Integral representations of one dimensional projections for multivariate stable densities

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Abstract

We consider the numerical evaluation of one dimensional projections of general multivariate stable densities introduced by Abdul-Hamid and Nolan (1998). In their approach higher order derivatives of one dimensional densities are used, which seem to be cumbersome in practice. Furthermore there are some difficulties for even dimensions. In order to overcome these difficulties we obtain the explicit finite-interval integral representation of one dimensional projections for all dimensions. For this purpose we utilize the imaginary part of complex integration, whose real part corresponds to the derivative of the one dimensional inversion formula. We also give summaries on relations between various parameterizations of stable multivariate density and its one dimensional projection.

1 Introduction

Stable distributions are known as the limiting distributions of the general central limit theorem and the time invariant distributions for Lévy processes. Because of these probabilistically very important properties, many studies have been carried out on their theoretical aspects. Comprehensive books have been published (Araujo and Giné (1980) or Christoph and Wolf (1993)). Furthermore applications of stable distributions to heavy-tailed data have been growing over past few decades. These applications have appeared in many fields like finance, Internet traffic or physics. Especially in time series, the observations from the causal stable ARMA or fractional stable ARIMA model can be considered as a multivariate stable random vector. We refer to Uchaikin and Zolotarev (1999), Rachev (2003) or Adler et al. (2003) for these applications. As various multivariate heavy tailed data have become available in these fields, the treatments of multivariate stable distributions are needed. However, the multivariate stable distributions require handling of their spectral measures which is computationally difficult.

We briefly review some important recent progresses concerning the multivariate stable distributions. Concerning the calculation of the integral with respect to the spectral measure, Byczkowski et al. (1993) have approximated the spectral measure by a discrete spectral measure which is uniformly close to the original spectral measure. They also have given several instructive graphs of the stable densities. Concerning the approximations, Davydov and Nagaev (2002) have outlined possible directions
although their results are still theoretical. For the estimation of the parameters, see Pivato and Seco (2003) and Davydov and Paulauskas (1999) and for the hypothesis testing see Mittnik et al. (1999). Although there are many other researches, accurate calculations of the multivariate stable densities are clearly of basic importance.

For the calculations of the multivariate stable densities, the method of Abdul-Hamid and Nolan (1998) based on one dimensional projection is very promising. In the approach of Abdul-Hamid and Nolan (1998), the basic ingredient is a function $g_{\alpha,d}(v,\beta)$ defined by

$$
g_{\alpha,d}(v,\beta) = \begin{cases} 
\frac{1}{(2\pi)^d} \int_0^\infty \cos \left( vu - \left( \beta \tan \frac{\pi \alpha}{2} \right) u^\alpha \right) u^{d-1} e^{-u^\alpha} du, & \alpha \neq 1, \\
\frac{1}{(2\pi)^d} \int_0^\infty \cos \left( vu + \frac{2}{\pi} \beta u \log u \right) u^{d-1} e^{-u} du, & \alpha = 1,
\end{cases}
$$

(1)

where $\alpha$ is the characteristic exponent of the stable distribution, $d$ is the dimension and $\beta$ corresponds to the skewness parameter.

Evaluating $g_{\alpha,d}(v,\beta)$ using this very definition is not very practical because the integrand is infinitely oscillating and changing its sign. Therefore an alternative evaluation of $g_{\alpha,d}(v,\beta)$ is desirable.

In this paper, we obtain explicit finite-interval integral representations of $g_{\alpha,d}(v,\beta)$, which do not require the derivatives of one dimensional densities. The integrand in our representation is well-behaved without infinite oscillations. Furthermore, our integral representation covers all dimensions and all parameter values, except for an exceptional case of $\alpha \leq 1, \beta = 1$ and even dimension, where we need an additional term which is easily tractable.

The calculation of $g_{\alpha,d}(v,\beta)$ corresponds to the evaluation of the equation (2.2.18) of Zolotarev (1986), which is stated without an explicit proof. For odd dimension, the real part of the equation is equivalent to $g_{\alpha,d}(v,\beta)$ and for even dimension, the imaginary part is equivalent to $g_{\alpha,d}(v,\beta)$. Complete derivations of the equation (2.2.18) of Zolotarev (1986) is also useful for the numerical evaluations of the derivatives of one dimensional symmetric densities investigated in Matsui and Takemura (2006).

This paper is organized as follows. For the rest of this section we prepare preliminary results based on Abdul-Hamid and Nolan (1998) and Nolan (1998). In Section 2 we derive finite-interval integral representations of one dimensional projections. In Section 3 we summarize relations between various parameterizations of the multivariate stable density and its one dimensional projection as propositions. Some discussions and directions for further studies are given in Section 4. Some proofs are postponed to Section 5.

1.1 Definitions and preliminary results

Let $\mathbf{X} = (X_1, X_2, \ldots, X_d)$ denote a $d$-dimensional $\alpha$-stable random vector with the characteristic function

$$
\Phi_{\alpha,d}(\mathbf{t}) = E \exp(i\langle \mathbf{t}, \mathbf{x} \rangle),
$$

where $\mathbf{t} = (t_1, t_2, \ldots, t_d)$ and $\langle , \rangle$ denotes the usual inner product. Among several definitions of stable distributions, we give the spectral representation of $\Phi_{\alpha,d}(\mathbf{t})$. This requires integration over the unit sphere $S^d = \{ s : \|s\| = 1 \}$ in $\mathbb{R}^d$ with respect to a spectral measure $\Gamma$ (Theorem 2.3.1 of Samorodnitsky and Taqqu 1994, Theorem 14.10 of Sato 1999). The spectral definition is given as
\[
\Phi_{\alpha,d}(t) = \begin{cases} 
\exp \left( - \int_{\mathbb{R}^d} |\langle t, s \rangle|^\alpha \left( 1 - i \text{sign}(\langle t, s \rangle) \tan \frac{\alpha \pi}{2} \right) \Gamma(ds) + i\langle t, \mu \rangle \right), & \alpha \neq 1, \\
\exp \left( - \int_{\mathbb{R}^d} |\langle t, s \rangle| \left( 1 + i \frac{\alpha}{\pi} \text{sign}(\langle t, s \rangle) \log |\langle t, s \rangle| \right) \Gamma(ds) + i\langle t, \mu \rangle \right), & \alpha = 1.
\end{cases}
\] (2)

The pair \((\Gamma, \mu)\) is unique. The definition (2) corresponds to Zolotarev’s (A) representation and we use the notation \(S_{\alpha,d}(\Gamma, \mu)\) for this definition following [Nolan (1998)].

