GENERALIZED NEWMAN PHENOMENA AND DIGIT CONJECTURES ON PRIMES

VLADIMIR SHEVELEV

Abstract. We prove that the ratio of the Newman sum over numbers multiple of a fixed integer which is not multiple of 3 and the Newman sum over numbers multiple of a fixed integer divisible by 3 is o(1) when the upper limit of summing tends to infinity. We also discuss a connection of our results with a digit conjecture on primes.

1. Introduction

Denote for $x, m \in \mathbb{N}$

\begin{equation}
S_m(x) = \sum_{0 \leq n < x, n \equiv 0 \pmod{m}} (-1)^{s(n)},
\end{equation}

where $s(n)$ is the number of 1’s in the binary expansion of $n$. Sum (1) is a Newman digit sum.

From the fundamental paper of A.O.Gelfond [4] it follows that

\begin{equation}
S_m(x) = O(x^\lambda), \quad \lambda = \frac{\ln 3}{\ln 4}.
\end{equation}

The case $m = 3$ was studied in detail [5], [2], [7].

So, from the Coquet’s theorem [2], [1] it follows that

\begin{equation}
-\frac{1}{3} + \frac{2}{\sqrt{3}} x^\lambda \leq S_3(3x) \leq \frac{1}{3} + \frac{55}{3} \left(\frac{3}{65}\right) x^\lambda.
\end{equation}

with a microscopic improvement [7]

\begin{equation}
\frac{2}{\sqrt{3}} x^\lambda \leq S_3(3x) \leq \frac{55}{3} \left(\frac{3}{65}\right) x^\lambda, \quad x \geq 2,
\end{equation}

and moreover,
These estimates give the most exact modern limits of the so-called Newman phenomena. Note that M. Drmota and M. Skalba [3] using a close function $(S_m(x))$ proved that if $m$ is a multiple of 3 then for sufficiently large $x$, 

\[ S_m(x) > 0, \quad x \geq x_0(m). \]

In this paper we study a general case for $m \geq 5$ (in the cases of $m = 2$ and $m = 4$ we have $|S_m(n)| \leq 1$).

To formulate our results put for $m \geq 5$

\[ \lambda_m = 1 + \log_2 b_m, \]

\[ \mu_m = \frac{2b_m + 1}{2b_m - 1}, \]

where

\[ b_m^* = \begin{cases} 
\sin(\frac{\pi}{3}(1 + \frac{2}{m}))(\sqrt{3} - \sin(\frac{\pi}{3}(1 + \frac{2}{m}))), & \text{if } m \equiv 0 \pmod{3} \\
\sin(\frac{\pi}{3}(1 - \frac{1}{m}))(\sqrt{3} - \sin(\frac{\pi}{3}(1 - \frac{1}{m}))), & \text{if } m \equiv 1 \pmod{3} \\
\sin(\frac{\pi}{3}(1 + \frac{1}{m}))(\sqrt{3} - \sin(\frac{\pi}{3}(1 + \frac{1}{m}))), & \text{if } m \equiv 2 \pmod{3}.
\end{cases} \]

Directly one can see that

\[ \frac{\sqrt{3}}{2} > b_m \geq \begin{cases} 
0.86184088 \ldots, & \text{if } (m, 3) = 1, \\
0.85559967 \ldots, & \text{if } (m, 3) = 3,
\end{cases} \]

and thus,

\[ \lambda_m < \lambda \]

and

\[ 3.73205080 \ldots < \mu_m \leq \begin{cases} 
3.76364572 \ldots, & \text{if } (m, 3) = 1, \\
3.81215109 \ldots, & \text{if } (m, 3) = 3.
\end{cases} \]

Below we prove the following results.
Theorem 1. If \((m, 3) = 1\) then

\[ |S_m(x)| \leq \mu_m x^{\lambda_m}. \]  

Theorem 2. (Generalized Newman phenomena). If \(m > 3\) is a multiple of 3 then

\[ \left| S_m(x) - \frac{3}{m} S_3(x) \right| \leq \mu_m x^{\lambda_m}. \]  

Using Theorem 2 and (5) one can estimate \(x_0(m)\) in (6). E.g., one can prove that \(x_0(21) < e^{985}\).

