Noncolliding Squared Bessel Processes

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Abstract

We consider a particle system of the squared Bessel processes with index $\nu > -1$ conditioned never to collide with each other, in which if $-1 < \nu < 0$ the origin is assumed to be reflecting. When the number of particles is finite, we prove for any fixed initial configuration that this noncolliding diffusion process is determinantal in the sense that any multitime correlation function is given by a determinant with a continuous kernel called the correlation kernel. When the number of particles is infinite, we give sufficient conditions for initial configurations so that the system is well defined. There the process with an infinite number of particles is determinantal and the correlation kernel is expressed using an entire function represented by the Weierstrass canonical product, whose zeros on the positive part of the real axis are given by the particle-positions in the initial configuration. From the class of infinite-particle initial configurations satisfying our conditions, we report one example in detail, which is a fixed configuration such that every point of the square of positive zero of the Bessel function $J_\nu$ is occupied by one particle. The process starting from this initial configuration shows a relaxation phenomenon converging to the stationary process, which is determinantal with the extended Bessel kernel, in the long-term limit.

Keywords Noncolliding diffusion process, Squared Bessel process, Fredholm determinants, Entire functions, Weierstrass canonical products, Infinite particle systems

1 Introduction

Let $\mathcal{M}$ be the space of nonnegative integer-valued Radon measures on $\mathbb{R}$, which is a Polish space with the vague topology. We say $\xi_n \in \mathcal{M}, n \in \mathbb{N} \equiv \{1, 2, \ldots\}$ converges to $\xi \in \mathcal{M}$ weakly in the vague topology, if $\lim_{n \to \infty} \int_{\mathbb{R}} \varphi(x) \xi_n(dx) = \int_{\mathbb{R}} \varphi(x) \xi(dx)$ for any $\varphi \in C_0(\mathbb{R})$, where $C_0(\mathbb{R})$ is the set of all continuous real-valued functions with compact supports in $\mathbb{R}$. Any element of $\mathcal{M}$ can be represented by $\sum_{i \in \Lambda} \delta_{x_i}(\cdot)$ with a sequence of points in $\mathbb{R}$, $x = (x_i)_{i \in \Lambda}$, satisfying $\#\{x_i : x_i \in I\} < \infty$ for any compact subset $I \subset \mathbb{R}$, and with a

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countable set (an index set) $\Lambda$. We call an element $\xi$ of $\mathcal{M}$ an unlabeled configuration, and a sequence of points $x \in \Lambda$ a labeled configuration. For $A \subset \mathbb{R}$, we write the restriction of $\xi \in \mathcal{M}$ on $A$ as $(\xi \cap A)(\cdot) = \sum_{i \in A} x_i \delta_{x_i}(\cdot)$. Let $\mathbb{R}_+ = \{ x \in \mathbb{R} : x \geq 0 \}$ and define $\mathcal{M}^+ = \{(\xi \cap \mathbb{R}_+)(\cdot) : \xi(\cdot) \in \mathcal{M} \}$. In the present paper we consider a one-parameter family of $\mathcal{M}^+$-valued processes with a parameter $\nu > -1$,  
\[ \Xi(\nu)(t, \cdot) = \sum_i \delta_{X_i^{(\nu)}(t)}(\cdot), \quad t \in [0, \infty), \]  
(1.1)
describing a particle system of squared Bessel processes with index $\nu > -1$ (BESQ$^{(\nu)}$) interacting with each other by long-ranged repulsive forces, such that $X_i^{(\nu)}(t)$’s satisfy the SDEs 
\[ dX_i^{(\nu)}(t) = 2\sqrt{X_i^{(\nu)}(t)} dB_i(t) + 2(\nu + 1) dt \]
\[ + \frac{1}{4} \sum_{j,j \neq i} X_i^{(\nu)}(t) - X_j^{(\nu)}(t) dt, \quad i = 1, 2, \ldots, \quad t \in [0, \infty) \]  
(1.2)
with a collection of independent standard Brownian motions (BMs), $\{ B_i(t), i \in \mathbb{N} \}$, and, if $-1 < \nu < 0$, with a reflection wall at the origin. Note that for the BM in $\mathbb{R}^d$, $B(t) = (B_1(t), \ldots, B_d(t)), d \in \mathbb{N}$, the square of its distance from the origin, $X(t) \equiv |B(t)|^2 = \sum_{i=1}^d B_i(t)^2$, solves the SDE, 
\[ dX(t) = 2\sqrt{X(t)} dB(t) + 2(\nu + 1) dt \]
with 
\[ \nu = \frac{d}{2} - 1 \]  
(1.3)
where $B(t)$ is a standard BM which is different from $\widetilde{B}_i(t), 1 \leq i \leq d$ [24, 1]. We give the initial configuration of the process $\xi(\cdot) = \Xi(\nu)(0, \cdot) = \sum_i \delta_{x_i}(\cdot)$ and the process is denoted by $(\Xi(\nu)(t), \mathbb{P}_\nu)$. When the number of particles is finite, $\xi_N(\mathbb{R}_+) = N < \infty$, the process $(\Xi(\nu)(t), \mathbb{P}_\nu)$ is realized in the following systems.

(i) When $\nu \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$, i.e., when the corresponding dimension $d$ given by (1.3) is a positive even integer, $(\Xi(\nu)(t), \mathbb{P}_\nu)$ is realized as the eigenvalue process of the Laguerre process [20]. Let $M(t)$ be an $(N + \nu) \times N$ matrix, whose entries are independent complex BMs having the real and imaginary parts given by independent standard BMs, and set $L(t) = M(t)^* M(t)$. The $N \times N$ matrix-valued process $L = (L(t))_{t \in [0, \infty)}$ is called the Laguerre process. The matrix $L(t)$ is Hermitian and positive definite, and its $N$ eigenvalues satisfy (1.2) with $i = 1, 2, \ldots, N$. When the entries of $M(t)$ are independent standard real BMs, the matrix-valued process $(M(t)^T M(t))_{t \in [0, \infty)}$ is called the Wishart process [3], and thus $L$ is also called the complex Wishart process. The eigenvalue processes of the real and complex Wishart processes are related with the random matrix theory [23, 9] for the chiral Gaussian ensembles studied in the high energy physics (see [15] and references therein).
(ii) Let $\mathcal{H}(N)$ be the space of $N \times N$ Hermitian matrices. And let $\mathfrak{sp}(2N, \mathbb{C})$ and $\mathfrak{so}(2N, \mathbb{C})$ be the symplectic Lie algebra and the orthogonal Lie algebra, having $2N \times 2N$-matrix representations, respectively. If the $2N \times 2N$ matrix is in the space $\mathcal{H}_C(2N) \equiv \mathcal{H}(2N) \cap \mathfrak{sp}(2N, \mathbb{C})$ or in $\mathcal{H}_D(2N) \equiv \mathcal{H}(2N) \cap \mathfrak{so}(2N, \mathbb{C})$, its eigenvalues are given by $N$ pairs of positive and negative ones with the same absolute value, $\{(\lambda_i, -\lambda_i) : \lambda_i \geq 0, 1 \leq i \leq N\}$. Consider the $\mathcal{H}_C(2N)$-valued and the $\mathcal{H}_D(2N)$-valued Brownian motions. The dynamics of positive eigenvalues of them are described by [1, 2] with $\nu = 1/2 (d = 3)$ for the former case and with $\nu = -1/2 (d = 1)$ for the latter case, respectively [15]. The pairing of positive and negative eigenvalues simulates the particle-hole symmetry in the energy space of the Bogoliubov-de Gennes formalism of superconductivity and these processes are related with the random matrix theory studied in the solid-state physics [1].

(iii) Let $p^{(\nu)}(t, y|x), y \in \mathbb{R}^+$, be the transition probability density for BESQ$^{(\nu)}$, $\nu > -1$,

$$p^{(\nu)}(t, y|x) = \begin{cases} \frac{1}{2t} \left( \frac{y}{x} \right)^{\nu/2} \exp \left( -\frac{x + y}{2t} \right) I_\nu \left( \sqrt{xy/t} \right), & t > 0, x > 0, \\ \frac{y^\nu}{(2t)^{\nu+1}\Gamma(\nu+1)} e^{-y/2t}, & t > 0, x = 0, \\ \delta(y - x), & t = 0, x \in \mathbb{R}^+, \end{cases} \quad (1.4)$$

if $-1 < \nu < 0$, the origin is assumed to be reflecting [24, 4], where $I_\nu(x)$ is the modified Bessel function of the first kind defined by

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+1)\Gamma(n+1+\nu)} \left( \frac{x}{2} \right)^{2n+\nu} \quad (1.5)$$

with the Gamma function $\Gamma(z) = \int_0^\infty e^{-u}u^{z-1}du, \; \Re u > 0$. First we consider the following Karlin-McGregor determinant [14],

$$f_N^{(\nu)}(t, y|x) = \det_{1 \leq i, j \leq N} \left[ p^{(\nu)}(t, y_i|x_j) \right], \quad t \geq 0, \quad x = (x_i)_{i=1}^N, \; y = (y_i)_{i=1}^N \in \mathbb{W}_N^+, \quad (1.6)$$

where $\mathbb{W}_N^+ = \{ x = (x_1, \ldots, x_N) \in \mathbb{R}^N : 0 \leq x_1 < \cdots < x_N \}$. The transition probability density of the $N$-particle system of BESQ$^{(\nu)}$ conditioned never to collide with each other, which we call the noncolliding BESQ$^{(\nu)}$, is given by the $h$-transform of (1.6),

$$p_N^{(\nu)}(t, y|x) = h_N(y) f_N^{(\nu)}(t, y|x) \frac{1}{h_N(x)} \quad (1.7)$$

with the harmonic function given by the Vandermonde determinant [11, 20, 15]

$$h_N(x) = \det_{1 \leq i, j \leq N} [x_i^{j-1}] = \prod_{1 \leq i < j \leq N} (x_j - x_i). \quad (1.8)$$
It is easy to confirm that $p_N^{(\nu)}(t, \cdot | x)$ satisfies the following backward Kolmogorov equation

\[
\frac{\partial}{\partial t} u(t, x) = 2 \sum_{i=1}^{N} x_i \frac{\partial^2}{\partial x_i^2} u(t, x) + 2(\nu + 1) \sum_{i=1}^{N} \frac{\partial}{\partial x_i} u(t, x)
\]

\[
+ 4 \sum_{i=1}^{N} \sum_{j=1 \atop j \neq i}^{N} \frac{x_i - x_j}{x_i - x_j} \frac{\partial}{\partial x_i} u(t, x), \quad t \geq 0, \quad x \in \mathbb{W}_N^+,
\]

and it implies that the process $(\Xi^{(\nu)}(t), \mathbb{P}_x^{\nu})$, $\nu > -1$, is realized as the noncolliding BESQ$^{(\nu)}$. We put

\[
\mathfrak{M}_0^+ = \{ \xi \in \mathfrak{M}^+ : \xi(\{x\}) \leq 1 \text{ for any } x \in \mathbb{R} \}.
\]

We see that $\Xi^{(\nu)}(t) \in \mathfrak{M}_0^+$, $\forall t > 0$.

In the present paper, we call the process $(\Xi^{(\nu)}(t), \mathbb{P}_x^{\nu})$ the noncolliding BESQ$^{(\nu)}$. See \[28\] \[16\] \[21\] for related noncolliding diffusion processes.

