HYPERBOLIC L-SPACE KNOTS AND THEIR FORMAL SEMIGROUPS

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Abstract. For an L-space knot, the formal semigroup is defined from its Alexander polynomial. It is not necessarily a semigroup. That is, it may not be closed under addition. There exists an infinite family of hyperbolic L-space knots whose formal semigroups are semigroups generated by three elements. In this paper, we give the first infinite family of hyperbolic L-space knots whose formal semigroups are semigroups generated by five elements.

1. Introduction

A knot in the 3–sphere is called an L–space knot if it admits a Dehn surgery yielding an L–space. An L–space Y is a rational homology 3-sphere with the simplest Heegaard Floer homology, that is, rank $\widehat{HF}(Y) = |H_1(Y; \mathbb{Z})|$. Typical examples are the knots admitting lens space surgeries, such as torus knots. There are several known constraints for L–space knots [25, 26]. Such a knot $K$ is fibered, and its Alexander polynomial $\Delta_K(t)$ has the form of

$$\Delta_K(t) = 1 - t^{a_1} + t^{a_2} - \cdots + t^{a_{2k}},$$

where $0 < a_1 < a_2 < \cdots < a_{2k}$ for some $k$, and $a_{2k}$ equals to twice of the knot genus. Also, it is known that $a_1 = 1$ by [11]. In general, it seems that there remained little to be clarified about the distribution of $a_i$. See [32, 33] for lens space surgery case.

Wang [35] introduced the formal semigroup $S$ for an L–space knot $K$. It is a set of nonnegative integers defined from the formal power series expansion

$$\frac{\Delta_K(t)}{1-t} = \sum_{s \in S} t^s \in \mathbb{Z}[[t]].$$

The form of (1.1) implies that $0 \in S$. Hence, if a formal semigroup is a semigroup, then it is a monoid. Nevertheless, we use the term “formal semigroup” in deference to previous research.

Essentially, [29, 37] discuss the same notion precedentely. In [29], it is called the support of the Turaev torsion. We remark that $\Delta_K(t)/(t-1)$ is usually called the Reidemeister–Milnor torsion or Turaev torsion in literatures.

For example, a torus knot of type $(2, 7)$ has the Alexander polynomial $1 - t + t^2 - t^3 + t^4 - t^5 + t^6$. Hence $S = \{0, 2, 4\} \cup \mathbb{Z}_{\geq 6}$. Another typical example is the $(-2, 3, 7)$–pretzel knot. Its Alexander polynomial is $1 - t + t^3 - t^4 + t^5 - t^6 + t^7 - t^9 + t^{10}$, so $S = \{0, 3, 5, 7, 8\} \cup \mathbb{Z}_{\geq 10}$. (In this paper, we use the notation $\mathbb{Z}_{\geq m}$ for the set of integers bigger than or equal to $m$.)

In general, a formal semigroup is a subset of $\mathbb{Z}_{\geq 0}$, and we use the addition as the binary operation. As seen from the above example, a formal semigroup is not

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necessarily closed under the addition. It is an easy and known fact that a torus knot of type \((p, q)\) \((p, q > 0)\) has the formal semigroup \(\langle p, q \rangle = \{ap + bq \mid a, b \geq 0\}\), which is a semigroup (see [7], Example 1.10). Also, the formal semigroup of an iterated torus L–space knot is a semigroup [35]. However, this is not the case for the \((-2, 3, 7)\)–pretzel knot, because \(3 \in S\) but \(6 \notin S\). More generally, it is straightforward to verify that any hyperbolic Montesinos L–space knot has the formal semigroup which is not a semigroup. Because such a knot is known to be the \((-2, 3, 2n + 1)\)–pretzel knot with \(n \geq 3\) by [4], and its Alexander polynomial given by [12] immediately implies that \(3 \in S\) but \(6 \notin S\). Also, we checked that most of hyperbolic Berge knots have formal semigroups which are not semigroups. Hence it is not too much to say that the formal semigroup of a hyperbolic L–space knot is less likely to become an actual semigroup.

In [35, Question 2.8], Wang asked if there exists an L–space knot which is not an iterated torus knot and whose formal semigroup is a semigroup. As explained in [2], the author found two hyperbolic L–space knots \(K_{8,201}\) and \(K_{9,449}\) whose formal semigroups are actual semigroups. Indeed, they are the only knots among 630 hyperbolic L–space knots listed by Dunfield. (This list is found in [1].) There were two unclassified knots in Dunfield’s data, but they are confirmed to be L–space knots by [3]. The formal semigroups of these two are not semigroups.) On the other hand, Baker and Kegel [2] show that \(K_{9,449}\) is the only knot among Dunfield’s list that is not the closure of a positive braid, and try to generalize it to an infinite family of hyperbolic L–space knots \(\{K_n\}\), where \(K_1 = K_{9,449}\). The author also found that their knots give the first infinite family of hyperbolic L–space knots whose formal semigroups are semigroups. More precisely, the formal semigroup of \(K_n\) is \(\langle 4, 4n + 2, 4n + 5 \rangle\). See [2] for more details.

