THE $\alpha$-STABLE TIME-CHANGED FRACTIONAL ORNSTEIN-UHLENBECK PROCESS

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Abstract. We consider the fractional Ornstein-Uhlenbeck process, solution of a stochastic differential equation driven by the fractional Brownian motion, and we study its time-changed version, obtained via an inverse $\alpha$-stable subordinator. We focus on the convergence of the probability density function as the Hurst index $H \to 1/2$. The generalized fractional Fokker-Planck equation for such process is introduced and the class of subordinated solutions of such equation is studied, providing some uniqueness-isolation results and studying the convergence as $H \to 1/2$.

Keywords: fractional Brownian motion, stable subordinator, fractional Caputo derivative, fractional Fokker-Planck equation.

1. Introduction

The Ornstein-Uhlenbeck (OU) process is a standard process in the application context. However, its covariance with a fast decay and the fact that the strong Markov property holds makes it to be unrealistic in situations in which memory plays a crucial role. For this reason, in [10], the fractional OU (fOU) process has been introduced as the solution of the fractional Brownian motion (fBm)-driven equation

$$dU_H(t) = -\frac{1}{\theta}U_H(t)dt + \sigma dB_H(t)$$

where $\theta, \sigma \in \mathbb{R}^+$ and $B_H(t)$ is a fBm with Hurst parameter $H \in (0, 1)$. Such kind of process exhibits long or short-range dependence in function of the Hurst parameter (see [10, 14]). In the context of the applications, memory phenomena occurs for instance in the financial market, hence different kind of noise have to be implemented to describe them (see for instance [1]). Thus, in this direction, the study of the fractional Cox-Ingersoll-Ross process, that can be expressed as the square of a fOU process until it reaches 0, has to be carried on, in particular referring to the hitting time at 0 of the fOU process (see [20, 21]). On the other hand, for instance in the field of theoretical neuroscience, one can propose some different kind of noise to generate some memory effects, that are typical of neurons of the prefrontal cortex (see [23]). For this reason, different kind of noise can be introduced in the model to reproduce memory (see [5] and references therein). Moreover, one has to face the fact that stimuli can be stochastic. Thus, in [3] we studied a fOU process with stochastic drift, considering how such drift comes into play in modifying the behaviour of the covariance.

From another perspective, memory has been also introduced by changing the time scale from a deterministic one to a stochastic one. This is for instance the
case of \cite{16, 17}, where the adjective \textit{fractional} follows from the fact that the usual Kolmogorov equations admit a fractional derivative in time if we apply a \textit{time-change} to the process, in the sense that one composes the process with the inverse of a stable subordinator. In particular in \cite{17}, what here we could call an \( \alpha \)-stable time-changed OU process has been introduced and its Kolmogorov equations have been studied. Moreover, in \cite{16}, the correlation structure is exposed, showing that also this approach leads to a long-range dependent process. A further generalization has been achieve in \cite{12}, where a general subordinator is considered in place of a stable one. This approach also revealed to be interesting in applications, as for instance in neuroscience, where such a time-changed process could lead to heavy tailed distribution for hitting times (see \cite{6}) and then better describe the behaviour of some unusual neurons (see \cite{7}).

In \cite{3} we introduced a time-changed fOU process, i.e. a process obtained by considering the composition between the fOU process \( U_H \) and the inverse of a general subordinator, and studied some of its properties, together with its generalized Fokker-Planck equation. In this work we want to focus on a particular case of the latter process, that is to say the \( \alpha \)-stable time-changed fOU process, obtained by using an \( \alpha \)-stable subordinator in place of a general one. For such process, we are able to explicit more properties, such as the differentiability with respect to time of the density. Moreover, in the spirit of \cite{3}, we also study the dependence of the density with respect to the Hurst index. In particular, we also study the dependence of the operators, involved in the construction of the Fokker-Planck equation, applied to the density with respect to the Hurst index, exploiting the convergence of all the operators involved to the respective version for \( H = 1/2 \).

However, since we start from a Fokker-Planck equation obtained by exploiting the Gaussian nature of the process \( U_H \), what we obtain for \( H = 1/2 \) is not the well-known Kolmogorov equation for the time-changed OU process (see \cite{12, 17}), but an equation that takes into account the non-homogeneity in time of the Fokker-Planck equation, giving us a different operator in place of the generator of the OU process. However, this operator and the generator of the OU process coincide when applied to the density of the \( \alpha \)-stable time-changed OU process.

Finally, we study the generalized Fokker-Planck in abstract for a particular class of functions obtained by means of the inverse subordinator, focusing on uniqueness and convergence of the solutions.

The paper is structured as follows:

- In Section 2 we give some preliminaries concerning the property of the \( \alpha \)-stable subordinator and its inverse. Moreover, we recall some concepts from fractional calculus;
- In Section 3 we introduce the process, recall some properties we achieved in \cite{4} and then we prove the differentiability of the density with respect to time. Moreover, in this section we prove also the uniform convergence of the density to the one of the time-changed OU process as \( H \to 1/2 \);
- In Section 4 we show that the density is a classical solution of the generalized Fokker-Planck equation introduced in \cite{4}. Moreover, we show that all the operators involved in the definition of the equation converge to the respective ones for \( H = 1/2 \) when applied to the density of the time-changed fOU;
Finally, in Section 3 we study the Fokker-Planck equation working on solutions of the form \( v_{\alpha}(x, t) = E[v(x, E_{\alpha}(t))] \) where \( E_{\alpha} \) is the inverse of an \( \alpha \)-stable subordinator. In particular we study uniqueness of the classical solutions of such equations, isolation of the mild solutions and convergence of the solutions.

2. Preliminaries

2.1. The \( \alpha \)-stable subordinator and its inverse. Let us recall the definition of subordinator, as given in [9, Chapter 3]. A subordinator \( \sigma(t) \) is an increasing (and hence positive) Lévy process. In our case we will consider only \( \alpha \)-stable subordinators \( \sigma_{\alpha}(t) \) for \( \alpha \in (0, 1) \), i.e. subordinators whose Laplace transform equals
\[
E[e^{-\lambda \sigma_{\alpha}(t)}] = e^{t \lambda^\alpha}.
\]
Let us also define the inverse \( \alpha \)-stable subordinator \( E_{\alpha}(t) \) as
\[
E_{\alpha}(t) := \inf\{y > 0 : \sigma_{\alpha}(y) > t\}.
\]
As shown in [19], \( \sigma_{\alpha}(t) \) and \( E_{\alpha}(t) \) are absolutely continuous random variables for any \( t > 0 \). Let us denote by \( g_{\alpha}(x) \) the probability density function of \( \sigma_{\alpha}(1) \) and by \( f_{\alpha}(x, t) \) the probability density function of \( E_{\alpha}(t) \). Then it has been shown in [19] that
\[
f_{\alpha}(x, t) = \frac{t}{\alpha} x^{-1 - \frac{1}{\alpha}} g_{\alpha}(tx^{-\frac{1}{\alpha}}).
\]
In particular, \( \sigma_{\alpha} \) is an almost surely strictly increasing pure jump process, while \( E_{\alpha} \) is increasing, continuous and admits plateaus whereas \( \sigma_{\alpha} \) admits jumps.

Let us also recall (see [19, Equation 10]) that the Laplace transform of \( f_{\alpha}(x, t) \) in \( t \) has a form
\[
\mathcal{L}_{t \to \lambda}[f_{\alpha}(x, t)] = \lambda^{\alpha - 1} e^{-x \lambda^\alpha}.
\]
The following lemma contains the result established in [4], where its proof is contained in Remarks 4.2 and 4.4, therefore now is omitted.

Lemma 2.1. Let us fix \( H \in (\frac{1}{\alpha}, 1) \). Then \( E[E_{\alpha}^{-H}(t)] < +\infty \) for any \( t > 0 \). Moreover, for any \( n > 1 \) it holds that \( E[E_{\alpha}^{-nH}(t)] = +\infty \).

2.2. Fractional integral and fractional derivatives. Let us recall some definitions given in [15]. Following [15] formula 2.1.1, given a suitable function \( f : (0, +\infty) \to \mathbb{R} \) and fixed \( \alpha \in (0, 1) \), for any \( t > 0 \), we define the fractional integral of order \( \alpha \) of the function \( f \) as
\[
\mathcal{I}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau) d\tau.
\]
Moreover, referring to [15] Formula 2.1.5, for any suitable function \( f : (0, +\infty) \to \mathbb{R} \) and fixed \( \alpha \in (0, 1) \), for any \( t > 0 \), we define the Riemann-Liouville fractional derivative of order \( \alpha \) of the function \( f \) as
\[
\mathcal{D}^\alpha f(t) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t (t - \tau)^{-\alpha} f(\tau) d\tau = \frac{d}{dt} \mathcal{I}^{1-\alpha} f(t).
\]
Since it holds \( \mathcal{I}^\alpha \mathcal{I}^\beta = \mathcal{I}^{\alpha+\beta} \) and \( \mathcal{I}^1 f(t) = \int_0^t f(\tau) d\tau \), we also have
\[
\mathcal{D}^\alpha \mathcal{I}^\alpha f(t) = \frac{d}{dt} \mathcal{I}^{1-\alpha} \mathcal{I}^\alpha f(t) = \frac{d}{dt} \mathcal{I}^1 f(t) = f(t).
\]
Finally, referring to [15] Formula 2.4.4, for any suitable function \( f : [0, +\infty) \to \mathbb{R} \) and fixed \( \alpha \in (0, 1) \), for any \( t > 0 \), we define the **Caputo fractional derivative of order** \( \alpha \) of the function \( f \) as

\[
{\mathcal{D}}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} (f(\tau) - f(0)) d\tau = \mathcal{D}^\alpha [f(\cdot) - f(0)](t).
\]

Let us denote by \( \mathcal{D}(\partial^\alpha) \) the domain of such fractional derivative. It is easy to see that \( C^1(\mathbb{R}) \subset \mathcal{D}(\partial^\alpha) \) and for any \( f \in C^1(\mathbb{R}) \) it holds that

\[
\partial^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f'(\tau) d\tau = \mathfrak{D}^{1-\alpha} f'(t).
\]

Using this representation, we get immediately that for \( f \in C^1(\mathbb{R}) \)

\[
\mathcal{D}^\alpha \mathcal{D}^\alpha f(t) = \mathfrak{D}^{1-\alpha} \mathfrak{D}^{1-\alpha} f(t) = \mathfrak{D}^1 f(t) = f(t) - f(0).
\]

Whenever we consider a function \( f : (t, x) \in (0, +\infty) \times X \to f(t, x) \in Y \) where \( X, Y = \mathbb{R} \) or \( \mathbb{C} \), we denote by \( \partial^\alpha_t \) the Caputo derivative with respect to the variable \( t \). If \( f \) and \( \partial^\alpha f \) are Laplace transformable, then it is easy to see (as stated in [15] Formula 2.4.62) that, denoting by \( \mathcal{L}_{t \to \lambda}[f(t)](\lambda) = \tilde{f}(\lambda) \), we get the equality

\[
\mathcal{L}_{t \to \lambda}[\partial^\alpha f(t)](\lambda) = \lambda^\alpha \tilde{f}(\lambda) - \lambda^{\alpha-1} f(0).
\]

3. **The \( \alpha \)-stable time-changed fractional Ornstein-Uhlenbeck process**

Let \( (\Omega, \mathcal{F}, P) \) be a complete probability space supporting all stochastic processes that will be considered below. Let us fix Hurst index \( H \in (\frac{1}{2}, 1) \) and consider a fractional Brownian motion \( B_H = \{ B_H(t), t \geq 0 \} \) with Hurst index \( H \), i.e. a centered Gaussian process with covariance function given by

\[
\mathbb{E}[B_H(t)B_H(s)] = 1/2(t^{2H} + s^{2H} - |t-s|^{2H}), s, t \in \mathbb{R}^+.
\]

Let us also fix some number \( \theta > 0 \) and introduce the fractional Ornstein-Uhlenbeck process (defined in [19]) as

\[
U_H(t) = e^{-\theta t} \int_0^t e^{\theta \tau} dB_H(\tau), t \geq 0.
\]

Now we can define the \( \alpha \)-stable time-changed fractional Ornstein-Uhlenbeck process by considering a fractional Ornstein-Uhlenbeck process \( U_H(t) \), together with an independent inverse \( \alpha \)-stable subordinator \( E_\alpha(t) \), and defining

\[
U_{H,\alpha}(t) := U_H(E_\alpha(t)).
\]

This is a particular case of the time-changed fractional Ornstein-Uhlenbeck process introduced in [4]. Thus we can use the results on such paper to express some properties of \( U_{H,\alpha} \). Let us denote by \( V_{n,H}(t) := \mathbb{E}[|U_H(t)|^n] \) and \( V_{n,H,\alpha}(t) := \mathbb{E}[|U_{H,\alpha}(t)|^n] \). In particular,

\[
V_{2,H}(t) = e^{-2\theta t} \int_0^t \int_0^t e^{\theta |u-v|} |u-v|^{2H-2} duds, t \geq 0.
\]

Let us recall some properties of \( V_{2,H,\alpha} \) (see [4] Lemma 3.1).

**Proposition 3.1.**

(i) \( V_{2,H,\alpha}(t) \) is finite for any \( t > 0 \) and \( n \in \mathbb{N} \).

(ii) It holds that

\[
V_{2,H,\alpha}(t) = \int_0^{+\infty} V_{2,H}(s)f_\alpha(s,t)ds.
\]
(iii) $V_{2n,H,\alpha}(t)$ is increasing in $t$ for any $n \in \mathbb{N}$ and

$$
\lim_{t \to +\infty} V_{2n,H,\alpha}(t) = V_{2n,H}(\infty) = \left(\frac{2\theta^{2H} \Gamma(2H)}{\sqrt{\pi}}\right)^n \Gamma \left(\frac{2n+1}{\alpha}\right).
$$

**Remark 3.2.** The fact that in property (iii) the asymptotic value does not depend on $\alpha$ is strictly connected to the nature of the time-change. Indeed, $E_\alpha(t)$ acts as a delay in the time-scale of $U_H(t)$, hence we expect $U_{H,\alpha}(t)$ to have the same asymptotic behaviour, despite behaving quite differently on the whole trajectories.

We will also need the following limits for $V_{2,H}(t)$ and its derivative. They can be obtained by [3, Equation 29] and [4, Lemma 5.2].

**Lemma 3.3.** Function $V_{2,H}$ satisfies the relations

$$
\lim_{t \to 0^+} \frac{V_{2,H}(t)}{t^{2H}} = 1 \quad \text{and} \quad \lim_{t \to +\infty} V_{2,H}(t) = \theta^{2H} \Gamma(2H).
$$

Moreover, $V_{2,H} \in C^1(0, +\infty)$ and its derivative satisfies the relations

$$
\lim_{t \to 0^+} \frac{V'_{2,H}(t)}{t^{2H-1}} = 2H \quad \text{and} \quad \lim_{t \to +\infty} e^{\frac{t}{\theta} t^{2-2H}} V'_{2,H}(t) = 2H(2H-1)\theta.
$$

Now our goal is to investigate the smoothness of $U_{H,\alpha}(t)$. Concerning the absolute continuity of $U_{H,\alpha}(t)$, we have the following results.

**Proposition 3.4.** Let $p_H(x, t)$ be the probability density function of $U_H(t)$. Then $U_{H,\alpha}(t)$ is absolutely continuous for any fixed $t > 0$ and its probability density is given by

$$
 p_{H,\alpha}(x, t) = \int_0^{+\infty} p_H(x, s)f_\alpha(s, t)ds = \frac{t}{\alpha} \int_0^{+\infty} p_H(x, s)s^{-1-\frac{1}{\alpha}}g_\alpha(ts^{-\frac{1}{\alpha}})ds.
$$

Moreover, denoting $\mathcal{P}_H(x, \lambda)$ and $\mathcal{P}_{H,\alpha}(x, \lambda)$ respectively the Laplace transform of $p_H(x, \cdot)$ and $p_{H,\alpha}(x, \cdot)$, it holds

$$
\mathcal{P}_{H,\alpha}(x, \lambda) = \lambda^{\alpha-1}\mathcal{P}_H(x, \lambda^\alpha).
$$

Finally, for any $x \in \mathbb{R}$ it holds

$$
\lim_{t \to +\infty} p_{H,\alpha}(x, t) = \frac{1}{\sqrt{2\pi \theta^{2H} \Gamma(2H)}} e^{-\frac{\lambda^2 x^2}{2\theta^{2H} \Gamma(2H)}}.
$$

**Proof.** By Lemma 2.1 we know that $E[E_{\alpha}^{-1}(t)] < +\infty$, hence, by [3 Proposition 4.1], we know that $U_{H,\alpha}$ admits a characteristic function $\varphi_{H,\alpha} \in L^1(0, +\infty)$ such that

$$
\varphi_{H,\alpha}(\lambda, t) = \int_0^{+\infty} \varphi_H(\lambda, s)f_\alpha(s, t)ds
$$

where $\varphi_H$ is the characteristic function of $U_H$. By using Lévy inversion theorem one obtains

$$
p_{H,\alpha}(x, t) = \int_0^{+\infty} p_H(x, s)f_\alpha(s, t)ds
$$

and then, by using Equation (2.1) we also have

$$
p_{H,\alpha}(x, t) = \frac{t}{\alpha} \int_0^{+\infty} p_H(x, s)s^{-1-\frac{1}{\alpha}}g_\alpha(ts^{-\frac{1}{\alpha}})ds.
$$
Concerning the asymptotics, let us first consider the change of variable \( U = \int_0^1 e^{-\lambda t} \int_0^{+\infty} p_H(x, s) f_\alpha(s, t) ds dt \)
\[ = \int_0^{+\infty} p_H(x, s) \int_0^{+\infty} f_\alpha(s, t) e^{-\lambda t} dt ds \]
\[ = \lambda^{\alpha-1} \int_0^{+\infty} p_H(x, s) e^{-s\lambda^\alpha} ds = \lambda^{\alpha-1} \tilde{p}_H(x, \lambda^\alpha) \]
where we used Fubini theorem to change the order of the integrals (since the integrand is positive) and Equation (2.2) to evaluate the Laplace transform of \( f_\alpha(s, t) \).

Concerning the asymptotics, let us first consider the change of variable \( ty^{-\frac{\alpha}{2}} = w \) to obtain
\[ p_{H, \alpha}(x, t) = \int_0^{+\infty} p_H \left( x, \left( \frac{t}{w} \right)^{\alpha} \right) g_\alpha(w) dw. \]  
(3.2)

Now, let us recall that \( U_H(t) \) is a centered Gaussian process with variance \( V_{2, H}(t) \), and consequently
\[ p_H(x, t) = \frac{1}{\sqrt{2\pi V_{2, H}(t)}} e^{-\frac{x^2}{2V_{2, H}(t)}}. \]
(3.3)

Fix \( x \neq 0 \) and consider the auxiliary function \( h_1(t) = (2\pi t)^{-\frac{\alpha}{2}} e^{-\frac{t}{2}} \), such that \( p_H(x, t) = h_1(V_{2, H}(t)) \). We have \( \lim_{t \to +\infty} h_1(t) = \lim_{t \to 0^+} h_1(t) = 0 \), while \( h_1(t) \) is continuous and positive in \((0, +\infty)\). Hence we can define the constant \( C_1 = \max_{t \in (0, +\infty)} h_1(t) \). Thus we have
\[ p_H \left(x, \left( \frac{t}{w} \right)^{\alpha} \right) \leq C_1, \]
(3.4)

hence we can use dominated convergence theorem to achieve
\[ \lim_{t \to +\infty} p_{H, \alpha}(x, t) = \int_0^{+\infty} \lim_{t \to +\infty} p_H \left( x, \left( \frac{t}{w} \right)^{\alpha} \right) g_\alpha(w) dw = \frac{1}{\sqrt{2\pi g^{2H} \Gamma(2H)}} e^{-\frac{w^2}{2g^{2H} \Gamma(2H)}}. \]

Concerning \( x = 0 \), it follows from Proposition 3.3 that
\[ \lim_{t \to 0^+} \frac{V_{2, H}(t)}{t^{2H} \wedge 1} = 2H \text{ and } \lim_{t \to +\infty} \frac{V_{2, H}(t)}{t^{2H} \wedge 1} = H \theta^{2H} \Gamma(2H), \]
hence the function \( t \mapsto \frac{V_{2, H}(t)}{t^{2H} \wedge 1} \) is continuous and positive on \((0, +\infty)\) and \( C_2(H) = \inf_{t \in (0, +\infty)} \frac{V_{2, H}(t)}{t^{2H} \wedge 1} > 0 \). Let us set \( C_3(H) = \frac{1}{\sqrt{2\pi C_2(H)}} \) and suppose \( t > t_0 > 0 \) to achieve
\[ p_H \left( 0, \left( \frac{t}{w} \right)^{\alpha} \right) \leq C_3(H) \frac{1}{\sqrt{(w \wedge 1)^{2H}}} =: h_2(w). \]
(3.4)

To show that \( h_2(w) g_\alpha(w) \) is in \( L^1(0, +\infty) \) let us split the integral to achieve
\[ \int_0^{+\infty} h_2(w) g_\alpha(w) dw = C_3(H) \left( \int_0^{t_0} g_\alpha(w) dw + t_0^{H\alpha} \int_{t_0}^{+\infty} w^{H\alpha} g_\alpha(w) dw \right). \]
The first integral is finite since it is controlled by 1, while the second integral is controlled by \( \mathbb{E}[\sigma_\alpha(1)^{H\alpha}] \) which is finite since \( H\alpha < \alpha \). Thus we can use
dominated convergence theorem to achieve
\[
\lim_{t \to +\infty} p_{H,\alpha}(0,t) = \int_0^{+\infty} \lim_{t \to +\infty} p_H \left(0, \left(\frac{t}{w}\right)^\alpha\right) g_\alpha(w) dw = \frac{1}{\sqrt{2\pi H} t^{2H} \Gamma(2H)}. 
\]

On the other hand, we have \( \mathbb{E}[E_0^{-nH}(t)] = +\infty \) for any \( n > 1 \), thus we cannot use \([4, \text{Corollary 4.3}]\) to deduce that \( p_{H,\alpha}(x,t) \) is differentiable in \( x \).

