KNASTER’S PROBLEM FOR ALMOST \((\mathbb{Z}_p)^k\)-ORBITS

R.N. KARASEV AND A.YU. VOLOVIKOV

Abstract. In this paper some new cases of Knaster’s problem on continuous maps from spheres are established. In particular, we consider an almost orbit of a \(p\)-torus \(X\) on the sphere, a continuous map \(f\) from the sphere to the real line or real plane, and show that \(X\) can be rotated so that \(f\) becomes constant on \(X\).

1. Introduction

In [7] the following conjecture (Knaster’s problem) was formulated.

**Conjecture 1.** Let \(S^{d-1}\) be a unit sphere in \(\mathbb{R}^d\). Suppose we are given \(m = d - k + 1\) points \(x_1, \ldots, x_m \in S^{d-1}\) and a continuous map \(f : S^{d-1} \to \mathbb{R}^k\). Then there exists a rotation \(\rho \in SO(d)\) such that

\[
f(\rho(x_1)) = f(\rho(x_2)) = \cdots = f(\rho(x_m)).
\]

In papers [6, 4] it was shown that for certain sets \(\{x_1, \ldots, x_m\} \subset S^{d-1}\) Knaster’s conjecture fails, such counterexamples exist for every \(k > 2\), for \(k = 2\) and \(d \geq 5\), for \(k = 1\) and \(d \geq 67\).

Still it is possible to prove Knaster’s conjecture in some particular cases of sets. In [10] the set of points was some orbit of the action of a \(p\)-torus \(G = (\mathbb{Z}_p)^k\) on \(\mathbb{R}[G]\) for \(k = 1\) and on \(\mathbb{R}[G] \oplus \mathbb{R}\) for \(k = 2\). Here we prove some similar results, the set of points being a \((\mathbb{Z}_p)^k\)-orbit minus one point.

The group algebra \(\mathbb{R}[G]\) is supposed to have left \(G\)-action, unless otherwise stated. Considered as a \(G\)-representation, \(\mathbb{R}[G]\) may have a \(G\)-invariant inner product. In fact, the space of invariant inner products has the dimension equal to the number of distinct irreducible \(G\)-representations in \(\mathbb{R}[G]\) (for a commutative \(G\)), for a \(p\)-torus \(G = (\mathbb{Z}_p)^k\) the dimension of this space is \(p^{k+1}/2\) for odd \(p\), and \(p^k\) for \(p = 2\).

**Definition 1.** Denote \(I[G] \subset \mathbb{R}[G]\) the \(G\)-invariant subspace in \(\mathbb{R}[G]\) consisting of

\[
\sum_{g \in G} \alpha_g g, \quad \text{with} \quad \sum_{g \in G} \alpha_g = 0.
\]
Note that its orthogonal complement (w.r.t. any $G$-invariant inner product) is the one-dimensional space with trivial $G$-action.

In the sequel we consider a $p$-torus $G = (\mathbb{Z}_p)^k$ and denote $q = p^k$.

**Theorem 1.** Let $S^{q-2}$ be the unit sphere of $I[G]$ w.r.t. some $G$-invariant inner product, denoted by $(\cdot, \cdot)$. Then conjecture [4] holds for $k = 1$, the rotations w.r.t. $(\cdot, \cdot)$, and the set $Gx \setminus \{x\}$, where $x \in S^{q-2}$ is any point.

**Theorem 2.** Let $S^{q-1}$ be the unit sphere of $\mathbb{R}[G]$ w.r.t. some $G$-invariant inner product $(\cdot, \cdot)$. Then conjecture [4] holds for $k = 2$, $q$ odd, the rotations w.r.t. $(\cdot, \cdot)$, and the set $Gx \setminus \{x\}$, where $x \in S^{q-1}$ is any point.

