On ancient periodic solutions to Axially-Symmetric Navier-Stokes Equations

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March 1, 2019

Abstract

An old problem asks whether bounded mild ancient solutions of the 3 dimensional Navier-Stokes equations are constants. While the full 3 dimensional problem seems out of reach, in the works [4, 10], the authors expressed their belief that the following conjecture should be true. For incompressible axially-symmetric Navier-Stokes equations (ASNS) in three dimensions: bounded mild ancient solutions are constant. Understanding of such solutions could play useful roles in the study of global regularity of solutions to the ASNS.

In this article, we essentially prove this conjecture in the special case that \(u\) is periodic in \(z\) and time variables. To the best of our knowledge, this seems to be the first result on this conjecture without unverified decay condition. It also shows that periodic solutions are not models of possible singularity or high velocity region. Some partial result in the non-periodic case is also given.

1 Introduction

The classical Liouville theorem, stating that bounded harmonic functions in \(\mathbb{R}^n\) are constants, has been extended to many other elliptic and parabolic equations in different setting, and numerous applications have been found. The following is one of them. To study singularity formation for solutions of nonlinear equations, one often blows up the solution near singularity. This procedure often results in a bounded solution which exists in whole space or, in the case of evolution equations, a bounded solution which exists in the time interval \((-\infty, 0]\). These kind of solutions are often referred to as ancient solutions. Information on ancient solutions reveals singularity structure or the lack of singularity of the original solutions, and the behavior of solutions in regions with high value. For the 3 dimensional incompressible Navier-Stokes equation, one can also carry out this procedure. However, unless one imposes some extra decay

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conditions, whether or not Liouville theorem holds seems to be a question beyond reach. In fact, the question is still wide open even for the stationary case. See a recent paper [9] e.g. The full stationary problem seems intractable since it contains, as a special case, another old unsolved problem concerning D solutions, which asks whether a 3 dimensional stationary solutions with finite Dirichlet energy and vanishing at infinity is identically 0. Although, comparing with boundedness of solutions, this extra condition seems very restrictive at the first glance, it has not offered any help, even in the axially symmetric case.

However the authors of the papers [4, 10] expressed their belief that the following conjecture should be true. For incompressible axially-symmetric Navier-Stokes equations (ASNS) in three dimensions: bounded mild ancient solutions are constant.

We mention that in the papers [4] and [3], the authors proved Liouville theorem for ASNS under the critical assumption that $|v(x, t)| \leq C/|x'|$ where $v$ is the velocity, \( x = (x^1, x^2, x^3) \) and \( x' = (x^1, x^2, 0) \) is the Cartesian coordinates. See also an extension to the case that $v$ is in $BMO^{-1}$ in [5] and further improvements in [11].

Let $v$ be the velocity and $v_r$, $v_z$ and $v_\theta$ be the component of $v$ is the cylindrical coordinates of \( \{e_r, e_z, e_\theta\} \) respectively. Suppose $v$ is independent of the angle $\theta$, then ASNS takes the form of

\[
\begin{align*}
\left(\Delta - \frac{1}{r^2}\right) v_r - (b \cdot \nabla) v_r + \frac{v_r^2}{r} - \frac{\partial p}{\partial r} - \frac{\partial v_r}{\partial r} &= 0, \\
\left(\Delta - \frac{1}{r^2}\right) v_\theta - (b \cdot \nabla) v_\theta - \frac{\partial v_r v_\theta}{r} - \frac{\partial v_\theta}{\partial t} &= 0, \\
\Delta v_z - (b \cdot \nabla) v_z - \frac{\partial p}{\partial z} - \frac{\partial v_z}{\partial t} &= 0, \\
\frac{1}{r} \frac{\partial (rv_\theta)}{\partial r} + \frac{\partial v_\theta}{\partial z} &= 0,
\end{align*}
\]

where $b(x, t) = v_r e_r + v_z e_z$ and the last equation is the divergence-free condition. Here, $\Delta$ is the cylindrical scalar Laplacian and $\nabla$ is the cylindrical gradient field:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}, \quad \nabla = \left( \frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \theta}, \frac{\partial}{\partial z} \right).$$

Observe that the equation for $v_\theta$ does not depend on the pressure. Let $\Gamma = rv_\theta$, one sees that the function $\Gamma$ satisfies

$$\Delta \Gamma - (b \cdot \nabla) \Gamma - \frac{2}{r} \frac{\partial \Gamma}{\partial r} - \frac{\partial \Gamma}{\partial t} = 0, \quad \text{div} b = 0. \quad (1.2)$$

The first theorem of the paper is:

**Theorem 1.1.** Let $v = v_\theta e_\theta + v_r e_r + v_z e_z$ be a bounded (mild) ancient solution to the ASNS such that $\Gamma = rv_\theta$ is bounded. Then the following conclusions are true.

(a). Suppose $v$ is periodic in $z$ and $t$. Then $v$ is a constant.

(b). Suppose $v$ is periodic in $z$ and stationary in $t$. Then $v$ is a constant.

A remark is in order for the assumption that the function $\Gamma = rv_\theta$ is bounded. It is well known that $\Gamma$ satisfies the maximum principle. Hence if the initial condition of a
Cauchy problem satisfies the condition, then it will be satisfied for all time. Since $\Gamma$ is scaling invariant, ancient solutions arising from blow up procedure will also satisfy the condition, with perhaps a different constant due to shift of the axis during the blow up. So this condition is essentially not a restriction for the study of singularity. In a recent paper [1], periodic stationary solutions are treated under Leray’s D condition, namely $\int |\nabla v(x)|^2 dx < \infty$, with no assumption on $\Gamma$.

Actually we will also prove the following theorem, which includes Theorem 1.1 as a special case. Here, instead assuming $v$ is periodic in $t$, we assume certain convergence property of $\Gamma$ when $r \to \infty$. As shown in Lemma 2.3 below, this assumption is automatically true for solutions which are periodic in $z$ and $t$.

**Theorem 1.2.** Let $v = v_\theta e_\theta + v_r e_r + v_z e_z$ be a bounded (mild) ancient solution to the ASNS such that $\Gamma = rv_\theta$ is bounded. Suppose (a). $v$ is periodic in $z$. (b). There exist $\sigma_0 \in (0, 1)$ and a sequence $r_i \to \infty$ such that

$$|\limsup_{i \to \infty} \Gamma^2(r_i, z, t) - \limsup_{r \to \infty} \Gamma^2| \leq \sigma_0 \limsup_{r \to \infty} \Gamma^2$$

(1.3)

uniformly for $z$ and $t$. Then $v = ce_z$ where $c$ is a constant.

Here and later

$$\limsup_{r \to \infty} \Gamma = \limsup_{r \to \infty} \sup_{z,t} \Gamma(r, z, t).$$

(1.4)

It will be shown in Section 2 that if $v$ is any ancient solution such that $\Gamma$ is bounded, then $\limsup_{r \to \infty} \Gamma = \sup \Gamma$. Moreover, there exists a sequence $p_i \equiv (r_i, z_i, t_i)$ with $r_i \to \infty$ and $t_i \to -\infty$ such that $\Gamma$ converges to $\sup \Gamma$ in a fixed parabolic cube centered at $p_i$, in $C^{2,1,\alpha}$ topology.

For the definition of mild solutions, one can consult the paper [4]. Roughly speaking, these solutions satisfy certain integral equations involving the heat and Stokes kernel, ruling out the so called parasitic solutions such as $v = a(t)$ and $P = -a'(t) \cdot x$ where $a(t)$ is any smooth vector field depending only on time.

Next we present a theorem that deals with non-periodic case under an extra convergence condition on $\Gamma$. It can be regarded as a step forward in proving the full conjecture in [4]. Note that no decay condition is made on $v_r$ and $v_z$. See also a related result [7].

**Theorem 1.3.** Let $v = v_\theta e_\theta + v_r e_r + v_z e_z$ be a bounded (mild) ancient solution to the ASNS such that $\Gamma = rv_\theta$ is bounded. Then the following conclusions are true.

(a). Suppose, for a small number $\epsilon \in (0, 1)$, depending only on $\|v\|_\infty$, the following condition holds: when $r$ is large,

$$|\Gamma^2(r, z, t) - \limsup_{r \to \infty} \Gamma^2| \leq \frac{\epsilon}{r} \limsup_{r \to \infty} \Gamma^2$$

(1.5)

uniformly for $z$ and $t$. Then $v = ce_z$ where $c$ is a constant.

(b). Suppose $v_r \leq 0$ and, for some $\epsilon > 0$ depending only on $\|v\|_\infty$, and large $r$,

$$|\Gamma^2(r, z, t) - \limsup_{r \to \infty} \Gamma^2| \leq \epsilon \limsup_{r \to \infty} \Gamma^2$$

(1.6)

uniformly for $z$ and $t$. Then $v = ce_z$ where $c$ is a constant.
The rest of paper is organized as follows. In Section 2, we will prove a number of preliminary results on the behavior of the functions $v_r$ and $\Gamma = rv_\theta$ near infinity. Theorems 1.1 and 1.2 will be proved in Section 3. Theorem 1.3 will be proved in Section 4. We will apply a weighted energy method for the function $\Gamma = rv_\theta$, exploiting the special structure of the equation and the fact that $\Gamma = 0$ at the $z$ axis. It is well known that the usual energy method will run into the difficulty of insufficient decay of solutions. The new idea of the proof lies in the construction of a special weight function, part of which is constructed by hand and part of which is constructed by solving an auxiliary PDE. We will also need a new observation that for $z$ periodic ancient solutions, $v_r$ converges to 0 uniformly as $r \to \infty$.

