On the Adversarial Robustness of Causal Algorithmic Recourse

Ricardo Dominguez-Olmedo1,2  Amir-Hossein Karimi1,3  Bernhard Schölkopf1,3

1Max-Planck-Institute for Intelligent Systems, Tübingen, Germany
2University of Tübingen, Germany
3ETH Zürich, Switzerland

Abstract

Algorithmic recourse seeks to provide actionable recommendations for individuals to overcome unfavorable outcomes made by automated decision-making systems. Recourse recommendations should ideally be robust to reasonably small uncertainty in the features of the individual seeking recourse. In this work, we formulate the adversarially robust recourse problem and show that recourse methods offering minimally costly recourse fail to be robust. We then present methods for generating adversarially robust recourse in the linear and in the differentiable case. To ensure that recourse is robust, individuals are asked to make more effort than they would have otherwise had to. In order to shift part of the burden of robustness from the decision-subject to the decision-maker, we propose a model regularizer that encourages the additional cost of seeking robust recourse to be low. We show that classifiers trained with our proposed model regularizer, which penalizes relying on unactionable features for prediction, offer potentially less effortful recourse.

1 Introduction

Machine learning (ML) classifiers are increasingly being used for consequential decision-making in domains such as justice and finance (e.g., granting pretrial bail or loan approval) [1]. The need to preserve human agency despite the rise in automatic decisions faced by individuals has motivated the study of algorithmic recourse, which aims to empower individuals by providing them with actionable recommendations to reverse unfavourable algorithmic decisions [37]. Prior works have argued that for recourse to warrant trust, the decision-maker must commit to reversing an unfavourable decision upon the decision-subject fully adopting their prescribed recourse recommendations [40, 38, 12]. We argue that if algorithmic recourse is indeed to be treated as a contractual agreement, then recourse recommendations must be robust to plausible uncertainties arising in the recourse process.

For instance, consider a bank that commits to approving the loan of an individual if they increase their salary by some amount. However, by the time the individual achieves the prescribed salary increase the country’s economic situation has slightly worsened and the classifier still deems the individual likely to default on the loan. Shielding the recourse recommendation against uncertainty ex-post by nonetheless granting the loan may be detrimental to both the bank (e.g., monetary loss) and the individual (e.g., bankruptcy and inability to secure future loans), while breaking the recourse promise would negate the effort exerted by the individual and erode trust in the decision maker. We therefore argue for the necessity of ensuring that recourse recommendations are ex-ante robust to uncertainty.

In this work, we direct our focus towards robustifying recourse recommendations against uncertainty in the features of the individual seeking recourse. Such uncertainty may arise due to the temporal nature of recourse (e.g., some features may not be static [37]), and/or the presence of noise [8, 41], adversarial manipulation [17, 35] and other misrepresentations or errors [42]. Previous works on the
robustness of recourse with respect to uncertainty in the features of the decision-subject have studied
whether the cost of recourse is robust [39, 34], that is, to what extent are similar individuals assigned
recourse recommendations with similar cost. In contrast, we focus on the validity of recourse, and
seek recourse recommendations which remain valid (i.e. lead to favourable classification outcomes)
for all plausible individuals similar to the individual seeking recourse. We refer to this notion of
robustness as the adversarial robustness of recourse, in order to distinguish it from other robustness
considerations previously studied in the recourse literature (e.g., robustness with respect to changes to
the decision-making classifier [29, 36, 3]), and as a reference to the adversarial robustness literature,
which considers robustness of prediction precisely against uncertainty in the features of the data.

We study the adversarial robustness of recourse from the lens of causality [26]. Causal recourse
views recourse recommendations as causal interventions on the features of the decision-subject [14],
and therefore presents a more faithful account of how the features of the individual change as the
individual acts on its recourse recommendations, provided that the underlying structural causal model
is known or can be approximated reasonably-well [13]. In this work, we consider the problem
of robustifying recourse against uncertainty in the features of the individual seeking recourse. In
Section 2, we discuss the different sources of uncertainty present in the recourse process and relate
them with previous works on the robustness of recourse. In Section 3, we model the uncertainty in
the features of the individual seeking recourse by leveraging a counterfactual notion of similarity between
individuals which explicitly considers the causal relationships between features. In Section 4, we
formally define the adversarially robust recourse problem, we show that minimum-cost recourse fails
to be adversarially robust, and we derive sufficient conditions for the existence of adversarially robust
recourse. In Section 5 we discuss how to generate adversarially robust recourse in the linear and in
the differentiable case. In Section 6, we theoretically motivate a model regularizer that encourages
the additional cost of seeking robust recourse to be low. Finally, in Section 7 we empirically evaluate
the proposed methods in both the causal and non-causal recourse setting.

2 Background and related work

2.1 Structural causal models, interventions, and counterfactuals

We assume that the data-generating process of the features $X = \{X_1, \ldots, X_n\}$ of individuals
$x \in \mathcal{X}$ is characterised by a known structural causal model (SCM) [26] $\mathcal{M} = (\mathcal{S}, \mathcal{P}_U)$. The
structural equations $\mathcal{S} = \{X_i := f_i (X_{pa(i)}, U_i)\}_{i=1}^n$ describe the causal relationship between
any given feature $X_i$, its direct causes $X_{pa(i)}$ and some exogenous variable $U_i$ as a deterministic
function $f_i$. The exogenous variables $U \in \mathcal{U}$, which are distributed according to some probability
distribution $P_U$, represent unobserved background factors which are responsible for the variations
observed in the data. We assume that the causal graph $\mathcal{G}$ implied by the SCM, with nodes $X \cup U$
and edges $\{(v, X_i): v \in X_{pa(i)} \cup U_i, i \in [1, n]\}$, is acyclic. The SCM $\mathcal{M}$ then implies a unique
observational distribution $p_X$ over the features $X$. Moreover, the structural equations $\mathcal{S}$ induce a
mapping $\mathcal{S} : \mathcal{U} \rightarrow \mathcal{X}$ between exogenous and endogenous variables. Under the assumption that
the exogenous variables are mutually independent (causal sufficiency), if there exists some inverse
mapping $\mathcal{S}^{-1} : \mathcal{X} \rightarrow \mathcal{U}$ such that $\mathcal{S}(\mathcal{S}^{-1}(x)) = x \ \forall x \in \mathcal{X}$, then the endogenous variables
corresponding to some individual $x \in \mathcal{X}$ are uniquely identifiable by $U|x = \mathcal{S}^{-1}(x)$.

SCMs allow for modelling and evaluating the effect of interventions on the system which the SCM
models. Hard interventions $do(X_\mathcal{I} = \theta)$ [26] fix the values of a subset $\mathcal{I} \subseteq [d]$ of features $X_\mathcal{I}$ to some
$\theta \in \mathbb{R}^{|\mathcal{I}|}$ by altering the structural equations of the intervened upon variables $S_{\mathcal{I}}^{do(X_\mathcal{I} = \theta)} = X_\mathcal{I} := \theta,$
while preserving the rest of the structural equations $S_i^{do(X_\mathcal{I} = \theta)} = S_i \ \forall i \notin \mathcal{I}$. Consequently, hard
interventions sever the causal relationship between an intervened upon variables and all of its ancestors
in the causal graph. Soft interventions, on the other hand, may modify the structural equations in a
more general manner [15]. In particular, additive interventions perturb the features $X$ with some
perturbation vector $\Delta \in \mathbb{R}^n$ while preserving all causal relationships. Additive interventions alter
the structural equations according to $\mathcal{S}^{\Delta} = \{X_i := f_i (X_{pa(i)}, U_i) + \Delta_i\}_{i=1}^n$ [7].

Moreover, SCMs imply distributions over counterfactuals, allowing to reason about what would
have happened under certain hypothetical interventions all else being equal. Under the aforementioned
assumptions, the counterfactual $x^{CF}$ pertaining to some observed factual individual
$x \in \mathcal{X}$ under some hypothetical hard intervention $do(X_\mathcal{I} = \theta)$ (resp. soft intervention $\Delta$) can
We use the notation \( x \) (e.g., race) or bounded (e.g., age), only feasible actions should be recommended. The action independently validates the recourse, as highlighted in Equation 1. Some well-studied sources of uncertainty in the data \cite{42}. Regarding the classifier presence of noise \cite{8,41}, adversarial manipulation \cite{17,35} and other misrepresentations or errors in the literature has focused on uncertainty in the inputs classification setting naturally extend to algorithmic recourse. A great deal of the robust classification uncertainties may arise throughout the recourse process, as depicted in Figure 1, and may alter the validation of recourse, as highlighted in Equation 1. Since certain features may be immutable (e.g., race) or bounded (e.g., age), only feasible actions should be recommended. The action feasibility set \( \mathcal{F}(x) \) captures the set of feasible actions available to the individual \( x \). Ideally, recourse recommendations should incur the least amount of effort possible for decision-subjects. The cost function \( c(x, a) \) models the effort required by an individual \( x \) to implement the recourse action \( a \). Finding the minimum-cost recourse action for some individual \( x \) is equivalent to solving the following optimization problem:

\[
\arg\min_{a = \text{do}(X_\mathcal{I} = x_\mathcal{I} + \theta)} c(x,a) \quad \text{s.t.} \quad a \in \mathcal{F}(x) \land \left\langle h_x, \left( \mathcal{CIF} \left( \hat{x}_\mathcal{I}, a, \hat{M}_\mathcal{I} \right) \right) \right\rangle = 1 \tag{1}
\]

The non-causal recourse setting is equivalent to the causal recourse setting under the independently manipulable features (IMF) assumption, that is, if no causal relationships exist between features. Under such assumption, \( \mathcal{CIF} \left( x, \text{do}(X = x + \theta) \right) = x + \theta \).

