Research Article

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The H-force sets of the graphs satisfying the condition of Ore’s theorem

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Abstract: Let $G$ be a Hamiltonian graph. A nonempty vertex set $X \subseteq V(G)$ is called a Hamiltonian cycle enforcing set (in short, an H-force set) of $G$ if every $X$-cycle of $G$ (i.e., a cycle of $G$ containing all vertices of $X$) is a Hamiltonian cycle. For the graph $G$, $h(G)$ (called the H-force number of $G$) is the smallest cardinality of an H-force set of $G$. Ore’s theorem states that an $n$-vertex graph $G$ is Hamiltonian if $d(u) + d(v) \geq n$ for every pair of nonadjacent vertices $u, v$ of $G$. In this article, we study the H-force sets of the graphs satisfying the condition of Ore’s theorem, show that the H-force number of these graphs is possibly $n$, or $n - 2$, or $\frac{n}{2}$ and give a classification of these graphs due to the H-force number.

Keywords: H-force set, H-force number, Ore’s theorem, weak closure

MSC 2010: 05C07, 05C45

1 Terminology and introduction

In this article, we study simple graphs without loops or parallel edges. For terminology and notations not defined here we refer the reader to [1]. Let $G$ be a graph with $n$ vertices. The vertex set and the edge set of $G$ are denoted by $V(G)$ and $E(G)$, respectively. For a subset $X \subseteq V(G)$, the cardinality of $X$ is denoted by $|X|$. A subgraph of $G$ induced by a subset $X$ is denoted by $G[X]$. In addition, $G - X = G[V(G) - X]$. Let $S$ be a subset of $V(G)$ or a subgraph of $G$ and $v$ is not in $S$. The set $N_G(v) = \{x | vx \in E(G), x \in S\}$ and $d_G(v) = |N_G(v)|$. Let $\delta(G)$ denote the minimum degree of $G$.

A graph $G$ is said to be $k$-connected, if $V(G) \geq k + 1$ and $G - S$ is connected for each $X \subseteq V(G)$ with $|S| \leq k - 1$. The subset $S \subseteq V(G)$ is called an independent set of $G$ if any two vertices in $S$ are nonadjacent. In addition, the subset $S$ is called the largest independent set of $G$ if there is no independent set $S'$ such that $|S'| > |S|$.

For two disjoint graphs $G$ and $H$, $G \cup H$ denotes the graph such that every vertex of $G$ is adjacent to every vertex of $H$. A complete graph with $n$ vertices is denoted by $K_n$. $K_{m,n}$ is a complete bipartite graph with two parts $V_1, V_2$ and $|V_1| = m, |V_2| = n$.

A cycle of $G$ is called the Hamiltonian cycle, if it contains all vertices of $G$. The graph $G$ is said to be Hamiltonian if it has a Hamiltonian cycle. It is well known that the problem to check whether a graph has a Hamiltonian cycle or not is NP-complete. The various kinds of sufficient conditions to imply a graph to be Hamiltonian have been studied widely. More details can be found in [2–5].

Let $G$ be a Hamiltonian graph with $n$ vertices. For a subset $X \subseteq V(G)$, an $X$-cycle of $G$ is a cycle containing all vertices of $X$. A nonempty vertex set $X \subseteq V(G)$ is called a Hamiltonian cycle enforcing set (in short, an H-force set) of $G$ if every $X$-cycle of $G$ is a Hamiltonian cycle. For the graph $G$, $h(G)$ denotes the smallest cardinality of an H-force set of $G$ and is called the H-force number of $G$. The subset $X$ is called the minimum
H-force set of $G$ if $X$ is an H-force set with $|X| = h(G)$, i.e., $X$ is an H-force set and there is no H-force set $X'$ with $|X'| < |X|$.

The H-force set and H-force number were introduced by Fabrici et al. in [6]. These concepts play an important role in the Hamiltonian problems. In [6], the authors also investigated the H-force numbers of the bipartite graphs, the outerplanar Hamiltonian graphs and so on. Recently, these concepts were generalized to digraphs by Zhang et al. in [7] and to hypertournaments by Li et al. in [8], and they gave a characterization of the minimum H-force sets of locally semicomplete digraphs and hypertournaments and obtained their H-force numbers.

If $d(u) + d(v) \geq n$ for every pair of nonadjacent vertices $u$ and $v$ of $G$, we say that $G$ satisfies the condition of Ore’s theorem. For convenience, we call a graph satisfying the condition of Ore’s theorem an OTG. In [9], Ore proved that any OTG is Hamiltonian. In this article, we study the H-force sets and H-force number of the OTGs.

