Companion problems in quasispin and isospin

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Abstract

We note that the same mathematical results apply to problems involving quasispin and isospin, but the problems per se are different. In the quasispin case, one deals with a system of identical fermions (e.g. neutrons) and address the problem of how many seniority conserving interactions there are. In the isospin case, one deals with a system of both neutrons and protons and the problem in question is the number of neutron-proton pairs with a given total angular momentum. Other companion problems are also discussed.

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I. INTRODUCTION

This work is based on the observation that the same mathematical results have been used to solve two different problems. One of the problems involves a system of identical fermions and the other, a system of mixed protons and neutrons. In the first case, the quasispin formalism is used to address the problem of the number of seniority conserving interactions there are, and in the second case, isospin considerations are used to simplify the expression for the number of proton-neutron pairs of a given angular momentum. In both problems, it is found useful to obtain the eigenvalues and eigenfunctions of a unitary Racah coefficient. This can be done in a direct manner, but in both cases it is possible to obtain the eigenvalues without explicit diagonalization.

II. FIRST COMPANION PROBLEM

A. Quasispin

We here follow the works of Rosensteel and Rowe [1, 2]. They have constructed ‘useful solvable and partially solvable shell model Hamiltonians’ which conserve seniority. New numerical techniques were presented for computing irreps of the USp(2j + 1) algebra. We will however focus on the part of their work where they find novel ways to solve problems which have been previously discussed and/or worked out by de Shalit and Talmi [3], Talmi [4], Lawson [5], French [6], French and MacFarlane [7], Ginocchio and Haxton [8]—the number of seniority non-conserving interactions and constraints on seniority conserving interactions. This will be most relevant to the isospin part problem that we will later consider.

The quasispin formalism was introduced by Kerman [9] and quickly developed by Kerman, Lawson and MacFarlane [10]. For quasispin $S = 1/2$ the operators are creation and destruction operators

$$
\begin{align*}
(S, M_S) \\
(1/2, 1/2) & \quad a^+_j m \\
(1/2, -1/2) & \quad (-1)^{j-m} a_{j,-m}
\end{align*}
$$

(2.1)

We now suppress the single particle index $j$ in the following equations. For $S = 1$ we have
\[(1,1) \quad A_{LM} = \frac{1}{\sqrt{2}}[a^+a]_{LM} \]
\[(1,-1) \quad B_{LM} = \frac{1}{\sqrt{2}}[aa]_{LM} \]
\[(1,0) \quad C_{LM} = \frac{1}{2}([a^+a]_{LM} + [aa]_{LM}) = [a^+a]_{LM} - \sqrt{\frac{3}{2} \delta_{L,0}} \quad (2.2) \]

For the two particle interaction, we need quasispin operators of rank 0 and 2. The interaction can be written as
\[
\hat{V} = -\frac{1}{4} \sum_{J \text{ even}} \sqrt{2J+1} V^J (A_J B_J)_0, \quad (2.3)
\]
where \(V^J = \langle j^2, J M | V | j^2, J M \rangle\). For compactness the authors define \(Z_J = (A_J B_J)_0\).

The \(S = 0\) and \(S = 2\) quasispin operators with \(M_S = 0\) are
\[
(111 - 1|S0)(AB)_0 + (11 - 1|S0)(BA)_0 + (1100|S0)(CC)_0. \quad (2.4)
\]

More specifically, Rosensteel and Rowe define them as
\[
X^0(J) = (A_J B_J)_0 - (C_J C_J)_0 + (B_J A_J)_0 \quad (2.5)
\]
\[
X^2(J) = (A_J B_J)_0 + 2(C_J C_J)_0 + (B_J A_J)_0. \quad (2.6)
\]