2.2 The causal recourse problem

Consider the classification setting where a classifier \( h : \mathcal{X} \to \{0,1\} \) is used to assign either favourable or unfavourable outcomes to individuals \( x \in \mathcal{X} \) (e.g., loan approval). Algorithmic recourse aims to provide unfavourably classified individuals a set of recommendations which if acted upon would result the individual being favourably classified \cite{37,40}. We adopt the causal view of recourse introduced by Karimi et al. \cite{14} and model recourse actions as a hard interventions on the features of the individual seeking recourse, that is, \( a = \text{do}(X_\mathcal{I} = x_\mathcal{I} + \theta) \). We consider this additive form, rather than \( a = \text{do}(X_\mathcal{I} = \theta) \) as Karimi et al. \cite{14}, to explicitly allow for the uncertainty in the factual individual \( x \) to propagate to the recourse action \( a \). For a recourse action \( a \) to be considered valid, the corresponding counterfactual individual must be favourably classified, that is, \( h(\text{CIF}(x, a, \hat{M})) = 1 \). Since certain features may be immutable (e.g., race) or bounded (e.g., age), only feasible actions should be recommended. The action feasibility set \( \mathcal{F}(x) \) captures the set of feasible actions available to the individual \( x \). Ideally, recourse recommendations should incur the least amount of effort possible for decision-subjects. The cost function \( c(x, a) \) models the effort required by an individual \( x \) to implement the recourse action \( a \). Finding the minimum-cost recourse action for some individual \( x \) is equivalent to solving the following optimization problem:

\[
\arg\min_{a = \text{do}(X_\mathcal{I} = x_\mathcal{I} + \theta)} c(x,a) \quad \text{s.t.} \quad a \in \mathcal{F}(x) \land \left\langle h_x, \left( \mathcal{CIF} \left( \hat{x}_\mathcal{I}, a, \hat{M}_\mathcal{I} \right) \right) \right\rangle = 1 \tag{1}
\]

2.3 Uncertainties in the recourse process and robustness

Uncertainties may arise throughout the recourse process, as depicted in Figure 1, and may alter the validity of recourse, as highlighted in Equation 1. Some well-studied sources of uncertainty in the classification setting naturally extend to algorithmic recourse. A great deal of the robust classification literature has focused on uncertainty in the inputs \( x \) at inference time, which may arise due to the presence of noise \cite{8,41}, adversarial manipulation \cite{17,35} and other misrepresentations or errors in the data \cite{42}. Regarding the classifier \( h \), the optimization problem solved for model training often does not have unique optimal solution and multiple models may perform equally well in the
training data [4, 32]. Moreover, the temporal nature of recourse introduces a unique challenge: the circumstances under which recourse is generated may change by the time the individual is able to implement their prescribed recourse. For instance, the distribution over inputs itself may change at inference time, under phenomena such as data-set shift [19, 28] or for tasks pertaining out of distribution generalization [9, 20]. From a causal perspective, changes in the observational data distribution are a consequence of changes to the underlying SCM [5].

Indeed, the data-generation process characterised by the SCM $\mathcal{M}$ may be imperfectly known [39] or may dynamically change over time to some other SCM $\mathcal{M} \in \mathcal{U}_\mathcal{M}$, where $\mathcal{U}_\mathcal{M}$ is the uncertainty set over future SCMs. Consequently, the counterfactual individual resulting from the prescribed recourse intervention may also change. Furthermore, decision-makers may have to periodically retrain their models to prevent performance degradation due to the distribution shift resulting from a change in the SCM, producing further uncertainty over the future classifier $\hat{h} \in \mathcal{U}_h$ [29, 36]. Finally, it may be unreasonable to expect the individual $x$ to not suffer changes outside of its control over a extended period of time (e.g., suffering an accident causing a decrease in savings) [38], leading to uncertainty in the future individual $\hat{x} \in \mathcal{U}_x$. Thus, acting on the prescribed recourse may not lead to favourable classification due to changes to the SCM $\mathcal{M}$, classifier $\hat{h}$, and/or factual individual $\hat{x}$.

2.4 Related work

We now draw connections with existing literature on the robustness of recourse. Previous works have identified that small changes to the features of the decision-subject $x$ may result in recourse recommendations with very different costs being offered. Slack et al. [34] show that for gradient-based recourse methods it is possible to maliciously train a classifier such that small perturbations to the features of an individual drastically alter the cost of the generated recourse, and von Kügelgen et al. [39] show that “counterfactual twins” obtained by intervening on sensitive attributes (e.g., race, gender) may be assigned recourse actions with drastically different cost. While these works study the robustness of the cost of recourse, we instead focus in the robustness in the validity of recourse, that is, finding a recourse action which remains valid under uncertainty in the individual $x$.

Other works have considered the problem of generating recourse actions which remain valid under uncertainty in the classifier $h$. Pawelczyk et al. [23] show that recourse actions which place the counterfactual individual in regions of the feature space with large data support are more robust under predictive multiplicity compared to minimum-cost recourse actions. However, recourse actions with large data support may be unnecessarily costly. In contrast, our approach seeks counterfactual individuals which are sufficiently far from the decision-boundary to be robust but not overtly so, thus ensuring robust recourse while maintaining a relatively low cost of recourse. Another line of work has considered the robustness of recourse with respect to changes to the classifier in response of dataset shift. Rawal et al. [30] show that recourse actions are typically not robust to such model changes, and Upadhyay et al. [36] aim to mitigate this issue by generating recourse with a minimax optimization procedure where the cost the recourse is minimized subject to the recourse action being valid under adversarial changes to the classifier $h$. In contrast, we focus on generating recourse under uncertainty in the individual $x$ rather than the classifier $h$, and we consider the causal recourse setting.

Finally, Karimi et al. [13] consider the setting where the underlying SCM is not know and thus must be approximated, and propose a recourse method to generate recourse recommendations which have low probability of being invalid due to the misspecification of the underlying SCM. Our work is tangential to Karimi et al. [13] and both approaches can be used in tandem.

3 Counterfactual similarity

In the adversarial robustness literature, uncertainty in the features of some observed data point $x$ is often modelled by an $\epsilon$-ball of uncertainty $B(x) = \{x + \delta : \|\delta\| \leq \epsilon\}$ around $x$ [17, 2], where the norm $\|\cdot\|$ characterizes some notion of similarity $d(x, y) = \|x - y\|$. Intuitively, small changes $\delta$ to the data point $x$ result in similar data points $x' = x + \delta$. The uncertainty set $B(x)$ can therefore be interpreted as a neighbourhood of plausible data points similar to the observed $x$.

From a causal perspective, such feature changes $\delta$ are equivalent to additive interventions on the features $x$ under the IMF assumption, that is, if not causal relationships exist between features. However, in the causal recourse setting we assume to know such causal relationships, which are fully
specified by the assumed SCM $M$. We argue that explicitly considering these causal relationships can potentially provide more informative neighbourhoods of individuals.

**Definition 1** ($\epsilon$-Neighbourhood of counterfactually similar individuals). For some similarity norm $\|\cdot\|$ and SCM $M$, the $\epsilon$-neighbourhood of counterfactually similar individuals to some individual $x$ is defined as the set of counterfactual individuals under all possible $\epsilon$-small additive interventions

$$B(x) = \{\text{CF}(x, \Delta, M) \mid \|\Delta\| \leq \epsilon\} \quad (2)$$

As a motivating example, consider the SCM $M$ with features $X_1 = U_1$ and $X_2 = X_1 + U_2$ respectively denoting the income and savings of some individual $x$. Figure 2a illustrates the observational and counterfactual neighbourhoods of similar individuals for the 2-norm similarity metric $\|\cdot\|_2$. Observe that under the counterfactual neighbourhood, the individual $x$ is more similar to some individual $\tilde{x}$ with higher income and higher savings than to some other individual $\tilde{\tilde{x}}$ with higher income but lower savings, since the latter is not well explained by the SCM $M$ and thus its circumstances may substantially differ from those of $x$ (e.g. has a much larger number of individuals dependent on them, resulting in lower savings despite its higher income). Therefore, we argue that counterfactual neighbourhoods can be more informative than observational neighbourhoods, since the causal relationships between features are explicitly considered.

While we solely consider counterfactuals resulting from additive interventions, we recognize that other interventions could be considered, such as hard interventions or interventions on the distribution over exogenous variables $P_U$. In the particular case of additive noise models, additive interventions are equivalent to shifts in the exogenous variables $U$. We leave possible extensions for future work.

4 The adversarially robust recourse problem

We consider the problem of generating recourse recommendations which are robust to uncertainty in the features of the individual seeking recourse. We draw inspiration from the adversarial robustness literature, and require robust recourse actions $a$ to remain valid for every plausible individual in the uncertainty set $B(x)$ of individuals similar to the individual seeking recourse. Equivalently, as illustrated in Figure 2b, for every plausible factual individual $x' \in B(x)$ in the uncertainty set, the counterfactual resulting from the recourse intervention $a$ must be favourably classified.

