CUBATURE FORMULAS ON COMBINATORIAL GRAPHS

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Abstract. Many contemporary applications, for example, cataloging of galaxies, document analysis, face recognition, learning theory, image processing, operate with a large amount of data which is often represented as a graph embedded into a high dimensional Euclidean space. The variety of problems arising in contemporary data processing requires development on graphs such topics of the classical harmonic analysis as Shannon sampling, splines, wavelets, cubature formulas. The goal of the paper is to establish cubature formulas on finite combinatorial graphs. The results have direct applications to problems that arise in connection with data filtering, data denoising and data dimension reduction.

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1. Introduction

Many contemporary applications, for example, cataloging of galaxies, document analysis, face recognition, learning theory, image processing, (see [8] for references) operate with a large amount of data which is often represented as a combinatorial graph embedded into a high dimensional Euclidean space. In such situation any processing of data depends of ones ability to perform Harmonic analysis of functions (signals) defined on graphs and not on the regular Euclidean spaces. By Harmonic analysis on graphs we understand not only the traditional analog of Fourier analysis as the spectral analysis of the Laplace-Beltrami operator. The variety of problems arising in contemporary data processing requires development on graphs such classical themes as Shannon sampling, splines, wavelets, cubature formulas. All these topics are definitely underdeveloped in the setting of graphs. The goal of the paper is to establish cubature formulas on finite graphs. The present article utilizes our previous work on Shannon sampling and splines on combinatorial graphs [3]-[7]. Our results can find applications to problems that arise in connection with data filtering, data denoising and data dimension reduction.

Given a real-valued function \( f \) on a set of vertices \( V \) of a combinatorial graph its "integral" is \( \sum_{v \in V} f(v) \). Our goal is to develop a set of rules (cubature formulas)
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which allow for approximate or exact evaluation of integrals by using values of a function on a subset \( U \subset V \) of vertices.

In section 2 we review our results [4] about variational interpolating spines on graphs and describe an algorithm which allows an effective computation of variational splines. In section 3 by using interpolating splines we develop a set of cubature formulas. Theorem 3.1 shows that these formulas are exact on the set of variational splines. Theorem 3.2 provides an approximate cubature formula for all functions on the graph and Theorem 3.4 shows that this cubature formula is asymptotically exact for bandlimited functions of low bandwidths. At the end of section 3 it is proved that cubature formulas based on interpolating splines are essentially optimal (Theorem 3.5).

As it is explained in the Example after Theorem 3.4 for a cycle graph of 1000 vertices a set of about 670 ”uniformly” distributed vertices is sufficient to have asymptotically exact cubature formulas for linear combinations of the first 290 eigenfunctions (out of 1000) of the corresponding combinatorial Laplace operator.

It is worth to note that all results of section 3 which provide errors of approximation of integrals of functions on \( V \) through their values on a \( U \subset V \) reflect
1) geometry of \( U \) which is inherited into the quantity \( \sqrt{|V| - |U|} = \sqrt{|S|} \) and into the Poincare constant \( \Lambda \) (see section 3 for definitions),
2) smoothness of functions which is measured in terms of combinatorial Laplace operator.

In section 4 we develop a different set of cubature formulas which are exact on appropriate sets of bandlimited functions. The results in this section are formulated in the language of frames and only useful if it is possible to calculate dual frames explicitly. Since in general it is not easy to compute a dual frame we finish this section by explaining another approximate cubature formula which is based on the so-called frame algorithm. As the formula (4.24) shows the rate of convergence of this cubature rule depends solely of the way subset \( U \) (from which all the samples are taken) is embedded into entire graph \( V \). To be more specific it depends on the ratio of the number \( K_0(U) \) of ”in going” edges to the number \( D_0(U) \) of ”out going” edges.

2. Variational (polyharmonic) splines on graphs

We consider finite or infinite and in this case countable connected graphs \( G = (V(G), E(G)) \), where \( V(G) = V \) is its set of vertices and \( E(G) = E \) is its set of edges. We consider only simple (no loops, no multiple edges) undirected unweighed graphs. A number of vertices adjacent to a vertex \( v \) is called the degree of \( v \) and denoted by \( d(v) \). We assume that degrees of all vertices are bounded from above and we use notation

\[
d(G) = \max_{v \in V} d(v).
\]

The space \( L_2(G) \) is the Hilbert space of all real valued functions \( f : V \rightarrow \mathbb{R} \) with the following inner product

\[
\langle f, g \rangle = \sum_{v \in V} f(v)g(v)
\]
and the following norm
\[ \|f\| = \|f\|_0 = \left( \sum_{v \in V} |f(v)|^2 \right)^{1/2}. \]

Let \( A \) be the adjacency matrix of \( G \) and \( D \) be a diagonal matrix whose entrees on main diagonal are degrees of the corresponding vertices. Then we consider the following version of the discrete Laplace operator on \( G \)
\[(2.1) \quad L = D - A,
\]
or explicitly
\[ Lf(v) = \sum_{u \sim v} (f(v) - f(u)), f \in L^2(G), \]
where notation \( u \sim v \) means that \( u \) and \( v \) are adjacent vertices.

