ON MEAN CURVATURE FLOW OF SINGULAR RIEMANNIAN FOLIATIONS: NON COMPACT CASES

MARCOS M. ALEXANDRINO, LEONARDO F. CAVENAGHI, AND ICARO GONÇALVES

Abstract. In this paper we investigate the mean curvature flow (MCF) of a regular leaf of a closed generalized isoparametric foliation as initial datum, generalizing previous results of Radeschi and the first author. We show that, under bounded curvature conditions, any finite time singularity is a singular leaf, and the singularity is of type I. We also discuss the existence of basins of attraction, how cylinder structures can affect convergence of basic MCF of immersed submanifolds and make a few remarks on MCF of non-closed leaves of generalized isoparametric foliation.

1. Introduction

A singular foliation $\mathcal{F}$ on a complete Riemannian manifold $M$ is called singular Riemannian foliation (SRF) if every geodesic perpendicular to one leaf is perpendicular to every leaf it meets, see [12, page 189]. Recall that a leaf of a singular Riemannian foliation is called regular if it has maximal dimension, and singular otherwise. In addition, if the mean curvature vector field along regular leaves is basic, the foliation is called generalized isoparametric foliation.

A typical example of a generalized isoparametric foliation is the partition of a Riemannian manifold into the connected components of the orbits of an isometric action (the homogenous examples). Other classical examples are the families of isoparametric foliations on Euclidean and symmetric spaces. In addition, all examples of SRF with closed leaves on Euclidean or round sphere are generalized isoparametric, and there are infinitely many nonhomogeneous examples in these spaces, see [14]. For more detailed information on generalized isoparametric foliations see Sections 1 and 2 of [3].

In [3], Radeschi and the first author studied the mean curvature flow of a regular leaf of a generalized isoparametric foliation $\mathcal{F}$ as initial datum assuming that the ambient space $M$ is compact as well the leaves of $\mathcal{F}$. They
proved that any finite time singularity is a singular leaf, and the singularity is of type I, generalizing results of Liu–Terng [11] and Koike [10].

Recall that a smooth family of immersions $\varphi_t: L_0 \to M$, $t \in [0, T)$ is called a solution of the mean curvature flow (MCF for short) if $\varphi_t$ satisfies the evolution equation

$$\frac{d}{dt} \varphi_t(x) = H(t, x),$$

where $H(t, x)$ is the mean curvature of $L(t) := \varphi_t(L_0)$ at $x$. We say that the MCF $\varphi_t$ has initial datum $L_0$. By abuse of notation, we will often identify $\varphi_t$ with its image $L(t)$, and we will talk about the MCF flow $L(t)$. For more details on MCF, see e.g., [7].

In this paper we generalize [3], dropping the condition of compactness of $L$ and $M$, replacing it with other weaker conditions.

**Theorem 1.1.** Let $(M, F)$ be a generalized isoparametric foliation with closed leaves on a complete manifold $M$ so that $M/F$ is compact. Let $L_0 \in F$ be a regular leaf of $M$ and let $L(t)$ denote the mean curvature flow evolution of $L_0$ with maximal interval of existence $[0, T)$. Assume that $T < \infty$. Then the following statements hold:

(a) $L(t)$ converges (in the leaf space sense) to a singular leaf $L_T$ of $F$.

(b) If the curvature of $M$ is bounded and the shape operator along each leaf is bounded, then for each $p \in L(0)$ the line integral of MCF $\varphi_t(p)$ converges to a point of $L_T$. In addition the singularity is of type I, i.e.,

$$\limsup_{t \to T^-} \|A_t\|^2(T - t) < \infty,$$

where $\|A_t\|$ is the sup norm of the second fundamental form of $L(t)$.

**Remark 1.2.** Since the leaves of a SRF are locally equidistant (recall [5]), item (a) of the above theorem implies that $L(t)$ converges to a singular leaf $L_T$ in the Gromov-Hausdorff sense (recall [13, Chapter 10]). In addition, under bounded curvature conditions (i.e., $M$ has bounded curvature and the shape operator of each leaf of $F$ on $M$ is bounded) Lemma 1.1 implies that, for each $\epsilon$, we can find a small $r_0$ so that the metric projection $\rho: \text{Tub}_{r_0}(L_T) \to L_T$ restricted to $L(t) \subset \text{Tub}_{r_0}(L_T)$ turns to be an $\epsilon$-isometry i.e.,

$$|d(x, y) - d(\rho(x), \rho(y))| < \epsilon$$

for each $x, y \in L(t)$.

One of the key observations behind the proof of Theorem 1.1 is the following useful fact; see Lemma 2.1.

**Lemma 1.3.** For each SRF $F$ with closed leaves on a complete manifold $M$, and a singular leaf $L_q$, one can find for a tubular neighborhood $U$ of $L_q$ and a (Sasaki) metric $g^0$ so that the restricted foliation $F|_U$ turns to be a generalized isoparametric foliation on $(U, g^0)$, where the principal curvatures associated to each basic vector field along each regular leaf are constant.
Remark 1.4. From the proof of the above lemma, one can check that the holonomy foliation (with compact holonomy) restricted to the unit bundle and others more elaborate examples presented in [1] fulfill the hypothesis of Theorem 1.1.

As in [3], the main idea of the proof of item (a) of Theorem 1.1 is to assure the existence of basins of attraction. More precisely, we have that for each singular leaf $L_q \in \mathcal{F}$ there exists a small tubular neighborhood $\text{Tub}_\epsilon(L_q)$ so that for each regular leaf $L(t_0)$ contained in $\text{Tub}_\epsilon(L_q)$, the MCF $t \to L(t)$ continues to stay in $\text{Tub}_\epsilon(L_q)$ for $t > t_0$. In addition if $L(t_0) \subset \text{Tub}_\epsilon(L_q)$ we have $T < \infty$; see Lemma 3.2 for details.

As we remark in Section 5, Lemma 3.2 can be adapted to the case of SRF with non-closed leaves. As a simple application, we can assure convergence of MCF of $t \to L(t)$ when $T < \infty$ and $M$ is compact, see Proposition 5.1.

Under bounded curvature conditions, another adaption of Lemma 3.2 can also be useful to prove the convergence of MCF of an immersed submanifold $N$ contained in a regular leaf as initial datum, when the MCF of $N$ can be extended to a basic flow of $\mathcal{F}$, see Proposition 6.1. This adaption of Lemma 3.2 and hence the proof of Proposition 6.1 will follow direct from the estimate in Lemma 6.5, an interesting remark of immersion theory that we could not find in the classical literature. It states that given a Riemannian manifold $(M, g)$ with bounded curvature and an embedded submanifold $L$ with bounded shape operator, we can have a control of the trace of the shape operator $A(r)$ of a immersed submanifold $N \subset \partial \text{Tub}_\epsilon(L)$ with respect to the gradient of the distance $r$ to $L$, as long as, we have a well defined tubular neighborhood $\text{Tub}_\epsilon(L)$ of $L$, for a small $\epsilon$.

This paper is divided as follows. In Section 2 we prove Lemma 1.3 that will be important in the proof of Theorem 1.1 presented in Sections 3 and 4. In Section 5 we remark a few results on MCF of SRF with non-closed leaves, and Proposition 5.1 is presented. Finally, in Section 6 we prove Proposition 6.1 via the estimate in Lemma 6.5. In the appendix we also present Lemma 7.1 concerning the behavior of the distribution $\mathcal{T}$ used to define the Sasaki metric in Lemma 1.3. This lemma play a role in the proof of item (b) of Theorem 1.1 and may be relevant in future studies of SRF under bounded curvature conditions.