**Definition 2** (Adversarially robust recourse problem). For some uncertainty set $B(x)$, the minimum-cost recourse action which remains valid for all plausible individuals $x' \in B(x)$ is given by

$$\arg\min_{a = \text{do}(X_2 = x_2 + \theta)} \max_{x' \in B(x)} c(x', a) \quad \text{s.t.} \quad a \in \mathcal{F}(x') \land h(\text{CF}(x', a)) = 1 \quad (3)$$
In this section, we first show that under mild conditions, minimum-cost recourse is fragile to arbitrarily small uncertainty in the features of the individual seeking recourse. Then, we derive sufficient conditions for the existence of adversarially robust recourse, summarized in Table 1.

4.1 Recourse is fragile under mild conditions

Intuitively, minimum-cost recourse actions place the counterfactual individual arbitrarily close to the decision boundary of the classifier, since pushing the counterfactual further into the favourably classifier region of the feature space would incur additional cost. Consequently, under arbitrarily small uncertainty in the features of the individual seeking recourse, some of the plausible counterfactuals may be unfavourably classified, as illustrated in Figure 2b. Theorem 1 states mild conditions under which minimum-cost recourse actions are indeed fragile.

Theorem 1. Let $a^*$ be the solution to the recourse optimization problem stated in Equation 1. If

1. The cost function $c(x, do(X_T = x_T + \theta))$ is strictly convex in $\theta$ with minimum $\theta = 0$
2. $do(X_T = x_T + \theta) \in \mathcal{F}(x) \implies do(X_T = x_T + t\theta) \in \mathcal{F}(x)$ for all $0 < t < 1$
3. The SCM $\mathcal{M}$ is an additive noise model [26].

Then for any $\epsilon > 0$ there exists $x' \in B(x) = \{ CF(x; \Delta) \mid \| \Delta \| \leq \epsilon \}$ such that $h(CF(x', a^*)) = 0$, that is, the recourse action $a^*$ is fragile to arbitrarily small levels of uncertainty $\epsilon > 0$.

The conditions in Theorem 1 are often assumed by previous works. The first condition requires that larger changes to the features imply strictly more effort from the individual, which is satisfied by the most widely used cost functions, namely weighted p-norms [13] and percentile costs [37]. The second condition states that if it is feasible to change a feature by some amount, then it must also be feasible to change that feature to a lesser degree, which is satisfied for the box actionability constrains commonly assumed in the recourse literature (e.g., features are unbounded, bounded or immutable [12]). The third condition is assumed by causal recourse approaches which estimate the underlying SCM $\mathcal{M}$ from data [13], and also holds in the non-causal recourse setting (i.e. under the IMF assumption). Therefore, in the settings commonly considered by the algorithmic recourse literature, recourse methods seeking minimum-cost recourse generate recourse recommendations which are fragile to even arbitrarily small uncertainty in the features of the individual seeking recourse.

4.2 Sufficient conditions for the existence of robust recourse

The conditions required for the existence of robust recourse are strictly more restrictive than those required for the existence of standard recourse, since all plausible counterfactuals must be favourably classified rather than solely the one corresponding to the factual $x$. Example 1, illustrated in Appendix A.2, shows that even under the strong assumption that all features are actionable and that there exists recourse for every individual $x \in \mathcal{X}$, robust recourse may not exist for any individual $x \in \mathcal{X}$.

Example 1. Consider $x \in \mathbb{R}^2$, $h(x) = \sin(2\gamma \pi^{-1} x_2) \geq 0$ for $0 < \gamma < \epsilon$ and the uncertainty set $B(x) = \{ x + \Delta \mid \| \Delta \|_2 \leq \epsilon \}$. Whilst there exists some recourse recommendation for all $x \in \mathbb{R}^2$, there does not exist any adversarially robust recourse recommendation for any $x \in \mathbb{R}^2$.

The above example relies on the fact that the classifier does not produce robust predictions for any $x \in \mathcal{X}$, and therefore no counterfactual can remain valid (i.e. favourably classified) in the presence of uncertainty. This hints to some relation between robustness of prediction and robustness of recourse. In particular, we show that for recourse to exist, the classifier must be minimally robust in the sense that there must exist at least one individual $x^+ \in \mathcal{X}$ such that $h(x^+) = 1$ is robustly classified.

Lemma 1. If all features are actionable and there exists some $x^+ \in \mathcal{X}$ such that $h(x') = 1$ for all $x' \in B(x^+)$, then there exists some adversarially robust recourse recommendation for all $x \in \mathcal{X}$.

In order to relax the condition that all features must be actionable, we restrict ourselves to the case where both the classifier and the SCM are linear. Then, the existence of at least one actionable and unbounded feature is sufficient to guarantee the universal existence of robust recourse. Intuitively, the decision-maker can require arbitrarily large changes to an actionable and unbounded feature such that all plausible counterfactuals are favourably classified (e.g., increase savings for loan approval).
Table 1: Sufficient conditions for the existence of robust recourse.

| Classifier $h$                        | Actionability constraints | SCM $\mathcal{M}$ | Existence of recourse | Existence of robust recourse |
|----------------------------------------|---------------------------|-------------------|-----------------------|-----------------------------|
| $\exists x^+ \in \mathcal{X}$ s.t. $h(x^+) = 1$ | All features actionable   | Any               | Guaranteed (Ustun et al. [37]) | Not guaranteed (Example 1)   |
| $\exists x^+ \in \mathcal{X}$ s.t. $h(x') = 1$  
$\forall x' \in B(x^+)$ | All features actionable   | Any               | Guaranteed (Ustun et al. [37]) | Guaranteed (Lemma 1)         |
| Linear                                 | $\exists X_j$ actionable and unbounded | Linear            | Guaranteed (Lemma 2)     | Guaranteed (Lemma 2)         |
| Any                                    | All bounded, $\geq 1$ immutable | Any               | Not guaranteed (Ustun et al. [37]) | Not guaranteed (Follows directly) |

**Lemma 2.** For a linear classifier $h(x) = \langle w, x \rangle \geq b$ and an SCM $\mathcal{M}$ with linear structural equations, if there exists a feature $X_j$ such that $X_j$ is actionable and unbounded and $w_j \neq 0$, then there exists at least one adversarially robust recourse action for all $x \in \mathcal{X}$.

If all features are bounded and there exists at least one immutable feature, then as per Ustun et al. [37] Remark 3, it is not possible to guarantee the universal existence of recourse even in the linear case, and therefore it is also not possible to guarantee the universal existence of adversarially robust recourse.

## 5 Generating adversarially robust recourse

We now present methods to solve the adversarially robust recourse problem introduced in the previous Section. We consider both the linear and the differentiable case.

### 5.1 The linear case

For a linear classifier $h(x) = \langle w, x \rangle \geq b$ and linear SCM, generating robust recourse is equivalent to generating standard recourse for a modified classifier $h'(x) = \langle w, x \rangle \geq b'$ whose “acceptance threshold” is sufficiently increased (i.e. $b' \geq b$) such that all plausible counterfactuals remain above the acceptance threshold of the original classifier, as illustrated in Figure 3.

**Theorem 2.** Let $h(x) = \langle w, x \rangle \geq b$ be a linear classifier, $\mathcal{M}$ an SCM with linear structural equations, and $B(x) = \{\text{CF}(x, \Delta) | \|\Delta\| \leq \epsilon\}$ the uncertainty set of plausible individuals. If the feasibility set is invariant to perturbations to $x$, that is, $\forall x' \in B(x) : \mathcal{F}(x) = \mathcal{F}(x')$, then the adversarially robust recourse problem is equivalent to the following optimization problem

$$\min_{a=\text{do}(X_I=x_I+\theta)} c(x, a) \quad \text{s.t.} \quad a \in \mathcal{F}(x) \land \langle w, \text{CF}(x, a) \rangle \geq b + \|J_{\Delta x}^T w\|^* \epsilon$$  \hspace{1cm} (4)

where $\|\cdot\|^*$ denotes the dual norm of $\|\cdot\|$ and $J_{\Delta x}$ denotes the Jacobian of the interventional mapping resulting from hard-intervening on features $X_I$.

**Corollary 1.** Under the conditions of Theorem 2 and additionally under the IMF assumption, the adversarially robust recourse problem for the classifier $h(x) = \langle w, x \rangle \geq b$ is equivalent to the standard recourse problem for the modified classifier $h'(x) = \langle w, x \rangle \geq b + \|w\|^* \epsilon$.

