INITIAL VALUE PROBLEMS IN QUANTUM FIELD THEORY
AND THE FEYNMAN PATH INTEGRAL REPRESENTATION
OF SCHWINGER’S CLOSED TIME PATH FORMALISM

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Schwinger’s Closed-Time Path (CTP) formalism is an elegant way to insure causality for initial value problems in Quantum Field Theory. Feynman’s Path Integral on the other hand is much more amenable than Schwinger’s differential approach (related to the Schwinger Dyson equations) for non-perturbative (in coupling constant) expansions such as the large-N expansion. By marrying the CTP formalism with a large-N expansion of Feynman’s Path Integral approach, we are for the first time able to study the dynamics of phase transitions in quantum field theory settings. We review the Feynman Path Integral representation for the generating functional for the Green’s functions described by an initial density matrix. We then show that the large-N expansion for the path integral forms a natural non-perturbative framework for discussing phase transitions in quantum field theory as well as for giving a space time description of a heavy ion collision. We review results for a time evolving chiral phase transition and the degradation of an electric field due to pair production from strong electric fields.

1 Introduction

In this talk I would like to review work done on initial value problems over the past 8 years and explain why large-N expansions coupled with Schwinger’s closed time formulation for the Green’s functions is a good starting point for understanding the dynamics of many systems. One of the technical issues in doing initial value problems in Quantum Field Theory is to insure causality of the update equations. This problem is completely solved by using the Closed Time Path (CTP) Green’s functions of the Schwinger-Mahanthappa-Keldysh formalism. On the other hand, we want to have a starting point for calculations which allows us to encapsulate non-perturbative phenomena such as the phase structure of strongly-interacting systems. The Path Integral when expanded about self-consistently determined mean fields does the required bookkeeping for doing systematic expansions in non-perturbative (in coupling constant) domains.
2 Path Integral Formulation of Initial Value Problems-CTP formalism

The preservation of causality is guaranteed by the CTP formalism which was first discussed by Schwinger in the context of quantum Brownian motion. The application to field theory was by Mahanthappa et. al. and amplification of these ideas to quantum transport theory was elaborated and developed by Keldysh. The original formulation was in a Schwinger Dyson equation formulation. This was later reformulated as a Path Integral by Zhou et. al, R. D. Jordan and Calzetta and Hu. The Green’s functions that occur in the CTP formalism are also the ones that naturally occur in a study of inclusive particle production.

The generating functional for the Green’s functions is obtained by considering the “in” state matrix element in the presence of external sources, and inserting a complete set of late time $t'$ states. In this way one can express the original initial time matrix element as a product of transition matrix elements from 0 to $t'$ and the time reversed (complex conjugate) matrix element from $t'$ to 0. Since each term in this product is a transition matrix element of the usual or time reversed kind, standard path integral representations for each may be introduced.

$$e^{iW[J_+, J_-]} = Z_{in}[J_+, J_-] = \int [D\Psi] \langle in|\psi\rangle J_- \langle \psi|in\rangle J_+$$

$$= \int [D\Psi] \langle in| T^* \exp \left[ -i \int_0^{t'} dt J_- (t) \phi(t) \right] |\Psi, t'\rangle \times$$

$$\langle \Psi, t'| T \exp \left[ i \int_0^{t'} dt J_+ (t) \phi(t) \right] |in\rangle$$

(1)

so that, for example,

$$\delta^4 W_{in}[J_+, J_-] = \left. \frac{\delta^4 W_{in}[J_+, J_-]}{\delta J_+(t_3) \delta J_+(t_4) \delta J_-(t_1) \delta J_-(t_2)} \right|_{J_+ = J_0}$$

$$= \langle in| T^* \{\phi(t_1)\phi(t_2)} T \{\phi(t_3)\phi(t_4)} |in \rangle$$

(2)

Here $\phi(t) = \phi(x, t)$ and we are suppressing the coordinate dependence and the integration over the spatial volume in what follows for notational simplicity. Since the time ordering in eq. (1) is forward (denoted by $T$) along the time path from 0 to $t'$ in the second transition matrix element, but backward (denoted by $T^*$) along the path from $t'$ to 0 in the first matrix element. Thus the name: closed time path generating functional. If we deform the backward
and forward directed segments of the path slightly in opposite directions in
the complex $t$ plane, the symbol $\mathcal{T}_C$ may be introduced for path ordering along
the full closed time contour, $\mathcal{C}$.

