A rigorous treatment of
the lattice renormalization problem of $f_B$

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Abstract
The $B$-meson decay constant can be measured on the lattice using a $1/m_b$ expansion. To relate the physical quantity to Monte Carlo data one has to know the renormalization coefficient, $Z$, between the lattice operators and their continuum counterparts. We come back to this computation to resolve discrepancies found in previous calculations. We define and discuss in detail the renormalization procedure that allows the (perturbative) computation of $Z$. Comparing the one-loop calculations in the effective Lagrangian approach with the direct two-loop calculation of the two-point $B$-meson correlator in the limit of large $b$-quark mass, we prove that the two schemes give consistent results to order $\alpha_s$. We show that there is, however, a renormalization prescription ambiguity that can have sizeable numerical consequences. This ambiguity can be resolved in the framework of an $O(a)$ improved calculation, and we describe the correct prescription in that case. Finally we give the numerical values of $Z$ that correspond to the different types of lattice approximations discussed in the paper.

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1. Introduction

The study of heavy flavours has attracted a lot of interest in the last few years, both from the experimental and theoretical point of view. Many interesting properties of matrix elements can be derived on rather general assumptions, using the spin symmetry that emerges in the limit in which the mass of the heavy quark goes to infinity [1], [2].

Beyond symmetry arguments, the only method based on first principles that can be used to predict physical quantities, such as the $B$-meson decay constant, or the universal form factor for current matrix elements between heavy mesons, is lattice QCD, complemented with a $1/m_b$ expansion of the heavy-quark propagator, as proposed in ref. [1].

In this scheme, the heavy-quark field is no longer explicitly present on the lattice, but it is replaced by a static colour source, and one has only to consider correlation functions with gauge and light-quark fields. The latter ones can be evaluated in lattice QCD by Monte Carlo simulations in the standard way. This method is the only one applicable to the $B$ meson since the $b$ quark is too heavy to live even on lattices with the smallest lattice spacing presently attainable in numerical simulations ($a^{-1}$ of the order of 3 to 4 GeV). In the phenomenological analysis, one then has to interpolate between the infinite mass limit computed in this way and the results obtained with the standard approach for quarks lighter than $a^{-1}$ [3].

From now on we will concentrate on the method based on the infinite mass limit. We will call it for short in the following the static quark approach.

As usual, we have to deal with the problem of lattice renormalization to relate numbers effectively measured on the lattice to the corresponding physical quantities. Two methods have been used for the calculation of the necessary renormalization constants [4], [5]. Apparently the two approaches have led to different numerical results. One of the purposes of this paper is to understand and to solve this discrepancy.

Let us focus on the computation of $f_B$, the decay constant of a $B$-meson. From the current-current correlation function:

$$P(t) = \int dx \, \langle J_{qb}^\dagger(0,0) J_{qb}(x,t) \rangle,$$  \hspace{1cm} (1.1)

where

$$J_{qb}(x) = \bar{b}(x)\gamma_0\gamma_5 q(x),$$ \hspace{1cm} (1.2)
we can extract the value of $f_B$ by looking at its large $t$ behaviour:

$$P(t) \sim \frac{1}{2M_B} |<0|J_{qb}(0,0)|\bar{B}>|^2 e^{-M_B t},$$

(1.3)

where $M_B$ is the $B$-meson mass.

The presence of the heavy $b$ field prevents a direct lattice evaluation of the expectation value in (1.1). The nice idea of ref. [1] is to replace, in the limit $m_b \to \infty$, the $b$ and $\bar{b}$ fields in eq. (1.1) by the static $b$ propagator:

$$S_b(x', 0; x, t) = \mathcal{P}_x \left( \frac{t}{0} \right) \delta(x - x') \left[ \theta(t) \; e^{-\frac{1}{2} (M_b t + \gamma_0)} + \theta(-t) \; e^{\frac{1}{2} (M_b t + \gamma_0)} \right],$$

(1.4)

where $m_b$ is the $b$-quark mass\(^1\) and

$$\mathcal{P}_x \left( \frac{t_2}{t_1} \right) = P \exp \left[ ig \int_{t_1}^{t_2} dt' A_0^\alpha(x, t') T_\alpha \right],$$

(1.5)

is a path ordered product along a temporal line with $T_\alpha$ the usual Hermitian colour matrices. One formally gets in this way $(t > 0)$:

$$P(t) \sim P_{\text{stat}}(t) \equiv e^{-m_b t} \langle \text{Tr} \left\{ 1 - \frac{\gamma_0}{2} \gamma_0 \gamma_5 q(0, t) \; \bar{q}(0, 0) \; \gamma_0 \gamma_5 P_0(t) \right\} \rangle.$$ 

(1.6)

Except for the $c$-number factor $e^{-m_b t}$, the operator whose vacuum expectation value is taken in (1.6) depends only on light quarks and gauge fields. It is a gauge-invariant quantity, as expected. On the lattice the Green function corresponding to the bracket in eq. (1.6) is:

$$P_L(\tau) = \langle \text{Tr} \left( \frac{1 + \gamma_0}{2} S_L^\mu(0, \tau; 0, 0) U_0(1) U_0(2) \ldots U_0(\tau - 1) \right) \rangle.$$ 

(1.7)

where $\tau = t/a$, $U_\mu(x)$ is the link matrix pointing from the site $x$ in the direction $\mu$, and $S_L^\mu$ is the light-quark lattice propagator. The naïve $a \to 0$ limit transforms (1.7) into (1.6) and thus a Monte Carlo estimate of (1.7) would give an estimate of (1.1), i.e. of $f_B$. However, the breakdown of the static approximation for the propagator (1.4) at high frequencies (larger than $m_b$) and, as usual, the lattice granularity induce renormalization effects, so that we expect a relation of the type:

$$P(t) \sim P_{MC}(\tau) \; e^{-E_B a \tau},$$

(1.8)

\(^1\) As we shall see, we do not need a precise definition of $m_b$. 

2
where $P_{MC}$ is the numerical value of $P_L$ [eq. (1.7)] as measured in Monte Carlo simulations; $E_B$ is introduced to match the time dependence of the two sides of (1.8); its role and significance will be discussed in more detail later. The renormalization constant $Z$ is computed by comparing in perturbation theory the corresponding continuum and lattice quantities\(^2\).

To this end two methods have been proposed. The first one (BLP method) \([4]\) consists in a direct perturbative comparison of the correlation functions (1.1) and (1.7). To order $g^2$ this amounts to evaluating the two-loop diagrams of Fig. 1\(^3\). It will also be convenient to compute (1.6) as an intermediate step, which involves considering the graphs of Fig. 1 in the case of a static continuum quark source. The contributions to $Z$ coming from the diagrams of Fig. 1 labelled by $b$, $c$ and $d$ will be called $Z_{light}$, $Z_{vertex}$ and $Z_{heavy}$ respectively. To the order we work we have

$$Z = Z_{light}Z_{vertex}Z_{heavy}. \quad (1.9)$$

The EH method of ref. \([5]\) relies on the remark that the dynamics of a static (colour triplet) quark source can be represented by the effective Euclidian Lagrangian:

$$\mathcal{L}_{EH} = B^\dagger (\partial_0 + igA_0) B, \quad (1.10)$$

where the $B$ ($B^\dagger$) field annihilates (creates) a static quark. It should be noted that a fixed four-momentum ($m_b, 0$) has been removed from the momentum of the heavy quark. A completely independent static field should be introduced to describe processes involving heavy antiquarks. The $B$ field is a two-component spinor that describes spin-up and spin-down static $b$ quarks.

