Low-temperature Dephasing and Renormalization in Model Systems

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We investigate low-temperature dephasing in several model systems, where a quantum degree of freedom is coupled to a bath. Dephasing, defined as the decay of the coherence of initial non-equilibrium states, also influences the dynamics of equilibrium correlation and response functions, as well as static interference effects. In particular, in the latter case, dephasing should be distinguished from renormalization effects. For illustration, and because of its relevance for quantum state engineering in dissipative environments, we first reconsider dephasing in spin-boson models.

Next we review Caldeira-Leggett models, with applications, e.g., to persistent currents in mesoscopic rings. Then, we analyze the more general problem of a particle which interacts with a quantum field $\hat{V}(t, \mathbf{r}(t))$, the fluctuations of which are characterized by a dielectric function $\epsilon(\omega, \mathbf{k})$. Finally, we compare this model, both the formulation as well as the results, to the problem of interacting electrons in a diffusive conductor.

I. INTRODUCTION

A particle prepared in a non-equilibrium state and interacting with an environment usually relaxes to equilibrium. The decay of the off-diagonal elements of its density matrix is denoted as dephasing. More generally, any state of the particle described by mixture with density matrix $\hat{\rho} \neq \hat{\rho}^2$ can be interpreted as manifestation of dephasing. This applies for reduced density matrices, obtained after tracing out the environment, also in equilibrium and even in the ground state of the total system. We further note that certain correlation and response functions of the particle decay, a fact which may be related to dephasing as well. Again this decay is observable although the expectation values are evaluated in the ground state of the total system. All these manifestations of dephasing have the same origin: the interaction and entanglement of the particle with an environment, combined with the reduction of the description to a subsystem of the total system. The interaction usually has further consequences, incl. relaxation, dissipation, as well as renormalization effects. These effects, while closely related, must be carefully distinguished from each other.

In this article we consider several model systems, which in part can be analyzed exactly, with the idea to illustrate different manifestations of dephasing and the distinction to relaxation and renormalization effects. First we review spin-boson models \textsuperscript{4, 5}. Depending on the spectrum of the bath one finds dephasing, manifest as the decay of the coherence of an initial non-equilibrium state, even at $T = 0$. We further show that equilibrium correlation and response functions decay on a time scale which coincides with the dephasing time \textsuperscript{3}. On the other hand, some effects of the coupling to the bath can be interpreted as renormalization effects. The results are relevant, e.g., in the context of quantum manipulations of quantum systems in a dissipative environment \textsuperscript{6}.

Next we review Caldeira-Leggett (CL) models \textsuperscript{7} of a particle coupled linearly to a bath of oscillators and arrive at similar conclusions. As an application we consider persistent currents in mesoscopic rings with interactions \textsuperscript{1, 2, 3, 4, 5}. We then analyze the problem of a particle which interacts with a quantum field $\hat{V}$ as described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} - e\hat{V}(t, \mathbf{r}) + \hat{H}_{\text{env}}(\hat{V}, \hat{\psi}).$$

The field $\hat{V}(t, \mathbf{r})$ fluctuates due to the coupling to further environment degrees of freedom $\psi$. Its fluctuations are characterized by a frequency- and wave-vector-dependent dielectric function $\epsilon(\omega, \mathbf{k})$, thus generalizing the effect of the bath in the CL model. In spite of the differences we find results similar to those mentioned above, including a finite dephasing time at $T = 0$.

The results for the dephasing time of the last model coincide with those derived in Ref. \textsuperscript{12} (GZ) for the problem of interacting electrons in a diffusive conductor. Indeed, the interaction between electrons can be accounted for by a fluctuating field $\hat{V}(t, \mathbf{r})$, however, since the electrons are indistinguishable one has to account for the Pauli principle. An appropriately generalized formulation of the problem has been presented by GZ \textsuperscript{12}. They argued that the modifications do not yield qualitative changes for the dephasing time. This conclusion has been challenged; for a discussion see, e.g., Refs. \textsuperscript{13, 14}.

II. SPIN-BOSON MODEL

The spin-boson model has been studied extensively before \textsuperscript{4, 5}; the analysis of the following section is partially based on work presented in Ref. \textsuperscript{4}. The spin-boson model describes a two-state quantum system coupled to a bath of oscillators with Hamiltonian

$$\hat{H} = -\frac{\Delta E}{2}\hat{\sigma}_z + \hat{X}(\cos \theta \hat{\sigma}_z - \sin \theta \hat{\sigma}_x) + \sum_k \hbar \omega_k \hat{a}_k^\dagger \hat{a}_k.$$ 

Here $\Delta E$ is the bare energy splitting between the levels of the two-state system. The bath operator $\hat{X} = \sum_k \hat{a}_k^\dagger \hat{a}_k.$
The function \(\sum_k c_k (\hat{a}_k + \hat{a}_k^+)\) couples ‘longitudinally’ to \(\hat{\sigma}_z\) and ‘transverse’ to \(\hat{\sigma}_x\), depending on the angle \(\theta\). In thermal equilibrium the Fourier transform of the symmetrized correlation function of this operator,

\[
S_X(\omega) \equiv \langle [\hat{X}(t), \hat{X}(t')]_+ \rangle = 2\hbar J_s(\omega) \coth \frac{\hbar \omega}{2k_B T},
\]

depends on the bath spectral density \(J_s(\omega) \equiv (\pi/\hbar) \sum_k c_k^2 \delta(\omega - \omega_k)\). At low frequencies it typically follows a power-law up to a high-frequency cutoff \(\omega_c\),

\[
J_s(\omega) = \frac{\pi}{2} \hbar \alpha \omega_0^{1-s} \omega^s \Theta(\omega_c - \omega).
\]