The following is an easy observation on the H-force sets of a graph.

**Proposition 1.1.** [6] Let $G$ be a Hamiltonian graph. If $C$ is a non-Hamiltonian cycle of $G$, then any H-force set of $G$ contains a vertex of $V(G) \setminus V(C)$.

To present Theorem 1.2, we consider a special class of graphs, namely,

$$
\psi_{2m+1} = \{Z_m \cup (K_m^c + \{u\}) | Z_m \text{ is a graph with } m \text{ vertices},
$$

where $K_m^c$ is a set of $m$ vertices (also as the complement of the complete graph $K_m$) and $u$ is another single vertex; furthermore, the edge set of $Z_m \cup (K_m^c + \{u\})$ consists of $E(Z_m)$ and $\{xy | x \in V(Z_m) \text{ and } y \in K_m^c \cup \{u\}\}$ (Figure 1). In [2], Li et al. investigated the graphs satisfying $d(u) + d(v) \geq n - 1$ for every pair of vertices $u$, $v$ with $d(u, v) = 2$ and proved the following.

**Theorem 1.2.** [2] Let $G$ be a 2-connected graph with $n \geq 3$ vertices. If $d(u) + d(v) \geq n - 1$ for every pair of vertices $u$, $v$ with $d(u, v) = 2$, then $G$ is a Hamiltonian graph, unless $n$ is odd and $G \in \psi_n$.

By Theorem 1.2, we obtain immediately the following corollary.

**Corollary 1.3.** Let $G$ be a 2-connected graph with $n \geq 3$ vertices. If $d(u) + d(v) \geq n - 1$ for every pair of nonadjacent vertices $u$, $v$ of $G$, then $G$ is a Hamiltonian graph, unless $n$ is odd and $G \in \psi_n$.

![Figure 1: $G \in \psi_{2m+1}$](image)

**2 H-force sets of the OTGs**

**Lemma 2.1.** Let $G$ be an OTG with $n$ vertices and $X \subseteq V(G)$ be a vertex subset. If $d(u) + d(v) \geq n + 1$ for some pairs of nonadjacent vertices $u$ and $v$, then

1. $X$ is an H-force set of $G$ if and only if $X$ is an H-force set of $G + uv$.
2. $X$ is the minimum H-force set of $G$ if and only if $X$ is the minimum H-force set of $G + uv$, namely, $h(G) = h(G + uv)$.
The H-force sets of the graphs satisfying the condition of Ore's theorem

Proof. First, we prove (1). Note that a cycle of the graph $G$ is always a cycle of the graph $G + uv$. It is easy to see that if $X$ is an H-force set of $G + uv$, then $X$ is an H-force set of $G$.

Suppose that $X$ is an H-force set of $G$, but not an H-force set of $G + uv$. Let $C$ be the longest non-Hamiltonian cycle containing all vertices of $X$ in $G + uv$ and $H$ be the subgraph induced by $V(C)$ in $G$, i.e., $H = G[V(C)]$. Clearly, the cycle $C$ contains the edge $uv$. Assume that there are $a (\geq 1)$ vertices apart from $V(C)$. Then, $H$ contains $n - a$ vertices. Let $C = v_1v_2 \ldots v_{n-a}v_1$ with $u = v_1$ and $v = v_{n-a}$. Then, $v_1v_2 \ldots v_{n-a}$ is a $(u, v)$-path of $G$ containing all vertices of $X$. Let $S = \{v_i | \exists i \in E(H)\}$, $T = \{v_i | \forall v \in E(H)\}$, where the subscripts are taken modulo $n - a$. Since $v_{n-a} \notin S \cup T$, we have $|S \cup T| < n - a = |V(H)|$.

Recall that $C$ is the longest non-Hamiltonian cycle containing all vertices of $X$ in $G + uv$. For $a > 1$, the vertices $u$ and $v$ have no common adjacent vertex apart from $V(C)$. So,

$$d_H(u) + d_H(v) \geq d(u) + d(v) - a \geq n + 1 - a = n - a + 1 = |V(H)| + 1.$$

For $a = 1$, the common adjacent vertex of $u, v$ in $G - C$ is possibly the unique vertex of $G - C$. So,

$$d_H(u) + d_H(v) \geq d(u) + d(v) - 2 \geq n + 1 - 2 = n - 1 = |V(H)|.$$

In both cases, $d_H(u) + d_H(v) \geq |V(H)|$. Since

$$d_H(u) + d_H(v) = |S| + |T| = |S \cup T| + |S \cap T| \geq |V(H)|$$

and $S \cup T < |V(H)|$, then $|S \cap T| > 0$. Let $v_1 \in S \cap T$. Then, there is a non-Hamiltonian cycle $v_1v_2 \ldots v_{n-a}$ $v_{n-a-1} \ldots v_{n-a}v_1$ containing all vertices of $X$ in $G$ (Figure 2). This contradicts with the fact that $X$ is an H-force set of $G$. Thus, if the subset $X$ is an H-force set of $G$, $X$ is an H-force set of $G + uv$.

