Torsion Invariants of Combed 3-Manifolds with Boundary Pattern and Legendrian Links

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Abstract. We extend Turaev’s definition of torsion invariants of 3-dimensional manifolds equipped with non-singular vector fields, by allowing (suitable) tangency circles to the boundary, and manifolds with non-zero Euler characteristic. We show that these invariants apply in particular to (the exterior of) Legendrian links in contact 3-manifolds. Our approach uses a combinatorial encoding of vector fields, based on standard spines. In this paper we extend this encoding from closed manifolds to manifolds with boundary.

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Reidemeister torsion is a classical yet very vital topic in 3-dimensional topology, and it was recently used in a variety of important developments. To mention a few, torsion is a fundamental ingredient of the Casson-Walker-Lescop invariants (see e.g. [12]), and more generally of the perturbative approach to quantum invariants (see e.g. [11]). Relations have been pointed out between torsion and hyperbolic geometry [20]. Turaev’s torsion of non-singular vector fields on 3-manifolds [22] has been recognized to have deep connections with some 3-dimensional versions of the Seiberg-Witten invariants [16], [23]. It is also worth recalling that vector fields (and framings), have also been used by Kuperberg [10] to construct new invariants of different nature, based on Hopf algebras more general than those employed for quantum invariants. (There are reasons to speculate that also Kuperberg’s invariants should have a torsion content and relations with Turaev’s work, but we do not insist on this.)

In this paper, using (actually, improving) our theory of branched standard spines [2] and building on [22], we extend Turaev’s definition of torsion by allowing vector fields to have (appropriate) tangency circles to the boundary. Moreover, we do not require the manifolds to have zero Euler characteristic (an assumption which is at the base of Turaev’s theory). To be precise, we accept any compact oriented manifold with (possibly empty) boundary, and non-singular vector fields with the only restriction

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that their orbits should be tangent to the boundary “from inside” (i.e. along circles of concave apparent contour). Equivalence is given by homotopy through fields of the same type. Recall that Turaev never accepts tangency to the boundary, and the most important applications in [23] are given when the boundary is empty or a union of tori with the field pointing inwards on them.

One of the topological situations in which we are able to define torsion invariants arises quite naturally when one considers Legendrian links in contact 3-manifolds. Remark in particular that our torsions are defined also for homologically non-trivial Legendrian links, when the usual invariants, such as the rotation number (Maslov index), are not defined.

Our definition of torsion is based on a combinatorial encoding of non-singular concave vector fields on manifolds with boundary. The topological-geometric counterpart of this encoding is the theory of standard spines with the further structure of branched surface. The foundations of this theory, together with the combinatorial encoding of vector fields in the special case of closed manifolds, were provided in [2]. The main technical result of the present paper is the extension of this encoding to the case with boundary. We also show that one of the combinatorial moves taken into account in [2] can be discarded. This implies the rather significant fact that the branched versions of the basic 2-to-3 Matveev-Piergallini move are actually sufficient.

If a concave vector field \( v \) on a manifold \( M \) is encoded by a branched spine \( P \), in order to compute torsions we consider a cell complex \( X(P) \), whose support is obtained by collapsing the whole \( \partial M \) to an internal point. Moreover we use \( v \) to construct a “canonical spider” in \( X(P) \), and then we use this spider to determine a fundamental family of cells in the universal cover of \( X(P) \). The fact that we are able to accept manifolds with non-zero Euler characteristic essentially depends on the fact that our complex naturally arises as a pointed space, and the spider is connected, with head at the basepoint. We recover the situation considered by Turaev as a variation on our basic definition, by leaving uncollapsed the boundary components on which the field points inwards. The use of branched standard spines, in connection with torsion, allows a considerable simplification of the proof of invariance. All our proofs will be direct combinatorial arguments. Even if obviously inspired to Turaev’s [22], our work is essentially self-contained.

We conclude the introduction by pointing out an interesting subtlety which arises when dealing with torsions (see Subsection 1.3 for exact statements). We start by recalling that, in general, the torsion of a complex \( X \), which depends on a ring homomorphism \( \varphi : \mathbb{Z}[\pi_1(X)] \to \Lambda \), is an element of \( \mathcal{K}_1(\Lambda) \) which is only well-defined up the action of \( \varphi(\pi_1(X)) \). Turaev’s main achievement in [22] is the proof that this action can be disposed of when \( X \) is a manifold with a vector field defined on it. However, it may still be the case that the automorphism group of \( X \) acts non-trivially on torsions. We show in the present paper that when dealing with Legendrian links this action can sometimes be neglected, which leads to a sharper version of the invariant.
1 Torsion(s) of a branched spine

In this section we will briefly recall the notion of branched spine, we will describe how to associate a certain cell complex $X(P)$ to each branched spine $P$, and we will show that the branching of $P$ allows to define a canonical “spider” in $X(P)$. We will then define a torsion $\tau^\varphi(P, h) \in K_1(\Lambda)$ for a (suitable) representation $\varphi$ of $\pi_1(X)$ into the multiplicative group of a (suitable) ring $\Lambda$, where $K_1(\Lambda)$ denotes the Whitehead group of $\Lambda$ and $h$ is a $\Lambda$-basis of the $\varphi$-twisted homology module of $X(P)$, which is assumed to be free. Later we will describe some variations on the definitions of $X(P)$ and $\tau^\varphi(P, h)$.

The first subsection establishes various notations used extensively in this paper.

1.1 Reminder on branched spines

A quasi-standard polyhedron $P$ is a finite connected 2-dimensional polyhedron with singularity of stable nature (triple lines and points where 6 non-singular components meet). Such a $P$ is called standard if all the components of the natural stratification given by singularity are open cells. Depending on dimension, we will call the components vertices, edges and regions. A screw-orientation for $P$ is a screw-orientation on all its edges, with the obvious compatibility at vertices. A branching on $P$ is an orientation for each region in $P$, such that no edge is induced the same orientation three times. See [2] for careful definitions of all these notions.

A standard polyhedron with a fixed oriented branching and screw-orientation will be called branched polyhedron, and denoted typically by $P$. We will not use specific notations for the extra structures: they will be considered to be part of $P$. Unless the contrary is explicitly stated, by “manifold” we will mean a connected, oriented, compact 3-dimensional manifold, with or without boundary. Using the Hauptvermutung, we will freely intermingle the differentiable, piecewise linear and topological viewpoints. Homeomorphisms will always respect orientations.

All vector fields mentioned in this paper will be non-singular, and they will be termed just fields for the sake of brevity. On a manifold $M$ we will consider concave fields, namely non-singular fields which are simply tangent to $\partial M$ only along a finite union $\Gamma$ of circles, with the property that near tangency points orbits are contained in $\text{Int}(M)$. Note that $\Gamma$ splits $\partial M$ into a white portion $W$, on which the field points inwards, and a black portion $B$, on which the field points outwards.

It turns out [1] that a branched polyhedron with a screw-orientation is automatically the spine of a manifold, which is unique by [3], so we will often replace the term ‘polyhedron’ by ‘spine’. The following result, proved in [3], is the starting point of our constructions.

**Proposition 1.1.** To every branched spine $P$ there corresponds a manifold $M(P)$ with non-empty boundary and a concave field $v(P)$ on $M(P)$. The pair $(M(P), v(P))$ is well-
defined up to homeomorphism. Moreover an embedding $i : P \rightarrow \text{Int}(M(P))$ is defined, and has the property that $v(P)$ is positively transversal to $i(P)$.

The topological construction which underlies this proposition is actually quite simple, and it is illustrated in Fig. 1. Concerning the last assertion of the proposition, note that the branching allows to define an oriented tangent plane at each point of $P$.

Even if it is not necessary now, we inform the reader that suitable restrictions of the map $P \mapsto (M(P), v(P))$ defined above are surjective. One of the main achievements of [2] was the introduction of an equivalence relation on branched spines which makes this map injective, when restricted to manifolds bounded by $S^2$ (essentially, closed manifolds). In Section 3 we will provide a substantial improvement of this result.

**Spines and ideal triangulations.** We remind the reader that an ideal triangulation of a manifold $M$ with non-empty boundary is a partition $T$ of $\text{Int}(M)$ into open cells of dimensions 1, 2 and 3, induced by a triangulation $T'$ of the space $Q(M)$, where:

1. $Q(M)$ is obtained from $M$ by collapsing each component of $\partial M$ to a point;
2. $T'$ is a triangulation only in a loose sense, namely self-adjacencies and multiple adjacencies of tetrahedra are allowed;
3. The vertices of $T'$ are precisely the points of $Q(M)$ which correspond to components of $\partial M$.

It turns out ([13], [18], [15]) that there exists a natural bijection between standard spines and ideal triangulations of a 3-manifold. Given an ideal triangulation, the corresponding standard spine is just the 2-skeleton of the dual cellularization, as illustrated in Figure 2. The inverse of this correspondence will be denoted by $P \mapsto T(P)$. It will be convenient in the sequel to call centre of a cell of $P$ the only point in which the cell meets the simplex of $T(P)$ dual to it.

Now let $P$ be a branched spine. First of all, we can realize $T(P)$ in such a way that its edges are orbits of the restriction of $v(P)$ to $\text{Int}(M(P))$, and the 2-faces are
unions of such orbits. Being orbits, the edges of $\mathcal{T}(P)$ have a natural orientation, and the branching condition implies (as remarked in $[8]$) that on each tetrahedron of $\mathcal{T}(P)$ exactly one of the vertices is a sink and one is a source.

It is quite interesting to remark that not only the edges, but also the faces and the tetrahedra of $\mathcal{T}(P)$ have natural orientations. For tetrahedra, we just restrict the orientation of $M(P)$. For faces, we first note that the edges of $P$ have a natural orientation (the prevailing orientation induced by the incident regions). Now, we orient a face of $\mathcal{T}(P)$ so that the algebraic intersection in $M(P)$ with the dual edge is positive.

