Effects of Vector Norm Types in Maximum Stiffness Problem of Simple Truss Structures

Kook Keong Choong¹, Irene Wee Ling Loh², Jae-Yeol Kim³ and Joowon Kang*⁴

¹Associate Professor, D. Eng., School of Civil Engineering, University Sains, Malaysia  
²Solectron Sdn Bhd, Penang, Malaysia  
³Associate Professor, Ph.D., Department of Architectural Engineering, Hyupsung University, Korea  
⁴Associate Professor, Ph.D., School of Architecture, Yeungnam University, Korea

Abstract
The objective of maximum stiffness problem is to minimize the displacement. For this purpose, an optimization analysis is needed in order to obtain the structural shape with maximum stiffness. Displacement for structures with multiple degrees of freedom which has been modeled using finite element method is expressed in the form of a vector. In order to measure the size of vector quantity, a vector norm is used. The use of different types of vector norms might yield different results in maximum stiffness problem with shape as the design variable. This paper studies the effect of types of vector norms used in the outcome of analysis in maximum stiffness problem. Three types of vector norms, namely Euclidean, absolute value and maximum value norms, are studied. The results show that Euclidean norm is the most effective norm to be used.

Keywords: vector norms; maximum stiffness; truss structures

1. Introduction
Optimization involves finding the most appropriate design satisfying a certain prescribed objective or sets of objectives under certain constraints. Its application can be seen in various fields, such as medicine, acoustics, building science, optics, economics and engineering. A compilation of research work on shape and topology optimization by Mackerle¹ has shown the volume of work conducted. As a testimony of the application of optimization in diverse fields, an encyclopedia of optimization has also been published². Research works related to optimization of truss structures can be dealing with proposal of methods of analysis, studies on more efficient methods of numerical analysis in optimization and investigations on the optimal solutions for problems in different fields.

Optimization of truss structures forms one of the main focus of research in structural engineering both in terms of topology, shape and cross-section optimization³-¹⁵. The majority of the works deal with proposal of methods of optimization⁵, ⁸-¹⁰, ¹²-¹⁵. The more complicated problems of multi-objectives and multiple constraints type of optimization problem are studied in³, ⁷. Procedures of analysis for handling discrete design variables or design variables with different nature have also been studied¹⁰, ¹¹. Other than that, verification of applicability of available method of optimization to different engineering problem has also been reported¹⁴. In terms of criteria, the above mentioned works could be seen to be dealing with minimum weight⁴, ⁸, ⁹, ¹¹, maximum stiffness¹⁴ and minimum cost⁶. In problem involving maximization of stiffness, strain energy is commonly adopted as a judging measure where maximization of overall structural stiffness corresponds to minimization of strain energy. The objective of maximum stiffness problem is to minimize the overall deformation which could be represented by relative displacement. Displacement for structures with multiple degrees of freedom modeled using finite element method is expressed in the form of vector. Hence, the 'size' of displacement vector can be used as a possible judging measure in maximum stiffness problem. In order to measure the 'size' of a vector quantity, vector norm is used. Then¹⁶ has initiated a study on the characteristic of vector and matrix norms in shape optimization analysis. It is found that vector norm is more suitable to be used compared to matrix norm from the view point of numerical analysis. In the work by Then¹⁶, exact optimization analysis was not carried out. The author merely calculated the variation of norm of displacement vector versus the design variable in order to determine the shape providing maximum stiffness.

*Contact Author: Joowon Kang, Associate Professor, Ph.D., School of Architecture, Yeungnam University, Dae-dong, Gyeongsan City, 712-749 Korea  
Tel : +82-53-810-2429 Fax: +82-53-810-4625  
E-mail : kangj@ynu.ac.kr  
(Received October 8, 2008; accepted March 10, 2009)
The objective of this study is to carry out a basic investigation on the effect of types of vector norms in shape optimization of simple truss structures to achieve maximum stiffness. A description on the basic equations involved in maximum stiffness problem is first presented. Steepest descent method adopted to find the solution is explained next. This is then followed by the analysis results carried out on four simple planar truss problems, the discussion and the conclusion.

