A Parallel Algorithm to Test Chordality of Graphs

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1 Introduction

A graph $G$ is chordal if each cycle of size greater than 3 in $G$ has a chord, that is an edge between two non-adjacent vertices on the cycle. We present a simple parallel algorithm to test chordality of graphs which is based on the parallel Lexicographical Breadth-First Search algorithm. In total, the algorithm takes time $O(N)$ on $N$-threads machine and it performs work $O(N^2)$, where $N$ is the number of vertices in a graph. Our implementation of the algorithm uses a GPU environment Nvidia CUDA C. The algorithm is implemented in CUDA 4.2 and it has been tested on Nvidia GeForce GTX 560 Ti of compute capability 2.1. At the end of the thesis we present the results achieved by our implementation and compare them with the results achieved by the sequential algorithm.

This thesis is organized as follows. Section 2 is an introduction to the parallel programming using the GPU environment Nvidia CUDA C. Section 3 introduces the basic graph definitions used throughout the paper. Then it provides an overview of the graph theory related to the LexBFS algorithm and chordal graphs. Section 4 introduces the LexBFS algorithm and its two most known implementations. Section 5 provides the sequential algorithm to test chordality of graphs and the analysis of its correctness and time complexity. In section 6 we present our parallel LexBFS algorithm and a parallel algorithm to test chordality of graphs. In section 7 we give the performance results of our parallel implementation compared to the sequential algorithm. In section 8 we discuss our results and the possible further work.
2 CUDA Programming

CUDA (Compute Unified Device Architecture) is a general-purpose parallel computing architecture for Nvidia GPUs. We present the main features of CUDA C used in our implementation. For more details, we recommend NVIDIA CUDA C Programming Guide [4] and CUDA C Best Practices Guide [5].

CUDA C extends the C/C++ programming model to the heterogeneous programming model which operates on the CPU called the host, and on the GPU called the device. In CUDA, a kernel is a function executed in parallel by many threads on the device. A thread is a sequence of executions. The threads are grouped into blocks which are grouped into a grid. Each thread has a unique identifier in a grid. It can be computed within a kernel through a combination of the built-in variables: threadIdx, blockIdx and gridIdx.

All the threads may access data from the local, shared, constant, texture and global memory. To learn the texture memory and the constant memory see [4]. The local memory is a private memory of a thread. The shared memory is common to all threads within the same block and its lifetime is the same as the block. All threads have access to the same global memory.

The CUDA architecture allows to synchronize executions of the threads in one block by using the _syncthreads function. It works as a lock: the threads, which reach that point in the code, wait for other threads which have not done it yet.

One of the methods to synchronize the threads between blocks, is to split the computations in the synchronization points and to run each of that piece as a separate kernel. We use this method in our work.

3 Background

3.1 Basic graph definitions

We introduce the following terminology to be used throughout this thesis. Let \( G = (V, E) \) be an undirected graph with the vertex set \( V \) and the edge set \( E \), where \( E \) consists of unordered pairs of vertices in \( V \). We denote the size of \( V \) by \( N \) and the size of \( E \) by \( M \). If \( (u, v) \in E \) then we abbreviate it to \( uv \). We use \( N_x \) to denote the neighborhood of a vertex \( x \in V \) excluding \( x \).

Let \( \mathbb{N} \) be the set of the natural numbers and let \( \text{label}_x \) be the label of \( x \), where \( \text{label}_x \) is a string over the alphabet \( \mathbb{N} \). We use \( \circ \) to denote
the concatenation operator for labels.

A bijection $\pi = \{1, 2, \ldots, N\} \rightarrow V$ is called an ordering of $G$. Let $\pi^{-1}$ denote the inverse of $\pi$ and thus $\pi^{-1}(v)$ is the index of $v$ in the ordering of $G$. Let $\pi = v_1, \ldots, v_N$ be the ordering of $G$. We use $LN_{v_i}$ to denote the neighborhood of $v_i$ in the subgraph induced by $v_1, \ldots, v_{i-1}$.

We say that an ordering $\pi$ of $G$ is a BFS order if it is generated by the well-known BFS algorithm (see for example [7]). We present this algorithm in the next chapter. Note that a graph can have many different BFS orderings.

A graph $G$ is chordal if each cycle of size greater than 3 in $G$ has a chord, that is an edge between two non-adjacent vertices on the cycle. A vertex $x$ is simplicial if $N_x$ induces a clique. An order $v_1, v_2, \ldots, v_k$ is a perfect elimination order if, for each $i$, $v_i$ is a simplicial vertex in the graph induced by $v_1, v_2, \ldots, v_{i-1}$.

3.2 Overview

The LexBFS algorithm, in addition to the recognition of chordal graphs, has many other applications. The LexBFS algorithm is used as a part of many graph algorithms such as recognizing interval graphs, or computing transitive orientation of comparability graphs, co-comparability graphs and interval graphs.

An orientation of an undirected graph $G$ is a directed graph which is created by assigning a direction to each edge. An orientation of edges is acyclic if it does not contain a directed cycle. An orientation of edges is transitive for all $x, y, z$, if $x \rightarrow y$ is an edge and $y \rightarrow z$ is an edge then $x \rightarrow z$ is also an edge. A comparability graph is an undirected graph that has an acyclic transitive orientation on edges. A co-comparability graph is a graph $G$ whose complement $\overline{G}$ is a comparability graph. An interval graph $G$ is an undirected graph that is the intersection graph of intervals on the real line, i.e. $G = (V, E)$, where $V = \{I_1, I_2, \ldots, I_n\}$, $\forall_i I_i$ is an interval on the real line and $(I_i, I_j) \in E \Leftrightarrow I_i \cap I_j \neq \emptyset$.