2. Explicit formula for \(S_m(N)\)

We have

\[ S_m(N) = \sum_{n=0, m \mid n}^{N-1} (-1)^{s(n)} = \frac{1}{m} \sum_{t=0}^{m-1} \sum_{n=0}^{N-1} (-1)^{s(n)} e^{2\pi i (\frac{nt}{m})} = \]  

\[ = \frac{1}{m} \sum_{t=0}^{m-1} \sum_{n=0}^{N-1} e^{2\pi i (\frac{1}{m} n + \frac{1}{2} s(n))}. \]  

Note that the interior sum has the form

\[ F_{\alpha}(N) = \sum_{n=0}^{N-1} e^{2\pi i (\alpha n + \frac{1}{2} s(n))} \ 0 \leq \alpha < 1. \]  

Lemma 1. If \(N = 2^{\nu_0} + 2^{\nu_1} + \ldots + 2^{\nu_r}, \ \nu_0 > \nu_1 > \ldots > \nu_r \geq 0\), then

\[ F_{\alpha}(N) = \sum_{h=0}^{r} e^{2\pi i (\alpha \sum_{j=0}^{h-1} 2^{\nu_j} + \frac{1}{2}) \prod_{k=0}^{\nu_h-1} (1 + e^{2\pi i (\alpha 2^k + \frac{1}{2})})}, \]  

where as usual \(\sum_{j=0}^{1-1} = 0, \ \prod_{k=0}^{1-1} = 1.\)

Proof. Let \(r = 0\). Then by (16)

\[ F_{\alpha}(N) = \sum_{n=0}^{N-1} (-1)^{s(n)} e^{2\pi i \alpha n} = 1 - \sum_{j=0}^{\nu_0-1} e^{2\pi i \alpha 2^j} + \]  

\[ + \sum_{0 \leq j_1 < j_2 \leq \nu_0-1} e^{2\pi i \alpha (2^{j_1} + 2^{j_2})} - \ldots = \prod_{k=0}^{\nu_0-1} (1 - e^{2\pi i \alpha 2^k}), \]
which corresponds to (17) for \( r = 0 \).

Assuming that (17) is valid for every \( N \) with \( s(N) = r + 1 \) let us consider

\[
N = 2^{\nu_r} a + 2^{\nu_r+1}
\]

where \( a \) is odd, \( s(a) = r + 1 \) and \( \nu_{r+1} < \nu_r \). Let

\[
N_1 = 2^{\nu_r} a + 2^{\nu_r+1}
\]

Notice that for \( n \in [0,2^{\nu_r+1}) \) we have

\[
s(N + n) = s(N) + s(n).
\]

Therefore,

\[
F_\alpha(N_1) = F_\alpha(N) + \sum_{n=N}^{N_1-1} e^{2\pi i (an + \frac{1}{2} s(n))} =
\]

\[
= F_\alpha(N) + \sum_{n=0}^{2^{\nu_r+1}-1} e^{2\pi i (an + \frac{1}{2} s(N) + s(n))} =
\]

\[
= F_\alpha(N) + e^{2\pi i (aN + \frac{1}{2} s(N))} \sum_{n=0}^{2^{\nu_r+1}-1} e^{2\pi i (s(n))}.
\]

Thus, by (17) and (18),

\[
F_\alpha(N_1) = \sum_{h=0}^{r} e^{2\pi i (\alpha (\sum_{j=0}^{h-1} 2^{\nu_j} + \frac{h}{2}) + \nu_{h-1}^{-1})} \prod_{k=0}^{\nu_{h-1}^{-1}} (1 + e^{2\pi i (\alpha 2^k + \frac{1}{2})}) +
\]

\[
+ e^{2\pi i (\alpha (\sum_{j=0}^{r} 2^{\nu_j} + \frac{r+1}{2}) + \nu_{r+1}^{-1})} \prod_{k=0}^{\nu_{r+1}^{-1}} (1 + e^{2\pi i (\alpha 2^k + \frac{1}{2})}) =
\]

\[
= \sum_{h=0}^{r+1} e^{2\pi i (\alpha (\sum_{j=0}^{h-1} 2^{\nu_j} + \frac{h}{2}) + \nu_{h-1}^{-1})} \prod_{k=0}^{\nu_{h-1}^{-1}} (1 + e^{2\pi i (\alpha 2^k + \frac{1}{2})}) \square
\]

