Superposition Principle and the Problem of the Additivity of the Energies and Momenta of Distinct Electromagnetic Fields

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Abstract

In this paper we prove in a rigorous mathematical way (using the Clifford bundle formalism) that the energies and momenta of two distinct and otherwise arbitrary free Maxwell fields (of finite energies and momenta) that are superposed are additive and thus that there is no incompatibility between the principle of superposition of fields and the principle of energy-momentum conservation, contrary to some recent claims. Our proof depends on a noticeable formula for the energy-momentum densities, namely, Riesz formula $\star T^a = \frac{1}{2} \star (F \theta^a F)$, which is valid for any electromagnetic field configuration $F$ satisfying Maxwell equation $\partial F = 0$.

1 Introduction

In this paper we analyze the compatibility of the principle of superposition for Maxwell fields and the principle of energy-momentum conservation. More exactly, we prove in a rigorous mathematical way, using the Clifford bundle formalism, that the energy and momentum of two distinct and otherwise arbitrary Maxwell fields (of finite energy) are additive. Let us now describe precisely our problem.

In a given inertial frame $I = \partial/\partial x^0 \in \sec TM$ in Minkowski spacetime (see Appendix for some details) with a natural adapted coordinates $\{x^\mu\}$ in Einstein-
Lorentz-Poincaré gauge\footnote{We use units where the velocity of light $c$ has the numerical value $1$ and so the timelike coordinate $x^0 = t$.}, we have two identical antennas that are put on at time $t = -\tau$ and which are able to produce two distinct electromagnetic fields, which we take for simplicity, as being of the same time duration $\tau$. So, at time $t = 0$ we have two electromagnetic field configurations denoted $F_1(0, x)$ and $F_2(0, x)$, which, of course, have compact support in a region $\mathcal{R} \subset \mathbb{R}^3$ (the rest space of the inertial frame) and which we suppose are moving in opposite directions (to fix the ideas the $z$-direction). We suppose moreover that the two antennas are separated by a distance $D > \tau$, which means that one does not affect the other during the time they are generating the arbitrary electromagnetic field configurations $F_1(0, x)$ and $F_2(0, x)$. These fields can then be taken as Cauchy data for Maxwell equations and at any time $t$ satisfy Maxwell equation in free space
\[ \partial F_1 = 0, \quad \partial F_2 = 0. \tag{1} \]

Let $d$ be the distance between the wave fronts of the two pulses at $t = 0$ measured along the $z$ axis. Now, at time $t = (d + \tau)/2$ the pulses $F_1(t, x)$ and $F_2(t, x)$ (which will be always diffracted in relation to the initial configurations $F_1(0, x)$ and $F_2(0, x)$) which move with the velocity of light $c = 1$ fill the same region of space and generate a total electromagnetic field
\[ F(t, x) = F_1(t, x) + F_2(t, x), \tag{2} \]
which satisfy also the free Maxwell equation (with appropriate initial conditions) due to the principle of superposition valid for linear partial differential equations,
\[ \partial F = 0. \tag{3} \]

In the Clifford bundle formalism the energy-momentum densities $\star T^a \in \sec \wedge^3 T^* M \hookrightarrow \sec \mathcal{C}ℓ(M, η)$ ($a = 0, 1, 2, 3$) of an electromagnetic field configuration $F$ is given by Riesz formula\footnote{No misprint here. Maxwell equation in free space $\partial F = 0$ is the equation of motion for an electromagnetic field configuration $F \in \sec \wedge^2 T^* M \hookrightarrow \sec \mathcal{C}ℓ(M, η)$, where $\mathcal{C}ℓ(M, η)$ is the Clifford bundle of differential forms. See the Appendices for explanation of the symbols and the main definitions and [19] for a detailed exposition.} (see Appendix)
\[ \star T^a = \frac{1}{2} \star (F^a F), \tag{4} \]
and the energy-momentum $P^n_F$ of the field configuration at time $t = t$ is given by
\[ P^n_F = \int_{B_2} \star T^a, \tag{5} \]

\footnote{Notice that $T_a = T_{ab} \theta^b$, where $T_{ab}$ are the components of the usual energy-momentum tensor of Maxwell theory. Discussions about the appropriateness for the use of the usual energy-momentum tensor for the description of energy-momentum propagation are given in [15] [17].}
where $B_2$ is contained in the constant time hypersurface $t = t$ in Minkowski spacetime.

Now, due to Eq. (2) we have

$$
\star \mathcal{T}^a = \frac{1}{2} \star (F\theta_a \tilde{F}) = \frac{1}{2} \star ((F_1 + F_2)\theta^a(F_1 + F_2))
$$

$$
= \frac{1}{2} \star (F_1\theta^a \tilde{F}_1 + F_2\theta^a \tilde{F}_2 + F_1\theta^a \tilde{F}_2 + F_2\theta^a \tilde{F}_1)
$$

$$
= \star \mathcal{T}_1^a + \star \mathcal{T}_2^a + \star \mathcal{K}^a
$$

where

$$
\star \mathcal{T}_1^a = \frac{1}{2} \star (F_1\theta^a \tilde{F}_1), \quad \star \mathcal{T}_2^a = \frac{1}{2} \star (F_2\theta^a \tilde{F}_2),
$$

$$
\star \mathcal{K}^a = \frac{1}{2} \star (F_1\theta^a \tilde{F}_2 + F_2\theta^a \tilde{F}_1).
$$

Then we have that

$$
P_F^a = \int_{B_2} \star \mathcal{T}^a = \int_{B_2} \star \mathcal{T}_1^a + \int_{B_2} \star \mathcal{T}_2^a + \int_{B_2} \star \mathcal{K}^a.
$$

We want to prove that the energy and momentum of the field configuration $F_1$ and $F_2$ at time $t = t$ (and indeed at any time) is additive, i.e., is given by

$$
P_F^a = P_{F_1}^a + P_{F_2}^a, \quad a = 0, 1, 2, 3,
$$

with

$$
P_{F_1}^a = \int_{B_2} \star \mathcal{T}_1^a, \quad P_{F_2}^a = \int_{B_2} \star \mathcal{T}_2^a.
$$