The free propagator in position space is given by:

$$S_{EH}(x, y) = \langle B(x)B^\dagger(y) \rangle = \theta(x_0 - y_0) \delta(x - y), \quad (1.11)$$

\(^2\) For simplicity, we will use a massless light quark, since none of our conclusions depends on this choice \([4], [5]\). The Feynman gauge is used everywhere. Continuum renormalizations have been performed, as in ref. \([4]\), in the $\overline{MS}$ scheme supplemented by an on-shell mass renormalization.

\(^3\) In order to avoid redundancy, we use the same graphic representation for the three cases we shall encounter. According to the situation, the heavy line must be interpreted as a continuum quark propagator (1.1), a lattice static quark source (1.7) or a continuum static quark source (1.6). The context should make it clear which interpretation is to be used in each case.
and in momentum space by:

\[ S_{EH}(p) = \frac{i}{p_0 + i\epsilon}. \]  

Equation (1.10) can be formally obtained starting from the Dirac action of a particle of mass \( m_b \) and integrating out the "small components" of the Dirac field in the limit \( m_b \to \infty \).

The lattice-discretized version of the Lagrangian (1.10) can be chosen in Euclidean space to be [5]:

\[ L^L_{EH} = B^\dagger(n) \left( B(n) - \left[ U_0(n-\hat{0}) \right]^\dagger B(n-\hat{0}) \right), \]  

with \( n \equiv (n, \tau) \), leading to the free propagator for the lattice static source:

\[ S^L_{EH}(p) = \frac{1}{-\frac{1}{a}(e^{ip_0a} - 1) + i\epsilon}. \]  

In an external gauge field the exact static-quark propagator coming from the action (1.10) [resp. (1.13)] leads indeed to eq. (1.6) [resp. (1.7)].

In the presence of a "local mass" term \( M a B^\dagger(n)B(n) \) the free propagator coming from (1.13) becomes

\[ <B(n)B^\dagger(0)> = e^{-Ma(\tau+1)}\theta(\tau)\delta_{n,0}. \]  

The main reason to introduce an effective action for the heavy quark is to reduce the determination of the renormalization coefficient \( Z \) to the usual calculation of vertex and self-energy corrections, i.e. of the diagrams of Fig. 2. In this way, to order \( g^2 \), it will be sufficient to compute one-loop graphs, whereas, in the BLP method, one has to deal with two-loop calculations.

It should be noted however that one has to face, in the effective-action method, the problem of principles that the correlation function we are interested in appears as a product of two operators at different times, but at the same space position. Since the effective theory is not Lorentz-covariant, the renormalization procedure is not straightforward and it is not obvious that complications, coming from an interplay between infrared singular behaviour in time and ultraviolet divergences in space, will not be present. In this work we shall prove the important fact that indeed such complications do not arise in the one-loop calculation of \( Z \). In spite of that, the two papers [5] and [4] give different results. The second purpose of
this paper is then to study in detail the nature of the normalization condition that determines $Z$.

We will see that, as a matter of fact, the difference between the values of $Z$ quoted by different authors is due to terms that originate, in the language of ref. [5], from the freedom in the choice of the expression for the effective heavy-quark action on the lattice, while in the language of ref. [4] they can be seen to come out from an intrinsic (multiplicative) ambiguity in the form of the normalization condition that determines $Z$ (see also [6]). Although these terms vanish as $a \to 0$, in the present-day simulations they are numerically not negligible. We shall prove, however, that problems related to this ambiguity can be eliminated, to leading order in $a$, by a consistent use of an $O(a)$-improved action for the light quarks [7], together with the appropriate form of the normalization condition.

The paper is organized as follows. In section 2 we discuss in detail the nature of the normalization condition that allows us to relate, for large $t$, the physical Green function $P(t)$ to the Monte Carlo measured quantity $P_{MC}$. The detailed comparison of the approaches of refs. [4] and [5] is presented in section 3, where the controversial calculation of the self-energy of the heavy-quark line is discussed. We show that numerically the difference between the two approaches lies in the way the “wave-function” renormalization constant of the heavy line is extracted from the relevant Feynman diagrams. In particular we notice that in the scheme of ref. [5] the result depends upon the way the heavy-quark action is discretized on the lattice.

section 4 addresses the problem of the potentially dangerous interplay between infrared (in time) and ultraviolet (in space) singularities in the vertex diagram. We rigorously prove that, to the order at which we work, this phenomenon does not occur, and the two methods lead to the same result for $Z_{\text{vertex}}$. Along the way we correct an error present in ref. [4]. In section 5 we give the final numerical values of $Z$.

2. The normalization condition

The value of $Z$ in (1.8) can be obtained from the comparison of the large $t$ behaviour of the non-perturbative ratio

$$R_{NP} = \frac{<J_{qb}^\dagger(0,0)J_{qb}(x,t)>_{\text{phys}}}{P_{MC}^t} \sim \frac{\frac{f^2_{MB}}{2}e^{-MBt}}{C_{MC}e^{-Bt}}(2.1)$$
to the corresponding perturbative one

\[ R_{PT} = \frac{\langle J_{\tau}^T(0,0) J_{\tau}(x,t) \rangle_{LT}}{P_L(t)_{LT}}. \]  

(2.2)

In eq. (2.1) \( B \) is a mass parameter fitted from the (expected) time exponential behaviour of \( P_{MC}(\tau) \):

\[ P_{MC}(\tau) \sim C_{MC} e^{-B\tau}. \]  

(2.3)

To any order in perturbation theory the behaviour of the r.h.s. of eq. (2.2) is, actually, a polynomial in \( t \). However in the large \( t \) limit, this polynomial is the \( g^2 \) expansion of a single exponential. In the following, for brevity, we will simply refer to it as an exponential.

With this in mind, one can evaluate \( Z \) through the following two-step procedure:

1) given the quantity, indicated as \( R_{NP} \) in (2.1), one computes in perturbation theory exactly the corresponding ratio of Green functions, \( R_{PT} \);
2) once the large \( t \) behaviours of \( R_{NP} \) and \( R_{PT} \) have been removed, the two remaining constant quantities are declared to be equal in the limit of vanishing lattice spacing. In formula (2.2), having defined \( M_{PT} \) so as to perturbatively remove the exponential \( t \) behaviour of \( R_{PT} \), i.e.

\[ \lim_{a \tau \to t \to \infty} e^{M_{PT}t} R_{PT}(t) = \text{const}, \]  

(2.4)

the normalization condition can be expressed as:

\[ \frac{f_2^2 M_B}{C_{MC}} = \lim_{a \to 0} \lim_{a \tau \to t \to \infty} e^{M_{PT}t} R_{PT}(t) \equiv Z. \]  

(2.5)

For future reference we write:

\[ M_{PT} = \delta_{PT} - \sigma, \]  

(2.6)

where, by definition, \( \sigma \) is the perturbative part of \( B \):

\[ B = \sigma + \delta_{NP}. \]  

(2.7)

We note that, although \( \sigma \) is, as we shall see, linearly divergent as \( a \to 0 \), the non-perturbative quantity \( \delta_{NP} \) is regular, i.e.

