INDUCED RIGID STRING ACTION FROM FERMIONS

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Abstract

From the Dirac action on the world sheet, an effective action is obtained by integrating over the 4-dimensional fermion fields pulled back to the world sheet. This action consists of the Nambu-Goto area term with right dimensionful constant in front, extrinsic curvature action and the topological Euler characteristic term.

1. QCD strings, in contrast to the fundamental strings, are described by an action

\[ S = T \int \sqrt{g} \, d^2 \xi + \alpha \int \sqrt{g} \, |H|^2 \, d^2 xi, \]  \hspace{1cm} (1)

where $T$ is the string tension (of dimension $\text{length}^{-2}$), $\alpha$ is a dimensionless constant - measure of the stiffness of QCD flux tubes, $H^i = \frac{1}{2} g^{\alpha \beta} H_{\alpha \beta}^i$; and $|H|^2 = H^i H^i$ with sum over $i$, the extrinsic curvature of the string world

\hspace{1cm}
sheet. Here and in what follows \( g_{\alpha \beta} \) is the induced metric on the world sheet which will be regarded as an immersion in \( \mathbb{R}^4 \). The above action (1) has been proposed for describing QCD strings by Polyakov [1] and independently by Kleinert [2]. By considering the second term alone, it has been shown [1,2,3] that the dimensionless coupling \( \alpha \) is asymptotically free. Further, by evaluating the 1-loop effective action for minimal and harmonic immersions in the instanton background [4], the partition function for the action (1) has been shown to be that of a modified 2-dimensional Coulomb system. Invoking the running coupling constant \( \alpha_R \), the system at low momentum region prefers to be in a phase with long range correlations, thereby suggesting confinement of the flux tubes. Recently, Kleinert and Chervyakov [5] have argued comparing the high temperature behaviour of QCD string (1) with that of large-N QCD results of Polchinski [6] that the stiffness parameter \( \alpha \) has to be negative.

2. The action (1) thus appears promising to describe QCD strings in the non-critical dimensions and a derivation of (1) will throw more insight into the issues involved. Wiegmann [7] has demonstrated first that the action of spinning string requires the extrinsic geometry of the embedding of the world sheet into the flat background and later [8] showed that such an action can be derived from the covariant action of Neveu-Schwarz-Ramond and Green-Schwarz super strings by the integration over the fermionic fields. Polyakov [9] has considered the expectation value of Wilson loops in compact \( U(1) \) gauge theory in the presence of monopoles and obtained the action (1) in the large loop limit. It will be important to examine the non-supersymmetric analogue of [8] without recourse to string actions and the present work is devoted to this issue. This is along the lines of and inspired by the work of Rajasekaran and Srinivasan [10] who showed by integrating over the fermion fields in the Dirac action \( \int d^4x \bar{\psi} \left\{ \partial + A \right\} \psi \), where \( A_\mu = A_\mu^a \lambda^a \) the gluon field, one obtains the full lagrangian for QCD. The field strength \( F^a_{\mu \nu} \) is induced by the fermion integration.

3. The Dirac action induced on a 2-dimensional surface \( \Sigma \) immersed in
$R^4$ is

$$\int \mathcal{L}_D d^2 \xi = \int \Sigma d^2 \xi \sqrt{g} \frac{i}{2} g^{\alpha \beta} \bar{q} \{ \gamma_\alpha \partial_\beta - \partial_\beta \gamma_\alpha \} q,$$  \hspace{1cm} (2)

where the Dirac spinor $q$ is in $R^4$ and $\gamma_\alpha$ is the pull-back of the euclidean Dirac gamma matrices $\gamma_\mu$ ($\mu = 1,2,3,4$)

$$\gamma_\alpha = \partial_\alpha X^\mu \gamma_\mu.$$  \hspace{1cm} (3)

