A plastic flow theory for amorphous materials

V.I. Marchenko, and Chaouqi Misbah
P.L. Kapitza Institute for Physical Problems, RAS, 119334, Kosugina 2, Moscow, Russia
Lab. de Spectrométrie Physique, Université Joseph Fourier (CNRS) Grenoble I, B.P. 87, Saint-Martin d’Hères, 38402 Cedex, France
(Dated: December 22, 2008)

Starting from known kinematic picture for plasticity, we derive a set of dynamical equations describing plastic flow in a Lagrangian formulation. Our derivation is a natural and a straightforward extension of simple fluids, elastic and viscous solids theories. These equations contain the Maxwell model as a special limit. We discuss some results of plasticity which can be described by this set of equations. We exploit the model equations for the simple examples: straining of a slab and a rod. We find that necking manifests always itself (not as a result of instability), except if the very special constant-velocity stretching process is imposed.

PACS numbers: 62.20.F-, 46.05.+b, 46.35.+z
Keywords:

Plastic materials exhibit several features which are not present in the usual liquids or solids. Their dynamics consist in a nontrivial mixture of liquid-like and solid-like behaviors. Understanding plasticity in metal industry and, in general, in technology, is of a paramount importance. Nonetheless, to date no universal dynamical equations describing plastic materials like Navier-Stokes equations for fluids, and Lamé equations for elastic solids, are available. Under strain, plastic material may exhibit elastic-like behaviors, yield stress, flowing behaviors, nonlinear engineering strain-stress relation, and so on [1]. A major goal in material science is the description of these phenomena in terms of dynamical evolution equations for relevant variables, namely the velocity, stress, and the analogues of strain.

There has been important contributions to the theory of plasticity, especially for crystalline materials in terms of dislocations [2–4]. However, there is no need to evoke dislocations (if ever this notion has a meaning) for amorphous materials, and thus the question arises of how a corresponding theory can be build at the continuum level. This question has known recently an upsurge of interest [5]. An essential issue when addressing the question of plasticity is the distinction between crystalline solids and amorphous materials. Elastically deformed monocrystals are in a metastable state. Their plastic flow takes place only upon creation of dislocations, and is thus a nonlinear process. Conversely, plastic flow of amorphous materials should occur, in principle, linearly with respect to the applied stress. In crystals, an additional field, namely dislocation density, is introduced which couples to the elastic as well as to plastic distortions [3, 4]. It is proposed here that one can derive a plastic continuum theory for amorphous materials, without evoking neither dislocation density, nor an internal variable that is distinct from the variables describing usual kinematics of plasticity. The evolution equations can be written in a closed form in terms of the elastic and plastic distortions only. The concept of distortion [2] used here was introduced by Kröener and Rieder in 1956 [6], and will constitute our basic definition of the plastic flow variable. The present contribution aims to introduce the most simple dynamical approach by following Kröener and Rieder [6].

If \( u_k \) denotes the \( k^{th} \) component of the displacement field, then the total distortion tensor reads \( \partial_i u_k \). For a purely elastic solid, and in the small deformation regime, the symmetrized part \( u_k = (\partial_i u_k + \partial_k u_i)/2 \) is the strain tensor. If plastic flow is involved, the total distortion tensor \( \partial_i u_k \) is taken as a sum [2] of the plastic flow contribution \( u_k^{pl} \), and the elastic part \( u_k \) (see [2])

\[
\partial_i u_k = w_k^{pl} + w_{ik}. \tag{1}
\]

The symmetrical part of the elastic distortion defines the strain tensor

\[
u_{ik} = \frac{1}{2}(w_{ik} + w_{ki}). \tag{2}
\]

Note that definition (2) is different from \((\partial_i u_k + \partial_k u_i)/2\) which is valid only in absence of plasticity. The constrain \( u_{ik}^{pl} = 0 \) is usually supposed. This is true [2] if plastic flow occurs without destroying continuity of the material. Then from Eqs.(1,2) it follows \( u_{ii} = \partial_i u_i \).

