DECOMPOSED RICHELOT ISOGENIES OF JACOBIAN VARIETIES OF CURVES OF GENUS 3

TOSHIYUKI KATSURA

ABSTRACT. For a non-singular projective curve $C$ of genus 3 defined over an algebraically closed field of characteristic $p \neq 2$, we give a necessary and sufficient condition that the Jacobian variety $J(C)$ has a decomposed Richelot isogeny outgoing from it and we determine the structures of decomposed ones.

1. INTRODUCTION

Isogeny-based cryptosystem, for example SIKE (Supersingular Isogeny Key Encapsulation), is one of the candidates of post-quantum cryptosystem. The system of supersingular elliptic curves is now well examined and achieves a great success (cf. Jao-De Feo [12] and Costello [5], for instance). As a next step, it is natural for researchers to investigate higher genus cases. In the case of genus 2, many such trials are already done (cf. Takashima [20], Castryck–Decru–Smith [3] and Costello–Smith [6], for instance) and we know now fairly well the structure of graph of superspecial curves of genus 2 for $(2, 2)$-isogenies (cf. Ibukiyama–Katsura–Oort [11], Katsura–Takashima [14], Florit–Smith [7] and Jordan–Zaytman [13]). As for the case of genus 3, Richelot isogenies outgoing from the Jacobian varieties of hyperelliptic curves with tractable isotropic subgroups are studied (cf. Smith [19], for instance), and also Richelot isogenies outgoing from products of 3 elliptic curves are very well analyzed (cf. Howe–Leprévost- Poonen [10]), but general cases seem not to be well examined yet.

In this paper, we examine the decomposed Richelot isogenies outgoing from the Jacobian varieties $J(C)$ of non-singular projective curves $C$ of genus 3 defined over an algebraically closed field $k$ of characteristic $p \neq 2$. A Richelot isogeny is a $(2, 2, 2)$-isogeny outgoing from the Jacobian variety $J(C)$ (cf. Definition 2.5). Note that decomposed Richelot isogenies (cf. Definition 2.5) play important roles to analyze the security of cryptosystems constructed by isogeny graph (see Costello–Smith [6]). From a theoretical point of view, it is also interesting to examine when Jacobian varieties have decomposed
Richelot isogenies. In this paper, we show the following two theorems (for the definition of long automorphism of order 2, see Section 3).

**Theorem I.** Let $C$ be a non-singular projective curve of genus 3. Then, there exists a decomposed Richelot isogeny outgoing from $J(C)$ if and only if $C$ has a long automorphism of order 2.

**Theorem II.** Let $C$ be a non-singular projective curve of genus 3 with a long automorphism $\sigma$ of order 2. We set $E = C/\langle \sigma \rangle$. Then, $E$ is an elliptic curve. Let $f : C \longrightarrow C/\langle \sigma \rangle = E$ be the quotient morphism, $N_f : J(C) \longrightarrow E$ be the induced homomorphism and $f^* : E \cong J(E) \longrightarrow J(C)$ be the pull-back by $f$.

1. If $C$ is hyperelliptic with hyperelliptic involution $\iota$, then $\{E, C/\langle \sigma \circ \iota \rangle\}$ is a set of an elliptic curve and a curve of genus 2. The target of the decomposed Richelot isogeny outgoing from $J(C)$ related to $\sigma$ is isomorphic to $J(E) \times J(C/\langle \sigma \circ \iota \rangle)$, the product of Jacobian varieties.
2. If $C$ is non-hyperelliptic, then $f^*$ is injective. Moreover, $A = \text{Ker } N_f$ is an irreducible abelian surface, and there exist three étale coverings $\tilde{A}$ of $A$ of degree 2 such that the targets of the decomposed Richelot isogenies outgoing from $J(C)$ related to $\sigma$ are isomorphic to $(E, O) \times (\tilde{A}, \Xi)$. Here, $\Xi$ is a principal polarization on $\tilde{A}$.
3. If the Jacobian variety $J(C)$ has a completely decomposed Richelot isogeny, then $C$ is a Howe curve of genus 3. The automorphism group $\text{Aut}(C)$ of $C$ contains a subgroup $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with generators $\sigma, \tau$ such that the three curves $C/\langle \sigma \rangle, C/\langle \tau \rangle$ and $C/\langle \sigma \circ \tau \rangle$ are elliptic curves, and the target of the completely decomposed Richelot isogeny outgoing from $J(C)$ related to $\sigma$ and $\tau$ is isomorphic to $(C/\langle \sigma \rangle, O) \times (C/\langle \tau \rangle, O) \times (C/\langle \sigma \circ \tau \rangle, O)$.

We explain the outline of our paper. In this paper, the genus of curves $C$ which we treat is always 3, if otherwise mentioned. In Section 2, we prepare some lemmas which we use later. These lemmas are somehow known (cf. Birkenhake–Lange [1], for instance), but to explain our setting precisely, we give full proofs for them. In Section 3, we examine the relation between long automorphisms of order 2 and decomposed Richelot isogenies. In Section 4, we treat the case of hyperelliptic curves of genus 3, and give a criterion for the existence of decomposed Richelot isogenies. In Section 5, we prepare some results on automorphisms of non-hyperelliptic curves of genus 3. In Section 6, we examine the structure of Howe curves of genus 3 and show that the Jacobian varieties of Howe curves have completely decomposed Richelot isogenies. This part is essentially known in Howe–Leprévote–Poonen [10] from the dual viewpoint of ours. In Section 7, we treat non-hyperelliptic curves of genus 3 and we show how non-hyperelliptic curves of genus 3 with long automorphism of order 2 make decomposed Richelot isogenies. Finally,
summarizing our results, we prove Theorems I and II. Similar decompositions of Jacobian varieties are also investigated in Lombardo-García-Ritzenthaler-Sijsling [16]. The difference is that their results are up to isogeny and our results are up to isomorphism.

The author thanks Katsuyuki Takashima for useful discussions and for giving the author much information on cryptography, and Everett Howe for useful comments and the information on the paper [10]. He also thanks the referee for his careful reading and for giving the author many advices.

Notation and conventions. For an abelian variety \( A \) and divisors \( D, D' \) on \( A \), we use the following notation.

- \( O \) : the zero point of \( A \),
- \( id_A \) : the identity of \( A \),
- \( \iota_A \) : the inversion of \( A \), i.e. the multiplication-by-(-1),
- \( \hat{A} = \text{Pic}^0(A) \) : the dual (Picard variety) of \( A \),
- \( \text{NS}(A) \) : the Néron-Severi group of \( A \),
- \( D \sim D' \) : linear equivalence,
- \( D \approx D' \) : algebraic equivalence.

For a vector space \( V \) and a group \( G \) which acts on \( V \), we denote by \( V^G \) the invariant subspace of \( V \). Sometimes, a Cartier divisor and the associated invertible sheaf will be identified.

2. Preliminaries

Let \( k \) be an algebraically closed field of characteristic \( p \neq 2 \). In this section, we introduce some notation and lemmas which we will use later.