We highlight the importance of this result: if the conditions for Theorem 2 are satisfied, then generating recourse recommendations with respect to the modified classifier $h'$ guarantees that recourse is adversarially robust. Conveniently, one may use any given recourse generation method from the rich corpus on algorithmic recourse in order to enforce, together with adversarial robustness, other desiderata such as large data-support [11, 23] or fairness constrains [10, 39].
5.2 The differentiable case

Similarly to Wachter et al. [40], we consider the objective function

$$\mathcal{L}(x, a, \lambda) = c(x, a) + \lambda \ell (h(\text{CF}(x, a)), 1)$$

where $\ell$ is the binary cross entropy loss. The adversarially robust recourse problem is then equivalent to the following unconstrained penalty problem

$$\max_{\lambda \geq 0} \min_{a = \text{do}(X_I = x_I + \theta)} \max_{x' \in B(x)} \mathcal{L}(x', a, \lambda)$$

For the outer maximization problem in Equation 6, we adopt a common approach in the literature [25, 13] and, starting with a sufficiently small $\lambda > 0$, we iteratively increase $\lambda$ until some recourse action is found. Intuitively, small values of $\lambda$ favour low-cost recourse actions, whereas iteratively increasing $\lambda$ places a growing emphasis on crossing the classifier’s decision boundary in order for the counterfactual individual to be favourably classified. For the minimization problem in Equation 6, we use projected gradient descent. The actionability constraints typically considered in the algorithmic recourse literature (e.g. whether features are actionable and bounded [37]) amount to box constrains, and thus projecting to $\mathcal{F}(x)$ is trivial. The inner maximization problem in Equation 6 can be readily solved using projected gradient ascent [17] over the uncertainty set $B(x) = \{ \text{CF}(x, \Delta) \ | \ ||\Delta|| \leq \epsilon \}$. To reduce computational run-time, we instead maximize the first order Taylor approximation of $\mathcal{L}$ with respect to $x$, which has the following solution in closed form:

$$x^* = \text{CF} \left( x, \frac{g\epsilon}{\|g\|} \right) \text{ s.t. } g = \nabla_{\Delta} \mathcal{L} (\text{CF}(x, \Delta), a, \lambda)$$

We summarize the proposed optimization procedure in Algorithm 1. Note that if there is no uncertainty in the features of the individual, that is $B(x) = \{ x \}$, then the optimization procedure is precisely that of Wachter et al. [40] and Karimi et al. [13], for causal and non-casual recourse respectively. Our approach is also similar to Upadhyay et al. [36], with the difference that they consider an inner maximization problem over perturbations to the classifier $h$ rather than the individual $x$, they treat $\lambda$ as a fixed hyperparameter, and they consider the non-casual recourse setting.

6 A model regularizer to reduce the additional cost of recourse

In the previous sections, we have assumed a fixed classifier for which robust recourse must be generated. Then, to ensure that recourse recommendations are robust, individuals are asked to make more effort than they would have otherwise had to. Consequently, the burden of immunizing recourse...
against uncertainty falls solely on the decision-subjects. We argue, however, that robust recourse desiderata could be directly embedded into the training of the classifier. Satisfying such desiderata may come at a cost in predictive accuracy, thus shifting part of the burden of robust recourse from the decision-subject to the decision maker. In this section, we first restrict ourselves to the linear case in order to theoretically motivate a regularization penalty to reduce the additional cost of robust recourse. We then extend this regularization penalty to the differentiable case by drawing inspiration from local linearity regularization [27], a popular technique from the adversarial robustness literature.

6.1 An upper bound on the additional cost of robust recourse

We restrict ourselves to the linear case in order to derive an upper bound on the additional cost of robust recourse under certain actionability assumptions. For some recourse action \( a = \text{do}(X_I = x_I + \theta) \), we provide an expression for the increase in action magnitude \( \beta \) required for the more effortful recourse action \( a' = \text{do}(X_I = x_I + (1 + \beta \epsilon)\theta) \) to be a robust recourse action.

**Theorem 3.** Let \( h \) be a linear classifier \( h(x) = \langle w, x \rangle \geq b \), \( M \) an SCM with linear structural equations, \( x \in \mathcal{X} \) a negatively classified individual for which there exists some recourse action \( a = \text{do}(X_I = x_I + \theta) \), and \( B(x) = \{ \mathbb{C}(\mathbb{F}, \Delta) \ | \ |\Delta|| \leq \epsilon \} \). Then, there exists some constant

\[
\beta = \frac{\|J_g^{T}w\|^*}{\langle J_g^{T}w, \theta \rangle}
\]  

(8)

such that for \( a' = \text{do}(X_I = x_I + (1 + \beta \epsilon)\theta) \) it holds that \( \mathbb{C}(\mathbb{F}, \Delta) = 1 \forall x' \in B(x) \). If \( a' \) is a feasible action \( a' \in \mathcal{F}(x) \), then it follows that \( a' \) is an adversarial robust recourse action.

**Corollary 2.** Under the assumptions of Theorem 3 and additionally under the assumption that the cost function is subadditive, then the additional cost incurred by robustifying action \( a \) is

\[
\frac{c(x, a') - c(x, a)}{c(x, a)} \leq \beta \epsilon
\]  

(9)

Consequently, \( \beta \epsilon \) constitutes an upper bound on the additional cost of recourse incurred by the decision-subject as a result of seeking robust recourse. Observe that \( \beta \) (Equation 8) depends on the weights of the classifier \( w \), and thus it may be possible to regularize \( w \) such that the upper bound on the additional cost of recourse \( \beta \epsilon \) is reduced. For simplicity, we henceforth make the IMF assumption, such that \( J_g^{T} \) is the identity matrix. Let \( A \) (resp. \( U \)) be the set of features which are actionable (resp. unactionable) and \( m_A \in [0, 1]^n \) (resp. \( m_U \in [0, 1]^n \)) the mask vector such that \((m_A)_i = 1 \iff i \in A \) (resp. \((m_U)_i = 1 \iff i \in U \)). It is then possible decompose \( \beta \) into the weights corresponding to actionable features and those corresponding to unactionable features:

\[
\beta = \frac{\|w\|^*}{\langle w, \theta \rangle} = \frac{\|m_A \circ w + m_U \circ w\|^*}{\langle m_A \circ w, \theta \rangle} = \frac{\|m_A \circ w\|^* + \|m_U \circ w\|^*}{\langle m_A \circ w, \theta \rangle}
\]  

(10)

where \( \circ \) denotes the elementwise product. Consequently, reducing the dual norm \( \|m_U \circ w\|^* \) of the classifier weights corresponding to the unactionable features directly reduces the value of \( \beta \), that is, the upper bound on the additional cost of robust recourse. In particular, we propose to use the unactionability penalty \( \|m_U \circ w\|^* \) as a regularization term during training, inducing the learning bias “the classifier should rely more strongly on actionable features for its predictions”.

6.2 Actionable local linearity regularization

We consider classifiers of the form \( h(x) = g(x) \geq b \), where \( g(x) \) is a differentiable function. With the aim of reducing the additional cost of robust recourse, we propose the following regularizer.

\[
\mathcal{R}(x) = \mu \|m_U \circ \nabla_x g(x)\|^* + \gamma \max_{\|\delta\| \leq \epsilon} |g(x + \delta) - \langle \delta, \nabla_x g(x) \rangle - g(x)|
\]  

(11)

which we denote as the actionable locally linear regularizer (ALLR). The first term corresponds to the previously motivated actionability penalty for the linear approximation \( h'(x') = \langle \nabla_x g(x), x' \rangle \geq b' \) of the classifier \( h \) around \( x \), where \( b' = b + \langle \nabla_x g(x), x \rangle - g(x) \). The second term, inspired by the
locally linear regularizer (LLR) [27], encourages the function \( g \) to behave linearly near \( x \), such that the linear classifier \( h' \) is a reasonably accurate approximation of \( h \) around \( x \). Intuitively, the first term aims to reduce the upper bound on the additional cost of robust recourse for the linear classifier \( h' \) which locally approximates \( h \), while the second term aims to ensure that such upper bound is reasonably accurate for the actual classifier \( h \). The hyperparameters \( \mu, \gamma \in \mathbb{R} \) determine the strength of regularization. The classifier is then trained by minimizing the following objective

\[
E_{(x,y) \sim p(x,y)} [\ell(h(x), y) + R(x)]
\]

where \( \ell \) is some loss function (e.g., binary cross-entropy) and \( p(x, y) \) is the training data distribution.

7 Experimental results

Firstly, we empirically validate the method proposed in Section 5.2 for generating adversarially robust recourse in the differentiable case. Secondly, we empirically show that classifiers trained with the proposed actionable locally linear regularizer (ALLR) potentially offer recourse at a lower cost.

For the causal recourse setting, we consider one semi-synthetic dataset and one real-world dataset. Firstly, we consider a 7-variable semi-synthetic SCM introduced by Karimi et. al [13] and inspired by the German Credit UCI dataset [21], where individuals are classified as having either high or low credit risks. We consider two features as actionable: education and income. Some of the structural equations in the SCM are non-linear. Secondly, we consider the Adult dataset [21], which contains census data from 1994 on over 45,000 individuals with the prediction target being individuals have a yearly salary greater than $50,000. We consider the causal graph assumed in Nabi and Shpitser [22], and fit the structural equations as 1-layer MLPs. We use 6 features, out of which we consider two actionable: the education level and the number of weekly working hours.

For the non-causal recourse setting, we consider two real-world datasets. The COMPAS dataset [16] contains information on over 10,000 criminal defendants in Broward County, Florida, and the prediction target is whether the defendants reoffended in a two year period. We use 9 features, out of which we consider one actionable: the number of previous crimes. Lastly, we use a dataset (denoted as Bail for consistency with Ress et al. [31]) containing the information of over 4,000 individuals released from prison in 1978 in the state of North Carolina [33], where the target variable is whether the individuals reoffended in a two year period. We use 15 features, out of which we consider two actionable: the number of years of school coursed, and the number of reported prison rule violations.

For the considered datasets, all actionable features and all of its causal descendants are real-valued, with the exception of “education” in the Adult data set, which we assumed to be real-valued. We standardize all real-valued features. We generate recourse for negatively classified individuals from the test data, and we use as the cost function the \( \ell_1 \) norm of the prescribed feature change, that is \( c(x, a = do(X_T = x_T + \theta)) = \|\theta\|_1 \). For the classifiers, we consider a logistic regression (LR) model, and a 3-layer MLP with 100 hidden units and tanh activation. We define the uncertainty set \( B(x) \) with respect to the 2-norm, that is, \( B(x) = \{(\text{CIF}(x, \Delta)) \mid \|\Delta\|_2 \leq \epsilon\} \).

