TURBULENT STRESSES IN LOCAL SIMULATIONS OF RADIATION-DOMINATED ACCRETION DISKS, AND THE POSSIBILITY OF THE LIGHTMAN–EARDLEY INSTABILITY

SHIGENOBU HIROSE1, OMER BLAES2, AND JULIAN H. KROLIK3
1 Institute for Research on Earth Evolution, JAMSTEC, Yokohama, Kanagawa 236-0001, Japan
2 Department of Physics, University of California, Santa Barbara, CA 93106, USA
3 Department of Physics and Astronomy, Johns Hopkins University, Baltimore, MD 21218, USA

ABSTRACT

We present the results of a series of radiation MHD simulations of a local patch of an accretion disk, with a fixed vertical gravity profile but with different surface mass densities and a broad range of radiation to gas pressure ratios. Each simulation achieves a thermal equilibrium that lasts for many cooling times. After averaging over times that are long compared to a cooling time, we find that the vertically integrated stress is approximately proportional to the vertically averaged total thermal (gas plus radiation) pressure. We map out—for the first time on the basis of explicit physics—the thermal equilibrium relation between stress and surface density: the stress decreases (increases) with increasing surface mass density when the simulation is radiation (gas) pressure dominated. The dependence of stress on surface mass density in the radiation pressure dominated regime suggests the possibility of a Lightman–Eardley inflow instability, but global simulations or shearing box simulations with much wider radial boxes will be necessary to confirm this and determine its nonlinear behavior.

Key words: accretion, accretion disks—instabilities—MHD—X-rays: binaries

Online-only material: color figures

1. INTRODUCTION

It has long been known that models of optically thick, geometrically thin accretion disks based on the alpha stress prescription of Shakura & Sunyaev (1973) are subject to thermal and inflow (“viscous”) instabilities when the vertically averaged radiation to gas pressure ratio exceeds 3/2 (Lightman & Eardley 1974; Shibazaki & Hoshii 1975; Shakura & Sunyaev 1976). Such radiation pressure dominated accretion disks are expected to be relevant for luminous active galactic nuclei and quasars as well as for thermal states of X-ray binaries. However, with one possible exception, there has never been clear observational evidence, or even observational motivation, for the existence of these instabilities in these sources. This is in marked contrast to the cases of dwarf novae and soft X-ray transients, where thermal instabilities in the disk associated with hydrogen ionization, not radiation pressure, are central to explaining the observed outbursts (Lasota 2001). The X-ray binary GRS 1915+105 does exhibit recurrent outburst behavior that has been modeled as being due to radiation pressure driven instabilities (Belloni et al. 1997), but it is far from clear that this is the correct explanation. This source is the brightest among Galactic black hole X-ray binaries, and spends considerable time at super-Eddington luminosities (Done et al. 2004). Other black hole X-ray binaries commonly reach high enough Eddington ratios for instabilities to exist according to standard accretion disk theory, but do not exhibit similar variability.

It is now widely suspected that accretion stresses in black hole accretion disks are due to turbulence related to the nonlinear growth of the magnetorotational instability (MRI; Balbus & Hawley 1998). It is computationally feasible to perform thermodynamically consistent, radiation MHD simulations of this turbulence in local patches of accretion disks. These stratified shearing box simulations fully capture grid-scale numerical losses of energy as heat and also account for radiative heat losses within the flux-limited diffusion approximation (Hirose et al. 2006). Such simulations have now been performed for a broad range of radiation to gas pressure ratios, and in each case an approximate thermal equilibrium has been established lasting for many cooling times (Hirose et al. 2006; Krolik et al. 2007; Hirose et al. 2009). No sign of the radiation pressure thermal instability is present, even at radiation to gas pressure ratios well above the instability threshold of the standard alpha model (Hirose et al. 2009; see also Turner 2004).

Thermal stability exists because the stress—pressure relation assumed by the standard alpha model is only established on timescales longer than the thermal time.4 The causal direction of the relation is from stress to pressure, not from pressure to stress. Turbulence is chaotic and results in a highly fluctuating dissipation rate. It is that dissipation that ultimately changes the pressure, but that pressure response is only established after a thermal time. An upward fluctuation in pressure does not result in an upward stress response on this timescale as the causal direction is the other way round. Hence, there is no positive feedback loop on the thermal timescale that would result in a thermal runaway (Hirose et al. 2009).

This still leaves open the question of the slower inflow instability. Mass and angular momentum conservation imply that radial mass transport in an accretion disk with a local turbulent stress is governed by the equation (Lightman & Eardley 1974; Lynden-Bell & Pringle 1974)

$$\frac{\partial \Sigma}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left[ \frac{1}{\ell'} \frac{\partial}{\partial r} \left( r^2 W_{r\phi} \right) \right],$$

where \( \Sigma \) is the local surface mass density, \( r \) is the radius, \( \ell' \equiv d\ell/dr \) is the radial derivative of specific angular momentum, and \( W_{r\phi} \) is the vertically integrated turbulent stress. For geometrically thin disks, the inflow time is much longer...

