A note on the Brush Numbers of Mycielski Graphs, \( \mu(G) \)

(Johan Kok, Susanth C, Sunny Joseph Kalayathankal)

Abstract

The concept of the brush number \( b_r(G) \) was introduced for a simple connected undirected graph \( G \). The concept will be applied to the Mycielskian graph \( \mu(G) \) of a simple connected graph \( G \) to find \( b_r(\mu(G)) \) in terms of an optimal orientation of \( G \). We prove a surprisingly simple general result for simple connected graphs on \( n \geq 2 \) vertices, namely:

\[
b_r(\mu(G)) = b_r(\mu^\rightarrow(G)) = 2 \sum_{i=1}^{n} d_{\mu^\rightarrow(G)}^r(v_i).
\]

Keywords: Brush number, Mycielskian graph

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1 Introduction

For a general reference to notation and concepts of graph theory see [1]. For ease of self-containeness we shall briefly introduce the concepts of brush numbers and Mycielskian graphs.

Affiliation of author:
Johan Kok (Tshwane Metropolitan Police Department), City of Tshwane, Republic of South Africa
e-mail: kokkiek2@tshwane.gov.za

Susanth C (Department of Mathematics, Vidya Academy of Science and Technology), Thalakkottukara, Thrissur-680501, Republic of India
e-mail: susanthc@yahoo.com

Sunny Joseph Kalayathankal (Department of Mathematics, Kuriakose Elias College), Mannanam, Kottayam-686561, Kerala, Republic of India
e-mail: sunnyjoseph2014@yahoo.com
1.1 The brush number of a graph $G$

The concept of the brush number $b_r(G)$ of a simple connected graph $G$ was introduced by McKeil [3] and Messinger et. al. [5]. The problem is initially set that all edges of a simple connected undirected graph $G$ is dirty. A finite number of brushes, $\beta_G(v) \geq 0$ is allocated to each vertex $v \in V(G)$. Sequentially any vertex which has $\beta_G(v) \geq d(v)$ brushes allocated may clean the vertex $v$ and send exactly one brush along a dirty edge and in doing so allocate an additional brush to the corresponding adjacent vertex (neighbour). The reduced graph $G' = G - vu_{v \in E(G), \beta_G(v) \geq d(v)}$ is considered for the next iterative cleaning step. Note that a neighbour of vertex $v$ in $G$ say vertex $u$, now have $\beta_{G'}(u) = \beta_G(u) + 1$.

Clearly for any simple connected undirected graph $G$ the first step of cleaning can begin if and only if at least one vertex $v$ is allocated, $\beta_G(v) = d(v)$ brushes. The minimum number of brushes that is required to allow the first step of cleaning to begin is, $\beta_G(u) = d(u) = \delta(G)$. Note that these conditions do not guarantee that the graph will be cleaned. The conditions merely assure at least the first step of cleaning.

If a simple connected graph $G$ is orientated to become a directed graph, brushes may only clean along an out-arc from a vertex. Cleaning may initiate from a vertex $v$ if and only if $\beta_G(v) \geq d^+(v)$ and $d^-(v) = 0$. The order in which vertices sequentially initiate cleaning is called the cleaning sequence in respect of the orientation $\alpha_i$. The minimum number of brushes to be allocated to clean a graph for a given orientation $\alpha_i(G)$ is denoted $b_r^{\alpha_i}$. If an orientation $\alpha_i$ renders cleaning of the graph undoable we define $b_r^{\alpha_i} = \infty$. An orientation $\alpha_i$ for which $b_r^{\alpha_i}$ is a minimum over all possible orientations is called optimal.

Now, since the graph $G$ having $\epsilon(G)$ edges can have $2^{\epsilon(G)}$ orientations, the optimal orientation is not necessary unique. Let the set $\mathcal{A} = \{\alpha_i | \alpha_i \text{ an orientation of } G\}$.

**Lemma 1.1.** For a simple connected directed graph $G$, we have that:

$$b_r(G) = \min_{\alpha_i \in \mathcal{A}} \left( \sum_{v \in V(G)} \max \{0, d^+(v) - d^-(v)\} \right) = \min_{\alpha_i} b_r^{\alpha_i}.$$  

**Proof.** See [7]. \hfill $\Box$

Although we mainly deal with simple connected graphs it is easy to see that for set of simple connected graphs $\{G_1, G_2, G_3, ..., G_n\}$ we have that, $b_r(\cup_{i=1}^n G_i) = \sum_{i=1}^n b_r(G_i)$. 

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1.2 Mycielskian graph $\mu(G)$ of a graph, $G$

Mycielski [6] introduced an interesting graph transformation in 1955. The transformation can be described as follows:

1. Consider any simple connected graph $G$ on $n \geq 2$ vertices labelled $v_1, v_2, v_3, ..., v_n$ and edge set $E(G)$.
2. Consider the extended vertex set $V(G) \cup \{x_1, x_2, x_3, ..., x_n\}$ and add the edges $\{v_ix_j, v_jx_i\}$ iff $v_iv_j \in E(G)$.
3. Add one more vertex $w$ together with the edges $\{wx_i|\forall i\}$.

The transformed graph (Mycielskian graph of $G$ or Mycielski $G$) denoted $\mu(G)$, is the simple connected graph with $V(\mu(G)) = V(G) \cup \{x_1, x_2, x_3, ..., x_n\} \cup \{w\}$ and $E(\mu(G)) = E(G) \cup \{v_ix_j, v_jx_i\}$ iff $v_iv_j \in E(G) \cup \{wx_i|\forall i\}$.

