Giving freedom and physical meaning to the effective parameters of metamaterials for all frequencies

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Metamaterial effective parameters may exhibit freedom from dispersion constraints owing to their loss of physical meaning outside a subset of frequencies and wave numbers ($\omega, k$). For instance, effective parameters $\mu_{\text{eff}}$ and $\epsilon_{\text{eff}}$ can have a negative imaginary part for passive metamaterial systems, or may not tend to unity when analytically continued to high frequencies. We characterize this freedom through generalized Kramers-Kronig relations, and allocate alternative meaning to the effective parameters that remains valid for all $(\omega, k)$. There exists several alternative definitions for $\mu_{\text{eff}}$ or $\epsilon_{\text{eff}}$, thereby giving different frequency variations for high frequencies, while nevertheless converging to the same dispersion for long wavelengths.

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I. INTRODUCTION

The concept of a metamaterial is a powerful one. Complex electromagnetic systems are treated as simple, effectively continuous media with effective homogeneous fields, for which electromagnetic properties unlike those found in any conventional media may emerge. These properties are described by effective parameters $\mu_{\text{eff}}(\omega)$ and $\epsilon_{\text{eff}}(\omega)$ which represent the effective permeability and permittivity, respectively, as seen by macroscopic fields in the long wave limit. Analytic expressions of such parameters have been derived for several systems, including arrays of split-ring cylinders [1] in which magnetism is realized from non-magnetic conductors, and loaded transmission lines [2] which realize a left handed medium.

While homogenization procedures allow us to treat metamaterials as effectively continuous media, the resulting effective parameters $\mu_{\text{eff}}$ and $\epsilon_{\text{eff}}$ often need to be treated with greater care than the conventional parameters $\mu$ and $\epsilon$ for continuous media. First of all, one must make sure that the intricacies of the medium, such as bianisotropy, magnetoelectric coupling, and lattice effects, have been properly accounted for through a rigorous homogenization procedure [3, 4]. When this is not the case, non-physical artifacts such as violations of causality, reciprocity and passivity have been shown to emerge in the effective parameters even for long wavelengths [4]. Secondly, one must also pay attention to the limited frequency bandwidth in which the parameters are physically meaningful [5, 6]. When overlooked, it is possible to attribute non-physical behavior (e.g. anomalous dispersion) to the effective parameters for high frequencies, as has been pointed out in [6]. It is this second limitation that is of interest in the context of this article; it turns out that the effective parameters $\mu_{\text{eff}}$ and $\epsilon_{\text{eff}}$ as a result do not necessarily adhere to the same dispersion constraints that apply for the conventional parameters $\mu$ and $\epsilon$. The aim of this paper is therefore to show that the loss of physical meaning of the effective parameters $\mu_{\text{eff}}$ and $\epsilon_{\text{eff}}$ under certain conditions allows for new freedom in dispersions that are not present in conventional materials.

This paper concerns itself with effective parameters $\mu_{\text{eff}}$ and $\epsilon_{\text{eff}}$ for metamaterials that locally relate the fields of macroscopic electromagnetism $\mathbf{H}_{\text{av}}$ and $\mathbf{D}_{\text{av}}$ to the fields $\mathbf{B}_{\text{av}}$ and $\mathbf{E}_{\text{av}}$ for long wavelengths, i.e. $kd \ll 1$, where $k$ is the wave number and $d$ represents the characteristic size of the relevant inclusions. For simplicity, the effective parameters are assumed to be scalar throughout the paper. We make the following distinction between conventional and effective parameters:

| $\mu, \epsilon$ | Local permeability and permittivity of a conventional medium. |
| $\mu_{\text{eff}}, \epsilon_{\text{eff}}$ | Effective parameters which yield the effective local permeability and permittivity of the metamaterial in the long wave limit. |