2. Formulation of Maximum Stiffness Problem

Using the displacement method of analysis, the equation of equilibrium can be expressed as follows:

\[ f = K d \]  
(1)

where \( f \) is the external load vector, \( K \) is the global stiffness matrix and \( d \) is the displacement vector. Under the condition of constant material properties, \( K \) will be a function of shape, represented by vector \( x \). Considering the above fact, Equation (1) could be re-expressed as

\[ f = K(x)d \]  
(2)

From Equation (2), it can be seen that basic equation in the analysis of maximum stiffness problem for a multiple d.o.f. system is given by the following equation:

\[ d = K^{-1}(x)f \]  
(3)

With reference to Equation (3), the statement of maximum stiffness problem could be stated as follows: For a prescribed external load vector \( f \), find the shape \( x \) which maximizes stiffness matrix \( K \) or equivalently find the shape \( x \) which minimizes the displacement vector \( d \).

Considering the latter statement where minimization of \( d \) is involved, means of measuring the 'size' of a vector quantity is needed. Such measurement could be achieved by means of vector norm. Hence, the objective function \( g \) to be used in the optimization analysis carried out in a maximum stiffness problem with shape \( x \) as the design variable could be expressed as follows:

\[ g = \| d(x) \| \]  
(4)

where \( \| d(x) \| \) denotes the norm of displacement vector \( d \).

2.1 Vector Norm

A norm is a scalar function \( \| v \| \) defined for every vector \( v \) in some vector space \( \mathbb{R}^n \), real or complex, and possessing the following three characteristics:

a. \( \| v \| \geq 0 \) for all \( v \in \mathbb{R}^n \)  
   (Non-negative)

b. \( \| \alpha v \| = |\alpha| \| v \| \) for all scalars \( \alpha \) and \( v \in \mathbb{R}^n \)  
   (Homogeneous)

c. \( \| v + w \| \leq \| v \| + \| w \| \) for all \( v, w \in \mathbb{R}^n \)  
   (Triangle inequality)

(5)

Three vector norms which satisfy Equation (5) and studied in this paper are:

a. \( \| v \|_2 = \left( \sum_{i=1}^{n} v_i^2 \right)^{1/2} \) : Euclidean norm

b. \( \| v \|_1 = \sum_{i=1}^{n} |v_i| \) : Absolute value norm

c. \( \| v \|_{\infty} = \max_i |v_i| \) : Maximum value norm

where right subscript represents \( i^{th} \) component of the corresponding variable. Euclidean, absolute value and maximum value norms are sometimes denoted as \( l_2 \), \( l_1 \) and \( l_{\infty} \) norms, respectively.

2.2 Steepest Descent Method

Iterative methods are used in optimization problems to find the minimum (or maximum) of an objective function. In this study, the optimum solution corresponding to shape with maximum stiffness is obtained by using the steepest decent method. In steepest descent method, the local minimum for a multivariable function of the form \( g: \mathbb{R}^n \rightarrow \mathbb{R} \) is sought.

Solution of a system of nonlinear equations coincides with a minimum of \( g(x) \) where:

\[ g(x) = \left( \sum_{i=1}^{n} h_i(x) \right)^2 \]  
(7)

where \( x = (x_1, x_2, \ldots, x_n) \). Since the objective function of this study is norm of displacement vector as shown in Equation (4), it can be seen that the function of \( n \) variable \( \sum_{i=1}^{n} h_i(x) \) in Equation (7) corresponds to Euclidean, absolute value or maximum value norms of displacement vector \( d \).

The steepest descent algorithm starts by assuming initial shape \( x^{(0)} \) where a right superscript in parenthesis denotes iterative step with step \( \theta \) corresponding to initial step. An improved shape say \( x^{(1)} \) is found from \( x^{(0)} \) and so on. The solution \( x^{(k)} \) \( (k=0,1,2,\ldots) \) is the best possible shape if it minimizes the objective function. The iterative procedures adopted are summarized as follows:

a. A starting initial shape \( x^{(0)} \) \( (k=0) \) is selected and \( g(x^{(k)}) \) \( (k=0) \) is evaluated.

b. The gradient vector is then evaluated using \( \nabla g(x^{(k)}) \). Analysis is terminated if
norm of gradient vector, \( z^{(0)} = \| z^{(1)} \| = 0 \).

