ARRANGEMENTS WITH LOW DEGREE LOGARITHMIC VECTOR FIELDS

STEFAN O. TOHANEANU

ABSTRACT. In these notes we study hyperplane arrangements having at least one logarithmic derivation of degree one, not a constant multiple of the Euler derivation, or having at least one logarithmic derivation of degree two not a multiple of a degree one derivation. We prove that the first case is equivalent to the hyperplane arrangement being a product of smaller hyperplane arrangements. In the second case we present a computational lemma that leads to a full classification of hyperplane arrangements of rank 3 having a quadratic logarithmic derivation.

1. INTRODUCTION

Let $\mathcal{A}$ be a central essential hyperplane arrangement in $V$ a vector space of dimension $k$ over $\mathbb{K}$ a field of characteristic zero. Let $R = Sym(V^*) = \mathbb{K}[x_1, \ldots, x_k]$ and fix $\ell_i \in R$, $i = 1, \ldots, n$ the linear forms defining the hyperplanes of $\mathcal{A}$. After a change of coordinates, assume that $\ell_i = x_i$, $i = 1, \ldots, k$.

A logarithmic derivation (or logarithmic vector field) of $\mathcal{A}$ is an element $\theta \in Der(R)$, such that $\theta(\ell_i) \in \langle \ell_i \rangle$, for all $i = 1, \ldots, n$. Picking the standard basis for $Der(R)$, i.e., $\partial_1 := \partial_{x_1}, \ldots, \partial_k := \partial_{x_k}$, $\theta$ is written as

$$\theta = \sum_{i=1}^{k} P_i \partial_i,$$

where $P_i \in R$ are homogeneous polynomials of the same degree denoted $\text{deg}(\theta)$. The set of logarithmic derivations form an $R$–module, and whenever this module is free one says that the hyperplane arrangement is free.

In general, every central hyperplane arrangement has a logarithmic derivation, the Euler derivation:

$$\theta_E = x_1 \partial_1 + \cdots + x_k \partial_k.$$

There exists a one-to-one correspondence between logarithmic derivations not multiples of $\theta_E$ and the first syzygies on the Jacobian ideal of $\mathcal{A}$, which

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is the ideal of $R$ generated by the (first order) partial derivatives of the defining polynomial of $A$. Therefore, we are interested in hyperplane arrangements that have a linear, respectively quadratic, syzygy on its Jacobian ideal.

The concepts of logarithmic derivations and freeness were originally defined for any divisor, not necessarily a union of hyperplanes. Related to our topic, [1, Section 3] studies the freeness of divisors that have lots of special linear syzygies (called Euler vector fields) on their Jacobian ideal. A few months before, [3] shows that if one homogenizes a weighted-homogeneous polynomial, the new divisor obtained has an Euler vector field (see Lemma 2.4). Both papers go into more depths of analyzing freeness and other homological properties of different classes of divisors (quite far away from hyperplane arrangements), and for those interested in free divisors, a very current topic of active research, we recommend these papers and the important citations from within.

In these notes we do not discuss the freeness of the hyperplane arrangements we study. We are more interested in the geometry of the configuration of points that are dual to the hyperplanes of an arrangement that has a linear syzygy, or a quadratic syzygy on its Jacobian ideal. In Section 2 we discover that in the linear syzygy case, the points dual to hyperplanes sit on the variety of an edge ideal of a complete multipartite graph, this allowing to prove a result which seems to be known in the hyperplane arrangements community, yet to this day we are not aware of any (other) proof: a hyperplane arrangement has a linear syzygy if and only if it is the product of two smaller rank hyperplanes (Theorem 2.1). In Section 3 we study the case of hyperplane arrangements with a quadratic minimal syzygy. We also obtain that the dual points lie on an interesting variety, though its description is not even close to the nice combinatorial case of the linear syzygy. Nevertheless, using this description we are able to classify all rank 3 hyperplane arrangements having a quadratic minimal linear syzygy on their Jacobian ideal (Theorem 3.3).