### 3.1. Differentiability of \( p_{H,\alpha}(x,t) \) in \( t > 0 \)

In this section, we will focus on the differentiability of \( p_{H,\alpha}(x,t) \) with respect to time. In particular, we have the following result.

**Proposition 3.5.** For any \( \alpha \in (0,1) \) and \( x \neq 0 \) the function \( p_{H,\alpha}(x,\cdot) \in C^1(0,+\infty) \).

**Proof.** Let us recall (3.1) and (3.2) which imply that
\[
p_{H,\alpha}(x,t) = \int_0^{+\infty} p_H(x,z) \frac{1}{\alpha} \left( t - \frac{1}{\alpha} \right) g_\alpha(tz^{-\frac{1}{\alpha}}) dz = \int_0^{+\infty} p_H \left( x, \left( \frac{t}{w} \right)^\alpha \right) g_\alpha(w) dw.
\]

Now we want to show that we could differentiate under the sign of the integral. To do this, first of all fix \( t > 0 \) and consider an interval \([t_1, t_2]\) such that \( t \in [t_1, t_2] \). To use the differentiation under the sign of integral, we will show that for \( t \in [t_1, t_2] \) and \( w \in (0, +\infty) \) the function \( \frac{\partial}{\partial t} p_H(x, \left( \frac{t}{w} \right)^\alpha) \) is bounded by a constant independent of \( t \) and \( w \). Then, we will need to split the domain \((0, +\infty)\) into two intervals: \((0, t]\) and \((t, +\infty)\) and study the upper bound as \( w \) belongs to these different intervals.

Let us first recall that \( p_H(x,t) \) is expressed in formula (3.3). Now fix \( t > 0 \) and observe that
\[
\frac{d}{dt} p_H \left( x, \left( \frac{t}{w} \right)^\alpha \right) = \alpha (\alpha - 1) w^{-\alpha - 1} \left( \frac{\partial}{\partial y} p_H(x,y) \right) |_{y=\left( \frac{t}{w} \right)^\alpha}.
\]

and
\[
\frac{\partial}{\partial y} p_H(x,y) = \frac{1}{2} V'_2 H(y) \frac{1}{\sqrt{2\pi V_2 H(y) (V_2 H(y))^2}} [x^2 - V_2 H(y)] e^{-\frac{x^2}{2V_2 H(y)}}.
\]

Let us also observe that
\[
|x^2 - V_2 H(y)| \leq x^2 + V_2 H(\infty) =: M(x,H).
\]

Now let us consider \( y \in (0,1] \). Asymptotics given in Lemma [3.3] implies that
\[
\lim_{y \to 0^+} \frac{V'_2 H(y)}{y^H - 2\pi V_2 H(y)} = \lim_{y \to 0^+} \frac{1}{\sqrt{2\pi}} \frac{V'_2 H(y)}{V_2 H(y)} \frac{1}{y^{2H-1}} \frac{1}{\sqrt{V_2 H(y)}} \frac{1}{y^{H-1}} = 4H^2 \frac{1}{\sqrt{4\pi H}}.
\]

and \( y \in (0,1] \mapsto \frac{V'_2 H(y)}{y^{H-1} 2\pi V_2 H(y)} \) is a continuous function, hence there exists a constant \( C_1(H) = \sup_{y \in (0,1)] \frac{V'_2 H(y)}{y^{H-1} 2\pi V_2 H(y)} > 0 \) such that
\[
0 \leq \frac{V'_2 H(y)}{\sqrt{2\pi V_2 H(y)}} \leq C_1(H) y^{H-1} \quad \forall y \in (0,1].
\]

Since \( V_2 H(y) \approx y^{2H} \) as \( y \to 0 \), there exists also two positive constants \( C_2(H) \) and \( C_3(H) \) such that
\[
C_2(H) y^{2H} \leq V_2 H(y) \leq C_3(H) y^{2H} \quad \forall y \in (0,1].
\]
Thus we have for any \( y \in [0, 1] \)
\[
\left| \frac{\partial}{\partial y} p_H(x, y) \right| \leq \frac{M(x, H)C_1(H)}{2C_2^2(H)} y^{-3H-1} e^{-\frac{x^2}{2C_3(H)}} w^{2\alpha}.
\]

Recall that \( t \in [t_1, t_2] \). We have that \( (\frac{1}{w})^\alpha < 1 \) if and only if \( w > t \). Thus, for \( w \in (t, +\infty) \) we have
\[
\left| \frac{d}{dt} p_H \left( x, \left( \frac{t}{w} \right)^\alpha \right) \right| \leq \frac{M(x, H)C_1(H)}{2C_2^2(H)} t^{-3\alpha} w^\alpha \exp \left( -\frac{x^2}{2C_3(H)t^{2\alpha}} w^{2\alpha} \right)
\]
\[
\leq \frac{M(x, H)C_1(H)}{2C_2^2(H)} t^{-3\alpha} w^\alpha \exp \left( -\frac{x^2}{2C_3(H)t^{2\alpha}} w^{2\alpha} \right).
\]

Now let us observe that
\[
\lim_{w \to +\infty} w^\alpha e^{-\frac{x^2}{2C_3(H)t^{2\alpha}} w^{2\alpha}} = 0,
\]
hence we can define
\[
C_4(x, H, \alpha, t_1) := \sup_{w \in (t_1, +\infty)} w^\alpha e^{-\frac{x^2}{2C_3(H)t^{2\alpha}} w^{2\alpha}}
\]
and then, for \( w > t \), we have
\[
w^\alpha e^{-\frac{x^2}{2C_3(H)t^{2\alpha}} w^{2\alpha}} \leq C_4(x, H, \alpha, t_1).
\]

We have that
\[
\left| \frac{d}{dt} p_H \left( x, \left( \frac{t}{w} \right)^\alpha \right) \right| \leq C_5(x, H, \alpha, t_1, t_2)
\]
where
\[
C_5(x, H, \alpha, t_1, t_2) := \frac{\alpha M(x, H)C_1(H)C_4(x, H, \alpha, t_3)}{2C_2^2(H)t_1^{-3\alpha+1}}.
\]

Now let us consider \( y \in [1, +\infty) \). As before, by Lemma 5.3, we know that
\[
\lim_{y \to +\infty} \frac{V'_{2,H}(y)}{V_{2,H}(y)} = \frac{2H(2H-1)\theta}{\sqrt{2\pi V_{2,H}(\infty)}}
\]
and \( y \in [1, +\infty) \mapsto \frac{V'_{2,H}(y)}{\sqrt{2\pi V_{2,H}(y)}} \) is a continuous function, hence there exists a constant \( C_6(H) = \sup_{y \in (1, +\infty)} \frac{V'_{2,H}(y)}{\sqrt{2\pi V_{2,H}(y)}} \) such that
\[
\frac{V'_{2,H}(y)}{\sqrt{2\pi V_{2,H}(y)}} \leq C_6(H)y^{-2H} e^{-\frac{\theta}{y}}
\]
while, being \( V_{2,H} \) increasing, we have
\[
V_{2,H}(1) \leq V_{2,H}(y).
\]

Thus, for \( y \in [1, +\infty) \) we have
\[
\left| \frac{\partial}{\partial y} p_H(x, y) \right| \leq \frac{M(x, H)C_6(H)}{2(V_{2,H}(1))^2} y^{2-2H} e^{-\frac{\theta}{y}}.
\]
We have that \((\frac{1}{w})^\alpha \geq 1\) if and only if \(w \leq t\). Thus, for \(w \leq t\) we have
\[
\left| \frac{d}{dt} \left( x, \left( \frac{t}{w} \right) ^\alpha \right) \right| \leq \frac{\alpha M(x, H) C_\alpha(H)}{2(V_2, H(1))^2} \left( \frac{3\alpha - 2H\alpha - 1}{w} \right)^\alpha 2H\alpha - 3\alpha e^{-\alpha \theta w^{-\alpha}} \leq \frac{\alpha M(x, H) C_\alpha(H)}{2(V_2, H(1))^2} \left( \frac{3\alpha - 2H\alpha - 1}{w} \right)^\alpha 2H\alpha - 3\alpha e^{-\alpha \theta w^{-\alpha}},
\]
where
\[
t_{\text{max}} = \begin{cases} t_1 & \text{if } (3 - 2H)\alpha - 1 < 0, \\ t_2 & \text{if } (3 - 2H)\alpha - 1 \geq 0. \end{cases}
\]
Now let us observe that
\[
\lim_{w \to 0} w^{(2H-3)} e^{-\alpha \theta w^{-\alpha}} = 0,
\]
hence we can define
\[
C_7(H, t_2, \theta, \alpha) = \sup_{w \in (0, t_2)} w^{(2H-3)} e^{-\alpha \theta w^{-\alpha}}
\]
to obtain for any \(w \leq t\)
\[
w^{(2H-3)} e^{-\alpha \theta w^{-\alpha}} \leq C_7(H, t_2, \theta, \alpha).
\]
Thus we have
\[
\left| \frac{d}{dt} \left( x, \left( \frac{t}{w} \right) ^\alpha \right) \right| \leq C_8(H, t_1, t_2, \theta, \alpha, x),
\]
where
\[
C_8(H, t_1, t_2, \theta, \alpha, x) := \frac{\alpha M(x, H) C_\alpha(H)}{2(V_2, H(1))^2} \left( t_{\text{max}} \right)^{(3-2H)\alpha - 1} C_7(H, t_2, \theta, \alpha).
\]
Finally let us define
\[
C_9(x, H, \alpha, \theta, t_1, t_2) := \max\{C_8(x, H, \alpha, t_1, t_2), C_8(x, t_1, t_2, \alpha, H, \theta)\}
\]
to obtain
\[
\left| \frac{d}{dt} \left( x, \left( \frac{t}{w} \right) ^\alpha \right) \right| \leq C_9(x, H, \alpha, \theta, t_1, t_2)
\]
for \(w \in (0, +\infty)\). Thus we have that for any \(t > 0\) the derivative is bounded in a neighbourhood of \(t\) and, being \(g_\alpha(w)\) a probability density function, we can use differentiation under integral sign, concluding the proof.

\[\square\]

3.2. Density \(p_{H, \alpha}\) as a function of \(H\). Let us consider the behavior of the density \(p_{H, \alpha}\) around the limit point 1/2. Recall that \(U_{\frac{H}{2}}(t)\) is a classical Ornstein-Uhlenbeck process. Corresponding time-changed process \(U_{\frac{H}{2}}(t)\) and Kolmogorov equation for its density have been widely studied in \([16, 17]\). Let us denote by \(p_{\frac{H}{2}, \alpha}(x, t)\) its density. Note that variance of the initial process \(U_{H}(t)\) has the form
\[
V_{2, H}(t) = H \left( \int_0^t e^{-\frac{\theta}{2} z^{2H-1}} dz + e^{-\frac{2\alpha}{3} H} \int_0^t e^{\frac{\theta}{2} z^{2H-1}} dz \right),
\]
and so is continuous for \((H, t) \in [\frac{1}{2}, 1) \times [0, +\infty)\) (see \([3, \text{Theorem 1}]\)). This also leads to the fact that \(\lim_{H \to \frac{1}{2}^+} p_H(x, t) = p_{\frac{1}{2}}(x, t)\) for any \((x, t) \in \mathbb{R} \times (0, +\infty)\).

By using this observation, we can show the following convergence result.
Theorem 3.6. It holds that
\[
\lim_{H \to \frac{1}{2}^+} p_{H,\alpha}(x, t) = p_{\frac{1}{2},\alpha}(x, t)
\]
for any \( t \in (0, +\infty) \) and for any \( x \in \mathbb{R} \). Moreover, for any compact set \( K \subset \mathbb{R} \setminus \{0\} \), \( p_{H,\alpha} \to p_{\frac{1}{2},\alpha} \) uniformly in \( K \times [0, +\infty) \) as \( H \to \frac{1}{2} \).

Proof. We have already shown in equation (3.4) that for any \( x \in \mathbb{R} \setminus \{0\}, t \geq 0 \) and \( w > 0 \) it holds
\[
p_H \left( x, \left( \frac{t}{w} \right)^\alpha \right) \leq C_1
\]
where the constant \( C_1 \) does not depend on \( H \). Hence we can use dominated convergence theorem to achieve
\[
\lim_{H \to \frac{1}{2}^+} \int_0^\infty p_{H,\alpha}(x, t) dw = \int_0^\infty p_{\frac{1}{2}}(x, t) dw = p_{\frac{1}{2},\alpha}(x, t).
\]

Concerning \( x = 0 \), let us consider the function \( h_3(t, H) = \frac{V_{2, H}(t)}{2\pi H} \) on \((0, +\infty) \times [\frac{1}{2}, 1] \). \( h_3 \) can be extended by setting
\[
h_3(0, H) = 2H \quad h_3(+\infty, H) = H \Gamma(2H),
\]
and we get a continuous function on \([0, +\infty] \). In particular we have that \( C_2(H) = \min_{t \in [0, +\infty]} h_3(t, H) > 0 \) By Berge’s Maximum Theorem (see [8, Page 116]) we have that \( C_2 : [\frac{1}{2}, 1] \to \mathbb{R}^+ \) is continuous. In particular, since we want to study the behaviour as \( H \to \frac{1}{2}^+ \), we can suppose \( H \leq \frac{3}{4} \). Let us define \( C_3 = \min_{H \in [\frac{1}{2}, \frac{3}{4}]} C_2(H) > 0 \). Define \( C_4 = \frac{1}{\sqrt{2\pi C_3}} \) to achieve
\[
p_H \left( 0, \left( \frac{t}{w} \right)^\alpha \right) \leq \frac{C_4}{\sqrt{\left( \frac{t}{w} \right)^{2H\alpha} + 1}}
\]
Finally, let us observe that if \( \frac{t}{w} \leq 1 \) then \( \left( \frac{t}{w} \right)^{2H\alpha} \geq \left( \frac{t}{w} \right)^{\frac{3}{2} \alpha} \). Thus we have
\[
p_H \left( 0, \left( \frac{t}{w} \right)^\alpha \right) \leq \begin{cases} C_4 w^{\frac{3}{2} \alpha} & w \geq t, \\ C_4 & w \leq t, \end{cases}
\]
where the RHS is independent of \( H \) and integrable when multiplied by \( g_\alpha(w) \). Hence we can use dominated convergence theorem to achieve \( \lim_{H \to \frac{1}{2}^+} p_{H,\alpha}(0, t) = p_{\frac{1}{2},\alpha}(0, t) \) for any \( t > 0 \).

Now let us show the uniform convergence. Consider \( K \subset \mathbb{R} \setminus \{0\} \) a compact set. Let us suppose, without loss of generality, that \( K = [a, b] \) for some \( a, b > 0 \). Define the function
\[
p(x, t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}
\]
for \((x, t) \in K \times (0, +\infty) \). We can extend it by continuity to \((x, t) \in K \times [0, +\infty) \) by setting \( p(x, 0) = 0 \). Let us differentiate \( p \) with respect to \( x \) and \( t \). We have
\[
\frac{\partial p}{\partial t}(x, t) = \frac{1}{2\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}} \left( \frac{x^2 - t}{t} \right) \quad \frac{\partial p}{\partial x}(x, t) = -\frac{x}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}}.
\]
Let us consider the vector

\[ q(x, t) = \left( \frac{x^2 - t}{t}, -2x \right) \]

to obtain that \( \nabla p(x, t) = \frac{1}{2\sqrt{2\pi t^2}} e^{-\frac{x^2}{2t}} q(x, t) \). Obviously,

\[ |q(x, t)| = \sqrt{\left( \frac{x^2 - t}{t} \right)^2 + 4t^2 x^2} \]

and therefore

\[ |\nabla p(x, t)| = \frac{\sqrt{\left( \frac{x^2 - t}{t} \right)^2 + 4t^2 x^2}}{2\sqrt{2\pi t^2}} e^{-\frac{x^2}{2t}}. \]

Observe that we can extend \( |\nabla p(x, t)| \) by continuity to \( K \times [0, +\infty) \) by putting

\[ |\nabla p(x, 0)| = 0. \]

Let us now fix a compact \( K_2 = [0, T] \) for some \( T > 0 \) and let \((x, t), (y, s) \in K \times K_2\). Since \( K \times K_2 \) admits convex interior, we can apply Lagrange’s Theorem to show that there exists a point \((z, \tau) \in [(x, t), (y, s)]\) (where \([(x, t), (y, s)]\) is the segment connecting \((x, t)\) to \((y, s)\)) such that

\[ p(x, t) - p(y, s) = \langle \nabla p(z, \tau), (x - y, t - s) \rangle. \]

Taking the absolute value and using Cauchy-Schwartz inequality we have

\[ |p(x, t) - p(y, s)| \leq |\nabla p(z, \tau)||[(x - y, t - s)]. \]

Finally, since we have shown that \( |\nabla p(x, t)| \) is continuous in \( K \times K_2 \) (that is compact), we can take the maximum, achieving

\[ |p(x, t) - p(y, s)| \leq \left( \max_{(z, \tau) \in K \times K_2} |\nabla p(z, \tau)| \right) |[(x - y, t - s)]. \]

Now let us observe that the function \( H \mapsto V_{2, H}(+\infty) \) is continuous on the interval \([\frac{1}{2}, 1]\) for any \( \theta > 0 \). Therefore we can introduce finite values

\[ T = \max_{H \in [\frac{1}{2}, 1]} V_{2, H}(+\infty) \quad \text{and} \quad C_5(K) = \max_{(z, \tau) \in K \times K_2} |\nabla p(z, \tau)|, \]

accompanied by the compact set \( K_2 = [0, T] \). Observe also that

\[ p(x, V_{2, H}(t)) = p_H(x, t). \]

Thus, we have

\[ |p_H(x, t) - p_+(x, t)| = |p(x, V_{2, H}(t)) - p(x, V_{2, +}(t))| \]

\[ \leq C_5(K)|V_{2, H}(t) - V_{2, +}(t)| \]

\[ = C_5(K)|V_{2, H}(t) - V_{2, +}(t)| \]

\[ \leq C_5(K) \left\| V_{2, H}(t) - V_{2, +}(t) \right\|_{L_\infty(0, +\infty)}. \]

Taking the supremum as \((x, t) \in K \times [0, +\infty)\) we have

\[ \left\| p_H - p_+ \right\|_{L_\infty(K \times [0, +\infty))} \leq C_5(K) \left\| V_{2, H} - V_{2, +} \right\|_{L_\infty(0, +\infty)}. \]
Now let us observe that for any $t \in [0, +\infty)$ and $x \in K$ we have
\[
|p_{H,\alpha}(x, t) - p_{\frac{1}{2},\alpha}(x, t)| \leq \int_{0}^{+\infty} |p_{H}(x, s) - p_{\frac{1}{2}}(x, s)| f_{\alpha}(s, t)ds
\]
\[
\leq \left\|p_{H} - p_{\frac{1}{2}}\right\|_{L^{\infty}(K \times (0, +\infty))}
\leq C_{5}(K) \left\|V_{2,H} - V_{2,\frac{1}{2}}\right\|_{L^{\infty}(0, +\infty)}
\]
and then, taking the supremum, we have
\[
(3.6) \quad \left\|p_{H,\alpha}(x, t) - p_{\frac{1}{2},\alpha}(x, t)\right\|_{L^{\infty}(K \times [0, +\infty))} \leq C_{5}(K) \left\|V_{2,H} - V_{2,\frac{1}{2}}\right\|_{L^{\infty}(0, +\infty)}.
\]

Finally, let us recall that $V_{2,H}(t)$ is continuous for $(H, t) \in [\frac{1}{2}, 1] \times [0, +\infty)$ where we define $V_{2,H}(+\infty) = \theta^{H} H^{2} T(2H)$. Thus in particular it is uniformly continuous for $(H, t) \in [\frac{1}{2}, 1] \times [0, +\infty]$ and then $V_{2,H}(t) \to V_{2,\frac{1}{2}}(t)$ uniformly. Thus we can take the limit as $H \to \frac{1}{2}^{+}$ in Equation (3.6) to obtain
\[
\lim_{H \to \frac{1}{2}^{+}} \left\|p_{H,\alpha}(x, t) - p_{\frac{1}{2},\alpha}(x, t)\right\|_{L^{\infty}(K \times [0, +\infty))} = 0,
\]
concluding the proof. \hfill \square

One can prove that also the moments converge uniformly.

**Proposition 3.7.** It holds \( \lim_{H \to \frac{1}{2}^{+}} V_{n,H,\alpha}(t) = V_{n,\frac{1}{2},\alpha}(t) \). Moreover for $n \geq 2$ the convergence is uniform in $[0, +\infty)$, while for $n = 1$ the convergence is uniform in any set of the form $[t_{0}, +\infty)$ for $t_{0} > 0$.