Formulas (15)-(17) give an explicit expression for \( S_m(N) \) as a linear combination of the products of the form

\[
(19) \quad \prod_{k=0}^{\nu_{h-1}^{-1}} \left(1 + e^{2\pi i (\alpha 2^k + \frac{1}{2})}\right), \quad \alpha = \frac{t}{m}, \quad 0 \leq t \leq m - 1.
\]

**Remark 1.** On can extract (17) from a very complicated general Gelfond formula [4], however, we prefer to give an independent proof.
3. Proof of Theorem 1

Note that in (17)

\[ r \leq \nu_0 = \left\lfloor \frac{\ln N}{\ln 2} \right\rfloor. \]

By Lemma 1 we have

\[ |F_\alpha(N)| \leq \sum_{\nu_h=\nu_0,\nu_1,\ldots,\nu_r} \left| \prod_{k=1}^{\nu_h} \left( 1 + e^{2\pi i(\alpha 2^{k-1} + \frac{1}{2})} \right) \right| \leq \]

\[ \leq \sum_{h=0}^{\nu_0} \left| \prod_{k=1}^{h} \left( 1 + e^{2\pi i(\alpha 2^{k-1} + \frac{1}{2})} \right) \right|. \]

Furthermore,

\[ 1 + e^{2\pi i(2^{k-1}\alpha + \frac{1}{2})} = 2 \sin(2^{k-1}\alpha \pi)(\sin(2^{k-1}\alpha \pi) - i \cos(2^{k-1}\alpha \pi)) \]

and, therefore,

\[ \left| 1 + e^{2\pi i(2^{k-1}\alpha + \frac{1}{2})} \right| \leq 2 \left| \sin(2^{k-1}\alpha \pi) \right|. \]

According to (21) let us estimate the product

\[ \prod_{k=1}^{h} (2|\sin(2^{k-1}\alpha \pi)|) = 2^h \prod_{k=1}^{h} |\sin(2^{k-1}\alpha \pi)|, \]

where by (15)

\[ \alpha = \frac{t}{m}, \quad 0 \leq t \leq m - 1. \]

Repeating arguments of [4], put

\[ |\sin(2^{k-1}\alpha \pi)| = t_k. \]

Considering the function

\[ \rho(x) = 2x\sqrt{1 - x^2}, \quad 0 \leq x \leq 1, \]
we have

\[ t_k = 2t_{k-1} \sqrt{1 - t_{k-1}^2} = \rho(t_{k-1}). \]  

Note that

\[ \rho'(x) = 2(\sqrt{1 - x^2} - \frac{x^2}{\sqrt{1 - x^2}}) \leq -1 \]

for \( x_0 \leq x \leq 1 \), where

\[ x_0 = \frac{\sqrt{3}}{2} \]

is the only positive root of the equation \( \rho(x) = x \).

Show that either

\[ t_k \leq \sin \left( \frac{\pi}{m} \left\lfloor \frac{m}{3} \right\rfloor \right) = \sin \left( \frac{\pi}{m} \left\lceil \frac{2m}{3} \right\rceil \right) = g_m < \frac{\sqrt{3}}{2} \]

or simultaneously \( t_k > g_m \) and

\[ t_k t_{k+1} \leq \max_{0 \leq l \leq m-1} \left( \left| \sin \frac{l\pi}{m} \right| \left( \sqrt{3} - \left| \sin \frac{l\pi}{m} \right| \right) \right) = \]

\[ = \begin{cases} 
   \left( \sin \left( \frac{\pi}{m} \left\lfloor \frac{m}{3} \right\rfloor \right) \right) \left( \sqrt{3} - \sin \left( \frac{\pi}{m} \left\lfloor \frac{m}{3} \right\rfloor \right) \right), & \text{if } m \equiv 1 \pmod{3} \\
   \left( \sin \left( \frac{\pi}{m} \left\lceil \frac{m}{3} \right\rceil \right) \right) \left( \sqrt{3} - \sin \left( \frac{\pi}{m} \left\lceil \frac{m}{3} \right\rceil \right) \right), & \text{if } m \equiv 2 \pmod{3} 
\end{cases} = h_m < \frac{3}{4}. \]

Indeed, let for a fixed values of \( t \in [0, m-1] \) and \( k \in [1, n] \)

\[ t^{2k-1} \equiv l \pmod{m}, \ 0 \leq l \leq m - 1. \]

Then

\[ t_k = \left| \sin \frac{l\pi}{m} \right|. \]

Now distinguish two cases: 1) \( t_k \leq \frac{\sqrt{3}}{2} \) 2) \( t_k > \frac{\sqrt{3}}{2} \).