1.2 Basic definition of torsion

Let $P$ be a branched spine.

**Definition (triangulated complex associated to $P$).** Let $X(P) = M(P)/\partial(M(P))$ and let $x_0(P)$ be the point which corresponds to $\partial(M(P))$. Since $X(P) \setminus \{x_0(P)\} \cong \text{Int}(M(P))$, $P$ naturally embeds in $X(P)$. Moreover, a field $\overline{\pi}(P)$ is defined on $X(P) \setminus \{x_0(P)\}$, and the closure in $X(P)$ of each infinite half-orbit of $\overline{\pi}(P)$ is obtained by adding $x_0(P)$. The ideal triangulation $\mathcal{T}(P)$ of $M(P)$ induces a triangulation $\mathcal{T}(P)$ of $X(P)$, with only vertex $x_0(P)$ and open positive-dimensional simplices which correspond to those of $\mathcal{T}(P)$ and are unions of orbits of $\overline{\pi}(P)$.

**Remark 1.2.** (i) The Euler characteristic of $X(P)$ is given by

$$\chi(X(P)) = 1 - (1/2) \cdot \chi(\partial(M(P))) = 1 - \chi(M(P)) = 1 - \chi(P).$$

(ii) Stipulating $x_0(P)$ to be positive, and using the remarks made above, we see that also in $\mathcal{T}(P)$ all the simplices have a natural orientation.

**Definition (spider associated to $P$).** We define $s(P)$ as the singular 1-chain in $X(P)$ obtained as $\sum_c \alpha_c$, where $c$ runs over the centres of cells of $P$, and $\alpha_c$ is the closure of the positive orbit of $\overline{\pi}(P)$ which starts at $c$. Note that the final endpoint of each $\alpha_c$ is $x_0(P)$. 

5
Remark 1.3. Let \( \varepsilon(c) = (-1)^d \) if \( c \) is the centre of a \( d \)-cell. Then

\[
\partial \left( \sum_c \varepsilon(c) \cdot \alpha_c \right) = (1 - \chi(X(P))) \cdot x_0(P) + \sum_c \varepsilon(c) \cdot c.
\]

So, if \( \chi(X(P)) = 0 \), the chain \( \sum_c \varepsilon(c) \alpha_c \) is precisely a spider (or Euler chain) in Turaev’s terminology \([22]\). But our definition makes sense also for \( \chi(X(P)) \neq 0 \), so our situation is indeed more general. We have a preferred basepoint which is part of the structure from the beginning, and the “error” in the boundary of our spider is automatically located at the basepoint. This fact will be crucial also below.

For the sake of brevity, in the sequel we will denote \( \pi_1(X(P), x_0(P)) \) just by \( \pi \). We denote now by \( (\tilde{X}(P), \tilde{x}_0(P)) \) a universal cover of \( (X(P), x_0(P)) \). The reason for considering pointed spaces is that any two such covers are canonically isomorphic, and all our constructions will obviously be equivariant under this isomorphism. On \( \tilde{x}_0(P) \) we consider the action of \( \pi \) defined using the basepoint \( \tilde{x}_0(P) \). We denote by \( \tilde{T}(P) \) the \( \pi \)-invariant lifting of \( T(P) \) to \( \tilde{X}(P) \). We will consider in the sequel the complex \( C_*^{\text{cell}}(\tilde{X}(P); \mathbb{Z}) \) of integer chains in \( \tilde{X}(P) \) which are cellular with respect to \( \tilde{T}(P) \).

In a natural way, \( C_*^{\text{cell}}(\tilde{X}(P); \mathbb{Z}) \) is a complex of \( \mathbb{Z}[\pi] \)-modules. Moreover each \( C_1^{\text{cell}}(\tilde{X}(P); \mathbb{Z}) \) is free, and a free basis is determined by the choice of an ordering for the \( i \)-simplices in \( \tilde{T}(P) \) and one lifting for each of them (as remarked, orientations are canonical).

Definition (lifted spider and free basis). We define \( \tilde{s}(P) \) as the singular 1-chain \( \sum_c \tilde{\alpha}_c \) in \( \tilde{X}(P) \), where \( \tilde{\alpha}_c \) is the only lifting of \( \alpha_c \) with second endpoint \( \tilde{x}_0(P) \). We choose \( \tilde{x}_0(P) \) as preferred lifting of \( x_0(P) \). For a positive-dimensional simplex of \( \tilde{T}(P) \) dual to a cell with centre \( c \), we choose as preferred lifting the one which contains the first endpoint of \( \tilde{\alpha}_c \). If \( \sigma \) is an ordering of the simplices in \( \tilde{T}(P) \), we denote by \( g_i(P, \sigma) \) the free \( \mathbb{Z}[\pi] \)-basis of \( C_i^{\text{cell}}(\tilde{X}(P); \mathbb{Z}) \) obtained from \( \sigma \) and these preferred liftings.

We briefly review now the general algebraic machinery used to define torsions \([17]\). We consider a ring \( \Lambda \) with unit, with the property that if \( n \) and \( m \) are distinct positive integers then \( \Lambda^n \) and \( \Lambda^m \) are not isomorphic as \( \Lambda \)-modules. We recall that the Whitehead group \( K_1(\Lambda) \) is defined as the Abelianization of \( GL_\infty(\Lambda) \). Moreover, \( K_1(\Lambda) \) is the quotient of \( K_1(\Lambda) \) under the action of \( -1 \in GL_1(\Lambda) = \Lambda_+ \subset \Lambda \).

Given a free \( \Lambda \)-module \( M \) and two finite bases \( b = (b_k) \), \( b' = (b'_k) \) of \( M \), the assumption on \( \Lambda \) guarantees that \( b \) and \( b' \) have the same number of elements, so there exists an invertible square matrix \( (\lambda^b_k) \) such that \( b'_k = \sum \lambda^b_kb_h \). We will denote by \( [b'/b] \) the image of \( (\lambda^b_k) \) in \( K_1(\Lambda) \).

Definition (twisted homology and chain basis). Going back to the topological situation, let us consider now a group homomorphism \( \varphi : \pi \to \Lambda_+ \), and its natural extension \( \tilde{\varphi} : \mathbb{Z}[\pi] \to \Lambda \) (a ring homomorphism). We can define now the twisted chain complex \( C_*^{\varphi}(P) \), where \( C_1^{\varphi}(P) \) is defined as \( \Lambda \otimes_{\tilde{\varphi}} C_1^{\text{cell}}(\tilde{X}(P); \mathbb{Z}) \), and the boundary operator is induced from the ordinary boundary. Note that \( C_1^{\varphi}(P) \) is a free \( \Lambda \)-module, and each \( \mathbb{Z}[\pi] \)-basis of \( C_1^{\text{cell}}(\tilde{X}(P); \mathbb{Z}) \) determines a \( \Lambda \)-basis of \( C_1^{\varphi}(P) \). We will denote
by $g_i(\mathcal{P}, \sigma)$ the $\Lambda$-basis of $C_i(\mathcal{P})$ corresponding to $g_i(\mathcal{P}, \sigma)$, and by $H_i(\mathcal{P})$ the $i$-th homology group of the complex $C_i(\mathcal{P})$. The canonical isomorphism which exists between two pointed universal covers of $(X(\mathcal{P}), x_0(\mathcal{P}))$ induces an isomorphism of the corresponding homology groups, so $H_i(\mathcal{P})$ is intrinsically defined.

Our assumptions on $\Lambda$ easily imply the following:

**Lemma 1.4.** If $H_i(\mathcal{P}) = 0$ for all $i$ then $\chi(\mathcal{P}) = 0$.

**Definition (torsion — acyclic case).** Assume that $H_i(\mathcal{P}) = 0$. Then we can apply the general definition of torsion of an acyclic chain complex of free $\Lambda$-modules with assigned bases. We briefly review this definition, confining ourselves to the case where the boundary modules are free (in general, a stable basis should be used). So, let $\mathcal{b}_i$ be a finite subset of $C_i(\mathcal{P})$ such that $\partial \mathcal{b}_i$ is a basis of $\partial C_i(\mathcal{P})$. The complex being acyclic, $(\partial \mathcal{b}_{i+1}) \cdot \mathcal{b}_i$ is now a basis of $C_i(\mathcal{P})$, so we can compare it with $g_i(\mathcal{P}, \sigma)$. We define

$$\tau_0^\varphi(\mathcal{P}, \sigma) = \prod_{i=0}^{3} \left[ (\partial \mathcal{b}_{i+1}) \cdot \mathcal{b}_i \big/ g_i(\mathcal{P}, \sigma) \right]^{(-1)^i} \in K_1(\Lambda).$$

(Independence of the $\mathcal{b}_i$’s and invariance under isomorphism of pointed universal covers is readily checked.) Of course $\sigma$ is responsible of at most a sign change, so $\tau^\varphi(\mathcal{P}) = \pm \tau_0^\varphi(\mathcal{P}, \sigma) \in K_1(\Lambda)$ is well-defined.

**Definition (torsion — general case).** It follows from Lemma 1.4 that $C_i(\mathcal{P})$ is often not acyclic. It is a general fact that torsion can be defined also in this case, provided the homology $\Lambda$-modules are free and have assigned bases. Namely, if $\mathcal{h}_i$ is a $\Lambda$-basis of $H_i(\mathcal{P})$, we replace $(\partial \mathcal{b}_{i+1}) \cdot \mathcal{b}_i$ in the above formula by $(\partial \mathcal{b}_{i+1}) \cdot \tilde{\mathcal{h}}_i \cdot \mathcal{b}_i$, where $\tilde{\mathcal{h}}_i$ is a lifting of $\mathcal{h}_i$ to $C_i(\mathcal{P})$. So, we have a torsion $\tau^\varphi(\mathcal{P}, \mathcal{h}) \in K_1(\Lambda)$ when the $\Lambda$-modules $H_i(\mathcal{P})$ are free with basis $\mathcal{h}$.

