REGULARITY RESULTS FOR SHORTEST BILLIARD TRAJECTORIES IN CONVEX BODIES IN $\mathbb{R}^n$

STEFAN KRUPP AND DANIEL RUDOLF

Abstract. We derive properties of closed billiard trajectories in convex bodies in $\mathbb{R}^n$. Building on techniques introduced by K. and D. Bezdek we establish two regularity results for length minimizing closed billiard trajectories: one for billiard trajectories in general convex bodies, the other for billiard trajectories in the special case of acute convex polytopes. Moreover, we attach particular importance to various examples, also including examples which show the sharpness of the first regularity result.

1. Introduction

In this paper we analyze closed billiard trajectories in convex bodies in $\mathbb{R}^n$. The billiard trajectories are meant to be Euclidean which for bouncing points locally means: The angle of reflection equals the angle of incidence. This local reflection rule can be seen as consequence of the global least action principle. For a planar billiard table boundary this principle means that a billiard trajectory segment $(p_{j-1}, p_j, p_{j+1})$ minimizes the Euclidean length in the space of all paths connecting $p_{j-1}$ and $p_{j+1}$ via a reflection at the billiard table boundary.

From the geometric optics point of view, Euclidean billiards describe the wave propagation in a medium which is not only homogeneous and isotropic but also contains perfectly reflecting mirrors.

There is generally much interest into the study of billiards: Problems in almost every mathematical field can be related to problems in mathematical billiards, see for example [7], [8] and [10] for comprehensive surveys. Euclidean billiard trajectories in the plane have been investigated intensively. Nonetheless, so far not much is known about Euclidean billiard trajectories on higher-dimensional "tables".

The aim of this paper is to establish two regularity results for length minimizing closed Euclidean billiard trajectories in higher-dimensional convex bodies and to show how these results can be used to calculate (manually and computationally) these trajectories for certain classes of convex polytopes.

Let us precisely define closed Euclidean billiard trajectories based on the above mentioned least action principle.

Definition 1.1. Let $T \subset \mathbb{R}^n$ be a convex body, i.e. a compact convex set with nonempty interior, which from now on we call the billiard table. We say that a closed polygonal line with pairwise distinct vertices $p_1, ..., p_m$ on the boundary of $T$ (denoted by $\partial T$), is a closed billiard trajectory in $T$ if for every $j \in \{1, ..., m\}$ there is a $T$-supporting hyperplane $H_j$ through $p_j$ such that $p_j$ minimizes

$$||\overline{p}_j - p_{j-1}|| + ||p_{j+1} - \overline{p}_j||$$

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over all \( p_\bar{j} \in H_j \), where \( p_{m+1} = p_1 \) and \( p_0 = p_m \), and if \( p_j \) is not contained in the line segment connecting \( p_{j-1} \) and \( p_{j+1} \). We encode this closed billiard trajectory by \( (p_1, \ldots, p_m) \) and call its vertices bouncing points. Its length is given by

\[
\ell((p_1, \ldots, p_m)) = \sum_{j=1}^{m} ||p_{j+1} - p_j||.
\]

We call a boundary point \( p \in \partial T \) smooth if there is a unique \( T \)-supporting hyperplane through \( p \). We say that \( \partial T \) is smooth if every boundary point is smooth.

We remark that the notion of billiard trajectories is usually used for classical trajectories, i.e. for trajectories with bouncing points in smooth boundary points (billiard table gangs) while they stop in nonsmooth boundary points (billiard table holes). Definition \[1.1\] generalizes this classical billiard reflection rule to nonsmooth boundaries. To the author’s knowledge the papers [3], [6] and [4] in order were among the first suggesting a detailed study of these generalized billiard trajectories.

This definition implies the local billiard reflection rule: The angle of reflection equals the angle of incidence. In this generalized version the angle of incidence/reflection is the angle enclosed by the incoming/reflecting billiard trajectory and the hyperplane which appears within the minimization in \((1)\). Since there are infinitely many different \( T \)-supporting hyperplanes through nonsmooth boundary points of the billiard table \( T \), the billiard reflection rule may produce different bouncing points following two already known consecutive ones.

In order to state the first regularity result let

\[
N_T(p) := \{ n \in \mathbb{R}^n : \langle n, y - p \rangle \leq 0 \text{ for all } y \in T \}
\]

be the outer normal cone at \( T \) in the point \( p \in \partial T \). Then the first regularity result reads as follows:

**Theorem 1.2.** Let \( T \subset \mathbb{R}^n \) be a billiard table and \( p = (p_1, \ldots, p_m) \) a length minimizing closed billiard trajectory in \( T \). Further, let \( T \cap V \) be the smallest affine section of \( T \) containing \( p \). Then, it follows \( \dim V = m - 1 \) and

\[
\dim (N_T(p_j) \cap V_0) = 1
\]
for all $j \in \{1, \ldots, m\}$, where $V_0$ is the vector subspace that is parallel to the affine subspace $V$.

From $\dim V \leq n$ it follows that $m \leq n + 1$. Furthermore, $\dim V = m - 1$ implies that $p$ is maximally spanning, i.e.

$$\dim (\text{conv}\{p_1, \ldots, p_m\}) = m - 1.$$ 

In fact, (3) is a regularity result: If $m = n + 1$, meaning that $V = \mathbb{R}^n$, then $p$ is regular, i.e. all bouncing points of $p$ are smooth boundary points of $T$; in other words: $N_T(p_j)$ is one-dimensional for all $j \in \{1, \ldots, m\}$. For $n = 2$ this means that every length minimizing billiard trajectory in $T$ has either two or three bouncing points, while in the latter case all of them are smooth boundary points of $T$.

Some special cases of Theorem 1.2 were already known: In [4] it has been proven for length minimizing closed billiard trajectories that $m$ is bounded from above by $n + 1$. In [2] it was shown the two-dimensional case, while in [1] it has been proven that every length minimizing closed billiard trajectory in $T \subset \mathbb{R}^n$ with $n + 1$ bouncing points is regular.

For the second regularity result we introduce the following definition: Let $P \subset \mathbb{R}^n$ be a convex polytope, i.e. a bounded intersection of finitely many closed half-spaces of $\mathbb{R}^n$ with nonempty interior. Let $F_1, \ldots, F_k$ be the facets, i.e. the $(n - 1)$-dimensional faces, of $P$ where we denote by $q_1, \ldots, q_k$ the inward unit normal vectors to them. For $i, j \in \{1, \ldots, k\}$ with $i \neq j$ and $\dim (F_i \cap F_j) = n - 2$ let $\gamma_{ij} \in (0, \pi)$ be the angle enclosed by $q_i$ and $q_j$. Then $\alpha_{ij} = \pi - \gamma_{ij}$ is called the dihedral angle between $F_i$ and $F_j$. The dihedral angle $\alpha_{ij}$ is called acute when $\alpha_{ij} \in (0, \frac{\pi}{2})$. We call $P$ acute if all dihedral angles of $P$ are acute.

Then the second regularity result is:

**Theorem 1.3.** Let $P \subset \mathbb{R}^n$ be an acute convex polytope that plays the role of the billiard table. Then every length minimizing closed billiard trajectory in $P$ is maximally spanning, regular and has $n + 1$ bouncing points.

This second regularity result is an improvement of Theorem 1.1 in [1] for the special case of acute convex polytopes. There it has been proven that every length minimizing closed billiard trajectory in the more general class of acute convex bodies in $\mathbb{R}^n$ is regular and has at most $n + 1$ bouncing points.
Since every dihedral acute angled simplex (cf. [9] for a survey on the simplices’
dihedral angles) is an acute convex polytope the same regularity result holds for
these simplices (this is already known from Corollary 1.2 in [1]). In order to empha-
size the significance of Theorem 1.3 it remains to understand the difference between
acute convex polytopes and acute simplices: Is there any acute polytope which is
not a simplex? We discuss this question at the end of Section 4.

In this paper we attach particular importance to various examples. These in-
clude counterexamples for what at first seem rather natural conclusions concern-
ing (length minimizing) closed billiard trajectories within sections of the billiard table.
Furthermore, we provide some examples showing the sharpness of Theorem 1.2.

More precisely, we provide examples for the following statements:

(i) The length minimality of closed billiard trajectories in \( T \subset \mathbb{R}^n \) is not in-
variant under going to sections of \( T \) containing the billiard trajectory (cf. Example A). The length minimality may not even be locally preserved (cf. Example B).

(ii) The length minimality of closed billiard trajectories in \( T \subset \mathbb{R}^n \) is not invari-
ant under going to smallest sections of \( T \) containing the billiard trajectory (cf. Example C). The length minimality may not even be locally preserved (cf. Example D).

(iii) A length minimizing closed billiard trajectory in \( T \subset \mathbb{R}^n \) may not be reg-
ular within the smallest section of \( T \) containing the billiard trajectory (cf. Example E). This can even appear for the unique length minimizing closed
billiard trajectory (cf. Example F).

(iv) A length minimizing closed billiard trajectory in \( T \subset \mathbb{R}^n \) can have bouncing
points in vertices as well as in more than 0-dimensional faces of \( T \) (cf. Examples E and F).

Let us briefly present the structure of this paper. In Section 2 we discuss prop-
erties of closed billiard trajectories in convex bodies. In Section 3 and 4 we prove
our first and second regularity result, respectively. In Section 5 we provide various
examples concerning above listed statements.

2. Properties of closed billiard trajectories

We start by recalling the statements of the following useful Lemma for which for
a convex body \( T \subset \mathbb{R}^n \) we first define

\[ F(T) := \{ F : F \text{ is a set of points in } \mathbb{R}^n \text{ that cannot be translated into } \hat{T}\}, \]

where \( \hat{T} \) denotes the interior of \( T \) in \( \mathbb{R}^n \), and write for the sake of simplicity
\((p_1, ..., p_m) \in F(T)\) while we actually mean \( \{p_1, ..., p_m\} \in F(T)\).