This is done to make clear the fact that the effective parameters of metamaterials as local ($k$-independent) parameters may lose their physical meaning outside a restricted subset of frequencies and wave numbers $(\omega, k)$: In general, we can only define local effective parameters in the long wavelength regime $kd \ll 1$ [1, 2]. When assuming eigenmodal propagation this naturally also implies restrictions upon $\omega$. Furthermore, metamaterial models often make the assumption of quasi-static interactions, which may become invalid with increasing $\omega$. Hence the relevant subset may be classified as $k \leq k_{\text{max}}$ and $\omega \leq \omega_{\text{max}}$, where $k$-dependence in $\mu_{\text{eff}}$ and $\epsilon_{\text{eff}}$ is here assumed negligible below $k_{\text{max}}$, and $\omega_{\text{max}}$ is found according to the dispersion relation (Fig. 1).

Source-driven systems need in general not adhere to the dispersion relation. Hence, any combination of $\omega$ and $k$ is in principle possible provided the necessary arrangement of sources. Physical $\mu_{\text{eff}}$ and $\epsilon_{\text{eff}}$ may be defined for all $\omega$ if the medium is source driven to keep $k < k_{\text{max}}$, and are obtained from an appropriate homogenization procedure [4]. Therefore, meaningful local effective parameters...
The Kramers-Kronig relations relate the real and imaginary parts of a square-integrable complex valued function $\chi(\omega)$ which is analytic for $\text{Im} \omega > 0$. As a result of causality and other physically reasonable assumptions, the electric and magnetic susceptibilities of conventional media $\chi_e(\omega) = \epsilon(\omega) - 1$ and $\chi_m(\omega) = \mu(\omega) - 1$, respectively, are usually assumed to fulfill these requirements. Accordingly, by use of Cauchy’s integral theorem one therefore finds

$$\text{Re}\mu(\omega) = 1 + \frac{2P}{\pi} \int_0^\infty \frac{x\text{Im}\mu(x)}{x^2 - \omega^2} \, dx \quad (1a)$$
$$\text{Im}\mu(\omega) = \frac{2\omega P}{\pi} \int_0^\infty \frac{\text{Re}\mu(x) - 1}{x^2 - \omega^2} \, dx, \quad (1b)$$

where $P$ represents the principle value. The conditions upon $\chi_m(\omega)$ translate into demanding that the permeability $\mu(\omega) \to 1$ as $\omega \to \infty$ and that $\mu(\omega)$ must be analytic for $\text{Im} \omega \geq 0$.

When instead wanting to apply the Kramers-Kronig relations to metamaterial effective parameters, $\mu_{\text{eff}}(\omega)$ and $\epsilon_{\text{eff}}(\omega)$, one faces the interesting problem that there exists a number of metamaterial systems in which the asymptotic forms of their effective parameters do not fulfill the conditions outlined above: $\mu_{\text{eff}}(\omega), \epsilon_{\text{eff}}(\omega) \not\to 1$ as $\omega \to \infty$. For instance, in the well-known split ring cylinder metamaterial proposed by Pendry, the analytic continuation of the effective parameter derived there gives

$$\mu_{\text{eff}}(\omega) \to 1 - F, \quad \text{as } \omega \to \infty, \quad (2)$$

where $F$ represents the filling factor of the cylinder in a unit cell. By violating the assumptions of (1), which are generally taken to describe the class of possible dispersions in conventional media, this serves as a tell-tale sign of some additional dispersion freedom inherent to the split ring cylinder metamaterial.

One way to quantify additional freedom for metamaterial responses is to generalize the Kramers-Kronig relations for varying asymptotic forms of the effective parameters $\mu_{\text{eff}}(\omega), \epsilon_{\text{eff}}(\omega)$ as $\omega \to \infty$. Before doing so, it is instructive to first investigate the asymptotic forms that may be expected in passive metamaterials. In the doctoral thesis of Otto Brune from 1931 an argument was given which will here be modified to show that the three possible asymptotic forms consonant with passivity are

$$O(\mu_{\text{eff}}), O(\epsilon_{\text{eff}}) \to 1, \omega^{-1}, \text{ or } \omega^{-2}, \quad \text{as } \omega \to \infty, \quad (3)$$

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1 For clarity in the following discussions we have assumed a $\mu$ which approaches unity for large $\omega$, before any loss of physical meaning.

