An interpretation of the infrared singularity of the effective electromagnetic field

K. A. Kazakov and V. V. Nikitin

Department of Theoretical Physics, Physics Faculty, Moscow State University - 119991, Moscow, Russian Federation

received 22 July 2010; accepted in final form 11 December 2010
published online 21 January 2011

PACS 12.20.-m – Quantum electrodynamics
PACS 11.10.Wx – Finite-temperature field theory
PACS 11.15.Bt – General properties of perturbation theory

Abstract – The problem of infrared divergence of the effective electromagnetic field produced by elementary particles is revisited using the nonequilibrium model of an electron interacting with low-temperature photons. A new scheme of infrared regularization is proposed which allows us to factorize the infrared contributions in multi-loop diagrams, and to sum the corresponding infinite series. The obtained results suggest that the infrared singularity of the effective field can be interpreted as a thermalization of the electron. It is shown that this thermalization is negligible in actual field measurements as it is completely dominated by the usual quantum spreading.

Introduction. – It is well known that calculations in quantum field theories involving massless particles are plagued by the presence of infrared divergences. Unlike the case of ultraviolet singularities which can in general be consistently “subtracted” by an appropriate redefinition of the parameters of the theory, there is no unique recipe to deal with/interpret the infrared-infinite contributions. As far as the scattering matrix method is applicable, a general prescription is to sum the scattering cross-sections over suitable classes of initial and/or final states. In quantum electrodynamics or quantum gravity, for instance, the standard resolution of the “infrared catastrophe” is to sum overall final states containing arbitrary number of real soft photons or gravitons [1] (in Yang-Mills theories, the procedure is more intricate [2]). However, the problem persists beyond the scope of the S-matrix, in particular, it presents a serious obstacle to the use of the effective field methods which are of vital importance in investigating such issues as the spontaneous symmetry breaking in the electroweak theory, the quark-gluon plasma effects in quantum chromodynamics, particle creation and inflation in quantum cosmology, etc.

Specifically, it is known that the vertex form factors of the massive field quanta, used in constructing the effective (mean) fields of elementary particles, are infrared-divergent. Though the same form factors appear in the scattering amplitudes, the above-mentioned prescription of the S-matrix theory does not apply to the effective fields. Indeed, the very statement of the problem in the two cases is quite different. The Bloch-Nordsieck theorem states that infrared divergences in the radiative corrections to the scattering cross-sections exactly cancel those due to the emission of real soft photons. At the same time, real photons do not appear at all in the calculation of the static effective field (at zero temperature). By this reason, the effective field formalism can be applied, strictly speaking, only to the fields produced by classical sources. If, for instance, the charged particle mass is sufficiently large, radiative corrections to its interaction with the electromagnetic field can be neglected, thus putting the question about their divergence aside. In fact, assumptions of this kind underlie the classic calculation [3] of the effective static field of a point source and its generalizations [4]. However, this limitation looks rather unsatisfactory from the theoretical point of view, especially in the light of measurability of the electromagnetic field, established long ago [5].

This problem is sharpened at finite temperatures, because the heat-bath effects introduce new low-energy singularities to the photon propagator. These additional singularities are generally believed to worsen infrared properties of the Feynman integrals. At the same time, inclusion of the finite-temperature effects into consideration is a question of principle: no matter how small temperature is, it is never zero exactly, and the question whether $T \approx 0$ can be replaced by $T = 0$ can be answered only by investigating the general nonzero-temperature case. In other words, continuity of this sort, being a
necessary physical requirement for the very possibility to neglect the heat-bath effects, is to be proved rather than postulated. Previous investigations of the problem have been aimed mainly at generalizing the Bloch-Nordsieck and Kinoshita-Lee-Nauenberg theorems to nonzero temperatures [6]. As was already mentioned, their results do not apply to the effective fields.

