Asymptotic space-time behavior of HTL gauge propagator

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The asymptotic behavior as \( t \to \infty \) and \( r \to \infty \) of the hard-thermal-loop propagator \( D^{\mu \nu}(t, \vec{r}) \) is computed in the Coulomb gauge. The asymptotic falloff is always a power law though generally different in the deep time-like and space-like regions. The contributions of quasiparticle poles and Landau branch cuts are computed. The most difficult calculation is the contribution of the branch cut in the transverse propagator \( D^{ij}(t, \vec{r}) \). For QED this produces a leading behavior of order \( T/r \) in both the time-like and space-like regions. The inclusion of a magnetic mass so as to describe QCD makes the leading behavior \( 1/(Tr)^{3} \), thus improving the infrared convergence. The asymptotic space-like behavior of all contributions (longitudinal and transverse, poles and cuts) is confirmed by also computing in the Euclidean formalism and analytically continuing. The results are compared will those for free gauge bosons at finite temperature.

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I. INTRODUCTION

The hard-thermal-loop approximation to high temperature gauge theories developed by Braaten and Pisarski provides a consistent perturbative method for calculating processes in which one or more of the external momenta are of order \( gT \). An excellent review of the entire subject and its applications is given by Le Bellac. For an SU(N) gauge theory with \( N_{f} \) flavors of massless quarks in the fundamental representation, in the high-temperature deconfined phase there is an effective thermal mass for the gluon

\[
m_{g}^{2} = \left( N + \frac{N_{f}}{2} \right) g^{2}T^{2}.
\]

For QED at high temperatures \( \pi T \gg m_{e} \), there is an effective thermal mass for the photon

\[
m_{g} = eT/3.
\]

However, these effective masses do not screen the electric and magnetic fields produced by all time-dependent currents. The situation is much more subtle. The phenomena of dynamical screening applies only to currents with frequencies below \( m_{g} \). When the frequency satisfies \( k_{0} < m_{g} \) then the momentum-space propagator has poles at complex values of the wave vector \( k \). The imaginary part of \( k \) produces the screening. In a different frequency regime, this is the reason that AM radio signals are exponentially attenuated in the earth’s ionosphere. In the hard-thermal-loop approximation static magnetic fields with \( k_{0} = 0 \) are not screened. Only if \( 0 < k_{0} < m_{g} \) does the pole contribution to the hard-thermal-loop propagator, \( D^{\mu \nu \text{pole}}(k_{0}, \vec{r}) \), fall exponentially at large \( r \). However, there are also branch cuts in the momentum-space propagator. How the cut contribution, \( D^{\mu \nu \text{cut}}(k_{0}, \vec{r}) \), behaves at large \( r \) has not been investigated.

Recent calculations of Boyanovsky, de Vega, et al have provided complementary information about the asymptotic behavior of the hard-thermal-loop propagator at large time and fixed fixed wave-vector \( k \). Those authors were interested in non-equilibrium effects and found it convenient to use the retarded propagator, \( D^{\mu \nu}_{R}(t, \vec{k}) \). Being the Fourier transform of the fixed frequency propagator, it necessarily contains the effects of all frequencies. They discovered that the Landau branch cuts are extremely important for the asymptotic behavior of \( D^{\mu \nu}_{R}(t, \vec{k}) \) and found power-law falloff rather than exponential. Similar results for current-current correlators were obtained by Arnold and Yaffe.

This paper will examine the asymptotic behavior in both time and space of the hard-thermal-loop propagator \( D^{\mu \nu}(t, \vec{r}) \). The propagator considered will be in the Coulomb gauge and it will be the time-ordered propagator, not the retarded. The asymptotic behavior in the limit \( t \to \infty, r \to \infty \) with \( t/r \) fixed is controlled by the region in which the energy \( k_{0} \) and momentum \( k \) are both of order \( m_{g} \), which is precisely the region in which the momentum-space propagator differs significantly from the free propagator. Reaching the asymptotic regime will generally require \( t \gg 1/m_{g} \) and \( r \gg 1/m_{g} \). The region far from the space-time origin with \( r/t < 1 \) will be called the deep time-like region and that with \( r/t > 1 \) will be called the deep space-like region.

It is difficult to anticipate the asymptotic behavior. The case of a free, massive vector boson is an interesting comparison. In the deep time-like region that propagator falls like a power, \( 1/t^{3/2} \); in the deep space-like region it falls exponentially, \( (T/r) \exp(-Mr) \).

For the hard-thermal-loop propagator the one might expect the quasiparticle poles to produce an asymptotic dependence like that of a massive boson. In the the deep time-like region this is correct, i.e. \( 1/t^{3/2} \). However, in the deep space-like region the quasiparticle pole contributions to both the longitudinal and transverse propagators produces power-law fall-off. The failure of the quasiparticle poles to produce an exponential fall-off confirms that the electric and magnetic fields produced by localized, time-dependent currents are not generally screened since those sources can contain frequencies higher than \( m_{g} \).
It is even more difficult to anticipate the effect of the branch cuts in the hard thermal loop propagator since free propagators can offer no guide. The most surprising result is the effect produced by the branch cut in the transverse propagator. The asymptotic behavior in either the deep space-like or time-like regions is the same:

\[ *D^{ij}_{\text{cut}}(t, \vec{r}) \rightarrow -\frac{iT}{8\pi \tau} (\delta^{ij} + i\vec{r} \vec{\tau}) + \ldots \] (1.3)

In the deep space-like region the $T/\tau$ behavior is not surprising since that is the leading behavior of free thermal propagators. However in the deep time-like region the $T/\tau$ behavior is very non-trivial. In contrast, free propagators in the deep time-like region fall exponentially.

In QCD the inclusion of a magnetic screening mass will change the above effect of the branch cut in the transverse propagator. This change will be calculated. In QED there is no magnetic mass and Eq. (1.3) applies.

The paper develops systematically the mathematical machinery necessary for the analysis. There are, of necessity, many contributions to evaluate. There are two propagators, longitudinal and transverse. Each propagator has a pole and a cut to consider. Each contribution must be evaluated in the time-like and in the space-like limits. The emphasis throughout is on extracting the asymptotic power-law behavior of the propagators. Exponentially small corrections are always omitted.

Section II examines the simple example of the free photon propagator in Feynman gauge. Section III summarizes how gauge boson propagators, free or interacting, are expressed in terms of spectral functions. The real subject of the paper, the asymptotic behavior of the hard-thermal-loop propagator, is confronted in Secs. IV and V which treat the longitudinal and the transverse propagators, respectively. Section VI provides a detailed summary of the results. For many readers it may be best to read the results in Sec. VI before reading how they were calculated.

There are four appendices. Appendix A displays the hard-thermal-loop spectral functions in other gauges. Appendix B proves the lemma that is used in the text to determine the asymptotic behavior in the deep space-like region. Appendix C computes the asymptotic behavior of $*D^{\mu\nu}(t, \vec{r})$ in the deep space-like region by using the Euclidean time formalism. The sum of the static and non-static contributions agree exactly with the sum of the pole and cut contributions that are computed directly in Minkowski space in Secs. IV and V. Appendix D contains details of the most difficult calculation, the asymptotic behavior of $*D^{ij}_{\text{cut}}(t, \vec{r})$ in the deep time-like region.

II. SIMPLE EXAMPLE

Before confronting the problem of hard-thermal-loop propagators it is useful to examine the much simpler problem of thermal propagators for free gauge bosons, i.e. those without hard-thermal-loop resummation. The reason for this example is to illustrate two points that will be important later. However, the free thermal propagators are important in their own right in two different contexts: (i) In the low temperature regime of QED, $\pi T \ll m_e$, no resummation is necessary for the photon propagator. (ii) In the high temperature regime of QED or QCD the ghost propagators do not require resummation and the free thermal form applies.

The free propagator is simple to compute in the Feynman gauge and provides two useful comparisons with the hard-thermal-loop results that will come later. In the Euclidean time formalism the Feynman gauge propagator is

\[ D_\mu^\mu(\tau, \vec{r}) = g^{\mu\nu}iT \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k} \cdot \vec{r} - i\omega_n \tau}}{\omega_n^2 + k^2}, \] (2.1)

where $\omega_n = 2\pi n T$ and $-\beta \leq \tau \leq \beta$. The poles at $k = \pm i\omega_n$ allow the momentum integration to be done by Cauchy’s theorem:

\[ D_\mu^\mu(\tau, \vec{r}) = g^{\mu\nu}iT \sum_{n=1}^{\infty} \cos(\omega_n \tau) e^{-\omega_n \pi r}. \]

The time-ordered propagator in real time is obtained by the replacement $\tau \rightarrow it$:

\[ D_{\mu\nu}(t, \vec{r}) = g^{\mu\nu}iT \sum_{n=1}^{\infty} \cos(\omega_n t) e^{-\omega_n \pi r}. \]

In the space-like region $r > t$ each term in the sum is exponentially small. Therefore the asymptotic behavior in the deep space-like region is $iT/4\pi \tau$. However, in the time-like region $t > r$ the terms in the series grow exponentially with $n$. To evaluate the behavior in the time-like region it is necessary to first sum the series in the space-like region (the region of convergence) and then continue that finite result to the time-like domain. Since the series is geometrical, the summation is elementary and gives

\[ D_{\mu\nu}(t, \vec{r}) = g^{\mu\nu}iT \sum_{n=1}^{\infty} \frac{1}{e^{2\pi T(r-t)} - 1} + \frac{1}{e^{2\pi T(r+t)} - 1}. \]

When $r \rightarrow \infty$ this still behaves as $1/r$ plus exponentially small corrections. The surprise is that when $t \rightarrow \infty$ this is exponentially small. More generally, for $r \rightarrow \infty$ and $t \rightarrow \infty$ with a fixed ratio $r/t < 1$, the propagator is exponentially small.

There are two lessons to be drawn. (i) The behavior in the deep space-like region can be easily calculated using the Euclidean formalism. (ii) The behavior in the deep time-like region can be quite different from the deep space-like. Unless one is able to perform the sum over frequencies, the Euclidean propagator cannot provide information about the asymptotic behavior of the Minkowski propagator in the deep time-like region.
This example is misleading in one respect. For the free propagator the leading space-like behavior, $T/r$, comes entirely from the static $n = 0$ mode. For the hard-thermal-loop propagators the leading behavior of $\mathcal{D}_{00}(t, r)$ will come from the non-static modes.

III. GENERAL STRUCTURE

Before getting to the HTL propagators, it is useful to display the underlying structure of the gauge boson propagator that holds whatever the approximation. The basic two-point functions or thermal Wightman functions are

$$
\mathcal{D}_{\mu\nu}^0(x) = -\text{i} \text{Tr} \left[ \mathcal{D}(x) A^\mu(x) A^\nu(0) \right],
\mathcal{D}_{\mu\nu}^I(x) = -\text{i} \text{Tr} \left[ \mathcal{D}(0) A^\mu(x) A^\nu (0) \right],
$$

where $\mathcal{D} = \exp(-\beta H)/\text{Tr}[\exp(-\beta H)]$ is the density operator at temperature $T = 1/\beta$.

**Time-ordered propagator.** The propagator that will be calculated later is the time-ordered propagator:

$$
\mathcal{D}_{\mu\nu}^>(t, \vec{r}) = \theta(t) \mathcal{D}_{\mu\nu}^0(t, \vec{r}) + \theta(-t) \mathcal{D}_{\mu\nu}^I(t, \vec{r}).
$$

What is actually known from hard-thermal-loop calculations is the propagator in momentum space or equivalently the spectral function since the time-ordered propagator can be expressed as

$$
\mathcal{D}_{\mu\nu}^0(k_0, \vec{k}) = \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \frac{\rho_{\mu\nu}(\tau, \vec{k})}{k_0 - \tau + \text{i}\epsilon} - \frac{\rho_{\mu\nu}(0, \vec{k})}{e^{\beta k_0} - 1}.
$$

Knowledge of the momentum-space propagator immediately yields the spectral function since

$$
\text{Im} \mathcal{D}_{\mu\nu}^0(k_0, \vec{k}) = -\rho_{\mu\nu}(k_0, \vec{k}) \left[ \frac{1}{2} + \frac{1}{e^{\beta k_0} - 1} \right].
$$

The Fourier transform of Eq. (3.1) implies that the two-point functions in momentum space are

$$
\mathcal{D}_{\mu\nu}^< (k_0, \vec{k}) = -i \rho_{\mu\nu}(k_0, \vec{k}) \left[ \frac{1}{2} + \frac{1}{e^{\beta k_0} - 1} \right],
\mathcal{D}_{\mu\nu}^< (k_0, \vec{k}) = -i \rho_{\mu\nu}(k_0, \vec{k}) \left[ \frac{1}{e^{\beta k_0} - 1} \right].
$$

Knowing the spectral function allows one to calculate the two-point functions in space-time by Fourier transforming:

$$
\mathcal{D}_{\mu\nu}^>(t, \vec{r}) = -i \int \frac{d^4k}{(2\pi)^4} e^{\text{i} \vec{k} \cdot \vec{r} - \text{i} k_0 t} \rho_{\mu\nu}(k_0, \vec{k}) \left( \frac{1}{1 - e^{-\beta k_0}} \right),
\mathcal{D}_{\mu\nu}^< (t, \vec{r}) = -i \int \frac{d^4k}{(2\pi)^4} e^{\text{i} \vec{k} \cdot \vec{r} - \text{i} k_0 t} \rho_{\mu\nu}(k_0, \vec{k}) \left( \frac{1}{e^{\beta k_0} - 1} \right).
$$

The asymptotic behavior of this Fourier transform is the subject of this paper. For the asymptotic behavior the time will always be positive (and large) so that the time-ordered propagator is the same as the thermal Wightman function.