For another definition, we give a multivariate version of Zolotarev’s (M) parameterization, which is also defined in [Nolan (1998)]. We use notation \(S_{\alpha,d}^M(\Gamma, \mu_0)\).

\[
\Phi_{\alpha,d}(t) = \begin{cases} 
\exp \left( - \int_{\mathbb{R}^d} |\langle t, s \rangle|^\alpha \left( 1 + i \text{sign}(\langle t, s \rangle) \tan \frac{\alpha \pi}{2} (|\langle t, s \rangle|^\alpha - 1) \right) \Gamma(ds) + i\langle t, \mu_0 \rangle \right), & \alpha \neq 1, \\
\exp \left( - \int_{\mathbb{R}^d} |\langle t, s \rangle| \left( 1 + i \frac{2}{\pi} \text{sign}(\langle t, s \rangle) \log |\langle t, s \rangle| \right) \Gamma(ds) + i\langle t, \mu_0 \rangle \right), & \alpha = 1.
\end{cases}
\] (3)

In the equation (3), we avoid the discontinuity at \(\alpha = 1\) and the divergence of the mode as \(\alpha \to 1\) which are observed in the definition (2) in the non-symmetric case. The difference of (M) parameterization from (A) representation is only

\[\mu_0 = \mu - \tan \frac{\pi \alpha}{2} \int_{\mathbb{R}^d} s \Gamma(ds), \quad \alpha \neq 1.\]

Now consider inverting the characteristic function \(\Phi_{\alpha,d}(t)\) to evaluate the multivariate stable density. The useful idea of [Nolan (1998)] and [Abdul-Hamid and Nolan (1998)] is to perform this inversion with respect to the polar coordinates. The projection of an \(\alpha\)-stable random vector onto one dimensional stable random variable is explained as follows. Here we follow the arguments of Ex.2.3.4 of [Samorodnitsky and Taqqu (1994)]. For any \(t \in \mathbb{R}^d\) consider the characteristic function \(E^x \exp(iu(t, x))\) of a linear combination \(Y = \sum_i t_i X_i\), i.e.,

\[
E^x \exp(iu(t, x - \mu)) = \begin{cases} 
\exp \left( -|u|^{\alpha} \int_{\mathbb{R}^d} |\langle t, s \rangle|^{\alpha} \Gamma(ds) - i \text{sign}(u) \tan \frac{\alpha \pi}{2} \int_{\mathbb{R}^d} \text{sign}(\langle t, s \rangle) |\langle t, s \rangle|^\alpha \Gamma(ds) \right), & \alpha \neq 1, \\
\exp \left( -|u| \int_{\mathbb{R}^d} \langle t, s \rangle \Gamma(ds) + i \frac{2}{\pi} \text{sign}(u) \int_{\mathbb{R}^d} (|u| + \log |\langle t, s \rangle|) \Gamma(ds) \right), & \alpha = 1.
\end{cases}
\]

Then, from the definition below, \(E^x \exp(iu(t, x))\) is considered as a characteristic function of one dimensional stable distribution with the following parameters.

\[
\sigma(t) = \left( \int_{\mathbb{R}^d} |\langle t, s \rangle|^\alpha \Gamma(ds) \right)^{1/\alpha},
\] (4)

\[
\beta(t) = (\sigma(t))^{-\alpha} \int_{\mathbb{R}^d} \text{sign}(\langle t, s \rangle) |\langle t, s \rangle|^\alpha \Gamma(ds),
\] (5)

\[
\mu(t) = \begin{cases} 
0, & \alpha \neq 1, \\
- \frac{2}{\pi} \int_{\mathbb{R}^d} \langle t, s \rangle \log |\langle t, s \rangle| \Gamma(ds), & \alpha = 1.
\end{cases}
\] (6)

Regarding \(u\) as the length and \(t\) as the angle, we transform the \(d\)-dimensional rectangular integral into the polar coordinate integral. In the following theorem of [Nolan (1998)], multivariate stable densities of (A) representation are projected onto one dimensional densities in (A) representation. Since in the original theorem [Nolan (1998)] there are some trivial typographical errors, we correct them in the formula (1) and the following theorem.
Theorem 1.1 [Nolan (1998)] Let $0 < \alpha < 2$ and $X = (X_1, X_2, \ldots, X_d)$ be an $\alpha$-stable random vector $S_{\alpha,d}(\Gamma, \nu)$ with $d \geq 2$. Then the density $f_{\alpha,d}(x)$ of $X$ is given as

$$f_{\alpha,d}(x) = \begin{cases} \int_{s_{\alpha,d}} g_{\alpha,d} \left( \frac{x - v, s}{\sigma(s)}, \beta(s) \right) (\sigma(s))^{-d} \, ds, & \alpha \neq 1, \\ \int_{s_{\alpha,d}} g_{1,\alpha} \left( \frac{x - v, s - \mu(s) - (2/\pi)\beta(s)\sigma(s)\log \sigma(s)}{\sigma(s)}, \beta(s) \right) (\sigma(s))^{-d} \, ds, & \alpha = 1, \end{cases}$$

where $g_{\alpha,d}(v, \beta)$ is given in (1).

