One dimensional systems with singular perturbations.

J.J. Alvarez\textsuperscript{1}, M. Gadella\textsuperscript{2}, L.M. Glasser\textsuperscript{3}, L.P. Lara\textsuperscript{4}, L.M. Nieto\textsuperscript{2}.
\textsuperscript{1}Departamento de Informática, E.U. de Informática, Universidad de Valladolid, 40005 Segovia, Spain
\textsuperscript{2}Departamento de FTAO, University of Valladolid, 47071 Valladolid, Spain
\textsuperscript{3}Department of Physics, Clarkson University, Potsdam, NY 13699-5820, USA
\textsuperscript{4}Departamento de Sistemas, FRRO, Zevallos 1345, Rosario, Argentina.
E-mail: jjalvarez@infor.uva.es, manuelgadella1@gmail.com, laryg@clarkson.edu, lplara@fceia.unr.edu.ar, luismi@metodos.fam.cie.uva.es

Abstract. This paper discusses some one dimensional quantum models with singular perturbations. Eventually, a mass discontinuity is added at the points that support the singular perturbations. The simplest model includes an attractive singular potential with a mass jump both located at the origin. We study the form of the only bound state. Another model exhibits a hard core at the origin plus one or more repulsive deltas with mass jumps at the points supporting these deltas. We study the location and the multiplicity of these resonances for the case of one or two deltas and settle the basis for a generalization. Finally, we consider the harmonic oscillator and the infinite square well plus a singular potential at the origin. We see how the energy of bound states is affected by the singular perturbation.

1. Introduction
In this brief review, we shall introduce some one dimensional models showing singular interactions type Dirac delta and its derivative plus mass jumps at the points supporting these singularities. The motivation for this study is in one side to construct solvable models in basic quantum mechanics and on the other to use these simple models as building blocks for more complicated but still solvable models showing interesting features like resonances. Then, it seems quite natural that the first situation under our consideration is the simplest that includes the above mentioned ingredients: one singularity at the origin, in this case a linear combination of a delta and its first derivative, plus a mass jump at the same point.

The singular potential is introduced by making use of the formalism of self adjoint extensions of symmetric operators, which has been extensively studied in the literature \cite{1, 2, 3}. In physics models, singular potentials have been studied by several authors \cite{4}. Hamiltonians with position dependent mass have been object of study in numerous publications \cite{6, 7}. A particular case of particular interest appears when the dependence of the mass with the position is constant except for a jump at one or several points. This is a situation of physical interest. For instance in one dimensional models in which a particle moves along a wire formed by two distinct materials that join abruptly at some point \cite{8}.

In the simplest problem under our consideration mentioned above, we use an attractive delta in order to study the resulting bound state. With no surprise we obtain that its energy depends...
on the chosen self adjoint extension.

Now, assume a one dimensional free particle being allowed to move on the positive semiaxis only. This implies the existence of an impenetrable barrier at the origin (also called a hard core). If we add a potential formed by two repulsive Dirac deltas located at points \( a, b > 0 \), then the system shows resonances, which can be double (double poles of the \( S \) matrix) in the case of two deltas [5]. Then, it is quite interesting to study the influence of the mass jumps in the behavior of the resonances and on their multiplicity.

In the last section of the present review, we include a brief discussion on the influence of a singular interaction on two well know exactly solvable models: the harmonic oscillator and the infinite square well, both in one dimension. Now, the singularity can be either an attractive or a repulsive delta plus its derivative, although no discontinuity in the mass is included in these latter cases.

2. Delta potential with a mass jump
To start with, let us consider the most general form of a one dimensional free Hamiltonian with non constant mass. It is given by

\[
K = \frac{1}{2} m^\alpha(x) p m^\beta(x) p m^\alpha(x),
\]

with \( 2\alpha + \beta = -1 \) and \( m(x) \) is the function that gives the mass in terms of the position. In this paper, we are going to assume that the mass is constant save for discontinuities at some given points (mass jumps). A mass jump at the origin gives the following function \( m(x) \):

\[
m(x) = m_1 H(-x) + m_2 H(x) = \begin{cases} m_1 & \text{if } x < 0, \\ m_2 & \text{if } x > 0, \end{cases}
\]

(2)

where \( H(x) \) is the Heaviside step function.

