Comparative performance of novel nodal-to-edge finite elements over conventional nodal element for electromagnetic analysis

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ABSTRACT
In nodal-based finite element method (FEM), degrees of freedom are associated with the nodes of the element whereas, for edge FEM, degrees of freedom are assigned to the edges of the element. Edge element is constructed based on Whitney spaces. Nodal elements impose both tangential and normal continuity of vector or scalar fields across interface boundaries. But in edge elements, only tangential continuity is imposed across interface boundaries, which is consistent with electromagnetic field problems. Therefore the required continuities in the electromagnetic analysis are directly obtained with edge elements whereas in nodal elements they are attained through potential formulations. Hence, while using edge elements, field variables are directly calculated but with nodal elements, post-processing is required to obtain the field variables from the potentials. Here, we present the finite element formulations with the edge element as well as with nodal elements. Thereafter, we have demonstrated the relative performances of different nodal and edge elements through a series of examples. All possible complexities like curved boundaries, non-convex domains, sharp corners, non-homogeneous domains have been addressed in those examples. The robustness of edge elements in predicting the singular eigen values for the domains with sharp edges and corners is evident in the analysis. A better coarse mesh accuracy has been observed for all the edge elements as compared to the respective nodal elements. Edge elements are also not susceptible to mesh distortion.

1. Introduction
Several numerical methods have been developed to solve the partial differential equations (PDE) in studying various engineering and scientific applications. Among the mesh-based methods, Finite Element Method (FEM), Finite Domain Time Domain (FDTD) have gained a wide popularity due to their application to complex structures. However, there are other powerful alternative methods such as meshfree methods and Isogeometric Analysis (IGA). In [1], authors used Deep Neural Networks (DNNs) as a function approximation.
machines to solve PDE in computational mechanics. In the literature, a wide variety of electromagnetic wave problems have been solved by using several computational techniques such as FEM [2], FDTD [3], Method of Moments [4], meshfree/meshless methods [5], nonlocal operator method (NOM) [6], Isogeometric analysis [7] etc. Some of the applications of electromagnetic waves includes wireless satellite communication and optics, radar, remote sensing, bioelectromagnetics, electromagnetic forming [8–10], etc. In the transmission of electromagnetic waves, waveguide structures are used to transmit the waves with minimum losses of energy. Eigen mode of the wave propagation is required to study the wave propagation in waveguides and eigenvalues to determine the resonance frequencies in cavities.

FEM has been widely used in solving various electromagnetic field problems, such as eigenvalue analysis, scattering, and radiation analysis of interior and exterior domains, and so on [2,3]. The domain of interest can be discretized with finite elements which include either nodal or edge elements to implement the FEM. However, when nodal elements are used, eigenvalue analysis shows spurious modes during eigen analysis of some specific domains [11–15]. Vector field problems require a special type of formulations due to their special continuity requirements at material interfaces. In [16–20], the potential formulation is used in nodal framework. But this potential formulation failed in eliminating the problem of spurious eigenvalues in sharp corner objects. In [21], mixed finite element formulation was adopted and this was successful in the case of sharp corner objects, inhomogeneous domains in 2D. However, this method failed in the case of three-dimensional curved objects. In potential formulation, field variables cannot be obtained directly and post-processing is required. However, this drawback is overcome by edge element as it provides the required continuity automatically in its formulation. Also, because of proper satisfaction of the continuity requirement, singular eigenvalues are also predicted accurately for domains with sharp corners and edges. This provides the theoretical motivation to adopt edge element.

Whitney presented a revolutionary method to address the aforementioned limitations in the 1980s by using employed edge elements, in which degrees of freedom are assigned to the edges of the finite element rather than the nodes. These elements have been constructed using curl-conforming bases. So, these elements possess tangential continuity and normal discontinuity at material interfaces. In [22], Webb mentioned various important properties of edge elements. The theoretical concept, properties, and development of edge elements were published in [14,23]. The construction of higher order edge elements can be done in two different ways, namely, Hierarchical and Interpolatory. Different higher order edge elements were constructed using a hierarchical approach in [24–27]. In [28–30], various higher order interpolatory elements were developed and used to analyze various field problems. In hierarchical type of edge elements, within the same discretized domain both h and p-refinements are possible, whereas in interpolatory type, only p-refinement is allowed. These vector elements were used in eigenvalue analysis for various domains in [12,13,25,31–36]. In [34,35], authors presented a novel conversion algorithm that converts the nodal mesh data to edge element data. In [37], the author presented a detailed description of the conversion technique for different order edge elements.

We have presented the article in the following manner. In Section 2, mathematical formulation of Maxwell’s electromagnetic wave equation, variational and FEM formulation in both nodal and edge element is given. The relative performance of nodal and edge elements
has been compared using benchmark examples in Section 3. The effect of mesh distortion on edge elements along with the computational cost in numerical analysis has also been presented in Section 3. We have concluded in Section 4.

2. Mathematical formulation

2.1. Maxwell’s equations in electrodynamics

The governing differential equations for electromagnetic analysis are Maxwell’s equations, given in the strong form as [38]

\[
\begin{align*}
\frac{\partial B}{\partial t} + \nabla \times E &= 0, \\
\nabla \cdot B &= 0, \\
\frac{\partial D}{\partial t} - \nabla \times H &= -j, \\
\nabla \cdot D &= \rho,
\end{align*}
\]

where the electric and magnetic fields are given by \( E \) and \( H \), the electric displacement (electric flux) is \( D \), the magnetic induction (magnetic flux) is \( B \), the charge density is \( \rho \), and the current density is \( j \). The following constitutive relations complement the above governing equations

\[
\begin{align*}
D &= \epsilon E, \\
B &= \mu H,
\end{align*}
\]

where \( \mu \) and \( \epsilon \) are the magnetic permeability and electric permittivity, respectively. Considering that \( \epsilon \) and \( \mu \) are independent of time and substituting the constitutive relations into Equations (1a), (1c) and (1d), and also after eliminating \( H \) we get

\[
\frac{\epsilon}{\mu} \frac{\partial^2 E}{\partial t^2} + \frac{\partial j}{\partial t} + \nabla \times \left( \frac{1}{\mu} \nabla \times E \right) = 0.
\]

We get the compatibility condition from Equations (1c), (2a) and (1d) as

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0.
\]

We have the boundary condition as on conducting boundary, \( E \times n = 0 \). Also, across the material interface, both \( E \times n \) and \( H \times n \) are continuous as there is no impressed surface currents.