For an abelian variety \( A \) and a divisor \( D \) on \( A \), we have a homomorphism

\[
\Phi_D : A \longrightarrow \text{Pic}^0(A) = \hat{A} \\
x \mapsto T^*_x D - D
\]

(cf. Mumford [18]). Here, \( T_x \) is the translation by \( x \in A \). We put \( K(D) = \text{Ker} \Phi_D \). We know that \( \Phi_D \) is an isogeny if \( D \) is ample.

Let \( C \) be a non-singular projective curve of genus \( g \geq 1 \) defined over \( k \). We denote by \( J(C) \) the Jacobian variety of \( C \), and by \( \Theta \) the principal polarization on \( J(C) \) given also by \( C \). We have a natural immersion (up to translation)

\[
\alpha_C : C \hookrightarrow J(C) = \text{Pic}^0(C)
\]

By the abuse of terminology, we sometimes denote \( \alpha_C(C) \) by \( C \). The morphism \( \alpha_C \) induces a homomorphism

\[
\alpha^*_C : J(C) = \text{Pic}^0(J(C)) \longrightarrow \text{Pic}^0(C) = J(C).
\]

Lemma 2.1 (Birkenhake–Lange [11]). \( \alpha^*_C = -\Phi^{-1}_\Theta \).
Proof. We identify the image $\alpha_C(C')$ with $C$. As is well-known, we have $(\Theta \cdot C) = g$. Therefore, the invertible sheaf $\mathcal{O}_{J(C)}(\Theta)|_C$ gives a divisor $\delta$ of degree $g$ on $C$. For $x \in J(C)$, $x$ is an invertible sheaf on $C$, which we denote by $\mathcal{L}$. Then, we have

$$
\alpha_C^*(\Phi_{\Theta}(x)) = (T_x^*(\Theta) - \Theta)|_C
= \mathcal{L}^{-1} \otimes \delta \otimes \delta^{-1}
= \mathcal{L}^{-1} = -x
$$

This means $\alpha_C^* \circ \Phi_{\Theta} = -\text{id}_{J(C)}$. Since $\Theta$ is a principal polarization, $\Phi_{\Theta}$ is an isomorphism. Therefore, we have $\alpha_C^* = -\Phi_{\Theta}^{-1}$. □

Let $f : C \to C'$ be a morphism of degree 2 from $C$ to a non-singular projective curve $C'$ of genus $g' \geq 1$. For an invertible sheaf $\mathcal{O}_{C}(\sum m_iP_i) \in J(C)$ ($P_i \in C$, $m_i \in \mathbb{Z}$), the homomorphism $N_f : J(C) \to J(C')$ is defined by

$$
N_f(\mathcal{O}_{C}(\sum m_iP_i)) = \mathcal{O}_{C'}(\sum m_if(P_i)).
$$

Then, by suitable choices of $\alpha_C$ and $\alpha_{C'}$, we have a commutative diagram

$$
\begin{array}{ccc}
C & \xrightarrow{\alpha_C} & J(C) \\
\downarrow f & & \downarrow \uparrow N_f \\
C' & \xleftarrow{\alpha_{C'}} & J(C')
\end{array}
$$

Lemma 2.2. $\Phi_{\Theta} \circ f^* = \hat{N}_f \circ \Phi_{\Theta'}$.

Proof. We have a diagram

$$
\begin{array}{ccc}
J(C) & \xleftarrow{\alpha_C} & J(C) \\
\uparrow f^* & & \uparrow \uparrow N_f^* \\
J(C') & \xrightarrow{\alpha_{C'}} & J(C')
\end{array}
$$

Therefore, using Lemma 2.1, we have

$$
\begin{array}{ccc}
J(C) & \xrightarrow{\Phi_{\Theta}} & J(\hat{C}) \\
\uparrow f^* & & \uparrow \uparrow N_f^* \\
J(C') & \xrightarrow{\Phi_{\Theta'}} & J(\hat{C}')
\end{array}
$$

Therefore, we have $\Phi_{\Theta} \circ f^* = N_f^* \circ \Phi_{\Theta'}$. Since $N_f^* = \hat{N}_f$, we complete our proof. □

Lemma 2.3. $(f^*)^*(\Theta) \approx 2\Theta'$.  

4
Proof. By definition, we have \( N_f \circ f^* = [2]_{J(C')}. \) Therefore, we have \( \hat{f}^* \circ \hat{N}_f = [2]_{J(\hat{C}')} \). Using Lemma 2.2 we have

\[
\Phi_{2\Theta'} = [2]_{J(\hat{C}')} \circ \Phi_{\Theta'} \\
= \hat{f}^* \circ \hat{N}_f \circ \Phi_{\Theta'} \\
= \hat{f}^* \circ \Phi_{\Theta} \circ f^* \\
= \Phi_{(f^*)^*(\Theta)}.
\]

Therefore, we have \( (f^*)^*(\Theta) \approx 2\Theta' \). \( \square \)

**Definition 2.4.** Let \( A_i \) be abelian varieties with principal polarizations \( \Theta_i \) \((i = 1, 2, \ldots, n)\), respectively. The product \((A_1, \Theta_1) \times (A_2, \Theta_2) \times \ldots \times (A_n, \Theta_n)\) means the principally polarized abelian variety \( A_1 \times A_2 \times \ldots \times A_n \) with principal polarization \( \Theta_1 \times A_2 \times A_3 \times \ldots \times A_n + A_1 \times \Theta_2 \times A_3 \times \ldots \times A_n + \ldots + A_1 \times A_2 \times \ldots \times A_{n-1} \times \Theta_n \).

**Definition 2.5.** Let \( C \) be a non-singular projective curve of genus \( g \geq 2 \), and \( J(C) \) be the Jacobian variety of \( C \). We denote by \( \Theta \) the canonical principal polarization of \( J(C) \). Let \( A \) be an abelian variety of dimension \( g \) with principal polarization \( D \), and \( f : J(C) \rightarrow A \) be an isogeny. The isogeny \( f \) is called a Richelot isogeny if \( 2\Theta \approx f^*(D) \). A Richelot isogeny \( f \) is said to be decomposed if there exist two principally polarized abelian varieties \((A_i, \Theta_i) \) \((i = 1, 2)\) such that \( (A, D) \cong (A_1, \Theta_1) \times (A_2, \Theta_2) \). A decomposed Richelot isogeny is said to be completely decomposed if there exist elliptic curves \( E_i \) with zero point \( O_i \) \((i = 1, 2, \ldots, g)\) such that \( (A, D) \cong (E_1, O_1) \times (E_2, O_2) \times \ldots \times (E_g, O_g) \).

3. SOME LEMMAS ON AUTOMORPHISMS

**Lemma 3.1.** Let \( C \) be a non-singular projective curve of genus \( g \geq 2 \), and \( \sigma \) be an automorphism of \( C \) of order \( n < \infty \) such that the induced automorphism on \( H^0(C, \Omega^1_C) \) is trivial. Then, \( \sigma \) is the identity morphism.

