On the thermodynamic limit of form factor expansions of dynamical correlation functions in the massless regime of the XXZ spin $1/2$ chain.

Karol K. Kozlowski

Univ Lyon, Ens de Lyon, Univ Claude Bernard, CNRS, Laboratoire de Physique, F-69342 Lyon, France.

Abstract

This work constructs a well-defined and operational form factor expansion in a model having a massless spectrum of excitations. More precisely, the dynamic two-point functions in the massless regime of the XXZ spin-$1/2$ chain are expressed in terms of properly regularised series of multiple integrals. These series are obtained by taking, in an appropriate way, the thermodynamic limit of the finite volume form factor expansions. The series are structured in such a way allowing one to identify directly the contributions to the correlator stemming from the conformal-type excitations on the Fermi surface and those issuing from the massive excitations (deep holes, particles and bound states). The obtained form factor series opens up the possibility of a systematic and exact study of asymptotic regimes of dynamical correlation functions in the massless regime of the XXZ spin $1/2$ chain. Furthermore, the assumptions on the microscopic structure of the model’s Hilbert space that are necessary so as to write down the series appear to be compatible with any model -not necessarily integrable- belonging to the Luttinger liquid universality class. Thus, the present analysis provides also the phenomenological structure of form factor expansions in massless models belonging to this universality class.

Introduction

It is usually impossible to compute explicitly, in a closed form the correlation functions, be it dynamic or static ones, in a one-dimensional quantum model. The best one can hope for, and this is in fact enough for most of the applications, is to come up with an effective theory allowing one to grasp the salient features of the correlators. In doing so, the few cases of quantum integrable models where the calculations are amenable to an end may serve as a guiding principle, be it for building some aspects or testing the domain of applicability of such a theory. The computations of correlation functions in quantum integrable models has a rather long history which goes back to the early 60s when Lieb, Mattis and Schultz [52] computed, in closed form, the static correlators of the XY chain. Their approach was generalised or adapted to many other free fermion equivalent integrable models; the bottom line being that the correlators in such models are representable in terms of Toeplitz determinants or Painlevé transcendent [4, 24, 54]. Such expressions can be considered as explicit enough in that they allow for a...
rather thorough analysis of the various physically interesting asymptotic regimes of the correlators [53, 68], this even including a rather extensive characterisation of the dynamic response functions [5, 66]. Starting from the mid 90s new ideas led to explicit, multiple integral based, representations for the correlators in the XXZ spin-1/2 chain, be it the static [23, 55], the dynamic [54], the static at finite temperature [18] or even the dynamic at finite temperature [60] ones. Some of the obtained representations for the static correlation functions could even be analysed so as to extract, starting from first principles, the leading long-distance asymptotic behaviour of the two point functions in the massless regime of the chain [28].

Notwithstanding, the very existence of the above representations - just as the possibility to analyse these- is closely related to the integrable structure of the model and hence, rather far away from the kind of representations usually dealt with in physics. Indeed, for a general model in finite volume, it appears convenient to characterise correlators by means of expansions into form factor series or closely related objects. Such series are obtained by introducing the closure relation between each of the local operators building the correlator. Form factor expansions have the advantage of separating the dynamic (distance and space dependence) part of a correlator from its operator content (specific dependence on a given operator). This property predestines form factor expansions to be powerful tools for studying numerous properties of a model, this provided that one can make sense out of their thermodynamic, viz. infinite volume, limit. It appears that such series are well-defined for models having a massive spectrum in the thermodynamic limit. In such case, the form factors -matrix elements of local operators taken between two excited states of the model- go, with an appropriate integer power in the volume, to a form factor density what allows one to replace, in the thermodynamic limit, the summation over excited states by integrations over the continua of excited states. This property, along with a system of axioms [25, 27] on the properties that ought to be satisfied by a form factor in an integrable massive quantum field theory in 1+1 dimensions allowed for a construction of form factor expansions for numerous instances of such models, see e.g. [3, 65]. Owing to the mentioned splitting between the dynamical and operator parts, the obtained series were extremely effective for studying the long-distance behaviour of the correlators.

The situation becomes however problematic in the case of massless models. Indeed, it is expected [8] -and these expectations have been confirmed for numerous massless quantum integrable models [2, 29, 31, 48, 61]-, that the form factors have a non-uniform large-volume behaviour: the latter differs depending on whether one of the excited state contains massless excitations or not. In particular form factors involving massless excitations do exhibit a non-integer power-law behaviour in the volume what makes it impossible to take the thermodynamic limit of a finite volume form factors series solely in terms of multiple integrals. Some attempts were made to obtain form factor expansions in certain massless quantum field theories by taking an appropriate large energy limit of the massive form factors [11, 55, 59]. However, although systematic, the procedure was not applicable to numerous operators of interest; the reason being that, for such operators, it produces non-integrable form factor densities what renders the associated form factor series divergent. This problem takes its origin in the lack of an appropriate regularising treatment of the vicinity of the massless models [11]. The cluster property satisfied by the form factors of primary operators which allows for an effective exponentiation of the series in the infrared limit [64], indicates that it should be possible to make sense out of the mentioned massless limit, at least in the case of primary operators. However, obtaining explicit closed expressions through such an approach does not seem so evident and has not been achieved so far. Still, the mentioned clustering structure was used in [50, 51] to obtain the first two terms present in the massless form factor expansion of certain specific operators. Thus, at least in its present state of the art, this approach was not able to produce a clear, systematic and operational procedure allowing to re-absorb the various local divergences related to the presence of massless modes and re-sum explicitly the original massive form factor series into an explicit massless form factor expansion.

A completely different approach for dealing with form factors series in massless integrable models was pioneered in [39] for the case of two-point functions in free fermion equivalent models. The idea was to start from a model in finite volume where there is no problem to define a form factor expansion and then to re-sum the
expansion over the excited states in terms of a finite size matrix. The latter, in the thermodynamic limit, goes to a Fredholm determinant of an operator $id + V$, with $V$ being an integrable integral operator $[21]$. The technique appeared rather general and turned out to be applicable to dynamical correlation functions at finite temperature $[9]$, and multi-point correlators $[62]$ in free fermion equivalent models. The Fredholm determinant based representations allowed for an extensive and very effective characterisation of the asymptotic regimes of the free fermionic correlators by means of solving auxiliary Riemann–Hilbert problems. For the price of introducing certain auxiliary quantum fields $[37]$, the so-called dual fields, it was argued that, at least formally, one can still perform the summation over all excited states, even in the case of interacting integrable models. Then, the correlators are expressed in terms of an average, over an auxiliary Hilbert space, of a dual field valued Fredholm determinant of an integrable integral operator. The presence of dual fields in the kernel of the integrable integral operator turned out to be an important issue for dealing with such representations. In particular, it posed serious problems when trying to extract physically interesting asymptotic regimes of the correlators out of such representations and only partial results could be obtained $[22, 63]$. The works $[44, 49]$ proposed a more effective method of resumming a form factor series expansion for interacting integrable models; the latter was based on the concept of multi-dimensional deformation flows techniques. This led to explicit representations of two-point functions in the non-linear Schrödinger model in terms of objects that can be interpreted as multidimensional generalisations of a Fredholm determinant. The latter construction allowed for a rather thorough and explicit analysis of the long-distance and large-time asymptotics of the correlators in this model $[44, 49]$ by means of solving an auxiliary Riemann–Hilbert problem $[43]$.

One can raise two criticisms. The resummation techniques were strongly relying on the integrable structure of the model. Furthermore, the approach circumvented the issue of defining directly an infinite volume form factor series of a massless model. Thus, despite some of its successes such as checking some of the universality based predictions for the asymptotics of correlation functions in massless integrable models, these analysis didn’t allow to get a sufficient feeling of the very structure of a form factor expansion in a massless model, what could in principle allow for building up a phenomenological description of such series in the case of not exactly solvable, \textit{i.e.} generic, massless models.

Recently, the work $[30]$ proposed a method allowing one to sum up directly, in the long-distance regime, the expansion of a static two-point function over the so-called critical form factors $[2]$ in massless integrable models such as the Bose gas or the spin-1/2 XXZ chain. The method was generalised to the cases of the large-distance behaviour of form factor series representing static multi-point $[33]$ correlation function of these models or static two-point functions in higher rank (multi-species) integrable model $[48]$. The approach could have also been conformed to deal with dynamical correlation functions $[32]$ this in large-time and distance regime and in the case of the non-linear Schrödinger model. These summations allowed for a relatively easy check of the conformal field theory/non-linear Luttinger liquid based predictions for various asymptotic behaviours of the correlators in these models. It also allowed to identify the microscopic mechanism, \textit{viz.} the very structure of a model’s Hilbert space, that it responsible for the model to belong to the Luttinger liquid universality class $[47]$. However, the summations could only have been done in the asymptotic regime, what allowed to argue certain simplifications in the form factor series.

The purpose of the present work is to push further the observations of $[30, 32]$ so as to develop techniques allowing one to build explicit form factor expansions for the massless regime of the XXZ spin-1/2 chain, this in the thermodynamic limit of the model and for almost any regime of the distance and time. The analysis strongly relies on the results of $[45]$ where the author obtained the large-volume behaviour of the form factors of local operators in the XXZ chain taken between the ground state and any excited state of the model. This work generalises the earlier analysis $[29, 31]$ of such form factor where only certain particle-hole excited states

\footnote{\textit{i.e.} expectation values of local operators taken between the ground state and the low-lying excited states which exhibit a conformal structure}
were considered. The main result of the paper is a well-defined infinite volume form factor expansion for the dynamic two-point functions in the massless regime of the XXZ spin-1/2 chain. This series is effective in that, as will be demonstrated in a forthcoming publication, it allows one to extract rather straightforwardly the large-time and long-distance asymptotic behaviour of the two-point functions. Since the series has a very close connection with the model’s spectrum, it also allows one to show that, in the special case of static correlators, the bound states contribute to the large-distance asymptotics on the level of corrections exponentially small in the distance. Finally, as will be shown in a subsequent work, the series also allows one to extract the so-called edge exponents characterising the power-law behaviour of the dynamic response functions in the vicinity of the multi-particle or bound state excitation thresholds. This analysis will show that the non-linear Luttinger liquid approach misses part of the dynamic response functions’ singularity thresholds lines and thus the form of the associated edge exponents.

The assumptions on the microscopic structure of the model’s Hilbert space that are necessary so as to write down the infinite volume form factor expansion in the massless regime of the XXZ chain appear, in fact, to be satisfied by any model -not necessarily integrable- belonging to the Luttinger liquid universality class. Thus, the results of present paper also provides one with the phenomenological form taken by a form factor expansions in a massless model belonging to this universality class.

The paper is organised as follows. Section 1 introduces the model of interest and some notations. This section also outlines the main result obtained in the paper. Section 2 discusses the microscopic structure of the Hilbert space associated with the massless regime of the XXZ spin-1/2 chain. Subsection 2.1 discusses the parametisation of the eigenstates. Subsection 2.2 discusses the structure of the spectrum above the ground state in the large volume limit. Subsection 2.3 discusses the large-volume behaviour of form factors. Section 3 is devoted to the computation of the thermodynamic limit of the form factor expansion of dynamical two-point functions in the massless regime of the XXZ spin-1/2 chain. Various preliminary transformations are implemented in Subsections 3.1-3.2 while the per se thermodynamic limit of the form factor expansion is obtained in Subsection 3.3. This series is recast within the momentum representation in Subsection 3.4 and the dynamic response functions are computed in Subsection 3.5. Finally, Section 4 discusses briefly how the present result provides one with the phenomenological form of form factor expansions in massless non-integrable models belonging to the Luttinger liquid universality class. The paper contains three appendices which gather some technical results of interest to the analysis. Appendix A discusses the description of the observables in the XXZ spin-1/2 chain. Subappendix A.1 discusses the various solutions of linear integral equations which parametrisate the spectrum of the model. Subappendix A.2 recalls the existence conditions of the bound states. Appendix B evaluates the action of an operator that arises in the course of taking the thermodynamic limit of the form factor series. Finally, Appendix C evaluates an integral which arises in the context of computing the dynamical response functions.

1 Main results

1.1 The model

The XXZ spin-1/2 chain refers to a system of interacting spins in one dimension described by the Hamiltonian

$$H = J \sum_{a=1}^{L} \left\{ \sigma_{a}^{x} \sigma_{a+1}^{x} + \sigma_{a}^{y} \sigma_{a+1}^{y} + \cos(\zeta) \sigma_{a}^{z} \sigma_{a+1}^{z} \right\} - \frac{h}{2} \sum_{a=1}^{L} \sigma_{a}^{z}.$$  (1.1)

$H$ is an operator on the Hilbert space $\mathcal{H}_{XXZ} = \bigotimes_{a=1}^{L} \mathcal{H}_{a}$ with $\mathcal{H}_{a} \simeq \mathbb{C}^{2}$. The matrices $\sigma^{\gamma}$, $\gamma = x, y, z$ are the Pauli matrices and the operator $\sigma_{a}^{\gamma}$ acts as the Pauli matrix $\sigma^{\gamma}$ on $\mathcal{H}_{a}$ and as the identity on all the other spaces. The Hamiltonian depends on three coupling constants:

- $J > 0$ which represents the so-called exchange interaction ;
• $\cos(\zeta)$ which parametrises the anisotropy in the coupling between the spins in the longitudinal and transverse directions;

• $h > 0$ which measures the intensity of the overall longitudinal magnetic field that is applied to the chain.

Throughout this work I shall focus on the following massless anti-ferromagnetic regime of the chain: $-1 < \cos(\zeta) < 1$, i.e. $\zeta \in ]0; \pi[$, and $h_c = 8J\cos^2(\zeta/2) > h > 0$. I will assume periodic boundary conditions, viz. $\sigma^{\gamma}_{a+L} = \sigma^{\gamma}_{a}$.

Given a spin operator $O_1$, acting on $h_1$, the translation invariance of the chain ensures that the time and space evolved operator $O_{m+1}(t)$ takes the form

$$O_{m+1}(t) = e^{imP+ilt} \cdot O_1 \cdot e^{-ilt-imP}, \quad (1.2)$$

where $P$ is the momentum operator and hence, $e^{P}$ is the translation operator by one-site.

At zero temperature, the finite volume two-point functions are given by the ground state $|\Omega\rangle$ expectation value:

$$\langle \Omega | \sigma^{\gamma}_{m+1} \sigma^{\gamma}_{m+1} | \Omega \rangle. \quad (1.3)$$

Here and in the following, the operator $\sigma^{\gamma}_{i}$ is defined by $\sigma^{\gamma}_{i} = (\sigma^{\gamma}_{i})^\dagger$ where $^\dagger$ stands for the Hermitian conjugation. The presumably existing infinite volume limit of the two-point function is the object of main interest to this work. It will be denoted as

$$\langle \sigma^{\gamma}_{1}(t) \sigma^{\gamma}_{m+1} \rangle = \lim_{L \to \infty} \langle \Omega | \sigma^{\gamma}_{1}(t) \sigma^{\gamma}_{m+1} | \Omega \rangle \quad (1.4)$$

### 1.2 The main result

The main result of this work consists in constructing a well-defined, viz. free of any divergencies, form factor series expansion for the thermodynamic limit of the dynamical two-point functions $\langle \sigma^{\gamma}_{1}(t) \sigma^{\gamma}_{m+1} \rangle$ in the massless regime of the XXZ spin 1/2 chain. The obtained form factor expansion is valid in the region where $v = m/t \neq \pm v_F$, with $v_F$ the Fermi velocity of the model. It is given by the series of multiple integrals:

$$\langle \sigma^{\gamma}_{1}(t) \sigma^{\gamma}_{m+1} \rangle = (-1)^{m_F} \sum_{m \in \mathbb{Z}} \prod_{r \in \mathbb{R}} \left\{ \int_{\gamma^{(r)}} \frac{d\nu^{(r)}}{\nu! \cdot (2\pi)^{\nu}} \right\} \cdot \left\{ \int_{\gamma^{(\nu)}} \frac{d\mu^{(\nu)}}{\mu! \cdot (2\pi)^{\mu}} \cdot F^{(\gamma)}(\nu) \right\} \prod_{i = \pm} \left\{ \frac{e^{imF_{iF}}}{} \right\}$$

$$\times \prod_{r \in \mathbb{R}} \prod_{a = 1}^{n_r} \left\{ e^{imP_{iF}(\nu_{a}^{(r)})-i\epsilon_{i}(\nu_{a}^{(r)})} \right\} \cdot \prod_{a = 1}^{n_{\mu}} \left\{ e^{imP_{iF}(\mu_{a})-i\epsilon_{i}(\mu_{a})} \right\} \cdot \left( 1 + \delta_{m,m'} \right) \quad (1.5)$$

This formula demands some explanations.

- The series contains the relativistic combination of time and distance

$$m_{F} = m - \sqrt{m^2 - v_{F}^2} \quad (1.6)$$

---

1This way of writing down the two-point function has been chosen so as to ensure that the obtained representations for the correlators involve combinations of the dressed energies $e_a$ and momenta $p_a$ of the excitation in the form of a difference $mp_a - te_a$. This plays only a cosmetic role in the analysis and the usually used two point function correlator $\langle \Omega | \sigma^{\gamma}_{1} \sigma^{\gamma}_{m+1} | \Omega \rangle$ can be recovered by the substitution $t \to -t$. Note also that the relative sign in the $\delta m$ dependent terms in (1.2) issues from the choice of the sign of the Hamiltonian and of the anisotropy $\cos(\zeta)$ in (1.1).

2This limit exists for $t = 0$ as follows by putting together the results of [35][41].
The sum in (1.5) runs through all possible choices of integer $s$:

$$s_z = 0 \quad \text{and} \quad s_\pm = \mp 1 \quad .$$

More precisely, $-s_y$ corresponds to the relative to the ground state longitudinal spin of the excited states that are connected to the ground state by the operator $\sigma_1^y$.

- The sum in (1.5) runs through all possible choices of integers
  $$n = (\{n_r\}_{r \in R}, n_h, \ell_v)$$
  parametrising the various types of massive excitations. $n_r$ counts the various types, labelled by $r \in R \subset \mathbb{Z}$, of bound state excitations. Here $1 \in R$ and more specifically $n_1$ counts the so-called particle excitations. $n_h$ counts the hole excitations. Finally, $\ell_v \in \mathbb{Z}$ are the Umklapp deficiencies which encode the difference between the numbers of massless particles and holes forming in the swarm of zero-energy excitations lying on the left ($+$) and right ($-$) Fermi boundary of the model. These integers are subject to the constraints

$$n_h = \sum_{r=\pm} \ell_v + \sum_{r \in R} n_r \quad .$$

- The rapidities of the various excitations evolve on curves $\varphi_{r}^{(\delta)}$ and $\varphi_{h}^{(\delta)}$. For $r \geq 2$ these curves are either given by $\mathbb{R}$ or by $\mathbb{R} + i\pi/2$, this depending on the value of $r$ and they are $\delta$ independent. The particle, resp. hole, rapidity curve $\varphi_{r}^{(\delta)} = \varphi_{p}^{(\delta)}$, resp. $\varphi_{h}^{(\delta)}$, coincides with $[\mathbb{R} \setminus [-q; q]] \cup [-\mathbb{R} + i\pi/2]$, resp. $[-q; q]$, with the exception of a vicinity of $\pm q$. There, the curves stay at a distance of the order $\delta$ from $\pm q$ and avoid this point, in such a way that the associated particle, resp. hole, oscillatory in the distance and time exponential has modulus smaller than one. These curves are depicted in Figure 6. Here $[-q; q]$ corresponds to the Fermi zone of the model.

- The particle, resp. hole, excitations carry a dressed momentum $p_1(v_a^{(1)})$, resp. $p_1(\mu_a)$, and a dressed energy $\varepsilon_1(v_a^{(1)})$, resp. $\varepsilon_1(\mu_a)$. The $r$-bound state excitations, $r \in \mathbb{N} \setminus \{1\}$, carry a dressed momentum $p_r(v_a^{(r)})$ and a dressed energy $\varepsilon_r(v_a^{(r)})$.

- $F^{(y)}(\mathfrak{Y})$ corresponds to the form factor density squared of the operator $\sigma_1^y$ taken between the ground state and an excited state whose massive excitations are parametrised by $\mathfrak{Y}$.

- $\theta_2^2(\mathfrak{Y})$ corresponds to the critical exponent governing the large-distance long-time decay associated with the excitations characterised by the collection of rapidities $\mathfrak{Y}$.

- $\tau_{\delta,m}(\mathfrak{Y})$ is a remainder that is controlled as

$$\tau_{\delta,m}(\mathfrak{Y}) = O(\delta \ln |\delta| + \sum_{|m| \geq \pm} (\delta^2|m_\ell| + \delta \ln |m_\ell| + e^{-\delta|m_\ell|})) \quad (1.11)$$

with $m_\ell$ given in (1.6). Although the remainder is not provided explicitly in this work, it can, in principle, be computed, order-by-order up to the desired precision in powers of $1/m_\ell$ and $\ln |m_\ell|/m_\ell$.  

6
The form factor series depends on an auxiliary control parameter \( \delta > 0 \) which is arbitrary and can be taken as small as necessary. This dependence manifests itself on the level of the remainder \( \nu_{s,m,l}(q) \) and in the way the integration curves \( \mathcal{C}^{(\delta)}_{1} \) and \( \mathcal{C}^{(\delta)}_{h} \) are deformed in the vicinity of the endpoints \( \pm q \) of the Fermi zone.

### 1.3 Comments

The parameter \( \delta \) emerges in the course of the analysis as a means to separate between the massive and massless constituents of the spectrum. Indeed, due to a qualitatively different structure of the form factors associated with these two kinds of excitations, each of these demands a separate and completely different treatment. The magnitude of this separation is arbitrary what means that, on the level of (1.5), the parameter \( \delta \) can be taken as small as necessary. However, it cannot be set directly to zero. Indeed, the form factor densities \( \mathcal{F}^{(\gamma)}(\delta) \) exhibit non-integrable singularities when a particle’s rapidity \( \nu^{(1)} \) or a hole’s rapidity \( \mu_{h} \) approaches one of the endpoints \( \pm q \) of the Fermi zone \([−q ; q]\). Furthermore, the integration curve \( \mathcal{C}^{(\delta)}_{p} \), resp. \( \mathcal{C}^{(\delta)}_{h} \), approaches \([\mathbb{R} \backslash [−q ; q]] \cup [−\mathbb{R} + i\pi/2], \text{resp. } [−q ; q] \), when \( \delta \to 0^+ \). Then one can argue that any given multiple particle-hole integral produces, in the \( \delta \to 0^+ \) limit, logarithmically diverging contributions of the form \( P(\ln \delta) \) with \( P \) a polynomial depending on the integral. These divergences are reminiscent of the problems with dealing correctly with the infrared divergencies of a theory. It is important to stress nonetheless that the correlator \( \langle \sigma^{\gamma}_{1}(t) \sigma^{\gamma}_{m+1} \rangle \) does not depend on \( \delta \) and thus, obviously, has a well-defined \( \delta \to 0^+ \) limit. However, obtaining the latter on the level of the series would demand non-trivial resummations that would end up in destroying the very structure of the form factor expansion which is at the root of its usefulness for applications. Thus the presence of a regularising parameter \( \delta \) appears necessary if one wants to preserve the salient features of a form factor expansion.

I refer to the core of the paper so as to obtain a deeper understanding of the expansion and its constituents, Section 2.2 in what concerns the spectrum, Section 2.3 in what concerns the large volume behaviour of the form factors of local operators and, finally, Section 3.3 where the above representation is obtained.

Finally, one should mention that the above series can be recast in the momentum representation, c.f. Section 3.4 in particular equation (3.88). Also, the form factor expansion (1.5) allows one to compute the dynamical response function subordinate to the two-point function

\[
\mathcal{F}^{(\gamma)}(k, \omega) = \sum_{m \in \mathbb{Z}} \int_{\mathbb{R}} \langle \sigma^{\gamma}_{1}(t) \sigma^{\gamma}_{m+1} \rangle \cdot e^{i(\omega t - km)} \, dt.
\]

The explicit expression of \( \mathcal{F}^{(\gamma)}(k, \omega) \) can be found in (3.71)-(3.72).

