Research Article

Imed Abid*, Sami Baraket, and Rached Jaidane

On a weighted elliptic equation of \( N \)-Kirchhoff type with double exponential growth

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Abstract: In this work, we study the weighted Kirchhoff problem

\[
\begin{align*}
-\mathbf{g} \left( \int_B \sigma(x)|\nabla u|^N \, dx \right) \text{div}(\sigma(x)|\nabla u|^{N-2} \nabla u) &= f(x, u) \quad \text{in } B, \\
u > 0 &\quad \text{in } B, \\
u = 0 &\quad \text{on } \partial B,
\end{align*}
\]

where \( B \) is the unit ball of \( \mathbb{R}^N \), \( \sigma(x) = \left( \log \left( \frac{e^x}{|x|} \right) \right)^{N-1} \), the singular logarithm weight in the Trudinger-Moser embedding, and \( g \) is a continuous positive function on \( \mathbb{R}^+ \). The nonlinearity is critical or subcritical growth in view of Trudinger-Moser inequalities. We first obtain the existence of a solution in the subcritical exponential growth case with positive energy by using minimax techniques combined with the Trudinger-Moser inequality. In the critical case, the associated energy does not satisfy the condition of compactness. We provide a new condition for growth, and we stress its importance to check the compactness level.

Keywords: nonlocal Kirchhoff equation, Trudinger-Moser’s inequality, critical double exponential nonlinearities, mountain pass theorem

MSC 2020: 35J20, 35J25, 35J60

1 Introduction

In this article, we consider the following non-local weighted problem:

\[
\begin{align*}
L_{(N, \sigma, g)} = -\mathbf{g} \left( \int_B \sigma(x)|\nabla u|^N \, dx \right) \text{div}(\sigma(x)|\nabla u|^{N-2} \nabla u) &= f(x, u) \quad \text{in } B \\
u > 0 &\quad \text{in } B, \\
u = 0 &\quad \text{on } \partial B,
\end{align*}
\]

where \( B = B(0, 1) \) is the unit open ball in \( \mathbb{R}^N \), \( f(x, t) \) is a radial function with respect to \( x \), and the weight \( \sigma(x) \) is given by

* Corresponding author: Imed Abid, Higher Institute of Medical Technologies of Tunis, University of Tunis El Manar, Tunis, Tunisia, e-mail: imed.abid@istmt.utm.tn

Sami Baraket, Rached Jaidane: Faculty of Sciences of Tunis, University of Tunis El Manar, Tunis, Tunisia, e-mail: smbaraket@yahoo.fr, rachedjaidane@gmail.com

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\[ \sigma(x) = \left( \log \left( \frac{e}{|x|} \right) \right)^{N-1} \]  

and \( g: \mathbb{R}^+ \to \mathbb{R}^+ \) is a positive continuous function which will be specified later.

In 1883, Kirchhoff studied the following parabolic problem:

\[ \rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{P_0}{h} + \frac{E}{2L} \int_0^L \left( \frac{\partial u}{\partial x} \right)^2 \, dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \]

for free vibrations of elastic strings. The parameters in equation (3) have the following meanings: \( L \) is the length of the string, \( h \) is the area of cross-section, \( E \) is the Young's modulus of the material, \( \rho \) is the mass density, and \( P_0 \) is the initial tension.

These kinds of problems have physical motivations. Indeed, the Kirchhoff operator \( \int \nabla G u \, dx \) also appears in the nonlinear vibration equation, namely,

\[ \frac{\partial^2 u}{\partial t^2} - G \left( \int_B |\nabla u|^2 \, dx \right) \operatorname{div}(\nabla u) = f(x, u) \text{ in } B \times (0, T) \]

\[ u > 0 \text{ in } B \times (0, T) \]

\[ u = 0 \text{ on } \partial B \]

\[ u(x, 0) = u_0(x) \text{ in } B \]

\[ \frac{\partial u}{\partial t}(x, 0) = u_1(x) \text{ in } B \]

which have received the attention of several researchers, mainly as a result of the work of Lions [1].

We mention that non-local problems also arise in other areas, e.g., biological systems where the function \( u \) describes a process that depends on the average of itself (e.g., population density), see, e.g., [2,3] and references therein.

In the non-weighted case, i.e., when \( \sigma(x) \equiv 1 \) and when \( N = 2 \), problem (1) can be seen as a stationary version of the evolution problem (4).

Recently, Xiu et al. [4] studied the following singular nonlocal elliptic problem:

\[ \left\{ \begin{array}{l}
-M \int_{\mathbb{R}^N} |x|^{-ap} |\nabla u|^p \, dx \operatorname{div}(|x|^{-ap} |\nabla u|^{p-2} \nabla u) = h(x)|u|^{p-2} u + H(x)|u|^{q-2} u, \\
u(x) \to 0 \text{ as } |x| \to +\infty,
\end{array} \right. \]

where \( x \in \mathbb{R}^N \), \( M(t) = \tilde{b} + \tilde{a}t, \tilde{a}, \tilde{b} > 0 \), \( a < \frac{N-p}{p} \), \( h(x) \) and \( H(x) \) are nonnegative function. They proved that this problem has infinitely many solutions by variational methods and the genus theorem.

In order to motivate our study, we begin by giving a brief survey on Trudinger-Moser inequalities. In the past few decades, Moser gives the famous result about the Trudinger-Moser inequality [5,6]; many applications take place as in conformal deformation theory on manifolds, the study of the prescribed Gauss curvature and mean field equations. After that, a logarithmic Trudinger-Moser inequality was used in a crucial way in [7] to study the Liouville equation of the form

\[ \left\{ \begin{array}{l}
-\Delta u = \lambda \frac{e^u}{\int_{\Omega} e^u} \text{ in } \Omega \\
u = 0 \text{ on } \partial \Omega,
\end{array} \right. \]
where \( \Omega \) is an open domain of \( \mathbb{R}^N \), \( N \geq 2 \), and \( \lambda \) a positive parameter. Equation (5) has a long history and has been derived in the study of multiple condensate solution in the Chern-Simons-Higgs theory \([8,9]\) and also it appeared in the study of Euler flow \([10–13]\).

Later, the Trudinger-Moser inequality was improved to weighted inequalities \([14,15]\). The influence of the weight in the Sobolev norm was studied as the compact embedding \([16,17]\).

When the weight is of logarithmic type, Calanchi and Ruf \([18]\) extended the Trudinger-Moser inequality and gave some applications when \( N = 2 \) and for prescribed nonlinearities. After that, Calanchi et al. \([19]\) considered more general nonlinearities and proved the existence of radial solutions.

We point out that recently, in the case \( g(t) = 1 \), Deng et al. \([20]\) have proved the existence of a nontrivial solution for the following boundary value problem:

\[
\begin{aligned}
-\text{div}(w(x) \nabla u(x))^{N-2} &\nabla u(x) = f(x, u) \quad \text{in } B \\
&\quad \text{on } \partial B,
\end{aligned}
\]

where \( B \) is the unit ball in \( \mathbb{R}^N \), \( N \geq 2 \), the radial positive weight \( w(x) \) is of logarithmic type, the function \( f(x, u) \) is continuous in \( B \times \mathbb{R} \) and has critical double exponential growth.

Also recently, de Figueiredo and Severo \([21]\) studied the following problem:

\[
\begin{aligned}
-\text{div}(|\nabla u|^2) &\nabla u = f(x, u) \quad \text{in } \Omega \\
&\quad \text{in } \Omega \\
&\quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( \Omega \) is a smooth-bounded domain in \( \mathbb{R}^2 \), the nonlinearity \( f \) behaves like \( \exp(\alpha|t|^2) \) as \( t \to +\infty \), for some \( \alpha > 0 \). The authors proved that this problem has a positive ground state solution. The existence result was proved by combining minimax techniques and Trudinger-Moser inequalities.

Inspired by the last two works, we investigate our problem by adapting weighted Sobolev space setting. We use the Trudinger-Moser inequality to study and prove the existence of solutions of (1).

Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain and \( \sigma \in L^1(\Omega) \) be a nonnegative function. We define the weighted Sobolev space as follows:

\[
W_0^{1,N}(\Omega, \sigma) = \text{closure} \left\{ u \in C_0^\infty(\Omega) \middle| \int_B |\nabla u|^N \sigma(x) \, dx < \infty \right\},
\]

we will limit our attention to radial functions and then consider the subspace,

\[
\mathcal{W} = W_{0,\text{rad}}^{1,N}(B, \sigma) = \text{closure} \left\{ u \in C_0^{\infty}(B) \middle| \int_B |\nabla u|^N \sigma(x) \, dx < \infty \right\},
\]

equipped with the norm

\[
||u|| = \left( \int_B |\nabla u|^N \sigma(x) \, dx \right)^\frac{1}{N}.
\]

The choice of the weight and the space \( \mathcal{W} = W_{0,\text{rad}}^{1,N}(B, \sigma) \) are motivated by the following exponential inequalities.

**Theorem 1.1.** \([15]\) Let \( \sigma \) be given by (2), then

\[
\int_B \exp \left( e^{u(x)^2} \right) \, dx < +\infty, \quad \forall u \in \mathcal{W},
\]
\[
\sup_{u \in W^1_0(B)} \int_B \exp\left(\beta e^{\frac{u^+}{2N^2}}\right) dx < +\infty \iff \beta \leq N,
\]

(7)

where \(\omega_{N-1}\) is the area of the unit sphere \(S^{N-1}\) in \(\mathbb{R}^N\).