For a finite set of positive integers \(\{p_1, p_2, \ldots, p_k\}\),

\[
\langle p_1, p_2, \ldots, p_k \rangle = \{a_1p_1 + a_2p_2 + \cdots + a_kp_k \mid a_i \in \mathbb{Z}_{\geq 0}\}
\]

is a semigroup under the addition. Since we include 0 as a coefficient, the identity element 0 is excluded from generators. For such a semigroup, the rank is defined to be the minimal cardinality of a generating set.

Thus, the formal semigroup of the hyperbolic L–space knot \(K_n\) in [2] is a semigroup of rank three. On the other hand, the cabling formula of [35] implies that the semigroup of an iterated torus L–space knot can have arbitrarily high rank. Then it is natural to ask the following.

**Question 1.1.** Does there exist a hyperbolic L–space knot whose formal semigroup is a semigroup with arbitrarily high rank?

The purpose of the present paper is to construct a new family of hyperbolic L–space knots whose formal semigroups are semigroups of rank 5.

**Theorem 1.2.** There exists an infinite family of hyperbolic L–space knots whose formal semigroups are semigroups of rank 5.

The formal semigroup of an L–space knot is related to its knot Floer complex (see [16]). However, the meaning of closedness under addition in the formal semigroup seems to be missing.
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Figure 1. The braid $\beta_n$. The knot $K_n$ is the closure of $\beta_n$.

2. THE FAMILY OF HYPERBOLIC L–SPACE KNOTS

For any integer $n \geq 1$, let $\beta_n$ be the 6–braid defined as

$$\beta_n = (\sigma_3 \sigma_2 \sigma_1 \sigma_3 \sigma_2 \sigma_1 \sigma_3)^{2n+1} \sigma_3 \sigma_2 \sigma_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \sigma_2,$$

where $\sigma_i$ is the standard generator in the 6–strand braid group. See Fig. 1. Let $K_n$ be the knot obtained as the closure of $\beta_n$.

Since $\beta_n$ is positive, $K_n$ is fibered ([31]), and its genus $g(K_n)$ is $9n + 7$ as seen from an Euler characteristic calculation

$$\text{(number of strand)} - \text{(number of crossing)} = 6 - (9(2n + 1) + 10) = 1 - 2g(K_n).$$

We remark that $\beta_0$ can be defined, but $K_0$ is the $(2,11)$–cable of the trefoil. Hence this is excluded from our interest.

Theorem 1.2 is the direct consequence of the next theorem.

Theorem 2.1. For $n \geq 1$, let $K_n$ be the knot defined as above. Then we have:

1. $K_n$ is a hyperbolic L–space knot;
2. its formal semigroup is a semigroup $\langle 6, 6n+4, 6n+8, 12n+11, 12n+15 \rangle$.

The proof of Theorem 2.1 is divided in the remaining sections. In Section 3 we prove that $(18n+22)$–surgery on $K_n$ yields an L–space by using the Montesinos trick. In Section 4 we calculate the Alexander polynomial and the formal semigroup. Finally, we prove that $K_n$ is hyperbolic in Section 5.

3. MONTESINOS TRICK AND L–SPACE SURGERY

In this section, we prove that each $K_n$ admits a Dehn surgery yielding an L–space by using the Montesinos trick. The Montesinos trick is the standard tool introduced by [21]. For a strongly invertible link $L$ in $S^3$, the resulting manifold by Dehn surgery on $L$ is described as the double branched cover of some link $\ell$. The surgery corresponds to a tangle replacement. For details, see [24, 36].

For $K_n$, Fig. 2 shows a surgery diagram of $K \cup C_1 \cup C_2$, where performing $-1/n$–surgery on $C_1$ and $1/2n$–surgery on $C_2$ changes $K$ to $K_n$. This diagram can be changed into a strongly invertible position as illustrated in Fig. 3. We remark that $r$–surgery on $K$ corresponds to $(18n+r)$–surgery on $K_n$.

Theorem 3.1. For $n \geq 1$, $(18n+22)$–surgery on $K_n$ yields an L–space.

Proof. For the link in Fig. 3 take a quotient by the involution around the axis. Note that the surgery coefficient on $K$ is 22. In the diagram of Fig. 3 the blackboard framing (or writhet) of $K$ is 19. Hence, 22–surgery on $K$ corresponds to the tangle replacement by the 3–tangle.
Figure 2. A surgery diagram $K \cup C_1 \cup C_2$. Performing $-1/n$–surgery on $C_1$ and $1/2n$–surgery on $C_2$ changes $K$ to our $K_n$.

Figure 3. A strongly invertible position of the link $K \cup C_1 \cup C_2$ with an axis.

Rational tangle replacements corresponding to the surgeries on $K, C_1, C_2$ yield a link $\ell$, whose double branched cover is the result of $(18n + 22)$–surgery on $K_n$. Figures 4, 5, 6 and 7 show the deformation of $\ell$.