**Retarded propagator.** For certain applications the retarded propagator is of interest [10,11]. In space-time the retarded propagator is

$$
\mathcal{D}_{\mu\nu}^R(t, \vec{r}) = \theta(t) \left[ \mathcal{D}_{\mu\nu}^>(t, \vec{r}) - \mathcal{D}_{\mu\nu}^<(t, \vec{r}) \right].
$$

In momentum space this becomes

$$
\mathcal{D}_{\mu\nu}^R(k_0, \vec{k}) = \int_{-\infty}^{\infty} \frac{d\sigma}{2\pi} \frac{\rho_{\mu\nu}(\sigma, \vec{k})}{k_0 - \sigma + \text{i}\epsilon}.
$$

Knowing the spectral function allows computation of the space-time dependence by

$$
\mathcal{D}_{\mu\nu}^R(t, \vec{r}) = -i \int \frac{d^4k}{(2\pi)^4} e^{\text{i} \vec{k} \cdot \vec{r} - \text{i} k_0 t} \rho_{\mu\nu}(k_0, \vec{k}).
$$

Comparison of Eq. (3.3) with Eq. (3.5) shows that the time-ordered propagator contains the Bose-Einstein function but the retarded propagator does not. As $t \to \infty$ the time-ordered propagator is more sensitive to the low frequency part of the spectral function.

**Imaginary time propagator.** The imaginary-time, or Euclidean, propagator already used in Sec. II is ordered in the variable $\tau$:

$$
\mathcal{D}_{\mu\nu}^E(\tau, \vec{r}) = \theta(\tau) \mathcal{D}_{\mu\nu}^>(-i\tau, \vec{r}) + \theta(-\tau) \mathcal{D}_{\mu\nu}^<(i\tau, \vec{r}),
$$

where $-\beta \leq \tau \leq \beta$. If one computes the Euclidean propagator for $0 \leq \tau \leq \beta$ and then analytically continues $\tau$ to the imaginary axis ($\tau \to it$), the result will be $\mathcal{D}_{\mu\nu}^< (t, \vec{r})$. Thus analytic continuations in time relates the Euclidean propagator to the time-ordered propagator as used in sample calculation of Sec. II. By contrast, if one knows the Euclidean propagator as a function of frequency, $\mathcal{D}_{\mu\nu}^E(\omega_n, \vec{r})$, then the continuation $\omega_n \to i(k_0 - \epsilon)$ will give the retarded propagator $\mathcal{D}_{\mu\nu}^R(k_0, \vec{r})$.

**HTL spectral functions in Coulomb gauge.** Calculations will be done in the Coulomb gauge, which is used extensively in applications of hard-thermal-loops [1]. In this gauge the spectral function has the tensor structure

$$
\rho^{00}(k_0, \vec{k}) = \rho_{t}(k_0, k),
\rho^{ij}(k_0, k) = (\delta^{ij} - \hat{k}^i \hat{k}^j) \rho_t(k_0, k),
$$

with six components vanishing: $\rho^{0j} = \rho^{ij} = 0$. The longitudinal and transverse spectral functions, introduced by Pisarski [1], both have the form

$$
\frac{1}{2\pi} \rho_s(k_0, k) = Z_s \left[ \delta(k_0 - \omega_s) - \delta(k_0 + \omega_s) \right] + \theta(k^2 - k_0^2) \beta_s(k_0, k),
$$

where $s$ denotes either $\ell$ or $t$. Both $\rho_{t}$ and $\rho_{s}$ are odd functions of $k_0$. The Dirac delta functions come from the pole in the momentum-space propagator at the quasiparticle energies $\pm \omega_s(k)$. The residues of those poles are the real functions $Z_s(k)$. The continuum contribution $\beta_s$ come from the discontinuity across the Landau branch cut in the momentum space propagators.
IV. LONGITUDINAL PROPAGATOR: $^*D^{00}(x)$

The main subject is the computation of the asymptotic behavior of the hard-thermal-loop, time-ordered propagator $^*D^\mu\nu(t,\vec{r})$ which for positive $t$ is the same as the Wightman function $^*D^\mu\nu_t(t,\vec{r})$. This section treats $^*D^\mu\nu_t(t,\vec{r})$, the Wightman function for gauge bosons with time-like polarizations in the Coulomb gauge. In momentum space it is determined by the longitudinal spectral function:

$$^*D^{00}_{\ell}(k, t) = -i \rho_{\ell}(k_0, k) \frac{1}{1 - e^{-\beta k_0}}. \tag{4.1}$$

The spectral function is

$$\frac{1}{2\pi} \rho_{\ell}(k_0, k) = Z_\ell [\delta(k - \omega_\ell) - \delta(k + \omega_\ell)] + \theta(k^2 - k_0^2) \beta_\ell(k_0, k),$$

where the residue function is

$$Z_\ell = \frac{\omega_\ell (\omega_\ell^2 - k^2)}{k^2(3m_g^2 - \omega_\ell^2 + k^2)^2},$$

and $m_g$ is the thermal gluon mass. The longitudinal spectral function for the cut is

$$\beta_\ell(k_0, k) = \frac{2}{3m_g^2} \frac{x}{N_\ell(x, k)}, \tag{4.2}$$

where $x = k_0/k$ and the denominator is

$$N_\ell(x, k) = \left[ \frac{2k^2}{3m_g^2} + 2 - x \ln \left( \frac{1 + x}{1 - x} \right) \right]^2 + [\pi x]^2. \tag{4.3}$$

The longitudinal spectral function satisfies various sum rules. The one that will be useful subsequently is

$$\int_{-k}^k dk_0 \beta_\ell(k_0, k) - \frac{3m_g^2}{k^2(3m_g^2 + k^2)^2} = 2Z_\ell \frac{\omega_\ell}{\omega_\ell} = 2Z_\ell \frac{\omega_\ell}{\omega_\ell} \tag{4.4}$$

To compute the Fourier transform of Eq. (4.1) it is efficient to organize the calculation into two steps:

$$^*D^{00}_{\ell}(t, k) = -i \int \frac{d^3 k}{(2\pi)^3} e^{ik \cdot \vec{r}} F_\ell(t, k),$$

$$F_\ell(t, k) = \int_{-\infty}^\infty dk_0 \frac{e^{-ik_0 t}}{2\pi} \rho_{\ell}(k_0, k).$$

The integral over the solid angles of $\hat{k}$ can be done directly:

$$^*D^{00}_{\ell}(t, k) = -i \int \frac{d^3 k}{2\pi r^3} e^{ik \cdot \vec{r}} F_\ell(t, k), \tag{4.5}$$

The function $F_\ell$ is a sum of the pole and cut contributions $F_\ell = F_{\ell, \text{pole}} + F_{\ell, \text{cut}}$, where

$$F_{\ell, \text{pole}}(t, k) = Z_\ell \left[ \frac{e^{-i\omega_\ell t}}{1 - e^{-\beta \omega_\ell}} + \frac{e^{i\omega_\ell t}}{\alpha \beta \omega_\ell - 1} \right], \tag{4.6}$$

$$F_{\ell, \text{cut}}(t, k) = \int_{-k}^k dk_0 \frac{e^{-ik_0 t}}{1 - e^{-\beta \omega_\ell}} \beta_\ell(k_0, k). \tag{4.7}$$

The Wightman function at fixed three-momentum is

$$^*D^{00}_{\ell}(t, k) = -iF_\ell(t, k).$$

A. Asymptotic time-like behavior of $^*D^{00}_{\ell}(x)$

The deep time-like region requires $t \to \infty$ and $r \to \infty$ with a fixed ratio $t/r > 1$.

1. Pole contribution.

The contribution of the quasiparticle pole is

$$^*D^{00}_{\ell, \text{pole}}(t, k) = -i \int_0^\infty dk k \sin(kr) \times Z_\ell \left[ \frac{e^{-i\omega_\ell t}}{1 - e^{-\beta \omega_\ell}} - \frac{e^{i\omega_\ell t}}{1 - e^{\beta \omega_\ell}} \right].$$

Extending the integration to negative $k$ gives

$$^*D^{00}_{\ell, \text{pole}}(t, k) = \frac{1}{4\pi^2 r} \int_{-\infty}^\infty dk k Z_\ell \left[ \frac{e^{ikr - i\omega_\ell t}}{1 - e^{-\beta \omega_\ell}} - \frac{e^{-ikr + i\omega_\ell t}}{1 - e^{\beta \omega_\ell}} \right].$$

The asymptotic behavior of this will be computed by the method of stationary phase. Define

$$\phi(k) = kr - \omega_\ell(k)t. \tag{4.8}$$

At large values of $r$ and $t$ the functions $e^{\pm i\phi}$ oscillate rapidly except at special values of $k$. Let $\overline{k}$ be the value of the wave vector at which the phase is stationary:

$$0 = \left[ \frac{d\phi}{dk} \right]_{\overline{k}} = r - \left[ \frac{d\omega_\ell}{dk} \right]_{\overline{k}} t. \tag{4.9}$$

This requires that the group velocity of the quasiparticle satisfy

$$\left[ \frac{d\omega_\ell}{dk} \right]_{\overline{k}} = \frac{r}{t} < 1.$$

The group velocity of the longitudinal dispersion relation is zero at $k = 0$ and grows monotonically with $k$, approaching $1$ as $k \to \infty$. Therefore there is always a real $\overline{k}$ which satisfies this. Near the stationary point the phase is

$$\phi(k) = \phi(\overline{k}) - \frac{1}{2}(k - \overline{k})^2 t \left[ \frac{d^2\omega_\ell}{dk^2} \right]_{\overline{k}} + \ldots.$$
The acceleration $d^2\omega / dt^2$ is always positive. To evaluate the contribution of the stationary phase point, the $k$ contour must be deformed into the complex plane so that it passes through the real value of $k$ on a path of steepest descent. For the first integrand, $\exp(+i\phi)$, this requires $k = k_0 + e^{-i\pi/4}$ where $s$ is real and gives a simple Gaussian integral:

$$\int dk e^{i\phi} = e^{i\phi} - i/4 \int ds e^{-s^2t/2}. $$

For the second integrand, $\exp(-i\phi)$, the correct path is $k = k_0 + e^{i\pi/4}s$. Performing the two Gaussian integrations gives

$$*\mathcal{D}^{00}_\text{pole}(t, r) = \frac{-i}{(2\pi)^2} \int_{-\infty}^{\infty} dk \int_{-1}^{1} dx e^{ik(r-xt)} I_t(x, k)$$

The phase $\phi(x, k) = k(r-xt)$ has vanishing first derivatives with respect to $x$ and $k$ at the stationary point $\tau = r/t$, $k_0 = 0$. In order to perform a Gaussian integration about this stationary point it is convenient to change to new variables $u$ and $\theta$ defined by

$$x = \frac{r}{t} + \frac{1}{(m_g t)^{1/2}} \frac{u e^{i\theta - i\pi/4}}{e^{-\pi/4}}$$

$$k = \frac{(m_g t)^{1/2}}{2} u e^{-i\theta - i\pi/4},$$

where $u$ is positive and $\theta$ is between 0 and $2\pi$. We are concerned with the limit $m_g t \rightarrow \infty$. As long as $u \ll (m_g t)^{1/2}$ then the variables $x$ and $k$ are only slightly extended into the complex plane. With this substitution the complex phase factor becomes a Gaussian:

$$e^{ik(r-xt)} = e^{-u^2}. $$

This is exact. The Jacobian of the transformation gives $dk dx = u du d\theta (2/t)$. The cut contribution becomes

$$*\mathcal{D}^{00}_\text{cut}(t, r) = \frac{-1}{2\pi^2 r t} \int_{0}^{u_{\text{max}}} u du \int_{0}^{2\pi} d\theta e^{-u^2 I_t(x, k)}.$$
B. Asymptotic space-like behavior of $^*D^{00}(x)$

The limit $t \to \infty$ and $r \to \infty$ with fixed ratio $t/r < 1$ is the deep space-like region. As is obvious from the analysis of the pole and cut contributions in Sec. A, when $r/t > 1$ there will be no points of stationary phase for real values of $k$ and $k_0$. Stationary phase points at complex values of $k$ or $k_0$ will produce terms that fall exponentially. The largest effects will instead be endpoint contributions that come from the region of small $k$ in the integral

$$^*D^{00}_< (t, r) = \frac{-i}{2\pi r} \int_0^\infty dk \ k \sin(kr) F_\ell(t, k).$$

The endpoint contributions are evaluated using the lemma proven in Appendix B which depends upon $F_\ell(t, k)$ being an even function of $k$ that is infinitely differentiable on the the real $k$ axis. Using repeated integration by parts it is possible to show that if $F_\ell(t, k)$ is finite at $k = 0$ then the integral falls exponentially as $r \to \infty$. If however $F_\ell(t, k) \to c_0/k^2$ with $c_0$ a constant as $k \to 0$ then the integral is of order $1/r$ with exponentially small corrections. Specifically

$$\lim_{r \to \infty} \int_0^\infty dk \ k \sin(kr) F_\ell(t, k) = \frac{c_0\pi}{2r} + \text{exp small}.$$

This will be used here and again in Sec. V B.