Let \(N'\) be the Hölder conjugate of \(N\) that is \(N' = \frac{N}{N-1}\). In view of inequalities (6) and (7), we say that \(f\) has subcritical growth at \(+\infty\), if
\[
\lim_{s \to +\infty} \frac{|f(x, s)|}{\exp(N e^{as N})} = 0, \quad \forall \alpha > 0 \quad \text{uniformly in } x \in B
\]

and \(f\) has critical growth at \(+\infty\), if there exists some \(\alpha_0 > 0\), such that
\[
\lim_{s \to +\infty} \frac{|f(x, s)|}{\exp(N e^{as N})} = +\infty, \quad \forall \alpha < \alpha_0 \quad \text{uniformly in } x \in B.
\]

In this article, we consider problem (1) with subcritical or critical growth nonlinearities \(f(t, x)\). Furthermore, we suppose that \(f(t, x)\) satisfies the following hypotheses:

\((H_1)\) The nonlinearity \(f : B \times \mathbb{R} \to \mathbb{R}\) is positive, continuous, radial in \(x\), and \(f(x, t) = 0\) for \(t \leq 0\).

\((H_2)\) There exist \(t_0 > 0\) and \(M_0 > 0\), such that for all \(t > t_0\) and for all \(x \in B\), we have
\[
0 < F(x, t) \leq M_0 f(x, t),
\]

where \(F(x, t) = \int_0^t f(x, s) ds\).

\((H_3)\) For each \(x \in B\), \(\frac{f(x, t)}{t^{N-1}}\) is increasing for \(t > 0\).

\((H_4)\) In the critical case, there exists a constant \(\gamma_0\) with \(\gamma_0 > \frac{1}{a^2 - 2\gamma_0 \frac{\alpha_0 - 1}{a^2 - 1}}\) such that
\[
\lim_{t \to +\infty} \frac{f(x, t) t}{\exp(N e^{a_0 t N})} \geq \gamma_0 \quad \text{uniformly in } x \in B.
\]

The condition \((H_2)\) implies that for any \(\epsilon > 0\), there exists a real \(t_\epsilon > 0\), such that
\[
F(x, t) \leq \epsilon f(x, t), \quad \forall |t| > t_\epsilon \quad \text{uniformly in } x \in B.
\]

(8)

Also, we have that condition \((H_3)\) leads to
\[
\lim_{t \to -0} \frac{f(x, t)}{t^\theta} = 0 \quad \forall 0 \leq \theta < 2N - 1 \quad \text{uniformly in } x \in B.
\]

(9)

The condition asymptotic \((H_4)\) would be crucial to identify the minmax level of the energy associated with problem (1). We give an example of \(f\). Let \(f(t) = F(t)\), with \(F(t) = \frac{t^{1/2}}{2N + 2} + t^\tau \exp(N e^{a_0 t N})\), with \(\tau > 2N\). A simple calculation shows that \(f\) verifies conditions \((H_1)\), \((H_2)\), \((H_3)\), and \((H_4)\).

We define the function \(G(t) = \int_t^0 g(s) ds\). The function \(g\) is continuous on \(\mathbb{R}^+\) and verifies

\((G_1)\) There exists \(g_0 > 0\), such that \(g(t) \geq g_0\) for all \(t \geq 0\) and
\[
G(t + s) \geq G(t) + G(s) \quad \forall s, t \geq 0.
\]

\((G_2)\) \(\frac{g(t)}{t}\) is nonincreasing for \(t > 0\).

The assumption \((G_2)\) implies that \(\frac{g(t)}{t} \leq g(1)\) for all \(t \geq 1\). Then, one has \(g(t) \leq g(1)t\) for \(t \geq 1\).
Another consequence of \((G_2)\) is that a simple calculation shows that
\[
\frac{1}{N} G(t) - \frac{1}{2N} g(t) t \text{ is nondecreasing for } t \geq 0.
\]
So, one has
\[
\frac{1}{N} G(t) - \frac{1}{2N} g(t) t \geq 0, \quad \forall t \geq 0. \tag{10}
\]
A typical example of a function \(g\) fulfilling conditions \((G_0)\) and \((G_2)\) is given by
\[
g(t) = g_0 + at, \quad g_0, a > 0.
\]
Another example is given by \(g(t) = 1 + \ln(1 + t)\).

The major difficulty in this problem lies in the concurrence between the growths of \(g\) and \(f\).

It will be said that \(u\) is a solution to problem (1), if \(u\) is a weak solution in the following sense.

**Definition 1.1.** A function \(u\) is called a solution to problem (1) if \(u \in \mathcal{W}\) and
\[
g(\|u\|^N) \int_B (\sigma(x)|\nabla u|^2 - 2\nabla u \nabla \varphi) \, dx = \int_B f(x, u) \varphi \, dx, \quad \text{for all } \varphi \in \mathcal{W}.
\]

The energy functional, also known as the Euler-Lagrange functional associated with (1), is defined by
\[
\mathcal{J} : \mathcal{W} \to \mathbb{R}
\]
\[
\mathcal{J}(u) = \frac{1}{N} G(\|u\|^N) - \int_B F(x, u) \, dx. \tag{11}
\]

It is quite clear that finding weak solutions to problem (1) is equivalent to finding non-zero critical points of the functional \(\mathcal{J}\) over \(\mathcal{W}\).

In the subcritical exponential growth case, we will prove the following result.

**Theorem 1.2.** Let \(f(x, t)\) be a function that has a subcritical growth at \(+\infty\) and satisfies \((H_1), (H_2),\) and \((H_3)\). Assume that \(g\) satisfies \((G_0)\) and \((G_2)\). Then, problem (1) has a non-trivial radial solution.

In the context of the critical double exponential growth, the study of problem (1) becomes more difficult than in the subcritical case. Our Euler-Lagrange function is losing compactness at a certain level. To overcome this lack of compactness, we choose test functions, which are extremal for the Trudinger-Moser inequality (7). Our result is as follows.

**Theorem 1.3.** Assume that \(f(x, t)\) has a critical growth at \(+\infty\) and satisfies conditions \((H_1), (H_2), (H_3),\) and \((H_4)\). Assume that \(g\) satisfies \((G_0)\) and \((G_2)\). Then, problem (1) has a nontrivial solution.

This article is organized as follows. In Section 2, we give some useful lemmas for the compactness analysis. In Section 3, we prove that the functional \(\mathcal{J}\) satisfies the two geometric properties. Section 4 is devoted to estimate the minimax level of the energy. We conclude with the proofs of Theorems 1.2 and 1.3 in Section 5.

We shall use the notation \(\|u\|_p\) for the norm in the Lebesgue space \(L^p(B)\). We will also use the Sobolev weighted space defined by
\[
W^{2,p}(\Omega, \sigma) = \{u \in L^p(B) \text{ such that } D^a u \in L^p(B, \sigma) \text{ for all } 1 \leq |a| \leq 2\},
\]
equipped with the norm
\[
\|u\|_{W^{2,p}(B, \sigma)} = \left(\|u\|^p_p + \sum_{1 \leq |a| \leq 2} \|D^a u\|^p_{L^p(B, \sigma)}\right)^{\frac{1}{p}}.
\]
and where the Lebesgue weighted space,

\[ L^p(B, \sigma) = \left\{ u : B \to \mathbb{R} \text{ measurable} \mid \int_B \sigma(x)|u|^pdx < \infty \right\}, \]

is endowed with the norm \( \|u\|_{p, \sigma} = \left( \int_B \sigma(x)|u|^pdx \right)^{\frac{1}{p}} \).

## 2 Preliminaries for the variational formulation

In this section, we will present a number of technical lemmas for our future use. We begin with the radial lemma.

**Lemma 1.** [15] Let \( u \) be a radially symmetric function in \( C^1_0(B) \). Then, we have

\[ |u(x)| \leq \frac{1}{\omega_{N-1}^{\frac{1}{N}}} \log \frac{\epsilon}{|x|} \|u\|, \]

where \( \omega_{N-1} \) is the area of the unit sphere \( S^{N-1} \in \mathbb{R}^N \).

The second important lemma is given as follows:

**Lemma 2.** [22] Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain and \( f : \overline{\Omega} \times \mathbb{R} \) be a continuous function. Let \( \{u_n\}_n \) be a sequence in \( L^1(\Omega) \) converging to \( u \) in \( L^1(\Omega) \). Assume that \( f(x, u_n) \) and \( f(x, u) \) are also in \( L^1(\Omega) \). If

\[ \int_{\Omega} |f(x, u_n)|udx \leq C, \]

where \( C \) is a positive constant, then

\[ f(x, u_n) \rightharpoonup f(x, u) \quad \text{in} \ L^1(\Omega). \]

In an attempt to prove a compactness condition for the energy \( \mathcal{E} \), we need a Lions-type result [23] about an improved TM-inequality when we deal with weakly convergent sequences and double exponential case.

