHZ_{A'B'C'}^r) = \langle \sigma_{A'B'C'}^r, GHZ_{A'B'C'}^r \rangle = \sum_{r=0}^{7} p_r(\sigma_{A'B'C'}^r, GHZ_{A'B'C'}^r) \geq \sum_{r=0}^{7} p_r G(\epsilon') = q.
\]

(C7)

Furthermore, the spectrum of \(\sigma_{A'}\) is the same as \(\sigma_{A'}^r\) because of

\[
\sigma_{A'} = Tr_{R_1R_2R_3}(\sigma_{A'R_1} R_{R_1R_2R_3}) = Tr_{R_1R_2R_3}(\sigma_{A'R_1} \otimes \sigma_{B'R_2} \otimes \sigma_{C'R_3}) = \sum_r Tr_{B'C'R_1R_2R_3}(R'_{R_1R_2R_3}(\sigma_{A'R_1} \otimes \sigma_{B'R_2} \otimes \sigma_{C'R_3})) = \sum_r p_r Tr_{B'C'}(\sigma_{A'B'C'}) = \sum_r p_r \sigma_{A'} = \sigma_{A'}^r,
\]

where we use that \(\sum_{r=0}^{N-1} R_{R_1R_2R_3} = I\). Next, we will bound the spectrum of \(\sigma_{A'}^r\). One can always find a pure state \(\sigma_{A'B'C'}^r\) to achieve the upper and lower bounds. Without loss of generality, let \(\sigma_{A'B'C'}^r = \alpha|000\rangle + \beta|111\rangle\). By inequality Eq. (C7), \(0.5 < q \leq 1\) and \(\alpha^2 + \beta^2 = 1\), one has \(\frac{1-2\sqrt{q(1-q)}}{2} \leq \alpha^2 \leq \frac{1+2\sqrt{q(1-q)}}{2}\). Thus, \(\text{spectrum}(\sigma_{A'}) = \text{spectrum}(\sigma_{A'}^r) \in [\frac{1-2\sqrt{q(1-q)}}{2}, \frac{1+2\sqrt{q(1-q)}}{2}]\). One can write the spectrum of \(\sigma_{A'}\) as

\[
\text{spectrum}(\sigma_{A'}) = \left\{ \frac{1 - \eta_{A'}}{2}, \frac{1 + \eta_{A'}}{2} \right\},
\]
where $0 \leq \eta_{A'} \leq 2\sqrt{q(1-q)} < 1$. The same bound on $\eta_{B'}$ and $\eta_{C'}$ will be obtained in a similar way for

$$\text{spectrum}(\sigma_{B'}) = \left\{ \frac{1 - \eta_{B'}}, \frac{1 + \eta_{B'}}{2} \right\}$$

and

$$\text{spectrum}(\sigma_{C'}) = \left\{ \frac{1 - \eta_{C'}}, \frac{1 + \eta_{C'}}{2} \right\}.$$ 

Now, the detailed form of Choi states are:

$$\chi_{A'R_1}^T = (\sigma_{A'}^{-1/2} \otimes I)\sigma_{A'R_1}(\sigma_{A'}^{-1/2} \otimes I),$$

$$\chi_{B'R_2}^T = \frac{2}{1 + \eta_{B'}}\sigma_{B'R_2} + \sigma_{B_2} \otimes \left(I - \frac{2}{1 + \eta_{B'}}\sigma_{B'}\right),$$

$$(C8)\qquad \chi_{C'R_3}^T = \frac{2}{1 + \eta_{C'}}\sigma_{C'R_3} + \sigma_{C_3} \otimes \left(I - \frac{2}{1 + \eta_{C'}}\sigma_{C'}\right),$$

where $\sigma_{A'} = T_{R_1} \sigma_{A'R_1}, \sigma_{B'} = T_{R_2} \sigma_{B'R_2}, \sigma_{C'} = T_{R_3} \sigma_{C'R_3}$, and $\sigma_{R_2} = T_{R_2' \sigma_{B'R_2}, \sigma_{R_3} = T_{R_3' \sigma_{C'R_3}}$. As the spectrum($\sigma_{B'}$) = \{1-\eta_{B'}, 1+\eta_{B'}\} and spectrum($\sigma_{C'}$) = \{1-\eta_{C'}, 1+\eta_{C'}\} are bounded by $0 \leq \eta_{B'} \leq 2\sqrt{q(1-q)}$, $0 \leq \eta_{C'} \leq 2\sqrt{q(1-q)}$, the $\sigma_{R_2} \otimes (I - \frac{2}{1 + \eta_{B'}}\sigma_{B'})$ are positive semidefinite. Thus, one has

$$\chi_{A'R_1}^T \otimes \chi_{B'R_2}^T \otimes \chi_{C'R_3}^T \geq \chi_{A'R_1}^T \otimes \frac{2}{1 + \eta_{B'}}\sigma_{B'R_2} \otimes \frac{2}{1 + \eta_{C'}}\sigma_{C'R_3}.$$ 