Abdul-Hamid and Nolan (1998) expressed $g_{\alpha,d}(v, \beta)$ as an integral over a finite interval, utilizing the integral expressions of one dimensional densities. They explained the method when $d$ is odd as follows. From Theorem 2.2.3 of Zolotarev (1986), the one dimensional density $g_{\alpha,1}(v, \beta)$ can be written as

$$g_{\alpha,1}(v, \beta) = \int_{a}^{b} h_{g}(\theta; \alpha, \beta, v) d\theta,$$

where $-\pi/2 \leq a < b \leq \pi/2$ and $h_{g}$ is a somewhat complicated but explicit function. Then, for odd $d$, differentiating $g_{\alpha,1}$ $d$ times with respect to $v$ gives the representation

$$g_{\alpha,d}(v, \beta) = c(d) \int_{a}^{b} \frac{\partial^{d-1} h_{g}(\theta; \alpha, \beta, v)}{\partial v^{d-1}} d\theta,$$

where $c(d)$ is some constant. However, the computation of $\partial^{d-1} h_{g}(\theta; \alpha, \beta, v)/\partial v^{d-1}$ seems to be cumbersome. Furthermore, they did not give the corresponding expression for even $d$. In the following, we obtain explicit finite-interval integral representations of $g_{\alpha,d}(v, \beta)$, which do not require the derivatives of one dimensional densities.

2 Finite-interval integral representations of projected one dimensional functions

In this section we obtain the finite-interval integral representations of $g_{\alpha,d}(v, \beta)$ in (1). It is based on the function $h^{n}(x; \alpha, \beta)$ defined below in (1), which corresponds to the $n$-th order derivative of one dimensional inversion formula including the imaginary part.

We first prepare some notations which are the same as in Zolotarev (1986).

$$K(\alpha) = \alpha - 1 + \text{sign}(1 - \alpha), \quad \alpha \neq 1,$$

$$\theta = \beta K(\alpha)/\alpha, \quad \alpha \neq 1,$$

$$U_{\alpha}(\varphi; \theta) = \left( \frac{\sin \frac{\pi}{2}(\varphi + \theta)}{\cos \frac{\pi}{2} \varphi} \right)^{\alpha/(1-\alpha)} \frac{\cos \frac{\pi}{2}(\alpha - 1)\varphi + \alpha \theta}{\cos \frac{\pi}{2} \varphi}, \quad \alpha \neq 1,$$

$$U_{1}(\varphi; \beta) = \frac{\pi + \beta \varphi}{2 \cos \frac{\pi}{2}} \exp \left( \frac{\pi}{2} \left( \varphi + \frac{1}{\beta} \right) \tan \frac{\pi}{2} \varphi \right),$$

$$r(\varphi) = \begin{cases} \left( \frac{\sin \alpha(\varphi + \pi/2)}{x \cos \varphi} \right)^{1/(1-\alpha)}, & \alpha \neq 1, \\ \exp(-x/\beta + (\varphi + \pi/(2\beta)) \tan \varphi), & \alpha = 1, \end{cases}$$

4
\[ \tau = \begin{cases} \frac{(\alpha/x)^{1/(1-\alpha)}}{\exp(-x-1)}, & \alpha \neq 1, \\ \exp(-x-1), & \alpha = 1, \end{cases} \]

\[ V_n(\varphi) = r^n \left( \frac{\pi}{2} \right)^{\alpha/2} \left\{ r' \left( \frac{\pi}{2} \varphi \right) \sin \left( \frac{\pi}{2} n + 1 \right)(\varphi + 1) + r \left( \frac{\pi}{2} \varphi \right) \cos \left( \frac{\pi}{2} n + 1 \right)(\varphi + 1) \right\}, \]

\[ W_n(\varphi) = r^n \left( \frac{\pi}{2} \varphi \right) \left\{ r' \left( \frac{\pi}{2} \varphi \right) \sin \left( \frac{\pi}{2} n + 1 \right)(\varphi + 1) - r' \left( \frac{\pi}{2} \varphi \right) \cos \left( \frac{\pi}{2} n + 1 \right)(\varphi + 1) \right\}. \]

We now define a function \( h^n(x; \alpha, \beta) \) as

\[ h^n(x; \alpha, \beta) = \frac{1}{\pi} \int_0^\infty (iz)^n \exp \left( izx + \psi(z; \alpha, -\beta) \right) dz, \tag{7} \]

where

\[ \psi(z; \alpha, \beta) = \begin{cases} -z^\alpha \exp(-iz \theta), & \alpha \neq 1, \\ -\pi - i\beta \log z, & \alpha = 1, \end{cases} \]

is the logarithm of the characteristic function of stable distributions which are analytically extended from the semi-axis \( \text{Re} \ z > 0 \) into the complex plane. Then, \( g_{\alpha,d}(v, \beta) \) in \( (1) \) is expressed in terms of \( h^n(x; \alpha, \beta) \) as follows. Note that \( \beta \) in \( g_{\alpha,d}(v, \beta) \) is not equal to \( \beta \) in \( h^n(x; \alpha, \beta) \), because \( \beta \) in \( h^n(x; \alpha, \beta) \) is based on one dimensional (B) parameterization. Therefore, we need the following definitions,

\[ \beta_B = \frac{2}{\pi K(\alpha)} \arctan \left( \beta \tan \frac{\pi \alpha}{2} \right), \quad \theta_B = \beta_B K(\alpha)/\alpha, \quad \alpha \neq 1. \]

**Proposition 2.1** Define \( x = (\cos(\pi/2\alpha \theta_B))^{1/\alpha} v \) and \( y = \pi/2v + \beta \log(\pi/2) \). Then a representation of \( g_{\alpha,d}(v, \beta) \) in \( (1) \) in terms of \( h(\cdot; \alpha, \beta) \) is given as follows:

\[ g_{\alpha,d}(v, \beta) = \begin{cases} \text{Re} \left\{ \frac{(\cos(\pi/2\alpha \theta_B))^{d/\alpha}}{2^d \pi^{d-1}} h^{d-1}(|x|; \alpha, \beta_B^*), \right\} & \alpha \neq 1, \\ \text{Re} \left\{ \frac{\pi}{4\pi^{d-1}} h^{d-1}(y^*; \alpha, |\beta|), \right\} & \alpha = 1, \end{cases} \]

where \( \beta_B^* = \beta_B \text{ sign } x \) and \( y^* = y \text{ sign } \beta \).