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4 In a prescient comment, Lightman & Eardley (1974) suggested that the alpha prescription might only be valid on slow timescales, which they identified as being of the order of the inflow time and longer.
than the thermal time, so both thermal and vertical hydrostatic equilibrium should be maintained over the timescales associated with mass transport. Assuming the disk is optically thick and cools through radiative losses (which is the case for all of our stratified shearing box simulations so far; Hirose et al. 2006; Krolik et al. 2007; Hirose et al. 2009), then radiative, hydrostatic, and thermal equilibrium imply that the vertically averaged stress in a radiation pressure dominated disk is simply given by

$$\tau_r = \frac{c \Omega^2}{\kappa r \Omega'},$$

(2)

where $\Omega$ is the angular velocity in the disk, $c$ is the speed of light, and $\Omega' \equiv d\Omega/dr$ is the shear (Shakura & Sunyaev 1976). The opacity $\kappa$ is generally dominated by electron scattering in this regime, and is therefore constant. Hydrostatic equilibrium implies that the disk half-thickness is $H \sim 2P/(\Omega^2 \Sigma)$, where $P$ is the midplane pressure. Hence, at a particular radiation pressure dominated radius,

$$W_r \sim 2H \tau_r \propto \frac{P}{\Sigma}. \quad (3)$$

A standard alpha-disk model with $\tau_r = \alpha P$ implies from Equation (2) that $P$ is independent of $\Sigma$, so $\partial(W_r) / \partial \Sigma < 0$. Equation (1) therefore represents a diffusion equation with a negative diffusion coefficient. Unstable growth of surface density enhancements and rarefactions would therefore result (Lightman & Eardley 1974; Lightman 1974a, 1974b).

Like the fictitious thermal instability, however, the reality of the inflow instability has always been questionable. For example, as pointed out by Lightman & Eardley (1974) themselves, a stress proportional to the gas pressure alone would produce an inflow stable disk. On the other hand, the stratified shearing box simulations appear to be consistent with total thermal pressure scaling with stress on suprathermal timescales (Hirose et al. 2009), a fact that we will demonstrate much more explicitly in this paper.

A plot of vertically integrated stress $W$ versus surface density for a range of thermal equilibria at a fixed radius within a disk would suggest inflow stability or instability depending on whether the slope is positive or negative, respectively. Because thermal equilibrium implies that the local radiation flux $F$ emerging from each face of the disk is given by $F = W_r \Omega^2 / \Omega'$, a plot of $F$ (or effective temperature) versus surface density may be used in the same way. Such “S-curve” plots are commonly used to investigate the hydrogen ionization driven disk instabilities in dwarf novae and soft X-ray transients (e.g., Smak 1984; Lasota 2001).

Using radiation MHD simulations of stratified shearing boxes, we have now mapped out the stress versus surface density thermal equilibrium curve for a wide range of radiation to gas pressure ratios at a fixed radius in a disk around a stellar mass black hole. This is the first time that this curve has been drawn on the basis of explicit physical mechanisms, rather than the phenomenological estimates. While there are large fluctuations which produce an inherent scatter in the thermal equilibrium curve, the results are consistent with the standard alpha-disk model. The inflow instability might therefore be present even in MRI turbulent disks.

This paper is organized as follows. In Section 2, we provide a brief overview of the numerical parameters and properties of the simulations. In Section 3, we discuss the stress–pressure relation and demonstrate that average total thermal pressure, rather than a pressure which singles out gas pressure as being special in some way, is best correlated with average stress. In Section 4, we summarize the resulting thermal equilibria on a stress–surface density diagram, and discuss the possible implications. We summarize our conclusions in Section 5.

2. SIMULATIONS

The radiation MHD equations for stratified shearing boxes, and the numerical methods we use to solve them, have been described in detail by Hirose et al. (2006) and Hirose et al. (2009), and references therein. Grid-scale losses of mechanical and magnetic energy are fully captured as heat in the gas, and gas and radiation exchange heat through Planck mean-free-free absorption and emission and Compton scattering. Radiation transport is treated through flux-limited diffusion.

All the simulations were run with an angular velocity $\Omega = 190 \text{ s}^{-1}$, corresponding to a radius of $30GM/c^2$ around a 6.62 $M_\odot$ Schwarzschild black hole. Different total surface mass densities were chosen for each simulation in order to map out the stress–surface density relation. Table 1 summarizes the parameters of each simulation. The $x$, $y$, and $z$ axes correspond to the radial, azimuthal, and vertical directions, respectively. The simulations were initialized in hydrostatic and thermal equilibrium under an assumed initial vertical profile of dissipation per unit volume proportional to the density divided by the square root of the optical depth measured from the nearest surface. Once the simulation starts, the dissipation is thereafter self-consistently determined from the turbulent dynamics. Each simulation started with a weak magnetic field consisting of a twisted azimuthal flux tube located at the center of the box. We refer the reader to Hirose et al. (2009) for more details.