Usual elasticity can be presented in a Lagrangian formulation by introducing an energy and a dissipative function. The elastic energy of a solid is given by

\[
\frac{\lambda}{2} u_{ii}^2 + \mu u_{ik}^2, \tag{3}
\]

where \( \lambda, \mu \) are Lamé coefficients, and the kinetic energy is \( \rho v^2/2 \), where \( \rho \) is the density of the material. The dissipation function reads for a viscous solid

\[
\eta^k \left( \dot{u}_{ik} - \frac{1}{\delta} \delta_{ik} u_{ilt} \right)^2 + \frac{\zeta}{2} u_{il}^2 \tag{4}
\]
In the presence of a plastic flow one must introduce a new additional dissipative part related to plasticity. The system is described by three independent tensors, namely $u_{ik}$ and $u_{ik}^{\text{pl}}$ which are the symmetric (+), and antisymmetric (−) parts of the plastic distortion tensor, respectively. From the basic kinematic relation (1) and the strain tensor definition (2) it follows

$$u_{ik}^{\text{pl}+} = \frac{1}{2}(\partial_i u_k + \partial_k u_i) - u_{ik}. \quad (5)$$

We expect the dissipation to consist of a quadratic form of these quantities, that we write as

$$2\alpha \ddot{u}_{ik} u_{ik}^{\text{pl}+} + \eta \left( u_{ik}^{\text{pl}+} \right)^2 + \gamma \left( u_{ik}^{\text{pl}−} \right)^2 \quad (6)$$

This is the dissipation corresponding to plastic flow. The total dissipation must be definite positive in order to guarantee stability. This is fulfilled if $\eta^s > 0$, $\gamma^s > 0$, $\eta^p > \alpha^2$, $\gamma > 0$. Note that there is only one dilatational viscosity constant $\zeta$. All other constants are related with shear motions. As we will see below in the liquid limit the constant $\eta$ is the usual viscosity.

The strategy now consists in performing variations of the total Lagrangian with respect to the independent variables. Variation with respect to $\delta u$, with $\delta u_{ik}^{\text{pl}−} = 0$, yields, upon using, $2\alpha u_{ik} = \partial_i \delta u_k + \partial_k \delta u_i$, the momentum conservation law

$$\tilde{p} \delta u = \partial_i \delta \sigma_{ik} \quad (7)$$

with the stress tensor $\sigma_{ik}$ consisting of the sum of the usual elastic part as well as the dissipative part with the usual solid viscosity terms, and an additional plastic term

$$\sigma_{ik} = \lambda u_{il} \delta_{ik} + 2\mu u_{ik} +$$

$$+ 2\eta^s \dot{u}_{ik} + \left( \zeta - \frac{2}{3} \eta^s \right) \dot{u}_{ik} \dot{u}_{ll} + 2\alpha \ddot{u}_{ik}^{\text{pl}+}. \quad (8)$$

Variation with respect to $\delta u_{ik}^{\text{pl}−}$, with $\delta u = 0$, and $\delta u_{ik}^{\text{pl}−} = 0$ (in that case $\delta u_{ik} = -\delta u_{ik}^{\text{pl}−}$) provides us with

$$\delta \sigma_{ik} = \alpha \left( \dot{u}_{ik} - \frac{2}{3} \dot{u}_{ik} \dot{u}_{ll} + 2\eta \dot{u}_{ik}^{\text{pl}−} \right), \quad (9)$$

here $\delta \sigma_{ik}$ is the traceless part of $\sigma_{ik}$.

Finally, variation with respect to $\delta u_{ik}^{\text{pl}−}$, with $\delta u = 0$, and $\delta u_{ik}^{\text{pl}−} = 0$ (in that case $\delta u_{ik} = 0$) leads to $u_{ik}^{\text{pl}−} = 0$ (a direct consequence of the absence of dissipation for rigid rotation; note also that energy does not depend on that mode). Note that if we require a continuous single-valued solution for the displacement field $u$, it can be shown that $\eta \equiv \nabla \times (\nabla \times (w^{\text{pl}−}))$ (called the incompatibility tensor) must be orthogonal to the permutation tensor[7].

An important remark is in order. Differentiating (1) with respect to time one obtains

$$\partial_k v_i + \partial_i v_k = -2j_{ik}^{\text{pl}} + 2\dot{u}_{ik}. \quad (10)$$

where $j_{ik}^{\text{pl}} \equiv -u_{ik}^{\text{pl}−}$, is the plastic current. This equation (see also [2]), apart from the (conventional) minus sign in front of $j_{ik}^{\text{pl}}$, bears resemblance with Eq.(3) of the seminal work of Ref. [5]. There is, however, some difference. Indeed, Eq.(3) of Ref. [5] uses the kinematic condition (10), plus Hooke’s law, where $u_{ik}$ is assumed to be related to the stress tensor by

$$u_{ik} = \frac{\sigma_{ik}}{2\mu} - \frac{p}{2K} \delta_{ik}. \quad (11)$$

where $p = -\sigma_{kk}/2$, is the pressure, and $K$ and $\mu$ are the compressibility and the shear modul (note that a 2D geometry is assumed in Ref. [5]). We do not postulate here a Hooke’s relation, since in our theory it has been possible to put both elastic and plastic contributions within the total distortion tensor $\partial_i u_k$. The relation between $u_{ik}$ and $\sigma_{ik}$ follows here as a consequence of the Lagrangian formulation, and the relationship between these two quantities is provided by (8) (showing that, in our theory, a measure of the stress is a combination of elastic and plastic deformations). Note finally that in contrast to crystal plasticity where $u_{ik}^{\text{pl}−}$ is non zero only at the dislocation line [2], this notion looses its meaning for amorphous materials.