The aim of this letter is to propose a physical interpretation of the infrared divergence of the effective electromagnetic field produced by an elementary charged particle. Namely, we shall evaluate the effective Coulomb field of a free electron that was in an arbitrary state in the past, and show that this field vanishes upon account of the infrared radiative corrections to all orders of the perturbation theory. We will argue that this result signifies the existence of a peculiar spreading of the charge which is inherently irreversible, in the sense that it makes impossible the preparation of a spatially localized electron state at finite times by operating with arbitrary single-electron states in the remote past. In other words, to prepare a localized state requires inclusion of initial nonequilibrium photons or other charged particles.

Our consideration applies equally to the zero- and nonzero-temperature cases, and begins in the next section with a description of the physical model to be investigated.

We then introduce an infrared regularization of the model, which represents a modification of the usual momentum cutoff method, and prove that the proposed scheme admits factorization of the infrared radiative contributions to the effective field to all orders of the perturbation theory. This result is used in the last section to demonstrate that the Coulomb field of the electron vanishes at any given position in the limit of removed cutoff, in a way that respects the total charge conservation. We argue that interpreted in terms of the electron density matrix, this field nullification can be described as an electron thermalization through its interaction with photons.

The model. – Consider the electromagnetic field produced by a nonrelativistic electron of mass $m$, which is on average at rest and interacts with the virtual as well as real photons in equilibrium at finite temperature $T < m$. Since the field strength is linear with respect to the electromagnetic potential, it is sufficient to find the mean value of the latter,

$$ A^\mu_{\text{eff}}(x) = N^{-1} \text{Tr}(A_\mu(x)e^{-\beta H_\text{e}}\varrho), \quad N = \text{Tr}(e^{-\beta H_\text{e}}\varrho), \quad \beta = 1/T. \quad (1) $$

Here $A_\mu(x)$ is the Heisenberg picture operator of the electromagnetic potential, $H_\text{e}$ the Hamiltonian of free photons, $\varrho$ the electron density matrix, and the trace is over all photon states as well as the single-electron states. Actually, the consideration can be restricted to the 0-component of the effective potential, because the other components are suppressed by the small factor $\Delta q/m$, where $\Delta q$ is the particle momentum variance. Although this fact is quite obvious under the nonrelativistic conditions we have chosen, it follows formally from the existence of the gauge in which the photon propagator is diagonal (the Feynman gauge), and the gauge independence of the field strength (see below).

To evaluate the effective field, we use the general framework of the real time approach [7] according to which the right-hand side of eq. (1) can be written in the interaction picture

$$ A^\mu_{\text{eff}}(x) = N^{-1} \text{Tr}(T_e \exp \left( \int_0^\infty d\tau L_I(\tau) A_\mu(x) \right) e^{-\beta H_\text{e}}\varrho). \quad (2) $$

where $L_I$ is the interaction Lagrangian, the $x^0$-integration is along the standard Schwinger-Keldysh time-contour $C$ running from $t_i$ to $t_f > x^0$ and back, and $T_e$ denotes the operator ordering along this contour. The conventional limit $t_i \to -\infty$ is directly related to the infrared problem and will be discussed in the last section. As usual, we assume adiabatic switching off of the interaction in this limit; accordingly, the electron evolves freely in the remote past, so that its 4-momentum $q^\mu$ satisfies $q^2 = m^2$, while the radiative corrections to its density matrix at finite times, including those due to interaction with the real heat-bath photons, are represented by the loop diagrams built according to the standard rules of the real-time formalism. In particular, the momentum space field propagators, rendered $2 \times 2$ matrices by the $T_e$-ordering, take the form

$$ D^{(ij)}(k) = M(k) \begin{pmatrix} D_F(k) & B(k_0, k) \\ B(-k_0, k) & -D_F(k) \end{pmatrix} M(k), $$

where the matrix indices $i,j$ take the value $1,2$ for fields on the forward (backward) branch of the contour $C$, the step function $\theta(-k_0)$ is taken with the plus (minus) sign for the photon (electron), the tilde symbolizes the special operation of complex conjugation with respect to which the Dirac matrices are real, $D_F(k)$ is the usual vacuum Feynman propagator, and $B(k_0, k) = -2\pi i \theta(k_0) n(k) \delta(k^2)$. In the present case of a single-electron system, one has