\[ \lim_{a \to 0} B - \sigma = \lim_{a \to 0} \delta_{NP} = \text{finite} \]  

(2.8)
Two observations are in order here. The first one is that the normalization condition (2.5) has to be imposed in the continuum limit, i.e. among finite quantities in the limit $a \to 0$ and $a\tau = t$ fixed. The second, and more important one, is that eqs. (2.4) and (2.5) can fix $Z$ only up to a multiplicative factor (going to 1 as $a \to 0$). In fact the condition (2.5) does not forbid multiplying $R_{NP}$ by a time-independent constant, provided we similarly multiply $R_{PT}$ by the corresponding perturbative factor, as prescribed by the normalization procedure described above. For instance we can equally well replace eq. (2.5) by:

$$
\lim_{a \to 0} \frac{f_B^2 M_B}{e^{-x_B B} C_{MC}} = \lim_{a \to 0} \lim_{\tau = t \to \infty} \frac{e^{M_{PT} t} R_{PT}(t)}{e^{-x_B \sigma}},
$$

(2.9)

where $x$ is any (small) real number. Equation (2.9), for $x = 1$, is essentially the normalization condition imposed in ref. [5] (see section 3), while for $x = 0$ we get the prescription of ref. [4].

The exponential $\exp(-x_B \sigma)$ in the r.h.s. of eq. (2.9) is introduced to compensate for the corresponding factor, $\exp(-x_B B)$, present in the l.h.s. This is in agreement with our prescription and it is necessary because, as we have said, $B$ is (perturbatively) linearly divergent, so $aB$ does not vanish as $a \to 0$, while $a(B - \sigma)$ does.

Using eq. (2.5) or (2.9), one can extract $f_B$ from Monte Carlo data. The reason why in practice the two choices, $x = 0$ and $x = 1$, lead to different values for $f_B$ is then obvious. Since $P_{MC}$ and, consequently, $B$ is only known at finite values of $a$, the two exponentials in eq. (2.9) do not compensate exactly. If $\sigma$ is taken to one-loop in perturbation theory, one has:

$$
\exp[xa(B - \sigma)] = 1 + x[O(a) + O(g^4)] \neq 1. \tag{2.10}
$$

As a consequence, the actual value of $f_B$, extracted from Monte Carlo data, depends on the detailed form of the normalization condition. From eq. (2.9) one gets:

$$
(f_B)_x^2 = (f_B)_{x=0}^2 \exp[xa(B - \sigma)] \tag{2.11}
$$

where, using the definition of $Z$ given by eq. (2.5), one has

$$
\frac{M_B}{2} (f_B)_{x=0}^2 = Z C_{MC}. \tag{2.12}
$$

For $x = 1$, eq. (2.11) becomes

$$
(f_B)_{EH} = (f_B)_{BLP} \exp \left[ \frac{a(B - \sigma)}{2} \right]. \tag{2.13}
$$
which gives the relation between the value of $f_B$ extracted from Monte Carlo data following the approach of ref. [5], called here $(f_B)_{EH}$, and $(f_B)_{BLP}$, the same quantity as evaluated within the approach of ref. [4].

Strictly speaking, there is no compelling reason to prefer $x = 0$ to $x = 1$, or, for that matter, to any other value of $x$. We shall see, in fact, in subsection 3.4 that exactly the same line of arguments, which “naturally” led to the choice $x = 1$ in ref. [5], can also be used to arrive “naturally” at the value $x = -1$.

The question is then whether there is any criterion to fix this problem univocally. It is not difficult to see that this can be done if we decide to improve the light-quark Wilson action [7] employed in the Monte Carlo simulation of $P_{MC}$ and in the corresponding perturbative calculations of $Z$ [8], [9].

In this case, in fact, since eq. (2.8) maintains its validity, the only consistent way of eliminating all $O(a)$ terms from the lattice evaluation of $f_B$ is to take $x = 0$, that is to use the normalization condition (2.5). If we do so, we remain with only $O(a g^2) + O(g^4)$ corrections.

3. Heavy-quark renormalization

In this section we want to compare in detail the approaches of refs. [4] and [5] and to clarify the origin of the discrepancy between the two results by discussing the calculation of the graphs of Figs. 1 (d) and 2 (c) which give rise to the controversial contribution, $Z_{heavy}$, to $Z$.

3.1. BLP evaluation of $Z_{heavy}$

With the BLP method the graphs of Fig. 1 (d) have been computed in ref. [4]. In the Feynman gauge, their contributions to (1.7), (1.6) and (1.1) are respectively:

\[
P^d_L(t) = -\frac{4}{3} g^2 \int_{-\pi}^{+\pi} \frac{d^4k}{(2\pi)^4} \frac{1 - \cos(k_0t/a)}{1 - \cos(k_0)} \frac{1}{2 \sum(1 - \cos(k_\lambda))} \frac{1}{1 - \cos(k_0)} P^\text{tree}_L(t) \]

\[
\sim_{t \to \infty} \left\{ \frac{g^2}{3\pi^2} \left[ \ln \left( \frac{t}{2a} \right) + \frac{\gamma_E}{2} + \frac{F_{0000} + F_{0001}}{4} + 1 \right] - \sigma a \tau \right\} P^\text{tree}_L(t) \tag{3.1}
\]

\[
P^d_{\text{stat}}(t) = e^{-m_b t} \frac{g^2}{3\pi^2} \left[ \ln \left( \frac{\mu t}{2} \right) + 1 + \gamma_E \right] P^\text{tree}(t), \tag{3.2}
\]

\[
P^d(t) = e^{-m_b t} \frac{g^2}{3\pi^2} \left[ \ln \left( \frac{\mu t}{2} \right) + \frac{1}{2} \ln \left( \frac{m_b}{\mu} \right) + \gamma_E \right] P^\text{tree}(t). \tag{3.3}
\]
where, on the lattice:

\[
P_{L}^{\text{tree}}(t) = \int_{-\pi}^{+\pi} \frac{d^4k}{(2\pi)^4} \frac{1}{2} \times \text{Tr} \left[ (1 + \gamma_0) \frac{-i \sum_\mu \gamma_\mu \sin p_\mu + ma + r \sum_\nu (1 - \cos p_\nu)}{\sum_\mu \sin^2 p_\mu + (ma + r \sum_\nu (1 - \cos p_\nu))^2} \right] \quad (3.4)
\]

and in the continuum

\[
P_{\text{tree}}(t) = \frac{3}{\pi^2 t^3}. \quad (3.4')
\]

In (3.1), \(F_{0000}\) and \(F_{0001}\) are the numerical constants defined in ref. [10]: \(F_{0000} \simeq 4.369\) and \(F_{0001} \simeq 1.311\); \(\gamma_E = 0.5772\) is the Euler constant, and

\[
\sigma = \frac{g^2}{24\pi^3} \frac{1}{a} \int_{-\pi}^{+\pi} d^3k \frac{1}{\sum_j (1 - \cos k_j)} \quad (3.5)
\]

\[
\simeq \frac{g^2}{4\pi^2} \cdot 6.65
\]

is the linearly-divergent one-loop radiative correction to the energy of the static source. In the following we will often write \(1 - a\sigma = e^{-a\sigma} + O(g^4)\).