Introducing the relative permeability and relative permittivity \( \mu_r := \mu / \mu_0 \) and \( \epsilon_r := \epsilon / \epsilon_0 \), where \( \mu_0 \) and \( \epsilon_0 \) are the permeability and permittivity for the vacuum, Equation (3) can be written in the frequency domain as

\[
\nabla \times \left( \frac{1}{\mu_r} \nabla \times E \right) - k_0^2 \epsilon_r E = -i \omega \mu_0 j.
\]
where \( k_0 = \omega / c \) is the wave number, \( c = 1 / \sqrt{\varepsilon_0 \mu_0} \), and \( i = \sqrt{-1} \). Considering \( j \) to be zero, Equation (4) becomes

\[
\nabla \times \left( \frac{1}{\mu_r} \nabla \times E \right) = k_0^2 \varepsilon_r E.
\]

which governs the eigenvalue problem.

### 2.2. Variational statement in the nodal framework

For homogeneous domains, we get wrong multiplicities of eigenvalues and for inhomogeneous domains, we get spurious values using regularized formulations in the nodal framework [39]. It is partially overcome in the regularized potential formulation [40], which is fairly robust in all convex domains, homogeneous or inhomogeneous. Only, it fails to find the singular eigenvalues for a non-convex domain, owing to the penalty term.

Replacing \( E \) by \( A + \nabla \phi \) in Equation (5), we have

\[
\nabla \times \left( \frac{1}{\mu_r} \nabla \times A \right) = k_0^2 \varepsilon_r A + k_0^2 \varepsilon_r \nabla \phi,
\]

where \( \phi \) and \( A \) are scalar and vector potentials. We get the required variational statement after adding the penalty term [40] as

\[
\int_{\Omega} \frac{1}{\mu_r} (\nabla \times A_\delta) \cdot (\nabla \times A) \, d\Omega + \int_{\Omega} \frac{1}{\varepsilon_r \mu_r} (\nabla \cdot A_\delta) [\nabla \cdot (\varepsilon_r A)] \, d\Omega
\]

\[
= k_0^2 \int_{\Omega} \varepsilon_r A_\delta \cdot A \, d\Omega + k_0^2 \int_{\Omega} \varepsilon_r A_\delta \cdot \nabla \phi.
\]

Multiplying Equation (1d) by the variation \( \phi_\delta \), replacing \( E \) by \( A + \nabla \phi \), and considering no charge for eigenanalysis we have

\[
k_0^2 \int_{\Omega} \nabla \phi_\delta \cdot (\varepsilon_r A) \, d\Omega + k_0^2 \int_{\Omega} \nabla \phi_\delta \cdot (\varepsilon_r \nabla \phi) \, d\Omega = 0
\]

On parts of the boundary where \( E \times n = 0 \), we specify \( \phi \) and \( A \times n \) to be zero.

### 2.3. FEM formulation in nodal framework

The vector potential \( A \) and its variation \( A_\delta \) are discretized as

\[
A = N\hat{A},
\]

\[
A_\delta = N\hat{A}_\delta,
\]

leading to

\[
\nabla \times A = B\hat{A}, \quad \nabla \cdot A = B_p\hat{A},
\]

\[
\nabla \times A_\delta = B\hat{A}_\delta, \quad \nabla \cdot A_\delta = B_p\hat{A}_\delta.
\]
Similarly, for the scalar potential $\phi$, we have
\[
\phi = N_\phi \hat{\phi}, \quad \nabla \phi = B_\phi \hat{\phi},
\]
\[
\phi_\delta = N_\phi \hat{\phi}_\delta, \quad \nabla \phi_\delta = B_\phi \hat{\phi}_\delta.
\]
where
\[
N = \begin{bmatrix}
N_1 & 0 & 0 & N_2 & 0 & 0 & \ldots \\
0 & N_1 & 0 & 0 & N_2 & 0 & \ldots \\
0 & 0 & N_1 & 0 & 0 & N_2 & \ldots
\end{bmatrix}, \quad N_\phi = \begin{bmatrix} N_1 & N_2 & N_3 & \ldots \end{bmatrix},
\]
\[
B = \begin{bmatrix}
0 & -\frac{\partial N_1}{\partial z} & -\frac{\partial N_1}{\partial y} & 0 & -\frac{\partial N_2}{\partial z} & -\frac{\partial N_2}{\partial y} & \ldots \\
\frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_2}{\partial x} & \ldots \\
\frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & 0 & -\frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & 0 & \ldots
\end{bmatrix},
\]
\[
B_p = \begin{bmatrix}
\frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_3}{\partial y} & \ldots \\
\frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \ldots \\
\frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial z} & \frac{\partial N_3}{\partial y} & \ldots
\end{bmatrix},
\]
\[
B_\phi = \begin{bmatrix}
\frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \ldots \\
\frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \ldots \\
\frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial z} & \frac{\partial N_3}{\partial z} & \ldots
\end{bmatrix}.
\]

The discretized forms of Equations (7) and (8) are given by
\[
\begin{bmatrix} K_{AA} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{A} \\ \hat{\phi} \end{bmatrix} = k_0^2 \begin{bmatrix} M_{AA} & M_{A\phi} \\ M_{\phi A} & M_{\phi\phi} \end{bmatrix} \begin{bmatrix} \hat{A} \\ \hat{\phi} \end{bmatrix},
\]
where
\[
K_{AA} = \int_{\Omega} \frac{1}{\mu_r} \left[ B^T B + B_p^T B_p \right] d\Omega, \quad (10a)
\]
\[
M_{AA} = \int_{\Omega} \epsilon_r N^T N d\Omega, \quad (10b)
\]
\[
M_{A\phi} = \int_{\Omega} \epsilon_r N^T B_\phi d\Omega, \quad (10c)
\]
\[
M_{\phi A} = \int_{\Omega} \epsilon_r B_\phi^T N d\Omega, \quad (10d)
\]
\[
M_{\phi\phi} = \int_{\Omega} \epsilon_r B_\phi^T B_\phi d\Omega. \quad (10e)
\]
2.4. **FEM formulation in edge element framework**

The variational formulation of Equation (5) can be derived as

\[
\int_{\Omega} \frac{1}{\mu_{r}} (\nabla \times \mathbf{E}) \cdot (\nabla \times \mathbf{E}) \, d\Omega = k_{0}^{2} \int_{\Omega} \epsilon_{r} \mathbf{E} \cdot \mathbf{E} \, d\Omega. \tag{11}
\]

where the boundary conditions \( \mathbf{E} \times \mathbf{n} \) and \( \mathbf{H} \times \mathbf{n} \) are specified on the surfaces \( \Gamma_{e} \) and \( \Gamma_{h} \) of the domain respectively. For the conducting boundary, we have \( \mathbf{E} \times \mathbf{n} = \mathbf{0} \) and for eigen analysis, we have \( \mathbf{H} = \mathbf{0} \).