1. Pole contribution

The behavior of $F_\ell^{\text{pole}}(t, r)$ shown in Eq. (4.6) at small $k$ depends upon the behavior of the quasiparticle residue $Z_\ell$. In the low momentum limit $\omega_\ell \to m_g$ and $Z_\ell \to m_g/2k^2$ so that

$$k \to 0 : \quad F_\ell^{\text{pole}}(t, 0) = \frac{m_g}{2k^2} c(t),$$

where

$$c(t) = \frac{e^{-im_\ell t}}{1 - e^{-\beta m_g}} + \frac{e^{im_\ell t}}{e^{\beta m_g} - 1}.$$

Applying the lemma gives the asymptotic behavior

$$^*D^{00 \text{pole}}(t, r) \to \frac{-im_\ell}{8\pi r} c(t) + \text{exp small}.$$

2. Cut contribution

For the cut, one needs the small momentum behavior of Eq. (4.3): 

$$k \to 0 : \quad F_\ell^{\text{cut}}(t, k) \to T \int_{-k}^k dk_0 \frac{\beta_\ell(k_0, k)}{k_0} [1 + O(k_0^2)].$$

The value of this integral is given by the sum rule in Eq. (1.4). At small momentum, $Z_\ell/\omega_\ell \to 1/(2k^2)$, which makes the right hand side finite at $k = 0$. Using the low momentum behavior of the longitudinal quasiparticle energy, $\omega_\ell^2 \to m_g^2 + 3k^2/5 + \ldots$, in the residue $Z_\ell$ gives for the low momentum limit:

$$k \to 0 : \quad F_\ell^{\text{cut}}(t, k) \to \frac{4}{15} \frac{T}{m_g^2} + O(k^2).$$

Since this is finite, by the lemma displayed in Eq. (4.16) $^*D^{00 \text{cut}}(t, r)$ does not behave as $1/r$ in the deep space-like region but instead falls exponentially with $r$. The only part of $^*D^{00}(t, r)$ that does not fall exponentially is the pole contribution already given in Eq. (4.18). In Appendix C the same result is obtained using the Euclidean time formulation.

V. TRANSVERSE PROPAGATOR: $^*D^{ij}(x)$

The analysis of the asymptotic behavior of the transverse components of the HTL propagator parallels that of Sec. IV. In the Coulomb gauge the transverse components of the momentum-space Wightman function are

$$^*D^{ij}_\ell(k_0, k) = -i \rho_\ell(k_0, k) \frac{1 - e^{-\beta k_0}}{1 - e^{-\beta k_0}} (\delta^{ij} - \hat{k}^i \hat{k}^j).$$

Because the spectral integral for the momentum-space propagator in Eq. (3.2) has poles in $k_0$ at the quasiparticle energies $\pm \omega_i(k)$ and a branch cut from $k_0 = -k$ to $k_0 = k$, the transverse spectral function has the structure

$$\frac{1}{2\pi} \rho_\ell(k_0, k) = Z_\ell \left[ \delta(k_0 - \omega_i) - \delta(k_0 + \omega_i) \right] + \theta(k^2 - k_0^2) \beta_\ell(k_0, k).$$

The residue function is

$$Z_\ell = \frac{\omega_i(\omega_i^2 - k^2)}{3m_g^2 \omega_i^2 - (\omega_i^2 - k^2)^2}.$$ 

The explicit form of the spectral function for the cut is

$$\beta_\ell(k_0, k) = \frac{4}{3m_g^2} \frac{x(1 - x^2)}{N_\ell(x, k)},$$

where $x = k_0/k$ and the denominator is

$$N_\ell(x, k) = \left[ \frac{4k^2}{3m_g^2} (1 - x^2) + 2x^2 + x(1 - x^2) \ln \left( \frac{1 + x}{1 - x} \right)^2 \right] + \left[ \pi x (1 - x^2) \right]^2.$$ 

The spectral function for the cut satisfies various sum rules. The one that will be useful here is
\[ \int_{-k}^{k} dk_0 \beta_i(k_0, k) = \frac{1}{k^2} + \frac{2Z_i}{\omega_i}. \quad (5.4) \]

The Fourier transform of \( \ast D^i_j(k_0, k) \) will be written
\[ \ast D^i_j(t, \vec{r}) = -i \int \frac{d^3k}{(2\pi)^3} (\delta^{ij} - \hat{k}_i \hat{k}_j) e^{ik \cdot \vec{r}} F_i(t, k), \quad (5.5) \]
\[ F_i(t, k) = \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \frac{e^{-ik_0t}}{1 - e^{-\beta \omega}} \beta_i(k_0, k). \]

Here \( F_i \) denotes the sum of pole and cut contributions, \( F_i = F_i^{\text{pole}} + F_i^{\text{cut}}, \)
\[ F_i^{\text{pole}}(t, k) = Z_i \left[ \frac{e^{-i\omega t}}{1 - e^{-\beta \omega}} + \frac{e^{i\omega t}}{e^{\beta \omega} - 1} \right], \quad (5.6) \]
\[ F_i^{\text{cut}}(t, k) = \int_{-k}^{k} dk_0 \frac{e^{-ik_0t}}{1 - e^{-\beta \omega}} \beta_i(k_0, k). \quad (5.7) \]

As in the previous section, it will be important to know the behavior of these two functions at small momentum. As \( k \to 0 \) the quasiparticle energy \( \omega \to m_g \) and therefore the residue \( Z_i \to 1/2m_g \). This implies
\[ k \to 0: \quad F_i^{\text{pole}}(t, 0) = \frac{1}{2m_g} c(t), \quad (5.8) \]
with \( c(t) \) the same function as in Eq. (4.17). For the cut contribution, the sum rule in Eq. (5.4) and the fact that \( Z_i \to 1/2m_g \) implies that
\[ k \to 0: \quad F_i^{\text{cut}}(t, k) \to \frac{T}{k^2} - \frac{T}{m_g^2} + O(k^2). \quad (5.9) \]

As \( k \to 0 \) the right hand side diverges as \( 1/k^2 \), which will make the subsequent analysis more difficult.

Angular integrations and tensor structure: It is convenient to first deal with the tensor structure by focusing on the angular integrations. The most general rotationally covariant tensor is
\[ \ast D^i_j(t, \vec{r}) = \left( -\delta^{ij} + 3\hat{r}_i \hat{r}_j \right) G(t, r) \text{ nonumber} \quad (5.10) \]
\[ + (\delta^{ij} - \hat{r}_i \hat{r}_j) H(t, r). \quad (5.11) \]

By contracting Eq. (5.3) with \( \hat{r}_i \hat{r}_j \) the angular integration over the solid angle of \( k \) can be performed with the result
\[ G(t, r) = \frac{1}{2} \hat{r}_i \hat{r}_j \ast D^i_j(t, \vec{r}) \]
\[ = \frac{i}{2\pi^2 r} \int_0^{\infty} dk \frac{\partial}{\partial r} \left( \sin kr \right) F_i(t, k). \quad (5.12) \]

The contraction of Eq. (5.5) with \( \delta_{ij} \) leads to
\[ H(t, r) = \frac{1}{2} \delta_{ij} \ast D^i_j(t, \vec{r}) \]
\[ = -\frac{i}{2\pi^2 r} \int_0^{\infty} dk k \sin(kr) F_i(t, k). \quad (5.13) \]

The integrands of Eqs. (5.12) and (5.13) both behave at small \( k \) like \( k^2 F_i(t, k) \). The previous observations imply that as \( k \to 0, k^2 F_i^{\text{pole}}(t, k) \to 0 \) and \( k^2 F_i^{\text{cut}}(t, k) \to T \). Consequently the integrations for \( G \) and \( H \) are convergent at small \( k \). By using the relation
\[ \left[ \frac{\partial^2}{\partial r^2} + \frac{2\partial}{r \partial r} + k^2 \right] \left( \frac{\sin kr}{kr} \right) = 0, \]
it is easy to show that the two coefficient functions are related by
\[ \frac{\partial}{\partial r} \left( r^2 G(t, r) \right) = r^2 H(t, r). \quad (5.14) \]

It will often be easier to compute the integral for \( H(t, r) \) in Eq. (5.13) and then use this relation to compute \( G(t, r) \). Of course it will not determine any part of \( G \) that behaves like \( 1/r^3 \).

The asymptotic behaviors of \( G \) and \( H \) will be computed in this section, but the results will not be substituted into Eq. (5.1) for examination. That examination and discussion will be deferred to Sec. VI and the impatient reader interested in the answers rather than the computations is encouraged to turn directly to Sec. VI.

### A. Asymptotic time-like behavior of \( \ast D^i_j(x) \)

The deep time-like region denotes the limit in which \( t \to \infty, r \to \infty \) at a fixed ratio satisfying \( t/r > 1 \).

1. Pole contribution

The asymptotic behavior of the pole contribution to the transverse propagator is controlled by a point of stationary phase. From Eqs. (6.6) and (6.13) the contribution of the quasiparticle pole is
\[ H^{\text{pole}}(t, r) = \frac{-i}{2\pi^2 r} \int_0^{\infty} dk k \sin kr \times Z_i \left[ \frac{e^{-i\omega t}}{1 - e^{-\beta \omega}} + \frac{e^{i\omega t}}{e^{\beta \omega} - 1} \right]. \]

One can extend the integration to negative \( k \) so that
\[ H^{\text{pole}}(t, r) = \frac{-1}{4\pi^2 r} \int_{-\infty}^{\infty} dk k Z_i \left[ \frac{e^{i kr - i\omega t}}{1 - e^{-\beta \omega}} - \frac{e^{-i kr + i\omega t}}{e^{\beta \omega} - 1} \right]. \]

As before, the method of stationary phase [14,15] will be used to compute the asymptotic value of this integral. Define
\[ \phi(k) = kr - \omega t(k) \]
At large values of \( r \) and \( t \) the functions \( e^{\pm ik} \) oscillate rapidly except at special values of \( k \). Let \( \vec{k} \) be the value of the wave vector at which the phase is stationary. This
requires that the group velocity of the transverse quasiparticle satisfy

$$\left[ \frac{d \omega_z}{dk} \right]_r = \frac{r}{t} < 1.$$  \hspace{1cm} (5.16)

The group velocity of the transverse dispersion relation is zero at \( k = 0 \) and grows monotonically with \( k \), approaching 1 as \( k \to \infty \). Therefore there is always a real \( k \) which satisfies the stationary phase condition. Near the stationary point, the phase is

$$\phi(k) = \phi(k) - \frac{1}{2} (k - \bar{k})^2 t \left[ \frac{d^2 \omega_z}{dk^2} \right]_r + \ldots.$$  

The acceleration \( d^2 \omega_z / dk^2 \) is always positive. The \( k \) contour must be deformed into the complex plane so that it passes through the real value of \( \bar{k} \) on a path of steepest descent. For the first integrand, \( \exp (+i\phi) \), this requires \( k = \bar{k} + e^{-i\pi/4} s \) where \( s \) is real. For the second integrand, \( \exp (-i\phi) \), the correct path is \( k = \bar{k} + e^{i\pi/4} s \). Performing the two Gaussian integrations gives the asymptotic behavior:

$$H_{\text{pole}}(t, r) \to \frac{e^{-i\bar{k} r}}{r} \left[ \frac{e^{-i\bar{k} r}}{e^{-i\bar{k} r}} - \frac{e^{-i\bar{k} r}}{e^{i\bar{k} r}} \right]$$  

$$\times \frac{1}{(2\pi)^{3/2}} \left[ \frac{e^{i\phi-i\pi/4}}{1-e^{-i\phi-i/4}} - \frac{e^{-i\phi+i\pi/4}}{e^{i\phi+i/4}} \right]$$

Since the ratio \( r/t \) is fixed, this falls like \( t^{-3/2} \).

The tensor structure of the transverse propagator in Eq. (5.11) depends on both \( H \) and \( G \). Using Eq. (5.14) gives the asymptotic behavior of \( G_{\text{pole}} \):

$$G_{\text{pole}}(t, r) \to \frac{i}{r^2} \left[ \frac{e^{i\phi-i\pi/4}}{e^{-i\phi-i/4}} - \frac{e^{-i\phi+i\pi/4}}{e^{i\phi+i/4}} \right]$$

$$\times \frac{1}{(2\pi)^{3/2}} \left[ \frac{e^{i\phi-i\pi/4}}{1-e^{-i\phi-i/4}} - \frac{e^{-i\phi+i\pi/4}}{e^{i\phi+i/4}} \right]$$

To confirm that this is the dominant term in \( G_{\text{pole}} \) it is necessary to recognize that the most rapid dependence on \( r \) comes from the phase, \( \bar{\phi} = \bar{k} r - \omega_z(\bar{k}) t \). The \( r \) derivative of \( G_{\text{pole}} \) comes from three sources: dependence of the phase \( \bar{\phi} \) on \( r \), from the \( 1/r^2 \) power dependence, and from the implicit dependence of the stationary point \( \bar{k} \) on \( r \):

$$\frac{\partial G_{\text{pole}}}{\partial r} = \frac{d \bar{\phi}}{dr} \frac{\partial G_{\text{pole}}}{\partial \bar{\phi}} - 2 \frac{d G_{\text{pole}}}{dr} + \frac{d \bar{k}}{dr} \frac{\partial G_{\text{pole}}}{\partial \bar{k}}.$$  

Since \( d \bar{\phi}/dr = \bar{k} \) the first term is larger than the second at large \( r \). The first term is also larger than the third term because dependence of \( \bar{k} \) on \( r \) is very slow: \( dk/dr = 1/(\bar{k}^2) \), which is of order \( 1/r \). This confirms Eq. (5.18).