Assume that $\xi_N \in \mathfrak{M}^+$ with $\xi_N(\mathbb{R}_+) = N \in \mathbb{N}$. For any $M \in \mathbb{N}$ and any time sequence $0 < t_1 < \cdots < t_M < \infty$, the formula \[1.7\] and the Markov property of the system give the multitime probability density of $(\Xi^{(\nu)}, \mathbb{P}_x^{\nu})$ as \[11\] \[17\]

\[
p^{\nu}_\xi^N(t_1, \xi^{(1)}_N; \ldots; t_M, \xi^{(M)}_N) = h_N(x^{(M)}) \prod_{m=1}^{M-1} f^{(\nu)}_N(t_{m+1} - t_m; x^{(m)} \cdot | x^{(m)}) \frac{f^{(\nu)}(t_1, x^{(1)} \cdot | x)}{h_N(x)} (1.10)
\]

with $\xi_N(\cdot) = \sum_{i=1}^{N} \delta_{x_i}(\cdot) \in \mathfrak{M}^+$, $0 \leq x_1 \leq x_2 \leq \cdots \leq x_N$ for the initial configuration and $\xi^{(m)}_N(\cdot) = \sum_{i=1}^{N} \delta_{x^{(m)}_i}(\cdot) \in \mathfrak{M}_0^+$, $x^{(m)} = (x^{(m)}_1, \ldots, x^{(m)}_N) \in \mathbb{W}_N^+$ for the configurations at times $t_m, 1 \leq m \leq M$. In \[1.10\], if some of $x_i$’s in $x$ coincide, the factor $f^{(\nu)}_N(t_1, x^{(1)} \cdot | x)/h_N(x)$ is interpreted using l’Hôpital’s rule.

For $x^{(m)} = (x^{(m)}_1, \ldots, x^{(m)}_N_N) \in \mathbb{W}_N^+$ with $\xi^{(m)}_N(\cdot) = \sum_{i=1}^{N} \delta_{x^{(m)}_i}(\cdot)$ and $N \in \{1, 2, \ldots, N\}$, we put $x^{(m)}_N = (x^{(m)}_1, \ldots, x^{(m)}_N) \in \mathbb{W}_N^+$, $1 \leq m \leq M$. For a sequence $(N_m)^{M}_{m=1}$ of positive integers less than or equal to $N$, we define the $(N_1, \ldots, N_M)$-multitime correlation function by

\[
\rho^{\nu}_\xi^N(t_1, x^{(1)}_N; \ldots; t_M, x^{(M)}_N)
\]

\[
= \int_{\prod_{m=1}^{M} \mathbb{R}^{N_N - N_m} \prod_{m=1}^{M} \mathbb{N}} \left( \frac{1}{(N - N_m)!} \right) \rho^{\nu}_\xi^N(t_1, \xi^{(1)}_N; \ldots; t_M, \xi^{(M)}_N) \prod_{m=1}^{M} \frac{1}{(N - N_m)!} (1.11)
\]

which is symmetric in the sense that $\rho^{\nu}_\xi^N(\ldots; t_m, \sigma(x^{(m)}_{N_m}); \ldots) = \rho^{\nu}_\xi^N(\ldots; t_m, x^{(m)}_{N_m}; \ldots)$ with $\sigma(x^{(m)}_{N_m}) \equiv (x^{(m)}_{\sigma(1)}, \ldots, x^{(m)}_{\sigma(N_m)})$ for any permutation $\sigma \in S_{N_m}, 1 \leq \forall m \leq M$. For any $M \in \mathbb{N}$, $f_m \in C_0(\mathbb{R}_+), \theta_m \in \mathbb{R}, 1 \leq m \leq M, 0 < t_1 < \cdots < t_M < \infty$, the Laplace transform of \[1.10\]
is considered as a functional of $\chi(x) = (\chi_1(x), \ldots, \chi_M(x))$, where $\chi_m(x) \equiv e^{\theta_m f_m(x)} - 1, 1 \leq m \leq M, x \in \mathbb{R}_+$:

$$G^\xi_N[\chi] = \int_{\mathbb{R}_+^M} \prod_{m=1}^M dx^{(m)} p^\xi_N(t_1, \xi^{(1)}_N; \ldots; t_M, \xi^{(M)}_N) \exp\left\{ \sum_{m=1}^M \theta_m \int_{\mathbb{R}} f_m(x) \xi^{(m)}_N(dx) \right\}$$

$$= \mathbb{E}^\xi_N\left[ \exp\left\{ \sum_{m=1}^M \theta_m \int_{\mathbb{R}} f_m(x) \Xi^{(\nu)}(t_m, dx) \right\} \right]. \quad (1.12)$$

It is the generating function of multitime correlation functions, since if we expand it with respect to $\chi_m(x^{(m)})$'s, $p^\xi_N$'s appear as coefficients in terms:

$$G^\xi_N[\chi] = \sum_{N_1=0}^N \cdots \sum_{N_M=0}^N \prod_{m=1}^M \frac{1}{N_m!} \int_{\mathbb{R}_+^{N_m}} \prod_{m=1}^M \chi_m(x^{(m)}_N) p^\xi_N(t_1, x^{(1)}_{N_1}; \ldots; t_M, x^{(M)}_{N_M}). \quad (1.13)$$

In the present paper, first we prove that, for any fixed initial configuration $\xi_N \in \mathcal{M}^+$ with $\xi_N(\mathbb{R}_+) = N \in \mathbb{N}$, there is a function $K^\xi_N(s, x; t, y)$, which is continuous with respect to $(x, y) \in (0, \infty)^2$ for any fixed $(s, t) \in [0, \infty)^2$, and that the function (1.12) is given by the Fredholm determinant in the form

$$G^\xi_N[\chi] = \det_{1 \leq m, n \leq M} \delta_{mn} \delta(x - y) + K^\xi_N(t_m, x; t_n, y) \chi_n(y). \quad (1.14)$$

By definition of Fredholm determinant (see Eq. (4.8) in Section 4.1), (1.14) means that any multitime correlation function is given by a determinant

$$p^\xi_N(t_1, x^{(1)}_{N_1}; \ldots; t_M, x^{(M)}_{N_M}) = \det_{1 \leq i \leq N_m, 1 \leq j \leq N_n} K^\xi_N(t_m, x^{(m)}_i; t_n, x^{(n)}_j). \quad (1.15)$$

The function $K^\xi_N$ is called the correlation kernel and it determines the finite dimensional distributions of the process $(\Xi^{(\nu)}(t), \mathbb{P}^\xi_N)$ through (1.15). It is an extension of determinantal (Fermion) point process of distributions studied by Soshnikov [26] and Shirai and Takahashi [25] to the cases on $\mathbf{T} \times \mathbb{R}_+$ with $\mathbf{T} = \{t_1, \ldots, t_M\}, M \in \mathbb{N}, 0 < t_1 < \cdots < t_M < \infty$. See [12] for variety of examples of determinantal point processes. We express this result by simply saying that the noncolliding BESQ$^{(\nu)}$ is determinantal with a correlation kernel $K^\xi_N$ for any $\xi_N \in \mathcal{M}^+$ with $\xi_N(\mathbb{R}_+) = N \in \mathbb{N}$ (Theorem 2.1).

Next we consider the infinite-particle limits. For $\xi \in \mathcal{M}^+$ with $\xi(\mathbb{R}_+) = \infty$, when $K^\xi_{\nu_0(0, L)}$ converges to a continuous function as $L \to \infty$, the limit is written as $K^\xi$. If $\sup_{x, y \in I} |K^\xi_{\nu_0(0, L)}(s, x; t, y)| < \infty, \forall L > 0$ for any $(s, t) \in (0, \infty)^2$ and any compact interval $I \subset (0, \infty)$, we can obtain the convergence of generating functions of multitime correlation...
functions, $G_\nu^{\xi(0,L)}(x) \to G_\nu^{\xi}(x)$, as $L \to \infty$. It implies $\mathbb{P}_\nu^{\xi(0,L)} \to \exists \mathbb{P}_\nu^{\xi}$ as $L \to \infty$ in the sense of finite dimensional distributions weakly in the vague topology. In this case, we say that the noncolliding BESQ$^{(\nu)}(\Xi^{(\nu)}(t), \mathbb{P}_\nu^{\xi})$ with an infinite number of particles $\xi(\mathbb{R}_+) = \infty$ is well defined with the correlation kernel $K^{\xi}$ [19]. We will give sufficient conditions so that the process $(\Xi^{(\nu)}(t), \mathbb{P}_\nu^{\xi})$ is well defined, in which the correlation kernel is generally expressed using a double integral of an entire function represented by the Weierstrass canonical product having zeros on $\text{supp } \xi$, where $\text{supp } \xi = \{ x \in \mathbb{R} : \xi(\{x\}) > 0 \}$ (Theorem 2.2). As an application of this theorem, we will study the following example of infinite particle system, which is a non-equilibrium dynamics exhibiting a relaxation phenomenon.

Consider the Bessel function

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1+\nu)} \left(\frac{z}{2}\right)^{2n+\nu}.$$  \hspace{1cm} (1.16)

It is an analytic function of $z$ in a cut plane. The function $J_\nu(z)/z^\nu$ is an entire function. As usual we define $z^\nu$ to be $\exp(\nu \log z)$, where the argument of $z$ is given its principal value;

$$z^\nu = \exp \left[ \nu \left\{ \log |z| + \sqrt{-1} \text{arg}(z) \right\} \right], \quad -\pi < \text{arg}(z) \leq \pi. \hspace{1cm} (1.17)$$

The function $J_\nu(z)$ is analytically continued outside this range of arg($z$) so that the relation

$$J_\nu(e^{m\pi\sqrt{-1}}z) = e^{m\pi\sqrt{-1}}J_\nu(z) \hspace{1cm} (1.18)$$

holds [29]. If $\nu > -1$, $J_\nu(z)$ has an infinite number of pairs of positive and negative zeros with the same absolute value, which are all simple. We write the positive zeros of $J_\nu(z)$ arranged in ascending order of the absolute value as

$$0 < j_{\nu,1} < j_{\nu,2} < j_{\nu,3} < \cdots.$$  \hspace{1cm}

Explicitly $J_\nu(z)$ is expressed using the infinite product of the Weierstrass primary factors of genus zero as (see Chapter XV of [29]),

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \prod_{i=1}^{\infty} \left(1 - \frac{z^2}{j_{\nu,i}^2}\right). \hspace{1cm} (1.19)$$

The configuration in which every point of the square of positive zero of $J_\nu(z)$ is occupied by one particle, denoted by

$$\xi^{(2)}_{J_\nu}(\cdot) = \sum_{i=1}^{\infty} \delta_{j_{\nu,i}}(\cdot), \hspace{1cm} (1.20)$$

satisfies the conditions of Theorem 2.2. We will determine the correlation kernel of the noncolliding BESQ$^{(\nu)}$ starting from $\xi^{(2)}_{J_\nu}$ explicitly (Theorem 2.3 (i)) and prove that the process shows a relaxation phenomenon to a stationary process,

$$(\Xi^{(\nu)}(t+\theta), \mathbb{P}_\nu^{\xi_{J_\nu}}) \to (\Xi^{(\nu)}(t), \mathbb{P}_{J_\nu}) \quad \text{as} \quad \theta \to \infty.$$
weakly in the sense of finite dimensional distributions (Theorem 2.3 (ii)). Here \((\Xi^{(\nu)}(t), \mathbf{P}_{J_\nu})\)

is the equilibrium dynamics, which is determinantal with the correlation kernel

\[
K_{J_\nu}(t - s, y|x) = \begin{cases} 
\int_0^1 du e^{-2u(s-t)} J_\nu(2\sqrt{ux})J_\nu(2\sqrt{uy}) & \text{if } s < t \\
\frac{J_\nu(2\sqrt{x})\sqrt{y}J'_\nu(2\sqrt{y}) - \sqrt{x}J'_\nu(2\sqrt{x})J_\nu(2\sqrt{y})}{x - y} & \text{if } t = s \\
\int_1^\infty du e^{-2u(s-t)} J_\nu(2\sqrt{ux})J_\nu(2\sqrt{uy}) & \text{if } s > t,
\end{cases}
\]  