As mentioned earlier, regardless of whether a singularity forms, solutions of a Cauchy problem in a high velocity region is approximated by an ancient solution. Perhaps it is helpful to compare the current result with another nonlinear parabolic system, namely the 3 dimensional Ricci flow. Perelman [8] showed that the typical model for high curvature region is $S^2 \times \mathbb{R}$ which is periodic in the $S^2$ part. In contrast, Theorem 1.1 shows periodic solutions are not models of high velocity region for ASNS.

2 Preliminaries

In this section, we present some necessary preliminary results. We will frequently use the following convergence result, which holds for general ancient solutions.

**Lemma 2.1 (Sliding Property).** Let $v$ be a bounded ancient mild solution to the axially-symmetric Navier-Stokes equations (ASNS) with $|\Gamma| \lesssim 1$. Let $(x_n, t_n)$ be any sequence with $x_n = (r_n, 0, z_n)$ such that $r_n \to \infty$. Then, up to a subsequence, $v$ uniformly converges to a constant vector on the parabolic cube $Q_R(x_n, t_n) = \{(x, t) \mid |x - x_n| < R, 0 < t_n - t < R^2\}$ for any given $R > 0$. Moreover, up to a further subsequence, $\Gamma$ uniformly converges to a constant on $Q_R(x_n, t_n)$.

**Proof.** The conclusion is known in the literature. See the proof of Theorem 1.1 in [6]. But for completeness, let’s give a simple proof here. Define

$$v^{(n)}(x, t) = v(x_n + x, t_n + t), \quad p^{(n)}(x, t) = p(x_n + x, t_n + t).$$

Here $p$ is the pressure. Clearly, $(v^{(n)}, p^{(n)})$ is still a bounded ancient mild solution to the Navier-Stokes equations and thus smooth. Hence, its weak limit $(v^{(\infty)}, p^{(\infty)})$ (up to a subsequence) is still a bounded ancient mild solution to the Navier-Stokes equations and thus smooth. Moreover, the convergence from $(v^{(n)}, p^{(n)})$ to its limit $(v^{(\infty)}, p^{(\infty)})$ is locally strong in $C^{2k, k}_{loc}$ for any $k \geq 0$.

Now consider any parabolic cube $Q_R = B_R(0) \times [-R^2, 0]$. It is clear that, up to subsequence, there holds

$$\left\{ \begin{array}{l}
e (x + x_n) \to e_1, \quad e_\theta (x + x_n) \to e_2, \\
v^{(n)}(x, t) \cdot e_\theta (x + x_n) = \frac{\sqrt{y_1^2 + y_2^2} e_\theta (y, t_n + t) - e_\theta (y)}{\sqrt{y_1^2 + y_2^2}} \bigg|_{y = x_n + x} \to 0, \quad (x, t) \in Q_R, \end{array} \right.$$
where $y = x_n + x$, $e_1 = (1, 0, 0)^T$ and $e_1 = (0, 1, 0)^T$. Hence, we have

$$v^{(∞)} \cdot e_2 = 0 \quad \text{on } Q_R.$$ 

Write $x = x_1 e_1 + x_2 e_2 + x_3 e_z$ with $e_z = (0, 0, 1)^T$. Due to

$$\partial_{x_2} v^{(n)}(x, t) = \partial_{x_2} v(x + x, t_n + t) = \partial_{y_2} v(y, t_n + t)$$

$$= \frac{y_2}{r(y)} \partial_r v(y, t_n + t) + \frac{y_1}{r(y)} \partial_\theta v(y, t_n + t)$$

$$= \frac{y_2}{r(y)} \partial_r v(y, t_n + t) + \frac{y_1}{r(y)} v_r(y, t_n + t) e_\theta(y)$$

$$- \frac{y_1}{r(y)} r(y) v_r(y, t_n + t) e_r(y) \to 0 \quad \text{on } Q_R,$$

we conclude that $v^{(∞)}$ is independent of $x_2$ on $Q_R$. Here $r(y)$ is the distance from $y$ to the $z$ axis. Since $R$ is arbitrary, we can conclude that

$$(v^{(∞)} \cdot e_1, v^{(∞)} \cdot e_z)$$

is a bounded ancient mild solution to the two-dimensional Navier-Stokes equations. Using the Liouville theorem in [4], we see that $v^{(∞)}$ must be a constant vector (independent of time). This proves the convergence for $v$.

Next we turn to $Γ$. Define

$$Γ^{(n)}(x, t) = Γ(x_n + x, t_n + t).$$

By using the smoothness of $v$ and the fact that $|Γ| \lesssim 1$, it is easy to derive that $Γ$ is smooth. Hence, up to a further subsequence, one has

$$Γ^{(n)} \to Γ^{(∞)}, \quad |Γ^{(∞)}| \lesssim 1,$$

and the convergence is locally strong in $C^{k,2k}_{loc}$ for any $k \geq 0$. Moreover, one can similarly derive that $Γ^{(∞)}$ is independent of $x_2$. From the equation

$$\partial_t Γ + b \cdot ∇Γ + \frac{1}{r} ∂_r Γ = (∂^2_r + ∂^2_\theta)Γ,$$  

we deduce

$$\partial_t Γ^{(n)} + v^{(n)} \cdot ∇Γ^{(n)} + \frac{y}{r^2} \bigg|_{y=x_n+x} \cdot (\partial_1, \partial_2, 0)^T Γ^{(n)} = \Delta Γ^{(n)},$$

one sees that

$$\partial_t Γ^{(∞)} + v^{(∞)} \cdot ∇Γ^{(∞)} = (∂^2_1 + ∂^2_2)Γ^{(∞)}.$$ 

Note that $v^{∞}$ is a constant vector. So one can convert the above equation into the standard heat equation by a change of variable. The standard Liouville theorem for the heat equation implies that $Γ^{(∞)}$ must be a constant. We have proved the lemma. □
Next we prove the following lemma concerning periodic ancient solutions in $z$ only, which shows that the radial velocity $v_r$ converges to 0 when $r \to \infty$ uniformly, in $z$ and $t$.

**Lemma 2.2.** Let $v$ be a bounded ancient solution to the ASNS. Then the following is true.

- if $v$ is mild and periodic in $z$ only, then $v_r \to 0$ uniformly for $(z, t) \in \mathbb{T}^1 \times (-\infty, 0)$ as $r \to \infty$.

- if $v$ is steady and periodic in $z$, then $v_r \to 0$ uniformly in $z \in \mathbb{T}^1$ as $r \to \infty$. Here $\mathbb{T}^1$ is the unit circle and we assume the period in $z$ is $2\pi$ for simplicity.

**Proof.** We will only present the proof for the time-dependent case, since the steady case is similar and easier. We use the argument by contradiction. Suppose the conclusion is not true. Then there exists $c_0 \neq 0$ and a sequence of $(r_n, z_n, t_n)$ with $t_n$ such that

$$\lim_{n \to \infty} v_r(P_n) = c_0, \quad P_n = (r_n, 0, z_n, t_n).$$

Here $0$ is the angle in the cylindrical system.

Pick $R$ large enough such that $\mathbb{T}^1 \subset \{|z - z_n| \leq \frac{R}{2}\}$. By using the Sliding property in Lemma 2.1 one sees that the solution $v$ converges to a constant on the parabolic ball $Q_R(P_n)$ with radius $R$, centering at $P_n$. This means that

$$|v_r| \geq \frac{|c_0|}{2} \quad \text{on} \ Q_R(P_n)$$

as $n$ is large enough.

Now let $L_\theta$ be the angular stream function which solves$^1$

$$\nabla \times (L_\theta e_\theta) = v_r e_r + v_z e_z.$$  

Clearly, $\frac{L_\theta}{r}$ is a smooth function which is locally bounded. Moreover, one has

$$v_r = -\partial_z L_\theta.$$  

Note that for each $(r, t)$, $L_\theta$ is a smooth function in $z \in \mathbb{T}^1$. Hence, there exists $z = z(r, t)$ such that $L_\theta(r, z(r, t), t)$ attains its maximal value and

$$v_r(r, z(r, t), t) = 0.$$  

Hence, one also has

$$v_r(\widetilde{P}_n) = 0, \quad \text{for some point} \quad \widetilde{P}_n = (\tilde{r}_n, 0, z(\tilde{r}_n, t_n), t_n) \in Q_R(P_n).$$

By (2.2) and (2.3), we arrive at a contradiction. Hence the conclusion of the lemma is true.

$^1$One can truncate the right hand side so that it is smooth and divergence-free. Then $L_\theta/r$ is well-defined and is bounded locally.
The next lemma states that, in the periodic case, as \( r \to \infty \), \( \Gamma = \Gamma(r, z, t) \) converges uniformly to one constant.

**Lemma 2.3.** Let \( v \) be a bounded ancient solution to the ASNS. Then the following conclusions are true.

- if \( v \) is mild and periodic in \( z \) and \( t \): \( z \in \mathbb{T}^1 = [0, 2\pi] \) and \( t \in \beta\mathbb{T}^1, \beta > 0 \). Then there exists a constant \( c \) such that \( \Gamma \to c \) uniformly for \( (z, t) \in \mathbb{T}^1 \times \beta\mathbb{T}^1 \) as \( r \to \infty \).