**Lemma 2.1** (Lemmata 2.1, 2.2 and 2.3 in [3]).
(i) Let \( T \subset \mathbb{R}^n \) be a convex
set and \( F \) a finite set of at least \( n+1 \) points in \( \mathbb{R}^n \). Then there is a
translate of \( T \) that covers \( F \) if and only if every choice of \( n+1 \) points of \( F \)
can be covered by a translate of \( T \).

(ii) Let \( T \subset \mathbb{R}^n \) be a convex body and \( F \) a set of points \( p_1, ..., p_m \in \mathbb{R}^n \). Then
\( F \in F(T) \) is equivalent to: There is a translate \( T' \) of \( T \) and there are closed
half-spaces \( H^+_1, ..., H^+_m \) of \( \mathbb{R}^n \) such that

- \( p_j \in \partial H^+_j \) for all \( j \in \{1, ..., m\} \),
- \( T' \cap \partial H^+_j = \emptyset \) for all \( j \in \{1, ..., m\} \).
\begin{itemize}
  \item $T' \subseteq H_j^+$ for all $j \in \{1, \ldots, m\}$,
  \item $\bigcap_{j=1}^m H_j^+$ is nearly bounded, i.e. it lies between two parallel hyperplanes.
\end{itemize}

(iii) Let $T \subset \mathbb{R}^n$ be a billiard table and $p = (p_1, \ldots, p_m)$ a closed billiard trajectory in $T$. Then we have $p \in F(T)$.

We note that (2.1i) can be equivalently (by contraposition applied on $\bar{T}$) expressed by: Let $T \subset \mathbb{R}^n$ be a convex set. A set $F$ of at least $n + 1$ points in $\mathbb{R}^n$ is in $F(T)$ if and only if there is a choice of $n + 1$ points out of $F$ that is in $F(T)$.

Furthermore, we recall the following characterisation of length minimizing closed billiard trajectories:

\textbf{Theorem 2.2} (Theorem 1.1 and Lemma 2.4 in [4]). Let $T \subset \mathbb{R}^n$ be a billiard table. Then every length minimizing closed billiard trajectory in $T$ has at most $n + 1$ bouncing points. Moreover, every length minimizing closed billiard trajectory in $T$ is a length minimizing element of $F(T)$ and, conversely, every length minimizing element of $F(T)$ can be translated to a length minimizing closed billiard trajectory in $T$.

We note that Theorem 2.2 is an existence result for closed billiard trajectories in arbitrary billiard tables: Let $T \subset \mathbb{R}^n$ be any billiard table. Then there is a closed billiard trajectory in $T$ (with at most $n + 1$ bouncing points). This can be easily concluded by a compactness argument applied on the set of closed polygonal lines in $F(T)$ combined with the $d_H$-continuity, i.e. continuity with respect to the Hausdorff distance $d_H$, of the length functional.

We continue by stating the following property of closed billiard trajectories:

\textbf{Proposition 2.3.} Let $T \subset \mathbb{R}^n$ be a billiard table, $p = (p_1, \ldots, p_m)$ a closed billiard trajectory in $T$ and $T \cap V$ an affine section of $T$ containing $p$. Then $p$ is a closed billiard trajectory in $T \cap V$.

\textbf{Proof.} Since $p = (p_1, \ldots, p_m)$ is a closed billiard trajectory in $T$ there are $T$-supporting hyperplanes $H_1, \ldots, H_m$ in $\mathbb{R}^n$ through $p_1, \ldots, p_m$ such that $p_j$ minimizes

\begin{equation}
\|\bar{p}_j - p_{j-1}\| + \|p_{j+1} - \bar{p}_j\|
\end{equation}

over all $\bar{p}_j \in H_j$ for all $j \in \{1, \ldots, m\}$. Since $T \cap V$ contains $p$ it follows that $p_j$ minimizes (4) over all $\bar{p}_j \in H_j \cap V$ for all $j \in \{1, \ldots, m\}$. This implies that $p$ is a billiard trajectory in $T \cap V$. \hfill $\square$

Clearly, the converse is not true: We can imagine an affine section $T \cap V$ of $T$ that can be translated into $T$. Then every closed billiard trajectory $p$ in $T \cap V$ can be translated into $T$. But by Lemma 2.1(iii) then $p$ cannot be a closed billiard trajectory in $T$.

In Section 5 (cf. Examples A, B, C and D) we will see that generally the length minimality of a closed billiard trajectory in $T$ is not invariant under going to (smallest) affine sections of $T$ containing the closed billiard trajectory.

For what follows it will be useful to reformulate the billiard reflection rule in the sense of the following Proposition 2.4. For that we denote the $(n-1)$-dimensional unit sphere of $\mathbb{R}^n$ by $S^{n-1}$.

\textbf{Proposition 2.4.} Let $T \subset \mathbb{R}^n$ be a billiard table. A closed polygonal line with vertices $p_1, \ldots, p_m$ on $\partial T$ is a closed billiard trajectory in $T$ if and only if there are
vectors \( n_1, \ldots, n_m \in S^{n-1} \) such that

\[
\begin{align*}
& p_{j+1} - p_{j} = \lambda_j n_j, \quad \lambda_j > 0, \\
& n_{j+1} - n_{j} \in -N_T(p_{j+1})
\end{align*}
\]

is fulfilled for all \( j \in \{1, \ldots, m\} \) (here we write \( p_{m+1} = p_1 \) and \( n_{m+1} = n_1 \)).

**Figure 3.** The visualization of (5).

**Proof.** Let \( p = (p_1, \ldots, p_m) \) be a closed polygonal line with vertices on \( \partial T \). Let us assume there are vectors \( n_1, \ldots, n_m \in S^{n-1} \) which together with \( p \) fulfill (5). Then for all \( j \in \{1, \ldots, m\} \) there is a unit vector \( n_T(p_{j+1}) \in N_T(p_{j+1}) \) such that

\[
\begin{align*}
& p_{j+1} - p_{j} = \lambda_j n_j, \quad \lambda_j > 0, \\
& n_{j+1} - n_{j} = -\mu_j n_T(p_{j+1}), \quad \mu_j > 0,
\end{align*}
\]

holds. We define \( H_1, \ldots, H_m \) to be the \( T \)-supporting hyperplanes in \( \mathbb{R}^n \) through \( p_1, \ldots, p_m \) which are normal to \( n_T(p_1), \ldots, n_T(p_m) \). Then the following holds for all \( j \in \{1, \ldots, m\} \):

\[
\nabla_{\bar{p}_j = p_j} (||\bar{p}_j - p_{j-1}|| + ||p_{j+1} - \bar{p}_j||) = \frac{p_j - p_{j-1}}{||p_j - p_{j-1}||} - \frac{p_{j+1} - p_j}{||p_{j+1} - p_j||}
\]

\[
= \frac{\lambda_j n_{j-1}}{||\lambda_j n_{j-1}||} - \frac{\lambda_j n_j}{||\lambda_j n_j||} = n_{j-1} - n_j = \mu_j n_T(p_j).
\]

Therefore, for all \( j \in \{1, \ldots, m\} \) \( p_j \) extremizes

\[
||\bar{p}_j - p_{j-1}|| + ||p_{j+1} - \bar{p}_j||
\]

over all \( \bar{p}_j \in H_j \) near \( p_j \). Since for all \( j \in \{1, \ldots, m\} \) (6) is a convex function with respect to \( \bar{p}_j \) it follows for all \( j \in \{1, \ldots, m\} \) that \( p_j \) is a global minimizer of (6) over all \( \bar{p}_j \in H_j \). Therefore the billiard reflection rule is fulfilled in \( p_j \) for all \( j \in \{1, \ldots, m\} \).

Eventually \( p \) is a closed billiard trajectory in \( T \).

Conversely, let us assume \( p = (p_1, \ldots, p_m) \) is a closed billiard trajectory in \( T \). Then there are \( T \)-supporting hyperplanes \( H_1, \ldots, H_m \) in \( \mathbb{R}^n \) through \( p_1, \ldots, p_m \) such
that for all \( j \in \{1, \ldots, m\} \) \( p_j \) minimizes (6) over all \( \bar{p}_j \in H_j \). By Lagrange’s multiplier theorem this means
\[
\frac{p_j - p_{j-1}}{||p_j - p_{j-1}||} - \frac{p_{j+1} - p_j}{||p_{j+1} - p_j||} = \mu_j n_T(p_j), \quad \mu_j \in \mathbb{R},
\]
where \( n_T(p_j) \) is the outer unit vector normal to \( H_j \), for all \( j \in \{1, \ldots, m\} \). Scalar multiplication with \( n_T(p_j) \) gives \( \mu_j > 0 \) for all \( j \in \{1, \ldots, m\} \). If we define
\[
(7) \quad n_j := (p_{j+1} - p_j)/\lambda_j, \quad \lambda_j := ||p_{j+1} - p_j||, \quad j \in \{1, \ldots, m\},
\]
and consider \( n_T(p_j) \in N_T(p_j) \) for all \( j \in \{1, \ldots, m\} \), then (6) is fulfilled for \( p \) together with the unit vectors \( n_1, \ldots, n_m \) defined in (7).

The proof of Proposition 2.4 shows even more: A closed polygonal line with vertices \( p_1, \ldots, p_m \) on \( \partial T \) is a closed billiard trajectory in \( T \) with \( H_1, \ldots, H_m \) the \( T \)-supporting hyperplanes which are associated to the billiard reflection rule if and only if there are vectors \( n_1, \ldots, n_m \in S^{n-1} \) such that
\[
\begin{aligned}
p_{j+1} - p_j &= \lambda_j n_j, \quad \lambda_j > 0, \\
n_j + n_{j+1} &= \mu_j n_{H_{j+1}}, \quad \mu_j > 0
\end{aligned}
\]
is fulfilled for all \( j \in \{1, \ldots, m\} \) where we denoted the outer unit normal vectors at \( H_1, \ldots, H_m \) by \( n_{H_1}, \ldots, n_{H_m} \).

The following rather obvious Proposition is needed within the proof of Theorem 1.2. It follows immediately from within the proof of Proposition 2.4.

