Delta-epsilon functions and uniform continuity on metric spaces

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Abstract
Under certain general conditions, an explicit formula to compute the greatest delta-epsilon function of a continuous function is given. From this formula, a new way to analyze the uniform continuity of a continuous function is given. Several examples illustrating the theory are discussed.

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1 Introduction
Directly or indirectly most results in mathematical analysis use the concept of continuity in order to extend a property of a function \( f \) that is satisfied at a point \( p \) to a property satisfied in a neighborhood of \( p \). A well known example that illustrates that fact is the Inverse function theorem, which we recall here:

Roughly speaking, the Inverse function theorem states that a continuously differentiable mapping \( f : \mathbb{R}^n \to \mathbb{R}^n \) is invertible on a certain open ball \( B(p, \delta) \) where the linear transformation \( f'(p) \) is invertible.

In this example, the invertibility of \( f' \) at \( p \) is extended to an open ball via the continuity of the mapping \( x \to f'(x) \) at \( p \). Now, we notice that in the statement of the Inverse function theorem, nothing is said about the size of the ball where the function \( f \) is invertible. However, it is not difficult to check, see for instance [2], that the radius of the open ball depends on the norm of the linear transformation \( [f'(p)]^{-1} \) and on the positive number delta appearing in the definition of continuity of the function \( x \to f'(x) \) at the point \( p \). More exactly, for \( 2\lambda|[f'(p)]^{-1}| = 1 \), then \( \delta \) is such that

\[
\text{If, } |x - p| < \delta, \quad \text{then, } |f'(x) - f'(p)| < \lambda.
\]

So, inspired by the previous discussion, we would like to deal with the following questions:

1. Let \( X, Y \) be metric spaces, \( p \in X \) and \( f : X \to Y \) a continuous function at \( p \), and let \( \epsilon > 0 \) be fixed, which is the greatest positive \( \delta(p, \epsilon) \) such that the epsilon-delta definition of continuity is satisfied?

Is there a formula to compute it?

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The previous questions has been solved indirectly in [1] under simple assumptions on the function $f$. In fact, considering
\[ f : [b, \infty) \to [f(b), \infty) \] an increasing bijective function,
(1)
$p \in [b, \infty), \epsilon > 0$ fixed, $p_0 = f^{-1}(f(b) + \epsilon)$, $\delta_1(p, \epsilon) = p - f^{-1}(f(p) - \epsilon)$, and $\delta_2(p, \epsilon) = f^{-1}(f(p) + \epsilon) - p$,
then the positive number
\[
\delta(p, \epsilon) = \begin{cases} 
\delta_2(p, \epsilon) & b \leq p < p_0 \\
\min\{\delta_1(p, \epsilon), \delta_2(p, \epsilon)\} & p_0 \leq p
\end{cases}
\] (2)
satisfies the definition of continuity of $f$ at the point $p$. More precisely
\[
\text{If, } x \in [b, \infty) \text{ and } |x - p| < \delta(p, \epsilon), \text{ then } |f(x) - f(p)| < \epsilon.
\] (3)

In addition, $\delta(p, \epsilon)$ is the maximum positive number satisfying the previous condition. One of the aims of this paper is to generalize the formula (2) to the case of functions defined on metric spaces.

Now, it is also shown in [1], that the function $p \in [b, \infty) \to \delta(p, \epsilon)$ in (2) provides a way to study the uniform continuity of increasing bijective functions defined on unbounded intervals. Indeed, it is proven in [1] that an increasing bijective function $f : [b, \infty) \to [f(b), \infty)$ is not uniformly continuous on $[b, \infty)$, if and only if, there exists $\epsilon_0 > 0$, such that
\[
\inf_{x \in [b, \infty)} \delta(x, \epsilon_0) = 0.
\]
Thus, other natural questions we are interested in are the followings:

2. Is it possible to study the uniform continuity of a continuous function $f : X \to Y$ in terms of the function $p \in X \to \delta(p, \epsilon)$? Are there other mathematical problems where the function $p \in X \to \delta(p, \epsilon)$ is useful?

Finally, it is important to point out that the question 1 and 2 are related to the concept of modulus of continuity which has been extensively used in approximation theory, see for instance [3], [4] and the references therein.