2 The particular asymptote assumes a constant conductivity in the cylinders. Even for a physical conductivity it is possible to have $\mu_{\text{eff}} \not\to 1$ for $\omega \to \infty$ when $k$ is fixed. Following the comments made in the introduction, a $\mu_{\text{eff}}(\omega)$ that does not approach unity may therefore be an entirely physical response for a source-driven Pendry medium for small $k$. 
under the condition that \( \omega \) remains in the frequency bandwidth where \( \mu_{\text{eff}}(\omega) \) and \( \epsilon_{\text{eff}}(\omega) \) represent effective permeability and permittivity, respectively. In practical terms, this means that if the long wavelength regime of the physical model underlying \( \mu_{\text{eff}}(\omega) \) or \( \epsilon_{\text{eff}}(\omega) \) is extended indefinitely, for instance by reducing the length scales of the inclusions, then it follows that \( \mu_{\text{eff}}(\omega) \) and \( \epsilon_{\text{eff}}(\omega) \) must take one of the forms given by (3).

For a passive system, the passivity condition (as derived in [2]) when extended to the upper complex half-plane for the permeability of a system becomes \( \text{Im} \, \omega \mu > \text{Im} \, \omega \mu |_{\omega \rightarrow \infty} \). We characterize the possible asymptotic forms of \( \mu_{\text{eff}} \) by assuming

\[
O(\omega \mu_{\text{eff}}) \xrightarrow{|\omega| \rightarrow \infty} \omega^n, \quad (4)
\]

where \( n \in \mathbb{Z} \). Then, by expressing \( \omega^n = |\omega|^n \exp(in\theta) \), one may write

\[
O(\text{Im} \, \omega \mu_{\text{eff}}) \xrightarrow{|\omega| \rightarrow \infty} |\omega|^n \sin(n\theta). \quad (5)
\]

Now in order that \( \text{Im} \, \omega \mu_{\text{eff}} > 0 \) for \( \text{Im} \, \omega > 0 \), it may be seen from (5) that the values of \( n \) are restricted to \( n \in \{-1, 0, 1\} \), so as to avoid a sign change for \( \theta \in [0, \pi] \). Hence, on the basis of passivity the parameter \( \mu_{\text{eff}} \) must have one of the asymptotic forms (3). Equivalent considerations on \( \epsilon_{\text{eff}}(\omega) \) give the same result.

Where may we expect to find parameters which exhibit the asymptotic forms (3)? All three forms may in fact be visualized in terms of transmission-line metamaterials, which serve as helpful testbeds for the construction of systems with desired effective parameters by elementary circuit theory. For the 1D unit cell displayed in Fig. 2 with series impedance \( Z'(\omega) \) and the shunt admittance \( Y'(\omega) \) per unit length, the homogenization procedure is as follows: Imposing an equivalence between the telegrapher’s equations and Maxwell’s equations for a homogeneous, isotropic medium, one finds

\[
\mu_{\text{eff}}(\omega) = \frac{Z'}{i\omega}, \quad (6a)
\]
\[
\epsilon_{\text{eff}}(\omega) = \frac{Y'}{i\omega}. \quad (6b)
\]

If the impedance and admittance of the transmission line for instance are taken to represent common lumped circuit elements such as inductors \( Z_L = -i\omega L \), resistors \( Z_R = R \) or capacitors \( Z_C = -1/i\omega C \), then it is clear from (6a) and (6b) that the asymptotic forms (3) follow upon analytic continuation for \( \omega \rightarrow \infty \).