**Proposition 2.5.** Let \( T \subset \mathbb{R}^n \) be a billiard table and \( p = (p_1, \ldots, p_m) \) a closed billiard trajectory in \( T \). Then for every \( j \in \{1, \ldots, m\} \) there is only one \( T \)-supporting hyperplane through \( p_j \) for which the billiard reflection rule in \( p_j \) is fulfilled.

**Proof.** This claim follows from the fact that the outer unit vector \( n_T(p_j) \) normal to \( H_j \) is uniquely determined by the condition
\[
\frac{p_j - p_{j-1}}{||p_j - p_{j-1}||} - \frac{p_{j+1} - p_j}{||p_{j+1} - p_j||} = \mu_j n_T(p_j), \quad \mu_j \neq 0,
\]
which arises from Lagrange’s multiplier theorem as within the converse implication of the proof of Proposition 2.4.

The next two Propositions make a statement on the positional relationship of the hyperplanes which determine the billiard reflection rule.

**Proposition 2.6.** Let \( T \subset \mathbb{R}^n \) be a billiard table, \( p = (p_1, \ldots, p_m) \) a closed billiard trajectory in \( T \) and \( T \cap V \) the smallest affine section of \( T \) containing \( p \). Then the convex cone spanned by the outer unit vectors
\[
n_T(p_1), \ldots, n_T(p_m)
\]
which are normal to the \( T \)-supporting hyperplanes \( H_1, \ldots, H_m \) through \( p_1, \ldots, p_m \) associated to the billiard reflection rule is \( V_0 \), where \( V_0 \) is the vector subspace underlying \( V \).

**Proof.** Since \( p \) is a closed billiard trajectory in \( T \) and \( T \cap V \) an affine section of \( T \) containing \( p \) the vectors
\[
(8) \quad p_2 - p_1, \ldots, p_m - p_{m-1}, p_1 - p_m
\]
are all in $V_0$. Since $T \cap V$ is the smallest affine section containing $p$ it follows that
the convex cone spanned by the vectors in $\{8\}$ actually is $V_0$.

From the proof of Proposition 2.4 it follows that there are $n_1, \ldots, n_m \in S^{n-1}$ (cf. [7] for the definition) such that
\[
\begin{cases}
  p_{j+1} - p_j = \lambda_j n_j, \quad \lambda_j > 0, \\
  n_{j+1} - n_j = -\mu_{j+1} n_T(p_{j+1}), \quad \mu_{j+1} > 0,
\end{cases}
\]
holds for all $j \in \{1, \ldots, m\}$.

From
\[
p_{j+1} - p_j = \lambda_j n_j, \quad \lambda_j > 0,
\]
for all $j \in \{1, \ldots, m\}$ it follows that the convex cone spanned by $n_1, \ldots, n_m$ is $V_0$ and
the same is true for the convex cone spanned by $n_2 - n_1, \ldots, n_m - n_{m-1}, n_1 - n_m$.

Then
\[
n_{j+1} - n_j = -\mu_{j+1} n_T(p_{j+1}), \quad \mu_{j+1} > 0,
\]
for all $j \in \{1, \ldots, m\}$ implies that the convex cone spanned by $n_T(p_1), \ldots, n_T(p_m)$
is $V_0$. $\square$

**Proposition 2.7.** Let $T \subset \mathbb{R}^n$ be a billiard table, $p = (p_1, \ldots, p_m)$ a closed billiard
trajectory in $T$ and $T \cap V$ the smallest affine section of $T$ containing $p$. Let $H_1^+, \ldots, H_m^+$ be the $T$-supporting half-spaces of $\mathbb{R}^n$ which are bounded by the hyperplanes $H_1, \ldots, H_m$ through $p_1, \ldots, p_m$ which are related to the billiard reflection rule.
Further, let $W$ be the orthogonal complement to $V$. Then we can write
\[
H_j = (H_j \cap V) \oplus W \text{ and } H_j^+ = (H_j^+ \cap V) \oplus W,
\]
and have that
\[
\bigcap_{j=1}^m (H_j^+ \cap V) \text{ is bounded in } V, \quad \bigcap_{j=1}^m H_j^+ \text{ is nearly bounded in } \mathbb{R}^n.
\]

**Proof.** By Proposition 2.6 the convex cone spanned by the outer unit vectors normal to $H_1, \ldots, H_m$ is the vector subspace $V_0$ that underlies the affine subspace $V$. This implies on the one hand that we can write
\[
(9) \quad H_j = (H_j \cap V) \oplus W \text{ and } H_j^+ = (H_j^+ \cap V) \oplus W
\]
for all $j \in \{1, \ldots, m\}$ and on the other hand that
\[
(10) \quad \bigcap_{j=1}^m (H_j^+ \cap V)
\]
is bounded in $V$. The latter fact implies that there are hyperplanes $H$ and $H + d$, $d \in V_0$, in $V$ such that $\bigcap_{j=1}^m (H_j^+ \cap V)$ lies in-between. With $\bigcap_{j=1}^m (H_j^+ \cap V)$ this implies that
\[
\bigcap_{j=1}^m H_j^+ = \bigcap_{j=1}^m ((H_j^+ \cap V) \oplus W) = \left( \bigcap_{j=1}^m (H_j^+ \cap V) \right) \oplus W
\]
lies between the hyperplanes $H \oplus W$ and $H + d \oplus W$ in $\mathbb{R}^n$ and therefore
\[
H_1^+ \cap \ldots \cap H_m^+
\]
is nearly bounded in \( \mathbb{R}^n \).

Considering Lemma 2.1(ii) we note that with Proposition 2.7 we have subsequently provided a proof of Lemma 2.1(iii).

The following Proposition is a preparation for the proof of Theorem 1.2.

**Proposition 2.8.** Let \( T \subset \mathbb{R}^n \) be a billiard table, \( p = (p_1, ..., p_m) \) a closed billiard trajectory in \( T \) and \( T \cap V \) the smallest affine section of \( T \) containing \( p \). Then there is a selection \( \{i_1, ..., i_{\dim V + 1}\} \subset \{1, ..., m\} \) such that

\[
\{p_{i_1}, ..., p_{i_{\dim V + 1}}\} \in F(T).
\]

**Proof.** If \( \dim V = n \) then the claim follows immediately by Lemma 2.1(i) & (iii). This is also the case when \( m = \dim V + 1 \).

Let \( \dim V \leq \min\{n - 1, m - 2\} \). Since \( p \) is a closed billiard trajectory in \( T \) there are \( T \)-supporting hyperplanes \( H_1, ..., H_m \) through \( p_1, ..., p_m \) for which the billiard reflection rule is fulfilled. Proposition 2.7 implies on the one hand that we can write

\[
H_j = (H_j \cap V) \oplus W \quad \text{and} \quad H_j^+ = (H_j^+ \cap V) \oplus W
\]

for all \( j \in \{1, ..., m\} \), where \( W \) is the orthogonal complement to \( V \) and \( H_1^+, ..., H_m^+ \) are the closed half-spaces defined by \( \partial H_j^+ = H_j \) and \( T \subset H_j^+ \) for all \( j \in \{1, ..., m\} \), and on the other hand that

\[
\bigcap_{j=1}^{m} (H_j^+ \cap V)
\]

is bounded in \( V \). Then there is a selection \( \{i_1, ..., i_{\dim V + 1}\} \subset \{1, ..., m\} \) such that

\[
\bigcap_{j=1}^{\dim V + 1} (H_{i_j}^+ \cap V)
\]

is nearly bounded in \( V \).

Indeed, let us assume this is not the case. Then it follows by Lemma 2.1(ii) that for every selection \( \{i_1, ..., i_{\dim V + 1}\} \subset \{1, ..., m\} \)

\[
\{p_{i_1}, ..., p_{i_{\dim V + 1}}\}
\]

can be translated into the interior of \( T \cap V \). By Lemma 2.1(i) this implies that \( \{p_1, ..., p_m\} \) can be translated into the interior of \( T \cap V \). But again with Lemma 2.1(ii) this is a contradiction to the fact that (12) is bounded in \( V \) and the claim is proven.

We conclude that (13) lies between two parallel hyperplanes in \( V \), say \( H \) and \( H + d \) where \( d \) is an element of the vector subspace \( V_0 \) that underlies the affine subspace \( V \). By applying (11) it follows that

\[
\bigcap_{j=1}^{\dim V + 1} H_{i_j}^+ = \bigcap_{j=1}^{\dim V + 1} ((H_{i_j}^+ \cap V) \oplus W) = \left( \bigcap_{j=1}^{\dim V + 1} (H_{i_j}^+ \cap V) \right) \oplus W
\]

lies between the two parallel hyperplanes \( H \oplus W \) and \( (H + d) \oplus W \) in \( \mathbb{R}^n \), i.e. it is nearly bounded in \( \mathbb{R}^n \). By Lemma 2.1(ii) it follows that

\[
\{p_{i_1}, ..., p_{i_{\dim V + 1}}\} \in F(T).
\]

\[\square\]

\(^1\)One also could produce a contradiction by applying Proposition 2.3 and Lemma 2.1(iii).
We remark that the statement of Proposition 2.8 is not true when requiring \( p \) just to be a closed polygonal line in \( F(T) \) (and not a closed billiard trajectory in \( T \)). To see this we consider the following example in \( \mathbb{R}^5 \) which has been communicated to us by A. Abbondandolo:

We start from four convex bodies \( K_1, K_2, K_3, K_4 \) in \( \mathbb{R}^3 \) with the following two properties:

(a) the intersection of all of them is empty;
(b) the intersection of any three of them has non-empty interior.

One has these examples because Helly’s theorem is sharp. Then we consider the four vertices of a square in \( \mathbb{R}^2 \):

\[
v_1 = (0,0), \ v_2 = (1,0), \ v_3 = (1,1), \ v_4 = (0,1).
\]

Now let \( T \) be the convex hull in \( \mathbb{R}^5 = \mathbb{R}^2 \times \mathbb{R}^3 \) of the union of the following four sets:

\[
\{v_1\} \times K_1, \ \{v_2\} \times K_2, \ \{v_3\} \times K_3, \ \{v_4\} \times K_4.
\]

\( T \) projects onto the square, but (a) implies that each section of \( T \) that is parallel to \( \mathbb{R}^2 \times \{0\} \) has area smaller than 1.