**Proof.** We have a morphism \( f : C \rightarrow C/\langle \sigma \rangle \) of degree \( n \). Since the induced action \( \sigma^* \) of \( \sigma \) on \( H^0(C, \Omega^1_C) \) is trivial, we have

\[
H^0(C, \Omega^1_C) = H^0(C, \Omega^1_C)^{\sigma^*} \cong H^0(C/\langle \sigma \rangle, \Omega^1_{C/\langle \sigma \rangle})
\]

Therefore, the genus of \( C/\langle \sigma \rangle \) is equal to \( g \). By the Hurwitz formula, we have \( 2(g - 1) = 2n(g - 1) + \delta \) with an integer \( \delta \geq 0 \). Therefore, we have \( n = 1 \) and \( \delta = 0 \). This means \( \sigma \) is the identity morphism. \( \square \)

**Lemma 3.2.** Let \( C \) be a non-singular projective curve of genus \( g \geq 3 \). If \( C \) has an automorphism \( \sigma \) of order 2 such that the induced automorphism on \( H^0(C, \Omega^1_C) \) is the multiplication by \(-1\), then \( C \) is a hyperelliptic curve and \( \sigma \) is the hyperelliptic involution.
Proof. Since $H^0(C/\langle \sigma \rangle, \Omega^1_{C/\langle \sigma \rangle}) \cong H^0(C, \Omega^1_C)^{\langle \sigma \rangle} = \{0\}$, we see that the genus of $C/\langle \sigma \rangle$ is 0. Therefore, we have the morphism $C \longrightarrow C/\langle \sigma \rangle \cong \mathbb{P}^1$ of degree 2. Therefore, $C$ is hyperelliptic and $\sigma$ is the hyperelliptic involution. □

Lemma 3.3. Let $A$, $A_1$ and $A_2$ be abelian varieties, and let $f : A_1 \times A_2 \longrightarrow A$ be an isogeny. Let $\sigma$ be an automorphism of $A$ such that $\sigma \circ f = f \circ (id_{A_1} \times \iota_{A_2})$ and $\Theta$ be a polarization of $A$ such that $\sigma^* \Theta \approx \Theta$. Then,

$$(A_1 \times A_2, f^* \Theta) \cong (A_1, f|_{A_1}^* \Theta) \times (A_2, f|_{A_2}^* \Theta).$$

Proof. Since $\sigma^* \Theta \approx \Theta$, we have

$$(id_{A_1} \times \iota_{A_2})^* (f^* \Theta) \approx (f^* \Theta).$$

Therefore, we have $\Phi_{(id_{A_1} \times \iota_{A_2})}^* (f^* \Theta) = \Phi_{f^* \Theta}$ and we have a commutative diagram

$$
\begin{array}{ccc}
A_1 \times A_2 & \xrightarrow{\Phi_{f^* \Theta}} & \hat{A_1} \times \hat{A_2} \\
(id_{A_1} \times \iota_{A_2}) \downarrow & & \uparrow & \iota_{\hat{A}_1} \times \iota_{\hat{A}_2} \\
A_1 \times A_2 & \xrightarrow{\Phi_{f^* \Theta}} & \hat{A_1} \times \hat{A_2}
\end{array}
$$

(3.1)

We express $\Phi_{f^* \Theta}$ as a matrix

$$
\begin{pmatrix}
\varphi_1 & \varphi_2 \\
\varphi_3 & \varphi_4
\end{pmatrix}
$$

(where $\varphi_1 \in \text{Hom}(A_1, \hat{A}_1)$, $\varphi_2 \in \text{Hom}(A_2, \hat{A}_1)$, $\varphi_3 \in \text{Hom}(A_1, \hat{A}_2)$ and $\varphi_2 \in \text{Hom}(A_2, \hat{A}_2)$). Then, the diagram (3.1) says

$$
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
\varphi_1 & \varphi_2 \\
\varphi_3 & \varphi_4
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
= \begin{pmatrix}
\varphi_1 & \varphi_2 \\
\varphi_3 & \varphi_4
\end{pmatrix}.
$$

Therefore, we have $-\varphi_2 = 0$ and $-\varphi_3 = 0$. Hence, we have $\varphi_2 = 0$ and $\varphi_3 = 0$. This means $\Phi_{f^* \Theta} = \Phi_{f|_{A_1}^* \Theta} \times \Phi_{f|_{A_2}^* \Theta}$, and we complete our proof. □

Definition 3.4. Let $C$ be a non-singular projective curve of genus $g \geq 2$ and $\sigma$ be an automorphism of $C$ of order 2. The automorphism $\sigma$ of $C$ is said to be a long automorphism if the $g$ eigenvalues of the induced action of $\sigma$ on $H^0(C, \Omega^1_C)$ are given by $1, -1, -1, \cdots, -1$ (the number of $-1$ is $g - 1$).

Remark 3.5. In case $C$ is a non-singular projective curve of genus 2, this definition of long automorphism coincides with the definition of the long element in Katsura–Takashima [14] (see also Ibukiyama-Katsura-Oort [11]).

Definition 3.6. For a polarized abelian variety with polarization $\Theta$, we denote by $\text{Aut}(A, \Theta)$ the group of automorphisms of $A$ which preserve the polarization $\Theta$. 

6
Lemma 3.7. Let $C$ be a non-singular projective curve of genus $g \geq 2$, and $(J(C), \Theta)$ is the Jacobian variety of $C$ with the canonical principal polarization $\Theta$. If the Jacobian variety $J(C)$ of $C$ has a decomposed Richelot isogeny outgoing from $J(C)$, then there exists an automorphism of order 2 in $\text{Aut}(J(C), \Theta)$ which is not the inversion.

Proof. By assumption, we have a Richelot isogeny

\[ \pi : J(C) \longrightarrow J(C)/G \]

such that $G$ is a maximal isotropic subgroup of $J(C)[2]$ with respect to $2\Theta$, and that $J(C)/G$ has a decomposed principal polarization $\Theta'$ with $\pi^*\Theta' = 2\Theta$. This means that there exist two principally polarized abelian varieties $(A_1, \Theta_1)$ and $(A_2, \Theta_2)$ such that $(J(C)/G, \Theta') \cong (A_1, \Theta_1) \times (A_2, \Theta_2)$. Since $\Theta$ is a principal polarization, we have an isomorphism $\varphi : J(C) \cong J(C)$. By a similar reason, we have $J(C)/G \cong (J(C)/G)$. Using these isomorphisms, we identifies $J(C)$ (resp. $J(C)/G$) with $\hat{J}(C)$ (resp. $(J(C)/G)$). Dualizing (3.2), we have

\[ \eta = \hat{\pi} : J(C)/G \longrightarrow J(C). \]

Here, we have $J(C)/G \cong A_1 \times A_2$ with principal polarization $\Theta'$ such that $\eta^*(\Theta) \sim 2\Theta'$. The kernel $\text{Ker} \eta$ is an isotropic subgroup of $(A_1 \times A_2)[2]$ with respect to the divisor $2\Theta'$.

