Noncompact Heisenberg spin magnets from high-energy QCD

I. Baxter

Q-operator and Separation of Variables

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Abstract:

We analyze a completely integrable two-dimensional quantum-mechanical model that emerged in the recent studies of the compound gluonic states in multi-color QCD at high energy. The model represents a generalization of the well-known homogenous Heisenberg spin magnet to infinite-dimensional representations of the $SL(2, \mathbb{C})$ group and can be reformulated within the quantum inverse scattering method. Solving the Yang-Baxter equation, we obtain the $R$-matrix for the $SL(2, \mathbb{C})$ representations of the principal series and discuss its properties. We explicitly construct the Baxter $Q$-operator for this model and show how it can be used to determine the energy spectrum. We apply Sklyanin’s method of the Separated Variables to obtain an integral representation for the eigenfunctions of the Hamiltonian. We demonstrate that the language of Feynman diagrams supplemented with the method of uniqueness provide a powerful technique for analyzing the properties of the model.
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1. Introduction

In this paper, we study a completely integrable quantum-mechanical model of $N$ interacting spinning particles in two-dimensional space. The model can be thought of as a generalization of the well-known periodic one-dimensional XXX Heisenberg spin chain magnet $^1$ $^2$ $^3$ – each particle defines the site of the spin chain and the interaction occurs between spins of two neighboring particles. The spin operators associated with each particle are the generators of an infinite-dimensional principal series representation of the $SL(2, \mathbb{C})$ group and that is why we shall refer to the model as a noncompact spin magnet.

The motivation for studying such models comes from two different and seemingly unrelated physical problems. The first of them has to do with the Regge phenomena in QCD. As was shown in $^4$ $^5$, the high-energy behavior of the scattering amplitudes in the generalized leading-logarithmic approximation is governed by the contribution of the color-singlet compound states, built from an arbitrary number ($N = 2, 3, \ldots$) of gluons and defined as ground states of the effective QCD Hamiltonian. The effective Hamiltonian acts on the two-dimensional transverse coordinates of the gluons and exhibits remarkable properties of integrability $^6$ $^7$. Namely, in the limit of the multi-color QCD, the wave function of the $N$–gluon compound state turns out to be identical to the ground state of noncompact Heisenberg spin chain model with the number of sites equal to the number of gluons, $N$. The quantum space in each site of this model is parameterized by two-dimensional gluon transverse coordinates and corresponds to the spin $s = 0$ principal series representation of the $SL(2, \mathbb{C})$ group. At $N = 2$ the ground state is known as the BFKL Pomeron $^4$. At $N = 3$ it gives rise to the Odderon $^8$ and for higher $N$ the corresponding ground states define the unitarity corrections to the scattering amplitudes. In spite of a lot of efforts, the solution of the Schrödinger equation for $N \geq 3$ gluon compound states still represents an outstanding theoretical problem in QCD.

Another motivation for studying the noncompact spin magnets comes from the problem of solving the quantum completely integrable models defined on an infinite-dimensional phase space. The best known example of such models is given by the one-dimensional Toda chain model $^9$. One of their main features is that they can not be solved using the conventional Algebraic Bethe Ansatz (ABA) method $^1$ $^2$ $^3$ and one has to rely on more advanced methods $^0$ $^1$. The noncompact two-dimensional $SL(2, \mathbb{C})$ spin magnets belong to the same class of models and their solution represents a theoretical challenge. As we will show in this paper, these two-dimensional models have many features in common with the known one-dimensional integrable models. Namely, they admit the same $R$–matrix representation as XXX Heisenberg compact spin chain and part of their spectrum can be reconstructed using the ABA approach. At the same time, similar to the Toda model, the quantum space of the system is infinite-dimensional and, in general, the eigenstates do not admit the highest weight representation. This means that the ABA approach can not provide the full set of the eigenstates of the model and, therefore, it is not complete.

Our approach to solving the Schrödinger equation for the $SL(2, \mathbb{C})$ spin magnet relies on the methods of the Baxter $Q$–operator $^0$ and the Separation of Variables (SoV) developed by Sklyanin $^1$. The former allows to determine the energy spectrum of the model, while the latter provides an integral representation of the corresponding eigenfunctions. In recent years, both methods have been developed and successfully applied to solving the Toda chain model $^1$ $^2$ $^3$ and high spin generalizations of the homogenous XXX Heisenberg $SL(2, \mathbb{R})$ spin magnet $^1$. In this paper, we shall extend these results to the $SL(2, \mathbb{C})$ Heisenberg spin magnets.
The central point of our analysis is the representation of different operators in the model (the $R$–matrix, Hamiltonian, Baxter $Q$–operator etc.) by their integral kernels defined on the two-dimensional plane. In this way, remarkable integrability properties of the model that are usually expressed in the form of operator identities, like Yang-Baxter equations, are translated into (complicated) integral relations between the corresponding kernels. Adopting the language of the two-dimensional Feynman integrals, one can associate the kernels with particular Feynman diagrams and develop the diagrammatical representation for the above identities. One of the main findings of this paper is that, using the diagrammatical approach, one can establish integrability properties of the $SL(2, \mathbb{C})$ spin magnet without doing any actual calculations by applying two elementary identities between the Feynman diagrams known in QCD as the “uniqueness relations” \cite{15, 16}. Following this approach, we find the solution of the Yang-Baxter equation for the $R$–matrix corresponding to the $SL(2, \mathbb{C})$ principal series representation, construct the Baxter $Q$–operator and the unitary transformation to the Separated Variables and, finally, establish different relations between them that allow to solve the model.

The paper is organized as follows. In Section 2 we introduce the Hamiltonian of the model and show that it admits the $R$–matrix representation. We obtain the expression for the $R$–matrix by solving the Yang-Baxter equation for the principal series $SL(2, \mathbb{C})$ representation and discuss its general properties. This allows to prove a complete integrability of the model and establish its symmetry properties. Section 3 is devoted to the construction of the Baxter $Q$–operator for the $SL(2, \mathbb{C})$ spin magnet. Our approach is similar to the one used before for the Toda chain \cite{12} and the homogenous $SL(2, \mathbb{R})$ Heisenberg spin magnets \cite{14}. Examining the properties of the obtained expressions, we establish the relation between the transfer matrices of the model and the product of the Baxter $Q$–operators. This relation allows to express the Hamiltonian of the model in terms the $Q$–operator and reduce the original Schrödinger equation to the problem of finding the solutions to the Baxter equation on the eigenvalues of the $Q$–operator under the additional conditions imposed by the analytical properties of the $Q$–operator and its asymptotic behavior at infinity. In Section 5, we apply Sklyanin’s method of Separation of Variables to obtain an integral representation for the eigenfunctions of the Hamiltonian. We construct the unitary transformation to the separated variables and demonstrate its relation with the Baxter $Q$–operators. We obtain the quantization conditions on the separated variables and calculate the integration measure on the space of the eigenstates in the SoV representation. Concluding remarks are presented in Section 6. Appendix A provides a detailed description of the method of uniqueness which is intensively used throughout the paper. In Appendix B, we describe the properties of the $SL(2, \mathbb{C})$ transfer matrices including the fusion identities and their relation with the Baxter $Q$–operators. The relation between the obtained integral representation of the eigenstates and the Algebraic Bethe Ansatz method is discussed in the Appendix C.

2. The quantum noncompact spin chain model

2.1. Definition of the model

The noncompact spin chain model is a quantum mechanical system of $N$ interacting particles on a two-dimensional $(x, y)$–plane. In high-energy QCD, this plane corresponds to transverse gluonic degrees of freedom. It becomes convenient to define the position of the particles on the
We associate with each particle a pair of mutually commuting holomorphic and antiholomorphic spin operators, \( S^{(k)}_\alpha \) and \( \bar{S}^{(k)}_\alpha \), satisfying the standard commutation relations \( [S^{(k)}_\alpha, S^{(\alpha)}_\beta] = i\varepsilon_{\alpha\beta\gamma}S^{(k)}_\gamma \) and similarly for \( \bar{S}^{(k)}_\alpha \). They act on the quantum space of the \( k \)-th particle, \( \Psi^{(s_k, \bar{s}_k)} \), and can be represented as the following differential operators

\[
S_0^{(k)} = z_k \partial_{z_k} + s_k, \quad \bar{S}_0^{(k)} = \bar{z}_k \partial_{\bar{z}_k} + \bar{s}_k,
\]

so that \((S^{(k)})^2 = (S_0^{(k)})^2 + (S_+^{(k)}S_-^{(k)} + S_-^{(k)}S_+^{(k)})/2 = s_k(s_k-1)\) and similar for the antiholomorphic Casimir operator \((\bar{S}^{(k)})^2\). The spin operators defined in this way are the generators of the unitary principal series representation of the \( SL(2, \mathbb{C}) \) group

\[
\Psi(z_k, \bar{z}_k) \rightarrow \Psi'(z_k, \bar{z}_k) = \psi(z_k + d)\bar{z}_k)\frac{2s_k}{2\bar{s}_k}\psi(z_k, \bar{z}_k),
\]

where

\[
z' = \frac{az_k + \bar{b}}{cz_k + d}, \quad \bar{z}' = \frac{\bar{a}z_k + \bar{b}}{\bar{c}z_k + \bar{d}}
\]  

with \( k = 1, ..., N, \) \( ad - bc = \bar{a}d - \bar{b}c = 1 \) and the complex parameters \( s_k \) and \( \bar{s}_k \) specified below (see Eq. (2.13)). In what follows we shall assume that the model is homogenous and the particles have the same spin, \( s_k = s \) and \( \bar{s}_k = s \) for \( k = 1, ..., N \).

The Hamiltonian of the model, \( \mathcal{H}_N \), describes the interaction between \( N \) noncompact \( SL(2, \mathbb{C}) \) spins attached to the particles and it has the following general form

\[
\mathcal{H}_N = H_N + \overline{H}_N, \quad [H_N, \overline{H}_N] = 0,
\]

where \( H_N \) and \( \overline{H}_N \) act on the holomorphic and antiholomorphic coordinates of the particles, respectively, and therefore commute. The (anti)holomorphic Hamiltonian is given by the sum of two-particle Hamiltonians describing the nearest neighbor interaction between the corresponding (anti)holomorphic spins with periodic boundary conditions

\[
H_N = \sum_{k=1}^{N} H(J_{k,k+1}), \quad \overline{H}_N = \sum_{k=1}^{N} H(J_{k,k+1}), \quad H(J) = \psi(1 - J) + \psi(J) - 2\psi(1)
\]

with \( \psi(x) = d\ln\Gamma(x)/dx \) being the Euler function and \( J_{N,N+1} = J_{1,1} \). Here, \( J_{k,k+1} \) and \( \overline{J}_{k,k+1} \) are defined through the Casimir operators for the sum of the spins \[3\]

\[
J_{k,k+1}(J_{k,k+1} - 1) = (S^{(k)} + S^{(k+1)})^2
\]

with \( S^{(N+1)}_\alpha = S^{(1)}_\alpha \), and \( \overline{J}_{k,k+1} \) is defined similarly.

This paper is devoted to solving the Schrödinger equation

\[
\mathcal{H}_N \Psi(\bar{z}_1, \bar{z}_2, ..., \bar{z}_N) = E_N \Psi(\bar{z}_1, \bar{z}_2, ..., \bar{z}_N)
\]
with the eigenstates $\Psi(z_1, ..., z_N)$ being single-valued functions on the plane $z = (z, \bar{z})$, normalizable with respect to the $SL(2, \mathbb{C})$ invariant scalar product

$$\|\Psi\|^2 = \int d^2z_1 d^2z_2 ... d^2z_N |\Psi(z_1, z_2, ..., z_N)|^2$$

(2.9)

with $d^2z = dx dy = dz d\bar{z}/2$. The equation (2.8) has appeared for the first time in the analysis of the high-energy asymptotics of the partial waves in the multi-color QCD [3]. It was found [3, 4], that it possesses the set of mutually commuting conserved charges whose number is large enough for the Schrödinger equation (2.8) to be completely integrable.

The total spin of the $N-$particle system is one of the conserved charges. Indeed, the Hamiltonian (2.6) is a function of two-particle Casimir operators and, therefore, it commutes with the operators of the total spin $S_x = \sum_k S_x^{(k)}$ and $S_y = \sum_k S_y^{(k)}$, acting on the quantum space of the system $V_N \equiv V^{(s_1, \bar{s}_1)} \otimes V^{(s_2, \bar{s}_2)} \otimes ... \otimes V^{(s_N, \bar{s}_N)}$. This implies that the eigenstates can be classified according to the irreducible representations of the $SL(2, \mathbb{C})$ group, $V^{(h, \bar{h})}$, parameterized by the spins $(h, \bar{h})$. The eigenstates $\Psi(z)$, belonging to $V^{(h, \bar{h})}$ are labelled by the center-of-mass coordinate $z_0$ and they can be chosen to have the following $SL(2, \mathbb{C})$ transformation properties

$$\Psi(z'_k - z'_0) = (cz_0 + d)^2h(\bar{c}z_0 + \bar{d})^{2\bar{h}} \prod_{k=1}^N (cz_k + d)^{2s_k}\bar{(\bar{c}\bar{z}_k + \bar{d})^{2\bar{s}_k}}\Psi(z_k - z_0)$$

(2.10)

with $z_0$ and $\bar{z}_0$ transformed in the same way as $z_k$ and $\bar{z}_k$, Eq. (2.4). As a consequence, they diagonalize the Casimir operators corresponding to the total spin of the system, $S^2 = S_x^2 + (S_+S_- + S_-S_+)/2$,

$$(S^2 - h(h - 1)) \Psi(z_1, z_2, ..., z_N) = (S^2 - \bar{h}(\bar{h} - 1)) \Psi(z_1, z_2, ..., z_N) = 0 .$$

(2.11)

The complex parameters $(s_k, \bar{s}_k)$ and $(h, \bar{h})$ entering (2.7) and (2.10) parameterize the irreducible $SL(2, \mathbb{C})$ representations. For the principal series representation they satisfy the conditions [17]

$$s_k - \bar{s}_k = n_{s_k} , \quad s_k + (\bar{s}_k)^* = 1$$

(2.12)

and have the following form

$$s_k = \frac{1 + n_{s_k}}{2} + i\nu_{s_k} , \quad \bar{s}_k = \frac{1 - n_{s_k}}{2} + i\nu_{s_k}$$

(2.13)

with $\nu_{s_k}$ being real and $n_{s_k}$ being integer or half-integer. The spins $(h, \bar{h})$ are given by similar expressions with $n_{s_k}$ and $\nu_{s_k}$ replaced by $n_h$ and $\nu_h$, respectively. The parameter $n_{s_k}$ has the meaning of the two-dimensional Lorentz spin of the particle, whereas $\nu_{s_k}$ defines its scaling dimension. To see this, one performs a $2\pi-$rotation of the particle on the plane, $z \rightarrow z \exp(2\pi i)\bar{z}$ and $\bar{z} \rightarrow \bar{z} \exp(-2\pi i)$ and finds from (2.10) that the wave function acquires a phase $\Psi(z_k, \bar{z}_k) \rightarrow (-1)^{2n_{s_k}} \Psi(z_k, \bar{z}_k)$. For half-integer $n_{s_k}$ it changes the sign and the corresponding unitary representation is spinor. Similarly, to define the scaling dimension, $s + \bar{s} = 1 + 2i\nu_{s_k}$,

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4. Throughout the paper we indicate by bar symbol the variables belonging to the antiholomorphic sector. Notice that, in general, the corresponding variables in two sectors are not complex conjugated to each other.

5. Since the unitary representations labeled by the spins $(s, \bar{s})$ and $(1-s, 1-\bar{s})$ are unitary equivalent and are related to each other through the intertwining relation (see Eq. (3.76) below) one can choose $n_{s_k}$ in (2.13) to be nonnegative.
one performs the transformation $z \to \lambda z$ and $\bar{z} \to \lambda \bar{z}$. We recall that for the homogenous spin chain one takes $s_k = s$ and $\bar{s}_k = \bar{s}$ for arbitrary $k$. In what follows, for the sake of simplicity, we will not consider the spinor $SL(2, \mathbb{C})$ representations and choose $n_{s_k}$ in (2.13) to be nonnegative integer.

We notice that the holomorphic and antiholomorphic spin generators (2.2) as well as the Casimir operators (2.7) are conjugated to each other with respect to the scalar product (2.9) (see footnote to (2.7))

$$\left[S^{(k)}_{\alpha}\right]^\dagger = -\bar{S}^{(k)}_{\alpha}, \quad [J_{k,k+1}]^\dagger = 1 - \bar{J}_{k,k+1}. \quad (2.14)$$

This implies that $H^1_N = \bar{H}_N$ in Eq. (2.3) and, as a consequence, the Hamiltonian (2.3) is hermitian on the space of functions endowed with the scalar product (2.9), $\mathcal{H}^1_N = \mathcal{H}_N$. Thus, the energy $E_N$ in (2.8) is real and the corresponding eigenstates are orthogonal to each other with respect to (2.9). Since the Euler $\psi-$function has poles at negative integer arguments, the holomorphic and antiholomorphic Hamiltonians, Eq. (2.6), are unbounded operators. One can verify however [21] that, thanks to the properties of the principal $SL(2, \mathbb{C})$ series, Eqs. (2.12) and (2.13), the poles are cancelled in their sum (2.3) leading to the Hamiltonian $\mathcal{H}_N$ which is bounded from below.

2.2. $R-$matrix

Let us show that the model defined in the previous section can be described using the $R-$matrix approach [4]. For this we will need the expression for the $R-$matrix acting on the tensor product of two $SL(2, \mathbb{C})$ representations, $V^{(s_1, \bar{s}_1)} \otimes V^{(s_2, \bar{s}_2)}$. The general expression for the $R-$matrix is well known for the Heisenberg spin magnets in the case of arbitrary higher spin $SL(2, \mathbb{R})$ representations [3, 4] and to the best of our knowledge its $SL(2, \mathbb{C})$ analog has not been discussed in the literature.

To find the $R-$matrix for the infinite-dimensional $SL(2, \mathbb{C})$ representations, we introduce the Lax operators in the holomorphic and antiholomorphic sectors

\[
L_s(u) = u + i(\sigma \cdot S) = \begin{pmatrix} u + iS_0 & iS_- \\ iS_+ & u - iS_0 \end{pmatrix},
\]

\[
\bar{L}_{\bar{s}}(\bar{u}) = \bar{u} + i(\sigma \cdot \bar{S}) = \begin{pmatrix} \bar{u} + i\bar{S}_0 & i\bar{S}_- \\ i\bar{S}_+ & \bar{u} - i\bar{S}_0 \end{pmatrix}
\]

(2.15)

with $u$ and $\bar{u}$ being arbitrary complex parameters and $\sigma_\alpha$ being Pauli matrices. These operators act on the quantum space $V^{(s, \bar{s})}$ and coincide with similar expressions for the Lax operator in the Heisenberg spin chain model [1, 2]. We define the $R-$matrix by requiring that the Lax operators have to satisfy the commutation relations

\[
L_{s_1}(v)L_{s_2}(v + u)R_{(s_1, \bar{s}_1),(s_2, \bar{s}_2)}(u, \bar{u}) = R_{(s_1, \bar{s}_1),(s_2, \bar{s}_2)}(u, \bar{u})L_{s_2}(v + u)L_{s_1}(v),
\]

\[
\bar{L}_{\bar{s}_1}(\bar{v})\bar{L}_{\bar{s}_2}(\bar{v} + \bar{u})R_{(s_1, \bar{s}_1),(s_2, \bar{s}_2)}(u, \bar{u}) = R_{(s_1, \bar{s}_1),(s_2, \bar{s}_2)}(u, \bar{u})\bar{L}_{\bar{s}_2}(\bar{v} + \bar{u})\bar{L}_{\bar{s}_1}(\bar{v}).
\]

(2.16)

Here, the operator $R_{(s_1, \bar{s}_1),(s_2, \bar{s}_2)}(u, \bar{u})$ depends on two spectral parameters, $u$ and $\bar{u}$, and it acts on the tensor product $V^{(s_1, \bar{s}_1)} \otimes V^{(s_2, \bar{s}_2)}$ labeled by the $SL(2, \mathbb{C})$ spins $(s_1, \bar{s}_1)$ and $(s_2, \bar{s}_2)$. 

6
2.2.1. Yang-Baxter equation

Solving the Yang-Baxter equation (2.16), we use the method proposed in [19, 20]. It is based on the representation of the $R$–operator by its integral kernel

$$[R_{(s_1, s_2, s_3)}(u, \bar{u})\Psi](z_1, z_2) \equiv \int d^2w_1 \int d^2w_2 R_{u, \bar{u}}(z_1, z_2 | w_1, \bar{w}_2) \Psi(w_1, \bar{w}_2),$$ (2.17)

where $z_i = (z_i, \bar{z}_i)$ and $\bar{w}_i = (w_i, \bar{w}_i)$ are two-dimensional vectors with the (anti)holomorphic coordinates defined in (2.1) and $\Psi(w_1, \bar{w}_2)$ is an arbitrary test function belonging to $V^{(s_1, z_1)} \otimes V^{(s_2, z_2)}$. To find the kernel, $R_{u, \bar{u}}(z_1, z_2 | w_1, \bar{w}_2)$, we require that the $R$–matrix defined in this way has to satisfy the Yang-Baxter equations (2.16) and, in addition, $R_{u, \bar{u}}(z_1, z_2 | w_1, \bar{w}_2)$ has to be a single-valued function on the two-dimensional plane. As we shall see in a moment, the latter condition imposes constraints on the possible values of the spectral parameters $u$ and $\bar{u}$.

The dependence of the kernel of the $R$–matrix on the holomorphic coordinates is fixed by the first relation in (2.10). Applying its both sides to the same test function $\Psi(z_1, z_2)$, one substitutes the Lax operators and the $R$–operator by their expressions, Eqs. (2.13) and (2.17), and integrates by parts in the r.h.s. using the identity

$$[R(u, \bar{u})S^{(1)}_{\alpha}(z_1, z_2) = \int d^2w_1 \int d^2w_2 R_{u, \bar{u}}(z_1, z_2 | w_1, \bar{w}_2) \left( S^{(s_1)}_{\alpha}(w_1) \Psi(w_1, \bar{w}_2) \right)$$

$$= -\int d^2w_1 \int d^2w_2 \left( S^{(1-s_1)}_{\alpha}(w_1) R_{u, \bar{u}}(z_1, z_2 | w_1, \bar{w}_2) \right) \Psi(w_1, \bar{w}_2).$$ (2.18)

Here, $S^{(s_1)}_{\alpha}(w_1)$ and $S^{(1-s_1)}_{\alpha}(w_1)$ denote the differential operators acting on the holomorphic coordinates $w_1$. They and given by Eqs. (2.2) with the spin $s_k$ replaced by $s_1$ and $1 - s_1$, respectively. In this way, the first relation in (2.16) can be replaced by the following matrix equation

$$(v + i\sigma \cdot S^{(s_1)}(z_1))(v + u + i\sigma \cdot S^{(s_2)}(z_2)) R_{u, \bar{u}}(z_1, z_2 | w_1, \bar{w}_2) = (u + v - i\sigma \cdot S^{(1-s_2)}(w_2))(v - i\sigma \cdot S^{(1-s_1)}(w_1)) R_{u, \bar{u}}(z_1, z_2 | w_1, \bar{w}_2),$$ (2.19)

which leads to an overcompleted system of the differential equations on the kernel $R_{u, \bar{u}}$. It turns out that the system has a unique solution. The simplest way to find it is [20] to project the both sides of (2.19) by the vector $(-w_2, 1)$ from the left and by the vector $(1, z_2)$ from the right and, then, match the coefficients in front of powers of $u$ and $v$. One obtains two first-order differential equations on $R_{u, \bar{u}}$, which fix the dependence of the kernel on the holomorphic coordinates $z_1$ and $w_1$ up to arbitrary prefactor depending on the spectral parameters. Repeating similar analysis for the second equation in (2.16), we restore the dependence of the kernel on the antiholomorphic coordinates. The resulting expression for the solution to (2.16) is given by the product of the holomorphic and antiholomorphic kernels

$$R_{u, \bar{u}}(\bar{z}_1, \bar{z}_2 | \bar{w}_1, \bar{w}_2) = \rho_{\alpha}(u, \bar{u})$$

$$\times \left( w_2 - z_1 \right)^{\text{iu}-s_1+s_2-1} \left( z_1 - z_2 \right)^{\text{iu}+s_1-s_2-1} \left( w_1 - w_2 \right)^{\text{iu}+s_1+s_2-1} \left( z_2 - w_1 \right)^{\text{iu}+s_1-s_2-1}$$

$$\times \left( w_2 - \bar{z}_1 \right)^{\text{iu}-s_1+s_2-1} \left( \bar{z}_1 - \bar{z}_2 \right)^{\text{iu}+s_1-s_2-1} \left( \bar{w}_1 - \bar{w}_2 \right)^{\text{iu}+s_1+s_2-1} \left( \bar{z}_2 - \bar{w}_1 \right)^{\text{iu}+s_1-s_2-1}$$ (2.20)

Here, we assume that there exists the region of the spectral parameters, $u$ and $\bar{u}$, in which the integral is convergent. For $u$ and $\bar{u}$ outside this region the kernel of the $R$–operator is defined by the analytical continuation.
with the normalization factor $\rho_R(u, \bar{u})$ being an arbitrary function of the spectral parameters. The obtained expression for the kernel of the $R-$matrix can be translated into the language of two-dimensional Feynman diagrams as shown in Fig. 1. As we will demonstrate below, this diagrammatic representation becomes extremely useful in solving the model.

Examining the expression for the kernel, Eq. (2.20), we notice that $R_{u,\bar{u}}(\vec{z}_1, \vec{z}_2 | \vec{w}_1, \vec{w}_2)$ acquires a nonzero monodromy as $\vec{z}_1$ encircles the points $\vec{z}_2$ and $\vec{w}_2$ on the plane and similarly for other arguments of the $R-$matrix. For the kernel to be a single-valued function, the corresponding monodromies should cancel in the r.h.s. of (2.20). In general, for the function of the form $(w - z_1)^\alpha(\bar{w} - \bar{z}_1)^{\bar{\alpha}}$ this condition amounts to the Lorentz spin $\alpha - \bar{\alpha}$ to be integer. Applying the same condition to (2.20) and taking into account the expressions for the spins $s_1$ and $s_2$, Eqs. (2.13) with $n_{s_1}$ and $n_{s_2}$ integer, we find that the spectral parameters $u$ and $\bar{u}$ have to satisfy the additional condition

$$i(u - \bar{u}) = n$$

(2.21)

with $n$ being an integer.

Using the expression for the kernel (2.20) it becomes straightforward to show that the $R-$matrix defined by Eqs. (2.17) and (2.20) satisfies the Yang-Baxter equation

$$R_{12}(u, \bar{u})R_{13}(v, \bar{v})R_{23}(v - u, \bar{v} - \bar{u}) = R_{23}(v - u, \bar{v} - \bar{u})R_{13}(v, \bar{v})R_{12}(u, \bar{u}),$$

(2.22)

where we used a shorthand notation for $R_{k\ell n}(u, \bar{u}) \equiv R_{(s_k, \bar{s}_k),(s_\ell, \bar{s}_\ell)}(u, \bar{u})$. Going over to the integral representation (2.17), the product of the $R-$operators in the l.h.s. of (2.22) has the following kernel

$$\left[ R_{12}(u, \bar{u})R_{13}(v, \bar{v})R_{23}(v - u, \bar{v} - \bar{u}) \right](\vec{z}_1, \vec{z}_2, \vec{z}_3 | \vec{w}_1, \vec{w}_2, \vec{w}_3) =$$

$$\int d^2y_1 d^2y_2 d^2y_3 R_{u,\bar{u}}(\vec{z}_1, \vec{z}_2 | \vec{y}_1, \vec{y}_2)R_{v,\bar{v}}(\vec{y}_1, \vec{z}_3 | \vec{w}_1, \vec{w}_3)R_{v-u,\bar{v}-\bar{u}}(\vec{y}_2, \vec{y}_3 | \vec{w}_2, \vec{w}_3).$$

(2.23)

One finds similar expression for the r.h.s. of (2.22). The proof of (2.22) can be carried out diagrammatically as shown in Fig. 2, without doing any calculations. Replacing each $R-$matrix by a square (see Fig. 1) one represents the l.h.s. of (2.22) by the left Feynman diagram in the lower line in Fig. 2. This diagram consists of three squares and each pair of squares shares a common vertex. The central triangle formed by the sides of three squares turns out to be
“unique” (see Appendix A for the definitions) and it can be transformed into the “star” diagram, using the uniqueness relation (A.8). Then, three new stars with the centers located in the vertices of the central triangle are also unique and can be transformed back into triangles giving rise to a hexagon diagram. Performing the same set of transformations on the r.h.s. of (2.22) (the left diagram in the upper line in Fig. 2), one arrives at the same hexagon diagram, thus proving the Yang-Baxter equation.

Following similar procedure one can show that the $R$–operator (2.20) satisfies the $T$–inversion relation (see Fig. 3)

$$
\rho_R(u, \bar{u}) = \frac{\pi^4}{a(s_1 - s_2 + iu) a(\bar{s}_2 - \bar{s}_1 + i\bar{u}) a(s_1 - s_2 - iu) a(\bar{s}_2 - \bar{s}_1 - i\bar{u})} \times \mathbb{1},
$$

where the function $a(x)$ is defined in (A.3) and $\mathbb{1}$ stands for the identity operator on $V^{(s_1, \bar{s}_1)} \otimes V^{(s_2, \bar{s}_2)}$. Defining the normalization factor $\rho_R(u, \bar{u})$ we fix, for later convenience, the coefficient in front of the identity operator in Eq.(2.24) to be equal to unity

$$
\rho_R(u, \bar{u}) = \frac{1}{\pi^2} a(s_1 - s_2 + iu) a(\bar{s}_2 - \bar{s}_1 - i\bar{u}).
$$

2.2.2. Eigenvalues of the $R$–matrix

It follows from the Yang-Baxter equation (2.16) that the $R$–matrix commutes with the sum the $SL(2, \mathbb{C})$ spins and, therefore, it is invariant under the $SL(2, \mathbb{C})$ transformations (2.3) and (2.4)

$$
[R_{12}(u, \bar{u}), S^{(1)}_\alpha + S^{(2)}_\alpha] = [R_{12}(u, \bar{u}), \bar{S}^{(1)}_\alpha + \bar{S}^{(2)}_\alpha] = 0.
$$

To see this, one takes the limit in (2.16) as $v \to \infty$ for $u$ fixed and similarly for the antiholomorphic sector.
This implies that the $R$–operator is a function of the Casimir operators $J_{12}$ and $\tilde{J}_{12}$ defined in (2.7). In order to find the explicit form of this function or, equivalently, to calculate the eigenvalues of the $R$–matrix, one has to decompose the tensor product $V^{(s_1,\bar{s}_1)} \otimes V^{(s_2,\bar{s}_2)}$ over the irreducible components $V^{(h,\bar{h})}$ and take into account that the operators, $J_{12}$, $\tilde{J}_{12}$ and $R_{12}(u,\bar{u})$ are diagonal on $V^{(h,\bar{h})}$ simultaneously.