Using,

$$\{ \gamma_\mu, \gamma_\nu \} = 2 \delta_{\mu \nu}$$

it follows

$$\{ \gamma_\alpha, \gamma_\beta \} = 2 g_{\alpha \beta}. \hspace{1cm} (4)$$

By writing

$$-\frac{i}{2} \int_\Sigma \sqrt{g} d^2 \xi g^{\alpha \beta} (\partial_\beta \bar{q}) \gamma_\alpha q = \frac{i}{2} \int_\Sigma \partial_\beta (\sqrt{g} g^{\alpha \beta} \bar{q} \gamma_\alpha q) d^2 \xi$$  

$$+ \frac{i}{2} \int_\Sigma \sqrt{g} d^2 \xi g^{\alpha \beta} \bar{q} (\partial_\beta \gamma_\alpha q + \frac{i}{2} \int_\Sigma d^2 \xi g^{\alpha \beta} \bar{q} \gamma_\alpha \partial_\beta q$$  

$$- \frac{i}{2} \int_\Sigma d^2 \xi \partial_\beta \{ \sqrt{g} g^{\alpha \beta} \bar{q} \gamma_\alpha q \}$$

and using

$$\partial_\beta \gamma_\alpha = \partial_\beta X^\mu \gamma_\mu = \Gamma^\delta_{\alpha \beta} \partial_\delta X^\mu \gamma_\mu + H^\mu_{\alpha \beta} \gamma_\mu,$$

where we have made use of the Gauss equation for immersion, viz.,

$$\partial_\beta \partial_\alpha X^\mu = \Gamma^\delta_{\beta \alpha} \partial_\delta X^\mu + H^\mu_{\beta \alpha} \hspace{1cm} (5)$$

with the affine connections $\Gamma^\delta_{\beta \alpha}$ determined by $g_{\alpha \beta}$ and

$$H^\mu_{\alpha \beta} = H^i_{\alpha \beta} \mu, \hspace{1cm} (6)$$

where $H^i_{\alpha \beta}$'s are components of the second fundamental form along the two normals $N^{i\mu}$ ($i = 1,2$) to the surface $\Sigma$, Eqn.2 becomes

$$\int \mathcal{L}_D d^2 \xi = \int \Sigma \sqrt{g} d^2 \xi i g^{\alpha \beta} \bar{q} \{ \gamma_\alpha \partial_\beta + \frac{1}{2} H^i_{\alpha \beta} N^{i\mu} \gamma_\mu \} q.$$  \hspace{1cm} (7)
where we have used $\nabla_\beta (\sqrt{g} g^{\alpha \beta}) = 0$, $\nabla_\beta$ being the covariant derivative on $\Sigma$. We introduce the pull-back of $\gamma_\mu$ onto the normal frame by

$$n_i = N^\mu_i \gamma_\mu,$$

and it follows from $\{\gamma_\mu, \gamma_\nu\} = 2 \delta_{\mu \nu}$ that

$$\{n_i, n_j\} = 2 \delta_{ij}$$

$$\{n_i, \gamma_\alpha\} = 0.$$  \hspace{1cm} (9)

Then Eqn.7 becomes

$$\int L D d^2 \xi = \int_\Sigma \sqrt{g} d^2 \xi |g^{\alpha \beta} \bar{q} \{\gamma_\alpha \partial_\beta + \frac{1}{2} n_i H^i_{\alpha \beta}\} q.$$  \hspace{1cm} (10)

In this way, the Dirac fermion couples to the world sheet only through the extrinsic geometry of the surface $\Sigma$. This agrees with [7,8] and Sedrakyan and Stora [11]. However, instead of rewriting (10) using spin(n) operators as in [11], we use (10) here, along the lines of [7,8].

4. We employ generalized Gauss map $G$ [12,13,14] to describe the immersion of $\Sigma$ in $R^4$. Then,

$$G : \Sigma \rightarrow G_{2,4} \simeq SO(4)/\{SO(2) \times SO(2)\} \simeq Q_2,$$  \hspace{1cm} (11)

where the complex quadric $Q_2$ is taken as a model for the Grassmannian $G_{2,4}$ [12,13]. Then the tangent vector(s) $\partial_\xi X^\mu$ is identified as a point in $Q_2$, up to a multiplicative complex function $\psi$, namely,

$$\partial_\xi X^\mu = \psi \Phi^\mu,$$

$$\partial_\bar{\xi} X^\mu = \bar{\psi} \bar{\Phi}^\mu.$$  \hspace{1cm} (12)

In (12), $\Phi^\mu \Phi^\mu = 0$. This drives the induced metric $g_{\alpha \beta}$ to be in the conformal gauge,

$$g_{zz} = g_{\bar{z} \bar{z}} = 0,$$

$$g_{z \bar{z}} = |\psi|^2 |\Phi|^2,$$

$$\sqrt{g} = |\psi|^2 |\Phi|^2,$$

$$g^{\bar{z} \bar{z}} = 1/(|\psi|^2 |\Phi|^2).$$  \hspace{1cm} (13)
where \(| \Phi |^2 = \Phi^\mu \Phi^\mu\). Using (13), the Dirac action (10) is written as

\[
\int G_D d^2 \xi = \int_{\Sigma} \sqrt{g} (dz \wedge d\bar{z})^i \frac{1}{\sqrt{g}} \left\{ \frac{1}{\sqrt{g}} \left( \partial_z X^\mu \partial_{\bar{z}} + \partial_{\bar{z}} X^\mu \partial_z \right) + H^i N^\mu_i \right\} \gamma_\mu q,
\]