**Lemma 3.** Let \( \{u_k\}_k \) be a sequence in \( \mathcal{W} \). Suppose that \( \|u_k\| = 1 \), \( u_k \rightharpoonup u \) weakly in \( \mathcal{W} \), \( u_k(x) \to u(x) \) a.e. \( x \in B, \forall u_k(x) \to \forall u(x) \) a.e. \( x \in B \) and \( u \neq 0 \). Then

\[ \sup_k \int_B \exp \left( Ne^{p \omega_B^{\frac{1}{N}} |u_k|^p} \right)dx < +\infty \]

for all \( 1 < p < U \) where \( U \) is given by:

\[ U = \begin{cases} (1 - \|u\|^N)^{\frac{1}{N}} & \text{if } \|u\| < 1 \\ +\infty & \text{if } \|u\| = 1, \end{cases} \]

**Proof.** For \( a, b \in \mathbb{R}, q > 1 \). If \( q' \) is a conjugate, i.e., \( \frac{1}{q} + \frac{1}{q'} = 1 \), we have, by Young’s inequality, that

\[ e^{a+b} \leq \frac{1}{q} e^{qa} + \frac{1}{q'} e^{qb}, \]

and so
\[ \exp(Ne^{a+b}) \leq \exp\left(\frac{Ne^{qa}}{q} + \frac{Ne^{qb}}{q} \right). \]

Therefore,
\[ \exp(Ne^{a+b}) \leq \frac{1}{q} \exp(Ne^{qa}) + \frac{1}{q} \exp(Ne^{qb}). \]

Also, we have
\[ (1 + a)^q \leq (1 + \varepsilon)a^q + \left(1 - \frac{1}{(1 + \varepsilon)^{q-1}}\right), \quad \forall \varepsilon > 0 \quad \forall q > 1. \]

So, we obtain
\[ |u_k|^N = |u_k - u + u|^N \leq (|u_k - u| + |u|)^N \leq (1 + \varepsilon)|u_k - u|^N + \left(1 - \frac{1}{(1 + \varepsilon)^{q-1}}\right)|u|^N. \]

This implies that
\[ \int_B \exp\left(\frac{1}{q} Ne^{pq\varepsilon^2(1+\varepsilon)|u_k|^N}\right) dx \leq \frac{1}{q} \int_B \exp(\frac{1}{q} Ne^{pq\varepsilon^2(1+\varepsilon)|u|^N}) dx + \frac{1}{q} \int_B \exp\left(\frac{1}{q} Ne^{pq\varepsilon^2(1+\varepsilon)|u_k-u|^N}\right) dx, \]
for any \( p > 1. \) From (6), the last integral is finite. To complete the proof, we have to prove that for every \( p \) such that \( 1 < p < U, \)
\[ \sup_k \int_B \exp\left(\frac{1}{q} Ne^{pq\varepsilon^2(1+\varepsilon)|u_k-u|^N}\right) dx < +\infty, \quad (12) \]
for some \( \varepsilon > 0 \) and \( q > 1. \)

In the following, we suppose that \( |u| < 1, \) and in the case of \( |u| = 1, \) the proof is similar. When
\[ |u| < 1 \]
and
\[ p < \frac{1}{(1 - |u|^N)^{1+1}}, \]
there exists \( \nu > 0, \) such that
\[ p(1 - |u|^N)^{\frac{1}{1+1}}(1 + \nu) < 1. \]

On the other hand, by Brezis-Lieb’s lemma [24] we have
\[ \|u_k - u\|^N = \|u_k\|^N - \|u\|^N + o(1) \quad \text{where} \quad o(1) \to 0 \quad \text{as} \quad k \to +\infty. \]

Then,
\[ \|u_k - u\|^N = 1 - \|u\|^N + o(1), \]
and so,
\[ \lim_{k \to +\infty} \|u_k - u\|^N = 1 - \|u\|^N, \]
that is,
\[ \lim_{k \to +\infty} \|u_k - u\|^N = (1 - \|u\|^N)^{\frac{1}{1+1}}. \]
Therefore, for every $\varepsilon > 0$, there exists $k_\varepsilon \geq 1$ such that
\[ |u_k - u|^N \leq (1 + \varepsilon)^{(1 - |u|^N)^{\frac{1}{r}}}, \quad \forall k \geq k_\varepsilon. \]

If we take $q = 1 + \varepsilon$ with $\varepsilon = \sqrt{1 + \nu - 1}$, then $\forall k \geq k_\varepsilon$, we have
\[ pq(1 + \varepsilon)|u_k - u|^N \leq 1. \]

Consequently,
\[
\int_B \exp\left(Ne^{\frac{1}{pq}(1+\varepsilon)|u_k-u|^N}\right)dx \leq \int_B \exp\left(Ne^{\frac{1}{pq}|u-u|^N}\right)dx
\]
\[
\leq \int_B \exp\left(Ne^{\frac{1}{pq}|u|^N}\right)dx
\]
\[
\leq \sup_{|u|\leq 1} \int_B \exp\left(Ne^{\frac{1}{pq}|u|^N}\right)dx < +\infty.
\]

Now, (12) follows from (7). This completes the proof. \qed

3 The mountain pass geometry of the energy

Since the nonlinearity $f$ is critical or subcritical at $+\infty$, there exist $a, C > 0$ positive constants and there exists $t_2 > 1$ such that
\[ |f(x, t)| \leq C \exp\left(e^{at^N}\right), \quad \forall |t| > t_2. \]

So the functional $\mathcal{J}$ given by (11) is well defined and of class $C^1$.

In order to prove the existence of nontrivial solution to problem (1), we will prove the existence of nonzero critical point of the functional $\mathcal{J}$ by using the theorem introduced by Ambrosetti and Rabinowitz in [25] (mountain pass theorem) without the Palais-Smale condition.

**Theorem 3.1.** [25] Let $E$ be a Banach space and $J : E \to \mathbb{R}$ a $C^1$ functional satisfying $J(0) = 0$. Suppose that there exist $\rho, \beta > 0$, and $e \in E$ with $\|e\| > \rho$ such that
\[
\inf_{\|u\| = \rho} J(u) \geq \beta \quad \text{and} \quad J(e) \leq 0.
\]

Then, there is a sequence $(u_n) \subset E$ such that
\[ J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0,
\]
where
\[ c = \inf_{y \in \Gamma} \max_{[0,1]} J(y(t)) \geq \beta
\]
and
\[ \Gamma = \{ y \in C([0,1], E) \text{ such that } y(0) = 0 \text{ and } y(1) = e \}. \]

The number $c$ is called mountain pass level or minimax level of the functional $J$.

Before starting the proof of the geometric properties for the functional $\mathcal{J}$, it follows from the continuous embedding $\mathcal{W} \hookrightarrow L^q(B)$ for all $q \geq 1$, that there exists a constant $C > 0$ such that $\|u\|_{L^q} \leq C\|u\|$, for all $u \in \mathcal{W}$. 
In the next lemmas, we prove that the functional $\mathcal{J}$ has the mountain pass geometry of Theorem 3.1.

**Lemma 4.** Suppose that $f$ has critical or subcritical growth at $+\infty$. In addition, if $(H_1)$, $(H_2)$, and $(G_1)$ hold, then there exist $\rho, \beta > 0$ such that $\mathcal{J}(u) \geq \beta$ for all $u \in \mathcal{W}$ with $\|u\| = \rho$.

**Proof.** It follows from (9) that there exists $\delta_0 > 0$
\[
F(x, t) \leq \varepsilon |t|^N \quad \text{for } |t| < \delta_0.
\]
From (13), for all $q > N$, there exists a positive constant $\delta_1 > 0$ such that
\[
F(x, t) \leq C|t|^q \exp\big(e^{at}|t|^N\big), \quad \forall |t| > \delta_1.
\]
Using the continuity of $F$, we obtain
\[
F(x, t) \leq \varepsilon |t|^N + C|t|^q \exp\big(e^{at}|t|^N\big) \quad \text{for } t \in \mathbb{R}.
\]
We obtain from $(G_1)$,
\[
\mathcal{J}(u) \geq \frac{g_0}{N}\|u\|^N - \varepsilon \int_B |u|^N \, dx - C\int_B |u|^q \exp\big(e^{at}|u|^N\big) \, dx.
\]
From the Hölder inequality, we obtain
\[
\mathcal{J}(u) \geq \frac{g_0}{N}\|u\|^N - \varepsilon \int_B |u|^N \, dx - C\left(\int_B \exp\left(Ne^{at}|u|^N\right) \, dx\right)^{\frac{1}{q}}\|u\|_{N,q}^q.
\]
If we choose $u \in \mathcal{W}$ such that
\[
a\|u\|^N \leq \omega_{N-1}^{\frac{1}{N-1}},
\]
then from Theorem 1.1, we obtain
\[
\int_B \exp\left(Ne^{at}|u|^N\right) \, dx = \int_B \exp\left(Ne^{at}\frac{|u|}{(1+|u|)^a}\right) \, dx < C,
\]
with $C$ not depending on $u$.