7.1 Generating adversarially robust recourse

We first evaluate whether the method proposed in Section 5.2 generates recourse actions which are robust to uncertainty in the features of the individual seeking recourse. To do so, we assess whether the generated recourse \( a \) is valid (e.g., leads to favourable classification outcomes) for all plausible individuals in the uncertainty set \( x^f \in B(x) \). In particular, we search for the smallest additive intervention \( \Delta^* \) on the features of the individual such that \( a \) is no longer a valid recourse action

\[
\Delta^* = \arg\min_{\Delta} \|\Delta\|_2 \quad \text{s.t.} \quad h(\text{CIF}(x^f, a)) = 0, \ x^f = \text{CIF}(x, \Delta)
\]

If the magnitude of such intervention \( \Delta^* \) is smaller than the level of uncertainty \( \epsilon \) we aim to robustify recourse against, that is \( \|\Delta^*\|_2 \leq \epsilon \), we can assert that the recourse action \( a \) is fragile. For our experiments, we use the method proposed in Section 5.2 to generate recourse recommendations against different magnitudes of uncertainty \( \epsilon \in \{0, 0.01, 0.1, 0.5\} \), where \( \epsilon = 0 \) corresponds to the standard non-robust recourse setting. For each generated recourse action, we use the Carlini & Wagned (C&W) attack [6], a popular method from the adversarial robustness literature, to search for the smallest intervention \( \Delta^* \) which invalidates the recourse action. When assessing the robustness of standard recourse (\( \epsilon = 0 \)), we assume that there exists a small amount of uncertainty \( \epsilon = 0.01 \).
The experimental results are presented in Figure 4. Firstly, we observe that standard recourse may be invalidated by very small amounts of uncertainty in the order of $\epsilon = 10^{-6}$, and consequently a large portion of the recourse actions may be fragile. While we have showed that minimum-cost recourse actions are guaranteed to be fragile, in practice due to computational constrains the recourse problem is not solved exactly (e.g., optimization may be stopped after some maximum number of iterations) and the recourse action found may not be the minimum-cost recourse action.

For recourse that has been robustified against reasonably small amounts of uncertainty $\epsilon \in \{0.01, 0.1\}$, the magnitude of the smallest additive intervention required to invalidate the recourse action is no lower than $\epsilon$, and therefore the generated recourse actions are robust. For large amounts of uncertainty (i.e. $\epsilon = 0.5$), the first-order approximation made for solving the inner maximization problem may not be accurate, and consequently some of the offered recourse actions can be fragile. This can be remedied by solving the inner maximization problem with greater accuracy (e.g., using multi-step projected gradient ascent), at the expense of greater computational cost. Finally, we observe that robustifying recourse against larger uncertainty $\epsilon$ results in more costly recourse actions.

For completeness, in Appendix B.1 we consider two other popular adversarial attack techniques, the projected gradient descent attack [17] and the DeepFool attack [18]. In Appendix B.1 we also present results for the non-causal recourse setting for the COMPAS and Bail datasets.

### 7.2 Actionable local linearity regularization: reducing the cost of recourse

We empirically evaluate whether classifiers trained with the proposed ALLR regularizer offer lower cost recourse compared to classifiers trained with empirical risk minimization (ERM), the standard model training setting considered in the algorithmic recourse literature. Since our proposed ALLR regularizer enforces the inductive bias “the classifier should rely more strongly on the actionable features”, we also train as a baseline comparison classifiers which only use actionable features (AF). Such case corresponds to ALLR with arbitrarily large strength of regularization $\gamma$.

For each of the three different model training approaches discussed (ERM, AF, ALLR), we train five different classifiers using different train and test data splits. We then generate standard recourse and...
Figure 5: Mean cost of robust recourse and classification accuracy for each of the three model training approaches considered. Classifiers trained with ALLR offer lower-cost recourse compared to classifiers trained with ERM, while maintaining higher classification accuracy than AF classifiers.

robust recourse ($\epsilon = 0.1$) for 100 negatively classified individuals. We compute the mean cost of recourse as well as the percentage of individuals for which recourse was found.

We present the experimental results in Figure 5. Compared to ERM, our proposed regularizer reduces the mean cost of recourse by up to 60%, while classification accuracy may only decrease by up to 2%, and in some cases it does not decrease at all. While the proposed regularizer is theoretically motivated for reducing the additional cost of robust recourse, we observe that the cost of standard recourse also decreases for all but one of the datasets considered. We also observe that using only actionable features (AF), equivalent to ALLR with arbitrarily large regularization, can result in even greater reductions in cost of recourse compared to both ERM and ALLR. However, both classification accuracy and the percentage of individuals for which recourse is found may significantly decrease. Therefore, we argue that ALLR provides the most favourable trade-off between classification accuracy and cost of recourse. Consequently, ALLR can be an effective technique in regulating the burden of (robust) recourse between the decision-maker and the decision-subject.

For completeness, in Appendix B.2 we compare our proposed regularizer against other related model-training approaches (L2 regularization, local linearity regularization [27] and adversarial training [35, 17]). None of these approaches reduces cost of recourse to an extent comparable to ALLR. On the contrary, we find evidence that model training approaches which promote robustness of prediction produce classifiers that offer recourse at much greater cost. We leave the study of the relation between robustness of prediction and robustness of recourse for future work.

8 Conclusion

Uncertainty in the recourse process is inevitable. Previously suggested ex-post solutions to mitigate the effect of uncertainty in the recourse process may result in negative outcomes for both the decision-maker and the individual. We instead adopt an ex-anti approach to robustness of recourse by requiring the recourse recommendations to be robust to uncertainty in the features of the individual seeking recourse. We show that, in practice, minimum-cost recourse is fragile to arbitrarily small uncertainty in the features of the individual. To address this, we formulate the adversarially robust recourse problem, and present methods to generate adversarially robust recourse in both the linear and differentiable case. Finally, in order to regulate the burden of robustness between the decision-maker and the decision-subject, we theoretically motivate a model regularizer that encourages the additional
cost of seeking robust recourse to be low. We empirically show that classifiers trained with our proposed model regularizer, which penalizes relying on unactionable features for prediction, offer potentially less effortful recourse in both the causal and the non-causal recourse setting.

References

[1] Solon Barocas, Moritz Hardt, and Arvind Narayanan. Fairness and machine learning. fairmlbook.org. URL: http://www.fairmlbook.org, 2019.

[2] Dimitris Bertsimas, Jack Dunn, Colin Pawlowski, and Ying Daisy Zhuo. Robust classification. INFORMS Journal on Optimization, 1(1):2–34, 2019.

[3] Emily Black, Zifan Wang, Matt Fredrikson, and Anupam Datta. Consistent counterfactuals for deep models. arXiv preprint arXiv:2110.03109, 2021.

[4] Leo Breiman. Statistical modeling: The two cultures. Statistical science, 16(3):199–231, 2001.

[5] Peter Bühlmann. Invariance, causality and robustness. Statistical Science, 35(3):404–426, 2020.

[6] Nicholas Carlini and David Wagner. Towards evaluating the robustness of neural networks. In 2017 IEEE symposium on security and privacy (sp), pages 39–57. IEEE, 2017.

[7] Frederick Eberhardt and Richard Scheines. Interventions and causal inference. Philosophy of science, 74(5):981–995, 2007.

[8] Alhussein Fawzi, Seyed-Mohsen Moosavi-Dezfooli, and Pascal Frossard. Robustness of classifiers: from adversarial to random noise. Advances in Neural Information Processing Systems, 29:1632–1640, 2016.

[9] Amir-Hossein Karimi, Bernhard Schölkopf, and Isabel Valera. Algorithmic recourse: from counterfactual explanations to interventions. In Proceedings of the 2021 ACM Conference on Fairness, Accountability, and Transparency, pages 353–362, 2021.

[10] Vivek Gupta, Pegah Nokhiz, Chitadeep Dutta Roy, and Suresh Venkatasubramanian. Equalizing recourse across groups. arXiv preprint arXiv:1909.03166, 2019.

[11] Shalmali Joshi, Oluwasanmi Koyejo, Warut Vijitbenjaronk, Been Kim, and Joydeep Ghosh. Towards realistic individual recourse and actionable explanations in black-box decision making systems. arXiv preprint arXiv:1907.09615, 2019.
[18] Seyed-Mohsen Moosavi-Dezfooli, Alhussein Fawzi, and Pascal Frossard. Deepfool: a simple and accurate method to fool deep neural networks. In Proceedings of the IEEE conference on computer vision and pattern recognition, pages 2574–2582, 2016.

[19] Jose G Moreno-Torres, Troy Raeder, Rocio Alain-Rodriguez, Nitesh V Chawla, and Francisco Herrera. A unifying view on dataset shift in classification. Pattern recognition, 45(1):521–530, 2012.

[20] Krikamol Muandet, David Balduzzi, and Bernhard Schölkopf. Domain generalization via invariant feature representation. In International Conference on Machine Learning, pages 10–18. PMLR, 2013.

[21] Patrick M Murphy. Uci repository of machine learning databases. ftp:/pub/machine-learning-databaseonics.uci.edu, 1994.

[22] Razieh Nabi and Ilya Shpitser. Fair inference on outcomes. In Proceedings of the AAAI Conference on Artificial Intelligence, volume 32, 2018.

