LOCALLY LIPSCHITZ CONTRACTIBILITY OF ALEXANDROV SPACES AND ITS APPLICATIONS

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Abstract. We prove that any finite dimensional Alexandrov space with a lower curvature bound is locally Lipschitz contractible. As applications, we obtain a sufficient condition for solving the Plateau problem in an Alexandrov space considered by Mese and Zulkowski.

1. Introduction

Alexandrov spaces are naturally appeared in the collapsing and convergence theory of Riemannian manifolds and played important roles in Riemannian geometry. In the paper, when we say simply an Alexandrov space, it means that an Alexandrov space of curvature bounded from below locally and of finite dimension. Their fundamental properties of such spaces were well studied in [BGP]. There is a remarkable study of topological structures for Alexandrov spaces by Perelman [P Alx2]. There, the topological stability theorem was proved which states that, if two compact Alexandrov spaces of the same dimension are very close in the Gromov-Hausdorff topology, then they are homeomorphic to each other. Further, it implies that for any point in an Alexandrov space, its small open ball is homeomorphic to its tangent cone. In particular, an open ball of small radius with respect to its center is contractible. It is expected by geometers that the corresponding statements replacing homeomorphic by bi-Lipschitz homeomorphic could be proved. Until now, we did not know any Lipschitz structure of an Alexandrov space around singular points. A main purpose of this paper is to prove that any finite dimensional Alexandrov space with a lower curvature bound is strongly locally Lipschitz contractible in the sense defined later. For short, SLLC denotes this property. The SLLC-condition is a strong version of the LLC-condition introduced in [Y] (cf. Remark 4.5).

We define the strongly locally Lipschitz contractibility. We denote by $U(p,r)$ an open ball centered at $p$ of radius $r$ in a metric space.

Definition 1.1. A metric space $X$ is strongly locally Lipschitz contractible, for short SLLC, if for every point $p \in X$, there exists $r > 0$
and a map 
\[ h : U(p, r) \times [0, 1] \to U(p, r) \]
such that \( h \) is homotopy from \( h(\cdot, 0) = id_{U(p, r)} \) to \( h(\cdot, 1) = p \), and it is Lipschitz, i.e., there exists \( C, C' > 0 \) such that

\[ d(h(x, s), h(y, t)) \leq Cd(x, y) + C'|s - t| \]
for every \( x, y \in U(p, r) \) and \( s, t \in [0, 1] \), and for every \( r' < r \), the image of \( h \) restricted to \( U(p, r') \times [0, 1] \) is \( U(p, r') \).

We call such a ball \( U(p, r) \) a Lipschitz contractible ball and \( h \) a Lipschitz contraction on \( U(p, r) \).

A main result in the present paper is the following.

**Theorem 1.2.** Any finite dimensional Alexandrov space is strongly locally Lipschitz contractible.

In [Y], a weaker form of Theorem 1.2 was conjectured.

For metric spaces \( P \) and \( X \) and possibly empty subsets \( Q \subset P \) and \( A \subset X \), we denote by \( f : (P, Q) \to (X, A) \) a map from \( P \) to \( X \) with \( f(Q) \subset A \). Two maps \( f \) and \( g \) from \( (P, Q) \) to \( (X, A) \) are homotopic (resp. Lipschitz homotopic) to each other if there exists a continuous (resp. Lipschitz) map

\[ h : (P \times [0, 1], Q \times [0, 1]) \to (X, A) \]
such that \( h(x, 0) = f(x) \) and \( h(x, 1) = g(x) \) for all \( x \in P \). Then, we write \( f \sim g \) (resp. \( f \sim_{Lip} g \)). Let us denote by

\[ [(P, Q), (X, A)] \text{ and } [(P, Q), (X, A)]_{Lip} \]
the set of all homotopy classes of continuous maps from \( (P, Q) \) to \( (X, A) \) and the set of all Lipschitz homotopy classes of Lipschitz maps from \( (P, Q) \) to \( (X, A) \), respectively.

Let us consider a Lipschitz simplicial complex which means that it is a metric space and admits a triangulation such that each simplex is a bi-Lipschitz image of a simplex in a Euclidean space. For precise definition, see Section 4.

**Corollary 1.3.** Let \( P \) be a finite Lipschitz simplicial complex and \( Q \) a possibly empty subcomplex of \( P \). Let \( X \) be an Alexandrov space and \( A \) an open subset of \( X \). Then, a natural map from \( [(P, Q), (X, A)]_{Lip} \) to \( [(P, Q), (X, A)] \) is bijective.

For a metric space \( X \) and a point \( x_0 \in X \) and \( k \in \mathbb{N} \), we define the \( k \)-th Lipschitz homotopy group \( \pi^Lip_k(X, x_0) \) by \( \pi^Lip_k(X, x_0) = [(S^k, \ast), (X, x_0)]_{Lip} \) as sets, where \( \ast \in S^k \) is an arbitrary point, equipped with a group operation as in the usual homotopy groups.

**Corollary 1.4.** For an Alexandrov space \( X \) and a point \( x_0 \in X \) and \( k \in \mathbb{N} \), a natural map

\[ \pi^Lip_k(X, x_0) \to \pi_k(X, x_0). \]
1.1. Application: the Plateau problem. By Mese and Zulkowski [MZ], the Plateau problem in an Alexandrov space was considered as follows. Let $W^{1,2}(D^2, X)$ denote the $(1, 2)$-Sobolev space from $D^2$ to an Alexandrov space $X$ in the sense of the Sobolev space of a metric space target defined by Korevaar and Schoen [KS]. Giving a closed Jordan curve $\Gamma$ in $X$, we set
\[ F_\Gamma := \{ u \in W^{1,2}(D^2, X) \cap C(D^2, X) ; \quad u|_{\partial D^2} \text{ parametrizes } \Gamma \text{ monotonically} \}. \]
They defined the area $A(u)$ of a Sobolev map $u \in W^{1,2}(D^2, X)$. Under these settings, the Plateau problem is stated as follows.
\[ \text{The Plateau problem.} \quad \text{Find a map } u \in W^{1,2}(D^2, X) \text{ such that } A(u) = \inf \{ A(v) | v \in F_\Gamma \}. \]
They obtained
\[ \text{Theorem 1.5 ([MZ]).} \quad \text{Let } X \text{ be a finite dimensional compact Alexandrov space and } \Gamma \text{ be a closed Jordan curve in } X. \text{ If } F_\Gamma \neq \emptyset, \text{ then there exists a solution of the Plateau problem.} \]
For an Alexandrov space, any condition of $\Gamma$ for implying $F_\Gamma \neq \emptyset$ was not known. As an application of Theorem 1.2, we can obtain such a condition of $\Gamma$.

Corollary 1.6. Let $\Gamma$ be a rectifiable closed Jordan curve in an Alexandrov space $X$. If $\Gamma$ is topologically contractible in $X$, then $F_\Gamma \neq \emptyset$.

1.2. Application: simplicial volume. In [Y, Theorem 0.5], the second author proved, assuming an LLC-condition on an Alexandrov space, an inequality between the Gromov’s simplicial volume and the Hausdorff measure of it. As an immediate consequence of Theorem 1.2, we obtain
\[ \text{Corollary 1.7 (cf. [G], [Y]).} \quad \text{Let } X \text{ be a compact orientable } n\text{-dimensional Alexandrov space without boundary of curvature } \geq \kappa \text{ for } \kappa < 0. \text{ Then, } \|X\| \leq n! (n - 1)^n \sqrt{-\kappa^n} \mathcal{H}^n(X). \]
Here, $\|X\|$ is the Gromov’s simplicial volume which is the $\ell_1$-norm of the fundamental class of $X$, and $\mathcal{H}^n$ denotes the $n$-dimensional Hausdorff measure. For precise terminologies, we refer [G] and [Y].
Further, if we assume “a lower Ricci curvature bound” for $X$ in the sense of Bacher and Sturm [BS], then we obtain the following.
\[ \text{Theorem 1.8.} \quad \text{Let } X \text{ be a compact orientable } n\text{-dimensional Alexandrov space without boundary. Let } m \text{ be a locally finite Borel measure on } X \text{ with full support which is absolutely continuous with respect to} \]

$\mathcal{H}^n$. If the metric measure space $(X, m)$ satisfies the reduced curvature-dimension condition $\text{CD}^*(K, N)$ locally for $K, N \in \mathbb{R}$ with $N \geq 1$ and $K < 0$, then

$$\|X\| \leq n! \sqrt{-(N-1)K} \mathcal{H}^n(X).$$

Theorem 1.8 is new even if $X$ is a manifold, because a reference measure $m$ can be freely chosen.

**Organization.** We review fundamental properties of Alexandrov spaces in Section 2. In particular, we recall the theory of the gradient flow of distance functions on an Alexandrov space established by Perelman and Petrunin [PP]. In Section 3 we prove that the distance function from a metric sphere at each point in an Alexandrov space is regular on a much smaller concentric punctured ball. Then, using the flow of it, we prove Theorem 1.2. In Section 3 we recall precise terminologies in the applications in the introduction, and prove Corollaries 1.3, 1.4 and 1.7. In Section 5 we note that our proof given in Section 3 also works for infinite dimensional Alexandrov spaces whenever the space of directions is compact. In Section 6, we recall several notions of a lower Ricci curvature bound on metric space together with a Borel measure and their relation. By using the Bishop-Gromov type volume growth inequality, we prove Theorem 1.8.

2. Preliminaries

This section consists of just a review of the definition of Alexandrov spaces and somewhat detailed review of the gradient flow theory of semiconcave functions on Alexandrov spaces. For precisely, we refer [BGP], [BB] or [Pt sem].

We recall the definition of Alexandrov spaces.

**Definition 2.1** (cf. [BB], [BGP]). Let $\kappa \in \mathbb{R}$. We call a complete metric space $X$ an *Alexandrov space of curvature $\geq \kappa$* if it satisfies the following.

1. $X$ is a geodesic space, i.e., for every $x$ and $y$ in $X$, there is a curve $\gamma : [0, |x, y|] \to X$ such that $\gamma(0) = x$ and $\gamma(|x, y|) = y$ and the length $L(\gamma)$ of $\gamma$ equals $|x, y|$. Here, $|x, y|$ denotes the distance between $x$ and $y$. We call such a curve $\gamma$ a geodesic between $x$ and $y$, and denote it by $xy$.