This problem is a nontrivial one, and has been not discussed in the literature in an appropriate and satisfactory way according to our view. For example, in [11] (written in 1980) the author said that he found the problem discussed in only two ([6, 24]) out of 50 textbooks he has examined. Moreover, from a few papers published in the literature, we found some good ideas, but none offers a rigorous solution for the problem. Worse, some papers and books [2] [9, 10] have very odd and/or dubious statements. Indeed, in [2] it is said that Eq. (8) implies in non conservation of energy-momentum. The statement about non conservation of energy-momentum is also done by the author of [9] [10] who says that results of recent experiments [7] endorse his statement. In [8] the double slit interference with monochromatic waves [4] is analyzed and it is said that energy-momentum is conserved only after spatial average. On the other hand, e.g., [4] shows that Eq. (9) is the correct one for the case of two plane waves.
waves moving in opposite directions, but since waves of this kind (which do not have compact support) have infinite energy when the integration in Eq. (8) is done in all space, his approach cannot in any way be considered satisfactory. In [1] it is proposed that energy-momentum tensors that differ from an exact differential must be considered equivalent. This is a good idea, if it could be proved (something which has not been done in [1]) that

\[ \star K^a = -d \star E^a \]

for some \( \star E^a \in \text{sec} \bigwedge^2 T^* M \hookrightarrow \text{sec} \mathcal{C}(M, \eta) \) which goes to zero at spatial infinity at time \( t = t \), since in this case we can write using Stokes theorem that

\[ \int_{B_2} \star K^a = - \int_{B_2} d \star E^a = - \int_{\partial B_2} \star E^a = 0 \] (11)

In the Section 2 we show that this is indeed the case for our problem. In Section 3 we prove that the energies and momenta of two different superposed electromagnetic field configurations are indeed additive. In Section 4 we present our conclusions. The paper have 4 Appendices. Appendix A introduces the concept of Clifford bundles, some important Clifford algebra identities, the Hodge star operator as an algebraic operation, and the Dirac operator acting on sections of the Clifford bundle. Appendix B presents Maxwell equation \( \partial F = J \) and the noticeable formula for the energy-momentum densities \( \star T^a = \frac{1}{2} \star (F \theta^a F) \). In Appendix C we describe the main features of the standard cylinder in Minkowski spacetime need in the applications of the Stokes theorem in the main text and in Appendix D we recall for completeness the generalized Green’s formula for differential forms.

2 Proof that \( \star K^a = -d \star E^a \)

From Maxwell theory it follows (see Appendix) that for any free electromagnetic field configuration that

\[ \delta T^a = -\partial_\alpha T^a = 0. \]

Since we obviously have \( \partial_\alpha T^a_1 = \partial_\alpha T^a_2 = 0 \), we necessarily must have that \( \partial_\alpha K^a = 0 \), i.e.,

\[ \partial_\alpha \frac{1}{2} \left( F_1 \theta^a \bar{F}_2 + F_2 \theta^a \bar{F}_1 \right) = 0. \] (12)

To show that this is indeed the case, first, observe that

\[ F_1 \theta^a \bar{F}_2 = \left\langle F_1 \theta^a \bar{F}_2 \right\rangle_1 + \left\langle F_1 \theta^a \bar{F}_2 \right\rangle_3, \]

\[ F_2 \theta^a \bar{F}_1 = \left\langle F_2 \theta^a \bar{F}_1 \right\rangle_1 + \left\langle F_2 \theta^a \bar{F}_1 \right\rangle_3. \] (13)

When the previous equations are added, the terms \( \left\langle F_1 \theta^a \bar{F}_2 \right\rangle_3 \) and \( \left\langle F_2 \theta^a \bar{F}_1 \right\rangle_3 \) cancel and we have

\[ F_1 \theta^a \bar{F}_2 + F_2 \theta^a \bar{F}_1 = \left\langle F_1 \theta^a \bar{F}_2 + F_2 \theta^a \bar{F}_1 \right\rangle_1. \] (14)
Returning to Eq. (12) we see that we need only to calculate \( \partial_j \left( F_1 \theta^a \tilde{F}_2 + F_2 \theta^a \tilde{F}_1 \right)_1 \).

We have,

\[
\partial_j \left( F_1 \theta^a \tilde{F}_2 + F_2 \theta^a \tilde{F}_1 \right) = \left( \partial \left( F_1 \theta^a \tilde{F}_2 \right) + \partial \left( F_2 \theta^a \tilde{F}_1 \right) \right)_0
\]

or

\[
\partial_j \left( F_1 \theta^a \tilde{F}_2 + F_2 \theta^a \tilde{F}_1 \right)_1
= \left( \partial^b D_{cb} \left( F_1 \theta^a \tilde{F}_2 \right) + \partial^b D_{cb} \left( F_2 \theta^a \tilde{F}_1 \right) \right)_0
\]

On the other hand, from the Eq. (1) we have

\[
\theta^b D_{cb} F_1 = \partial F_1 = 0 \quad \text{and} \quad \theta^b D_{cb} F_2 = \partial F_2 = 0
\]

and recalling that \( \theta^a = \delta^a_\mu dx^\mu \) and \( c_{cb} = \delta^\mu_\nu \partial / \partial x^\mu \), we have that \( D_{cb} \theta^a = 0 \).