This manuscript will be divided as follows, in the section 2 we deal with the question 1 more exactly, for continuous functions defined on certain metric spaces, an explicit formula to compute the greatest positive number $\delta(p, \epsilon)$ is given, see theorem 2.1 and its corollaries. On the other hand, in the section 3 the question 1 is discussed. Specifically, the theorems 2.1 and 2.2 proved in [1] are generalized, see theorems 4 and 5. Additionally, some examples are discussed in the section 4.

2 A formula to compute delta-epsilon numbers

In this section, we give an explicit formula to compute delta-epsilon numbers of continuous functions.

**Definition 1.** Let $X, Y$ be metric spaces, $f : X \to Y$ a continuous function at $p \in X$, and $\epsilon > 0$. A positive number $\delta$ is said to be a delta-epsilon number for $f$ at $p$, if $\delta$ satisfies the $\epsilon$-$\delta$ definition of continuity of $f$ at the point $p$. In other words, $\delta$ is such that
\[
\text{if } x \in X \text{ and } d_X(x, p) < \delta, \text{ then } d_Y(f(x), f(p)) < \epsilon.
\] (4)
The following theorem provides a theoretical formula to compute the greatest delta-epsilon number for a wide class of functions. Namely,

**Theorem 1.** Let $f : X \to Y$ be a continuous function on $X$, $p \in X$ and $\epsilon > 0$, then we have:

1. If $f^{-1}(S[f(p), \epsilon]) \neq \emptyset$, then the quantity
$$
\delta(p, \epsilon) = \text{dist}(p, f^{-1}(S[f(p), \epsilon])),
$$
is well defined and represents a positive number. Here, $S[f(p), \epsilon]$ denotes the sphere with center at $f(p)$ and radius $\epsilon$, that is to say, $S[f(p), \epsilon] = \{ y \in Y | d_Y(f(p), y) = \epsilon \}$.

2. Furthermore, if the open ball $B(p, \delta(p, \epsilon))$ is path-connected then the number $\delta(p, \epsilon)$ is a delta-epsilon number for $f$ at $p$. More exactly, for every $x \in X$ such that, $d_X(x, p) < \delta(p, \epsilon)$, then $d_Y(f(x), f(p)) < \epsilon$.

3. $\delta(p, \epsilon)$ is the greatest delta-epsilon number at $p$.

4. Finally, if we define the set $D_{p, \epsilon}$ as:
$$
D_{p, \epsilon} = \{ \beta \in \mathbb{R}^+ | (\forall x \in X)(d_X(x, p) < \beta \Rightarrow d_Y(f(x), f(p)) < \epsilon) \},
$$
then, $\delta(p, \epsilon) = \max D_{p, \epsilon}$ and of course $D_{p, \epsilon} = (0, \delta(p, \epsilon)]$.

**Proof:** The proof of this theorem proceed as follows:

1. We first observe that since $f^{-1}(S[f(p), \epsilon])$ is a nonempty set, then the number
$$
\delta(p, \epsilon) = \inf \{ d_X(x, p) | x \in X, \; d_Y(f(x), f(p)) = \epsilon \},
$$
is well defined. Now, if $\delta(p, \epsilon) = 0$, there exists a sequence $x_n \in X$ so that $\lim d_X(x_n, p) = 0$ with $\lim d_Y(f(x_n), f(p)) = \epsilon$. Being $f$ continuous at $p$, we can conclude that $\lim d_Y(f(x_n), f(p)) = 0$, since $\epsilon > 0$, we have a contradiction. Thus, $\delta(p, \epsilon)$ have to be a positive number.

2. Let $x \in X$, we want to prove that,
$$
\text{if, } \; d_X(x, p) < \delta(p, \epsilon) \; \text{ then } \; d_Y(f(x), f(p)) < \epsilon.
$$
In fact, because of the definition of $\delta(p, \epsilon)$, clearly $d_Y(f(x), f(p)) \neq \epsilon$. So, to finish the proof of our statement, we must show that the inequality $d_Y(f(x), f(p)) > \epsilon$ is not possible. Now, we argue by contradiction. If $d_Y(f(x), f(p)) > \epsilon$, since the open ball $B(p, \delta(p, \epsilon))$ is path-connected, there exists a continuous function $\gamma : [0, 1] \to B(p, \delta(p, \epsilon))$ such that $\gamma(0) = p$ and $\gamma(1) = x$. Therefore, the function $g : [0, 1] \to \mathbb{R}$ given by $g(t) = d_Y(f(\gamma(t)), f(p))$ is continuous and satisfies that $g(0) = 0$ and $g(1) > \epsilon$, so by the intermediate value theorem, there exists $t_0 \in (0, 1)$ such that $g(t_0) = d_Y(f(\gamma(t_0)), f(p)) = \epsilon$. Thus, $\gamma(t_0)$ satisfies that $d_X(\gamma(t_0), p) < \delta(p, \epsilon)$ and $d_Y(f(\gamma(t_0)), f(p)) = \epsilon$, the last affirmation contradicts the definition of $\delta(p, \epsilon)$. Hence, we can conclude that the number $\delta(p, \epsilon)$ given in (5) is a delta-epsilon number for $f$ at $p$. As we wanted to prove.