2. The distribution $\mathcal{T}$ and the Sasaki metric

As discussed in [1,2], given a closed leaf $L_q$ we can find a $\mathcal{F}$-saturated tubular neighborhood $U = \text{Tub}_\epsilon(L_q)$ of $L_q$ and a subfoliation $\mathcal{F}_\ell \subset \mathcal{F}|_U$ (the linearized foliation) that is the maximal infinitesimal homogenous subfoliation of $\mathcal{F}$. In other words, if $\rho : U \to L_q$ is the metric projection, and $S_p = \rho^{-1}(p)$ is the slice (i.e., $S_p := \exp_p(v_p L \cap B_\epsilon(0))$, then $\mathcal{F}_\ell = S_p \cap \mathcal{F}$ is the maximal homogenous subfoliation of the infinitesimal foliation $\mathcal{F}_p = S_p \cap \mathcal{F}$. The infinitesimal foliation $\mathcal{F}_p$ turns to be a SRF on the Euclidean space $(S_p, g_p)$ if we identify $S_p$ via the exponential map with an open set.
of $\nu_{p}(L_q)$ with the flat metric $g_p$. In addition, we can find a distribution $\mathcal{T}$ homothetic invariant (with respect to $T$) that is tangent to $\mathcal{F}^\ell$ and extends $T(L_q)$.

Set $U^0 := \exp^{-1}(U)$. The distribution and the foliation on $U^0$ defined by the pullback of $\mathcal{T}$ and $\mathcal{F}^\ell$ through the normal exponential map will also be denoted by $\mathcal{T}$ and $\mathcal{F}^\ell$. Let $g^0$ be the associated Sasaki metric, i.e., the metric so that $\mathcal{T}$ is orthogonal to $\nu(L_q)$, the foot point projection $\rho^0 : (\nu(L_q), g^0) \to (L_q, g)$ is a Riemannian submersion and the fibers $\nu_{p}(L_q)$ have the flat metric $g_p$. The foliation $\mathcal{F}^0 := (\exp^\nu)^{-1}(\mathcal{F}|U)$ turns to be a SRF with respect to $g^0$ on $U^0$ and by homothetic transformation it can be extended to $\nu(L_q)$. Let us denote $\nabla^0$ the Riemannian connection associated to $g^0$.

Now we present a useful application of the above discussion.

**Lemma 2.1.** Let $\mathcal{F}$ be a SRF with closed leaves and $L_q$ be a singular leaf. Consider the SRF $\mathcal{F}^0$ on $(\nu(L_q), g^0)$ defined above. Then

(a) $A^{0}_{\xi}|_{\mathcal{T}} = 0$, where $\xi$ is a normal vector field along a regular leaf $L_x$.

(b) The principal curvatures associated to basic vector fields along regular leaves of $\mathcal{F}^0$ are constant. In particular $\mathcal{F}^0$ is a generalized isoparametric foliation.

(c) The principal directions associated to non-zero curvatures are tangents to the fibers of $\nu(L_q)$.

(d) $\nabla^0\xi|_{\mathcal{T}} = 0$, if $\xi$ is the gradient of the distance function $r(x) = d^0(L_q, x)$ i.e., the distance between $x$ and $L_q$ with respect to the metric $g^0$.

**Proof.** Once $g^0$ is a Sasaki metric, the fibers of $\nu(L_q)$ are totally geodesics and isometric to each other. Therefore the space $T_x(\nu_{p(x)}(L_q) \cap L_x)$ is $A^{0}_{\xi}$-invariant and hence the distribution $\mathcal{T}$ is also $A^{0}_{\xi}$-invariant. Also recall that the principal curvatures associated to basic vector fields along regular leaves of the infinitesimal foliation $\mathcal{F}^0 \cap \nu_{p(x)}(L_q)$ are constant; see [4, Remark 3.2]. These facts together imply that items (b) and (c) of the lemma will be proved once we have checked item (a).

Given vector fields $X_1$ and $X_2$ on $L_q$, consider their lifts $X^\tau_1$ tangent to $\mathcal{T}$ and $\xi$ a normal vector field along $L_x$. Let us also denote $\nabla^b$ the induced Riemannian connection on $L_q$. As $\rho^0$ is a Riemannian submersion we have

\[
\nabla^0_{X^\tau_1}X^\tau_2 = (\nabla^b_{X_1}X_2)^\tau + \frac{1}{2}[X^\tau_1, X^\tau_2]^\nu.
\]

Since $\mathcal{T}$ is tangent to $L_x$ and $\xi$ is orthogonal to $L_x$ we infer that:

\[
\nabla^0(\xi, (\nabla^b_{X_1}X_2)^\tau) = 0.
\]

Since the (possible nointegrable) distribution $\mathcal{T}$ is tangent to $\mathcal{F}^\ell \subset \mathcal{F}$, we have that $[X_1^\tau, X_2^\tau]$ is tangent to $\mathcal{F}^\ell$ and hence:

\[
\nabla^0(\xi, \frac{1}{2}[X_1^\tau, X_2^\tau]^\nu) = 0.
\]
From Eq. (2.1) (2.2) (2.3) we conclude that
\begin{equation}
\label{eq:2.4}
g^0(A^0_\bar{x}(X^1_\tau), X^2_\tau) = g^0(\xi, \nabla^0_{X^1_\tau}X^2_\tau) = 0.
\end{equation}
This finishes the proof of item (a) and hence the proof of items (b) and (c), as discussed above.

In order to prove item (d), consider the geodesic variations
\begin{equation}
\label{eq:2.5}
f(s, t) = \exp_\beta(t)(s\xi(t))
\end{equation}
so that
\begin{equation}
\label{eq:2.6}
\frac{\partial f}{\partial t}(s, 0) = J(s) \in T.
\end{equation}
here \( t \to \beta(t) \) is a curve in \( L_q \) and \( t \to \xi(t) \) is a unit normal field along \( L_q \).
Set \( \gamma(s) := f(s, 0) \).
Since \( T \) is orthogonal to the totally geodesic submanifold \( \nu_q(L) \), we conclude from Eq. (2.5) that
\begin{equation}
\label{eq:2.7}
\nabla_0^0 \frac{\partial f}{\partial t}(s, 0) = \frac{\partial f}{\partial s}(s, 0) = \frac{\partial f}{\partial t}(s, 0) = J'(s) \in T_{\gamma(s)}.
\end{equation}
Equation (2.6) and item (a) imply item (d) of the lemma.

3. Proof of item (a) of Theorem 1.1

3.1. A new estimate of the shape operator. In this section we generalize a estimate of [3] that will allow us to prove item (a) of Theorem 1.1

Let us start by fixing some notations that will be used in the proof of Lemma 3.1. Given the original metric \( g \) on \( M \), the metric on a neighborhood \( U^0 := \exp^{-1}(\text{Tub}_\epsilon(L_q)) \) of the null section of \( \nu(L_q) \) defined by the pullback of \( g \) via the normal exponential map will also be denoted by \( g \). Let \( \nabla \) be the Riemannian connection associated to \( g \) on \( U^0 \). Consider the connection \( \nabla^0 \) associated to \( g^0 \) and set \( \omega := \nabla - \nabla^0 \). Consider an orthonormal basis \( \{e_\alpha\} \) of \( T_xL_x \) with respect to the original metric \( g \) so that \( e_\alpha \in T_x(\nu_{\rho(x)}(L_q) \cap L_x) \) (for \( \alpha = 1 \cdots k \)) and \( e_l \in T_xL_x \) (for \( l = k + 1 \cdots \dim \mathcal{F} \)).

**Lemma 3.1.** Let \( L_q \) be a closed singular leaf. Then there exist a radius \( \epsilon \) and constant \( c_1 > 0 \) such that in the tubular neighborhood \( \text{Tub}_\epsilon(L_q) \) the following equation holds true:
\begin{equation}
\label{eq:3.1}
-\frac{k}{r(x)} - c_1 \leq \text{tr}(A\nabla r)_x \leq -\frac{k}{r(x)} + c_1,
\end{equation}
where \( r(x) = d(L_q, x) \) is the distance between the regular points \( x \in L(t) \subset \text{Tub}_\epsilon(L_q) \) and the singular leaf \( L_q \) and \( k = \dim \mathcal{F} - \dim L_q \).