In the next lemma, we prove that the double branched cover of the link $\ell$ is an L–space, so the proof is complete. □

**Lemma 3.2.** The double branched cover of the link $\ell$ is an L–space.

**Proof.** First, det $\ell = 18n + 22$. This is easily calculated from the Goeritz matrix for the checkerboard coloring of the diagram of $\ell$ in Fig. 7. Let $c$ be the crossing
Figure 4. A deformation of $\ell$, where a rectangle box means horizontal half-twists. The integer indicates the number of half-twists, which is right-handed if it is positive, or left-handed otherwise.

Figure 5. Continued from Figure 4.

as indicated in Fig. 7. Then we have $\ell_0$ and $\ell_\infty$ by smoothing $c$ as shown in Fig. 8 and it is also a direct calculation to see $\det \ell_0 = 4n + 5$ and $\det \ell_\infty = 14n + 17$ from Figs. 9 and 10. Hence $\det \ell = \det \ell_0 + \det \ell_\infty$ holds.

By [27, Proposition 2.1], the triple of thee double branched covers of $\ell$, $\ell_0$ and $\ell_\infty$ forms a triad. Thus [26, Proposition 2.1] claims that if the double branched covers of $\ell_0$ and $\ell_\infty$ are L–spaces, then so is the double branched cover of $\ell$.

We can see that $\ell_0$ is the Montesinos knot $M(-\frac{n+1}{2n+1}, \frac{1}{2}, \frac{2}{4n-1})$ as shown in Fig. 9. When $n = 1$, $\ell_0$ is the knot $8_20$, which is non-alternating, but quasi-alternating. Then the double branched cover is an L–space by [27]. If $n > 1$, then $\ell_0$ is not quasi-alternating by [14]. Nevertheless, we can show the following.
Claim 3.3. The double branched cover of $\ell_0$ is an L-space.

Proof of Claim 3.3. The double branched cover of $\ell_0$ is the Seifert fibered space $M(0; -\frac{n+1}{2n+1}, \frac{1}{2}, \frac{2}{4n-1}) = M(-1; \frac{n}{2n+1}, \frac{1}{2}, \frac{2}{4n-1})$. (We use the convention of [20], which is the same as [2].)

Theorem 1.1 of [20] combined with [19] claims that such a Seifert fibered space $M(-1; r_1, r_2, r_3)$ ($1 \geq r_1 \geq r_2 \geq r_3 > 0$) is an L-space if and only if there are no relatively prime integers $m > a > 0$ such that $mr_1 < a < m(1-r_2)$ and $mr_3 < 1$.

First, assume $n = 1$. Then $r_1 = 2/3$, $r_2 = 1/2$ and $r_3 = 1/3$, so $1 - r_2 = 1/2$. Since $r_1 > 1 - r_2$, we have no solution $m$ satisfying $mr_1 < m(1 - r_2)$. 
Suppose $n \geq 2$. Then $r_1 = 1/2$, $r_2 = n/(2n + 1)$ and $r_3 = 2/(4n - 1)$, so $1 - r_2 = (n + 1)/(2n + 1)$.
Figure 11. The resolution $\ell_{\infty 0}$ is a Montesinos link.

We assume that there are coprime integers $m$ and $a$ such that $m/2 < a < m(n+1)/(2n+1)$ and $2m/(4n-1) < 1$. Then the first gives

$$0 < 2a - m < \frac{m}{2n+1},$$

and the second gives $m < 2n - 1/2$. Combining these yields

$$0 < 2a - m < \frac{4n-1}{4n+2} < 1.$$ 

Since $a$ and $m$ are integers, this is a contradiction. □

For the other resolution $\ell_{\infty}$, we further perform two smoothings at the crossing $d$ as shown in Fig. 10. Then we have a link $\ell_{\infty 0}$ and a knot $\ell_{\infty \infty}$ as shown there. In particular, a direct calculation on Fig. 10 (or, Figs. 11 and 12) shows that

$$\det \ell_{\infty 0} = 4n + 14$$

and $\det \ell_{\infty \infty} = 10n + 3$. Hence $\det \ell_{\infty} = \det \ell_{\infty 0} + \det \ell_{\infty \infty}$ holds.

Claim 3.4. The double branched covers of $\ell_{\infty 0}$ and $\ell_{\infty \infty}$ are $L$-spaces.

Proof of Claim 3.4. The link $\ell_{\infty 0}$ is the Montesinos link $M(\frac{1}{2n+1}, -\frac{1}{2}, \frac{2n+3}{4n+8})$ as shown in Fig. 11. Although this is not quasi-alternating by [14], we can show that the double branched cover is an $L$-space as before. The double branched cover is the Seifert fibered space $M(0; \frac{1}{2n+1}, -\frac{1}{2}, \frac{2n+3}{4n+8}) = M(-1; \frac{1}{2n+1}, \frac{1}{2}, \frac{2n+3}{4n+8})$. As in the proof of Claim 3.3, set $r_1 = 1/2$, $r_2 = (2n+3)/(4n+8)$ and $r_3 = 1/(2n+1)$.