Figure 2: A nonhamiltonian cycle $v_1v_2 \ldots v_{n-a}v_{n-a-1} \ldots v_1$.

By (1), it is easy to see that (2) holds.

Lemma 2.1 motivates the following definition. The weak closure of an OTG $G$ is the graph obtained from $G$ by recursively joining pairs of nonadjacent vertices whose degree sum is at least $n + 1$ until no such pair remains. We denote the weak closure of $G$ by $C_w(G)$. The idea of the weak closure follows the well-known Bondy-Chvátal's closure of a graph. If we join pairs of nonadjacent vertices with the degree sum at least $n - a$ until no such pair remains, then we obtain the closure of a graph, denoted by $C(G)$. Bondy and Chvátal proved that $C(G)$ is well defined. That means, if $G_1$ and $G_2$ are two graphs obtained from $G$ by recursively joining pairs of nonadjacent vertices whose degree sum is at least $n$ until no such pair remains, then $G_1 = G_2$. By the similar argument, we can see that $C_w(G)$ is well defined.

Lemma 2.1 implies that the H-force set and H-force number of an OTG $G$ are the H-force set and H-force number of $C_w(G)$, respectively. We can obtain the H-force set and H-force number of $G$ by studying the weak closure $C_w(G)$ of $G$. Clearly, for an OTG $G$, the weak closure $C_w(G)$ is a complete graph or a graph satisfying $d_{C_w(G)}(u) + d_{C_w(G)}(v) = n$ for every pair of nonadjacent vertices $u$ and $v$.

Theorem 2.2. Let $G$ be an OTG with $n \geq 5$ vertices and $X$ be the minimum H-force set of $G$. Let $C_w(G)$ be the weak closure of $G$ and $S$ be the largest independent set of $C_w(G)$. Then,

(1) the H-force number $h(G)$ $\in \left\{ \frac{n}{2}, n - 2, n \right\}$, and

$$X = \begin{cases} V(G) \setminus \{x, y\}, & \text{if } h(G) = n - 2, \\ S, & \text{if } h(G) = \frac{n}{2}, \\ V(G), & \text{if } h(G) = n, \end{cases}$$

where $x, y \in V(G)$ with $d_{C_w(G)}(x) = n - 1$ and $d_{C_w(G)}(y) = n - 1$. 


**Proof.** According to Lemma 2.1, it is sufficient to consider the minimum $H$-force set and $H$-force number of the weak closure $C_w(G)$. For convenience, let $G_w = C_w(G)$.

Obviously, the minimum $H$-force set of $G$ is $V(G)$ and $h(G) = n$ if $G_w$ is a complete graph.

So assume that $G_w$ is not a complete graph and satisfies $d_{G_w}(u) + d_{G_w}(v) = n$ for every pair of non-adjacent vertices $u$ and $v$ in $G_w$. Clearly, $2 \leq \delta(G_w) \leq \frac{n}{2}$.

We consider the following two cases.

**Case 1:** $2 \leq \delta(G_w) < \frac{n}{2}$.

Let $\delta(G_w) = a$ and $d_{G_w}(u) = \delta(G_w) = a$ for some $u \in V(G_w)$. Then, $d_{G_w}(u) < \frac{n}{2}$. We use the following notations:

$$U = \{v \in V(G_w) - u : uv \in E(G_w)\}, \quad W = V(G_w) - U - \{u\}.$$ 

Obviously, $|U| = a, |W| = n - a - 1$ (Figure 3).

**Figure 3:** $G_w$ and $G_{11} \in \varphi_1$.

**Claim A.**

(i) For each vertex $w \in W$, $d_{G_w}(w) = n - a$.

(ii) $W$ induces a complete subgraph.

(iii) For each vertex $w \in W$, $d_U(w) = 2$.

Let $w \in W$ and $x, y \in U$ be the vertices adjacent to $w$ in $U$.

(iv) If $U - \{x, y\} \neq \emptyset$, then $d_{U}(p) = a$ for each vertex $p \in U - \{x, y\}$.

(v) If $U - \{x, y\} \neq \emptyset$, then $x$ and $y$ are adjacent to every vertex of $U - \{x, y\}$

(vi) $x$ and $y$ are adjacent to every vertex of $U - \{x, y\}$ if $U - \{x, y\} \neq \emptyset$.