Otherwise, a normalized gradient vector is computed using \( \tilde{z}^{(k)} = \frac{-\nabla g(x^{(k)})}{\|\nabla g(x^{(k)})\|} \).

c. Step size \( \alpha^{(k)} \) for the search along the direction of \( z^{(k)} \) is next computed through minimization of \( g(\alpha^{(k)}) = g(x^{(k)} + \alpha^{(k)} \tilde{z}^{(k)}), \alpha^{(k)} > 0 \).

d. Shape vector is then updated using \( x^{(k+1)} = x^{(k)} + \alpha^{(k)} \tilde{z}^{(k)} \).

e. \( g(x^{(k)}) \) is next evaluated. Analysis is terminated if any of the stopping criteria stated in the following section is satisfied. Otherwise, update \( x^{(k)} = x^{(k+1)} \) and \( k = k + 1 \) are carried out and steps b to e are repeated.

Solution for line search problem with quadratic fit obtained through 'three-point pattern' is used to determine step size of \( \alpha \) in each iterative step. The procedures are described as follows:

a. The following single-variable function at the end of iterative step \( k \) is considered:

\[
\hat{h}(\alpha) = g(x^{(k)} - \alpha \nabla g(x^{(k)}))
\]  

(8)

Value of \( \alpha \) which minimizes \( h(\alpha) \) is desired.

b. Three numbers \( \alpha_1 < \alpha_2 < \alpha_3 \) that are hopefully close to the value of \( \alpha \) which minimizes \( h \) are first chosen.

c. A quadratic polynomial \( Q(x) \) that interpolates \( h \) at \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) is next constructed. \( \hat{\alpha} \) within [\( \alpha_1, \alpha_3 \)] which minimizes \( Q(\hat{\alpha}) \) is computed. \( Q(\hat{\alpha}) \) is used as approximation for the minimum value of \( h(\alpha) \).

d. New iterate of shape vector \( x \) for approximating the minimal value of \( g \) is determined using \( \hat{\alpha} \) as follows:

\[
x^{(k+1)} = x^{(k)} - \hat{\alpha} \nabla g(x^{(k)})
\]  

(9)

e. Since \( g(x^{(k)}) \) is available, \( \alpha_1 = 0 \) is first chosen in order to minimize the computational time. The maximum value for \( \alpha_1 \) is set as \( \alpha_1 = 1 \). A value of \( \alpha_3 \) satisfying the condition \( h(\alpha_3) < h(\alpha_1) \) is then computed. If \( h(\alpha_3) > h(\alpha_1) \), \( \alpha_1 \) is reduced by 1/2. This process is repeated until \( h(\alpha_1) < h(\alpha_1) \).

f. \( \alpha_3 = \alpha_2 / 2 \) is then set. The minimum value of \( Q \) within [\( \alpha_2, \alpha_3 \)] occurs either at the critical point of \( Q \) or at the right endpoint \( \alpha_3 \) since by assumption \( Q(\alpha_3) = h(\alpha_3) < Q(\alpha_2) = h(\alpha_2) \). The computed value of \( \alpha \) is then adopted as \( \alpha^{(k+1)} \) in the subsequent iterative step.