It is possible to express the plastic distortion tensor in terms of other quantities. From Eqs.(8-9) we may express $\sigma_{ik}$ in terms of $u_{ik}$ and its time derivative. It is convenient to split the stress tensor $\sigma_{ik}$ into a traceless and a pressure-like term (actually the trace of $\sigma_{ik}$):

$$\sigma_{ik} = \frac{1}{3} \delta_{ik} \sigma_{ll} + \tilde{\sigma}_{ik} \quad (11)$$

The trace has a usual elastic (including the dissipative part) form

$$\sigma_{ll} = (3\lambda + 2\mu)u_{ll} + 3\zeta \dot{u}_{ll}. \quad (12)$$

The traceless parts of the stress tensor are connected with each other and with spatial gradients of the velocity by the following two relations

$$\eta(\gamma - \alpha)\tilde{\sigma}_{ik} = 2\eta \mu \dot{u}_{ik} + 2(\eta^s - \alpha^2) \dot{u}_{ik}, \quad (13)$$

$$\tilde{\sigma}_{ik} + 2(\gamma - \alpha) \dot{u}_{ik} = \eta \left( \partial_i v_k + \partial_k v_i - \frac{2}{3} \delta_{ik} \delta_{ij} v_j \right). \quad (14)$$

The first relation is obtained by expressing $\tilde{\omega}_{ik}^{\text{pl}}$ from Eq. (9) and inserting the resulting relation into (8). The second one follows from Eq. (9) by using relation (5). The set of Eqs. (7,11-14) defines space-time evolution of displacement vector $u$, strain $u_{ik}$ and the stress tensor $\sigma_{ik}$. This constitutes a complete set of equations for the three (vectorial and tensorial) quantities $u$, $u_{ik}$ and $\sigma_{ik}$ (we could, of course, alternatively use other quantities like $u_{ik}^{\text{pl}+}$). Note that Eq. (14) has some similarity with the Maxwell model, used to describe plasticity with a yield
The set of Eqs. (7,13,14) reads:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, w_{xy}, w_{yx}, \text{and } \alpha_{xy}. \]

Then the set of Eqs. (7,13,14) reads:

\[ \rho \frac{\partial u}{\partial x} + \frac{2\eta \mu u_{xy} + 2(\eta \tau - \alpha^2)u_{xy}}{\eta - \alpha}; \quad (16) \]

\[ \mu u_{xy} + (\eta - 2\alpha + \alpha^2)u_{xy} = \frac{\eta - \alpha}{\eta - 2} \frac{\partial u}{\partial y}. \quad (17) \]

It is then found that each field is a linear combination of the complex modes \( \propto \exp(-i\omega t + ixy) \), where \( \kappa \) is defined by

\[ \kappa^2 = (\kappa' + i\kappa'')^2 = \frac{i\xi}{\eta} \left( 1 - \frac{i\omega(\eta - \alpha)^2}{\eta \mu - i\omega(\eta \tau - \alpha^2)} \right). \quad (18) \]

For example, the displacement field is \( u_x = u \cos(\omega t - \kappa' y) \exp(-i\kappa'' y) \). The low frequency limit recovers a known Stokes result for a shear viscous mode in liquids (see 224 in [9]). Elastic solid behavior (an emission of shear sound) corresponds to the limit of high plastic viscosity \( \eta \rightarrow \infty \).

We would like to point out some results that can be captured analytically in some special limit. The long time behavior of a slab under tension, is expected to be dominated by plastic flow. Ultimately, the plastic flow should look-like a hydrodynamical flow. Let us concentrate on this limit. Consider a plate (or a rod) of a plastic material with free surfaces (Fig. 1). This is a similar geometry to that treated in Ref. [5]. The plate is stretched along the x direction. For a flat geometry we have obtained an exact solution with the plate thickness \( h(t) \) that depends only on the \( t \) variable. This type of solution exists only in the case where the stretching occurs at a given \textit{constant} velocity. Let us first motivate the solution on the basis of symmetries. Because of the axial symmetry with respect to the y axis at \( x = 0 \), \( v_x \) must be zero on that line. For constant \( h \) there is a simple solution that fulfills that symmetry, \( v_x = cx \), where \( c(t) \) is for the moment an arbitrary function of time. From incompressibility condition we have \( v_y = -cy + g(x,t) \), where \( g \) is \textit{a priori} an arbitrary function of \( x \) and \( t \). Symmetry with respect the middle line \( y = 0 \), enforces \( g = 0 \).