2. $X$ has curvature $\geq \kappa$, i.e., for every $p, q, r \in X$ (with $|p, q| + |q, r| + |r, p| < 2\pi/\sqrt{\kappa}$ if $\kappa > 0$) and every $x$ in a geodesic $qr$ between $q$ and $r$, taking a comparison triangle $\Delta \tilde{p}q\tilde{r} = \Delta pqr$ in a simply-connected complete surface $\mathbb{M}_\kappa$ of constant curvature $\kappa$, and corresponding point $\tilde{x}$ in $\tilde{q}\tilde{r}$, we have

$$|p, x| \geq |\tilde{p}, \tilde{x}|.$$
We simply say that a complete metric space $X$ is an *Alexandrov space* if it is a geodesic space and for any $p \in X$, there exists a neighborhood $U$ of $p$ and $\kappa \in \mathbb{R}$ such that $U$ has curvature $\geq \kappa$ as the condition (2), i.e., any triangle in $U$ (whose sides contained in $U$) is not thinner than its comparison triangle in $M_\kappa$.

If $X$ is compact, then it has a uniform lower curvature bound. Throughout in the paper, we do not need a uniform lower curvature bound, since we are mainly interested in a local property. It is known that if $X$ has a uniform lower curvature bound, say $\kappa$, then $X$ has curvature $\geq \kappa$ (\cite{BGP}).

2.1. *Semiconcave functions.* In this subsection, we refer \cite{Pt sem} and \cite{Pt QC}.

**Definition 2.2.** Let $I$ be an interval and $\lambda \in \mathbb{R}$. We say a function $f : I \to \mathbb{R}$ to be $\lambda$-concave if the function

$$
\bar{f}(t) = f(t) - \frac{\lambda}{2} t^2
$$

is concave on $I$. Namely, for any $t < t' < t''$ in $I$, we have

$$
\frac{\bar{f}(t') - \bar{f}(t)}{t' - t} \geq \frac{\bar{f}(t'') - \bar{f}(t')}{t'' - t'}.
$$

We say a function $f : I \to \mathbb{R}$ to be $\lambda$-concave in the barrier sense if for any $t_0 \in \text{int} I$, there exist a neighborhood $I_0$ of $t_0$ in $I$ and a twice differentiable function $g : I_0 \to \mathbb{R}$ such that

$$
g(t_0) = f(t_0), \quad g \geq f \quad \text{and} \quad g'' \leq \lambda \quad \text{on} \quad \text{int} I.
$$

**Lemma 2.3** (cf. \cite{Pt QC}). Let $f : I \to \mathbb{R}$ be a continuous function on an interval $I$ and $\lambda \in \mathbb{R}$. Then the following are equivalent.

1. $f$ is $\lambda$-concave in the sense of Definition 2.2.
2. For any $t_0 \in I$, there is $A \in \mathbb{R}$ such that

$$
f(t) \leq f(t_0) + A(t - t_0) + \frac{\lambda}{2}(t - t_0)^2
$$

for any $t \in I$.
3. $f$ is $\lambda$-concave in the barrier sense.

**Proof.** By considering $f(t) - (\lambda/2) t^2$, we may assume that $\lambda = 0$.

Let us prove the implication (1) $\Rightarrow$ (2). Let us take $t_0 \in I$ to be not the maximum number of $I$. By the concavity of $f$, the value

$$
A = \lim_{\varepsilon \to 0^+} \frac{f(t_0 + \varepsilon) - f(t_0)}{\varepsilon}
$$

is well-defined. And, the concavity of $f$ implies

$$
f(t) \leq f(t_0) + A(t - t_0).
$$

When $t_0 \in I$ is the maximum value of $I$, then replacing $A$ to the limit $\lim_{\varepsilon \to 0^+}(f(t_0 - \varepsilon) - f(t_0))/\varepsilon$, we obtain the inequality as same as above.
The implication (2) ⇒ (3) is trivial.
Let us assume that \( f \) satisfies (3). Let us take \( t_0 \) in the interior of \( I \). Then there exists a twice differentiable function \( g : I \to \mathbb{R} \) such that

\[
g(t_0) = f(t_0), \quad g \geq f \quad \text{and} \quad g'' \leq 0.
\]

Hence, for any \( t' < t_0 < t \), we have

\[
\frac{f(t) - f(t_0)}{t - t_0} \leq g(t) - g(t_0) \leq \frac{g(t_0) - g(t')}{t_0 - t'} \leq \frac{f(t_0) - f(t')}{t_0 - t'}.
\]

Therefore, \( f \) is concave.

Let \( X \) be a geodesic space and \( U \) be an open subset of \( X \). Let \( f : U \to \mathbb{R} \) be a function. We say that \( f \) is \( \lambda \)-concave on \( U \) if for every geodesic \( \gamma : I \to U \), the function \( f \circ \gamma : I \to \mathbb{R} \) is \( \lambda \)-concave on \( I \). For a function \( g : U \to \mathbb{R} \), we say that \( f \) to be \( g \)-concave if for any \( p \in U \) and \( \varepsilon > 0 \), there is an open neighborhood \( V \) of \( p \) in \( U \), such that \( f \) is \( (g(p) + \varepsilon) \)-concave on \( V \). We say that \( f : U \to \mathbb{R} \) is \( g \)-concave in the barrier sense if for any \( p \in U \) and \( \varepsilon > 0 \), there exists an open neighborhood \( V \) of \( p \) in \( U \) such that for every geodesic \( \gamma \) contained in \( V \), \( f \circ \gamma \) is \( (g(p) + \varepsilon) \)-concave in the barrier sense. By a similar argument to the proof of Lemma 2.3, \( f \) is \( g \)-concave if and only if \( f \) is \( g \)-concave in the barrier sense.

From now on, we fix an Alexandrov space \( X \). We use results and notions on Alexandrov spaces obtained in [BGP], and we refer [BBI]. \( T_p X \) denotes the tangent cone of \( X \) at \( p \), and \( \Sigma_p X \) denotes the space of directions of \( X \) at \( p \).

For any \( \lambda \)-concave function \( f : U \to \mathbb{R} \) on an open subset \( U \) of \( X \) and \( p \in U \), and \( \delta > 0 \), a function \( f_\delta : \delta^{-1}U \to \mathbb{R} \) is defined by the same function \( f_\delta = f \) on the same domain \( \delta^{-1}U = U \) as sets. Since the metric of \( \delta^{-1}U \) is the metric of \( U \) multiplied by \( \delta^{-1} \), \( f_\delta \) is \( \delta^2 \lambda \)-concave on \( \delta^{-1}U \).

In addition, if \( f \) is Lipschitz near \( p \), then the blow-up \( d_p f : T_p X \to \mathbb{R} \), namely the limit with respect to some sequence \( \delta_i \to 0 \),

\[
\lim_{i \to \infty} f_{\delta_i} : \lim_{i \to \infty} (\delta_i^{-1}U, p) \to \mathbb{R}
\]

is 0-concave on \( T_p X \). \( d_p f \) is called the differential of \( f \) at \( p \). Note that the differential of locally Lipschitz semiconcave function always exist and does not depend on the choice of a sequence \( (\delta_i) \). Actually, \( d_p f(\xi) \) is calculated by

\[
d_p f(\xi) = \lim_{t \to 0^+} \frac{f(\exp_p(t\xi)) - f(p)}{t}
\]

if \( \xi \in \Sigma'_p \) is a geodesic direction, where \( \exp_p(t\xi) \) denotes the geodesic starting from \( p \) with the direction \( \xi \).
2.2. Distance functions as semiconcave functions. For any real number \( \kappa \), let us define “trigonometric functions” \( \text{sn}_\kappa \) and \( \text{cs}_\kappa \) by the following ODE.

\[
\begin{align*}
\text{sn}_\kappa''(t) + \kappa \text{sn}_\kappa(t) &= 0, \quad \text{sn}_\kappa(0) = 0, \quad \text{sn}_\kappa'(0) = 1; \\
\text{cs}_\kappa''(t) + \kappa \text{cs}_\kappa(t) &= 0, \quad \text{cs}_\kappa(0) = 1, \quad \text{cs}_\kappa'(0) = 0.
\end{align*}
\]

They are explicitly represented as follows.

\[
\text{sn}_\kappa(t) = \sum_{n=0}^{\infty} \frac{(-\kappa)^n}{(2n+1)!} t^{2n+1} = \begin{cases} 
\frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa} t) & \text{if } \kappa < 0 \\
\frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa} t) & \text{if } \kappa > 0 \\
1 & \text{if } \kappa = 0
\end{cases}
\]

\[
\text{cs}_\kappa(t) = \text{sn}_\kappa'(t) = \sum_{n=0}^{\infty} \frac{(-\kappa)^n}{(2n)!} t^{2n} = \begin{cases} 
\frac{\cos(\sqrt{\kappa} t)}{\sqrt{\kappa}} & \text{if } \kappa > 0 \\
1 & \text{if } \kappa = 0 \\
\cosh(\sqrt{-\kappa} t) & \text{if } \kappa < 0
\end{cases}
\]

These functions are elementary for the space form \( \mathbb{M}_\kappa \) in the sense that they satisfy the following. Let us take any points \( p, q, r \in \mathbb{M}_\kappa \) with \( |pq| + |qr| + |rp| < 2 \text{ diam } \mathbb{M}_\kappa \), and set \( \theta := \angle qpr \). Let \( \gamma \) be the geodesic \( pr \) with \( \gamma(0) = p \) and \( \gamma([p, r]) = r \). We set \( \ell(t) = |q, \gamma(t)| \). When \( \kappa \neq 0 \), the cosine formula states

\[
\text{cs}_\kappa(\ell(t)) = \text{cs}_\kappa[pq|cs_t + \kappa \text{ sn}_\kappa|pq|\text{ sn}_\kappa t \cos \theta.
\]

Then, we have

\[
(\text{cs}_\kappa(\ell(t)))'' + \kappa \text{ cs}_\kappa(\ell(t)) = 0. \tag{2.1}
\]

**Lemma 2.4** (cf. [PP]). The distance function \( d_A \) from a closed subset \( A \) in an Alexandrov space \( X \) of curvature \( \geq \kappa \) is \( (\text{cs}_\kappa(d_A)/\text{sn}_\kappa(d_A)) \)-concave on \( (X - A) \cap \{d_A < \frac{\pi}{2\sqrt{\kappa}}\} \). Here, if \( \kappa \leq 0 \), then we consider \( \frac{\pi}{2\sqrt{\kappa}} \) as \( +\infty \), and if \( \kappa = 0 \), then we consider \( \text{cs}_\kappa(d_A)/\text{sn}_\kappa(d_A) \) as \( 1/d_A \).