Gilmore and Hoffman [1] proved that a graph is an interval graph if and only if it is a chordal graph and a co-comparability graph (Figure 1).

Figure 1: From left: $C_4$ is a co-comparability graph and it is not an interval graph. The next graph is a chordal graph and it is not an interval graph.
Habib, McConnell, Paul and Viennot [2] gave a $O(N + M\log N)$ algorithm for the transitive orientation of a comparability graph and a $O(N + M)$ algorithm to recognize interval graphs. Both of them use the LexBFS algorithm. It can be proved [2] that if $G$ is a co-comparability graph and $\pi = v_1, v_2, \ldots, v_n$ is a LexBFS order of $G$ then there exists a transitive orientation of $\overline{G}$ such that $v_n$ is a sink/source of the orientation. Moreover if $G$ is a comparability graph and $\pi = v_1, v_2, \ldots, v_n$ is a LexBFS order of $\overline{G}$ then there exists a transitive orientation of $G$ such that $v_n$ is a sink/source of the orientation.

4 Lexicographic Breadth-First Search

The Lexicographic Breadth-First Search (LexBFS) algorithm was introduced by D. Rose, R. Tarjan and G. Lueker in 1976 for finding a perfect elimination order, if any exists. The LexBFS algorithm is a restriction of the widely used Breadth-First Search (BFS) algorithm in the following sense: each possible order of vertices produced by LexBFS is a BFS order. The difference between them is that the LexBFS algorithm additionally assigns labels to nodes and then in each step of the algorithm chooses a node, whose label is lexicographically the largest.

4.1 Characterization of BFS and LexBFS orderings

We present and compare two characterizations of the vertex orderings that can by obtained by the BFS algorithm and the LexBFS algorithm.

Let $x$ be vertex of a graph $G$ and let $N_x$ denote its neighborhood. Let $Q$ be a FIFO queue. We present an equivalent version of Tarjan’s BFS algorithm:

\[
\text{Breadth-Frist Search algorithm}
\]

\[
\text{BFS()}
\]

\[
\text{for } x = 1 \text{ to } n \text{ do } \pi^{-1}(x) = 0
\]

\[
Q = \emptyset
\]

\[
\text{for } i = 1 \text{ to } n \text{ do }
\]

\[
\text{if } Q.\text{nonEmpty()} \text{ then } x = Q.\text{dequeue()}
\]

\[
\text{else } x = \text{any node s.t. } \pi^{-1}(x) = 0
\]

\[
\pi^{-1}(x) = i, \pi(i) = x
\]

\[
\text{for each } y \in N(x) \text{ s.t. } \pi^{-1}(y) = 0 \text{ do }
\]

\[
\text{if } y \notin Q \text{ then } Q.\text{enqueue}(y)
\]
Property B. If \( a < b < c \), \( ac \in E \) and \( ab \notin E \) then exists \( d < a \) such that \( db \in E \).

Lemma 4.1. \( \pi \) is a BFS order \( \iff \pi \) satisfies the B-property.

Proof. (\( \Rightarrow \)) Let \( \pi \) be a BFS order and let \( Q \) be a FIFO queue used by the algorithm. We assume that the nodes of the graph \( G \) are renumbered according to \( \pi \).

\[
\begin{array}{c}
\text{a} \\
\text{b} \\
\text{c}
\end{array}
\]

During the algorithm \( a \) was visited before \( b \) which was visited before \( c \). When the algorithm visits \( a \) then it adds \( c \) to \( Q \), because of \( ac \in G \), and it does not add \( b \) to \( Q \), because of \( ab \notin G \). Then \( b \) can be first in \( Q \) before \( c \) if and only if the algorithm had visited some \( d \) before it visited \( a \) and \( db \in G \) then the algorithm had added \( b \) to \( Q \) before it added \( c \) to \( Q \). In the same words, there exists \( d < a \) such that \( db \in G \).

\[
\begin{array}{c}
\text{d} \\
\text{a} \\
\text{b} \\
\text{c}
\end{array}
\]

(\( \Leftarrow \)) Let \( \pi_0 \) be an order satisfying a property \( B \). We want to show that \( \pi_0 \) is a BFS order. Let \( d \) be some vertex of graph \( G \). When the BFS algorithm visits \( d \) it pushes on a queue all neighbors of \( d \) which have not been visited yet. Then BFS pops the next vertex from a queue. Let \( a \) be another vertex of a graph \( G \). If the BFS algorithm visits \( d \) before \( a \) then all not visited neighbors of \( d \) are placed in \( \pi_0 \) before all not visited neighbors of \( a \). Hence it is sufficient to show that if \( d < a \) in order \( \pi_0 \), then all neighbors of \( d \) which are placed on to the right of \( d \) in \( \pi_0 \), lie before all neighbors of \( a \) (which are to the right of \( a \) in \( \pi_0 \)). But it is equivalent to property \( B \). See the figure above.

\[
\begin{array}{c}
\text{d} \\
\text{a} \\
\text{b} \\
\text{c}
\end{array}
\]

Let \( x \) be a vertex of a graph \( G \) and let \( N_x \) be the neighborhood of \( x \). Let \( label_x \) denote the label of \( x \). Let \( pQ \) be a priority queue of vertices with priority on lexicographically the largest label. Consider the following algorithm:
Lexicographic Breadth-Frist Search algorithm

LexBFS()
    for x = 1 to n do
        π⁻¹(x) = 0
        label_x = ∅
        pQ.enqueue(x)
    for i = 1 to n do
        x = pQ.dequeue() // label_x is lexicographically the largest
        π⁻¹(x) = i, π(i) = x
        for each y ∈ N_x s.t. π⁻¹(y) = 0 do
            label_x = label_x ◦ (n – i)

Property LB. If a < b < c, ac ∈ E and ab /∈ E then exists d < a such that db ∈ E and dc /∈ E.

