Critical behavior of systems with long-range interaction in restricted geometry

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Abstract

The present review is devoted to the problems of finite-size scaling due to the presence of long-range interaction decaying at large distance as $1/r^{d+\sigma}$, $\sigma > 0$. The attention is focused mainly on the renormalization group results in the framework of $O(n) \varphi^4$ - theory for systems with fully finite (block) geometry under periodic boundary conditions. Some bulk critical properties and Monte Carlo results also are reviewed. The role of the cutoff effects as well their relation with those originating from the long-range interaction is also discussed. Special attention is paid to the description of the adequate mathematical technique that allows to treat the long-range and short-range interactions on equal ground. The review closes with short discussion of some open problems.

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I. INTRODUCTION

Any system which has a finite size $L$ in at least one space dimension we will call a finite-size system or a system with restricted geometry. In such systems singularities in the thermodynamic functions at the critical point may occur only in the thermodynamic (bulk) limit, taken in at least $d_l$ dimensions ($d_l$ is the lower critical dimension). Mainly three specific geometries (with periodic boundary condition imposed along the finite dimensions) are of particular interest: i) the fully finite cube $L^d$, ii) the $d$-dimensional layer (or film) $L^1 \times \infty^{d-1}$ and iii) the infinitely long cylinder $L^{d-1} \times \infty$. Further in this review we will not discuss the latter two cases. So far, in the context of long-range (LR) interaction they are studied only in the spherical $n = \infty$ limit. For the corresponding results one may consult the review [1] and the recent monograph [2].

The geometry of real systems and Monte Carlo calculations usually correspond to the fully finite size case. During the last two decades the study of systems with restricted geometry has undergone an extensive development and gains still growing importance for the theory of critical phenomena [2, 3, 4, 5, 6]. Generally speaking the critical behavior depends essentially on geometry, boundary conditions, and on the universality class of the bulk system.

It is well known that the universality class to which the critical behavior of a bulk system at a second order phase transition belongs depends upon the dimensionality of the space $d$, the number of the components of the order parameter $n$, the symmetry of the Hamiltonian (either in spin-space or in coordinate space) and the interaction potentials [7, 8, 9, 10]. One of the most commonly studied interaction potentials, is the one corresponding to LR ferromagnetic interaction decaying algebraically with the spin separation $r$ as $r^{-d-\sigma}$. The parameter $\sigma > 0$ controls the range of the interaction. The investigation of such systems was initiated by Joyce in his paper on the phase transition in the ferromagnetic spherical model [11]. The interest in this type of interaction is tightly related to the exploration of the critical behavior of systems with restricted dimensionality in which no phase transition occurs otherwise (see [12, 13, 14] and references therein). Here the condition $\sigma > 0$ is needed to avoid an ill-defined thermodynamic limit. In the limit $\sigma \to 0^+$ the interaction (after appropriate renormalization) equals the equivalent-neighbors interaction [15].

The opposite case, corresponding to a LR interaction with $-d \leq \sigma \leq 0$, sometimes called also nonintegrable interaction, has been studied in connection with the so called ”non-extensive thermodynamics” (see [16] and references therein). In spite of the recent growing interest, the case of nonintegrable interaction is beyond the scope of the present study.

The LR interaction, $r^{-d-\sigma}$, enters the expressions of the theory only through its Fourier transform. The corresponding small $q$ expansion of the Fourier transform has the general form

$$v(q) = v_0 + v_2 q^2 + v_{\sigma|q|^\sigma} + w(q) \quad 0 < \sigma \neq 2,$$

(1)

with $w(q)/q^2 \to 0$, for $q \to 0$, i.e in the long-wavelength approximation. Note that the case $\sigma = 2$ leads to logarithmic factors in [11] which are not allowed to enter the analysis below. However, further on we will formally relate $\sigma = 2$, to the short-range (SR) interaction since then [11] is the Fourier transform of an interaction decaying exponentially with distance. Depending on whether $\sigma \leq 2$ in [11] we will speak about leading or subleading LR term respectively. Since the SR term $\sim q^2$ in [11] always exists the problem in the momentum space must be considered with necessity as an interplay between SR and LR effects.

This review is arranged as follows. In Section [11] we summarize the results on the critical
behavior of bulk systems with the LR interaction. Some general remarks on the finite-size scaling (FSS) for such systems are presented in Section III. We review some results on the finite-size critical behavior of systems with leading LR interaction in Section IV and for the subleading one in Section VI. Section V specially deals with Binder’s cumulant as an utilized tool to compare the theory to data from Monte Carlo simulations. Some remarks on the cutoff effects are devised in Section VII. In Section VIII we deliberately emphasize some unsolved issues. In A, we present some mathematical details in calculating lattice sums in the case of LR interaction.

II. BASIC RESULTS IN THE BULK CASE

The results concerning the bulk critical behavior in the simplest case of the spherical limit was analyzed in details in [17] (for a recent review, see Chapter 3 of [2]). These results were generalized to the $O(n)$ vector $\varphi^4$ model by means of perturbation theory in combination with the renormalization group (RG) technique near the upper critical dimension $d = 2\sigma$, lower critical dimension $d = \sigma$, and the $1/n$-expansion [27, 28]. Computer simulations also contributed to the exploration of the critical properties of such systems [29, 30, 31, 32, 33, 34, 35].