Suppose that there are coprime integers $m$ and $a$ such that $mr_1 < a < m(1-r_2)$ and $mr_3 < 1$. Then

$$0 < 2a - m < \frac{m}{2n+4} < \frac{2n+1}{2n+4} < 1,$$

a contradiction.
The link $\ell_\infty$ is the Montesinos knot $M(-\frac{1}{2}, \frac{n}{2n+1}, \frac{5}{10n+7})$ as shown in Fig. 12, which is not quasi-alternating by [14] again. The double branched cover is the Seifert fibered space $M(0; -\frac{1}{2}, \frac{n}{2n+1}, \frac{5}{10n+7}) = M(-1; \frac{1}{2}, \frac{n}{2n+1}, \frac{5}{10n+7})$.

Set $r_1 = 1/2$, $r_2 = n/(2n+1)$ and $r_3 = 5/(10n+7)$. Suppose that there are coprime integers $m$ and $a$ as above. Then

$$0 < 2a - m < \frac{m}{2n+1} < \frac{10n+7}{10n+5}.$$ 

Hence $2a - m = 1$.

Then $a < m(1 - r_2)$ implies $2n + 1 < m$. Combining with $mr_3 < 1$ gives

$$10n + 5 < 5m < 10n + 7,$$

which is impossible. □

By [26] Proposition 2.1 and [27] Proposition 2.1, the double branched cover of $\ell_\infty$ is an L-space, so is that of $\ell$. □

We remark that computer experiments suggest that the knot $K_n$ does not admit a nontrivial exceptional surgery. This situation brings us a difficulty to find candidates of slopes for L-space surgeries. Also, since $K_n$ has genus $9n + 7$, if $r$–surgery on $K_n$ yields an L-space, then $r \geq 2(9n + 7) - 1 = 18n + 13$ by [28]. In fact, it is known that any $r$ ($\geq 18n + 13$) yields an L-space. We selected the slope $18n + 22$ for our proof, but there might be a better slope for a proof.

4. ALEXANDER POLYNOMIALS AND FORMAL SEMIGROUPS

In this section, we calculate the Alexander polynomial $\Delta_{K_n}(t)$ of $K_n$, and its formal semigroup. For the former, we mimic the argument in [2] [6].
Figure 13. A modified surgery diagram of \( L = K \cup C_1 \cup C_2 \). This is simpler than the previous link in Figure 2.

**Theorem 4.1.** The Alexander polynomial of \( K_n \) is given as

\[
\Delta_{K_n}(t) = t^{6n+4} + \sum_{i=0}^{n} (A_1 + A_2 + A_3 + A_4 + A_5),
\]

where

\[
\begin{align*}
A_1 &= t^{6(n-i)} - t^{6(n-i)+1}, \\
A_2 &= t^{6(n+i)+6} - t^{6(n+i)+5}, \\
A_3 &= t^{6(n+i)+8} - t^{6(n+i)+7}, \\
A_4 &= t^{6(n+i)+10} - t^{6(n+i)+9}, \\
A_5 &= t^{6(2n+i)+14} - t^{6(2n+i)+13}.
\end{align*}
\]

**Proof.** Let \( L = K \cup C_1 \cup C_2 \) be the oriented link as shown in Fig. 13. We remark that this is modified from the link in Fig. 2 to reduce the number of crossing by changing the surgery coefficients. (This process is not critical, because we only need to calculate the multivariable Alexander polynomial.)

It has the multivariable Alexander polynomial

\[
\Delta_L(x, y, z) = (x^3 - 1)(x^5 y^4 z^2 + x^3 y^5 z^2 - x^3 y^4 z^2 + x^2 y^4 z^2 + x^4 y^2 z + x^3 y^3 z - x^3 y^2 z - x^2 y^3 z + x^2 y^2 z + x y^3 z + x^3 y - x^2 y + x^2 + y).
\]

(We used [15] for the calculation.)

Performing \(-1/(n+1)\)-surgery on \( C_1 \) and \( 1/(2n+2)\)-surgery on \( C_2 \) changes the link \( K \cup C_1 \cup C_2 \) to \( K_n \cup C_1^n \cup C_2^n \). Clearly, these links have homeomorphic exteriors. Hence the induced isomorphism of the homeomorphism on their homology groups relates the Alexander polynomials of two links. (See [9, 23].) Let \( \mu_K, \mu_{C_1}, \) and \( \mu_{C_2} \) be the homology classes of meridians of \( K, C_1 \) and \( C_2 \), respectively. We assume
that each (oriented) meridian has linking number one with the corresponding knot. Moreover, let \( \lambda_K, \lambda_{C_1} \), and \( \lambda_{C_2} \) be the homology classes of their oriented longitudes.

