Affine 7-brane Backgrounds and Five-Dimensional $E_N$ Theories on $S^1$

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Abstract

Elliptic curves for the 7-brane configurations realizing the affine Lie algebras $\widehat{E}_n$ ($1 \leq n \leq 8$) and $\widehat{E}_n$ ($n = 0, 1$) are systematically derived from the cubic equation for a rational elliptic surface. It is then shown that the $\widehat{E}_n$ 7-branes describe the discriminant locus of the elliptic curves for five-dimensional (5D) $\mathcal{N} = 1$ $E_n$ theories compactified on a circle. This is in accordance with a recent construction of 5D $\mathcal{N} = 1$ $E_n$ theories on the IIB 5-brane web with 7-branes, and indicates the validity of the D3 probe picture for 5D $E_n$ theories on $\mathbb{R}^4 \times S^1$. Using the $\widehat{E}_n$ curves we also study the compactification of 5D $E_n$ theories to four dimensions.
1 Introduction

In a series of papers \[1\]-\[10\] the 7-brane technology has been developed systematically in connection with the F-theory compactification on an elliptic $K3$ surface. It is particularly interesting that there exist the 7-brane configurations on which infinite symmetries of the affine Lie algebras $\hat{E}_N$ are realized \[8, 10\].

An interesting application of the 7-brane technology has been found recently by Hanany et al. in constructing five-dimensional (5D) $\mathcal{N} = 1$ theories on the brane web \[11\]. In 5D, Seiberg discovered non-trivial interacting $\mathcal{N} = 1$ superconformal theories with global $E_N$ symmetries \[12\]. It is quite difficult to obtain such $E_N$ theories in a conventional M-theory description of the brane web. In \[11\], however, it is shown that introducing 7-branes in the $(p, q)$ 5-brane web makes it possible to construct 5D $\mathcal{N} = 1$ theories with $E_N$ symmetries on the web. In this approach the affine property of the 7-branes plays a crucial role.

5D $\mathcal{N} = 1$ $E_N$ theories are known to arise upon compactifying M-theory on a Calabi-Yau threefold with a shrinking del Pezzo four-cycle \[13, 14\]. Minahan et al. obtained the elliptic curves for the Coulomb branch of 5D $E_N$ theories compactified on a circle \[15\]. Their construction of the curves is motivated by considering the local model of a singular Calabi-Yau threefold where a del Pezzo surface shrinks to zero size \[16\].

Since the 7-brane configurations are described in terms of elliptic curves \[17\] we expect that the curves for 5D $E_N$ theories may be derived from the 7-branes if their affine property is properly taken into account. We will find that this is indeed the case.

In this paper, after briefly reviewing the 7-branes with exceptional symmetries in section 2, we construct elliptic curves to describe the 7-branes realizing the affine algebras $\hat{E}_N$ ($1 \leq N \leq 8$) and $\tilde{E}_N$ ($N = 0, 1$) in section 3. We then show in section 4 that our curves coincide with those obtained in \[13\] for 5D $E_N$ theories. We also discuss the compactification of 5D $E_N$ theories down to four dimensions by reducing the curves for 5D theories to the known Seiberg-Witten curves for four-dimensional (4D) $\mathcal{N} = 2$ theories.
2 Affine 7-branes

The configurations of IIB 7-branes are described in terms of an elliptic curve

\[ y^2 = x^3 + f(z)x + g(z), \tag{2.1} \]

where \( f \) and \( g \) are certain polynomials in \( z \), and \( z \) is a complex coordinate of \( \mathbb{P}^1 \) which is the space transverse to the 7-branes [17]. The zeroes of the discriminant

\[ \Delta(z) = 4f(z)^3 + 27g(z)^2 \tag{2.2} \]

determine the transverse positions of the 7-branes. The \( SL(2, \mathbb{Z}) \) invariant is defined by

\[ J = \frac{4f(z)^3}{\Delta(z)}. \tag{2.3} \]

When \( f \) is of degree \( \leq 4k \) and \( g \) is of degree \( \leq 6k \), the cubic (2.1) defines a rational elliptic surface for \( k = 1 \) and an elliptic K3 surface for \( k = 2 \). Let \( I_{p,q} \) denote a type I fiber specified by a vanishing cycle \( p\alpha + q\beta \) where \( p, q \) are mutually prime integers and \( \alpha, \beta \) are homology cycles of a fiber torus. A fiber of type I has the \( A_0 \) singularity and its monodromy is given by \( SL(2, \mathbb{Z}) \) conjugate to

\[ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2.4} \]

Thus there exists generically a fiber \( I_{p,q} \) at the transverse position of a 7-brane. Following [9, 10] we will label such a 7-brane as \( X_{[p,q]} \).

Type IIB \((p,q)\) strings can end on a 7-brane \( X_{[p,q]} \). Since \((p,q)\) strings are obtained from the fundamental \((1,0)\) strings by the \( SL(2, \mathbb{Z}) \) transformation the monodromy matrix \( K_{[p,q]} \) associated to \( X_{[p,q]} \) turns out to be \[ K_{[p,q]} = \begin{pmatrix} 1 + pq & -p^2 \\ q^2 & 1 - pq \end{pmatrix}. \tag{2.5} \]

Given a 7-brane configuration \( X_1 \cdots X_{n-1}X_n \) we have the total monodromy matrix \( K = K_nK_{n-1}\cdots K_1 \). The value of \( \text{tr} \, K \) is relevant to the classification of 7-brane configurations since \( \text{tr} \, K \) is an \( SL(2, \mathbb{Z}) \) conjugation invariant [9, 10]. Classification of collapsible brane

\[ ^{\dag} \text{Following the convention of [9, 10] the monodromy matrix } K_{[p,q]} \text{ is the inverse of the usual monodromy matrix.} \]
Table 1: Kodaira classification, ADE singularities and 7-branes. \( n \geq 1 \) for \( I_n \) and \( I_n^* \).

The collapsible 7-branes are of physical importance since they represent the enhanced
gauge symmetries in the bulk. The string/junction configurations giving rise to the mass-
less gauge bosons are found in [5]. This is done by identifying the BPS junctions with
the root vectors of the finite Lie algebra which is determined by the singularity type in
Table 1. It is also important to consider non-collapsible 7-brane configurations beyond
the Kodaira classification. For instance \( D_n = A^n BC \) with \( 0 \leq n \leq 3 \) are relevant when
we describe the Seiberg-Witten solution for four-dimensional \( N = 2 \) \( SU(2) \) QCD with
\( N_f = n \) fundamental flavors in the D3 probe picture [19, 20].

Extending the list in Table 1 to non-collapsible configurations there appear infinite
series of 7-brane configurations [9] among which we are particularly interested in the \( E_N \)
and \( \tilde{E}_N \) series defined by

\[
E_N = A^{N-1} BC^2, \quad \tilde{E}_N = A^N X_{[2,-1]} C. \quad (2.6)
\]

It should be noted, however, that these two series are equivalent for \( N \geq 2 \), while
The finite Lie algebras associated to \( E_N \) are easily identified once we see how \( E_{8,7,6} \) yield the root vectors of \( E_{8,7,6} \). Corresponding to \( E_N \) with \( N \leq 5 \) we have \( E_5 = D_5, E_4 = A_4, E_3 = A_1 \oplus A_2, E_2 = A_1 \oplus u(1) \) and \( E_1 = A_1 \). Likewise one finds \( \tilde{E}_1 = u(1) \) for \( \tilde{E}_1 \) and no symmetry for \( \tilde{E}_0 \). \( E_N \) with \( N \geq 9 \) do not realize the finite Lie algebra.