2. Cut contribution

To analyze the cut contribution of the transverse propagator in the deep time-like region it is convenient to begin with the double integral representation

$$H_{\text{cut}}(t, r) = \frac{-i}{r} \int_{-\infty}^{\infty} \frac{dk}{(2\pi)^2} k \sin kr$$

$$\times \int_{-\infty}^{\infty} dk_0 e^{-ik_0 t} \beta(t, k_0, k),$$

which follows from Eqs. (5.7) and (5.13). As previously noted, the integral over \( k_0 \) approaches \( T/k^2 \) at small \( k \) and therefore the integration is convergent for small \( k \). Changing variables from \( k_0 \) to the phase variable \( x = k_0/k \) and replacing \( i \sin kr \) with \( \exp(ikr) \) gives

$$H_{\text{cut}}(t, r) = \frac{-1}{r(2\pi)^2} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dx e^{ik(r-x)} I_t(x, k)$$

$$I_t(x, k) = k^2 \beta_t(x, k) \frac{1}{1 - e^{-\beta_k x}}.$$  \hspace{1cm} (5.19)

This double integral is quite similar to the integral which arose for the longitudinal propagator in Eq. (4.11). The stationary phase point occurs at \( \bar{r} = r/t, \bar{k} = 0 \). The contribution of this stationary phase point will be of order \( 1/r^3 \), which was the final result for the longitudinal propagator in Eq. (4.14).

It is rather surprising that the stationary phase point does not give the largest term in the asymptotic expansion. The subtlety comes ultimately from the fact that there is no magnetic screening. There is a hand-waving way to get the correct leading behavior. Reference to Eq. (5.9) suggests making the split

$$F_t = \frac{T}{k^2} + \delta F_t \text{cut}(t, k).$$  

This gives an elementary integration for the leading term:

$$H_{\text{cut}}(t, r) = \frac{-i}{2\pi r} \int_{0}^{\infty} dk \sin(kr) \left[ \frac{T}{k^2} + \delta F_t \text{cut}(t, k) \right]$$

$$= \frac{i}{4 \pi r} - \frac{i}{4 \pi r} \int_{0}^{\infty} dk \sin(kr) \delta F_t \text{cut}(t, k).$$

It will turn out that \( -iT/4\pi r \) actually is the correct leading behavior. However this simple calculation does not really prove that because one cannot show that the contribution of the integral over \( \delta F_t \text{cut} \) is smaller than \( T/r \). The problem is that \( \delta F_t \text{cut} \) is too awkward to deduce the asymptotic behavior.

A more convincing analysis is to check qualitatively that there is a region of integration whose contribution is larger than the point of stationary phase. Thus we examine the integrand \( I_t \) of the double integral Eq. (5.19) in the region where \( x \ll 1 \) and \( |k| \ll m_g^2 \):

$$I_t(x, k) \bigg|_{\text{small } x, \text{small } k} \approx \frac{4T}{3m_g^2} \frac{k}{\left[ \frac{4k^2}{3m_g^2} \right]^{1/2} + |\pi x|^2},$$

in the region where \( x \ll 1 \) and \( |k| \ll m_g^2 \):
This part of the integrand is largest when $k^2/m_g^2$ is similar in magnitude to $x$. The large value of $I_1$ can be suppressed by oscillations in the phase factor $\exp[ik(r-xt)]$. However, at very small $k \ll 1/r$ these oscillations are negligible. Thus the dominant region is $k \sim m_g \sqrt{x} \ll 1/r$. The approximate value of the integrand here is very large: $I_1(x, k) \sim T m_g r^{-3}$ The effect of this small region on $H^\text{cut}$ can be crudely estimated as

$$H^\text{cut}(t, r) \sim \frac{1}{r} \int_0^{r^{-1}} dk \int_0^{(m_g r)^{-2}} dx \left[T m_g r^3\right] \sim \frac{T}{r}. \quad (5.20)$$

At sufficiently large $r$ this dominates the stationary phase contribution, $T/(m_g r^3)$. Note also that the $T/r$ contribution is independent of $m_g$ and thus independent of the coupling.

To perform a precise calculation of the leading and sub-leading behavior of $H^\text{cut}$, the essential step is to separate out the dominant contribution to the integrand by defining

$$I^{(1)}(x, k) = \frac{4T}{3m_g^2} \left[4k^2/m_g^2\right] + [\pi x] \quad (5.21)$$

$$I^{(2)}(x, k) = I_1(x, k) - I^{(1)}(x, k). \quad (5.22)$$

Obviously $I^{(1)}$ contains the most important behavior. With this split, $H^\text{cut} = H^{(1)} + H^{(2)}$ where for $n = 1$ or $2$ the integrals are given by

$$H^{(n)}(t, r) = \frac{-1}{r(2\pi)^2} \int_{-\infty}^{\infty} dk \int_{-1}^{1} dx \ e^{ik(r-xt)} I^{(n)}(x, k). \quad (5.23)$$

(1) The dominant contribution will come from $H^{(1)}$. This is computed by direct integration in Appendix D. That calculation gives the asymptotic behavior

$$H^{(1)}(t, r) \rightarrow -\frac{iT}{4\pi r} - \frac{4iT}{3\pi^2 m_g^2 r^3 \pi^2}, \quad (5.24)$$

with power-law corrections. Using this in Eq. (5.14) gives the asymptotic behavior of $G^{(1)}$ as

$$G^{(1)}(t, r) \rightarrow -\frac{iT}{8\pi r} + \frac{4iT}{3\pi^3 m_g^3 r^5 \pi^2}. \quad (5.25)$$

(2) The behavior of $H^{(2)}$ in the deep time-like region can be obtained from the same type of stationary-phase method used for $P^{(0)}_{\text{cut}, t}(r, r)$ in Sec. IV A. The change of variables in Eq. (4.12) gives

$$H^{(2)}(t, r) = \frac{-2}{rt(2\pi)^2} \int_0^{u_{\text{max}}} du \int_0^{2\pi} d\theta \ e^{-u^2} f^{(2)}(x, k). \quad (5.26)$$

At the stationary point, $\pi = r/t$ and $\bar{k} = 0$, the integrand $f^{(2)}(\pi, 0)$ vanishes. The leading term is obtained using Eq. (4.13):

$$H^{(2)}(t, r) \rightarrow -\frac{2}{rt(2\pi)^2} \left(-\frac{i\pi}{t}\right) \left[\partial^2 f^{(2)} / \partial x \partial k\right]_{\pi, 0} \quad (5.27)$$

Working this out explicitly gives

$$H^{(2)}(t, r) \rightarrow -\frac{i2T}{rt^2 3\pi m_g^2} \frac{d}{dx} \left[1 - \frac{x^2}{N_t(\pi, 0)} - \frac{1}{\pi^2 x^2}\right]. \quad (5.28)$$

Because of the subtracted term this is finite even at $\pi = 0$. From Eq. (5.14) the other coefficient $G^{(2)}$ must have the asymptotic behavior

$$G^{(2)}(t, r) \rightarrow \frac{2iT}{3\pi m_g^3 r^5} f(\pi). \quad (5.29)$$

where $f(\pi)$ satisfies

$$df(\pi) / d\pi = \frac{\pi}{N_t(\pi, 0)} - \frac{1}{\pi^2 x^2}. \quad (5.30)$$

Integrating this is not automatic because one does not have an initial condition. As $\pi \rightarrow 0$ the right hand side is of order $\pi^2$. Thus $f(\pi) \rightarrow f(0) + \mathcal{O}(\pi)$ as $\pi \rightarrow 0$. It is not obvious that $f(0)$ should vanish. (This would make $G^{(2)}$ finite at $r = 0$, but there is no reason to demand this since $G^{(1)}$ is not finite at $r = 0$.) Appendix D performs a stationary phase calculation of $G^{(2)}$ with the result

$$f(\pi) = \int_0^{\pi} dy / \left[N_t(y, 0)\right] - \frac{1}{\pi^2 y^2} \quad (5.31)$$

In the calculation of Appendix D the integration variable over $y$ is a remnant of the angular integration. Specifically, $y = \pi \cos \theta$ where $\bar{k} \cdot \bar{r} = \cos \theta$. Since the quantity in square brackets is finite at $y = 0$, one can integrate by parts to obtain

$$f(\pi) = \frac{\pi(1 - \pi^2)}{N_t(\pi, 0)} - \frac{1}{\pi^2 \pi} - \int_0^{\pi} dy / \left[N_t(\pi, 0)\right] - \frac{1}{\pi^2 y^2}. \quad (5.32)$$

In this form the entire integral of the last term $1/(\pi y)^2$ appears important. Because the expression in square brackets is finite at $y = 0$, one can replace the lower limit of zero by an infinitesimal value $y_0$ and take the limit $y_0 \rightarrow 0$ after performing the integration. Doing this gives

$$f(\pi) = \frac{\pi(1 - \pi^2)}{N_t(\pi, 0)} - \frac{2}{\pi \pi} \lim_{y_0 \rightarrow 0} \left[\int_0^{\pi} dy / \left[N_t(\pi, 0) - \frac{1}{\pi^2 y_0}\right]\right]. \quad (5.33)$$

In this form the divergence from the lower limit of the integration is canceled by the term $-1/(\pi y_0)^2$.

**Total cut contribution.** In the last form for $f(\pi)$ the term $-2/(\pi \pi)$ when substituted into Eq. (5.27) exactly cancels the order $1/\pi r^3$ term in Eq. (5.24). Consequently the sum of $G^{(1)}$ and $G^{(2)}$ is

$$G^\text{cut}(t, r) \rightarrow -\frac{iT}{8\pi r} + \frac{2iT}{3\pi m_g^3 r^5} g(\pi) \quad (5.34)$$

$$g(\pi) = \frac{\pi(1 - \pi^2)}{N_t(\pi, 0)} - \lim_{y_0 \rightarrow 0} \left[\int_0^{\pi} dy / \left[N_t(\pi, 0) - \frac{1}{\pi^2 y_0}\right]\right]. \quad (5.35)$$
The surviving terms of order $1/r^3$ come entirely from the stationary phase point. Similarly when $H^{(1)}$ and $H^{(2)}$ are added together the order $1/r^3$ term from Eq. (5.23) is canceled by the last term in Eq. (5.25) to give

$$H^{\text{cut}}(t, r) \to -\frac{it}{4\pi} + \frac{it^2}{2\pi^2m_g^2} \frac{\partial}{\partial x} \left[ 1 - \frac{\pi^2}{N_1(x, 0)} \right].$$  \hspace{1cm} (5.28)

The corrections to $1/r$ come from the stationary phase point.

Inclusion of a magnetic mass. It is widely believed that non-abelian gauge theories at high temperature generate a magnetic mass of order $g^2 T$ defined by the limit

$$\mu^2 = \lim_{k \to 0} \Pi_t(0, k).$$

It is not known how a magnetic mass would change the transverse spectral function at nonzero values of $k_0$ and $k$. The simplest possibility is that the occurrence of $k^2$ in the transverse spectral function is replaced by the sum $k^2 + \mu^2$. Thus the denominator of the new spectral function, denoted with a tilde, is

$$\tilde{N}_t(x, k) = \left[ \frac{4(k^2 + \mu^2)}{3m_g^2} - x^2 \right] + \left[ \pi x(1 - x^2) \right]^2.$$  \hspace{1cm} (5.29)

The function $H^{\text{cut}}$ becomes

$$\tilde{H}^{\text{cut}}(t, r) = \frac{-1}{r^2(2\pi)^2} \int_{-\infty}^{\infty} dk \int_{-1}^{1} dx \ e^{ik(r-xt)} \tilde{I}_t(x, k),$$

where the magnetic mass now occurs in the denominator of the spectral function:

$$\tilde{I}_t(x, k) = \frac{4k^2x(1-x^2)}{3m_g^2 \left[ 1 - e^{-3\pi kr} \right] N_1(x, k)}.$$  \hspace{1cm} (5.30)

Because of the magnetic mass $\tilde{I}_t(x, k)$ does not diverge at $x \sim \sqrt{k/m_g} \to 0$.

The asymptotic behavior of $\tilde{H}^{\text{cut}}$ in the deep time-like region can be obtained from the stationary-phase method used in Sec. III A. The change of variables in Eq. (4.12) gives

$$\tilde{H}^{\text{cut}}(t, r) = \frac{-2}{rt^2(2\pi)^2} \int_{0}^{u_{\text{max}}} udu \int_{0}^{2\pi} d\theta \ e^{-u^2} \tilde{I}_t(x, k).$$

At the stationary-phase point, $\tilde{I}_t(\tau, 0)$ vanishes and so according to Eq. (4.13) the leading term is

$$\tilde{H}^{\text{cut}}(t, r) \to \frac{-2}{rt^2(2\pi)^2} \left[ -\pi \right] \frac{\partial^2 \tilde{I}_t}{\partial x \partial k} \tau, 0.$$  \hspace{1cm} (5.31)

Working this out explicitly gives

$$\tilde{H}^{\text{cut}}(t, r) \to \frac{i2T}{rt^23\pi m_g^2} \frac{\partial}{\partial x} \left[ 1 - \pi x^2 \right].$$  \hspace{1cm} (5.32)

The asymptotic behavior of $\tilde{G}^{\text{cut}}$ is

$$\tilde{G}^{\text{cut}}(t, r) \to \frac{2iT}{3\pi m_g^2 r^2} \left[ \pi(1-x^2) - \int_0^\pi dy \frac{1-y^2}{N_1(y, 0)} \right].$$  \hspace{1cm} (5.33)

Thus in the presence of a magnetic mass the transverse propagator falls like $1/t^3$ in the deep time-like region.