Lemma 4.2. π is a LexBFS order ⇔ π satisfies LB-property.

Proof. (⇒) Let π be the LexBFS order and let pQ be a priority queue used by the LexBFS algorithm. We assume that the nodes of the graph G are renumbered according to π.

During the algorithm a was visited before b which was visited before c. Let b₁, b₂,... be the label of b and c₁, c₂,... be the label of c. Let i be the number of iteration and N be the number of vertices in G.

When the algorithm visits a, it concatenates the label of c with N – i, as ac ∈ E, and it does not concatenate the label of b with N – i, as ab /∈ E. Since the label of b is lexicographically larger than the label of c then there is an index j₀ in the labels such that ∀j < j₀ : b_j = c_j and b_j₀ > c_j₀. The index j₀ is the first index at which the labels were updated in different iterations.

During the algorithm b was updated before the label of c because b_j₀ > c_j₀ and in each iteration the number N – i decreases as i increases. The j₀ index exists if and only if the algorithm visited some d such that the algorithm concatenated the label of b with b_j₀ and it did not concatenate the label of c with b_j₀.

It must be that d < a because otherwise we would have:

1. if ad ∈ E then a < d < c < b, as the numbers N – i decreased and ac ∈ E, ab /∈ E.
2. if $ad \notin E$ then $a < c < d < b$, as $ac \in E$ and $ab \notin E$.

Both cases are in contradiction to $a < b < c$. Therefore, there exists $d < a$ such that $db \in E$ and $dc \notin E$.

\[\begin{array}{c}
d & a & b & c \\
\end{array}\]

($\Leftarrow$) Let $\pi_0$ be an order satisfying a property LB. We want to show that $\pi_0$ is a LexBFS order. Let $d$ be a vertex of graph $G$. When the LexBFS algorithm visits $d$ then it updates the labels of all not visited neighbors of $d$. Let $a$ be another vertex of $G$. If the LexBFS algorithm visits $d$ before $a$ then all neighbors of $d$ which are not adjacent to $a$ have labels greater than the labels of all neighbors of $a$, because the numbers added to the end of labels decrease for successive vertices. So in order to prove the claim we must show that if $d < a$ in $\pi_0$ then all neighbors of $d$, which are to the right of $d$ in $\pi_0$ and they are not adjacent to $a$, lie before all neighbors of $a$, which are to the right of $a$ in $\pi_0$. But again, it is equivalent to property LB. See the figure above.

\[\square\]

It is easy to see that the LB-property implies B-property so the LexBFS algorithm is a restriction of the BFS algorithm.

4.2 Two implementations

The first implementation of the LexBFS algorithm was proposed by D.J. Rose, R.E. Tarjan, G. S. Leuker in 1976 [3]. They use a double-linked list $L_k$ to store vertices of the same label $k$. All lists $L_k$ are stored in the list $L$ in descending order given by labels $k$. Additionally, each vertex $x$ has a pair of pointers the first of which is leading to the list $L_k$ containing $x$ and the second one is leading to the place of $x$ on the list $L_k$.

At the beginning of the algorithm, all the vertices have the same label $\emptyset$ and they are on the list $L_{\emptyset}$. There are two operations: getting a vertex $x$ with the lexicographically largest label and updating labels of all nodes adjacent to $x$. To perform the first operation the algorithm takes the first list $L_k$ from $L$ and then returns the first vertex from $L_k$. In the second operation, for each $y$ adjacent to $x$, the algorithm concatenates the label $k$ of the vertex $y$ with the number $(N - i)$. Then the algorithm removes $y$ from the list $L_k$ and inserts it to the list $L_{k\circ(N-i)}$, where $i$ is the iteration number. If the list $L_{k\circ(N-i)}$ does not existed in $L$ then the algorithm creates it. This implementation has the $O(N + M)$ time complexity.
The second implementation was proposed by Habib, McConnell, Paul and Viennot in 2000 [2] and it uses the partition refinement technique. Let $V$ be a doubly-linked list consisting of all vertices of $G$. Let $L$ be a doubly-linked list of classes of vertices. All vertices in a class occupy consecutive elements in $V$ and the class is represented by a pair of pointers to the first and the last element in the class. Each vertex $x$ has a pointer to the class containing $x$.

The LexBFS algorithm using partition refinement

```plaintext
LexBFS()
    $L$ - a single-element list of a class containing all vertices
    for $i = 1$ to $n$ do
        $x$ - the first element of the first class on the list $L$
        remove $x$ from $L$
        $\pi^{-1}(x) = i$, $\pi(i) = x$
    //partition
    for each class $C \in L$ do
        $C_x = C \cap N_x$
        $C_2 = C \setminus C_x$
        replace $C$ by $C_x, C_2$ in $L$
```

During the partition, each $y \in N_x$ is removed from an old class $C$ and it is inserted to some new class $C_x$. The partition procedure can be implemented in $O(|N_x|)$ time.

5 Chordal graphs

Before we present the algorithm to test chordality of graphs we prove a theorem introduced by D.J. Rose, R.E. Tarjan, G. S. Leuker [3].

**Theorem 5.1** (Rose, Tarjan, Leuker). A graph $G$ is chordal if and only if a LexBFS order of $G$ is a perfect elimination order.

**Proof.** ($\Rightarrow$) Let $G$ be a chordal graph and let $\pi = v_1, \ldots, v_n$ be its LexBFS order. We assume that the nodes of $G$ are renumbered according to $\pi$. We show that each vertex $v_i$ is simplicial in the graph induced by $v_1, \ldots, v_{i-1}$.

