SPINOR CLASS NUMBER FORMULAS FOR TOTALLY DEFINITE QUATERNION ORDERS

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Abstract. We present two class number formulas associated to orders in totally definite quaternion algebras in the spirit of the Eichler class number formula. More precisely, let \( F \) be a totally real number field, \( D \) be a totally definite quaternion \( F \)-algebra, and \( O \) an \( O_F \)-order in \( D \). We derive explicit formulas for the following two class numbers associated to \( O \):

1. the class number of the reduced norm one group with respect to \( O \), namely, the cardinality of the double coset space \( D^1 \backslash \hat{D}^1 / \hat{O}^1 \); and
2. the number of locally principal right \( O \)-ideal classes within the spinor class of the principal right \( O \)-ideals, that is, the cardinality of \( D^x \backslash (D^x \hat{D}^x / \hat{O}^x) \). Both class numbers depend only on the spinor genus of \( O \), hence the title of the present paper. The formulas work unconditionally for \( O_F \)-orders with nonzero Eichler invariants at all finite places of \( F \) (e.g. Eichler orders of arbitrary level), and are extendable to other classes of orders as long as certain local condition for optimal embeddings is satisfied. The proofs are made possible by optimal spinor selectivity for quaternion orders.

1. Introduction

Let \( F \) be a totally real field, and \( D \) be a totally definite quaternion \( F \)-algebra, that is, \( D \otimes_{F, \sigma} \mathbb{R} \) is isomorphic to the Hamilton quaternion algebra \( \mathbb{H} \) for every embedding \( \sigma : F \hookrightarrow \mathbb{R} \). The algebra \( D \) admits a canonical involution \( \alpha \mapsto \bar{\alpha} \) and \( \text{Nr}(\alpha) = \alpha \bar{\alpha} \) are respectively the reduced trace and reduced norm of \( \alpha \in D \). Let \( O_F \) be the ring of integers of \( F \), and \( O \) be an \( O_F \)-order (of full rank) in \( D \). The class number \( h(O) \) of \( O \) is the cardinality of the finite set \( \text{Cl}(O) \) of locally principal right \( O \)-ideal classes in \( D \). If we write \( \hat{O} \) for the profinite completion of \( O \), and \( \hat{D} \) for the ring of finite adeles of \( D \), then

\[
h(O) = |\text{Cl}(O)| = |D^x \backslash \hat{D}^x / \hat{O}^x|,
\]

and it can be computed by the well known Eichler class number formula [12 Corollaire V.2.5, p. 144]; see also [26 Theorem 2] and [45 Theorem 1.5].

The purpose of this paper is to present two class number formulas of a similar form for two different double coset spaces as stated in the abstract. Given a set \( X \subseteq \hat{D} \), we write \( X^1 \) for the subset of elements with reduced norm 1, that is,

\[
X^1 := \{ x \in X \mid \text{Nr}(x) = 1 \}.
\]

Under certain assumptions on \( O \) to be made clear later, we give explicit formulas in Corollary 3.4 and Theorem 3.8 respectively for the following two class numbers.
associated to $\mathcal{O}$:

\begin{equation}
(1.3) \quad h^1(\mathcal{O}) := |D^1/\overline{D^1}|, \quad h_{sc}(\mathcal{O}) := |D^x/(D^x \overline{D^x})/\overline{D^x}|.
\end{equation}

The first class number $h^1(\mathcal{O})$ can be interpreted as follows. Regard $D$ as a 1-dimensional right vector space over itself, equipped with the Hermitian form

\begin{equation}
(1.4) \quad \psi : D \times D \to D, \quad (x, y) \mapsto \bar{xy}.
\end{equation}

Two $\mathcal{O}_F$-lattices $I$ and $I'$ in $D$ are said to be isometric if there exists $\alpha \in D^1$ such that $I' = \alpha I$, and they are said to belong to the same genus if there exists $x \in \overline{D^1}$ such that $\hat{I}' = x\hat{I}$. Let $I$ be an $\mathcal{O}_F$-lattice whose associated left order $\mathcal{O}_I(I) := \{\alpha \in D \mid \alpha I \subseteq I\}$ coincides with $\mathcal{O}$ (e.g. $I = \mathcal{O}$ itself). Then $h^1(\mathcal{O})$ counts the number of isometric classes of $\mathcal{O}_F$-lattices belonging to the genus of $I$ in the quaternion Hermitian space $(D, \psi)$.

For the meaning of $h_{sc}(\mathcal{O})$, we recall the following notions from [10, §1].

**Definition 1.1.** Two $\mathcal{O}_F$-orders $\mathcal{O}$ and $\mathcal{O}'$ are in the same spinor genus if there exists $x \in D^x \overline{D^1}$ such that $\hat{\mathcal{O}}' = x\hat{\mathcal{O}}x^{-1}$. Similarly, two locally principal right $\mathcal{O}$-ideals $I$ and $I'$ are in the same spinor class if there exists $x \in D^x \overline{D^1}$ such that $\hat{I}' = x\hat{I}$.

If we denote by $\text{Cl}_{sc}(\mathcal{O})$ the set of locally principal right $\mathcal{O}$-ideal classes within the spinor class of $\mathcal{O}$ itself, then $h_{sc}(\mathcal{O}) = |\text{Cl}_{sc}(\mathcal{O})|$, and the subscript $sc$ stands for “spinor class”. Clearly, $\text{Cl}_{sc}(\mathcal{O})$ is a subset of $\text{Cl}(\mathcal{O})$. It will be shown in (2.11) that $h(\mathcal{O})$ can be expressed as a finite sum of several $h_{sc}(\mathcal{O}')$ for various orders $\mathcal{O}'$, so a class number formula for $h_{sc}(\mathcal{O})$ is naturally a refinement of the Eichler class number formula for $h(\mathcal{O})$.

If $D$ is indefinite (i.e. unramified at an infinite place of $F$) rather than totally definite as we have assumed, then $h^1(\mathcal{O}) = h_{sc}(\mathcal{O}) = 1$ by the strong approximation theorem [31, Theorem 7.7.5] [42, Theorem III.4.3]. This is precisely the reason why we focus on the totally definite case. In Lemma 2.11 we show that both $h^1(\mathcal{O})$ and $h_{sc}(\mathcal{O})$ depend only on the spinor genus of $\mathcal{O}$, hence the title of the present paper.

The proofs of both the formulas for $h^1(\mathcal{O})$ and $h_{sc}(\mathcal{O})$ rely on the spinor trace formula for optimal embeddings in Proposition 2.14 which is again a refinement of the classical trace formula for optimal embeddings [42, Theorem III.5.11]. Vignéras studied the spinor trace formula in its specialized form for Eichler orders in indefinite quaternion algebras in [42, Corollaire III.5.17]. However, it is pointed out by Chinburg and Friedman in [10, Remark 3.4] that there is a subtle phenomenon called “selectivity” that was overlooked by Vignéras, so her formula must be adjusted to account for the exceptional cases where selectivity does occur. See Voight [44, Theorem 31.1.7 and Corollary 31.1.10] for the corrected formula under the same assumption as Vignéras’s. The work by Chinburg and Friedman has inspired a large amount of subsequent research on selectivity. Unfortunately, most of the discussions [6, 11, 16, 21, 24, 38, 44] assume the Eichler condition [39, Definition 34.3] (i.e. the quaternion algebra is assumed to be indefinite), which has to be avoided for applications to totally definite quaternion algebras. As explained by Arenas-Carmona [3], a generalization to all quaternion algebras can be achieved by replacing “selectivity” with “spinor selectivity”. Unfortunately for us again, the emphasis of [3] is placed on (general) embeddings rather than optimal embeddings, rendering it not directly applicable to the present situation either.
For the above reasons we spend considerable amount of efforts on optimal spinor selectivity in Section 2.2. It should be mentioned that this section is largely expository, as most of the ideas are already present in the literature (with spinor selectivity by Arenas-Carmona [3] and optimal selectivity by Maclachlan [30] and Voight [44, Chapter 31], for example). Our new contribution to this topic mostly includes verifying certain local condition for optimal embeddings in Lemma 2.10 for the case when the Eichler invariant of the order is equal to $-1$.

At the moment, the aforementioned local condition has been verified only for local orders with nonzero Eichler invariant (see Definition 2.9). As a result, the spinor trace formula, and in turn our formulas for $h^1(\mathcal{O})$ and $h_{sc}(\mathcal{O})$, hold only for $\mathcal{O}_F$-orders whose Eichler invariant is nonzero at every finite place of $F$. These already incorporate a large class of orders including Eichler orders of arbitrary level. Moreover, our formulas are readily extendable to other classes of orders as long as the local condition is satisfied.

This paper is organized as follows. Section 2 focuses on the optimal spinor selectivity for quaternion orders and the spinor trace formula as mentioned. Section 3 constitutes the core part of this paper, deriving the explicit formulas for $h^1(\mathcal{O})$ and $h_{sc}(\mathcal{O})$. Among these two formulas, the priority is given to $h^1(\mathcal{O})$ since it is the more challenging one. Indeed, the proof for the formula of $h_{sc}(\mathcal{O})$ follows from the same argument as that for the Eichler class number formula, so it is merely sketch toward the end of Section 3. To compute $h^1(\mathcal{O})$, we make use of the Selberg trace formula for compact quotient. Instead of calculating each orbital integral individually, we reorganize the terms and connect them with the spinor trace formula. In Section 4, we work out a comprehensive example by computing the class numbers $h^1(\mathcal{O})$ and $h_{sc}(\mathcal{O})$ for every maximal order $\mathcal{O}$ in the quaternion algebra $D_{\infty_1, \infty_2}$ over the quadratic real fields $F = \mathbb{Q}(\sqrt{p})$, where $p \in \mathbb{N}$ is a prime number. When $p \equiv 3 \pmod{4}$, the class number formula for $h^1(\mathcal{O})$ is new (as far as we are aware), and the formula for $h_{sc}(\mathcal{O})$ recovers the proper class number formula of even definite quaternary quadratic forms of discriminant $4p$ within a fixed genus, which is first calculated by Ponomarev [35], and reproduced by Chan and Peters [13, §3] using another method.

2. Optimal spinor selectivity

In this section, we explain the main result of the optimal spinor selectivity and derive the spinor trace formula for optimal embeddings. At the moment, there is no need to assume that the quaternion algebra $D$ is totally definite, so $F$ is an arbitrary number field, whose ring of integers is denoted by $\mathcal{O}_F$.

Let $\mathcal{V}(F)$ be the set of all the places of $F$, and $\mathcal{V}_{\infty}(F)$ (resp. $\mathcal{V}_f(F)$) be the subset of infinite (resp. finite) places. The set $\mathcal{V}_f(F)$ is naturally identified with the set of nonzero prime ideals of $\mathcal{O}_F$ (i.e. the finite primes of $F$). If $\mathfrak{p} \in \mathcal{V}_f(F)$ is a finite prime and $M$ is a finite dimensional $F$-vector space or a finite $\mathcal{O}_F$-module, we write $M_{\mathfrak{p}}$ for the $\mathfrak{p}$-adic completion of $M$. In particular, $\mathcal{O}_F$ is the $\mathfrak{p}$-adic completion of $F$, whose $\mathfrak{p}$-adic discrete valuation is denoted by $\nu_{\mathfrak{p}} : F_\mathfrak{p}^\times \to \mathbb{Z}$. Let $\hat{\mathbb{Z}} = \lim_{\leftarrow} \mathbb{Z}/n\mathbb{Z} = \prod_{\mathfrak{p}} \mathbb{Z}_{\mathfrak{p}}$ be the profinite completion of $\mathbb{Z}$. If $X$ is a finitely generated $\hat{\mathbb{Z}}$-module or a finite dimensional $\mathbb{Q}$-vector space, we set $\hat{X} = X \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$. For example, $\hat{F}$ is the ring of finite adeles of $F$, and $\hat{\mathcal{O}}_F = \prod_{\mathfrak{p} \in \mathcal{V}_f(F)} \mathcal{O}_F$. Let $\mathcal{I}_F$ be the group of fractional ideals of $F$, which is a free abelian group generated by $\mathcal{V}_f(F)$. For each $\mathfrak{p} \in \mathcal{V}_f(F)$, we put $N(\mathfrak{p}) := |\mathcal{O}_F/\mathfrak{p}|$ and extend it to a multiplicative function on $\mathcal{I}_F$. 

We shall make frequent use of the “local-global” dictionary of $O_F$-lattices [19 Proposition 4.21]. Let $V$ be a finite dimensional $F$-vector space, and $G$ be a closed algebraic subgroup of $GL(V)$. We write $G(\hat{F})$ for the group of finite adelic points of $G$ [35 §5.1]. Let $\Lambda \subset V$ be an $O_F$-lattice (i.e. a finitely generated $O_F$-submodule that spans $V$ over $F$). For any $g = (g_p) \in G(\hat{F})$, there exists a unique $O_F$-lattice $\Lambda' \subset V$ such that $\Lambda'_p = g_p\Lambda_p$ for every finite prime $p$, so we put $g\Lambda := \Lambda'$.

2.1. The spinor classes of ideals and spinor genera of $O_F$-orders. Let $D$ be an arbitrary quaternion $F$-algebra. We write $\text{Ram}(D)$ for the set of ramified places of $D$, and $\text{Ram}_\infty(D)$ (resp. $\text{Ram}_f(D)$) for the set of the infinite (resp. finite) ramified places. From [12 Theorem III.3.1], $\text{Ram}(D)$ is a finite set with even cardinality, and $D$ is uniquely determined by $\text{Ram}(D)$ up to isomorphism. The product of all members of $\text{Ram}_f(D)$ is called the reduced discriminant of $D$ and denoted by $\mathfrak{d}(D)$.

Following [12 §III.4], we write $F^*_D$ for the subgroup of $F^\times$ consisting of the elements that are positive at each infinite place in $\text{Ram}_\infty(D)$. The Hasse-Schilling-Maass theorem [39 Theorem 33.15] [12 Theorem III.4.1] implies that

\begin{equation}
(2.1) \quad \text{Nr}(D^\times) = F^*_D.
\end{equation}

An $O_F$-order in $D$ is an $O_F$-lattice in $D$ that contains 1 and is closed under multiplication. Two $O_F$-orders in $D$ are of the same type if they are $O_F$-isomorphic (or equivalently, $D^\times$-conjugate). Two $O_F$-orders $\mathcal{O}$ and $\mathcal{O}'$ are said to be in the same genus if there exists $x \in \hat{D}^\times$ such that $\mathcal{O}' = x\mathcal{O}x^{-1}$, or equivalently, if $\mathcal{O}_p$ and $\mathcal{O}'_p$ are $O_{F_p}$-isomorphic for every $p \in \mathcal{V}_f(F)$. For example, given a fixed nonzero integral ideal $\mathfrak{n} \subseteq O_F$ coprime to $\mathfrak{d}(D)$, all Eichler orders of level $\mathfrak{n}$ form a single genus. Fix a genus $\mathcal{G}$ of $O_F$-orders in $D$ and a member $\mathcal{O} \in \mathcal{G}$. The normalizer of $\mathcal{O}$ in $\hat{D}^\times$ is denoted by $\mathcal{N}(\mathcal{O})$. Let $[\mathcal{O}] := \{\alpha\mathcal{O}\alpha^{-1} \mid \alpha \in D^\times\}$ be the type of $\mathcal{O}$, and $\text{Tp}(\mathcal{O})$ be the finite set of types of $O_F$-orders in $\mathcal{G}$, regarded as a pointed set with the base point $[\mathcal{O}]$. There is a canonical bijection

\begin{equation}
(2.2) \quad \text{Tp}(\mathcal{O}) \simeq D^\times/\hat{D}^\times/\mathcal{N}(\mathcal{O})
\end{equation}

sending the type $[\mathcal{O}]$ to the neutral double coset.

For each $O_F$-lattice $I$ in $D$, we write $\mathcal{O}_l(I)$ (resp. $\mathcal{O}_r(I)$) for its associated left (resp. right) order. More explicitly,

\begin{equation}
(2.3) \quad \mathcal{O}_l(I) := \{\alpha \in D \mid \alpha I \subseteq I\}, \quad \text{and} \quad \mathcal{O}_r(I) := \{\alpha \in D \mid I\alpha \subseteq I\}.
\end{equation}

If $I$ is a locally principal fractional right $O$-ideal, then its ideal class $\{\alpha I \mid \alpha \in D^\times\}$ is denoted by $[I]$. Recall that $\text{Cl}(\mathcal{O})$ denotes the finite set of locally principal right $O$-ideal classes. Once again, there is a canonical bijection

\begin{equation}
(2.4) \quad \text{Cl}(\mathcal{O}) \simeq D^\times/\hat{D}^\times/\mathcal{O}^\times
\end{equation}

sending the principal class $[\mathcal{O}]$ to the neutral double coset. It is well known that the class number $h(\mathcal{O}) = |\text{Cl}(\mathcal{O})|$ depends only on the genus $\mathcal{G}$ and not on the choice of $\mathcal{O} \in \mathcal{G}$.

The notions spinor genus of $O_F$-orders and spinor class of locally principal $O$-ideals have already appeared in Definition 1.1. When $D$ satisfies the Eichler condition, Brzezinski [10 Proposition 1.1] shows that each spinor genus of $O_F$-orders contains exactly one type, and the same argument shows that each spinor class of locally principal right $O$-ideals contains exactly one ideal class.
Notation 2.1. We write $\mathcal{O} \sim \mathcal{O}'$ if two $O_F$-orders $\mathcal{O}$ and $\mathcal{O}'$ are in the same spinor genus. The spinor genus of $\mathcal{O}$ is denoted by $[\mathcal{O}]_{\text{sc}}$.

Similarly, we write $I \sim I'$ if two locally principal right $\mathcal{O}$-ideals $I$ and $I'$ are in the same spinor class. The spinor class of $I$ is denoted by $[I]_{\text{sc}}$. For a fixed spinor class $[I]_{\text{sc}}$, the set of ideal classes in $[I]_{\text{sc}}$ is denoted by $\text{Cl}(\mathcal{O}, [I]_{\text{sc}})$. In other words,

$$\text{(2.10)} \quad \text{Cl}(\mathcal{O}, [I]_{\text{sc}}) := \{ [I'] \in \text{Cl}(\mathcal{O}) \mid [I'] \subseteq [I]_{\text{sc}} \}.$$  

For simplicity, we put

$$\text{(2.11)} \quad \text{Cl}_{\text{sc}}(\mathcal{O}) = \text{Cl}(\mathcal{O}, [\mathcal{O}]_{\text{sc}}), \quad \text{and} \quad h_{\text{sc}}(\mathcal{O}) = |\text{Cl}_{\text{sc}}(\mathcal{O})|.$$ 

The definition of $h_{\text{sc}}(\mathcal{O})$ matches with the one in (2.3), as shown in (2.7) below.

