Weighted entropy and optimal portfolios for risk-averse Kelly investments

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Abstract

Following a series of works on capital growth investment, we analyse log-optimal portfolios where the return evaluation includes ‘weights’ of different outcomes. The results are twofold: (A) under certain conditions, the logarithmic growth rate leads to a supermartingale, and (B) the optimal (martingale) investment strategy is a proportional betting. We focus on properties of the optimal portfolios and discuss a number of simple examples extending the well-known Kelly betting scheme.

An important restriction is that the investment does not exceed the current capital value and allows the trader to cover the worst possible losses.

The paper deals with a class of discrete-time models. A continuous-time extension is a topic of an ongoing study.

1 A Markovian model with a single risky asset

This paper is an initial part of a work on log-optimal portfolios influenced by a number of earlier publications, mainly by T. Cover and co-authors. Cf. Refs [1, 3] and [4], Chapter 6. Also see [10, 12, 18] and Ref [11]. Parts II and III. We also intend to use a recent progress in studying weighted entropies; cf. [14, 15, 16, 17]. A strong impact on the whole direction of this research was made by [8, 9] where a powerful methodology of a convex analysis has been developed (and elegantly presented) in a general form, leading – among other achievements – to existence of log-optimal portfolios. See Theorem 1 from [9]. In the present article, we attempt to go beyond the issue of existence and provide a specific form of the optimal strategy.

Let us discuss a finance-related context of this work. The sequential version of portfolio selection problem has received much attention in the literature not to speak about the financial practice, see [10, 12, 14, 15] and the references therein. A simple discrete-time model of a wide use in financial engineering is where the market consists of one riskless asset and one or more risky assets. (If the riskless asset produces a zero return, we can speak of risky assets only.) Investments are made at times \( n = 0, 1, \ldots \); the returns are recorded at subsequent times \( n = 1, 2, \ldots \).

We consider two investment schemes, showing that the results are valid for both schemes mutatis mutandis.

Scheme I: an investor signs a deal with a broker at the time \( n = 1 \) but the actual transaction happens at the moment \( n \) when the betting results become available.

Scheme II: at the moment \( n = 0 \) an investor transfers the required capital to a broker who invests this capital to buy shares or other risky assets.

In fact, Scheme II can be treated as a version of Scheme I, where the number of assets increases by 1.

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For convenience of presentation we switch freely between these two schemes keeping in mind that the results are valid in both cases with minor changes.

The mathematical problem under consideration is to characterize an optimal investment strategy/policy/portfolio (it will be convenient to use all these terms as equal in right, as with some other synonyms). Formally, we have to introduce the objective function and describe the class of strategies within which the optimization problem is solved. The setting for our study is probabilistic: we assume that, generally speaking, the returns are random, with known probabilistic laws. (In practice, it means that probabilistic/statistical features can be established, e.g., from available historic data.) More precisely, we deal with a random process of return values. For illustrative purposes we adopt for the most part a Markovian model of the return process(es) but also provide a mathematical result under quite general assumptions. (A number of features of the solution depends on the character and parameters of the return process.)

The objective function is introduced as the expected value of the weighted logarithmic return $S_n$ after $n$ trials. This is in line with the proposal going back to J. Kelly (Kelly investments, [7]) although our approach is based on some important modifications. In particular, we consider only cautious/risk-averse investment policies with a guaranteed cover of all possible losses. Here, a passive/restrained 0-strategy is a notable example, where the investor decides not to bet on the outcome of next trial. Pictorially speaking, the answer emerges from a comparison of the best adventurous bet and the zero-bet (and in most cases the 0-bet is preferable or the comparison is inconclusive, and the theory does not produce a formal answer).

In short, the results offered here can be summarized as follows: under certain conditions, an optimal policy is to invest a proportion of the current capital, regardless of the value of the capital achieved by the time of the investment decision. The proportion depends on the current state of the return process (and possibly on its history). As was said, in many cases the recommended proportion is 0 meaning no investment into a given trial.

This paper focuses on discrete-time models. A continuous-time version of our approach is currently under investigation and will be a topic of forthcoming publications. In this regard, we note an alternative approach propagated in [2] and, more recently, in [13]. An extensive bibliography of this field is available in [5].

We start with a basic discrete-time setting under Scheme I. An investor is betting on results $\varepsilon_n$ of subsequent random trials, $n = 0, 1, 2, \ldots$. Suppose that the $\varepsilon_n$ are generated by a Markov chain with a finite or countable state space $M$. The transition matrix is $P = (p(i,k), i,k \in M)$.

Let us introduce a return function $(i,k) \in M \times M \mapsto g(i,k)$ with real values. Then let us agree that if the investor stakes $c$ on the $n$th trial he/she wins $cg(i,k)$ if the result is $k$ preceded by the outcome $i$ of the $(n-1)$st trial (which you may know). (If $g(i,k) < 0$ you loose $c|g(i,k)|$.) When $g(i,k) = g(k)$, the return from the next trial is determined by the coming outcome regardless of the previous one(s). A more general setting, with a ‘long’ memory, is considered in the forthcoming sections. Let $Z_{n-1}$ be the investor’s fortune/capital after $n-1$ trials; we will assume that $Z_{n-1} > 0$ which will be justified below. ($Z_o > 0$ is an initial capital which can be dependent on $\varepsilon_0$, the initial-trial result.) Introduce the $\sigma$-algebras $\mathfrak{M}_0 = \sigma(Z_0, \varepsilon_0)$ and $\mathfrak{M}_n = \mathfrak{M}_0 \vee \sigma(\varepsilon_0)$, $n \geq 1$, and consider a sequence $\{C_n, n \geq 0\}$ where $C_n$ is $\mathfrak{M}_n$-measurable random variable (a predictable strategy): $C_n = C_n(Z_0, \varepsilon_0)$. Here and below, $\varepsilon_n = (\varepsilon_0, \ldots, \varepsilon_n)$ stands for a string of subsequent random variables (RVs) representing the first $(n+1)$ trial results. The dependence upon $Z_0$ will be omitted. The recursion for $Z_n$ is

$$Z_n = Z_{n-1} + C_{n-1}g(\varepsilon_{n-1}, \varepsilon_n) = Z_{n-1} \left[ 1 + \frac{C_{n-1}g(\varepsilon_{n-1}, \varepsilon_n)}{Z_{n-1}} \right];$$

it shows that $Z_n \in \mathfrak{M}_n$: $Z_n = Z_n(\varepsilon_n)$. For the return function $g$ we will use the acronym RF.

Next, let us consider another function, $(i,k) \mapsto \varphi(i,k) \geq 0$, representing a ‘utility’ value assigned to outcome $k$ when it succeeds outcome $i$. If $\varphi(i,k) \equiv 1$, all outcomes are treated entirely in terms of their returns, and if $\varphi(i,k)$ does not depend on $i$, the value does not take into account the history. We say that $\varphi$ is a weight function (WF); including one-step history $i$ agrees with the Markovian assumption for $\varepsilon_n$.

It will be always assumed that the WF and the RF are not identically 0; this includes the modified models below with more than one asset (where we deal with an RF vector $g$).
We wish to maximize, in \( C_0, \ldots, C_{n-1} \), the mean value \( \mathbb{E}S_n \) where
\[
S_n := \sum_{j=1}^{n} \varphi(\varepsilon_{j-1}, \varepsilon_j) \ln \frac{Z_j}{Z_{j-1}},
\]
and determine, when possible, a sequence of optimal strategies \( \{C_j^O\} \), within ‘natural’ classes \( \{C_j\} \) of predictable strategies defined by recursive inequalities (3) below:
\[
(C_0^O, \ldots, C_{n-1}^O) = \text{argmax} \left[ \mathbb{E}S_n : C_j \in C_j, \ 0 \leq j \leq n-1 \right].
\]
The classes \( C_j \) are described through conditions (a0)--(a2) or (a0)--(a3) listed in Eqn (6) below. Under our assumptions, the optimum is at a proportional betting, where \( C_j^O = D_{j-1}^O(\varepsilon_{j-1})Z_{j-1} \). Here \( Z_{j-1} \) is the capital after \( j-1 \) trials and \( D_{j-1}^O(i) \) is the proportionality coefficient indicating the fraction of the capital to be invested into the \( j \)th trial.
Quantity \( S_n/n \) can be considered as a weighted log-capital rate after \( n \) trials. When \( \varphi(i, k) \equiv 1 \), the sum in (2) becomes telescopic, and \( S_n \) equals \( \ln \frac{Z_n}{Z_0} \) (a standard quantity in financial calculations, particularly in relation to the Kelly-type investments).

The form of summation in Eqn (2) suggests the use of the weighted Kullback–Leibler (KL) entropy of the row probability vector \( (p(i, k), k \in M) \) relative to chosen ‘calibrating’ functions \( (i, k) \in M \times M \mapsto q_j(i, k) \) with \( q_j(i, k) > 0 \), \( j = 0, 1, \ldots \). To this end, set:
\[
\alpha_j(i) = \sum_{l \in M} \varphi(i, l) p(i, l) \ln \frac{p(i, l)}{q_j(i, l)}, \ i \in M, \ j \geq 0.
\]
For the definition and basic properties of weighted entropies, see [14] and references therein. Some applications of weighted entropies are discussed in [15, 16, 17].

The sum in (4) and similar series below are supposed to converge absolutely.

The choice of calibrating functions (CFs) \( q_j(i, k) \) is a part of the optimization procedure and is discussed below: see (6) and (7). We consider the random process (RP) of the cumulative weighted KL entropy
\[
A_n = \sum_{j=1}^{n} \alpha_j-1(\varepsilon_{j-1}), \ \text{with} \ \mathbb{E}A_n = \sum_{j=1}^{n} \mathbb{E}\alpha_j-1(\varepsilon_{j-1}).
\]

Also fix a value \( b > 0 \) (a proportional ruin threshold).
Let us summarize conditions on the class of policies and involved functions: \( \forall \ j \geq 0, \)
\[
(a0) \ C_j \in \mathcal{M}_j, \ (a1) \ 0 \leq C_j < Z_j, \ (a2) \ 1 + \frac{C_j g(\varepsilon_j, \varepsilon_{j+1})}{Z_j} \geq b, \ \text{and} \ (a3) \ C_j(\varepsilon_j^0) \sum_{l \in M} \varphi(\varepsilon_j, l) q_{j+1}(\varepsilon_j, l) g(\varepsilon_j, l) = 0.
\]

Also, \( \forall \ i \in M, \ j \geq 0, \) we assume the condition labelled as \( (q-p) \) in Eqn (7):
\[
(q - p) \sum_{l \in M} \varphi(i, l) [q_j(i, l) - p(i, l)] \leq 0.
\]

**Remark 1.1.** Recall, \( C_j = 0 \) means no investment into the result of the \( (j + 1) \)st trial. Note that \( C_j \equiv 0 \) is always a feasible choice: it yields \( S_{j+1} = S_j \). In some situations it gives an optimum (when the outlook of the results is not favorable for the investor). On the other hand, if we manage to verify that, \( \forall \ j, \) the sum \( \sum_{l \in M} \varphi(\varepsilon_j, l) q_{j+1}(\varepsilon_j, l) g(\varepsilon_j, l) = 0 \) then (a3) provides no additional restriction upon \( C_j \) and can be discarded. It means that the optimality can be achieved within the larger class of strategies satisfying (a0)--(a2) only. (Still, the optimum may be \( C_j = 0 \).) This aspect of the theory will be repeatedly stressed in various models and examples below.

We offer the following result.
Theorem 1.1. Suppose the recursion (1) holds true.

(a) Suppose a sequence of CFs $q_n$ is given, obeying (7). Take any sequence $\{C_n, n \geq 0\}$ of random variables (RVs) $C_n$ satisfying properties (a0)–(a3) in Eqn (3). Consider RVs $S_n$ and $A_n$ defined in (4) and (5). Then the sequence of differences $\{S_n - A_n, n \geq 1\}$ is a supermartingale; consequently, $E S_n \leq E A_n$ $\forall$ $n \geq 1$.

(b) To achieve equality $E S_n = E A_n$: the sequence $\{S_n - A_n\}$ is a martingale for a sequence of RVs $C_n$ satisfying (a0)–(a3) in (4) iff the following conditions (i)–(ii) hold.

(i) There exists a function $D : M \to \mathbb{R}$ such that, $\forall$ $i, k \in M$,

\begin{align*}
0 &\leq D(i) < 1, \quad (i2) \quad 1 + D(i)g(i, k) \geq b, \quad (i3) \quad D(i) \sum_{l \in M} \frac{p(i, l)\varphi(i, l)g(i, l)}{1 + D(i)g(i, l)} = 0, \\
\text{i.e., either (i3A)} &\quad \sum_{l \in M} \frac{p(i, l)\varphi(i, l)g(i, l)}{1 + D(i)g(i, l)} = 0 \text{ or (i3B)} D(i) = 0, \quad \text{and}
\end{align*}

\begin{align*}
(i4) &\quad \text{the CFs} \quad q_j \text{ are of the form} \quad q_j(i, k) = \frac{p(i, k)}{1 + D(i)g(i, k)}, \quad \text{independently of} \\
&\quad \text{in Eqn (10). Consider RVs} \quad S_n \text{ satisfying properties (a0)–(a3) in (4) iff, in addition to (i3), we have that} \\
&\quad D(i) = 0 \quad \text{or} \quad \sum_{l \in M} \frac{p(i, l)g(i, l)}{1 + D(i)g(i, l)} = 0, \quad i \in M. \quad (9)
\end{align*}

(c) Define the map $i \in M \mapsto D^O(i)$ as follows. Given $i$, consider Eqn (i3A): it has at most one solution $D(i) > 0$. If (i3A) has a solution $D(i) > 0$ obeying conditions (i1)–(i2), set $D^O(i) = D(i)$; otherwise $D^O(i) = 0$. Then the policy $C_{n-1}^O = D^O(\varepsilon_{n-1})Z_{n-1}$ yields the following value $E_n$ for the expectation $E S_n$:

\begin{align*}
E_n &\quad = \sum_{j=1}^{n} \beta_{j-1} \quad \text{where} \quad \beta_{j-1} = E \{\varphi(\varepsilon_{j-1}, \varepsilon_j) \ln \left[1 + D^O(\varepsilon_{j-1})g(\varepsilon_{j-1}, \varepsilon_j)\right]\}.
\end{align*}

Moreover, the value $E_n$ gives the maximum of $E S_n$ over all strategies satisfying conditions (a0)–(a3) in (4).

(d) Suppose that the map $i \in M \mapsto D^O(i)$ from assertion (c) is such that $D^O(i) > 0$ (so the alternative (i3A) holds), $\forall$ $i \in M$. Then the policy $C_{n-1}^O = D^O(\varepsilon_{n-1})Z_{n-1}$ maximises each summand $\beta_{j-1}$ in (10), and therefore yields the maximum of the whole sum $E S_n$, among strategies satisfying properties (a0)–(a2) in Eqn (4).

