Gravitational Waves from a Pulsar Kick Caused by Neutrino Conversions

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Abstract

It has been suggested that the observed pulsar velocities are caused by an asymmetric neutrino emission from a hot neutron star during the first seconds after the supernova collapse. We calculate the magnitude of gravitational waves produced by the asymmetries in the emission of neutrinos. The resulting periodic gravitational waves may be detectable by LIGO and LISA in the event of a nearby supernova explosion.

1 Introduction

It has been suggested that the proper motions of pulsars [1] may be caused by a slight asymmetry in the ejection of neutrinos [2, 3, 4], which carry away most of the energy of a supernova. In fact, it would take only a 1% asymmetry in the neutrino emission to achieve the observed neutron star velocities. Such an asymmetry could be caused by neutrino oscillations in matter, in a strong magnetic field. Although neutrino magnetic moments are negligible, the neutrino interactions with matter depend on the polarization of matter fermions [5]. The viability of the neutrino kick mechanism has been discussed at length in several papers [2, 3, 4]. This mechanism can work if there is a sterile neutrino with mass 1-20 keV and a small, $\sin \theta \sim 10^{-4}$, mixing with the electron neutrino. It is intriguing, that the same neutrino could make up the cosmological dark matter [4]. Unfortunately, detection

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of such a neutrino in a terrestrial experiment is probably not feasible at present because the mixing angle $\sin \theta$ is very small. However, in the event of a nearby supernova, the asymmetric emission of neutrinos from a rotating nascent neutron star can produce gravity waves that may be detectable.

In this paper we will describe the expected gravity waves (GW) from a pulsar kick due to neutrino conversions taking into account the rotation of the neutron star and the neutrino asymmetry associated with this mechanism. Detection of such a signal can help understand the origin of the pulsar kicks.

Gravitational waves from the supernova neutrinos have been considered in the literature\[6,7,8\]. However, the source geometry was assumed somewhat different from that specific to the neutrino kick mechanism. Epstein \[6\] has calculated the burst radiation assuming the distribution of neutrinos is roughly elliptical. The asymmetry was assumed to arise due to flattening by centripetal motion. Burrows and Hayes \[7\] performed a hydrodynamic simulation of a supernova with some imposed initial asymmetry. Cuesta \[8\] described the burst radiation from the sudden neutrino conversions. He has also proposed that the GWs themselves may be the source of the neutron star momentum kick, through their radiation back reaction \[9\].

However, there is an additional, and possibly stronger, source of gravitational radiation due to the asymmetric emission of neutrinos from a rapidly rotating neutron star. One can think of the escaping neutrinos as forming an off-centered rotating beam, at some angle to the rotation axis, superimposed on a spherically symmetric distribution. The symmetric component does not produce gravity waves, but the 1% of all neutrinos that make up the asymmetric “beam” are a source of a periodic GW signal.

\section{Gravity waves from a spinning neutrino beam}

We will adopt Epstein’s notation\[6\], with a few minor modifications. We define the nontensor potentials

$$h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu},$$

$$\theta_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h, \quad (h = \eta^{\alpha\beta} h_{\alpha\beta}),$$

where $g_{\mu\nu}$ is the metric. Indeces of all non-tensors are raised and lowered with the Minkowski metric $\eta^{\alpha\beta} = \delta_{\alpha\beta} = \text{diag}(-1,1,1,1)$. Also, we define $\cdot_{\nu} \equiv \partial/\partial x^\nu$, and take $G = c = 1$. This allows us to write the Einstein field equations as

$$\eta^{\alpha\beta} \theta^\mu_{\nu,\alpha\beta} = -16\pi \tau^\mu_{\nu}$$

\[3\]
with the gauge condition
\[ \theta^{\mu\nu,\nu} = 0. \tag{4} \]

Here we followed Epstein in defining
\[ \tau^{\mu\nu} = \tau^{\nu\mu} = T^{\mu\nu} + t^{\mu\nu}. \tag{5} \]

\( T^{\mu\nu} \) is the matter stress-energy tensor and \( t^{\mu\nu} \) is the suitably defined gravitational stress-energy pseudo-tensor.

Equation (3) is in the form of a wave equation in flat space. Its solution for outgoing wave boundary conditions is
\[ \theta^{\mu\nu}(t, x) = 4 \int \tau^{\mu\nu}(t - |x - x'|, x')|x - x'|^{-1} d^3 x' \tag{6} \]
at any field point \( x^\mu = (t, x) = (x^0, x^i), i, j, ... = 1, 2, 3. \) We will assume all gravitational fields are weak enough that we may evaluate equations (5) and (6) to leading order in \( \theta^{\mu\nu}. \)

Following Epstein [6], let us consider a source of the form
\[ \tau^{ij}(t, x) = n^i n^j r^{-2} \sigma(t - r) f(\Omega, t - r), \tag{7} \]
where \( n = x/r, \) \( r = |x|, \) \( f(\Omega, t) \geq 0, \) \( \int f(\Omega, t) d\Omega \equiv 1. \) This form describes neutrinos continuously released from \( x = 0 \) at the speed of light. \( \sigma(t) \) and \( f(\Omega, t) \) are the rate of energy loss (luminosity) and angular distribution, respectively. In the weak field limit, it is safe to treat all neutrinos as originating from \( x = 0 \) and to use \( t - r = \text{const} \) as an approximation to a null geodesic.

