On the mapping class groups of \(\#_r S^p \times S^p\) for \(p = 3, 7\)

Diarmuid J. Crowley

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Abstract

For \(M_r = \#_r S^p \times S^p\), \(p = 3, 7\), we calculate \(\pi_0 \text{Diff}(M_r)/\Theta_{2p+1}\) and \(E(M_r)\), respectively the group of isotopy classes of orientation preserving diffeomorphisms of \(M_r\) modulo isotopy classes with representatives which are the identity outside a \(2p\)-disc and the group of homotopy classes of orientation preserving homotopy equivalences of \(M_r\).

1 Introduction

Let \(p = 3\) or \(7\) and let \(M_r = \#_r S^p \times S^p\) be the \(r\)-fold connected sum of \(S^p \times S^p\). We consider the following mapping class groups of equivalence classes of isomorphisms of \(M_r\). Let \(\pi_0 \text{Diff}(M_r)\) be the group of isotopy classes of orientation preserving diffeomorphisms of \(M_r\), and let \(\text{Aut}(M_r) := \pi_0 \text{Diff}(M_r)/\Theta_{2p+1}\) be the quotient of \(\pi_0 \text{Diff}(M_r)\) by the subgroup of isotopy classes with representatives which are the identity outside a \(2p\)-disc. Let \(E(M_r)\) be the group of homotopy classes of orientation preserving homotopy equivalences of \(M_r\). Let \(\phi_r : H_p(M_r) \times H_p(M_r) \to \mathbb{Z}\) be the intersection form of \(M_r\) which is a nonsingular anti-symmetric hyperbolic form on \(H_p(M_r) \cong \mathbb{Z}^{2r}\). Let \(\text{Aut}(\phi_r)\) be the automorphism group of \(\phi_r\), the symplectic group. Taking the induced map on homology defines surjective homomorphisms \(H_p : \text{Aut}(M_r) \to \text{Aut}(\phi_r)\) and \(H_p : E(M_r) \to \text{Aut}(\phi_r)\) which fit into commuting extensions, the upper due to [Kre][Thm. 2], the lower [B][Thm. 8.14] and the diagram [B][Thm. 10.3],

\[
\begin{array}{cccccc}
0 & \text{Hom}(H_p(M_r), S\pi_p(SO_p)) & \xrightarrow{I} & \text{Aut}(M_r) & \xrightarrow{H_p} & \text{Aut}(\phi_r) & 1 \\
& J \circ \text{PD} & & & & \\
0 & H_p(M_r) \otimes S\pi_{2p}(S^p) & \xrightarrow{I} & E(M_r) & \xrightarrow{H_p} & \text{Aut}(\phi_r) & 1.
\end{array}
\]

Here \(E : \text{Aut}(M_r) \to E(M_r)\) is the obvious map which is onto by Proposition 1.6, the map \(J \circ \text{PD}\) is explained in Subsection 1.1. \(S : \pi_p(SO_p) \to \pi_p(SO_{p+1})\) and \(S : \pi_{2p}(S^p) \to \pi_{2p+1}(S^{p+1})\) are the stabilisation maps, \(S\pi_p(SO_p) \cong \mathbb{Z}\) and \(S\pi_{2p}(S^p) \cong \mathbb{Z}_{12}\) or \(\mathbb{Z}_{120}\) for \(p = 3\) or \(7\). In Corollary 3.5, we identify these extensions algebraically and as a consequence prove

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Corollary 1.4. If $\phi_r$ homotopy, on $\pi$ Remark 1.3. Despite Theorem 1.1 [F] and [Kry2] show that a consequence of the existence of maps of Hopf invariant one as we discuss below.

Proof. For $r > 1$ such an action would provide a splitting for $\mathcal{E}(M_r) \to \text{Aut}(\phi_r)$ which by Theorem 1.1 does not exist. For $r = 1$ and $p = 3$ such an action would provide a splitting for $\pi_0\text{Diff}(S^3 \times S^3) \to \text{Aut}(\phi_1)$ which by Remark 1.3 does not exist. 

The rest of the paper is organised as follows: we complete the introduction by recalling the results of [Kre] and [B] which establish the extensions in (1). We also recall how [Kry1] and [B] split the extensions mentioned in Remark 1.2 and explain why their methods fail in dimensions 6 and 14. In section 2 we recall and assemble the necessary algebraic facts about 1-cocycles, quadratic forms and group extensions which we require to identify the groups $\text{Aut}(M_r)$ and $\mathcal{E}(M_r)$ algebraically. Section 3 contains the topology: we construct a 1-cocycle on $\text{Aut}(M_r)$ called the global derivative and in Lemma 3.4 we apply some results from surgery theory to give an algebraic description of the global derivative. This leads to Corollary 3.5 where the extensions in (1) are identified with the algebraic extensions in (5) from Section 2. We thereby calculate $\text{Aut}(M_r)$ and $\mathcal{E}(M_r)$ and prove Theorem 1.1.