(1.21)



\((x, y) \in (0, \infty)^2\), where \(J'(z) = dJ(z)/dz\). Note that this kernel is temporally homogeneous but spatially inhomogeneous. Let \(\mu_{J_\nu}\) be the determinantal (Fermion) point process on \(\mathbb{R}_+\), in which, for any \(N \in \mathbb{N}\), \(N\)-point correlation function is given by

\[
\rho_\nu(x_N) = \det_{1 \leq i, j \leq N} \left[ K_{J_\nu}(x_i|x_j) \right], \quad x_N = (x_1, \ldots, x_N) \in (0, \infty)^N,
\]

with the Bessel kernel

\[
K_{J_\nu}(y|x) \equiv K_{J_\nu}(0, y|x) = \frac{J_\nu(2\sqrt{x})\sqrt{y}J'_\nu(2\sqrt{y}) - \sqrt{x}J'_\nu(2\sqrt{x})J_\nu(2\sqrt{y})}{x - y} = \frac{\sqrt{x}J_{\nu+1}(2\sqrt{x})J_\nu(2\sqrt{y}) - J_\nu(2\sqrt{x})\sqrt{y}J_{\nu+1}(2\sqrt{y})}{x - y}.
\]  

(1.22)

In particular, the density of particle at \(x \in (0, \infty)\) is given by

\[
\rho_\nu(x) = K_{J_\nu}(x|x) \equiv \lim_{y \to x} K_{J_\nu}(y|x) = (J_\nu(2\sqrt{x}))^2 + \left(1 - \frac{\nu^2}{4x}\right)(J_\nu(2\sqrt{x}))^2 = (J_\nu(2\sqrt{x}))^2 - J_{\nu+1}(2\sqrt{x})J_{\nu-1}(2\sqrt{x}).
\]  

(1.23)

The probability measure \(\mu_{J_\nu}\) is obtained in an \(N \to \infty\) limit called the hard-edge scaling-limit of the distribution of squares of eigenvalues of random matrices in the chiral Gaussian unitary ensemble \([23, 9]\). The process \((\Xi^{(\nu)}(t), \mathbf{P}_{J_\nu})\) is a reversible process with respect to \(\mu_{J_\nu}\). The correlation kernel (1.21) is called the extended Bessel kernel \([10, 27]\).

In the random matrix theory, three kinds of determinantal point processes of infinite particle systems have been well studied, the correlation kernels of which are given by (i) the sine kernel, (ii) the Airy kernel, and (iii) the Bessel kernel \([23, 9]\). They are obtained by taking (i) the bulk, (ii) the soft-edge, and (iii) the hard-edge scaling limits in the Gaussian unitary ensemble (GUE) for (i) and (ii) and in the chiral GUE for (iii), respectively. These three determinantal point processes have been extended to time-dependent versions so that
they describe equilibrium dynamics, which are reversible with respect to the determinantal point processes \[17\]. Corresponding to these three stationary processes, the present authors introduced three relaxation processes with infinite numbers of particles realized in (i) the Dyson model (the noncolliding Brownian motion) starting from \(Z\) \((i.e.\) the zeros of \(\sin(\pi z)\)) in \[19\], (ii) the Dyson model with drift terms starting from the Airy zeros in \[18\], and (iii) the noncolliding BESQ\((\nu)\) starting from the squares of positive zeros of \(J_\nu\) in the present paper. The scaling limits are performed by increasing the number of particles in the system \(N \to \infty\), while in our setting determinantal processes with infinite numbers of particles \(N = \infty\) are well constructed based on the theory of entire functions. In our relaxation processes in non-equilibrium we only have to wait for sufficiently long time to observe the three determinantal point processes. In order to give temporally inhomogeneous correlation kernels explicitly, we have reported the relaxation processes with the special initial configurations. The universality of the three determinantal point processes in a wide variety of fields of mathematics, physics, and others \[23, 9\] implies robustness of relaxation phenomena with respect to initial configurations. Mathematical justification of this fact will be reported in the future.

The present paper is organized as follows. In Section 2 preliminaries and main results are given. In Section 3 the properties of special functions used in this paper are given. Section 4 is devoted to proofs of results.

## 2 Preliminaries and Main Results

We introduce the following operations; for \(\xi(\cdot) = \sum_{i \in \Lambda} \delta_{x_i}(\cdot) \in M\),

- **(shift)** with \(u \in \mathbb{R}\), \(\tau_u \xi(\cdot) = \sum_{i \in \Lambda} \delta_{x_i+u}(\cdot)\),

- **(dilatation)** with \(c > 0\), \(c \circ \xi(\cdot) = \sum_{i \in \Lambda} \delta_{cx_i}(\cdot)\),

- **(square)** \(\xi^{(2)}(\cdot) = \sum_{i \in \Lambda} \delta_{x_i^2}(\cdot)\),

and for \(\xi(\cdot) = \sum_{i \in \Lambda} \delta_{x_i}(\cdot) \in M^+\),

- **(square root)** \(\xi^{(1/2)}(\cdot) = \sum_{i \in \Lambda} (\delta_{\sqrt{x_i}} + \delta_{-\sqrt{x_i}}(\cdot))\).

Note that the notation (1.20) states that this configuration is obtained as the square of the point-mass distribution on the positive zeros of \(J_\nu(z)\) denoted by \(\xi_{J_\nu}(\cdot) = \sum_{i=1}^{\infty} \delta_{j_{\nu,i}}(\cdot)\). We use the convention such that

\[
\prod_{x \in \xi} f(x) = \exp \left\{ \int_{\mathbb{R}} \xi(dx) \log f(x) \right\} = \prod_{x \in \text{supp } \xi} f(x)^{\xi((x))}
\]
for $\xi \in \mathcal{M}$ and a function $f$ on $\mathbb{R}$. For a multivariate symmetric function $g$ we write $g((x)_{x \in \xi})$ for $g((x_i)_{i \in \Lambda})$.

The transition probability density of BESQ$(\nu)$, given by (1.4), satisfies the Chapman-Kolmogorov equation

$$
\int_0^\infty dy p^{(\nu)}(t-s, z|y)p^{(\nu)}(s, y|x) = p^{(\nu)}(t, z|x),
$$

for $0 \leq s \leq t, x, z \in \mathbb{R}_+$. Here we define the modified Bessel function of the first kind on $\mathbb{C}$ as

$$
I_\nu(z) = \begin{cases} 
    e^{-\nu \pi \sqrt{-1/2}} J_\nu(e^{\pi \sqrt{-1/2}} z), & -\pi < \arg(z) \leq \pi/2, \\
    e^{3\nu \pi \sqrt{-1/2}} J_\nu(e^{-3\pi \sqrt{-1/2}} z), & \pi/2 < \arg(z) \leq \pi,
\end{cases}
$$

where $J_\nu$ is defined by (1.16) so that (1.18) holds [2]. This definition is consistent with (1.5), associated with the relation

$$
I_\nu(e^{m \pi \sqrt{-1}} z) = e^{\nu m \pi \sqrt{-1}} I_\nu(z).
$$

For $t \in \mathbb{R}$, we define

$$
p^{(\nu)}(t, y|x) = \begin{cases} 
    \frac{1}{2|t|} \left( \frac{y}{x} \right)^{\nu/2} \exp \left( -\frac{x+y}{2t} \right) I_\nu \left( \frac{\sqrt{xy}}{t} \right), & t \in \mathbb{R} \setminus \{0\}, x \in \mathbb{C} \setminus \{0\}, \\
    \frac{y^{\nu/2}}{(2|t|)^{\nu+1} \Gamma(\nu+1)} e^{-y/2t}, & t \in \mathbb{R} \setminus \{0\}, x = 0, \\
    \delta(y-x), & t = 0, x \in \mathbb{C},
\end{cases}
$$

$y \in \mathbb{C}$. Then by using the modified version of Weber’s integral of the Bessel functions (see Eqs. (3.1) and (3.2) in Section 3.1), (2.1) is extended to the following equations. For $0 \leq s \leq t, x, z \in \mathbb{C}$,

$$
\int_0^\infty dy p^{(\nu)}(-t, z|y)p^{(\nu)}(t-s, y|x) = p^{(\nu)}(-s, z|x),
$$

$$
\int_0^\infty dy p^{(\nu)}(t-s, z|y)p^{(\nu)}(-t, y|x) = p^{(\nu)}(-s, z|x).
$$

We define

$$
p_{J_\nu}(t, y|x) = \begin{cases} 
    \left( \frac{x}{y} \right)^{\nu/2} p^{(\nu)}(t, y|x), & t \in \mathbb{R}, x \in \mathbb{C} \setminus \{0\}, \\
    y^{-\nu/2} p^{(\nu)}(t, y|0), & t \in \mathbb{R}, x = 0,
\end{cases}
$$

$y \in \mathbb{C}$. When $t \geq 0, x, y \in \mathbb{R}_+$, it has the expression

$$
p_{J_\nu}(t, y|x) = \int_0^\infty du J_\nu(2\sqrt{ux}) J_\nu(2\sqrt{uy}) e^{-2ut}
$$

$$
= 2 \int_0^\infty dw w J_\nu(2w\sqrt{x}) J_\nu(2w\sqrt{y}) e^{-2w^2t}.
$$
The Chapman-Kolmogorov equation (2.1) and its extensions (2.5) and (2.6) are mapped to
\[ \int_0^\infty dy p_J(t - s, z|y)p_J(s, y|x) = p_J(t, z|x), \]  
(2.9)
\[ \int_0^\infty dy p_J(-t, z|y)p_J(t - s, y|x) = p_J(-s, z|x), \]  
(2.10)
\[ \int_0^\infty dy p_J(t - s, z|y)p_J(-t, y|x) = p_J(-s, z|x) \]  
(2.11)
for \( 0 \leq s \leq t, x, z \in \mathbb{C} \).

For \( \xi_N \in \mathcal{M}^+ \) with \( \xi_N(\mathbb{R}_+) = N \in \mathbb{N} \), we define the functions of \( z \in \mathbb{C} \),
\[ \Pi_0(\xi_N, z) = \prod_{x \in \xi_N \cap \{0\}^c} \left( 1 - \frac{z}{x} \right), \]  
(2.12)
\[ \Pi^{(\nu)}(\xi_N, z) = z^{\nu/2} \Pi_0(\xi_N, z). \]  
(2.13)
For \( a \in \mathbb{C} \) we also define
\[ \Phi_0(\xi_N, a, z) = \Pi_0(\tau_a \xi_N, z - a) = \prod_{x \in \xi_N \cap \{a\}^c} \left( 1 - \frac{z - a}{x - a} \right), \]  
(2.14)
\[ \Phi^{(\nu)}(\xi_N, a, z) = \begin{cases} \left( \frac{z}{a} \right)^{\nu/2} \Phi_0(\xi_N, a, z), & \text{if } a \neq 0, \\ \Pi^{(\nu)}(\xi_N, z), & \text{if } a = 0. \end{cases} \]  
(2.15)

**Theorem 2.1** (i) For any fixed configuration \( \xi_N \in \mathcal{M}^+ \) with \( \xi_N(\mathbb{R}_+) = N \in \mathbb{N} \), \( (\Xi^{(\nu)}(t), \mathbb{P}^{\xi_N}) \) is determinantal with the correlation kernel
\[ K^{\xi_N}_{\nu}(s, x; t, y) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi \sqrt{-1}} \int_{-\infty}^{-\varepsilon} dz' \int_{\Gamma_{\nu}(\xi_N)} dy' p^{(\nu)}(s, x|z') \frac{1}{y' - z} \Phi_0(\xi_N, z, y') p^{(\nu)}(-t, y'|y) \]  
\[ -1(s > t) p^{(\nu)}(s - t, x|y'), \quad (s, t) \in [0, \infty)^2, (x, y) \in (0, \infty)^2, \]  
(2.16)
where \( \Gamma_{\nu}(\xi_N) \) denotes a counterclockwise contour on the complex plane \( \mathbb{C} \) encircling the points in \( \text{supp } \xi_N \) on \( \mathbb{R}_+ \) but not the point \( y' \in (-\infty, -\varepsilon] \), and \( 1(\omega) \) is the indicator function of condition \( \omega \).