(vii) For any vertex $q \in U - \{x, y\}$, $q$ has no neighbour in $W$.

(viii) $x$ and $y$ are adjacent to every vertex of $W$.

(ix) $d_{G_w}(x) = d_{G_w}(y) = n - 1$ and $U$ induces a complete subgraph.

**Proof.**

(i) For the vertex $w \in W$, since $w$ is not adjacent to $u$, we see that $d_{G_w}(w) = n - d_{G_w}(u) = n - a$.

(ii) For the distinct vertices $w, w' \in W$, we have $d_{G_w}(w) = d_{G_w}(w') = n - a > \frac{n}{2}$, and hence $d_{G_w}(w) + d_{G_w}(w') > n$. So, $w$ and $w'$ are adjacent. By the choice of $w, w'$, $W$ induces a complete subgraph.

(iii) Since $|W| = n - a - 1$ and $W$ is a complete subgraph, we have

$$d_U(w) = d_{G_w}(w) - (n - a - 2) = (n - a) - (n - a - 2) = 2.$$

(iv) Since the vertex $p$ is not adjacent to $w$ and $d_{G_w}(w) = n - a$, we have $d_{G_w}(p) = n - d_{G_w}(w) = n - (n - a) = a$.

(v) If $p, q$ are a pair of non-adjacent vertices in $U - \{x, y\}$, then $d_{U}(p) + d_{U}(q) = n$. But $d_{G_w}(p) + d_{G_w}(q) = 2a < n$. So, $p$ and $q$ are adjacent and $U - \{x, y\}$ induces a complete subgraph.

(vi) By contradiction. Let $p$ be a vertex of $U - \{x, y\}$ such that $x$ is not adjacent to $p$. By (iv), we have $d_{G_w}(x) = n - d_{G_w}(p) = n - a$. For a vertex $w'$ in $W$, $d_{G_w}(w') = n - a$ by (i). Then, $d_{G_w}(x) + d_{G_w}(w') > n$. So, $x$ is adjacent to every vertex of $W$.

Recall that $|W| = n - a - 1$ and $d_{G_w}(x) = n - a$. Hence, $x$ has no neighbour in $U - \{x\}$. In particular, $x$ is not adjacent to $y$. So, $d_{G_w}(y) = n - d_{G_w}(x) = a$. Since the degree of any vertex of $U - \{x, y\}$ is also $a$, the vertex $y$ is adjacent to every vertex of $U - \{x, y\}$. So, $w$ is the only adjacent vertex of $y$ in $W$. Therefore,
$N_{G_w}(x) = \{u\} \cup W, \quad N_{G_w}(y) = \{u\} \cup (U - \{x, y\}) \cup \{w\}.$

In other words, $|\{x\} \cup (W - \{w'\})| \leq N_{G_w}(w')$ for any vertex $w' \in W - \{w\}. By \|(x) \cup (W - \{w'\})\| = n - a - 1$ and $d_{G_w}(w') = n - a$, we see that every vertex of $W - \{w\}$ has exactly one adjacent vertex in $U - \{x, y\}$.

Let $q \in U - \{x, y\}$ be arbitrary. By the arguments above, we see that $\{u\} \cup (U - \{x, q\}) \leq N_{G_w}(q)$. Since $d_{G_w}(q) = a$ (see (iv)) and $\{u\} \cup (U - \{x, q\}) = a - 1$, we have that $q$ has an adjacent vertex in $W - \{w\}$. So, $|U - \{x, y\}| = |W - \{w\}|$. But $a - 2 < n - a - 2$, a contradiction. Thus, $x$ is adjacent to every vertex of $U - \{x, y\}$. Similarly, $y$ is adjacent to every vertex of $U - \{x, y\}$.

(vii) For $q \in U - \{x, y\}$, claims (iv), (v) and (vi) imply that $q$ is not adjacent to any vertex of $W$.

(viii) Claims (iii) and (vii) show that every vertex of $W$ has two adjacent vertices in $U$, which are just $x$ and $y$.

(ix) By (vi) and (viii), $d_{G_w}(x) + d_{G_w}(y) \geq 2(|U - \{x, y\}| + |W| + |\{u\}|) = 2(n - 2) > n$. Then, $x$ and $y$ are adjacent and hence (ix) holds.

Now, $G_w$ is isomorphic to $G_1 = K_2 \vee (K_{a+1} + K_{n-a-1})$ (Figure 3), where $U' = (U - \{x, y\}) \cup \{u\}$. In particular, $d_{G_w}(x) = d_{G_w}(y) = n - 1$, $U'$ and $W$ induce complete subgraphs with $|U'| = a - 1$ and $|W| = n - a - 1$.