To obtain the detectable radiation we must find the transverse traceless (TT) part of the source. This can be done either with a projector or with a gauge transformation. Let us use the former. The detectable radiation in then
\[ \theta^{\mu\nu}_{TT}(t, x) = 4 \int \tau^{\mu\nu}_{TT}(t - |x - x'|, x')|x - x'|^{-1} d^3 x', \tag{8} \]
where
\[ \tau^{ij}_{TT} = (P^i_k P^j_l - \frac{1}{2} \delta^{ij} P_{kl}) \tau^{kl} \tag{9} \]
with
\[ P^i_j = \delta^i_j - \tilde{n}^i \tilde{n}_j, \quad \tilde{n} = \frac{x - x'}{|x - x'|}. \tag{10} \]
Here \( n \) is a unit vector in the \( x \) direction, \( n' \) is in the \( x' \) direction and \( \tilde{n} \) is in the \( x - x' \) direction. Let \( \theta \) be the angle between \( n \) and \( n' \), \( \psi \) is the angle between \( n \) and \( \tilde{n} \) and \( \psi + \theta \) is the angle between \( n' \) and \( \tilde{n} \).
Then assuming $\mathbf{x}$ is along the $z$-direction, and $\mathbf{n}'$ has polar coordinates $(\theta, \phi)$, we find

$$
(n'^x n'^x)_{TT} = \frac{1}{2} \sin^2(\psi + \theta) \left[ \cos^2 \phi (1 + \cos^2 \psi) - 1 \right], \quad (11)
$$

$$
(n'^y n'^y)_{TT} = \frac{1}{2} \sin^2(\psi + \theta) \left[ \sin^2 \phi (1 + \cos^2 \psi) - 1 \right], \quad (12)
$$

$$
(n'^z n'^z)_{TT} = \frac{1}{2} \sin^2(\psi + \theta) \sin \phi \cos \phi (1 + \cos^2 \psi), \quad (13)
$$

$$
(n'^z n'^x)_{TT} = \frac{1}{2} \sin^2 \psi \sin^2(\psi + \theta), \quad (14)
$$

$$
(n'^z n'^y)_{TT} = \frac{1}{2} \sin \psi \cos \psi \sin \phi \sin^2(\psi + \theta), \quad (15)
$$

$$
(n'^z n'^y)_{TT} = \frac{1}{2} \sin \psi \cos \psi \sin \phi \sin(\psi + \theta). \quad (16)
$$

Using the identities

$$
\sin \psi \frac{1}{r'} = \frac{\sin \theta}{|\mathbf{x} - \mathbf{x}'|} = \frac{\sin(\psi + \theta)}{r}, \quad (18)
$$

$$
|\mathbf{x} - \mathbf{x}'|^2 = r^2 + r'^2 - 2rr' \cos \theta, \quad (19)
$$

we write equation (5) as

$$
\tau^{ij}(t, \mathbf{x}) = n^i n^j r^{-2} \int_{-\infty}^{\infty} f(\Omega', t') \sigma(t') \delta(t - t' - r) dt'. \quad (20)
$$

Inserting this into equation (8) we obtain

$$
\theta^{ij}_{TT}(t, \mathbf{x}) = 4 \int_{-\infty}^{\infty} \int_{4\pi} \int_{0}^{\infty} \frac{(n'^i n'^j)_{TT} f(\Omega', t') \sigma(t')}{|\mathbf{x} - \mathbf{x}'|} \delta(t - |\mathbf{x} - \mathbf{x}'| - t' - r') dr' d\Omega' dt'. \quad (21)
$$

We can now perform the $r'$ integration using equations (18) and (19) to yield

$$
\theta^{ij}_{TT}(t, \mathbf{x}) = 4 \int_{-\infty}^{t-r} \int_{4\pi} \int_{0}^{\infty} \frac{(n'^i n'^j)_{TT} f(\Omega', t') \sigma(t')}{t - t' - r \cos \theta'} d\Omega' dt'. \quad (22)
$$