(ii) If \( \xi_N \in \mathcal{M}^+_0 \) with \( \xi_N(\mathbb{R}_+) = N \in \mathbb{N} \), the correlation kernel is given by
\[ K^{\xi_N}_{\nu}(s, x; t, y) = \int_0^\infty \xi_N(dx') \int_{-\infty}^0 dy' p^{(\nu)}(s, x|x') \Phi_0(\xi_N, x', y') p^{(\nu)}(-t, y'|y) \]  
\[ -1(s > t) p^{(\nu)}(s - t, x|y'), \quad (s, t) \in [0, \infty)^2, (x, y) \in (0, \infty)^2. \]  
(2.17)

Without changing any finite dimensional distributions of the process, the correlation kernel (2.17) can be replaced by
\[ K^{\xi_N}_{\nu}(s, x; t, y) = \int_0^\infty \xi_N(dx') \int_{-\infty}^0 dy' p_J(s, x|x') \Phi^{(\nu)}(\xi_N, x', y') p_J(-t, y'|y) \]  
\[ -1(s > t) p_J(s - t, x|y'), \quad (s, t) \in [0, \infty)^2, (x, y) \in (0, \infty)^2. \]  
(2.18)
Remark 1. If we consider a spatial distribution of particles at a single time $t \geq 0$ of the noncolliding BESQ($\nu$), we have a determinantal point process with the correlation function $K(t, x; t, y), (x, y) \in (0, \infty)^2$. In particular, if we set $s = t = 1/2$, the correlation kernel (2.16) is reduced to be the kernel of the perturbed chiral GUE of random matrices given in Proposition 5 by Desrosiers and Forrester [8]. More detail, see Remark 2 in Section 3.3.

For $L > 0, \alpha > 0$ and $\xi \in \mathcal{M}$ we put

$$M_\alpha(\xi, L) = \left(\int_{[-L, L] \setminus \{0\}} \frac{\xi(dx)}{|x|^\alpha}\right)^{1/\alpha}$$

and

$$M_\alpha(\xi) = \lim_{L \to \infty} M_\alpha(\xi, L),$$

if the limit finitely exists. We introduce the following conditions for configurations $\xi \in \mathcal{M}^+$. (C.A) (i) There exists $\alpha \in (1/2, 1)$ and $C_1 > 0$ such that $M_\alpha(\xi) \leq C_1$.

(ii) There exist $\beta > 0$ and $C_2 > 0$ such that

$$M_1(\tau_{-a} \xi) \leq C_2(|a| \lor 1)^{-\beta}, \quad \forall a \in \text{supp} \, \xi.$$

We denote by $\mathcal{X}^+$ the set of configurations satisfying the conditions (C.A), and put $\mathcal{X}_0^+ = \mathcal{X}^+ \cap \mathcal{M}_0^+$. For $\xi \in \mathcal{X}_0^+, a \in \mathbb{R}$ and $z \in \mathbb{C}$ we can define

$$\Phi_0(\xi, a, z) = \lim_{L \to \infty} \Phi_0(\xi \cap [a - L, a + L], a, z),$$

$$\Phi(\nu)(\xi, a, z) = \lim_{L \to \infty} \Phi(\nu)(\xi \cap [a - L, a + L], a, z), \quad \nu > -1.$$

Since $\Phi_0(\xi, a, z)$ has the expression of the Weierstrass canonical product of genus zero (see (2.14)), it is an entire function of a variable $z \in \mathbb{C}$ [22]. The set of zeros of $\Phi(\nu)(\xi, a, z)$ is given by (supp $\xi \cup \{0\}) \cap \{a\}^\nu$ and all zeros except 0 are simple for $\xi \in \mathcal{M}_0^+$.

Theorem 2.2 If $\xi \in \mathcal{X}_0^+$, the process $(\Xi(\nu)(t), \mathcal{P}_\xi)$ is well defined with the correlation kernel

$$K(\xi, s, t; y) = \int_0^\infty \xi(dx') \int_{-\infty}^0 dy' p^{(\nu)}(s, x|x') \Phi_0(\xi, x', y') p^{(\nu)}(-t, y'|y)$$

$$-\mathbf{1}(s > t) p^{(\nu)}(s - t, x|y), \quad (s, t) \in [0, \infty)^2, (x, y) \in (0, \infty)^2. \quad (2.19)$$

This correlation kernel $K(\xi)$ can be replaced by

$$K(\eta, s, t; y) = \int_0^\infty \xi(dx') \int_{-\infty}^0 dy' p\eta(s, x|x') \Phi(\xi, x', y') p\eta(-t, y'|y)$$

$$-\mathbf{1}(s > t) p\eta(s - t, x|y), \quad (s, t) \in [0, \infty)^2, (x, y) \in (0, \infty)^2 \quad (2.20)$$

without changing any finite dimensional distributions of the process.
In case $\xi(\mathbb{R}_+)=\infty$, Theorem 2.2 gives the noncolliding BESQ$^{(\nu)}$ with an infinite number of particles starting from the configuration $\xi \in \mathcal{X}^+_{00}$. It is easy to check that, if $(x, y) \in (0, \infty)^2$, \[
abla t \kappa^\xi_{\nu} (t; x)\kappa^\xi_{\nu} (t; y) dx dy \to \xi(dx) 1(x = y), \quad t \to 0 \quad \text{in the vague topology.}
\]

For $\gamma > 0$, we put \[g_\gamma (x) = x^\gamma, \quad x \in \mathbb{R}_+\]
and \[\eta_\gamma (\cdot) = \sum_{i=1}^{\infty} \delta_{g_\gamma (i)} (\cdot).\]
For any $\gamma > 1$ we can show by simple calculation that $\eta_\gamma$ satisfies \((C.A) (i)\) with any $\alpha \in \begin{array}{c} 1/\gamma, 1 \end{array}$ and some $C^1_1 = C^1_1 (\alpha) > 0$ depending on $\alpha$, and \((C.A) (ii)\) with any $\beta \in (0, \gamma - 1)$ and some $C^2_2 = C^2_2 (\beta) > 0$ depending on $\beta$. This implies that $\eta_\gamma$ is an element of $\mathcal{X}^+_{00}$ for any $\gamma > 1$.

More interesting example is given by the following theorem.

**Theorem 2.3** (i) The noncolliding BESQ$^{(\nu)}$ starting from $\xi^{(2)}_{\nu}, (\Xi^{(\nu)}(t), \mathcal{P}^{(2)}_{\nu})$, is well defined with the correlation kernel \[
\kappa_{J_{\nu}} (s, x; t, y) = \sum_{i=1}^{\infty} \int_{-\infty}^{0} dz p_{\nu} (s, x| J_{\nu,i}) \frac{2 J_{\nu,i}^2}{J_{\nu,i}^2 - z J_{\nu+1,i} J_{\nu,i}} p_{\nu} (-t, z| y) \]
\[ - 1 (s > t) p_{\nu} (s - t, x| y), \quad (s, t) \in [0, \infty)^2, (x, y) \in (0, \infty)^2. \quad (2.21)\]

(ii) Let $(\Xi^{(\nu)}(t), \mathcal{P}_{\nu})$ be the equilibrium dynamics, which is determinantal with the extended Bessel kernel \((1.21)\). Then, for $t \geq 0$ \[
(\Xi^{(\nu)}(t + \theta), \mathcal{P}^{(2)}_{\nu}) \to (\Xi^{(\nu)}(t), \mathcal{P}_{\nu}) \quad \text{as} \quad \theta \to \infty \quad (2.22)\]
weakly in the sense of finite dimensional distributions.

### 3 Some Properties of Special Functions

#### 3.1 Integral formulas of Bessel functions

The following integral formulas are known \[29, 2\]. For $\Re \nu > -1, p, a, b > 0, \[
\int_{0}^{\infty} du u e^{-u^2 a} J_{\nu}(au) J_{\nu}(bu) = \frac{1}{2p^2} \exp \left( -\frac{a^2 + b^2}{4p^2} \right) I_{\nu} \left( \frac{ab}{2p^2} \right), \quad (3.1)\]
\[
\int_{0}^{\infty} du u e^{-u^2 a} I_{\nu}(au) I_{\nu}(bu) = \frac{1}{2p^2} \exp \left( -\frac{a^2 + b^2}{4p^2} \right) I_{\nu} \left( \frac{ab}{2p^2} \right). \quad (3.2)\]
The equalities (2.1), (2.5) and (2.6) for $p(\nu)$ and (2.8)-(2.11) for $p_{\nu}$ are derived from the above integral formulas. In addition to them, the following equality is also derived from (3.1) with (2.2) and (2.8). For $t > 0$, $x, z > 0$
\[ \int_{-\infty}^{0} dy J_{\nu}(\sqrt{zy})p_{\nu}(-t, y|x) = e^{tz/2}J_{\nu}(\sqrt{zx}). \] (3.3)

Lemma 3.1 For any $i \in \mathbb{N}$, $z \neq j_{\nu,i}$
\[ \frac{j_{\nu,i}}{J_{\nu+1}(j_{\nu,i})} \frac{J_{\nu}(z)}{z^2 J_{\nu}(j_{\nu,i})} = \frac{1}{(J_{\nu+1}(j_{\nu,i}))^2} \int_{0}^{1} du uJ_{\nu}(zu)J_{\nu}(j_{\nu,i}u) \] (3.4)
\[ = \frac{1}{(J_{\nu+1}(j_{\nu,i}))^2} \int_{0}^{1} dw J_{\nu}(z\sqrt{w})J_{\nu}(j_{\nu,i}\sqrt{w}). \] (3.5)

Proof. The following formula is found on page 482 in [29]
\[ \int_{0}^{x} tJ_{\nu}(\alpha t)J_{\nu}(\alpha_0 t)dt = \frac{x}{\alpha^2 - \alpha_0^2} \left[ J_{\nu}(\alpha x)J_{\nu}(\alpha_0 x) - J_{\nu}(\alpha_0 x)J_{\nu}(\alpha x) \right]. \]
Set $\alpha_0 = j_{\nu,i}$ and $x = 1$. Since $J_{\nu}(j_{\nu,i}) = 0$ by definition of $j_{\nu,i}$’s, we have
\[ \int_{0}^{1} tJ_{\nu}(\alpha t)J_{\nu}(j_{\nu,i} t)dt = \frac{j_{\nu,i}}{\alpha^2 - j_{\nu,i}^2} J_{\nu}(\alpha)J_{\nu+1}(j_{\nu,i}). \]
Note that the Bessel function satisfies the relation $J'_{\nu}(z) = (\nu/z)J_{\nu}(z) - J_{\nu+1}(z)$. Then
\[ J'_{\nu}(j_{\nu,i}) = -J_{\nu+1}(j_{\nu,i}) \] (3.6)
and thus
\[ \int_{0}^{1} tJ_{\nu}(\alpha t)J_{\nu}(j_{\nu,i} t)dt = \frac{j_{\nu,i}}{j_{\nu,i}^2 - \alpha^2} J_{\nu}(\alpha)J_{\nu+1}(j_{\nu,i}). \]
Divide the both sides by $(J_{\nu+1}(j_{\nu,i}))^2$, we obtain (3.4). Eq. (3.5) is obtained from (3.4) by setting $u = \sqrt{w}$. \]