**Proof.** Let us first recall that $U_{H}(t)$, being a Gaussian process, admits the following equalities for its absolute moments:
\[
V_{n,H}(t) = \frac{2^{\frac{n}{2}} \Gamma \left( \frac{n+1}{2} \right)}{\sqrt{\pi}} (V_{2,H}(t))^{\frac{n}{2}}.
\]

Hence, we have that
\[
V_{n,H}(t) - V_{n,\frac{1}{2}}(t) = \frac{2^{\frac{n}{2}} \Gamma \left( \frac{n+1}{2} \right)}{\sqrt{\pi}} ((V_{2,H}(t))^{\frac{n}{2}} - (V_{2,\frac{1}{2}}(t))^{\frac{n}{2}}).
\]

Let us consider $H \in [\frac{1}{2}, 1]$, fix $V = \max_{H \in [\frac{1}{2}, 1]} V_{2,H}(\infty)$ and for $n \geq 2$ set $L(n) = \frac{n}{2} V^{\frac{n}{2}-1}$. Then observe that
\[
\left| (V_{2,H}(t))^{\frac{n}{2}} - (V_{2,\frac{1}{2}}(t))^{\frac{n}{2}} \right| \leq L(n) \left| V_{2,H}(t) - V_{2,\frac{1}{2}}(t) \right|.
\]

Taking the supremum, we have
\[
\left\| V_{n,H} - V_{n,\frac{1}{2}} \right\|_{L^{\infty}(0, +\infty)} \leq L(n) \left\| V_{2,H} - V_{2,\frac{1}{2}} \right\|_{L^{\infty}(0, +\infty)},
\]
and then we can take the limit as $H \to \frac{1}{2}^{+}$ to conclude that for any $n \geq 2$
\[
\lim_{H \to \frac{1}{2}^{+}} \left\| V_{n,H} - V_{n,\frac{1}{2}} \right\|_{L^{\infty}(0, +\infty)} = 0.
\]
The case \( n = 1 \) is different. In this case \( V_{1,H}(t) = \sqrt{\frac{2}{\pi}} \sqrt{V_{2,H}(t)} \). However we have
\[
|V_{1,H}(t) - V_{1,\frac{1}{2}}(t)| \leq \sqrt{\frac{2}{\pi}} \left| \sqrt{V_{2,H}(t)} - \sqrt{V_{2,\frac{1}{2}}(t)} \right| \leq \sqrt{\frac{2}{\pi}} \sqrt{|V_{2,H}(t) - V_{2,\frac{1}{2}}(t)|}.
\]
Now, taking the supremum, and using the fact that \( t \mapsto \sqrt{t} \) is increasing we have
\[
\left\| V_{1,H} - V_{1,\frac{1}{2}} \right\|_{L^\infty(0, +\infty)} \leq \sqrt{\frac{2}{\pi}} \left\| V_{2,H} - V_{2,\frac{1}{2}} \right\|_{L^\infty(0, +\infty)}.
\]
Taking the limit as \( H \to \frac{1}{2}^+ \) we have
\[
\lim_{H \to \frac{1}{2}^+} \left\| V_{1,H} - V_{1,\frac{1}{2}} \right\|_{L^\infty(0, +\infty)} = 0.
\]
Now, for any \( n \in \mathbb{N} \), we have
\[
V_{n,H,\alpha}(t) = \int_0^{+\infty} V_{n,H}(s)f_\alpha(s,t)ds,
\]
hence,
\[
|V_{n,H,\alpha}(t) - V_{n,\frac{1}{2},\alpha}(t)| \leq \left\| V_{n,H} - V_{n,\frac{1}{2}} \right\|_{L^\infty(0, +\infty)}.
\]
Taking the supremum, we get
\[
\left\| V_{n,H,\alpha} - V_{n,\frac{1}{2},\alpha} \right\|_{L^\infty(0, +\infty)} \leq \left\| V_{n,H} - V_{n,\frac{1}{2}} \right\|_{L^\infty(0, +\infty)}.
\]
Finally, taking the limit as \( H \to \frac{1}{2}^+ \), we conclude that
\[
\lim_{H \to \frac{1}{2}^+} \left\| V_{n,H,\alpha} - V_{n,\frac{1}{2},\alpha} \right\|_{L^\infty(0, +\infty)} = 0.
\]

4. The generalized Fokker-Planck equation

In this section we want to discuss a Fokker-Planck equation for \( p_{H,\alpha}(x,t) \). To do this, we need to recall some operators that were introduced in [4]. Denote \( \mathcal{H} = \{ \lambda \in \mathbb{C} : \Re(\lambda) > 0 \} \). First of all, let us define
\[
L_H u(x, \lambda) = \mathcal{L}_{t \to \lambda} [V_{2,H}(t)u(x,t)], \ \lambda \in \mathcal{H}, x \in I \subset \mathbb{R},
\]
for any function \( u : I \times (0, +\infty) \to \mathbb{R} \) such that \( L_H u \) is well-defined, and denote by \( \mathcal{D}(L_H, I) \) the domain of \( L_H \). We can actually apply \( L_H \) to functions \( u : (0, +\infty) \to \mathbb{R} \) without the dependence on \( x \). In such case, we denote by \( \mathcal{D}(L_H) \) the domain of \( L_H \). Moreover, fix \( c_1 < 0 < c_2 \) such that \( c_2 - c_1 < 1/\theta \). We can define the operator \( \widehat{L}_H \) for a function \( v : I \times \mathcal{H} \to \mathbb{C} \) such that \((c_2 + iz)^{1/\alpha - 1}v(x, (c_2 + iz)^{1/\alpha})\) does not depend on the representation of \((c_2 + iz)^{1/\alpha}\):
\[
\widehat{L}_H v(x, \lambda) = \frac{1}{4\pi^2} \int_0^{+\infty} e^{-\lambda^2t} \lim_{R \to +\infty} \int_{-\infty}^{+\infty} e^{(c_1+iw)t} \times
\]
\[
\times \int_{-R}^{R} \mathcal{L}_{t \to \lambda} [V_{2,H}(t)](c_1 - c_2 + i(w-z))(c_2 + iz)^{1/\alpha - 1}v(x, (c_2 + iz)^{1/\alpha})dwdt,
\]
denoting by \( \mathcal{D}(\widehat{L}_H, I) \) the set of functions for which \( \widehat{L}_H v \) is well-defined. As before, we denote by \( \mathcal{D}(\widehat{L}_H) \) the domain of \( \widehat{L}_H \) when applied to functions that do not depend on \( x \). There is a wide class of functions that belongs to \( \mathcal{D}(\widehat{L}_H) \). We will
explore them in Section [5].

Finally, let us define the operator $G_{\alpha,H}$ on a Laplace transformable function $v : I \times (0, +\infty) \to \mathbb{R}$ with Laplace transform $\mathcal{P}$ as

$$G_{\alpha,H} v(x,t) = \mathcal{L}_{\lambda \to t}^{-1} \left[ \lambda^{\alpha-1} \frac{\partial^2}{\partial x^2} \hat{L}_H \mathcal{P}(x,\lambda) \right],$$

and denote by $\mathcal{D}(G_{\alpha,H}, I)$ the set of functions for which $G$ is well-defined. We want to focus on the fractional Fokker-Planck equation

$$\partial_t^\alpha v(x,t) = \frac{1}{2} G_{\alpha,H} v(x,t) \quad (x,t) \in I \times (0, +\infty),$$

where subscript $t$ is used to specify that the respective operator acts in variable $t$.

**Definition 4.1.** We say that $v : I \times [0, +\infty) \to \mathbb{R}$ is a classical solution of (4.1) in $I \times [0,T]$ (eventually $T = +\infty$) if

- $v \in \mathcal{D}(G_{\alpha,H}, I)$
- $v(x, \cdot) \in \mathcal{D}(\partial_t^\alpha)$ for any $x \in I$
- Equation (4.1) holds for any $x \in I$ and almost any $t \in [0,T]$.

On the other hand, we can define a weaker form of solution.

**Definition 4.2.** We say that $v : I \times [0, +\infty) \to \mathbb{R}$ is a mild solution of the fractional Fokker-Planck equation (4.1) if, denoting $\mathcal{P}(x,\lambda) := \mathcal{L}_{\lambda \to t}[v(x,t)](\lambda)$,

- $v(x, \cdot)$ is Laplace transformable for any $x \in I$;
- $\mathcal{P} \in \mathcal{D}(\hat{L}_H, I)$;
- $\forall \lambda \in \mathcal{H}, \mathcal{P}(\cdot, \lambda) \in C^0(I)$;
- it holds

$$\lambda^\alpha \mathcal{P}(x,\lambda) - \lambda^{\alpha-1} v(x,0) = \frac{\lambda^{\alpha-1}}{2} \frac{\partial^2}{\partial x^2} \hat{L}_H \mathcal{P}(x,\lambda), \quad \forall x \in I.$$

Before showing that $p_{H,\alpha}(x,t)$ is both mild that classical solution of (4.1), let us recall some property of $L_H$, given in [11] Lemmas 6.1 and 6.7.

**Lemma 4.1.** We have $p_H(x,t), \frac{\partial p_H}{\partial x}(x,t) \in \mathcal{D}(L_H, \mathbb{R})$ and $\frac{\partial^2}{\partial x^2} p_H(x,t) \in \mathcal{D}(L_H, \mathbb{R} \setminus \{0\})$. Moreover, for any $x \in \mathbb{R} \setminus \{0\}$ and $\lambda \in \mathcal{H}$ such that $\Re(\lambda) > 0$, it holds

$$L_H \left( \frac{\partial}{\partial x} p_H \right)(x,\lambda) = \frac{\partial}{\partial x} L_H p_H(x,\lambda), \quad L_H \left( \frac{\partial^2}{\partial x^2} p_H \right)(x,\lambda) = \frac{\partial^2}{\partial x^2} L_H p_H(x,\lambda).$$

Finally, it holds that $\mathcal{P}_{H,\alpha}(x,\lambda) := \mathcal{L}_{\lambda \to t}[p_{H,\alpha}(x,t)](\lambda) \in \mathcal{D}(\hat{L}_H, \mathbb{R} \setminus \{0\})$ and $\hat{L}_H \mathcal{P}_{H,\alpha}(x,\lambda) = L_H p_{H,\alpha}(x,\lambda^\alpha)$.

Now we are ready to show the following result (which is just a specialization of [11] Theorem 7.1), but we rewrite the proof for completeness.

**Theorem 4.2.** $p_{H,\alpha}(x,t)$ is a mild solution of (4.1) in $(\mathbb{R} \setminus \{0\}) \times (0, +\infty)$.

**Proof.** Fix $x \neq 0$. First of all, let us recall that $p_H(x,t)$ is Laplace transformable and denote $\mathcal{P}_H(x,\lambda)$ its Laplace transform. Moreover, since $U_H(t)$ is a Gaussian process with variance $V_{2,H}(t)$, $p_{H}(x,t)$ is solution of the following Fokker-Planck equation:

$$\frac{\partial}{\partial t} p_{H}(x,t) = \frac{1}{2} V_{2,H}(t) \frac{\partial^2}{\partial x^2} p_{H}(x,t).$$
Taking the Laplace transform in both sides we have
\[ \lambda \mathcal{P}_H(x, \lambda) - p_H(x, 0) = \frac{1}{2} \mathcal{L}_H \left( \frac{\partial^2}{\partial x^2} p_H \right)(x, \lambda). \]

Now let us recall that for \( x \neq 0 \) it holds \( p_H(x, 0) = 0 \). Moreover, we can use Lemma 4.1 and substitute \( \lambda^\alpha \) in place of \( \lambda \) to obtain
\[ \lambda^\alpha \mathcal{P}_H(x, \lambda^\alpha) = \frac{1}{2} \frac{\partial^2}{\partial x^2} \hat{\mathcal{L}}_H \mathcal{P}_{H,\alpha}(x, \lambda). \]
Multiplying everything by \( \lambda^{\alpha - 1} \), we come to equality
\[ \lambda^\alpha \lambda^{\alpha - 1} \mathcal{P}_H(x, \lambda^\alpha) = \frac{1}{2} \frac{\partial^2}{\partial x^2} \hat{\mathcal{L}}_H \mathcal{P}_{H,\alpha}(x, \lambda), \]
that is equivalent to the following one:
\[ \lambda^\alpha \mathcal{P}_{H,\alpha}(x, \lambda) = \frac{1}{2} \frac{\partial^2}{\partial x^2} \hat{\mathcal{L}}_H \mathcal{P}_{H,\alpha}(x, \lambda), \]
whence the proof follows (since also \( \mathcal{P}_{H,\alpha}(x, 0) = 0 \)). \( \square \)

Now let us establish a stronger result concerning \( p_{H,\alpha} \). To do this, we need the following lemma, that is a direct consequence of [4, Lemma 7.6] and Proposition 3.5.

**Lemma 4.3.** Function \( \partial_t^\alpha p_{H,\alpha}(x, t) \) is well defined and is Laplace transformable in \( t \) for \( \lambda \in \mathcal{H} \).

Now we can prove the stronger result.

**Theorem 4.4.** Density \( p_{H,\alpha}(x, t) \) is a classical solution of \( (4.4) \) in \((\mathbb{R} \setminus \{0\}) \times (0, +\infty)\).

**Proof.** Let us first observe that \( p_{H,\alpha}(x, t) \) is bounded as \( x \neq 0 \), thus it is Laplace transformable for \( \lambda \in \mathcal{H} \). Moreover, \( p_{H,\alpha}(x, \cdot) \in C^1(0, +\infty) \) as \( x \neq 0 \), thus \( p_{H,\alpha}(x, \cdot) \in D(\partial_t^\alpha) \) as \( x \neq 0 \). Moreover, we have that both \( \partial_t^\alpha p_{H,\alpha}(x, t) \) and \( p_{H,\alpha}(x, t) \) are Laplace transformable, and consequently
\[ \mathcal{L}_{t \rightarrow \lambda}[\partial_t^\alpha p_{H,\alpha}(x, t)](\lambda) = \lambda^\alpha \mathcal{P}_{H,\alpha}(x, \lambda). \]
Hence in Equation (4.2) the left-hand side is the Laplace transform of a function and then also the right-hand side is the Laplace transform of a function. Thus we can take the inverse Laplace transform on both sides to obtain
\[ \partial_t^\alpha p_{H,\alpha}(x, t) = \frac{1}{2} \mathcal{G}_{\alpha,\mathcal{H}} p_{H,\alpha}(x, t). \]
\( \square \)

Actually, we have stronger regularity than Caputo differentiability: our function \( p_{H,\alpha}(x, t) \) is \( C^1 \) in \( t > 0 \). Hence we can try to transfer the non-locality in time to the space term. To do this, let us introduce the operator \( \mathcal{G}_H = \mathcal{D}^{1-\alpha} \mathcal{G}_{\alpha,\mathcal{H}} \) recalling that all operators act in \( t \) for \( x \) fixed. We state the following result.

**Corollary 4.5.** Density \( p_{H,\alpha}(x, t) \) is a classical solution in \((\mathbb{R} \setminus \{0\}) \times (0, +\infty)\) of the equation
\[ (4.4) \]
\[ \frac{\partial}{\partial t} p_{H,\alpha}(x, t) = \frac{1}{2} \mathcal{G}_H p_{H,\alpha}(x, t), \]
in the sense that for any \( x \neq 0 \) \( p_{H,\alpha}(x,t) \) satisfies equation (3.3) for almost all \( t \in (0, +\infty) \).

**Proof.** We know that \( p_{H,\alpha}(x,t) \) is a classical solution of (4.1), so that the following equality holds:
\[
\partial_t p_{H,\alpha}(x,t) = \frac{1}{2} G_{\alpha,H} p_{H,\alpha}(x,t).
\]
Now let us observe that \( p_{H,\alpha}(x,\cdot) \in C^1(0,+\infty) \) for any \( x \neq 0 \), therefore
\[
G_1^{-\alpha} \frac{\partial}{\partial t} p_{H,\alpha}(x,t) = \partial_t p_{H,\alpha}(x,t) = \frac{1}{2} G_{\alpha,H} p_{H,\alpha}(x,t).
\]
Since \( \frac{\partial}{\partial t} p_{H,\alpha}(x,t) \) is well-defined, \( G_1^{-\alpha} \frac{\partial}{\partial t} p_{H,\alpha}(x,t) \) belongs to the domain of the Riemann-Liouville derivative \( D_t^{-\alpha} \) and thus also \( \frac{1}{2} G_{\alpha,H} p_{H,\alpha}(x,t) \). Finally, applying \( G_1^{-\alpha} \) on both sides of the equation we conclude the proof. \( \square \)

**4.1. Asymptotics of operators involved in the Fokker-Planck equation.** Just as we did with \( p_{H,\alpha} \), we want to study the asymptotics of the operators \( L_H \) and \( G_{\alpha,H} \) as \( H \to \frac{1}{2}^+ \). To do this, let us first establish the following result.

**Proposition 4.6.** For any \( \varepsilon > 0 \) there exists a constant \( H_\varepsilon \in (\frac{1}{2},1) \) and a function \( C_\varepsilon : (\frac{1}{2}, H_\varepsilon] \to \mathbb{R}_+ \) such that \( \lim_{H \to \frac{1}{2}^+} C_\varepsilon(H) = 0 \) and
\[
|V_2'_{2,H}(t) - V_2'_{2,\frac{1}{2}}(t)| \leq C_\varepsilon(H) e^{-\frac{\theta}{2}} \quad \forall t \in [\varepsilon, +\infty), \quad \forall H \in \left(\frac{1}{2}, H_\varepsilon\right).
\]
Moreover, for any \( \varepsilon > 0 \) it holds that
\[
\limsup_{H \to \frac{1}{2}^+} \left\| V_2'_{2,H} - V_2'_{2,\frac{1}{2}} \right\|_{L^\infty(0,\varepsilon)} \leq 1.
\]

**Proof.** First of all, \( V_2'_{2,\frac{1}{2}}(t) = e^{-\frac{\theta}{2}} \), while, according to formula (5.3) from [1] as \( H \neq \frac{1}{2} \),
\[
V_2'_{2,H}(t) = 2H(2H - 1)e^{-\frac{\theta}{2}} \int_0^t e^{\frac{\theta}{2}} z^{2H-2}dz.
\]
Now, let us consider an alternative way of representation of \( V_2'_{2,H} \). Applying [13] Formula 3.383.1, we get the equality
\[
\int_0^t e^{\frac{\theta}{2}} z^{2H-2}dz = B(1,2H-1)t^{2H-1} {}_1F_1\left(2H-1;2H;\frac{t}{\theta}\right),
\]
where \( B \) is the Beta function that in our case can be simplified to
\[
B(1,2H-1) = \frac{\Gamma(1)\Gamma(2H-1)}{\Gamma(2H)} = \frac{1}{2H - 1},
\]
and \( {}_1F_1 \) is the hypergeometric sum of the form
\[
{}_1F_1(\alpha;\beta; t) = \sum_{k=0}^{+\infty} \frac{\Gamma(\alpha+k)\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta+k)} \frac{t^k}{k!}.
\]
Now, let us observe that the hypergeometric sum admits the representation
\[
{}_1F_1\left(2H-1;2H;\frac{t}{\theta}\right) = \sum_{k=0}^{+\infty} \frac{\Gamma(2H-1+k)\Gamma(2H)}{\Gamma(2H-1)\Gamma(2H+k)} \frac{t^k}{\theta^k k!} = \sum_{k=0}^{+\infty} \frac{2H-1+k}{2H-1+k} \frac{t^k}{\theta^k k!}.
\]
It means that the derivative can be rewritten as

$$V'_{2,H}(t) = 2H t^{2H-1} e^{-\frac{2t}{\theta}} \sum_{k=0}^{+\infty} \frac{2H-1}{2H-1+k} \frac{t^k}{\theta^k k!}$$

$$= 2H t^{2H-1} e^{-\frac{2t}{\theta}} + 2H(2H-1) t^{2H-1} e^{-\frac{2t}{\theta}} \sum_{k=1}^{+\infty} \frac{1}{2H-1+k} \frac{t^k}{\theta^k k!}.$$

Therefore the difference of the derivatives can be bounded as

$$|V'_{2,H}(t) - V'_{2,\frac{1}{2}}(t)| \leq e^{-\frac{\theta}{2} t} |2H t^{2H-1} - 1| + 2H(2H-1) e^{-\frac{\theta}{2} t} \sum_{k=1}^{+\infty} \frac{1}{2H-1+k} \frac{t^{2H-1+k}}{\theta^k k!}$$

$$= I_1(t, H) + I_2(t, H)$$

Let us first work with the series. Define the function $g(t) = t^{2H-1+k} e^{-\frac{\theta}{2} t} > 0$ and observe that

$$g'(t) = e^{-\frac{\theta}{2} t} \frac{t^{2H-2+k}}{\theta} (\theta(2H-1+k) - t).$$

It means that

$$g(t) \leq \theta^{2H-1+k} (2H-1+k)^{2H-1+k} e^{-(2H-1+k)},$$

and it immediately implies that

$$I_2(t, H) \leq 2H(2H-1) \theta^{2H-1} e^{-\frac{\theta}{2} t} \sum_{k=1}^{+\infty} (2H-1+k)^{2H-2+k} e^{-(2H-1+k)} \frac{1}{k!}.$$

Let us also recall that

$$k! \geq \sqrt{2\pi} e^{-k} k^{k+\frac{1}{2}},$$

and conclude that

$$I_2(t, H) \leq 2H(2H-1) \theta^{2H-1} e^{-(2H-1)} \sum_{k=1}^{+\infty} \left( \frac{2H-1+k}{k} \right)^k \frac{(2H-1+k)^{2H-2}}{k^{\frac{1}{2}}} \frac{1}{\sqrt{2\pi}}.$$

Furthermore, observe that

$$\left( \frac{2H-1+k}{k} \right)^k = \left( 1 + \frac{2H-1}{k} \right)^{\frac{k}{2H-1}}.$$  

Obviously, we have that $\lim_{k \to +\infty} \left( \frac{2H-1+k}{k} \right)^k = e^{2H-1}$. Thus, to make sure that the series converges, we only need to show that

$$2 - 2H + \frac{1}{2} > 1,$$

that is equivalent to the upper bound $H < \frac{3}{4}$. So, we understand that choosing $H > \frac{1}{2}$ and

$$C_1(H) = 2H(2H-1) \theta^{2H-1} e^{-(2H-1)} \sum_{k=1}^{+\infty} \left( \frac{2H-1+k}{k} \right)^k \frac{(2H-1+k)^{2H-2}}{k^{\frac{1}{2}}} \frac{1}{\sqrt{2\pi}},$$
we obtain that $I_2(t, H) \leq C_1(H)e^{-\frac{H}{t}}$ for any $H \in (\frac{1}{2}, H_\varepsilon]$, and, moreover, $C_1(H) \to 0$ as $H \to \frac{1}{2}$. Now let us consider $I_1(t, H)$. It is natural to distinguish three cases. If $t \in [0, \varepsilon]$, then we have

$$I_1(t, H) \leq |2Ht^{2H-1} - 1|$$

where $2Ht^{2H-1} - 1$ is an increasing function, hence

$$I_1(t, H) \leq \max\{1, |2H_\varepsilon^{2H-1} - 1|\}.$$

In such a case, calculating the supremum for $t \in [0, \varepsilon]$ in equation $46$, we obtain that

$$\|V_{2,H} - V_{2,H_\varepsilon}\|_{L^\infty(0,\varepsilon)} \leq \max\{1, |2H_\varepsilon^{2H-1} - 1|\} + C_1(H),$$

and taking the limit superior as $H \to \frac{1}{2}^+$, we ultimately come to the upper bound

$$\lim_{H \to \frac{1}{2}^+} \|V_{2,H} - V_{2,H_\varepsilon}\|_{L^\infty(0,\varepsilon)} \leq 1,$$

since $|2H_\varepsilon^{2H-1} - 1| \to 0$.

