Review on some constructions of cyclically ordered groups

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Abstract. We review some techniques of construction of new cyclically ordered groups from the old ones. Especially, we focus on the conditions of quotient group to be a cyclically ordered group.

1. Introduction
Historically, it was Rieger [1], who firstly introduced the concept of cyclically ordered group (cog). In brief, the concept is a generalization from a binary to a ternary relation. For this reason, the concept is evidently closely related to the concepts of ordered groups. As for prerequisites, the reader should be familiar with the theory of ordered set. Some basic facts about cog may be found in [2].

The important point to note here is a connection between cyclically and linearly ordered groups. The connection says that if we have a linear order in a group $G$, then we can induce a cyclic order in it. More deeply, we can apply this way to any group, provided it is finite and cyclic. This technique is what Fuchs used and exposed in [2].

If we have two groups, then we can construct the new one through its direct product. The operation is defined in a natural way. Accordingly, this fact may suggest the researchers to apply it in the context of cyclically ordered groups. In 1959, Swierczkowski [5] developed this approach to construct a new cyclically ordered groups from the old ones. Precisely, from two linearly ordered groups $K = [0,1)$ and $L$, it can be defined the cyclically ordered group $K \times L$. Some related results may be found in [3,4,7,8].

Recall that from given group $G$ and its normal subgroup $N$, we can construct the quotient group $G/N$. Note that the properties of $G/N$ depend on $G$ and $N$, in general. Moreover, there are some applications of the group quotient to identify certain properties of related groups. For example, if $G/Z(G)$ is cyclic (in the sense of groups), then $G$ is Abelian. The other one is that $G/Z(G)$ is isomorphic to the group of inner automorphism.

It is not our purpose to study many other techniques, but we wish to highlight some implications of many previous results. Especially, what is the necessary condition for a group $G$ and its normal subgroup $N$ to make $G/N$ as a cyclically ordered group. It will be shown that there are immediate implications from the properties of $G$ and $G/N$.

2. Preliminaries
In this section we summarize without proofs the relevant material on cyclically ordered group from [5]. I hope the reader is familiar with basic facts of groups.

Let $G$ be a group. The order $[x,y,z]$ in $G$ is said to be cyclic if for all $x,y,z,a,b$ in $G$, the followings are satisfied:

1. $[x,y,z] \Rightarrow x \neq y \neq z \neq x$ and if $x \neq y \neq z \neq x$ it must be $[x,y,z]$ or $[z,y,x]$. 

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ii. \([x, y, z] \Rightarrow [y, z, x]\).

iii. \([x, y, z]\) and \([y, u, z]\) implies \([x, u, z]\).

iv. \([x, y, z]\) implies \([axb, ayb, azb]\).

The group \(G\) with this relation is called a cyclically ordered group.

As stated previously, a cyclic order can be obtained from a linear order. Assume that \(G\) is a linearly ordered group with the linear order \(<\). We can define a relation \([x,y,z]\) in \(G\) if one of the followings are satisfied:

\[x < y < z, \text{ or } y < z < x, \text{ or } z < x < y.\]

If \(x \neq y \neq z \neq x\), of course one of the three conditions must be true. It is obvious that the next three conditions are satisfied, so the relation \([...,]\) defined a cyclic order in \(G\).

Here is an interesting construction of cyclic order in a finite and cyclic group. Assume that \(G\) is a cyclic group of order \(n\). If we assume that the generator is \(a\), we can write \(G = \{a^i : 0 < i < n-1\}\). It can be verified that for the elements \(x, y, z\) in \(G\), the order \([x,y,z]\) can be defined for the conditions \(i < j < k\), \(j < k < i\), or \(k < i < j\), where \(x = a^i, y = a^j, z = a^k\). [2]

It is worth pointing out that, by virtue the construction above, we can view every finite and cyclic group as a cyclically ordered group. It also should be noted that the different generator may define the other cyclic order. For example, the reader may consider the group \(Z_7\) which the generators 2 and 3 give the distinct orders.

The following example [5] is one of important constructions of a new cyclically ordered group from the old ones. It gives one technique to define a cyclic order in a direct product of linearly ordered groups.

**Example.** [5]

Assume that \(L\) is a linearly ordered group and \(K = [0,1]\), and consider the direct product \(K \times L\). For distinct elements \(u = (a, x), v = (b, y), w = (c, z)\), we set \([u, v, w]\) if some of the following conditions is satisfied:

(i) \([a, b, c]\);

(ii) \(a = b \neq c \text{ and } x < y\);

(iii) \(b = c \neq a \text{ and } y < z\);

(iv) \(c = a \neq b \text{ and } z < x\);

(v) \(a = b = c \text{ and } [x, y, z]\).

It can be verified that \(K \times L\) is a cyclically ordered group.

The reader may realize that this way applies for the direct product of any two cyclically ordered groups, with certain conditions.

**Proposition.** [9]

Let \(G\) and \(H\) be two cyclically ordered groups with their cyclic order \([\ldots]\) \(G\), \([\ldots]\) \(H\), respectively. If \(H\) has a positive cone \(H^+ = \{x \in H \mid x > 0\}\) and for any distinct elements \(x, y, z \in H\), the conditions \(\langle xy^\ast(-1), 0, xy^\ast(-1) \rangle \langle H, \langle yz^\ast(-1), 0, zy^\ast(-1) \rangle \langle H, \langle xz^\ast(-1), 0, xz^\ast(-1) \rangle \langle H \) hold, then the direct product \(G \times H\) is a cyclically ordered group.

**3. Results**

Now we come to the main part of this paper. In the previous sections we see some constructions of a new cyclically ordered groups. Here, we present some implications related to some conditions of the quotient groups.

**Proposition.** Let \((G, +)\) be a group, and \(N\) be a normal subgroup of \(G\).

1. If \(G/N = \{\bar{a} = a + N \mid a \in G\}\) is finite and cyclic, \(G/N\) is a cyclic ordered group.
2. If the group $G/N$ is isomorphic to a cyclically ordered group, then $G/N$ is a cyclically ordered group too.
3. If $G$ is a linearly ordered group, then $G/N$ is a cyclically ordered group.

**Proof.**
1. The condition is a direct implication of its finiteness and cyclicity.
2. If $G/N$ is isomorphic to a cyclically ordered group $G'$ through the isomorphism $f$, we can define the order $[a', b', c']$ in $G/N$ for the condition $[x, y, z]$ in $G'$, where $a' = f(x), b' = f(y), c' = f(z)$.
3. We can define $[a', b', c']$ in $G/N$ for the corresponding order $[a, b, c]$ in $G$.

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