**Proof.** We consider the case that \( \kappa \neq 0 \). Let us take any geodesic \( \gamma \) contained in \( (X - A) \cap \{d_A < \frac{\pi}{2\sqrt{\kappa}}\} \). We take \( x \) on \( \gamma \) and reparametrize \( \gamma \) as \( x = \gamma(0) \). We choose \( w \in A \) such that \( |Ax| = |wx| \). We set \( \ell(t) := |A, \gamma(t)| \). Let us take a geodesic \( \tilde{\gamma} \) and a point \( \tilde{w} \) in the \( \kappa \)-plane \( \mathbb{M}_\kappa \) such that \( |\tilde{w}\tilde{\gamma}(0)| = |wx| \) and \( \angle(\uparrow_x^\kappa, \tilde{\gamma}^+(0)) = \angle(\uparrow_x^\kappa, \gamma^+(0)) \). Let us set \( \ell(t) := |\tilde{w}, \tilde{\gamma}(t)| \). By the Alexandrov convexity, \( \ell(t) \leq \ell(\tilde{t}) \). Noticing the sign, we obtain

\[
\frac{-1}{\kappa} \text{cs}_\kappa(\ell) \leq -\frac{1}{\kappa} \text{cs}_\kappa(\tilde{\ell}).
\]

By the calculation (2.1), we have

\[
\left(\frac{-1}{\kappa} \text{cs}_\kappa(\ell)\right)'' \leq \text{cs}_\kappa(\ell)
\]
at \( t = 0 \) in the barrier sense. On the other hand, we can calculate the second derivative as

\[
(cs_\kappa \circ \ell(t))'' = -\kappa [cs_\kappa(\ell) \cdot (\ell')^2 + sn_\kappa(\ell) \cdot \ell'']
\]

in the barrier sense. Noticing that \( cs_\kappa(\ell) \geq 0 \) if \( \ell \leq \frac{\pi}{2\sqrt{\kappa}} \), we obtain

\[
sn_\kappa(\ell) \cdot \ell'' \leq cs_\kappa(\ell)
\]
at \( t = 0 \) in the barrier sense. It completes the proof of the lemma if \( \kappa \neq 0 \). When \( X \) has nonnegative curvature, taking a negative number \( \kappa \) as a lower curvature bound of \( X \) and tending \( \kappa \) to 0, we obtain

\[
\frac{cs_\kappa(d_A)}{sn_\kappa(d_A)} \to \frac{1}{d_A}.
\]
\[\square\]

2.3. Gradient flows. In this subsection, we refer [Pt sem], [Pt QG] and [PP].

For vectors \( v, w \) in the tangent cone \( T_pX \), setting \( o = o_p \) the origin of \( T_pX \), we define

\[
|v| = |o, v| \quad \text{and} \quad \langle v, w \rangle = \begin{cases} |v||w| \cos \angle vow & \text{if } |v|, |w| > 0 \\ 0 & \text{otherwise.} \end{cases}
\]

**Definition 2.5 ([PP], [Pt QG]).** Let \( f \) be a \( \lambda \)-concave function on an open subset \( U \) of \( X \). We say that a vector \( g \in T_pX \) at \( p \in U \) is a gradient of \( f \) at \( p \) if it satisfies

1. \( df_p(v) \leq \langle v, g \rangle \) for all \( v \in T_pX \);
2. \( df_p(g) = \langle g, g \rangle \).

We recall that \( g \) uniquely exists.

We say that \( f \) is regular at \( p \) if \( d_p f(v) > 0 \) for some \( v \in T_pX \), equivalently, \( |\nabla_p f| > 0 \). Otherwise, \( f \) is said to be critical at \( p \).

**Definition 2.6 ([PP], [Pt QG]).** Let \( f : U \to \mathbb{R} \) be a semiconcave function on an open subset \( U \) of an Alexandrov space. A Lipschitz curve \( \gamma : [0, a) \to X \) on an interval \( [0, a) \) is said to be a gradient curve on \( U \) if for any \( t \in [0, a) \) with \( \gamma(t) \in U \),

\[
\lim_{\varepsilon \to 0^+} \frac{f \circ \gamma(t + \varepsilon) - f \circ \gamma(t)}{\varepsilon}
\]

exists and it is equals to \( |\nabla f|^2(\gamma(t)) \).

Note that if \( f \) is critical at \( \gamma(t) \), the gradient curve \( \gamma \) for \( f \) satisfies that \( \gamma(t') = \gamma(t) \) for any \( t' \geq t \).

The (multi-valued) logarithm map \( \log_p : X \to T_pX \) is defined by if \( x \neq p \), then \( \log_p(x) = \{px \cdot \uparrow_p^x \} \), where \( \uparrow_p^x \) is a direction of a geodesic \( px \) and if \( x = p \), then \( \log_p(x) = o_p \). If \( \gamma \) is a gradient curve on \( U \), then for \( t \) with \( \gamma(t) \in U \), the forward direction

\[
\gamma^+(t) := \lim_{\varepsilon \to 0^+} \frac{\log(\gamma(t + \varepsilon))}{\varepsilon} \in T_{\gamma(t)}X
\]

exists and it is equals to the gradient \( \nabla f(\gamma(t)) \).
Proposition 2.7 ([KPT], [Pt sem], [Pt QG], [PP]). Let $\gamma$ and $\eta$ be gradient curve starting from $x = \gamma(0)$ and $y = \eta(0)$ in an open subset $U$ for a $\lambda$-concave function $f : U \to \mathbb{R}$, we obtain
\[ |\gamma(s)\eta(s)| \leq e^{\lambda s}|xy| \]
for every $s \geq 0$.

This proposition implies a gradient curve starting at $x \in U$ is unique on the domain of definition.

Theorem 2.8 ([Pt QG], [Pt sem], [PP]). For any open subset $U$ of an Alexandrov space, a semiconcave function $f$ on $U$ and $x \in U$, there exists a unique maximal gradient curve
\[ \gamma : [0, a) \to U \]
with $\gamma(0) = x$ for $f$, where $\gamma$ is maximal if for every gradient curve $\eta : [0, b) \to U$ for $f$ with $\eta(0) = x$, then $b \leq a$.

Definition 2.9 ([PP], [Pt QG]). Let $U$ be an open subset of an Alexandrov space $X$ and $f : U \to \mathbb{R}$ is semiconcave function. Let $\{[0, a_x)\}_{x \in U}$ be a family of intervals for $a_x > 0$. A map
\[ \Phi : \bigcup_{x \in U} \{x\} \times [0, a_x) \to U \]
is a gradient flow of $f$ on $U$ (with respect to $\{[0, a_x)\}_{x \in U}$) if for every $x \in U$, $\Phi(x, 0) = x$ and the restriction
\[ \Phi(x, \cdot) : [0, a_x) \to U \]
is gradient curve of $f$ on $U$.

A gradient flow $\Phi$ is maximal if each domain $[0, a_x)$ of the gradient curve is maximal.

By Theorem 2.8 and Proposition 2.7, a maximal gradient flow on $U$ always uniquely exists.

Let $\Phi$ be a gradient flow of a semiconcave function on an open subset $U$. By a standard argument, we obtain
\[ \Phi(x, s + t) = \Phi(\Phi(x, s), t) \]
for every $x \in U$ and $s, t \geq 0$, whenever the formula has the meaning.

3. Proof of Theorem 1.2

The purpose of this section is to prove Theorem 1.2. Let us fix a finite dimensional Alexandrov space $X$. As see before in Section 5, this construction makes a sense for an infinite dimensional Alexandrov space with additional assumption.

We first prove the following. Let us consider the distance function $f = d(S(p, R), \cdot)$ from a metric sphere $S(p, R) = \{q \in X \mid |pq| = R\}$. We may assume that a neighborhood of $p$ has curvature $\geq -1$ by
rescaling the metric of $X$ if necessary. By $B(p, R)$, we denote the closed ball centered at $p$ of radius $R$.

**Proposition 3.1.** For any $p \in X$ and $\varepsilon > 0$, there exists $R > 0$ and \( \delta_0 = \delta_0(\varepsilon, R) > 0 \) such that the distance function

$$ f = d(S(p, R), \cdot) $$

from the metric sphere $S(p, R)$ satisfies that for every $x \in B(p, \delta_0 R) - \{p\}$,

$$ d_x f(\uparrow_p x) > \cos \varepsilon. $$

In particular, $f$ is regular on $B(p, \delta_0 R) - \{p\}$.

**Remark 3.2.** In [Pelm], Perelman constructed a strictly concave function on a small neighborhood of each point by taking an average of composition of a strictly concave polynomial and distance functions. Kapovitch also constructed such a function in [K reg] and [K rest] based on Perelman’s construction. We explain their construction. For $\delta > 0$, let $\phi$ be a second degree polynomial such that $\phi'' \equiv -1/\delta$, $\phi'$ is positive and bounded on small interval $[R - 4\delta, R + 4\delta]$. Taking a sufficiently fine net $\{x_\alpha\}_{\alpha=1}^N$ in $S(p, R)$ and much finer net $\{x_{\alpha \beta}\}_{\beta=1}^{N_\alpha}$ near each $x_\alpha$, we define

$$ f_\alpha(x) = \frac{1}{N_\alpha} \sum_{\beta=1}^{N_\alpha} \phi(d(x_{\alpha \beta}, x)). $$

They proved that $f_\alpha$ is $(-c/\delta)$-concave on $B(p, \delta)$ in the sense of Definition 2.3, where $c$ is a positive constant independent on $\delta$. Here, we note that a $\lambda$-concave function is $(-\lambda)$-concave in their term. Then, $f = \min_\alpha f_\alpha$ is also $(-c/\delta)$-concave. In particular, it follows that any point has a convex neighborhood. One can prove that $f$ has a unique maximum at $p$ in $B(p, \delta)$. Immediately, by Proposition 2.7, the gradient flow $\Phi$ of $f$ satisfies $d(\Phi(x, t), p) \leq e^{-\frac{ct}{\delta}} d(x, p)$ for any $x \in B(p, \delta)$. Hence, the flow curve $\Phi(x, t)$ is tending to $p$ as $t \to \infty$. However, we do not know whether $\Phi(x, \cdot)$ reach $p$ at finite time. In particular, we do not know whether $df(\uparrow_p x)$ has a uniform positive lower bound in $B(p, \delta) - \{p\}$ as the conclusion of Proposition 3.1. Proposition 3.1 is a key in our paper, which implies an important Lemma 3.4 later.