From these results we get for the \(Z_{\text{heavy}}\) contribution to the total normalization constant \(Z\), defined in (1.9) and (2.5):

\[
Z_{\text{heavy}} - 1 \equiv \lim_{t \to \infty} e^{M_{a}t} \frac{P_{\text{tree}}(t) + P^{d}(t)}{\left( P_{L}^{\text{tree}}(t) + P_{L}^{d}(t) \right)} - 1
= \frac{g^2}{3\pi^2} \left\{ \ln(m_b a) + \frac{1}{2} \ln \left( \frac{m_b}{\mu} \right) + \frac{\gamma_E}{2} - \frac{F_{0000} + F_{0001}}{4} - 1 \right\} \quad (3.6)
\]

\[
\simeq \frac{g^2}{4\pi^2} \left\{ \frac{4}{3} \ln(m_b a) + \frac{2}{3} \ln \left( \frac{m_b}{\mu} \right) - 2.84 \right\},
\]

where \(e^{M_{a}t}\) is the appropriate factor necessary to compensate for the overall time exponential behaviour in the r.h.s. of the first line of eq. (3.6). In perturbation theory the multiplication by this factor simply amounts to subtracting out from the computed amplitudes all terms proportional to \(t\).
3.2. EH evaluation of $Z_{\text{heavy}}$

With the second method [5], the self-energy of the heavy quark, $\Sigma(p_0)_{EH}$, is computed directly with the Feynman rules deduced from the effective action (1.13), obtaining:

$$
\Sigma(p_0)_{EH} = \frac{4}{3} g^2 a \int_{-\pi}^{+\pi} \frac{d^4k}{(2\pi)^4} \frac{1}{4} \sum_{\mu} \sin^2 \left( \frac{k_\mu}{2} \right) e^{i(k_0 + 2ap_0)} - i \epsilon - \frac{2}{3} g^2 a \int_{-\pi}^{+\pi} \frac{d^4k}{(2\pi)^4} \frac{1}{4} \sum_{\mu} \sin^2 \left( \frac{k_\mu}{2} \right) .
$$

(3.7)

It is crucial to remark that if we insert $\Sigma(p_0)_{EH}$ into the external quark loop, we recover exactly the integral giving $P^d_L$ in eq. (3.1). So the two methods should agree. In ref. [5], the wave function renormalization was computed with the standard formula:

$$
Z_2 - 1 = -i \frac{\partial \Sigma}{\partial p_0} \bigg|_{p_0=0} .
$$

(3.8)

It is immediate to check that the radiative correction to the mass is the same as in (3.5), i.e.

$$
\sigma = \Sigma(p_0 = 0) .
$$

(3.9)

Using eq. (3.8), the result for the ratio of the wave function renormalization constant of the continuum dynamical $b$-quark [eq. (1.1)] to that of the lattice static $B$ field in (1.13) is:

$$
\frac{1 - i \frac{\partial \Sigma_{\text{cont}}}{\partial p_0}}{1 - i \frac{\partial \Sigma_{EH}}{\partial p_0} \bigg|_{p_0=0}} = Z_{\text{heavy}} (1 - a\sigma) \simeq Z_{\text{heavy}} e^{-a\sigma} ,
$$

(3.10)

with $Z_{\text{heavy}}$ given by eq. (3.6).

3.3. Comparison of the two approaches

To compare the results for $Z$ obtained in refs. [4] and [5], one has to include in the calculation the values of $Z_{\text{light}}$ and $Z_{\text{vertex}}$. As for $Z_{\text{light}}$, it is easy to see that the analog of the ratios in eqs. (3.6) and (3.10) for the light-quark self-energy diagrams, explicitly given in subsection 5.1 [Figs. 1 (b) and 2 (a)], yield the same results, both with the method of ref. [5] and with that of ref. [4]. The reason is that there is no ambiguity of the type we have discussed above for the heavy $b$ quark. In
fact, on the one hand the way the mass-counterterm for the light quark is introduced is univocally fixed by the form of the Wilson action that is being used, and, on the other hand, both in the Monte Carlo simulations and in the perturbative calculations the same expression for the light-quark action with the same mass renormalization condition (vanishing pion and quark masses respectively) is employed. Concerning the contribution of the vertex diagrams [Figs. 1 (c) and 2 (b)], we will show in section 4 that the results of refs. [4] and [5] for \( Z_{\text{vertex}} \) also coincide.

It then follows that the relation between \( Z \) in (1.8), as computed in ref. [4] and the current renormalization constant \( Z_J \) introduced in [5] is

\[
Z_J^{-2} = Z e^{-a\sigma}. \tag{3.11}
\]

Now we turn to the central issue: given a Monte Carlo data set, what is the prediction for the physical \( f_B \), and do the two methods agree?

Fitting the Monte Carlo data for large \( t \) through formula (2.3), in ref. [4] one arrives at the result

\[
\left. \frac{f_B^2 M_B}{2} \right|_{BLP} = Z C_{MC}. \tag{3.12}
\]

In ref. [5], the Monte Carlo data are fitted instead, in view of eq. (1.15), according to

\[
P_{MC} \sim A e^{-B(t+a)}. \tag{3.13}
\]

Comparing (2.3) and (3.13) one has

\[
C_{MC} = Ae^{-Ba}. \tag{3.14}
\]

For the physical correlation function (eq. (26) in [5]) one obtains:

\[
\left. \frac{f_B^2 M_B}{2} \right|_{EH} e^{-M_B a\tau} = Z_J^{-2} e^{-m_b a\tau} A e^{-Ba(\tau+1)} e^{-\Delta a(\tau+1)}. \tag{3.15}
\]

For consistency of our notation we have rewritten the factor \( 1/(1 + a\delta m/Z) \) of ref. [5] as \( e^{-a\Delta} \). By matching the large \( t \) behaviour of (3.15), one gets:

\[
\Delta = M_B - m_b - B \tag{3.16}
\]

and

\[
\left. \frac{f_B^2 M_B}{2} \right|_{EH} = Z_J^{-2} A e^{-Ba} e^{-\Delta a} = Z_J^{-2} A e^{-(M_B-m_b)a}. \tag{3.17}
\]

At this point the authors of ref. [5] identify \( M_B - m_b \) as the \( B \)-meson binding energy. They estimate \( \Delta \) in the limit \( a \to 0 \), by arguing that \( M_B - m_b \) must
converge to a constant in the continuum limit, i.e. \( \lim_{a \to 0} a(M_B - m_b) = 0 \), so they end up with:

\[
\frac{f_B^2 M_B}{2} \bigg|_{EH} = Z^{-2} A .
\] (3.18)

Remembering the relations (3.11) and (3.14), the comparison of (3.12) and (3.18) gives:

\[ f_B \big|_{EH} = f_B \big|_{BLP} \times e^{\frac{a(B-\sigma)}{2}} . \] (3.19)

This answers our central question: given a set of Monte Carlo data the number extracted for the physical \( f_B \) according to [5] [eq. (3.18)] will be that of [4] [eq. (3.12)] multiplied by \( e^{\frac{a(B-\sigma)}{2}} \).

Since \( a(B - \sigma) = a(-\Delta - \sigma + M_B - m_b) \) is \( O(g^4) + O(a g^0) \), it follows that both results agree in principle in the approximation in which we are working. However, taking a typical Monte Carlo slope \( aB = 0.7 \) (aB = 0.5) for \( \beta = 6 \) (\( \beta = 6.4 \)), eq. (3.18) gives a value for \( f_B \big|_{EH} \) that is larger by a factor \( \sim 1.3 \) (\( \sim 1.2 \)) than \( f_B \big|_{BLP} \) for the same Monte Carlo data set: the discrepancy has an obvious practical relevance and its origin should be fully understood.