Indeed, we take any \( w \in \mathbb{R}^3 \) and look at the section \( T \cap (\mathbb{R}^2 \times w) \). By the definition of \( T \) the points in this section are of the form \((v,w)\) with

\[
v = \sum_j \lambda_j v_j \quad \text{and} \quad w = \sum_j \lambda_j w_j,
\]

where \( w_j \) is in \( K_j \) and the \( \lambda_j \)'s are positive and add up to one. In particular, this section is contained in \( Q \times w \), where \( Q \) denotes the square with vertices \( v_j \). This section cannot contain all the four points \((v_j, w)\). In fact, assume that it contains the point \((v_1, w)\). Then (14) and the fact that \( v_1 \) is an extremal point of \( Q \) imply that \( w \) belongs to \( K_1 \), as all the \( \lambda_j \)'s with \( j > 1 \) must vanish in (14). Since any given \( w \) in \( \mathbb{R}^3 \) belongs to at most three of the \( K_j \)'s, the claim is proven. Being a closed set that is contained in \( Q \times w \) and does not contain \((v_j, w)\) for at least one \( j \), the section \( T \cap (\mathbb{R}^2 \times w) \) has area strictly smaller than 1. Since the area of the intersection of a convex body with \( \mathbb{R}^2 \times w \) is an upper semicontinuous function of \( w \), all the sections of \( T \) by planes parallel to \( \mathbb{R}^2 \times \{0\} \) have area less than \( A \) for some \( A < 1 \).

We choose as \( p_1, p_2, p_3, p_4 \) the points

\[
p_1 = (1-t)v_1, \ p_2 = (1-t)v_2, \ p_3 = (1-t)v_3, \ p_4 = (1-t)v_4
\]

for \( t > 0 \) so small that the area of the square with the vertices \( p_1, p_2, p_3, p_4 \) is larger than \( A \) (while smaller than 1). Then

\[
\{p_1, p_2, p_3, p_4\} \in F(T)
\]

because any translation of these points will enclose a square that is too big to be contained in a section of \( T \) parallel to \( \mathbb{R}^2 \times \{0\} \). However, any triplet from \( \{p_1, p_2, p_3, p_4\} \) is not in \( F(T) \): Consider without loss of generality the triplet \( \{p_1, p_2, p_3\} \). By (b), there is a point \( w \) in the interior of \( K_1 \cap K_2 \cap K_3 \). Then the section \( \mathbb{R}^2 \times \{w\} \) contains a translated copy of the triangle with vertices \( v_1, v_2, v_3 \) and \( \{p_1, p_2, p_3\} \) can be translated into the interior of such triangle, and hence into the interior of \( T \). This proofs the claim.

The next two statements, i.e. Lemma 2.9 and Proposition 2.10 give insights on how to translate sets of finitely many points on the boundary of convex polytopes.
Lemma [2.9] is the general version while Proposition [2.10] considers the bouncing points of closed billiard trajectories. The latter is the main ingredient for the proof of Theorem [1.3].

**Lemma 2.9.** Let \( P \subset \mathbb{R}^n \) be a convex polytope. Every set of \( m \leq n - 1 \) points on \( \partial P \) can be translated to a position where all the points still are on its original facets but at least one of them is a nonsmooth boundary point of \( P \).

**Proof.** To illustrate the argument we start with the case \( n = 3 \). The facets of \( P \) are two-dimensional and we assume to have two points \( p_1, p_2 \) in the interiors of two of the facets. There are two cases: either the \( P \)-supporting hyperplanes associated to the two facets have no intersection or their intersection is a one-dimensional straight line. The first case is trivial since we can choose any direction (parallel to the facets) for translating \( \{p_1, p_2\} \) without \( p_1, p_2 \) leaving the facets until at least one of these points is a nonsmooth boundary point of \( P \). For the second case there is one uniquely determined direction (up to orientation) given by the already mentioned one-dimensional straight line along which translating \( \{p_1, p_2\} \) is possible without \( p_1, p_2 \) leaving the facets until at least one of these points is a nonsmooth boundary point of \( P \).

We generalize this argument to higher dimensions: Let \( p_1, \ldots, p_m \), \( m \leq n - 1 \), be interior points of facets \( F_1, \ldots, F_m \) of \( P \). Let \( H_1, \ldots, H_m \) be the supporting hyperplanes of \( F_1, \ldots, F_m \) and suppose \( c_1, \ldots, c_m \in \mathbb{R}^n \) are chosen such that \( H_1 + c_1, \ldots, H_m + c_m \) are \((n-1)\)-dimensional vector subspaces of \( \mathbb{R}^n \). We conclude

\[
\dim \bigcap_{j=1}^{m} (H_j + c_j) = n - 1 - (m - 1) = n - m \geq 1 > 0.
\]

Now \( \{p_1, \ldots, p_m\} \) can be translated in directions given by vectors in

\[
\bigcap_{j=1}^{m} (H_j + c_j)
\]

while \( p_1, \ldots, p_m \) are not leaving \( F_1, \ldots, F_m \) until at least one of these points is a nonsmooth boundary point of \( P \). \( \square \)

**Proposition 2.10.** Let \( P \subset \mathbb{R}^n \) be a convex polytope. Every regular closed billiard trajectory in \( P \) with \( m \leq n \) bouncing points can be translated into a closed nonregular, i.e. not regular billiard trajectory.

**Proof.** Let \( p = (p_1, \ldots, p_m) \) be a regular closed billiard trajectory in \( P \) with \( m \leq n \) and let \( P \cap V \) be the smallest affine section of \( P \) containing \( p \).

If \( m \leq n - 1 \) then we apply Lemma [2.9] on the set of bouncing points \( \{p_1, \ldots, p_m\} \). Let \( p + c, c \in \mathbb{R}^n \), be the translated set in the sense of Lemma [2.9], i.e. at least one of the points \( p_1 + c, \ldots, p_m + c \) is a nonsmooth boundary point of \( P \). (We argue below that \( p + c \) is a closed billiard trajectory in \( P \).)

Let \( m = n \) and \( p_1, \ldots, p_m \) be interior points of facets \( F_1, \ldots, F_m \) of \( P \). Let \( H_1, \ldots, H_m \) be the supporting hyperplanes of \( F_1, \ldots, F_m \). Applying Proposition [2.6] we can choose \( c_1, \ldots, c_m \in V_0 \) (for instance as positive/negative multiples of the unit vectors normal to \( H_1, \ldots, H_m \)), where \( V_0 \) is the vector subspace of \( \mathbb{R}^n \) that underlies the affine subspace \( V \), such that \( H_1 + c_1, \ldots, H_m + c_m \) are \((n-1)\)-dimensional vector subspaces of \( \mathbb{R}^n \). Based on the fact that \( \{p_1, \ldots, p_m\} \) is a closed billiard trajectory
in $P$ we claim that
\begin{equation}
\dim \bigcap_{j=1}^{m} (H_j + c_j) > 0.
\end{equation}

Then, analogously to the proof of Lemma 2.9, \{p_1, ..., p_m\} can be translated in directions given by vectors in
\begin{equation}
\bigcap_{j=1}^{m} (H_j + c_j)
\end{equation}
while $p_1, ..., p_m$ are not leaving $F_1, ..., F_m$ until at least one of these points is a nonsmooth boundary point of $P$. Let $p + c$ with $c \in \mathbb{R}^n$ in (16) be this translate. (We argue below that $p + c$ is a closed billiard trajectory in $P$.)

We justify (15): Since $m = n$ the affine section $P \cap V$ of $P$ has dimension less or equal than $n-1$. Since $p$ is a closed billiard trajectory in $P$ it follows by Proposition 2.7 that we can write
\[
H_j = (H_j \cap V) \oplus W \quad \text{and} \quad (H_j \cap V) \perp W
\]
for all $j \in \{1, ..., m\}$, where we denote by $W$ the $(n - \dim V)$-dimensional orthogonal complement to $V$. Using that $c_1, ..., c_m$ where chosen to be vectors in $V_0$ we can write
\[
H_j + c_j = (H_j \cap V) \oplus W + c_j = ((H_j \cap V) + c_j) \oplus W \quad \text{[}((H_j + c_j) \cap V) \oplus W]\]
for all $j \in \{1, ..., m\}$ and therefore
\[
\dim \bigcap_{j=1}^{m} (H_j + c_j) = \dim \bigcap_{j=1}^{m} ((H_j \cap V) \oplus W + c_j) = \dim \bigcap_{j=1}^{m} (((H_j \cap V) + c_j) \oplus W)
\]
\[
= \dim \bigcap_{j=1}^{m} ((H_j \cap V) + c_j) + \dim (W) \geq \dim (W) = n - \dim V \geq 1.
\]

By using Proposition 2.4 we argue that $p + c$ (for $m \leq n - 1$ as well as for $m = n$) is a closed billiard trajectory in $P$: Since $p$ is a closed billiard trajectory in $P$ there are unit vectors $n_1, ..., n_m \in S^{n-1}$ such that
\[
\begin{cases}
    p_{j+1} - p_j = \lambda_j n_j, \quad \lambda_j > 0 \\
    n_{j+1} - n_j \in -N_P(p_{j+1})
\end{cases}
\]
is fulfilled for all $j \in \{1, ..., m\}$, where $N_P(p_1), ..., N_P(p_m)$ are one-dimensional since $p_1, ..., p_m$ are smooth boundary points of $P$. Then with
\[
N_P(p_j) \subseteq N_P(p_j + c)
\]
for all $j \in \{1, ..., m\}$ it follows that $p_1 + c, ..., p_m + c$ fulfill
\[
\begin{cases}
    (p_{j+1} + c) - (p_j + c) = p_{j+1} - p_j = \lambda_j n_j \\
    n_{j+1} - n_j \in -N_P(p_{j+1} + c) \subseteq -N_P(p_{j+1} + c)
\end{cases}
\]
for all $j \in \{1, ..., m\}$. Therefore Proposition 2.4 implies that $p + c$ is a closed billiard trajectory in $P$. \hfill \Box

A consequence of Proposition 2.10 is: If there is a length minimizing closed billiard trajectory in a convex polytope $P$ with less than $n + 1$ bouncing points then there always is a nonregular length minimizing closed billiard trajectory.
We briefly note that the opposite is not the case: There are convex polytopes in \( \mathbb{R}^n \) with closed inscribed nonregular billiard trajectories with \( \leq n \) bouncing points that cannot be translated into a regular one (cf. for instance the length minimizing closed billiard trajectory in Example F discussed in Section 5).