The projector onto the principal series spin–$(h,\bar{h})$ representation, $\Pi^{(h,\bar{h})}$, is defined as

$$
\Psi^{(h,\bar{h})}(\vec{z}) = \int d^2w_1 d^2w_2 \Pi^{(h,\bar{h})}(\vec{w}_1 - \vec{z}, \vec{w}_2 - \vec{z}) \Psi^{(s_1,\bar{s}_1),(s_2,\bar{s}_2)}(\vec{w}_1, \vec{w}_2),
$$

(2.27)

where $\Psi^{(h,\bar{h})}(\vec{z})$ belongs to $V^{(h,\bar{h})}$ and the kernel $\Pi^{(h,\bar{h})}(\vec{w}_1 - \vec{z}, \vec{w}_2 - \vec{z})$ is given by [17]

$$
\Pi^{(h,\bar{h})}(\vec{w}_1 - \vec{z}, \vec{w}_2 - \vec{z}) = (w_1 - w_2)^{h+s_1+s_2-2} (w_2 - z)^{s_2-h-s_1} (z - w_1)^{s_1-h-s_2} 
\times (\vec{w}_1 - \vec{w}_2)^{h+s_1+s_2-2} (\vec{w}_2 - \vec{z})^{\bar{s}_2-h-\bar{s}_1} (\vec{z} - \vec{w}_1)^{\bar{s}_1-h-\bar{s}_2}.
$$

(2.28)

Requiring this kernel to be a well-defined function on the plane, one finds that the possible values of the spins $(h, \bar{h})$ are given by general $SL(2,\mathbb{C})$ expressions (2.13)

$$
h = \frac{1 + n_h}{2} + i\nu_h, \quad \bar{h} = \frac{1 - n_h}{2} + i\nu_h
$$

(2.29)

with $\nu_h$ arbitrary real and $n_h$ integer. One can verify that the projectors $\Pi^{(h,\bar{h})}(\vec{z}_1 - \vec{z}_0, \vec{z}_2 - \vec{z}_0)$ are orthogonal to each other with respect to the scalar product (2.9) for different $\vec{z}_0$ and the $SL(2,\mathbb{C})$ spins $(h, \bar{h})$ such that $n_h \geq 0$. 
By the definition, the projector diagonalizes the Casimir operator \((S_1 + S_2)^2 = h(h - 1)\) and 
\((\bar{S}_1 + \bar{S}_2)^2 = \bar{h}(\bar{h} - 1)\) leading to
\[
\Pi^{(h,\bar{h})} J_{12} = h \Pi^{(h,\bar{h})}, \quad \Pi^{(h,\bar{h})} \bar{J}_{12} = \bar{h} \Pi^{(h,\bar{h})}.
\]  
(2.30)

To find the eigenvalue of the \(R\)–matrix we calculate the product of the operators \(\Pi^{(h,\bar{h})} R_{12}(u, \bar{u})\). Replacing the operators by their kernels, Eqs. (2.28) and (2.20), one gets
\[
\int d^2z_1 d^2z_2 \Pi^{(h,\bar{h})}(\bar{z}_1 - z, \bar{z}_2 - z) R_{u,\bar{u}}(z_1, \bar{z}_2 | \bar{w}_1, \bar{w}_2) = R_{h,\bar{h}}(u, \bar{u}) \Pi^{(h,\bar{h})}(\bar{w}_1 - z, \bar{w}_2 - z),
\]  
(2.31)
where the r.h.s. is fixed up to factor \(R_{h,\bar{h}}(u, \bar{u})\) by the \(SL(2, \mathbb{C})\) transformation properties (2.4). The calculation of the integral in the l.h.s. can be perform diagrammatically as shown in Fig. 4.

The corresponding diagram is obtained by gluing together the triangle \((\Pi^{(h,\bar{h})})\) and the square \((R_{u,\bar{u}})\) along the line connecting the points \(z_1\) and \(z_2\). In the resulting diagram, these points are the centers of two stars, which turn out to be unique. Subsequently applying the uniqueness “star-triangle” relations, one obtains the triangle diagram, which is equal to the projector multiplied by a \(c\)–valued factor \(R_{h,\bar{h}}(u, \bar{u})\) depending on the \(SL(2, \mathbb{C})\) spins \(h\) and \(\bar{h}\). Replacing them by the corresponding operators, Eq. (2.30), one obtains the operator form of the \(R\)–matrix
\[
R_{h,\bar{h}}(u, \bar{u}) = \frac{\Gamma(s_2 - s_1 + i\bar{u}) \Gamma(1 + s_1 - s_2 + i\bar{u})}{\Gamma(s_2 - s_1 - i\bar{u}) \Gamma(1 + s_1 - s_2 - i\bar{u})} \times \frac{\Gamma(1 - \bar{h} - i\bar{u}) \Gamma(\bar{h} - i\bar{u})}{\Gamma(1 - h + iu) \Gamma(h + iu)},
\]
(2.32)

Remarkably enough, the obtained expression for the \(R\)–matrix is factorized into the product of the holomorphic and antiholomorphic operators. Each of them formally coincides with the well-known expression for the \(R\)–matrix for high spin representations of the \(SL(2, \mathbb{R})\) group [18]. However, the important difference with the latter case is that the spectral parameters in two sectors, \(u\) and \(\bar{u}\), are not arbitrary anymore and have to satisfy the additional condition (2.21).

2.2.3. Unitary \(R\)–matrix

For the homogenous spin chain, \(s_1 = s_2 = s\) and \(\bar{s}_1 = \bar{s}_2 = \bar{s}\), the general expression for the \(R\)–matrix, Eq. (2.32), simplifies to
\[
R_{12}(u, \bar{u}) = \frac{\Gamma(i\bar{u}) \Gamma(1 + i\bar{u})}{\Gamma(-i\bar{u}) \Gamma(1 - i\bar{u})} \times \frac{\Gamma(1 - J_{12} - i\bar{u}) \Gamma(J_{12} - i\bar{u})}{\Gamma(1 - J_{12} + i\bar{u}) \Gamma(J_{12} + i\bar{u})}.
\]  
(2.33)

This operator acts on the tensor product \(V \otimes V\) with \(V \equiv V^{(s,\bar{s})}\) and has the following properties.

At \(u = \bar{u} = 0\) the \(R\)–matrix (2.33) coincides with the permutation operator \(P_{12}\)
\[
R_{12}(0, 0) = (-1)^{1 + J_{12} - J_{12}} = -P_{12},
\]  
(2.34)

which is defined on \(V \otimes V\) as
\[
P_{12} \Psi(z_1, \bar{z}_2) = \Psi(\bar{z}_2, z_1).
\]  
(2.35)

Since the spectral parameters have to satisfy the condition (2.21), the limit of (2.33) at \(u = \bar{u} = 0\) has to be calculated by putting \(u = \bar{u}\) and sending \(u \to 0\) afterwards.
To verify (2.34), one projects its both sides onto $\Pi^{(h, \bar{\bar{h}})}$ and takes into account (2.29).

Taking into account (2.14), one finds that the $R$–operator satisfies the following relations

$$[R_{12}(u, \bar{u})]^\dagger = R_{12}(\bar{u}^*, -u^*), \quad R_{12}(u, \bar{u})R_{12}(-u, -\bar{u}) = 1.$$  \hspace{1cm} (2.36)

According to (2.36), the $R$–matrix is not a unitary operator on $V \otimes V$ for arbitrary $u$ and $\bar{u}$ satisfying (2.21). However, there exists a region of the spectral parameters

$$\bar{u} = u^*,$$  \hspace{1cm} (2.37)

in which the unitarity holds

$$[R_{12}(u, \bar{u})]^\dagger R_{12}(u, \bar{u}) = 1.$$  \hspace{1cm} (2.38)

Combining (2.37) together with (2.21), we find that the $R$–matrix is a unitary operator on $V \otimes V$ for the spectral parameters of the general form

$$u = \nu - \frac{in}{2}, \quad \bar{u} = \nu + \frac{in}{2}$$  \hspace{1cm} (2.39)

with $\nu$ real and $n$ integer.

### 2.3. Complete integrability

Let us show that the $R$–matrix (2.33) defines a completely integrable quantum-mechanical system with the Hamiltonian given by Eqs. (2.5)–(2.6). Applying the $R$–matrix approach and following the standard procedure $[1, 2, 3]$, we define the family of the transfer matrices $T_N^{(s_0, \bar{s}_0)}(u, \bar{u})$ parameterized by the spins $(s_0, \bar{s}_0)$ and acting on the quantum space of the system $V^N$

$$T_N^{(s_0, \bar{s}_0)}(u, \bar{u}) = \text{Tr}_{(s_0, \bar{s}_0)} \left[ R_{(s_0, \bar{s}_0), (s_1, \bar{s}_1)}(u, \bar{u})R_{(s_1, \bar{s}_1), (s_2, \bar{s}_2)}(u, \bar{u})...R_{(s_N, \bar{s}_N)}(u, \bar{u}) \right].$$  \hspace{1cm} (2.40)

Here, the trace is taken over the auxiliary $SL(2, \mathbb{C})$ representation space $V^{(s_0, \bar{s}_0)}$. We recall that the spin chain is homogenous, so that $s_1 = ... = s_N = s$ and $\bar{s}_1 = ... = \bar{s}_N = \bar{s}$. The spins $(s_0, \bar{s}_0)$ have the form (2.13) and, in general, they are different from the spins $(s, \bar{s})$. Substituting the $R$–matrices in (2.40) by their integral representation, (2.17) and (2.20), one can evaluate the kernel of the transfer matrix $T_N^{(s_0, \bar{s}_0)}(u, \bar{u})$ as $N$–fold convolution of the $R$–kernel with periodic boundary conditions. The diagrammatical representation of the transfer matrix $T_N^{(s_0, \bar{s}_0)}(u, \bar{u})$ is shown in Fig. 4.

Invoking the standard arguments $[1, 2, 3]$, one finds from the Yang-Baxter equation, Eq. (2.22), that the transfer matrices form the family of mutually commuting $SL(2, \mathbb{C})$ invariant operators

$$[T_N^{(s_0, \bar{s}_0)}(u, \bar{u}), T_N^{(s_0', \bar{s}_0')}(v, \bar{v})] = [S_\alpha, T_N^{(s_0, \bar{s}_0)}(u, \bar{u})] = [\bar{S}_\alpha, T_N^{(s_0, \bar{s}_0)}(u, \bar{u})] = 0$$  \hspace{1cm} (2.41)

and, therefore, serve as the generating functions of the integrals of motion and the Hamiltonian of the model. The latter is obtained from the fundamental transfer matrix $T^{(s, \bar{s})}(u, \bar{u})$, for which the auxiliary space $V^{(s, \bar{s})}$ coincides with the quantum space of a single particle, $V$, and the $R$–matrices entering (2.40) are given by (2.33).

Examining the expansion of the fundamental transfer matrix $T^{(s, \bar{s})}(u, \bar{u})$ around the origin, $u = \bar{u} = 0$, and using the properties of the $R$–operators, Eqs. (2.33) and (2.34), one finds

$$\mathcal{H}_N = i \left[ \frac{d}{du} \ln T_N^{(s, \bar{s})}(u, u) \right]_{u=0} = H_{12} + ... + H_{N-1,N} + H_{N,1}.$$  \hspace{1cm} (2.42)
Figure 5: Diagrammatical representation of the transfer matrix $T_N^{(s_0, \bar{s}_0)}(u, \bar{u})$ defined in (2.40). The left- and rightmost vertices are located at the same point and integration over its position as well as over the position of the remaining internal vertices is implied.

where the two-particle Hamiltonian $H_{k,k+1}$ is given by

$$H_{12} = -iP_{12} \frac{d}{du} R_{(s_1, \bar{s}_1),(s_2, \bar{s}_2)}(u, \bar{u}) \bigg|_{u=0} = \psi(J_{12}) + \psi(1 - J_{12}) - 2\psi(1) + \psi(\bar{J}_{12}) + \psi(1 - \bar{J}_{12}) - 2\psi(1),$$

with the Casimir operators $J_{12}$ and $\bar{J}_{12}$ defined in (2.7). Here, we substituted the $R$–matrix by its operator expression (2.33) and used (2.34). Comparing (2.42) with (2.5) we conclude that two Hamiltonians are identical.

To identify the total set of the integrals of motion of the model, one constructs the auxiliary holomorphic monodromy matrix $T_N(u) = L_1(u)L_2(u)...L_N(u) = \begin{pmatrix} A_N(u) & B_N(u) \\ C_N(u) & D_N(u) \end{pmatrix}$ (2.44)

and similarly for antiholomorphic monodromy operator $\bar{T}_N(u)$. Replacing the Lax operators by their expressions $L_k \equiv L_{s_k}(u)$, Eq. (2.13), one obtains the operators $A_N, ..., D_N$ in the form of polynomials in the spectral parameter. Their asymptotics at large $u$ is given by

$$A_N(u) = u^N + iS_0 u^{N-1} + O(u^{N-2}), \quad B_N(u) = iS_-u^{N-1} + O(u^{N-2}),$$
$$D_N(u) = u^N - iS_0 u^{N-1} + O(u^{N-2}), \quad C_N(u) = iS_+ u^{N-1} + O(u^{N-2}),$$

where $S_\alpha = \sum_{k=1}^N S_\alpha^{(k)}$ is the total $SL(2, \mathbb{C})$ spin of $N$ particles. These operators act on the quantum space of the system and they are related to their antiholomorphic counterparts as

$$[A_N(u)]^\dagger = \bar{A}_N(u^*), \quad [B_N(u)]^\dagger = \bar{B}_N(u^*)$$

and similarly for the remaining operators $C_N$ and $D_N$.

Taking the trace of the monodromy matrix (2.44) we define the auxiliary transfer matrix

$$t_N(u) = A_N(u) + D_N(u) = 2u^N + q_2 u^{N-2} + ... + q_N$$

and similarly for $\bar{t}_N(\bar{u})$. Combining together (2.46) and (2.47) one finds that

$$[t_N(u)]^\dagger = \bar{t}_N(u^*), \quad q_k^\dagger = \bar{q}_k$$
for $k = 2, \ldots, N$. Here, the operators $q_k$ ($\bar{q}_k$) are given by certain linear combinations of the product of $k$ spin operators. They can be rewritten using (2.2) as $k$–th order differential operators acting on (anti)holomorphic coordinates. For instance,

$$ q_2 = -\sum_{k>n} 2(S^{(k)} S^{(n)}) = -S^2 + Ns(s-1) = \sum_{k>n} (z_k - z_n)^{2(1-s)} \partial_{z_k} \partial_{z_n} (z_k - z_n)^{2s} + 2Ns(s-1), $$

where $S^2 = h(h-1)$ is the Casimir operator corresponding to the total spin of $N$ particles, Eq. (2.11).

It follows from the Yang-Baxter equations (2.16) and (2.22) that the auxiliary transfer matrices $t_N(u)$ and $\bar{t}_N(\bar{u})$ commute with the transfer matrices $T^{(s_\alpha, s_\beta)}(v, \bar{v})$ and, in addition, satisfy the same relations (2.41). Together with Eqs. (2.47) and (2.43) these relations imply that the operators $(q_k, \bar{q}_k)$ for $k = 2, \ldots, N$ form the set of $2N - 2$ mutually commuting $SL(2, \mathbb{C})$ invariant integrals of motion

$$ [\mathcal{H}_N, q_k] = [\mathcal{H}_N, \bar{q}_k] = [\mathcal{H}_N, S_\alpha] = [\mathcal{H}_N, \bar{S}_\alpha] = 0 $$

and

$$ [q_k, q_n] = [\bar{q}_k, \bar{q}_n] = [q_k, S_\alpha] = [\bar{q}_k, \bar{S}_\alpha] = 0. $$

Two additional operators can be added to this set due to the $SL(2, \mathbb{C})$ invariance of the Hamiltonian (2.50). It is convenient to choose them as particular projections of the total spins $S_\alpha$ and $\bar{S}_\alpha$

$$ p = iS_- = -i \sum_{k=1}^N \partial_{z_k}, \quad \bar{p} = iS_- = -i \sum_{k=1}^N \partial_{\bar{z}_k}, $$

which have the meaning of the total momenta of $N$ particles. Then, the eigenstates of the Hamiltonian with a definite value of the momenta $\mathbf{p} = (p, \bar{p})$ are given by

$$ \Psi_{\mathbf{p}, (q, \bar{q})}(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_N) = \int d^2z_0 e^{i\mathbf{z}_0 \mathbf{p} + i\mathbf{z}_0 \bar{p}} \Psi(\bar{z}_1 - \bar{z}_0, \bar{z}_2 - \bar{z}_0, \ldots, \bar{z}_N - \bar{z}_0) $$

with $\bar{z}_0$ being the center-of-mass of the system.

Thus, the Schrödinger equation (2.8) possesses the set of $2N$ mutually commuting conserved charges, $\mathbf{p}$ and $\{q_k, \bar{q}_k\}$ ($2 \leq k \leq N$), and, therefore, is completely integrable. This implies that, firstly, the Hamiltonian of the model can be expressed as a function of the integrals of motion

$$ \mathcal{H}_N = \mathcal{H}_N(q_2, \bar{q}_2; \ldots; q_N, \bar{q}_N) $$

and, secondly, the wave function $\Psi_{\mathbf{p}, (q, \bar{q})}(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_N)$ can be defined as a simultaneous eigenstate of the integrals of motion. The corresponding energy levels $E_{(q, \bar{q})}$ can be obtained from (2.54) by replacing the operators $\{q_k, \bar{q}_k\}$ by their corresponding eigenvalues. Notice that the Hamiltonian can not depend on the momentum operator $\mathbf{p}$ due to the $SL(2, \mathbb{C})$ invariance (2.50). The explicit form of the dependence (2.54) will be established in the Section 3.5 using the method of the Baxter $Q$–operator.

2.3.1. Special case: $N = 2$

At $N = 2$ the Schrödinger equation (2.8) can be solved exactly. In this case, the Hamiltonian of the model is given by the two-particle kernel (2.6), $\mathcal{H}_2 = 2H(J_{12}) + 2H(\bar{J}_{12})$ and its spectrum
can be found by diagonalizing simultaneously the Casimir operators $J_{12}$ and $\tilde{J}_{12}$, as well as the momentum operators $p$ and $\tilde{p}$. Taking into account Eqs. (2.28) and (2.30) and putting $s_1 = s_2 = s$ one finds the eigenstates as

$$
\Psi_{\bar{h}, h} (\bar{z}_1, \bar{z}_2) = \int d^2 z_0 e^{iz_0p + iz_0\tilde{p}} \Psi_{h, \bar{h}} (z_1 - z_0, \bar{z}_2 - \bar{z}_0),
$$

$$
\Psi_{h, \bar{h}} (z_1 - z_0, \bar{z}_2 - \bar{z}_0) = [z_1 - z_2]^{h-2s} [\bar{z}_1 - \bar{z}_2]^{-\bar{h}} [z_2 - z_0]^{-h} \tag{2.55}
$$

where the notation was introduced for $[z_j - z_k]^{\alpha} \equiv (z_j - z_k)^{\alpha} (\bar{z}_j - \bar{z}_k)^{\bar{\alpha}}$. Here, the spins $h = (1 + n_h)/2 + i\nu_h$ and $\bar{h} = (1 - n_h)/2 + i\nu_h$ are the eigenvalues of the $SL(2, \mathbb{C})$ Casimir operators $J_{12}$ and $\tilde{J}_{12}$, Eq. (2.11). One verifies that (2.55) satisfies (2.10). The corresponding energy is equal to

$$
E_{N=2}(h, \bar{h}) = 4 \text{Re} [\psi(1 - h) + \psi(h) - 2\psi(1)] = 8 \text{Re} \left[ \psi \left( \frac{1 + |n_h|}{2} + i\nu_h \right) - \psi(1) \right]. \tag{2.56}
$$

Minimizing this expression with respect to integer $n_h$ and real $\nu_h$ one finds that the ground state corresponds to $h = \bar{h} = 1/2$, or equivalently $n_h = \nu_h = 0$

$$
E_{N=2}^{(0)} = -16 \ln 2. \tag{2.57}
$$

For $s = 0$ and $\bar{s} = 1$ the expressions (2.55) and (2.56) are well known in QCD as defining the spectrum of the two-gluon color singlet compound states. The ground state energy (2.57) determines the intercept of the BFKL Pomeron [4].

2.4. Discrete symmetries

The Hamiltonian (2.42) is invariant under the cyclic and mirror permutations of the particles. The generators of these transformations, $\mathbb{P}$ and $\mathbb{M}$, respectively, are defined as follows

$$
[\mathbb{P} \Psi] (\bar{z}_1, ..., \bar{z}_{N-1}, \bar{z}_N) = \Psi (\bar{z}_2, ..., \bar{z}_N, \bar{z}_1), \quad [\mathbb{M} \Psi] (\bar{z}_1, ..., \bar{z}_{N-1}, \bar{z}_N) = \Psi (\bar{z}_N, \bar{z}_{N-1}, ..., \bar{z}_1) \tag{2.58}
$$

so that $[\mathcal{H}_N, \mathbb{P}] = [\mathcal{H}_N, \mathbb{M}] = 0$. Obviously, the operators $\mathbb{P}$ and $\mathbb{M}$ are identical at $N = 2$. As we will show below, this symmetry allows to establish some general properties of the spectrum of the model.

By the definition (2.58), the operators $\mathbb{P}$ and $\mathbb{M}$ do not commute and satisfy the following relations

$$
\mathbb{P}^N = \mathbb{M}^2 = 1, \quad \mathbb{P}^\dagger = \mathbb{P}^{-1} = \mathbb{P}^{N-1}, \quad \mathbb{M}^\dagger = \mathbb{M}, \quad \mathbb{P} \mathbb{M} = \mathbb{M} \mathbb{P} = \mathbb{M}^N \mathbb{P}^{N-1} = \mathbb{M}^N \mathbb{P}^{N-1} \tag{2.59}
$$

or equivalently $(\mathbb{M} \mathbb{P})^2 = 1$. As a consequence, the operators $\mathbb{P}$ and $\mathbb{M}$ can not be diagonalized simultaneously and, therefore, the eigenvalues of the Hamiltonian could possess, in general, definite quantum numbers only with respect to one of them.

Let us examine the action of the permutations on the integrals of motion, $q_k$ and $\bar{q}_k$, or equivalently on the auxiliary transfer matrices $t_N(u)$ and $\bar{t}_N(\bar{u})$. Using the definition (2.44) and (2.47), we find that the transfer matrices are invariant under the cyclic permutations, $\mathbb{P}^\dagger t_N(u) \mathbb{P} = t_N(u)$, while under the mirror permutation they are transformed as

$$
\mathbb{M} t_N(u) \mathbb{M} = \text{tr} (L_N(u) L_2(u) L_1(u)) = (-1)^N \text{tr} (L_1(-u) L_2(-u) L_N(-u)) = (-1)^N t_N(-u), \tag{2.60}
$$
where the second relation follows from the property of the transposed Lax operator, Eq. (2.13),
\[ L^T(u) = -\sigma_2 L(-u)\sigma_2. \]
Replacing the auxiliary transfer matrix by its expression (2.47), we find
\[
\mathbb{M} q_k = (-1)^k q_k \mathbb{M}, \quad \mathbb{P} q_k = q_k \mathbb{P}
\] (2.61)
and similar relations hold for the antiholomorphic charges. Since the Hamiltonian is invariant under the mirror permutations, it has to satisfy the following relation as a function of the conserved charges, Eq. (2.54),
\[
\mathcal{H}(q_k, \bar{q}_k) = \mathbb{M} \mathcal{H}(q_k, \bar{q}_k) = \mathcal{H}(\mathbb{M} q_k, \mathbb{M} \bar{q}_k) = \mathcal{H}((-1)^k q_k, (-1)^k \bar{q}_k).
\] (2.62)
This implies that, firstly, the eigenstates of the Hamiltonian corresponding to two different sets of the quantum numbers, \(\{q_k, \bar{q}_k\}\) and \(\{(-1)^k q_k, (-1)^k \bar{q}_k\}\), have the same energy
\[
E_N(q_k, \bar{q}_k) = E_N((-1)^k q_k, (-1)^k \bar{q}_k)
\] (2.63)
and, secondly, all energy levels of the Hamiltonian except those with \(q_{2k+1} = \bar{q}_{2k+1} = 0\) (**k = 1, 2, ...**) are (at least) double degenerate and the corresponding wave functions are related as
\[
\Psi_{\bar{q},((-1)^k q_k,(-1)^k \bar{q}_k)}(\vec{z}_1, \vec{z}_2, ..., \vec{z}_N) = \mathbb{M} \Psi_{\bar{q},(q_k,\bar{q}_k)}(\vec{z}_1, \vec{z}_2, ..., \vec{z}_N) = \Psi_{\bar{q},(q_k,\bar{q}_k)}(\vec{z}_N, ..., \vec{z}_2, \vec{z}_1).
\] (2.64)

One concludes from (2.61), that among two operators, \(\mathbb{P}\) and \(\mathbb{M}\), only the first one commutes simultaneously with the Hamiltonian and the conserved charges, and therefore, it is diagonalized by the eigenstates \(\Psi_{\bar{q},(q,\bar{q})}\). The corresponding eigenvalues define the quasimomentum \(\theta\) of the state
\[
\mathbb{P} \Psi_{\bar{q},(q,\bar{q})}(\vec{z}_1, ..., \vec{z}_N) = e^{i\theta(q,\bar{q})} \Psi_{\bar{q},(q,\bar{q})}(\vec{z}_1, ..., \vec{z}_N).
\] (2.65)
The complete integrability of the model implies that the quasimomentum is a function of the total set of the integrals of motion \(\{q, \bar{q}\}\). Since \(\mathbb{P}^N = \mathbb{1}\), its eigenvalues are quantized as
\[
\theta(q, \bar{q}) = 2\pi \frac{k}{N}, \quad \text{for } k = 0, 1, ..., N - 1.
\] (2.66)
Applying the operator of mirror permutations, \(\mathbb{M}\), to the both sides of (2.63) and taking into account (2.64) and (2.59), we find that the quasimomentum of the eigenstate \(\Psi_{\bar{q},((-1)^k q_k,(-1)^k \bar{q}_k)}\) is equal to \(-\theta(q, \bar{q})\). This leads to the following relation
\[
\theta((-1)^k q_k, (-1)^k \bar{q}_k) = -\theta(q_k, \bar{q}_k).
\] (2.67)
As a consequence, the eigenstate of the Hamiltonian with the quantum numbers \(q_{2k+1} = \bar{q}_{2k+1} = 0\) (**k = 1, 2, ...**) has a vanishing quasimomentum and, therefore, is symmetric under the cyclic permutations of \(N\) particles.

Using the solutions to (2.63), one can construct the eigenstates of the operator of mirror permutations \(\mathbb{M}\)
\[
\Psi^{(\pm)}_{\bar{q},(q,\bar{q})} = \frac{1 \pm \mathbb{M}}{2} \Psi_{\bar{q},(q,\bar{q})}; \quad \mathbb{M} \Psi^{(\pm)}_{\bar{q},(q,\bar{q})} = \pm \Psi^{(\pm)}_{\bar{q},(q,\bar{q})}.
\] (2.68)
Although these states do not diagonalize the integrals of motion, \(\{q, \bar{q}\}\), they are the eigenstates of the Hamiltonian with the same energy \(E_N(q, \bar{q})\). We find from (2.59) and (2.63) that the operator of cyclic permutations \(\mathbb{P}\) acts on them as
\[
\mathbb{P} \Psi^{(\pm)}_{\bar{q},(q,\bar{q})} = \frac{1}{2} (e^{i\theta} \pm e^{-i\theta} \mathbb{M}) \Psi_{\bar{q},(q,\bar{q})}.
\] (2.69)
Notice that at $\theta = 0$ and $\theta = \pi$ the states $\Psi^{(\pm)}_{\vec{p}(\pm)}$ diagonalize the operator $\mathbb{P}$ and, therefore, have a definite parity with respect to the cyclic and mirror permutations simultaneously.

Defining the eigenstates of the model, one has to choose between two (equivalent) sets of the states, $\Psi_{\vec{p}(\pm)}$ and $\Psi^{(\pm)}_{\vec{p}(\pm)}$. The former set is consistent with the integrability properties of the model, while the latter is more suitable for high-energy QCD as it reveals the Bose properties of the corresponding $N$-gluon states.

3. Baxter $Q$–operator

Solving the Schrödinger equation (2.8), we shall apply the powerful method of the $Q$–operator \cite{10}. This method plays an important role in the theory of integrable models as it provides an alternative to the conventional Algebraic Bethe Ansatz. It is based on the existence of the operator $Q$ that acts on the quantum space of the model and satisfies a finite-difference Baxter operator relations. In contrast with the ABA, the method of the $Q$–operator does not assume the existence of highest weight representation for the eigenstates and, as a consequence, it has a wider range of applicability. Both methods become equivalent if the eigenvalues of the $Q$–operator are restricted to be polynomials in a spectral parameter. In this case, the Baxter equations are reduced to the Bethe equations on the roots of these polynomials. It is not obvious, however, whether polynomial $Q$–operators furnish all relevant physical solutions to the Baxter equation, or equivalently the ABA method is complete. This turns out to be the case for the $SL(2,\mathbb{R})$ Heisenberg spin magnets \cite{21,14}, while for the noncompact, $SL(2,\mathbb{C})$ spin chain one has to go beyond the class of polynomial solutions.