- if \( v \) is steady and periodic in \( z \), then there exists a constant \( c \) such that \( \Gamma \to c \) uniformly for \( z \in \mathbb{T}^1 \).

**Proof.** Again we just give a proof for the time dependent case. Pick a sequence \( r_i \to 0 \).

By Lemma 2.1 using the fact that the equations for \( v_r \) and \( v_z \) in ASNS reduce to a 2 dimensional NS of variables \( r, z \) at infinity and the 2 dimensional Liouville theorem, we can find a subsequence, still denoted by \( r_i \) with the following property. The sequence \( \Gamma(r_i, z, t) \) converges uniformly to a constant \( c_0 \). We now prove

\[
\lim_{r \to \infty} \Gamma(r, z, t) = c_0 \quad (2.4)
\]

uniformly. Since \( \lim_{i \to \infty} \Gamma(r_i, z, t) = c_0 \), for any \( \epsilon > 0 \), there exists integer \( N > 0 \), if \( i \geq N \), then \( |\Gamma(r_i, z, t) - c_0| < \epsilon \) for all \( z, t \). Pick any \( r > r_N \). Then there exists integers \( i, j > N \) such that \( r_i \leq r \leq r_j \). In the domain \([r_i, r_j] \times \mathbb{T}^1 \times \beta\mathbb{T}^1 \) for \( (r, z, t) \), the function \( \Gamma \) satisfies the parabolic maximum principle. Hence

\[
c_0 - \epsilon \leq \Gamma(r, z, t) \leq c_0 + \epsilon.
\]

This proves (2.4).

³ Proof of Theorems 1.1 and 1.2

In this section we prove Theorems 1.1 and 1.2. Observe that Theorem 1.1 is an immediate consequence of Theorem 1.2 and Lemma 2.3. So we just need to give a

**Proof of Theorem 1.2.** We will show that

\[
\lim_{r \to \infty} \Gamma(r, z, t) = 0 \quad (3.1)
\]

uniformly in \( z, t \). Then, by the paper [7], \( v = ce_z \). We use the method of contradiction.

The rest of the proof is divided into a few steps.

**Step 1.**

According to the maximum principle, the function \( \sup_{r>0,z} |\Gamma(r, z, t)| \) is non-increasing in time. Suppose (3.1) is false. Then

\[
\limsup_{r \to \infty, z, t \to \infty} |\Gamma| = c_0 \neq 0
\]
for some constant $c_0$. Here $t_\infty$ is either $-\infty$ or a finite negative number. We will eventually reach a contradiction. We can assume, without loss of generality, that $c_0 = 1$, otherwise we can multiply $\Gamma$ by a suitable constant.

First, we make the observation that

$$|\Gamma| \leq 1. \tag{3.2}$$

The reason is that

$$\limsup_{r \to \infty, z, t \to t_\infty} |\Gamma| = \sup_{r, z, t} |\Gamma|.$$ 

Otherwise there would be a bounded sequence $\{r_i\}$, $z_i$ and $t_i \to t_\infty$ such that

$$\lim_{i \to \infty} |\Gamma(r_i, z_i, t_i)| = \sup_{r, z, t} |\Gamma|.$$ 

Consider the translated sequence

$$\Gamma_i = \Gamma_i(r, z, t) = \Gamma(r, z, t + t_i).$$

Then we can find a subsequence of $\Gamma_i$ which converges, in $C^{2,1}_{loc}$ topology, to $\Gamma_\infty$ which is a bounded ancient solution of

$$\Delta \Gamma_\infty - \bar{b} \nabla \Gamma_\infty - \frac{2}{r} \partial_r \Gamma_\infty - \partial_t \Gamma_\infty = 0.$$ 

Here $\bar{b}$ is a bounded $C^{2,1}$ vector field. We can suppose that $r_i \to r_\infty < \infty$ and $z_i \to z_\infty$. Then $\Gamma_\infty$ reaches nonzero interior maximum away from the $z$ axis at the point $(r_\infty, z_\infty, 0)$. Hence $\Gamma_\infty$ is a nonzero constant by the maximum principle. This contradicts with the fact that $\Gamma_\infty = 0$ at the $z$ axis. This proves (3.2).

So now we can assume, for sequences $r_i \to \infty$, $z_i$ and $t_i \to t_\infty$ that

$$\lim_{i \to \infty} \Gamma(r_i, z_i, t_i) = 1 = \sup_{r, z, t} \Gamma.$$ 

For convenience, we consider the solution after translations in $z, t$, i.e. consider

$$v_i = v(r, z_i + z, t_i + t)$$

and

$$\Gamma_i = \Gamma(r, z_i + z, t_i + t).$$

For simplicity of presentation, we will drop the index $i$ for $\Gamma_i$ and $v_i$, unless stated otherwise.

Here we remark that, for any fixed large number $T$, by the sliding method in Lemma 2.1 we can prove that

$$\lim_{i \to \infty} \Gamma_i(r, z, t) = 1 \tag{3.3}$$

uniformly for $r, z, t$ such that $|r - r_i| \leq Z_0, |z| \leq Z_0$ and $t \in [-T, 0]$. However we can not yet prove this fact for all $t$, i.e. $T = -\infty$. Note that this argument works for any bounded ancient solution with $\Gamma$ being bounded.
Step 2.

Given three large positive numbers $r_0, R_0, R$, in increasing order, let $\lambda = \lambda(r)$ be the piecewise linear function

$$
\lambda = \begin{cases} 
  r, & r \in [0, r_0]; \\
  r_0 - \frac{r - r_0}{R_0 - r_0} (r - r_0), & r \in [r_0, R_0]; \\
  \delta, & r \geq R_0.
\end{cases} 
$$

(3.4)

Here, $\delta \in (0, 1]$ is a small positive number to be chosen later.

Eventually we will take $r_0 = r_i$ and $R_0 = r_{i+1}$ where $r_i$ is given in the statement of the theorem. Hence by the assumption of the theorem, we have

$$
|1 - \Gamma^2(r_0, z, t)| + |1 - \Gamma^2(R_0, z, t)| \leq \sigma_0 < 1, \quad (z, t) \in [-Z_0, Z_0] \times [-T, 0],
$$

(3.5)

when $i \to \infty$. We may also assume that $R_0 >> r_0$ by choosing $R_0 = r_{i+j}$ for a large $j$ if necessary.

For convenience, we write the equation for $\Gamma$ as a 2 dimensional one in terms of the variables $r, z$, namely

$$
\partial_r^2 \Gamma + \partial_z^2 \Gamma - v_r \partial_r \Gamma - v_z \partial_z \Gamma - \frac{1}{r} \partial_r \Gamma - \partial_t \Gamma = 0.
$$

(3.6)

We will use certain test function and the weighted volume element $d\mu = \lambda(r)drdz$ to test against the above equation. The following are the details. The calculation below is for general weight function $\lambda$ and is not restricted to periodic solutions.

Consider the domain

$$D = D_1 \cup D_2,$$

where

$$D_1 = \{(r, z, t) \mid 0 \leq r < R_0; -R \leq z \leq R, -T \leq t \leq 0\},$$

$$D_2 = \{(r, z, t) \mid R_0 \leq r \leq R; -R \leq z \leq R, -T \leq t \leq 0\}.$$  

Eventually, we take $R$ to be large integer multiples of the period $Z_0$.

Now we define two functions whose domains are $D_1$ and $D_2$ respectively. Let

$$\phi_1 = \frac{t}{-T},$$

defined on $D_1$. Let $\phi_2 = \phi_2(r, z, t)$ be the unique solution to the final time boundary value problem of the backward equation, which is well-posed.

$$
\begin{align*}
\partial_r^2 \phi_2 + \partial_z^2 \phi_2 + \frac{2\lambda'(r)}{\lambda(r)} \partial_r \phi_2 + v_r \partial_r \phi_2 + v_z \partial_z \phi_2 &+ \frac{1}{2} \left( \frac{\lambda'(r)}{\lambda(r)} - \frac{2}{r} \right) v_r + \left( \frac{\lambda'(r)}{r} \right)' \frac{1}{\lambda(r)} \phi_2 - A\phi_2 - \partial_t \phi_2 = 0, \quad (r, z) \in D_2, t \in [-T, 0]; \\
\phi_2(R_0, z, t) = \phi_1(z, t), \quad \phi_2(R, z, t) = 0, t \in [-T, 0]; \\
\phi_2 \text{ has period } Z_0 \text{ in } z; \\
\phi_2(r, z, 0) = 0, \quad (r, z) \in D_2.
\end{align*}
$$

(3.7)
Here $Z_0$ is the period of the ancient solution and $A$ is a positive constant to be chosen next. Since $\lambda = \delta$ in $D_2$, the above equation simplifies to

$$
\begin{aligned}
\begin{cases}
\partial_t^2 \phi_2 + \partial_z^2 \phi_2 + v_r \partial_r \phi_2 + v_z \partial_z \phi_2 \\
-\frac{1}{2}(\frac{1}{r}v_r + \frac{1}{r^2})\phi_2 - A\phi_2 + \partial_t \phi_2 = 0, \\
\phi_2(R_0, z, t) = \phi_1(z, t), \\
\phi_2(r, z, 0) = 0,
\end{cases}
\end{aligned}
$$

(3.8)

Note the equation is independent of $\delta$. Now we choose

$$A = \frac{1}{2R_0} \| v_r \|_\infty.$$

Since $r \geq R_0 > 1$, the coefficient of $\phi_2$ in the equation is non-positive. The standard parabolic equation theory tells us that the above problem has a unique solution such that $0 \leq \phi_2 \leq 1$ and $\| \nabla \phi_2 \|_\infty \leq A_0$. Here $A_0$ is a constant depending only on $C^1$ norm of $b$. Indeed the bound on $\phi_2$ is a consequence of the maximum principle and the gradient bound follows from Bernstein method in the interior and reflection method at the boundary.