The six-\(j\) symbol that we were referring to in the introduction enters when we try to express the \(X^S(J)\) in terms of the nucleon–nucleon interaction, or more simply the \(Z_J\). They find
\[
X^0(J) = -(M^\Omega - 2I)Z_J + \cdots, \quad (2.7)
\]
\[
X^2(J) = 2(M^\Omega + I)Z_J + \cdots, \quad (2.8)
\]
where \(+ \cdots\) refers to constants and terms linear in the number operator, and \(M^\Omega Z_J = \sum_{\gamma} Z_{\gamma} M^\Omega_{\gamma J}\), with
\[
M^\Omega_{\gamma J} = 2\sqrt{(2J+1)(2\gamma+1)} \binom{j \ j \ \gamma}{j \ j \ J}. \quad (2.9)
\]

They then state and prove ‘Proposition 1’: the eigenvalues of \(M^\Omega\) are equal to \(-1\) or 2. They do so by noting that, if the eigenvalues were other than those, then an eigenfunction of
$M^\Omega$ would simultaneously have quasispin $S = 0$ and $S = 2$, a contradiction. The projection operators for $S = 0$ and $S = 2$ are, then, respectively:

$$
P_0 = \frac{1}{3}(M^\Omega - 2I), \quad P_2 = \frac{1}{3}(M^\Omega + I). \quad (2.10)
$$

They denote by $p_1$ the number of linearly independent rotationally invariant quasispin scalars and by $p_2$, the quasispin rank 2 tensors. However, one of the $S = 2$ operators is the pairing interaction $X_0^2(0)$ and this does not mix states of different seniority. Hence, the number of seniority mixing interactions is $p_2 - 1$. They find that

$$
p_2 = \text{tr}P_2 = \frac{1}{3} \left( \frac{2j + 1}{2} + 2 \sum_{\text{even } J} (2J + 1) \left\{ \begin{array}{ccc} j & j & J \\ j & j & J \end{array} \right\} \right) = \left\lfloor \frac{2j + 3}{6} \right\rfloor, \quad (2.11)
$$

where the square brackets mean the integer part of what is inside. We will discuss this more in the next section. We can see that eq. (2.11) is the same result as Haxton and Ginocchio [8], but using a novel technique. Rosensteel and Rowe [2] further obtain the condition that an interaction $\hat{V}$ should satisfy to conserve seniority:

$$
(M^\Omega + I)V = \lambda(M^\Omega + I)Z_0. \quad (2.12)
$$

This yields the same conditions that are present in de Shalit and Talmi [3] and Talmi [4]. We should emphasize that Rosensteel and Rowe [1, 2] have other results which are new but that we are not focusing on here.

B. Isospin

Some of the topics in this section have been discussed in part in preprints [11, 12] and in conference proceedings [13].

In the single $j$-shell model, the nucleus $^{44}$Ti consists of two valence protons and two valence neutrons. We can make an association of QUASISPIN and ISOSPIN by relating the $(S = \frac{1}{2}, M_S = \frac{1}{2})$ creation operator with a proton and the $(S = \frac{1}{2}, M_S = -\frac{1}{2})$ destruction operator with a neutron.

The wave function of a state $\alpha$ of total angular momentum $I$ can be written as

$$
\Psi^{\alpha I} = \sum_{J_PJ_N} D^{\alpha I}(J_PJ_N)[(J^2_p)J_P(J^2_n)J_N]^I. \quad (2.13)
$$
Now we focus on $I = 0$ states, for which $J_P = J_N \equiv J$

$$
\Psi^{\alpha I=0} = \sum_{\text{even } J} D^{\alpha I=0}(JJ)[(J_J^2)J(J_{\alpha}^2)J]^0.
$$

We already see a resemblance of the wave function for this problem with the interaction for the quasispin problem. It can be shown that there are four angular momentum $I = 0$ states, three of which have isospin $T = 0$ and one has isospin $T = 2$.