In fact, the last theorem may be formulated a little stronger. For example, Theorem 5 (see below) gives the following statement. Let \( x \in S^{q-1} \) be as in the theorem, and let \( f_1, f_2 : S^{q-1} \to \mathbb{R} \) be two continuous functions. Then for some rotation \( \rho \) and two constants \( c_1, c_2 \)

\[
\forall g \in G \quad f_1(\rho(gx)) = c_1 \\
\forall g \in G, \ g \neq e \quad f_2(\rho(gx)) = c_2.
\]

2. **EQUIVARIANT COHOMOLOGY OF $G$-SPACES**

We consider topological spaces with continuous action of a finite group $G$ and continuous maps between such spaces that commute with the action of $G$. We call them $G$-spaces and $G$-maps.

Let us consider the group $G = (\mathbb{Z}_p)^k$ and list the results (mostly from [12]) that we need in this paper.

The cohomology is taken with coefficients in $\mathbb{Z}_p$, in the notation we omit the coefficients.

Consider the algebra of $G$-equivariant (in the sense of Borel) cohomology of the point $A_G = H^*_G(pt) = H^*(BG)$. For any $G$-space $X$ the natural map $X \to pt$ induces the natural map of cohomology $\pi^*_X : A_G \to H^*_G(X)$.

For a group $G = (\mathbb{Z}_p)^k$ the algebra $A_G$ (see [3]) has the following structure. For odd $p$, it has $2k$ multiplicative generators $v_i, u_i$ with dimensions dim $v_i = 1$ and dim $u_i = 2$ and relations

\[v_i^2 = 0, \quad \beta v_i = u_i.\]

Here we denote $\beta(x)$ the Bockstein homomorphism.

For a group $G = (\mathbb{Z}_2)^k$ the algebra $A_G$ is the algebra of polynomials of $k$ one-dimensional generators $v_i$.

The powerful tool of studying $G$-spaces is the following spectral sequence (see [5] [8]).

**Theorem 3.** There exists a spectral sequence with $E_2$-term

\[E_2^{x,y} = H^x(BG, \mathcal{H}^y(X, \mathbb{Z}_p)),\]

that converges to the graded module, associated with the filtration of $H^*_G(X, \mathbb{Z}_p)$. 
The system of coefficients $\mathcal{H}^0(X, \mathbb{Z}_p)$ is obtained from the cohomology $H^0(X, \mathbb{Z}_p)$ by the action of $G = \pi_1(BG)$. The differentials of this spectral sequence are homomorphisms of $H^*(BG, \mathbb{Z}_p)$-modules.

For every term $E_r(X)$ of this spectral sequence there is a natural map $\pi^*_r : A_G \to E_r(X)$.

**Definition 2.** Denote the kernel of the map $\pi^*_r$ by $\text{Ind}_G^r X$.

The ideal-valued index of a $G$-space was introduced in [3], the above filtered version was introduced in [11]. Remind the properties of $\text{Ind}_G^r X$, that are obvious by the definition. We omit the subscript $G$ when a single group is considered.

- If there is a $G$-map $f : X \to Y$, then $\text{Ind}^r X \supseteq \text{Ind}^r Y$.
- $\text{Ind}^{r+1} X$ may differ from $\text{Ind}^r X$ only in dimensions $\geq r$.
- $\bigcup_r \text{Ind}^r X = \text{Ind} X = \ker \pi_X^r : A_G \to H^*_G(X)$.

The first property in this list is very useful to prove nonexistence of $G$-maps. Following [12] we define a numeric invariant of this ideal filtering $\text{Ind}_G^r X$.

**Definition 3.** Put

$$i_G(X) = \max \{ r : \text{Ind}_G^r X = 0 \}.$$ 

It is easy to see that $i_G(X) \geq 1$ for any $G$-space $X$, $i_G(X) \geq 2$ for a connected $G$-space $X$, and $i_G(X)$ may be equal to $+\infty$. Moreover, for a $G$-space $X$ without fixed points, $G$-homotopy equivalent to a finite $G$-CW-complex, $i_G(X) \leq \dim X + 1$.

From the definition of $\text{Ind}_G^r X$ it follows that if there exists a $G$-map $f : X \to Y$, then $i_G(X) \leq i_G(Y)$ (the monotonicity property).

The definition of $i_G(X)$ can be further extended.