In this Section we shall construct a general (nonpolynomial) $Q$–operator for the homogeneous $SL(2,\mathbb{C})$ spin chain that we shall denote as $Q(u, \bar{u})$. This operator acts on the quantum space of the model $V \otimes \ldots \otimes V$, with $V \equiv V^{(s,\bar{s})}$, and depends on two spectral parameters $u$ and $\bar{u}$. As we will see in a moment, for $Q(u, \bar{u})$ to be a well-defined operator, these parameters have to satisfy the same condition (2.27) as those for the $R$–matrix. Following \cite{10}, we require that the operator $Q(u, \bar{u})$ has to satisfy the relations

- **Commutativity:**
  \[ [Q(u, \bar{u}), Q(v, \bar{v})] = 0. \quad (3.1) \]

- **$Q$–$t$ relations:**
  \[ [t_N(u), Q(u, \bar{u})] = [\bar{t}_N(\bar{u}), Q(u, \bar{u})] = 0. \quad (3.2) \]

- **Baxter equations:**
  \[
  t_N(u) Q(u, \bar{u}) = (u + is)^N Q(u + i, \bar{u}) + (u - is)^N Q(u - i, \bar{u}), \quad (3.3)
  \]
  \[
  \bar{t}_N(\bar{u}) Q(u, \bar{u}) = (\bar{u} + i\bar{s})^N Q(u, \bar{u} + i) + (\bar{u} - i\bar{s})^N Q(u, \bar{u} - i), \quad (3.4)
  \]

where $t_N(u)$ and $\bar{t}_N(\bar{u})$ are the auxiliary transfer matrices defined in (2.47).

According to (3.1), the Baxter $Q$-operator and the auxiliary transfer matrices, $t_N(u)$ and $\bar{t}_N(\bar{u})$, share the common set of the eigenfunctions

\[
Q(u, \bar{u}) \Psi_{\vec{p},\{q,\bar{q}\}}(\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_N) = Q_{\{q,\bar{q}\}}(u, \bar{u}) \Psi_{\vec{p},\{q,\bar{q}\}}(\vec{z}_1, \vec{z}_2, \ldots, \vec{z}_N), \quad (3.5)
\]
which, at the same time, are the solutions to the Schrödinger equation (2.8) The eigenvalues of the \( Q \)-operator, \( Q(u, \bar{u}) \), satisfy the same Baxter equations, Eqs. (3.3) and (3.4), with the auxiliary transfer matrices (2.47) replaced by their corresponding eigenvalues.

Our approach to constructing the \( Q \)-operator for the \( SL(2, \mathbb{C}) \) spin chain is inspired by previous works [12, 14], in which the Baxter \( Q \)-operator was constructed for the periodic Toda chain and the homogenous \( SL(2, \mathbb{R}) \) Heisenberg spin magnet. Namely, we shall represent the \( Q \)-operator by its integral kernel

\[
[Q(u, \bar{u}) \Psi](\tilde{z}_1, \tilde{z}_2, ..., \tilde{z}_N) = \int d^2 w_1 ... \int d^2 w_N Q_{u, \bar{u}}(\tilde{z}_1, \tilde{z}_2, ..., \tilde{z}_N| \bar{w}_1, \bar{w}_2, ..., \bar{w}_N) \Psi(\bar{w}_1, \bar{w}_2, ..., \bar{w}_N)
\]

and find the explicit form of the kernel by solving Eqs. (3.1)–(3.4).

3.1. Baxter equations

According to the definition (2.47), the auxiliary transfer matrix \( t_N(u) \) is a polynomial of degree \( N \) in the \( SL(2, \mathbb{C}) \) spin operators \( S^{(s)}_a \). Using (2.47), \( t_N(u) \) can be expressed as the \( N \)-order differential operator acting on the holomorphic coordinates \( z_k \). Its substitution into the l.h.s. of (3.3) leads to a complicated holomorphic differential equation on the kernel \( Q_{u, \bar{u}} \). One finds similar antiholomorphic equation from (3.4).

Instead of trying to solve the resulting differential equations we follow the approach developed in [12, 14]. It allows to find the exact solution to (3.3) and (3.4) by using the invariance of the auxiliary transfer matrix \( t_N(u) = \text{tr} T_N(u) \) under local (gauge) rotations of the Lax operators

\[
L_k(u) \rightarrow \tilde{L}_k(u) = M_k^{-1} L_k(u) M_{k+1}, \quad T_N(u) \rightarrow \tilde{T}_N(u) = M_1^{-1} T_N(u) M_1,
\]

where \( M_k \) are arbitrary \( 2 \times 2 \) matrices, such that \( M_{N+1} = M_1 \) and \( \det M_k \neq 0 \). Let us choose the matrices \( M_k \) as

\[
M_k = \begin{pmatrix} 1 & 0 \\ y_k & 1 \end{pmatrix}, \quad M_k^{-1} = \begin{pmatrix} 1 & 0 \\ -y_k & 1 \end{pmatrix}
\]

with \( y_1, ..., y_N \) being arbitrary gauge parameters. The matrix elements of the rotated Lax operator (3.7) are given by

\[
[\tilde{L}_k]_{11} = u + is + i(z_k - y_{k+1})\partial_{z_k}, \quad [\tilde{L}_k]_{22} = u - is - i(z_k - y_k)\partial_{z_k},
\]

\[
[\tilde{L}_k]_{21} = i(z_k - y_k)(z_k - y_{k+1})\partial_{z_k} + (u + is)(z_k - y_k) + (-u + is)(z_k - y_{k+1}),
\]

while \( [\tilde{L}_k]_{12} \) remains unchanged.

Let us now define the following function

\[
Y^{(s, \bar{s})}_{u, \bar{u}}(\vec{z}, \vec{y}) = \prod_{k=1}^{N} [z_k - y_k]^{-s-iu} [z_k - y_{k+1}]^{-\bar{s}+iu},
\]

where \( [z - y]^{-s+iu} \equiv (z - y)^{-s+iu}(\bar{z} - \bar{y})^{-\bar{s}+iu} \), \( \bar{y}_{N+1} = \bar{y}_1 \) and \( \vec{z} \equiv (\vec{z}_1, ..., \vec{z}_N) \). It depends on \( N \) auxiliary vectors \( \vec{y} \equiv (\vec{y}_1, ..., \vec{y}_N) \), as well as the spins \( (s, \bar{s}) \) and the spectral parameters \( (u, \bar{u}) \).

The unique feature of the function \( Y^{(s, \bar{s})}_{u, \bar{u}}(\vec{z}, \vec{y}) \) is that it is annihilated by the off-diagonal matrix element of the gauge transformed Lax operator,

\[
[\tilde{L}_k]_{21} Y^{(s, \bar{s})}_{u, \bar{u}}(\vec{z}, \vec{y}) = 0
\]

(3.11)
for $k = 1, ..., N$. Therefore, applying $Y_{u,\bar{u}}^{(s,\bar{s})}(\bar{z},\bar{y})$ to the matrix elements of the Lax operator $\tilde{L}_k$, one finds that the latter takes the form of the upper triangular matrix. The product of such matrices can be easily calculated leading to the monodromy operator of the form

$$
\tilde{T}_N(u) Y_{u,\bar{u}}^{(s,\bar{s})}(\bar{z},\bar{y}) = \begin{pmatrix}
\prod_{k=1}^N [\tilde{L}_k(u)]_{11} & \prod_{k=1}^N [\tilde{L}_k(u)]_{22} \\
0 & 0
\end{pmatrix}
Y_{u,\bar{u}}^{(s,\bar{s})}
= \begin{pmatrix}
(u - is)^N Y_{u-i,\bar{u}}^{(s,\bar{s})} & 0 \\
0 & (u + is)^N Y_{u+i,\bar{u}}^{(s,\bar{s})}
\end{pmatrix},
$$

(3.12)

where in the last relation we used the explicit expression for the diagonal elements of the Lax operator (3.9). Taking the trace in the both sides of this relation, we obtain the following relation for the transfer matrix $t_N = \text{tr} \tilde{T}_N(u)$

$$
t_N(u; S^{(s)}(z)) Y_{u,\bar{u}}^{(s,\bar{s})}(\bar{z},\bar{y}) = (u - is)^N Y_{u-i,\bar{u}}^{(s,\bar{s})}(\bar{z},\bar{y}) + (u + is)^N Y_{u+i,\bar{u}}^{(s,\bar{s})}(\bar{z},\bar{y}).
$$

(3.13)

Here, we indicated explicitly that the transfer matrix $t_N(u)$ is expressed in terms of the differential operators $S^{(s)}(z)$ acting on the $z-$coordinates of the particles and defined in (2.2). Remarkably enough, the relation (3.13) takes the form of the Baxter equation (3.3) and, as we shall see later in this Section, it allows to construct the $Q-$operator.

Repeating similar analysis for the transfer matrix in the antiholomorphic sector, $\tilde{t}_N(\bar{u})$, one can show that the same function (3.11) satisfies the antiholomorphic Baxter relation

$$
\tilde{t}_N(\bar{u}; \bar{S}^{(s)}(z)) Y_{u,\bar{u}}^{(s,\bar{s})}(\bar{z},\bar{y}) = (\bar{u} - is)^N Y_{u-i,\bar{u}}^{(s,\bar{s})}(\bar{z},\bar{y}) + (\bar{u} + is)^N Y_{u+i,\bar{u}}^{(s,\bar{s})}(\bar{z},\bar{y}).
$$

(3.14)

Notice that in order for the function $Y_{u,\bar{u}}^{(s,\bar{s})}(\bar{z},\bar{y})$ to be well-defined on the plane, the spectral parameters $u$ and $\bar{u}$ have to satisfy the condition (2.21).

### 3.1.1. Properties of the $Y-$function

Before we proceed with constructing the Baxter $Q-$operator, let us discuss some properties of the function $Y_{u,\bar{u}}^{(s,\bar{s})}(\bar{z},\bar{y})$, which will be used below.

Since the monodromy matrix $\tilde{T}_N(u)$ is related to the operator $T_N(u)$ by the gauge transformation (3.7), its matrix elements can be expressed, using (2.44), in terms of the operators $A_N(u)$, ..., $D_N(u)$ and the gauge parameter $y_1$. Substituting the resulting expressions into (3.12), we find that the function $Y_{u,\bar{u}}^{(s,\bar{s})}(\bar{z},\bar{y})$ satisfies the following relations for arbitrary $y_k$

$$
[y_1^2 B_N(u) + y_1 (A_N(u) - D_N(u)) - C_N(u)] Y_{u,\bar{u}}^{(s,\bar{s})}(\bar{z},\bar{y}) = 0
$$

(3.15)

and

$$
[A_N(u) + y_1 B_N(u)] Y_{u,\bar{u}}^{(s,\bar{s})}(\bar{z},\bar{y}) = (u - is)^N Y_{u-i,\bar{u}}^{(s,\bar{s})}(\bar{z},\bar{y}),
$$

$$
[D_N(u) - y_1 B_N(u)] Y_{u,\bar{u}}^{(s,\bar{s})}(\bar{z},\bar{y}) = (u + is)^N Y_{u+i,\bar{u}}^{(s,\bar{s})}(\bar{z},\bar{y}).
$$

(3.16)

We recall that the operators $A_N(u)$, ..., $D_N(u)$ act on the holomorphic $z-$coordinates of $N-$particles and do not depend on the gauge parameters $y_k$. 

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Let us examine the relations (3.13) and (3.16) in the limit \( y_1 \to \infty \). At large \( y_1 \), we use the definition (3.11) and expand the \( Y \)-function as

\[
Y^{(s, \bar{s})}_{\bar{u}, u}(\vec{z}, \bar{y}) = [-y_1]^{2s} \left\{ \Lambda^{(s, \bar{s})}_{\bar{u}, u}(\vec{z}_1, ..., \vec{z}_N|\bar{y}_2, ..., \bar{y}_N) + \mathcal{O}(1/y_1, 1/\bar{y}_1) \right\},
\]

with the leading term given by

\[
\Lambda^{(s, \bar{s})}_{\bar{u}, u}(\vec{z}_1, ..., \vec{z}_N|\bar{y}_2, ..., \bar{y}_N) = [z_1 - y_2]^{-s +iu} \left( \prod_{k=2}^{N-1} [z_k - y_k]^{-s -iu} [z_k - y_{k+1}]^{-s +iu} \right) [z_N - y_N]^{-s -iu}.
\]

Then, one finds from (3.15) and (3.16) that the function \( \Lambda^{(s, \bar{s})}_{\bar{u}, u} \), defined in this way, satisfies the following relations

\[
B_N(u)\Lambda^{(s, \bar{s})}_{\bar{u}, u}(\vec{z}|\bar{y}) = 0, \quad A_N(u)\Lambda^{(s, \bar{s})}_{\bar{u}, u} = (u + is)^N \Lambda^{(s, \bar{s})}_{\bar{u}, u+i}, \quad D_N(u)\Lambda^{(s, \bar{s})}_{\bar{u}, u} = (u - is)^N \Lambda^{(s, \bar{s})}_{\bar{u}, u}.
\]

Obviously, the same relations hold in the antiholomorphic sector. As we will show in Sect. 4.2, the function \( \Lambda^{(s, \bar{s})}_{\bar{u}, u}(\vec{z}|\bar{y}) \) becomes a building block in the construction of the unitary transformation to the Separated Variables.

3.1.2. The kernel of the \( \mathbb{Q} \)-operator

Taking into account the properties of the function \( Y^{(s, \bar{s})}_{\bar{u}, u} \), Eqs. (3.13) and (3.14), we look for the kernel of the Baxter \( \mathbb{Q} \)-operator in the form

\[
Q_{\bar{u}, u}(\vec{z}|\bar{w}) = \int d^2y Y^{(s, \bar{s})}_{\bar{u}, u}(\vec{z}, \bar{y}) Z_R(\bar{y}, \bar{w}),
\]

where \( d^2y = d^2y_1...d^2y_N \) and \( Z_R \) is assumed to be a well-defined function on the plane, independent on the spectral parameters \( u \) and \( \bar{u} \). Substituting this ansatz into (3.16) one finds that, thanks to Eqs. (3.13) and (3.14), the Baxter equations (3.3) and (3.4) are satisfied for arbitrary function \( Z_R(\bar{y}, \bar{w}) \).

To fix the form of the function \( Z_R(\bar{y}, \bar{w}) \) in (3.20) we require that the \( \mathbb{Q} \)-operator has to commute with the auxiliary transfer matrices, Eq. (3.2). It is convenient to represent the same condition in the form of the Baxter equations, Eqs. (3.3) and (3.4), with the transfer matrices and the \( \mathbb{Q} \)-operator interchanged in the l.h.s. of these equations. Applying the operator \( \mathbb{Q}(u, \bar{u}) t_N(u) \) to an arbitrary test function and replacing the \( \mathbb{Q} \)-operator by its integral representation, one integrates by parts to arrive at the relation analogous to (2.18)

\[
[\mathbb{Q}(u, \bar{u}) t_N(u) \Psi](\vec{z}) = \int d^2w \left( t_N(u; -S^{(1-s)}(w))Q_{\bar{u}, u}(\vec{z}|\bar{w}) \right) \Psi(\bar{w}),
\]

where \( d^2w \equiv d^2w_1...d^2w_N \) and the auxiliary transfer matrix in the r.h.s. is obtained from (2.47) and (2.44) by replacing the holomorphic \( SL(2, \mathbb{C}) \) generators in the expression for the Lax operators as \( S^{(s)}_\alpha \to -S^{(1-s)}_\alpha \). In this way, the condition (3.2) on the \( \mathbb{Q} \)-operator can be formulated as

\[
(-1)^N t_N(-u; S^{(1-s)}(w)) Q_{\bar{u}, u}(\vec{z}|\bar{w}) = (u + is)^N Q_{u+i, \bar{u}}(\vec{z}|\bar{w}) + (u - is)^N Q_{u-i, \bar{u}}(\vec{z}|\bar{w}),
\]

(3.22)
where we took into account that the auxiliary transfer matrix is a polynomial of degree \( N \) in the spectral parameter with the coefficient in front of \( u^k \) proportional to the \((N - k)\)th power of the spin operators. Similar relation holds for the antiholomorphic auxiliary transfer matrix \( \tilde{t}_N(-\bar{u}; \tilde{S}^{(1-s)}(\bar{w})) \).

Let us look for the solution to (3.22) in the form

\[
Q_{u,\bar{u}}(\bar{z} | \bar{w}) = [a(s + iu, \bar{s} - i\bar{u})]^{N} \tilde{Q}_{-u,-\bar{u}}(\bar{w} | \bar{z})
\]

(3.23)

with the functions \( a(x) \) defined in (A.3). Substituting this ansatz into (3.22) and changing the spectral parameter as \( u \to -u \), one finds that the function \( \tilde{Q}_{u,\bar{u}} \) satisfies the Baxter equation

\[
t_N(u, S^{(1-s)}(w)) \tilde{Q}_{u,\bar{u}}(\bar{w} | \bar{z}) = (u+i(1-s))^N \tilde{Q}_{u+i,\bar{u}}(\bar{w} | \bar{z}) + (u-i(1-s))^N \tilde{Q}_{u-i,\bar{u}}(\bar{w} | \bar{z}),
\]

(3.24)

which coincides with (3.13) once one changes the spin as \( s \to 1 - s \). Therefore, the general solution (3.23) to the Baxter equation (3.22) and its antiholomorphic counterpart can be written as

\[
Q_{u,\bar{u}}(\bar{z} | \bar{w}) = [a(s + iu, \bar{s} - i\bar{u})]^{N} \int d^2y Z_L(\bar{y}, \bar{z}) Y_{-u,-\bar{u}}^{(1-s,1-s)}(\bar{w}, \bar{y}),
\]

(3.25)

with \( Z_L(\bar{y}, \bar{z}) \) being a well-defined function on the plane, independent on the spectral parameters \( u \) and \( \bar{u} \).

Comparing two different representations for the kernel of the \( Q \)-operator, Eqs. (3.20) and (3.25), we notice that (3.20) fixes the dependence of \( Q_{u,\bar{u}}(\bar{z} | \bar{w}) \) on the \( z \)-coordinates, whereas (3.25) fixes its \( w \)-dependence. We find that (3.20) and (3.25) become compatible provided that \( Z_R \) is given by the following expressions

\[
Z_R^+(\bar{y}, \bar{w}) = \prod_{k=1}^{N}[w_{k-1} - y_k]^{2s-2}, \quad Z_R^- (\bar{y}, \bar{w}) = \prod_{k=1}^{N} \delta^2(w_k - y_k)[w_k - w_{k+1}]^{2s-1},
\]

(3.26)

with \( w_0 = w_N \) and \( w_{N+1} = w_1 \). Substitution of (3.26) into (3.20) yields two different expressions for the kernel of the \( Q \)-operator

\[
Q_{u,\bar{u}}^{(+)\bar{+}}(\bar{z} | \bar{w}) = \int d^2y \prod_{k=1}^{N}[z_k - y_k]^{-s-iu} [z_{k-1} - y_k]^{-s+iu} [w_{k-1} - y_k]^{2s-2}
\]

(3.27)

\[
= [a(2 - 2s, s + iu, \bar{s} - i\bar{u})]^{N} \prod_{k=1}^{N}[w_k - z_k]^{s-1+iu} [w_k - z_{k+1}]^{s-1-iu} [z_k - z_{k+1}]^{1-2s},
\]

and

\[
Q_{u,\bar{u}}^{(-)\bar{-}}(\bar{z} | \bar{w}) = \prod_{k=1}^{N}[z_k - w_k]^{-s-iu} [z_k - w_{k+1}]^{-s+iu} [w_k - w_{k+1}]^{2s-1}
\]

(3.28)

\[
= [a(1 - 2s, s + iu, \bar{s} - i\bar{u})]^{N} \int d^2y \prod_{k=1}^{N}[w_k - y_k]^{s-1+iu} [w_{k-1} - y_k]^{s-1-iu} [z_{k-1} - y_k]^{-2s}.
\]

Here, the second relation in both equations is obtained from the star-triangle identity (A.8). The diagrammatical representation of the kernels \( Q_{u,\bar{u}}^{(+)\bar{+}} \) and \( Q_{u,\bar{u}}^{(-)\bar{-}} \) is shown in Figs. 3 and 4, respectively.
Using this representation, one checks that the expressions (3.27) and (3.28) match into Eqs. (3.20) and (3.25) simultaneously.

The obtained expressions for the kernels, Eqs. (3.27) and (3.28), define two different operators, \( Q^+(u, \bar{u}) \) and \( Q^-(u, \bar{u}) \). By the construction, they commute with the auxiliary transfer matrices, Eq. (3.2), and satisfy the Baxter equation (3.3) and (3.4). For the kernels of these operators, Eqs. (3.27) and (3.28), to be well-defined on the plane one has to require that, similar to the \( R \)-matrix, the spectral parameters \( u \) and \( \bar{u} \) have to satisfy the condition (2.21). Finally, in order to identify \( Q^+(u, \bar{u}) \) and \( Q^-(u, \bar{u}) \) as the Baxter \( Q \)-operators we have to show that they fulfil the commutativity condition (3.1).

### 3.2. Commutativity condition

Let us show that the operators \( Q^+(u, \bar{u}) \) and \( Q^-(u, \bar{u}) \) with the kernels defined in (3.27) and (3.28) commute with each other for different values of the spectral parameters

\[
[Q^+(u, \bar{u}), Q^+(v, \bar{v})] = [Q^-(u, \bar{u}), Q^-(v, \bar{v})] = [Q^+(u, \bar{u}), Q^-(v, \bar{v})] = 0
\]

and satisfy the following exchange relations

\[
Q^+(u, \bar{u}) Q^-(v, \bar{v}) = Q^+(v, \bar{v}) Q^-(u, \bar{u}),
\]

\[
Q^-(u, \bar{u}) Q^+(v, \bar{v}) = Q^-(v, \bar{v}) Q^+(u, \bar{u}).
\]

We start with the product of two operators

\[
[Q^-(v, \bar{v}) Q^+(u, \bar{u})](\vec{z}| \vec{w}) = \int d^2y Q^{(v)}_{y, \bar{v}} (\vec{z}| \vec{y}) Q^{(u)}_{\bar{u}, \bar{w}} (\vec{y}| \vec{w})
\]

and substitute them by the corresponding Feynman diagrams. It becomes convenient to use the left diagram in Fig. 7 for the operator \( Q^-(v, \bar{v}) \) and the right diagram in Fig. 6 for the operator...
Figure 8: Diagrammatic representation of the kernel $X_{\bar{u}, u; v, \bar{v}}(\vec{z}, \vec{w})$ defined in Eq. (3.32) as a periodic chain of rhombuses. The rightmost vertex is identified with the leftmost one. To prove the commutativity relation (3.35), one inserts two lines with opposite indices into the left rhombus between the points $\vec{z}_1$ and $\vec{w}_1$.

$Q_+(u, \bar{u})$. Gluing together two sets of the triangles we notice that their common lines with the indices $(1-2s)$ and $(2s-1)$ annihilate each other and the resulting Feynman diagram takes the form shown in Fig. 8. The corresponding Feynman integral can be written in a simple form by introducing notation for the following function

$$X_{v, \bar{v}; u, \bar{u}}(\vec{z} | \vec{w}) = \left[a(s + iu, \bar{s} - i\bar{u})\right]^N \int d^2 y Y_{v, \bar{v}}^{(s, \bar{s})} (\vec{y}, \vec{z}) Y_{-u, -\bar{u}}^{(1-s, 1-\bar{s})} (\vec{y}, \vec{w}) ,$$

with the $Y-$functions defined in (3.10). In this way, one arrives at

$$[Q_+(u, \bar{u}) Q_-(v, \bar{v})](\vec{z} | \vec{w}) = \left[\pi a(2 - 2s)\right]^N X_{v, \bar{v}; u, \bar{u}}(\vec{z} | \vec{w}) .$$

(3.33)

Following the same steps, we now calculate the product of the same operators but in an opposite order, $[Q_+(u, \bar{u}) Q_-(v, \bar{v})](\vec{z} | \vec{w})$. This time we replace two operators by the left diagram in Fig. 6 and the right diagram in Fig. 7, respectively, and obtain two sets of the star-diagrams glued together through common vertices. Integration over the position of these vertices can be easily performed using (A.7) and it gives rise to the $\delta-$function connecting the centers of the star-diagrams. As a result, one arrives at the diagram, which is similar to the one shown in Fig. 8, leading to

$$[Q_+(u, \bar{u}) Q_-(v, \bar{v})](\vec{z} | \vec{w}) = \left[\pi a(2 - 2s)\right]^N X_{u, \bar{u}; v, \bar{v}}(\vec{z} | \vec{w}) .$$

(3.34)

Comparing (3.33) and (3.34) we notice that their r.h.s. differ from each other by interchanging the spectral parameters. Then, the commutativity of the $Q_+(u, \bar{u})$ and $Q_-(v, \bar{v})$ as well as the relations (3.30) follow from the following symmetry property of the $X-$function

$$X_{u, \bar{u}; v, \bar{v}}(\vec{z} | \vec{w}) = X_{v, \bar{v}; u, \bar{u}}(\vec{z} | \vec{w}) .$$

(3.35)

The proof of (3.35) is based on the uniqueness relations, Eq. (A.8), and it can be carried out using the diagrammatical representation of the function $X_{u, \bar{u}; v, \bar{v}}$. It also relies on the permutation identity shown in Fig. 9. To verify this identity it is sufficient to turn the “unique” triangles in the both sides of the relation into unique stars and check that the resulting diagrams coincide. Turning to (3.33), we use the diagrammatical representation of the function $X_{u, \bar{u}; v, \bar{v}}$ and insert
two additional propagators with the indices \( i(u-v) \) and \(-i(v-u)\), respectively, into one of the rhombuses as shown in Fig. 8. Since the sum of the indices vanishes, this transformation does not change the function. Subsequently applying the permutation identity (see Fig. 9), we move the line with the index \( i(u-v) \) to the right of the diagram until it returns to its initial position and annihilates the line with an opposite index, thanks to periodic boundary conditions along the chain of rhombuses in Fig. 8. In this way, one arrives at the initial diagram, in which the spectral parameters \( u \) and \( v \) are interchanged, thus proving the relation (3.35).

Let us now turn to the first two relations in (3.29) and examine the product

\[
[Q_+(u, \bar{u}) Q_+(v, \bar{v})](\vec{z} | \vec{w}) = [\pi a(2 - 2s)]^N \int d^2 y X_{v, \bar{v}; u, \bar{u}}(\bar{y} | \vec{z}) \prod_{k=1}^{N} [z_k - z_{k+1}]^{1-2s}[w_k - y_{k+1}]^{2(1-s)}.
\]

(3.36)

Due to the symmetry property (3.35), the r.h.s. of this relation is symmetric under permutation of the spectral parameters and, as a consequence, the operators \( Q_+(u, \bar{u}) \) and \( Q_+(v, \bar{v}) \) commute. The proof of the commutativity of the operators \( Q_-(u, \bar{u}) \) and \( Q_-(v, \bar{v}) \) goes along the same lines.

Finally, \( Q_+(u, \bar{u}) \) and \( Q_-(u, \bar{u}) \) are the \( SL(2,\mathbb{C}) \) invariant operators

\[
[ Q_+(u, \bar{u}), S_\alpha ] = [ Q_-(u, \bar{u}), S_\alpha ] = 0
\]

(3.37)

with \( S_\alpha = \sum_{k=1}^{N} S_\alpha^{(c)} \) being the total \( SL(2,\mathbb{C}) \) spin of the system. The same property implies that the kernels of the \( Q_- \)–operators have to transform under the \( SL(2,\mathbb{C}) \) transformations (2.4) as

\[
Q_{u, \bar{u}}^{(\pm)}(\vec{z}' | \vec{w}') = \left( \prod_{k=1}^{N} [cw_k + d]^{2-2s}[cz_k + d]^{2s} \right) Q_{u, \bar{u}}^{(\pm)}(\vec{z} | \vec{w}) .
\]

(3.38)

This relation can be verified using the explicit form of the kernels, Eqs. (3.27) and (3.28).