Let us define

$$
\alpha = \frac{t - t'}{r} - 1. \quad (23)
$$

Then we can write all the angles in terms of $\theta'$ and $\alpha$

$$
\sin \psi = \frac{r'}{|\mathbf{x} - \mathbf{x}'|} \sin \theta' = \frac{\alpha^2 + 2\alpha}{\alpha^2 + 2\alpha(1 - \cos \theta') + 2(1 - \cos \theta') \sin \theta'}. \quad (4)
$$
\[ \sin(\psi + \theta') = \frac{r}{|x - x'|} \sin \theta' = \frac{\sqrt{\alpha^2 + 2\alpha(1 - \cos \theta') + 2(1 - \cos \theta')}}{\alpha} \sin \theta'. \]

We can then write

\[ Q^{ij}(\alpha, t') = \int_{4\pi} \frac{(n'^i n'^j)_{TT}(\alpha, \Omega', f(\Omega', t') d\Omega',}{\alpha + 1 - \cos \theta'} \]

Next we write equation (22) as

\[ \theta^{ij}_{TT}(t, x) = 4r^{-1} \int_{-\infty}^{t-r} Q^{ij}(\alpha, t') \sigma(t') dt'. \quad (25) \]

Let us note that all of the dependence on \( t \) (not \( t' \)) is either in the integration limit, or in \( \alpha \). Thus, we can write the time rate of change of \( \theta^{ij} \) as

\[ \theta^{ij}_{\delta TT}(t, x) = 4r^{-1}Q^{ij}(0, t - r)\sigma(t - r) + 4r^{-2} \int_{-\infty}^{t-r} \frac{\partial Q^{ij}(\alpha, t')}{\partial \alpha} \sigma(t') dt'. \quad (26) \]

We see that there are two contributions. The first is due to the new neutrinos being ejected at \( t - r \), the ones whose effect is just becoming visible. The other, is due to the change in the effect of previously ejected neutrinos, as their distance and angle change. However, this second term is suppressed by an extra factor of \( r \), so it is negligible as long as the radiation lasts for a time that is short compared to the distance between the source and the observer. (In a supernova, the event is about 10 s and the distance is thousands of light years so we can safely ignore this term.)

For the first term, since \( \alpha = 0 \) and \( \psi = 0 \), the angular terms simplify greatly. All of the \( \theta^{2i}_{TT} \) terms drop out and we find

\[ (\theta^{xx}_{-0})_{TT}(x, t) = -2r^{-1}\sigma(t - r) \int_{4\pi} (1 + \cos \theta') \cos 2\phi' f(\Omega', t - r) d\Omega', \quad (27) \]

\[ (\theta^{yy}_{-0})_{TT} = 2r^{-1}\sigma(t - r) \int_{4\pi} (1 + \cos \theta') \sin 2\phi' f(\Omega', t - r) d\Omega'. \quad (28) \]

### 3 \( \delta \)-function Distribution

At this point we simply need to choose a distribution \( f(\Omega', t) \) to find the form of the radiation. The simplest is just a \( \delta \) function in a particular direction. In essence, we are modeling the slight neutrino excess to one side as a \( \delta \)-function jet. In reality the \( \delta \) function introduces some spurious results, but it also provides a basis for more accurate results. Let

\[ f(\Omega', t) = \delta(\cos \theta(t) - \cos \theta'(t))\delta(\phi(t) - \phi'). \quad (29) \]
Then we find
\[
\theta^{xx,0TT}(x,t) = -\theta^{yy,0TT}(x,t) = 2r^{-1}\sigma(t-r)(1 + \cos(\theta(t-r)))\cos 2\phi(t-r),
\]
\[
\theta^{xy,0TT} = 2r^{-1}\sigma(t-r)(1 + \cos(\theta(t-r)))\sin 2\phi(t-r).
\]

The most striking feature is that while the radiation is suppressed for \(\theta = \pi\) when the neutrino jet is away from the observer as expected, there is no suppression near \(\theta = 0\) when the jet is toward the observer. We will see later that this is one of the spurious effects of the \(\delta\)-function.

The neutrino jet’s motion could be quite complex. We will focus on one simplified possibility where the jet precesses at an angular frequency \(\omega\) around an axis that is inclined at an angle \(\alpha\) relative to the \(z\)-axis, and always makes an angle of \(\chi\) relative to that axis. For simplicity we will always assume that the axis of precession is in the \(x-z\)-plane. Rotating it away from this plane can be handled most easily by rotating the observer’s axes.

If \(\alpha = 0\) then the radiation is simply
\[
\theta^{xx,0TT}(x,t) = 2r^{-1}\sigma(t-r)(1 + \cos(\chi))\cos 2\omega(t-r),
\]
\[
\theta^{xy,0TT} = 2r^{-1}\sigma(t-r)(1 + \cos(\chi))\sin 2\omega(t-r).
\]

Which is just circularly polarized gravitational radiation at an angular frequency of \(2\omega\) which is exactly what we would expect.