1.1 Theorems of Kreck and Baues

Throughout this subsection $M$ is a closed, $(k - 1)$-connected almost-parallelisable, smooth 2k-manifold, $k \geq 3$. Let $\pi_0\text{Diff}(M)$ and let $\text{Aut}(M) := \pi_0\text{Diff}(M)/\Theta_{2k+1}$ be defined as for $M_r$ above. Let $\text{Aut}(\alpha_M)$ denote the group of automorphisms of $H_k(M) := H_k(M; \mathbb{Z})$ which preserve the extended quadratic form $(H_k(M), \lambda_M, \alpha_M)$ defined in [Wa]: $\lambda_M$ is the usual intersection form on $H_k(M)$ and $\alpha_M : H_k(M) \to \pi_{k-1}(SO_k)$ gives the isomorphism class of the normal bundle of an embedded sphere representing an element of $H_k(M)$. In [Kre][Thm. 2] Kreck a gives pair of exact sequences which compute $\pi_0\text{Diff}(M)$ and we adapt the sequences to the simpler case of $\text{Aut}(M)$:

\[ 0 \to \pi_0\text{SAut}(M) \xrightarrow{I} \text{Aut}(M) \xrightarrow{H_k} \text{Aut}(\alpha_M) \to 0, \]  

(2)

\[ 0 \to \pi_0\text{SAut}(M) \xrightarrow{\chi} \text{Hom}(H_k(M), S\pi_{k}(SO_k)) \to 0. \]  

(3)

Here $\pi_0\text{SAut}(M)$ is by definition the kernel of $H_k$. Elements $[f]$ of $\pi_0\text{SAut}(M)$ have representatives $f : M \cong M$ which fix $k$-spheres representing generators of $H_k(M)$ and the map $\chi$
is defined by taking the derivative of $f$ restricted to the normal bundle of these $k$-spheres: we shall return to this point in the proof of Lemma 3.4. The upper exact sequence of (1) follows by using $\chi$ from (3) to identify $\text{Ker}(H_p) \subset \text{Aut}(M_r)$ with $\text{Hom}(H_p(M_r), S\pi_p(SO_p))$ and from the fact that the extended quadratic form of $M_r$ is identical to the intersection form since $\pi_{p−1}(SO_p) = 0$.

We now discuss how maps of Hopf invariant one affect splitting results for $\text{Aut}(M)$: the key point is that if $c_j$ is the order of the cokernel of the stabilisation map $\pi_{4j−1}(SO_{4j−1}) \to \pi_{4j−1}(SO)$ then $c_1 = c_2 = 2$ and $c_j = 1$ for $j > 2$. Next we recall and correct a construction from [Kre] bearing $c_j$ in mind. Let $k = 4j − 1$. Given $[f] \in \pi_0\text{Aut}(M)$, let $M_f = M \times [0, 1]/(0, m) \sim (1, f(m))$ be the mapping torus of $M$, let $p_j(M_f) \in H^{4j}(M_f)$ be the $j$-th Pontrjagin class of $M_f$ and let $c : M_f \to \Sigma(M_+)$ be the map collapsing $M \times \{0\} \subset M_f$ to a point where $\Sigma(M_+)$ denotes the suspension of $M$ plus a disjoint point. As $H_k([f]) = \text{Id}$, the map $c$ induces an isomorphism $c^* : H^{4j}(\Sigma(M_+)) \cong H^{4j}(M_f)$. Identifying $H^{4j}(\Sigma(M_+)) = H^{4j−1}(M) = \text{Hom}(H_{4j−1}(M), \pi_{4j−1}(SO))$ Kreck defines

$$p([f]) := c^{−1}(p_{4j}(M_f)) \in \text{Hom}(H_{4j−1}(M), \pi_{4j−1}(SO)).$$

**Lemma 1.5.** (c.f. [Kre][Lemma 2]) Let $a_j = (3 − (−1)^j)/2$. The map $[f] \mapsto p([f])$ defines a homomorphism $\pi_0\text{Aut}(M) \to \text{Hom}(H_{4j−1}(M), \pi_{4j−1}(SO))$ such that

$$p([f]) = ±a_j c_j(2j − 1)! \chi([f]).$$

**Proof.** This is Lemma 2 of [Kre] restated for $\pi_0\text{Aut}(M)$ as opposed to $\pi_0\text{Diff}(M)$, the analogous subgroup of $\pi_0\text{Diff}(M)$, and corrected to include the factor $c_j$ which was omitted in [Kre]. There $S\pi_{4j−1}(SO_{4j−1})$ is identified with $\mathbb{Z}$ and then with $\pi_{4j−1}(SO)$: a move which is valid only if $c_j = 1$. Kreck’s proof of his Lemma 2 gives a correct proof of this lemma. □

In [Kry1][Ch. 3] Krylov extended the definition of $p$ to all of $\text{Aut}(M)$ and in [Kry1][Thm. 3.2] he used the extended function $p$ and [Kre][Lemma 2] to construct a splitting of (2). This elegant argument works perfectly when $c_j = 1$ and so for $k \neq 3, 7$ (2) splits. However, when $k = 3, 7$ the existence of $p$ is not enough to split (2) in general as we prove in Section 3.