We shall show that $X = V - \{x, y\} = U' \cup W$ is the minimum H-force set and hence $h(G_w) = n - 2$.

Obviously, $X$ is an H-force set because a cycle containing all vertices of $X$ must encounter $x$ and $y$. Suppose to the contrary that $X'$ is an H-force set of $G_w$ with $|X'| < |X|$. Clearly, there exists a vertex $z \in U'$ or $z \in W$ such that $z \notin X'$. Without loss of generality, assume $z \in U'$. Consider the subgraph $G_w - z$. For any pair of nonadjacent vertices $u \in U' - z$, $v \in W$ of $G_w - z$, we have

\[ d_{G_w-z}(u) + d_{G_w-z}(v) = d_{G_w}(u) - 1 + d_{G_w}(v) = n - 1. \]

Then, $G_w - z$ is an OTG and hence $G_w - z$ is a Hamiltonian graph. In other words, there is a non-Hamiltonian cycle containing all vertices of $X'$ in $G_w$, a contradiction. Thus, there is no such an H-force set $X'$ with $|X'| < |X|$. So, $X$ is the minimum H-force set and $h(G_w) = h(G) = n - 2$.\[\]

**Case 2:** $\delta(G_w) = \frac{n}{2}$ and $n$ is even.

**Subcase 2.1:** There is $v \in V(G_w)$ such that $G_w - v$ is not a Hamiltonian graph.

Let $H = G_w - v$. For arbitrary $x, y \in V(H)$, we have

\[ d_{H}(x) + d_{H}(y) \geq d_{G_w}(x) - 1 + d_{G_w}(y) - 1 = n - 2 = |V(H)| - 1. \]

By Corollary 1.3, either $H$ is not a 2-connected graph or $H \in \psi_{n-1}$ (Figure 1).

**Subcase 2.1.1:** $H = G_w - v$ is not a 2-connected graph.

Then, there exists a vertex $u$ such that $H - u$ is not connected.

**Claim B.**

(i) For any vertex $x \in V(G_w)$, if $d_{G_w}(x) > \frac{n}{2}$, then $d_{G_w}(x) = n - 1$.

(ii) $H - u$ has exactly two components, say $U$ and $W$.

(iii) For any vertices $x \in U$, $y \in W$, $d_{G_w}(x) = d_{G_w}(y) = \frac{n}{2}$.

(iv) $|U| = |W| = \frac{n}{2} - 1$.

(v) Both $U$ and $W$ induce complete subgraphs, and $d_{G_w}(u) = d_{G_w}(v) = n - 1$.

**Proof.**

(i) Recall that $\delta(G_w) = \frac{n}{2}$ and $d_{G_w}(u) + d_{G_w}(v) = n$ for nonadjacent vertices $u, v$. It is clear that (i) holds.

(ii) Suppose to the contrary that there are three components $U, W$ and $U'$ in $H - u$. So, $|U| + |W| \leq n - 3$.

For the vertices $x \in U$, $y \in W$, we see that

\[ n = d_{G_w}(x) + d_{G_w}(y) = d_{H}(x) + d_{H}(y) + d_{[U,V]}(x) + d_{[U,V]}(y) \leq |U| - 1 + |W| - 1 + 4 = |U| + |W| + 2 \leq n - 3 + 2 = n - 1, \]

a contradiction. Thus, $H - u$ has two components.
(iii) For \( x \in U, y \in W \), we have \( d_G(x) \neq n - 1 \) and \( d_G(y) \neq n - 1 \). By (i), \( d_G(x) = d_G(y) = \frac{n}{2} \).
(iv) Note that \( |U \cup W| = n - 2 \). Assume without loss of generality that \( |U| \leq \frac{n}{2} - 1 \). For any vertex \( x \in U \), we have
\[
\frac{n}{2} = d_G(x) \leq |U| - 1 + 2 = |U| + 1.
\]
Then, \( |U| \geq \frac{n}{2} - 1 \). So, \( |U| = |W| = \frac{n}{2} - 1 \).
(v) For any vertex \( x \in U \), \( d_G(x) = \frac{n}{2} \leq d(x) + 2 \leq |U| - 1 + 2 = \frac{n}{2} - 1 + 1 = \frac{n}{2} \). Then, \( d_G(x) = d(u) + 2 = (|U| - 1) + 2 \). This implies that \( U \) induces a complete subgraph and \( x \) is adjacent to \( u, v \). Similarly, \( W \) induces a complete subgraph and every vertex of \( W \) is adjacent to \( u, v \).
Furthermore, all vertices of \( U \) and \( W \) are the neighbours of \( u \) and \( v \). So, \( d_G(u) \geq n - 2, d_G(v) \geq n - 2 \) and hence \( d_G(u) + d_G(v) > n \). It means that \( u \) and \( v \) are adjacent. Thus, \( d_G(u) = d_G(v) = n - 1 \). \( \square \)