Then, Eq. (12) can be written as

\[
\partial_j \frac{1}{2} \left( F_1 \theta^a \tilde{F}_2 + F_2 \theta^a \tilde{F}_1 \right)_1 = \frac{1}{2} \left( \partial^b F_1 \theta^a D_{cb} \tilde{F}_2 + \partial^b F_2 \theta^a D_{cb} \tilde{F}_1 \right)_0
\]

Now we examine the term \( \left( \partial^b F_1 \theta^a D_{cb} \tilde{F}_2 + \partial^b F_2 \theta^a D_{cb} \tilde{F}_1 \right)_0 \). First observe that

\[
\partial^b \left( F_1 \theta^a D_{cb} \tilde{F}_2 \right) = \partial^b \left( F_1 \theta^a D_{cb} \tilde{F}_2 \right)_1 + \partial^b \left( F_1 \theta^a D_{cb} \tilde{F}_2 \right)_3
= \partial^b \left( F_1 \theta^a D_{cb} \tilde{F}_2 \right)_1 + \partial^b \wedge \left( F_1 \theta^a D_{cb} \tilde{F}_2 \right)_1
+ \partial^b \left( F_1 \theta^a D_{cb} \tilde{F}_2 \right)_3 + \partial^b \wedge \left( F_1 \theta^a D_{cb} \tilde{F}_2 \right)_3
\]

Then

\[
\left( \partial^b F_1 \theta^a D_{cb} \tilde{F}_2 \right)_0 = \partial^b \left( F_1 \theta^a D_{cb} \tilde{F}_2 \right)_1
= \left( F_1 \theta^a D_{cb} \tilde{F}_2 \right)_1 \wedge \theta^b
= \left( F_1 \theta^a \left( D_{cb} \tilde{F}_2 \right) \theta^b \right)_0
= \left( F_1 \theta^a \left( \tilde{F}_2 \tilde{\partial} \right) \theta^b \right)_0
\]

where we use the symbol \( \left( D_{cb} \tilde{F}_2 \right) \theta^b := \tilde{F}_2 \tilde{\partial} \). Since \( \tilde{F}_2 \tilde{\partial} \theta^b = \left( \partial \theta^b \right) = 0 \), we have \( \left( \partial^b F_1 \theta^a D_{cb} \tilde{F}_2 \right)_0 = 0 \). Analogously we get that \( \left( \partial^b F_2 \theta^a D_{cb} \tilde{F}_1 \right)_0 = 0 \), and thus

\[
\left( \partial^b F_1 \theta^a D_{cb} \tilde{F}_2 + \partial^b F_2 \theta^a D_{cb} \tilde{F}_1 \right)_0 = 0.
\]
Now, using Eq. (21) in Eq. (18) we have
\[
\partial \frac{1}{2} \left( F_1 \theta^a F_2 + F_2 \theta^a F_1 \right) = 0. \tag{22}
\]

We just proved that indeed, \( \delta K^a = 0 \), or what is the same, that
\[
d * K^a = 0, \tag{23}
\]
and since we are in Minkowski spacetime Poincaré’s lemma implies that the 3-form fields \( \star K^a \in \sec \Lambda^3 T^* M \to \sec \mathcal{C} \ell (M, \eta) \) must be exact, i.e.,
\[
\star K^a = -d * c^a, \tag{24}
\]
or
\[
\delta c^a = -K^a \tag{25}
\]

3 The Energies and Momenta of Two Different Superposed Electromagnetic Field Configurations are Additive.

In this section the standard cylinder of Minkowski spacetime and its boundary submanifolds (see Figure ) described in the Appendix will be used. We start our enterprise by recalling that since \( \star T^a = \star T_1^a + \star T_2^a + \star K^a \) and \( d * T^a = 0, \ d * T_1^a = 0, \ d * T_2^a = 0 \) and \( d * K^a = 0 \) we can use Stokes theorem to write
\[
0 = \int_N d * K^a = -\int_{B_1'} \star K^a + \int_{B_2'} \star K^a + \int_{B_3'} \star K^a
= -\int_{B_1} \star K^a + \int_{B_2} \star K^a + \int_{B_3} \star K^a, \tag{26}
\]
from where it follows that
\[
\int_{B_1} \star K^a = \int_{B_2} \star K^a + \int_{B_3} \star K^a. \tag{27}
\]

Now, if we take into account Eq. (7) which defines \( \star K^a \) we see immediately that \( \int_{B_3'} \star K^a = 0 \) because \( F_1|_{B_3'} = 0 \) and \( F_1|_{B_3} = 0 \). Also, since (obviously) \( \star K^a|_{B_1} = 0 \) it follows that \( \int_{B_1} \star K^a = 0 \) and we get that
\[
\int_{B_2} \star K^a = 0, \tag{28}
\]
even if \( \star K^a|_{B_2} \neq 0 \).
Using these results we can calculate the total energy of \( F = F_1 + F_2 \) containing in \( B_2 \). We have, taking into account Eq.\((6)\) and Eq.\((28)\) that

\[
\int_{B_2} \star T^a = \int_{B_2} \star (T_1^a + T_2^a) + \int_{B_2} \star K^a
\]

\[
= \int_{B_2} \star (T_1^a + T_2^a),
\]

i.e.,

\[
P_F = P_{F_1} + P_{F_2},
\]

as we wanted to prove.

4 Conclusions

In this paper we proved that the energy and momenta of two free electromagnetic field configurations (which satisfy at any instant of time a free Maxwell equation and for each finite instant of time have compact support in \( \mathbb{R}^3 \)) of finite energy are additive and thus there is no incompatibility between the principle of superposition of energy and the principle of energy-momentum conservation as suggested by some authors, quoted in Section \( 1 \).

We emphasize that our proof is made relatively simple due to the amazing power of the Clifford bundle formalism, and indeed we do not see how to do the calculations using the old Heaviside-Gibbs vector calculus, or even only the Cartan calculus of differential forms, since our proof depends deeply on the noticeable formula for the energy-momentum densities, namely \( \star T^a = \frac{1}{2} F \theta^a F \) which is valid for any electromagnetic field configuration \( F \) satisfying Maxwell equation \( \partial F = 0 \). We emphasize also that we have found there exist closed 2-forms \( \star W^a \in \sec \bigwedge^2 T^*M \rightarrow \sec \mathcal{O}(M, \eta) \) satisfying generalized Maxwell equations \( \partial W^a = T^a + \star Ma \). This result that will be explored in future publications.