Since $(A_2, \Theta_2)$ is a principally polarized abelian variety, we may assume (by a suitable translation of $\Theta_2$) $\iota_{A_2}^*(\Theta_2) = \Theta_2$. We set

\[ \bar{\tau} = \text{id}_{A_1} \times \iota_{A_2}. \]

Then, $\bar{\tau}$ is an automorphism of order 2 which is not the inversion of $A_1 \times A_2$. By the definition, we have

\[ \bar{\tau}^*(\Theta') = \Theta'. \]

Moreover, since $\text{Ker} \eta$ consists of elements of order 2 and $\bar{\tau}$ fixes the elements of order 2, $\bar{\tau}$ preserves $\text{Ker} \eta$. Therefore, $\bar{\tau}$ induces an automorphism $\tau$ of $J(C) \cong (J(C)/G)/\text{Ker} \eta \cong (A_1 \times A_2)/\text{Ker} \eta$. Therefore, we have the following diagram:

\[
\begin{array}{c}
A_1 \times A_2 \xrightarrow{\pi} A_1 \times A_2 \\
\eta \downarrow \quad \downarrow \eta \\
J(C) \quad \xrightarrow{\tau} \quad J(C).
\end{array}
\]

We have

\[ \eta^* \tau^* \Theta = \bar{\tau}^* \eta^* \Theta \sim \bar{\tau}^*(2\Theta') = 2\Theta'. \]

On the other hand, we have

\[ \eta^* \Theta \sim 2\Theta'. \]
Since \( \eta^* \) is an injective homomorphism from \( \text{NS}(J(C)) \) to \( \text{NS}(A_1 \times A_2) \), we have \( \Theta \approx \tau^* \Theta \). Therefore, \( \tau \) is an element of order 2 of the group \( \text{Aut}(J(C), \Theta) \). By definition, this is not the inversion \( \iota \) of \( J(C) \).

\[ \square \]

4. Hyperelliptic Curves of Genus 3

In this section, we assume that \( C \) is a hyperelliptic curve of genus 3. For the Jacobian variety \( J(C) \) of \( C \), we denote by \( \Theta \) the canonical principal polarization of \( J(C) \).

**Proposition 4.1.** If the Jacobian variety \( J(C) \) of \( C \) has a decomposed Richelot isogeny outgoing from \( J(C) \), then there exists a long automorphism of order 2 of \( C \).

**Proof.** In the proof of Lemma 3.7, we can take \( A_1 \) as an elliptic curve and \( A_2 \) as an abelian surface. We take an automorphism \( \bar{\tau} = \text{id}_{A_1} \times \iota_{A_2} \). Then, by Lemma 3.7, we have a long automorphism \( \tau \) of order 2 of \( J(C) \) which preserves the polarization \( \Theta \). For hyperelliptic curves, we have \( \text{Aut}(C) \cong \text{Aut}(J(C), \Theta) \), and we have \( H^0(C, \Omega^1_C) \cong H^0(J(C), \Omega^1_{J(C)}) \) with the compatible action of the group of automorphisms (see Milne [17]). Hence, \( \tau \) gives a long automorphism of order 2 of \( C \).

Let \( \sigma \) be a long automorphism of order 2 of a hyperelliptic curve \( C \) of genus 3, and \( \iota \) be a hyperelliptic inversion of \( C \). We set \( \tau = \sigma \circ \iota \). We have a morphism \( \varphi : C \longrightarrow \mathbb{P}^1 \cong C/\langle \iota \rangle \), and the automorphism \( \sigma \) induces an automorphism of \( \mathbb{P}^1 \). If \( \sigma \) has a fixed point in the ramification points of \( \varphi \), by a suitable choice of the coordinate \( x \) of \( \mathbb{A}^1 \subset \mathbb{P}^1 \), we may assume that \( \sigma \) has the fixed points at \( x = 0 \) and \( \infty \), and we may assume

\[ \sigma : x \mapsto -x; \quad y \mapsto y. \]

Then the ramification points are given by

\[ 0, 1, -1, \sqrt{a}, -\sqrt{a}, \sqrt{b}, -\sqrt{b}, \infty. \]

Here, \( a, b \) are mutually different and they are equal to neither 0 nor 1. The normal form of the curve \( C \) is given by

\[ y^2 = x(x^2 - 1)(x^2 - a)(x^2 - b). \]

Then, the action of \( \sigma \) on \( C \) is

\[ x \mapsto -x; \quad y \mapsto \pm \sqrt{-1} y. \]

Therefore, the order of \( \sigma \) is 4, a contradiction. Hence, \( \sigma \) has no fixed points on the ramification points. Therefore, the ramifications are given by

\[ 1, -1, \sqrt{a}, -\sqrt{a}, \sqrt{b}, -\sqrt{b}, \sqrt{c}, -\sqrt{c}, \]
and the normal form of the curve $C$ is given by

$$y^2 = (x^2 - 1)(x^2 - a)(x^2 - b)(x^2 - c).$$

Elements $x^2$ and $y$ are invariant under $\sigma$. We set $X = x^2$, $Y = y$. Then, the defining equation of the curve $C/\langle \sigma \rangle$ is given by

$$Y^2 = (X - 1)(X - a)(X - b)(X - c).$$

The curve $C/\langle \sigma \rangle$ is an elliptic curve. We set $E_\sigma = C/\langle \sigma \rangle$. We have a quotient morphism $f_1 : C \to E_\sigma$. Elements $x^2$ and $xy$ are invariant under $\tau$. We set $X = x^2$, $Y = xy$. Then, the defining equation of the curve $C/\langle \tau \rangle$ is given by

$$Y^2 = X(X - 1)(X - a)(X - b)(X - c).$$

The curve $C/\langle \tau \rangle$ is a curve of genus 2. We set $C_\tau = C/\langle \tau \rangle$. We have a quotient morphism $f_2 : C \to C_\tau$. Using these morphisms, we have a morphism

$$f = (f_1, f_2) : C \to E_\sigma \times C_\tau.$$

The morphism $f$ induces a homomorphism

$$N_f = (N_{f_1}, N_{f_2}) : J(C) \to E_\sigma \times J(C_\tau).$$

Note that

$$N_{f_1} \circ f_1^* = [2]_{E_\sigma}, \quad N_{f_2} \circ f_2^* = [2]_{J(C_\tau)}.$$

By our construction, we have

$$N_{f_1} \circ f_2^* = 0, \quad N_{f_2} \circ f_1^* = 0.$$

Therefore, we have

$$N_f \circ f^* = [2]_{E_\sigma \times J(C_\tau)}.$$

Dualizing the situation (4.1), we have

$$f^* : E_\sigma \times J(C_\tau) \to J(C).$$

**Theorem 4.2.** Let $C$ is a hyperelliptic curve of genus 3 with a long automorphism $\sigma$ of order 2. Then, the isogeny $N_f : J(C) \to E_\sigma \times J(C_\tau)$ is a decomposed Richelot isogeny.