From the Lemma 3 in the supplement material for Ref. [36], the inequality

$$\chi_{A'R_1}^T \geq s(\eta_{A'})\sigma_{A'R_1} - t(\eta_{A'})\frac{I}{2} \otimes \sigma_{R_1} \tag{C9}$$

holds, where $s(x) = \frac{2}{\sqrt{1 - x^2}}, t(x) = \frac{4}{\sqrt{1 - x^2}} - \frac{4}{1 + x}$ and $\sigma_{R_1} = T_{A'\sigma_{A'R_1}}$. Therefore, one has

$$\chi_{A'R_1}^T \otimes \chi_{B'R_2}^T \otimes \chi_{C'R_3}^T \geq (s(\eta_{A'})\sigma_{A'R_1} - t(\eta_{A'})\frac{I}{2} \otimes \sigma_{R_1})$$

$$\otimes \frac{2}{1 + \eta_{B'}}\sigma_{B'R_2} \otimes \frac{2}{1 + \eta_{C'}}\sigma_{C'R_3},$$

where the inequality is from Eq. (C9) and positive semidefinite matrices $\frac{2}{1 + \eta_{B'}}\sigma_{B'R_2}, \frac{2}{1 + \eta_{C'}}\sigma_{C'R_3}$. As

$$\langle \sigma_{R_1} \otimes \sigma_{R_2} \otimes \sigma_{R_3}, R_{R_1 R_2 R_3}^r \rangle$$

$$\rightarrow T_{R_1 R_2 R_3}((\sigma_{R_1} \otimes \sigma_{R_2} \otimes \sigma_{R_3}) \cdot R_{R_1 R_2 R_3}^r)$$

$$= T_{A'B'C'R_1 R_2 R_3}((\sigma_{A'R_1} \otimes \sigma_{B'R_2} \otimes \sigma_{C'R_3}) \cdot R_{R_1 R_2 R_3}^r)$$

$$= p_r T_{A'B'C'} \sigma_{A'B'C'} = p_r,$$

one has

$$\langle I \otimes \sigma_{R_1} \otimes I \otimes \sigma_{R_2} \otimes \sigma_{R_3}, G H Z_{A'B'C'}^r \otimes R_{R_1 R_2 R_3}^r \rangle = \frac{1}{2} \langle \sigma_{R_1} \otimes \sigma_{R_2} \otimes \sigma_{R_3}, R_{R_1 R_2 R_3}^r \rangle = \frac{p_r}{2}.$$ 

Then, one arrives at

$$\langle \chi_{A'R_1}^T \otimes \chi_{B'R_2}^T \otimes \chi_{C'R_3}^T, G H Z_{A'B'C'}^r \otimes R_{R_1 R_2 R_3}^r \rangle$$

$$\geq \frac{4s(\eta_{A'})p_r G(\epsilon') - t(\eta_{A'})p_r}{(1 + \eta_{B'})(1 + \eta_{C'})}.$$
The inequality is derived from the fact that the fidelity can only increase after tracing out the subsystem. Now, we can obtain

\[ Q(R, P) \geq \frac{1}{8} \sum_{r=0}^{7} (\lambda_{A'R_1}^T \otimes \lambda_{B'R_2}^T \otimes \lambda_{C'R_3}^T, GHZ_{A'B'C'}^r \otimes R_{R_1R_2R_3}^r) \]

\[ \geq \frac{1}{8} \left( 4s(\eta_{A'}) \sum_{r=0}^{7} p_r G(\epsilon') - t(\eta_{A'}) \sum_{r=0}^{7} p_r \right) \]

\[ \geq \frac{1}{8} \left( 4s(\eta_{A'}) q - t(\eta_{A'}) \right) \]

As \( 0 < q \leq 1 \), the numerator is positive. Hence, one obtains the result

\[ Q(R, P) \geq \frac{1}{2(1 + 2\sqrt{q(1-q)})^2} \left( \frac{2q - 1}{\sqrt{(1 - \eta_{A'}^2)}} + \frac{1}{1 + \eta_{A'}} \right) \]

\[ \geq \frac{1}{2(1 + 2\sqrt{q(1-q)})^2} \cdot \min_{u \in [0, 2\sqrt{q(1-q)}]} \left( \frac{2q - 1}{\sqrt{(1 - u^2)}} + \frac{1}{1 + u} \right). \]

Here, we have presented a lower bound for the quality of unknown joint measurement performed by Roy under certain white noise. The quality implies the ability that the unknown measurement try to simulate the ideal three-qubit GHZ-state measurement. Therefore, a robust self-testing statement for three-qubit GHZ-state measurement has been shown.