Now we compute

$$
- \int_{D_1} (\partial_r^2 \Gamma + \partial_z^2 \Gamma) \Gamma \phi_1^2 d\mu dt
= - \int_{D_1} (\partial_r^2 \Gamma + \partial_z^2 \Gamma) \Gamma \phi_1^2 \lambda(r) dr dz dt
= \int_{D_1} (|\partial_r \Gamma|^2 + |\partial_z \Gamma|^2) \phi_1^2 d\mu dt + \int_{-T}^0 \int_{-R}^R \int_0^{R_0} (\partial_r \Gamma) \Gamma \phi_1^2(z, t) \lambda(r) dr dz dt
+ \int_{-T}^0 \int_{-R}^R \int_0^{R_0} (\partial_z \Gamma) \Gamma 2\phi_1(z, t) \partial_z \phi_1(z, t) \lambda(r) dr dz dt - \int_{-T}^0 \int_{-R}^R (\partial_r \Gamma) \Gamma \phi_1^2(z, t) \lambda(r) \big|_{r=R_0} d\mu dt.
$$

Similarly

$$
- \int_{D_2} (\partial_r^2 \Gamma + \partial_z^2 \Gamma) \Gamma \phi_2^2 d\mu dt
= \int_{D_2} (|\partial_r \Gamma|^2 + |\partial_z \Gamma|^2) \phi_2^2 d\mu dt + \int_{-T}^0 \int_{-R}^R \int_{R_0}^R (\partial_r \Gamma) \Gamma \phi_2^2 \lambda(r) dr dz dt
+ \int_{-T}^0 \int_{-R}^R \int_{R_0}^R (\partial_z \Gamma) \Gamma 2\phi_2 \partial_z \phi_2 \lambda(r) dr dz dt + \int_{-T}^0 \int_{-R}^R \int_{R_0}^R (\partial_z \Gamma) \Gamma 2\phi_2 \phi_2 \lambda(r) dr dz dt
+ \int_{-T}^0 \int_{-R}^R (\partial_r \Gamma) \Gamma \phi_2^2 \lambda(r) \big|_{r=R_0} d\mu dt.
$$
Adding the previous two identities and noting the last two boundary terms cancel and also $\lambda' = 0$ when $r > R_0$, we obtain

\[ L \equiv \int_{D_1} (|\partial_r \Gamma|^2 + |\partial_z \Gamma|^2) \phi_1^2 d\mu dt + \int_{D_2} (|\partial_r \Gamma|^2 + |\partial_z \Gamma|^2) \phi_2^2 d\mu dt \]

\[ = - \int_{D_1} (\partial_r^2 \Gamma + \partial_z^2 \Gamma) \Gamma \phi_1^2 d\mu dt \]

\[ - \int_{D_1}^0 \int_{-R}^R \int_{-R_0}^{R_0} (\partial_r \Gamma) \Gamma 2 \phi_1(z) \partial_z \phi_1(z) \lambda(r) dr dz dt \]

\[ - \int_{D_2} (\partial_r^2 \Gamma + \partial_z^2 \Gamma) \Gamma \phi_2^2 d\mu dt \]

\[ - \int_{D_2}^0 \int_{-R}^R \int_{-R_0}^{R_0} (\partial_z \Gamma) \Gamma 2 \phi_2 \partial_z \phi_2 \lambda(r) dr dz dt \]

\[ - \int_{D_2} (\partial_r^2 \Gamma + \partial_z^2 \Gamma) \Gamma \phi_2^2 d\mu dt \]

\[ \equiv T_1 + \ldots + T_6. \tag{3.9} \]

In the rest of the proof we will find an upper bound for each $T_i$, $i = 1, \ldots, 6$.

**Step 3. Bound for $T_1$.**

From equation (3.6),

\[ T_1 = - \int_{-T}^0 \int_{-R} R \int_{-R_0}^{R_0} (v_r \partial_r \Gamma + v_z \partial_z \Gamma + \frac{1}{r} \partial_r \Gamma + \partial_t \Gamma) \Gamma \phi_1^2 \lambda(r) dr dz dt. \]

After integration by parts and using the divergence free property of $v_r e_r + v_z e_z$, we
Hence, we can deduce, after leaving $T_{12}$ direct computation shows, since $\phi_1(z, t)\lambda(r) dr dz dt$
\begin{align*}
T_1 &= -\frac{1}{2} \int_{-T}^{0} \int_{-R}^{0} \int_{0}^{R_0} \left[ v_r \partial_r (\Gamma^2 - 1) + v_z \partial_z (\Gamma^2 - 1) + \frac{1}{r} \partial_r (\Gamma^2 - 1) \right] \phi_1^2 (z, t) \lambda(r) dr dz dt \\
&= \frac{1}{2} \int_{-T}^{0} \int_{-R}^{0} \int_{0}^{R_0} (v_r \partial_r \phi_1^2 + v_z \partial_z \phi_1^2) (\Gamma^2 - 1) \lambda(r) dr dz dt \\
&\quad - \frac{1}{2} \int_{-T}^{0} \int_{-R}^{0} v_r (\Gamma^2 - 1) \phi_1^2 \lambda(r) \bigg|_{r=R_0} dz dt \\
&\quad + \frac{1}{2} \int_{-T}^{0} \int_{-R}^{0} \int_{0}^{R_0} v_r (\Gamma^2 - 1) \phi_1^2 \left( \lambda'(r) - \frac{\lambda(r)}{r} \right) dr dz dt \\
&\quad + \frac{1}{2} \int_{-T}^{0} \int_{-R}^{0} \int_{0}^{R_0} \Gamma^2 \phi_1^2 \left( \frac{\lambda(r)}{r} \right)' dr dz dt \\
&\quad - \frac{1}{2} \int_{-T}^{0} \int_{-R}^{0} \Gamma^2 \phi_1^2 \lambda(r) \bigg|_{r=R_0} dz dt \\
&\quad + \frac{1}{2} \int_{-T}^{0} \int_{-R}^{0} \int_{0}^{R_0} (\Gamma^2 - 1) \partial_t \phi_1^2 \lambda(r) dr dz dt \\
&\quad + \left( \frac{1}{2} \right) \int_{-R}^{0} \int_{0}^{R_0} (\Gamma^2 - 1) \phi_1^2 \bigg|_{t=-T}^{t=0} \lambda(r) dr dz \\
&\equiv T_{11} + \ldots + T_{17}.
\end{align*}

Notice that $\phi_1$ depends only on $t$. Therefore
\[ T_{11} = 0. \]

The term $T_{12}$ is a boundary ones which will be cancelled with a boundary term from integration on $D_2$, which is called $T_{42}$. $T_{14} \leq 0$ and $T_{15} \leq 0$. Also, since $\partial_t \phi_1 = -1/T$ and
\[ \int_{0}^{R_0} \lambda(r) dr = \delta (R_0 - r_0) + \frac{1}{2} R_0 r_0, \]
direct computation shows, since $\delta \in (0, 1]$ that
\[ T_{16} \leq \frac{1}{2} \| \Gamma^2 - 1 \|_\infty (R_0 r_0 + R_0 - r_0). \]

Similarly
\[ T_{17} \leq -\frac{1}{2} \inf [1 - \Gamma^2 (R_0, \cdot, -T)] R (R_0 r_0 + R_0 - r_0) \leq 0. \]

Hence, we can deduce, after leaving $T_{12}$ and $T_{13}$ alone for now, that
\begin{align*}
T_1 &\leq \frac{1}{2} \left( \| \Gamma^2 - 1 \|_\infty - \inf [1 - \Gamma^2 (R_0, \cdot, -T)] \right) R (R_0 r_0 + R_0 - r_0) + T_{12} \\
&\quad + \frac{1}{2} \int_{-T}^{0} \int_{-R}^{0} \int_{0}^{R_0} v_r (\Gamma^2 - 1) \phi_1^2 \left( \lambda'(r) - \frac{\lambda(r)}{r} \right) dr dz dt.
\end{align*}

(3.10)
Step 4. bounds for $T_2, ..., T_7$

First we bound $T_2$. By our choice of $\lambda$,

$$T_2 = -\frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{0}^{R_0} \partial_r (\Gamma^2 - 1) \phi_1^3 \lambda(r) dr dz dt$$

$$= -\frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{0}^{R_0} \partial_r (\Gamma^2 - 1) dr dz dt + \frac{r_0 - \delta}{2(R_0 - r_0)} \int_{-T}^{0} \int_{-R}^{R} \int_{r_0}^{R_0} \partial_r (\Gamma^2 - 1) dr dz dt$$

$$\leq \left[ -\frac{1}{3} + \left( \frac{1}{3} + \frac{2}{3(R_0 - r_0)} \right) \sup_{r = r_0, t \in [-T, 0]} (1 - \Gamma^2) \right] RT.$$  

(3.14)

Next

$$T_3 = 0$$  

(3.15)

since $\partial_z \phi_1 = 0$.