There is no ambiguity on the symbol $\sim$, since it does not make sense to talk about two distinct $O_F$-orders being in the same spinor class. Note that $I \sim I'$ implies that $\mathcal{O}_l(I) \sim \mathcal{O}_l(I')$, but the converse is not necessarily true. If $I = x\mathcal{O}$ for some $x \in \hat{D}$, then we put

$$\text{(2.7)} \quad \hat{D}([I]_{\text{sc}}) := D^x \hat{D}^1 x\hat{O}^x = D^x \hat{D}^1 \hat{O}^x x,$$

where the last equality follows from the fact that

$$\text{(2.8)} \quad \hat{D}^1 \hat{O}^x = \hat{D}^1 x\hat{O}^x x^{-1}, \quad \forall x \in \hat{D}.$$ 

Clearly, $\hat{D}([I]_{\text{sc}})$ depends only on the spinor class $[I]_{\text{sc}}$. The set $\text{Cl}(\mathcal{O}, [I]_{\text{sc}})$ may be described adelically as

$$\text{(2.9)} \quad \text{Cl}(\mathcal{O}, [I]_{\text{sc}}) \simeq D^x \hat{D}([I]_{\text{sc}})/\hat{O}^x \simeq D^x \hat{D}^1 x\hat{O}^x x^{-1} / (x\hat{O}^x x^{-1}),$$

where the last isomorphism is induced from the right multiplication of $\hat{D}([I]_{\text{sc}})$ by $x^{-1}$. If we set $\mathcal{O}' := \mathcal{O}_l(I) = x\mathcal{O} x^{-1}$, then there is a bijection

$$\text{(2.10)} \quad \text{Cl}(\mathcal{O}, [I]_{\text{sc}}) \simeq \text{Cl}(\mathcal{O}', [\mathcal{O}']_{\text{sc}}).$$

Moreover, we have a canonical surjection

$$D^1 \hat{D}^1 / \hat{O}^1 \rightarrow D^x \hat{D}^1 x\hat{O}^x x^{-1} / (x\hat{O}^x x^{-1}),$$

which is not injective in general. Therefore, $h^1(\mathcal{O}) \geq h_{\text{sc}}(\mathcal{O})$.

Let $\text{Scl}(\mathcal{O})$ be the set of spinor classes of locally principal right $\mathcal{O}$-ideals. From (2.10), we have

$$\text{(2.11)} \quad h(\mathcal{O}) = \sum_{[I]_{\text{sc}} \in \text{Scl}(\mathcal{O})} h_{\text{sc}}(\mathcal{O}_l(I)).$$

As to be shown in Lemma 2.1, $h_{\text{sc}}(\mathcal{O})$ depends only on the spinor genus of $\mathcal{O}$, so $h_{\text{sc}}(\mathcal{O}_l(I))$ does not depend on the choice of the representative $I$ of $[I]_{\text{sc}}$. There is a canonical identification

$$\text{(2.12)} \quad \text{Scl}(\mathcal{O}) \simeq (D^x \hat{D}^1) \hat{D}^x / \hat{O}^x \xrightarrow{\cong} F^x_B \hat{D}^x / \text{Nr}(\hat{O}^x),$$

which equips $\text{Scl}(\mathcal{O})$ with a group structure. Thus we call $\text{Scl}(\mathcal{O})$ the spinor class group. If $\mathcal{O}'$ is another member of $\mathcal{G}$, then $\hat{O}^x$ and $\hat{O}^x$ are $\hat{D}^x$-conjugate, so the group $F^x_B \hat{D}^x / \text{Nr}(\hat{O}^x)$ in (2.12) depends only on $\mathcal{G}$ and not on the choice of $\mathcal{O}$.
Let \( \text{SG}(\mathcal{O}) \) be the set \( \mathcal{G}/\sim \) of all spinor genera of \( O_F \)-orders in \( \mathcal{G} \), regarded as a pointed set with the base point \([\mathcal{O}]_{\text{sg}}\). There is an adelic description as follows \[ \text{SG}(\mathcal{O}) \simeq (D^x \backslash \hat{D}^x) / \mathcal{N}(\hat{\mathcal{O}}) \cong F_D^x \backslash \hat{F}^x / \mathcal{N}(\mathcal{N}(\hat{\mathcal{O}})). \] (2.13)

Once again, the group \( F_D^x \backslash \hat{F}^x / \mathcal{N}(\mathcal{N}(\hat{\mathcal{O}})) \) in (2.13) depends only on \( \mathcal{G} \), so it will be called the \textit{group of spinor genera in} \( \mathcal{G} \) and denoted by \( \mathcal{G}_{\text{sg}}(\mathcal{G}) \). Since \( \mathcal{N}(\mathcal{N}(\hat{\mathcal{O}})) \) is an open subgroup of \( \hat{F}^x \) containing \((\hat{F}^x)^2\), the group \( \mathcal{G}_{\text{sg}}(\mathcal{G}) \) is a \textit{finite} elementary 2-group \[ \text{Proposition 3.5}. \)

It should be emphasized that the bijection \( \text{SG}(\mathcal{O}) \simeq \mathcal{G}_{\text{sg}}(\mathcal{G}) \) in (2.13) depends on the choice of \( \mathcal{O} \). More canonically, the set of spinor genera \( \mathcal{G}/\sim \) is a principal homogeneous space over \( \mathcal{G}_{\text{sg}}(\mathcal{G}) \). Indeed, for any \( \mathfrak{a} \in \hat{F}^x \), we write \([\mathfrak{a}]\) for its image in \( \mathcal{G}_{\text{sg}}(\mathcal{G}) \) and pick \( x \in \hat{D}^x \) such that \( \mathcal{N}(x) = \mathfrak{a} \). Then there is a natural group action of \( \mathcal{G}_{\text{sg}}(\mathcal{G}) \) on \( \mathcal{G}/\sim \) as follows
\[
[a] \cdot [\mathcal{O}]_{\text{sg}} = [x\mathcal{O}x^{-1}]_{\text{sg}}, \quad \forall \mathcal{O} \in \mathcal{G}.
\]
(2.14)

Given \( \mathcal{O} \) and \( \mathcal{O}' \) in \( \mathcal{G} \), we define \( \rho(\mathcal{O}, \mathcal{O}') \) to be the unique element of \( \mathcal{G}_{\text{sg}}(\mathcal{G}) \) such that \( \rho(\mathcal{O}, \mathcal{O}') \cdot [\mathcal{O}]_{\text{sg}} = [\mathcal{O}']_{\text{sg}} \). Clearly, \( \rho(\mathcal{O}, \mathcal{O}') = \rho(\mathcal{O}', \mathcal{O}) \), since \( \mathcal{G}_{\text{sg}}(\mathcal{G}) \) is an elementary 2-group.

The identifications (2.2), (2.4), (2.12) and (2.13) fit in to a commutative diagram as follows
\[
\begin{array}{cccc}
D^x \backslash \hat{D}^x / \mathcal{N}(\hat{\mathcal{O}}) & \xrightarrow{\cong} & \text{Cl}(\mathcal{O}) & \xrightarrow{\Upsilon} & \text{Tp}(\mathcal{O}) & \xrightarrow{\cong} & D^x \backslash \hat{D}^x / \mathcal{N}(\hat{\mathcal{O}}) \\
\downarrow \mathcal{N} & & \downarrow \Upsilon & & \downarrow \text{Tp} & & \downarrow \mathcal{N} \\
F_D^x \backslash \hat{F}^x / \mathcal{N}(\hat{\mathcal{O}}) & \xrightarrow{\cong} & \text{SCI}(\mathcal{O}) & \xrightarrow{\Upsilon} & \text{SG}(\mathcal{O}) & \xrightarrow{\cong} & F_D^x \backslash \hat{F}^x / \mathcal{N}(\mathcal{N}(\hat{\mathcal{O}}))
\end{array}
\]
(2.15)

Here the two vertical maps in the middle are canonical projections, and \( \Upsilon : \text{Cl}(\mathcal{O}) \to \text{Tp}(\mathcal{O}) \) is defined as in \[ \text{Proposition 3.9} \] of \[ \text{Proposition 3.9} \].
(2.16)

The other \( \Upsilon : \text{SCI}(\mathcal{O}) \to \text{SG}(\mathcal{O}) \) is defined in the same way, hence so is the notation.

\textbf{Definition 2.2} (\[ \text{§2}, \text{Proposition 3.9} \]). \textit{The spinor genus field} of \( \mathcal{G} \) is the abelian field extension \( \Sigma_{\mathfrak{q}} / F \) corresponding to the open subgroup \( F_D^x \mathcal{N}(\mathcal{N}(\hat{\mathcal{O}})) \subseteq \hat{F}^x \) via the class field theory \[ \text{Theorem X.5} \].

The Artin reciprocity map establishes an isomorphism
\[
\mathcal{G}_{\text{sg}}(\mathcal{G}) \simeq \text{Gal}(\Sigma_{\mathfrak{q}} / F), \quad [a] \mapsto ([a], \Sigma_{\mathfrak{q}} / F).
\]
(2.17)

If \( K/F \) is an sub-extension of \( \Sigma_{\mathfrak{q}} \), then \([a], K/F \in \text{Gal}(K/F)\) is well-defined and just the restriction of \([a]/\Sigma_{\mathfrak{q}} / F \) to \( K/F \).

\textbf{Example 2.3.} \textit{Let} \( n \text{ be a nonzero ideal of } \mathcal{O}_F \text{ coprime to } \mathfrak{d}(\mathcal{D}), \text{ and } \mathcal{G}_n \text{ be the genus of Eichler orders of level } n \). \textit{For every } \( \mathcal{O} \in \mathcal{G}_n \), \textit{we have } \( \mathcal{N}(\hat{\mathcal{O}}^x) = \hat{O}_F^x \). \textit{Thus}
\[
\text{SCI}(\mathcal{O}) \simeq \hat{F}^x / (F_D^x \hat{O}_F^x),
\]
(2.18)

\textsuperscript{1}This field is often called the \textit{spinor class field} in the literature \[ \text{3, 4, 5} \], but that would be too confusing here.
which is independent of \( n \), so we put
\[(2.19) \quad h_D(F) := |\text{SCl}(O)| = |\hat{F}^\times / (F_D^\times \hat{O}_F^\times)|
\]
and call it the restricted class number of \( F \) with respect to \( D \) [42 §III.5, p. 88].

Next, we study the group \( \Phi_{sg}(n) := \Phi_{sg}(\mathcal{G}_n) \) of spinor genera in \( \mathcal{G}_n \) and the spinor genus field \( \Sigma_n := \Sigma_{g_n} \). According to [42 §II.1–2],
\[\text{Nr}(N(\hat{O})) = \{ a = (a_p) \in \hat{F}^\times \mid \text{for every } p \mid \mathfrak{d}(D), \nu_p(a_p) \equiv 0 \text{ or } \nu_p(n) \pmod{2}\}.\]
Thus \( \Sigma_n \) is the maximal abelian extension of \( F \) of exponent 2 that is unramified outside \( \text{Ram}_\infty(D) \) and splits completely at all of the following finite primes
\[(2.20) \quad \text{Ram}_f(D) \cup \{ p \in \mathcal{V}_f(F) \mid \nu_p(n) \equiv 1 \pmod{2}\}.
\]
In particular, \( \Sigma_n \) is a subfield of the narrow Hilbert class field of \( F \), and \( \Phi_{sg}(n) \) is a quotient of the narrow class group of \( F \). If \( n = O_F \), then \( \mathcal{G}_n \) is the genus of maximal orders in \( D \), and \( \Sigma_n \) is the field “\( K(B) \)” studied in [16, p. 39]. In general, \( \Sigma_n \) is the field “\( K(B) \)” studied in [21, p. 212], and a special case of the field “\( K(R) \)” in [29, p. 1429].

Let \( O \) and \( O' \) be two members of \( \mathcal{G}_n \). There exists an \( O_F \)-lattice \( I \) in \( D \) linking \( O \) and \( O' \) as in [42 §I.4], that is, \( O = O(I) \) and \( O' = \hat{O}(I) \). Then \( \rho(O, O') \in \Phi_{sg}(n) \) is represented by the fractional ideal \( \text{Nr}(I) \in \mathcal{I}_F \). If further \( n = O_F \) so that \( \mathcal{G}_n \) is the genus of maximal orders, then \( \rho(O, O') \in \Phi_{sg}(n) \) is represented by the distance ideal of \( O \) and \( O' \) (i.e. the index ideal \( \chi(O/\langle O \cap O' \rangle) \) as in [40 §III.1]); see [16, p. 34] or [30 (5), p. 2855].

### 2.2. Optimal spinor selectivity

Throughout this section, we assume that \( K/F \) is a quadratic field extension that is \( F \)-embeddable into \( D \). Let \( B \subseteq O_K \) be an \( O_F \)-order in \( K \) of conductor \( f(B) \). In other words, \( f(B) \) is the unique ideal of \( O_F \) such that \( B = O_F + f(B)O_K \). If \( B' \subseteq O_K \) is another \( O_F \)-order in \( K \), we put
\[(2.21) \quad f(B'/B) := f(B')^{-1}f(B) \in \mathcal{I}_F,
\]
and call it the relative conductor of \( B \) with respect to \( B' \). Indeed, if \( B \subseteq B' \), then \( f(B'/B) \) is the global version of the relative conductor defined in [42 §II.3, p. 44]. In general, \( f(B'/B) \) is a fractional ideal of \( F \).

Fix a genus \( \mathcal{G} \) of \( O_F \)-orders in \( D \) and a member \( O \in \mathcal{G} \). Let \( \text{Emb}(B, O) \) be the set of optimal embeddings of \( B \) into \( O \), that is
\[(2.22) \quad \text{Emb}(B, O) := \{ \varphi \in \text{Hom}_F(K, D) \mid \varphi(K) \cap O = \varphi(B) \}.
\]
The normalizer of \( N(O) \subseteq D^\times \) acts on \( \text{Emb}(B, O) \) from the right by \( \varphi \mapsto \alpha^{-1}\varphi\alpha \) for all \( \varphi \in \text{Emb}(B, O) \) and \( \alpha \in N(O) \). For every group \( G \) intermediate to \( O^\times \subseteq N(O) \), we put
\[(2.23) \quad m(B, O, G) := |\text{Emb}(B, O)/G|,
\]
which is a finite number thanks to the Jordan-Zassenhaus Theorem [19 Theorem 24.1, p. 534]. Similarly, for every \( p \in V_f(F) \) and every group \( G_p \) intermediate to \( O_p^\times \subseteq N(O_p) \), we put
\[(2.24) \quad m(B_p, O_p, G_p) := |\text{Emb}(B_p, O_p)/G_p|.
\]
Note that \( m(B_p, O_p, O_p^\times) \) depends only on the genus \( \mathcal{G} \) and not on the choice of \( O \in \mathcal{G} \). Since \( \mathcal{G} \) is fixed throughout, for simplicity we put
\[(2.25) \quad m_p(B) := m(B_p, O_p, O_p^\times).
\]
Example 2.4. Let $n \subseteq O_F$ be a nonzero square-free ideal coprime to $\mathfrak{d}(D)$, and $\mathcal{G} = \mathcal{G}_n$ be the genus of Eichler orders of level $n$ as in Example 2.3. Then according to [22] Theorem II.3.2,

$$m_p(B) = \begin{cases} 
1 - \left( \frac{p}{\mathfrak{d}} \right) & \text{if } p | \mathfrak{d}(D), \\
1 + \left( \frac{p}{\mathfrak{d}} \right) & \text{if } p | n, \\
1 & \text{otherwise,}
\end{cases} \tag{2.26}$$

where $\left( \frac{\mathfrak{d}}{p} \right)$ is the Eichler symbol [22, p. 94]. More explicitly, if $p \not| f(B)$, then $\left( \frac{\mathfrak{d}}{p} \right) = 1$, otherwise $\left( \frac{\mathfrak{d}}{p} \right) = \left( \frac{\mathfrak{d}}{p} \right)$, where $\left( \frac{\mathfrak{d}}{p} \right)$ is the Artin symbol, which takes value $1, 0, -1$ according to whether $p$ is split, ramified or inert in the extension $K/F$.

Moreover, if $\Delta(B, O) = 1$ if $m_p(B) \neq 0$.

We return to the general case where $\mathcal{G}$ is an arbitrary genus of $O_F$-orders. There is a well known trace formula [22] Theorem III.5.11 (cf. [18] Lemma 3.2.1) as follows:

$$\sum_{[I] \in \text{Cl}(\mathcal{O})} m(B, O_1(I), O_1(I)^\times) = h(B) \prod_{p \in V_f(F)} m_p(B). \tag{2.28}$$

where the product is well-defined since $m_p(B) = 1$ for all but finitely many $p$ thanks to (2.26). It follows from (2.28) that

$$\exists O_0 \in \mathcal{G} \text{ with } \text{Emb}(B, O_0) \neq \emptyset \iff m_p(B) \neq 0 \text{ for every } p \in V_f(F). \tag{2.29}$$

We define the symbol

$$\Delta(B, \mathcal{O}) = \begin{cases} 
1 & \text{if } \exists O' \text{ such that } O' \sim \mathcal{O} \text{ and } \text{Emb}(B, O') \neq \emptyset, \\
0 & \text{otherwise.}
\end{cases} \tag{2.30}$$

Clearly, $\Delta(B, \mathcal{O}) = 0$ if there exists $p \in V_f(F)$ such that $m_p(B) = 0$.

Definition 2.5. We say $B$ is optimally spinor selective (or selective for short) for the genus $\mathcal{G}$ of $O_F$-orders if $\{O \in \mathcal{G} \mid \Delta(B, O) = 1\}$ is a nonempty proper subset of $\mathcal{G}$. Suppose that $B$ is selective for $\mathcal{G}$. A spinor genus $[O]_{sg} \subseteq \mathcal{G}$ with $\Delta(B, O) = 1$ is said to be selected by $B$. Similarly, for a fixed $O_F$-order $O \in \mathcal{G}$, a spinor class $[I]_{sc} \in \text{Cl}(\mathcal{O})$ is said to be selected by $B$ if $\Delta(B, O_1(I)) = 1$.

To give a preliminary criterion for when a spinor genus is selected by $B$, we introduce some more notation. For each $F$-embedding $\varphi : K \hookrightarrow D$, consider the following sets

$$\mathcal{E}(\varphi, B, \mathcal{O}) := \{ \alpha \in D^\times \mid \varphi(K) \cap \alpha O \alpha^{-1} = \varphi(B) \}, \tag{2.31}$$
$$\tilde{\mathcal{E}}(\varphi, B, \mathcal{O}) := \{ g = (g_p) \in \tilde{D}^\times \mid \varphi(K) \cap g \tilde{O} g^{-1} = \varphi(B) \}, \tag{2.32}$$
$$\mathcal{E}_p(\varphi, B, \mathcal{O}) := \{ g_p \in D_p^\times \mid \varphi(K_p) \cap g_p O_p g_p^{-1} = \varphi(B_p) \}, \forall p \in V_f(F). \tag{2.33}$$

Given any other $F$-embedding $\varphi' : K \hookrightarrow D$ and $O_F$-order $O' \in \mathcal{G}$, we pick $\alpha \in D^\times$ and $x \in \tilde{D}$ such that $\varphi' = \alpha^{-1} \varphi \alpha$ and $O' = x O x^{-1}$. Then

$$\tilde{\mathcal{E}}(\varphi', B, O') = \tilde{\mathcal{E}}(\alpha^{-1} \varphi \alpha, B, x O x^{-1}) = \alpha^{-1} \tilde{\mathcal{E}}(\varphi, B, O) x^{-1}. \tag{2.34}$$

When $\varphi$ is fixed and clear from the context, we often drop it from the notation and simply write $\mathcal{E}(B, \mathcal{O}), \tilde{\mathcal{E}}(B, \mathcal{O})$ and $\mathcal{E}_p(B, \mathcal{O})$ instead.
Without lose of generality, we assume $m_p(B) \neq 0$ for every $p \in V_f(F)$ so that there exists $\mathcal{O} \in \mathcal{G}$ with $\text{Emb}(B, \mathcal{O}) \neq 0$. Fix such an $\mathcal{O}$ and an optimal embedding $\varphi \in \text{Emb}(B, \mathcal{O})$. For simplicity, we identify $K$ with its image $\varphi(K) \subset D$. Clearly, $\hat{\mathcal{E}}(\varphi, B, \mathcal{O})$ is left translation invariant by $\hat{K}^\times$ and right translation invariant by $\mathcal{N}(\hat{\mathcal{O}})$, so we have

$$\hat{\mathcal{E}} := \hat{\mathcal{E}}(\varphi, B, \mathcal{O}) \supseteq \hat{K}^\times \mathcal{N}(\hat{\mathcal{O}}). \tag{2.35}$$

The following lemma is a special case of \cite[Lemma 3.1]{3} (see also \cite[Lemma 2.3]{23}), except that one needs to replace “$G$-representations” by “optimal $G$-representations” and change the definition of the set “$X_{\Lambda|\mathcal{M}}$” there accordingly.