In the case of LR interaction, some RG predictions, e.g. using the $\epsilon$-expansion, can be verified in an ideal testing ground since the value of $\epsilon = 2\sigma - d$ or $\epsilon = d - \sigma$ would be small enough for integer values of the dimensionality. In this context the outcome of the computer simulations, obtained by means of the Monte Carlo method, concerned mainly Ising systems ($n = 1$) with classical (mean-field) critical behavior [29, 34], i.e. $0 < \sigma < d/2$ with $d = 1, 2, 3$, and nonclassical critical behavior [33, 34, 35], i.e. $d/2 < \sigma$ with $d = 1, 2$. Some comparisons with rigorous results were obtained for low dimensional systems in [12, 13] and for $d$-dimensional systems in [36].

It is worth mentioning that the critical behavior depends strongly upon the interaction parameter $\sigma$. When $\sigma < 2$ the expansion [11] has been used for detailed investigations of the critical behavior of $O(n)$ models including questions like the $\sigma, d$ and $n$ dependence of the critical exponents and critical amplitude ratios, as well as for determining the universal scaling functions. In this case, the LR term is leading and the critical exponents of the system are $\sigma$ dependent. By increasing $\sigma$, a crossover from LR critical behavior to SR one takes place. This issue has been the matter of a long-standing debate in the literature. As a measure for the crossover one may consider the Fisher exponent $\eta$.

First, it has been argued, that in the interval $d/2 < \sigma < 2$ and above the critical temperature $T_c$, the critical exponent $\eta$ is equal to $2 - \sigma$ with no corrections to order $\epsilon^2$ and $\epsilon^3$ (at least), see [13], and to $O(\epsilon^2)$, see [27, 28]. It is reasonable to believe that $\eta$ ”sticks” to this value to all order in $\epsilon$ [11]. So $\eta = \eta(\sigma)$ as a function of $\sigma$, is not continuous at the ”crossover point” $\sigma = 2$, since the SR exponent $\eta(2) \neq 0$.

Later it was pointed out, in the cases of $(4 - \epsilon)$ dimensions [12] and in $(2 + \epsilon)$ dimensions [25] that the crossover from LR to SR critical behavior is shifted and occurs at a ”critical” value of $\sigma$ given by $\sigma_c = 2 - \eta(2)$. As a result $\eta = \eta(\sigma)$ is a continuous function in $\sigma$ at $\sigma = 2$, since $\eta(\sigma_c) = \eta(2)$. Further support for this assertion was obtained in the framework of different perturbation schemes in [20, 21] and [23]. In terms of the RG language one can say that, by crossing the border point $\sigma_c$, the corresponding LR and SR fixed points exchange stabilities and in a non vanishing range $2 - \eta(2) < \sigma < 2$ LR perturbations are irrelevant (see also [20] where the problem in the more general case of $(n, d, \sigma)$ space was considered). This
statement, which can be determined also from more general considerations (see for example p.71 in [10]) seems to be accepted in the literature. However, some doubts arise since it conflicts with the opposite statement of [36] at least for the non Ising case \( n \geq 2 \) obtained on a most rigorous level. Furthermore there is the criticism in [22] against the use of \( (2-\sigma) \) as small perturbation parameter. If one reconsiders the problem with \( (2\sigma-d) \) as a small parameter in conjunction with the use of the perturbation scheme of [22], previous results that LR fixed point is stable up to \( \sigma = 2 \), are restored. A recent attempt to reconsider this issue using Monte Carlo simulations [35] shows unambiguously, a crossover at \( \sigma_c \), in agreement with [13,23], but there only the Ising case (\( n = 1 \)) is considered. Consequently, the situation seems to be settled only for the case \( n = 1 \).

When \( \sigma > 2 \) one usually considers the model as equivalent to a model with SR interaction, since it is widely accepted [18, 22, 23, 24] that, the LR terms (i.e. the third and etc. in Eq.(1)), do not contribute to the critical behavior of the bulk system, consequently this term has been always omitted in the computational analysis. Indeed, in this case, the critical exponents does not depend on the parameter \( \sigma \).

III. SOME GENERAL REMARKS ON THE FINITE-SIZE SCALING

Scaling is a central idea in critical phenomena near a continuous phase transition and in the field theory. In both cases the singular behavior emerges from the overwhelming large number of degrees of freedom, corresponding to the original cutoff scale, which need to be integrated out leaving behind long wavelength with smoothly varying momenta. This behavior is controlled by a dynamically generated length scale: the bulk correlation length \( \xi \). Such a fundamental idea is difficult to test theoretically because it requires the study of a huge number of interacting degrees of freedom. Experimentally, however, one hopes to be able to study scaling in finite systems near a second order phase transition. Namely the system is confined to a finite geometry and the FSS theory is expected to describe the behavior of the system near the bulk critical temperature (for a review on the FSS theory see [2, 3, 5]). In a few words the standard FSS is usually formulated in terms of only one reference length - the bulk correlation length \( \xi \). For a system with finite linear size \( L \), the main statements of the theory are:

i) The only relevant variable in terms of which the properties of the finite system depend in the neighborhood of the bulk critical temperature \( T_c \) is \( L/\xi \).

ii) The rounding of the thermodynamic function exhibiting singularities at the bulk phase transition in a given finite system sets in when \( L/\xi = O(1) \).