If we move \(\alpha\) away from the \(z\)-axis. Things get more complicated. By simply rotating the axes we can find
\[
cos \theta = \cos \alpha \cos \chi - \sin \alpha \sin \chi \cos \phi_0,
\]
\[
tan \phi = \frac{\sin \chi \sin \phi_0}{\sin \alpha \cos \chi + \cos \alpha \sin \chi \cos \phi_0},
\]
\[
cos 2\phi = \frac{1 - \tan^2 \phi}{1 + \tan^2 \phi},
\]
\[
\sin 2\phi = \frac{2\tan \phi}{1 + \tan^2 \phi},
\]
where we have let \(\phi_0 = \omega(t-r)\). Using these gives us gravitational radiation that is elliptically polarized and and contains components at all angular frequencies \(n\omega\) where \(n\) is an integer. The radiation is generally dominated by \(\omega\) and \(2\omega\) components, and in fact we will find that the higher frequency terms are spurious effects of the \(\delta\)-function.

The magnitude of any particular Fourier component is then
\[
\theta_{ij}^{TT} \approx \frac{2eL}{r \omega \sqrt{\tau}}
\]
$L$ is the luminosity of neutrinos, $e$ is the excess in one direction (about 1%), $\omega$ is the frequency of the radiation (not the frequency of precession this time), and $\tau$ is the length of time over which the neutrinos are released. (For burst radiation this would be a factor of $\sqrt{\frac{1}{2\pi\omega}}$).

4 Simple Dipole

The $\delta$-function is of course an idealization. The excess of neutrino energy in one direction is caused by the magnetic field and is expected to be of the form

$$f(\Omega', t) = \frac{1}{4\pi} \left( 1 + 3e \cos \tilde{\theta} \right).$$

This distribution is chosen so that it has only a uniform and a dipole contribution. The numerical factors are chosen so that $\int f d\Omega' = 1$ and $\int f \cos \tilde{\theta} d\Omega' = e$. Here $e$ is again the fractional excess of neutrino momentum in one direction compared to the other, and $\tilde{\theta}$ is the angle to the magnetic axis.

If the magnetic axis points in the $(\theta, \phi)$ direction, and we are looking in the $(\theta', \phi')$ direction, then it is easy enough to find

$$\cos \tilde{\theta} = \hat{n}(\theta, \phi) \cdot \hat{n}(\theta', \phi') = \sin \theta \sin \theta' (\cos \phi \cos \phi' + \sin \phi \sin \phi') + \cos \theta \cos \theta'. \quad (40)$$

If we now insert equations (39) and (40) into equations (27) and (28) each term of the integral over $\phi'$ will take one of the following three forms:

$$\int_0^{2\pi} \cos 2\phi' d\phi' = \int_0^{2\pi} \cos 2\phi' \sin \phi' d\phi' = \int_0^{2\pi} \cos 2\phi' \cos \phi' d\phi' = 0. \quad (41)$$

All three are zero, so the result for $\theta^{xx \cdot 0TT}$ and $\theta^{xy \cdot 0TT}$ is identically 0. This is a reflection of the fact that there is no gravitational dipole radiation.

The first term that will produce noticeable radiation is the quadrupole term which we will write in the form

$$f(\Omega', t) = \frac{\gamma}{4\pi} \cos^2 \tilde{\theta}. \quad (42)$$

This is actually not a true quadrupole as it has a monopole contribution, but that is irrelevant as monopoles will not create GWs. The factor of $4\pi$ is to remain consistent with equation (39). The factor of $\gamma$ represents the strength of this term. It is left general, but we would expect that if the dipole term is suppressed by the factor $e$, the quadrupole term should be suppressed by two factors of $e$ so that $\gamma \sim e^2$. 
If equation (42) is inserted into (27) and (28) then we get

$$\theta_{xx} (x,t) = 2r^{-1}\sigma(t-r) \frac{\gamma}{6} \sin^2 \theta \cos 2\phi,$$

(43)

$$\theta_{xy} (x,t) = 2r^{-1}\sigma(t-r) \frac{\gamma}{6} \sin^2 \theta \sin 2\phi.$$

(44)

This is nearly the same result as the $\delta$-function case except for the numerical factor and a $\sin^2 \theta$ instead of $1 + \cos \theta$. This small change results in suppressed radiation whenever the predominant neutrino direction is aligned with the observer’s line of sight. (Because it is a quadrupole term, there is no distinction between aligned and anti-aligned.) Even more interesting, if we allow the axis of rotation to be different from the $z$-axis as we did in the case of the $\delta$-function, we find that we only have constant terms and terms with angular frequency $\omega$ and $2\omega$. All of the higher frequency terms are absent. Unfortunately, it seems at first glance that $\gamma \sim e^2$ so this radiation is likely to be suppressed by two more orders of magnitude and may be unobservable.