Following Calzetta and Hu we introduce the path integral representation
for each transition matrix element in eq. (1) and obtain

$$Z [J_+, J_-, \rho] = \int [\mathcal{D}\varphi][\mathcal{D}\varphi'] \langle \varphi|\rho|\varphi' \rangle \int [\mathcal{D}\psi] \int [\mathcal{D}\phi_+] \int [\mathcal{D}\phi_-] \times$$

$$\exp \left[ i \int_0^\infty dt \left( L[\phi_+] - L[\phi_-] + J_+\phi_+ - J_-\phi_- \right) \right].$$

(3)

The double path integral over the fields $\phi_+$ and $\phi_-$ in (3) suggests that
we introduce a two component contravariant vector of field variables by

$$\phi^a = \left( \begin{array}{c} \phi_+ \\ \phi_- \end{array} \right); \quad a = 1, 2$$

(4)

One raises and lowers indices in this vector space with a $2 \times 2$ matrix with
indefinite signature, namely

$$c_{ab} = \text{diag} (+1, -1) = c^{ab}$$

(5)

so that, for example

$$J^a c_{ab} \Phi^b = J_+\phi_+ - J_-\phi_-.$$

(6)

The correlation functions are $2 \times 2$ matrices. The components of the two point
function is

$$G^{ab}(t, t') = \frac{\delta^2 W}{\delta J_a(t)\delta J_b(t')} \bigg|_{J=0}.$$

(7)

Explicitly, the components of this $2 \times 2$ matrix are

$$G^{11}(t, t') \equiv G_>(t, t') = i\text{Tr} \{ \rho \Phi(t)\bar{\Phi}(t') \} \bigg|_{\text{con}},$$

$$G^{12}(t, t') \equiv G_<(t, t') = \pm i\text{Tr} \{ \rho \bar{\Phi}(t')\Phi(t) \} \bigg|_{\text{con}},$$

$$G_F = G^{11}(t, t') = i\text{Tr} \{ \rho \mathcal{T}[\Phi(t)\bar{\Phi}(t')] \} \bigg|_{\text{con}} = \theta(t, t')G_>(t, t') + \theta(t', t)G_<(t, t')$$

$$G_{F*} = G^{22}(t, t') = i\text{Tr} \{ \rho \mathcal{T}^*[\Phi(t)\bar{\Phi}(t')] \} \bigg|_{\text{con}} = \theta(t', t)G_>(t, t') + \theta(t, t')G_<(t, t')$$

(8)

The $\pm$ refer to Bose (+) or Fermi (−). $\bar{\Phi} = \Phi^\dagger$ for charged Bosons and
$\bar{\Psi} = \Psi^\dagger \gamma^0$ for Dirac Fermions. An alternative generating functional uses the
Complex Path Ordered Form:

\[ Z_C[J, \rho] = \text{Tr} \left\{ \rho \left( T_C \exp \left[ i \int_C dt \phi(t) \right] \right) \right\} \]

\[ = \int [D\varphi^1] [D\varphi^2] \langle \varphi^1 | \rho | \varphi^2 \rangle \int_{\varphi^1}^{\varphi^2} [D\phi] \exp \left[ i \int_C (L[\phi] + J[\phi]) \right]. \]

(9)

This is identical in structure to the usual expression for the generating functional in the more familiar in-out formalism, The only difference is that path ordering according to the complex time contour \( C \) replaces the ordinary time ordering prescription. The propagator is now written in the equivalent form:

\[ G(t, t') = \Theta_C(t, t')G_>(t, t') + \Theta_C(t', t)G<_<(t, t') \]

(10)

where \( \Theta_C \) is the CTP complex contour ordered theta function (see \( \Theta_C \)).

With this definition of \( G(t, t') \), the Feynman rules are the ordinary ones, and matrix indices are not required. As we have shown previously this complex contour form allows one to give simple rules for constructing arbitrary graphs which preserve causality.