3. Numerical Examples

Four numerical examples corresponding to simple 2D truss problems have been analyzed. The first three examples are 2 d.o.f. problems and the fourth one is a 4 d.o.f. problem. In all four examples, only one design variable has been considered in order that the characteristics of the three vector norms could be studied clearly. The shape of the simple truss is considered to have converged to the maximum stiffness shape during the optimization analysis when anyone of the following stopping criterion has been satisfied:

a. \( E_a = z^{(k)} = \|\nabla g(x^{(k)})\| < 10^{-9} \)  

(10)

b. \( E_b = g(x^{(k)} - \alpha_3 \tilde{z}^{(k)}) - g(x^{(k)}) \) with \( \alpha_3 \leq 0.025 \)

(11)

c. \( E_c = \left( x^{(k+1)} - x^{(k)} \right) / x^{(k)} \leq 10^{-3} \)  

(12)

\( E_a, E_b \) and \( E_c \) correspond to convergence checking based on norm of gradient vector, satisfaction of objective function and closeness of two successive approximation of shape. The three errors in Equations (10) to (12) have been plotted versus iterations in order to study the efficiency of each of them in the shape optimization analysis to achieve convergence to the shape with maximum stiffness. Notations \( \text{norm}(l,2), \text{norm}(l,1) \) and \( \text{norm}(l,\infty) \) have been used to represent Euclidean \( l_2 \), absolute value \( l_1 \) and maximum value \( l_\infty \) norms, respectively, in the figures shown in the following sections.

3.1 Numerical example 1

Numerical example 1 is a simple 2 d.o.f. truss structure with two members and three joints as shown in Fig.1.

The structure is loaded by a vertical downward load of \( F = 10 \text{kN} \) at joint 2. Analysis data for this model of truss is as follows: \( EA = 160 \times 10^3 \text{kN} \cdot \text{m}, B = 400 \text{cm} \) and \( H/B = 0.3 \). x-coordinate of joint 2, \( x_2 \), has been selected as the design variable and assigned a series of initial values of \( x_2^{(0)} = 250 \text{cm} \) to \( 500 \text{cm} \). The result of analysis for the particular case of \( x_2^{(0)} = 500 \text{cm} \) is shown in Fig.2. Inset (a) and (b) show the initial shape with \( x_2 = 500 \text{cm} \) and converged shape with \( x_2 = 389.5 \text{cm} \), respectively.

3.2 Numerical Example 2

Numerical example 2 is also a simple 2 d.o.f. simple truss structure with three members and four joints as shown in Fig.3. The truss is loaded by a horizontal load of \( F = 10 \text{kN} \) at joint 2.
Analysis data for this model of truss are as follows: \( EA = 160 \times 10^3 \) kN, \( B = 400 \) cm and \( H/B = 0.5 \). \( y \)-coordinate of joint 4, \( y_4 \), has been selected as the design variable and assigned initial value of \( y_4^{(0)} = 160 \) cm to 250 cm. The result of analysis for the particular case of \( y_4^{(0)} = 250 \) cm is shown in Fig. 4.

Insets (a) and (b) show the initial assumed shape with \( y_4 = 250 \) cm and converged shape with \( y_4 = 200 \) cm, respectively.

### 3.3 Numerical Example 3

Numerical example 3 is again a simple 2 d.o.f. truss structure with two members and three joints as shown in Fig. 5. The truss is loaded with a load \( F = 50 \) kN that is parallel to member 1. Analysis data for this model of truss are as follows: \( EA = 10 \times 10^4 \) kN, \( H = 500 \) cm and \( H/B = 1 \). \( x \)-coordinate of node 3, \( x_3 \), has been selected as the design variable and assigned initial assumed values...
of $x_3^{(0)}=200\text{cm}$ to $1020\text{cm}$. Result of analysis for the case of $x_3^{(0)}=200\text{cm}$ using Euclidean and absolute value norms is shown in Fig.6. Insets (a) and (b) again show the initial assumed shape with $x_3=200\text{cm}$ and the shape with maximum stiffness with $x_3=1000\text{cm}$, respectively. Fig.7 shows the result of analysis for the two cases of $x_3=550\text{cm}$ and $700\text{cm}$ using Euclidean norm. Insets (a) and (b) show the initial assumed shape for the case of $x_3=550\text{cm}$ and $700\text{cm}$, respectively; whereas inset (c) shows the shape with maximum stiffness with $x_3=1000\text{cm}$.