Now let us consider $t \in [\varepsilon, 1]$. For this values of argument we have

$$I_1(t, H) \leq e^{-\frac{2H}{t}} \max\{2H_\varepsilon^{2H-1} - 1, 2H - 1\} := C_2(H)e^{-\frac{2H}{t}},$$

where $C_2(H) \to 0$ as $H \to \frac{1}{2}^+$. Finally, let us consider $t \in (1, +\infty)$. Let us observe that $2Ht^{2H-1} - 1 > 0$ if and only if $t^{2H-1} > \frac{1}{2H}$, where $2H > 1$. In particular this is achieved if $t > 1$. Therefore,

$$I_1(t, H) = e^{-\frac{t}{H}} f_H(t),$$

where

$$f_H(t) = (2Ht^{2H-1} - 1)e^{-\frac{t}{H}}.$$

Setting $f(+\infty) = 0$, we can state that $f_H$ is a continuous and non-negative function on the interval $[1, +\infty]$. So, we can search for a maximum within this set. Differentiating $f_H$, one can see that

$$f'_H(t) = \frac{e^{-\frac{t}{H}}}{\theta}(2H\theta(2H - 1)t^{2H-2} - 2Ht^{2H-1} + 1).$$

Denote by $t_{\text{max}}(H)$ the maximum point of $f_H$. Then, since $f_H(+\infty) = 0$, it is possible to conclude that either $t_{\text{max}}(H) = 1$, or

$$2H(\theta(2H - 1)t_{\text{max}}(H)^{-1} - 1)t_{\text{max}}(H)^{2H-1} + 1 = 0.$$

The latter equality is equivalent to the following one:

$$t_{\text{max}}(H)^{2H-1} = \frac{1}{2H(1 - \theta(2H - 1)t_{\text{max}}(H)^{-1})} = \frac{t_{\text{max}}(H)}{2H(t_{\text{max}}(H) - \theta(2H - 1))}.$$

If $t_{\text{max}}(H) = 1$, then, evidently,

$$f_H(t_{\text{max}}(H)) = (2H - 1)e^{-\frac{1}{H}},$$

and this value goes to 0 as $H \to \frac{1}{2}$. If $t_{\text{max}}(H) \neq 1$, then

$$f_H(t_{\text{max}}(H)) = \left( \frac{1}{1 - \frac{\theta(2H - 1)}{t_{\text{max}}(H)}} - 1 \right) e^{-\frac{t_{\text{max}}(H)}{\theta}}.$$
Now let us distinguish two cases. If \( \limsup_{H \to \frac{1}{2}^+} t_{\max}(H) = +\infty \), then
\[
\lim_{H \to \frac{1}{2}^+} \frac{\theta(2H - 1)}{t_{\max}(H)} = 0,
\]
and so \( \lim_{H \to \frac{1}{2}^+} f_H(t_{\max}(H)) = 0 \). If, on the contrary,
\[
\limsup_{H \to \frac{1}{2}^+} t_{\max}(H) \neq +\infty,
\]
then \( t_{\max}(H) \) is bounded in \( \left( \frac{1}{2}, H_\varepsilon \right] \) for some \( H_\varepsilon > \frac{1}{2} \) and then we have again \( \lim_{H \to \frac{1}{2}^+} f_H(t_{\max}(H)) = 0 \). Thus we can define \( C_3(H) = f_H(t_{\max}(H)) \) and \( C_4(H) = \max\{C_2(H), C_3(H)\} \). In such case, for any \( t \in [\varepsilon, +\infty) \) and \( H \in \left( \frac{1}{2}, H_\varepsilon \right] \) we have that
\[
|V_{2, H}(t) - V_{2, \frac{\varepsilon}{2}}(t)| \leq (C_4(H) + C_1(H))e^{-\frac{\varepsilon}{2}},
\]
concluding the proof, setting \( C_2(H) = C_4(H) + C_1(H) \).

Additionally, we need the following result.

Lemma 4.7. For any compact set \( K \subset \mathbb{R} \setminus \{0\} \) we have that \( \frac{\partial^2 p_{\mathbb{R}}}{{\partial x}^2} \to \frac{\partial^2 p}{\partial x^2} \) as \( H \to \frac{1}{2}^+ \) uniformly in \( K \times [0, +\infty) \).

Proof. Let us consider the heat kernel \( p(x, t) \) as given in equation \( (3.3) \). Its second derivative equals
\[
\frac{\partial^2 p}{\partial x^2}(x, t) = \frac{x^2 - t}{\sqrt{2\pi t^2}}e^{-\frac{x^2}{2t}} =: f(x, t).
\]
Now let us observe that
\[
\frac{\partial f}{\partial x}(x, t) = \frac{3xt - x^3}{\sqrt{2\pi t^2}}e^{-\frac{x^2}{2t}} , \text{ and } \frac{\partial f}{\partial t}(x, t) = \frac{1}{\sqrt{2\pi t^2}}e^{-\frac{x^2}{2t}} \left( \frac{3t^2 - 6x^2t + x^4}{2t} \right).
\]
It means that
\[
\nabla f(x, t) = \frac{1}{\sqrt{2\pi t^2}}e^{-\frac{x^2}{2t}} q(x, t),
\]
where
\[
q(x, t) = \left( 3xt - x^3, \frac{3t^2 - 6x^2t + x^4}{2t} \right).
\]
Further, the absolute value equals
\[
|\nabla f(x, t)| = \frac{1}{2\sqrt{2\pi t^2}}e^{-\frac{x^2}{2t}} \sqrt{4(3xt - x^3)^2t^2 + (3t^2 - 6x^2t + x^4)^2}.
\]
Now let us consider a compact set \( K \subset \mathbb{R} \setminus \{0\} \). We can extend by continuity \( |\nabla f(x, t)| \) on \( K \times [0, +\infty) \) by setting \( |\nabla f(x, 0)| = 0 \) for \( x \in K \). As we have already done before, consider \( T = \max_{H \in [\frac{1}{2}, 1]} V_{2, H}(\infty) \) and \( K_2 = [0, T] \). Finally, define
\[
C_1(K) = \max_{(x, t) \in K \times K_2} |\nabla f(x, t)|,
\]
in order to obtain that for each \( x \in K \) and \( t, s \in K_2 \) the inequality
\[
|f(x, t) - f(x, s)| \leq C_1(K)|t - s|
\]
holds. Now, observe that \( \frac{\partial^2 p_{\mathbb{R}}}{\partial x^2}(x, V_{2, H}(t)) = \frac{\partial^2 p}{\partial x^2} p_H(x, t) \), hence
\[
\left| \frac{\partial^2}{\partial x^2} P_{H}(x, t) - \frac{\partial^2}{\partial x^2} P_{\frac{1}{2}^+}(x, t) \right| \leq C_1(K)|V_{2, H}(t) - V_{2, \frac{\varepsilon}{2}}(t)| \leq C_1(K) \left\| V_{2, H} - V_{2, \frac{\varepsilon}{2}} \right\|_{L^\infty(0, +\infty)}.
\]
Taking the supremum in \((x,t)\) and then the limit as \(H \to \frac{1}{2}^+\), we finally get that

\[
\lim_{H \to \frac{1}{2}^+} \left\| \frac{\partial^2}{\partial x^2} p_H - \frac{\partial^2}{\partial x^2} p_{\frac{1}{2}} \right\|_{L^\infty(K \times [0, +\infty))} = 0,
\]

and the last relation concludes the proof.

Now we can establish the following convergence result for \(L_H\).

**Theorem 4.8.** Let \(K \subset \mathbb{R} \setminus \{0\}\) and \(\mathcal{K} \subset \mathcal{H}\) be compact sets. Then \(L_H p_H(x,\lambda) \to L_{\frac{1}{2}} p_{\frac{1}{2}}(x,\lambda)\) and \(L_H \left( \frac{\partial^2 p_H}{\partial x^2} \right)(x,\lambda) \to L_{\frac{1}{2}} \left( \frac{\partial^2 p_{\frac{1}{2}}}{\partial x^2} \right)(x,\lambda)\) uniformly in \(K \times \mathcal{K}\).

**Proof.** Let us show the result for \(L_H p_H\), since the proof for \(L_H \left( \frac{\partial^2 p_H}{\partial x^2} \right)\) is analogous. Let us fix \(K \subset \mathbb{R} \setminus \{0\}\). Moreover, without loss of generality, we can suppose that \(\mathcal{K} = [a,b] \times \{0\}\), i.e. we can assume that \(\lambda \in [a,b]\) and \(a > 0\). Let us also consider \(\varepsilon \in (0, 1)\). Then the following relations hold:

\[
L_H p_H(x,\lambda) - L_{\frac{1}{2}} p_{\frac{1}{2}}(x,\lambda) = \int_0^{+\infty} e^{-\lambda t}(V'_{2,H}(t)p_H(x,t) - V'_{2,\frac{1}{2}}(t)p_{\frac{1}{2}}(x,t))dt
\]

\[
= \int_0^{\varepsilon} e^{-\lambda t}V'_{2,H}(t)(p_H(x,t) - p_{\frac{1}{2}}(x,t))dt + \int_{\varepsilon}^{+\infty} e^{-\lambda t}(V'_{2,H}(t) - V'_{2,\frac{1}{2}}(t))p_{\frac{1}{2}}(x,t)dt
\]

Therefore we can bound the difference under consideration as follows:

\[
|L_H p_H(x,\lambda) - L_{\frac{1}{2}} p_{\frac{1}{2}}(x,\lambda)| \leq \int_0^{+\infty} e^{-\lambda t}V'_{2,H}(t)|p_H(x,t) - p_{\frac{1}{2}}(x,t)|dt
\]

\[
+ \int_0^{\varepsilon} e^{-\lambda t}|V'_{2,H}(t) - V'_{2,\frac{1}{2}}(t)|p_{\frac{1}{2}}(x,t)dt
\]

\[
+ \int_{\varepsilon}^{+\infty} e^{-\lambda t}|V'_{2,H}(t) - V'_{2,\frac{1}{2}}(t)|p_{\frac{1}{2}}(x,t)dt
\]

\[
\leq \left\| p_H - p_{\frac{1}{2}} \right\|_{L^\infty(K \times [0, +\infty))} \mathcal{L}_{1 \to 1}[V'_{2,H}(t)](\lambda)
\]

\[
+ \left\| V'_{2,H} - V'_{2,\frac{1}{2}} \right\|_{L^\infty([0,\varepsilon])} \int_0^{\varepsilon} e^{-\lambda t}p_{\frac{1}{2}}(x,t)dt
\]

\[
+ \left\| V'_{2,H} - V'_{2,\frac{1}{2}} \right\|_{L^\infty((\varepsilon, +\infty))} \mathcal{L}_{1 \to 1}[p_{\frac{1}{2}}(x,t)](\lambda).
\]

Let us observe that \(\lim_{t \to 0^+} e^{-\lambda t}p_{\frac{1}{2}}(x,t) = \lim_{t \to +\infty} e^{-\lambda t}p_{\frac{1}{2}}(x,t) = 0\), hence we can consider \(C_1(x,\lambda) = \max_{t \in [0, +\infty]} e^{-\lambda t}p_{\frac{1}{2}}(x,t)\). By Berge’s maximum theorem we know that \(C_1\) is continuous in \(K \times \mathcal{K}\), hence we can define \(C_2 = \max_{(x,\lambda) \in K \times \mathcal{K}} C_1(x,\lambda)\).
Therefore we can continue as follows:

\[ |L_H p_H (x, \lambda) - L_H p_{\frac{1}{2}}(x, \lambda)| \leq \| p_H - p_{\frac{1}{2}} \|_{L^\infty(K \times [0, +\infty))} \mathcal{L}_{t \to \lambda} [V'_{2, H}(t)](\lambda) + C_2 \varepsilon \left\| V'_{2, H} - V'_{2, \frac{1}{2}} \right\|_{L^\infty([0, \varepsilon])} + \left\| V'_{2, H} - V'_{2, \frac{1}{2}} \right\|_{L^\infty([\varepsilon, +\infty))}. \]

Moreover, \( \mathcal{L}_{t \to \lambda} [V'_{2, H}(t)](\lambda) \) is a continuous function of \( \lambda \) and \( \mathcal{L}_{t \to \lambda} [p_{\frac{1}{2}}(x, t)](\lambda) \) is a continuous function of \( (x, \lambda) \), hence we can define

\[ C_3 = \max_{\lambda \in K} \mathcal{L}_{t \to \lambda} [V'_{2, H}(t)](\lambda), \quad C_4 = \max_{(x, \lambda) \in K \times K} \mathcal{L}_{t \to \lambda} [p_{\frac{1}{2}}(x, t)](\lambda) \]
to achieve

\[ |L_H p_H (x, \lambda) - L_H p_{\frac{1}{2}}(x, \lambda)| \leq C_3 \left\| p_H - p_{\frac{1}{2}} \right\|_{L^\infty(K \times [0, +\infty))} + C_2 \varepsilon \left\| V'_{2, H} - V'_{2, \frac{1}{2}} \right\|_{L^\infty([0, \varepsilon])} + C_4 \left\| V'_{2, H} - V'_{2, \frac{1}{2}} \right\|_{L^\infty([\varepsilon, +\infty))}. \]

Applying inequality (3.6) we conclude that

\[ |L_H p_H (x, \lambda) - L_H p_{\frac{1}{2}}(x, \lambda)| \leq C_5(K) \left\| V_{2, H} - V_{2, \frac{1}{2}} \right\|_{L^\infty([0, +\infty))} + C_2 \varepsilon \left\| V'_{2, H} - V'_{2, \frac{1}{2}} \right\|_{L^\infty([0, \varepsilon])} + C_4 \left\| V'_{2, H} - V'_{2, \frac{1}{2}} \right\|_{L^\infty([\varepsilon, +\infty))} \]

and then

\[ \left\| L_H p_H - L_H p_{\frac{1}{2}} \right\|_{L^\infty(K \times K)} \leq C_5(K) \left\| V_{2, H} - V_{2, \frac{1}{2}} \right\|_{L^\infty([0, +\infty))} + C_2 \varepsilon \left\| V'_{2, H} - V'_{2, \frac{1}{2}} \right\|_{L^\infty([0, \varepsilon])} + C_4 \left\| V'_{2, H} - V'_{2, \frac{1}{2}} \right\|_{L^\infty([\varepsilon, +\infty))}. \]

Now we can take the limit superior as \( H \to \frac{1}{2}^+ \) (recalling also Proposition 4.6) to achieve

\[ \limsup_{H \to \frac{1}{2}^+} \left\| L_H p_H - L_H p_{\frac{1}{2}} \right\|_{L^\infty(K \times K)} \leq C_2 \varepsilon. \]

Since \( \varepsilon > 0 \) is arbitrary, we can take \( \varepsilon \to 0 \) obtaining finally

\[ \lim_{H \to \frac{1}{2}^+} \left\| L_H p_H - L_H p_{\frac{1}{2}} \right\|_{L^\infty(K \times K)} = 0. \]

\[ \Box \]

Now, to study the asymptotic of \( G_{\alpha, H} \) we have first to make some observation. First of all, we have to identify \( G_{\alpha, H} p_{\frac{1}{2}, \alpha} \). Indeed, for any \( C^2 \) function \( f \), let us define

\[ G_{OU} f(x) = \theta \frac{d}{dx} (xf(x)) + \frac{1}{2} \frac{d^2}{dx^2} f(x) \]
that is to say the Kolmogorov forward operator of the Ornstein-Uhlenbeck process $U_t$. It has been shown in [12] that $p_{\frac{1}{2},\alpha}(x,t)$ solves the following equation:

$$\partial_t \alpha p_{\frac{1}{2},\alpha}(x,t) = G^{OU} p_{\frac{1}{2},\alpha}(x,t).$$

On the other hand, we have that for any $x \in \mathbb{R} \setminus \{0\}$ and almost any $t > 0$ it holds

$$\partial_t \alpha p_{\frac{1}{2},\alpha}(x,t) = \frac{1}{2} \mathcal{G}_{\alpha,\frac{1}{2}} p_{\frac{1}{2},\alpha}(x,t),$$

hence for $x \in \mathbb{R} \setminus \{0\}$ and $t \in (0, +\infty)$ we have

$$\frac{1}{2} \mathcal{G}_{\alpha,\frac{1}{2}} p_{\frac{1}{2},\alpha}(x,t) = G^{OU} p_{\frac{1}{2},\alpha}(x,t).$$

Hence, if we show that $\partial_t \alpha p_{\frac{1}{2},\alpha}$ converges to $\partial_t \alpha p_{\frac{1}{2},\alpha}$, we have that $\mathcal{G}_{\alpha,H} p_{H,\alpha}$ converges (in a certain sense) towards $G^{OU} p_{\frac{1}{2},\alpha}$.

In order to realize this plan, let us prove the following result.

**Lemma 4.9.** For any compact set $K \subset \mathbb{R} \setminus \{0\}$ there exists a constant $C(K)$ such that

$$\sup_{x \in K} \left| \frac{\partial}{\partial t} p_{H,\alpha}(x,t) - \frac{\partial}{\partial t} p_{\frac{1}{2},\alpha}(x,t) \right| \leq C(K) \left( t^{\alpha-1} \left\| V_{2,\alpha} - V_{2,\frac{1}{2}} \right\|_{L^\infty(0, +\infty)} + \varepsilon t^{\alpha-1} \left( V_{2,\alpha} - V_{2,\frac{1}{2}} \right)_{L^\infty(0, t^\varepsilon, +\infty)} \right)$$

for any $t \in (0, +\infty)$.