[23] Martin Pawelczyk, Klaus Broelemann, and Gjergji Kasneci. On counterfactual explanations under predictive multiplicity. In Conference on Uncertainty in Artificial Intelligence, pages 809–818. PMLR, 2020.

[24] Martin Pawelczyk, Chirag Agarwal, Shalmali Joshi, Sohini Upadhyay, and Himabindu Lakkaraju. Exploring counterfactual explanations through the lens of adversarial examples: A theoretical and empirical analysis. arXiv preprint arXiv:2106.09992, 2021.

[25] Martin Pawelczyk, Sascha Bielawski, Johan Van den Heuvel, Tobias Richter, and Gjergji Kasneci. Carla: A python library to benchmark algorithmic recourse and counterfactual explanation algorithms. 35th Conference on Neural Information Processing Systems, 2021.

[26] Judea Pearl. Causality. Cambridge university press, 2009.

[27] Chongli Qin, James Martens, Sven Gowal, Dilip Krishnan, Krishnamurthy Dvijotham, Alhussein Fawzi, Soham De, Robert Stanforth, and Pushmeet Kohli. Adversarial robustness through local linearization. Advances in Neural Information Processing Systems, 32:13847–13856, 2019.

[28] Joaquin Quiñonero-Candela, Masashi Sugiyama, Neil D Lawrence, and Anton Schwaighofer. Dataset shift in machine learning. Mit Press, 2009.

[29] Kaivalya Rawal, Ece Kamar, and Himabindu Lakkaraju. Algorithmic recourse in the wild: Understanding the impact of data and model shifts. arXiv preprint arXiv:2012.11788, 2020.

[30] Kaivalya Rawal, Ece Kamar, and Himabindu Lakkaraju. Can i still trust you?: Understanding the impact of distribution shifts on algorithmic recourses. arXiv preprint arXiv:2012.11788, 2020.

[31] Alexis Ross, Himabindu Lakkaraju, and Osbert Bastani. Learning models for actionable recourse. Advances in Neural Information Processing Systems, 34, 2021.

[32] Cynthia Rudin. Stop explaining black box machine learning models for high stakes decisions and use interpretable models instead. Nature Machine Intelligence, 1(5):206–215, 2019.

[33] Peter Schmidt and Ann D Witte. Predicting recidivism in north carolina, 1978 and 1980. Inter-university Consortium for Political and Social Research, 1988.

[34] Dylan Slack, Sophie Hilgard, Himabindu Lakkaraju, and Sameer Singh. Counterfactual explanations can be manipulated. 35th Conference on Neural Information Processing Systems, 2021.

[35] Christian Szegedy, Wojciech Zaremba, Ilya Sutskever, Joan Bruna, Dumitru Erhan, Ian Goodfellow, and Rob Fergus. Intriguing properties of neural networks. In 2nd International Conference on Learning Representations, ICLR 2014, 2014.
[36] Sohini Upadhyay, Shalmali Joshi, and Himabindu Lakkaraju. Towards robust and reliable algorithmic recourse. *35th Conference on Neural Information Processing Systems*, 2021.

[37] Berk Ustun, Alexander Spangher, and Yang Liu. Actionable recourse in linear classification. In *Proceedings of the Conference on Fairness, Accountability, and Transparency*, pages 10–19, 2019.

[38] Suresh Venkatasubramanian and Mark Alfano. The philosophical basis of algorithmic recourse. In *Proceedings of the 2020 conference on fairness, accountability, and transparency*, pages 284–293, 2020.

[39] Julius von Kügelgen, Amir-Hossein Karimi, Umang Bhatt, Isabel Valera, Adrian Weller, and Bernhard Schölkopf. On the fairness of causal algorithmic recourse. *ICML 2021 Workshop on Algorithmic Recourse*, 2021.

[40] Sandra Wachter, Brent Mittelstadt, and Chris Russell. Counterfactual explanations without opening the black box: Automated decisions and the gdpr. *Harv. JL & Tech.*, 31:841, 2017.

[41] Huan Xu, Constantine Caramanis, and Shie Mannor. Robustness and regularization of support vector machines. *Journal of machine learning research*, 10(7), 2009.

[42] Stephan Zheng, Yang Song, Thomas Leung, and Ian Goodfellow. Improving the robustness of deep neural networks via stability training. In *Proceedings of the ieee conference on computer vision and pattern recognition*, pages 4480–4488, 2016.
A Proofs

A.1 Theorem 1

Let \( a^* = \text{do}(X_I = x_I + \theta^*) \) be the minimum-cost recourse action for a classifier \( h \) and an individual \( x \). Assume that \( a^* \) is a robust recourse action, that is, \( h(CF(x, \Delta), a^*) = 1 \) \( \forall \| \Delta \| \leq \epsilon \). Consider any \( I_j \) such that for all \( i \in I \), \( X_i \) is not a causal descendent of \( X_{I_j} \). Consider \( e_j \in \mathbb{R}^{|I_j|} \) such that \( (e_j)_j = 1 \) and \( (e_j)_j = 0 \ \forall i \neq j \). Then the action \( a = \text{do}(X_I = x_I - \theta^* + \alpha e_j \text{sign}(\theta_j)) \) is a valid recourse action, since \( h(CF(x, a)) = h(CF(x, e_j \text{sign}(\theta_j)), a^*) = 1 \) for any \( \alpha \leq \epsilon \), per the assumption that \( a^* \) is robust, and given that \( a \in \mathcal{F}(x) \) per assumption ii) in the Theorem. Furthermore, per assumption i) in the Theorem (strict convexity of the cost function), it must be that \( c(x, a) < c(x, a^*) \), which is a contradiction on \( a^* \) being a minimum-cost recourse action, and consequently the minimum-recourse action \( a^* \) must be fragile to perturbations \( x \).

A.2 Example 1

The shaded area is the favourably classified region of the feature space. While there exists recourse for every individual, there does not exist robust recourse for any individual.

![Diagram showing the shaded area as the favourably classified region.](image)

A.3 Lemma 1

Per assumption, there exists some \( x^+ \in \mathcal{X} \) such that \( h(x^+) = 1 \) for all \( x' \in B(x^+) \), where \( B(x^+) = \{CF(x^+, \Delta) \ | \ \| \Delta \| \leq \epsilon \} \). For any given individual \( x \), the action \( a = \text{do}(X = x + (x^+ - x)) \) results in the counterfactual individual \( x^{\text{CF}} = CF(x, a) = x^+ \). The action \( a \) is feasible, since all features are actionable. The action \( a \) is a recourse action, since \( h(x^{\text{CF}}) = h(x^+) = 1 \). Since the action \( a \) hard intervenes on all features, \( CF(CF(x, \Delta), a) = CF(CF(x, a), \Delta) = CF(x^+, \Delta) \), and consequently \( \{CF(CF(x, \Delta), a) \ | \ \| \Delta \| \leq \epsilon \} = \{CF(x^+, \Delta) \ | \ \| \Delta \| \leq \epsilon \} = B(x^+) \). It follows that \( a \) is a robust recourse action, since \( h(x') = 1 \) for all \( x' \in B(x^+) \).

A.4 Lemma 2

Per assumption, there exists some feature \( X_j \) such that \( X_j \) is actionable and unbounded, and \( X_j \) affects its causal descendants linearly. Consider the recourse action \( a = \text{do}(X_j = x_j + \theta) \) for \( \theta \in \mathbb{R} \). Per Theorem 2, we must find a recourse action such that \( \langle w, CF(x, a) \rangle \geq b' \). Due to the linearity assumptions on the SCM, \( CF(x, a) = x + \theta v \) for some \( v \in \mathbb{R}^n \). Then, \( \langle w, CF(x, a) \rangle = \langle w, x + \theta v \rangle = \langle w, x \rangle + \theta \langle w, v \rangle \). A robust recourse action is equivalent to any \( \theta \) such that \( \theta \langle w, v \rangle \geq b' - \langle w, x \rangle \). If \( \langle w, v \rangle \neq 0 \) (i.e., the non-trivial case where the weights of the classifier are not chosen adversarially to the SCM), then clearly it is possible to set \( \theta \) to have arbitrarily large magnitude and same sign as \( \langle w, v \rangle \), such that the inequality above is met. Since \( X_j \) is actionable and unbounded, \( a = \text{do}(X_j = x_j + \theta) \) is a feasible action. Consequently, \( a \) is a robust recourse action.
A.5 Theorem 2

The adversarially robust recourse problem is defined as
\[
\min_{a=do(X_I=x_I+\theta)} \max_{x' \in B(x)} c(x, a) \quad \text{s.t.} \quad a \in \mathcal{F}(x') \land h(CF(x', a)) = 1 \tag{14}
\]

Assuming \( h(x) = \langle w, x \rangle \geq b \) and \( \mathcal{F}(x) = \mathcal{F}(x') \forall x' \in B(x) \)
\[
\min_{a=do(X_I=x_I+\theta)} \max_{x' \in B(x)} c(x, a) \quad \text{s.t.} \quad a \in \mathcal{F}(x) \land \langle w, (CF(x', a)) \rangle \geq b \tag{15}
\]

For an action \( a \) to be robust feasible, the second constrain must hold for every \( x' \in B(x) \), that is,
\[
\left( \min_{x' \in B(x)} \langle w, (CF(x, a)) \rangle \right) \geq b \tag{16}
\]

Consequently, Equation 15 is equivalent to
\[
\min_{a=do(X_I=x_I+\theta)} c(a) \quad \text{s.t.} \quad a \in \mathcal{F}(x) \land \left( \min_{x' \in B(x)} \langle w, (CF(x', a)) \rangle \right) \geq b \tag{17}
\]