**Lemma 2.6.** Let $\mathcal{O} \in \mathcal{G}$ and $\varphi \in \text{Emb}(B, \mathcal{O})$ be fixed as above, and $\mathcal{O}'$ be an arbitrary $O_F$-order in $\mathcal{G}$. Recall that $\hat{\mathcal{O}}_{\text{sg}}(\mathcal{G}) = \hat{F}^\times / (F^\times D_{\mathcal{O}} \mathcal{N}(\mathcal{O}))$. Pick a representative $a \in \hat{F}^\times$ for $\rho(\mathcal{O}', \mathcal{O}) \in \hat{\mathcal{O}}_{\text{sg}}(\mathcal{G})$. Then $\Delta(B, \mathcal{O}') = 1$ if and only if $a \in F_D^\times \mathcal{N}(\hat{\mathcal{E}})$.

**Proof.** Clearly, the condition $a \in F_D^\times \mathcal{N}(\hat{\mathcal{E}})$ depends only on $[\mathcal{O}']_{\text{sg}}$ and not on the choice of $a \in \hat{F}^\times$. Choose $x \in \hat{D}^\times$ such that $\hat{\mathcal{O}}' = x\hat{O}x^{-1}$. Then $\rho(\mathcal{O}', \mathcal{O}) = [\mathcal{N}(x)]$, so we take $a = \mathcal{N}(x)$. By the Skolem-Noether theorem, we get

$$\Delta(B, \mathcal{O}') = 1 \iff \exists y \in F^\times \hat{D}^1 \text{ such that } \varphi \in \text{Emb}(B, y\mathcal{O}'y^{-1})$$

$$\iff \exists y \in F^\times \hat{D}^1 \text{ such that } yx \in \hat{\mathcal{E}}$$

$$\iff x \in F^\times \hat{D}^1 \hat{\mathcal{E}}$$

$$\iff \mathcal{N}(x) \in \mathcal{N}(F^\times \hat{D}^1 \hat{\mathcal{E}}) = F_D^\times \mathcal{N}(\hat{\mathcal{E}}). \qed$$

**Lemma 2.7** (Voight \cite[31.3.14]{31}). The set $F_D^\times \mathcal{N}(\hat{\mathcal{E}})$ is a subgroup of $\hat{F}^\times$ of index at most 2.

**Proof.** Similar to $F_D^\times$, we define $F_K^\times$ to be the subgroup of $F^\times$ consisting of the elements that are positive at each infinite place of $F$ that is ramified in $K/F$. The assumption that $K$ is $F$-embeddable into $D$ implies that

$$F_K^\times \subseteq F_D^\times. \tag{2.36}$$

It follows that there is a chain of inclusions called the *selectivity sandwich* by Voight:

$$F_K^\times \mathcal{N}(\hat{K}^\times) \subseteq F_D^\times \mathcal{N}(\hat{K}^\times) \mathcal{N}(\mathcal{N}(\hat{\mathcal{O}})) \subseteq F_D^\times \mathcal{N}(\hat{\mathcal{E}}) \subseteq \hat{F}^\times. \tag{2.37}$$

The first two terms of (2.37) correspond via the class field theory to $K$ and $K \cap \Sigma_{\mathcal{G}}$ respectively. In particular, we have

$$[\hat{F}^\times : F_K^\times \mathcal{N}(\hat{K}^\times)] = [K : F] = 2. \tag{2.38}$$

Since $F_D^\times \mathcal{N}(\hat{\mathcal{E}})$ is translation invariant under $F_D^\times \mathcal{N}(\hat{K}^\times) \mathcal{N}(\mathcal{N}(\hat{\mathcal{O}}))$, it is a union of cosets of the latter group, which has index at most 2 in $\hat{F}^\times$. Thus $F_D^\times \mathcal{N}(\hat{\mathcal{E}})$ itself must be a subgroup of $\hat{F}^\times$ of index less or equal to 2. \qed

Similarly, for each $p \in V_f(F)$, the local class field theory implies that the reduced norm of $\mathcal{E}_p := \mathcal{E}_p(\varphi, B, \mathcal{O})$ is a subgroup of $F_p^\times$ of index at most 3 thanks to the following chain of groups

$$\mathcal{N}(K_p^\times) \subseteq \mathcal{N}(K_p^\times) \mathcal{N}(\mathcal{N}(\mathcal{O}_p)) \subseteq \mathcal{N}(\mathcal{E}_p) \subseteq F_p^\times. \tag{2.39}$$
If $K$ is unramified at $p$ and $O_p \simeq M_2(O_{F_p})$, then $\text{Nr}(K_p^\times) \supseteq O_p^\times \cap \mathcal{N}(O_p) = F_p^\times O_p^\times$, and $\mathcal{E}_p = K_p^\times O_p^\times$ by (2.20), which implies that

\begin{equation}
\text{Nr}(K_p^\times) = \text{Nr}(K_p^\times) \cap \mathcal{N}(O_p) = \text{Nr}(\mathcal{E}_p).
\end{equation}

In particular, (2.40) holds for all but finitely many $p \in V_f(F)$. We also note that the groups $\text{Nr}(K_p^\times) \cap \mathcal{N}(O_p)$ and $\text{Nr}(\mathcal{E}_p)$ depend only on the genus $\mathcal{G}$ and not on the choice of $O$ (as long as we take a local optimal embedding $\varphi \in \text{Emb}(B_p, O_p)$ in (2.33)).

In light of (2.38) and Lemma 2.8, we see that

\begin{equation}
B \text{ is selective } \iff F_p^\times \text{Nr}(\hat{K}^\times) = F_p^\times \text{Nr}(\hat{E}).
\end{equation}

If $K \cap \Sigma = F$, then

\begin{equation}
F_p^\times \text{Nr}(\hat{K}^\times) \cap \mathcal{N}(\hat{O}) = F_p^\times \text{Nr}(\hat{E}) = \hat{F}^\times,
\end{equation}

and hence $B$ is not selective in this case.

**Lemma 2.8.** We have $K \subseteq \Sigma$ if and only if both of the following conditions hold:

(i) $F_p^\times = F_p^\times \cap \mathcal{N}(O_p)$, or equivalently by weak approximation, $K$ and $D$ are ramified at exactly the same (possibly empty) set of real places of $F$;

(ii) $\mathcal{N}(O_p) \subseteq \text{Nr}(K_p^\times)$ for every $p \in V_f(F)$.

**Proof.** Clearly, (i) and (ii) guarantee that $F_p^\times \text{Nr}(\hat{K}^\times) \supseteq F_p^\times \text{Nr}(\hat{E})$, and hence $K \subseteq \Sigma$.

Conversely, suppose that $K \subseteq \Sigma$, so the the first inclusion in (2.37) becomes an equality. By definition, $\Sigma$ splits completely at all the real places of $F$ that are unramified in $D$, and hence (i) necessarily holds. It follows from $F_p^\times = F_p^\times$ that

\begin{equation}
\text{Nr}(\mathcal{N}(\hat{O})) \subseteq F_p^\times \text{Nr}(\hat{K}^\times).
\end{equation}

If $p$ splits in $K$, then $\text{Nr}(K_p^\times) = F_p^\times$, so $\mathcal{N}(\mathcal{O}_p) \subseteq \text{Nr}(K_p^\times)$ automatically.

On the other hand, according to the local-global compatibility of class field theory [41 §6], for each $p$ with $(K/p) \neq 1$, there is a commutative diagram

\[
\begin{array}{ccc}
F_p^\times / \text{Nr}(K_p^\times) & \overset{\sim}{\longrightarrow} & \text{Gal}(K_p/F_p) \\
\downarrow & & \downarrow \\
\hat{F}^\times / F_p^\times \text{Nr}(\hat{K}^\times) & \overset{\sim}{\longrightarrow} & \text{Gal}(K/F)
\end{array}
\]

Given $a_p \in F_p^\times$, we have $a_p \in \text{Nr}(K_p^\times)$ if and only if $a = (\ldots, 1, a_p, 1, \ldots) \in F_p^\times \text{Nr}(\hat{K}^\times)$. Therefore, condition (ii) is necessary as well. \hfill \Box

As mentioned above, $B$ is selective for the genus $\mathcal{G}$ only if $K \subseteq \Sigma$. For the moment, suppose that this is the case so in particular $F_p^\times = F_p^\times$. The same proof as Lemma 2.8 shows that $F_p^\times \text{Nr}(\hat{K}^\times) = F_p^\times \text{Nr}(\hat{E})$ holds if and only if

\begin{equation}
\text{Nr}(K_p^\times) = \text{Nr}(\mathcal{E}_p) \text{ for every } p \in V_f(F).
\end{equation}

We already have $\text{Nr}(K_p^\times) = \text{Nr}(\mathcal{E}_p)$ at every $p \in V_f(F)$ thanks to Lemma 2.8 again. Thus it remains to discuss whether the equality

\begin{equation}
\text{Nr}(K_p^\times) \cap \mathcal{N}(O_p) = \text{Nr}(\mathcal{E}_p)
\end{equation}

\footnote{Since $\Sigma$ is an abelian extension of $F$, the intersection $K \cap \Sigma$ is well-defined and independent of the choice of $K$ into an algebraic closure of $F$.}
holds for every \( p \in \mathcal{V}_f(F) \) or not. For example, we have seen in \( (2.40) \) that the equality holds when \( K \) is unramified at \( p \) and \( O_p \cong M_2(O_{F_p}) \). In general, it is more desirable to treat \( (2.45) \) purely as a local question, discussed separately from the assumption \( K \subseteq \Sigma_g \). For that end we recall the notion of Eichler invariant in [9 Definition 1.8].

**Definition 2.9.** Let \( p \) be a finite prime of \( F \), \( \mathfrak{f}_p := O_p/p \) be the finite residue field of \( p \), and \( \mathfrak{f}_p' \) be the unique quadratic field extension. When \( O_p \not\cong M_2(O_{F_p}) \), the quotients of \( O_p \) by its Jacobson radical \( \mathfrak{N}(O_p) \) falls into the following three cases:

\[
O_p/\mathfrak{N}(O_p) \cong \mathfrak{f}_p \times \mathfrak{f}_p, \quad \mathfrak{f}_p, \quad \text{or} \quad \mathfrak{f}_p',
\]

and the *Eichler invariant* \( e_p(O) \) of \( O \) at \( p \) is defined to be \( 1, 0, -1 \) accordingly. As a convention, if \( O_p \cong M_2(O_{F_p}) \), then its Eichler invariant is defined to be \( 2 \).

For example, if \( p \in \text{Ram}_f(D) \) and \( O_p \) is maximal, then \( e_p(O) = -1 \). It is shown in [9 Proposition 2.1] that \( e_p(O) = 1 \) if and only if \( O_p \) is a non-maximal Eichler order (particularly, \( p \not\in \text{Ram}_f(D) \)).

**Lemma 2.10.** Suppose that \( e_p(O) \not= 0 \), then \( \text{Nr}(K_p^\infty) \text{Nr}(\mathcal{N}(O_p)) = \text{Nr}(\mathcal{E}_p) \) for all \( B_p \subseteq O_{K_p} \) optimally embeddable into \( O_p \).

**Proof.** Indeed, if \( e_p(O) = 1 \) or \( 2 \), then \( O_p \) is an Eichler order, and the equality \( (2.40) \) is a result of Voight [43]. Thus we focus on the case \( e_p(O) = -1 \).

Without lose of generality, we assume that \( K_p \) is a field extension of \( F_p \), otherwise \( \text{Nr}(K_p^\infty) = F_p^\infty \), and the equality follows trivially from \( (2.39) \). By [9 Proposition 3.1], \( O_p \) is a Bass order. Let \( L_p/F_p \) be the unique unramified quadratic field extension. It is shown in [11 Proposition 1.12] that \( \text{Emb}(O_{L_p}, O_p) \not= \emptyset \), which implies that \( \text{Nr}(O_p^\infty) = O_p^\infty \).

Thus if \( K_p/F_p \) is ramified, then \( \text{Nr}(K_p^\infty) \text{Nr}(\mathcal{N}(O_p)) = F_p^\infty \) and the equality is trivial again.

Let \( \mathfrak{d}(O_p) \) be the reduced discriminant of \( O_p \). From [9 Corollary 3.2], \( \nu_p(\mathfrak{d}(O_p)) \) is odd if and only if \( D_p \) is division. When \( D_p \) is division, there exists \( u \in \mathcal{N}(O_p) \) such that \( \nu_p(\mathfrak{d}(O_p)) \) is odd by [11 Theorem 2.2]. It follows that \( \text{Nr}(\mathcal{N}(O_p)) = F_p^\infty \) in this case and equality is trivial once more.

Lastly, assume that \( K_p = L_p \) and \( D_p \) splits over \( F_p \). In this case, \( \text{Nr}(\mathcal{N}(O_p)) = \text{Nr}(K_p^\infty) \) by [11 Theorem 2.2]. If \( B_p = O_{K_p} \), then \( \mathcal{N}(O_p) \) acts transitively on \( \text{Emb}(B_p, O_p) \) as shown in the start of the proof of [11 Theorem 3.3, p. 178]. In other words, \( \mathcal{E}_p = K_p^\infty \mathcal{N}(O_p) \), and the equality \( (2.45) \) holds. Now suppose that \( B_p \) is non-maximal, and \( B_p' \subseteq B_p \) is the unique \( O_{F_{p'}} \)-order in \( K_p \) such that \( f(B_p'/B_p) = pO_{F_p} \).

Let \( O_p' \supseteq O \) be the unique minimal overorder of \( O_p \). If \( O_p' \not\cong M_2(O_{F_p}) \), then it has Eichler invariant \(-1 \) again by [9 Corollary 3.2]. On the other hand, the equality \( \text{Nr}(\mathcal{N}(O_p')) = \text{Nr}(K_p^\infty) \) holds even if \( O_p' \cong M_2(O_{F_p}) \). Fix an optimal embedding \( \varphi_p : B_p \to O_p \), and put \( \mathcal{E}_p' := \mathcal{E}_p(\varphi_p, B_p', O_p') \). According to [11 Lemma 3.7], every optimal embedding \( B_p \to O_p \) extends to an optimal embedding \( B_p' \to O_p' \), which implies that \( \mathcal{E}_p' \subseteq \mathcal{E}_p' \). Thus to show that \( \text{Nr}(K_p^\infty) = \text{Nr}(\mathcal{E}_p') \), it is enough to show that \( \text{Nr}(K_p^\infty) = \text{Nr}(\mathcal{E}_p') \) for the pair \( (B_p', O_p') \). Iterating the above argument using the ascending chain of orders in [9 Corollary 3.2], we eventually arrive at a pair
(B_p^\alpha, O_p^\alpha) where either B_p^\alpha = O_{K_p} or O_p^\alpha \simeq M_2(O_{F_p}). It has already been shown that \( \text{Nr}(K_p^\alpha) = \text{Nr}(E_p^\alpha) \), and the lemma is proved. \( \square \)

Recall that the groups \( \text{Nr}(K_p^\alpha) \), \( \text{Nr}(\mathcal{N}(O_p)) \) and \( \text{Nr}(E_p) \) depend only on the genus \( \mathcal{G} \). In the next theorem, we no longer keep \( \mathcal{O} \) fixed with \( \text{Emb}(B, \mathcal{O}) \neq \emptyset \). The current optimal spinor selectivity theorem may be stated as follows.

**Theorem 2.11.** Let \( \mathcal{G} \) be a genus of \( O_F \)-orders in \( D \), and \( \mathcal{O} \) be an arbitrary \( O_F \)-order in \( \mathcal{G} \). Let \( K/F \) be a quadratic field extension that is \( F \)-embeddable into \( D \), and \( B \) be an \( O_F \)-order in \( K \). Suppose that \( m_p(B) \neq 0 \) for every \( p \in \mathcal{V}_f(F) \) so that \( B \) is optimally embeddable into some member of \( \mathcal{G} \). Let \( S \subseteq \mathcal{V}_f(F) \) be the following finite (possibly empty) set of places:

\[
S := \{ p \in \mathcal{V}_f(F) | (K/p) \neq 1 \text{ and } c_p(\mathcal{O}) = 0 \},
\]

Then \( B \) is optimally spinor selective for \( \mathcal{G} \) if and only if

\[
K \subseteq \Sigma_{\mathcal{G}}, \text{ and } \text{Nr}(K_p^\alpha) \text{Nr}(\mathcal{N}(O_p)) = \text{Nr}(E_p) \text{ for every } p \in S.
\]

If \( B \) is selective, then

1. for any two \( O_F \)-orders \( \mathcal{O}, \mathcal{O}' \in \mathcal{G} \),

\[
\Delta(B, \mathcal{O}') = (\rho(\mathcal{O}', \mathcal{O}), K/F) + \Delta(B, \mathcal{O}),
\]

where \( (\rho(\mathcal{O}', \mathcal{O}), K/F) \in \text{Gal}(K/F) \) is the Artin symbol as discussed right below (2.17), and the summation is taken inside \( \mathbb{Z}/2\mathbb{Z} \) with the canonical identification \( \text{Gal}(K/F) \simeq \mathbb{Z}/2\mathbb{Z} \);

2. exactly half of the spinor genera in the genus \( \mathcal{G} \) are selected by \( B \), and similarly, exact half of the spinor classes in \( \text{SCl}(\mathcal{O}) \) is selected by \( B \).

**Proof Theorem 2.11.** Combining Lemma 2.10 with the proof of Lemma 2.8 we see that (2.17) holds if and only if \( F^K \text{Nr}(\hat{K}^\times) = F^\times \text{Nr}(\hat{E}) \), so indeed (2.17) gives a criterion for the selectivity of \( B \).

Suppose next that \( B \) is selective for \( \mathcal{G} \). For the moment, let \( \mathcal{O} \in \mathcal{G} \) be an \( O_F \)-order with \( \text{Emb}(B, \mathcal{O}) \neq \emptyset \). Then (2.18) is just a restatement of Lemma 2.7. In other words, a spinor genus \( [\mathcal{O}']_{\text{sg}} \) is selected by \( B \) if and only if the restriction of the Galois element \( (\rho(\mathcal{O}', \mathcal{O}), K/F) \in \text{Gal}(K/F) \) on \( K \) is trivial. This criterion is just the usual one in the literature (e.g. [16, p. 34]), but now stated for optimal spinor selectivity. Since exactly half of the elements of \( \text{Gal}(\Sigma_{\mathcal{G}}/F) \) restrict trivially on \( K \) and \( \text{SG}(\mathcal{O}) \simeq \text{Gal}(\Sigma_{\mathcal{G}}/F) \), exactly half of the spinor genera in \( \text{SG}(\mathcal{O}) \) are selected by \( B \). Here \( \text{SG}(\mathcal{O}) \) is identified with the group \( \mathfrak{G}_{\text{sg}}(\mathcal{G}) \) as in (2.13).

To prove (2.48) for general \( \mathcal{O}, \mathcal{O}' \in \mathcal{G} \), we pick \( \mathcal{O}'' \in \mathcal{G} \) such that \( \text{Emb}(B, \mathcal{O}'') \neq \emptyset \). Then \( \rho(\mathcal{O}', \mathcal{O}) = \rho(\mathcal{O}', \mathcal{O}'')\rho(\mathcal{O}'', \mathcal{O}) \), and

\[
\Delta(B, \mathcal{O}') = (\rho(\mathcal{O}', \mathcal{O}''), K/F) + 1, \quad \Delta(B, \mathcal{O}) = (\rho(\mathcal{O}, \mathcal{O}''), K/F) + 1.
\]

Taking the difference between these two equalities, we obtain (2.48).