We discretize the fields and their variations in Equation (11) as

\[
\mathbf{E} = \mathbf{V}_{\hat{E}}, \quad \mathbf{E}_{\delta} = \mathbf{V}_{\hat{E}}_{\delta},
\]

\[
\nabla \times \mathbf{E} = \mathbf{B}_{\hat{E}}, \quad \nabla \times \mathbf{E}_{\delta} = \mathbf{B}_{\hat{E}}_{\delta},
\]

where \( \hat{E} \) is the values of \( \mathbf{E} \) at different edges, \( \hat{E}_{\delta} \) denote the respective variation of \( \hat{E} \). Edge shape functions matrix, \( \mathbf{V} \) and \( \mathbf{B} \)-matrix are given as

\[
\mathbf{V} = \begin{bmatrix} v_{1x} & v_{2x} & \ldots \\ v_{1y} & v_{2y} & \ldots \\ v_{1z} & v_{2z} & \ldots \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \frac{\partial v_{1z}}{\partial y} - \frac{\partial v_{1y}}{\partial z} & \frac{\partial v_{2z}}{\partial y} - \frac{\partial v_{2y}}{\partial z} & \ldots \\ \frac{\partial v_{1x}}{\partial z} - \frac{\partial v_{1z}}{\partial x} & \frac{\partial v_{2x}}{\partial z} - \frac{\partial v_{2z}}{\partial x} & \ldots \\ \frac{\partial v_{1y}}{\partial x} - \frac{\partial v_{1x}}{\partial y} & \frac{\partial v_{2y}}{\partial x} - \frac{\partial v_{2x}}{\partial y} & \ldots \end{bmatrix}.
\]

where \( v_{x}, v_{y} \) and \( v_{z} \) are the \( x, y, z \) components of each edge shape function.

After substituting the above discretizations into Equation (11) and using the arbitrariness of variations we get

\[
\mathbf{K} \mathbf{\hat{E}} = k_{0}^{2} \mathbf{M} \mathbf{\hat{E}}, \tag{12}
\]

where

\[
\mathbf{K} = \int_{\Omega} \frac{1}{\mu_{r}} \mathbf{B}^{T} \mathbf{B} \, d\Omega, \tag{13a}
\]

\[
\mathbf{M} = \int_{\Omega} \epsilon_{r} \mathbf{V}^{T} \mathbf{V} \, d\Omega \tag{13b}
\]

2.5. **Calculation of \( \nabla \xi, \nabla \eta \)**

\( \nabla \xi \) and \( \nabla \eta \) appear in different shape functions of various edge elements, hence we have to understand how they are found from the components of inverse Jacobian. We know that

\[
\begin{bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{bmatrix} = J \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} \implies \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial f}{\partial \xi} \\ \frac{\partial f}{\partial \eta} \end{bmatrix}, \tag{14}
\]
where Jacobian $J$ can be written as

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix},$$

and assume $J^{-1} = \Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix}$.

In two dimensions, natural coordinates are $\xi$ and $\eta$. For $f = \xi$ in Equation (14) we replace $\frac{\partial f}{\partial \xi} = 1$, $\frac{\partial f}{\partial \eta} = 0$.

$$\begin{bmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \xi}{\partial y} \end{bmatrix} = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \Gamma_{11} \\ \Gamma_{21} \end{bmatrix}.$$ 

Similarly, for $f = \eta$ we get

$$\begin{bmatrix} \frac{\partial \eta}{\partial x} \\ \frac{\partial \eta}{\partial y} \end{bmatrix} = \begin{bmatrix} \Gamma_{12} \\ \Gamma_{22} \end{bmatrix}.$$

### 2.6. Different edge elements

Different nodal elements used in this work are well known in the literature. We denote 4-node quadrilateral elements by Q4, 9-node quadrilateral elements by Q9, and 6-node triangular elements by T6. We are presenting different edge elements in this section. These elements include 4-edge quadrilateral, 12-edge quadrilateral and 8-edge triangular elements.

#### 2.6.1. Four-edge quadrilateral element

Figure 1(a) shows the quadrilateral edge element with four edges. The arrow directions are representing positive convention directions along those edges. We denote this element by EQ4. Four-edge shape functions $v_1$, $v_2$, $v_3$, and $v_4$ are given as

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} \frac{l_1}{4}(1-\eta)\nabla \xi \\ \frac{l_2}{4}(1+\eta)\nabla \xi \\ \frac{l_3}{4}(1-\xi)\nabla \eta \\ \frac{l_4}{4}(1+\xi)\nabla \eta \end{bmatrix},$$

where $l_1$, $l_2$, $l_3$, and $l_4$ are lengths of edges 1, 2, 3, and 4 respectively.

#### 2.6.2. Twelve-edge quadrilateral element

Figure 1(b) shows the higher order quadrilateral element with 12 edges. This element is denoted by EQ12. $v_1$, $v_2$, ..., $v_{11}$, and $v_{12}$ are the edge shape functions of edges 1 to 12.
respectively. These shape functions are given as