$$ D_F(k) = \frac{k + m}{m^2 - k^2 - i\epsilon} , \quad n(k) = 0 $$

for the electron, while for the photon,

$$ D_F(k) = \frac{d_{\mu\nu}(k)}{(k^2 + i\epsilon)^2} , \quad n(k) = \frac{1}{e^{\beta |k|} - 1} , \quad d_{\mu\nu}(k) = k^2 \eta_{\mu\nu} + (\xi - 1)k_\mu k_\nu, $$

$1$We use relativistic units $\hbar = c = 1$. Also, the Minkowski metric is $\eta_{\mu\nu} = \text{diag}(+1,-1,-1,-1)$. 61001-p2
where $\xi$ is an arbitrary gauge parameter ($\xi = 1$ is the Feynman gauge). Since gauge dependence is not a problem in quantum electrodynamics (in fact, gauge independence of the effective quantities such as Heisenberg-Euler Lagrangian, the mean field, etc. has been proved in quite general form [8]), we do not consider here the general case, and limit the gauge freedom to the above simple form in order to illustrate the usual cancelation of the gauge-dependent contributions. Upon transition to the momentum space, the effective potential takes the form

$$A_{\mu}^{\text{eff}}(x) = -e \sum_{\sigma, \sigma'} \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3p}{(2\pi)^3} e^{ipx} D^{(11)}_{\mu\nu}(p)$$

$$\times \partial_{\sigma\sigma'}(q, q+p) R(p, q) \bar{u}_{\sigma'}(q+p) \gamma^\mu u_{\sigma}(q),$$

where

$$D^{(11)}_{\mu\nu}(p) = d_{\mu\nu}(p) \left\{ \frac{1}{(p^2 + i0)^2} - n(p) \left[ \frac{1}{(p^2 + i0)^2} - \frac{1}{(p^2 + i0)^2} \right] \right\}$$

is the (11)-component of the photon propagator, and the scalar function $R(p, q)$ incorporates the radiative corrections to the effective field ($R(p, q) = 1$ in the tree approximation). As seen from eq. (3), the values of $q^0$ and $p^0$ are fixed by the mass-shell condition $q^2 = \sqrt{q^2 + m^2} > 0$ and the energy-momentum conservation. Since the field-producing charge is nonrelativistic, $p^0 = (q + p)^2/2m - q^2/2m$. The bispinor amplitudes and the momentum-space density matrix are normalized on unity:

$$\bar{u}_\sigma u_\sigma = 1, \sum_\sigma \left( \int \frac{d^3q}{(2\pi)^3} \partial_{\sigma\sigma}(q, q) \right) = 1.$$

Using the mass-shell identity $\bar{u}_{\sigma'}(q+p) u_{\sigma}(q) = 0$, it is readily seen from eq. (3) that the gauge-dependent part of the external propagator $D^{(11)}_{\mu\nu}(p)$ does not contribute to $A_{\mu}^{\text{eff}}$. As is well known, the notion of one-particle density matrix is of limited validity in relativistic quantum theory because of the possibility of pair creation. However, under the assumption $T \ll m$ the probability of this process is negligible, and the expression (3) shows that the quantity

$$\partial_{\sigma\sigma'}(q, q') \equiv \partial_{\sigma\sigma'}(q, q') R(q' - q, q), \quad q^2 = q'^2 = m^2,$$

is to be considered as an effective density matrix of the electron. If one discards the radiative corrections, and chooses the density matrix $\rho$ to describe electron state in which it is spatially localized near the point $x_0$ at time $t$, then at distances large compared to the characteristic length of the electron spreading, one can write $\partial_{\sigma\sigma'}(q, q + p) \approx \partial_{\sigma\sigma'}(q, q) e^{ipx_0}$, $p^0 \approx 0$, and also neglect $p$ in the bispinor amplitudes. Using the normalization conditions in eq. (3) then gives for the scalar potential