B. Asymptotic space-like behavior of $^\ast D^{(3)}(x)$

The limit $t \to \infty$, $r \to \infty$ with a fixed ratio $t/r < 1$ is the deep space-like region. As observed in Sec. IV B, there are no points of stationary phase for real values of $k$ and $k_0$. Stationary phase points at complex values of $k$ or $k_0$ will produce terms that fall exponentially. The largest effects are instead endpoint contributions that come from the region of small $k$ in the integral of Eq. (5.12):

$$G(t, r) = \frac{i}{2\pi^2r} \int_0^\infty dk \frac{\partial}{\partial r} \sin kr F_t(t, k).$$  \hspace{1cm} (5.34)

1. Pole contribution

Eq. (5.8) shows that at zero momentum, $F_t^{\text{pole}}(t, 0) = c(t)/2m_g$. The $r$ derivative may be taken outside the integral defining $G^{\text{pole}}$ so that

$$G^{\text{pole}}(t, r) = \frac{i}{2\pi^2 r} \frac{\partial}{\partial r} \left[ \frac{1}{r} \int_0^\infty dk \frac{\sin kr}{k} F_t(t, k) \right].$$

The integral over $k$ now satisfies the conditions of the lemma displayed in Eq. (4.16) and proven in Appendix D. Therefore in the deep space-like limit

$$G^{\text{pole}}(t, r) \to \frac{i}{2\pi^2 r} \frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\pi c(t)}{2m_g} \right] = -\frac{ic(t)}{4\pi m_g r^3}.$$  \hspace{1cm} (5.35)

The corrections to this are exponentially small. Since this is a pure $1/r^3$ behavior, by Eq. (5.14) the asymptotic behavior of the other coefficient function is

$$H^{\text{pole}}(t, r) \to \exp \text{small}.$$  \hspace{1cm} (5.36)

2. Cut contribution

For the contribution of the cut, the result in Eq. (5.9) motivates separating out the dominant behavior by defining

$$F_t^{\text{cut}}(t, r) = \frac{T}{k^2} + \delta F_t^{\text{cut}}(t, k),$$

where

$$\delta F_t^{\text{cut}}(t, k) = \frac{r}{rt^23\pi m_g^2} \frac{\partial}{\partial x} \left[ 1 - \pi x^2 \right].$$  \hspace{1cm} (5.37)

The limit $t \to \infty$, $r \to \infty$ with a fixed ratio $t/r < 1$ is the deep space-like region. As observed in Sec. IV B, there are no points of stationary phase for real values of $k$ and $k_0$. Stationary phase points at complex values of $k$ or $k_0$ will produce terms that fall exponentially. The largest effects are instead endpoint contributions that come from the region of small $k$ in the integral of Eq. (5.12):

$$G(t, r) = \frac{i}{2\pi^2r} \int_0^\infty dk \frac{\partial}{\partial r} \sin kr F_t(t, k).$$  \hspace{1cm} (5.38)

The integral over $k$ now satisfies the conditions of the lemma displayed in Eq. (4.16) and proven in Appendix D. Therefore in the deep space-like limit

$$G^{\text{pole}}(t, r) = \frac{i}{2\pi^2 r} \frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\pi c(t)}{2m_g} \right] = -\frac{ic(t)}{4\pi m_g r^3}.$$  \hspace{1cm} (5.39)

The corrections to this are exponentially small. Since this is a pure $1/r^3$ behavior, by Eq. (5.14) the asymptotic behavior of the other coefficient function is

$$H^{\text{pole}}(t, r) \to \exp \text{small}.$$  \hspace{1cm} (5.40)

For the contribution of the cut, the result in Eq. (5.9) motivates separating out the dominant behavior by defining

$$F_t^{\text{cut}}(t, r) = \frac{T}{k^2} + \delta F_t^{\text{cut}}(t, k),$$

where

$$\delta F_t^{\text{cut}}(t, k) = \frac{r}{rt^23\pi m_g^2} \frac{\partial}{\partial x} \left[ 1 - \pi x^2 \right].$$  \hspace{1cm} (5.41)
where \( \delta F_i^{\text{cut}}(t,k) \) is finite at \( k = 0 \). Using this in Eq. (5.32) gives
\[
G^{\text{cut}}(t,r) = \frac{i}{2\pi^2 r} \int_0^\infty dr \frac{\partial}{\partial r} \left( \frac{\sin kr}{kr} \right) \frac{T}{k^2}
+ \frac{i}{2\pi^2 r} \int_0^\infty dk \frac{\sin kr}{kr} \delta F_i^{\text{cut}}(t,k).
\]

Both these integrals are convergent at \( k = 0 \) and at \( k = \infty \). The first integral may be done exactly. For the second, the asymptotic behavior is given by the lemma, Eq. (4.10). The result is
\[
G^{\text{cut}}(t,r) \rightarrow \frac{-iT}{8\pi r} + \frac{iT}{4\pi m_g^2 r^3} + \exp \text{ small.} \tag{5.35}
\]

By Eq. (5.14) the other coefficient function has asymptotic behavior
\[
H^{\text{cut}}(t,r) \rightarrow \frac{-iT}{4\pi r} + \exp \text{ small.} \tag{5.36}
\]

**Inclusion of a magnetic mass.** The existence of a magnetic mass \( \mu \) would change the asymptotic space-like behavior of the transverse propagator. The contribution of the quasiparticle pole does not change appreciably. However, for the contribution of the branch cut the sum-rule in Eq. (5.32) changes to
\[
\int_{-\infty}^{\infty} dk \frac{\beta_i(k_0,k)}{k_0} = \frac{1}{k^2 + \Pi_\perp(0,k)} - \frac{2Z_i}{\omega_i},
\]
as argued by Pisarski \[3\]. In the limit of small \( k \) this gives
\[
 k \rightarrow 0 : \int_{-\infty}^{\infty} dk_0 \frac{\beta_i(k_0,k)}{k_0} \rightarrow \frac{1}{\mu^2} - \frac{1}{m_g^2} + \mathcal{O}(k^2).
\]

Therefore in Eq. (5.35), the term of order \( 1/r \) is absent and the coefficient of the \( 1/r^3 \) changes so that
\[
\tilde{G}^{\text{cut}}(t,r) \rightarrow \frac{iT}{4\pi r^3} \left[ \frac{1}{m_g^2} - \frac{1}{\mu^2} \right] + \exp \text{ small.} \tag{5.37}
\]

By Eq. (5.14)
\[
\tilde{H}^{\text{cut}}(t,r) \rightarrow \exp \text{ small.} \tag{5.38}
\]

Among the exponentially small terms is the familiar \( \exp(-\mu r) \). However, the dominant behavior is the \( 1/r^3 \) term derived here. In Appendix \[3\] the same results are obtained using the imaginary time approach.

**VI. SUMMARY AND DISCUSSION**

This section reviews the results obtained in the previous calculations and compares them with free propagators at finite temperature. The summary is self-contained and may be read independently of the previous sections. It is organized differently than the text. First comes the asymptotic time-like behavior of thermal propagators; then the asymptotic space-like behaviors. The propagators are always in the Coulomb gauge.

It is perhaps worth repeating that ghost propagators do not contain hard-thermal-loops and therefore are not resummed \[3\]. In the HTL approximation the ghost propagators are just the free thermal propagators. Their asymptotic behavior may be found in \[13\].

**A. Asymptotic time-like behavior of gauge boson propagators**

In the deep time-like region \( t > r \) and \( r \) are both large with a fixed ratio \( t/r > 1 \). For the free propagator \( t/r \) must be large compared to \( 1/2\pi T \). For the HTL propagator, they need to be large compared to \( 1/m_g \). The leading term in the asymptotic time-like behavior is displayed in the first column of Table I.

1. **Free, massless gauge boson.** In Coulomb gauge the scalar potential \( A^0(x) \) is instantaneous and consequently \( \mathcal{D}^{00}(t,r) = \delta(t)/4\pi r \). The exact space-time dependence of \( \mathcal{D}^{ij}(t,\vec{r}) \) can be found in \[13\]. In the deep time-like region \( t > r \gg 1/(2\pi T) \) the results are
\[
\mathcal{D}^{00}(t,r) = 0 \tag{6.1a}
\]
\[
\mathcal{D}^{ij}(t,\vec{r}) \rightarrow \exp \text{ small..} \tag{6.1b}
\]

Exponential fall-off in the time-like direction is characteristic of free massless bosons and was found in the Feynman gauge example in Sec. II. The same behavior occurs for spin 0, for spin 1 in any gauge, and for spin 2 gravitons \[13\]. The exponential fall-off can be understood in terms of the calculations performed in Secs. IV A and V A. The power-law behavior that occurs in the HTL propagators comes from a point of stationary phase. The stationary phase contribution always has exponentially small corrections, which have been consistently omitted. However, for free, massless bosons since \( k_0 = k \), there can be no value of \( k \) for which the phase factor \( \exp(ik(r-t)) \) is stationary. The entire answer consists of the exponentially small corrections.

**TABLE I. Leading asymptotic behavior of gauge boson propagators at finite temperature in the Coulomb gauge.**

| Type          | Time-like | Space-like |
|---------------|-----------|------------|
| Free massless | exp. small| 1/r        |
| HTL \( \mathcal{D}^{00} \) pole | \( 1/r^{3/2} \) | 1/r       |
| HTL \( \mathcal{D}^{00} \) cut | \( 1/r^3 \) | exp small |
| HTL \( \mathcal{D}^{ij} \) pole | \( 1/r^{3/2} \) | \( 1/r^3 \) |
| HTL \( \mathcal{D}^{ij} \) cut | \( 1/r^a \) | \( 1/r^b \) |

\( a \)This changes to \( 1/r^3 \) if there is a magnetic mass.
\( b \)This changes to \( 1/r^3 \) if there is a magnetic mass.
2. HTL longitudinal propagator. From Sec. IV A the asymptotic time-like behavior of the longitudinal gauge propagator for $t \gg 1/m_g$ is

$$D^{00}_{\text{pole}}(t, r) \rightarrow \frac{a^0_{\text{pole}}(t, r)}{t^{3/2}}, \quad \text{(6.2a)}$$

$$D^{00}_{\text{cut}}(t, r) \rightarrow -i \frac{a^0_{\text{cut}}(\overline{\tau})}{t^3}, \quad \text{(6.2b)}$$

where the coefficient functions are

$$a^0_{\text{pole}}(t, r) = -\frac{\mathcal{E}}{\mathcal{F}} \frac{Z_t}{\sqrt{(2\pi)^3} \mathcal{E}_t} \left[ \frac{e^{i\overline{\phi} - i\pi/4}}{1 - e^{-\overline{\phi}} - e^{i\overline{\phi} + i\pi/4}} \right],$$

$$a^0_{\text{cut}}(\overline{\tau}) = -\frac{T}{3\pi \frac{m^2}{\mathcal{F}} \mathcal{E}_t} \left[ \frac{1}{N_t(\overline{\tau}, 0)} \right].$$

The function $a^0_{\text{cut}}$ depends only on the variable $\overline{\tau} = r/t$. The function $a^0_{\text{pole}}$ also depends on the ratio $r/t$, since the stationary point is the value momentum $\overline{\mathcal{F}}$ at which the group velocity of the quasiparticle satisfies $d\omega_e/dk = r/t$. In addition $a^0_{\text{pole}}$ depends on $t$ and $r$ through the value of the phase at the stationary point: $\overline{\phi} = \overline{\mathcal{F}} - \omega_i(\overline{k})t$.