We turn now to the homotopy category and results of [B]. Let $E(M)$ be the group of homotopy classes of orientation preserving homotopy equivalences of $M$. Note that as we restrict our attention to orientation preserving maps we write $E(M)$ for what Baues calls $\mathcal{E}_+(M)$. Let $M^\bullet$ denote $M − D^{2k}$ so that $M^\bullet$ is a $k$-skeleton of $M$, $M^\bullet \simeq vS^k$ and $M$ is homotopy equivalent to $M^\bullet \cup_f D^{2k}$ where $f : S^{2k−1} \to M^\bullet$ is the attaching map for the top cell of $M$. The fundamental extension for $E(M)$ is the short exact sequence

$$0 \longrightarrow E(M | M^\bullet) \longrightarrow E(M) \longrightarrow E(M^\bullet, f) \longrightarrow 0 \quad (4)$$

where $E(M | M^\bullet)$ is the subgroup of homotopy classes containing representatives which are the identity on $M^\bullet$, $E(M^\bullet, f)$ is the group of homotopy classes of homotopy equivalences of $M^\bullet$ compatible with the attaching map $f$ and $E(M | M^\bullet)$ is an $E(M^\bullet, f)$-module whose underlying group is finite abelian by [B][Thm. 1.4]. The lower exact sequence of (1) is the fundamental extension for $M_r$ where $E(M_r | M_r^\bullet) \cong H \otimes S\pi_{2p}(Sp)$ by [B][Thm. 7.6] and $E(M_r^\bullet, f) \cong \text{Aut}(\phi_r)$. 

3
In [E][Thm. 10.3] Baues identifies the restriction of \( E : \text{Aut}(M_r) \to E(M_r) \) to Ker\((H_p)\) as a composition of Poincaré duality \( PD : \text{Hom}(H, S\pi_p(SO_p)) \to H \otimes S\pi_p(SO_p) \) with the \( J \)-homorphism \( J : S\pi_p(SO_p) \to S\pi_{2p}(S^p) \) applied to the second factor: \( \text{Id} \otimes J : H \otimes S\pi_p(SO_p) \to H \otimes S\pi_{2p}(S^p) \). In particular since \( J \) is surjective \((\text{Id} \otimes J) \circ PD \) is surjective and we have

**Proposition 1.6.** The homomorphism \( E : \text{Aut}(M_r) \to E(M_r) \) is surjective.

We now discuss how the existence of maps of Hopf invariant one effects Baues’ results on splitting in the fundamental extension. Call \( M \) \( \Sigma_1 \)-reducible if the suspension \( \Sigma M \) is homotopic to \( S^{2k+1} \vee \Sigma M^* \). For example \( M = S_r S^k \times S^k \) is \( \Sigma_1 \)-reducible with \( E(M \mid M^*) \cong S\pi_{2k}(\Sigma_2 S^k) \). Theorem 5.2 of [E] gives a criterion for when the fundamental extension of \( M \) splits: one requires that the inclusion \( S\pi_{2k}(\Sigma_2 S^k) \subset \pi_{2k+1}(\Sigma_2 S^{k+1}) \) admits a retraction of \( E(M^*, f) \)-modules. If \( k \geq 3 \) is odd and \( k \neq 3,7 \) this criterion holds but for \( M_r \) it fails: the inclusion \( \pi_{2p}(\Sigma_2 S^p) \to \pi_{2p+1}(\Sigma_2 S^{p+1}) \) is isomorphic to \( \oplus_{2r} \mathbb{Z} \to \oplus_{2r} (\mathbb{Z} \oplus \mathbb{Z}) \) where \( c = 12, 120 \) if \( p = 3, 7 \) and there is a retraction as abelian groups but not of \( E(M^*, f) \)-modules. The point being that \(-1\text{Id}\) acts by \(-1\) on \( \mathbb{Z}_c \subset \pi_{2p+1}(S^p) \) but sends \((1, 0) \in \pi_{2p+1}(S^p)\) to \((1, −1)\) where \((1, 0)\) is the Hopf map and \((0, 1)\) is a suitable generator of \( S\pi_{2p}(S^p) \): see [E][§9]. The case of \( M_1 = S^p \times S^p \) shows that the splitting criterion of [E][Thm. 5.2] is not a necessary condition for splitting.