$$A_0^{\text{eff}}(x) = -e \sum_{\sigma, \sigma'} \left( \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3p}{(2\pi)^3} e^{-ip(x-x_0)} \right)$$

$$\times \partial_{\sigma\sigma'}(q, q) \bar{u}_{\sigma'}(q) \gamma^0 u_{\sigma}(q) = \frac{e}{4\pi r},$$

where $r = |x - x_0|$, i.e., the Coulomb law. Our aim below is to show that account of the radiative corrections gives $A_0^{\text{eff}}(x) = 0$ at any finite distance.

$\lambda$-regularization. - Infrared singularities in the electromagnetic form factors and self-energy contributions to the effective field, as well as ultraviolet divergences require intermediate regularization. First of all, we introduce the usual infrared regulator $\lambda_0$ restricting all loop momenta to $k \rightarrow \Lambda_0$ in the nonzero temperature case, this cutoff is assumed to satisfy $\lambda_0 \ll T$, and also a momentum threshold $\Lambda$ such that $T \ll \Lambda \ll m$, which identifies the photons with $\lambda_0 \ll \Lambda \ll m$ as “soft.” As to the usual ultraviolet divergences, they are supposed to be regularized using some conventional means, say, the dimensional technique. Finally, it is necessary to regularize the diagrams involving self-energy insertions into the on-shell electron propagators, see fig. 1 (see footnote 2). For this purpose, we introduce the following smearing of the $\delta$-functions expressing conservation of the 4-momentum in the interaction vertices $\delta^4(k) \rightarrow \Delta_\lambda(k)$, where $\Delta_\lambda(w)$ satisfies

$$\int \frac{d^4w}{w_0^2 + w^2} \Delta_\lambda(w) = 1, \quad \Delta_\lambda(w) = \Delta_\lambda(-w),$$

$$\Delta_\lambda(w \neq 0) \rightarrow 0 \quad \text{as} \quad \lambda \rightarrow 0.$$

A convenient choice is

$$\Delta_\lambda(w) = \frac{1}{\pi^2 \lambda^2} \exp \left( -\frac{w_0^2 + w^2}{\lambda^2} \right).$$

It will be assumed in what follows that the parameter $\lambda$ characterizing the width of the smeared $\delta$-function satisfies $\lambda \ll \lambda_0$.

Next, to ensure convergence of the effective field in the limit $\lambda \rightarrow 0$ (with fixed $\lambda_0 \neq 0$), we have to introduce counterterms into Lagrangian. To be specific, let us consider diagrams of the type shown in fig. 1(a). The external line with the self-energy insertion contributes a factor

$$\int \frac{d^4u}{u^0} \Delta_\lambda(u) \Delta_\lambda(u) \frac{\delta + \Sigma + \frac{m^2}{2q(u + v) + i0}}{2q(u + v) + i0} \Sigma^{(11)}(q + u)$$

$$= \int \frac{d^4u}{2q(u + v) + i0} \Delta_\lambda(u) \frac{\delta + \Sigma + \frac{m^2}{2q(u + v) + i0}}{2q(u + v) + i0} \Sigma^{(11)}(q + u) + O(\lambda),$$

2Leaving the mass shell does not help in this respect, as it precludes factorization of the infrared contributions, to be proved in the next section. In addition to that, it makes the regularized expression explicitly gauge-dependent.
where
\[ \Delta^{(1)}_{\Lambda}(w) \equiv \frac{1}{16i} \int d^4 \xi \Delta^{(1)}_{\Lambda} \left( \frac{w-\xi}{2} \right) \Delta^{(1)}_{\Lambda} \left( \frac{w+\xi}{2} \right). \]