In case 1)

\[ t_k = \frac{\sqrt{3}}{2} \iff \frac{l\pi}{m} = \frac{r\pi}{3}, \ (r, 3) = 1 \]
and since $0 \leq l \leq m - 1$ then

$$m = \frac{3l}{r}, \quad r = 1, 2.$$ 

Because of the condition $(m, 3) = 1$, we have $t_k < \frac{\sqrt{3}}{2}$.

Thus, in (33)

$$l \in \left[0, \frac{m}{3}\right] \cup \left[\frac{2m}{3}, m\right]$$

and (30) follows.

In case 2) let $t_k > \frac{\sqrt{3}}{2} = x_0$. For $\varepsilon > 0$ put

$$1 + \varepsilon = \frac{t_k}{x_0} = \frac{2}{\sqrt{3}} \left| \sin(\pi 2^{k-1} \alpha) \right|,$$

such that

$$1 - \varepsilon = 2 - \frac{2}{\sqrt{3}} \left| \sin(\pi 2^{k-1} \alpha) \right|$$

and

$$1 - \varepsilon^2 = \frac{4}{3} \left| \sin(\pi 2^{k-1} \alpha) \right| \left( \sqrt{3} - \left| \sin(\pi 2^{k-1} \alpha) \right| \right).$$

By (27) and (34) we have

$$t_{k+1} = \rho(t_k) = \rho((1 + \varepsilon)x_0) = \rho(x_0) + \varepsilon x_0 \rho'(c),$$

where $c \in (x_0, (1 + \varepsilon)x_0)$.

Thus, according to (28) and taking into account that $\rho(x_0) = x_0$, we find

$$t_{k+1} \leq x_0(1 + \varepsilon)$$

while by (34)

$$t_k = x_0(1 + \varepsilon).$$

Now in view of (35) and (29)

$$t_k t_{k+1} \leq \left| \sin \pi 2^{k-1} \alpha \right| \left( \sqrt{3} - \left| \sin(\pi 2^{k-1} \alpha) \right| \right)$$

and according to (32), (33) we obtain that

$$t_k t_{k+1} \leq h_m,$$

where $h_m$ is defined by (31).
Notice that from simple arguments and according to (9)

\[ g_m \leq \sqrt{h_m} = b_m. \]

Therefore,

\[ \prod_{k=1}^{h} \left| \sin(\pi 2^{k-1} \alpha) \right| \leq (b_m^{\frac{h}{2}})^2 \leq b_m^{h-1}. \]

Now, by (21)- (22), for \( \alpha = \frac{t}{m}, \ t = 0, 1, ..., m - 1, \) we have

\[ \left| F_{\frac{t}{m}}(N) \right| \leq \sum_{h=0}^{\nu_0} \prod_{k=1}^{h} (1 + e^{2\pi i(\alpha 2^{k-1} + \frac{1}{2})}) \leq \sum_{h=0}^{\nu_0} 2^h \prod_{k=1}^{h} \left| \sin(2^{k-1} \alpha \pi) \right| \leq 1 + 2 \sum_{h=1}^{\nu_0} (2b_m)^{h-1} \leq 1 + \frac{2 (2b_m)^{\nu_0}}{2b_m - 1}. \]

Note that, according to (7) and (20)

\[ (2b_m)^{\nu_0} = 2^{\lambda_m \nu_0} \leq 2^{\lambda_m \log_2 N} = N^{\lambda_m}. \]

Thus,

\[ |F_{\frac{t}{m}}(N)| \leq 1 + \frac{2}{2b_m - 1} N^{\lambda_m} \leq \frac{2}{2b_m - 1 - \gamma_m} N^{\lambda_m}, \]

where \( \gamma_m \) is defined by the equality

\[ \frac{1}{2b_m - 1 - \gamma_m} - \frac{1}{2b_m - 1} = \frac{1}{2}. \]