We straightforwardly obtain from the Navier-Stokes equation the pressure field

\[ p = -p(\dot{c} + c^2) \frac{x^2}{2} + p(\dot{c} - c^2) \frac{y^2}{2} + f(t), \]

where \( f(t) \) is a function of time to be determined below. At the free surface the normal component of the stress (the tangential vanishes automatically) must vanish. This is easily computed from the above result by using the definition \( \sigma_{xy} = -p + 2\eta \partial_x v_y = -p - 2\eta \). Imposing \( \sigma_{yy} = 0 \) on the free surfaces at \( y = \pm h \), at any \( x \), we obtain \( \dot{c} + c^2 = 0 \). This provides us with \( c = (t - t_0)^{-1} \), where \( t_0 \) is a constant of integration. It is convenient to measure the time from the moment \( t_0 \), so we will set \( t_0 = 0 \). For the length of the strip \( L \), one has \( \dot{L} = cL \), so

\[ L(t) \]

\[ \text{FIG. 1: A sketch of the geometry under consideration.} \]
where $S = \pi a^2$ is a rod cross section area.

The above solutions exists only for a constant velocity stretching. The question thus naturally arises of what happens if an other process is imposed. This is what we would like to investigate now. Following Ref.[5], if the lateral boundary of the plate moves at a pre-determined strain rate $\dot{L}/L = \Omega = \text{const.}$, then our result shows that a homogeneous thinning of the strip is not possible. Thus a modulated strip prevails. This is a precursor of the necking problem. Thus necking appears here as natural phenomenon due to material flowing [10] whenever the stretching is not performed at a constant speed. The necking is not related with an instability [5], but rather a natural phenomenon due to material flowing [10] whenever an other process is imposed. This is what we have found that if the stretching velocity is not constant the ultimate stage is a modulated thickness, of necking type. In fact, Fig.1 represents the result of our numerical solution, that exhibits necking in the case of initially flat plate and $L/L = \text{const.}$ (same stretching law as in Ref. [5]). This behavior is found for various initial conditions, and (non constant) stretching laws. Thus necking seems to be a robust feature, which takes place whenever the stretching is non constant.

It should be mentioned that here our plastic equations have been written by disregarding the so-called objective derivative (we have used ordinary derivatives) for tensors. One alternative in order to confer an objective form to these equations, is to replace the time derivative of tensors by the so-called co-rotational derivative[11], as is done in [5]. We shall report on full numerics of our completed set of equations (7,11-14) in the future.

We acknowledge CNRS, Univ. J. Fourier, and CNES for financial support.

---

1. L.E. Malverne, Introduction to the Mechanics of a Continuous Medium (Prentice Hall, Inc. New Jersey, 1969); J. Lubliner, Plasticity Theory (Macmillan Publishing Company, New York, 1990); Unified Constitutive Laws of Plastic Deformation, edited by A.S. Krausz and K. Krausz (Academic Press, San Diego, 1996).
2. L.D. Landau and E.M. Lifshitz, Theory of Elasticity, Pergamon Press, Oxford (1987).
3. V. Bulatov, F. Abraham, L. Kubin, Devincare, and S. Yip, Nature 391, 669 (1998), and references therein.
4. I. Groma and B. Bakó, Phys. Rev. Lett. 84, 1487 (2000).
5. L.O. Eastgate, J.S. Langer, and L. Pechelnik, Phys. Rev. Lett. 90, 045506 (2003), and references therein.
6. E. Kröener and R. Rieder, Z. Phys. 145, 424 (1956).
7. A. Menzel and P. Steinmann, J. Phys. IV France 8, Pr8-239 (1998).
8. P. Saramito, J. Non-Newtonian Fluid Mech. 145, 1 (2007), and references therein.
9. L.D. Landau and E.M. Lifshitz, Fluid Mechanics, Butterworth-Heinemann, Oxford (2000).
10. J.W. Hutchinson and K.W. Neale, Acta Metall. 25, 839 (1977).
11. R. Bird, R. C. Armstrong, and O. Hassager, Dynamics of Polymeric Liquids, Vol. 1, Fluid Dynamics,” Wiley, New York (1977, 2nd edition 1987).