### 3.4. Further remarks

Let us insist once more on the content and the origin of the factor in (3.19). First, it has nothing to do with the computational method, namely the use of either direct computation of the correlation function or effective Lagrangians. Second, there is no discussion on the validity of the starting equation (1.8). Thirdly there is no discrepancy in the mathematics.

The point is that, in ref. [4], \( Z \) is computed (to second order of perturbation theory) by using the normalization condition (2.5), thus directly obtaining eq. (3.12). In this way, one must simply assume that an \( E_B \) exists, which matches the time dependence of both sides of eq. (1.8), without the drawback of having to give a problematic estimate of a linearly divergent quantity. As we have discussed in section 2, however, an apparently irrelevant modification of the normalization condition, such as the one made in eq. (2.9), leads to a non-negligible numerical ambiguity in the estimated value of \( f_B^2 \) given by the factor (2.10).

The approach of ref. [5], instead, relies on the use of an effective Lagrangian for the heavy quark, both in the way the large \( t \) dependence of the perturbative and the Monte Carlo expressions of \( P_L \) is parametrized and in the way the wave-function
renormalization of the heavy-quark line is computed. The authors of ref. [5] were, in fact, led by the form (1.15) of their free propagator to the parametrization (3.13); they were also driven by formula (3.8), which looks so familiar, to choose $Z_2$ as the quantity to be computed perturbatively.

The key observation at this point is that the effective action is merely an intermediate step used to compute the relative normalization between the physical observable and the quantity measured on the lattice. Neither quantity has anything to do with any effective action and the final result should not depend on it. However, it is easy to produce a variety of effective lattice actions whose continuum limit is (1.10) and which, by the use of the corresponding matching condition and of formula (3.8), lead to different values of $f_B$ for a given set of Monte Carlo data.

For instance instead of (1.13), one could as well choose for the discretized version of (1.10) the Lagrangian:

$$\mathcal{L}'_{EH} = B\uparrow(n)(U_0(n+\hat{0})B(n+\hat{0}) - B(n)) \, .$$

If a local mass term $M a B\uparrow(n) B(n)$ is added, one gets:

$$<B(n)B\uparrow(0)> = e^{-Ma(\tau-1)}\theta(\tau)\delta_{n,0} \, .$$

Notice the change $(\tau + 1) \rightarrow (\tau - 1)$ in eq. (3.21) with respect to eq. (1.15). This change propagates step by step from eq. (3.13), leading now to:

$$f_B'|_{EH} = f_B'|_{BLP} \times e^{-\frac{a(B-\sigma)}{2}} \, (3.22)$$

instead of (3.19).

Another interesting choice is to use an $O(a)$-improved effective Lagrangian for the heavy quark, i.e. to take:

$$\mathcal{L}_{imp} = B\uparrow(n) \left( \frac{3}{2} B(n) - 2U_0(n-\hat{0})\uparrow B(n-\hat{0}) \right. \left. + \frac{1}{2} U_0(n-\hat{0})\uparrow U_0(n-2\hat{0})\uparrow B(n-2\hat{0}) \right) \, . (3.23)$$

One checks that, if the static field is integrated out, the Lagrangian (3.23) leads exactly to the same correlator (1.7) as the Lagrangians (1.13) or (3.20), up to exponentially small terms, vanishing as $t \rightarrow \infty$.

In view of this argument, the authors of [8] and [9] claim correctly that there is no need to use an improved effective action for the heavy quark, if one uses
the normalization condition (2.5). However, if one insists on using eq. (3.8) to define $Z_{\text{heavy}}$ the precise form of the effective action does matter. In fact, the free propagator obtained from (3.23) is:

$$S_{\text{imp}}(p) = \frac{1}{-\frac{1}{a}(2e^{ip_0a} - \frac{1}{2}e^{2ip_0a} - \frac{3}{2}) + i\epsilon} .$$

(3.24)

It coincides with the continuum one [eq. (1.12)] up to $O(a^2)$, while (1.14) differs already at $O(a)$. We have computed $\Sigma_{\text{imp}}(p)$, the self-energy of the heavy line, with the “improved” effective Lagrangian (3.23) and derived the mass and wave function renormalization, by using formulae (3.8) and (3.9); $\Sigma_{\text{imp}}(0)$ is identical to (3.5), but the noteworthy fact is that, instead of eq. (3.10), one obtains

$$\left.\frac{1 - i\frac{\partial \Sigma_{\text{cont}}}{\partial p_0}}{1 - i\frac{\partial \Sigma_{\text{imp}}}{\partial p_0}}\right|_{p_0=0} = Z_{\text{heavy}} .$$

(3.25)

Following the steps of ref. [5], we end up this time with a result identical to (3.12) (the same as in ref. [4]), that is:

$$(f_B)_{\text{imp}}|_{EH} = (f_B)_{x=0} = f_B|_{BLP} .$$

(3.26)

Thus a determination of $Z$ that uses formula (3.8) can give the same result as (2.5), if one uses the Lagrangian (3.23).

From all this analysis, we can conclude that the use of the effective action approach by itself is not sufficient to lead to an unambiguous (in the sense we are discussing here) determination of $f_B$ from Monte Carlo simulations.

We have instead shown in section 2 that this can be achieved to $O(a)$ only if one consistently uses:

i) an $O(a)$ improved action for the light quarks.

ii) the normalization condition (2.5), in which the matching does not involve unknown $O(a)$ terms.

4. Vertex renormalization

In this section we discuss the evaluation of the contribution to $Z$ of the vertex diagrams [Fig. 1 (c)] and we rigorously prove that the effective Lagrangian method leads to the same result as the full-fledged two-loop calculation of ref. [4]. That is
to say, there is no dangerous interplay between IR and UV divergences. The direct evaluation [4] of the two-loop graph of Fig. 1 (c) gives:

\[
P_L^c(t) = \frac{4}{3} g^2 \frac{1}{t^3} \left( I_1^{latt}(\tau) + I_2^{latt}(\tau) \right),
\]

with

\[
I_1^{latt} = 3 \int_{-\pi}^{+\pi} \frac{dp_0}{2\pi} \int_{-\pi}^{+\pi} \frac{d^3p}{(2\pi)^3} e^{ip_0\tau} \frac{1 - e^{i p_0} + \Sigma(p)}{2\Delta_2(p_0, p_\tau)} \times 
\]

\[
e^{-i p_0} \int_{-\pi}^{+\pi} \frac{d^4k}{(2\pi)^4} \frac{e^{ik_0/2} 1}{e^{ik_0} - 1 \Delta_1(k)} \left\{ e^{-ik_0/2} \frac{1 - e^{i(p_0+k_0)} + \Sigma(p + k)}{\Delta_2(p_0 + k_0, p_\tau + k)} - e^{ik_0/2} \frac{1 - e^{i(p_0-k_0)} + \Sigma(p - k)}{\Delta_2(p_0 - k_0, p_\tau - k)} \right\}
\]