Operator(matrix) \( L \) is symmetric and positive definite. Let \( E_\omega(L) \) be the span of eigenvectors of \( L \) whose corresponding eigenvalues are \( \leq \omega \). The invariant subspace \( E_\omega(L) \) is the space of all vectors in \( L^2(G) \) on which \( L \) has norm \( \omega \). In other words \( f \) belongs to \( E_\omega(L) \) if and only if the following inequality holds
\[(2.2) \quad \|Ls f\| \leq \omega s \|f\|, s \geq 0. \]

**Variational Problem**

Given a subset of vertices \( U = \{u\} \subset V \), a sequence of real numbers \( y = \{y_u\} \in l^2, u \in U \), a natural \( k \), and a positive \( \varepsilon > 0 \) we consider the following variational problem:

Find a function \( Y \) from the space \( L^2(G) \) which has the following properties:
1) \( Y(u) = y_u, u \in U \),
2) \( Y \) minimizes functional \( Y \rightarrow \| (\varepsilon I + L)^k Y \| \).

We show that the above variational problem has a unique solution \( Y_{U,y,k,\varepsilon} \).

For the sake of simplicity we will also use notation \( Y_{\overline{\mathcal{Y}},k,\varepsilon} \) assuming that \( U \) and \( \varepsilon \) are fixed.

We say that \( Y_{\overline{\mathcal{Y}},k,\varepsilon} \) is a variational spline of order \( k \). It is also shown that every spline is a linear combination of fundamental solutions of the operator \( (\varepsilon I + L)^k \) and in this sense it is a polyharmonic function with singularities. Namely it is shown that every spline satisfies the following equation
\[(2.3) \quad (\varepsilon I + L)^{2k} Y_{\overline{\mathcal{Y}},k,\varepsilon} = \sum_{u \in U} \alpha_u \delta_u,
\]
where \( \{\alpha_u\}_{u \in U} = \left\{ \alpha_u(Y_{\overline{\mathcal{Y}},k,\varepsilon}) \right\}_{u \in U} \) is a sequence from \( l^2 \) and \( \delta_u \) is the Dirac measure at a vertex \( u \in U \). The set of all such splines for a fixed \( U \subset V \) and fixed \( k > 0, \varepsilon \geq 0 \), will be denoted as \( \mathcal{Y}(U,k,\varepsilon) \).

A fundamental solution \( F_{k}^u = F_{k,\varepsilon}^u \), \( u \in V \), of the operator \( (\varepsilon I + L)^k \) is the solution of the equation
\[(2.4) \quad (\varepsilon I + L)^{k} F_{k}^u = \delta_u, k \in \mathbb{N}, \]
where \( \delta_u \) is the Dirac measure at \( u \in V(G) \). It follows from \([23]\) that the following representation holds

\[
Y^\mathbf{y}_k = \sum_{u \in U} \alpha_u F^u_{2k}.
\]

It is shown in \([4]\) that for every set of vertices \( U = \{u\} \), every natural \( k \), every \( \varepsilon \geq 0 \), and for any given sequence \( \mathbf{y} = \{y_u\} \in l_2 \), the solution \( Y^\mathbf{y}_k \) of the Variational Problem has a representation

\[
Y^\mathbf{y}_k = \sum_{u \in U} y_u L^u_k,
\]

where \( L^u_k \) is the so-called Lagrangian spline, i.e., it is a solution of the same Variational Problem with constraints \( L^u_k(v) = \delta_{u,v} \), \( u \in U \), where \( \delta_{u,v} \) is the Kronecker delta. It implies in particular, that \( \mathcal{Y}(U, k, \varepsilon) \) is a linear set.

Given a function \( f \in L^2(G) \) we will say that the spline \( Y^f_k \) interpolates \( f \) on \( U \) if \( Y^f_k(u) = f(u) \) for all \( u \in U \).

**Algorithm for computing variational splines.**

The above results give a constructive way for computing variational splines. Suppose we are going to construct splines which have prescribed values on a subset of vertices \( U \subset V \).

1. One has to solve the following \(|U|\) systems of linear equations of the size \(|V| \times |V|\)

\[
(\varepsilon I + \mathcal{L})^k F^u_{k,\varepsilon} = \delta_u, \quad u \in U, \quad k \in \mathbb{N},
\]

in order to determine functions \( F^u_{k,\varepsilon} \).