Similarly, we have homology classes of meridians, \( \mu_{K_n}, \mu_{C_1^n} \) and \( \mu_{C_2^n} \) of \( K_n, C_1^n \) and \( C_2^n \). Then we have

\[
\mu_{K_n} = \mu_K, \quad \mu_{C_1^n} = \mu_{C_1} - (n+1)\lambda_{C_1}, \quad \mu_{C_2^n} = \mu_{C_2} + (2n+2)\lambda_{C_2}.
\]

Since \( \lambda_{C_1} = 6\mu_K \) and \( \lambda_{C_2} = 3\mu_K \),

\[
\mu_{C_1^n} = \mu_{C_1} - 6(n+1)\mu_K, \quad \mu_{C_2^n} = \mu_{C_2} + 6(n+1)\mu_K.
\]

Thus

\[
\mu_{K_n} = \mu_K, \quad \mu_{C_1} = \mu_{C_1^n} + 6(n+1)\mu_K, \quad \mu_{C_2} = \mu_{C_2^n} - 6(n+1)\mu_K.
\]

Hence we have the relation between the Alexander polynomials as

\[
\Delta_{K_n\cup C_1^n\cup C_2^n}(x, y, z) = \Delta_L(x, yx^{6(n+1)}, zx^{-6(n+1)}).
\]

On the other hand, since \( \text{lk}(K_n, C_1^n) = \text{lk}(K, C_2) = 3 \), the Torres condition \([34]\) gives

\[
\Delta_{K_n\cup C_1^n\cup C_2^n}(x, y, 1) = (x^3y^0 - 1)\Delta_{K_n\cup C_1^n}(x, y) = (x^3 - 1)\Delta_{K_n\cup C_1^n}(x, y).
\]

Similarly, since \( \text{lk}(K_n, C_1^n) = \text{lk}(K, C_1) = 6 \),

\[
\Delta_{K_n\cup C_1^n}(x, 1) = \frac{x^6 - 1}{x - 1} \Delta_{K_n}(x).
\]

Thus,

\[
\Delta_{K_n}(x) = \frac{x - 1}{x^6 - 1} \Delta_{K_n\cup C_1^n}(x, 1) = \frac{x - 1}{(x^6 - 1)(x^3 - 1)} \Delta_{K_n\cup C_1^n\cup C_2^n}(x, 1, 1).
\]

Then \([42]\) gives

\[
\Delta_{K_n}(t) = \frac{t - 1}{(t^6 - 1)(t^3 - 1)} \Delta_L(t, t^{6(n+1)}, t^{-6(n+1)}) = \frac{t^2(t - 1)}{t^6 - 1} \left( t^{18n+19} + t^{12n+15} + t^{12n+11} + t^{6n+8} + t^{6n+4} + 1 \right) = \frac{t^{18n+19} + t^{12n+15} + t^{12n+11} + t^{6n+8} + t^{6n+4} + 1}{t^5 + t^4 + t^3 + t^2 + t + 1}.
\]

(Recall that \( \equiv \) means equivalence up to units.)

Next, we calculate

\[
(t^5 + t^4 + t^3 + t^2 + t + 1)(t^{6n+4} + \sum_{i=0}^n (A_1 + A_2 + A_3 + A_4 + A_5)).
\]

First,

\[
(t^5 + t^4 + t^3 + t^2 + t + 1)t^{6n+4} = t^{6n+9} + t^{6n+8} + t^{6n+7} + t^{6n+6} + t^{6n+5} + t^{6n+4}.
\]
Next,
\[(t^5 + t^4 + t^3 + t^2 + t + 1) \sum_{i=0}^{n} A_1 = (t^5 + t^4 + t^3 + t^2 + t + 1) \sum_{i=0}^{n} (t^{6(n-i)} - t^{6(n-i)+1})
= (t^5 + t^4 + t^3 + t^2 + t + 1) \sum_{i=0}^{n} t^{6(n-i)}(1-t)
= (1 - t^6) \sum_{i=0}^{n} t^{6(n-i)}
= \sum_{i=0}^{n} t^{6(n-i)} - \sum_{i=0}^{n} t^{6(n-i)+6}
= 1 - t^{6n+6}.
\]

Similarly,
\[(t^5 + t^4 + t^3 + t^2 + t + 1) \sum_{i=0}^{n} A_2 = (t^5 + t^4 + t^3 + t^2 + t + 1) \sum_{i=0}^{n} (t^{6(n+i)} - t^{6(n+i)+5})
= (t^6 - 1) \sum_{i=0}^{n} t^{6(n+i)+5}
= \sum_{i=0}^{n} t^{6(n+i)+11} - \sum_{i=0}^{n} t^{6(n+i)+5}
= t^{12n+11} - t^{6n+5},
\]
\[(t^5 + t^4 + t^3 + t^2 + t + 1) \sum_{i=0}^{n} A_3 = (t^5 + t^4 + t^3 + t^2 + t + 1) \sum_{i=0}^{n} (t^{6(n+i)+8} - t^{6(n+i)+7})
= (t^6 - 1) \sum_{i=0}^{n} t^{6(n+i)+7}
= \sum_{i=0}^{n} t^{6(n+i)+13} - \sum_{i=0}^{n} t^{6(n+i)+7}
= t^{12n+13} - t^{6n+7},
\]
\[(t^5 + t^4 + t^3 + t^2 + t + 1) \sum_{i=0}^{n} A_4 = (t^5 + t^4 + t^3 + t^2 + t + 1) \sum_{i=0}^{n} (t^{6(n+i)+10} - t^{6(n+i)+9})
= (t^6 - 1) \sum_{i=0}^{n} t^{6(n+i)+9}
= \sum_{i=0}^{n} t^{6(n+i)+15} - \sum_{i=0}^{n} t^{6(n+i)+9}
= t^{12n+15} - t^{6n+9},
\]
Proof. Let $Hence