**Proof.** Since $\sigma$ induces an isomorphism from $J(C)$ to $J(C)$ and we may assume that this isomorphism is an automorphism of $J(C)$. We have a commutative diagram

$$
\begin{array}{ccc}
E_\sigma \times J(C_\tau) & \xrightarrow{id_{E_\sigma \times J(C_\tau)}} & E_\sigma \times J(C_\tau) \\
\uparrow f^* & & \downarrow f^* \\
J(C) & \xrightarrow{\sigma} & J(C) \\
\downarrow N_f & & \downarrow N_f \\
E_\sigma \times J(C_\tau) & \xrightarrow{id_{E_\sigma \times J(C_\tau)}} & E_\sigma \times J(C_\tau)
\end{array}
$$
Since $\sigma^*(\Theta) = \Theta$, using Lemma 3.3, we have

$$f^*(\Theta) \approx f_1^*(\Theta) \times J(C) + E_\sigma \times f_2^*(\Theta).$$

Therefore, by Lemma 2.3, we see

$$f^*(\Theta) \approx 2(O \times J(C)) + 2(E_\sigma \times C).$$

Dualizing this situation, we have

$$N_f^*(O \times J(C)) + (E_\sigma \times C) \approx 2\Theta.$$

This means that $N_f$ is a decomposed Richelot isogeny outgoing from $J(C)$. □

5. NON-HYPERELLIPTIC CURVES

In this section, we examine automorphisms of non-hyperelliptic curves.

Lemma 5.1. Let $C$ be a non-hyperelliptic curve of genus 3. Then, there exist no surjective morphisms from $C$ to curves of genus 2.

Proof. Let $C'$ be a non-singular projective curve of genus 2, and let $f : C \longrightarrow C'$ be a nontrivial morphism. We set $\deg f = n$. Then we have $n \geq 2$. If $n \geq 3$, by the Hurwitz formula, we have

$$2(3 - 1) = n \cdot 2(2 - 1) + \delta$$

with a non-negative integer $\delta$, which is impossible. If $n = 2$, we have $\delta = 0$. Therefore, $f$ is an étale covering. Therefore, there exists a non-trivial invertible sheaf $\mathcal{L}$ on $C'$ such that both $\mathcal{L} \otimes O$ and $f^* \mathcal{L}$ are trivial. Since $C'$ is of genus 2 and hyperelliptic, there exist two ramification points $P_1, P_2$ of the hyperelliptic covering over $\mathbb{P}^1$ such that $\mathcal{L} \cong O_{C'}(P_2 - P_1)$, and we have $f^*(\mathcal{L}) \cong O_C$. This means $f^*(P_2) - f^*(P_1) \sim 0$, that is, there exists a rational function $h$ on $C$ such that $(h) = f^*(P_2) - f^*(P_1)$. Since $n = 2$, we see the degree of the pole divisor of $h$ is 2 and we have a morphism $h : C \longrightarrow \mathbb{P}^1$ of degree 2, which contradicts the fact that $C$ is non-hyperelliptic. □

Corollary 5.2. Let $C$ be a non-hyperelliptic curve of genus 3, and $\sigma$ an automorphism of order 2. Then, the quotient curve $C/\langle \sigma \rangle$ is an elliptic curve.

Proof. Since $C$ is non-hyperelliptic, the possibility of the genus of the curve $C' = C/\langle \sigma \rangle$ is either 1 or 2. However, 2 is excluded by Lemma 5.1. □

We can also show the following corollary by the classification result in Lombardo-García-Ritzenthaler-Sijsling [16].

Corollary 5.3. Let $C$ be a non-hyperelliptic curve of genus 3 and $\sigma$ an automorphism of $C$ of order 2. Then, the eigenvalues of the action of $\sigma^*$ on $H^0(C, \Omega_C^1)$ are 1, −1, −1, that is, $\sigma$ is a long automorphism.
Proof. By Lemmas 3.1 and 3.2, we can exclude \{1, 1, 1\} and \{-1, -1, -1\}. Suppose the eigenvalues are 1, 1, −1. Then, we have
\[
\dim H^0(C/\langle \sigma \rangle, \Omega^1_{C/\langle \sigma \rangle}) = \dim H^0(C, \Omega^1_C) = 2,
\]
that is, the genus of the curve \(C/\langle \sigma \rangle\) is equal to 2, which is excluded by Lemma 5.1.

Proposition 5.4. Let \(C\) be a non-hyperelliptic curve of genus 3. If \(C\) has a decomposed Richelot isogeny outgoing from \(J(C)\), then there exists a long automorphism of order 2 of \(C\).

Proof. By Lemma 3.7, we have a long automorphism \(\tau\) of order 2 of \(J(C)\) which preserves the polarization \(\Theta\). For non-hyperelliptic curves, either \(\tau\) or \(-\tau\) is induced from an element of \(\text{Aut}(C)\) (cf. Milne [17]). We have an isomorphism \(H^0(C, \Omega^1_C) \cong H^0(J(C), \Omega^1_{J(C)})\) with the compatible actions of automorphisms in \(\text{Aut}(C)\). By Corollary 5.3, \(-\tau\) cannot become an automorphism of \(C\). Therefore, \(\tau\) comes from an automorphism of \(C\). Hence, this gives a long automorphism of order 2 of \(C\).

6. HOWE CURVES

Let \(E_1, E_2\) be two elliptic curves, and let \(f_1 : E_1 \to \mathbb{P}^1\), \(f_2 : E_2 \to \mathbb{P}^1\) be morphisms of degree 2. We consider the fiber product \(E_1 \times_{\mathbb{P}^1} E_2\):

\[
\begin{array}{ccc}
E_1 \times_{\mathbb{P}^1} E_2 & \xrightarrow{\pi_2} & E_2 \\
\pi_1 \downarrow & & \downarrow f_2 \\
E_1 & \xrightarrow{f_1} & \mathbb{P}^1.
\end{array}
\]

We denote by \(r\) the number of common ramification points of \(f_1\) and \(f_2\) (\(0 \leq r < 4\)). We exclude the case \(r = 4\). Because if \(r = 4\), there exists an isomorphism \(\varphi : E_1 \cong E_2\) such that \(f_2 \circ \varphi = f_1\) and the fiber product \(E_1 \times_{\mathbb{P}^1} E_2\) is not irreducible. We denote by \(C\) the non-singular projective model of \(E_1 \times_{\mathbb{P}^1} E_2\), and we denote by \(h : C \to E_1 \times_{\mathbb{P}^1} E_2\) the resolution of singularities. We call \(C\) a Howe curve (cf. Howe [9] and Kudo-Harashita-Senda [15]). Note that in case \(C\) is a curve of genus 3, then this curve \(C\) is historically called a Ciani curve (cf. Ciani [4]). There exist two automorphisms \(\sigma, \tau\) of order 2 of \(C\) such that \(C/\langle \sigma \rangle \cong E_1\) and \(C/\langle \tau \rangle \cong E_2\). It is clear that \(\langle \sigma, \tau \rangle \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\). We set \(h_1 = \pi_1 \circ h\). Then, the degree of \(h_1\) is 2. The genus of a Howe curve is given by the following proposition.

Proposition 6.1. The genus of \(C\) is equal to \(5 - r\).