In this first simple example the total Hamiltonian has the form \( K + W \), where \( W \) is the singular perturbation given by

\[
W(x) := -a\delta(x) + b\delta'(x),
\]

(3)

with \( a > 0, b \in \mathbb{R} \). This gives the following formal Hamiltonian:

\[
H = K + W = \frac{1}{2} m^\alpha(x) p m^\beta(x) p m^\alpha(x) - a\delta(x) + b\delta'(x),
\]

(4)

where the singular part has to be defined by using the formalism of self adjoint extensions of symmetric operators [13, 2, 3]. These self adjoint extensions are characterized by their domains. In the present case, the domain is specified by choosing, for the functions in this domain, convenient matching conditions at the origin. Then, functions in the domain \( \mathcal{D}(H) \) of \( H \) as in (4) are expected to have a discontinuity at the origin. Any wave function \( \psi(x) \in \mathcal{D}(H) \) should satisfy the Schrödinger equation for \( H \), which is

\[
K\psi(x) - a\delta(x)\psi(x) + b\delta'(x)\psi(x) = E\psi(x).
\]

(5)

Since \( \psi(x) \) is in general discontinuous at the origin, we need a prescription to determine expressions like \( \delta(x)\psi(x) \) and \( \delta'(x)\psi(x) \). The simplest uses the following averages [3]:

\[
\psi(x)\delta(x) = \frac{\psi(0+) + \psi(0-)}{2} \delta(x),
\]

\[
\psi(x)\delta'(x) = \frac{\psi(0+) + \psi(0-)}{2} \delta'(x) - \frac{\psi'(0+) + \psi'(0-)}{2} \delta(x),
\]

(6)
where \( \varphi(0+) \) and \( \varphi(0-) \) denote right and left limits of the function \( \varphi(x) \) discontinuous at the origin, respectively. Note that (5) is an equation in distributions.

Although this problem can be solved for arbitrary real values of \( \alpha \) and \( \beta \) with \( 2\alpha + \beta = -1 \) [9], here we shall discuss the simpler case in which the kinetic term of this linear combination are all equal to zero. The resulting expressions give the matching equation (5) with the kinetic term given in (7). To do it, we just write in (5) \( \psi \) which corresponds to the choice \( \alpha = 0, \beta = -1 \). Then, our goal is to solve the Schrödinger equation (5) with the kinetic term given in (7). To do it, we just write in (5) \( \psi(x) = f_1(x)H(-x) + f_2(x)H(x) \) and and use (6) and (7). Then, we obtain a linear combination of \( H(-x), H(x), \delta(x) \) and \( \delta(x) \), which is equal to zero. This is only possible if the coefficients of this linear combination are all equal to zero. The resulting expressions give the matching conditions for \( \psi(x) \) at the origin

\[
\begin{pmatrix}
\psi(0+) \\
\psi'(0+)
\end{pmatrix} = \begin{pmatrix}
\frac{h^2}{\hbar^2-m_2^2} & 0 \\
-\frac{ah^2(m_1+m_2)}{\hbar^2-m_2^2} & \frac{h^2}{\hbar^2-m_2^2}
\end{pmatrix} \begin{pmatrix}
\psi(0-) \\
\psi'(0-)
\end{pmatrix},
\]

the energy of the bound state,

\[
E = -\frac{\hbar^2}{2} \frac{(m_1+m_2)^2 a^2 \hbar^4}{[\hbar^2-m_1 b (\hbar^2-m_2 b) \sqrt{m_1} + (\hbar^2+m_1 b)(\hbar^2+m_2 b) \sqrt{m_2}]} \tag{9}
\]

and the wave function of the bound state

\[
\psi(x) = A \left[ (\hbar^2-m_2 b) e^{k_1 x} H(-x) + (\hbar^2+m_1 b) e^{-k_2 x} H(x) \right], \tag{10}
\]

with \( k_i = \sqrt{-2Em_i/\hbar^2} \), \( i = 1, 2 \) and \( A \) is a normalization constant [9].