Now the $T = 2$ state must be orthogonal to the $T = 0$ states (for $I = 0$):

$$
\sum_J D^{T=2}(JJ)D^{\alpha T=0}(JJ) = 0;
$$

we also have the normalization condition:

$$
\sum_J D^{T=2}(JJ)D^{T=2}(JJ) = 1.
$$

Since the $T = 2$ state is the double analog of a state for a system of identical nucleons (calcium isotopes), the unique $I = 0, T = 2$ wave function is known and leads to the result

$$
D^{I=0,T=2}(JJ) = (j^2 J j^2 J) j^4 0,
$$

where we have a two particle coefficient of fractional parentage (cfp) on the right hand side.

A useful identity can also relate the above to a one particle cfp:

$$
D^{I=0,T=2}(JJ) = (j^2 J j^3 J) j^3 J.
$$

However, a recursion formula, found on page 528 of de Shalit and Talmi [3], namely,

$$
n(j^{n-1} \alpha_0 J_0 j_j j_j \alpha_0 J_0 J)(j^{n-1} \alpha_1 J_1 j_j j_j \alpha_0 J_0 J) = \delta_{\alpha_1, \alpha_0} \delta_{J_1, J_0} + (n - 1) \sum_{\alpha_2 J_2} (-1)^{j_0 + J_1} \sqrt{(2J_0 + 1)(2J_1 + 1)} \times \left\{\begin{array}{c}
J_2 & j & J_1 \\
J & j & J_0
\end{array}\right\} (j^{n-2} j_J j_J) (j^{n-2} j_J j_J) \times (j^{n-2} \alpha_2 J_2 j_j j_j \alpha_0 J_0) j^{n-2} \alpha_2 J_2 j_j j_j \alpha_0 J_0 (j^{n-2} \alpha_2 J_2 j_j j_j \alpha_0 J_0) j^{n-2} \alpha_2 J_2 j_j j_j \alpha_0 J_0
$$

with $n$ set equal to $3$, leads to the following result, valid for $j$ values of $3/2, 5/2$ and $7/2$:

$$
\sqrt{(2J + 1)(2J' + 1)} \left\{\begin{array}{c}
J & j & J' \\
J & j & J
\end{array}\right\} = -\frac{\delta_{J,J'}}{2} + \frac{3}{2} (j^2 (Jj) j^3 Jv = 1)(j^2 (J'j) j^3 Jv = 1).
$$
This leads to the following:

\[
2 \sum_{J_p} \sqrt{(2J_p+1)(2J_{12}+1)} \begin{bmatrix} j & j & J_p \\ j & j & J_{12} \end{bmatrix} D(J_pJ_p) = -D(J_{12}J_{12}) \quad \text{for } T = 0 \quad (2.21)
\]

\[
= 2D(J_{12}J_{12}) \quad \text{for } T = 2. \quad (2.22)
\]

But this is an eigenvalue problem for the unitary 6\textit{j}-symbol and the eigenvalues are now shown to be −1 and 2 without explicit diagonalization. These are the same eigenvalues for the same operator that Rosensteel and Rowe state in their Proposition 1. The equations (2.21) and (2.22) have the same physical structure as (2.7) and (2.8).

In the isospin case, the vectors are the wave function components \(D(JJJ)\); in the quasispin case, the vectors are the \(Z(J)\)'s, i.e., \([[a^+a^+]^J[aa]^J]^0\). In the isospin problem, the practical application of this is to obtain the number of neutron–proton pairs in \(^{44}\text{Ti}\). The general expression is complicated, but for even \(J_{12}\) of a pair, the result using the 6\textit{j} eigenvalue equation for \(^{44}\text{Ti}\) is

\[
\# \text{ of nn pairs} = \# \text{ of np pairs} = \# \text{ of pp pairs} = |D(J_{12}J_{12})|^2, \quad (2.23)
\]

with \(J_{12} = 0, 2, 4\) and 6. Simple expressions for the number of \(J_{12} = 0\) pairs for \(^{46}\text{Ti}\) and \(^{48}\text{Ti}\) have also been obtained.