In eq. (5), the electron self-energy \( \Sigma^{(1)}(q) \) is taken on the mass shell. The standard renormalization prescription that the propagator poles be at the physical mass requires vanishing of this quantity, which can be met by introducing a counterterm (this prescription is usually realized along with the ultraviolet renormalization). The above expression shows that the \( \delta \)-functions appearing in the two-point counterterm vertices are to be regularized as \( \delta^4(q_1 - q_2) \). The counterterm diagram shown in fig. 1(b) then reads
\[ \int d^4 w \Delta^{(1)}_{\Lambda}(w) \frac{1}{2qw+i0}(q + \psi + m) \Sigma^{(1)}(q). \]

The remaining contribution coming from integration of the \( O(\lambda) \)-term in eq. (5) will be taken care of below when summing the infrared of many-loop diagrams. It is interesting to note that in the limit \( p \to 0 \), the counterterm diagrams cancel each other in the expression for the effective field, which can be verified directly using the above definitions. However, this property is violated at finite \( p \) by the Lorentz-noninvariant integration in eq. (6).

**Factorization of infrared contributions.** – Now that the finite-momenta contributions to the electron self-energy have been canceled by counterterms as discussed in the preceding section, it remains to take into account contributions of small virtual photon momenta. The general structure of diagrams to be considered is shown in fig. 2. In terms of the function \( R(p, q) \), the sum of all such diagrams can be written as a series
\[ R(p, q) = I_n \sum_{N=0}^{\infty} (e^2)^N I_N(p, q, \Lambda), \]

where \( I_n \) is the contribution of photons with momenta \( k > \Lambda \). Introducing abridged notation \( Q_{st}(k) = q_s^0 q_t^0 D^{(1)}_{st}(k) \), where \( s, t = 1, 2, q_1 = q, q_2 = q + p \), the functions \( I_N \) take the form
\[ I_N(p, q, \Lambda) = \frac{1}{iN} \sum_{n_1=0}^{N} \sum_{n_2=0}^{N-n_1} \int \prod_{i=1}^{n_2} \frac{d^4 k_i}{(2\pi)^4} Q_{12}(k_i) \times F_{n_1}^{n_2}(q_1, k_1, \ldots, k_{n_1}) \times F_{n_2}^{n_1}(q_2, k_1, \ldots, k_{n_2}) + 2\text{-terms}, \]

where \( N \) is the number of virtual photon lines, of which \( n_1 \) and \( n_2 \) reside on the incoming (outgoing) electron line, while the remaining \( n_0 = N - n_1 - n_2 \) connect the two electron lines; “2-terms” denotes the contribution of diagrams involving 2-vertices; finally, the factors \( F_{n_1}^{n_2} \) read \( (w_1 + \ldots + w_k = W_k) \)
\[ F_{n_1}^{n_2}(q_1, k_1, \ldots, k_{n_1}) = \int d^4 w_1 \cdots d^4 w_{2n+n_0} \]
\[ \prod_{i=1}^{n_1} \frac{\delta^{4}(w_{n_0+i+1})}{(2\pi)^4} Q_{ss}(k_{n_0+i+1}) \Delta(w_{2i-1} - k_{n_0+i}) \Delta(w_{2i} + k_{n_0+i+1}) \Delta(w_{2i+1} - k_1) \cdots \Delta(w_{2n+n_0} - k_{n_0}) \]
\[ \times \sum_{perm} \left[ \frac{1}{W_1 q_s + i0} \frac{1}{W_2 q_s + i0} \cdots \frac{1}{W_{2n+n_0} q_s + i0} \right], \]

where the sum is over all permutations of indices 1, 2, \ldots, \( 2n + n_0 \).