The spin-boson model has been studied mostly for baths with Ohmic spectrum \((s=1)\). In general, for dimensional reasons a frequency scale \(\omega_0\) has been introduced in [15, 16], although \(J_s(\omega)\) depends only the combination \(\omega_0^{1-s}\). We could choose \(\omega_0\) equal to the high frequency cutoff \(\omega_c\) of the bath [15], however for later discussion it is more convenient to distinguish both.

### A. Relaxation of a non-equilibrium state

Two different time scales describe the evolution in the spin-boson model. The dephasing time \(\tau_\varphi\) characterizes the decay of the off-diagonal elements of the spin’s reduced density matrix \(\rho(t)\) in the eigenbasis of \(H_0\). Frequently one encounters an exponential long-time dependence, \(\rho(t)_{12} \sim e^{-t/\tau_\varphi}\), but other decay laws may emerge as well. The second, the relaxation time scale \(\tau_{\text{rel}}\), characterizes how diagonal entries tend to their thermal equilibrium values, \(\rho_{ii}(t) - \rho_{ii}^{eq} \sim e^{-t/\tau_{\text{rel}}}\). Both times were evaluated in Refs. [15, 16] with the results

\[
\tau_{\text{rel}}^{-1} = \frac{1}{\hbar^2} \sin^2 \theta \ S_X(\omega = \Delta E/\hbar) ,
\]

\[
\tau_\varphi^{-1} = \frac{1}{2} \tau_{\text{rel}}^{-1} + \frac{1}{\hbar^2} \cos^2 \theta \ S_X(\omega = 0).
\]

For transverse coupling \(\propto \sin \theta\) the fluctuating field induces transitions between the eigenstates of the unperturbed system. For longitudinal coupling \(\propto \cos \theta\) it still contributes to dephasing, since it leads to fluctuations of the eigenenergies and, thus, to a random phase shift.

This is the origin of the second, “pure” dephasing term \(\Gamma_\varphi = S_X(\omega = 0)/\hbar^2\) in Eq. (1). For an Ohmic environment at \(T \neq 0\) one finds \(\Gamma_\varphi = 2\pi\alpha k_B T/\hbar\). At \(T = 0\), on the other hand, for most spectra the expression (1) yields a vanishing or divergent result for \(\Gamma_\varphi\), demonstrating the need for a more detailed analysis.

In the limit of purely longitudinal coupling, \(\theta = 0\), the analysis can be done exactly. Assuming a factorized initial density matrix one finds \(\rho_{12} \sim P_{\omega_0}(t)\). The function \(P_{\omega_0}(t)\) (known from the “P(E)”-theory [11, 12]) can be expressed as \(P_{\omega_0}(t) = e^{K(t)}\), with

\[
K(t) = \frac{1}{\hbar^2} \int_0^{\infty} \omega^2 \rho(\omega) \cos(\hbar \omega t/2k_B T) - i \sin \omega t\]

For an Ohmic bath \((s=1)\), finite temperatures, and \(t > h/k_B T\) it reduces to \(\text{Re} K(t) \approx -\pi \alpha 2\hbar T \omega_0 t\), consistent with Eq. (4). On the other hand, for lower temperatures or shorter times, \(1/\omega_c < t < h/k_B T\), one finds \(\text{Re} K(t) \approx -2\alpha \ln(\omega_c t)\), implying a power-law decay

\[
\rho_{12}(t) = (\omega_0 t)^{-2\alpha} e^{-i\Delta E t/h} \rho_{12}(0).
\]

Thus even at \(T = 0\) the off-diagonal elements of the density matrix decay in time. It should be noted that all oscillators up to the cutoff \(\omega_c\) contribute to this decay.

For sub-Ohmic baths \((0 < s < 1)\) with high density of low-frequency oscillators exponential dephasing is observed for all temperatures and times: \(\rho_{12} \propto \exp[-(\omega_0 t)^{1-s}]\) for \(t < h/k_B T\), while \(\rho_{12} \propto \exp[-\alpha t (\omega_0 t)^{1-s}]\) for \(t > h/k_B T\). Thus the dephasing rate is \(\Gamma_\varphi = \alpha^{1/(1-s)}\omega_0\) for \(T < \alpha^{1/(1-s)}\omega_0\) and \(\Gamma_\varphi \propto (\alpha T/\omega_0)^{1/(2-s)}\omega_0\) for \(T > \alpha^{1/(1-s)}\omega_0\).