On the other hand, $\|u\|_{N,q} \leq C\|u\|$, so
\[
\mathcal{J}(u) \geq \frac{g_0}{N}\|u\|^N - \varepsilon C\|u\|^N - C\|u\|^q = \|u\|^N\left(\frac{g_0}{N} - \varepsilon C_1 - C\|u\|^{q-N}\right),
\]
for all $u \in \mathcal{W}$ satisfying (14). Since $N < q$, we can choose $\rho = \|u\| \leq \omega_{N-1}^{\frac{1}{N-1}}$ and for fixed $\varepsilon$ such that $\frac{g_0}{N} - \varepsilon C_1 > 0$, there exists $\beta = \rho^N\left(\frac{g_0}{N} - \varepsilon C_1 - C\rho^{q-N}\right) > 0$ with $\mathcal{J}(u) \geq \beta > 0$. □

By the following lemma, we prove the second geometric property for the functional $\mathcal{J}$.

**Lemma 5.** Suppose that $(H_1)$, $(H_2)$, and $(G_2)$ hold. Then, there exists $e \in \mathcal{W}$ with $\mathcal{J}(e) < 0$ and $\|e\| > \rho$.

**Proof.** From condition $(G_2)$, for all $t \geq 1$, we have that
\[
G(t) \leq \frac{g(1)}{2} t^2.
\]
It follows from condition $(H_2)$, for all $t \geq t_0$
\[
f(x, t) = \frac{\partial}{\partial t} F(x, t) \geq \frac{1}{M_0} F(x, t).
\]
So

\[ F(x, t) \geq C e^{\omega t}, \quad \forall t \geq t_0. \]

In particular, for \( p > 2N \), there exist \( C_1 \) and \( C_2 \) such that

\[ F(x, t) \geq C_1 |t|^p - C_2 \quad \forall t \in \mathbb{R}, \ x \in B. \]

Next, one arbitrarily picks \( \bar{u} \in \mathcal{W} \) such that \( |\bar{u}| = 1 \). Thus from (15), for all \( t \geq 1 \),

\[ J(t\bar{u}) \leq \frac{g(1)}{2N} t^{2N} - C_1|\bar{u}|^p, \quad t^p - \frac{\omega_{N-1}}{N} C_2. \]

Therefore,

\[ \lim_{t \to +\infty} J(t\bar{u}) = -\infty. \]

We take \( e = t\bar{u} \), for some \( t > 0 \) large enough. So, Lemma 5 follows. \( \square \)

4 The minimax estimate of the energy

According to Lemmas 4 and 5, let

\[ d = \inf_{y \in \mathcal{E}} \max_{t \in [0, 1]} J(y(t)) > 0 \]

and

\[ \Lambda = \{ y \in C([0, 1], \mathcal{W}) \text{ such that } y(0) = 0 \quad \text{and} \quad J(y(1)) < 0 \}. \]

We are going to estimate the minimax value \( d \) of the functional \( J \). The idea is to construct a sequence of functions \( (v_n) \in \mathcal{W} \), and estimate \( \max\{J(v_n) : t \geq 0\} \). For this goal, let us consider the following Moser sequence:

\[ \psi_n(t) = \begin{cases} \frac{\log(1 + t)}{\log^p(1 + n)} & \text{if } 0 \leq t \leq n, \\ \frac{\log^p(1 + n)}{\log^p(1 + n)} & \text{if } t \geq n. \end{cases} \quad (16) \]

Let \( v_n(x) \) be the function defined by

\[ \psi_n(t) = \frac{1}{\omega_{N-1}^{N-1}} v_n(x), \quad \text{where } e^{-t} = |x|. \]

With this choice of \( \psi_n \), the sequence \( (v_n) \) is normalized since

\[ |v_n|^N = \frac{1}{\omega_{N-1}} \int_B \left| \nabla \psi_n \right|^N \left| \log \left( \frac{e}{|x|} \right) \right|^{N-1} dx = \int_0^{+\infty} \left| \psi(t)^N (1 + t)^{N-1} \right| dt = 1. \]

We have the following elementary crucial result.

**Lemma 6.** We have

\[ \lim_{n \to +\infty} \int_0^{+\infty} \exp(N e^{\psi_n} - N t) dt = \left( \frac{N + 1}{N} \right) e^{\psi_j}. \]

**Proof.** We make the changes of variable \( s = 1 + t \) and \( j = n + 1 \), so
\[ \int_0^{+\infty} \exp(Ne^{\psi(s)} - Nt)dt = \frac{e^N}{N} + \int_0^n \exp\left(Ne^{\psi(s)} - Nt\right)dt \]
\[ = \frac{e^N}{N} + \int_1^j \exp\left(Ns\left(\frac{\log s}{\log j}\right)^{\frac{1}{\alpha}} - N(s - 1)\right) ds \]
\[ = \frac{e^N}{N} + e^N \int_1^j \exp\left(Ns\left(\frac{\log s}{\log j}\right)^{\frac{1}{\alpha}} - Ns\right) ds. \]

We claim that
\[ \lim_{j \to +\infty} \int_1^j \exp\left(Ns\left(\frac{\log s}{\log j}\right)^{\frac{1}{\alpha}} - Ns\right) ds = 1. \]

Indeed, for any \( j > 4 \), we have
\[ \psi_j(s) = Ns\left(\frac{\log s}{\log j}\right)^{\frac{1}{\alpha}} - Ns \quad \text{with} \quad s \geq 1. \]

The interval \([1, j]\) is then divided as follows:
\[ [1, j] = \left[1, j^{\frac{1}{\alpha}}\right] \cup \left[j^{\frac{1}{\alpha}}, j - j^{\frac{1}{\alpha}}\right] \cup \left[j - j^{\frac{1}{\alpha}}, j\right]. \]

First, we consider the interval \( \left[1, j^{\frac{1}{\alpha}}\right]. \) Since
\[ \chi_{\left[1, j^{\frac{1}{\alpha}}\right]}(s)e^{\psi_j(s)} \leq e^{Nj^{\frac{1}{\alpha}} - Ns} \in L^1([1, +\infty)), \]
\[ \chi_{\left[1, j^{\frac{1}{\alpha}}\right]}(s)e^{\psi_j(s)} \to e^{N-Ns} \quad \text{for a.e} \quad s \in [1, +\infty), \text{as} \quad j \to +\infty, \]
then, using the Lebesgue-dominated convergence theorem, we obtain
\[ \lim_{j \to +\infty} \int_1^j \exp\left(Ns\left(\frac{\log s}{\log j}\right)^{\frac{1}{\alpha}} - Ns\right) ds = \lim_{j \to +\infty} \int_1^j \chi_{\left[1, j^{\frac{1}{\alpha}}\right]}(s)e^{\psi_j(s)} ds = \frac{1}{N}. \]

Now, we are going to study the limit of this integral on \( \left[j^{\frac{1}{\alpha}}, j - j^{\frac{1}{\alpha}}\right] \) and \( \left[j - j^{\frac{1}{\alpha}}, j\right], \) so we compute
\[ \psi\left(j^{\frac{1}{\alpha}}\right) = -Nj^{\frac{1}{\alpha}}\left(1 - j^{\frac{1}{\alpha}}\right) \]
and
\[ \psi\left(j^{\frac{1}{\alpha}}\right) = -j^{\frac{1}{\alpha}} \quad \text{for all} \quad j \geq 4. \quad (17) \]

We have also
\[ \psi\left(j - j^{\frac{1}{\alpha}}\right) = N \exp\left(\frac{1}{\log^{\frac{1}{\alpha}} j} \left[\log j + \log\left(1 - j^{\frac{1}{\alpha}}\right)^{\frac{1}{\alpha}}\right]\right) - N\left(j - j^{\frac{1}{\alpha}}\right) \]
\[ = N \exp\left(\log j + \frac{\log\left(1 - j^{\frac{1}{\alpha}}\right)^{\frac{1}{\alpha}}}{\log j}\right) - N\left(j - j^{\frac{1}{\alpha}}\right) \]
\[ \begin{align*}
&= N \left[ \exp \left( \log j \left( 1 - N' \frac{j^{2^N-N}} {\log j} + o \left( \frac{j^{2^N-N}} {\log j} \right) \right) \right) - j \right] + N_j \frac{j} {2^N-N} \\
&= N_j \left[ \exp \left( -N' \frac{j^{2^N-N}} {\log j} + o \left( \frac{j^{2^N-N}} {\log j} \right) \right) - 1 \right] + N_j \frac{j} {2^N-N}.
\end{align*} \]

Therefore, for every \( \varepsilon \in (0, 1) \), there exists \( j_0 \geq 1 \) such that

\[ \psi_j \left( j - j^{2^N-N} \right) \leq N_j \frac{j} {2^N-N} \left( 1 - (1 - \varepsilon)N' \right) \quad \text{for every } j \geq j_0. \]  

(18)

Let \( j \) be fixed and large enough. A qualitative study conducted on \( \psi_j \) in \([1, +\infty)\) shows that there exists a unique \( s_j \in (1, j) \) such that the derivative of \( \psi_j(s_j) = 0 \) and consequently

\[ \int_{j^{2^N-N}}^{j^{2^N-N}} e^{\psi_j(s)} ds \leq \left( j - 2j^{2^N-N} \right) e^{\max \left( \psi_j \left( j^{2^N-N} \right), \psi_j \left( j - j^{2^N-N} \right) \right) } . \]

