On isolated singularities for fractional Lane-Emden equation in the Serrin’s critical case

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Abstract

A challenging problem for nonlocal semilinear elliptic equation is the fractional Lane-Emden equation in the Serrin’s critical case. This is due to the various challenges and the lack of tools to analyze the isolated singularities. These difficulties come from the nonlocal setting where the ODE’s tools fail to provide useful information. Additionally, the computation of the fractional Laplacian for the involved logarithmic functions is tricky. Finally, it is not an easy task to show that the solutions are in the appropriate function space. In this paper, we solve the fractional Lane-Emden equation in the Serrin’s critical case for the fractional Laplacian by developing an innovative and self-contained approach that also applies to the classical setting (Laplacian).

We give a classification of the isolated singularities of positive solutions to the semilinear fractional elliptic equations

\[(E) \quad (-\Delta)^s u = u^{\frac{N}{N-2s}} \quad \text{in} \quad \Omega \setminus \{0\}, \quad u \geq 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega,\]

where \(s \in (0,1), \Omega\) is a bounded domain containing the origin in \(\mathbb{R}^N\) with \(N > 2s\) and \(\frac{N}{N-2s}\) is the Serrin’s critical exponent. We use an initial asymptotic at infinity to transform the critical case into a subcritical case where the underlying equation involves the fractional Hardy operator. The construction of singular solutions is based on the fact that some special functions are subsolutions of the original problem near the origin.

We also classify the non-removable singularities of the solutions of \((E)\) and show the existence of a sequence of isolated singular solutions parameterizing the coefficients of the second order blow up term. To the best of our knowledge, this idea has also been first used in this work.

Keywords: Isolated singularities; Fractional Laplacian; Serrin’s critical.

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1 Introduction

Let \(s \in (0,1), \Omega\) be a bounded \(C^2\) domain containing the origin in \(\mathbb{R}^N\) with \(N\) a positive integer such that \(N > 2s\) and

\[p^* = \frac{N}{N-2s}\]

is the Serrin’s critical exponent. The purpose of this paper is to classify the isolated singular positive solutions of the elliptic problem

\[\left\{ \begin{array}{ll}
(-\Delta)^s u = u^{p^*} & \text{in} \quad \Omega \setminus \{0\}, \\
u = h & \text{in} \quad \mathbb{R}^N \setminus \Omega,
\end{array} \right.\]

(1.1)
where \((-\Delta)^s\) is the fractional Laplacian defined by
\[
(-\Delta)^s u(x) = c_{N,s} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_r} \frac{u(x) - u(x + z)}{|z|^{N+2s}} \, dz,
\]
\(B_r(y) \subset \mathbb{R}^N\) is the ball of radius \(r > 0\) centered at \(y\). Here and in what follows, \(B_r = B_r(0)\),
\[
c_{N,s} = 2^{2s} \pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(1-s)}
\]
is the normalized constant, see [21], with \(\Gamma\) being the Gamma function. It is known that \((-\Delta)^s u(x)\) is well defined if \(u\) is twice continuously differentiable in a neighborhood of \(x\), and contained in the space \(L^1_1(\mathbb{R}^N) := L^1(\mathbb{R}^N, \frac{dx}{|x|^{N+2s}})\). We also note that for \(u \in L^1_1(\mathbb{R}^N)\) the fractional Laplacian \((-\Delta)^s u\) can also be defined as a distribution:
\[
((-\Delta)^s u, \varphi) = \int_{\mathbb{R}^N} u(-\Delta)^s \varphi \, dx \quad \text{for any} \quad \varphi \in C_c^\infty(\mathbb{R}^N).
\]
We then have
\[
\mathcal{F}((-\Delta)^s u) = |\cdot|^{2s} \hat{u} \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}
\]
in the sense of distributions, both \(\mathcal{F}(\cdot)\) and \(|\cdot|\) denote the Fourier transform.

When \(s = 1\) and \(N \geq 3\), the isolated singularities of the solutions of the related model equation
\[
\begin{cases}
-\Delta u = u^p & \text{in} \quad \Omega \setminus \{0\}, \\
u = 0 & \text{on} \quad \partial \Omega
\end{cases}
\tag{1.2}
\]
were studied extensively in the last decades. When \(p \in (1, \frac{N}{N-2})\), Lions in [28] classified the singular solutions by establishing a connection with the weak solutions of
\[
-\Delta u = u^p + k\delta_0.
\]
He showed that the positive solutions have the asymptotic behavior
\[
\lim_{|x| \to 0^+} u_k(x)|x|^{N-2} = c_N k
\]
for some \(k \geq 0\), and a normalized constant \(c_N > 0\). In the particular case \(k = 0\), the solutions have removable singularity at the origin. This is always the case when \(p \geq \frac{N}{N-2}\). Therefore this method fails to classify the singularities in this case. When \(p \in \left(\frac{N}{N-2}, \frac{N}{N-2}^2\right)\), the singularity of the solutions to (1.2) was studied by Gidas-Spruck in [24] by using analytic techniques from [4]. In this case, the positive singular solutions have the following type of singularity:
\[
u(x) = c_p |x|^{\frac{2}{p^2}}(1 + o(1)) \quad \text{as} \quad |x| \to 0^+
\]
with the coefficient \(c_p^{-1} = \frac{2}{p^2}((N-2)\frac{2}{p} - 1)\). When \(p = \frac{N}{N-2}\), the author in [3, 4] gave the following classification: Any positive solution \(u\) of (1.2) has a removable singularity at the origin or has the asymptotic behavior:
\[
u(x) = \mathcal{K}_1 |x|^{2-N}(-\ln |x|)^{-\frac{N-2}{2}} (1 + o(1)) \quad \text{as} \quad |x| \to 0^+,
\]
where
\[
\mathcal{K}_1 = \left(\frac{(N-2)^2}{2}\right)^{\frac{N-2}{2}}.
\]
Moreover, the existence of a singular solution is obtained by the Phase plane analysis. Using these solutions, Pacard in [29] constructed positive solutions with the prescribed singular set. Later on, Caffarelli-Gidas-Spruck in [27] (also see [27]) classified the singular solutions of (1.2) when \(p = \frac{N}{N-2}\). In this case, they satisfy
\[
u(x) = \varphi_D(\ln |x|)|x|^{-\frac{N-2}{2}}(1 + o(|x|)) \quad \text{as} \quad |x| \to 0^+,
\]
\( \varphi_D \) may be a constant or a periodic function. We refer to [5, 25, 31, 32] for more details for elliptic problems with general differential operators.

Due to the numerous applications of Dirichlet problems with nonlocal operators, there has been an increasing interest in this topic during the last years. A general setting of nonlinear nonlocal operators are studied in [11], where the authors connect the fractional problem with an underlying second order degenerated problem in a half space in a higher dimension. For regularity properties of the solutions of the fractional problem, the reader could see [11, 30, 34]. For blow-up analysis for nonlocal problems, see [5, 12, 14].

Motivated by [28], the isolated singularity of

\[
\begin{cases}
(-\Delta)^s u = u^p & \text{in } \Omega \setminus \{0\}, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega
\end{cases}
\tag{1.3}
\]

was studied in [13] for \( p \in (1, p^*) \) via the connection with the distributional solutions of

\[
\begin{cases}
(-\Delta)^s u = u^p + k\delta_0 & \text{in } \Omega \setminus \{0\}, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

where the positive solution with singularity can be described by fundamental solution of fractional Laplacian. As in the Laplacian case, this method does not apply to classify isolated singularities for the Lane-Emden equation with Serrin’s critical and super critical exponent, i.e. \( p \geq p^* \). When \( p \in (p^*, \frac{N+2s}{N-2s}) \), [8, 26] built the platform of the isolated singularity for positive solution to

\[
\begin{align*}
\text{div}(t^{1-2s}\nabla U) &= 0 & \text{in } B_1 \times (0,1) \\
\lim_{t \to 0} t^{1-2s} \partial_t U(x,t) &= U^p(x,0) & \text{on } B_1 \setminus \{0\},
\end{align*}
\tag{1.4}
\]

in the Sobolev critical case \( p = \frac{N+2s}{N-2s} \). The non-removable singular solutions of (1.3) behave as follows:

\[
c_2|x|^{-\frac{N+2s}{N-2s}} \leq u(x) \leq c_1|x|^{-\frac{N+2s}{N-2s}}
\]

for some \( c_1 > c_2 > 0 \). [27] gives a description of singular solutions for \( p \in (p^*, \frac{N+2s}{N-2s}) \)

\[
c_4|x|^{-\frac{N+2s}{p^*-p}} \leq u(x) \leq c_3|x|^{-\frac{N+2s}{p^*-p}}
\]

for some \( c_3 > c_4 > 0 \). For the Serrin case \( p = p^* \) [30] shows the bound

\[
b_1(|x|(-\ln |x|)^\frac{N-2s}{2})^{2s-N} \leq u(x) \leq b_2(|x|(-\ln |x|)^\frac{N-2s}{2})^{2s-N}
\]

for some \( 0 < b_1 < b_2 < +\infty \).

It is worth noting that (1.3) is a local degenerated problem. For the existence of isolated singular solutions, [11] constructed a sequence of isolated singular solutions of (1.3) with \( \Omega = \mathbb{R}^N \) and \( p \in (p^*, \frac{N+2s}{N-2s}) \), with fast decaying at infinity. (also see [27] for the fractional Yamabe problem with isolated singularities in the Sobolev critical case).

It is known that the isolated singularity of solutions to (1.3) in the Serrin’s critical case is totally different, since it doesn’t blow up with a negative-power function, therefore it is much more difficult to analyze the isolated singularities. The main challenges arise from the nonlinear property: One is that the solutions must remain in \( L^1(\mathbb{R}^N) \) in some type of scaling in the blow-up analysis; the second is the tricky calculation of the fractional Laplacian for the involved logarithmic functions; the last and the most challenging is to obtain the singular solutions for the fractional problem, while we don’t have phrase plane analysis for the existence.

Our aim in this paper is two-fold: the first is to classify the positive singular solutions of the fractional problem (1.3) in the Serrin’s critical case and the second is to show the existence of singular solutions. Let us first introduce some notations. For the critical case, we note that \( -\frac{2s}{p^*-1} = 2s - N \). For \( \tau \in (-N, 2s) \), we denote

\[
C_\tau (\tau) = 2^{2s}\frac{\Gamma(\frac{N+2s}{2})\Gamma(\frac{2s-\tau}{2})}{\Gamma(-\frac{\tau}{2})\Gamma(\frac{N-2s+\tau}{2})}
\tag{1.5}
\]
where $\nu > s$. Let
\[ C_s'(0) = -2^{2s-1} \frac{\Gamma\left(\frac{N}{2s}\right)\Gamma(s)}{\Gamma\left(\frac{N+2s}{2s}\right)} < 0. \] (1.6)

Let
\[ K_s = \left( - C_s'(0) \frac{N - 2s}{2s} \right)^{\frac{N-2s}{2s}}, \] (1.7)

then
\[ \lim_{s \to 1^-} K_s = \left( \frac{(N - 2)^2}{2} \right)^{\frac{N-2s}{s}} = K_1. \] (1.8)

See Appendix A for the details of the proof.

The classification of isolated singularities of (1.1) states as follows:

**Theorem 1.1** Let $B_1 \subset \Omega \subset B_{R_{0}}$ for $R_{0} \geq 1$, $h$ be a nonnegative function in $C^{\theta}(B_{2R_{0}}) \cap L^{1}_{s}(\mathbb{R}^{N})$ with $\theta > 2s$, and $u$ be a positive solution of (1.2), then $u$ has either a removable singularity at the origin or

\[ \frac{K_{s}}{C_{0}} \leq \liminf_{|x| \to 0^+} u(x)(|x|(-\ln|x|)^{\frac{1}{2s}})^{N-2s} \leq K_{s} \leq \limsup_{|x| \to 0^+} u(x)(|x|(-\ln|x|)^{\frac{1}{2s}})^{N-2s} \leq C_{0}K_{s}, \]

where $C_{0} \geq 1$ is the best constant in Harnack inequality.

Unlike the Serrin’s super critical case, the bound \( \lim_{|x| \to 0^+} (u(x)|x|^{-2s}) < +\infty \) derived by the blow-up analysis is not sharp. In order to improve the upper bound, we use the Liouville property of the fractional Poisson problem

\[ \begin{cases} \ (-\Delta)^{s}u = \frac{\nu}{|x|^{N-2s}}u + f & \text{in } B_{R_0} \setminus \{0\}, \\ u \geq 0 & \text{in } \mathbb{R}^{N} \setminus B_{R_0}, \end{cases} \]

where $\nu > 0$. For the lower bound, we first obtain a rough one: \( \lim_{|x| \to 0^+} (u(x)|x|^{-\tau}) = +\infty \). We then use some special auxiliary tools to improve it to:

\[ \liminf_{|x| \to 0^+} \left( u(x)(|x|(-\ln|x|)^{\frac{1}{\tau}})^{N-2s} \right) > 0. \]

The improvement of the bound is performed by considering $(-\Delta)^{s}w$ near the origin, where $w(x) = (|x|(-\ln|x|)^{\frac{1}{\tau}})^{2s-N}$ for $|x| > 0$ small, see the details in Proposition 2.1 below.

Because of the nonlocal property, we can’t transform our problem into an ODE and it is a challenging problem is to get the precise blow-up behavior:

\[ \lim_{|x| \to 0^+} u(x)(|x|(-\ln|x|)^{\frac{1}{\tau}})^{N-2s} = K_s. \]

To achieve this goal, develop new techniques have to be involved.

Our second purpose is to construct isolated singular solutions of (1.1).

**Theorem 1.2** Let $h = 0$ and $\Omega = B_1$, then there exists $k^{\ast} \in \mathbb{R}$ such that for any $k \in (-\infty, k^{\ast})$ problem (1.3) has a positive singular solution $u_k$ such that

\[ \lim_{|x| \to 0^+} \left( u_k(x) - K_s (|x|(-\ln|x|)^{\frac{1}{\tau}})^{2s-N}) \right) |x|^{-2s}(-\ln|x|)^{\frac{1}{\tau}} = k. \]

Notice that a positive solution of (1.3) with a removable singularity could be derived by Mountain Pass theorem. Our construction of singular solutions is based on the observation that some special functions could be sub-solutions near the origin and the approximation procedure is applied to obtain singular solutions by suitable estimates and scaling property.

The derivation of a sequence of isolated singular solutions parameterized in the second blow-up rate is totally new and our construction is also appropriate to the laplacian case.
Corollary 1.1 Let $p = \frac{N}{N-2}$, then there exists $k^* \in \mathbb{R}$ such that for any $k \in (-\infty, k^*)$ problem (1.2) has a positive singular solution $u_k$ such that

$$\lim_{|x| \to 0^+} \left( u_k(x) - K_1(|x|(-\ln |x|)^{\frac{1}{2}})^{2-N} \right) |x|^{N-2}(-\ln |x|)^{\frac{N}{2}} = k.$$ 

It is important to mention that the positive singular solutions in general bounded domain can be derived by approximation and scaling techniques. On the contrary, in the whole space problem (1.1) has no positive solution. More precisely:

Proposition 1.1 Problem

$$(-\Delta)^s u = u^p \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}$$

(1.9)

has no positive solutions.

The nonexistence in Proposition 1.1 isn’t new, [23, Theorem 1.3]. However, our methods for all the other results are new and self-contained. In fact, for the Serrin’s critical case, we use some initial asymptotic at infinity to transform the critical problem into a subcritical case for an underlying problem with the fractional Hardy operator.

The remainder of this paper is organized as follows. Section 2 includes basic tools for the fractional Laplacian, the fundamental calculations of fractional Laplacian on functions involving the logarithmic functions. In Section 3, we obtain the blow-up upper and lower bounds and we improve the isolated singularity to prove Theorem 1.1 in Section 4. Section 5 is devoted to the existence of isolated singular solutions and prove Theorem 1.2. Finally, we discuss the constants $C_s(0)$ and $K_s$, the radially symmetric for singular solutions and the nonexistence of positive solution of (1.1) in $\mathbb{R}^N \setminus \{0\}$.

2 Preliminary

2.1 Poisson problem

We start this section by recalling some important facts about the fractional Poisson problem

$$\begin{cases}
(-\Delta)^s u = f & \text{in} \quad \Omega \setminus \{0\}, \\
u = 0 & \text{in} \quad \mathbb{R}^N \setminus \Omega.
\end{cases}$$

(2.1)

Here and in what follows, for the sake of simplicity, we always set $B_1 \subset \Omega \subset B_{R_0}$ for some $R_0 \geq 1$.

Theorem 2.1 [25, Theorem 1.3] Let $f \in C^\theta_{\text{loc}}(\Omega \setminus \{0\})$ for some $\theta \in (0,1)$.

(i) If $f \in L^1(\Omega)$, then for every $k \in \mathbb{R}$ there exists a unique solution $u_k \in L^\infty_{\text{loc}}(\mathbb{R}^N \setminus \{0\}) \cap L^1_s(\mathbb{R}^N)$ of the problem (2.1) satisfying the distributional identity

$$\int_\Omega u_k(-\Delta)^s \xi \, dx = \int_\Omega f \xi \, dx + k\xi(0) \quad \text{for all} \quad \xi \in C^2_0(\Omega).$$

(2.2)

If moreover $f \in L^\infty(\Omega, |x|^\rho dx)$ for some $\rho < 2s$, then $u_k$ has the asymptotics

$$\lim_{x \to 0^+} u(x)|x|^{N-2s} = \frac{k}{c_{s,0}} \quad \text{with} \quad c_{s,0} := c_{N,s} \omega_{N-1} \int_0^1 \int_{B_1(0)} \frac{|z|^{2s-N-1}}{|c_1 - z|^{N+2s}}dzdt.$$

(2.3)

(ii) Assume that $f$ is nonnegative and $\int_\Omega f \, dx = +\infty$. Then problem (2.7) has no nonnegative distributional solution $u \in L^\infty_{\text{loc}}(\mathbb{R}^N \setminus \{0\}) \cap L^1_s(\mathbb{R}^N)$. 

5
Motivated by above theorem, we can obtain the nonexistence of solutions for

\[
\begin{cases}
(-\Delta)^s u = f & \text{in } \Omega \setminus \{0\}, \\
u = h & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]  

(2.4)

where \( h \in C^2(B_{2R_0}) \cap L^1_1(\mathbb{R}^N) \) is nonnegative.

**Theorem 2.2** Let \( f \) be a nonnegative function in \( C^0_{\text{loc}}(\Omega \setminus \{0\}) \) with \( \beta \in (0,1) \), and \( h \) be a nonnegative function in \( C^2(B_{2R_0}) \cap L^1_1(\mathbb{R}^N) \).

Then problem (2.4) has no positive solution, if

\[
\lim_{r \to 0^+} \int_{B_{r_0}(0) \setminus B_r(0)} f(x) \, dx = +\infty.
\]

 Particularly, the above assumption could be replaced by

\[
\liminf_{|x| \to 0^+} |x|^N (-\ln |x|) \nu > 0 \quad \text{for } \nu \leq 1.
\]

To prove Theorem 2.2, we need the following comparison principle.

