Semiclassical states for quantum cosmology

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In a metric variable based Hamiltonian quantization, we give a prescription for constructing semiclassical matter-geometry states for homogeneous and isotropic cosmological models. These "collective" states arise as infinite linear combinations of fundamental excitations in an unconventional "polymer" quantization. They satisfy a number of properties characteristic of semiclassicality, such as peaking on classical phase space configurations. We describe how these states can be used to determine quantum corrections to the classical evolution equations, and to compute the initial state of the universe by a backward time evolution.

I. INTRODUCTION

One of the foundational questions in cosmology is how a large universe described effectively by classical physics emerges from a small and highly quantum one. There are many facets to this question, ranging from a "theory of initial conditions" for the Wheeler-DeWitt equation to the origin of the vacuum state responsible for the emergence of density fluctuations, to how such fluctuations become classical. At present, there are no final answers to these questions.

While the relation between a quantum system and its classical counterpart has many facets, see [1] for a nice overview and [2] for a discussion in the context of quantum gravity, the notion of semiclassical state plays an important role. For a "standard" quantum system such as the harmonic oscillator there is a well-developed notion of semiclassical state, namely the "coherent state", characterized by properties such as minimum uncertainty, peakedness on a classical configuration, and the relationship to classical physics that arises via Ehrenfest theorems. For quantum gravity the WKB approximation has been the more common approach for exploring semiclassical physics, although there has been work in cosmology that uses a notion of semiclassical state [3].

In this paper we develop the semiclassical sector of quantum cosmology. To do this
we use a "polymer" quantization of Friedmann-Robertson-Walker (FRW) cosmology in the Arnowitt-Deser-Misner (ADM) canonical variables that was recently presented in [4]. (For a related discussion using connection-triad variables see Refs. [5, 6, 7].) We give a construction of semiclassical states, and show that these states have properties such as being peaked on a point in classical phase space, and satisfying minimal uncertainty relations.

We then outline how these states can be used in applications. First we discuss how to calculate quantum corrections to the classical FRW dynamics by calculating expectation values of the quantum dynamical equations in those coherent states. There are some options available for this, reflecting approaches one can take to gravitational dynamics.

The second application concerns the question of the initial state of the universe. The basic idea is to posit that the present state of the universe is described by a semiclassical state (to be described below), and then ask questions about the history of the universe by evolving this state backward (or forward) in time. Or, put the other way around, what quantum state when evolved for a sufficiently long time leads to the state that we observe today, i.e. a semiclassical state peaked on a flat FRW cosmology with some matter content? This requires a notion of time and its corresponding true Hamiltonian, which we obtain by fixing a time gauge. This provides a computational framework that allows one to compute the "initial state" of the universe.

The paper is structured as follows: In section II we describe the classical system. In section III we recall its quantization as developed in [4], with slight modifications to better suit our goals here, and then introduce semiclassical states. We prove several properties which are physical requirements for a semiclassical interpretation of these states. In the final section we present an outline of two interesting applications of these coherent states.

II. CLASSICAL THEORY

Our starting point is the ADM Hamiltonian action for general relativity minimally coupled to a massless scalar field

$$S = \frac{1}{8\pi G} \int d^3x dt \left( \tilde{\pi}^{ab} \dot{q}_{ab} + p_\phi \dot{\phi} - NH - N^a C_a \right)$$  \hspace{1cm} (1)

where the Hamiltonian and diffeomorphism constraints are

$$H = \frac{1}{\sqrt{q}} \left( \tilde{\pi}^{ab} \tilde{\pi}_{ab} - \frac{1}{2} \tilde{\pi}^2 \right) + \sqrt{q} \left( \Lambda - R(q) \right) + 8\pi G \left( \frac{1}{2} p_\phi^2 + \frac{1}{2} \sqrt{q} q^{ab} \partial_a \phi \partial_b \phi \right)$$  \hspace{1cm} (2)

$$C_a = D_c \tilde{\pi}^c_a + 8\pi G p_\phi \partial_a \phi,$$  \hspace{1cm} (3)