The tacit assumption is that all other reference lengths are irrelevant and will lead only to corrections towards the above picture. Moreover, the crucial point in the FSS theory is that we always assume: the finite size \( L \) of systems under consideration and the correlation length \( \xi \) are large in the microscopic scale. This means \( L \gg a \) and \( \xi \gg a \), where \( a \) is the lattice spacing. The scaling limit of a quantity is called its value when all corrections involving the ratios \( a/L \) and \( a/\xi \) are neglected. One postulates FSS limit as:

\[
\frac{\xi}{a} \rightarrow \infty, \quad \frac{L}{a} \rightarrow \infty, \quad \frac{L}{\xi} = \text{const.} \tag{2}
\]

In the literature there exist two conceptually different approaches concerning the scaling regime. One inspired from the condensed matter theory uses lattice language and models, in which \( a \) is fixed (while \( L \) and \( \xi \) are large, going to \( \infty \)). The other one is inspired from particle
physics. It uses continuum field theory models, when \( L \) and \( \xi \) are fixed (while \( a \) is infinitely small, going to zero). For studying a dimensionless quantity when all the dependence on \( a, L \) and \( \xi \) is through the ratios \( a/L \) and \( a/\xi \) the two approaches seem to be equivalent.

We shall be interested in the case where only the bulk system displays a phase transition. In such system there is no fixed point for finite \( L \) and so no crossover to any other fixed point as a function of \( L/\xi \) can takes place. This means that an \( L \)-independent RG procedure can be used. In the case of LR interaction there are additional reasons (related with possibilities of phase transitions in lowered dimensions) to study the case where finite system as well the bulk displays a phase transition. However this more sophisticated case has been studied only in the SR case [37].

So, we review the FSS properties of fully finite systems in the framework of the continuous field theory. The effects of the cutoff \( \Lambda = \pi/a \) related with a fixed will be discussed in some details below.

In case of LR interaction there are two aspects of the theory which should be mentioned here. First, in addition to the SR term \( |q|^2 \) in the propagators of the Feynman diagrams we have also the LR term \( |q|^{\sigma} \). The last term causes peculiar mathematical problems [38, 39, 40] concerning the evaluation of the lattice sums over \( q \). Among the various methods of accomplishing these summation some details of the method we prefer is presented in [A] where its advantages are also discussed. The second aspect arises specifically because the basic quantity of the theory \( \xi \) is unambiguously defined for SR interactions. For the LR case it has to be defined in a more appropriate way [41]. The standard definitions of the bulk correlation length are based on the asymptotic behavior of the pair correlation function at large distance. The two most commonly used definitions of the bulk correlation length are unambiguous in the case of exponential decay of the bulk pair correlation function \( G_\infty(R; T) \) with the distance \( R = |R| \). One defines

\[
\xi_1(T) = -\lim_{R\to\infty} \left[ R/\ln G_\infty(R; T) \right].
\]

Alternatively, one may consider the second moment of the bulk pair correlation function and define the effective correlation radius,

\[
\xi_2(T) = \left[ \sum_R R^2 G_\infty(R; T)/G_\infty(R; T) \right]^{1/2}.
\]

If the pair correlation function decays exponentially with distance, the two definitions are equivalent. A distinctive feature of the LR interaction is that the function \( G_\infty(R; T) \) decays as \( R^{-d+\sigma} \) when \( R \to \infty \), for \( T > T_c \) (see e.g. [17]) and both definitions yield \( \xi_1(T) = \xi_2(T) = \infty \). One can overcome this difficulty by following the approach proposed in [1] (see also p.138 of [2]) where instead of \( \xi \) some bulk characteristic length \( \lambda(T) \) is used. This characteristic length determines the length-scale of variation of the correlation function and diverges at the critical point.

As in the bulk limit \( \mathcal{O}(n) \) models are the most often used as laboratory tools on the basis of which one studies the scaling properties of finite-size systems. The exactly investigated cases are those corresponding to \( n = 1 \), i.e. for the Ising model (particular attention has been turned to the two dimensional case), and the limit \( n = \infty \), which includes the spherical model (see e.g. Chapter 5 in [2] and p.141 in [2]). It is commonly assumed that the last one is the only model which combines exact solubility even in the presence of a magnetic field and direct relevance to the physical reality. For that reason it is especially suitable
for testing the FSS hypotheses. For \( n \neq 1, \infty \) there are no exact results and the commonly preferable analytical method for the derivation of the properties of the corresponding models (like \( XY \), i.e. \( n = 2 \), and Heisenberg, i.e. \( n = 3 \)) is that of the RG theory. In addition an important amount of information for such systems is derived by numerical simulations, normally via Monte Carlo methods.

A systematic and controlled field-theoretical approach to the quantitative computation of the thermodynamic moments was proposed in the middle of 80’s \([42, 43]\) for studying FSS in the case of SR interaction. It is based on the idea of using a mode expansion, i.e. one treats the zero mode of the order parameter, which is equivalent to the magnetization, separately from the higher modes. The nonzero modes are traced over to yield an effective Hamiltonian for the zero mode. This method is used in combination with the loop expansion in the framework of the minimal subtraction scheme. The method is quite general and was proven to apply to a large extent to the investigation of FSS in systems with LR interaction and with finite number of components \( n \) of the spin vector \([44]\). However such systems have been the object of analytical investigations mainly recently \([32, 45, 46, 47]\). For example, it is possible to perform the quantitative computation of the thermodynamic moments, usually used in numerical analysis. These moments are related to the Binder’s cumulant \( B \) and to various thermodynamic functions like the susceptibility.

The present wisdom is that in the finite size case the results are quite different depending on whether the LR interaction is leading or subleading. So, below we will consider separately the cases of leading and subleading LR interaction.