**Lemma 2.1** Assume that \( O \) is a bounded, Lipschitz continuous domain containing the origin, \( f_i \in C^0_{\text{loc}}(\Omega \setminus \{0\}) \), \( h_i \in C^2(B_{2R_0}) \cap L^1_1(\mathbb{R}^N) \) and \( u_i \) with \( i = 1,2 \) are classical solutions of

\[
\begin{cases}
(-\Delta)^s u_i = f_i & \text{in } O \setminus \{0\}, \\
u_i = h_i & \text{in } \mathbb{R}^N \setminus O
\end{cases}
\]

(2.5)

and

\[
\limsup_{|x| \to 0^+} u_1(x)|x|^{N-2s} \leq \liminf_{|x| \to 0^+} u_2(x)|x|^{N-2s}.
\]

If \( f_1 \leq f_2 \) in \( O \setminus \{0\} \) and \( h_1 \leq h_2 \) in \( \mathbb{R}^N \setminus O \), then

\[
u_1 \leq \nu_2 \text{ in } O \setminus \{0\}.
\]

**Proof.** Let \( u = u_1 - u_2 \) and then

\[
(-\Delta)^s u \leq 0 \quad \text{in } O \setminus \{0\} \quad \text{and} \quad \limsup_{|x| \to 0^+} u(x)\Phi_s^{-1}(x) \leq 0,
\]

then for any \( \epsilon > 0 \), there exists \( r_\epsilon > 0 \) converging to zero as \( \epsilon \to 0 \) such that

\[
u \leq \epsilon \Phi_s \quad \text{in } B_{r_\epsilon}(0) \setminus \{0\},
\]

where \( \Phi_s(x) = |x|^{2s-N} \) is the fundamental solution of \( (-\Delta)^s \) in \( \mathbb{R}^N \).

We see that

\[
u = 0 < \epsilon \Phi_s \quad \text{in } \mathbb{R}^N \setminus O,
\]

then we have that \( u \leq \epsilon \Phi_s \) in \( O \setminus \{0\} \). By letting \( \epsilon \) go to zero, we have that \( u \leq 0 \) in \( O \setminus \{0\} \). \qed

**Proof of Theorem 2.2** By contradiction, we assume that \( u \) is a positive solution of (2.4). Let \( f_n(x) = (1 - \eta_0(nx))f(x) \), where \( \eta_0 : \mathbb{R}^N \to [0,1] \) is a radially symmetric, smooth function such that

\[
\eta_0 = 0 \quad \text{in } \mathbb{R}^N \setminus B_2 \quad \text{and} \quad \eta_0 = 1 \quad \text{in } B_1.
\]

Since \( f_n \) is Hölder continuous, it follows by [15] Theorem 4.1 that

\[
\begin{cases}
(-\Delta)^s v = f_n & \text{in } O \setminus \{0\}, \\
v = 0 & \text{in } \mathbb{R}^N \setminus O
\end{cases}
\]

\[
\lim_{|x| \to 0^+} v(x)|x|^{N-2s} = 0
\]

(2.6)
has unique solution \( v_n \) and

\[
0 < v_n \leq u \quad \text{in } \Omega \setminus \{0\}
\]

by Lemma 2.1 and the assumption that \( h \) is non-negative.

By the stability results [12, Theorem 2.4] and the regularity result [12, Theorem 2.1], the limit \( \{v_n\}_n \) exists, denoting \( v_\infty \), is a positive classical solution of

\[
\begin{aligned}
(-\Delta)^s v &= f \quad \text{in } \Omega \setminus \{0\}, \\
v &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{aligned}
\]

then we obtain a contradiction from Theorem 2.1 part (ii). \( \square \)

### 2.2 Estimates

For the calculations of the fractional Laplacian for the involved logarithmic functions, we need the following estimates. Recall that for \( \tau \in (-N, 2s) \)

\[
C_s(\tau) = 2^{2s} \frac{\Gamma\left(\frac{N+\tau}{2}\right)\Gamma\left(\frac{2s-\tau}{2}\right)}{\Gamma\left(\frac{\tau}{2}\right)\Gamma\left(\frac{N-2s+\tau}{2}\right)}
\]

and then

\[
C_s(\tau) = 0
\]

has two zero points \( 0, 2s - N \). Moreover, there holds

\[
(-\Delta)^s |\cdot|^\tau = C_s(\tau) |\cdot|^{-2s} \quad \text{in } \mathcal{S}'(\mathbb{R}^N)
\]

and

\[
C_s(\tau) = -\frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \frac{|e_1 + z|^\tau + |e_1 - z|\tau - 2}{|z|^{N+2s}} \, dz,
\]

where \( e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^N \).

**Lemma 2.2** [15, Lemma 2.3] The function \( C_s \), defined in (2.8), is strictly concave and uniquely maximized at the point \( \frac{2s-N}{2} \) with the maximal value \( 2^{2s-2^s} \frac{\Gamma\left(\frac{N+\tau}{2}\right)\Gamma\left(\frac{N-2s+\tau}{2}\right)}{\Gamma\left(\frac{\tau}{2}\right)\Gamma\left(\frac{N-2s+\tau}{2}\right)} \).

Moreover,

\[
C_s(\tau) = C_s(2s - N - \tau) \quad \text{for } \tau \in (-N, 2s)
\]

(2.10)

and

\[
\lim_{\tau \to -N} C_s(\tau) = \lim_{\tau \to 2s} C_s(\tau) = -\infty.
\]

(2.11)

Note that

\[
C_s(\tau) > 0 \quad \text{for } \tau \in (2s - N, 0),
\]

\[
C_s(\tau) < 0 \quad \text{for } \tau \in (0, 2s) \cup (-N, 2s - N)
\]

and

\[
C'_s(2s - N) = -C'_s(0) > 0 \quad \text{and} \quad C''_s(2s - N) = C''_s(0) < 0.
\]

For \( m \in \mathbb{R} \) denote \( v_m \) be a smooth, radially symmetric function such that

\[
v_m(x) = (-\ln |x|)^m \quad \text{for } 0 < |x| < \frac{1}{e^2} \quad \text{and} \quad v_m(x) = 0 \quad \text{for } |x| > 1.
\]

(2.12)

Moreover, \( v_m \) is non-increasing in \( |x| \) if \( m > 0 \). Denote

\[
w_m(x) = |x|^{2s-N} v_m(x) \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}.
\]

(2.13)
Proposition 2.1 Let $m \neq 0$ and

$$B_m = C'_s(0)m, \quad D_m = C''_s(0)\frac{m(m-1)}{2},$$

then $B_m > 0$ for $m < 0$, and there exist $r_0 \in (0, \frac{1}{c^5})$ and $c_5 > 0$ such that

$$\left| (-\Delta)^s w_m(x) - B_m |x|^{-N}(-\ln |x|)^{m-1} - D_m |x|^{-N}(-\ln |x|)^{m-2} \right| \leq c_5 |x|^{-N}(-\ln |x|)^{m-3}.$$  

In order to get estimate (2.15), we need the following lemmas.

Lemma 2.3 Let $m < 0$, then there exist $r_0 \in (0, \frac{1}{c_6})$ and $c_6 > 0$ such that for $0 < |x| < r_0$

$$|E_m(x) - 2B_m |x|^{-N}(-\ln |x|)^{m-1}| \leq c_6 |x|^{-N}(-\ln |x|)^m.$$  

Proof. A direct computation shows that

$$C'_s(\tau) = \frac{-c_{N,s}}{2} \int_{\mathbb{R}^N} \frac{|e_1 + z|^\tau \ln |e_1 + z| + |e_1 - z|^\tau \ln |e_1 - z|}{|z|^{N+2s}} \, dz$$

and

$$C''_s(\tau) = \frac{-c_{N,s}}{2} \int_{\mathbb{R}^N} \frac{(|e_1 + z|^\tau - 1) \ln |e_1 + z|}{|z|^{N+2s}} \, dz + \int_{\mathbb{R}^N} \frac{|e_1 - z|^{\tau - 1} \ln |e_1 - z|}{|z|^{N+2s}} \, dz + \int_{\mathbb{R}^N} \frac{\ln |e_1 - z| + \ln |e_1 + z|}{|z|^{N+2s}} \, dz$$

Henceforth

$$C'_s(0) = \frac{-c_{N,s}}{2} \int_{\mathbb{R}^N} \frac{\ln |e_1 - z| + \ln |e_1 + z|}{|z|^{N+2s}} \, dz$$
Therefore, we obtain some important equalities:

\[ C'_s(\tau) - C'_s(0) = c_{N,s} \int_{\mathbb{R}^N} \frac{(1 - |z|^7)\ln|z|}{|e_1 - z|^{N+2s}}dz, \]

\[ C''_s(\tau) - C''_s(0) = c_{N,s} \int_{\mathbb{R}^N} \frac{(1 - |z|^7)(-\ln|z|)^2}{|e_1 - z|^{N+2s}}dz \]

and

\[ c_{N,s} \int_{\mathbb{R}^N} \frac{(1 - |z|^{2s-N})(-\ln|z|)}{|e_1 - z|^{N+2s}}dz = C'_s(2s - N) - C'_s(0) = -2C'_s(0) > 0, \]

\[ c_{N,s} \int_{\mathbb{R}^N} \frac{(1 - |z|^{2s-N})(-\ln|z|)^2}{|e_1 - z|^{N+2s}}dz = C''_s(2s - N) - C''_s(0) = 0. \]

Let \(0 < |x| < r_0\) with \(r_0 > 0\) small enough, we see that

\[
\left|\frac{1}{\ln|z|}\right| \leq \frac{1}{2} \quad \text{for} \quad |z| < \frac{1}{\sqrt{r_0}} \text{ and}
\]

\[ 1 + \frac{-\ln|z|}{-\ln|x|} = -m - \frac{-\ln|z|}{-\ln|x|} + O\left(\frac{-\ln|z|}{-\ln|x|}^2\right). \]

Thus, we see that

\[
\left|\frac{-2C'_s(0)}{c_{N,s}} - \int_{\mathbb{R}^N \setminus B_{\frac{1}{\sqrt{r_0}}}(e_1)} \frac{(1 - |z|^{2s-N})(-\ln|z|)}{|e_1 - z|^{N+2s}}dz \right|
\]

\[
= \left|\int_{\mathbb{R}^N \setminus B_{\frac{1}{\sqrt{r_0}}}(e_1)} \frac{(1 - |z|^{2s-N})(-\ln|z|)}{|e_1 - z|^{N+2s}}dz \right|
\]

\[
\leq \int_{\mathbb{R}^N \setminus B_{\frac{1}{\sqrt{r_0}}}(e_1)} \frac{-\ln|z|}{|e_1 - z|^{N+2s}}dz
\]

\[
\leq c_7 (-\ln|x|)|x|^s
\]
and

\[
\left| \int_{B_{\sqrt{m}(e_1)}} \frac{(1 - |z|^{2s-N})(-\ln |z|)^2}{|e_1 - z|^{N+2s}} \, dz \right|
\]

\[
= \left| \int_{\mathbb{R}^N} \frac{(1 - |z|^{2s-N})(-\ln |z|)^2}{|e_1 - z|^{N+2s}} \, dz - \int_{\mathbb{R}^N \setminus B_{\sqrt{m}(e_1)}} \frac{(1 - |z|^{2s-N})(-\ln |z|)^2}{|e_1 - z|^{N+2s}} \, dz \right|
\]

\[
\leq \int_{\mathbb{R}^N \setminus B_{\sqrt{m}(e_1)}} \frac{(-\ln |z|)^2}{|e_1 - z|^{N+2s}} \, dz
\]

\[
\leq c_7 \left( \frac{1}{2} \ln |x| \right)^2 |x|^s
\]

by using (2.16).

As a consequence, we obtain that

\[
\int_{B_{\sqrt{m}(x)}} \frac{(|x|^{2s-N} - |y|^{2s-N})(v_m(x) - v_m(y))}{|x - y|^{N+2s}} \, dy
\]

\[
= 2B_m |x|^{-N} (-\ln |x|)^{m-1} \left( 1 + O \left( |x|^s (-\ln |x|) \right) \right). \tag{2.17}
\]

On the other hand, we see that

\[
\left| \int_{\mathbb{R}^N \setminus B_{\sqrt{m}(x)}} \frac{(|x|^{2s-N} - |y|^{2s-N})(v_m(x) - v_m(y))}{|x - y|^{N+2s}} \, dy \right|
\]

\[
= |x|^{2s-N} \left( (-\ln |x|)^m + 1 \right) \int_{\mathbb{R}^N \setminus B_{\sqrt{m}(x)}} \frac{1}{|x - y|^{N+2s}} \, dy
\]

\[
\leq c_8 |x|^{s-N} \left( (-\ln |x|)^m + 1 \right),
\]

where \( c_8 > 0 \) is independent of \( |x| \) and

\[
v_m(y) \leq v_m(x) \quad \text{for} \quad |y| > |x|.
\]

Together with (2.16), we obtain (2.15). Then we complete the proof. \(
\square
\)

**Lemma 2.4** There exist \( r_0 \in (0, \frac{1}{2r}] \) and \( c_9 > 0 \) such that for \( 0 < |x| < r_0 \)

\[
|F_m(x) + B_m |x|^{-N} (-\ln |x|)^{m-1} - D_m |x|^{-N} (-\ln |x|)^{m-2}| \leq c_9 |x|^{-N} (-\ln |x|)^{m-3}. \tag{2.18}
\]

**Proof.** For \( x \in B_{\frac{1}{2r}} \setminus \{0\} \), we see that

\[
(-\Delta)^s v_m(x) = \frac{C_{N,s}}{2} \int_{B_{\sqrt{m}}} \frac{2(-\ln |x|)^m - (-\ln |x + y|)^m - (-\ln |x - y|)^m}{|y|^{N+2s}} \, dy
\]

\[
+ C_{N,s} \int_{\mathbb{R}^N \setminus B_{\sqrt{m}}} \frac{(-\ln |x|)^m - v_m(x) - v_m(y)}{|y|^{N+2s}} \, dy,
\]

where

\[
0 < C_{N,s} \int_{\mathbb{R}^N \setminus B_{\sqrt{m}}} \frac{(-\ln |x|)^m - v_m(x + y)}{|y|^{N+2s}} \, dy
\]

\[
\leq C_{N,s} \int_{\mathbb{R}^N \setminus B_{\sqrt{m}}} \frac{(-\ln |x|)^m}{|y|^{N+2s}} \, dy
\]

\[
\leq c_{10} |x|^{-s} (-\ln |x|)^m.
\]

10
Moreover, there holds

\[
\frac{c_{N,s}}{2} \int_B \frac{2(- \ln |x|)^m - (- \ln |x + y|)^m - (- \ln |x - y|)^m}{|y|^{N + 2s}} dy
\]

\[
= \frac{c_{N,s}}{2} |x|^{-2s}(- \ln |x|)^m \int_B \frac{2 - (1 + \frac{-\ln z + \varepsilon_1}{-\ln |x|})^m - (1 + \frac{-\ln z - \varepsilon_1}{-\ln |x|})^m}{|z|^{N + 2s}} dz
\]

\[
= \frac{c_{N,s} m(m - 1)}{4} |x|^{-2s}(- \ln |x|)^{m-1} \int_B \frac{(\ln |e_1 - z|)^2 + (\ln |e_1 + z|)^2}{|z|^{N + 2s}} dz + |x|^{-2s}(- \ln |x|)^{m-3} O\left( \int_{\mathbb{R}^N} \frac{(\ln |e_1 - z|)^3 + (\ln |e_1 - z|)^3}{|z|^{N + 2s}} dz \right)
\]

\[
= m |x|^{-2s}(- \ln |x|)^{m-1} \left( -C_s''(0) - c_{N,s} \int_{\mathbb{R}^N \setminus B} \frac{\ln |e_1 - z|}{|z|^{N + 2s}} dz \right) + \frac{m(m - 1)}{2} |x|^{-2s}(- \ln |x|)^{m-2} (C_s''(0) - c_{N,s} \int_{\mathbb{R}^N \setminus B} \frac{(\ln |e_1 - z|)^2}{|z|^{N + 2s}} dz + |x|^{-2s}(- \ln |x|)^{m-3} O(1)
\]

\[
= |x|^{-2s}(- \ln |x|)^{m-1} \left( -mC_s''(0) + \frac{m(m - 1)}{2} C_s''(0)(- \ln |x|)^{-1} + (- \ln |x|)^{-2} O(1) \right).
\]

where we used $|\frac{-\ln z + \varepsilon_1}{-\ln |x|}| \ll 1$ when $|x|$ small enough and

\[
(1 + t)^m = 1 + mt + \frac{m(m - 1)}{2} t^2 + O(t^3).
\]

As a consequence, we conclude that

\[
\left| (-\Delta)^s v_m(x) + mC_s''(0)|x|^{-2s}(- \ln |x|)^{m-1} - \frac{m(m - 1)}{2} C_s''(0)|x|^{-2s}(- \ln |x|)^{m-2} \right|
\]

\[
\leq c_{11} |x|^{-2s}(- \ln |x|)^{m-3},
\]

which implies (2.18). We complete the proof. \( \Box \)

**Proof of Proposition 2.1.** It follows by Lemma 2.3 and Lemma 2.4 directly. \( \Box \)

From the proof of Lemma 2.4, we have the following corollary.

**Corollary 2.1.** Let $m \neq 0$ then there exist $r_0 \in (0, \frac{1}{m})$ and $c_{12} > 0$ such that for $0 < |x| < r_0$

\[
\left| (-\Delta)^s v_m(x) + B_m|x|^{-2s}(- \ln |x|)^{m-1} - D_m|x|^{-2s}(- \ln |x|)^{m-2} \right| \leq c_{12} |x|^{-2s}(- \ln |x|)^{m-3}. \tag{2.19}
\]

More generally, for $\tau \in (-N, 2s)$, $m \in \mathbb{R}$, let

\[
w_{\tau,m}(x) = |x|^\tau v_m(x) \quad \text{for} \quad x \in \mathbb{R}^N \setminus \{0\}.
\]

The same calculation implies that
Corollary 2.2 Let $\tau \in (-N, 2s)$, $m \in \mathbb{R}$, $|\tau| + |m| \neq 0$ and
\[
\mathcal{B}_{\tau, m} = C_{\delta}^2(\tau) m,
\]
then there exist $r_0 \in (0, \frac{1}{M})$ and $c_{13} > 0$ such that for $0 < |x| < r_0$
\[
\left|(-\Delta)^s w_{\tau, m}(x) - C_{\delta}(\tau) w_{\tau, m}(x) |x|^{-2s} - B_{\tau, m} |x|^{\tau - 2s}(- \ln |x|)^{m - 1}\right| \leq c_{13} |x|^{\tau - 2s}(- \ln |x|)^{m - 2}. \tag{2.20}
\]

2.3 Blow-up rate estimates
The following estimates play an important role in our classification of isolated singularities of (1.1). In what follows, we always let $r_0 \in (0, \frac{1}{M})$ be from Proposition 2.1.