Now we deal with the terms $T_4, ..., T_7$ which involve integrations on $D_2$ only. From equation (3.6),

$$T_4 = -\int_{-T}^{0} \int_{-R}^{R} \int_{0}^{R_0} (v_r \partial_r \Gamma + v_z \partial_z \Gamma + \frac{1}{r} \partial_r \Gamma + \partial_t \Gamma) \phi_2^2 \lambda(r) dr dz dt.$$  

After integration by parts and using the divergence free property of $v_r e_r + v_z e_z$, we deduce

$$T_4 = -\frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{0}^{R_0} \left[ v_r \partial_r (\Gamma^2 - 1) + v_z \partial_z (\Gamma^2 - 1) + \frac{1}{r} \partial_r (\Gamma^2 - 1) + \partial_t (\Gamma^2 - 1) \right] \phi_2^2 \lambda(r) dr dz dt$$

$$= \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{0}^{R_0} (v_r \partial_r \phi_2^2 + v_z \partial_z \phi_2^2) (\Gamma^2 - 1) \lambda(r) dr dz dt$$

$$+ \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \left. v_r (\Gamma^2 - 1) \phi_2^2 \lambda(r) \right|_{r = R_0} dz$$

$$+ \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{0}^{R_0} \left. v_r (\Gamma^2 - 1) \phi_2^2 \left( \frac{\lambda(r)}{r} - \frac{\lambda'(r)}{r^2} \right) \right|_{r = R_0} dr dz dt$$

$$+ \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{0}^{R_0} (\Gamma^2 - 1) \phi_2^2 \left( \frac{\lambda'(r)}{r} \right)' dr dz dt$$

$$+ \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{0}^{R_0} (\Gamma^2 - 1) \phi_2^2 \lambda(r) dr dz dt$$

$$\equiv T_{41} + \ldots + T_{47}.$$  

(3.16)
Notice that \( T_{42} \) will cancel with \( T_{12} \) when all terms are added. Also \( T_{45} \leq 0 \) and

\[
T_{47} \leq -\inf_{r \in [R_0, R]} (1 - \Gamma^2(r, \cdot, -T)) \int_{-R}^{R} \int_{R_0}^{R} \phi_2^2(r, z, -T) \lambda(r) dr dz \leq 0. \tag{3.17}
\]

Therefore

\[
T_4 \leq T_{41} + T_{42} + T_{43} + T_{44} + T_{46}.
\]

Next

\[
T_5 = -\frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{-R_0}^{R} \partial_r (\Gamma^2 - 1) \partial_r \phi_2^2 \lambda(r) dr dz dt
\]

\[
= \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{-R_0}^{R} (\Gamma^2 - 1) \partial_r^2 \phi_2^2 \lambda(r) dr dz dt - \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} (\Gamma^2 - 1) \partial_r \phi_2^2 \lambda(r) \bigg|_{R_0}^{R} \, dz dt,
\]

since \( \lambda(r) = \delta \) is a constant here. Finally

\[
T_6 = \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{-R_0}^{R} (\Gamma^2 - 1) \partial_x^2 \phi_2^2 \lambda(r) dr dz dt. \tag{3.18}
\]

Combining the bounds on \( T_i, \, i = 1, ..., 6 \), noticing cancellation of boundary terms \( T_{12} \) with \( T_{42} \), we find that

\[
L \leq \left[ -\frac{1}{3} + \left( \frac{1}{3} + \frac{r_0 - 1}{2(R_0 - r_0)} \right) \sup_{r = r_0, t \in [-T, 0]} (1 - \Gamma^2) \right] RT + \| \Gamma^2 - 1 \|_{\infty} R R_0 r_0
\]

\[
+ \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{-R_0}^{R} v_r (\Gamma^2 - 1) \phi_2^2 (z) \left( \lambda'(r) - \frac{\lambda(r)}{r} \right) \, dr dz dt
\]

\[
- \inf_{r \in [R_0, R]} (1 - \Gamma^2(r, \cdot, -T)) \int_{-R}^{R} \int_{R_0}^{R} \phi_2^2(r, z, -T) dr dz
\]

\[
+ \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{-R_0}^{R} \left[ \partial_r^2 \phi_2^2 + \partial_z^2 \phi_2^2 + v_r \partial_r \phi_2^2 + v_z \partial_z \phi_2^2 + 2 \frac{\lambda'(r)}{\lambda(r)} \partial_r \phi_2^2 \right.
\]

\[
+ v_r \left( \frac{\lambda'(r)}{\lambda(r)} - \frac{1}{r} \right) \phi_2^2 + \left( \frac{\lambda(r)}{r} \right)' \frac{1}{\lambda(r)} \phi_2^2 + \partial_t \phi_2^2 \left( \Gamma^2 - 1 \right) \lambda(r) dr dz dt
\]

\[
+ \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \left( \Gamma^2 - 1 \right) \partial_r \phi_2^2 \lambda(r) \bigg|_{r = R_0} \, dz dt.
\]

Since \( \Gamma^2 - 1 \leq 0 \) and \( \phi_2 \) is a solution to (3.8), the second from last integral in the above
inequality is nonpositive. Hence

\[
L \leq \left[ -\frac{1}{3} + \left( \frac{1}{3} + \frac{r_0 - 1}{2(R_0 - r_0)} \right) \sup_{r = r_0, t \in [-T, 0]} (1 - \Gamma^2) \right] RT \\
+ \frac{1}{2} \left( \|\Gamma^2 - 1\|_\infty - \inf[1 - \Gamma^2(R_0, \cdot, -T)] \right) R(R_0 r_0 + R_0 - r_0) \\
- \inf_{r \in [R_0, R]} (1 - \Gamma^2(r, \cdot, -T)) \int_{-R}^{R} \int_{-R}^{R} \phi_2^2(r, z, -T) dr dz \\
+ \frac{1}{2} \left[ \int_{-T}^{0} \int_{-R}^{R} \int_{-R}^{R_0} \left( \partial_r \phi_2^2 \lambda(r) \right) \left( \lambda'(r) - \frac{\lambda(r)}{r} \right) dr dz dt \right] \\
+ \frac{1}{2} \left[ \int_{-T}^{0} \int_{-R}^{R} \int_{-R}^{R_0} (\Gamma^2 - 1) \partial_r \phi_2^2 \lambda(r) |_{r = R_0} dz dt \right] \\
= \ldots + I_1 + I_2.
\]  

Here the last two integrals are denoted by \( I_1, I_2 \) respectively.

**Step 5.**

It remains to bound the two integrals \( I_1 \) and \( I_2 \).

First let us bound \( I_2 \). Observe that the coefficients of lower order terms of the equation (3.20) are bounded by \( \frac{1}{R_0} + \|v_r\|_\infty + \|v_z\|_\infty \) in \( D_2 \) and the boundary value of \( \phi_2 \) at \( r = R_0 \) is \( \phi_1 = -t/T \) which satisfies \( 0 \leq \phi_1 \leq 1 \) and \( |\partial_z \phi_1| = 0 \leq C/R \) and \( |\partial_t \phi_1| = 1/T \). By standard boundary gradient bound for parabolic equations, we know that

\[
|\partial_r \phi_2|_{r = R_0} \leq C_1, \quad C_1 = C_1(\|v_r\|_\infty, \|v_z\|_\infty).
\]

Hence, using \( \lambda = \delta \) here, we find:

\[
I_2 \leq C_1 \sup_{r = R_0, t \in [-T, 0]} (1 - \Gamma^2) RT \delta.
\]  

Finally

\[
I_1 = \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{-R}^{R_0} v_r (\Gamma^2 - 1) \phi_1^2(z, t) \left( \lambda'(r) - \frac{\lambda(r)}{r} \right) dr dz dt \\
= -\frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{-R}^{R_0} \partial_z (L_\theta(r, z) - L_\theta(r, 0)) (\Gamma^2 - 1) \phi_1^2(z, t) \left( \lambda'(r) - \frac{\lambda(r)}{r} \right) dr dz dt
\]

where \( L_\theta \) is the angular stream function. If one attempts to estimate \( I_1 \) in the first line of the above expression, then one would not be able to obtain the required bound. In the following we will use periodicity essentially to obtain better estimate.
After integration by parts, this becomes
\[
I_1 = \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{r_0}^{R_0} \nu_r (\Gamma^2 - 1) \phi_1^2(z,t) \left( \lambda'(r) - \frac{\lambda(r)}{r} \right) drdzdt \\
= \int_{-T}^{0} \int_{-R}^{R} \int_{r_0}^{R_0} (L_\theta(r,z) - L_\theta(r,0)) (\partial_z \Gamma) (\Gamma) \phi_1^2(z,t) \left( \frac{\lambda'(r)}{r} - \frac{\lambda(r)}{r} \right) drdzdt \\
+ \int_{-T}^{0} \int_{-R}^{R} \int_{r_0}^{R_0} (L_\theta(r,z) - L_\theta(r,0)) (\Gamma^2 - 1) \partial_z \phi_1(z,t) \phi_1(z,t) \left( \frac{\lambda'(r)}{r} - \frac{\lambda(r)}{r} \right) drdzdt.
\]