Let us show that the 2-terms do not contribute to the effective field in the \( \Lambda \)-regularization. Figure 3 depicts the general structure of the corresponding diagrams. Graphs with a 1-vertex appearing to the left of a 2-vertex can be omitted, because they involve the function \( D^{(12)}_{st}(q - k) \sim \theta(k_0 - q_0) \), and therefore do not contribute at small loop momenta. If a diagram has \( m > 1 \) vertices of the 2-type on the outgoing electron line, then the corresponding sum
over permutations reads
\[\sum_{\text{perm}} \left[ \frac{1}{W_1 q - i\alpha} \cdots \frac{1}{W_m q + i\alpha} \frac{1}{W_{m+1} q - i\alpha} \right] = 0,\]
where we explicitly performed the permutation over indices 1, \ldots, m. In the case \(m = 1\), expression in the square brackets takes the form
\[\left( \frac{1}{w_1 q + i\alpha} - \frac{1}{w_1 q - i\alpha} \right)^{-1} \left( \frac{1}{w_2 q + i\alpha} - \frac{1}{w_2 q - i\alpha} \right)^{-1} \left( \frac{1}{w_{m+1} q + i\alpha} - \frac{1}{w_{m+1} q - i\alpha} \right)^{-1} = 0,\]
This vanishes too, because \(w_i \neq 0\) in the \(\alpha\)-regularization, thus proving that the contribution of the 2-terms is zero.

This result permits factorization of the infrared contributions and allows us to sum the series (7). Indeed, the sum over permutations in eq. (9) reduces to the product \(\frac{1}{w_1 q + i0} \cdots \frac{1}{w_{m+1} q + i0} \), and the series summation can be carried out in a way similar to the standard treatment of the loop infrared divergences (see, e.g., [9]). Taking the limit \(\alpha \to 0\) (smearing of the \(\delta\)-functions removed), one finds
\[I_N = \sum_{n_1=0}^{N} \sum_{n_2=0}^{N-n_1} \frac{g_{11}^{n_1} g_{22}^{n_2} g_{12}^{n_1+n_2}}{n_1! n_2! n_{m+1}!} \left( \frac{g_{11} + g_{22} + 2g_{12}}{N!2^N} \right)^N,\]
where
\[g_{st} = \eta_s \eta_t \int \frac{d^4k}{(2\pi)^4} Q_{st}(k), \quad \eta_1 = 1, \quad \eta_2 = -1.\]
Putting this into eq. (7) yields
\[R(p, q) = \exp \left( \frac{\alpha}{2} \left( g_{11} + g_{22} + 2g_{12} \right) \right) I_\Lambda. \quad (10)\]

Using eq. (4), it is readily seen that upon substitution of \(g_{st}\) into eq. (10), all \(\xi\)-dependent terms cancel, thus proving \(\xi\)-independence of the result. Evaluation of these integrals in the case of small momentum transfer \(|p| \ll m\) gives
\[R(p, q) = \begin{cases} \exp \left( -\frac{\alpha p^2}{3\pi m^2} \ln \frac{\Lambda}{\lambda_0} \right) I_\Lambda, & T = 0, \\ \exp \left( -\frac{\alpha p^2}{3\pi m^2} \left( 2T - \ln \frac{\Lambda}{\lambda_0} \right) \right) I_\Lambda, & T \neq 0, \end{cases}\]
where \(\alpha = e^2/4\pi\) is the fine structure constant. We observe that in the case \(T = 0\), the infrared factor \(R(p, q)\) coincides with that appearing in the scattering amplitudes [9]. In this respect, the following circumstance should be emphasized. In the \(S\)-matrix theory, this factor disappears from the scattering cross-sections upon account of the soft photons radiated by the charges being scattered. The reason why it remains uncompensated in the present case is the fact that in the effective field formalism, there is no analog of the unobserved photons radiated by the charge. This is because in the \(S\)-matrix case, this radiation is not present in the \(in\)-state and appears only in the \(out\)-state as a result of the scattering, while the effective field is evaluated entirely over the given \(in\)-state (electron + thermal photons), and no \(out\)-state ever appears in the formalism.

