WAVELETS AND INFORMATION-PRESERVING TRANSFORMATIONS

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Abstract

The underlying mathematics of the wavelet formalism is a representation of the inhomogeneous Lorentz group or the affine group. Within the framework of wavelets, it is possible to define the “window” which allows us to introduce a Lorentz-covariant cut-off procedure. The window plays the central role in tackling the problem of photon localization. It is possible to make a transition from light waves to photons through the window. On the other hand, the windowed wave function loses analyticity. This loss of analyticity can be measured in terms of entropy difference. It is shown that this entropy difference can be defined in a Lorentz-invariant manner within the framework of the wavelet formalism.

1 Introduction

One of the still-unsolved problems in quantum mechanics is the transition from classical light waves to photons. Light waves are classical objects, and their quantum counterparts are photons. Then, are photons light waves? From the traditional theoretical point of view, the answer is NO [1]. However, this negative answer does not prevent us from examining how close photons are to waves by employing a new mathematical device called wavelet. The word “wavelet” is relatively new in physics [2, 3], but its concept was formulated in the 1960s [4]. The wavelet combines the traditional Fourier transformation with dilation or “squeeze” and translational symmetries. Since the squeezes and translations are the basic symmetries in the Poincaré group, and since the Fourier transformation is the standard language for the superposition principle, the wavelet formulation of light waves gives a covariant description of the superposition principle applicable to light waves.
Photons are relativistic particles requiring a covariant theoretical description. Wave functions in classical optics satisfy the superposition principle and can be localized, but they are not covariant under Lorentz transformations. The wavelet formalism makes light waves covariant, and this makes light waves closer to photons. Furthermore, the formalism allows us to make a quantitative analysis of the difference between these two clearly defined physical concepts. In this way, we can assert that photons are waves with a proper qualification [5].

In order to carry out this program, we need another important property of wavelets: translation symmetry. This symmetry allows us to introduce the concept of "window" [6, 7, 8, 9]. The window allows us to keep a function defined within a specified interval and let it vanish outside the interval or the window. This finite interval requires the concept of translation. From the mathematical point of view, this translational symmetry is very cumbersome and is often misunderstood by physicists. For instance, the translation does not commute with the Lorentz boost.

However, it is possible to define the order of transformations to preserve the information contained in the window [5]. The windowing process is a cut-off process, which leads to a loss of information. This information loss can be formulated in terms of the entropy change. It is shown that this entropy change can be formulated in a covariant manner. In this report, we give a brief review of the earlier work on this subject. Sections 2, 3, 4 consist of review of the recent paper by Han, Kim, and Noz [5]. We report a new result in Sec. 5. This section deals with entropy.

2 Light Waves and Wavelets

For light waves, we start with the usual expression

\[ F(z, t) = \frac{1}{\sqrt{2\pi}} \int g(k) e^{iku} dk , \]  

where \( u = (z - t) \). Even though light waves do not satisfy the Schrödinger equation, the very concept of the superposition principle was derived from the behavior of light waves. Furthermore, it was reconfirmed recently that light waves satisfy the superposition principle [10]. It is not difficult to carry out a spectral analysis on Eq.(1) and give a probability interpretation.

Before getting into the wavelet formalism, let us consider the expression

\[ A(z, t) = \int \frac{1}{\sqrt{2\pi \omega}} a(k) e^{iku} dk . \]  

This is the basic form we use in quantum electrodynamics, and is thus very familiar to us. We regard this as a classical quantity, with an understanding that it will become the photon field after second quantization. This is a covariant expression in the sense that
the norm
\[ \int \frac{|a(k)|^2}{2\pi\omega} \, dk. \tag{3} \]
is invariant under Lorentz transformations, because the integral measure \((1/\omega) \, dk\) is Lorentz-invariant. It is possible to give a particle interpretation to Eq. (2) after second quantization. However, \(A(z, t)\) cannot be used for the localization of photons. On the other hand, it is possible to give a localized probability interpretation to \(F(z, t)\) of Eq. (1), while it does not accept the particle interpretation of quantum field theory.