Proof. Let \(P \in \mathbb{P}^1\) be a common ramification point of \(f_1\) and \(f_2\). We can choose a coordinate \(x\) on \(\mathbb{A}^1 \subset \mathbb{P}^1\) such that \(P\) is given by \(x = 0\). Then, the equation of \(E_1\) (resp. \(E_2\)) around \(P\) is given by
\[
y_1^2 = u_1 x \quad (\text{resp.} \quad y_2^2 = u_2 x).
\]
Here, \( u_1 \) and \( u_2 \) are units at \( P \). We denote by \( \tilde{P} \) the point of the fiber product \( E_1 \times_P E_2 \) over \( P \). Then, around \( \tilde{P} \) the fiber product \( E_1 \times_P E_2 \) is defined by
\[
y_1^2 = u_1 x, \quad y_2^2 = u_2 x.
\]
Therefore, by eliminating \( x \), the equation around \( \tilde{P} \) is given by the equation
\[
u_2 y_1^2 = u_1 y_2^2.
\]
This means that \( \tilde{P} \) is a singular point with two branches. Therefore, on \( C \) \( \tilde{P} \) splits into two non-singular points and \( P \) is not a ramification point of \( h_1 \).

By the meaning of fiber product, the branch points of \( f_1 \) whose images by \( f_1 \) are not ramification points of \( f_2 \) are not ramification points of \( h_1 \), and the points on \( E_1 \) which are not branch points of \( f_1 \) and whose images by \( f_1 \) are ramification points of \( f_2 \) are ramification points of \( h_1 \). Therefore, on the curve \( C \), \( h_1 \) has \( 2(4 - r) \) branch points of index 2. Applying the Hurwitz formula to the morphism \( h_1 : C \rightarrow E_1 \), we have
\[
2(g(C) - 1) = 2 \cdot 2(g(E_1) - 1) + 2(4 - r).
\]
Since \( g(E_1) = 1 \), we have the result. \( \Box \)

The following two theorems are essentially known in Howe–Leprévost–Poonen [10].

**Theorem 6.2.** Let \( C \) be a Howe curve of genus 3. Then, there exists a completely decomposed Richelot isogeny outgoing from \( J(C) \).

**Proof.** We set \( E_3 = C/\langle \sigma \circ \tau \rangle \). Since \( C/\langle \sigma \rangle \cong E_1 \) (resp. \( C/\langle \tau \rangle \cong E_2 \)) is an elliptic curve, the eigenvalues of the action of \( \sigma \) (resp. \( \tau \)) on \( H^0(C, \Omega_C^1) \) are given by \( 1, -1, -1 \) (resp. \( -1, 1, -1 \)) with respect to a suitable choice of the basis of \( H^0(C, \Omega_C^1) \). Therefore, the eigenvalues of the action of \( \sigma \circ \tau \) on \( H^0(C, \Omega_C^1) \) are given by \( -1, -1, 1 \). Therefore, \( E_3 \) is an elliptic curve. We denote by \( \Theta \) the canonical principal divisor of \( J(C) \). By the universality of Jacobian variety we have an isogeny
\[
f : J(C) \rightarrow E_1 \times E_2 \times E_3.
\]
Then by a similar method to the one in Theorem 4.2 we have \( 2\Theta = f^*\{0\} \times E_2 \times E_3 + E_1 \times \{0\} \times E_3 + E_1 \times E_2 \times \{0\} \) and \( f \) is a completely decomposed Richelot isogeny. \( \Box \)

**Theorem 6.3.** Let \( C \) be a non-singular curve of genus 3. If there exists a completely decomposed Richelot isogeny outgoing from \( J(C) \), then \( C \) is a Howe curve of genus 3.

**Proof.** As in Propositions 4.1 and 5.4, we have two automorphisms \( \sigma, \tau \) of \( C \) of order 2 such that \( \sigma \circ \tau = \tau \circ \sigma \). The eigenvalues of the actions of \( \sigma \) and \( \tau \) are both given by one 1 and two \(-1\)'s. Therefore, \( E_1 = C/\langle \sigma \rangle \) and \( E_2 = C/\langle \tau \rangle \) are elliptic curves. The automorphism \( \tau \) (resp. \( \sigma \)) induces the
inversion of $E_1$ (resp. $E_2$), and $C/\langle \sigma, \tau \rangle \cong \mathbb{P}^1$. Then, considering the fiber product, we have a commutative diagram:

$$
\begin{array}{ccc}
E_1 \times_{\mathbb{P}^1} E_2 & \longrightarrow & E_2 \\
\downarrow & & \downarrow f_2 \\
E_1 & \longrightarrow & \mathbb{P}^1.
\end{array}
$$

Since we have morphisms $C \twoheadrightarrow E_1$ and $C \twoheadrightarrow E_2$, by the universality of fiber product there exists a morphism $f : C \twoheadrightarrow E_1 \times_{\mathbb{P}^1} E_2$. By the degree calculation of morphisms, we see $\deg f = 1$. Therefore, $C$ is birationally equivalent to $E_1 \times_{\mathbb{P}^1} E_2$ and $C$ is a Howe curve.

Many examples of Howe curves are known (cf. Howe–Leprévost–Poonen [10] and Brock [2]). We give here typical examples of a hyperelliptic Howe curve and a non-hyperelliptic one. The examples are well-known (cf. Lombardo-García-Ritzenthaler-Sijsling [16]), but such simple examples make our situation clearer.

**Example 6.4.** We consider the non-singular complete model $C$ of a curve defined by

$$
y^2 = x^8 - 1.
$$

The genus of $C$ is 3 and it has two automorphisms defined by

$$
\sigma : x \mapsto -x, \quad y \mapsto y; \quad \tau : x \mapsto \zeta/x, \quad y \mapsto \zeta^2 y/x^4.
$$

Here, $\zeta$ is a primitive eighth root of unity. Then, they are long automorphisms of order 2 with $\sigma \circ \tau = \tau \circ \sigma$. Therefore, by the proof of Theorem 6.3, $C$ is a hyperelliptic Howe curve.

**Example 6.5.** We consider the non-singular complete model $C$ of a Fermat curve defined by

$$
x^4 + y^4 = 1.
$$

The genus of $C$ is 3 and it has two automorphisms defined by

$$
\sigma : x \mapsto -x, \quad y \mapsto y; \quad \tau : x \mapsto x, \quad y \mapsto -y.
$$

Then, they are long automorphisms of order 2 with $\sigma \circ \tau = \tau \circ \sigma$. Therefore, $C$ is a non-hyperelliptic Howe curve.

7. **Non-hyperelliptic curves of genus 3 with long automorphism**

Let $C$ be a non-hyperelliptic curve of genus 3 with an automorphism $\sigma$ of order 2. By Corollary [5,2] the quotient curve $E = C/\langle \sigma \rangle$ is an elliptic curve,
and we have the quotient morphism \( f : C \to E \). As before, choosing an immersion \( \alpha = \alpha_C : C \hookrightarrow J(C) \) suitably, we have a commutative diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\alpha} & J(C) \\
f \downarrow & & \downarrow N_f \\
E & & 
\end{array}
\]

**Lemma 7.1.** \( f^* : J(E) \to J(C) \) is injective.

*Proof.* Suppose that \( f^* \) is not injective. We denote the zero element of \( E \) by \( O \). Since any element of \( J(E) \) is given by \( P - O \) with a suitable point \( P \in E \), there exists a point \( Q (Q \neq O) \) of \( E \) such that \( f^*(Q - O) \) is linearly equivalent to \( 0 \). This means there exists a rational function \( h \) on \( C \) such that \( (h) = f^*(Q) - f^*(O) \). Since \( f \) is degree 2, we have a morphism \( h : C \to \mathbb{P}^1 \) which is of degree 2. This contradicts the assumption that \( C \) is not hyperelliptic. \( \square \)