### 3.3. Properties of the \( Q_- \)–operator

In the previous Section we have constructed two different Baxter \( Q_- \)–operators, \( Q_+(u, \bar{u}) \) and \( Q_-(u, \bar{u}) \). Each of them satisfies the defining relations, Eqs. (3.1) – (3.4), as well as the additional relations (3.30) and (3.37). Let us show that these two operators are conjugated to each other with respect to the scalar product (2.3).
3.3.1. Conjugated $Q$–operator

The kernel of the operator $Q^\dagger(u, \bar{u})$ conjugated to the Baxter $Q$–operator is defined as follows

$$\left[\left[Q(u, \bar{u})\right]^\dagger\right] \Psi(\bar{z}_1, ..., \bar{z}_N) = \int d^2 w \left(Q_{u,\bar{u}}(\bar{w}_1, ..., \bar{w}_N | \bar{z}_1, ..., \bar{z}_N)\right)^* \Psi(\bar{w}_1, ..., \bar{w}_N).$$ (3.39)

Calculating $\left[Q_+(u, \bar{u})\right]^\dagger$, one substitutes its kernel, Eq. (3.27), into the r.h.s. of (3.39) and notices, using $(2.12)$, that $(Q_{u,\bar{u}}^{(s)}(\bar{w} | \bar{z}))^*$ turns out to be proportional to the kernel $Q_{u,\bar{u}^*}(\bar{z} | \bar{w})$ defined in (3.28). This leads to the following relation between the corresponding $Q$–operators

$$\left[Q_+(u, \bar{u})\right]^\dagger = (a(1 - 2s, s + i\bar{u}^*, s - iu^*)/\pi)^{-N} Q_-(\bar{u}^*, u^*).$$ (3.40)

The inverse relation looks like

$$\left[Q_-(u, \bar{u})\right]^\dagger = [(a(2 - 2s, s + i\bar{u}^*, s - iu^*)/\pi)^{-N} \bar{Q}_+(\bar{u}^*, u^*).$$ (3.41)

Since the operators $Q_{\pm}^\dagger(u, \bar{u})$ and $\bar{Q}_+(\bar{u}^*, u^*)$ differ from each other by c-valued coefficient function, they commute with each other for different values of the spectral parameters due to (3.29)

$$\left[\left[Q_{\pm}(u, \bar{u})\right]^\dagger, \bar{Q}_+(v, \bar{v})\right] = \left[\left[Q_{\pm}(u, \bar{u})\right]^\dagger, Q_{\pm}(v, \bar{v})\right] = 0.$$ (3.42)

Additional properties of the operators $Q_{\pm}^\dagger(u, \bar{u})$ follow from conjugation of the corresponding relations for the operators $Q_{\pm}(u, \bar{u})$. In particular, the conjugated Baxter equations (3.3) and (3.4) look as

$$t_N(u) \left[Q_{\pm}(\bar{u}^*, u^*)\right]^\dagger = (u - i(1 - s))^N \left[Q_{\pm}(\bar{u}^* - i, u^*)\right]^\dagger + (u + i(1 - s))^N \left[Q_{\pm}(\bar{u}^* + i, u^*)\right]^\dagger,$$

$$\bar{t}_N(\bar{u}) \left[Q_{\pm}(\bar{u}^*, u^*)\right]^\dagger = (\bar{u} - i(1 - \bar{s}))^N \left[Q_{\pm}(\bar{u}^*, u^* - i)\right]^\dagger + (\bar{u} + i(1 - \bar{s}))^N \left[Q_{\pm}(\bar{u}^*, u^* + i)\right]^\dagger.$$ (3.43)

Here, the spectral parameters $u$ and $\bar{u}$ are arbitrary complex numbers satisfying the condition (2.21).

The relations (3.40) – (3.43) can be further simplified if one imposes the additional condition on the spectral parameters, $\bar{u} = u^*$, leading to (2.33). As we will show in Sect. 4.4, it is for these values of the spectral parameters that the Baxter operator $Q(u, u^*)$ enters into the expression for the eigenstates of the Hamiltonian in the representation of the Separated Variables. Notice, however, that in the r.h.s. of the Baxter equations (3.43) the holomorphic and antiholomorphic arguments of the $Q$–operators are shifted by $\pm i$ independently and, therefore, one can not obtain from (3.43) a closed system of equations on the operator $Q(u, u^*)$.

3.3.2. Quasimomentum

The Baxter $Q$–operator commutes with the transfer matrices of the model and, therefore, it could serve as a generating function of the Hamiltonian and the integrals of motion. It is well-known from the theory of the compact spin magnets that the main observables (like Hamiltonian, quasimomentum, etc) are determined by the behaviour of the Baxter $Q$–function at the special values of the spectral parameter $u = \pm is$. As we will show below, the same formulae hold for the noncompact spin magnet.
Let us examine the operator \( Q_+(u, \bar{u}) \) with the spectral parameters given by \( u = \pm is \) and \( \bar{u} = \pm \bar{i}s \). Using the diagrammatical representation of this operator, Fig. 3, we find that the lines with the indices \( s \pm iu \) disappear and the integrals over the centers of the star diagrams take the form of (3.47) and give rise to the \( \delta \)-function

\[
Q^{(+)}_{is,i\bar{s}}(z|\bar{w}) = \rho_s^{-N} \prod_{k=1}^{N} \delta^{(2)}(z_k - w_k), \quad Q^{(+)}_{-is,-i\bar{s}}(\bar{z}|w) = \rho_s^{-N} \prod_{k=1}^{N} \delta^{(2)}(z_{k+1} - w_k), \tag{3.44}
\]

where \( \rho_s = 1/(\pi^2 a(2\bar{s}, 2 - 2s)) = (n_s^2 + 4\nu_s^2)/\pi^2 \). Going over to the operators one gets

\[
Q_+(is, is) = \rho_s^{-N} \times 1, \quad Q_+(-is, -i\bar{s}) = \rho_s^{-N} \times \mathbb{P}, \tag{3.45}
\]

where \( \mathbb{P} \) is the operator of the cyclic permutations defined in (2.58). Putting \( u = \pm is \) and \( \bar{u} = \pm \bar{i}s \) in (3.40) and taking into account (3.45), one finds that the \( a \)-factor in the r.h.s. of (3.40) vanishes and, as a consequence, the operator \( Q_-(\pm is^*, \pm is^*) \) is divergent. A careful analysis shows that

\[
Q_-(is^* + \epsilon, -is^* + \epsilon) = 1 \times \frac{(i\pi)^N}{\epsilon^N} \left[ 1 + \mathcal{O}(\epsilon) \right],
Q_-(is^* - \epsilon, is^* - \epsilon) = \mathbb{P}^{-1} \times \frac{(i\pi)^N}{\epsilon^N} \left[ 1 + \mathcal{O}(\epsilon) \right] \tag{3.46}
\]

as \( \epsilon \to 0 \), so that the \( Q_- \)-operator exhibits the \( N \)-th order pole. The detailed analysis of the analytical properties of the operators \( Q_+ \) will be performed in Sect. 3.5.

Using (3.45) and (3.46), one obtains the following relations for the operator of quasimomentum \( \theta(q, \bar{q}) \) defined in Eq. (2.63)

\[
\theta = -i \ln \mathbb{P} = i \ln \frac{Q_+(is, is)}{Q_+(-is, -i\bar{s})} = i \ln \frac{Q_-(is^* + \epsilon, -is^* + \epsilon)}{Q_-(is^* - \epsilon, is^* - \epsilon)} \bigg|_{\epsilon \to 0}. \tag{3.47}
\]

We recall that the eigenvalues of this operator have the general form (2.60) and satisfy the relation (2.67).

### 3.3.3. Properties of the eigenvalues

As we have seen in Sect. 2.4, the Hamiltonian of the model is invariant under the cyclic and mirror permutations generated by the operators \( \mathbb{P} \) and \( \mathbb{M} \), respectively. Let us now examine the action of these operators on the \( Q \)-operators. Since the \( Q \)-operators commute with each other for different values of the spectral parameters, it follows from (3.45) that

\[
[Q_\pm(u, \bar{u}), \mathbb{P}] = 0. \tag{3.48}
\]

The same property can be easily understood from Figs. 3 and 7 - the diagrams representing the \( Q \)-operators remain unchanged if one performs simultaneously the cyclic permutations of the \( z \)- and \( w \)-vertices.

The transformation properties of the \( Q \)-operators under mirror permutations become more complicated and can be expressed as

\[
Q_+(u, \bar{u}) = \mathbb{M} \mathbb{P}^{-1} Q_+(u, \bar{u}) \mathbb{M}, \quad Q_-(u, \bar{u}) = \mathbb{M} \mathbb{P} Q_-(u, \bar{u}) \mathbb{M}. \tag{3.49}
\]
To see this, one applies the operators $\mathbb{M}$ and $\mathbb{P}$ to the diagrams shown in Figs. 3 and 6 and finds that the diagrams describing the r.h.s. of the relations (3.49) look like mirror images of those shown in Figs. 3 and 6. To bring them back to the original form, one flips the whole diagram along the vertical axis and changes simultaneously the sign of the spectral parameters as indicated in the l.h.s. of (3.49). One can verify (3.49) by putting $u = -is$ and $u = i\bar{s}^*$ in the first and the second relations, respectively, and substituting the $\mathbb{Q}$–operators by their expressions (3.43) and (3.46).

Applying the both sides of (3.49) to the eigenstate $\Psi_{\bar{p},(q,\bar{q})}$ one finds the relation between the corresponding eigenvalues of the $\mathbb{Q}$–operators defined in (3.3)

$$
\mathbb{Q}_+(u, \bar{u}) = \mathbb{M}^{-1} \mathbb{P}^{-1} \mathbb{Q}_+(u, \bar{u}) \Psi_{\bar{p},(q,\bar{q})} = e^{-i\phi(-q,-\bar{q})} \mathbb{Q}_{\bar{q},-q}^{(+)\dagger}(u, \bar{u}) \Psi_{\bar{p},(q,\bar{q})},
$$

(3.50)

where $(-q, -\bar{q}) \equiv ((-1)^k q_k, (-1)^k \bar{q}_k)$. Here, in the second relation we used Eq. (2.64) and in third one Eqs. (2.65) and (2.67). Then, taking into account (2.65) we find

$$
\mathbb{Q}_{\bar{q},q}^{(+)}(-u, -\bar{u}) = e^{i\phi(q,\bar{q})} \mathbb{Q}_{\bar{q},q}^{(-)}(u, \bar{u}).
$$

(3.51)

In a similar manner, we find from the second relation in (3.49)

$$
\mathbb{Q}_{\bar{q},q}^{(-)}(-u, -\bar{u}) = e^{i\phi(q,\bar{q})} \mathbb{Q}_{\bar{q},q}^{(-)}(u, \bar{u}).
$$

(3.52)

Let us now consider the ratio of the operators

$$
\frac{\mathbb{Q}_+^{(u, \bar{u})}}{\mathbb{Q}_-^{(u, \bar{u})}} \equiv \rho_s^{-N/2} \times \mathbb{W}
$$

(3.53)

with $\rho_s = (n_s^2 + 4\nu_s^2)/\pi^2$. According to (3.30), the operator $\mathbb{W}$ defined in this way does not depends on the spectral parameters. The normalization factor is chosen in (3.53) in such a way that the operator $\mathbb{W}$ is unitary

$$
\mathbb{W}^\dagger \mathbb{W} = \rho_s^N \left[ \frac{\mathbb{Q}_+^{(u^*, u^*)}}{\mathbb{Q}_-^{(u^*, u^*)}} \right] \frac{\mathbb{Q}_+^{(u, \bar{u})}}{\mathbb{Q}_-^{(u, \bar{u})}} = \mathbb{I}, \quad \mathbb{W} \Psi_{\bar{p},(q,\bar{q})} = e^{i\omega_{q,\bar{q}}} \Psi_{\bar{p},(q,\bar{q})}.
$$

(3.54)

Here, we used the fact that the operator $\mathbb{W}$ does not depend on the spectral parameters and applied the identity (3.40). Since the eigenstates $\Psi_{\bar{p},(q,\bar{q})}(\bar{z})$ diagonalize the operators $\mathbb{Q}_\pm$, they also diagonalize $\mathbb{W}$. Then, we obtain from (3.53) and (3.40)

$$
\frac{\mathbb{Q}_{\bar{q},q}^{(+)}(u, \bar{u})}{\mathbb{Q}_{\bar{q},q}^{(-)}(u^*, u^*)} = e^{i\omega_{q,\bar{q}}} [a(s + iu, \bar{s} - i\bar{u}) e^{i\varphi_s}]^N,
$$

(3.55)

with $e^{i\varphi_s} = [a(1 - 2s, 2 - 2s)]^{1/2}$, or $\varphi_s = \arg[\Gamma(1 - n_s + 2i\nu_s)\Gamma(-n_s + 2i\nu_s)]$.

We shall use the relation (3.53) below to fix the phase of $\mathbb{Q}_{\bar{q},q}^{(+)}(u, u^*)$ at $\bar{u} = u^*$ and to calculate the residues of $\mathbb{Q}_{\bar{q},q}^{(+)}(u, \bar{u})$ at its poles. For instance, putting $u = \pm (i\bar{s}^* - \epsilon)$ and $\bar{u} = \pm (is^* - \epsilon)$ in (3.53) and applying (3.43), we calculate the residue of $\mathbb{Q}_{\bar{q},q}^{(+)}$ at the $N$–th order pole in $\epsilon$ as

$$
\mathbb{Q}_{\bar{q},q}^{(+)}(-i\bar{s}^* + \epsilon, -is^* + \epsilon) = e^{i\omega_{q,\bar{q}}} C_s^N \frac{\epsilon}{\epsilon^N} [1 + O(\epsilon)],
$$

$$
\mathbb{Q}_{\bar{q},q}^{(+)}(is^* - \epsilon, is^* - \epsilon) = e^{-i\omega_{q,\bar{q}}} C_s^N \frac{\epsilon}{\epsilon^N} [1 + O(\epsilon)]
$$

(3.56)

with $C_s = i\pi \rho_s^{-1/2} = i[(n_s^2 + 4\nu_s^2)/\pi^2]^{-1/2}$ and $s = 1 - s^*$. 

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3.3.4. Reduction formulae

The operator $Q_+(u, \bar{u})$ acts on the quantum space of the system $V_1 \otimes ... \otimes V_N$, with $V_k \equiv V^{(s, s')}$, and depends explicitly on the number of particles $N$. There exist the relations connecting the Baxter operators defined for the different number of particles.

To obtain the reduction relations, one examines the kernel of the operator, $Q^{(+)q}(\bar{z}|\bar{w})$, in the limit when two particles with coordinates, say, $\bar{z}_N$ and $\bar{z}_1$ approach each other on the plane, $|\bar{z}_N - \bar{z}_1| \to 0$. Using the diagrammatical representation of the kernel – the left diagram in Fig. 4, one finds that in this limit two lines connecting the center of the left unique star diagram with the points $\bar{z}_N$ and $\bar{z}_1$ merge and produce a single line with the index $2s$. Then, the resulting integral over the center of the star takes the form \((3.59)\) and gives rise to $\delta(\bar{z}_N - \bar{w}_N)$

\[
Q^{(+)q}(\bar{z}_1, ..., \bar{z}_{N-1}, \bar{z}_N|\bar{w}_1, ..., \bar{w}_{N-1}, \bar{w}_N) \bigg|_{\bar{z}_N=\bar{z}_1} = \pi^2 a(2s, 2(1-s)) \delta(\bar{z}_N - \bar{w}_N) Q^{(+)q}(\bar{z}_1, ..., \bar{z}_{N-1}|\bar{w}_1, ..., \bar{w}_{N-1}) .
\]

Here, the $Q_+-$operator in the r.h.s. acts on the quantum space of $N-1$ particles, $V_1 \otimes ... \otimes V_{N-1}$. One can continue the reduction procedure by putting any pair of the nearest neighbors atop of each other, $\bar{z}_k = \bar{z}_{k+1}$. Each time the number of particles is reduced by 1 until one reaches the limit $\bar{z}_1 = \bar{z}_2 = ... = \bar{z}_N$ when the $N-$particle Baxter $Q-$operator is degenerated into the identity operator. Similar reduction formula holds for the kernel of the $Q_-$operator, Eq. \((3.28)\), but in this case one merges the right arguments of the kernel

\[
Q^{(-)q}(\bar{z}_1, ..., \bar{z}_{N-1}, \bar{z}_N|\bar{w}_1, ..., \bar{w}_{N-1}, \bar{w}_N) \bigg|_{\bar{w}_N=\bar{w}_1} = \pi a(2(1-s), s + iu, s - i\bar{u}) \delta(\bar{z}_N - \bar{w}_N) Q^{(-)q}(\bar{z}_1, ..., \bar{z}_{N-1}|\bar{w}_1, ..., \bar{w}_{N-1}) .
\]

Here, the additional factor comes from the $B-$factor in the r.h.s. of Fig. 4.

Let us now put $\bar{z}_1 = \bar{z}_N$ in the both sides of the eigenproblem \((3.3)\) and apply the relation \((3.57)\)

\[
Q^{(N)}_{q,q}(u, \bar{u}) \Psi_p^{(q,q)}(\bar{z}_1, ..., \bar{z}_{N-1}, \bar{z}_1) = \frac{\pi^2}{n_r^2 + 4v_s^2} \times \int d^2 w \ Q^{(+)q}(\bar{z}_1, ..., \bar{z}_{N-1}|\bar{w}_1, ..., \bar{w}_{N-1}) \Psi_p^{(q,q)}(\bar{w}_1, ..., \bar{w}_{N-1}, \bar{z}_1) , \tag{3.59}
\]

where the superscript $(N)$ in the l.h.s. indicates that $Q^{(N)}_{q,q}(u, \bar{u})$ is the eigenvalue of the $N-$particle Baxter operator. Using the property of completeness of the solutions to the eigenproblem \((3.3)\), we may expand $\Psi_p^{(q,q)}(\bar{w}_1, ..., \bar{w}_{N-1}, \bar{z}_1)$ over the eigenstates of the $(N-1)-$particle Baxter operator, $\Psi_{p_{(q,q')}}^{(N-1)}(\bar{w}_1, ..., \bar{w}_{N-1})$ with the expansion coefficients depending on $\bar{z}_1$. After its substitution into \((3.59)\), the integral in the r.h.s. is replaced by the sum over the eigenvalues $Q^{(N-1)}_{q,q'}$. Then, comparing the $\bar{z}-dependence$ of the both sides of the resulting relation one arrives at overcompleted system of equations on the expansion coefficients. Its consistency requirements provide the quantization conditions on the integrals of motion $q_k$ and $\bar{q}_k$ \([24]\). Solving the system one finds the expansion coefficients and, as a byproduct, reconstructs the expansion of the eigenfunctions and the eigenvalues of the $N-$particle Baxter equation, $Q^{(N)}_{q,q}(u, \bar{u})$, over those corresponding to the system of $(N-1)-$particles \([24]\).
3.3.5. Eigenvalues at \( N = 2 \)

The eigenproblem for the Baxter operator \( Q_+(u, \bar{u}) \) can be solved exactly for the system of \( N = 2 \) particles. In this case, the eigenfunctions of \( Q_+(u, \bar{u}) \) are given by (2.55). They are parameterized by the pair of the \( SL(2, \mathbb{C}) \) spins \( (h, \bar{h}) \) and by the total momentum \( p \), or equivalently the center-of-mass coordinate \( \bar{z}_0 \). Their substitution into (3.5) yields

\[
\int d^2w_1 d^2w_2 Q_{u,\bar{u}}(\bar{z}_1, \bar{z}_2|\bar{w}_1, \bar{w}_2) \Psi_{h,\bar{h}}(\bar{w}_1 - \bar{z}_0, \bar{w}_2 - \bar{z}_0) = Q_{h,\bar{h}}(u, \bar{u}) \Psi_{h,\bar{h}}(\bar{z}_1 - \bar{z}_0, \bar{z}_2 - \bar{z}_0),
\]

where the function \( \Psi_{h,\bar{h}}(\bar{z}_1, \bar{z}_2) \) was defined in (2.55). Following the diagrammatical approach, we represent the integral entering the l.h.s. of this relation as the Feynman diagram shown in Fig. 10a. To arrive at this diagram we replaced the kernel of the \( Q \)-operator by the right diagram in Fig. 6 at \( N = 2 \) and represented the eigenfunction \( \Psi_{h,\bar{h}}(\bar{w}_1 - \bar{z}_0, \bar{w}_2 - \bar{z}_0) \) as a (nonunique) triangle with the vertices at the points \( \bar{w}_1, \bar{w}_2 \) and \( \bar{z}_0 \).

The both sides of the relation (3.60) depend on two vectors \( \bar{z}_1 - \bar{z}_0 \) and \( \bar{z}_2 - \bar{z}_0 \) that we may choose to our convenience. One possible choice could be \( \bar{z}_1 \to \infty, \bar{z}_0 = 0 \) and \( \bar{z}_2 = \bar{z}_2 = 1 \). In this limit, four lines connecting the point \( \bar{z}_i \) with the rest of the diagram can be effectively removed and the resulting diagram takes the form shown in Fig. 10b. The corresponding Feynman integral defines the eigenvalues of the Baxter operator \( Q_+(u, \bar{u}) \) at \( N = 2 \)

\[
Q_{h,\bar{h}}^{(N=2)}(u, \bar{u}) = \int d^2w_1 d^2w_2 \frac{(\pi a(2 - 2s, s + iu, \bar{s} - i\bar{u}))^2}{[w_1]^{1-s+iu}[w_2]^{1-s-iu}((w_1 - 1)^h((w_2 - 1)^h[([w_1 - 1])^h(w_2 - 1)])^{2s-h}).
\]

One can find another (equivalent) representation for the same eigenvalue by choosing in (3.60) \( \bar{z}_0 \to \infty, \bar{z}_2 = 0 \) and \( z_1 = \bar{z}_1 = 1 \). In this case, one can remove two lines attached to the point \( \bar{z}_0 \). The resulting diagram takes the form shown in Fig. 10c. By the construction, \( Q_{h,\bar{h}}^{(N=2)}(u, \bar{u}) \) satisfies the Baxter equations, Eqs. (3.3) and (3.4) for \( N = 2 \), with the auxiliary transfer matrices given by (2.47)

\[
t_2(u) = 2u^2 - h(h - 1) + 2s(s - 1)
\]

and \( \bar{t}_2(\bar{u}) \) is given by a similar expression in the antiholomorphic sector.

The two-dimensional integral in (3.61) has a striking resemblance to expressions for the correlation functions in two-dimensional conformal field theories and, therefore, can be evaluated using the powerful technique developed in [25]. The resulting expression for (3.61) is given by the sum over the product of holomorphic and antiholomorphic “conformal blocks”

\[
Q_{h,\bar{h}}^{(N=2)}(u, \bar{u}) = c(s, h) \frac{\Gamma(1 - \bar{s} - i\bar{u})\Gamma(1 - \bar{s} + i\bar{u})}{\Gamma(\bar{s} - i\bar{u})\Gamma(\bar{s} + i\bar{u})} \times \left\{ Q_0(u) (Q_0(\bar{u}^*)^* + (-1)^n Q_0(-u) (Q_0(\bar{u}^*))^* \right\},
\]

where \( h = (1 + n_h)/2 + i\nu_h, \bar{h} = 1 - h^* \) and \( Q_0(u) \) is given by one-dimensional contour integral that can be expressed in terms of hypergeometric series as

\[
Q_0(u) = 3F_2\left(\begin{array}{c} 2s - h, 2s - 1 + h, s + iu \\ 2s, 2s \end{array}\right)^1. \]

The constant \( c(s, h) \) in (3.63) does not depend on the spectral parameter and it is fixed by the normalization condition (3.43), \( Q_{h,\bar{h}}(is, i\bar{s}) = \pi^4/(n_s^2 + 4\nu_s^2)^2 \).
Figure 10: The eigenvalue of the Baxter $Q$–operator at $N = 2$: (a) general diagrammatic representation; (b) reduction at $\vec{z}_0 = 1$, $\vec{z}_1 \to \infty$ and $\vec{z}_2 = 0$; (b) reduction at $\vec{z}_0 = \infty$, $\vec{z}_1 = 1$ and $\vec{z}_2 = 0$.

As a matter of fact, the expression (3.64) has already appeared in the analysis of the Baxter equation for the noncompact spin chains. It was introduced for the first time in [7, 21] as the solution to the holomorphic Baxter equation for the $\text{SL}(2, \mathbb{C})$ magnet of spin $s = 1$ and $\bar{s} = 0$. Later, the same expression was identified [14] as the solution to Baxter equation for the Heisenberg $\text{SL}(2, \mathbb{R})$ magnet of arbitrary spin $s$ and the total spin of the system, $h - 2s$, being nonnegative integer. In the latter case, $Q_0(u)$ is reduced to a polynomial of degree $h - 2s$ in $u$, which turns out to be identical to the Hahn orthogonal polynomial.

The detailed analysis of the properties of the obtained expressions, Eqs. (3.63) and (3.64), and their generalization to higher $N$ will be presented elsewhere [24].

3.4. Relation to the Hamiltonian

There exists the relation between the Hamiltonian of the model, (2.42), and the Baxter $Q$–operator. It allows to replace the original Schrödinger equation (2.8) by the spectral problem (3.5) and express the energy $E_N$ in terms of the eigenvalues of the operator $Q_+(u, \bar{u})$.

The above relation follows, in its turn, from a more general relation between the $\text{SL}(2, \mathbb{C})$ transfer matrices of arbitrary spin, $T^{(s_0, \bar{s}_0)}(u, \bar{u})$, defined in (2.40), and the Baxter $Q$–operator (see Appendix B for details). The simplest way to establish this relation is to compare the Feynman diagrams describing the transfer matrix and the product of $Q$–operators, Figs. 3 and 8, respectively. It is easy to see that, up to redefinition of the spins and the spectral parameters, two diagrams are identical. In this way, we obtain the following relation

$$T_N^{(s_0, \bar{s}_0)}(u, \bar{u}) \sim X(u + i(1 - s_0), \bar{u} + i(1 - \bar{s}_0); u + is_0, \bar{u} + i\bar{s}_0),$$

(3.65)

which is valid for arbitrary $\text{SL}(2, \mathbb{C})$ spins $(s_0, \bar{s}_0)$ and the spectral parameters $u$ and $\bar{u}$. Here, the notation was introduced for the operator $X(u, \bar{u}; v, \bar{v})$ with the kernel $X_{u, \bar{u}; v, \bar{v}}(\vec{z}, \vec{w})$ defined in (3.32). Taking into account the relation between the $X$–operator and the Baxter $Q$–operators, Eqs. (3.33) and (3.34), we obtain the expression for the transfer matrix in terms of the operators $Q_\pm$

$$T_N^{(s_0, \bar{s}_0)}(u, \bar{u}) = \rho_T^{(s_0, \bar{s}_0)}(u, \bar{u}) \times Q_-(u + i(1 - s_0), \bar{u} + i(1 - \bar{s}_0)) Q_+(u + is_0, \bar{u} + i\bar{s}_0),$$
where the spectral parameters satisfy the relation (2.21) and the normalization factor \( \rho_T^{(s_0,s_0)}(u, \bar{u}) \) can be calculated from Eqs. (2.27) and (3.32) as
\[
\rho_T^{(s_0,s_0)}(u, \bar{u}) = \left[ \frac{1}{\pi^3 a(2 - 2s)} \frac{a(s + s + i\bar{u}, s - s - i\bar{u})}{a(s - s + i\bar{u}, s + s - i\bar{u})} \right]^N, \tag{3.67}
\]

The following comments are in order.

According to (2.42), the Hamiltonian of the model is given by the logarithmic derivative of the fundamental transfer matrix \( T^{(s,s)}(u, \bar{u}) \) at \( u = \bar{u} = 0 \). Since the spectral parameters have to satisfy the condition (2.21), to calculate the Hamiltonian it becomes sufficient to analyze the transfer matrix for real values of the spectral parameters \( u = \bar{u} = u^* \). In this case, one puts \( s_0 = s \) in (3.66) and substitutes the \( Q_+ \) operator by its expression in terms of \( (Q_+)^\dagger \) operator, (3.40), to find (see also (3.2))
\[
T_N^{(s,s)}(u, u) = \rho_Q(u) [Q_+(u - is, u - i\bar{s})]^\dagger Q_+(u + is, u + i\bar{s})
\]
\[
= \tilde{\rho}_Q u^{2N} [Q_+(u - i\bar{s}^*, u - is^*)]^\dagger Q_+(u + i\bar{s}^*, u + is^*)
\]
with \( s^* = 1 - s \) and the normalization factors
\[
\rho_Q(u) = \left( \frac{a(1 - 2\bar{s} + i\bar{u}, 1 - i\bar{u})}{a(2 - 2s - i\bar{u}, 1 + i\bar{u})} \right)^N \tilde{\rho}_Q, \quad \tilde{\rho}_Q = \left( \frac{\nu_s^2 + 4\nu_s^2}{\pi^4} \right)^N.
\]

One verifies, using (3.43), that \( T_N^{(s,s)}(0, 0) = (-1)^N \mathbb{P}^{-1} \). Together with (2.42) this leads to \( T_N^{(s,s)}(u, u) = (-1)^N \exp(-i\theta_N - iu\mathcal{H}_N + O(u^2)) \). We find the expression for the Hamiltonian \( \mathcal{H}_N \) by substituting (3.68) into this relation
\[
\mathcal{H}_N = \varepsilon_N + i \frac{d}{du} \ln Q_+(u + is, u + i\bar{s}) \bigg|_{u=0} - \left[ i \frac{d}{du} \ln Q_+(u - is, u - i\bar{s}) \bigg|_{u=0} \right]^\dagger, \tag{3.69}
\]
where the notation was introduced for the additive normalization constant
\[
\varepsilon_N = i \frac{d}{du} \ln \rho_Q(u) \bigg|_{u=0} = 2N \Re \left[ \psi(2s) + \psi(2 - 2s) - 2\psi(1) \right]. \tag{3.70}
\]

Eq. (3.69) establishes the relation between the Hamiltonian and the Baxter \( Q \)–operator. We recall that the latter satisfies the Baxter equations, Eqs. (3.3) and (3.4), and depends on the integrals of motion \( \{q, \bar{q}\} \). Then, Eq. (3.69) provides the explicit form of the dependence (2.54).

Applying the eigenstate \( \Psi_{\rho(q, \bar{q})} \) to the both sides of (3.69) we calculate the energy as
\[
E_N(q, \bar{q}) = \varepsilon_N + i \left( \ln Q_{q, \bar{q}}(is, i\bar{s}) \right)' + i \left( \ln [Q_{q, \bar{q}}(-is, -i\bar{s})]^* \right)', \tag{3.71}
\]
where prime denotes derivative with respect to the spectral parameter and \( Q_{q, \bar{q}}(u, \bar{u}) \equiv Q_{q, \bar{q}}^+(u, \bar{u}) \) stands for the eigenvalue of the operator \( Q_+(u, \bar{u}) \). We can further simplify this expression by using the properties of the eigenvalues (3.31)
\[
E_N(q, \bar{q}) = \varepsilon_N + i \left( \ln Q_{q, \bar{q}}(is, i\bar{s}) \right)' - i \left( \ln [Q_{-q, -\bar{q}}(is, i\bar{s})]^* \right)', \tag{3.72}
\]

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where \((-q, -\bar{q}) \equiv ((-1)^k q_k, (-1)^k \bar{q}_k)\). Taking into account that \(E_N(q, \bar{q})\) is invariant under the replacement \((q, \bar{q}) \to (-q, -\bar{q})\) (see Sect. 2.4) one gets from the last relation the expression for the energy

\[
E_N(q, \bar{q}) = \varepsilon_N - \Im \frac{d}{du} \ln [Q_{q, \bar{q}}(u + is, u + i\bar{s}) Q_{-q, -\bar{q}}(u + i\bar{s}, u + is)] \bigg|_{u=0},
\]

(3.73)

which is explicitly real. This expression was obtained from the first relation in (3.68). Starting from the second relation in (3.68), one can obtain another representation for the energy

\[
E_N(q, \bar{q}) = -\Im \frac{d}{du} \ln \left[ u^{2N} Q_{q, \bar{q}}(u + is^*, u + is^*) Q_{-q, -\bar{q}}(u + is^*, u + is^*) \right] \bigg|_{u=0}.
\]

(3.74)

We notice that, according to (3.50), the \(Q\)-function entering this expression has the \(N\)-th order pole at \(u = 0\). The factor \(u^{2N}\) compensates this pole and, as a consequence, the energy is determined by the subleading \(O(\epsilon)\) terms in the r.h.s. of (3.56).