**Proof.** Let us first observe that, by Proposition 3.5, we know that

$$\frac{\partial}{\partial t} p_{H,\alpha}(x,t) = \int_0^\infty \frac{\partial}{\partial t} p_{H} \left( x, \left( \frac{t}{w} \right)^\alpha \right) g_\alpha(w) dw.$$

Now let us observe that for $t > 0$

$$\frac{\partial}{\partial t} p_{H} \left( x, \left( \frac{t}{w} \right)^\alpha \right) = \frac{\alpha t^{\alpha-1} w^{-\alpha}}{2\sqrt{2\pi}} f \left( x, V_{2,H} \left( \left( \frac{t}{w} \right)^\alpha \right), V_{2,H}^t \left( \left( \frac{t}{w} \right)^\alpha \right) \right)$$

where

$$f(x,t,w) = \frac{w(x^2 - t)}{t^{\frac{3}{2}}} e^{-\frac{x^2}{2t}}.$$

Concerning the derivatives of $f$, we have

$$\frac{\partial f}{\partial x}(x,t,w) = \frac{w}{t^{\frac{3}{2}}} \left[ 3xt - x^3 \right] e^{-\frac{x^2}{2t}},$$

$$\frac{\partial f}{\partial t}(x,t,w) = \frac{w}{t^{\frac{3}{2}}} \left[ 3t^2 - 6xt^2 + x^4 \right] e^{-\frac{x^2}{2t}},$$

$$\frac{\partial f}{\partial w}(x,t,w) = \frac{x^2 - t}{t^{\frac{3}{2}}} e^{-\frac{x^2}{2t}},$$

hence

$$\nabla f(x,t,w) = \frac{e^{-\frac{x^2}{2t}}}{t^{\frac{3}{2}}} q(x,t,w)$$

where

$$q(x,t,w) = \left( \frac{(3xt - x^3)w}{t}, \frac{(3t^2 - 6xt^2 + x^4)w}{2t^2}, x^2 - t \right).$$
In particular it holds
\[ |q(x, t, w)| = \sqrt{4(3x^2 - x^4)2t^2w^2 + (3t^2 - 6x^2t + x^4)2w^2 + 4(x^2 - t)^2t^4} \]
and then
\[ |\nabla f(x, t, w)| = \frac{e^{-\frac{x^2}{2t^2}}}{2t^{\frac{3}{2}} \sqrt{4(3x^2 - x^4)2t^2w^2 + (3t^2 - 6x^2t + x^4)2w^2 + 4(x^2 - t)^2t^4}}. \]

In particular we can extend \( |\nabla f(x, t, w)| \) by continuity for \( t = 0 \), by setting \( |\nabla f(x, 0, w)| = 0 \). Now let us define
\[ T = \max_{H \in [\frac{1}{2}, 1]} V_{2, H}(+\infty) \quad \text{and} \quad W = \max_{H \in [\frac{1}{2}, 1]} \max_{t \in (0, +\infty)} V'_{2, H}(t) \]
which are well-defined by Berge’s maximum theorem. Let us also define \( K_2 = [0, T] \) and \( K_3 = [0, W] \). Finally, let us recall that \( \sqrt{x^2 + y^2} \leq |x| + |y| \). Thus, if we define
\[ C_1(K) = \max_{(x, t, w) \in K \times K_2 \times K_3} |\nabla f(x, t, w)| \]
we have
\[ \left| \frac{\partial}{\partial t} p_{H}(x, \left(\frac{t}{w}\right)^\alpha) - \frac{\partial}{\partial t} p_{\frac{1}{2}, 1}(x, \left(\frac{t}{w}\right)^\alpha) \right| = \frac{\alpha t^{\alpha-1}w^{-\alpha}}{2\sqrt{2\pi}} f(x, V_{2, H}(\left(\frac{t}{w}\right)^\alpha), V'_{2, H}(\left(\frac{t}{w}\right)^\alpha)) - f(x, V_{\frac{1}{2}, 1}(\left(\frac{t}{w}\right)^\alpha), V'_{\frac{1}{2}, 1}(\left(\frac{t}{w}\right)^\alpha)) \]
\[ \leq \frac{\alpha t^{\alpha-1}w^{-\alpha}}{2\sqrt{2\pi}} C_1(K) \left( \left\| V_{2, H} - V_{\frac{1}{2}, 1} \right\|_{L^\infty(0, +\infty)} + \left\| V'_{2, H}(\left(\frac{t}{w}\right)^\alpha) - V'_{\frac{1}{2}, 1}(\left(\frac{t}{w}\right)^\alpha) \right\| \right). \]

Now let us observe that
\[ \left| \frac{\partial}{\partial t} p_{H,\alpha}(x, t) - \frac{\partial}{\partial t} p_{\frac{1}{2}, \alpha}(x, t) \right| \leq \int_0^{+\infty} \left| \frac{\partial}{\partial t} p_{H}(x, \left(\frac{t}{w}\right)^\alpha) - \frac{\partial}{\partial t} p_{\frac{1}{2}, 1}(x, \left(\frac{t}{w}\right)^\alpha) \right| g_\alpha(w)dw \]
\[ \leq \frac{\alpha t^{\alpha-1}}{2\sqrt{2\pi}} C_1(K) \left\| V_{2, H} - V_{\frac{1}{2}, 1} \right\|_{L^\infty(0, +\infty)} \int_0^{+\infty} w^{-\alpha} g_\alpha(w)dw \]
\[ + \frac{\alpha t^{\alpha-1}}{2\sqrt{2\pi}} C_1(K) \int_0^{+\infty} w^{-\alpha} \left| V'_{2, H}(\left(\frac{t}{w}\right)^\alpha) - V'_{\frac{1}{2}, 1}(\left(\frac{t}{w}\right)^\alpha) \right| g_\alpha(w)dw. \]

Now let us recall (see [19]) that as \( w \to 0^+ \) we have
\[ g_\alpha(w) \sim C_\alpha \left(\frac{\alpha}{2}\right)^{\frac{1-\frac{\alpha}{2}}{1-\alpha}} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\alpha}{2}+1)} e^{-|1-\alpha|\left(\frac{\alpha}{2}\right)^{\frac{\alpha}{2}}}, \]
thus, since \( \alpha \in (0, 1) \) and \( \alpha - 1 < 0 \), we have that \( w^{-\alpha} g_\alpha(w) \) is integrable in \((0, +\infty)\).

Thus we can define
\[ C_2(K) = \frac{\alpha}{2\sqrt{2\pi}} C_1(K) \int_0^{+\infty} w^{-\alpha} g_\alpha(w)dw \]
Now let us consider \( I_2(t) \). First of all, let us consider the change of variables \( s = (\frac{\alpha}{\pi})^\alpha \) to obtain
\[
I_2(t) = \int_0^{+\infty} t^{-\alpha} s|V'_{2,H}(s) - V'_{2,2}(s)| s(t^2 - \frac{t}{\alpha}) ds = t^{-\alpha} \int_0^{+\infty} s|V'_{2,H}(s) - V'_{2,2}(s)| f_\alpha(s,t) ds.
\]
Now let us consider \( \varepsilon > 0 \) and let us split the integral in the following way:
\[
I_2(t) = t^{-\alpha} \left( \int_0^{t^\varepsilon} s|V'_{2,H}(s) - V'_{2,2}(s)| f_\alpha(s,t) ds + \int_{t^\varepsilon}^{+\infty} s|V'_{2,H}(s) - V'_{2,2}(s)| f_\alpha(s,t) ds \right)
\]
\[
\leq t^{-\alpha} \left( |V'_{2,H} - V'_{2,2}|_{L^\infty(0,t^\varepsilon)} \int_0^{t^\varepsilon} s f_\alpha(s,t) ds + |V'_{2,H} - V'_{2,2}|_{L^\infty(t^\varepsilon, +\infty)} \int_{t^\varepsilon}^{+\infty} s f_\alpha(s,t) ds \right)
\]
and then
\[
\frac{\alpha t^{\alpha-1}}{2\sqrt{2\pi}} I_2(t) \leq \frac{\alpha}{2\sqrt{2\pi}} \left( \|V'_{2,H} - V'_{2,2}\|_{L^\infty(0,t^\varepsilon)} \int_0^{t^\varepsilon} s f_\alpha(s,t) ds + \|V'_{2,H} - V'_{2,2}\|_{L^\infty(t^\varepsilon, +\infty)} \int_{t^\varepsilon}^{+\infty} s f_\alpha(s,t) ds \right)
\]
\[
= \frac{\alpha}{2\sqrt{2\pi}} \left( \|V'_{2,H} - V'_{2,2}\|_{L^\infty(0,t^\varepsilon)} I_3(t) + \|V'_{2,H} - V'_{2,2}\|_{L^\infty(t^\varepsilon, +\infty)} I_4(t) \right).
\]
Let us first focus on \( I_3(t) \). We have
\[
I_3(t) \leq \varepsilon t^\alpha.
\]
Concerning \( I_4 \), we have
\[
I_4(t) \leq \mathbb{E}[E_\alpha(t)].
\]
Thus we have
\[
\frac{\alpha t^{\alpha-1}}{2\sqrt{2\pi}} I_2(t) \leq \frac{\varepsilon \alpha t^{\alpha-1}}{2\sqrt{2\pi}} \left( \|V'_{2,H} - V'_{2,2}\|_{L^\infty(0,t^\varepsilon)} + \|V'_{2,H} - V'_{2,2}\|_{L^\infty(t^\varepsilon, +\infty)} \right) \mathbb{E}[E_\alpha(t)].
\]

Now we are ready to show that \( \mathcal{G}_{\alpha,H} p_{H,\alpha}(x,t) \) converges uniformly to \( 2 \mathcal{G}^{OU} p_{\frac{1}{2},\alpha}(x,t) \).

**Theorem 4.10.** Let \( K_1 \subset \mathbb{R} \setminus \{0\} \) be a compact set and \( K_2 = [T_1, T_2] \) for some \( 0 < T_1 < T_2 \). Then, in \( K_1 \times K_2 \), \( \mathcal{G}_{\alpha,H} p_{H,\alpha}(x,t) \) converges towards \( 2 \mathcal{G}^{OU} p_{\frac{1}{2},\alpha}(x,t) \) uniformly as \( H \to \frac{1}{2}^+ \).
Proof. Let us observe that
\[ \partial_t^\alpha p^{H,\alpha}(x, t) = \frac{1}{2} \mathcal{G}_{\alpha,H} p^{H,\alpha}(x, t) \quad \text{and} \quad \partial_t^\alpha p^\pm_{\pm,\alpha}(x, t) = \frac{1}{2} \mathcal{G}_{\alpha,\pm} p^\pm_{\pm,\alpha}(x, t) \]
hence we only have to show that \( \partial_t^\alpha p^{H,\alpha}(x, t) \) converges uniformly to \( \partial_t^\alpha p^\pm_{\pm,\alpha}(x, t) \).
Let us write \( \partial_t^\alpha = \mathcal{J}_t^{1-\alpha} \frac{\partial}{\partial t} \) to obtain
\[
|\partial_t^\alpha p^{H,\alpha}(x, t) - \partial_t^\alpha p^\pm_{\pm,\alpha}(x, t)| \leq \mathcal{J}_t^{1-\alpha} \left| \frac{\partial}{\partial t} p^{H,\alpha}(x, t) - \frac{\partial}{\partial t} p^\pm_{\pm,\alpha}(x, t) \right|
\]
\[
\leq C(K) \left( \|V_{2, H} - V_{2, \pm}\|_{L^\infty(0, +\infty)} \mathcal{J}_t^{1-\alpha} t^{\alpha-1} \right.
\]
\[
+ \varepsilon \mathcal{J}_t^{1-\alpha} (t^{\alpha-1} \|V'_{2, H} - V'_{2, \pm}\|_{L^\infty(0, +\infty)})
\]
\[
+ \mathcal{J}_t^{1-\alpha} \left( \frac{E[E_\alpha(t)]}{t} \|V'_{2, H} - V'_{2, \pm}\|_{L^\infty(\varepsilon T_1^\alpha, +\infty)} \mathcal{J}_t^{1-\alpha} \right)
\]
\[
\leq C(K) \left( \|V_{2, H} - V_{2, \pm}\|_{L^\infty(0, +\infty)} \mathcal{J}_t^{1-\alpha} t^{\alpha-1} \right.
\]
\[
+ \varepsilon \|V'_{2, H} - V'_{2, \pm}\|_{L^\infty(0, \varepsilon T_2^\alpha)} \mathcal{J}_t^{1-\alpha} \mathcal{J}_t^{1-\alpha} \left( \frac{E[E_\alpha(t)]}{t} \right)
\]
In particular we have \( \mathcal{J}_t^{1-\alpha} t^{\alpha-1} = \Gamma(\alpha) \) and, since \( \frac{E[E_\alpha(t)]}{t} \) is non negative, we have:
\[
|\partial_t^\alpha p^{H,\alpha}(x, t) - \partial_t^\alpha p^\pm_{\pm,\alpha}(x, t)| \leq C(K) \Gamma(\alpha) \left( \|V_{2, H} - V_{2, \pm}\|_{L^\infty(0, +\infty)} \right.
\]
\[
+ \varepsilon \|V'_{2, H} - V'_{2, \pm}\|_{L^\infty(0, \varepsilon T_2^\alpha)} \mathcal{J}_t^{1-\alpha} \mathcal{J}_t^{1-\alpha} \left( \frac{E[E_\alpha(t)]}{t} \right)
\]
Let us show that \( \mathcal{J}_{T_2}^{1-\alpha} \left( \frac{E[E_\alpha(t)]}{t} \right) \) is a finite quantity. Let us observe that
\[ \mathcal{L}_{\lambda \to t}[E[E_\alpha(t)]] = \lambda^{\alpha-1} \int_0^t \frac{se^{-s\lambda}}{\lambda} ds = \lambda^{-\alpha-1} \]
hence, by Karamata’s tauberian theorem, we know that, as \( t \to 0^+ \),
\[ \int_0^t E[E_\alpha(s)] ds \sim \frac{t^{\alpha+1}}{\Gamma(\alpha + 1)} \]
By monotone density theorem (since \( E[E_\alpha(s)] \) is increasing), we have as \( t \to 0^+ \)
\[ E[E_\alpha(t)] \sim \frac{t^{\alpha}}{\Gamma(\alpha)} \]
thus
\[ \frac{E[E_\alpha(t)]}{t} \sim \frac{t^{\alpha-1}}{\Gamma(\alpha)} \]
and we conclude that \( \mathcal{J}_{T_2}^{1-\alpha} \frac{E[E_\alpha(t)]}{t} \) is finite. Let us denote it by \( C_1(T_2) \) and \( C_2(K, T_2) = \max\{C_1(T_2), C(K)\Gamma(\alpha)\} \).
We have
\[
\left| \partial_t^{\alpha} p_{H,\alpha}(x,t) - \partial_t^{\alpha} p_{\frac{1}{2},\alpha}(x,t) \right| \leq C_2(K,T_2) \left( \left\| V_{2,H} - V_{2,\frac{1}{2}} \right\|_{L^\infty(0,\infty)} + \varepsilon \left\| V_{2,H}' - V_{2,\frac{1}{2}}' \right\|_{L^\infty(0,\varepsilon T_2)} + \left\| V_{2,H}' - V_{2,\frac{1}{2}}' \right\|_{L^\infty(\varepsilon T_2^\alpha,\infty)} \right).
\]

Taking the supremum in $K \times [0,\infty)$ we have
\[
\left\| \partial_t^{\alpha} p_{H,\alpha} - \partial_t^{\alpha} p_{\frac{1}{2},\alpha} \right\|_{L^\infty(K \times (0,\infty))} \leq C_2(K,T_2) \left( \left\| V_{2,H} - V_{2,\frac{1}{2}} \right\|_{L^\infty(0,\infty)} + \varepsilon \left\| V_{2,H}' - V_{2,\frac{1}{2}}' \right\|_{L^\infty(0,\varepsilon T_2)} + \left\| V_{2,H}' - V_{2,\frac{1}{2}}' \right\|_{L^\infty(\varepsilon T_2^\alpha,\infty)} \right).
\]

Now, taking the limit superior as $H \to \frac{1}{2}^+$ we have
\[
\limsup_{H \to \frac{1}{2}^+} \left\| \partial_t^{\alpha} p_{H,\alpha} - \partial_t^{\alpha} p_{\frac{1}{2},\alpha} \right\|_{L^\infty(K \times (0,\infty))} \leq C_2(K,T_2) \varepsilon \limsup_{H \to \frac{1}{2}^+} \left\| V_{2,H}' - V_{2,\frac{1}{2}}' \right\|_{L^\infty(0,\varepsilon T_2)} \leq C_2(K,T_2) \varepsilon.
\]

Finally, taking the limit as $\varepsilon \to 0^+$ we conclude the proof.

**Remark 4.11.** Let us recall that the operators $G_{\alpha,\frac{1}{2}}$ and $2G^{OU}_{\alpha}$ coincide only when acting on $p_{\frac{1}{2},\alpha}$ for any $\alpha \in (0,1]$.

### 5. Subordinated solutions of the fractional Fokker-Planck equation

In this Section we want to investigate a particular class of solutions of Equation (4.1) of which the density $p_{H,\alpha}$ is the main representative. In particular we will show some general results concerning this kind of solutions, such as uniqueness and convergence, that will also hold for $p_{H,\alpha}$. To do this, we first need to introduce some operators.

#### 5.1. Subordination.

**Definition 5.1.** The $\alpha$-subordination operator $S_\alpha$ on $L^\infty(0,\infty)$ is defined as
\[
S_\alpha v(t) = \int_0^t v(s) f_\alpha(s,t) ds, \quad \forall v \in L^\infty(0,\infty).
\]

We denote $v_\alpha(t) = S_\alpha v(t)$ and we call it a $\alpha$-subordinated function. Let us observe, in particular, that $v_\alpha(t) = E[v(E_\alpha(t))]$ by definition.

First of all, let us show some properties concerning the continuity of the operator $S_\alpha$. 

Proposition 5.1. The operator \( S_\alpha : L^\infty(0,+\infty) \to L^\infty(0,+\infty) \) is continuous with 
\[ \| S_\alpha v \|_{L^\infty(0,+\infty)} \leq \| v \|_{L^\infty(0,+\infty)} \]
thus \( S_\alpha \) is continuous and \( \| S_\alpha \|_{L^\infty(0,+\infty),L^\infty(0,+\infty)} \leq 1 \). Proof is concluded by observing that \( S_\alpha 1 = 1 \).

Let us set \( \mathcal{L}_\alpha = S_\alpha(L^\infty(0,\infty)) \). Now we can express ourselves concerning the Laplace transform of \( \alpha \)-subordinated functions.

Proposition 5.2. Let \( v_\alpha(t) = S_\alpha v(t) \). Then \( v_\alpha(t) \) is Laplace transformable with abscissa of convergence \( \text{abs}(v_\alpha) = 0 \) and
\[ \mathcal{L}_{t\to \lambda}[v_\alpha(t)](\lambda) = \lambda^{\alpha-1} \mathcal{L}_{t\to \lambda}[v(t)](\lambda^\alpha). \]

Proof. Let us first observe that being \( v, v_\alpha \in L^\infty(0,\infty) \), they are Laplace transformable for \( \lambda \in \mathcal{H} \). Denoting by \( \mathcal{F} \) and \( \mathcal{F}_\alpha \) the Laplace transform of \( v \) and \( v_\alpha \), we immediately get (by a simple application of Fubini’s theorem) that
\[ \mathcal{F}_\alpha(\lambda) = \lambda^{\alpha-1} \int_0^{+\infty} v(s)e^{-\lambda^\alpha s}ds = \lambda^{\alpha-1}\mathcal{F}(\lambda^\alpha). \]

Remark 5.3. From the Proposition 5.2 we obtain also that
\[ \mathcal{F}(\lambda) = \lambda^{\frac{1}{\alpha}}\mathcal{F}_\alpha(\lambda^{\frac{1}{\alpha}}), \]
thus, in particular, \( \mathcal{F}_\alpha(\lambda) \) is such that \( \lambda^{\frac{1}{\alpha}}\mathcal{F}_\alpha(\lambda^{\frac{1}{\alpha}}) \) is independent of the representation of the power \( \lambda^{\frac{1}{\alpha}} \).

Remark 5.3 suggests us that the Laplace transform of subordinated functions could belong to the domain of \( \hat{L}_H \). Indeed we have the following result.

Proposition 5.4. Let \( v_\alpha(t) = S_\alpha v(t) \). Then \( v \in \mathcal{D}(L_H) \), \( \mathcal{F}_\alpha \in \mathcal{D}(\hat{L}_H) \) and
\[ \hat{L}_H\mathcal{F}_\alpha(\lambda) = L_Hv(\lambda^\alpha). \]

We omit the proof, that can be achieved as in [4] Lemma 6.7].

Now let us state an important property of the operator \( S_\alpha \).

Proposition 5.5. \( S_\alpha \) is injective.

Proof. Since \( S_\alpha \) is linear, we only have to prove that \( \text{Ker}(S_\alpha) = \{0\} \), i.e. \( S_\alpha v = 0 \) if and only if \( v = 0 \). To do this, let us take the Laplace transform on both sides of the identity \( S_\alpha v = 0 \). Let us consider \( \lambda > 0 \) real, without loss of generality, we have
\[ \lambda^{\alpha-1} \mathcal{L}_{t\to \lambda}[v(t)](\lambda^\alpha) = 0. \]

Being \( \lambda > 0 \) any real number, we have
\[ \mathcal{L}_{t\to \lambda}[v(t)](\lambda) = 0. \]

Since the Laplace transform is injective, we have \( v \equiv 0 \), concluding the proof. □
Remark 5.6. Proposition 5.3 ensures that if we state that \( v_\alpha \in \mathcal{L}_\alpha \), then there exists a unique \( v \in L^\infty(0, +\infty) \) such that \( v_\alpha = S_\alpha v \).

In the following we will need also another operator. First of all, let us recall that \( V_{2,H} \in L^\infty(0, +\infty) \). Thus, if \( v \in L^\infty(0, +\infty) \), \( V_{2,H} v \in L^\infty(0, +\infty) \). Then we can define \( S_{\alpha,H} : L^\infty(0, +\infty) \rightarrow L^\infty(0, +\infty) \) as

\[
S_{\alpha,H} v = S_\alpha(V_{2,H} v).
\]

**Proposition 5.7.** The operator \( S_{\alpha,H} : L^\infty(0, +\infty) \rightarrow L^\infty(0, +\infty) \) is continuous with \( \|S_{\alpha,H}\|_{L^\infty(0, +\infty)} \leq \|V_{2,H}\|_{L^\infty} \) and injective. Moreover, it holds

\[
\lambda^{\alpha-1} L_{t-\lambda}[S_{\alpha,H} v(t)](\lambda) = \lambda^\alpha v(\lambda^\alpha).
\]

**Proof.** First of all, it is easy to see that

\[
\|S_{\alpha,H} v\|_{L^\infty} \leq \|V_{2,H}\|_{L^\infty} \|v\|_{L^\infty}.
\]

Moreover, Equation (5.2) follows from Proposition 5.2.

Finally, since \( S_{\alpha,H} \) is linear, to show that it is injective we only need to show that \( \text{Ker}(S_{\alpha,H}) = \{0\} \). Thus let us consider \( v \in L^\infty(0, +\infty) \) such that \( S_{\alpha,H} v = 0 \). Taking the Laplace transform of both sides we obtain

\[
\lambda^{\alpha-1} L_{t-\lambda}[V_{2,H} v](\lambda) = 0.
\]

Thus in particular, for any \( \lambda > 0 \), \( L_{t-\lambda}[V_{2,H} v](\lambda) = 0 \). Since \( L \) is injective, then \( V_{2,H} v \equiv 0 \). Moreover, as \( V_{2,H}(t) > 0 \) for any \( t \in (0, +\infty) \), one has \( v \equiv 0 \). \( \square \)

The following proposition is a direct consequence of dominated convergence theorem and the fact that \( V_{2,H} \in L^\infty \).