### 3.4 Numerical Example 4

Numerical example 4 is a 4 d.o.f. truss structure with three links supported at the two free joints by

![Fig.5. Numerical Example 3](image)

![Fig.6. Error $E_a$ Versus Iterations with Initial Value of $x_3 = 200\text{cm}$ using Euclidean Norm and Absolute Value Norm](image)

![Fig.7. Error $E_a$ Versus Iterations with Initial Value of $x_3 = 550\text{cm}$ and $700\text{cm}$ using Euclidean Norm](image)
two linear springs as shown in Fig.8. Two vertical downward loads, with magnitude F each, act at joints 2 and 3. Analysis data for this truss model are as follows: B=10m, H=0.4B, F/AE=1x10^-6 and k=0.08AE where k=stiffness of spring. Since the problem is symmetry, x-coordinates of joints 3 and 2 are related as follows: 
\[ x_3 = B - x_2 \]
x_2 has been selected as the design variable and assigned initial values of 
\[ x_2^{(0)} = -5.5m \text{ to } 18.8m \]
The result of analysis for the case of \( x_2 = 4.9m \) is shown in Fig.9.

Insets (a) and (b) show the initial assumed shape with \( x_2 = 4.9m \) and converged shape with \( x_2 = 0 \) representing a shape where position of the two vertical springs coincides with member 1 and 3.

4. Discussion

Although convergence was achieved in all analysis using the three vector norms in numerical example 1, 2 and 4, each vector norm yields different trends in the process of iterations as shown in Figs.2., 4. and 9. Analysis using absolute value norm need more iterations in order to achieve convergence compared to that using Euclidean or maximum value norms. Hence, analysis using Euclidean or maximum value norms give faster convergence than analysis using absolute value norm in numerical example 1, 2 and 4. In numerical example 4, it can be seen from Fig.9. that error \( E_b \) shows an initial trend of increase in the first two iterative steps for the case of absolute value norm. Such characteristic is due to the definition of \( E_b \) used which is obtained during the process of determination of step size in line search as explained earlier.

Although analysis using maximum value norm converges at almost the same time as Euclidean norm in numerical example 1, 2, and 4, the optimization analysis using maximum value norm failed to converge in numerical example 3. Norm of displacement vector according to the definition of Euclidean, absolute value and maximum value norms have been computed for assigned values of \( x_3 \) ranging from 0 to 5000cm for numerical example 3. The result is shown in Fig.10. Existence of a point with discontinuity in slope at \( x_3 = 1000cm \) on the curve of \( l_\infty \)-norm as can be clearly seen in Fig.10. Such a discontinuity which is the cause of convergence failure occurs as a result of change in displacement component selected for minimization:

\[ l_x = \|u_x\| \text{ when } x_3 < 1000cm \text{ and } l_x = \|v_2\| \text{ when } x_3 > 1000cm \]

For the case where \( l_1 \)-norm was used, it was observed that convergence was achieved for all analysis with \( x_3^{(0)} < 500cm \) as illustrated in Fig.6 for \( x_3^{(0)} = 200cm \). However, the converged shape (with \( x_3 = 500cm \)) was different from the one obtained in the case where \( l_2 \)-norm (with \( x_3 = 1000cm \)) was used as shown in insets (b) and (c) of Fig.6. Apart from that, it was also observed that analysis using \( x_3^{(0)} > 500cm \) showed instant convergence regardless of the initial values of \( x_3 \) when \( l_1 \)-norm was used. The above observation is again caused by the existence of a point with discontinuity in slope at \( x_3 = 500cm \) on the curve.
of $l_1$-norm after which the slope of the curve remains zero regardless of the value of $x_3$ as illustrated in Fig.10. The peculiar characteristic of $l_1$-norm could be due to the configuration of numerical example 3 where up-until the shape with $x_3=500$cm, the displacement of joint 2 is dominated by x-displacement $u_2$. For shapes with $x_3>500$cm, decrease in magnitude of $u_2$ is accompanied by proportionate increase in magnitude of $v_2$. As $l_1$-norm is summation of absolute value of all displacement components, the aforementioned characteristic of displacement of joint 2 leads to the result of constant $l_1$-norm for $x_3>500$cm. As analysis using $l_1$-norm yielded different converged shapes depending on initial assumed value assigned, it is found to be not an appropriate norm to be used for the case of numerical example 3. Only analysis using Euclidean norm converges smoothly to the shape with maximum stiffness regardless of the initial assumed value used for $x_3$ as shown in Figs.6. and 7. Hence, it can be said that use of $l_1$- and $l_\infty$ norms are not suitable for numerical example 3. Use of $l_2$-norm is the most appropriate.