Kramers-Kronig relations generalized for the three asymptotic forms (3) can be derived by expressing the effective parameter as

\[
\mu_{\text{eff}}(\omega) = a + \mu_a(\omega) \quad (7)
\]

where \( a = \lim_{\omega \rightarrow \infty} \mu_{\text{eff}}(\omega), \mu_a(\omega) \) is square-integrable and \( \mu_{\text{eff}}(\omega) \) is analytic for \( \text{Im} \, \omega > 0 \), and then applying Cauchy’s integral theorem. From the symmetry property \( \mu_{\text{eff}}(-\omega) = \mu_{\text{eff}}^*(\omega^*) \) it follows that the constant \( a \) is real (this can also be seen from the previous argument concerning (4) and (5)). This gives the following expressions

\[
\text{Re} \, \mu_{\text{eff}}(\omega) = a + \frac{2P}{\pi} \int_0^\infty \frac{x \text{Im} \mu_a(x)}{x^2 - \omega^2} \, dx \quad (8a)
\]
\[
\text{Im} \, \mu_{\text{eff}}(\omega) = -\frac{2\omega P}{\pi} \int_0^\infty \frac{\text{Re} \mu_a(x)}{x^2 - \omega^2} \, dx. \quad (8b)
\]

Despite the effective parameters generally loosing physical meaning in the sense of representing effective permeability or permittivity above a finite frequency \( \omega_{\text{max}} \), the analytic continuation of the parameter for frequencies above this is used in order to generalize (1) for the three asymptotic forms (3). Notice that in the event that \( \lim_{\omega \rightarrow \infty} \mu_{\text{eff}}(\omega) = 1 \) (8a) and (8b) reduce to the conventional Kramers-Kronig relations (where \( \mu_a(\omega) = \chi_m(\omega) \)).

**FIG. 2:** Unit cell of a 1D transmission line with series impedance \( Z'(\omega) \) and shunt admittance \( Y'(\omega) \) per unit length.

An example of the additional freedom present in metamaterial systems is observed when applying the generalized Kramers-Kronig relations (8) to Pendry’s split-ring cylinder metamaterial, where the cylinders are made of a non-magnetic metal, such as aluminum or copper [1]. Upon applying a time-varying field, induced current flows on the cylinder surfaces and thereby leads to an emergent magnetic response, with \( \mu_{\text{eff}}(\omega) \neq 1 \). At zero frequency, however, only the intrinsic material properties matter, meaning that \( \mu_{\text{eff}}(0) = 1 \). Now, if we were to search for such emergent magnetism in a conventional, continuous medium which obeys (1), it turns out that we would necessarily be looking for a gain medium! This is observed from (1a) with \( \omega = 0 \):

\[
\text{Re} \, \mu(0) = 1 + \frac{2P}{\pi} \int_0^\infty \frac{\text{Im} \mu(x)}{x} \, dx. \quad (9)
\]

---

3 The new asymptotic forms arise only when neglecting the intrinsic capacitance, \( C_0 \), and inductance, \( L_0 \), of the transmission line. For long wavelengths, however, where (6a) and (6b) correspond to the permeability and permittivity of the system, these are often negligible.
The only way to have \( \text{Re} \mu(0) = 1 \) for our system is for the integral to equal zero, thereby implying that \( \text{Im} \mu(\omega) < 0 \) for some frequencies. The additional freedom inherent in the metamaterial arrangement is therefore made visible: While a continuous medium would need to display gain in order to have this property of emergent magnetism, Pendry’s split-ring cylinder metamaterial achieves this while being passive. In [1] the effective parameter corresponding to Pendry’s medium is derived to be

\[
\mu_{\text{eff}}(\omega) = 1 + \frac{\omega^2 F}{\omega_0^2 - \omega^2 - i\omega \Gamma} \tag{10}
\]

where \( F \) is the volume fraction of the interior of the cylinder in the unit cell, \( \omega_0 \) is the resonance frequency determined by the conductivity and capacitance, and \( \Gamma \) is the response width determined by the conductivity and cylinder radius [12]. Equation (10) shows that \( \mu_{\text{eff}}(0) = 1 \) while \( \text{Im} \mu_{\text{eff}}(\omega) \geq 0 \) for all positive frequencies. By use of the generalized Kramers-Kronig relation [8a] where \( a = 1 - F \) according to [2] we find

\[
\text{Re} \mu_{\text{eff}}(0) = 1 - F + \frac{2P}{\pi} \int_0^\infty \frac{\text{Im} \mu_a(x)}{x} \text{d}x. \tag{11}
\]

Evidently, one need not assume that \( \text{Im} \mu_{\text{eff}}(\omega) = \text{Im} \mu_a(\omega) < 0 \) in order that \( \mu_{\text{eff}}(0) = 1 \).