**Proof of Proposition 3.1.** Since the tangent cone $T_p X$ is isometric to the metric cone $K(\Sigma_p)$ over the space of directions $\Sigma_p$, there exists a positive constant $R$ satisfying the following.

$$ (3.2) \quad \text{For any } v \in \Sigma_p, \text{ there is } q \in S(p, R) \text{ such that } \angle(v, \uparrow_p q) \leq \varepsilon. $$
From now on, we set \( S := S(p, R) \). For any \( x \in S(p, \delta R) \), fixing a direction \( \uparrow^x_p \in x' \), let us take \( q_1, q_2 \in S \) such that

\[
|x, q_1| = |x, S| := \min_{q \in S} |x, q|;
\]

\[
\angle xpq_2 = \angle (\uparrow^x_p, \uparrow^q_2) = \angle (\uparrow^x_p, S') := \min_{v \in S'} \angle (\uparrow^x_p, v).
\]

By the condition (3.2), we have

\[
\tilde{\angle} xpq_2 \leq \angle xpq_2 \leq \varepsilon.
\]

Then, by the law of sine, we obtain

\[
\sin \tilde{\angle} pxq_2 = \frac{\sinh R}{\sinh |xq_2|} \sin \tilde{\angle} xpq_2 \leq \frac{\sinh R}{\sinh R(1 - \delta)} \sin \varepsilon.
\]

On the other hand, by the law of cosine, we obtain

\[
\cosh |xq_2| = \cosh \delta R \cosh R - \sinh \delta R \sinh R \cos \tilde{\angle} xpq_2 \\
\leq \cosh \delta R \cosh R - \sinh \delta R \sinh R \cos \varepsilon
\]

and

\[
- \sinh \delta R \sinh |xq_2| \cos \tilde{\angle} pxq_2 = \cosh R - \cosh \delta R \cosh |xq_2| \\
\geq \cosh R \{1 - \cosh^2 \delta R\} + \sinh R \sinh \delta R \cos \varepsilon.
\]

Therefore, if \( \delta \) is smaller than some constant, then

\[
- \cos \tilde{\angle} pxq_2 > 0.
\]

By (3.5) and (3.6), we obtain

\[
\tilde{\angle} pxq_2 \geq \pi - (1 + \tau(\delta))\varepsilon.
\]

Next, let us consider the point \( q_1 \) taken as (3.3). Then, it satisfies

\[
\tilde{\angle} xpq_1 = \min_{q \in S} \tilde{\angle} xpq \leq \min_{q \in S} \angle xpq \leq \varepsilon.
\]

By a similar argument as \( q_1 \) instead of \( q_2 \), we obtain

\[
\tilde{\angle} pxq_1 \geq \pi - (1 + \tau(\delta))\varepsilon.
\]

By the quadruple condition with (3.7) and (3.8), we obtain

\[
\tilde{\angle} q_1xq_2 \leq 2\pi - \tilde{\angle} pxq_1 - \tilde{\angle} pxq_2 \leq (2 + \tau(\delta))\varepsilon.
\]

If \( \delta \) is small with respect to \( \varepsilon \), then we obtain

\[
|q_1q_2| \leq 3R\varepsilon.
\]

Therefore, we obtain

\[
\tilde{\angle} q_1pq_2 \leq 4\varepsilon.
\]

For any \( y \in px - \{p, x\} \), we set \( q_3 = q_3(y) \in S \) such that

\[
|y, q_3| = |y, S|.
\]

By an argument as above, we obtain

\[
\tilde{\angle} pyq_3 \geq \pi - (1 + \tau(|py|/R))\varepsilon > \pi - 2\varepsilon.
\]
Then, we have
\[ \tilde{\angle} yq_3 < 2\varepsilon. \]
By the Gauss-Bonnet’s theorem, if \( y \) is near \( x \), then
\[ \tilde{\angle} yq_3 > \pi - 3\varepsilon. \]

By the first variation formula, we obtain
\[
\frac{df_x(\uparrow_p x)}{dt} = \lim_{x \to y, y \to x} \frac{|Sy| - |Sx|}{|xy|} \geq \liminf_{x \to y, y \to x} \frac{|q_3y| - |q_3x|}{|xy|} \geq \cos 3\varepsilon.
\]
This completes the proof. \( \square \)

We fix \( \delta_0 \) as in the conclusion of Proposition \( \text{3.1} \) and fix \( \delta \leq \delta_0 \).

**Lemma 3.3.** For any \( x \in B(p, \delta R) - \{p\} \), we have
\[ \angle(\nabla_x f, \uparrow_p x) < \varepsilon \]
and \( |\nabla_x f, \uparrow_p x| < \sqrt{2}\varepsilon \).

**Proof.** By Proposition \( \text{3.1} \), we have
\[ df_x(\uparrow_p x) > \cos \varepsilon. \]
By the definition of the gradient, we obtain
\[ df_x(\uparrow_p x) \leq |\nabla_x f| \cos \angle(\nabla_x f, \uparrow_p x) \leq \cos \angle(\nabla_x f, \uparrow_p x). \]
Therefore, we have \( \angle(\nabla_x f, \uparrow_p x) < \varepsilon \).

Since \( f \) is 1-Lipschitz, \( |\nabla f| \leq 1 \). And, by the above inequality,
\[ |\nabla f| = \max_{\xi \in \Sigma_x} df_x(\xi) \geq df_x(\uparrow_p x) > \cos \varepsilon. \]
Then, we obtain
\[ |\nabla_x f, \uparrow_p x|^2 < |\nabla f|^2 + 1 - 2|\nabla f| \cos \varepsilon \leq 2\sin^2 \varepsilon. \]
Therefore, \( |\nabla f, \uparrow_p x| < \sqrt{2}\varepsilon \).

Let us consider the gradient flow \( \Phi_t \) of \( f = d(S, \cdot) \).

**Lemma 3.4.** For every \( x \in B(p, \delta R) \),
\[ |\Phi_t(x), p| \leq |x, p| - \cos \varepsilon \cdot t, \]
whenever this formula has meaning. In particular, for any \( t \geq \delta R/\cos \varepsilon \), we have \( \Phi_t(x) = p \).

**Proof.** Let us set \( \gamma(t) = \Phi_t(x) \) the gradient curve for \( f \) starting from \( \gamma(0) = x \). If \( \gamma(t_0) \neq p \), then
\[ \frac{d}{dt} \Big|_{t=t_0+} |\Phi_t(x), p| = -\langle \nabla_{\gamma(t_0)} f, \uparrow_p \gamma(t_0) \rangle < -\cos \varepsilon. \]
Integrating this, we have
\[ |\Phi_{t_0}(x), p| - |x, p| \leq -\cos \varepsilon \cdot t_0. \]
This completes the proof. \( \square \)
Finally, we estimate the Lipschitz constant of the flow $\Phi$ on $B(p, \delta R)$. Let us recall that $f$ is $\lambda$-concave on $B(p, \delta R)$ for some $\lambda$. By Lemma 2.4, $\lambda$ can be given as follows.

$$\frac{\cosh(f)}{\sinh(f)} \leq \frac{\cosh R}{\sinh(R(1 - \delta))} = \lambda.$$ 

By Proposition 2.7, for any $x, y \in B(p, \delta R)$,

$$|\Phi(x, t), \Phi(y, t)| \leq e^{\lambda t} |xy|.$$ 

Since $f$ is 1-Lipschitz, for $x \in B(p, \delta R)$ and $t' < t$, we have

$$|\Phi(x, t), \Phi(x, t')| \leq \int_{t'}^{t} \left| \frac{d}{ds} \Phi(x, s) \right| ds \leq \int_{t'}^{t} |\nabla f(\Phi(x, s))| ds \leq t - t'.$$

Therefore, we obtain the following.

**Lemma 3.5.** For any $x, y \in B(p, \delta R)$ and $t \geq s \geq 0$,

$$|\Phi(x, s), \Phi(y, t)| \leq e^{\lambda s} |x, y| + t - s.$$

Note that, by Lemma 3.4, setting $\ell = \delta_0 R / \cos \varepsilon$, $e^{\lambda \ell}$ can be bounded from above by a constant arbitrary close to 1 if we choose $\delta_0$ and $R$ so small.

By Lemma 3.5, we obtain a Lipschitz homotopy $\varphi : B(p, \delta_0 R) \times [0, 1] \to B(p, \delta_0 R)$ with $\varphi(\cdot, 1) = p$ defined by $\varphi(x, t) = \Phi(x, \ell t)$ for $(x, t) \in B(p, \delta_0 R) \times [0, 1]$.

4. **Proof of Applications**

4.1. **Proof of Corollaries 1.3 and 1.4.** Let $V$ be a metric space and $U$ be a subset of $V$ and $p \in V$. We say that $U$ is Lipschitz contractible to $p$ in $V$ if there exists a Lipschitz map

$$h : U \times [0, 1] \to V$$

such that

$$h(x, 0) = x \text{ and } h(x, 1) = p$$

for any $x \in U$. We call such an $h$ a Lipschitz contraction from $U$ to $p$ in $V$. We say that $U$ is Lipschitz contractible in $V$ if $U$ is Lipschitz contractible to some point in $V$.

**Lemma 4.1.** Let $U$ be Lipschitz contractible in a metric space $V$. For any Lipschitz map $\varphi : S^{n-1} \to U$, there exists a Lipschitz map $\tilde{\varphi} : D^n \to V$ such that $\tilde{\varphi}|_{S^{n-1}} = \varphi$.