2. Linear logarithmic derivations

Keeping the notations from the beginning of Introduction, let us assume that $A$ has a linear logarithmic derivation, not a constant multiple of $\theta_E$:

$$\theta = L_1 \partial_1 + \cdots + L_k \partial_k,$$

\footnote{For more details about this, and in general about the theory of hyperplane arrangements, the first place to look is the landmark book of Orlik and Terao, [2].}
where \( L_j \) are some linear forms in \( R \). Because \( \theta(x_i) = a_ix_i, \ i = 1, \ldots, k \) for some constants \( a_i \in \mathbb{K} \), then
\[
L_i = a_ix_i, \ i = 1, \ldots, k,
\]
and not all \( a_i \)'s are equal to each-other (otherwise we’d get a constant multiple of \( \theta_E \)).

For \( i \geq 4 \), suppose \( \ell_i = p_{1,i}x_1 + \cdots + p_{k,i}x_k, p_{j,i} \in \mathbb{K} \). The logarithmic condition \( \theta(\ell_i) = \lambda_i\ell_i, \ i \geq 4, \lambda_i \in \mathbb{K} \) translates into
\[
\begin{bmatrix}
a_1 \\
\vdots \\
a_k
\end{bmatrix}
\cdot
\begin{bmatrix}
p_{1,i} \\
\vdots \\
p_{k,i}
\end{bmatrix}
=
\lambda_i
\begin{bmatrix}
p_{1,i} \\
\vdots \\
p_{k,i}
\end{bmatrix}.
\]

Therefore the points in \( \mathbb{P}^{k-1} \) dual to the hyperplanes \( \ell_i \) sit on the scheme with defining ideal \( I \) generated by the \( 2 \times 2 \) minors of the matrix
\[
\begin{bmatrix}
a_1x_1 & a_2x_2 & \cdots & a_kx_k \\
x_1 & x_2 & \cdots & x_k
\end{bmatrix}.
\]

Obviously
\[
I = \langle \{(a_i - a_j)x_ix_j : i \neq j\} \rangle,
\]
and this is the edge (graph) ideal of a complete multipartite simple graph on vertices \( 1, \ldots, k \); two vertices \( u \) and \( v \) belong to the same partition iff \( a_u = a_v \).

**Theorem 2.1.** A hyperplane arrangement \( A \subset \mathbb{P}^{k-1} \) of rank \( k \) has a linear logarithmic derivation not a constant multiple of the Euler derivation if and only if \( A = A_1 \times A_2 \). Moreover, if this happens, then \( A = A_1 \times \cdots \times A_s \) where \( A_i \subset \mathbb{P}^{|P_i|-1}, i = 1, \ldots, s \) and \( P_1, \ldots, P_s \) is the partition of the complete multipartite graph we’ve seen above.

**Proof.** Suppose that \( A \) has a logarithmic derivation not a multiple of \( \theta_E \). Let \( I \) be the ideal we considered above.

For a simple graph \( G \), a minimal vertex cover is a subset of vertices of \( G \), minimal under inclusion, such that every edge of \( G \) has at least one vertex in this subset.

By [5], since \( I \) is the edge ideal of a simple graph \( G \) (complete multipartite), all the minimal primes of \( I \) are generated by subsets of variables corresponding to the minimal vertex covers of \( G \). Also, since \( I \) is generated by square-free monomials, it must be a radical ideal, hence it is equal to the intersection of its minimal primes.

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\(^2\)Greg Burnham, an REU student of Jessica Sidman, attributes this well known result to Rafael Villarreal, so we decided to use the same citation.
Claim: If $G$ is a complete multipartite graph with partition $P_1, \ldots, P_s$, then the minimal vertex covers of $G$ are $V(G) - P_i, i = 1, \ldots, s$.

Let $S \subset V(G)$ be a minimal vertex cover. Let $v \in S$, and suppose $v \in P_1$. Consider $G' = G - \{v\}$ the graph obtained from $G$ by deleting the vertex $v$ (and all the edges incident to $v$). Then $G'$ is a complete multipartite graph with partition $P'_1 := P_1 - \{v\}, P_2, \ldots, P_s$. By induction, all the minimal vertex covers of $G'$ are:

$$V(G') - P'_1, V(G') - P_2, \ldots, V(G') - P_s.$$ 

$S - \{v\}$ is a vertex cover for $G'$ so it must contain one of the minimal vertex covers above. Now adding back the vertex $v$ we see that $S$ contains one of the $V(G) - P_i$. Since $V(G) - P_1$ is a vertex cover and since $S$ is minimal, then they must be equal.