2 Brush Numbers of Mycielskian Graphs

In general we have that if $\beta_G(v)$ at a particular cleaning step has $\beta_G'(v) > d_G(v)$, exactly $\beta_G'(v) - d_G(v)$ brushes are left redundant and can clean along new edges linked to vertex $v$ if such are added through transformation of the graph $G$. It is known that for $b_r(G)$ an optimal orientation exists and brushes may only clean along out-arcs of a vertex. Construct the following directed Mycielskian graph of $G$, denoted $\mu^\rightarrow(G)$.

1. Consider any simple connected graph $G$ on $n \geq 2$ vertices labelled $v_1, v_2, v_3, ..., v_n$ and edge set $E(G)$.
2. Orientate $G$ corresponding to an optimal orientation associated with $b_r(G)$, denoted $G^\rightarrow_{b_r(G)}$.
3. Consider the extended vertex set $V(G) \cup \{x_1, x_2, x_3, ..., x_n\}$ and add the arcs $\{(v_i, x_j), (v_j, x_i)\}$ iff $v_iv_j \in E(G)$.
3. Add one more vertex $w$ together with the arcs $\{(x_i, w)|\forall i\}$.

Knowing that after adding an edge $e$ (or arc) to a graph $G$ we have $b_r(G + e) \geq b_r(G)$ enables us to determine the brush number of the directed Mycielskian graph, $\mu^\rightarrow(G)$.

Theorem 2.1. (Tshegofatso’s theorem) For a simple connected graph $G$ on, $n \geq 2$ vertices the brush number of the Mycielskian graph of $G$ is given by:
Proof. Allocating the \( b_r(G) \) brushes to the corresponding vertices of \( G \) implies that the same allocations to \( G_{b_r(G)} \) will ensure cleaning \( G_{b_r(G)} \) with minimum brushes. Now consider the directed Mycielski \( G, \mu^\rightarrow(G) \).

Consider any vertex \( v \in V(G) \). Note that \( d_{G_{b_r(G)}}(v) = d_{G_{b_r(G)}}(v) + d_{G_{b_r(G)}}(v) \).

**Case 1:** Assume \( d_{G_{b_r(G)}}(v) = d_{G_{b_r(G)}}(v) \). Clearly zero brushes are initially allocated to \( v \) and at some iterative cleaning step exactly \( d_{G_{b_r(G)}}(v) \) brushes reaches \( v \). These brushes will exit from \( v \) along the \( d_{G_{b_r(G)}}(v) \) arcs if and only if a minimum of \( d_{G_{b_r(G)}}(v) = d_{G_{b_r(G)}}(v) + d_{G_{b_r(G)}}(v) = 2d_{G_{b_r(G)}}(v) \) brushes are added to \( v \) to clean the \( 2d_{G_{b_r(G)}}(v) \) arcs linking \( v \) with \( 2d_{G_{b_r(G)}}(v) \) vertices \( x_i \in \{ x_1, x_2, x_3, ..., x_n \} \).

So it follows that for all vertices satisfying this case we have the partial minimum sum of brushes, \( \sum_{v \in V(G)} d_{G_{b_r(G)}}(v) = d_{G_{b_r(G)}}(v) d_{G_{b_r(G)}}(v) \).

**Case 2:** Assume \( d_{G_{b_r(G)}}(v) < d_{G_{b_r(G)}}(v) \). Clearly a minimum of \( d_{G_{b_r(G)}}(v) - d_{G_{b_r(G)}}(v) \) brushes must be added to \( v \) to clean all out-arcs from \( v \) in \( G^\rightarrow \). In addition a minimum of \( d_{G_{b_r(G)}}(v) + 2(d_{G_{b_r(G)}}(v) - d_{G_{b_r(G)}}(v)) \) brushes must be allocated to \( v \) to clean the \( d_{G_{b_r(G)}}(v) - d_{G_{b_r(G)}}(v) \) arcs linking \( v \) with vertices \( x_i \in \{ x_1, x_2, x_3, ..., x_n \} \).

It follows that the minimum number of additional brushes is given by:

\[
2(d_{G_{b_r(G)}}(v) + d_{G_{b_r(G)}}(v)) + 2(d_{G_{b_r(G)}}(v) - d_{G_{b_r(G)}}(v)) = 2d_{G_{b_r(G)}}(v).
\]

So it follows that for all vertices satisfying this case we have the partial minimum sum.
of brushes, $2 \sum_{v \in V(G)} d^-_{G_{br}(G)}(v) < d^+_{G_{br}(G)}(v) d^+_{G_{br}(G)}(v)$.

**Case 3:** Assume $d^-_{G_{br}(G)}(v) > d^+_{G_{br}(G)}(v)$. The proof follows similar to Case 2.

Since all cases have been settled and all vertices are accounted for, the result:

$$b_r(\mu(G)) = b_r(\mu^->(G)) = 2 \sum_{v \in V(G)} d^-_{G_{br}(G)}(v) = d^+_{G_{br}(G)}(v) +$$

$$2 \sum_{v \in V(G)} d^-_{G_{br}(G)}(v) < d^+_{G_{br}(G)}(v) d^+_{G_{br}(G)}(v) + 2 \sum_{v \in V(G)} d^-_{G_{br}(G)}(v) > d^+_{G_{br}(G)}(v) d^+_{G_{br}(G)}(v) =$$

$$2 \sum_{i=1}^{n} d^+_{G_{br}(G)}(v_i)$$

follows conclusively.

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2 The reader can formalise the proof of Case 3.