Lemma 2.5 Let $g \in C^3_{\text{loc}}(\Omega \setminus \{0\})$, with $\beta \in (0, 1)$, be a nonnegative function such that for some $m < 0$
\[
g(x) \geq |x|^{-N}(- \ln |x|)^{m - 1} \quad \text{in} \quad B_{r_0} \setminus \{0\}.
\]
Let $u_g$ be a positive solution of problem
\[
\begin{cases}
(-\Delta)^s u = g & \text{in} \quad \Omega \setminus \{0\}, \\
u \geq 0 & \text{in} \quad \mathbb{R}^N \setminus \Omega.
\end{cases}
\tag{2.21}
\]
Then there exists $r \in (0, r_0]$ such that
\[
u_g(x) \geq B_m |x|^{2s-N}(- \ln |x|)^{m} \quad \text{in} \quad B_r \setminus \{0\}.
\]

Proof. Recall that $w_m$ is defined in (2.13), has compact support in $B_1$ and
\[
w_m(x) = |x|^{2s-N}(- \ln |x|)^{m} \quad \text{in} \quad B_{\frac{1}{r_0}} \setminus \{0\}
\]
by Proposition 2.1, there exists $r_1 \in (0, r_0]$ such that
\[
(-\Delta)^s w_m \leq B_m |x|^{-N}(- \ln |x|)^{m - 1} + D_m |x|^{-N}(- \ln |x|)^{m - 2}
+ O(1)|x|^{-N}(- \ln |x|)^{m - 3}
\leq B_m |x|^{-N}(- \ln |x|)^{m - 1} \quad \text{for} \quad x \in B_{r_1} \setminus \{0\},
\]
where $B_m > 0$ and $D_m < 0$ for $m < 0$.
Note that $u_g$ is positive and continuous in $\Omega \setminus \{0\}$, then $u_g \geq 0$ in $\mathbb{R}^N \setminus B_{r_0}$. By the lower bound of $g$ there exists $t_0 = B_m > 0$ such that
\[
(-\Delta)^s t_0 u_g \geq t_0 g(x) \geq t_0 |x|^{-N}(- \ln |x|)^{m - 1}
\geq (-\Delta)^s w_m
\geq (-\Delta)^s (w_m - w_m(r_1)) \quad \text{in} \quad B_{r_1} \setminus \{0\}
\]
and
\[
\liminf_{|x| \to 0^+} u_g \Phi_\delta^{-1}(x) \geq 0 = \lim_{|x| \to 0^+} (w_m - w_m(r_1)) \Phi_\delta^{-1}(x),
\]
\[
t_0 u_g \geq 0 \geq (w_m - w_m(r_1)) \quad \text{in} \quad \mathbb{R}^N \setminus B_{r_1}.
\]
Then by Lemma 2.1, we have that
\[
t_0 u_g \geq w_m - w_m(r_1) \quad \text{in} \quad \Omega \setminus \{0\}.
\]
This completes the proof. \qed
Lemma 2.6 Let $g \in C^{2\beta}_{\loc}(\Omega \setminus \{0\})$, with $\beta \in (0,1)$, be a nonnegative function such that
\[ g(x) \leq |x|^{-N}(-\ln |x|)^{m-1} \quad \text{in } B_r \setminus \{0\}. \]

Let $h \in C(B_{2R_0}) \cap L^1_1(\mathbb{R}^N)$ be a nonnegative function such that
\[ h = 0 \quad \text{in } B_{\frac{r}{2}}. \]

Let $u_g$ be a positive solution of the problem:
\[
\begin{aligned}
&(-\Delta)^s u = g & \text{in } \Omega \setminus \{0\}, \\
u = h & \text{in } \mathbb{R}^N \setminus \Omega,
\end{aligned}
\]
\[\lim_{|x| \to 0^+} u(x)|x|^{N-2s} = 0.\]

Then for any $\epsilon \in (0,B_m)$, there exists $r \in (0,r_0]$ such that
\[ u_g(x) \leq (B_m + \epsilon)|x|^{2s-N}(-\ln |x|)^m \quad \text{in } B_r \setminus \{0\}. \]

Proof. Let $U_0(x) = -(-\Delta)^s h(x)$ for $x \in B_{r_0}$. A direct computation shows that for $x \in B_{r_0}$
\[
0 < U_0(x) = c_{N,s} \int_{\mathbb{R}^N \setminus B_{\frac{r}{2}}} \frac{h(y)}{|x-y|^{N+2s}} dy 
\leq c_{N,s} \int_{\mathbb{R}^N \setminus B_{\frac{r}{2}}} \frac{h(y)}{|y-r_0|^{N+2s}} dy
\]
by using the fact that
\[ |x-y| \geq |y| - r_0 \quad \text{for } |x| \leq r \text{ and } |y| \geq \frac{1}{2}, \]
and taking into account that
\[ w_m(x) = |x|^{2s-N}(-\ln |x|)^m \quad \text{in } B_{\frac{r}{2}} \setminus \{0\} \]
and that by Proposition [2.14] for given $\epsilon \in (0,B_m)$, there exists $r_2 \in (0,r_0]$ such that
\[
(-\Delta)^s w_m \geq B_m |x|^{-N}(-\ln |x|)^{m-1} - c_{13}|x|^{-N}(-\ln |x|)^{m-2} 
\geq (B_m - \epsilon)|x|^{-N}(-\ln |x|)^{m-1} \quad \text{for } x \in B_{r_2} \setminus \{0\}. \]

Note that $u_g$ is positive and continuous in $\Omega \setminus \{0\}$, then $u_g \leq \varphi_0$ in $\Omega \setminus B_{r_2}$ for some $\varphi_0 > 0$. By the upper bound of $g$ there exists $t_1 = (B_m - 2\epsilon) > 0$ such that
\[
(-\Delta)^s t_1 (u_g - \varphi_0 - h) \leq t_1 (g(x) - U_0) 
\leq (B_m - \epsilon)|x|^{-N}(-\ln |x|)^{m-1} 
\leq (-\Delta)^s w_m 
\leq (-\Delta)^s (w_m - w_m(r_1)) \quad \text{in } B_{r_2} \setminus \{0\},
\]
where we used that $U_0$ is bounded in $B_{r_2}$. Moreover, we have that
\[
\liminf_{|x| \to 0^+} (u_g(x) - \varphi_0 - h(x))\Phi_s^{-1}(x) = 0 = \lim_{|x| \to 0^+} w_m \Phi_s^{-1}(x)
\]
and
\[ t_1 (u_g - \varphi_0 - h) = t_1 (u_g - \varphi_0) \leq 0 \leq w_m \quad \text{in } \mathbb{R}^N \setminus B_{r_2}. \]
Then by Lemma [2.1] we have that
\[ t_1 (u_g - \varphi_0 - h) \leq w_m \quad \text{in } \Omega \setminus \{0\}. \]
This completes the proof. \[\square\]

13
Lemma 2.7 Let \( g \in C^\beta_{\text{loc}}(\Omega \setminus \{0\}) \), with \( \beta \in (0,1) \), be a nonnegative function such that there exists \( \tau \in (2s-N, +\infty) \) such that
\[
g(x) \leq |x|^{\tau - 2s} \quad \text{in } B_{r_0} \setminus \{0\}.
\]
Let \( h \in C(B_{2r_0}) \cap L^1_*(\mathbb{R}^N) \) be a nonnegative function such that
\[
h = 0 \quad \text{in } B_{3r_0}.
\]
Let \( u_g \) be a positive solution of problem
\[
\begin{cases}
(\Delta)^s u \leq g & \text{in } \Omega \setminus \{0\}, \\
u = h & \text{in } \mathbb{R}^N \setminus \Omega, \\
\lim_{|x| \to 0^+} u(x)|x|^{N-2s} = 0.
\end{cases}
\tag{2.24}
\]
Then
\begin{itemize}
\item[(i)] for \( \tau \in (2s-N,0) \) there exists \( c_4 \geq 0 \) such that
\[
u_g(x) \leq c_4|x|^{\tau} \quad \text{in } \Omega \setminus \{0\};
\]
\item[(ii)] for \( \tau \in (0,2s) \), there exists \( c_5 \geq 0 \) such that
\[
u_g(x) \leq c_5 \quad \text{in } \Omega \setminus \{0\}.
\end{itemize}

Proof. Recall that \( U_0(x) = -(\Delta)^s h(x) \) for \( x \in B_{r_0} \), which is bounded by (2.23).

(i) For \( \tau \in (2s-N,0) \), we have that
\[
(\Delta)^s |x|^{\tau} = \mathcal{C}(\tau)|x|^{\tau-2s} \quad \text{in } \mathbb{R}^N \setminus \{0\},
\]
where \( \mathcal{C}(\tau) > 0 \).

Note that there exists \( t_4 > 0 \) such that
\[
(\Delta)^s t_4 u_g - h \leq t_4 (g(x) + U_0) \leq \mathcal{C}(\tau)|x|^{\tau-2s} = (\Delta)^s |x|^{\tau} \quad \text{in } \Omega \setminus \{0\}
\]
and
\[
\lim_{|x| \to 0^+} u_g(x)|x|^{N-2s} = 0, \quad u_g = h \quad \text{in } \mathbb{R}^N \setminus \Omega.
\]

Then by Lemma 2.1 we have that
\[
t_4 u_g(x) \leq |x|^{\tau} \quad \text{for } x \in \Omega \setminus \{0\}.
\]

(ii) Take
\[
\tilde{w}_3(x) = (2R_0)^{\tau} - |x|^{\tau} \quad \text{in } \mathbb{R}^N \setminus \{0\}
\]
and
\[
\omega_3(x) = (2R_0)^{\tau} - |x|^{\tau} \quad \text{in } B_{2R_0} \setminus \{0\}, \quad \omega_3(x) = 0 \quad \text{in } \mathbb{R}^N \setminus B_{R_0}.
\]

Direct computation shows that for \( x \in B_{R_0} \setminus \{0\} \)
\[
(\Delta)^s \omega_3(x) = (\Delta)^s \tilde{w}_3(x) + (\Delta)^s (w_3 - \tilde{w}_3)(x)
\]
\[
\geq -\mathcal{C}(\tau)|x|^{\tau - 2s} - c_{N,s}(2R_0)^{\tau} \int_{\mathbb{R}^N \setminus B_{2R_0}} \frac{1}{|x-y|^{N+2s}}dy
\]
\[
\geq -\mathcal{C}(\tau)|x|^{\tau - 2s} - c_{N,s}(2R_0)^{\tau - 2s} \int_{\mathbb{R}^N \setminus B_1} \frac{1}{|x-z|^{N+2s}}dz
\]
\[
\geq -\mathcal{C}(\tau)|x|^{\tau - 2s} - 2^{N+\tau} c_{N,s}R_0^{-2s} \int_{\mathbb{R}^N \setminus B_1} \frac{1}{|z|^{N+2s}}dz,
\]
\[14\]
where $-C_s(\tau) > 0$ for $\tau \in (0, 2s)$, $\tilde{e}_x = \frac{x}{|x|}$ and $|\tilde{e}_x - z| \leq \frac{|z|}{2}$. Thus, there exists $r \in (0, 1)$ such that for $x \in B_r \setminus \{0\}$

$$(-\Delta)^s w_3(x) \geq -\frac{1}{2} C_s(\tau)|x|^{-2s}.$$  

Consequently there exists $t_4 > 0$ such that

$$(-\Delta)^s t_4 (u_g - h) \leq t_4 (g + U_0) \leq -\frac{1}{2} C_s(\tau)|x|^{-2s} \leq (-\Delta)^s w_3 \quad \text{in} \quad B_r \setminus \{0\}$$

and

$$w_3 \geq (2^\tau - 1)R_0^\tau \geq t_4 (u_g - h) \quad \text{in} \quad \Omega \setminus B_r,$$

since $\Omega \subset B_{R_0}$. Together with

$$\lim_{|x| \to 0^+} u_g(x)|x|^{N-2s} = 0, \quad u_g = h \quad \text{in} \quad \mathbb{R}^N \setminus B_r,$$

it implies by Lemma 2.1 that

$$t_4 (u_g - h) \leq w_3 \quad \text{for any} \quad x \in B_r \setminus \{0\}.$$  

Therefore, we obtain that

$$u_g(x) \leq c_{15} \quad \text{for any} \quad x \in \Omega \setminus \{0\}.$$  

This completes the proof. \hfill \Box

3 Isolated singularity

In this section, we provide rough bounds for the isolated singular solution of (1.1).

**Theorem 3.1** Let $u$ be a positive solution of (1.1) verifying

$$\lim_{|x| \to 0^+} u(x) = +\infty,$$

then

**upper bound** : $\limsup_{|x| \to 0^+} u(x)|x|^{N-2s} < +\infty$

and for any $\tau \in (2s - N, 0)$

**lower bound** : $\lim_{|x| \to 0^+} u(x)|x|^{-\tau} = +\infty.$

3.1 Upper bound

**Proposition 3.1** Let $h \in C^\theta(B_{2R_0}) \cap L^1(\mathbb{R}^N)$ with $\theta > 2s$, and $u$ be a nonnegative solution of (1.1), then there exist $r_1 \in (0, \frac{1}{2})$ and $c_{16} > 0$ such that

$$u(x) \leq c_{16}|x|^{2s-N}, \quad \forall x \in B_{r_1} \setminus \{0\}. \quad (3.1)$$

In order to prove Proposition 3.1 we need following lemma.

**Lemma 3.1** Assume that $h \in C^\theta(B_{2R_0}) \cap L^1(\mathbb{R}^N)$ with $\theta > 2s$, and $u$ is a nonnegative classical solution of (1.1) replaced $p^*$ by $p > 1$. Then $u^p \in L^1(\Omega, \rho dx)$ and there exists a uniform $c_{17} > 0$ independent of $u$ such that

$$\int_{\Omega} u^p \rho(x)^p dx < c_{17}, \quad (3.2)$$

where $\rho(x) = \text{dist}(x, \partial \Omega)$.  

15
Proof. Recall that $U_0(x) = -(\Delta)^+ h(x)$ for $x \in \Omega$, which is uniformly bounded in $\Omega$. Let $w = u - h$ in $\mathbb{R}^N$, then we have that

$$(-\Delta)^+ w = u^p + U_0 \quad \text{in} \quad \Omega \setminus \{0\}, \quad w = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega$$

From Theorem 2.1, we have that $w^p - U_0 \in L^1(\Omega)$, so is $u$, thanks to the boundedness of $U_0$. Moreover, we have that

$$\int_\Omega w(-\Delta)^+ \xi \, dx = \int_\Omega (u^p + U_0) \xi \, dx \quad \text{for all} \quad \xi \in C_0^\infty(\Omega).$$

Let $(\lambda_1, \xi_1)$ be the first eigenvalue and related positive eigenfunction of $(-\Delta)^+$ in $\Omega$ with zero Dirichlet boundary condition, i.e.

$$(-\Delta)^+ \xi_1 = \lambda_1 \xi_1 \quad \text{in} \quad \Omega, \quad \xi_1 = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \Omega.$$

The existence and properties could see [33, Proposition 9]. In fact, we have that $\lambda_1 > 0$, $\xi_1$ is positive, $\xi_1 \in C_{loc}^2(\Omega) \cap C^s(\mathbb{R}^N)$ and $\xi(x) \sim \rho^s(x)$ as $\rho(x) \to 0$. Moreover, for some $c_{18} > 1$, there holds

$$\frac{1}{c_{18}} \rho^s \leq \xi_1 \leq c_{18} \rho^s \quad \text{in} \quad \Omega.$$

Using $\xi_1$ as a test function, we have that

$$\int_\Omega w^p \xi_1 \, dx + \int_\Omega U_0 \xi_1 \, dx = \lambda_1 \int_\Omega w \xi_1 \, dx \leq \lambda_1 \left( \int_\Omega w^p \xi_1 \, dx \right)^\frac{1}{p} \left( \int_\Omega \xi_1 \, dx \right)^{1 - \frac{1}{p}},$$

which implies

$$\int_\Omega w^p \rho^s \, dx \leq c_{17},$$

where $c_{17} > 0$ depends on $h$. □

Proof of Proposition 3.1. Suppose by contradiction that there exists a sequence of points $\{x_k\} \subset B_{r_0} \setminus \{0\}$ and a sequence of solutions $u_k$ of (1.1) (we can take the same function if $\lim \sup_{|x| \to 0^+} |x|^N u(x) = +\infty$) such that $|x_k| \to 0^+$ as $k \to +\infty$ and

$$|x_k|^{-2s} u_k(x_k) \to +\infty \quad \text{as} \quad k \to +\infty.$$

We can choose $x_k$ again such that

$$|x_k|^{-2s} u_k(x_k) = \max_{x \in \Omega \setminus B_{|x_k|}} |x_k|^{-2s} u_k(x) \to +\infty \quad \text{as} \quad k \to +\infty \quad (3.3)$$

by the fact that the mapping $r \mapsto \max_{x \in \Omega \setminus B_r} \sigma(x)^{-2s} u_k(x)$ is nondecreasing.