With the claim above one gets

$$I = I(G) = \cap_{i=1}^s \langle \{x_v : v \in V(G) - P_i\} \rangle.$$ 

The points dual to the hyperplanes of $A$ are in the zero set (the variety) of $I$. If $[p_1, \ldots, p_k] \in \hat{V}(I)$, then there is $1 \leq j \leq s$ with $p_i = 0$, for all $v \in V(G) - P_j$. Then the linear form dual to this point belongs to $\mathbb{K}[x_v, v \in P_j]$, so it defines a hyperplane in $\mathbb{P}^{|P_j| - 1}$.

Since $A$ has full rank, each component must contain at least one of these points, and therefore we can group the linear forms accordingly to the components their dual points belong to.

For the converse, suppose $A = A_1 \times A_2$, with $A_1 \subset \mathbb{P}^{p-1}$ and $A_2 \subset \mathbb{P}^{k-p-1}$. Then, the defining polynomials of $A$, and $A_i, i = 1, 2$ satisfy the relation:

$$F(x_1, \ldots, x_k) = F_1(x_1, \ldots, x_p) \cdot F_2(x_{p+1}, \ldots, x_k).$$

Taking the partial derivatives with respect to $x_1, \ldots, x_p$, and using Euler’s formula one obtains the linear syzygy on the partial derivatives of $F$:

$$\deg(F_2)x_1F_{x_1} + \cdots + \deg(F_2)x_pF_{x_p} = \deg(F_1)x_{p+1}F_{x_{p+1}} + \cdots + \deg(F_1)x_kF_{x_k}.$$ 

And this gives a logarithmic derivation not a constant multiple of $\theta_E$. □

**Example 2.2.** It is immediate that the existence of a linear logarithmic derivation not a constant multiple of the Euler derivation is equivalent to the Jacobian ideal of the defining polynomial of $A$ having a linear syzygy. On this idea, Remark 3.3 in [4] shows that a line arrangement in $\mathbb{P}^2$ has a linear syzygy on the Jacobian ideal if and only if the arrangement consists of a pencil of lines and a line at infinity. This can be also seen immediately from the theorem above.
A complete multipartite graph on three vertices has either the partition \( P_1 = \{1\}, P_2 = \{2\}, P_3 = \{3\} \) or \( P_1 = \{1, 2\}, P_2 = \{3\} \). In the first case, the ideal \( I \) considered above is \( I = \langle x_1, x_2 \rangle \cap \langle x_1, x_3 \rangle \cap \langle x_2, x_3 \rangle \), leading to the arrangement with defining polynomial \( F = x_1x_2x_3 \). In the second case, \( I = \langle x_3 \rangle \cap \langle x_1, x_2 \rangle \). The second component is just the point \([0, 0, 1]\) which is dual to \( x_3 = 0 \). The first component contains only points of the form \([a, b, 0]\) which are dual to linear forms in \( \mathbb{K}[x_1, x_2] \); all of these lines pass through the point \([0, 0, 1]\), hence the pencil.

### 3. Quadratic Logarithmic Derivations

In this section we consider hyperplane arrangements with quadratic logarithmic derivations, not multiple of a linear logarithmic derivation.

Let \( \mathcal{A} \) be as before, with
\[
\ell_i = x_i, \quad i = 1, \ldots, k
\]
and
\[
\ell_j = p_{1,j}x_1 + \cdots + p_{k,j}x_k, \quad j \geq k + 1.
\]

Let \( \theta = Q_1 \partial_1 + \cdots + Q_k \partial_k \) be a quadratic logarithmic derivation, \( Q_i \in R := \mathbb{K}[x_1, \ldots, x_k] \) quadratic homogeneous polynomials (assumed to have no common divisor to eliminate some simple cases already discussed in the previous Section).