**Proposition 5.8.** Let \( v : I \times \mathbb{R}^+ \rightarrow \mathbb{R} \) be such that:

- For fixed \( x \in I \), \( v(x, \cdot) \in L^\infty(0, +\infty) \);
- For fixed \( t \in \mathbb{R}^+ \), \( v(\cdot, t) \in C^0(I) \).

Then, for fixed \( t \in \mathbb{R}^+ \), \( S_{\alpha,H} v(\cdot, t) \) and \( S_{\alpha,H} v(\cdot, t) \) belong to \( C^0(I) \). In particular \( S_{\alpha,H} \) and \( S_{\alpha,H} \) can be defined as \( S_{\alpha,H} : L^\infty(\mathbb{R}^+; C^0(I)) \rightarrow L^\infty(\mathbb{R}^+; C^0(I)) \).

### 5.2. Isolation of subordinated mild solutions

For general mild solutions we are not able to actually show uniqueness. However, we can show a form of isolation, in the sense that, with respect to some partial order of functions, solutions cannot be compared. To do this, we need the following Theorem.

**Theorem 5.9.** Let for some \( a \in \mathbb{R} \) functions \( v_\alpha(x, t) \) and \( w_\alpha(x, t) \) be two subordinated mild solutions of \( \{1\} \) on \([a, b] \times [0, +\infty) \) (with \( b > a \), eventually \( b = +\infty \)) satisfying the conditions:

- \( v_\alpha, w_\alpha \in L^\infty([a, b] \times [0, +\infty)) \);
- \( v_\alpha(x, 0) = w_\alpha(x, 0) \);
- \( v - w \geq 0 \);
- For any \( x \in [a, b] \) the difference \( (v - w)(x, \cdot) \) is increasing in \( [0, \varepsilon] \) and decreasing in \( [M, +\infty) \) for some \( 0 < \varepsilon \leq M \);
- It holds that \( L_H v(\alpha, \lambda) = L_H w(\alpha, \lambda) \) and \( \frac{\partial}{\partial \alpha} L_H v(\alpha, \lambda) = \frac{\partial}{\partial \alpha} L_H w(\alpha, \lambda) \) for any \( \alpha \in \mathcal{H} \).

Then \( v = w \) on \([a, b] \times [0, +\infty) \).
Proof. Let us consider \(v_\alpha, w_\alpha\) and \(v, w\) (that are uniquely defined since \(S_\alpha\) is injective) such that \(v_\alpha = S_\alpha v\) and \(w_\alpha = S_\alpha w\). Since \(v_\alpha, w_\alpha\) are mild solutions of (4.1) and all the operators involved in these calculations are linear, we have that \(v_\alpha - w_\alpha\) and \(w_\alpha - v_\alpha\) are mild solutions of (4.1). Let us recall that \((v - w)(x, t)\) is increasing in \([0, \varepsilon]\) and decreasing in \([\varepsilon, M]\). Furthermore, since \(v_\alpha - w_\alpha\) is a mild solution of (4.1) with \((v_\alpha - w_\alpha)(x, 0) = 0\), we have that

\[
\lambda^\alpha(\overline{v}_\alpha - \overline{w}_\alpha)(x, \lambda) = \frac{\lambda^\alpha - 1}{2} \frac{\partial^2}{\partial x^2} \hat{L}_H(\overline{v}_\alpha - \overline{w}_\alpha)(x, \lambda),
\]

and then

\[
(5.3) \quad 2\lambda^\alpha(\overline{v}(x, \lambda^\alpha) - \overline{w}(x, \lambda^\alpha)) = \frac{\partial^2}{\partial x^2} \hat{L}_H(\overline{v}(x, \lambda^\alpha) - \overline{w}(x, \lambda^\alpha)).
\]

Let us define

\[
f(x, \lambda) = \frac{\partial}{\partial x} \hat{L}_H(\overline{v}_\alpha - \overline{w}_\alpha)(x, \lambda) = \frac{\partial}{\partial x} \hat{L}_H(v - w)(x, \lambda^\alpha)
\]

to re-write Equation (5.3) as

\[
(5.4) \quad \begin{cases}
\frac{\partial f}{\partial x}(x, \lambda) = 2\lambda^\alpha(\overline{v}(x, \lambda^\alpha) - \overline{w}(x, \lambda^\alpha)) \\
\frac{\partial}{\partial x} \hat{L}_H(v - w)(x, \lambda^\alpha) = f(x, \lambda).
\end{cases}
\]

Defining the function \(g : [\varepsilon, M] \times \mathcal{H} \to \mathbb{C}^2\) by

\[
g(x, \lambda) = (\hat{L}_H(v - w)(x, \lambda^\alpha), f(x, \lambda))
\]

we can re-write Equation (5.4) in vector form as

\[
\frac{\partial}{\partial x} g(x, \lambda) = (f(x, \lambda), 2\lambda^\alpha(\overline{v}(x, \lambda^\alpha) - \overline{w}(x, \lambda^\alpha))).
\]

For the latter function we have

\[
g(x, \lambda) = \int_{a}^{x} \frac{\partial}{\partial x} g(s, \lambda) ds,
\]

whereas \(g(a, \lambda) = (0, 0)\). The latter formula is valid since \(\frac{\partial}{\partial x} g(s, \lambda)\) is continuous in \(s\). Indeed, \(v_\alpha - w_\alpha\) is a mild solution of (4.1) equation, this fact implies the continuity at least of the Laplace transform and of \(d/dx \hat{L}_H(v - w)(x, \lambda^\alpha)\) in \(x\). Now, taking the absolute value, we get the bound

\[
(5.5) \quad |g(x, \lambda)| \leq \int_{a}^{x} \left| \frac{\partial}{\partial x} g(s, \lambda) \right| ds.
\]

Now let us consider separately \(\hat{L}_H(v - w)(x, \lambda^\alpha)\). We have

\[
(5.6) \quad \hat{L}_H(v - w)(x, \lambda^\alpha) = \int_{0}^{\varepsilon} e^{-\lambda^\alpha t}(v - w)(x, t)V_{2,H}'(t) dt + \int_{\varepsilon}^{M} e^{-\lambda^\alpha t}(v - w)(x, t)V_{2,H}'(t) dt + \int_{M}^{\infty} e^{-\lambda^\alpha t}(v - w)(x, t)V_{2,H}'(t) dt =: I_1(x, \lambda^\alpha) + I_2(x, \lambda^\alpha) + I_3(x, \lambda^\alpha).
\]

Let us first consider \(I_2(x, \lambda^\alpha)\). We have

\[
I_2(x, \lambda^\alpha) \geq \left( \min_{t \in [\varepsilon, M]} V_{2,H}'(t) \right) \int_{\varepsilon}^{M} e^{-\lambda^\alpha t}(v - w)(x, t) dt := C_1 \int_{\varepsilon}^{M} e^{-\lambda^\alpha t}(v - w)(x, t) dt.
\]
Concerning $I_1(x, \lambda^\alpha)$, we have
\[
I_1(x, \lambda^\alpha) = \frac{1 - e^{-\lambda^\alpha t}}{\lambda^\alpha} \int_0^\varepsilon (v - w)(x, t)V'_2H(t)d \left( \frac{1 - e^{-\lambda^\alpha t}}{1 - e^{-\lambda^\alpha \varepsilon}} \right),
\]
where $d \left( \frac{1 - e^{-\lambda^\alpha t}}{1 - e^{-\lambda^\alpha \varepsilon}} \right)$ is a probability measure on $[0, \varepsilon]$. Hence we can use Chebyshev’s integral inequality (see [22 Chapter IX, Equation (1.1)]), since we can suppose $\varepsilon$ is small enough to have $v - w$ and $V'_2H$ comonotone, to achieve
\[
I_1(x, \lambda^\alpha) \geq \frac{1 - e^{-\lambda^\alpha \varepsilon}}{\lambda^\alpha} \int_0^\varepsilon (v - w)(x, t)d \left( \frac{1 - e^{-\lambda^\alpha t}}{1 - e^{-\lambda^\alpha \varepsilon}} \right) \int_0^\varepsilon V'_2H(t)d \left( \frac{1 - e^{-\lambda^\alpha t}}{1 - e^{-\lambda^\alpha \varepsilon}} \right).
\]
In particular we have
\[
\int_0^\varepsilon V'_2H(t)d \left( \frac{1 - e^{-\lambda^\alpha t}}{1 - e^{-\lambda^\alpha \varepsilon}} \right) = \frac{\lambda^\alpha}{1 - e^{-\lambda^\alpha \varepsilon}} \int_0^\varepsilon e^{-\lambda^\alpha t}V'_2H(t)dt := \frac{\lambda^\alpha}{1 - e^{-\lambda^\alpha \varepsilon}} C_2(\lambda)
\]
and
\[
\int_0^\varepsilon (v - w)(x, t)d \left( \frac{1 - e^{-\lambda^\alpha t}}{1 - e^{-\lambda^\alpha \varepsilon}} \right) = \frac{\lambda^\alpha}{1 - e^{-\lambda^\alpha \varepsilon}} \int_0^\varepsilon e^{-\lambda^\alpha t}(v - w)(x, t)dt.
\]
Thus we have
\[
I_1(x, \lambda^\alpha) \geq C_3(\lambda) \int_0^\varepsilon e^{-\lambda^\alpha t}(v - w)(x, t)dt,
\]
where $C_3(\lambda) = \frac{\lambda^\alpha C_2(\lambda)}{1 - e^{-\lambda^\alpha \varepsilon}}$.

Now let us consider $I_3(x, \lambda^\alpha)$. We have
\[
I_3(x, \lambda^\alpha) = \frac{e^{-\lambda^\alpha M}}{\lambda^\alpha} \int_M^{+\infty} (v - w)(x, t)V'_2H(t)d \left( \frac{e^{-\lambda^\alpha M} - e^{-\lambda^\alpha t}}{e^{-\lambda^\alpha M}} \right),
\]
where $d \left( \frac{e^{-\lambda^\alpha M} - e^{-\lambda^\alpha t}}{e^{-\lambda^\alpha M}} \right)$ is a probability measure on $[M, +\infty]$. Hence we can use Chebyshev’s integral inequality (since we can suppose $M$ is big enough to have $v - w$ and $V'_2H$ comonotone) to achieve
\[
I_3(x, \lambda^\alpha) \geq \frac{e^{-\lambda^\alpha M}}{\lambda^\alpha} \int_M^{+\infty} (v - w)(x, t)d \left( \frac{e^{-\lambda^\alpha M} - e^{-\lambda^\alpha t}}{e^{-\lambda^\alpha M}} \right) \int_M^{+\infty} V'_2H(t)d \left( \frac{e^{-\lambda^\alpha M} - e^{-\lambda^\alpha t}}{e^{-\lambda^\alpha M}} \right).
\]
In particular, we have
\[
\int_M^{+\infty} V'_2H(t)d \left( \frac{e^{-\lambda^\alpha M} - e^{-\lambda^\alpha t}}{e^{-\lambda^\alpha M}} \right) = \frac{\lambda^\alpha}{e^{-\lambda^\alpha M}} \int_M^{+\infty} e^{-\lambda^\alpha t}V'_2H(t)dt := \frac{\lambda^\alpha}{e^{-\lambda^\alpha M}} C_4(\lambda),
\]
and
\[
\int_M^{+\infty} (v - w)(x, t)d \left( \frac{e^{-\lambda^\alpha M} - e^{-\lambda^\alpha t}}{e^{-\lambda^\alpha M}} \right) = \frac{\lambda^\alpha}{e^{-\lambda^\alpha M}} \int_M^{+\infty} e^{-\lambda^\alpha t}(v - w)(x, t)dt.
\]
Thus we have
\[
I_3(x, \lambda^\alpha) \geq C_5(\lambda) \int_M^{+\infty} e^{-\lambda^\alpha t}(v - w)(x, t)dt,
\]
where $C_5(\lambda) = \frac{\lambda^\alpha C_4(\lambda)}{e^{-\lambda^\alpha M}}$.

Substituting these relations into Equation (5.6) and defining $C_6(\lambda) = \min\{C_1, C_3(\lambda), C_5(\lambda)\}$, we achieve the following relations:
\[
L_H(v - w)(x, \lambda^\alpha) \geq C_6(\lambda)(\overline{v} - \underline{w})(x, \lambda^\alpha).
\]
Now we can go back to Equation (5.5). On the one hand, we have

\[ |g(x, \lambda)| = \sqrt{(L_H(v-w)(x, \lambda^\alpha))^2 + f^2(x, \lambda)} \]

\[ \geq \sqrt{\frac{C_2(\lambda)}{4\lambda^{2\alpha}} (2\lambda^\alpha(\overline{v} - \overline{w})(x, \lambda^\alpha))^2 + f^2(x, \lambda)} \]

\[ \geq C_7(\lambda) \sqrt{(2\lambda^\alpha(\overline{v} - \overline{w})(x, \lambda^\alpha))^2 + f^2(x, \lambda)} = C_7(\lambda)h(x, \lambda), \]

where

\[ h(x, \lambda) = \sqrt{(2\lambda^\alpha(\overline{v} - \overline{w})(x, \lambda^\alpha))^2 + f^2(x, \lambda)}. \]

On the other hand, we know that the derivative equals

\[ \left| \frac{\partial}{\partial x} g(x, \lambda) \right| = \sqrt{(2\lambda^\alpha(\overline{v} - \overline{w})(x, \lambda^\alpha))^2 + f^2(x, \lambda)} = h(x, \lambda). \]

Finally, defining \( C_8(\lambda) = 1/C_7(\lambda) \) and recalling (5.4), we have

\[ h(x, \lambda) \leq C_8(\lambda) \int_a^x h(y, \lambda)dy. \]

By Gronwall’s Lemma we conclude that \( h(x, \lambda) = 0 \) and then

\[ (\overline{v} - \overline{w})(x, \lambda^\alpha) = 0. \]

Since the Laplace transform is injective, we conclude that \( v = w \) and then \( v_\alpha = w_\alpha. \]

In particular, to guarantee some form of uniqueness, we needed to introduce some border conditions typical of Cauchy problems. We were actually working with the following Cauchy problem:

\begin{equation}
\begin{aligned}
\lambda^\alpha \overline{v}_\alpha(x, \lambda) - \lambda^{\alpha-1} v_\alpha(x, 0) &= \frac{\lambda^{\alpha-1}}{\lambda^2} \frac{\partial^2}{\partial x^2} \overline{L}_H \overline{v}_\alpha(x, \lambda), \quad \forall (x, \lambda) \in [a, b] \times \mathcal{H}, \\
v_\alpha(x, 0) &= f(x) \quad x \in [a, b] \\
\overline{L}_H \overline{v}_\alpha(a, \lambda) &= g_1(\lambda) \quad \lambda \in \mathcal{H}; \\
\frac{\partial}{\partial x} \overline{L}_H \overline{v}_\alpha(a, \lambda) &= g_2(\lambda) \quad \lambda \in \mathcal{H};
\end{aligned}
\end{equation}

Let us denote \( \mathcal{S}_H(f, g_1, g_2, \alpha) \) the set of the subordinated mild solutions of (5.7). Now we define a partial order on it

**Definition 5.2.** Given \( v, w \in \mathcal{S}_H(f, g_1, g_2, \alpha, I) \) with \( v_\alpha = \mathcal{S}_\alpha v, w_\alpha = \mathcal{S}_\alpha w \) and \( I = [a, b] \) (eventually \( b = +\infty \)), we denote \( w_\alpha \preceq v_\alpha \) if and only if:

- \( w \leq v \) in \([a, b] \times \mathcal{H} \);
- There exist \( \varepsilon, M > 0 \) such that \( \forall x \in [a, b], v(x., \cdot) \) is increasing in \([0, \varepsilon]\)
- and decreasing in \([M, +\infty]\).

This partial order is well defined due to the injectivity of the operator \( \mathcal{S}_\alpha \).

To shorten the notation, let us fix \( \mathcal{S}_H := \mathcal{S}_H(f, g_1, g_2, \alpha, I) \). We have a parset \((\mathcal{S}_H, \preceq)\). An upset of \( \mathcal{S}_H \) is a set \( U \) such

\[ \forall x \in U, \forall y \in \mathcal{S}_H : (x \preceq y \Rightarrow y \in U). \]

In particular for any element \( u \in \mathcal{S}_H \), \( u \uparrow \) is the upset of \( \mathcal{S}_H \) generated by \( u \), that is to say \( u \uparrow = \{ y \in \mathcal{S}_H : u \preceq y \} \). Let us finally denote \( \tau_H := \{ U \subseteq \mathcal{S}_H : U \text{ is an upset} \} \). Obviously \( \mathcal{S}_H, \emptyset \in \tau_H \). Let us consider two upsets \( U_1, U_2 \). Then \( U_1 \cap U_2 \) is still an upset. Indeed, if \( y \in \mathcal{S}_H \) is such that there exists \( x \in U_1 \cap U_2 \) such that \( x \preceq y \), then \( y \in U_1 \) and \( y \in U_2 \), thus \( y \in U_1 \cap U_2 \). On the other hand, if
we consider a family of upsets \( \{U_\alpha\}_{\alpha \in A} \), then obviously \( \bigcup_{\alpha \in A} U_\alpha \) is still an upset. Thus we have that \((S_H, \tau_H)\) is a topological space. In particular \(\tau_H\) is called the upset topology induced by \(\leq\) (see [11]).

So, what do we have is a structure of \(S_H\) as a poset and, at the same time, as a topological space whose topology describes the behaviour of the partial ordering. However, the topology we introduced is trivial: indeed, we have the following Corollary.

**Corollary 5.10.** The space \((S_H, \tau_H)\) is discrete.

**Proof.** To show that \((S_H, \tau_H)\) is discrete we have to show that for any \(v_\alpha \in S_H\) it holds \(\{v_\alpha\} \in \tau_H\). Let us consider \(v_\alpha \uparrow\) and let us suppose \(w_\alpha \in v_\alpha \uparrow\). Then \(v_\alpha \leq w_\alpha\) that is to say \(w - v \geq 0\), is increasing in some \([0, \varepsilon]\) and decreasing in some \([M, +\infty)\). Thus, by Theorem 5.9 we have \(v_\alpha = w_\alpha\) and then \(v_\alpha \uparrow = \{v_\alpha\}\), concluding the proof.

In particular what we have shown is that subordinated mild solutions of (5.7) are isolated in their upset order topology. Thus, even if we do not have a real notion of uniqueness, we can say that subordinated mild solutions cannot be dominated one by another. This situation is quite different if we ask just a bit more of regularity to subordinated mild solutions.

### 5.3. Regularity of subordinated mild solutions

To show the results that will follow, we need to recall the corresponding Fokker-Planck equation for \(\alpha = 1\) and the notion of mild solution for it.

**Definition 5.3.** We say that \(v : I \times [0, +\infty) \to \mathbb{R}\) is a classical solution of

\[
\frac{\partial}{\partial t} v(x, t) = \frac{1}{2} V_{2,H}'(t) \frac{\partial^2}{\partial x^2} v(x, t)
\]

in \(I \times [0, T]\) (eventually \(T = +\infty\)) if

- For fixed \(t > 0\) \(v(\cdot, t) \in C^2(\hat{I})\);
- For fixed \(x \in I\), \(\frac{\partial}{\partial t} v(x, t)\) exists for almost every \(t > 0\) and belongs to \(L^1_{\text{loc}}(0, +\infty)\),
- Equation (5.8) holds for any \(x \in \hat{I}\) and almost every \(t \in (0, T)\).

**Definition 5.4.** We say that \(v : I \times [0, +\infty) \to \mathbb{R}\) is a mild solution of the fractional Fokker-Planck equation (5.8) if, denoting \(v(x, \lambda) := \mathcal{L}_{t \to \lambda}[v(x, t)](\lambda)\),

- \(v(x, \cdot)\) is Laplace transformable for any \(x \in I\);
- \(v \in D(\hat{L}_H, I)\);
- \(\forall \lambda \in \mathcal{H}, \mathcal{P}(\cdot, \lambda) \in C^0(I)\);
- it holds

\[
\lambda \mathcal{P}(x, \lambda) - v(x, 0) = \frac{1}{2} \frac{\partial^2}{\partial x^2} \hat{L}_H \mathcal{P}(x, \lambda), \; \forall x \in I.
\]

With the latter definition in mind, we can show the following result.

**Proposition 5.11.** Let \(v_\alpha = S_\alpha v\). Then the following properties are equivalent:

1. \(v_\alpha\) is a mild solution of (5.1);
2. \(v\) is a mild solution of (4.4).
Proof. Let us observe that the first three assumptions in Definitions 4.2 and 5.4 coincide. Let us show that (1) implies (2). To do that, just observe that, since $v_\alpha$ is a subordinated mild solution of (1.1), equation (4.2) holds, that is to say, by also using Proposition 5.2 and the fact that, by definition $v_\alpha(x,0) = v(x,0)$, for any $x \in I$ and $\lambda \in \mathcal{H}$

\[
\lambda^{\alpha-1}(\lambda^\alpha v(x,\lambda^\alpha) - v(x,0)) = \frac{\lambda^{\alpha-1}}{2} \frac{\partial^2}{\partial x^2} L_H v(x,\lambda^\alpha).
\]

Without loss of generality, we can suppose $\lambda > 0$ is real. Then we can divide last relation by $\lambda^{\alpha-1}$ and write $\lambda$ in place of $\lambda^\alpha$, obtaining

\[
\lambda^{\alpha-1} v(x,\lambda) - v(x,0) = \frac{1}{2} \frac{\partial^2}{\partial x^2} L_H v(x,\lambda)
\]

that is equation (5.9) for $v$. The converse is shown in the same way. \[\square\]

Now, using this Proposition, we aim to show that if $v_\alpha = S_\alpha v$ is a subordinated mild solution of (1.1) such that for any $t \in \mathbb{R}^+$ the function $x \in I \mapsto v(x,t) \in \mathbb{R}$ is $C^0(I)$, then $v_\alpha$ is also a classical solution of (1.1).