3. HTL transverse propagator. From Sec. V A the transverse gauge propagator in the deep time-like region has the following asymptotic behavior:

$$D^{ij}_{\text{pole}}(t, r) \rightarrow \frac{b^i_{\text{pole}}(t, r)}{t^{3/2}} (-\delta^{ij} + 3\delta^i r^j)$$

$$+ \frac{b^i_{\text{pole}}(t, r)}{t^{3/2}} (\delta^{ij} - \delta^i r^j), \quad \text{(6.3a)}$$

$$D^{ij}_{\text{cut}}(t, r) \rightarrow -i \frac{T}{8\pi r} (\delta^{ij} + \hat{r}^i \hat{r}^j)$$

$$- i \frac{b^i_{\text{cut}}(\overline{\tau})}{t^3} (\delta^{ij} - \delta^i r^j)$$

$$- i \frac{b^i_{\text{cut}}(\overline{\tau})}{t^3} (\delta^{ij} - \delta^i r^j). \quad \text{(6.3b)}$$

for times $t \gg 1/m_g$. The hard-thermal-loop propagators do not contain a magnetic mass. Therefore Eqs. (5.30) and (5.31) apply to high temperature QED. For high temperature QCD the inclusion of a magnetic mass changes the asymptotic behavior as discussed in the next paragraph. The coefficients for the pole contribution are

$$b^i_{\text{pole}}(t, r) = -\frac{\mathcal{E}}{\mathcal{F}} \frac{Z_t}{\sqrt{(2\pi)^3} \mathcal{E}_t} \left[ \frac{e^{i\overline{\phi} - i\pi/4}}{1 - e^{-\overline{\phi}} - e^{i\overline{\phi} + i\pi/4}} \right],$$

$$b^i_{\text{pole}}(t, r) = -\frac{\mathcal{E}}{\mathcal{F}} \frac{Z_t}{\sqrt{(2\pi)^3} \mathcal{E}_t} \left[ \frac{e^{-i\overline{\phi} - i\pi/4}}{1 - e^{-\overline{\phi}} - e^{i\overline{\phi} + i\pi/4}} \right].$$

These separately on $t$ and $r$ through the phase $\overline{\phi}$. The coefficients for the cut contributions depend only on the ratio $\overline{\tau} = r/t$:

$$b^i_{\text{cut}}(\overline{\tau}) = \frac{2T}{3\pi \frac{m^2}{\mathcal{F}} \mathcal{E}_t} \left\{ \lim_{\gamma \rightarrow 0} \left[ \int_{y_0 - \gamma}^{\overline{\mathcal{F}}} dy N_t(y, 0) - \frac{1}{\pi^2 y_0} \right] \right\}.$$

$$b^i_{\text{cut}}(\overline{\tau}) = -\frac{2T}{3\pi \frac{m^2}{\mathcal{F}} \mathcal{E}_t} \left\{ \frac{1}{\overline{\mathcal{F}}} \right\}.$$

In Sec. V A the calculation of the leading term, $T/r$, in $D^{ij}_{\text{cut}}(t, r)$ is lengthy. However, it has a profound implication for the behavior of electrons. As shown by Pisaschi the electron propagator dressed with one HTL photon has a damping rate that diverges on the electron mass shell [13]. In space-time this comes from the $T/r$. Blaizot and Iancu showed that summing the most infrared-sensitive contributions to the electron propagator produces a time dependence of the form $\exp[-\alpha T t \ln(t)]$, rather than exponential [15]. Boyanovsky, de Vega, et al found the same behavior in scalar QED [16].

4. HTL transverse propagator with magnetic mass. For QCD the HTL propagators must be modified so as to include a magnetic mass $\mu^2$. This does not change the asymptotic behavior of the pole contribution. As shown in Sec. V A it does change the cut contribution by eliminating the term of order $T/r$ and modifying the coefficients of the $1/r^3$ terms. Using Eqs. (5.30) and (5.31) gives

$$D^{ij}_{\text{cut}}(t, r) \rightarrow -i \frac{\hat{b}^i_{\text{cut}}(\overline{\tau})}{t^3} (\delta^{ij} + 3\delta^i \hat{r}^j)$$

$$- i \frac{\hat{b}^i_{\text{cut}}(\overline{\tau})}{t^3} (\delta^{ij} - \delta^i \hat{r}^j). \quad \text{(6.4)}$$

The coefficients depend on the magnetic mass through the denominator of the transverse spectral function $N_t$ given in Eq. (5.24):

$$\hat{b}^i_{\text{cut}}(\overline{\tau}) = \frac{2T}{3\pi \frac{m^2}{\mathcal{F}} \mathcal{E}_t} \left\{ \int_{0}^{\overline{\tau}} dy N_t(y, 0) - \frac{1 - \tau^2}{N_t(\overline{\tau}, 0)} \right\}.$$

The more rapid fall-off in Eq. (6.4) than for QED implies that the quark damping rates are infrared finite [15].

B. Asymptotic space-like behavior of gauge boson propagators

The deep space-like region requires $t$ and $r$ both large with a fixed ratio of $t/r$ satisfying $t/r < 1$. For the free propagator $r$ and $t$ must be large compared to $1/2\pi T$; for the HTL propagator they must be large compared to $1/m_g$. The leading term in the asymptotic space-like behavior is displayed in the second column of Table I.
1. **Free, massless gauge boson.** In the Coulomb gauge \( \mathcal{D}^{00}(t, r) = \delta(t)/4\pi r \). The asymptotic behavior of the transverse propagator in the deep space-like region can be found in (6.6a)

\[
\mathcal{D}^{00}(t, r) = 0
\]

(6.5a)

\[
\mathcal{D}^{ij}(t, \vec{r}) \rightarrow \frac{-iT}{8\pi r} (\delta^{ij} + \hat{r}^i \hat{r}^j) + \frac{iT}{8\pi r^3} (\delta^{ij} + 3\hat{r}^i \hat{r}^j). 
\]

(6.5b)

There are omitted terms of order 1/\( r^2 \) and 1/\( r^3 \).

2. **HTL longitudinal propagator.** The asymptotic space-like behavior of the longitudinal gauge propagator computed in Sec. IV B can be summarized as follows:

\[
\mathcal{D}_{00}^{\text{pole}}(t, r) \rightarrow -i\frac{m_g c(t)}{8\pi r} 
\]

(6.6a)

\[
\mathcal{D}_{00}^{\text{cut}}(t, r) \rightarrow \text{exp. small} 
\]

(6.6b)

where

\[
c(t) = \frac{e^{-im_g t}}{1 - e^{-\beta m_g}} - \frac{e^{im_g t}}{1 - e^{\beta m_g}} 
\]

(6.7)

The cut contribution is exponentially small as \( r \to \infty \). In Appendix C the same results are obtained in the imaginary time formalism. In that approach the leading and the subleading terms are obtained in the imaginary time formalism. From that viewpoint, the static \( n = 0 \) mode is exponentially small because of Debye screening. The non-static contributions give exactly the same asymptotic behavior as the pole contribution in Eq. (6.6a).

3. **HTL transverse propagator.** From Sec. IV B the asymptotic behavior of the transverse propagator in the deep space-like region is given by

\[
\mathcal{D}^{ij \text{ pole}}(t, \vec{r}) \rightarrow -i\frac{c(t)}{8\pi m_g r^3} (-\delta^{ij} + 3\hat{r}^i \hat{r}^j) 
\]

(6.8a)

\[
\mathcal{D}^{ij \text{ cut}}(t, \vec{r}) \rightarrow \frac{iT}{8\pi r} (\delta^{ij} + \hat{r}^i \hat{r}^j) + \frac{iT}{4\pi m_g^2 r^3} (-\delta^{ij} + 3\hat{r}^i \hat{r}^j). 
\]

(6.8b)

It is rather unexpected that the leading term \( T/r \) comes entirely from the branch cut in the transverse propagator. The subleading term \( 1/(T r^3) \) comes both from the branch cut and from the pole. In Appendix C both the leading and the subleading terms are obtained in the imaginary time formalism. In that approach the leading \( T/r \) term comes from the static \( n = 0 \) mode. The subleading contribution comes from summing over the non-static modes. This applies to high temperature QED, where there is no magnetic mass.

4. **HTL transverse propagator with a magnetic mass.** For QCD a magnetic mass \( \mu^2 \) must be included. The asymptotic behavior of the pole contribution does not change. As shown in Sec. V B, the term of order \( T/r \) in the cut contribution is absent and the asymptotic behavior becomes

\[
\mathcal{D}^{ij \text{ pole}}(t, \vec{r}) \rightarrow -i\frac{c(t)}{8\pi m_g r^3} (-\delta^{ij} + 3\hat{r}^i \hat{r}^j) 
\]

(6.9a)

\[
\mathcal{D}^{ij \text{ cut}}(t, \vec{r}) \rightarrow \frac{iT}{4\pi r^3} \left[ \frac{1}{m_g^2} - \frac{1}{\mu^2} \right] (-\delta^{ij} + 3\hat{r}^i \hat{r}^j). 
\]

(6.9b)

Exactly the same asymptotic behavior is obtained using the imaginary time formalism in Appendix C. In that approach the \( 1/\mu^2 \) term comes from the static \( n = 0 \) mode and the other terms come from the nonstatic modes.

C. **Light-cone behavior**

The asymptotic behaviors computed for \( t/r > 1 \) and for \( t/r < 1 \) do not allow any conclusion about the behavior of the HTL propagators on the light cone, \( t/r = 1 \). To deduce the light-cone behavior, consider the inverse Fourier integral over space-time:

\[
\mathcal{D}_\mu(0, \vec{k}) = \int d^4 x e^{ikx} \mathcal{D}_\mu^\ast(t, \vec{r}). 
\]

(6.10)

After performing the angular integrations, the phase can be expressed as

\[
k_0 t - kr = \frac{1}{2}(k_0 + k)(t - r) + \frac{1}{2}(k_0 - k)(t + r). 
\]

In the limit \( k_0 + k \to \infty \) with \( k_0 - k \) fixed the integration is dominated by the light cone \( t - r \to 0 \) At large energy and momentum the HTL propagator reduces to the free thermal propagator in the Coulomb gauge \( \mathcal{D}_\mu^\ast (0, \vec{k}) \). Thus

\[
\lim_{k_0 + k \to \infty} \mathcal{D}_\mu^\ast (0, \vec{k}) = -i\frac{2\pi\epsilon(k_0) \delta(k_0^2 - k^2)}{1 - e^{-\beta k_0}} (\delta^{ij} - \hat{k}^i \hat{k}^j). 
\]

Since \( \beta k_0 \to \infty \), the limit is actually independent of temperature. Because the momentum-space limit \( k_0 + k \to \infty \) with \( k_0 - k \) fixed is controlled by the light cone \( t - r \to 0 \) this implies that the HTL propagator in space-time should have the same light-cone behavior as the free propagator in space-time. Near the light cone the singular behavior of the free Coulomb gauge propagator is

\[
t \to r: \quad \mathcal{D}_\mu^\ast (0, \vec{k}) = -i\frac{\epsilon(k_0) \delta(k_0^2 - k^2)}{2\pi^2 r(t - r)} (\delta^{ij} - \hat{r}^i \hat{r}^j) 
\]

\[
-\frac{i}{8\pi^2 r} \ln(t - r) (\delta^{ij} + 3\hat{r}^i \hat{r}^j) + \ldots, 
\]

with terms that are finite at \( t = r \) omitted. Therefore the HTL propagator \( \mathcal{D}_\mu^\ast (0, \vec{k}) \) in the Coulomb gauge should have these same singularities at \( t = r \).

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**APPENDIX A: SPECTRAL FUNCTION IN OTHER GAUGES**

All calculations in the paper are done in the Coulomb gauge and the spectral function has the structure given in Eq. [3.6]. In any other gauge the various components of \( \rho^{\mu\nu}(K) \) can also be expressed in terms of the longitudinal and transverse spectral functions.

A particularly simple choice is the temporal gauge, \( A^0 = 0 \). There are seven vanishing components of the spectral function: \( \rho^{00} = \rho^{ij} = \rho^0 = 0 \). The only nonvanishing components are

\[
\rho^{ij}(K) = (\delta^{ij} - \hat{k}^i\hat{k}^j)\rho_i(K) + \frac{k^ik^j}{k_0^2}\rho_i(K).
\]

In the Landau gauge the various components of the spectral function are

\[
\rho^{00}(K) = \frac{k^4}{(K^2)^2}\rho_i(K)
\]

\[
\rho^{ij}(K) = \frac{k_0^2k^ik^j}{(K^2)^2}\rho_i(K)
\]

\[
\rho^{ij}(K) = \frac{k_0^2k^ik^j}{(K^2)^2}\rho_i(K) + (\delta^{ij} - \hat{k}^i\hat{k}^j)\rho_i(K).
\]

In more general covariant gauges the spectral function has an additional term \( K^\mu K^\nu \delta^{\nu}(K^2) \) that must be added to the Landau gauge result.

**APPENDIX B: LEMMA ON ASYMPTOTIC SPACE-LIKE BEHAVIORS**

In the deep space-like region, \( r \to \infty \) with a fixed ratio \( r/t > 1 \), there are no points of stationary phase on the real \( k \) axis. For the pole contributions the stationary phase condition \( d\omega/dk = r/t \) cannot be satisfied for real \( k \). For the cut contributions the stationary phase condition \( k_0/k = r/t \) cannot be satisfied since \( k_0 < k \) on the Landau cut. In the complex \( k \) plane there can be points of stationary phase but they will give contributions that fall exponentially with \( r \). However, the leading asymptotic behavior will typically be a power-law falloff that is not related to the stationary phase points. The generic integral to be investigated is

\[
\mathcal{D}(t, r) = \int_0^\infty dk \frac{\sin kr}{kr} f(t, k).
\]

The function \( f(t, k) \) will always be an even function of \( k \) and infinitely differentiable on the real \( k \) axis. The lemma will show that asymptotic behavior is controlled by the endpoint \( k = 0 \). Specifically in the deep space-like region

\[
\mathcal{D}(t, r) \to \frac{\pi}{2r} f(t, 0) + \text{exp. small.} \quad (B2)
\]

This is the form in which the lemma is used in Sec. V B. However in Sec. IV B the naturally occurring integral is of the form

\[
\mathcal{D}(t, r) = \frac{1}{r} \int_0^\infty dk k \sin kr F(t, k).
\]

By renaming \( F(t, k) = f(t, k)/k^2 \) it is clear that the asymptotic behavior of the integral will be \( c_0\pi/2r \) where \( F(t, k) \to c_0/k^2 \) as \( k \to 0 \). It is this second form that occurs naturally in Sec. IV B.