Then $\text{semigroup } T$ of $n$ of $6, 6$ so $12$ $n \equiv 0 (\text{mod } 2)$, so $12$ $\Delta$ $12$, $− 1$

These show that (4.2) is equal to $T$ $\leq 4$ + $t$ $n$ $\Delta$ $t$ $n$ $\Delta$

Hence, as a formal power series, $\text{Theorem 4.3.}$ $\text{Proof.}$ $\text{Lemma 4.2.}$ For $n \geq 1$, let $T = \langle 6, 6n + 4, 6n + 8, 12n + 11, 12n + 15 \rangle$. The the semigroup $T$ has rank 5.

Proof. Let $G$ be a generating set of $T$. It suffices to show that $\{6, 6n + 4, 6n + 8, 12n + 11, 12n + 15\} \subset G$.

Since $6$ is the minimal nonzero element of $T$, we need $6 \in G$. Except the multiples of $6, 6n + 4$ is the minimal element of $T$ and $6n + 8$ is the next, so $6n + 4, 6n + 8 \in G$.

Assume $12n + 11 = 6a + (6n + 4)b + (6n + 8)c$ for $a, b, c \geq 0$. Then $1 \equiv 0$ (mod 2), so $12n + 11 \not\in \{6, 6n + 4, 6n + 8\}$. Hence $12n + 11 \in G$.

Finally, assume $12n + 15 = 6a + (6n + 4)b + (6n + 8)c + (12n + 11)d$ for $a, b, c, d \geq 0$. Then $1 \equiv d$ (mod 2), so $d \neq 0$. In fact, $d = 1$. We have $4 = 6a + (6n + 4)b + (6n + 8)c$.

Since $n \geq 1$, this is impossible. Hence $12n + 15 \in G$. □

Theorem 4.3. The formal semigroup $S$ of $K_n$ is a semigroup of rank 5:

$S = \langle 6, 6n + 4, 6n + 8, 12n + 11, 12n + 15 \rangle$.

Proof. By Theorem 4.1

$\Delta_{K_n}(t) = t^{6n+4} + \sum_{i=0}^{n} (A_1 + A_2 + A_3 + A_4 + A_5)$.

Hence, as a formal power series,

$\Delta_{K_n}(t) \frac{t^{6n+4}}{1-t} = \frac{t^{6n+4}}{1-t} + \sum_{i=0}^{n} \left( \frac{A_1}{1-t} + \frac{A_2}{1-t} + \frac{A_3}{1-t} + \frac{A_4}{1-t} + \frac{A_5}{1-t} \right)$

$= \sum_{j=0}^{\infty} t^j + \sum_{i=0}^{n} (t^{6(n-i)} - t^{6(n+i)+5} - t^{6(n+i)+7} - t^{6(n+i)+9} - t^{6(2n+i)+13})$

$= \sum_{j=0}^{\infty} t^{6n+4+j} + \sum_{i=0}^{n} t^{6(n-i)} - t^{6n+5} \sum_{i=0}^{n} (t^{6i} + t^{6i+2} + t^{6i+4}) - \sum_{i=0}^{n} t^{12n+13+6i}$.

Then

$S = \mathbb{Z}_{\geq 6n+4} \cup B - C - D$,

where $B = \{0, 6, 12, \ldots, 6n\}$, $C = \{6n+5, 6n+7, 6n+9, \ldots, 12n+5, 12n+7, 12n+9\}$ and $D = \{12n + 13, 12n + 19, \ldots, 18n + 13\}$. 


Let $\mathcal{T} = \langle 6, 6n + 4, 6n + 8, 12n + 11, 12n + 15 \rangle$. We need to show that $\mathcal{S} = \mathcal{T}$.

First, if $m \geq 18n + 14$, then $m \in \mathcal{S}$. Thus $\mathbb{Z}_{\geq 18n+14} \subseteq \mathcal{S}$. To show that $\mathbb{Z}_{\geq 18n+14} \subseteq \mathcal{T}$, it suffices to verify that $18n + 14, 18n + 15, \ldots, 18n + 19 \in \mathcal{T}$, since $6 \in \mathcal{T}$. This follows from

$$18n + 14 \equiv 6n + 8, 18n + 15 \equiv 12n + 15, 18n + 16 \equiv 6n + 4,$$

$$18n + 17 \equiv 12n + 11, 18n + 18 \equiv 6 \pmod{6},$$

$$18n + 19 = (6n + 8) + (12n + 11).$$

Next, the set $\mathbb{Z}_{<18n+14}$ of nonnegative integers less than $18n + 14$ is decomposed into the congruence classes of modulo 6, $\mathbb{Z}_0, \mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_5$, where $\mathbb{Z}_i = \{m \mid 0 \leq m < 18n + 14, m \equiv i \pmod{6}\}$. Then $B \subseteq \mathbb{Z}_0, D \subseteq \mathbb{Z}_1$, and $C \subseteq \mathbb{Z}_3 \cup \mathbb{Z}_4 \cup \mathbb{Z}_5$.