**Definition 4.** Define the index of a cohomology class $\alpha \in A_G$ on a $G$-space $X$

$$i_G(\alpha, X) = \max \{ r : \alpha \notin \text{Ind}_G^r X \}.$$ 

It may equal $+\infty$ if $\alpha \notin \text{Ind}_G X$.

It is clear from the definition that either $i_G(\alpha, X) = +\infty$, or $i_G(\alpha, X) \leq \dim \alpha$ and $i_G(\alpha, X) \leq \dim X + 1$ (for a finite $G$-CW-complex). Moreover, for any $G$-map $f : X \to Y$ we have the monotonicity property

$$i_G(\alpha, X) \leq i_G(\alpha, Y).$$

### 3. Reformulations

We reformulate Theorems [1] and [2] in a more general way.

**Theorem 4.** Let $S^{q-2}$ be the unit sphere of $I[G]$ w.r.t. some $G$-invariant inner product, and let $f : S^{q-2} \to \mathbb{R}$ be some continuous function. Consider $x \in S^{q-2}$, the vector $v = \sum_{g \in G} g \in \mathbb{R}[G]$ and some other vector $w \in \mathbb{R}[G]$, non-collinear to $v$.

Then for some rotation $\rho \in SO(q-1)$ the vector $\sum_{g \in G} f(\rho(gx))g \in \mathbb{R}[G]$ is in the linear span of $v$ and $w$. 
Theorem 4 follows from this theorem in the following way. Put \( w = e \in \mathbb{R}[G] \). Then by Theorem 4 there exists a rotation \( \rho \) such that for some \( \alpha, \beta \in \mathbb{R} \)
\[
\forall g \in G, \ g \neq e, \ f(\rho(gx)) = \alpha, \ f(\rho(x)) = \alpha + \beta.
\]
That is exactly the statement of Theorem 4.

**Theorem 5.** Let \( S^{q-1} \) be the unit sphere of \( \mathbb{R}[G] \) w.r.t. some \( G \)-invariant inner product, and let \( f : S^{q-1} \rightarrow \mathbb{R}^2 \) be some continuous map with coordinates \( f_1, f_2 \). Let \( q \) be odd. Consider \( x \in S^{q-1} \), the vectors \( v = \sum_{g \in G} g \cdot 0 \in \mathbb{R}[G] \oplus \mathbb{R}[G] \), \( u = 0 \oplus \sum_{g \in G} g \in \mathbb{R}[G] \oplus \mathbb{R}[G] \) and some other vector \( w \in \mathbb{R}[G] \oplus \mathbb{R}[G] \), non-coplanar to \( v, u \).

Then for some rotation \( \rho \in SO(q) \) the vector
\[
\sum_{g \in G} f_1(\rho(gx))g \oplus \sum_{g \in G} f_2(\rho(gx))g \in \mathbb{R}[G] \oplus \mathbb{R}[G]
\]
is in the linear span of \( v, u, w \).

Again, Theorem 2 (and its stronger version in the remark after Theorem 2) follows from this theorem by taking a vector \( w = e \oplus 0 \), similar to the previous remark.

**4. Proof of Theorem 4 in the case of odd \( q \)**

In this section \( q = p^k \), \( p \) is an odd prime, \( G = (\mathbb{Z}_p)^k \). Define for any \( \rho \in SO(q-1) \)
\[
\phi(\rho) = \sum_{g \in G} f(\rho(gx))g \in \mathbb{R}[G].
\]
For any \( h \in G \) we have
\[
\phi(\rho \circ h^{-1}) = \sum_{g \in G} f(\rho(h^{-1}g(x)))g = \sum_{g \in G} f(\rho(h^{-1}g(x)))hh^{-1}g = \sum_{g \in G} f(\rho(g(x)))hg.
\]
Thus the map \( \phi : SO(q-1) \rightarrow \mathbb{R}[G] \) is a \( G \)-map for the left action of \( G \) on \( SO(q-1) \) by right multiplications by \( g^{-1} \in G \), and for the standard left action of \( G \) on \( \mathbb{R}[G] \).