\[
I_2^{latt} = 3 \sum_j \int_{-\pi}^{+\pi} \frac{dp_0}{2\pi} \int_{-\pi}^{+\pi} \frac{d^3p}{(2\pi)^3} e^{ip_0\tau} \frac{-i \sin(p_\tau)}{2\Delta_2(p_0, p_\tau)} \times 
\]

\[
e^{-i p_0} \int_{-\pi}^{+\pi} \frac{d^4k}{(2\pi)^4} \frac{e^{ik_0/2} 1}{e^{ik_0} - 1 \Delta_1(k)} \left\{ e^{-ik_0/2} \frac{-i \sin(p_\tau + k_j)}{\Delta_2(p_0 + k_0, p_\tau + k)} - e^{ik_0/2} \frac{-i \sin(p_\tau - k_j)}{\Delta_2(p_0 - k_0, p_\tau - k)} \right\}
\]

where the \(\Delta\)'s come from the gluon and fermion lattice propagators:

\[
\Delta_1(k) = \frac{1}{2} \sum_\lambda (1 - \cos(k_\lambda))
\]

\[
\Delta_2(p_0, p) = \sum_\lambda \sin^2(p_\lambda) + \left( \sum_\lambda [1 - \cos(p_\lambda)] \right)^2.
\]

As we said, we have chosen here to work with a massless light quark, since the renormalization constants are independent of the mass [4], [5]. No mass has been
given to the gluon since the quantities we are dealing with are infrared-finite after the singularities among individual terms have been cancelled.

To determine $Z_{\text{vertex}}$, we have to study the behaviour of $I_{1}^{\text{latt}}$ and $I_{2}^{\text{latt}}$ for large $t$, at fixed $a$, i.e. when $\tau$ goes to $\infty$. The integrations over the temporal components, $p_0$ and $k_0$, are performed exactly, by closing the contour in the complex plane. Oscillating factors are turned into exponentially damped ones, easier to handle. The large-$\tau$ behaviour of the resulting spatial integrals is analysed by means of Lebesgue’s lemma. We find (see appendix A for a sketch of the proof):

$$
I_{1}^{\text{latt}}(\tau) \sim \frac{3}{\pi^2} \int_{-\pi}^{+\pi} d^{4}k \frac{e^{ik_0} - 1 - \sum j(1 - \cos(k_j))}{2\Delta_1(k)\Delta_2(k_0, k)} \frac{1 - e^{ik_0\tau}}{1 - e^{ik_0}} + C_1 + o(\ln(\tau)/\tau) \ ,
$$

$$
I_{2}^{\text{latt}}(\tau) \sim C_2 + o(\ln(\tau)/\tau) \ ,
$$

where $C_1$ and $C_2$ are some ($\tau$-independent) continuum-like integrals. A similar analysis can be performed in the continuum for $P_{\text{stat}}^c(t)$ [eq. (1.6)], using dimensional regularization ($d = 4 - \epsilon$) to control its ultraviolet behaviour, and gives:

$$
P_{\text{stat}}^c(t) = \frac{4}{3} g^2 \frac{1}{\mu^3} \left( I_{1}^{\text{stat}}(\mu t) + I_{2}^{\text{stat}}(\mu t) \right) ,
$$

with

$$
I_{1}^{\text{stat}}(\mu t) \sim \frac{3}{\pi^2} \mu^\epsilon \int_{-\infty}^{+\infty} \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}} \frac{e^{ik_0t} - 1}{\frac{1}{2}(k^2)^2} + C_1 + o(\epsilon) \ ,
$$

$$
I_{2}^{\text{stat}}(\mu t) \sim C_2 + o(\epsilon) .
$$

The constants $C_1$ and $C_2$ are the same as in eq. (4.4). The decomposition into the terms appearing in eqs. (4.4) and (4.6) were performed in such a way that the dependence upon the regularization scheme was isolated in the first terms, leaving $C_1$ and $C_2$ renormalization-scheme independent quantities.

Using the $\overline{MS}$ renormalization scheme in the continuum, we find that the contribution to the ratio of (1.6) to (1.7) coming from this graph (see appendix B) is:
\[
\lim_{t \to \infty} e^{M_b t} \frac{P^{\text{tree}}(t) + P^{\text{stat}}_L(t)}{P^{\text{tree}}_L(t) + P^{\text{stat}}_R(t)} - 1
= \frac{g^2}{8\pi^2} \left\{ \frac{2 \ln(\mu a) + \gamma_E - F_{0000} - \frac{1}{2\pi} \int_{-\pi}^{\pi} d^3k \frac{1}{\Delta_2(0,k)} }{3 + 4.29} \right\},
\]

where the factor \( e^{M_b t} \) plays the same role as the similar exponential in eq. (3.6). The result (4.7) coincides with the value obtained from the evaluation of the one-loop vertex graph of ref. [5]. This indeed shows that there are at this order no complications due to the problem of the renormalizability of a non-covariant effective theory. In ref. [4] this computation was actually performed with a non-vanishing light-quark mass. The results agree, apart from the \( 1/2\pi \int d^3k \Delta^{-1}_2(0,k) \) term, which is present in eq. (4.7) but was erroneously forgotten in ref. [4].

5. Results

In this section we complete our discussion on the evaluation of \( Z \), by giving the expression of \( Z_{\text{light}} \), and we compare the values of \( f_B \) obtained with different normalization conditions, with the number one would obtain by using an \( O(a) \)-improved light-fermion action [in which case it is the form (2.5) of the normalization condition that, for consistency, has to be employed].

5.1. Contribution of \( Z_{\text{light}} \)

The last piece of the puzzle is the contribution to the renormalization constant due to the graph 1 (b) which is the usual wave-function renormalization for a Wilson fermion [11], [12], [13]. Numerically one obtains:

\[
e^{M_b t} \frac{P^{\text{tree}}(t) + P^{\text{stat}}_b(t)}{P^{\text{tree}}_L(t) + P^{\text{stat}}_R(t)} - 1 \underset{t \to \infty}{\sim} - \frac{g^2}{4\pi^2} \left\{ \frac{2 \ln(\mu a)}{3} + 4.29 \right\},
\]

where, once again, \( e^{M_b t} \) is the appropriate factor introduced to make the ratio in the l.h.s. go to a constant as \( t \to \infty \).
To reach the final result, we need the contribution to the ratio $P_{\text{stat}}/P_L$ coming from the d-type graphs in Fig. 1 [in analogy with eqs. (4.7) and (5.1)]. One has:

\[
e^{M_T} \left( \frac{P_{\text{tree}}(t) + P_{\text{stat}}^d(t)}{P_L^\text{tree}(t) + P_L^d(t)} \right) - 1 \sim \frac{g^2}{4\pi^2} \left\{ \frac{4 \ln(\mu a)}{3} + \frac{2}{3} \gamma_E - \frac{F_{0000} + F_{0001}}{3} \right\}
\]

(5.2)

\[
\approx \frac{g^2}{4\pi^2} \left\{ \frac{4}{3} \ln(\mu a) - 1.51 \right\}
\]

The relation between the two continuum correlation functions (1.1) and (1.6) is already known [4], [5]; the result is the same in the two methods and it has also been checked in the temporal gauge ($A_0 = 0$)[14]. It can be written:

\[
e^{M_T} \frac{P(t)}{P_{\text{stat}}(t)} \bigg|_{P_T} - 1 \sim - \frac{g^2}{2\pi^2} \ln \left[ \frac{\mu e^{2/3}}{m_b} \right]
\]