We denote

$$\phi_k(x) := \left( \frac{|x_k|}{2} - |x - x_k| \right)^{-2s} u_0(x) \quad \text{for} \quad |x - x_k| \leq \frac{|x_k|}{2}.$$

Let $\bar{x}_k$ be the maximum point of $\phi_k$ in $B_{|x_k|/2}(x_k)$, that is,

$$\phi_k(\bar{x}_k) = \max_{|x - x_k| \leq \frac{|x_k|}{2}} \phi_k(x).$$

Set

$$\nu_k = \frac{1}{2} \left( \frac{|x_k|}{2} - |\bar{x}_k - x_k| \right),$$

16
then $0 < 2\nu_k < \frac{|x_k|}{2}$ and
\[ \frac{|x_k|}{2} - |x - x_k| \geq \nu_k \text{ for } |x - x_k| \leq \nu_k. \]
By the definition of $\phi_k$, for any $|x - x_k| \leq \nu_k$,
\[ (2\nu_k)^{N-2s}u_0(\bar{x}_k) = \phi_k(\bar{x}_k) \geq \phi_k(x_k) \geq (\nu_k)^{N-2s}u_0(x), \]
which implies that
\[ 2^{N-2s}u_0(\bar{x}_k) \geq u_0(x) \text{ for any } |x - x_k| \leq \nu_k. \]
Moreover, we see that
\[ |\bar{x}_k|^{N-2s}u_0(\bar{x}_k) \geq (2\nu_k)^{N-2s}u_0(\bar{x}_k) = \phi_k(\bar{x}_k) \geq \phi_k(x_k) \geq \left(\frac{|x_k|}{2}\right)^{N-2s}u_0(x_k) \rightarrow +\infty \text{ as } k \rightarrow +\infty \]
by the fact that $|\bar{x}_k| \geq \frac{|x_k|}{2} \geq 2\nu_k$. Denote
\[ W_k(y) = \frac{1}{u_k(\bar{x}_k)}u_k(u_k(\bar{x}_k) - \frac{1}{\nu_k} y - \bar{x}_k), \quad \forall y \in \Omega_k \setminus \{X_k\}, \]
where
\[ \Omega_k := \left\{ y \in \mathbb{R}^N : u_k(\bar{x}_k) - \frac{1}{\nu_k} y - \bar{x}_k \in \Omega \right\} \]
and
\[ X_k = u_k(\bar{x}_k) \frac{1}{\nu_k} x_k. \]
Note that
\[ |X_k| = (u_k(\bar{x}_k)|\bar{x}_k|^{N-2s})^{\frac{1}{N+2s}} \rightarrow +\infty \text{ as } k \rightarrow +\infty. \]
Thus, we have that for $y \in \Omega_k \setminus \{X_k\}$
\[ (-\Delta)^s W_k(y) = \frac{1}{u_k(\bar{x}_k)}(-\Delta)^s u_k(u_k(\bar{x}_k) - \frac{1}{\nu_k} y - \bar{x}_k) \]
\[ = \frac{1}{u_k(\bar{x}_k)} u_k^\nu (u_k(\bar{x}_k) - \frac{1}{\nu_k} y - \bar{x}_k) \]
\[ = W_k^\nu(y), \]
that is
\[ (-\Delta)^s W_k(y) = W_k^\nu(y) \quad \text{for} \quad x \in \Omega_k \setminus \{X_k\}. \tag{3.4} \]
We claim that there is $c_{18} > 0$ independent of $k$ such that
\[ \|W_k\|_{L^1(\mathbb{R}^N)} \leq c_{18} \]
and for any $\epsilon > 0$, there exists $k_1 > 0$ and $R > 0$ such that
\[ \int_{\mathbb{R}^N \setminus B_R(0)} W_k(y)(1 + |y|)^{-N-2s}dy \leq \epsilon. \tag{3.5} \]
In fact, since $|\bar{x}_k| \rightarrow 0$, we see that
\[ 0 \leq \int_{\mathbb{R}^N \setminus \Omega_k} W_k(y)(1 + |y|)^{-N-2s}dy \]
\[ = \frac{1}{u_k(\bar{x}_k)} \int_{\mathbb{R}^N \setminus \Omega_k} u_k(u_k(\bar{x}_k) - \frac{1}{\nu_k} y - \bar{x}_k)(1 + |y|)^{-N-2s}dy \]
\[ = \frac{1}{u_k^p(\bar{x}_k)} \int_{\mathbb{R}^N \setminus \Omega} h(z) (u_k(\bar{x}_k) \frac{1}{|z - \bar{x}_k|^s} + |z - \bar{x}_k|)^{-N-2s} dz \]
\[ \leq \frac{1}{u_k^p(\bar{x}_k)} \int_{\mathbb{R}^N} h(z)(1 + |z - \bar{x}_k|)^{-N-2s} dz \]
\[ \leq \frac{2}{u_k^p(\bar{x}_k)} \int_{\mathbb{R}^N} h(z)(1 + |z|)^{-N-2s} dz \rightarrow 0 \quad \text{as} \ k \rightarrow +\infty. \]

Taking \( r_k = |\bar{x}_k| u_k(\bar{x}_k) \frac{1}{\|\bar{x}_k\|} \rightarrow +\infty \) as \( k \rightarrow +\infty \), we obtain that
\[
0 \leq \int_{B_{r_k}(x_k)} W_k(y)(1 + |y|)^{-N-2s} dy \\
\leq \frac{r_k^{-N-2s} u_k(\bar{x}_k)}{u_k(x_k)} \int_{B_{r_k}(x_k)} u_k(\bar{x}_k) \frac{1}{|\bar{x}_k - y|^s} dy \\
= \frac{r_k^{-N-2s} u_k(\bar{x}_k)}{u_k(x_k)} \int_{B_{\frac{1}{2}r_k}} u_k(z) dz \\
\leq \frac{r_k^{-N-2s} u_k(\bar{x}_k)}{u_k(x_k)} \int_{B_{\frac{1}{2}r_k}} u_k^p(z) dz \left( \int_{B_{\frac{1}{2}r_k}} dz \right)^{1 - \frac{1}{p}} \\
= c_{19} r_k^{-N-2s} u_k(\bar{x}_k) \int_{B_{\frac{1}{2}r_k}} |\bar{x}_k|^{2s} \left( \int_{B_{\frac{1}{2}r_k}} u_k^p(z) dz \right)^{\frac{1}{p}} \\
\leq c_{19} r_k^{-N} \rightarrow 0 \quad \text{as} \ k \rightarrow +\infty
\]
by Lemma 3.1.

Moreover, \( W_k(y) \leq 1 \) in \( \Omega_k \setminus B_{r_k} \), then
\[
\int_{\Omega_k \setminus B_{r_k}(x_k)} W_k(y)(1 + |y|)^{-N-2s} dy \leq \int_{\Omega_k \setminus B_{r_k}(x_k)} (1 + |y|)^{-N-2s} dy \leq \int_{\mathbb{R}^N} (1 + |y|)^{-N-2s} dy
\]
and
\[
\int_{(\Omega_k \setminus B_{r_k}(x_k)) \setminus B_R} W_k(y)(1 + |y|)^{-N-2s} dy \leq \int_{\mathbb{R}^N \setminus B_R} (1 + |y|)^{-N-2s} dy \leq c_{20} R^{-2s},
\]
where \( c_{20} > 0 \) is independent of \( k \). Then (3.8) holds true and the claim is proved.

Note that \( 0 < W_k \leq 2^{N-2s} \) in \( B_{\hat{r}_k} \), where
\[ \hat{r}_k = \nu_k m_k \rightarrow +\infty \quad \text{as} \ k \rightarrow +\infty, \]
then for any \( R > 0 \), there exists \( k_R \) for any \( k \geq k_R \)
\[
\|W_k\|_{C^{2s,\alpha}(B_{\hat{r}_k})} \leq c_{20} \left( \|W_k\|_{L^1(\mathbb{R}^N)} + \|W_k\|_{L^\infty(B_{2\hat{r}_k})} + \|W_k^p\|_{L^\infty(B_{2\hat{r}_k})} \right) \\
\leq c_{20} \left( \|W_k\|_{L^1(\mathbb{R}^N)} + 2^{N-2s} \right),
\]
where \( \alpha \in (0, s) \) and \( c_{20} > 0 \).

Since \( 0 < W_k \leq c_0 \) in \( B_{r_k} \), so for any \( R > 0 \), there exists \( k_R \) for \( k \geq k_R \)
\[
\|W_k\|_{C^{2s,\alpha}(B_{R})} \leq \|W_k\|_{L^1(\mathbb{R}^N)} + \|W_k^p\|_{L^\infty(B_{r_k})},
\]

18
where $\alpha \in (0, s)$.

Since $R$ is arbitrary, up to subsequence, there exists a nonnegative function $W_\infty \in L^\infty(\mathbb{R}^N)$ such that as $k \to +\infty$

$$W_k \to W_\infty \quad \text{in} \quad C^{2s+\alpha'}_{loc}(\mathbb{R}^N),$$
$$W_k \to W_\infty \quad \text{in} \quad L^1_s(\mathbb{R}^N),$$
$$0 < W_\infty \leq 2^{N-2s}$$

for some $\alpha' 

(3.6)

For any $\epsilon > 0$, by $(3.6)$, we have that

$$\frac{1}{c_{N,s}} (-\Delta)^s W_k(x) = \text{p.v.} \int_{B_R} \frac{W_k(x) - W_k(y)}{|x-y|^{N+2s}} dy + W_k(x) \int_{\mathbb{R}^N \setminus B_R} \frac{1}{|x-y|^{N+2s}} dy$$
$$- \int_{\mathbb{R}^N \setminus B_R} \frac{W_k(y)}{|x-y|^{N+2s}} dy$$

$$=: E_{1,k}(x) + E_{2,k}(x) - E_{3,k}(x), \quad \text{respectively.}$$

Furthermore, by $(3.5)$ we have that

$$\left| E_{1,k}(x) - \text{p.v.} \int_{B_R} \frac{w_\infty(x) - w_\infty(y)}{|x-y|^{N+2s}} dy \right| < \frac{\epsilon}{3}$$

and there exists $R_1 > 0$ such that for $R > R_1$

$$\left| E_{2,k}(x) - w_\infty(x) \int_{\mathbb{R}^N \setminus B_R} \frac{1}{|x-y|^{N+2s}} dy \right| < \frac{\epsilon}{3}.$$

Therefore, we conclude that

$$0 < E_{3,k}(x) \leq \int_{\mathbb{R}^N \setminus B_R} W_k(y)(1 + |y|)^{-N-2s} dy < \frac{\epsilon}{3}$$

for $k > 0$ and $R$ large enough.

Note that $0 < W_k \leq 2^{N-2s}$ in $B_{\tilde{r}_k}$, where

$$\tilde{r}_k = \nu_k u_k(\tilde{x}_k) \frac{1}{\nu_k^{1-2s}} = (\nu_k^{N-2s} u_k(\tilde{x}_k))^{\frac{1}{1-2s}} \to +\infty \quad \text{as} \quad k \to +\infty,$$

Then for any $R > 0$, there exists $k_R$ for any $k \geq k_R$

$$\|W_k\|_{C^{2s+\alpha'}(B_R)} \leq c_{20} \left(\|W_k\|_{L^1_s(\mathbb{R}^N)} + \|W_k^p\|_{L^\infty(B_{\tilde{r}_k})}\right) \leq c_{20} \left(\|W_k\|_{L^1_s(\mathbb{R}^N)} + 2^{N-2s}\right), \quad (3.7)$$

where $\alpha \in (0, s)$ and $c_{20} > 0$. Therefore, we conclude that

$$\lim_{k \to +\infty} (-\Delta)^s W_k(x) = (-\Delta)^s W_\infty(x)$$

and $W_\infty$ is a classical solution of

$$(-\Delta)^s W_\infty = W_\infty^p \quad \text{in} \quad \mathbb{R}^N$$

satisfying

$$0 \leq W_\infty \leq 2^{N-2s}.$$  

Since $W_\infty(0) = 1$, then $w_\infty > 0$ in $\mathbb{R}^N$, thanks to the nonnegative property of $W_\infty$. By $[16]$ Theorem 3 or $[17]$ Theorem 4.5, problem $(3.8)$ has no bounded positive solution since $p^* \in (1, \frac{N+2s}{N-2s})$. This completes the proof.

The upper bound in Proposition 3.1 is to obtain the Harnack inequality for singular solution of $(1.1)$.  

19
Proposition 3.2 Let \( u \) be a nonnegative solution of (1.1). Then there exists \( C_0 > 0 \) independent of \( u \) such that for all \( r \in (0, r_0) \)

\[
\sup_{x \in B_{2r} \setminus B_r} u(x) \leq C_0 \left( \inf_{x \in B_{2r} \setminus B_r} u(x) + \|u\|_{L^1_2(\mathbb{R}^N)} \right).
\]

If additionally \( u \) is singular at the origin, then for all \( r \in (0, r_0) \)

\[
\sup_{x \in B_{2r} \setminus B_r} u(x) \leq C_0 \inf_{x \in B_{2r} \setminus B_r} u(x). \tag{3.9}
\]

**Proof.** Note that \( B_{r_0} \subset \Omega \) for some \( r_0 > 0 \) and without loss of the generality, we set \( r_0 = 1 \). Now for fixed \( x_0 \in \mathbb{R}^N \) verifying \(|x_0| = \frac{1}{2}\), from there exists \( C > 0 \) independent of \( u_0 \) such that for any \( t \in (0, \frac{1}{t_1}) \)

\[
\sup_{x \in B_t(x_0)} u_0(x) < C.
\]

Let

\[
w(x) = u_0(x) (1 - \eta_0(4x)) - h(x) \text{ in } \mathbb{R}^N,
\]

where \( \eta_0 \) be a smooth function such that \( \eta_0 = 1 \) in \( B_1(0) \) and \( \eta_0 = 0 \) in \( \mathbb{R}^N \setminus B_2 \), we recall that \( h = 0 \) in \( B_4 \). Then

\[
(-\Delta)^s w = |x|^s u_0^{p^* - 1} w + (-\Delta)^s h + (-\Delta)^s (u_0 \eta_0(4x)) \text{ for } x \in B_t(x_0),
\]

where \( (-\Delta)^s h > 0 \) in \( B_1 \). Note that \( 0 < u_0^{p^* - 1} \leq c_{15} \) for some \( c_{15} > 0 \) independent of \( u_0 \) and

\[
0 < (-\Delta)^s h(x) + (-\Delta)^s (u_0(x) \eta_0(4x)) \leq c_{22} \left( \|h\|_{L^1_2(\mathbb{R}^N)} + \|u_0\|_{L^1_2(\mathbb{R}^N)} \right) \leq 2c_{22} \|u_0\|_{L^1(\mathbb{R}^N)}.
\]

Then \([55, \text{Theorem } 1.1]\) (also see \([10, \text{Theorem } 11.1]\)) implies that

\[
\sup_{x \in B_t(x_0)} u_0(x) \leq C_1 \left( \inf_{x \in B_t(x_0)} u_0(x) + \|u_0\|_{L^1_2(\mathbb{R}^N)} \right),
\]

which infers

\[
\sup_{x \in B_{2r} \setminus B_{r_0}} u_0(x) \leq C_1 \left( \inf_{x \in B_{2r} \setminus B_{r_0}} u_0(x) + \|u_0\|_{L^1_2(\mathbb{R}^N)} \right) \tag{3.10}
\]

by finite covering argument, the scaling property and the upper bound of \( u_0 \).

Now we do the scaling:

\[
u_r(x) = r^{N-2s} u_0(rx)
\]

for \( r \in (0, \frac{1}{t_1}) \). Then \( u_r \) also verifies \([11, \text{Theorem } 1.1]\) and from Proposition \([6, \text{Theorem } 1.1]\) we have

\[
 r^{2s} u_r(x)^{p^* - 1} \leq C \text{ for } r < |x| < 2r,
\]

where \( C \) is dependent of \( t \).

It follows by \([3,10]\) that

\[
\sup_{x \in B_{2r} \setminus B_r} u_0(x) \leq C_0 \left( \inf_{x \in B_{2r} \setminus B_r} u_0(x) + \|u_0\|_{L^1_2(\mathbb{R}^N)} \right).
\]

Thanks to

\[
\lim_{|x| \to 0^+} u_0(x) = +\infty,
\]

we obtain \([3,9]\). This completes the proof. \(\square\)
3.2 Lower bound

**Proposition 3.3** Let $u$ be a nonnegative solution of (1.1) with non-removable singularity

\[ \lim_{|x| \to 0^+} u(x) = +\infty. \]

Then for any $\tau \in (2s - N, 0)$

\[ \lim_{|x| \to 0^+} u(x)|x|^{-\tau} = +\infty. \quad (3.11) \]

**Proof.** By contradiction, we suppose that (1.1) has a solution $u_0$ such that

\[ \liminf_{|x| \to 0^+} u_0(x)|x|^{-\tau} < +\infty \]

for $\tau \in (2s - N, 0)$. This gives

\[ \limsup_{|x| \to 0^+} u_0(x)|x|^{-\tau} < +\infty \]

by Harnack inequality (3.9).

Note that for some $r > 0$ and $d_0 > 0$

\[ u_0^p(x) \leq d_0 |x|^\tau p^* \quad \text{for } x \in B_r \setminus \{0\}, \]

where

\[ \tau p^* + 2s < N. \]

Let $\tau_0 = \tau > 2s - N$ and

\[ \tau_1 := p^* \tau_0 + 2s, \]

then

\[ \tau_1 - \tau_0 = \frac{2s}{N - 2s} \left( \tau - (N - 2s) \right) > 0. \]

If $\tau_1 > 0$, by Lemma 2.7 part (ii), we know that

\[ u_0(x) \leq c_{23} \quad \text{for } x \in B_r \setminus \{0\} \]

which ends the proof since

\[ \lim_{|x| \to +\infty} u_0(x) = +\infty. \]

If $\tau_1 \in (2s - N, 0]$, by Lemma 2.7 part (i), we have that

\[ u_0(x) \leq d_1 |x|^\tau_1 \quad \text{for } x \in B_{\tau_0}(0) \setminus \{0\}. \]

Iteratively, we recall that

\[ \tau_j := p\tau_{j-1} + \theta + 2s, \quad j = 1, 2, \ldots. \]

Note that

\[ \tau_1 - \tau_0 = (p - 1)\tau_0 + \theta + 2s > 0 \]

thanks to $\tau_0 > \frac{2s + \theta}{p - 1}$.

If $\tau_j p + \theta + 2s \leq \tau_+ (s, \mu)$ the proof is complete, otherwise, it follows by Lemma 2.7 that

\[ u_0(x) \geq d_{j+1} |x|^\tau_{j+1}, \]

where

\[ \tau_{j+1} = p^* \tau_j + 2s < \tau_j. \]

We claim that $\{\tau_j\}_j$ is an increasing sequence and there exists $j_0 \in \mathbb{N}$ such that

\[ \tau_{j_0} \leq 0 \quad \text{and} \quad \tau_{j_0 - 1} > 0. \quad (3.12) \]
In fact, for $\tau_0 > 2s - N$,

$$\tau_j - \tau_{j-1} = p^*(\tau_{j-1} - \tau_{j-2}) = (p^*)^{j-1}(\tau_1 - \tau_0) \to +\infty \text{ as } j \to +\infty,$$

which implies that the sequence $\{\tau_j\}$ is increasing.

Thus there exists $j_0 \in \mathbb{N}$ such that

$$\tau_{j_0} \leq 0, \quad \tau_{j_0+1} > 0$$

and

$$(-\Delta)^s u_0(x) \leq c_{23} d_{j_0} \rho^s |x|^{\tau_{j_0}+1-2s} \text{ in } B_{r_0}(0) \setminus \{0\},$$

then $\sup_{x \in \Omega \setminus \{0\}} u_0(x) < +\infty$ and which contradicts the fact that $u_0$ has non-removable singularity at the origin. □

With the help of Proposition 3.1 and Proposition 3.3 we are in a position to show Theorem 3.1

Proof of Theorem 3.1 The upper bound and lower bound follow Proposition 3.1 and Proposition 3.3 respectively. We complete the proof. □

4 Improved singularity

4.1 Important estimates

For $R > 0$, let $u_0$ be a positive solution of

$$\begin{cases}
(-\Delta)^s u = u^{p^*} & \text{in } B_R \setminus \{0\}, \\
u = 0 & \text{in } \mathbb{R}^N \setminus B_R, \\
\lim_{|x| \to 0^+} u = +\infty.
\end{cases}$$

(4.1)

In this section, we will improve the isolated singularity of $u_0$ at the origin.

Theorem 4.1 For any $R > 0$, any positive solution $u_0$ of (4.1) is radially symmetric, strictly decreasing with respect to $|x|$.

The proof of Theorem 4.1 is addressed in Appendix B.

We consider the function

$$w_0 = v_{-m_0} u_0 \text{ in } \mathbb{R}^N,$$

(4.2)

where $u_0$ is a positive solution of (4.1); $v_{-m_0}$ be a smooth, radially symmetric function non-increasing with respect to $|x|$ satisfying (2.12) with

$$m_0 = -\frac{N - 2s}{2s} < 0.$$

Direct computation shows that $w_0$ verifies

$$\begin{cases}
(-\Delta)^s w_0 = v_{-m_0} u_0^{p^*} - L_s u_0 + Q_1 u_0 & \text{in } B_R \setminus \{0\}, \\
w_0 = 0 & \text{in } \mathbb{R}^N \setminus B_R,
\end{cases}$$

(4.3)

where

$$Q_1(x) = (-\Delta)^s v_{-m_0}(x)$$

and

$$L_s u_0(x) = c_{N,s} \int_{\mathbb{R}^N} \frac{(u_0(x) - u_0(y))(v_{-m_0}(x) - v_{-m_0}(y))}{|x - y|^{N+2s}} dy.$$
Lemma 4.1 Let $u_0$ be a nonnegative classical solution of (4.1), $w_0$ be given in (4.2) and

$$F(x) = v_{-m_0}w_0^+ - L_s u_0 + Q_1 u_0 \quad \text{in } B_R. \quad (4.4)$$

For any $\kappa_0 \in (0, (-B_{-m_0})^{-\gamma - r})$, there exists $\bar{r} \in (0, \min\{\frac{R}{2}, \frac{1}{\kappa_0}\})$ such that

$$u_0(x) \leq \kappa_0 |x|^{2\gamma - N}(-\ln |x|)^{-m_0} \quad \text{in } 0 < |x| < \bar{r},$$

then

$$F(x) \leq 0 \quad \text{for } x \in B_R \setminus \{0\} \quad (4.5)$$

and $u_0$ is bounded.

