Zero discord implies classicality

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The "classical-quantum" (cq) discord of a bipartite state $\rho^{AB}$ is the smallest difference between the mutual information $S(\rho^{AB})$ of $\rho$ and that of $\rho$ after a measurement channel is applied on the A system. Relating zero discord to the strong subadditivity of the Von Neumann entropy, Datta proved that a state has zero cq discord if and only if it can be written in the form $\sum_i p_i |i\rangle \langle i| \otimes \rho^B$ for $p_i$ a probability distribution, $|i\rangle$ a basis of the A system and $\rho^B$ states of the B system. We provide a simple proof of that same result using directly a theorem of Petz on channels that leave unchanged the relative entropy of two given states.

Various measures have been proposed to quantify the classicality or equivalently the quantumness in a multipartite state. One of them, "discord", is the theme of this note. Limiting ourselves to a bipartite system $AB$, given an orthonormal basis $A = \{|a\rangle\}_{a \in A}$ of $\mathcal{H}_A$, any state $\rho^{AB}$, i.e. density operator, can be represented by a block matrix with blocks $\rho_{aa}^B$, corresponding to $\rho^{AB} = \sum_{aa'} |a\rangle \langle a'| \otimes \rho_{aa'}^B$. The state is said to be "classical-quantum" (cq) if there is such a basis for which $\rho_{aa}^B$ is properly chosen. It is clear that if $\rho$ is a cq state, then $\rho$ is a basis corresponding to the discord. The following theorem thus implies that if the $\rho_{aa}^B$ satisfy strong subadditivity of quantum entropy with equality.

**Theorem** Let $\mathcal{D} = \mathcal{D}_A$. If $I(\mathcal{D}(\rho^{AB})) > I(\rho^{AB})$ then $\rho$ can be block diagonalized in some basis of $\mathcal{H}_A$.

**Proof.** The equality $I(\mathcal{D}(\rho^{AB})) = I(\rho^{AB})$ is equivalent to $S(\rho^{AB}) = S(\rho^A \otimes \rho^B)$. A theorem of Petz states that if a channel $\mathcal{E}$ is such that $S(\mathcal{E}(\rho)) = S(\rho)$ then there exists $\mathcal{E}_a$ such that $\mathcal{E}_a(\rho) = \rho$ and moreover $\mathcal{E}_a(Y) = \sigma^{1/2} \mathcal{E}_a(Y) \sigma^{-1/2}$ where $\sigma$ is the adjoint of $\mathcal{E}$. Letting $\rho^{AB} = \sum_{aa'} |a\rangle \langle a'| \otimes \rho_{aa'}^B$, $\rho_a = \text{tr}[\rho_{aa'}^B \otimes \rho^B]$ and $\sigma = \rho^A \otimes \rho^B$ gives $\mathcal{D}(\sigma) = \sum_a \rho_a |a\rangle \langle a| \otimes \rho^B$ and $\mathcal{D}(\rho^{AB}) = \sum_a \rho_a |a\rangle \langle a| \otimes \rho^B$. It follows that $(\mathcal{D}(\sigma))^{-1/2} = \sum_{a \in A} \rho_a^{-1/2} |a\rangle \langle a| \otimes \rho^B$ and

$$
\rho^{AB} = \mathcal{E}_a(\rho^{AB}) = \sum_{a \in A} \rho_a^{-1/2} |a\rangle \langle a| \otimes \rho^B
$$

and $(|a\rangle \langle a| \otimes 1_B) \rho^{AB} (|a\rangle \langle a| \otimes 1_B) = \sum_{a' \in A} \rho_{a'a'}^{1/2} |a'\rangle \langle a'| \otimes \rho^B$ if $\rho_{aa'} \neq 0$ then

$$
\rho_{a'a'} = \sum_{a' \neq a} \rho_{a'a'} \rho_{a'a'}' \rho_{a'a'} = \frac{|\langle a'| \rho_{a'}^{1/2} |a\rangle|^2}{\rho_{aa} - |\langle a'| \rho_{a'}^{1/2} |a\rangle|^2}
$$