Note that (9) is not symmetric with respect to the interchange of \( m_1 \) and \( m_2 \), but is symmetric with respect to the interchange of the notions of left and right for which we have to interchange the value of the masses as well as the sign of \( b \). In the equal masses limit, \( m_1 = m_2 = m \), the value of the energy of the unique bound state is the same which has been obtained by a direct calculation in [10]. This is:

\[
E = -\frac{1}{2} \frac{ma^2 \hbar^6}{(\hbar^4 + m^2 b^2)^2}. \tag{11}
\]

In the next section, we shall study the conditions under which the Hamiltonian \( H = K + V \) is self adjoint. One unexpected consequence of these conditions is that \( b \) in (3,4) cannot be arbitrary, although for \( m_1 \neq m_2 \) it must have the form [9]:

\[
b^2 = \frac{\hbar^4}{m_1^2 + m_2^2 + m_1 m_2}. \tag{12}
\]

For \( m_1 = m_2 \) this restriction makes no sense [9]. This conditions also imply that if \( b = 0 \), then \( m_1 = m_2 \). The presence of the mass jump plus the delta at the origin requires the presence of the term \( b \delta'(x) \). Without this term with \( b \) given by (11), the proposed model has no solution [9].
3. Self adjointness of the Hamiltonian

In order to define the Hamiltonian $H$ given in the previous section, we start with a domain for the kinetic term $K$ in which $K$ be symmetric with equal deficiency indices. One such domain could be

$$\mathcal{D} := \{\psi(x) \in W^2_2(\mathbb{R}) / \psi(0) = \psi'(0) = 0\},$$

(13)

where $W^2_2(\mathbb{R})$ is the linear space of functions $\psi(x) : \mathbb{R} \to \mathbb{C}$, $\mathbb{C}$ being the field of complex numbers, such that i.) $\psi(x)$ admits a first continuous derivative; ii.) the second derivative of $\psi(x)$ exists almost everywhere and iii.) both $\psi(x)$ and its second derivative $\psi''(x)$ are square integrable, i.e.,

$$\int_{-\infty}^{\infty} \{|\psi(x)|^2 + |\psi''(x)|^2\} \, dx < \infty.$$  

(14)

The adjoint of $K$, $K^\dagger$, has domain $\mathcal{D}^* \equiv W^2_2(\mathbb{R}/\{0\})$. The functions in $W^2_2(\mathbb{R}/\{0\})$ satisfy the same properties of the functions in $W^2_2(\mathbb{R})$ except that they and their derivatives may admit a finite jump at the origin. It is straightforward that $K$ has deficiency indices $(2, 2)$ and therefore it has an infinite number of self-adjoint extensions. These self-adjoint extensions are restrictions of $K^\dagger$ to certain subdomains, which are determined by the matching conditions at the origin [13, 2, 3]. For any two $\varphi, \psi \in \mathcal{D}^*$, we have:

$$\langle \psi | K^\dagger \varphi \rangle = \mathcal{G} + \langle K^\dagger \psi | \varphi \rangle,$$

(15)

where

$$\mathcal{G} = \frac{\hbar^2}{2m_2} [\psi^s(0+)\varphi'(0+) - \psi''^s(0+)\varphi(0+)] - \frac{\hbar^2}{2m_1} [\psi^s(0-)\varphi'(0-) - \psi''^s(0-)\varphi(0-)].$$

(16)

If we define $T$ as

$$\left(\begin{array}{c}
\psi(0+)
\
\psi'(0+)
\end{array}\right) = T \left(\begin{array}{c}
\psi(0-)
\
\psi'(0-)
\end{array}\right),$$

(17)

we have

$$\mathcal{G} = (\psi^s(0-), \psi''^s(0-))[\mathcal{M}_1 + T^\dagger \mathcal{M}_2 T] \left(\begin{array}{c}
\varphi(0-)
\
\varphi'(0-)
\end{array}\right),$$