3. Proof of Theorem 1.2

In order to prove Theorem 1.2 it will be useful to formulate the following Lemma:

**Lemma 3.1.** Let \( H_1^+, ..., H_k^+ \) be half-spaces of \( \mathbb{R}^{d \geq 2} \) such that

\[
H_1^+ \cap ... \cap H_k^+
\]

is bounded in \( \mathbb{R}^d \). Let \( n_1, ..., n_k \) be the outer (with respect to \( H_1^+, ..., H_k^+ \)) unit vectors normal to \( H_1, ..., H_k \). The following holds for every \( j \in \{1, ..., k\} \): There is an \( \varepsilon_j > 0 \) such that

\[
H_j^{\text{pert},+} \cap \left( \bigcap_{i=1, i \neq j}^k H_i^+ \right)
\]

is bounded in \( \mathbb{R}^d \) for all \( H_j^{\text{pert}} := \partial(H_j^{\text{pert},+}) \) whose outer unit normal vector is an element of \( S^{d-1} \cap B_{\varepsilon_j}(n_j) \), where by \( B_{\varepsilon_j}(n_j) \) we denote the \( d \)-dimensional ball of radius \( \varepsilon_j \) fixed in \( n_j \).

**Proof.** The statement is equivalent to the following one: Let \( n_1, ..., n_k \in S^{d-1} \) be unit vectors with 0 in the interior of the convex hull \( \text{conv}\{n_1, ..., n_k\} \). Then for every \( j \in \{1, ..., k\} \) there is an \( \varepsilon_j > 0 \) such that 0 is in the interior of

\[
\text{conv}\{n_1, ..., n_{j-1}, n_j^{\text{pert}}, n_{j+1}, ..., n_k\}
\]

for every

\[
n_j^{\text{pert}} \in S^{d-1} \cap B_{\varepsilon_j}(n_j).
\]

But this is clear since for every \( j \in \{1, ..., k\} \) the fact

"0 is in the interior of \( \text{conv}\{n_1, ..., n_k\}" \]

is invariant under small perturbations of \( n_j \). \( \square \)

We come to the proof of Theorem 1.2:

**Proof of Theorem 1.2** Let \( p = (p_1, ..., p_m) \) be a length minimizing closed billiard trajectory in \( T \) and \( T \cap V \) the smallest affine section containing \( p \), i.e. \( \dim V \leq \min\{n, m-1\} \).

Then by Proposition 2.8 there is a selection \( \{i_1, ..., i_{\dim V+1}\} \subseteq \{1, ..., m\} \) with

\[
\{p_{i_1}, ..., p_{i_{\dim V+1}}\} \in F(T).
\]

Without loss of generality we can assume \( i_1 < ... < i_{\dim V+1} \) and define the closed polygonal line

\[
\tilde{p} := (p_{i_1}, ..., p_{i_{\dim V+1}}).
\]

For \( m > \dim V + 1 \) it follows \( \ell(\tilde{p}) < \ell(p) \). But with Theorem 2.2 this is a contradiction to the minimality of \( p \). Therefore it follows \( \dim V = m - 1 \).

Let us denote by \( H_1^+, ..., H_m^+ \) the closed half-spaces defined by \( \partial H_j^+ = H_j \) and \( T \subseteq H_j^+ \) for all \( j \in \{1, ..., m\} \), where by \( H_1, ..., H_m \) we denote the \( T \)-supporting
hyperplanes through \( p_1, \ldots, p_m \) which are related to the billiard reflection rule in these points. By Proposition 2.7 we conclude that

\[
H_j = (H_j \cap V) \oplus W \quad \text{and} \quad H_j^+ = (H_j^+ \cap V) \oplus W, \quad j \in \{1, \ldots, m\},
\]

where \( W \) is the orthogonal complement to \( V \), and

\[
\bigcap_{j=1}^m (H_j^+ \cap V) \text{ bounded in } V \left( \bigcap_{j=1}^m H_j^+ \text{ nearly bounded in } \mathbb{R}^n \right).
\]

Let \( n_1, \ldots, n_m \) be the outer unit vectors normal to \( H_1, \ldots, H_m \). Then it follows by (17) that

\[
n_j \in N_T(p_j) \cap V_0
\]

and therefore

\[
\dim (N_T(p_j) \cap V_0) \geq 1
\]

for all \( j \in \{1, \ldots, m\} \).

Let us assume there is an \( i \in \{1, \ldots, m\} \) such that

\[
\dim (N_T(p_i) \cap V_0) > 1.
\]

Noting

\[
N_T(p_i) \cap V_0 \subseteq N_{T \cap V}(p_i)
\]

it follows

\[
\dim (N_T(p_i) \cap V_0 \cap N_{T \cap V}(p_i)) > 1,
\]

and because of Lemma 3.1 (for \( d = n \) and \( k = m \)) we can find a unit vector

\[
n_i^{pert} \in N_T(p_i) \cap N_{T \cap V}(p_i)
\]

with \( n_i^{pert} \neq n_i \) such that

\[
H_i^{pert, +} \cap \left( \bigcap_{j=1, j \neq i}^m (H_j^+ \cap V) \right)
\]

remains bounded in \( V \) where we denote by \( H_i^{pert, +} \) the closed half-space of \( V \) that contains \( T \cap V \) and which is bounded by \( H_i^{pert} \) which is the hyperplane in \( V \) through \( p_i \) that is normal to \( n_i^{pert} \). Since by Proposition 2.5 the billiard reflection rule in \( p_i \) (as bouncing point of the closed billiard trajectory \( p \) in \( T \cap V \), cf. Proposition 2.3) is no longer fulfilled with respect to the perturbed hyperplane \( H_i^{pert} \), the bouncing point \( p_i \) can be moved along \( H_i^{pert} \), say to \( p_i^{pert} \), in order to reduce the length of the polygonal line segment \((p_{i-1}, p_i, p_{i+1})\). We define the closed polygonal line

\[
\tilde{\rho} := (p_1, \ldots, p_i, p_i^{pert}, p_{i+1}, \ldots, p_3),
\]

for which with the boundedness of (19) in \( V \) we conclude \( \tilde{\rho} \in F(T \cap V) \) by Lemma 2.1(2). Now we argue that \( \tilde{\rho} \in F(T) \). With the boundedness of (19) in \( V \) it follows with

\[
H_i^{pert} := H_i^{pert} \oplus W, \quad H_i^{pert, +} := H_i^{pert, +} \oplus W
\]

and (17) the nearly boundedness of

\[
H_i^{pert, +} \cap \left( \bigcap_{j=1, j \neq i}^m H_j^+ \right)
\]

in \( \mathbb{R}^n \).
Indeed, when the intersection in (19) is bounded in $V$ then there is a hyperplane $H$ in $V$ such that the intersection lies between $H$ and $H + d$ for an appropriate $d \in V_0$. Then it follows with (17) and (20) that

$$H_i^{\text{pert},+} \cap \left( \bigcap_{j=1, j \neq i}^{m} H_j^+ \right) = \left( H_i^{\text{pert},+} \oplus W \right) \cap \left( \bigcap_{j=1, j \neq i}^{m} ((H_j^+ \cap V) \oplus W) \right)$$

lies between the hyperplanes $H \oplus W$ and $(H + d) \oplus W$.

Since $H_i^{\text{pert}}$ is a $T$-supporting hyperplane through $p_i$ (what follows from the fact that by (18) its outer unit normal vector $n_i^{\text{pert}}$ is an element of $N_T(p_i)$) we conclude that $T$ is a subset of the intersection in (21). Then it follows from the nearly boundedness (in $\mathbb{R}^n$) of the intersection in (21) together with Lemma 2.1(ii) that $\tilde{p} \in F(T)$. By referring to Theorem 2.2 from $\ell(\tilde{p}) < \ell(p)$ we derive a contradiction to the minimality of $p$.

Therefore:

$$\dim (N_T(p_i) \cap V_0) = 1.$$  \hfill \square

4. PROOF OF THEOREM 1.3

Before we prove Theorem 1.3 we need the following Lemma whose proof was brought to the authors’ attention by A. Balitskiy:

**Lemma 4.1.** Let $P \subset \mathbb{R}^n$ be an acute convex polytope. Then for every nonsmooth boundary point $p$ of $P$ and for every ray $\rho \subset N_P(p)$ there is a section $N_P(p) \cap \tau$ by a two-dimensional plane $\tau \supset \rho$ that contains an angle greater than $\pi/2$.

**Proof.** Let $p$ be a nonsmooth boundary point of $P$. Then $p$ lies in the relative interior of a $k$-face $F_k$, $0 \leq k \leq n - 2$. For a ray $\rho \subset N_P(p)$ we would like to find a two-dimensional plane $\tau \supset \rho$ such that $N_P(p) \cap \tau$ is an obtuse angle. Equivalently, we need to place $P$ in an acute dihedral angle $H_1^+ \cap H_2^+$ (bounded by hyperplanes $H_1$ and $H_2$) such that $p \in H_1 \cap H_2$ and $(H_1 \cap H_2) \perp \rho$. Take $H_1$ to be the spanning hyperplane of any facet $\mathcal{F}$ containing $F_k$. Let $\ell$ be the one-dimensional subspace normal to $\mathcal{F}$ (and $H_1$). Consider the cylinder $\mathcal{F} + \ell$. The acuteness of dihedral angles of $P$ implies that the cylinder contains $P$. Let $H_2$ be a supporting hyperplane for the cylinder at $p$ such that $(H_1 \cap H_2) \perp \rho$. Since $\partial(\mathcal{F} + \ell) \cap P$ is the relative boundary of $\mathcal{F}$ one can tilt $H_2$ slightly with respect to $H_1 \cap H_2$ such that the dihedral angle containing $P$ becomes acute. \hfill \square

**Proof of Theorem 1.3.** The proof uses Theorem 1.2, Proposition 2.10 and Lemma 4.1 combined with the proof of Theorem 3.3 in [1] which is based on the fact that for any closed nonregular billiard trajectory $p = (p_1, ..., p_m)$ in $P$ there is a closed polygonal line $\tilde{p} = (\tilde{p}_1, ..., \tilde{p}_{m+1}) \in F(P)$ with $\ell(\tilde{p}) < \ell(p)$.