5 Realistic quadrupole from an off-centered distribution

The best chance of observing such gravitational waves is from an object with a large quadrupole moment. The pulsar kick mechanism based on neutrino conversions [2, 3, 4] predicts an asymmetric neutrino emission whose strength and direction are determined by the magnetic field inside the neutron star. The magnetic fields deep inside the neutron stars are believed to grow during the first seconds after the supernova collapse ($\alpha - \Omega$ dynamo effect [10]) because of convection. The cooling of the outer regions of the star creates the temperature and entropy gradient that causes convection in these outer regions. It is likely, therefore, that the strongest magnetic field develops away from the central region and that the overall magnetic field has a large non-dipole component. Since the neutrino emission is affected by this magnetic field, the asymmetric part of the neutrinos can form an off-centered beam. This, in turn creates a source for GW with a large quadrupole moment.

Here we consider two cases, which correspond to resonant and off-resonant conversions of neutrinos in Refs. [3] and [4], respectively.

5.1 Resonant conversions

Let us consider a $B$-field of the form

$$B = \hat{k} B_0 \left( \frac{\rho}{\rho_0} \right)^2.$$

(45)
Here $\rho$ is the distance from some magnetic axis, $B_0$ is the magnetic field at the center of the star and $\rho_0$ is the distance from the magnetic axis to the center of the star. This is the simplest model that has no singularity in the current at the magnetic axis and represents charged particles circling with a constant angular velocity so that the current increases linearly with $\rho$. This model is based on cylindrical symmetry so it is a simplification for our case, but it should be sufficient for an order of magnitude estimate.

According to Ref. [2, 3], the radius at which neutrinos escape can be described as

$$r = r_0 + \delta r,$$  \hspace{1cm} (46)

$$2 \frac{\partial N_e}{\partial r} \delta r = -e \left( \frac{3N_e}{\pi^4} \right)^{1/3} \frac{k \cdot B}{k}.$$  \hspace{1cm} (47)

The only change from the reference is that I’ll allow $B$ to vary with distance from the magnetic axis. For simplicity I’ll place the magnetic axis in the $x - z$ plane. Then we can find $\delta$ as a function of $\tilde{\theta}$ and $\tilde{\phi}$:

$$\delta r = \delta_0 \cos \tilde{\theta} [1 - 2x \sin \tilde{\theta} \cos \tilde{\phi} + x^2 \sin^2 \tilde{\theta}];$$  \hspace{1cm} (48)

$\delta_0$ is the same as $\delta$ in the reference (except with the assumed constant B-field replaced by the B field at the origin.) The value $x$ is simply the ratio of the unperturbed escape radius to the distance between the magnetic axis and the center of the star $x = \frac{r_0}{\rho_0}$. Both distances can be a significant fraction of the star’s overall size and are not correlated so $x$ is most likely of order 1 and can be greater or less than 1.

The first term is the one discussed in Ref.[2, 3] it is responsible for the momentum kick and as demonstrated causes no gravitational radiation. The last term is mainly $l = 3$ and also doesn’t contribute to the gravity waves. The middle term however, is a quadrupole term and does create gravity waves.

Its worth noting that while this is actually a measure of how much the neutrinosphere deviates from the sphere, it is assumed that the neutrinos, responding to their environment, will have an energy spectrum that mimics the radius distribution.

$$f(\Omega', t) = \frac{1}{4 \pi} \left( 1 + 3e(1 + \frac{2}{3}x^2)^{-1} \cos \tilde{\theta} [1 - 2x \sin \tilde{\theta} \cos \tilde{\phi} + x^2 \sin^2 \tilde{\theta}] \right).$$  \hspace{1cm} (49)

This distribution has been chosen to have a large monopole term, and a perturbing term of the same form as equation (48). Like equation (48) it has been normalized so that $\int \int f d\Omega' = 1$ and $\int f \cos \tilde{\theta} d\Omega' = e$ where $e$ represents the fractional momentum excess in one direction.

The next task is to find $\cos \tilde{\theta} \sin \tilde{\theta} \cos \tilde{\phi}$ in terms of $(\theta, \theta', \phi, \phi')$. As I already discussed, $\cos \tilde{\theta}$ is the projection of the $\hat{r}'$ ($\hat{n}'$) vector (the integration direction) onto the $\hat{r}$ ($\hat{n}$)vector.
(the direction of the magnetic field axis). Essentially, the radial direction is the $z$ direction for the tilde variables, and $\cos \tilde{\theta}$ is the projection onto this axis.