In the super-Ohmic regime \((s > 1)\) after an initial decay on time scale \(\omega_c^{-1}\), the exponent \(\text{Re} K(t)\) saturates at a finite value \(\text{Re} K(\infty) = -\alpha(\omega_c/\omega_0)^{s-1}\), and the off-diagonal element \(\rho_{12}\) stays constant for \(t < h/k_B T\). At longer times, \(t > h/k_B T\), if \(s < 2\) an exponential decay is observed, \(\rho_{12}(t) \propto \exp[-\alpha T (\omega_0 t)^{1-s}]\), whereas for \(s \geq 2\) there is almost no additional decay.

### B. Renormalization effects

Above we assumed factorized initial conditions: the bath was prepared in the equilibrium state characterized by temperature \(T\), while the spin was prepared in an arbitrary initial state. Thus dephasing is to be expected even at vanishing bath temperature. This initial conditions can be achieved in principle by applying sudden pulses to rotate the spin. In a real experiment, however, the preparation pulse takes a finite time, \(\tau_p\), during which the bath oscillators partially adjust to the changing spin state. This will modify the results for dephasing [3].

For example, a \(\pi/2\)-pulse, which transforms the state \(|\uparrow\rangle \rightarrow \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle)\), is accomplished by applying a field \(\hat{H}_p = \hbar \omega_p \hat{\sigma}_x\) for a time \(\tau_p = \pi/2\omega_p\). In this case oscillators with high frequencies, \(\omega_k \gg \omega_p\), follow the spin adiabatically, while those with low frequency, \(\omega_k \ll \omega_p\), do not change their state. Assuming that the oscillators can be split into these two groups, we arrive at an initial state where only the low-frequency oscillators are factorized from the spin \(\frac{1}{\sqrt{2}} (|\uparrow\rangle \otimes |g_1\rangle + |\downarrow\rangle \otimes |g_1\rangle) \otimes |g_1\rangle\).

Here the superscripts ‘hi’ and ‘l’ refer to high and low frequencies and the states \(|g_1\rangle\) are the ground states of the Hamiltonians \(\hat{H}_{1/l} \equiv \sum_k \hbar \omega_k \hat{a}_k^+ \hat{a}_k \pm \hat{X}\).

For the off-diagonal element of the density matrix we now obtain \(\rho_{12} = Z(\omega_c, \omega_p) P_{\omega_0}(t)\). Here \(Z(\omega_c, \omega_p) = \langle |\hat{g}_1^\uparrow| |\hat{g}_1^\downarrow|\rangle\) describes the effect of high-frequency bath oscillators and should be interpreted as a renormalization, while the factor \(P_{\omega_0}(t) = (|\hat{g}_1^\uparrow\rangle e^{-i\hat{H}_p t/\hbar} |\hat{g}_1^\downarrow\rangle)\), which reduces to the same form as \(P_{\omega_0}(t)\) described before except...
that the high-frequency cutoff is reduced to $\omega_p$, describes dephasing due to low-frequency modes.

The criterion to distinguish between both is the fact that renormalization effects are reversible, as illustrated by a continuation of the Gedanken experiment: After the preparation $\pi/2$-pulse we allow for a free evolution of the system for some time $t$, when the state evolves as
\[
\frac{d}{dt} \left( e^{i\Delta E t/2} |\uparrow\rangle \otimes |g^b\rangle \right) + e^{-i\Delta E t/2} |\downarrow\rangle \otimes |g^h\rangle \otimes e^{-iH_s t/\hbar} |g^s_{\uparrow\downarrow}\rangle. \]

Then we apply a $(-\pi/2)$-pulse (also of width $\pi/2\omega_p$) and measure $\bar{\sigma}_x$. Without dissipation the result would be $\langle \bar{\sigma}_x \rangle = \cos(\Delta E t)$, with dissipation we obtain $\frac{\hbar}{i} \langle \bar{\sigma}_x \rangle = \Re \left[ \mathcal{P}_{\omega_p}(t) e^{-i\Delta E t} \right]$. I.e., the amplitude of the coherent oscillations is reduced by the factor $\mathcal{P}_{\omega_p}(t)$, associated with slow oscillators. It describes dephasing, since there is no way to reverse the time evolution contained in this factor. In contrast, the factor $Z(\omega_c, \omega_p)$ does not appear in the final signal. It originates from the overlap of the high-frequency oscillator wave functions, $|g^b\rangle$ and $|g^h\rangle$. They have been manipulated adiabatically, and this effect can be reversed. This effect is properly described by the concept of renormalization.

The distinction between both effects can also be demonstrated if we discuss the Gedanken experiment using renormalized spins, $|\uparrow\rangle \equiv |\uparrow\rangle |g^b\rangle$ and $|\downarrow\rangle \equiv |\downarrow\rangle |g^h\rangle$, and reduce the high-frequency cutoff of the bath to $\omega_p$.