where $\tilde{\pi} = \tilde{\pi}^{ab} q_{ab}$, $R(q)$ is the Ricci scalar of the spatial metric $q_{ab}$, and $\Lambda$ is the cosmological constant.
A reduction to the flat homogeneous isotropic case may be done by writing a parametrization for the canonical pair \((q_{ab}, \tilde{\pi}^{ab})\). For FRW cosmology a suitable choice is

\[ q_{ab} = a^2(t) e_{ab} \]

\[ \tilde{\pi}^{ab} = \frac{p_a(t)}{2a} e^{ab} \]

where \(e_{ab}\) is the flat Euclidean metric. Plugging this into the ADM 3+1 action gives the reduced action

\[ S^{\text{Red}} = \int dt \frac{1}{8\pi G} \left( p_a \dot{a} + p_\phi \dot{\phi} - N H_R \right) \]

where the reduced Hamiltonian constraint, or equivalently the Friedman equation in canonical coordinates, is

\[ H_R = -\frac{3}{8} \frac{p_a^2}{|a|} + |a|^{3}\Lambda + 8\pi G \frac{p_\phi^2}{2|a|^3} = 0. \]

The fundamental Poisson bracket relations are

\[ \{a, p_a\} = 8\pi G, \quad \{\phi, p_\phi\} = 1. \]

The topology of the reduced phase space (for gravity and matter) is \(R^2 \times R^2\). In \(c = 1\) units the gravitational phase space variables each have dimension length.

The configuration and translation variables \(a\) and

\[ U_\lambda(p_a) = \exp \left( i\lambda p_a / L \right) \]

satisfy the algebra

\[ \{a, U_\lambda\} = \frac{8\pi G}{L} i\lambda U_\lambda. \]

As the classical limit for the coherent states constructed below is obtained by sending a dimensionless parameter \(t\) to zero (the correct implementation of the textbook style \(\hbar \to 0\) limit), it turns out to be useful to work with the dimensionless variables \(\tilde{a} = \frac{a}{L}\) and \(\tilde{p}_a = \frac{p_a}{L}\), for which the Poisson bracket becomes

\[ \{\tilde{a}, \tilde{U}_\lambda\} = \frac{8\pi G}{L^2} i\lambda \tilde{U}_\lambda, \]

where \(\tilde{U}_\lambda = \exp(i\lambda \tilde{p}_a)\). From here onwards we drop the tilde and use the dimensionless variables. This is the basic bracket that will be realized as a commutator in the quantum theory.

Another observable of interest is the inverse scale factor \(1/a\), which may be represented by classical identities of the type \[8\]

\[ \frac{1}{|a|} = -\frac{4L^4}{(8\pi G)^2 \lambda^2} \left( U^*_\lambda \left\{ U_\lambda, \sqrt{|a|} \right\} \right)^2. \]
Such expressions are useful in that a representation of the variables $a$ and $U_\lambda$ leads, via the right hand side, to a realization of inverse scale factor and curvature operators that are well-defined even on the state corresponding to the classical singularity.

For a dynamical system with a Hamiltonian constraint there are two ways to proceed to quantization– either via a reduced Hamiltonian obtained from a time gauge-fixing, or by a Hamiltonian constraint operator. The former has the advantage that it "solves" the problem of time at the classical level by an explicit deparametrization, but it leaves open the question of unitary equivalence of quantum theories obtained from different time gauges. The latter is aimed at obtaining fully gauge invariant states in which the problem of time must still be addressed in some way before dynamical processes can be described.

In vacuum gravity a gauge choice is a suitable function of the canonical coordinates and momenta $t = f(q, \pi)$. The functions $f$ may be subdivided into three classes – intrinsic, extrinsic or mixed depending on whether $f$ depends on the 3-metric $q$, its conjugate momentum $\pi$, or on both. If there is matter coupling, there is the additional possibility of choosing the time (and other coordinate gauges) by choosing functions $f$ that depend only on the matter variables. When a canonical gauge fixing condition is used, the requirement that it be preserved in time leads to equations that fix the lapse and shift functions.

For the covariant Einstein equations, gauge choices are made by fixing directly the lapse and shift. For example the choice $N = \text{constant}$ is often made in cosmology. A canonical choice for gravity coupled to a scalar field that has been much studied is $t = \phi$ \cite{3, 10}, especially in relation to the problem of curvature singularity avoidance. We exhibit here the canonical gauge-fixing conditions corresponding to the covariant choices $N = 1$ ("Hubble time") and $N = a(t)$ (conformal time), and derive the corresponding reduced Hamiltonians. We do the same for the gauge condition $\phi = t$.