To do this, we first need the following Lemma.

Lemma 5.12. For any $\alpha \in (0,1]$, let $v_\alpha(x,t) = S_\alpha v(x,t)$ be a mild solution of (1.1) in $I \times \mathbb{R}^+$. Suppose that for fixed $t \in \mathbb{R}^+$, $v(\cdot,t) \in C^0(I)$. Moreover suppose that $v_\alpha$ belongs to the domain of $G_{\alpha,H}$ and $G_{\alpha,H} v_\alpha(\cdot,t) \in C^0(I)$ for fixed $t \in \mathbb{R}^+$, where $G_{1,H} = \mathcal{L}_{\lambda^{-1}} \left[ \frac{\partial^2}{\partial x^2} L_H v(x,\lambda) \right]$. Then $S_{\alpha,H} v(\cdot,t) \in C^2(I)$ for any fixed $t \in \mathbb{R}^+$ and

\[
G_{\alpha,H} v_\alpha(x,t) = \frac{\partial^2}{\partial x^2} S_{\alpha,H} v(x,t).
\]

Proof. First of all, since we are working with Laplace transforms, without loss of generality, we can suppose $\lambda \in \mathbb{R}$. Let us denote by $C^2(I)$ the Banach space of functions that are 2 times derivable in the $x$ variable with continuous derivative, equipped with the norm:

\[
\|f\|_{C^2(I)} = \sum_{j=0}^{\infty} \left\| \frac{d^j}{dx^j} f \right\|_{L^\infty}
\]

where $\frac{d^j}{dx^j} f = f$. Moreover, the operator

\[
\frac{d^2}{dx^2} : C^2(I) \rightarrow C^0(I)
\]

is a closed operator. Since $v_\alpha$ is a mild solution of (1.1), we have that, for fixed $\lambda > 0$, $\lambda^{\alpha-1} L_H v(x,\lambda^\alpha) \in C^2(I)$. Moreover, since $v_\alpha$ belongs to the domain of $G_H$, then $\frac{\partial^2}{\partial x^2} \lambda^{\alpha-1} L_H v(x,\lambda^\alpha)$ is the Laplace transform of some function $G_{\alpha,H} v_\alpha(x,t)$. In particular, let us consider $G_{\alpha,H} v_\alpha(x,t)$ as a function with values on the Banach space $C^0(I)$, in the sense that

\[
G_{\alpha,H} v_\alpha : t \in \mathbb{R}^+ \mapsto (x \in I \mapsto G_{\alpha,H} v_\alpha(x,t) \in \mathbb{R}) \in C^0(I)
\]

The same can be done for $\frac{\partial^2}{\partial x^2} \lambda^{\alpha-1} L_H v(x,\lambda^\alpha)$, in the sense that

\[
\frac{\partial^2}{\partial x^2} \lambda^{\alpha-1} L_H v : \lambda \in \mathbb{R}^+ \mapsto (x \in I \mapsto \frac{\partial^2}{\partial x^2} \lambda^{\alpha-1} L_H v(x,\lambda^\alpha) \in \mathbb{R}) \in C^0(I).
\]
Now let us consider $S_{\alpha,H}v(x,t)$. Since for fixed $t > 0$ $v(\cdot,t) \in C^0(I)$, then, by Proposition 5.8, $S_{\alpha,H}v(\cdot,t) \in C^0(I)$. Moreover, by Proposition 5.7, it holds
\[
\mathcal{L}_{t-\lambda}[S_{\alpha,H}v(x,t)] = \lambda^{-1}L_Hv(x,\lambda^a).
\]
Since $v_\alpha$ is a mild solution of (4.1), we know that, for fixed $\lambda > 0$, $\lambda^{-1}L_Hv(\cdot,\lambda^a) \in C^2(I)$. Thus, again by [2, Proposition 1.7.6], we have that $S_{\alpha,H}v(x,t) \in C^2(I)$ and
\[
\frac{\partial^2}{\partial x^2}S_{\alpha,H}v(x,t) = G_{\alpha,H}v_\alpha(x,t).
\]

Now we are ready to show the following result.

**Theorem 5.13.** Let $v_\alpha(x,t) = S_{\alpha}v(x,t)$ be a mild solution of (4.1) in $I \times \mathbb{R}^+$. Suppose that for fixed $t \in \mathbb{R}^+$, $v(\cdot,t) \in C^0(I)$, $v \in \mathcal{D}(G_{1,H},I)$, $G_{1,H}v(\cdot,t) \in C^0(I)$. Moreover, suppose that for any fixed $x \in I$ it holds $G_{1,H}v(x,\cdot) \in L^\infty(0,\infty)$. Then $v_\alpha$ is a classical solution of (4.1).

**Proof.** Let us first observe that the function $\frac{\partial^2}{\partial x^2}v(x,\cdot)$ is well defined by Lemma 5.12, thus the hypothesis on its integrability is legitimate. Now, by Proposition 5.11, we have
\[
\lambda\bar{\nu}(x,\lambda) - v(x,0) = \frac{1}{2} \frac{\partial^2}{\partial x^2}L_Hv(x,\lambda).
\]
Dividing everything by $\lambda$ we have
\[
\bar{\nu}(x,\lambda) = \frac{1}{\lambda}v(x,0) + \frac{1}{2\lambda} \frac{\partial^2}{\partial x^2}L_Hv(x,\lambda).
\]
Observe that $\frac{1}{\lambda}v(x,0)$ is the Laplace transform of $v(x,0)$. Moreover, by Lemma 5.12, we have that $V_{1,H}^t(t)v(\cdot,t) = S_{1,H}v(\cdot,t) \in C^2(I)$. Thus $v(\cdot,t) \in C^2(I)$ and then
\[
\frac{\partial^2}{\partial x^2}L_Hv(x,\lambda) = L_H\left(\frac{\partial^2}{\partial x^2}v\right)(x,\lambda).
\]
On the other hand, $G_{1,H}v(x,\cdot) \in L^\infty(0,\infty)$, hence $F(x,t) = \frac{1}{2}\int_0^t G_{1,H}v(x,s)ds$ is well defined. Taking the Laplace transform, we have
\[
\bar{F}(x,\lambda) = \frac{1}{2\lambda} \frac{\partial^2}{\partial x^2}L_Hv(x,\lambda) = \frac{1}{2\lambda}L_H\left(\frac{\partial^2}{\partial x^2}v\right)(x,\lambda).
\]
Thus we have $\bar{\nu}(x,\lambda) = \mathcal{L}_{t-\lambda}[v(x,0) + F(x,t)]$. By injectivity of the Laplace transform we achieve:
\[
v(x,t) = v(x,0) + \frac{1}{2} \int_0^t G_{1,H}v(x,s)ds = v(x,0) + \frac{1}{2} \int_0^t V_{2,H}^t(s)\frac{\partial^2}{\partial x^2}v(x,s)ds.
\]
In particular $v(x,\cdot)$ is absolutely continuous and for almost all $t \in (0,\infty)$
\[
\frac{\partial}{\partial t}v(x,t) = \frac{1}{2} V_{2,H}^t(t)\frac{\partial^2}{\partial x^2}v(x,t)
\]
hence $v$ is a classical solution of Equation 5.3. Now let us observe that $\frac{1}{2} V_{2,H}^t(t)\frac{\partial^2}{\partial x^2}v(x,t)$ belongs to $L^\infty(0,\infty)$, hence also $\frac{\partial}{\partial t}v(x,t)$ belongs to $L^\infty(0,\infty)$ and then we can apply $S_{\alpha}$ to it.
Now let us observe that, since $S_\alpha : L^\infty \to L^\infty$, we have that $v_\alpha(x, t) - v(x, 0)$ is Laplace transformable. Let us consider the function

$$\nu(t) = \frac{t^{-\alpha}}{\Gamma(1 - \alpha)} \chi(0, +\infty)(t)$$

that is Laplace transformable with Laplace transform $\nu(\lambda) = \lambda^{\alpha-1}$. Moreover, since $\nu(t) \in L^1_{loc}(0, +\infty)$ and $v_\alpha(x, \cdot) \in L^\infty(0, +\infty)$, the function $\nu * (v_\alpha(x, \cdot) - v(x, 0))(t)$ is well defined and

$$L_{t\to\lambda}[\nu * (v_\alpha(x, \cdot) - v(x, 0))(t)] = \lambda^{\alpha-1} \mu_\alpha(x, \lambda) - \lambda^{\alpha-2} v(x, 0).$$

On the other hand, we have, since $\partial_t v(x, t)$ belongs to $L^\infty(0, +\infty)$ for fixed $x \in I$, so does $S_\alpha(\partial_t v)(x, t)$ and

$$L_{t\to\lambda} \left[ \int_0^t S_\alpha \left( \frac{\partial}{\partial t} v \right)(x, s) ds \right] = \lambda^{2\alpha-2} \mu(x, \lambda) - \lambda^{\alpha-2} v(x, 0)
= \lambda^{\alpha-1} \mu_\alpha(x, \lambda) - \lambda^{\alpha-2} v(x, 0) = L_{t\to\lambda}[\nu * (v_\alpha(x, \cdot) - v(x, 0))(t)]$$

and then

$$\frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - \tau)^{-\alpha} (v_\alpha(x, \tau) - v(x, 0)) d\tau = \int_0^t S_\alpha \left( \frac{\partial}{\partial t} v \right)(x, \tau) d\tau.$$

In particular, the RHS is absolutely continuous and we can differentiate both terms to obtain

$$\partial_t^\alpha v_\alpha(x, t) = S_\alpha \left( \frac{\partial}{\partial t} v \right)(x, t).$$

On the other hand, we have, since $v_\alpha$ is mild solution of (4.1),

$$L_{t\to\lambda} \left[ S_\alpha \left( \frac{\partial}{\partial t} v \right)(x, t) \right] = \lambda^{\alpha-1} \mu_\alpha(x, \lambda) - \lambda^{\alpha-1} v(x, 0) = \lambda^{\alpha-1} \frac{\partial^2}{\partial x^2} L_H v(x, \lambda^\alpha)$$

and then

$$S_\alpha \left( \frac{\partial}{\partial t} v \right)(x, t) = \frac{1}{2} G_{\alpha, H} v_\alpha(x, t)$$

concluding the proof.

5.4. **Uniqueness of subordinated classical solutions.** In this Subsection we will show a weak maximum principle for subordinated classical solutions of equation (4.1), and then we will use this result to show uniqueness of subordinated classical solutions.

To do this, we first need the following preliminary result.

**Lemma 5.14.** Let $v_\alpha(x, t) = S_\alpha v(x, t) v(t)$ is an absolute maximum (minimum) point for $v_\alpha$ if and only if it is an absolute maximum (minimum) point for $S_{\alpha, H} v$.

**Proof.** Let us first show that if $(x_0, t_0)$ is an absolute maximum point for $S_{\alpha, H} v$ then it is also for $v_\alpha$. To do this, let us observe that $V_{2, H}'$ can be extended by continuity on $[0, +\infty)$ by setting $V_{2, H}'(0) = V_{2, H}'(+\infty) = 0$. Moreover, since $V_{2, H}$ is...
increasing, $V'_{α,H}$ is positive. Let us denote $M = \max_{t ∈ [0, +∞)} V'_{α,H}(t) > 0$. We have, for any $(x, t) ∈ I × [0, +∞)$

$$0 ≤ S_{α,H} v(x, t_0) - S_{α,H} v(x, t) = \int_0^{+∞} V'_{α,H}(s)(v(x_0, s)f_α(s, t_0) - v(x, s)f_α(s, t))ds$$

\[ \leq M \int_0^{+∞} (v(x_0, s)f_α(s, t_0) - v(x, s)f_α(s, t))ds \]

\[ = M(v_α(x_0, t_0) - v_α(x, t)), \]

thus $(x_0, t_0)$ is an absolute maximum point for $v_α$.

Now let us suppose $(x_0, t_0)$ is an absolute maximum point for $v_α$. Fix $(x, t) ∈ I × [0, +∞)$. Then we have

$$\int_0^{+∞} (v(x_0, s)f_α(s, t_0) - v(x, s)f_α(s, t))ds = v_α(x_0, t_0) - v_α(x, t) ≥ 0.$$

Let us suppose that there exists an increasing sequence $R_n$ and a decreasing sequence $\varepsilon_n > 0$ such that $\lim_{n→+∞} R_n = +∞$, $\lim_{n→+∞} \varepsilon_n = 0$ and

$$\int_{\varepsilon_n}^{R_n} (v(x_0, s)f_α(s, t_0) - v(x, s)f_α(s, t))ds ≥ 0.$$

Now let us observe that $V'_{α,H}(s) > 0$ for any $s ∈ (0, +∞)$. In particular let us denote $m_n = \min_{s ∈ [\varepsilon_n, R_n]} V'_{α,H}(s) > 0$ for any $n ∈ N$. Now fix $n ∈ N$ and observe that

$$S_{α,H} v(x, t_0) - S_{α,H} v(x, t) = \int_0^{+∞} V'_{α,H}(s)(v(x_0, s)f_α(s, t_0) - v(x, s)f_α(s, t))ds$$

\[ = \int_{\varepsilon_n}^{R_n} V'_{α,H}(s)(v(x_0, s)f_α(s, t_0) - v(x, s)f_α(s, t))ds \]

\[ + \int_{\varepsilon_n}^{+∞} V'_{α,H}(s)(v(x_0, s)f_α(s, t_0) - v(x, s)f_α(s, t))ds \]

\[ + \int_{\varepsilon_n}^{R_n} V'_{α,H}(s)(v(x_0, s)f_α(s, t_0) - v(x, s)f_α(s, t))ds \]

\[ ≥ \int_{\varepsilon_n}^{R_n} V'_{α,H}(s)(v(x_0, s)f_α(s, t_0) - v(x, s)f_α(s, t))ds \]

\[ + m_n \int_{\varepsilon_n}^{R_n} (v(x_0, s)f_α(s, t_0) - v(x, s)f_α(s, t))ds \]

\[ + \int_{\varepsilon_n}^{+∞} V'_{α,H}(s)(v(x_0, s)f_α(s, t_0) - v(x, s)f_α(s, t))ds \]

\[ + \int_{\varepsilon_n}^{R_n} V'_{α,H}(s)(v(x_0, s)f_α(s, t_0) - v(x, s)f_α(s, t))ds \]

Now, since for fixed $(x, t)$ it holds $V'_{α,H}(s)(v(x_0, s)f_α(s, t_0) - v(x, s)f_α(s, t)) ∈ L^1$ by the fact that $v$ belongs to the domain of $L_H$, then, taking the limit as $n → +∞$ we achieve

$$S_{α,H} v(x, t_0) - S_{α,H} v(x, t) ≥ 0.$$
Now let us suppose such sequences do not exist. Then the only possible case is given by
\[ \int_0^{+\infty} (v(x_0, s)f_\alpha(s, t_0) - v(x, s)f_\alpha(s, t_0))ds = 0. \]
In particular, since we do not have any couple of sequences such that \( \int_{\varepsilon_n}^{R_n} (v(x_0, s)f_\alpha(s, t_0) - v(x, s)f_\alpha(s, t_0))ds \geq 0 \) with \( \varepsilon_n \to 0 \) and \( R_n \to +\infty \), it means that there exists \( \varepsilon_0 \) and \( R_0 \) such that for any \( \varepsilon < \varepsilon_0 \) and \( R > R_0 \)
\[ \int_{\varepsilon}^{R} (v(x_0, s)f_\alpha(s, t_0) - v(x, s)f_\alpha(s, t_0))ds < 0 \]
Since \( V_{2,H}'(s) \to 0 \) as \( s \to 0^+ \) and \( s \to +\infty \), we can choose \( \varepsilon_0 \) and \( R_0 \) such that \( m = \min_{s \in [\varepsilon_0, R_0]} V_{2,H}(s) < 1 \). Now let us consider a sequence \( \varepsilon_n \to 0 \) such that \( \varepsilon_n < \varepsilon_0 \) and \( R_n \to +\infty \) such that \( R_n > R_0 \). Then we have, since \( \int_{\varepsilon_n}^{R_n} (v(x_0, s)f_\alpha(s, t_0) - v(x, s)f_\alpha(s, t_0))ds < 0 \) and \( m_n = \min_{s \in (\varepsilon_n, R_n)} V_{2,H}(s) < 1 \),
\[ S_{\alpha,H}v(x_0, t_0) - S_{\alpha,H}v(x, t) = \int_0^{+\infty} V_{2,H}'(s)(v(x_0, s)f_\alpha(s, t_0) - v(x, s)f_\alpha(s, t_0))ds \]
\[ = \int_{\varepsilon_n}^{R_n} V_{2,H}'(s)(v(x_0, s)f_\alpha(s, t_0) - v(x, s)f_\alpha(s, t_0))ds \]
\[ + \int_{\varepsilon_n}^{R_n} V_{2,H}'(s)(v(x_0, s)f_\alpha(s, t_0) - v(x, s)f_\alpha(s, t_0))ds \]
\[ + \int_{\varepsilon_n}^{+\infty} V_{2,H}'(s)(v(x_0, s)f_\alpha(s, t_0) - v(x, s)f_\alpha(s, t_0))ds \]
\[ \geq \int_{\varepsilon_n}^{R_n} V_{2,H}(s)(v(x_0, s)f_\alpha(s, t_0) - v(x, s)f_\alpha(s, t_0))ds \]
\[ + \int_{\varepsilon_n}^{+\infty} (v(x_0, s)f_\alpha(s, t_0) - v(x, s)f_\alpha(s, t_0))ds \]
\[ + \int_{\varepsilon_n}^{R_n} V_{2,H}'(s)(v(x_0, s)f_\alpha(s, t_0) - v(x, s)f_\alpha(s, t_0))ds \]
\[ \geq \int_{\varepsilon_n}^{R_n} V_{2,H}(s)(v(x_0, s)f_\alpha(s, t_0) - v(x, s)f_\alpha(s, t_0))ds \]
\[ + \int_{\varepsilon_n}^{R_n} (v(x_0, s)f_\alpha(s, t_0) - v(x, s)f_\alpha(s, t_0))ds \]
\[ + \int_{\varepsilon_n}^{+\infty} V_{2,H}'(s)(v(x_0, s)f_\alpha(s, t_0) - v(x, s)f_\alpha(s, t_0))ds. \]
Finally, taking the limit as \( n \to +\infty \), we have \( S_{\alpha,H}v(x_0, t_0) - S_{\alpha,H}v(x, t) \geq 0 \). In this case, changing the roles of \( (x_0, t_0) \) and \( (x, t) \), we have in particular \( S_{\alpha,H}v(x_0, t_0) = S_{\alpha,H}v(x, t) \).
To prove the statement for the minimum, just substitute \( v \) with \( -v \).

For the next Theorem, we will follow the lines of [18, Theorem 2], adapting the proof to our non-standard case.

**Theorem 5.15 (Weak maximum principle).** Let \( v_\alpha \) a subordinated classical solution in \([a, b] \times [0, +\infty)\) of Equation (4.1) such that for any \( t \in \mathbb{R}^+ \), it holds
Since $v_m$ for any constant $C$

**Proof.** First of all, let us observe that if $v_a$ is a subordinated solution of \( (4.1) \), then, for any constant $C$, $v_a + C$ is still a subordinated solution. Indeed we easily have $S_\alpha C = C$ and then, being $S_\alpha$ linear, $v_a + C = S_\alpha (v + C)$. In particular it holds, being $L_H$ a linear operator,

$$G_{\alpha,H}(v_a + C)(x,t) = L^{-1}_\lambda \frac{\partial^\alpha}{\partial x^\alpha} L_H(v(x,\cdot) + C)(\lambda)$$

$$= L^{-1}_\lambda \frac{\partial^\alpha}{\partial x^\alpha} L_H v(x,\lambda) + \frac{\partial}{\partial x^\alpha} L_H C(\lambda)$$

$$= L^{-1}_\lambda \frac{\partial^\alpha}{\partial x^\alpha} L_H v(x,\lambda) = G_{\alpha,H} v_a(x,t).$$

On the other hand, we also have $\partial^\alpha_t (v_a + C)(x,t) = \partial^\alpha_t (v_a)(x,t)$ since the Caputo derivative of a constant is 0. Hence we have shown that $v_a + C$ is still a subordinated solution of \( (4.1) \).

Hence we can suppose, without loss of generality, that $v_a$ is positive in $\Omega$. Indeed, if it is not the case, being $v_a$ a classical solution of \( (4.1) \), it must be in $C^0$ and then we can take $m = \min_{(x,t) \in \Omega} v_a(x,t)$. In this case we work with $v_a + C$ in place of $v_a$ where $C$ is any real constant such that $C > -m$.

Since $v_a$ is a classical solution of \( (4.1) \) it belongs to the domain of $G_{\alpha,H}$ and it is also a mild solution of \( (4.1) \), hence, by Lemma 5.12 we know that

$$G_{\alpha,H} v_a(x,t) = \frac{\partial^2}{\partial x^2} S_{\alpha,H} v(x,t)$$

where $v$ is uniquely defined since $S_\alpha$ is injective.

Now let us suppose that $v_a$ admits a maximum point $(x_0,t_0)$ belonging to $\hat{\Omega} \cup (\{a\} \times \{T\})$. Let $M = \max_{(x,t) \in \partial_\Omega} v_a(x,t)$ and suppose, by contradiction, that $M < v_a(x_0,t_0)$. Consider $\varepsilon = v_a(x_0,t_0) - M > 0$ and define

$$w_\alpha(x,t) = v_a(x,t) + \frac{\varepsilon}{2} S_\alpha \left( \frac{T - t}{T} \chi_{[0,T]}(t) \right), \quad \forall(x,t) \in \Omega,$$

where $\chi_{[0,T]}(t)$ is the indicator function of the interval $[0,T]$.