We can use eq. (2.20) to cast some limited insight into the result of Rosensteel and Rowe [eq. (2.11)]. This result involve

\[
\text{SUM}6\textit{j} = \sum_{\text{even } J} (2J + 1) \begin{bmatrix} j & j & J \\ j & j & J \end{bmatrix} \quad (2.24)
\]

For \(j \leq 7/2\) we can use eq. (2.21) to evaluate this. We have the following result for the cfp’s:

\[
\sum_J |(j^2(J||j^3j)|^2 = 1. \quad (2.25)
\]

This condition comes from the fact that the wave function for three identical particles is normalized to unity. Hence, for \(j = 3/2, 5/2\) and \(7/2\), we have the following result:

\[
\text{SUM}6\textit{j} = -\frac{2j+1}{4} + \frac{3}{2}, \quad (2.26)
\]

i.e., 0.5, 0 and −0.5 for the three \(j\) values above. This formula does not work for \(j = 1/2\) because we cannot have 3 neutrons in the \(s_{1/2}\) shell; we can, however, use the same recursion
formula (2.19) with \( n = 2 \) in this case to find that \( \text{SUM}6j = -0.5 \) (as one can just look it up). Eq. (2.26) does not work for \( j = 9/2 \) or higher because in that case there is more than one state of a given angular momentum; e.g., for \( J = j = 9/2 \) there are two states, one with seniority 1 and the other with seniority 3.

A more general result has been obtained by Zhao, Arima, Ginocchio and Yoshinaga [17]. The pattern \((-0.5, 0.5, 0)\) for \( j = s_{1/2}, p_{3/2} \) and \( d_{5/2} \) respectively repeats itself, i.e., the same trio of results holds for \((f_{7/2}, g_{9/2}, h_{11/2})\), etc. This was shown by the above authors [17] simply by equating the left-hand side of eq. (2.11) with the right-hand side, and relying on the proof in ref. [8] that \([2j + 1]/6\) is indeed the number of states of different seniority. For completeness, we give also two other references by this group ([18], [19]).

(Whereas the sum in \( \text{SUM}6j \) is over even angular momenta, it was pointed out to us by I. Talmi that the sum over all angular momenta is much easier to obtain. Indeed the result is explicitly given in a work by J. Schwinger “On Angular Momentum”, eqs. (4.27) and (4.28). The sum over all \( J \) is zero for half-integer \( j \) and one for integer \( j \) [20]).

We can use isospin arguments to derive the left hand side Rosensteel–Rowe relation [eq. (2.11)]. We consider a system of one proton and two neutrons in a single \( j \)-shell.

We use isospin variables \( p \) and \( n \) for the proton and neutron. The basis states are, then,

\[
\psi^I[J_0] = \frac{1}{\sqrt{3}}(1 - P_{12} - P_{13}) \left[ j(1) j(2) j(3) \right]^I p(1)n(2)n(3). \tag{2.27}
\]

We introduce a simplified hamiltonian \( V = \sum_{i<j}(a + bt(i) \cdot t(j)) \), where \( a \) and \( b \) are constants.

We now evaluate the trace

\[
\text{tr}[J_0] = \sum_{J_0 \text{ even}} \langle \psi^I[J_0] V \psi^I[J_0] \rangle, \tag{2.28}
\]

and we find

\[
\text{tr}[J_0] = \sum_{J_0 \text{ even}} \left[ (3a - \frac{b}{4}) + b(2J_0 + 1) \left\{ \begin{array}{c} j \\
  j \\
  J_0 
\end{array} \right\} \right] = \left( 3a - \frac{b}{4} \right) \frac{2j + 1}{2} + b \sum_{J_0 \text{ even}} (2J_0 + 1) \left\{ \begin{array}{c} j \\
  j \\
  J_0 
\end{array} \right\}. \tag{2.29}
\]