Consider first the time gauge $t = ka^2/p_a$ where $k$ is a constant. The requirement that this condition be preserved in time is

$$\dot{t} = 1 = \left\{ \frac{ka^2}{p_a}, NH \right\},$$  \hspace{1cm} (13)

which gives (for $\Lambda = 0$)

$$N = sgn(a)k^{-1} \left( -\frac{9}{8} - \frac{12\pi G p_a^2}{a^2 p_a^2} \right)^{-1}.$$  \hspace{1cm} (14)

Solving the Hamiltonian constraint strongly gives

$$\frac{8\pi G p_a^2}{a^2 p_a^2} = \frac{3}{4},$$  \hspace{1cm} (15)

so we get $N = -sgn(a)4/9k$. Thus $k = -sgn(a)9/4$ gives $N = 1$. The main point here is the observation that Hubble time gauge corresponds to a canonical time choice proportional to $a^2/p_a$. 
The reduced Hamiltonian is proportional to the variable canonically conjugate to the time choice, evaluated on the solution of the Hamiltonian constraint. For the Hubble time gauge this is

\[ h = \pm \frac{2}{3t} \sqrt{\frac{8\pi G\rho}{3}}. \]  

(16)

Consider next the time gauge \( t = ka/p_a \) where \( k \) is again a constant. This leads to the lapse function

\[ N = \text{sgn}(a) \frac{a}{k} \left( -\frac{9}{8} - \frac{12\pi G\rho^2}{a^2 p_a^2} \right)^{-1}, \]  

(17)

which gives \( N = a \) for \( k = -9/4 \). This leads to the FRW metric written in conformal time. The corresponding reduced Hamiltonian is

\[ h = \pm \frac{1}{t} \sqrt{\frac{8\pi G\rho}{3}}, \]  

(18)

which is proportional to the Hubble time Hamiltonian. This is an accident of homogeneity and the fact that \( ht \sim ap_a \) for both of these time choices. Note also that the Hamiltonian gauge conditions corresponding to these commonly used covariant gauge choices are a mixture of the geometrodynamic coordinates and momenta.

Finally for the \( t = \phi \) gauge, the lapse function is

\[ N = \text{sgn}(a) \frac{a^3}{8\pi G \rho}, \]  

(19)

and the reduced Hamiltonian is

\[ h = \pm \sqrt{\frac{3}{32\pi G}} ap_a, \]  

(20)

which is time independent.

**III. QUANTIZATION**

We summarize briefly the quantization of this model presented in [4]. The definition of basic variables used here is slightly different in that we use dimensionless phase space variables for quantization, which leads to a dimensionless parameter \( t = (l_P/L)^2 \) in the operator expressions (\( l_P \) is the Planck length and \( L \) is an external scale). This parameter is then utilized in the construction of semiclassical states.

The (kinematical) Hilbert space on which the basic variables are realized has a basis given by the kets \( |\mu\rangle \equiv |\exp(i\mu p_a)\rangle \), where the quantum numbers \( \mu \in \mathbb{R} \), with the inner product

\[ \langle \mu | \nu \rangle = \delta_{\mu\nu}. \]  

(21)
The basic variables are represented by

\[ \hat{a}|\mu\rangle = 8\pi t \mu |\mu\rangle, \quad (22) \]
\[ \hat{U}_\lambda |\mu\rangle = |\mu - \lambda\rangle, \quad (23) \]

which gives the commutator

\[ [\hat{a}, \hat{U}_\lambda] = -8\pi t \lambda \hat{U}_\lambda. \quad (24) \]

The (kinematical) Hilbert space is not separable since, unlike the Schrödinger representation, the inner product is such that configuration variable eigenstates are normalizable. As a consequence, the infinitesimal generators of translations, i.e. operators corresponding to \( p_a \), do not exist in this Hilbert space. This is the essential difference from the Schrödinger representation, and it leads to a fundamental inherent lattice structure at the quantum level.