The series (2.6) are very interesting since they admit the affine extension [8, 10]. The affine configurations are constructed by adding a single 7-brane to (2.6)

\[
\hat{E}_N = A^{N-1}BC^2X_{[3,1]}, \quad \hat{\tilde{E}}_N = A^N X_{[2,-1]}CX_{[4,1]},
\]

(2.7)

where \( \hat{E}_N \) and \( \hat{\tilde{E}}_N \) are equivalent for \( N \geq 2 \) as in the finite case. Note also the equivalence relation

\[
\hat{E}_N = A^{N-1}BC^2X_{[3,1]} = A^N -1BCBC.
\]

(2.8)

There now exists a BPS loop junction which goes around the 7-branes. This junction is identified with the imaginary root so that the root system associated to (2.7) turns out to be the affine root system. Thus the 7-branes (2.7) realize infinite dimensional symmetries of the affine algebras \( \hat{E}_N \) and \( \hat{\tilde{E}}_N \). Note that the monodromy matrices for \( \hat{E}_N \) and \( \hat{\tilde{E}}_N \) are identical

\[
K(\hat{E}_N) = K(\hat{\tilde{E}}_N) = \begin{pmatrix} 1 & 9 - N \\ 0 & 1 \end{pmatrix}.
\]

(2.9)

We refer to [9, 10] for further extensive explanation of 7-brane configurations.

### 3 Elliptic curves for affine 7-branes

Let us concentrate on \( \hat{E}_N \) with \( N \leq 9 \), \( \hat{E}_1 \) and \( \hat{E}_0 \). These exceptional series of 7-branes have an intimate relation with the del Pezzo surface [11]. The correspondence is summarized in Table 2. It is not difficult to figure out how a basis for the two-cycles
in the del Pezzo is translated into a basis of the junction lattice on the corresponding 7-branes. On top of this, the $\hat{E}_9$ configuration, which contains 12 7-branes, corresponds to the ninth del Pezzo surface which is an elliptic manifold over $\mathbb{P}^1$ described by (2.1) with
\begin{align}
 f(z) &= \sum_{i=0}^{4} a_i z^i, \\
g(z) &= \sum_{i=0}^{6} b_i z^i. 
\end{align}

The ninth del Pezzo, dubbed also $\frac{1}{2}K3$, has appeared in the literature on the F-theory description of six-dimensional non-critical strings [21]-[26]. This surface is not the blow-up of $\mathbb{P}^2$ at generic nine points, but the position of the ninth point is fixed by the other eight points.

Starting with the $\hat{E}_9$ curve (2.1) with (3.1) we now wish to construct $\hat{E}_N$ curves for $N \leq 8$ in a systematic way. For generic values of $a_i$, $b_i$ the 7-branes are located in the finite region on the $z$-plane. Our basic idea to obtain $\hat{E}_N$ is to remove $(9-N)$ $A$-branes from $\hat{E}_9$ and send them infinitely far away from the rest. If we place the removed 7-branes $A^{9-N}$ at $z = 0$, they are described as the $A_{8-N}$ singularity at $z = 0$ (see Table I). The appearance of the series of $A$-type singularities makes the derivation of $\hat{E}_N$ curves systematic. In fact, according to the Tate algorithm, the $\hat{E}_9$ curve should obey the condition
\begin{align}
 \text{ord}_z(J) &= -(9-N), \\
 \text{ord}_z(f) &\equiv 0 \pmod{2}, 
\end{align}

where $\text{ord}_z(\bullet)$ means the vanishing order of $\bullet$ in $z$. Writing the discriminant as
\begin{align}
 \Delta &= \sum_{i=0}^{12} \Delta_i(a, b) z^i, 
\end{align}

we see that the condition is satisfied by imposing $\Delta_i = 0$ for $0 \leq i \leq 8 - N$. Then we set $z = 1/u$ and $(x, y) \rightarrow (u^4 x, u^6 y)$ to express the curve in such a way that the $\hat{E}_N$ branes are located around $u = 0$.

First of all, to obtain the $\hat{E}_8$ curve requires
\begin{align}
 \Delta_0 &= 4a_0^3 + 27b_0^2 = 0 
\end{align}

from which we take
\begin{align}
 a_0 &= -3/L^4, \\
b_0 &= 2/L^6. 
\end{align}
Here $L$ is regarded as a scale parameter which will play an important role later. Thus the generic $\hat{E}_8$ curve reads
\[ y^2 = x^3 + \left(-\frac{3u^4}{L^4} + \sum_{i=0}^{3} a_{4-i} u^i\right) x + \left(\frac{2u^6}{L^6} + \sum_{i=0}^{5} b_{6-i} u^i\right). \] (3.6)

Next, the $\hat{E}_7$ curve is obtained from $\Delta_0 = 0$ as well as $\Delta_1 = 0$. The latter yields
\[ b_1 = -\frac{a_1}{L^2}. \] (3.7)

The generic $\hat{E}_7$ curve is then written as
\[ y^2 = x^3 + \left(-\frac{3u^4}{L^4} + \sum_{i=0}^{3} a_{4-i} u^i\right) x + \left(\frac{2u^6}{L^6} - \frac{a_1}{L^2} u^5 + \sum_{i=0}^{4} b_{6-i} u^i\right). \] (3.8)

Repeating this procedure we see that $\Delta_j = 0$ for $2 \leq j \leq 6$ vanish if and only if
\begin{align*}
    b_2 &= \frac{1}{12L^2} \left(a_1^2 L^4 - 12a_2\right), \\
    b_3 &= \frac{1}{216L^2} \left(a_1^3 L^8 + 36a_1 a_2 L^4 - 216a_3\right), \\
    b_4 &= \frac{1}{1728L^2} \left(a_1^4 L^{12} + 24a_1^2 a_2 L^8 + 144a_2^2 L^4 + 288a_1 a_3 L^4 - 1728a_4\right), \\
    b_5 &= \frac{L^2}{10368} \left(a_1^5 L^{12} + 24a_1^3 a_2 L^8 + 144a_1 a_2^2 L^4 + 144a_1^2 a_3 L^4 + 1728a_2 a_3 + 1728a_1 a_4\right), \\
    b_6 &= \frac{L^2}{373248} \left(7a_1^6 L^{16} + 180a_1^4 a_2 L^{12} + 1296a_1^2 a_2^2 L^8 + 864a_1^3 a_3 L^8 + 1728a_2^3 L^4 \\
    &\quad + 10368a_1 a_2 a_3 L^4 + 5184a_2^2 L^4 + 31104a_3^2 + 62208a_2 a_4\right). \quad (3.9)
\end{align*}

Thus the generic curves of type $\hat{E}_N$ for $N = 6, 5, \ldots, 2$ are explicitly written down by substituting (3.7), (3.9) for $b_i$ ($0 \leq i \leq 8 - N$) into (3.6).