**Proof.** Let \( \xi \) be the gradient of the distance function \( r \) with respect to \( g \) (or with respect to \( g^0 \) that gives the same gradient for the function \( r \)).
\[ \text{tr} A_\xi = - \sum_{n=1}^{\dim F} g(\nabla e_n \xi, e_n) \]

\[ = - \sum_{\alpha=1}^{k} g(\nabla^0 e_\alpha \xi, e_\alpha) - \sum_{l=k+1}^{\dim F} g(\nabla_e^0 \xi, e_l) \]

\[ - \sum_{n=1}^{\dim F} g(\omega(e_n) \xi, e_n) \]

Now let us examine each of the above terms. We know from Euclidean geometry that

\[ - \sum_{\alpha=1}^{k} g(\nabla^0 e_\alpha \xi, e_\alpha) = - \sum_{\alpha=1}^{k} g(1^{e_\alpha}, e_\alpha) = -\frac{k}{r}. \]

For \( k + 1 \leq l \leq \dim F \) set \( e_l = e^\prime_l + e^\tau_l \) where \( e^\prime_l \in T_x(\nu(\rho(x)) (L_q) \cap L_x) \) and \( e^\tau_l \in \mathcal{T} \). Also recall from Lemma 2.1 that \( \nabla^0 \xi \mid T = 0 \). Then

\[ - \sum_{l=k+1}^{\dim F} g(\nabla^0 e_l \xi, e_l) = - \sum_{l=k+1}^{\dim F} g(\nabla^0 e^\prime_l \xi, e_l) - \sum_{l=k+1}^{\dim F} g(\nabla^0 e^\tau_l \xi, e_l) \]

\[ = - \frac{1}{r} \sum_{l=k+1}^{\dim F} g(e^\prime_l, e_l) + 0 = 0, \]

where the last equality follows from the fact that \( g(X, e_l) = 0 \) for each \( X \in T_x(\nu(\rho(x)) (L_q) \cap L_x) \).

From the equations above we infer that

\[ (3.2) \quad \text{tr} A_\xi = -\frac{k}{r} - \text{tr} \omega(\cdot) \xi. \]

Equation (3.2) implies that \( \text{tr} \omega(\cdot) \xi \) is basic. On the other hand, in a relative compact neighborhood of \( q \)

\[ (3.3) \quad -c_1 \leq -\text{tr} \omega(\cdot) \xi \leq c_1. \]

Now Eq. (3.1) follows from (3.2) and (3.3). \( \square \)

3.2 Revised proof. Once we have proved Lemma 3.1, the proof of item (a) of Theorem 1.1 follows from the same arguments as in [3]. For the sake of completeness let us briefly recall these arguments extracted from [3].

Lemma 3.2 (Basins of attraction [3]). Let \( F \) be a generalized isoparametric foliation with closed leaves on a complete Riemannian manifold \( M \). Let \( L_q \) be a singular leaf. Then there exists a neighborhood \( \text{Tub}_r(L_q) \) around \( L_q \) with radius \( r \) small enough such that if the initial data \( L(t_0) \subset \text{Tub}_r(L_q) \) then the following properties hold true:
(a) Let \( L(t) \) be the MCF with initial data \( L(t_0) \) and \( r(t) \) the distance between \( L(t) \) and \( L_q \). Then
\[
C^2_2(t - t_0) \leq r^2(t_0) - r^2(t) \leq C^2_2(t - t_0),
\]
where \( C_1 \) and \( C_2 \) are positive constants that depend only on \( \text{Tub}_p(L_q) \).

(b) \( T < \infty \) and \( L(t) \subset \text{Tub}_p(L_q) \) for all \( t > t_0 \).

(c) If \( L(t) \) converges to \( L_q \) then
\[
C_1 \sqrt{T - t} \leq r(t) \leq C_2 \sqrt{T - t}.
\]

Proof. We start with a small \( \epsilon_0 \) so that the distance function \( r(x) = d(L_q, x) \) with respect to a singular leaf \( L_q \) is smooth on \( \text{Tub}_p(L_q) \setminus L_q \). Let \( p \in L(t_0) \) and consider the solution of the MCF \( \varphi \) with initial condition \( p \) i.e., the curve \( t \to \varphi_t(p) \) such that \( \frac{d}{dt} \varphi_t(p) = H(t) \). Then we have
\[
r'(t) = \frac{d}{dt} r \circ \varphi_t(p)
= \langle \nabla r, \varphi_t'(p) \rangle
= \langle \nabla r, H(t) \rangle
= \text{tr}(A \nabla r).
\]

From Lemma 3.1 we have:
\[
-\frac{k}{r} - c_1 \frac{r}{r} \leq \text{tr}(A \nabla r) \leq -\frac{k}{r} + c_1 \frac{r}{r}.
\]

Now we chose \( \epsilon < \min\{\epsilon_0, \frac{k}{c_1} \} \) and define the constants \( C_1, C_2 \) by the equations
\[
\begin{cases}
C^2_1 = k - \epsilon \cdot c_1, \\
C^2_2 = k + \epsilon \cdot c_1.
\end{cases}
\]

The above equations imply
\[
-\frac{C^2_2}{2r(t)} \leq r'(t) \leq -\frac{C^2_1}{2r(t)}
\]
or, equivalently, \(-C^2_2 \leq (r^2(t))' \leq -C^2_1 \). Integrating this equation we get
\[
C^2_2(t - t_0) \leq r^2(t_0) - r^2(t) \leq C^2_2(t - t_0)
\]
for \( t > t_0 \) closer to \( t_0 \) and hence for every \( t > t_0 \). This conclude the proof of item (a). Items (b), (c) follow directly from item (a).

Let \( \pi : M \to M/\mathcal{F} \) be the canonical projection. Since \( t \to \pi(L(t)) \) is contained in a compact set and \( T \) is finite, the limit set of \( t \to \pi(L(t)) \) cannot be contained in the regular stratum and thus it must be contained in the singular one. Let \( L_q \) be a leaf in the limit set, and consider a sequence \( t_n \subset [0, T) \) so that \( t_n \to T \) and \( \pi(L(t_n)) \to \pi(L_q) \). Given small \( \epsilon \), Lemma 3.2 implies that there exists \( t_{n_0} \) so that if \( t > t_{n_0} \) then \( L(t) \in \text{Tub}_p(L_q) \). The arbitrariness of \( \epsilon \) implies that \( \pi(L(t)) \) converges to \( \pi(L_q) \).
4. Proof of item (b) of Theorem 1.1

4.1. New estimate of the shape operator under bounded curvature conditions. In this section we generalize estimates in [3] and these will allow us to prove item (b) of Theorem 1.1. We are going to use the same convention for local frame established in Section 3.1.

Lemma 4.1. Let $L_q$ be a closed singular leaf and assume that there exists a tubular neighborhood of $L_q$ with bounded curvature, i.e., $-k_1 \leq K \leq k_1$ for a positive constant $k_1$. Then, reducing the tubular neighborhood if necessarily, there exists $t_0 > 0$ so that for $t_0 < t < T$ we have:

\[ \|H(t)\| \leq C_1 \|A^0(t)\|_0 + C_2. \]

Proof. Consider the $(1,1)$-tensor field $G$ defined as $g(X,Y) = g^o(GX,Y)$. For $y$ close to $L(t)$, let $r^o(x) = d^o(x,y)$ be the distance function with respect to the metric $g^o$. A direct calculation implies that

\[ \nabla r^0 = G^{-1}\nabla^0 r^0. \]

Using the fact that $r^0$ is a $\mathcal{F}$-basic function, and $U^o$ is a saturation of a relative compact neighborhood of $q$, it is straightforward to check the following properties:

Claim 4.2.