**Proof.** By the definition, there exist $p \in V$ and a Lipschitz map

$$h : U \times [0, 1] \to V$$

such that

$$h(x, 0) = x \text{ and } h(x, 1) = p$$
for any \( x \in U \). We define a map
\[
\varphi_1 : S^{n-1} \times [0,1] \to V
\]
by \( \varphi_1 = h \circ (\varphi \times \text{id}) \). Then, \( \varphi_1 \) is Lipschitz with Lipschitz constant \( \leq \text{Lip}(h) \cdot \max\{1, \text{Lip}(\varphi)\} \). We define a map
\[
\varphi_2 : D^n \times \{1\} \to V
\]
by \( \varphi_2(v,1) = p \) for all \( v \in D^n \). And we consider a space
\[
Y = S^{n-1} \times [0,1] \cup D^n \times \{1\}
\]
equipped with length metric with respect to a gluing \( S^{n-1} \times \{1\} \ni (v,1) \mapsto (v,1) \in \partial D^n \times \{1\} \). Now we define a map
\[
\varphi_3 : Y \to V
\]
by
\[
\varphi_3 = \begin{cases} 
\varphi_1 \text{ on } S^{n-1} \times [0,1] \\
\varphi_2 \text{ on } D^n \times \{1\}
\end{cases}
\]
This is well-defined. Then, \( \varphi_3 \) is \( \text{Lip}(\varphi_1) \)-Lipschitz. Indeed, for \( x \in S^{n-1} \times [0,1] \) and \( y \in D^n \times \{1\} \), we have
\[
|\varphi_3(x),\varphi_3(y)| = |\varphi_3(x),p|.
\]
Let \( \bar{x} \in S^{n-1} \times \{1\} \) be the foot of perpendicular segment from \( x \) to \( S^{n-1} \times \{1\} \). We note that \( |x,\bar{x}| \leq |x,y| \) and \( \varphi_3(\bar{x}) = p \). Then, we obtain
\[
|\varphi_3(x),p| = |\varphi_3(x),\varphi_3(\bar{x})| = |\varphi_1(x),\varphi_1(\bar{x})| \\
\leq \text{Lip}(\varphi_1)|x,\bar{x}| \leq \text{Lip}(\varphi_1)|x,y|.
\]
Obviously, there exists a bi-Lipschitz homeomorphism
\[
f : D^n \to Y
\]
with \( f(0) = (0,1) \in D^n \times \{1\} \) preserving the boundaries in the sense that it satisfies \( f(v) = (v,0) \in S^{n-1} \times \{0\} \) for any \( v \in S^{n-1} \). Then, we obtain a Lipschitz map \( \tilde{\varphi} := \varphi_3 \circ f \) satisfying the desired condition. \( \square \)

**Definition 4.2.** We say that a metric space \( Y \) is a **Lipschitz simplicial complex** if there exists a triangulation \( T \) of \( Y \) satisfying the following.

For each simplex \( S \in T \), there exists a bi-Lipschitz homeomorphism \( \varphi_S : \Delta^{\dim S} \to S \). Here, the simplex \( \Delta^{\dim S} \) is a standard simplex equipped with the Euclidean metric and \( S \) has the restricted metric of \( Y \). And, we say that such a triangulation \( T \) is a **Lipschitz triangulation** of \( Y \). The dimension of \( Y \) is given by \( \dim Y = \sup_{S \in T} \dim S \). We only deal with \( Y \) of \( \dim Y < \infty \).

A Lipschitz simplicial complex \( Y \) is called **finite** if it has a Lipschitz triangulation consisting of finitely many elements.

Note that a subdivision, for instance, the barycentric one, of a Lipschitz triangulation is also a Lipschitz triangulation.
Proposition 4.3. Let $X$ be an SLLC space, $Y$ be a Lipschitz simplicial complex and $f : Y \to X$ be a continuous map. Then, there exists a homotopy

$$h : Y \times [0, 1] \to X$$

from $h_0 = f$ such that $h_1$ is Lipschitz on each simplex of $Y$.

Further, if $f$ is Lipschitz on a subcomplex $A$ of $Y$, then a homotopy $h$ can be chosen so that it is relative to $A$. Namely, it satisfies $h(a, t) = a$ for any $a \in A$ and $t \in [0, 1]$.

Proof. If $\dim Y = 0$, then we set $h(x, t) = f(x)$ for $x \in Y$ and $t \in [0, 1]$. Then, $h$ is the desired homotopy.

We assume that the assertion holds for $\dim Y \leq k - 1$. First, we prove that for any $f : \Delta^k \to X$, there exists a homotopy $h : \Delta^k \times [0, 1] \to X$ from $h_0 = f$ to a Lipschitz map $h_1$. Taking a subdivision if necessary, let us take a finite Lipschitz triangulation $T$ of $\Delta^k$ satisfying the following. For any $k$-simplex $E \in T$, there exists an open subset $U_E$ of $X$ which is a Lipschitz contractible ball such that $f(E) \subset U_E$. For any simplex $F \in T$ of $\dim F \leq k - 1$, we set

$$U_F = \bigcap_{F \subset E \in T} U_E.$$ 

This is an open subset of $X$. Let us denote by $Z$ a $(k - 1)$-skeleton of $\Delta^k$ with respect to $T$. By an inductive assumption, there exists a homotopy

$$h : Z \times [0, 1] \to X$$

from $h_0 = f|_Z$ such that for every simplex $F$ of $Z$,

- $h_1|_F$ is Lipschitz;
- $h(F \times [0, 1]) \subset U_F$;
- if $f|_F$ is Lipschitz, then $h_t|_F = f|_F$ for any $t$.

Let $E$ be a $k$-simplex of $\Delta^k$ with respect to $T$. We denote by $h^{\partial E}$ the restriction of $h$ to $\partial E \times [0, 1]$. Then, the image of $h^{\partial E}$ is contained in $\bigcup_{T \supseteq E} U_F \subset U_E$. Since a pair $(E, \partial E)$ has the homotopy extension property, there exists a homotopy

$$h^E : E \times [0, 1] \to U_E$$

from $f|_E$ which is an extension of $h^{\partial E}$. Then, $h^E_1$ is Lipschitz on $\partial E$. For another $k$-simplex $E'$ of $\Delta^k$ with common face $E \cap E'$,

$$h^E_t = h^{E'}_t$$

on $E \cap E'$ for all $t$. Since $U_E$ is Lipschitz contractible ball, by Lemma 4.1, there is a homotopy

$$\bar{h}^E : E \times [0, 1] \to X$$
relative to \( \partial E \) from \( \tilde{h}_0^E = h_1^E \) to a Lipschitz map \( \tilde{h}_1^E : E \to X \). Let us define a homotopy \( \hat{h}^E : E \to X \) by

\[
\hat{h}^E(x, t) = \begin{cases} 
    h^E(x, t) & \text{if } t \in [0, 1/2]; \\
    \tilde{h}(x, t) & \text{if } t \in [1/2, 1]. 
\end{cases}
\]

And we define \( \hat{h} : \triangle^k \times [0, 1] \to X \) by

\[
\hat{h}(x, t) = \hat{h}^E(x, t)
\]

for \( x \in E \in T \). Then, \( \hat{h}_0 = f \) and \( \hat{h}_1 \) is Lipschitz.

Next, we consider a continuous map \( f : Y \to X \) from a Lipschitz simplicial complex \( Y \) of \( \dim Y = k \). Let \( Z \) be a \((k - 1)\)-simplex of \( Y \). By an inductive assumption, there exists a homotopy

\[
h : Z \times [0, 1] \to X
\]

from \( h_0 = f|_Z \) and \( h_1 \) is Lipschitz on every simplex of \( Z \). From now on, let us denote by \( E \) a \( k \)-skeleton of \( Y \). For any \( E \subset Y \), by using the homotopy extension property for \( (E, \partial E) \) and Lemma 4.1, we obtain a homotopy

\[
h^E : E \times [0, 1] \to X
\]

which is an extension of \( h|_{\partial E \times [0, 1]} \) with \( h_0^E = f|_E \). Since \( h_1^E \vert_{\partial E} = h_1 \vert_{\partial E} \) is Lipschitz, there exists a homotopy

\[
\tilde{h}^E : E \times [0, 1] \to X
\]

relative to \( \partial E \) from \( \tilde{h}_0^E = h_1^E \) to a Lipschitz map \( \tilde{h}_1^E \). We set \( \tilde{h}(x, t) = h(x, 1) \) for \( x \in Z \) and \( t \in [0, 1] \). And, we define a homotopy \( \hat{h} : Y \times [0, 1] \to X \) by

\[
\hat{h}(x, t) = \begin{cases} 
    h(x, 2t) & \text{if } x \in Z \text{ and } t \in [0, 1/2] \\
    \tilde{h}(x, 2t - 1) & \text{if } x \in Z \text{ and } t \in [1/2, 1] \\
    h^E(x, 2t) & \text{if } x \in E \subset Y \text{ and } t \in [0, 1/2] \\
    \tilde{h}(x, 2t - 1) & \text{if } x \in E \subset Y \text{ and } t \in [1/2, 1] 
\end{cases}
\]

Then, \( \hat{h}_0 = f \) and \( \hat{h}_1 \) is Lipschitz on every simplex.

\[\square\]

**Corollary 4.4.** Let \( Y \) be a Lipschitz simplicial complex, \( X \) be an SLLC space and \( f : Y \to X \) be a continuous map. Let \( T \) be a Lipschitz triangulation of \( Y \) and \( \{U_F \mid F \in T\} \) be a family of open subsets of \( X \) with the following property.

- \( f(F) \subset U_F \) for \( F \in T \);
- \( U_F \subset U_E \) for \( F, E \in T \) with \( F \subset E \).

Then, there exists a homotopy \( h : Y \times [0, 1] \to X \) from \( h_0 = f \) such that for every \( F \in T \),

- \( h_1 \) is Lipschitz on \( F \);
- \( h(F \times [0, 1]) \subset U_F \);
- if \( f \) is Lipschitz on \( F \), then \( h_t = f \) on \( F \) for all \( t \).
For instance, fixing $\varepsilon > 0$ and setting $U_F$ an $\varepsilon$-neighborhood of $f(F)$ for every $F \in T$, the family $\{U_F \mid F \in T\}$ satisfies the assumption of Corollary 4.4.

Proof of Corollary 4.4. If $\dim Y = 0$, the assertion is trivial. We assume that Corollary 4.4 holds when $\dim Y \leq k - 1$ for $k \geq 1$. Let $Y$ be a Lipschitz simplicial complex with $\dim Y = k$ and $T$ be a Lipschitz triangulation of $Y$. Let us take a family $\{U_F \mid F \in T\}$ of open subsets as the assumption of Corollary 4.4. By inductive assumption, there exists a homotopy $h : Y^{(k-1)} \times [0,1] \to X$ from $h_0 = f|_{Y^{(k-1)}}$ and $h_1$ is Lipschitz on each $F \in T$ of dim $\leq k - 1$, and $h_t(F) \subset U_F$ for all $t$. Let us denote by $E$ a $k$-simplex in $T$. By Proposition 4.3, there exists a homotopy $h_E : E \times [0,1] \to U_E$ from $h_E^0 = f|_E$ to a Lipschitz map $h_E^1$ such that $h_E^t = h_t$ on $\partial E$ for all $t$. Then, a concatenation map

$$\hat{h}(x,t) = \begin{cases} h(x,t) & \text{if } x \in Y^{(k-1)}, \\ h^E(x,t) & \text{if } x \in E. \end{cases}$$

is a desired homotopy.