![Figure 4: G_{12} \notin \varphi_1.](image)

In subcase 2.1.1, \( G_w \equiv G_{12} \) (Figure 4), in which \( U \) and \( W \) are complete subgraphs with \( |U| = |W| = \frac{n}{2} - 1 \). By the same argument of the case when \( 2 \leq \delta(G_w) < \frac{n}{2} \) and \( G_w \equiv G_{11} \), we also have \( V - \{u, v\} \) is the minimum \( H \)-force set and \( h(G_w) = h(G) = n - 2 \).

**Subcase 2.1.2:** \( H = G_w - v \in \varphi_{n-1} \).

Let \( H = G_w - v = Z_m \cup (K_{m+1}^c) \), where \( Z_m \) is a graph with \( m \) vertices. Clearly, \( |V(H)| = n - 1 = 2m + 1 \), namely, \( m = 2m + 2 \). Recall that \( n \geq 5 \). So, \( m \geq 2 \).

**Claim C.** If there is at least one edge in \( Z_m \), then
(i) \( Z_m \) is a complete subgraph.
(ii) The vertex \( v \) is adjacent to all vertices of \( Z_m \).
(iii) The vertex \( v \) is adjacent to all vertices of \( K_{m+1}^c \).
(iv) \( G_w \equiv G_{21} = K_{m+1}^c \cup K_{m+1} \) (Figure 5). Furthermore, \( X = V(K_{m+1}^c) \) is the minimum \( H \)-force set and \( h(G) = h(G_w) = \frac{n}{2} \).

**Proof.**
(i) Let \( x \) be an endpoint of the edge of \( Z_m \). Then, \( d_G(x) \geq m + 2 \). For arbitrary \( y \in Z_m - x \), since \( d_G(y) \geq m + 1 \), we have \( d_G(x) + d_G(y) \geq 2m + 3 > n \). This implies that \( x \) is adjacent to the remaining vertices of \( Z_m \).

Furthermore, for any pair of \( p, q \in Z_m - x \), we have \( d_G(p) \geq m + 2, d_G(q) \geq m + 2 \), and hence \( d_G(p) + d_G(q) \geq 2m + 4 > n \), which implies that \( p \) and \( q \) are adjacent. Thus, \( Z_m \) is a complete subgraph.
(ii) For a vertex \( z \in Z_m \),
\[
d_G(v) + d_G(z) \geq \frac{n}{2} + (n - 2) = m + 1 + 2m > 2m + 2 = n.
\]
Thus, \( v \) is adjacent to all vertices of \( Z_m \).
(iii) Since \( \delta(G_w) = \frac{n}{2}, d_G(x) \geq \frac{n}{2} = m + 1 = |V(Z_m)| + 1 \) for any vertex \( x \in K_{m+1}^c \). So, \( x \) is adjacent to \( v \). Thus, \( v \) is adjacent to all vertices of \( K_{m+1}^c \).
(iv) By the arguments above, it is obvious \( G_w \equiv G_{21} = K_{m+1}^c \cup K_{m+1} \).
Let $X = V(K_{m+1}^c)$. For any $x \in X$, it is obvious that $G_w - x$ is an OTG and hence it is Hamiltonian. By Proposition 1.1, the minimum $H$-force set contains all vertices of $V(K_{m+1}^c)$. In addition, it is clear that $X = V(K_{m+1}^c)$ is an $H$-force set of $G_w$. So, $X = V(K_{m+1}^c)$ is the minimum $H$-force set and $h(G) = h(G_w) = \frac{n}{2}$.

Claim D. If there is no edge in $Z_m$, namely, $Z_m = K_{m}^c$, then

(i) $v$ has no neighbour in $Z_m$.

(ii) $v$ is adjacent to all vertices of $K_{m+1}^c$.

(iii) $G_w \cong K_{\frac{n}{2}}^c$ (Figure 5). Furthermore, $X = V(K_{m+1}^c)$ is the minimum $H$-force set and $h(G) = h(G_w) = \frac{n}{2}$.

Proof.