**The interpretation.** — The above expressions for \(R(p, q)\) tell us that the Coulomb field of the electron vanishes at any finite \(x\). Indeed, it is seen from eq. (3) that this field is determined by the Fourier components with \(|p| \sim 1/|x - x_0|\), while the infrared exponent in the function \(R(p, q)\) tends to zero for \(p \neq 0\) in the limit \(\lambda_0 \to 0\). On the other hand, this exponent equals unity for \(p = 0\), which expresses the electric charge conservation. In fact, it is not difficult to verify the validity of the Gauss law in infinite space: Recalling that \(\partial A/\partial t\) is negligible in the nonrelativistic limit, one sees that integration of the electric field over an infinitely remote sphere is equivalent in \(p\)-representation to multiplying by \(p^2\) the integrand in eq. (3) with \(\mu = 0\), followed by taking the limit \(p \to 0\). The resulting expression
\[\lim_{p \to 0} \int \frac{d^3q}{(2\pi)^3} \langle p' | D_{00}^{(11)}(p) g_{\sigma\sigma'}(q, q + p) R(p, q) u_{\sigma'}(q + p) u_{\sigma}(q) e \int \frac{d^3q}{(2\pi)^3} g_{\sigma\sigma}(q, q)\]
is equal to the electron charge by virtue of the normalization condition for \(g\).

The natural physical interpretation of this field nullification is that the interaction with soft photons causes an electron to spread over infinite space so that the charge density becomes infinitely small everywhere. An essential difference of this spreading from the usual one is that it takes place independently of the particular form of the electron density matrix, while in nonrelativistic quantum mechanics, the density matrix of a free electron can always be chosen so as to describe a state which is spatially localized at any given time instant. As was already mentioned, the expression (3) for the effective electric field suggests that the matrix \(g_{\sigma\sigma'}(q, q')\) is to be considered as an effective density matrix of the electron, which incorporates the effects of its interaction with soft photons. In terms of this matrix, the electron spreading can be described as its thermalization. Indeed, the fact that the function \(R(q' - q, q)\) vanishes for \(q \neq q'\) in the limit \(\lambda_0 \to 0\) means that the effective density matrix becomes diagonal, and hence time-independent in this limit, signifying that the electron is driven near an equilibrium with photons.

61001-p5
It is remarkable that this thermalization occurs even at zero temperature, the difference from the case $T \neq 0$ being only quantitative: the power dependence on the infrared cutoff at $T = 0$ switches to an exponential dependence at finite temperatures. Yet, this difference is important from the practical point of view. To assess the influence of the infrared exponent, we note that any actual measurement naturally sets an infrared cutoff specific to the given experimental situation. In particular, the total duration of the experiment, $\tau$, cuts off the photon energy at $\sim \hbar/\tau$. This implies that instead of taking the formal limit $t_i \to -\infty$, eq. (2) is to be considered at finite $t_i$ such that $t - t_i \sim \tau$, which regularizes the energy-integrations in the Feynman integrals. Furthermore, the finite fundamental speed of interaction propagation effectively confines the system to a box with the linear dimension $c\tau$, thereby cutting off all momenta at $\lambda_0 \sim \hbar/ct$. Noting also that any field measurement is meaningful only at distances of the Compton length, $\tau \gtrsim l_c = \hbar/mc$, and replacing the threshold $\Lambda$ by $mc$, we see that in the zero-temperature case, the infrared exponent becomes important when

$$\frac{\alpha}{3\pi} \ln \frac{mc^2r}{\hbar} \sim 1.$$  

The corresponding time $\tau \sim 10^{34} \text{ s}$ far exceeds the age of the Universe. However, things change at finite temperatures. In experiments using cathode-ray tubes, for instance, $r$ ranges approximately from $1 \mu m$ to 1 cm, and the corresponding

$$\tau \sim \frac{3\pi\hbar}{2\alpha T} \left( \frac{r}{l_c} \right)^2$$

ranges from $10^2$ to $10^{10}$ seconds, at room temperature.