Since obviously $0 \leq \frac{T - t}{T} \leq 1$ we have

$$v_a(x,t) \leq w_\alpha(x,t) \leq v_a(x,t) + \frac{\varepsilon}{2}, \quad \forall(x,t) \in \Omega.$$

In particular, for any $(x,t) \in \partial_\Omega$, it holds

$$w_\alpha(x_0,t_0) \geq v_a(x_0,t_0) = \varepsilon + M$$

$$\geq \varepsilon + v_a(x,t) \geq \frac{\varepsilon}{2} + w_\alpha(x,t).$$

In particular, being $(x_0,t_0) \not\in \partial_\Omega$, it holds

$$\max_{(x,t) \in \partial_\Omega} w_\alpha(x,t) > \max_{(x,t) \in \Omega} w_\alpha(x,t).$$
Let \((x_1, t_1)\) be such that \(w_\alpha(x_1, t_1) = \max_{(x,t) \in \Omega} w_\alpha(x, t)\). Then \((x_1, t_1) \in \hat{\Omega} \cup ((a, b) \times \{T\})\).

Now let us observe that
\[
v_\alpha(x, t) = w_\alpha(x, t) - \frac{\varepsilon}{2} S_\alpha \left( \frac{T - t}{T} \chi_{[0,T]}(t) \right).
\]

First of all, let us consider \(\partial_t^\alpha v_\alpha(x, t)\). We have
\[
\partial_t^\alpha v_\alpha(x, t) = \partial_t^\alpha w_\alpha(x, t) - \frac{\varepsilon}{2} \partial_t^\alpha S_\alpha \left( \frac{T - t}{T} \chi_{[0,T]}(t) \right),
\]
thus we need to determine \(\partial_t^\alpha S_\alpha \left( \frac{T - t}{T} \chi_{[0,T]}(t) \right)\).

Finally, substituting last identity in Equation (5.10) we have
\[
\partial_t^\alpha v_\alpha(x, t) = \partial_t^\alpha w_\alpha(x, t) + \frac{\varepsilon}{2T} \int_0^T f_\alpha(s, t)ds.
\]

Now let us focus on \(G_{\alpha,H} v_\alpha(x, t)\). Since \(v_\alpha\) is a subordinated mild solution of \(\mathcal{L}_{t-\lambda}\), we know, by Lemma 5.12, we have
\[
G_{\alpha,H} v_\alpha(x, t) = \frac{\partial^2}{\partial x^2} S_{\alpha,H} v(x, t).
\]
On the other hand, we have, defining for \((x, t) \in \Omega\)
\[
  w(x, t) = v(x, t) + \frac{\varepsilon(T - t)}{2T},
\]
\[w_\alpha(x, t) = S_\alpha w(x, t).\]
Moreover, we have
\[
v(x, t) = w(x, t) - \frac{\varepsilon(T - t)}{2T},
\]
thus Equation (5.14) becomes (since \(S_{\alpha, H} \left( \frac{\varepsilon(T - t)}{2T} \frac{\chi_{[0, T]}(t)}{x, t} \right)\) does not depend on \(x\)
\[
  G_{\alpha, H} v_\alpha(x, t) = \frac{\partial^2}{\partial x^2} S_{\alpha, H} w(x, t).
\]
Now let us rewrite Equation (4.1) as
\[
  \partial_t^\alpha w(x, t) + \frac{\varepsilon}{2T} \int_0^T f_\alpha(s, t_1) ds - \frac{1}{2} \frac{\partial^2}{\partial x^2} S_{\alpha, H} w(x, t) = 0.
\]
By [18, Theorem 1] we know that \(\partial_t^\alpha w_\alpha(x_1, t_1) \geq 0\) on the other hand, by Lemma 5.14 we know that \((x_1, t_1)\) is maximum point for \(S_{\alpha, H} w(x, t)\) hence
\[
  \frac{1}{2} \frac{\partial^2}{\partial x^2} S_{\alpha, H} w(x_1, t_1) \leq 0.
\]
Thus, considering the LHS of Equation (5.16) in \((x_1, t_1)\), we get
\[
  \partial_t^\alpha w_\alpha(x_1, t_1) + \frac{\varepsilon}{2T} \int_0^T f_\alpha(s, t_1) ds - \frac{1}{2} \frac{\partial^2}{\partial x^2} S_{\alpha, H} w(x_1, t_1) \geq \frac{\varepsilon}{2T} \int_0^T f_\alpha(s, t_1) ds > 0
\]
which is a contradiction.
Hence \(M = v(x_0, t_0)\) and \(\max_{(x, t) \in \partial_\Omega} v_\alpha(x, t) = \max_{(x, t) \in \Omega} v_\alpha(x, t)\) concluding the proof.

From the weak maximum principle for Equation (4.1), we obtain the continuous dependence of the solution with respect to the boundary-initial datum and then uniqueness of the classical solutions for finite time-intervals.

**Corollary 5.16.** Let \(\Omega = [a, b] \times [0, T]\) for some \(T > 0\). Let us consider the problems:
\[
\begin{cases}
  \partial_t^\alpha v^{(i)}_\alpha(x, t) = \frac{1}{2} G_{\alpha, H} v^{(i)}_\alpha(x, t), & (x, t) \in \Omega \\
  v^{(i)}_\alpha(a, t) = f_i(t), & t \in [0, T] \\
  v^{(i)}_\alpha(b, t) = g_i(t), & t \in [0, T] \\
  v^{(i)}_\alpha(x, 0) = h_i(x), & x \in [a, b]
\end{cases}
\]
for \(i = 1, 2\), where \(f_i(0) = h_i(a)\) and \(g_i(0) = h_i(b)\). Suppose both of the problems admit a subordinated solution \(v^{(i)}_\alpha\) with \(i = 1, 2\). Then
\[
\|v^{(1)}_\alpha - v^{(2)}_\alpha\|_{L^\infty(\Omega)} = \max\{\|f_1 - f_2\|_{L^\infty(0, T)} , \|g_1 - g_2\|_{L^\infty(0, T)} , \|h_1 - h_2\|_{L^\infty(a, b)}\}.
\]
In particular if a Problem of the form (5.17) admits a subordinated solution, it is the unique subordinated solution.
Proof. We have that $v_\alpha^{(1)} - v_\alpha^{(2)}$ is a subordinated solution of

$$\begin{align*}
\partial_t^\alpha (v_\alpha^{(1)} - v_\alpha^{(2)})(x, t) &= \frac{1}{2} G_{\alpha, H}(v_\alpha^{(1)} - v_\alpha^{(2)})(x, t) \quad (x, t) \in \Omega \\
(v_\alpha^{(1)} - v_\alpha^{(2)})(a, t) &= (f_1 - f_2)(t) \quad t \in [0, T] \\
(v_\alpha^{(1)} - v_\alpha^{(2)})(b, t) &= (g_1 - g_2)(t) \quad t \in [0, T] \\
(v_\alpha^{(1)} - v_\alpha^{(2)})(x, 0) &= (h_1 - h_2)(x) \quad x \in [a, b].
\end{align*}$$

By Theorem 5.15 we have

\[ \max_{(x,t) \in \Omega} (v_\alpha^{(1)} - v_\alpha^{(2)})(x, t) = \max_{(x,t) \in \partial \Omega} \max_{t \in [0, T]} \{ (f_1 - f_2)(t), (g_1 - g_2)(t), (h_1 - h_2)(x) \} \]

\[ \min_{(x,t) \in \Omega} (v_\alpha^{(1)} - v_\alpha^{(2)})(x, t) = \min_{(x,t) \in \partial \Omega} \min_{t \in [0, T]} \{ (f_1 - f_2)(t), (g_1 - g_2)(t), (h_1 - h_2)(x) \}. \]

Observing that

\[ \left\| v_\alpha^{(1)} - v_\alpha^{(2)} \right\|_{L^\infty(0)} = \max_{(x,t) \in \Omega} \{ \max_{t \in [0, T]} (v_\alpha^{(1)} - v_\alpha^{(2)})(x, t), - \min_{t \in [0, T]} (v_\alpha^{(1)} - v_\alpha^{(2)})(x, t) \} \]

\[ = \max_{t \in [0, T]} \{ \max_{x \in [a,b]} (f_1 - f_2)(t), \max_{x \in [a,b]} (g_1 - g_2)(t), \min_{x \in [a,b]} (h_1 - h_2)(x) \}, \]

\[ - \min_{t \in [0, T]} \{ \min_{x \in [a,b]} (f_1 - f_2)(t), \min_{x \in [a,b]} (g_1 - g_2)(t), \max_{x \in [a,b]} (h_1 - h_2)(x) \} \]

\[ = \max_{t \in [0, T]} \{ \max_{x \in [a,b]} (g_1 - g_2)(t), - \min_{x \in [a,b]} (g_1 - g_2)(t) \}, \]

\[ \max_{t \in [0, T]} \{ \max_{x \in [a,b]} (h_1 - h_2)(x), - \min_{x \in [a,b]} (h_1 - h_2)(x) \} \]

\[ = \max \left\{ \| f_1 - f_2 \|_{L^\infty(0, T)}, \| g_1 - g_2 \|_{L^\infty(0, T)}, \| h_1 - h_2 \|_{L^\infty(a,b)} \right\}, \]

we conclude the proof.

Concerning uniqueness, it follows from the previous estimate with $f_1 = f_2$, $g_1 = g_2$, $h_1 = h_2$. \[\square\]

5.5. Convergence of subordinated solutions. In this Subsection we focus on the question of convergence of subordinated solutions as $H \rightarrow \frac{1}{2}$. First of all, we show that if a sequence of subordinated mild solution converges in a certain sense to a function, then this function is a subordinated mild solution itself. Finally, we also show that if we are under hypotheses of Theorem 5.13 then also the limit satisfies the same hypotheses.

**Theorem 5.17.** Let us consider a sequence $H_n \rightarrow \frac{1}{2}$ and $v_{n, \alpha} \in S_{H_n}$ for each $n \in \mathbb{N}$. Let us suppose there exists a function $v : [a, b] \times [0, +\infty) \rightarrow \mathbb{R}$ such that, setting $v_{n, \alpha} = S_{\alpha} v_n$, $v_n \rightarrow v$ in $L^\infty(\mathbb{R}^+; C^2(I))$, where $I = [a, b]$, that is to say

\[ \lim_{n \rightarrow +\infty} \sup_{t \in (0, +\infty)} \| v_n(\cdot, t) - v(\cdot, t) \|_{C^2(I)} = 0. \]

Then $v_{n, \alpha} = S_{\alpha} v$ is a subordinated mild solution of Equation (1.1) for $H = 1/2$.

If additionally $v_n \in D(G_{1, H_n}, I)$, $G_{1, H_n} v_n(\cdot, t) \in C^0(I)$ and $G_{1, H_n} v_n(x, \cdot) \in L^\infty(0, +\infty)$, then $v_{n, \alpha}$ are classical solutions of (1.1) and $v_{\alpha}$ is a classical solution of (1.1) for $H = 1/2$.

**Proof.** Let us first observe that, since $S_{\alpha}$ is a bounded linear operator preserving $C^i(I)$ for any $i \geq 0$, $v_{n, \alpha} \rightarrow v_{\alpha}$ in $L^\infty(\mathbb{R}^+; C^2(I))$. Now, recalling (4.2) and
Propositions 5.2 and 5.4 let us consider

\[
\left| \lambda^\alpha \pi_\alpha(x, \lambda) - \lambda^{\alpha-1} v_n(x, 0) - \frac{\lambda^{\alpha-1}}{2} \frac{\partial^2}{\partial x^2} L_{\frac{1}{2}} v(x, \lambda^\alpha) \right| \\
\leq \left| \lambda^{\alpha-1} \pi(x, \lambda^\alpha) - \lambda^{\alpha-1} \pi(x, \lambda^\alpha) \right| \\
+ \left| \lambda^{\alpha-1} v_{n, \alpha}(x, 0) - \lambda^{\alpha-1} v_n(x, 0) \right| \\
+ \frac{\lambda^{\alpha-1}}{2} \frac{\partial^2}{\partial x^2} \left( L_{H_n} v_n(x, \lambda^\alpha) - L_{\frac{1}{2}} v(x, \lambda^\alpha) \right),
\]

where the inequality is obtained by subtracting \( \lambda^{\alpha-1} \pi_n(x, \lambda^\alpha) - \lambda^{\alpha-1} v_{n, \alpha}(x, 0) - \frac{\lambda^{\alpha-1}}{2} \frac{\partial^2}{\partial x^2} L_{H_n} v_n(x, \lambda^\alpha) \) which is actually 0.

Let us work with the third part of the upper bound. We have

\[
\left| \frac{\lambda^{\alpha-1}}{2} \frac{\partial^2}{\partial x^2} \left( L_{H_n} v_n(x, \lambda^\alpha) - L_{\frac{1}{2}} v(x, \lambda^\alpha) \right) \right| \leq \left| \frac{\lambda^{\alpha-1}}{2} \frac{\partial^2}{\partial x^2} \left( L_{H_n} v_n(x, \lambda^\alpha) - L_{\frac{1}{2}} v(x, \lambda^\alpha) \right) \right| \\
+ \frac{\lambda^{\alpha-1}}{2} \frac{\partial^2}{\partial x^2} \left( L_{\frac{1}{2}} v_n(x, \lambda^\alpha) - L_{\frac{1}{2}} v(x, \lambda^\alpha) \right) \\
:= \frac{\lambda^{\alpha-1}}{2} (I_1(x) + I_2(x)).
\]

Let us first work with \( I_1(x) \). We have, since \( \frac{\partial^2}{\partial x^2} \) is a closed operator and \( L_H \) is actually a Laplace transform,

\[
I_1(x) = \int_0^{t+\infty} e^{-\lambda^\alpha t} (V'_{2,H_n}(t) - V'_{2,\frac{1}{2}}(t)) \frac{\partial^2}{\partial x^2} v_n(x, t) dt.
\]

Fix \( \varepsilon > 0 \) and split the integral. We have, by Proposition 4.6

\[
I_1(x) = \int_0^e e^{-\lambda^\alpha t} (V'_{2,H_n}(t) - V'_{2,\frac{1}{2}}(t)) \frac{\partial^2}{\partial x^2} v_n(x, t) dt \\
+ \int_e^{t+\infty} e^{-\lambda^\alpha t} (V'_{2,H_n}(t) - V'_{2,\frac{1}{2}}(t)) \frac{\partial^2}{\partial x^2} v_n(x, t) dt \\
\leq \left\| V'_{2,H_n} - V'_{2,\frac{1}{2}} \right\|_{L^\infty(0, \varepsilon)} \left\| v_n \right\|_{L^\infty(\mathbb{R}^+; C^2(I))} \frac{1 - e^{-\lambda^\alpha \varepsilon}}{\lambda^\alpha} + C_\varepsilon(H_n) \left\| v_n \right\|_{L^\infty(\mathbb{R}^+; C^2(I))}.
\]

Since \( v_n \rightarrow v \) in \( L^\infty(\mathbb{R}^+; C^2(I)) \), there exists a constant \( K \) such that \( \left\| v_n \right\|_{L^\infty(\mathbb{R}^+; C^2(I))} \leq K \). Thus we have

\[
I_1(x) \leq K \left( \left\| V'_{2,H_n} - V'_{2,\frac{1}{2}} \right\|_{L^\infty(0, \varepsilon)} \frac{1 - e^{-\lambda^\alpha \varepsilon}}{\lambda^\alpha} + C_\varepsilon(H_n) \right).
\]

Concerning \( I_2(x) \), we have, recalling that \( V'_{2,\frac{1}{2}}(t) = e^{-\frac{\lambda^\alpha}{2} t} \),

\[
I_2(x) \leq \left\| v_n - v \right\|_{L^\infty(\mathbb{R}^+; C^2(I))}.
\]

Thus we have

\[
\left| \lambda^\alpha \pi_\alpha(x, \lambda) - \lambda^{\alpha-1} v_n(x, 0) - \frac{\lambda^{\alpha-1}}{2} \frac{\partial^2}{\partial x^2} L_{\frac{1}{2}} v(x, \lambda^\alpha) \right| \\
\leq \lambda^{\alpha-1} \left\| v_n - v \right\|_{L^\infty(\mathbb{R}^+; C^2(I))} + \lambda^{\alpha-1} \left\| v_{n, \alpha}(\cdot, 0) - v_{\alpha}(\cdot, 0) \right\|_{C^2(I)} \\
+ K \lambda^{\alpha-1} \left( \left\| V'_{2,H_n} - V'_{2,\frac{1}{2}} \right\|_{L^\infty(0, \varepsilon)} \frac{1 - e^{-\lambda^\alpha \varepsilon}}{\lambda^\alpha} + C_\varepsilon(H_n) \right) + \frac{\lambda^{\alpha-1}}{2} \left\| v_n - v \right\|_{L^\infty(\mathbb{R}^+; C^2(I))}.
\]
and then we have, for fixed $\lambda$,
\[
\left\| \lambda^\alpha \tau_\alpha(x, \lambda) - \lambda^{-1} v_\alpha(x, 0) - \frac{\lambda^{\alpha-1}}{2} \frac{\partial^2}{\partial x^2} L^{\frac{1}{2}} v(x, \lambda^\alpha) \right\|_{L^\infty(a, b)}
\leq \lambda^{-1} \left\| v_n - v \right\|_{L^\infty(\mathbb{R}^+, C^2(I))} + \lambda^{-1} \left\| v_{n, \alpha} - v_\alpha \right\|_{C^2(I)}
+ \frac{K\lambda^{\alpha-1}}{2} \left( \left\| V'_{2, H_n} - V'_{2, \theta} \right\|_{L^\infty(0, \varepsilon)} + \frac{1 - e^{-\lambda^\alpha \varepsilon}}{\lambda^\alpha} + C_\varepsilon(H_n) \right) + \frac{\lambda^{\alpha-1}}{2} \left\| v_n - v \right\|_{L^\infty(\mathbb{R}^+, C^2(I))}.
\]

Taking the limit superior as $n \to +\infty$ and recalling that $\limsup_n \left\| V'_{2, H_n} - V'_{2, \theta} \right\|_{L^\infty(0, \varepsilon)} \leq 1$ (by Proposition 4.6), we obtain
\[
\left\| \lambda^\alpha \tau_\alpha(x, \lambda) - \lambda^{-1} v_\alpha(x, 0) - \frac{\lambda^{\alpha-1}}{2} \frac{\partial^2}{\partial x^2} L^{\frac{1}{2}} v(x, \lambda^\alpha) \right\|_{L^\infty(a, b)}
\leq \frac{K\lambda^{\alpha-1}}{2} - e^{-\lambda^\alpha \varepsilon}.
\]

Finally, taking $\varepsilon \to 0$, we have
\[
\lambda^\alpha \tau_\alpha(x, \lambda) - \lambda^{-1} v_\alpha(x, 0) - \frac{\lambda^{\alpha-1}}{2} \frac{\partial^2}{\partial x^2} L^{\frac{1}{2}} v(x, \lambda^\alpha) = 0.
\]

Now, under the additional hypotheses, we know that $v_{n, \alpha}$ are classical solutions. We know that $v \in L^\infty(\mathbb{R}^+, C^2(I))$. Now let us observe that, if it exists, then
\[
G_{1, \frac{2}{\theta}} v(x, \lambda) = L^{\frac{1}{2}} \left[ \frac{\partial^2}{\partial x^2} L^{\frac{1}{2}} v(x, \lambda) \right].
\]

However, since $V'_{2, \theta}(t) = e^{-\frac{2}{\theta} t}$, we have that
\[
L^{\frac{1}{2}} v(x, \lambda) = \tau \left( x, \lambda + \frac{2}{\theta} \right).
\]

In particular, since $v \in L^\infty(\mathbb{R}^+, C^2(I))$, then $v$ and $\frac{\partial^2}{\partial x^2} v$ are Laplace transformable. Being $\frac{\partial^2}{\partial x^2}$ a closed operator we have, for fixed $\lambda \in \mathcal{H}$
\[
\frac{\partial^2}{\partial x^2} \left( x, \lambda + \frac{2}{\theta} \right) = L_{t \to \lambda} \left( \frac{\partial^2}{\partial x^2} v(x, t) \right) \left( \lambda + \frac{2}{\theta} \right) = L_{t \to \lambda} \left( e^{-\frac{2}{\theta} t} \frac{\partial^2}{\partial x^2} v(x, t) \right)(\lambda)
\]

hence we have that $v \in G_{1, \frac{2}{\theta}}$ and
\[
G_{1, \frac{2}{\theta}} v(x, \lambda) = e^{-\frac{2}{\theta} t} \frac{\partial^2}{\partial x^2} v(x, t).
\]

Moreover, since $v \in L^\infty(\mathbb{R}^+, C^2(I))$, then for fixed $x \in I$, $t \mapsto e^{-\frac{2}{\theta} t} \frac{\partial^2}{\partial x^2} v(x, t)$ belongs to $L^\infty(0, +\infty)$, thus, by Theorem 6.13, we have that $v_\alpha$ is a classical solution of (4.1) with $H = 1/2$.

\section*{Acknowledgements}
This research is partially supported by MIUR - PRIN 2017, project Stochastic Models for Complex Systems, no. 2017JFFHSH, by Gruppo Nazionale per il Calcolo Scientifico (GNCS-IndAM), by Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA-IndAM). The research of the 2nd author was partially supported by ToppForsk project nr. 274410 of the Research Council of Norway with title STORM: Stochastics for Time-Space Risk Models.
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