For \( i = 1, \ldots, k \), since \( \theta(x_i) = L_i x_i \) for linear form
\[
L_i = b_{1,i}x_1 + \cdots + b_{k,i}x_k, \quad b_{u,i} \in \mathbb{K},
\]
then \( Q_i = L_i x_i, \quad i = 1, \ldots, k \).

Similarly to the previous section, we will analyse the dual points to each hyperplane in \( \mathcal{A} \), and in fact the configuration of these points if \( \mathcal{A} \) has a quadratic logarithmic derivation. The next result gives the first insights into this regard.

**Lemma 3.1.** Let \( \mathcal{A} \) be a hyperplane arrangement with a quadratic logarithmic derivation. If \( V(\ell_j) \in \mathcal{A} \), where \( \ell_j = p_{1,j}x_1 + \cdots + p_{k,j}x_k \) with \( p_{u,j}, p_{v,j} \neq 0 \), then
\[
[p_{1,j}, \ldots, p_{k,j}] \in V(I_{u,v}),
\]
where \( I_{u,v} \) in the ideal of \( R \) generated by the following \( k - 1 \) elements:
\[
x_u(b_{v,u} - b_{v,v}) + x_v(b_{u,v} - b_{u,u}),
\]
and
\[
x_u x_v(b_{w,u} - b_{w,v}) + x_v x_w(b_{u,w} - b_{u,u}) - x_u x_w(b_{v,w} - b_{v,v}), \quad w \neq u, v.
\]
Proof. Suppose \( p_{1,j} \neq 0 \) and \( p_{2,j} \neq 0 \).

We have that \( \theta(\ell_j) = \ell_j(A_{1,j}x_1 + \cdots + A_{k,j}x_k), A_{i,j} \in \mathbb{K}, \) leading to

\[
L_1x_1p_{1,j} + \cdots + L_kx_kp_{k,j} = (p_{1,j}x_1 + \cdots + p_{k,j}x_k)(A_{1,j}x_1 + \cdots + A_{k,j}x_k).
\]

Identifying coefficients, out of all the equations one obtains the following equations relevant to our calculations:

\[
p_{1,j}(b_{1,1} - A_{1,j}) = 0
\]
\[
p_{2,j}(b_{2,2} - A_{2,j}) = 0
\]

and

\[
p_{1,j}(b_{2,1} - A_{2,j}) + p_{2,j}(b_{1,2} - A_{1,j}) = 0
\]
\[
p_{1,j}(b_{u,1} - A_{u,j}) + p_{u,j}(b_{1,u} - A_{1,j}) = 0, \ u \geq 3
\]
\[
p_{2,j}(b_{u,2} - A_{u,j}) + p_{u,j}(b_{2,u} - A_{2,j}) = 0, \ u \geq 3.
\]

Since \( p_{1,j}, p_{2,j} \neq 0 \), we have \( A_{1,j} = b_{1,1} \) and \( A_{2,j} = b_{2,2} \), from the first equations, and from the second group of equations we have

\[
p_{1,j}(b_{2,1} - b_{2,2}) + p_{2,j}(b_{1,2} - b_{1,1}) = 0
\]

and for all \( u \geq 3 \)

\[
A_{u,j} = b_{u,1} + \frac{p_{u,j}}{p_{1,j}}(b_{1,u} - b_{1,1})
\]
\[
= b_{u,2} + \frac{p_{u,j}}{p_{2,j}}(b_{2,u} - b_{2,2}).
\]

From these one obtains that the dual point to the line \( \ell_j = 0 \), belongs to the ideal of \( R \) generated by

\[
x_1(b_{2,1} - b_{2,2}) + x_2(b_{1,2} - b_{1,1})
\]

and

\[
\{x_1x_2(b_{u,1} - b_{u,2}) + x_2x_u(b_{1,u} - b_{1,1}) - x_1x_u(b_{2,u} - b_{2,2})\}_{u \geq 3}.
\]

□

If \( j \geq k + 1 \), \( \ell_j \) has at least two non-zero coefficients. If \( 1 \leq i \leq k \), \( \ell_i = x_i \) and the dual point to this hyperplane belongs to \( V(I_{u,v}) \) for any \( u, v \neq i \). Summing up we obtain the following:

**Corollary 3.2.** If a hyperplane arrangement \( \mathcal{A} \) has a quadratic logarithmic derivation then the points dual to the hyperplanes of \( \mathcal{A} \) lie on the variety

\[
\bigcup_{1 \leq u < v \leq k} V(I_{u,v}), \text{ where each ideal } I_{u,v} \text{ is defined as in Lemma 3.7}
\]
3.1. The case of line arrangements in $\mathbb{P}^2$. In this subsection we classify the line arrangements in $\mathbb{P}^2$ having a quadratic logarithmic derivation not a multiple of a linear logarithmic derivation. In what follows $\mathcal{A}$ has defining linear forms $\ell_1 = x, \ell_2 = y, \ell_3 = z$, and $\ell_i = \alpha_i x + \beta_i y + \gamma_i z, i \geq 4$.

From Corollary 3.2 we have the points dual to the lines, meaning $[\alpha_i, \beta_i, \gamma_i]$, sitting on $V(I_{xy}) \cup V(I_{xz}) \cup V(I_{yz})$,

where

$$I_{xy} = \langle x(b_{2,1} - b_{2,2}) + y(b_{1,2} - b_{1,1}), xyz(b_{3,1} - b_{3,2}) + yz(b_{1,3} - b_{1,1}) - xz(b_{2,3} - b_{2,2}) \rangle$$

$$I_{xz} = \langle x(b_{3,1} - b_{3,3}) + z(b_{1,3} - b_{1,1}), xz(b_{2,1} - b_{2,3}) + yz(b_{1,2} - b_{1,1}) - xy(b_{3,2} - b_{3,3}) \rangle$$

$$I_{yz} = \langle y(b_{3,2} - b_{3,3}) + z(b_{2,3} - b_{2,2}), yz(b_{1,2} - b_{1,3}) + xz(b_{2,1} - b_{2,3}) - xy(b_{3,1} - b_{3,3}) \rangle.$$

Denote $b_{2,1} - b_{2,2} = a_1, b_{1,2} - b_{1,1} = b_1, b_{3,1} - b_{3,3} = a_2, b_{1,3} - b_{1,1} = c_2, b_{3,2} - b_{3,3} = b_3, b_{2,3} - b_{2,2} = c_3$. Then our ideals of interest become:

$$I_{xy} = \langle a_1 x + b_1 y, y(a_2 x + c_2 z) - x(b_3 y + c_3 z) \rangle$$

$$I_{xz} = \langle a_2 x + c_2 z, z(a_1 x + b_1 y) - x(b_3 y + c_3 z) \rangle$$

$$I_{yz} = \langle b_3 y + c_3 z, z(a_1 x + b_1 y) - y(a_2 x + c_2 z) \rangle.$$

**Theorem 3.3.** Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2$, having a quadratic syzygy on the Jacobian ideal of its defining polynomial, not a multiple of a linear syzygy. Then, up to a change of coordinates, $\mathcal{A}$ is one of the following three types of arrangements with defining polynomial:

1. $F = xyz(x + y) \prod_j (y + t_j z), t_j \neq 0$.
2. $F = xyz(x + y + z) \prod_j (y + t_j z), t_j \neq 0$.
3. $F = xyz(x + y + z)(x + z)(y + z)$.

**Proof.** A couple of observations are in place:

- The point $[0, 0, 1]$ (dual to $\ell_3$) is in $V(I_{xy})$, the point $[0, 1, 0]$ (dual to $\ell_2$) is in $V(I_{xz})$, and the point $[1, 0, 0]$ (dual to $\ell_1$) is in $V(I_{yz})$.
- If the zero locus of any of the three ideals contains more than 3 points, then the corresponding ideal will have codimension 1, and hence will be generated by the linear generator (if the coefficients of this are not zero). This comes from the fact that if the codimension of such an ideal is 2, then the zero locus will be a finite set of points, and since the ideal is generated by a linear form and a quadric, by Bézout’s theorem we can have at most 2 points in this zero locus (exactly 2 if the line and the conic intersect transversally).