C. Correlation functions

In the limit $\theta = 0$ we can also calculate exactly the linear response of $\bar{\sigma}_x$ to a weak field $H_1 = -(1/2)\delta B_x(t)\sigma_x$:
\[
\chi(t) = \frac{i}{\hbar} \Theta(t) \langle \bar{\sigma}_x(t), \bar{\sigma}_x(0) \rangle. \tag{8}
\]

Using the equilibrium density matrix
\[
\rho^eq = \frac{|\uparrow\rangle \langle \uparrow| \otimes \rho^1 + e^{-\beta \Delta E} |\downarrow\rangle \langle \downarrow| \otimes \rho^1}{1 + e^{-\beta \Delta E}}, \tag{9}
\]

where $\rho^1 \propto \exp(-\beta H_1)$ is the bath density matrix adjusted to the spin state $|\uparrow\rangle$, and similar for $\rho^1$, we obtain the imaginary part of the Fourier transform of the susceptibility, describing dissipation,
\[
\chi''(\omega) = \frac{P(\hbar \omega - \Delta E) + e^{-\beta \Delta E} P(\hbar \omega + \Delta E)}{2(1 + e^{-\beta \Delta E})} - ...(-\omega). \tag{10}
\]

For an Ohmic bath ($s = 1$) at $T = 0$ and positive values of $\omega$ we obtain
\[
\chi''(\omega) = \Theta(\hbar \omega - \Delta E) \frac{e^{-2\gamma \alpha(\hbar \omega - \Delta E)^{2\alpha-1}}}{2\Gamma(2\alpha)(\hbar \omega_c)^{2\alpha}}. \tag{11}
\]

We observe that the dissipative part $\chi''$ has a gap $\Delta E$, corresponding to the minimum energy needed to flip the spin, and a power-law behavior as $\omega$ approaches the threshold. This behavior of $\chi''(\omega)$ parallels the orthogonality catastrophe scenario. It implies that the ground states of the bath for different spin states, $|g^1\rangle$ and $|g^h\rangle$, are macroscopically orthogonal. The form of the response function for $s = 1$ is also known from the X-ray absorption in metals. For sub-Ohmic spectra, as $s$ decreases the $T = 0$ shape of $\chi''(\omega)$ gradually evolves towards a bell shape with width given by the dephasing rate. At $s = 0$ it becomes a Lorentzian, which corresponds to the exponential decay of $|\mathcal{P}(t)|$ in this case.

As $\chi''(\omega)$ characterizes the dissipation it is interesting to distinguish again the roles of high and low frequency oscillators. We use the spectral decomposition at $T = 0$,
\[
\chi''(\omega) = \pi \sum_{\nu} |\langle 0 | \sigma_x | \nu \rangle|^2 \left[ \delta(\omega - E_{\nu}) - \delta(\omega + E_{\nu}) \right],
\]

where $\nu$ denotes the eigenstates of the system. These are $|\uparrow\rangle n_\uparrow$ and $|\downarrow\rangle n_\downarrow$, where $|n_\uparrow\rangle$ denote the excited oscillator states of the Hamiltonians $H_{\uparrow\downarrow}$. The ground state is $|\uparrow\rangle n_\uparrow$, and the only excited states contributing to $\chi''(\omega)$ are $|\downarrow\rangle n_\downarrow$ with $H_{\downarrow} n_\downarrow = (\omega - \Delta E) n_\downarrow$. I.e. all oscillators with frequencies $\omega_k > \omega - \Delta E$ have to be in the ground state. Thus we find for $\omega_p > \omega - \Delta E$
\[
\chi''(\omega_p)(\omega_c, \omega_p) = Z^2(\omega_c, \omega_p) \chi''(\omega).
\tag{12}
\]

To interpret this result we note that for a model including a $g$-factor in the coupling to transverse fields, $H_1 = -(g/2)B_z(t)\sigma_z$, the energy absorption is proportional to $g^2$. Thus, the response function (12) of the spin at frequencies $\omega < \omega_p + \Delta E$ coincides with the one of a model with $g = Z(\omega_c, \omega_p)$ and cutoff $\omega_p$. Again, this property of the high-frequency oscillators is naturally associated with a renormalization phenomenon.

III. PARTICLE PLUS OSCILLATOR BATH

In this section we consider an exactly solvable special case of the Caldeira-Leggett model [13], namely a free particle coupled linearly to a bath of oscillators,
\[
\hat{H} = \frac{\hat{P}^2}{2m} + \sum_k \left[ \frac{\hat{P}_k^2}{2m_k} + \frac{m_k \omega_k^2}{2} \left( \hat{R}_k - \frac{c_k}{m_k \omega_k^2} \right)^2 \right]. \tag{13}
\]

The properties of this model strongly depend on the frequency spectrum of the bath. Here we consider mostly the Ohmic case with
\[
J(\omega) \equiv \frac{\pi}{2} \sum_k \frac{c_k^2}{m_k \omega_k} \delta(\omega - \omega_k) = m \gamma \omega \theta(\omega_c - \omega), \tag{14}
\]
where $\gamma$ is the damping rate in the equation of motion derived from (13) in the classical limit.