Assume by contradiction that some $v_i$ is not simplicial. Then there exist $a, b \in \pi$ such that $a < b < v_i$ both adjacent to $v_i$ and not adjacent themselves.
Because \( \pi \) satisfies LB-property then there exists some \( c \in \pi \) such that \( c < a \) and \( c \) is adjacent to \( b \) and it is not adjacent to \( v_i \). Note that \( ca \notin E \) because \( G \) is chordal and otherwise we would have a chordless cycle \((c,a,v_i,b)\).

\[
\begin{array}{c}
\text{c} \\
\text{a} \\
\text{b} \\
v_i
\end{array}
\]

Now we have got \( c < a < b < v_i, \ ca \notin E \) and \( cb \in E \). Again, we use the LB-property in respect to \( c, a, b \) and obtain \( d \in \pi: \ d < c, \ da \in E, \ db \notin E \). Moreover \( d \) is not adjacent to \( c \) because of the cycle \((d,c,b,v_i,a)\) and chordality of \( G \).

\[
\begin{array}{c}
\text{d} \\
\text{c} \\
\text{a} \\
\text{b} \\
v_i
\end{array}
\]

Next time we can apply the LB-property to the vertices \( d, c, a \). This step can be repeated infinitely, thus contradicting the general assumption that \( G \) is finite.

So we have proved that each vertex \( v_i \) is simplicial in the graph induced by \( v_1, \ldots, v_{i-1} \), that is the order \( v_1, \ldots, v_n \) is the perfect elimination order.

(\( \Leftarrow \)) It suffices to prove that if graph \( G \) has any perfect elimination order then it is chordal.

Let \( \pi = v_1, \ldots, v_n \) be a perfect elimination order of \( G \). Assume that in \( G \) there is a cycle \( C \) of length \( \geq 4 \) and let \( v_i \in C \) be the vertex of the greatest index in \( \pi \). Let \( a \) and \( b \) be vertices adjacent to \( v_i \) in the cycle \( C \). Because \( a \) and \( b \) are on the left from \( v_i \) in \( \pi \) then they are adjacent as \( \pi \) is a perfect elimination order. Therefore \( C \) has the chord \( ab \), which finishes the proof.

\[\square\]

### 5.1 Maximum Cardinality Search

In 1984 Robert E. Tarjan and M. Yannakakis introduced in [6] the Maximum Cardinality Search algorithm (MCS) as an alternative method for finding a perfect elimination ordering of chordal graphs. The Maximum Cardinality Search instead of strings uses natural numbers as labels for vertices. In each iteration, the algorithm chooses a new vertex of the largest label, that is the vertex whose neighborhood in the graph induced by the nodes chosen so far is the largest among all vertices have not chosen yet. The MCS algorithm has a \( O(N + M) \) time implementation [6].
Let $G$ be a graph, $pQ$ be a priority queue. For each $x \in V(G)$ $\text{label}_x \in \mathbb{N}$. We present Tarjan and Yannakakis’s MCS algorithm:

| Maximum Cardinality Search algorithm |
|--------------------------------------|
| MCS()                                |
| for $x$ = 1 to $n$ do                |
| $\pi^{-1}(x) = 0$                    |
| $\text{label}_x = 0$                |
| $pQ$.enqueue($x$)                    |
| for $i$ = 1 to $n$ do               |
| $x = pQ$.dequeue()                  |
| $\pi^{-1}(x) = i$, $\pi(i) = x$    |
| for each $y \in N_x$ such that $\pi^{-1}(y) = 0$ do |
| $\text{label}_x = \text{label}_x + 1$ |

Robert E. Tarjan and M. Yannakakis proved the following theorem [6].

**Theory 5.2.** $G$ is a chordal graph if and only if MCS-order of $G$ is a perfect elimination order.

### 5.2 Algorithm to test chordality

Theorem 5.1. gives us the following tool to test if a given graph is chordal. First we run the LexBFS algorithm to produce a LexBFS order. Next we check if the LexBFS order is the perfect elimination order.

Let $\pi = v_1, \ldots, v_n$ be an order returned by the LexBFS algorithm. For each $v_i$ let $LN_{v_i} \subset N_{v_i}$ be vertices adjacent to $v_i$ on the left from $v_i$ in $\pi$ and let $p_{v_i} \in LN_{v_i}$ be the right most vertex in $LN_{v_i}$.

The algorithm presented below tests if $\pi$ is a perfect elimination order. This is performed by checking if for each $v_i$ it holds that $LN_{v_i} - \{p_{v_i}\} \subset LN_{p_{v_i}}$. The correctness of such the approach is proved later.

To test if $\pi$ is a perfect elimination order we only need to check if for each $v_i$ is $LN_{v_i} - \{p_{v_i}\} \subset LN_{p_{v_i}}$. 

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Test if a LexBFS order is a perfect elimination order

chordalityTest()
    for $x = 1$ to $n$ do $p_x = 0$
    for each $y \in N_x$ that $\pi^{-1}(y) < \pi^{-1}(x)$ do
        $LN_x.add(y)$
        if $\pi^{-1}(y) > \pi^{-1}(p_x)$ then $p_x = y$
    for $x = 1$ to $n$ do
        for each $y \in N_x$ do
            $visited_y = 1$
        for each $y \in N_x$ do
            if $p_y = x$ then
                // check if $LN_y - \{x\} \subset LN_x$
                for each $z \in LN_y$ such that $z \neq x$ do
                    if $visited_z \neq 1$ then
                        return false
                for each $y \in N_x$ do
                    $visited_y = 0$
        return true

5.3 Correctness and complexity of algorithm

We prove that the algorithm for testing if a LexBFS order is a perfect elimination order is correct.