The proof goes through several cases enforced by these two bullets.
CASE 1: Suppose $n_1 \geq 2$ dual points have the first two coordinates different than zero. Then, from Lemma 3.1 these points belong to $V(I_{xy})$. From the two bullets above, $\text{codim}(I_{xy}) = 1$.

CASE 1.1: Suppose $a_1 \neq 0$ and $\text{codim}(I_{xy}) = 1$.

If $b_1 = 0$ then these $n_1$ points will be on $V(a_1x)$, hence their first coordinate will be zero. Contradiction. So $b_1 \neq 0$, and these $n_1$ points have homogeneous coordinates $[b_1, -a_1, t]$, for some $t \in \mathbb{K}$.

Also, $\text{codim}(I_{xy}) = 1$ implies $a_1x + b_1y$ divides $y(a_2x + c_2z) - x(b_3y + c_3z)$, which is true if and only if $a_2 = b_3, c_3 = -a_1w, c_2 = b_1w$, for some $w \in \mathbb{K}$.

We can have $t = 0$, leading to the point $[b_1, -a_1, 0]$, and the corresponding dual linear form $b_1x - a_1y$; or $t \neq 0$ and in this last situation $[b_1, -a_1, t] \in V(I_{xz}) \cap V(I_{yz})$ (as all the coordinates of this point are different than zero, and from Lemma 3.1). So $a_2b_1 + c_2t = 0$, leading to $wt = -a_2$.

CASE 1.1.1: $a_2 = 0$.

If $t \neq 0$, then $w = 0$ and so $a_2 = b_3 = c_2 = c_3 = 0$. This leads to $I_{xy} = \langle a_1x + b_1y \rangle, I_{xz} = I_{yz} = \langle z(a_1x + b_1y) \rangle$.

Looking at $\ell_i = \alpha_i x + \beta_i y + \gamma_i z, i \geq 4$, if $\gamma_u = 0$, for some $u$, then none of the corresponding $\alpha_u$ or $\beta_u$ can be zero, as we would obtain $\ell_1$ or $\ell_2$. So the dual point $[\alpha_u, \beta_u, 0]$ has the first two coordinates $\neq 0$, the setup of CASE 1. Hence $a_1\alpha_u + b_1\beta_u = 0$, equivalently obtaining the linear form $b_1x - a_1y$; same situation when $t = 0$.

If $\gamma_u \neq 0$, then we obtain again that the first two coordinates of the dual points of $\ell_i, i \geq 4$ must satisfy the equation $a_1x + b_1y = 0$.

This leads to the only possibility of $A$ having the defining polynomial: $F = xyz(b_1x - a_1y) \prod(b_1x - a_1y + \gamma_j z), \gamma_j \neq 0$. After an appropriate change of coordinates one gets

$$F = x(x + y)yz \prod(y + \gamma_j z), \gamma_j \neq 0,$$

which is a type (1) arrangement.

CASE 1.1.2: $a_2 \neq 0$.

Then $w \neq 0$ and $t = -a_2/w \neq 0$. We are still in the situation $b_1 \neq 0$, see the beginning of CASE 1.1. So $c_2 = b_1w \neq 0$.

First possibility. Suppose there exists another (dual) point, different than $[0, 1, 0]$ and $[b_1, -a_1, -a_2/w]$, and with the first and last coordinate different than zero. Then, by Lemma 3.1 this point belongs to $V(I_{xz})$, and hence it has homogeneous coordinates $[c_2, t', -a_2] = [b_1, t'/w, -a_2/w]$, for some $t' \in \mathbb{K}$.
Similarly as before, one must have

\[ \text{CASE 1.2} \]

By the second bullet, \( \text{codim}(I_{xz}) = 1 \), and hence \( a_2x + c_2z \) divides \( z(a_1x + b_1y) - x(b_3y + c_3z) \). Since \( c_2 \neq 0 \), one gets that \( a_1 = b_3 \) and \( b_1 = a_2w', c_3 = -c_2w', \) for some \( w' \in \mathbb{K} \).