Lastly, we show that exactly half of the spinor classes in \( \text{SCl}(\mathcal{O}) \) are selected by \( B \). From the commutative diagram (2.15), the natural map

\[
\Upsilon : \text{SCl}(\mathcal{O}) \to \text{SG}(\mathcal{O}), \quad [I]_{\text{sc}} \mapsto [[O(I)]_{\text{sg}}
\]

is a surjective homomorphism. By definition, \( [I]_{\text{sc}} \) is selected by \( B \) if and only if \( [[O(I)]_{\text{sg}} \) is selected by \( B \). Since exactly half of the spinor genus in \( \text{SG}(\mathcal{O}) \) are selected by \( B \), the desired result for \( \text{SCl}(\mathcal{O}) \) follows. \( \square \)
Example 2.12. We look at the case of Eichler orders more closely, so keep the assumptions of Theorem 2.11 and suppose further that \( G = \mathcal{G}_n \) as in Example 2.3. In this case, the set \( S \) in (2.40) is empty. Thus \( B \) is selective if and only if \( K \subseteq \Sigma_n \). The same arguments as those in [29, Proposition 5.11] show that \( K \subseteq \Sigma_n \) if and only if both of the following conditions hold:

(a) the extension \( K/F \) and the algebra \( D \) are unramified at every \( p \in \mathcal{V}_f(F) \) and ramify at exactly the same (possibly empty) set of infinite places;

(b) if \( p \) is a finite prime of \( F \) with \( \nu_p(n) \equiv 1 \pmod{2} \), then \( p \) splits in \( K \).

Since \( K \) is assumed to be \( F \)-embeddable into \( D \), we have \( (\frac{K}{F}) \neq 1 \) for every \( p \in \text{Ram}_f(D) \), but \( \Sigma_n \) splits completely at every \( p \in \text{Ram}_f(D) \), so the inclusion \( K \subseteq \Sigma_n \) implies that \( \text{Ram}_f(D) = \emptyset \). The remaining parts follow from Lemma 2.8 and the description of \( \Sigma_n \) in Example 2.3.

Suppose further that \( n \) is square-free and \( K \subseteq \Sigma_n \). Then \( (\frac{K}{F}) = 1 \) for every \( p|n \).

From (2.20), \( m_p(B) \neq 0 \) for every \( O_F \)-order \( B \) in \( K \) and every finite prime \( p \) of \( F \). It follows from the above discussion that every \( O_F \)-order \( B \) in \( K \) is selective. Let \( B, B' \) be two \( O_F \)-orders in \( K \) and \( O, O' \in \mathcal{G}_n \) be two Eichler orders of level \( n \). Formula (2.48) may be enhanced to

\[
\Delta(B', O') = \left( j(B/B') \rho(O', O), K/F \right) + \Delta(B, O).
\]

To prove (2.49), first suppose that \( \text{Emb}(B, O) \neq \emptyset \). A pure local construction as in the proof of [30, Theorem 3.3] produces another Eichler order \( O'' \in \mathcal{G}_n \) such that \( \text{Emb}(B', O'') \neq \emptyset \) and \( \rho(O, O'') = j(B/B') \), where the equality is taken inside \( \mathcal{G}_{sg}(n) \). Thus from (2.48), we have

\[
\Delta(B', O') = \left( \rho(O', O''), K/F \right) + 1 = \left( j(B/B') \rho(O', O), K/F \right) + \Delta(B, O).
\]

The general case for arbitrary \( O', O \in \mathcal{G}_n \) follows from the special case as in the proof of Theorem 2.11.

We give a brief (and far from complete) historical account of the research on selectivity. Chevalley [15] initiated the study of selectivity questions for a central simple algebra \( D \) over a number field \( F \). When \( D \) is the matrix algebra \( M_n(F) \), and \( K \) is a maximal subfield of \( D \), he gave a formula in terms of the Hilbert class field of \( F \) for the precise ratio of the types of maximal orders of \( D \) containing an isomorphic copy of \( O_K \) to the total number of types of maximal orders. The interest in selectivity is rekindled by the influential paper of Chinburg and Friedman [16], who studied selectivity for maximal orders in quaternion algebras. Their work inspired a series of generalization by Guo and Qin [21], Maclachlan [30], Linowitz [29], Arenas-Carmona [3, 4, 5], Arenas et al. [2], and the list is too extensive to exhaust here. In a different setting, the spinor class (genus) field is introduced by Estes and Hsia [20] and used to study representations of spinor genera of quadratic lattices by many authors: Hsia, Shao and Xu [23], Hsia [22], Chan and Xu [14], Xu [17], etc. These two lines of investigation merge on the selectivity questions, as explained in [3].

2.3. The spinor trace formula. Let \( D \) be a quaternion \( F \)-algebra as before, \( \mathcal{G} \) be a genus of \( O_F \)-orders in \( D \), and \( O \) be a member of \( \mathcal{G} \). Suppose that \( K/F \) is a quadratic field extension that is \( F \)-embeddable into \( D \), and \( B \) is an \( O_F \)-order in \( K \).
Recall the the trace formula
\[
\sum_{[I] \in \text{Cl}(\mathcal{O})} m(B, \mathcal{O}_I, \mathcal{O}_I) = h(B) \prod_{p \in V_f(F)} m_p(B).
\]

Fix a spinor class \([J]_{sc} \in \text{Cl}(\mathcal{O})\) of locally principal right \(\mathcal{O}\)-ideals. Rather than summing over all of \(\text{Cl}(\mathcal{O})\), what happens if the left hand side of \((2.50)\) is summed over the set \(\text{Cl}(\mathcal{O}, [J]_{sc})\) of ideal classes in \([J]_{sc}\)? The surjectivities of the \(Y_i\)'s in the commutative diagram \((2.15)\) imply that
\[
\sum_{[I] \in \text{Cl}(\mathcal{O}, [J]_{sc})} m(B, \mathcal{O}_I, \mathcal{O}_I) = 0 \iff \Delta(B, \mathcal{O}_I(J)) = 0,
\]
so the question is closely related to selectivity.

Let \(\mathcal{O}' \in \mathcal{G}\) be another member of \(\mathcal{G}\), and \(M'\) be a locally principal right \(\mathcal{O}'\)-ideal with \(\mathcal{O}_I(M') = \mathcal{O}\). The map \(I \mapsto IM'\) induces a bijection between locally principal right ideals of \(\mathcal{O}\) and those of \(\mathcal{O}'\), and it preserves ideal classes, spinor classes, and associated left orders. Therefore,
\[
\sum_{[I] \in \text{Cl}(\mathcal{O}, [J]_{sc})} m(B, \mathcal{O}_I, \mathcal{O}_I) = \sum_{[I'] \in \text{Cl}(\mathcal{O}', [J]_{sc}')} m(B, \mathcal{O}_I, \mathcal{O}_I).
\]
We leave it as an exercise to show that if \(\mathcal{O}' \sim \mathcal{O}\), then
\[
\sum_{[I] \in \text{Cl}(\mathcal{O}, [J]_{sc})} m(B, \mathcal{O}_I, \mathcal{O}_I) = \sum_{[I'] \in \text{Cl}(\mathcal{O}', [J]_{sc}')} m(B, \mathcal{O}_I, \mathcal{O}_I).
\]

Let us reformulate the summation in \((2.51)\) adelically. Fix an embedding \(\varphi : K \rightarrow D\) and identify \(K\) with its image in \(D\) as before. We put
\[
\widehat{\mathcal{E}}(B, \mathcal{O}, [J]_{sc}) := \widehat{\mathcal{D}}([J]_{sc}) \cap \widehat{\mathcal{E}}(B, \mathcal{O}),
\]
where \(\widehat{\mathcal{D}}([J]_{sc})\) is the coset of \(D^\times \widehat{D}^1 \widehat{\mathcal{O}}^\times\) defined in \((2.7)\).

**Lemma 2.13.** We have
\[
\sum_{[I] \in \text{Cl}(\mathcal{O}, [J]_{sc})} m(B, \mathcal{O}_I, \mathcal{O}_I) = |K^\times \widehat{\mathcal{E}}(B, \mathcal{O}, [J]_{sc})| / |\widehat{\mathcal{O}}^\times|.
\]

**Proof.** The lemma can be obtained by the same method as in the proof of [42, Theorem III.5.11]. We provide a brief sketch for the convenience of the reader.

Let \(\{g_1, \ldots, g_r\} \subseteq \widehat{D}^1\) be a complete set of representatives of the double coset space \(D^\times \backslash \widehat{D}([J]_{sc}) / \widehat{\mathcal{O}}^\times\). Then \(\{I_i := g_i \mathcal{O} \mid 1 \leq i \leq r\}\) forms a complete set of representatives of \(\text{Cl}(\mathcal{O}, [J]_{sc})\) by \((2.11)\). Put \(\mathcal{O}_i := \mathcal{O}_I(I_i)\) for each \(1 \leq i \leq r\). There is a canonical bijection
\[
K^\times \backslash \widehat{\mathcal{E}}(B, \mathcal{O}_i) \rightarrow \text{Emb}(B, \mathcal{O}_i), \quad K^\times \alpha \mapsto \alpha^{-1} \varphi \alpha.
\]

On the other hand, consider the map
\[
\Phi : \prod_{i=1}^r K^\times \backslash \widehat{\mathcal{E}}(B, \mathcal{O}_i) / \mathcal{O}_i^\times \rightarrow K^\times \backslash \widehat{\mathcal{E}}(B, \mathcal{O}_i),
\]
\[
K^\times \alpha \mathcal{O}_i^\times \mapsto K^\times \alpha g_i \mathcal{O}_i^\times.
\]

It is straightforward to check by definition that \(\Phi\) is a well-defined bijection, and the lemma follows. \(\Box\)
Now we are ready to prove the spinor trace formula.

**Proposition 2.14.** Suppose that one of the following conditions holds:

- $m_p(B) = 0$ for some $p \in \mathcal{V}_f(F)$;
- $K \cap \Sigma_{\mathcal{O}} = F$;
- $B$ is selective for $\mathcal{O}$.

Then for each $[J]_{sc} \in \text{SCI}(\mathcal{O})$, we have

$$
(2.59) \sum_{[I] \in \text{Cl}(\mathcal{O}, [J]_{sc})} m(B, \mathcal{O}_l(I), \mathcal{O}_l(I)^\times) = \frac{2s(\mathcal{Y}, K) \Delta(B, \mathcal{O}_l(J))h(B)}{\lvert \text{SCI}(\mathcal{O}) \rvert} \prod_p m_p(B).
$$

where $s(\mathcal{Y}, K)$ takes the value 1 or 0 depending on whether $K \subseteq \Sigma_{\mathcal{O}}$ or not.

**Remark 2.15.** The only case for which the assumptions of this proposition fail is when

$$(*) \quad K \subseteq \Sigma_{\mathcal{O}} \text{ but } \exists p \in \mathcal{V}_f(F) \text{ such that } e_p(\mathcal{O}) = 0, \text{ and } \text{Nr}(K_p^\times) \neq \text{Nr}(\mathcal{E}_p).$$

So far we do not even have a single example of a pair $(B, \mathcal{O})$ for which $(*)$ holds. Clearly, if $e_p(\mathcal{O}) \neq 0$ for every $p \in \mathcal{V}_f(F)$, e.g. in the case when $\mathcal{O}$ is an Eichler order of arbitrary level, then the assumptions of the proposition hold automatically. If $D$ is further assumed to satisfy the Eichler condition, then $\text{Cl}(\mathcal{O}, [J]_{sc})$ is a singleton with the unique member $[J]$, and we recover the formula in Voight [14], Corollary 31.1.10.

**Proof.** In light of (2.50) and statement (2) of Theorem 2.11, the proposition is equivalent to the following statement: under the assumptions, if $\Delta(B, \mathcal{O}_l(J)) = \Delta(B, \mathcal{O}_l(J'))$ for $[J]_{sc}$ and $[J']_{sc}$ in $\text{SCI}(\mathcal{O})$, then

$$
(2.60) \sum_{[I] \in \text{Cl}(\mathcal{O}, [J]_{sc})} m(B, \mathcal{O}_l(I), \mathcal{O}_l(I)^\times) = \sum_{[I'] \in \text{Cl}(\mathcal{O}, [J']_{sc})} m(B, \mathcal{O}_l(I'), \mathcal{O}_l(I')^\times).
$$

Without lose of generality, we assume that $m_p(B) \neq 0$ for every $p \in \mathcal{V}_f(F)$, and $\Delta(B, \mathcal{O}_l(J)) = 1$, for otherwise the proposition is trivially true. Let $\mathcal{O}_0 \in \mathcal{O}$ be a member with $\text{Emb}(B, \mathcal{O}_0) \neq \emptyset$. By (2.32), we may further assume that $\mathcal{O} = \mathcal{O}_0$. It remains to show that (2.60) holds for $J' = O$.

Fix an optimal embedding $\varphi \in \text{Emb}(B, \mathcal{O})$ and identify $K$ with $\varphi(K) \subseteq D$ as before. The assumptions of the proposition guarantee that

$$
(2.61) \quad F_D^\times \text{Nr}(\hat{K}^\times) \text{Nr}(\mathcal{N}(\hat{\mathcal{O}})) = F_D^\times \text{Nr}(\hat{\mathcal{E}}(B, \mathcal{O})).
$$

Indeed, if $K \cap \Sigma_{\mathcal{O}} = F$, then both sides are equal to $F_D^\times$ as shown in (2.32). If $B$ is selective for $\mathcal{O}$, then both sides are equal to $F_D^\times \text{Nr}(\hat{K}^\times)$.

Write $J = z\mathcal{O}$ for some $z \in \hat{D}^\times$. Then $\mathcal{O}_l(J) = z\mathcal{O}z^{-1}$. By Lemma 2.6, the assumption $\Delta(B, \mathcal{O}_l(J)) = 1$ implies that $\text{Nr}(z) \in F_D^\times \text{Nr}(\hat{\mathcal{E}}(B, \mathcal{O}))$. Thanks to (2.61), there exist

$$
(2.62) \quad y \in D^\times \hat{D}^1, \quad k \in \hat{K}^\times \quad \text{and} \quad u \in \mathcal{N}(\hat{\mathcal{O}})
$$

such that $yz = ku$. Thus we have

$$
(2.63) \quad D([J]_{sc}) = D^\times \hat{D}^1 yz\hat{\mathcal{O}}^\times = D^\times \hat{D}^1 ku\hat{\mathcal{O}}^\times = kD^\times \hat{D}^1 \hat{\mathcal{O}}^\times u.
$$
Plugging in (2.54), we obtain
\[
\hat{\cal E}(B, \cal O, [J_{sc}]) = (kD^\times \hat{D}^\times u) \cap \hat{\cal E}(B, \cal O)
\]
(2.64)
\[= k \left(D^\times \hat{D}^\times \cap \hat{\cal E}(B, \cal O)\right)u = k\hat{\cal E}(B, \cal O, [\cal O_{sc}])u.
\]

Lastly, observe that the map \(x \mapsto kxu\) for \(x \in \hat{\cal E}(B, \cal O, [\cal O_{sc}])\) induces a bijection
\[K^\times \backslash \hat{\cal E}(B, \cal O, [\cal O_{sc}])/\hat{\cal O}^\times \rightarrow K^\times \backslash \hat{\cal E}(B, \cal O, [J_{sc}])/\hat{\cal O}^\times.
\]
It follows from Lemma 2.13 that (2.60) holds for \(J' = \cal O\). The proposition is proved.

\[
\Box
\]

3. The spinor class number formulas

We keep the notation and assumptions of the previous section. Suppose further that \(F\) is a totally real number field, and \(D\) is a totally definite quaternion \(F\)-algebra. Let \(\cal O\) be an \(O_F\)-order in \(D\). The goal of this section is to obtain class number formulas for
\[
h^1(\cal O) = |D^\times \backslash \hat{D}^1/\hat{\cal O}^1|, \quad \text{and} \quad h_{sc}(\cal O) = |D^\times \backslash (D^\times \hat{D}^1)/\hat{\cal O}^\times|.
\]
(3.1)

In fact, the majority of this section is devoted to the computation of \(h^1(\cal O)\), which is much more involved than that of \(h_{sc}(\cal O)\). The proof of the formula for \(h_{sc}(\cal O)\) follows exactly the same line of argument as that for the Eichler class number formula, so it is postponed toward the end of this section.

First, we observe that both \(h^1(\cal O)\) and \(h_{sc}(\cal O)\) are constant on each spinor genus of \(O_F\)-orders.

Lemma 3.1. Let \(\cal O, \cal O' \subset D\) be two \(O_F\)-orders in same spinor genus. Then
\[h^1(\cal O) = h^1(\cal O'), \quad \text{and} \quad h_{sc}(\cal O) = h_{sc}(\cal O').
\]
Proof. Pick \(\alpha \in D^\times\) and \(y \in \hat{D}^1\) such that \(\cal O' = (\alpha y)\cal O(\alpha y)^{-1}\). The right multiplication of \(\hat{D}^1\) by \(y^{-1}\) induces a bijection
\[
D^\times \backslash \hat{D}^1/\hat{\cal O}^1 \rightarrow D^\times \backslash \hat{D}^1/(y\hat{\cal O}^1 y^{-1}), \quad [x] \mapsto [xy^{-1}].
\]
(3.2)

Since \(\hat{D}^1\) (resp. \(D^1\)) is normal in \(\hat{D}^\times\) (resp. \(D^\times\)), conjugation by \(\alpha\) induces a bijection
\[
D^\times \backslash \hat{D}^1/(\alpha y\hat{\cal O}^1 y^{-1}) \rightarrow D^\times \backslash \hat{D}^1/(\alpha y\hat{\cal O}^1 y^{-1}\alpha^{-1}) = D^\times \backslash \hat{D}^1/\hat{\cal O}^\times , \quad [x] \mapsto [\alpha x \alpha^{-1}].
\]
The equality \(h^1(\cal O) = h^1(\cal O')\) follows immediately. Lastly, we have \(D^\times \hat{D}^1 \hat{\cal O}^\times = D^\times \hat{D}^1(\alpha y)\hat{\cal O}^\times (\alpha y)^{-1}\) by (2.8). Thus the equality \(h_{sc}(\cal O) = h_{sc}(\cal O')\) is induced from the right multiplication of \(D^\times \hat{D}^1 \hat{\cal O}^\times\) by \((\alpha y)^{-1}\) as in (3.2).

\[
\Box
\]

3.1. The class number formula for \(h^1(\cal O)\). Let \(F^\times_+\) be the group of totally positive elements of \(F^\times\), and \(O_{F^+}^\times := F^\times_+ \cap O_F^\times\). The subgroup of \(O_{F^+}^\times\) consisting of all perfect squares in \(O_F^\times\) is denoted by \(O_{F^+}^{\times 2}\). Since \(D\) is totally definite, \(\text{Nr}(D^\times) = F^\times_+\) by (2.1). It follows that \(h_D(F) = h^+(F)\), the narrow class number of \(F\). More precisely, \(h^+(F) = |\text{Pic}^+(O_F)|\), where \(\text{Pic}^+(O_F) \simeq F^\times_+ \hat{\cal O}_F^\times\) is the narrow class group of \(F\).

Lemma 3.2. Let \(\text{Scl}(\cal O)\) be the spinor class group of \(\cal O\) in (2.72). Then
\[
|h(D)| = h(F)|\hat{\cal O}_F^\times : \text{Nr}(\hat{\cal O}^\times)|[O_{F^+}^\times \cap \text{Nr}(\hat{\cal O}^\times) : O_{F^+}^{\times 2}].
\]
(3.3)
Proof. A straightforward calculation shows that
\[
\frac{[\text{Cl}(O)]}{h^+(F)} = [F^\times_+ \hat{O}_F^\times : F^\times_+ \text{Nr}(\hat{O}^\times)] = \frac{[\hat{O}_F^\times : \text{Nr}(\hat{O}^\times)]}{[O^\times_{F,+} : (O^\times_{F,+} \cap \text{Nr}(\hat{O}^\times))]}.
\]

On the other hand, \(h^+(F) = h(F)[O^\times_{F,+} : O^\times_F] \) by [18] Lemma 11.6. Plugging this into (3.4), we obtain (3.3) by observing that \((O^\times_{F,+} \cap \text{Nr}(\hat{O}^\times)) \supseteq O^\times_F\).