IV. LEADING LR INTERACTION

Recall that the critical exponents depend upon the parameter \( \sigma < 2 \) controlling the interaction range. The FSS properties (under periodic boundary conditions) in the spherical limit are well established. It has been found that the scaling properties of the system with SR interaction remains valid also in the case of LR interaction (for a review see e.g Chapters 4 and 5 in \([2]\) and references therein). For finite \( n \) a limited number of recent numerical results \([30, 32, 34]\), as well as few analytical works \([32, 34, 44, 45, 46]\) became available.

The nonlocal character of the LR interaction has been an obstacle in investigating the critical behavior by means of numerical methods. For that purpose different algorithms have been developed. Using the Ewald method to evaluate the energy of a given configuration, the critical behavior of such systems for different values of \( d, n \) and \( \sigma \) has been investigated \([30]\). Another approach, analogous to FSS, where the range of the interaction is cutoff at a certain value, whence the name ”finite-range scaling”, is also used \([31]\). This method has been tested essentially on the one dimensional Ising model in the classical (mean-field) as well as in the nonclassical critical regimes. Results of the aforementioned approaches have been judged comparable to the theoretical predictions. These treatments required special efforts that restrict the consideration to very small systems. Recently, the problem was resolved by the use of cumulative probabilities within the Wulff cluster algorithm \([29, 34]\). This method has been applied mainly to Ising systems in the mean-field regime. The FSS hypotheses have been tested in the nonclassical regime \([32, 34]\). The common conclusion of all these approaches is that in the case \( d = \sigma \) such systems exhibit a Kosterlitz-Thouless-like phase transition.

Analytically the nonclassical case has been studied first for \( n = 1 \) \([32]\) and for \( n \geq 1 \) \([45, 46]\). It has been found that, as for the bulk systems, the critical behavior depends on
the small parameter $\epsilon = 2\sigma - d$. The scaling properties of the finite system are not altered by the presence of the LR interaction and can be written as

$$\mathcal{X} = L^{-\gamma_x/\nu} \mathcal{F}(tL^{1/\nu}),$$

where $\gamma_x$ is the critical exponent of the observable $\mathcal{X}$ and $t = (T - T_c)/T_c$. Some thermodynamic quantities, like for instance, the shift of the critical temperature due to the finite size effects, the susceptibility and the Binder’s cumulant at the bulk critical temperature $T_c$ [32, 45, 46] and above it, [45, 46] as a function of $\sqrt{\epsilon}$ have been obtained. The $d$ dependence of the finite size properties has been considered in details in reference [46] by means of a method, which consists of the use of the minimal subtraction scheme applied to a fixed space dimensionality [48].

It has been shown that the critical behavior of the system is dominated by its bulk critical behavior away from the critical domain and that the FSS is relevant in the vicinity of the critical point. A distinctive feature of the LR case is that for $\epsilon \ll 1$ the finite-size corrections are not exponentially small as in the SR case; they vary instead as power-in-L law, in both the spherical $n = \infty$ [40, 41] and $n \neq \infty$ cases [45, 46].

In [44] the behavior of the coupling constants during the crossover from LR to SR interactions i.e. the limit $\sigma \to 2$, was considered. To one-loop order it has been concluded that the renormalized values of the temperature and the coupling constant are continuous functions of $\sigma$.

V. BINDER’S CUMULANT

As it was mentioned above a quantity of central interest in the study of the critical behavior of finite size systems is the Binder’s cumulant ratio $B$. Here we present the analytical result for $B$ ($B = 1 - M_4/3M_2^2$, where $M_{2n}$ is the $n$-th moment; for some details, see [42] and also [32, 45]). Close to the critical point we have

$$B = 1 - \frac{n}{12} \frac{\Gamma^2 \left(\frac{1}{4}n\right)}{\Gamma^2 \left(\frac{1}{4}(n + 2)\right)} \left\{ 1 - z \left[ \frac{\Gamma \left(\frac{1}{4}(n + 6)\right)}{\Gamma \left(\frac{1}{4}(n + 4)\right)} + \frac{\Gamma \left(\frac{1}{4}(n + 2)\right)}{\Gamma \left(\frac{1}{4}n\right)} - \frac{2}{3} \Gamma \left(\frac{1}{4}(n + 4)\right) \right] \right\} + \frac{z^2}{\Gamma^2 \left(\frac{1}{4}(n + 4)\right)} \left[ \frac{\Gamma \left(\frac{1}{4}(n + 6)\right)}{\Gamma \left(\frac{1}{4}(n + 4)\right)} + \frac{3}{2} \frac{\Gamma \left(\frac{1}{4}(n + 4)\right)}{\Gamma \left(\frac{1}{4}(n + 2)\right)} - n - 1 \right] + O \left(z^3\right) \right\} \right\} \right\} . \tag{5}

At the fixed point $z$ is given by [45] ($\epsilon = 2\sigma - d$)

$$z = \sqrt{\frac{n + 8}{\epsilon}} \left[ y - \frac{\epsilon}{2\sigma} y \left( 1 - \frac{n - 4}{n + 8} \ln y \right) + 2^\sigma - 1 - \frac{\epsilon n + 2}{n + 8} \Gamma(\sigma) F_{2\sigma,\sigma}(y) \right] - \frac{\epsilon}{2} \frac{2^\sigma - 1}{y \Gamma(\sigma) F_{2\sigma,\sigma}'(y)} . \tag{6}

Here, we have introduced the scaling variable $y = tL^{1/\nu}$ and the function

$$F_{d,\sigma}(y) = \int_0^\infty dx x^{d-1} E_{\frac{d}{2}} \left( \frac{\theta d}{2 \pi} \right) \left[ \theta^d \left( x/\pi \right) - 1 - \frac{\pi}{x} \right] . \tag{7}

In Eq.(7) $E_{\alpha,\beta}(x)$ and $\theta(x/\pi)$ are the generalized Mittag-Leffler and the reduced Jacobi $\theta_4$ functions, respectively (see also [3] for details). The function $F_{d,\sigma}(y)$ is well known [2] in the theory of FSS.