One could get the wrong impression that the presence of a regularisation parameter \( \delta \) and of a non-explicit remainder \( \nu_{s,m,l}(q) \) limits the usefulness of the obtained form factor expansion. First of all, I stress that within the setting developed in the core of the work, \( \nu_{s,m,l}(q) \) can be computed, at least in principle, to an arbitrary precision. However, the estimates (1.11) are already enough for practical applications. As a matter of fact, in its present form, the series is fully operational on a technical level. It will be show in forthcoming publications that, by choosing suitable values for \( \delta \), the series allows one, by solely relying on saddle-point techniques, to access the asymptotic regimes of the two-point functions: their long-time and large-distance asymptotics on the one hand and, on the other hand, the power law-behaviour in the frequency that characterises the non-smooth behaviour of the dynamical response functions in the vicinity of the edges of the excitations thresholds -in particular to access to the so-called edge exponents-. 

---

**Footnotes**

1. The form factor series depends on an auxiliary control parameter \( \delta > 0 \) which is arbitrary and can be taken as small as necessary. This dependence manifests itself on the level of the remainder \( \nu_{s,m,l}(q) \) and in the way the integration curves \( \mathcal{C}^{(\delta)}_{1} \) and \( \mathcal{C}^{(\delta)}_{h} \) are deformed in the vicinity of the endpoints \( \pm q \) of the Fermi zone.

2. The explicit expression of \( \mathcal{F}^{(\gamma)}(k, \omega) \) can be found in (3.71)-(3.72).

3. One could get the wrong impression that the presence of a regularisation parameter \( \delta \) and of a non-explicit remainder \( \nu_{s,m,l}(q) \) limits the usefulness of the obtained form factor expansion. First of all, I stress that within the setting developed in the core of the work, \( \nu_{s,m,l}(q) \) can be computed, at least in principle, to an arbitrary precision. However, the estimates (1.11) are already enough for practical applications. As a matter of fact, in its present form, the series is fully operational on a technical level. It will be show in forthcoming publications that, by choosing suitable values for \( \delta \), the series allows one, by solely relying on saddle-point techniques, to access the asymptotic regimes of the two-point functions: their long-time and large-distance asymptotics on the one hand and, on the other hand, the power law-behaviour in the frequency that characterises the non-smooth behaviour of the dynamical response functions in the vicinity of the edges of the excitations thresholds -in particular to access to the so-called edge exponents-.
2 The microscopic structure of the XXZ chain’s Hilbert space

2.1 Parametrisation of the Eigenstates

The Eigenstates and energies of the XXZ spin-1/2 chain can be obtained within the Bethe Ansatz. In this approach, the Eigenstates $|\Upsilon\rangle$ and associated energies $\tilde{E}_\Upsilon$ are parametrised by a set $\Upsilon$ consisting of auxiliary parameters, the so-called Bethe roots $[6, 58]$. In the following, I will reserve the notation $|\Omega\rangle$ for the ground state of $\mathcal{H}$ and denote by $\Omega$ the collection of Bethe roots characterising the ground state.

One expects that, in the thermodynamic limit $L \to +\infty$, only Eigenstates having a finite excitation energy relatively to the ground state, i.e. such that $\lim_{L \to +\infty} \tilde{E}_\Upsilon - \tilde{E}_\Omega < +\infty$, should contribute to the zero-temperature correlation functions of the model. I will thus only discuss the structure of these Eigenstates.

According to the general Bethe Ansatz reasonings, the ground state $|\Omega\rangle$ is described by a collection of $N$ real parameters, the ground state Bethe roots. The value of $N$ is fixed by the magnetic field $h$, $h > 0$. When $L \to +\infty$, one has that $N \to +\infty$ in such a way that $N/L \to D \in [0; 1/2]$ and the value of $D$ is uniquely determined by $h$. The associated ground state Bethe roots then form a dense distribution on a symmetric segment $[-q; q]$ whose endpoint $q$ is fixed by $h$ [15]. The interval $[-q; q]$ is called the Fermi zone of the model.

Eigenstates of the XXZ spin-1/2 chain having a finite relative excitation energy above the ground state can be seen as a collection of dressed excitations above the Fermi zone $[-q; q]$ of the model [7, 12, 13, 16, 20] which take place in a spin $s$ sector relatively to the ground state. These excitations can be gathered into three classes: the particles, the holes and the bound states. Any bound state excitation is necessarily massive. However, the particle or hole excitations can be either massless or massive.

There exist several branches of bound state excitations in the XXZ chain. They are labelled by an integer $r$. In the Bethe Ansatz language, an “$r$-bound state” corresponds to a collection of Bethe roots forming a string of length $r \geq 2$ and, hence, is called an $r$-string [6]. For given $\zeta \in ]0; \pi [$ there may or may not exist an upper bound $r_{\text{max}}$ on the allowed length of a string. This depends on whether $\zeta/\pi$ is rational or not [66]. In the following, if there is no upper bound, one should simply take $r_{\text{max}} = +\infty$. Furthermore, for fixed $\zeta$, an $r$-string may only arise when $r$ takes values in a subset $\mathbb{R}_r$ of $[2 : r_{\text{max}}]$. The fact that an $r$ string exists or not depends on the continued fraction decomposition of $\zeta/\pi$. The conditions fixing the allowed lengths of strings were first argued by Suzuki and Takahashi in [67] and, subsequently, Korepin [38] argued the form of these conditions on the basis of the normalisability of the wave function for the Thirring model. Korepin’s arguments were applied to the XXZ chain simultaneously by Hida [19] and by Fowler, Zotos in [17]. However, per se, these arguments only hold in the ferromagnetic sector $D = 0$, i.e. for the model at $h$ large enough. Other fine effects come into play when $N/L$ has non-zero limit. The author [42] corrected the form of these conditions by taking rigorously the large-$L$ limit. This correct form of the constraints for the existence of an $r$ bound state is recalled in Appendix A.2.

When $L$ is large, a given excited state $|\Upsilon\rangle$ having a finite excitation energy relatively to the ground state can thus be parametrised by its spin $s_\Upsilon$ relatively to the ground state and by the collection of rapidities of the various excitations:

$$\mathcal{R}_\Upsilon = \left\{ \left( \lambda^{(1)}_{m_1} \right)^{r_1}_{p_1} \cup \left( \lambda^{(2)}_{m_2} \right)^{r_2}_{p_2} \cup \cdots \right\} \cup \left( \left( \nu^{(1)}_{d_1} \right)^{r_1}_{d_1} \right)_{r \in \mathbb{R}_r}. \tag{2.1}$$

For finite $L$, the rapidities in (2.1) satisfy to the higher level Bethe Ansatz equations

$$\tilde{\xi}_1(\lambda^{(1)}_{m_1}|\mathcal{R}_\Upsilon) = \frac{2\pi}{\mathcal{L}} m_1,$$
$$\tilde{\xi}_1(\lambda^{(2)}_{m_2}|\mathcal{R}_\Upsilon) = \frac{2\pi}{\mathcal{L}} m_2,$$
$$\tilde{\xi}_2(\nu^{(1)}_{d_1}|\mathcal{R}_\Upsilon) = \frac{2\pi}{\mathcal{L}} d_1. \tag{2.2}$$

$\tilde{\xi}_a$ are the so-called counting functions [10, 14, 36]. When $a = 1$, they are associated with the particle-holes rapidities and, more generally, when $a = r \geq 2$ with the $r$-strings rapidities. The equations (2.2) involve three

---

1 See [44] for a more precise discussion of this phenomenon. Notably, in this work, the property has been proven to hold for free fermion equivalent models.
types of integers $m_a, m'_a$ and $d^{(r)}_a$ which belong to the sets

$$m_a \in [\!\! [-M_-; M_+ \!] \!\! ] \setminus [\!\! [1 ; N + s_\Upsilon \!] \!\! ], \quad m'_a \in [\!\! [1 ; N + s_\Upsilon \!] \!\! ], \quad d^{(r)}_a \in [\!\! [-M^{(r)}_+ ; M^{(r)}_- \!] \!\! ] .$$

(2.3)

The integers $M_a, M^{(r)}_a$ and $N$ all go linearly with $L$ to infinity.

A given excited state $| \Upsilon \rangle$ in the spin $s_\Upsilon$ sector above the ground state corresponds to the choice of the integers $n_p^{(tot)}, n_h^{(tot)}$ and $n_r, r \in \mathcal{R}_a$, satisfying to the constraint

$$n_h^{(tot)} = n_p^{(tot)} + \sum_{r \in \mathcal{R}_a} n_r, \quad n_h^{(tot)} \in [0 ; \kappa_L], \tag{2.4}$$

and then to the choice of increasing sequences of integers

$$m_1 < \cdots < m_{n_p^{(tot)}}, \quad m'_1 < \cdots < m'_{n_h^{(tot)}}, \quad \text{and} \quad d^{(r)}_1 < \cdots < d^{(r)}_{n_r},$$

(2.5)

such that each of the involved integers belongs to its respective domain as given in (2.3). Here, $\kappa_L$ is some arbitrary sequence such that $\kappa_L/\sqrt{L} \rightarrow +\infty$ and $\kappa_L/L \rightarrow 0$. It is believed that only excited states such that $0 \leq n_h^{(tot)} \leq \kappa_L$ may have a finite thermodynamic limit of their excitation energy above the ground state. The rapidities arising in (2.1) have different origins:

- $\lambda^{(h)}_a$ are the rapidities of the hole excitations. In the $L \rightarrow +\infty$ limit and when $a$ runs through $[\!\! [1 ; N + s_\Upsilon \!] \!\! ]$, $\lambda^{(h)}_a$ runs through $[-q ; q]$.

- $\lambda^{(p)}_a$ are the rapidities of the particle excitations. When $L \rightarrow +\infty$ and $a$ runs through $[\!\! [-M_- ; M_+ \!] \!\! ] \setminus [\!\! [1 ; N + s_\Upsilon \!] \!\! ]$, the rapidities vary on $\{ -\mathbb{R} + i\pi/2 \} \cup \{ \mathbb{R} \setminus [-q ; q] \}$. Here $-\mathbb{R}$ indicates that the set is skimmed through along the opposite orientation.

- $\nu^{(r)}_a$ are the rapidities of the $r$-strings. An $r$-string is characterised by a definite parity $\delta_r$, which is either 0 or 1. If the $r$-string has zero parity, then its rapidity is real valued, whereas, if it has parity one, its rapidity belongs to $\mathbb{R} + i\pi/2$. When $L \rightarrow +\infty$ and $d^{(r)}_a$ runs through $[\!\! [-M^{(r)}_+ ; M^{(r)}_- \!] \!\! ]$, the $r$-string rapidity varies on $s_r \mathbb{R} + \delta_r i\pi/2$. Here $s_r = 1$ or $-1$ and encodes the orientation along which the set is skimmed through.

Given any configuration $\mathcal{R}$ of parameters -not necessarily solving (2.2)-

$$\mathcal{R} = \left\{ \{ \lambda^{(p)}_a \}'^{(tot)}_i \cup \{ \lambda^{(h)}_a \}'^{(tot)}_i \cup \{ \nu^{(r)}_a \}'^{(tot)}_{i=1} \right\}_{r \in \mathcal{R}_a} \tag{2.6}$$

such that the cardinalities of the respective sets satisfy (2.4), the counting functions takes the form

$$\tilde{F}_r(\omega \, | \, \mathcal{R}) = p_r(\omega) - \frac{1}{L} \tilde{F}_r(\omega \, | \, \mathcal{R}) + \delta_{r,1} \frac{N + 1}{2L}. \tag{2.7}$$

The functions $p_r$ have the interpretation of the dressed momenta of the excitations associated with $\tilde{F}_r$. They are defined as solutions to linear integral equations, c.f. Appendix A.1 for more details. The functions $\tilde{F}_r$ appearing

\[\footnote{Since $n_h^{(tot)}$ is finite, this necessarily implies that, for the given excited state under consideration, $\{ r \in \mathcal{R}_a : n_r \neq 0 \}$ is a finite set. This is a non-trivial constraint when $\zeta/\pi$ is irrational as the set of allowed string lengths $\mathcal{R}_a$ is unbounded.} \]
in (2.7) depend on $\mathbb{R}$ and are analytic functions of their arguments provided that these stay in a neighbourhood of the domains of condensation of the associated rapidities. They are such that
\[
|\hat{F}_r(\omega \mid \mathbb{R})| \leq C \cdot (n_{p}^{(\text{tot})} + n_{h}^{(\text{tot})} + \sum_{r \in \mathbb{N}_d} n_r),
\] (2.8)
uniformly in the rapidities contained in $\mathbb{R}$ and in $\omega$ belonging to a neighbourhood of $\mathbb{R} \cup \{ \mathbb{R} + i\pi/2 \}$. Furthermore, they obey reduction properties when some of the rapidities of the particles or holes become close to the Fermi boundaries $\pm q$. More precisely, assume that the particle and hole rapidities present in $\mathbb{R}$ partition as
\[
\{ \lambda_{a}^{(p)} \}_{n_{p}^{(\text{tot})}} \cup \{ \nu_{a}^{(1)} \}_{n_{r}^{(\text{tot})}} \quad \text{and} \quad \{ \lambda_{a}^{(h)} \}_{n_{h}^{(\text{tot})}} \cup \{ \mu_{a} \}_{n_{h}^{(\text{tot})}}.
\] (2.9)
The rapidities building up the sets $\mathbb{Y}_{a}^{(p)}$, resp. $\mathbb{Y}_{a}^{(h)}$, are at most within a distance $\delta$ of the endpoints $\pm q$ of the Fermi zone:
\[
|\lambda_{a}^{(\nu)} - q| \leq \delta \quad \text{and} \quad |\mu_{a}^{(\nu)} - q| \leq \delta.
\] (2.10)
Here, $\delta > 0$ is some fixed, sufficiently small parameter that is $L$-independent. In their turn, the "bulk" rapidities $\mu_{a}$ and $\nu_{a}^{(1)}$ are all uniformly away from the endpoints of the Fermi zone
\[
|\mu_{a} - q| > \delta \quad \text{and} \quad |\nu_{a}^{(1)} - q| > \delta \quad \text{for} \quad \nu \in \{ \pm \}.
\] (2.11)
Up to precision $\delta$, the rapidities in $\mathbb{Y}_{a}^{(p/h)}$ can be thought of as collapsing on the right (+), resp. left (−), endpoint of the Fermi zone. The integers
\[
\ell_{a} = n_{p}^{(\nu)} - n_{h}^{(\nu)}
\] (2.12)
encode the discrepancy between the numbers of particles and holes collapsing on either of the endpoints of the Fermi zone.

If the decomposition (2.9) holds, the rapidities gathered in $\mathbb{R}$ split
\[
\mathbb{R} = \mathbb{Y} \cup \bigcup_{\nu = \pm} \{ \mathbb{Y}_{a}^{(p)} \cup \mathbb{Y}_{a}^{(h)} \}
\] (2.13)
into a collection of macroscopic variables of the massive modes
\[
\mathbb{Y} = \{ \mathbb{Y}_{a}^{(p)} : \{ \mathbb{Y}_{a}^{(h)} \}_{a = 1} \} \cup \{ \ell_{a} \} \text{ with } \mathbb{R} = \mathbb{R}_{\text{at}} \cup \{ 1 \}
\] (2.14)
and a collection of rapidities $\mathbb{Y}_{a}^{(p/h)}$ of the particles and holes belonging to a neighbourhood of the Fermi endpoints, viz. those giving rise to the massless part of the spectrum.

Then, given the decomposition (2.13) subordinate to $\delta > 0$ as in (2.10), the building blocks of the counting functions satisfy
\[
\hat{F}_r(\omega \mid \mathbb{R}) = \hat{F}_r(\omega \mid \mathbb{Y}) + O(\delta)
\] (2.15)
what entails that
\[
\hat{\xi}_r(\omega \mid \mathbb{R}) = \hat{\xi}_r(\omega \mid \mathbb{Y}) + O(\delta).
\] (2.16)
2.2 The spectrum at large-$L$

The relative to the ground state excitation energy and momentum of an excited state |$\Upsilon$\rangle parametrised by the rapidities (2.1) and located in the -$s_{\Upsilon}$ spin sector takes the form

$$\hat{\mathcal{E}}_{\Upsilon;\Omega} = \mathcal{E}(\mathcal{R};\Upsilon) + O(L^{-1}) \quad \text{and} \quad \hat{\mathcal{P}}_{\Upsilon;\Omega} = \mathcal{P}(\mathcal{R};\Upsilon) + \pi s_{\Upsilon} + O(L^{-1})$$

(2.17)

where, given $\mathcal{R}$ as in (2.6), one has

$$\mathcal{E}(\mathcal{R}) = \sum_{a=1}^{n_{\mathcal{R}}^{(0)}} E_1(\lambda_a^{(p)}) - \sum_{a=1}^{n_{\mathcal{R}}^{(0)}} E_1(\lambda_a^{(h)}) + \sum_{r \in \mathcal{R}_{st}} \sum_{a=1}^{n_r} E_r(\nu_{a}^{(r)})$$

(2.18)

$$\mathcal{P}(\mathcal{R}) = \sum_{a=1}^{n_{\mathcal{R}}^{(0)}} p_1(\lambda_a^{(p)}) - \sum_{a=1}^{n_{\mathcal{R}}^{(0)}} p_1(\lambda_a^{(h)}) + \sum_{r \in \mathcal{R}_{st}} \sum_{a=1}^{n_r} p_r(\nu_{a}^{(r)})$$

(2.19)

The functions $e_a$, resp. $p_a$, are the dressed energies and momenta of the individual excitations. They are defined as solutions to linear integral equation, c.f. Appendix A.1. Finally, $p_F = p_1(q)$ is the Fermi momentum.

The form taken by the excitation energy $\hat{\mathcal{E}}_{\Upsilon;\Omega}$ and the properties of the dressed energies $e_r$, c.f. Appendix A.1, ensure that

- the bound state excitations are massive, i.e. $e_r(\lambda) > c_r > 0$ for some constant $c_r$ on $\mathbb{R} + \delta_0i\pi/2$, for $r \in \mathcal{R}_{st}$;

- the particle-hole excitation have a massless and a massive branch: one has

$$e_1 > 0 \quad \text{on} \quad [\mathbb{R} + i\frac{\pi}{2}] \cup [\mathbb{R} \setminus [-q : q]] \quad \text{and} \quad e_1 < 0 \quad \text{on} \quad [-q : q].$$

In particular $e_1(\pm q) = 0$ so that the massless excitations will correspond to particles, resp. holes, whose rapidities $\lambda_a^{(p)}$, resp. $\lambda_a^{(h)}$, collapse, in the thermodynamic limit, on the endpoints of the Fermi zone. Such a collapse is achieved if the integer $m_a$, resp. $m'_a$, in (2.2) is of the type $N + o(L)$ -meaning that the rapidity collapses on $q$- or $1 + o(L)$ -meaning that the rapidity collapses on $-q$. Hence, the distinction made in (2.10)-(2.11)

For further purposes, it is convenient to introduce a special notation for the ratio

$$v = \frac{m}{\tau}$$

(2.20)

of the distance to time as well as for the combination

$$\mathcal{U}(\mathcal{R}, v) = \mathcal{P}(\mathcal{R}) - \frac{1}{v} \cdot \mathcal{E}(\mathcal{R})$$

(2.21)

of the excitation momenta and energies which can be recast as

$$\mathcal{U}(\mathcal{R}, v) = \sum_{a=1}^{n_{\mathcal{R}}^{(0)}} u_1(\lambda_a^{(p)}, v) - \sum_{a=1}^{n_{\mathcal{R}}^{(0)}} u_1(\lambda_a^{(h)}, v) + \sum_{r \in \mathcal{R}_{st}} \sum_{a=1}^{n_r} u_r(\nu_{a}^{(r)}, v).$$

(2.22)

There, I agree upon

$$u_r(\lambda, v) = p_r(\lambda) - \frac{e_r(\lambda)}{v}.$$
If the configuration of rapidities in $\mathcal{R}$ decomposes as \((2.13)\), then the function $\mathcal{U}(\mathcal{R}, \nu)$ simplifies to

$$
\mathcal{U}(\mathcal{R}, \nu) = \mathcal{V}(\mathcal{R}, \nu) + \mathcal{O}(\delta)
$$

(2.24)

where

$$
\mathcal{V}(\mathcal{R}, \nu) = \sum_{r \in \mathcal{R}} \sum_{a=1}^{n_r} u_r(\nu_a(r), \nu) - \sum_{a=1}^{n_{\delta}} u_1(\mu_a, \nu) + \sum_{\nu = \pm} \nu \ell_\nu p_F.
$$

(2.25)

Above, I have introduced the Fermi momentum $p_F = p_1(q)$.

I refer to [45] where all this is discussed at length.

### 2.3 The form factors at large-$L$

The analysis to come will build on the explicit form taken by the large-volume behaviour of the form factors of the local operators $\sigma^\gamma_{\nu}$, $\gamma \in \{\pm, \cdot\}$, namely the expectation values $\langle [T|\sigma^\gamma_{\nu}|\Omega]\rangle$. The large volume behaviour of $\langle [T|\sigma^\gamma_{\nu}|\Omega]\rangle$ when $|\Omega\rangle$ is an excited state as described in Section \((2.1)\) and $|\Omega\rangle$ the ground state has been determined in [45] on the basis of a rigorous analysis. One concludes that there exists an $L$-dependent function $\mathcal{F}(\gamma)$ of the rapidities $\mathcal{R}$ associated with the excited state $|\gamma\rangle$, c.f. \((2.1)\), such that

$$
\langle [r|\sigma^\gamma_{\nu}|\Omega]\rangle^2 = \mathcal{F}(\gamma)(\mathcal{R}_\gamma) \cdot \left(1 + \mathcal{O}\left(\ln \frac{L}{\ell}\right)\right).
$$

(2.26)

The explicit expression of $\mathcal{F}(\gamma)(\mathcal{R})$ is rather cumbersome and will be of no use in the following. It can be found in [45]. For further purposes, it is enough to know that it is an analytic function on some small neighbourhood of the region where the various rapidities evolve. Furthermore, for all $r$-string rapidities and particle or hole rapidities uniformly away from $\pm q$, this neighbourhood may be taken $L$-independent.

The large-$L$ expansion given above is not uniform in respect to the position of the particle-hole rapidities $\mathcal{R}_\gamma$ given in \((2.1)\), or equivalently in respect to the integers arising in \((2.2)\). It can be further simplified if one provides additional information on the proximity of the particle-hole rapidities to the endpoints $\pm q$ of the Fermi zone. This non-uniformness has to do with the fact that the particle or hole excitations on the Fermi boundaries $\pm q$ correspond to the massless excitations of the model.

In order to state the form of the simplified expression, consider a collection of rapidities $\mathcal{R}$, not necessarily solving \((2.2)\), which partitions as in \((2.13)\). It is convenient to re-parametrise the rapidities in $\mathcal{R}_\gamma^{(p/h)}$ as

$$
\mathcal{R}_\gamma^{(p/h)} = \left\{ \nu_L [\mathcal{E}_\gamma(\mu \mid \mathcal{Y}) - N_\gamma], \nu \in \mathcal{Y}_\gamma^{(p/h)} : \nu [N_\gamma - \left(\mathcal{E}_\gamma(\mu \mid \mathcal{Y})\right)] - 1 \right\}.
$$

(2.27)

Then, given the partitioning \((2.13)\), one gets

$$
\mathcal{F}(\gamma)(\mathcal{R}) = \prod_{\nu \in \{\pm\}} \mathcal{F}(\nu\mid \mathcal{Y}) : \frac{\mathcal{F}(\nu\mid \mathcal{Y}) \cdot \left(1 + \mathcal{O}\left(\ln \frac{L}{\ell} + \delta \ln \delta\right)\right)}{\prod_{a=1}^{n_{\delta}} \left[L_\gamma \mathcal{E}_\gamma(\mu_a \mid \mathcal{Y})\right] \cdot \prod_{r \in \mathcal{R}} \prod_{a=1}^{n_r} \left[L_\gamma \mathcal{E}_\gamma(v_a(r) \mid \mathcal{Y})\right]}
$$

(2.28)

where the counting functions are as given in \((2.2)\). The $: :$ symbol appearing in the first factor corresponds to an operator normal-like ordering; its precise meaning will be discussed below. The form factor $\mathcal{F}(\gamma)(\mathcal{R})$ factorises into three parts:

- $\mathcal{F}(\gamma)(\mathcal{Y})$ which should be thought of as the form factor density squared associated with the massive modes of the model;
• \( \tilde{\delta}_n(\gamma | 3^\nu \), which should be thought of as the form factor density squared that is associated with the massless excitations of the model;

• the denominator containing the counting functions which corresponds to the densities of the massive modes parametrised by \( \gamma \).