2. Let \( \delta_{w,v} \) be the Kronecker delta. One has to solve \(|U|\) linear system of the size \(|U| \times |U|\) to determine coefficients \( \alpha^w_u \)

\[
\delta_{w,\gamma} = \sum_{u \in U} \alpha^w_u F^u_{k,\varepsilon}(\gamma), \quad w, \gamma \in U.
\]

3. It gives the following representation of the corresponding Lagrangian spline

\[
L^w_{k,\varepsilon} = \sum_{u \in U} \alpha^w_u F^u_{k,\varepsilon}, \quad w \in U.
\]

4. Every spline \( Y^y_{s,\varepsilon} \in \mathcal{Y}(U, s, \varepsilon) \) which takes prescribed values \( \mathbf{y} = \{y_w\}, w \in U \), can be written explicitly as

\[
Y^y_{s,\varepsilon} = \sum_{w \in W} y_w L^w_{s,\varepsilon}.
\]

**3. Cubature formulas associated with variational interpolating splines**

**Theorem 3.1.** In the same notations as above for every subset of vertices \( U = \{u\} \) and every \( k \in \mathbb{N}, \varepsilon > 0 \), there exists a set of weights \( \theta_u = \theta_u(U, k, \varepsilon) \in \mathbb{R}, u \in U \), such that for every spline \( Y^U_y \) that takes values \( Y^U_y(u) = y_u, u \in U \), the following exact formula holds

\[
\sum_{v \in V} Y^U_y(v) = \sum_{u \in U} y_u \theta_u
\]
Proof. According to the above if $L_{U,u}^{k,\varepsilon}$ is the Lagrangian spline, i.e.
$$L_{U,u}^{k,\varepsilon}(v) = \delta_{u,v}$$
for $u \in U$, $v \in V$, then the following representation holds
$$Y_{U,f}^{k,\varepsilon}(v) = \sum_{u \in U} f(u)L_{U,u}^{k,\varepsilon}(v), \ v \in V. \tag{3.2}$$
By introducing scalars
$$\theta_u = \theta_u(U, k, \varepsilon) = \sum_{v \in V} L_{U,u}^{k,\varepsilon}(v)$$
for a subset $S \subseteq V$ (finite or infinite) the notation $L_2(S)$ will denote the space of all functions from $L_2(G)$ with support in $S$:
$$L_2(S) = \{ \phi \in L_2(G), \phi(v) = 0, v \in V \setminus S \}. \tag{3.3}$$
**Definition 1.** We say that a set of vertices $S \subseteq V$ is a Λ-set if for any $\phi \in L_2(S)$ it admits a Poincare inequality with a constant $\Lambda = \Lambda(S) > 0$
$$\|\phi\| \leq \Lambda \|L_2 \phi\|, \ \phi \in L_2(S). \tag{3.4}$$
The infimum of all $\Lambda > 0$ for which $S$ is a Λ-set will be called the Poincare constant of the set $S$ and denoted by $\Lambda(S)$.

The following Theorem gives a cubature rule that allows to compute the integral
$$\sum_{v \in V} f(v)$$
by using only values of $f$ on a smaller set $U$.

**Theorem 3.2.** For every set of vertices $U \subseteq V$ for which $S = V \setminus U$ is a Λ-set and for any $\varepsilon > 0$, $k = 2^l$, $l \in \mathbb{N}$, there exist weights $\theta_u = \theta_u(U, k, \varepsilon) \in \mathbb{R}$ such that for every function $f \in L_2(G)$,
$$\left| \sum_{v \in V} f(v) - \sum_{u \in U} f(u)\theta_u \right| \leq 2\sqrt{|S|} \Lambda^k \| (\varepsilon I + L)^k f \|. \tag{3.5}$$

Proof. If $f \in L_2(G)$ and $Y_{k,\varepsilon}^{U,f}$ is a variational spline which interpolates $f$ on a set $U = V \setminus S$ then
$$\left| \sum_{v \in V} f(v) - \sum_{u \in V} Y_{U,k,\varepsilon}^{U,f}(v) \right| \leq \sum_{u \in S} \left| f(v) - Y_{k,\varepsilon}^{U,f}(v) \right| \leq \sqrt{|S|} \| f - Y_{k,\varepsilon}^{U,f} \|. \tag{3.6}$$
Since $S$ is a Λ- set we have
$$\| f - Y_{k,\varepsilon}^{U,f} \| \leq \Lambda \| L \left( f - Y_{k,\varepsilon}^{U,f} \right) \|. \tag{3.7}$$
For any $g \in L_2(G)$ the following inequality holds true
$$\| L g \| \leq \| (\varepsilon I + L) g \|. \tag{3.8}$$
Thus one obtains the inequality
$$\| f - Y_{k,\varepsilon}^{U,f} \| \leq \Lambda \| (\varepsilon I + L) \left( f - Y_{k,\varepsilon}^{U,f} \right) \|. \tag{3.9}$$
The following lemma holds true [3].
Lemma 3.3. If $A$ is a bounded self-adjoint positive definite operator in a Hilbert space $H$ and for an $\varphi \in H$ and a positive $a > 0$ the following inequality holds true
\[ \|\varphi\| \leq a\|A\varphi\|, \]
then for the same $\varphi \in H$, and all $k = 2^l$, $l = 0, 1, 2, \ldots$ the following inequality holds
\[ \|\varphi\| \leq a^k\|A^k\varphi\|. \]