Remark 4.5. We note that Proposition 4.3 and Corollary 4.4 above can be also proved assuming $X$ is just LLC instead of SLLC. Here, we say that a metric space $X$ is locally Lipschitz contractible, for short LLC, if for any $p \in X$ and $\varepsilon > 0$, there exist $r \in (0,\varepsilon]$ and a Lipschitz contraction $\varphi$ from $U(p,r)$ to $p$ in $U(p,\varepsilon)$. We also remark that Corollaries 1.3 and 1.4 are true if $X$ is just LLC.

Let us start to prove Corollaries 1.3 and 1.4.

Proof of Corollaries 1.3 and 1.4. Let us take a finite Lipschitz simplicial complex pair $(P,Q)$, possibly $Q$ is empty. We prove Corollaries 1.3 and 1.4 assuming $X$ to be SLLC. Let $A$ be an open subset in $X$. Let us consider a continuous map $f : (P,Q) \to (X,A)$. By Corollary 4.4 and Theorem 1.2 we obtain a homotopy

$$\varphi : (P,Q) \times [0,1] \to (X,A)$$

from $\varphi_0 = f$ to a Lipschitz map $\varphi_1 : (P,Q) \to (X,A)$. Here, we note that since $A$ is open in $X$, the homotopy $\varphi_t$ can be chosen so that $\varphi_t(Q) \subset A$. Then, we obtain a corresponding

$$C((P,Q),(X,A)) \ni f \mapsto \varphi_1 \in \operatorname{Lip}((P,Q),(X,A)),$$

where $C(*,**)$ (resp. $\operatorname{Lip}(*,**)$) denotes the set of all continuous (resp. Lipschitz) maps from $*$ to $**$.

Let us consider two continuous maps $f$ and $g$ from $(P,Q)$ to $(X,A)$ such that they are homotopic to each other. From the correspondence
(4.1), we obtain Lipschitz maps $f'$ and $g'$ from $(P, Q)$ to $(X, A)$ which are homotopic to $f$ and $g$, respectively. Connecting these homotopies, we obtain a homotopy

$$H : (P, Q) \times [0, 1] \to (X, A)$$

between $H(\cdot, 0) = f'$ and $H(\cdot, 1) = g'$. Now, we consider a Lipschitz simplicial complex $\tilde{P} = P \times [0, 1]$ and a subcomplex $\tilde{R} = P \times \{0, 1\}$. Then, the map $H$ is Lipschitz on $\tilde{R}$. Hence, by Proposition 4.3, we obtain a homotopy $\tilde{H} : \tilde{P} \times [0, 1] \to X$ relative to $\tilde{R}$ from $\tilde{H}(\cdot, 0) = H$ to a Lipschitz map $\tilde{H}(\cdot, 1)$. Then, $\tilde{H}(\cdot, 1)$ is a Lipschitz homotopy between $f'$ and $g'$. Therefore, we conclude that the corresponding (4.1) sends a homotopy to a Lipschitz homotopy. It completes the proof of Corollary 1.3.

Let us consider a pointed $n$-sphere $(S^n, p_0)$ and an Alexandrov space $X$ with point $x_0 \in X$. Then, for any map $f : (S^n, p_0) \to (X, x_0)$, the restriction $f|_{\{p_0\}}$ is always Lipschitz. Hence, by an argument as above and Proposition 4.3, we obtain the conclusion of Corollary 1.4.

□

4.2. Plateau problem. We first recall the definition of the Sobolev space of metric space target, to state the setting of Plateau problem in an Alexandrov space as in the introduction, referring [KS] and [MZ]. For a complete metric space $X$ and a domain $\Omega$ in a Riemannian manifold having compact closure, a function $u : \Omega \to X$ is said to be $L^2$-map if $u$ is Borel measurable and for some (and any) point $p_0 \in X$, the integral

$$\int_{\Omega} |u(x), p_0|^2 d\mu$$

is finite, where $\mu$ is the Riemannian volume measure. The set of all $L^2$-maps from $\Omega$ to $X$ denotes $L^2(\Omega, X)$. We recall the definition of the energy of $u \in L^2(\Omega, X)$. For any $\varepsilon > 0$, we set $\Omega_\varepsilon = \{x \in \Omega \mid d(\partial \Omega, x) > \varepsilon\}$ and define an approximate energy density $e^u_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ by

$$e^u_\varepsilon(x) = \frac{1}{\omega_n} \int_{S(x, \varepsilon)} \frac{d(u(x), u(y))^2}{\varepsilon^2} ds \varepsilon^{n-1}.\$$

Here, $n = \dim \Omega$, $S(x, \varepsilon)$ is the metric sphere around $x$ with radius $\varepsilon$ and $\sigma$ is the surface measure on it. By [KS] 1.2iii, we obtain

$$\int_{\Omega_\varepsilon} e^u_\varepsilon(x) d\mu \leq C\varepsilon^{-2}.$$

Let us take a Borel measure $\nu$ on the interval $(0, 2)$ satisfying

$$\nu \geq 0, \nu((0, 2)) = 1, \int_0^2 \lambda^{-2} d\nu(\lambda) < \infty.$$
An averaged approximate energy density $\nu e^u_\varepsilon(x)$ is defined by

$$
\nu e^u_\varepsilon(x) = \begin{cases} 
\int_0^2 e^u_\lambda(x) d\nu(\lambda) & \text{if } x \in \Omega_2 \\
0 & \text{otherwise}
\end{cases}
$$

Let $\mathcal{C}_c(\Omega)$ be the set of all continuous function on $\Omega$ with compact support. We define a functional $E^u_\varepsilon : \mathcal{C}_c(\Omega) \to \mathbb{R}$ by

$$
E^u_\varepsilon(f) := \int_{\Omega} f(x) \nu e^u_\varepsilon d\mu(x).
$$

Then, the energy of $u$ is defined by

$$
E^u = \sup_{f \in \mathcal{C}_c(\Omega), 0 \leq f \leq 1} \limsup_{\varepsilon \to 0} E^u_\varepsilon(f).
$$

The $(1,2)$-Sobolev space is defined as

$$
W^{1,2}(\Omega, X) = \{ u \in L^2(\Omega, X) \mid E^u < \infty \}.
$$

We start to prove Corollary 1.6.

**Proof of Corollary 1.6** Let $\Gamma$ be a rectifiable closed Jordan curve in an Alexandrov space $X$, which is topologically contractible. Since the rectifiability of $\Gamma$, we can take a Lipschitz monotonic parametrization $\gamma : S^1 \to \Gamma$.

By the contractibility of $\Gamma$, there exists a continuous map $h : \Gamma \times [0,1] \to X$

such that $h(\cdot,0) = id_{\Gamma}$ and $h(\cdot,1) = p$ for some $p \in X$. We define a map $f : S^1 \times [0,1] \to X$ by $f(x,t) = h(\gamma(x), t)$. Further, we set $f(y,1) = p$ for $y \in D^2$. By taking reparametrization of $f : S^1 \times [0,1] \cup D^2 \times \{1\} \to X$, we obtain a continuous map $g : D^2 \to X$

such that $g|_{\partial D^2} = \gamma$.

By Proposition 1.3, there exists a homotopy $\tilde{h} : D^2 \times [0,1] \to X$

relative to $\partial D^2$ such that $\tilde{h}(\cdot,0) = g$ and $\tilde{h}(\cdot,1)$ is Lipschitz. Thus, we obtain the Lipschitz map $\tilde{g} = \tilde{h}(\cdot,1)$ such that $\tilde{g}|_{\partial D^2} = \gamma$. By the definition of the energy, we obtain

$$
E(\tilde{g}) \leq \text{Lip}(\tilde{g})^2 < \infty.
$$

Here, $\text{Lip}(\tilde{g})$ is the Lipschitz constant of $\tilde{g}$. Therefore, we conclude $\tilde{g} \in \mathcal{F}_\Gamma$. \[\square\]
5. A note on the infinite dimensional case

It is known that the (Hausdorff) dimension of an Alexandrov space is nonnegative integer or infinite. There are only few works of infinite dimensional Alexandrov spaces. It is not known whether an infinite dimensional Alexandrov space is locally contractible.

When we consider an Alexandrov space of possibly infinite dimension, we somewhat generalize Definition 2.1 as follows. A complete metric space $X$ is called an Alexandrov space if it is a length metric space and satisfies the quadruple condition locally. Here, a complete metric space $X$ is length if every two points $p, q \in X$ and any $\varepsilon > 0$, there exists a point $r \in X$ satisfying $\max\{|pr|, |rq|\} < |pq|/2 + \varepsilon$. Since a length metric space has no geodesic in general, to define a notion of a lower curvature bound, we change the triangle comparison condition by the quadruple condition. Here, an open subset $U$ of a length space $X$ satisfies the quadruple condition modeled on the $\kappa$-plane $M_\kappa$ if for every distinct four points $p_0, p_1, p_2$ and $p_3$ in $U$, we obtain

$$\tilde{\angle}p_1p_0p_2 + \tilde{\angle}p_2p_0p_3 + \tilde{\angle}p_3p_0p_1 \leq 2\pi,$$

where $\tilde{\angle} = \tilde{\angle}_\kappa$ denotes the comparison angle modeled on $M_\kappa$.

By the standard argument, any geodesic triangle (if it exists) in an Alexandrov space of possibly infinite dimension satisfies the triangle comparison condition. It is known that finite dimensional Alexandrov spaces are proper metric space, in particular, by Hopf-Rinow theorem, they are geodesic spaces.

Plaut [Pl] proved that an Alexandrov space of infinite dimension is an “almost” geodesic space. Precisely,

**Theorem 5.1 ([Pl]).** Let $X$ be an Alexandrov space of infinite dimension. For any $p \in X$, a subset $J_p \subset X$ defined by

$$J_p = \bigcap_{\delta > 0}\{q \in X - \{p\} \mid \text{there exists } x \in X - \{p, q\} \text{ with } \tilde{\angle}pqx > \pi - \delta\}$$

is dense $G_\delta$ subset in $X$, and for every $q \in J_p$, there exists a unique geodesic connecting $p$ and $q$.