In addition, from (17) and (18) with \( \varepsilon = \frac{1} {N'} \), we obtain

\[ \max \left( \psi_j \left( j^{2^N-N} \right), \psi_j \left( j - j^{2^N-N} \right) \right) \leq -j^{2^N-N}, \]

as condition that \( j \) is large enough. Hence, there exists \( \tilde{j} \geq 1 \) such that

\[ \int_{j^{2^N-N}}^{j^{2^N-N}} e^{\psi_j(s)} ds \leq \left( j - 2j^{2^N-N} \right) e^{-j^{2^N-N}} \quad \text{for all } j \geq \tilde{j}. \]

Therefore,

\[ \lim_{j \to +\infty} \int_{j^{2^N-N}}^{j^{2^N-N}} \exp \left( Ne \left( \frac{\psi_j(s)} {2^N-N} \right) - Ns \right) ds = 0. \]

Finally, we will study the limit on the interval \( \left[ j - j^{2^N-N}, j \right] \). We mention that for a fixed \( j \geq 1 \) large enough, \( \psi_j \) is a convex function on \( \left[ j - j^{2^N-N}, +\infty \right) \), and \( \psi_j(j) = 0 \), so we can obtain this estimate

\[ \psi_j(s) \leq \frac{j - s} {j^{2^N-N}} \psi_j \left( j - j^{2^N-N} \right), \quad s \in \left[ j - j^{2^N-N}, j \right]. \]

On the other hand, in view of (17) and (18), if \( \varepsilon \in (0, \frac{1} {N'}) \) and \( j \geq j_0 \), we have

\[ \psi_j(s) \leq N(1 - (1 - \varepsilon)N')(j - s), \quad s \in \left[ j - j^{2^N-N}, j \right]. \]  

(19)

Furthermore, using the fact that \( \psi_j \) is convex on \( \left[ j - j^{2^N-N}, +\infty \right) \) and \( \psi_j(j) = N' \), we obtain

\[ \psi_j(s) \geq \phi_j(j)(s - j) = N'(s - j), \quad s \in \left[ j - j^{2^N-N}, j \right]. \]

(20)

Then, by bringing together (19) and (20), we deduce

\[ \frac{1} {N'} \leq \lim_{j \to +\infty} \int_{j - j^{2^N-N}}^{j} e^{\psi_j(s)} ds \leq \frac{1} {N(1 - (1 - \varepsilon)N')}. \]

By tending \( \varepsilon \) to zero, we obtain
\[
\lim_{j \to +\infty} \int_{j^\frac{1}{N-1}} \exp \left( \frac{\mu_{N-1}}{N} - Ns \right) \, ds = \frac{1}{N}.
\]

So, our claim is proved and the lemma follows. \(\square\)

Finally, we give the desired estimate.

**Lemma 7.** Assume that if \((G_1), (G_2), (H_1), (H_2),\) and \((H_4)\) hold, then

\[
d < \frac{1}{N} G \left( \frac{\omega_{N-1}}{a_0^{-1}} \right).
\]

**Proof.** We have \(v_0 \geq 0\) and \(\|v_0\| = 1\). Then, from Lemma 5, \(J(tv_0) \to -\infty\) as \(t \to +\infty\). As a consequence,

\[
d \leq \max_{t \geq 0} J(tv_0).
\]

We argue by contradiction and suppose that for all \(n \geq 1\),

\[
\max_{t \geq 0} J(tv_0) \geq \frac{1}{N} G \left( \frac{\omega_{N-1}}{a_0^{-1}} \right).
\]

Since \(J\) possesses the mountain pass geometry, for any \(n \geq 1\), there exists \(t_n > 0\) such that

\[
\max_{t \geq 0} J(tv_0) = J(t_nv_0) \geq \frac{1}{N} G \left( \frac{\omega_{N-1}}{a_0^{-1}} \right).
\]

Using the fact that \(F(x, t) \geq 0\) for all \((x, t) \in B \times \mathbb{R}\), we obtain

\[
G(t_nv_0) \geq \frac{1}{N} G \left( \frac{\omega_{N-1}}{a_0^{-1}} \right).
\]

On one hand, the condition \((G_1)\) implies that \(G : [0, +\infty) \to [0, +\infty)\) is an increasing bijection. So

\[
t_n \geq \frac{\omega_{N-1}}{a_0^{-1}}. \tag{21}
\]

On the other hand,

\[
\frac{d}{dt} f(tv_0) \bigg|_{t=t_n} = g(t_n) t_n^{N-1} - \int_B f(x, t_nv_0) v_0 \, dx = 0,
\]

that is,

\[
g(t_n) t_n^{N-1} = \int_B f(x, t_nv_0) v_0 \, dx. \tag{22}
\]

Now, we claim that the sequence \(t_n\) is bounded in \((0, +\infty)\).

Indeed, it follows from \((H_4)\) that for all \(\varepsilon > 0\), there exists \(t_\varepsilon > 0\) such that

\[
f(x, t| t \geq (\psi_0 - \varepsilon) \exp(\mu_{N-1} t) \quad \forall \|x\| \geq t_\varepsilon \quad \text{uniformly in } x \in B. \tag{23}
\]

From (16) and (22), we have

\[
g(t_n) t_n^{N-1} = \int_B f(x, t_nv_0) v_0 \, dx \geq \omega_{N-1} \int_n^{+\infty} f \left( e^{\varepsilon}, t_n \frac{\psi_0}{\omega_{N-1}} \right) t_n \frac{\psi_0}{\omega_{N-1}} \, e^{-N\varepsilon} \, ds.
\]
Also,

\[ t_n \frac{\psi_n}{\omega^{N-1}_n} = t_n \left( \frac{\log(1 + n)}{\omega^{N-1}_n} \right)^{\frac{1}{\rho'}} \geq \left( \frac{\log(1 + n)}{\alpha_0} \right)^{\frac{1}{\rho'}}, \]

then, it follows from (23) that for all \( \varepsilon > 0 \), there exists \( n_0 \) such that for all \( n \geq n_0 \)

\[ g(t_n^N) t_n^N \geq \omega_{N-1} \varepsilon \int \frac{\omega_{N-1}}{\omega_{N-1}} \exp \left( \frac{\alpha_0}{\omega_{N-1}^N} \right) ds, \]

that is,

\[ g(t_n^N) t_n^N \geq \frac{\omega_{N-1}}{N} (\varepsilon - \varepsilon) \exp \left( \frac{\alpha_0}{\omega_{N-1}^N} \log(1 + n) \right). \]

Using the condition \((G_2)\), we obtain

\[ g(1) t_n^N \geq \frac{\omega_{N-1}}{N} (\varepsilon - \varepsilon) \exp \left( \frac{\alpha_0}{\omega_{N-1}^N} \log(1 + n) \right). \]

From (25), we obtain for \( n \) large enough

\[ 1 \geq \frac{\omega_{N-1}}{N} (\varepsilon - \varepsilon) \exp \left( \frac{\alpha_0}{\omega_{N-1}^N} \log(1 + n) \right). \]

Therefore, \((t_n)\) is bounded in \( R \). Now, suppose that

\[ \lim_{n \to +\infty} t_n^N > \frac{\omega_{N-1}}{\alpha_0^{N-1}}. \]

For \( n \) large enough, \( t_n^N > \frac{\omega_{N-1}}{\alpha_0^{N-1}} \) and in this case, the right-hand side of inequality (25) will give the unboundedness of the sequence \((t_n)\). Since \((t_n)\) is bounded, we obtain

\[ \lim_{n \to +\infty} t_n^N = \frac{\omega_{N-1}}{\alpha_0^{N-1}}. \]

Now, we are going to estimate the expression in (22). So let

\[ B_{N^+} = \{ x \in B ; \quad t_n \varepsilon_n(x) \geq t_c \} \quad \text{and} \quad B_{N^-} = \{ x \in B ; \quad t_n \varepsilon_n(x) < t_c \}. \]

We have

\[ g(t_n^N) t_n^N \geq (\varepsilon - \varepsilon) \int_{B_{N^+}} \exp(\omega_{N-1} \varepsilon_n) dx + \int_{B_{N^-}} f(x, t_n \varepsilon_n) t_n \varepsilon_n dx, \]

then

\[ g(t_n^N) t_n^N \geq (\varepsilon - \varepsilon) \int_B \exp(\omega_{N-1} \varepsilon_n) dx - (\varepsilon - \varepsilon) \int_{B_{N^-}} \exp(\omega_{N-1} \varepsilon_n) dx + \int_{B_{N^-}} f(x, t_n \varepsilon_n) t_n \varepsilon_n dx. \]

The sequence \((\varepsilon_n)\) converges to 0 in \( B \) and \( \chi_{B_{N^-}} \) converges to 1 a.e. in \( B \). By using the dominated convergence theorem, we obtain

\[ \lim_{n \to +\infty} \int_{B_{N^-}} f(x, t_n \varepsilon_n) t_n \varepsilon_n dx = 0 \]
and
\[
\lim_{n \to +\infty} \int_{B_{n-1}} \exp(\alpha N_{\alpha}^{\omega N}) dx \leq \frac{\alpha N_{\alpha}^{\omega N}}{N} e^N.
\]

We also have
\[
\lim_{n \to +\infty} \int_{B} \exp\left(\alpha N_{\alpha}^{\omega N}\right) dx = \lim_{n \to +\infty} \int_{0}^{+\infty} \exp(\alpha N_{\alpha}^{\omega N} - N t) dt.
\]

Then using (21) and Lemma 6, we obtain
\[
\lim_{n \to +\infty} \int_{B} \exp(\alpha N_{\alpha}^{\omega N}) dx \geq \lim_{n \to +\infty} \int_{B} \exp\left(\alpha N_{\alpha}^{\omega N}\right) dx = \alpha N_{\alpha}^{\omega N} \left(\frac{N + 1}{N}\right) e^N.
\]

Passing to the limit in (26), we obtain
\[
\left(\frac{\alpha N_{\alpha}^{\omega N}}{a_0^{\omega N}}\right) \left(\frac{\alpha N_{\alpha}^{\omega N}}{a_0^{\omega N}}\right) \geq (Y_0 - \epsilon) \alpha N_{\alpha}^{\omega N} e^N,
\]

for all \(\epsilon > 0\). So,
\[
Y_0 \leq \frac{1}{a_0^{\omega N} e^N} \left(\frac{\alpha N_{\alpha}^{\omega N}}{a_0^{\omega N}}\right),
\]

which contradicts the condition \((H_6)\). Hence, the lemma is proved. \(\square\)

5 Proof of main results

First, we begin by some crucial lemmas.