Proof of this proposition is given in Section 3. Although the imaginary number \( i \) is present in the above formulas, it is for notational convenience and we do not have to write \( \text{Re}h(\cdot) \) or \( \text{Im}h(\cdot) \) depending on the dimension \( d \).

By Proposition 2.1 it suffices to express \( h^n(x; \alpha, \beta) \) as a finite-interval integral. We now give our main theorem. We mention that the following results are partially stated in [Zolotarev 1986] for the real case without proofs and the equation (2.2.34) of Zolotarev (1986) is not complete. Therefore we give the complete results including the calculation of imaginary parts. The full proof is given in Section 5.

**Theorem 2.1** Let \( x > 0 \) in the case \( \alpha \neq 1 \) and \( \beta > 0 \) in the case \( \alpha = 1 \). Then the finite-interval integral representations of \( h^n(x; \alpha, \beta) \) are as follows.

(a) \( (\alpha \neq 1, \beta \neq 1) \) or \( (\alpha > 1, \beta = 1) \)

\[ \text{Re} \ h^n(x; \alpha, \beta) = \frac{1}{2} \int_{-\theta}^{\theta} \exp \left( -x^{\alpha/(\alpha-1)} U_\alpha(\varphi; \theta) \right) V_n(\varphi) d\varphi. \]
\[ \text{Im } h^n(x; \alpha, \beta) = \frac{1}{2} \int_{-\theta}^{1} \exp \left( -x^{\alpha/(\alpha-1)} U_\alpha(\varphi; \theta) \right) W_n(\varphi) d\varphi. \]

(b) \( \alpha = 1, \beta \neq 1 \)

\[ \text{Re } h^n(x; 1, \beta) = \frac{1}{2} \int_{-1}^{1} \exp \left( -e^{-x/\beta} U_1(\varphi; \beta) \right) V_n(\varphi) d\varphi. \]

\[ \text{Im } h^n(x; 1, \beta) = \frac{1}{2} \int_{-1}^{1} \exp \left( -e^{-x/\beta} U_1(\varphi; \beta) \right) W_n(\varphi) d\varphi. \]

(c) \( \alpha < 1, \beta = 1 \)

Re \( h^n(x; \alpha, 1) \) is the same as in (a).

\[ \text{Im } h^n(x; \alpha, 1) = \frac{1}{2} \int_{-\theta}^{1} \exp \left( -x^{\alpha/(\alpha-1)} U_\alpha(\varphi; \theta) \right) W_n(\varphi) d\varphi - \frac{1}{\pi} \int_{0}^{\tau} \exp(x r - r^\alpha) r^n \, dr. \]

(d) \( \alpha = 1, \beta = 1 \)

Re \( h^n(x; 1, 1) \) is the same as in (b).

\[ \text{Im } h^n(x; 1, 1) = \frac{1}{2} \int_{-1}^{1} \exp \left( -e^{-x/\beta} U_1(\varphi; \beta) \right) W_n(\varphi) d\varphi - \frac{1}{\pi} \int_{0}^{\tau} \exp(x r + r \log r) r^n \, dr. \]

Interestingly, the case \( \beta = 1 \) is different from other values of the parameters. As suggested by Zolotarev (1986), the representations of Theorem 2.1 are essentially different from the \( n \)-th order derivative of the integral representations of densities given in the equation (2.2.18) of Zolotarev (1986).

### 3 Relations between representations of multivariate stable densities and one dimensional projections

In this section, we present propositions concerning the projections of a stable random vector onto one dimensional stable random variable. The idea of projection is useful for the purpose of the computation of the multivariate densities. There exist two representations (A) and (M) for the multivariate stable distributions, whereas three representations (A), (B) and (M) are known for general one dimensional stable distributions. The projection of the multivariate representation (A) onto one dimensional representation (A) is given in Theorem 1.1, which is obtained by Nolan (1998). Although Nolan (1998) mentioned the projection of (M) onto (M) as a theorem without proof, his is different from ours. Therefore we describe the projection of the multivariate representation (M) onto the projected one dimensional representation (M) with a precise proof. Note that the idea of projection entirely belongs to Nolan (1998). Further we present results for all projections without proofs by adding the other 4 projections. Note that for \( \alpha = 1 \) multivariate representations (A) and (M) and one dimensional representations (A) and (M) are the same. Therefore we omit statements on the projections (A \( \to \) M), (M \( \to \) A) and (M \( \to \) B) for \( \alpha = 1 \).
3.1 One-dimensional projections in various representations

Here we present propositions on various projections. Note that the parameters $\sigma(t)$, $\beta(t)$ and $\mu(t)$ used in this section have already been defined by (4), (5) and (6) in Section 1.1, respectively.

**Proposition 3.1 (The projection $M \to M$)** Let $0 < \alpha < 2$ and $X = (X_1, X_2, \ldots, X_d)$ be an $\alpha$-stable random vector $S_{a,d}(\Gamma, \nu)$ with $d \geq 2$. Then, the density $f_{\alpha,d}(x)$ is given as follows.

$$f_{\alpha,d}(x) = \int_{\mathbb{R}^d} g_{\alpha,d}^M \left( \frac{(x - \nu, s) - \mu_M^M(s)}{\sigma(s)}, \beta(s) \right) (\sigma(s))^{-d} \, ds,$$

where

$$g_{\alpha,d}^M(v, \beta) = \frac{1}{(2\pi)^d} \int_0^\infty \cos \left( vu + \left( \beta \tan \frac{\pi \alpha}{2}(u - u^\alpha) \right) \right) u^{d-1} e^{-u^\alpha} \, du$$

and

$$\mu_M^M(t) = \tan \frac{\pi \alpha}{2} \left( \beta(t)\sigma(t) - \int_{\mathbb{R}^d} (t, s) \Gamma(ds) \right).$$