One can verify [24] that at \(N = 2\) the substitution of (3.68) into Eqs. (3.73) and (3.74) leads to the well-known expression for the energy (2.56).

### 3.5. Analytical properties

Let us examine the analytical properties of the Baxter operator, \(Q_+(u, \bar{u})\), and its eigenvalues, \(Q_{\bar{q}, q}(u, \bar{u})\), as functions of the spectral parameters \(u\) and \(\bar{u}\). We remind that the spectral parameters have to satisfy the condition \(i(u - \bar{u}) = n\) with \(n\) being integer. It becomes convenient to parameterize their possible values as \(u = -in/2 + \lambda\) and \(\bar{u} = in/2 + \lambda\) with \(\lambda\) being arbitrary complex. The operator \(Q_+(\lambda - in/2, \lambda + in/2)\) becomes a function of integer \(n\) and continuous complex \(\lambda\).

In this Section we shall determine analytical properties of \(Q_+(\lambda - in/2, \lambda + in/2)\) on the complex \(\lambda\)-plane for fixed \(n\). To start with, we consider Eq. (3.5) and substitute the \(Q\)-operator by its integral representation with the kernel given by the first relation in (3.27)

\[
\int d^2y \prod_{k=1}^{N} [z_k - y_{k-1}]^{-s-iu} [z_{k-1} - y_k]^{-s+iu} \tilde{\Psi}_{\bar{q}, q}(\vec{y}) = Q_{\bar{q}, q}(u, \bar{u}) \Psi_{\bar{q}, q}(\vec{z}),
\]

(3.75)

where the notation was introduced for the function

\[
\tilde{\Psi}_{\bar{q}, q}(\vec{y}) = \int d^2w \Psi_{\bar{q}, q}(\vec{w}) \prod_{k=1}^{N} [w_k - y_k]^{2s-2} = \left( \frac{n^2 + 4\nu^2}{\pi^2} \right)^{-N/2} [U \Psi_{\bar{q}, q}(\vec{z})] (\vec{y}).
\]

(3.76)

This relation is well-known as the \(SL(2, \mathbb{C})\) intertwining transformation with \(U\) being the corresponding unitary operator [17]. It maps the state \(\Psi^{(s, \bar{s})}(\vec{z})\) belonging to the space \(V \otimes \ldots \otimes V\), with \(V \equiv V^{(s, \bar{s})}\), into the unitary equivalent state \(\Psi^{(1-s, 1-\bar{s})}(\vec{z}) = [U \Psi^{(s, \bar{s})}] (\vec{z})\).

The singularities of the eigenvalues \(Q_{\bar{q}, q}(\lambda - in/2, \lambda + in/2)\) in \(\lambda\) are in one-to-one correspondence with divergences of the integral in the l.h.s. of (3.75). The latter originate from two different integration regions: \(|\vec{y}_k - \vec{z}_{k+1}| \to 0\) and \(|\vec{y}_k - \vec{z}_{k-1}| \to 0\). In the first region, one expands the integrand in (3.75) in powers of \(\vec{w}_k \equiv \vec{y}_k - \vec{z}_{k+1}\), assuming that \(\tilde{\Psi}_{\bar{q}, q}(\vec{y}) - \tilde{\Psi}_{\bar{q}, q}(\vec{z}) \sim \sum_k c_k w_k^{m_s - 1} w_k^{m_{\bar{s}} - 1}\), and calculates the l.h.s. of (3.75) as the product of integrals of the form

\[
\int_{|w_k|<\epsilon} d^2w_k w_k^{-s-iu} w_k^{-\bar{s}-i\bar{u}} w_k^{m_s - 1} w_k^{m_{\bar{s}} - 1} \sim \frac{2\pi \delta_{s+\bar{s}=m_k}}{s + iu - m_k}
\]

(3.77)

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Figure 11: The position of poles (fat points) of the Baxter operator $Q_+(\lambda - in/2, \lambda + in/2)$ in the complex $\lambda-$plane at $n = 0$. The $SL(2, \mathbb{C})$ spin of particles, $s = (1 + n_s)/2 + i\nu_s$, is chosen in such a way that $n_s = 0$ and $\nu_s > 0$. Two squares correspond to the values of the spectral parameters $(u, \bar{u}) = (\pm is, \pm i\bar{s})$.

with $m_k$ and $\bar{m}_k$ being positive integer. Thus, integration over each $\tilde{y}_k \sim \tilde{w}_{k+1}$ in (3.75) brings a simple pole in the spectral parameter located at $s + iu - m_k = \bar{s} + i\bar{u} - \bar{m}_k = 0$. Taking their product we find that the resulting $N-$fold integral over $\tilde{y}_k \sim \tilde{z}_k$ we find another (infinite) set of poles located at $s - iu - m_k = \bar{s} - i\bar{u} - \bar{m}_k = 0$.

Combining together the contribution from both regions, we conclude that the eigenvalue of the Baxter $Q-$operator, $Q_{(q,\bar{q})}(u, \bar{u})$, is an analytical function on the complex plane except the infinite set of points

$$\{ u_m^+ = i(s - m), \bar{u}_{\bar{m}}^+ = i(\bar{s} - \bar{m}) \} ; \quad \{ u_m^- = -i(s - m), \bar{u}_{\bar{m}}^- = -i(\bar{s} - \bar{m}) \}$$

with $m, \bar{m} = 1, 2, \ldots$, at which it has the poles of the order not higher than $N$. Using the above parameterization of the spectral parameters, $u = -in/2 + \lambda$ and $\bar{u} = in/2 + \lambda$, we obtain that the poles can be enumerated by a pair of positive integers $(m, \bar{m})$

$$n_{m,\bar{m}}^\pm = \mp(n_s + \bar{m} - m), \quad \lambda_{m,\bar{m}}^\pm = \mp \left( \nu_s + i \frac{m + \bar{m} - 1}{2} \right)$$

with $s = (1 + n_s)/2 + i\nu_s$ and $m, \bar{m} = 1, 2, \ldots$. Then, for arbitrary integer $n = n_{m,\bar{m}}^\pm$ the function $Q_{(q,\bar{q})}(\lambda - in/2, \lambda + in/2)$ has the $N-$th order poles situated along the imaginary axis in the $\lambda-$plane at the points $\lambda = \lambda_{m,\bar{m}}^\pm$ as shown in Fig. 11. These two infinite sets of poles lie on the different sides from the real axis and do not overlap. Since $|\text{Im} \lambda_{m,\bar{m}}^\pm| \geq 1/2$, the eigenvalues of the Baxter $Q-$operator do not have poles within the strip $-1/2 < \text{Im} \lambda < 1/2$.

As was shown in the previous section, different observables of the model – the Hamiltonian and quasimomentum, Eqs. (3.73), (3.74) and (3.47), are expressed in terms of the Baxter operator $Q_+$ and its derivatives defined at the special points $(u = \pm is, \bar{u} = \pm i\bar{s})$ and $(u = \pm is^*, \bar{u} = \pm is^*)$. In the former case, the points belong to the region of analyticity of the $Q_+$-operator (see Fig. 11).
and, therefore, the above operators are well-defined. In the latter case, the \( N \)-th order pole of the Baxter \( Q_+ \)-operator is compensated by the factor \( u^{2N} \) entering the second relation in (3.68).

Analyzing the singularities of the \( Q_+ \)-operator we have used the integral representation of the kernel defined by the left diagram in Fig. 3. One could use instead another representation, corresponding to the right diagram on the same figure. In this way, one finds that the Feynman integrals have poles at \( u = \pm i(s + m) \) and \( \bar{u} = \pm i(\bar{s} + \bar{m}) \) with \( m \) and \( \bar{m} \) positive integer, that are different from (3.78). In addition, one has to take into account that the prefactor \( A(u, \bar{u}) \) also generates singularities. One finds that \( A(u, \bar{u}) \) produces the poles at the points (3.78) and, at the same time, it vanishes at \( u = \pm i(s + m) \) and \( \bar{u} = \pm i(\bar{s} + \bar{m}) \), thus compensating the singularities of the Feynman integral.

Summarizing the analytical properties of the Baxter operator, we expand \( Q_+(u, \bar{u}) \) into the sum over poles (3.73)

\[
Q_+(\lambda - in/2, \lambda + in/2) = \sum_{m,m=1}^{\infty} \delta_{n,n,m,m} \left[ \frac{1}{(\lambda - \lambda^+_m,m)^N} + \frac{i \mathbb{R}^+_m,m}{(\lambda - \lambda^+_m,m)^{N-1}} + \cdots \right] + \delta_{n,n,m,m} \left[ \frac{1}{(\lambda - \lambda^-_m,m)^N} + \frac{i \mathbb{E}^+_m,m}{(\lambda - \lambda^-_m,m)^{N-1}} + \cdots \right],
\]

where the operators \( \mathbb{R}^\pm_{m,m} \) and \( \mathbb{E}^\pm_{m,m} \) define the corresponding residues and ellipses denote the contribution of the \( k \)-th order poles \( (k \leq N - 2) \).

The operators \( \mathbb{R}^\pm_{m,m} \) and \( \mathbb{E}^\pm_{m,m} \) share all properties of the operator \( Q_+(u, \bar{u}) \). In particular, they commute with each other and are diagonalized by the eigenstate \( \Psi_{\lambda}(\bar{z}, z) \). Replacing the operators \( \mathbb{R}^\pm_{m,m} \) and \( \mathbb{E}^\pm_{m,m} \) by their corresponding eigenvalues, \( R^\pm_{m,m}(q, \bar{q}) \) and \( E^\pm_{m,m}(q, \bar{q}) \), respectively, one obtains from (3.80) the pole expansion of the eigenvalues of the \( Q_+ \)-operator, \( Q_{q,\bar{q}}(u, \bar{u}) \). The properties of the eigenvalues, Eq. (3.51), are translated into the following relations

\[
R^\pm_{m,m}(q, \bar{q}) = (-1)^N e^{i\theta_{q,\bar{q}}} R^\pm_{m,m}(-q, -\bar{q}) , \quad E^\pm_{m,m}(q, \bar{q}) = -E^\mp_{m,m}(-q, -\bar{q}).
\]

Similarly, substituting \( u = u^+_m + \epsilon \) and \( \bar{u} = \bar{u}^+_m + \epsilon \) into (3.55) and matching \( Q_{q,\bar{q}}(u^+_m + \epsilon, \bar{u}^+_m + \epsilon) \) into (3.80), one arrives at

\[
R^\pm_{m,m}(q, \bar{q}) = e^{i\theta_{q,\bar{q}}} C^N_{m,m} \left[ Q_{q,\bar{q}}(i(s + \bar{m} - 1), i(\bar{s} + m - 1)) \right]^* \\
E^\pm_{m,m}(q, \bar{q}) = N\varepsilon_{m,m} - i \frac{d}{d\epsilon} \ln \left[ Q_{q,\bar{q}}(i(s + \bar{m} - 1), i(\bar{s} + m - 1) + \epsilon) \right] \bigg|_{\epsilon=0}.
\]

Here, \( C^N_{m,m} = -C_s/[\pi^2 B(m, 1 - 2s) B(\bar{m}, 1 - 2\bar{s})] \) is expressed in terms of the Euler \( B \)-function, the constant \( C_s \) is defined in (3.56) and \( \varepsilon_{m,m} = \psi(1 - 2s + m) + \psi(1 - 2\bar{s} + \bar{m}) - \psi(m) - \psi(\bar{m}) - \psi(1 - 2\bar{s}) + \psi(2\bar{s}) \). We verify, using (3.82) and (3.43), that \( R^\pm_{m,1}(q, \bar{q}) = e^{i\theta_{q,\bar{q}}} C^N_{s} \) in accordance with (3.56).

Remarkably enough, the spectrum of the model can be found from the pole expansion of the \( Q_+ \)-operator, Eq. (3.80). Indeed, according to (3.83), the quasimomentum \( \theta_{q,\bar{q}} \) can be expressed in terms of the residue functions \( R^\pm_{1,1}(\pm q, \pm \bar{q}) \). Substituting (3.80) into (3.74), we obtain that the energy \( E_N(q, \bar{q}) \) is related to the residue functions \( E^\pm_{1,1}(\pm q, \pm \bar{q}) \)

\[
E_N(q, \bar{q}) = -\text{Re} \left[ E^\pm_{1,1}(q, \bar{q}) + E^\mp_{1,1}(-q, -\bar{q}) \right] = \text{Re} \left[ E^+_1(q, \bar{q}) + E^-_1(-q, -\bar{q}) \right].
\]

Making use of (3.82) it is easy to see that this expression coincides with (3.73).
3.6. Asymptotic behavior

As we have seen in the previous Section, the eigenvalues of the $Q$–operators, $Q_{q,q}^{(s)}(u,\bar{u})$, are meromorphic functions of the spectral parameters, $u = \lambda - in/2$ and $\bar{u} = \lambda + in/2$, with a (infinite) series of poles in $\lambda$ located parallel to the imaginary axis outside the strip $|\text{Im}\lambda| < 1/2$. Let us find their asymptotic behavior at large $\lambda$ along the horizontal axis, $\text{Re}\lambda \to \infty$ for $\text{Im}\lambda = \text{fixed}$. In this limit, we may neglect the imaginary part of $\lambda$ and choose the spectral parameter to be real, $\lambda = \nu$, with $\nu \to \infty$.

We substitute $u = - in/2 + \nu$ and $\bar{u} = in/2 + \nu$ into the relation (3.33) and examine its both sides in the limit of large real $\nu$. Since the $Q$–operator commutes with the $SL(2,\mathbb{C})$–generators, Eq. (3.37), it can not depend on the momentum operator $\vec{p}$ and the same is true for its eigenvalues, $Q_{q,q}^{(s)}(u,\bar{u})$. Using this property we put $\vec{p} = 0$ in (3.33). The reason for this particular choice of the momentum is that the state $\Psi_{\vec{p}=0,(q,q)}(\vec{z})$ possesses additional symmetry properties – it diagonalizes the operators $S_0$ and $\bar{S}_0$, and at the same time it is annihilated by the operators $p = iS_-$ and $\bar{p} = i\bar{S}_-$

$$S_- \Psi_{(q,q)}^{(0)}(\vec{z}) = 0, \quad (S_0 - h) \Psi_{(q,q)}^{(0)}(\vec{z}) = 0 \quad (3.84)$$

and similar relations hold in the antiholomorphic sector. These properties imply that, firstly, $\Psi_{(q,q)}^{(0)}(\vec{z})$ is translation invariant and, as a consequence, it depends on the differences of the coordinates and, secondly, it has a definite scaling dimension

$$\Psi^{(0)}(z_k + \epsilon, \bar{z}_k + \bar{\epsilon}) = \Psi^{(0)}(z_k, \bar{z}_k), \quad \Psi^{(0)}(\lambda z_k, \bar{\lambda} z_k) = \lambda^{\nu N_s} \bar{\lambda}^{-N_s} \Psi^{(0)}(z_k, \bar{z}_k). \quad (3.85)$$

Let us determine the asymptotic behavior of the operator $Q_+(u,\bar{u})$ on the space of functions $\Psi^{(0)}(\vec{z})$ satisfying the conditions (3.85). Using integral representation for the $Q_+$–operator, Eq. (3.6), with the kernel given by the second relation in (3.27), one arrives at the following relation in the limit $\nu \to \infty$

$$[Q_+(u,\bar{u})\Psi^{(0)}](\vec{z}) \nu \to \infty \quad \nu^{2N(1-s-\bar{s})} c(s,\bar{s}) \prod_{k=1}^{N} [z_k - z_{k+1}]^{1-2s} \quad (3.86)$$

$$\times \int d^2 w \prod_{k=1}^{N} [w_k - z_k]^{-1+\frac{\nu}{2}+i\nu} [w_k - z_{k+1}]^{-1-\frac{\nu}{2}-i\nu} \Psi^{(0)}(\vec{w})$$

with $c(s,\bar{s}) = [\pi a(2-2s)(-1)^{n_s}]^N$. Examining the asymptotics of the integrand at large $\nu$, one finds that the dominant contribution comes from two different integration regions:

$$\text{(I)} : \quad \vec{w}_k = O(\nu), \quad (\text{II}) : \quad \vec{w}_k - \vec{w}_{k+1} = O(1/\nu) \quad (3.87)$$

with $k = 1, \ldots, N$, in which the $\nu$–dependence of the integrand cancels out in the product of the $z$–dependent factors.

In the first region, at large $\vec{w}_k$, one rescales the integration variables as $w_k = y_k \nu$ and $\vec{w}_k = \vec{y}_k \nu$ and applies the identity

$$[\nu y_k - z_k]^{i\nu} [\nu y_k - z_{k+1}]^{-i\nu} \sim \exp(-i(z_k - z_{k+1})/y_k - i(\bar{z}_k - \bar{z}_{k+1})/\bar{y}_k) \equiv e^{-2i \text{Re}((z_k - z_{k+1})/y_k)}. \quad (3.88)$$

Taking into account the scaling properties of the functions $\Psi^{(0)}$, Eq. (3.85), we obtain from (3.86)

$$[Q_+(u,\bar{u})\Psi^{(0)}]_1(\vec{z}) \nu \to \infty \quad \nu^{h - \bar{h} - N(s+\bar{s})} \times c(s,\bar{s}) \Psi^{(0)}_1(\vec{z}), \quad (3.89)$$
where the subscript (I) refers to the integration region in (3.87). Here, the notation was introduced for the function $\tilde{\Psi}^{(0)}_I$ depending on the differences of the coordinates $z_k - z_{k+1}$. It is obtained from the function $\Psi^{(0)}(\tilde{z})$ through the transformation generated by the operator $S_I$ defined as

$$
\tilde{\Psi}^{(0)}_I(\tilde{z}) = [S_I \Psi^{(0)}](\tilde{z})
$$

where $Dy = d^2y_1...d^2y_N \delta(\sum_{i=1}^N \tilde{y}_i/N)$ is the integration measure on the space of translation invariant functions. According to (3.90), the operator $S_I$ transforms $\Psi^{(0)}(\tilde{z})$ into another translation invariant function. Then, it follows from (3.89) that the contribution of the region (I) to the asymptotic behavior of the operator $Q_+(u, \bar{u})$ on the space of the functions satisfying (3.83) is given by

$$
[Q_+(\nu - in/2, \nu + in/2)]_{I} \nu^{\rightarrow \infty} \nu^{i(h - N(s + \bar{s})} \times c(s, \bar{s}) S_I
$$

with the operator $S_I$ defined in (3.90).

In the second region, Eq. (3.87), all $\bar{w}_k$ approach the same point $\bar{z}_0$, but its position on the plane can be arbitrary. It is convenient to introduce the “center-of-mass” coordinate $z_0$ and define the relative coordinates $\bar{y}_k$

$$
z_0 = \frac{1}{N}(\bar{w}_1 + ... + \bar{w}_N), \quad \bar{y}_k = \nu(\bar{w}_k - \bar{z}_0) = \mathcal{O}(\nu^0),
$$

such that $\sum_k \bar{y}_k = 0$. Changing the integration variables in (3.88) from $(\bar{w}_1, ..., \bar{w}_N)$ to $(\bar{z}_0, \bar{y}_1, ..., \bar{y}_N)$, one applies the identity

$$
[z_0 + y_k/\nu - z_k]^{i\nu}[z_0 + y_{k-1}/\nu - z_k]^{-i\nu} \sim \exp(-i(y_k - y_{k-1})/(z_k - z_0)) \quad (3.93)
$$

and uses the translation invariance of the function $\Psi^{(0)}$ to find after some algebra

$$
[Q_+(u, \bar{u})\Psi^{(0)}]_{II}(\tilde{z}) \nu^{\rightarrow \infty} \nu^{i-1h+1-h-N(s+\bar{s})} \times c(s, \bar{s}) \Psi^{(0)}_II(\tilde{z}). \quad (3.94)
$$

Here, $\Psi^{(0)}_II(\tilde{z})$ is translation invariant function related to $\Psi^{(0)}(\tilde{z})$ through the transformation generated by the second operator $S_{II}$ defined as

$$
\Psi^{(0)}_II(\tilde{z}) = [S_{II} \Psi^{(0)}](\tilde{z})
$$

where the measure $Dy$ defined in (3.90). Then, the contribution of the region (II) to the asymptotic behavior of the $Q_+$-operator can be expressed as

$$
[Q_+(\nu - in/2, \nu + in/2)]_{II} \nu^{\rightarrow \infty} \nu^{1-h+1-h-N(s+\bar{s})} \times c(s, \bar{s}) S_{II}. \quad (3.96)
$$

We notice that this relation looks similar to (3.91) and it can be obtained from the latter by replacing the conformal spin of the state, $h \rightarrow 1 - h$ and $\bar{h} \rightarrow 1 - \bar{h}$, and substituting the operator
\(S_1\) by its counterpart \((3.93)\). Moreover, comparing \((3.90)\) and \((3.95)\) one finds that the kernels of the operators are related to each other as

\[
S_1(z | y) = M \prod_{k=1}^{N} [z_k - z_{k+1}]^{1-2s} S_{II}(y | z) \prod_{k=1}^{N} [y_k - y_{k+1}]^{2s-1}
\]

(3.97)

with the operator of mirror permutations \(M\) defined in \((2.58)\).

Finally, combining together two contributions, Eqs. \((3.91)\) and \((3.96)\), we obtain the following asymptotic behavior of the operator \(Q_+\)

\[
Q_+(\nu - in/2, \nu + in/2) \overset{\nu \to \infty}{=} c(s, \bar{s}) \left[ \nu^{h - N(s + \bar{s})} S_1 + \nu^{1 - h + 1 - h - N(s + \bar{s})} S_{II} \right].
\]

(3.98)

We would like to stress that this relation is valid on the space of functions satisfying the conditions \((3.83)\). One finds the properties of the operators \(S_{I/II}\) by requiring that the asymptotic expressions for the \(Q-\)operators have to satisfy the relations established in Sect. 3.2. In particular, it follows from \((3.29)\) and \((3.42)\) that these operators commute with each other and with the \(Q-\)operators

\[
[S_1, Q_\pm(u, \bar{u})] = [S_{II}, Q_\pm(u, \bar{u})] = [S_1, S_{II}] = 0.
\]

(3.99)

Let us compare \((3.98)\) with the general expression for the asymptotic behavior of the eigenvalues of the operator \(Q_+(\nu - in/2, \nu + in/2)\) that follows from \((3.55)\)

\[
Q_{(q, \bar{q})}^{(\pm)}(\nu - in/2, \nu + in/2) \overset{\nu \to \infty}{=} \left| Q_{(q, \bar{q})}^{(\pm)} \right| e^{iN(\nu + i\nu_s)/2} \nu^{-2N\nu_s}
\]

(3.100)

with \(s = (1 + n_s)/2 + i\nu_s\) and the phase \(\varphi_s\) defined in \((3.53)\). Matching \((3.98)\) into this expression we require that the \(\nu-\)dependent part of the phase should not depend on the conformal spin of the state \(h = (1 + n_h)/2 + i\nu_h\) and \(\bar{h} = 1 - h^*\). This condition is satisfied provided that the eigenvalues of the operators \(S_{I/II}\) have the same absolute value and differ only by a phase

\[
\left[ S_{I/II} \Psi_{q, \bar{q}}^{(0)}(\bar{z}_1, \bar{z}_2, ..., \bar{z}_N) = Y_{q, \bar{q}} e^{i(\Omega_{q, \bar{q}} \pm \Theta_{q, \bar{q}})} \Psi_{q, \bar{q}}^{(0)}(\bar{z}_1, \bar{z}_2, ..., \bar{z}_N). \right.
\]

(3.101)

As a consequence, the corresponding eigenvalues of the \(Q_+ -\)operator \((3.98)\) have the following asymptotic behavior

\[
Q_{q, \bar{q}}^{(\pm)}(\nu - in/2, \nu + in/2) \overset{\nu \to \infty}{=} 2^{1-N(1+2i\nu_s)} c(s, \bar{s}) Y_{q, \bar{q}} e^{i\Omega_{q, \bar{q}} \cos (\Theta_{q, \bar{q}} + 2\nu_h \ln \nu)}
\]

(3.102)

with \(Y_{q, \bar{q}}, \Theta_{q, \bar{q}}\) and \(\Omega_{q, \bar{q}}\) being real. Its comparison with \((3.100)\) yields (up to an additive \((q, \bar{q})-\)independent correction to \(\Omega_{q, \bar{q}}\) coming from the normalization factors)

\[
\Omega_{q, \bar{q}} = \frac{1}{2} w_{q, \bar{q}}, \quad |Q_{(q, \bar{q})}^{(\pm)}| \sim \nu^{1-N} \cos (\Theta_{q, \bar{q}} + 2\nu_h \ln \nu).
\]

(3.103)

We recall that \(w_{q, \bar{q}}\) was defined in \((3.55)\) as eigenvalue of the operator \(\mathbb{W}\).

Since eigenvalues of the \(Q-\)operator do not depend on the momentum of the state \(\bar{p}\), the asymptotic behavior \((3.102)\) holds for arbitrary eigenstate \(\Psi_{\bar{p}, (q, \bar{q})}(\bar{z})\). This is in distinction with \((3.98)\) that is valid only on the space of functions satisfying the conditions \((3.83)\).

\[\text{This implies, in particular, that the states belonging to this space have an infinite norm} \quad (2.9) \quad \text{and therefore one cannot define the operators conjugated to} \quad S_{I/II} \quad \text{with respect to the scalar product} \quad (2.9)\].

37
In this Section, we have constructed the Baxter $Q-$operator and have shown that the main physical observables of the system, like Hamiltonian, quasimomentum operator, transfer matrices etc, can be expressed in terms of a single operator $Q_+(u, \bar{u})$. Thus, the problem of finding the energy spectrum of the model is reduced to solving the second order finite-difference Baxter equations, Eqs. \((3.3)\) and \((3.4)\), on the eigenvalues of this operator. The construction of the corresponding eigenstates $\Psi_{\vec{p}, \{q, \bar{q}\}}(\vec{z}_1, ..., \vec{z}_N)$ will be the subject of the next Section. We would like to stress that solving the Baxter equations on the eigenvalues of the $Q-$operator, we have to impose the additional conditions on their solutions that follow from the analytical properties of the $Q-$operator, Eq. \((3.80)\), and its asymptotic behavior at infinity, Eq. \((3.102)\). For a general solution to the Baxter equation, depending on the total set of the integrals of motion, $\{q, \bar{q}\}$, these conditions can be satisfied provided that the values of $\{q, \bar{q}\}$ are quantized. The explicit form of the quantization conditions will be discussed in the forthcoming publication \([24]\).

4. Separation of Variables

For the system of $N = 2$ particles, the eigenstates can be found exactly, Eq. \((2.53)\), due to the $SL(2, \mathbb{C})$ invariance of the Hamiltonian. For arbitrary $N$, due to complete integrability of the model, the functions $\Psi_{\vec{p}, \{q, \bar{q}\}}(\vec{z})$ can be defined as simultaneous eigenstates of the total set of the integrals of motion, $\vec{p}$ and $(q_k, \bar{q}_k)$ with $k = 2, ..., N$. The latter are given by the $k-$th order differential operators acting on holomorphic and antiholomorphic coordinates of the particles. Instead of going through the solution of the resulting system of the differential equations on $\Psi_{\vec{p}, \{q, \bar{q}\}}(\vec{z})$ we shall employ the method of the Separation of Variables (SoV) developed by Sklyanin \([11]\). It allows to find the integral representation for the eigenstates of the model by going over to the representation of the separated coordinates $\vec{x} = (\vec{x}_1, ..., \vec{x}_{N-1})$ \([11]\)

$$\Psi_{\vec{p}, \{q, \bar{q}\}}(\vec{z}) = \int d\vec{x} \mu(\vec{x}_1, ..., \vec{x}_{N-1}) U_{\vec{p}, \vec{x}_1,...,\vec{x}_{N-1}}(\vec{z}_1, ..., \vec{z}_N) \left( \Phi_{\{q, \bar{q}\}}(\vec{x}_1, ..., \vec{x}_{N-1}) \right)^*,$$

where $U_{\vec{p}, \vec{x}}(\vec{z})$ is the kernel of the unitary operator corresponding to this transformation

$$U_{\vec{p}, \vec{x}_1,...,\vec{x}_{N-1}}(\vec{z}_1, ..., \vec{z}_N) = \langle \vec{z}_1, ..., \vec{z}_N | \vec{p}, \vec{x}_1, ..., \vec{x}_{N-1} \rangle,$$

and $\Phi_{\{q, \bar{q}\}}(\vec{x}_1, ..., \vec{x}_{N-1})$ is the eigenstate of the Hamiltonian in the SoV representation

$$\Phi_{\{q, \bar{q}\}}(\vec{x}_1, ..., \vec{x}_{N-1}) \delta(\vec{p} - \vec{p}') = \langle \Psi_{\vec{p}, \{q, \bar{q}\}} | \vec{p}', \vec{x}_1, ..., \vec{x}_{N-1} \rangle = \int d\vec{z} U_{\vec{p}', \vec{x}_1,...,\vec{x}_{N-1}}(\vec{z}) \left( \Psi_{\vec{p}, \{q, \bar{q}\}}(\vec{z}) \right)^*.$$

Here, we introduced the standard notation for the bra and ket vectors on the quantum space of the system and defined the eigenstates in the different representations by projecting them out onto the corresponding basis of the states, $\langle \Psi_{\vec{p}, \{q, \bar{q}\}} | \vec{p}', \vec{x}_1, ..., \vec{x}_{N-1} \rangle$ and $\langle \Psi_{\vec{p}, \{q, \bar{q}\}} | \vec{z}_1, ..., \vec{z}_N \rangle$.