Now, we consider the Nehari manifold associated with the functional \(J\), namely,
\[
\mathcal{N} = \{u \in \mathcal{W} : J'(u)u = 0, u \neq 0\},
\]
and the number \(c = \inf_{u \in \mathcal{N}} J(u)\). We have the following lemmas.

Lemma 8. Assume that the condition \((H_3)\) holds, then for each \(x \in B\),
\[
tf(x, t) - 2NF(x, t) \text{ is increasing for } t \geq 0.
\]

Proof. Assume that \(0 < t < s\). For each \(x \in B\), we have
\[
tf(x, t) - 2NF(x, t) = \frac{f(x, t)}{t^{2N-1}} t^{2N} - 2NF(x, s) + 2N \int_{t}^{s} f(x, v) dv
\]
\[
< \frac{f(x, s)}{s^{2N-1}} s^{2N} - 2NF(x, s) + \frac{f(x, s)}{s^{2N-1}} (s^{2N} - t^{2N})
\]
\[
= sf(x, s) - 2NF(x, s). \quad \square
\]

Lemma 9. If \((G_2)\) and \((H_3)\) are satisfied, then \(d \leq c\).

Proof. Let \(\bar{u} \in \mathcal{N}, \bar{u} > 0\) and consider the function \(\psi : (0, +\infty) \to \mathbb{R}\) defined by \(\psi(t) = J(t\bar{u})\). \(\psi\) is differentiable and we have
We have \( g(||u||^N)||u||^N = \int_B f(x, u) dx \). Hence, 
\[
\psi'(t) = t^{2N-||u||^N} \left( g(T^N ||u||^N) - g(||u||^N) \right) + t^{2N-1} \int_B \left( f(x, u) - f(x, tu) \right) ||u||^N dx.
\]

We have that \( \psi'(1) = 0 \). We also have by conditions \((G_2)\) and \((H_2)\) that \( \psi'(t) > 0 \) for all \( 0 < t < 1 \) and \( \psi'(t) < 0 \) for all \( t > 1 \). It follows that 
\[
\mathcal{J}(\bar{u}) = \max_{t \geq 0} \mathcal{J}(t\bar{u}).
\]

We define the function \( \lambda : [0, 1] \to \mathcal{W} \) such that \( \lambda(t) = t\bar{u} \), with \( \mathcal{J}(t\bar{u}) < 0 \). We have \( \lambda \in \Lambda \), and hence 
\[
d \leq \max_{t \in [0,1]} \mathcal{J}(\lambda(t)) = \max_{t \geq 0} \mathcal{J}(t\bar{u}) = \mathcal{J}(\bar{u}).
\]

Since \( \bar{u} \in \mathcal{N} \) is arbitrary, then \( d \leq c \). \(\Box\)

### 5.1 Proof of Theorems 1.2 and 1.3

Since \( \mathcal{J} \) possesses the mountain pass geometry, there exists \( u_n \in \mathcal{W} \) such that 
\[
\mathcal{J}(u_n) = \frac{1}{N} G(||u_n||^N) - \int_B F(x, u_n) dx \to d, \quad n \to +\infty
\]
and 
\[
|\mathcal{J}'(u_n)\phi| = |g(||u_n||^N) \int_B \sigma(x)|\nabla u_n|^N-2 \nabla \phi dx - \int_B f(x, u_n)\phi dx| \leq \varepsilon_n ||\phi||, \quad \phi \in \mathcal{W}, \quad \varepsilon_n \to 0, \quad n \to +\infty.
\]

By (27), for all \( \varepsilon > 0 \), there exists a constant \( C > 0 \) 
\[
\frac{1}{N} G(||u_n||^N) \leq C + \int_B F(x, u_n) dx.
\]

From (8), we have 
\[
\frac{1}{N} G(||u_n||^N) \leq C + \int_{|u_n| \leq t_\varepsilon} F(x, u_n) dx + \varepsilon \int_B f(x, u_n) u_n dx.
\]

From (28) and (10), we obtain 
\[
\frac{1}{2^N} G(||u_n||^N) ||u_n||^N \leq \frac{1}{N} G(||u_n||^N) \leq C_1 + \varepsilon \varepsilon_n ||u_n|| + \varepsilon g(||u_n||^N) ||u_n||^N,
\]
for some constant \( C_1 > 0 \). Using the condition \((G_1)\), for all \( \varepsilon \) such that \( 0 < \varepsilon < \frac{1}{2^N} \) we obtain 
\[
g \left( \frac{1}{2^N} - \varepsilon \right) ||u_n||^N \leq C_1 + \varepsilon \varepsilon_n ||u_n||,
\]
and we deduce that the sequence \((u_n)\) is bounded in \( \mathcal{W} \). As a consequence, there exists \( u \in \mathcal{W} \) such that, up to subsequence, \( u_n \to u \) weakly in \( \mathcal{W} \), \( u_n \to u \) strongly in \( L^q(B) \), for all \( q \geq 1 \).
Furthermore, we have from (28) and (8) that

\[ 0 < \int_B f(x, u_n)u_n \leq C \]

and

\[ 0 < \int_B F(x, u_n) \leq C. \]

Since by Lemma 2, we have

\[ f(x, u_n) \to f(x, u) \quad \text{in } L^1(B) \quad \text{as } n \to +\infty, \]

then, it follows from \((H_2)\) and the generalized Lebesgue-dominated convergence theorem that

\[ F(x, u_n) \to F(x, u) \quad \text{in } L^1(B) \quad \text{as } n \to +\infty. \]

So,

\[ \lim_{n \to +\infty} G(\|u_n\|^N) = N(d + \int_B F(x, u)dx). \]

Next, we are going to make some claims. **Claim 1.** \(\nabla u_n(x) \to \nabla u(x)\) a.e \(x \in B\). Indeed, for any \(\eta > 0\), let \(\mathcal{A}_\eta = \{x \in B, |u_n - u| \geq \eta\}\). For all \(t \in \mathbb{R}\), for all positive \(c > 0\), we have

\[ ct \leq Ne^t + \frac{c^2}{N}. \]

It follows that for \(t = \omega N^{\frac{1}{N} - 1} \left(\frac{|u_n - u|}{|u_n - u|}\right)^N\), \(c = \frac{1}{N^{\frac{1}{N} - 1}}\|u_n - u\|^N\), we obtain

\[ |u_n - u|^N \leq Ne^{\frac{2}{N^{\frac{1}{N} - 1}}} \left(\frac{|u_n - u|}{|u_n - u|}\right)^N + \frac{1}{N} \|u_n - u\|^2 \leq Ne^{\frac{1}{N^{\frac{1}{N} - 1}}} \left(\frac{|u_n - u|}{|u_n - u|}\right)^N + C_1(N), \]

where \(C_1(N)\) is a constant depending only on \(N\) and the upper bound of \(\|u_n\|\). So, if we denote by \(L(\mathcal{A}_\eta)\) the Lebesgue measure of the set \(\mathcal{A}_\eta\), we obtain

\[ L(\mathcal{A}_\eta) = \int_{\mathcal{A}_\eta} e^{\|u_n - u\|^N} e^{-|u_n - u|^N} dx \leq e^{-\|u_n\|^N} \int_{\mathcal{A}_\eta} \exp \left( Ne^{\frac{1}{N^{\frac{1}{N} - 1}}} \left(\frac{|u_n - u|}{|u_n - u|}\right)^N + C_1(N) \right) dx \]

\[ \leq e^{-\|u_n\|^N} e^{C_1(N)} \int_{\mathcal{B}} \exp \left( Ne^{\frac{1}{N^{\frac{1}{N} - 1}}} \left(\frac{|u_n - u|}{|u_n - u|}\right)^N \right) dx \]

\[ \leq e^{-\|u_n\|^N} C_2(N) \to 0 \quad \text{as } \eta \to +\infty, \]

where \(C_2(N)\) is a positive constant depending only on \(N\) and the upper bound of \(\|u_n\|\). It follows that

\[ \int_{\mathcal{A}_\eta} |\nabla u_n - \nabla u|dx \leq Ce^{-\frac{1}{N^{\frac{1}{N} - 1}}} \left( \int_{\mathcal{B}} |\nabla u_n - \nabla u|^2 \sigma(x)dx \right)^{\frac{1}{2}} \to 0 \quad \text{as } \eta \to +\infty. \]