Then since the SCM \( \mathcal{M} \) is linear
\[
\text{CF}(\text{CF}(x, \Delta), a) = S^a (S^{-1} (x')) = S^a (S^{-1} (S^a (S^{-1} (x)))) = S^a (S^{-1} (S (S^{-1} (x) + \Delta))) = S^a (S^{-1} (x) + \Delta)
\]
\[
= S^a (S^{-1} (x)) + S^a (\Delta) = \text{CF}(x, a) + J_{\text{CF}} \Delta
\]

where \( J_{\text{CF}} \) denotes the Jacobian of the interventional mapping \( S^T \). Then
\[
\min_{x' \in B(x)} \langle w, (CF(x, a)) \rangle = \min_{\|\Delta\| \leq \epsilon} \langle w, (CF(x, a)) + J_{\text{CF}} \Delta \rangle = \langle w, (CF(x, a)) \rangle + \min_{\|\Delta\| \leq \epsilon} \langle w, J_{\text{CF}} \Delta \rangle = \langle w, (CF(x, a)) \rangle - \|J_{\text{CF}}^T w\|^{\star} \epsilon
\]

Consequently the optimization problem in Equation 17 reduces to
\[
\min_{a=do(X_I=x_I+\theta)} c(x, a) \quad \text{s.t.} \quad a \in \mathcal{F}(x) \land \langle w, (CF(x, a)) \rangle \geq b + \|J_{\text{CF}}^T w\|^{\star} \epsilon \tag{19}
\]

The corollary follows directly, since under the IMF assumption \( J_{\text{CF}} = I \), and then Equation 20 resembles the definition of the recourse problem in Equation 1 for the classifier
\[
h(x) = \langle w, x \rangle \geq b + \|w\|^{\star} \epsilon \tag{21}
\]

A.6 Theorem 3

Per Theorem 2, the robust recourse action \( a' = do(X_I = x_I + (1 + \beta \epsilon)\theta) \) must satisfy
\[
\langle w, CF(x, a') \rangle \geq b + \|J_{\text{CF}}^T w\|^{\star} \epsilon
\]
\[
\tag{22}
\]

Since the SCM is linear, \( \text{CF}(x, a') = x + J_{\text{CF}} (1 + \beta \epsilon)\theta \). Then,
\[
\langle w, \text{CF}(x, a') \rangle = \langle w, x + (1 + \beta \epsilon)J_{\text{CF}} \theta \rangle = \langle w, x + J_{\text{CF}} \theta \rangle + \beta \epsilon \langle w, J_{\text{CF}} \theta \rangle \geq b + \beta \epsilon \langle w, J_{\text{CF}} \theta \rangle \tag{23}
\]
where the last inequality follows by assumption that $a$ is a recourse action for $h(x) = \langle w, x \rangle \geq b$. Consequently, if
\[
\beta = \frac{\| J_{x \theta}^T w \|^*}{\langle w, J_{x \theta}^T \theta \rangle}
\]
then Equation 23 satisfies the robust recourse condition in Equation 22.

By assumption that $a$ is a recourse action then $\langle w, J_{x \theta} \rangle > 0$. Then $0 < \beta < \infty$. Consequently, if $a' \in F(x)$, the action $a' = do(X_I = x_I + (1 + \beta c)\theta)$ is a robust recourse action.

B Full experimental results

B.1 Validity of the methods proposed for generating robust recourse

Consider the function $f(\Delta) = h(CF(CF(x, \Delta), a))$, which maps additive interventions $\Delta$ on the factual individual to the classification outcome of the counterfactual individual under the prescribed recourse intervention $a$. From an adversarial robustness viewpoint, the optimization problem in Equation 13 is equivalent to searching for the smallest adversarial perturbation $\Delta^*$ (i.e., intervention of the features of $x$) which flips the output of $f$ from $f(0) = 1$ to $f(0 + \Delta^*) = 0$ (i.e., invalidates the recourse action $a$). Consequently, we propose to search for $\Delta^*$ by leveraging adversarial attack techniques form the adversarial robustness literature.

Projected Gradient Descent (PGD) Attack [17] We minimize $\min_{\|\Delta\|_2 \leq \gamma} \ell(x^{CF}|\Delta, 0)$ using projected gradient descent, where $\ell$ is the binary cross-entropy loss. This attack aims to find a $\gamma$-small perturbation such that the counterfactual individual is negatively classified. We consider three different strengths of attack $\gamma \in \{0.01, 0.1, 0.5\}$. In Table 2 we report the success rate of the attack for each $\gamma$ considered, that is, the rate at which recourse is invalidated by the PGD attack. We observe that generating adversarially robust recourse with respect to $\epsilon$-large perturbations guards recourse against $\epsilon$-large PGD attacks, with the exception of the largest perturbation, $\epsilon = 0.5$. We note that this perturbation is remarkably large, given that features are standardized. We hypothesize that the fragility stems from the first-order Taylor series approximation made to solve the inner maximization in Equation 7, which is only locally accurate. Consequently, the inner maximization should be solved exactly with projected gradient descent when guarding against very large perturbations.

C&W attack [6] We minimize $\min_{\Delta} \epsilon \cdot \ell(x^{CF}|\Delta, 0) + \|\Delta\|_2$ with 15 binary search steps over hyperparameter $\epsilon$, where $\ell$ is the binary cross-entropy loss. This attack aims to find the smallest perturbation to the individual such that the corresponding counterfactual individual is negatively classified, and thus provides some notion of distance between the counterfactual and the classifier’s decision boundary in the non-causal setting. In Table 2 we report the mean and minimum perturbation found by the C&W attack. We observe that generating adversarially robust recourse with respect to $\epsilon$-large perturbations places the counterfactual individual approximately $\epsilon$-far from the decision boundary, with the exception of very large $\epsilon$, where as previously discussed there are inaccuracies in solving the inner maximization when generating robust recourse. Note that in the standard recourse setting, that is $\epsilon = 0$, counterfactual may be arbitrarily close to the decision boundary.

DeepFool attack [18] The perturbation is updated by $\Delta_{t+1} = \Delta_t - \frac{f(x^{CF}|\Delta_t)}{\|g\|_2} g$, where $f$ maps to the logits of the classifier and $g = \nabla_{\Delta} f(x^{CF}|\Delta)|_{\Delta = \Delta_t}$. Similarly to the C&W attack, DeepFool attacks aim to find the smallest perturbation such that the counterfactual individual is negatively classified. The results of the DeepFool attack, shown in Table 2, are comparable to those of the C&W attack.

B.2 Comparing ALLR against other model training approaches

L2 regularization (L2) For linear classifiers, ALLR is equivalent to a norm penalty on the unactionable weights of the classifiers. We consider L2 regularization in order to determine to what extent penalizing all weights may impact the cost of recourse. We do not observe a significant difference.

Adversarial training (AT) [35] $\min_{\theta} \mathbb{E} \left[ \max_{x' \in B(x)} \ell(h_{\theta}(x'), y) \right]$ The loss is minimized subject to adversarial or “worst case” perturbations to the input. This approach is widely used in the adversarial robustness literature to train classifiers with robust predictions. We include AT in our experiments to assess if robustness of prediction facilitates robustness of recourse, and as a comparison to LLR.
δ along which the linearity condition is maximally violated is computed. Intuitively, the linearity of the loss is

Local linearity regularization (LLR) of prediction naturally hinders the existence of low cost recourse recommendations.

Table 2: Validity of the proposed methods for generating adversarially robust recourse.