A. Equilibrium density matrix at $T = 0$.

For an Ohmic spectrum (14) in the ground state of the total system (particle plus bath) the re-
duced density matrix of the particle $\hat{\rho}(x_1 - x_2) = \int dR_k \Psi_0(x_1, R_k)\Psi_0^*(x_2, R_k)$, is found to be

$$\hat{\rho}(x_1 - x_2) = \exp\left[-(x_1 - x_2)^2/2L_\phi^2\right].$$

(15)

I.e., the density matrix decays if $|x_1 - x_2|$ exceeds a certain length $L_\phi$, which for an Ohmic spectrum is

$$L_\phi = \sqrt{\frac{\pi \hbar}{m \gamma \ln(\omega_c/\gamma)}}.$$

(16)

This decay can be observed in equilibrium interference experiments, e.g., in the persistent current of a particle in a ring threaded by a magnetic flux. For a free particle the amplitude of the current decays with increasing ring radius $R$ as $I \propto 1/R^2$. If the particle is coupled to an Ohmic bath, this amplitude decays exponentially [5, 7], $I \propto \exp(-R^2/L_\phi^2)$, on the scale given by $L_\phi$.

If the bath spectrum has a gap at low frequencies, then with increasing radius the persistent current decreases rapidly, but beyond some radius it crosses over to the $1/R^2$-dependence characteristic for free particles. Examples are provided by the models studied in Refs. 3, 4, 10. In this case, the effect of the environment can be interpreted as a renormalization of the particle mass. On the other hand, for an Ohmic bath [4, 5, 7, 17, 18] such an interpretation is not possible. In order to get further insight and to distinguish between dephasing and renormalization effects one should analyze the behavior of other physical quantities, such as, e.g., fluctuations [6] or the real-time decay of non-equilibrium states. Below we will provide arguments why we interpret the reduction of the persistent current as evidence of dephasing and justify denoting the length scale $L_\phi$ as ‘dephasing length’.

B. Relaxation of a non-equilibrium state

Above we illustrated effects of the bath in the ground state of the total system. To examine the question whether they are related to dephasing processes, we consider the relaxation of an excited state. We start from a factorized initial state $\rho_{\text{total}} = \hat{\rho}_{\text{particle}}^{(0)}\hat{\rho}_{\text{bath}}(T = 0)$, where initially the particle is in a superposition of two plane waves with opposite momenta:

$$\hat{\rho}_{\text{particle}}^{(0)} = \psi(x_1)\psi^*(x_2), \quad \psi(x) = \frac{e^{ikx} + e^{-ikx}}{\sqrt{2}}.$$

(17)

The time evolution of the reduced density matrix of the particle can be expressed by an influence functional [3, 4, 5].

$$\rho_{\text{reduced}}(t, x_1, x_2) = \int J(t, x_1, x_2, x_1^{(0)}, x_2^{(0)}) \times \hat{\rho}_{\text{particle}}^{(0)}(x_1^{(0)}, x_2^{(0)}) dx_1^{(0)} dx_2^{(0)},$$

which in turn can be written as a path integral

$$J = \int \mathcal{D}x_1 \mathcal{D}x_2 e^{[S_0[x_1] - S_0[x_2] - S_{R}[x_1, x_2] + iS_{\phi}[x_1, x_2]]}.$$

(18)

Here $S_0$ is the action of a free particle, while $S_R$ and $S_{\phi}$ are associated with the bath. For the sake of brevity we do not present explicit forms of $J$ and the actions $S_{R/\phi}$. We only note that the path integral (18) can be evaluated exactly with the result

$$\rho_{\text{reduced}}(t, x, x) = 1 + e^{-F(t)} \cos[2kx],$$

(19)

where

$$F(t) = \frac{2k^2}{\pi m \gamma} \int_0^t dt_1 \int_0^t dt_2 \int_0^{\omega_c} d\omega \hbar \omega \coth \frac{\hbar \omega}{2T} \times \cos[\omega(t_1 - t_2)](1 - e^{-\gamma t_1})(1 - e^{-\gamma t_2}).$$

(20)

The exponent $F(t)$ describes the suppression of interference, i.e., dephasing. We note that it contains only $\coth \hbar \omega/2T$, rather than the combination $\coth \hbar \omega/2T - 1$ expected from Golden Rule type arguments. The function $F(t)$ arises as the imaginary part of the action $S_{\phi}$ evaluated on the saddle-point paths $x_{1,2}$. It is, thus, affected by $S_R$ through the damping of these paths, which leads to the factors $1 - \exp[-\gamma t_{1,2}]$ in (21). The role of $S_R$ has been discussed recently also in Refs. 13, 14.

The dephasing time is naturally defined from the condition $F(\tau_{\phi}) = 1$. For long and short times we find

$$F(t) = \frac{4k^2}{\pi m \gamma} \times \left\{ \begin{array}{ll} (\gamma t)^2 \ln \omega_c t & \text{for } \gamma t \ll 1, \\ \pi \ln(\omega_c t) & \text{for } \gamma t \gg 1. \end{array} \right.$$

(21)

Now we distinguish two cases:

1) $k \gg 1/L_\phi$. In this case the short-time asymptotic is sufficient to determine the dephasing time. We find

$$\tau_{\phi} \approx \frac{1}{v} \sqrt{\frac{\pi \hbar}{2m \gamma \ln(\pi \hbar \omega_c^2/4m \gamma v^2)}}, \quad \text{with } v = \frac{\hbar k}{m}.$$  

(22)

Comparing to Eq. (16) we observe $L_\phi \sim v \tau_{\phi}$, i.e., there exists a simple relation between the dephasing time associated with the relaxation of a non-equilibrium state and the ‘dephasing length’ $L_\phi$ found as a ground state property. For later use we also note that $\tau_{\phi}$ can be obtained directly from $S_{\phi}$, since for $\gamma t \ll 1$ we have $F(t) \approx \frac{1}{\gamma} S_{\phi}(t, v', -v')$. (In this limit the real part of the action $S_R$ [18] has no effect on $\tau_{\phi}$. On the other hand, it may be important in other contexts, e.g., for the evaluation of the relaxation rates [11, 14].)