2.1 Inclusive Dilepton Production

Using LSZ reductions formula, we can write the expression for the inclusive production of dileptons in a form which relates this quantity to an off mass shell Green’s function exactly of the form given by Schwinger’s CTP generating functional. One finds:

\[ \frac{E_k E'_k}{m^4} \frac{d^0 N}{d^3 k} = \frac{d^0 N}{d^3 k'} \frac{d^0 N}{d^3 k'} \]

(11)

Here \( P_1 \) and \( P_2 \) represent the two colliding particles at \( t = t_0 \).
3 Large $N$

Here we derive the causal evolution equation to order $1/N$ in the large $N$ expansion for the $O(N)$ model. The details are in reference 8. Following Coleman, Jackiw and Politzer we write the Lagrangian as:

$$\tilde{L}_{cl}[\Phi, \chi] = -\frac{1}{2}\Phi_i(\partial_\mu \partial^\mu + \chi)\Phi_i + \frac{N}{\lambda} \chi \left(\frac{\chi}{2} + \mu^2\right)$$

(12)

where $i = 1, \ldots, N$ and $\chi$ is given by:

$$\chi = -\mu^2 + \frac{\lambda}{2N} \Phi_i \Phi_i.$$  

(13)

If $\mu^2 > 0$, spontaneous symmetry breaking at the classical level. The generating functional for all Graphs is given by:

$$Z[j, K] = \int d\phi d\chi \exp\{iS[\phi, \chi] + i \int [j\phi + K\chi]\}$$

(14)

Perform the Gaussian integral over the field $\phi$ and then Legendre transform to obtain the effective action $\Gamma$ we obtain:

$$\Gamma_{eff}[\phi, \chi] = S_{cl}[\phi, \chi] + \frac{i\hbar}{2} \text{Tr} \ln G^{-1}[\chi] + \frac{i}{2N} \text{Tr} \ln D^{-1}[\phi, \chi].$$

(17)

Expanding the $\chi$ integral by stationary phase leads to the $1/N$ expansion. Keeping terms up to quadratic in $\chi$, perform the ensuing Gaussian integral

$$G^{-1}[\chi](x,y) = -\frac{1}{\lambda} \delta^4(x,y) - \phi(x)G[\chi](x,y)\phi(y) + \frac{i}{2} G[\chi](x,y)G[\chi](y,x).$$

(18)

The field equation for $\phi$ up to order $1/N$ is

$$(\partial_\mu \partial^\mu + \chi(x))\phi(x) - \frac{2}{N} \int_0^t dt_d d^3\vec{y} \text{Im} [G>[x,y)D>[x,y)] \phi(y) = 0.$$  

(19)
\[ \chi(x) = -\mu^2 + \frac{\lambda}{2}\phi^2(x) - \frac{i\lambda}{4}[G_>(x,x) + G_< (x,x)] + \frac{\lambda}{N}\text{Im}\int_0^t dt_1 d^3\vec{x}_1 \int_0^{t_1} dt_2 d^3\vec{x}_2 [G_>(x, x_1) - G_< (x, x_1)] \times \left[ \tilde{\Sigma}_<(x_1, x_2) G_>(x_2, x) - \tilde{\Sigma}_>(x_1, x_2) - G_< (x_2, x) \right], \] (20)

with \( \tilde{\Sigma} \) given by

\[ \tilde{\Sigma}(x_1, x_2) = D(x_1, x_2) \left[ iG(x_1, x_2) - \phi(x_1) \phi(x_2) \right] \] (21)

Here we have displayed the causal structure that one gets from the CTP Green’s functions.

4 Summary of Applications

Because of space limitations, I will just give the references to the applications of these ideas here.

Our first studies were on the quantum back reaction problem which completed Schwinger’s classic work on pair production from strong external fields. We determined the time evolution of the Electric field from the semiclassical Maxwell equation with the current determined by the expectation value of the quantum electromagnetic current resulting from pair production. This problem was studied both for a homogeneous plasma as well as for one that corresponded to boost invariant kinematics appropriate for heavy ion collisions.

We studied the dynamics of the chiral phase transition in the \( O(4) \) sigma model, both for a boost invariant kinematical situation as well as for a radially expanding plasma of pions. We showed how to calculate the inclusive distribution of pions as well as dileptons from the expanding plasma. For the theoretical case of the unbroken \( O(4) \) model we discussed the theoretical question of decoherence, entropy production and determining the complete density matrix in the mean field approximation.

We studied inhomogeneous initial conditions on the fields as well as solitonic initial conditions. Lastly we have been studying in quantum mechanical scenarios the accuracy of including the \( 1/N \) corrections to the mean field theory results. In quantum mechanics, the \( 1/N \) expansion can only be used for large \( N \), with \( g/N < 1/70 \) because the order \( 1/N \) effective potential is not defined for larger \( g/N \) at small \( x \). For these large values of \( N \) keeping the non leading corrections to the large \( N \) approximation significantly improves the accuracy.
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