By contrast $\sin \tilde{\theta} \cos \tilde{\phi}$ is the projection onto the $x$-axis for the tilde variables. Since $\hat{r}$ is the $\tilde{z}$-direction, the $\tilde{x}$-direction has to be in the $\tilde{\theta}, \tilde{\phi}$ plane, but it can be any linear combination of these. So I’ll let it be

$$\tilde{x} = \cos \alpha \hat{\theta} + \sin \alpha \hat{\phi}. \quad (50)$$

In terms of the standard $(x, y, z)$ directions we can write

$$\hat{\theta} = \begin{pmatrix} \cos \theta \cos \phi \\ \cos \theta \sin \phi \\ -\sin \theta \end{pmatrix}, \quad (51)$$

$$\hat{\phi} = \begin{pmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{pmatrix}, \quad (52)$$

$$\hat{r}' = \begin{pmatrix} \sin \theta' \cos \phi' \\ \sin \theta' \sin \phi' \\ \cos \theta' \end{pmatrix}. \quad (53)$$

Which means

$$\sin \tilde{\theta} \cos \tilde{\phi} = \cos \alpha (\cos \theta \sin \theta' \cos (\phi - \phi') - \sin \theta \cos \theta') + \sin \alpha (\sin \theta' \sin (\phi' - \phi)), \quad (54)$$

$$\cos \tilde{\theta} = \sin \theta \sin \theta' \cos (\phi - \phi') - \cos \theta \cos \theta'. \quad (55)$$

Substituting this into equation (49) and in turn into the GW equations (27 and 28) we get

$$\begin{pmatrix} \theta^{xx,0}_{TT}(x, t) \\ \theta^{xy,0}_{TT}(x, t) \end{pmatrix} = -\frac{2e\sigma(t-r)}{r \left(1 + \frac{3}{5} x^2 \right)} \sin \theta \begin{pmatrix} \cos \theta \cos \alpha \cos 2\phi - \sin \alpha \sin 2\phi \\ \cos \theta \sin \alpha \sin 2\phi + \sin \alpha \cos 2\phi \end{pmatrix}$$

$$= -\frac{2e\sigma(t-r)}{r \left(1 + \frac{3}{5} x^2 \right)} \sin \theta \sqrt{\cos^2 \theta \cos^2 \alpha + \sin^2 \alpha} \begin{pmatrix} \cos (2\phi + \delta) \\ \sin (2\phi + \delta) \end{pmatrix}. \quad (56)$$

Here $\tan \delta = \tan \alpha \sec \theta$.

Clearly there is radiation and there is no extra suppression compared to the dipole term. There is now a wider variety of radiation producing motions. Not only can we allow $\theta$ and $\phi$ to change based on rotation about some axis, but $\alpha$ can also change. However, if we assume a simple rotation of angular frequency $\omega$ then $\alpha$ will be linked to the rotation angle. There can be a phase difference between them $\alpha = \phi_0 + \epsilon$, but $\epsilon$ goes to 0 if the magnetic and
rotation axes are coplanar or \( \pi/2 \) when maximally skewed. There is now an extra factor of the rotation angle which adds significant radiation at a frequency of \( 3\omega \) as well as \( \omega \) and \( 2\omega \).

The maximally skewed case is to be expected if the neutrino kick is also driving the neutron star’s rotation as in [4]. This could also suppress the translational kick and make the gravity waves larger than expected relative to the proper motion.

5.2 Off-resonant neutrino conversions

If the active-to-sterile neutrino conversions occur off-resonance, the sterile neutrinos are emitted from the entire core of the star [4]. Such emission is only possible after the electroweak matter potential is driven to zero by neutrino oscillations [12]. Therefore, in this case, the sterile emission can begin several seconds after the onset of the supernova. More importantly, this emission may last for a period of time that could be as long as 10 s, but can also be much shorter. This depends on the neutrino mass and mixing parameters.

The emission of neutrinos in the solid angle \( d\Omega \) is \( \frac{dN}{d\Omega} = N_0(1 + \epsilon \cos \tilde{\theta}) \). The dependence on the magnetic field is more complicated than in previous models. However, we can get a reasonable order of magnitude estimate by parametrizing the emission in the same way as (48). Specifically the gravity wave generating term will take the form

\[
-2N_0\epsilon_0x \sin \tilde{\theta} \cos \tilde{\theta} \cos \phi.
\] (57)

This must be integrated over all radii where neutrinos are emitted. Assuming they are uniformly emitted throughout the core (i.e. no explicit dependence on radius) the final form will be

\[
-\frac{3}{2}N_0\epsilon_0x_{max} \sin \tilde{\theta} \cos \tilde{\theta} \cos \phi
\] (58)

Except for the extra factor of \( 3/4 \) the result is the same as for the resonant case and will yield a similar magnitude signal.