3.2 Fourier-Bessel expansion
As shown in Section 18.24 in [29], any continuous function $f(x) \in L^2(0, 1)$ has the expansion
\[ f(x) = \sum_{i=1}^{\infty} a_i J_{\nu}(j_{\nu,i} x) \quad \text{with} \quad a_i = \frac{2}{(J_{\nu+1}(j_{\nu,i}))^2} \int_{0}^{1} uf(u)J_{\nu}(j_{\nu,i} u)du. \]
That is,
\[ f(x) = \int_{0}^{1} du uf(u) \sum_{i=1}^{\infty} \frac{2 J_{\nu}(j_{\nu,i} x) J_{\nu}(j_{\nu,i} u)}{(J_{\nu+1}(j_{\nu,i}))^2}, \quad f \in L^2(0, 1), \]
which is called the Fourier-Bessel expansion. Set \( u = \sqrt{y} \) and then replace \( x \) by \( \sqrt{x} \), we have

\[
    f(\sqrt{x}) = \int_0^1 dy f(\sqrt{y}) \sum_{i=1}^{\infty} \frac{J_\nu(j_{\nu,i}\sqrt{x})J_\nu(j_{\nu,i}\sqrt{y})}{(J_{\nu+1}(j_{\nu,i}))^2}.
\]

In other words, the functions \( \{J_\nu(j_{\nu,i}\sqrt{x})/J_{\nu+1}(j_{\nu,i})\}, i \in \mathbb{N} \) form an orthonormal basis for \( f \in L^2(0, 1) \) and the completeness is also established;

\[
    \sum_{i=1}^{\infty} \frac{J_\nu(j_{\nu,i}\sqrt{x})J_\nu(j_{\nu,i}\sqrt{y})}{(J_{\nu+1}(j_{\nu,i}))^2} = \delta(x-y), \quad x, y \in (0, 1). 
\]  

(3.7)

3.3 Multiple orthogonal polynomials

Fix a configuration \( \xi_N \in \mathcal{M}^+ \) with \( \xi_N(\mathbb{R}^+) = N \in \mathbb{N} \).

In the present paper the multiple orthogonal polynomials associated with the modified Bessel function \( I_\nu \) indexed by \( \xi_N \) are defined by the following [6, 7, 21, 8].

**Type I:** The *multiple orthogonal polynomials of the type I* are the set of functions

\[
    \left\{ A^{(\nu)}_{\xi_N}(y, x) : x \in \text{supp } \xi_N, \text{polynomial of } y \text{ of degree } \xi_N(x) - 1 \right\} 
\]  

(3.8)

such that, if we set

\[
    Q^{(\nu)}_{\xi_N}(y) = \sum_{x \in \text{supp } \xi_N} A^{(\nu)}_{\xi_N}(y, x) \frac{1}{2} \left( \frac{y}{x} \right)^{\nu/2} I_\nu(\sqrt{xy})e^{-(x+y)/2}, 
\]  

(3.9)

then

\[
    \int_0^\infty dy Q^{(\nu)}_{\xi_N}(y)y^i = \left\{ \begin{array}{cc}
    0, & i = 0, 1, \ldots, \xi_N(\mathbb{R}^+) - 2 \\
    1, & i = \xi_N(\mathbb{R}^+) - 1.
    \end{array} \right. 
\]  

(3.10)

**Type II:** The *multiple orthogonal polynomial of the type II* is the monic polynomial of degree \( \xi_N(\mathbb{R}^+) \),

\[
    P^{(\nu)}_{\xi_N}(y) = y^{\xi_N(\mathbb{R}^+)} + \mathcal{O}(y^{\xi_N(\mathbb{R}^+)-1}) 
\]  

(3.11)

such that for each \( x \in \text{supp } \xi_N \)

\[
    \int_0^\infty dy P^{(\nu)}_{\xi_N}(y)y^i \frac{1}{2} \left( \frac{y}{x} \right)^{\nu/2} I_\nu(\sqrt{xy})e^{-(x+y)/2} = 0, \quad 0 \leq i \leq \xi_N(x) - 1. 
\]  

(3.12)

The following integral representations have been obtained by Desrosiers and Forrester [8].
Lemma 3.2  The functions $Q_{\xi_N}^{(\nu)}(y)$ and $P_{\xi_N}^{(\nu)}(y)$ have the following integral representations,

$$Q_{\xi_N}^{(\nu)}(y) = \frac{1}{2\pi \sqrt{-1}} \int_{\Gamma(\xi_N)} dz \frac{1}{2} \left( \frac{y}{z} \right)^{\nu/2} I_{\nu}(\sqrt{yz})e^{-(y+z)/2} \prod_{x \in \xi_N} \frac{1}{z-x},$$  \hspace{1cm} (3.13)

$$P_{\xi_N}^{(\nu)}(y) = \int_{-\infty}^{0} dw \frac{1}{2} \left( \frac{w}{y} \right)^{\nu/2} I_{\nu}(-\sqrt{yw})e^{(y+w)/2} \prod_{x \in \xi_N} (w-x),$$  \hspace{1cm} (3.14)

where $\Gamma(\xi_N)$ denotes a counterclockwise contour on the complex plane $\mathbb{C}$ encircling the points in supp $\xi_N$ on $\mathbb{R}_+$. 

Remark 2. The present definition of multiple orthogonal polynomials associated with the modified Bessel function $I_\nu$ is slightly different from that given by Desrosiers and Forrester [8]. Moreover, since they have used the function $\sum_{n=0}^{\infty} z^n/\{\alpha,n!\}$ in order to express the polynomials instead of $I_\nu(z)$, our expressions (3.13) and (3.14) seem to be quite different from their functions. The identity $I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} F_1(\nu+1,\frac{z^2}{4})$ is established, however, and then we can see that Lemma 3.2 given above is equivalent with Proposition 6 of [8]. More precisely speaking, if we write the orthogonal polynomials in [8] as $Q_{\eta}^{\text{DF}}(y)$ and $P_{\eta}^{\text{DF}}(y)$, where the parameters $\alpha = \nu$, $\eta = (n_1,\ldots,n_D)$, and $a = b^\eta$, we have the correspondence

$$Q_{\xi_N}^{(\nu)}(y) = 2^{-|\eta|} Q_{\eta}^{\text{DF}} \left( \frac{y}{2} \right), \quad P_{\xi_N}^{(\nu)}(y) = 2^{|\eta|} P_{\eta}^{\text{DF}} \left( \frac{y}{2} \right),$$  \hspace{1cm} (3.15)

for $\xi_N(\cdot) = \sum_{i=1}^{N} \delta_{x_i}(\cdot) = \sum_{i=1}^{D} n_i \delta_{2y_i}(\cdot)$ and $\xi_N(\mathbb{R}_+) = N = \sum_{i=1}^{D} n_i = |\eta|$. 

We write $\xi_N(\cdot) = \sum_{i=1}^{N} \delta_{x_i}(\cdot)$ with $\mathbf{x} = (x_i)_{i=1}^{N}$ such that $0 \leq x_1 \leq x_2 \leq \cdots \leq x_N$. Then we set

$$\xi_{N,0}(\cdot) \equiv 0 \quad \text{and} \quad \xi_{N,i}(\cdot) = \sum_{k=1}^{i} \delta_{x_k}(\cdot), \quad 1 \leq i \leq N.$$ 

By this definition $\xi_{N,i}(\mathbb{R}_+) = i$, $0 \leq i \leq N$ and $\xi_{N,i}(\{x\}) \leq \xi_{N,i+1}(\{x\}), \forall x \in \mathbb{R}_+, 0 \leq i \leq N - 1$. We define

$$M_i^{(\nu,+)}(y;\xi_N) = Q_{\xi_{N,i+1}}^{(\nu)}(y), \quad M_i^{(\nu,-)}(y;\xi_N) = P_{\xi_{N,i}}^{(\nu)}(y), \quad 0 \leq i \leq N - 1.$$  \hspace{1cm} (3.16)

By the orthogonality relations (3.10), (3.12) and the above definitions, we can prove the biorthonormality

$$\int_{0}^{\infty} dy M_i^{(\nu,-)}(y;\xi_N) M_k^{(\nu,+)}(y;\xi_N) = \delta_{ik}, \quad 0 \leq i, k \leq N - 1.$$  \hspace{1cm} (3.17)
Lemma 3.3 Let $N \in \mathbb{N}, \xi_N \in \mathcal{R}^+ \text{ with } \xi_N(\mathbb{R}^+) = N$. For $0 \leq t, x, y \in \mathbb{R}^+, 0 \leq i \leq N - 1$,

$$
\int_{0}^{\infty} dt \, p^{(\nu)}(t - s, y | x) M_i^{(\nu, +)} \left( \frac{x}{s} \bigg| \frac{1}{s} \triangleleft \xi_N \right) = \left( \frac{s}{t} \right)^{i+1} M_i^{(\nu, +)} \left( \frac{y}{t} \bigg| \frac{1}{t} \triangleleft \xi_N \right) \tag{3.18}
$$

$$
\int_{0}^{\infty} dy \, M_i^{(\nu, -)} \left( \frac{y}{t} \bigg| \frac{1}{t} \triangleleft \xi_N \right) p^{(\nu)}(t - s, y | x) = \left( \frac{s}{t} \right)^{i} M_i^{(\nu, -)} \left( \frac{x}{s} \bigg| \frac{1}{s} \triangleleft \xi_N \right) \tag{3.19}
$$

Proof. By definition (3.16) and Lemma 3.2

$$
M_i^{(\nu, +)} \left( \frac{x}{s} \bigg| \frac{1}{s} \triangleleft \xi_N \right) = \frac{1}{2 \pi \sqrt{-1}} \int_{\Gamma(1/\xi_N)} dz \frac{1}{2} \frac{1}{z^{\nu/2}} \left( \frac{x}{s} \right)^{\nu/2} I_\nu \left( \sqrt{\frac{zx}{s}} \right) e^{-z/2-x/(2s)} \prod_{a \in \xi_N, i+1} \frac{1}{(z - a/s)}. \tag{3.20}
$$

By setting $z = w/s$ and by the definition (2.4) of $p^{(\nu)}$, we find

$$
M_i^{(\nu, +)} \left( \frac{x}{s} \bigg| \frac{1}{s} \triangleleft \xi_N \right) = \frac{1}{2 \pi \sqrt{-1}} \int_{\Gamma(1/\xi_N)} dw \frac{1}{2s} \left( \frac{x}{w} \right)^{\nu/2} I_\nu \left( \frac{\sqrt{wx}}{s} \right) e^{-(w+x)/(2s)} \prod_{a \in \xi_N, i+1} \frac{1}{(w - a)}. \tag{3.20}
$$

Therefore the LHS of (3.18) is

$$
\int_{\Gamma(1/\xi_N)} dw \prod_{a \in \xi_N, i+1} \frac{1}{(w - a)} \int_{0}^{\infty} dx \, p^{(\nu)}(t - s, y | x) M_i^{(\nu, +)}(s, x | w).
$$

By the Chapman-Kolmogorov equation (2.1), it equals to

$$
\int_{\Gamma(1/\xi_N)} dw \prod_{a \in \xi_N, i+1} \frac{1}{(w - a)} p^{(\nu)}(t, y | w).
$$

Then, comparing with the expression (3.20), we have (3.18).