The interested reader is referred to [4] for more details, and to [11, 12] for other applications of this type of quantization.

With the representation (23) an inverse scale factor operator is readily constructed using (12). It is diagonal in the basis with eigenvalue given by

\[ \frac{1}{|a|} |\mu\rangle = \frac{1}{2\pi \lambda^2 t} \left( |\mu|^{1/2} - |\mu - \lambda|^{1/2} \right)^2 |\mu\rangle. \quad (25) \]

Although the (kinematical) Hilbert space used in this quantization is not separable, the dynamics selects a separable subspace, once an initial state has been chosen. Thus all computations are naturally restricted to separable subspaces, and this extends to the semiclassical sector constructed below. An example is given by the span of the vectors

\[ |m\rangle \equiv |\mu_0 + m\mu\rangle, \quad (26) \]

where \( m \) is an integer, and \( \mu \) and \( \mu_0 \) are arbitrary real numbers; \( \mu_0 \) may be viewed as the "origin" of a lattice with spacing given by \( \mu \). In order to utilize this subspace, we must work with operators that do not take us out of it. Since all operators are constructed from \( a \) and \( U_\lambda \), this is accomplished by working only with those \( U_\lambda \)'s adapted to the subspace, i.e. we must set \( \lambda = \mu \). In the following we set \( \mu_0 = 0 \) and work in the subspace \( |m\lambda\rangle \).

**IV. SEMICLASSICAL STATES**

Coherent states for a particle in the Schrödinger representation are of the form

\[ \psi \sim \frac{1}{\sqrt{t}} e^{-(x-x_0)^2/2t + i x_0 p_0}. \quad (27) \]

These are peaked at the classical phase space point \((x_0, p_0)\) – in this sense they are semiclassical. The peaking properties may be seen by computing the expectation values of the \( \hat{x} \) and
\[ \hat{p} \] operators in this state. They also have the additional property that they are eigenstates of the operator \[ \hat{x} - i \hat{p} \].

We would like to construct semiclassical states for FRW cosmology in the representation described above, motivated by the same considerations. Such states have been discussed in loop quantum gravity [13], where the holonomy of a connection based on a spatial loop is the analog of the translation variable \( U_\lambda \). In particular the case of the \( U(1) \) gauge theory coherent states discussed there is similar to what we require.

These considerations motivate the definition of states

\[ |\alpha, \beta\rangle_{t,\lambda} = \frac{1}{C} \sum_{m=-\infty}^{\infty} e^{-\frac{1}{4}(\lambda m)^2} e^{i\lambda \alpha} e^{im\lambda \beta} |m\rangle. \]  

(28)

The normalization constant \( C > 0 \) is given by the convergent sum

\[ C^2 = \sum_{m=-\infty}^{\infty} e^{-t\lambda^2 m^2} e^{2\alpha \lambda m}. \]  

(29)

The real parameters \( \alpha \) and \( \beta \) correspond to a classical configuration in the same sense as the parameters \( x_0, p_0 \) in the state (27), as we now show.

A first check is to verify that the states (28) are eigenstates of an operator analogous to the annihilation operator \( \hat{A} = \hat{x} + i \hat{p} \) for a Schrödinger particle. However, since the momentum operator is not directly represented in this quantization, the closest we have is the exponential \( e^A \equiv e^{x+ip} \), which is represented by the operator

\[ e^{\gamma \hat{a}} \hat{U}_\lambda(\hat{p}_a), \]  

(30)

where the parameter \( \gamma \) is determined by the condition that the state (28) is an eigenstate of it. It is straightforward to verify that

\[ e^{(\lambda/8\pi)\hat{a}} \hat{U}_\lambda|\alpha, \beta\rangle_{t,\lambda} = e^{t\lambda^2 / 2} e^{\lambda(\alpha+i\beta)} |\alpha, \beta\rangle_{t,\lambda}. \]  

(31)

This result suggests that the expectation values of operators \( \hat{O}(\hat{a}, \hat{U}_\gamma) \) are peaked at the corresponding classical phase space functions \( O(a, p_a) \) in the limit \( t \to 0 \). That this is in fact the case is established by direct calculation of expectation values. The limit \( t \to 0 \) requires use of the Poisson re-summation formula

\[ \sum_{m=-\infty}^{\infty} f(sm) = \frac{2\pi}{s} \sum_{m=-\infty}^{\infty} \tilde{f}\left(\frac{2\pi m}{s}\right), \]  