We apply this lemma with $A = \varepsilon I + L$, $a = \Lambda$ and $\varphi = f - Y_{k,\varepsilon}^{U,f}$. It gives the inequality
\[ (3.9) \quad \left\| f - Y_{k,\varepsilon}^{U,f} \right\| \leq \Lambda^k \left\| (\varepsilon I + L)^k \left( f - Y_{k,\varepsilon}^{U,f} \right) \right\| \]
for all $k = 2^l$, $l = 0, 1, 2, \ldots$. Using the minimization property of $Y_{k,\varepsilon}^{U,f}$ we obtain
\[ \left\| f - Y_{k,\varepsilon}^{U,f} \right\| \leq 2\Lambda^k \left\| (\varepsilon I + L)^k f \right\|, k = 2^l, l \in \mathbb{N}. \]
Together with (3.5) it gives
\[ (3.10) \quad \left| \sum_{v \in V} f(v) - \sum_{u \in U} Y_{k,\varepsilon}^{U,f}(v) \right| \leq 2\sqrt{|S|\Lambda^k\| (\varepsilon I + L)^k f \|}, k = 2^l, l \in \mathbb{N}. \]

By applying the Theorem 3.1 we finish the proof. □

The Bernstein inequality (2.2) and the last Theorem imply the following result.

Corollary 3.1. For every set of vertices $U \subset V$ for which $S = V \setminus U$ is a $\Lambda$-set and for any $\varepsilon > 0$, $k = 2^l$, $l \in \mathbb{N}$, there exist weights $\theta_u = \theta_u(U, k, \varepsilon) \in \mathbb{R}$ such that for every function $f \in E_\omega(L)$, the following inequality holds
\[ (3.11) \quad \left| \sum_{v \in V} f(v) - \sum_{u \in U} f(u)\theta_u \right| \leq 2\gamma^k \sqrt{|S|\| f \|}, \]
where $\gamma = \Lambda(\omega + \varepsilon)$, $k = 2^l$, $l \in \mathbb{N}$.

If in addition the following condition holds
\[ 0 < \omega < \frac{1}{\Lambda} - \varepsilon \]
and $f \in E_\omega(L)$ then this Corollary imply the following Theorem.

Theorem 3.4. If $U$ is a subset of vertices for which $S = V \setminus U$ is a $\Lambda$-set then for any $0 < \varepsilon < 1/\Lambda$, $k = 2^l$, $l \in \mathbb{N}$, there exist weights $\theta_u = \theta_u(U, k, \varepsilon) \in \mathbb{R}$ such that for every function $f \in E_\omega(L)$, where
\[ 0 < \omega < \frac{1}{\Lambda} - \varepsilon, \]
the following relation holds
\[ (3.12) \quad \left| \sum_{v \in V} f(v) - \sum_{u \in U} f(u)\theta_u \right| \to 0, \]
when $k = 2^l \to \infty$. 
Example 1.
Consider the cycle graph $C_{1000}$ of 1000 vertices. The Laplace operator $\mathcal{L}$ as it defined in (2.1) has one thousand eigenvalues which are given by the formula
\[ \lambda_k = 2 - 2 \cos \frac{2\pi k}{1000}, \quad k = 0, 1, \ldots, 999 \] (see [1]).

It is easy to verify that every single vertex in $C_{1000}$ is a $\Lambda = \sqrt{6}$-set. It is also easy to understand that if closures of two vertices do not intersect i.e.,
\[ (v_j \cup \partial v_j) \cap (v_i \cup \partial v_i) = \emptyset, \quad v_j, v_i \in C_{1000}, \]
then their union $v_j \cup v_i$ is also a $\Lambda = \sqrt{6}$-set. It implies, that one can remove from $C_{1000}$ every third vertex and on the remaining set of 670 the formula (3.12) will be true for the span of about 290 first eigenfunctions of $\mathcal{L}$.

Example 2.
One can show [3] that if $S = \{v_1, v_2, \ldots, v_N\}$ consists of $|S| \leq 998$ successive vertices of the graph $C_{1000}$ then it is a $\Lambda$-set with
\[ \Lambda = \frac{1}{2} \left( \sin \frac{\pi}{2|S| + 2} \right)^{-2}. \]

It implies for example that on a set of 100 uniformly distributed vertices of $C_{1000}$ the formula (3.12) will be true for every function in the span of about 40 first eigenfunctions of $\mathcal{L}$.

It is worth to note that the above formulas are optimal in the sense it is described bellow.

Definition 2. For the given $U \subset V, f \in L^2(G), k \in \mathbb{N}, \varepsilon \geq 0, K > 0$, the notation $Q(U, f, k, \varepsilon, K)$ will be used for a set of all functions $h$ in $L^2(G)$ such that

1) $h(u) = f(u), u \in U$,

and

2) $\| (\varepsilon I + \mathcal{L})^k h \| \leq K$.