We define for \(\eta > 0\), the truncation function used in [26]

\[ T_\eta(s) = \begin{cases} s & \text{if } |s| < \eta \\ \eta - \frac{s}{|s|} & \text{if } |s| \geq \eta. \end{cases} \]
If we take $\varphi = T_\eta(u_n - u) \in \mathcal{W}$, then $\nabla \varphi = \chi_{B \setminus \mathcal{A}_\eta} \nabla (u_n - u)$. Considering $\varphi$ in (28) we obtain

$$
\left| g(\|u_n\|^N) \int_{B \setminus \mathcal{A}_\eta} \sigma(x)[|\nabla u_n|^{N-2}\nabla u_n - |\nabla u|^{N-2}\nabla u]. (\nabla u_n - \nabla u) dx \right|
\leq \left| g(\|u_n\|^N) \int_{B \setminus \mathcal{A}_\eta} \sigma(x)|\nabla u_n|^{N-2}\nabla u. (\nabla u_n - \nabla u) dx \right| + \int_B f(x, u_n) T_\eta(u_n - u) dx + \varepsilon_n\|T_\eta(u_n - u)\|.$$

Since $u_n \rightharpoonup u$ weakly in $\mathcal{W}$, then $\left| \int_B f(x, u_n) T_\eta(u_n - u) dx \right| \to 0$. By (29) and the Lebesgue-dominated convergence theorem, we obtain

$$
\int_B f(x, u_n) T_\eta(u_n - u) dx \to 0 \text{ as } n \to +\infty.
$$

Using the well-known inequality,

$$
\langle |x|^{N-2} x - |y|^{N-2} y, x - y \rangle \geq 2^{2-N} |x - y|^N \quad \forall x, y \in \mathbb{R}^N, \quad N \geq 2,
$$

$\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^N$ and the fact that $0 < g_0 \leq g(\|u_n\|^N)$, one has

$$
\int_{B \setminus \mathcal{A}_\eta} \sigma(x)|\nabla u_n - \nabla u|^{N} dx \to 0.
$$

Therefore,

$$
\int_{B \setminus \mathcal{A}_\eta} |\nabla u_n - \nabla u|^{N} dx \leq \left( \int_{B \setminus \mathcal{A}_\eta} \sigma(x)|\nabla u_n - \nabla u|^{N} dx \right)^{\frac{1}{N}} \left( \mathcal{L}(B \setminus \mathcal{A}_\eta) \right)^{\frac{1}{N}} \to 0 \text{ as } n \to +\infty. \quad (33)
$$

From (32) and (33), we deduce that

$$
\int_B |\nabla u_n - \nabla u| dx \to 0 \text{ as } n \to +\infty.
$$

Therefore, $\nabla u_n(x) \to \nabla u(x)$ a.e $x \in B$ and claim 1 is proved.

**Claim 2.** At this stage, we affirm that $u \neq 0$. Indeed, we argue by contradiction and suppose that $u \equiv 0$.

Therefore, $\int_B F(x, u_n) dx \to 0$, and consequently, we obtain

$$
\frac{1}{N} G(\|u_n\|^N) \to d < \frac{1}{N} G\left( \frac{w_{N-1}}{a_0^{N-1}} \right). \quad (34)
$$

First, we claim that there exists $q > 1$ such that

$$
\int_B |f(x, u_n)|^{q} dx \leq C. \quad (35)
$$

By (28), we have

$$
\left| g(\|u_n\|^N)\|u_n\|^N - \int_B f(x, u_n) u_n dx \right| \leq C\varepsilon_n.
$$
So,
\[ g(\|u_n\|^N}\|u_n\|^N) \leq C\epsilon_n + \left( \int_B |f(x, u_n)|^q \right)^{\frac{1}{q}} \left( \int_B |u_n|^q \right)^{\frac{1}{q}}, \]
where \( q' \) is the conjugate of \( q \). Since \((u_n)\) converges to 0 in \( L^q(B) \)
\[ \lim_{n \to +\infty} g(\|u_n\|^N\|u_n\|^N) = 0. \]
From the condition \((G_1)\), we obtain
\[ \lim_{n \to +\infty} \|u_n\|^N = 0, \]
then \( u_n \to 0 \) in \( \mathcal{W} \). Therefore, \( \mathcal{J}(u_n) \to 0 \), which is in contradiction with \( d > 0 \).

For the proof of claim (35), since \( f \) has critical growth, for every \( \epsilon > 0 \) and \( q > 1 \), there exists \( t_\epsilon > 0 \) and \( C > 0 \) such that for all \( |t| \geq t_\epsilon \), we have
\[ |f(x, t)|^q \leq C \exp\left( Ne^{\alpha_0(1+\epsilon)|t|^N}\right). \]

Consequently,
\[ \int_B |f(x, u_n)|^q \, dx = \int_{|u_n| \leq \left(\frac{\alpha_0}{N}\right)} |f(x, u_n)|^q \, dx + \int_{|u_n| > \left(\frac{\alpha_0}{N}\right)} |f(x, u_n)|^q \, dx \]
\[ \leq \omega_{N-1} \max_{|t| = \left(\frac{\alpha_0}{N}\right)} |f(x, t)|^q + C \int_B \exp(\alpha_0(1+\epsilon)|u|^N) \, dx. \]

Since \((G^{-1}(Nd))^{\frac{1}{N-1}} < \frac{\alpha_0}{N}\), there exists \( \eta \in (0, \frac{1}{2}) \) such that \((G^{-1}(Nd))^{\frac{1}{N-1}} = (1 - 2\eta)\frac{\alpha_0}{N}\). From (34), \( \|u_n\|^N \to (G^{-1}(Nd))^{\frac{1}{N-1}} \), so there exist \( n_\eta \in \mathbb{N} \) such that \( \alpha_0\|u_n\|^N \leq (1 - \eta)\omega_{N-1}^{1/N} \), for all \( n \geq n_\eta \). Therefore,
\[ \alpha_0(1 + \epsilon) \left( \frac{|u_n|}{\|u_n\|} \right)^N \|u_n\|^N \leq (1 + \epsilon)(1 - \eta) \left( \frac{|u_n|}{\|u_n\|} \right)^N \omega_{N-1}^{1/N}. \]

We choose \( \epsilon > 0 \) small enough to obtain
\[ (1 + \epsilon)(1 - \eta) < 1, \]
hence the second integral is uniformly bounded in view of (7).

**Claim 3.** \( g(\|u\|^N)\|u\|^N \geq \int_B f(x, u) \, du \). We proceed by contradiction and we suppose that \( g(\|u\|^N)\|u\|^N < \int_B f(x, u) \, du \). Hence, \( \mathcal{J}'(u) < 0 \). The function \( \psi : t \to \psi(t) = \mathcal{J}'(tu)u \) is positive for \( t \) small enough. Indeed, from (9) and the critical (resp subcritical) growth of the nonlinearity \( f \), for every \( \epsilon > 0 \), for every \( q > N \), there exist positive constants \( C \) and \( \alpha_0 \) such that
\[ |f(x, t)| \leq C|t^{N-1} + C|t^q \exp\left(e^{\frac{C|t|^N}{N}}\right), \quad \forall (t, x) \in \mathbb{R} \times B. \]

Then, using the condition \((G_1)\), the last inequality, and the Hölder inequality, we obtain
\[ \psi(t) = g(t^N\|u\|^N)t^{N-1}\|u\|^N - \int_B f(x, tu) \, du \]
\[ \geq g_0 t^{N-1}\|u\|^N - \epsilon t^{N-1} \int_B u^N \, dx - C \left( \int_B \exp\left( Ne^{\alpha_0\|u\|^N}\right) \, dx \right) \left( \int_B u^N \, dx \right)^{\frac{1}{N}}. \]
In view of (7) the integral \( \int_B \exp(Ne^{\alpha N u^p}) \, dx = \int_B \exp \left( Ne^{\frac{\alpha N}{e^{\alpha N/|u|}} |u|^p} \right) \, dx < \infty \), provided \( t \leq \frac{\alpha N}{e^{\alpha N/|u|}} \). Using the radial Lemma 1, we obtain \( ||u||^p_N \leq C ||u||^q \). Then,

\[
\psi(t) \geq g_0 t^{N-1} ||u||^N - C_0 t^N - ||u||^N - C_0 ||u||^q = ||u||^N t^{N-1} \left( g_0 - C_0 t^{q-(N-1)} ||u||^{q-N} \right).
\]

We choose \( \varepsilon > 0 \), such that \( g_0 - C_0 \varepsilon > 0 \) and since \( \eta \in (0, 1) \) such that \( \psi(\eta u) = 0 \). Therefore, \( \eta u \in \mathcal{N} \). Using (10), the result of Lemma 7, the semicontinuity of norm, and Fatou’s lemma, we obtain

\[
\begin{aligned}
\int_B \int_B |\nabla u|^N |\nabla v|^N &= \int_B \int_B f(x, u) v \, dx \\
&\geq \liminf_{n \to \infty} \left[ \frac{1}{N} G(||u||^N) - \frac{1}{2N} G(||u||^N) ||u||^N + \frac{1}{2N} \int_B (f(x, u) u_n - 2NF(x, u_n)) \, dx \right] \\
&\leq \lim_{n \to \infty} \left[ \mathcal{J}(u_n) - \frac{1}{2N} \mathcal{J}'(u_n) u_n \right] = d,
\end{aligned}
\]

which is absurd and Claim 3 is well established.