Since \(D\) is totally definite, for any noncentral element \(\gamma \in D\), the field extension \(F(\gamma)/F\) is a CM-extension (i.e., a totally imaginary quadratic extension). For simplicity, an \(O_F\)-order \(B\) in a CM-extension \(K/F\) is called a CM \(O_F\)-order. Let \(\mu(B)\) be the group of roots of unity in \(B\). We put
\[
\mathcal{B}^1 := \{B \mid B \text{ is a CM } O_F\text{-order with } |\mu(B)| > 2\},
\]
where two CM \(O_F\)-orders are regarded as the same if they are \(O_F\)-isomorphic. Note that \(\mathcal{B}^1\) is a subset of the finite set \(\mathcal{B}\) studied in [25] §3. In particular, \(\mathcal{B}^1\) itself is a finite set, which can also be seen as follows. Let \(B\) be a member of \(\mathcal{B}\) with fraction field \(K\). For any \(\gamma \in \mu(B) \setminus \{\pm 1\}\), the minimal polynomial of \(\gamma\) over \(F\) is of the form
\[
P_b(T) := T^2 - bT + 1 \in F[T],
\]
where \(b = \text{Tr}_{K/F}(\gamma) \in O_F\). Necessarily \(4 - b^2 \in F^+_\times\) since \(K/F\) is a CM-extension. This leads to the consideration of the set
\[
\mathcal{T} := \{b \in O_F \mid 4 - b^2 \in F^+_\times\},
\]
which is finite since \(O_F\) is discrete in \(F \otimes \mathbb{Q} \mathbb{R}\). For each \(b \in \mathcal{T}\), put \(K_b := F[T]/(P_b(T))\). As \(b\) ranges over \(\mathcal{T}\), the ordered pair \((K_b, \mathcal{T})\) ranges over all \((F\text{-isomorphism classes of) CM-extensions } K/F\) with \(|\mu(K)| > 2\) and a marked root of unity of order \(> 2\). Let \(\mathcal{B}^1_b\) be the finite set of \(O_F\)-orders in \(K_b\) as follows:
\[
\mathcal{B}^1_b := \{B \subseteq K_b \mid O_F[\mathcal{T}] \subseteq B \subseteq O_{K_b}\}.
\]
Clearly, \(\mathcal{B}^1 := \bigcup_{b \in \mathcal{T}} \mathcal{B}^1_b\), which implies that \(\mathcal{B}^1\) is a finite set. In fact, the fiber of the canonical map
\[
\bigcup_{b \in \mathcal{T}} \mathcal{B}^1_b \to \mathcal{B}^1
\]
over each \(B \in \mathcal{B}^1\) has exactly \((|\mu(B)| - 2)/2\) elements.

Next, we recall that concept of mass following the exposition in [51] §2. Let \(G^1\) be the semisimple algebraic \(\mathbb{Q}\)-group that represents the functor
\[
R \mapsto G^1(R) = \{g \in (D \otimes \mathbb{Q} R)^\times \mid \text{Nr}(g) = \hat{g}g = 1\}.
\]
for any commutative \(\mathbb{Q}\)-algebra \(R\). Thus \(\hat{D}^1 = G^1(\hat{\mathbb{Q}})\) is a locally compact unimodular group, and \(\hat{O}^1\) is an open compact subgroup of \(\hat{D}^1\). We normalize the Haar measure \(dx\) on \(\hat{D}^1\) such that
\[
\text{Vol}(\hat{O}^1) = \int_{\hat{D}^1} 1_{\hat{O}^1}(x)dx = 1,
\]
where \(1_{\hat{O}^1}\) is the characteristic function of \(\hat{O}^1\). From [42] Lemma V.1.1, \(O^1 = D^1 \cap \hat{O}^1\) is a finite group, so \(D^1 = G^1(\mathbb{Q})\) is discrete in \(\hat{D}^1\). Thus the equivalent conditions in [51] Proposition 2.1 are satisfied. In particular, \(D^1\) is cocompact in \(\hat{D}^1\), which is also clear by [35] Theorem 5.2. Let \(x_1, \cdots, x_s \in \hat{D}^1\) be a complete
set of representatives for the double coset space \( D^1 \setminus \hat{D}^1 / \hat{O}^1 \), where \( s := h^1(O) \). For each \( 1 \leq i \leq s \), we write \( O_i \) for the \( O_F \)-order \( D \cap x_i \hat{O}_i^{-1} \). By definition, \( \text{Mass}(G^1, \hat{O}^1) \) is the following weighted sum

\[
\text{Mass}(G^1, \hat{O}^1) := \sum_{i=1}^{s} |D^1 \cap x_i \hat{O}_i^{-1}|^{-1} = \sum_{i=1}^{s} |O_i|^{-1}.
\]

Thanks to [51, Lemma 2.2], we have

\[
\text{Mass}(G^1, \hat{O}^1) = \text{Vol}(D^1 \setminus \hat{D}^1),
\]

where \( D^1 \) is equipped with the counting measure. For simplicity, we put

\[
\text{Mass}^1(O) := \text{Mass}(G^1, \hat{O}^1).
\]

From (3.13) and the normalization of the Haar measure (3.11), \( \text{Mass}^1(O) \) depends only on the genus of \( O \).

To write down an explicit formula for \( \text{Mass}^1(O) \), we introduce some new notation. Let \( \mathfrak{d}(O) \) be the reduced discriminant \([27, \S\text{I.4}, \text{p. 24}]\) of \( O \), which is a product of local reduced discriminants:

\[
\mathfrak{d}(O) = \prod_{p \in V_f(F)} \mathfrak{d}(O_p).
\]

Recall that \( \mathfrak{d}(O_p) = O_{F_p} \) if and only if \( O_p \simeq M_2(O_{F_p}) \), which holds for all but finitely many \( p \in V_f(F) \). In particular, the product in (3.16) is finite. Combining [51, Theorem 3.2, Corollary 3.8 and Corollary 4.3], we obtain

\[
\text{Mass}^1(O) = \frac{|\zeta_F(-1)| N(O) \prod_{p \mid \mathfrak{d}(O)} 1 - N(p)^{-2}}{2^n |O_F^\times : \text{Nr}(O^\times)| \prod_{p \mid \mathfrak{d}(O)} 1 - e_p(O) N(p)^{-1}},
\]

where \( n = [F : \mathbb{Q}] \), and \( e_p(O) \) is the Eichler invariant in Definition [29]. Here \( \zeta_F(s) \) is the Dedekind zeta-function of \( F \), whose special values at all negative odd integers are rational [53, Theorem, p. 59]. It is easy to see from the functional equation [27, \S\text{XIII.3, Theorem 2}] that \( \text{sgn}(\zeta_F(-1)) = (-1)^n \).

One of the main results of this section is the following theorem, which connects the class number \( h^1(O) \) with sums of the form \( \sum m(B, O_I(I), O_I(I)^\times) \) over \( \text{Cl}_{sc}(O) \) as studied in Section [2.3].

**Theorem 3.3.** Let \( D \) be a totally definite quaternion algebra over a totally real field \( F \), and \( O \) be an \( O_F \)-order in \( D \). Then

\[
h^1(O) = 2 \text{Mass}^1(O)
\]

\[
= \frac{u(O)}{4} \sum_{B \in \mathcal{B}^1} \frac{[\mu(B)]-2}{w(B)} \sum_{[I] \in \text{Cl}_{sc}(O)} m(B, O_I(I), O_I(I)^\times).
\]

where \( w(B) := |B^\times : O_F^\times| \) and

\[
u(O) := \left( \frac{(O_F^\times, \text{Nr}(O^\times)) : O_F^\times}{2} \right).
\]

Note that the index \( u(O) \) is closely related to the size of the spinor class group \( \text{SCI}(O) \) by Lemma [3.2]. Let us put

\[
M(B) := \frac{h(B)}{w(B)} \prod_{p \in V_f(F)} m_p(B).
\]
as in [42 §V.2, p. 143]. We obtain an explicit formula for $h^1(O)$ by combining Theorem 3.3 with Proposition 2.14.

**Corollary 3.4.** Suppose that the assumptions of Proposition 2.14 hold for every $B \in \mathcal{B}^1$ (e.g. if $e_\mathfrak{p}(O) \neq 0$ for every $\mathfrak{p} \in \mathcal{V}(F)$). Then

\begin{equation}
    h^1(O) = 2\text{Mass}^1(O) + \frac{1}{4\text{det}(F)(\mathcal{O}_K^\times : \text{Nm}(\mathcal{O}_K^\times))} \sum_{B \in \mathcal{B}^1} 2^{\text{rank}(B)} \Delta(B, O)(|\mu(B)| - 2) \cdot M(B).
\end{equation}

The first step of the proof of Theorem 3.3 is to apply the Selberg trace formula for compact quotients. We refer to [6 §1] and [34 §5] for brief introductions. Let $H$ be a locally compact totally disconnected topological group (a group of $td$-type as in [12 §1]). We further assume that $H$ is unimodular with a Haar measure $dx$, normalized so that $\text{Vol}(U) = 1$ for a fixed open compact subgroup $U \subseteq H$ as in (3.14). If $H_1$ is a unimodular closed subgroup of $H$ with Haar measure $dy$, then by an abuse of notation, we still write $dx$ for the induced right $H$-invariant measure $\mathcal{B}$, Corollary 4, §[III.4] on the homogeneous space $H_1 \backslash H$, which is characterized by the following integration formula

\begin{equation}
    \int_H f(x)dx = \int_{H_1 \backslash H} \int_{H_1} f(yx)dydx, \quad \forall f \in C^\infty_c(H).
\end{equation}

Here $C^\infty_c(H)$ denotes the space of locally constant $\mathbb{C}$-valued functions on $H$ with compact support. If $V \subseteq H$ is an open subset of $H$, we write $\text{Vol}(V; H_1 \backslash H) \in [0, \infty]$ for the volume of the canonical image of $V$ in $H_1 \backslash H$. In other words,

\begin{equation}
    \text{Vol}(V; H_1 \backslash H) := \int_{H_1 \backslash H} 1_{H_1 \backslash V}(x)dx.
\end{equation}

Now let $H_2 \subseteq H_1$ be another unimodular closed subgroup with Haar measure $dz$. Suppose that $U \subseteq H$ is an open compact subgroup. Then for any $x \in H$, we have

\begin{equation}
    \text{Vol}(xU; H_2 \backslash H) = \text{Vol}(H_1 \cap xUx^{-1}; H_2 \backslash H_1) \cdot \text{Vol}(xU; H_1 \backslash H).
\end{equation}

For example, if $H_2$ is the trivial group, then we get

\begin{equation}
    \text{Vol}(xU; H_1 \backslash H) = \frac{\text{Vol}(U; H)}{\text{Vol}(H_1 \cap xUx^{-1}; H_1)},
\end{equation}

which will be used frequently below. If $x \in H_1$, then (3.23) simplifies into

\begin{equation}
    \text{Vol}(xU; H_2 \backslash H) = \text{Vol}(x(H_1 \cap U); H_2 \backslash H_1) \cdot \text{Vol}(U; H_1 \backslash H).
\end{equation}

Suppose that $\Gamma$ is a discrete cocompact subgroup of $H$. Denote by $L^2(\Gamma \backslash H)$ the Hilbert space of $\mathbb{C}$-valued square-integrable functions on $\Gamma \backslash H$. For any $f \in C^\infty_c(H)$, we form an operator $R(f) : L^2(\Gamma \backslash H) \to L^2(\Gamma \backslash H)$ by

\begin{equation}
    (R(f)\phi)(y) = \int_H f(x)\phi(xy)dx = \int_H f(y^{-1}x)\phi(x)dx
\end{equation}

for all $y \in H$ and $\phi \in L^2(\Gamma \backslash H)$. It is clear from (3.26) that $R(1_U)$ is the projection on the $U$-invariant subspace $L^2(\Gamma \backslash H)^U = L^2(\Gamma \backslash H/U)$. Therefore,

\begin{equation}
    \text{Tr}(1_U) = \dim_{\mathbb{C}} L^2(\Gamma \backslash H/U) = |\Gamma \backslash H/U|.
\end{equation}
For any $\gamma \in \Gamma$ and any subset $\Omega$ in $H$, we write $\Omega_\gamma$ for the centralizer of $\gamma$ in $\Omega$. Let \{\gamma\} be the $\Gamma$-conjugacy class of $\gamma \in \Gamma$, and \{\Gamma\} be the set of all conjugacy classes of $\Gamma$. Applying the Selberg trace formula [6, p. 9] to $f = \mathbb{1}_U$, we obtain

\begin{equation}
\text{Tr}(\mathbb{1}_U) = \sum_{(\gamma) \in \{\Gamma\}} \int_{\Gamma \setminus H} \mathbb{1}_U(x^{-1}\gamma x)dx
\end{equation}

Here each $\Gamma_\gamma$ is equipped with the counting measure, and $dx$ is the induced right $H$-invariant measure on $\Gamma \setminus H$ as in (3.21). It is well known that the right hand side of (3.28) is a finite sum; see [52, Proposition 8.4].

Applying (3.27) and (3.28) in the case that \{\gamma\} reduces to matching the second line of (3.17) with the first term in the right hand side of (3.17). Thus the proof of Theorem 3.3 is reduced to matching the second line of (3.17) with the contributions of the noncentral classes in $\{D_1\}$.

Assume that $\gamma \in D_1 \setminus \{\pm 1\}$ for the rest of this section. Let us put

\begin{equation}
\hat{E}^1(\gamma, \mathcal{O}) := \{ x \in \hat{D}_1 \mid x^{-1}\gamma x \in \hat{O}_1 \}, \quad \text{and} \quad K_\gamma := F(\gamma)
\end{equation}

so that

\begin{equation}
\int_{\Gamma \setminus H} \mathbb{1}_U(x^{-1}\gamma x)dx = \text{Vol}(\hat{E}^1(\gamma, \mathcal{O}); K_\gamma \setminus \hat{D}_1).
\end{equation}

Since $\hat{E}^1(\gamma, \mathcal{O})$ is an open subset of $\hat{D}_1$, the volume is nonzero if and only if $\hat{E}^1(\gamma, \mathcal{O}) \neq \emptyset$. The latter condition implies that $b = \text{Tr}(\gamma) \in T$ as in (3.27), or equivalently, $\gamma \in \mu(K_\gamma)$ by [15, Lemma 1.6]. For each $b \in T$, we put

\begin{equation}
\Gamma(b) := \{ \gamma \in D_1 \mid \text{Tr}(\gamma) = b \},
\end{equation}

which forms a single $D^x\Gamma$-conjugacy class in $D_1$. Given $\gamma \in \Gamma(b)$, the $F$-embedding

\begin{equation}
\varphi_\gamma : K_b \hookrightarrow D \quad \tilde{T} \mapsto \gamma
\end{equation}

identifies $K_b$ with $K_\gamma$. For each $B \in \mathcal{B}_1$, we define

\begin{equation}
\hat{E}^1(\gamma, B, \mathcal{O}) := \{ x \in \hat{D}_1 \mid \varphi_\gamma(K_b) \cap x\tilde{O}x^{-1} = \varphi_\gamma(B) \} = \hat{E}(\varphi_\gamma, B, \mathcal{O}) \cap \hat{D}_1,
\end{equation}

where $\hat{E}(\varphi_\gamma, B, \mathcal{O})$ is defined in (2.31). There is a $K_\gamma^1$-equivariant decomposition

\begin{equation}
\hat{E}^1(\gamma, \mathcal{O}) = \bigsqcup_{B \in \mathcal{B}_1} \hat{E}^1(\gamma, B, \mathcal{O}),
\end{equation}

which implies that

\begin{equation}
\int_{\Gamma \setminus H} \mathbb{1}_U(x^{-1}\gamma x)dx = \sum_{B \in \mathcal{B}_1} \text{Vol}(\hat{E}^1(\gamma, B, \mathcal{O}); K_\gamma^1 \setminus \hat{D}_1).
\end{equation}

Grouping all those $\Gamma$-conjugacy classes \{\gamma\} with the same minimal polynomial together in the right hand side of (3.28), we obtain
Lemma 3.6. is a unimodular closed normal subgroup of $H$.

Lemma 3.7. where the intersection is taken within $\hat{x}$.

Pick where the last equality follows from Hasse-Minkowski Theorem [35, Theorem 1.6].

is unimodular.

Proposition 3.5. For each $b \in T$ and $B \in \mathcal{B}_b$, we have

$$
\sum_{(\gamma) \in \Gamma(b)} \text{Vol}(\tilde{E}^1(\gamma, B, O); K_\gamma \backslash \hat{D}) = \frac{u(O)}{2u(B)} \sum_{[I] \in \text{Cl}_b(O)} m(B, O(I), O(I)^*) \cdot \text{Vol}(\tilde{E}^1(\gamma, B, O); K_\gamma \backslash \hat{D}).
$$

Note that the right hand side of this equality depends only on (the isomorphism class of) $B$ and not on $b \in T$. Thus Theorem 5.3 follows directly from Proposition 3.5 and the remark around (3.9).

To prove this proposition, we study a new set of groups

$$(3.39) \quad H := \hat{D}^\times / \hat{O}_F^\times, \quad U := \hat{O}_F^\times / \hat{O}_F^\times, \quad \Omega := D^\times \hat{O}_F^\times / \hat{O}_F^\times \simeq D^\times / O_F^\times.$$

Once again, $H$ is a unimodular group of td-type, and we normalize the Haar measure on $H$ such that $\text{Vol}(U) = 1$. By [32] Theorem V.1.2, $O_F^\times / O_F^\times$ is a finite group, which implies that $\Omega$ is discrete cocompact in $H$. We define

$$(3.40) \quad C := \hat{D}^1 \cap \hat{O}_F^\times = \{ \pm 1 \} \cdot \text{Vol}(F),$$

where the intersection is taken within $\hat{D}^\times$. Then $C$ is a compact subgroup of $\hat{D}^1$, and the image of $\hat{D}^1$ in $H$ is canonically isomorphic to $\hat{D}^1 / C$.

Lemma 3.6. $(D^\times \hat{D}^1) \cap \hat{O}_F^\times = O_F^\times C$.

Proof. Clearly, $(D^\times \hat{D}^1) \cap \hat{O}_F^\times \supseteq O_F^\times C$. For any $x \in (D^\times \hat{D}^1) \cap \hat{O}_F^\times$, we have

$$\text{Nr}(x) \in \text{Nr}(D) \cap \hat{O}_F^{\times 2} = F_+^\times \cap \hat{O}_F^{\times 2} = O_F^{\times 2},$$

where the last equality follows from Hasse-Minkowski Theorem [35, Theorem 1.6].

Pick $\xi \in O_F^\times$ such that $\xi^2 = \text{Nr}(x)$. Then $\xi^{-1} x \in \hat{D}^1 \cap \hat{O}_F^\times = C$. It follows that $(D^\times \hat{D}^1) \cap \hat{O}_F^\times \subseteq O_F^\times C$, and the lemma is proved.

Lemma 3.7. The following group

$$(3.41) \quad H_1 := (D^\times \hat{D}^1 \hat{O}_F^\times) / \hat{O}_F^\times \simeq (D^\times \hat{D}^1) / (O_F^\times C)$$

is a unimodular closed normal subgroup of $H$. Moreover,

$$(3.42) \quad U_1 := (\hat{O}_F^\times \hat{O}_F^\times) / \hat{O}_F^\times \simeq \hat{D}^1 / C$$

is an open compact subgroup of $H_1$. In particular, $\hat{D}^1 / C$ is open in $H_1$ as well.