The last term is 0 since \( \partial_z \phi_1 = 0 \). Since \( L_\theta \) is periodic with period \( \Gamma_\theta \), we know, for all \( z \in [-R, R] \),
\[
|L_\theta(r,z) - L_\theta(r,0)| \leq \sup_{r \in [r_0, R_0]} |v_r(r, \cdot)| Z_0 = Z_0 \sup_{r \in [r_0, R_0]} |v_r| = o(1),
\]
where we have used the choice \( R = NZ_0 \) for a large positive integer \( N \), and also Lemma [2.2]. Therefore
\[
I_1 \leq \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{r_0}^{R_0} |\partial_z \Gamma|^2 \phi_1^2 \lambda(r) drdzdt \\
+ 2 \int_{-T}^{0} \int_{-R}^{R} \int_{r_0}^{R_0} (L_\theta(r,z) - L_\theta(r,0))^2 \Gamma^2 \phi_1^2 \left( \frac{\lambda'(r)}{r} - \frac{\lambda(r)}{r} \right)^2 \frac{1}{\lambda(r)} drdzdt \\
\leq \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{r_0}^{R_0} |\partial_z \Gamma|^2 \phi_1^2 \lambda(r) drdzdt \\
+ R \int_{-T}^{0} \int_{r_0}^{R_0} \left( \frac{\lambda'(r)}{r} - \frac{\lambda(r)}{r} \right)^2 \frac{1}{\lambda(r)} drdt Z_0 \sup_{r \in [r_0, R_0]} |v_r|.
\]

Recall from [3.4] that, for \( r \in [r_0, R_0] \),
\[
\left( \frac{\lambda'(r)}{r} - \frac{\lambda(r)}{r} \right)^2 \frac{1}{\lambda(r)} = \frac{r_0^2 (R_0 - \delta)^2}{r^2 (R_0 - r_0) (R_0 - r_0) - (r_0 - \delta) (r - r_0)}.
\]

By direct computation
\[
\int_{r_0}^{R_0} \left( \frac{\lambda'(r)}{r} - \frac{\lambda(r)}{r} \right)^2 \frac{1}{\lambda(r)} dr \\
= \int_{r_0}^{R_0+\delta/2} \frac{1}{\lambda(r)} dr + \int_{(R_0+\delta/2)}^{R_0} \frac{1}{\lambda(r)} dr \leq \frac{C}{\delta},
\]
if \( R_0 \geq r_0 \ln R_0 \). Hence
\[
I_1 \leq \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{r_0}^{R_0} |\partial_z \Gamma|^2 \phi_1^2(z,t) \lambda(r) drdz + \delta^{-1} RT Z_0 \sup_{r \in [r_0, R_0]} |v_r|. \tag{3.22}
\]
Now we substitute, (3.21) and (3.22) into (3.20) to obtain

\[
L \leq \frac{1}{2} \int_{-T}^{0} \int_{D_1} (|\partial_r \Gamma|^2 + |\partial_z \Gamma|^2) \phi_1^2 d\mu + \int_{-T}^{0} \int_{D_2} (|\partial_r \Gamma|^2 + |\partial_z \Gamma|^2) \phi_2^2 d\mu \\
= -\frac{1}{3} RT + \left( \frac{1}{3} + \frac{r_0 - 1}{2(R_0 - r_0)} \right) \sup_{r = r_0, t \in [-T, 0]} (1 - \Gamma^2) RT \\
+ \frac{1}{2} R(R_0 r_0 + R_0 - r_0) \\
+ C_1 \delta RT + Z_0 \sup_{r \in [r_0, R_0]} |v_r| \delta^{-1} RT. \tag{3.23}
\]

Step 6. Under the assumption (1.3) in the theorem, we choose \( r_0 = r_i \) for large \( i \). Then there exists a positive number \( \sigma_0 \) such that

\[
\sup_{r = r_0, t \in [-T, 0]} (1 - \Gamma^2) < \sigma_0 < 1.
\]

Choosing \( R_0 \) much large than \( r_0 \), we find that

\[
\frac{1}{2} \int_{-T}^{0} \int_{D_1} (|\partial_r \Gamma|^2 + |\partial_z \Gamma|^2) \phi_1^2 d\mu + \int_{-T}^{0} \int_{D_2} (|\partial_r \Gamma|^2 + |\partial_z \Gamma|^2) \phi_2^2 d\mu \\
\leq -\frac{1 - \sigma_0}{2} RT + \frac{1}{2} R(R_0 r_0 + R_0 - r_0) \\
+ C_1 \delta RT + Z_0 \sup_{r \in [r_0, R_0]} |v_r| \delta^{-1} RT. \tag{3.24}
\]

Now we fix \( \delta \) so that \( C_1 \delta = (1 - \sigma_0)/4 \). Then the right hand side of the last inequality is negative when \( R >> R_0 r_0 \) a and \( T >> R_0 r_0 \), which is a contradiction since the left hand side is nonnegative. Here we also used the fact that \( v_r \to 0 \) when \( r \to \infty \) uniformly, due to Lemma 2.2. This contradiction shows that

\[
\lim_{r \to \infty} \Gamma = 0
\]

uniformly in \( z \) and \( t \). As mentioned earlier, from [7] we can conclude \( v = ce_z \).
4 Proof of Theorem 1.3

In this section, we will use the idea in the previous section to treat non-periodic ancient solutions under an extra assumption on the convergence rate of $\Gamma$. In this section, we take the weight function $\lambda$ to be

$$\lambda = \begin{cases} 
  r, & r \in [0, r_0]; \\
  r_0 - \frac{r_0 - 1}{R_0 - r_0}(r - r_0), & r \in [r_0, R_0]; \\
  1, & r \geq R_0.
\end{cases} \tag{4.1}$$

Again, we consider the domain

$$D = D_1 \cup D_2,$$

where

$$D_1 = \{(r, z, t) \mid 0 \leq r < R_0; -R \leq z \leq R, -T \leq t \leq 0\},$$

$$D_2 = \{(r, z, t) \mid R_0 \leq r \leq R; -R \leq z \leq R, -T \leq t \leq 0\}.$$

Let

$$\phi_1 = \xi(z) \frac{t}{-T},$$

where $\xi = \xi(z)$ is a smooth cut-off function on $[-R, R]$ such that $0 \leq \xi_1 \leq 1$, $|\xi'| \leq C/R$, and $\xi_1(z) = 1$ when $z \in [-R/2, R/2]$.

Let $\phi_2 = \phi_2(r, z, t)$ be the unique solution to the final time boundary value problem of the backward equation, which is well-posed.

$$\begin{align*}
  \partial_r^2 \phi_2 + \partial_z^2 \phi_2 + & \frac{2\lambda(r)}{\lambda(\Gamma)} \partial_r \phi_2 + v_r \partial_r \phi_2 + v_z \partial_z \phi_2 \\
  + & \frac{1}{2} \left( \frac{\lambda'(r)}{\lambda(r)} - \frac{r}{r} \right) v_r + \left( \frac{\lambda'(r)}{r} \right) v_z \phi_2 - A \phi_2 + \partial_t \phi_2 = 0, \quad (r, z) \in D_2; \\
  \phi_2(R_0, z, t) = \phi_1(z, t), & \quad \phi_2(R, z, t) = 0, t \in [-T, 0]; \quad \phi_2 = 0 \quad \text{if} \quad z = R, -R; \\
  \phi_2(r, z, 0) = 0.
\end{align*} \tag{4.2}$$

Here again we take $A = \frac{1}{2r_0} \|v_r\|_\infty$.

Since $r \geq R_0 > 1$ and $\lambda'$ is bounded, standard parabolic equation theory tells us that the above problem has a unique solution such that $0 \leq \phi_2 \leq 1$ and $\|\nabla \phi_2\|_\infty \leq A_0$. Here $A_0$ is a constant depending only on $C^1$ norm of $b$.

Now we compute

$$- \int_{D_1} (\partial_r^2 \Gamma + \partial_z^2 \Gamma) \Gamma \phi_2^2 d\mu dt$$

$$= - \int_{D_1} (\partial_r^2 \Gamma + \partial_z^2 \Gamma) \Gamma \phi_2^2 \lambda(r) dr dz dt$$

$$= \int_{D_1} (|\partial_r \Gamma|^2 + |\partial_z \Gamma|^2) \phi_2^2 d\mu dt + \int_{-T}^0 \int_{-R}^R \int_{0}^{R_0} (\partial_r \Gamma) \Gamma \phi_2^2(z, t) \lambda'(r) dr dz dt$$

$$+ \int_{-T}^0 \int_{-R}^R \int_{0}^{R_0} (\partial_z \Gamma) \Gamma 2 \phi_1(z, t) \partial_r \phi_1(z, t) \lambda(r) dr dz dt - \int_{-T}^0 \int_{-R}^R (-\partial_r \Gamma) \Gamma \phi_2^2(z, t) \lambda(r) \Big|_{r=R_0} dz.$$
Similarly

\[- \int_{D_2} (\partial_r^2 \Gamma + \partial_z^2 \Gamma) \phi_2^2 d\mu dt \]

\[= \int_{D_2} (|\partial_r \Gamma|^2 + |\partial_z \Gamma|^2) \phi_2^2 d\mu + \int_{-T}^{0} \int_{-R}^{R} \int_{R_0}^{R} (\partial_r \Gamma) \phi_2^2 \lambda'(r) dr dz dt \]

\[+ \int_{-T}^{0} \int_{-R}^{R} \int_{R_0}^{R} (\partial_z \Gamma) \phi_2^2 \lambda'(r) dr dz dt + \int_{-T}^{0} \int_{-R}^{R} \int_{R_0}^{R} (\partial_z \Gamma) \phi_2^2 \phi_2 \lambda(r) dr dz dt \]

\[+ \int_{-T}^{0} \int_{-R}^{R} (\partial_r \Gamma) \phi_2^2 \lambda(r) \bigg|_{r=R_0} dr dz dt. \]