The expectation value of \( V \) for an \( A \) body system with total isospin \( T \) is:

\[
\langle V \rangle = \frac{A(A - 1)}{2} a + \frac{b}{2} T(T + 1) - \frac{3}{8} bA, \tag{2.30}
\]
which, for $A = 3$, becomes

$$\langle V \rangle = 3a + \frac{b}{2}T(T+1) - \frac{9}{8}b.$$  \hspace{1cm} (2.31)

We choose $a$ and $b$ so that $\langle V \rangle = 0$ for $T = 1/2$ and $\langle V \rangle = 1$ for $T = 3/2$. We find $a = 1/6$, $b = 2/3$. With this choice, $\text{tr}[J_0]$ becomes the number of $T = 3/2$ states of angular momentum $I$, which is also the same as the number of states of angular momentum $I$ for a system of 3 identical particles.

We then obtain $\text{tr}[J_0] = p_2$, as in eq. (2.11), namely

$$\frac{1}{3} \left( \frac{2j+1}{2} + 2 \sum_{J_0 \text{ even}} (2J_0 + 1) \left\{ J J J \right\} \right).$$  \hspace{1cm} (2.32)

The above result supports the theme of this work that there is an interrelationship between quasispin and isospin.

### III. SECOND COMPANION PROBLEM

In the realm of identical fermions, e.g. neutrons, the Pauli Principle imposes a severe restriction on the number of states that are allowed. For example, in the calcium isotopes, if we limit ourselves to one single $j$-shell $j = f_{7/2}$, then the allowed states for $^{43}\text{Ca}$ are $I = 3/2, 5/2, 7/2, 9/2, 11/2$ and $15/2$, all occurring only once; while for $^{44}\text{Ca}$ the allowed angular momentum–seniority combinations are $I = 0, v = 0, I = 2, 4, 6$ all with $v = 2$, and $I = 2, 4, 5$ and $8$ with seniority $v = 4$. All states of given $(I, v)$ occur only once.

As discussed in de Shalit and Talmi \cite{3} and Talmi \cite{4}, if one is foolish enough to try to calculate a coefficient of fractional parentage for a non-existent state, one gets zero. But this can produce useful results. For example, in $^{43}\text{Ca}$ there is no $I = 13/2$ state of the $f_{7/2}^3$ configuration. This leads to the result for a certain $6j$-symbol:

$$\left\{ \frac{7}{2} \frac{7}{2} \frac{4}{2} \right\} = 0.$$  \hspace{1cm} (3.1)

This result can be easily generalized to

$$\left\{ \frac{j}{3j-4} \frac{j}{2} \frac{(2j-3)}{2j-1} \right\} = 0.$$  \hspace{1cm} (3.2)
The companion problem deals with the isospin $T = \frac{1}{2}$ states in $^{43}\text{Sc}$ (or the mirror $^{43}\text{Ti}$). This is in contrast to the original problem which deals with $T = \frac{3}{2}$ states in $^{43}\text{Ca}$. For $^{43}\text{Sc}$ one requires knowledge of both the proton–neutron interaction and the neutron–neutron interaction. In terms of isospin, the $nn$ system must have isospin 1, but we can have both $T = 0$ and $T = 1$ $np$ states. Indeed, in the single $j$-shell the $np$ states with even total angular momenta $J$ have isospin $T = 1$, whereas those with odd $J$ value have $T = 0$.

Robinson and Zamick [14, 15, 16] posed the question of what happens if the two-body $T = 0$ matrix elements are set equal to zero (or, what amounts to the same thing, a constant). The motivation is contained in the references and will not be repeated here. It was found that the result of this was twofold.

In $^{43}\text{Sc}$ the basis states can be classified by $(J_N, j)$, the angular momentum of the two neutrons and that of the single proton. The wave function can be written $\Psi^{\alpha I} = \sum_{J_N \text{even}} C^{\alpha I}(J_N, J)[(j^2)^{J_N} j]^I$. For a certain set of states, a dynamical symmetry was found. These $T = \frac{1}{2}$ states had angular momenta $I = 1/2, 13/2, 17/2$ and $19/2$. For these states, when the two-body matrix elements were set to zero, the wave functions had quantum numbers $(J_N, j)$, i.e., one $13/2$ state was $(4, 7/2)$ and the other $(6, 7/2)$. Furthermore, there were degeneracies. The $I = 1/2$ and $13/2$ states were degenerate as were the $17/2$ and $19/2$ states.