An amusing phenomenon occurs when we require $\Delta_7 = 0$. We find $\Delta_7$ in the factorized form, and hence $\Delta_7 = 0$ yields either
\[ a_3 = -\frac{L^4}{12} a_1 \left(a_1^2 L^4 + 12a_2\right) \] (3.10)
or
\[ a_4 = -\frac{L^4}{576} \left(a_1^4 L^8 + 16a_1^2 a_2 L^4 + 48a_2^2 + 48a_1 a_3\right). \] (3.11)

Two curves obtained from (3.10) and (3.11) are identified with the generic $\hat{E}_1$ and $\hat{E}_1$ curves, respectively. Branching of the sequence at rank one nicely corresponds to the properties of $\hat{E}_N$ and $\hat{E}_N$ branes at $N = 1$. 
Inspecting the discriminant, it is observed that the \( \hat{E}_1 \) curve admits the further degeneration, \textit{i.e.} the \( A_8 \) singularity at \( u = \infty \). We find that putting
\[
a_2 = -\frac{L^4}{8}a_1^2 \tag{3.12}
\]
in the \( \hat{E}_1 \) curve gives the \( \hat{E}_0 \) curve.

We note here that the curves obtained so far admit the \( GL(2, \mathbb{C}) \) transformation
\[
u \to c_1 \nu + c_2, \quad x \to \frac{x}{(c_3 \nu + c_4)^2}, \quad y \to \frac{y}{(c_3 \nu + c_4)^3}. \tag{3.13}
\]
By virtue of this, let us see that the expressions of the curves for \( \hat{E}_1, \hat{E}_1, \hat{E}_0 \) are greatly simplified. In the \( \hat{E}_1 \) curve, in view of the transformation (3.13) we put
\[
a_1 = \frac{12}{L^3 p}(16 + p^2), \quad a_2 = -\frac{6}{L^2 p^2}(512 - 160p^2 + 3p^4), \quad a_4 = -\frac{3}{p^2}(-192 + p^2)(1024 - 256p^2 + p^4). \tag{3.14}
\]
We next apply (3.13) with \( c = (c_1, c_2, c_3, c_4) = (\frac{L}{\sqrt{3}}, 0, 0, \frac{1}{\sqrt{3}}) \) and shift \( x \). The \( \hat{E}_1 \) curve is then expressed as
\[
y^2 = x^3 + \left( u^2 - 2u \left( p + \frac{16}{p} \right) + p^2 - 224 \right)x^2 + \frac{65536}{p^2} x. \tag{3.15}
\]
In the \( \hat{E}_1 \) curve, we put
\[
a_1 = \frac{12}{L^3 p}(16 + p^2), \quad a_2 = -\frac{6}{L^2 p^2}(512 + 96p^2 + 3p^4), \quad a_3 = \frac{12}{Lp}(3584 + 48p^2 + p^4). \tag{3.16}
\]
We next apply (3.13) with \( c = (\frac{L}{\sqrt{3}}, 0, 0, \frac{1}{\sqrt{3}}) \) and shift \( x \). The \( \hat{E}_1 \) curve is then expressed as
\[
y^2 = x^3 + \left( u^2 - 2u \left( p + \frac{16}{p} \right) + 32 + p^2 \right)x^2 + 4096 \left( \frac{u}{p} - \frac{16}{p^2} - 1 \right)x + \frac{4194304}{p^2}. \tag{3.17}
\]
In the \( \hat{E}_0 \) curve, we put
\[
a_1 = 0, \quad a_3 = -\frac{8}{3L}. \tag{3.18}
\]
We next apply (3.13) with \( c = (\sqrt{3}L, 0, 0, \sqrt{3}) \) and shift \( x \). The curve now reads

\[
y^2 = x^3 + 9u^2 x^2 - 24ux + 16
\]

with the discriminant \( \Delta = 6912(u^3 + 1) \). Three zeroes of \( \Delta \) at \( u = e^{i\pi(2j+1)/3} \) with \( j = 0, 1, 2 \) are the positions of three 7-branes of \( \widehat{E}_0 \). Evaluating the monodromy \( M_j \) at \( u = e^{i\pi(2j+1)/3} \) we obtain \( M_0 = K_{[1,1]}, M_1 = K_{[-1,2]}, M_2 = K_{[2,-1]} \). This is in agreement with the monodromy of \( \widehat{E}_0 = X_{[2,-1]}CX_{[4,1]} = X_{[2,-1]}X_{[-1,2]}C \). The \( \mathbb{Z}_3 \) symmetry acting on the \( u \)-plane is observed by noting that \( STM_j(ST)^{-1} = M_{j+1(\text{mod} 3)} \) where \( S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) and \( (ST)^3 = 1 \).

To summarize, our derivation of the \( \widehat{E}_N \) as well as \( \widehat{E}_N \) curves is depicted in the diagram

\[
\begin{align*}
A_7 & \to A_8 \\
0 & \to A_0 \to A_1 \to A_2 \to A_3 \to A_4 \to A_5 \to A_6 \to A_7 \\
\widehat{E}_9 & \to \widehat{E}_8 \to \widehat{E}_7 \to \widehat{E}_6 \to \widehat{E}_5 \to \widehat{E}_4 \to \widehat{E}_3 \to \widehat{E}_2 \to \widehat{E}_1
\end{align*}
\]

Here the upper sequence stands for the coalescence flow of the \( A \)-type singularities (i.e. \( A^{8-N} \)-branes) at \( u = \infty \) while the lower sequence stands for the degeneration flow of the \( \widehat{E}_N \) configurations. As we will see in section 4 this diagram corresponds to the renormalization group flows among 5D \( \mathcal{N} = 1 \) supersymmetric \( E_N \) theories.

### 3.1 Massless curves

In order to see more explicitly the correspondence between the \( \widehat{E}_N \) curves and the \( \widehat{E}_N \) branes we now study the degeneration of the curves. For each \( \widehat{E}_N \) there is a maximally collapsible sub-configuration of 7-branes \( [11] \). Correspondingly the \( \widehat{E}_N \) curve has a degeneration which realizes the \( E_N \) singularity. We describe such degeneration by adjusting the parameters \( a_i, b_i \). The curves obtained in this way will be referred to as the massless curves. Our results are then as follows:

- **Massless \( \widehat{E}_8 \):**
In order to have the $E_8$ singularity at $u = 0$, we must set $a_i = 0$ ($i = 4, 3, 2, 1$) and $b_i = 0$ ($i = 6, 5, 4, 3, 2$). Then we obtain the massless $\hat{E}_8$ curve

$$y^2 = x^3 - \frac{3}{L^4} u^4 x + \left( \frac{2}{L^6} u^6 + b_1 u^5 \right)$$  \hspace{1cm} (3.21)

with the discriminant

$$\Delta = 27 b_1 u^{10} \left( 4u + b_1 L^6 \right) / L^6. \hspace{1cm} (3.22)$$

The form of $\Delta$ indicates that 10 zeroes at $u = 0$ represent the coalescing $E_8 = A^7BC^2$ branes and a zero at $u = -b_1 L^6 / 4$ a 7-brane $X_{[3,1]}$. Since $\hat{E}_8$ is not collapsible $X_{[3,1]}$ keeps a finite distance from the collapsible $E_8$.