We now show that the compactness of the space of directions at some point implies the Lipschitz contractibility around the point.

**Proposition 5.2.** Let $X$ be an Alexandrov space of infinite dimension. Suppose that there exists a point $p \in X$ such that the space of directions $\Sigma_p$ at $p$ is compact. Then, the following are true.

(i) The pointed Gromov-Hausdorff limit of scaling space $(rX, p)$ as $r \to \infty$ exists and it is isometric to the cone over $\Sigma_p$.

(ii) $\Sigma_p$ is a geodesic space.

(iii) $X$ is proper.

(iv) There exists $R_0 > 0$ depending on $p$ such that for every $R \leq R_0$, $U(p, R)$ is Lipschitz contractible to $p$ in itself.
Proof. (i). Let $K = K(\Sigma_p)$ be the Euclidean cone over $\Sigma_p$ and $B$ be the unit ball around the origin $o$. Let $J_p$ be the set defined in Theorem 5.1. For any $\varepsilon > 0$, we take a finite $\varepsilon$-net $\{v_\alpha\}_\alpha \subset B$. We may assume that every $v_\alpha$ is contained in $K(\Sigma'_p) - \{o\}$. Namely, there exists $r > 0$ such that for every $\alpha$, there is a geodesic $\gamma_\alpha$ starting from $p$ having the direction $v_\alpha |v_\alpha|$ with length at least $r$. Let $x_\alpha \in B(o,r)$ be taken as $x_\alpha = \gamma_\alpha(r|v_\alpha|)$. Then, $\{x_\alpha\}_\alpha$ is an $\varepsilon$-net in $1/r B(p,r)$. Indeed, for any $x \in B(p,r) \cap J_p$, setting $v = \log_p(x) \in K(\Sigma_p)$, and then $1/r v \in B$. Then, there exists $\alpha$ such that $|v_\alpha, 1/r v| \leq \varepsilon$. Therefore, $|rv_\alpha, v| \leq r\varepsilon$. We may assume that a lower curvature bound of $X$ is less than or equals to 0. Then $\exp_p : B(o,r) \cap \text{dom}(\exp_p) \to B(p,r)$ is 1-Lipschitz, where $\text{dom}(\exp_p)$ is the domain of $\exp_p$. Therefore, $|x_\alpha, x|_X \leq r\varepsilon$.

Let us retake $r$ to be small so that $|x_\alpha, x_\beta| \leq |v_\alpha, v_\beta| \leq \varepsilon$. Then, the map $v_\alpha \mapsto x_\alpha$ implies a $C\varepsilon$-approximation between $B$ and $1/r B(p,r)$ for any small $r$. Here, $C$ is a constant not depending on any other term. Therefore, the pointed space $(1/r X, p)$ is Gromov-Hausdorff converging to $(K(\Sigma_p), o)$ as $r \to 0$.

(ii) obviously holds by (i) and (iii). We prove (iii). Let us consider any closed ball $B(p,r)$ centered at $p$. Let us take any sequence $\{x_i\} \subset B(p,r)$. We take $y_i \in B(p,r) \cap J_p$ such that $|x_i, y_i| \leq 1/i$. Then, $v_i = \log_p(y_i) \in B(o,r) \subset T_p X$ is well-defined. By (i), $T_p X$ is proper. Hence, there exists a converging subsequence $\{v_{n(i)}\}_i$ of $\{v_i\}_i$. Since $\exp_p$ is Lipschitz, $\{x_{n(i)}\}_i$ is converging.

We recall that the proof of Theorem 1.2 started from the assertion (3.2) in Proposition 3.1. The assertion (i) guarantees (3.2). Therefore, one can prove (iv) in the same way as the proof of Theorem 1.2.

6. AN ESTIMATION OF SIMPLICIAL VOLUME OF ALEXANDROV SPACES

In this section, we consider an Alexandrov space having a lower Ricci curvature bounds, and we prove an estimation of the simplicial volume of such a space as stated in Theorem 1.8. The original form of Theorem 1.8 was proved by Gromov [G] when $X$ is a Riemannian manifold with a lower Ricci curvature bound.

The original Gromov’s proof was depending on the well-known Bishop-Gromov volume inequality. For an Alexandrov space of curvature $\geq \kappa$ by some $\kappa \in \mathbb{R}$, its Hausdorff measure is known to satisfy the Bishop-Gromov type volume growth estimate. The second author’s proof of Corollary 1.7 was depending on this volume growth estimate (Y). It is known that all several natural generalized notions of a lower Ricci
curvature bound induce a volume growth estimate. Among them, the condition named local reduced curvature-dimension condition introduced by Bacher and Sturm ([BS]) can be used to prove Theorem 1.8. For completeness, we explain it as follows.

6.1. Several conditions of lower Ricci curvature bound. We recall several generalized notions of a lower bound of Ricci curvature defined on a pair of a metric space and a Borel measure on it. For their theory, history and undefined terms appearing in the following, we refer [S], [S2], [BS], [CS], [Oh] and their references.

In this section, we denote by $M$ a complete separable metric space. By $P_2(M)$ we denote the set of all Borel probability measures $\mu$ on $M$ with finite second moment. A metric called the $L^2$-Wasserstein distance $W_2$ is defined on $P_2(M)$. Let us fix a locally finite Borel measure $m$ on $M$. Such a pair $(M,m)$ is called a metric measure space. Let us denote by $P_\infty(M,m)$ the subset of $P_2(M)$ consisting of all measures which are absolutely continuous in $m$ and have bounded support.

From now on, $K$ and $N$ denote real numbers with $N \geq 1$. For $\nu \in P_\infty(M,m)$ with density $\rho = d\nu/dm$, its Rényi entropy with respect to $m$ is given by

$$S_N(\nu|m) := -\int_M \rho^{1-1/N} \, dm = -\int_M \rho^{-1/N} \, d\nu.$$  

For $t \in [0,1]$, a function $\sigma_{K,N}^{(t)} : (0,\infty) \to [0,\infty)$ is defined as

$$\sigma_{K,N}^{(t)}(\theta) = \begin{cases} +\infty & \text{if } K\theta^2 \geq N\pi^2 \\ \frac{\sin(K\theta/N)}{\sin(K/N)} & \text{if else.} \end{cases}$$

And, we set $\tau_{K,N}^{(t)}(\theta) = t^{1/N} \sigma_{K,N-1}^{(t)}(\theta)^{(N-1)/N}$.

**Definition 6.1 ([BS], [CS], [S2]).** Let $K$ and $N$ be real numbers with $N \geq 1$. Let $(M,m)$ be a metric measure space.

We say that $(M,m)$ satisfies the reduced curvature-dimension condition $CD^*(K,N)$ locally – denoted by $CD_{\text{loc}}^*(K,N)$ – if for any $p \in M$ there exists a neighborhood $M(p)$ such that for all $\nu_0, \nu_1 \in P_\infty(M,m)$ supported $M(p)$, denoting those densities by $\rho_0, \rho_1$ with respect to $m$, there exist an optimal coupling $q$ of $\nu_0$ and $\nu_1$ and a geodesic $\Gamma : [0,1] \to P_\infty(M,m)$, parametrized proportionally to arclength, connecting $\nu_0 = \Gamma(0)$ and $\nu_1 = \Gamma(1)$ such that

$$S_N(\Gamma(t)|m) \leq -\int_{M \times M} \left[ \sigma_{K,N'}^{(t-1)}(d(x_0,x_1))\rho_0^{-1/N'}(x_0) \\
+ \sigma_{K,N'}^{(t)}(d(x_0,x_1))\rho_1^{-1/N'}(x_1) \right] \, dq(x_0,x_1)$$

holds for all $t \in [0,1]$ and all $N' \geq N$. 
We say that \((M, m)\) satisfies the curvature-dimension condition \(CD(K, N)\) locally – denoted by \(CD_{\text{loc}}(K, N)\) – if it satisfies \(CD_{\text{loc}}^*(K, N)\) with \(\sigma_{K,N}'(s)\) replaced by \(\tau_{K,N}'(s)\) for each \(s \in [0, 1]\) and \(N' \geq N\).

The (global) conditions \(CD^*(K, N)\) and \(CD(K, N)\) are also defined as similar to and imply corresponding local ones.

From the identical inequality \(\tau_{K,N}(t) \geq \sigma_{K,N}(t)\), \(CD(K, N)\) (resp. \(CD_{\text{loc}}(K, N)\)) induces \(CD^*(K, N)\) (resp. \(CD^*_{\text{loc}}(K, N)\)). Further, it is known that the local CD-conditions are equivalent in the following sense: When a mathematical condition \(\phi(K)\) is given for each \(K \in \mathbb{R}\), we say that a mathematical object \(P\) satisfies \(\phi(K-)\) if \(P\) satisfies \(\phi(K')\) for all \(K' < K\).

**Theorem 6.2** ([BS, Proposition 5.5]). Let \(K, N \in \mathbb{R}\) with \(N \geq 1\) and let \((M, m)\) be a metric measure space. Then, \((M, m)\) satisfies \(CD_{\text{loc}}^*(K-, N)\) if and only if it satisfies \(CD_{\text{loc}}(K-, N)\).

There is another notion of a lower Ricci curvature bound in metric measure spaces which is called the measure contraction property, denoted by \(\text{MCP}(K, N)\). Since we do not use its theory to prove Theorem 1.8 in this paper, omit its definition. For the definition and theory, we refer [Oh] and [S2].

A metric measure space \((M, m)\) is called non-branching if \(M\) is a geodesic space and is non-branching in the sense that for any four points \(x, y, z_1, z_2\) in \(M\), if \(y\) is a common midpoint of \(x\) and \(z_1\) and of \(x\) and \(z_2\), then \(z_1 = z_2\). It is known that a non-branching metric measure space satisfying \(CD(K, N)\) satisfies \(\text{MCP}(K, N)\). Recently, Cavalleti and Sturm proved

**Theorem 6.3** ([CS, Theorem 1.1]). Let \((M, m)\) be a non-branching metric measure space. Let \(K, N \in \mathbb{R}\) with \(N \geq 1\). If \((M, m)\) satisfies \(CD_{\text{loc}}(K, N)\), then it satisfies \(\text{MCP}(K, N)\).