How does this fact relate to the original $^{43}\text{Ca}$ problem? When we ask why the matrix element $\langle [4, 7/2]^{13/2} V [6, 7/2]^{13/2} \rangle$ vanishes, we find that the expression involves the $6j$-symbol

$$\begin{bmatrix} 7/2 & 7/2 & 4 \\ 13/2 & 7/2 & 6 \end{bmatrix},$$

which was shown to vanish because there was no $13/2$ state of the $f_{7/2}^3$ configuration in $^{43}\text{Ca}$. The vanishing of the $13/2$ cfp also leads to a diagonal condition which leads to the degeneracy of the states $[4, 7/2]^{13/2}$ and $[4, 7/2]^{1/2}$, as well as $[6, 7/2]^{17/2}$ and $[6, 7/2]^{19/2}$.

Note that we have a partial dynamical symmetry. States of the other angular momenta, i.e. $I = 3/2, 5/2, 7/2, 9/2, 11/2$ and $15/2$ do not behave in this way. We now see what is happening—this partial dynamical symmetry for $T = 1/2$ states in $^{43}\text{Sc}$ applies only to angular momenta which do not occur for the $f_{7/2}^3$ configuration of $^{43}\text{Ca}$.

We can see this more clearly by comparing the $I = 13/2$ and $I = 15/2$ states in $^{43}\text{Sc}$. For both cases the basis states are $[4, 7/2]^I$ and $[6, 7/2]^I$. For $I = 13/2$, both states have $T = 1/2$. However, for $I = 15/2$, one state has $T = 1/2$ and one has $T = 3/2$. The latter
state is an analog state of the unique $I = 15/2$ state in $^{43}\text{Ca}$. For $I = 15/2$ the two states can be written as

$$\Psi_1 = a[4, 7/2] + b[6, 7/2] \quad (3.3)$$
$$\Psi_2 = -b[4, 7/2] + a[6, 7/2]. \quad (3.4)$$

Let the first state be the $T = 3/2$ state. Because it is the analog of a state in $^{43}\text{Ca}$, we can easily show that $a$ and $b$ are coefficients of fractional parentage:

$$a = (j^2 4j|j^3 15/2) = \sqrt{\frac{5}{22}}, \quad (3.5)$$
$$b = (j^2 6j|j^3 15/2) = \sqrt{\frac{17}{22}}. \quad (3.6)$$

This is independent of what (isospin conserving) interaction is chosen. The second state, with isospin $T = 1/2$, must be orthogonal to the first $T = 3/2$ state, so its components also involve the same cfp’s. The point is that here also the interaction cannot change the wave function, and so we do not get a partial dynamical symmetry.

To summarize this section, the vanishing of a $6j$-symbol enters into two companion problems—why certain angular momenta cannot occur for an $(f^3_{7/2})$ $T = 3/2$ configuration and why, for these same angular momenta, the $T = 1/2$ states can be classified by the quantum numbers $(J_N, j)$, and further why states with the same $(J_N, j)$ but different angular momenta are degenerate.

As a brief summary of both sections, we have shown that the same mathematical relations can be used to solve problems involving systems of neutrons and protons, and systems of identical particles. With the exception of the result of eq. (2.32), the problems are quite different. Our distinctive contributions have been for the systems of mixed neutrons and protons involving isospin and we have established connections with the work on identical particles of Haxton, Ginocchio, Rosensteel, Rowe, Zhao, Arima, and Yoshinaga, as well as de Shalit, Talmi and Racah. Perhaps in the near future other examples will emerge.

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