Remarkable feature of \((1.1)\) is that its substitution into \((2.8)\) leads to the Schrödinger equation for $\Phi_{\{q, \bar{q}\}}(\vec{x})$ that takes the form of multi-dimensional Baxter equation (see Eq. \((1.13)\) below). Its solution is given by the product of the eigenvalues of the Baxter $Q-$operator with the spectral parameters equal to holomorphic and antiholomorphic components of the separated coordinates $\vec{x}_k = (x_k, \bar{x}_k)$

$$\left( \Phi_{\{q, \bar{q}\}}(\vec{x}_1, ..., \vec{x}_{N-1}) \right)^* \sim Q(x_1, \bar{x}_1) ... Q(x_{N-1}, \bar{x}_{N-1}).$$

\(^{10}\)Here, it proves convenient to define the transformation $\Phi \to \Psi$ to be anti-linear.
In this Section, we shall construct explicitly the SoV transformation, Eqs. (4.1)–(4.4), establish the quantization conditions on the possible values of the separated coordinates \( \vec{x}_k \) and calculate the integration measure \( \mu(\vec{x}_1, ..., \vec{x}_{N-1}) \) entering (4.1).

### 4.1. Representation of the Separated Variables

Defining the representation of the Separated Variables for the noncompact \( SL(2, \mathbb{C}) \) magnet, we follow closely the Sklyanin’s approach \[1\]. This approach is based on the properties of the auxiliary monodromy operator \( T_N(u) \) defined in (2.44) and, in particular, the off-diagonal component \( B_N(u) \) and its antiholomorphic counterpart. One finds from the Yang-Baxter equations on the monodromy operator that the operator \( B_N(u) \) satisfies the following relations \[1, 2, 3, 11\]

\[
[B_N(u), B_N(v)] = [S_-, B_N(u)] = 0, \quad [S_0, B_N(u)] = -B_N(u) \tag{4.5}
\]

with \( S_\alpha \) being the total \( SL(2, \mathbb{C}) \) holomorphic spin.

We recall that, according to Eq. (2.45), \( B_N(u) \) is given by a polynomial in \( u \) of degree \( N - 1 \) with the operator valued coefficients. Commutativity property (4.3) implies that the operator coefficients commute with each other and, therefore, can be diagonalized simultaneously. The eigenvalues of the operator \( B_N(u) \), being polynomials in \( u \), can be parameterized by their zeros \( x_1, ..., x_{N-1} \). Denoting the corresponding eigenfunction as \( U_{\vec{p},\vec{x}}(\vec{z}_1, ..., \vec{z}_N) \) and taking into account (2.45), one gets

\[
B_N(u)U_{\vec{p},\vec{x}}(\vec{z}_1, ..., \vec{z}_N) = p(u - x_1) ... (u - x_{N-1})U_{\vec{p},\vec{x}}(\vec{z}_1, ..., \vec{z}_N) \tag{4.6}
\]

and similarly in the antiholomorphic sector

\[
\bar{B}_N(\bar{u})U_{\vec{p},\vec{x}}(\vec{z}_1, ..., \vec{z}_N) = \bar{p}(\bar{u} - \bar{x}_1) ... (\bar{u} - \bar{x}_{N-1})U_{\vec{p},\vec{x}}(\vec{z}_1, ..., \vec{z}_N). \tag{4.7}
\]

Here, \( \vec{x} = (\vec{x}_1, ..., \vec{x}_{N-1}) \), with \( \vec{x}_k = (x_k, \bar{x}_k) \), denotes the zeros of the eigenvalues of the operators \( B_N(u) \) and \( \bar{B}_N(\bar{u}) \).

The common eigenfunctions of the operators \( B_N(u) \) and \( \bar{B}_N(\bar{u}) \) define the basis on the quantum space of the system, \( U_{\vec{p},\vec{x}}(\vec{z}_1, ..., \vec{z}_N) \), which is parameterized by the momentum \( \vec{p} \) and the set of zeros \( \vec{x} \). Using this basis, one may expand the eigenstates of the Hamiltonian, \( \Psi_{\vec{p},(q,d)}(\vec{z}) \) over the states \( U_{\vec{p},\vec{x}}(\vec{z}_1, ..., \vec{z}_N) \) to arrive at the expansion similar to (1.1) \[1\]. As was shown by Sklyanin \[1\], the resulting expansion defines the representation of the Separated Variables.

Since the eigenfunction in the separated coordinates, Eq. (4.4), is symmetric under permutation of any pair \( \vec{x}_k \) and \( \vec{x}_j \), we impose the same condition on the solutions to (4.6) and (4.7)

\[
U_{\vec{p},\vec{x}_1...\vec{x}_k...\vec{x}_{N-1}}(\vec{z}) = U_{\vec{p},\vec{x}_1...\vec{x}_j...\vec{x}_k...\vec{x}_{N-1}}(\vec{z}). \tag{4.8}
\]

The operators \( B_N(u) \) and \( \bar{B}_N(\bar{u}) \) are conjugated to each other with respect to the \( SL(2, \mathbb{C}) \) scalar product, Eq. (2.46). Together with (4.6) and (4.7) this implies that the same relation holds between their eigenvalues leading to

\[
x_k^* = \bar{x}_k, \quad k = 1, 2, ..., N - 1. \tag{4.9}
\]

\[1\]Taking into account (2.46), one notices that this basis consists of the eigenstates of two mutually commuting hermitian operators, \( B(u) + B(u^*) \) and \( i(B(u) - B(u^*)) \), and therefore it is expected to be complete.
together with \( p^* = \bar{p} \). This becomes the first constraint on the possible values of the separated coordinates. As a consequence of (4.3), the solutions to (1.6) and (1.7) corresponding to different \( x_k \) and \( \bar{p} \) are orthogonal to each other with respect to the scalar product (2.9)

\[
\langle \vec{x}', \vec{p}' | \vec{x}, \vec{p} \rangle = \int d^2 z U_{\vec{p}, \vec{x}}(z_1, ..., z_N) \left( U_{\vec{p}, \vec{x}}(z_1, ..., z_N) \right)^* \delta^{(2)}(p - p') \{ \delta(x - x') + \cdots \} \frac{\mu^{-1}(\vec{x})}{(N - 1)!}, \tag{4.10}
\]

where ellipses denote the sum of terms involving all permutations of the vectors inside the set \( \vec{x} = (\vec{x}_1, ..., \vec{x}_{N-1}) \) that are needed to restore the symmetry property (1.8). To give the meaning to the \( \delta(x - x') \) one has to specify the possible values of the separated coordinates (see Eqs. (4.33) and (4.43) below).

### 4.1.1. Sklyanin’s operator zeros

Following [11], one can interpret parameters \( x_k \) in (4.3) as eigenvalues of certain operators \( \hat{x}_k \) and represent the operator \( B_N(u) \) as

\[
B_N(u) = i S_-(u - \hat{x}_1) \cdots (u - \hat{x}_{N-1}). \tag{4.11}
\]

One also defines similar operators \( \hat{x}_k \) in the antiholomorphic sector, so that \( B_N(\hat{x}_k) = \hat{B}_N(\hat{x}_k) = 0 \). According to (4.4), the operator zeros \( \hat{x}_k \), defined in this way, satisfy the commutation relations

\[
[\hat{x}_k, \hat{x}_j] = [S_-, \hat{x}_k] = [S_0, \hat{x}_k] = 0 \tag{4.12}
\]

and, therefore, they can be diagonalized simultaneously

\[
\hat{x}_k U_{\vec{p}, \vec{x}}(\vec{z}) = x_k U_{\vec{p}, \vec{x}}(\vec{z}), \quad i S_- U_{\vec{p}, \vec{x}}(\vec{z}) = p U_{\vec{p}, \vec{x}}(\vec{z}). \tag{4.13}
\]

To find the Schrödinger equation on the expansion coefficients \( \Phi_{\{q, \bar{q}\}}(\vec{x}_1, ..., \vec{x}_{N-1}) \) in (1.1) we introduce the operators \( X_j^\pm \)

\[
X_j^+ = A_N(\hat{x}_j \leftrightarrow u), \quad X_j^- = D_N(\hat{x}_j \leftrightarrow u), \tag{4.14}
\]

where \( \hat{x}_j \leftrightarrow u \) stands for the substitution of the spectral parameter by the operator from the right [11] (see Eq. (4.20) below). The operators \( X_j^\pm \) and their antiholomorphic counterparts \( \hat{X}_j^\pm, j = 1, ..., N - 1 \), have the following remarkable properties [11]

\[
X_k^\pm \hat{x}_j = (\hat{x}_j \mp i \delta_{jk}) X_k^\pm, \quad \hat{X}_k^\pm \hat{x}_j = (\hat{x}_j \mp i \delta_{jk}) \hat{X}_k^\pm \tag{4.15}
\]

and

\[
X_k^\pm U_{p, \vec{x}}(\vec{z}) = (x_k \pm i s)^N U_{\vec{p}, \vec{x}}(\vec{z}) \hat{e}_k(\vec{z}), \quad \hat{X}_k^\pm U_{p, \vec{x}}(\vec{z}) = (\hat{x}_k \pm i \bar{s})^N U_{\vec{p}, \vec{x}}(\vec{z}) \hat{e}_k(\vec{z}), \tag{4.16}
\]

\[^{12}\text{In [11], similar operators, } \hat{X}_j^\pm, \text{ were defined by substituting } u = \hat{x}_j \text{ in (4.14) from the left. Both sets of the operators are related to each other as } \hat{X}_j^\pm = X_j^\pm \prod_{j \neq k} (\hat{x}_k - \hat{x}_j \pm i)/(\hat{x}_k - \hat{x}_j) \text{ and it is more convenient to our purposes to use the operators } X_j^\pm.\]
where the vectors \( \vec{e}_k \) and \( \vec{e}_k \) add a unit to the holomorphic and antiholomorphic components of the vector \( \vec{x}_k = (x_k, \bar{x}_k) \), respectively,
\[
\vec{x} \pm i \vec{e}_k = (\bar{x}_1, ..., \bar{x}_k \pm i \bar{e}_k, ..., \bar{x}_N), \quad \vec{e}_k = (1, 0),
\]
and similar for \( \vec{x} \pm i \vec{e}_k \) with \( \bar{e}_k = (0, 1) \). Then, the equation on \( \Phi_{\{q, \bar{q}\}}(\vec{x}_1, ..., \vec{x}_{N-1}) \) follows from the following operator identity
\[
\langle \Psi \rangle_{\{q, \bar{q}\}} | t_N(\hat{x}_k \leftrightarrow u) | \vec{p}', \vec{x} \rangle = t_N(x_k) \langle \Psi \rangle_{\{q, \bar{q}\}} | \vec{p}', \vec{x} \rangle = (x_k + is)^N \langle \Psi \rangle_{\{q, \bar{q}\}} | \vec{p}', \vec{x} + i \vec{e}_k \rangle + (x_k - is)^N \langle \Psi \rangle_{\{q, \bar{q}\}} | \vec{p}', \vec{x} - i \vec{e}_k \rangle,
\]
where, in the first relation, we took into account that \( \Psi \) satisfies the multi-dimensional Baxter equation and similar for \( \vec{p}, \vec{x} \). Thus, in the representation, defined by the functions \( U_{\vec{p}, \vec{x}}(\vec{z}) \), the Schrödinger equation on the states \( \Psi_{\{q, \bar{q}\}}(\vec{z}) \) becomes equivalent to the system of separated, one-dimensional finite difference equations on \( \Phi_{\{q, \bar{q}\}}(\vec{x}) \). It remains unclear, however, what are the possible values of separated coordinates \( \vec{x} \) and how does the integration measure \( \mu(\vec{x}) \) in (1.1) look like. Both questions will be answered in the next Section.

One can find the recurrence relations on the integration measure \( \mu(\vec{x}) \) by taking into account the properties of the operator \( A_N(u) \) under conjugation, Eq. (2.46). Since \( A_N(u) \) is a polynomial in \( u \) of degree \( N \), it can be uniquely reconstructed from its values at \( u = \hat{x}_k \) with \( k = 1, ..., N - 1 \), Eq. (1.14), and the asymptotic behavior at large \( u \), Eq. (2.45),
\[
A_N(u) = (u + is_0) + \sum_k \frac{N-1}{N} \prod_{j=1}^{N-1} (u - \hat{x}_j) + \frac{N-1}{N} \prod_{j=1}^{N-1} \frac{u - \hat{x}_j}{\hat{x}_k - \hat{x}_j}.
\]
Writing similar expansion for \( A_N(\bar{u}) \) and imposing the condition \( A_N(u) = [A_N(u^*)]^\dagger \), one obtains the relation between the operators \( \hat{X}_k^+ \) and \( \hat{X}_k^+ \)
\[
(\hat{X}_k^+) = \hat{X}_k^+ \prod_{j \neq k} \frac{\hat{x}_k - \hat{x}_j \pm i}{\hat{x}_k - \hat{x}_j}.
\]
Here, we also included the relation between \( \hat{X}_k^- \) and \( \hat{X}_k^+ \) that comes from the analysis of the operator \( D_N(u) \). Combining together (1.21), (1.13) and (1.16), one obtains
\[
(\hat{X}_k^+) U_{\vec{p}, \vec{x}}(\vec{z}) = (x_k \pm is)^N \prod_{j \neq k} \frac{x_k - x_j \pm i}{x_k - x_j} U_{\vec{p}, \vec{x} \pm i} \vec{e}_k(\vec{z})\].

\[\text{13}\] Since \( \vec{x} \pm i \vec{e}_k \) does not satisfy (1.9), arriving at this equation one assumes certain analyticity properties of the function \( \Phi_{\{q, \bar{q}\}}(\vec{x}) \) outside the region (4.3) that will be discussed below.
Then, examining the identity
\[
\int d^2 \bar{z} \left( \bar{X}_k U_{\bar{p}}, \bar{x} (\bar{z}) \right)^* U_{\bar{p}}, \bar{x} (\bar{z}) = \int d^2 \bar{z} \left( U_{\bar{p}}, \bar{x} (\bar{z}) \right)^* \left( \bar{X}_k \right)^\dagger U_{\bar{p}}, \bar{x} (\bar{z}) ,
\]
we calculate its both sides using \((4.16), (4.13)\) and \((4.21)\), and compare the coefficients in front of \(\delta (\bar{x} - \bar{x}' \pm i \epsilon_k)\) in the resulting expressions. In this way, we arrive at the following finite-difference equation on the integration measure
\[
\frac{\mu (\bar{x} \pm i \epsilon_k)}{\mu (\bar{x})} = \prod_{j \neq k} \frac{x_k - x_j \pm i}{x_k - x_j} .
\]
This relation constrains the dependence of the measure on the holomorphic coordinates \(x_k\). One gets similar relation in the antiholomorphic sector by replacing \(x_k \rightarrow \bar{x}_k\).

The solutions to the equations \((4.19)\) and \((4.24)\) are defined up to multiplication by an arbitrary periodic function \(f(\bar{x})\), such that \(f(\bar{x} \pm i \epsilon_k) = f(\bar{x})\). The algebraic approach described in this Section, does not allow to fix this ambiguity and, therefore, the construction of the SoV transformation remains incomplete. To avoid this obstacle, we shall construct in the next Section the explicit expression for the basis functions \(U_{\bar{p}}, \bar{x} (\bar{z})\).

### 4.2. Construction of the SoV transformation

The transition functions to the SoV representation, \(U_{\bar{p}}, \bar{x} (\bar{z})\), are simultaneous solutions to the system of equations \((4.6)\)–\((4.10)\). Let us start with the conditions \((4.6)\) and \((4.7)\). Since \(B_N(u)\) is a polynomial of degree \(N - 1\) in the spectral parameter \(u\), its general expression at arbitrary \(u\) can be reconstructed from its special values at \(N - 1\) distinct points that we choose as \(u = x_k\). Then, Eqs. \((4.6)\) and \((4.7)\) become equivalent the system of equations
\[
B_N(x_k) U_{\bar{p}}, \bar{x} (\bar{z}_1, ..., \bar{z}_N) = \bar{B}_N(\bar{x}_k) U_{\bar{p}}, \bar{x} (\bar{z}_1, ..., \bar{z}_N) = 0 , \quad k = 1, 2, ..., N - 1 .
\]
One of the reasons for this particular choice of the spectral parameters is that similar equations have already appeared in our analysis of the \(Q\)–operator, Eqs. \((3.13)\) and \((3.18)\).

Indeed, using \((3.18)\) and \((3.19)\), we find that the function
\[
U_{\bar{p}}, \bar{x} (\bar{z}) = \int d^2 y \Psi_{x_1, \bar{x}_1}^{(s, \bar{s})} (\bar{z}_1, ..., \bar{z}_N | \bar{y}_2, ..., \bar{y}_N) Z_{\bar{p}, \bar{x}_2, ..., \bar{x}_{N-1}} (\bar{y})
\]
\[
= \lim_{y_1 \rightarrow \infty} [y_1]^{-2s} \int d^2 y Y_{x_1, \bar{x}_1}^{(s, \bar{s})} (\bar{z}_1, ..., \bar{z}_N | \bar{y}_1, \bar{y}_2, ..., \bar{y}_N) Z_{\bar{p}, \bar{x}_2, ..., \bar{x}_{N-1}} (\bar{y})
\]
satisfies the relations \((4.25)\) at \(k = 1\) for arbitrary weight function \(Z(\bar{y})\) and \(\bar{y} = (\bar{y}_2, ..., \bar{y}_N)\). It remains to show that the same relations hold for \(k \geq 2\) and, in addition, to ensure the symmetry property \((4.8)\). It is easy to see that the former requirement is equivalent to the latter property combined with \(B_N(x_k) U_{\bar{p}, \bar{x}} (\bar{z}) = \bar{B}_N(\bar{x}_k) U_{\bar{p}, \bar{x}} (\bar{z}) = 0\). Thus, to satisfy the equations \((4.25)\) for \(k \geq 2\), it is enough to require that the integral in r.h.s. of \((4.26)\) should be symmetric under permutations \(\bar{x}_1 \leftrightarrow \bar{x}_k\). This condition imposes constraints on the possible form of the weight function \(Z(\bar{y})\) in \((4.20)\). Supposing that such weight function exists, one applies the last two relations in Eq. \((3.19)\) to obtain
\[
A_N(x_k) U_{\bar{p}, \bar{x}} (\bar{z}) = (x_k + is)^N U_{\bar{p}, \bar{x} + i \epsilon_k} (\bar{z}) ,
\]
\[
D_N(x_k) U_{\bar{p}, \bar{x}} (\bar{z}) = (x_k - is)^N U_{\bar{p}, \bar{x} - i \epsilon_k} (\bar{z}) ,
\]
where the vector $\vec{e}_k$ was defined in (1.17). These relations coincide with the operator identities (1.16).

As a hint to finding the solution for $Z_{\vec{p},\vec{x}_1,...,\vec{x}_{N-1}}$, we notice a similarity between (4.26) and the definition of the kernel $X_{v,\vec{u},\vec{w}}(\vec{z} | \vec{w})$, Eq. (3.32). For $v = x_1$ and $\vec{v} = \vec{x}_1$ both expressions involve the same $Y-$function but defined for different values of $\vec{y}_1$. Most importantly, the $X-$function is symmetric under permutation of the spectral parameters, Eq. (3.35), and it is this property that we want to impose on (4.26). Matching the second relation in (4.26) into (3.32) we choose the weight function as

$$Z_{\vec{p},\vec{x}_1,...,\vec{x}_{N-1}}(\vec{y}) = (a(s + ix_2, \bar{s} - i\bar{x}_2))^{N-1} \int dw \Lambda^{(1-s,1-s)}_{x_2,\bar{x}_2}(\vec{y}_2, ..., \vec{y}_N | \vec{w}_3, ..., \vec{w}_N)Z_{\vec{p},\vec{x}_3,...,\vec{x}_{N-1}}(\vec{w}).$$

(4.28)

Substituting this ansatz into (4.26) one finds that the dependence of $U_{\vec{p},\vec{z}}(\vec{z})$ on $\vec{x}_1$ and $\vec{x}_2$ is factorized into the following integral

$$\left[ \Lambda^{(s,\bar{s})}_{N-1}(\vec{x}_1)\Lambda^{(1-s,1-\bar{s})}_{N-2}(\vec{x}_2) \right] (\vec{z}_1, ..., \vec{z}_N | w_3, ..., \vec{w}_N) \equiv \int d^2y \Lambda^{(s,\bar{s})}_{x_1,\bar{x}_1}(\vec{z}_1, ..., \vec{z}_N | \vec{y}_2, ..., \vec{y}_N) \Lambda^{(1-s,1-\bar{s})}_{x_2,\bar{x}_2}(\vec{y}_2, ..., \vec{y}_N | \vec{w}_3, ..., \vec{w}_N),$$

(4.29)

where subscript in the l.h.s. refers to the number of right arguments of the $\Lambda-$function. We would like to stress that the number of left arguments of the function $\Lambda^{(s,\bar{s})}_{x_1,\bar{x}_1}$ is larger by 1 than the number of its right arguments. Therefore, in the convolution of two $\Lambda-$functions this difference increases to 2. Diagrammatical representation of (4.29) is shown in Fig. 12. The symmetry of (4.26) under permutation of $\vec{x}_1$ and $\vec{x}_2$ comes from the following remarkable property of (4.29)

$$\left[ \Lambda^{(s,\bar{s})}_{N-1}(\vec{x}_1)\Lambda^{(1-s,1-\bar{s})}_{N-2}(\vec{x}_2) \right] = \left( \frac{a(s + i\bar{x}_1, \bar{s} - i\vec{x}_1)}{a(s + i\vec{x}_2, \bar{s} - i\bar{x}_2)} \right)^{N-1} \left[ \Lambda^{(s,\bar{s})}_{N-1}(\vec{x}_2)\Lambda^{(1-s,1-\bar{s})}_{N-2}(\vec{x}_1) \right],$$

(4.30)

which can be considered as a generalization of the similar property of the $X-$function, Eq. (3.35). Replacing $(s, \bar{s}) \to (1-s, 1-\bar{s})$ in (4.30), one obtains another useful relation

$$\left[ \Lambda^{(1-s,1-\bar{s})}_{N-1}(\vec{x}_1)\Lambda^{(s,\bar{s})}_{N-2}(\vec{x}_2) \right] = \left( \frac{a(s + i\bar{x}_1, \bar{s} - i\vec{x}_1)}{a(s + i\vec{x}_2, \bar{s} - i\bar{x}_2)} \right)^{1-N} \left[ \Lambda^{(1-s,1-\bar{s})}_{N-1}(\vec{x}_2)\Lambda^{(s,\bar{s})}_{N-2}(\vec{x}_1) \right],$$

(4.31)

which is valid for arbitrary $N$, $\vec{x}_1$ and $\vec{x}_2$.

The proof of (4.30) and (4.31) goes along the same lines as those for the $X-$function, Eq. (3.35), and it is based on the permutation identities shown in Figs. 8 and 13. Namely, one inserts two auxiliary lines with the indices $\pm i(x_1 - x_2)$ into the leftmost rhombus in Fig. 12 and moves one of the lines to the right of the diagram, systematically applying the permutation identity, Fig. 4. In contrast with the previous case, the chain of rhombuses is not periodic and two auxiliary lines can not annihilate with each other and remain attached to the right- and leftmost rhombuses. Nevertheless, each of these lines disappears due to the identity shown in Fig. 13.

Applying the identities (4.30) and (4.31), it becomes straightforward to write the expression for the transition function $U_{\vec{p},\vec{z}}(\vec{z})$ which is consistent with (4.26) and is symmetric under permutation of any pair of the separated coordinates, Eq. (4.8),

$$U_{\vec{p},\vec{z}}(\vec{z}) = c_N(\vec{z}) (\vec{p}^2)^{(N-1)/2} \int d^2w_N e^{2\bar{p}\bar{w}_N} U_{\vec{z}}(\vec{z}; \vec{w}_N),$$

(4.32)
Figure 12: Diagrammatical representation of the convolution of two $\Lambda$–functions defined in Eqs. (4.29).

\[
\begin{align*}
\Lambda^{(s,\bar{s})(s',\bar{s}')}_{N-1}(\vec{x}_{N-1}) = & \Lambda^{(1-s,1-\bar{s})(s',\bar{s}')}_{N-2}(\vec{x}_{N-2}) \Lambda^{(s,\bar{s})(s',\bar{s}')}_{N-3}(\vec{x}_{N-3}) \cdots \Lambda^{(s,\bar{s})(s',\bar{s}')}_{1}(\vec{x}_1) \\
& (\vec{z}_1, ..., \vec{z}_N, |\vec{w}_N|)
\end{align*}
\] (4.33)

for even $N$, and

\[
\begin{align*}
\Lambda^{(s,\bar{s})(s',\bar{s}')}_{N-1}(\vec{x}_{N-1}) = & \Lambda^{(1-s,1-\bar{s})(s',\bar{s}')}_{N-2}(\vec{x}_{N-2}) \Lambda^{(s,\bar{s})(s',\bar{s}')}_{N-3}(\vec{x}_{N-3}) \cdots \Lambda^{(s,\bar{s})(s',\bar{s}')}_{1}(\vec{x}_1) \\
& (\vec{z}_1, ..., \vec{z}_N, |\vec{w}_N|)
\end{align*}
\] (4.34)

for odd $N$. Here, the convolution involves the product of $(N-1)$ functions $\Lambda^{(s,\bar{s})(s',\bar{s}')}_{N-k}(\vec{x}_k)$ with alternating spins $(s, \bar{s})$ and $(1-s, 1-\bar{s})$. The following comments are in order.

Since the $\Lambda$–functions are translation invariant, their convolution in (4.33) and (4.34) depends on the differences $\vec{z}_k - \vec{w}_N$ and, as a consequence, $(iS_- - p)U_{\vec{p},\vec{x}}(\vec{z}) = 0$ in agreement with (4.13). The additional factor $(\vec{p}^2)^{(N-1)/2}$ in (4.32) restores the scaling dimension of $U_{\vec{p},\vec{x}}$.

Throughout this section we have tacitly assumed that $\Lambda^{(s,\bar{s})(s',\bar{s}')}_{x_k,\bar{x}_k}(\vec{z}_1, ..., \vec{z}_N, |\vec{y}_2, ..., \vec{y}_N)$ is a well-defined function of the $\vec{z}$– and $\vec{y}$–vectors on the plane. As we have seen in Sect. 2.2.1, this condition constrains the possible values of the spectral parameters, Eq. (2.21). Repeating the same analysis for the function $\Lambda^{(s,\bar{s})(s',\bar{s}')}_{x_k,\bar{x}_k}$ defined in (3.18), we find that the separated coordinates have to satisfy the relation (2.21) with $u$ and $\bar{u}$ replaced by $x_k$ and $\bar{x}_k$, respectively. Together with (4.3) this leads to the following quantization conditions on the separated coordinates

\[
x_k = \nu_k - \frac{in_k}{2}, \quad \bar{x}_k = \nu_k + \frac{in_k}{2}
\] (4.35)

with $\nu_k$ real and $n_k$ integer. We would like to remind that it is for these values of the spectral parameters that the $R$–matrix is a unitary operator, Eq. (2.38) and (2.39).

\[\text{Since the separated variables are dimensionless (see Eq. (4.13)), it follows from (4.10) that the scaling dimension of } U_{\vec{p},\vec{x}} \text{ is equal to } (N-1).\]
Figure 14: Diagrammatic representation of the transition function, $U_{\vec{p}, \vec{x}}(\vec{z}_1, ..., \vec{z}_N)$. The momentum $\vec{p}$ flows into the diagram through the top vertex $\vec{w}_N$. The indices $\alpha_k$ and $\beta_k$ parameterize the corresponding two-dimensional propagators (see Eq. (A.1)) and are defined differently for even and odd $k$: $\alpha_{2k} = 1 - i x_{2k}$, $\beta_{2k} = 1 + i x_{2k}$ and $\alpha_{2k+1} = s - i x_{2k+1}$, $\beta_{2k+1} = s + i x_{2k+1}$.

Integration over the positions of $(N + 1)(N - 2)/2$ intermediate vertices is tacitly assumed.