A Clifford Bundles

Let \((M, \eta, D, \tau_\eta, \uparrow)\) be Minkowski spacetime. \((M, \eta)\) is a four dimensional space oriented (by the volume form \( \tau_\eta \)) and time oriented (by the equivalence relation \( \uparrow \), see \[19\]) Lorentzian manifold, with \( M \simeq \mathbb{R}^4 \) and \( \eta \in \sec T^0_2 M \) is a Lorentzian metric of signature \((1, 3)\). \( T^a M \) \([TM]\) is the cotangent [tangent] bundle. \( T^* M = \cup_{x \in M} T_x^* M, \ T^1 M = \cup_{x \in M} T_x M, \) and \( T^a_x M \simeq T_x^* M \simeq \mathbb{R}^{1,3} \), where \( \mathbb{R}^{1,3} \) is the Minkowski vector space. \( D \) is the Levi-Civita connection of \( \eta \), i.e., \( D\eta = 0, \ R(D) = 0 \). Also \( T(D) = 0, \ R \) and \( T \) being respectively the torsion and curvature tensors. Let \( \eta \in \sec T^0_2 M \) be the metric on the cotangent bundle associated to \( \eta \in \sec T^0_2 M \). The Clifford bundle of differential forms \( \mathcal{O}(M, \eta) \) is the bundle of algebras, i.e., \( \mathcal{O}(M, \eta) = \cup_{x \in M} \mathcal{O}(T_x^* M) \),

\[8\]We observe that the same problem occur for all linear field theories, and indeed in reference \([11, 12, 21]\) we have some discussion of the problem for sound (and other elastic) waves.
where \( \forall x \in M, \mathcal{O}(T^*_x M) = \mathbb{R}_{1,3} \), the so called \textit{spacetime algebra}. Recall also that \( \mathcal{O}(M, \eta) \) is a vector bundle associated to the \textit{orthonormal frame bundle} \( P_{SO(1,3)}(M) \), i.e., \( \mathcal{O}(M, \eta) = P_{SO(1,3)}(M) \times_{ad} \mathbb{R}_{1,3} \) (see details in, e.g., \cite{16, 13, 19}). For any \( x \in M, \mathcal{O}(T^*_x M) \) is a linear space over the real field \( \mathbb{R} \). Moreover, \( \mathcal{O}(T^*_x M) \) is isomorphic to the Cartan algebra \( \wedge T^*_x M \) of the cotangent space and \( \wedge T^*_x M = \sum_{k=0}^{\infty} \wedge^k T^*_x M \), where \( \wedge^k T^*_x M \) is the \( \binom{n}{k} \)-dimensional space of \( k \)-forms. Then, sections of \( \mathcal{O}(M, \eta) \) can be represented as a sum of non homogeneous differential forms. Let \( \{ x^\mu \} \) be coordinates in Einstein-Lorentz-Poincaré gauge for \( M \) and let \( \{ e_\mu = \partial / \partial x^\mu \} \in \sec FM \) (the frame bundle) be an orthonormal basis for \( TM \), i.e., \( \eta(e_\mu, e_\nu) = \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \). Let \( \gamma^\nu = dx^\nu \in \sec \bigwedge^1 T^* M \hookrightarrow \sec \mathcal{O}(M, \eta) \) where \( \gamma = (\nu = 0, 1, 2, 3) \) such that the set \( \{ \gamma^\nu \} \) is the dual basis of \( \{ e_\mu \} \), and of course, \( \eta(\gamma^\mu, \gamma^\nu) = \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1) \). We introduce moreover the notations \( \theta^a = \delta^a_\mu dx^\mu \) and \( e_a = \delta^a_\mu \partial / \partial x^\mu \). We say that \( \{ e_a \} \) is a section of the orthonormal frame bundle \( P_{SO(1,3)}(M) \) and its dual basis \( \{ \theta^a \} \) a section of the orthonormal coframe bundle (denoted \( P_{SO(1,3)}(M) \)).

A.1 Clifford Product

The fundamental \textit{Clifford product} (in what follows to be denoted by juxtaposition of symbols) is generated by \( \theta^a \theta^a + \theta^b \theta^b = 2\eta^{ab} \) and if \( C \in \mathcal{O}(M, \eta) \) we have

\[
\mathcal{C} = s + v_a \theta^a + \frac{1}{2!} b_{ab} \theta^a \theta^b + \frac{1}{3!} a_{abc} \theta^a \theta^b \theta^c + p \theta^5 , \tag{30}
\]

where \( \tau_n := \theta^5 = \theta^0 \theta^1 \theta^2 \theta^3 = dx^0 dx^1 dx^2 dx^3 \) is the volume element and \( s, v_a, b_{ab}, a_{abc}, p \in \sec \bigwedge^0 T^* M \hookrightarrow \sec \mathcal{O}(M, \eta) \).

Let \( A_r, B_s \in \sec \bigwedge^r T^* M \hookrightarrow \sec \mathcal{O}(M, \eta) \). For \( r = s = 1 \), we define the \textit{scalar product} as follows:

For \( a, b \in \sec \bigwedge^1 T^* M \hookrightarrow \sec \mathcal{O}(M, \eta) \),

\[
a \cdot b = \frac{1}{2} (ab + ba) = \eta(a, b). \tag{31}
\]

We define also the \textit{exterior product} \( (\forall r, s = 0, 1, 2, 3) \) by

\[
A_r \wedge B_s = (A_r B_s)_{r+s},
\]

\[
A_r \wedge B_s = (-1)^s B_s \wedge A_r, \tag{32}
\]

where \( \langle k \rangle \) is the component in \( \bigwedge^k T^* M \) (projection) of the Clifford field. The exterior product is extended by linearity to all sections of \( \mathcal{O}(M, \eta) \).