\[
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
v_5 \\
v_6 \\
v_7 \\
v_8 \\
v_9 \\
v_{10} \\
v_{11} \\
v_{12}
\end{bmatrix} = \begin{bmatrix}
\frac{-l_1}{2} \eta(\eta - 1)(\xi - 0.5) \nabla \xi \\
\frac{l_2}{2} \eta(\eta - 1)(\xi + 0.5) \nabla \xi \\
\frac{l_3}{2}(\eta^2 - 1)(\xi - 0.5) \nabla \xi \\
-\frac{l_4}{2}(\eta^2 - 1)(\xi + 0.5) \nabla \xi \\
-\frac{l_5}{2} \eta(\eta + 1)(\xi - 0.5) \nabla \xi \\
\frac{l_6}{2} \eta(\eta + 1)(\xi + 0.5) \nabla \xi \\
-\frac{l_7}{2} \xi(\xi - 1)(\eta - 0.5) \nabla \eta \\
\frac{l_8}{2}(\xi^2 - 1)(\eta - 0.5) \nabla \eta \\
-\frac{l_9}{2} \xi(\xi + 1)(\eta - 0.5) \nabla \eta \\
\frac{l_{10}}{2} \xi(\xi - 1)(\eta + 0.5) \nabla \eta \\
-\frac{l_{11}}{2}(\xi^2 - 1)(\eta + 0.5) \nabla \eta \\
\frac{l_{12}}{2} \xi(\xi + 1)(\eta + 0.5) \nabla \eta
\end{bmatrix}
\]

where \(l_1, l_2, \ldots, l_{11},\) and \(l_{12}\) are lengths of edges 1, 2, \ldots, 11, and 12 respectively.

**2.6.3. Eight-edge triangular element**

Figure 1(c) shows the triangular edge element with eight edges. ET8 is used to denote this edge element in this work. \(v_1, v_2, \ldots, v_7,\) and \(v_8\) are the edge shape functions of edges 1,
2, ..., 7, and 8 edges respectively. These edge shape functions can be given as

\[
\begin{aligned}
\mathbf{v} &= \left\{ 
\begin{array}{c}
v_1 \\
v_2 \\
v_3 \\
v_4 \\
v_5 \\
v_6 \\
v_7 \\
v_8 \\
\end{array}
\right. \\
&= \left\{ 
\begin{array}{c}
l_1(4\xi - 1)(\xi \nabla \eta - \eta \nabla \xi) \\
l_2(4\eta - 1)(\xi \nabla \eta - \eta \nabla \xi) \\
l_3(4\eta - 1)(\eta \nabla \alpha - \alpha \nabla \eta) \\
l_4(4\alpha - 1)(\eta \nabla \alpha - \alpha \nabla \eta) \\
l_5(4\alpha - 1)(\alpha \nabla \xi - \xi \nabla \alpha) \\
l_6(4\xi - 1)(\alpha \nabla \xi - \xi \nabla \alpha) \\
4l_7\eta(\alpha \nabla \xi - \xi \nabla \alpha) \\
4l_8\xi(\eta \nabla \alpha - \alpha \nabla \eta)
\end{array}
\right. \\
\end{aligned}
\]

where \(l_1, l_2, \ldots, l_7,\) and \(l_8\) are lengths of edges 1, 2, ..., 7, and 8 respectively.

Partial derivatives of edge shape functions \(v_1, v_2,\) etc. with respect to \(\xi\) and \(\eta\) i.e. \(\frac{\partial v_1}{\partial \xi}, \frac{\partial v_1}{\partial \eta}, \ldots\) can be found using Mathematica [41] for EQ4 and EQ12 whereas for ET8 they can be obtained by Finite difference method (FDM).

\[
\frac{\partial v_1}{\partial \xi} = \lim_{\Delta \xi \to 0} \frac{v_1(\xi + \Delta \xi, \eta) - v_1(\xi, \eta)}{\Delta \xi} 
\]

(15)

\[
\frac{\partial v_1}{\partial \eta} = \lim_{\Delta \eta \to 0} \frac{v_1(\xi, \eta + \Delta \eta) - v_1(\xi, \eta)}{\Delta \eta} 
\]

(16)

3. Numerical examples

3.1. Comparative analysis of mesh convergence study between nodal elements and edge elements

We have done some comparative performance study among three nodal elements Q4, Q9, and T6 and three edge elements EQ4, EQ12, and ET8 through several standard benchmark examples. For all the problems discussed in this section, we have assumed \(\epsilon_r = \mu_r = 1.0.\)

In the following examples, we have compared edge and nodal elements with respect to a certain error percentage with analytical benchmark values. In that bar-diagram comparison, each bar represents the minimum no. of required FDOF for that particular element to attain an error percentage less than the predefined scale.

There are black crosses in some bar diagrams which signify that it is not possible to generate less than 10% error with that element with available computational resources. We have mentioned best possible result for those cases. Such situations mostly occur with nodal element which further establish the better performance of edge elements.

3.1.1. Square domain

A square domain with a side length of \(\pi\) is considered. The square domain’s sides/boundaries are all perfectly conducting. Analysis data of different nodal and edge elements, like no. of free degree of freedom (FDOF), i.e. total no. of equations, is presented in Table 1.
Table 1. Analysis data of different nodal and edge elements for the square domain problem.

| Nodal element | No. of elements | No. of FDOF/equations | Edge element | No. of elements | No. of FDOF/equations |
|---------------|-----------------|-----------------------|--------------|-----------------|-----------------------|
| Element type  |                 |                       |              |                 |                       |
| Q4            | 64              | 231                   | EQ4          | 196             | 364                   |
| T6            | 98              | 663                   | ET8          | 112             | 530                   |
| Q9            | 64              | 855                   | EQ12         | 100             | 760                   |

Table 2. $k_0^2$ on the square domain for different elements.

| Analytical Benchmark | Nodal element | Edge element |
|----------------------|---------------|--------------|
|                      | Q4            | Q9           | T6           | EQ4           | EQ12         | ET8          |
| 1                    | 1.012916      | 0.999843     | 0.999919     | 1.004203      | 1.000013    | 0.999990     |
| 2                    | 2.025832      | 1.999264     | 2.000128     | 2.008407      | 2.000027    | 2.000150     |
| 4                    | 4.209548      | 3.998554     | 4.000663     | 4.067583      | 4.000850    | 3.999844     |
| 5                    | 5.222465      | 4.997901     | 5.003591     | 5.071786      | 5.000863    | 5.000285     |
| 8                    | 8.419096      | 7.966779     | 8.033095     | 8.135166      | 8.001700    | 8.008889     |
| 9                    | 10.080293     | 9.007072     | 9.024805     | 9.344778      | 9.009435    | 8.998040     |
| 10                   | 11.093208     | 10.002567    | 10.038396    | 10.348982     | 10.009449   | 10.003717    |
| 13                   | 14.289839     | 13.015767    | 13.088936    | 13.412361     | 13.010285   | 13.014842    |
| 16                   | 19.453669     | 16.064853    | 16.145861    | 17.100354     | 16.051302   | 15.989497    |
| Number of computed zeros | –             | 77           | 245          | 221            | 45           | 121          | 65            |