Under the Lorentz boost:
\[ z' = (\cosh \eta)z + (\sinh \eta)t, \quad t' = (\sinh \eta)z + (\cosh \eta)t, \tag{4} \]
the variables \(u\) and \(k\) become \(e^{-\eta}u\) and \(e^{\eta}k\) respectively. Thus, the Lorentz boost is a squeeze transformation in the phase space of \(u\) and \(k\) \([11]\). The expression given in Eq. (1) is not covariant if \(g(k)\) is a scalar function, because the measure \(dk\) is not invariant. If \(g(k)\) is not a scalar function, what is its transformation property? It was shown by Han, Kim, and Noz \([12]\) that we can solve this covariance problem by replacing \(F(u)\) and \(g(k)\) by \(F'(u)\) and \(g'(k)\) respectively defined as
\[ F'(u) = \sqrt{\frac{p}{\sigma}} F(u), \quad g'(k) = \sqrt{\frac{\sigma}{p}} g(k), \tag{5} \]
where \(p\) is the average value of the momentum:
\[ p = \frac{\int k|g(k)|^2 \, dk}{\int |g(k)|^2 \, dk}, \tag{6} \]
which becomes \(p = e^{\eta}p\) under the Lorentz boost. Then the functions of Eq. (3) will satisfy Parseval’s equation:
\[ \int |F'(u)|^2 \, du = \int |g'(k)|^2 \, dk \tag{7} \]
in every Lorentz frame without the burden of carrying the multipliers as given in Eq. (5). We can simplify the above cumbersome procedure by introducing the form
\[ G(u) = \frac{1}{\sqrt{2\pi p}} \int g(k)e^{iku} \, dk. \tag{8} \]
where the procedure for the Lorentz boost is to replace \(p\) by \(e^{\eta}p\), and \(k\) in \(g(k)\) by \(e^{-\eta}\). As is shown in Ref. \([12]\), this is a squeeze transformation. This is precisely the wavelet form for the localized light wave, and this definition is consistent with the form given in earlier papers on wavelets \([2, 4]\).

We are quite familiar with the expression of Eq. (1) for wave optics, and with that of Eq. (2) for quantum electrodynamics. The above expression satisfies the same superposition principle as Eq. (1), and has the same covariance property as Eq. (2). It is quite
similar to both Eq.(1) and Eq.(2), but they are not the same. The difference between
$F(u)$ of Eq.(1) and the wavelet $G(u)$ is insignificant. Other than the factor $\sqrt{\sigma}$ where
$\sigma$ has the dimension of the energy, the wavelet $G(u)$ has the same property as $F(u)$ in
every Lorentz frame [12].

However, there is still a significant difference between $G(u)$ of Eq.(8) and $A(u)$ of
Eq.(2). In Eq.(2), the factor $1/\sqrt{\omega}$ is inside the integral and is a variable. Thus the
superposition principle is not applicable to $A(u)$ with $a(k)$ as a spectral function. On
the other hand, in Eq.(8), the factor $1/\sqrt{p}$ is constant. Thus, the superposition principle
is applicable to $G(u)$ with $g(u)$ as a spectral function.

This difference disappears if the spectral functions $g(k)$ and $a(k)$ are delta functions.
The difference becomes non-trivial as the widths of these functions increase. If the
widths can be controlled, the difference can also be controlled. In so doing, it is essential
that the spectral functions vanish for $k = 0$ and for infinite values of $k$. We can achieve
this goal by using the concept of window.

3 Windows

Here, the window means that the spectral function is non-zero within a finite interval,
and vanishes everywhere else. There is a tendency to regard this type of cut-off procedure
as an approximation. On the other hand, we have to keep in mind that physics is an
experimental science. When we measure in laboratories, we do not measure functions,
but we take data points from which functions are constructed. It is well known that
functions so constructed can never be unique. This is the limitation of accuracy in
physics.