For \( A_r = a_1 \wedge ... \wedge a_r, B_r = b_1 \wedge ... \wedge b_r \), the scalar product is defined here as follows,

\[
A_r \cdot B_r = (a_1 \wedge ... \wedge a_r) \cdot (b_1 \wedge ... \wedge b_r) = \left| \begin{array}{cccc}
a_1 \cdot b_1 & ... & a_1 \cdot b_r \\
\ldots & \cdots & \ldots & \cdots \\
a_r \cdot b_1 & \ldots & a_r \cdot b_r \\
\end{array} \right|. \tag{33}
\]
We agree that if \( r = s = 0 \), the scalar product is simple the ordinary product in the real field.

Also, if \( r \neq s \), then \( \mathcal{A}_r \cdot \mathcal{B}_s = 0 \). Finally, the scalar product is extended by linearity for all sections of \( \mathcal{C}(M, \eta) \).

For \( r \leq s \), \( \mathcal{A}_r = a_1 \wedge ... \wedge a_r \), \( \mathcal{B}_s = b_1 \wedge ... \wedge b_s \) we define the left contraction by

\[
\mathcal{A}_r \llcorner \mathcal{B}_s = \sum_{i_1 < ... < i_r} \epsilon^{i_1 ... i_r} (a_1 \wedge ... \wedge a_r) \cdot (b_{i_1} \wedge ... \wedge b_{i_r}) \sim b_{i_r+1} \wedge ... \wedge b_s
\]

where \( \sim \) is the reverse mapping (reversion) defined by

\[
\sim: \text{sec} \bigwedge^p T^*M \ni a_1 \wedge ... \wedge a_p \mapsto a_p \wedge ... \wedge a_1
\]

and extended by linearity to all sections of \( \mathcal{C}(M, \eta) \). We agree that for \( \alpha, \beta \in \text{sec} \bigwedge^0 T^*M \) the contraction is the ordinary (pointwise) product in the real field and that if \( \alpha \in \text{sec} \bigwedge^0 T^*M \to \mathcal{C}(M, \eta) \), \( \mathcal{A}_r \in \text{sec} \bigwedge^r T^*M \to \mathcal{C}(M, \eta) \), \( \mathcal{B}_s \in \text{sec} \bigwedge^s T^*M \to \mathcal{C}(M, \eta) \) then \( \langle \alpha \mathcal{A}_r \rangle \llcorner \mathcal{B}_s = \mathcal{A}_r \llcorner \langle \alpha \rangle \mathcal{B}_s \). Left contraction is extended by linearity to all pairs of elements of sections of \( \mathcal{C}(M, \eta) \), i.e., for \( \mathcal{A}, \mathcal{B} \in \text{sec} \mathcal{C}(M, \eta) \)

\[
\mathcal{A} \cdot \mathcal{B} = \sum_{r,s} \langle \mathcal{A} \rangle_{r-\llcorner} \langle \mathcal{B} \rangle_s, \quad r \leq s.
\]

It is also necessary to introduce the operator of right contraction denoted by \( \llcorner \). The definition is obtained from the one presenting the left contraction with the imposition that \( r \geq s \) and taking into account that now if \( \mathcal{A}_r \in \text{sec} \bigwedge^r T^*M \leftrightarrow \mathcal{C}(M, \eta) \), \( \mathcal{B}_s \in \text{sec} \bigwedge^s T^*M \leftrightarrow \mathcal{C}(M, \eta) \) then \( \mathcal{A}_r \llcorner (\alpha \mathcal{B}_s) = (\alpha \mathcal{A}_r) \llcorner \mathcal{B}_s \).

The main formulas used in the present paper can be obtained (details may be found in [19]) from the following ones (where \( a \in \text{sec} \bigwedge^1 T^*M \leftrightarrow \text{sec} \mathcal{C}(M, \eta) \)):

\[
a \mathcal{B}_s = a \llcorner \mathcal{B}_s + a \wedge \mathcal{B}_s, \quad \mathcal{B}_s a = \mathcal{B}_s \wedge a + \mathcal{B}_s a,
\]

\[
a \llcorner \mathcal{B}_s = \frac{1}{2} (a \mathcal{B}_s - (-1)^s \mathcal{B}_s a),
\]

\[
\mathcal{A}_r \llcorner \mathcal{B}_s = (-1)^{r(s-r)} \mathcal{B}_s \mathcal{A}_r,
\]

\[
a \wedge \mathcal{B}_s = \frac{1}{2} (a \mathcal{B}_s + (-1)^s \mathcal{B}_s a),
\]

\[
\mathcal{A}_r \mathcal{B}_s = \langle \mathcal{A}_r \mathcal{B}_s \rangle_{[r-s]} + \langle \mathcal{A}_r \mathcal{B}_s \rangle_{[r-s]+2} + ... + \langle \mathcal{A}_r \mathcal{B}_s \rangle_{[r+s]}
\]

\[
= \sum_{k=0}^m \langle \mathcal{A}_r \mathcal{B}_s \rangle_{[r-s]+2k}
\]

\[
\mathcal{A}_r \cdot \mathcal{B}_r = \mathcal{B}_r \cdot \mathcal{A}_r = \bar{\mathcal{A}}_r \llcorner \mathcal{B}_r = \mathcal{A}_r \llcorner \bar{\mathcal{B}}_r = \langle \bar{\mathcal{A}}_r \mathcal{B}_r \rangle_0 = \langle \mathcal{A}_r \bar{\mathcal{B}}_r \rangle_0.
\]
\[ \langle AB \rangle_r = (-1)^{(r-1)/2} \langle B \tilde{A} \rangle_r, \]
\[ \langle A_r B_s \rangle_r = \langle B_s A_r \rangle_r = (-1)^{(s-1)/2} \langle B_s A_r \rangle_r, \]
\[ \langle A B C_r \rangle_q = (-1)^{e} \langle C_r B_s A_r \rangle_q, \]
\[ \varepsilon = \frac{1}{2}(q^2 + r^2 + s^2 + t^2 - q - r - s - t) \] (38)