The square of eigenvalues for each type of element is given in Table 2 along with analytical results reported in [12]. All of these elements have produced correct eigenvalues along with the correct multiplicities. For all the elements the first non-zero eigenvalue has occurred after a certain number of zero eigen values at the machine precision level which signifies the approximation of null space. We have noticed a significant difference between nodal and edge elements in the mesh convergence analysis. With edge elements, we have found a far better coarse mesh accuracy. This fact is presented in bar diagrams in Figure 2. We have compared Q9 and EQ12 in Figure 2(a) on a scale of less than 1% error. Q4 and EQ4 elements are compared for less than 6% error in Figure 2(b). A scale of less than 0.2% error is chosen in Figure 2(c) to compare T6 and ET8 elements. From Figure 2, we can see that we can attain the required level of accuracy for all five eigen values with a coarser edge element mesh than the required nodal element mesh for both quadrilateral and triangular elements.

3.1.2. Circular shape domain

Here, the domain is a circle of radius unity and Figure 3 shows the discretized domain. The circular domain has a perfectly conducting boundary. No. of elements for different meshes of various element types are given in Table 3. The results are found with these meshes
Figure 2. Comparative study of numerical performance of nodal elements with edge elements in predicting eigenvalues for square domain.

and they are closely matching with the benchmark values presented in [3,42]. For all the elements squared of the obtained eigenvalues are listed in Table 4. The required degrees of freedom required to get less than 7% error for various elements has been presented in Figure 4.

3.1.3. L-shaped domain

The L-shaped domain is obtained by deleting one quadrant from the square domain of side \( \pi \) which has been considered in the previous example (Section 3.1.1). Here, the domain is a non-convex domain with sharp corners. In numerical analysis, conducting boundary conditions are applied to all the boundaries of domain. The discretized L shape domain with Q9 elements is shown in Figure 5. The L-shaped domain is discretized with different
Table 3. Analysis data of different nodal and edge elements for the circular shape domain problem.

| Element type | No. of elements | No. of FDOF/equations |
|--------------|-----------------|-----------------------|
| Q9 & T6      | 256             | 2979                  |
| EQ12 & ET8   | 600             | 4720                  |

Table 4. $k_0^2$ on the circular domain for different elements (bracketed values show the multiplicity).

| Analytical Benchmark | Nodal element | Edge element |
|----------------------|---------------|--------------|
|                      | Q9 & T6       | EQ4 & ET3    | EQ12 & ET8   |
| 3.391122 (2)         | 3.388867      | 3.410866     | 3.380563     |
| 9.329970 (2)         | 9.318499      | 9.443712     | 9.329356     |
| 14.680392 (1)        | 14.668347     | 14.768032    | 14.756723    |
| 17.652602 (2)        | 17.609686     | 17.643890    | 17.662206    |
| 28.275806 (2)        | 28.161350     | 28.568164    | 28.262108    |
| 28.419561 (2)        | 28.376474     | 29.300835    | 28.343555    |
| 41.158640 (2)        | 40.896578     | 43.394573    | 41.404578    |
| 44.970436 (2)        | 44.858738     | 45.351002    | 44.973322    |
| 49.224256 (1)        | 49.098151     | 49.599825    | 49.656871    |
| 56.272502 (2)        | 55.747995     | 60.581974    | 56.956837    |
| 64.240225 (2)        | 64.023638     | 65.163560    | 64.271251    |

Number of computed zeros
- 992 401 450

mesh sizes for different elements, which is given in Table 5. Square of the eigenvalues are listed in Table 6 for all the elements along with standard values reported in [12].

As nodal elements cannot capture the singular eigen value (0.591790), we have compared the accuracy of squared of the obtained eigenvalues in Figure 6 from the second eigen frequency. All the nodal elements are not able to capture the singular eigen value, as well as all of them, generate one spurious eigen value, whereas all the edge elements
Figure 4. Minimum number of FDOF required for less than 7% error in predicting eigenvalues for the circular domain.

Figure 5. Discretized L shape domain.

Table 5. Analysis data of different nodal and edge elements for the L shape domain problem.

| Nodal element | Edge element |
|---------------|--------------|
| Element type  | No. of elements | No. of FDOF/ equations | Element type | No. of elements | No. of FDOF/ equations |
| Q4            | 768           | 2481                     | EQ4          | 432           | 805                     |
| T6            | 384           | 2481                     | ET8          | 384           | 1856                    |
| Q9            | 192           | 2481                     | EQ12         | 192           | 1457                    |

are able to predict the singular eigen value properly and they do not generate the spurious eigen value. We have found better coarse mesh accuracy with edge elements which is depicted in bar diagrams in Figure 6. We have compared Q9 and EQ12 elements in Figure 6(a) on a scale of less than 6% error. Q4 and EQ4 elements are compared on the scale of
Table 6. $k^2_0$ on the L-shaped domain for different elements.

| Benchmark | Nodal element | Edge element |
|-----------|---------------|--------------|
|           | Q4            | Q9           | T6           | EQ4  | EQ12 | ET8  |
| 0.591790  | –             | –            | –            | 0.596170 | 0.597191 | 0.596538 |
| 1.432320  | 1.479654      | 1.507641     | 1.481174     | 1.434491 | 1.432148 | 1.432253 |
|           | 1.620830      | 2.277630     | 1.623752     | –     | –     | –    |
| 4.005540  | 4.012869      | 3.997995     | 3.998716     | 3.781632 | 3.815079 | 3.999995 |
| 4.005540  | 4.012869      | 3.998150     | 3.998852     | 4.022899 | 4.000132 | 4.000019 |
| 4.613200  | 4.649897      | 4.645948     | 4.636011     | 4.418905 | 4.427041 | 4.616022 |
| 5.067330  | 5.577478      | 5.960861     | 5.561194     | 4.946604 | 4.939201 | 5.090472 |
| 7.955130  | 8.025738      | 7.990559     | 7.995340     | 7.821276 | 7.806319 | 8.000462 |
| 8.647370  | 9.404155      | 9.563233     | 9.336718     | 8.731623 | 8.649600 | 8.671775 |
| 9.481660  | 9.597178      | 9.838923     | 9.528452     | 9.575278 | 9.457106 | 9.460954 |
| 11.426100 | 12.380478     | 13.563361    | 12.304280    | 11.421820 | 11.349739 | 11.534996 |
| 14.448600 | 14.808709     | 14.771906    | 14.700390    | 14.611556 | 14.451696 | 14.542718 |
| 16.086200 | 16.206662     | 15.974127    | 15.984640    | 16.368790 | 16.008200 | 16.000365 |

Number of computed zeros

Table 7. Analysis data of different nodal and edge elements for the cracked circular domain problem.

| Nodal element | Edge element |
|---------------|--------------|
| Element type  | No. of elements | No. of FDOF/equations | Element type  | No. of elements | No. of FDOF/equations |
| Q9 & T6       | 256          | 3030                   | EQ4 & ET3     | 384          | 723                   |
|               |              |                        | EQ12 & ET8    | 400          | 3161                  |

6% error in Figure 6(b) whereas T6 and ET8 elements are compared for less than 4% error in Figure 6(c).