3. If $\alpha$ is such that $\delta(p, \epsilon) < \alpha$ then by definition of $\delta(p, \epsilon)$ there exists $x \in X$ so that $\delta(p, \epsilon) \leq d_X(x, p) < \alpha$ with $d_Y(f(x), f(p)) = \epsilon$, so $\alpha$ is not a delta-epsilon number for $f$ at $p$.

4. Now we proceed to prove item [4]. From items [1] and [2] we deduce that $\delta(p, \epsilon) \in D_{p, \epsilon}$. From item [3] we obtain that any other number greater than $\delta(p, \epsilon)$ is not in $D_{p, \epsilon}$. Hence, we can conclude that $\delta(p, \epsilon) = \max D_{p, \epsilon}$. This finishes the proof of the theorem.
**Theorem 1.** Similarly, it is shown that \( f < \epsilon < \beta \)

**Proof:** Since there exists a path \( \delta - \epsilon \) number for \( f \) at \( p \). In fact, if there exists \( x_0 \in B(p, \delta(p, \epsilon)) \) such that \( d_Y(f(x_0), f(p)) > \epsilon \), then, the function \( h \) given by

\[
h : B(p, \delta(p, \epsilon)) \to \mathbb{R}, \quad h(x) = d_Y(f(x), f(p))
\]

is continuous and satisfies \( h(p) = 0 < \epsilon < h(x_0) \), then by the intermediate value theorem, there exists \( x_1 \in B(p, \delta(p, \epsilon)) \) such that \( d_Y(f(x), f(p)) = \epsilon \), which contradicts the definition of \( \delta(p, \epsilon) \).

The following results give us sufficient conditions to compute delta-epsilon numbers for points \( p, x \in X \) which are connected by a path and such that \( f(p) \neq f(x) \). More accurately,

**Corollary 1.** Let \( f : X \to Y \) be a continuous function on \( X \), suppose that there exist \( p, x \in X \) such that \( d_Y(f(x), f(p)) =: \beta > 0 \) and there exists a path connecting the points \( p \) and \( x \), then for every \( \epsilon \) such that \( 0 < \epsilon \leq \beta \) we have that \( f^{-1}(S[f(p), \epsilon]) \neq \emptyset \) and \( f^{-1}(S[f(x), \epsilon]) \neq \emptyset \). Particularly, for every \( \epsilon \) satisfying \( 0 < \epsilon \leq \beta \), the numbers \( \delta(p, \epsilon) \) and \( \delta(x, \epsilon) \) given by the formula (5) are well defined and positives.

**Proof:** Since there exists a path \( \gamma : [0, 1] \to X \) connecting \( p \) and \( x \), then the function \( g(t) : [0, 1] \to \mathbb{R} \) given by \( d_Y(f(\gamma(t), f(p))) \) is continuous and satisfies \( [0, \beta] \subset g((0, 1]) \). Thus, for every \( \epsilon \) such that \( 0 < \epsilon < \beta \) there exists \( t_0 \in (0, 1) \) satisfying \( d_Y(f(\gamma(t_0), f(p))) = \epsilon \), which proves that \( f^{-1}(S[f(p), \epsilon]) \neq \emptyset \). Similarly, it is shown that \( f^{-1}(S[f(x), \epsilon]) \neq \emptyset \). The rest of the proof follows from the item 1 in the theorem 1.