**A.2 Hodge Star Operator**

Let \( \star \) be the Hodge star operator, i.e., the mapping

\[ \star : \bigwedge^k T^* M \to \bigwedge^{4-k} T^* M, \quad A_k \mapsto \star A_k \]

where for \( A_k \in \text{sec} \bigwedge^k T^* M \hookrightarrow \text{Cℓ}(M, \eta) \)

\[ [B_k \cdot A_k] \tau_\eta = B_k \wedge \star A_k, \quad \forall B_k \in \text{sec} \bigwedge^k T^* M \hookrightarrow \text{sec} \text{Cℓ}(M, \eta). \] (39)

\( \tau_\eta = \theta^p \in \bigwedge^4 T^* M \) is a standard volume element. Then we can verify that

\[ \star A_k = \tilde{A}_k \tau_\eta = \tilde{A}_k \theta^5. \] (40)

**A.3 Dirac Operator**

Let \( d \) and \( \delta \) be respectively the differential and Hodge codifferential operators acting on sections of \( \text{sec} \bigwedge^k T^* M \hookrightarrow \text{Cℓ}(M, \eta) \). If \( A_p \in \text{sec} \bigwedge^p T^* M \hookrightarrow \text{sec} \text{Cℓ}(M, \eta) \), then

\[ \delta A_p = (-1)^p \star^{-1} d \star A_p, \quad \text{with} \quad \star^{-1} \star = \text{id}. \]

The Dirac operator acting on sections of \( \text{Cℓ}(M, g) \) is the invariant first order differential operator

\[ \partial = \theta^a D_{e_a}. \] (41)

**A.3.1 \( D_{e_a} A \)**

The reciprocal basis of \( \{\theta^b\} \) is denoted \( \{\theta_a\} \) and we have \( \theta_a \cdot \theta_b = \eta_{ab} \) \((\eta_{ab} = \text{diag}(1, -1, -1, -1))\). Also,

\[ D_{e_a} \theta^b = -\omega^b_{ac} \theta^c = -\omega^b_{ac} \theta_c, \] (42)

with \( \omega^b_{ac} = -\omega^c_{ab} \), and \( \omega^b_{ac} = \eta^b_{ak} \omega^k_{ca} \eta_{cl}, \omega_{abc} = \eta_{ad} \omega_{d bc} = -\omega_{cba} \). Defining

\[ \omega_a = \frac{1}{2} \omega^b_{ac} \theta_b \wedge \theta_c \in \text{sec} \bigwedge^2 T^* M \hookrightarrow \text{sec} \text{Cℓ}(M, \eta), \] (43)

we have (by linearity) that for any \( A \in \text{sec} \bigwedge T^* M \hookrightarrow \text{sec} \text{Cℓ}(M, \eta) \)

\[ D_{e_a} A = \partial_{e_a} A + \frac{1}{2}[\omega_a, A], \] (44)

where \( \partial_{e_a} \) is the Pfaff derivative\(^9\).

---

\(^9\)E.g., if \( A = \frac{1}{p!} A_{i_1 \ldots i_p} \theta^{i_1} \ldots \theta^{i_p} \) then \( \partial_{e_a} A = \frac{1}{p!} [\theta_a (A_{i_1 \ldots i_p}) \theta^{i_1} \ldots \theta^{i_p}] \).

---
A.3.2 $\partial = d - \delta$

Using Eq. (44) we can easily show the very important result:

$$\partial A = \partial \wedge A + \partial \lrcorner A = dA - \delta A,$$

$$\partial \wedge A = dA, \quad \partial \lrcorner A = -\delta A. \quad (45)$$

B Maxwell Equation

Eq. (45) permit us to write the Maxwell equations

$$dF = 0, \quad \delta F = -J \quad (46)$$

for $F \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}(M, \eta)$ as a single equation (Maxwell equation),

$$\partial F = J. \quad (47)$$

B.1 The Noticeable Riesz Formula $T_a = \frac{1}{2} F \theta_a \tilde{F}$

We now prove that the energy-momentum densities $*T_a$ of the Maxwell field can be written in the Clifford bundle formalism as

$$*T_a = \frac{1}{2} * (F \theta_a \tilde{F}) \in \sec \bigwedge^3 T^*M \hookrightarrow \sec \mathcal{C}(M, \eta). \quad (48)$$

To derive Eq. (48) we start from the Maxwell Lagrangian

$$L_m = \frac{1}{2} F \wedge * F, \quad (49)$$

where $F = \frac{1}{2} F_{ab} \theta^a \wedge \theta^b := \frac{1}{2} F_{ab} \theta^{ab} \in \sec \bigwedge^2 TM \hookrightarrow \sec \mathcal{C}(M, \eta)$ is the electromagnetic field. Now, denoting by $\delta$ the variational symbol we can easily verify that

$$\delta * \theta^{ab} = \delta \theta^c \wedge [\theta_{c\lrcorner} * \theta^{ab}].$$

Moreover, in general $\delta$ and $*$ do not commute. Indeed, for any $A_p \in \sec \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}(M, \eta)$ we have

$$[\delta, *} A_p = \delta * A_p - * \delta A_p = \delta \theta^a \wedge (\theta_{a\lrcorner} * A_p) - * [\delta \theta^a \wedge (\theta_{a\lrcorner} A_p)]. \quad (50)$$

The formula $T_a = \frac{1}{2} F \theta_a \tilde{F}$ has been first obtained (but, not using the algebraic derivatives of the Lagrangian density) by M. Riesz in 1947 [18] and it has been rediscovered by Hestenes in 1996 [5] (which also does not use the algebraic derivatives of the Lagrangian density). Algebraic derivatives of homogenous form fields has been described, e.g., in Thirring’s book [23].