It is maybe appropriate here to remark that the choice of a basis $\mathcal{h}$ of $H_i(\mathcal{P})$ and the definition of $\tau^\varphi(\mathcal{P}, \mathcal{h})$ implicitly assume a description of the universal cover of $X(\mathcal{P})$, which is typically undoable in practical cases. However, if one starts from a representation of $\pi$ into the units of a commutative ring $\Lambda$, one can use from the very beginning the maximal Abelian rather than the universal cover, which makes computations more feasible.

**Definition (sign-refined torsion).** An enhancement of the definition of torsion, due to Turaev, applies in our situation. As remarked above, the sign ambiguity in the definition of torsion is only due to the ordering $\sigma$ of the cells of $X(\mathcal{P})$. This ambiguity can be removed by considering a homological orientation $\sigma$ of $X(\mathcal{P})$, namely an orientation of all the spaces $H_i(X(\mathcal{P}); \mathbb{R})$. We briefly recall how this construction goes. Using $\sigma$, we get bases of the $C_i^{\text{cell}}(X(\mathcal{P}); \mathbb{R})$’s. Now we choose bases of the $H_i(X(\mathcal{P}); \mathbb{R})$’s compatible with $\sigma$, and we compute the torsion of $C_i^{\text{cell}}(X(\mathcal{P}); \mathbb{R})$ using these bases, thus getting $a \in K_1(\mathbb{R}) = \mathbb{R}_+$. It is easily checked that $\text{sgn}(a) \cdot \tau_0^\varphi(\mathcal{P}, \sigma) \in K_1(\Lambda)$ is now independent of $\sigma$. Hence we get a torsion $\tau^\varphi_0(\mathcal{P}, \mathcal{h}, \sigma) \in K_1(\Lambda)$. Of course, in the acyclic case, $\mathcal{h}$ is omitted.
Manifolds with white boundary components (Turaev’s torsion). It could happen that in $\partial(M(P))$ there are some white boundary components, i.e. components on which the field $v(P)$ points inwards. In this case we can modify the definition of the triangulated complex associated to $P$ by identifying together only the boundary components which are not white. We denote by $X_w(P)$ the space thus obtained, by $x_{w,0}(P)$ the basepoint obtained by collapsing the non-white boundary components, and by $W(P)$ the (homeomorphic) image in $X_w(P)$ of the union of white boundary components. A field $\pi_w(P)$ is naturally induced by $v(P)$ on $X_w(P)\setminus(W(P)\cup\{x_{w,0}(P)\})$, and a spider $s_w(P)$ with head in $x_{w,0}(P)$ can be defined exactly as above.

Now, if we consider the universal covering $\tilde{X}_w(P)$ with a basepoint $\tilde{x}_{w,0}$, we can still lift the spider by locating its head in $\tilde{x}_{w,0}$, but now the legs of the spider only determine liftings of the cells of $X_w(P)\setminus W(P)$. So the spider defines a basis of the relative chain complex $C_\text{rel}(\tilde{X}_w(P),\tilde{W}(P);\mathbb{Z})$ as a $\mathbb{Z}[\pi_1(X_w(P))]$-module. Now the construction proceeds exactly as above, and allows to define a torsion $\tau^\varphi_w(P)\in\overline{K}_0(\Lambda)$ for every homomorphism $\varphi: \pi_1(X_w(P))\to\Lambda$ such that the twisted chain complex $C^\varphi_\text{rel}(X_w(P),W(P))$ is acyclic. Following the scheme mentioned above, one can also define refined torsions $\tau^\varphi_{w,0}(P,h,o)\in K_1(\Lambda)$.

An interesting special case of the definition of $\tau^\varphi_{w,0}$ is when $\partial(M(P))$ consists of a (possibly empty) union of white tori together with one sphere with black-white splitting consisting of two discs. In this case $X_w(P)$ is a combed manifold $N$, closed or bounded by white tori. Moreover our spider defines a combinatorial Euler structure in Turaev’s sense (see Remark 1.3), and one sees that torsions coincide. We will see below in Section 3 that every combed manifold bounded by white tori arises as $X_w(P)$ for some $P$, so we actually do cover all situations considered by Turaev. See Subsection 4.1 for further comments.

### 1.3 How to use torsions to compare objects

A theoretical problem arises when one wants to use torsions to distinguish objects. We carefully describe this problem in our setting, but the phenomenon is general.

We start with the following remark. Let $P_0$ and $P_1$ be branched standard spines, and let $f: P_0 \to P_1$ be a homeomorphism which preserves all the structures of branched spine. For $i = 0, 1$, consider the combed manifold $(M(P_i), v(P_i))$ determined by $P_i$. As already mentioned, this manifold is only determined up to homeomorphism, but we fix a definite representative, together with an embedding of $P_i$ into $M(P_i)$ satisfying the usual conditions. Now $f$ extends to a homeomorphism $M(P_0) \to M(P_1)$ well-defined up isotopy, which induces a homeomorphism $F: X(P_0) \to X(P_1)$ again determined up to isotopy. Of course we have

$$
\tau^\varphi_0(P_0, h, o) = \tau^F_0(\varphi)(P_1, F_*(h), F_*(o))
$$

for all torsions $\tau^\varphi_0(P_0, h, o)$. However, it may in principle happen for some other home-
omorphism \( g : M(P_0) \to M(P_1) \) inducing \( G : X(P_0) \to X(P_1) \) that
\[
\tau^\phi_0(P_0, h, o) \neq \tau^{G_*(\phi)}_0(P_1, G_*(h), G_*(o)).
\]

To describe the situation in a different way, suppose that we have branched spines \( P_0 \) and \( P_1 \), we already know that \( M(P_0) \) and \( M(P_1) \) are homeomorphic, and we want to use torsions to decide whether \( P_0 \) and \( P_1 \) are isomorphic or not. (Of course, this is not a good example, since the isomorphism problem for branched spines is easy, while the homeomorphism problem for manifolds is hard, but more appropriate examples will arise later.) The above remark implies that torsions cannot be used directly, because an action of the automorphism (actually, mapping class) group of \( M(P_0) = M(P_1) \) has to be taken into account. Therefore, we have an analogue of the Teichmüller vs. moduli space situation: the basic definition of torsion involves a “marking” of the manifold, so, to get a marking-independent torsion, one has to quotient out under an action of the mapping class group. We will privilege in this paper this moduli-type approach, but we will mention in Remark 4.1 and Subsection 4.3 situations where a marking is natural, so a Teichmüller-type approach is feasible.

2 Invariance under sliding

The fundamental move for standard spines, which allows to obtain from each other any two spines of the same manifold, is the Matveev-Piergallini move \([14], [19]\), which corresponds, in terms of the dual ideal triangulation, to the so-called 2-to-3 move. Both versions of the move are illustrated in Fig. 3. We will consider to be positive the move in the direction which increases the number of vertices (or tetrahedra).

In \([4]\) we have shown that if \( P \) is a branched standard spine and \( P' \) is a spine obtained from \( P \) by a positive MP-move then also \( P' \) can be given the structure of a branched spine (sometimes not in a unique way). Some of the branched MP-moves induce, at least locally, an essential modification of the black-white splitting of the boundary of the corresponding manifold. All other MP-moves, which we have called sliding moves, do not change up to isomorphism the pair \((M, v)\) associated to the spine. Moreover the moves can be realized in \((M, v)\) as continuous modifications of one spine into another one, through spines positively transversal to \( v \), with exactly one non-standard spine along the modification. If one takes into account orientations...
there are exactly 16 different sliding moves, but the essential phenomena are only those described in Fig. 4.

In [2] we have based on these sliding moves (together with an extra 0-to-2 move) a combinatorial presentation of combed closed manifolds. In Section 3 we will substantially improve this result to include the case with boundary, and we will also show that the extra move can be disposed of. This improvement will allow us to define torsions in various topologically relevant situations, thanks to the following result to which the present section is devoted:

**Theorem 2.1.** All torsions are invariant under sliding moves.

In view of the warning given in Subsection 1.3 the assertion that torsion is invariant under sliding moves must be interpreted with some care: as we have remarked above, slidings can be physically realized as continuous modifications inside combed manifolds. Using such a “physical model” for a sliding move $P_0 \to P_1$, we can consider a “common model” for the (abstractly homeomorphic) spaces $X(P_0)$ and $X(P_1)$, so it makes sense to say that torsions are actually the same.

**Proof of 2.1.** We start with invariance of the basic version $\tau^c(P, \mathcal{H})$ of torsion. As mentioned in the introduction, our proof mimics the more general proof of Turaev [22], but it is self-contained. The reason why our proof is technically easier is that we only need to deal with some definite local modifications of the cell complex structure.