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FIG. 1: The cumulant ratio $B$ as a function of the interaction range $\sigma$ for $d = 1$. The case $0 < \sigma < d/2$ corresponds to classical (mean-field) regime. The Monte Carlo results follow from [28] for $0 < \sigma \leq 0.5$ and from [32] for $0.5 < \sigma \leq 1$.

Finally, let us notice that one can easily see that the expression (6) for $z$ as a function of $y$ verifies the FSS hypotheses and, consequently, all the thermodynamic functions, which indeed are $z$ dependent, do [45]. At the critical temperature $T_c$ (i.e. $t = 0$, and so $y = 0$), we obtain

$$z_0 = \sqrt{\epsilon} \left[ \frac{n + 2}{\sqrt{n + 8}} \sqrt{\frac{\Gamma(\sigma)}{2\pi^\sigma}} F_{2\sigma,\sigma}(0) + O(\epsilon) \right],$$

(8)

where the coefficient $F_{2\sigma,\sigma}(0)$ appearing in the right hand side of (8) can be evaluated analytically as well as numerically for different values of the interparticle interaction range $\sigma$

$$F_{2\sigma,\sigma}(0) = \begin{cases} 2\zeta(1/2), & \sigma = 1/2, \\ 4\zeta(1/2)\beta(1/2), & \sigma = 1, \\ -4.82271993, & \sigma = 3/2, \\ -8 \ln 2, & \sigma = 2. \end{cases}$$

(9)

Here $\zeta(x)$ is the Riemann zeta function with $\zeta(1/2) = -1.460354508...$ and $\beta(x)$ is the analytic continuation of the Dirichlet series:

$$\beta(x) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell + 1)^x},$$

with $\beta(1/2) = 0.667691457...$. Note that the function $F_{2\sigma,\sigma}(0)$ increases as the parameter $\sigma$ vanishes.

Numerical values for the Binder’s cumulant ratio $z$ can be obtained by replacing the value of $z_0$ form [33] and taking some specific values of the small parameter $\epsilon$. The behavior of the universal constant $B$ for the cases $d = 1, 2$ is presented in Figures 1 and 2 respectively.
Note that the scaling variable $z$ is proportional to $\sqrt{\epsilon}$ as it was found previously (see [42] for example) in the case of SR forces. Furthermore for $n = 1$ it is in full agreement with the result obtained in reference [32] devoted to the exploration of FSS in $O(n)$ systems with LR interaction. Notice that in [32] (see also [34]) the mathematically ensuing pertinent integrals have to be evaluated only numerically, due to the choice of a parametrization, e.g. \( (\Lambda 9) \), that does not reduce the $d$-dimensional problem to the effective one-dimensional one. The approach proposed in [45] based on the parametrization (\(\Lambda 10\)) is more efficient in the sense that the corresponding final expressions can be handled by analytical means [49].

It has been found [32], using the Monte Carlo method, that the amplitude ratio $Q = M_2^2/M_4$ (which is related to $B$) is a linear function of $\epsilon$ (see also Fig.1 and Fig.2), while the analytical result [8] shows an expansion in powers of $\sqrt{\epsilon}$ [32, 45]. It is possible that higher orders in $\epsilon$ could improve the result. Better agreement could also be obtained by using the method developed in [46].

VI. SUBLEADING LR INTERACTION

Let us first note that the most prominent example for such kind of interaction in the case $d = 3$ is the van der Waals interaction $1/r^6$ for which we have $\sigma = 3$. One can easily see that in this case the Fourier transform is indeed of the type [11].

The fact that the critical behavior is not affected by the LR interaction if $\sigma > 2$ is well established in the bulk case. Analytically, in the framework of $n = \infty$ model it has been shown [50] that such a statement is incorrect for finite-size systems because of finite-size contributions due to the subleading, $\sigma > 2$, term in the interaction. It has been shown also that the same remains true for a finite number of component by means of the RG techniques [47]. The authors of [47] have investigated the FSS behavior of a fully finite $O(n)$ system with periodic boundary conditions and in the presence of a LR interaction that does not alter the SR exponents of its critical behavior. The small $|q|$ expansion of the Fourier transform of the interaction $v(q)$ is supposed to be of the form [11] with $2 < \sigma < 4$. For such a system, it
has been demonstrated that all the thermodynamic functions can be expressed in a scaling form as

$$X = L^{-\gamma_x/\nu} F(tL^{1/\nu}, bL^{2-\sigma-n}),$$

where $b$ is a model (nonuniversal) constant. Note that one needs two scaling variables in order to describe in a proper way the finite size behavior of these quantities.

At the critical point the Binder’s cumulant is given \(^{17}\) by \(^{5}\) with

$$z_0 = -\sqrt{\epsilon} \frac{\sqrt{32}}{\pi} \frac{n + 2}{\sqrt{n} + 8} \left[ \ln 2 + b\omega_{\sigma}L^{2-\sigma} + \mathcal{O}(\epsilon) \right], \quad (10)$$

where $\omega_{\sigma} = (2\pi)^{\sigma-2}(1 - 4^{\sigma/2-1})\zeta(1 - \sigma/2)\zeta(2 - \sigma/2)$. So, in this case $B$ is not an universal constant. The behavior of $B$ as a function of system size $L$ is presented in Fig. 3. Equation \(^{10}\) is a generalization for the case under consideration of the result obtained in \(^{42}\) for SR interaction.