2) $k \ll 1/L_\phi$. This case is governed by the long-time asymptotics of $F(t)$. The classical relaxation, which is influenced by $S_R$, is strong. The interference pattern decays as a power-law with exponentially long dephasing time $\tau_{\phi} \approx \omega_c^{-1} \exp[\hbar^2 m^2 \gamma/(4m^2 v^2)]$. No simple relation between $\tau_{\phi}$ and $L_\phi$ can be established in this limit.

C. Correlation functions

The dephasing in time can be observed also in the decay of equilibrium correlation functions, such as $C(t) =...
\[ \langle a_1(x) a_2(x) \rangle \text{ of coordinates. For translationally invariant systems we can express it by Fourier transforms} \]
\[ C(t) = \int \frac{dk}{2\pi} \delta_1(k) a_{2}(-k) K(k,t), \] depending on the correlator
\[ K(k,t) = (e^{ikx(t)}e^{-ikx(0)}). \] (23)

For a free particle \( K(k,t) = \exp[-i \frac{\hbar k^2}{2m} t] \) evolves with a pure phase factor, while a decay signals a dephasing process. In the presence of the bath we find an expression analogous to that appearing in the \( P(E) \)-theory of
\[ K(t,k) = \exp \left\{ -\frac{\hbar k^2 \gamma}{\pi m} \int_0^\infty \frac{dk}{\omega} \right\} \]
\[ \times \left\{ \coth \frac{\hbar \omega}{2T} \left(1 - \cos \omega t + i \sin \omega t \right) \right\}. \] (24)

At \( T = 0 \) it reduces to
\[ |K_0(t,k)|^2 \approx \left\{ \begin{array}{ll}
\left( \frac{1}{\sqrt{v}} \right) \sqrt{\pi h/m \gamma} \ln L_{\omega} & \text{for } k \gg 1/L_{\omega} \\
\exp \left\{ \frac{\hbar k^2 \gamma}{2 \pi m \gamma} \ln \left( \gamma t + 0.57722 \right) \right\} & \text{for } k \ll 1/L_{\omega}.
\end{array} \right. \] (25)

Again we find a simple relation \( L_{\omega} \sim \nu \tau_\omega \) for \( k \gg 1/L_{\omega} \), but no simple relation in the opposite limit.

To illustrate the respective roles of high- and low-frequency oscillators, it is useful to consider the coupling to only one oscillator. In this case one finds at \( T = 0 \)
\[ K(t) = \exp \left\{ -i \frac{\hbar k^2 \gamma}{1 + \alpha \frac{2}{m}} \right\} \]
\[ -\alpha \hbar k^2 \left( 1 - e^{-i\Omega t} \right) \] (27)

where \( \alpha = c^2/m M \omega^4 \) is the coupling strength, and \( \omega \) and \( \Omega = \sqrt{\omega^2 + \alpha} \) are the bare and renormalized frequency of the oscillator, respectively. The function \( K(t) \) now displays the phenomenon of beating. If the bath contains many oscillators the beating adds up to an incoherent decay. Clearly this effect cannot be interpreted in terms of renormalization of the particle mass. On the other hand, if the detector is not sensitive to the position of the mass is important and one can consider the system “particle+oscillator” as a single particle, analogue to a molecule with the mass \( M + m \). This effect can be regarded as a mass renormalization. Finally, we note that the effect of oscillators with very low frequencies can never be interpreted as a mass renormalization.

**IV. PARTICLE IN A QUANTUM FIELD**

Next we consider a particle coupled to an electromagnetic environment as described by the Hamiltonian \( [3] \).

In the classical limit the fluctuations of the field \( V(t,\mathbf{r}) \) are characterized by the dielectric function \( \epsilon(\omega, \mathbf{k}) \). In general the fluctuations are produced by a quantum environment and, hence, the field should be treated as a quantum field itself. Therefore, within the path integral formulation on the Keldysh contour, the fluctuating field on the forward part of the contour \( V_1(t,\mathbf{r}) \) is to be distinguished from the one on the backward part \( V_2(t,\mathbf{r}) \). Both fields are Gaussian distributed with correlators
\[ \langle V_1(t,\mathbf{r}) V_2(0,0) \rangle = \hbar I(t,\mathbf{r}) + \frac{i\hbar}{2} \left( 1 + 1 \right) R(t,\mathbf{r}) + \\
\left( 1 + 1 \right) R(-t,-\mathbf{r}), \] (28)

where \( I(\omega, \mathbf{k}) = \text{Im} \left( \frac{4\pi\alpha}{k^2 \epsilon(\omega, \mathbf{k})} \right) \) and \( R(\omega, \mathbf{k}) = \frac{4\pi\alpha}{k^2 \epsilon(\omega, \mathbf{k})} \). We observe that \( \text{Im}(1/\epsilon(\omega, \mathbf{k})) \) generalizes the spectral density \( J(\omega) \) in CL-like models. In what follows we concentrate on the Drude model of a normal metal with dielectric function
\[ \epsilon(\omega, \mathbf{k}) = \frac{4\pi \sigma}{\omega + D k^2}. \] (29)

Similar models have been discussed by Weiss [3], Guinea [8, 19], and Cohen [20].