The form factor density of the massive modes \( F^{(\gamma)}(\gamma) \) is a smooth function of \( \gamma \). Its explicit expression can be found in [45]. The only properties that will be needed is that

i) it has at least a double zero when some rapidities of the same type (i.e. particle, hole or \( r \)-string) coincide;

ii) it decays as \( e^{+2r \nu a} \) when the real part of a rapidity \( \nu^{(r)} a \) goes to \( \pm \infty \);

iii) it has power-law singularities at \( \pm q \) in respect to the massive particle-hole variables \( v_a^{(1)} \) and \( \mu_a \);

iv) it extends to an analytic function of any of the variables present in \( \gamma \), where each rapidity belongs to an \( L \)-independent, sufficiently small, neighbourhood of the domain where the corresponding excitation rapidities live. The size of this neighbourhood depends on \( \delta \) in what concerns the particle-hole rapidities. The form factor density is also analytic in a neighbourhood of \( \infty \), with the exception of the lines

\[
v_a^{(1)} \in -\infty ; -q[+i\zeta] \quad \text{and} \quad v_a^{(r)} \in -\infty ; -q[+i\omega_{\pm} \zeta] \quad \text{for} \quad r \in \mathbb{N}_{\text{st}} \quad \text{and \ modulo \ } i\pi \tag{2.29}
\]

This upon agreeing that \( \gamma = \min(\eta - \pi[\eta/\pi], \pi - \eta + \pi[\eta/\pi]) \). On the above rays, the form factor has a jump discontinuity which takes the form

\[
\mathcal{F}^{(\gamma)}(\gamma_{a,s}^+) = \mathcal{F}^{(\gamma)}(\gamma_{a,s}^-) \quad . \tag{2.30}
\]

This boundary value problem involves two sets of variables:

\[
\gamma_{a,s}^+ = \left\{ \{\mu_a\}_{a=1}^{n_a} ; \{v_a^{(r)}\}_{a=1}^{n_a} \right\} ; \{\ell_a\} ; v_a^{(r)} = x+i\omega_{\pm} \zeta \quad , \tag{2.31}
\]

\[
\gamma_{a,s}^- = \left\{ \{\mu_a\}_{a=1}^{n_a} ; \{v_a^{(r)}\}_{a=1}^{n_a} \right\} ; \{\ell_a\} ; v_a^{(r)} = x+i\omega_{\pm} \zeta \quad , \tag{2.32}
\]

with \( x < -q \). All the rapidities in \( \gamma_{a,s}^{\pm} \) are taken generic with the exception of \( v_b^{(s)} \) which ought to be specialised as stated. Above, \( \sigma \in \{\pm 1\} \) if \( s \geq 2 \) and \( \sigma = 1 \) if \( s = 1 \). Furthermore, one has

\[
u_a^r = -\text{sgn}(\pi + 2\pi[\mu_a/2\pi]) \cdot (r + \sigma) \zeta \cdot (1 - \delta_{\sigma,-\delta_{r,1}}) \quad . \tag{2.33}
\]

The first three properties, put all together, can be summarised within the representation

\[
\mathcal{F}^{(\gamma)}(\gamma) = \prod_{r \in \mathbb{N}_{\text{st}}} \prod_{a \neq b} (v_a^{(r)} - v_b^{(r)}) \cdot D_{n_1;n_0}(\{v_a^{(1)}\}_{1}^{n_1} ; \{\mu_a\}_{1}^{n_0}) \cdot \mathcal{F}^{(\gamma)}(\gamma) \tag{2.34}
\]

where one has

\[
D_{n_1;n_0}(\{v_a^{(1)}\}_{1}^{n_1} ; \{\mu_a\}_{1}^{n_0}) = \prod_{a=1}^{n_1} \left( \frac{v_a^{(1)} + q}{v_a^{(1)} - q} \right)^{2\theta_{\gamma}(v_a^{(1)} | \gamma)} \prod_{a=1}^{n_0} \left( \frac{\mu_a - q}{\mu_a + q} \right)^{2\theta_{\gamma}(\mu_a | \gamma)} \times \prod_{\nu = \pm} \left\{ \frac{1}{\prod_{a=1}^{n_1} (\mu_a - v_\nu)} \right\}^{2\nu} \frac{m}{\prod_{a=1}^{n_1} \prod_{b=1}^{n_0} (v_a^{(1)} - v_b^{(1)})} \tag{2.35}
\]

\[
\times \prod_{\nu = \pm} \left\{ \frac{1}{\prod_{a=1}^{n_1} \prod_{b=1}^{n_0} (v_a^{(1)} - \mu_b)} \right\}^{2\nu} \frac{m}{\prod_{a=1}^{n_1} \prod_{b=1}^{n_0} (v_a^{(1)} - \mu_b)} \tag{2.35}
\]

\[
\times \prod_{\nu = \pm} \left\{ \frac{1}{\prod_{a=1}^{n_1} \prod_{b=1}^{n_0} (v_a^{(1)} - \mu_b)} \right\}^{2\nu} \frac{m}{\prod_{a=1}^{n_1} \prod_{b=1}^{n_0} (v_a^{(1)} - \mu_b)} \tag{2.35}
\]

\[
\times \prod_{\nu = \pm} \left\{ \frac{1}{\prod_{a=1}^{n_1} \prod_{b=1}^{n_0} (v_a^{(1)} - \mu_b)} \right\}^{2\nu} \frac{m}{\prod_{a=1}^{n_1} \prod_{b=1}^{n_0} (v_a^{(1)} - \mu_b)} \tag{2.35}
\]

\[
\times \prod_{\nu = \pm} \left\{ \frac{1}{\prod_{a=1}^{n_1} \prod_{b=1}^{n_0} (v_a^{(1)} - \mu_b)} \right\}^{2\nu} \frac{m}{\prod_{a=1}^{n_1} \prod_{b=1}^{n_0} (v_a^{(1)} - \mu_b)} \tag{2.35}
\]
Above, the exponents \( \vartheta(\omega \mid \mathcal{Y}) \) are related to the opposite of the thermodynamic limit of the shift function associated with the massive excitation \( \mathcal{Y} \). They take the explicit form

\[
\vartheta(\omega \mid \mathcal{Y}) = \sum_{a=1}^{n_h} \phi_1(\omega, \mu_a) + \frac{2}{\pi} \mathcal{Y} \mathcal{Z}(\omega) - \sum_{r \in \mathbb{R}} \sum_{a=1}^{n_h} \phi_r(\omega, \nu_{a,r}^{(r)}) - \sum_{\nu' \in \{\pm\}} \ell_{\nu'} \phi_1(\omega, \nu' q) .
\] (2.36)

Here \( Z \) is the dressed charge \( \text{(A.12)} \) and \( \phi_r \) the dressed phase \( \text{(A.11)} \) of an \( r \)-string. Note that, for generic parameters, the function \( \mathcal{F}_{\mathrm{reg}}^{(\gamma)}(\mathcal{Y}) \) appearing in the decomposition \( \text{(2.34)} \) does not vanish when some rapidities coincide or approach the endpoints \( \pm q \) of the Fermi zone. Note also that the singularities at \( \pm q \) in \( \text{(2.34)} \) that are present in the \( D \)-factor are reminiscent of the fact that the form factor \( \mathcal{F}^{(\gamma)}(\mathcal{R}) \) does not admit a uniform in the hole and particle rapidity, large-\( L \) asymptotics. Therefore, these singularities do not correspond to singularities of \( \mathcal{F}^{(\gamma)}(\mathcal{R}) \) since, by construction, the rapidities \( \mu_a \) and \( \nu_{a,r}^{(r)} \) are at least at distance \( \delta \) from \( \nu q \).

The discrete form factor density \( \mathcal{G}_\nu(\mathcal{Y} \mid 3^\nu) \) associated with the massless modes of the model solely depends on the rapidities of the massive modes through \( \vartheta_r(\mathcal{Y}) \) which takes the form

\[
\vartheta_r(\mathcal{Y}) = \vartheta(\nu q \mid \mathcal{Y}) - \nu \ell_{\nu} .
\] (2.37)

This discrete form factor reads

\[
\mathcal{G}_\nu(\mathcal{Y} \mid 3^\nu) = \prod_{\mu \in \Gamma''(\nu)} e^{\delta_\nu(\mu)} \prod_{\mu \in \Gamma''(\nu)} e^{-\delta_\nu(\mu)} \cdot \frac{G^2(1 - \nu \vartheta_r(\mathcal{Y}) - \nu \ell_{\nu} - \ell_{\nu})}{G^2(1 - \nu \vartheta_r(\mathcal{Y}) - \nu \ell_{\nu})} \times R(3^\nu \mid -\nu \vartheta_r(\mathcal{Y}) - \nu \ell_{\nu} - \ell_{\nu}) \cdot \left( \frac{2\pi}{L} \right)^{(\nu_{\vartheta}(\mathcal{Y}) + \nu_{\ell})^2} .
\] (2.38)

\( 3^\nu \) has been defined in \( \text{(2.27)} \), while \( \delta_\nu(\mu) \) is the differential operator

\[
\delta_\nu(\mu) = \sum_{\nu' \neq \pm} \left\{ \phi_1(\nu' q, \mu) - \phi_1(\nu' q, \nu q) \right\} \cdot \frac{\partial}{\partial Y_{\nu'}} \bigg|_{Y_{\nu'}=0} .
\] (2.39)

Note that the only explicit, \( i.e. \) non-rescaled as in the definition of \( 3^\nu \), dependence in \( \text{(2.38)} \) on the rapidities contained in \( \Gamma''(\nu/h) \) is through the action of the operator \( \delta_\nu \). The \( \ast \) symbol in \( \text{(2.28)} \) is relatively to the ordering that has to be imposed on the action of this operator: all the derivatives should appear to the left of the expression in \( \text{(2.28)} \) prior to be computed and the \( \nu_s \to 0 \) limit should be taken at the very end of the calculations.

Finally, the ratio of Barnes functions \( G \) is a normalisation pre-factor while \( R(3^\nu \mid -\nu \vartheta_r(\mathcal{Y}) - \ell_{\nu}) \) contains the non-trivial part of the discrete form factor. Given a parameter \( \nu \) and for \( 3 = \{ [p a_{1}], [h a_{1}] \} \), it takes the form:

\[
R(3 \mid \nu) = \left( \frac{\sin(\pi \nu)}{\pi} \right)^{2n_\nu} \prod_{a < b} (h_a - h_b)^2 \cdot \prod_{a < b} (p_a - p_b)^2 \cdot \prod_{a=1}^{n_h} \frac{\Gamma^2(1 + p_a + \nu)}{\Gamma^2(1 + p_a)} \cdot \prod_{a=1}^{n_h} \frac{\Gamma^2(1 + h_a - \nu)}{\Gamma^2(1 + h_a)} .
\] (2.40)

It is important for further purposes to state that the exponent \( \vartheta_r(\mathcal{Y}) \) defined in \( \text{(2.37)} \) and the reduced momentum-energy combination \( \mathcal{H}(\mathcal{Y}, \nu) \) given in \( \text{(2.25)} \) satisfy jump discontinuities that are identical to those satisfied by the from factor density \( \mathcal{F}^{(\gamma)}(\mathcal{Y}) \) \( \text{(2.30)} \), namely:

\[
\vartheta_r(\mathcal{Y}^s_{\sigma, s}) = \vartheta_r(\mathcal{Y}^s_{\sigma, s}) \quad \text{and} \quad \mathcal{H}(\mathcal{Y}^s_{\sigma, s}, \nu) = \mathcal{H}(\mathcal{Y}^s_{\sigma, s}, \nu)
\] (2.41)

with \( s \in \mathbb{R}, \sigma \in \{ \pm 1 \} \) and where I have used the parametrisation \( \text{(2.32)} \). Note also that although it follows from \( \text{(A.7)} \) that \( \mathcal{H}(\mathcal{Y}; \nu) \) has cuts along \( v_{a,r}^{(r)} \in \mathbb{R}^+ \pm i \xi/2 \mathrm{modulo} i \pi \), the function \( \exp \{ i \mathcal{H}(\mathcal{Y}; \nu) \} \) is continuous across these lines.
3 The form factor expansion in the thermodynamic limit

3.1 The auxiliary series $\hat{C}_1^{(γ)}(m, t)$

The two-point functions admit the form factor series expansion over all the excited states $γ$ of the model

$$\langle Ω | σ_1^γ(t)σ_{m+1}^γ | Ω \rangle = \sum_{γ} e^{i mE_{γ} - itE_{γ} Ω}. |\langle γ | σ_{m+1}^γ | Ω \rangle|^2.$$ (3.1)

Since the Hilbert space of the model is finite dimensional, the form factor expansion is well defined when $L$ is finite. However, convergence related subtleties can arise when sending $L \to +\infty$ on the level of the form factor series. To avoid some of these complications and regularise the calculation of the thermodynamic limit of the two-point function, one should consider the two-point function in the distributional sense with, furthermore, $t$ being continued to the upper half-plane. This improves significantly the convergence properties of the series. In particular, the expressions that follow should be understood in the sense of a $\Im(t) \to 0^+$ limit.

The presumably existing infinite volume limit of the two-point function will be denoted as $\lim_{L \to +\infty} \langle Ω | σ_1^γ(t)σ_{m+1}^γ | Ω \rangle$. (3.2)

As already discussed, one expects that, in the thermodynamic limit $L \to +\infty$,

\begin{itemize}
  \item[i)] only those Eigenstates having a finite thermodynamic limit of the relative excitation energy above the ground state $\hat{E}_{γ} Ω$ should contribute to the sum over the excited states defining the form factor expansion of the correlator. This means that, effectively speaking, the sum over the excited states boils down to a summation over all integers $n_p^{(tot)}, n_{h}^{(tot)}$ and $n_r$ subject to the constraint (2.4) and then to a summation over all the possible choices of integers (2.5) belonging to the sets (2.3).

  \item[ii)] Only the leading large-$L$ behaviour of the summands in (3.1) will contribute to the thermodynamic limit of the two-point function.
\end{itemize}

The two above statements were shown to hold true for a generalised free fermion model in [44] and various supporting arguments and checks, in the case of the interacting non-linear Schrödinger model, were given in [44 49].

Upon dropping the terms in (3.1) that are irrelevant for the thermodynamic limit, one gets

$$\lim_{L \to +\infty} \langle σ_1^γ(t)σ_{m+1}^γ(t) \rangle = \lim_{L \to +\infty} \langle Ω | σ_1^γ(t)σ_{m+1}^γ | Ω \rangle = \hat{C}_1^{(γ)}(m, t).$$ (3.3)

where $\hat{C}_1^{(γ)}(m, t)$ contains the summation over the excited states whose excitation integers $n_p^{(tot)}, n_{h}^{(tot)}$, $n_r$, collected in $n = (n_p^{(tot)}, n_h^{(tot)}, \{n_r\}_{r \in \mathbb{N}_a})$, belong to

$$\mathbb{Z}_{tot} = \{ (n_p^{(tot)}, n_h^{(tot)}, \{n_r\}_{r \in \mathbb{N}_a}) : n_p^{(tot)} + \sum_{r \in \mathbb{N}_a} n_r \in \mathbb{Z}_L \}$$ (3.4)

and with rapidities $\mathcal{R}_γ$ as in (2.1) and solving the system of equations (2.2):

$$\hat{C}_1^{(γ)}(m, t) = (-1)^{m_{σγ}} \sum_{n \in \mathbb{Z}_{tot}} \sum_{\mathcal{R}_γ} \hat{C}^{(γ)}(\mathcal{R}_γ) \cdot e^{im\mathcal{R}_γ \cdot r}.$$ (3.5)

Note that $s_γ$ appearing in (3.4)-(3.5) corresponds to the pseudo-spin of the operator $σ_1^γ$ which has been defined in (1.7). Finally, the function $\mathcal{U}(\mathcal{R}_γ, v)$ appearing in (3.5) has been defined in (2.22).
By using the analyticity of the summand, the summation over the excited states in (3.5) can be recast in terms of contour integrals by means of the multidimensional residue theorem $[1]$:

$$
\tilde{C}^{(y)}_1(m, t) = (-1)^{mn_y} \sum_{n \in \mathbb{Z}_{tot}} \prod_{r \in \mathbb{R}_{tot}} \left\{ \oint_{\Gamma^{(r)}_{tot}} D^{(y)}_n \gamma^{(r)} \right\} \cdot \oint_{\Gamma^{(p)}_{tot}} D^{(h)}_n A^{(h)} \cdot \tilde{H}(\mathbb{R}) \cdot \tilde{\gamma}^{(y)}(\mathbb{R}) \cdot e^{in \mathbb{R}(\mathbb{R}, v)}.
$$

(3.6)

Above, $\mathbb{R}$ contains all the integration variables as in (2.6), while the integration measures read

$$
D^{(y)}_n \gamma^{(r)} = \prod_{a=1}^{n_y} \left( \frac{1}{L_{\xi_1}^{(a)}(\gamma^{(r)}_{a} | \mathbb{R})} - 1 \right) \cdot \frac{d\gamma^{(r)}_{a}}{n_y! (2\pi)^{n_y}}, \quad D^{(h)}_n A^{(p)} = \prod_{a=1}^{n_p} \left( \frac{1}{L_{\xi_1}^{(a)}(\lambda^{(p)}_{a} | \mathbb{R})} - 1 \right) \cdot \frac{d\lambda^{(p)}_{a}}{n_p! (2\pi)^{n_p}}.
$$

(3.7)

and $D^{(h)}_n A^{(h)}$ is obtained from $D^{(h)}_n A^{(p)}$ by the exchange of superscripts $p \leftrightarrow h$. The counting functions arising in the definition of the measures are as defined in (2.7). $\tilde{H}(\mathbb{R})$ is there so as to compensate for the Jacobian arising from the computation of the multidimensional residues

$$
\tilde{H}(\mathbb{R}) = \prod_{a=1}^{n_p} \left( \frac{1}{L_{\xi_1}^{(a)}(\lambda^{(p)}_{a} | \mathbb{R})} - 1 \right) \cdot \prod_{a=1}^{n_y} \left( \frac{1}{L_{\xi_1}^{(a)}(\gamma^{(r)}_{a} | \mathbb{R})} - 1 \right) \cdot \left\{ \prod_{r \in \mathbb{R}_{tot}} \prod_{a=1}^{n_r} \left( \frac{1}{L_{\xi_1}^{(a)}(\nu^{(r)}_{a} | \mathbb{R})} - 1 \right) \right\} \cdot \det \left[ D\varphi \right],
$$

(3.8)

where $D\varphi$ is the differential of the map:

$$
\varphi : r \mapsto \left( \left( \frac{1}{L_{\xi_1}^{(a)}(\lambda^{(p)}_{a} | \mathbb{R})} - 1 \right)_{a=1}^{n_p}, \left( \frac{1}{L_{\xi_1}^{(a)}(\gamma^{(r)}_{a} | \mathbb{R})} - 1 \right)_{a=1}^{n_y}, \left( \left( \frac{1}{L_{\xi_1}^{(a)}(\nu^{(r)}_{a} | \mathbb{R})} - 1 \right)_{a=1}^{n_r} \right)_{r \in \mathbb{R}_{tot}} \right).
$$

(3.9)

Here, $r = (\lambda^{(p)}, \lambda^{(h)}, \gamma^{(r)}, \ldots, \nu^{(r)})$ with $\mathbb{R}_{tot} = \{r_1, \ldots, r_{\max}\}$ is the vector whose entries are the $n_{p/h}$-dimensional vectors

$$
\lambda^{(p/h)} = (\lambda^{(p/h)}_{1}, \ldots, \lambda^{(p/h)}_{n_{p/h}})
$$

(3.10)

and the $n_r$-dimensional vectors $\gamma^{(r)} = (\gamma^{(r)}_{1}, \ldots, \nu^{(r)}_{n_r})$ whose coordinates issue from the elements of the respective sets in $\mathbb{R}$. Note that the order of choosing the coordinates is irrelevant in that all the functions of interest are symmetric in the coordinates of $\lambda^{(p)}$, $\lambda^{(h)}$, or any of the $\gamma^{(r)}$ taken singly. In the following, such an identification between vectors and a collection of sets $\mathbb{R}$ will be made sometimes, this without further notice.

The integration (3.6) runs along two kinds of contours:

- The contour $\Gamma^{(r)}_{tot} = (\Gamma^{(r)}_{tot})^{n_r}$ is associated with a summation over the possible choices of rapidities for the $r$-bound states. The contour $\Gamma^{(r)}$ is depicted in Figure 1 and consists of a small counterclockwise loop around $\mathbb{R} + i \mathbb{R} \delta_r$.

- The integration domain $\Gamma^{(p)}_{tot} \ast \Gamma^{(h)}_{tot}$ is associated with a summation over all the possible choices of the hole, resp. particle, rapidities. It corresponds to an integration over an $n_{p/h}^{(tot)} + n_h^{(tot)}$ real dimensional sub-manifold of $\mathbb{C}^{n_{p/h}^{(tot)} + n_h^{(tot)}}$. To define it, one first fixes a given value of the $r$-string rapidities $\{\nu^{(r)}_{a}\}_{a=1}^{n_{r}}$. Then one introduces the map

$$
\psi : (\lambda^{(p)}, \lambda^{(h)}) \mapsto \left( \tilde{\xi}_1(\lambda^{(p)}_{1} | \mathbb{R}), \ldots, \tilde{\xi}_1(\lambda^{(p)}_{n_{p/h}} | \mathbb{R}), \tilde{\xi}_1(\lambda^{(h)}_{1} | \mathbb{R}), \ldots, \tilde{\xi}_1(\lambda^{(h)}_{n_{h}} | \mathbb{R}) \right).
$$

(3.11)
where the $r$-string rapidities entering in the definition of $\mathcal{R}$ are given by the above choice and where the vectors $\lambda^{(\hat{p}/\hat{h})}$ are as in (3.10) and built from the respective variables present in $\mathcal{R}$. Then, $\Gamma^{(\hat{p})}_{\text{tot}} \ast \Gamma^{(\hat{h})}_{\text{tot}}$ is defined as the pre-image

$$
\Gamma^{(\hat{p})}_{\text{tot}} \ast \Gamma^{(\hat{h})}_{\text{tot}} = \psi^{-1}\left((\mathcal{C}^{(\hat{p})})^{\star}_{\text{tot}} \times (\mathcal{C}^{(\hat{h})})^{\star}_{\text{tot}}\right)
$$

(3.12)

where the one-real dimensional contours $\mathcal{C}^{(\hat{p}/\hat{h})}$ are as defined on Figure 2. Although the submanifold $\Gamma^{(\hat{p})}_{\text{tot}} \ast \Gamma^{(\hat{h})}_{\text{tot}}$ is not a Carthesian product, effectively speaking and owing to the Carthesian product structure in the target space of $\psi$, one can think of each particle, resp. hole, rapidity as being integrated on the one-dimensional contour depicted in Figure 3. In fact, such a reduction occurs in the large-$L$ limit upon replacing $\xi_{\hat{i}}$ by $\rho_{1}$.

![Figure 1: Contour $\Gamma^{(r)} = \Gamma^{(r)}_+ \cup \Gamma^{(r)}_-$, $r \in \mathcal{R}_{\text{st}}$.](image1.png)

![Figure 2: Contours $\mathcal{C}^{(p)} = \bigcup_{\nu = \pm} \left(\mathcal{C}^{(p)}_{1,\nu} \cup \mathcal{C}^{(p)}_{0,\nu}\right)$ and $\mathcal{C}^{(h)} = \bigcup_{\nu = \pm} \mathcal{C}^{(h)}_{\nu}$ arising in the definition of $\Gamma^{(\hat{p})}_{\text{tot}}$ and $\Gamma^{(\hat{h})}_{\text{tot}}$. Here $N_+ = N + 1 + s_{\gamma}$ and $\delta > 0$ is some fixed but sufficiently small parameter. The contour $\bigcup_{\nu = \pm} \mathcal{C}^{(p)}_{1,\nu}$ corresponds to integrating around solutions to $e^{\nu \xi_{\hat{i}}(\omega;\mathcal{R})} - 1 = 0$ that are very close to $\mathbb{R}$ while the contour $\bigcup_{\nu = \pm} \mathcal{C}^{(h)}_{\nu}$ corresponds to integrating around solutions that are close to $\mathbb{R} + i \frac{\pi}{2}$. These contours are infinitesimally close to each other as indicate the $\pm \delta$ shifts.](image2.png)

### 3.2 The auxiliary series $\widehat{C}^{(p)}_2(m, t)$

The series of multiple contour integrals for $\widehat{C}^{(p)}_1(m, t)$ has still to be transformed so as to be able to compute its thermodynamic limit. To start with, I shall argue that one can drop several terms that should not contribute to its thermodynamic limit.