Putting everything together we have the conditions

\[ a_1 = a_2 = b_3 \neq 0 \quad \text{and} \quad b_1 = a_2w', c_3 = -c_2w', c_3 = -a_1w, c_2 = b_1w. \]

Second possibility. Suppose in addition to the First possibility above, there is an extra dual point with the last two coordinates different than zero, and different than \( [b_1, -a_1, -a_2/w] \). This extra point must have coordinates \( [t'', c_3, -b_3] = [t''/w, -a_1, -a_2/w] \) (from the conditions expressed above). Similarly as before, one must have \( t'' = 0 \), and therefore this extra point is \( [0, -a_1w, -a_1] \).

i. If First and Second possibilities occur, then the defining polynomial of \( \mathcal{A} \) is of the form \( xyz(c_2x - wa_1y - a_1z)(c_2x - a_1z)(-a_1wy - a_1z). \)

After an appropriate change of coordinates one gets

\[ F = xyz(x + y + z)(x + z)(y + z), \]

which is the braid arrangement \( A_3 \), or type (3) in our statement.

ii. If just the First possibility occurs, then the defining polynomial of \( \mathcal{A} \) is

\[ xyz(c_2x - wa_1y - a_1z)(c_2x - a_1z), w \neq 0, \]

an arrangement of type (1) of 5 lines.

iii. If none of the possibilities occur, then one obtains the defining polynomial of \( \mathcal{A} \) is

\[ xyz(c_2x - wa_1y - a_1z), w \neq 0, \]

an arrangement of type (2) of 4 lines.

CASE 1.2: Suppose \( a_1 = b_1 = 0 \) and \( \text{codim}(I_{xy}) = 1 \). Then,

\[ I_{xy} = \langle y(a_2x + c_2z) - x(b_3y + c_3z) \rangle \]

\[ I_{xz} = \langle a_2x + c_2z, x(b_3y + c_3z) \rangle \]

\[ I_{yz} = \langle b_3y + c_3z, y(a_2x + c_2z) \rangle. \]

CASE 1.2.1: Suppose one of the \( n_1 \) points also has the third coordinate different than zero. So this point is of the form \( [\alpha, \beta, \gamma] \), with \( \alpha, \beta, \gamma \neq 0. \).
Lemma 3.1 implies that these coordinates must satisfy also the equations
\[ a_2 \alpha + c_2 \gamma = 0 \]
\[ b_3 \beta + c_3 \gamma = 0. \]

**Situation i.** If \( a_2 \neq 0 \) and \( b_3 \neq 0 \), then this point must be \([c_2/a_2, c_3/b_3, -1]\). Also \( c_2, c_3 \neq 0 \).

If there is another dual point with the first and last coordinate not equal to zero, and different than this point, then, from the two bullets at the beginning of the proof, \( \text{codim}(I_{xz}) = 1 \) leading to \( a_2 x + c_2 z \) dividing \( x(b_3 y + c_3 z) \). But under the conditions \( a_2, b_3, c_2, c_3 \neq 0 \), this is impossible.

For this situation we obtain \( \mathcal{A} \) with defining polynomial
\[ F = xyz(c_2 x/a_2 + c_3 y/b_3 - z) \prod_j (\alpha_j x + \beta_j y), \alpha_j, \beta_j \neq 0, \]
which, after a change of coordinates is a type (2) arrangement in our statement.

**Situation ii.** If \( a_2 \neq 0 \) and \( b_3 = 0 \), then \( c_3 = 0 \) and \( c_2 \neq 0 \). Then our ideals are
\[ I_{xy} = I_{yz} = (y(a_2 x + c_2 z)), I_{xz} = (a_2 x + c_2 z). \]

Looking at \( \ell_i = \alpha_i x + \beta_i y + \gamma_i z, i \geq 4 \), if \( \beta_u = 0 \), for some \( u \), then the corresponding \( \alpha_u, \gamma_u \neq 0 \), as we would obtain \( \ell_1 \) or \( \ell_3 \). So this point is of the form \( [\alpha_u, 0, \gamma_u] \in V(I_{xz}) \); therefore \( a_2 \alpha_u + c_2 \gamma_u = 0 \), and therefore obtaining \( \ell_u = c_2 x - a_2 z \).