3.1.4. Cracked circular domain

Here, the domain is a circle of radius unity but it is with a crack as shown in Figure 7. All the boundaries of the computational domain are subjected to perfectly conducting boundary conditions.

The discretization of meshed domain with different types of elements is shown in Table 7. The results are found with these meshes and are listed in Table 8 along with the benchmark values from [3,42]. Numerical results are compared and they are closely matching with the benchmark values. As nodal elements cannot capture the singular eigenvalue (1.358390), we have compared the accuracy of the obtained eigenvalues in Figure 8 from the second eigen frequency. The degrees of freedom required to get an error less than 4% is shown in Figure 8. With Q9 & T6 elements, we have obtained error of 10.1% with 3030 FDOF for second eigen value.

3.1.5. Curved L-shaped domain

This example is taken from [43]. Here, the domain has three straight and three circular sides of radii 1, 2, and 3, and Figure 9 shows the discretized domain. All the sides of the domain are perfectly conducting. This is one of the challenging problem as the domain is
Figure 6. Comparative study of numerical performance of nodal elements with edge elements in predicting eigenvalues for L-shape domain.

curved, non-convex along with sharp corner. Details of the meshed domain for different nodal and edge elements are given in Table 9.

For all the elements, squared of the obtained eigenvalues are listed in Table 10 along with the benchmark values from [43]. All the nodal elements are not able to capture the singular eigen value (1.818571) (due to the presence of a sharp corner) as well as all of them generate one spurious eigen value. Whereas all the edge elements are able to predict the singular eigen value properly and they do not generate the spurious eigen value. Therefore we have compared accuracy from the second eigenvalue in Figure 10. We have found better coarse mesh accuracy with edge elements which is depicted in bar diagrams in Figure 10. We have compared Q9 and EQ12 elements in Figure 10(a) on a scale of less than 2.5% error. A scale of 6.5% error is chosen in Figure 10(b) to compare Q4 and EQ4. T6 and ET8 elements are compared for less than 8% error in Figure 10(c). With 657 FDOF of Q9 element, we have obtained 10.4% error for the second eigen value and 21.7% error for the fifth eigen value.
An error of 11.3% is obtained for Q4 element with 909 FDOF for the fifth eigen value. With T6 element for the fifth eigen value we get an error of 10.6% with 657 FDOF.

### 3.1.6. Inhomogeneous L shape domain

In this example, the efficacy of the edge elements for the inhomogeneous domain becomes evident. This example is considered from [12]. Figure 11 represents the domain where relative permittivities are 1 and 5 for grey and the light green regions respectively. Here, the walls of the domain are assumed to be perfectly conducting. Discretization details of the domain for various elements are presented in Table 11. For all the elements squared of the obtained eigenvalues are listed in Table 12. All the nodal elements are not able to capture the singular eigen value (0.175980) as well as all of them generate one spurious eigen value.

Whereas all the edge elements are able to predict the singular eigen value properly and they do not generate the spurious eigen value, thus we have compared the accuracy of
Figure 8. Minimum number of FDOF required for less than 4% error in predicting eigenvalues for the cracked circular domain.

Figure 9. Discretized curved L shape domain.

elements from the second eigen value in Figure 12. We have found better coarse mesh accuracy with edge elements which is depicted in bar diagrams in Figure 12. We have compared Q9 and EQ12 elements in Figure 12(a) on a scale of less than 6% error. Q4 and EQ4 elements are compared on a scale of 10% error in Figure 12(b) whereas T6 and ET8 elements are compared for less than 5% error in Figure 12(c).

With Q9 elements for 2481 FDOF there is 17% error for the fifth eigen value. For EQ12 elements, we obtained error of 11.8% with 1457 FDOF for the third eigen value while for Q4 element we get 17.1% error for 2481 FDOF with the fifth eigen value. Error of 16.5% is obtained with T6 element for 2481 FDOF for the fifth eigen value.
3.2. Comparative study of mesh convergence analysis with $L_2$-norm error between nodal and edge elements

In this section, we have conducted a comparative analysis of numerical accuracy in terms of $L_2$ error norm with mesh convergence, between nodal and edge elements in 2D. For this study, we have chosen four different numerical examples such as square, curved L shape, L shape and inhomogeneous L shape geometries. For each problem, five different meshes of nodal and edge elements are used to perform the convergence analysis. Table 13 shows the detailed mesh analysis data for all four domains. $L_2$-norm error is calculated between numerical and analytical benchmark values as

$$L_2\text{-norm error} = \sqrt{\frac{\sum_{i=1}^{N} [(\text{FEM value})_i - (\text{Analytical value})_i]^2 \times m_i}{\sum_{i=1}^{N} [(\text{Analytical value})_i]^2 \times m_i}},$$ (17)
where \( N \) is the maximum no. of nonsingular eigen values obtained with the coarsest mesh, \( m_i \) is the multiplicity of one particular eigen value.

\( L_2 \)-norm error values of both nodal and edge elements are plotted along with free degrees of freedom (FDOF) for all the problems, as shown in Figure 13. The error values are compared between nodal and edge elements for each problem. For example, in a square domain problem with 32 elements (196 equations), ET8 edge element predicts eigenvalues with 0.025% error, whereas T6 nodal element achieve 0.045% error with the finest mesh we have chosen, i.e. of 98 elements (663 equations). In the case of Q12 element, eigen frequencies are predicted with 0.26 % error by using 36 elements (180 equations). But, for the same number of nodal elements 495 equations are solved to achieve 0.23 % of error prediction.