We set \( \text{Ker } N_f = A \). We denote by \( i_A \) the natural immersion of \( A \) into \( J(C) \):

\[
i_A : A \hookrightarrow J(C).
\]

**Lemma 7.2.** \( \alpha(C) \cdot A = 2 \).

*Proof.* For the zero point \( O \in E \), we have

\[
\alpha(C) \circ A = \deg (\alpha^{-1} \circ N_f^{-1}(O)) = \deg f^{-1}(O) = 2.
\]

\( \square \)

**Lemma 7.3.** \( A \) is irreducible.

*Proof.* Since \( \alpha(C) \cdot A = 2 \), the curve \( \alpha(C) \) will intersect \( A \) with two points. If \( A \) is not irreducible, then considering the Stein factorization, we have a fiber space such that \( \alpha(C) \) is a section of the fiber space. However, since \( J(C) \) is an abelian variety, the base curve is an elliptic curve. Therefore, the curve of genus 3 cannot become a section. \( \square \)

For the canonical principal polarization \( \Theta \) of \( J(C) \), we set \( D = A \cap \Theta \). Then, \( D \) is a divisor on the abelian surface \( A \).

**Lemma 7.4.** \( i_A^*(\Theta) = D \) and \( D^2 = 4 \).

*Proof.* The former part comes from the definition. By Matsusaka’s theorem on the characterization of Jacobian variety, we have \((1/2)!\Theta^2 \approx \alpha(C)\). Therefore, we have

\[
D^2 = (\Theta \cdot (\Theta \cdot A)) = (\Theta^2 \cdot A) = 2(\alpha(C) \cdot A) = 4.
\]

\( \square \)

By the identification of \( E \) with \( \hat{E} \), we can regard \( f^* \) as the natural immersion \( i_E : E \hookrightarrow J(C) \).

**Lemma 7.5.** \( f^*(\Theta) \approx 2O \).
Proof. This follows from Lemma 2.3.

Lemma 7.6. Let $L$ be an ample divisor on an abelian surface $A$ with $|K(L)| = 4$. Then, $K(L) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Proof. Suppose that $K(L) \cong \mathbb{Z}/4\mathbb{Z}$. Since $e^L$ is alternating, for a generator $\zeta \in K(L)$ we have $e^L(\zeta, \zeta) = 1$, which contradicts the fact that $e^L$ is a non-degenerate pairing on $K(L)$ (cf. Mumford [18]).

Lemma 7.7. Let $L$ be an ample divisor on an abelian surface $A$. Then, $K(L)$ cannot be isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

Proof. Suppose that $K(L) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. Then, the subgroup $G \cong \mathbb{Z}/4\mathbb{Z}$ of $K(L)$ is an isotropic subgroup with respect to the pairing $e^L$ as in the proof of Lemma 7.6. Therefore, we have a principal divisor $\Xi$ on $A/G$ and a commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\Phi_L} & \hat{A} \\
\pi \downarrow & & \uparrow \hat{\pi} \\
A/G & \xrightarrow{\Phi_\Xi} & \hat{A}/G
\end{array}
$$

Note that $\Phi_\Xi$ is an isomorphism. Since $K(L) \cong \text{Ker} \Phi_L \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ and Ker $\pi \cong G \cong \mathbb{Z}/4\mathbb{Z}$, we see that Ker $\hat{\pi}$ is dual to Ker $\pi$ (cf. Mumford [18]). By abuse of notation, we denote by $E$ the image of $f^\ast$. Then, we have a homomorphism

$$i_E + i_A : E \times A \longrightarrow J(C).$$

Lemma 7.8. $\Phi_{(i_E+i_A)\ast} \Theta = \Phi_{i_E} \Theta \times \Phi_{i_A} \Theta$.

Proof. On $E$ $\sigma$ acts as the identity and on $A$ $\sigma$ acts as the inversion $i_A$ of $A$. Therefore, we have a commutative diagram

$$
\begin{array}{ccc}
E \times A & \xrightarrow{i_E \times i_A} & E \times A \\
\downarrow i_E + i_A & & \downarrow i_E + i_A \\
J(C) & \xrightarrow{\sigma} & J(C)
\end{array}
$$

and since $\sigma^\ast \Theta \approx \Theta$, we get our result by Lemma 3.3.

Corollary 7.9. $\Phi_{(i_E+i_A)\ast} \Theta = \Phi_{2O} \times \Phi_D$.

Proof. This follows from Lemmas 7.4, 7.5 and 7.8.

Since $D^2 = 4$, we have $|K(D)| = ((D^2)/2)^2 = 4$. Therefore, by Lemma 7.6 we see $K(D) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Therefore, we have a homomorphism $\varphi : \hat{A} \longrightarrow A$ such that $\Phi_D \circ \varphi = [2]_\hat{A}$. Since Ker $\varphi \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, we have three elements of order 2 in Ker $\varphi$. We take one of them, say $a \in \text{Ker} \varphi$, $a \neq 0$. Then, we have the following homomorphisms:

$$[2]_\hat{A} : \hat{A} \longrightarrow \hat{A}/\langle a \rangle \xrightarrow{\pi} A \xrightarrow{\Phi_D} \hat{A}.$$
We set $\tilde{A}_a = \hat{A}/(a)$. Using this decomposition of the homomorphism $[2]_{\tilde{A}}$, we have a diagram

$$
\begin{array}{c}
\tilde{A}_a \\
\downarrow \pi \\
A
\end{array}
\xrightarrow{\Phi_D} 
\begin{array}{c}
\hat{A}_a \\
\uparrow \hat{\pi}
\end{array}
\xrightarrow{\Phi} 
\hat{A}.
$$

Since $(\pi^*D)^2 = (\deg \pi)(D^2) = 8$, we have $\deg \Phi_D = ((\pi^*D)^2)/2 = 16$. Therefore, we have $|K(\pi^*D)| = 16$. Since $K(\pi^*D) \supset \ker \Phi_D \lhd \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $K(\pi^*D)$ is isomorphic to either $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. By Lemma 7.1, we conclude

$$K(\pi^*D) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$ 

Namely, we have $K(\pi^*D) \cong \ker [2]_{\tilde{A}_a}$. By Mumford [18, Section 23, Theorem 3], we see that there exists a principal divisor $\Xi$ on $A$ such that $\pi^*D \approx 2\Xi$. Hence, we have the following theorem.

**Theorem 7.10.** Let $C$ be a non-hyperelliptic curve of genus 3 with an automorphism $\sigma$ of order two. Then, related to the automorphism $\sigma$, there exist three decomposed Richelot isogenies outgoing from the Jacobian variety $J(C)$.