6 Signal Magnitude

We are now ready to consider the magnitude of the radiation. In a typical supernova, neutrinos carry away about \( 10^{53} \) ergs over a period of several seconds. So \( \sigma \) is approximately the average luminosity \( \sigma = L = 10^{52}(10s/\tau) \) erg/s where \( \tau \) is the total time of neutrino emission. The magnitude of the gravity waves is then

\[
\theta_{\delta T T} (t) \approx \left( \frac{2e \times 10^{52} \text{erg/s}}{\tau} \right) \left( \frac{10s}{\tau} \right) F(\theta, t).
\] (59)

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\( F(\theta_i, t) \) is the function of angles appearing in equation (56). \( \theta_i \) merely represents the collection of angles.

To be compared with experiment, this result needs to be integrated with respect to time and Fourier transformed. In the Fourier transform all three major components \((\omega, 2\omega, \text{and } 3\omega)\) have amplitudes of order 1, and integrating is equivalent to dividing by the angular frequency.

The units must be changed also. The Gaussian noise must be Fourier transformed in a way that gives it units of \( \text{Hz}^{-1/2} \). To adjust for this we must divide by the square root of the frequency for a burst source, or multiply by the square root of \( \tau \) for a periodic source [13]. Since neutron stars have typical frequencies of several kHz, and the burst is expected to last a few seconds, we should generally treat this as a periodic source. However, in the case of lower frequency signals detectable by LISA, the burst description may be more appropriate.

The final result is

\[
\theta_{TT}(f) \approx 10^{-24} \text{Hz}^{-1/2} \left( \frac{e}{.01} \right) \left( \frac{10 \text{s}}{\tau} \right)^{1/2} \left( \frac{1 \text{kpc}}{r} \right) \left( \frac{1 \text{kHz}}{f} \right)
\]  

for periodic signals, or

\[
\theta_{TT}(f) \approx 3 \times 10^{-22} \text{Hz}^{-1/2} \left( \frac{e}{.01} \right) \left( \frac{10 \text{s}}{\tau} \right) \left( \frac{1 \text{kpc}}{r} \right) \left( \frac{1 \text{Hz}}{f} \right)^{3/2}
\]  

for a burst signal. In both cases \( f \) represents the frequency of the radiation, not the star rotation. (Thus, \( f \) will be 1, 2, or 3 times the frequency of star rotation.)

LIGO is most sensitive in the 10 - 1000 Hz range, neutron stars rotating at this rate are likely to undergo several rotations in the neutrino ejection time and therefore equation (60) is most appropriate in describing LIGO’s sensitivity to gravity waves figure [4] LISA by contrast is most sensitive in the .001 - .1 Hz range. Neutron stars rotating at this rate are likely to undergo only part of a rotation, or at most a few rotations in the neutrino ejection time. Thus, equation (61) is more appropriate in describing its sensitivity figure [2].

From the figures it appears that LISA is more likely to see this GW signal, being sensitive to supernovas as far as several Mpc. Unfortunately, LISA’s frequency range is very low and it would only see GWs from very slowly rotating neutrino beams.

## 7 Gravitational Energy

The total power and energy carried by the gravity wave are simply

\[
P = |\theta_{TT}^\mu|^2, \tag{62}
\]

\[
\Delta E = |\theta_{TT}^\mu|^2 \tau, \tag{63}
\]
Figure 1: Gravity wave intensity from a pulsar kick caused by neutrino oscillations. The LIGO sensitivity is shown for reference. For near supernovas with the appropriate rotation frequency, the signal should be observable. Please note this shows the magnitude at a given frequency. The actual signal is not continuous, but would be concentrated at three closely spaced frequencies, $f$, $2f$, and $3f$. 
Figure 2: Gravity wave intensity from a pulsar kick caused by neutrino oscillations. The LISA sensitivity is shown for reference. Note that for near supernovas with the appropriate rotation frequency, the signal can be observable. Please note this shows the magnitude at a given frequency. The actual signal is not continuous, but would be concentrated at three closely spaced frequencies, $f$, $2f$, and $3f$. 
where $\theta^{ij}_{\alpha\beta\gamma}$ can be given by any of the several equations for this quantity as appropriate. Inserting equation (59) this means that the energy carried away by gravity waves will be about $0.1 \text{ erg/cm}^2$ or a total of about $10^{43} \text{ erg}$.

By comparison, Müller and Janka [14] find a GW energy of $E^\nu_{GW} \sim [10^{-10} - 10^{-13}] M_\odot c^2 \sim [10^{44} - 10^{41}] \text{ erg}$, for a luminosity of $L^\nu \sim 10^{53} \text{ erg/s}$. Cuesta [9] gets a value of $E^\nu_{GW} \sim 10^{-4} M_\odot c^2 \sim [10^{50}] \text{ erg}$ with a luminosity of $L^\nu \sim 10^{56} \text{ erg/s}$. (The neutrino luminosity is important because gravitational luminosity scales with $L^2_\nu$ and the total energy scales like $L^2_\nu \tau \sim L^\nu$.) Thus, Müller and Janka’s estimate is of roughly the same size as mine, Cuesta’s is about 3 orders of magnitude higher.