Proof. (a) Write:

\begin{align*}
E \{S_n - A_n)\} | W_{n-1} \} &\quad = S_n - A_{n-1} \\
+ E \left\{\varphi(\varepsilon_{n-1}, \varepsilon_n) \ln \left[1 + \frac{C_{n-1}(\varepsilon_{n-1}, \varepsilon_n)}{Z_{n-1}}\right]\right\} | W_{n-1} \} - \alpha_{n-1}(\varepsilon_{n-1}).
\end{align*}
Next, represent
\[
\begin{align*}
\mathbb{E} \left\{ \varphi(\varepsilon_{n-1}, \varepsilon_n) \ln \left( 1 + \frac{C_{n-1} g(\varepsilon_{n-1}, \varepsilon_n)}{Z_{n-1}} \right) \right\} n_{n-1} - \alpha_{n-1} \\
= \sum_{l \in M} \varphi_n(\varepsilon_{n-1}, l) p(\varepsilon_{n-1}, l) \ln \left( 1 + \frac{C_{n-1} g(\varepsilon_{n-1}, l)}{Z_{n-1}} \right) \\
- \sum_{l \in M} \varphi_n(\varepsilon_{n-1}, l) p(\varepsilon_{n-1}, l) \ln \frac{p(\varepsilon_{n-1}, l)}{q_n(\varepsilon_{n-1}, l)} \\
= \sum_{l \in M} \varphi_n(\varepsilon_{n-1}, l) p(\varepsilon_{n-1}, l) \ln \left( 1 + \frac{C_{n-1} g(\varepsilon_{n-1}, l)/Z_{n-1}}{p(\varepsilon_{n-1}, l)/q_n(\varepsilon_{n-1}, l)} \right) \\
= \sum_{l \in M} \varphi_n(\varepsilon_{n-1}, l) p(\varepsilon_{n-1}, l) \ln \frac{h_n(\varepsilon_{n-1}, l)}{p(\varepsilon_{n-1}, l)} \\
\leq \sum_{l \in M} \varphi_n(\varepsilon_{n-1}, l) p(\varepsilon_{n-1}, l) \left[ \frac{h_n(\varepsilon_{n-1}, l)}{p(\varepsilon_{n-1}, l)} - 1 \right] 1(p(\varepsilon_{n-1}, l) > 0) \\
= \sum_{l \in M} \varphi_n(\varepsilon_{n-1}, l) \left[ h_n(\varepsilon_{n-1}, l) - p(\varepsilon_{n-1}, l) \right] \leq 0.
\end{align*}
\]

Here \( h_n(\varepsilon_{n-1}, k) := q_n(\varepsilon_{n-1}, k) \left[ 1 + \frac{C_{n-1} g(\varepsilon_{n-1}, k)}{Z_{n-1}} \right] \), \( k \in M \).

The final inequality in (11) holds since
\[
\sum_{l \in M} \varphi(\varepsilon_{n-1}, l) h_n(\varepsilon_{n-1}, l) = \sum_{l \in M} \varphi(\varepsilon_{n-1}, l) q_n(\varepsilon_{n-1}, l) + \frac{C_{n-1}}{Z_{n-1}} \sum_{l \in M} \varphi(\varepsilon_{n-1}, l) q_n(\varepsilon_{n-1}, l) g(\varepsilon_{n-1}, l) \leq \sum_{l \in M} \varphi(\varepsilon_{n-1}, l) p(\varepsilon_{n-1}, l),
\]

because of property (a3) in Eqn (8) and (q–p) in Eqn (7).

(b) To guarantee the martingale property, we have to reach equality in the inequalities in (11). The first inequality becomes equality iff \( h_n = p \) which yields (14) in Eqn (8). The second inequality (i.e., the inequality in (12)) gives equality iff \( I \) the relation (a3) from (6) holds true (for \( C_n(\varepsilon_n) = D(\varepsilon_n)Z_n \)), and (II) the bound (q–p) in Eqn (7) becomes equality. After the substitution from (14), both properties (I) and (II) are equivalent to (33) from Eqn (8). The inequalities (11) and (12) in Eqn (8) are the same as (a1) and (a2) in Eqn (8).

The deduction of Eqn (9) is direct, from the above considerations.

(c) The proof of this assertion is straightforward.

(d) Assuming, for a map \( i \in M \to D(i) \), the alternative (i3A) in (8) means that, with CFS q_n defined by (14), ∀ sequence of policies \( C_n \) satisfying (a6)–(a2) we also have property (a3) and therefore the supermartingale inequality \( \mathbb{E} S_n \leq \mathbb{E} A_n \), \( n \geq 1 \). Setting \( C_n = D(\varepsilon_{n-1})Z_n \) yields a martingale equation \( \mathbb{E} S_n = E_n \), where \( E_n \) is as in (10).

Consider the optimization problems emerging from Eqn (10):

maximise the objective function \( \beta_{j-1} := \mathbb{E} \left\{ \varphi(\varepsilon_{j-1}, \varepsilon_j) \ln \left[ 1 + d(\varepsilon_{j-1})g(\varepsilon_{j-1}, \varepsilon_j) \right] \right\} \)

in variables \( d(i) \in \mathcal{I}(i), i \in M \),

where the feasibility interval \( \mathcal{I}(i) = \left\{ \tilde{d} \in [0, 1] : 1 + \tilde{d}g(i, k) \geq b \ \forall \ k \in M \right\} \).

Here \( j = 1, \ldots, n \) (which corresponds with the summand \( \beta_{j-1} \) in (10)).

Observe that the Hessian matrix \( H(\beta_{j-1}) = \left( \partial^2 \beta_{j-1} / \partial d(i) \partial d(i') \right) \) is non-positive definite:

\[
H(\beta_{j-1}) = -\text{diag} \left( \mathbb{E} \left\{ 1(\varepsilon_{j-1} = i) \varphi(i, \varepsilon_j) \left[g(i, \varepsilon_j)\right]^2 \left[1 + d(i)g(i, \varepsilon_j) \right]^2 \right\}, i \in M \right).
\]

It implies that each \( \beta_{j-1} \) is a concave function in \( d(i) \in \mathcal{I}(i), i \in M \). The values \( d(i) = D(i), i \in M \), give a zero of the gradient vectors \( \nabla \beta_{j-1} \) of functions \( \beta_{j-1} \), hence the maximum for the terms \( \beta_{j-1} \) from
Equation (11). If, in addition, \( D(i) \in I(i) \) \( \forall i \in M \) (which is what we assume in this statement) then \( \forall n \geq 1 \) the sum \( \mathbb{E}A_n \) attains the maximum value among the sequences \( \{C_n\} \) satisfying (a0)–(a2).

**Remarks. 1.2.** Terminologically, the requirement (q–p) in Equation (7) is referred to as a weighted q,p-dominance condition. As was stressed earlier, condition (a0) in (10) means predictability. The inequalities (a1) and (a2) are called, respectively, a sustainability condition and a no-ruin condition. (In fact, the bound (a1) is not used in the proof of statement (a), only (a2).) The relation (a3) is called the (a1,a2)–balance condition. In Equations (8): (i1) is called a D-sustainability condition, and (i2) a D-no-ruin condition. (In particular, the version (i3A)) is, arguably, the most serious restriction from the point of applications; in all examples under consideration it is a condition that specifies the optimum. Computationally, it requires solving (a system of) equations involving rational functions. See examples below.

**1.4.** The return function \( g \) (and, in fact, the weight function \( \varphi \)) can also be a part of an optimization procedure, with obvious (although tedious) changes in assertions (c,d) of Theorem 1.1.

**1.5.** As can be seen from the proof, a key role in Theorem 1.1(a) is played by the Gibbs inequality 1.5.

**Example 1.1: IID trials.** In the case of IID trials, \( p(i,k) = p(k) \) does not depend on \( i \). Assume that \( g(i,k) = g(k) \), \( \varphi(i,k) = \varphi(k) \) and \( q_j(i,k) = q(k) \). Also choose \( b \in (0,1) \). We obtain the following form of relations (3) and (8):

\[
\forall j \geq 0 : \quad (a0) \ C_j \in W_j, \quad (a1) \ 0 \leq C_j < Z_n, \quad (a2) \ 1 + \frac{C_j(\varepsilon_0^j)g(\varepsilon_{j+1}^j)}{Z_j(\varepsilon_0^j)} \geq b, \\
\quad \text{and} \quad (a3) \text{ if the sum } \sum_{l \in M} \varphi(l)q(l)g(l) \neq 0 \text{ then } C_j(\varepsilon_0^j) = 0; \\
(q-p) \sum_{l \in M} \varphi(l)[q(l) - p(l)] \leq 0; \\
\text{and} \\
(i) \exists \ (i1) \text{ a constant } D \in [0,1] \text{ such that } (i2) \ 1 + Dg(k) \geq b, \quad k \in M, \\
(i3) \ D \sum_{l \in M} \frac{\varphi(l)g(l)p(l)}{1 + Dg(l)} = 0, \quad \text{and} \quad (i4) \ q(k) = \frac{p(k)}{1 + Dg(k)}, \quad k \in M, \\
\text{with the alternatives } (i3A) \sum_{l \in M} \varphi(l)g(l)p(l) = 0 \quad \text{and } (i3B) \ D = 0.
\]

In the IID case, the weighted KL entropy \( \alpha(i) = \alpha \) does not depend on \( i \): \( \alpha = \sum_{l \in M} \varphi(l)p(l) \ln \frac{p(l)}{q(l)} \), see (4). Consequently, if \( \mu \leq 0 \) for some CF \( q \) satisfying (q–p) in (14) then the risk-averse trader would restrain from investments.

Define

\[
D^0 = \begin{cases} 
D, & \text{if } D \in (0,1) \text{ is the solution to (i3A) which satisfies (i2),} \\
0, & \text{if } D = 0 \text{ is the only solution to (i3) satisfying (i1)–(i2).}
\end{cases}
\]

According to Theorem 1.1(c), the policies \( C_j^0 = D^0Z_j, j \geq 0 \), give the maximum of \( \mathbb{E}S_n, n \geq 1 \), among the strategies satisfying (a0)–(a3). Moreover, according to Theorem 1.1(d), if \( D^0 > 0 \) then the policies \( C_j^0 = D^0Z_j \) maximise \( \mathbb{E}S_n \) over the strategies satisfying (a0)–(a2). The maximal value in both cases is

\[
E_n = n \sum_{l \in M} \varphi(l)p(l) \ln [1 + D^0g(l)] \geq 0.
\]
To illustrate further, take the case $M = \{0, 1\}$ (two outcomes), with $k = 0, 1$ and probabilities $p(0), p(1)$. Here the martingale property occurs iff we can find a constant $D$ such that

\[(i)\, D \in [0, 1], \quad (ii)\, 1 + Dg(k) \geq b, \quad (iii)\, D \left[ \frac{\varphi(1)g(1)p(1)}{1 + Dg(1)} + \frac{\varphi(0)g(0)p(0)}{1 + Dg(0)} \right] = 0.\]

Next, Eqn (iii) is solved by

\[D = \frac{-\varphi(1)g(1)p(1) + \varphi(0)g(0)p(0)}{g(0)g(1)\varphi(1)p(1) + \varphi(0)p(0)}, \quad \text{which should obey (i), (ii).}\]

To simplify, take $g(1) = -g(0) = \gamma > 0$. Also, set $\varphi(1) = \varphi(0) = 1$ (no preference). A general form of the WF can be easily incorporated in the argument, which will be also true in all examples below. Then the optimal policy is $C^O_n = D^O Z_n$ where

\[D^O = \begin{cases} \begin{array}{ll}
\frac{p(1) - p(0)}{\gamma}, & \text{if } \frac{b}{2} \leq p(0) \leq p(1) \text{ and } \gamma \geq p(1) - p(0), \\
0, & \text{otherwise}.
\end{array} \end{cases} \]  

(17)

Formula (17) defines a typical Kelly investment scheme $\mathbb{M}$. The maximal growth rate for $\mathbb{E} S_n$ takes the form

\[E_n = n1 \left( \frac{b}{2} \leq p(0) \leq p(1) \leq \gamma + p(0) \right) \{ p(1) \ln [1 + p(1) - p(0)] + p(0) \ln [1 - p(1) + p(0)] \}. \]

If $D^O > 0$, the optimality holds over the strategies $\{C_j\}$ with properties (a1,2) $0 \leq C_j \leq (1 - b) Z_j$, $\forall j \geq 0$. If $D^O = 0$ then the optimality holds among the strategies satisfying, in addition, the property (a3) that $C_j(\varepsilon_0^j)[p(1) - p(0)] = 0$. (Note that (a3) yields no restriction when, for instance, $p(1) = p(0) = 1/2$.) More generally, if $g(1) = \gamma_1 > 0, g(0) = -\gamma_2 < 0$ then in the top line of (17) we obtain

\[D^O = \frac{p(1)}{\gamma_2} - \frac{p(0)}{\gamma_1}, \quad \text{if } \frac{b}{2} \leq p(0) \leq p(1) \text{ and } 1 - p(1) + \frac{\gamma_2}{\gamma_1}p(0) \geq b, \]

with obvious changes in the maximal growth rate.

Example 1.2: A two-state Markov chain. In the Markov case, when the trader observes the current state $i$, he/she uses the similar optimization procedure for the $i$th row of the $2 \times 2$ transition matrix $P = (p(i, j))$. Again suppose for simplicity that $M = \{0, 1\}$, the WF $\varphi(i, j) = 1$ and the RF $g$ has $g(1) = -g(0) = \gamma > 0$. Also suppose that $b \in (0, 1)$ is given. Then an analog of the previous picture emerges. Namely, set, for $i = 1, 0$,

\[D^O(i) = \begin{cases} \begin{array}{ll}
\frac{p(i, 1) - p(i, 0)}{\gamma}, & \text{if } \frac{b}{2} \leq p(i, 0) \leq p(i, 1), \text{ and } \gamma \geq p(i, 1) - p(i, 0), \\
0, & \text{otherwise}.
\end{array} \end{cases} \]  

(18)

The policy $C^O_n = D(\varepsilon_n)Z_n(\varepsilon_n^0)$ is optimal, under similar constraints. That is, if $D^O(i) > 0$ for both $i = 0, 1$ then the maximum is attained over strategies satisfying (a0) $C_j \in \mathbb{M}_j$ and (a1,2) $0 \leq C_j \leq (1 - b) Z_j$, $\forall j \geq 0$. Otherwise, if $D^O(i) = 0$ for some $i$ then it is among the strategies obeying (a0)–(a1,2) plus property (a3): $C_j(\varepsilon_0^j)[p(\varepsilon_n, 1) - p(\varepsilon_n, 0)] = 0 \, \forall \, j \geq 0$.