The vertical box height $L_z$ of the simulations was chosen so that two conditions would be satisfied (Hirose et al. 2009). First, the total surface mass density changes by less than a few percent due to vertical mass loss and mass creation by the density floor of the simulation. Second, the MRI is always well resolved in the midplane regions. These conditions were checked a posteriori. The upper and lower photospheres are always within the simulation domain, although the optically thin regions do not necessarily have a tremendous vertical extent. This should not significantly affect the emergent flux, which is what really matters. Because the radiation energy density in the optically thin regions is nearly independent of height, the radiation energy density at the photospheres should depend only weakly on box height. It is this radiation energy density that provides the effective boundary condition for the radiation diffusion equation in the optically thick regions where the vast majority of the dissipation occurs, so the emergent flux should be well determined. Nevertheless, we warn the reader that we have not demonstrated numerical convergence with respect to variations in the simulation box dimensions.

Six of the simulations (0211b, 0519b, 1112a, 1126b, 0520a, and 0320a) were initialized as radiation pressure dominated, while two (090304a and 090423a) started out as gas pressure dominated. (Due to an initialization error, simulations 090304a and 090423a actually had an angular velocity 1.6% smaller than the others: 187 rad s$^{-1}$ rather than 190 rad s$^{-1}$. We do not believe this significantly affects our conclusions, as the fluctuations in stress and radiative cooling are considerably larger than this.) The results of simulations 1112a and 1126b were extensively discussed in Hirose et al. (2009).

All of the simulations share many of the properties that we discussed in detail in previous papers on radiation MHD...
Figure 1. Time history of the box-integrated radiation internal energy (red), gas internal energy (green), magnetic energy (blue), and turbulent kinetic energy (black) for each of the radiation pressure dominated simulations. (A color version of this figure is available in the online journal.)

Table 1

| Simulation  | $\Sigma$ (g cm$^{-2}$) | $H$ (cm) | Duration (orbits) | Box Dimensions ($L_x/H \times L_y/H \times L_z/H$) | Grid Zones ($N_x \times N_y \times N_z$) |
|------------|------------------------|----------|------------------|-----------------------------------------------|-----------------------------------------------|
| 0211b      | $5.43 \times 10^4$     | $5.83 \times 10^6$ | 264              | $0.3375 \times 1.35 \times 6.3$               | $48 \times 96 \times 896$                    |
| 0519b      | $7.48 \times 10^4$     | $4.37 \times 10^6$ | 403              | $0.3375 \times 1.35 \times 6.3$               | $48 \times 96 \times 896$                    |
| 1112a      | $1.06 \times 10^5$     | $1.46 \times 10^6$ | 610              | $0.45 \times 1.8 \times 8.4$                 | $48 \times 96 \times 896$                    |
| 1126b      | $1.06 \times 10^5$     | $1.46 \times 10^6$ | 611              | $0.45 \times 1.8 \times 8.4$                 | $48 \times 96 \times 896$                    |
| 0520a      | $1.24 \times 10^5$     | $1.17 \times 10^6$ | 603              | $0.54 \times 2.16 \times 10.08$              | $48 \times 96 \times 896$                    |
| 0320a      | $1.52 \times 10^5$     | $7.28 \times 10^5$ | 426              | $0.6 \times 2.4 \times 11.2$                 | $48 \times 96 \times 896$                    |
| 090304a    | $5.00 \times 10^4$     | $3.13 \times 10^5$ | 600              | $0.625 \times 2.5 \times 10$                 | $32 \times 64 \times 512$                    |
| 090423a    | $2.00 \times 10^4$     | $2.10 \times 10^5$ | 262              | $0.5 \times 2.0 \times 8.0$                  | $32 \times 64 \times 512$                    |

The subphotospheric regions consist of a magnetorotational turbulent zone in the midplane regions which is supported against gravity by gas and radiation pressure gradients. Further out, magnetic forces dominate or contribute substantially; and Parker instability dynamics, rather than MRI turbulence, appear to control the structure of the outer layers.

Simulations of stratified shearing boxes (Hirose et al. 2006; Krolik et al. 2007; Blaes et al. 2007; Hirose et al. 2009). The subphotospheric regions consist of a magnetorotational turbulent zone in the midplane regions which is supported against gravity by gas and radiation pressure gradients. Further out, magnetic forces dominate or contribute substantially; and Parker instability dynamics, rather than MRI turbulence, appear to control the structure of the outer layers.