We first need to recall the subdivision technique. Consider a branched spine $P$ and the corresponding complex $X(P)$ with triangulation $\mathcal{T}(P)$, and let $\mathcal{D}$ be a subdivision of $\mathcal{T}(P)$. We will mainly be interested in the case where also $\mathcal{D}$ is a triangulation (possibly with multiple and self-adjacencies). A subdivided spider $s_D(P)$ can be defined as $\sum_p \beta_p$, where $\{p\}$ is a collection of one interior point for each simplex of $\mathcal{D}$, and $\beta_p$ is (the closure of) the orbit of $v(P)$ which starts at $p$ and reaches $x_0(P)$. The reader can easily check that the choice of $\{p\}$ is inessential, so we omit it from the notation. Our definition of subdivided spider is somewhat easier than Turaev’s general one, thanks to our choice of taking the simplices of $\mathcal{T}(P)$ to be unions of orbits of $v(P)$. Now consider
the initial data $\varphi, \Lambda, h, 0$ which allow to define a torsion $\tau^{\varphi}_{\psi}(P, h, 0)$. We can define the
$\Lambda$-modules $C^{\psi, \varphi}_{*}(P)$ using cellular chains with respect to the lifting of $D$ to $\tilde{X}(P)$, and
we can use the spider $s_{D}(P)$ to construct preferred $\Lambda$-bases of these modules. Now one easily checks that there exists a canonical isomorphism $H^{\psi, \varphi}_{*}(P) \cong H^{\psi}_{*}(P)$, so we can use the same symbol $h$ for a $\Lambda$-basis of
$H^{\psi, \varphi}_{*}(P)$, and define a torsion $\tau^{\psi, \varphi}_{*}(P, h)$ exactly as we have done in the previous section. The next result is a simplified version
of Lemma 3.2.3 in [22].

**Proposition 2.2.** $\tau^{\psi, \varphi}_{*}(P, h) = \tau^{\psi}_{*}(P, h)$.

We will prove this proposition only in the more specific situation we are actually
interested in. So, let us consider a sliding move $P_0 \to P_1$, physically realized inside a
manifold $M$, so that the pointed complexes $X(P_0)$ and $X(P_1)$ are both identified to
$X = M/\partial M$. Note that $X$ comes with two distinct triangulations $T(P_0)$ and $T(P_1)$. Figure 5 shows the obvious simplest common subdivision $D$ of $T(P_0)$ and $T(P_1)$. We confine ourselves to showing that $\tau^{\psi, D}_{*}(P_j, h) = \tau^{\psi}_{*}(P_j, h)$ for $j = 0, 1$.

**Proof of 2.2.** Note first that the definition of subdivided spider leads to the following
very natural rule: a simplex of $D$ lying in a simplex $S$ of $T(P)$ is lifted in $\tilde{X}(P)$ to the
only preimage which lies in the lifting of $S$ determined by $P$. Now, this rule makes sense
also for more general subdivisions than triangulations, in particular for cell complexes,
so we will use them. One easily sees that the subdivisions of $T(P_0)$ and $T(P_1)$ into $D$
can be expressed as combinations of the following elementary transformations:

1. An edge is subdivided into two edges by adding a vertex;
2. A square is subdivided into two triangles by adding a diagonal;
3. The inverse of the transformation which removes a triangle, thus replacing the two
   adjacent tetrahedra by one polyhedron with 5 vertices, 9 edges, and 6 triangular
   faces.

We are left to prove that torsion is invariant under these transformations. In all three
cases, the proof goes as follows:

(i) consider data $b_i, \tilde{b}_i, g_i, i = 0, \ldots, 3$, which allow to compute $\tau$ before subdivision;
(ii) describe new data $b_i', h_i', g_i'$ for the subdivided complex;

(iii) analyze the matrices $((\partial b_{i+1}) \cdot h_i \cdot b_i)/g_i$ and $((\partial b_{i+1}') \cdot h_i' \cdot b_i')/g_i'$ to show that they are the same in $T_1(\Lambda)$.

Note that this proves that torsion is unchanged “term by term”, not only globally.

We only deal with transformation 3, leaving the other cases to the reader. Denote by $Q$ the polyhedron which is split into tetrahedra $T_1, T_2$ by addition of a triangle $\Delta$. Then, in a natural way, $g_0 = g_0, g_1' = g_1$, and $g_2'$ is obtained from $g_2$ by adding the lifting of $\Delta$ which lies in the lifting of $Q$; to get $g_3'$ from $g_3$, we need to remove the lifting of $Q$ and add the liftings of $T_1$ and $T_2$ which lie in it. For the lifted homology bases, we have $h_i' = h_i$ for $i = 0, 1, 2$, and $h_3'$ is obtained from $h_3$ by replacing each occurrence of $Q$ with $T_1 + T_2$. Similarly, $b_i' = b_i$ except for $i = 3$, and $b_3'$ is obtained from $b_3$ by replacing each occurrence of $Q$ with $T_1 + T_2$, and then adding $T_2$. The transition matrices are unchanged in dimensions 0 and 1, while in dimensions 2 and 3, with obvious meaning of symbols, we have:

$$
((\partial b_3') h_3 b_3')/g_3' = \left( \begin{array}{ccc}
\partial b_3/g_2 & b_2/g_2 & b_2/g_2 \\
0 \cdots 0 & 1 & 0 \cdots 0 \\
\cdots & \cdots & \cdots
\end{array} \right),
$$

$$
(h_3' b_3')/g_3' = \left( \begin{array}{ccc}
\tilde{h}_3/(g_3 \setminus \{\tilde{Q}\}) & b_3/(g_3 \setminus \{\tilde{Q}\}) & 0 \\
0 & 0 & 0 \\
\tilde{h}_3/\tilde{Q} & b_3/\tilde{Q} & 1
\end{array} \right).
$$

When $\Lambda$ is a field, one immediately gets the conclusion by taking determinants. For the general case one needs to recall the definition of $T_1(\Lambda)$, and the conclusion follows anyway.

Our next step is again a simplified version of a more general result. With the same notation as above, consider the subdivided spiders $s_D(p) = \sum_p \beta_p^{(i)}$. Using Remark 1.3 it is easy to see that $\sum_p \varepsilon(p)(\beta_p^{(0)} - \beta_p^{(1)})$ is a cycle, where $\varepsilon(p) = (-1)^d$, and $d$ is the dimension of the cell of which $p$ is the centre. We denote by $h(P_0, P_1)$ the class in $H_1(X(P); \mathbb{Z})$ of this cycle.

**Proposition 2.3.** If $h(P_0, P_1) = 0$ then $\tau^\varphi(P_0, h) = \tau^\varphi(P_1, h)$.

**Proof of 2.3.** By the previous result, it is enough to show that $\tau^{\varphi,D}(P_0, h) = \tau^{\varphi,D}(P_1, h)$. With obvious meaning of symbols, we have

$$
\tau^{\varphi,D}(P_1, h) = \pm \prod_{i=0}^3 \left[ ((\partial b_{i+1}) h_i b_i)/g_i^{\varphi,D}(P_1) \right]^{-1}.
$$
\[
\begin{align*}
&= \pm \prod_{i=0}^{3} \left( (\partial b_{i+1} \bar{b}_i, b_i) / g_i^{\varphi, D}(P_0) \right)^{(-1)^i} \cdot \left[ g_i^{\varphi, D}(P_0) / g_i^{\varphi, D}(P_1) \right]^{(-1)^i} \\
&= \tau^{\varphi, D}(P_0, \mathfrak{h}) \cdot \left( \prod_{i=0}^{3} \left[ g_i^{\varphi, D}(P_0) / g_i^{\varphi, D}(P_1) \right]^{(-1)^i} \right)
\end{align*}
\]

To compute the last correction factor, we start by remarking that the homomorphism \( \varphi : \pi \to \Lambda \) induces another one \( \varphi' : \pi \to \overline{\mathbb{K}}_1(\Lambda) \). Since \( \overline{\mathbb{K}}_1(\Lambda) \) is an Abelian group, \( \varphi' \) induces a homomorphism \( \varphi'' : H_1(X) \to \overline{\mathbb{K}}_1(\Lambda) \). Now, let us denote by \( \tilde{e}_p^{(k)} \) the lifting in \( \tilde{X} \) of the cell of \( \mathcal{D} \) centred in \( p \), where \( k = 0, 1 \) and the lifting is determined by \( s_\mathcal{D}(P_k) \). Note that \( \tilde{e}_p^{(0)} = \gamma_p \cdot \tilde{e}_p^{(1)} \), with \( \gamma_p = [(\beta_p^{(1)})^{-1} \cdot \beta_p^{(0)}] \in \pi_1(X, x_0) \). This implies that \( [g_i^{\varphi, D}(P_0) / g_i^{\varphi, D}(P_1)] \) is the image in \( \overline{\mathbb{K}}_1(\Lambda) \) of a diagonal matrix with entries \( \varphi(\gamma_p) \), as \( p \) varies in the centres of \( i \)-cells of \( \mathcal{D} \). It easily follows that

\[
\prod_{i=0}^{3} \left[ g_i^{\varphi, D}(P_0) / g_i^{\varphi, D}(P_1) \right]^{(-1)^i} = \varphi''(h(P_0, P_1))
\]

whence the conclusion.

The next result concludes the proof of invariance.

**Proposition 2.4.** For all sliding moves \( P_0 \rightarrow P_1 \) we have \( h(P_0, P_1) = 0 \).

**Proof of 2.4.** Instead of treating in detail all the moves, we confine ourselves to the description of a general framework, and then we apply the method to one instance of the move, the other cases being similar. Recall that Fig. 5 describes the portion \( R \) where the move takes place, with portions of the triangulations \( \mathcal{T}(P_0), \mathcal{T}(P_1) \) and their common subdivision \( \mathcal{D} \). To be precise, the figure shows an “unfolded version” \( R' \) of the portion \( R \), because in \( R \) all the “external” vertices of the figure are identified together (with the point \( x_0 \)), so for example each edge of \( \mathcal{T}(P_0) \) and \( \mathcal{T}(P_1) \) represents a (probably non-trivial) element of \( \pi_1(X, x_0) \). Note that also some external edges or faces could be glued together. The basic idea of the proof is to lift the cycle \( \sum_p \varepsilon(p)(\beta_p^{(0)} - \beta_p^{(1)}) \) to a 1-chain in \( R' \), and show that the result is again a cycle. Since \( R' \) is contractible, the conclusion easily follows.