Adding the previous two identities and noting the last two boundary terms cancel and also \( \lambda' = 0 \) when \( r > R_0 \), we obtain

\[L \equiv \int_{D_1} (|\partial_r \Gamma|^2 + |\partial_z \Gamma|^2) \phi_1^2 d\mu dt + \int_{D_2} (|\partial_r \Gamma|^2 + |\partial_z \Gamma|^2) \phi_2^2 d\mu dt \]

\[= - \int_{D_1} (\partial_r^2 \Gamma + \partial_z^2 \Gamma) \phi_1^2 d\mu dt - \int_{D_2} (\partial_r^2 \Gamma + \partial_z^2 \Gamma) \phi_2^2 d\mu dt \]

\[\begin{align*}
&\quad - \int_{-T}^{0} \int_{-R}^{R} \int_{R_0}^{R} (\partial_r \Gamma) \phi_1 \partial_r \phi_1 \lambda(r) dr dz dt - \int_{D_2} (\partial_z \Gamma) \phi_2 \phi_1^2 (z, t) \lambda'(r) dr dz dt \\
&\quad + \int_{-T}^{0} \int_{-R}^{R} \int_{R_0}^{R} (\partial_z \Gamma) \phi_2 \phi_1 \phi_2 \lambda(r) dr dz dt - \int_{D_2} (\partial_z \Gamma) \phi_2^2 \phi_1^2 \lambda(r) dr dz dt \\
&\quad - \int_{-T}^{0} \int_{-R}^{R} \int_{R_0}^{R} (\partial_z \Gamma) \phi_2 \partial_z \phi_2 \lambda(r) dr dz dt - \int_{D_2} (\partial_z \Gamma) \phi_2^2 \partial_z \phi_2 \lambda(r) dr dz dt \end{align*}
\]

\[\equiv T_1 + \ldots + T_6. \quad (4.3)\]

In the rest of the proof we will find an upper bound for each \( T_i \), \( i = 1, \ldots, 6 \).

**Step 3. bound for \( T_1 \).**

From equation \( (3.6) \),

\[T_1 = - \int_{-T}^{0} \int_{-R}^{R} \int_{0}^{R_0} \left( v_r \partial_r \Gamma + v_z \partial_z \Gamma + \frac{1}{r} \partial_r \Gamma + \partial_t \Gamma \right) \phi_1^2 \lambda(r) dr dz dt. \]

After integration by parts and using the divergence free property of \( v_r e_r + v_z e_z \), we
deduce

\[
T_1 = -\frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{0}^{R_0} \left[ v_r \partial_r (\Gamma^2 - 1) + v_z \partial_z (\Gamma^2 - 1) + \frac{1}{r} \partial_r \Gamma^2 + \partial_t (\Gamma^2 - 1) \right] \phi_1^2(z, t) \lambda(r) dr dz dt
\]

\[
= \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{0}^{R_0} (v_r \partial_r \phi_1^2 + v_z \partial_z \phi_1^2) (\Gamma^2 - 1) \lambda(r) dr dz dt
\]

\[
\underbrace{-\frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} v_r (\Gamma^2 - 1) \phi_1^2 \lambda(r) \bigg|_{r=R_0} dz dt}_{T_{11}}
\]

\[
\underbrace{+ \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{0}^{R_0} v_r (\Gamma^2 - 1) \frac{\lambda^2}{r} \left( \lambda(r) - \frac{\lambda(r)}{r} \right) }_{T_{12}} dr dz dt
\]

\[
\underbrace{+ \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{0}^{R_0} \Gamma^2 \phi_1^2 \frac{\lambda(r)}{r} }_{T_{13}} dr dz dt
\]

\[
\underbrace{- \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \frac{\phi_1^2 \lambda(r)}{r} \bigg|_{r=R_0} dz dt}_{T_{14}}
\]

\[
\underbrace{+ \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{0}^{R_0} (\Gamma^2 - 1) \partial_t \phi_1^2 \lambda(r) dr dz dt}_{T_{16}}
\]

\[
\underbrace{+ \left( -\frac{1}{2} \right) \int_{-R_0}^{R_0} (\Gamma^2 - 1) \phi_1^2 \bigg|_{t=-T} \lambda(r) r dz}_{T_{17}}
\]

\[\equiv T_{11} + \ldots + T_{17}.\] (4.4)

Notice that

\[|T_{11}| \leq \|v_z\|_\infty \frac{CT}{R} RR_0 \|\lambda\|_\infty = CR_0 r_0 T \|v_z\|_\infty \|\Gamma^2 - 1\|_\infty.\]

The term \(T_{12}\) is a boundary ones which will be cancelled with a boundary term from integration on \(D_2\), which is called \(T_{42}\). \(T_{14} \leq 0\) and \(T_{15} \leq 0\). Also, since

\[|\partial_t \phi_1| \leq 1/T\]

and

\[
\int_{0}^{R_0} \lambda(r) dr = \frac{1}{2} (R_0 - r_0) + \frac{1}{2} R_0 r_0,
\]

direct computation shows

\[T_{16} \leq \frac{1}{2} \|\Gamma^2 - 1\|_\infty R (R_0 r_0 + R_0 - r_0).\] (4.5)

Similarly

\[T_{17} \leq -\frac{1}{2} \inf[1 - \Gamma^2(R_0, \cdot, -T)] R (R_0 r_0 + R_0 - r_0) \leq 0.\] (4.6)
Hence, we can deduce, after leaving $T_{12}$ and $T_{13}$ along for now, that

$$
T_1 \leq CR_0 r_0 \|v_z\|_\infty T + \frac{1}{2} (\|\Gamma^2 - 1\|_\infty - \inf[1 - \Gamma^2(R_0, \cdot, -T)]) R(R_0 r_0 + R_0 - r_0) + T_{12} + \frac{1}{2} \int_0^T \int_{-R}^R \int_{-R}^R v_r (\Gamma^2 - 1) \phi_1^2 \left( \lambda'(r) - \frac{\lambda(r)}{r} \right) dr dz dt.
$$

(4.7)

**Step 4. bounds for $T_2, \ldots, T_7$**

First we bound $T_2$. By our choice of $\lambda$,

$$
T_2 = \frac{1}{2} \int_0^T \int_{-R}^R \int_0^R \partial_r (\Gamma^2 - 1) \phi_1^2 \lambda(r) dr dz dt
$$

$$
= \frac{1}{2} \int_0^T \int_{-R}^R \int_0^{r_0} \partial_r (\Gamma^2 - 1) dr dz dt + \frac{r_0 - 1}{2(R_0 - r_0)} \int_0^T \int_{-R}^R \int_{r_0}^R \partial_r (\Gamma^2 - 1) dr dz dt
$$

$$
\leq \left[ -\frac{1}{3} + \left( \frac{1}{3} + \frac{r_0 - 1}{3(R_0 - r_0)} \right) \sup_{r = r_0, t \in [-T, 0]} (1 - \Gamma^2) \right] RT.
$$

(4.8)

Next

$$
T_3 \leq C \|\partial_z \Gamma\|_\infty r_0 R_0 T
$$

(4.9)

since $|\partial_z \phi_1| \leq C/R$.

Now we deal with the terms $T_4, \ldots, T_7$ which involve integrations on $D_2$ only. From equation [3.6],

$$
T_4 = -\int_0^T \int_{-R}^R \int_{R_0}^R (v_r \partial_r \Gamma + v_z \partial_z \Gamma + \frac{1}{r} \partial_r \Gamma + \partial_t \Gamma) \phi_2^2 \lambda(r) dr dz dt.
$$

After integration by parts and using the divergence free property of $v_r e_r + v_z e_z$, we
deduce

\[ T_4 = -\frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{R_0}^{R} \left[ v_r \partial_r (\Gamma^2 - 1) + v_z \partial_z (\Gamma^2 - 1) + \frac{1}{r} \partial_r (\Gamma^2 - 1) + \partial_t (\Gamma^2 - 1) \right] \phi_2^2 \lambda(r) dr dz dt \]

\[ = \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{R_0}^{R} (v_r \partial_r \phi_2^2 + v_z \partial_z \phi_2^2)(\Gamma^2 - 1) \lambda(r) dr dz dt \]

\[ + \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} v_r (\Gamma^2 - 1) \phi_2^2 \lambda(r) \bigg|_{r=R_0} dz \]

\[ + \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{R_0}^{R} \left[ \lambda'(r) - \frac{\lambda(r)}{r} \right] dr dz dt \]

\[ + \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{R_0}^{R} (\Gamma^2 - 1) \partial_t \phi_2^2 \lambda(r) dr dz dt \]

\[ \equiv T_{41} + \ldots + T_{47}. \]  \hspace{1cm} (4.10)

Notice that \( T_{42} \) will cancel with \( T_{12} \) when all terms are added. Also \( T_{45} \leq 0 \) and

\[ T_{47} \leq - \inf_{r \in [R_0, R]} (1 - \Gamma^2(r, \cdot, -T)) \int_{-R}^{R} \int_{R_0}^{R} \phi_2^2(r, z, -T) \lambda(r) dr dz \leq 0. \]  \hspace{1cm} (4.11)

Therefore

\[ T_4 \leq T_{41} + T_{42} + T_{43} + T_{44} + T_{46}. \]