Figure 1 shows the thermal and turbulent energy contents in the box as a function of time for each of the radiation pressure dominated simulations. The thermal histories of the more gas-dominated simulations are shown in Figure 2. Defining the instantaneous thermal time as the total radiation and gas internal energy divided by the emergent radiative flux on both vertical faces of the box, the thermal time averaged over the duration of each of the simulations ranges from a minimum of six orbits for 090423a to a maximum of 24 orbits for 0519b. All the simulations have reached an approximate thermal equilibrium, albeit with long timescale fluctuations. There is no evidence for the thermal instability predicted by classic alpha-disk models (Shakura & Sunyaev 1976), in spite of the fact that the time-averaged ratio of vertically averaged radiation pressure to vertically averaged gas pressure is as high as 70 in the case of simulation 0519b.

3. THE STRESS–PRESSURE RELATION

A number of authors have suggested alternative stress prescriptions in which the accretion stress is proportional to gas pressure or some combination of gas and radiation pressures, rather than total thermal pressure (gas plus radiation), in part to stabilize the radiation pressure dominated portion of black hole accretion disks (Sakimoto & Coroniti 1981; Stella & Rosner 1984; Burm 1985; Szuszkiewicz 1990; Merloni & Fabian 2002; Merloni 2003). Sakimoto & Coroniti (1989) have even argued that standard alpha-disk models are inconsistent...
in the radiation pressure dominated regime, as magnetic fields that are strong enough to provide accretion stress would be too buoyant to be retained by the disk. Their argument has two flaws, however. First, they assumed that the magnetic field consisted of discrete flux tubes, rather than being more continuously distributed throughout the plasma. Second, they were unaware at the time of magnetic field generation by the MRI. Note that there is no indication in the energy histories shown in Figure 1 that the magnetic and turbulent kinetic energies are limited by the gas internal energy. Indeed, in the two most radiation pressure dominated simulations (0211b and 0519b), the magnetic energy almost always exceeds the gas internal energy, and even the turbulent kinetic energy can occasionally be larger than the gas internal energy.

Based on the behavior of the stress to thermal pressure ratio within simulation 1112a, we argued in the past that a prescription where stress and total thermal pressure are proportional to each other (on timescales longer than the thermal time) is a superior description of the simulation behavior than alternatives where the pressure is taken to be just the gas pressure alone or the geometric mean of radiation and gas pressure, respectively.

For each of our simulations, we computed the vertically averaged stress \( \langle \tau_{r\phi,av} \rangle \) and the box average of various measures of pressure \( P_{av} \): the sum of the radiation pressure \( P_{rad} \) and gas pressure \( P_{gas} \), the geometric mean \( (P_{rad} P_{gas})^{1/2} \) of these pressures, and just the gas pressure. We then computed the time average of the ratio of these spatial averages of stress and pressure, \( \alpha \equiv \langle \tau_{r\phi,av} / P_{av} \rangle \), and compared them to the time average of the ratio of spatially averaged radiation and gas pressure, \( \langle P_{rad,av} / P_{gas,av} \rangle \). These time averages were computed over the entire duration of each simulation, excluding the first 10 orbits as the MRI was still in its growth phase.

The results are shown in Figure 3. The black points show the results for \( P_{av} \) defined as the total pressure, and are clearly closest to being independent of the radiation to gas pressure ratio. (A linear fit to the dependence of these values of \( \alpha \) on the radiation to gas pressure ratio gives a slightly negative slope which is consistent with zero within one standard deviation of the slope determination.) A weighted average of these values gives \( \tilde{\alpha} = 0.018 \pm 0.002 \). The green curve shows what one would obtain if stress and total pressure were proportional with this ratio, but we redefined \( \alpha \) in terms of the geometric mean of radiation and gas pressure, i.e.,

\[
\left\langle \frac{\tau_{r\phi,av}}{P_{av}} \right\rangle \simeq \tilde{\alpha} \left[ \frac{P_{rad,av}}{P_{gas,av}} \right]^{1/2} + \frac{P_{rad,av}}{P_{gas,av}} \right]^{-1/2}.
\]

The blue curve shows the same thing for the gas pressure stress prescription,

\[
\left\langle \frac{\tau_{r\phi,av}}{P_{av}} \right\rangle \simeq \tilde{\alpha} \left( 1 + \frac{P_{rad,av}}{P_{gas,av}} \right).
\]

These curves clearly explain the trends seen in the data for these alternative stress prescriptions.