Viz., assume that the Markov chain (MC) is in the stationary regime, with stationary probabilities $\pi(1), \pi(0)$. Then the maximal growth of $\mathbb{E} S_n$ is

\[E_n = \pi(1)1 \left( \frac{b}{2} \leq p(1, 0) \leq p(1, 1) \leq \gamma + p(1, 0) \right) \times \left\{ p(1, 1) \ln [1 + p(1, 1) - p(1, 0)] + p(1, 0) \ln [1 - p(1, 1) + p(1, 0)] \right\} \times \pi(0)1 \left( \frac{b}{2} \leq p(0, 0) \leq p(0, 1) \leq \gamma + p(0, 0) \right) \times \left\{ p(0, 1) \ln [1 + p(0, 1) - p(0, 0)] + p(0, 0) \ln [1 - p(0, 1) + p(0, 0)] \right\} \times \pi(0)1 \left( \frac{b}{2} \leq p(0, 0) \leq p(0, 1) \leq \gamma + p(0, 0) \right) \times \left\{ p(0, 1) \ln [1 + p(0, 1) - p(0, 0)] + p(0, 0) \ln [1 - p(0, 1) + p(0, 0)] \right\}. \]
The last observation can be extended to general MCs. Indeed, suppose the trial MC starts with an invariant distribution \((\pi(i), i \in M)\). In this case the statements \((c,d)\) of Theorem 1.1 assert that the maximum for \(ES_n\) is given as \(n \sum_{i,k \in M} \pi_i p(i,k) \varphi(i,k) \ln \left[ 1 + D^0(i)g(i,k) \right] \). Note that we do not need assumptions of irreducibility or aperiodicity: the invariant distribution is not assumed to be unique.

## 2 Markov trials with a riskless asset

In this section we switch from Scheme I to II. Consider the situation where the trials affect several assets/returns, say, two, described by RFs \(g(0)\) and \(q\). Assume that the return \(g(0)\) is riskless: \(g^{(0)}(i,k) = 1 + \rho\) where \(\rho > 0\) represents the interest rate, while the other asset, with a RF \(g\), is risky. As above, suppose that the trial results \(\varepsilon_0, \varepsilon_1, \ldots\) form an MC with a finite/countable state space \(M\) and a transition probability matrix \(P = (p_{ik} = p(i,k), i, k \in M)\). Fix a WF \((i,k) \mapsto \varphi(i,k) \geq 0\), a CF \((i,k) \mapsto q_j(i,k) > 0\) and a no-ruin parameter value \(b > 0\). Set:

\[
g^*(i,k) = g(i,k) - (1 + \rho), \quad i, k \in M. \quad (19)
\]

We make a convention that, at times \(n = 0, 1, \ldots\), a part \(C_n\) of the current capital \(Z_n\) is invested in the riskless asset. It means that, compared with \((1)\), we have the recursion: for \(n+1\),

\[
Z_n = (1 + \rho)(Z_{n-1} - C_{n-1}) + C_{n-1}g(\varepsilon_{n-1}, \varepsilon_n) = Z_{n-1} \left[ 1 + \rho + \frac{C_{n-1}}{Z_{n-1}} g^*(\varepsilon_{n-1}, \varepsilon_n) \right]. \quad (20)
\]

Next, for given \(b > 0\) and CFs \((i,k) \mapsto q_j(i,k) > 0\), we consider strategies/policies \(C_j, j \geq 0\), such that

\[
(a0) \quad C_j \in \mathcal{W}_j, \quad (a1) \quad 0 \leq C_j < Z_j, \quad (a2) \quad 1 + \rho + \frac{C_j g^*(\varepsilon_j, \varepsilon_{j+1})}{Z_j} \geq b,
\]

and \((a3) \quad C_j(\varepsilon_{j,l}) \sum_{l \in M} \varphi(\varepsilon_j, l)q_{j+1}(\varepsilon_j, l)g^*(\varepsilon_j, l) = 0. \quad (21)\]

Following the same pattern as before, set:

\[
S_n = \sum_{j=1}^{n} \varphi(\varepsilon_{j-1}, \varepsilon_j) \ln \frac{Z_j}{Z_{j-1}} = \sum_{j=1}^{n} \varphi(\varepsilon_{j-1}, \varepsilon_j) \ln \left[ 1 + \rho + \frac{C_{j-1} g^*(\varepsilon_{j-1}, \varepsilon_j)}{Z_{j-1}} \right].
\]

and

\[
A_n = \sum_{j=1}^{n} \alpha_{j-1}(\varepsilon_{j-1}) \quad \text{where} \quad \alpha_{j-1}(\varepsilon_{j-1}) = \mathbb{E} \left[ \varphi(\varepsilon_{j-1}, \varepsilon_j) \ln \frac{p(\varepsilon_{j-1}, \varepsilon_j)}{q_{j}(\varepsilon_{j-1}, \varepsilon_j)} \right] \mathcal{W}_{j-1}.
\]

Again we are interested in maximizing the mean value \(\mathbb{E}S_n\) in \(C_0, \ldots, C_{n-1}\). Assume the condition that, \(\forall \ i \in M, j \geq 0,\)

\[
(q - p) \sum_{l \in M} \varphi(i,l) [(1 + \rho)q_j(i,l) - p(i,l)] \leq 0. \quad (22)
\]

Assumptions \((21)\) mimic \((0)\) while \((22)\) mimics \((7)\); terminological parallels are also notable here.

**Theorem 2.1.** Adopt Eqns \((19) - (22)\). Then:

(a) The sequence \(\{S_n - A_n, n \geq 1\}\) is a supermartingale and hence \(\mathbb{E}S_n \leq \mathbb{E}A_n \quad \forall \ n \geq 1\).

(b) To achieve equality \(\mathbb{E}S_n = \mathbb{E}A_n\): the sequence \(\{S_n - A_n\}\) is a martingale for a sequence of RVs \(C_j\) satisfying \((21)\) iff the following conditions \((i)-(ii)\) are fulfilled.
(i) There exists a function \( D : M \to \mathbb{R} \) such that, \( \forall \; i, k \in M \),

\[ \begin{align*}
(1) & \quad 0 \leq D(i) < 1, \quad (2) \quad 1 + \rho + D(i)g^*(i, k) \geq b, \\
(3) & \quad D(i) \sum_{l \in M} \frac{\varphi(i, l)p(i, l)g^*(i, l)}{1 + \rho + D(i)g^*(i, l)} = 0, \; \text{i.e., either} \\
(3A) & \quad D(i) \sum_{l \in M} \frac{\varphi(i, l)p(i, l)g^*(i, l)}{1 + \rho + D(i)g^*(i, l)} = 0 \quad \text{or} \quad (3B) \quad D(i) = 0, \quad \text{and} \\
(4) & \quad \text{the CFSs } q_j \text{ have the form } q_j(i, k) = \frac{p(i, k)}{1 + \rho + D(i)g^*(i, k)}, \\
\text{independently of } j \geq 0.
\end{align*} \]

(ii) \( \forall \; n \geq 1 \), the policy \( C_{n-1} \) produces a proportional investment portfolio: after the \((n-1)\)st trial the amount \( C_{n-1}(\varepsilon_n^{n-1}) = D(\varepsilon_n)Z_{n-1} \) goes to the asset with return \( g \) and \( [1 - D(\varepsilon_n)]Z_{n-1} \) to the riskless return.

(c) Define the map \( i \in M \mapsto D^O(i) \) as follows. Given \( i \), consider Eqn (3A); it has at most one solution \( D(i) > 0 \). If (3A) has a solution \( D(i) > 0 \) obeying conditions (i1)–(i2), set \( D^O(i) = D(i) \); otherwise \( D^O(i) = 0 \). Then the policy \( C^O_{n-1} = D^O(\varepsilon_n)Z_{n-1} \) yields the following value \( E_n \) for the expectation \( ES_n \):

\[ E_n = \sum_{j=1}^{\infty} \beta_{j-1} \quad \text{where} \quad \beta_{j-1} = E\left\{ \varphi(\varepsilon_{j-1}, \varepsilon_j) \ln \left( 1 + \rho + D(\varepsilon_{j-1})g^*(\varepsilon_{j-1}, \varepsilon_j) \right) \right\}. \] (24)

The maximum of \( E_n \) in (24) over maps \( i \in M \mapsto D^O(i) \) satisfying (i1)–(i3) from (23) gives the maximum of \( ES_n \) over the portfolios \( \{C_i\} \) satisfying properties (a0)–(a3) in Eqn (21).

(d) Suppose that the map \( i \in M \mapsto D^O(i) \) from assertion (c) is such that \( D^O(i) > 0 \) (so the alternative (3A) holds) \( \forall \; i \in M \). Then the policy \( C^O_{n-1} = D^O(\varepsilon_n)Z_{n-1} \) maximizes each summand \( \beta_{j-1} \) in (10), and therefore yields the maximum of the whole sum \( ES_n \), among strategies satisfying properties (a0)–(a2) in (21).

**Proof.** (a) We follow the same pattern as in Theorem 1.1(a). Write:

\[ \mathbb{E}\left\{ (S_n - A_n) | \mathfrak{M}_{n-1} \right\} = S_{n-1} - A_{n-1} + \mathbb{E}\left\{ \left[ \varphi(\varepsilon_{n-1}, \varepsilon_n) \ln \left( 1 + \rho + \frac{C_{n-1}g^*(\varepsilon_{n-1}, \varepsilon_n)}{Z_{n-1}} \right) \right] | \mathfrak{M}_{n-1} \right\} - \alpha_{n-1}(\varepsilon_{n-1}). \]

Next, represent

\[ \begin{align*}
\mathbb{E}\left\{ \left[ \varphi(\varepsilon_{n-1}, \varepsilon_n) \ln \left( 1 + \rho + \frac{C_{n-1}g^*(\varepsilon_{n-1}, \varepsilon_n)}{Z_{n-1}} \right) \right] | \mathfrak{M}_{n-1} \right\} &= \sum_{l \in M} \varphi(\varepsilon_{n-1}, l) p(\varepsilon_{n-1}, l) \ln \left( 1 + \rho + \frac{C_{n-1}g^*(\varepsilon_{n-1}, l)}{Z_{n-1}} \right) \\
&\quad - \sum_{l \in M} \varphi(\varepsilon_{n-1}, l)p(\varepsilon_{n-1}, l) \ln \left( \frac{p(\varepsilon_{n-1}, l)}{q_n(\varepsilon_{n-1}, l)} \right) \\
&= \sum_{l \in M} \varphi(\varepsilon_{n-1}, l)p(\varepsilon_{n-1}, l) \ln \left( 1 + \rho + \frac{C_{n-1}g^*(\varepsilon_{n-1}, l)}{Z_{n-1}} \right) \frac{h_n(\varepsilon_{n-1}, l)}{p(\varepsilon_{n-1}, l)} \\
&\quad - \left[ \sum_{l \in M} \varphi(\varepsilon_{n-1}, l)p(\varepsilon_{n-1}, l) \left[ \frac{h_n(\varepsilon_{n-1}, l)}{p(\varepsilon_{n-1}, l)} - 1 \right] \right] 1(p(\varepsilon_{n-1}, l) > 0) \\
&\quad \geq \sum_{l \in M} \varphi(\varepsilon_{n-1}, l) \left[ h_n(\varepsilon_{n-1}, l) \right] - \rho_{n-1}(\varepsilon_{n-1}, l) \leq 0.
\end{align*} \] (25)

Here \( h_n(\varepsilon_{n-1}, k) := q_n(\varepsilon_{n-1}, k) \left[ 1 + \rho + \frac{C_{n-1}g^*(\varepsilon_{n-1}, k)}{Z_{n-1}} \right], \; k \in M. \)
The final inequality in (25) holds since
\[
\sum_{l \in M} \varphi(\varepsilon_{n-1}, l) b_n(\varepsilon_{n-1}, l) = (1 + \rho) \sum_{l \in M} \varphi(\varepsilon_{n-1}, l) q_n(\varepsilon_{n-1}, l)
\]
\[+ \frac{C_{n-1}}{Z_{n-1}} \sum_{l \in M} \varphi(\varepsilon_{n-1}, l) q_n(\varepsilon_{n-1}, l) g^*(\varepsilon_{n-1}, l) \leq \sum_{l \in M} \varphi(\varepsilon_{n-1}, l) p(\varepsilon_{n-1}, l),
\]
(26)
because of properties (a3) in Eqn (21) and (q–p) in Eqn (22).
(b) As before, the martingale property emerges when we have equality in the inequalities in (25). The analysis of these situations proceeds as in the proof of Theorem 1.1(b).
(c,d). The proof of these assertions does not differ from the proof of their counterparts in Theorem 1.1. □

Example 2.1: IID trials. Consider IID trials in the context of Scheme II. As in Example 1.1, we now have \(p(i, k) = p(k)\) and work with \(g(i, k) = g(k)\), \(\varphi(i, k) = \varphi(k)\) and \(q_j(i, k) = q(k)\). Choose \(b \in (0, 1 + \rho)\). As in (11), set: \(g^*(k) = g(k) - (1 + \rho)\). Eqns (20) and (21) read:
\[
Z_n = (1 + \rho)(Z_{n-1} - C_{n-1}) + C_{n-1}g(\varepsilon_{n-1}) = Z_{n-1} \left[ 1 + \rho + \frac{C_{n-1}g^*(\varepsilon_{n-1})}{Z_{n-1}} \right]
\]
where
(a0) \(C_{n-1} \in \mathcal{W}_{n-1}\), (a1) \(0 \leq C_{n-1} < Z_{n-1}\), (a2) \(1 + \rho + \frac{C_{n-1}g^*(\varepsilon_{n-1})}{Z_{n-1}} \geq b\), and
\[
C_{n-1} \sum_{l \in M} \varphi(l)q(l)g^*(l) = 0 \quad \text{(meaning that at least one factor is 0)}.
\]

We now define:
\[
S_n = \frac{n}{\sum_{j=1}^n \varphi(\varepsilon_j) \ln \frac{Z_j}{Z_{j-1}}} = \sum_{j=1}^n \varphi(\varepsilon_j) \ln \left[ 1 + \rho + \frac{C_jg^*(\varepsilon_j)}{Z_j} \right]
\]
and
\[
A_n = n\alpha \quad \text{where} \quad \alpha = \mathbb{E} \left[ \varphi(\varepsilon_1) \ln \frac{p(\varepsilon_1)}{q(\varepsilon_1)} \right].
\]
Again we are interested in maximizing the mean value \(\mathbb{E}S_n\) in \(C_0, \ldots, C_{n-1}\). Eqns (22) and (28) take the form, respectively,
\[
(q - p) \sum_{l \in M} \varphi(l)[(1 + \rho)q(l) - p(l)] \leq 0, \quad \text{and}
\]
(i1) \(\exists\) a constant \(D \in [0, 1]\) such that \(i2\) \(1 + \rho + Dg^*(k) \geq b\),
\[
(i3) D \sum_{l \in M} \frac{\varphi(l)p(l)g^*(l)}{1 + \rho + Dg^*(l)} = 0, \quad \text{and} \quad (i4) \quad \text{the CF is} \quad q(k) = \frac{p(k)}{1 + \rho + Dg^*(k)}, \quad k \in M.
\]
The expression for \(\mathbb{E}S_n\) in Eqn (24) is
\[
E_n = n\beta \quad \text{where} \quad \beta = \mathbb{E} \left\{ \varphi(\varepsilon_1) \ln \left[ 1 + \rho + Dg^*(\varepsilon_1) \right] \right\},
\]
which should be maximized over all choices of \(D\) satisfying (i1)–(i3).
As before, to make the conditions more explicit, we take \(M = \{0, 1\}\). To simplify, set \(g(1) = -g(0) = \gamma > 0\) and \(\varphi(1) = \varphi(0) = 1\), with \(g^*(1) = \gamma - (1 + \rho)\), \(g^*(0) = -\gamma - (1 + \rho)\). Take \(0 < b \leq 1 + \rho\). Re-write Eqn (25):
\[
(q - p) \quad (1 + \rho)[q(1) + q(0)] \leq 1;
\]
(i1) \(\exists D \in [0, 1]\) such that \(i2\) \(D \leq \frac{1 + \rho - b}{\gamma + 1 + \rho}\),
\[
(i3) D \left[ \frac{p(1)(\gamma - 1 - \rho)}{1 + \rho + D(\gamma - 1 - \rho)} - \frac{p(0)\gamma + 1 + \rho}{1 + \rho - D(\gamma + 1 + \rho)} \right] = 0, \quad \text{and}
\]
(i4) the CF values are \(q(1) = \frac{p(1)}{1 + \rho + D(\gamma - 1 - \rho)}, \quad q(0) = \frac{p(0)}{1 + \rho - D(\gamma + 1 + \rho)}\).
Now, Eqn (33) in (20) can be solved explicitly:

\[ D = 0 \text{ or } D = D_0 \text{ where } D_0 = (1 + \rho) \frac{\gamma [p(1) - p(0)] - (1 + \rho)}{\gamma^2 - (1 + \rho)^2}. \]  

(30)

We see that the non-trivial solution \( D > 0 \) emerges if either (A) \( \gamma [p(1) - p(0)] > 1 + \rho \) or (B) \( \gamma < 1 + \rho \).