First of all, it is easy to convince oneself that the below pointwise limit holds

$$
\lim_{L \to +\infty} \left[ \hat{H}(\mathcal{R}) \right] = 1.
$$

(3.13)

It is thus reasonable to expect that one can replace $\hat{H}(\mathcal{R})$ by 1 in the series expansion for $\widehat{C}^{(p)}_1(m, t)$ without altering the value of its thermodynamic limit. This assumption will be made in the following.
Figure 3: Contours \( \Gamma_r^{(p)} = \Gamma_r^{(p)+0} \cup \Gamma_r^{(p)+1} \cup \Gamma_r^{(p)-0} \cup \Gamma_r^{(p)-1} \) and \( \Gamma_r^{(h)} = \Gamma_r^{(h)+0} \cup \Gamma_r^{(h)-0} \). Here, the endpoints of integration are defined as \( q_p^- = p_1^{-1}(\frac{2\pi a}{4\pi}), \ \hat{q}_p^- = p_1^{-1}(\frac{2\pi a}{4\pi}(N_+ - \frac{1}{2})), \ \hat{q}_p^+ = p_1^{-1}(\frac{2\pi a}{4\pi}(N_+ - \frac{1}{2})) \).

Second, observe that the one-dimensional \( r \)-string contour integral can be decomposed as

\[
\int_{\Gamma(r)} \hat{\xi}_a'(|\mathcal{R}g|) \cdot d^\nu v = -\int_{\Gamma(r)} \hat{\xi}_a'(|\mathcal{R}g|) \cdot d^\nu v + \sum_{\epsilon = \pm} \int_{\Gamma_r^{(\epsilon)}} \hat{\xi}_a'(|\mathcal{R}g|) \cdot d^\nu v - \int_{\Gamma_r^{(\epsilon)}} \hat{\xi}_a'(|\mathcal{R}g|) \cdot d^\nu v \quad (3.14)
\]

where the contours \( \Gamma_r^{(\epsilon)} \) are defined in Figure[1]

\[
s_r = \text{sgn}[p_r] \quad (3.15)
\]

and \( f \) is some holomorphic function inside of \( \Gamma(r) \) that, furthermore, decays at least as \( e^{-2|\mathcal{R}g|} \) at \( \infty \). Here, \( v \) corresponds to any of the \( r \)-string variables contained in \( \mathcal{R} \). All the other variables present in \( \mathcal{R} \) are assumed to belong to their respective domains of integration, as appearing in (3.6). It is then straightforward to estimate the large-\( L \) behaviour of the sum over \( \epsilon = \pm \) in (3.16) by using the properties of the dressed momentum \( p_r \) and the form of the counting function. One finds that this sum produces \( O(L^{-1}) \) corrections. This estimate is uniform in respect to all the other variables present in \( \mathcal{R} \). It is readily seen that the same property does hold for a \( n_r \)-dimensional analogue:

\[
\int_{\Gamma(r)} \prod_{a=1}^{n_r} \left\{ \frac{\hat{\xi}_a'(|\mathcal{R}g|)}{e^{i\epsilon L \hat{\xi}_a(|\mathcal{R}g|) - 1}} \right\} f(v) \cdot d^nu v = (-1)^r \int_{\Gamma(r)} \prod_{a=1}^{n_r} \left\{ \frac{\hat{\xi}_a'(|\mathcal{R}g|)}{e^{i\epsilon L \hat{\xi}_a(|\mathcal{R}g|) - 1}} \right\} f(v) \cdot d^nu v + O(L^{-1}) \quad (3.16)
\]

Here, one should understand that \( v = (v_1, \ldots, v_{n_r}) \) corresponds to the \( r \)-string rapidities present in \( \mathcal{R} \) and all other variables present in \( \mathcal{R} \) belong to their respective domains of integration that appear in (3.6).

Quite similarly, by using the Cartesian product structure of \( \Gamma_{tot}^{(p)} \# \Gamma_{tot}^{(h)} \) in the target space of \( \psi \) and the lower bounds (A.10) on the dressed momentum, one can decompose the particle-hole integrals as

\[
\int_{\Gamma_{tot}^{(p)} \# \Gamma_{tot}^{(h)}} \prod_{a=1}^{n_a^{(tot)}} \left\{ \frac{\hat{\xi}_a'(|\mathcal{R}g|)}{e^{i\epsilon L \hat{\xi}_a(|\mathcal{R}g|) - 1}} \right\} \cdot \prod_{a=1}^{n_h^{(tot)}} \left\{ \frac{\hat{\xi}_a'(|\mathcal{R}g|)}{e^{i\epsilon L \hat{\xi}_a(|\mathcal{R}g|) - 1}} \right\} \cdot f(\lambda^{(p)}, \lambda^{(h)}) \cdot d^{nu \lambda^{(p)}} \cdot d^{nu \lambda^{(h)}} = \int_{(\Gamma_{tot}^{(p)} \# \Gamma_{tot}^{(h)})}\prod_{a=1}^{n_a^{(tot)}} \left\{ -\hat{\xi}_a'(|\mathcal{R}g|) \right\} \cdot \prod_{a=1}^{n_h^{(tot)}} \left\{ -\hat{\xi}_a'(|\mathcal{R}g|) \right\} \cdot f(\lambda^{(p)}, \lambda^{(h)}) \cdot d^{nu \lambda^{(p)}} \cdot d^{nu \lambda^{(h)}} + O(L^{-1}) \quad (3.17)
\]
where the submanifold \((\Gamma_{\text{tot}}^{(p)} \ast \Gamma_{\text{tot}}^{(h)})_+\), giving rise to the leading contribution is defined as being the pre-image
\[
(\Gamma_{\text{tot}}^{(p)} \ast \Gamma_{\text{tot}}^{(h)})_+ = \psi^{-1}
\left((\mathcal{C}_{\text{tot}}^{(p)} \cup \mathcal{C}_{\text{tot}}^{(h)})_{+0}^{\to} \times (\mathcal{C}_{\text{tot}}^{(h)})_{-0}^{\to}
\right)
\]
by the map \(\psi\) introduced in (3.11). In terms of the "model" contour depicted in Figure 3 one can think, in a leading order in \(L\) approximation, of the contour \((\Gamma_{\text{tot}}^{(p)} \ast \Gamma_{\text{tot}}^{(h)})_+\) as the Cartesian product of the one-dimensional contours \(\mathcal{C}_{\text{tot}}^{(p)}_{+0} \cup \mathcal{C}_{\text{tot}}^{(p)}_{-1}\) relativeto the particle rapidities and of the one-dimensional contours \(\mathcal{C}_{\text{tot}}^{(h)}\) relativeto the hole rapidities as depicted in Figure 3.

Clearly, the precise bounds of both remainders in (3.16) and (3.17) do depend on the function \(f\) as well as on the number of integrations involved. It seems reasonable to assume that the class of functions involved in the effective form factor series \(\tilde{C}_1^{(r)}(m, t)\) makes all such remainder uniform and thus allows one to drop these when computing the thermodynamic limit of \(\tilde{C}_1^{(r)}(m, t)\). I word, I shall assume that it is licit to make the below replacements in \(\tilde{C}_1^{(r)}(m, t)\) without altering the value of its thermodynamic limit:

- \(\Gamma^{(r)} \leftrightarrow \Gamma^{(r)}_{+}\), \(\Gamma^{(p)} \ast \Gamma^{(h)}_{\text{tot}} \leftrightarrow (\Gamma^{(p)}_{\text{tot}} \ast \Gamma^{(h)}_{\text{tot}})_+\);
- \(\mathcal{D}^{\nu}(v^{(r)}) \leftrightarrow \mathcal{D}^{\nu}_{\text{red}}(v^{(r)}) = \prod_{a=1}^{n} \left(-L_{\gamma_a}^{(v^{(r)})}(\gamma)\right) \cdot \frac{\delta_{\nu}^{(v^{(r)})}}{n_{\gamma_a}(2\pi)^p}\);
- \(\mathcal{D}^{(l)}_{\text{red}}(p) \leftrightarrow \mathcal{D}^{(l)}_{\text{red}}(p) = \prod_{a=1}^{n} \left(-L_{\gamma_a}^{(p)}(\gamma)\right) \cdot \frac{\delta^{(l)}_{\text{red}}}{n_{\gamma_a}(2\pi)^p}\) and likewise with \(p \leftrightarrow h\).

On the basis of the above discussion, one expects that
\[
\lim_{L \to +\infty} \tilde{C}_1^{(r)}(m, t) = \lim_{L \to +\infty} \tilde{C}_2^{(r)}(m, t) \quad (3.19)
\]
where
\[
\tilde{C}_2^{(r)}(m, t) = (-1)^{m_{\gamma}} \sum_{n \in \mathbb{Z}_{\text{tot}}} \prod_{\gamma_{\text{tot}}} \left(\int \mathcal{D}^{\nu}_{\text{red}}(v^{(r)}) \cdot \int \mathcal{D}^{(l)}_{\text{red}}(p) \cdot \mathcal{D}^{(l)}_{\text{red}}(h) \cdot \tilde{\mathcal{F}}^{(r)}(\gamma) \cdot e^{im_{\gamma}(\mathcal{R}, v)}\right). \quad (3.20)
\]

It is convenient to slightly re-organise the contours of integration in (3.20) by deforming them back to the contours where the physical rapidities, solving the higher-level Bethe Ansatz equations, condense. Some care should however be taken in the treatment of the part associated to the particle-hole rapidities. There, one should treat slightly differently the vicinity of the endpoints of the Fermi zone: the curves linking the endpoint of the Fermi zone to some points at distance \(C\delta\), for some \(C > 0\), should be deformed into a path along which \(e^{im_{\gamma}(\lambda, v)}\)
- in what concerns the particles- and \(e^{-im_{\gamma}(\lambda, v)}\) - in what concerns the holes- is decaying. Observe that for
\[
|v| > v_F \quad \text{one has} \quad u_{1}^{(r)}(\pm q, v) = p_{1}^{(r)}(q) \cdot \left(1 + \frac{v_F}{v}\right) > 0, \quad \text{where} \quad v_F = \frac{e_{\gamma}(q)}{p_{1}^{(r)}(q)}, \quad (3.21)
\]
and hence \(e^{im_{\gamma}(\lambda, v)}\), resp. \(e^{-im_{\gamma}(\lambda, v)}\), is decaying when \(\lambda\) moves from \(\pm q\) in the direction of positive, resp. negative, imaginary parts. Likewise, for
\[
v_F > v > 0 \quad \text{one has} \quad \mp u_{1}^{(r)}(\pm q, v) > 0 \quad (3.22)
\]
\(^{1}\text{Note that, in the large-}\L\text{ limit, }\mathcal{C}_{\text{tot}}^{(p)}_{+\gamma} \text{ is mapped onto }\Gamma_{\toh}^{(p)} \text{ due to } p_{1}^{(r)}(\lambda + i\pi/2) < 0.\)
and hence the situation remains unchanged in what concerns $-q$ but the directions of decay are inverted around $q$. Finally, for

$$0 > v > -v_F \quad \text{one has} \quad \pm u'_i(\pm q, v) > 0$$

(3.23)

the situation remains as in the first case in what concerns $q$ but the directions of decay are inverted around $-q$.

The analiticity of the integrand allows one to deform the contours in (3.20) as

- $\Gamma^{(r)}_{s, r} \rightarrow \mathcal{C}_r$ where $\mathcal{C}_r = s_r \mathbb{R} + i \delta s_{r} \frac{\pi}{T}$;

- $(\Gamma^{(p)}_{\text{tot}} * \Gamma^{(h)}_{\text{tot}})_+ \rightarrow \mathcal{C}_{p/\text{tot}} * \mathcal{C}_{h/\text{tot}}$ where

$$\mathcal{C}_{p/\text{tot}} * \mathcal{C}_{h/\text{tot}} = \psi^{-1}\left( (\mathcal{C}_{p, \text{cos}})^{(+)} \times (\mathcal{C}_{h, \text{co}})^{(+)} \right)$$

(3.24)

with $\psi$ as in (3.11) and the contours $\mathcal{C}_{p/h; \text{co}}$ are as defined in Figure 4.

![Figure 4: Particle $\mathcal{C}_{p; \text{cos}}$ -in blue- and hole $\mathcal{C}_{h; \text{co}}$ -in orange- contours in the target space and plotted for the three regimes of the velocity $v = m/t$ appearing from bottom to top $|v| > v_F$, then $v_F > v > 0$ and finally $0 > v > -v_F$. Here $N_+ = N + 1 + s_v$. The contour $\mathcal{C}_{p; \text{cos}}$, resp. $\mathcal{C}_{h; \text{co}}$, partitions into three parts $\mathcal{C}_{p; \text{cos}} = \mathcal{C}_{p; \text{cos}}^{(+)} \cup \mathcal{C}_{p; \text{cos}}^{(0)} \cup \mathcal{C}_{p; \text{cos}}^{(+)}$, resp. $\mathcal{C}_{h; \text{co}} = \mathcal{C}_{h; \text{co}}^{(-)} \cup \mathcal{C}_{h; \text{co}}^{(0)} \cup \mathcal{C}_{h; \text{co}}^{(+)}$. The contours $\mathcal{C}_{p/h; \text{co}}^{(0)}$ are depicted in dotted lines while the contours $\mathcal{C}_{p/h; \text{co}}^{(0)}$ are depicted in solid lines. $\mathcal{C}_{p/h; \text{co}}^{(0)}$ appears to the right while $\mathcal{C}_{p/h; \text{co}}^{(0)}$ to the left.](image)

Doing so, recasts $\tilde{C}_2^{(r)}(m, t)$ as

$$\tilde{C}_2^{(r)}(m, t) = (-1)^{m+\gamma} \sum_{\mathcal{C}_{p/\text{tot}} \cap \mathcal{C}_{h/\text{tot}}} \left\{ \int D_{\text{red}}^{(p)} \left(\mathcal{C}_r^{(r)}\right) \right\} \cdot \int D_{\text{red}}^{(p)} A^{(p)} \cdot D_{\text{red}}^{(h)} A^{(h)} \cdot \mathcal{F}(\mathcal{R}) \cdot e^{i\mathbb{R}(\mathcal{R}, v)}. \quad (3.25)$$

(3.25)
3.3 The thermodynamic limit of $\hat{C}_2^{(r)}(m, t)$

One more step is necessary so as to be able to take the thermodynamic limit of $\hat{C}_2^{(r)}(m, t)$, namely decompose the contours of integration for the particle-hole rapidities into their parts that are "close", with precision $\delta$, to $\pm q$ and the parts that are at least at distance $C\delta$, for some $C > 0$. This will allow to make a sharp distinction between the "massive" and "massless" parts of the particle-hole spectrum and hence make use of the more refined large-$C$ asymptotics of the form factor $\hat{C}_2^{(r)}(\gamma)$ given in (2.28). The decomposition of the particle and hole contours can be done by breaking up the contours $\mathbb{C}_{p/h, \infty}$ in the target space as

$$\mathbb{C}_{p/h, \infty} = \mathbb{C}_{p/h, \infty}^{(-)} \cup \mathbb{C}_{p/h, \infty}^{(0)} \cup \mathbb{C}_{p/h, \infty}^{(+)}.$$  

The contours appearing on the rhs are given in Figure 4. $\mathbb{C}_{p/h, \infty}^{(0)}$ are depicted in full lines whereas $\mathbb{C}_{p/h, \infty}^{(\pm)}$ are depicted in dotted lines.

Let $f$ be a symmetric function of $\lambda^{(p)}$ and $\lambda^{(h)}$ taken singly and consider the integral

$$I_{\hat{A}^{(p)}, \hat{A}^{(h)}}^{\text{red}}[f] = \int_{\hat{A}^{(p)}} \mathbb{D}_{\text{red}}^{\hat{A}^{(p)}} \cdot \mathbb{D}_{\text{red}}^{\hat{A}^{(h)}} f(\hat{A}^{(p)}, \hat{A}^{(h)}).$$  

When decomposing each contour arising in the Cartesian product decomposition as in (3.26) and summing up over all possible decompositions, owing to the symmetry of the function $f$, it is enough to sum up over all the possible decompositions of the original set of integration variables into

$$\{\lambda_a^{(p)}\}^{\text{red}}_1 = \{\lambda_a^{(p)}\}^{\text{red}}_1 \cup \{\nu_a^{(p)}\}_1^{\text{red}} \cup \{\lambda_a^{(h)}\}^{\text{red}}_1 \quad \text{and} \quad \{\lambda_a^{(h)}\}^{\text{red}}_1 = \{\lambda_a^{(h)}\}^{\text{red}}_1 \cup \{\mu_a^{(h)}\}_1^{\text{red}} \cup \{\mu_a^{(h)}\}^{\text{red}}_1$$

where $\lambda_a^{(p)}$, resp. $\mu_a^{(h)}$, correspond to integrations, in the target space, over $\mathbb{C}_{p/h, \infty}^{(p)}$, resp. $\mathbb{C}_{h, \infty}^{(h)}$, and $\nu_a^{(p)}$, resp. $\mu_a^{(h)}$, to an integration over $\mathbb{C}_{p/h, \infty}^{(0)}$, resp. $\mathbb{C}_{h, \infty}^{(0)}$. Such a sum is weighted by a multi-nomial coefficient and the integration runs through

$$\hat{C}_{n_p, n_h} = \psi^{-1}\left((\mathbb{C}_{p, \infty}^{(+)} - \mathbb{C}_{h, \infty}^{(-)}) \times (\mathbb{C}_{p, \infty}^{(0)} - \mathbb{C}_{h, \infty}^{(0)}) \times (\mathbb{C}_{h, \infty}^{(-)} - \mathbb{C}_{p, \infty}^{(0)}) \times (\mathbb{C}_{p, \infty}^{(0)} - \mathbb{C}_{h, \infty}^{(-)}) \right).$$

with $\psi$ as in (3.11). One eventually gets

$$I_{\hat{A}^{(p)}, \hat{A}^{(h)}}^{\text{red}}[f] = \sum_{\{\lambda_a^{(p)}\}^{\text{red}}_1 \cup \{\nu_a^{(p)}\}_1^{\text{red}} \cup \{\lambda_a^{(h)}\}^{\text{red}}_1} \int_{\hat{A}_1^{(p)}} \mathbb{D}_{\text{red}}^{\hat{A}_1^{(p)}} \cdot \mathbb{D}_{\text{red}}^{\hat{A}_1^{(h)}} f(\hat{A}_1^{(p)}, \hat{A}_1^{(h)}).$$

Here, the integration measure $\mathbb{D}_p^{\hat{A}}$, resp. $\mathbb{D}_h^{\hat{A}}$, etc., is obtained from $\mathbb{D}_p^{\hat{A}}$ upon the substitution $\lambda^{(p)} \leftrightarrow \lambda^{(h)}$, resp. $\lambda^{(h)} \leftrightarrow \lambda^{(p)}$ etc. Also, above and in the following, one should understand that $\lambda^{(p)} = (\lambda^+, \nu^{(p)} \nu^{(h)})$, $\lambda^{(h)} = (\mu^+, \mu^{(h)} \mu^{(p)})$.

The variables $\{\lambda_a^{(p)}\}^{\text{red}}_1$ and $\{\mu_a^{(h)}\}^{\text{red}}_1$ are at a distance $O(\delta)$ from $\nu q$. This means that, up to $1 + O(\delta/L)$ corrections, one can replace the variables $Y$ in the functions $\hat{C}_1^r(\gamma | Y)$ occurring in each of the $D$ measures by variables $\gamma$ given in (2.14), i.e. by the functions $\hat{C}_1^r(\gamma | \gamma)$. Again, up to such corrections, one can make the same replacement in the function $\Psi$ defining the integration contour $\hat{C}_{n_p, n_h}$. Thus introduce

$$\Psi : (\nu^{(1)}_1, \cdots, \nu^{(p)}_n, \mu_1, \cdots, \mu_n, \ell_0) \mapsto \left(\hat{C}_1(\nu^{(1)}_1 | \gamma), \cdots, \hat{C}_1(\nu^{(p)}_n | \gamma), \hat{C}_1(\mu_1 | \gamma), \cdots, \hat{C}_1(\mu_n | \gamma)\right).$$
and define
\[ \mathcal{G}_{n_p, n_h}^{(\delta)} = \Psi^{-1} \left( -\mathcal{C}_{p, \infty}^{(0)} \right)^{n_p} \times \left( -\mathcal{C}_{h, \infty}^{(0)} \right)^{n_h} \].

Here, the \(-\) sign means that the contours ought to be endowed in the opposite orientation. Also, for fixed \(\nu\), define the one-dimensional contours
\[ \mathcal{G}_\nu^\nu \cdot \mathcal{G}_\nu^\nu \]
and
\[ \mathcal{G}_\nu^\nu \cdot \mathcal{G}_\nu^\nu \]
where the \(\pm\) prefactor in the interval indicates the orientation,
\[ \varepsilon_\nu^\nu = \text{sgn}(\nu_1(\nu q; \nu)) \], \( N_+ = N + 1 + s_\nu \), \( N_- = 0 \).

All of this allows one to recast the original integral as
\[ I_{n_p, n_h}^{n_p, n_h} [f] = \sum_{n_p, n_h} \int \mathcal{D}_\nu^\nu \cdot \mathcal{D}_\mu^\mu \cdot \prod_{\nu \in \mathbb{Z}} \left\{ \int \mathcal{D}_\nu^\nu \cdot \mathcal{D}_\mu^\mu \right\} \cdot f(A^{(p)}, A^{(h)})(1 + O(\frac{1}{L})) \] (3.36)

Above, I have introduced the measure
\[ \mathcal{D}_\nu^\nu \cdot \mathcal{D}_\mu^\mu = \prod_{\nu \in \mathbb{Z}} \left\{ L_\nu^\nu (\mu_\nu^\nu | \nu) \right\} \cdot \frac{d\nu_\nu}{2\pi} \] (3.37)
and likewise with \(h \leftrightarrow p\) and \(\lambda \leftrightarrow \mu\). Note the absence of the \(-\) sign in front of \(L_\nu^\nu\). It has been absorbed into the change of orientation in the contours \(\mathcal{G}_\nu^\nu\) as compared to \(\mathcal{C}_\nu^\nu\).

The measures involving \(\nu_1^{(1)}\) and \(\mu_\nu\) are defined analogously. Assuming the summability of the remainder, one can apply this result to the series representing \(\tilde{C}_2^{(\nu)}(m, t)\), hence recasting it as
\[ \tilde{C}_2^{(\nu)}(m, t) = (\pm)^{m_p} \sum_{n_p, n_h} \sum_{n_p, n_h} \prod_{\nu \in \mathbb{Z}} \left\{ \int \mathcal{D}_\nu^\nu \cdot \mathcal{D}_\mu^\mu \right\} \cdot f(A^{(p)}, A^{(h)})(1 + O(\frac{1}{L})) \] (3.38)

The summation in the series defining \(\tilde{C}_2^{(\nu)}(m, t)\) runs through all the positive integers \(n_p, (n_r)_{r \in \mathbb{Z}}\) and the integers \(\ell_\nu = n_p - n_h\) gathered in \(n = (n_p, (n_r)_{r \in \mathbb{Z}}, \ell_\nu)\) which runs through the set
\[ \Xi = \left\{ (n_p, (n_r)_{r \in \mathbb{Z}}, \ell_\nu) : n_h = \sum_{\nu \in \mathbb{Z}} \ell_\nu + \sum_{r \in \mathbb{Z}} r n_r \ , \ n_p + n_h + n_\nu \in \mathbb{N} \ , \ n_h \in \mathbb{N} \right\} \] (3.39)

Also, one sums up over all positive integers \(n_\nu\) satisfying to \(n_\nu - n_\nu = \ell_\nu\).
Observe that owing to the properties of the contours, the collection of rapidities $R$ appearing in each multidimensional integral of the series (3.38) partitions exactly as in (2.13). Hence, one can replace $\hat{F}(\gamma)(R)$ in (3.38) by its large-$L$ asymptotics taking into account the more refined information on the locii of the rapidities of the particles and holes and whose explicit form is given in (2.28). These asymptotics contain $O(\delta \ln \delta)$ corrections which correspond to dropping the dependence of the form factor on certain "massless" rapidities $\lambda^\nu$ and $\mu^\nu$. The factorisation (2.28) of the asymptotics of $\hat{F}(\gamma)(R)$ allows one to take a partial thermodynamic limit of $\hat{C}(\gamma)$, namely send $L \to +\infty$ in the part of the series (3.38) that is relative to the "massive" modes parametrised by $Y$. In order to write down the result, it is enough to observe that, owing to the form taken by the counting function (2.7), the joint particle-hole $n_p + n_h$ real dimensional manifold of integration $\hat{C}(\gamma)$ factorises into a simple Cartesian product in the thermodynamic limit:

$$\lim_{L\to+\infty} \{\hat{C}(\gamma)\} = \left(\hat{C}_p(\delta)^{n_p}\right) \times \left(\hat{C}_h(\delta)^{n_h}\right)$$

(3.40)

where the contours $\hat{C}_p/h$ are as depicted in Figure 6.