**Corollary 2.** Let \( f : X \to Y \) be a continuous function on \( X \), suppose that there exist \( p, x \in X \) such that \( d_Y(f(x), f(p)) =: \beta > 0 \), and there exists a path connecting the points \( p \) and \( x \), and if for every \( \epsilon \) with \( 0 < \epsilon \leq \beta \), the open balls \( B(p, \delta(p, \epsilon)), B(x, \delta(x, \epsilon)) \) are path-connected, then the numbers \( \delta(p, \epsilon), \delta(x, \epsilon) \) are delta-epsilon numbers for \( f \) at \( p \) and \( x \) respectively.

**Proof:** The proof of this result follows from the corollary 1 and the proof of the item 2 in the theorem 1.

The following theorem allows us to compute delta-epsilon numbers in a neighborhood of a point \( p \) which admits a delta-epsilon number. Namely,

**Theorem 2.** Let \( f : X \to Y \) be a continuous function on \( X \), suppose that there exist \( p, x \in X \) such that \( d_Y(f(x), f(p)) =: \beta > 0 \), suppose that the open ball \( B(p, \delta(p, \beta)) \) is path-connected and that there exists a path connecting the points \( p \) and \( x \). Then, for every \( \epsilon \), with \( 0 < \epsilon < \beta \), there exists \( \delta \) satisfying \( 0 < \delta \leq \delta(p, \beta) \), such that if \( d_X(q, p) < \delta \) then the numbers \( \delta(q, \epsilon) \) given in (6) are well defined and positives. If in addition, the open balls \( B(q, \delta(q, \epsilon)) \) are path-connected, then for all \( q \in B(p, \delta) \), the numbers \( \delta(q, \epsilon) \) are delta-epsilon numbers.

**Proof:** We divide the proof of this theorem into two parts:

1. First all of, we shall prove that there exists \( \delta \) with \( 0 < \delta \leq \delta(p, \beta) \) and such that if \( d_X(q, p) < \delta \) then \( \epsilon < d_Y(f(x), f(q)) \). In fact, since \( f^{-1}(S[f(p), \beta]) \neq \emptyset \) and the open ball \( B(p, \delta(p, \beta)) \) is path-connected, then from the theorem 1 we conclude that the number \( \delta(p, \beta) \) is the maximum delta-epsilon number at \( p \). On the other hand, since \( f \) is continuous at \( p \) and \( \beta - \epsilon \) is positive, there exists \( \delta > 0 \) such that if \( d_X(q, p) < \delta \) then \( d_Y(f(q), f(p)) < \beta - \epsilon < \beta \). So, since \( \delta(p, \beta) \) is the
maximum delta-epsilon number at $p$, we deduce that $\delta \leq \delta(p, \beta)$. Now, by taking $q \in B(p, \delta)$ and from triangular inequality, we obtain that

$$
\beta = d_Y(f(x), f(p)) \leq d_Y(f(x), f(q)) + d_Y(f(q), f(p))
< d_Y(f(x), f(q)) + \beta - \epsilon,
$$

so, if $d_X(q, p) < \delta$ then $\epsilon < d_Y(f(x), f(q))$. As we wanted to show.

2. Final part, the conclusion of the proof. As each point $q$ in the ball $B(p, \delta)$ can be connected with $x$ using a path and since $\epsilon < d_Y(f(x), f(q))$, then the corollary leads us to conclude that $f^{-1}(S[f(q), \epsilon]) \neq \emptyset$. So the numbers $\delta(q, \epsilon)$ are well defined in the ball $B(p, \delta)$. Finally, since the open balls $B(q, \delta(q, \epsilon))$ are path-connected then the item in the theorem allows us to conclude that the numbers $\delta(q, \epsilon)$ are delta-epsilon numbers. This finishes the proof of the theorem.

Finally, the next corollary gives us sufficient conditions to calculate delta-epsilon numbers in all of the domain of the function $f$.

**Corollary 3.** Let $f : X \to Y$ be a nonconstant continuous function defined on a metric space $X$. If for all $p \in X$ and $r > 0$ the open balls $B(p, r)$ are path-connected, then there exists $\beta > 0$ such that, the delta-epsilon numbers $\delta(p, \epsilon)$ are well defined on the set $X \times (0, \beta)$.