(18)

with

$$\mathcal{M}_i = \frac{\hbar^2}{2m_i} \left(\begin{array}{cc}
0 & 1
\-1 & 0
\end{array}\right), \quad i = 1, 2.$$  

(19)

From (15), one sees that the domains of the self adjoint extensions of $K$ are those that make $\mathcal{G} \equiv 0$, nontrivially, i.e., allowing non zero limit values of the wave function and its first derivative at the origin. Therefore, according to (18) and taking into account on the arbitrariness of the functions $\psi(x)$ and $\varphi(x)$ the domains of the self adjoint extensions of $K$ are those characterized by the following property of $T$:

$$\mathcal{M}_1 = T^\dagger \mathcal{M}_2 T.$$  

(20)

Note that because of (17), once we have defined $T$ satisfying (20), we have defined a domain and vice versa. Therefore, (20) gives a necessary and sufficient condition for the domain characterized by $T$ be a domain of a self adjoint extension of $K$.

In the case of the Hamiltonian $H = K + W$ studied in the previous section, matrix $T$ was given in equation (8). It is straightforward to show that it satisfies (20) and that consequently $H$ as in (4) is well defined as a legitimate self adjoint extension of $K$. Furthermore, equation (20) restricts the possible values of $b$ to one (and the same with the opposite sign), which in the case of $K$ given by (7) is precisely (12).
4. A one dimensional model of resonances with mass jumps

The next example of one dimensional potential with a singular perturbation is given by

\[ H = K + V_0 + \gamma \delta(x - a), \quad a > 0, \]  

(21)

where \( K \) and \( V_0 \) have the following form:

\[ K = \begin{cases} -\frac{k^2}{2m_1} \frac{d^2}{dx^2} & \text{if } 0 < x < a \\ -\frac{k^2}{2m_2} \frac{d^2}{dx^2} & \text{if } x > a \end{cases}, \quad V_0(x) = \begin{cases} \infty & \text{if } x \leq 0 \\ 0 & \text{if } x > 0 \end{cases}. \]

(22)

Thus, \( H \) defines an impenetrable barrier supported on the negative real semiaxis (hard core) and has a repulsive delta and a mass jump at the point \( x = a \). The simplest and most natural matching conditions at the point \( x = a \) that gives account for our situation would be the following:

\[ \psi(a-) = \psi(a+) = \psi(a), \quad \frac{1}{m_2} \psi'(a+) - \frac{1}{m_1} \psi'(a-) = \frac{2\gamma}{\hbar^2} \psi(a), \]

(23)

since this is the minimal generalization of the matching conditions defining a repulsive delta interaction at \( x = a \) that includes the mass jump. Formulas (23) can be written in matrix form as in (17) with 0 replaced by \( a \). In this case, matrix \( T \) is quite simple

\[ T = \begin{pmatrix} 1 & 0 \\ 2\gamma m_1/m_2 & \gamma m_2/m_1 \end{pmatrix}. \]

(24)

This \( T \) in (24) satisfies (20) and therefore, defines a self adjoint Hamiltonian \( H \).

In the case of a repulsive delta, the model has no bound states, but due to the presence of the hard core, it has resonances. The situation is similar to the constant mass case. If we denote by \( I \) and \( II \) the regions \( 0 < x < a \) and \( a < x \) respectively, the solution of the Schrödinger equation corresponding to the Hamiltonian in (21) with boundary conditions \( \psi_I(0) = 0 \) and \( \psi'_I(0) = 1 \) is

\[ \psi_I(x) = \frac{1}{k_1} \sin(k_1 x), \quad \psi_{II}(x) = Ce^{ik_2 x} + De^{-ik_2 x}, \]

(25)

with \( k_i = \sqrt{2m_i E/\hbar^2}, i = 1, 2 \). Purely outgoing boundary condition will give resonances. These conditions imply the existence of outgoing wave only, which means that \( D(k) = 0 \) with \( k = k_1 = k_2\sqrt{m_1/m_2} \). Resonances come in pairs of poles. In fact, if \( k_0 \) is a solution of \( D(k) = 0 \), then \(-k_0^*\) is also a solution [11]. If we introduce the new variables \( z \) and \( \xi \), defined as \( z = ka, m_2 = m_1\xi^2, \xi > 0 \), and if we choose units in such a way that \( h^2 = 2\gamma a m_1 \), equation \( D = 0 \) gives [11]:

\[ (z + i\xi) \sin z + i\xi z \cos z = 0. \]

(26)

Solutions in \( z \) of this equation are the resonances of the model. In Figs. 1 and 2, we plot the zeroes of the real part of (26) (solid curves) and the zeroes of its imaginary part (dashed curves) for two different values of \( \xi = \sqrt{m_2/m_1} \). Thus, intersections of solid and dashed lines represent resonances.