To isolate the 1/r contribution, write the original integral Eq. (B1) as \( \mathcal{D} = \mathcal{D}_f + \mathcal{D}_g \) where

\[
\mathcal{D}_f(t, r) = \int_0^\infty dk \frac{\sin kr}{kr} e^{-ak} f(t, 0) \quad (B3a)
\]

\[
\mathcal{D}_g(t, r) = \int_0^\infty dk \frac{\sin kr}{kr} g(t, k), \quad (B3b)
\]

with \( g(t, k) \) defined by

\[
g(t, k) = \frac{1}{k} [f(t, k) - e^{-ak} f(t, 0)]. \quad (B4)
\]

The parameter \( \alpha \) exactly cancels in the sum \( \mathcal{D}_f + \mathcal{D}_g \).

(1) The integral \( \mathcal{D}_f \) is elementary and gives

\[
\mathcal{D}_f(t, r) = f(t, 0) \frac{1}{r} \tan^{-1}(r/\alpha).
\]

Since \( \alpha \) is a fixed parameter and \( r \to \infty \), one can expand the inverse tangent to get

\[
\mathcal{D}_f(t, r) = f(t, 0) \frac{1}{r} \left[ \frac{\pi}{2} - \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{2\ell + 1} \left( \frac{\alpha}{r} \right)^{2\ell+1} \right]. \quad (B5)
\]

(2) To investigate \( \mathcal{D}_g \) it is useful to use repeated integration by parts. First write \( \mathcal{D}_g \) as

\[
\mathcal{D}_g(t, r) = \frac{1}{r^2} \int_0^\infty dk \frac{\cos kr}{d^2} g(t, k).
\]

An integration by parts gives

\[
\mathcal{D}_g(t, r) = \frac{1}{r^2} g(t, 0) + \frac{1}{r^2} \int_0^\infty dk \cos kr \frac{dg(t, k)}{dk}.
\]

There is no surface term from the upper limit since \( g(t, \infty) = 0 \). Next write \( \cos kr = r^{-1}d\sin kr/dk \) and integrate by parts. Because of \( \sin kr \) there will be no contribution at \( k = 0 \):
\[ D_g(t, r) = \frac{1}{r^2} g(t, 0) - \frac{1}{r^3} \int_0^\infty dk \sin(kr) \frac{d^2g(t, k)}{dk^2}. \]

Continuing this process gives

\[ D_g(t, r) = \sum_{\ell=0}^n \frac{(-1)^\ell}{r^{2\ell+2}} \left[ \frac{d^{2\ell+1} g}{dk^{2\ell+1}} \right]_{k=0} + \frac{(-1)^n}{r^{2n+3}} \int_0^\infty dk \sin(kr) \frac{d^{2n+2} g}{dk^{2n+2}}. \]

From the definition of \( g(t, k) \) the even derivatives are

\[ \left[ \frac{d^{2\ell} g}{dk^{2\ell}} \right]_{k=0} = \frac{1}{2\ell + 1} \left( \frac{d^{2\ell+1} f}{dk^{2\ell+1}} \right)_{k=0} + \alpha^{2\ell+1} f(t, 0). \]

The first term vanishes since \( f(t, k) \) is an even function of \( k \). Therefore

\[ D_g(t, r) = f(t, 0) \frac{\pi}{2r} \sum_{\ell=0}^n \frac{(-1)^\ell}{2\ell + 1} \left( \frac{\alpha}{r} \right)^{2\ell+1} \frac{d^{2\ell+1} g}{dk^{2\ell+1}} + \frac{(-1)^n}{r^{2n+3}} \int_0^\infty dk \sin(kr) \frac{d^{2n+2} g}{dk^{2n+2}}. \]

Adding together \( D_f \) and \( D_g \) results in a cancellation of the powers \( r^{-2}, r^{-4}, \ldots, r^{-2n-2} \) and gives

\[ D(t, r) = f(t, 0) \frac{\pi}{2r} \sum_{\ell=0}^n \frac{(-1)^\ell}{2\ell + 1} \left( \frac{\alpha}{r} \right)^{2\ell+1} \frac{d^{2\ell+1} g}{dk^{2\ell+1}} - f(t, 0) \frac{1}{r} \sum_{\ell=n+1}^\infty \frac{(-1)^\ell}{2\ell + 1} \left( \frac{\alpha}{r} \right)^{2\ell+1} \frac{d^{2\ell+1} g}{dk^{2\ell+1}}. \]

This formula is exact. The right hand side is independent of the choice of \( \alpha \) and of \( n \). (For example, one could set \( \alpha = 0 \) and eliminate the summation.) The first correction to \( 1/r \) is of order \( 1/r^{2n+3} \). Since \( f(t, k) \) is infinitely differentiable, \( n \) may be chosen arbitrarily large. This proves that the corrections to \( 1/r \) fall faster than any power of \( 1/r \), i.e. fall exponentially with \( r \). The asymptotic behavior is therefore as given in Eq. (B6). It is, of course, possible for \( f(t, 0) \) to vanish. In that case the leading behavior falls exponentially with \( r \).

**APPENDIX C: ASYMPTOTIC SPACE-LIKE BEHAVIOR USING IMAGINARY TIME**

The analysis of the hard-thermal-loop propagator in the deep space-like region showed that both the longitudinal and transverse propagators fall like \( 1/r \). In the longitudinal case the coefficient of \( 1/r \) depends on the thermal gluon mass \( m_g \) and on time. This contribution comes entirely from the longitudinal quasiparticle pole. For the transverse propagator the \( 1/r \) contribution is the same as for a free, thermal field theory. Rather surprisingly, this simple contribution comes entirely from the cut in the transverse propagator.

This Appendix computes the asymptotic space-like behavior of the longitudinal and transverse propagators using the imaginary time formalism. The results obtained are identical with Eqs. (4.3), (6.8), and (11.3). The asymptotic behavior of the longitudinal propagator, which comes from the quasiparticle pole in the real-time method, will come from all the non-static modes in the imaginary-time method. The asymptotic behavior of the transverse propagator, which comes from the branch cut in the real-time method, will come from entirely from the static mode of the imaginary-time method.

1. Asymptotic space-like behavior of \( \ast D^{00}(t, r) \)

In imaginary time the propagator for the \( A^0 \) component of the vector potential in the Coulomb gauge is

\[ \ast D^{00}_E(\tau, r) = i T \sum_{n=\infty}^{\infty} \int \frac{d^3 k}{(2\pi)^3} \frac{\sin(\vec{k} \cdot \vec{r} - i\omega_n \tau)}{k^2 + i\Pi_\tau(\omega_n, k)}, \]  \hfill (C1)

where \( \omega_n = 2\pi n T \) and \( -\beta \leq \tau \leq \beta \). The denominator depends on \( \omega_n \) only through the self-energy. Separate this into the static, \( n=0 \), mode and the nonstatic modes

\[ D^{00}_E(\tau, r) = D^{00}_{n=0}(\tau, r) + D^{00}_{n \neq 0}(\tau, r). \] \hfill (C2)

The static value, \( \Pi_\tau(0, k) = 3m_g^2 \), produces the standard Debye screening result:

\[ D^{00}_{n=0}(\tau, r) = i T \int \frac{d^3 k}{(2\pi)^3} \frac{\sin(\vec{k} \cdot \vec{r} - \omega_n \tau)}{k^2 + 3m_g^2} = i T \frac{\omega_n}{4\pi} e^{-m_g r \sqrt{3}}. \]

This exponentially small contribution is not the dominant one.

The nonstatic terms will turn out to dominate at large separations. The reason is that for \( n \neq 0 \) the propagator denominator vanishes at small \( k \):

\[ k \to 0 \quad k^2 + \Pi_\tau(i\omega_n, k) \to k^2 \left( 1 + \frac{m_g^2}{\omega_n^2} \right) + O(k^4). \]

Therefore the leading nonstatic contribution in the deep space-like region is

\[ D^{00}_{n \neq 0}(\tau, r) \to i T \int \frac{d^3 k}{(2\pi)^3} \frac{e^{-\vec{k} \cdot \vec{r}}}{k^2} \sum_{n=\infty}^{\infty} \frac{\omega_n^2}{\omega_n^2 + m_g^2} e^{-i\omega_n \tau}. \]

The integration gives \( 1/(4\pi r) \) and the sum is elementary. The sum converges for \( \tau \) real and gives

\[ D^{00}_{n \neq 0}(\tau, r) \to \frac{-im_g}{8\pi r} \left[ \frac{e^{-m_g \tau}}{1 - e^{-\beta m_g}} + \frac{e^{m_g \tau}}{e^{\beta m_g} - 1} \right]. \]

This can be continued from Euclidean time \( \tau \) to real time \( t \) by the replacement \( \tau \to it \). Thus use the function

\[ c(t) = \frac{e^{-im_g t}}{1 - e^{-\beta m_g}} + \frac{e^{im_g t}}{e^{\beta m_g} - 1}, \] \hfill (C3)

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that was introduced in Eq. (4.14). The space-like asymptotic behavior of the Minkowski two-point function is

$$D^0_{>}(t, r) \sim -\frac{im_g \phi(t)}{8\pi r} + \text{exp. small.} \quad (C4)$$

This coincides with Eq. (6.6). In the real-time calculation this result comes entirely from the quasiparticle pole.

2. Asymptotic space-like behavior of $^*D^{ij}(x)$

The transverse propagator in Euclidean time is

$$^*D^{ij}_{E}(\tau, \vec{r}) = -iT \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot \vec{r} - i\omega_n \tau}}{\omega_n^2 + k^2 + \Pi_t(i\omega_n, k)}.$$ 

As before, separate out the static, $n = 0$ mode from the others:

$$^*D^{ij}_{E} = ^*D^{ij}_{n=0} + ^*D^{ij}_{n \neq 0}.$$ 

Because the transverse self-energy vanishes at $n = 0$ the static propagator is the same as the free propagator:

$$^*D^{ij}_{n=0}(\tau, \vec{r}) = -iT \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot \vec{r}}}{k^2} (\delta^{ij} - \hat{k}^i \hat{k}^j).$$

This will turn out to be the dominant term. It coincides with the cut contribution in Eq. (6.8).

The remaining sum over non-zero modes can be written

$$^*D^{ij}_{n \neq 0}(\tau, \vec{r}) = (\delta^{ij} \nabla^2 + \nabla^i \nabla^j) W_{n \neq 0}(\tau, r),$$

where the new function is

$$W_{n \neq 0}(\tau, r) = -iT \sum_{n \neq 0} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot \vec{r} - i\omega_n \tau}}{\omega_n^2 + k^2 + \Pi_t(i\omega_n, k)}.$$ 

The dominant contribution at large $r$ comes from $k \to 0$. In this limit $\Pi_t(i\omega_n, 0) = m_g^2$ and thus

$$W_{n \neq 0}(\tau, r) \sim -iT \sum_{n \neq 0} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot \vec{r} - i\omega_n \tau}}{\omega_n^2 + m_g^2}.$$ 

The sum is convergent for real $\tau$ and gives

$$W_{n \neq 0}(\tau, r) \sim -iT \frac{1}{4\pi r} \left[ \frac{1 - e^{-m_g \tau}}{m_g^2} - \frac{e^{m_g \tau}}{2m_g^2} \right].$$

The continuation from Euclidean $\tau$ to real $t$ requires the replacement $\tau \rightarrow it$, so that

$$W_{n \neq 0}(t, r) \sim \frac{iT}{4\pi r} \left[ \frac{1}{m_g^2} - \frac{c(t)}{2m_g T} \right].$$

Performing the spatial derivatives gives for the asymptotic behavior of the non-static propagator

$$^*D^{ij}_{n \neq 0}(t, \vec{r}) \sim \frac{iT}{4\pi r^3} \left[ \frac{1}{m_g^2} - \frac{c(t)}{2m_g T} \right] \left( -\delta^{ij} + 3\hat{r}^i \hat{r}^j \right). \quad (C6)$$

The leading contribution is the static term, Eq. (C5). This sub-leading term agrees with Eq. (6.8). In the real-time calculation the sub-leading term comes both from the quasiparticle pole and from the branch cut.

Inclusion of a magnetic mass. A magnetic mass eliminates the $1/r$ contribution of the static mode but introduces a new $1/r^3$ static contribution. To see this, define

$$W_{n=0}(r) = -iT \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot \vec{r}}}{k^2 + \Pi_t(0, k)}.$$ 

The angular integrations give

$$W_{n=0}(r) = -iT \int_0^\infty \frac{dk}{2\pi^2} \frac{\sin kr}{kr} \frac{1}{k^2 + \Pi_t(0, k)}.$$ 

As $r \to \infty$ this integral is dominated by the region $k \to 0$. One expects that $\Pi_t(0, k) \to \mu^2 + k^2$, in the limit $k \to 0$. The asymptotic behavior is

$$W_{n=0}(r) \sim \frac{iT}{4\pi r^3} \frac{1}{k^2 + \mu^2} \to \frac{-iT}{4\pi r^3} \text{exp. small.}$$

The exponential fall-off is of the form $\exp(-\mu r)$. Differentiating this gives for the static part of the transverse propagator

$$^*D^{ij}_{n=0}(r) \sim \frac{-iT}{4\pi r^3} \mu^2 (-\delta^{ij} + 3\hat{r}^i \hat{r}^j).$$

Adding the non-static contribution from Eq. (C6) gives

$$^*D^{ij}(t, \vec{r}) \sim \frac{iT}{4\pi r^3} \left[ -\frac{1}{\mu^2} + \frac{1}{m_g^2} \frac{c(t)}{2m_g T} \right] (-\delta^{ij} + 3\hat{r}^i \hat{r}^j). \quad (C7)$$

This agrees with the real-time result in Eq. (6.9).