Hence, we find

\[ \gamma_m = \frac{(2b_m - 1)^2}{2b_m + 1} \]

and, consequently, by (8),

\[ |F_{\frac{t}{m}}(N)| \leq \frac{2b_m + 1}{2b_m - 1} N^{\lambda_m} = \mu_m N^{\lambda_m}. \]

Thus, the theorem follows from (15).■

4. Proof of Theorem 2.

Select in (15) the summands which correspond to \( t = 0, \ \frac{m}{3}, \ \frac{2m}{3}. \)

We have

\[ mS_m(N) = \sum_{n=0}^{N-1} \left( e^{\pi i s(n)} + e^{2\pi i(\alpha + \frac{1}{2}s(n))} + e^{2\pi i(\alpha n + \frac{1}{2}s(n))} \right) + \]

\[ + \sum_{t=1, t \neq \frac{m}{3}, \frac{2m}{3}}^{m-1} \sum_{n=0}^{N-1} e^{2\pi i(\frac{t}{m} n + \frac{1}{2}s(n))}. \]
Since the chosen summands do not depend on $m$ and for $m = 3$ the latter sum is empty then we find

\[(37) \quad mS_m(N) = 3S_3(N) + \sum_{t=1, t \neq \frac{m}{3}, \frac{2m}{3}}^{m-1} \sum_{n=0}^{N-1} e^{2\pi i \left(\frac{t}{m}n + \frac{1}{2}s(n)\right)}.
\]

Further, the last double sum is estimated by the same way as in Section 3 such that

\[(38) \quad \left| S_m(N) - \frac{3}{m} S_3(N) \right| \leq \mu_m N^{\lambda_m} \square.
\]

**Remark 2.** Notice that from elementary arguments it follows that if $m \geq 5$ is a multiple of 3 then

\[
\left( \sin \frac{\pi}{m} \left\lfloor \frac{m-1}{3} \right\rfloor \right) \left( \sqrt{3} - \sin \frac{\pi}{m} \left\lfloor \frac{m-1}{3} \right\rfloor \right) \leq \left( \sin \frac{\pi}{m} \left\lceil \frac{m+1}{3} \right\rceil \right) \left( \sqrt{3} - \sin \frac{\pi}{m} \left\lceil \frac{m+1}{3} \right\rceil \right).
\]

The latter expression is the value of $b_m^2$ in this case (see (9)).

**Example.** Let us find some $x_0$ such that $S_{21}(x) > 0$ for $x \geq x_0$.

Supposing that $x$ is multiple of 3 and using (4) we obtain that

\[S_3(x) \geq \frac{2}{3^{\lambda + \frac{1}{2}}} x^{\lambda}.
\]

Therefore, putting $m = 21$ in Theorem 2, we have

\[S_{21}(x) \geq \frac{1}{l} S_3(x) - \mu_{21} x^{\lambda_{21}} \geq \frac{2}{7 \cdot 3^{\lambda + \frac{1}{2}}} x^{\lambda} - \mu_{21} x^{\lambda_{21}}.
\]

Now, calculating $\lambda$ and $\lambda_m$ by (2) and (8), we find a required $x_0$:

\[x_0 = (3.5 \cdot 3^{\lambda + \frac{1}{2}} \mu_{21})^{\frac{1}{\lambda_{21}}} = e^{984.839...}.
\]

**Corollary.** For $m$ which is not a multiple of 3, denote $U_m(x)$ the set of the positive integers not exceeding $x$ which are multiples of $m$ and not multiples of 3. Then

\[\sum_{n \in U_m(x)} (-1)^{s(n)} = -\frac{1}{m} S_3(x) + O(x^{\lambda_m}).
\]

In particular, for sufficiently large $x$ we have
\[ \sum_{n \in U_m(x)} (-1)^{s(n)} < 0. \]

**Proof.** Since

\[ |U_m(x)| = S_m(x) - S_{3m}(x) \]

then the corollary immediately follows from Theorems 1, 2.

5. **On Newman sum over primes**

In [6] we put the following binary digit conjectures on primes.