Let $\pi$ be a perfect elimination order. We show that then the algorithm returns true. Let $v$ be some node of $G$. There are two cases:

1. $p_v = 0$. Then $LN_v$ is empty and the algorithm does not return false.
2. $p_v = u$, for some $u$. Then algorithm checks if $LN_v - \{u\} \subset LN_u$.

Because all nodes of set $LN_v - \{u\}$ are on the left of $u$ then they are candidates to members of $LN_u$. It remains to show that they are
adjacent to $u$. Because $u = p_v$ then $u \in LN_v$ and $LN_v$ is a clique (because $\pi$ is a perfect elimination order) therefore all vertices of $LN_v - \{u\}$ are adjacent to $u$ and $LN_v - \{u\} \subseteq LN_u$. So the algorithm never return false, hence it returns true in the end.

Let $\pi$ be not a perfect elimination order. We show that then the algorithm returns false. We assume that vertices of $G$ are renumbered according to $\pi$. As $\pi$ is not a perfect elimination order, there exists some node in $\pi$ that its left neighborhood does not induce a clique. Let $v$ be the first such node in $\pi$ and let $p_v = u$ for some $u$. Because $u < v$ in $\pi$ then $LN_u$ is a clique (as $v$ is the first vertex in $\pi$ for which $LN_v$ is not a clique). Therefore $LN_v - \{u\} \not\subseteq LN_u$ and the algorithm returns false.

The time complexity of the algorithm is determined by the nested loops. Let $v$ be some node of $G$. Note that the size of $LN_v$ is $O(|N_v|)$. Let us see how many times the algorithm scans $N_v$.

1. marking visited array
2. looking for $p_v$
3. unmarking visited array
4. for $u$ such that $p_u = v$ and for each $v$ there is at most one such vertex $u$

It means that each list $N_v$ is read at most four times which gives $O(M)$ time for the whole graph. Summing up with the time of producing the LexBFS order, the test of chordality takes $O(N + M)$ time.

6 Parallel algorithm

Testing chordality of graphs has two steps: finding a LexBFS order and checking if the LexBFS order is a perfect elimination order. To parallelize the chordality test we need to parallelize each of these steps separately.

6.1 Parallel LexBFS

In our approach to the parallel version of LexBFS, the main loop of algorithm runs on the CPU and during each iteration $i$ two task are performed on the GPU. The first one is choosing the vertex $v$ with the lexicographically largest label and the second one is concatenating labels of vertices adjacent to $v$ with $N - i$. Both jobs are performed by $N$ threads assigned to $N$ vertices in a graph.

We use the following data structures. The graph $G$ is stored in an adjacency matrix $Adj$. A linked list $L$ is a list of sets $L_k$. Each set $L_k$ includes all
vertices whose labels are equal $k$. We identify the label of a set with the label of nodes in that set. These sets form a partition of the vertex set, defined by means of labels. $L$ is sorted lexicographically ascending. As $L$ is a linked list, each set of $L$ has a pointer next leading to the next set in the order, next = NULL in the last set in $L$. The order array stores the order of nodes computed by the algorithm. We say that a node is active if it has not processed yet. (see Figure 3.)

Figure 2: LexBFS algorithm

Figure 3: List $L$ including sets: $L_{k1}$, $L_{k2}$, $L_{k3}$, $L_{k4}$, $L_{k5}$. The last set on the list is $L_{k5}$. 

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At the beginning of algorithm all nodes of $G$ are active and they have the same label $\emptyset$. It means that the list $L$ has only one set $L_\emptyset$ consisting of all nodes of $G$ and $\text{next}$ leads to $\text{NULL}$.

In a sequential version of algorithm, to find the vertex with the lexicographically largest label, the algorithm returns any vertex belonging to the last set on $L$. As the last set on $L$ is characterized by the pointer $\text{next}$ equal $\text{NULL}$, then this procedure can be performed in parallel by $N$ threads as follows.

Let $x$ be the vertex assigned to the thread $th_x$. Let $L_x$ be the set including $x$. Let $\text{current}$ be a global variable shared by all threads. For each thread $th_x$ in parallel do:

$$\text{if } L_x.\text{next} = \text{NULL} \text{ then } \text{current} \leftarrow x$$

After this procedure the $\text{current}$ variable stores a vertex whose label is lexicographically the largest. Note that if there is more than one such vertex then we cannot predict which one will be stored in $\text{current}$.

Let $i$ be the iteration of the main loop in which $\text{current}$ has the lexicographically largest label. Let $y$ be some neighbor of $\text{current}$, $l_y$ be the label of $y$ and let $y$ be in the set $L_y$.

The update operation concatenates label $l_y$ in back with number $N - i$. Note that the number $N - i$ has not appeared in any label so far and it is the smallest among all numbers occurring in the labels.

Next, the algorithm removes $y$ from $L_y$ and inserts it to the new set containing the nodes with the label $l_y \circ (N - i)$. If the new set has not existed yet then the algorithm creates it.

Let us look closely at the operation of creating a new set. Let $A$ and $B$ be two sets of nodes on the list $L$ containing nodes labeled $l_A$ and $l_B$ respectively. Assume that $l_A < l_B$, i.e. $A$ comes before $B$ in $L$.

**Lemma 6.1.** If $j$ is a number of iteration during which the new set containing nodes with the label $l_A \circ (N - j)$ is created then $l_A < (l_A \circ (N - j)) < l_B$ in $L$.

**Proof.** Each label is a string of numbers. Let $l_A = \{a_1, a_2, a_3, \ldots, a_{|l_A|}\}$, $l_A \circ (N - j) = \{a_1, a_2, a_3, \ldots, a_{|l_A|}, N - j\}$ and $l_B = \{b_1, b_2, b_3, \ldots, b_{|l_B|}\}$.