### III. VISUALIZING DISPERSION-FREEDOM BY GIVING \( \mu_{\text{eff}}(\omega) \) PHYSICAL MEANING FOR ALL FREQUENCIES

When generalizing the Kramers-Kronig relation to the three asymptotic forms [3] as we have done above, we implicitly allow for an analytic continuation of the effective parameter \( \mu_{\text{eff}}(\omega) \) to large frequencies even though the parameter might not have any physical meaning outside the long wave limit. In this section, rather than generalize the Kramers-Kronig relations, we shall instead redefine the parameter \( \mu_{\text{eff}}(\omega) \) in a way that ensures its physical meaning for all frequencies. This implies attributing to \( \mu_{\text{eff}}(\omega) \) a physical meaning other than that of effective permeability in general, which in the long wavelength regime nevertheless coincides with the physical effective permeability. By doing so the Kramers-Kronig relations [1] or [8] can be understood to relate the real and imaginary parts of parameters with intuitive physical meaning for all frequencies.

We choose to attribute the following physical meaning to the parameter \( \mu_{\text{eff}}(\omega) \) of an arbitrary metamaterial, as given from the relation

\[
B_{\text{av}}^d(\omega) = \mu_0 \mu_{\text{eff}}(\omega) H_{\text{av}}^d(\omega) \tag{12}
\]

where

\[
\begin{align*}
B_{\text{av}}^d &= \mu_0 \bar{H} + \frac{1}{V} \int_V M(r) d^3r \tag{13a} \\
H_{\text{av}}^d &= \bar{H} + \frac{i\omega}{V} \int_V \frac{r \times P(r)}{2} d^3r \tag{13b} \\
\bar{H} &= \frac{1}{V} \int_V H(r) e^{-ik \cdot r} d^3r. \tag{13c}
\end{align*}
\]

Here \( B_{\text{av}}^d(\omega) \) and \( H_{\text{av}}^d(\omega) \) generally represent the dominant terms of the averaged field expansions according to the first principle’s homogenization approach outlined in [4]. \( M(r) \) represents the microscopic induced magnetization, \( P(r) \) represents the microscopic induced electrical polarization, and \( V \) represents the volume of the unit cell. Again, we stress that the parameter \( \mu_{\text{eff}}(\omega) \) under the definition [12] does not represent the permeability of the system in general. The physical meaning of \( \mu_{\text{eff}}(\omega) \) by [12] is now given by the field quantities [13a]-[13c], and this concrete meaning is kept for all \((\omega,k)\) in general, while additionally obtaining the particular meaning of approximating the system’s effective permeability in the long wave limit \( k \rightarrow 0 \), according to the homogenization procedure outlined in [4].

A rigorous method to calculate the \( \mu_{\text{eff}}(\omega) \) in [12] for all frequencies is proposed in [4], where electric and magnetic dyadic Green’s functions are summed to determine the interaction fields between the inclusions. The complexity involved in calculating these sums in the dynamic regime makes such a treatment lie beyond the scope of the current discussion. Nevertheless, in order to highlight some of the features of this alternative definition when applied to Pendry’s split-ring cylinder metamaterial, we shall pursue a qualitative approach. The next section, however, will present an exact treatment of an analogous effective parameter definition on a less complicated system. The 1D photonic crystal.