Thus, when we deal with a localized distribution, the window is a very useful math-
ematical device. Because of the inherent gap between a function and a collection of
data points, we do have the freedom of choosing a windowed function for a localized
distribution. The problem here is the question of covariance. Let us choose a window
in one Lorentz frame. How would this window look in a different Lorentz frame? Is this
window going to preserve all the information given in the initial frame.

In order to answer these questions, we have to examine the translation symmetry
of wavelets, particularly the translation combined with squeeze transformations. The
problem here is that this affiance symmetry can sometimes lead to information preserving
windows and sometimes non-preserving windows. Let us look at this problem closely.

The problem is caused by the fact that squeeze transformations do not commute
with translations. In the present case, the squeeze corresponds to a multiplication of
the variable $k$ by a real constant, while the translation is achieved by an addition of a
real number. To a given number, we can add another number, and we can also multiply
it by another real number. This combined mathematical operation is called the affine
transformation [13]. Since the multiplication does not commute with addition, affine
transformations can only be achieved by matrices. We can write the addition of \( b \) to \( x \) as
\[
\begin{pmatrix} x' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} .
\] (9)
This results in \( x' = x + b \). This is a translation. We can represent the multiplication of \( x \) by \( e^{\eta} \) as
\[
\begin{pmatrix} x' \\ 1 \end{pmatrix} = \begin{pmatrix} e^{\eta} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} ,
\] (10)
which leads to \( x' = e^{\eta}x \). This is a squeeze transformation.

We are interested in combined transformations. The translation does not commute with the squeeze. If the squeeze precedes the translation, we shall call this the affine transformation of the first kind, and the transformation takes the form
\[
x' = e^{\eta}x + b .
\] (11)
If the translation is made first, we shall call this the affine transformation of the second kind, and the transformation takes the form
\[
x' = e^{\eta}(x + b) .
\] (12)
The distinction between the first and second kinds is not mathematically precise, because the translation subgroup of the affine group is an invariant subgroup. We make this distinction purely for convenience. Whether we choose the first kind or second kind depends on the physical problem under consideration. For a covariant description of light waves, the affine transformation of the second kind is more appropriate, and the inverse of Eq. (12) is
\[
x = e^{-\eta}x' - b = e^{-\eta}(x' - e^{\eta}b) .
\] (13)
Therefore, the transformation of a function \( f(x) \) corresponding to the vector transformation of Eq. (12) is
\[
f \left( e^{-\eta}x - b \right) = f \left( e^{-\eta}(x - e^{\eta}b) \right) .
\] (14)
Next, does this translation lead to a problem in normalization as the squeeze did? The normalization integral does not depend on the translation parameter \( b \), but it does depend on the multiplication parameter \( \eta \). Indeed,
\[
\int |f(e^{-\eta}x - b)|^2 dx = e^{\eta} \int |f(x - b)|^2 dx .
\] (15)
In order to preserve the normalization under the affine transformation, we can introduce the form
\[
e^{-\eta/2}f(e^{-\eta}x - b) .
\] (16)
This is the wavelet form of the function \( f(e^{-\eta}x - b) \). This is of course the wavelet form of the second kind. The wavelet of the first kind will be
\[
e^{-\eta/2}f \left( e^{-\eta}(x - b) \right) .
\] (17)
Both the first and second kinds of wavelet forms are implicitly discussed in the literature \[2\]. Here, we are using this concept in constructing an information-preserving window under Lorentz boosts.

With this preparation, we can allow the function to be nonzero within the interval

\[ a \leq x \leq a + w , \tag{18} \]

while demanding that the function vanish everywhere else. The parameter \( w \) determines the size of the window. The window can be translated or expanded/contracted according to the operation of the affine group. We can now define the window of the first kind and the window of the second kind. Both windows can be translated according to the transformation given in Eq.\([18]\). The window of the first kind is not affected by the scale transformation. However, the size and location of the window of the second kind becomes affected by the scale transformation according to Eq.\([12]\). We can choose either of these two windows depending on our need. The window of the first kind is useful when we describe an observer with a fixed scope.