1. $\nabla r^0$ is basic;
2. $g(\nabla r^0, \nabla^0 r^0)$ is constant along regular leaves;
3. $c_1 < \sqrt{g(\nabla r^0, \nabla^0 r^0)} < c_2$ on $U_0$, where $c_i$ is a constant that does not depend on $\nabla r^0$.

From Eq. (4.2) and $(\nabla^0 r^0 G^{-1}) = -G^{-1}(\nabla^0 G)G^{-1}$ we have:

\[ \nabla e_n \nabla^0 r^0 = G^{-1}\nabla^0 e_n \nabla^0 r^0 - G^{-1}(\nabla^0 e_n G)G^{-1}\nabla^0 r^0. \]

Since $\nabla e_n \nabla^0 r^0 = \nabla e_n \nabla^0 r^0 + \omega(e_n) \nabla^0 r^0$ we have from Eq. (4.3) that:

\[ -g(H(t), \nabla^0 r^0) = -\text{tr}A_{\nabla^0 r^0} = \sum_n g(\nabla e_n \nabla^0 r^0, e_n) \]
\[ = \sum_n g(\nabla^0 e_n \nabla^0 r^0, e_n) + \sum_n g(\omega(e_n) \nabla r^0, e_n) \]
\[ = \sum_n g^0(\nabla^0 e_n \nabla^0 r^0, e_n) - \sum_n g(G^{-1}(\nabla^0 e_n G)G^{-1}\nabla^0 r^0, e_n) \]
\[ + \sum_n g(\omega(e_n) \nabla r^0, e_n). \]

In what follows we are going to prove that there exists $c_3$ so that

\[ |g^0(\nabla^0 e_n \nabla^0 r^0, e_n)| < c_3 \|A^0(t)\|_0 \]

Set $B := -\sum_n g(G^{-1}(\nabla^0 e_n G)G^{-1}\nabla^0 r^0, e_n) + \sum_n g(\omega(e_n) \nabla r^0, e_n)$. Note that $B$ is well defined along the regular leaves (its definition does not depend on
the frame \( \{e_n\} \). The fact that the mean curvature and \( \|A^0(t)\|_0 \) are basic and Eq. (1.1) will imply that \( \mathcal{B} \) is bounded along each regular leaf and hence (since is bounded on relative compact a neighborhood of \( q \)) bounded on the regular stratum of \( U^0 \) i.e., \( |\mathcal{B}| < c_4 \). These equations will then imply that
\[
(4.5) \quad |g(H(t), \nabla r^0)| \leq c_3 \|A^0(t)\|_0 + c_4.
\]

The arbitrariness of \( r^0 \), Eq. (4.5) and item (c) of Claim 4.2 allow us to infer Eq. (4.1). Let us now prove Eq. (4.4).

As in the previous lemma, we denote \( X^\nu \) the \( g^0 \)-projection of a vector \( X \) onto the fibers of \( \nu(L_q) \). By using Lemma 2.1 we can check:
\[
(4.6) \quad g^0(\nabla_{c_l} r^0, e_l) = g^0(\nabla_{c_l} r^0, e_l, X^\nu).
\]
Writing \( X^\nu = \sum_{\beta} g^0(X, e^0_\beta) e^0_\beta \), where \( \{e^0_\beta\} \) is a \( g^0 \)-orthonormal basis of principal directions of \( A^0_{\nu_\rho,0} \), it is easy to verify the next equation:
\[
(4.7) \quad |g^0(\nabla_{X^\nu} r^0, X^\nu)| \leq \|A^0_{\nu_\rho,0}\| h g^0(X^\nu, X^\nu).
\]

**Claim 4.3.** There exist constants \( c_5, c_6 \) (that depends only on radius \( \epsilon \) and the bounded curvature) so that \( c_6 g(X^\nu, X^\nu) \leq g^0(X^\nu, X^\nu) \leq c_5 g(X^\nu, X^\nu) \), for every \( X^\nu \in T_x(\nu_{\rho(x)}(L_q)) \).

In fact, consider \( W \in \nu_{\rho(x)}(L_q) \) so that \( g(rW, rW) = g^0(rW, rW)_{\rho(x)} = 1 \) and \( \frac{X^\nu}{\|X^\nu\|} = \frac{J(r)}{\|J(r)\|} \) where \( J(s) = d(\exp_{\rho(x)})_{sv}(sW) \) is the associated Jacobi field. Since \( \exp_{\rho(x)}^0 = \exp_{\rho(x)} \), we have that \( J(s) = J^0(s) \) and hence
\[
g^0\left( \frac{X^\nu}{\|X^\nu\|}, \frac{X^\nu}{\|X^\nu\|} \right) = g^0\left( \frac{J(r)}{\|J(r)\|}, \frac{J(r)}{\|J(r)\|} \right) = g^0\left( \frac{J^0(r)}{\|J^0(r)\|}, \frac{J^0(r)}{\|J^0(r)\|} \right) = \frac{1}{\|J^0(r)\|^2}.
\]
Claim 4.3 follows from Rauch’s theorem [6, Chapter 10], that assures \( \frac{1}{\sqrt{c_5}} \leq \|J(r)\| \leq \frac{1}{\sqrt{c_6}} \).

From Lemmas 6.4 and 7.1 and Claim 4.3 we know that if \( g(e_1, e_1) = 1 \) then \( g^0(e_1^0, e_1^0) \) is bounded. This fact, Eq. (4.6), (4.7) and Claim 4.3 imply Eq. (4.4), which concludes the proof, as discussed before. We stress that the condition of bounded shape operators along the leaves has been used in Lemma 7.1.

**Remark 4.4.** The proof of Lemma 4.1 also implies that, for each \( \tilde{q} \) in the singular leaf \( L_q \), there exists a (relative compact) neighborhood \( V \) of the point \( \tilde{q} \) and constants \( C_1 \) and \( C_3 \) so that
\[
(4.8) \quad \|A_x(t)\| \leq C_1 \|A^0_x(t)\|_0 + C_3.
\]
for \( x \in V \). Here, it is important to stress that the constant \( C_3 \) may depend on the neighborhood \( V \) of \( \tilde{q} \) and may not be defined on tubular neighborhood of \( L_q \) in the case where \( L_q \) is not compact.
4.2. Revised proof. Once we have proved Lemma 4.1 and Eq. (4.8), the proof of Item (b) of Theorem 1.1 follows from a small adaptation of [3] as we now review.

Given a tubular neighborhood $\text{Tub}_\varepsilon(L_q)$, we define $r_\Sigma : \text{Tub}_\varepsilon(L_q) \to \mathbb{R}$ as the distance between $L_x$ and the singular strata $\Sigma$, and $f : \text{Tub}_\varepsilon(L_q) \to \mathbb{R}$ as the distance between $L_x$ and its focal set. By abuse of notation, we set $r_\Sigma(t) := r_\Sigma(L(t))$ and $f(t) = f(L(t))$. As proved in [3 Proposition 3.6] $r_\Sigma(t) \geq Cr(t)$ for $r(t) = d(L(t), L_q)$. From Lemma 3.2 we can infer that $r_\Sigma(t) \geq C\sqrt{T - t}$. As proved in [3 Proposition 3.7], there exists a constant $\sigma \in (0, 1)$ such that $f(p) \geq \sigma r_\Sigma(p)$ for every regular point $p \in M$. These results hold on a tubular neighborhood of $L_q$ for the original metric $g$ and metric $g^0$. Putting these results together we get
\begin{equation}
(4.9) \quad f^0(t) \geq C\sqrt{T - t},
\end{equation}
where $f^0(t)$ is the distance between $L_x$ and its focal set with respect to $g^0$. On the other hand, by Lemma 2.1 we infer
\begin{equation}
(4.10) \quad \|A^0(t)\|_0 = \frac{1}{f^0(t)}.
\end{equation}
Combining Eq. (4.9) and (4.10) we have that
\begin{equation}
(4.11) \quad \|A^0(t)\|_0 \sqrt{T - t} \leq C_3
\end{equation}
holds on $\text{Tub}_\varepsilon(L_q)$. Eq. (4.11) and Lemma 4.1 imply
\begin{equation}
(4.12) \quad \|H(t)\|_0 \sqrt{T - t} \leq C_4.
\end{equation}