**Proof.** We see that the functions $v_{-m_0}, u_0$ are radially symmetric and decreasing with respect to $|x|$ by the definition of $v_{-m_0}$ and Theorem 4.1. Thus, we have that

$$(u_0(x) - u_0(y))(v_{-m_0}(x) - v_{-m_0}(y)) > 0 \quad \text{if } |y| \neq |x|$$

and

$$-L_s u_0 < 0.$$  

From the upper bound of $u_0$ in our assumption, we have that for $0 < |x| < \min\{\frac{R}{2}, \frac{1}{\kappa_0}\}$

$$v_{-m_0}(x)w_0^+ \leq \kappa_0^+ (-\ln |x|)^{-N}.$$  

Using (2.13), we have that

$$\left|(-\Delta)^s v_{-m_0} - B_{-m_0} |x|^{-2\gamma}(-\ln |x|)\frac{2\gamma}{2\gamma - 1}\right| \leq c_{24} |x|^{-2\gamma}(-\ln |x|)^{\frac{2\gamma}{2\gamma - 2}}, \quad \forall 0 < |x| < r_0.$$  

Thus, for any $\epsilon > 0$, there exists $\bar{r} \in (0, \min\{\frac{R}{2}, r_0\})$ such that

$$Q_1(x) \leq (B_{-m_0} + \epsilon)|x|^{-2\gamma}(-\ln |x|)\frac{2\gamma}{2\gamma - 1} \quad \text{for } 0 < |x| < \bar{r}, \quad (4.6)$$

where $B_{-m_0} < 0$. This implies that for $0 < |x| < \bar{r}$

$$Q_1(x)u_0(x) \leq \kappa_0(B_{-m_0} + \epsilon)(-\ln |x|)^{-N}.$$  

Note that $v_{-m_0}w_0^+ + Q_1 u_0 - L_s u_0 \leq \kappa_0^{\gamma - 1} + B_{-m_0} + \epsilon)(-\ln |x|)^{-1}|x|^{-N}, \quad x \in B_R \setminus \{0\}$

by an appropriate choice of $\epsilon$ could taking $\kappa_0^{\gamma - 1} + B_{-m_0} + \epsilon < 0$ and that $\kappa_0 < (-B_{-m_0})^{-\gamma - r}$. Thus, we have that $F \leq 0$ in $B_r \setminus \{0\}$ and $F_+ = \max\{F, 0\}$ is bounded in $B_R \setminus \{0\}$, where $F_+ = \max\{F, 0\}$.

Denote $\mathcal{G}_s[f]$ the Green operator of $f \in L^1(B_R)$ by

$$\mathcal{G}_s[f](x) = \int_{B_R} G_s(x, y)f(y)dy \quad \text{for } x \in B_R,$$

where $G_s(\cdot, \cdot)$ is the Green kernel of $(-\Delta)^s$ subject to the zero Dirichlet condition in $\mathbb{R}^N \setminus B_R$. There is some $c_{24} > 0$ independent of $R$ such that $G_s(x, y) \leq c_{24}|x - y|^{2\gamma - N}$ for $x \neq y$. Here we refer to [13] for the properties of Green kernel. Thus $\mathcal{G}_s[F_+]$ is bounded.

From Lemma 2.1 we have that

$$0 \leq w_0(x) \leq \mathcal{G}_s[F_+](x).$$

Consequently,

$$u_0(x) \leq (-\ln |x|)^{-m_0} \quad \text{in } B_R \setminus \{0\}.$$
For some \( \tau' \in (2s - N, 0) \) and \( c_{24} > 0 \), we have:

\[
    u_0(x) \leq c_{24}|x|^\tau' \quad \text{in} \quad B_{r} \setminus \{0\}. \tag{4.7}
\]

Now we prove that \( u_0 \) is bounded. If not, we can assume that

\[
    \limsup_{|x| \to 0^+} u_0(x) = +\infty,
\]

which implies by the Harnack inequality Proposition 3.2 that

\[
    \lim_{|x| \to 0^+} u_0(x) = +\infty
\]

From Theorem 3.1, we have that

\[
    \liminf_{|x| \to 0^+} u_0(x)|x|^{-\tau'} = +\infty,
\]

which contradicts (4.7). So \( u_0 \) is bounded.

To improve the upper bound, we need to consider the Liouville type theorem for the fractional Poisson problem with a weak Hardy potential

\[
    \begin{cases}
        (-\Delta)^{s} u = \frac{\nu}{|x|^{2s-N}} u + f & \text{in} \quad B_{R_1} \setminus \{0\}, \\
        u \geq 0 & \text{in} \quad \mathbb{R}^N \setminus B_{R_1},
    \end{cases} \tag{4.8}
\]

where \( R_1 \in (0, \frac{1}{\nu}) \) and \( f : B_{R_1} \setminus \{0\} \to \mathbb{R} \) is Hölder continuous locally in \( B_{R_1} \setminus \{0\} \).

**Lemma 4.2** Let \( \nu > 0 \), and \( f \) be a nonnegative function such that \( f \in C^2_{\text{loc}}(B_{R_1} \setminus \{0\}) \) for some \( \beta \in (0, 1) \). The homogeneous problem (4.8) has no positive solution if

\[
    \liminf_{|x| \to 0^+} f(x) |x|^{N} (-\ln |x|)^{1+\frac{\nu}{(s-\beta)(N)}} > 0. \tag{4.9}
\]

**Proof.** By contradiction, we assume that problem (4.8) has a positive solution \( u_0 \).

From the assumption (4.9), we take

\[
    m_1 = \frac{\nu}{C_{\beta}(0)} \in (-\infty, 0). \tag{4.10}
\]

Let us set:

\[
    \liminf_{|x| \to 0^+} f(x) |x|^{N} (-\ln |x|)^{1-m_1} = 2k_0,
\]

then there exists \( r_1 \in (0, \min\{r_0, R_0\}) \) such that

\[
    f(x) \geq \tilde{k}_0 |x|^{-N} (-\ln |x|)^{m_1-1} \quad \text{for} \quad 0 < |x| < r_1,
\]

where \( B_{m_1} > 0 \) for \( m_1 < 0 \). Let \( W_0 \) be the solution of

\[
    \begin{cases}
        (-\Delta)^{s} u = \tilde{k}_0 |x|^{-N} (-\ln |x|)^{m_1-1} \chi_{B_{r_1}} & \text{in} \quad B_{R_1} \setminus \{0\}, \\
        u = 0 & \text{in} \quad \mathbb{R}^N \setminus B_{R_1},
    \end{cases} \tag{4.11}
\]

where \( \chi_{B_{r_1}} \) is the characterized function of \( B_{r_1} \). By direct comparison with \( w_{m_1}(R_0^{-1} \cdot) \) and Lemma 3.4 with \( m = m_1 \) and \( D_{m_1} < 0 \), by re-choice of \( r_1 \) if necessary, we have that

\[
    W_0(x) \geq \frac{\tilde{k}_0}{B_{m_1}} |x|^{2s-N} (-\ln |x|)^{m_1} \quad \text{for} \quad 0 < |x| < r_1.
\]

Then Lemma 3.1 implies that

\[
    u_0(x) \geq W_0(x) \geq \frac{\tilde{k}_0}{B_{m_1}} |x|^{2s-N} (-\ln |x|)^{m_1} \quad \text{for} \quad 0 < |x| < r_1
\]

24
and now let

$$k_0 = \frac{\bar{k}_0}{B_{m_1}}$$

then

$$\frac{\nu}{|x|^{2s(-\ln |x|)}}u_0(x) \geq k_0 \nu |x|^{-N}(-\ln |x|)^{m_1-1} \quad \text{for } 0 < |x| < r_1.$$ 

So we have that

$$(-\Delta)^{s} u_0 \geq (\bar{k}_0 + k_0 \nu)|x|^{-N}(-\ln |x|)^{m_1-1} \quad \text{for } 0 < |x| < r_1.$$ 

Then by Lemma 2.4 again, we have that

$$u_0 \geq k_1 W_0 \quad \text{for } 0 < |x| < R_0,$$

where

$$k_1 = \frac{\bar{k}_0 + k_0 \nu}{B_{m_1}} = k_0 + \frac{\nu}{B_{m_1}} k_0$$

and then for $0 < |x| < r_1$

$$\frac{\nu}{|x|^{2s(-\ln |x|)}}u_0(x) \geq k_1 k_0 |x|^{-N}(-\ln |x|)^{m_1-1}.$$ 

By repeating the above procedure, we have that

$$u_0 \geq k_n W_0 \quad \text{for } 0 < |x| < r_1,$$

where

$$k_n = k_0 + \frac{\nu}{B_{m_1}} k_{n-1}.$$ 

Note that

$$k_n - k_{n-1} = \frac{\nu}{B_{m_1}} (k_{n-1} - k_{n-2}) = \left(\frac{\nu}{B_{m_1}}\right)^n k_0 = k_0,$$

where $\frac{\nu}{B_{m_1}} = 1$ by the choice of $m_1$ in (4.10) and $B_{m_1} = C_s(0)m_1$. Then

$$k_n \to +\infty \quad \text{as } n \to +\infty,$$

which implies that $u_0$ blows up in $B_{r_1}$ and we obtain a contradiction. \qed

### 4.2 Isolated singularity of (1.1)

**Lemma 4.3** Let $h$ be a nonnegative function in $C^{\theta}(B_{2R_0}) \cap L^1_s(\mathbb{R}^N)$ with $\theta > 2s$ and $u_0$ be a positive singular solution of (1.1) such that

$$\limsup_{|x|\to 0^+} u_0(x) (|x|(-\ln |x|)^\frac{\theta}{2s})^{N-2s} < +\infty.$$ 

Then for $r_1 \in (0, \frac{1}{e}]$, there exists $m > 0$ such that problem (4.1) with $r = r_1$ has a positive singular solution $u_1$ such that

$$u_0(x) - (-\ln |x|)^{m+1} \leq u_1(x) \leq u_0(x) \quad \text{in } B_{r_1} \setminus \{0\}.$$ 

**Proof.** From the upper bound assumption, there exists $c_{25} > 0$ such that

$$u_0(x) \leq c_{25} |x|^{2s-N}(-\ln |x|)^{m_0} \quad \text{for } 0 < |x| < r_1.$$ 

25
Let 
\[ v_0 = u_0 \text{ in } \mathbb{R}^N. \]
We denote \( v_n \) the solution of
\[
\begin{cases}
(\Delta)^s v_n = v_{n-1}^p - v_n^p & \text{in } B_{r_1} \setminus \{0\}, \\
v_n = 0 & \text{in } \mathbb{R}^N \setminus B_{r_1}, \\
\lim_{|x| \to 0^+} v_n(x)|x|^{N-2s} = 0.
\end{cases}
\]
(4.12)

Obviously, we have that
\[
(-\Delta)^s (v_1 - v_0) \leq 0
\]
and the comparison principle implies that \( 0 < v_1 \leq v_0 \text{ in } \mathbb{R}^N \setminus \{0\} \). Inductively, the mapping
\[ n \in \mathbb{N} \mapsto v_n \]
is decreasing. Therefore, by stability results \cite{12} Theorem 2.4 (also see \cite{11} Lemma 4.3 for bounded sequence) and the regularity result \cite{12} Theorem 2.1, the limit of \( \{v_n\} \), \( v_\infty \) exists,
\[
\lim_{n \to +\infty} v_n(x) = v_\infty(x) \text{ for any } x \in \mathbb{R}^N \setminus \{0\},
\]
then \( v_\infty \) is a solution of (4.11) and it verifies that
\[ 0 \leq v_\infty \leq v_0 \text{ in } \mathbb{R}^N \setminus \{0\}. \]

Let \( \nu_1 = v_0 - v_1 \), then
\[
(-\Delta)^s \nu_1 \leq 0 \text{ in } B_{r_1} \setminus \{0\}
\]
and
\[
\nu_1 = u_0 \text{ in } \mathbb{R}^N \setminus B_{r_1}, \quad \lim_{|x| \to 0^+} \nu_1(x)|x|^{N-2s} = 0.
\]
Then there exists \( M_0 > 0 \) depending on \( r_1 \) such that
\[ 0 < \nu_1(x) \leq M_0. \]

For \( n = 2, 3, \ldots \), denote
\[ \nu_n = v_{n-1} - v_n, \]
then
\[
(-\Delta)^s \nu_n(x) = v_{n-1}^p(x) - v_n^p(x) \\
\leq v_0^p - v_{n-1}^p(x) \nu_{n-1}(x) \leq c_{25}^{p-1}|x|^{-2s}(-\ln |x|)^{-1}\nu_{n-1}(x).
\]
In particular, we have that
\[
(-\Delta)^s \nu_2(x) \leq c_{25}^{p-1}M_0|x|^{-2s}(-\ln |x|)^{-1}.
\]
Let \( r_1 \leq \frac{1}{4} \) and then for \(-\ln r_1 \geq 2 \) and \( \epsilon_0 > 0 \), there exists \( m > 0 \) such that
\[
(-\Delta)^s \nu_2(x) \leq \epsilon_0|x|^{-2s}(-\ln |x|)^m \text{ for } 0 < |x| < r_1.
\]
Here \( m > 0 \) is chosen such that
\[ \epsilon_0 \geq (-\ln r_1)^{-m+1}c_{25}^{p-1}M_0. \]

Note that for \( 0 < |x| < r_1 \)
\[
(-\Delta)^s v_{m+1}(x) = -B_{m+1}|x|^{-2s}(-\ln |x|)^m + O(1)|x|^{-2s}(-\ln |x|)^m-1 \\
\geq (-B_{m+1} - \epsilon_0)|x|^{-2s}(-\ln |x|)^m,
\]

26
where $-B_{m+1} > 3\epsilon_0 > 0$ since $m + 1 > 0$.

Therefore, we have that
\[
0 < \nu_2(x) \leq M_1 (-\ln |x|)^{m+1},
\]
where
\[
M_1 = \frac{\epsilon_0}{-B_{m+1} - \epsilon_0} < \frac{1}{2}.
\]

By repeating the above process, we can obtain that
\[
0 < \nu_n(x) \leq M_n (-\ln |x|)^{m+1},
\]
where
\[
M_n = M_{n-1} \frac{\epsilon_0}{B_{m+1} - \epsilon_0} = \left( \frac{\epsilon_0}{B_{m+1} - \epsilon_0} \right)^n.
\]

Since
\[
0 < \frac{\epsilon_0}{B_{m+1} - \epsilon_0} < \frac{1}{2},
\]
then
\[
u_0 \geq v_\infty(x) \geq v_0(x) - \sum_{n=1}^{+\infty} \nu_n(x)
\geq v_0(x) - \sum_{n=1}^{+\infty} \left( \frac{\epsilon_0}{B_1 - \epsilon_0} \right)^n (-\ln |x|)^{m+1}
\geq v_0(x) - (-\ln |x|)^{m+1}.
\]

Which completes the proof. \[\Box\]

Proof of Theorem 1.1: Our proof is divided into two parts.

Part I: the Ball $\Omega = B_1$ and $h \equiv 0$. Let $u_0$ be a positive solution of (1.1) and recall that
\[
K_s = B_{\frac{N-2s}{2}}.
\]

Set
\[
k = \liminf_{|x| \to 0^+} u_0(x) (|x|(-\ln |x|)^{\frac{1}{2}})^{N-2s}.
\]
(4.13)

If $k = 0$, from the Harnack inequality, we have that
\[
\lim_{|x| \to 0^+} u_0(x) (|x|(-\ln |x|)^{\frac{1}{2}})^{N-2s} = 0
\]
and Lemma 4.1 implies that $u_0$ is bounded.

So if $u_0$ has a non-removable singularity at the origin, the Harnack inequality implies
\[
\lim_{|x| \to 0^+} u_0(x) = +\infty,
\]
then we have that $k > 0$.

Now we claim
\[
k \leq K_s.
\]

In fact, if
\[
k \in (K_s, +\infty],
\]

letting
\[
\epsilon_0 = \frac{k - K_s}{2K_s},
\]

27
then by Lemma 2.3, there exists \( r_1 \in (0, \frac{1}{s}) \) such that

\[
u(x) \geq K_s \left( 1 + \epsilon_0 \right) (|x|(- \ln |x|) \frac{1}{4})^{2s-N} \quad \text{for} \quad 0 < |x| < r_1,
\]

then for \( 0 < |x| < r_1 \)

\[
u \geq K^* \left( 1 + \epsilon_0 \right)^p \left( |x|(- \ln |x|) \frac{1}{4} \right)^{-N} = B_{m_o} \left( 1 + \epsilon_0 \right)^p \left( |x|(- \ln |x|) \frac{1}{4} \right)^{-N}.
\]

Therefore, \( u_0 \) verifies that

\[
\begin{align*}
(-\Delta)^s u_0 &= B_{m_o} \quad \text{in} \quad B_{r_0} \setminus \{0\}, \\
u_0 &\geq 0 \quad \text{in} \quad \mathbb{R}^N \setminus B_{r_0},
\end{align*}
\]

where

\[
f(x) = \left( u_0^{p^* - 1} - \frac{B_{m_o}}{|x|^{2s(- \ln |x|)}} \right) u_0(x) \geq \epsilon_0 |x|^{-N} (- \ln |x|)^{m_o - 1}.
\]

Then

\[
\liminf_{|x| \to 0^+} f(x)|x|^N (- \ln |x|)^{1-m_o} > 0
\]

and a contradiction arises Lemma 4.2 with \( \nu = B_{m_o} \), from which problem 4.14 has no such positive solution. Therefore we obtain that \( k \leq K_s \).