When $tL^{1/\nu} \gg 1$, it was found that the susceptibility approaches its bulk value not in an exponential-in-$L$, as it was commonly believed to be the case for systems with short-range critical exponents, but in a power-in-$L$ way of the order of $bL^{-(d+\sigma)}$. The last goes beyond the standard formulation of the finite-size scaling, but is completely consistent with the intrinsic large-distance power-law behavior of the correlations in systems with subleading LR interactions. In the spherical limit such large-distance power-law behavior is shown exactly in \(^{51}\).

For an arbitrary value of $\sigma$, away from the critical point, i.e. in the region $tL^{1/\nu} \gg 1$, the Binder’s cumulant ratio is given by $B = 1 - \frac{1}{3} \left(1 + \frac{2}{n}\right)$, with finite size correction falling off in a power law. This result corresponds to a $n$-dimensional Gaussian distribution for $n$ independent components of the vector variable. Obviously, all the values lie in the interval from $B = 0$ (Ising model, $n = 1$) to $B = 2/3$ (spherical model, $n = \infty$).
VII. LATTICE AND CONTINUUM MODELS AND THE EFFECT OF CUTOFF

In the above, we have concentrated our attention on how to separate the size dependence in the continuum (scaling) limit, where the linear size as well as the correlation length tend to infinity, but keeping their ratio a finite quantity. In this theory the cutoff is sent to infinity and the lattice spacing completely disappears. This is precisely the regime in which one expects FSS to hold \[42, 43\] and this statement, as it was pointed out, does not depend on the range of the interaction.

In order to clarify the effect of a (sharp) cutoff it is sufficient to consider the pure SR case. The finite cutoff effects violate the FSS, see \[52, 53\] and references therein. The result is a system with finite-size behavior with leading nonscaling term going as an inverse power law in \(L\) that depends also on \(L\). In \[52\] this is shown to be tightly related to lattice effects in the system in conjunction with the long wavelength approximation. Indeed this is not in conflict with the arguments proposed in \[42, 43\] where the limit of infinite cutoff is considered. Later, in \[50\] one argues: i) the violation of scaling is related with the second term in (A4) which in the case \(a \neq 0\) can not be omitted in the region \(L/\xi \gg 1\), ii) the effect of the sharp cutoff is similar to that found if one considers the effect of subleading LR interaction. Obviously, this effective LR nature of the finite-size behavior should generate difficulties in the analysis of the Monte Carlo data from simulations not only for SR, but for LR systems as well. The last follows directly from (A13). Recently, the reconsideration of the problem in \[54\] has shown that the source of these obstacles is the artificial cusp-like singularity

\[
\left. \frac{\partial v(q)}{\partial q_i} \right|_{q_i = \Lambda} \neq \left. \frac{\partial v(q)}{\partial q_i} \right|_{q_i = -\Lambda}, \quad i = 1, \ldots, d
\]

at the border of the Brillouin zone (\(\Lambda \sim 1/a\)). As a result, if one treats properly the effects generated by the momentum cutoff in the Fourier transform of the interaction potential both, the lattice and continuum models may produce results in mutual agreement independent of the cutoff scheme \[54\].

VIII. UNSOLVED PROBLEMS

First, an area where the bulk RG predictions \[19, 23\] must be tested by means of numerical methods is the case of \(n > 1\). The strong indication \[36\] that exactly here some problems exist, remains still valid. The numerical results of \[35\] shed some light on the Ising case \(n = 1\) but do not solve the longstanding controversy about the boundary between LR and SR critical behaviors as a function of \(\sigma\).

In this review the FSS properties of systems with no crossover from the bulk fixed point to any other as a function of \(L/\xi\) are considered. Let us note that within the SR models this issue is studied in \[37\]. The consideration of this crossover in the LR is missing. On the other hand, e.g. systems with slab and cylinder geometries and \(n \neq \infty\) have been studied only in the SR case. The first one is related to the statistical-mechanical Casimir effect in fluctuating systems \[55\] and recently there has been an upsurge in theoretical investigations \[2, 6, 53, 54\]. The second one deals with the so called transfer matrix (or Hamiltonian) formulation of the problem \[42\] (see also Chapter 36.4 of \[9\]). A study of both cases in the context of the LR interactions is a provocation for the theory and is of undoubted experimental interest.

Some problems exist with the comparison of the analytical results and the numerical
data. It has been shown\cite{32} (see also \cite{34}) using Monte Carlo simulation (for \( n = 1 \)), that the Binder’s cumulant ratio is linear in \( \epsilon \). The analytical evaluation \cite{5} for the \( \mathcal{O}(n) \) symmetric \( \varphi^4 \) model, however, showed that it is linear in \( \sqrt{\epsilon} \). A possible way to resolve this controversy between the Monte Carlo method and the analytical results is to carry on finite-size calculations to higher loop order\cite{32}. This could ameliorate the analytical results, which would be comparable to those obtained by numerical simulations. However we would like to mention that higher loop corrections that are dealt with through the minimal subtraction scheme and the \( \epsilon \)-expansion are not done, even for the more simple case of SR interaction. We have witnessed that two-loop calculations can be applied in many other investigations like the problems taking into account disorder effects in finite size systems\cite{56}. Here, if \( n = 1 \), the one-loop fixed point is degenerate and the first interesting results can be obtained only if one considers two loops.