Our results are therefore most consistent with the standard alpha prescription involving total thermal pressure. Recent axisymmetric global radiation MHD simulations by Ohsuga et al. (2009) also reach a similar conclusion. We emphasize, however, that such a prescription should still be treated with caution. There is no obvious reason that \( \alpha \) should be a universal constant, and we do not know what physics is determining its value in our simulations. It is possible, for example, that \( \alpha \) is a function of \( \Omega \) or \( M \), as we have not varied these parameters at all; global simulations Hawley & Krolik (2001, 2002) have shown that it can be a function of radius when the underlying orbital dynamics change. In fact, it is rather surprising that it is as constant as it is when defined in terms of total thermal pressure. Moreover, as we emphasized in Hirose et al. (2009), the fact that stress is in any way related to pressure is really because turbulent dissipation heats the plasma. Only when averaged over height and averaged over many thermal times does the standard alpha prescription provide an adequate description of the fact that pressure is correlated with stress.
4. THE STRESS–SURFACE DENSITY RELATION

For each simulation, we computed the vertically integrated stress as a function of time, and then time averaged this over the simulation’s duration, again ignoring the first 10 orbits. The results are plotted as a function of surface density in Figure 4. Vertical error bars indicate the standard deviations of the instantaneous fluctuations in stress about the mean. (Due to mass loss and an imposed density floor, the surface density also fluctuates, but by at most 2% in all the simulations.)

The right-hand axis indicates the effective temperature of the radiation leaving each vertical face of the box. We have also computed versions of the diagram by time averaging the radiation flux leaving both faces of the box and computing the effective temperature, and time averaging the volume-integrated dissipation rate. All three versions are almost identical, as they must be, given the approximate thermal equilibrium that has been established in each simulation.

The curves in Figure 4 show the results predicted by standard alpha-disk models with stress scaling with total pressure, but with internal structure parameters based on the vertical structure observed in the simulations themselves. These parameters are defined in Appendix A and listed in Table 2. The simulation data are clearly consistent with the alpha-disk model, although there are variations in the internal structure parameters that, when averaged over all the simulations, produce an alpha-disk curve that is almost identical to the local limit of Equation (1), where $q = -d \ln \Omega / d \ln r$ and $|x| \ll r$. If $\partial W_{xy} / \partial \Sigma < 0$ under conditions of thermal equilibrium, we would then expect inflow instability. Stratified shearing box simulations sufficiently wide in the radial direction might therefore manifest this instability in the radiation pressure dominated regime. To see this effect may require radial box widths much larger than the vertical pressure scale height, and that stress scale with local pressure on timescales longer than the local thermal time. In any case, such simulations would enable a determination of whether the inflow instability is real, and what its minimum and fastest growing wavelengths are.

If the instability does manifest itself in MRI turbulence, such simulations would also help determine its nonlinear outcome. Of course, a number of hydrodynamic simulations have been done of the combined thermal/inflow instability of radiation-dominated alpha disks (e.g., Honma et al. 1991; Szuszkiewicz & Miller 1998). However, the time evolution of these simulations

![Figure 4](image_url)

**Figure 4.** Time-averaged, vertically integrated stress as a function of surface mass density for each simulation. The right-hand axis shows the corresponding effective temperature of the radiation leaving each vertical face of the box. The solid curve shows the prediction of the vertically integrated alpha-disk model with internal structure parameters averaged over all the simulations. The dashed line shows the prediction of the radiation pressure dominated alpha-disk model with internal structure parameters averaged over the radiation pressure dominated simulations. The dot-dashed line shows the prediction of the gas pressure dominated alpha-disk model with internal structure parameters measured from the gas pressure dominated simulations. (See Appendix A for the equations used to define the internal structure parameters.)
is dominated by the faster thermal instability, which we now know to be fictitious. As far as we are aware, the only simulations that have ever been attempted of the radiation-dominated inflow instability on its own were done by Lightman (1974b) himself, who numerically solved the alpha-disk mass diffusion equation, assuming that thermal equilibrium is strictly maintained. All his simulations therefore started with a surface density less than $\Sigma_{\text{crit}}$. The resulting evolution rapidly produced clumping of the surface density up to $\Sigma_{\text{crit}}$ with optically thin rarefied regions in between. Both of these conditions violated the assumptions on which the mass diffusion equation is based, and the numerical calculation had to be stopped. As Lightman (1974a) pointed out, surface densities exceeding $\Sigma_{\text{crit}}$ cannot be in thermal equilibrium within the assumptions of the alpha model, because heat generation always exceeds cooling through vertical radiative diffusion. On the other hand, if the characteristic radial size of the clumps is as small as the disk scale height, radial heat transport is likely to be important in the nonlinear outcome. Radiation MHD simulations with radially wide shearing boxes could address all these issues.

Radiation MHD simulations could also clear up another question clouding prediction of the outcome of this putative instability: the negative slope of the stress–surface density relation on the radiation-dominated branch implies growing surface density fluctuations only if local thermal equilibrium with purely vertical heat flow is maintained everywhere while the instability tries to develop. It could be that subtleties in the behavior of MRI turbulence preclude this from happening, just as the thermodynamics of the turbulence prevented the thermal instability from manifesting itself.