We conclude the introduction by correcting statements in [E] pertaining to splitting the fundamental extension for \( M_r \). Note that since every \( M_r \) has an orientation reversing map of order two we may confine our attention to \( E(M_r) \). Theorem 5.2 of [E] computes \( E(S_k \times S^k) \) and is correct as stated but the proof of splitting given fails for \( n = 3, 7 \) in which case a correct proof is given in Corollary 3.3 below and can also be found in [Kry2] for \( k = 3 \). The final sentence of [E][Thm. 8.14] should be altered to exclude the dimensions \( k = 3, 7 \) and the correct statement is found in Corollary 3.3 since for \( k = 3 \) or 7 the only \( \Sigma_1 \)-reducible \((k − 1)\)-connected Poincaré complexes of dimension \( 2k \) are homotopy equivalent to \( M_r \) for some \( r \).

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## 2 1-cocycles, quadratic forms and groups extensions

In this section we develop the necessary algebraic preliminaries required to describe \( \text{Aut}(M_r) \): we first review the definition of a 1-cocycle, then we recall some basic facts about quadratic forms and groups extensions and then we construct some nontrivial extensions of the symplectic group.

Let \( S \) be a group with identity \( \text{Id} \) which acts on the right on an abelian group \( X \). A 1-cocycle is a map \( s : S \to X \) satisfying

\[
s(AB) = s(A) \cdot B + s(B) \quad \forall A, B \in S.\]

Note that necessarily \( s(\text{Id}) = 0 \in X\).
Definition 2.1. The set of 1-cocycles, $Z^1(S;X)$, is an abelian group under pointwise addition. Every $x \in X$ defines a 1-cocycle by

$$s(x)(A) := x \cdot A - x$$

and the set of such 1-cocycles forms a subgroup $B^1(S;X) \subset Z^1(S;X)$. The first cohomology group of $S$ with coefficients in $X$ is by definition

$$H^1(S;X) := Z^1(S;X)/B^1(S;X).$$

We shall use the above definition in the following context from the introduction: $H$ is a free abelian group of rank $2r$, $\phi : H \times H \rightarrow \mathbb{Z}$ is a nonsingular, anti-symmetric bilinear form (here suppress the subscript $r$ on $\phi$), $S = \text{Aut}(\phi)$ is the symplectic group and $X = H^*$ where $H^* = \text{Hom}(H,\mathbb{Z})$ is the dual of $H$ where $\text{Aut}(\phi)$ acts on the right of $H^*$ by composition:

$$\alpha : H^* \times \text{Aut}(\phi) \longrightarrow H^*, \quad (x, A) \longmapsto x \cdot A := x \circ A.$$

Our interest in 1-cocycles springs from the need to understand subgroups and quotients of the semi-direct product defined by this action, which is called the Jacobi group

$$\Gamma(\phi) := H^* \times_\alpha \text{Aut}(\phi), \quad (x, A) \cdot (y, B) = (x \cdot B + y, AB).$$

Given a cyclic group $C$ of order $c$ we shall also consider the case where $X = H_c^* := H^* \otimes C$ with the induced $\text{Aut}(\phi)$ action and the Jacobi group $\Gamma(\phi, C) := H_c^* \times_\alpha \text{Aut}(\phi)$ with the obvious surjection $\Gamma(\phi) \rightarrow \Gamma(\phi, C)$. We shall identify $H^* = H^*_c$ and $\Gamma(\phi, \mathbb{Z}) = \Gamma(\phi)$.

Next recall that a quadratic refinement of the hyperbolic form $\phi : H \times H \rightarrow \mathbb{Z}$ is a function $\psi : H \rightarrow \mathbb{Z}_2$ such that

$$\psi(v + w) = \psi(v) + \psi(w) + \bar{\phi}(v,w)$$

where $\bar{\phi}$ is the mod 2 reduction of $\phi$. Let $\Psi$ be the set of all such $\psi$ refining $\phi$. The symplectic group $\text{Aut}(\phi)$ acts on $\Psi$ by $\psi \cdot A = \psi \circ A$ and we shall use this action to construct a 1-cocycle on $\text{Aut}(\phi)$ with values in $H^*_2 = H^* \otimes \mathbb{Z}_2$.

Lemma 2.2. (i) For all $\psi_0, \psi_1 \in \Psi$ the function $\psi_1 - \psi_0$ belongs to $H^*_2$ and moreover $H^*_2$ acts freely and transitively on $\Psi$ by addition of functions.

(ii) The orbits of the action of $\text{Aut}(\phi)$ on $\Psi$ are the isomorphism classes of quadratic refinements of $\phi$. For each $r$ there are two orbits which are detected by the Arf invariant, $|\Psi_0| = 2^{2r-1} + 2 r^{-1}$, $|\Psi_1| = 2^{2r-1} - 2 r^{-1}$.

(iii) Fixing $\psi \in \Psi$ defines a 1-cocycle $s(\psi) : \text{Aut}(\phi) \rightarrow H^*_2$

$$s(\psi)(A) = \psi \cdot A - \psi.$$

(iv) For all $\bar{x} \in H^*_2$, $s(\psi + \bar{x}) = s(\psi) + s(\bar{x})$.  