As it is discussed in the previous section in the long wavelength approximation\cite{11}, if one goes beyond the continuum field theory, the lattice effects in conjunction with the sharp cutoff generate nonuniversal finite-size terms. The suggestion that such effects are artificial in the case of SR interaction\cite{54} seems to be true also in the LR case. How to avoid the problem in the framework of a concrete cutoff scheme in the case of LR interaction is still an open problem.

In recent years considerable attention has been paid to the critical dynamics of systems with LR interaction\cite{57}. It would be interesting to extend the corresponding results to the finite-size case. Let us note that for the case of the SR interaction the theory of finite size effects in critical dynamics was developed in\cite{58}(see also Chapter 36.6 of\cite{9}).

The present review is devoted to the classical critical phenomena. However, another interesting field is closely related to the extensively investigated field of quantum critical points, i.e. phase transitions occurring at zero-temperature\cite{59}. Let us just recall that in systems showing quantum critical behavior the temperature plays two different roles. For temperatures low enough, quantum effects are essential. In this case the temperature affects the geometry to which the system is confined adding a “new” size to the Euclidean space-time coordinate system. By raising the temperature, the system is driven away from the quantum criticality. At high temperatures, however, the size in the “imaginary-time” direction becomes irrelevant in comparison with all length scales in the system. In this case we have a classical system in \( d \) dimensions and the temperature is just a coupling constant in the classical critical behavior. In this context we find it useful to investigate the quantum models, with LR interaction, e.g. considered in\cite{60, 61} in the large \( n \) limit or in\cite{62} in the mean field and one-loop RG theory.

\section*{APPENDIX A: THE ORIGIN OF THE MATHEMATICAL DIFFICULTIES IN EVALUATING THE LATTICE SUMS}

In order to introduce the reader to the problems let us first consider in some details the SR case on a \( d \)-dimensional hypercubic lattice \( \mathbb{Z}^d \) with \( N = N_0^d \) sites and periodic boundary conditions. The sites are given by

\[ r = a(n_1, \ldots, n_d), \]

where \( n_i \) ranges over all distinct integer values \( \text{mod}N_0 \), for \( i = 1, \ldots, d \).
Let us consider the dimensionless expression, entering most of the mathematical analysis of the scaling properties of finite systems

\[ G(N_0, a|d) = \frac{m^{2-d}}{(aN_0)^d} \sum_{n_1=-N_0/2}^{N_0/2-1} \cdots \sum_{n_d=-N_0/2}^{N_0/2-1} \frac{1}{m^2 + |q|^2} \]

\[ = \frac{1}{(amN_0)^d} \int_0^\infty \exp(-t) \left\{ \sum_{n=-N_0/2}^{N_0/2-1} \exp \left[ -\left( \frac{2\pi n}{amN_0} \right)^2 t \right] \right\}^d dt, \quad (A2) \]

where \( q_i = \frac{2\pi n}{amN_0} \) and \( N_0 \) is even. For the sum

\[ Q_{N_0}(t) = \left( \frac{1}{amN_0} \right) \sum_{n=-N_0/2}^{N_0/2-1} \exp \left[ -\left( \frac{2\pi n}{amN_0} \right)^2 t \right], \quad (A3) \]

using the result of [50], we have

\[ Q_{N_0}(t) \approx \frac{1}{\sqrt{4\pi t}} \left[ \text{erf} \left( \frac{\pi t^{1/2}}{am} \right) - \frac{2\pi^2 t}{3} \frac{1}{am} \exp \left[ -\pi^2 t \left( \frac{1}{am} \right)^2 \right] \right] \]

\[ + \frac{1}{\sqrt{\pi t}} \left\{ \sum_{l=1}^\infty \exp[-l^2(ambN_0)^2/4t] \right\}, \quad (A4) \]

valid in the large \( N_0 \) asymptotic regime. The first and the second terms in the above equation are size independent and are precisely the infinite volume limit of \( Q_{N_0}(t) \). The ultraviolet divergences which may appear in the theory when the lattice spacing vanishes are related with the first term, for which one must perform the required subtractions (see, e.g. p. 177 of [9]). The second term depends on the ratio \( \xi/a \equiv 1/ma \) and so if \( a \to 0 \) or \( \xi \to \infty \) it is of order

\[ O \left( (\xi/a)te^{-\pi^2 t(\xi/a)^2} \right). \quad (A5) \]

In the continuum limit such terms are exponentially small and must be omitted.

The different finite-size regimes are governed by the ratio \( \xi/L \) in the third term. They are independent of the microscopic details, e.g. lattice spacing \( a \). When \( \xi \gg a \), the universal properties can be described by a continuous field theory. Let us recall that the finite linear dimension \( L = N_0 a \), in the case of continuous finite volume means that \( a \to 0 \) and simultaneously \( N_0 \to \infty \).

In the continuum limit, since \( \text{erf}(\pi t^{1/2}/am) = 1 \), we end up with the result

\[ G(N_0, 0|d) = G(\infty, 0|d)|_{L=\infty} + g(L|d), \quad (A6) \]

where the size dependence is contained in the function

\[ g(L|d) = \frac{1}{\pi^{d/2}} \int_0^\infty \frac{\exp(-t)}{t^{d/2}} \sum_{l(d)=1}^\infty \exp \left( -m^2 l(d)^2 L^2/4t \right) dt \]

\[ = \frac{1}{(4\pi)^{d/2}} \int_0^\infty \frac{\exp(-t)}{t^{d/2}} \left\{ \left[ \theta \left( \frac{m^2 L^2}{4\pi t} \right) \right]^d - 1 \right\} dt, \quad (A7) \]
and where
\[ \theta(x) = \sum_{n=-\infty}^{\infty} \exp(-\pi xn^2). \] (A8)

The bulk part \( G(\infty, 0|d)|_{L=\infty} \) contains poles at \( d = 2, 4, \ldots \). The finite size correction \( g(L|d) \) combined with the corresponding contribution from the counterterm yields \([42, 43]\) (to one-loop order) a finite-size shift of the "mass term" \( m^2 \).