**Proof:** According to the corollary to show this result, it is necessary to find out a positive number $\beta$ such that for every $p \in X$ there exists $x \in X$ satisfying that $d_Y(f(p), f(x)) = \beta$. In fact, since $f$ is a nonconstant function, then the diameter of $f(X)$ is positive, namely, $\text{diam}(f(X)) > R$ for some $R > 0$. So, there exist $a, b \in X$ with $R/2 < d_Y(f(a), f(b))$. Now, for $p \in X$, we have that

$$
R/2 < d_Y(f(a), f(b)) \leq d_Y(f(a), f(p)) + d_Y(f(p), f(b)),
$$

thus, we can conclude that either $R/4 < d_Y(f(a), f(p))$ or $R/4 < d_Y(f(p), f(b))$. On the other hand, since $X$ is path-connecting, then there exists $x \in X$ such that $d_Y(f(x), f(p)) = R/4$. Finally, the proof of the corollary follows from direct application of the corollaries by taking $\beta := R/4$.

Now, we establish some properties of the delta-epsilon numbers.

**Theorem 3.** Let $f : X \to Y$ be a nonconstant continuous function defined on a compact metric space $X$. Suppose that for all $p \in X$, $r > 0$ the open balls $B(p, r)$ are path-connected, let $\beta$ be the positive number obtained in the corollary then the function $\delta : X \times (0, \beta) \to \mathbb{R}^+$ defined by $(p, \epsilon) \to \delta(p, \epsilon)$ satisfies the following properties:

1. For all $p \in X$ and $\epsilon > 0$ there exists $x \in X$ such that $\delta(p, \epsilon) = d_X(p, x)$, and $d_Y(f(p), f(x)) = \epsilon$.
2. If $a < b < \beta$, then $\delta(p, a) \leq \delta(p, b)$.
3. $\lim_{n \to \infty} \delta(p, \epsilon - 1/n) = \delta(p, \epsilon)$.
4. Let $x_n$ a sequence in $X$ such that $\lim_{n \to \infty} x_n = p$, then for all $r > 0$, there exists $n_0 \in \mathbb{N}$, so that for all $n > n_0$

$$
\delta(p, \epsilon - r) - r \leq \delta(x_n, \epsilon) \leq \delta(p, \epsilon + r) + r.
$$

**Proof:**
1. Since \( f \) is a nonconstant continuous function, and \( X \) is a compact path-connected set, then \( f^{-1}(S[f(p), \epsilon]) \) is a nonempty compact set that does not contain \( p \). So the distance between \( p \) and \( f^{-1}(S[f(p), \epsilon]) \) is reached at some point \( x \in f^{-1}(S[f(p), \epsilon]) \). That is to say,

\[
\delta(p, \epsilon) = \text{dist}(p, f^{-1}(S[f(p), \epsilon])) = d_X(p, x),
\]

with \( d_Y(f(p), f(x)) = \epsilon \).

2. In terms of the notation of the theorem \([1]\) since \( a < b \), then \( \delta(p, a) \in D_{p,b} \), and since \( \delta(p, b) = \max D_{p,b} \), then we conclude that \( \delta(p, a) \leq \delta(p, b) \).

3. From the item \([2]\) the sequence \( \delta(p, \epsilon - 1/n) \) is increasing and bounded from above by \( \delta(p, \epsilon) \). On the other hand, from the item \([4]\) there exists a sequence \( x_n \in X \) such that

\[
\delta(p, \epsilon - 1/n) = d_X(p, x_n), \quad \text{and} \quad d_Y(f(p), f(x_n)) = \epsilon - \frac{1}{n}.
\]

Now, since \( X \) is compact, then there exists \( q \in X \) such that \( \lim_{k \to \infty} x_{n_k} = q \) where \( x_{n_k} \) is a subsequence of the sequence \( x_n \). So, by the continuity of the function \( f \), we have that \( \lim_{n \to \infty} \delta(p, \epsilon - 1/n) = d_X(p, q) \) with \( d_Y(f(p), f(q)) = \epsilon \), then we have that \( d_X(p, q) \leq \delta(p, \epsilon) \leq d_X(p, q) \), thus, we obtain that \( \lim_{n \to \infty} \delta(p, \epsilon - 1/n) = \delta(p, \epsilon) \). This finishes the proof of item \([3]\).

4. Since \( f \) is continuous at \( p \in X \) and \( \lim_{n \to \infty} f(x_n) = f(p) \), then \( \lim_{n \to \infty} d_Y(f(x_n), f(p)) < \epsilon \). Now, we prove that \( \delta(p, \epsilon - r) - r \in D_Z, \epsilon \). In fact, let \( y \in X \) with \( d_X(y, x_n) < \delta(p, \epsilon - r) - r \), then from triangular inequality, we obtain that \( d_X(y, p) < \delta(p, \epsilon - r) \). Finally, \( d_Y(f(y), f(x_n)) \leq d_Y(f(y), f(p)) + d_Y(f(p), f(x_n)) < \epsilon \). This finishes the proof of the first inequality of item \([4]\). The proof of the second inequality can be done similarly.