It is easy to verify that every set $Q(U, f, k, \varepsilon, K)$ is convex, bounded, and closed. It implies that the set of all integrals of functions in $Q(U, f, k, \varepsilon, K)$ is an interval i.e.
\[ [a, b] = \left\{ \sum_{v \in V} h(v) : h \in Q(U, f, k, \varepsilon, K) \right\} \]

The optimality result is the following.

Theorem 3.5. For every set of vertices $U \subset V$ and for any $\varepsilon > 0, k = 2^l, l \in \mathbb{N}$, if $\theta_u = \theta_u(U, k, \varepsilon) \in \mathbb{R}$ are the same weights that appeared in the previous statements, then for any $g \in Q(U, f, k, \varepsilon, K)$
\[ \sum_{u \in U} g(u) \theta_u = \frac{a + b}{2}, \]
where $[a, b]$ is defined in (3.13).
Proof. We are going to show that for a given function $f$ the interpolating spline $Y_{k,\varepsilon}^{U,f}$ is the center of the convex, closed and bounded set $Q(U, f, k, \varepsilon, K)$ for any $K \geq \left\| (\varepsilon I + \mathcal{L})^k Y_{k,\varepsilon}^{U,f} \right\|$. In other words it is sufficient to show that if $Y_{k,\varepsilon}^{U,f} + h \in Q(U, f, k, \varepsilon, K)$ for some function $h$ then the function $Y_{k,\varepsilon}^{U,f} - h$ also belongs to the same intersection.

Indeed, since $h$ is zero on the set $U$ then according to (2.3) one has

$$\langle (\varepsilon I + \mathcal{L})^k Y_{k,\varepsilon}^{U,f}, (\varepsilon I + \mathcal{L})^k h \rangle = \langle (\varepsilon I + \mathcal{L})^{2k} Y_{k,\varepsilon}^{U,f}, h \rangle = 0.$$ 

But then

$$\left\| (\varepsilon I + \mathcal{L})^k \left( Y_{k,\varepsilon}^{U,f} + h \right) \right\| \leq K$$

and because $Y_{k,\varepsilon}^{U,f} + h$ and $Y_{k,\varepsilon}^{U,f} - h$ take the same values on $U$ the function $Y_{k,\varepsilon}^{U,f} - h$ belongs to $Q(U, f, k, \varepsilon, K)$. From here the Theorem follows. □

Corollary 3.2. Fix a function $f \in L^2(G)$ and a set of vertices $U \subset V$ for which $S = V \setminus U$ is a $\Lambda$-set. Then for any $\varepsilon > 0$, $k = 2^l$, $l \in \mathbb{N}$, for the same set of weights $\theta_u = \theta_u(U, k, \varepsilon) \in \mathbb{R}$ that appeared in the previous statements the following inequalities hold for every function $g \in Q(U, f, k, \varepsilon, K)$,

$$\left| \sum_{v \in V} g(v) - \sum_{u \in U} f(u) \theta_u \right| \leq \sqrt{|S|} \Lambda^k \text{diam} Q(U, f, k, \varepsilon, K).$$

Proof. Since $f$ and $g$ coincide on $U$ from (3.5) and (3.9) we obtain the inequality

$$\left| \sum_{v \in V} g(v) - \sum_{v \in V} Y_{k,\varepsilon}^{U,f}(v) \right| \leq \sqrt{|S|} \Lambda^k \left\| (\varepsilon I + \mathcal{L})^k \left( f - Y_{k,\varepsilon}^{U,f} \right) \right\|$$

By the Theorem 3.5 the following inequality holds

$$\left\| (\varepsilon I + \mathcal{L})^k \left( Y_{k,\varepsilon}^{U,f} - g \right) \right\| \leq \frac{1}{2} \text{diam} Q(U, f, k, \varepsilon, K)$$

for any $g \in Q(U, f, k, \varepsilon, K)$. The last two inequalities imply the Corollary. □

4. Cubature formulas for bandlimited functions

We introduce another set of cubature formulas which are exact on some sets of bandlimited functions.

Theorem 4.1. If $U$ is a subset of vertices for which $S = V \setminus U$ is a $\Lambda$-set then there exist weights $\sigma_u = \sigma_u(U) \in \mathbb{R}$, $u \in U$, such that for every function $f \in E_\omega(\mathcal{L})$, where

$$0 < \omega < \frac{1}{\Lambda},$$

...
the following exact formula holds

\( \sum_{v \in V} f(v) = \sum_{u \in U} f(u) \sigma_u, \ U = V \setminus S, \)

**Proof.** First, we show that the set \( U \) is a uniqueness set for the space \( E_\omega(L) \), i.e. for any two functions from \( E_\omega(L) \) the fact that they coincide on \( U \) implies that they coincide on \( V \).