5. Conclusion

Effect of three types of vector norms on the analysis process of maximum stiffness problem has been studied by means of four simple planar truss examples. It has been found that Euclidean norm is the most suitable vector norm to be used compared to maximum value norm and absolute value norm because all optimization analysis showed smooth convergence to the shape with maximum stiffness where Euclidean norm is used. There is a possibility that point with discontinuity in slope might be encountered in the cases of absolute value and maximum value norms. Such discontinuity in slope might lead to non-convergence or convergence to different shapes depending on the initial assumed shapes.

Acknowledgements

This research was supported by the Yeungnam University research grants in 2007.

References

1) Jaroslav Mackerle (2003) Topology and shape optimization of structures using FEM and BEM: A bibliography (1999–2001), Finite Elements in Analysis and Design, 39(3), pp.243-253.
2) Encyclopedia of Optimization, ed.Christodoulos A. Floudas and Panos M. Pardalos, Kluwer Academic Publisher, 2001.
3) Romanas Karkauskas and Arnolds Norkus (2006) Truss optimization under stiffness, stability constraints and random loading. Mechanics Research Communications, 33(2), pp.177-189.
4) Panagiotis A. Makris, Christopher G. Provatidis and Demetrios T. Venetanos (2006) Structural optimization of thin-walled tubular trusses using a virtual strain energy density approach. Thin-Walled Structures, 44(2), pp.235-246.
5) Vedat Togan and Ayge T. Daloglu (2006) Optimization of 3d trusses with adaptive approach in genetic algorithms. Engineering Structures, 28(7), pp.1019-1027.
6) Simon Sih, Miroslav Premrov and Stojan Kravanja (2005) Optimum design of plane timber trusses considering joint flexibility. Engineering Structures, 27(1), pp.145-154.
7) Guan-Chun Luh and Chung-Huei Chueh (2004) Multi-objective optimal design of truss structure with immune algorithm. Computers & Structures, 82(11-12), pp.829-844.
8) Panagiotis A. Makris and Christopher G. Provatidis (2002) Weight minimisation of displacement-constrained truss structures using a strain energy criterion. Computer Methods in Applied Mechanics and Engineering, 191(19-20), pp.2187-2205.
9) D. Wang, W. H. Zhang and J. S. Jiang (2002) Truss shape optimization with multiple displacement constraints. Computer Methods in Applied Mechanics and Engineering, 191(33) pp.3597-3612.
10) Lluis Gil and Antoni Andreu (2001) Shape and cross-section optimization of a truss structure. Computers & Structures, 79(7), pp.681-689.
11) W. H. Tong and G. R. Liu (2001) An optimization procedure for truss structures with discrete design variables and dynamic constraints. Computers & Structures, 79(2), pp.155-162.
12) B. Y. Duan, A. B. Templeman and J. J. Chen (2000) Entropy-based method for topological optimization of truss structures. Computers & Structures, 75(5), pp.539-550.
13) Makoto Ohsaki (1998) Simultaneous optimization of topology and geometry of a regular plane truss. Computers & Structures, 66(1), pp.69-77.
14) M. Beekers and C. Fleury (1997) A primal-dual approach in truss topology optimization. Computers & Structures, 64(1-4), pp.77-88.
15) M. P. Saka and M. Ulker (1992) Optimum design of geometrically nonlinear space trusses. Computers & Structures, 42(3), 289-299.
16) Then, S. E. (2002). "A study about characteristics of vector and matrix norms in shape optimization analysis", MSc Dissertation, School of Civil Engineering, Universiti Sains Malaysia.