In order to investigate the FSS properties of systems with LR interaction, one can use a suitable mathematical method allowing to simplify the analytical calculations. In the above case of SR interaction it has been possible to replace the summand in eq. (A2) by its Laplace transform
\[ \sum_{q} \frac{1}{m^2 + |q|^2} = \int_{0}^{\infty} dt \exp(-m^2 t) \left[ \sum_{q} \exp(-q^2 t) \right]^d, \] (A9)

where \( q \) and \( \bar{q} \) are \( d \)-dimensional and one-dimensional discrete vectors, respectively. This is the so called Schwinger parametric representation. The aim of this replacement is two fold: i) to reduce the \( d \)-dimensional sum to the corresponding effective one-dimensional one, and ii) to give to the dimensionality \( d \) the status of a continuous variable. In the case with the \( \sigma \) term leading in (1), one cannot just use the Schwinger representation in its familiar form. In\([39]\)] the following generalization of (A9) has been suggested
\[ \sum_{q} \frac{1}{m^2 + |q|^\sigma} = m^{\frac{4-2\sigma}{\sigma}} \int_{0}^{\infty} dt Q_\sigma(m^{4/\sigma} t) \left[ \sum_{q} \exp(-q^2 t) \right]^d, \] (A10)

where the function \( Q_\sigma(t) \) for \( 0 < \sigma < 2 \) is given by
\[ Q_\sigma(t) = \int_{0}^{\infty} dy \exp(-ty) \tilde{Q}_\sigma(y), \] (A11)

and
\[ \tilde{Q}_\sigma(y) = \frac{1}{\pi} \frac{\sin(\sigma \pi/2) y^{\sigma/2}}{1 + 2y^{\sigma/2} \cos(\sigma \pi/2) + y^{\sigma}}. \] (A12)

From (A10) and (A11) for the summand in (A10) one obtains
\[ \frac{1}{m^2 + |q|^\sigma} = \int_{0}^{\infty} dt \tilde{Q}_\sigma \left( \frac{t}{m^{4/\sigma}} \right) \frac{1}{tm^2 + |q|^2}. \] (A13)

The integral representation (A13) illustrates the mathematical difficulty appearing in the LR case. It is shown that by an additional integration the problem can be effectively reduced to the SR case \([39]\)]. The behavior of \( \tilde{Q}_\sigma(x) \) as a function of \( \sigma \) (see \( [44] \)) shows another aspect of the nonlocal character of the LR interaction \( 1/r^{d+\sigma} \).

The function \( Q_\sigma(x) \) is related \([63]\) to the entire function of the Mittag-Leffler type defined by the power series
\[ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0. \] (A14)

For a more recent review on these functions see \([64]\)]. It has been demonstrated \([63]\) that
\[ Q_\sigma(x) = x^{\sigma/2 - 1} E_{\sigma/2,\sigma/2}(-x^{\sigma/2}). \] (A15)
FIG. 4: The dependence of kernel (A12) as a function of \( x \) for different interparticle interaction range \( \sigma = 1.50; 1.75; 1.90 \) and 2.00.

and all the specific features of the finite size properties of systems with LR interaction are a result of the analytical properties of the Mittag-Leffler type functions (for some examples, see e.g. Chapters 5 and 6 of [2]).

In the case with both SR and LR terms the corresponding expressions are significantly more complicated. When the LR term is of the peculiar type \( \sigma = 2 - 2\alpha \) (and the parameter \( \alpha \to 0^+ \)) considered in the context of the bulk crossover the following replacement of the SR propagator takes place [23]

\[
\frac{1}{m^2 + |q|^2} \rightarrow \sum_{l=0}^{\infty} \frac{(-v_\sigma)^l |q|^{2(l-1)\alpha}}{(m^2 + |q|^2)^{1+l}}.
\]

(A16)

In order to reduce the problem of evaluating the asymptotic behavior of the sum over \( q \) to the corresponding one-dimensional sum in the right hand side of (A16), the following identity has been used [44]

\[
\frac{|q|^{2(l-1)\alpha}}{(m^2 + |q|^2)^{1+l}} = \frac{1}{\Gamma(1+l\alpha)} \int_0^\infty dx x^{l\alpha} \, {}_1F_1(1+l; 1+l\alpha; -m^2) \exp(-|q|^2 x).
\]

(A17)

Here, \( {}_1F_1 \) is the degenerate hypergeometric function (Kumer’s function). The same identity (A17) has been used in [50] to analyze the finite-size behavior of a propagator with SR and subleading, i.e. \( \sigma > 2 \), LR interaction (treated as a perturbation) where terms similar to the r.h.s. of (A16) also appear. In the last case an expression for the entire propagator (i.e. SR and LR interactions treated on equal ground) can be obtained via contour integration on the complex plane [50].

The identities (A10) and (A17) demonstrate that LR case can be effectively reduced to the SR case with an integration over an additional parameter. So, all the mathematical machinery developed for the SR case may be used without further complications.
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