Figure 5: Particle $\hat{C}_p(\delta)$ -in blue- and hole $\hat{C}_h(\delta)$ -in orange- contours in the vicinity of $R$ and in the thermodynamic limit. The contours are plotted for the three regimes of the velocity $v = m/t$ appearing from bottom to top $|v| > v_F$, $v_F > v > 0$ and $0 > v > -v_F$. The contour $\hat{C}_p(\delta)$ and $\hat{C}_h(\delta)$ start at the points $\lambda_{\pm 1} = p_{\pm 1}^{-1}(\pm i q + \epsilon_1 \delta) = \pm q + O(\delta)$, where $\epsilon_1 = +1$ and $\epsilon_1 = -1$, and then, over a distance of the order of $\delta$ they joint with the real axis.

Thus, since

$$\lim_{L\to+\infty} \{\hat{C}(\gamma)(m,t)\} = \langle \sigma_1^\gamma(t)\sigma_{m+1}^\gamma \rangle,$$

(3.41)
one obtains the below representation for the thermodynamic limit of the two-point function

\[ \langle \sigma^\gamma_s(t) \sigma^\gamma_{m+1} \rangle = (-1)^{m \gamma s} \sum_{m \in \mathbb{Z}} \sum_{r \in \mathbb{N}} \left\{ \int \frac{d^p \nu^\gamma_s}{n_r! (2\pi)^{n_r}} \prod_{a=1}^{n_r} \left[ e^{\text{i} \text{imag}(\nu^\gamma_s) \lambda_a v_a} \right] \right. \cdot \left. \int \frac{d^p \mu}{n_h! (2\pi)^{n_h}} \prod_{a=1}^{n_h} \left[ e^{-\text{i} \text{imag}(\mu \nu)} \right] \right. \]

\[ \times \mathcal{F}^\gamma(\nu) \cdot \lim_{L \to \infty} : \left\{ \prod_{\nu = \pm} \mathcal{B}^\nu_{\ell_v} \left[ \mathcal{S}_v(\nu) \mid \nu \right] : \left( 1 + O(\delta \ln \delta) \right) \right\}. \]  

(3.42)

Here, I remind that \( \mathbb{N} = \mathbb{N}_0 \cup \{1\} \) and that the functions \( u_s(\mu, \nu) \) have been defined in (2.23). Also, one should understand that

\[ \mathcal{C}^{(s)}_1 = \mathcal{C}^{(p)}_p \quad \text{and} \quad \mathcal{C}^{(s)}_r = \mathcal{C}_r \text{ for } r \in \mathbb{N}_0. \]  

(3.43)

Finally, the summation runs through all positive integers \( n_h \) and \( \{n_r \}_{r \in \mathbb{N}} \) and relative integers \( \ell_v \), gathered in \( n = \{n_h, \{n_r \}_{r \in \mathbb{N}}, \ell_v\} \) which runs through

\[ \mathcal{Z} = \left\{ \{n_h, \{n_r \}_{r \in \mathbb{N}}, \ell_v \} : n_h = \sum_{v \in \{\pm\}} \ell_v + \sum_{r \in \mathbb{N}} r n_r \right\}. \]  

(3.44)

Finally, (3.42) involves the functionals \( \mathcal{B}^{(s)}_{\ell_v} \). These act on the tower of discrete form factors \( \mathcal{S}_v \), associated with all the possible choices of the "collapsing" rapidities \( \lambda^s, \mu^p \) that are compatible with a given choice of \( \nu \). In taking the thermodynamic limit, it is assumed that one can first evaluate the action of the functionals and only then compute the effect of the \( : \nu : \) ordering. It remains to provide an explicit definition of the functional \( \mathcal{B}^{(s)}_{\ell_v} \). Given \( \ell \in \mathbb{N} \), it acts on a collection of functions \( f_{n_p,n_h}(\{\lambda_1\}_{v=1}^{n_p}; \{\mu_1\}_{v=1}^{n_h}) \) of two sets of variables \( \{\lambda_1\}_{v=1}^{n_p} \) and \( \{\mu_1\}_{v=1}^{n_h} \) with \( n_p, n_h \in \mathbb{N} \) such that the numbers \( n_p \) of \( \lambda \) and \( n_h \) of \( \mu \) variables are constrained by the relation \( n_p - n_h = \ell \). The action of \( \mathcal{B}^{(s)}_{\ell_v} \) takes the explicit form:

\[ \mathcal{B}^{(s)}_{\ell_v} [f_x \mid \nu] = \sum_{n_p,n_h = \ell} \int \frac{d^p \lambda}{n_p!(2\pi)^{n_p}} \prod_{a=1}^{n_p} \left[ L^s_{\ell_v}(\lambda_a \mid \nu) \right] \frac{d^p \mu}{n_h!(2\pi)^{n_h}} \prod_{a=1}^{n_h} \left[ L^s_{\ell_v}(\mu_a \mid \nu) \right] \cdot f_{n_p,n_h}(\{\lambda_1\}_{v=1}^{n_p}; \{\mu_1\}_{v=1}^{n_h}) \cdot \mathcal{F}^s(\nu) \cdot \mathcal{O}(\delta \ln \delta) \]  

(3.45)

Note that \( \mathcal{B}^{(s)}_{\ell_v} \) acts on the \( \mathcal{Y}_p^{(s)} \) variables gathered in \( \mathcal{Y}_p^{(s)} \) as in (2.27).

The action of \( \mathcal{B}^{(s)}_{\ell_v} \) on \( \mathcal{S}_v \), after the \( : \nu : \) ordering and up to some \( \delta \) and \( m, t \) dependent corrections, is computed in Appendix [B], c.f. (B.1). This result entails the below expression for the thermodynamic limit of the form factor expression of the two-point function:

\[ \langle \sigma^\gamma_s(t) \sigma^\gamma_{m+1} \rangle = (-1)^{m \gamma s} \sum_{m \in \mathbb{Z}} \left\{ \int \frac{d^p \nu^\gamma_s}{n_r! (2\pi)^{n_r}} \prod_{a=1}^{n_r} \left[ e^{\text{i} \text{imag}(\nu^\gamma_s) \lambda_a v_a} \right] \right. \cdot \left. \int \frac{d^p \mu}{n_h! (2\pi)^{n_h}} \prod_{a=1}^{n_h} \left[ e^{-\text{i} \text{imag}(\mu \nu)} \right] \right. \]

\[ \times \left( \mathcal{F}^\gamma(\nu) \cdot \mathcal{O}(\delta \ln \delta) \right) \cdot \mathcal{O}(\delta \ln \delta) \cdot \mathcal{O}(\delta \ln \delta) \]

\[ \times \left( 1 + O(\delta \ln \delta) + \sum_{\nu = \pm} \left[ \delta \text{imag}(\nu) + \delta \text{imag}(\nu) \right] \right). \]  

(3.46)

Here,

\[ m_v = \nu m - \nu \ell t \]  

(3.47)
and it is assumed that $v = m/t \neq \pm v_F$. The function $\mathcal{Y}(\eta, v)$ has been introduced in (2.25). $E$ is as in (3.44). Finally, $\delta > 0$ is a control parameter that can be taken as small as necessary, but finite nonetheless. Indeed, owing to the singular behaviour of $F^{(\gamma)}(\eta)$ described by (2.34) when rapidities $\nu^{(1)}$ of the particles or rapidities $\mu$ of the holes approach the endpoints of the Fermi zone $\pm q$, each individual particle-hole integral diverges in the $\delta \rightarrow 0^+$ limit. Furthermore, the a priori control on the remainder blows up in the $\delta \rightarrow 0$ limit. Taken as a whole, the series has of course a well-defined $\delta \rightarrow 0^+$ limit, but taking this limit on the level of (3.46) would demand additional resummations, along the line of ideas developed in [44, 49]. This is beyond the scope of the present analysis and, anyway, doing so would spoil the very structure of the form factor series (3.46) which makes it so useful for studying asymptotic regimes of correlation functions.

### 3.4 The dressed momentum picture

There is a final transformation of (3.46) that is necessary so as to put it in a form appropriate for further applications. Indeed, owing to the jump conditions (2.30)-(2.32) satisfied by the form factor density $F^{(\gamma)}(\eta)$, those (2.41) satisfied by the exponent $\theta_\nu(\eta)$ (2.37) and by the combination of energy and momentum $\mathcal{Y}(\eta, v)$ (2.25), the integrand in each multiple integral is discontinuous in respect to $\nu^{(1)}$ when this variable passes from $-\infty + i\pi/2$ to $-\infty$. The form of the jump conditions does however allow one to re-organise the series in such a form that the continuity between the mentioned points is achieved. Having a continuous, and in fact smooth, integrand along the integration contour is an important ingredient for the asymptotic analysis of the singular structure of dynamic response functions in the vicinity of the excitation thresholds.

One starts by partitioning the contour $C^{(\gamma)}_1 = C^{(\gamma)}_p$ as $C^{(\gamma)}_1 = C^{(\gamma)}_{1:1} \cup C^{(\gamma)}_{1:R}$ where

$$
C^{(\gamma)}_{1:1} = C^{(\gamma)}_1 \cap \left\{ R(\lambda) \leq -q , \ |F(\lambda)| < \epsilon \right\} \quad \text{and} \quad C^{(\gamma)}_{1:R} = C^{(\gamma)}_1 \setminus C^{(\gamma)}_{1:1}.
$$

(3.48)

Here $\epsilon$ is taken small enough. Upon splitting the integration domain $C^{(\gamma)}_1$ as above and then using the symmetry of the integrand, one recasts (3.46) as

$$
\langle \sigma^{(\gamma)}_1 (t) \sigma^{(\gamma)}_{m+1} \rangle = (-1)^{m_2} \sum_{m \in \mathbb{Z}} \prod_{A \in \mathcal{L}, \nu \in \mathbb{R}} \left\{ \int_{(\nu^{(\gamma)}_\nu)^n} \frac{d^{m_1} \nu^{(\gamma)}_\nu}{n_1! \cdot (2\pi)^{m_1}} \right\} \cdot \prod_{A \in \mathcal{L}, \nu \in \mathbb{R}} \left\{ \int_{(\nu^{(\gamma)}_\nu)^n} \frac{d^{m_2} \nu^{(\gamma)}_\nu}{n_1! \cdot (2\pi)^{m_2}} \right\}
$$

$$
\times \frac{F^{(\gamma)}(\eta^{(\gamma)}_{L,R}) \cdot e^{im\eta^{(\gamma)}_{L,R}}}{\prod_{\nu \in \mathbb{Z}} \left[ -i m \right]^{(1/2)}(\eta^{(\gamma)}_{L,R})} \cdot \left( 1 + O(\delta \ln \delta + \sum_{\nu \in \mathbb{Z}} \delta^2 |m_1| + \delta \ln |m_1| + e^{-|m_1|}) \right) .
$$

(3.49)

There, the integration variables $\eta^{(\gamma)}_{L,R}$ and the set $\mathcal{L}_{L,R}$ over which the various integers are being summed up, take the form

$$
\mathcal{L}_{L,R} = \left\{ n \in (n_h, n_{1:1}, n_{1:R}, \{ n_r \}_{r \in \mathbb{R}_d}, \ell_\nu) : n_h = \sum_{\nu \in \{ \pm \}} \ell_\nu + n_{1:1} \ + n_{1:R} + \sum_{r \in \mathbb{R}_d} n_r \right\}
$$

(3.50)

$$
\eta^{(\gamma)}_{L,R} = \left\{ (\mu^{(1)}_a)^{n_h} \cdot (\nu^{(1)}_{a^{(1:L)}})^{n_{1:1}} \cup (\nu^{(1:R)}_{a^{(1:L)}})^{n_{1:R}} \cup \{ (\nu^{(\gamma)}_d)^{n_r} \}_{r \in \mathbb{R}_d} ; \{ \ell_\nu \} \right\} .
$$

(3.51)

Next one shifts the $\ell_\nu$ summation variables as $\ell_\nu \rightarrow \ell_\nu + m_{1:1} \cdot u^{(\gamma)}_\nu$ where I remind that $u^{(\gamma)}_\nu = -\text{sgn}(\pi - 2\zeta)$ as introduced in (2.33). I stress that this change of variables is compatible with the constraint on the summation integers appearing in $\mathcal{L}_{L,R}$ in that it does not alter them. It then remains to observe that

$$
n_{1:1} = \sum_{A \in \mathcal{L}_{L,R}} \sum_{a=1}^{n_{1:1}^{(1:A)}} 1_{\ell_{1:1}, (\nu^{(1:A)}_a)}
$$

(3.52)
where \( \mathbf{1}_A \) is the indicator function of the set \( A \), and then carry out backwards the contour decomposition so as to get

\[
\langle \sigma^\gamma_r(\ell) \sigma^\gamma_{m+1} \rangle = (-1)^{m\gamma} \sum_{\mathbf{m} \in \mathbb{Z}^r} \prod_{r \in \mathbb{R}} \left\{ \int_{(x_h^{(r)})^n} \frac{d^{n_r} \ell_r}{n_r! \cdot (2\pi)^{n_r}} \right\} \cdot \int_{(x_h^{(r)})^n} \frac{d^n \mu}{n_h! \cdot (2\pi)^{n_h}}
\]

\[
\times F^{(\gamma)}(\gamma) \cdot e^{im(\gamma)} \left( \mathbf{1}_{\mathbb{R}^{(\gamma)}} \cdot \left( 1 + O(\delta \ln \delta + \sum_{i = \pm} \left\{ \delta^2 |m_i| + |\ln |m_i|| + e^{-|m_i|} \right) \right) \right) \tag{3.53}
\]

In this series of multiple integrals expansion, the integration variables take the form

\[
\gamma = \left\{ \mu_a^{(u)} \right\} \in \mathbb{R}^r \cdot \left\{ \ell_u + \sum_{a = 1}^{n_1} \mathbf{1}_{\mathbb{R}^{(\gamma)}} \right\} \tag{3.54}
\]

and the summation is as given in (3.44). The jump conditions (2.30) and (2.41) satisfied by the various building blocks of the integrand then ensure that the latter is a smooth function in respect to variables \( \nu_a^{(1)} \) moving along \( \mathcal{C}_1^{(\delta)} \).

The additional shifts in respect to the integers \( \ell_u \), which are present in \( \mathcal{Y}(\gamma, v) \) can be reabsorbed by re-defining the dressed momentum of the particles as:

\[
\tilde{P}_1(\lambda) = p_1(\lambda) + 2\pi \mu u^{(1)} \mathbf{1}_{\mathbb{R}^{(\gamma)}} + 2\pi \mathbf{1}_{\mathbb{R}^{(\gamma)}} \mathbf{1}_{\mathbb{R}^{(\gamma)}} \tag{3.55}
\]

Note that adding a shift by \( 2\pi \) does not alter the expression since the function \( p_1 \) only appears in the combination \( e^{im(\gamma)} \). The matter is that due to the presence of the additional shifts, the function \( \tilde{P}_1 \) is continuous along \( \mathcal{C}_1^{(\delta)} \) and, in fact, a diffeomorphism onto \( \tilde{P}_1(\mathcal{C}_1^{(\delta)}) \).

It then remains to implement the changes of variables

\[
\text{for } r \in \mathbb{R}_{\text{st}} \text{, } k_a^{(r)} = p_r(\nu_a^{(r)}) \text{ and } \left\{ \begin{array}{l} k_a^{(1)} = \tilde{P}_1(\nu_a^{(1)}) \\ t_a = P_1(\mu_a) \end{array} \right. \tag{3.56}
\]

This transforms the oriented integration contours into the oriented curves

\[
\mathcal{A}_h = p_1(\mathcal{C}_h^{(\delta)}) \text{, } \mathcal{A}_1 = \tilde{P}_1(\mathcal{C}_1^{(\delta)}) \text{ and } \mathcal{A}_r = p_r(\mathcal{C}_r) \tag{3.57}
\]

The resulting series takes the form

\[
\langle \sigma^\gamma_r(\ell) \sigma^\gamma_{m+1} \rangle = (-1)^{m\gamma} \sum_{\mathbf{m} \in \mathbb{Z}^r} \prod_{r \in \mathbb{R}} \left\{ \int_{(x_h^{(r)})^n} \frac{d^{n_r} k_r^{(r)}}{n_r! \cdot (2\pi)^{n_r}} \prod_{a = 1}^{n_r} \left[ e^{im u_a(\nu_a^{(r)})} \right] \right\} \cdot \int_{(x_h^{(r)})^n} \frac{d^n t}{n_h! \cdot (2\pi)^{n_h}} \prod_{a = 1}^{n_h} \left[ e^{-im u_t(\nu_t)} \right]
\]

\[
\times F^{(\gamma)}(\gamma) \cdot \left\{ \int_{r > \pm} \left[ e^{im(\gamma)} \right] \left( 1 + O(\delta \ln \delta + \sum_{i = \pm} \left\{ \delta^2 |m_i| + |\ln |m_i|| + e^{-|m_i|} \right) \right) \right\} \tag{3.58}
\]

There, I have introduced

\[
\mathfrak{R} = \left\{ t_a^{(u)} \right\} \cdot \left\{ \left[ k_a^{(r)} \right]_{a = 1}^{n_r} \right\} \cdot \left\{ \ell_u \right\} \tag{3.59}
\]
Furthermore, the functions arising in the integrand take the form
\[
\mathcal{F}^{(\gamma)}(\mathbb{R}) = \mathcal{F}^{(\gamma)}(\mathbb{P}^{-1}(\mathbb{R})) \prod_{r \in \mathbb{N}_a} \prod_{a=1}^{n_r} \left\{ \frac{1}{p_r' \circ p_r^{-1}(k_{a}^{(\nu)})} \right\} \prod_{a=1}^{n_\nu} \left\{ \frac{1}{p_\nu' \circ p_\nu^{-1}(t_{a})} \right\} \prod_{a=1}^{n_\nu} \left\{ \frac{1}{p_\nu' \circ p_\nu^{-1}(k_{a}^{(\nu)})} \right\}
\]
(3.60)
\[
\Delta_a(R) = \mathcal{J}_a^{2}(\mathbb{P}^{-1}(\mathbb{R}))
\]
and
\[
\mathbb{P}^{-1}(\mathbb{R}) = \left\{ \{p^{-1}_{1}(t_{a})\}^{n_1}_{1}; \{(\mathcal{P}_{1}^{-1}(k_{a}^{(\nu)})\}^{n_\nu}_{1} \cup \{(p^{-1}_{\nu}(k_{a}^{(\nu)})\}^{n_\nu}_{a} \right\} \left\{ \ell_{\nu} + \nu \sum_{a=1}^{n_1} \mathcal{J}_{a}(\mathcal{P}_{1}^{-1}(k_{a}^{(\nu)})) \right\} .
\]
(3.61)

Finally, I agree upon
\[
u_r(k, \nu) = k - \frac{\varepsilon_r(k)}{\nu} \quad \text{with} \quad \varepsilon_1 = \varepsilon_1 \circ \mathcal{P}_{1}^{-1} \quad \text{and} \quad \varepsilon_\nu = \varepsilon_\nu \circ p_\nu^{-1}, \quad r \in \mathbb{N}_{\text{st}}.
\]
(3.62)

I stress that the Jacobian in (3.60) appears without the absolute value since the orientation of the contours in (3.57) is preserved. In particular, within such a construction, the intervals \( \mathcal{I}_h, \mathcal{I}_r \) are skimmed through from the smallest to the largest element.

### 3.5 The dynamic response functions

Observe that when a particle \( k_{a}^{(1)} \) or a hole \( t_{a} \) momentum approaches one of the endpoints of its respective domain of integration, then the associated oscillatory phases take complex values hence generating an exponential decay in \( m, t \to +\infty \). On the one hand, the presence of complex valued oscillatory phases is not that convenient for taking the space and time Fourier transform. On the other hand, as will be confirmed below, such integrations only contribute as higher order corrections in \( m_{\nu} \). Thus prior to computing the dynamic response functions, it appears convenient to recast the form factor series in such a way that the integrations outside of real intervals are included into corrections.

For such a purpose, by using the holomorphicity of the integrands, one slightly deforms the contours \( \mathcal{I}_h \) and \( \mathcal{I}_1 \) according to Fig. 8 what ends up with the replacements
\[
\mathcal{I}_h \leftarrow \mathcal{Z}_h^{(L)} \cup \mathcal{I}_h^{(e)} \cup \mathcal{Z}_h^{(R)}, \quad \mathcal{I}_1 \leftarrow \mathcal{Z}_1^{(L)} \cup \mathcal{I}_1^{(e)} \cup \mathcal{Z}_1^{(R)}.
\]
(3.63)

Above, \( \epsilon > 0 \) can be taken as small as necessary,
\[
\mathcal{I}_h^{(e)} = \left\{ -p_{F} + \epsilon : p_{F} - \epsilon \right\}, \quad \mathcal{I}_1^{(e)} = \left\{ 2\pi - p_{F} - \epsilon - 2p_{F}\text{sgn}(\pi - 2z) \right\}
\]
(3.64)
while the curves \( \mathcal{Z}_h^{(L/R)}, \mathcal{Z}_1^{(L/R)}, \) joint the left/right endpoints of \( \mathcal{I}_h^{(e)}, \mathcal{I}_1^{(e)}, \) to the nearby endpoints of the original intervals \( \mathcal{I}_h, \mathcal{I}_1, \) this by solely passing in the complex half-plane where this endpoint belongs to. As a consequence, it follows that
\[
|e^{i\mu_1(t, \nu)}| < 1 \quad \text{for} \quad t \in \text{Int}(\mathcal{Z}_h^{(L/R)}) \quad \text{and, likewise,} \quad |e^{i\mu_1(k, \nu)}| < 1 \quad \text{for} \quad k \in \text{Int}(\mathcal{Z}_1^{(L/R)}).
\]
(3.65)

By following an analogous procedure to the one described in the previous section, one reorganises the form factor
expansion given in (3.58) as

$$
\langle \sigma^\gamma_m(t)\sigma^\gamma_{m+1} \rangle = (-1)^{m_R} \sum_{m=\text{part}} \prod_{r=\text{h}} \left\{ \int \frac{d^n k^{(r)}}{n^1_r \cdot (2\pi)^{n_r}} \prod_{a=1}^{n_r} \left[ e^{i m_R (k^{(r)} \cdot v)} \right] \right\} 
\times \int \frac{d^n k^{(L)}}{n^1_L \cdot (2\pi)^{n_L}} \prod_{a=1}^{n_L} \left[ e^{i m_L (k^{(L)} \cdot v)} \right] \cdot \int \frac{d^n k^{(R)}}{n^1_R \cdot (2\pi)^{n_R}} \prod_{a=1}^{n_R} \left[ e^{i m_R (k^{(R)} \cdot v)} \right] 
\times W_{n^R_m} \left[ \mathcal{F}(\mathcal{R}_{\text{part}}) \cdot \prod_{v=\pm} \left( \frac{e^{i m_R (p_F)} \cdot \mathcal{R}_{\text{part}}}{1 - i m_v} \right) \cdot \left( 1 + O(\delta \ln \delta + \sum_{v=\pm} \delta^2 |m_v| + \delta |\ln |m_v|| + e^{-|m_v|}) \right) \right].
$$

(3.66)

where

$$
\mathcal{Z}_{\text{part}} = \left\{ n \in \{n^L_h, n^R_h, n^L_h, n^L_R, n^L_1, n^R_1\}, (n_v)_{v=\pm}, \mathcal{R}_{\text{part}} : n_h = \sum_{v=\pm} n_v + \sum_{A=L,R} n^A_h - n^A_h \right\}
$$

and

$$
\mathcal{R}_{\text{part}} = \left\{ \left\{ l_{v}^{(L)}(1)_{v=1} \cup \left\{ l_{v}^{R}(1)_{v=1} \cup \left\{ l_{v}^{R}(1)_{v=1} \right\} \right\}, \left\{ k_{v}^{(L)}(1)_{v=1} \cup \left\{ k_{v}^{(R)}(1)_{v=1} \right\} \cup \left\{ k_{v}^{(R)}(1)_{v=1} \right\} \right\} \right\}_{v=\pm, l_{v}^{(R)}} \right\}.
$$

Figure 6: Particle $\mathcal{Z}_{1}^{(L)} \cup \mathcal{J}_{1}^{(e)} \cup \mathcal{Z}_{1}^{(R)}$-in blue- and hole $\mathcal{Z}_{h}^{(L)} \cup \mathcal{J}_{h}^{(e)} \cup \mathcal{Z}_{h}^{(R)}$-in orange- deformed contours in the momentum representation. The contours are plotted for the three regimes of the velocity $v = m/t$ appearing from bottom to top $|v| > v_F, v_F > v > 0$ and $0 > v > -v_F$. The contour $\mathcal{C}_{p}^{(L)}$ and $\mathcal{C}_{p}^{(R)}$ start at the points $p_{\pm 1/2}$ such that $\mathcal{Z}(p_{\pm 1/2}) = i\epsilon_1/\ell\delta$, where $\epsilon_1 = +1$ and $\epsilon_1 = -1$, and then, over a distance of the order of $\epsilon$ they joint with the real axis. Finally, one has $p_{\text{max}} = 2\pi - p_{-F} - 2p_{F} \text{sgn}(\pi - 2\zeta)$. 

\[6.5\]
\( \mathcal{W}_{n_h}^{(e), n_l} \) is an integral operator acting on the variables \( \{ t_n^{(e)} \} \) and \( \{ k_n^{(1, e)} \} \) present in \( \mathcal{R}_\text{part} \) as

\[
\mathcal{W}_{n_h}^{(e), n_l} [f] = \prod_{A \in (L, R)} \left\{ \int \frac{d^n k^{(1, A)}}{2 \pi^n} \prod_{a=1}^{n_h(A)} \left[ e^{i \text{Im} u_n^{(1, A)} (t_n^{(e)}, \tau)} \right] \right\} 
\times \prod_{A \in (L, R)} \left\{ \int \frac{d^n t^{(e)}}{2 \pi^n} \prod_{a=1}^{n_l(A)} \left[ e^{-i \text{Im} u_n^{(2)} (t_n^{(e)}, \tau)} \right] \right\} f \left( t_n^{(L)}, \tau \right) f \left( t_n^{(R)}, \tau \right) .
\]  

(3.67)

The functions occurring in the argument of \( \mathcal{W}_{n_h}^{(e), n_l} \) in (3.66) exhibit power-law singularities when the particle, resp. hole, momenta approach \( p_F \) or \( 2 \pi - p_F - 2 p_F \text{sgn}(\pi - 2 \zeta) \), resp. \( \mp p_F \), as can be inferred from the local behaviour of form factor densities in rapidity space (2.34)–(2.35). These information and a straightforward application of Watson lemma in the vicinity of the points \( \pm (p_F - \epsilon) \), \( p_F + \epsilon \), \( 2 \pi - p_F - 2 p_F \text{sgn}(\pi - 2 \zeta) - \epsilon \) along with the exponential decay of the exponents of oscillatory phases outside of these endpoints on \( \gamma_{h, l} \) ensure that, whenever \( n_h^{(A)} \neq 0 \) or \( n_l^{(A)} \neq 0 \) for some \( A \in \{ L, R \} \), the second line of (3.66) behaves as most as \( O \left( \sum |m_{\nu}|^{-1} \right) \).