## 3 Uniform continuity and \( \delta-\epsilon \) functions

In this section, we extend the concept of \emph{delta-epsilon function} introduced in \([1]\) and use it to study the relationship between continuity and uniform continuity.

\textbf{Definition 2.} Let \( X \) be a nonempty set, a function \( f : X \to Y \) is called uniformly continuous on \( X \), if for every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for every \( x, y \in X \) with \( d_X(x, y) < \delta \), then \( d_Y(f(x), f(y)) < \epsilon \).

\textbf{Definition 3.} Let \( X \) be a nonempty set, \( f : X \to Y \) a continuous function. Let \( \epsilon > 0 \) fixed, we say that a function \( g_\epsilon : X \to \mathbb{R}^+ \) is a \emph{delta-epsilon function} for \( f \), if \( g_\epsilon(x) \) is a delta-epsilon number for \( f \) at \( x \in X \).

\textbf{Example 1.} Let \( f : X \to Y \) be a nonconstant continuous function defined on a metric space \( X \), and suppose that for all \( p \in X \) and \( r > 0 \) the open balls \( B(p, r) \subset X \) are path-connected. Then from the corollary \([3]\) and the item \([3]\) in the theorem \([1]\) we have that for any \( \epsilon \in (0, \beta) \) fixed, the function \( \delta(\cdot, \epsilon) : X \to \mathbb{R}^+ \), given by

\[
\delta(x, \epsilon) = \text{dist}(x, f^{-1}(S[f(x), \epsilon])), \quad (10)
\]

is a delta-epsilon function for \( f \) that is greater that any other delta-epsilon function for \( f \).

The following theorem gives a characterization of the uniform continuity concept in terms of delta-epsilon functions,
Theorem 4. Let $X$ a nonempty set and $f : X \to Y$ a continuous function. Then, $f$ is continuous uniformly on $X$ if and only if there exists a family $\{g_\epsilon\}_{\epsilon > 0}$ of delta-epsilon functions for $f$ such that,
\[
\eta_\epsilon := \inf_{x \in X} g_\epsilon(x) > 0,
\]for every $\epsilon > 0$.

Proof: If $f : X \to Y$ is continuous uniformly on $X$, then for $\epsilon > 0$ there exists $\delta > 0$, such that for every $x, y \in X$ with $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \epsilon$. Thus, the constant function $g_\epsilon : X \to \mathbb{R}^+$, $g_\epsilon(x) = \delta$, is a delta-epsilon function for $f$ that clearly satisfies the condition (11).

Conversely, let $\{g_\epsilon\}_{\epsilon > 0}$ a family of delta-epsilon functions for the continuous function $f$ that satisfies the condition (11). Hence, for $\epsilon > 0$ and $x, y \in X$, we have that if $d_X(x, y) < \eta_\epsilon \leq g_\epsilon(x)$, then, since $f$ is continuous at $x$ and $g_\epsilon(x)$ satisfies the continuity definition at $x$, we can conclude that $d_Y(f(x), f(y)) < \epsilon$.

Remark 2. Roughly speaking, the previous theorem says us that a continuous function $f$ that admits a family of constant delta-epsilon functions $\{\eta_\epsilon\}_{\epsilon > 0}$ is continuous uniformly.

Remark 3. In terms of the theorem 4, to show that a continuous function $f$ is not continuous uniformly, we must verify that any family $\{g_\epsilon\}_{\epsilon > 0}$ of delta-epsilon functions has an element $g_{\epsilon_0}$ such that, $\inf_{x \in X} g_{\epsilon_0}(x) = 0$, what seems to be a difficult task. For a certain class of functions the following theorem simplify this work.

Now, we give a characterization of the concept of uniform continuity in terms of the optimal delta epsilon function given in (10).