References

[1] Nicolas Brunner, Daniel Cavalcanti, Stefano Pironio, Valerio Scarani, and Stephanie Wehner. Bell nonlocality. Rev. Mod. Phys., 86(2):419, 2014.
[2] Dominic Mayers and Andrew Yao. Self testing quantum apparatus. arXiv preprint quant-ph/0307205, 2003.
[3] Tzyh Haur Yang and Miguel Navascués. Robust self-testing of unknown quantum systems into any entangled two-qubit states. Phys. Rev. A, 87(5):050102(R), 2013.
[4] Anand Natarajan and Thomas Vidick. Robust self-testing of many-qubit states. arXiv preprint arXiv:1610.03574, 2016.
[5] Matthew McKague, Tzyh Haur Yang, and Valerio Scarani. Robust self-testing of the singlet. J. Phys. A: Math. Theor., 45(45):455304, 2012.
[6] Cédric Bamps and Stefano Pironio. Sum-of-squares decompositions for a family of Clauser-Horne-Shimony-Holt-like inequalities and their application to self-testing. Phys. Rev. A, 91(5):052111, 2015.
[7] Ivan Šupić, Remigiusz Augusiak, Alexia Salavrakos, and Antonio Acín. Self-testing protocols based on the chained Bell inequalities. New J. Phys., 18(3):035013, 2016.
[8] Ivan Šupić, Daniel Cavalcanti, and Joseph Bowles. Device-independent certification of tensor products of quantum states using single-copy self-testing protocols. arXiv preprint arXiv:1909.12759, 2019.
[9] Suchetana Goswami, Bihalan Bhattacharya, Debarshi Das, Souradeep Sasmal, C Jebaratnam, and A. S. Majumdar. One-sided device-independent self-testing of any pure two-qubit entangled state. Phys. Rev. A, 98(2):022311, 2018.
[10] Shubhayan Sarkar, Debashis Saha, Jędrzej Kaniewski, and Remigiusz Augusiak. Self-testing quantum systems of arbitrary local dimension with minimal number of measurements. npj Quantum Inf., 7(1):1–5, 2021.
11. Kishor Bharti, Maharshi Ray, Antonios Varvitsiotis, Naqueeb Ahmad Warsi, Adán Cabello, and Leong-Chuan Kwek. Robust Self-Testing of Quantum Systems via Non-contextuality Inequalities. *Phys. Rev. Lett.*, 122(25):250403, 2019.
12. Xinhui Li, Yukun Wang, Yunguang Han, Sujuan Qin, Fei Gao, and Qiaoyan Wen. Self-Testing of Symmetric Three-Qubit States. *IEEE J. Sel. Areas Commun.*, 38(3):589–597, 2020.
13. Andrea Coladangelo. Generalization of the Clauser-Horne-Shimony-Holt inequality self-testing maximally entangled states of any local dimension. *Phys. Rev. A*, 98(5):052115, 2018.
14. Tim Coopmans, Jędrzej Kaniewski, and Christian Schaffner. Robust self-testing of two-qubit states. *Phys. Rev. A*, 99(5):052123, 2019.
15. Jędrzej Kaniewski. Self-testing of binary observables based on commutation. *Phys. Rev. A*, 95(6):062323, 2017.
16. Armin Tavakoli, Jędrzej Kaniewski, Tamás Vértesi, Denis Rosset, and Nicolas Brunner. Self-testing quantum states and measurements in the prepare-and-measure scenario. *Phys. Rev. A*, 98(6):062307, 2018.
17. Xingyao Wu, Jean-Daniel Bancal, Matthew McKague, and Valerio Scarani. Device-independent parallel self-testing of two singlets. *Phys. Rev. A*, 93(6):062121, 2016.
18. Tzyh Haur Yang, Tamás Vértesi, Jean-Daniel Bancal, Valerio Scarani, and Miguel Navascués. Robust and Versatile Black-Box Certification of Quantum Devices. *Phys. Rev. Lett.*, 113(4):040401, 2014.
19. Nikolai Miklin, Jakub J Borkała, and Marcin Pawłowski. Semi-device-independent self-testing of unsharp measurements. *Phys. Rev. Research*, 2(3):033014, 2020.
20. C Jebarathinam, Jui-Chen Hung, Shin-Liang Chen, and Yeong-Cherng Liang. Maximal violation of a broad class of Bell inequalities and its implication on self-testing. *Phys. Rev. Research*, 1(3):033073, 2019.
21. Nikolai Miklin and Michał Oszmaniec. A universal scheme for robust self-testing in the prepare-and-measure scenario. *Quantum*, 5:424, 2021.
22. Kishor Bharti, Maharshi Ray, Zhen-Peng Xu, Masahito Hayashi, Leong-Chuan Kwek, and Adán Cabello. Graph-Theoretic Framework for Self-Testing in Bell Scenarios. *arXiv preprint arXiv:2104.13035*, 2021.
23. Andrea Coladangelo, Koon Tong Goh, and Valerio Scarani. All pure bipartite entangled states can be self-tested. *Nat. Commun.*, 8(1):15485, 2017.
24. Xingyao Wu, Yu Cai, Tzyh Haur Yang, Huy Nguyen Le, Jean-Daniel Bancal, and Valerio Scarani. Robust self-testing of the three-qubit W state. *Phys. Rev. A*, 90(4):042339, 2014.
25. Matthew McKague. Self-testing graph states. In *Conference on Quantum Computation, Communication, and Cryptography*, pages 104–120. Springer, 2011.
26. I Šupić, A Coladangelo, R Augusiak, and A Acín. Self-testing multipartite entangled states through projections onto two systems. *New J. Phys.*, 20(8):083041, 2018.
27. Pavel Sekatski, Jean-Daniel Bancal, Sebastian Wagner, and Nicolas Sangouard. Certifying the Building Blocks of Quantum Computers from Bell’s Theorem. *Phys. Rev. Lett.*, 121(18):180505, 2018.
28. Matthew McKague and Michele Mosca. Generalized self-testing and the security of the 6-state protocol. In *Conference on Quantum Computation, Communication, and Cryptography*, pages 113–130. Springer, 2010.
29. Carl A Miller and Yaoyun Shi. Robust Protocols for Securely Expanding Randomness and Distributing Keys Using Untrusted Quantum Devices. *Journal of the ACM (JACM)*, 63(4):33, 2016.
[30] Joseph Bowles, Ivan Šupić, Daniel Cavalcanti, and Antonio Acín. Device-Independent Entanglement Certification of All Entangled States. Phys. Rev. Lett., 121(18):180503, 2018.