Let $\gamma(t) := \varphi_1(p)$ be the integral curve of $H$ starting at $p$. Define $h : [0, 1) \to [0, T)$ as $h(s) := T - T(1 - s)^2$ and set $\beta(s) := \gamma(h(s))$. As consequence of Eq. (4.12) we have $\|\beta'(s)\| < \infty$. Since $L(t)$ converges to $L_q$ (in the leaf space sense) and $\beta$ has finite length, we conclude that $\beta$ converges to a point $q$, i.e., $\gamma$ converges to a point $q$. This fact, Eq. (4.11) and (4.8) imply that
\begin{equation}
(4.13) \quad \|A_x(t)\|_0 \sqrt{T - t} \leq C_5,
\end{equation}
for $x$ close to $q$. Equation (4.13) implies that the convergence is type I.

5. Remarks on MCF of non-closed regular leaf.

As proved in [2], if $\mathcal{F} = \{L\}$ is a SRF then $\overline{\mathcal{F}} = \{\overline{L}, L \in \mathcal{F}\}$ (i.e, partition of $M$ into the closures of the leaves of $\mathcal{F}$) is also a SRF. This was the so called Molino’s conjecture.

Note that mean curvature of $\overline{L_q}$ does not necessarily coincide with the mean curvature of $L_q$ and hence it would make sense to ask if we can say something about the MCF of a regular (non-closed) leaf as initial datum.

As we are going to explain, a part from a small generalization in the semi-local model presented in Section 2 (see [2] and [1]), the proofs of Lemmas 3.1 and 3.2 also hold for SRF with non-closed leaves. In this case the singular leaf $L_q$ can be replaced by its closure $\overline{L_q}$. 
Let $B$ be a closed submanifold of $M$ saturated by leaves of $\mathcal{F}$ with the same dimension (e.g., $B = \mathcal{L}$ or $B$ is the minimal stratum). Consider $U := \text{Tub}_\epsilon(B)$ the tubular neighborhood and $\rho : U \to B$ the metric projection. Again, via normal exponential map, we can identify $U$ with a neighborhood of the null section $B, \mathcal{F}|_U$ with a foliation on the normal bundle $\nu(B)$ of $B$, and the map $\rho$ with the foot point map $\nu(B) \to B$.

There exists 3 homothetic distributions $(\mathcal{K}, \mathcal{T}, \mathcal{N})$ so that:

- $\mathcal{K} = \ker \rho^*,$
- $\mathcal{T}$ extends $\mathcal{T}\mathcal{F}|_B$ and is everywhere tangent to the leaves of $\mathcal{F}$,
- $\mathcal{N}$ extends the normal space of $\mathcal{T}\mathcal{F}|_B$,
- $TM = \mathcal{T} \oplus \mathcal{N} \oplus \mathcal{K}$.

Let $g^0$ be the “Sasaki” metric on the fiber bundle $\mathcal{K} \to B$ with respect to the homothetic distribution $\mathcal{T} \oplus \mathcal{N}$. Note that in this case the fibers $\nu(B)$ are not totally geodesic, because the product is compatible with the metric only if we derive in the direction of $\mathcal{T}$. The same proof of Lemma 2.1 allow us to conclude that

$$g^0(\nabla g_0 \xi |_{\mathcal{T}}, X) = 0 \quad \forall \quad X \in TxLx,$$

where $\xi = \nabla g_0 r$, for $r(x) = d^0(L_q, x)$. Equation (5.1) allow us to infer the analogous to Lemma 3.1 and to Lemma 3.2, if we replace the singular leaf $L_q$ with its closure $\overline{L_q}$.

In the particular case where $M$ is compact, we can use this adaptation of Lemma 3.2 to infer the convergence of MCF.

**Proposition 5.1.** Let $M$ be a compact Riemannian manifold and $\mathcal{F}$ be a generalized isoparametric foliation on $M$, with possible non-closed leaves. Assume that the MCF $t \to L(t)$ of a regular leaf $L(0)$ as initial datum has $T < \infty$. Then $t \to L(t)$ must converge to the closure of a singular leaf. In addition, if for each $x \in M$ all leaves of the infinitesimal foliation $\mathcal{F}_x$ are compact (i.e., if $\mathcal{F}$ is infinitesimal compact) then for each $p \in L(0)$ the line integral of MCF $\varphi_t(p)$ converges to a point of $L_T$ and singularity is of type I.

**Proof.** Since $M$ is a compact Riemannian manifold, we know that the singular strata $\Sigma$ (i.e., the union of singular leaves) is also compact, because it is closed in $M$. Therefore, one can cover $\Sigma$ with a finite union of small tubular neighborhoods $\{ \text{Tub}_\epsilon(\overline{L_q}) \}$ (the basins of attraction). This property, the fact that the mean curvature is bounded on the precompact set $M \setminus \cup \text{Tub}_\epsilon(\overline{L_q})$ and the the arbitrariness choice of $\epsilon$ imply that limit set of $t \to \pi(L(t))$ must be contained in $\Sigma$ when $T < \infty$. Therefore we can also follow the same argument of Section 3.2 and conclude the convergence in the leaf sense, i.e., $\pi(L(t))$ converges to a singular point of $M/\mathcal{F}$, where $\pi : M \to M/\mathcal{F}$ is the canonical projection.

Now we assume that $\mathcal{F}$ is infinitesimal compact. In order to check the type I convergence, let us consider a finite open cover $\{ \mathcal{O}_n \}$ of the compact manifold $B = \overline{L_q}$. Here $\{ \mathcal{O}_n \}$ denotes the tubular neighborhood of a plaque
$P_n \subset B$ as defined in [3]. Note that the discussion introduced in [3] still holds in the neighborhood $O_n$ because $\mathcal{F}$ is infinitesimal compact.

Since $L(t)$ converges in a leaf sense to $B$, we can find $t_0$ so that the integral line $\alpha(t) = \varphi_t(p)$ is contained in $\cup_{n} O_n$ for $t > t_0$. For each neighborhood $O_n$ we know that there exists an open set $I_n \subset (t_0, T)$ so that $\|A(t)\| \leq \|A'(t)\| + \tilde{C}$ for $t \in I_n$. Here $A'(t)$ is the shape operator with respect to the flat metric on $O_n$, see [3] for details. Also as explained in [3], $\|A'(t)\| \sqrt{T-t} < C$ for all $t \in I_n$. Therefore we have that $\|A(t)\| \sqrt{T-t} < C$ for all $t \in I_n$. Set $C := \max_n \{C_n\}$. Hence $\|A(t)\| \sqrt{T-t} < C$ for all $t \in (t_0, T)$. From this, as explained at the end of Section 4.2, we infer the type I convergence and that the integral line $\alpha$ converges to a point.

□

Remark 5.2. It is possible to check that there exists a metric $g^0$ so that $\mathcal{F}|_U$ is a generalized isoparametric foliation, when $\mathcal{F}|_B$ is a generalized isoparametric foliation (with the respect to the original metric).

6. Cylinder structure, bounded geometry and MCF

In this section we present a generalization of item (a) of Theorem 1.1.

**Proposition 6.1.** Let $\mathcal{F}$ be a SRF with closed leaves on a complete manifold $(M, g)$. Assume that

1. $M$ has bounded sectional curvature;
2. the shape operator along each leaf $L \in \mathcal{F}$ is bounded;
3. $M/\mathcal{F}$ is compact.