We remark now that a branching on a standard spine can be encoded just by an orientation of the edges of the dual triangulation. This orientation is defined by the requirement that the algebraic intersection between a region of the spine and the dual edge should be positive. Edge-orientations coming from branchings are characterized by the property that no triangle has a circular orientation of its edges (see also [8]). Recall as well that the edges are orbits of the field defined by the spine, and orientations match. More importantly, edge-orientations allow us to describe the orbits of the field also on the open triangles and tetrahedra of the triangulation. The rules are illustrated in Fig. 6.
Now, each sliding move will be encoded by a pair of matching patterns of orientations of the edges of $\mathcal{T}(P_0)$ and $\mathcal{T}(P_1)$, both defining fields on the common portion they triangulate. Note that non-zero contributions to $h(P_0, P_1)$ can only come from simplices of $D$ which are not shared with both $\mathcal{T}(P_0)$ and $\mathcal{T}(P_1)$. This rules out all simplices outside the portion $R'$ and those on its boundary. We are left to deal with internal simplices, namely 1 vertex, 5 edges, 9 faces and 6 tetrahedra. To carry out the program outlined above, we must lift to $R'$ the corresponding loops $(-1)^d \cdot (\beta_p^{(0)} - \beta_p^{(1)})$ and show that the sum of the boundaries is always null.

As promised, we carry out calculations in one example. The first sliding move of Fig. 4 translates in terms of edge-orientations as described in Fig. 7, where again we are showing the “unfolded version”. In the same figure we introduce some useful notation for the subdivided triangulation. Note again that $a = b = c = d = e = x_0$ in $X$. To analyze the contributions of the internal simplices to the boundary of the lifted chain, we need to determine, for both fields, the targets of the orbits starting at centres of simplices. This is done in the next tables.

| Simplex $\sigma$ | $v$ | $va$ | $vb$ | $vc$ | $vd$ | $ve$ | $vab$ | $vad$ | $vae$ | $vcb$ |
|------------------|----|-----|-----|-----|-----|-----|------|------|------|------|
| $(−1)^{\dim(\sigma)}$ | +1 | −1 | −1 | −1 | −1 | −1 | +1 | +1 | +1 | +1 |
| End $\beta_p^{(0)}(\sigma)$ | $d$ | $d$ | $d$ | $d$ | $d$ | $d$ | $d$ | $d$ | $d$ | $d$ |
| End $\beta_p^{(1)}(\sigma)$ | $c$ | $c$ | $c$ | $c$ | $d$ | $c$ | $c$ | $d$ | $c$ | $c$ |
| Boundary | $d - c$ | $c - d$ | $c - d$ | $c - d$ | $0$ | $c - d$ | $d - c$ | $0$ | $d - c$ | $d - c$ |
The sum of the bottom rows of the two tables is null, which gives the desired conclusion. This eventually proves the invariance of $\tau^\varepsilon(P, h)$. The sign-refined version $\tau^\varepsilon_0(P, h, o)$ is dealt with analogously, with a little more effort. A similar argument works also for the modified versions of torsion $\tau_w$ and $\tau_{w,0}$. One only needs to note that not all the vertices which appear Fig. 5 are identified to the basepoint $x_0$, but the vertices which are endpoints of orbits indeed are, so the proof proceeds exactly as above.

2.1 The canonical spider and the Euler class

Let us recall that our definition of torsion of a spine was based on a spider obtained by integrating the field in the positive direction starting from the centres of the cells. However, as the reader can easily check, another definition is obtained along the same lines but integrating the field in the negative rather than positive direction. The problem naturally arises to compare these constructions. We will now prove that the difference of the positive and the negative spiders, when lifted to $M(P)$, is a very natural object, namely the Poincaré dual of the Euler class of the plane distribution normal to the field. See Subsection 4.1 for further comments on the meaning of this result.

For a precise statement, we need to introduce some notation. Given a branched spine $P$, we denote by $\sum c \tilde{\alpha}_+^c$ the natural lifting to $M(P)$ of the spider defined in Subsection 1.2. Note that this is not quite a spider any more, because the head is exploded into a set of points scattered on $\partial(M(P))$. We define $\sum c \tilde{\alpha}_-^c$ in a similar way, integrating the field in the negative direction. So, each difference $\tilde{\alpha}_+^c - \tilde{\alpha}_-^c$ represents an arc with both ends on $\partial(M(P))$. Now, with the usual meaning of symbols, we have:

**Proposition 2.5.** $[\sum c \varepsilon(c) \cdot (\tilde{\alpha}_+^c - \tilde{\alpha}_-^c)] = \text{PD}(\mathcal{E}(v(P)^\perp)) \in H_1(M(P), \partial(M(P)); \mathbb{Z})$.

**Proof of 2.5.** In [3], inspired by [8], we have introduced a canonical cochain $c_P$ for the Euler class of a field encoded by a branched spine $P$. To get $c_P$, one considers on $P$, near the singular set $S(P)$, the tangent field $\mu_P$ which is transversal to $S(P)$ and points from the locally two-sheeted area to the locally one-sheeted area. The value of $c_P$ on a region $R$ is the index of the extension of $\mu_P$ to $R$. Noting that, at each vertex, $\mu_P$ is tangent to the boundary of exactly two (opposite) regions, one sees that $c_P(R) = 1 - n(R)/2$, where $n(R)$ is the number (with multiplicity) of vertices of $R$ at which $\mu_P$ is tangent to $\partial R$. Contributions of a vertex to the regions incident to it are shown on the left in Fig. 8.
Now, in $\sum_{c} \varepsilon(c) \cdot (\tilde{\alpha}_{c}^{+} - \tilde{\alpha}^{-})$, arcs coming from vertices and edges of $P$ are not in general position with respect to $P$, so we modify them by slightly pushing along $\mu_{P}$.

The value on a region $R$ is now given by a sum of contributions: one $+1$, coming from $R$ itself, some $-1$’s, coming from edges, and some $+1$’s, coming from vertices. If one halves the contributions of edges and localizes the halves at the ends, one has that the values of the cochain are expressed just as for $c_{P}$, namely $+1$ plus some contributions coming from vertices. The latter are shown on the right in Fig. 8. Since contributions are actually the same, the proof is complete.

\section{Branched spines of combed manifolds with boundary pattern}

In this section we will extend the main results of Chapter 5 of [2] from the closed to the bounded case. Namely, we will show that compact manifolds with concave combings (see below for the precise definitions) are combinatorially described by (suitable) branched spines up to sliding. We first show that one of the sliding moves considered in [2] is essentially generated by the other moves.

\subsection{The snake move}

In Section 2, when proving invariance of torsions, we have not dealt with the extra move which, together with the branched MP-moves, was defined in [2] to be a generator of the sliding calculus. This is the “snake move”, described in Fig. 9 (actually, taking into account orientations, there are two versions of the move). Our reason for not treating the move was the next Proposition 3.1 (see also Corollary 3.4).

In the rest of this section we will denote by $\mathcal{P}$ the set of (isomorphism classes of) branched spines, and by $\mathcal{R}$ the subset of those which are “rigid” from the point of view.
view of the branched MP-moves, i.e. the spines to which no such move applies. An explicit description of $\mathcal{R}$ is given in the proof of the next result. In the statement we only emphasize the most important consequences of this description. It will be convenient to use the terminology introduced in [2]: we call trivial and denote by $S^2_{\text{triv}}$ the bicoloration of $S^2$ which consists of one black disc and one white disc.

**Proposition 3.1.**  
(i) For every bicolorated surface $\Sigma$ there are at most two spines $P \in \mathcal{R}$ such that $\partial(M(P)) \cong \Sigma$.

(ii) If two elements of $P \setminus \mathcal{R}$ are related through branched MP-moves and snake moves, they are also related through branched MP-moves only.

**Proof of 3.1.** In this proof we will find convenient to use the graphic representation of branched spines introduced in [2]. We will skip most details, giving only the main points. We start by listing rigid spines. Note first that if a negative branched MP-move applies to a spine then also a positive one does, so we only need to consider positive rigidity. The spines with one vertex, shown in Fig. 10, are of course rigid. Using [2] one easily checks that $\partial(M(P))$ is $S^2_{\text{triv}}$ for the first two spines, and $S^2_{\text{triv}} \sqcup S^2_{\text{triv}}$ for the other two.

Now we turn to rigid spines with more than one vertex. Rigidity implies that all edges with distinct endpoints should appear as on the left in Fig. 11. It is not hard to deduce that rigid spines come in a sequence $P^\text{rig}_1, P^\text{rig}_2, \ldots$ as shown in the rest of Fig. 11, where $P^\text{rig}_k$ has $2k$ vertices, and $\partial(M(P^\text{rig}_k))$ is the union of $S^2_{\text{triv}}$ together with $k$ copies of $S^2_{\text{white}}$ and $k$ copies of $S^2_{\text{black}}$. This classification proves (i).

To show (ii) we must prove that:
(ii-a) Sequences which contain rigid spines can be replaced by sequences which do not.

(ii-b) If two non-rigid spines are related by one snake move then they are also related by a sequence of branched MP-moves.

For (ii-a), we note that the result of a positive snake move is never rigid. So if a rigid spine \( P \) appears in a sequence of moves then around \( P \) we see \( P_1 \overset{\mu_1}{\longrightarrow} P \overset{\mu_2}{\longrightarrow} P_2 \), with \( \mu_1 \) and \( \mu_2 \) positive snake moves. Since all edges of a spine survive through a snake move, there is a version \( \tilde{\mu}_2 \) of \( \mu_2 \) which applies to \( P_1 \) and a version \( \tilde{\mu}_1 \) of \( \mu_1 \) which applies to \( P_2 \), and the result is the same. So we have \( P_1 \overset{\tilde{\mu}_2}{\longrightarrow} P \overset{\tilde{\mu}_1}{\longrightarrow} P_2 \), and now all the spines involved are non-rigid.