Search for degenerate resonances.- A resonance of order \( n \) is given by a pole of \( n \)-th order of the \( S \)-matrix or, equivalently, by a zero of \( n \)-th order of \( D(k) \). Resonances of order \( n \geq 2 \) are called multiple pole resonances or degenerate resonances [5]. Note that a necessary condition for the existence of a degenerate resonance is the existence of a simultaneous solution \( k_0 \) of the transcendent equations \( D(k) = D'(k) = 0 \). Thus, if there is no such a solution, no degenerate resonances may exist.
Figure 1. Resonances for $\xi = \sqrt{m_2/m_1} = 1.2$. Dots represent resonances in the limit of equal masses ($\xi = 1$). Here $k = \mu + i\nu$.

Figure 2. Resonances for $\xi = \sqrt{m_2/m_1} = 1.5$. Dots represent resonances in the limit of equal masses ($\xi = 1$). Here $k = \mu + i\nu$.

Then, to search for degenerate resonances, we first need to derive the left hand side of (26) with respect to $k$ and then equalize to zero the resulting expression. It can be shown [11] that both equations do not have simultaneous solutions and that therefore, no degenerate resonances may exist in this model. Consequently, a jump of mass along a delta barrier never has the effect of a double delta barrier [5] in the sense that it does not produce degenerate resonances.

We would like to remark that, in opposition with the situation described in Section 2, derivatives of the delta are not necessary in the present case. In the model described in Section 2, the derivative of the delta was necessary because it naturally appeared when we obtained the second derivative of the wave function with a discontinuity at the origin. In this other case, matching conditions (23) impose a jump in the derivative of the wave function and not on the wave function itself. Thus, the derivative of the delta cancels out when we calculate the explicit form of the Schrödinger equation in this case. In any case and for further consideration, one may also include a perturbation of the type $a\delta(x - a) + b\delta'(x - a)$ along with a mass discontinuity at $x = a$.

4.1. More than one singular point

Once we have solved the problem of finding the resonances of the previous model, we may investigate the situation in which more than one delta is present in the potential. In addition, we consider respective mass jumps at the points supporting the deltas. Thus, if the singular potential is given by $\gamma_1\delta(x - a_1) + \gamma_2\delta(x - a_2) + \ldots + \gamma_n\delta(x - a_n)$, with $0 < a_1 < a_2 < \ldots < a_n$, the function of the mass $m(x)$ should be given by

\[
  m(x) := \begin{cases} 
  m_1 & \text{if } 0 < x < a_1 \\
  m_2 & \text{if } a_1 < x < a_2 \\
  \ldots & \ldots & \ldots \ldots \\
  m_{n+1} & \text{if } a_n < x 
  \end{cases}
\]  

(27)

The generalization of the kinetic term $K$ as in (5) to this situation is obvious. Now, in order to obtain the self adjoint Hamiltonian with the proposed singular potential, we need to determine
matching conditions at all points in the sequence \( \{a_1, a_2, \ldots, a_n\} \). This matching conditions will be given by matrix relations, so that if \( \psi(x) \) is a wave function on the domain of self adjointness of \( H \), it should fulfill at each \( a_i, i = 1, 2, \ldots, n \), the following relation:

\[
\begin{pmatrix}
\psi_{i+1}(a_i) \\
\psi'_{i+1}(a_i)
\end{pmatrix}
= T_i(a_i)
\begin{pmatrix}
\psi_i(a_i) \\
\psi_i'(a_i)
\end{pmatrix},
\]