• Massless $\hat{E}_7$:

Setting $a_i = 0$ ($i = 4, 3, 2$) and $b_i = 0$ ($i = 6, 5, 4, 3, 2$) we obtain

$$y^2 = x^3 + \left( -\frac{3}{L^4} u^4 + a_1 u^3 \right) x + \left( \frac{2}{L^6} u^6 - \frac{a_1}{L^2} u^5 \right)$$  \hspace{1cm} (3.23)

with the discriminant

$$\Delta = -a_1^2 u^9 \left( 9u - 4a_1 L^4 \right) / L^4. \hspace{1cm} (3.24)$$

Thus there exist coalescing $E_7 = A^6BC^2$ branes at $u = 0$ and $X_{[3,1]}$ at $u = 4a_1 L^4 / 9$.

• Massless $\hat{E}_6$:

Setting $a_i = 0$ ($i = 4, 3, 2$) and $b_i = 0$ ($i = 6, 5, 4, 3$) we obtain

$$y^2 = x^3 + \left( -\frac{3}{L^4} u^4 + a_1 u^3 \right) x + \left( \frac{2}{L^6} u^6 - \frac{a_1}{L^2} u^5 + \frac{a_1^2 L^2}{12} u^4 \right)$$  \hspace{1cm} (3.25)

with the discriminant

$$\Delta = -a_1^3 u^8 \left( 8u - 3a_1 L^4 \right) / 16. \hspace{1cm} (3.26)$$

Thus there exist coalescing $E_6 = A^5BC^2$ branes at $u = 0$ and $X_{[3,1]}$ at $u = 3a_1 L^4 / 8$.

• Massless $\hat{E}_5$:

Setting $a_i = 0$ ($i = 4, 3$), $a_2 = -\frac{L^4}{48} a_1^2$ and $b_i = 0$ ($i = 6, 5, 4$) we obtain

$$y^2 = x^3 + \left( -\frac{3}{L^4} u^4 + a_1 u^3 - \frac{L^4 a_1^2}{48} u^2 \right) x + \left( \frac{2}{L^6} u^6 - \frac{a_1}{L^2} u^5 + \frac{5}{48} a_1^2 L^2 u^4 + \frac{a_1^3 L^6}{864} u^3 \right)$$  \hspace{1cm} (3.27)
with the discriminant
\[ \Delta = -3a_4^4 L^4 u^7 \left(3u - a_1 L^4\right)/256. \] (3.28)

Thus there exist 7 coalescing 7-branes at \( u = 0 \) which realize \( E_5 = D_5 \). This is seen from the equivalence
\[ \hat{E}_5 = A^4 B C^2 X_{[3,1]} = A^5 X_{[2,-1]} C X_{[4,1]} = D_5 X_{[4,1]}, \] (3.29)
where \( X_{[4,1]} \) is located at \( u = a_1 L^4/3 \).

- Massless \( \hat{E}_4 \):

Here we encounter a somewhat different situation from the above. Writing the discriminant of the \( \hat{E}_4 \) curve as
\[ \Delta = \sum_{i=0}^{7} \Delta_i(a, b) u^i \] (3.30)
we first require \( \Delta_0 = 4a_3^4 + 27b_6^2 = 0 \). This is obeyed by taking
\[ a_4 = -3/T^4, \quad b_6 = 2/T^6, \] (3.31)
where, in addition to \( L \), another scale parameter \( T \) has appeared. We can then set at most \( \Delta_j = 0 \) for \( 0 \leq j \leq 4 \) by tuning
\[
\begin{align*}
b_5 &= \frac{a_3}{T^2}, \\
a_1 &= \frac{36}{L^3 T}, \\
a_3 &= -\frac{2}{LT^3} \left(a_1 L^3 T - 18\right), \\
a_2 &= -\frac{1}{12L^2 T^2} \left(a_1^2 L^6 T^2 + 12a_3 LT^3 - 12a_1 L^3 T + 72\right),
\end{align*}
\] (3.32)
from which the massless curve is obtained as
\[
y^2 = x^3 + \left( -\frac{3}{L^4} u^4 + \frac{36}{L^3 T} u^3 - \frac{42}{L^2 T^2} u^2 - \frac{36}{LT^3} u - \frac{3}{T^4} \right) x \\
+ \left( \frac{2}{L^6} u^6 - \frac{36}{L^5 T} u^5 + \frac{150}{L^4 T^2} u^4 + \frac{150}{L^3 T^4} u^2 + \frac{36}{LT^5} u + \frac{2}{T^6} \right) \] (3.33)
with the discriminant
\[ \Delta = -186624 u^5 \left(T^2 u^2 - 11LT u - L^2\right)/(LT)^7. \] (3.34)

The existence of 5 coalescing 7-branes at \( u = 0 \) is understood by noting that \( \hat{E}_4 = A^3 B C^2 X_{[3,1]} \) is equivalent to \( B^5 X_{[-2,3]} C \). Hence we have the \( A_4 \) singularity at the origin.
in agreement with the relation $E_4 = A_4$. Two 7-branes $X_{[-2,3]}$ and $C$ stay at a distance from the collapsed $B^5$ at $u = 0$, reflecting the fact that both $\hat{E}_4$ and $E_4$ are non-collapsible configurations.

• Massless $\hat{E}_3$:

We proceed in parallel with the $\hat{E}_4$ case. The maximal degeneracy of the curve is achieved by taking

$$a_4 = -\frac{3}{T^4}, \quad b_6 = \frac{2}{T^6}, \quad a_3 = -\frac{6}{LT^3}, \quad a_2 = 0, \quad a_1 = \frac{12}{L^3T}.$$  (3.35)

The massless curve turns out to be

$$y^2 = x^3 + \left(-\frac{3}{L^4}u^4 + \frac{12}{L^3T}u^3 - \frac{6}{LT^3}u - \frac{3}{T^4}\right) x$$

$$+ \left(\frac{2}{L^6}u^6 - \frac{12}{L^5T}u^5 + \frac{12}{L^4T^2}u^4 + \frac{14}{L^3T^3}u^3 + \frac{3}{L^2T^4}u^2 + \frac{6}{LT^5}u + \frac{2}{T^6}\right)$$  (3.36)

with the discriminant

$$\Delta = -729u^3 (uT - 4L) (L + 2uT)^2 / (L^6T^9).$$  (3.37)

As in the previous case, the structure of the discriminant is again understood by showing that $\hat{E}_3 = A^2BC^2X_{[3,1]}$ is equivalent to $X^2_{[2,-1]}X^{3}_{[-1,1]}C$ in accordance with the fact that $E_3 = A_1 \oplus A_2$.