Set

\[
k = \limsup_{|x| \to 0^+} u_0(x) \left( |x|(- \ln |x|) \frac{1}{4} \right)^{N-2s}.
\]

Let us prove that

\[k \geq K_s.
\]

In fact, if

\[k < K_s,
\]

letting

\[\epsilon_1 = \min \left\{ \frac{K_s - k}{2K_s}, \frac{1}{4} \right\},
\]

then by Lemma 2.9 there exists \( r_1 \in (0, \frac{1}{s}) \) such that

\[
0 < |x| < r_1
\]

then for \( 0 < |x| < r_1 \)

\[
u \geq K^* \left( 1 - \epsilon_1 \right)^p \left( |x|(- \ln |x|) \frac{1}{4} \right)^{-N}.
\]

Set

\[s_0 \in \left( 0, \frac{p^* - 1}{4} \right),
\]

then for \( r_1 \) small enough,

\[
(-\Delta)^s w_{m_o} \geq B_{m_o} \left( 1 - \epsilon_1 \right)^{s_0} |x|^{-N} (- \ln |x|)^{-\frac{1}{4}} \quad \text{for} \quad 0 < |x| < r_1,
\]

since \( D_{m_o} < 0 \). Then by Lemma 2.6 there exists \( r_2 \in (0, r_1) \) such that

\[
u \leq B_{m_o} \left( 1 - \epsilon_1 \right)^{p^* - s_0} \left( |x|(- \ln |x|) \frac{1}{4} \right)^{2s-N} \quad \text{for} \quad 0 < |x| < r_2,
\]

28
and then
\[ u_0^+(x) \leq K_s p^* (1 - \epsilon_1)^s ((-\ln|x|)^{-s_0})^{-N}. \]

By repeating the above procedure, there exists \( r_n > 0 \) such that
\[ u_0 \leq K_s (1 - \epsilon_1)^s ((-\ln|x|)^{-s_0})^{2s-N} \text{ for } 0 < |x| < r_n, \]
where
\[ t_0 = p^* \text{ and } t_n = p^*(t_{n-1} - s_0). \]

Note that
\[ t_n - t_{n-1} = p^*(t_{n-1} - t_{n-2}) = (p^*)^{n-1}(t_1 - t_0) = (p^*)^n(p^* - s_0 - 1) \to +\infty \text{ as } n \to +\infty, \]
then there exists \( n_1 \in \mathbb{N} \) such that \( K_s (1 - \epsilon_1)^{n_1} < \kappa \), which contradicts (4.15) when \( n \geq n_1 \).

Now we conclude that
\[ 0 < \liminf_{|x| \to 0^+} u(x)|(-\ln|x|)^{\frac{1}{s}}|^{N-2s} \leq K_s \leq \limsup_{|x| \to 0^+} u(x)|(-\ln|x|)^{\frac{1}{s}}|^{N-2s} \leq +\infty. \]

However, if \( \limsup_{|x| \to 0^+} u(x)|(-\ln|x|)^{\frac{1}{s}}|^{N-2s} = +\infty \), the Harnack inequality (5.9) implies that
\[ \liminf_{|x| \to 0^+} u(x)|(-\ln|x|)^{\frac{1}{s}}|^{N-2s} = +\infty. \]

Therefore, by the Harnack inequality
\[ \frac{K_s}{C_0} \leq \liminf_{|x| \to 0^+} (|x|(-\ln|x|)^{\frac{1}{s}})^{N-2s} \leq K_s \leq \limsup_{|x| \to 0^+} (|x|(-\ln|x|)^{\frac{1}{s}})^{N-2s} \leq K_s C_0. \]

By the scaling technique, the classification of singularities for positive solutions of (1.1) holds in any ball \( B_r \).

**Part II: General domain.** In general bounded domain \( \Omega \)
\[ B_1 \subset \Omega \subset BR_0. \]

Let \( u_0 \) be a positive solution of (1.1) such that
\[ \lim_{|x| \to 0^+} u_0(x) = +\infty. \]

Note that the argument of the upper bound doesn’t use the radially symmetric property of Theorem 4.1 so we have that
\[ k_0 \leq \limsup_{|x| \to 0^+} u_0(x)|x|^{N-2s}(-\ln|x|)^{-m_0} < +\infty. \]

By contradiction, we set
\[ \liminf_{|x| \to 0^+} u_0(x)|x|^{N-2s}(-\ln|x|)^{-m_0} = 0. \]

By Lemma 4.3, problem (1.1) with \( R = R_0 \) has a positive singular solution \( u_1 \) such that
\[ u_0(x) - c_m (-\ln|x|)^{m+1} \leq u_1(x) \leq u_0(x), \quad x \in B_{R_0} \setminus \{0\} \]
for some \( m > 0 \) and \( c_m > 0 \). Then there holds
\[ \limsup_{|x| \to 0^+} u_1(x)|x|^{N-2s}(-\ln|x|)^{-m_0} \geq K_s \]
and
\[ \liminf_{|x| \to 0^+} u_1(x)|x|^{N-2s}(-\ln|x|)^{-m_0} = 0, \]
which are impossible with our above argument when \( \Omega = BR_0. \)
5 Existence of singular solutions

5.1 First singular solution

From Proposition 2.1 with \( m = m_0 = -\frac{N - 2s}{2s} < 0 \),

\[
B_{m_0} = m_0 C_1(0) > 0, \quad D_{m_0} = \frac{m_0(m_0 - 1)}{2} C_2''(0) < 0,
\]

recalling that

\[
w_{m_0}(x) = |x|^{2s-N}(-\ln |x|)^{m_0} \quad \text{for } x \in B_{\frac{1}{m_0}} \setminus \{0\},
\]

there exists \( r_1 \in (0, r_0) \) such that for \( 0 < |x| < r_1 \)

\[
(-\Delta)^s w_{m_0}(x) = B_{m_0} |x|^{-N}(-\ln |x|)^{m_0-1} + D_{m_0} |x|^{-2s}(-\ln |x|)^{m_0-2}
\]

\[
+ O(1) |x|^{-N}(-\ln |x|)^{m_0-3}
\]

\[
\leq B_{m_0} |x|^{-N}(-\ln |x|)^{m_0-1}.
\]

Recall that

\[
\mathcal{K}_s = B_{\frac{N-2s}{2s}},
\]

then \( w_{m_0} \) verifies that

\[
(-\Delta)^s(\mathcal{K}_s w_{m_0}) \leq (\mathcal{K}_s w_{m_0})^p \quad \text{in } B_{r_1} \setminus \{0\}. \tag{5.1}
\]

Proposition 5.1 Let \( N > 2s \), then for \( r \in (0, r_1] \) problem

\[
\begin{cases}
(-\Delta)^s u = u^p & \text{in } B_r \setminus \{0\}, \\
u = 0 & \text{in } \mathbb{R}^N \setminus B_r
\end{cases} \tag{5.2}
\]

has a positive singular solution \( u_r \), verifying for some \( m > 0 \)

\[
-K_s r^{2s-N}(-\ln r)^m \leq u_r(x) = \mathcal{K}_s |x|^{2s-N}(-\ln |x|)^m \leq C_{27} |x|^{2s-N}(-\ln |x|)^{-m-1},
\]

where \( C_{27} > 0 \) is independent of \( r \).

Moreover, the mapping \( r \in (0, r_1] \mapsto u_r \) is increasing.

Proof. Let

\[
v_0 = \mathcal{K}_s w_{m_0} \quad \text{in } \mathbb{R}^N
\]

and we first show that

\[
\begin{cases}
(-\Delta)^s u = u^p & \text{in } B_{r_1} \setminus \{0\}, \\
u = v_0 & \text{in } \mathbb{R}^N \setminus B_{r_1}
\end{cases} \tag{5.3}
\]

has a solution \( u_{r_1} \), verifying

\[
\lim_{|x| \to 0^+} u_{r_1}(x) |x|^{-2s}(-\ln |x|)^{-m} = \mathcal{K}_s.
\]

To this end, we set

\[
\epsilon_1 \in \left(0, \min \left\{ -\frac{D_{m_0}}{4}, \frac{1}{2} \frac{B_{m_0-1} - B_{m_0}}{B_{m_0-1} + 1} \right\}\right),
\]

where \( B_{m_0-1} - B_{m_0} = -C'_1(0) > 0 \). For \( r_1 > 0 \), we re-choose it smaller if necessary, there holds that for \( 0 < |x| < r_1 \)

\[
(-\Delta)^s v_0(x) \leq K_s^p |x|^{-N}(-\ln |x|)^{m_0-1} + K_s(D_{m_0} + \epsilon_1) |x|^{-2s}(-\ln |x|)^{m_0-2}
\]

\[
= v_0^p(x) + K_s(D_{m_0} + \epsilon_1) |x|^{-2s}(-\ln |x|)^{m_0-2},
\]

where we recall that \( D_{m_0} = \frac{m_0(m_0-1)}{2} C_2''(0) < 0 \).
Inductively, we denote \( v_n \), (also use the notation \( v_{r,n} \) if \( r_1 \) is replaced by \( r \)) the solution of

\[
\begin{cases}
(\Delta)^s v_n = v_{n-1}^p & \text{in } B_{r_1} \setminus \{0\}, \\
v_n = v_0 & \text{in } \mathbb{R}^N \setminus B_{r_1}, \\
\lim_{|x| \to 0^+} v_n(x)|x|^{N-2s} = 0.
\end{cases}
\]

Obviously, we have that

\[ (-\Delta)^s (v_1 - v_0) \geq 0 \]

and the comparison principle implies that \( v_1 \geq v_0 \) in \( \mathbb{R}^N \setminus \{0\} \). Inductively, the mapping

\[ n \in \mathbb{N} \mapsto v_n \]

is increasing.

Set \( \nu_1 = v_1 - v_0 \), then

\[ (-\Delta)^s \nu_1 \leq \mathcal{K}_s(-D_{m_0} + \epsilon)|x|^{-2s}(\ln |x|)^{m_0-2} \]

in \( B_{r_1} \setminus \{0\} \)

and

\[ \nu_1 = 0 \quad \text{in } \mathbb{R}^N \setminus B_r, \quad \lim_{|x| \to 0^+} \nu_1(x)|x|^{N-2s} = 0. \]

By the comparison Principle, we have that

\[ 0 < \nu_1(x) \leq Z_1|x|^{2s-N}(\ln |x|)^{m_0-1}, \]

where

\[ Z_1 = \frac{\mathcal{K}_s(-D_{m_0} + \epsilon_1)}{B_{m_0-1} - \epsilon_1}. \]

For \( n = 2, 3, \ldots \), denote

\[ \nu_n = v_n - v_{n-1} \quad \text{in } \mathbb{R}^N \setminus \{0\}. \]

If \( p^* \leq 2 \)

\[ (-\Delta)^s \nu_2'(x) = v_1^p - v_0^p \leq v_0^p - 1)(v_1(x) \]

\[ \leq (v_0 + v_1)^p - 1)(v_1(x) \]

\[ \leq v_0^{p-1}(x)nu_1(x) + \nu_1^p(x); \]

if \( p^* > 2 \)

\[ (-\Delta)^s \nu_2'(x) \leq (v_0 + v_1)^p - 1)(v_1(x) \]

\[ \leq v_0^{p-1}(x)nu_1(x) + 2^p v_0^{p-2}(x)nu_1^2(x) + 2^p \nu_1^p(x). \]

In the case \( p^* \leq 2 \), for \( 0 < |x| < r_1 \) one has that

\[ 0 < \nu_2(x) \leq Z_1 \frac{B_{m_0}}{B_{m_0-1} - \epsilon_1}|x|^{2s-N}(-\ln |x|)^{m_0-1} \]

\[ \leq Z_2|x|^{2s-N}(-\ln |x|)^{m_0-1}, \]

where

\[ Z_2 = Z_1 \left( \frac{B_{m_0}}{B_{m_0-1} - \epsilon_1} + \epsilon_1 \right) \]
and we used the fact that 
\[(m_0 - 1)p^* + 1 < m_0 - 1.\]

In the case \(p^* > 2\), for \(0 < |x| < r_1\)
\[
0 < \nu_2(x) \leq Z_1 \frac{B_{m_0}}{B_{m_0 - 1} - \epsilon_1} |x|^{2s-N} (-\ln |x|)^{m_0 - 1} + 2^p Z_1^2 |x|^{2s-N} (-\ln |x|)^{m_0 - 2} \]
\[
+ 2^{p^*} \left( \frac{Z_1^{p^*}}{B_{(m_0-1)p^* + 1} - \epsilon_1} \right) |x|^{2s-N} (-\ln |x|)^{(m_0-1)p^* + 1} \]
\[
\leq Z_2 |x|^{2s-N} (-\ln |x|)^{m_0 - 1}.
\]

By iterating the above process, we can obtain that
\[
0 < \nu_n(x) \leq Z_n |x|^{2s-N} (-\ln |x|)^{m_0 - 1},
\]
where
\[
Z_n = Z_{n-1} \left( \frac{B_{m_0}}{B_{m_0 - 1} - \epsilon_1} + \epsilon_1 \right) = Z_1 \left( \frac{B_{m_0}}{B_{m_0 - 1} - \epsilon_1} + \epsilon_1 \right)^n.
\]
Note that
\[
0 < \frac{B_{m_0}}{B_{m_0 - 1} - \epsilon_1} + \epsilon_1 \]
\[
= 1 - \frac{B_{m_0 - 1} - B_{m_0} - \epsilon_1 B_{m_0 - 1} + \epsilon_1 + \epsilon_1^2}{B_{m_0 - 1} - \epsilon_1} \]
\[
\leq 1 - \frac{(B_{m_0 - 1} - B_{m_0})(1 - \frac{B_{m_0 - 1}}{2B_{m_0 - 1} + 1}) + \epsilon_1}{B_{m_0 - 1} - \epsilon_1} < 1,
\]
where the last inequality holds by the choice of \(\epsilon_1\). Then we have that
\[
v_0 \leq v_n(x) \leq v_0(x) + \sum_{n=1}^{\infty} \nu_n(x)
\]
\[
\leq v_0(x) + Z_1 \left( \sum_{n=1}^{\infty} \left( \frac{B_{m_0}}{B_{m_0 - 1} - \epsilon_1} + \epsilon_1 \right) \right) |x|^{2s-N} (-\ln |x|)^{m_0 - 1}
\]
\[
< v_0(x) + \bar{v}(x),
\]
where
\[
\bar{v}(x) = Z_1 \left( \sum_{n=1}^{\infty} \left( \frac{B_{m_0}}{B_{m_0 - 1} - \epsilon_1} + \epsilon_1 \right) \right) |x|^{2s-N} (-\ln |x|)^{m_0 - 1} \quad \text{for } x \in B_{r_1} \setminus \{0\}.
\]

Therefore, by stability results [13, Theorem 2.4] (also see [11, Lemma 4.3] for bounded sequence) and the regularity result [12, Theorem 2.1], the limit of \(\{v_n\}\) exists, denoting \(v_{r_1, \infty}\),
\[
\lim_{n \to +\infty} v_n(x) = v_{r_1, \infty}(x) \quad \text{in } \mathbb{R}^N \setminus \{0\},
\]
then \(v_{r_1, \infty}\) is a solution of \([5,3]\) and it verifies that
\[
v_0 \leq v_{r_1, \infty} \leq v_0 + \bar{v} \quad \text{in } \mathbb{R}^N \setminus \{0\},
\]
which implies that
\[
\lim_{|x| \to 0^+} v_{r_1, \infty}(x) |x|^{N-2s} (-\ln |x|)^{-m_0} = K_x.
\]

32
Corollary 5.1

Let \( r \leq r_1 \),

\[
\begin{cases}
(-\Delta)^s v_{r,n} = v_{r,n-1}^p & \text{in } B_r \setminus \{0\}, \\
v_{r,n} = v_0 & \text{in } \mathbb{R}^N \setminus B_r, \\
\lim_{|x| \to 0^+} v_{r,n}(x)|x|^{N-2s} = 0
\end{cases}
\]

has a unique solution \( v_{r,n} \). By the comparison principle, there holds

\[ v_{r,n} \leq v_n \quad \text{in } \mathbb{R}^N \setminus \{0\} \]

and then the limit of \( \{ v_{r,n} \} \) exists, denoting \( v_{r,\infty} \),

\[ v_{r,\infty} \leq v_{r,\infty} \quad \text{in } \mathbb{R}^N \setminus \{0\}. \]

Thanks to the nonnegative property of \( v_0 \), the function \( v_{r,\infty} \), the solution of (5.2), is a super solution of (5.3) for \( r \in (0, r_1) \). Moreover, \( v_{r_1,0} - v_{r_1,0}(r_1) \) is a sub solution of (5.2). Repeat above arguments, \( v_{r_1,0} \) has a solution \( u_{r_1} \) such that

\[ v_{r_1,0} - v_{r_1,0}(r_1) \leq u_{r_1} \leq v_{r,\infty} \leq v_{r,\infty} + v \quad \text{in } B_r \setminus \{0\}. \]

For \( 0 < r \leq r_1 \), comparison principle implies that if \( v_{r,1} \leq v_{r,1} \) and inductively, we obtain that \( v_{r,\infty} \leq v_{r,\infty} \) and the order could be kept, that is, then

\[ u_r \leq u_{r_1} \quad \text{for } r \leq r_1. \]

This completes the proof.

\[ \square \]

Corollary 5.1 Let \( N > 2 \), then for \( r \in (0, \frac{1}{2r^2}) \) problem

\[
\begin{cases}
-\Delta u = u^{\frac{N-2}{2}} & \text{in } B_r \setminus \{0\}, \\
u = 0 & \text{on } \partial B_r
\end{cases}
\]

has a positive singular solution \( u_r \) verifying for some \( m > 0 \)

\[
-K_1 r^{2-N}(-\ln r)^\frac{N-2}{2} \leq u_r(x) - K_1 |x|^{2-N}(-\ln |x|)^\frac{N-2}{2} \leq c_{27} |x|^{2-N}(-\ln |x|)^{-\frac{N}{2}},
\]

where \( c_{27} > 0 \) is independent of \( r \).

Moreover, the mapping \( r \in (0, \frac{1}{2r^2}) \mapsto u_r \) is increasing.

Proof. Let

\[ v_0(x) = |x|^{2-N}(-\ln |x|)^\frac{N-2}{2} \quad \text{in } B_{\frac{1}{2r^2}} \setminus \{0\} \]

and direct computation shows that

\[
-\Delta v_0 = (-\ln |x|)^\frac{N-2}{2}(-\Delta)|x|^{2-N}
-2\nabla |x|^{2-N} \cdot \nabla (-\ln |x|)^\frac{N-2}{2} - |x|^{2-N}(-\Delta)(-\ln |x|)^\frac{N-2}{2}
= -\frac{(N-2)^2}{2} |x|^{-N}(-\ln |x|)^{-\frac{N}{2}} - \frac{N(N+2)}{4} |x|^{-N}(-\ln |x|)^{-\frac{N}{2}},
\]

which gives a sub solution of

\[
\begin{cases}
-\Delta u = u^{\frac{N}{2}} & \text{in } B_r \setminus \{0\}, \\
u = v_0 & \text{on } \partial B_r
\end{cases}
\]

for \( r \in (0, \frac{1}{2r^2}) \).

By a similar iterative procedure of Proposition 5.1, we can construct a desired solution \( u_r \).

\[ \square \]
5.2 Infinitely many solutions

Lemma 5.1 Let \( u_r \) be the solution of (5.2) obtained in Proposition 5.1, then
\[
\left| u_r(x) - K_s |x|^{2s-N}(-\ln |x|)^{m_0} \right| \leq c_{29} |x|^{2s-N}(-\ln |x|)^{(m_0-1)p^*-1} \quad \text{for } x \in B_{\mathbb{R}} \setminus \{0\},
\]
where \( c_{29} > 0 \) depends on \( r \) and \( (m_0-1)p^* + 1 < m_0 - 1 \).