We examine each congruence class.

- $\mathbb{Z}_0 \subseteq \mathcal{S}$ and $\mathbb{Z}_0 \subseteq \mathcal{T}$. Thus $\mathbb{Z}_0 \cap \mathcal{S} = \mathbb{Z}_0 \cap \mathcal{T} = \mathbb{Z}_0$.
- $\mathbb{Z}_1 \cap \mathcal{S} = \emptyset$.
- $\mathbb{Z}_2 \cap \mathcal{S} = \{6n + 8, 6n + 14, \ldots, 18n + 8\} \subseteq \mathcal{T}$.
- $\mathbb{Z}_3 \cap \mathcal{S} = \{12n + 15, 12n + 21, \ldots, 18n + 9\} \subseteq \mathcal{T}$.
- $\mathbb{Z}_4 \cap \mathcal{S} = \{6n + 4, 6n + 10, \ldots, 18n + 10\} \subseteq \mathcal{T}$.
- $\mathbb{Z}_5 \cap \mathcal{S} = \{12n + 11, 12n + 17, \ldots, 18n + 11\} \subseteq \mathcal{T}$.

Hence $\mathcal{S} \cap \mathbb{Z}_{<18n+14} \subseteq \mathcal{T}$.

Conversely, let $m \in \mathbb{Z}_1$. If $m \in \mathcal{T}$, then we need to use $12n + 11$ or $12n + 15$ to yield $m$. Since $m \leq 18n + 13$, either

1. $m = (12n + 11) + r$ and $r \leq 6n + 2$, $r \equiv 2 \pmod{6}$, or
2. $m = (12n + 15) + r$ and $r \leq 6n - 2$, $r \equiv 4 \pmod{6}$.

However, there is no $r \in \mathcal{T}$ satisfying these. Hence $\mathbb{Z}_1 \cap \mathcal{T} = \emptyset$.

Let $m \in \mathbb{Z}_2 \cap \mathcal{T}$. If $m < 6n + 8$, then $m \leq 6n + 2$. So, there is no such element in $\mathcal{T}$. Hence $\mathbb{Z}_2 \cap \mathcal{S} = \mathbb{Z}_2 \cap \mathcal{T}$.

Similarly, let $m \in \mathbb{Z}_4 \cap \mathcal{T}$. If $m < 6n + 4$, then $m \leq 6n - 2$. But there is no such element in $\mathcal{T}$. Hence $\mathbb{Z}_4 \cap \mathcal{S} = \mathbb{Z}_4 \cap \mathcal{T}$.

Let $m \in \mathbb{Z}_3 \cap \mathcal{T}$. Again, we need to use $12n + 11$ or $12n + 15$ to yield $m$. Then, we have either

3. $m = (12n + 11) + r$ and $r \leq 6n - 2$ and $r \equiv 4 \pmod{6}$, or
4. $m = (12n + 15) + r$ and $r \leq 6n - 6$ and $r \equiv 0 \pmod{6}$.

For (3), there is no $r \in \mathcal{T}$. For (4), $r \in \{0, 6, 12, \ldots, 6n - 6\}$, so $m \in \mathbb{Z}_3 \cap \mathcal{S}$. Hence $\mathbb{Z}_3 \cap \mathcal{S} = \mathbb{Z}_3 \cap \mathcal{T}$.

Finally, let $m \in \mathbb{Z}_5 \cap \mathcal{T}$. Again, we have either

5. $m = (12n + 11) + r$ and $r \leq 6n$ and $r \equiv 0 \pmod{6}$, or
6. $m = (12n + 15) + r$ and $r \leq 6n - 4$ and $r \equiv 2 \pmod{6}$.

For (6), there is no $r \in \mathcal{T}$. For (5), $r \in \{0, 6, 12, \ldots, 6n\}$, so $m \in \mathbb{Z}_5 \cap \mathcal{S}$. Hence $\mathbb{Z}_5 \cap \mathcal{S} = \mathbb{Z}_5 \cap \mathcal{T}$.

5. **Hyperbolicity**

In this section, we prove that our knot $K_n$ is a hyperbolic knot for any $n \geq 1$ by using the fact that $K_n$ has tunnel number one.

**Lemma 5.1.** For $n \geq 1$, $K_n$ has tunnel number one, hence $K_n$ is prime.
Figure 14. The unknotting tunnel $\gamma$ and the series of isotopy of $N(K_n \cup \gamma)$.

Proof. Figure 14 shows an unknotting tunnel $\gamma$ for $K_n$. The series of isotopy as illustrated in Fig. 14 indicates that the outside of a regular neighborhood of $K_n \cup \gamma$ is a genus two handlebody. □

Theorem 5.2. For $n \geq 1$, $K_n$ is a hyperbolic knot.

Proof. By Theorem 4.3, the formal semigroup of $K_n$ is a semigroup of rank 5. Since the formal semigroup of a torus knot is a semigroup of rank two (see [7]), $K_n$ is not a torus knot.