• Massless $\hat{E}_2$:

We need to have the $A_1$ singularity at $u = 0$. For this the coefficients of $u$ and $u^0$ in the discriminant should vanish. At first sight it looks quite difficult to find such a solution. If we eliminate $a_4$ from these two equations, however, the resulting expression takes a remarkable factorized form. As a result we see that $a_3$ and $a_4$ should be

$$a_3 = \frac{L^4}{576}a_1 \left(a_1^2L^4 - 24a_2\right),$$

$$a_4 = -\frac{L^4}{27648} \left(25a_1^4L^8 + 240a_1^2a_2L^4 + 576a_2^2\right).$$  (3.38)

Rescaling the variables as $a_2 = \frac{k}{24}a_1^2L^4$, $u \rightarrow \frac{1}{24}a_1L^4$ and $x \rightarrow \frac{1}{576}a_1^2L^6$, we find the massless curve in the form

$$y^2 = x^3 + (-12(k + 5)^2 - 24(k - 1)u + 24ku^2 + 24u^3 - 3u^4)x$$
\[+16(k + 5)^3 + 48(k - 1)(k + 5)u + 12(49 + 18k + 5k^2)u^2
\]
\[+40(3k + 1)u^3 - 24(k - 2)u^4 - 24u^5 + 2u^6 \quad (3.39)\]

with the discriminant

\[\Delta = 11664(k + 3)^3u^2(200 + 80k + 8k^2 + (72 + 40k)u + (21 - k)u^2 - 4u^3). \quad (3.40)\]

The symmetry on the brane is expected to be \(E_2 = A_1 \oplus u(1)\). In fact, two zeroes at \(u = 0\) represent two collapsed \(C\)-branes in \(\hat{E}_2 = ABC^2X_{[3,1]}\), being described as the \(A_1\) singularity. One can find a junction with zero asymptotic charges and self-intersection \(-4\) which thus generates the \(u(1)\) factor although this junction is not associated with coalescing branes.

The \(\hat{E}_2\) curve has the singular fibers \(A_1 + 3A_0 + A_6\) in total, and is not completely massless in the sense that the \(u(1)\) factor is carried by the non-collapsible junction. We note that there are two possible choices of \(k\) to make \(\Delta\) factorized more:

\[k = -5, \quad (H_1 + A_0 + A_0 + A_6)\]
\[k = 51, \quad (A_1 + H_0 + A_0 + A_6). \quad (3.41)\]

In each case, however, the singularity type remains unchanged.

- Massless \(\hat{E}_1\):

  Setting \(a_1 = 0\) and \(a_4 = -L^4a_2^2/48\) we obtain

  \[y^2 = x^3 + \left( -\frac{3}{L^4} u^4 + a_2u^2 - \frac{L^4}{48}a_2^2 \right)x + \left( \frac{2}{L^6} u^6 - \frac{a_2}{L^2} u^4 + \frac{5}{48} L^2 a_2^2 u^2 + \frac{1}{864} L^6 a_2^3 \right) \quad (3.42)\]

  with the discriminant

  \[\Delta = -3L^4a_2^4u^2 \left( 3u^2 - a_2L^4 \right)/256. \quad (3.43)\]

  Since \(\hat{E}_1 = BC^2X_{[3,1]}\) we have two coalescing \(C\)-branes at \(u = 0\), realizing the \(A_1\) singularity, and hence \(E_1 = A_1\).

- \(\hat{E}_1\) and \(\hat{E}_0\):

  These cases do not admit further degeneration.
4 Five-dimensional $E_N$ theories on $S^1$

In this section we consider $5D \mathcal{N} = 1$ supersymmetric $SU(2)$ gauge theory with $N_f$ quark hypermultiplets. The vector multiplet consists of a vector field, a real scalar and a spinor, and the hypermultiplet contains four real scalars and a spinor. $N_f$ quark hypermultiplets are in the doublet of $SU(2)$. Note that in 5D the bare gauge coupling $1/g_0^2$ has mass dimension one. When the bare quark masses vanish the global flavor symmetry is $SO(2N_f)$. There also exists a global $U(1)_I$ symmetry generated by the current $j = \ast \text{Tr}(F \wedge F)$ which is conserved in 5D. Hence the massless microscopic theory possesses the global symmetry $SO(2N_f) \times U(1)_I$.

Seiberg found that for $N_f \leq 8$ non-trivial interacting superconformal field theories appear in the limit $g_0 \to \infty$ [12]. At these strongly-coupled fixed points, surprisingly, the global symmetry is enhanced to $E_{N_f+1}$. Moreover there exist two theories with global $E_1$ and $\tilde{E}_1$ symmetry, respectively, for $N_f = 0$. The $\tilde{E}_1$ theory further flows down to the $\tilde{E}_0$ theory with no global symmetry. This class of 5D theories with exceptional global symmetry is shown to appear when M-theory is compactified on a Calabi-Yau threefold with a vanishing four-cycle realized by del Pezzo surfaces [13, 14]. The correspondence between global symmetries and del Pezzo surfaces is precisely the one presented in Table 2.

The brane web construction of various 5D theories has been considered in the literature [27]-[32] though the $SU(2)$ theories with exceptional symmetries were not obtained in the M-theory 5-brane approach. In a recent interesting paper, however, it is found that introducing 7-branes in the $(p, q)$ 5-brane web makes it possible to realize exceptional global symmetries on the web [11].

As discussed in [33, 34, 25, 28, 15, 29, 30] the Coulomb branch of 5D theories on $S^1$ is described in terms of complex curves where the dependence on the bare quark masses enters through trigonometric functions. In particular, the curves for the $E_N$ theories on $S^1$ have been derived explicitly in [15]. Their derivation of the curves is motivated by the analysis of the local model of a shrinking del Pezzo four-cycle in a Calabi-Yau manifold [16].

It turns out that our $\hat{E}_N$ curves to describe the affine 7-brane backgrounds are equiv-
alent to those obtained in [14] where $u$ is a moduli parameter of the Coulomb branch. It is straightforward to compare the curves in the massless limit. First of all, for massless $\tilde{E}_8$, applying the transformation (3.13) with $c = (\frac{L}{\sqrt{3}}, 0, 0, -\frac{18\sqrt{3}}{b_1L^5})$ one obtains

$$y^2 = x^3 - \frac{u^4}{3}x - 2u^5 + \frac{2u^6}{27}$$

(4.1)

which is rewritten as

$$\tilde{E}_8 : y^2 = x^3 + u^2x^2 - 2u^5$$

(4.2)

by a shift in $x$.

Similarly, the massless curves for $\tilde{E}_N$ ($7 \geq N \geq 2$) are transformed into

$$\tilde{E}_7 : y^2 = x^3 + u^2x^2 + 2u^3x,$$

$$\tilde{E}_6 : y^2 = x^3 + u^2x^2 - 2iu^3x - u^4,$$

$$\tilde{E}_5 : y^2 = x^3 + (u^2 - 4u)x^2 + 4u^2x,$$

$$\tilde{E}_4 : y^2 = x^3 + (u^2 - 6iu + 11)x^2 + (40 - 40iu)x + (48 - 64iu),$$

$$\tilde{E}_3 : y^2 = x^3 + (u^2 + 10u - 23)x^2 + 128(1 - u)x,$$

$$\tilde{E}_2 : y^2 = x^3 + (u^2 + 18iu + 47)x^2 + 512(iu - 1)x - 65536.$$

(4.3)

Here the transformation (3.13) has been utilized with $c = (\frac{L}{\sqrt{3}}, 0, 0, c_4)$ where

$$c_4 = \frac{6\sqrt{3}}{L^3a_1}, \ -\frac{6i\sqrt{3}}{L^3a_1}, \ \frac{8\sqrt{3}}{L^3a_1}, \ \frac{iT}{\sqrt{3}}$$

(4.4)

for $N = 7, 6, 5, 4$, respectively, and $c = (\frac{L}{\sqrt{3}}, \frac{L}{\sqrt{3}}, 0, -\frac{4T}{\sqrt{3}})$ for $N = 3$. For $N = 2$ the curve has been obtained from (3.39) by a specialization $k = 5$, a transformation (3.13) with $c = (\frac{1}{\sqrt{3}}, \frac{5i}{\sqrt{3}}, 0, -\frac{2i}{\sqrt{3}})$ and a shift in $x$.