This behaviour can then be included into the corrections, hence leading to the below form of the massless form factor expansion for two-point functions

\[
\langle \sigma^\gamma(t) \sigma^\gamma_{m+1}(t) \rangle = (-1)^{m_y} \sum_{r \in \mathbb{R}} \left\{ \int \frac{d^n k^{(r)}}{2 \pi^n} \prod_{a=1}^{n_h(r)} \left[ e^{i \text{Im} u_n^{(1, r)} (t_n^{(r)}, \tau)} \right] \right\} f \left( t_n^{(L)}, \tau \right) f \left( t_n^{(R)}, \tau \right) \times \mathcal{F}^{(r)}(\mathfrak{R}) \cdot \prod_{t=\pm} \left\{ \left[ e^{i \text{Im} u_n^{(1, r)} (t_n^{(r)}, \tau)} \right] \cdot \left( 1 + O(\delta \ln \delta + \sum_{t=\pm} [\delta^2 |m_{\nu}| + \delta \ln |m_{\nu}| + e^{-|m_{\nu}|/\delta} + \frac{1}{|m_{\nu}|}] \right) \right\} .
\]  

(3.68)

Here, \( \mathcal{F}^{(r)}(\mathfrak{R}) \) is almost in good form so as to allow for the computation of the time and space Fourier transform of \( \langle \sigma^\gamma(t) \sigma^\gamma_{m+1}(t) \rangle \). One only needs to put the remainder terms into a more uniform form. Recall that the parameter \( \delta > 0 \) can be set to the desired value. In particular, one can take

\[
\delta = \frac{C}{[m_{\nu} + |m_{\nu}| + 1]^{1 - \tau/2}}
\]  

(3.69)

with \( C > 0 \) and small enough so as to ensure a "minimal" smallness of \( \delta \) and where \( 0 < \tau < 1 \) is arbitrary. Substituting this in the remainder and using that one has \( e^{-|m_{\nu}|/\delta} = O([|m_{\nu}|\delta]^{-\tau}) \) in the regime of interest, one gets that

\[
O(\delta \ln \delta + \sum_{t=\pm} [\delta^2 |m_{\nu}| + \delta \ln |m_{\nu}| + e^{-|m_{\nu}|/\delta} + \frac{1}{|m_{\nu}|}] = O\left( \sum_{t=\pm} \frac{1}{|m_{\nu}|^{1 - \tau}} \right).
\]  

(3.70)

Once that the remainders have been replaced as above, the dynamic response functions given in (1.12) can be computed by using the results gathered in Appendix [C] Proposition [C.1] \( \text{Per se} \), the application of that result demands a slightly better control on the asymptotic expansion of the remainder than the one given just above. However, it is relatively clear that by pushing further the techniques of asymptotic expansions developed in this work one would indeed get a remainder that indeed takes the desired form, as given in (C2). Then, the dynamic response function takes the form

\[
\mathcal{J}^{(r)}(k, \omega) = \sum_{n \in \mathbb{Z}} \mathcal{J}_n^{(r)}(k, \omega)
\]  

(3.71)
where the summation runs through \( \mathbf{n} = (n_h, n_1, n_{r_1}, \ldots, n_{r_{n_w}}, \ell_+, \ell_-) \) with \( \mathcal{R}_{st} = \{ r_1, \ldots, r_{n_w} \} \). The summand and \( \mathcal{A}_n^{(y)}(k, \omega) \) corresponds to the contribution to the dynamic response function of all the excited states that are characterised by fixed Umklapp integers \( \ell_+ \) and \( \ell_- \) having \( n_h \) holes in the bulk of the Fermi zone, \( n_1 \) particles lying uniformly away from the endpoints of the Fermi zone and having \( n_r \) \( r \)-string excitations, with \( r \in \mathcal{R}_{st} \). It takes the form

\[
\mathcal{A}_n^{(y)}(k, \omega) = \int_{(\mathcal{A}_n^{(y)})^{n_h}} d^n k \cdot \prod_{r \in \mathcal{R}} \left\{ \int_{(\mathcal{A}_n^{(y)})^{n_r}} d^n k^{(r)} \right\} \cdot \mathcal{F}^{(y)}(\mathcal{R}) \sum_{s \in \mathcal{S}} \prod_{\nu = \pm} \left\{ \Xi(\bar{c}_n(\mathcal{R}; s)) \cdot \left[ \Xi(\bar{c}_n(\mathcal{R}; s)) \right]^{\Delta_+(\mathcal{R})-1} \right\}
\times \left( 1 + O\left( \sum_{i=\pm} |\bar{c}_n(\mathcal{R}; s)|^{1-r} \right) \right). \tag{3.72}
\]

The integrand of (3.72) is built up from two contributions. The first one is smooth in \( \mathcal{R} \) and corresponds to a dressing of the form factor density

\[
\mathcal{F}^{(y)}(\mathcal{R}) = \frac{(2\pi)^2 \cdot \mathcal{F}^{(y)}(\mathcal{R}) \cdot [2\mathcal{F}]^{-\Delta_+(\mathcal{R})-\Delta_-(\mathcal{R})+1}}{n_h! (2\pi)^{n_h} \cdot \prod_{r \in \mathcal{R}} [n_r! (2\pi)^{n_r} \cdot \Gamma(\Delta_+(\mathcal{R})) \cdot \Gamma(\Delta_-(\mathcal{R}))]} \tag{3.73}
\]

with \( \mathcal{F}^{(y)}(\mathcal{R}), \Delta_+(\mathcal{R}), \Delta_-(\mathcal{R}) \) defined resp. in (3.60) and below it. The second contribution introduces singularities in the integrand as it is the one responsible for the existence of an edge singular behaviour of the spectral functions. This contributions is given in terms of powers of the functions

\[
\Xi(\mathcal{R}; s) = \omega - \mathcal{E}(\mathcal{R}) + \nu \mathcal{F} [k - \mathcal{P}(\mathcal{R}) + 2\pi s] \tag{3.74}
\]

in which

\[
\mathcal{P}(\mathcal{R}) = \sum_{r \in \mathcal{R}} n_r k^{(r)} + p_F \sum_{\nu = \pm} \nu f_\nu + \pi s_\gamma - \sum_{a=1}^{n_h} t_a \tag{3.75}
\]

and

\[
\mathcal{E}(\mathcal{R}) = \sum_{r \in \mathcal{R}} \sum_{a=1}^{n_r} \epsilon_a(k^{(r)}_a) - \sum_{a=1}^{n_h} \epsilon_1(\mu_a). \tag{3.76}
\]

I remind that \( \epsilon_a \) has been defined in (3.62). Also, the integrations runs along the intervals \( \mathcal{A}_n^{(e)} \) and \( \mathcal{A}_1^{(e)} \) given in (3.64) while \( \mathcal{A}_r^{(e)} = \mathcal{A}_r \) for \( r \in \mathcal{R}_{st} \). I stress again that \( \epsilon > 0 \) and \( \tau > 0 \) are control parameters which are arbitrary but should be taken finite so that the controls on the remainders do not blow up.

Finally, one should observe that the summation over \( s \) in (3.72) simply translates the fact that the spectral function is a 2\( \pi \) periodic function of \( k \), owing to the discrete nature of the XXZ chain.

## 4 Phenomenological form of massless form factor expansions for a model belonging to a Luttinger liquid universality class

I will now argue that the form taken by the form factor expansions in the massless regime of the XXZ chain obtained above captures the full structure of a form factor expansion arising in a massless one dimensional model belonging to the Luttinger liquid universality class. As will be shown in forthcoming publications, it is this
structure that is responsible for all the universal features shared by models belonging to this class. Hence having at one’s disposal the general phenomenological form of a form factor expansion in a model belonging to the Luttinger liquid universality class may lead to an even deeper insight into the universal properties of such models.

Given a massless one-dimensional model, one expects that, when the volume gets large enough, the excited states can be interpreted as being built up from various species of particle excitations and/or from hole excitations located inside the Fermi zone of the model. I will assume in the following that the model has one Fermi zone, as in the XXZ chain case, although the case of several Fermi zones can be treated analogously. Given that there is one Fermi zone, there will be one species of particles which can generate excitations right above that zone and hence that has a massless spectrum. In fact, the massless part of the spectrum corresponds to making excitations located directly on the endpoints of the Fermi zone.

The various elementary excitations can be parametrised by collections of their rapidities, similarly to (2.1). Also, it is natural to expect that these rapidities will encapsulate enough data so as to fully parametrise the observables of the model associated with a given excited state. For a model in finite volume, the rapidities will be quantised. The quantisation equations will take the generic multi-variable form as in (2.2). Still, for a generic model, the functions $\xi_a$ are unknown; thus the quantisation equations can take, in principle, any form and do not carry as such any data. However, if one adds the physical requirement that the elementary excitations should propagate as free independent particles in the infinite volume limit, then the functions realising the quantisation equations should take the form (2.7). The $p_a$’s will then be some unexplicit model dependent functions in one variable and the $1/L$ corrections involving $F_a$ will introduce the dependence on the other rapidities.

Owing to the multi-particle species interpretation of the model’s spectrum of excitations, the relative to the ground state excitation energy and momentum will have the form given in (2.17), (2.18), (2.19). The difference with the XXZ chains is that the momentum and energy associated with the various elementary excitation species are now unknown, model dependent, functions. Imagine a situation where a particle and a hole both collapse onto a given endpoint of the Fermi zone. To the leading order in their distance to the Fermi zone, one should not be able to distinguish between this situation and the one where this particle and hole are both absent in the excited state. Thus, regarding to massless excitations, be it particle or hole ones, only the relative differences $\ell_\pm = n_\pm^p - n_\pm^h$ between the number of particles $n_\pm^p$ and holes $n_\pm^h$ collapsing on a given Fermi boundary should matter. This reasoning means that upon a splitting of the excitation’s rapidities into massive and massless modes as in (2.13)-(2.14), the quantisation functions should enjoy the property (2.16).

Until this stage the reasonings should capture the structure of a very large class, if not all, of massless one-dimensional models. What however fixes the universality class is the structure of the form factors of local operators. Indeed it was unravelled in [47] that a model belongs to the Luttinger liquid universality class if and only if the matrix elements of its local operators taken between two massless excited states† take a very specific form. In the special case of ground to excited states, the arguments developed in the work [47] means that the form factors squared involving solely excitations on the left and/or right Fermi boundary are given by

$$
\prod_{\nu \in \{\pm\}} \left[ \delta_{\nu}(\mathbf{Y}) \right] = \left(4.1\right).
$$

These building blocks take exactly the same form as for the XXZ chain (2.38), see [47] for more details. The function $\theta_\nu(\mathbf{Y})$ arising in these expression is now some non-explicit, model dependent, function. Also, the differential operators $g_\nu$, which involve the dependence on the massless rapidities collapsing on the endpoints of the Fermi zone will take a much more involved form than (2.39). However, due to the indistinguishability phenomenon discussed earlier when particle and hole excitations collapse on the Fermi boundaries, one may reasonably expect

† viz. the excited states whose energies differs from the one of the ground state by $1/L$ corrections, what corresponds to rapidities collapsing onto the endpoints of the Fermi zone
that $q_0 = O(\delta/L)$, exactly as for the XXZ spin chain. This closes the discussion of the contribution of the massless modes to the form factors.

It remains to discuss the structure induced by the massive modes. It appears reasonable to assume that such modes lead to a smooth structure of form factors; such a behaviour is confirmed by the structure of form factors in massive quantum integrable field theories or those of massive excitations in the XXZ spin-1/2 chain. This means that if an excited state contains, on top of a swarm of massless excitations, $n_{\text{exc}}$ massive excitations, then the form factor should be weighted by an additional power of the volume $(1/L)^{n_{\text{exc}}}$ and be multiplied by a massive excitation form factor density function $F(\gamma(Y))$. One can reasonably assume $F(\gamma(Y))$ to be a smooth function of the massive excitations' rapidities. The latter will be some non-explicit function whose details depend on the model. In fact, in order for this prefactor to have a true interpretation in terms of a form factor density, this contribution should also be weighted by the density of each massive mode which corresponds, to the leading order in $L$, to the derivative of the quantisation function in respect to its principal variable, exactly as in (2.28). I nonetheless do stress that the explicit presence of this factor is not that important since it can be incorporated in the definition of the form factor density $F(\gamma(Y))$.

Once that all these ingredients relative to the microscopic structure of the Hilbert space are set, it is evident that one can re-do the steps outlined in the core of the paper. Indeed, the analysis was not relying at all on the specific properties of the functions that were dealt with. It is true that some of the steps of the analysis built on the use of complex variables techniques. One way to deal with the issue is to assume that, in a general model, the observables -excitation energies, momenta, form factor densities,...- will be analytic functions of the massive modes’ rapidities, at least in some small vicinity of the curves where the quantised rapidities condense. Otherwise, in the smooth case, one will have to modify certain arguments by using local polynomial approximations allowing one to bypass the lack of analyticity, but globally the final result should still hold. More precisely, with such a structure at hand, equation (3.46) appears to grasp the correct functional form of a form factor expansion in a massless model belonging to the Luttinger liquid universality class. As it was explained in Section 3.4, some more properties of the model -the structure of cuts of the form factors in the complex plane- are necessary so as to pass onto the momentum picture. It is hard to imagine how to postulate these for a generic model, especially that these will quite probably depend on the fine details of a model. Still, it seems not so crazy to expect that the form factor densities would be smooth functions of the various excitation momenta in the dressed momentum representation, viz. when one parametises each excitation in terms of its associated momentum. Hence, a representation in the spirit of (3.58) should also hold, with the sole exception that one ought to substitute the appropriate to the model and operator expression -usually unknown- for the form factor density of the massive modes. This entails the functional form of the dynamic response functions since the passage from that representation to the one given in (3.72) is rather direct.

5 Conclusion

The present work developed techniques which allow one, starting from the large-volume behaviour of form factors of local operators in the massless regime of the XXZ chain, to write down the thermodynamic limit of a form factor expansion of the zero-temperature dynamical two-point functions in this model. In particular, in the thermodynamic limit, it carries a constructive and explicit renormalisation of the form factor series which stems from the dressing of the interactions by the swarm of massless models. The method developed in this work appears quite general and uses a rather weak input on the details of the model other than the typical features of its universality class, the Luttinger liquid. The latter property allows one to advocate a certain universal form taken by the form factor expansions in one-dimensional models, not necessary integrable, belonging to this universality class.

The obtained massless form factor series depends on a control parameter $\delta$ which provides one with a scale allowing one to separate between the massive and massless modes of the model. Although leading to certain
unexplicit terms, this dependence is already enough so as to extract all the physically interesting properties of the two-point functions such as the long-distance and large-time asymptotic behaviour of dynamic two-point functions or the singular structure of the dynamic response functions. The techniques for achieving this goal and the corresponding results will appear in forthcoming publications.

One should note that there would be no principal obstacle to obtain similar representations for the XXX chain or for the massless regime of the XXZ corresponding to \( \cos(\zeta) > 1 \) and \( h_{c;1} > h > h_{c;2} \) where \( h_{c;1} \) is the upper, resp. \( h_{c;2} \) is the lower, critical magnetic field. Also, with a little more work, one could generalise the results of [45] so as to obtain the large-volume behaviour of the XXZ chain’s form factors corresponding to local operators taken between two excited states above the ground state. The adaptation of the techniques developed in the present work along with the formulae that were obtained in [33] would then allow to conform the present technique so as to construct massless form factor series representation for multi-point dynamical correlation functions of the XXZ chain.

Acknowledgment

K.K.K. acknowledges support from a CNRS and ENS de Lyon. The author is indebted to M. Brockmann, J.-S. Caux, F. Göhmann, J.M. Maillet, G. Niccoli for stimulating discussions on various aspects of the project.

Appendix

A Observables in the infinite XXZ chain

A.1 Solutions to the linear integral equations

The observables describing the thermodynamic limit of the spin-1/2 XXZ chain are characterised by means of a collection of functions solving linear integral equations. These equations are driven by the integral kernel

\[
K(\lambda \mid \eta) = \frac{\sin(2\eta)}{2\pi \sinh(\lambda + i\eta) \sinh(\lambda - i\eta)}. \tag{A.1}
\]

To introduce all of the functions of interest to this work, one starts by defining the \( Q \)-dependent dressed energy which allows one to construct the Fermi zone of the model. It is defined as the solution to the linear integral equation

\[
\varepsilon(\lambda \mid Q) + \int_{-Q}^{Q} K(\lambda - \mu \mid \zeta) \varepsilon(\lambda \mid Q) \cdot d\mu = h - 4\pi J \sin(\zeta) K(\lambda \mid \frac{1}{2}\zeta). \tag{A.2}
\]

The endpoint of the Fermi zone is defined as the unique [15] positive solution \( q \) to \( \varepsilon(q \mid q) = 0 \). Then, the function \( \varepsilon_1(\lambda) = \varepsilon(\lambda \mid q) \) corresponds to the dressed energy of the particle-hole excitations of the model. The functions

\[
\varepsilon_r(\lambda) = h - 4\pi J \sin(\zeta) K(\lambda \mid \frac{1}{2}\zeta) - \int_{-q}^{q} K_r(\lambda - \mu) \varepsilon_1(\mu) \cdot d\mu \tag{A.3}
\]

with \( r \geq 2 \) and

\[
K_r(\lambda) = K\left( \lambda \mid \frac{1}{2}\zeta(r+1) \right) + K\left( \lambda \mid \frac{1}{2}\zeta(r-1) \right) \tag{A.4}
\]
correspond to the dressed energies of the $r$-bound state excitations. For any $0 < \zeta < \pi/2$ and under some additional constraints for $\pi/2 < \zeta < \pi$, one can show [42] that for any $r \in \mathbb{N}_{st}$, there exists $c_r > 0$ such that $\varepsilon_r(\lambda + i\delta_r\pi/2) \geq c_r > 0$ for any $\lambda \in \mathbb{R}$. Still, one expects a on the basis of a numerical investigation of the solutions to (A.3) that this lower bound holds throughout the whole massless regime $0 < \zeta < \pi$, irrespectively of the mentioned additional conditions on $\zeta$. Also, I recall that $\delta_r$ is such that the $r$-string condensates on $\mathbb{R} + i\delta_r\pi/2$.

In order to introduce the dressed momenta of the $r$-bound states and of the particle-hole excitations, I first need to define the $r$-bound state bare phases $\theta_r$:

\[ \theta_r(\lambda) = \theta(\lambda | \frac{1}{2}\zeta(r + 1)) + \theta(\lambda | \frac{1}{2}\zeta(r - 1)) \quad \text{for} \quad r \geq 2 \quad \text{and} \quad \theta_1(\lambda) = \theta(\lambda | \zeta) \quad \text{(A.5)} \]

with

\[ \theta(\lambda | \eta) = 2\pi \int_{\Gamma_\lambda} K(\mu - 0^+ | \eta) \cdot d\mu . \quad \text{(A.6)} \]

The contour of integration appearing above corresponds to the union of two segments $\Gamma_\lambda = [0; i\Omega(\lambda)] \cup [i\Omega(\lambda) ; \lambda]$ and the $-0^+$ prescription indicates that the poles of the integrand at $\pm i\eta + i\pi\mathbb{Z}$ should be avoided from the left.

Then, the function

\[ p_r(\lambda) = \theta(\lambda | \frac{1}{2}\zeta) - \int_{-\eta}^{\eta} \theta_r(\lambda - \mu)p'_r(\mu) \cdot \frac{d\mu}{2\pi} \]

\[ + \pi \ell_r(\zeta) + p_F \sum_{\sigma = \pm} (1 - \delta_{r,\sigma} - \delta_{r,1}) \text{sgn}(1 - 2 \cdot \frac{\zeta + r\pi}{2\pi}) \cdot I_{A_{\sigma,r}}(\lambda) , \quad \text{(A.7)} \]

extended by $i\pi$-periodicity to $\mathbb{C}$, corresponds to the dressed momentum of the $r$-bound states. Above, I have introduced

\[ \ell_r(\zeta) = 1 - r + 2\lfloor \frac{r + 1}{2} \rfloor \quad \text{and} \quad m_r(\zeta) = 2 - r - \delta_{r,1} + 2 \sum_{\nu = \pm} \lfloor \frac{\zeta + r\pi}{2\pi} \rfloor . \quad \text{(A.8)} \]

Furthermore, $I_A$ stands for the indicator function of the set $A$, and

\[ \tilde{\eta} = \eta - \pi \lfloor \frac{\eta}{\pi} \rfloor \quad \text{while} \quad A_{\sigma,r} = \left\{ \lambda \in \mathbb{C} : \frac{\eta}{2} \geq |\Omega(\lambda)| \geq \min\left(\frac{\zeta + r\pi}{2\pi}, \pi - \frac{\zeta + r\pi}{2\pi}\right) \right\} . \quad \text{(A.9)} \]

In order to obtain $p_r$, one should first solve the linear integro-differential equation for $p_1$ and then use $p_1$ to define $p_r$ by (A.7). $p_1$ corresponds to the dressed momentum of the particle or hole excitations and $p_F = p_1(q)$ corresponds to the Fermi momentum.