This issue is closely related to the question of exactly what the gas-dominated and radiation-dominated branches of equilibria in Figure 4 truly represent. If this were a low-dimensional dynamical system like the standard alpha disk, then the fact that both branches are thermally stable would imply an unstable equilibrium branch in between. However, a stratified shearing box with real turbulence is not a low-dimensional dynamical system—the spatial dependence of gas density, pressure, velocity, magnetic field, and radiation pressure all influence its evolution; and Figure 4 is probably better viewed as a projection of a very complicated dynamical phase space. In our experience, simulations that are initialized far away from the equilibrium branches do not undergo steady heating or cooling, but instead wander chaotically due to the fluctuating character of the turbulence. Whether there is a true unstable thermal equilibrium branch between the gas and radiation-dominated branches would be very well masked by this stochasticity even if the vertically integrated stress was the only significant dynamical variable; the much higher dimensionality of the real system makes it essentially impossible to test whether such a branch exists on the basis of simulation data. It is conceivable that a local perturbation in surface density would require considerable time to reach the thermal equilibrium branch, and would in any case fluctuate about that branch. It is an open question whether or not this evolution would be conducive to inflow instability.

It is also noteworthy that hydrostatic and radiative equilibrium necessarily enforce a characteristic stress on the radiation-dominated branch (Equation (2); Shakura & Sunyaev 1976). There is no such constraint on the gas-dominated branch, and in fact hydrostatic equilibrium is not even needed to derive the relation between stress and surface density in the alpha model on this branch (see Appendix A). It is possible that these differences may also be relevant to evolution on the inflow timescale in the gas and radiation pressure dominated regimes in real MRI turbulence.

5. CONCLUSIONS

We have completed vertically stratified, local radiation MHD simulations of magnetorotational turbulence with fixed vertical gravity over a range of surface mass densities. All of the simulations reach a thermal equilibrium, but with continued long-term fluctuations in the internal energy content. We have confirmed earlier work (Turner 2004; Hirose et al. 2009) that the radiation pressure dominated thermal instability predicted by the standard alpha-disk model does not exist, even though the box- and time-averaged radiation to gas pressure ratios in the new simulations are as high as 70.

However, we also find that, when averaged over many thermal times, the vertically integrated total thermal pressure (i.e., radiation plus gas pressure) is well correlated with the vertically integrated stress. Neither the vertically integrated gas pressure nor the geometric mean of gas and radiation pressure exhibits such a good correlation. The same simulation data therefore yield a thermal equilibrium relation between surface density and (long) time-averaged vertically integrated stress that follows the one predicted by the traditional alpha

### Table 2

| Simulation | $\Sigma$ (g cm$^{-2}$) | $a$ | $\xi$ | $H_{\text{gas}}/H_{\text{P}}$ | $H_{\text{rad}}/H_{\text{P}}$ |
|------------|------------------------|-----|------|-----------------------------|-----------------------------|
| 0211bb     | $5.43 \times 10^4$     | 0.019 ± 0.005 | 5.4 ± 1.2 | 0.952 | 0.629 |
| 0519bb     | $7.48 \times 10^4$     | 0.015 ± 0.004 | 5.6 ± 1.3 | 0.949 | 0.630 |
| 1112a      | $1.06 \times 10^5$     | 0.023 ± 0.007 | 4.5 ± 1.1 | 0.934 | 0.624 |
| 1126b      | $1.06 \times 10^5$     | 0.020 ± 0.006 | 4.7 ± 1.0 | 0.948 | 0.632 |
| 0520a      | $1.24 \times 10^5$     | 0.016 ± 0.006 | 4.6 ± 1.0 | 0.926 | 0.629 |
| 0320a      | $1.52 \times 10^5$     | 0.015 ± 0.004 | 4.1 ± 0.8 | 0.990 | 0.645 |
| 090304a    | $5.00 \times 10^5$     | 0.028 ± 0.009 | 2.8 ± 0.6 | 1.090 | 0.680 |
| 090423a    | $2.00 \times 10^5$     | 0.028 ± 0.007 | 2.3 ± 0.4 | 1.127 | 0.685 |
| Mean$_{\text{gas}}^a$ | 0.017 ± 0.002 | 4.7 ± 0.4 | 0.950 | 0.631 |
| Mean$_{\text{rad}}^b$ | 0.028 ± 0.006 | 2.5 ± 0.4 | 1.190 | 0.683 |
| Mean$_{\text{all}}^c$ | 0.018 ± 0.002 | 3.4 ± 0.3 | 0.990 | 0.644 |

Notes.