By definition (3.16) and Lemma 3.2

$$
M_i^{(\nu, -)} \left( \frac{y}{t} \bigg| \frac{1}{t} \triangleleft \xi_N \right) = \int_{-\infty}^{0} dw \frac{1}{2} w^{\nu/2} \left( \frac{t}{y} \right)^{\nu/2} e^{y/2t} e^{-w/2} I_\nu \left( -\sqrt{w/t} \right) \prod_{a \in (1/t) \triangleleft \xi_N, i} (w - a).
$$

By setting $w = u/t$ and by the definition (2.4) of $p^{(\nu)}$, we have the expression

$$
M_i^{(\nu, -)} \left( \frac{y}{t} \bigg| \frac{1}{t} \triangleleft \xi_N \right) = t^{-i} \int_{-\infty}^{0} du \, p^{(\nu)}(-t, u | y) \prod_{a \in \xi_N, i} (u - a). \tag{3.21}
$$
Therefore the LHS of (3.19) is
\[
t^{-i} \int_{-\infty}^{0} du \prod_{a \in \xi_N,i} (u - a) \int_{0}^{\infty} dy \, p^{(r)}(-t, u|y)p^{(r)}(t - s, y|x).
\]
By the extension of the Chapman-Kolmogorov equation [25], it equals to
\[
t^{-i} \int_{-\infty}^{0} du \prod_{a \in \xi_N,i} (u - a)p^{(r)}(-s, u|x) = \left( \frac{s}{t} \right)^i M_i^{(v,-)} \left( \frac{x}{s}, \frac{1}{s} \circ \xi_N \right),
\]
where the expression (3.21) was used. This completes the proof of (3.19).

**Lemma 3.4** Let \( y = (y_i)_{i=1}^{N} \in \mathbb{W}_{\mathbb{N}}^+. \) For any \( \xi_N(\cdot) = \sum_{i=1}^{N} \delta_{x_i}(\cdot) \in \mathcal{M}^{+} \) with a labeled configuration \( x = (x_i)_{i=1}^{N} \) such that \( x_1 \leq x_2 \leq \cdots \leq x_N, \)
\[
\frac{1}{h_N(x)} \det_{1 \leq i,j \leq N} \left[ \frac{1}{2} \left( \frac{y_j}{x_i} \right)^{\nu/2} e^{-(x_i+y_j)/2 I_\nu(\sqrt{x_i y_j})} \right] = \det_{1 \leq i,j \leq N} \left[ M_{i-1}^{(v,+)}(y_j; \xi_N) \right]. \tag{3.22}
\]
Here, when some of the \( x_i \)'s coincide, we interpret the LHS using l'Hôpital's rule.

**Proof.** By the multilinearity of determinant
\[
\frac{1}{h_N(x)} \det_{1 \leq i,j \leq N} \left[ \frac{1}{2} \left( \frac{y_j}{x_i} \right)^{\nu/2} e^{-(x_i+y_j)/2 I_\nu(\sqrt{x_i y_j})} \right] = \det_{1 \leq i,j \leq N} \left[ \frac{1}{2} \left( \frac{y_j}{x_i} \right)^{\nu/2} e^{-(x_i+y_j)/2 I_\nu(\sqrt{x_i y_j})} \prod_{k=1}^{i-1} \frac{1}{x_i - x_k} \prod_{1 \leq k \leq i, k \neq \ell} (x_\ell - x_k) \right].
\]
By definition (3.16) with (3.13) of Lemma 3.2 when \( \xi_N \in \mathcal{M}_0^+, \xi_N(\mathbb{R}^+) = N, \)
\[
M_{i-1}^{(v,+)}(y; \xi_N) = \frac{1}{2\pi \sqrt{-1}} \int_{\Gamma(\xi_N,i)} dz \frac{1}{2} \left( \frac{y}{z} \right)^{\nu/2} e^{-(z+y)/2 I_\nu(\sqrt{z y})} \prod_{\nu \in \xi_N,i} (z - x) = \frac{1}{2\pi \sqrt{-1}} \int_{\Gamma(\xi_N,i)} dz \frac{1}{2} \left( \frac{y}{z} \right)^{\nu/2} e^{-(z+y)/2 I_\nu(\sqrt{z y})} \prod_{k=1}^{i-1} (z - x_k) = \sum_{\ell=1}^{i} \frac{1}{2} \left( \frac{y}{x_\ell} \right)^{\nu/2} e^{-(x_\ell+y)/2 I_\nu(\sqrt{x_\ell y})} \prod_{1 \leq k \leq i, k \neq \ell} (x_\ell - x_k). \tag{3.23}
\]
1 \( \leq i \leq N, y \in \mathbb{R}^+ \). Then (3.22) is proved for \( \xi_N \in \mathcal{M}_0^+. \) When some of the \( x_i \)'s coincide, the LHS of (3.22) is interpreted using l'Hôpital’s rule and in the RHS of (3.22) \( M_{i-1}^{(v,+)}(y_j; \xi_N) \) should be given by the first expression of (3.23). Then (3.22) is valid for any \( \xi_N \in \mathcal{M}^+ \) with \( \xi_N(\mathbb{R}^+) = N \in \mathbb{N}. \)
4 Proofs of Theorems

4.1 Proof of Theorem 2.1

In this subsection we give a proof of Theorem 2.1. Assume that $\xi_N \in \mathfrak{M}^+$ with $\xi_N(\mathbb{R}_+) = N \in \mathbb{N}$. Define

$$\phi^{(\nu, \pm)}_i(t, x; \xi_N) \equiv t^{-(i+1)} M^{(\nu, \pm)}_i \left( \frac{x}{t}; \frac{1}{t} \circ \xi_N \right), \quad (4.1)$$

$$\phi^{(\nu, -)}_i(t, x; \xi_N) \equiv t^i M^{(\nu, -)}_i \left( \frac{x}{t}; \frac{1}{t} \circ \xi_N \right), \quad (4.2)$$

$0 \leq i \leq N-1, t > 0, x \in \mathbb{R}$. From Lemma 3.3, the following relations are derived.

Lemma 4.1 For $\xi_N \in \mathfrak{M}_0$ with $\xi_N(\mathbb{R}) = N \in \mathbb{N}$, $0 \leq t_1 \leq t_2$,

$$\int_0^\infty dx_2 \phi^{(\nu, -)}_i(t_2, x_2; \xi_N) p^{(\nu)}(t_2 - t_1, x_2|x_1) = \phi^{(\nu, -)}_j(t_1, x_1; \xi_N), \quad 0 \leq i \leq N - 1, (4.3)$$

$$\int_0^\infty dx_1 \phi^{(\nu)}(t_2 - t_1, x_2|x_1) \phi^{(\nu, +)}_i(t_1, x_1; \xi_N) = \phi^{(\nu, +)}_j(t_2, x_2; \xi_N), \quad 0 \leq i \leq N - 1, (4.4)$$

$$\int_0^\infty dx_1 \int_0^\infty dx_2 \phi^{(\nu, -)}_i(t_2, x_2; \xi_N) p^{(\nu)}(t_2 - t_1, x_2|x_1) \phi^{(\nu, +)}_j(t_1, x_1; \xi_N) = \delta_{ij}, \quad 0 \leq i, j \leq N - 1. \quad (4.5)$$

Remark 3. The equation (4.5) is obtained from the combination of (4.3) and (4.4). It should be emphasize the fact that the biorthonormality (3.17) is obtained by just taking the limit $t_2 - t_1 \to 0$ in (4.5). For the purpose in this paper, the connection to the theory of multiple orthogonal polynomials [6, 7, 3, 21, 8] is not so essential. We can think that the integrals (3.20) and (3.21) define the functions $\phi^{(\nu, \pm)}_i(t, x; \xi)$ and $\phi^{(\nu, -)}_i(t, x; \xi)$ through (4.1) and (4.2), respectively. As shown in the proof of Lemma 3.3, (4.3) and (4.4) are readily obtained by applying the Chapman-Kolmogorov equation (2.1) and its extension (2.5).

By definitions (3.11), (3.16), and (4.2), we can see that $\phi^{(\nu, -)}_i(t, x; \xi_N)$ is a monic polynomial of $x$ of degree $i$, which is independent of $t$. By this fact and Lemma 3.4, the multitime probability density (1.10) is expressed as

$$p_N^{\xi} \left( t_1; \xi_1^{(1)}; \ldots; t_M, \xi_M^{(M)} \right) = \det_{1 \leq i, j \leq N} \left[ \phi^{(\nu, -)}_{i-1}(t_M, x_M^{(M)}; \xi_N) \right]$$

$$\times \prod_{m=1}^{M-1} f_N^{(\nu)}(t_{m+1} - t_m; x_{m+1}^{(m)}|x^{(m)}) \det_{1 \leq k, \ell \leq N} \left[ \phi^{(\nu, +)}_{k-1}(t_1, x_1^{(1)}; \xi_N) \right]. \quad (4.6)$$

for $\xi_N \in \mathfrak{M}^+$ with $\xi_N(\mathbb{R}_+) = N \in \mathbb{N}$. By the argument given in Section 4 in [17], the expression (4.6) with Lemma 4.1 leads to the Fredholm determinantal expression for the
generating function of multitime correlation functions \( G_{\nu}^{\xi} [\chi] \),

\[
G_{\nu}^{\xi} [\chi] = \det_{1 \leq m, n \leq M} \left[ \delta_{mn} \delta(x - y) + \tilde{S}^{m,n}(x, y; \xi_N) \chi_n(y) \right],
\]

where

\[
\tilde{S}^{m,n}(x, y; \xi_N) = S^{m,n}(x, y; \xi_N) - 1(m > n) \rho^{(\nu)}(t_m - t_n, x|y)
\]

with

\[
S^{m,n}(x, y; \xi_N) = \sum_{i=0}^{N-1} \phi_i^{(\nu,+)}(t_m, x; \xi_N) \phi_i^{(\nu,-)}(t_n, y; \xi_N)
\]

\[
= \frac{1}{t_m} \sum_{i=0}^{N-1} \left( \frac{t_n}{t_m} \right)^{i/2} M_i^{(\nu,+)} \left( \frac{x}{t_m} \circ \xi_N \right) M_i^{(\nu,-)} \left( \frac{y}{t_n} \circ \xi_N \right).
\]

Here the Fredholm determinant is defined by the following expansion

\[
\det_{1 \leq m, n \leq M} \left[ \delta_{mn} \delta(x - y) + \tilde{S}^{m,n}(x, y; \xi_N) \chi_n(y) \right]
\]

\[
= \prod_{N_1=0}^{N} \cdots \prod_{N_M=0}^{N} \prod_{m=1}^{N_m} \frac{1}{N_m!} \int_{\mathbb{R}^{N_1}} \cdots \int_{\mathbb{R}^{N_M}} dx_i^{(1)} \cdots dx_i^{(M)}
\]

\[
\times \prod_{m=1}^{M} \prod_{i=1}^{N_m} \chi_m \left( x_i^{(m)} \right) \det_{1 \leq m_1, n_1, \ldots, n_n \leq M} \left[ \tilde{S}^{m,n}(x_i^{(m)}, x_j^{(n)}; \xi_N) \right].
\]

The following invariance of finite dimensional distributions of determinantal processes will be used.

**Lemma 4.2** Let \((\Xi(t), \mathbb{P}^{\xi})\) and \((\tilde{\Xi}(t), \tilde{\mathbb{P}}^{\xi})\) be the \(\mathcal{M}^+\)-valued processes, which are determinantal with correlation kernels \(K\) and \(\tilde{K}\), respectively. If there is a function \(G(s, x)\), which is continuous with respect to \(x \in (0, \infty)\) for any fixed \(s \in [0, \infty)\), such that

\[
K(s, x; y) = \frac{G(s, x)}{G(t, y)} \tilde{K}(s, x; t, y), \quad (s, t) \in [0, \infty)^2, \quad (x, y) \in (0, \infty)^2,
\]

then

\[
(\Xi(t), \mathbb{P}^{\xi}) = (\tilde{\Xi}(t), \tilde{\mathbb{P}}^{\xi})
\]

in the sense of finite dimensional distributions.