One can show [42], under similar conditions on $\zeta$ as for the dressed energy, that, for any $\lambda \in \mathbb{R}$

\[ |p'_1(\lambda + i\delta_r\frac{\pi}{2})| > 0 \quad \text{when} \quad r \in \mathbb{N}_{st} \quad \text{and} \quad \min\left(p'_1(\lambda), -p'_1(\lambda + i\frac{\pi}{2})\right) > 0 . \quad \text{(A.10)} \]

Again, a numerical investigation indicates that (A.10) does, in fact, hold irrespectively of the value of $\zeta$.

Finally, the $r$-bound dressed phase is defined as the solution to

\[ \phi_r(\lambda, \mu) = \frac{1}{2\pi} \theta_r(\lambda - \mu) - \int_{-\eta}^{\eta} K(\lambda - v) \phi_r(v, \mu) \cdot dv + \frac{m_r(\zeta)}{2} \quad \text{(A.11)} \]

\(^1\text{when } r = 1, \text{ both options } \delta_1 \in (0,1) \text{ are possible.}\)
and the dressed charge solves
\[ Z(\lambda) + \int_{-q}^{q} K(\lambda - \mu) Z(\mu) \cdot d\mu = 1. \] (A.12)
The dressed charge is related to the dressed phase by the identities [40]:
\[ \phi_1(\lambda, q) - \phi_1(\lambda, -q) + 1 = Z(\lambda) \quad \text{and} \quad 1 + \phi_1(q, q) - \phi_1(-q, q) = \frac{1}{Z(q)}. \] (A.13)
It is easy to see that the difference of boundary values of the dressed phase
\[ \Delta \phi_{\nu}^{\rho}(\lambda, \mu) = \phi_{\nu}(\lambda, \mu + i\pi \sigma, \zeta - i0^+) - \phi_{\nu}(\lambda, \mu + i\pi \sigma, \zeta + i0^+), \] (A.14)
with \( f_\zeta \) defined below of (2.29), satisfies to the linear integral equation
\[ \Delta \phi_{\nu}^{\rho}(\lambda, \mu) + \int_{-q}^{q} K(\lambda - \nu) \Delta \phi_{\nu}^{\rho}(\nu, \mu) \cdot d\nu = u_{\nu}^{\rho} 1_{[\mu; +\infty]}(\lambda) \quad \lambda, \mu \in \mathbb{R}. \] (A.15)
Here, \( u_{\nu}^{\rho} \) is as given in (2.33). In particular, for \( \mu < -q \), it holds that
\[ \Delta \phi_{\nu}^{\rho}(\lambda, \mu) = u_{\nu}^{\rho} \cdot Z(\lambda) \quad \text{for} \quad \lambda \in [-q; q]. \] (A.16)

A.2 The bound states
It has been proven in [42] that for \( \pi/2 < \zeta < \pi \) an \( r \) bound state centred on \( \mathbb{R} + i\pi/2, s \in \{0, 1\} \), exists if and only if the below constraints are all simultaneously satisfied:
\[ (-1)^s \sin[k\zeta] \cdot \sin[(r - k)\zeta] > 0 \quad \text{for} \quad k = 1, \ldots, r - 1. \] (A.17)
These are precisely the conditions argued earlier by Suzuki-Takahashi [67] and, subsequently, in [17, 19, 38]. However, for \( 0 < \zeta < \pi/2 \), the work [42] proved that an \( r \) bound state centred on \( \mathbb{R} - i\zeta, \pi/2 \), exists if and only if the below constraints are all simultaneously satisfied:
\[ (-1)^s \cdot \sin\left(\frac{\pi \zeta}{\pi - \zeta}(k - p)\right) \cdot \sin\left(\frac{\pi \zeta}{\pi - \zeta}(r - k + p - \kappa_r - 1)\right) > 0 \] (A.18)
for \( k = 1, \ldots, r - 2 \) and \( k \in [w_p + 1 ; w_{p+1} - 1] \). Above, \( \kappa_r = [(r - 1)\zeta/\pi] \) and
\[ w_p = \left\lfloor (p - \frac{\zeta}{\pi} + (r - 1)\frac{\pi}{(\pi - \zeta)}) \right\rfloor. \] (A.19)

B The operator \( \widehat{B}_{\ell\nu}^{(\delta)} \) and restricted sums
The present appendix is devoted to the computation of the action of the functional \( \widehat{B}_{\ell\nu}^{(\delta)} \) on the discrete form factor \( \widehat{\sigma}_\ell(\nu | \nu) \), this up to some \( \delta, m \) and \( L \) dependent corrections. The main result of this computation is the representation
\[ \lim_{L \to +\infty} \prod_{\nu = \pm} \widehat{B}_{\ell\nu}^{(\delta)} \left[ \widehat{\sigma}_\ell(\nu | \nu) | \nu \right] = \prod_{\nu = \pm} \left\{ \frac{e^{i\nu m_\nu \zeta}}{[-1 - m_\nu]^{\nu, \nu}} \right\} \cdot \left( 1 + O\left( \sum_{\nu = \pm} \delta |m_\nu| + \delta^2|m_\nu| + e^{-\delta|m_\nu|} \right) \right) \] (B.1)
where \( m_o = \nu m - \nu F \).

In the handlings that follow, I will assume that the remainders are such that it is allowed to carry out the various interchanges of symbols. One first focuses on \( \hat{B}^{(v)}_{t_o}[\tilde{g}_p(\mathcal{Y} | \star | \mathcal{Y})] \). Then, the first step consists in changing the variables in (3.43) as

\[
\frac{1}{\sqrt{2\pi}} \tilde{\xi}_1(\mu_o | \mathcal{Y}) = t_o \quad \text{and} \quad \frac{1}{\sqrt{2\pi}} \tilde{\xi}_1(\lambda_o | \mathcal{Y}) = k_o .
\]  

(B.2)

Upon inserting the explicit expression for the discrete form factor \( \tilde{g}_p \) given in (2.38), this yields

\[
\hat{B}^{(v)}_{t_o}[\tilde{g}_p(\mathcal{Y} | \star | \mathcal{Y})] = \frac{G^2(1 - \nu \theta_v(\mathcal{Y}) - \nu \nu_v - \ell_v)}{G^2(1 - \nu \theta_v(\mathcal{Y}) - \nu \nu_v)} \cdot \left( \frac{2\pi}{L} \right) .
\]

\[
\times \sum_{n_p - n_o = \nu(L_o - N_o)} \int \frac{d^p k}{n_p} \int_{t_o} \int_{t_o} \left( \int \frac{d^h}{n_o} \right) \left[ L e^{i m_u(k_v, \mathcal{Y})} \tilde{g}_p(k_v) \right] \cdot \left[ L e^{-i m_u(t_v, \mathcal{Y})} \tilde{g}_p(t_u) \right] \cdot \mathcal{R}(3 | -\nu \theta_v(\mathcal{Y}) - \nu \nu_v - \ell_v) .
\]

(B.3)

Here, the function \( \mathcal{R} \) is as defined in (2.40) and depends on the collection of integration variables

\[
3 = \left\{ \{ \nu(L_o - N_o) \} \nu_o : \{ \nu(N_o - L_o) - 1 \} \nu_o \right\} .
\]  

(B.4)

Also, the integration in (B.3) runs over the intervals

\[
\mathcal{I}_p^{(v)} = \nu \left[ \frac{N_o - \nu / 2}{L} ; \frac{N_o - \nu / 2}{L} + i \frac{\delta \nu}{2} \right] \quad \text{and} \quad \mathcal{I}_h^{(v)} = -\nu \left[ \frac{N_o - 3\nu / 4}{L} ; \frac{N_o - 3\nu / 4}{L} - i \frac{\delta \nu}{2} \right]
\]

(B.5)

where I remind that \( s_o^{\nu} = \text{sgn}(u^1(\nu q, \nu)) \). Finally, I have set

\[
\tilde{u}_1(s, \nu) = u_1 \left( \tilde{\xi}_1^{-1}(2\pi s | \mathcal{Y}), \nu \right) \quad \text{and} \quad \tilde{g}_p(s) = g_p \left( \tilde{\xi}_1^{-1}(2\pi s | \mathcal{Y}) \right).
\]

(B.6)

At this stage, it remains to shift the integration variables by \( \frac{1}{2} N_o \) and expand the functions appearing in the exponents around 0. One has

\[
\tilde{g}_p \left( \frac{N_o}{2} + s \right) = O(\delta \sum_{\nu_o = \pm} \partial_{\nu_o})
\]

(B.7)

Above, \( O(\delta \sum_{\nu_o = \pm} \partial_{\nu_o}) \) refers to the presence of some function of the differential operators \( \partial_{\nu_o} \) such that each derivative is always preceded by a prefactor which is, at most, of the magnitude \( O(\delta) \). Also, one has the expansion

\[
m \tilde{u}_1 \left( \frac{N_o}{2} + s, \nu \right) = m \tilde{u}_1 \left( \frac{N_o}{2}, \nu \right) + m s \cdot \tilde{u}_1 \left( \frac{N_o}{2}, \nu \right) + O(\delta^2 |m| + |t|)
\]

\[
= ump_F + ms \cdot \tilde{u}_1 \left( \frac{N_o}{2}, \nu \right) + O(\delta^2 + \frac{1}{L} \sum_{\nu_o = \pm} |m_o|).
\]

(B.8)

Here, one obtains the second line by noting that

\[
m \tilde{u}_1 \left( \frac{N_o}{2}, \nu \right) = ump_F + O \left( \frac{|m| + |t|}{L} \right)
\]

(B.9)

and by using that \( 2m = m_+ - m_- \) and \( -2\nu F t = m_+ + m_- \).
All these simplifications yield

\[
\tilde{\mathcal{B}}^{(v)}_{\ell_v} \left[ \tilde{\mathcal{G}}_v(y \mid +) \right. = e^{iu m_{\ell_v} p F} \left. \frac{G^2(1 - u \phi(y)) - u \phi_v - \ell_v}{G^2(1 - u \phi_v(y)) - u \phi_v v} \right] \cdot (2\pi) \left( \frac{\theta_v(y) + v}{L} \right)^2 \\
\times \sum_{n_p \neq n_v = \ell_v} \frac{L u_{p+n}^v}{n_p! n_v!} \int \frac{d^{n_p} k}{(2\pi)^{n_p}} \int \frac{d^{n_v} t}{(2\pi)^{n_v}} \exp \left\{ i m \bar{u} \left( \frac{n_v}{2}, v \right) \left[ \sum_{a=1}^{n_p} k_a - \sum_{a=1}^{n_v} t_a \right] \right\} \cdot (v)^{n_p} \cdot (-v)^{n_v} \\
\times \mathcal{R} \left( 3 \mid -u \phi_v(y) - u \phi_v - \ell_v \right) \cdot \left( 1 + O(\delta^2 \sum_{u=\pm} |m_u| + \frac{1}{2} \sum_{u=\pm} |m_u| + \delta \sum_{v' \neq \pm} \partial_{v'} \right) \right). \tag{B.10}
\]

In (B.10), the argument \( 3 \) of \( \mathcal{R} \) reads

\[
3 = \left\{ \nu L k_a \right\}^{n_p}_1 \left\{ -u L t_a - 1 \right\}^{n_v}_1.
\]

Even though the remainders involving differential operators appear in the rhs of (B.10), one should bear in mind that they will act on the lhs of the series once that the \( \ast : \ast \) order is imposed.

Recall that, in fact, one is only interested in computing the thermodynamic limit of the action of the functionals \( \tilde{\mathcal{B}}^{(v)}_{\ell_v} \). Thus, one can add additional terms to the integrals that will presumably not change the value of the thermodynamic limit, much in the spirit of the replacements that have been discussed in Section 3.2. Indeed, the below replacement in \( k \) or \( t \)-based one-dimensional integrals only produces \( O(L^{-1}) \) corrections

\[
\frac{\mu}{4L} \int dk \leftrightarrow \int \frac{d^{n_p} k}{(2\pi)^{n_p}} \frac{L s^v}{e^{2\i u L k_a} - 1} \left( 1 + O(L^{-1}) \right) + L s^v \int \frac{d^{n_v} t}{(2\pi)^{n_v}} \tag{B.12}
\]

and

\[
-\frac{\nu L}{4L} \int dt \leftrightarrow \int \frac{d^{n_v} t}{(2\pi)^{n_v}} \frac{L s^v}{1 - e^{-2\i u L t_a}} \left( 1 + O(L^{-1}) \right) - L s^v \int \frac{d^{n_p} k}{(2\pi)^{n_p}}. \tag{B.13}
\]

The contours \( \Gamma^{(v)}_{\mu/n} \) arising in (B.12)–(B.13) are depicted in Figure 7 and

\[
\eta_{\mu}(v) = \begin{cases} \uparrow \text{ if } s^v_{\mu} = + & \text{ and } \eta_{\nu}(v) = \begin{cases} \downarrow \text{ if } s^v_{\nu} = + \\uparrow \text{ if } s^v_{\nu} = - \end{cases} \end{cases} \tag{B.14}
\]

It appears convenient to introduce the integration measures

\[
\mathcal{D}^{n_p}_{u} k = \prod_{a=1}^{n_p} \left\{ \frac{L s^v}{e^{2\i u L k_a} - 1} \right\} \frac{d^{n_p} k}{n_p!} \quad \text{and} \quad \mathcal{D}^{n_v}_{u} t = \prod_{a=1}^{n_v} \left\{ \frac{L s^v}{1 - e^{-2\i u L t_a}} \right\} \frac{d^{n_v} t}{n_v!}. \tag{B.15}
\]

The contour substitutions (B.12) and (B.13) introduce a new partitioning of the integration variables in (B.10).
Figure 7: Contours $\Gamma^{(+)}_{p/h}$ -depicted in the bottom graph- and $\Gamma^{(-)}_{p/h}$ -depicted in the top graph-. One has $\Gamma^{(\nu,\gamma)}_{p/h} = \Gamma^{(\nu)}_{p/h} \cap \{\Re + i\delta/2\pi\}$ and $\Gamma^{(\nu,\delta)}_{p/h} = \Gamma^{(\nu)}_{p/h} \cap \{\Re - i\delta/2\pi\}$.

what recasts the functional as

$$\tilde{B}^{(\nu)}_{\ell_p} \left[ \delta_{\nu}(\mathcal{Y} | +) | \mathcal{Y} \right] = e^{i\text{Im} \ell_p p} \frac{G^2(1 - \nu \theta_{\nu}(\mathcal{Y})) - \nu v_{\nu} - \ell_{\nu}}{G^2(1 - \nu \theta_{\nu}(\mathcal{Y}) - \nu v_{\nu})} \left( \frac{2\pi}{L} \right)^{\theta_{\nu}(\mathcal{Y}) + v_{\nu}^2}$$

$$\times \sum_{\gamma_{p,h}} \sum_{\gamma_{p,h}} \sum_{\gamma_{p,h}} \sum_{\gamma_{p,h}} \frac{d\gamma_{p,h}}{\gamma_{p,h}} \frac{d\gamma_{p,h}}{\gamma_{p,h}} \int d^{2\nu} x \frac{(\dot{\gamma}_{p,h})^{\nu}}{\nu!} \int d^{2\nu} y \frac{(-\dot{\gamma}_{p,h})^{\nu}}{\nu!} \times \exp \left\{ i m \bar{\gamma}_{i} \left[ \sum_{a=1}^{n_{\nu}} k_{a} \gamma_{p,h} \gamma_{p,h} - \sum_{a=1}^{n_{\nu}} k_{a} \gamma_{p,h} \gamma_{p,h} \right] \right\}$$

$$\times \mathcal{R}(\gamma_{x,y} \cup \gamma_{k,t} | - \nu \theta_{\nu}(\mathcal{Y}) - \nu v_{\nu} - \ell_{\nu}) \cdot \left( 1 + O(\delta^2 \sum_{\nu=\pm} |m_{\nu}| + \frac{1}{2} \sum_{\nu=\pm} |m_{\nu}| + \delta \sum_{\nu=\pm} \partial_{v_{\nu}} \right). \quad (B.16)$$

Here, I have introduced two sets of variables

$$\gamma_{x,y} = \left\{ \gamma_{x,y_{1}}^{\nu} : \{ - \nu L y_{a} - 1 \}^{\nu} \right\} \quad \text{and} \quad \gamma_{k,t} = \left\{ \nu_{L} k_{a}^{\nu} : \{ - \nu L t_{a} - 1 \}^{\nu} \right\} \quad (B.17)$$

and the union of sets $\gamma_{x,y} \cup \gamma_{k,t}$ means joining together the $k$ and $x$ variables as well as the $t$ and $y$ variables. The contour integrals over $\Gamma^{(\nu)}_{\nu} \text{ and } \Gamma^{(\nu)}_{\nu}$ can be evaluated by taking the residues at

$$k_{a} = \frac{\nu}{L} p_{a}, \quad t_{a} = -\frac{\nu}{L} (h_{a} + 1) \quad \text{with} \quad p_{a}, h_{a} \in \mathbb{N} \quad (B.18)$$

and by using the symmetry of the integrand. Also, by using the analyticity of the integrand, one can deform the integration domains as

$$\Gamma^{(\nu,\gamma)}_{p/h} \leftrightarrow -\nu \bar{\gamma}_{a}^{\nu} \left[ i \bar{\gamma}_{a}^{\nu} \frac{\delta}{\delta \gamma_{a}^{\nu}} ; i \bar{\gamma}_{a}^{\nu} \right] \quad \text{and} \quad \Gamma^{(\nu,\delta)}_{h} \leftrightarrow -\nu \bar{\gamma}_{a}^{\nu} \left[ i \bar{\gamma}_{a}^{\nu} \frac{\delta}{\delta \gamma_{a}^{\nu}} ; -i \bar{\gamma}_{a}^{\nu} \right] \quad (B.19)$$
where the $-\nu s_\nu^u$ pre-factor corresponds to the orientation of the interval. Thus, eventually, one gets

$$\mathcal{B}^{(v)}_{\ell_v} \left[ \mathcal{S}_\nu(y \mid + | y) \right] = e^{i \text{Im} \ell_v \nu} \frac{G^2(1 - \nu \theta(y))}{G^2(1 - \nu \theta(y) - \nu \ell_v)} \cdot \left( \frac{2\pi}{L} \right)^{\nu(n + m)} \cdot \left( \nu \ell_v \right)^{\nu(n + m)} \times \sum_{n_\nu, m_\nu} \sum_{n_{v+h}} \int \frac{d^p x}{n_1!} \frac{(-i\nu s_\nu^u L)^{n_\nu}}{(\nu L)^{n_\nu}} \exp \left( -m \left[ \bar{a}_1'(\nu \ell_v, v) \right] \cdot \left[ \sum_{a=1}^{n_\nu} x_a + \sum_{a=1}^{n_v} y_a \right] \right)$$

$$\sum_{Z_{p,h} : n_p - n_h = l_v} e^{i \mathcal{H}_\nu \bar{a}_1'(\nu \ell_v, v)} \mathcal{J}(Z_{p,h}) \cdot \mathcal{R}(\bar{3}_{xy} \cup Z_{p,h} \mid -\nu \theta_2(y) - \nu \nu - \nu \ell_v) \times \left( 1 + O\left( \delta^2 \sum_{i=1}^{\nu} |m_i| + \frac{1}{\nu} \sum_{i=1}^{\nu} |m_i| \cdot \delta \sum_{\nu=1}^{\nu} \partial_{\nu} \right) \right). \quad (B.20)$$

Above, it is undercurrent that

$$Z_{p,h} = \{ (p_\nu)_{n_\nu}, (h_a)_{n_a} \} \text{ and } \bar{3}_{xy} = \{ (\nu L\nu x_a)_{n_\nu}, (\nu \nu y_a - 1)_{n_a} \}$$ \quad (B.21)

as well as

$$\mathcal{J}(Z_{p,h}) = \sum_{a=1}^{n_p} p_a + \sum_{a=1}^{n_h} (h_a + 1). \quad (B.22)$$

Finally, the summation in (B.20) runs through all the possible choices of the set $Z_{p,h}$, viz. of $p_a, h_a \in \mathbb{N}$ such that $n_p - n_h = l_v$ and $p_1 < \cdots < p_{n_p}$, resp. $h_1 < \cdots < h_{n_h}$. Straightforward estimates and the lower bound $x_a, y_a \geq \delta/2\pi$ ensures that one has the large-$L$ behaviour

$$\mathcal{R}(\bar{3}_{xy} \cup Z_{p,h} \mid v) = \left( \frac{L}{2\pi} \right)^{n_\nu + 2n_v + l_v} \mathcal{D}_0(Z_{x,y} \mid v + l_v) \cdot \left( -i\nu s_\nu^u L \right)^{n_\nu + n_v} \mathcal{R}(Z_{p,h} \mid v) \cdot \prod_{a=1}^{n_p} \varphi_{x,y}(v, p_a) \prod_{a=1}^{n_h} \varphi_{x,y}(v, -h_a - \frac{1}{L}) \left( 1 + O \left( \frac{n_\nu + n_v}{\delta L} \right) \right). \quad (B.23)$$

Here $Z_{x,y} = \{ (x_a)_{n_a} ; (y_a)_{n_a} \}$.

$$\mathcal{D}_0(Z_{x,y} \mid v) = (-1)^{n_v} \cdot \left( 2i\nu s_\nu^u L \right)^{n_\nu + 2n_v} \cdot \left( \frac{\sin(\pi v)}{\pi} \right)^{2n_v} \cdot \prod_{a=1}^{n_v} \frac{x_a^{2v}}{n_a} \cdot \prod_{a,b}^{n_v} (x_a - x_b)^2 \cdot \prod_{a,b}^{n_v} (y_a - y_b)^2 \quad (B.24)$$

and

$$\varphi_{x,y}(v) = \prod_{a=1}^{n_v} \left( 1 - \frac{w}{i\nu s_\nu^u x_a} \right)^2 \quad \prod_{a=1}^{n_v} \left( 1 + \frac{w}{i\nu s_\nu^u y_a} \right)^2. \quad (B.25)$$
One is now in position to take a partial \( L \to +\infty \) limit of the series. By using that \( \overline{a}'(N/L, \nu) = 2\pi vm_{\nu} \), with \( m_{\nu} = nm - nf \) one obtains

\[
\mathcal{B}^{(\nu)}_{\ell_{\nu}} \left[ \mathcal{G}_{\nu}(\mathcal{Y} \mid +) \mid \mathcal{Y} \right] = e^{im\ell_{\nu}PF} \sum_{n_{\nu}+n_{y_{\nu}} = n_{\nu}} \sum_{n_{x_{\nu}}+n_{l_{\nu}} = n_{l_{\nu}}} \int \frac{d^{n_{x_{\nu}}}x_{a}}{n_{x_{\nu}}!} \int \frac{d^{n_{l_{\nu}}}l_{a}}{n_{l_{\nu}}!} \exp \left\{ -2\pi [m_{\nu}] \cdot \left[ \sum_{a=1}^{n_{x_{\nu}}} x_{a} + \sum_{a=1}^{n_{l_{\nu}}} l_{a} \right] \right\}
\]

\[
\times D_{\nu}(\mathcal{Z}_{x,y} \mid -nu_{\nu} + nu_{\nu}) \cdot \mathcal{G}_{\nu}(\mathcal{Z}_{x,y}) \left( 1 + O\left( \delta^{2} \sum_{i=1}^{n_{y_{\nu}}} |m_{i}| + \delta \sum_{v'_{\nu} = \pm} \partial_{v'_{\nu}} \right) \right). \tag{B.26}
\]

Above, I have introduced

\[
\mathcal{G}_{\nu}(\mathcal{Z}_{x,y}) = \frac{G^{2}(1 - nu_{\nu}(\mathcal{Y}) - nu_{\nu} - \ell_{\nu})}{G^{2}(1 - nu_{\nu}(\mathcal{Y}) - nu_{\nu})} \cdot \lim_{L \to +\infty} \left( \frac{2\pi}{L} \right)^{(nu_{\nu} + nu_{\nu} + n_{\nu})^{2}}
\]

\[
\times \sum_{\mathcal{Z}_{p,h}} e^{\nu_{\nu}(\frac{N}{L})} \mathcal{J}(\mathcal{Z}_{p,h}) \cdot R(\mathcal{Z}_{p,h} \mid nu_{\nu}(\mathcal{Y}) - nu_{\nu} - l_{\nu}) \cdot \frac{n_{\nu}}{\prod_{a=1}^{n_{\nu}} \varphi_{\nu}(\frac{p}{m_{\nu}})} \cdot \frac{1}{\prod_{a=1}^{n_{\nu}} \varphi_{\nu}(\frac{p}{m_{\nu}})} \right). \tag{B.27}
\]

One can convince oneself that the term in (B.26) corresponding to \( n_{x} \) \( x \)-integrations and \( n_{y} \) \( y \)-integrations can be bounded by \( e^{-m_{\nu}d(n_{x} + n_{y})} \) provided that \( \delta|m_{\nu}| > c > 0 \) for some \( c > 0 \).