5
Proof. The first two statements are well known so we omit their proof as well as the definition of the Arf invariant. The third and fourth statements are easily checked. For example:

\[ s(\psi)(AB) = \psi \cdot (AB) - \psi = (\psi \cdot A - \psi) \cdot B + \psi \cdot B - \psi = s(\psi)(A) \cdot B + s(\psi)(B). \]

For a cyclic group \( C \) which is of even order \( c \) or infinite, \( c = \infty \), the short exact sequence of \( \text{Aut}(\phi) \)-modules

\[ 0 \longrightarrow 2H^*_c \xrightarrow{Q_c} H^*_c \xrightarrow{P_c} H^*_2 \longrightarrow 0 \]

gives rise to the is a long exact sequence in cohomology, where we write \( S = \text{Aut}(\phi) \),

\[ \ldots \longrightarrow H^1(S;2H^*_c) \xrightarrow{Q_c} H^1(S;H^*_c) \xrightarrow{P_c} H^1(S;H^*_2) \xrightarrow{\partial_c} H^2(S;2H^*_c) \longrightarrow \ldots . \]

By Brown [Br] [Ch. IV] the group \( H^2(\text{Aut}(\phi);2H^*_c) \) stands in bijective correspondence with isomorphism classes of group extensions

\[ 0 \longrightarrow 2H^*_c \xrightarrow{i} G \xrightarrow{\pi} \text{Aut}(\phi) \longrightarrow 1. \]

Now for \( x \in H^*_c \), let \( \bar{x} \in H^*_2 \) be the reduction of \( x \) mod 2. By [Br][Ex. 2, Ch. IV, §3, p. 91], the extension defined by \( \partial_c[s(\psi)] \) is the extension

\[ 0 \longrightarrow 2H^*_c \xrightarrow{i} \Gamma(\psi, C) \xrightarrow{\pi} \text{Aut}(\phi) \longrightarrow 1 \tag{5} \]

where

\[ \Gamma(\psi, C) := \{(x, A) \in H^*_c \times_{\alpha} \text{Aut}(\phi) | \bar{x} = s(\psi)(A) \} \subset \Gamma(\phi, C), \tag{6} \]

and \( \pi(x, A) = A \). The fact that \( s(\psi) \) is a 1-cocycle ensures the \( \Gamma(\psi, C) \) is a group. We shall write \( \Gamma(\psi) \) for \( \Gamma(\psi, \mathbb{Z}) \) and \( P_\infty : \Gamma(\psi) \rightarrow \Gamma(\psi, C) \) for the obvious surjection.

**Proposition 2.3.** If \( c \) is divisible by 4 or \( c = \infty \) then the extension

\[ 0 \longrightarrow 2H^*_c \xrightarrow{i} \Gamma(\psi, C) \xrightarrow{\pi} \text{Aut}(\phi_r) \longrightarrow 1 \]

splits if and only if \( r = 1 \): i.e. if and only if \( H \cong \mathbb{Z}_2 \).

**Proof.** By the above discussion we must show that \( \partial_c[s(\psi)] = 0 \in H^2(\text{Aut}(\phi);2H^*_c) \) if and only if \( r = 1 \). This happens if and only if there is a 1-cocycle \( s_c : \text{Aut}(\phi) \rightarrow H^*_c \) which reduces to \( s(\psi) \) mod 2. First we note that by Lemma 2.2 (iv) \( [s(\psi)] = 0 \) if and only if there exists \( \bar{x} \in H^*_2 \) such that \( s(\psi + \bar{x}) \) is identically zero. But for any \( \chi \in \Psi \), \( s(\chi) \) is identically zero if and only if the orbit \( \chi \cdot \text{Aut}(\phi) \) contains precisely one element. By Lemma 2.2 (ii) this happens only for the Arf invariant 1 quadratic refinement when \( r = 1 \).

We conclude that \( \Gamma(\psi, C) \rightarrow \text{Aut}(\phi) \) splits if \( r = 1 \) and also that we can complete the proof by showing that \( [s(\psi)] = 0 \) if and only if \( [s(\psi)] \) lies in the image of \( P_c : H^1(\text{Aut}(\phi), H^*_c) \rightarrow H^1(\text{Aut}(\phi), H^*_2) \). We do this now. For any quadratic form \( \psi \in \Psi \) refining \( \phi \) and any \( v \in H \)

\[ 0 = \psi(0) = \psi(v - v) = \psi(v) + \psi(-v) + \tilde{\phi}(v, -v) = \psi(v) + \psi(-v) \in \mathbb{Z}_2. \]
Hence $-\text{Id}$ is an automorphism of every $\psi$ and $s(\psi)(-\text{Id}) = 0$ for all $\psi \in \Psi$. Thus if $s_c \in Z^1(\text{Aut}(\phi), H^*_c)$ maps to $s(\psi)$, $s_c(-\text{Id}) = 2x$ for some $x \in H^*_c$. But for any 1-cocycle $s$ with values in $H^*_c$,