The relation \(\text{(4.9)}\) is called the **gauge transformation** and \(\text{(4.10)}\) is said to be the **gauge invariance** of the determinantal processes \([18]\).
Proof of Theorem 2.1. Inserting the integral representations for $M^{(\nu, \pm)}$, given by (3.20) and (3.21), into (4.7), the kernel $S_{m,n}$ is written as

$$S_{m,n}(x, y; \xi_N) = \frac{1}{2\pi \sqrt{-1}} \int_{-\infty}^{0} du \int_{\Gamma_{\nu}(t_m^{-1} \circ \xi_N)} dz \ p^{(\nu)}(t_m, x|z)p^{(\nu)}(-t_n, u|y)$$

$$\times \sum_{k=0}^{N-1} \frac{\prod_{j=1}^{k}(u-a_j)}{\prod_{j=1}^{k+1}(z-a_j)},$$

where $\xi_N(\cdot) = \sum_{i=1}^{N} \delta_{a_i}(\cdot)$. For $z_1, z_2 \in \mathbb{C}$ with $z_1 \notin \{x_1, \ldots, x_N\}$, the following identity holds,

$$\sum_{k=0}^{N-1} \frac{\prod_{j=1}^{k}(z_2-x_j)}{\prod_{j=1}^{k+1}(z_1-x_j)} = \left( \prod_{j=1}^{N} \frac{z_2-x_j}{z_1-x_j} - 1 \right) \frac{1}{z_2-z_1}.$$

By this identity, we have

$$S_{m,n}(x, y; \xi_N) = \frac{1}{2\pi \sqrt{-1}} \int_{-\infty}^{0} du \int_{\Gamma_{\nu}(t_m^{-1} \circ \xi_N)} dz \ p^{(\nu)}(t_m, x|z)p^{(\nu)}(-t_n, u|y)$$

$$\times \frac{1}{u-z} \left( \prod_{j=1}^{N} \frac{u-a_j}{z-a_j} - 1 \right).$$

Note that

$$\frac{1}{2\pi \sqrt{-1}} \int_{-\infty}^{\varepsilon} du \int_{\Gamma_{\nu}(t_m^{-1} \circ \xi_N)} dz \ p^{(\nu)}(t_m, x|z)p^{(\nu)}(-t_n, u|y) \frac{1}{u-z} = 0, \quad \forall \varepsilon > 0,$$

since by definition (2.14)

$$p^{(\nu)}(t_m, x|z) = \frac{x^{\nu}}{(2t_m)^{\nu+1}} e^{-(x+z)/2t_m} \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+1)\Gamma(n+1+\nu)} \left( \frac{x}{2t_m} \right)^{n} z^{n}$$

is an entire function with respect to $z$. Then (2.16) is obtained. When $\xi_N \in \mathbb{M}_0^+$, the Cauchy integrals are performed in (2.16) and (2.17) is obtained. If we use (2.7) and (2.15), we have the equality

$$K^{\xi_N}_{s, x} (s, x; t, y) = \left( \frac{x}{y} \right)^{\nu/2} K^{\xi_N}_{s, t} (s, x; t, y), \quad (s, t) \in [0, \infty)^2, (x, y) \in (0, \infty)^2.$$

Then, by Lemma 4.2, the correlation kernel $K^{\xi_N}_{s, x}$ can be replaced by $K^{\xi_N}_{s, t}$ without changing any finite dimensional distributions. This completes the proof.
4.2 Proof of Theorem 2.2

In this subsection we give a proof of Theorem 2.2. For \( \zeta \in \mathcal{M} \), \( L > 0 \), we define

\[
M(\zeta, L) = \int_{[-L,L]\setminus\{0\}} \frac{\zeta(dx)}{x}
\]

and put

\[
M(\zeta) = \lim_{L \to \infty} M(\xi, L),
\]

if the limit finitely exists. In an earlier paper we proved the following lemma (Lemma 4.4 in [19]).

Lemma 4.3 Assume that \( \zeta \in \mathcal{M} \) satisfies the following conditions:

(C.1) there exists \( \hat{C}_0 > 0 \) such that \( |M(\zeta)| < \hat{C}_0 \),
(C.2) (i) there exist \( \hat{\alpha} \in (1, 2) \) and \( \hat{C}_1 > 0 \) such that \( M_\alpha(\zeta) \leq \hat{C}_1 \), (ii) there exist \( \hat{\beta} > 0 \) and \( \hat{C}_2 > 0 \) such that

\[
M_1(\tau_{-\alpha^2}(\zeta^{(2)})) \leq \hat{C}_2(|a| \vee 1)^{-\hat{\beta}}, \quad \forall a \in \text{supp} \, \zeta.
\]

Then there exists \( \hat{C}_3 = \hat{C}_3(\alpha, \beta, \hat{C}_0, \hat{C}_1, \hat{C}_2) > 0 \) and \( \theta \in (\hat{\alpha} \vee (2 - \hat{\beta}), 2) \) such that

\[
|\Phi_0(\zeta, a, \sqrt{-1} y)| \leq \left[ \hat{C}_3\left( (|y| \vee 1)^{\hat{\beta}} + (|a| \vee 1)^{\theta} \right) \right], \quad \forall y \in \mathbb{R}, \quad \forall a \in \text{supp} \, \zeta.
\]

Here we use the following version.

Lemma 4.4 For any \( \xi \in \mathcal{X}_0^+ \), there exist \( C_3 = C_3(\alpha, \beta, C_1, C_2) > 0 \) and \( \theta \in (\alpha \vee (1 - \beta), 1) \) such that

\[
|\Phi_0(\zeta, a, y)| \leq \exp\left[ C_3\left( (|a| \vee 1)^{\theta} + (|y| \vee 1)^{\theta} \right) \right], \quad \forall y < 0, \quad \forall a \in \text{supp} \, \zeta.
\]

Proof. First we consider the case that \( a = 0 \in \xi \). Let \( \zeta = \xi^{(1/2)} \). Then the equalities

\[
\Phi_0(\zeta, 0, z) = \prod_{x \in \xi \cap \{0\}^c} \left( 1 - \frac{z}{x} \right) = \prod_{x \in \xi \cap \{0\}^c} \left( 1 - \frac{\sqrt{z}}{\sqrt{x}} \right) \left( 1 + \frac{\sqrt{z}}{\sqrt{x}} \right)
\]

\[
= \prod_{x' \in \xi \cap \{0\}^c} \left( 1 - \frac{\sqrt{z}}{x'} \right) = \Phi_0(\zeta, 0, \sqrt{z}) \quad (4.12)
\]

hold, if the products finitely exist. Since \( M(\zeta) = 0 \) by the definition \( \zeta = \xi^{(1/2)} \), \( \zeta \) satisfies the condition (C.1) in Lemma 4.3. By assumption of the present lemma, that is, \( \xi \) satisfies the condition (C.A), the condition (C.2) is satisfied by \( \zeta \) with \( \hat{\alpha} = 2\alpha \) and \( \hat{\beta} = 2\beta \), since

\[
M_a(\zeta) = (M_{2a}(\zeta))^2 \quad \text{and} \quad M_1(\tau_{-a}(\zeta)) = M_1(\tau_{-(\sqrt{a})^2}(\zeta)) \quad a \in \text{supp} \, \zeta.
\]

Then, by Lemma 4.3, there exist \( \hat{\beta} \in (\hat{\alpha} \vee (2 - \hat{\beta}), 2) = (2\alpha \vee 2(1 - \beta), 2) \) and \( \hat{C}_3 < \infty \) such that \( \Phi_0(\zeta, 0, \sqrt{-z}) \leq \exp\left[ \hat{C}_3(|z| \vee 1)^{\hat{\beta}} \right], z > 0 \). Since (4.12) is established, it implies

\[
\Phi_0(\zeta, 0, -z) \leq \exp\left[ \hat{C}_3(|z| \vee 1)^{\hat{\beta}} \right] \quad (4.13)
\]
with \( \theta = \hat{\theta}/2 \).

Next we consider the case that \( a \in \text{supp } \xi \) and \( a \neq 0 \). For \( y < 0 \), the equality

\[
\Phi_0(\xi, a, y) = \Phi_0(\xi, 0, y) \Phi_0(\xi \cap \{0\}^c, a, 0) \left( \frac{y}{a} \right)^{\xi(\{0\})} \frac{a}{a-y}
\]

is valid. Since \( y < 0 \) is assumed

\[
\left| \left( \frac{y}{a} \right)^{\xi(\{0\})} \frac{a}{a-y} \right| = \left| \frac{a}{a-y} \right| \vee \left| \frac{y}{a-y} \right| \leq 1.
\]

On the other hand,

\[
\Phi_0(\xi \cap \{0\}^c, a, 0) = \prod_{a \in \xi \cap \{0\}^c} \left( 1 + \frac{a}{x-a} \right) \leq \exp \left\{ \int_{\mathbb{R}} (\tau_{-a}) (dx) \left| \frac{a}{|x|} \right| \right\} = \exp \left\{ |a| M_1(\tau_{-a}) \right\}.
\]

By the condition (C.A) (ii) of \( X_{a+0}^0 \), it is bounded from the above by \( \exp \{ C_2 (|a| \vee 1)^{-\beta} |a| \} = \exp \{ C_2 (|a| \vee 1)^{1-\beta} \} \). Combining this with (4.13), the proof is completed.

**Proof of Theorem 2.2.** Note that \( \xi \cap [0, L], L > 0 \) and \( \xi \) satisfy (C.A) with the same constants \( C_1, C_2 \) and indices \( \alpha, \beta \). By Lemma 4.4 we see that there exists \( C_3 > 0 \) such that

\[
|\Phi_0(\xi \cap [0, L], x', y')| \leq \exp \left\{ C_3 \left\{ (|x'| \vee 1)^{\theta} + (|y'|^{\theta} \vee 1) \right\} \right\},
\]

\( \forall L > 0, \forall x' \in \text{supp } \xi \cap [0, L], \forall y' < 0 \) with \( \theta \in (\alpha \vee (1-\beta), 1) \). Therefore, since for any \( x' \in \text{supp } \xi, y' < 0 \)

\[
\Phi_0(\xi \cap [0, L], x', y') \to \Phi_0(\xi, x', y'), \quad L \to \infty,
\]

we can apply Lebesgue’s convergence theorem to (2.18) and obtain

\[
\lim_{L \to \infty} \mathbb{K}_{\nu}^{\xi(\{0\},L)}(s, x; t, y) = \mathbb{K}_{\nu}^{\xi}(s, x; t, y).
\]

Since for any \( (s, t) \in (0, \infty)^2 \) and any compact interval \( I \subset (0, \infty) \)

\[
\sup_{x, y \in I} \left| \mathbb{K}_{\nu}^{\xi(\{0\},L)}(s, x; t, y) \right| < \infty,
\]

we can obtain the convergence of generating functions for multitime correlation functions, \( G_{\nu}^{\xi(\{0\},L)}[\chi] \to G_{\nu}^{\xi}[\chi] \), as \( L \to \infty \). It implies \( \mathbb{P}_{\nu}^{\xi(\{0\},L)} \to \mathbb{P}_{\nu}^{\xi} \) as \( L \to \infty \) in the sense of finite dimensional distributions. By virtue of the gauge invariance of determinantal processes (Lemma 2.2), the proof is completed.

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4.3 Proof of Theorem 2.3

In this subsection we give a proof of Theorem 2.3. First we prove a lemma.