6.2. **Bishop-Gromov volume growth estimate.** Let \((M, m)\) be a metric measure space and \(x \in \text{supp}(m)\). We set

\[
v_x(r) := m(B(x, r)).
\]

For \(K, N \in \mathbb{R}\) with \(N > 1\), we define

\[
\bar{v}_{K,N}(r) = \int_0^r \text{sn}^{N-1}_{K/(N-1)}(t) \, dt.
\]

A metric measure space \((M, m)\) satisfies the Bishop-Gromov volume growth estimate \(\text{BG}(K, N)\) if for any \(x \in \text{supp}(m)\), the function

\[
v_x(r) / \bar{v}_{K,N}(r)
\]

is nonincreasing in \(r \in (0, \infty)\), (with \(r \leq \pi \sqrt{(N-1)/K}\) if \(K > 0\)).

Since \(\bar{v}_{K,N}(r)\) is continuous in \(K\), \(\text{BG}(K-, N)\) implies \(\text{BG}(K, N)\). The Bishop-Gromov volume growth estimate is implied by several lower Ricci curvature bounds, for instance the measure contraction property.
Theorem 6.4 ([Oh Theorem 5.1], [S2 Remark 5.3]). If \((M, m)\) satisfies MCP\((K, N)\), then it satisfies BG\((K, N)\).

Summarizing above facts, we can use the following implication: Let \(K, N \in \mathbb{R}\) with \(N \geq 1\). For a non-branching metric measure space \((M, m)\),

\[
\begin{align*}
\text{CD}_{\text{loc}}^{\ast}(K, N) & \implies \text{CD}_{\text{loc}}^{\ast}(K-, N) \iff \text{CD}_{\text{loc}}(K-, N) \\
& \implies \text{MCP}(K-, N) \implies \text{BG}(K-, N) \implies \text{BG}(K, N)
\end{align*}
\]

holds.

6.3. Universal covering space with lifted measure. Let \(X\) be a semi-locally simply connected space. Then, there is a universal covering \(\pi : Y \to X\). In addition, if \(X\) is a length space, then \(Y\) is also considered as a length space. The map \(\pi\) becomes a local isometry.

In addition, we assume that \((X, m)\) is a proper metric measure space. Let \(\mathcal{V}\) be the family of all open sheets of the universal covering \(\pi : Y \to X\). We define a set function \(m_Y : \mathcal{V} \to [0, \infty]\) by

\[m_Y(V) = m(\pi(V)).\]

One can naturally extend \(m_Y\) to a Borel measure on \(Y\). We also write its measure as \(m_Y\), and call it the lift of \(m\). Since \(m\) is locally finite, so is \(m_Y\).

In general, for a geodesic \(\Gamma : [0, 1] \to \mathcal{P}_2(M)\), if \(\Gamma(0)\) and \(\Gamma(1)\) are supported on \(U(x, r)\) for some \(x \in X\) and \(r > 0\), then \(\Gamma(t)\) is supported on \(U(x, 2r)\) for every \(t \in (0, 1)\) ([S, Lemma 2.11]). Therefore, we obtain

Proposition 6.5 (cf. [BS Theorem 7.10]). The local (reduced) curvature-dimension condition is inherited to the lift. Namely, let \(K, N \in \mathbb{R}\) with \(N \geq 1\) and let \((X, m)\) and \((Y, m_Y)\) be as above. If \((X, m)\) satisfies CD_{loc}^{\ast}(K, N) (resp. CD_{loc}^{\ast}(K, N)), then \((Y, m_Y)\) also satisfies CD_{loc}^{\ast}(K, N) (resp. CD_{loc}^{\ast}(K, N)).

6.4. Proof of Theorem 1.8

Proof of Theorem 1.8 Let \(X\) be an \(n\)-dimensional compact orientable Alexandrov space without boundary. Let \(m\) be a locally finite Borel measure on \(X\) with full support. We assume that \((X, m)\) satisfies CD_{loc}^{\ast}(K, N) for \(K < 0\) and \(N \geq 1\). By Proposition 6.5 the universal covering \(Y\) of \(X\) with lift \(m_Y\) of \(m\) also satisfies CD_{loc}^{\ast}(K, N). And, \(Y\) is an \(n\)-dimensional Alexandrov space. Since \(m\) has full support, so is \(m_Y\). By the implication (6.1), \((Y, m_Y)\) satisfies BG\((K, N)\). Therefore, as mentioned in the preface of this section, the proof of original Gromov’s theorem relying on the Bishop-Gromov volume comparison works in our setting (cf. [G, §2] [Y, Appendix]). Hence, we can prove Theorem 1.8. We recall such an argument. For undefined terms appearing and for facts used in the following argument, we refer [G] and [Y].
Let $\mathcal{M}$ (resp. $\mathcal{M}_+$) be the Banach space (resp. the set) of all finite signed (resp. positive) Borel measure on $Y$, equipped with the norm $\|\mu\| = \int_Y d|\mu| \in [0, \infty)$ for $\mu \in \mathcal{M}$. Due to the general theory established in [G, §2] and [X, Appendix], if a differentiable averaging operator $S : Y \to \mathcal{M}_+$ exists, then for any $\alpha \in H_n(X)$,

$$
(6.2) \quad \|\alpha\|_1 \leq n! (\mathcal{L}[S])^n \text{mass}(\alpha)
$$

holds. Here, the value $\mathcal{L}[S]$ is defined as follows. For $y \in Y$,

$$
\mathcal{L}_y = \limsup_{z \to y} \frac{\|S(z) - S(y)\|}{d(z, y)} \quad \text{and} \quad \mathcal{L}[S] = \sup_{y \in Y} \frac{\mathcal{L}_y}{\|S(y)\|}.
$$

We recall a concrete construction of a differentiable averaging operator. For $R > 0$ and $y \in Y$, we set $S_R(y) \in \mathcal{M}_+$ as

$$
S_R(y) = \mathbb{1}_{B(y, R)} \cdot m_Y.
$$

Here, $\mathbb{1}_A$ is the characterizing function of $A \subset Y$. For every $\epsilon > 0$, we define $S_{R, \epsilon} : Y \to \mathcal{M}_+$ by

$$
S_{R, \epsilon}(y) = \frac{1}{\epsilon} \int_{R - \epsilon}^R S_R(y) \, dR'.
$$

Its norm is $\|S_{R, \epsilon}(y)\| = \frac{1}{\epsilon} \int_{R - \epsilon}^R v_y(R') \, dR'$ and is not less than $v_y(R - \epsilon)$.

Here, $v_y(r) = m_Y(B(z, r))$ for $z \in Y$ and $r > 0$. Given a Lipschitz function $\psi = \psi_{R, \epsilon} : [0, \infty) \to [0, 1]$ defined as

$$
\psi(t) = \begin{cases} 
1 & \text{if } t \leq R - \epsilon \\
(R - t)/\epsilon & \text{if } t \in [R - \epsilon, R] \\
0 & \text{if } t \geq R,
\end{cases}
$$

we can write $S_{R, \epsilon}(y) = \psi(d(y, \cdot)) m_Y$ for any $y \in Y$.

We can check $S_{R, \epsilon}$ is a differentiable averaging operator as follows. Since $m_Y$ is $\pi_1(X)$-invariant, the maps $S_R$ and $S_{R, \epsilon}$ are $\pi_1(X)$-equivariant. Since $m$ is absolutely continuous in $\mathcal{H}_X^n$, so is $m_Y$ in $\mathcal{H}_Y^n$. One can check that $S_{R, \epsilon}$ is differentiable at $m_Y$-almost everywhere with respect to the differentiable structure of $Y$, where, the differentiable structure on Alexandrov spaces are defined by Otsu and Shioya [OS]. Indeed, the differential $D_yS_{R, \epsilon}(\gamma^+(0))$ of $S_{R, \epsilon}$ at $y$ along a geodesic $\gamma$ starting from $y = \gamma(0)$ is calculated as

$$
(D_yS_{R, \epsilon}(\gamma^+(0))) (A) = \frac{1}{\epsilon} \int_{A \cap (y; R - \epsilon, R)} \cos \angle(\hat{y}_{\gamma}, \gamma^+(0)) \, dm_Y(z)
$$

for any Borel set $A \subset Y$, where $A(z; r, r')$ is the annulus around $z \in Y$ of radii between $r$ and $r'$ for $r \leq r'$.

To estimate $\mathcal{L}[S_{R, \epsilon}]$, we use the Bishop-Gromov volume growth estimate as follows. We obtain

$$
\mathcal{L}(S_{R, \epsilon})_y = \sup_{\xi \in \Sigma_y} \|D_yS_{R, \epsilon}(\xi)\| \leq \frac{m_Y(A(y; R - \epsilon, R))}{\epsilon}
$$
It follows from $\mathcal{B}(K,N)$, 
\[
\mathcal{L}(S_{R',\epsilon})(y) \leq \frac{\psi_{y}(R) - \psi_{y}(R - \epsilon)}{\epsilon \cdot \psi_{y}(R - \epsilon)} \leq C_{K,N}(R,\epsilon).
\]
Here, setting \(\bar{\psi}(R') = \bar{\psi}_{K,N}(R') = \int_{0}^{R'} \frac{N-1}{K/(N-1)}(t) dt\),
\[
C_{K,N}(R,\epsilon) := \bar{\psi}(R) - \bar{\psi}(R - \epsilon) \epsilon \cdot \bar{\psi}(R - \epsilon).
\]

Since \(\text{mass}([X]) = \mathcal{H}^{n}(X)\) \(\text{[Y] Theorem 0.1}\), by using (6.2) and by tending \(\epsilon \to 0\) and \(R \to \infty\), we obtain
\[
\|X\| \leq n! \sqrt{-K(N-1)^{n}} \mathcal{H}^{n}(X).
\]
It completes the proof of Theorem 1.8. \(\square\)

**Remark 6.6.** Due to Petrunin [Pt ALVS] and Zhang and Zhu [ZZ], it is known that for \(n\)-dimensional Alexandrov space \(X\) of curvature \(\geq \kappa\), the metric measure space \((X, \mathcal{H}^{n})\) satisfies the curvature-dimension condition CD((\(n-1\))\(\kappa,n\)). Therefore, Corollary 1.7 is implied by Theorem \([LS]\) via [Pt ALVS] and [ZZ].

If there exists a compact orientable \(n\)-dimensional Alexandrov space \(X\) without boundary of curvature \(\geq \kappa\) with \(\kappa < 0\) which has nonnegative Ricci curvature with respect to some reference measure \(m\) so that \(m \ll \mathcal{H}^{n}\) and \(\text{supp}(m) = X\), then Theorem \([LS]\) yields \(\|X\| = 0\).

**Acknowledgments.** The first author is supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists.

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