If \( f, g \in E_\omega(L) \) then \( f - g \in E_\omega(L) \) and according to the inequality (2.2) the following holds true

\( \| L(f - g) \| \leq \omega \| f - g \| \).

If \( f \) and \( g \) coincide on \( U = V \setminus S \) then \( f - g \) belongs to \( L^2(S) \) and since \( S \) is a \( \Lambda \)-set then we will have

\( \| f - g \| \leq \Lambda \| L(f - g) \|, \ f - g \in L^2(S). \)

Thus, if \( f - g \) is not zero and \( \omega < 1/\Lambda \) we have the following inequalities

\( \| f - g \| \leq \Lambda \| L(f - g) \| \leq \Lambda \omega \| f - g \| < \| f - g \|, \)

which contradict the assumption that \( f - g \) is not identical zero. Thus, the set \( U \) is a uniqueness set for the space \( E_\omega(L) \).

It implies that there exists a constant \( C = C(U, \omega) \) for which the following Plancherel-Polya inequalities hold true

\( \left( \sum_{u \in U} | f(u) |^2 \right)^{1/2} \leq \| f \| \leq C \left( \sum_{u \in U} | f(u) |^2 \right)^{1/2} \)

for all \( f \in E_\omega(L) \). Indeed, the functional

\( ||| f ||| = \left( \sum_{u \in U} | f(u) |^2 \right)^{1/2} \)

defines another norm on \( E_\omega(L) \) because the condition \( ||| f ||| = 0, f \in E_\omega(L) \), implies that \( f \) is identical zero on entire graph. Since in finite-dimensional situation any two norms are equivalent we obtain existence of a constant \( C \) for which (4.4) holds true.

Let \( \delta_v \in L^2(G) \) be a Dirac measure supported at a vertex \( v \in V \). The notation \( \vartheta_v \) will be used for a function which is orthogonal projection of the function

\( \frac{1}{\sqrt{d(v)}} \delta_v \)

on the subspace \( E_\omega(L) \). If \( \varphi_0, \varphi_1, \ldots, \varphi_j(\omega) \) are orthonormal eigenfunctions of \( L \) which constitute an orthonormal basis in \( E_\omega(L) \) then the explicit formula for \( \vartheta_v \) is

\( \vartheta_v = \sum_{j=0}^{(j)} \varphi_j(v) \varphi_j. \)

In these notations the Plancherel-Polya inequalities (1.4) can be written in the form

\( \sum_{u \in U} | \langle f, \vartheta_u \rangle |^2 \leq \| f \|^2 \leq C^2 \sum_{u \in U} | \langle f, \vartheta_u \rangle |^2, \)

where \( f, \vartheta_u \in E_\omega(L) \) and \( \langle f, \vartheta_u \rangle \) is the inner product in \( L^2(G) \). These inequalities mean that if \( U \) is a uniqueness set for the subspace \( E_\omega(L) \) then the functions
\(\{\vartheta_u\}_{u \in U}\) form a frame in the subspace \(E_\omega(\mathcal{L})\) and the tightness of this frame is \(1/C^2\). This fact implies that there exists a frame of functions \(\{\Theta_u\}_{u \in U}\) in the space \(E_\omega(\mathcal{L})\) such that the following reconstruction formula holds true for all \(f \in E_\omega(\mathcal{L})\)

\[
f(v) = \sum_{u \in U} f(u) \Theta_u(v), \quad v \in V.
\]

(4.7)

By setting \(\sigma_u = \sum_{v \in V} \Theta_u(v)\) one obtains (4.1).

Unfortunately this approach does not give any information about constant \(C\) in (4.6) and it make realization of the Theorem 4.1 problematic. We are going to utilize another approach to the Plancherel-Polya-type inequality which was developed in our paper [7] and which produces explicit constant.

For any \(U\) which is a subset of vertices of \(G\) we introduce the following operator

\[
cl^0(U) = U, \quad cl(U) = S \cup \partial U, \quad cl^m(U) = cl\left(cl^{m-1}(U)\right), \quad m \in \mathbb{N}, U \subset V.
\]

We will use the following notion of the relative degree.

**Definition 3.** For a vertex \(v \in cl^m(U)\) we introduce the relative degree \(d_m(v)\) as the number of vertices in the boundary \(\partial(cl^m(U))\) which are adjacent to \(v\):

\[
d_m(v) = \text{card}\{w \in \partial(cl^m(U)) : w \sim v\}.
\]

For any \(S \subset V\) we introduce the following notation

\[
D_m = D_m(U) = \sup_{v \in cl^m(U)} d_m(v).
\]

**Definition 4.** For a vertex \(v \in \partial(cl^m(U))\) we introduce the quantity \(k_m(v)\) as the number of vertices in the set \(cl^m(U)\) which are adjacent to \(v\):

\[
k_m(v) = \text{card}\{w \in cl^m(U) : w \sim v\}.
\]

For any \(U \subset V\) we introduce the following notation

\[
K_m = K_m(U) = \inf_{v \in \partial(cl^m(U))} k_m(v).
\]

For a given set \(U \subset V\) and a fixed \(n \in \mathbb{N}\) consider a sequence of closures \(U, cl(U),..., cl^n(U), n \in \mathbb{N}\).