(5.3)

We insist once more on the fact that all the exponential factors in eqs. (3.6), (4.7), (5.1), (5.2) and (5.3) are defined so as to insure the constancy of the corresponding ratios of correlation functions as $t \to \infty$. The values of the mass coefficients are totally irrelevant for the purpose of computing $Z$, because the only thing one actually has to do is to neglect in the calculations all the terms that are proportional to $t$. Combining eqs. (4.7), (5.1), (5.2) and (5.3), we obtain the result:

\[
e^{M_{P_T}} \frac{P(t)}{P_{\text{stat}}(a, t)} \bigg|_{P_T} = Z - 1 = \frac{g^2}{4\pi^2} \left[ 2 \ln \left( m_b e^{-2/3} \right) - 13.59 \right]
\]

(5.4)

5.2. Numerical evaluation of $Z$

The relation of the physical value of $f_B$ to the measured lattice quantity

\[
f_B^{\text{latt}} \equiv \left( \frac{2C_{MC}}{M_B} \right)^{1/2}
\]

extracted from the behaviour of $P_{MC}(\tau)$ at large $t$ [eq. (2.3)] is then, from eq. (5.4):

\[
f_B = f_B^{\text{latt}} \left\{ 1 + \frac{g^2}{8\pi^2} \left[ 2 \ln \left( m_b e^{-2/3} \right) - 13.59 \right] \right\}
\]

(5.5)

Numerically, if we take $m_b = 4.5$ GeV and $\beta = 6/g^2 = 6.0, 6.2, 6.4$ with $a^{-1} = 2.3, 2.9, 3.7$ GeV respectively, which are typical values for the Monte Carlo simulations used in $B$ mesons studies [15], we see that in all cases the logarithmic correction is negligible and we obtain the final result:

\[
f_B = 0.83 f_B^{\text{latt}}
\]

(5.6)
Note that the prescription of ref. [5] [i.e. \( x = 1 \) in eq. (2.11)], would lead to much larger values for the renormalization constant \( Z \). As a consequence, one would obtain \( f_B \sim (0.97 \text{ to } 1.07) f_B^{\text{latt}} \) for typical Monte Carlo slopes of \( 0.5a^{-1} \) to \( 0.7a^{-1} \). With the choice \( x = -1 \) [eq. (3.22)] the value of \( f_B \) would be lowered to \( f_B \sim (0.63 \text{ to } 0.70) f_B^{\text{latt}} \).

Recently the renormalization constants of operators involving heavy quarks have been computed [8] by using for the light quarks a nearest-neighbour \( O(a) \) improved action. In this theory [7] \( O(a) \) corrections are absent from on-shell hadronic matrix elements. Using the normalization condition (2.5), the net result for \( Z \) amounts to replacing the constant 13.59 in (5.5) by 10.08. Then to first order in perturbation theory one gets:

\[
f_B = 0.87 f_B^{\text{latt-imp}} ,
\]

where \( f_B^{\text{latt-imp}} \) is the Monte Carlo value measured in simulations which employ the same improved action as is used in the corresponding perturbative calculations. We stress again that in this case the prescription (2.5) \([x = 0 \text{ in (2.10)}]\) is compulsory in order not to loose \( O(a) \) improvement.

Two numerical calculations of \( f_B \) with a static \( b \)-quark have been performed in the literature [16], [17] at \( \beta = 6 \), using for the renormalization coefficient the value 0.8, essentially in agreement with the estimate (5.6)\(^4\). A systematic comparison with the results on meson decay constants for the case of propagating quarks [3] shows a fair agreement with the static points, the latter having however a tendency to lie a bit too high. With the EH normalization condition \([x = 1 \text{ in eq. (2.9)}]\), the situation would be much worse, since the static points would be raised by about 30%. Of course this pattern has to be checked at higher values of \( \beta \)\(^5\).

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\(^4\) It is argued in refs. [16] and [17] that the value 0.8 is in essential agreement with the results of both [4] and [5]. This statement probably refers to the value 1.22 found for \( Z_f \) in [5]. However, as we discussed in section 3, this apparent agreement leads in fact to quite different values for \( f_B \).

\(^5\) The authors of ref. [17] claim that the static \( f_B \) decreases sensibly with increasing \( \beta \).
Appendix A.

This appendix is devoted to a sketch of the steps needed to go from eq. (4.2) to eq. (4.4). Let us study only \( I_{1\text{latt}} \), the analysis for \( I_{2\text{latt}} \) being similar. After the integrations over \( p_0 \) and \( k_0 \), we get:

\[
\mathcal{I}_{1\text{latt}} = \int_{-\pi}^{+\pi} \frac{d^3 p}{(2\pi)^3} \frac{A(\xi) - Z(\xi)}{A(\xi) - Z(\xi)} (Z(\xi))^\tau \int_{-\pi}^{+\pi} \frac{d^3 k}{(2\pi)^3} \frac{Z_1(k)}{A(k+\xi) - Z(\xi)Z_1(k)} \times \left\{ \frac{1}{Z_1(k)} \right\} - \frac{1}{1-Z_1(k)} \times \left( \frac{1}{1-Z_1(k)} - \Theta(1-R) \frac{1-(R)^\tau}{1-R} \right) \}
\]

where

\[
A(k) = 1 + \sum_j (1 - \cos(k_j))
\]

\[
Z_1(k) = A(k) - \sqrt{A(k)^2 - 1}
\]

\[
Z(k) = 1 + B(k) - \sqrt{(1 + B(k))^2 - 1}
\]

\[
B(k) = \frac{\sum_j \sin^2(k_j) + (1 - A(k))^2}{2A(k)}
\]

\[
R = \frac{Z(k+\xi)}{Z(\xi)}
\]

and \( \Theta \) is the usual Heaviside distribution.

We are interested in the behaviour of \( \mathcal{I}_{1\text{latt}} \) when \( \tau \to \infty \). It is possible to show that the integration over \( p \) gives no problem, essentially because it is protected by the exponentially damping factor \( Z(\xi)^\tau \). To illustrate the method we have used, we shall not consider the whole expression (A.1), which is rather long, but a simpler quantity; this offers however, a degree of mathematical complexity that is similar to that of the integrals we encounter in the actual computation, namely:
\[ J = \int_{-\pi}^{\pi} \frac{d^3k}{(2\pi)^3} \frac{1}{Z(k + \frac{x}{\tau}) - Z(\frac{x}{\tau})Z_1(k)} \frac{1}{Z_1(k)} - rac{1 - (Z_1(k))^\tau}{1 - Z_1(k)} . \] (A.3)

The problem about eq. (A.3) is that, since the integral diverges when \( \tau \to \infty \), we cannot exchange the limit and the integration. To study the behaviour of \( J \) as \( \tau \to \infty \), it is convenient to first add and subtract from the integrand its value at \( p = 0 \); \( J \) is then decomposed into a sum \( J_1 + J_2 \) with:

\[ J_1 = \int_{-\pi}^{\pi} \frac{d^3k}{(2\pi)^3} \frac{1}{Z(k) - Z_1(k)} \frac{1}{Z_1(k)} - \frac{1 - (Z_1(k))^\tau}{1 - Z_1(k)} \]

\[ J_2 = \int_{-\pi}^{\pi} \frac{d^3k}{(2\pi)^3} \frac{1}{Z(k)} - \frac{1}{Z_1(k)} - \frac{1 - (Z_1(k))^\tau}{1 - Z_1(k)} \times \]

\[ \frac{1}{Z_1(\frac{x}{\tau}) - Z_1(\frac{x}{\tau})} - \frac{1}{Z_1(\frac{x}{\tau})} - \frac{1 - (Z_1(\frac{x}{\tau}))^\tau}{1 - Z_1(\frac{x}{\tau})} . \] (A.4)