(i) By contradiction. Let $v$ be adjacent to some $x \in V(Z_m)$. Let $y$ be another vertex of $Z_m$. Then, $d_{G_w}(x) + d_{G_w}(y) \geq m + 2 + m + 1 > n$ and hence $x$ and $y$ are adjacent, which contradicts with $Z_m = K_{m}^c$.

(ii) Since $\delta(G_w) = \frac{n}{2}$, we have $v$ is adjacent to all vertices of $K_{m+1}^c$.

(iii) By (i) and (ii), it is easy to see that $G_w \cong K_{\frac{n}{2}}^c$. Clearly, the largest independent set $V(K_{m+1}^c)$ is the minimum $H$-force set and $h(G) = h(G_w) = \frac{n}{2}$.

Subcase 2.2: For any vertex $v \in V(G_w)$, the subgraph $G_w - v$ is Hamiltonian.

In this subcase, Proposition 1.1 yields that the minimum $H$-force set is $V(G)$ and $h(G) = h(G_w) = n$.

In [10], Dirac proved Theorem 2.3.

**Theorem 2.3.** [10] If the minimum degree $\delta(G)$ of $G$ is at least $\frac{n}{2}$, then $G$ is Hamiltonian.

Clearly, the graph satisfying the condition of Dirac’s theorem is an OTG. By Theorem 2.2, we obtain immediately the following results on the $H$-force sets of graphs satisfying the condition of Dirac’s theorem.

**Corollary 2.4.** Let $G$ be a graph satisfying $\delta(G) \geq \frac{n}{2}$ and $X$ be the minimum $H$-force set of $G$. Let $G_w = C_w(G)$ be the weak closure of $G$ and $S$ be the largest independent set of $G_w$. Then,

1. The $H$-force number $h(G) \in \left\{ \frac{n}{2}, n - 2, n \right\}$.
2. $X = \begin{cases} V(G) - \{x, y\}, & \text{if } h(G) = n - 2, \\ S, & \text{if } h(G) = \frac{n}{2}, \\ V(G), & \text{if } h(G) = n, \end{cases}$

where $x, y \in V(G)$ with $d_{G_w}(x) = n - 1$ and $d_{G_w}(y) = n - 1$.

**3 Classification of the OTGs**

To present Theorem 3.1, we define several special classes of graphs, namely,
\[ \varphi_1 = \left\{ K_2 \lor (K_m + K_{n-m-2}) \mid 1 \leq m < \frac{n}{2} \right\} \text{ (Figures 3 and 4)}, \]

\[ \varphi_2 = \left\{ G_{21}, K_{\frac{n}{2}} \right\} \text{ (Figure 5)}, \]

\[ \varphi_3 = \left\{ \left\{ Z_{n-m} \lor K_m \mid n \text{ is even, } 0 \leq m < \frac{n}{2} \text{ and } d(x) = \frac{n}{2} \text{ for } x \in Z_{n-m} \right\} \right\} \setminus \left\{ G_{12}, K_{\frac{n}{2}} \right\} \]

\[ \cup \{ G \mid G \text{ is a complete graph} \} \text{ (Figure 6)}, \]

where \( Z_{n-m} \) is a graph with \( n-m \) vertices.

Figure 6: \( G_{31} \in \varphi_1 \) and \( G_{32} \in \varphi_3 \).

Let \( C_w \) be the class of the weak closures of all OTGs. Clearly, \( \varphi_1, \varphi_2, \varphi_3 \) are all the subsets of \( C_w \). By Theorem 2.2, we can give a partition of the class \( C_w \) and obtain the following theorem.

**Theorem 3.1.** The class \( C_w \) of the weak closures of all OTGs has a partition

\[ C_w = \varphi_1 \cup \varphi_2 \cup \varphi_3 \]

due to the H-force set, where \( \varphi_1, \varphi_2, \varphi_3 \) are defined above. Let \( G_w \in C_w \) be arbitrary. Then,

1. \( h(G_w) = n - 2 \) if and only if \( G_w \in \varphi_1 \).
2. \( h(G_w) = \frac{n}{2} \) if and only if \( G_w \in \varphi_2 \).
3. \( h(G_w) = n \) if and only if \( G_w \in \varphi_3 \).

**Proof.** By case 1 and subcase 2.1.1 of Theorem 2.2, statement (1) holds. Also, by subcase 2.1.2, statement (2) holds. We show (3) as follows.

"\( \Rightarrow \)" By the proof of Theorem 2.2, we know that \( h(G_w) = n - 2 \) if \( 2 \leq \delta(G_w) < \frac{n}{2} \). So, if \( h(G_w) = n \), then \( \delta(G_w) = \frac{n}{2} \) (n is even) or \( G_w \) is a complete graph. In the latter case, \( G_w \in \varphi_3 \) and we are done.