Figure 10. Comparative study of numerical performance of nodal elements with edge elements in predicting eigenvalues for curved L shape domain.
Figure 11. Inhomogeneous L shape domain.

Table 12. $k^2$ on the inhomogeneous L shape domain for different elements.

| Benchmark | Nodal element | Edge element |
|-----------|---------------|--------------|
|           | Q4    | Q9    | T6    | EQ4    | EQ12   | ET8    |
| 0.175980  | 0.410869 | –     | –     | 0.176192 | 0.176090 | 0.176085 |
| 0.398080  | 0.973806 | 0.973085 | 0.970759 | 0.397985 | 0.397393 | 0.397395 |
| –         | 1.85034  | 1.78645 | 1.775821 | –     | 0.410993 | –     |
| 0.964840  | 0.998372 | 1.003343 | 0.994894 | 0.976496 | 0.973764 | 0.980699 |
| 0.978740  | 1.785034 | 1.783645 | 1.775821 | 1.529238 | 1.521521 | 1.522974 |
| 1.524310  | 1.816265 | 1.973428 | 1.807250 | 1.768005 | 1.758150 | 1.761401 |
| 2.274180  | 2.306054 | 2.292383 | 2.293151 | 2.252808 | 2.233477 | 2.293116 |
| 2.389530  | 2.563009 | 2.694188 | 2.545494 | 2.395908 | 2.381229 | 2.412659 |
| 3.394090  | 3.428865 | 3.380034 | 3.382655 | 3.424851 | 3.385157 | 3.384739 |
| 3.397400  | 3.433245 | 3.384257 | 3.386683 | 3.427972 | 3.381373 | 3.388719 |
| 3.468500  | 3.695923 | 3.664321 | 3.663456 | 3.664173 | 3.630283 | 3.641584 |
| 3.664270  | 3.906853 | 4.096015 | 3.873663 | 3.686256 | 3.655079 | 3.658646 |
| –         | 827    | 827    | 827    | 19    | 56    | 15    |

For 256 Q4 and EQ4 elements, numerical results are predicted by EQ4 elements with more accurately (2.7 % error) than Q4 elements (4.8 % error).

We want to assess the robustness of edge and nodal elements in predicting eigenvalues for the curved objects with sharp edges and corners. For that reason, convergence analysis has been carried out for curved L shape domain. From Figure 13(b), when comparing EQ12 with Q9, for almost the same degrees of freedom, i.e. 264 FDOF (Q9) and 261 FDOF (EQ12), Q9 element predict eigenvalues with an error of 16.54 %, whereas EQ12 element achieve almost more than ten times more accuracy with 1.4 % error. Similarly, among ET8 and T6, ET8 is more accurate for almost equivalent FDOF whereas EQ4 has better accuracy than Q4 for meshes with close FDOF.
We have observed similar trends for other two examples, namely, homogeneous L shape domain (Figure 13 c) and inhomogeneous L shape domain (Figure 13 d). For all three types of elements, namely, linear quadrilateral, quadratic quadrilateral, and quadratic triangular elements; the edge elements have almost one order lower error norm than that of corresponding nodal elements having equivalent FDOF. Here FDOF is same as the no. of equations, hence it can be taken as an equivalent measure of computational cost.

3.3. Performance analysis for distorted mesh

To analyze the performance of the distorted mesh, we have considered the standard curved L shape problem. To obtain the eigenvalues, the curved L shape domain is discretized using a uniform and deformed mesh made of 198 Q4 and EQ4 elements, 75 Q9 and EQ12 elements and 72 T6 and ET8 elements. The distorted meshes of different elements are shown...
Table 13. Analysis data of different nodal and edge elements for $L_2$-norm error convergence analysis.

| Square domain | No. of equations/ FDOF | Total no. of elements | No. of equations/ FDOF | Total no. of elements | No. of equations/ FDOF |
|---------------|------------------------|-----------------------|------------------------|-----------------------|------------------------|
|               | Q4 EQ4                 | Q9 EQ12               | T6 ET8                 |
| 16            | 63 24                  | 16 135 112            | 18 135 120             |
| 36            | 96 60                  | 25 351 180            | 32 231 196             |
| 64            | 135 112                | 36 495 264            | 50 351 264             |
| 144           | 260 180                | 49 663 480            | 72 495 380             |
| 256           | 480 231                | 64 855 760            | 98 663 560             |
| Curved L shape domain | Q4 EQ4 | Q9 EQ12 | T6 ET8 |
| 24            | 98 93                  | 27 177 84             | 24 177 144             |
| 60            | 219 163                | 36 261 112            | 36 288 261             |
| 96            | 717 295                | 48 381 264            | 54 432 381             |
| 168           | 813 367                | 84 501 480            | 72 501 448             |
| 270           | 909 560                | 108 760 657           | 96 680 657             |
| L shape domain | Q4 EQ4 | Q9 EQ12  | T6 ET8 |
| 192           | 483 345                | 48 657 345            | 96 657 448             |
| 300           | 962 551                | 75 1005 551           | 150 1005 710           |
| 432           | 1425 805               | 108 1425 805          | 216 1425 1032          |
| 588           | 1817 1107              | 147 1917 1107         | 294 1917 1414          |
| 768           | 2081 1457              | 192 2381 1457         | 384 2381 1856          |
| Inhomogeneous L shape domain | Q4 EQ4 | Q9 EQ12 | T6 ET8 |
| 192           | 483 345                | 48 657 345            | 96 657 448             |
| 300           | 962 551                | 75 1005 551           | 150 1005 710           |
| 432           | 1425 805               | 108 1425 805          | 216 1425 1032          |
| 588           | 1817 1107              | 147 1917 1107         | 294 1917 1414          |
| 768           | 2081 1457              | 192 2381 1457         | 384 2381 1856          |

in Figures 14(a), 14(b) and 14(c). All of the elements' results are presented in Table 14. The results of a distorted mesh of all the elements are almost the same as the results of a uniform mesh.