Let us turn to (ii-b). The proof results from three steps, to describe which we introduce in Figure 12 another move, called vertex move, whose unbranched version was already considered in \([14]\) and \([19]\). Again, taking into account orientations, there are two versions of the move (for each vertex type), but we will ignore this detail.

Step 1: if \( v \) is a vertex of a spine \( P \), \( e \) is any one of the edges incident to \( v \), \( P_v \) is obtained from \( P \) via the vertex move at \( v \), and \( P_e \) is obtained from \( P \) via the snake move on \( e \), then \( P_v \) and \( P_e \) are related by MP-moves. This is proved by an easy case-by-case analysis. It turns out that two MP-moves (a positive and a negative one) are always sufficient.

Step 2: let \( v, P \) and \( P_v \) be as above. If \( P \) and \( P_v \) are related by MP-moves, the same is true for \( P \) and any spine obtained from \( P \) by a snake move. To see this, use step 1 to successively transform vertex moves into snake moves and conversely, until the desired snake move is reached.

Step 3: if \( P \) is non-rigid then there exists a vertex \( v \) such that \( P \) and \( P_v \) are related by MP-moves. The vertex \( v \) is chosen to be an endpoint of an edge to which the positive MP-move applies. The argument is again a long case-by-case one, which refines in a branched context the argument given by Piergallini in \([19]\). The sequence always consists of three positive moves followed by a negative one. This concludes the proof.

Figure 12: The vertex move.
3.2 Generalized combed calculus

Recall from Subsection 1.1 that a concave field $v$ on a compact 3-manifold $M$ is one which is tangent to $\partial M$ only "from inside", along some simple curves. We will denote by $\text{Comb}^\partial$ the set of all such pairs $(M, v)$, viewed up to homeomorphism of $M$ and homotopy of $v$ through concave fields. Note that the black-white splitting of $\partial M$ evolves isotopically during a homotopy of $v$, so we can associate to $[M, v] \in \text{Comb}^\partial$ a well-defined boundary pattern. A class $[M, v] \in \text{Comb}^\partial$ is called a combing on the homeomorphism class of the manifold $M$. For a technical reason we rule out from $\text{Comb}^\partial$ the set of those classes $[M, v]$ such that $\partial M$ contains components of the type $S^2_{\text{triv}}$. This is actually not a serious restriction, because each $S^2_{\text{triv}}$ component can be capped off by a $B^3_{\text{triv}}$ (the 3-ball with constant field), and the result is well-defined up to homotopy. Note that we do accept pairs $(M, v)$ with $M$ closed, and pairs in which $v$ has no tangency at all to $\partial M$.

Let us denote now by $\mathcal{V}$ the set of pairs $(M, v)$ where $v$ is concave and traversing, i.e. such that all orbits are segments with both ends on $\partial M$, and in $\partial M$ there is exactly one $S^2_{\text{triv}}$ component. These pairs will be viewed up to homeomorphism of $M$ and homotopies of $v$ through concave traversing fields. By the construction recalled in Subsection 1.1 we can associate to every branched spine a manifold with a concave traversing field. We will denote by $\text{Obj}^\partial$ the set of (isomorphism classes of) those spines which give rise to elements of $\mathcal{V}$. Given $[P] \in \text{Obj}^\partial$, if we cap off the only $S^2_{\text{triv}}$ in $\partial(M(P))$ by a $B^3_{\text{triv}}$, we obtain a well-defined element of $\text{Comb}^\partial$.

**Theorem 3.2.** The map $r^\partial : \text{Obj}^\partial \to \text{Comb}^\partial$ thus defined is surjective, and the equivalence relation defined by $r^\partial$ on $\text{Obj}^\partial$ is generated by standard sliding moves.

**Remark 3.3.** The following interpretation of the surjectivity of $r^\partial$ is perhaps useful. Note first that the dynamics of a field, even a concave one, can be very complicated, whereas the dynamics of a traversing field (in particular, $B^3_{\text{triv}}$) is simple. Surjectivity of $r^\partial$ means that for any (complicated) concave field there exists a sphere $S^2$ which splits the field into two (simple) pieces: a standard $B^3_{\text{triv}}$ and a concave traversing field. Actually, a 1-parameter version of this statement also holds (see Remark 3.7): we will need it to show that the $r^\partial$-equivalence is the same as the sliding equivalence.

**Remark 3.4.** By Proposition 3.1, in Theorem 3.2 we could remove from $\text{Obj}^\partial$ the set $\mathcal{R}$ of MP-rigid spines and forget the snake move, leaving the rest of the statement unchanged.

The proof of Theorem 3.2 is an extension of the argument given in Chapter 5 of [2], and it is based on the following technical notion, which extends ideas originally due to Ishii [4]. Let $v$ be a concave field on $M$. Let $B_1, \ldots, B_k$ be the black components of
the splitting of $\partial M$, \emph{i.e.} the regions on which $v$ points outwards. A \emph{normal section} for $(M, v)$ is a compact surface $\Sigma$ with boundary, embedded in the interior of $M$, with the following properties:

1. $v$ is transverse to $\Sigma$;
2. $\Sigma$ has exactly $k + 1$ components $\Sigma_0, \ldots, \Sigma_k$, with $\Sigma_0 \cong D^2$;
3. For $i > 0$, the projection of $B_i$ on $\Sigma$ along the orbits of $-v$ is well-defined and yields a homeomorphism between $B_i$ and a surface $B'_i$ contained in the interior of $\Sigma_i$, with $\Sigma_i \setminus B'_i$ being a collar on $\partial \Sigma_i$;
4. Each positive half-orbit of $v$ meets either the interior of some $B_i$ (where it stops), or the interior of some $\Sigma_i$;
5. $\partial \Sigma$ meets itself generically along $v$ (\emph{i.e.} each orbit of $v$ meets $\Sigma$ at most two consecutive times on $\partial \Sigma$, and, if so, transversely);
6. Let $P_\Sigma$ be the union of $\Sigma$ with all the orbit segments starting on $\partial \Sigma$ and ending on $\Sigma$. Then $\Sigma$, which is a quasi-standard polyhedron by the previous point, is actually standard.

The next two lemmas show that normal sections of $(M, v)$ correspond bijectively to spines $P$ such that $r^\partial([P_\Sigma]) = [M, v]$. The proof of surjectivity of $r^\partial$ and the discussion of its non-injectivity will be based on these lemmas.

**Lemma 3.5.** If $(M, v)$, $\Sigma$ and $P_\Sigma$ are as above, then $P_\Sigma$ can be given a structure of branched spine such that $r^\partial([P_\Sigma]) = [M, v]$.

\textbf{Proof of 3.5.} We orient $\Sigma$ so that $v \cap \Sigma$ (by default $M$ is oriented). Every region of $P_\Sigma$ contains some open portion of $\Sigma$, so it can be oriented canonically. With the obvious screw-orientation, this turns $P_\Sigma$ into a branched spine.

We show that $r^\partial([P_\Sigma]) = [M, v]$ by embedding the abstract manifold $M(P_\Sigma)$ in $M$, in such a way that the field carried by $P_\Sigma$ on $M(P_\Sigma) \subset M$ is just the restriction of $v$. By construction, $M \setminus M(P_\Sigma)$ will consist of a copy of $B^3_{triv}$, together with a collar on $\partial M$ which can be parametrized as $(\partial M) \times [0, 1]$ in such a way that $v$ is constant in the $[0, 1]$-direction. This easily implies that $r^\partial([P_\Sigma]) = [M, v]$ indeed.

We illustrate the embedding of $M(P_\Sigma)$ in $M$ pictorially in one dimension less. Figure 13 shows how $\Sigma_0$ gives rise to a $B^3_{triv}$. In the figure we describe $v$ by dotted lines, $\Sigma$ by thick lines, portions of $P_\Sigma \setminus \Sigma$ by thin lines, and $\partial(M(P_\Sigma))$ by a thick dashed line. Note also that the portions of $P_\Sigma \setminus \Sigma$ have been slightly modified so to become positively transversal to $v$, which allows us to represent the branching as usual, \emph{i.e.} as a $C^1$ structure on $P_\Sigma$.

Figure 14 shows the collar based on a component of $\partial M$. We use the same con-
Figure 13: The trivial ball.

Figure 14: Collar on a boundary component.
ventions as in the previous figure, and in addition we represent the black and white components of $\partial M$ by thick and thin lines respectively. This description concludes the proof.

**Lemma 3.6.** Let $[P] \in \text{Obj}^0$ and $r^\partial([P]) = [M, v] \in \text{Comb}^0$, with $P$ embedded in $(M, v)$ according to the geometric description of $r^\partial$. Let $\Sigma$ be obtained from $P$ as suggested (in one dimension less) in Fig. 15. Then $\Sigma$ is a normal section of $(M, v)$, and $P_\Sigma$ is isomorphic to $\Sigma$.

**Proof of 3.6.** The construction suggested by Fig. 15 is obviously the inverse of the construction in the proof of Lemma 3.5.