$$s(-A) = -s(A) + s(-\text{Id}) \quad \text{and} \quad s(-A) = s(-\text{Id}) \cdot A + s(A)$$

and so $2s(A) = -(s(-\text{Id}) \cdot A - s(-\text{Id}))$. Hence $2(s_c(A) + s(x)(A)) = 0$. It follows that $s_c(A) + s(x)(A) \in 2H^*_c$: recall that $c$ is divisible by 4 or $c = \infty$. Thus $[s_c] = Q_c[s_{2c}]$ for some $[s_{2c}] \in H^1(\text{Aut}(\phi), 2H^*_c)$ and so $[s(\psi)] = P_c(Q_c([s_{2c}])) = 0$. \hfill $\square$

3 The global derivative and $\text{Aut}(M_r)$

Recall that $M_r = \sharp_p S^p \times S^p$ for $p = 3, 7$. In this section we set $H = H_p(M_r)$ and identify $\pi_p(SO) = \mathbb{Z}$, $\pi_p(SO_p) = 2\mathbb{Z}$ and $\text{Hom}(H, S\pi_p SO_p) = 2H^*$ so that the upper extension of $\mathbb{I}$ is isomorphic to

$$0 \longrightarrow 2H^* \xrightarrow{I} \text{Aut}(M_r) \xrightarrow{H_p} \text{Aut}(\phi_r) \longrightarrow 1.$$ 

For every manifold $M_r$ there there are stable framings of $\tau(M_r)$, the tangent bundle of $M_r$, which are bundle isomorphisms $F : \tau(M_r) \oplus \varepsilon(M_r) \cong \varepsilon^{2p+1}(M_r)$ where $\varepsilon^k(M_r) = M_r \times \mathbb{R}^k$ is the trivial rank $k$ vector bundle over $M_r$. We begin this section by showing how a stable framing of the tangent bundle of $M_r$ defines a 1-cocycle $s_F : \text{Aut}(M_r) \to H^*$. As we mentioned in Subsection [14] this idea goes back to [Kry2][Ch.3]. We take a different approach from [Kry2] using what we call the global derivative.

For a bundle $E$ over $M_r$ we write $E_m$ for the fibre of $E$ over $m$. Fix a stable framing $F$ of $M_r$ and for each $m \in M_r$ let $F_m$ be the isomorphism $(\tau(M_r) \oplus \varepsilon(M_r))_m \cong \mathbb{R}^{2p+1}$ defined by $F$. Given a diffeomorphism $f : M_r \cong M_r$ let $D(f) : \tau(M_r) \cong \tau(M_r)$ denote the derivative of $f$ with $D_m(f) : \tau(M_r)_m \cong \tau(M_r)_{f(m)}$. Now for each $m \in M_r$ define $s_F(m, f) \in GL_{2p+1}(\mathbb{R})$, by

$$s_F(m, f) := F_{f(m)} \circ (D_m(f) \oplus \text{Id}) \circ (F_m)^{-1} : \mathbb{R}^{2p+1} \cong \mathbb{R}^{2p+1}.$$ 

Choosing a retraction from $GL_{2p+1}(\mathbb{R})$ to $SO(2p + 1)$ and composing with the inclusion $SO(2p + 1) \hookrightarrow SO$, we obtain a continuous map $s_F(f) : M \to SO$. Evidently, the homotopy class of $s_F(f)$ is independent of the isotopy class of $f$ and so we obtain a map $s_F : \pi_0 \text{Diff}(M_r) \to [M_r, SO]$ called the global derivative. Elementary obstruction theory shows that $[M_r, SO] \cong [M^*_r, SO] \cong H^0(M_r; \pi_p(SO)) = H^*$. If $f$ is the identity outside a 2p-disc then $s_F([f])$ will be constant over a $p$-skeleton of $M$ and hence $s_F$ descends to a map $s_F : \text{Aut}(M_r) \to H^*$. Note that the set of homotopy classes $[M_r, SO]$ is a group under pointwise composition in $SO$ and $[M_r, SO] = H^*$ is an $\text{Aut}(M_r)$ module under the right action of by pre-composition of functions. Thus for $x \in [M_r, SO] \cong H^*$ and $[g] \in \text{Aut}(M_r)$, $x \circ [g] = x \cdot H_p([g])$.