Proof. Since the quotient $\hat{D}^\times / \hat{D}^\times \simeq \hat{F}^\times$ is abelian, any subgroup of $H$ containing $\hat{D}^1 / C$ is normal, and the reduced norm map induces an isomorphism

$$(3.43) \quad H_1 / H \simeq \hat{F}^\times / (F_+^\times \hat{O}_F^{\times 2}).$$

By [35, Theorem 5.2], there is a compact fundamental set $Z$ for $D^1$ in $\hat{D}^1$. Then $H_1 = \Omega \hat{Z}$, where $\hat{Z}$ denotes the canonical image of $Z$ in $H$, and hence $H_1$ is closed by [27, Lemma 1, §X.2]. Now it follows from [8, Proposition B.2.2] that $H_1$ is unimodular.
On the other hand, the reduced norm map induces a bijection
\[ K \] 
Piecing together the above bijections, we obtain
\[ \text{Vol} (\text{Nr}(\hat{\phi}^\times); \hat{F}^x/(\hat{F}^x \times O_F^{x^2})) = \text{Vol}(\hat{\phi}^\times; H_1 \setminus H) = 1/u(O) \]

We start from the right hand side of (3.38) and make a proof of Proposition 3.5.

Clearly, \( U_1 \) is a compact subgroup of \( H_1 \). To show that it is open, it is enough to show that \( U_1 \) is of finite index in the open subgroup \( H_1 \cap U \). We calculate
\[
\begin{align*}
[H_1 \cap U : U_1] &= \left[ \left( (D^x \hat{B}^1 \hat{O}^\times_F) \cap \hat{O}^\times_F \right) : \hat{O}^\times_F \right] \\
&= \left[ \left( (D^x \hat{B}^1) \cap \hat{O}^\times_F \right) : \hat{O}^\times_F \right] \\
&= \left[ \left( (D^x \hat{B}^1) \cap \hat{O}^\times : ((D^x \hat{B}^1) \cap \hat{O}^\times \cap \hat{O}^\times_F) \right) \right] \\
&= \left[ \left( (D^x \hat{B}^1) \cap \hat{O}^\times : \hat{O}^\times O_F^\times \right) \right] \\
&= \left[ \text{Nr} ((D^x \hat{B}^1) \cap \hat{O}^\times) : \text{Nr}(\hat{O}^\times O_F^\times) \right] \\
&= \left[ (\hat{O}^\times_F^1 + \hat{\text{Nr}}(\hat{O}^\times)) : O_F^{x^2} \right] = u(O). \\
\end{align*}
\]

We normalize the Haar measure on \( H_1 \) such that \( \text{Vol}(U_1) = 1 \). From (3.24),
\[
(3.44) \quad \text{Vol}(\text{Nr}(\hat{\phi}^\times); \hat{F}^x/(\hat{F}^x \times O_F^{x^2})) = \text{Vol}(\hat{\phi}^\times; H_1 \setminus H) = 1/u(O)
\]

with respect to the induced measure on \( H_1 \setminus H \).

Let us fix \( \gamma_0 \in \Gamma(b) \) and put \( K_0 := K_{\gamma_0} = F(\gamma_0) \). There is a bijection
\[
(3.45) \quad K_0^\times \setminus D^x \mapsto \Gamma(b), \quad K_0^\times \alpha \mapsto \alpha^{-1} \gamma_0 \alpha,
\]

which gives rise to a bijection
\[
(3.46) \quad K_0^\times \setminus D^x / D^1 \to \{ \Gamma(b) \}.
\]

On the other hand, the reduced norm map induces a bijection
\[
(3.47) \quad K_0^\times \setminus D^x / D^1 \overset{\gamma}{\to} \hat{F}^x / \text{Nr}(K_0^\times).
\]

Piecing together the above bijections, we obtain
\[
(3.48) \quad F^x / \text{Nr}(K_0^\times) \simeq \{ \Gamma(b) \}.
\]

According to Lemma 3.6 we have
\[
D^x \cap (\hat{D}^1 \hat{O}^\times_F) = D^x \cap (\hat{D}^1 (O_F^\times C)) = D^x \cap (\hat{D}^1 O_F^\times) = (D^x \cap \hat{D}^1 O_F^\times) = D^1 O_F^\times.
\]

It follows that there is a canonical isomorphism
\[
(3.49) \quad K_0^\times \setminus (D^x \hat{D}^1 \hat{O}^\times_F)/(\hat{D}^1 \hat{O}^\times_F) \simeq K_0^\times \setminus D^x / (D^1 O_F^\times) = K_0^\times \setminus D^x / D^1.
\]

Proof of Proposition 3.5. We start from the right hand side of (3.38) and make a series of reductions to eventually arrive at the left. Fix \( \gamma_0 \in \Gamma(b) \) as above and put \( \varphi_0 := \varphi_{\gamma_0} \) as defined in (3.33). According to Lemma 2.13
\[
(3.50) \quad \sum_{[l] \in \text{Cl}_{sc}(O)} m(B, O_l(I), O_l(I)^\times) = |(K_0^\times \hat{O}^\times_F) \setminus \hat{E}(B, O, [O]_{sc})|/\hat{O}^\times_F|,
\]

where as defined in (2.54),
\[
(3.51) \quad \hat{E}(B, O, [O]_{sc}) = D^x \hat{D}^1 \hat{O}^\times \cap \hat{E}(\varphi_0, B, O) = (D^x \hat{D}^1 \cap \hat{E}(\varphi_0, B, O)) \hat{O}^\times_F.
\]

Let \( H = \hat{D}^x / \hat{O}^\times_F \) and \( U = \hat{O}^\times / \hat{O}^\times_F \) as in (3.39). Put \( H_2 := K_0^\times / O_F^\times \subset D^x / O_F^\times \) and equip it with the counting measure. Let \( y_1, \ldots, y_r \in D^x \hat{D}^1 \) be a complete set of representatives for the double coset space in (3.50) so that
\[
(3.52) \quad H \supset \hat{E}(B, O, [O]_{sc}) = \bigcup_{i=1}^r H_2 y_i U.
\]
Thanks to (3.24), we have
\[
\text{Vol}(y_i U; H_2 \setminus H) = \frac{\text{Vol}(U; H)}{\text{Vol}(H_2 \cap y_i U y_i^{-1}; H_2)} = \frac{1}{\text{Vol}(B^x/O_F^x; H_2)} = \frac{1}{w(B)}
\]
for each $1 \leq i \leq r$. It follows that
\[
\frac{1}{w(B)} \sum_{[I] \in \mathcal{Cl}_m(O)} m(B, O, (I), O(I)^\times) = \text{Vol}(\hat{E}(B, O, [O]_{sc}); H_2 \setminus H).
\]

On the other hand, we apply (3.25) and (3.24) to obtain
\[
\text{Vol}(y_i U; H_2 \setminus H) = \frac{1}{u(O)} \text{Vol}(y_i (H_1 \cap U); H_2 \setminus H_1).
\]

Summing both sides over all $1 \leq i \leq r$, we get
\[
u(O) \text{Vol}\left(\hat{E}(B, O, [O]_{sc}); H_2 \setminus H\right) = \text{Vol}\left((\hat{E}(B, O, [O]_{sc}) \cap D^x \hat{D}^1 \hat{O}^x_F)/\hat{O}^x_F; H_2 \setminus H_1\right)
\]
\[
= \text{Vol}\left((\hat{E}(\varphi_0, B, O) \cap D^x \hat{D}^1 \hat{O}^x_F)/\hat{O}^x_F; H_2 \setminus H_1\right)
\]
\[
= \text{Vol}\left((\hat{E}(\varphi_0, B, O) \cap D^x \hat{D}^1 \hat{O}^x_F)/\hat{O}^x_F; H_2 \setminus H_1\right).
\]

Let \( \{\alpha_j\}_{j \in A} \) be a complete set of representatives for \( K_0^x \setminus D^x/D^1 \), where \( A := F_+^x/\text{Nr}(K_0^x) \) is regarded as an index set. Then set \( \{\gamma_j := \alpha_j^{-1} \gamma_0 \alpha_j \mid j \in A\} \) forms a complete set of representatives for \( \{\Gamma(b)\} \). According to (3.29), we have
\[
D^x \hat{D}^1 \hat{O}^x_F = \bigcup_{j \in A} (K_0^x \hat{O}^x_F)\alpha_j \hat{D}^1.
\]

It follows from (3.24) and (3.29) that
\[
\hat{E}(\varphi_0, B, O) \cap D^x \hat{D}^1 \hat{O}^x_F = \bigcup_{j \in A} K_0^x \hat{O}^x_F \alpha_j (\hat{E}(\alpha_j^{-1} \varphi_0 \alpha_j, B, O) \cap \hat{D}^1)
\]
\[
= \bigcup_{j \in A} K_0^x \hat{O}^x_F \alpha_j \hat{E}^1(\gamma_j, B, O),
\]

This implies that
\[
u(O) \sum_{[I] \in \mathcal{Cl}_m(O)} m(B, O, (I), O(I)^\times)
\]
\[
= \text{Vol}\left((\hat{E}(\varphi_0, B, O) \cap D^x \hat{D}^1 \hat{O}^x_F)/\hat{O}^x_F; H_2 \setminus H_1\right)
\]
\[
= \sum_{j \in A} \text{Vol}(\alpha_j \hat{E}^1(\gamma_j, B, O) \hat{O}^x_F/\hat{O}^x_F; H_2 \setminus H_1)
\]
\[
= \sum_{j \in A} \text{Vol}(\hat{E}^1(\gamma_j, B, O) \hat{O}^x_F/\hat{O}^x_F; (K_0^x \alpha_j/O_F^x) \setminus H_1)
\]
\[
= \sum_{\{\gamma\} \in \{\Gamma(b)\}} \text{Vol}(\hat{E}^1(\gamma, B, O) \hat{O}^x_F/\hat{O}^x_F; (K_0^x \alpha_j/O_F^x) \setminus H_1),
\]

where \( K_0^x/O_F^x \) is equipped with the counting measure for every \( \{\gamma\} \in \{\Gamma(b)\} \). Since \( \hat{E}^1(\gamma, B, O) \hat{O}^x_F/\hat{O}^x_F \) is contained in the open subgroup \( \hat{D}^1 \hat{O}^x_F/\hat{O}^x_F \) of \( H_1 \),
\[
\text{Vol}(\hat{E}^1(\gamma, B, O) \hat{O}^x_F/\hat{O}^x_F; (K_0^x \alpha_j/O_F^x) \setminus H_1)
\]
\[
= \text{Vol}(\hat{E}^1(\gamma, B, O) \hat{O}^x_F/\hat{O}^x_F; (K_0^x \hat{O}^x_F) \setminus (K_0^x \hat{O}^x_F \hat{D}^1)).
\]
A similar proof as that of Lemma 3.6 shows that
\[ (3.61) \quad K_1 \hat{\mathcal{O}}_F \cap \hat{D}^1 = K_1 C. \]
Thus we have a right \( \hat{D}^1 \)-equivariant bijection
\[ (3.62) \quad (K_1 \hat{\mathcal{O}}_F \backslash K_1 \hat{\mathcal{O}}_F) \sim (K_1 \hat{\mathcal{O}}_F \backslash \hat{D}^1), \]
which is also measure preserving once the Haar measure on \( K_1 C \) is normalized so that \( \text{Vol}(C) = 1 \). Indeed, the canonical images of \( \hat{D}^1 \) on both sides have the same volume. Thus
\[ \text{Vol}(\hat{\mathcal{E}}^1(\gamma, B, \mathcal{O}) \hat{\mathcal{O}}_F / \hat{\mathcal{O}}_F; (K_1 \hat{\mathcal{O}}_F \backslash K_1 \hat{\mathcal{O}}_F) \hat{D}^1) = \text{Vol}(\hat{\mathcal{E}}^1(\gamma, B, \mathcal{O}); (K_1 \hat{\mathcal{O}}_F \backslash \hat{D}^1)). \]
Since \( \hat{\mathcal{E}}^1(\gamma, B, \mathcal{O}) \) is left invariant under \( K_1 C \), we have
\[ \text{Vol}(\hat{\mathcal{E}}^1(\gamma, B, \mathcal{O}); (K_1 \hat{\mathcal{O}}_F \backslash \hat{D}^1)) = \text{Vol}(K_1 \hat{\mathcal{O}}_F \backslash (K_1 \hat{\mathcal{O}}_F))^{-1} \text{Vol}(\hat{\mathcal{E}}^1(\gamma, B, \mathcal{O}); K_1 \hat{\mathcal{O}}_F \backslash \hat{D}^1) \]
\[ = 2 \text{Vol}(\hat{\mathcal{E}}^1(\gamma, B, \mathcal{O}); K_1 \hat{\mathcal{O}}_F \backslash \hat{D}^1). \]
Combining the above calculations with (3.59), we obtain (3.38) and the proposition is proved. \( \square \)

3.2. The class number formula for \( h_{sc}(\mathcal{O}) \). Similar to the case of \( h^1(\mathcal{O}) \), we first work out a mass formula. Recall that the mass of \( \mathcal{O} \) is defined as
\[ (3.63) \quad \text{Mass}(\mathcal{O}) := \sum_{[I] \in \text{Cl}(\mathcal{O})} \frac{1}{|\mathcal{O}_I(1)^{\times} : \mathcal{O}_F^{\times}|}. \]
For each spinor ideal class \( [J]_{sc} \in \text{Cl}(\mathcal{O}) \), it is natural to define a spinor mass
\[ (3.64) \quad \text{Mass}(\mathcal{O}, [J]_{sc}) := \sum_{[I] \in \text{Cl}(\mathcal{O}, [J]_{sc})} \frac{1}{|\mathcal{O}_I(1)^{\times} : \mathcal{O}_F^{\times}|}. \]
Write \( J = x\mathcal{O} \) for some \( x \in \hat{\mathcal{O}}^{\times} \), and keep the notation of the set of groups in (3.39). By [48] Lemma 5.1.1, we have \( \text{Mass}(\mathcal{O}) = \text{Vol}(\Omega \backslash H) \), and the same proof shows that
\[ \text{Mass}(\mathcal{O}, [J]_{sc}) = \text{Vol}(D^{\times} \hat{D}^1 x \hat{\mathcal{O}}^{\times} / \hat{\mathcal{O}}_F^{\times}; \Omega \backslash H) = \text{Vol}(D^{\times} \hat{D}^1 x \hat{\mathcal{O}}^{\times} x^{-1} / \hat{\mathcal{O}}_F^{\times}; \Omega \backslash H) \]
\[ = \text{Vol}(D^{\times} \hat{D}^1 \hat{\mathcal{O}}^{\times} / \hat{\mathcal{O}}_F^{\times}; \Omega \backslash H) = \text{Mass}(\mathcal{O}, [\mathcal{O}]_{sc}), \]
where we have applied the equality \( \hat{D}^1 \hat{\mathcal{O}}^{\times} = \hat{D}^1 x \hat{\mathcal{O}}^{\times} x^{-1} \) in (2.8). Therefore
\[ (3.65) \quad \text{Mass}(\mathcal{O}, [J]_{sc}) = \frac{\text{Mass}(\mathcal{O})}{|\text{Cl}(\mathcal{O})|}. \]
For simplicity, we put \( \text{Mass}_{sc}(\mathcal{O}) := \text{Mass}(\mathcal{O}, [J]_{sc}) \) since it does not depend on the choice of \( [J]_{sc} \). It is first shown by Körner [26] (cf. [51] Corollary 4.3] that
\[ (3.66) \quad \text{Mass}(\mathcal{O}) = \frac{h(F)[\xi_F(-1)]|N(\mathcal{O})|}{2^{[F:Q]-1}} \prod_{p | \mathcal{O}} \frac{1 - N(p)^{-2}}{1 - e_p(\mathcal{O}) N(p)^{-1}}. \]
We conclude that
\[ (3.67) \quad \text{Mass}_{sc}(\mathcal{O}) = \frac{|\xi_F(-1)| N(\mathcal{O})}{2^{[F:Q]-1} [\mathcal{O} F^\times : \text{Nr}(\mathcal{O}^{\times})]} \prod_{p | \mathcal{O}} \frac{1 - N(p)^{-2}}{1 - e_p(\mathcal{O}) N(p)^{-1}}, \]
where \( u(\mathcal{O}) = [O_{F^\times} \cap \text{Nr}(\hat{\mathcal{O}}^{\times}) : O_F^{\times}] \) as before.
Let \( \mathcal{B} \) be the set of CM \( O_F \)-orders \( B \) with \( w(B) > 1 \), which is again finite set by [28, §3] (see also [48, §3.3] and [34, footnote, p. 91]). Clearly, \( \mathcal{B} \) contains \( \mathcal{B}^1 \) as a subset.

**Theorem 3.8.** Suppose that the assumption of Proposition 2.14 holds for every \( B \in \mathcal{B} \) (e.g. if \( e_p(O) \neq 0 \) for every \( p \in \mathcal{V}_f(F) \)). Then

\[
(3.68) \quad h_{sc}(O) = \text{Mass}_{sc}(O) + \frac{1}{2|\text{Cl}(O)|} \sum_{B \in \mathcal{B}} 2^{s(B,K)} \Delta(B,O)(w(B) - 1)M(B).
\]

**Proof.** The proof relies on exactly the same argument as those for [32, Corollary V.2.5] and [48, Theorem 1.5], so we merely provide a sketch here. Let \( h = |\text{Cl}(O)| \), and \( I_1, \ldots, I_h \) be a complete set of representatives for the full ideal class set \( \text{Cl}(O) \). Arrange the ideals so that \( [I_i] \in \text{Cl}_c(O) \) for each \( 1 \leq i \leq r \), where \( r = h_{sc}(O) \). For every ideal integral \( a \subseteq O_F \), there is an \( h \times h \) integral matrix \( \mathfrak{B}(a) = (\mathfrak{B}_{ij}(a)) \in M_h(\mathbb{Z}) \) called the Brandt matrix associated to \( a \) as in [48, Definition 3.1.3]. Each entry \( \mathfrak{B}_{ij}(a) \) is non-negative, and all the diagonal entries \( \mathfrak{B}_{ii}(a) = 0 \) unless \( a \) is principal and generated by a totally positive element. We define the spinor Brandt matrix associated to \( a \) as

\[
(3.69) \quad \mathfrak{B}_{sc}(a) := (\mathfrak{B}_{ij}(a))_{1 \leq i, j \leq r}.
\]

In other words, \( \mathfrak{B}_{sc}(a) \) is just the upper-left \((r \times r)\)-block of \( \mathfrak{B}(a) \). Suppose that \( a \subseteq O_F \) is a principal integral ideal generated by a totally positive element \( a \). Fix a complete set \( \mathcal{I} = \{\epsilon_1, \ldots, \epsilon_s\} \) of representatives for the finite elementary 2-group \( O_{F,+}^2 \). For each CM \( O_F \)-order \( B \), we define a finite set

\[
(3.70) \quad T_{B,a} := \{x \in B \setminus O_F \mid N_{K/F}(x) = \epsilon a \text{ for some } \epsilon \in \mathcal{I}\},
\]

where \( K \) is the fractional field of \( B \). Let \( \mathcal{B}_a \) be the set of CM \( O_F \)-orders \( B \) with \( T_{B,a} \neq \emptyset \). For example, if \( a = O_F \), then \( \mathcal{B}_a \) coincides with the set \( \mathcal{B} \) considered above. In general, \( \mathcal{B}_a \) is always a finite set, which can be prove in a similar way as the finiteness of \( \mathcal{B}_1 \) discussed below [36]. Suppose that the assumptions of Proposition 2.14 hold for every \( B \in \mathcal{B}_a \). Mimicking the proof of the Eichler trace formula [42, Proposition 2.4] (cf. [48, Theorem 3.3.7]), we obtain a trace formula for \( \text{Tr}(\mathfrak{B}_{sc}(a)) \) as follows:

\[
(3.71) \quad \text{Tr}(\mathfrak{B}_{sc}(a)) = \delta_a \text{Mass}_{sc}(O) + \frac{1}{4|\text{Cl}(O)|} \sum_{B \in \mathcal{B}_a} 2^{s(B,K)} \Delta(B,O)M(B)T_{B,a},
\]

where \( \delta_a \) takes the value 1 or 0 depending on whether \( a \) is the square of a principal ideal or not. Indeed, (3.71) can be proved in exactly the same way as [48, Theorem 3.3.7]. The only adjustment needed is to apply the spinor trace formula (2.14) and the spinor mass (3.65) when summing up the diagonal entries \( \mathfrak{B}_{ii}(a) \) in [48, (3.15)]. Lastly, note that \( \mathfrak{B}_{ii}(a) = 1 \) for every \( 1 \leq i \leq h \) when \( a = O_F \), so \( h_{sc}(O) = \text{Tr}(\mathfrak{B}_{sc}(O_F)) \). One shows that \( |T_{B,O_F}| = 2(w(B) - 1) \), and Theorem 3.8 is proved.