**Proof.** Using the notation above, we consider the isogeny

$$\tilde{\rho} : E \times \tilde{A}_a \xrightarrow{\text{id}_E \times \pi} E \times \tilde{A}_a \xrightarrow{i_{E+\tilde{A}}^A} J(C).$$

Then, we have $\tilde{\rho}^*\Theta = 2(O \times \tilde{A}_a + E \times \Xi)$. Therefore, there exists a homomorphism $\rho : J(C) \rightarrow E \times \tilde{A}_a$ such that $\rho \circ \tilde{\rho} = [2]_{E \times \tilde{A}_a}$ and $\rho^*(O \times \tilde{A}_a + E \times \Xi) = 2\Theta$. We have 3 possibilities for the choice of $a$. □

**Remark 7.11.** The decomposition, up to isogeny, of the Jacobian variety of a curve of genus 3 with automorphism of order 2 into a product of an elliptic curve and the Jacobian variety of a curve of genus 2 is studied and classified in Lombardo, García, Ritzenthaler and Sijsling [16]. They give the concrete equations for the curves of genus 2. Although the subject of the paper is not Richelot isogeny, their result is closely related to Theorem 7.10.

Now, we are ready to show Theorems I and II. As for Theorem I, the result that if $C$ has a long automorphism of order 2, then we have a decomposed Richelot isogeny outgoing from $J(C)$ is shown in Theorem 4.2 in case $C$ is hyperelliptic, and in Theorem 7.10 in case $C$ is non-hyperelliptic. The converse follows from Proposition 4.1 in case $C$ is hyperelliptic, and from Proposition 5.4 in case $C$ is non-hyperelliptic. As for Theorem II, the result in (1) on the target of the decomposed Richelot isogeny outgoing from $J(C)$ is given in Theorem 4.2. The injectivity of $f^*$ in (2) is proved in Lemma
7.1 and the irreducibility of \( A = \text{Ker } N_f \) is proved in Lemma 7.3. The structure of decomposed Richelot isogeny of the Jacobian variety \( J(C) \) of non-hyperelliptic curve \( C \) of genus 3 with an automorphism of order 2 in (2) is given in Theorem 7.10 with Corollary 5.2, which is the most important result in this paper. The former part of (3) is proved in Theorem 6.3 and the latter part of (3) is proved in Theorem 6.2.

Finally, we examine the number of superspecial curves of genus 3 whose Jacobian varieties \( J(C) \) have decomposed Richelot isogenies outgoing from \( J(C) \).

**Proposition 7.12.** Let \( k \) be an algebraically closed field of characteristic \( p > 2 \). Then, the asymptotic behavior of the rate of superspecial curves \( C \) of genus 3 whose Jacobian varieties \( J(C) \) have decomposed Richelot isogenies outgoing from \( J(C) \) to the superspecial curves of genus 3 is given by

\[
\frac{1260}{p^2}.
\]

**Proof.** By Brock [2] and Hashimoto [8], the main term of the number of superspecial curves of genus 3 is given by

\[
\frac{(p - 1)(p - 9)(p - 11)(p^3 + 20p^2 - 349p - 3200)}{1451520}.
\]

As for the main term of the number of superspecial curves of genus 3 with long automorphism of order 2, by Brock [2] it is given by

\[
\frac{(p - 1)(p - 9)(p^2 - 3p - 82)}{1152}.
\]

Therefore, we get our asymptotic behavior of the rate.

**REFERENCES**

[1] C. Birkenhake and H. Lange, Complex Abelian varieties, Springer-Verlag Berlin Heidelberg 1980.

[2] B. W. Brock, Superspecial curves of genera two and three, PhD thesis, Princeton Univ., 1993.

[3] W. Castryck, T. Decru and B. Smith, Hash functions from superspecial genus-2 curves using Richelot isogenies, Number-Theoretic Methods in Cryptology, 2019 (Nut MiC 2019), J. Math. Crypt. 14 (2020), 268–292.

[4] E. Ciani, I varii tipi possibili di quartiche piane piú volte omologico-armoniche, Palermo Rendi.,13 (1899), 347–373.

[5] C. Costello, Supersingular isogeny key exchange for beginners, in Selected Areas in Cryptography, SAC2019, 21–50.

[6] C. Costello and B. Smith, The supersingular isogeny problem in genus 2 and beyond, International Conference on Post-Quantum Cryptography, PQCrypto 2020: Post-Quantum Cryptography, Lecture Notes in Computer Science, 12100 (2020), 151–168.

[7] E. Florit and B. Smith, An atlas of the Richelot isogeny graph, IACR Cryptol. ePrint Arch. 2021: 13 (2021)
[8] K. Hashimoto, Class numbers of positive definite ternary quaternion hermitian forms, Proc. Japan Acad. 59 Ser. A (1983), 490–493.
[9] E. W. Howe, Quickly constructing curves of genus 4 with many points, Frobenius distributions: Lang-Trotter and Sato-Tate conjectures (D. Kohel and I. Shparlinski, eds.), Contemp. Math., vol. 663, Amer. Math. Soc., Providence, RI, 2016, pp. 149–173.
[10] E. W. Howe, F. Leprévost and B. Poonen, Large torsion subgroups of split Jacobians of curves of genus two or three, Forum Math. 12 (2000), 315–364.
[11] T. Ibukiyama, T. Katsura and F. Oort, Supersingular curves of genus two and class numbers, Comp. Math. 57 (1986), 127–152.
[12] D. Jao and L. De Feo, Towards quantum-resistant cryptosystems from supersingular elliptic curve isogenies, in “Proceedings of Post-Quantum Cryptography, 4th International Workshop, PQCrypto 2011, Taipei, Taiwan, November 29–December 2, 2011 (edited B. Yang)”, Lecture Notes in Computer Science, vol. 7071, Springer, 2011, pp. 19–34.
[13] B. W. Jordan and Y. Zaytman, Isogeny graphs of superspecial abelian varieties and generalized Brandt matrices, ArXiv:2005.09031v2[math.NT].
[14] T. Katsura and K. Takashima, Counting Richelot isogenies between superspecial abelian surfaces, in “Proceedings of the Fourteenth Algorithmic Number Theory Symposium (ANTS-XIV)” (edited by Steven Galbraith), Open Book Series 4, Mathematical Sciences Publishers, Berkeley, 2020, pp. 283–300.
[15] M. Kudo, S. Harashita and H. Senda, The existence of supersingular curves of genus 4 in arbitrary characteristic, Research in Number Theory, Vol. 6, Issue 4, Article number: 44 (2020).
[16] D. Lombardo, E. L. García, C. Ritzenthaler and J. Sijsling, Decomposed Jacobians via Galois covers, arXiv:2003.07774v1.
[17] J. S. Milne, Jacobian varieties, in “Arithmetic Geometry” (edited by G. Cornell and J. H. Silverman), Springer-Verlag, New York Berlin London Paris Tokyo, 1986, 167–212.
[18] D. Mumford, Abelian Varieties, Oxford Univ. Press, London/New York, 1970.
[19] B. Smith, Isogenies and the discrete logarithm problem in Jacobians of genus 3 hyperelliptic curves, J. Cryptography 22, (2009), 505–529.
[20] K. Takashima, Efficient algorithms for isogeny sequences and their cryptographic applications, in “Mathematical Modelling for Next-Generation Cryptography”, CREST Crypto-Math Project, 2017, 97–114.

Graduate School of Mathematical Sciences, The University of Tokyo, Meguro-ku, Tokyo 153-8914, Japan

Email address: tkatsura@ms.u-tokyo.ac.jp