Next, taking \( q \) as in (i4) yields equality in (q–p) which is equivalent to (i3).

Thus, it remains to check the relations (i1)–(i2) for \( D \) specified in (20):

\[ 0 \leq (1 + \rho) \frac{\gamma [p(1) - p(0)] - (1 + \rho)}{\gamma^2 - (1 + \rho)^2} \leq 1 \land \frac{1 + \rho - b}{\gamma + 1 + \rho}. \]  

(31)

If, for given \( p(1), p(0), \gamma \) and \( \rho \), the inequalities in (31) are satisfied then we have two solutions for \( D \) specified in (20). Next, we have to compare \( \beta = \mathbb{E}_\varphi(z_1) \ln \left[ 1 + \rho + D_0 g(z_1) \right] \) for \( D = D_0 \) and \( \beta = \ln (1 + \rho) \mathbb{E}_\varphi(z_1) \) for \( D = 0 \): the larger solution identifies the optimizer \( D^0 \) giving a maximal value for \( \mathbb{E}_S_n \). Viz., such a comparison shows that \( D^0 = 0 \) in the situation (B) \( \gamma < 1 + \rho \), in which case the investor allocates all capital to the riskless asset.

3 Markov trials with several risky assets

Now assume that, within the remits of the investment Scheme I, we have to make a choice between \( K \) risky assets, with a vector of individual RFs \( g = (g^{(1)}, \ldots, g^{(K)}) \). As before, let the trial results \( \varepsilon_n \) be generated by an MC with states \( i, k \) from space \( M \), finite or countable, and with transition probabilities \( p(i, k) \). As before, the RF value \( g^{(s)}(i, k) \) gives the return from asset \( s \) at the time (or immediately after) the \( n \)th trial when the outcome is \( k \) preceded by outcome \( i \) at the \((n-1)\)st trial. Here we consider a sequence \( \{C_n, n \geq 0\} \) where \( C_n = (C_n^{(1)}, \ldots, C_n^{(K)}) \) is an \( \mathbb{M}_{n} \)-measurable random \( K \)-dimensional vector (a predictable policy/strategy portfolio). The recursion for \( Z_n \) is similar to (1), with replacing scalar random variables by random vectors (RVs):

\[ Z_n = Z_{n-1} + C_{n-1} g^{(1)}(\varepsilon_{n-1}, \varepsilon_n) = Z_{n-1} \left[ 1 + \frac{C_{n-1} g^{(1)}(\varepsilon_{n-1}, \varepsilon_n)}{Z_{n-1}} \right], \quad n \geq 1. \]  

(32)

Here and below,

\[ C_{n-1} g^{(s)}(\varepsilon_{n-1}, \varepsilon_n) = \sum_{s=1}^{K} C_{n-1}^{(s)} g^{(s)}(\varepsilon_{n-1}, \varepsilon_n). \]

Also, \( |C_j| = \sum_{s=1}^{K} C_{n-1}^{(s)} = C_{n-1} \mathbb{1} \) where \( \mathbb{1} = (1, \ldots, 1) \), and we write \( C_{\varphi} \supseteq 0 \) if \( C_{\varphi}^{(s)} \geq 0 \quad \forall \ s \). A similar convention is in place for other expressions of an analogous structure.

As above, we set \( S_n := \sum_{j=1}^{n} \varphi(\varepsilon_{j-1}, \varepsilon_j) \ln \frac{Z_j}{Z_{j-1}} \) and aim at maximizing the mean value \( \mathbb{E} S_n \) in \( C_0, \ldots, C_{n-1} \), under certain restrictions. Fix \( b > 0 \) and re-write the conditions outlining the portfolio classes under consideration: \( \forall \ j \geq 0, \)

\[ (a0) \quad C_j \in \mathbb{M} \text{ (predictability), } \quad (a1) \quad C_j \geq 0, \quad |C_j| < Z_j \text{ (sustainability),} \]

\[ (a2) \quad 1 + \frac{C_j g^{(\varepsilon_j, \varepsilon_{j+1})}}{Z_j} \geq b \text{ (no ruin), and} \]

\[ (a3) \quad \sum_{l \in M} \varphi(\varepsilon_j, l) q_{j+1}(\varepsilon_j, l) \left[ C_j(g^{(l)}(\varepsilon_j, l)) \right] = 0 \text{ (weighted (q,g)-balance).} \]  

(33)

We also assume, \( \forall \ i \in M \), the (q–p) bound (7).

We again want to determine, when possible, a sequence of optimal strategies. For \( \alpha_j(i) \) and \( A_n \) we follow the definitions from Eqns (4) and (5).

It is instructive to observe that the vector case can be treated as scalar after we fix the fractions \( h^{(s)}_j := C_j^{(s)} / |C_j|, \ 1 \leq s \leq K \), and introduce the ‘weighted’ RFs: \( \mathcal{g}(i, k) = \sum_{s=1}^{K} h^{(s)}_j g^{(s)}(i, k) \). Such a view will be useful when one considers examples; see below.
Consider the following conditions (i1)–(i4) which are vector counterparts of their scalar predecessors from (33). For convenience, we use the same labelling system as above.

(i) There exists a map \( i \in M \mapsto D(i) \) where vector \( D(i) = (D^{(i)}(i), \ldots, D^{(K)}(i)) \) is such that \( \forall i, k \in M, \)

\[
\begin{align*}
(1) & \quad D(i) \geq 0, \text{ and } |D(i)| < 1 \quad \text{(D-sustainability)}, \\
(2) & \quad 1 + D(i) \cdot g(i, k) \geq b \quad \text{(D-no-ruin)}, \\
(3) & \quad \sum_{l \in M} \varphi(i, l) p(i, l) \frac{D(i) \cdot g(i, l)}{1 + D(i) \cdot g(i, l)} = 0 \quad \text{(WE (D,g)-balance), and} \\
(4) & \quad \text{the CFs } q_j \text{ are } q_j(i, k) = \frac{p(i, k)}{1 + D(i) \cdot g(i, k)}, \\
\end{align*}
\]

independently of \( j \geq 0 \) (q-representation).

Here the analog of the first alternative in (i3) is that \( D(i) \) satisfies a system of equations:

\[
\begin{align*}
(i3A) & \quad \sum_{l \in M} p(i, l) \varphi(i, l) g(s)(i, l) \frac{D(i) \cdot g(i, l)}{1 + D(i) \cdot g(i, l)} = 0 \\
& \quad \forall i \in M \text{ and } 1 \leq s \leq K \quad \text{(strong WE (D,g)-balance)}.
\end{align*}
\]

Cf. (i3A) in Eqn (33).

**Theorem 3.1.** Assume the above setting (32)–(34). The following assertions hold true.

(a) Take any sequence \( \{C_j, j \geq 0\} \) obeying (a0)–(a3) in (33). Then the sequence \( \{S_n - A_n, n \geq 1\} \) is a supermartingale; hence \( \mathbb{E} S_n \leq \mathbb{E} A_n \) \( \forall n \geq 1 \).

(b) To reach equality \( \mathbb{E} S_n = \sum_{j=1}^{n} \mathbb{E} a(\varepsilon_{j-1}) \): the sequence \( \{S_n - A_n\} \) is a martingale for a sequence of RVs \( \{C_j\} \), satisfying (a0)–(a3) iff the additional properties (i), (ii) below are fulfilled.

(i) There exists a map \( i \in M \mapsto D(i) \) where vector \( D(i) = (D^{(i)}(i), \ldots, D^{(K)}(i)) \) is such that \( \forall i, k \in M, \) properties (i1)–(i3) in (34) are fulfilled, and the CFs \( q_j \) are as in (i4).

(ii) The portfolio vectors \( C_j \) have components \( C_j(s)(\varepsilon_j) = D^{(s)}(\varepsilon_j) Z_j, 1 \leq s \leq K, j \geq 0 \). That is, the prespecified fractions of the capital value \( Z_j \) are invested in the available returns.

(c) Suppose there exists a map \( i \in M \mapsto D(i) = (D^{(i)}(i), \ldots, D^{(K)}(i)) \) fulfilling conditions (i1)–(i3) in Eqn (34). Let \( D \) stand for the array of values \( D^{(i)}(i), i \in M, 1 \leq s \leq K \), and define the quantity \( E_n = E_n(D) \) by

\[
E_n = \sum_{j=1}^{n} \beta_{j-1} \text{ where } \beta_{j-1} = \mathbb{E}\{\varphi(\varepsilon_{j-1}, \varepsilon_j) \ln \left[ 1 + D(\varepsilon_{j-1}) \cdot g(\varepsilon_{j-1}, \varepsilon_j) \right] \}. \quad (36)
\]

Consider the optimization problem

\[
\text{maximise } E_n(D) \text{ subject to (i1)–(i3)}. \quad (37)
\]

Let \( D^* = \arg \max E_n \) be a (possibly, non-unique) optimizer, and \( E_n^* = E_n(D^*) \) denote the optimal value for (37). Then \( E_n^* \) defines the maximum of the expectation \( \mathbb{E} S_n \) among all portfolios \( \{C_j\} \) satisfying the properties (a0)–(a3) in Eqn (33). The optimizer \( D^* \) written as a collection of vectors \( D^*(i), i \in M, \) yields a proportional investment portfolio where \( C_j(\varepsilon_j^n) = D^*(\varepsilon_j Z_j(\varepsilon_j)). \)

(d) Suppose there exists a map \( i \in M \mapsto D(i) \) fulfilling conditions (i1)–(i2) and (i3A) in Eqns (34) and (33), respectively. Then such a map is unique, and the proportional investment portfolio \( C^{D^*}_{n-1} = D^*(\varepsilon_{j-1}) Z_{j-1} \) maximises each summand \( \beta_{j-1} \) in (36). Therefore, it yields the maximum of the whole sum \( \mathbb{E} S_n \), among strategies satisfying properties (a0)–(a2) in Eqn (33).

**Proof.** (a) We just repeat the argument from the proof of Theorem 1.1(a) for the vector case. Here

\[
\mathbb{E}\{(S_n - A_n)|\mathcal{M}_{n-1}\} = S_{n-1} - A_{n-1} + \mathbb{E}\left\{\varphi(\varepsilon_{n-1}, \varepsilon_n) \ln \left( 1 + \frac{C_{n-1}(\varepsilon_{n-1}, \varepsilon_n)}{Z_{n-1}} \right) \right\}|\mathcal{M}_{n-1} = \alpha_{n-1}(\varepsilon_{n-1}),
\]

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and we write
\[
E \left\{ \left[ \varphi(\varepsilon_{n-1}, \varepsilon_n) \ln \left( 1 + \frac{C_{n-1}g(\varepsilon_{n-1}, \varepsilon_n)}{Z_{n-1}} \right) \right] \bigg| \mathcal{M}_{n-1} \right\} - \alpha_{n-1}
\]
\[
= \sum_{l \in M} \varphi_n(\varepsilon_{n-1}, l) p(\varepsilon_{n-1}, l) \ln \left( 1 + \frac{C_{n-1}g(\varepsilon_{n-1}, l)}{Z_{n-1}} \right)
\]
\[
- \sum_{l \in M} \varphi(\varepsilon_{n-1}, l) p(\varepsilon_{n-1}, l) \ln \frac{p(\varepsilon_{n-1}, l)}{q(\varepsilon_{n-1}, l)}
\]
\[
= \sum_{l \in M} \varphi(\varepsilon_{n-1}, l) p(\varepsilon_{n-1}, l) \left[ 1 + \frac{C_{n-1}g(\varepsilon_{n-1}, l)Z_{n-1}}{q(\varepsilon_{n-1}, l)} \right]
\]
\[
= \sum_{l \in M} \varphi(\varepsilon_{n-1}, l) p(\varepsilon_{n-1}, l) \left[ h_n(\varepsilon_{n-1}, l) \frac{1}{p(\varepsilon_{n-1}, l)} - 1 \right] \mathbf{1}(p(\varepsilon_{n-1}, l) > 0)
\]
\[
= \sum_{l \in M} \varphi(\varepsilon_{n-1}, l) \left[ h_n(\varepsilon_{n-1}, l) - p(\varepsilon_{n-1}, l) \right] \leq 0.
\]

Here \(h_n(\varepsilon_{n-1}, k) := q_n(\varepsilon_{n-1}, k) \left[ 1 + \frac{C_{n-1}g(\varepsilon_{n-1}, k)}{Z_{n-1}} \right], k \in M\). The final inequality in (38) holds since
\[
\sum_{l \in M} \varphi(\varepsilon_{n-1}, l) h_n(\varepsilon_{n-1}, l) = \sum_{l \in M} \varphi(\varepsilon_{n-1}, l) q_n(\varepsilon_{n-1}, l)
\]
\[
\quad + \frac{1}{Z_{n-1}} \sum_{l \in M} \varphi(\varepsilon_{n-1}, l) q_n(\varepsilon_{n-1}, l) C_{n-1} \cdot g(\varepsilon_{n-1}, l) \leq \sum_{l \in M} \varphi(\varepsilon_{n-1}, l) p(\varepsilon_{n-1}, l),
\]

because of property (q–p) in Eqn (7).

(b) The proof of this assertion is reduced to the analysis of the equality cases in (38). It does not differ from assertion (b) in Theorem 1.1.

The proof of assertion (c) is again a straightforward inspection.

For (d), we can repeat the argument used in the proof of Theorem 1.1(d). It yields only some notational complications without making the situation different in principle. \(\square\)

Theorems 2.1 and 3.1 can be combined into a statement about Scheme II, with one riskless and several risky assets. The assumption is that the trials results \(\varepsilon_j, j \geq 0\), are still generated by an MC on \(M\) with transition probabilities \(p(i, k)\), and we have \(K+1\) assets with RFs \(g^{(0)}\) and \(g = (g^{(1)}, \ldots, g^{(K)})\).