- $^a$ Averaged over the six radiation pressure dominated simulations, i.e., excluding 090304a and 090423a.
- $^b$ Averaged over the gas pressure dominated simulations 090304a and 090423a.
- $^c$ Averaged over all eight simulations.
model: \( W_{\phi} \propto \Sigma^{-1} \). Consequently, if thermal equilibrium is maintained on the long timescales associated with radial mass transport by the turbulent stresses, these local results suggest that the nonlocal clumping of mass associated with the classic Lightman & Eardley (1974) inflow instability might suggest that the nonlocal clumping of mass associated with mass transport by the turbulent stresses, these local results is maintained on the long timescales associated with radial

The midplane pressure in the simulations is always dominated by gas and radiation pressure,

\[ P(0) = \frac{1}{2} \Omega^2 \Sigma H_{\rho1}, \quad (A1) \]

where \( H_{\rho1} \) is a density scale height defined through the first vertical moment of the density distribution,

\[ H_{\rho1} = \frac{1}{2} \sum_0^\infty \rho(z) z^2. \quad (A2) \]

The midplane pressure in the simulations is always dominated by gas and radiation pressure,

\[ P(0) = \frac{1}{2} \mu T(0) + \frac{\Sigma k T(0)}{2\mu H_{\rho0}}, \quad (A3) \]

where \( T(0) \) is the midplane temperature and \( \mu = 0.6 \) atomic mass units is the mean particle mass assumed in the simulations. We have eliminated the midplane density by defining the zeroth vertical moment of the density distribution,

\[ H_{\rho0} \equiv \frac{1}{2} \rho(0) \int_{-\infty}^{\infty} \rho(z) z^2. \quad (A4) \]

The fundamental assumption of an alpha-disk model is that the vertically integrated stress \( W_{\phi} \) is \( \alpha \) times the vertical integral of the thermal pressure \( P_{th} \). We may therefore write

\[ W_{\phi} = 2\alpha H_P P(0), \quad (A5) \]

where the thermal pressure scale height has been defined as

\[ H_P \equiv \frac{1}{2} P(0) \int_{-\infty}^{\infty} P_{th}(z) d\zeta. \quad (A6) \]

Thermal equilibrium implies that the flux \( F = \sigma T^4 \) emerging from each face of the disk is given by

\[ F = -\frac{1}{2} W_{\phi} \frac{d\Omega}{dr} = \frac{3}{4} \Omega W_{\phi}. \quad (A7) \]

Finally, the fact that most of the accretion power ultimately escapes vertically through radiative diffusion in the simulations motivates us to write the emergent flux as

\[ F = \xi \frac{2 \alpha c T(0)^4}{3 \kappa \Sigma}, \quad (A8) \]

where the opacity is dominated by electron scattering in our simulations, \( \kappa \approx 0.33 \). We have introduced a parameter \( \xi \) which is usually taken to be approximately unity in alpha-disk models. As we discuss below, however, our measured values of \( \xi \) in the simulation are substantially greater than unity. This is probably due to the fact that the dissipation profile peaks off the midplane, and that a non-negligible fraction of the accretion power is transported away from the midplane by mechanical motions.

Equations (A1), (A3), (A5), (A7), and (A8) can be combined to give the relationship between emergent flux or vertically integrated stress and surface density. In the gas and radiation pressure dominated limits, the result is

\[ F = \frac{3}{4} \Omega W_{\phi} = \left[ \frac{\frac{3}{4} \kappa}{2^{\frac{3}{4}}} \left( \frac{\alpha T}{\mu} \right)^4 \left( \frac{H_P}{H_{\rho0}} \right)^4 \right]^{1/3} \Sigma^{5/3}, \quad (A9) \]

This gives the usual result that the alpha disk is inflow stable if gas pressure dominated, but inflow unstable if radiation pressure dominated. Note that the hydrostatic equilibrium equation (A1) is not needed to derive the gas pressure dominated relation.

If neither pressure dominates at the midplane, the following equation can be used to derive the relationship between flux or stress and surface density:

\[ 5\tilde{F}^{3/4} - 2\tilde{\Sigma}^{3/4} - 3\tilde{F}^{5/4}\tilde{\Sigma}^{1/2} = 0, \quad (A10) \]

where \( \tilde{\Sigma} \equiv \Sigma/\Sigma_{crit}, \quad \tilde{F} \equiv F/F_{crit} \). \footnote{Note that the expression for the maximum surface density consistent with the thermal equilibrium that was first derived by Lightman (1974a).