When $\alpha = 1$, (A) parametrization in Theorem 1.1 and (M) parametrization in Proposition 3.1 are the same. It is easy to confirm $f_{1,d}(x) = f_{1,d}(x)$ and $g_{1,d}^M(v, \beta) = g_{1,d}(v, \beta)$ only by notational change. Note that at $\alpha = 1$, $\mu_M^M(t)$ becomes

$$\mu_M^M(t) = \frac{2}{\pi} \left( \beta(t)\sigma(t) \log \sigma(t) - \int_{\mathbb{R}^d} (t, s) \log |(t, s)| \Gamma(ds) \right) = \mu(s) + (2/\pi) \beta(s)\sigma(s) \log \sigma(s).$$

**Proposition 3.2 (The projection $A \to B$)** Under the same conditions and notations as in Theorem 1.1, the density $f_{\alpha,d}(x)$ is given as follows.

$$f_{\alpha,d}(x) = \begin{cases} \int_{\mathbb{R}^d} \frac{\sigma(t)}{\sigma_B^2(s)} \beta_B(s) \left( \sigma_B(s) \right)^{-d} \, ds, & \alpha \neq 1, \\ \int_{\mathbb{R}^d} \frac{\sigma(t)}{\sigma_B^2(s)} \beta_B(s) \left( \sigma_B(s) \right)^{-d} \, ds, & \alpha = 1, \end{cases}$$

where

$$g_{\alpha,d}^B(v, \beta) = \begin{cases} \frac{1}{(2\pi)^d} \int_0^\infty \cos \left( vu - u^\alpha \sin \left( \frac{\pi}{2} K(\alpha) \beta(t) \right) \right) u^{d-1} e^{-u^\alpha \cos(\pi/2K(\alpha)\beta(s))} \, du, & \alpha \neq 1, \\ \frac{1}{(2\pi)^d} \int_0^\infty \cos \left( vu + \beta u \log u \right) u^{d-1} e^{-\pi/2u} \, du, & \alpha = 1, \end{cases}$$

and

$$\sigma_B(t) = \begin{cases} \frac{\sigma(t)}{\left( \cos \left( \frac{\pi}{2} K(\alpha)\beta(t) \right) \right)^{1/\alpha}}, & \alpha \neq 1, \\ \frac{2}{\pi} \sigma(t), & \alpha = 1, \end{cases}$$

$$\beta_B(t) = \frac{2}{\pi K(\alpha)} \arctan \left( \beta(t) \tan \frac{\pi \alpha}{2} \right),$$

$$\mu_B(t) = \sigma_B(t)\beta(t) \log \sigma_B(t) - \frac{2}{\pi} \int_{\mathbb{R}^d} (t, s) \log |(t, s)| \Gamma(ds).$$
Proposition 3.3 (The projection \((A \rightarrow M)\)) Under the same conditions and relations as in Theorem 2.1, the density \(f_{\alpha,d}(x)\) is given as follows.

\[
f_{\alpha,d}(x) = \int_{\mathbb{R}^d} g_{\alpha,d}^M \left( \frac{\langle x - \nu, s \rangle - \mu_M(s)}{\sigma(s)}, \beta(s) \right) (\sigma(s))^{-d} ds, \quad \alpha \neq 1,
\]

where

\[
\mu_M(t) = \sigma(t) \beta(t) \tan \frac{\pi\alpha}{2}.
\]

Proposition 3.4 (The projection \((M \rightarrow A)\)) Under the same conditions and relations as in Proposition 3.1, the density \(f_{\alpha,d}^M(x)\) is given as follows.

\[
f_{\alpha,d}^M(x) = \int_{\mathbb{R}^d} g_{\alpha,d}^A \left( \frac{\langle x - \nu, s \rangle - \mu_A^M(s)}{\sigma(s)}, \beta_A(s) \right) (\sigma(s))^{-d} ds, \quad \alpha \neq 1,
\]

where

\[
\mu_A^M(t) = \tan \frac{\pi \alpha}{2} \int_{\mathbb{R}^d} (t, s) \Gamma(ds).
\]

Proposition 3.5 (The projection \((M \rightarrow B)\)) Under the same conditions and relations as in Proposition 3.1, the density \(f_{\alpha,d}^M(x)\) is given as follows.

\[
f_{\alpha,d}^M(x) = \int_{\mathbb{R}^d} g_{\alpha,d}^B \left( \frac{\langle x - \nu, s \rangle - \mu_A^M(s)}{\sigma_B(s)}, \beta_B(s) \right) (\sigma_B(s))^{-d} ds, \quad \alpha \neq 1.
\]

3.2 Finite-integral representations for other representations

In Zolotarev (1986) only the analytic extension of the characteristic function defined by (B) representation is considered and Theorem 2.1 corresponds to (B) representation of stable distributions. Therefore, we have introduced the parameters \(\beta_B\) and \(\theta_B\) in the finite integral-representation for (A) of Proposition 2.1. Although it is possible to derive (A) or (M) representation versions of Theorem 2.1 utilizing the results in Nolan (1997), we have to consider the analytic extension of the characteristic function defined by (A) or (M) representation which needs very complicated arguments. Therefore, in this paper, we confine our results to the case of (B) representation and for (A) and (M) representation, we utilize Proposition 2.1 and 3.6. Note that, accordingly, for \(g_{\alpha,d}^B(v, \beta)\) in Proposition 3.7 we do not need to use the extra parameters like \(\beta_B\) or \(\theta_B\) other than that defined in Theorem 2.1.