(14)

where (3) and (8) have been used and \(H^i = \frac{1}{2} g^{\alpha\beta} H^i_{\alpha\beta}\). The Dirac operator is of the form

\[
\gamma_\mu (D^\mu + A^\mu),
\]

(15)

where

\[
D^\mu = \frac{1}{\sqrt{g}} \left\{ \partial_z X^\mu \partial_{\bar{z}} + \partial_{\bar{z}} X^\mu \partial_z \right\},
\]

\[
A^\mu = H^i N^\mu_i.
\]

(16)

5. The effective action obtained by integrating over the fermions is

\[
\Gamma = \frac{1}{2} Tr \ln[\gamma \cdot (D + A)]^2,
\]

(17)

where the trace is over the \(n\)-dimensional gamma matrices and over the 2-dimensional surface \(\Sigma\). It is seen that

\[
[\gamma \cdot (D + A)]^2 = D_\mu D^\mu + D_\mu A^\mu + A_\mu A^\mu + \frac{1}{2} [\gamma_\mu, \gamma_\nu] \{ \frac{1}{2} [D^\mu, D^\nu] + D^\mu A^\nu + A^\mu D^\nu \},
\]

(18)

since \(A^\mu D_\mu = 0\) on account of \(\partial_z X^\mu N^i_\mu = 0\). The \(D_\mu D^\mu\) term using (13) is

\[
D_\mu D^\mu = \frac{1}{\sqrt{g}} \left( \partial_z X^\mu \partial_{\bar{z}} + \partial_{\bar{z}} X^\mu \partial_z \right) \left( \partial_\bar{z} X^\mu \partial^\bar{z} + \partial^\bar{z} X^\mu \partial_\bar{z} \right)
\]

\[
= \frac{|\psi|^2 |\Phi|^2}{\sqrt{g}} \left( \partial_\bar{z} \partial^\bar{z} + \partial_\bar{z} \partial^\bar{z} + \Gamma^\bar{z}_2 \partial^\bar{z} + \Gamma^\bar{z}_2 \partial^\bar{z} \right),
\]

\[
= \nabla_\alpha \nabla^\alpha.
\]
The term \( D_\mu A^\mu \) in (18) will be evaluated using the Weingarten equation for the normals [15]

\[
\partial_\alpha N^{i\mu} = -H^i_\alpha \sqrt{g} \partial_\delta X^\mu + (N^{ju} \partial_\alpha N^{i\nu}) N^{j\mu}
\]  

(20)
as

\[
D_\mu A^\mu = \frac{H^i}{\sqrt{g}} \left( \partial_\alpha X^\mu \partial_\delta N^i_\mu + \partial_\delta X^\mu \partial_\alpha N^i_\mu \right)
= -\frac{H^i}{\sqrt{g}} \left( H^i_\alpha \sqrt{g} \partial_\delta + H^i_\delta \sqrt{g} \partial_\alpha \right)
= -\frac{H^i}{\sqrt{g}} \left( H^i_{zz} + H^i_{z\bar{z}} \right),
\]

where repeated use of the relations (13) have been made. Now from \( H^i = \frac{1}{2} g^{\alpha \beta} H^i_\alpha \beta \) and (13), it follows \( H^i = \frac{1}{2} g^{zz} (H^i_{zz} + H^i_{z\bar{z}}) \) and so,

\[
D_\mu A^\mu = -2H^i H^i = -2 |H|^2.
\]  

(21)
The term \( A_\mu A^\mu \) is simply \( H^i H^i = |H|^2 \) and thus,

\[
[\gamma \cdot (D + A)]^2 = \nabla_\alpha \nabla^\alpha - |H|^2 + \frac{1}{4} [\gamma_\mu, \gamma_\nu] \{ [D^\mu, D^\nu] + (D^\mu A^\nu) \}
= \nabla_\alpha \nabla^\alpha + Y,
\]  

(22)
thereby defining \( Y \).