Assume that $K_n$ is a satellite knot for a contradiction. By Lemma 5.1, $K_n$ has tunnel number one. Then Morimoto and Sakuma’s classification [22] tells us that $K_n$ has a torus knot $T(p, q)$ as its companion. Since $K_n$ has bridge number at most 6, the companion has bridge number at most three [30]. More precisely, either the companion is 3–bridge and the wrapping number of the pattern is two, or the companion is 2–bridge and the wrapping number is two or three.

By Theorem 3.1, $K_n$ is an L–space knot. Then [10] implies that the pattern knot is also an L–space knot. Furthermore, [5, Theorem 1.17] claims that the pattern is braided in the pattern solid torus. In particular, the wrapping number coincides with the winding number there.

We divide the argument into two cases.

Case 1. Suppose that $K_n$ has a companion $T(3, q)$ ($|q| > 3$) and a braided pattern knot $P$. Then the wrapping number and winding number of $P$ are equal to two. This means that $K_n$ is a 2–cable of $T(3, q)$. However, the cabling formula of [35] shows that its formal semigroup has rank three. Hence this case is impossible.

Case 2. Suppose that $K_n$ has a companion $T = T(2, q)$ ($|q| \geq 3$) and a braided pattern knot $P$. We may assume that $q > 0$ by taking the mirror image of $K_n$, if necessary. If the wrapping number is two, then $K_n$ is a 2–cable, which is a
contradiction again. Hence the wrapping number and the winding number are equal to three. For the Alexander polynomials, we have

\[ \Delta_{K_n}(t) = \Delta_T(t^3) \Delta_P(t) \]

(see [8]). Here, \( \Delta_T(t^3) = 1 - t^3 + t^6 - \cdots + t^{3(q-1)} \). Also, this implies \( g(K_n) = 3g(T) + g(P) \) (see [13, Lemma 2.6]).

By [17, Theorem 3.1], the only closed 3–braids which are L–space knots are torus knots and twisted torus knots \( T(3, t; 2, s) \) with \( ts > 0 \). Here, \( T(3, t; 2, s) \) is obtained from \( T(3, t) \) by adding \( s \) full twists on two adjacent strings. Again, a 3–cable is excluded.

We need to recall the construction of [22] of tunnel number one satellite knots. Let \( k_1 \cup k_2 \) be a 2–bridge link in \( S^3 \). We remark that each component \( k_i \) is unknotted. The exterior of \( k_2 \) is a solid torus \( J \) containing \( k_1 \) in its interior. Here, the longitude of \( J \) is the meridian of \( k_2 \). For the companion \( T \), consider the homeomorphism \( f \) from \( J \) to the regular neighborhood \( N(T) \) of \( T \), which sends the longitude of \( J \) to the \((1, 2q)\)-curve on \( \partial N(T) \). This \((1, 2q)\)-curve corresponds to a regular fiber of the Seifert fibration in the exterior of \( T \). Then the image \( f(k_1) \) gives our \( K_n \).

Since the pattern knot \( P \) is defined so as to preserve the preferred longitudes of \( J \) and \( N(T) \), \( P \) is obtained from \( k_1 \) in \( J \) by adding \( 2q \)–full twists. Conversely, if we add \((−2q)\)-full twists on \( P \), then the result is unknotted.

By the classification of twisted torus knots which are unknotted in [18], \( T(3, 2; 2, −1), T(3, 2; 2, −2), T(3; 1, 2, −1) \) and their mirror images \( T(3, −2; 2, 1), T(3, −2; 2, 2), T(3; −1, 2, 1) \) give all 3–strand twisted torus knots, which are unknotted. Thus Table 1 is the list of possible pattern knot \( P \) with genus. (Each knot has a positive braid presentation, so its genus is calculated as in Section 2.) Since \( P \) is an L–space knot and not a 3–cable, (1), (2) and (3) are excluded by [17]. (In fact, (1) gives a 3–cable.)

Recall that \( g(K_n) = 9n + 7 \) and \( g(T) = (q − 1)/2 \). If \( g(P) = 6q − 1 \), then \( 9n + 7 = 3(q − 1)/2 + 6q − 1 \), so \( 18n + 14 = 3(q − 1) + 12q − 2 \). Then \( 18n + 14 = −2 \) (mod 3), a contradiction. Thus (4) remains. For this case, \( 9n + 7 = 3(q − 1)/2 + 6q − 2 \) gives \( 6n + 7 = 5q \). Then \( n \equiv 3 \) (mod 5). Set \( n = 5m + 3 \) (\( m \geq 0 \)). Then \( q = 6m + 5 \).

We have \( \Delta_{K_n}(−1) = \Delta_T(−1) \Delta_P(−1) \), and \( \Delta_{K_n}(−1) = 10n + 11 = 50m + 41 \) from Theorem [14]. However, \( \Delta_T(−1) = q = 6m + 5 \). Then \( 6m + 5 \) does not divide \( 50m + 41 \), a contradiction.

Thus we have shown that our \( K_n \) is hyperbolic.

Proof of Theorem 2.1. By Theorems 8.1 and 5.2, \( K_n \) is a hyperbolic L–space knot. Its formal semigroup is described in Theorem 4.3. \( \square \)
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