Please, do not confuse the variational symbol $\delta$ with the symbol $\delta$ of the Hodge coderivative.
Multiplying both members of Eq. (50) with $A_p = F$ on the right by $F \wedge$ we get

$$F \wedge \delta * F = F \wedge \delta (\theta_a \cdot F) + \delta (\theta_a \cdot F) \wedge \delta (\theta_a \cdot F).$$

Next we sum $\delta F \wedge * F$ to both members of the above equation obtaining

$$\delta (F \wedge * F) = 2 \delta F \wedge * F + \delta \theta_a \wedge \left[ F \wedge (\theta_a \cdot F) - (\theta_a \cdot F) \wedge * F \right].$$

Then, it follows (see, [19, 20] for details) that if $\frac{1}{12} \delta \theta_a = - \xi \theta_a$, for some diffeomorphism generated by the vector field $\xi$ that

$$\star T_a = \frac{\partial L_m}{\partial \theta_a} = \frac{1}{2} \left[ F \wedge (\theta_a \cdot F) - (\theta_a \cdot F) \wedge * F \right].$$

Now,

$$(\theta_a \cdot F) \wedge * F = - \left[ (\theta_a \cdot F) \cdot F \right] = - \left[ (\theta_a \cdot F) \cdot F \right] \tau_\eta$$

and using also the identity [19]

$$(\theta_a \cdot F) \wedge * F = \theta_a (F \cdot F) \tau_\eta - F \wedge (\theta_a \cdot F).$$

we get

$$\frac{1}{2} \left[ F \wedge (\theta_a \cdot F) - (\theta_a \cdot F) \wedge * F \right] = \frac{1}{2} \left[ \theta_a (F \cdot F) \tau_\eta - (\theta_a \cdot F) \wedge * F - (\theta_a \cdot F) \wedge * F \right]$$

$$= \frac{1}{2} \left\{ \theta_a (F \cdot F) \tau_\eta - 2(\theta_a \cdot F) \wedge * F \right\}$$

$$= \frac{1}{2} \left\{ \theta_a (F \cdot F) \tau_\eta + 2[(\theta_a \cdot F) \cdot F] \tau_\eta \right\}$$

$$= \star \left( \frac{1}{2} \theta_a (F \cdot F) + (\theta_a \cdot F) \cdot F \right) = \frac{1}{2} \star (F \theta_a \tilde{F}),$$

where in writing the last line we used the identity

$$\frac{1}{2} F n \tilde{F} = (\theta_a \cdot F) \cdot F + \frac{1}{2} n (F \cdot F), \quad (51)$$

whose proof is as follows:

$$(\theta_a \cdot F) \cdot F + \frac{1}{2} n (F \cdot F) = \frac{1}{2} \left[ (n \cdot F) F - F (n \cdot F) \right] + \frac{1}{2} n (F \cdot F)$$

$$= \frac{1}{4} [n F F - F n F - F n F + F F n] + \frac{1}{2} n (F \cdot F)$$

$$= - \frac{1}{2} F n F + \frac{1}{4} [-2 n (F \cdot F) + n (F \wedge F) + (F \wedge F) n] + \frac{1}{2} n (F \cdot F)$$

$$= - \frac{1}{2} F n F + \frac{1}{2} n (F \cdot F) + \frac{1}{2} n (F \wedge F) + \frac{1}{2} n (F \cdot F)$$

$$= - \frac{1}{2} F n F = \frac{1}{2} F n \tilde{F}.$$  

$^{12} \xi$ denotes the Lie derivative in the direction of the vector field $\xi$. 

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valid for any $n \in \sec \bigwedge^1 T^* M \hookrightarrow \sec \mathcal{C}^\ell(M, \eta)$ and $F \in \sec \bigwedge^2 T^* M \hookrightarrow \sec \mathcal{C}^\ell(M, \eta)$.

For completeness and presentation of some more tricks of the trade we detail the proof that $\mathcal{T}_a \cdot \theta_b = \mathcal{T}_b \cdot \theta_a$.

\[ \mathcal{T}_a \cdot \theta_b = -\frac{1}{2} (\theta_a F \theta_b)_0 - \frac{1}{2} (\theta_a (F \wedge F) \theta_b)_0 \]

For completeness and presentation of some more tricks of the trade we detail the proof that $\mathcal{T}_a \cdot \theta_b = \mathcal{T}_b \cdot \theta_a$.

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For completeness and presentation of some more tricks of the trade we detail the proof that $\mathcal{T}_a \cdot \theta_b = \mathcal{T}_b \cdot \theta_a$.

\[ \mathcal{T}_a \cdot \theta_b = -\frac{1}{2} (\theta_a F \theta_b)_0 - \frac{1}{2} (\theta_a (F \wedge F) \theta_b)_0 \]

For completeness and presentation of some more tricks of the trade we detail the proof that $\mathcal{T}_a \cdot \theta_b = \mathcal{T}_b \cdot \theta_a$.

\[ \mathcal{T}_a \cdot \theta_b = -\frac{1}{2} (\theta_a F \theta_b)_0 - \frac{1}{2} (\theta_a (F \wedge F) \theta_b)_0 \]

For completeness and presentation of some more tricks of the trade we detail the proof that $\mathcal{T}_a \cdot \theta_b = \mathcal{T}_b \cdot \theta_a$.

\[ \mathcal{T}_a \cdot \theta_b = -\frac{1}{2} (\theta_a F \theta_b)_0 - \frac{1}{2} (\theta_a (F \wedge F) \theta_b)_0 \]

For completeness and presentation of some more tricks of the trade we detail the proof that $\mathcal{T}_a \cdot \theta_b = \mathcal{T}_b \cdot \theta_a$.

\[ \mathcal{T}_a \cdot \theta_b = -\frac{1}{2} (\theta_a F \theta_b)_0 - \frac{1}{2} (\theta_a (F \wedge F) \theta_b)_0 \]

For completeness and presentation of some more tricks of the trade we detail the proof that $\mathcal{T}_a \cdot \theta_b = \mathcal{T}_b \cdot \theta_a$.