The curves (1.2) and (4.3) are in precise agreement with the massless limit of the curves in [15]. For $\tilde{E}_1$ and $\tilde{E}_1$, we take the massive curves in the form of (3.15) and (3.17), respectively. Setting $p = e^{i\lambda}$ we see that they agree with the corresponding curves in [15] with $\lambda$ being the $U(1)$ mass parameter. For massive $\tilde{E}_{N \geq 2}$ we have not yet make a detailed comparison, though we are sure that they will certainly agree according to the general singularity theory.

In [13] the $\tilde{E}_N$ curves for $N \leq 7$ are derived from the massive $\tilde{E}_8$ curve by decoupling a large mass along with the scaling. The renormalization group flows among the 5D $E_N$
degree \( \hat{E}_8 \) \( \hat{E}_7 \) \( \hat{E}_6 \) \( \hat{E}_{N \leq 5}, \hat{E}_1, \hat{E}_0 \)

| \( q_y \) | 15 | 9 | 6 | 3 |
| \( q_x \) | 10 | 6 | 4 | 2 |
| \( q_u \) | 6 | 4 | 3 | 2 |

Table 3: Degree of variables

theories follow from this analysis. Our derivation of the generic \( \hat{E}_N \) curves is simpler since what one had to do is to generate the successive coalescence flows of the \( A \)-type singularities at \( u = \infty \). In order to discuss the compactification of non-critical \( E \)-strings, however, expressing deformation parameters \( a_i, b_i \) in terms of the Wilson lines is useful. This is carried out in [15] by investigating the instanton expansion of the prepotential. We suspect that this can also be performed in a representation theoretic way as was done for the case of finite \( ADE \) symmetries on 7-branes [35].

4.1 Compactification to four dimensions

We now wish to discuss the compactification of 5D \( E_N \) theories to four dimensions. For this purpose the degree \( q_i \) (mass dimension) is assigned to \( y, x, u \) in the curves, see Table 3. Then the scale parameters \( L \) and \( T^{-1} \) have mass dimension one. We are thus led to identify \( L = 1/R_5 \) where \( R_5 \) is the radius of \( S^1 \) along which the fifth dimension is curled up. This \( R_5 \)-dependence of the \( u^4 \) and \( u^6 \) terms in the curve (3.6) agrees with the one fixed in [25]. The compactification limit \( R_5 \to 0 \) can be taken directly in the massless \( \hat{E}_{8,7,6} \) curves. We obtain

\[
\begin{align*}
\hat{E}_8 : & \quad y^2 = x^3 + b_1 u^5, \\
\hat{E}_7 : & \quad y^2 = x^3 + a_1 u^3 x, \\
\hat{E}_6 : & \quad y^2 = x^3 + \frac{\alpha^2}{12} u^4, 
\end{align*}
\]

(4.5)

where we have set \( a_1 = \alpha/L \) for \( \hat{E}_6 \) with \( \alpha \) being kept fixed when \( L \to \infty \). These are in the form of the \( E_6 \) singularities which describe the \( \mathcal{N} = 2 \ E_6 \) fixed points in four dimensions [36, 35]. This indicates that 5D \( E_{8,7,6} \) theories on critical compactify to 4D
critical $E_{8,7,6}$ theories without adjusting any relevant parameters. In view of the 7-brane configurations the compactification limit is taken by decoupling the brane $X_{[3,1]}$ from the $\mathbf{E}_N$ branes, leaving the $\mathbf{E}_N$ branes as the background to describe 4D theory.

The situation changes when we consider 5D $E_{N\leq 5}$ theories. Taking the limit $R_5 \to 0$ of the massless curves does not work in finding 4D theories. Recall that $E_N$ theories have the $N_f = N - 1$ flavors. Thus, upon compactification, 5D $E_{N\leq 5}$ theories are expected to reduce to 4D $\mathcal{N} = 2$ $SU(2)$ QCD with $N_f$ flavors whose Coulomb branch is described in terms of the Seiberg-Witten (SW) curve [37]. To see this explicitly, we decouple $C$ and $X_{[3,1]}$-branes simultaneously from the branes $\mathbf{E}_{N_f+1} = A^{N_f}BC^2X_{[3,1]}$ as $R_5 \to 0$, leaving the branes $D_{N_f} = A^{N_f}BC$ for 4D theory. This implies that one has to take the scaling limit by turning on suitable mass parameters in the theory. In the following we show how to obtain the massless SW curves for 4D theories. For this, we recall that, in the massless case, the relevant singularity structures on the $u$-plane are given by $D_4$ for $N_f = 4$, $A_3 + A_0$ for $N_f = 3$, $A_1 + A_1$ for $N_f = 2$, $A_0 + A_0 + A_0$ for $N_f = 1$ and $A_0 + A_0$ for $N_f = 0$. These are realized by taking the scaling limit in the massive curves for 5D theories as follows:

- $\mathbf{E}_5$ curve:
  This case is rather simple. Setting $a_i = 0 \,(i = 1, 3, 4)$, $b_i = 0 \,(i = 4, 5, 6)$ and letting $L \to \infty$ we obtain
  \[ y^2 = x^3 + a_2u^2x. \]  \hspace{1cm} (4.6)

This is the $D_4$ singularity for the massless $N_f = 4$ theory in 4D. The original $N_f = 4$ SW curve is recovered via a change of variables as shown in [33].