so that each diagonal block is a convex combination of the others. Thus, for all the extremal states $\rho_d^B$ of the convex hull of the $\rho_{a'a'}$, $|\langle a' | \rho_{a'a'}^{1/2} |a\rangle|^2 = 0$ if $\rho_{a'a'} \neq 0$. If we consider the non extremal states, the extremal ones do not appear in their convex combination and we may apply the same argument to their convex hull, and so on, eventually getting that $\langle a' | \rho_{a'a'}^{1/2} |a\rangle = 0$ if $\rho_{a'a'} \neq 0$. Grouping together the $a$ with equal $\rho_{a'a'}$ gives a partition $A_1, A_2, \ldots, A_k$ of $A$ and Eq. (1) becomes

$$
\rho^{AB} = \sum_{i=1}^k P_i \otimes \rho_{a_i} = \rho_{a_1}^{1/2} \sum_{a \in A_1} |a\rangle \langle a| \rho_{a_1}^{1/2}
$$

with $\langle a'| P_i |a\rangle = 0$ if either $a' \in A \setminus A_i$ or $a \in A \setminus A_i$. Letting $\mathcal{H}_A = \text{Span} \{|a\rangle : a \in A_i\}$, that implies that $\text{Supp}(P_i) \subseteq \mathcal{H}_i$; the supports of the $P_i$ being pairwise orthogonal, the $P_i$ can be simultaneously diagonalized, letting $\rho^{AB}$ block diagonal.

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Optimizing over all measurement maps

In this section, if ε is defined on the first system only then, when applied also on B, it is ε ⊗ 1_B that is meant.

A POVM M = {M_m}_{m ∈ M} is a family of operators such that 0 ≤ M_m ≤ 1_A and ∑_{m ∈ M} M_m = 1_A. For any measurement procedure on \( \mathcal{H}_A \) with outputs in M, there is a POVM such that the probability of output m given the state \( ρ \) is \( \text{tr}(M_m ρ) \). [6]

Following [5], we call measurement map associated to M the channel defined by \( \mathcal{M}_M(X) = ∑_{m ∈ M} \text{tr}(M_m X) M_m \). If \( ρ \) is a state of \( \mathcal{H}_A \) then \( \mathcal{M}_M(ρ) \) is state of \( \mathcal{H}_M \) with “standard basis” \( \{ |m⟩⟩m \}_{m ∈ M} \).

A POVM \( \mathbf{M}' = \{ M'_{m'} \}_{m' ∈ M'} \) with outputs in \( M' \) is a refinement of \( \mathbf{M} = \{ M_m \}_{m ∈ M} \) if \( M'_{m'} = ∑_{m' = m'} M_{m'} \). \( \mathbf{M}' \) corresponds to the output \( p(x') \) each time \( \mathbf{M}' \) output \( x' \), i.e. \( \mathbf{M} \) groups the outputs of \( \mathbf{M}' \) using \( p \), and sums their probabilities. If \( \mathbf{M}' \succeq \mathbf{M} \) then there is a channel \( ε \) s.t. \( ε \circ \mathcal{M}_M = \mathcal{M}_{\mathbf{M}'} \). Let \( \mathcal{E}(X) = ∑_{m' ∈ \mathbf{M}'} A_{m'} X A_{m'}^† \) where \( A_m = ∑_{m'' ∈ \mathbf{M}} M_{m''} M_{m''}^* \). \( A_m \) is extremal iff \( ∑ M_{m''} A_{m''} = 1_{\mathcal{H}_A} \) and \( \mathcal{E}(X) = ∑_{m'' ∈ \mathbf{M}} \text{tr}(M_{m''} X M_{m''}^*) |m''⟩⟩m'' \).

The set of POVMs with output set \( M \) is convex. Since \( \mathcal{M}_M \) is linear in \( \mathbf{M} \) and the relative information is convex in its inputs, any POVM with outputs in \( M \) and for which \( I(\mathcal{M}_M(ρ)) \) is optimal must be extremal [7]. Letting \( M_m = ∑_{n', n''} |e_{m'}⟩⟩n' |e_{m''}⟩⟩n'' \) be a spectral decomposition of \( M_m \), \( \mathbf{M} \) is extremal iff \( ∑ M_{m''} A_{m''} \) operators \( |e_{m'}⟩⟩n' \) for \( 1 ≤ n, n' ≤ d_M \) are linearly independent [8], so that there is at most \( d_M^2 \) operators \( |e_{m'}⟩⟩n' \) where \( d_M = \dim \mathcal{H}_A \). Let \( \mathbf{M}' \) be the refinement of \( \mathbf{M} \) with those POVM elements; \( \mathbf{M}' \) is extremal, of rank one, with at most \( d_M^2 \) elements. Moreover, since \( \mathcal{M}_M = \mathcal{M}_{\mathbf{M}'} \) for some \( \mathbf{M}' \) and since \( I(\mathcal{E}(\mathcal{M}_M(ρ))) ≤ I(\mathcal{E}(\mathcal{M}_{\mathbf{M}'}(ρ))) \), it follows that the optimum mutual information is obtained considering only rank 1 POVMs indexed by a set \( \mathbf{M}' \) of size \( d_M^2 \).