The normalization factor $c_N(\vec{x})$ in (4.32) ensures the symmetry property (4.8) of the expression (4.32). It compensates the additional factors in the r.h.s. of (4.30) and (4.31) and is given for $N \geq 3$ by

$$c_N(\vec{x}) = \prod_{k=1}^{[(N-1)/2]} [a(s + i x_{2k}, \bar{s} - i \bar{x}_{2k})]^{N-k} \prod_{k=1}^{[N/2-1]} [a(s + i x_{2k+1}, \bar{s} - i \bar{x}_{2k+1})]^k$$

(4.36)

while $c_2(\vec{x}_1) = 1$. Using the definition of the $a-$function, Eq. (A.3), together with the relations, $\bar{s} = 1 - s^*$ and $\bar{x}_k = x_k^*$, one finds that the r.h.s. of (4.36) is a phase factor, $|c_N(\vec{x})| = 1$. Its explicit expression at $N = 3, 4$ looks as follows

$$c_3(\vec{x}_1, \vec{x}_2) = [a(s + i x_2, \bar{s} - i \bar{x}_2)]^2,$$

$$c_4(\vec{x}_1, \vec{x}_2, \vec{x}_3) = [a(s + i x_2, \bar{s} - i \bar{x}_2)]^3 [a(s + i x_3, \bar{s} - i \bar{x}_3)]^1.$$

(4.37)

The expressions (4.33) and (4.34) can be represented as the “Pyramide du Louvre” diagram shown in Fig. 14. This diagram consists of $(N - 1)$-rows, in accordance with the total number of $\Lambda-$functions in Eqs. (4.33) and (4.34). The $k$-th row of the pyramid is built from the lines carrying the indices $\alpha_k$ and $\beta_k$ that depend on the separated coordinate $\vec{x}_k$ and are defined in Fig. 14. It is attached to the $(k + 1)$-th row through $(N - k)$-vertices. At $N = 2$ the expression for the pyramid looks like

$$U_{\vec{p}, \vec{x}_1}(\vec{z}) = |\vec{p}| \int d^2 w_2 e^{2i \vec{p} \cdot \vec{w}_2} U_{\vec{x}_1}(\vec{z}; \vec{w}_2),$$

$$U_{\vec{x}_1}(\vec{z}; \vec{w}_2) = [z_1 - w_2]^{s-ix_1}[z_2 - w_2]^{s+ix_1}.$$

(4.38)
At \( N = 3 \) the corresponding expression for \( U_{\vec{x}_1,\vec{x}_2}(\vec{z};\vec{w}_3) \) is equal to the product of two (nonunique) star diagrams, which can be expressed in terms of \( _2F_1 \)-hypergeometric series. We do not present here its explicit form, since it is more convenient to our purposes to use the integral representation (4.32).

\[ \mu(\vec{x}) \]

**4.3. Integration measure**

The transition functions \( U_{\vec{p},\vec{x}}(\vec{z}) \) defined in (4.32) are the eigenstates of two mutually commuting hermitian operators, \( B(u) + \bar{B}(u^*) \) and \( i(B(u) - \bar{B}(u^*)) \), Eqs. (4.4) and (4.7). As such, they should be orthogonal to each other with respect to the \( SL(2, \mathbb{C}) \) scalar product (2.9). The general form of their orthogonality condition, consistent with the symmetry properties (4.8), is given by (4.10) and it involves the integration measure \( \mu(\vec{x}) \). To find \( \mu(\vec{x}) \) we substitute the transition function (4.32) into the orthogonality condition (4.10) and match the result of integration into the r.h.s. of (4.10).

Let us evaluate the scalar product (4.11) using the diagrammatical representation of \( U_{\vec{p},\vec{x}}(\vec{z}) \) shown in Fig. 14. The conjugated function \( (U_{\vec{p},\vec{x}}(\vec{z}))^* \) is represented by the same pyramid diagram, in which the exponents \( \alpha_k \) and \( \beta_k \) are replaced by the corresponding conjugated exponents

\[ \alpha_k \rightarrow \bar{\alpha}_k^* = 1 - \alpha_k, \quad \beta_k \rightarrow \bar{\beta}_k^* = 1 - \beta_k. \]

(4.39)

It becomes convenient to flip horizontally the conjugated pyramid diagram, so that the point \( \vec{w}_N \) will be located at the bottom of the diagram and the points \( \vec{z}_k \) at the top. The scalar product

\[ \langle \vec{x}, \vec{w}_N | \vec{x}', \vec{w}_N' \rangle = \int d^2 z U_{\vec{p},\vec{x}}(\vec{z};\vec{w}_N)(U_{\vec{p}',\vec{x}}'(\vec{z};\vec{w}_N'))^* \]

(4.40)

is obtained by sewing the pyramid and its conjugated counterpart through the points \( \vec{z}_1, ..., \vec{z}_N \). The resulting Feynman diagram has the form of a big rhombus built out of \( (N - 1)^2 \)-elementary rhombuses (see Fig. 15 below) with the top and bottom vertices located at the points \( \vec{w}_N \) and \( \vec{w}_N' \), respectively. Notice that the indices \( \alpha_k \) and \( \beta_k \) in the upper and lower part of this rhombus depend on the separated coordinates \( \vec{x} \) and \( \vec{x}' \), respectively.

Let us first calculate (4.40) at \( N = 2 \). Substituting (4.38) into (4.40) one obtains the rhombus diagram shown in Fig. 15. Integration over \( \vec{z}_1 \) and \( \vec{z}_2 \) can be easily performed using the chain

\[ \text{Figure 15: Diagrammatical representation of the scalar product of two pyramids at } N = 2. \]
the permutation identity shown in Fig. 9, until they meet and annihilate each other. In this way, which can be moved horizontally through the diagram in the direction of each other, thanks to the previous case, producing two vertical lines with the indices $z, \bar{z}$ attached to them. At the same time, only two lines are joined at the points $z, \bar{z}$, $\vec{w}^1, \vec{w}^2$, leading to the expression independent on $z, \bar{z}$ relation (A.6), leading to the expression independent on $z, \bar{z}$, $\vec{w}^1, \vec{w}^2$, $\langle \vec{x}_1, \vec{w}_2 | \vec{x}_1', \vec{w}_2' \rangle \sim |\vec{w}_2 - \vec{w}_2'|^0$. Its Fourier transformation with respect to $\vec{w}_2$ and $\vec{w}_2'$ leads to a singular expression. To identify the corresponding generalized function, one has to regularize the underlying two-dimensional Feynman integrals. The singularities appear due to the fact that the sum of four indices, corresponding to the sides of the rhombus, coincides with the space-dimension, $d_\varepsilon = 2$. This suggests to regularize the Feynman integrals by shifting the indices as

$$(\alpha_k, \bar{\alpha}_k) \rightarrow (\alpha_k - \varepsilon, \bar{\alpha}_k - \varepsilon), \quad (\beta_k, \bar{\beta}_k) \rightarrow (\beta_k - \varepsilon, \bar{\beta}_k - \varepsilon)$$

with $\varepsilon$ real positive. The sum of four indices inside the regularized rhombus becomes $2 - 4 \varepsilon$ and the Feynman integrals are well-defined. Their calculation is based on the chain relation (A.6) and it leads to

$$\langle \vec{x}_1, \vec{w}_2 | \vec{x}_1', \vec{w}_2' \rangle_\varepsilon = \pi^2 [w_2 - w_2']^{4\varepsilon} a(s - ix_1 - \varepsilon, 1 - s + ix_1' + \varepsilon, 1 + i(\vec{x}_1 - \vec{x}_1') + 2\varepsilon) \times a(s + ix_1 - \varepsilon, 1 - s - ix_1' + \varepsilon, 1 - i(\vec{x}_1 - \vec{x}_1') + 2\varepsilon),$$

where the subscript $\varepsilon$ in the l.h.s. indicates the regularization. Going over to the momentum representation and using (A.2), we calculate the Fourier transform of (4.42) and obtain the expression involving the factor $a(-4\varepsilon) = \Gamma(1 + 4\varepsilon)/\Gamma(-4\varepsilon)$ that vanishes as $\varepsilon \rightarrow 0$. To get a nonzero result, the smallness of $a(-4\varepsilon)$ should be compensated by one of the factors in (4.42). Examining (4.42) one finds that this happens at $\vec{x}_1 = \vec{x}_1'$, since $a(1 + 2\varepsilon) = \Gamma(-2\varepsilon)/\Gamma(1 + 2\varepsilon) \sim 1/\varepsilon$.\footnote{This has to do with the fact that (4.40) should be understood as a distribution, that is one has to integrate the r.h.s. of (4.40) with a smooth function of $\vec{x}$ and perform the integration over $\vec{x}$ afterwards.} Carefully examining the limit $\varepsilon \rightarrow 0$ one obtains

$$\lim_{\varepsilon \rightarrow 0} a(-4\varepsilon, 1 + i(\vec{x}_1 - \vec{x}_1') + 2\varepsilon, 1 - i(\vec{x}_1 - \vec{x}_1') + 2\varepsilon) = 2\pi \delta_{n_1, n_1'} \delta(\nu_1 - \nu_1') \equiv 2\pi \delta(x_1 - x_1'),$$

where $x_1 = \nu_1 - in_1/2$ and $x_1' = \nu_1' - in_1'/2$. Finally, combining together Eqs. (4.42) and (4.43), we obtain the orthogonality condition at $N = 2$

$$\langle \vec{x}_1', \vec{p}' | \vec{x}_1, \vec{p} \rangle = \lim_{\varepsilon \rightarrow 0} \frac{2\pi^2}{\bar{\rho}^2} \int d^2w_2 d^2w_2' e^{2i\bar{\rho} \vec{w}' - 2i\bar{\rho} \vec{w}} \langle \vec{x}_1, \vec{w}_2 | \vec{x}_1', \vec{w}_2' \rangle_\varepsilon = (2\pi)^2 \delta^{(2)}(p - p') \delta(x_1 - x_1') \frac{\pi^4}{2}.$$\footnote{There exist other possibilities to get the factor $\sim 1/\varepsilon$ by putting, for instance, $\vec{x}_1 = i(1 - \bar{s})$ for $x_1 \neq -is$, but the corresponding values of the separated coordinates do not satisfy the quantization condition (4.33).}

Its comparison with (1.11) yields the expression for the measure at $N = 2$

$$\mu(x_1) = 2/\pi^4.$$ \hspace{1cm} (4.45)

The calculation of the scalar product at higher $N$ goes along the same lines and is shown schematically in Fig. 10. Examining the rhombus diagram shown in Fig. 10a, one notices that all sewing vertices $z_2, ..., z_{N-1}$, except the left- and rightmost ones, $z_1$ and $z_N$, respectively, have four lines attached to them. At the same time, only two lines are joined at the points $z_1$ and $z_N$. The integration over these two points can be performed using the chain relation, similar to the previous case, producing two vertical lines with the indices $\pm i(x_1 - x_1')$ (see Fig. 10b), which can be moved horizontally through the diagram in the direction of each other, thanks to the permutation identity shown in Fig. 4, until they meet and annihilate each other. In this way, one arrives at the diagram shown in Fig. 10c. It differs from Fig. 10a in that two sewing vertices
Figure 16: Diagrammatic calculation of the scalar product of two pyramids. The fat points indicate the vertices that can be integrated out using the chain relation shown in Fig. [28].

$\vec{z}_1$ and $\vec{z}_N$ disappeared and the spectral parameters $\vec{x}_1$ and $\vec{x}_1'$ were interchanged. We notice that in this diagram there are already four vertices which can be integrated out using the chain relation (A.6). Further simplification amounts to repetition of the steps that we just described. Each next step reduces the number of vertices in the diagram and interchanges the spectral parameters $\vec{x}_k \leftrightarrow \vec{x}_j$. Continuing this procedure, one obtains the diagram shown in Fig. [28d], in which $(N - 1)$ elementary rhombuses are aligned along the vertical axis and are attached to their neighbors through the common vertices. Each of these rhombuses corresponds to the Feynman diagram in Fig. [13] defining the scalar product at $N = 2$, Eq. (4.42). The only difference with the previous case is that the separated variables parameterizing the $k$–th rhombus from the top are given by $\vec{x}_{N-k}$ and $\vec{x}_k'$ instead of $\vec{x}_1$ and $\vec{x}_1'$. We expect that the chain of rhombuses should produce the contribution $\sim \prod_{k=1}^{N} \delta(\vec{x}_{N-k} - \vec{x}_k')$. To calculate it carefully, one has to regularize the Feynman integrals according to (4.39). Applying (4.42), we replace each rhombus by a single line with the index $-4\varepsilon$ (see Fig. [28e]) and integrate over their $(N - 2)$–common vertices using the chain relation (A.6). Going over to the limit $\varepsilon \to 0$ and taking into account the identity (4.33), we combine together different factors coming from (4.36) and (4.42), perform the Fourier transformation (4.32) and finally obtain the following relation

$$
\lim_{\varepsilon \to 0} \langle \vec{x}, \vec{p} | \vec{x}', \vec{p}' \rangle = \frac{(2\pi)^N}{\prod_{j \neq k}} (p_j - p_j') \delta^{(2)}(x_1 - x_1') \delta^{(2)}(x_2 - x_2') \ldots \delta^{(2)}(x_{N-1} - x_{N-1}') 
$$

$$
\times \prod_{j \neq k}^{N-1} a(1 + i(x_k - x_j), 1 - i(x_k - x_j)) \ . \quad (4.46)
$$

In arriving at this relation we integrated out, using the chain relation (A.6), all vertices of the pyramid except the top and bottom vertices. Each integrated vertex brought the factor $\sim \pi a(1 + i(x_k - x_{N-j}))$ and their product appeared in the second line in (4.46).

Comparing (4.46) with the general expression for the scalar product, Eq. (4.10), we notice that it is not symmetric under permutations of the separated variables $\vec{x}_k \leftrightarrow \vec{x}_j$. The reason for this is that performing the calculation of (4.46) we have tacitly assumed that $\vec{x}_1$ is different from $\vec{x}_1', \vec{x}_2', \ldots, \vec{x}_{N-2}'$, thus eliminating all terms in (4.10) except the one entering (4.46). Indeed,
as we have seen in the case of the $N = 2$ scalar product, the limit $x_1 \to x'_1$ requires the regularization of the exponents \(\text{I.}41\). Although this regularization makes all Feynman integrals well defined, it transforms the unique Feynman diagrams (stars and triangles) into nonunique ones and, therefore, it does not allow to apply the permutation identity (see Fig. 1). Since in our calculation we applied the permutation identity to all pairs of the separated coordinates \((\vec{x}_1, \vec{x}_{N-k})\), $k = 2, ..., N - 1$, except \((\vec{x}_1, \vec{x}_{N-1})\) we have to assume that $|\vec{x}_1 - \vec{x}_{N-1}| \neq 0$.

Matching \(\text{I.46}\) into \(\text{I.10}\) we obtain the following expression for the integration measure

$$
\mu(\vec{x}) = \frac{2\pi^{-N^2}}{(N-1)!} \prod_{j,k=1}^{N-1} a(1 + i(x_k - x_j))^{-2} = \frac{2\pi^{-N^2}}{(N-1)!} \prod_{j,k=1}^{N-1} |\vec{x}_k - \vec{x}_j|^2,
$$

(4.47)

which is valid for $N \geq 3$. Here, $|\vec{x}_k - \vec{x}_j|^2 = (x_k - x_j)(\vec{x}_k - \vec{x}_j)$ and the relations \(\text{A.3}\) and \(\text{I.3}\) have been taking into account. Replacing $\vec{x}_k$ by their quantized values \(\text{I.35}\) one gets

$$
\mu(\vec{x}) = \frac{2\pi^{-N^2}}{(N-1)!} \prod_{j,k=1}^{N-1} \left[ (\nu_k - \nu_j)^2 + \frac{1}{4} (n_k - n_j)^2 \right].
$$

(4.48)

We verify that the obtained expression for the integration measure satisfies the relation \(\text{I.24}\).

Having determined the integration measure in the SoV representation, we can use \(\text{I.10}\) to write the completeness condition

$$
\int d^2p \int d\vec{x} \mu(\vec{x}) \left( U_{\vec{p},\vec{x}}(\vec{z}_1, ..., \vec{z}_N) \right)^* U_{\vec{p}',\vec{x}}(\vec{z}_1, ..., \vec{z}_N) = (2\pi)^N \delta^{(2)}(z - z'),
$$

(4.49)

where the integration over quantized values of the separated variables, $\int d\vec{x}$, implies the summation over integer $n_k$ and integration over continuous $\nu_k$ defined in \(\text{I.35}\)

$$
\int d\vec{x} = \prod_{k=1}^{N-1} \left( \sum_{n_k=-\infty}^{\infty} \int_{-\infty}^{\infty} d\nu_k \right).
$$

(4.50)

Since the eigenstates $\Psi_{\vec{p},(q,q')}(\vec{z})$ are orthogonal to each other with respect to the $SL(2, \mathbb{C})$ scalar product \(\text{I.9}\), the same should be true for the functions $\Phi_{(q,q')}(\vec{x}_1, ..., \vec{x}_{N-1})$ in the SoV representation. To obtain the corresponding orthogonality condition one substitutes \(\text{I.11}\) into \(\text{I.2}\) and uses Eq. \(\text{I.10}\). In this way one arrives at the Plancherel formula

$$
\langle \Psi_{\vec{p}',(q,q')} | \Psi_{\vec{p},(q',q')} \rangle = (2\pi)^N \delta(\vec{p} - \vec{p}') \langle \Phi_{(q,q')} | \Phi_{(q',q')} \rangle = (2\pi)^N \delta(\vec{p} - \vec{p'}) \delta(q - q'),
$$

(4.51)

where the notation was introduced for the norm of states in the SoV representation

$$
\langle \Phi_{(q,q')} | \Phi_{(q',q')} \rangle \equiv \int d\vec{x} \mu(\vec{x}) \left( \Phi_{(q,q')}(\vec{x}_1, ..., \vec{x}_{N-1}) \right)^* \Phi_{(q',q')}(\vec{x}_1, ..., \vec{x}_{N-1}) = \delta(q - q').
$$

(4.52)

Here, $\Phi_{(q,q')}(\vec{x}_1, ..., \vec{x}_{N-1})$ is a completely symmetric function of the separated variables $\vec{x}_k$, satisfying the multi-dimensional Baxter equations, Eq. \(\text{I.13}\), in the holomorphic and antiholomorphic sectors. As we will show in Sect. 4.4, the solution of these equations is factorized into the product of the eigenvalues of the $Q-$operators that have been studied in Sect. 3. Therefore, substitution of \(\text{I.4}\) into \(\text{I.52}\) yields the orthogonality condition for the solutions of the Baxter equation. Its explicit form depends on the normalization factor present in the r.h.s. of \(\text{I.4}\) and it will be established in the next Section.
4.4. Relation to the Baxter $Q$–operator

The construction of the SoV representation, performed in Sect. 4.2, is similar in many respects to that of the Baxter $Q$–operators in Sect. 3. In addition, the eigenfunctions in the SoV representation, $\Phi_{Q}^{u, \bar{u}}(\vec{x}_1, ..., \vec{x}_{N-1})$, and the eigenvalues of the Baxter $Q$–operators satisfy similar finite-difference Baxter equations. This suggests [27] that there should exist the relation between $Q_{\pm}(u, \bar{u})$ given by Eqs. (3.27) and (3.28). In this Section, we shall establish such relation (see Eq. (4.54) below) and use it to prove (4.4).

To start with, let us consider the product of $N - 1$ Baxter operators $Q_-(u, \bar{u})$

$$Q_{\vec{x}_1, ..., \vec{x}_{N-1}}(\vec{z} | \vec{w}) = [Q_-(x_1, \bar{x}_1) \ldots Q_-(x_{N-1}, \bar{x}_{N-1})](\vec{z} | \vec{w}) = \int d^2\vec{y} \ldots d^2\vec{y}' Q_{\vec{x}_1, \vec{y}_1}^{(-)}(\vec{z} | \vec{y}') \ldots Q_{\vec{x}_{N-1}, \vec{y}_{N-1}}^{(-)}(\vec{y} | \vec{w}), \quad (4.53)$$

where $\vec{y} = (\vec{y}_1, ..., \vec{y}_N)$ and the kernel $Q_{\vec{x}_1, \vec{y}_1}^{(-)}(\vec{z} | \vec{y})$ was defined in (3.28). Using the diagrammatical representation of the $Q_{\vec{x}_1, \vec{y}_1}^{(-)}(\vec{z} | \vec{y})$, the left diagram in Fig. 4 we associate (4.53) with the Feynman diagram, in which $(N - 1)$–periodic chains of the unique triangles (one for each $Q_-$) are glued together through their common vertices as shown in the left diagram in Fig. 14. Comparing this diagram with the one shown in Fig. 14, we notice their similarity – the lowest row defining the $\vec{z}$–dependence look alike, as well as their difference – the number of vertices is higher in the former diagram; the latter diagram does not have periodic boundary conditions along each row. In addition, we notice that $Q_{\vec{x}}(\vec{z} | \vec{w}_1, ..., \vec{w}_{N-1}, \vec{w}_N)$ depends on the $(N - 1)$–additional arbitrary vectors $\vec{w}_1, ..., \vec{w}_{N-1}$ as compared to $U_{\vec{x}}(\vec{z} | \vec{w}_N)$, defined in (3.28) and (4.34). As we will see in a moment, in order to obtain the transition function $U_{\vec{x}}(\vec{z} | \vec{w}_N)$ from (4.53), these vectors have to be chosen in a particular way – one has to put $\vec{w}_1 = \vec{w}_2 = ... = \vec{w}_{N-1}$ and take the limit $w_1 \to \infty$ afterwards

$$U_{\vec{p}}(\vec{z}) = c_p \lim_{w_1 \to \infty} [w_1]^{2(N-1)(1-s)} \int d^2w_N e^{ip\vec{w}N+ip\vec{w}_N} Q_{\vec{x}}(\vec{z} | \vec{w}_1, ..., \vec{w}_1, \vec{w}_N). \quad (4.54)$$

Here, $c_p$ is the normalization factor depending only on the momentum $\vec{p}$.

The proof of (1.54) is based on the reduction formulae obtained in Sect. 3.3.4 and it goes as follows. The dependence of $Q_{\vec{x}_1, ..., \vec{x}_{N-1}}(\vec{z} | \vec{w})$ on $\vec{w}_1, ..., \vec{w}_{N-1}$ is carried by the rightmost $Q$–kernel in (4.53), or equivalently by the $N$ unique triangles in the upper row in Fig. 17. At $\vec{w}_1 = \vec{w}_2 = ... = \vec{w}_{N-1}, (N - 2)$ of these triangles shrink into $\delta$–functions (see Eq. (A.9)), leading to significant simplification of the corresponding kernel. Indeed, systematically applying the reduction formula (3.58) one finds that

$$Q_{\vec{x}_{N-1}, \vec{y}_{N-1}}^{(-)}(\vec{y} | \vec{w}) \sim \delta(\vec{w}_1 - \vec{y}_1)\delta(\vec{w}_1 - \vec{y}_2)\ldots\delta(\vec{w}_1 - \vec{y}_{N-2}).$$

Substituting this expression into (4.53), one finds that the adjacent kernel $Q_{\vec{x}_{N-2}, \vec{y}_{N-2}}^{(-)}(\vec{y} | \vec{y})$ is evaluated at $\vec{y}_1 = \vec{y}_2 = ... = \vec{y}_{N-2}$ and the same reduction formulae can be applied again reducing the number of unique triangles in the row next to the upper one by $N - 3$. Repeating this procedure one finds that the substitution $\vec{w}_1 = \vec{w}_2 = ... = \vec{w}_{N-1}$ in the upper, $(N - 1)$–th row of the diagram, creates an avalanche of simplifications throughout the whole diagram, in which $(k - 1)$ unique triangles disappear in the $k$–th row ($k = 2, 3, ..., N - 1$) as shown in Fig. 17. The resulting diagram still depends on the arbitrary vector $\vec{w}_1$, which defines the position of the vertex of the right- and leftmost unique triangles in each row. In the limit $\vec{w}_1 \to \infty$, these triangles scale as a power of $[w_1]^{-2(1-s)}$. Combining together the factors coming from $(N - 1)$ rows of the diagram, one finds that the r.h.s. of (4.54)
Figure 17: Reduction of the product of \((N - 1)\) Q-operators, Eq. (4.53), into the pyramid diagram. The \(k\)-th row of triangles in the left diagram is given by Fig. 7 for \(u = x_k\). The triangles shown by the dash-dotted lines collapse into the \(\delta\)-functions at \(\vec{w}_1 = \vec{w}_2 = \ldots = \vec{w}_{N-1}\) due to the reduction formulae. The triangles shown by the dashed lines are replaced by a power of \([w_1]\) as \(\vec{w}_1 \to \infty\). The corresponding Feynman diagram has the form shown in the r.h.s. of Fig. 17. Still, it differs from the pyramid diagram, Fig. 14, by additional horizontal lines with the indices \((2s - 1)\).

To demonstrate the equivalence between two different diagrams, Figs. 17 and 14, we apply the same trick as was used in Sect. 3.2 in establishing the commutativity condition for the Baxter Q-operators (see Fig. 8). Namely, we start with the diagram in Fig. 14 and insert two horizontal lines with the opposite indices, \((2s - 1)\) and \((1 - 2s)\), into all rhombuses with the indices \((\alpha_1, \beta_1, \alpha_2, \beta_2)\), in the lower part of the diagram. Applying the permutation identity, Fig. 9, we move \((N - 2)\) auxiliary lines with the indices \((2s - 1)\) vertically to the upper part of the diagram until they either reach the boundary of the pyramid and disappear in virtue of the permutation identity, Fig. 13, or end up in the rhombus with the indices \((\alpha_{N-2}, \beta_{N-2}, \alpha_{N-1}, \beta_{N-1})\) just below the top vertex \(\vec{w}_N\). In the latter case, one gets rid of the auxiliary line by applying the star-triangle relation to the unique triangle with the indices \((\alpha_{N-1}, \beta_{N-1}, 2s - 1)\). Up to the additional line with the index \(2(1 - s)\) attached to the top of the pyramid, the resulting diagram looks like the original pyramid with one row less. The latter has been replaced by the lowest row of the diagram in Fig. 17. Applying the same transformations to the reduced pyramid with \((N - 2)\)-rows, we reconstruct one more layer of the diagram Fig. 17. We continue this procedure iteratively \(N - 3\) times until we arrive at the reduced pyramid containing a single rhombus. Inserting, as before, two lines with the indices \((1 - 2s)\) and \((2s - 1)\) into the rhombus, we get rid of the latter line by applying the uniqueness relation to the triangle containing the top of the pyramid. The resulting diagram coincides with the right diagram in Fig. 17, in which the chain of the additional lines with the index \(2(1 - s)\) is attached to its top vertex \(\vec{w}_N\). The total number of lines in this chain, \([((N - 1)/2]\), is equal to the number of rhombuses in the original pyramid, in which two opposite vertices are located on the vertical axis going through the top vertex \(w_N\). Going over to the momentum representation, the whole chain is reduced to a power of the momenta \(((-1)^{s\alpha}p^{1-2s}p^{1-2s})^{((N - 1)/2)}\) and one recover the relation (4.54). Carefully collecting all factors and comparing the expression for the r.h.s. of (4.54) with Eq. (4.32), we calculate the
normalization constant in (4.54) as
\[ c_{\vec{p}} = \left( (-1)^n p^{1-2s} \bar{p}^{1-2s} \right)^{(N-1)/2} \left[ \pi a(2(1-s)) \right]^{-(N-1)(N-2)/2}. \]

It comes about from applying the reduction formulæ (3.58) to \((N-1)(N-2)/2\) collapsing unique triangles in Fig. [17] and compensates the difference of the scaling dimensions in the both sides of the relation (4.54).

The relation (4.54) between \(U_{\vec{p},\vec{d}}(\vec{z})\) and the Baxter \(Q\)-operator can be rewritten in the operator form by introducing the special state \(|\omega_{\vec{z}_0,\vec{p}}\rangle\), which depends on two arbitrary vectors, \(\vec{z}_0\) and \(\vec{p}\), and is defined in the coordinate representation as follows [14]
\[ \langle \vec{z}|\omega_{\vec{z}_0,\vec{p}} \rangle = c_{\vec{p}} [z_0]^{2(N-1)(1-s)} e^{ip\vec{z}_N+i\vec{p}\vec{z}_N} \prod_{k=1}^{N-1} \delta(\vec{z}_k - \vec{z}_0). \]

Using this state, one finds from (4.54)
\[ U_{\vec{p},\vec{d}}(\vec{z}) = \lim_{\vec{z}_0 \to \infty} \langle \vec{z}|Q_{-}(x_1, \vec{x}_1) \ldots Q_{-}(x_{N-1}, \vec{x}_{N-1}) | \omega_{\vec{z}_0,\vec{p}} \rangle. \]

Many properties of the SoV representation follow from this relation. In particular, the symmetry property (4.8) comes from the commutativity of the \(Q\)-operators. Substituting (4.57) into (4.3), we take into account that the eigenstates \(\Psi_{\vec{p},\{q,\bar{q}\}}(\vec{z})\) diagonalize the operator \(Q_{-}(u, \bar{u})\)
\[ \Phi_{\{q,\bar{q}\}}(\vec{d}) \delta(\vec{p} - \vec{p}') = \langle \Psi_{\vec{p},\{q,\bar{q}\}} | \{Q_{-}(x_1, \vec{x}_1) \ldots Q_{-}(x_{N-1}, \vec{x}_{N-1}) | \omega_{\vec{z}_0=\infty,\vec{p}'},\rangle = c_{\Phi} \delta^{(2)}(p - p') \prod_{k=1}^{N-1} Q_{q,\bar{q}}(x_k, \vec{x}_k), \]

where the normalization factor is defined as \(c_{\Phi} = \int d^2p \langle \Psi_{\vec{p},\{q,\bar{q}\}} | \omega_{\vec{z}_0=\infty,\vec{p}'} \rangle\) and \(Q_{q,\bar{q}}^{(c)}(u, \bar{u})\) stands for the eigenvalue of the operator \(Q_{-}(u, \bar{u})\). This relation replaces the ansatz (4.4) and it provides the expression for \(\Phi_{\{q,\bar{q}\}}(\vec{d})\) that automatically satisfies the multi-dimensional Baxter equation (4.19). Applying the conjugation relations (2.40), one expresses the wave function in the separated coordinates, \(\Phi_{\{q,\bar{q}\}}(\vec{d})\) in terms of the eigenvalues \(Q_{q,\bar{q}} \equiv Q_{q,\bar{q}}^{(c)}(u, \bar{u})\) (up to irrelevant normalization factor \(c_{\Phi}a(1-2s)/\pi)^{N(N-1)}\) as
\[ \Phi_{\{q,\bar{q}\}}(\vec{d}) = \prod_{k=1}^{N-1} \left[ a(s + ix_k, \bar{s} - i\bar{x}_k) \right]^N (Q_{q,\bar{q}}(x_k, \vec{x}_k))^*. \]

We recall that the possible values of the separated coordinates, \(x_k = \nu_k - in_k/2\) and \(\bar{x}_k = x_k^\ast\), are parameterized by integer \(n_k\) and real \(\nu_k\), Eq. (4.35). The explicit form of \(\Phi_{\{q,\bar{q}\}}(\vec{d})\) at \(N = 2\) can be found from (3.63).