4. The Quaternion Algebra \( D = D_{\infty_1,\infty_2} \) over \( F = \mathbb{Q}(\sqrt{p}) \)

Throughout this section, \( p \in \mathbb{N} \) is a prime number, \( F = \mathbb{Q}(\sqrt{p}) \), and \( D = D_{\infty_1,\infty_2} \) is the totally definite quaternion \( F \)-algebra that splits at all finite places of \( F \). Let \( \mathcal{G} \) be the genus of maximal \( O_F \)-orders in \( D \), and \( O \in \mathcal{G} \) be an arbitrary member. We aim to write down both \( h^1(O) \) and \( h_{sc}(O) \) as explicitly as possible.
From \(\text{(3.10)}\) and Corollary \(\text{(3.24)}\), \(h^1(\mathbb{O})\) takes the form

\[
(4.1) \quad h^1(\mathbb{O}) = \frac{1}{2} \zeta_F(-1) + \frac{1}{4h(F)} \sum_{B \in \mathcal{B}^1} 2^{s(\mathfrak{g},K)} \Delta(B, \mathbb{O})(|\mu(B)| - 2) h(B)/w(B).
\]

where \(K\) is the field of fractions of \(B\), \(s(\mathfrak{g},K)\) is defined in Proposition \(\text{(2.14)}\) and \(\Delta(B, \mathbb{O})\) in \(\text{(2.30)}\). Here \(\mathcal{B}^1\) denotes the set of CM \(O_F\)-orders with \(|\mu(K)| > 2\) as defined in \(\text{(3.10)}\), so in particular \(|\mu(K)| > 2\). According to \(\text{(4.4)}\) or \(\text{(28)}\), if \(p \geq 7\), then \(K = F(\sqrt{-1})\) or \(F(\sqrt{-3})\). The primes \(p \in \{2, 3, 5\}\) will be handled separately. For simplicity, we put

\[
K_m := F(\sqrt{-m}) \quad \text{for} \quad m \in \{1, 3\}.
\]

Recall that \(F^\times\) denotes the group of totally positive elements of \(F^\times\), and \(O_{F,+} := F^\times_+ \cap O_F^\times\). Let \(\varepsilon \in O_F^\times\) be the fundamental unit of \(F\). By \([1, \text{Theorem 14.34}]\) that the Gauss genus group of \(\Sigma\) is isomorphic (via the Artin reciprocity map) to the group of spinor genera

\[
\Theta_{sg} := \Theta_{sg}(\mathcal{G}) \simeq \hat{F}^\times/(F^\times_+ \hat{O}_F^\times),
\]

which is canonically identifiable with the Gauss genus group \([17, \text{Definition 14.29}]\)

\[
\text{Pic}^+(O_F)/\text{Pic}^+(O_F)^2.
\]

It is well known \([17, \text{Theorem 14.34}]\) that the Gauss genus group of \(F\) has order \(2^{t-1}\), where \(t\) is the number of finite primes ramified in \(F/Q\), so in our case

\[
|\Theta_{sg}| = \begin{cases} 
1 & \text{if } p \not\equiv 3 \pmod{4}, \\
2 & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\]

If \(p \equiv 3 \pmod{4}\), then there are two distinct spinor genera of maximal \(O_F\)-orders, and \(\Sigma\) is a quadratic extension of \(F\). In this case, \(K_1 = F(\sqrt{-1})\) is a CM-extension of \(F\) unramified at all the finite places, so we have

\[
\Sigma = K_1 \quad \text{if} \quad p \equiv 3 \pmod{4}
\]

by the description of \(\Sigma\) in Example \(\text{(2.3)}\).

To distinguish the two distinct spinor genera of maximal orders when \(p \equiv 3 \pmod{4}\), we make use of \(D_{p,\infty} = \mathbb{Q}(\sqrt{-p})\), the unique quaternion \(\mathbb{Q}\)-algebra ramified exactly at \(p\) and \(\infty\). Since the theory is interesting in its own right, we keep the prime \(p \in \mathbb{N}\) arbitrary for the moment. Clearly, \(D = D_{p,\infty} \otimes \mathbb{Q} F\). Moreover, the maximal orders of \(D_{p,\infty}\) and \(D\) are related by the following lemma:

**Lemma 4.1** \(\text{(28, Lemma 2.11)}\). For every maximal \(\mathbb{Z}\)-order \(\mathfrak{o}\) in \(D_{p,\infty}\), there exists a unique maximal \(O_F\)-order \(\mathcal{M}(\mathfrak{o})\) in \(D = D_{p,\infty} \otimes \mathbb{Q} F\) containing \(\mathfrak{o} \otimes \mathbb{Z} O_F\).
Clearly, \(\mathfrak{o} = \mathcal{M}(\mathfrak{o}) \cap D_{p,\infty}\). Thanks to the uniqueness, each \(\mathcal{M}(\mathfrak{o})\) is symmetric in the sense of [37, §1]. It is shown in [37, Corollary, §2, p. 130] that when \(p \equiv 1\) (mod 4), the map \(\mathcal{M}\) establishes a one-to-one correspondence between the maximal orders of \(D_{p,\infty}\) and the symmetric maximal orders of \(D\). The proof hinges on the fact that \(p\) is the only prime ramified in \(F\), so it applies to the case \(p = 2\) as well. For each \(p \equiv 3\) (mod 4), we exhibit a symmetric maximal order \(\mathcal{O}_{24} \subset D\) in [1,23] with \(\mathcal{O}_{24} \cap D_{p,\infty}\) non-maximal, so the cited corollary is not directly extendable to any prime \(p \equiv 3\) (mod 4).

Let \(\text{Tp}(D)\) be the set of types of maximal \(O_F\)-orders in \(D\), and define \(\text{Tp}(D_{p,\infty})\) similarly. From Lemma 4.1 there is a well-defined map:

\[
\mathcal{M} : \text{Tp}(D_{p,\infty}) \rightarrow \text{Tp}(D), \quad [\mathfrak{o}] \mapsto [\mathcal{M}(\mathfrak{o})].
\]

If \(p = 2\), then \(|\text{Tp}(D_{p,\infty})| = |\text{Tp}(D)| = 1\). When \(p \equiv 1\) (mod 4), the map \(\mathcal{M}\) in (4.7) is studied in detail by Ponomarev [37], and it is shown that

- \(\mathcal{M}\) is one-to-one on the subset of \(\text{Tp}(D_{p,\infty})\) consisting of those conjugacy classes \([\mathfrak{o}]\) where \(\mathfrak{o}\) does not have a principal two-sided prime ideal above \(p\), and \(\mathcal{M}\) is at most two-to-one on the complement;
- \(|\text{img}(\mathcal{M})| = |\text{Tp}(D_{p,\infty})| = (h(-p) - 1 + \left(\frac{2}{p}\right)) / 4|.

Here for each non-square \(d \in \mathbb{Z}\), we write \(h(d)\) for the class number of the quadratic field \(\mathbb{Q}(\sqrt{d})\).

However, we have a different purpose in mind for the map \(\mathcal{M}\). We claim that all members of \(\text{img}(\mathcal{M})\) belong to the same spinor genus. Indeed, for two maximal \(\mathbb{Z}\)-orders \(\mathfrak{o}\) and \(\mathfrak{o}'\) in \(D_{p,\infty}\), there exists an \(x \in \hat{D}_{p,\infty}\) such that \(\mathfrak{o}' = x\mathfrak{o}x^{-1}\). The uniqueness in Lemma 4.1 implies that

\(\mathcal{M}(\mathfrak{o}') = (x \otimes 1)\mathcal{M}(\mathfrak{o})(x \otimes 1)^{-1}\).

We have \(\text{Nr}(x \otimes 1) = \text{Nr}(x) \in \mathbb{Q}^\times\), which projects trivially into \(\mathfrak{S}_\mathbb{Z}\). Therefore, \(\mathcal{M}(\mathfrak{o}) \sim \mathcal{M}(\mathfrak{o}')\), and the claim is verified.

**Definition 4.2.** Suppose that \(p \equiv 3\) (mod 4). A maximal \(O_F\)-order \(\mathcal{O}\) in \(D\) is said to belong to the principal spinor genus if \(\mathcal{O} \sim \mathcal{M}(\mathfrak{o})\) for a maximal \(\mathbb{Z}\)-order \(\mathfrak{o}\) in \(D_{p,\infty}\), otherwise it is said to belong to the nonprincipal spinor genus. We denote the set of types of maximal orders belonging to the principal spinor genus by \(\text{Tp}_0(D)\) and its complement \(\text{Tp}(D) \setminus \text{Tp}_0(D)\) by \(\text{Tp}_1(D)\).

**4.1. The class number formula for** \(h^1(\mathcal{O})\) **when** \(p \neq 3\) (mod 4). **According to** [23, §7], **all maximal** \(O_F\)-orders in \(D\) **belong to the same spinor genus, and** \(\Sigma = F\), **which implies that**

\[
(4.8) \quad s(\mathcal{G}, K) = 0 \quad \text{and} \quad \Delta(B, \mathcal{O}) = 1 \quad \forall B \in \mathcal{B}^1, \mathcal{O} \in \mathcal{G}.
\]

First suppose that \(p \equiv 1\) (mod 4) and \(p \geq 13\). The CM \(O_F\)-orders \(B\) with \(w(B) > 1\) are listed in [23, §7]. In particular, \(\mathcal{B}^1 = \{O_{K_1}, O_{K_2}\}\). To apply (4.1), we gather the relevant data in the following table.

| \(B\) | \(|\mathfrak{m}(B)|\) | \(w(B)\) | \(h(B)/h(F)\) |
|------|----------------|------|----------------|
| \(O_{K_1}\) | 4 | 2 | \(h(-p)/2\) |
| \(O_{K_2}\) | 6 | 3 | \(h(-3p)/2\) |
Plugging the preceding table into (4.1), we get

\[
(4.9) \quad h_1(\mathcal{O}) = \frac{1}{2} \zeta_F(-1) + \frac{1}{8} h(-p) + \frac{1}{6} h(-3p) \quad \text{if } p \equiv 1 \pmod{4} \quad \text{and } p \geq 13.
\]

Coincidentally, \(h_1(\mathcal{O})\) is equal to the type number \(|Tp(D)|\) by [50, §4, (4.10)] (see also [48, Remark 1.4]). This quantity is also known to be the proper class number of even definite quaternary quadratic forms of discriminant \(p\) [13, p. 85]. In this setting the formula is first computed by Kitaoka [25], and also by Ponomarev [36, Corollary, p. 31] using another method.

Now suppose that \(p = 2\). According to [48, §6.2.7], the members of \(B_1\) and the relative invariants are given by the following table

| \(B\)         | \(|\mu(B)|\) | \(|w(B)|\) | \(h(B)/h(F)\) |
|---------------|-------------|------------|---------------|
| \(O_{K_1}\)  | 8           | 4          | 1             |
| \(\mathbb{Z}[\sqrt{2}, \sqrt{-1}]\) | 4           | 2          | 1             |
| \(O_{K_3}\)  | 6           | 3          | 1             |

Recall that \(\zeta_F(-1) = 1/12\) (ibid.), so \(h_1(\mathcal{O}) = 1\) when \(p = 2\).

Next, suppose that \(p = 5\). According to [48, §6.2.9], we have

| \(B\)         | \(|\mu(B)|\) | \(|w(B)|\) | \(h(B)/h(F)\) |
|---------------|-------------|------------|---------------|
| \(O_{K_1}\)  | 4           | 2          | 1             |
| \(O_{K_3}\)  | 6           | 3          | 1             |
| \(\mathbb{Z}[\zeta_{10}]\) | 10          | 5          | 1             |

where \(\zeta_{10}\) denotes a primitive 10th root of unity. Recall that \(\zeta_F(-1) = 1/30\) (ibid.), so \(h_1(\mathcal{O}) = 1\) when \(p = 5\).

4.2. The class number formula for \(h_1(\mathcal{O})\) when \(p \equiv 3 \pmod{4}\). In this case, there are two spinor genera of maximal \(O_F\)-orders in \(D\), the principal one and the nonprincipal one as characterized in Definition 4.2. According to Theorem 2.11 and (4.6), an CM \(O_F\)-order \(B \in B_1\) is selective if and only if its fractional field is \(K_1 = F(\sqrt{-1})\). The most tricky part of the calculation for \(h_1(\mathcal{O})\) is to work out the value of \(\Delta(B, \mathcal{O})\) when \(B \subseteq K_1\).

Regard \(\mathfrak{G}_{sg}\) as a quotient of Pic\(^+\)(\(O_F\)) (and in turn of \(I_F\)). The unique nontrivial element of \(\mathfrak{G}_{sg}\) is represented by the principal ideal \(\sqrt{p}O_F\). If \(\mathcal{O}\) and \(\mathcal{O}'\) are two maximal orders, then \(\rho(\mathcal{O}, \mathcal{O}') \in \mathfrak{G}_{sg}\) is represented by the distance ideal between \(\mathcal{O}\) and \(\mathcal{O}'\) as explained in Example 2.3. For every narrow ideal class in Pic\(^+\)(\(O_F\)), there is an integral ideal \(a \subseteq O_F\) coprime to \(p\) representing this class. If we identify \(\mathfrak{G}_{sg}\) with the multiplicative group \(\{\pm 1\}\), then the canonical projection map Pic\(^+\)(\(O_F\)) \to \mathfrak{G}_{sg}\) is identified with the unique nontrivial genus character [17, §14G, p. 150]

\[
(4.10) \quad \chi : \text{Pic}^+(O_F) \to \{\pm 1\},
\]

which can be expressed in terms of the Kronecker symbol [17, §10D] as follows:

\[
(4.11) \quad \chi([a]_+) = \left(\frac{-p}{N(a)}\right), \quad \forall [a]_+ \in \text{Pic}^+(O_F) \quad \text{with } \gcd(a, p) = 1.
\]

For example, if \(q\) is the unique dyadic prime of \(F\), then

\[
(4.12) \quad \chi([q]_+) = \left(\frac{-p}{2}\right) = \left(\frac{q}{2}\right) = \begin{cases} 1 & \text{if } p \equiv 7 \pmod{8}, \\ -1 & \text{if } p \equiv 3 \pmod{8}. \end{cases}
\]
This can be re-interpreted as follows. By [32 Lemma 3] or [54 Lemma 3.2(1)], there exists \( \theta \in F^\times \) such that \( \varepsilon = 2\theta^2 \), so \( q = 2\theta F \). From (4.12), we have \( N_{F/Q}(\theta) = \frac{1}{2} (\frac{q}{\theta}) \) (cf. the proof of [28 Lemma 6.2.6]).

Given a maximal order \( \mathcal{O} \subset D \), the quotient group \( \mathcal{O}^\times/O_F^\times \) is a finite group called the reduced unit group of \( \mathcal{O} \). Let \( Tp^d(D) \) be the subset of \( Tp(D) \) consisting those \( \mathcal{O} \) such that \( \mathcal{O}^\times/O_F^\times \) is non-cyclic. From [28 §8.2], we have

\[
Tp^d(D) = \{ [\mathcal{O}_8], [\mathcal{O}_{24}], [\mathcal{O}_6] \} \quad \text{if} \ p \geq 7.
\]

If \( p = 3 \), then \( Tp(D) = \{ [\mathcal{O}_8], [\mathcal{O}_{24}] \} \).

To write down these orders explicitly, we adopt the following convention. If \( L \) is a field with \( \text{char}(L) \neq 2 \), and \( \mathfrak{A} = \left( \frac{\alpha}{\mathfrak{A}} \right) \) is a quaternion \( L \)-algebra for some \( a,b \in L^\times \), then \( \{1,i,j,k\} \) denotes the standard \( L \)-basis of \( \mathfrak{A} \) subjected to the following multiplication rules:

\[
i^2 = a, \quad j^2 = b, \quad \text{and} \quad k = ij = -ji.
\]

According to [42 Exercise III.5.2], when \( p \equiv 3 \pmod{4} \), \( D_{p,\infty} \) may be presented as \( 1 (\frac{-1-p}{q}) \), with the standard basis denoted by \( \{1,i,j,k\} \) to emphasize that \( j_i^2 = k_j^2 = -p \). Then \( D = D_{p,\infty} \otimes_{Q} F \) is presented as \( \left( \frac{-1}{q} \right) \) with standard basis \( \{1,i,j,k\} \), where \( j = j_p/\sqrt{p} \) and \( k = k_p/\sqrt{p} \). The conjugation \( \sigma \in \text{Gal}(F/Q) \) extends to a \( Q \)-linear automorphism of \( D \) as follows

\[
\sigma : \sqrt{p} \mapsto -\sqrt{p}, \quad i \mapsto i, \quad j \mapsto -j, \quad k \mapsto -k.
\]

Following [37 §1], a maximal order \( \mathcal{O} \subset D \) is called symmetric if \( \sigma(\mathcal{O}) = \mathcal{O} \) (or equivalently, \( \sigma(\mathcal{O}) = \mathcal{O} \)). Given a symmetric order \( \mathcal{O} \), we write \( \mathcal{O}^\sigma := \{ \alpha \in \mathcal{O} \mid \sigma \alpha = \alpha \} \), which is naturally an order in \( D_{p,\infty} \).

4.3. By [42 Exercise III.5.2] again, \( \mathcal{O}_2 := \mathbb{Z}[i, (1 + j_p)/2] \) is a maximal order in \( D_{p,\infty} = \left( \frac{-1-p}{q} \right) \). In fact, if \( p = 3 \), then \( \mathcal{O}_2 \) is the unique maximal order in \( D_{p,\infty} \) up to \( D_{p,\infty} \)-conjugation. If \( p \geq 7 \), then by [42 Proposition V.3.2], \( \mathcal{O}_2 \) is the unique (up to conjugation) maximal order with \( \mathcal{O}_2^\times/\{\pm 1\} \sim \mathbb{Z}/2\mathbb{Z} \), hence the subscript.

Following [28 Definition-Lemma 5.4], we define

\[
O_8 := OF(j) + iOF(j) = OF + OFi + OF\sqrt{p} + \frac{j}{2} + OF\sqrt{p}i + \frac{k}{2} \subset \left( \frac{-1}{F} \right).
\]

Then \( O_8^\times/O_F^\times \) is a dihedral group of order 8 (resp. 24) if \( p \geq 7 \) (resp. \( p = 3 \)), and \( O_8 \) is the unique maximal order (up to conjugation) with this property by [28 Proposition 5.7]. Clearly, \( O_8 \supseteq O_2 \otimes \mathbb{Z} \), so \( O_8 = M(O_2) \) by Lemma 4.1. It follows from Definition 4.2 that \( O_8 \) belongs to the principal spinor genus.