(32)

where \( s \) is a real parameter, \( f \) is function on the real line, and \( \tilde{f} \) is its Fourier transform

\[ \tilde{f}(k) = \int_{-\infty}^{\infty} dx \ f(x) e^{-ikx}. \]  

(33)
As illustrative examples, let us compute in the state (28) the expectation value of \( \hat{a} \), \( \hat{U}_\lambda \), and of an expression for the momentum operator. The latter in this quantization is represented by

\[
\hat{p}_\lambda^a \equiv \frac{1}{2i\lambda} \left( \hat{U}_\lambda - \hat{U}_\lambda^\dagger \right).
\]

Let us first note that the normalization constant for the semiclassical states (29) may be rewritten using (32) as

\[
C^2 = \sqrt{\frac{\pi}{\lambda^2 t}} e^{\alpha^2 t/2} \left( 1 + 2 \sum_{m \neq 0} \cos \left( \frac{2\pi m \alpha}{\lambda t} \right) e^{-\pi^2 m^2 / t\lambda^2} \right).
\]

This form of the result facilitates taking the \( t \to 0 \) limit since the terms in the sum get damped to zero. Using this we obtain

\[
\langle \hat{a} \rangle = 8\pi\alpha \left( 1 + 2 \sum_{m \neq 0} \cos \left( \frac{2\pi m \alpha}{\lambda t} \right) e^{-\pi^2 m^2 / t\lambda^2} \right),
\]

\[
\langle \hat{U}_\lambda \rangle = e^{i\lambda \beta} e^{-t\lambda^2 / 4} \left( 1 + 2 \sum_{m \neq 0} \cos \left( \frac{2\pi m \alpha}{\lambda t} \right) \left( 1 + \frac{\lambda}{2\alpha} \right) e^{-\pi^2 m^2 / t\lambda^2} \right).
\]

Note that these expressions have the limits expected of semiclassical states:

\[
\lim_{t \to 0} \langle \hat{a} \rangle = 8\pi\alpha,
\]

\[
\lim_{t \to 0} \langle \hat{U}_\lambda \rangle = e^{i\lambda \beta}.
\]

Equation (37) together with the definition (34) gives the expectation value

\[
\langle \hat{p}_\lambda^a \rangle = \frac{\sin(\beta \lambda)}{\lambda} e^{-t\lambda^2 / 4} \left( 1 + 2 \sum_{m \neq 0} \cos \left( \frac{2\pi m \alpha}{\lambda t} \right) \left( 1 + \frac{\lambda}{2\alpha} \right) e^{-\pi^2 m^2 / t\lambda^2} \right),
\]

where the Poisson re-summation formula has been used in the last step. This formula has the limits

\[
\lim_{t \to 0} \langle \hat{p}_\lambda^a \rangle = \sin(\beta \lambda) / \lambda,
\]

\[
\lim_{\lambda \to 0} \langle \hat{p}_\lambda^a \rangle = \beta.
\]

The first shows that the semiclassical state on the lattice is peaked at the corresponding phase space value. The second shows that the continuum limit of the momentum expectation value has the appropriate peaked value in this state, even though only the translation operators exist in the representation we are using for the quantum theory.

It is also possible to define \( \lambda \) dependent creation and annihilation operators in this quantization via

\[
\hat{A}^\lambda \equiv \hat{a} - i\hat{p}_\lambda^a
\]
and its adjoint. From the above result for the expectation value of $\hat{p}_a^\lambda$ it follows that

$$\lim_{t \to 0} \langle \hat{A}^\lambda \rangle = \alpha + i\beta,$$

(44)

and hence that the semiclassical state with $\alpha = \beta = 0$ may be compared to the usual oscillator vacuum in this alternative quantization. This ”vacuum” $|\alpha = 0, \beta = 0\rangle$ may be viewed as a ”collective” state in the sense that it is an infinite linear combination of suitably weighed elementary states $|m\rangle$. The states resulting from repeated action of $A^\lambda t$ on this vacuum similarly provide a correspondence with the excited oscillator states. This idea may also be applied to a related quantization of the scalar field [15], and the gravity scalar field model in spherical symmetry [16] to gain more insight into how the usual background dependent Fock quantization of the scalar field is related to the present one.