We briefly rephrase the argument on which is based the proof of Theorem 3.3. in [1]: Let $p = (p_1, ..., p_m)$ be any nonregular closed billiard trajectory in $P$. Then there is a nonsmooth boundary point $p_i$ of $P$. The acuteness of $P$ guarantees with Lemma 1.1 the existence of a two-dimensional plane $\tau$ containing the ray emanating from $p_i$ with direction $n_P(p_i) \in N_P(p_i)$ opposite to the bisector of the polygonal
exists a closed polygonal line with Theorem 1.2 this implies the stated regularity result.

in $P$ (cf. Lemma 2.1(ii)). This guarantees how at polygonal segment $p_i$.

Thus, if we replace the billiard trajectory segment $(p_{i-1}, p_i, p_{i+1})$ with $H_q$ then we have $\ell(p) < \ell(q)$. Since $n_P(p_i)$ is a positive combination of $n_i^t$ and $n_i^r$ the normals at the vertices of $p_i$ still surround the origin (cf. Lemma 2.1(iii)). This guarantees $\tilde{p} \in F(T)$.

Let us assume there is a length minimizing closed billiard trajectory $p = (p_1, \ldots, p_m)$ in $P$ with $m \leq n$. Then with Proposition 2.10 $p$ can be translated within $P$ to a nonregular closed billiard trajectory $p + c$, $c \in \mathbb{R}^n$. Then, as shown above, there exists a closed polygonal line $\tilde{p} = (\tilde{p}_1, \ldots, \tilde{p}_{m+1}) \in F(P)$ with $\ell(\tilde{p}) < \ell(p + c) = \ell(p)$. But because of the minimality of $p$ this is a contradiction to Theorem 2.2. Together with Theorem 1.2 this implies the stated regularity result. □

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure4.png}
\caption{The billiard trajectory segment $(p_{i-1}, p_i, p_{i+1})$ is replaced by the polygonal line segment $(p_{i-1}, p_i^*, p_{i+1}^*, p_{i+1})$.}
\end{figure}
Theorem 1.3 and geometrical considerations make us formulate the following conjecture:

**Conjecture.** Every acute convex polytope in $\mathbb{R}^n$ is a simplex.

The conjecture is true for $n = 2$. Indeed, the formula for the sum of the interior angles of any convex polygon $P$ with $k \geq 3$ edges is $(k - 2)\pi$. We conclude

$$\min_{P \text{ convex polygon with } k \text{ vertices}} \max_{\alpha \text{ interior angle of } P} \alpha = \frac{(k - 2)\pi}{k}$$

(22)

Therefore every acute convex polygon must have three edges, i.e. a simplex.

It remains an open problem whether this approach can be generalized to higher dimensions (below we will prove the conjecture for $n = 3$ using a different argument).

Conversely, a sufficient condition for a counterexample, i.e. an acute convex polytope which is not a simplex, is that the sum of all dihedral angles is greater than $\binom{n}{2}\pi$. For that we recall that Gaddum proved in [5] that the sum of the $\binom{n+1}{2}$ dihedral angles of a simplex in $\mathbb{R}^n$ lies between $\lfloor \frac{n^2-1}{4} \rfloor \pi$ and $\binom{n}{2}\pi$ and the sum can take any value in this range.

Another sufficient condition for a counterexample is that the sum of all dihedral angles is greater or equal than $\frac{k\pi}{2}$, where $k$ is the number of dihedral angles. Otherwise, if there is an acute simplex with dihedral angle sum greater or equal than $\frac{k\pi}{2}$, then, using the argument in (22) generalized to higher dimensions, there is at least one angle greater or equal $\frac{\pi}{2}$. This is a contradiction to the acuteness. Since $k$ has to be greater or equal than $\binom{n+1}{2}$ (simplices are minimizing the number of dihedral angles over all convex polytopes) we get that a sufficient condition for a counterexample is that the sum of the dihedral angles is greater than $\binom{n+1}{2}\frac{\pi}{2} = \frac{n}{2}(n + 1)n$ which for $n \geq 4$ is even smaller than $\binom{n}{2}\pi = \frac{n}{2}n(n-1)$ (and equal for $n = 3$). This implies an improvement of Gaddum’s result for the special case of dihedral acute angled simplices:

**Proposition 4.2.** Let $\Delta_n \subset \mathbb{R}^n$ be an acute simplex, i.e. dihedral acute angled simplex. Then the sum of the $\binom{n+1}{2}$ dihedral angles lies between $\lfloor \frac{n^2-1}{4} \rfloor \pi$ and $\binom{n+1}{2}\frac{\pi}{2}$.

Let us briefly argue that the conjecture is also true for $n = 3$: Let $P$ be any acute convex polytope in $\mathbb{R}^3$. Let $F_1$ be a facet of $P$ and $G_1, \ldots, G_k$ the facets of $F_1$ (without loss of generality we have $G_j \cap G_{j+1} \neq \emptyset$ for all $j \in \{1, \ldots, k\}$, where we write $G_{k+1} = G_1$). Assume $F_1$ is not a simplex, then $k \geq 4$. Without loss of generality we assume $F_1 \cap F_2 = G_1, \ldots, F_1 \cap F_k = G_k$, where $F_2, \ldots, F_k$ are other (pairwise distinct) facets of $P$. All dihedral angles between $F_1$ and $F_2, \ldots, F_k$ are acute. If one of the angles enclosed by $G_j$ and $G_{j+1}$ is not acute, then also the dihedral angle between $F_{j+1}$ and $F_{j+2}$ - which is greater or equal than the angle between $G_j$ and $G_{j+1}$ - are not acute (we write $F_{k+2} = F_2$). This would be contradiction to the fact that $P$ is acute. Therefore, $F_1$ is a simplex, i.e. $k = 3$. Now we argue that $F_1, F_2, F_3, F_4$ are the only facets of $P$: Let us assume there is another facet $F_5$. Then, necessarily $F_5$ has nonempty intersection with $F_2, F_3, F_4$ and since the dihedral angles between $F_1$ and $F_2, F_3, F_4$ are all acute it follows that the orthogonal projection of $F_5$ onto the hyperplane supporting $F_1$ is
closed polygonal lines in $F$. This implies that at least one of the dihedral angles between $F_5$ and $F_2$, $F_3$, $F_4$ is not acute. Again, this would be a contradiction to the fact that $P$ is acute. This implies that $F_1$, $F_2$, $F_3$ and $F_4$ are the only facets of $P$ and therefore $P$ is a simplex.

Regardless of the fact that (for general dimension) some technical details need to be clarified, this proof-method also seems promising for proving the conjecture for general dimension.

5. Examples

We begin by noting that in Proposition 2.3 in general the length minimality of a closed billiard trajectory is not invariant under going to affine sections of the billiard table containing this trajectory.

More precisely, let $T \subset \mathbb{R}^3$ be a billiard table and $p = (p_1, ..., p_m)$ a length minimizing closed billiard trajectory in $T$. If $T \cap V$ is an affine section of $T$ containing $p$ then $p$ may not be a length minimizing closed billiard trajectory in $T \cap V$.

In the following we denote the $(x_i, x_j)$-plane of $\mathbb{R}^3$ by $X_{i,j}$ (for $i, j \in \{1, 2, 3\}, i \neq j$).

**Example A:** Let $T \subset \mathbb{R}^3$ be the convex hull of the points

$$(-\frac{1}{2}, 0, 0), (\frac{1}{2}, 0, 0), (0, 0, 1), (-\frac{1}{2}, -\frac{1}{2}, 0), (\frac{1}{2}, -\frac{1}{2}, 0), (0, -\frac{1}{2}, 1).$$

One checks that $p = (p_1, p_2)$ with

$$p_1 = (0, 0, \frac{3}{2}) \quad \text{and} \quad p_2 = (0, -\frac{1}{2}, \frac{3}{2})$$

is a length minimizing closed billiard trajectory in $T$ with $\ell(p) = 1$. We define $T \cap V$ to be the affine section of $T$ where

$$V := X_{1,2} + (0, 0, \frac{2}{3}).$$

One checks that $\tilde{p} = (\tilde{p}_1, \tilde{p}_2)$ with

$$\tilde{p}_1 = (-\frac{1}{8}, -\frac{1}{4}, \frac{3}{2}) \quad \text{and} \quad \tilde{p}_2 = (\frac{1}{8}, -\frac{1}{4}, \frac{3}{2})$$

is a length minimizing closed billiard trajectory in $T \cap V$ with $\ell(\tilde{p}) = \frac{1}{2} < \ell(p)$. Therefore $p$ is not a length minimizing closed billiard trajectory in $T \cap V$. $\Box$

In the above described situation $p$ may not even locally minimize the length of closed polygonal lines in $F(T \cap V)$.