**Proof of 3.2.** We start with the proof of surjectivity. So, let us consider a combed manifold $(M, v)$, subject to the usual restrictions. By Lemma 3.5 it is natural to try to construct a normal section for $(M, v)$. Let $B_1, \ldots, B_k$ be the black regions in $\partial M$. Slightly translate each $B_i$ along $-v$, getting $B'_i$. Add to each $B'_i$ a small collar normal to $v$, getting $\Sigma_i$ (if $\partial B_i = \emptyset$, we set $\Sigma_i = B'_i$). Select finitely many discs $\{D_n\}$ disjoint from each other and from all the $\Sigma_i$’s, such that all positive orbits of $v$, except for the small segments between $B'_i$ and $B_i$, meet $(\bigcup_{i \geq 1} \Sigma_i) \cup (\bigcup D_n)$ in some interior point. Connect the $D_n$’s together by strips normal to $v$ and disjoint from $\bigcup_{i \geq 1} \Sigma_i$, getting a disc $\Sigma_0$. Up to a generic small perturbation, the surface $\Sigma = \bigcup_{i \geq 0} \Sigma_i$ satisfies all axioms of a normal section for $(M, v)$, except axiom 3.3.

Now, even if it is not standard, $P_\Sigma$ can be defined, and the proof of Lemma 3.5 shows that it is a quasi-standard branched spine of $(M \setminus B^3, v)$. In particular, $P_\Sigma$ is connected and its singular locus is non-empty. Under these assumptions, it is not too hard to see that there exists a sequence of (abstract) quasi-standard sliding moves which turns $P_\Sigma$ into a standard spine. If we physically realize these moves within $M$, preserving transversality to $v$, the result is a standard branched spine $P$ such that $r^\partial([P]) = [M, v]$.

We are left to show that if $r^\partial([P_0]) = r^\partial([P_1])$ then $P_0$ and $P_1$ are sliding-equivalent. By the definition of $\text{Comb}^0$ and $r^\partial$, using also the above lemmas, there exists a manifold $M$ and a homotopy $(v_t)$ of concave fields on $M$, such that $P_0$ and $P_1$ are defined by normal sections $\Sigma^{(0)}$ and $\Sigma^{(1)}$ of $(M, v_0)$ and $(M, v_1)$ respectively.
We prove that $P_0$ and $P_1$ are sliding-equivalent first in the special case where $v_0 = v_1 = v$. The general case will be an easy consequence. For $j = 0, 1$, let $\Sigma^{(j)} = \bigcup_{i \geq 0} \Sigma_i^{(j)}$. Proceeding as in the above proof of surjectivity, for each black region $B_i$ of $\partial M$, we consider a collared negative translate $\Sigma_i$ of $B_i$. We choose $\Sigma_i$ so close to $B_i$ that $\Sigma_i \cap \Sigma^{(j)} = \emptyset$, and the negative integration of $v$ yields a homeomorphism from $\Sigma_i$ to a subset of $\Sigma^{(j)}$.

**Step I.** For $j = 0, 1$, there exists a disc $D_j$ such that $D_j \cup (\bigcup_{i \geq 1} \Sigma_i)$ is a normal section of $(M, v)$, and the associated branched spine is sliding-equivalent to $P_j$. To prove this, we temporarily drop the index $j$. We first isotope each $\Sigma_i$, without changing the associated spine, until it contains $\Sigma_i$, as suggested in Fig. 16.

Note that if $\partial B_i = \emptyset$ we automatically have $\Sigma_i = \Sigma_i$. Otherwise, we concentrate on one of the annuli $A$ of which $\Sigma_i \setminus \Sigma_i$ consists. Note that we cannot just shrink $A$ leaving the rest of the section unchanged, because we could spoil axiom 4 of the definition of normal section. To actually shrink $A$ we first need to “insulate” it, toward the positive direction of $v$, by adding to the disc $\Sigma_0$ a strip normal to $v$. Figure 17 suggests how to do this.

As we modify $\Sigma_0$ as suggested, it is clear that we keep having a “quasi-normal” section, i.e. all axioms except 6 hold. Moreover the corresponding quasi-standard branched spines are obtained from each other by quasi-standard sliding moves. To conclude we apply, as above, the fact that a quasi-standard branched spine is sliding-equivalent to a standard one, and the technical result established in 2, according to which standard spines which are equivalent under quasi-standard sliding moves are also equivalent under standard sliding moves. This proves Step I.

The conclusion will now follow quite closely the argument in 2.
Figure 18: Transformation of a disc into a disjoint one.

**Step II.** There exist discs $D_j'$ and $D_j''$ such that $D_j' \cup (\bigcup_{i \geq 1} \Sigma_i)$ is a normal section of $(M, v)$ for $j = 0, 1$, and $D_0 \cap D_0' = D_0' \cap D_1' = D_1' \cap D_1 = \emptyset$. Choosing a metric on $M$, one can construct $D_0'$ and $D_1'$ by first taking many very small discs almost orthogonal to $v$, and then connecting these discs by strips transversal to $v$.

**Step III.** Conclusion in the case $v_0 = v_1$. If we connect $D_0'$ and $D_1'$ by a strip orthogonal to $v$, we get a bigger disc $\tilde{D}_0'$ such that $\tilde{D}_0' \cup (\bigcup_{i \geq 1} \Sigma_i)$ is still a normal section of $(M, v)$. We can actually imagine a dynamical process, in which $D_0'$ is first enlarged to $\tilde{D}_0'$, and then is reduced to $D_1'$, as in Fig. [18]. If the transformation is chosen generic enough, at all times axioms [1, 2, 3] and [4] will hold, and axiom [5] will hold at all but finitely many times. This means that the corresponding branched spines are related by quasi-standard sliding moves. Similarly, we can replace $D_0'$ first by $D_1'$ and then by $D_1$. Using the facts quoted above, the conclusion follows.

We are left to deal with the general case, where $(v_t)$ is a non-constant homotopy. It is then sufficient to take a partition $0 = t_0 < t_1 < \ldots < t_n = 1$ of $[0, 1]$, fine enough that $(M, v_{t_{k-1}})$ and $(M, v_{t_k})$ admit a common normal section which gives rise to isomorphic branched spines.

**Remark 3.7.** Along the lines of the previous proof we have established the following topological fact, whose statement does not involve spines. Let $(M, v) \in \text{Comb}_{\partial}$, if we choose $[P] \in (r^{\partial})^{-1}([M, v])$, a representation $\varphi : \pi_1(X(P)) \to \Lambda_*$, a $\Lambda$-basis $\mathfrak{h}$ of $H^k_r(P)$ (assuming it to be free) and a homological orientation $\sigma$ of $X(P)$, then we...

### 4 Torsions of combings and links

In this section we will combine the results previously obtained, to define torsions in some topologically relevant situations. We will also give some hints on how to carry out computations.

#### 4.1 Torsion of a combed manifold

As a direct application of what we have proved, we see that for $[M, v] \in \text{Comb}^0$, if we choose $[P] \in (r^{\partial})^{-1}([M, v])$, a representation $\varphi : \pi_1(X(P)) \to \Lambda_*$, a $\Lambda$-basis $\mathfrak{h}$ of $H^k_r(P)$ (assuming it to be free) and a homological orientation $\sigma$ of $X(P)$, then we...
have a torsion $\tau_0^\varphi(P, h, o) \in K_1(\Lambda)$ whose equivalence class under the natural action of the mapping class group of $M$ depends on $[M, v]$, $h$ and $o$ only. If the homological orientation is omitted, the invariant takes values in $K_1(\Lambda)$. When in $(M, v)$ there are white boundary components, the definition of $\tau_{w,0}^\varphi(P, h, o)$ also applies.

**Remark 4.1.** The reason why we are forced to let mapping class groups act on torsions is that in the definition of $\text{Comb}^\partial$ we have considered manifolds to be defined only up to homeomorphism. Focusing on a certain marked manifold one can neglect the action, obtaining a Teichmüller-type theory of torsions. Namely, if a manifold $M$ and a black-white boundary pattern $P$ are fixed, we can consider the set $\text{Comb}(M, P)$ of concave non-singular vector fields on $M$ matching $P$, viewed up to homotopy. Now $X = M/(\partial M \cup \{\ast\})$, and for $[v] \in \text{Comb}(M, P)$, $\varphi : \pi_1(X) \to \Lambda$, $h$ and $o$ as usual, a torsion $\tau_0^\varphi([v], h, o)$ is well-defined, without taking further quotients.

As already remarked, $\tau_{w,0}^\varphi(P, h, o)$ includes the cases considered by Turaev in [22] and [23], namely manifolds which are closed or bounded by tori with field pointing inwards. However, it is not completely obvious that the invariants are exactly the same. We will now explain more carefully the relation of our construction with Turaev’s work. For the sake of simplicity, we confine ourselves to the closed case.

Turaev’s definition of torsion goes as follows. He first defines torsions for combinatorial Euler structures on triangulations, and then he describes a universal procedure to map bijectively the set of combinatorial Euler structures on any given triangulation of a manifold $M$ onto the set of smooth Euler structures on $M$ (i.e. the set of equivalence classes of non-singular vector fields on $M$, under homotopy and local modifications). Our construction goes in the opposite direction. We start with a vector field and we use it, together with a spine, to produce a combinatorial Euler structure (actually, a spider) on the ideal triangulation dual to the spine.

If we first apply our procedure and then Turaev’s one, we obtain by composition a map from vector fields to smooth Euler structures, and it is not completely obvious that this map is the canonical projection. The point is that the set of Euler structures is in a natural way an affine space over $H_1(M; \mathbb{Z})$, and it is conceivable that the two approaches lead to different choices of the origin. Nonetheless, since both constructions are universal and very natural, our guess is that indeed the map described is the projection, or minus it.

We view Proposition 2.5 as an evidence supporting our guess that the map described above is the canonical projection. More precisely, assuming the map to be the projection, Proposition 2.5 is a formal consequence of a result in [22], according to which the cycle which appears in the proposition is the Poincaré dual of the relative characteristic class of $v$ with respect to $-v$, i.e. the Euler class of $v$. We also remark that Proposition 2.5 could be used as a basis for a direct proof that also our generalized torsions, in the acyclic closed case, satisfy the duality property established in Section 2.7 of [23].
Another point deserves some emphasis. As mentioned, Turaev’s torsions are (by construction) invariant under local modifications of the field. It is conceivable that the same is true for our generalized torsions, but we believe that a direct proof, using only spines, may be hard.