6. We follow Hawking [16] for the evaluation of the determinant using the heat-kernel method in curved space. The proper-time regularization method will be used to write (17) as

\[
\Gamma = -\frac{1}{2} Tr \int_0^\infty \frac{ds}{s} \exp \left[ -s \left[ \gamma \cdot (D + A) \right]^2 \right],
\]  

(23)
after subtracting the divergent part [16]. In the coincidence limit [16,17], we have

\[
\Gamma = \lim_{\xi \to \xi'} -\frac{1}{2} Tr \int_0^\infty \frac{ds}{s} <\xi | \exp \left[ -s \left[ \gamma \cdot (D + A) \right]^2 \right] | \xi' > \sqrt{g} d^2 \xi
\]  

(24)
It is to be noted here that the measure $\sqrt{g} \, d^2\xi$ is in the coincident limit and Hawking [16] places $\sqrt{g}$ in the Minakshisundaram-Seeley coefficients after the coincident limit taken. The heat-kernel has a short distance expansion [17] which in view of the fact that the operator here is 2-dimensional gives,

$$\Gamma = -\frac{1}{2} \int \sqrt{g} \, d^2\xi \text{Tr} \left( \int_0^\infty \frac{ds}{4\pi s} \exp -\frac{\sigma}{2s} \{a_0 + a_1 s + a_2 s^2 + \cdots \} \right)$$

(25)

where $\sigma$ is the world function [18] which is half the square of the geodesic distance between $\xi$ and $\xi'$. The terms in (22) other than the Laplacian will be contained in $a_1$ onwards (see [17]). The first term in (25) can be integrated to give $\frac{2}{\pi \sigma}$. Denote $-\frac{1}{\pi \sigma}$ by $I_0$. The second term after $t = 1/s$ substitution gives $\ell n(\frac{2\mu}{\sigma})a_1$, where $\mu$ is introduced to make the logarithm dimensionless. Terms $a_2$ onwards vanish when $\sigma \to 0$. Such terms involve $\int_0^\infty dss^n \exp (-\sigma/(2s))$ with $n = 0, 1, 2$ etc, and they are $\frac{\pi^{n+1}}{2^{n+1}}\Gamma(-n-1)$. Then (25) becomes,

$$\Gamma = I_0 \int \sqrt{g} \, d^2\xi + \ell n(\frac{2\mu}{\sigma}) \int \sqrt{g} d^2\xi \text{Tr} a_1.$$  

(26)

From the expression (22) and from [17], it follows that

$$\text{Tr} a_1 = \frac{R}{6} - 4 |H|^2,$$

(27)

where $R$ is the scalar curvature of the surface $\Sigma$. Thus the effective action, after setting $4\ell n\frac{2\mu}{\sigma}$ by $I_R$ and $\frac{2}{3}\ell n(\frac{2\mu}{\sigma})$ by $I_E$, becomes,

$$\Gamma = I_0 \int \sqrt{g} \, d^2\xi - I_R \int \sqrt{g} |H|^2 d^2\xi + I_E \int \sqrt{g} Rd^2\xi.$$  

(28)

7. Thus we have shown that starting from the Dirac action (2) on $\Sigma$ for free fermions, the effective action obtained by integrating the fermion fields consists of the Nambu-Goto area term, the extrinsic curvature term and the Euler characteristic. The constants in front of them are divergent as $\sigma \to 0$. It is important to see that the first term in (29) which is the NG action has divergent constant $I_0 (= -\frac{1}{\pi \sigma})$. Since $\sigma$ is the half the square of the geodesic distance between the coincident points $\xi$ and $\xi'$, $\sigma$ has the dimensions of $(\text{length})^2$ and so $I_0$ has $(\text{length})^{-2}$ dimensions. In string
theory [19], the coefficient in front of the NG term indeed has the dimensions of \((\text{length})^{-2}\) (in \(\hbar = c = 1\) units, used here as well). If one starts with a string action, then \(I_0\) can be legitimately absorbed in the string tension. The coefficient in front of the second term in (29), the extrinsic curvature action, is dimensionless and can be absorbed in the stiffness parameter. This demonstration gives a legitimacy for introducing the extrinsic curvature term in the string action. The importance of the extrinsic curvature term in QCD strings, in providing smoothness for the world sheet in lattice calculations [20], in obtaining physical effects [21], providing the correct high temperature behaviour [5] and indicating confinement of QCD flux tubes [4] are well known.

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