If \( \beta_u \neq 0 \), since \( [\alpha_u, \beta_u, \gamma_u] \in V(I_{xy}) \cup V(I_{xz}) \cup V(I_{yz}) \), we obtain \( a_2 \alpha_u + c_2 \gamma_u = 0 \), as well.

In this situation one obtains
\[ F = xyz(c_2 x - a_2 z) \prod_j (c_2 x + \beta_j y - a_2 z), \beta_j \neq 0, \]
which after a change of coordinates is an arrangement of type (1) in the statement.

**Situation iii.** If \( a_2 = 0 \) and \( b_3 \neq 0 \), then \( c_2 = 0 \) and \( c_3 \neq 0 \). This is a similar situation as **Situation ii**.

**Situation iv.** If \( a_2 = b_3 = 0 \), then \( c_2 = c_3 = 0 \), and with \( a_1 = b_1 = 0 \) (the setup of CASE 1.2), one obtains \( L_1 = L_2 = L_3 \), leading to \( xF_x + yF_y + zF_z = 0 \), contradiction.

CASE 1.2.2: Suppose all the \( n_1 \) points have the last coordinate equal to zero. Then, since they are points on \( V(I_{xy}) \), one must have \( a_2 = b_3 \).

Also, if the linear forms different than these \( n_1 \) are \( \ell_1 = x, \ell_2 = y \) and \( \ell_3 = z \), then \( \mathcal{A} \) is a pencil of lines and a line at infinity, which is the case of
Section 2 about arrangements with linear syzygies. Since we exclude this particular case, we can assume that there must exist a point with the first and last coordinate not zero. This extra point must be in $V(I_{xz})$, so it must satisfy the equations

$$a_2x + c_2z = 0$$
$$a_2xy + c_3xz = 0.$$

So there can exist only one such extra point: $[c_2, c_3, -a_2]$. In this case, the defining polynomial looks like:

$$F = xyz(x + c_3y + z) \prod_j (\alpha_j x + \beta_j y), \alpha_j, \beta_j \neq 0,$$

which after a change of coordinates is of type (1) if $c_3 = 0$, and it is of type (2) if $c_3 \neq 0$.

**CASE 2:** Suppose that we have exactly one dual point with the first two coordinates nonzero, exactly one point with the first and last coordinates nonzero, and exactly one point with the last two coordinates nonzero. Then it is not difficult to see that we obtain a type (3) arrangement.

**Remark 3.4.** In the setup of Theorem 3.3, using [4, Proposition 3.6] one obtains that the singular locus of $\mathcal{A}$ lies on a cubic curve. First observe that in all the three types presented this is indeed the case, the cubic being a union of three lines.

Second, let us consider an arrangement of 5 generic lines. The singular locus consists of 10 points which are in sufficiently general position such that there is no cubic passing through all of them; every time one requires for a cubic to pass to such a point the dimension of the space of cubics drops by one, starting with the dimension of plane cubics being equal to 10. So $\mathcal{A}$ cannot have as a subarrangement an arrangement of 5 generic lines.

If this second observation would give a less computational and more inspiring proof for Theorem 3.3 we would be happy to see it.

We end with a remark regarding a possible generalization of Theorem 3.3 to hyperplane arrangements in arbitrary number of variables and having a quadratic minimal syzygy (i.e., quadratic minimal logarithmic derivation).

**Remark 3.5.** The proof of Theorem 2.1 is based on the primary decomposition of a certain edge ideal. The similar ideal of interest in the case of quadratic logarithmic derivation is $I_A := \bigcap \{I_{u,v} \mid 1 \leq u < v \leq k\}$, where each ideal $I_{u,v}$ is defined as in Lemma 3.1. It would be really interesting to be able to follow the same approach and find the primary decomposition of $I_A$. This way
the proof of Theorem 3.3 would become really short, similar to Example 2.2.

Does the ideal $I_A$ have a meaning beyond the Corollary 3.2?

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Department of Mathematics, University of Idaho, Moscow, Idaho 83844-1103, USA

E-mail address: tohaneanu@uidaho.edu