Therefore, the only term in (B.26) not giving rise to an exponential behaviour in \( \delta|m_{\nu}| \) corresponds to taking \( n_{x} = 0 \) and \( n_{x} = n_{y} = 0 \). For that particular choice, one has that \( \varphi_{\nu}(\nu_{\nu}) = 1 \) and then the series in (B.27) corresponds to a so-called restricted sum. When \( \ell = 0 \), these appeared for the first time in the context of harmonic analysis on \( GL_{\infty} \) \cite{26, 57} and, for any \( \ell \in \mathbb{Z} \), in the study of the large-\( \nu \) and low-temperature expansion of correlation function in the non-linear Schrödinger model \cite{46}. It has been shown in \cite{26} for \( \ell = 0 \) and in \cite{30} for general \( \ell \), see also \cite{33, 47} for other proofs, that the following summation identity holds

\[
\frac{G^{2}(1 + n - \ell)}{G^{2}(1 + n)} \left( \frac{2\pi}{L} \right)^{n} \sum_{\mathcal{Z}_{p,h} : n_{p} - n_{h} = \ell} e^{\frac{n}{2} \mathcal{J}(\mathcal{Z}_{p,h})} \cdot R(\mathcal{Z}_{p,h} \mid n - \ell) = \left( \frac{2\pi/L}{1 - e^{2\pi/L}} \right)^{n}. \tag{B.28}
\]

The summation identity allows one to compute readily the \( L \to +\infty \) limit of \( \mathcal{G}_{\nu}(([\emptyset]; \{\emptyset\})) \), hence leading to

\[
\mathcal{B}^{(\nu)}_{\ell_{\nu}} \left[ \mathcal{G}_{\nu}(\mathcal{Y} \mid +) \mid \mathcal{Y} \right] = e^{im\ell_{\nu}PF} \left[ -im\ell_{\nu} \right] (d_{\nu}(\mathcal{Y}) + v_{\nu})^{2} \left( 1 + O\left( \delta^{2} \sum_{i=1}^{n_{y_{\nu}}} |m_{i}| + \delta \sum_{v'_{\nu} = \pm} \partial_{v'_{\nu}} \right) \right). \tag{B.29}
\]

It then remains to take the product and compute the effect of the operator ordering \( \ast \). Due to the form of the leading term, its action only produces logarithmic corrections, viz. the operator ordering amounts to the substitution

\[
O(\delta) \sum_{v'_{\nu} = \pm} \rightarrow O(\delta) \sum_{v'_{\nu} = \pm} \ln |m_{i}|, \tag{B.30}
\]

what entails the claim.
C A discrete/continuous Fourier transform

The purpose of this appendix is to compute the discrete time and space Fourier transform that arises in the context of studying spectral functions:

\[
\mathcal{T}[W](k, \omega) = \sum_{m \in \mathbb{Z}} \int_{\mathbb{R}} e^{i(\omega t - km)} W(m, -) \frac{\partial}{\partial \nu} \right|_{\nu = \nu_F(t - i0^+)} \right|_{m \nu = \nu_F(t - i0^+)} dt \quad \text{with} \quad m \nu = \nu_F(t - i0^+) \, .
\] (C.1)

In this representation, \( W \) is a function of two variables that is analytic in \( \mathbb{H}^+ \times \mathbb{H}^+ \) and that admits the large variable asymptotic expansion, uniform on \( \mathbb{H}^+ \times \mathbb{H}^+ \) of the type

\[
W(\varphi_+, \varphi_-) = 1 + \sum_{i=\pm} V_i(\varphi_\nu) + \sum_{a=1}^{n_\nu} a_{a} \cdot \prod_{v=\pm} \{ A_{a_v}(\ln [-i\varphi_\nu]) - i\varphi_\nu \} + O\left( \prod [\varphi_\nu]^{-\alpha_v} \right) \] (C.2)

where

\[
V_\nu(\varphi_\nu) = \sum_{a=1}^{n_v} v^{(a)}_\nu \cdot \left. Q_a^{(a)}(\ln [-i\varphi_\nu]) \right|_{[-i\varphi_\nu]} + O\left( [\varphi_\nu]^{-\gamma_v} \right) . \] (C.3)

Here \( A^{(a)}_a \), \( Q^{(a)}_a \) are some polynomials. The constants controlling the power-law decay are such that

\[0 < \alpha_1 < \cdots < \alpha_{n_v} \quad \text{and} \quad 0 < \gamma_1 < \cdots < \gamma_{n_v} \, . \] (C.4)

Furthermore, the largest coefficients satisfy

\[\gamma_{n_v} + \delta_v > 1 \quad \text{and} \quad \alpha_{n_v} + \delta_v > 1 . \] (C.5)

**Proposition C.1.** Under the above assumptions, and provided that for some \( 0 < \tau < 1 \),

\[
W(\varphi_+, \varphi_-) = 1 + O\left( \sum_{v=\pm} \frac{1}{[\varphi_\nu]^{1-\tau}} \right) \] (C.6)

it holds, in the sense of distributions, and for \( \delta_v \geq 0 \),

\[
\mathcal{T}[W](k, \omega) = \frac{(2\pi)^2}{2\nu_F \Gamma(\delta_v) \Gamma(\delta_-)} \sum_{m \in \mathbb{Z}} \prod_{v=\pm} \left\{ \Xi[\omega + \nu \nu_F(k + 2\pi n)] \cdot \left( \omega + \nu \nu_F(k + 2\pi n) \right)^{\delta_v - 1} \right\} \\
\times \left\{ 1 + \sum_{v=\pm} O\left( \left[ \omega + \nu \nu_F(k + 2\pi n) \right]^{1-\tau} \right) \right\} \] (C.7)

**Proof —**

To start with one observes that \( \mathcal{T}(k, \omega) \) can be recast as

\[
\mathcal{T}[W](k, \omega) = \sum_{m \in \mathbb{Z}} \int_{\mathbb{R}} e^{i(\omega t - km)} \delta(x - m) \prod_{v=\pm} \left( i\nu F(t - i0^+) - \nu x \right) W(x - \nu F(t - i0^+), -x - \nu F(t - i0^+)) \cdot dt \, dx \]

\[= \sum_{m \in \mathbb{Z}} \int_{\mathbb{R}} e^{i(\omega t - m + 2\pi \Omega x)} \prod_{v=\pm} \left( i\nu F(t - i0^+) - \nu x \right) W(x - \nu F(t - i0^+), -x - \nu F(t - i0^+)) \cdot dt \, dx \]

\[= \sum_{m \in \mathbb{Z}} \prod_{v=\pm} \left\{ \frac{e^{-i\varphi_\nu, \Omega m}}{-i[\varphi_\nu + i0^+]} \right\} W(\varphi_+ + i0^+, \varphi_- + i0^+) \cdot \frac{d\varphi_+ d\varphi_-}{2\nu_F} . \] (C.8)
Above, I have set
\[ \Omega_{\nu, n} = \frac{\omega + \nu \nu_F (k + 2\pi n)}{2\nu_F}. \]  
\[ \text{(C.9)} \]

In the intermediate steps one uses the Fourier series expansion of the Dirac Comb: \( \sum_{m \in \mathbb{Z}} \delta(x - m) = \sum_{m \in \mathbb{Z}} e^{2i\pi mx} \) and, subsequently, changes the variables to \( \varphi_\nu = -(\nu_F t - \nu x) \). Since \( W \) is analytic and bounded on \( \mathbb{H}^+ \times \mathbb{H}^+ \), it follows by deforming the integrals to +i\( \infty \), that, whenever \( \Omega_{\nu, n} < 0 \), the integral vanishes. Hence, one has that
\[ T[W](k, \omega) = \frac{1}{2\nu_F} \sum_{\nu \in \mathbb{Z}} \prod_{\nu = \pm} \left\{ \Xi(\Omega_{\nu, n}) \cdot [\Omega_{\nu, n}]^{\delta_{\nu, 1}} \right\} \int \prod_{\nu = \pm} \left\{ \frac{e^{-i\varphi_\nu}}{(-i[\varphi_\nu + i0^+]^{\delta_{\nu}})} \right\} W \left( \frac{\varphi_\nu}{\Omega_{\nu, n}} + i0^+, \frac{\varphi_\nu}{\Omega_{\nu, n}} + i0^+ \right) \frac{d\varphi_+ d\varphi_-}{2\nu_F}. \]  
\[ \text{(C.10)} \]

At this stage it remains to deform the integration contours to the lines \( \{ \mathbb{R} + i\epsilon_+ \} \times \{ \mathbb{R} + i\epsilon_- \} \) for some \( \epsilon_\pm > 0 \) small enough and then inject the asymptotic expansion \( \text{(C.2)} \) into \( \text{(C.10)} \). This yields
\[ T[W](k, \omega) = \frac{1}{2\nu_F} \sum_{\nu \in \mathbb{Z}} \prod_{\nu = \pm} \left\{ \Xi(\Omega_{\nu, n}) \cdot [\Omega_{\nu, n}]^{\delta_{\nu, 1}} \right\} \times \left\{ J(\delta_+) J(\delta_-) + \sum_{\nu = \pm} \sum_{a=1}^n \nu_a^{(\nu)} \cdot J(\delta_{\nu a}) \cdot Q_{\nu a}^{(\nu)}(\partial_{\nu a}) \cdot \left\{ [\Omega_{\nu, n}]^{\nu_{\nu a}} \cdot J(\delta_{\nu a} + \gamma_{\nu a}) \right\} + \sum_{\nu = \pm} \prod_{\nu = \pm} \left\{ J(\delta_{\nu a}) \cdot \left\{ [\Omega_{\nu, n}]^{\nu_{\nu a}} \cdot J(\delta_{\nu a} + \alpha_{\delta_{\nu a}}) \right\} \right\} \right\} \]  
\[ \text{(C.11)} \]

\( J(\delta) \) are given as one dimensional integrals
\[ J(\delta) = \int_{\mathbb{R}} \frac{e^{-i\varphi}}{(-i[\varphi + i0^+]^{\delta})} \cdot d\varphi = \frac{2\pi}{\Gamma(\delta)} \]  
\[ \text{(C.12)} \]

where the explicit result comes from deforming the contour to a loop around \(-i\mathbb{R}\) and recognising the definition of the Gamma function. Finally, the remainder \( \tau(\Omega_{+, n}, \Omega_{-, n}) \) is readily seen to be bounded as
\[ |\tau(\Omega_{+, n}, \Omega_{-, n})| \leq C \prod_{\nu = \pm} \left\{ \Xi(\Omega_{\nu, n}) \cdot [\Omega_{\nu, n}]^{\delta_{\nu, 1}} \right\} \times \left\{ \sum_{\nu = \pm} \int_{\mathbb{R}} \frac{[\Omega_{\nu, n}]^{\mu_{\nu, a}}}{|\varphi + i\epsilon|^{n_{\nu, a} + \gamma_{\nu, a}}} \cdot d\varphi + \prod_{\nu = \pm} \int_{\mathbb{R}} \frac{[\Omega_{\nu, n}]^{\mu_{\nu, a}}}{|\varphi + i\epsilon|^{n_{\nu, a} + \alpha_{\delta_{\nu a}}}} \cdot d\varphi \right\}. \]  
\[ \text{(C.13)} \]

The integrals converge due to the conditions \( \text{(C.5)} \) and the bounds \( \text{(C.6)} \) then allow one to provide an estimate on all the power laws appearing in the expansions \( \text{(C.11)} \).

References

[1] I.A. Aizenberg and A.P. Yuzhakov, *Integral representations and residues in multidimensional complex analysis*, Graduate Texts in Mathematics, vol. 58, American Mathematical Society, 1978.

[2] M. Arikawa, M. Kabrach, G. Müller, and K. Wiele, "Spinon excitations in the XX chain: spectra, transition rates, observability." J. Phys. A: Math. Gen 39 (2006), 10623–10640.
[3] H. M. Babujian, A. Foerster, and M. Karowski, "The Form Factor Program: a Review and New Results - the Nested SU(N) Off-Shell Bethe Ansatz.", SIGMA, Proc. of the O’Raifeartaigh Symposium on Non-Perturbative and Symmetry Methods in Field Theory (June 2006, Budapest, Hungary) 2 (2006), 082.

[4] E. Barouch and B.M. McCoy, "Statistical mechanics of XY-model.2. Spin-correlation functions.", Phys. Rev. A 3 (1971), 786–804.

[5] H. Beck, J.C. Bonner, G. Müller, and H. Thomas, "Quantum spin dynamics of the antiferromagnetic linear chain in zero and nonzero magnetic field.", Phys. Rev. B 24 (1981), 1429–1467.

[6] H. Bethe, "Zür Theorie der Metalle: Eigenwerte und Eigenfunktionen der linearen Atomkette.", Zeitschrift für Physik 71 (1931), 205–226.

[7] N.M. Bogoliubov, A.G. Izergin, and V.E. Korepin, "Quantum inverse scattering method, correlation functions and algebraic Bethe Ansatz.", Cambridge monographs on mathematical physics, 1993.

[8] J.L. Cardy, "Operator content of two-dimensional conformally invariant theories.", Nucl. Phys. B 270 (1986), 186–204.

[9] F. Colomo, A.G. Izergin, V.E. Korepin, and V. Tognetti, "Temperature correlation functions in the XX0 Heisenberg chain.", Teor. Math. Phys. 94 (1993), 19–38.

[10] H.J. de Vega and F. Woynarowich, "Method for calculating finite size corrections in Bethe Ansatz systems-Heisenberg chains and 6-vertex model.", Nucl. Phys. B 251 (1985), 439–456.

[11] G. Delfino, G. Mussardo, and P. Simonetti, "Correlation Functions Along a Massless Flow.", Phys. Rev. D 51 (1995), 6620–6624.

[12] J. des Cloizeaux and M. Gaudin, "Anisotropic linear magnetic chain.", J. Math. Phys. 7 (1966), 1384–1400.

[13] J. des Cloizeaux and J.J. Pearson, "Spin-wave spectrum of the antiferromagnetic linear chain.", Phys. Rev. 128 (1962), 2131–2135.

[14] C. Destri and J.H. Lowenstein, "Analysis of the Bethe Ansatz equations of the chiral invariant Gross-Neveu model.", Nucl. Phys. B 205 (1982), 369–385.

[15] M. Dugave, F. Göhmann, and K.K. Kozlowski, "Functions characterizing the ground state of the XXZ spin-l/2 chain in the thermodynamic limit.", SIGMA 10 (2014), 043, 18 pages.

[16] L.D. Faddeev and L.A. Takhtadzhan, "What is the spin of a spin wave?", Phys. Lett. A 85 (1981), 375–377.

[17] M. Fowler and X. Zotos, "Quantum sine-Gordon thermodynamics:the Bethe Ansatz method.", Phys. Rev. B 24 (1981), 2634–2639.

[18] F. Göhmann, A. Klümper, and A. Seel, "Integral representations for correlation functions of the XXZ chain at finite temperature.", J. Phys. A: Math. Gen. 37 (2004), 7625–7652.

[19] K. Hida, "Rigorous derivation of the distribution of the eigenstates of the quantum Heisenberg-Ising chain with XY-like anisotropy.", Phys.Lett. A 84 (1981), 338–340.

[20] N. Ishimura and H. Shiba, "Effect of the magnetic field on the des Cloizeaux-Pearson spin wave spectrum.", Prog. Theor. Phys. 57 (1977), 1862–1873.
[21] A.R. Its, A.G. Izergin, V.E. Korepin, and N.A. Slavnov, "Differential equations for quantum correlation functions.", Int. J. Mod. Physics B4 (1990), 1003–1037.

[22] A.R. Its and N.A. Slavnov, "On the Riemann-Hilbert approach to the asymptotic analysis of the correlation functions of the Quantum Nonlinear Schrödinger equation. Non-free fermion case.", Theor. Math. Phys. 119:2 (1990), 541–593.

[23] M. Jimbo and T. Miwa, "QKZ equation with | q | =1 and correlation functions of the XXZ model in the gapless regime.", J. Phys. A 29 (1996), 2923–2958.

[24] M. Jimbo, T. Miwa, Y. Mori, and M. Sato, "Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent.", Physica D 1 (1980), 80–158.

[25] M. Karowski and P. Weisz, "Exact form factors in (1 + 1)-dimensional field theoretic models with soliton behaviour.", Nucl. Phys. B 139 (1978), 455–476.

[26] S. Kerov, G. Olshanski, and A. Vershik, "Harmonic analysis on the infinite symmetric group. A deformation of the regular representation.", Comptes Rend. Acad. Sci. Paris, Sér I 316 (1993), 773–778.

[27] A.N. Kirillov and F.A. Smirnov, "A representation of the current algebra connected with the SU (2)-invariant Thirring model.", Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. 198 (1987), 506–510.

[28] N. Kitanine, K.K. Kozlowski, J.-M. Maillet, N.A. Slavnov, and V. Terras, "Algebraic Bethe Ansatz approach to the asymptotics behavior of correlation functions.", J. Stat. Mech: Th. and Exp. 04 (2009), P04003.

[29] , "On the thermodynamic limit of form factors in the massless XXZ Heisenberg chain.", J. Math. Phys. 50 (2009), 095209.

[30] , "A form factor approach to the asymptotic behavior of correlation functions in critical models.", J. Stat. Mech. : Th. and Exp. 1112 (2011), P12010.

[31] , "Thermodynamic limit of particle-hole form factors in the massless XXZ Heisenberg chain.", J. Stat. Mech. : Th. and Exp. 1105 (2011), P05028.

[32] , "Form factor approach to dynamical correlation functions in critical models.", J. Stat. Mech. 1209 (2012), P09001.

[33] N. Kitanine, K.K. Kozlowski, J.-M. Maillet, and V. Terras, "Long-distance asymptotic behaviour of multipoint correlation functions in massless quantum integrable models.", J. Stat. Mech. 1405 (2014), P05011.

[34] N. Kitanine, J.-M. Maillet, N.A. Slavnov, and V. Terras, "Dynamical correlation functions of the XXZ spin-1/2 chain.", Nucl. Phys. B 729 (2005), 558–580.

[35] N. Kitanine, J.-M. Maillet, and V. Terras, "Correlation functions of the XXZ Heisenberg spin-1/2 chain in a magnetic field.", Nucl. Phys. B 567 (2000), 554–582.

[36] A. Klümper and M.T. Batchelor, "An analytic treatment of finite-size corrections of the spin-1 antiferromagnetic XXZ chain.", J. Phys. A: Math. Gen. 23 (1990), L189.

[37] T. Kojima, V.E. Korepin, and N.A. Slavnov, "Determinant representation for dynamical correlation functions of the quantum nonlinear Schrödinger equation.", Comm. Math. Phys. 188 (1997), 657–689.
[38] V.E. Korepin, "Direct calculation of the S-matrix in the massive Thirring model.", Theor. Math. Phys. 41 (1979), 169–189.

[39] V.E. Korepin and N.A. Slavnov, "The time dependent correlation function of an impenetrable Bose gas as a Fredholm minor I.", Comm. Math. Phys. 129 (1990), 103–113.

[40] , "The new identity for the scattering matrix of exactly solvable models.", Eur. Phys. J. B 5 (1998), 555–557.

[41] K.K. Kozlowski, "On condensation properties of Bethe roots associated with the XXZ chain.", ArXiV:math-ph/1508.05741.

[42] , "On string solutions to the Bethe equations for the XXZ chain: a rigorous approach.", to appear.

[43] , Riemann–Hilbert approach to the time-dependent generalized sine kernel.", Adv. Theor. Math. Phys. 15 (2011), 1–89.

[44] , "Large-distance and long-time asymptotic behavior of the reduced density matrix in the non-linear Schrödinger model.", Ann. Henri-Poincaré 16 (2015), 437–534.

[45] , "Form factors of bound states in the XXZ chain.", J. Phys. A: Math. Theor. Special Issue "Emerging talents" 50 (2017), 184002.

[46] K.K. Kozlowski, J.-M. Maillet, and N. A. Slavnov, "Low-temperature limit of the long-distance asymptotics in the non-linear Schrödinger model.", J.Stat.Mech. (2011), P03019.

[47] K.K. Kozlowski and J.M. Maillet, "Microscopic approach to a class of 1D quantum critical models.", J. Phys. A: Math and Theor. Baxter anniversary special issue 48 (2015), 484004.

[48] K.K. Kozlowski and E. Ragoucy, "Asymptotic behaviour of two-point functions in multi-species models.", Nucl. Phys. B (2016).

[49] K.K. Kozlowski and V. Terras, "Long-time and large-distance asymptotic behavior of the current-current correlators in the non-linear Schrödinger model.", J. Stat. Mech.: Th. and Exp. (2011), P09013.

[50] F. Lesage and H. Saleur, "Form-factors computation of Friedel oscillations in Luttinger liquids.", J. Phys. A: Math. Gen. 30 (1997), L457–L463.

[51] F. Lesage, H. Saleur, and S. Skorik, "Form factors approach to current correlations in one-dimensional systems with impurities.", JNucl. Phys. B 474 (1996), 602–640.

[52] E.H. Lieb, D.C. Mattis, and T.D. Schultz, "Two dimensionnal Ising model as a soluble problem of many fermions.", Rev. Mod. Phys. 36 (1964), 856–871.

[53] B.M. McCoy, "Spin correlation functions in the XY model.", Phys. Rev. 173 (1968), 531–541.

[54] B.M. McCoy, J.H.H. Perk, and R.E. Shrock, "Time-dependent correlation functions of the transverse Ising chain at the critical magnetic field.", Nucl. Phys. B 220 (1983), 35–47.

[55] P. Mejean and F.A. Smirnov, "Form Factors for Principal Chiral Field Model with Wess-Zumino-Novikov-Witten Term.", Int. J. Mod. Phys. A 12, 3383 (1997) 12 (1997), 3383–3395.
[56] G. Müller and R.E. Shrock, "Dynamic correlation functions for one-dimensional quantum-spin systems: new results based on a rigorous approach.", Phys. Rev. B 29 (1984), 288–301.

[57] G. Olshanski, "Point processes and the infinite symmetric group. Part I: The general formalism and the density function.", In: The orbit method in geometry and physics: in honor of A. A. Kirillov (C. Duval, L. Guieu, and V. Ovsienko, eds.), Birkhäuser, Verlag, Basel, Prog. in Math. 213 (2003).

[58] R. Orbach, "Linear antiferromagnetic chain with anisotropic coupling.", Phys. Rev. 112 (1958), 309–316.

[59] B. Ponsot, "Massless N=1 super-sinh-Gordon: form factors approach.", Phys. Lett. 575 (2003), 131–136.

[60] K. Sakai, "Dynamical correlation functions of the XXZ model at finite temperature.", J. Phys. A: Math Theor 40 (2007), 7523–7542.

[61] N.A. Slavnov, "Non-equal time current correlation function in a one-dimensional Bose gas.", Theor. Math. Phys. 82 (1990), 273–282.

[62] , "Differential equations for multipoint correlation functions in a one-dimensional impenetrable Bose gas.", Theor. Math. Phys. 106 (1996), 131–142.

[63] , "Integral equations for the correlation functions of the quantum one-dimensional Bose gas.", Theor. Math. Phys. 121 (1999), 1358–1376.

[64] F.A. Smirnov, "Reductions of the sine-Gordon model as a perturbation of minimal models of conformal field theory.", Nucl. Phys. B 337 (1990), 156–180.

[65] , "Form factors in completely integrable models of quantum field theory.", Advanced Series in Mathematical Physics, vol. 14, World Scientific, 1992.

[66] M. Takahashi, "Thermodynamics of one dimensional solvable models.", Cambridge university press, 1999.

[67] M. Takahashi and M. Suzuki, "One-dimensional anisotropic Heisenberg model at finite temperatures.", Prog. Theor. Phys. 48 (1972), 2187–2209.

[68] C.A. Tracy and H.G. Vaidya, "Transverse time-dependent spin correlation functions of the one-dimensional XY model at zero temperature.", Physica A 92 (1978), 1–41.

46