Let $k$ be the smallest index such that $\forall i < k : a_i = b_i$ and $a_k \neq b_k$.

There are two cases:

1. $k \leq |l_A|$ and $k \leq |l_B|$ 
   Then $a_k < b_k$ and this two numbers determine $l_A \circ (N - j) < l_B$.

2. $|l_A| < |l_B|$ and $k = |l_A| + 1$ 
   Then $l_A$ is a prefix of $l_B$ and concatenation also gives $l_A \circ (N - j) < l_B$.
because \((N - j)\) is less than all numbers in \(l_B\) so in particular smaller than \(b_k\).

Note that always \(l_B < l_B \circ (N - j)\) because of \(|l_B| < |l_B \circ (N - j)|\).

The lemma gives us the following observation:

**Observation 6.2.** When a new set is created for vertices from a given set \(S\) then it should be inserted between \(S\) and its successor on the list \(L\).

Based on observation 6.1., for each new set we can determine its place in the list without any additional list traversal or label comparisons.

How many new sets are created during one iteration? The answer is: at most one for each old one. Indeed if \(y\) and \(z\) are neighbors of current vertex and belonging to some set \(S\) then their labels are equal both before and after concatenating them with \(N - i\).

Since in our algorithm updating labels is performed in parallel, it could happen that for some new label several threads would simultaneously create several new sets and then insert them to the list. In order to avoid such a mistake we use synchronization between performed instructions.
Let $i$ be the number of iteration and let $current$ be a vertex with lexicographically the largest label chosen during $i$ iteration. Let $x$ be the vertex assigned to the thread $th_x$. Let $x$ belong to set $L_x$. Let $oldNext_x$ and $newNext_x$ be private variables of the thread $th_x$. For all threads $th_x$ in parallel do:

1. if $x$ is not active or $x$ is not adjacent to $current$ then stop
2. $oldNext_x \leftarrow L_x.next$
3. create a new set $newNext_x$
4. synchronization: wait for other threads
5. set pointers:
   - $L_x.next = newNext_x$
   - $newNext_x.next = oldNext_x$
6. synchronization: wait for other threads
7. insert $x$ to $L_x.next$

Each thread creates its new set $newNext_x$ and inserts it to the order. After that for each node $x$, $newNext_x.next = oldNext_x$ but only for one of them we will have $L_x.next = newNext_x$. See Figure 5.

Figure 5: Insert a new set $newNext_x$ into the list $L$

Note that we cannot predict which $newNext_x$ will be in $L_x.next$ so synchronization is performed to all threads read the same $L_x.next$ inserted after $L_x$ in the list. The new sets of other threads are forgotten.

Now let us look at removing vertices from the sets. After this operation some sets can be empty. To get a vertex of the lexicographically largest label in the list, we take the last set on the list and this set cannot be empty. Therefore after each update operation, when removing vertices from sets is performed, we remove all empty sets from the list.
Before we show the procedure of removing the empty sets, consider the following lemma.

**Lemma 6.3.** If, after the update operation, the set $L_k$ is empty then its successor on the list, if exists, is nonempty.

**Proof.** Let $L_k$ be a non-empty set before the update operation. If after the update operation $L_k$ is empty then from Observation 6.2, we know that the set $L_k.next$ includes all vertices which were in set $L_k$ before the update. 

The observation that for each empty set, its predecessor and successor are non-empty means that all empty sets can be removed in parallel in one time. The removal of one set requires changing only two links of two adjacent non-empty sets. It is correct because it does not require any additional traversing through the list and it is independent from removing other empty sets.

To find out which sets are empty, we use an additional counter array. For each set in the list, counter stores 1 if the set includes at least one node or 0 otherwise. At the beginning all slots of counter are 0.

Let $x$ be the vertex assigned to the thread $th_x$. Let $x$ be in set $L_x$. Let $i$ be the number of iteration. Let current be the vertex with lexicographically the largest label during $i$ iteration. Let oldNext$ _x$ and newNext$ _x$ be private variables of the thread $th_x$. For all threads $th_x$ in parallel do:

1. if $x$ is not active then stop
2. $counter[L_x] \leftarrow 1$
3. $oldNext_x \leftarrow L_x.next$
4. synchronization: wait for other threads
5. if $counter[oldNext_x] = 1$ then stop
6. set $L_x.next \leftarrow oldNext_x.next$

We use synchronization between instructions to make sure that counting vertices and removing empty sets are correct. Otherwise some sets could be removed despite they are not empty and pointers could be changed improperly.
In our implementation, getting the vertex of lexicographically the largest label and updating the labels of all adjacent vertices are performed in four stages. Each stage run on the GPU and they are synchronized - the new one does not start until the last one has not finished. Because we use the CUDA language to implement the algorithm we use the term \textit{kernel} instead of \textit{stage}. Each kernel is executed by $N$ threads, one for each vertex.

At the beginning of algorithm all vertices have lexicographically the same label then we set \textit{current} to random vertex 1. Next, in each iteration of for loop, one vertex is choosen to the LexBFS order.

The first kernel adds \textit{current} vertex to the LexBFS order and marks it as non-active. Next, the first part of the labels updating is performed: setting counters of sets on 0 and saving the \textit{next} pointers of each set.

In the second kernel, the new sets are inserted to the list $L$.

In the third kernel, each vertex that is active and adjacent to \textit{current} is moved to the next set. Next, the counting is performed: for each active vertex, the counter of its set is made equal 1. At the end, the first part of deleting the empty set is performed: saving the \textit{next} pointers of each set.

The last kernel deletes all empty sets and chooses the new \textit{current} vertex.