**Example B:** Let $T \subset \mathbb{R}^3$ be the convex hull of the points

$$(0, 0, 0), (4, 0, 0), (0, -4, 0), (\frac{16}{3}, -\frac{16}{3}, 0), (0, 0, 8), (0, -4, 8).$$

One checks that $p = (p_1, p_2)$ with

$$p_1 = (0, 0, 4) \quad \text{and} \quad p_2 = (\frac{8}{5}, -\frac{16}{5}, 4)$$

is a length minimizing closed billiard trajectory in $T$ with $\ell(p) = \frac{16}{\sqrt{5}}$. If we define

$$V := X_{1,2} + (0, 0, 4)$$

then $T \cap V$ is an affine section of $T$ containing $p$ and is given by the vertices

$$(0, 0, 4), (2, 0, 4), (\frac{8}{5}, -\frac{16}{5}, 4), (0, -4, 4).$$

By slightly moving $p_2$ clockwise along $\partial(T \cap V)$ (we denote this slightly perturbed $p_2$ by $\tilde{p}_2$) the closed polygonal line $\tilde{p} = (p_1, \tilde{p}_2)$ is in $F(T \cap V)$ but not in $F(T)$ (since $\pi_{1,2}(\tilde{p}_2)$ is in the interior of $\pi_{1,2}(T)$). Additionally one has $\ell(\tilde{p}) < \ell(p)$. Therefore $p$ does not locally minimize the length of closed polygonal lines in $F(T \cap V)$. $\Box$
The length minimality of a closed billiard trajectory is not invariant under going to affine sections of the billiard table containing this trajectory.

The same can be shown for the smallest affine sections containing length minimizing closed billiard trajectories: Let \( T \subset \mathbb{R}^n \) be a billiard table and \( p = (p_1, ..., p_m) \) a length minimizing closed billiard trajectory in \( T \). Let \( T \cap V \) be the smallest affine section of \( T \) containing \( p \). Then \( p \) may not be a length minimizing closed billiard trajectory in \( T \cap V \).

**Example C:** Let \( T \subset \mathbb{R}^3 \) be the convex hull of the points

\[
(0, 0, 0), (1, 0, 0), \left( \frac{1}{2}, 0, \frac{\sqrt{3}}{2} \right), (0, -2, 0), (1, -2, 0).
\]

One checks that \( p = (p_1, p_2, p_3) \) with

\[
p_1 = \left( \frac{1}{2}, -1, 0 \right), p_2 = \left( \frac{1}{4}, -1, \frac{\sqrt{3}}{4} \right), p_3 = \left( \frac{3}{4}, -1, \frac{\sqrt{3}}{4} \right)
\]
is a length minimizing closed billiard trajectory in $T$ (it is the Fagnano triangle of the affine section $T \cap X_{1,3}$ translated by $(0, -1, 0)$). If we define
\[ V := X_{1,3} + (0, -1, 0) \]
then $T \cap V$ is the smallest affine section of $T$ containing $p$. One checks that $\tilde{p} = (p_1, \tilde{p}_2)$ with
\[ \tilde{p}_2 = \left( \frac{1}{2}, -1, \frac{\sqrt{3}}{4} \right) \]
is a length minimizing closed billiard trajectory in $T \cap V$ (with $\tilde{p} \notin F(T)$) but $\ell(\tilde{p}) < \ell(p)$. Therefore $p$ is not a length minimizing closed billiard trajectory in $T \cap V$. \hfill $\square$

Again, in the situation described in Example C $p$ may not even locally minimize the length of closed polygonal lines in $F(T \cap V)$.

**Example D:** We consider $T$, $V$ and $p = (p_1, p_2, p_3)$ from Example C. We slightly move $p_2$ clockwise along $\partial T \cap V$. We denote this slightly perturbed $p_2$ by $\tilde{p}_2^\delta = (\frac{1}{2} + \delta, -1, \frac{\sqrt{3}}{4})$, $\delta > 0$ small. The closed polygonal line $\tilde{p}_2^\delta = (\tilde{p}_1, \tilde{p}_2^\delta, \tilde{p}_3)$ with $\tilde{p}_1 = p_1$ and $\tilde{p}_3 = p_3$ fulfills $\tilde{p}_2^\delta \in F(T \cap V)$, $\tilde{p}_2^\delta \notin F(T)$, $\ell(\tilde{p}_2^\delta) < \ell(p)$ (for small $\delta > 0$) and $\tilde{p}_2^\delta$ converges with respect to the Hausdorff distance to $p$ for $\delta \to 0$. \hfill $\square$

We conclude this Section by illustrating in which sense the statement of Theorem 1.2 is sharp: We recall from the introduction for what we want to give examples:

(iii) A length minimizing closed billiard trajectory in $T \subset \mathbb{R}^n$ may not be regular within the smallest section of $T$ containing the billiard trajectory. This can even appear for the unique length minimizing closed billiard trajectory.

(iv) A length minimizing closed billiard trajectory in $T \subset \mathbb{R}^n$ can have bouncing points in vertices as well as in the interior of more than 0-dimensional faces of $T$.

For the first statement within (iii) we consider Example E, for the second Example F. Examples E and F are also suitable in order to prove the statements in (iv): for the first we refer to Example E ($p_2^a$ as bouncing point of the length minimizing closed billiard trajectory $p_2$ is a vertex of $T_\varepsilon$), for the second to Example F ($p_3$ as bouncing point of the unique length minimizing closed billiard trajectory $p$ is an interior point of an one-dimensional face of $T \subset \mathbb{R}^3$).

We remark that Examples E and F both are convex polytopes. Nevertheless, this examples proving the statements in (iii) and (iv) are not restricted to convex polytopes. One can check that both Example E as well as Example F can be made strictly convex without losing the characteristics utilized within the proofs of (iii) and (iv).

**Example E:** Let $T_\varepsilon \subset \mathbb{R}^3$, $\varepsilon > 0$ small, be the convex polytope given by the vertices
\[
(0, 0, 0), \left( -\frac{1}{2}, 0, \frac{\sqrt{3}}{2} \right), \left( \frac{1}{2}, 0, \frac{\sqrt{3}}{2} \right), (0, -2, 0), \left( 0, -2, \frac{\sqrt{3}}{2} \right),
\]
\[
\left( -\frac{1}{2} + \frac{\sqrt{3}}{\varepsilon}, -2, \frac{\sqrt{3}}{2} - \varepsilon \right), \left( \frac{1}{2} - \frac{\sqrt{3}}{\varepsilon}, -2, \frac{\sqrt{3}}{2} - \varepsilon \right).
\]

We claim that for sufficiently small $\varepsilon > 0$ the length minimizing closed billiard trajectories in $T_\varepsilon$ are given by $p_2^a = (p_1^2, p_2^a, p_3^a)$ with
\[
p_1^a = \left( -\frac{1}{4}, -a, \frac{\sqrt{3}}{4} \right), p_2^a = \left( \frac{1}{4}, -a, \frac{\sqrt{3}}{4} \right), p_3^a = \left( 0, -a, \frac{\sqrt{3}}{4} \right).
Figure 7. Example E.

and $a \in [0, 2]$. Moreover, we claim that $p^2$ is not regular within the smallest affine section of $T_\varepsilon$ containing $p^2$.

Indeed, for all $a \in [0, 2]$ $p^a$ is contained in the affine section $T_\varepsilon \cap V^a$ of $T_\varepsilon$, where

$$V^a := X_{1,3} + (0, -a, 0),$$

and is subset of the equilateral triangle $\Delta^a_1$ given by the vertices

$$(0, -a, 0), \left(-\frac{1}{2}, -a, \frac{\sqrt{3}}{2}\right), \left(\frac{1}{2}, -a, -\frac{\sqrt{3}}{2}\right).$$

For all $a \in [0, 2]$ holds the following: It is $\ell(p^a) = \frac{3}{2}$ and $p^a$ is coinciding with the Fagnano triangle of $\Delta^a_1$ which is the unique length minimizing closed billiard trajectory in $\Delta^a_1$. Note that the Fagnano triangle is the only regular closed billiard trajectory in $\Delta^a_1$. The next longer closed billiard trajectories in $\Delta^a_1$ have two bouncing points and length $\sqrt{3}$. By construction $p^a$ is a closed billiard trajectory in $T_\varepsilon$ as well as in $T_\varepsilon \cap V^a$: The hyperplanes in $\mathbb{R}^3$, respectively in $V^a$, related to the billiard reflection rule are the one which are normal to the bisectors of the polygonal line segments $(p^1_1, p^2_2, p^3_3)$, $(p^2_1, p^3_2, p^1_3)$ and $(p^3_1, p^1_2, p^2_3)$. Since

$$d_H(T_\varepsilon \cap V^a, \Delta^a_1) \to 0 \text{ for } \varepsilon \to 0$$

we conclude for sufficiently small $\varepsilon > 0$ that $p^a$ is the unique length minimizing closed billiard trajectory in $T_\varepsilon \cap V^a$.

We show that all other closed billiard trajectories in $T_\varepsilon$ have a length greater than $\frac{3}{2}$: Whenever we consider a closed billiard trajectory in $T_\varepsilon$ with one bouncing point on the front-facet $T_\varepsilon \cap V^0$ and with another one on the back-facet $T_\varepsilon \cap V^2$ of $T_\varepsilon$, then it has a length greater or equal 4. For every other closed billiard trajectory
closed billiard trajectory in $T_\varepsilon$, we have $$\pi_{1,3}(p') + (0, -2, 0) \in F(T_\varepsilon \cap V^2),$$ where by $\pi_{1,3}$ we denote the orthogonal projection onto $X_{1,3}$. This follows from the fact that $T_\varepsilon \cap V^b \subset T_\varepsilon \cap V^a$ for $0 \leq a < b \leq 2$. But this implies $$\ell(p') \geq \ell(\pi_{1,3}(p') + (0, -2, 0)) \geq \ell(p^2) = \frac{3}{2}.$$ $p^a$ is regular in $T_\varepsilon \cap V^a$ for all $a \in [0, 2)$. This is due to the fact that $T_\varepsilon \cap V^a$ is the smallest affine section of $T_\varepsilon$ containing $p^a$ and the normal cones $N_{T_\varepsilon \cap V^j}(p^j)$ are one-dimensional for all $j \in \{1, 2, 3\}$ and all $a \in [0, 2)$. In contrast to that, $p^2$ is not regular in $T_\varepsilon \cap V^2$: The normal cone $N_{T_\varepsilon \cap V^2}(p^2)$ is two-dimensional, i.e. $p^2$ is a nonsmooth boundary point of $T_\varepsilon \cap V^2$.