**Claim 4.** \( u > 0 \). Indeed, since \((u_n)\) is bounded, up to a subsequence, \( ||u|| \to \rho > 0 \). In addition, \( \mathcal{J}'(u_n) \to 0 \) leads to

\[
g(\rho |u|) \int_B |\nabla u|^N |\nabla v|^N |v|^2 \, dx = \int_B f(x, u) v \, dx, \quad \forall v \in \mathcal{W}.
\]

By taking \( v = u^r \), with \( w = \max(\pm w, 0) \), we obtain \( ||u||^N = 0 \) and so \( u = u^r \geq 0 \). Since the nonlinearity has critical growth at \(+\infty\) and from the Trudinger-Moser inequality (7), \( f(.,u) \in L^p(B) \), for all \( p \geq 1 \). So, by elliptic regularity \( u \in W^{2,p}(B, \sigma) \), for all \( p \geq 1 \). Therefore, by Sobolev imbedding \( u \in C^{1,\gamma}(B) \).

Let us define \( B_0 = \{ x \in B : u(x) = 0 \} \). The set \( B_0 = \emptyset \). Indeed, suppose by contradiction that \( B_0 \neq \emptyset \). Since \( f(x, u) \geq 0 \), by the Harnack inequality (see [16], Theorem 1.9), we can deduce that \( B_0 \) is an open and closed set of \( B \). In virtue of the connectedness of \( B \), we reach a contradiction. Hence, Claim 4 is proved.

We affirm that \( \mathcal{J}(u) = d \). Indeed, by Claim 3, (10), and Lemma 8, we obtain

\[
\mathcal{J}(u) \geq \frac{1}{N} G(||u||^N) - \frac{1}{2N} G(||u||^N) ||u||^N + \frac{1}{2N} \int_B (f(x, u) - 2NF(x, u)) \, dx \geq 0.
\]

(36)

Now, using the semicontinuity of the norm and (30), we obtain,

\[
\mathcal{J}(u) \leq \frac{1}{N} \liminf_{n \to \infty} G(||u_n||^N) - \int_B F(x, u) \, dx = d.
\]

Suppose that

\[
\mathcal{J}(u) < d.
\]

Then

\[
||u||^N < \rho^N.
\]

(37)

In addition,

\[
\frac{1}{N} G(\rho^N) = \frac{1}{N} \lim_{n \to \infty} G(||u_n||^N) = \left( d + \int_B F(x, u) \, dx \right).
\]

(38)
which means that

\[ \rho^N = G^{-1} \left( Nd + N \int_{B} F(x, u) \, dx \right). \]

Set

\[ v_n = \frac{u_n}{\| u_n \|} \quad \text{and} \quad v = \frac{u}{\rho}. \]

We have \( |v_n| = 1, v_n \rightarrow v \) in \( W, v \neq 0, \) and \( |v| < 1. \) So, by Lemma 3, we obtain

\[ \sup_n \int_{B} \exp \left( Ne^\frac{v_n}{\rho^N} \right) \, dx < \infty, \]

for \( 1 < p < (1 - |v|^N) \frac{1}{\pi^2}. \)

By (27), (30), and (38), we have the following equality:

\[ Nd - NF(u) = G(\rho^N) - G(|u|^N). \]

From (36), Lemma 7, and the last equality, we obtain

\[ G(\rho^N) \leq Nd + G(|u|^N) < G \left( \frac{\omega_{N-1}}{a_0^{N-1}} \right) + G(|u|^N). \]

Now, using the condition (G), one has

\[ \rho^N < G^{-1} \left( G \left( \frac{\omega_{N-1}}{a_0^{N-1}} \right) + G(|u|^N) \right) \leq \frac{\omega_{N-1}}{a_0^{N-1}} + |u|^N. \quad (39) \]

Since

\[ \rho^N = \left( \frac{\rho^N - |u|^N}{1 - |v|^N} \right)^{\frac{1}{N-1}}, \]

we deduce from (39) that

\[ \rho^N < \left( \frac{\omega_{N-1}}{a_0^{N-1}} \right)^{\frac{1}{N-1}}. \quad (40) \]

On one hand, we have this estimate \( \int_{B} |f(x, u_n)|^q \, dx < C. \) Indeed, for \( \varepsilon > 0, \)

\[ \int_{B} |f(x, u_n)|^q \, dx = \int_{\{ |u_n| \leq \varepsilon \}} |f(x, u_n)|^q \, dx + \int_{\{ |u_n| > \varepsilon \}} |f(x, u_n)|^q \, dx \]

\[ \leq \omega_{N-1} \max_{B \setminus [-\varepsilon, \varepsilon]} |f(x, \ell)|^q + C \int_{B} \exp(\alpha |u_n|) \, dx \]

\[ \leq C_\varepsilon + C \int_{B} \exp(\alpha |u_n| |u_m|) \, dx \leq C \]

if we have \( \alpha_0 (1 + \varepsilon) |u_n| \leq \frac{1}{\pi^2}, \) with \( 1 < p < (1 - |v|^N) \frac{1}{\pi^2}. \)

From (40), there exists \( \delta \in (0, \frac{1}{2}) \) such that \( \rho^N = (1 - 2\delta) \left( \frac{\omega_{N-1}}{a_0^{N-1}} \right)^{\frac{1}{N-1}}. \)
Since \( \lim_{n \to \infty} ||u_n||^N = \rho^N \) then, for \( n \) large enough
\[
a_0(1 + \varepsilon)||u_n||^N \leq (1 + \varepsilon)(1 - \delta)||u_n||^N \left( \frac{1}{1 - ||u||^N} \right)^{\frac{1}{N-1}}.
\]
We choose \( \varepsilon > 0 \) small enough such that \( (1 + \varepsilon)(1 - \delta) < 1 \), which means
\[
a_0(1 + \varepsilon)||u_n||^N < \omega_{N-1}^N \left( \frac{1}{1 - ||u||^N} \right)^{\frac{1}{N-1}}
\]
and so, the sequence \( (f(x, u_n)) \) is bounded in \( L^q \), \( q > 1 \). Using the Hölder inequality, we deduce that
\[
\left| \int_B f(x, u_n)(u_n - u) \, dx \right| \leq \left( \int_B |f(x, u_n)|^q \, dx \right)^\frac{1}{q} \left( \int_B |u_n - u|^q \, dx \right)^\frac{1}{q} \\
\leq C \left( \int_B |u_n - u|^q \, dx \right)^\frac{1}{q} \to 0 \text{ as } n \to +\infty,
\]
where \( \frac{1}{q} + \frac{1}{q'} = 1 \). Since \( J'(u_n)(u_n - u) = o_n(1) \), it follows that
\[
g(||u||^N) \int_B (\sigma(x)|\nabla u_n|^{N-2} \nabla u_n \cdot (\nabla u_n - \nabla u)) \, dx \to 0.
\]
On the other side,
\[
g(||u||^N) \int_B \sigma(x)|\nabla u_n|^{N-2} \nabla u_n \cdot (\nabla u_n - \nabla u) \, dx \\
= g(||u||^N)||u||^N - g(||u||^N) \int_B \sigma(x)|\nabla u_n|^{N-2} \nabla u_n \cdot \nabla u \, dx.
\]
Passing to the limit in the last equality, we obtain
\[
g(\rho^N)\rho^N - g(\rho^N)||u||^N = 0,
\]
therefore \( ||u|| = \rho \). This is in contradiction with (37). Therefore, \( J(u) = d \). So, \( u \) is a solution of problem (1). The proof of Theorem 1.3 is complete.

**Proof of Theorem 1.2.** In the subcritical case, since \( u_n \) is bounded, there exist \( M > 0 \) and subsequences such that
\[
\begin{align*}
||u_n|| &\leq M \quad \text{in } \mathcal{W} \\
u_n &\rightharpoonup u \quad \text{weakly in } \mathcal{W} \\
u_n &\rightarrow u \quad \text{strongly in } L^q(B) \quad \forall q \geq 1 \\
u_n(x) &\rightarrow u(x) \quad \text{almost everywhere in } B.
\end{align*}
\]
Since \( f \) is subcritical at \( +\infty \), there exists a constant \( C_M > 0 \) such that
\[
f(x, s) \leq C_M \exp\left( \frac{s^\frac{1}{N-1}}{w^\frac{1}{N-1}} \right), \quad \forall (x, s) \in B \times (0, +\infty).
\]
Using the Hölder inequality
It is easy to check that \( J(u) = d \). Also, \( u \) is a solution of (1). This completes the proof of Theorem 1.2. \( \square \)

**Remark 5.1.** The solution \( u \) is also called a ground state solution of problem (1).

**Remark 5.2.** By a slight modification of the previous proof, we can prove that the functional \( J \) satisfies the Palais-Smale condition at all levels \( d \in \mathbb{R} \) for the subcritical case. However, in the critical case, \( J \) satisfies the Palais-Smale condition at all levels \( d < \frac{1}{N} \left( \frac{\omega_{N-1}}{\omega_0} \right)^{\frac{N}{N-1}} \).

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