At the end of the LexBFS algorithm, the \textit{order} array stores the LexBFS order.
Parallel Lexicographic Breadth-Frist Search algorithm

parallelLexBFS()

    current ← 1
    for time ← 1 to n do
        kernel1()
        kernel2()
        kernel3()
        kernel4()

kernel1()

    if x is active then
        oldNext_x ← L_x.next
        counter[L_x] ← 0
        create newNext_x
        if x is current then
            order[time] ← x
            mark x as non-active

kernel2()

    if x is active and x is adjacent to current then
        //inserting new sets to list
        L_x.next ← newNext_x
        newNext_x.next ← oldNext_x

kernel3()

    if x is active and x is adjacent to current then
        //moving to new sets
        move x to L_x.next
        if x is active then
            counter[L_x] ← 1
            oldNext_x ← L_x.next

kernel4()

    if x is active then
        if counter[L_x] = 0 then
            //deleting empty sets
            L_x.next ← oldNext_x.next
        if L_x.next = NULL then
            //updating current
            current ← x
6.2 Parallel test for perfect elimination order

Now we are given the LexBFS order $\pi$. The second step of testing chordality of graphs is checking if the LexBFS order is the perfect elimination order, that is, if for each vertex $x$, the neighborhood of $x$ on the left in the order forms a clique.

Let $LN_x \subset N_x$ be the set of all nodes adjacent to $x$ that lie on the left of $x$ in $\pi$ and let $p_x$ be the right most node in $LN_x$. In the sequential version of the algorithm, for each node $x$ we check if $LN_x - \{p_x\} \subset LN_{p_x}$. Now we do this in parallel for all nodes.

The algorithm has two kernels. The first one, for each $x$, in parallel computes the left neighborhood $LN_x$ and the right most vertex in $LN_x$. In the second kernel, each thread $th_x$ processes the left neighborhood of $x$ and the left neighborhood of $p_x$. If some left neighbor of $x$, different from $p_x$, is not a left neighbor of $p_x$ then $th_x$ marks the global variable $flag$ on false. At the end, if $flag$ is true then the order is the perfect elimination order.

| Parallel Test for Perfect Elimination Order |
|-------------------------------------------|
| //run on the cpu                           |
| parallelTestPEO()                          |
|   flag ← true                             |
|   preparationLNandP()                     |
|   testing()                               |
|     if flag = true then return YES        |
|     else return NO                         |

| //run on the gpu                           |
| preparationLNandP()                       |
|   $p_x$ ← 0                               |
|   for each $y$ adjacent to $x$ do         |
|     if $order^{-1}(y) < order^{-1}(x)$ then |
|     $LN_x$.insert($y$)                     |
|     if $order^{-1}(y) > order^{-1}(p_x)$ then |
|     $p_x$ ← $y$                           |

| //run on the gpu                           |
| testing()                                 |
|   for each $y$ adjacent to $x$ do         |
|     if $y$ ∈ $LN_x$ and $y \notin LN_{p_x}$ then |
|     $flag$ ← false                        |
6.3 Details of the parallel implementation

After reading the input, the algorithm copies the adjacency matrix of the graph to the array on the device memory. During the LexBFS algorithm, no other memory transfer between host and device is performed.

Since the algorithm does not compare any labels, the label concatenation can be omitted. Instead of this, the algorithm assigns to the new sets the numbers which have not appeared yet. In each iteration, there are at most \( N \) new sets, hence during the whole algorithm there are at most \( N^2 \) new sets. Since each set has a pointer to the next set on a list then to store all pointers, the algorithm uses an array of size \( N^2 \), which is indexed by the numbers of the sets.

Our parallel implementation of the LexBFS algorithm uses the following arrays:

1. the \( \text{Adj} \) array of size \( N^2 \) for the boolean adjacency matrix.
2. the \( \text{label} \) array of size \( N \) for the integer labels of sets.
3. the \( \text{order} \) array of size \( N \) for the integer indices of vertices in the LexBFS order.
4. the \( \text{next} \_\text{label} \) array of size \( N^2 \) for the integer indices of the next sets.
5. the \( \text{old} \_\text{next} \_\text{label} \) auxiliary array of size \( N \) for the saved values from the \( \text{next} \_\text{label} \) array, one value for each vertex.
6. the \( \text{counter} \) array of size \( N^2 \) for the boolean flags for recognizing if a set is empty.
7. the \( \text{current} \) variable for the integer number of vertex whose label is lexicographically the largest.

During an iteration, the algorithm processes one \( \text{current} \) vertex which is assigned to one row of every 2-dimensional array and all threads process that row in one time. See figure below.
Because each current is unique and unrepeatable throughout the LexBFS algorithm then each row is visited only once. Therefore, in order to reduce the amount of the device memory used by our algorithm, we use the 2-dimensional Adj array for two purposes: first as the adjacency matrix, next as the counter array.

The second part of the chordality algorithm uses two arrays: the Adj array and the order array. Because the Adj array is overwritten after the LexBFS algorithm then the algorithm copies again the adjacency matrix from the host memory to the Adj array on the device memory.

7 Tests and results

We introduce the following terminology to be used in this section. A graph $G = (V, E)$ with the vertex set $V$ of size $N$ is sparse if the size of $E$ is $\theta(N)$. A graph $G$ is dense if the size of $E$ is $\theta(N^2)$. We consider the following classes of graphs:

1. Cliques on $N$-vertices, for $N \in \{1000, 2000, 3000, \ldots, 10000\}$.
2. Dense random graphs on 10000 vertices.
3. Sparse random graphs on 10000 vertices.
4. Trees on 10000 vertices.
5. Chordal random graphs on 10000 vertices.