Hereafter we shall use the definition [12] under the assumption of eigenmodal propagation in the metamaterial, where \( k = \omega/e_{\text{eff}}\mu_{\text{eff}}/c \rightarrow \omega/c \) as \( \omega \rightarrow \infty \). In Pendry’s split-ring cylinder metamaterial, the wave propagation is orthogonal to each cylinder axis, and both \( B_{\text{av}}^d \) and \( H_{\text{av}}^d \) are parallel. Thus \( \mu_{\text{eff}}(\omega) \) is a scalar which according to [12] may be expressed as the ratio of the field quantities

\[
\mu_{\text{eff}}(\omega) = \frac{B_{\text{av}}^d(\omega)}{\mu_0 H_{\text{av}}^d(\omega)}. \tag{14}
\]

Assuming that the cylinders in Pendry’s medium have thin walls of a non-magnetic metal, Eq. [13a] and [13b] become

\[
\begin{align*}
B_{\text{av}}^d &= \mu_0 \bar{H} \tag{15a} \\
H_{\text{av}}^d &= \bar{H} - M_{\text{net}}, \tag{15b}
\end{align*}
\]

where \( M_{\text{net}} = F J_{\text{net}} \) is the net magnetization density, \( F \) is the volume fraction of the cylinder in a unit cell and

\[
J_{\text{net}} = \frac{1}{2\pi} \int_0^{2\pi} J(r,\phi) |_{r=R} d\phi, \tag{16}
\]
represents the net current per cylinder length flowing around the cylinders with radius $R$. Using (15a) and (15b) one may find

$$\mu_{\text{eff}}(\omega) = \frac{1}{1 - M_{\text{net}}/\bar{H}},$$

(17)

by insertion into (14). If the interaction between the cylinders is modeled quasi-statically, straightforward application of Faraday’s law [11,12] gives

$$M_{\text{net}}^{q.s.}(\omega) = \frac{\omega^2 \bar{H} F}{\omega_0^2 - \omega^2(1 - F) - i\omega \Gamma},$$

(18)

where $\omega_0$ is the resonance frequency determined by the cylinder radius and capacitance, and $\Gamma$ is the response width determined by the conductivity and cylinder radius. Inserting (15a) into (17) gives the Pendry response [10], as derived in [14], which has the asymptotic form (2) (plotted here in Fig. 3a). In contrast to the asymptotic limit of (18), it is clear that a proper evaluation of the current for dynamic frequencies should give $M_{\text{net}} = F J_{\text{net}}(\omega) \to 0$ as $\omega \to \infty$, since the conductivity of a metal will tend to zero. Since $\bar{H}$ will be non-zero in this limit under eigenmodal propagation, this in turn implies that the asymptote of $\mu_{\text{eff}}(\omega)$ according to the definition (14) should approach unity, in contrast to (10). This leads to the observation that our current definition of $\mu_{\text{eff}}(\omega)$ by (14) will fulfill the conventional Kramers-Kronig relations,

$$\text{Re} \mu_{\text{eff}}(\omega) = 1 + \frac{2\pi}{\pi} \int_{0}^{\infty} \frac{x \text{Im} \mu_{\text{eff}}(x)}{x^2 - \omega^2} \, dx,$$

(19a)

$$\text{Im} \mu_{\text{eff}}(\omega) = -\frac{2\omega}{\pi} \int_{0}^{\infty} \frac{\text{Re} \mu_{\text{eff}}(x) - 1}{x^2 - \omega^2} \, dx,$$

(19b)

provided that the resulting $\mu_{\text{eff}}(\omega)$ is analytic.

Although an exact treatment would now seek to calculate $M_{\text{net}}(\omega)$ for dynamic frequencies, we shall here avoid elaborate calculations, and rather search for an arbitrary $M_{\text{net}}^{q.s.}(\omega)$ which approximates $M_{\text{net}}^{q.s.}(\omega)$ for small $\omega$, and at the same time gives the asymptote $M_{\text{net}}^{q.s.}(\omega) \to 1$ as $\omega \to \infty$, while being analytic in the upper complex half-plane. This ensures the correct solution for $M_{\text{net}}(\omega)$ in the quasi-static regime and vacuum limit, though not in the remaining intermediate frequency bandwidth. To obtain a parameter which approximates these properties, we may for instance set

$$M_{\text{net}}^{\text{arb.}}(\omega) = M_{\text{net}}^{q.s.}(\omega) \frac{\omega_r^2}{\omega_r^2 - \omega^2 - i\omega \Gamma_m},$$

(20)

where we multiply the quasi-static solution (18) by a Lorentzian resonance function with resonance outside the quasi-static limit $\omega_r \gg \omega_0$ and width $\Gamma_m$. Inserting (20) with suitable parameters into (17) gives the plot shown in Fig. 3b.