**Lemma 4.5** For \( z \neq j_{\nu,i}, i \in \mathbb{N} \)

\[
\Phi^{(\nu)}(\xi_{j_{\nu,i}}, (j_{\nu,i})^2, z^2) = \frac{2j_{\nu,i}}{(j_{\nu,i})^2 - z^2} \frac{J_{\nu}(z)}{J_{\nu+1}(j_{\nu,i})},
\]  

(4.14)

**Proof.** By the expression (1.19) of \( J_{\nu}(z) \) and the definition (2.15) of \( \Pi^{(\nu)} \), we have

\[
J_{\nu}(z) = \frac{2^{-\nu}}{\Gamma(\nu + 1)} \Pi^{(\nu)}(\xi_{j_{\nu,i}}, z^2).
\]  

(4.15)

Put \( \Pi^{(\nu)'}(\xi, z^2) \equiv (d/dz) \Pi^{(\nu)}(\xi, z^2) \). Then

\[
J_{\nu}'(j_{\nu,i}) = \frac{2^{-\nu}}{\Gamma(\nu + 1)} \Pi^{(\nu)'}(\xi_{j_{\nu,i}}, j_{\nu,i}^2).
\]  

(4.16)

We see that, if \( x' \in \xi \),

\[
\frac{1}{1 - z^2/(x')^2} \Pi^{(\nu)}(\xi, z^2) = z^{\nu} \prod_{x \in \xi \cap \{x'\}^c} \left( 1 - \frac{z^2}{x^2} \right).
\]

Since

\[
\Pi^{(\nu)'}(\xi, z^2) = \nu \frac{1}{z} \Pi^{(\nu)}(\xi, z^2) + z\nu \int_{\mathbb{R}^+} \xi(dx) \left( -\frac{2z}{x^2} \right) \prod_{y \in \xi \cap \{x'\}^c} \left( 1 - \frac{z^2}{y^2} \right),
\]

if \( x' \in \xi \),

\[
\Pi^{(\nu)'}(\xi, (x')^2) = -\frac{2}{x'} (x')^{\nu} \prod_{x \in \xi \cap \{x'\}^c} \left( 1 - \frac{(x')^2}{x^2} \right).
\]

Therefore, for \( x' \in \xi \),

\[
\frac{1}{1 - z^2/(x')^2} \Pi^{(\nu)}(\xi, z^2) = -\frac{x'}{2} \left( \frac{z}{x'} \right)^{\nu} \prod_{x \in \xi \cap \{x'\}^c} \frac{1 - z^2/x^2}{1 - (x')^2/x^2}.
\]  

(4.17)

Since

\[
1 - \frac{z^2}{x^2 - a^2} = \frac{x^2 - z^2}{x^2 - a^2} = \frac{1 - z^2/x^2}{1 - a^2/x^2}
\]

for \( x \neq a \), combining (4.17) with the definition (2.15) gives

\[
\Phi^{(\nu)}(\xi, (x')^2, z^2) = \left( \frac{z}{x'} \right)^{\nu} \prod_{x \in \xi \cap \{x'\}^c} \frac{1 - z^2/x^2}{1 - (x')^2/x^2}
\]

\[
= \frac{2x'}{z^2 - (x')^2} \Pi^{(\nu)'}(\xi, (x')^2), \quad x' \in \xi.
\]  

(4.18)
Setting \( \xi = \xi J \) and \( x' = j_{\nu,i} \) in (4.18), we have

\[
\Phi^{(\nu)}(\xi_J^{(2)}, (j_{\nu,i})^2, z^2) = \frac{2j_{\nu,i}}{z^2 - (j_{\nu,i})^2} J_{\nu}(z),
\]

by (4.15) and (4.16). If we use (3.6), (4.14) is obtained. 

**Proof of Theorem 2.3.** We set \( \xi = \xi_J^{(2)} \) in \( K_{\xi_J} \) and obtain the kernel

\[
K_{\nu}(s, x; t, y) = \sum_{i=1}^{\infty} \int_{-\infty}^{0} dz p_{\nu}(s, x | j_{\nu,i}) \Phi^{(\nu)}(\xi_J^{(2)}, (j_{\nu,i})^2, z^2) p_{\nu}(-t, z | y)
- \mathbf{1}(s > t) p_{\nu}(s - t, x | y).
\]

By Lemmas 3.1 and 4.5,

\[
\Phi^{(\nu)}(\xi_J^{(2)}, (j_{\nu,i})^2, z^2) = \frac{2j_{\nu,i}}{(j_{\nu,i})^2 - z} J_{\nu+1}(j_{\nu,i}) = \frac{1}{(j_{\nu+1}(j_{\nu,i}))^2} \int_{0}^{1} du J_{\nu}(\sqrt{uw})J_{\nu}(\sqrt{uj_{\nu,i}}).
\]

By the integral formula (3.3), the first term of the RHS of (4.19) equals

\[
\sum_{i=1}^{\infty} p_{\nu}(s, x | (j_{\nu,i})^2) \frac{1}{(j_{\nu+1}(j_{\nu,i}))^2} \int_{0}^{1} du e^{ut/2} J_{\nu}(\sqrt{uy})J_{\nu}(2\sqrt{wx})\sum_{i=1}^{\infty} J_{\nu}(2\sqrt{wj_{\nu,i}})J_{\nu}(\sqrt{uj_{\nu,i}})(j_{\nu+1}(j_{\nu,i}))^2.
\]

Since \( s > 0 \), we can use the expression (2.8) for \( p_{\nu}(s, x | (j_{\nu,i})^2) \) and the above is written as

\[
\int_{0}^{1} du \int_{0}^{\infty} dw e^{ut/2 - 2ws} J_{\nu}(\sqrt{uy})J_{\nu}(2\sqrt{wx}) \sum_{i=1}^{\infty} \frac{J_{\nu}(2\sqrt{wj_{\nu,i}})J_{\nu}(\sqrt{uj_{\nu,i}})}{(j_{\nu+1}(j_{\nu,i}))^2}.
\]

From the completeness (3.7) of the orthonormal basis \( \{ J_{\nu}(j_{\nu,i} \sqrt{x})/J_{\nu+1}(j_{\nu,i}), i \in \mathbb{N} \} \) in \( L^2(0, 1) \), the above gives

\[
K_{\nu}(s, x; t, y) = K_{\nu}(s, x; t, y) + R(s, x; t, y)
\]

with the extended Bessel kernel (1.21) and

\[
R(s, x; t, y) = \int_{0}^{1} du \int_{1}^{\infty} dw e^{ut/2 - 2ws} J_{\nu}(\sqrt{uy})J_{\nu}(2\sqrt{wx})
\times \sum_{i=1}^{\infty} \frac{J_{\nu}(2\sqrt{wj_{\nu,i}})J_{\nu}(\sqrt{uj_{\nu,i}})}{(j_{\nu+1}(j_{\nu,i}))^2}.
\]

Since for any fixed \( s, t > 0 \)

\[
|R(s + \theta, x : t + \theta, y)| \to 0 \quad \text{as} \quad \theta \to \infty
\]

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uniformly on any compact subset of \((x, y) \in \mathbb{R}^2_+\),

\[
K_{J_n}(s + \theta, x; t + \theta) \to K_{J_n}(s, x; t, y)
\quad \text{as} \quad \theta \to \infty
\]

holds in the same sense. Hence we obtain (2.22). This completes the proof. \(\blacksquare\)

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**References**

[1] Altland, A., Zirnbauer, M. R.: Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structure. Phys. Rev. B **55**, 1142-1161 (1997)

[2] Andrews, G. E., Askey, R., Roy, R.: Special functions. Cambridge University Press, Cambridge (1999)

[3] Bleher, P. M., Kuijlaars, A. B.: Integral representations for multiple Hermite and multiple Laguerre polynomials. Ann. Inst. Fourier. **55**, 2001-2014 (2005)

[4] Borodin, A. N., Salminen, P.: Handbook of Brownian Motion – Facts and Formulae. 2nd edn. Birkhäuser, Basel (2002)

[5] Bru, M. F.: Wishart process. J. Theoret. Probab. **3**, 725-751 (1991)

[6] Coussement, E., Van Assche, W.: Asymptotics of multiple orthogonal polynomials associated with the modified Bessel functions of the first kind. J. Comput. Appl. Math. **153**, 141-149 (2003)

[7] Coussement, E., Van Assche, W.: Multiple orthogonal polynomials associated with the modified Bessel function of the first kind. Const. Approx. **19**, 237-263 (2003)

[8] Desrosiers, P., Forrester, P. J.: A note on biorthogonal ensembles. J. Approx. Theory **152**, 167-187 (2008)

[9] Forrester, P. J.: Log-gases and Random Matrices. London Mathematical Society Monographs, Princeton University Press, Princeton (2010)

[10] Forrester, P. J., Nagao, T., Honner, G.: Correlations for the orthogonal-unitary and symplectic-unitary transitions at the hard and soft edges. Nucl. Phys. **B553[PM]**, 601-643 (1999)

[11] Grabiner, D. J.: Brownian motion in a Weyl chamber, non-colliding particles, and random matrices. Ann. Inst. H. Poincaré, Probab. Stat. **35**, 177-204 (1999)

[12] Hough, J. B., Krishnapur, M., Peres, Y., Virág, B.: Zeros of Gaussian Analytic Functions and Determinantal Point Processes. University Lecture Series, Amer. Math. Soc., Providence (2009)

25
[13] Ismail, M. E. H.: Classical and Quantum Orthogonal Polynomials in One Variable. Cambridge University Press, Cambridge (2005)

[14] Karlin, S., McGregor, J.: Coincidence probabilities. Pacific J. Math. 9, 1141-1164 (1959)

[15] Katori, M., Tanemura, H.: Symmetry of matrix-valued stochastic processes and noncolliding diffusion particle systems. J. Math. Phys. 45, 3058-3085 (2004)

[16] Katori, M., Tanemura, H.: Infinite systems of noncolliding generalized meanders and Riemann-Liouville differintegrals. Probab. Theory Relat. Fields 138, 113-156 (2007)

[17] Katori, M., Tanemura, H.: Noncolliding Brownian motion and determinantal processes. J. Stat. Phys. 129, 1233-1277 (2007)

[18] Katori, M., Tanemura, H.: Zeros of Airy function and relaxation process. J. Stat. Phys. 136, 1177-1204 (2009)

[19] Katori, M., Tanemura, H.: Non-equilibrium dynamics of Dyson’s model with an infinite number of particles. Commun. Math. Phys. 293, 469-497 (2010)

[20] König, W., O’Connell, N.: Eigenvalues of the Laguerre process as non-colliding squared Bessel process. Elec. Commun. Probab. 6, 107-114 (2001)

[21] Kuijlaars, A. B., Martínez-Finkelshtein, A., Wielonsky, F.: Non-intersecting squared Bessel paths and multiple orthogonal polynomials for modified Bessel weight. Commun. Math. Phys. 286, 217-275 (2009)

[22] Levin, B. Ya.: Lectures on Entire Functions, Translations of Mathematical Monographs, vol.150. Amer. Math. Soc., Providence (1996)

[23] Mehta, M. L.: Random Matrices, 3rd edn. Elsevier, Amsterdam (2004)

[24] Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion. 3rd edn. Springer, New York (1998)

[25] Shirai, T., Takahashi, Y.: Random point fields associated with certain Fredholm determinants I: fermion, Poisson and boson point process. J. Funct. Anal. 205, 414-463 (2003)

[26] Soshnikov, A.: Determinantal random point fields. Russian Math. Surveys 55, 923-975 (2000)

[27] Tracy, C. A., Widom, H.: Differential equations for Dyson processes. Commun. Math. Phys. 252, 7-41 (2004)

[28] Tracy, C. A., Widom, H.: Nonintersecting Brownian excursions, Ann. Appl. Probab. 17, 953-979 (2007)

[29] Watson, G. N.: A Treatise on the Theory of Bessel Functions. 2nd edn. Cambridge University Press, Cambridge (1944)