Let us call \( J \) the integrand in \( J_2 \). From (A.2) we see that the quantities \( Z \) and \( Z_1 \) are smaller than 1; thus we have the bound:

\[ |J| \leq \frac{1}{\tau^3} \left| \frac{1}{Z(\frac{x}{\tau})} - \frac{1}{Z_1(\frac{x}{\tau})} \right| + \left( 1 - Z(\frac{x}{\tau}) \right) \frac{1}{Z(\frac{x}{\tau})} - \frac{1}{Z_1(\frac{x}{\tau})} - \frac{1 - (Z_1(\frac{x}{\tau}))^\tau}{1 - Z_1(\frac{x}{\tau})} . \] (A.5)

From (A.2), after some work, it is possible to show that there exists a set of constants \( \alpha_j \) such that:

\[ \frac{1}{Z(q)} - \frac{1}{Z(q')} \leq \alpha_1 |q| - |q'| \]

\[ \frac{1}{Z(q)} - Z_1(q') \leq \frac{\alpha_2}{|q| + |q'|} \]

\[ \frac{1 - (Z_1(q))^\tau}{1 - Z_1(q)} \leq \frac{\alpha_3}{\alpha_4 |q|} \]

\[ (1 - Z(q)) \leq \alpha_5 |q| . \] (A.6)

It then follows that the absolute value of \( J \) is bounded by an integrable function of \( k \) and, by Lebesgue’s theorem, that we can exchange the limit and the integration to get:

\[ \lim_{\tau \to \infty} J_2 = \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} \frac{1 - e^{-|k|}}{4 |k|^3} \left| k \right| - \left| k + p \right| - \left| p \right| . \] (A.7)
When the actual integral (A.1) is analysed in this way, it turns out that the terms that are analogous to $J_2$ are responsible for the appearance of the constant $C_1$ in eq. (4.4). The subtracted terms (analog to $J_1$) give:

$$\int_{-\pi}^{+\pi} \frac{d^3k}{(2\pi)^3} \frac{Z_1(k)}{A(k)} \frac{1}{Z(k) - Z_1(k)} \left( \frac{A(k) - Z(k)}{Z(k) - Z_1(k)} \right) \left( 1 - (Z(k))^\top \right) - \frac{A(k) - Z_1(k)}{Z_1(k) - Z_1(k)} \left( 1 - Z_1(k) \right)^\top \right), \tag{A.8}$$

which can be rewritten in a simpler form if we introduce an integration over $k_0$:

$$\int_{-\pi}^{+\pi} \frac{d^4k}{(2\pi)^4} e^{ik_0} - 1 - \sum_j (1 - \cos(k_j)) \frac{1 - e^{ik_0\tau}}{2\Delta_1\Delta_2} \frac{1 - e^{ik_0}}{1 - e^{ik_0}} \tag{A.9}$$

leading to the first term in the r.h.s. of (4.4).

**Appendix B.**

In this appendix, we shall derive formula (4.7). The zeroth-order graph 1 (a) is equal to $3/\pi^2t^{-3}$, and together with eqs. (4.4) and (4.6), we find from our definitions

$$e^{M_{ct}} \frac{P_{tree}^{ct}(t) + P_{stat}^{ct}(t)}{P_{tree}^{ct}(t)} - 1 \sim \frac{4}{3}g^2 \left\{ \int_{-\pi}^{+\pi} \frac{d^4k}{(2\pi)^4} e^{ik_0} - 1 - \sum_j (1 - \cos(k_j)) \frac{1 - e^{ik_0\tau}}{2\Delta_1(k)\Delta_2(k_0, k)} \frac{1 - e^{ik_0}}{1 - e^{ik_0}} \right\} \tag{B.1}$$

If we use the identity:

$$\frac{1}{\Delta_2} = \frac{1}{4\Delta_1} + \frac{\sum \sin^4(k_\lambda)}{\Delta_1\Delta_2} - \frac{\Delta_1}{\Delta_2} \tag{B.2}$$
the lattice integral in eq. (B.1) can be rewritten as:

\[
I_L = \int_{-\pi}^{+\pi} \frac{d^4 k \ e^{i k_0 \tau} - 1 - \sum_j (1 - \cos(k_j))}{2 \Delta_1(k) \Delta_2(k, k)} \frac{1 - e^{i k_0 \tau}}{1 - e^{i k_0}}
\]

\[
= \int_{-\pi}^{+\pi} \frac{d^4 k}{(2\pi)^4} \frac{1 - e^{i k_0 \tau}}{8 \Delta_1^2} + \frac{1}{2} \left( \frac{\sum \sin^4(k_j)}{\Delta_1^2 \Delta_2} - \frac{1}{\Delta_2} \right) \left( 1 - e^{i k_0 \tau} \right) (B.3)
\]

\[
+ \frac{1}{2} \left( \frac{\sum (1 - \cos(k_j))}{\Delta_1 \Delta_2} - \frac{1}{\Delta_2(0, k)} \right) \frac{1 - e^{i k_0 \tau}}{1 - e^{i k_0}} + \int_{-\pi}^{+\pi} \frac{d^3 k}{(2\pi)^3} \frac{1}{2 \Delta_2(0, k)}.
\]

When \(\tau\) is large, the factor \(e^{i k_0 \tau}\) gives no contribution in the second and the third terms in the second equality of eq. (B.3). In the first term, the denominator can be exponentiated and one arrives at some combination of Bessel functions. This method was discussed in appendix C of ref. [4] and leads to the formula:

\[
\int_{-\pi}^{+\pi} \frac{d^4 k}{(2\pi)^4} \frac{1 - e^{i k_0 \tau}}{8 \Delta_1^2} \sim \tau \to \infty \frac{1}{8\pi^2} \left( 2 \ln \left( \frac{\tau}{2} \right) + F_{00000} + \gamma_E \right). \quad (B.4)
\]

The computation of the continuum integral in eq. (B.1) is straightforward and gives:

\[
I_C = \mu^{-\infty} \int_{-\infty}^{+\infty} \frac{d^4 k \ e^{i k_0 t} - 1}{(2\pi)^4 - \epsilon} \frac{1}{\frac{1}{2} (k^2)^2}
\]

\[
= \frac{1}{8\pi^2} \left( \frac{2}{\epsilon} + \ln(4\pi) + 2 \ln \left( \frac{\mu t}{2} \right) + \gamma_E \right). \quad (B.5)
\]

From eqs. (B.3), (B.4) and (B.5), one finally obtains eq. (4.7).
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**Figure Captions**

Fig. 1: The four contributions of order $g^2$ to $P(t)$, $P_{\text{stat}}(t)$ and $P_L(\tau)$ as defined in eqs (1.1), (1.6) and (1.7) respectively (in order to avoid redundancy, we use the same graphic representation for the three cases).

Fig. 2: The graphs contributing to $P(t)$ in an effective-action formulation.