One observes that the effective parameter $\mu_{\text{eff}}(\omega)$ is essentially equal to that arising from the quasi-static model for low frequencies in Fig. 3a while achieving the asymptote $\mu_{\text{eff}}(\omega) \to 1$ for $\omega \to \infty$. For the intermediate regime one observes a large negative value of $\text{Im} \mu_{\text{eff}}(\omega)$. Although the particular form $\mu_{\text{eff}}(\omega)$ does not represent the actual ratio (14) owing to the arbitrary choice we have made for $M_{\text{net}}(\omega)$ by (20), we note that the Kramers-Kronig relations (19) nevertheless predict that the exact solution for $\mu_{\text{eff}}(\omega)$ will display $\text{Im} \mu_{\text{eff}}(\omega) < 0$ somewhere in this intermediate region. Our definition of $\mu_{\text{eff}}(\omega)$ according to (14) allows us to make sense of what a negative $\text{Im} \mu_{\text{eff}}(\omega)$ means: Since it does not occur in the long wave region, it does not represent gain (as this is nonsensical for a passive system), but merely represents a phase difference between the average field quantities $B_{\text{av.m}}^0(\omega)$ and $H_{\text{av.m}}^0(\omega)$ greater than $\pi$. We therefore observe how one is able to obtain an intuitive physical understanding of the occurrence of $\text{Im} \mu_{\text{eff}}(\omega) < 0$ in passive media on the basis of the alternative definition (14).

There exists a variety of possible definitions of $\mu_{\text{eff}}(\omega)$ which attain the particular meaning of the effective permeability in the limit $k \to 0$. Since these will generally be approximations of the effective permeability in the long wavelength regime, each definition of $\mu_{\text{eff}}(\omega)$ may exhibit small, perhaps negligible, deviations from one another here. It follows that each of their respective continuations to higher frequencies, though unique, may nevertheless deviate significantly from one another, as we observe from Fig. 3a and Fig. 3b. Such plurality in frequency behavior shall be discussed further in the following example system.

IV. CASE EXAMPLE: 1D PHOTONIC CRYSTAL

The previous section redefined the effective parameter $\mu_{\text{eff}}(\omega)$ in terms of a quantity that is defined for all $\{\omega, k\}$. Since the effective parameter was only treated quantitatively, we shall here present a simpler case example where an alternative definition of the effective parameter for all frequencies is easily evaluated. The 1D Bragg stack under eigenmodal propagation. We choose as our parameter $n_{\text{eff}}$, which will give the effective refractive index in the long wave limit. As we shall see, our parameter will depend on the coordinate $z$ along the axis of periodicity for frequencies outside the long wavelength regime. Hence, it will be possible to have a large number of different effective frequency variations for $n_{\text{eff}}$ at large frequencies, each corresponding to different values of $z$, which all converge to the same dispersion in the long wave limit. Some of these may exhibit negative imaginary parts for some frequencies, while others may only have positive imaginary parts.

Consider the 1D photonic crystal as shown in Fig. 4. We imagine placing a current sheet at $z = 0$ in the layer of lowest refractive index $n_1$ in which the current alternates as $J_y = J_0 \hat{x}$ (with the harmonic time variation suppressed). The magnetic field in the low index layer is
then

$$H = -\frac{J_0}{2} e^{ik_1 z} \hat{y} \quad \text{for} \quad 0 < z < \frac{d_1}{2},$$

where $k_1 = n_1 \omega / c$ is the wave number in layer 1. Noting that this field must be continuous over the interfaces, it follows that for $\omega \to 0$ the magnetic field becomes uniform over the unit cell of the 1D photonic crystal.