**Theorem 4.2.** In the same notations as above if the condition

\[
cl^n(U) = V
\]

(4.9) and the inequality

\[
\omega < \frac{1}{4} \left( \sum_{j=0}^{n-1} \frac{1}{K_j} \prod_{i=j+1}^{n-1} \left( \frac{2D_i}{K_i} + 1 \right) \right)^{-1}
\]

(4.10)

hold, then for any \(f \in E_\omega(\mathcal{L})\) the next inequality takes place

\[
\frac{1}{1-\gamma} \left( \sum_{u \in U} |f(u)|^2 \right)^{1/2} \leq \|f\| \leq \frac{1}{1-\gamma} \left( \prod_{i=0}^{n-1} \left( \frac{2D_i}{K_i} + 1 \right) \right)^{1/2} \left( \sum_{u \in U} |f(u)|^2 \right)^{1/2},
\]

(4.11)
where

\[ \gamma = 2^{\omega^{1/2}} \left( \frac{1}{\sum_{j=0}^{n-1} \prod_{i=j+1}^{n-1} \left( \frac{2D_i}{K_i} + 1 \right)} \right)^{1/2} < 1. \]

**Theorem 4.3.** If \( U \) is a subset of vertices for which condition (4.9) is satisfied then there exists a set of weights \( \mu_u \in \mathbb{R}, u \in U \), such that for any \( f \in E_\omega(\mathcal{L}) \), where \( \omega \) satisfies (4.10) the following exact formula holds

\[ \sum_{v \in V} f(v) = \sum_{u \in U} f(u) \mu_u. \]

**Proof.** The previous Theorem shows that \( U \) is a uniqueness set for the space \( E_\omega(\mathcal{L}) \), which means that every \( f \) in \( E_\omega(\mathcal{L}) \) is uniquely determined by its values on \( U \). This fact implies existence of a frame \( \{ \phi_u \}, u \in U, \) in \( E_\omega(\mathcal{L}) \) such that

\[ \frac{1}{1 - \gamma} \left( \sum_{u \in U} \| (f, \phi_u) \|^2 \right)^{1/2} \leq \| f \| \leq \left( \sum_{u \in U} \| (f, \phi_u) \|^2 \right)^{1/2}, \]

where \( \gamma \) as in (4.12).

Now, by using a frame \( \{ \Phi_u \} \) which is dual to the frame \( \{ \phi_u \} \) and by setting \( \mu_u = \sum_{v \in V} \Phi_u(v) \) we obtain the Theorem. \( \square \)

All the results in this section are only useful if it is possible to calculate dual frame explicitly. Since it is not a simple task we are going to offer a certain way around which is based on the so-called frame algorithm \([2]\), Ch. 5.

Let \( \{ e_j \} \) be a frame in a Hilbert \( H \) space with frames bounds \( A, B \), i.e.

\[ A\| f \|^2 \leq \sum_j \| (f, e_j) \|^2 \leq B\| f \|^2, \quad f \in H. \]

Given a relaxation parameter \( 0 < \lambda < \frac{2}{A+B} \), set \( \delta = \max\{|1 - \lambda A|, |1 - \lambda B|\} < 1. \)

Let \( f_0 = 0 \) and define recursively

\[ f_n = f_{n-1} + \lambda S(f - f_{n-1}), \]

where \( S \) is the frame operator which is defined on \( H \) by the formula

\[ Sf = \sum_{j} \langle f, e_j \rangle e_j. \]

In particular, \( f_1 = \lambda Sf = \lambda \sum_j \langle f, e_j \rangle e_j \). Then \( \lim_{n \to \infty} f_n = f \) with a geometric rate of convergence, that is,

\[ \| f - f_n \| \leq \delta^n \| f \|. \]

Note, that for the choice \( \lambda = \frac{2}{A+B} \) the convergence factor is

\[ \delta = \frac{B - A}{A + B}. \]
Let us go back to our situation in which we consider a subset of vertices \( U \subset V \). Just for the sake of simplicity we will consider the case when

\[
U = U \cup \partial U = V.
\]

It can be useful to remind that in this case the relative degree \( d_0(v) \) is the number of vertices in the boundary \( \partial U \) which are adjacent to a \( v \in U \):

\[
d_0(v) = \text{card} \{ w \in \partial U : w \sim v \}
\]

and

\[
D_0 = D_0(U) = \sup_{v \in U} d_0(v).
\]

For a vertex \( v \in \partial U \), the quantity \( k_0(v) \) is the number of vertices in the set \( U \) which are adjacent to \( v \):

\[
k_0(v) = \text{card} \{ w \in U : w \sim v \}
\]

and

\[
K_0 = K_0(U) = \inf_{v \in \partial U} k_0(v).
\]