**Lemma 3.1.** The map $s_F : \text{Aut}(M_r) \to H^*$ is a 1-cocycle with $s_F \circ I = \text{Id}_{2H^*}$. 

7
Proof. Let $f, g : M_r \cong M_r$ be diffeomorphisms. Applying the chain rule at $m \in M_r$ we have

$$s_F(m, f \circ g) = F_{(g(m))} \circ (D_m(f \circ g) \oplus \text{Id}) \circ (F_m)^{-1}$$

$$= F_{g(m)} \circ (D_m(g) \oplus \text{Id}) \circ (D_m(f) \oplus \text{Id}) \circ (F_m)^{-1}$$

$$= F_{g(m)} \circ (D_m(g) \oplus \text{Id}) \circ (F_m)^{-1} \circ F_g(m) \circ (D_m(g) \oplus \text{Id}) \circ (F_m)^{-1}$$

$$= s_F(g(m), f) \circ s_F(m, g).$$

As the group structure on $[M_r, SO] \cong H^*$ is point-wise composition we have

$$s_F([f \circ g]) = s_F([f]) \circ [g] + s_F([g]) = s_F([f]) \cdot H_p([g]) + s_F([g]).$$

For the second statement, note that $s_F$ is a homomorphism on the image of $I(2H^*)$. We do this now by choosing so called “Dehn twists” to represent a basis of $I(2H^*)$. Choose a symplectic basis $\{u_1, v_1, \ldots, u_r, v_r\}$ for $H = H_p(M_r)$, let $S^p \times D^p \subset M_r$ represent $u_j \in H$ and let $\alpha \in S\pi_p(SO_p) = \mathbb{Z}$. Choose an appropriate smooth map $(D^p, S^{p-1}) \to (SO(p + 1), \text{Id})$ representing $\alpha$ so that $f_\alpha$ defined by $(x, y) \mapsto (x, \alpha(x)y)$ on $S^p \times D^p$ and extension by the identity to the rest of $M_r$ is a diffeomorphism. It is easy to check that $s_F([f_\alpha])(u_i) = 0$ and $s_F([f_\alpha])(v_i) = \delta_{ij} \alpha$ where $\delta_{ij}$ is the Kronecker delta. This is precisely the element of $2H^*$ defined by $\chi([f_\alpha])$ where $\chi$ is the map of [3] from Subsection [4.1]. The same argument holds also for the basis vectors $v_i$ and we are done.

Recall from Section [2] that $\Gamma(\phi_r) = H^* \times \alpha \text{Aut}(\phi_r)$ is the Jacobi group of $\phi_r$.

**Lemma 3.2.** Together $s_F$ and the projection $H_p$ define an injective homomorphism

$$s_F \times H_p : \text{Aut}(M_r) \to \Gamma(\phi_r)$$

$$[f] \mapsto (s_F([f]), H_p([f])).$$

**Proof.** Injectivity is clear since $\text{Ker}(H_p) = \text{Im}(I)$ and $s_F \circ I = \text{Id}_{2H^*}$ by Lemma [3.4]. That $(s_F \times H_p)$ is a homomorphism follows from the 1-cocyle property of $s_F$ and the definition of the Jacobi group:

$$(s_F \times H_p)([f \circ g]) = (s_F([f]) \cdot H_p([g]) + s_F([g]), H_p([f \circ g])) = (s_F([f]), H_p([f])) \cdot (s_F([g]), H_p([g])).$$

We wish to characterise the image of $s_F \times H_p$ in $\Gamma(\phi_r)$ and for this we use a small piece of the apparatus of surgery theory. As well as $s_F$, a stable framing $F$ on $M_r$ defines a bundle map $(\bar{c}, c) : \tau(M_r) \oplus \varepsilon(M_r) \to \varepsilon^{2p+1}(S^{2p+1})$ to the trivial bundle over $S^{2p}$ which covers the collapse map $c : M_r \to S^{2p}$. The map $(\bar{c}, c)$ is called a degree one normal map with respect to the tangent bundle in [Lee][Definition 3.50] and in [Lee][§4.4] it is proven that $(\bar{c}, c)$ equips $\pi_p(M_r) \cong H_p(M_r) = H$ with a quadratic form $\psi(\bar{c}, c) : H \to \mathbb{Z}_2$ which we shall denote by $\psi_F$. The quadratic form $\psi_F$ refines $\phi_r : H \times H \to \mathbb{Z}$ and is defined as follows: the choice of framing $F$ defines a unique regular homotopy class of immersions $V : S^p \to M_r$ for each $v \in H$ and $\psi_F(v) \in \mathbb{Z}_2$ is the number of self-intersections of a generic representative of this regular homotopy class counted modulo 2.

If we modify the stable framing $F$ we may modify the quadratic form $\psi_F$ as follows: any two stable framings $F_0$ and $F_1$ differ up to homotopy by a homotopy class $x = x(F_0, F_1) \in [M_r, SO] \cong H^*$. If $\bar{x} \in H_2^*$ is the mode 2 reduction of $x$ then we have
**Lemma 3.3.** Let $F_{0}$ and $F_{1}$ be stable framings of $M_{r}$ as above. Then $\psi_{F_{0}} = \psi_{F_{1}} + \bar{x}$.

*Proof.* This is a standard, if subtle, result of surgery: see [Le][Thm. 4.2(b)] and the discussion preceeding it for a proof. Of course Levine uses normal maps over the stable normal bundle but the bijective correspondence between these and tangential normal maps is covered in detail in [Le][Lemma 3.51].