The evolution of the density matrix can be described again by an influence functional
\[ J = \int D\mathbf{r}_1 D\mathbf{r}_2 e^{i\int_0^t dt' \frac{\mathbf{r}_1(t') - \mathbf{r}_2(t')}{2}} \]
\[ \times \left\{ e^{i \int_0^t dt' (e V_1(t', \mathbf{r}_1) - e V_2(t', \mathbf{r}_2))} \right\}_V \]
\[ > 0,1 \]
\[ \int D\mathbf{r} e^{-S_0[\mathbf{r}]+S_{\text{int}}[\mathbf{r}]}, \] where \( S_{\text{int}}[\mathbf{r}] = \int_0^t dt' \int_0^{\beta} d\mathbf{k} \sum_{\omega} \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \]
\[ \times \frac{4\pi}{k^2 \epsilon(i |\omega| \mathbf{k})} e^{-i\omega(t + t' - \mathbf{k}) + i\mathbf{k}(\mathbf{r}(t) - \mathbf{r}(t'))}. \] (30)

As an example we consider again the persistent current of a particle on a ring with fluctuations characterized by
a Drude dielectric function \( \varepsilon(\omega) \). This problem has been addressed in Ref. [9]. A decay of the persistent current has been found for radii exceeding a length scale, \( R > L_\varphi \),

\[
L_\varphi \sim \ell(k_F l)^2.
\]

(31)

Here \( \ell \) is the mean free path of electrons in the metal, and \( k_F \) the Fermi wave vector.

### B. Decay of a non-equilibrium state

Continuing as in previous sections, we consider the time evolution of a non-equilibrium state with factorized initial density matrix. The initial state of the particle is again assumed to be a sum of two counter-propagating plane waves \( \psi_0(r) \propto e^{ikr} + e^{-ikr} \). The visibility of the resulting interference pattern decays in time as \( e^{-F(t)} \). The exact solution of this problem is not possible, but for short times we find

\[
F(t) \approx S_I[v t', -v t']/\hbar \approx t/\tau_\varphi,
\]

(32)

where the effective velocity is \( v = \hbar k/m \), and

\[
\frac{1}{\tau_\varphi} \approx \frac{2e^2 v^2}{\pi \hbar} \int_0^\infty \frac{dk}{k} \int_0^{kv} d\omega \operatorname{Im} \left( \frac{-1}{\epsilon(\omega, k)} \right) \coth \frac{\hbar \omega}{2T}.
\]

(33)

We stress that this form is valid only if the effective velocity \( v \) is high enough, as only in this case the expression \([23]\) is applicable in the whole interval \( 0 < t < \tau_\varphi \).

The Drude formula gives \( \operatorname{Im} \left( -1/\epsilon(\omega, k) \right) = \omega/(4\pi\sigma) \) for \( k < k_{\text{max}} \). Hence Eq. \([34]\) reduces for \( T = 0 \) to

\[
\frac{1}{\tau_\varphi} = \frac{e^2 v k_{\text{max}}^2}{8\pi^2\hbar\sigma},
\]

(34)

which allows us to define a velocity-independent dephasing length

\[
L_\varphi = v\tau_\varphi = \frac{8\pi^2\hbar\sigma}{e^2 k_{\text{max}}^2}.
\]

(35)

If we choose \( k_{\text{max}} = 1/\ell \), where \( \ell \) is the mean free path, thus accounting for the fact that the Drude expression is not valid at shorter length scales, we recover \([22]\). I.e., as in the previous models, we observe a close relation between dephasing in real time and the reduction of interference effects in equilibrium.

For completeness, we mention that the correlation function \( K(t) = \langle e^{i\mathbf{k}'(t)} e^{-i\mathbf{k}(0)} \rangle \) also decays at the same time scale \( \tau_\varphi \).

## V. INTERACTING ELECTRONS

The model considered in the previous section captures essential features of the problem of interacting electrons in disordered conductors. Both the particle in the example studied above and the electron propagating in a disordered conductor interact with a quantum field \( \tilde{V}(t, \mathbf{r}) \) (in the latter case produced by all other electrons). On the other hand, an important and necessary extension is due to the property that the interacting electron is indistinguishable from those producing the fluctuating quantum field, and the Pauli principle has to be obeyed. A general path integral formulation for this problem of interacting electrons, which accounts for the Pauli principle, was formulated in Ref. \([11]\) (GZ). Here we do not attempt to repeat this derivation; we merely proceed with a summary of the main conclusions:

The description of GZ yields effective actions \( S_R \) and \( S_I \) similar to those found above. More precisely, \( S_I \) is unchanged, while the function \( R \) in the expression for \( S_R \) should be multiplied by \( 1 - 2f(H_0(p, \mathbf{r})) \), where \( f(\xi) \) is the Fermi function and \( H_0 \) the Hamiltonian of a non-interacting electron. This fact has a transparent physical interpretation. Since at low temperature the factor \( 1 - 2f(\xi) \) approaches \( \text{sign}(\xi - \mu) \) it effectively implies an energy-dependent dissipation: Above the Fermi level \( (\xi > \mu) \) the electron can lose energy due to the interaction with the bath formed by all other electrons, whereas the effect of the same bath for \( \xi < \mu \) is to push up the holes to the Fermi surface.