The asset with RF \(g^{(0)}\) is riskless: \(g^{(0)}(i, k) = 1 + \rho\) with \(\rho > 0\). The assets with RFs \(g^{(1)}, \ldots, g^{(K)}\) are risky: they can bring profit or loss \(g^{(s)}(i, k), i, k \in M, 1 \leq s \leq K\). We again adopt the Scheme II: the capital not invested in the risky assets is put in the riskless return.

Observe that the model under consideration emerges as a special case of the previous model if we include \(g^{(0)}\) into vector \(g\), increasing its dimension from \(K\) to \(K+1\). However, the explicit use of the form \(\rho^{(0)}(k) = 1 + \rho\) of the riskless asset makes the presentation less abstract.

As before, let \(\mathbf{C}_{n-1}\) denote the random vector \((C^{(n-1)}, \ldots, C^{(K)})\) where \(C^{(s)}\) stands for the amount of investment in the \(s\)th risky asset before the \(n\)th trial, \(n \geq 1\). The recursion for the capital value becomes
\[
Z_n = (1 + \rho)(Z_{n-1} - |C_{n-1}|) + \mathbf{C}_{n-1} g^{*(n-1), \varepsilon_n} = Z_{n-1} \left[ 1 + \rho + \frac{C_{n-1}g^{*(n-1), \varepsilon_n}}{Z_{n-1}} \right].
\]

Here and below in this section:
\[
g^{*(i, k)} = \left( g^{(1)}(i, k) - (1 + \rho), \ldots, g^{(K)}(i, k) - (1 + \rho) \right) = g(i, k) - (1 + \rho) \mathbf{1}.
\]

Cf. Eqns (19), (20) and (32).
The assumption on the sequence of portfolios \( \{ \underline{C}_j \} \) and functions \( g, \varphi \) and \( q_j \) are: \( \forall \ j \geq 0, \)

\[
(a0) \quad \underline{C}_j \in \mathbb{R}, \quad (a1) \quad \underline{C}_j \geq 0, \quad |\underline{C}_j| < Z_j, \quad (a2) \quad 1 + \rho + \frac{C_j g^s(\varepsilon_j, \varepsilon_{j+1})}{Z_j} \geq b, \\
and \quad (a3) \quad \sum_{i \in \mathcal{M}} \varphi(\varepsilon_j, l)q_{j+1}(\varepsilon_j, l) \left[ C_j(\varepsilon_j, l)g^s(\varepsilon_j, l) \right] = 0,
\]

(42)

again citing sustainability, no-ruin and balance conditions. The dominance condition (22) will also be used.

The quantity of interest is still the mean value \( E\mathbb{S}_n \). The goal is to maximise \( E\mathbb{S}_n \) in \( \underline{C}_1, \ldots, \underline{C}_{n-1} \) subject to restrictions (42), with or without condition (a3).

To this end, we list the adapted conditions (i1)–(i4):

(i) There exists a map \( i \mapsto D(i) \) where vector \( D(i) = (D(1)(i), \ldots, D(K)(i)) \) is such that \( \forall i, k \in M, \)

\[
(i1) \quad D(i) \geq 0, \quad \text{and} \quad |D(i)| < 1 \quad \text{(D-sustainability)}, \\
(i2) \quad 1 + \rho + D(i) \cdot g^s(i, k) \geq b \quad \text{(D-no-ruin)}, \\
(i3) \quad \sum_{i \in \mathcal{M}} \varphi(i, l)p(i, l) \frac{D(i) \cdot g^s(i, l)}{1 + \rho + D(i) \cdot g^s(i, l)} = 0 \quad \text{(WE (D,g)-balance), and} \\
(i4) \quad \text{the CFs} \ q_j \text{ \ are } \ q_j(i, k) = \frac{1 + \rho + D(i) \cdot g^s(i, k)}{p(i, k)},
\]

independently of \( j \geq 0 \) (q-representation),

with the alternative in (i3A) as follows:

\[
(i3A) \quad \sum_{i \in \mathcal{M}} p(i, l) \frac{\varphi(i, l)g^s(i, l)}{1 + \rho + D(i) \cdot g^s(i, l)} = 0 \\
\forall i \in M \text{ and } 1 \leq s \leq K \quad \text{(strong WE (D,g)-balance).}
\]

(44)

Cf. (i3A) in Eqn (34).

The cumulative weighted KL entropy process \( \{ A_n \} \) is defined as in Eqs (1) and (2).

The combined statement is Theorem 3.2 below. We omit its proof as it repeats that of Theorems 2.1 and 3.1.

**Theorem 3.2.** Assuming the above setting, we obtain the following assertions.

(a) Take any sequence \( \{ \underline{C}_j \}, \ j \geq 0 \) obeying (a0)–(a3) in (42). Then the sequence \( \{ S_n - A_n, n \geq 1 \} \) is a supermartingale; hence \( E\mathbb{S}_n \leq E\mathbb{A}_n \ \forall n \geq 1. \)

(b) To reach equality \( E\mathbb{S}_n = \sum_{j=1}^n \mathbb{E}\varphi(e_{j-1}) \): the sequence \( \{ S_n - A_n \} \) is a martingale for a sequence of RVs \( \{ \underline{C}_j \} \), satisfying (a0)–(a3) iff the additional properties (i), (ii) below are fulfilled.

(i) There exists a map \( i \mapsto D(i) \) where vector \( D(i) = (D(1)(i), \ldots, D(K)(i)) \) is such that \( \forall i, k \in M, \) properties (i1)–(i3) in (43) are fulfilled, and the CFs \( q_j \) are as in (i4).

(ii) The portfolio vectors \( \underline{C}_j \) have components \( C_j(l)(\varepsilon_j) = D(s)(\varepsilon_j)Z_j, \ 1 \leq s \leq K, \ j \geq 0. \) That is, the prescribed fractions of the capital value \( Z_j \) are invested in the available returns while the rest is put in the riskless asset.

(c) Suppose there exists a map \( i \mapsto D(i) = (D(1)(i), \ldots, D(K)(i)) \) fulfilling conditions (i1)–(i3) in Eqn (43). As before, let \( D \) stand for the array of values \( D(s)(i), i \in M, 1 \leq s \leq K, \) and define the quantity \( E\mathbb{S}_n = E\mathbb{S}_n(D) \) by

\[
E\mathbb{S}_n = \sum_{j=1}^n \beta_j \mathbb{E} \left\{ \varphi(\varepsilon_{j-1}, \varepsilon_j) \ln \left[ 1 + \rho + D(\varepsilon_{j-1}) \cdot g^s(\varepsilon_{j-1}, \varepsilon_j) \right] \right\}.
\]

(45)

Consider the optimization problem

maximise \( E\mathbb{S}_n(D) \) subject to (i1)–(i3).

(46)

Take an optimizer \( D^* = \arg \max E\mathbb{S}_n \) (possibly, non-unique) and let \( E\mathbb{S}_n^* = E\mathbb{S}_n(D^*) \) denote the optimal value for (46). Then \( E\mathbb{S}_n^* \) defines the maximum of the expectation \( E\mathbb{S}_n \) among all portfolios \( \{ \underline{C}_j \} \).
Example 3.1: IID trials with two risky assets. In this example we adopt Scheme I. As was noted, for IID trials, \( p(i, k) = p(k) \). We again take \( \varphi(i, k) = \varphi(k) \), \( q(i, k) = q(k) \) and set \( g(i, k) = g(k) \) where \( g(k) = (g^{(1)}(k), g^{(2)}(k)) \). Recall, the portfolio has the form \( C_n = (C_n^{(1)}, C_n^{(2)}) \). Conditions \( (i) \), \( (ii) \), \( (iii) \) and \( (iv) \) are summarized as

\[
(q - p) \sum_{l \in M} \varphi(l) [q(l) - p(l)] \leq 0;
\]

and \( (a0) \sum_{i \in M} C_i \geq 0 \), \( |C_i| < Z_j \), \( (a2) 1 + \frac{C_i \cdot g(k)}{Z_j} \geq b \),

and \( (a3) \sum_{i \in M} \varphi(l) q(l) [C_i - g(l)] = 0 \), \( \forall k \in M, j \geq 0 \).

Therefore, it yields the maximum of the whole sum \( \mathbb{E} S_n \), among strategies satisfying properties \( (a0) \)–\( (a2) \) in Eqn (12).

$$\beta(D) := \sum_{i \in M} \varphi(l) p(l) \ln [1 + D \cdot g(l)]$$

in \( D = (D^{(1)}, D^{(2)}) \), subject to (i1)–(i3). Let \( D^O \) stand for the optimizer: 

$$\beta(D) \in \arg \max \beta(D)$$

Note that the Hessian \( 2 \times 2 \)-matrix \( H(\beta) \) is non-positive definite as its determinant is 0 and the trace is negative:

$$H(\beta) = \left( \begin{array}{cc} \frac{\partial^2 \beta(D)}{\partial D^{(ss)} \partial D^{(ss')}} & s, s' = 1, 2 \\ \frac{\varphi(l) p(l)}{(1 + D \cdot g(l))^2} \begin{pmatrix} g^{(1)}(l)^2 & g^{(1)}(l) g^{(2)}(l) \\ g^{(1)}(l) g^{(2)}(l) & g^{(2)}(l)^2 \end{pmatrix} \end{array} \right).$$

It shows that \( D \mapsto \beta(D) \) is a concave function over the polygon \( \mathbb{D} \) extracted by conditions (i1) and (i2):

$$\mathbb{D} := \left\{ D = (D^{(1)}, D^{(2)}) : D^{(1)}, D^{(2)} \geq 0, D^{(1)} + D^{(2)} < 1, 1 + D \cdot g(k) \geq b \forall k \in M \right\}.$$
\[ \sum_{l \in M} \varphi(l)p(l)g(l) = 0. \] Then the optimal proportion vector \( D^O \) is \((0,0)\) (no investment by the risk-averse trader). On the other hand, if \( \sum_{l \in M} \varphi(l)g(1)(l)p_l > 0 \) or \( \sum_{l \in M} \varphi(l)g(2)(l)p_l > 0 \) then we can look at the gradient values \( \nabla \beta(D) \) at \( D = (1,0) \) and \( D = (1,0) \) (investments into a single asset).

To illustrate further, consider again the case \( M = \{0,1\} \), with outcomes \( k = 0,1 \) and two probabilities \( p(0), p(1) \). Without loss of generality, assume \( p(1) \geq p(0) \). Next, for sake of simplicity, let us again take \( \varphi(0) = \varphi(1) = 1 \). Further, set \( g^{(1)}(1) = -g^{(1)}(0) = \gamma_1 > 0 \) and \( g^{(2)}(0) = -g^{(2)}(1) = \gamma_2 > 0 \), and fix \( b \in (0,1) \).

In line with Theorem 3.1(c,d), we seek to solve the optimization problem

\[
\max \beta(D) := p(1) \ln \left[ 1 + D^{(1)}_1 - D^{(2)}_2 \right] + p(0) \ln \left[ 1 - D^{(1)}_1 + D^{(2)}_2 \right]
\]
subject to

\[
1 + D^{(1)}_1 - D^{(2)}_2 \geq b, \quad 1 - D^{(1)}_1 + D^{(2)}_2 \geq b, \quad \text{and} \quad D^{(1)}_1 + D^{(2)}_2 < 1,
\]

\[
p(1) \frac{D^{(1)}_1 - D^{(2)}_2}{1 + D^{(1)}_1 - D^{(2)}_2} - p(0) \frac{D^{(1)}_1 - D^{(2)}_2}{1 - D^{(1)}_1 + D^{(2)}_2} = 0.
\]

The last equation (coming from (33)) can be reduced to two alternative ones:

\[
(A) \quad D^{(1)}_1 - D^{(2)}_2 = p(1) - p(0) \quad \text{or} \quad (B) \quad D^{(1)}_1 - D^{(2)}_2 = 0.
\]

When \( 1 - b \geq p(1) - p(0) \), these equations specify two (parallel) segments inside the quadrilateral

\[
D := \left\{ D = (D^{(1)}, D^{(2)}): D^{(1)}_1 + D^{(2)}_2 = 0, \quad D^{(1)}_1 + D^{(2)}_2 < 1, \quad 1 - b \leq D^{(1)}_1 - D^{(2)}_2 \leq 1 - b \right\}.
\]

Note that \( \beta(D) \) depends only on \( D^{(1)}_1 - D^{(2)}_2 \). Further, function \( D \in D \mapsto \alpha(D) \) is concave, and for the gradient vector of \( \beta(D) \) we have:

\[
\nabla \beta(D) = \left( \frac{p(1)\gamma_1}{1 + D^{(1)}_1 - D^{(2)}_2} - \frac{p(0)\gamma_1}{1 - D^{(1)}_1 + D^{(2)}_2}, \frac{p(1)\gamma_2}{1 + D^{(1)}_1 - D^{(2)}_2} + \frac{p(0)\gamma_2}{1 - D^{(1)}_1 + D^{(2)}_2} \right),
\]

with

\[
\nabla \beta(D) = 0 \quad \text{iff} \quad D^{(1)}_1 - D^{(2)}_2 = p(1) - p(0), \quad \text{i.e.,} \quad D \text{ lies in segment (A) in (50)}.
\]

Note that \( \beta(D) \) depends only on \( D^{(1)}_1 - D^{(2)}_2 \). Further, function \( D \in D \mapsto \alpha(D) \) is concave, and for the gradient vector of \( \beta(D) \) we have:

\[
\nabla \beta(D) = \left( \frac{p(1)\gamma_1}{1 + D^{(1)}_1 - D^{(2)}_2} - \frac{p(0)\gamma_1}{1 - D^{(1)}_1 + D^{(2)}_2}, \frac{p(1)\gamma_2}{1 + D^{(1)}_1 - D^{(2)}_2} + \frac{p(0)\gamma_2}{1 - D^{(1)}_1 + D^{(2)}_2} \right),
\]

with

\[
\nabla \beta(D) = 0 \quad \text{iff} \quad D^{(1)}_1 - D^{(2)}_2 = p(1) - p(0), \quad \text{i.e.,} \quad D \text{ lies in segment (A) in (50)}.
\]

4 A general discrete-time setting

The constructions developed so far show that the Markovian setting is not necessary for the main result. In this section we attempt to propose a general background where the proposed techniques still works. Here we again deal with results \( \varepsilon_n \) of subsequent random trials, \( n = 0,1,\ldots \). No specific condition upon the joint distribution is assumed, apart from a dominance condition (involving a given WF); see Eqn (58) below. Each \( \varepsilon_n \) is a random element in a standard measure space \((\mathcal{X}_n, \mathcal{X}_n, \mu_n)\). We suppose that a random string \( \varepsilon^*_n = (\varepsilon_0, \ldots, \varepsilon_n) \) has a joint probability density or probability mass function (PD/MF) \( f_n(\varepsilon^*_n) \) relative to reference measures \( \mu_0 \equiv \prod_{j=0}^n \mu_j \), on \( \times \mathcal{X}_j \):

\[
P(\varepsilon^* \in A) = \int_A f_n(\varepsilon^*_n) \, d\mu_n(\varepsilon^*_n), \quad \varepsilon^*_n = (x_1, \ldots, x_n) \in \times_{j=1}^n \mathcal{X}_j \quad \text{and} \quad A \subseteq \times_{j=1}^n \mathcal{X}_j.
\]
A conditional PD/MF $f_n(x_n|x_{n-1}^{-1})$ will be also used, with

$$f_n(x_n|x_{n-1}^{-1})f_{n-1}(x_{n-1}^{-1}) = f_n(x_n)$$ and $\int_{X_n} f_n(x_n|x_{n-1}^{-1})d\mu_n(x_n) = 1,$ $f_n$-a.s. \hspace{1cm} (53)

When necessary, standard properties of measures $\mu_n$ are assumed by default (completeness, $\sigma$-finiteness).