Proof. From Proposition 5.1 problem (5.2) has a positive singular solution \( u_r \) verifying
\[
\left| u_r(x) - K_s |x|^{2s-N}(-\ln |x|)^{m_0} \right| \leq c_{30} |x|^{2s-N}(-\ln |x|)^{m_0-1} \quad \text{for } x \in B_{\mathbb{R}} \setminus \{0\},
\]
where \( c_{30} > 0 \) depends on \( r \).

Then we have for \( x \in B_{\mathbb{R}} \setminus \{0\} \):
\[
\left| u_p^r - K_p^r |x|^{-N}(-\ln |x|)^{m_0-1} \right| \leq c_{31} |x|^{-N}(-\ln |x|)^{(m_0-1)p^*}
\]
and
\[
\left| (-\Delta)^s (u_r - K_s w_{m_0})(x) \right| = \left| u_p^r(x) - K_p^r |x|^{-N}(-\ln |x|)^{m_0-1} \right| \leq c_{31} |x|^{-N}(-\ln |x|)^{(m_0-1)p^*},
\]
which, by Lemma 2.9, implies that
\[
\left| u_r - K_s w_{m_0} \right| \leq c_{32} |x|^{2s-N}(-\ln |x|)^{(m_0-1)p^*+1}.
\]
This completes the proof. □

Proof of Theorem 5.2 Lemma 5.1 shows that for any \( r \leq r_1 \), problem (5.2) has a positive singular solution \( u_r \) verifying that for \( 0 < |x| < r \)
\[
\left| u_r(x) - K_s |x|^{2s-N}(-\ln |x|)^{m_0} \right| \leq c_{29} |x|^{2s-N}(-\ln |x|)^{(m_0-1)p^*+1}.
\]

Next we perform the following scaling:
\[
u_t(x) = t^{2s-N} u_r(t^{-1}x) \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}.
\]

Then \( u_t \) is a singular solution of
\[
\left\{
\begin{array}{ll}
(-\Delta)^s u = u^p^* & \text{in } B_{tr} \setminus \{0\}, \\
u = 0 & \text{in } \mathbb{R}^N \setminus B_{tr}.
\end{array}
\right.
\]

Note that for \( |x| > 0 \) sufficiently small,
\[
K_s |x|^{2s-N} (\ln l - \ln |x|)^{m_0} = K_s |x|^{2s-N}(-\ln |x|)^{m_0} + K_s m_0 (\ln l) |x|^{2s-N}(-\ln |x|)^{m_0-1} + O(1) |x|^{2s-N}(-\ln |x|)^{m_0-2}.
\]

From (5.7), we have that as \( |x| \to 0^+ \)
\[
u_t(x) - K_s |x|^{2s-N}(-\ln |x|)^{m_0} = K_s m_0 (\ln l) |x|^{2s-N}(-\ln |x|)^{m_0-1} + O(1) |x|^{2s-N}(-\ln |x|)^{\max\{m_0-2,(m_0-1)p^*+1\}},
\]
where \( \max\{m_0-2,(m_0-1)p^*+1\} < m_0 - 1 \).
If we choose \( l > 0 \) such that \( rl = 1 \), then

\[
\begin{cases}
-\Delta u = u^p & \text{in } B_1 \setminus \{0\}, \\
u = 0 & \text{in } \mathbb{R}^N \setminus B_1
\end{cases}
\] (5.9)

has a sequence of solutions \( \{u_l\}_l \) satisfying that

\[ u_l(x) - \mathcal{K}_s|x|^{2s-N}(-\ln |x|)^{m_0} = \mathcal{K}_s m_0(l) |x|^{2s-N}(-\ln |x|)^{m_0-1}(1 + o(1)) \quad \text{for } |x| \to 0^+, \]

where \( m_0 - 1 = -\frac{N}{2s} \). This completes the proof. \( \square \)

**Proof of Corollary 1.1.** From Corollary 5.1

\[
\begin{cases}
-\Delta u = u^{\frac{N}{N-2s}} & \text{in } B_r \setminus \{0\}, \\
u = 0 & \text{on } \partial B_r.
\end{cases}
\] (5.10)

has a solution \( u_r \) for \( r \in (0, \frac{1}{c_{27}}) \) such that

\[-\mathcal{K}_s r^{2s-N}(-\ln r)^{\frac{N}{2s}} \leq u_r(x) - \mathcal{K}_s |x|^{2s-N}(-\ln |x|)^{\frac{N}{2s}} \leq c_{27} |x|^{2s-N}(-\ln |x|)^{-\frac{N}{2s}},\]

where \( c_{27} > 0 \) is independent of \( r \).

For \( r \in (0, \frac{1}{c_{27}}) \), we do the scaling

\[ u_t(x) = l^{N-2} u_r(l^{-1} x), \quad x \in B_{rl} \setminus \{0\}. \]

The rest of the proof is omitted here. \( \square \)

### A Appendix: The constant \( C'_s(0) \) and \( \mathcal{K}_s \)

We show that

\[ C'_s(0) = -2^{2s-1} \frac{\Gamma\left(\frac{N}{2}\right) \Gamma(s)}{\Gamma\left(\frac{N-2s}{2}\right)}. \] (A.11)

In fact,

\[
\ln C_s(\tau) = 2s \ln 2 + \ln \Gamma\left(\frac{N+\tau}{2}\right) + \ln \left(\frac{2s-\tau}{2}\right) - \ln \Gamma\left(-\frac{\tau}{2}\right) - \ln \Gamma\left(-\frac{N-2s+\tau}{2}\right)
\]

and then

\[
C'_s(\tau) = C_s(\tau) \left(\frac{1}{2} \psi\left(-\frac{N+\tau}{2}\right) - \frac{1}{2} \psi\left(\frac{2s-\tau}{2}\right) + \frac{1}{2} \psi\left(-\frac{\tau}{2}\right) - \frac{1}{2} \psi\left(-\frac{N-2s+\tau}{2}\right)\right)
\]

\[
= \frac{2^{2s-1}}{\Gamma\left(-\frac{\tau}{2}\right)} \left(\frac{\Gamma\left(\frac{N+\tau}{2}\right) \Gamma\left(\frac{2s-\tau}{2}\right)}{\Gamma\left(\frac{N-2s+\tau}{2}\right)} - \frac{\Gamma\left(\frac{N+\tau}{2}\right) \Gamma\left(\frac{2s-\tau}{2}\right)}{\Gamma\left(-\frac{\tau}{2}\right)} \right)
\]

\[
+ \frac{2^{2s-1}}{\Gamma\left(-\frac{\tau}{2}\right)} \frac{\Gamma\left(\frac{N+\tau}{2}\right) \Gamma\left(\frac{2s-\tau}{2}\right)}{\Gamma\left(-\frac{\tau}{2}\right)} \psi\left(-\frac{\tau}{2}\right)
\]

where \( \psi \) is the Digamma function, i.e. \( \psi(t) = \frac{\Gamma'(t)}{\Gamma(t)} \). Note that the term

\[
\psi\left(-\frac{N+\tau}{2}\right) - \psi\left(-\frac{N-2s+\tau}{2}\right) - \psi\left(-\frac{2s-\tau}{2}\right)
\]

is uniformly bounded as \( s \to 1^+ \)
By properties of Gamma function and Digamma function, we have that
\[
\frac{1}{\Gamma(-\frac{\tau}{2})} = -\frac{\tau}{2} \frac{1}{\Gamma(1 - \frac{\tau}{2})}
\]
and
\[
\psi(-\frac{\tau}{2}) = \psi(1 - \frac{\tau}{2}) + \frac{\tau}{2}
\]
then
\[
\lim_{\tau \to 0} \frac{\psi(-\frac{\tau}{2})}{\Gamma(-\frac{\tau}{2})} = -1
\]
and
\[
\lim_{\tau \to 0} C_s(\tau) = 2^{2s-1} \lim_{\tau \to 0} \frac{\Gamma(N + \frac{\tau}{2}) \Gamma(\frac{2s-\tau}{2})}{\Gamma(N - \frac{\tau}{2})} \lim_{\tau \to 0} \frac{1}{\Gamma(-\frac{\tau}{2})} \left( \psi\left(\frac{N + \tau}{2}\right) - \psi\left(\frac{2s-\tau}{2}\right) \right)
\]
Next we prove that
\[
\lim_{s \to 1^-} K_s = \left(\frac{(N - 2)^2}{2}\right)^{\frac{N - 2s}{2}}.
\]
\[\text{(A.12)}\]
Notice that
\[
K_s = \left(2^{2s-1} \frac{\Gamma(N)}{\Gamma(N - \frac{2s}{2})} \frac{N - 2s}{2s}\right)^{\frac{N - 2s}{2}}
\]
where
\[
\lim_{s \to 1^-} \left(2^{2s-1} \frac{\Gamma(N)}{\Gamma(N - \frac{2s}{2})} \frac{N - 2s}{2s}\right) = (N - 2) \frac{\Gamma(N)}{\Gamma(N - 2s)} = (N - 2) \frac{\Gamma(N - 2s)}{\Gamma(N - 2s)} = \frac{(N - 2)^2}{2}.
\]
\[\text{B Appendix: Radial symmetry}\]
Our method of moving planes is motivated by [22] to deal with the solution \(u\) has possibly singular at the origin. For the moving planes of integral equations, we refer to [16,17]. For singular solutions, we use a direct moving plane method from [22] and we need the following variant Maximum Principle for small domain.

**Lemma B.1** [22, Corollary 2.1] Let \(O\) be an open and bounded subset of \(\mathbb{R}^N\). Suppose that \(\varphi : \Omega \to \mathbb{R}\) is in \(L^\infty(O)\) and \(w \in L^\infty(\mathbb{R}^N)\) is a classical solution of
\[
\begin{cases}
-(-\Delta)^s w(x) \leq \varphi(x) w(x), & x \in O, \\
\quad w(x) \geq 0, & x \in O^c.
\end{cases}
\]
Then there is \(\delta > 0\) such that whenever \(|O^-| \leq \delta\), \(w\) has to be non-negative in \(O\), where \(O^- = \{x \in O \mid w(x) < 0\}\).
Now, we use the moving plane method to show the radial symmetry and monotonicity of positive solutions to equation (B.1). For simplicity, we denote

\[ \Sigma_\lambda = \{ x = (x_1, x') \in B_1 \mid x_1 > \lambda \}, \]  
**(B.2)** 

\[ u_\lambda(x) = u(x_\lambda) \quad \text{and} \quad w_\lambda(x) = u_\lambda(x) - u(x), \]  
**(B.3)**

where \( \lambda \in (0, 1) \) and \( x_\lambda = (2\lambda - x_1, x') \) for \( x = (x_1, x') \in \mathbb{R}^N \) and \( B_1 = B_1 \setminus \{0\} \). For any subset \( A \) of \( \mathbb{R}^N \), we write \( A_\lambda = \{ x_\lambda : x \in A \} \).

On the contrary, suppose that \( \Sigma^-_\lambda = \{ x \in \Sigma_\lambda \mid w_\lambda(x) < 0 \} \neq \emptyset \) for \( \lambda \in (0, 1) \). Let us define

\[ w^-_\lambda(x) = \begin{cases} w_\lambda(x), & x \in \Sigma^-_\lambda, \\ 0, & x \in \mathbb{R}^N \setminus \Sigma^-_\lambda \end{cases} \]  
**(B.4)**

and

\[ w^-_\lambda(x) = \begin{cases} 0, & x \in \Sigma^-_\lambda, \\ w_\lambda(x), & x \in \mathbb{R}^N \setminus \Sigma^-_\lambda. \end{cases} \]  
**(B.5)**

Hence, \( w^-_\lambda(x) = w_\lambda(x) - w^+_\lambda(x) \) for all \( x \in \mathbb{R}^N \). It is obvious that \( (2\lambda, 0, \ldots, 0) \not\in \Sigma^-_\lambda \), since \( \lim_{|x| \to 0^+} u(x) = +\infty \).

**Lemma B.2** Assume that \( \Sigma^-_\lambda \neq \emptyset \) for \( 0 < \lambda < 1 \), then \[ (-\Delta)^{s} w^-_\lambda(x) \leq 0, \quad \forall x \in \Sigma^-_\lambda. \]  
**(B.6)**

**Proof.** By direct computation, for \( x \in \Sigma^-_\lambda \), we have

\[ (-\Delta)^{s} w^-_\lambda(x) = \int_{\mathbb{R}^N} \frac{w^-_\lambda(x) - w^-_\lambda(z)}{|x - z|^{N+2s}} dz - \int_{\mathbb{R}^N \setminus \Sigma^-_\lambda} \frac{w_\lambda(z)}{|x - z|^{N+2s}} dz. \]

We estimate these integrals separately. Since \( u = 0 \) in \((B_1)_\lambda \setminus B_1 \) and \( u_\lambda = 0 \) in \( B_1 \setminus (B_1)_\lambda \), then

\[ I_1 = \int_{(B_1)_\lambda \setminus B_1} \frac{u_\lambda(z)}{|x - z|^{N+2s}} dz \]

\[ = \int_{(B_1)_\lambda \setminus B_1} \frac{u_\lambda(z)}{|x - z|^{N+2s}} dz - \int_{B_1 \setminus (B_1)_\lambda} \frac{u(z)}{|x - z|^{N+2s}} dz \]

\[ = \int_{(B_1)_\lambda \setminus B_1} u_\lambda(z) \left( \frac{1}{|x - z|^{N+2s}} - \frac{1}{|x - z_\lambda|^{N+2s}} \right) dz \geq 0, \]

since \( u_\lambda \geq 0 \) and \( |x - z_\lambda| > |x - z| \) for all \( x \in \Sigma^-_\lambda \) and \( z \in (B_1)_\lambda \setminus B_1 \).

In order to decide the sign of \( I_2 \) we observe that \( w_\lambda(z_\lambda) = -w_\lambda(z) \) for any \( z \in \mathbb{R}^N \). Then,

\[ I_2 = \int_{\Sigma_\lambda \setminus \Sigma^-_\lambda} \frac{w_\lambda(z)}{|x - z|^{N+2s}} dz \]

\[ = \int_{\Sigma_\lambda \setminus \Sigma^-_\lambda} \frac{w_\lambda(z)}{|x - z|^{N+2s}} dz + \int_{\Sigma_\lambda \setminus \Sigma^-_\lambda} \frac{w_\lambda(z)}{|x - z_\lambda|^{N+2s}} dz \]

\[ \geq 0, \]

37
since \( w_\lambda \geq 0 \) in \( \Sigma_\lambda \setminus \Sigma_\lambda^- \) and \( |x - z_\lambda| > |x - z| \) for all \( x \in \Sigma_\lambda^- \) and \( z \in \Sigma_\lambda \setminus \Sigma_\lambda^- \).

Finally, since \( w_\lambda(z) < 0 \) for \( z \in \Sigma_\lambda^- \), we deduce
\[
I_3 = \int_{(\Sigma_\lambda^-)\lambda} \frac{w_\lambda(z)}{|x - z|^{N+2s}} dz = \int_{\Sigma_\lambda^-} \frac{w_\lambda(z)}{|x - z|^{N+2s}} dz
\]
\[
= - \int_{\Sigma_\lambda^-} \frac{w_\lambda(z)}{|x - z|^{N+2s}} dz \geq 0.
\]

The proof is complete. \( \square \)

Now we are ready to prove Theorem 4.1.

**Proof of Theorem 4.1.** Our purpose is to show the radial symmetry and decreasing monotonicity and we divide the proof into four steps.

**Step 1:** We prove that if \( \lambda \) is close to 1, then \( w_\lambda > 0 \) in \( \Sigma_\lambda \). First we show that \( w_\lambda \geq 0 \) in \( \Sigma_\lambda \), i.e. \( \Sigma_\lambda^- \) is empty. By contradiction, we assume that \( \Sigma_\lambda^- \neq \emptyset \). Now we apply \((B.6)\) and linearity of the fractional Laplacian to obtain that, for \( x \in \Sigma_\lambda^- \),
\[
(-\Delta)^s w_\lambda^+(x) \geq (-\Delta)^s w_\lambda(x) = (-\Delta)^s u_\lambda(x) - (-\Delta)^s u(x).
\]

Combining with \((B.7)\) and \((B.4)\), for \( x \in \Sigma_\lambda^- \), we have
\[
(-\Delta)^s w_\lambda^+(x) \geq (-\Delta)^s u_\lambda(x) - (-\Delta)^s u(x)
\]
\[
= -u_\lambda^+(x) + u^+(x) = -\varphi(x)w_\lambda^+(x),
\]
where
\[
\varphi(x) = \frac{(u_\lambda(x))^{p^*} - (u(x))^{p^*}}{u_\lambda(x) - u(x)}, \quad \forall x \in \Sigma_\lambda^-.
\]

For \( x \in \Sigma_\lambda^- \subset \Sigma_\lambda \subset \mathbb{R}^N \setminus B_\lambda(0) \), \( u_\lambda(x) < u(x) \). Moreover, there exists \( M_\lambda > 0 \) such that
\[
\|u\|_{L^\infty(\mathbb{R}^N \setminus B_\lambda(0))} \leq M_\lambda.
\]

Due to \( h \in C^1(\mathbb{R}_+) \), there exists \( c_{31} > 0 \) dependent of \( \lambda \) such that
\[
\|\varphi\|_{L^\infty(\Sigma_\lambda^-)} \leq c_{31}. \quad (B.8)
\]

Note that \( M_\lambda \to \infty \) as \( \lambda \to 0 \), since \( \lim_{|x| \to 0^+} u(x) = \infty \).

Therefore, for \( x \in \Sigma_\lambda^- \) and then
\[
-(-\Delta)^s w_\lambda^+(x) \leq \varphi(x)w_\lambda^+(x), \quad \forall x \in \Sigma_\lambda^-.
\]

Moreover, \( w_\lambda^+ = 0 \) in \( (\Sigma_\lambda^-)^c \). Choosing \( \lambda \in (0,1) \) close enough to 1 we have \( |\Sigma_\lambda^-| \) is small and we apply Lemma \((B.1)\) to obtain that
\[
w_\lambda = w_\lambda^+ \geq 0 \quad \text{in} \quad \Sigma_\lambda^-,
\]
which is impossible. Thus,
\[
w_\lambda \geq 0 \quad \text{in} \quad \Sigma_\lambda.
\]

Now we claim that for \( 0 < \lambda < 1 \), if \( w_\lambda \geq 0 \) and \( w_\lambda \neq 0 \) in \( \Sigma_\lambda \), then \( w_\lambda > 0 \) in \( \Sigma_\lambda \). Assuming the claim is true, we complete the proof. Since the function \( u \) is positive in \( B_1 \) and \( u = 0 \) on \( \partial B_1 \), \( w_\lambda \) is positive on \( \partial B_1 \cap \partial \Sigma_\lambda \) and then \( w_\lambda \neq 0 \) in \( \Sigma_\lambda \).