were we have denoted by \( \psi_i(x) \) the solution of the Schrödinger equation between \( a_{i-1} \) and \( a_i \), \( a_0 = 0 \). Note that \( \psi_i(x) = C_i e^{ik_i x} + D_i e^{-ik_i x} \), where \( k_i = \sqrt{2m_i E/\hbar^2} \). Then,

\[
\begin{pmatrix}
\psi_i(a_i) \\
\psi_i'(a_i)
\end{pmatrix}
= \begin{pmatrix}
 e^{ik_i a_i} & e^{-ik_i a_i} \\
 ik_i e^{ik_i a_i} & -ik_i e^{-ik_i a_i}
\end{pmatrix}
\begin{pmatrix}
 C_i \\
 D_i
\end{pmatrix}
= \mathcal{M}_i(a_i)
\begin{pmatrix}
 C_i \\
 D_i
\end{pmatrix}.
\]

Matrices \( \mathcal{M}_i(a_i) \) are defined in (29). With the help of (29) and (28), we can derive the following recurrence relation:

\[
\begin{pmatrix}
 C_{i+1} \\
 D_{i+1}
\end{pmatrix}
= \mathcal{M}_i^{-1}(a_i) T_i(a_i) \mathcal{M}_i(a_i)
\begin{pmatrix}
 C_i \\
 D_i
\end{pmatrix},
\]

The matrices, \( T_i(a_i) \) have the following form:

\[
T_i(a_i) := \begin{pmatrix}
 1 & 0 \\
 \gamma i n_{i+1} & m_{i+1}/m_i
\end{pmatrix}.
\]

The simplest case with \( n \neq 1 \) is \( n = 2 \). Then, we have two deltas and their respective mass jumps. The half line \( x > 0 \) is divided into three regions. The wave function in each region is expected when to the two deltas we add the mass jumps. Indeed this is the case and numerical estimations have shown the presence of doubly degenerate resonances at the time that deny the existence of resonances of order higher than two [12]. Also, preliminary results have indicated the existence of resonances of order at most three in the case of three deltas with their respective mass jumps at the same points.
5. Exactly solvable models with singular interactions

In this last Section, we shall analyze both the one dimensional harmonic oscillator and the infinite square well with a singular perturbation of the type \(-a\delta(x) + b\delta'(x)\). The coefficient \(a\) could be either positive or negative, and results will be slightly different in each case, and \(b\) is real. Thus, let us take the following Hamiltonian

\[
H = -\frac{d^2}{dx^2} + V_0(x) + W(x), \quad W(x) = -a\delta(x) + b\delta'(x), \quad a > 0, \quad (34)
\]

and either

\[
V_0(x) = \frac{\kappa}{2} x^2, \quad \text{or} \quad V_0(x) = \begin{cases} 
0 & \text{if } |x| < c \\
\infty & \text{if } |x| \geq c 
\end{cases} \quad (35)
\]

In the first case we have a harmonic oscillator and in the second an infinite square well. The singular potential is supported at the origin. In both models no resonances can be expected, so that we shall focus our attention on the behavior of bound states.

This study can be performed in two equivalent manners. Either we may solve directly the Schrödinger equation for each case or we apply the Green function method. Let us briefly outline the second method.

Let us consider a one-dimensional quantum system whose dynamical evolution is governed by two different Hamiltonian operators

\[
H_0 = -\frac{d^2}{dx^2} + V_0(x), \quad H = H_0 + W(x), \quad (36)
\]

the first one is the regular free Hamiltonian \(H_0\) and the second one the perturbed Hamiltonian \(H\) (units are usually chosen such that \(\hbar = 2m = 1\)). Let us assume that \(H_0\) admits a Green function given by \(G_0(x, x', E)\) (the dimension of \(x\) and \(x'\) is here irrelevant). Then, \(\psi(x)\) represents a bound state of the total Hamiltonian \(H\) with energy \(E\) if and only if the Lippman-Schwinger bound state equation

\[
\psi(x) = \int G_0(x, x', E) W(x') \psi(x') dx'. \quad (37)
\]

is satisfied. In consequence, bound states of the total Hamiltonian \(H\) should satisfy the equation given by

\[
\psi(x) = \int G_0(x, x', E) [-a\delta(x') + b\delta'(x')] \psi(x') dx'. \quad (38)
\]