We clearly see why the argument used in the proof of Theorem 1.2 does not work: Indeed, it is hardly possible to construct a closed polygonal line $\tilde{p}^2 \in F(T_\varepsilon \cap V^2)$ with $\ell(\tilde{p}^2) < \ell(p^2)$ while guaranteeing $\tilde{p}^2 \in F(T_\varepsilon)$.

**Example F:** Let $T_\varepsilon \subset \mathbb{R}^3$, $\varepsilon > 0$ small, be the convex polytope given by the vertices

$$(0, 0, 0), \left(-\frac{1}{2} + \frac{\varepsilon}{\sqrt{3}}, 0, \frac{\sqrt{3}}{2} - \varepsilon\right), \left(-\frac{1}{2} + \frac{\varepsilon}{\sqrt{3}}, 0, \frac{\sqrt{3}}{2} - \varepsilon\right), \left(\frac{1}{2} - \frac{\varepsilon}{\sqrt{3}}, 0, \frac{\sqrt{3}}{2} - \varepsilon\right),$$

$$(0, -2, 0), \left(-\frac{1}{2} + \frac{\varepsilon}{\sqrt{3}}, -2, \frac{\sqrt{3}}{2} - \varepsilon\right), \left(\frac{1}{2} - \frac{\varepsilon}{\sqrt{3}}, -2, \frac{\sqrt{3}}{2} - \varepsilon\right), \left(\frac{1}{2} - \frac{\varepsilon}{\sqrt{3}}, -2, \frac{\sqrt{3}}{2} - \varepsilon\right).$$

We claim that for sufficiently small $\varepsilon > 0$ the unique length minimizing closed billiard trajectory in $T_\varepsilon$ is given by $p = (p_1, p_2, p_3)$ with

$$p_1 = \left(-\frac{1}{2}, -1, \frac{\sqrt{3}}{2}\right), p_2 = \left(\frac{1}{2}, -1, \frac{\sqrt{3}}{2}\right), p_3 = \left(0, -1, \frac{\sqrt{3}}{2}\right).$$

Moreover, we claim that $p_3$ is not regular within the smallest affine section of $T_\varepsilon$ containing $p$.

Indeed, $p$ is contained in the affine section $T_\varepsilon \cap V$ with

$$V := X_{1,3} + (0, -1, 0).$$

$T_\varepsilon \cap V$ is given by the vertices

$$(0, -1, 0), \left(-\frac{1}{2} + \frac{\varepsilon}{\sqrt{3}}, -1, \frac{\sqrt{3}}{2} - \varepsilon\right), \left(-\frac{1}{2} + \frac{\varepsilon}{\sqrt{3}}, -1, \frac{\sqrt{3}}{2} - \varepsilon\right),$$

$$(0, -1, \frac{\sqrt{3}}{2}), \left(\frac{1}{2} - \frac{\varepsilon}{\sqrt{3}}, -1, \frac{\sqrt{3}}{2} - \varepsilon\right), \left(\frac{1}{2} - \frac{\varepsilon}{\sqrt{3}}, -1, \frac{\sqrt{3}}{2} - \varepsilon\right)$$

and is subset of the equilateral triangle $\Delta_1$ given by the vertices

$$(0, -1, 0), \left(-\frac{1}{2}, -1, \frac{\sqrt{3}}{2}\right), \left(\frac{1}{2}, -1, \frac{\sqrt{3}}{2}\right).$$

$p$ is coinciding with the Fagnano triangle of $\Delta_1$ which is the unique length minimizing closed billiard trajectory in $\Delta_1$ (with length $\frac{3}{2}$). By construction $p$ is a closed billiard trajectory in $T_\varepsilon$ as well as in $T_\varepsilon \cap V$: The hyperplanes in $\mathbb{R}^3$, respectively in $V$, related to the billiard reflection rule are the one which are normal to
the bisectors of the polygonal line segments \((p_1, p_2, p_3), (p_2, p_3, p_1)\) and \((p_3, p_1, p_2)\).

Since
\[
d_H(T_\varepsilon \cap V, \Delta_1) \to 0 \quad \text{for} \quad \varepsilon \to 0
\]
we conclude for sufficiently small \(\varepsilon > 0\), as in Example E, that \(p\) is the unique length minimizing closed billiard trajectory in \(T_\varepsilon \cap V\).

We show that all other closed billiard trajectories in \(T_\varepsilon\) have a length greater than \(\frac{3}{2}\).

For that we first define the following facets of \(T_\varepsilon\): Let \(F_1\) be the facet given as the intersection of \(T_\varepsilon\) and the hyperplane through the points
\[
\left(-\frac{1}{2} + \frac{\varepsilon}{2\sqrt{3}}, 0, \frac{\varepsilon}{2}\right), \left(\frac{1}{2} - \frac{\varepsilon}{2\sqrt{3}}, 0, \frac{\varepsilon}{2} - \varepsilon\right), \left(\frac{1}{2} - \frac{\varepsilon}{2\sqrt{3}}, -2, \frac{\varepsilon}{2}\right),
\]
and let \(F_2\) be the facet given as the intersection of \(T_\varepsilon\) and the hyperplane through the points
\[
\left(-\frac{1}{2} + \frac{\varepsilon}{2\sqrt{3}}, 0, \frac{\varepsilon}{2}\right), \left(\frac{1}{2} - \frac{\varepsilon}{2\sqrt{3}}, -2, \frac{\varepsilon}{2}\right), \left(-\frac{1}{2} + \frac{\varepsilon}{2\sqrt{3}}, -2, \frac{\varepsilon}{2} - \varepsilon\right).
\]

Then we begin to argue: Again, as in Example E, we first notice that whenever we consider a closed billiard trajectory in \(T_\varepsilon\) with one bouncing point on the front-facet \(T_\varepsilon \cap X_{1,3}\) and another one on the back-facet \(T_\varepsilon \cap (X_{1,3} + (0, -2, 0))\) of \(T_\varepsilon\), then it has a length greater or equal than 4.

Using the properties of (length minimizing) closed billiard trajectories from Section 2 we can check that for any other closed billiard trajectory \(p'\) in \(T_\varepsilon\), which is coming into question for minimizing the length, we have the following two cases:
First case: $p'$ is a closed billiard trajectory in $T_\varepsilon$ which is contained in $T_\varepsilon \cap (V + (0, 1 - a, 0))$ for an $a \in [0, 2]$. Then it follows

$$\pi_{1,3}(p') \in F(\pi_{1,3}(T_\varepsilon)).$$

This is due to the fact that in this situation by Proposition \[2.6\] and the geometric position of $F_1$ and $F_2$ there can be no bouncing point of $p'$ on $(F_1 \cup F_2) \setminus (F_1 \cap F_2)$. Therefore every bouncing point of $p'$ is on $\partial(\pi_{1,3}(T_\varepsilon)) + (0, -a, 0)$ for a proper $a \in [0, 2]$. Then, with $\pi_{1,3}(T_\varepsilon \cap V) \subset \pi_{1,3}(T_\varepsilon)$ we have $\ell(p') \geq \ell(p)$. Further, together with the fact that $p$ is a unique minimizer of $\Delta_1$ it follows $\ell(p') > \ell(p)$ for $p' \neq p$.

Second case: $p'$ is a closed billiard trajectory in $T_\varepsilon$ which is not contained in $T_\varepsilon \cap (V + (0, 1 - a, 0))$ for any $a \in [0, 2]$ but which has a bouncing point on $F_1 \cup F_2$. Without loss of generality we assume it is on $\bar{F}_1$. Because of Proposition \[2.6\] all other bouncing points of $p'$ can only be contained in the back-facet $T_\varepsilon \cap (X_{1,3} + (0, -2, 0))$ in a way that $\pi_{1,3}(p') \in F(T_\varepsilon \cap V)$. For $\varepsilon \to 0$ this implies

$$\ell(p') \geq \ell(\pi_{1,3}(p')) > \ell(p)$$

for $p' \neq p$. Similarly it can be argued for $p'$ on $\bar{F}_2$ together with the front-facet $T_\varepsilon \cap X_{1,3}$. For $p'$ on $F_1 \cap F_2$ both the back- and the front-facet come into question while the argument remains the same.

$p$ is not regular in $T_\varepsilon \cap V$ since the normal cone $N_{T_\varepsilon \cap V}(p_3)$ is two-dimensional, i.e. $p_3$ is a nonsmooth boundary point of $T_\varepsilon \cap V$.

Again, we clearly see why the argument used in the proof of Theorem \[1.2\] does not work. The $T_\varepsilon$-supporting hyperplane through $p_3$ for which the billiard reflection rule is fulfilled is

$$H_3 := X_{1,2} + \left(0, 0, \frac{\sqrt{3}}{2}\right).$$

The only way of perturbing $H_3$ to $H^\text{pert}_3$ as required within the proof of Theorem \[1.2\] is by tilting it around the axis through the points

$$\left(-\frac{1}{2} + \frac{\varepsilon}{\sqrt{3}}, 0, \frac{\sqrt{3}}{2}\right) \text{ and } \left(\frac{1}{2} - \frac{\varepsilon}{\sqrt{3}}, -2, \frac{\sqrt{3}}{2}\right).$$

But in any case

$$H_1^+ \cap H_2^+ \cap H^\text{pert,+}_3$$

cannot be nearly bounded in $\mathbb{R}^3$ (when $H_1$ and $H_2$ denote the uniquely determined $T_\varepsilon$-supporting hyperplanes through $p_1$ and $p_2$). Therefore it is not possible to construct a closed polygonal line $\bar{p} \in F(T_\varepsilon \cap V)$ with $\ell(\bar{p}) \leq \ell(p)$ and $\bar{p} \neq p$ while guaranteeing $\bar{p} \in F(T_\varepsilon)$.

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Stefan Krupp, Universität zu Köln, Mathematisches Institut, Weyertal 86-90, D-50931 Köln, Germany.

E-mail address: krupp@math.uni-koeln.de

Daniel Rudolf, Ruhr-Universität Bochum, Fakultät für Mathematik, Universitätssstraße 150, D-44801 Bochum, Germany.

E-mail address: daniel.rudolf@ruhr-uni-bochum.de