Next, suppose that a sequence $\{g_n, n \geq 1\}$ of real-valued return functions is given, where $g_n : (x_{n-1}^{-1}, x_n) \in X_1^{-1} \times X_n \mapsto \mathbb{R}$. For $n = 1$ we deal with $g_1(x_1).$ As above, if you stake $c_n$ on the $n$th trial, you win $c_n g_n(x_n^{-1}, x_n)$ if the outcome is $x_n \in X_n$ preceded by the string $x_{n-1}^{-1}$. That is, you make a profit when $c_n g_n(x_n^{-1}, x_n) > 0$ and incur a loss when $c_n g_n(x_n^{-1}, x_n) < 0.$

As before, $Z_0 > 0$ stands for an initial (random) capital. For a given $n \geq 1$, let $Z_n > 0$ denote the capital after $n$ trials. Then the recursion similar to $\Pi$ emerges:

$$Z_n = Z_{n-1} + C_{n-1} g_n(x_0^{-1}, \varepsilon_n) = \frac{Z_{n-1}}{C_{n-1}} + \frac{C_{n-1} g_n(x_0^{-1}, \varepsilon_n)}{Z_{n-1}}, \hspace{1cm} n \geq 1. \hspace{1cm} (54)$$

Formally, we assume that the probability space under consideration is $(\Omega, \mathcal{F}, \mathbb{P})$ as defined above. The sample set $\Omega$ is the Cartesian product $\mathbb{R}_+ \times \times X_n$ where $\mathbb{R}_+ = (0, \infty)$ equipped with the product $\sigma$-algebra $\mathcal{F} = \mathcal{B}(\mathbb{R}_+) \times \times X_n$ and the filtration $\mathcal{F}_n = \mathcal{B} \times \times X_n$. Here $\mathcal{B} = \mathcal{B}(\mathbb{R}_+)$ is the Borel $\sigma$-algebra in $\mathbb{R}_+$. Adopting the set-up from Sect II, $\mathcal{F}_0 = \sigma(Z_0, \varepsilon_0)$ and $\mathcal{F}_n = \mathcal{F}_n \vee \sigma(\varepsilon_n)$ for $n \geq 1$. Then $\omega \in \Omega$ is a pair $(z_0, x_0^\infty)$ where $x_0^\infty$ is a sequence $x_n : n \geq 1$ where $x_n \in X_n$. All random elements above and in the sequel are defined as functions of $\omega \in \Omega$, subject to standard measurability assumptions. A probability measure $\mathbb{P}$ is given by a compatible family of joint PD/MF $f_n(z_0, x_0^\infty)$ with respect to $\nu \times \mu_n^0$ where $\nu$ is a chosen measure on $(\mathbb{R}_+, \mathcal{B})$ (typically, a counting measure on a finite or countable subset or a Lebesgue measure). Once more we assume that the random variable $C_{n-1} = C_{n-1}(x_0^{-1})$ is $\mathcal{F}_{n-1}$-measurable, i.e., yields a predictable strategy. That is, the stake in the $n$th trial is based on the results of trials $1, \ldots, n-1$. Then $Z_{n-1} = Z_{n-1}(x_0^{-1})$ is $\mathcal{F}_{n-1}$-measurable.

As in the previous sections, we consider the weighted logarithmic growth $S_n$ after $n$ trials:

$$S_n := \sum_{j=1}^n \varphi_j(x_j^{-1}, \varepsilon_j) \ln \frac{Z_j}{Z_{j-1}}. \hspace{1cm} (55)$$

Cf. Eqn \(52\). The goal is the same: to maximize, in $\{C_0, \ldots, C_{n-1}\}$, the mean-value $\mathbb{E} S_n$ (and to identify maximizers). Here we deal with general WFs $\varphi_j : x_j^{-1} \mapsto \varphi_j(x_j^{-1}, x_j) \geq 0$ depending on the current outcome $x_j$ and the vector of preceding outcomes $x_j^{-1}$. Again, we can think that $\varphi_j(x_j^{-1}, x_j)$ represents a ‘utility’ value of outcome $x_j$ (given that it succeeds an outcome sequence $x_j^{-1}$ from previous trials), and it is taken into account when we calculate $S_n$. As above, when $\varphi_j \equiv 1$, the sum \(55\) becomes telescopic and equal to $\ln \frac{Z_N}{Z_0}$.

The maximization procedure involves a sequence of a.s. positive CFs $q_j(x_j^{-1}, x_j), j \geq 0,$ figuring in Eqs \(56\) - \(58\) below. Like we said, typically, the function $q_j$ will be a (conditional) PDF relative to the reference measure $\mu_j$ on $X_j$. Define RVs $\alpha_j = \alpha_j(x_0^j)$ and $A_n = A_n(x_0^n)$ by

$$\alpha_j = \int_{X_{j+1}} \varphi_{j+1}(x_0^j, x_{j+1}) f_{j+1}(x_{j+1}|x_0^j) \ln f_{j+1}(x_{j+1}|x_0^j) d\mu_{j+1}(x_{j+1})$$

$$= \mathbb{E} \left[ \varphi_{j+1}(x_0^j, \varepsilon_{j+1}) \ln \frac{f_{j+1}(x_{j+1}|x_0^j)}{q_{j+1}(x_0^j, \varepsilon_{j+1})} \right]_{\mathcal{V}^J}$$ and $A_n := \sum_{j=1}^n \alpha_j.$ \hspace{1cm} (56)

Here the RV $\alpha_n$ represents the weighted KL entropy of the conditional PD/MF $f_n(\cdot|x_0^{n-1})$ relative to $q_n(x_0^{n-1}, \cdot)$. Cf. Eqn \(3\). The RV $A_n$ yields the cumulative weighted KL.

\*\*All functions figuring throughout the paper are assumed measurable, with a specific indication of the $\sigma$-algebra when necessary.
Fix \( b > 0 \). As before, it is convenient to summarize the assumptions about RVs \( \{C_j\} \): \( \forall \ j \geq 0 \),

\[
\text{(a0) } C_j \in \mathbb{M}_j, \quad \text{(a1) } 0 \leq C_j < Z_j, \quad \text{(a2) } 1 + \frac{C_j g_{j+1}(e_0^{j+1}, \varepsilon_{j+1})}{Z_j} \geq b, \quad \text{and} \quad \text{(a3) } C_j(e_0^j) \int X_{j+1} \phi_j(x_0, x_{j+1}) q_{j+1}(e_0^j, x_{j+1}) \, d\mu_{j+1}(x_{j+1}) = 0,
\]

referred to, respectively, as predictability, sustainability, no-ruin and \((q, g)\)-balance conditions.

As was stressed, Eqns (57)–(59) are assumed denoted by the equation

\[
(i4) \quad \int X_{j+1} \phi_j(x_0^j, x_{j+1}) f_j(x_j | x_0^{j+1}) \, d\mu_j(x_j) = \mathbb{E}[\phi_j(x_0^{j+1}) | \mathbb{W}_{j-1}].
\]

The conditions (i1)–(i4) are re-written as follows:

(i) \( \forall \ j \geq 0, \quad \exists a \text{ RV } D_j(e_0^j) \text{ with the properties} \)

\[
\text{(i1) } 0 \leq D_j(e_0^j) < 1 \text{ (D-sustainability), (i2) } 1 + D_j(e_0^j) g_j(e_0^j, \varepsilon_{j+1}) \geq b \text{ (D-no-ruin),} \]

\[
\text{(i3) } D_j(e_0^j) \int X_{j+1} \phi_j(x_0^j, x_{j+1}) f_{j+1}(x_{j+1} | e_0^j, x_{j+1}) \, d\mu_{j+1}(x_{j+1}) = 0 \quad \text{(WE (D,g)-balance), with alternatives} \]

\[
\text{(i3a) } \int X_{j+1} \phi_j(x_0^j, x_{j+1}) f_{j+1}(x_{j+1} | e_0^j, x_{j+1}) \, d\mu_{j+1}(x_{j+1}) = 0, \quad \text{(i3b) } D_j(e_0^j) = 0, \quad \text{and} \quad \text{(i4) } q_j(e_0^j, \varepsilon_j) = \frac{f_j(e_0^j, \varepsilon_j)}{1 + D_{j-1}(e_0^{j-1}) g_j(e_0^{j-1}, \varepsilon_j)} \quad \text{(q-representation).} \]

As was stressed, Eqns (57)–(59) are assumed \( f_j \text{-a.s.} \) Such a convention is also extended to similar relations below. All integrals involved are supposed to converge absolutely. As before, we can interpret these conditions as implemented by a risk-averse trader.

**Theorem 4.1.** Assume we are given CFs \( q_j > 0 \), WFs \( \varphi_n \geq 0 \) and RFs \( g_n \) with values in \( \mathbb{R} \) satisfying (58). Assume the recursion (54). Consider the RVs \( S_n \) and \( \Lambda_n \) from Eqns (55) and (56). Then:

(a) For any sequence of RVs \( \{C_j, j \geq 0\} \) satisfying (57), the sequence \( \{S_n - \Lambda_n, n \geq 1\} \) is a supermartingale. Consequently, \( \mathbb{E}S_n \leq \mathbb{E}S_n \).

(b) To achieve equality \( \mathbb{E}S_n = \mathbb{E}S_n \): the sequence \( \{S_n - \Lambda_n, n \geq 1\} \) is a martingale for a sequence \( \{C_j\} \) satisfying (57) iff \( \forall \ j \geq 0 \) iff the properties (i), (ii) below hold true.

(i) \( \forall \ j \geq 0, \exists a \text{ RV } D_j(e_0^j) \text{ such that the relations (i1)–(i3)} \text{ in Eqn (59)} \text{ hold true, and CFs } q_j \text{ are given by the equation (i4).} \)

(ii) The strategy \( C_j, j \geq 0 \) yields a proportional investment: \( C_j(e_0^j) = D_j(e_0^j) Z_j \).

Furthermore, the CF \( q_j \) from (i4) has \( \int X_{j+1} q_j(x_0^j, x_{j+1}) \, d\mu_j(x_j) = 1 \) (i.e., determines a PD/MF) iff, in addition to (33), we have that, \( \forall j \geq 1, \)

\[
D_{j-1}(e_0^{j-1}) = 0 \quad \text{or} \quad \int X_j \frac{f_j(x_j | e_0^{j-1}) g_j(e_0^{j-1}, x_j)}{1 + D_{j-1}(e_0^{j-1}) g_j(e_0^{j-1}, x_j)} \, d\mu_j(x_j) = 0.
\]

(c) Suppose that \( \forall \ j \geq 0, \exists a \text{ RV } D_j(e_0^j) \text{ satisfying (i1)–(i3).} \text{ Construct the RVs } D_j^O(e_0^j) \text{ in the following manner. Set } D_j^O(e_0^j) = D_j(e_0^j) \text{ if the alternative (i3A) is fulfilled and the value } D_j(e_0^j) > 0 \text{ (such a value, if it exists, is unique), and } D_j^O(e_0^j) = 0 \text{ otherwise. Then the policies } C_j^O = D_j^O(e_0^j) Z_j \text{ yield the expectations } \mathbb{E}S_n = E_n \text{ where} \)

\[
E_n = \sum_{j=1}^{n} \beta_{j-1}, \text{ with } \beta_{j-1} = \mathbb{E}\left\{ \varphi_j(e_0^{j-1}, \varepsilon_j) \ln \left[ 1 + D_{j-1}^O(e_0^{j-1}) g_j(e_0^{j-1}, \varepsilon_j) \right] \right\}.
\]

It is a maximal value of \( \mathbb{E}S_n \) among the strategies satisfying properties (a1)–(a3) in Eqn (57).
(d) Under the assumptions adopted in (c), suppose that \( \forall \; j \geq 0 \), the RV \( D_j^O(x_j^0) > 0 \) (so, the alternative (33A) is fulfilled with \( D_j(x_j^0) > 0 \)). Then setting \( C_j^O = D_j^O Z_j \) yields the policies that maximize the mean value \( \mathbb{E}S_n \) over the strategies satisfying properties (a1)–(a2) in Eqn (67).

**Proof of Theorem 4.1.** We still follow the previously established pattern. (a) Write:

\[
\mathbb{E}\left\{ (S_n - A_n) \bigg\mid \mathcal{W}_{n-1} \right\} = S_{n-1} - A_{n-1} + \mathbb{E}\left\{ \varphi_n \ln \left( 1 + \frac{C_n g_n(x_n^0)}{Z_{n-1}} \right) \bigg\mid \mathcal{W}_{n-1} \right\} - \alpha_n.
\]

Next, represent

\[
\mathbb{E}\left\{ \varphi_n(x_n^0, \varepsilon_{n-1}) \ln \left( 1 + \frac{C_n g_n(x_n^0, \varepsilon_{n-1})}{Z_{n-1}} \right) \bigg\mid \mathcal{W}_{n-1} \right\} - \alpha_n
\]

\[
= \int_{\mathcal{X}_n} \varphi_n(x_n^0, \varepsilon_{n-1}) f_n(x_n|x_n^0) \ln \left( 1 + \frac{C_n g_n(x_n^0, \varepsilon_{n-1})}{Z_{n-1}} \right) d\mu_n(x_n)
\]

\[
= \int_{\mathcal{X}_n} \varphi_n(x_n^0, \varepsilon_{n-1}) f_n(x_n|x_n^0) \frac{1}{q_n(x_n^0)} d\mu_n(x_n)
\]

\[
= \int_{\mathcal{X}_n} \varphi_n(x_n^0, \varepsilon_{n-1}) f_n(x_n|x_n^0) \ln \left( 1 + \frac{C_n g_n(x_n^0, \varepsilon_{n-1})}{Z_{n-1}} \right) d\mu_n(x_n)
\]

\[
\leq \int_{\mathcal{X}_n} \varphi_n(x_n^0, \varepsilon_{n-1}) f_n(x_n|x_n^0) \left( 1 + \frac{C_n g_n(x_n^0, \varepsilon_{n-1})}{Z_{n-1}} \right) d\mu_n(x_n)
\]

Here \( h_n(x_n^0, \varepsilon_{n-1}) := q_n(x_n^0, \varepsilon_{n-1}) \left( 1 + \frac{C_n g_n(x_n^0, \varepsilon_{n-1})}{Z_{n-1}} \right), \; x_n \in \mathcal{X}_n \). The final inequality in (62) holds since, almost surely,

\[
\int_{\mathcal{X}_n} \varphi_n(x_n^0, \varepsilon_{n-1}) h_n(x_n^0, \varepsilon_{n-1}) d\mu_n(x_n) = \int_{\mathcal{X}_n} \varphi_n(x_n^0, \varepsilon_{n-1}) q_n(x_n^0, \varepsilon_{n-1}) d\mu_n(x_n)
\]

\[
+ \frac{C_n g_n(x_n^0, \varepsilon_{n-1})}{Z_{n-1}} \int_{\mathcal{X}_n} \varphi_n(x_n^0, \varepsilon_{n-1}) q_n(x_n^0, \varepsilon_{n-1}) g_n(x_n^0, \varepsilon_{n-1}) d\mu_n(x_n)
\]

\[
\leq \int_{\mathcal{X}_n} \varphi_n(x_n^0, \varepsilon_{n-1}) f_n(x_n|x_n^0) d\mu_n(x_n),
\]

due to (q-f) in (58).