Proposition 3.6 Let \(g_{\alpha,d}^M(v, \beta)\) be as in Proposition 3.1. Define \(x = (\cos(\pi/2\alpha\theta_B))^{1/\alpha}(v + \tan(\pi/2\alpha\theta_B))\) and \(y\) as in Proposition 2.1. Then the representation of \(g_{\alpha,d}^M(v, \beta)\) using function \(h(\cdot; \alpha, \beta)\) is as follows.

\[
g_{\alpha,d}^M(v, \beta) = \begin{cases} \text{Re} \left( \frac{\cos(\pi/2\alpha\theta_B))^{1/\alpha}}{2^d(\pi i)^d-1} h_{d-1}(|x|; \alpha, \beta_B^*) \right), & \alpha \neq 1, \\ g_{1,d}(v, \beta), & \alpha = 1, \end{cases}
\]

where \(\beta_B^* = \beta \text{ sign } x\).
**Proposition 3.7** Let \( g_{\alpha,d}^B(v,\beta) \) be as in Proposition 3.2. Then the representation of \( g_{\alpha,d}^B(v,\beta) \) using function \( h(\cdot;\alpha,\beta) \) is as follows.

\[
g_{\alpha,d}^B(v,\beta) = \begin{cases} 
\text{Re} \frac{1}{2 (\pi i)^d-1} h^{d-1}(|v|; \alpha, \beta^*), & \alpha \neq 1, \\
\text{Re} \frac{1}{2 (\pi i)^d-1} h^{d-1}(v^*; 1, |\beta|), & \alpha = 1,
\end{cases}
\]

where \( \beta^* = \beta \sign v \) and \( v^* = v \sign \beta \).

### 4 Some discussions and future works

In this paper we focused on one dimensional projections \( g \) of the general multivariate stable density. This is only one step in calculating the density itself. We need to substitute parameters which are functions on the unit sphere \( s \in \mathbb{S}^d \) into \( g \) and then we need to integrate over the unit sphere to evaluate the density itself. We presented improvements in evaluation of \( g \). Improvements of other steps are also important for the evaluation of the multivariate density. By further careful examinations of the integrand of the function \( h^n(x; \alpha, \beta) \), we might find some useful regularities like one dimensional finite integral representations stated in Section 3 of Nolan (1997).

For our future work we consider showing various properties like tail dependencies or relations between the spectral measure and tails. Theoretically the asymptotic estimates of multivariate stable densities are obtained by Watanabe (2000) or Hiraba (2003). Relations between these theoretical results and our representations or numerical results are of our next concern.

Furthermore, we can also consider the expansions or the asymptotic expansions of functions \( g \)'s for the boundary values. Many expansions of densities are found in Zolotarev (1986). The method of the expansions may be directly applied to the projected functions \( g \)'s.

### 5 Proofs

In this section, we give the proofs of Proposition 2.1, Theorem 2.1 and Theorem 3.1.

#### 5.1 Proof of Proposition 2.1

For \( \alpha \neq 1 \), define

\[
I g_{\alpha,d}(v,\beta) = \frac{1}{(2\pi)^d} \int_0^\infty \sin \left( vu - \left( \beta \tan \left( \frac{\alpha}{2} \right) u \right) \right) u^{d-1} e^{-u^\alpha} du.
\]

Then by simple notational change we have

\[
g_{\alpha,d}(v,\beta) + i I g_{\alpha,d}(v,\beta) = \frac{1}{(2\pi)^d} \int_0^\infty e^{iuv - i(\beta \tan \left( \frac{\alpha}{2} \right) u)} u^{d-1} e^{-u^\alpha} du
\]

\[
= \frac{1}{2^d (\pi i)^{d-1} \pi} \int_0^\infty (iu)^{d-1} e^{iu^\alpha} e^{-u^\alpha} \{1 + i \tan \left( \frac{\alpha}{2} \theta_B \right) \} du
\]

\[
= \frac{1}{2^d (\pi i)^{d-1} \pi} \int_0^\infty (iu)^{d-1} e^{iu^\alpha} e^{-u^\alpha} / \cos \left( \frac{\alpha}{2} \theta_B \right) e^{i(\pi/2) \theta_B} du.
\]
Replacing \( u \) by \( u = (\cos (\frac{\pi}{2} \alpha \theta_B))^{1/\alpha} t \), we have
\[
\frac{1}{2^d(\pi i)^{d-1}} \left( \cos \left( \frac{\pi}{2} \alpha \theta_B \right) \right)^{\alpha/d} \frac{1}{\pi} \int_0^\infty (it)^{d-1} e^{it(\cos (\frac{\pi}{2} \theta_B))^{d/\alpha} v} e^{-t^\alpha e^{i(\pi/2) \alpha \theta B}} dt
\]
\[
= \frac{1}{2^d(\pi i)^{d-1}} \left( \cos \left( \frac{\pi}{2} \alpha \theta_B \right) \right)^{\alpha/d} \frac{1}{\pi} \int_0^\infty (it)^{d-1} \exp(itx + \psi(t; \alpha, -\beta_B)) dt
\]
\[
= \frac{1}{2^d(\pi i)^{d-1}} \left( \cos \left( \frac{\pi}{2} \alpha \theta_B \right) \right)^{\alpha/d} h^{d-1}(x; \alpha, \beta_B).
\]

Taking the real part of this equation, we obtain the desired result when \( x > 0 \). When \( x < 0 \), replacing \( v \) by \( -v \) and \( \beta \) by \( -\beta \), we can obtain the result in a similar fashion. Note that \( g_{\alpha,d}(-v, -\beta) = g_{\alpha,d}(v, \beta) \).

For \( \alpha = 1 \), define
\[
I_{g_1,d}(v, \beta) = \frac{1}{(2\pi)^d} \int_0^\infty \sin \left( vu + \frac{2}{\pi} \beta u \log u \right) u^{d-1} e^{-u} du.
\]

What we need is to calculate \( g_{1,d}(v, \beta) + iI_{g_1,d}(v, \beta) \) and to consider the real part of this equation. Since the proof is similar to the proof of the case \( \alpha \neq 1 \), we omit the rest of the proof for \( \alpha = 1 \).