[31] Flavio Baccari, Remigiusz Augusiak, Ivan Šupić, and Antonio Acín. Device-Independent Certification of Genuinely Entangled Subspaces. Phys. Rev. Lett., 125(26):260507, 2020.

[32] Owidiusz Makuta and Remigiusz Augusiak. Self-testing maximally-dimensional genuinely entangled subspaces within the stabilizer formalism. New J. Phys., 23(4):043042, 2021.

[33] Irénée Frérot and Antonio Acín. Coarse-Grained Self-Testing. Phys. Rev. Lett., 127(24):240401, Dec 2021.

[34] Alexandru Gheorghiu, Petros Wallden, and Elham Kashefi. Rigidity of quantum steering and one-sided device-independent verifiable quantum computation. New J. Phys., 19(2):023043, 2017.

[35] Matthew McKague. Interactive proofs for BQP via self-tested graph states. arXiv preprint arXiv:1309.5675, 2013.

[36] Marc Olivier Renou, Jędrzej Kaniewski, and Nicolas Brunner. Self-Testing Entangled Measurements in Quantum Networks. Phys. Rev. Lett., 121(25):250507, 2018.

[37] Jean-Daniel Bancal, Nicolas Sangouard, and Pavel Sekatski. Noise-Resistant Device-Independent Certification of Bell State Measurements. Phys. Rev. Lett., 121(25):250506, 2018.

[38] Ivan Šupić and Joseph Bowles. Self-testing of quantum systems: a review. Quantum, 4:337, 2020.

[39] Antonio Acín, Serge Massar, and Stefano Pironio. Randomness versus Nonlocality and Entanglement. Phys. Rev. Lett., 108(10):100402, 2012.

[40] Marek Zukowski, Anton Zeilinger, Michael A Horne, and Aarthur K Ekert. “Event-ready-detectors” Bell experiment via entanglement swapping. Phys. Rev. Lett., 71(26):4287–4290, 1993.

[41] Man-Duen Choi. Completely positive linear maps on complex matrices. Linear Algebra Appl., 10(3):285–290, 1975.

[42] Xinhui Li, Yu Cai, Yunguang Han, Qiaoyan Wen, and Valerio Scarani. Self-testing using only marginal information. Phys. Rev. A, 98(5):052331, 2018.

[43] Jean-Daniel Bancal, Miguel Navascués, Valerio Scarani, Tamás Vértesi, and Tzyh Haur Yang. Physical characterization of quantum devices from nonlocal correlations. Phys. Rev. A, 91(2):022115, 2015.

[44] Lieven Vandenberghe and Stephen Boyd. Semidefinite programming. SIAM Rev., 38(1):49–95, 1996.

[45] Miguel Navascués, Stefano Pironio, and Antonio Acín. A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations. New J. Phys., 10(7):073013, 2008.