On the other hand, the window of the second kind is covariant and defines the information-preserving boundary conditions \([3]\). The width of this window is proportional to the average momentum, and the ratio \( w/p \) is a Lorentz-invariant quantity, where \( w \) is the width of the window. Indeed, this information-preserving window will play an important role in the photon localization problem.

### 4 Photon Localization Problem

Let us go back to Eq.\((2)\). \( A(u) \) is a classical amplitude, and it becomes a photon field after second quantization. If it is to be localized, \( a(k) \) must have a non-zero distribution, and \( A(u) \) is therefore a polychromatic \([14, 15, 16]\). \( A(u) \) of Eq.\((2)\) and \( G(u) \) of Eq.\((8)\) are numerically equal if

\[ a(k) = \sqrt{\frac{k}{p}} g(k) , \tag{19} \]

where the window is defined over a finite interval of \( k \) which does not include the point \( k = 0 \). It is thus possible to jump from the wavelet \( G(u) \) to the photon field \( A(u) \) using the above equation. Furthermore, since \( k \) and \( p \) have the same Lorentz-transformation property, this relation is Lorentz-invariant. Both \( a(k) \) and \( g(k) \) can be regarded as distribution functions which can be constructed from experimental data.

However, the above equality does not say that \( a(k) \) is equal to \( g(k) \). The intensity distribution of the localized light wave is not directly translated into the photon-number distribution. This is the quantitative difference between wavelets and photons. As we stated before, this difference becomes insignificant when the window becomes narrow. However, as the window becomes narrower, the wavelet becomes more wide-spread. The
wavelet then becomes non-localizable. This is why we are saying that photons are not localizable.

However, there are no rules saying that \( a(k) \) and \( g(k) \) should be the same. As long as we can transform from one expression to other as in Eq. (19), the transition from wavelets to photons can be carried out within the window. This point needs further investigation in the future.

## 5 Entropy Formulation of the Information Loss

We now introduce the concept of entropy to deal with the information loss due the windowing process. We use in this report the standard form for the entropy:

\[
S = - \int \rho(k) \ln[\rho(k)] dk, \tag{20}
\]

where \( \rho(k) \) is the probability distribution function, with the normalization condition

\[
\int \rho(k) dk = 1. \tag{21}
\]

If the Lorentz boost transforms \( k \) into \( e^\eta k \), the distribution becomes widespread for positive values of \( \eta \). The normalization integral becomes

\[
\int e^{-\eta} \rho(e^{-\eta} k) dk = 1. \tag{22}
\]

This normalization condition is form-invariant and is valid for all normalizable probability distribution functions. The Lorentz-boosted entropy takes the form

\[
S' = - \int e^{-\eta} \rho(e^{-\eta} k) \ln[e^{-\eta} \rho(e^{-\eta} k)] dk, \tag{23}
\]

which becomes

\[
S' = - \int \rho(k) \ln[\rho(k)] dk + \eta, \tag{24}
\]

The effect of the Lorentz boost is very simple. The boost simply add the parameter \( \eta \) to the original expression:

\[
S' - S = \eta. \tag{25}
\]

The entropy difference between the analytic and windowed distribution functions is

\[
\Delta S = - \int \{ \rho_A(k) \ln[\rho_A(k)] - \rho_W(k) \ln[\rho_W(k)] \} dk, \tag{26}
\]

where \( \rho_A(k) \) and \( \rho_W(k) \) are the probability distributions in the analytic and windowed forms respectively. The integration of this expression will produce a number. The question then is whether this is a Lorentz-invariant quantity. Let us go back to Eq. (25). The Lorentz-transformed entropy of the analytic form will produce \( \eta \), and so will the
windowed form. They will cancel each other. Thus the expression for the entropy difference given in Eq. (26) is a Lorentz-invariant expression.

It is shown in Ref. [5] that an information-preserving window can be defined. In this report, we have shown that the information loss due to the windowing process can also be defined in terms of a the Lorentz invariance of the entropy difference.

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