This integral equation is easily solvable in our case. In fact if we use the short notation:

\[
G_0^{ij}(x, x') := \frac{\partial^{i+j}}{\partial x^i \partial x'^j} G_0(x, x', E), \quad i, j = 0, 1 \quad (39)
\]

and

\[
A := G_0(0+, 0) = G_0(0-, 0), \quad B := G_0^{01}(0+, 0) = G_0^{10}(0-, 0), \\
C := G_0^{01}(0-, 0) = G_0^{10}(0+, 0), \quad D := G_0^{11}(0+, 0) = G_0^{11}(0-, 0), \quad (40)
\]

we can insert (39) and (40) into (38). In order to obtain the product of the delta and its derivative times the wave function \(\psi(x')\), we use equations (6). Thus, we obtain an homogeneous system of equations in the unknowns \(\psi(0\pm)\) and \(\psi'(0\pm)\), where \(\varphi(0+)\) and \(\varphi(0-)\) are the right and left limits of the function \(\varphi(x)\) at the origin. This systems admits a solution if and only if the
determinant of the coefficients vanishes. This is a determinant of a $4 \times 4$ matrix, which can be simplified to
\[
\begin{vmatrix}
2 & -1 + \frac{b}{2} (B - C) & 0 \\
-1 & 1 + \frac{a}{2} A + \frac{b}{2} C & \frac{b}{2} A \\
0 & \frac{a}{2} (B + C) + bD & 1 + \frac{b}{2} (B + C)
\end{vmatrix} = 0.
\] (41)

This equation (41) can give the energy of bound states for the cases under our study, described by either Hamiltonian in (34) and (35).

The Green function of the harmonic oscillator is given by
\[
G_0(y, y', E) = \frac{D_\nu(y_>) D_\nu(y_<)}{2D_\nu(0) D'_\nu(0)},
\] (42)

where $D_\nu(y)$ is the parabolic cylinder equation with index $\nu = (2E/\omega - 1)/2$ and we use the standard notation $y_>, y_< = \max, \min(y, y')$. This function, which admits an integral representation, is described in [13]. Using the identities (40), we arrive at
\[
A = -\frac{D_\nu(0)}{2D'_\nu(0)}, \quad B = -C = -\frac{1}{2}, \quad D = \frac{D'_\nu(0)}{2D_\nu(0)} = -\frac{1}{4A}.
\] (43)

For odd oscillator levels, the wave functions of the free oscillator are odd and therefore, they vanish at the origin. In this case, $D_\nu(0) = 0 \iff A = 0$. Here, equation $A = 0$ replaces to (41) and therefore, it should give the bound states for the odd oscillator levels plus the singular potential. Since $A = 0$ is equivalent to $D_\nu(0) = 0$, which happens if and only if $E = \omega[(2n + 1)1/2]$ ($\hbar = 1$), it results that the energy of odd levels does not change under the action of the singular perturbation. For even levels, let us go back to (41). Since now, $B = -C$ this gives $1 + \alpha A - \beta^2 AD = 0$, which together with (43) this yields to
\[
\frac{D_\nu(0)}{D'_\nu(0)} = \frac{4 + \beta^2}{2\alpha}.
\] (44)

Then, using the properties of the parabolic cylinder functions [13], we obtain:
\[
\frac{\Gamma[(1 - 2E/\omega)/4]}{\sqrt{2}\Gamma[(3 - 2E/\omega)/4]} = \frac{4 + \beta^2}{2\alpha} = \frac{4\omega^{1/2} + b^2\omega^{-1/2}}{2a}.
\] (45)

Here, we obtain the following result: the odd oscillator levels remain invariant while the even levels are lowered in energy if $a > 0$ and raised if $a < 0$. It is interesting to remark that the higher the level the smaller is the effect of the perturbation.