It is also possible to see that the wave function corresponding to the state (28) is peaked in the same way as the one for the oscillator. Recall that the basis elements in which we write the semiclassical state are configuration eigenstates with wave function $e^{-i\lambda mp}$. Therefore the momentum space wave function corresponding to the state (28) is

$$\psi(\alpha,\beta)(p) = \frac{1}{C} \sum_{m=-\infty}^{\infty} e^{-\frac{1}{2}(\lambda m)^2} e^{i\lambda m\alpha} e^{im(\lambda\beta-p)}.$$

(45)

This gives the momentum space probability distribution

$$\frac{1}{C^2} |\psi(\alpha,\beta)(p)|^2 = \sqrt{\frac{4\pi}{\lambda^2 t}} e^{-(\rho-\beta)^2/2t} \frac{1 + 2 \sum_{m\neq 0} \cos \left(\frac{2\pi m \alpha}{\lambda t}\right) e^{-\pi^2 m^2/\lambda^2 t} - 2\rho m \beta/\lambda t |^2}{1 + 2 \sum_{m\neq 0} \cos \left(\frac{2\pi m \alpha}{\lambda t}\right)} e^{-\pi^2 m^2/\lambda^2 t} \cos \left(\frac{2\pi m \alpha}{\lambda t}\right).$$

(46)

It is evident from this expression that in the limit $t \to 0$ the distribution is peaked at the momentum value $\beta$, just as the position space oscillator wave function (27) is peaked at $x_0$. This is because the sums in the numerator and denominator are damped to zero in this limit.

So far we have verified that the peaking property of the expectation value holds for the basic phase space variables $a$ and $U^\lambda(p_a)$. This is expected to be true also for any function of the basic variables. For example, the explicit calculation for the expectation value of the inverse scale factor (12) gives

$$\lim_{t \to 0} \langle \hat{1}/a \rangle = \frac{1}{8\pi \alpha},$$

(47)

which is the inverse of $\langle \hat{a} \rangle$ (38) in this limit.

All the above properties establish that the states defined in eqn. (28) have the required semiclassical properties.
V. APPLICATIONS AND DISCUSSION

In this section we propose two applications of semiclassical states to cosmology. The first concerns computing quantum corrections to classical dynamics and makes use of the Heisenberg interpretation. The second concerns implementing an idea to obtain the wave function at early times by evolving a semiclassical state backward in time, and uses the Schrödinger representation. Their implementation will appear elsewhere.

A. Quantum corrections to classical dynamics

One of the expectations from a quantum theory of gravity is that it provide a mechanism for the emergence of a classical spacetime in an appropriate limit, and a procedure for computing quantum corrections to classical equations. An immediate application would be to cosmology, which is perhaps the only arena where a quantum gravity theory may be testable.

A possible approach for computing quantum corrections to classical cosmological equations is suggested by the peaking results for semiclassical states proven above. The basic idea is to obtain the Heisenberg equations of motion for the relevant observables, and compute the expectation values of the commutator terms in the semiclassical states. In the $t \rightarrow 0$ limit the resulting equation would give the classical equations because of the results of the last section.

There are two ways that this idea can be implemented — with or without a time gauge fixing. In the former case the Hamiltonian $\hat{h}$ corresponding to the time gauge fixing is derived at the classical level and converted to an operator. The quantum corrected equations are then postulated to be

$$\dot{a} = \langle \alpha, \beta | \left[ \hat{a}, \hat{h} \right] | \alpha, \beta \rangle$$

for the scale factor, with similar equations for the scalar field and the conjugate momenta. The right hand side may be expanded in powers of the parameter $t$ to give the classical term and its corrections order by order.