Theorem 5. Let $f : X \to Y$ be a nonconstant continuous function defined on a metric space $X$. Suppose that for all $p \in X$ and $r > 0$ the open balls $B(p, r)$ are path-connected. Then, the following three conditions are equivalents:

1. $f$ is not uniformly continuous on $X$.
2. There exists $\epsilon_0$ such that, $\inf_{x \in X} \delta(x, \epsilon_0) = 0$.
3. There exist $\epsilon_0$ and sequences $x_n, y_n \in X$, such that, $\lim_{n \to \infty} d_X(x_n, y_n) = 0$ and $d_Y(f(x_n), f(y_n)) = \epsilon_0$.

Proof:

- First, we prove that 1 implies 2. It is clear that if $f$ is not continuous uniformly on $X$, then from the theorem 4 the family of delta-epsilon functions $\{\delta(\cdot, \epsilon)\}_{\epsilon \in (0, \beta)}$ must have an element satisfying the condition 2.

- Now, we prove that 2 implies 1. Let $\{\rho_\epsilon\}_{\epsilon > 0}$ be a family of delta-epsilon functions for $f$. Then, from the item 3 of theorem 4 and item 2 of theorem 3 we have that

$$
\rho_\epsilon(x) \leq \delta(x, \epsilon) \leq \delta(x, \epsilon_0), \quad \text{for all } x \in X, \quad 0 < \epsilon \leq \epsilon_0.
$$

Hence, from the condition 2 above, we obtain that $\inf_{x \in X} \rho_\epsilon(x) = 0$. So, from theorem 4 we can conclude that $f$ is not continuous uniformly on $X$. 

• Next, we prove that $\square$ implies $\square$. Since $\inf_{x \in X} \delta(x, \epsilon_0) = 0$, then for all $n \in \mathbb{N}$, there exists $x_n \in X$ such that $0 < \delta(x_n, \epsilon_0) < 1/n$. By the definition of $\delta(x_n, \epsilon_0)$ there exist $y_n \in X$, satisfying, $0 < \delta(x_n, \epsilon_0) \leq d_X(x_n, y_n) < 1/n$ and $d_Y(f(x_n), f(y_n)) = \epsilon_0$. Thus, we obtain two sequences of elements $x_n, y_n \in X$, such that, $\lim_{n \to \infty} d_X(x_n, y_n) = 0$ and $d_Y(f(x_n), f(y_n)) = \epsilon_0$.

• Finally, we prove that $\square$ implies $\square$. If the condition $\square$ holds, then, we deduce that $0 < \delta(x_n, \epsilon_0) \leq d_X(x_n, y_n)$. Hence, $\lim_{n \to \infty} \delta(x_n, \epsilon_0) = 0$, what implies that $\inf_{x \in X} \delta(x, \epsilon_0) = 0$. As we wanted to show.

4 Examples

In this section, from some specific examples, we illustrate the theory developed in sections $\square$ and $\square$. The first example is elementary, however, it explains our principal results clearly.

Example 2. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$, according to the theorem $\square$ it is clear that

$$f^{-1}(S[f(p), \epsilon]) \supset \{\sqrt{p^2 + \epsilon}, -\sqrt{p^2 + \epsilon}\} \neq \emptyset,$$

then, the hypothesis 1 in the theorem $\square$ is satisfied. After some calculations, we obtain that:

$$\delta(p, \epsilon) = \text{dist}(p, f^{-1}(S[f(p), \epsilon])) = \sqrt{p^2 + \epsilon} - |p|.$$ 

Now, since $B(p, \delta(p, \epsilon))$ is connected, then, we can conclude that the numbers $\delta(p, \epsilon)$ are maximum delta-epsilon numbers for $f$, that is:

$$\text{if } |x - p| < \sqrt{p^2 + \epsilon} - |p|, \text{ then } |x^2 - p^2| < \epsilon.$$ 

In addition, we obtain that

$$D_{p, \epsilon} = \left(0, \sqrt{p^2 + \epsilon} - |p|\right).$$

On the other hand, as the function $p \to \delta(p, \epsilon)$ is even a decreasing for $p > 0$, then, we deduce that for $M > 0$

$$\inf_{p \in B(0,M)} \delta(p, \epsilon) \geq \sqrt{M^2 + \epsilon} - M > 0,$$

thus, from the theorem $\square$, $f$ is continuous uniformly over bounded domains. In contrast, due to the fact that

$$\lim_{|p| \to \infty} \delta(p, \epsilon) = 0,$$

then, from the theorem $\square$ we obtain that $f$ is not continuous uniformly over unbounded domains.