For the infinite square well, the Green function is given by [14]
\[
G_0(x, x', k^2) = \frac{\cos[2kc - k|x-x'|] - \cos[k(x + x')]}{2k \sin(2kc)},
\] (46)

where $E = k^2$. Now, a similar analysis that we did for the harmonic oscillator produces the following results [14]: the energy levels with odd wave function remain unaltered while the energy levels with even wave function are either lowered ($a > 0$) or raised ($a < 0$).

**Acknowledgments**

Partial financial support is acknowledged to the Spanish Junta de Castilla y León (Project GR224) and the Ministry of Science and Innovation (Project MTM2009-10751).
References

[1] Albewerio S, Gesztesy F, Hoegh-Krohn R and Holden H 1988 *Solvable Models in Quantum Mechanics* (Springer, Berlin)

[2] Albewerio S and Kurasov P 2000 *Singular perturbations of differential operators. Solvable Schrödinger type operators* (London Mathematical Society Lecture Notes 271, Cambridge UP, Cambridge, UK)

[3] Kurasov P 1996 Jour. Math. Anal. Appl. 201 297

[4] Seba P 1986 Rep. Math. Phys. 24 111

[5] Coutinho FAB, Nogami Y and Fernando Pérez J 1997 J. Phys. A: Math. Gen. 30 3937

[6] Christiansen P L, Arnbak H C, Zolotaryuk A V, Ermakov V N and Gaididei Y B 2003 J. Phys. A: Math. Gen. 36, 7589

[7] Toyama F M and Nogami Y, J. 2007 Phys. A: Math. Theor. 40 F685

[8] Albewerio S, Dabrowski L and Kurasov P 1998 Lett. Math. Phys. 45 33

[9] Filöp T and Tsutsui I 2000 Phys. Lett. A 264 366

[10] Hejcik P and Cheon T 2006 Phys. Lett. A 356 290

[11] Coutinho FAB, Nogami Y and Fernando Pérez J 2005 J. Phys. A: Math. Gen., 38, 7509

[12] Hernández E, Jáuregui A and Mondragón A 2000 Journal of Physics A: Mathematical and General, 33 4507

[13] Antoniou I E et al. 2001 Chaos, Solitons and Fractals, 12 2719

[14] Hernández E, Jáuregui A and Mondragón A, Physical Review E, 72 026221

[15] Moiseyev N and Lefebvre R 2001 Phys. Rev. A 64 052711

[16] Cruz S C Y, Negro J and Nieto L M Phys. Lett. A 2007 369 400

[17] Ganguly A and Nieto L M 2007 J. Phys. A: Math. Theor. 40 7265

[18] Ganguly A, Kuru S Negro J and Nieto L M 2006 Phys. Lett. A, 360 228

[19] Ganguly A, Ioffe M V and Nieto L M 2006 J. Phys. A: Math. Gen., 39 14659

[20] Gadella M, Kuru S and Negro J 2007 Physics Letters A, 362, 265

[21] Lévy-Leblond J M 1995 Phys. Rev. A 52 1845

[22] Morrow R A and Brownstein K R 1984 Phys. Rev. B 30 678

[23] Enevold G T and Hemmer P C 1986 J. Phys. C 11 L1193

[24] Thomsen J, Enevold G T and Hemmer P C 1989 Phys. Rev. B 39 12783

[25] Enevold G T, Hemmer P C and Thomsen J 1990 Phys. Rev. B 42 3485

[26] Gadella M, Heras F J H, Negro J and Nieto L M 2009 J. Phys. A: Math. Theor. 42 465207.

[27] Gadella M, Negro J and Nieto L M 2009 Phys. Lett. A 373 1310

[28] Alvarez J J, Gadella M, Heras F J H, Nieto L M 2009 Phys. Lett. A 373 4022

[29] Alvarez J J, Gadella M, Nieto L M 2010 A study of resonances in a one-dimensional model with singular Hamiltonian and mass jumps Submitted

[30] Abramovich M and Stegun I A 1970 *Handbook of Mathematical Functions* (Dover, New York)

[31] Gadella M, Glasser M L and Nieto L M 2010 The infinite square well with a singular perturbation Submitted