If working without a time gauge fixing, the Hamiltonian constraint operator would be used with an arbitrary lapse function to obtain the evolution equation. This requires a definition of the Hamiltonian constraint operator $\hat{H}_R$ coming from the classical expression \(7\). The square of the momentum in this constraint may be realized by the operator

$$\left( \hat{p}_a^\lambda \right)^2 = \frac{1}{\lambda^2} \left( 2 - \hat{U}_\lambda - \hat{U}_\lambda^\dagger \right)$$

This, together with the operator corresponding to $1/|a|$ given in Eqn. \(12\) gives an expression for the first term in the Hamiltonian constraint operator (with a choice of operator ordering).
The semiclassical state peaked on a classical solution of the constraint satisfies

$$\lim_{t \to 0} \langle \hat{H}_R \rangle = 0 + O(t^\sigma),$$  \hfill (50)

where the power $\sigma > 0$ of the first quantum correction to expectation value may depend on the choice of operator ordering in the $p_2^2/a$ factor in the constraint \( \mathbf{7} \). Without the limit, this equation gives quantum corrections to the Friedman equation. Similarly, corrections to the Hamiltonian evolution are obtained by computing the right hand sides of lapse dependent equations such as

$$\dot{a} = N \langle \alpha, \beta | \left[ \hat{a}, \hat{H}_R \right] | \alpha, \beta \rangle.$$  \hfill (51)

We emphasize that this procedure is quite different from what is usually called ”the semiclassical approximation” in quantum gravity, which treats gravity classically and matter quantum mechanically. The central difference is that here the matter and gravity variables are treated at the same level – the full state used to compute quantum corrections is the tensor product of the matter and gravity semiclassical states. It is however possible, and quite straightforward to obtain this usual and more limited approximation from our more general procedure by taking the $t \to 0$ limit in the expectation values for only those terms that contain the gravitational variables. This effectively makes gravity variables classical, with the matter and interaction parts receiving the $t$ dependent quantum corrections, with an implicit choice of ”vacuum” defined by the matter semiclassical state.

B. Initial state of the Universe

Semiclassical states may be used as a ”present time condition” for the cosmological state of the Universe. This is a reasonable assumption because observations suggest that an FRW model provides a good large scale description. This state may be evolved into the past or the future using the Hamiltonian operator obtained by a time gauge fixing. It is apparent from the form of the Hamiltonian that such evolution leads, after some time steps, to a new state that is not of the form \( \mathbf{28} \). An initial state can be tracked to early times by following the evolution of the probability density \( \mathbf{46} \).

It is perhaps easiest to implement this procedure using the time independent Hamiltonian obtained from the $\phi = t$ gauge. After this gauge fixing the canonical variables are the pair $(a, p_a)$ with Hamiltonian $h \sim ap_a$. A time step evolution of an initial state using a simple scheme such as

$$\psi(t + \Delta t) = \left( I + i\hbar \Delta t \hat{h} \right) \psi(t)$$  \hfill (52)

may be implemented numerically to see how the state evolves to the past and future, and also to obtain an idea of the degree of coherence that is retained by evolution.
Although such evolution is unitary by construction, numerical implementation restores it only up to some order in the time step $\Delta t$. For example for the simple explicit scheme given above, unitarity is not exact with violations of order $\Delta t^2$. There are known implicit schemes whose unitary behaviour is much better. An example is provided by a modified Crank-Nicholson method where the Schrödinger equation is discretised as

$$\frac{i}{\Delta t} [\psi(t + \Delta t) - \psi(t)] = \frac{\hbar}{2} [\psi(t + \Delta t) + \psi(t)].$$  \hspace{1cm} (53)

This time stepping scheme remains useful if the gauge fixing is such that the Hamiltonian has explicit time dependence.

There are other physical situations where these semiclassical states may be used in cosmology. One of these is the question of quantum gravity corrections to the spectrum of density perturbations. There has been an initial exploration of this question without coherent states [17], where a quantum gravity corrected FRW scalar wave equation is obtained by replacing inverse scale factor terms by the eigenvalue of the corresponding operator (12) in a basis state. In the energy regime where this calculation is normally done, spacetime is approximately classical. Therefore it would be interesting to do such a calculation with the expectation value taken in the appropriate semiclassical state, and expanded to the desired order in the Planck length. This would give controlled corrections to the usual quantum-fields-on-a-classical-background semiclassical approximation. Work on developing these applications is in progress.

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[18] After this work was completed, we became aware of ref. [14] where there is a related discussion of an approximate momentum operator in this representation.