**Lemma 3.4.** Let $F$ be a stable framing of $M_{r}$ with associated quadratic form $\psi_{F}$. Then

1. $s_{F} = s(\psi_{F}) \circ H_{p} \mod 2$,
2. $s_{F} \times H_{p} : \text{Aut}(M_{r}) \cong \Gamma(\psi_{F}) \subset \Gamma(\phi)$.

*Proof.* The second statement follows immediately from the first, Lemma 3.2 and the definition of $\Gamma(\psi_{F})$ in (6) so we prove the first. Given $[f] \in \text{Aut}(M_{r})$ we write $f_{*} : H \cong H$ for $H_{p}(\{f\})$. We must show for all $[f]$ and for all $v \in H$ that $\psi_{F}(f_{*}(v)) - \psi_{F}(v) = s_{F}([f])(v) \mod 2$. Let $V : S^{p} \to M_{r}$ be an immersion which represents $v$ and which lies in the regular homotopy class prescribed by $F$. As $f$ is a diffeomorphism, $V$ and $f \circ V$ have the same number of double points and so $\psi_{F}(f_{*}(v)) - \psi_{F}(v)$. By Lemma 3.3 and the discussion preceeding it, we see that $\psi_{F}(f_{*}(v)) - \psi_{F}(v)$ equals the difference mod 2 of the framings prescribed by $F \circ D(f)$ and $F$ over the homotopy class $v$. That is, $\psi_{F}(f_{*}(v)) - \psi_{F}(v) = s_{F}([f])(v) \mod 2$.

**Corollary 3.5.** A stable framing $F$ defines isomorphims from the extensions of (7)

$$
\begin{array}{cccccc}
0 & \longrightarrow & \text{Hom}(H, S\pi_{p}(SO_{p})) & \overset{I}{\longrightarrow} & \text{Aut}(M_{r}) & \overset{H_{p}}{\longrightarrow} & \text{Aut}(\phi_{r}) & \longrightarrow & 1 \\
& & \downarrow \text{J}\text{OPD} & & \downarrow \varepsilon & & \downarrow \text{=} & & \\
0 & \longrightarrow & H \otimes S\pi_{2p}(S^{p}) & \overset{I}{\longrightarrow} & \mathcal{E}(M_{r}) & \overset{H_{p}}{\longrightarrow} & \text{Aut}(\phi_{r}) & \longrightarrow & 1
\end{array}
$$

to the extensions of (7) with $C = \mathbb{Z}$ or $C = \mathbb{Z}_{c}$

$$
\begin{array}{cccccc}
0 & \longrightarrow & 2H^{*} & \overset{i}{\longrightarrow} & \Gamma(\psi_{F}) & \overset{H_{p}}{\longrightarrow} & \text{Aut}(\phi_{r}) & \longrightarrow & 1 \\
& & \downarrow \text{P}_{\infty} & & \downarrow \text{=} & & \downarrow \text{=} & & \\
0 & \longrightarrow & 2H^{*} & \overset{i}{\longrightarrow} & \Gamma(\psi_{F}, \mathbb{Z}_{c}) & \overset{H_{p}}{\longrightarrow} & \text{Aut}(\phi_{r}) & \longrightarrow & 1.
\end{array}
$$

where $c = 12$ or $120$ as $p = 3$ or $7$. Hence the extension for $\text{Aut}(M_{r})$ corresponds to $\partial_{\infty}[s(\psi_{F})] \in H^{2}(\text{Aut}(\phi_{r}); 2H^{*})$ and the extension for $\mathcal{E}(M_{r})$ corresponds to $\partial_{c}[s(\psi_{F})] \in H^{2}(\text{Aut}(\phi_{r}); 2H^{*}_{c})$. In particular the extensions above split if and only if and only if $r = 1$.

*Proof.* By Lemma 3.4 (2) $s_{F} \times H_{p} : \text{Aut}(M_{r}) \cong \Gamma(\psi_{F})$ is an isomorphism commuting with $H_{p}$ and $\pi$ and this proves the corollary for $\text{Aut}(M_{r})$ since by definition $\Gamma(\psi_{F}) \rightarrow \text{Aut}(\phi_{r})$ is the extension defined by $\partial_{\infty}[s(\psi_{F})]$. The corollary for $\mathcal{E}(M_{r})$ follows from Proposition 1.6 which states that $\mathcal{E} : \text{Aut}(M_{r}) \rightarrow \mathcal{E}(M_{r})$ is onto and we see that the kernel of $\mathcal{E}$ is $I(2cH^{*}) \subset I(2H^{*})$. This identifies $\mathcal{E}(M_{r})$ with $\Gamma(\psi_{F}, \mathbb{Z}_{c})$. By Proposition 2.3 each of these extensions split if and only if $r = 1$. 

9
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Diarmuid Crowley
School of Mathematical Sciences
University of Adelaide
Australia, 5005.

E-mail address: diarmuidc23@gmail.com