Furthermore, the GW signal in this paper can be distinguished from other GWs caused by neutrino emission from a neutron star, by its strong periodic nature. This signal will be highly concentrated at three main frequencies $f$, $2f$, and $3f$ (see section 6). By contrast, other signals have their power spread out over a large, fairly continuous range of frequencies. In certain ideal situations individual cycles of the radiation may be recognizable. This should allow the periodic signal to be distinguished from other processes including those that contain more total energy.

8 Conclusions

We have shown that, in the event of a nearby supernova, gravitational waves caused by an asymmetric neutrino emission may be strong enough to be detected by LIGO and LISA. We have assumed that the asymmetry in outgoing neutrinos is as large as 1%, as suggested by the observed pulsar velocities. As can be seen from the figures, quickly rotating neutron stars may be seen by LIGO advanced, and more slowly rotating ones by LISA. This signal also has the distinguishing feature of having large radiation components at $f$, $2f$, and $3f$. This could be used for recognizing gravity waves from rotating neutrino beams. This mechanism could also be recognized by its short time duration and possible coincidence with neutrino and optical signals from the supernova.

If such a signal is detected and identified, it would have profound implications for one’s understanding of the supernova, as well as neutrino physics. In particular, LIGO and LISA may be able to test the existence of a keV sterile neutrino, which has been suggested as the explanation of both the pulsar kicks and the cosmological dark matter.
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References

[1] See, *e.g.*, A. G. Lyne and D. R. Lorimer, Nature 369 (1994) 127; J. M. Cordes and D. F. Chernoff, Astrophys. J. **505**, 315 (1998); B. M. S. Hansen and E. S. Phinney, Mon. Not. R. Astron. Soc. **291**, 569 (1997); C. Fryer, A. Burrows, and W. Benz, Astrophys. J. **496**, 333 (1998).

[2] A. Kusenko and G. Segrè, Phys. Rev. Lett. **77** (1996) 4872; A. Kusenko and G. Segrè, Phys. Rev. **D58** (1999) 1302; M. Barkovich, J. C. D’Olivo, R. Montemayor and J. F. Zanella, Phys. Rev. D **66**, 123005 (2002).

[3] A. Kusenko and G. Segrè, Phys. Lett. B **396**, 197 (1997)

[4] G. M. Fuller, A. Kusenko, I. Mocioiu and S. Pascoli, arXiv:astro-ph/0307267.

[5] J. F. Nieves and P. B. Pal, Phys. Rev. **D40** 1693 (1989); J. C. D’Olivo, J. F. Nieves and P. B. Pal, *ibid.*, 3679 (1989); J. C. D’Olivo, J. F. Nieves and P. B. Pal, Phys. Rev. Lett., **64**, 1088 (1990); V. B. Semikoz and J. W. F. Valle, Nucl. Phys. B **425**, 651 (1994); *ibid.*, 485, 545 (1997); S. Esposito and G. Capone, Z. Phys. C **70**, 55 (1996); A. Goyal, Phys. Rev. D **59**, 101301 (1999); H. Nunokawa, V. B. Semikoz, A. Yu. Smirnov, and J. W. F. Valle Nucl. Phys. B **501**, 17 (1997); S. Pastor, V. B. Semikoz, J. W. F. Valle, Astropart. Phys. **3**, 87 (1995)

[6] Epstein, R. 1978, ApJ, 223, 1037.

[7] A. Burrows and J. Hayes, Phys. Rev. Lett. **76**, 352 (1996).

[8] A. Burrows, J. Hayes, B. A. Fryxell, ApJ 450, 830 (1995).

[9] Cuesta, H.J.M. 2000, ApJ, 544, L61.

[10] Cuesta, H.J.M. 2002, Phys Rev D, vol 65, 061503.

[10] Ya. B. Zeldovich, A.A. Ruzmaikin, and D.D. Sokoloff, *Magnetic fields in astrophysics*, Gordon and Breach, New York, 1983.
[11] H. Spruit and E. S. Phinney, Nature 393, 139 (1998).

[12] K. Abazajian, G. M. Fuller and M. Patel, Phys. Rev. D 64, 023501 (2001).

[13] K.S. Thorne *Gravitational Radiation*, Chapter 9, Hawking and Israel, 300 Years of Gravitation, (Cambridge University Press, 1987).

[14] E. Müller and H.-Th. Janka, Astron. Astrophys. 317, 140 (1997).