The above picture accounts for the difference between the many-body Fermi system and that of a quantum particle distinguishable from its environment. However, as GZ stress this difference is unimportant for the dephasing effect produced by the interaction. The latter effect is due to quantum noise rather than dissipation and it is described by the term \( S_I \) which is not sensitive to the Pauli principle. Accordingly, the expression for \( \tau_\varphi \) which they derive \([11]\) coincides with the one found in the previous section if one sets \( v = v_F \). The Pauli principle may influence the quantum corrections to the classical electron action. GZ had argued \([11]\) that they only determine the pre-exponent, and thus are irrelevant for the dephasing time. On the other hand, von Delft recently conjectured \([13]\) that this effect may be responsible for the discrepancy between the conclusion of GZ and others \([21]\). In response, GZ \([13]\] explicitly analyzed the quantum corrections, thus confirming their earlier conclusions.

## VI. SUMMARY

We have studied the effect of interactions and coupling to baths in several model systems. This includes spin-boson models, a particle interacting with a Caldeira-Leggett bath, as well as more general baths with space and time dependent fluctuation. In these models, in contrast to the scattering problems discussed by Imry \([22]\), we have demonstrated that dephasing effects are observable at low temperature: (i) the decay of a non-equilibrium initial state, (ii) the decay of certain equilibrium correlation and response functions, and (iii)
equilibrium properties: e.g., the suppression of persistent currents.

Although details depend on the model, e.g. the spectrum of the bath, our main conclusions are: (a) In all the models considered we find a reduction of interference effects due to the coupling to the bath down to zero temperature. (b) Similarly we find that dephasing, understood as the decay of non-equilibrium initial states, persists down to zero temperature. (c) Response and correlation functions decay on the same time scales, even if evaluated in the ground state. (d) In some cases the distinction between dephasing and renormalization effects is ambiguous. An example is the reduction of equilibrium interference effects. However, we observe that the coupling to the bath which produces the dephasing in nonequilibrium situations determines in the same combination of parameters the reduction of interference effects.

For this reason we associate these effects with dephasing as well. (e) We observe similarities in formulation and results between the model for a particle in a fluctuating field and the problem of interacting electrons in disordered metals.

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[1] A.J. Leggett et al., Rev. Mod. Phys. 59, 1 (1987).
[2] U. Weiss, Quantum Dissipative Systems World Scientific, Singapore, Second Edition (1999).
[3] A. Shnirman, Yu. Makhlin, and G. Schön, to be published in the Proceedings of the NATO ARW “Quantum Noise in Mesoscopic Physics”, Delft, June 2002.
[4] A. Shnirman, Yu. Makhlin, and G. Schön, to be published in Physica Scripta, 2002.
[5] A.O. Caldeira, and A.J. Leggett, Physica A121, 587 (1983); 130, 374 (1985).
[6] P. Cedraschi, V. Ponomarenko, and M. Büttiker, Phys. Rev. Lett. 84, 346 (2000); Ann. Phys. 289, 1 (2001).
[7] F. Marquardt and C. Bruder, Phys. Rev. B65, 125315 (2002).
[8] F. Guinea, Phys. Rev. B65, 205317 (2002).
[9] D.S. Golubev, C.P. Herrero and A.D. Zaikin, cond-mat/0205540.
[10] D. Loss and T. Martin, Phys. Rev. B47, 4619 (1993).
[11] D.S. Golubev, and A.D. Zaikin, Phys. Rev. B59, 9195 (1999).
[12] D.S. Golubev, and A.D. Zaikin, cond-mat/0208140.
[13] see the article of J. von Delft, this volume.
[14] F. Marquardt, cond-mat/0207692.
[15] M. H. Devoret et al, Phys. Rev. Lett. 64, 1824 (1990).
[16] see e.g. G.-L. Ingold and Yu. V. Nazarov, in *Single Charge Tunneling*, Eds. H. Grabert and M.H. Devoret, Plenum 1992, Chapter 2.
[17] D.S. Golubev, and A.D. Zaikin, Physica B255, 164 (1998).
[18] D. Loss and K. Mullen, Phys. Rev. B 43, 13252 (1991).
[19] F. Guinea, cond-mat/0207381.
[20] D. Cohen, Phys. Rev. E55, 1422 (1997).
[21] I.L. Aleiner, B.L. Altshuler, and M.R. Gershenson, Waves in Random Media 9, 201 (1999).
[22] Y. Imry, cond-mat/0202041.