APPENDIX D: Asymptotic Time-Like Behavior of $^*D^{ij}_{\text{cut}}(t, \vec{r})$

The tensor structure of the transverse Wightman function is

$$^*D^{ij}_{>}(t, \vec{r}) = (-\delta^{ij} + 3\hat{r}^i \hat{r}^j) G(t, r) \text{ nonumber} \quad (D1)$$

$$+ (\delta^{ij} - \hat{r}^i \hat{r}^j) H(t, r). \quad (D2)$$
The contribution of the Landau branch cut is particularly delicate to calculate. In Sec. V A the cut contribution was split into a sum of two parts:

\[ G^\text{cut}(t, r) = G^{(1)}(t, r) + G^{(2)}(t, r), \]
\[ H^\text{cut}(t, r) = H^{(1)}(t, r) + H^{(2)}(t, r), \]
in which \( G^{(1)} \) and \( H^{(1)} \) were of order 1/r plus order 1/r^3, whereas \( G^{(2)} \) and \( H^{(2)} \) were entirely of order 1/r^3. This appendix contains the actual calculations of \( H^{(1)} \) and \( G^{(2)} \) that were not presented in Sec. V A.

1. Asymptotic behavior of \( H^{(1)}(t, r) \)

The leading behavior of the cut contribution comes from the integral \( H^{(1)} \), defined in Sec. V A as

\[ H^{(1)}(t, r) = \frac{-1}{r(2\pi)^2} \int_{-\infty}^{\infty} dk \int_{-1}^{1} dx \ e^{ik(r-xt)} I^{(1)}(x, k), \]

with \( I^{(1)} \) given in Eq. (5.21). To compute \( H^{(1)} \) it is convenient to split the integration over \( x \) into three parts,

\[ H^{(1)}(t, r) = \frac{-1}{r(2\pi)^2} (A + B + C), \tag{D3} \]

with \( A, B, C \) defined as

\[ A = \int_{-\infty}^{\infty} dk \int_{0}^{\infty} dx \ e^{ik(r-xt)} I^{(1)}(x, k), \]
\[ B = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx \ e^{ik(r-xt)} I^{(1)}(x, k), \]
\[ C = \int_{-\infty}^{\infty} \int_{-\infty}^{1} dx \ e^{ik(r+xt)} I^{(1)}(x, k). \]

At \( \varpi = r/t \) the phase changes sign. The \( k \) integrations will be performed using Cauchy’s theorem, since the only singularities in \( k \) of the integrand are simple poles. The location of the four poles are

\[ k_j = e^{i\pi(2j-1)/4} m \left( \frac{3\pi x}{4} \right)^{1/2}, \]

where \( k_1, k_2, k_3, k_4 \) are located in the first, second, third, and fourth quadrants, respectively. The function \( I^{(1)} \) can be written in terms of partial fractions as

\[ I^{(1)}(x, k) = \frac{3m^2 T}{16} \sum_{j=1}^{4} \frac{1}{(k - k_j) k_j^2}. \]

(A) In integral \( A \), the quantity \( r - xt \) is positive so that the integration over \( k \) can be computed by adding a semicircle of infinite radius in the upper-half of the complex \( k \) plane. Evaluating the \( k \) integration by Cauchy’s theorem gives

\[ A = \frac{T}{2} \int_{0}^{\varpi} \frac{dx}{x} \left[ e^{ik_2(r-xt)} - e^{ik_2(r-xt)} \right] \]

Using the values of \( k_1 \) and \( k_2 \) converts this to

\[ A = iT \int_{0}^{\varpi} \frac{dx}{x} e^{-m\sqrt{\varpi}(r-xt)} \sin \left[ m\sqrt{\varpi}(r-xt) \right], \]

where \( m = m \sqrt{3\pi}/8 \). For \( r \to \infty \) and \( t \to \infty \) with \( r/t < 1 \), the integrand falls exponentially except at the two endpoints \( x = 0 \) and \( x = \varpi \). The contribution of these endpoints will be called \( A_0 \) and \( A\varpi \) and will fall like a power of \( r \). Thus with exponentially small contributions omitted, one can say \( A = A_0 + A\varpi \).

To compute \( A_0 \) put \( x = \varepsilon^2 z^2 \) where

\[ \varepsilon = \frac{1}{ar} = \frac{1}{m g r} \left( \frac{8}{3\pi} \right)^{1/2}. \tag{D4} \]

The contribution of the lower limit is

\[ A_0 = 2iT \int_{0}^{\varpi} \frac{dz}{z} e^{-z + \varepsilon^2 z^2 / \varpi} \sin \left( z - \varepsilon^2 z^3 / \varpi \right). \]

Expanding the integrand to order \( \varepsilon^2 \) and integrating gives

\[ A_0 = iT \left[ \frac{\pi}{2} + \frac{\varepsilon^2}{\varpi} + O(\varepsilon^4) \right]. \tag{D5} \]

To compute the contribution of the upper endpoint, \( x = \varpi \), return to the exact expression for \( A \) and change variables to \( x = \varpi(1 + \varepsilon z) \). Then

\[ A\varpi = iT\varepsilon \int_{0}^{\varpi} \frac{dz}{1 - \varepsilon z} e^{-z \sqrt{\varpi(1 + \varepsilon z)}} \sin \left( z \sqrt{\varpi(1 - \varepsilon z)} \right). \]

Expanding the integrand to order \( \varepsilon^2 \) and integrating gives

\[ A\varpi = iT \left[ \frac{\varepsilon}{2\varpi^{3/2}} + \frac{\varepsilon^2}{\varpi} + O(\varepsilon^4) \right]. \tag{D6} \]

(B) In integral \( B \), the quantity \( r - xt \) is always negative. Therefore the \( k \) contour can be closed in the lower half-plane and evaluated by Cauchy’s theorem. The poles at \( k_3 \) and \( k_4 \) contribute and give

\[ B = -\frac{T}{2} \int_{\varpi}^{1} \frac{dx}{x} \left[ e^{ik_3(r-xt)} - e^{ik_4(r-xt)} \right]. \]

Substituting the value of the roots \( k_3 \) and \( k_4 \) gives

\[ B = iT \int_{0}^{\varpi} \frac{dz}{x} e^{m \sqrt{\varpi}(r-xt)} \sin \left[ m \sqrt{\varpi}(r-xt) \right] \]

Note that \( r - xt \leq 0 \) in this integral. The integrand is exponentially suppressed except at the lower endpoint \( x = \varpi \). This contribution \( B\varpi \) will fall like a power of \( r \). To compute this, put \( x = \varpi(1 + \varepsilon z) \) so that

\[ B\varpi = -iT\varepsilon \int_{0}^{\varpi} \frac{dz}{1 + \varepsilon z} e^{-z(1 + \varepsilon^2 z)^{1/2}} \sin \left[ z(1 + \varepsilon z)^{1/2} \right]. \]
Expanding this for small $\varepsilon$ and integrating gives
\[ B_{\varepsilon} = iT \left[ -\frac{\varepsilon}{2\pi} \right] + \mathcal{O}(\varepsilon^3) \]. \quad (D7)

(C) In integral $C$, since $r + xt$ is always positive, the contour can be closed in the upper-half plane and evaluated with Cauchy’s theorem:
\[ C = \frac{T}{2} \int_0^1 \frac{dx}{x} \left[ e^{ik_1(r+xt)} - e^{ik_2(r+xt)} \right]. \]

Using the values of $k_1$ and $k_2$ gives
\[ C = iT \int_0^1 \frac{dx}{x} e^{-m\sqrt{r}(r+xt)} \sin \left[ m\sqrt{r}(r+xt) \right], \]
where $m = m \sqrt{3\pi}/8$. The integrand falls exponentially except at the lower endpoint $x = 0$. To compute this contribution, put $x = \varepsilon^2 z^2$ to obtain
\[ C_0 = 2iT \int_0^{\infty} \frac{dz}{z} e^{-\varepsilon^2 z^2} \sin \left[ \varepsilon^2 z^2 \right]. \]

Expanding in $\varepsilon$ and integrating gives
\[ C_0 = iT \left[ \frac{\pi}{2} - \frac{2\varepsilon^2}{\pi} + \mathcal{O}(\varepsilon^4) \right]. \quad (D8) \]

**Total for $H^{(1)}$.** The sum of the endpoint contributions from $x \approx 0$ is
\[ A_0 + C_0 = iT \left[ \pi + \mathcal{O}(\varepsilon^4) \right]. \quad (D9) \]

The contributions from $x \approx \infty$ sum to
\[ A_\infty + B_\infty = iT \left[ \frac{2\varepsilon^2}{\pi} + \mathcal{O}(\varepsilon^3) \right]. \quad (D10) \]

It is quite likely that the $\mathcal{O}(\varepsilon^3)$ terms actually cancel in this sum, but it will not matter below. Using these in Eq. (D3) gives for the asymptotic behavior of $H^{(1)}$:
\[ H^{(1)}(t, r) \to -\frac{iT}{4\pi r} - \frac{4iT}{3\pi^3 m^2 r^3} + \mathcal{O} \left( \frac{1}{r^4} \right). \quad (D11) \]

This value is used in Eq. (5.23).

**2. Asymptotic behavior of $G^{(2)}(t, r)$**

In Sec. V A the asymptotic value of $H^{(2)}$ was computed by the method of stationary phase. This did not completely determine $G^{(2)}$ and so it too must be computed by the method of stationary phase.

It is convenient to start with the original definition of $G$ in Eq. (5.12). If the angular integration over $z = k \cdot \hat{r}$ is not performed, the cut contribution is
\[ G^{\text{cut}}(t, r) = -\frac{i}{16\pi^2} \int_0^1 dz (1 - z^2) G^{\text{cut}}(t, r, z), \quad (D12) \]
where
\[ G^{\text{cut}}(t, r, z) = \int_{-\infty}^{\infty} dk \int_{-1}^{1} dx e^{ik(r-xt)} kI_1(t, k). \]

with $I_1$ defined in Eq. (5.19). If the $z$ integration is performed at this stage it will change $kI_1$ to $I_1/k$ and therefore prevent convergence at $k = 0$. Separating $I_1 = I^{(1)} + I^{(2)}$ as given in Eq. (5.22) gives
\[ G^{(2)}(t, r, z) = \int_{-\infty}^{\infty} dk \int_{-1}^{1} dx e^{ik(r-xt)} kI^{(2)}(t, k). \]

The integrand $kI_2$ vanishes like $k^2$ at small $k$. The stationary phase point is $\hat{r} = rz/t$, $\hat{k} = 0$. Making the variable change analogous to Eq. (4.12) gives
\[ G^{(2)}(t, r, z) = \frac{2}{t} \int_{\pi}^{\pi} udu \int_{0}^{2\pi} d\theta e^{-u^2} kI^{(2)}(t, k). \]
The Gaussian integration can be performed using Eq. (4.13) to obtain the asymptotic behavior
\[ G^{(2)}(t, r, z) \to \frac{-\pi}{t^3} \left[ 2\pi \frac{\partial^2 [kI^{(2)}(x, k)]}{\partial x\partial k} \right]_{\hat{r}, \hat{k}}. \]

This simplifies to
\[ G^{(2)}(t, r, z) \to \frac{-8\pi T}{3m^2 r^3} \frac{\partial^2}{\partial x^2} \left[ 1 - \frac{\hat{r}^2}{\hat{r}^2} \right] \frac{1}{N(t, 0)} - \frac{1}{(\hat{r})^2}. \quad (D13) \]
The stationary point depends on the angular variable $z$ since $\hat{r} = rz/t$ with $\hat{r} = r/t < 1$. When $G^{(2)}$ is substituted into Eq. (D12) the integration over $z$ must be performed. It is convenient to change integration variables from $z$ to $y = \hat{r}z$ so that
\[ G^{(2)}(t, r) = -\frac{i}{16\pi^2} \int_{-\infty}^{\infty} dy \frac{1 - y^2}{\hat{r}^2} G^2(t, r, z). \]

Using the asymptotic form for $G^{(2)}$ gives
\[ G^{(2)}(t, r) \to \frac{iT}{3m^2 r^3} \int_{0}^{\pi} dy \frac{\partial^2}{\partial y^2} \left[ \frac{1 - y^2}{N(t, 0)} - \frac{1}{(\hat{r}y)^2} \right]. \quad (D14) \]
The quantity in square brackets is an even function of $y$ that is finite at $y = 0$. The first $y$ derivative vanishes at $y = 0$. Therefore one can integrate by parts to obtain
\[ G^{(2)}(t, r) \to \frac{2iT}{3m^2 r^3} \int_{0}^{\pi} dy \frac{\partial}{\partial y} \left[ \frac{1 - y^2}{N(t, y, 0)} - \frac{1}{(\hat{r}y)^2} \right]. \]

This is the result used in Eq. (5.24).
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