5.2 Proof of Theorem 2.1

The function \( h^0(x; \alpha, \beta) \), which can be regarded as the inversion formula including the imaginary part, has the following representation ((2.2.20) of Zolotarev (1986)):
\[
h^0(x; \alpha, \beta) = \frac{1}{\pi} \int_0^\infty \exp (ixz + \psi(z; \alpha, -\beta)) dz.
\]

Differentiating \( h^0(x; \alpha, \beta) \) \( n \) times with respect to \( x \), we have
\[
h^n(x; \alpha, \beta) = \frac{1}{\pi} \int_0^\infty (iz)^n \exp (izx + \psi(z; \alpha, -\beta)) dz.
\]

For the calculation of \( h^n(x; \alpha, \beta) \) we consider the same contour as in Zolotarev (1986), which is
\[
\Gamma = \left\{ z : \operatorname{Im} (izx + \psi(z; \alpha, -\beta)) = 0, \quad \frac{\pi}{2} k \leq \arg z \leq \frac{\pi}{2} \right\},
\]
where
\[
k = \begin{cases} 
-\theta & \text{if } \alpha \neq 1, \\
-1 & \text{if } \alpha = 1, \beta \neq -1, \\
1 & \text{if } \alpha \leq 1, \beta = -1.
\end{cases}
\]

Since readers can refer to Zolotarev (1986) if necessary, the details of the contour are omitted. From Lemma 2.2.3 of Zolotarev (1986), we only have to calculate the integration along the contour \( \Gamma \).

(a) \((\alpha \neq 1, \beta \neq 1)\) or \((\alpha > 1, \beta = 1)\). Direct calculation gives
\[
h^n(x; \alpha, \beta) = \frac{1}{\pi} \Re \int_{\Gamma} \exp (izx + \psi(z; \alpha, -\beta)) (iz)^n dz
\]
\[
\frac{1}{\pi} \text{Im} \int_{\Gamma} \exp \left( izx + \psi(z; \alpha, -\beta) \right) (iz)^n \, dz
\]
\[
= \frac{1}{\pi} \int_{\Gamma} \exp \left\{ \text{Re} \left( izx + \psi(z; \alpha, -\beta) \right) \right\} \text{Re} \left\{ (iz)^n \, dz \right\}
\]
\[
+ \frac{1}{\pi} \int_{\Gamma} \exp \left\{ \text{Re} \left( izx + \psi(z; \alpha, -\beta) \right) \right\} \text{Im} \left\{ (iz)^n \, dz \right\}
\]
\[
= \frac{1}{\pi} \int_{\Gamma} \exp(-W(\varphi)) \left\{ \text{Re} \left\{ (iz)^n \, dz \right\} + \text{Im} \left\{ (iz)^n \, dz \right\} \right\},
\]
(8)

where
\[
W(\varphi) = \begin{cases} 
  x^{\alpha/(\alpha-1)} U_\alpha(2\varphi/\pi; \theta), & \alpha \neq 1, \\
  \exp(-x/\beta) U_1(2\varphi/\pi; \beta), & \alpha = 1.
\end{cases}
\]

For details for derivation of \( W(\varphi) \), see p.76 of Zolotarev (1986). Replacing \( z \) by \( z = re^{i\varphi} \) gives
\[
(iz)^n = (re^{i(\varphi+\pi/2)})^n
\]
\[
= r^n e^{in(\varphi+\pi/2)}
\]
\[
= r^n (\cos n(\varphi + \pi/2) + i \sin n(\varphi + \pi/2)).
\]

Since
\[
d(r \cdot \sin \varphi) = r' \sin \varphi \, d\varphi + r \cos \varphi \, d\varphi,
\]
\[
d(r \cdot \cos \varphi) = r' \cos \varphi \, d\varphi - r \sin \varphi \, d\varphi,
\]
\[
dz = r' \cos \varphi \, d\varphi - r \sin \varphi \, d\varphi + i \{ r' \sin \varphi + r \cos \varphi \} \, d\varphi.
\]

Combining the equations above, we have
\[
\text{Re} \ (iz)^n \, dz = r^n \left( r' \sin(n+1) \left( \varphi + \frac{\pi}{2} \right) + r \cos(n+1) \left( \varphi + \frac{\pi}{2} \right) \right) \, d\varphi,
\]
\[
\text{Im} \ (iz)^n \, dz = r^n \left( r \sin(n+1) \left( \varphi + \frac{\pi}{2} \right) - r' \cos(n+1) \left( \varphi + \frac{\pi}{2} \right) \right) \, d\varphi.
\]

From these equations and the relations of angles and the contour \( \Gamma \) on p.76 of Zolotarev (1986), (8) becomes
\[
\text{Re} \ h(x; \alpha, \beta) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp \left( -x^{\alpha/(\alpha-1)} U_\alpha(2\varphi/\pi; \theta) \right) \times r^n \left( r' \sin(n+1) \left( \varphi + \frac{\pi}{2} \right) + r \cos(n+1) \left( \varphi + \frac{\pi}{2} \right) \right) \, d\varphi
\]
\[
= \frac{1}{2} \int_{-\theta}^{\theta} \exp \left( -x^{\alpha/(\alpha-1)} U_\alpha(\varphi; \theta) \right) V_n(\varphi) \, d\varphi
\]
and
\[
\text{Im} \ h(x; \alpha, \beta) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \exp \left( -x^{\alpha/(\alpha-1)} U_\alpha(2\varphi/\pi; \theta) \right)
\]
\[
\times r^n \left( r \sin(n+1) \left( \varphi + \frac{\pi}{2} \right) - r' \cos(n+1) \left( \varphi + \frac{\pi}{2} \right) \right) d\varphi
\]
\[
= \frac{1}{2} \int_{-\theta}^{1} \exp \left( -x^{\alpha/(\alpha-1)} U_{\alpha}(\varphi, \theta) \right) W_{n}(\varphi) d\varphi.
\]

(b) \((\alpha = 1, \beta \neq 1)\). Since similar results hold in this case, we omit the proof.