We test two implementations. The sequential implementation is the Habib, McConnell, Paul and Viennot algorithm presented in [2], which use a static memory allocation. For each class, we also present the time excluding the input reading and the dynamic allocation of the device memory. For the parallel implementation, the time of reading the input and the dynamic allocation on the GPU is many times greater than the remaining time of the algorithm. However algorithm cannot be implemented without these operations.
7.1 Cliques

Figure 6 presents timing results for cliques. For graphs of size smaller than 1000, the sequential version is faster. When vertices number is 10000, the parallel implementation is two times faster than the sequential one.

Figure 6: Cliques

| N   | GPU without input and memory allocation time | CPU without input time |
|-----|---------------------------------------------|------------------------|
| 1000| 0.4                                         | 1.3                    |
| 2000| 0.8                                         | 4.9                    |
| 3000| 1.4                                         | 11.1                   |
| 4000| 2.1                                         | 19.5                   |
| 5000| 2.7                                         | 30.5                   |
| 6000| 3.7                                         | 44.0                   |
| 7000| 4.4                                         | 60.0                   |
| 8000| 5.4                                         | 77.8                   |
| 9000| 6.6                                         | 99.0                   |
| 10000| 7.8                                        | 121.9                  |
| 11000| 8.9                                        | 147.0                  |
7.2 Dense graphs

Figure 7 presents timing results for dense random graphs. For each test the parallel implementation is almost two times faster than the sequential implementation.

![Figure 7: Dense random graphs: $N = 10000$, $M = O(N^2)$]

|                | GPU without input and memory allocation time | CPU without input time |
|----------------|---------------------------------------------|------------------------|
| test1          | 9.0                                         | 107.4                  |
| test2          | 8.9                                         | 108.6                  |
| test3          | 8.9                                         | 106.2                  |
| test4          | 8.9                                         | 106.7                  |
| test5          | 8.9                                         | 107.2                  |
7.3 Sparse graphs

We have tested our implementation on sparse random graphs which $M = 20N$. The parallel implementation is slower than the sequential implementation. (Figure 8)

Figure 8: Sparse random graphs: $N = 10000$, $M = 20N$

|        | the time (ms) |          |          |
|--------|---------------|----------|----------|
|        | GPU            | CPU      |          |
|        | without input and memory allocation time | without input time |          |
| test1  | 11.3           | 92.7     | 0.9      | 71.6     |
| test2  | 11.2           | 91.5     | 0.7      | 71.0     |
| test3  | 11.2           | 91.1     | 0.8      | 71.1     |
| test4  | 11.2           | 90.9     | 0.8      | 71.1     |
| test5  | 11.2           | 92.0     | 0.8      | 72.0     |
7.4 Trees

The results for trees are very similar to the results for sparse random graphs (Figure 9).

Figure 9: Trees: N=10000

|       | GPU without input and memory allocation time | CPU without input time |
|-------|---------------------------------------------|------------------------|
| test1 | 7.2                                         | 86.9                   |
| test2 | 7.3                                         | 87.6                   |
| test3 | 7.4                                         | 87.6                   |
| test4 | 7.3                                         | 86.6                   |
| test5 | 7.3                                         | 88.0                   |
| test6 | 7.5                                         | 87.7                   |
| test7 | 7.2                                         | 87.6                   |

|       | GPU without input and dynamic memory allocation time | CPU with input time |
|-------|---------------------------------------------|---------------------|
| test1 | 7.2                                         | 86.9                |
| test2 | 7.3                                         | 87.6                |
| test3 | 7.4                                         | 87.6                |
| test4 | 7.3                                         | 86.6                |
| test5 | 7.3                                         | 88.0                |
| test6 | 7.5                                         | 87.7                |
| test7 | 7.2                                         | 87.6                |
7.5 Chordal graphs

Figure 10 presents timing results for chordal random graphs, including dense and sparse graphs. Only for sparse graphs the parallel implementation is slower. On this figure it is easy to see that the parallel implementation is stable - the time of algorithm is independent from the number of edges, in contrast to the sequential implementation.

Figure 10: Chordal random graphs, \(N=10000\)

|       | GPU | CPU |
|-------|-----|-----|
|       | without input and memory allocation time | without input time |
| test1 | 7.2  | 92.0 | 19.2 | 92.8 |
| test2 | 7.9  | 99.0 | 66.4 | 146.5 |
| test3 | 7.9  | 99.2 | 68.9 | 149.2 |
| test4 | 7.6  | 98.1 | 62.8 | 142.5 |
| test5 | 7.8  | 95.7 | 42.4 | 120.2 |
| test6 | 7.4  | 90.0 | 12.8 | 86.0  |
| test7 | 7.5  | 90.7 | 13.1 | 85.4  |
| test8 | 7.4  | 90.1 | 11.7 | 83.8  |
8 Conclusion and Future Work

The main result of this paper is the parallel algorithm to test chordality of graphs based on our own efficient parallel version the LexBFS algorithm. For a graph $G$ of $N$ vertices and $M$ edges, the algorithm takes the $O(N)$ time and performs the $O(N^2)$ work on the $N$-threads machine. We use the CUDA multithreads architecture to implement these algorithms.

Our parallel implementation achieves best results for cliques and dense graphs. For graphs of 1000 and more vertices, the parallel algorithm is significantly faster than our fast sequential implementation and for graphs of 10000 vertices, the parallel implementation is two times faster than the sequential version. For trees, sparse graphs and small graphs (less than 1000 vertices) the sequential algorithm outperforms the parallel one. However, for this kind of data the parallel implementation is stable, the execution time is independent of the size of a graph.

It would be interesting if the parallel LexBFS algorithm could be used as a core for efficient parallel testing of interval graphs. Further research could be also made towards parallel implementation of the MCS algorithm.
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