Denote for any \( g \in G \) by \( L_g = \langle v, gw \rangle \subset \mathbb{R}[G] \) the 2-dimensional subspaces. Assume the contrary: that is the image of \( \phi \) does not intersect \( \bigcup_{g \in G} L_g \). So \( \phi \) maps \( SO(q-1) \) to the space \( Y = \mathbb{R}[G] \setminus \bigcup_{g \in G} L_g \). The natural projection \( \pi : Y \rightarrow \mathbb{R}[G]/\langle v \rangle = V \) gives a homotopy equivalence between \( Y \) and \( V \setminus \bigcup_{g \in G} \mathbb{R}[\pi(gw)] \), the latter space is homotopically \( q - 2 \)-dimensional sphere without several points, hence it is a wedge of \( q - 3 \)-dimensional spheres. \( G \) acts on \( Y \) without fixed points, so \( i_G(Y) \leq q - 2 \).

In [10] it was shown that \( i_G(SO(q-1)) = q - 1 \) w.r.t. the considered \( G \)-action. Here we give a short explanation. In the spectral sequence of Theorem 3 all multiplicative generators of \( H^*(SO(q-1), \mathbb{Z}_p) \) are transgressive, because they are pullbacks of the transgressive generators of \( H^*(SO(q-1), \mathbb{Z}_p) \) in the spectral sequence of the fiber bundle \( \pi_{SO(q-1)} : ESO(q-1) \rightarrow BSO(q-1) \). So the first nonzero \( \text{Ind}_G \mathbb{R}[SO(q-1)] \) corresponds to the first nonzero characteristic class of the \( G \)-representation \( I[G] \) in the cohomology ring \( A_G \). It was shown in [11] that this is the Euler class of \( I[G] \) of dimension \( q - 1 \).

So we have a contradiction with the monotonicity of \( i_G(X) \).
5. Proof of Theorem 5

Similar to the previous proof, we consider the $G$-map $\phi : SO(q) \to R[G] \oplus R[G]$, given by the formula

$$\phi(\rho) = \sum_{g \in G} f_1(\rho(g(x)))g \oplus \sum_{g \in G} f_2(\rho(g(x)))g \in R[G] \oplus R[G].$$

Take the composition $\psi = \pi \cdot \phi$ with the projection $\pi : R[G] \oplus R[G] \to I[G] \oplus I[G] = V$. Assume the contrary: that is the map $\phi$ does not intersect the linear span of $u$ and $v$ in $R[G] \oplus R[G]$ and $\psi$ does not intersect the linear span of $gw$ for any $g \in G$ in $V$, it means that the image of $\psi$ is in the space $Y = V \setminus \bigcup_{g \in G} R\pi(gw)$.

Let $e \in A_G$ be the Euler class of $V$. From the spectral sequence of Theorem 3 it is obvious that $I_G(e, V \setminus \{0\}) = 2q - 2$, because the spectral sequence for $V \setminus \{0\}$ has the only nontrivial differential that kills the Euler class $e$. Since $Y \subset V \setminus \{0\}$, then $i_G(e, Y) < +\infty$. Similar to the previous proof, the space $Y$ is homotopically a wedge of $2q - 4$-dimensional spheres, so $i_G(e, Y) \leq \dim Y + 1 = 2q - 3$.

In [10] it was shown that $i_G(e, SO(q)) = 2q - 2$, because $e$ is in the image of the transgression in the spectral sequence and $e$ is not contained in the ideal of $A_G$, generated by the characteristic classes of $SO(q)$ of lesser dimension. So we again have a contradiction with the monotonicity of $i_G(e, X)$.

6. Proof of Theorem 4 in the case of even $q$

In this section $q = 2^k$, $G = (Z_2)^k$. We use the notation from the odd case in Section 4. Note that the case $q = 2$ is trivial, and if $q \geq 4$ then $G$ acts on $I[G]$ by transforms with positive determinant, so the group $SO(q - 1)$ can be considered as the configuration space.

Assume the contrary: the image $\phi(SO(q - 1))$ is in $Y = R[G] \setminus \bigcup_{g \in G} L_g$.

Denote the Stiefel-Whitney classes of $I[G]$ in $A_G$ by $w_k$. We need the following lemma, stated in [10], based on results from [2, 9].