Equations (A9) and (A10) are what we use to plot the alpha-disk predictions in Figure 4. We measured the dimensionless parameters describing the internal structure of the disk as follows. Using the time series of horizontally averaged data from

\[ \Sigma_{crit} = \left[ \frac{2^{10}}{(9)^{5/10} \mu} \left( \frac{\kappa}{\alpha} \right)^7 \left( \frac{H_{\rho1} H_{\rho0}^3}{H_P^4} \right)^{1/8} \right] \quad (A11) \]

and

\[ F_{crit} = \frac{12c^2 \kappa^2 \Omega}{25 \alpha k^2 \Sigma_{crit} \left( \frac{H_{\rho1}}{H_P} \right)}. \quad (A12) \]

Apart from the extra dimensionless factors of \( \xi \) and the pressure and density scale heights, Equation (A11) agrees with the expression for the maximum surface density consistent with the thermal equilibrium that was first derived by Lightman (1974a).
each simulation, we measured $\alpha$ as the ratio of the vertically integrated stress to vertically averaged pressure, and $\xi$ in terms of the ratio of emerging flux (averaged over both faces of the disk) to midplane radiation energy density. We then averaged both of these quantities over time. We measured the scale height ratios $H_{\rho_0}/H_P$ and $H_{\rho_1}/H_P$ from the time and horizontally averaged vertical profiles of density and thermal pressure from each simulation. The results are shown in Table 2.

The scale height ratio parameters are remarkably constant across all the simulations. The numerical values of these parameters are close to what one would obtain from simple analytic equilibria in a gravitational field that increases linearly with height. An $n = 3$ polytrope (adiabatic, radiation pressure dominated) has $H_{\rho_0}/H_P = 1.125$ and $H_{\rho_1}/H_P \approx 0.67$, while an $n = 3/2$ polytrope (adiabatic, gas pressure dominated) has $H_{\rho_0}/H_P = 1.2$ and $H_{\rho_1}/H_P \approx 0.69$. Note that both ratios are slightly larger in the gas pressure dominated polytrope, in agreement with the trend that we see in the simulations. These simple models do not agree exactly with the simulations due in part to nonzero entropy gradients. In the simulations, the entropy generally decreases away from the midplane, so that the density must decrease faster for a given pressure decrease compared to an adiabatic profile. Hence, the ratio of density scale height to pressure scale height is smaller than that of an adiabatic profile, and this is why the ratios we measure in the simulations are slightly smaller than the polytropic ratios.

In contrast to the scale height ratios, the stress parameter $\alpha$ shows a little more scatter. The parameter $\xi$ clearly increases as the simulations become more radiation pressure dominated.

**APPENDIX B**

**THE RADIAL MASS DIFFUSION EQUATION FOR THE SHEARING BOX**

The radial mass diffusion equation can be derived from the equations of the shearing box (Hawley et al. 1995) as follows. Define the azimuthally averaged surface density at some radius $x$ and time $t$ as

$$\Sigma(x, t) \equiv \frac{1}{L_y L_z} \int_{x-\Delta x/2}^{x+\Delta x/2} dx' \int_{-L_z/2}^{L_z/2} dy \int_{-L_y/2}^{L_y/2} dz \rho(x', y, z, t),$$

and the mass-weighed vertical and azimuthal average of some quantity $X$ as

$$\langle X \rangle(x, t) = \frac{1}{L_y L_z \Delta x} \int_{x-\Delta x/2}^{x+\Delta x/2} dx' \int_{-L_z/2}^{L_z/2} dy \int_{-L_y/2}^{L_y/2} dz \rho(x', y, z, t) X(x', y, z, t).$$

Here, we have also performed a radial average over a length scale $\Delta x$ of the order of an assumed radial coherence length of the turbulence, which is presumably of the order of the disk scale height. Apart from the use of Cartesian coordinates in a box, these definitions are exactly the same as the vertical integrations and averages used to derive the alpha-disk equations from the MHD equations by Balbus & Papaloizou (1999).

Applying this averaging and vertical integration to the mass continuity and y-momentum equations of the shearing box, we obtain

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\Sigma(v_y)) = 0$$

(B3)

and

$$\frac{\partial}{\partial t}(\Sigma(v_y)) + \frac{\partial}{\partial x}(\Sigma(v_y)(v_y)) + \frac{\partial W_{xy}}{\partial x} = -2\Omega \langle v_y \rangle,$$

(B4)

where $W_{xy}$ is the vertically integrated and azimuthally averaged Maxwell and Reynolds stress,

$$W_{xy}(x, t) = \frac{1}{L_y} \int_{-L_z/2}^{L_z/2} dy \int_{-L_y/2}^{L_y/2} dz \left( -\frac{B_x B_y}{4\pi} + \rho v_x \delta v_y \right).$$

(B5)

Radial derivatives are defined through differences over the length scale $\Delta x$. We have also assumed zero mass and Poynting flux through the vertical boundaries, and $\langle \delta v_y \rangle = 0$. Equations (B3) and (B4) can then be combined into a radial mass diffusion equation,

$$\frac{\partial \rho}{\partial t} = \frac{1}{(2 - q)\Omega} \frac{\partial^2}{\partial x^2} W_{xy},$$

(B6)

where $q \equiv -d \ln \Omega / d \ln r$ is the shear parameter.

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