As a result, we get the supermartingale inequality

\[
\mathbb{E}\left\{ (S_n - A_n) \bigg\mid \mathcal{W}_{n-1} \right\} \leq S_{n-1} - A_{n-1}.
\]

(b) For the martingale property we need to attain equalities in Eqn (62). The first inequality becomes equality iff

\[
\frac{h_n(x_n^0, \varepsilon_{n-1})}{f_n(x_n|x_n^0)} - 1 = 1 \left( f_n(x_n^0, \varepsilon_{n-1}) > 0 \right) = 0, \; \mu_n-a.s., \; i.e.,
\]

\[
q_n(x_n^0, \varepsilon_{n-1}) \left( 1 + \frac{C_n g_n(x_n^0, \varepsilon_{n-1})}{Z_{n-1}} \right) = f_n(x_n^0, \varepsilon_{n-1}),
\]

which yields representation (44) in Eqn (69). The second equality, achieved in (63), follows from (58) after substituting (44) and is equivalent to (33). Equation (60) is also established by using (44).

Finally, properties (c,d) follows (a) and (b). □

**Remarks.** 4.1. Assumption (59) is meaningful when functions \( x_n \in \mathcal{X}_n \mapsto g_n(x_n^0, x_n) \) are lower-bounded for a.a. \( x_n^0 \).

4.2. Condition (a1) is not used in the proof of assertion (a), only (a2) is relevant there.

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4.3. Taking \( q_n(x_n, z_n^{(n-1)}) = f_n(x_n, z_n^{(n-1)}) \) leads to the case \( S_n = 0 \).

4.4. As earlier, the staple of the proof of Theorem 4.1 is the Gibbs inequality for weighted entropies; see [11, 17]. It is similar to the standard Gibbs inequality (cf. [4, 6]) but requires additional assumptions (58) and (59).

4.5. The optimization problem emerging in assertion (c) is an example of a control problem. The connections with specific facts and methods, including the Bellman equation seem fruitful and should be explored in forthcoming works.

The next step is to provide an assertion covering Scheme II (a riskless asset and a collection of risky assets) in a general setting. The trial RF \( \{ \varepsilon_n, n \geq 0 \} \) has the same (general) nature as before, and we continue using the same concepts and notations whenever possible. By the time of the \( n \)th trial, we now have \( K(n) + 1 \) assets under management, which generates RFs \( g_n^{(0)} \) and \( g_n = \left( g_n^{(1)}, \ldots, g_n^{(K(n))} \right) \) at the \( n \)th trial. We assume that the RF \( g_n \) (relatively) riskless and has the form

\[
g_n^{(0)}(x_n^{(n-1)}) = 1 + \rho_{n-1}, \quad \rho_{n-1} = \rho_{n-1}(x_n^{(n-1)}) \geq 0
\]

(66)

The remaining RFs are risky, with values \( g_n^{(s)}(x_n^{(n-1)}, x_n), 1 \leq s \leq K(n) \), depending on the outcomes \( x_n^{(n-1)} \) of the previous trials (which you know) and the outcome \( x_n \) of the \( n \)th trial (which is unknown at the time of the \( n \)th decision). We set the \( \mathbb{R}_+ = (1, \ldots, 1) \) (with \( K(n) \) entries altogether) and

\[
g_n^* = g_n - (1 + \rho_{n-1}) \mathbb{1}_n = \left( g_n^{(1)} - (1 + \rho_n), \ldots, g_n^{(K(n))} - (1 + \rho_n) \right).
\]

(66)

Similarly to (66), the recursion for the capital value is taken to be

\[
\begin{align*}
Z_n &= (1 + \rho_{n-1})(Z_{n-1} - |C_{n-1}|) + \left[ C_{n-1} \cdot g_n^*(\varepsilon_{n-1}, \varepsilon_n) \right] \\
&= Z_{n-1} + \left[ C_{n-1} \cdot g_n^*(\varepsilon_{n-1}, \varepsilon_n) \right], \quad n \geq 1.
\end{align*}
\]

(67)

Here the vector \( C_j = \left( C_j^{(1)}, \ldots, C_j^{(K(n))} \right) \) represents a random portfolio which determines the investment at the time after the \( j \)th trial, and \( C_j \cdot g_{j+1}^*(\varepsilon_j, \varepsilon_{j+1}) = \sum_{s=1}^{K(j)} C_j^{(s)} \left[ g_j^{(s)}(\varepsilon_{j-1}^s, \varepsilon_j) - (1 + \rho_j) \right] \). The norm \( |C_j| = K(j) \sum_{s=1}^{K(j)} C_j^{(s)} = C_j \cdot 1_j \) shows the total size of the investment. Given a sequence of CFs \( (x_{j-1}^n, x_j) \mapsto q_j(x_{j-1}^n, x_j) > 0 \) and the threshold value \( b > 0 \), we work under the following assumptions:

\[
\begin{align*}
(a0) \quad C_j \in \mathbb{W}_j, \quad (a1) \quad C_j \geq 0, \quad |C_j| < Z_j, \quad \text{and} \quad (a2) 1 + \rho_j(\varepsilon_j^0) + \frac{C_j^{(1)}(\varepsilon_j^0) \cdot g_{j+1}^*(\varepsilon_j^0, \varepsilon_{j+1})}{Z_j(\varepsilon_j^0)} \geq b, \quad \text{and} \quad (a3) \quad \int_{X_{j+1}} \varphi_{j+1}(\varepsilon_j, x_{j+1}) q_{j+1}(\varepsilon_j^0, x_{j+1}) \frac{C_j^{(1)}(\varepsilon_j^0) \cdot g_{j+1}^*(\varepsilon_j, x_{j+1})}{Z_j(\varepsilon_j^0)} \, d\mu_{j+1}(x_{j+1}) = 0.
\end{align*}
\]

(68)

As before, we want to maximize the mean value \( \mathbb{E} S_n \). Here, as before,

\[
S_n := \sum_{j=1}^{n} \varphi_j(\varepsilon_{j-1}^j, \varepsilon_j) \log \frac{Z_j(\varepsilon_j^0)}{Z_j(\varepsilon_{j-1}^j)}
\]

and \( (x_{j-1}^n, x_j) \in X_0^n \times X_j \mapsto \varphi_j(x_{j-1}^n, x_j) \geq 0 \) is a given sequence of WF\( s \varphi_j, j \geq 1 \). The maximization problem is in \( \{ C_j, j \geq 1 \} \) subject to restrictions (a0)–(a2) or (a0)–(a3) in (68).

As usual, we assume the \((q,f)\)-dominance property, that \( \forall j \geq 0, \)

\[
(q - 1) \int_{X_{j+1}} \varphi_{j+1}(\varepsilon_j^0, x_{j+1}) \left[ 1 + \rho_j(\varepsilon_j^0) \right] q_{j+1}(\varepsilon_j^0, x_{j+1}) - f_{j+1}(x_{j+1} | \varepsilon_j^0) \right) \leq 0.
\]

(69)
(i) \( \forall \ j \geq 0 \ \exists \) a random vector \( \mathbf{D}_j(\varepsilon_{0}^j) \) where \( \mathbf{D}_j(\varepsilon_{0}^j) = (D_j^{(1)}(\varepsilon_{0}^j), \ldots, D_j^{(K(n))}(\varepsilon_{0}^j)) \), such that

\[
(1) \mathbf{D}_j(\varepsilon_{0}^j) \geq 0 \text{ and } |\mathbf{D}_j(\varepsilon_{0}^j)| := \sum_{i=1}^{K(n)} D_j^{(i)}(\varepsilon_{0}^j) < 1, \\
(2) \ 1 + \rho_j(\varepsilon_{0}^j) + \left[ D_j(\varepsilon_{0}^j) \cdot g_j^{*+1}(\varepsilon_{0}^j, x_{j+1}) \right] \geq b, \\
(3) \int_{x_{j+1}} f_{j+1}(x_{j+1}|\varepsilon_{0}^j) \varphi_{j+1}(\varepsilon_{0}^j, x_n) \left[ D_j(\varepsilon_{0}^j) \cdot g_j^{*+1}(\varepsilon_{0}^j, x_{j+1}) \right] d\mu_{j+1}(x_{j+1}) = 0, \\
(4) \text{the CFs have the form } q_{j+1}(\varepsilon_{0}^j, x_{j+1}) = \frac{f_{j+1}(x_{j+1}|\varepsilon_{0}^j)}{1 + \rho_j(\varepsilon_{0}^j) + \mathbf{D}_j(\varepsilon_{0}^j) \cdot g_j^{*+1}(\varepsilon_{0}^j, x_{j+1})}.
\]  

The behavior of \( \mathbb{E}S_n \) is characterized through the cumulative weighted KL entropy RP \{\( A_n, n \geq 1 \)\} where \( A_n \) and \( \alpha_j(\varepsilon_{j-1}) \) are as in Eqn (69).

**Theorem 4.2.** Adopt the above setting in Eqns (68)–(70). Then the following assertions hold true.

(a) Take any sequence of RVs \( \mathbb{C}_j = \left( C_n^{(1)}, \ldots, C_j^{(K(n))} \right) \in \mathcal{M}_j, j \geq 0, \) satisfying (68). Then the sequence \( \{S_n - A_n, n \geq 1\} \) is a supermartingale; hence \( \mathbb{E}S_n \leq \mathbb{E}A_n \forall \ n \geq 1. \)

(b) For equality \( \mathbb{E}S_n = \sum_{j=1}^{n} \mathbb{E}a(\varepsilon_{j-1}) \): the sequence \( \{S_n - A_n, n \geq 1\} \) is a martingale for predictable RVs \( \mathbb{C}_j, j \geq 0, \) satisfying (68) iff the additional properties (i), (ii) below are fulfilled.

(i) \( \exists \) a sequence of RVs \( \mathbf{D}_j(\varepsilon_{0}^j) = (D_j^{(1)}(\varepsilon_{0}^j), \ldots, D_j^{(K(n))}(\varepsilon_{0}^j)) \) such that \( \forall \ j \geq 0, \) properties (i1)–(i4) in Eqn (70) hold true.

(ii) \( \forall \ j \geq 0, \) the policy \( \mathbb{C}_j \) produces a proportional investment: after the \( j \)th trial the amount \( \mathbb{C}_j(\varepsilon_{0}^j) = \mathbf{D}_j(\varepsilon_{0}^j)Z_j \) goes to the assets with the RF \( q_j \) whereas \( Z_j - |D_j|Z_j \) is allocated to the riskless return.

(c) Suppose that \( \forall \ j \geq 0, \) \( \exists \) a RV \( \mathbf{D}_j(\varepsilon_{0}^j) = (D_j^{(1)}(\varepsilon_{0}^j), \ldots, D_j^{(K(n))}(\varepsilon_{0}^j)) \) fulfilling conditions (i1)–(i3) in Eqn (70). Let \( \mathbf{D} \) stand for a sequence of RVs \( \{\mathbf{D}_j\} \). Then, for the corresponding proportional investment portfolios \( \mathbb{C}_j \), the expectation \( \mathbb{E}E_n(\mathbf{D}) = \mathbb{E}S_n \) has the form

\[
E_n(\mathbf{D}) = \sum_{j=0}^{n-1} \beta_{j-1} \text{ where } \beta_{j-1} = \mathbb{E}\left\{ \varphi(\varepsilon_{j-1}, \varepsilon_j) \ln \left[ 1 + \rho + D_{j-1}(\varepsilon_{j-1}^k) \cdot g^*(\varepsilon_{j-1}^k, \varepsilon_j) \right] \right\}.
\]  

Then the maximum of \( E_n(\mathbf{D}) \) over the RVs satisfying (i1)–(i3) yields the maximum of the mean value \( \mathbb{E}S_n \) over the strategies satisfying (a0)–(a3) in Eqn (69).

(d) Suppose that \( \forall \ j \geq 0, \) \( \exists \) a RV \( \mathbf{D}_j(\varepsilon_{0}^j) = (D_j^{(1)}(\varepsilon_{0}^j), \ldots, D_j^{(K(n))}(\varepsilon_{0}^j)) \) with all entries \( D_j^{(i)}(\varepsilon_{0}^j) > 0, \) fulfilling conditions (i1)–(i3) in Eqn (70). Then the maximum of \( E_n(\mathbf{D}) \) from Eqn (71) yields the maximum mean value \( \mathbb{E}S_n \) among the strategies satisfying conditions (a0)–(a2) in Eqn (68).

We omit the proof of Theorem 4.2 as it does not contain new elements compared with the preceding statements.

**Example 4.1: Uniform IID trials.** As an illustration, consider again the case of IID trials, within the remits of Scheme I, with a single asset. Suppose \( \mathcal{X}_n = \mathbb{R} \) and \( \mu_n \) is the Lebesgue measure. Take \( f_n(x_n|x_{n-1}^n) = f(x_n), \) i.e., assume the results of the trials are independent and distributed with PDF \( f: \varepsilon_n \sim f. \) Take a WF \( \varphi(x) \geq 0, \) a bounded RF \( q(x) \) and a CF \( q(x) > 0. \) Also fix \( b \in (0, 1). \) As in the above IID examples, the weighted KL entropy \( \alpha_j \) from Eqn (69) becomes a constant:

\[
\alpha_{j-1} = \alpha = \int \varphi(x)f(x) \ln \left( \frac{f(x)}{q(x)} \right) dx = \mathbb{E}\left[ \varphi(\varepsilon_1) \ln \left( \frac{f(\varepsilon_1)}{q(\varepsilon_1)} \right) \right].
\]  

The martingale condition (i4) in Theorem 4.1(b) is that \( q(x) = \frac{f(x)}{1 + Dg(x)} \) where, in accordance with (i1) and (i2) from Eqn (69), \( D \in [0, 1] \) is a constant such that \( 1 + Dg(x) \geq b, \) for \( f \)-a.a. \( x \in \mathbb{R}. \) In
should be maximised in the variable $D$ i.e., when $E[D]$. Addition, (i3) requires that $\phi(D) = \int \varphi(x)f(x)\ln[1 + Dg(x)]dx = E[\varphi(\epsilon_1)\ln[1 + Dg(\epsilon_1)]]$ should be maximised in the variable $D$, subject to the above restrictions (i1)–(i3). Set:

$$D_+ = \max\left\{D \in [0, 1] : 1 + Dg(x) \geq b \forall x \in \text{supp } f\right\}.$$  

Then $D \in [0, D_+] \mapsto \beta(D)$ is a concave function, with the derivative $\frac{d}{dD}\beta(D)^1 = E[\frac{\varphi(\epsilon_1)g(\epsilon_1)}{1 + Dg(\epsilon_1)}]$. Thus, we are interested in a stationary point $D_0$, where $E[\frac{\varphi(\epsilon_1)g(\epsilon_1)}{1 + D_0g(\epsilon_1)}] = 0$. At $D = 0$ we have $\beta(D) = 0$ and $\frac{d}{dD}\beta(D)^2 = \int \varphi(x)f(x)g(x)dx := E[\varphi(\epsilon_1)g(\epsilon_1)]$. We see that if $E[\varphi(\epsilon_1)g(\epsilon_1)] = 0$ then the optimal proportion $D^0$ from Theorem 4.1(c) equals 0, i.e., a cautious trader would not invest in the IID market where on average there is no profit/loss. (Let alone the case where the loss exceeds the profit.) Otherwise, i.e., when $E[\varphi(\epsilon_1)g(\epsilon_1)] > 0$, the trader looks at the value $D_0$; if $D_0 \leq D_+$, we set $D^0 = D_0$. Otherwise, when $D_0 > D_+$, the theory does not give a formal answer.