It is however needed to prove that there is actually a POVM for which the optimum is realized. A POVM \( \{ e_m⟩⟩m \} \) of rank 1 on \( \mathcal{H}_A \) defines an embedding \( 1 : \mathcal{H}_A → \mathcal{H}_M \) by \( m = ∑_{m ∈ M} n)⟩⟩m \). An easy calculation shows that \( 1^† 1 = 1_A \) if and only if \( |e_m⟩⟩e_m \) is a POVM. Conversely, any embedding \( 1 : \mathcal{H}_A → \mathcal{H}_M \) is defined by a POVM: from \( 1(φ) = ∑_{m} |m⟩⟩m 1(φ) \) it follows that \( |e_m⟩⟩e_m = |m⟩⟩m \).

Finally, the probability of measuring \( m \) with the POVM defined by \( |e_m⟩⟩e_m \) given \( ρ \) is the same as the probability of measuring \( |m⟩⟩m \) in the standard basis of \( \mathcal{H}_M \) given the state \( tρ^† \). The probability of measuring \( m \) given \( ρ \)

where \( U(\mathcal{H}_A) = U(d_A^2) \) is the unitary group on \( \mathcal{H}_A \). Since \( U(d_A^2) \) is compact, since its action is continuous, and since \( S(ρ_A^0) - S(ρ_A) \) is continuous in \( ρ \) (and \( U ⊗ 1_B \) leaves \( ρ_B \) fixed), there is a \( U \) for which the optimum is realized and sup may be replaced by max.

Zeroing conjugate off diagonal entries of off diagonal blocks

Since \( I(ρ^{A:B}) \) can never be less than \( I(ψ(ρ^{A:B})) \), one way to prove that equality implies that \( ρ^{A:B} \) is block diagonal might be to show that if it were not, \( I(ψ(ρ^{A:B})) \) could be decreased by zeroing non zero conjugate entries not on the block diagonal. To proceed [8] made the bold statement that if conjugate non zero entries that are neither on the block diagonal, nor on any diagonal of the blocs, are replaced by 0, then the entropy of the matrix strictly increases. That implies that \( I(ρ^{A:B}) \) decreases since then \( ρ_A \) and \( ρ_B \) are left unchanged, only \( S(ρ_A) \) is modified in \( I(ρ^{A:B}) = S(ρ_A) + S(ρ_B) \). Here is a Python 3 program that takes a two qubit density operator (thus a 4 x 4 matrix comprising four 2 x 2 blocks), returns its eigenvalues and its Von Neumann entropy and does the same on the matrix obtained after zeroing the (00) and (11) entries. The matrix has entropy 1.7555 and after zeroing the two conjugate entries, the entropy decreases to 1.7546 instead of increasing. The entropy also decreases if we choose the two other possible entries, (01) and (10). It is unclear how we could ever force the entropy to strictly increase by such methods.

From math import log, e
From numpy import array, linalg

def spec(m):
    return linalg.eigvalsh(m)
def S(m):
    sp = spec(m)
    return sum(-p*log(p,2) for p in sp if p != 0)
m = array([[0.25, 0.14, -0.02, -0.01],
            [0.14, 0.25, -0.01, -0.02],
            [-0.02, -0.01, 0.25, 0.14],
            [-0.01, -0.02, 0.14, 0.25]])
for i in range(2):
    print("The eigenvalues of\n%6.4f", % (S(m[i]))
print("are %s \% (spec(B))")
print("with Von Neumann entropy %6.4f\% (S(B))")
B[0][3]=0
B[3][0]=0