Let $N$ be an immersed submanifold contained in a regular leaf. Assume that the dimension of $N$ is greater than the dimension of singular leaves, that the mean curvature flow $N(t)$ is a restriction of a basic flow (with respect to $\mathcal{F}$) on the regular stratum and its maximal interval $[0, T)$ has $T < \infty$. Then $N(t)$ converges to a singular leaf $L$ in the leaf space sense.

Again the main idea is to deduce the existence of basins of attraction, i.e, Lemma 3.2. The key observation behind the proof of this adaption of Lemma 3.2 is the estimate in Lemma 6.5 which we believe can be useful in the context of immersion theory.

6.1. **Comparisons lemmas.** Here we consider two triples $(M_1, L_1, \gamma_1)$ and $(M_2, L_2, \gamma_2)$, where $(M_i, g_i)$ is a Riemannian manifold so that $\dim M_1 = \dim M_2$, $L_i$ is an embedded submanifold of $M_i$ so that $\dim L_1 = \dim L_2$ and $\gamma_i$ is a unit speed geodesic orthogonal to $L_i$ at $\gamma_i(0)$. We also assume that $U_i$ is a tubular neighborhood of $L_i$ of radius $\epsilon_0$ so that the distance function $r_i : U_i \setminus L_i \to \mathbb{R}$ defined as $r_i(x) = d_i(L_i, x)$ is smooth. In particular we are assuming that $\gamma_i|_{[0, \epsilon_0]}$ does not contain a focal point of $L_i$.

It is not difficult to adapt classical arguments of index lemma to conclude the next result, cf. [8] Chapter 2, Theorem A.

**Lemma 6.2.** Assume that:
The constant \( C \) are such that there exists an isomorphism \( \theta(\epsilon) : \nu_1(\epsilon) \rightarrow \nu_2(\epsilon) \) (for \( \epsilon < \epsilon_0 \)) so that

\[
\text{Hess} r_2(\theta(\epsilon)X_1, \theta(r)X_1) \leq \text{Hess} r_1(X_1, X_1).
\]

Here \( \nu_i(s) \) denotes the space of normal vectors to \( \gamma_i(s) \).

**Remark 6.3.** Let \( \mathcal{V}_1 \) be the vector spaces of differentiable vector fields orthogonal to \( \gamma_i \) starting tangent to \( T_{\gamma_i(0)}L_i \). The isomorphism defined above is an isomorphism between \( \mathcal{V}_1 \). In fact \( \theta : \mathcal{V}_1 \rightarrow \mathcal{V}_2 \) is defined as follows: let \( t \to \{e_{i,m}(t)\} \) be the parallel transport along \( \gamma_i \) of an orthogonal basis where \( e_{i,\alpha} \in \nu_{\gamma_i(0)}L_i, e_{i,t} \in T_{\gamma_i(0)}L_i \) and \( e_0 = \gamma_i(0) \). For each \( V \in \mathcal{V}_1 \) written as \( V = \sum_{m=1}^{\dim M-1} f_m e_{1,m} \), we set \( \theta(V) = \sum_{m=1}^{\dim M-1} f_m e_{2,m} \).

We intend to apply the above lemma to compare the submanifold \( L \) with a fiber of a warped product. During this kind of calculation, we will need to understand what happens with \( \theta(X_1) \) when \( X_1 \) is a vector perpendicular to a slice \( S_2 \) or when it is tangent to \( S_1 \). This will be related to the next result, which roughly speaking, assures us that, under bounded curvature conditions, if a parallel vector field \( e_{1,l_0} \) along \( \gamma_1 \) starts tangent to \( L_1 \), then (for small time \( s \)) the parallel vector field \( e_{2,l_0} := \theta(e_{1,l_0}) \) has small projection into the slice \( S_2 \). The next result is a direct application of classical Rauch’s theorem, see [6 Chapter 10].

**Lemma 6.4.** Assume that \( \sup_{e_{1i}(s) \in \nu_1(s)} K_1(e_1, \gamma_i(1)) \leq \inf_{e_{2i}(s) \in \nu_2(s)} K_2(e_2, \gamma_i(2)) \) where \( \nu_i(s) \) is the space of normal vectors to \( \gamma_i(s) \) and \( \|e_i\| = 1 \). Let \( e_{l_0} \) be a parallel unit vector field along \( \gamma_i \) so that \( e_{l_0}(0) \in T_{\gamma_i(0)}L_i \), for \( i = 1,2 \).

Let \( J_{l} \) be a Jacobi field along \( \gamma_i \) so that \( J_l(0) = 0, J'_l(0) \in T_{\gamma_i(0)}S_i \), where \( S_i := \exp_{\gamma_i(0)}(\nu_{\gamma_i(0)}L_i \cap B_1(0)) \) is a slice at \( \gamma_i(0) \) and \( \|J_{l}(0)\| = \|J_{l}(0)\| \).

Then there exists a constant \( C \) such that, for \( 0 < s < \epsilon \), we have:

\[
\|g_2(e_{2,l_0}(s), J_2/s)\| \leq C s^2.
\]

The constant \( C \) depends only on \( \sup_{[0,R]} \|R_2\| \) and \( \sup_{[0,R]} \|J_1/s\| \) (and in particular does not depend on frames).

**6.2. The proof of Proposition 6.1.** We start by proving a lemma that we believe can be useful in the context of immersion theory.

**Lemma 6.5.** Let \( (M, g) \) be a Riemannian manifold with bounded sectional curvature, i.e., there is a constant \( \Lambda \in \mathbb{R} \) such that \( |K_g| \leq \Lambda \). Let \( L \) be an embedded submanifold with bounded shape operator. Assume that there exists a well defined tubular neighborhood \( \text{Tub}_{c_0}(L) \) for some \( c_0 \). Then, given a positive integer number \( k \), reducing \( c_0 \) if necessary, there exist positive constants \( C, D, c_1 \) and \( c_2 \) so that for each \( \epsilon < \epsilon_0 \) and for each immersed submanifold \( N \subset \partial \text{Tub}_{c_0}(L) \) so that \( \dim N = \dim L + k \) we have:

\[
-\frac{D}{r(x)} - c_2 \leq \text{tr}(A \nabla r) \leq -\frac{C}{r(x)} + c_1.
\]
where $A_{\nabla r}$ is the shape operator of the immersed submanifold $N \subset \partial \text{Tub}_{\epsilon}(L)$ and $r(x) = d(L, x)$ is the distance between $L$ and $x \in N$.

**Remark 6.6.**

1. Lemma 6.5 can be thought as a natural generalization of [13, Chapter 6, Theorem 27].
2. In the particular case where $N$ coincides with the cylinder, the above lemma gives an estimate of mean curvature of cylinders.

**Proof of Lemma 6.5.** In what follows we are going to prove that $\text{tr}(A_{\nabla r}) \leq -\frac{C}{r^n(x)} + c$, i.e., the part of the lemma that will be used in the proof of Proposition 6.1.

We start with a tubular neighborhood $\text{Tub}_{\epsilon}(L)$ around $L$ such that the map $p \mapsto d(p, L)$ is smooth on $\text{Tub}_{\epsilon}(L) \setminus L$. Assume that $\epsilon$ is such that $\gamma$ has no focal points to $L$ on $\text{Tub}_{\epsilon}(L)$ and denote by $s \mapsto \gamma(s)$ an arc length parametrized geodesic that is perpendicular to $L$ at $s = 0$ and that realizes the distance between $L$ and $x \in N$, i.e., $r(x) = d(x, L) = s$.