The properties of \(\Phi_{\{q,\bar{q}\}}(\vec{d})\) can be established using the results for the functions \(Q_{q,\bar{q}}(u, \bar{u})\) obtained in Sect. 3. The analytical properties of \(\Phi_{\{q,\bar{q}\}}(\vec{d})\) on the complex \(\nu_k\)-plane follow from Eq. (1.80) and Fig. [1]. In particular, \(\Phi_{\{q,\bar{q}\}}(\vec{d})\) does have any singularities on the real

\[ 17 \text{Notice that this state is not normalizable with respect to the SL}(2, \mathbb{C}) \text{ scalar product (2.8) and, therefore, it does not belong to the quantum space of the system. Keeping this in mind, we shall use the same scalar product notation for the convolution of the wave function } \omega_{\vec{z}_0,\vec{p}}(\vec{z}) \text{ with an arbitrary test function, } \langle \Psi|\omega_{\vec{z}_0,\vec{p}}(\vec{z}) \rangle = \int d^2z \langle \Psi(\vec{z}) |^* \omega_{\vec{z}_0,\vec{p}}(\vec{z}), \text{ assuming that the integral is convergent.} \]
\( \nu_k - \)axis and, applying (3.102), one finds its the asymptotic behavior at large \( \nu_k \) and fixed \( n_k \) as 
\[
\Phi\{q,\bar{q}\}(\vec{x}) = O(\nu_k^{-N(1+2i\nu_s)}),
\]
leading to \( \mu(\vec{x})|\Phi\{q,\bar{q}\}(\vec{x})|^2 \sim 1/\nu_k^2 \). These properties are in accord with the fact that \( \Phi\{q,\bar{q}\}(\vec{x}) \) should be normalizable with respect to the scalar product (4.52).

Substituting (4.59) into (4.1), we obtain the integral representation of the eigenstates of the system, \( \Psi_{\vec{p},\{q,\bar{q}\}}(\vec{z}) \). As shown in the Appendix C, this representation can be rewritten, thanks to remarkable properties of the transition function \( U_{\vec{p},\vec{x}}(\vec{z}) \), in a concise form

\[
\Psi_{\vec{p},\{q,\bar{q}\}}(\vec{z}) = N^{-1} \prod_{k=1}^{N-1} Q_{q,\bar{q}}(\hat{x}_k, \hat{\bar{x}}_k) \Omega_{\vec{p}}(\vec{z}), \tag{4.60}
\]

where

\[
\Omega_{\vec{p}}(\vec{z}) = \int d^2 z_0 e^{2i\vec{p} \cdot \vec{z}_0} \prod_{k=1}^{N} (z_k - z_0)^{-2s}(\bar{z}_k - \bar{z}_0)^{-2\bar{s}} \tag{4.61}
\]
is the so-called pseudovacuum state and \( Q_{q,\bar{q}}(\hat{x}_k, \hat{\bar{x}}_k) \) stands for the eigenvalue of the Baxter operator \( Q_{q,\bar{q}}(u, \bar{u}) \) with the spectral parameters substituted by the operator zeros. The representation (4.60) is well-known in the theory of the \( SL(2, \mathbb{R}) \) spin magnets. There, the relevant solutions to the Baxter equation are given by polynomials in the spectral parameters and (4.60) becomes equivalent to the highest-weight representation for the eigenstates in the ABA approach.

Eq. (4.60) provides a generalization of the above representation to the noncompact \( SL(2, \mathbb{C}) \) magnets, for which the ABA is not applicable and the Baxter equations have nonpolynomial solutions for \( Q_{q,\bar{q}}(u, \bar{u}) \).

As a byproduct of (4.59), we substitute the relation (4.59) into the scalar product (4.52) and obtain the orthogonality condition on the space of solutions to the Baxter equation

\[
\int d\vec{x} \, \mu(\vec{x}) \prod_{k=1}^{N-1} Q_{q',\bar{q}'}(x_k, \bar{x}_k) (Q_{q,\bar{q}}(x_k, \bar{x}_k))^* = \delta(q - q'), \tag{4.62}
\]

where the integration measure \( \int d\vec{x} \, \mu(\vec{x}) \) was defined in (4.47) and (4.50).

5. Conclusions

In this paper, we have studied completely integrable quantum mechanical model of \( N \) interacting spinning particles in two-dimensional space that has previously appeared in high-energy QCD as describing the properties of the \( N \)–gluon compound states in the multi-color limit. Applying the quantum inverse scattering method, we identified this model as a generalization of one-dimensional homogenous XXX Heisenberg spin magnet to infinite-dimensional principal series representation of the \( SL(2, \mathbb{C}) \) group.

The model has a number of properties in common with two-dimensional conformal field theories \[28\]. Introducing holomorphic and antiholomorphic coordinates on a two-dimensional plane one finds that the the kernels of the \( R \)–matrix and the Baxter \( Q \)–operator are factorized into the product of two functions depending separately on the holomorphic and antiholomorphic coordinates, while the Hamiltonian is given by the sum of two commuting operators acting in the different sectors. In spite of this, the Hamiltonian dynamics of the model can not be reduced to independent dynamics in the holomorphic and antiholomorphic sectors. The interaction between
two sectors is ensured by the condition for the kernels of the operators and their eigenstates to be well-defined functions on a two-dimensional plane. This condition leads to the additional constraints on the properties of the model. In particular, it leads to the quantization of the integrals of motion of the system, the separated coordinates as well as the spectral parameters of the \( R \)-matrix and the Baxter \( Q \)-operator. Using the global \( SL(2, \mathbb{C}) \) invariance of the model, one can represent the eigenstates of the \( N \)-particle Hamiltonian, \( \Psi_{q,\bar{q}}(\vec{z}_1 - \vec{z}_0, \vec{z}_2 - \vec{z}_0, \ldots, \vec{z}_N - \vec{z}_0) \), as correlators of \( N + 1 \) quasi-primary fields in a two-dimensional conformal field theory. The latter are given by the sum of products of the holomorphic and antiholomorphic conformal blocks. This structure is general enough and, as was shown in Sect. 3.3.5, it also holds for the eigenvalues of the Baxter \( Q \)-operator. This, in turn, suggests that the Baxter \( Q \)-operators constructed in this paper should be related to similar operators defined in the conformal field theory \[29\].

We have argued in this paper that all integrability properties of the model are encoded in the properties of the \( Q \)-operator, satisfying the Baxter relations. Namely, the Hamiltonian as well as transfer matrices are expressed in terms of a single operator \( Q(u, \bar{u}) \) and, therefore, their spectrum is determined by the eigenvalues of the \( Q \)-operator. In addition, going over through Sklyanin’s construction of the Separated Variables we have demonstrated that the unitary transformation to these variables is also determined by the same operator. This leads to the integral representation for the eigenstates of the model with the wave function in the separated coordinates given by the product of the eigenvalues of the \( Q \)-operator.

Thus, the spectral problem for the noncompact \( SL(2, \mathbb{C}) \) spin magnet turns out to be equivalent to the problem of finding the eigenvalues of the Baxter \( Q \)-operator. The latter are defined as solutions to the Baxter equations, (3.3) and (3.4), supplemented by additional conditions on their analytical properties and asymptotic behavior at infinity that have been established in Sects. 3.5 and 3.6, respectively. The solution satisfying these conditions was found in the simplest case of the system of \( N = 2 \) particles. Constructing solutions to the Baxter equation for \( N \geq 3 \), we find that in order to satisfy the above conditions the values of the integrals of motion have to be constrained. The explicit form of the corresponding quantization conditions is known only for the polynomial solutions of the Baxter equation \[21, 30, 22\]. Their generalization to arbitrary, nonpolynomial solutions will be presented elsewhere \[24\].

**Acknowledgements**

This work was supported by the grant of Spanish Ministry of Science (A.M.) and by the grant 00-01-005-00 of the Russian Foundation for Fundamental Research (A.M. and S.D.).

**Note added**

When this paper was written, we learned about the recent work by H. J. de Vega and L.N. Lipatov, in which similar results were obtained – the unitary transformation to the separated variables was constructed, using different approach, for the \( SL(2, \mathbb{C}) \) magnet of spin \((s = 1, \bar{s} = 0)\) with \( N = 2 \) and \( N = 3 \) sites; the explicit solution to the Baxter equation was found at \( N = 2 \) and its generalization to higher \( N \) was discussed.
Appendix: Uniqueness relations

Analyzing the noncompact spin magnet we encounter various two-dimensional integrals over the positions of particles which we represent in terms of the Feynman diagrams. Such interpretation turns out to be very useful as it allows to apply powerful methods of calculation the Feynman integrals well-known in perturbation QCD. More precisely, our calculations are heavily based on the “method of uniqueness” [13, 16]. This method has been originally developed for the calculation of \( d \)-dimensional Feynman integral and we shall use its simplified version at \( d = 2 \). In this Appendix we collect all necessary formulae - the chain and the star-triangle relations, which form the basis of the method of uniqueness. Remarkable feature of the method is that subsequently applying these two relations one is able to obtain analytical expressions for complicated Feynman integrals without doing any integration.

Throughout the paper we use the following compact notation for the “propagator”

\[
\frac{1}{[z-w]^\alpha} = \frac{1}{(z-w)^\alpha (\bar{z} - \bar{w})^\bar{\alpha}} = \frac{(z-w)^{\alpha - \bar{\alpha}}}{|z-w|^{2\alpha}} = \frac{(-1)^{-\bar{\alpha}}}{|w-z|^\alpha},
\]

with \( \alpha - \bar{\alpha} = n \), and represent it graphically by the arrow directed from \( w \) to \( z \) and with the index \( \alpha \) attached to it as shown in Fig. 1. Going over to the Fourier transformation we define the propagator in the momentum representation

\[
\int d^2w e^{i(q \bar{w} + \bar{q} w)} [w]^\alpha = \frac{\pi}{\alpha - \bar{\alpha}} a(\alpha) \frac{1}{q^{1-\alpha} \bar{q}^{1-\bar{\alpha}}}. \tag{A.2}
\]

Here, the notation was introduced for the function

\[
a(\alpha) = \frac{\Gamma(1-\bar{\alpha})}{\Gamma(\alpha)}, \quad a(\bar{\alpha}) = \frac{\Gamma(1-\alpha)}{\Gamma(\bar{\alpha})}, \quad a(\alpha, \beta, ...) = a(\alpha) a(\beta) ... . \tag{A.3}
\]

It has the following properties

\[
a(\alpha) a(1-\bar{\alpha}) = 1, \quad \frac{a(1+\alpha)}{a(\alpha)} = -\frac{1}{\alpha \bar{\alpha}}, \quad a(\alpha) = (-1)^{\alpha - \bar{\alpha}} a(\bar{\alpha}) = \frac{(-1)^{\alpha - \bar{\alpha}}}{a(1-\alpha)}, \tag{A.4}
\]

where in the last relation \( \alpha - \bar{\alpha} \) has to be integer. Taking the limit of (A.2) for \( \alpha = \bar{\alpha} = i\varepsilon \) as \( \varepsilon \to 0 \) we obtain the following convenient representation for the delta function

\[
\delta^{(2)}(w) = \lim_{\varepsilon \to 0} \frac{a(i\varepsilon) \frac{1}{w}^{1-\varepsilon}}{\pi}. \tag{A.5}
\]

The method of uniqueness is based on the following two relations shown diagrammatically in Fig. 18 and 19

- Chain relation:

\[
\int d^2w \frac{1}{[x-w]^{\beta} [w-z]^{\bar{\gamma}}} = \pi (-1)^{\gamma-\bar{\gamma}} a(\alpha, \beta, \gamma) \frac{1}{[x-z]^{\alpha+\beta-1}}, \tag{A.6}
\]

where \( \gamma = 2 - \alpha - \beta \) and \( \bar{\gamma} = 2 - \bar{\alpha} - \bar{\beta} \). At \( \alpha + \beta = \bar{\alpha} + \bar{\beta} = 2 \) this relation can be further simplified using (A.5) as

\[
\int d^2w \frac{1}{[x-w]^{2-\alpha} [w-z]^{\bar{\alpha}}} = \pi^2 a(\alpha, 2-\alpha) \delta^{(2)}(x-z). \tag{A.7}
\]
• Star-triangle relation:

\[
\int d^2 w \frac{1}{[z-w]^\alpha [x-w]^\beta [y-w]^\gamma} = \frac{\pi a(\alpha, \beta, \gamma)}{[x-z]^{1-\gamma} [z-y]^{1-\beta} [y-x]^{1-\alpha}}
\]  

(A.8)

with \( \alpha + \beta + \gamma = \bar{\alpha} + \bar{\beta} + \bar{\gamma} = 2 \).

Figure 18: The chain relation.

\[\begin{array}{ccc}
\alpha & \beta & \alpha + \beta - 1 \\
\gamma & 1 - \beta & 1 - \gamma \\
1 - \alpha & \bar{\alpha} + \bar{\beta} + \bar{\gamma} = 2
\end{array}\]

Figure 19: The uniqueness star-triangle relation.

The l.h.s. of (A.8) is described by the “star” diagram in which three propagators are glued together at the same point \( \vec{w} \). The r.h.s. of (A.8) correspond to the “triangle” diagram. We shall call the star and triangle diagrams unique if, firstly, the sums of the holomorphic indices attached to three propagators in these diagrams coincides with the sum of the antiholomorphic indices and, secondly, it is equal to 2 and 1, respectively. Then, the star-triangle relation allows to replace the unique star diagram by unique triangle and vice versa as shown in Fig. 19.

In the limit \( \vec{y} \to \vec{x} \), when the end-points of the unique star diagram approach each other, or equivalently the line in the unique triangle diagram with the index \( 1 - \alpha \) shrinks into a point, one finds from (A.7)

\[
\int d^2 w \frac{1}{[z-w]^\alpha [x-w]^\beta [y-w]^\gamma} \xrightarrow{\vec{y} \to \vec{x}} \pi^2 a(\alpha, 2 - \bar{\alpha}) \delta^{(2)}(z - x) .
\]  

(A.9)

To prove (A.6) it is sufficient to take the Fourier transform of the both sides of the relation w.r.t. \( \vec{x} \) and apply (A.2). The relation (A.8) follows from (A.6). To see this one replaces the coordinates in (A.6) as \( w \to 1/(w - y), x \to 1/(x - y), z \to 1/(z - y) \) and similarly for the antiholomorphic coordinates. After a simple calculation one reproduces Eq. (A.8).

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Appendix: Relation between \(Q\)--operator and transfer matrices

It is well-known in the theory of compact spin magnets that the transfer matrices of high spin are related to those of the lowest spin through the fusion relations [15, 31]. Moreover, solving the hierarchy of the fusion relations one can express the transfer matrices of arbitrary spin in terms of the Baxter \(Q\)--operator [29, 32, 22]. In this Appendix, we shall generalize these relations to the noncompact \(SL(2, \mathbb{C})\) spin chains.

Our starting point is the relation between the transfer matrices and the product of \(Q\)--operator, Eq. (3.66). Using this relation we can establish some general properties of the transfer matrices.

Since the \(SL(2, \mathbb{C})\) representations of the spins \((s, \bar{s})\) and \((1 - s, 1 - \bar{s})\) are unitary equivalent, we expect that the corresponding transfer matrices should be related to each other. Indeed, one finds using (3.66) together with (3.67) that the transfer matrices \(T^{(s_0, \bar{s}_0)}(u, \bar{u})\) and \(T^{(1 - s_0, 1 - \bar{s}_0)}(u, \bar{u})\) differ by a \(c\)--valued coefficient function

\[
\frac{T^{(s_0, \bar{s}_0)}(u, \bar{u})}{T^{(1 - s_0, 1 - \bar{s}_0)}(u, \bar{u})} = \frac{\rho_T^{(s_0, \bar{s}_0)}(u, \bar{u})}{\rho_T^{(1 - s_0, 1 - \bar{s}_0)}(u, \bar{u})} = \left[\frac{a(s - 1 + s_0 + iu, \bar{s} - \bar{s}_0 - i\bar{u})}{a(s - s_0 + iu, \bar{s} - 1 + \bar{s}_0 - i\bar{u})}\right]^N.
\]

Here, we should not worry about ordering of the transfer matrices due their commutativity.

Making use of (3.49), we obtain the expression for the transfer matrix (3.66) entirely in terms of the \(Q_+\)--operator in two different forms

\[
T^{(s_0, \bar{s}_0)}(u, \bar{u}) = \rho_Q^{(s_0, \bar{s}_0)}(u, \bar{u}) \times [Q_+(\bar{u}^* - is_0, u^* - i\bar{s}_0)]^\dagger Q_+(u + is_0, \bar{u} + i\bar{s}_0) \quad (B.2)
\]

\[
\rho_Q^{(s_0, \bar{s}_0)}(u, \bar{u}) = [Q_+(\bar{u}^* - i(1 - s_0), u^* - i(1 - \bar{s}_0))]^\dagger Q_+(u + i(1 - s_0), \bar{u} + i(1 - \bar{s}_0))
\]

Here, the normalization factors are given by

\[
\rho_Q^{(s_0, \bar{s}_0)}(u, \bar{u}) = \left[(-1)^{2i(u-u)} \frac{n_s^2 + 4\nu^2}{\pi^4} a(s - 1 + s_0 + iu, \bar{s} - \bar{s}_0 - i\bar{u})\right]^N.
\]

\[
\rho_Q^{(s_0, \bar{s}_0)}(u, \bar{u}) = \left[(-1)^{2i(u-u)} \frac{n_s^2 + 4\nu^2}{\pi^4} a(s_0 - s + iu, \bar{s} - \bar{s}_0 - i\bar{u})\right]^N.
\]

It is straightforward to verify that (B.2) satisfies the intertwining relation (B.1) and the normalization factors \(\rho_T^{(s_0, \bar{s}_0)}, \rho_Q^{(s_0, \bar{s}_0)}\) and \(\rho_T^{(s_0, \bar{s}_0)}, \rho_Q^{(s_0, \bar{s}_0)}\) are related to each other as

\[
\rho_Q^{(s_0, \bar{s}_0)}(u, \bar{u}) = \rho_T^{(1 - s_0, 1 - \bar{s}_0)}(u, \bar{u}) \div \rho_T^{(s_0, \bar{s}_0)}(u, \bar{u}) / \rho_T^{(1 - s_0, 1 - \bar{s}_0)}(u, \bar{u}).
\]

Using (B.2) we can find the relation between the transfer matrix and its conjugate counterpart. Conjugating the both sides of the first relation in (B.2) we can either match it into itself using the relations (3.49), or into the second relation in (B.2). In this way, we obtain two equivalent representations

\[
[T^{(s_0, \bar{s}_0)}(u, \bar{u})]^\dagger = M T^{(s_0, \bar{s}_0)}(-\bar{u}^*, -u^*) M \times \frac{(\rho_Q^{(s_0, \bar{s}_0)}(u, \bar{u}))^*}{\rho_Q^{(s_0, \bar{s}_0)}(-\bar{u}^*, -u^*)} \quad (B.5)
\]

\[
T^{(s_0, \bar{s}_0)}(\bar{u}^* + i, u^* + i) \times \frac{(\rho_Q^{(s_0, \bar{s}_0)}(u, \bar{u}))^*}{\rho_Q^{(s_0, \bar{s}_0)}(\bar{u}^* + i, u^* + i)}. \quad (B.6)
\]
Finally, let us show that the transfer matrices of different spin satisfy nonlinear fusion relations. To this end, one considers the product of two transfer matrices of different spins as well as different values of the spectral parameters and expresses it in terms of the $Q-$operators using (B.2). The resulting expressions involve four $Q-$operators including two conjugated operators. Using their commutativity, we may regroup them into two pairs of $Q \dagger Q-$operators and apply the relations (B.2) to convert each pair into the transfer matrix. In this way, we arrive at the following fusion relation

\[
\hat{T}^{(s_0+i\bar{\sigma},\bar{s}_0-i\bar{\sigma})}(u+i\bar{\sigma},\bar{u}+\bar{\sigma}) \hat{T}^{(s_0+i\bar{\sigma},\bar{s}_0-i\bar{\sigma})}(u-i\bar{\sigma},\bar{u}+\bar{\sigma}) = \hat{T}^{(s_0-i\sigma,\bar{s}_0+i\sigma)}(u+i\sigma,\bar{u}+\bar{\sigma}) \hat{T}^{(s_0-i\sigma,\bar{s}_0+i\sigma)}(u-i\sigma,\bar{u}+\bar{\sigma}),
\]

where $\hat{T}^{(s_0)}(u) = T^{(s_0,\bar{s}_0)}(u,\bar{u})/\rho^{(s_0,\bar{s}_0)}(u,\bar{u})$. Another fusion relation follows from the Baxter equation (B.3). Multiplying its both sides by conjugated $Q-$operator and applying (B.2) we find

\[
t_N(u) \hat{T}^{(s_0,\bar{s}_0)}(u,\bar{u}) = (u+i\sigma)^N \hat{T}^{(s_0+i\frac{1}{2},\bar{s}_0)}(u-i\frac{1}{2},\bar{u}) + (u-i\sigma)^N \hat{T}^{(s_0-i\frac{1}{2},\bar{s}_0)}(u+i\frac{1}{2},\bar{u}),
\]

and similar relation in the antiholomorphic sector

\[
t_N(\bar{u}) \hat{T}^{(s_0,\bar{s}_0)}(u,\bar{u}) = (\bar{u}+i\bar{\sigma})^N \hat{T}^{(s_0,\bar{s}_0+i\frac{1}{2})}(\bar{u}+i\frac{1}{2},u) + (\bar{u}-i\bar{\sigma})^N \hat{T}^{(s_0,\bar{s}_0-i\frac{1}{2})}(\bar{u}+i\frac{1}{2},u).
\]

There is the following important difference between these relations and (B.7). According to the definition of the transfer matrix $T^{(s_0,\bar{s}_0)}(u,\bar{u})$, Eq. (2.40), the spins $(s_0,\bar{s}_0)$ parameterize the auxiliary space $V^{(s_0,\bar{s}_0)}$ and for the $SL(2, C)$ representation of the principal series they satisfy the conditions (2.12). We notice that for the transfer matrices in the r.h.s. of (B.8) and (B.9) these conditions are not satisfied and, therefore, they do not belong to the same family of the transfer matrices of the $SL(2, C)$ principal series. At the same time, appropriately choosing the parameters $(\sigma, \bar{\sigma})$ and $(\bar{\sigma}, \bar{\sigma})$, one finds from (B.4) the fusion relations between the transfer matrices within the same family.

C Appendix: Bethe Ansatz representation of the eigenstates

In this appendix we prove the relation (4.60) which establishes the correspondence between (4.1) and the representation of the eigenstates in the Algebraic Bethe Ansatzz [1, 2, 3]. The ABA representation is based on the existence of the special pseudovacuum state $\Omega(\bar{z})$, which is annihilated by the spin operators of all particles (2.2)

\[
\Omega(\bar{z}) = \prod_{k=1}^{N} z_k^{-2s} \bar{z}_k^{-2\bar{s}}, \quad S_+^{(k)} \Omega(\bar{z}) = S_-^{(k)} \Omega(\bar{z}) = 0.
\]

The remarkable feature of this state is [1, 2, 3] that it brings the Lax operators (2.13) to a upper-triangle form and, as a consequence, diagonalizes the auxiliary transfer matrices (2.41)

\[
t_N(u) \Omega(\bar{z} - \bar{z}_0) = \left[(u+is)^N + (u-is)^N\right] \Omega(\bar{z} - \bar{z}_0)
\]
with $\vec{z}_0$ being arbitrary vector reflecting the $SL(2, \mathbb{C})$ invariance of $t_N(u)$. The operator $\hat{t}_N(\vec{u})$ satisfies similar relation. Then, the complete integrability of the model implies that $\Omega(\vec{z} - \vec{z}_0)$ diagonalizes the Baxter $Q-$operators and, as a consequence, the Hamiltonian of the model. This does not mean, however, that the pseudovacuum state can be identified as one of the eigenstates of the model. Examining its $SL(2, \mathbb{C})$ transformation properties, Eqs. (2.10) and (2.4), one verifies that $\Omega(\vec{z} - \vec{z}_0)$ does not belong to the quantum space of the system (see footnote to Eq. (4.50)). Nevertheless, one can use its properties to establish different relations including Eq. (4.60).

Going over to the momentum representation, we define the state

$$\Omega_{\vec{p}}(\vec{z}_1, ..., \vec{z}_N) = c^\Omega_{\vec{p}} \int d^2 z_0 e^{2ip\vec{z}_0} \Omega(\vec{z} - \vec{z}_0)$$

with the normalization factor $c^\Omega_{\vec{p}} = (-1)^{s_1} p^{2s-1} \rho^{2s-1}/(\pi^3 a(2(1-s)))$ chosen for the later convenience. Let us examine its convolution with the transition function defined in (4.57)

$$\int d^2 z (\Omega_{\vec{p}}(\vec{z}))^* U_{\vec{p}'}(\vec{x})(\vec{z}) = \lim_{\vec{z}_0 \to \infty} \langle \Omega_{\vec{p}} | Q_-(x_1, \bar{x}_1) ... Q_-(x_{N-1}, \bar{x}_{N-1}) | \omega_{\vec{z}_0, \vec{p}'} \rangle$$

with the state $\omega_{\vec{z}_0, \vec{p}'}(\vec{z})$ defined in (4.50). Then, substituting (C.3) into (C.4) and calculating the convolution $\int d^2 z (\Omega(\vec{z} - \vec{z}_0))^* U_{\vec{p},\vec{x}}(\vec{z})$, we represent the kernel of the $Q-$operator by the right diagram in Fig. 4. The integral over the intermediate points $\vec{z}_k$ takes the form (A.7) and it gives rise to the $\delta-$functions which put the centers of the star diagrams at the same point $\vec{z}_0$. The resulting diagram is given by the product of propagators connecting $\vec{z}_0$ with the points $\vec{z}_k$ and it coincides, up to prefactor, with $\Omega_{\vec{p}}$. In this way, one gets

$$\langle \Omega_{\vec{p}} | Q_-(u, \bar{u}) = [\pi a(2 - 2s, s + iu, \bar{s} - i\bar{u})]^N \langle \Omega_{\vec{p}} | \equiv A(u, \bar{u}) \langle \Omega_{\vec{p}} | .$$

As it was expected, the pseudovacuum state diagonalizes the Baxter operator. One can verify that its eigenvalue satisfies the Baxter equation (4.3) with $t_N(u)$ replaced by (4.2).

Substituting (C.5) into (C.4) one replaces the $Q-$operators by their corresponding eigenvalues and reduces the r.h.s. of (C.4) to the convolution of the pseudovacuum state with the function $\omega_{\vec{z}_0, \vec{p}'}$ defined in (4.50). A simple calculation leads to

$$\lim_{\vec{z}_0 \to \infty} \langle \Omega_{\vec{p}} | \omega_{\vec{z}_0, \vec{p}'} = c_{\vec{p}'} \delta^{(2)}(p - p') .$$

Here, the normalization factor $c_{\vec{p}'}$ is given by (4.55). Finally, one calculates (C.4) as

$$\int d^2 z (\Omega_{\vec{p}}(\vec{z}))^* U_{\vec{p}',\vec{x}}(\vec{z}) = c_{\vec{p}'} \delta^{(2)}(p - p') \prod_{k=1}^{N-1} A(x_k, \bar{x}_k) .$$

One can arrive at the same relation using the diagrammatical representation of the l.h.s. as the pyramid diagram, Fig. [14], with the additional lines carrying the indices $2(1-s)$ attached to its end-points $\vec{z}_k$ and joined at the same point $\vec{z}_0$.

Using the completeness condition, Eq. (4.49), we find from (C.7)

$$c^*_{\vec{p}'} \int d\vec{x} \mu(\vec{x}) U_{\vec{p}',\vec{x}}(\vec{z}) \prod_{k=1}^{N-1} (A(x_k, \bar{x}_k))^* = (2\pi)^{-N} \Omega_{\vec{p}}(\vec{z}) .$$
Here, the normalization factor in the l.h.s. compensates the difference in the scaling dimensions of \( U_{\vec{p},\vec{x}} \) and \( \Omega_{\vec{p}} \). Let us consider an arbitrary operator \( f(\hat{x}_k, \bar{\hat{x}}_k) \), depending on the Sklyanin’s operator zeros defined in Sect. 4.1.1, and apply it to the both sides of the last relation. Since \( U_{\vec{p},\vec{x}} \) diagonalizes the operators zeros, Eq. (4.13), this operator can be replaced in the l.h.s. by its eigenvalue \( f(x_k, \bar{x}_k) \). Then, choosing \( f(x_k, \bar{x}_k) \) as the wave function in the separated coordinates \( \Phi_{\{q,\bar{q}\}}(\vec{x}) \), Eq. (4.59), we obtain from (4.1) (up to an overall normalization factor depending on the momentum \( \vec{p} \))

\[
\Psi_{\vec{p},\{q,\bar{q}\}}(\vec{z}) = (2\pi)^N c_{\vec{p}}^\ast \int d\mu(\vec{x}) U_{\vec{p},\vec{x}}(\vec{z}) \prod_{k=1}^{N-1} (A(x_k, \bar{x}_k))^r Q_{q,\bar{q}}(x_k, \bar{x}_k) = \prod_{k=1}^{N-1} Q_{q,\bar{q}}(x_k, \bar{x}_k) \Omega_{\vec{p}}(\vec{z}).
\]

(C.9)

Here, \( Q_{q,\bar{q}}(x_k, \bar{x}_k) \) stands for the eigenvalue of the Baxter operator with the spectral parameters substituted by the operator zeros.

The relation (C.9) generalizes the well-known highest-weight representation of the eigenstates of compact spin chain magnets in the Algebraic Bethe Ansatz

\[
|\Psi_{l}^{ABA}⟩ = B(\lambda_1) \ldots B(\lambda_l) |\Omega_{\vec{p}}⟩
\]

(C.10)

with \( B(u) \) being the off-diagonal element of the monodromy matrix, Eq. (2.44), and \( \lambda_1, \ldots, \lambda_l \) satisfying the Bethe equations. Using (4.11), one can show that two representations, Eqs. (C.9) and (C.10), are equivalent if the eigenvalues of the Baxter \( Q \)-operators are given by polynomials of degree \( l \) in the spectral parameter \( u \). Going beyond the class of polynomial solutions to the Baxter equation, one finds [7, 21] that (C.10) is not applicable while the representation (C.9) remains valid.

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