Therefore, as $\omega \to 0$ the transfer function

$$G \equiv \frac{H(z)}{H(0^+)}.$$  \hfill (22)

of the fields approaches that of a continuous medium given by

$$G = e^{in_{\text{eff}} \frac{z}{z}}.$$  \hfill (23)

in terms of an effective index of refraction $n_{\text{eff}}$. Motivated by the form of (23), we obtain our definition:

$$n_{\text{eff}}(\omega) = \frac{\omega}{\mu_{\text{eff}}} \ln G$$  \hfill (24)

where $G$ is given by (22). Now our parameter $n_{\text{eff}}(\omega)$ is in general a quantity proportional to the logarithm of the transfer function in the medium; a quantity that has physical meaning for all frequencies. The subscript $\omega$ indicates that the transfer function evaluated at each value of $\omega$ yields a different frequency variation. All of these converge to the same $n_{\text{eff}}(0)$ as $\omega \to 0$, for which the analytic value may be calculated to be

$$n_{\text{eff}}(0) = \sqrt{n_1^2 d_1 + n_2^2 d_2} / d_1 + d_2.$$  \hfill (25)

Provided the transfer function $G$ is not equal to zero in the upper complex half-plane, our parameter $n_{\text{eff}}(\omega)$ is analytic there while having the asymptote $n_{\text{eff}}(\omega) \to 1$, and therefore obeys the conventional Kramers-Kronig relations.

Figure 5 displays two possible variations as given by (21) where $z$ is chosen within a layer of the low index $n_1$ in Fig. 5a and where $z$ is chosen within a layer of the higher index $n_2$ in Fig. 5b. The indexes $n_1$ and $n_2$ have been assumed constant with respect to frequency for simplicity. This means that for $\omega \to \infty$ one has $n_{\text{eff}}(\omega) \to n_{\infty} \neq 1$. As a side remark, we note that a similar effective parameter definition for the effective refractive index with a similar plot as that in Fig. 5a is proposed in [11]. Figures 5a and 5b give slightly different variations: Fig. 5a has Im $n_{\text{eff}}(\omega) \geq 0$ for all $\omega$, and...
while Fig. 5b displays \( \text{Im} n_{z,\text{eff}}(\omega) < 0 \) for a small bandwidth. In light of the physical definition of \( n_{z,\text{eff}}(\omega) \), we notice that the negative imaginary part in Fig. 5b has nothing to do with gain: A negative value of \( \text{Im} n_{z,\text{eff}}(\omega) \) in Fig. 5b corresponds to a transfer function greater than unity. This occurs as a result of a Fabry-Perot interference occurring between the low and high index layers, leading to a local accumulation of field in the high index layer where the transfer function is evaluated.

It is interesting that one can achieve several different variations in the effective parameter for the same metamaterial, as demonstrated by Fig. 5a and Fig. 5b. This corresponds to the multiple ways in which one may define a physical quantity that is meaningful for all frequencies, which at the same time approximates the refractive index of the medium for small frequencies.

**CONCLUSIONS**

In this article we have identified examples of metamaterial effective parameters that do not follow conventional dispersion constraints, as represented by the Kramers-Kronig relations. This freedom in dispersion has been explained in terms of the limited validity of the parameters \( \mu_{\text{eff}} \) and \( \epsilon_{\text{eff}} \) as representing the effective permeability and permittivity of the medium. Such correspondence is generally only ensured in a subset of frequencies and wave numbers \( (\omega, k) \). The asymptotic forms which effective parameters may take in passive metamaterials when analytically continued to high frequencies have been investigated, for which generalized Kramers-Kronig relations have been derived. These serve to characterize the dispersion freedom attainable in passive metamaterials. The alternative possibility of defining metamaterial parameters for all \( (\omega, k) \) by allocating alternative physical meaning to the parameters, has also been presented. Such an approach is helpful in understanding the dispersion freedom in metamaterial parameters; in two cases the occurrence of negative imaginary parts in the parameters of two passive metamaterials has been given an intuitive physical explanation.

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