(c) \((\alpha < 1, \beta = 1)\). Here we must consider additional contour \(\Gamma^* = \{z : \text{Re } z = 0, -\tau \leq \text{Im } z \leq 0\}\) as stated in p.76 of [Zolotarev 1986]. The contour \(\Gamma^*\) satisfies
\[
\text{Im } ((izx + \psi(z;\alpha,-1)) = 0
\]
for \(z \in \Gamma^*. The integral along \(\Gamma^*\) becomes
\[
\frac{1}{\pi} \int_{\Gamma^*} \exp(ix + \psi(z;\alpha,-1)) (iz)^n dz = \frac{1}{\pi} \int_{\Gamma^*} \exp(\text{Re } (izx + \psi(z;\alpha,-1))) (iz)^n dz.
\]
Replacing \(z = re^{-\frac{\pi i}{\beta}}\) gives
\[
\frac{1}{\pi} \int_{0}^{r} \exp(\text{Re } (izx + \psi(z;\alpha,-1))) r^n (-i) dr = -\frac{i}{\pi} \int_{0}^{r} \exp \left( -W \left( -\frac{\pi}{2} \right) \right) r^n dr,
\]
where
\[
W \left( -\frac{\pi}{2} \right) = xr \sin \left( -\frac{\pi}{2} \right) + r^\alpha \cos \alpha \left( -\frac{\pi}{2} + \frac{\pi}{2} \theta \right)
\]
for \(\alpha \neq 1\) and
\[
W \left( -\frac{\pi}{2} \right) = xr \sin \left( -\frac{\pi}{2} \right) + r \log r \sin \left( -\frac{\pi}{2} \right) + \left( -\frac{\pi}{2} + \frac{\pi}{2} \cos \left( -\frac{\pi}{2} \right) \right)
\]
for \(\alpha = 1\).

5.3 Proof of Proposition 3.1

Without loss of generality, we assume \(\nu = 0\) throughout the proof. The characteristic function \(\Phi_{\alpha,d}^M(t)\) can be written as
\[
\Phi_{\alpha,d}^M(t) = \exp \left( -\int_{S^d} |\langle t, s \rangle|^\alpha \Gamma(ds) + i \tan \frac{\pi\alpha}{2} \int_{S^d} \text{sign}(t, s) |\langle t, s \rangle|^\alpha \Gamma(ds) \right.
\]
\[
- i \tan \frac{\pi\alpha}{2} \int_{S^d} \text{sign}(t, s) |\langle t, s \rangle| \Gamma(ds) \right)
\]
\[
= \exp \left( -(\sigma(t))^{\alpha} + i(\sigma(t))^{\alpha} \tan \frac{\pi\alpha}{2} \beta(t) - i \tan \frac{\pi\alpha}{2} \int_{S^d} \langle t, s \rangle \Gamma(ds) \right)
\]
\[
= \exp \left( -\left( \sigma(t) \right)^{\alpha} + i \left( \sigma(t) \right)^{\alpha} \tan \frac{\pi\alpha}{2} \beta(t) - i \tan \frac{\pi\alpha}{2} \int_{S^d} \langle t, s \rangle \Gamma(ds) \right)
\]
\[
= \exp \left( -(\sigma(t))^{\alpha} + i(\sigma(t))^{\alpha} \tan \frac{\pi\alpha}{2} \beta(t) - i \tan \frac{\pi\alpha}{2} \int_{S^d} \langle t, s \rangle \Gamma(ds) \right)
\]
\[
\exp \left( -\sigma(t)^\alpha - i\sigma(t)^\alpha \tan \frac{\pi\alpha}{2} \beta(t) \left( (\sigma(t))^{1-\alpha} - 1 \right) + i\mu_M^M(t) \right) = \exp (\psi^M(t)).
\]

Then, the inversion formula gives
\[
f_{\alpha,d}^M(x) = (2\pi)^{-d} \int e^{-i\langle x, t \rangle} \exp (-\psi^M(t)) \, dt.
\]

Note that for a positive real number \( r > 0 \) and a vector \( s \in \mathbb{R}^d \), \( \mu_M^M(rs) = r\mu_M^M(s) \), \( \sigma(rs) = r\sigma(s) \) and \( \beta(rs) = \beta(s) \). Putting \( t = rs \) where \( r > 0 \) and \( s \in S^d \), we obtain
\[
f_{\alpha,d}^M(x) = (2\pi)^{-d} \int_{S^d} \int_0^\infty \exp \left[ -i\langle x, s \rangle r - (r\sigma(s))^\alpha - i(r\sigma(s))^\alpha \tan \frac{\pi\alpha}{2} \beta(s) \left( (r\sigma(s))^{1-\alpha} - 1 \right) 
\right.
\]
\[
+ir\mu_M^M(s)] r^{d-1} dr ds.
\]

Furthermore, replacing \( r \) by \( \mu = r\sigma(s) \), we get
\[
f_{\alpha,d}^M(x) = (2\pi)^{-d} \int_{S^d} \int_0^\infty \exp \left[ -i\langle x, s \rangle \mu - \mu_M^M(s) \sigma(s) - \left\{ u^\alpha + iu^\alpha \tan \frac{\pi\alpha}{2} \beta(s) \left( u^{1-\alpha} - 1 \right) \right\} \right]
\]
\[
u^{d-1}(\sigma(s))^{-d} du ds.
\]

Taking the real part of the integrand, we obtain
\[
f_{\alpha,d}^M(x) = \int_{S^d} g_{\alpha,d}^M \left( \frac{\langle x, s \rangle - \mu_M^M(s)}{\sigma(s)}, \beta(s) \right) (\sigma(s))^{-d} \, ds.
\]

Note that the same conclusion holds if we let \( \alpha \to 1 \) in the proof due to continuity at \( \alpha = 1 \). The direct but tedious calculation gives the representation \( \mu_M^M(t) \) at \( \alpha = 1 \). \( \square \)

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