**Lemma 1.** The only nonzero Stiefel-Whitney classes of $I[G]$ are $w_{q - 2^l} \in A_G$ ($l = 0, \ldots, k$), the classes $w_{q - 2^l}$ ($l = 0, \ldots, k - 1$) are algebraically independent and form a regular sequence, hence $w_{q - 1}$ is nonzero and not contained in the ideal of $A_G$, generated by $w_k$ with $k < q - 1$.

Similar to the proof of Theorem 5 in Section 5 we find that $i_G(w_{q - 1}, Y) \leq \dim Y + 1 = q - 2$.

Now we apply the spectral sequence of Theorem 3 to the $G$-space $SO(q - 1)$. The action of $G$ on $SO(q - 1)$ is the restriction of action of $SO(q - 1)$ on itself, the latter group being connected, hence $G$ acts trivially on $H^*(SO(q - 1), Z_2)$.

The results of [11] imply that the differentials in this spectral sequence are generated by transgressions that send the primitive (in terms of [11]) elements of $H^*(SO(q - 1), Z_2)$ to the Stiefel-Whitney classes $w_k$ (see Proposition 23.1 in [11]). Thus Lemma 1 implies that $i_G(w_{q - 1}, SO(q - 1)) = q - 1$, and the existence of the $G$-map $\phi$ contradicts the monotonicity of $i_G(w_{q - 1}, X)$. 
References

[1] A. Borel. Sur la cohomology des espaces fibrés principaux et des espace homogènes de groupes de Lie compact. // Ann. Math., 57, 1953, 115–207.

[2] L.E. Dickson. A fundamental system of invariants of the general modular linear group with a solution of the form problem. // Trans. Amer. Math. Soc., 12(1), 1911, 75–98.

[3] E. Fadell, S. Husseini. An ideal valued cohomological index theory with applications to Borsuk-Ulam and Bourgin-Yang theorems. // Ergod. Th. & Dynam. Sys. 8, 1988, 73–85.

[4] A. Hinrichs, C. Richter. The Knaster problem: More counterexamples. // Israel Journal of Mathematics, 145(1), 2005, 311–324.

[5] Wu Yi Hsiang. Cohomology theory of topological transformation groups. Springer Verlag, 1975.

[6] B.S. Kashin, S.J. Szarek. The Knaster problem and the geometry of high-dimensional cubes. // Comptes Rendus Mathematique, 336(11), 2003, 931–936.

[7] B. Knaster. Problem 4. // Colloq. Math., 30, 1947, 30–31.

[8] J. McCleary. A user’s guide to spectral sequences. Cambridge University Press, 2001.

[9] H. Mïi. Modular invariant theory and cohomology algebras of symmetric groups. // J. Fac. Sci. U. Tokyo, 22, 1975, 319–369.

[10] A.Yu. Volovikov. A Bourgin-Yang-type theorem for $Z_{np}$-action (In Russian). // Mat. Sbornik, 183(2), 1992, 115–144; translation in Russian Acad. Sci. Sb. Math., 76(2), 1993, 361–387.

[11] A.Yu. Volovikov. On the index of G-spaces (In Russian). // Mat. Sbornik, 191(9), 2000, 3–22; translation in Sbornik Math., 191(9), 2000, 1259–1277.

[12] A.Yu. Volovikov. Coincidence points of maps of $Z_{np}$-spaces. // Izvestiya: Mathematics, 69(5), 2005, 913–962.

E-mail address: r.n_karasev@mail.ru

Roman Karasev, Dept. of Mathematics, Moscow Institute of Physics and Technology, Institutskiy per. 9, Dolgoprudny, Russia 141700

E-mail address: a_volov@list.ru

Alexey Volovikov, Department of Mathematics, University of Texas at Brownsville, 80 Fort Brown, Brownsville, TX, 78520, USA

Alexey Volovikov, Department of Higher Mathematics, Moscow State Institute of Radio-Engineering, Electronics and Automation (Technical University), Pr. Vernadskogo 78, Moscow 117454, Russia