- $\mathbf{E}_4$ curve:
  In the massive curve, let us first take
  \[ a_4 = -\frac{3}{T^4}, \quad a_3 = -\frac{2}{LT^3} \left(L^3T a_1 - 18 \right), \quad b_6 = \frac{2}{T^6}, \quad b_5 = -\frac{a_3}{T^2}, \]  \hspace{1cm} (4.7)
  then we are left with $a_1, a_2$. Note that $a_2$ is a dimensionless parameter. Inspecting the curve it is observed that the desired limit is obtained if we set
  \[ a_1 = \frac{\alpha}{L^2}, \quad a_2 = -\frac{\alpha^2}{12} + \frac{\beta}{TL}. \]  \hspace{1cm} (4.8)
Now, letting $L \to \infty$ yields the curve with the $A_3 + A_0$ singularity. Here the scale parameter $T$ survives the compactification limit, and hence we identify it with the QCD dynamical scale, $T^{-1} \propto \Lambda_3$. If we now put $\alpha = 2, \beta = 6, T = -i\sqrt{3}\Lambda_3$ and make a shift in $x$, the massless $N_f = 3$ SW curve follows

$$y^2 = x^2(x - u) - \frac{\Lambda_3^2}{64}(x - u)^2. \quad (4.9)$$

• $\hat{E}_3$ curve:

We take $a_4 = -3/T^4, b_6 = 2/T^6, a_3 = -1/T^2$ and set

$$a_1 = \frac{\alpha}{L^2}, \quad a_2 = -\frac{\alpha^2}{12} + \frac{\beta}{(TL)^2}. \quad (4.10)$$

Letting $L \to \infty$ yields the curve with the $A_1 + A_1$ singularity. The scale parameter $T$ is again converted into the QCD scale $\Lambda_2$. If we now put $\alpha = 2, \beta = -6, T = 2\sqrt{3}/\Lambda_2$ and make a shift in $x$ as well as in $u$, the massless $N_f = 2$ SW curve follows

$$y^2 = x^2(x - u) - \frac{\Lambda_2^4}{64}(x - u). \quad (4.11)$$

• $\hat{E}_2$ curve:

Setting $a_3 = a_4 = 0$ and

$$a_1 = \frac{\alpha}{L^2}, \quad a_2 = -\frac{\alpha^2}{12} + \frac{\beta}{(TL)^3}, \quad (4.12)$$

we let $L \to \infty$. The resulting curve exhibits the singularity $A_0 + A_0 + A_0$. Putting $\alpha = 2, \beta = i3\sqrt{3}/4$ and $T = 1/\Lambda_1$ we obtain the massless $N_f = 1$ SW curve

$$y^2 = x^2(x - u) - \frac{\Lambda_1^4}{64}. \quad (4.13)$$

• $\hat{E}_1$ curve:

Setting $a_4 = 1/(4T^4)$ and

$$a_1 = \frac{\alpha}{L^2}, \quad a_2 = -\frac{\alpha^2}{12} + \frac{\beta}{(TL)^4}, \quad (4.14)$$

we let $L \to \infty$. The resulting curve exhibits the singularity $A_0 + A_0$. Putting now $\alpha = 2, T = 1/\Lambda_0$ ($\beta$ remains arbitrary) yields

$$y^2 = x^2(x - u) + \frac{\Lambda_0^4}{4}x. \quad (4.15)$$
This is the SW curve for $\mathcal{N} = 2$ $SU(2)$ pure Yang-Mills theory.

**$\hat{\mathcal{E}}_1$ curve:**

We set $a_3 = 0$ and (4.14), and then let $L \to \infty$, yielding the curve with the $A_0 + A_0$ singularity. Putting now $\alpha = 2, \beta = -9/2, T = 1/A_0$ we get the SW curve (4.15) as in the previous case.

Thus we have seen that there exists the compactification limit of 5D $\mathcal{N} = 1$ $E_{N \leq 5}$ theories on $S^1$ down to 4D $\mathcal{N} = 2$ $SU(2)$ QCD with $N_f = N - 1$ flavors. For $N \leq 4$ the compactification limit is taken as the scaling limit as prescribed in (4.8). The appearance of the QCD scale $\Lambda_{N-1}$ reflects the fact that the $E_N$ brane configurations for $N \leq 4$ are not collapsible. It will also be possible to derive the massive $SU(2)$ SW curves from the massive $\hat{E}_N$ curves, see ref. [23] for earlier related computations in view of 6D non-critical strings.

5 Discussion

Starting with a rational elliptic surface for $\hat{E}_9$ we have constructed the the elliptic curves corresponding to the affine 7-branes $\hat{E}_N$ ($1 \leq N \leq 8$) and $\hat{E}_N$ ($N = 0, 1$). The brane picture is quite efficient in working out these curves explicitly. We have then shown that the curves for the affine 7-brane backgrounds describe the Coulomb branch of the 5D $E_N$ theories compactified on a circle. The result indicates that the idea of the D3-brane probe is valid for the description of 5D $E_N$ theories on $R^4 \times S^1$; both a probe D3-brane and background 7-branes extend over the bulk $R^4$, and the dependence on $S^1$ is encoded in the “affinizing” 7-brane $X_{[3,1]}$. The BPS states arise from $(p, q)$ strings/junctions stretching between the D3-brane and 7-branes. Thus there will exist BPS states with non-zero magnetic gauge charge $q \neq 0$ in 5D theory on $R^4 \times S^1$. On the other hand, the BPS states of the 5D $E_N$ theory in the bulk $R^5$ are represented as the 5-7 strings with charges $(p, q) = (2n_e, 0)$ where $n_e$ is the Cartan charge of 5D $SU(2)$ gauge group [11]. The level $k$ of the Kac-Moody algebra realized on the affine exceptional 7-branes (2.8) is related to $q$ through $k = -q$ [8] [10]. Therefore only the finite part of $\hat{E}_N$ is relevant to the enumeration of BPS states in the bulk 5D theory. As just mentioned above, however, the 5D theory on $S^1$ enables to carry BPS states with $q \neq 0$, and hence the affine property may play a
manifest role. In fact, interesting affine $E_N$ structures have already been revealed in the context of non-critical $E$-strings [24, 15, 26]. It will be worth pursuing further the idea in the framework of the 7-brane setup.

Mathematically speaking, what we have investigated in this paper is the problem of understanding the moduli of rational elliptic surfaces (2.1), which is a space of polynomials

$$f(z) = \sum_{i=0}^{4} a_i z^i, \quad g(z) = \sum_{i=0}^{6} b_i z^i$$  \hspace{1cm} (4.16)

divided by (3.13). Since the moduli space consists of various branches with different singularity structures, it is important to consider the interrelation (degeneration-coalescence) among the strata [38, 39]. The brane picture provides a natural description of such degeneration-coalescence structures.

The whole construction presented in this paper may be summarized in the degeneration-coalescence diagram. The degeneration diagram turns out to be

$$\hat{E}_1 \rightarrow \hat{E}_0$$

$$\hat{E}_8 \rightarrow \hat{E}_7 \rightarrow \hat{E}_6 \rightarrow \hat{E}_5 \rightarrow \hat{E}_4 \rightarrow \hat{E}_3 \rightarrow \hat{E}_2 \rightarrow \hat{E}_1$$

$$E_8 \rightarrow E_7 \rightarrow E_6 \rightarrow (E_5 \rightarrow E_4 \rightarrow E_3 \rightarrow E_2 \rightarrow E_1)$$

$$D_4 \rightarrow (D_3 \rightarrow D_2 \rightarrow D_1 \rightarrow D_0)$$

$$H_2 \rightarrow H_1 \rightarrow H_0$$

(4.17)

In this diagram, where each arrow stands for the flow generated by sending a single 7-brane to infinity, the $\hat{E}$-sequence represents the 5D $N = 1$ $E_N$ theories. Then the down arrows from the $\hat{E}$ to the $E$-sequence correspond to the compactification to four dimensions. The flavor number $N_f = N - 1$ is preserved along down arrows for each $N$. The $E$-sequence as well as the $D$- and $H$- sequences are 4D $N = 2$ theories with various global symmetries where the theories put in the brackets are not classified as non-trivial fixed points.