**Proposition 4.4.** Suppose that \( p \equiv 3 \pmod{4} \). Let \( \mathcal{O} \) (resp. \( \mathcal{O}' \)) be a maximal \( OF \)-order in \( D \) belonging to the principal (resp. nonprincipal) spinor genus. If \( p = 3 \), then \( h^1(\mathcal{O}) = h^1(\mathcal{O}') = 1 \). If \( p \geq 7 \), then

\[
h^1(\mathcal{O}) = \frac{\zeta_F(-1)}{2} + \left( 11 - 3\left( \frac{2}{p} \right) \right) \frac{h(-p)}{8} + \frac{h(-3p)}{6},
\]

\[
h^1(\mathcal{O}') = \frac{\zeta_F(-1)}{2} + \left( 3 - 3\left( \frac{2}{p} \right) \right) \frac{h(-p)}{8} + \frac{h(-3p)}{6}.
\]
Proof. Recall that a CM $O_F$-order $B$ is selective for the genus $\mathcal{G}$ of maximal orders in $D$ if and only if $B$ is contained in $K_1$. We focus on the values of $\Delta(B, \mathcal{O})$ and $\Delta(B, \mathcal{O}')$ for $B \subseteq K_1$, since the rest of the data required by (4.11) are pretty much routine (see [28 §7] and [49 §3]). Since exactly one of the two spinor genera is selected by $B$, we have

\begin{equation}
\Delta(B, \mathcal{O}) + \Delta(B, \mathcal{O}') = 1 \quad \text{if } B \subseteq K_1.
\end{equation}

We first treat the case $p \geq 7$. Define

\begin{equation}
B_{1,2} := O_F + qO_{K_1} = \mathbb{Z} + \mathbb{Z}\sqrt{p} + \mathbb{Z}\sqrt{-1} + \mathbb{Z}(1 + \sqrt{-1})(1 + \sqrt{p})/2,
\end{equation}

where $q = 2\mathfrak{d}O_F$ is the unique dyadic prime of $F$. According to [28 §7], we have

\begin{equation}
\mathcal{G}^1 = \{O_F[\sqrt{-1}], \quad B_{1,2}, \quad O_{K_1}, \quad O_{K_3}\} \quad \text{if } p \geq 7.
\end{equation}

It is clear from (4.16) that $O_8 \cap F(i) = O_F[i] \simeq O_F[\sqrt{-1}]$, and $O_8 \cap F(j) = O_F(j) \simeq O_{K_1}$. Since $O_8$ belongs to the principal spinor genus, we have

\begin{align}
(4.20) & \quad \Delta(O_F[\sqrt{-1}], \mathcal{O}) = 1, \quad \Delta(O_{K_1}, \mathcal{O}) = 1, \quad \text{and hence} \\
(4.21) & \quad \Delta(O_F[\sqrt{-1}], \mathcal{O}') = 0, \quad \Delta(O_{K_1}, \mathcal{O}') = 0 \quad \text{by (4.17)}.
\end{align}

Applying (2.49) with $B = O_{K_1}$ and $B' = B_{1,2}$, we obtain from (4.12) that

\begin{equation}
\Delta(B_{1,2}, \mathcal{O}) = \frac{1}{2} \left(1 + \left(\frac{2}{p}\right)\right), \quad \Delta(B_{1,2}, \mathcal{O}') = \frac{1}{2} \left(1 - \left(\frac{2}{p}\right)\right).
\end{equation}

The data required by (4.11) are assembled in the following table (see [28 §7]):

| $B$ | $|\mu(B)|$ | $w(B)$ | $h(B)/h(F)$ | $s(\mathcal{G}, K)$ | $\Delta(B, \mathcal{O})$ | $\Delta(B, \mathcal{O}')$ |
|-----|-------------|--------|-------------|----------------------|------------------|------------------|
| $O_F[\sqrt{-1}]$ | 4 | 2 | $(2 - \frac{2}{p})h(-p)$ | 1 | 1 | 0 |
| $B_{1,2}$ | 4 | 4 | $(2 - \frac{2}{p})h(-p)$ | 1 | $\frac{1}{2}(1 + \left(\frac{2}{p}\right))$ | $\frac{1}{2}(1 - \left(\frac{2}{p}\right))$ |
| $O_{K_1}$ | 4 | 4 | $h(-p)$ | 1 | 1 | 0 |
| $O_{K_3}$ | 6 | 3 | $h(-3p)/2$ | 0 | 1 | 1 |

The class number formulas for $p \geq 7$ in the proposition now follow by a straightforward calculation.

Next, suppose that $p = 3$. According to [28 Theorem 1.6], we have $|\mathcal{T}(D)| = 2$, and hence every spinor genus of maximal orders contains precisely one type. Necessarily, $\mathcal{O}$ is $D^\times$-conjugate to $O_8$, and $\mathcal{O}'$ is $D^\times$-conjugate to $O_{24}$ to be given in (4.23) below. Note that $K_1 = K_3 = \mathbb{Q}(\sqrt{3}, \sqrt{-1})$ in this case. Let $B_{1,3} := \mathbb{Z}[\sqrt{3}, (1 + \sqrt{-3})/2]$ as in [49 §6.2.6], which has conductor $\sqrt{3}O_F$ in $O_{K_1}$. Applying (2.49) again (or using the explicit form of $O_{24}$ below), we obtain

\begin{equation}
\Delta(B_{1,3}, \mathcal{O}) = 0, \quad \Delta(B_{1,3}, \mathcal{O}') = 1.
\end{equation}

The relevant data required by (4.11) is now given by the following table

| $B$ | $|\mu(B)|$ | $w(B)$ | $h(B)/h(F)$ | $s(\mathcal{G}, K)$ | $\Delta(B, \mathcal{O})$ | $\Delta(B, \mathcal{O}')$ |
|-----|-------------|--------|-------------|----------------------|------------------|------------------|
| $O_F[\sqrt{-1}]$ | 4 | 2 | $(2 - \frac{2}{3})h(-p)$ | 1 | 1 | 0 |
| $B_{1,3}$ | 4 | 4 | $(2 - \frac{2}{3})h(-p)$ | 1 | $\frac{1}{2}(1 + \left(\frac{2}{3}\right))$ | $\frac{1}{2}(1 - \left(\frac{2}{3}\right))$ |
| $O_{K_1}$ | 4 | 4 | $h(-p)$ | 1 | 1 | 0 |
| $O_{K_3}$ | 6 | 3 | $h(-3p)/2$ | 0 | 1 | 1 |
Recall that $\zeta_F(-1) = 1/6$ by [18, §6.2.6] again, we get $h^1(\mathcal{O}) = h^1(\mathcal{O'}) = 1$ when $p = 3$. \hfill $\square$

We write down a representative for the remaining two types of maximal orders $[\mathcal{O}_{24}]^+ \cap \mathcal{O}_6$ in $T_{p}^\ell(D)$ and determine which spinor genus each of them belongs to.

4.5. Let us identify $K_1 = F(\sqrt{-1})$ with the subfield $F(i) \subseteq D = \left(\frac{1}{p} - 1\right)$, and the $O_F$-order $B_{1,2}$ is identified in turn with a suborder $B \subseteq O_{F(i)}$. Consider the order

$$O_{24} := B + B(1 + i + i + k)/2 \subseteq D.$$  

Then $O_{24}^\prime/O_{F}^\prime$ is isomorphic to the symmetric group $S_4$, and $O_{24}$ is the unique maximal order up to conjugation with this property by [28, Proposition 5.1]. We leave it as an exercise to show that $O_{24}$ is symmetric, and

$$(4.24) \quad O_{24}^\prime = \mathbb{Z} + Zi + \mathbb{Z}\frac{(1 + i)(1 + j + k)}{2} \subset O_{23}.$$

Therefore, $O_{24}$ is not of the form $\mathcal{M}(\mathcal{O})$ for any maximal order $\mathcal{O} \subset D_{p,\infty}$.

From [28, Remark 5.8], $O_8 \cap O_{24}$ is an Eichler order of level $q$, and hence $\rho(O_8, O_{24}) \in \mathfrak{G}_{sg}$ is represented by $[q]_+$. It follows from (4.12) that $O_{24}$ belongs to the principal spinor genus if and only if $p \equiv 7 \pmod{8}$. In particular, if $p = 3$, then $O_{24}$ belongs to the non-principal spinor genus. By our construction, $O_{24} \cap F(i) \simeq B_{1,2}$, which verifies (4.22) from another angle.

4.6. Suppose that $p \geq 7$. According to [28, Theorem 1.6], there is a unique maximal order $O_6$ in $D$ up to conjugation with $O_{E}^\prime/O_{F}^\prime$ isomorphic to the dihedral group of order 6. Different from the cases for $O_8$ or $O_{24}$, the maximal order $O_6$ does not admit a unique form [28, §8.2], so we separate the discussion into cases.

First suppose that $p \equiv 7 \pmod{12}$, and write $3O_F = pp$. Since $2e \in F^{\times 2}$, we have $e \equiv -1 \pmod{3}$. From [28, Corollary 6.2.6], $O_{6}$ may be taken to be the maximal order in either [28 (6.3.8)] or [28 (6.3.9)] depending on whether $p \equiv 7 \pmod{24}$ or not. The proofs of [28, Proposition 6.2.2] and [6.2.5] imply that there is a suitable $\alpha \in D^\times$ such that $(\alpha O_6\alpha^{-1}) \cap O_{24}$ is an Eichler order of level $p$ (or $p$ for another suitable choice of $\alpha \in D^\times$). Therefore, $\rho(O_6, O_{24}) \in \mathfrak{G}_{sg}$ is represented by $[p]_+$, which is equal to $[p]_+^{-1}$ in $\text{Pic}^+(O_F)$. We have

$$(4.25) \quad \chi([p]_+) = \chi([p]_+) = \left(-\frac{p}{3}\right) = -1.$$ 

Therefore, when $p \equiv 1 \pmod{3}$, $O_6$ never belongs to the same spinor genus as $O_{24}$. Since $O_{24}$ belongs to the principal spinor genus if and only if $\left(\frac{p}{3}\right) = 1$, we see that $O_6$ belongs to the principal spinor genus if $p \equiv 19 \pmod{24}$, and the nonprincipal one if $p \equiv 7 \pmod{24}$.

Next, suppose that $p \equiv 11 \pmod{12}$. From [28, Proposition 5.6], we may take

$$(4.26) \quad O_6 = O_F + O_F \frac{i + k}{2} + O_F \frac{1 + j}{2} + O_F \left(\frac{\sqrt{p}j + k}{3}\right) \subseteq \left(\frac{-1, -3}{F}\right).$$
On the other hand, there is a unique (up to conjugation) maximal $\mathbb{Z}$-order $\mathcal{O}$ in $D_{p,\infty}$ with $\mathcal{O}_3/\{\pm 1\} \simeq \mathbb{Z}/3\mathbb{Z}$ by [21] Proposition V.3.2. In fact, we may take
\begin{equation}
\mathcal{O}_3 = \mathbb{Z} + \mathbb{Z} \frac{1 + j}{2} + \mathbb{Z} \frac{i_p(1 + j)}{2} + \mathbb{Z} \frac{(-1 + i_p)j}{3} \subseteq \left(\frac{-p,-3}{\mathbb{Q}}\right),
\end{equation}
where we again put a subscript $p$ on $i$ to emphasize that $i_p^2 = -p$. Indeed, this is just the maximal $\mathbb{Z}$-order $O(3, -1)$ in the notation of [24], p. 182. One immediately checks that $\mathcal{O}_3 \otimes_{\mathbb{Z}} O_F \subset \mathcal{O}_6$, and hence $\mathcal{O}_6 = M(\mathcal{O}_3)$. Therefore, $\mathcal{O}_6$ always belongs to the principal spinor genus when $p \equiv 11 \pmod{12}$.

### 4.3. The class number $h_{\text{sc}}(\mathcal{O})$ and the spinor type numbers

We return to the assumption that $p \in \mathbb{N}$ is an arbitrary prime. There is a close relation between the class $h_{\text{sc}}(\mathcal{O})$ and the spinor type number as follows.

**Lemma 4.7.** For each maximal order $\mathcal{O}$ in $D$, let $\text{Tp}_{\text{sg}}(\mathcal{O})$ be the type of maximal orders in the spinor genus of $\mathcal{O}$. Then the following map is bijective:
\begin{equation}
\Upsilon : \text{Cl}_{\text{sc}}(\mathcal{O}) \to \text{Tp}_{\text{sg}}(\mathcal{O}), \quad [I] \mapsto [\mathcal{O}_l(I)].
\end{equation}

**Proof.** Adelically, this map is given by the canonical projection
\begin{equation}
D^\times \backslash (D^\times \hat{D}^1 \hat{\mathbb{O}}^\times) / \hat{\mathbb{O}}^\times \to D^\times \backslash (D^\times \hat{D}^1 \mathcal{N}(\hat{\mathcal{O}})) / \mathcal{N}(\hat{\mathcal{O}}).
\end{equation}

Since $D = D_{\infty_1,\infty_2}$ splits at all the finite places and $\mathcal{O}$ is maximal, we have $\mathcal{N}(\hat{\mathcal{O}}) = \hat{F}^\times \hat{\mathbb{O}}^\times$. Thus to show the map in (4.29) is bijective, it is enough to show that
\begin{equation}
(D^\times \hat{D}^1 \mathcal{N}(\hat{\mathcal{O}})) / \mathcal{N}(\hat{\mathcal{O}}) \simeq (D^\times \hat{D}^1 \hat{\mathbb{O}}^\times) / (F^\times \hat{\mathbb{O}}^\times).
\end{equation}

According to [18] Corollary 18.4, $h(p) = h(F)$ is odd, so $F^\times_p \cap \hat{F}^\times \hat{\mathbb{O}}_F = F^\times 2O_F$. Therefore, if $x \in D^\times \hat{D}^1 \hat{\mathbb{O}}^\times \cap \hat{F}^\times \hat{\mathbb{O}}^\times$, then $\text{Nr}(x) \in (F^\times \hat{O}_F \cap \hat{F}^\times \hat{O}_F) = F^\times 2O_F$, and hence there exists $a \in F^\times$ such that $\text{Nr}(ax) \in \hat{O}_F$. On the other hand, any element $y \in \hat{F}^\times \hat{\mathbb{O}}^\times$ with $\text{Nr}(y) \in \hat{O}_F$ necessarily lies in $\hat{\mathbb{O}}^\times$. It follows that
\begin{equation}
D^\times \hat{D}^1 \hat{\mathbb{O}}^\times \cap \hat{F}^\times \hat{\mathbb{O}}^\times = F^\times \hat{\mathbb{O}}^\times,
\end{equation}
which in turn implies (4.30). The lemma is proved. \hfill \square

When $p \not\equiv 3 \pmod{4}$, all maximal orders in $D$ belong to the same spinor genus, and hence $h_{\text{sc}}(\mathcal{O}) = |\text{Tp}(D)|$ by Lemma 1.4. Therefore,
\begin{equation}
h_{\text{sc}}(\mathcal{O}) = h^1(\mathcal{O}) \quad \text{if} \quad p \not\equiv 3 \pmod{4}.
\end{equation}

Indeed, if $p \equiv 1 \pmod{4}$ and $p \geq 13$, then the equality follows from the remark below (4.3). If $p = 2$ or $5$, then $|\text{Tp}(D)| = 1$, and (4.31) follows from the explicit calculations in Section 4.1.

Next, if $p = 3$, then each spinor genus contains exactly one type of maximal orders as pointed out in the proof of Proposition 4.3 Thus $h_{\text{sc}}(\mathcal{O}) = 1$ for every maximal order $\mathcal{O} \subset D$ when $p = 3$.

Last, assume that $p \equiv 3 \pmod{4}$ and $p \geq 7$. It is known [20] Theorem 1.2] that
\begin{equation}
|\text{Tp}(D)| = \frac{\zeta_F(-1)}{2} + \left(13 - 5\left(\frac{2}{p}\right)\right)\frac{h(-p)}{8} + \frac{h(-2p)}{4} + \frac{h(-3p)}{6}.
\end{equation}

Lemma 4.7 provides the means to compute the spinor type numbers $|\text{Tp}_0(D)|$ and $|\text{Tp}_1(D)|$. A priori, these numbers, when interpreted in terms of proper class numbers of even definite quaternary quadratic forms of discriminant $4p$ within a fixed
Suppose that \( p \equiv 3 \pmod{4} \) and \( p \geq 7 \). Let \( \mathcal{T}_0(D) \) and \( \mathcal{T}_1(D) \) be the type sets of the two spinor genera of maximal orders in Definition 4.2. Then

\[
|\mathcal{T}_0(D)| = \frac{\zeta_F(-1)}{4} + \left( 17 - \frac{2}{p} \right) \frac{h(-p)}{16} + \frac{h(-2p)}{8} + \frac{h(-3p)}{12},
\]

\[
|\mathcal{T}_1(D)| = \frac{\zeta_F(-1)}{4} + \left( 9 - \frac{2}{p} \right) \frac{h(-p)}{16} + \frac{h(-2p)}{8} + \frac{h(-3p)}{12}.
\]

**Proposition 4.8.** Suppose that \( p \equiv 3 \pmod{4} \) and \( p \geq 7 \). Let \( \mathcal{T}_0(D) \) and \( \mathcal{T}_1(D) \) be the type sets of the two spinor genera of maximal orders in Definition 4.2. Then

\[
|\mathcal{T}_0(D)| = \frac{\zeta_F(-1)}{4} + \left( 17 - \frac{2}{p} \right) \frac{h(-p)}{16} + \frac{h(-2p)}{8} + \frac{h(-3p)}{12},
\]

\[
|\mathcal{T}_1(D)| = \frac{\zeta_F(-1)}{4} + \left( 9 - \frac{2}{p} \right) \frac{h(-p)}{16} + \frac{h(-2p)}{8} + \frac{h(-3p)}{12}.
\]

**Proof.** Let \( \mathcal{O} \) (resp. \( \mathcal{O}' \)) be a maximal \( O_F \)-order in the principal (resp. nonprincipal) spinor genus. Combining (4.3) with the spinor mass formula (3.67), we get

\[
\text{Mass}_{sc}(\mathcal{O}) = \text{Mass}_{sc}(\mathcal{O}') = \frac{1}{4} \zeta_F(-1).
\]

To apply the class number formula (3.68), we list the set \( \mathcal{B} \) of CM \( O_F \)-orders \( B \) with \( w(B) > 1 \) as in [28] [§8.2] (see also [19] [§3]):

\[
\mathcal{B} = \{ O_F(\sqrt{-1}), B_{1,2}, O_F(\sqrt{-2}), O_F(\sqrt{-3}), O_F(\sqrt{-7}) \}.
\]

Here we have used the fact that \( O_F(\sqrt{-2}) \) coincides with the maximal order \( O_{K_2} \) in \( K_2 := F(\sqrt{-2}) \) by [49] Proposition 2.6 (see also [28] Lemma 7.2.5). Since \( K_2 \nsubseteq \Sigma \), the order \( O_{K_2} \) is not selective for the genus \( \mathcal{G} \). In other words, we have

\[
s(\mathcal{G}, K_2) = 0, \quad \Delta(O_{K_2}, \mathcal{O}) = \Delta(O_{K_2}, \mathcal{O}') = 1.
\]

It is also known by [39] [§2.10] that

\[
h(O_{K_2})/h(F) = h(-2p) \quad \text{and} \quad w(O_{K_2}) = 2.
\]

The remaining required numerical invariants associated to each \( B \neq O_{K_2} \) have already been listed in the previous subsection. Plugging the above data into (3.68), we obtain the class number formulas for \( |\text{Cl}_{sc}(\mathcal{O})| \) and \( |\text{Cl}_{sc}(\mathcal{O}')| \). By Lemma 4.7 these are precisely the desired type numbers \( |\mathcal{T}_0(D)| \) and \( |\mathcal{T}_1(D)| \).

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