The following lemmas allow us to study uniform continuity of radial functions defined on normed vector spaces,

Lemma 1. Let $I \subset \mathbb{R}$ be a nondegenerate interval, $g : I \to \mathbb{R}$ an increasing bijective function. Then the numbers

$$\delta(p, \epsilon) = \min\{g^{-1}(g(p) + \epsilon) - p, p - g^{-1}(g(p) - \epsilon)\},$$

are maximum delta-epsilon numbers for the function $g$. Here, $\epsilon$ has to be chosen in such a way that some of the numbers $g^{-1}(g(p) + \epsilon) - p, p - g^{-1}(g(p) - \epsilon)$ be well defined.
**Proof:** Under these hypothesis, we can conclude that the function $g$ is continuous and nonconstant. Hence, since $I$ is connected, then the corollary implies the conclusion of the lemma. We point out that this lemma is exactly the lemma 1.2 in [1].

**Lemma 2.** Let $V$ be a normed vector space, $I \subset \mathbb{R}^+$ be a nondegenerate interval, $g : I \to \mathbb{R}$ a nonconstant continuous function and

$$\Omega = \{ x \in V \mid ||x|| \in I \},$$

if we define $f : \Omega \to \mathbb{R}$ by $f(x) = g(||x||)$ then, for $p \in \Omega$, we have that $\delta(||p||, \epsilon) \in D_{p, \epsilon}$. That is, the maximum delta-epsilon number $\delta(||p||, \epsilon)$ for $g$ at $||p||$ is a delta-epsilon number for the radial function $f$ at $p$.

**Proof:** First of all, we note that since $g$ is nonconstant and it is defined on an interval, then the corollary allows us to conclude that the numbers $\delta(||p||, \epsilon)$ are well defined in the sense of the formula (10). Now, suppose that $||x - p|| < \delta(||p||, \epsilon)$, then, $||x|| - ||p|| < \delta(||p||, \epsilon)$, and thus,

$$|f(x) - f(p)| = |g(||x||) - g(||p||)| < \epsilon,$$

as we wanted to prove.

**Example 3.** Let $V$ be a normed vector space, if we consider the function $f : V \to \mathbb{R}$, defined by

$$f(p) = e^{||p||},$$

then, from the example 6 in [1] and the previous lemma, we can infer that the family

$$\eta(p, \epsilon) = \ln(e^{||p||} + \epsilon) - ||p||,$$

is a family of delta-epsilon functions for $f$. In addition, since for $p \neq 0$, the function

$$t \to g(t) = e^{t||p||},$$

is an homeomorphism from $[0, \infty)$ to $[1, \infty)$, then from the lemma we obtain that

$$\delta(t, \epsilon) = \frac{\ln(e^{t||p||} + \epsilon) - t||p||}{||p||},$$

is a family of maximum delta-epsilon functions for $g$. Now, as

$$\lim_{t \to \infty} \frac{\ln(e^{t||p||} + \epsilon) - t||p||}{||p||} = 0,$$

we conclude that

$$\inf_{t \in [0, \infty)} \delta(t, \epsilon) = 0,$$

and from the theorem the function $g$ is not continuous uniformly on $[0, \infty)$. As a result of this analysis, we also obtain that the function $f$ is not continuous uniformly on any domain containing the set $\{tp \mid t \in [0, \infty)\} \subset V$. On the other hand, it is not difficult to show that the delta-epsilon functions given in (13) satisfy

$$\inf_{p \in B(0, M)} \eta(p, \epsilon) \geq \ln(e^M + \epsilon) - M > 0,$$

hence, from the theorem we deduce that $f$ is continuous uniformly on bounded domains.
**Example 4.** Similarly, from the example 7 in [1] and the lemma [2] we can deduce that the function 
\[ f : \mathbb{V} - \{0\} \to \mathbb{R} \]
defined by
\[ f(p) = \ln(||p||), \]
has the following family of delta-epsilon functions:
\[ \eta(p, \varepsilon) = ||p|| (1 - e^{-\varepsilon}), \] (14)
thus, from the theorem [4] we obtain that for any \( M > 0 \), \( f \) is continuous uniformly on the domain \( \mathbb{V} - B(0, M) \). On the other hand, following the ideas of the previous example it is possible to conclude that \( f \) is not continuous uniformly on \( \mathbb{V} - \{0\} \).

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