**Conjecture 1.** For all \( n \in \mathbb{N}, \ n \neq 5, 6 \)

\[ \sum_{p \leq n} (-1)^{s(p)} \leq 0, \]

where the summing is over all primes not exceeding \( n \).
Moreover, by the observations, \( \sum_{p \leq n} (-1)^{s(p)} < 0 \) beginning with \( n = 31. \)

**Conjecture 2.**

\[ \lim_{n \to \infty} \frac{\ln \left( -\sum_{p \leq n} (-1)^{s(p)} \right)}{\ln n} = \frac{\ln 3}{\ln 4}. \]

A heuristic proof of Conjecture 2 was given in [8]. For a prime \( p \), denote \( V_p(x) \) the set of positive integers not exceeding \( x \) for which \( p \) is the least prime divisor. Show that the correctness of Conjectures 1 (for \( n \geq n_0 \)) follows from the following very plausible statement, especially in view of the above estimates.

**Conjecture 3.** For sufficiently large \( n \) we have

\[ \left| \sum_{5 \leq p \leq \sqrt{n}} \sum_{j \in V_p(n), j > p} (-1)^{s(j)} \right| < \sum_{j \in V_3(n)} (-1)^{s(j)} = S_3(n) - S_6(n). \]

Indeed, in the "worst case" (really is not satisfied) in which for all \( n \geq p^2 \)

\[ \sum_{j \in V_3(n), j > p} (-1)^{s(j)} < 0, \quad p \geq 5. \]

we have a decreasing but positive sequence of sums
\[
\sum_{j \in V_3(n), j > 3} (-1)^{s(j)} + \sum_{j \in V_3(n), j > 5} (-1)^{s(j)}, \\
\ldots, \sum_{j \in V_3(n), j > 3} (-1)^{s(j)} + \sum_{5 \leq p < \sqrt{n}} \sum_{j \in V_p(n), j > p} (-1)^{s(j)} > 0.
\]

Hence, the "balance condition" for odd numbers \[8\]
\[
\left| \sum_{j \leq n, j \text{ is odd}} (-1)^{s(j)} \right| \leq 1
\]
must be ensured permanently by the excess of the odious primes. This explains Conjecture 1.

It is very interesting that for some primes \(p\) most likely indeed \[40\] is satisfied for all \(n \geq p^2\). Such primes we call "resonance primes". Our numerous observations show that all resonance primes not exceeding 1000 are:

\[
11, 19, 41, 67, 107, 173, 179, 181, 307, 313, 421, 431, 433, 587, \\
601, 631, 641, 647, 727, 787.
\]

In conclusion, note that for \(p \geq 3\) we have

\[
\lim_{n \to \infty} \frac{|V_p(n)|}{n} = \frac{1}{p} \prod_{2 \leq q < p} \left(1 - \frac{1}{q}\right)
\]
such that

\[
\lim_{n \to \infty} \left( \sum_{p \geq 3} \frac{|V_p(n)|}{n} \right) = \frac{1}{2}.
\]

Thus, using Theorems 1, 2 in the form

\[
S_m(n) = \begin{cases} 
\omega(S_3(n)), & (m, 3) = 1 \\
\frac{3}{m} S_3(n)(1 + o(1)), & 3|m
\end{cases}
\]
and inclusion-exclusion for \(p \geq 5\), we find

\[
\sum_{j \in V_p(n)} (-1)^{\sigma(j)} = -\frac{3}{3p} \prod_{2 \leq q < p, q \neq 3} \left(1 - \frac{1}{q}\right) S_3(n)(1 + o(1)) =
\]

\[
-\frac{3}{2p} \prod_{2 \leq q < p} \left(1 - \frac{1}{q}\right) S_3(n)(1 + o(1)).
\]
Now in view of (5) we obtain the following absolute result as an approximation of Conjectures 1, 2.

**Theorem 3.** For every prime number $p \geq 5$ and sufficiently large $n \geq n_p$ we have

$$\sum_{j \in V_p(n)} (-1)^{s(j)} < 0$$

and, moreover,

$$\lim_{n \to \infty} \frac{\ln \left(- \sum_{j \in V_p(n)} (-1)^{s(j)}\right)}{\ln n} = \frac{\ln 3}{\ln 4}.$$ 

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**Departments of Mathematics, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel. e-mail: shevelev@bgu.ac.il**