Now we prove the claim. Suppose on the contrary that there exists \( x_0 \in \Sigma_\lambda \) such that \( w_\lambda(x_0) = 0 \), i.e. \( u_\lambda(x_0) = u(x_0) \). Then
\[
(-\Delta)^s w_\lambda(x_0) = (-\Delta)^s u_\lambda(x_0) - (-\Delta)^s u(x_0) = 0. \quad (B.9)
\]
On the other hand, let $K_\lambda = \{(x_1, x') \in \mathbb{R}^N \mid x_1 > \lambda\}$. Noting $w_\lambda(z) = -w_\lambda(z)$ for any $z \in \mathbb{R}^N$ and $w_\lambda(x_0) = 0$, we deduce

$$(-\Delta)^s w_\lambda(x_0) = -\int_{K_\lambda} \frac{w_\lambda(z)}{|x_0 - z|^{N+2s}} dz - \int_{\mathbb{R}^N \setminus K_\lambda} \frac{w_\lambda(z)}{|x_0 - z|^{N+2s}} dz$$

$$= -\int_{K_\lambda} \frac{w_\lambda(z)}{|x_0 - z|^{N+2s}} dz - \int_{K_\lambda} \frac{w_\lambda(z_{\lambda})}{|x_0 - z_{\lambda}|^{N+2s}} dz$$

$$= -\int_{K_\lambda} w_\lambda(z) \left(\frac{1}{|x_0 - z|^{N+2s}} - \frac{1}{|x_0 - z_{\lambda}|^{N+2s}}\right) dz.$$ 

The fact $|x_0 - z_{\lambda}| > |x_0 - z|$ for $z \in K_\lambda$, $w_\lambda(z) \geq 0$ and $w_\lambda(z) \neq 0$ in $K_\lambda$ yield

$$(-\Delta)^s w_\lambda(x_0) < 0,$$

which contradicts (B.9), completing the proof of the claim.

**Step 2:** We prove $\lambda_0 := \inf\{\lambda \in (0, 1) \mid w_\lambda > 0 \text{ in } \Sigma_\lambda\} = 0$. Were it not true, we would have $\lambda_0 > 0$. Hence, $w_{\lambda_0} \geq 0$ in $\Sigma_{\lambda_0}$ and $w_{\lambda_0} \neq 0$ in $\Sigma_{\lambda_0}$. The claim in Step 1 implies $w_{\lambda_0} > 0$ in $\Sigma_{\lambda_0}$.

Next we claim that if $w_\lambda > 0$ in $\Sigma_\lambda$ for $\lambda \in (0, 1)$, then there exists $\epsilon \in (0, \lambda/4)$ such that $w_{\lambda_0} > 0$ in $\Sigma_{\lambda_0}$, where $\lambda_0 = \lambda - \epsilon > 3\lambda/4$. This claim directly implies that $\lambda_0 = 0$, which contradicts to the fact $\lambda_0 > 0$.

Now we prove the claim. Let $D_\mu = \{x \in \Sigma_\lambda \mid dist(x, \partial \Sigma_\lambda) \geq \mu\}$ for $\mu > 0$ small. Since $w_\lambda > 0$ in $\Sigma_\lambda$ and $D_\mu$ is compact, there exists $\mu_0 > 0$ such that $w_\lambda \geq \mu_0$ in $D_\mu$. By the continuity of $w_\lambda(x)$, for $\epsilon > 0$ small enough and $\lambda_\epsilon = \lambda - \epsilon$, we have that $w_{\lambda_\epsilon}(x) \geq 0$ in $D_\mu$. Therefore, $\Sigma_{\lambda_\epsilon} \subset \Sigma_{\lambda_0} \setminus D_\mu$ and $|\Sigma_{\lambda_\epsilon}|$ is small if $\epsilon$ and $\mu$ are small. Using (B.9) and proceeding as in Step 1, we have for all $x \in \Sigma_{\lambda_0}$ that

$$(-\Delta)^s w_{\lambda_\epsilon}^+(x) = (-\Delta)^s w_{\lambda_\epsilon}(x) - (-\Delta)^s u(x) - (-\Delta)^s w_{\lambda_\epsilon}^-(x)$$

$$\geq (-\Delta)^s w_{\lambda_\epsilon}(x) - (-\Delta)^s u(x)$$

$$= \varphi(x) w_{\lambda_\epsilon}^+(x).$$

By (B.8), if $\lambda_\epsilon > 3\lambda/4$, $\varphi(x)$ is controlled by some constant dependent of $\lambda$.

Since $w_{\lambda_\epsilon}^- = 0$ in $(\Sigma_{\lambda_0})^c$ and $|\Sigma_{\lambda_\epsilon}|$ is small, for $\epsilon$ and $\mu$ small, Lemma (B.1) implies that $w_{\lambda_\epsilon} \geq 0$ in $\Sigma_{\lambda_0}$. Combining with $\lambda_\epsilon > 0$ and $w_{\lambda_\epsilon} \neq 0$ in $\Sigma_{\lambda_0}$, we obtain $w_{\lambda_\epsilon} > 0$ in $\Sigma_{\lambda_0}$. The proof of the claim is finished.

**Step 3:** By Step 2, we have $\lambda_0 = 0$, which implies that $u(-x_1, x') \geq u(x_1, x')$ for $x_1 \geq 0$. Using the same argument from the other side, we conclude that $u(-x_1, x') \leq u(x_1, x')$ for $x_1 \geq 0$ and then $u(-x_1, x') = u(x_1, x')$ for $x_1 \geq 0$. Repeating this procedure in all directions we see that $u$ is radially symmetric.

Finally, we prove $u(r)$ is strictly decreasing in $r \in (0, 1)$. Let us consider $0 < x_1 < \tilde{x}_1 < 1$ and let $\lambda = \frac{x_1 + \tilde{x}_1}{2}$. As proved above we have

$$w_\lambda(x) > 0 \text{ for } x \in \Sigma_{\lambda}.$$ 

Then

$$0 < w_\lambda(\tilde{x}_1, 0, \ldots, 0) = u_\lambda(\tilde{x}_1, 0, \ldots, 0) - u(\tilde{x}_1, 0, \ldots, 0)$$

$$= u(x_1, 0, \ldots, 0) - u(\tilde{x}_1, 0, \ldots, 0),$$

i.e $u(x_1, 0, \ldots, 0) > u(\tilde{x}_1, 0, \ldots, 0)$. From the radial symmetry of $u$ and decreasing in the direction $\frac{x_1}{|x|}$, we can conclude the monotonicity of $u$. \qed

39
C Non-existence in the whole space

Let $\mathcal{L}_\mu^s$ be the fractional Hardy operator defined by

$$\mathcal{L}_\mu^s = (-\Delta)^s + \frac{\mu}{|x|^{2s}}$$

for $s \in (0, 1)$ and

$$\mu \geq \mu_0 := -2^2 \frac{\Gamma^2(N+2s)}{\Gamma^2(N-2s)}.$$

It is shown in [15] that for $\mu \geq \mu_0$ the equation

$$\mathcal{L}_\mu^s u = 0 \text{ in } \mathbb{R}^N \setminus \{0\}$$

has two distinct radial solutions

$$\Phi_{s,\mu}(x) = \begin{cases} |x|^{\tau_-(s,\mu)} & \text{if } \mu > \mu_0 \\ |x|^{\frac{N-2s}{2} \ln \left(\frac{1}{|x|}\right)} & \text{if } \mu = \mu_0 \end{cases} \text{ and } \Gamma_{s,\mu}(x) = |x|^{\tau_+(s,\mu)},$$

where $\tau_-(s,\mu) \leq \tau_+(s,\mu)$ verifies that

$$\tau_-(s,\mu) + \tau_+(s,\mu) = 2s - N \quad \text{for all } \mu \geq \mu_0,$$

$$\tau_-(s,\mu_0) = \tau_+(s,\mu_0) = \frac{2s-N}{2}, \quad \tau_-(s,0) = 2s - N, \quad \tau_+(s,0) = 0,$$

$$\lim_{\mu \to +\infty} \tau_-(s,\mu) = -N \quad \text{and} \quad \lim_{\mu \to +\infty} \tau_+(s,\mu) = 2s.$$

For simplicity, we put $\tau_+ = \tau_+(s,\mu)$, $\tau_- = \tau_-(s,\mu)$.

**Theorem C.1** Let $\mu > \mu_0$ and $p \in (1, p_\mu^*]$, where

$$p_\mu^* = 1 + \frac{2s}{-\tau_-(s,\mu)}.$$

Then problem

$$(-\Delta)^s u = u^p \quad \text{in } \mathbb{R}^N \setminus \{0\} \quad \text{(C.1)}$$

has no positive solution.

Note that the mapping $\mu \in [\mu_0, +\infty) \mapsto p_\mu^*$ is strictly decreasing. Particularly, $p_\mu^* = \frac{N}{N-2s}$ for $\mu = 0$ and $p_\mu^* = \frac{N+2s}{N-2s}$ for $\mu = \mu_0$.

C.1 Basic estimates

**Lemma C.1** Let $\mu \geq \mu_0$, $\Omega$ is a bounded domain containing the origin and nonnegative function $f \in C_\text{loc}^\beta(\mathbb{R}^N \setminus \bar{\Omega})$ for some $\beta \in (0, 1)$. The homogeneous problem

$$\begin{cases} \mathcal{L}_\mu^s u \geq f & \text{in } \mathbb{R}^N \setminus \bar{\Omega}, \\ u \geq 0 & \text{in } \bar{\Omega} \end{cases} \quad \text{(C.2)}$$

has no positive solution if

$$\lim_{r \to +\infty} \int_{B_r(0) \setminus B_{r_0}(0)} f(x)|x|^{2s-\tau_+ - N} \, dx = +\infty.$$

Particularly, the above assumption could be replaced by

$$\liminf_{|x| \to +\infty} f(x)|x|^{2s-\tau_+} > 0.$$
Proof. Let \( u_f \) be positive solution of (C.2) and denote by
\[
O^\sharp = \left\{ x \in \mathbb{R}^N : \frac{x}{|x|^2} \in O \right\}
\]
and \( u^\sharp(x) = |x|^{2s-N}u_f\left(\frac{x}{|x|^2}\right) \) for \( x \in O^\sharp \).

(C.3)

Clearly, \((\mathbb{R}^N \setminus \{0\})^\sharp = \mathbb{R}^N \setminus \{0\}\) and the Kelvin transformation is the following
\[
(-\Delta)^s u^\sharp(x) = |x|^{-2s-N}((-\Delta)^s u_f)\left(\frac{x}{|x|^2}\right) \text{ for } x \in O^\sharp.
\]

(C.4)

Therefore, \( u^\sharp \) is a super solution of
\[
\mathcal{L}_\mu^s u^\sharp(x) \geq |x|^{-2s-N} \tilde{f}(x) \text{ in } O^\sharp,
\]
where \( \tilde{f}(x) = f\left(\frac{x}{|x|^2}\right) \).

Note that there exists \( r_1 > 0 \) such that
\[
B_{r_1}(0) \subset O^\sharp \cup \{0\}.
\]

Note that for \( r > \frac{1}{r_1} > 0 \)
\[
\int_{B_{r_1}(0) \setminus B_{\frac{1}{r_1}}(0)} f\left(\frac{x}{|x|^2}\right)|x|^{-2s-N+s} \, dx = \int_{B_{r_1}(0) \setminus B_{\frac{1}{r_1}}(0)} f(y)|y|^{2s-s-N} \, dy
\]
\[
\to +\infty \text{ as } r \to +\infty.
\]

The contradiction follows by \[15, \text{Theorem 1.3}\]. We complete the proof. \(\square\)

Lemma C.2 Let \( \mu \geq \mu_0, \theta \in \mathbb{R} \) and \( u_0 \) be a positive solution of (1.9) in \( \mathbb{R}^N \setminus \Omega \), then
\[
\lim_{|x| \to +\infty} u_0(x)|x|^{N-2s} > 0.
\]

Proof. Let \( f = Qu^p \), then there exists \( \varepsilon_0 > 0 \) and \( r_1 > r_2 > 0 \) such that
\[
f \geq \varepsilon_0 \text{ in } B_{r_1} \setminus B_{r_2}.
\]

Our problem reduces to
\[
\begin{cases}
(-\Delta)^s u \geq f & \text{in } \mathbb{R}^N \setminus \Omega, \\
u \geq 0 & \text{in } \Omega.
\end{cases}
\]

(C.6)

Let
\[
v^\sharp(x) = |x|^{2s-N}u\left(\frac{x}{|x|^2}\right) \text{ for } x \in O^\sharp,
\]
and direct computation shows that
\[
(-\Delta)^s v^\sharp(x) = |x|^{-2s-N}f\left(\frac{x}{|x|^2}\right) \text{ in } B_1 \setminus \{0\}.
\]

Then Maximum principle shows that
\[
v^\sharp(x) \geq c_{32} \text{ in } B_{\frac{1}{r_1}} \setminus \{0\},
\]
which implies that
\[
u_0\left(\frac{x}{|x|^2}\right) \geq c_{32}|x|^{N-2s} \text{ in } B_{\frac{1}{r_1}} \setminus \{0\}
\]
and then
\[
u_0(x) \geq c_{32}|x|^{2s-N} \text{ in } \mathbb{R}^N \setminus B_2,
\]
where \( \tau_+ + \tau_- = 2s - N \). \(\square\)
Lemma C.3 Let $2s + \theta > 0$, $\tau_0 < 0$

$$p \in \left[1, 1 + \frac{2s + \theta}{-\tau_0}\right)$$

and \{\tau_j\}_j be the sequence generated by

$$\tau_j = 2s + \theta + p\tau_{j-1} \quad \text{for} \quad j = 1, 2, 3 \cdots.$$

Then \{\tau_j\}_j is an increasing sequence of numbers and for any $\bar{\tau} > \tau_0$ there exists $j_0 \in \mathbb{N}$ such that

$$\tau_{j_0} \geq \bar{\tau} \quad \text{and} \quad \tau_{j_0} - 1 < \bar{\tau}.$$

Proof. For $p \in \left(0, 1 + \frac{2s + \theta}{-\tau_0}\right)$, we have that

$$\tau_1 - \tau_0 = 2\alpha + \tau_0(p - 1) > 0$$

and

$$\tau_j - \tau_{j-1} = p(\tau_{j-1} - \tau_{j-2}) = p^{j-1}(\tau_1 - \tau_0),$$

which imply that the sequence \{\tau_j\}_j is increasing and our conclusions are obvious. □

Lemma C.4 Assume that $\mu > \mu_0$, $O_r = \mathbb{R}^N \setminus B_1$, the nonnegative function $g \in C^\beta_{\text{loc}}(O_r)$ for some $\beta \in (0, 1)$ and there exist $\tau \in (\tau_-, \tau_+)$, $c_{33} > 0$ and $r_3 > 0$ such that

$$g(x) \geq c_{33}|x|^\tau - 2s \quad \text{in} \quad O_{r_3}.$$

Let $u_g$ be a positive solution of problem

$$\mathcal{L}_\mu u \geq g \quad \text{in} \quad O_r, \quad u \geq 0 \quad \text{in} \quad \mathbb{R}^N \setminus O_r,$$

then there exists $c_{34} > 0$ such that

$$u_g(x) \geq c_{34}|x|^\tau \quad \text{in} \quad O_{r_3}.$$

Proof. For $\tau \in (\tau_-, \tau_+)$, we have that

$$\mathcal{L}_\mu^* |x|^\tau = b_\tau(x)|x|^\tau - 2s \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},$$

where $b_\tau(x) > 0$.

In the case $O = B_1 \setminus \{0\}$, we use the function

$$w(x) = |x|^\tau - |x|^{\tau_+} \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}$$

as a sub solution

$$\mathcal{L}_\mu^* u = c_\tau(x)|x|^\tau - 2s \quad \text{in} \quad B_1 \setminus \{0\}, \quad u \leq 0 \quad \text{in} \quad \mathbb{R}^N \setminus B_1,$$

where $c_\tau(x) > 0$. Then our argument follows by comparison principle.

When $O = \mathbb{R}^N \setminus B_1$, we use the function

$$w(x) = |x|^\tau - |x|^{\tau_-} \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}$$

as a sub solution

$$\mathcal{L}_\mu^* u = c_\tau(x)|x|^\tau - 2s \quad \text{in} \quad \mathbb{R}^N \setminus B_1, \quad u \leq 0 \quad \text{in} \quad B_1$$

and the left is standard. □
C.2 Nonexistence

Proof of Theorem C.1. By contradiction, we assume that (C.1) has a positive solution \( u_0 \). From Lemma C.3, we have that
\[
u_0(x) \geq d_0|x|^{2s-N} \quad \text{in} \quad \mathbb{R}^N \setminus B_2.
\]

Step 1: We first show the nonexistence of
\[
(-\Delta)^s u + \frac{\mu}{|x|^{2s}} = u^p \quad \text{in} \quad \mathbb{R}^N \setminus \{0\},
\]
under the assumption that \( \mu \in [0, \mu_0) \) and \( p \in (0, p^*_\mu) \), where
\[
p^*_{\mu} = 1 + \frac{2s}{\tau_-}.
\]

Let \( \tau_0 = \tau_- \), which verifies that for \( x \in \mathbb{R}^N \setminus \bar{B}_r \),
\[
L^s_\mu u_0(x) \geq d_0^p|x|^{|\tau_0|^{-2s}} = d_1 \quad \text{in} \quad \mathbb{R}^N \setminus \bar{B}_r,
\]
where
\[
\tau_1 := \tau_0 + 2s.
\]

If \( p\tau_0 \geq \tau_+ - 2s \), then
\[
u_0^p \geq d_0^p|x|^{|\tau_0|^{-2s}}
\]
and a contradiction follows by Lemma C.1. We are done.

If not, by Lemma C.4, we have that
\[
u_0(x) \geq d_1|x|^{\tau_1} \quad \text{in} \quad \mathbb{R}^N \setminus B_r.
\]

Iteratively, we recall that
\[
\tau_j := p\tau_{j-1} + \theta + 2s, \quad j = 1, 2, \ldots
\]

Note that for \( p \in (0, p^*_{\theta, \mu}) \)
\[
\tau_1 - \tau_0 = (p-1)\tau_0 + \theta + 2s > 0.
\]

If \( \tau_{j+1} = \tau_j + \theta + 2s \in (\tau_-, \tau_+) \), it following by Theorem C.4 that
\[
u_0(x) \geq d_{j+1}|x|^{\tau_{j+1}},
\]
where \( \tau_{j+1} = \tau_0 + 2s + \theta > \tau_0 \).

If \( p\tau_{j+1} + \theta \geq \tau_- \), we are done by lemma C.4. In fact, this iteration could stop by finite times since \( \tau_j \to +\infty \) as \( j \to +\infty \) if \( p \geq 1 \).

Step 2. Nonexistence in the critical case \( \mu > \mu_0 \)
\[
\mu = p^*_{\theta, \mu} \}
\]

Here we can assume that \( \mu - \sigma_0 \geq \mu_0 \), otherwise, we only take a smaller value for \( \sigma_0 \). So we can write problem (1.1) as following
\[
L^s_{\mu-\sigma_0} u_0 \geq \frac{1}{2} u_0^p \quad \text{in} \quad \mathbb{R}^N \setminus B_r,
\]
which the critical exponent
\[
p^*_{\mu-\sigma_0} = 1 + \frac{2s}{\tau_-(s, \mu - \sigma_0)} > 1 + \frac{2s}{\tau_+(s, \mu)} = p^*_{\mu},
\]
where \( \mu \in (\mu_0, +\infty) \to \tau_+(s, \mu) \) is strictly increasing. Thus a contradiction comes from step 1 for (C.11). □

Proof of Proposition 1.1. It is the particular case \( \mu = 0 \) and \( p = p^* \) in Theorem 1.1. □

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