| Robust recourse | PGD | Adversarial attacks | DeepFool |
|-----------------|-----|---------------------|-----------|
|                | $\epsilon$ | Found | Mean cost | $\gamma=0.01$ | $\gamma=0.1$ | $\gamma=0.5$ | Mean | Min | Mean | Min |
| **German** (Causal) | 0 | 1.00 | 1.33 ± 0.99 | 0.93 | 1.00 | 1.00 | 0.00 | 0.00 | 0.00 | 0.00 |
|                  | 0.01 | 1.00 | 1.35 ± 0.99 | 0.94 | 1.00 | 1.00 | 0.02 | 0.01 | 0.01 | 0.01 |
|                  | 0.10 | 1.00 | 1.56 ± 0.99 | 0.00 | 0.00 | 1.00 | 0.11 | 0.10 | 0.10 | 0.10 |
|                  | 0.50 | 1.00 | 2.47 ± 0.98 | 0.00 | 0.00 | 0.00 | 0.50 | 0.46 | 0.50 | 0.49 |
| **Adult** (Causal) | 0 | 1.00 | 2.06 ± 1.33 | 0.99 | 1.00 | 1.00 | 0.00 | 0.00 | 0.00 | 0.00 |
|                  | 0.01 | 1.00 | 2.09 ± 1.34 | 0.00 | 1.00 | 1.00 | 0.02 | 0.01 | 0.01 | 0.01 |
|                  | 0.10 | 1.00 | 2.34 ± 1.41 | 0.00 | 0.00 | 1.00 | 0.10 | 0.10 | 0.10 | 0.10 |
|                  | 0.50 | 1.00 | 3.47 ± 1.63 | 0.00 | 0.00 | 0.08 | 0.48 | 0.38 | 0.50 | 0.41 |
| **COMPAS** | 0 | 1.00 | 0.88 ± 0.92 | 1.00 | 1.00 | 1.00 | 0.00 | 0.00 | 0.00 | 0.00 |
|                  | 0.01 | 1.00 | 0.89 ± 0.92 | 0.00 | 1.00 | 1.00 | 0.02 | 0.01 | 0.01 | 0.01 |
|                  | 0.10 | 1.00 | 1.02 ± 0.92 | 0.00 | 0.00 | 1.00 | 0.10 | 0.10 | 0.10 | 0.10 |
|                  | 0.50 | 1.00 | 1.60 ± 0.92 | 0.00 | 0.00 | 0.00 | 0.51 | 0.51 | 0.50 | 0.50 |
| **Bail** | 0 | 1.00 | 3.40 ± 3.63 | 1.00 | 1.00 | 1.00 | 0.00 | 0.00 | 0.00 | 0.00 |
|                  | 0.01 | 1.00 | 3.46 ± 3.63 | 0.00 | 1.00 | 1.00 | 0.01 | 0.01 | 0.01 | 0.01 |
|                  | 0.10 | 1.00 | 4.02 ± 3.63 | 0.00 | 0.00 | 1.00 | 0.10 | 0.10 | 0.10 | 0.10 |
|                  | 0.50 | 1.00 | 6.52 ± 3.63 | 0.00 | 0.00 | 0.00 | 0.50 | 0.50 | 0.50 | 0.50 |
| **German** (Causal) | 0 | 1.00 | 0.66 ± 0.39 | 0.12 | 0.89 | 1.00 | 0.03 | 0.00 | 0.03 | 0.00 |
|                  | 0.01 | 1.00 | 0.68 ± 0.40 | 0.00 | 0.71 | 1.00 | 0.04 | 0.01 | 0.04 | 0.01 |
|                  | 0.10 | 1.00 | 0.85 ± 0.43 | 0.00 | 0.00 | 1.00 | 0.13 | 0.10 | 0.13 | 0.10 |
|                  | 0.50 | 0.92 | 2.13 ± 2.91 | 0.00 | 0.00 | 0.03 | 0.53 | 0.37 | 0.56 | 0.41 |
| **Adult** (Causal) | 0 | 1.00 | 2.55 ± 1.70 | 0.00 | 0.97 | 1.00 | 0.02 | 0.01 | 0.01 | 0.01 |
|                  | 0.01 | 1.00 | 2.96 ± 1.92 | 0.00 | 0.00 | 1.00 | 0.10 | 0.10 | 0.10 | 0.10 |
|                  | 0.50 | 0.97 | 4.63 ± 2.39 | 0.00 | 0.00 | 0.01 | 0.49 | 0.46 | 0.53 | 0.50 |
| **COMPAS** | 0 | 1.00 | 0.89 ± 0.81 | 0.12 | 1.00 | 1.00 | 0.04 | 0.00 | 0.04 | 0.00 |
|                  | 0.01 | 1.00 | 0.90 ± 0.81 | 0.00 | 1.00 | 1.00 | 0.05 | 0.01 | 0.05 | 0.01 |
|                  | 0.10 | 1.00 | 1.02 ± 0.81 | 0.00 | 0.01 | 1.00 | 0.14 | 0.10 | 0.14 | 0.10 |
|                  | 0.50 | 1.00 | 1.56 ± 0.81 | 0.00 | 0.00 | 0.12 | 0.52 | 0.47 | 0.60 | 0.50 |
| **Bail** | 0.00 | 0.89 | 1.91 ± 4.83 | 0.54 | 1.00 | 1.00 | 0.02 | 0.00 | 0.01 | 0.00 |
|                  | 0.01 | 0.88 | 1.96 ± 4.85 | 0.00 | 1.00 | 1.00 | 0.03 | 0.01 | 0.02 | 0.01 |
|                  | 0.10 | 0.88 | 3.00 ± 5.52 | 0.00 | 0.00 | 1.00 | 0.12 | 0.10 | 0.11 | 0.10 |
|                  | 0.50 | 0.82 | 6.22 ± 6.08 | 0.00 | 0.00 | 0.59 | 0.50 | 0.44 | 0.53 | 0.49 |

For non-linear classifiers, AT significantly increases the cost of recourse. Given the similarities between algorithmic recourse and adversarial examples [24], it seems likely that ensuring robustness of prediction naturally hinders the existence of low cost recourse recommendations.

**Local linearity regularization (LLR)** [27] Our proposed regularizer is inspired by LLR. This regularizer encourages the loss to behave linearly near the training data. Intuitively, the linearity of the loss is measured by $g(x, \delta) = \ell(h(x + \delta), y) - \langle \delta, \nabla_x \ell(h(x), y) \rangle - \ell(h(x), y)$. At each step, the direction along which the linearity condition is maximally violated is computed $\delta_{LLR} = \max_{\|\delta\| \leq \gamma} g(x, \delta)$. Then, the regularizer consists on a penalty on such violation, as well as a penalty on the gradient along $\delta_{LLR}$, that is, $\min_\theta \mathbb{E} [\ell(h_\theta(x), y) + \mu \langle \delta_{LLR}, \nabla_x \ell(x, y) \rangle + \gamma g(x, \delta_{LLR})]$. The results for LLR are comparable to those of AT, since both methods aim to ensure robustness of prediction.
Table 3: Cost of recourse and cost of robust recourse for different model training approaches.

| Method       | Cost increase | Recourse | Robust recourse |
|--------------|---------------|----------|-----------------|
|              | Accuracy      | Mean cost| Found           | Mean cost          | Mean  | Max  |
| ERM          | 0.74          | 1.08     | 0.83            | 1.00               | 1.27   | 0.85 |
|              | 0.73          | 1.12     | 0.84            | 1.00               | 1.31   | 0.84 |
|              | 0.74          | 1.14     | 0.83            | 1.00               | 1.33   | 0.83 |
|              | 0.67          | 0.69     | 0.52            | 1.00               | 0.78   | 0.52 |
|              | 0.73          | 1.03     | 0.78            | 1.00               | 1.21   | 0.78 |
| AT           | 0.82          | 2.45     | 1.52            | 1.00               | 2.81   | 1.58 |
|              | 0.82          | 2.52     | 1.50            | 1.00               | 2.87   | 1.57 |
|              | 0.82          | 2.46     | 1.42            | 1.00               | 2.78   | 1.50 |
|              | 0.78          | 1.88     | 1.09            | 1.00               | 2.08   | 1.13 |
|              | 0.81          | 1.86     | 1.04            | 1.00               | 2.06   | 1.08 |
| L2           | 0.67          | 1.09     | 1.09            | 1.00               | 1.21   | 1.09 |
| Adult (Causal) | 0.67        | 1.06     | 1.00            | 1.00               | 1.18   | 1.08 |
|              | 0.67          | 1.14     | 1.11            | 1.00               | 1.27   | 1.11 |
|              | 0.64          | 1.18     | 1.08            | 1.00               | 1.28   | 1.08 |
|              | 0.67          | 1.13     | 1.08            | 1.00               | 1.25   | 1.08 |
| ALLR         | 0.67          | 1.00     | 4.80            | 1.00               | 5.77   | 5.20 |
|              | 0.67          | 1.00     | 4.92            | 1.00               | 5.49   | 4.80 |
|              | 0.67          | 1.00     | 4.10            | 1.00               | 4.70   | 4.38 |
|              | 0.64          | 1.47     | 1.57            | 1.00               | 1.57   | 1.57 |
|              | 0.66          | 1.00     | 1.77            | 1.00               | 1.91   | 1.72 |
| Bail         | 0.86          | 0.71     | 0.45            | 1.00               | 0.89   | 0.48 |
| German (Causal) | 0.86      | 0.65     | 0.46            | 1.00               | 0.82   | 0.48 |
|              | 0.85          | 0.73     | 0.50            | 1.00               | 0.92   | 0.65 |
|              | 0.76          | 0.57     | 0.35            | 1.00               | 0.72   | 0.37 |
|              | 0.85          | 1.00     | 0.61            | 1.00               | 0.77   | 0.40 |
| MLP          | 0.82          | 3.34     | 3.11            | 0.92               | 3.89   | 3.51 |
|              | 0.82          | 3.52     | 5.45            | 0.72               | 4.21   | 7.52 |
|              | 0.82          | 11.21    | 19.89           | 0.74               | 12.60  | 23.65 |
|              | 0.78          | 1.93     | 1.31            | 0.92               | 2.15   | 1.43 |
|              | 0.82          | 2.22     | 1.42            | 1.00               | 2.42   | 1.49 |
| COMPAS       | 0.67          | 1.15     | 1.23            | 1.00               | 1.27   | 1.22 |
|              | 0.67          | 1.33     | 1.33            | 0.99               | 1.46   | 1.34 |
|              | 0.66          | 1.63     | 1.60            | 0.92               | 1.75   | 1.57 |
|              | 0.64          | 1.21     | 1.12            | 0.96               | 1.30   | 1.11 |
|              | 0.65          | 1.23     | 1.20            | 0.99               | 1.34   | 1.21 |
| Bail         | 0.68          | 3.32     | 4.33            | 0.92               | 4.23   | 4.47 |
|              | 0.68          | 4.12     | 4.52            | 0.91               | 4.69   | 4.40 |
|              | 0.68          | 4.22     | 4.79            | 0.86               | 4.88   | 4.85 |
|              | 0.64          | 0.91     | 1.05            | 0.84               | 1.02   | 1.07 |
|              | 0.67          | 1.95     | 2.37            | 0.94               | 2.15   | 2.31 |