\[ \mathcal{T}_a \cdot \theta_b = -\frac{1}{2} (\theta_a F \theta_b)_0 - \frac{1}{2} (\theta_a (F \wedge F) \theta_b)_0 \]
B.2 Enter New Maxwell Like Equations \( d \ast W^a = - \ast T^a \),
\[ d \ast W^a = \ast M^a \]
Let \( \ast T^a = \frac{1}{2} \ast (F^a \tilde{F}) \in \text{sec} \bigwedge^3 T^* M \hookrightarrow \text{sec} Cl(M, \eta) \) be the energy-momentum densities of a free electromagnetic field configuration \( F \in \text{sec} \bigwedge^2 T^* M \hookrightarrow \text{sec} Cl(M, \eta) \) \((\partial F = 0)\). As we already know, we have
\[ - \delta T^a = \partial \ast T^a = 0. \quad (55) \]
Eq. (55) is equivalent to \( d \ast T^a = 0 \) and since we are in Minkowski spacetime there must exist \( W^a \in \text{sec} \bigwedge^2 T^* M \hookrightarrow \text{sec} Cl(M, \eta) \) such that
\[ - T^a = \delta W^a. \quad (56) \]
Of course, we must also have
\[ d W^a = \ast M^a, \quad (57) \]
for some \( M^a \in \text{sec} \bigwedge^1 T^* M \hookrightarrow \text{sec} Cl(M, \eta) \). Eqs. (56) and (57) may be written as \( \partial W^a = \ast M^a + T^a \). In another publication [14] we determine the explicit form of the \( W^a \) and the \( M^a \).

C Standard Cylinder \( N \) in Minkowski Spacetime and its Boundary Submanifolds

Let \( N \) be the standard cylinder (Figure 1 at the end of the paper) [22] in Minkowski spacetime described in the Einstein-Lorentz-Poincaré coordinates \( \{x^\mu\} \) naturally adapted to an inertial frame \( I = \partial/\partial x^0 \) by
\[ N = \left\{ (x^0, x^1, x^2, x^3) \mid \sum_{i=1}^{3} x^i x^i \leq r', 0 \leq x^0 \leq t \right\} \quad (58) \]
The boundary manifolds of \( N \) are the following submanifolds of \( M \),
\[ B'_1 = \left\{ (0, x^1, x^2, x^3) \mid \sum_{i=1}^{3} x^i x^i < r' \right\} \]
\[ B'_2 = \left\{ (t, x^1, x^2, x^3) \mid \sum_{i=1}^{3} x^i x^i < r' \right\} \]
\[ B'_3 = \left\{ (x^0, x^1, x^2, x^3) \mid \sum_{i=1}^{3} x^i x^i = r', 0 < x^0 < t \right\} \]
\[ C'_1 = \left\{ (0, x^1, x^2, x^3) \mid \sum_{i=1}^{3} x^i x^i = r' \right\} \]
\[ C'_2 = \left\{ (t, x^1, x^2, x^3) \mid \sum_{i=1}^{3} x^i x^i = r' \right\}, \quad (59) \]
where $B'_3$ is a timelike hypersurface and the other four are spacelike hypersurfaces. We define also the manifolds $B_1 \subset B'_1$ and $B_2 \subset B'_2$

\[
B_1 = \left\{ (0, x^1, x^2, x^3) \mid \sum_{i=1}^{3} x^i x^i < r_1, r_1 << r' \right\},
\]

\[
B_2 = \left\{ (t, x^1, x^2, x^3) \mid \sum_{i=1}^{3} x^i x^i < r_2, r_2 << r' \right\},
\]

which contain respectively (see Figure 1) the field configurations $F(0, x) = F_1(0, x) + F_2(0, x)$ and $F(t, x) = F_1(t, x) + F_2(t, x)$.

We denote the interior of $N$ by $U'$ and also introduce the submanifold $U \subset U'$ (Figure 1). Table 1 collects \[22\] the main features of the above submanifolds, necessary for the integrations (appearing in Stokes theorem) performed in the main text.

| Submanifold | Topology | Orientation | Closure | Causal Character |
|-------------|----------|-------------|---------|-----------------|
| $U$         | $\mathbb{R}^4$ | from $M$     | $U^- = U \bigcup_{i=1}^{2} B_i \bigcup_{i=1}^{2} C_i$ | timelike         |
| $U'$        | $\mathbb{R}^4$ | from $M$     | $U'^- = U' \bigcup_{i=1}^{2} B'_i \bigcup_{i=1}^{2} C'_i$ | timelike         |
| $B_i$ ($i = 1, 2$) | $\mathbb{R}^3$ | from $U$     | $B_i^- = B_i \bigcup_{i=1}^{2} C_i$ | spacelike        |
| $B'_i$ ($i = 1, 2$) | $\mathbb{R}^3$ | from $U'$     | $B'_i^- = B'_i \bigcup_{i=1}^{2} C'_i$ | spacelike        |
| $B_3$       | $\mathbb{R} \times S^2$ | from $U$     | $B_3^- = B_3 \bigcup_{i=1}^{2} C_i \bigcup_{i=1}^{2} C_2$ | timelike        |
| $B'_3$      | $\mathbb{R} \times S^2$ | from $U'$     | $B'_3^- = B'_3 \bigcup_{i=1}^{2} C'_i \bigcup_{i=1}^{2} C_2$ | timelike        |
| $C_i$ ($i = 1, 2$) | $S^2$ | from $B_i$, not $B_3$ | $C_i^- = C_i$ | spacelike        |
| $C'_i$ ($i = 1, 2$) | $S^2$ | from $B'_i$, not $B'_3$ | $C'_i^- = C'_i$ | spacelike        |

Table 1. Main Features of the Submanifolds $N, B'_i, B_i, C'_i$ and $C_i$

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Figure 1: Standard Cylinder $N$ in Minkowski Spacetime, its Boundary Manifolds and the Field Configurations $F_1(0,x)$, $F_2(0,x)$ and $F(t,x) = F_1(t,x) + F_2(t,x)$