Now, we want to assess the influence of mesh distortion on numerical performance of finite elements quantitatively. For this, we are defining one metric $M_d$, a measure of distortion in the entire domain, as $M_d = n_s \times \frac{p_s}{100}$, where $p_s$ is % of shift of the nodes to introduce distortion, $n_s$ is the total no. of shifted nodes in the entire domain. In our numerical experimentation, four different cases are considered which are presented in Table 15 along with the corresponding values of $M_d$. Among total no. of shifted nodes $n_s$, half of the nodes are shifted in radial direction and the remaining points are shifted in $\theta$ direction. We have chosen curved L shape domain for the analysis and the domain is discretized with 336 Q4 and EQ4 elements. Figure 15 represents distorted meshes for all four cases mentioned in Table 15. For all the meshes, 628 (EQ4 elements) and 1125 (Q4 elements) FDOF/equations are solved for the eigen frequencies. In each case, for first four nonsingular eigenvalues, the percentage of error is calculated with benchmark values available from [43]. For each eigenvalue, the percentage of error values is plotted in Figure 16 with the distortion metric parameters for both nodal and edge elements. Figure 16 shows that the numerical performance of both nodal and edge elements is not affected due to mesh distortion.
3.4. Computational cost analysis

In this section, we have compared the required computational cost along with no. of required free degree of freedoms and $L_2$ error norm for Q9 and EQ12 elements while solving the problem of curved L shape domain. Different mesh refinements such as 27, 36, 48, 84, and 108 Q9 and EQ12 elements are used to discretize the computational domain in our numerical analysis and Table 16 represents respective no. of free degree of freedoms. Our computational codes are run on a Dell vostro model, using Intel Core i3 4.1 GHz processor with 4 GB RAM. We have plotted the computational time in seconds required to achieve the certain $L_2$-norm error along with total free degrees of freedom as shown in Figure 17. We can see that with EQ12 we achieve 0.02 $L_2$ error norm with around 670 FDOF in 1.3 s, whereas with almost same no. of FDOF in Q9 we have achieved $L_2$ error norm as 0.17 in 3.2 s. Also from Figure 17, we can notice that with Q9 element we can achieve $L_2$ norm of

![Convergence plots in $L_2$-norm with increasing mesh refinement in predicting eigenvalues for four different domains.](image-url)
Figure 14. Curved L-shaped domain discretized with (a) distorted Q4 elements, (b) distorted Q9 elements and (c) distorted T6 elements.

0.15 in computational time of 4.2 s whereas with EQ12 with in computational time of 0.4 s we can achieve $L_2$ norm of 0.04.

4. Conclusion

In this work, a detailed performance comparison between nodal and edge elements has been presented. Various solved examples include all possible complexities like curved boundaries, non-convex domains, sharp corners, distorted meshes, and non-homogeneous domains. In every case, edge elements have shown better coarse mesh accuracy than nodal elements. We have observed in many cases that, to achieve the same level of accuracy, the required no. of equations with edge element is less than half of that with nodal elements. For the non-convex domains with sharp corners, nodal elements cannot predict the singular eigen value which is well predicted by all the edge elements. In addition, for such domains, nodal elements predict one additional spurious eigen value which is not present.
Table 14. $k_0^2$ on the curved L shape domain for normal and distorted meshes of lower order higher order of nodal and edge elements.

|                  | Nodal element (Q4) | Edge element (EQ4) |
|------------------|--------------------|--------------------|
|                  | Normal | Distorted | Normal | Distorted |
| Q4: 198 elements (681 FDOF) |        |          | 1.810926 | 1.810926 |
|                  | 3.490576 | 3.729079 | 3.508771 | 3.508770 |
|                  | 5.084991 | 5.085667 | --      | --       |
|                  | 10.065602 | 10.148647 | 10.141604 | 10.141603 |
|                  | 10.111886 | 10.407295 | 10.348577 | 10.348577 |
|                  | 12.435537 | 13.902881 | 12.501734 | 12.501734 |
| Number of computed zeros | 65      | 65       | 227     | 227      |
| Q9: 75 elements (1005 FDOF) |        |          | 1.813849 | 1.816383 |
|                  | 3.490576 | 3.799074 | 3.490505 | 3.495839 |
|                  | 6.828452 | 6.843561 | --      | --       |
|                  | 10.065602 | 10.058195 | 10.067992 | 10.088585 |
|                  | 10.111886 | 10.245292 | 10.113374 | 10.140025 |
|                  | 12.435537 | 15.085305 | 12.429564 | 12.443714 |
| Number of computed zeros | 335     | 335      | 261     | 261      |
| T6: 72 elements (501 FDOF) |        |          | 1.803444 | 1.799088 |
|                  | 3.490576 | 3.751064 | 3.489667 | 3.484439 |
|                  | 5.118928 | 5.109720 | --      | --       |
|                  | 10.065602 | 10.063864 | 10.070665 | 10.059423 |
|                  | 10.111886 | 10.212374 | 10.110666 | 10.110596 |
|                  | 12.435537 | 13.824826 | 12.414018 | 12.407989 |
| Number of computed zeros | 167     | 167      | 117     | 117      |

Table 15. Mesh distortion metrics for different cases.

|                  | % of shift ($p_i$) | No. of nodes shifted ($n_i$) | $p_i \times 100$ | Distortion metric, $M_d=n_i \times p_i \times 100$ |
|------------------|-------------------|-----------------------------|------------------|---------------------------------------------|
| Case I           | 20                | 8                           | 0.2              | 1.6                                         |
| Case II          | 30                | 8                           | 0.3              | 2.4                                         |
| Case III         | 20                | 16                          | 0.2              | 3.2                                         |
| Case IV          | 30                | 16                          | 0.3              | 4.8                                         |

with edge elements. Also, we have observed that mesh distortion does not affect the performance of both nodal and edge elements. Edge elements are performing better with less computational cost than nodal elements.

Disclosure statement

No potential conflict of interest was reported by the author(s).
Figure 15. Discretized curved L-shaped domains with different distortion metrics.

Table 16. Analysis data of Q9 and EQ12 elements for curved L shape problem.

| Total no. of elements | No. of equations/FDOF | Q9 | EQ12 |
|-----------------------|-----------------------|----|------|
| 27                    |                       | 381| 192  |
| 36                    |                       | 501| 260  |
| 48                    |                       | 657| 352  |
| 84                    |                       | 1125| 628 |
| 108                   |                       | 1425| 816 |

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Figure 16. Comparative study of convergence analysis with different distortion metrics for curved L shape domain meshed with Q4 and EQ4 elements.

Figure 17. Computational cost and convergence analysis of curved L shape domain meshed with Q9 and EQ12 elements.

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