To be more specific, consider an IID case with uniformly distributed IID trial results, where $f \sim U[a_1, a_2]$, with $a_1 < 0 < a_2$. Thus, $f(x) = 1(a_1 \leq x \leq a_2)/a$ where $a = a_2 - a_1$. Take $\varphi(x) \equiv 1$ (no preference); examples of a non-constant WF can be also incorporated into the argument that follows. Fix constants $b > 0$ and $\delta_\pm, \gamma_\pm \in \mathbb{R}$ and consider a piece-wise linear RF

$$g(x) = \begin{cases} \delta_+x + \gamma_+, & 0 \leq x \leq a_2, \\ \delta_-x + \gamma_-, & a_1 \leq x < 0. \end{cases}$$

For definiteness, assume that $g(x) \geq 0$ for $0 \leq x \leq a_2$ and $g(x) \leq 0$ for $a_1 \leq x \leq 0$; this boils down to $\gamma_+ \geq 0$ and $a_2\delta_+ + \gamma_+ \geq 0$ and the opposite inequalities for $\delta_-$ and $\gamma_-$ with $a_1$ replacing $a_2$. (The case $\delta_\pm = 0$ was effectively treated in Example 1.1.) For the mean of the RF we obtain

$$E[g(\epsilon)] = \frac{1}{a} \left[a_2\gamma_+ + a_1\gamma_- + \frac{a_2^2\delta_-}{2} - \frac{a_2^2\delta_-}{2}\right].$$

To guarantee the martingale condition, we have to take $q(x)$ as in (i4): $q(x) = \frac{f(x)}{1 + Dg(x)}$ where $D$ obeys (i1)–(i2): $0 \leq D \leq D_+$ where

$$D_+ = \max\left\{D \in [0, 1] : 1 + D\gamma_- \geq b \text{ and } 1 + D(a_1\delta_- + \gamma_-) \geq b\right\}.$$  

Then, referring to (ii1), we want to maximise in $D$ the function $D \mapsto \beta(D)$. Here

$$\beta(D) = \frac{1}{a} \int_{a_1}^{0} \ln \left\{1 + D[x\delta_- + \gamma_-]\right\}dx + \frac{1}{a} \int_{a_1}^{a_2} \ln \left\{1 + D[x\delta_+ + \gamma_+]\right\}dx,$$

with $\beta(0) = 0$. A direct calculation yields

$$\frac{d}{dD}\beta(D) = \frac{1}{a} \left(\int_{a_1}^{0} \frac{(\gamma_- + x\delta_-)dx}{1 + D\gamma_- + xD\delta_-} + \int_{0}^{a_2} \frac{(\gamma_+ + x\delta_+)dx}{1 + D\gamma_+ + xD\delta_+}\right)$$
with \( \frac{d}{dD} \beta(D) \big|_{D=0} = \mathbb{E} g(\varepsilon) \). This allows us to specify the above routine in terms of \( a_{1,2}, \delta_{\pm} \) and \( \gamma_{\pm} \). Namely, if \( \mathbb{E} g(\varepsilon) \leq 0 \) then the optimal policy is \( D^O = 0 \) (no investment), while if \( \mathbb{E} g(\varepsilon) > 0 \) then \( D^O = D_0 \) whenever \( D_0 \leq D_+ \). Here \( D_0 > 0 \) is the zero of the derivative \( \frac{d}{dD} \beta(D) \) solving a transcendental equation

\[
a - \frac{1}{D_0 \delta_+} \ln \frac{1 + D_0 \gamma_+}{1 + D_0 \gamma_+ + D_0 a_2 \delta_+} + \frac{1}{D_0 \delta_-} \ln \frac{1 + D_0 \gamma_-}{1 + D_0 \gamma_- + D_0 a_2 \delta_-} = 0.
\]

As was stated earlier, for \( D_0 > D_+ \) the theory does not yield a formal answer.

A similar methodology is applicable in examples where the PDF \( f \) and RF \( g \) are piece-wise polynomial functions.

**Example 4.2: Gaussian IID trials.** Another example is where the trial results \( \varepsilon_n \) are IID and have a normal PDF: \( f \sim \mathcal{N}(0, \sigma^2) \). Here

\[
f(x) = \frac{\exp(-x^2/(2\sigma^2))}{\sqrt{2\pi}}, \quad x \in \mathbb{R}.
\]

Adopt Scheme I, again with a single asset. We have a choice of RFs \( g \) and WFs \( \varphi \) where one can do explicit calculations. Viz, take

\[
g(x) = \begin{cases} 
    a_1 x + a_2, & x > 0, \\
    -a_3, & x < 0,
\end{cases}
\]

where \( a_1, a_2 \geq 0, a_3 > 0 \). Then the WF can be taken piece-wise polynomial, e.g.,

\[
\varphi(x) = \begin{cases} 
    x^2 \theta_+ + x \gamma_+ + \delta_+, & x > 0, \\
    x^2 \theta_- + x \gamma_- + \delta_-, & x < 0.
\end{cases}
\]

The mean value \( \mathbb{E} \varphi(\varepsilon_1) g(\varepsilon) \) can be computed as a polynomial in variables \( a_{1,2,3}, \theta_{\pm}, \gamma_{\pm} \) and \( \delta_{\pm} \).

Let us also fix \( b \in (0,1) \). For the martingale CF \( q(x) := \frac{f(x)}{1 + Dg(x)} \), the weighted KL entropy \( \beta = \beta(D) \) becomes

\[
\beta = \ln (1 - Da_3) \int_{-\infty}^{0} f(x) \varphi(x) dx + \int_{0}^{\infty} f(x) \varphi(x) \ln [1 + D(a_1 x + a_2)] dx,
\]

which is defined for \( 0 \leq D \leq D_+ := 1 \land [(1 - b)/a_3] \). Here the derivative

\[
\frac{d}{dD} \beta(D) = -\frac{a_3}{1 - Da_3} \int_{-\infty}^{0} f(x) \varphi(x) dx + \int_{0}^{\infty} \frac{f(x) \varphi(x)(a_1 x + a_2)}{1 + D(a_1 x + a_2)} dx,
\]

again with \( \frac{d}{dD} \beta(D) \big|_{D=0} = \mathbb{E} \varphi(\varepsilon_1) g(\varepsilon_1) \). The argument from the previous example can be still used to specify the optimal proportion \( D^O \).

**Example 4.3: IID trials with linear/logarithmic risky RFs.** In this example, we assume the trial results \( \varepsilon_n \) are IID and uniformly distributed on the interval \([-1,1] \). Thus, \( \mathcal{X}_n = [-1,1] \), the reference measures \( \mu_n \) are Lebesgue’s, and the PDF \( \ell_n(x_n|x_{n-1}) = \frac{1}{2} \land x_{n-1} \in [0,1]^{n-1} \) and \( x_n \in [0,1] \).

Adopt Scheme II and suppose that we have one riskless asset with the RF \( g^{(0)}(x) = 1 + \rho \) and two risky assets, with

\[
g^{(1)}(x) = -\gamma x \quad \text{and} \quad g^{(2)}(x) = -\theta \ln (1 - x),
\]

\[
g^*(x) = (1 - \gamma x - (1 + \rho), -\theta \ln (1 - (1 + \rho)), -1 \leq x \leq 1.
\]

Here \( \rho \geq 0, \gamma > 0 \) and \( \theta > 0 \) are given parameters. (Additional bounds for \( \rho, \gamma \) and \( \theta \) will appear as simplifying assumptions below.) Also fix \( b > 0 \) and a WF \( x \in [-1,1] \mapsto \varphi(x) \geq 0 \).

The inequalities (i1), (i2) in (70) take now the form

\[(i1) \quad D^{(1)}, D^{(2)} \geq 0 \quad \text{and} \quad D^{(1)} + D^{(2)} < 1 \quad \text{(D-sustainability)},
\]

\[(i2) \quad (1 + \rho)(1 - D^{(1)} - D^{(2)}) - D^{(1)} \gamma x - D^{(2)} \theta \ln (1 - x) \geq b, -1 \leq x \leq 1 \quad \text{(D-no-ruin)}.\]
The function \( x \in [-1, 1] \mapsto (1 + \rho)(1 - D^{(1)} - D^{(2)}) - D^{(1)} \gamma x - D^{(2)} \theta \ln (1 - x) \) in the LHS of (i2) has the second derivative \( \frac{D^{(2)} \theta}{(1 - x)^2} \geq 0 \), so is convex. Its minimum on \([-1, 1]\) is attained at \( x = 1 - \frac{D^{(2)} \theta}{D^{(1)} \gamma} \) if \( D^{(2)} \theta \leq 2D^{(1)} \gamma \) and at \( x = -1 \) if \( D^{(2)} \theta > 2D^{(1)} \gamma \). So, we can re-write the condition in the form

(i2) \[
\begin{cases}
(1 + \rho)(1 - D^{(1)} - D^{(2)}) - D^{(1)} \gamma \left( 1 - \frac{D^{(2)} \theta}{D^{(1)} \gamma} \right) - D^{(2)} \theta \ln \frac{D^{(2)} \theta}{D^{(1)} \gamma} \geq b, & \text{if } D^{(2)} \theta \leq 2D^{(1)} \gamma, \\
(1 + \rho)(1 - D^{(1)} - D^{(2)}) + D^{(1)} \gamma - D^{(2)} \theta \ln 2 \geq b, & \text{if } D^{(2)} \theta \geq 2D^{(1)} \gamma.
\end{cases}
\]

For definiteness, let us focus on the bottom line in (i2) and consider the following domain \( \mathbb{D} \) in the \( D^{(1)}, D^{(2)} \)-plane:

\[
\mathbb{D} = \left\{ \begin{array}{l}
D = (D^{(1)}, D^{(2)}): D^{(1)}, D^{(2)} \geq 0, D^{(1)} + D^{(2)} < 1, \\
\quad \quad \quad \quad 2D^{(1)} \gamma \leq D^{(2)} \theta \leq \frac{(1 + \rho)(1 - D^{(1)}) - b + D^{(1)} \gamma}{\ln 2 + (1 + \rho)\theta} \end{array} \right\}.
\]

Suppose that

\[0 \lor 1 + \rho - \ln 2 - \frac{1 + \rho}{\theta} < b < 1 + \rho.\]

Then \( \mathbb{D} \) is non-empty and represents a triangle or quadrilateral with vertices at \( \mathbb{D} = 0 = (0, 0) \) (the origin) and \( \mathbb{D} = \left( 0, \frac{1 + \rho - b}{\ln 2 + (1 + \rho)\theta} \right) \) (a point in the \( D^{(2)} \)-axis); the remaining two or one lie(s) on the straight lines where \( 2D^{(1)} \gamma = D^{(2)} \theta \) and \( D^{(2)} \theta = \frac{(1 + \rho)(1 - D^{(1)}) - b + D^{(1)} \gamma}{\ln 2 + (1 + \rho)\theta} \).

Next, consider condition (iii) in (i3): \[
\int_{-1}^{1} \varphi(x) \left\{ D^{(1)}(\gamma x + 1 + \rho) + D^{(2)}[\theta \ln (1 - x) + 1 + \rho] \right\} dx = 0 \text{ (WE D,g-balance).} \tag{74}
\]

In the strong form it becomes a system of two equations, for the maximum of the concave function \( (D^{(1)}, D^{(2)}) \mapsto \mathbb{E} \ln [1 + \rho + D \cdot g^*(\varepsilon)]: \)

\[
\int_{-1}^{1} \frac{\varphi(x)(\gamma x + 1 + \rho)}{1 + \rho - D^{(1)}(\gamma x + 1 + \rho) - D^{(2)}[\theta \ln (1 - x) + 1 + \rho]} dx = \int_{-1}^{1} \frac{\varphi(x)[\theta \ln (1 - x) + 1 + \rho]}{1 + \rho - D^{(1)}(\gamma x + 1 + \rho) - D^{(2)}[\theta \ln (1 - x) + 1 + \rho]} dx = 0. \tag{75}
\]

Obviously, \( \mathbb{D} = \emptyset \), i.e., \( D^{(1)} = D^{(2)} = 0 \) is a solution to (74). If it is the only solution in \( \mathbb{D} \) then we have \( \mathbb{D}^O = \emptyset \) (no investment in the risky assets, with \( \mathbb{E} \mathbb{S}_n = n \ln (1 + \rho) \)). It yields the optimum over the portfolio sequences \( C_j = (C^{(1)}_j, C^{(2)}_j) \) such that, \( \forall j \geq 0, \)

(a0) \( C_j \in \mathfrak{M}_j, \quad (a1) \quad C^{(1)}_j(\varepsilon^0_j), C^{(2)}_j(\varepsilon^0_j) \geq 0, \quad C^{(1)}_j(\varepsilon^0_j) + C^{(2)}_j(\varepsilon^0_j) < Z_j, \)

(a2) \( 1 + \rho - C^{(1)}_j(\varepsilon^0_j)(\gamma x + 1 + \rho) + C^{(2)}_j(\varepsilon^0_j)[\theta \ln (1 - x) + 1 + \rho] \geq b, \quad -1 \leq x \leq 1, \)

and (a3) \( \int_{-1}^{1} \varphi(y) \left\{ C^{(1)}_j(\varepsilon^0_j)(-\gamma y - 1 - \rho) + C^{(2)}_j(\varepsilon^0_j)[-\theta \ln (1 - y) - 1 - \rho] \right\} dy = 0. \tag{76} \)

An alternative possibility may occur if we have a solution to (75) lying in \( \mathbb{D} \). In this case we set \( \mathbb{D}^O \) to be the solution of (75) and achieve the maximum of \( \beta(D) = \mathbb{E} \ln [1 + \rho D \cdot g^*(\varepsilon)] \) in \( \mathbb{D} \) at \( D = D^O \).

Then \( \mathbb{E} \mathbb{S}_n = n \beta(D^O) \) yields the optimum among all portfolio sequences \( \mathcal{C}_j = (C^{(1)}_j, C^{(2)}_j) \) satisfying, \( \forall j \geq 0 \), the requirements (a0)–(a2), without (a3). Note that still \( \mathbb{D}^O = \emptyset \) if we have that

\[
\int_{-1}^{1} \varphi(x)(\gamma x + 1 + \rho)dx = \int_{-1}^{1} \varphi(x)[\theta \ln (1 - x) + 1 + \rho]dx = 0.
\]
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