Next

\[ T_5 = -\frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{R_0}^{R} \partial_r (\Gamma^2 - 1) \partial_r \phi_2^2 \lambda(r) dr dz dt \]

\[ = \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{R_0}^{R} (\Gamma^2 - 1) \partial_r^2 \phi_2^2 \lambda(r) dr dz dt - \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} (\Gamma^2 - 1) \partial_r \phi_2^2 \lambda(r) \bigg|_{R_0}^{R} dz dt, \]  \hspace{1cm} (4.12)

since \( \lambda(r) = 1 \) here. Finally

\[ T_6 = \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{R_0}^{R} (\Gamma^2 - 1) \partial_z^2 \phi_2^2 \lambda(r) dr dz dt. \]  \hspace{1cm} (4.13)
Combining the bounds on $T_i$, $i = 1, ..., 6$, noticing cancellation of boundary terms $T_{12}$ with $T_{42}$, we find that

$$L \leq \left[ -\frac{1}{3} + \left( \frac{1}{3} + \frac{r_0 - 1}{2(R_0 - r_0)} \right) \sup_{r = r_0, t \in [-T, 0]} (1 - \Gamma^2) \right] RT$$

$$+ CR_0 r_0 \| v_z \|_\infty \| \Gamma^2 - 1 \|_\infty T + \| \Gamma^2 - 1 \|_\infty R R_0 r_0$$

$$+ \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{0}^{R_0} v_r (\Gamma^2 - 1) \phi_1^2 (z) \left( \lambda'(r) - \frac{\lambda(r)}{r} \right) dr dz dt$$

$$+ \inf_{r \in [R_0, R]} (1 - \Gamma^2 (r, \cdot, -T)) \int_{-R}^{R} \int_{0}^{R_0} \phi_2^2 (r, z, -T) dr dz$$

$$+ \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} \int_{0}^{R_0} v_r (\Gamma^2 - 1) \phi_2^2 (z) \left( \lambda'(r) - \frac{\lambda(r)}{r} \right) dr dz dt$$

$$+ \frac{1}{2} \int_{-T}^{0} \int_{-R}^{R} (\Gamma^2 - 1) \partial_r \phi_2^2 \lambda(r) \big|_{r = R_0} dz dt$$

$$\equiv ... + I_1 + I_2.$$  \tag{4.14}

Here the last two integrals are denoted by $I_1$, $I_2$ respectively.

**Step 5.**

It remains to bound the two integrals $I_1$ and $I_2$.

First let us bound $I_2$. Observe that the coefficients of lower order terms of the equation \eqref{12} are bounded by $\frac{1}{R_0} + \| v_r \|_\infty + \| v_z \|_\infty$ in $D_2$ and the boundary value of
\(\phi_2\) at \(r = R_0\) is \(\phi_1\) which satisfies \(0 \leq \phi_1 \leq 1\) and \(|\partial_z \phi_1| \leq C/R\) and \(|\partial_t \phi_1| \leq 1/T\). By standard boundary gradient bound for parabolic equations, we know that
\[|\partial_r \phi_2|_{r=R_0} \leq C_1, \quad C_1 = C_1(\|v_r\|_\infty, \|v_z\|_\infty).\]

Hence
\[I_2 \leq C_1 \sup_{r=R_0, t \in [-T, 0]} (1 - \Gamma^2)RT. \quad (4.15)\]

Finally
\[I_1 = \frac{1}{2} \int_{-T}^0 \int_{-R}^R \int_{r_0}^{R_0} v_r(\Gamma^2 - 1)\phi^2_2(z, t) \left(\frac{\lambda(r)}{r} - \frac{\lambda(r)}{r^2}\right) drdzdt.\]

By direct computation, we see that
\[\left|\lambda'(r) - \frac{\lambda(r)}{r}\right| \leq 2\frac{r_0}{r}\]
when \(R_0 \gg r_0\). So under the assumption (a) of the theorem, i.e. \(|\Gamma^2(r, z, t) - 1| \leq \epsilon/r\), we have
\[|I_1| \leq CRT \sup_{r_0 \leq r \leq R_0} |v_r| \int_{r_0}^{R_0} \frac{\epsilon}{r^2} dr \leq C\epsilon RT \sup_{r_0 \leq r \leq R_0} |v_r|.\]

\[I_1 \leq C\epsilon RT. \quad (4.16)\]

Now we substitute, (4.15) and (4.16) into (4.14) to obtain
\[L \leq \frac{1}{3}RT + \left(\frac{1}{3} + \frac{r_0 - 1}{2(R_0 - r_0)}\right) \sup_{r=r_0, t \in [-T, 0]} (1 - \Gamma^2)RT + C R_0 r_0 \|v_z\|_\infty T\|\Gamma^2 - 1\|_\infty
\[+ \frac{1}{2} \left(\|\Gamma^2 - 1\|_\infty - \inf_{r \in [R_0, R]} [1 - \Gamma^2(R_0, \cdot, -T)]\right) R(R_0 r_0 + R_0 - r_0) + C \|\partial_z \Gamma\|_\infty r_0 R_0 T
\[+ \frac{1}{2} \int_{-T}^0 \int_{D_1} \int_{D_2} (|\partial_r \Gamma|^2 + |\partial_z \Gamma|^2) \phi^2_2 d\mu + \int_{-T}^0 \int_{D_2} (|\partial_r \Gamma|^2 + |\partial_z \Gamma|^2) \phi^2_2 d\mu
\[\leq -\frac{1}{3}RT + \left(\frac{1}{3} + \frac{r_0 - 1}{2(R_0 - r_0)}\right) \sup_{r=r_0, t \in [-T, 0]} (1 - \Gamma^2)RT + C \|\Gamma^2 - 1\|_\infty R_0 r_0 \|v_z\|_\infty T
\[+ \frac{1}{2} \left(\|\Gamma^2 - 1\|_\infty - \inf_{r \in [R_0, R]} [1 - \Gamma^2(R_0, \cdot, -T)]\right) R(R_0 r_0 + R_0 - r_0) + C \|\partial_z \Gamma\|_\infty r_0 R_0 T
\[- \inf_{r \in [R_0, R]} \int_{-R}^R \int_{R_0}^R \phi^2_2(r, z, -T) drdz
\[+ C_1 \sup_{r=R_0, t \in [-T, 0]} (1 - \Gamma^2)RT + C\epsilon RT. \quad (4.17)\]
Therefore
\[
\frac{1}{2} \int_{-T}^{0} \int_{D_1} (|\partial_r \Gamma|^2 + |\partial_z \Gamma|^2) \phi_1^2 \, d\mu + \int_{-T}^{0} \int_{D_2} (|\partial_r \Gamma|^2 + |\partial_z \Gamma|^2) \phi_2^2 \, d\mu \\
\leq -\frac{1}{6} RT + C\|\Gamma^2 - 1\|_{\infty} R_0 r_0 \|v_z\|_{\infty} T \\
+ \frac{1}{2} \left(\|\Gamma^2 - 1\|_{\infty} - \inf [1 - \Gamma^2(R_0, \cdot, -T)] \right) R(R_0 r_0 + R_0 - r_0) + C\|\partial_z \Gamma\|_{\infty} r_0 R_0 T \\
- \inf_{r \in [R_0, R]} (1 - \Gamma^2(r, \cdot, -T)) \int_{-R}^{R} \int_{R_0}^{R} \phi_2^2(r, z, -T) \, dr \, dz \\
+ C\epsilon RT,
\] (4.18)
when \( R_0 >> r_0 \).

Under assumption (b) of the theorem, i.e. \( v_r \leq 0 \), we see that \( I_1 \leq 0 \) since \( \Gamma^2 - 1 \leq 0 \) and \( \lambda' - \frac{\lambda}{t} \leq 0 \). Using also the assumption that \( (1 - \Gamma^2(r, z, t)) \leq \epsilon \) for large \( r \), we find

\[
0 \leq L \leq -\frac{1}{3} RT + \left(\frac{1}{3} + \frac{r_0 - 1}{2(R_0 - r_0)}\right) \epsilon RT + CR_0 r_0 \|v_z\|_{\infty} T \|\Gamma^2 - 1\|_{\infty} \\
+ \frac{1}{2} \left(\|\Gamma^2 - 1\|_{\infty} - \inf [1 - \Gamma^2(R_0, \cdot, -T)] \right) R(R_0 r_0 + R_0 - r_0) + C\|\partial_z \Gamma\|_{\infty} r_0 R_0 T \\
- \inf_{r \in [R_0, R]} (1 - \Gamma^2(r, \cdot, -T)) \int_{-R}^{R} \int_{R_0}^{R} \phi_2^2(r, z, -T) \, dr \, dz \\
+ C_1 \epsilon RT.
\] (4.19)

**Step 6.**

Under either assumptions (a) or (b) of the theorem, both (4.18) and (4.19) is impossible when \( R \) is sufficiently large and \( T >> R_0 r_0, R_0 >> r_0 \) and \( \epsilon \) sufficiently small. Hence \( \lim_{r \to \infty} \Gamma = 0 \). As explained in the periodic case, this shows \( v_{\theta} = 0 \) and \( v = ce_z \).

**Acknowledgement**

Z.L. was supported by NSFC (grant No. 11725102) and National Support Program for Young Top-Notch Talents. Q.S.Z. wishes to thank the Simons Foundation for its support, and for Fudan University for its hospitality during his visit.

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