Corresponding to the degeneration flows (4.17), we have the coalescence flows among the singularity structures on the opposite side of the base $\mathbb{P}^1$. These flows are shown in
the following coalescence diagram:

\[
\begin{array}{cccccccc}
A_0 & \rightarrow & A_1 & \rightarrow & A_2 & \rightarrow & A_3 & \rightarrow & A_4 & \rightarrow & A_5 & \rightarrow & A_6 & \rightarrow & A_7 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_0 & \rightarrow & H_1 & \rightarrow & H_2 & \rightarrow & H_4 & \rightarrow & H_5 & \rightarrow & H_6 & \rightarrow & H_7 & \rightarrow & H_8 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
D_4 & \rightarrow & D_5 & \rightarrow & D_6 & \rightarrow & D_7 & \rightarrow & D_8 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
E_6 & \rightarrow & E_7 & \rightarrow & E_8 \\
\end{array}
\]

\[A_7 \rightarrow A_8\]

Each entry of this diagram represents the singularity type.

In order to analyze the BPS spectrum of 5D $E_N$ theories on $\mathbb{R}^4 \times S^1$ one turns to the study of the junction lattice on the 7-branes. For this purpose it is desired to have a further understanding of how the junction lattice is described in terms of geometry of rational elliptic surfaces. This amounts to analyzing the sections of the massive $\hat{E}_n$ curves based on the brane picture and its relation to the Mordell-Weil lattice \[40\]. The analysis will also become interesting from the standpoint of the F-theory/heterotic duality in eight dimensions.

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**Note added**

Just after this paper was submitted, an interesting paper by A. Sen and B. Zwiebach appeared \[41\] in which they have also constructed the elliptic curves for the affine 7-branes. Their construction substantially overlaps with ours in section 3.
References

[1] A. Johansen, Phys. Lett. B395 (1997) 36, hep-th/9608180.

[2] M.R. Gaberdiel and B. Zwiebach, Nucl. Phys. B518 (1998) 151, hep-th/9709013.

[3] Y. Imamura, Phys. Rev. D58 (1998) 106005, hep-th/9802189.

[4] M.R. Gaberdiel, T. Hauer and B. Zwiebach, Nucl. Phys. B525 (1998) 117, hep-th/9801203.

[5] O. DeWolfe and B. Zwiebach, Nucl. Phys. B541 (1999) 509, hep-th/9804210.

[6] O. DeWolfe, T. Hauer, A. Iqbal and B. Zwiebach, Nucl. Phys. B534 (1998) 261, hep-th/9805220.

[7] A. Iqbal, Self-Intersection Number of BPS Junctions in Backgrounds of Three and Seven-Branes, hep-th/9807117.

[8] O. DeWolfe, Nucl. Phys. B550 (1999) 622, hep-th/9809026.

[9] O. DeWolfe, T. Hauer, A. Iqbal and B. Zwiebach, Uncovering the Symmetries on \([p,q]\) 7-branes: Beyond the Kodaira Classification, hep-th/9812028.

[10] O. DeWolfe, T. Hauer, A. Iqbal and B. Zwiebach, Uncovering Infinite Symmetries on \([p,q]\) 7-branes: Kac-Moody Algebras and Beyond, hep-th/9812209.

[11] O. DeWolfe, A. Hanany, A. Iqbal and E. Katz, JHEP 9903 (1999) 006, hep-th/9902179.

[12] N. Seiberg, Phys. Lett. B388 (1996) 753, hep-th/9608111.

[13] D.R. Morrison and N. Seiberg, Nucl. Phys. B483 (1997) 229, hep-th/9609070.

[14] M.R. Douglas, S. Katz and C. Vafa, Nucl. Phys. B497 (1997) 155, hep-th/9609071.

[15] J.A. Minahan, D. Nemeschansky and N.P. Warner, Nucl. Phys. B508 (1997) 64, hep-th/9705237.
[16] W. Lerche, P. Mayr and N.P. Warner, Nucl. Phys. B499 (1997) 125, hep-th/9612085.

[17] C. Vafa, Nucl. Phys. B469 (1996) 403, hep-th/9602022.

[18] K. Kodaira, Ann. Math. 77 (1963) 563; Ann. Math. 78 (1963) 1.

[19] A. Sen, Nucl. Phys. B475 (1996) 562, hep-th/9605150; Phys. Rev. D55 (1997) 2501, hep-th/9608005.

[20] T. Banks, M. Douglas and N. Seiberg, Phys. Lett. B387 (1996) 278, hep-th/9603199.

[21] D.R. Morrison and C. Vafa, Nucl. Phys. B476 (1996) 437, hep-th/9603161.

[22] O.J. Ganor, Nucl. Phys. B479 (1996) 197, hep-th/9607020.

[23] O.J. Ganor, Nucl. Phys. B488 (1997) 223, hep-th/9608109.

[24] A. Klemm, P. Mayr and C. Vafa, in the proceedings of the conference “Advanced Quantum Field Theory” (in memory of Claude Itzykson), hep-th/9607131.

[25] O.J. Ganor, D.R. Morrison and N. Seiberg, Nucl. Phys. B487 (1997) 93, hep-th/9610251.

[26] J.A. Minahan, D. Nemeschansky, C. Vafa and N.P. Warner, Nucl. Phys. B527 (1998) 581, hep-th/9802168.

[27] O. Aharony and A. Hanany, Nucl. Phys. B504 ((1997) 239, hep-th/9704170.

[28] B. Kol, 5d Field Theories and M Theory, hep-th/9705031.

[29] A. Brandhuber, N. Itzhaki, J. Sonnenschein, S. Theisen and S. Yankielowicz, Phys. Lett. B415 (1997) 127, hep-th/9709010.

[30] O. Aharony, A. Hanany and B. Kol, JHEP 9801 (1998) 002, hep-th/9710116.

[31] B. Kol and J. Rahmfeld, JHEP 9808 (1998) 006, hep-th/9801067.

[32] N.C. Leung and C. Vafa, Adv. Theor. Math. Phys. 2 (1998) 91, hep-th/9711013.

[33] N. Nekrasov, Nucl. Phys. B531 (1998) 323, hep-th/9609219.
[34] A. Lawrence and N. Nekrasov, Nucl. Phys. B513 (1998) 239, hep-th/9706025.

[35] M. Noguchi, S. Terashima and S.-K. Yang, N=2 Superconformal Field Theory with ADE Global Symmetry on a D3-brane Probe, hep-th/9903215, to appear in Nucl. Phys. B.

[36] J.A. Minahan and D. Nemeschansky, Nucl. Phys. B482 (1996) 142, hep-th/9608047. Nucl. Phys. B489 (1997) 24, hep-th/9610076.

[37] N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19, hep-th/9407087. Nucl. Phys. B431 (1994) 484, hep-th/9408099.

[38] R. Miranda, Math. Ann. 255 (1981) 379.

[39] R. Miranda and U. Persson, Math. Z. 193 (1986) 537.

[40] M. Fukae, Y. Yamada and S.-K. Yang, to appear.

[41] A. Sen and B. Zwiebach, Stable Non-BPS States in F-theory, hep-th/9907164.