Let $B := \{e_n\}$ be an orthonormal frame consisting of tangent vectors to $N$ at $\gamma(s) = x$. Hence,

\begin{equation}
- \text{tr}(A_{\nabla r}) = \sum_{n=1}^{\dim N} g(\nabla e_n \nabla r, e_n) = \sum_{n=1}^{\dim N} \text{Hess} r(e_n, e_n).
\end{equation}

Let $S^d_R$ denote the sphere of radius $R$ with the round metric where $d$ stands for the codimension of $L$ on $M$. Let $s \mapsto \gamma_2(s)$ be an arc length geodesic in $S^d_R$ starting at north pole $q_N = \gamma_2(0) \in S^d_R$ and set $q_0 := \gamma_2(-s_0) \in S^d_R$.

Consider $S^d_R \times L$ endowed with the metric of warped product

$\tilde{g} = g_{S^d_R} \times e^{2\phi} g_L$,

where $g_L$ is the induced metric on $L$ from $g$ and

$\phi(p) := \lambda d^2(p, q_0), \lambda < 0, q_0 \in S^d_R$.

By applying the explicit calculations of sectional curvature of warped product metrics presented in [9, Proposition 2.2.2, pg 59], and choosing $|\lambda|$ and $R$ big enough and $q_0$ close enough to the north pole $q_N \in S^d_R$, one can check the next claim.

**Claim 6.7.** There exist real numbers $R > 0$ and $\lambda < 0$ and a point $q_0 \in S^d_R$ so that

$(M_1 := (\text{Tub}_{\epsilon}(L), g), L_1 := L, \gamma_1 := \gamma)\quad (M_2 := (S^d_R \times L, \tilde{g}), L_2 := \{q_N\} \times L, \gamma_2)$

fulfill all the requirements of Lemma 6.5.
Let $U_2 := \text{Tub}_0(L_2)$ be a smooth tubular neighborhood of $L_2$ on $M_2$ and consider the distance function
\[
r_2 : U_2 \setminus L_2 \to \mathbb{R},
\]
\[
r_2 : p \mapsto d_2(p, L_2),
\]
For each $n \in \{1, \ldots, \dim N\}$,
\[
\theta(e_n) = e_n^\top + e_n^\perp, \ e_n \in B,
\]
where $e_n^\top$ denotes the component that is tangent to $L$ and $e_n^\perp$ the component that is tangent to the base of the warped product.

By the comparison Lemma 6.2

(6.3) $\text{Hess} \ r(e_n, e_n) \geq \text{Hess} \ r_2(e_n^\top, e_n^\top) + \text{Hess} \ r_2(e_n^\perp, e_n^\perp)$.

The base of the warped product is the round sphere, thus

(6.4) $\text{Hess} \ r_2(e_n^\perp, e_n^\perp) = \frac{1}{R} \cot \left( \frac{r_2}{R} \right) \| e_n^\perp \|^2,
$

furthermore, from [9, Proposition 2.2.2, pg 59]

(6.5) $\text{Hess} \ r_2(e_n^\top, e_n^\top) = 2\lambda e^{2\lambda(s + s_0)}(s + s_0)\| e_n^\top \|^2, \ \lambda < 0.$

Hess $r(e_n, e_n) \geq 2\lambda(e_0 + s_0)e^{2\lambda(e_0 + s_0)}\| e_n^\top \|^2 + \frac{1}{R} \cot \left( \frac{r_2}{R} \right) \| e_n^\perp \|^2.$

Summing up and using the definition of $\text{tr}(A_{\nabla r})$ one has
\[
\text{tr}(A_{\nabla r}) \leq -2\lambda(e_0 + s_0)e^{2\lambda(e_0 + s_0)}\sum_n \| e_n^\top \|^2 - \frac{1}{R} \cot \left( \frac{r_2}{R} \right) \sum_n \| e_n^\perp \|^2.
\]

and hence, we conclude
\[
\text{tr}(A_{\nabla r}) \leq -2\lambda(e_0 + s_0)e^{2\lambda(e_0 + s_0)}\sum_n \| e_n^\top \|^2 - \frac{1}{R} \cot \left( \frac{r_2}{R} \right) \sum_n \| e_n^\perp \|^2.
\]

Writing $\cot \left( \frac{r_2}{R} \right) = \frac{R}{r} - O(r)$, where $O(r) > 0$, we have,

(6.7) $\text{tr}(A_{\nabla r}) \leq -2\lambda(e_0 + s_0)e^{2\lambda(e_0 + s_0)}\sum_n \| e_n^\top \|^2 - \frac{1}{R} \left( \frac{R}{r} - O(e_0) \right) \sum_n \| e_n^\perp \|^2.$

Since, for every $n \in \{1, \ldots, \dim N\}$, one has $\| e_n^\top \|, \| e_n^\perp \| \leq 1$, we define
\[
c_1 := -2\lambda(e_0 + s_0)e^{2\lambda(e_0 + s_0)} \dim N + \frac{1}{R} O(e_0) \dim N.
\]

to infer that
\[
\text{tr}(A_{\nabla r}) \leq c_1 - \frac{1}{R} \sum_n \| e_n^\perp \|^2.
\]

Now, reducing $e_0$ if necessary, we evoke Lemma 6.4 to conclude that there is $C > 0$ such that
\[
\sum_n \|e_n^+\|^2 \geq \sum_\alpha \|e_\alpha^+\|^2 \geq C,
\]
therefore,
\[
\text{tr}(A_{\nabla r}) \leq c_1 - \frac{C}{r(x)},
\]
finishing the proof of the main part of the lemma that will be used in the proof of Proposition 6.1. The proof of \(-D r(x) - c_2 \leq \text{tr}(A_{\nabla r})\) can be done in a similar way, considering a warped metric on \(\mathbb{H}^d_R \times L\).
\(\square\)

From now on, we let \((M, \mathcal{F}, N, g)\) be a setup just as in the hypothesis of Proposition 6.1. Starting by recalling that the MCF \(t \rightarrow N(t)\) of \(N\) preserves the dimension of \(N(t)\) for \(t < T\). This fact, and the same argument in Section 3.2 allow us to reduce the proof of Proposition 6.1 to check again the existence of basins of attraction, i.e., an adaptation of Lemma 3.2. It is a straightforward consequence of Lemma 6.5 with \(L = L_q\) and \(N = N(t)\) that there exists \(\epsilon > 0\) such that
\[
\begin{align*}
(1) \text{ there is a constant } C_1 > 0 \text{ depending only on the } &\text{Tub}_e(L_q) \text{ such that if } N(t_0) \in \text{Tub}_e(L_q) \text{ for some } t_0 > 0, \text{ then } r^2(t_0) - r^2(t) \geq C_1^2(t - t_0), \forall t \in [t_0, T), \text{ and } T < \infty. \\
(2) N(t) \subset \text{Tub}_e(L_q) \forall t \in [t_0, T).
\end{align*}
\]

7. Appendix: Inclination of the distribution \(\mathcal{T}\)

In this section we present the proof of Lemma 7.1. Roughly speaking, this lemma and Lemma 6.4 assure us that, under bounded curvature conditions and small \(\epsilon\), the “inclination” between the distribution \(\mathcal{T}\) (defined in Section 2) and a slice \(S_\epsilon(\bar{q})\) is bounded, and this bound does not depend on \(\bar{q} \in L_q\).

**Lemma 7.1.** Assume that \(M\) has bounded curvature and the shape operator of each leaf of \(\mathcal{F}\) on \(M\) is bounded. Then for each leaf \(L_q\) and small \(\epsilon > 0\) there exists a radius \(r_0\) with the following property: if

- \(X\) is a unit vector tangent to \(L_q\) at a point \(q\),
- \(\gamma\) is an unit speed geodesic orthogonal to \(L_q\) at \(\bar{q} = \gamma(0)\),
- \(s \rightarrow J(s)\) is the (unique) Jacobi field along \(\gamma\) so that \(J(0) = X\) and \(J(s) \in \mathcal{T}_\gamma(s)\),
- \(s \rightarrow \overline{e}(s)\) is a parallel vector field along \(\gamma\) so that \(\overline{e}(0)\) is normal to \(L_q\) and \(\|\overline{e}\| = 1\).