Assume that \( G_w \) is not a complete graph. Then, \( \delta(G_w) = \frac{n}{2} \) and \( d_{G_w}(u) + d_{G_w}(v) = n \) for every pair of non-adjacent vertices \( u, v \). So, the degree of any vertex \( u \in V(G_w) \) is either \( \frac{n}{2} \) or \( n - 1 \). Set \( U = \{ v \mid d_{G_w}(v) = n - 1 \} \) and \( m = |U| \).

When \( U \neq \emptyset \), we claim that \( 1 \leq m < \frac{n}{2} \). Clearly, \( 1 \leq m \leq \frac{n}{2} \). Note that \( d_{G_w}(w) = \frac{n}{2} \) for any vertex \( w \in V(G_w) - U \). If \( m = \frac{n}{2} \), then \( G_w \cong G_{21} \) and \( h(G_w) = \frac{n}{2} \), a contradiction. So, \( 1 \leq m < \frac{n}{2} \). Since \( h(G_{12}) = n - 2 \), we have \( G_w \notin G_{12} \). Thus, \( G_w \in \varphi_3 \).

When \( U \) is an empty set, then \( m = 0 \) and \( G_w \) is a \( \frac{n}{2} \)-regular graph. Since \( h(K_{\frac{n}{2}}) = \frac{n}{2} \), we have \( G_w \notin K_{\frac{n}{2}} \). Thus, \( G_w \in \varphi_3 \).

"\( \Leftarrow \)" It is not difficult to check that \( \varphi_1 \cap \varphi_3 = \emptyset \) and \( \varphi_2 \cap \varphi_3 = \emptyset \). Let \( G_w \in \varphi_3 \) be a weak closure of an OTG. So, \( h(G_w) \neq n - 2 \) or \( \frac{n}{2} \). According to Theorem 2.2, \( h(G_w) = n \).

To present Corollary 3.2, we define several special classes of graphs, namely,

\[ \mathcal{S}_i = \{ G \mid G \text{ is an OTG, } G_w \in \varphi_i \}, \quad i = 1, 2, 3. \]
Let $\mathcal{G}$ be the class of all OTG’s. Clearly, $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ are all the subsets of $\mathcal{G}$. By Theorem 3.1, we can give a partition of the class $\mathcal{G}$ and obtain the following corollary.

**Corollary 3.2.** The class $\mathcal{G}$ of all OTGs has a partition

$$
\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3
$$

due to the weak closure, where $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ are defined above. Let $G \in \mathcal{G}$ be arbitrary. Then,

$$
h(G) = \begin{cases} 
  n - 2, & \text{if } G \in \mathcal{G}_1, \\
  \frac{n}{2}, & \text{if } G \in \mathcal{G}_2, \\
  n, & \text{if } G \in \mathcal{G}_3.
\end{cases}
$$

### 4 Algorithm to check the minimum H-force set of an OTG

**Input:** A graph $G = (V, E)$.

**Output:** If $G$ is an OTG, return the H-force number $h(G)$ and the minimum H-force set $X$. If $G$ is not an OTG, return “not an OTG”.

1. For each pair of nonadjacent vertices $u, v$, check $d(u) + d(v)$. If there is $uv \notin E$ with $d(u) + d(v) < n$, then return “not an OTG”.
2. Make the weak closure $G_w$ of $G$.
3. For any $x \in V(G)$, check $d_{G_w}(x)$.
   1. If there are exactly two vertices $u$ and $v$ with $d_{G_w}(u) = d_{G_w}(v) = n - 1$ and $G_w \notin \varphi_1$, return $h(G) = n - 2$ and $X = V(G) \setminus \{u, v\}$.
   2. If there are exactly two vertices $u$ and $v$ with $d_{G_w}(u) = d_{G_w}(v) = n - 1$ and $G_w \in \varphi_3$, return $h(G) = n$ and $X = V(G)$.
4. If any $v \in V(G_w)$, $d_{G_w}(v) = \frac{n}{2}$ and $G_w \cong K_{\frac{n}{2}, \frac{n}{2}}$, return $h(G) = \frac{n}{2}$.
5. Find a partition set $X$ of $K_{\frac{n}{2}, \frac{n}{2}}$, return $X$.

**Theorem 4.1.** For an OTG, the minimum H-force set and the H-force number can be found in time $O(n^3)$.

**Proof.** The complexity follows from the fact that Step 1 can be performed in time $O(n^2)$, Step 2 in time $O(n^3)$ and Step 3 in time $O(n^2)$.

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