Thus, sufficiently slow experiments involving free electrons may require taking into account the electron thermalization. Still, this effect is completely negligible as far as one considers the electron electric field itself, because of the usual nonrelativistic spreading. Indeed, let $a$ denote characteristic length of the electron wave function before the field measurement. The subsequent free-electron evolution according to the Schrödinger equation leads to the wave function spreading, the characteristic time being $ma^2/\hbar$ (i.e., $a^2$ grows with time approximately as $\hbar t/m$). Since a meaningful measurement in any case requires $a \lesssim r$, it follows from the above expressions that the electron thermalization might affect its field during the experiment only under the condition

$$\frac{\hbar}{\alpha T} \left( \frac{r}{l_c} \right)^2 \lesssim \frac{mc^2}{\hbar},$$

or $T \gtrsim mc^2/\alpha$. But the latter is opposite to the general assumption $T \ll mc^2$ underlying our consideration of the single-electron picture.

We arrive at the conclusion that the infrared singularity in the electromagnetic field produced by a free electron is negligible in the description of processes driven by the Coulomb interaction. In particular, it follows from the above discussion that the electron thermalization can be completely discarded in the formulation of the asymptotic conditions for the scattering experiments involving charged particles. However, since this thermalization modifies the electron density matrix, it can in principle affect sufficiently slow processes, in particular, those sensitive to changes in the quantum entropy of electron states.

Further details of the calculation including comparison of different regularization schemes can be found in [10].

REFERENCES

[1] Bloch F. and Nordsieck A., Phys. Rev., 37 (1937) 54; Yennie D. R., Frautschi S. C. and Suura H., Ann. Phys. (N.Y.), 13 (1961) 379; Weinberg S., Phys. Rev., 140 (1965) B515; Grammer G. and Yennie D. R., Phys. Rev. D, 8 (1973) 4332.

[2] Kinoshita T., J. Math. Phys., 3 (1962) 650; Lee T. D. and Nauenberg M., Phys. Rev., 133 (1964) B1549; Sterman G. and Weinberg S., Phys. Rev. Lett., 39 (1977) 1416.

[3] Serber R., Phys. Rev., 48 (1935) 49; Uhling A. E., Phys. Rev., 48 (1935) 55.

[4] Bialynicki-Birula I. and Bialynicka-Birula Z., Quantum Electrodynamics (Pergamon, Oxford) 1975; Kaminski J. Z., J. Phys. A: Math. Gen., 16 (1983) 2587.

[5] Bohr N. and Rosenfeld L., K. Dan. Vidensk. Selskab. Math.-Fys. Medd., 12 (1933) 3; Phys. Rev., 78 (1950) 794; DeWitt B. S., in Gravitations: An Introduction to Current Research, edited by Witten L. (Wiley, New York) 1962, p. 266.

[6] Tryon E. P., Phys. Rev. Lett., 32 (1975) 1139; Donohue J. F., Holstein B. R. and Robinett R. W., Ann. Phys. (N.Y.), 164 (1985) 233; Altiherr T., Phys. Lett. B, 262 (1991) 314; Manjavizde J., hep-th/9510251 preprint (1995); Weldon H. A., Phys. Rev. D, 49 (1991) 1579; Nucl. Phys. A, 566 (1994) 581c; Indumathi D., Ann. Phys. (N.Y.), 263 (1998) 310; Muller A., hep-th/9912240 preprint (1999).

[7] Schwinger J., J. Math. Phys., 2 (1961) 407; Keldysh L. V., Sov. Phys. JETP, 20 (1964) 1018; Landsman N. P. and van Weert Ch. G., Phys. Rep., 145 (1987) 141.

[8] Fradkin E. S., Tr. Fiz. Inst. Akad. Nauk SSSR, 29 (1965) 7; Fukuda R. and Kugo T., Phys. Rev. D, 13 (1976) 3469; Lavrov P. M., Tyutin I. V. and Voronov B. L., Yad. Fiz., 36 (1982) 498 (English Translation: Sov. J. Nucl. Phys., 36 (1982) 292).

[9] Weinberg S., The Quantum Theory of Fields, Vol. 1 (Cambridge University Press) 1995, Chapt. 13.

[10] Kazakov K. A. and Nikitin V. V., arXiv:0910.5937 preprint (2009).