4.2 Torsion of a (pseudo-)Legendrian link

We will denote by Leg the set of all equivalence classes of triples \((M, v, L)\) where \(M\) is closed, \(v\) is a field on \(M\), and \(L\) is a link embedded in \(M\) and transversal to \(v\). The equivalence relation is generated by homeomorphisms of manifolds and homotopy of \(v\) through fields transverse to \(L\). Note that if one allows \(L\) to move isotopically during the homotopy of \(v\), the same equivalence relation is defined.

For \((M, v, L)\) as above, we will call \(L\) a pseudo-Legendrian link in \((M, v)\). Our terminology is due to the following example, which also serves as the main motivation for the definition. If \(\xi\) is a cooriented contact structure on \(M\) and \(L\) is a Legendrian link in \((M, \xi)\), viewed up to Legendrian isotopy, then an element \([M, \xi^+, L] \in \text{Leg}\) is well-defined, where \(\xi^+\) is any field positively transversal to \(\xi\).

Lemma 4.2. A map \(D : \text{Leg} \rightarrow \text{Comb}^\partial\) can be defined as follows: given \([M, v, L]\), consider a small open regular neighbourhood \(N\) of \(L\), and set \(D([M, v, L]) = [M \setminus N, v]\).

Proof of 4.2. If \(N\) is small, \(v\) has exactly two concave tangency circles on each boundary component of \(\partial(M \setminus N)\). Independence of \(N\) is immediate, and independence of the representative of \([M, v, L]\) follows from the fact that any homotopy of \(v\) through fields transversal to \(L\) can be replaced by a homotopy which is constant near \(L\).

This lemma implies that for \([M, v, L] \in \text{Leg}\) we can define torsion invariants \(\tau_0^\partial(P, h, o)\) for any branched spine \(P\) of \(D([M, v, L])\). As usual, one should take into account the action of a mapping class group (but see also Subsection 4.3).

Remark 4.3. The map \(D : \text{Leg} \rightarrow \text{Comb}^\partial\) defined in the lemma is neither surjective nor injective. The image of \(D\) consists precisely of pairs \([B, w]\) such that \(\partial B\) consists of tori, and each torus is split into a white and a black annulus. The reason for non-injectivity of \(D\) is that several non-isomorphic combed Dehn fillings on such a manifold \([B, w]\) can be compatible with the black-white splitting.

Remark 4.4. Instead of considering pseudo-Legendrian links in closed combed manifolds, we may have taken them in manifolds with non-empty boundary and concave combings. All definitions and constructions are easily adapted.
Remark 4.5. Since every (pseudo-)Legendrian link has a natural framing, there is a map which associates to an element of Leg a pair consisting of a combing and an isotopy class of framed link on the same manifold. One easily sees that this map is not injective. However, it is surjective. For instance, given a combing and a framed link, one can realize the link as a Legendrian one in the unique overtwisted contact structure transversal to the combing. Note that, by the work of Bennequin [4], one cannot use tight structures.

Remark 4.6. The flexibility of overtwisted contact structures suggests there could be essentially no difference between pseudo-Legendrian links in a combed 3-manifold and Legendrian links in the unique overtwisted contact structure homotopic to the combing. To be precise, it is maybe possible use the techniques of Eliashberg [7] to answer in the positive to the following question: for $j = 0, 1$ let $\xi_j$ be an overtwisted contact structure on $M$, and let $L_j$ be Legendrian with respect to $\xi_j$; if $[M, \xi_j^+, L_0] = [M, \xi_j^+, L_1]$ in Leg, does there exists an isotopy of $M$ which carries $\xi_0$ to $\xi_1$ and $L_0$ to $L_1$? The content of [7] is a positive answer for empty links.

Remark 4.7. As it often happens with link invariants, our torsions are actually invariants of the complement of a Legendrian link, rather than the Legendrian embedding itself (compare also with the non-injectivity of $D$ already pointed out above). Therefore, it is clear that there will be many inequivalent elements of Leg with the same invariants. In general, the typical situation in which one uses invariants of exteriors is to compare links in the same manifold. In our setting, we can for instance use torsions to distinguish different Legendrian links in one and the same combed manifold. A refinement of this situation will be discussed in the next subsection.

4.3 An embedding-refined of torsion for links

As already mentioned, one can take a Teichmüller-type and a moduli-type approach to torsions. We will describe now a situation where a canonical marking arises, so that the former approach is more natural, i.e. the action on torsions of the mapping class group can be neglected.

Let us fix a definite closed manifold $M$. Let $v_0$ and $v_1$ be fields on $M$, and consider links $L_0$ and $L_1$ transversal to $v_0$ and $v_1$ respectively, so that framings are naturally defined on $L_0$ and $L_1$. We will assume that $L_0$ and $L_1$ are isotopic in $M$, and we will define torsions which potentially can prove that they are not isotopic as framed links.

Let $E(L_i)$ be the complement in $M$ of an open regular neighbourhood of $L_i$, chosen so that $v_i$ behaves on each torus boundary in the usual way (with two parallel non-trivial tangency lines). Consider a branched spine $P_i$ which represents $(E(L_i), v_i|_{E(L_i)})$ in the sense of Theorem 3.2, and remark that $X(P_i)$ is homeomorphic to $E(L_i)/(\partial E(L_i))$. By
assumption, there is a homeomorphism $f : E(L_0) \to E(L_1)$ which is the restriction of an automorphism of $M$ isotopic to the identity. Such an $f$ defines a homeomorphism $F : X(P_0) \to X(P_1)$.

**Proposition 4.8.** If there exists a continuous family $\{v_t\}_{t \in [0,1]}$ of fields on $M$ and an isotopy of links $\{L_t\}_{t \in [0,1]}$ with $L_t$ transversal to $v_t$ for all $t$, then for all choices of $\varphi, h, o$ for $X(P_0)$ we have:

$$\tau_0^\varphi(P_0, h, o) = \tau_0^{F_{\varphi}}(P_1, F_*(h), F_*(o)).$$

**Proof of 4.8.** Let $\{Q_i\}_{i=0}^k$ be a sequence of branched spines, such that $Q_0 \cong P_0$, $Q_k \cong P_1$, and $Q_{i-1} \to Q_i$ is a sliding move. This move can actually be realized in $M$, so $X(Q_{i-1})$ and $X(Q_i)$ can be identified using the quotient of an isotopy of the complements. The composite homeomorphism $X(P_0) \to X(P_1)$ is isotopic to $F$, and the conclusion follows from Theorem 2.1 and the accompanying discussion.

### 4.4 Computational hints

**Spine of a Legendrian link.** As already mentioned in Remark 4.7, the typical setting in which one imagines to use invariants to distinguish Legendrian links is when a certain closed combed manifold $(M, v)$ is given, and one restricts to links in $(M, v)$. In this situation, we can also fix a certain branched spine $P$ of $(M, v)$, and use it to construct spines of link complements. We first note that on $P$ we can consider link diagrams, requiring crossings to be away from $S(P)$. Among link diagrams, it is natural to call $C^1$ those which never meet $S(P)$ by going from one sheet to the other sheet of the locally two-sheeted area. A $C^1$ link projection on $P$ naturally defines a Legendrian link in $(M, v)$. Conversely, since the complement of $P$ in $M$ is an open ball, one sees that every Legendrian link in $(M, v)$ is represented by a $C^1$ diagram on $P$. With a little more effort the following can be established.

**Lemma 4.9.** Every Legendrian link in $(M, v)$ is represented by a $C^1$ diagram on $P$ without crossings.

Now, the tunnel-digging process which takes from a Legendrian link to an element of $\text{Comb}^\partial$ can be easily carried over at the level of spines, as suggested in Fig. 19. This is particularly easy when there are no crossings, but it works in general. Note that the spine which results from the digging may occasionally be non-standard, but it is standard as soon as the diagram is complicated enough (e.g. if there are both a crossing and an intersection with $S(P)$).
Boundary operators. The key point for the computation of torsion is the knowledge of the boundary operators in the twisted chain complex $C^*_\tau(P)$. The first step to determine these operators is to describe the universal cover of $X(P)$, or the maximal Abelian cover when $\Lambda$ is commutative. Assuming the cover to be perfectly understood, the boundaries in $C^*_\tau(P)$ are just liftings of those in $C^*_\text{cell}(X(P);\mathbb{Z})$. We will now show that the complex $C^*_\text{cell}(X(P);\mathbb{Z})$ admits a very easy description, which seems to indicate that complete calculations should be feasible at least in some cases, and may be implemented on a computer.

In $X(P)$, we will denote by $\hat{R}$ (respectively $\hat{e}$, $\hat{v}$) the edge (respectively triangle, tetrahedron) of $\mathcal{T}(P)$, dual to a region (respectively edge, vertex) of $P$. First of all, since there is only one vertex, we have $\partial\hat{R} = 0$ for all $R$. Next, we have $\partial\hat{e} = \hat{R}_1 + \hat{R}_2 - \hat{R}_0$, where $R_0, R_1, R_2$ are the regions incident to $e$, numbered so that $R_1$ and $R_2$ induce on $e$ the same orientation. (Of course $R_0, R_1, R_2$ need not be different from each other, so the formula may actually have some cancellation.) Finally, $\partial\hat{v} = \hat{e}_1 + \hat{e}_2 - \hat{e}_3 - \hat{e}_4$, where $e_1, e_2$ are the edges which (with respect to the natural orientation) are leaving $v$, and $e_3, e_4$ are those which are reaching it. (Again, there could be repetitions.)

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