We consider the Hilbert space \( H = E_\omega(\mathcal{L}) \), where

\[
\omega < \frac{K_0(U)}{4}.
\]

The set \( U \) is a uniqueness set for the space \( E_\omega(\mathcal{L}) \) and the corresponding frame in it is \( \{ \phi_u \}, u \in U \). The frame operator \( S \) takes the following form

\[
Sf = \sum_{u \in U} \langle f, \phi_u \rangle \phi_u, \quad f \in E_\omega(\mathcal{L}),
\]

and

\[
\langle f, \phi_u \rangle = f(u), \quad f \in E_\omega(\mathcal{L}), \quad u \in U.
\]

Thus the recurrence sequence (4.15) takes the form \( f_0 = 0 \), and

\[
f_n = f_{n-1} + \lambda \sum_{u \in U} (f - f_{n-1})(u) \phi_u,
\]

We are ready to state the following fact which provides a reconstruction method for functions in \( E_\omega(\mathcal{L}) \) from their values on \( U \).

**Theorem 4.4.** Under the assumptions (4.18), (4.19) for \( f \in E_\omega(\mathcal{L}) \) the following inequality holds for all natural \( n \)

\[
\|f - f_n\| \leq \delta^n \|f\|,
\]

where the convergence factor is

\[
\delta = \frac{D_0(U)}{D_0(U) + K_0(U)}.
\]

**Proof.** The frame inequality (4.12) takes the form

\[
\frac{1}{1 - \gamma} \left( \sum_{u \in U} |f(u)|^2 \right)^{1/2} \leq \|f\| \leq \frac{1}{1 - \gamma} \sqrt{\frac{2D_0(U)}{K_0(U)} + 1} \left( \sum_{u \in U} |f(u)|^2 \right)^{1/2},
\]

where \( f \in E_\omega(\mathcal{L}) \), \( f(u) = \langle f, \phi_u \rangle, u \in U \), and

\[
\gamma = 2\sqrt{\omega/K_0(U)} < 1.
\]
It shows that the frame bounds of the frame \( \{ \phi_u \} \) are
\[
A = (1 - \gamma)^2 \frac{K_0(U)}{2D_0(U) + K_0(U)}, \quad B = (1 - \gamma)^2.
\]

For \( f \in E_\omega(L) \) and \( f_n \) defined in (4.15) one has the inequality (4.16) and according to (4.17) the convergence factor is
\[
(4.24) \quad \delta = \frac{D_0(U)}{D_0(U) + K_0(U)}.
\]

The Theorem is proved. \( \square \)

Since
\[
\delta = \frac{D_0(U)}{D_0(U) + K_0(U)} = \frac{1}{1 + \frac{K_0(U)}{D_0(U)}}
\]

it becomes clear that the best convergence occurs for such sets \( U \) which maximize the ratio of the number \( K_0(U) \) of "in going" edges to the number \( D_0(U) \) of "out going" edges.

By setting
\[
\nu_u = \sum_{v \in V} \phi_u(v), \quad u \in U,
\]
we obtain the recurrence sequence for corresponding "integrals"
\[
\sum_{v \in V} f_0(v) = 0,
\]
\[
\sum_{v \in V} f_1(v) = \lambda \sum_{u \in U} f(u)\nu_u,
\]
\[
\sum_{v \in V} f_n(v) = \sum_{v \in V} f_{n-1}(v) + \lambda \sum_{u \in U} (f - f_{n-1})(u)\nu_u.
\]

Thus, the "integral" \( \sum_{v \in V} f_n(v) \) for every \( f_n \) is expressed in terms of weights \( \{ \nu_u \}, \quad u \in U \), and values of \( f \) on the uniqueness set \( U \).

We are ready to state the following fact which provides an approximate cubature formula for functions in \( E_\omega(L) \).

**Theorem 4.5.** Under the assumptions (4.18), (4.19) for \( f \in E_\omega(L) \) and \( \sum_{v \in V} f_n(v) \) defined in (4) we have
\[
\left| \sum_{v \in V} f(v) - \sum_{v \in V} f_n(v) \right| \leq \delta^n \sqrt{|V|} \| f \|
\]
where the convergence factor is
\[
\delta = \frac{D_0(U)}{D_0(U) + K_0(U)}.
\]

The proof is obvious:
\[
\left| \sum_{v \in V} f(v) - \sum_{v \in V} f_n(v) \right| \leq \sum_{v \in V} |f(v) - f_n(v)| \leq \sqrt{|V|} \| f - f_n \| \leq \delta^n \sqrt{|V|} \| f \|.
\]

**Example 3.**

If one will consider the cycle graph \( C_{1000} \) and will remove from it every third vertex then the set \( U \) will contain about 670 remaining vertices and the space \( E_\omega(L) \)
will have dimension 290 (see Example 1). In this case $D_0(U) = 1$, $K_0(U) = 2$ and the convergence factor $\delta = 1/3$.

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