Then \(||J(s)|| - 1| < \epsilon\) and \(|g(J(s), \overline{e})| < \epsilon\) for each \(s \in [0, r_0]\).

**Proof.** Our strategy consists in bounding the inicial conditions of \(J\) by comparing them with inicial conditions of a Jacobi field along a geodesic \(\gamma_0\) starting at \(q\), stressing that they do not depend on \(\bar{q} \in L_q\).

We first assume that \(\bar{q}\) and \(q\) are in the same plaque of \(L_q\) and that exists a unit vector field \(\bar{X}\) on this plaque so that \(\bar{X}(\bar{q}) = X\). We can then extend
this vector field $\vec{X}$ to be tangent to the leaves near the plaque of $L_q$ and linearize this vector field, producing a linearized vector field $\vec{X}^\ell$; see [1, 2]. Let $\varphi$ be the flow of $\vec{X}^\ell$. As explained in [1, 2] this flow has the following properties:

(a) The flow $\varphi$ sends fiber to fiber of the normal bundle $\nu(L_q)$;
(b) $\varphi_t : \nu_p(L_q) \to \nu_{\varphi_t(p)}(L_q)$ is a linear isometry;
(c) For each $p \in L_q$ near $\tilde{q}$ we have $\vec{X}(p) = \frac{d}{dt}\varphi_t(p)|_{t=0}$.

Let us denote $t_0$ the time so that $\tilde{q} = \varphi_{t_0}(q)$ and $\gamma_0$ the geodesic defined as $\gamma = \varphi_{t_0} \circ \gamma_0$. Define the geodesic variations $f(t, s) = \varphi_t \circ \gamma_0(s)$ and the associated Jacobi field $J_t(\cdot) = \frac{\partial}{\partial s} f(t, \cdot)$. Hence $J = J_{t_0}$. Also note that $J_t(s) = d\varphi_t J_0(s)$. Using property (b) described above, we can infer:

\begin{equation}
\tag{7.1}
d\varphi_{t_0} \left( \nabla \frac{\partial}{\partial s} J_0(0) \right)^\nu = \left( \nabla \frac{\partial}{\partial s} J_{t_0}(0) \right)^\nu,
\end{equation}

where $(\cdot)^\nu$ is the normal component. In fact, since $\varphi_t : \nu_p(L_q) \to \nu_{\varphi_t(p)}(L_q)$ is a linear isometry we have that $\frac{\partial}{\partial s} f(t, 0) = \exp(t \xi) \frac{\partial}{\partial s} f(0, 0)$ where the matrix exponential $\exp(t \xi)$ is written with respect to some parallel frame (with respect to the normal connection $\nabla^\nu$) along the curve $t \to f(t, 0)$.

From Eq. (7.1) we infer:

\begin{equation}
\tag{7.2}
\left\| \left( \nabla \frac{\partial}{\partial s} J_0(0) \right)^\nu \right\| = \left\| \left( \nabla \frac{\partial}{\partial s} J_{t_0}(0) \right)^\nu \right\|
\end{equation}

On the other hand, note that each Jacobi field $\tilde{J}_0$ along $\gamma_0$ tangent to $\mathcal{T}$ is unique determined by $\tilde{J}_0(0)$ and hence, for each of such Jacobi field with $\|\tilde{J}_0(0)\| = 1$, we can infer that there exists a constant $c_1$ such that

\begin{equation}
\tag{7.3}
\left\| \left( \nabla \frac{\partial}{\partial s} \tilde{J}_0(0) \right)^\nu \right\| \leq \left\| \nabla \frac{\partial}{\partial s} \tilde{J}_0(0) \right\| \leq c_1.
\end{equation}
Also recall that for each Jacobi field $\tilde{J}_t$ along $\gamma_t$ that is $L_q$-Jacobi field one has

\begin{equation}
(7.4) \quad \left(\frac{\nabla}{\partial s} \tilde{J}_t(0)\right) \top = -A_{\gamma_t(0)} \tilde{J}_t(0),
\end{equation}

where $(\cdot)^\top$ is the tangent component. The fact that the shape operator along $L_q$ is bounded by $\|A\|$ and Eq. $(7.2)$, $(7.3)$ and $(7.4)$ imply:

\begin{equation}
(7.5) \quad \left\| \left(\frac{\nabla}{\partial s} J_\alpha(0)\right) \right\| \leq \sqrt{\|A\|^2 + c_1^2}.
\end{equation}

It is important to note that the constant $c_2 := \sqrt{\|A\|^2 + c_1^2}$ does not depend on the point $\tilde{q}$. We infer then that for Jacobi fields $\tilde{J}$ along $\gamma$ with $\|\tilde{J}(0)\| = 1$ so that $\tilde{J}$ are tangent to $T$ have inicial conditions bounded by constants that do not depend on the point $\tilde{q}$.

Now consider a parallel frame $\{\omega_m\}$ along $\gamma$ so that $\omega_1(0) \in T_{\tilde{q}}L_q$ and $\omega_0(0) \in v_qL_q$. Let $y_m$ the functions so that $J_t(s) = \sum_m y_m(s) \omega_m$. Then the Jacobi equation can be written (in this frame) as $y''(s) + R(s)y(s) = 0$. Set $\mathcal{R}(q, q, s) = (\tilde{q}, R(s)q, 1) \in \mathcal{X}(\mathbb{R}^{2n+1})$.

**Claim 7.2.** Consider the flow $\varphi^s$ of the vector field $\mathcal{R}$, the compact set $I := [0] \times \{0\} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ that contains the inicial conditions, and $r_1$ so that $I \subset B_{r_1}(0 \times [0, r_1]) \subset \mathbb{R}^{2n+1}$. Then we can find a time $s_1$ so that $\varphi^s(x) \subset B_{r_1}(0 \times [0, r_1])$ for each $s \in [0, s_1]$ and $x \in I$. In particular, since the curvature is bounded and $B_{r_1}(0 \times [0, r_1])$ is compact, there exists a constant $c_3$ (that does not depend on $\tilde{q}$) so that $|\mathcal{R} \circ \varphi^s(x)| < c_3$.

It follows from Claim 7.2 that

\[ |\varphi^s(x) - \varphi^0(x)| \leq \left| \int_0^s \frac{d}{dt} \varphi^t(x) dt \right| \leq \int_0^s |\mathcal{R} \circ \varphi^t(x)| dt \leq sc_3. \]

for $x \in I$. Set $\pi_1(q, q, s) = q$. The above equation and triangle inequality imply that there exists $r_0$ so that for $0 < s < r_0$ we have for all $x \in I$

\begin{equation}
(7.6) \quad 1 - \epsilon \leq |\pi_1(\varphi^s(x))| \leq 1 + \epsilon
\end{equation}

\begin{equation}
(7.7) \quad \left| \langle \varphi^s(x), e_\alpha \rangle \right| \leq \frac{\epsilon}{k}.
\end{equation}

Eq. $(7.6)$ and $(7.7)$ conclude the proof of the lemma in the case where $\tilde{q}$ is the same plaque of $q$. The general case follows direct from compositions of flows of linearized flows.
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(Alexandrino) Universidade de São Paulo, Instituto de Matemática e Estatística, Rua do Matão 1010,05508 090 São Paulo, Brazil
E-mail address: marcosmalex@yahoo.de, m.alexandrino@usp.br

(Cavenaghi) Universidade de São Paulo, Instituto de Matemática e Estatística, Rua do Matão 1010,05508 090 São Paulo, Brazil
E-mail address: leonardofcavenaghi@gmail.com, kvenagui@ime.usp.br

(Gonçalves) Centro de Matemática, Computação e Cognição, Universidade Federal do ABC, 09.210-170, Santo André, Brazil.
E-mail address: icaro.goncalves@ufabc.edu.br