Casimir energy in finite expanding universes: a prelude

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We present the Green’s functions that are the solutions of the massive Klein-Gordon equation for a scalar field with non-minimal coupling to gravity for several static and expanding cosmological models. An important feature of such functions is that they can be used to study the appearance of a Casimir energy in models of the universe with compact spatial sections.

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I. INTRODUCTION

In the description of the universe based on General Relativity, the idea of curved manifolds is essential. Einstein’s equations are mathematical relations between the curvature of spacetime and the matter and energy, which act as a source of the curvature and can be seen as a first step in the study of the structure of the physically possible models of the universe. Given the curvature of a spacetime, it is important to characterize its global properties, such as its topology. Since General Relativity provides no clues about these properties, it is valid to include in the list of possible models of the universe, those in which the space does not have the simplest topology.

A space with non-trivial topology may be compact (i.e., finite), regardless of its curvature. This means that the universe may present certain peculiarities due to its finite size, such as periodic spatial patterns. The observation of any of these peculiarities may indicate the global topology of the universe, so that it is very important to know what kind of mark a determined topology may imprint on the visible universe. Cosmic topology, which deals with this issue, has been the subject of a growing interest, reflected in the publication of recent reviews [1, 2] and introductory texts [3, 4, 5].

An interesting physical effect due to the finiteness of the space is the appearance of a Casimir energy. The simplest case where this energy, measurable experimentally, appears is between two parallel reflecting planes in the Minkowski vacuum. However, this case can be generalized. The Casimir energy can be seen to be created by the imposition of periodic boundary conditions on a generic field propagating in the vacuum and can be studied theoretically with a scalar field, by means of Green’s functions $G(x, x')$ for the Klein-Gordon equation [6]. The procedure begins with the classical energy-momentum tensor associated with the field, obtained from the Green’s function through a second order differential operator $D_{\mu\nu}^{x, x'}$,

$$T_{\mu\nu}(x, x') = D_{\mu\nu}^{x, x'}G(x, x').$$

From this, we obtain the mean value of the field in the vacuum, such that

$$\langle 0 | T_{\mu\nu} | 0 \rangle = \lim_{x \to x'} T_{\mu\nu}(x, x').$$

In general, when speaking of the Casimir energy, it is to the finite part of the 00 component of this mean value that one refers (see, e.g., [7, 8]).

As seen in some recent works [7, 8], the same kind of study can be done in models of the universe with non-trivial topology. However, these studies have been limited, until now, to static models of the universe. Nevertheless, there is not, a priori, any reason not to include models with expansion. What is important to notice is that a fundamental step in this process is to write, for each kind of expansion of the universe, the specific Green’s functions for the Klein-Gordon equation.

The purpose of this work is to present the Green’s functions found as solutions of the generalized Klein-Gordon equation for a scalar field of mass $m$ with non-minimal coupling $\xi$ to the gravitational field for static and expanding cosmological models. In the expanding cases, the functions presented are valid for a group of cosmological models, parametrized by the curvature scalar $k$ and by an index $n$. Such a group of models includes, for example, the de Sitter solution.

The Green’s functions for universes with non-trivial topology are obtained from the general solutions by noting that the Green’s function for a generic finite space, $\mathcal{M} = M/\Gamma$, is

$$G_{\mathcal{M}}(x, x') = \sum_{\Gamma} G_{\tilde{\mathcal{M}}}[x, \Gamma(x')]$$

where $\Gamma$ are the elements of the group which defines $\mathcal{M}$, each one represented as an operator that identifies a point $P$ on the finite space $\tilde{\mathcal{M}}$ and its image $P'$ onto the infinite covering space $\tilde{\mathcal{M}}$, i.e., $P' = \Gamma(P) \equiv P$, with $P' \in \tilde{\mathcal{M}}$. 

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and $P \in \mathcal{M}$. For 3-dimensional spaces of constant curvature the covering space $\mathcal{M}$ can be [1]

- the spherical space $S^3$, of positive curvature;
- the flat space $E^3$, of null curvature;
- or the hyperbolical space $H^3$, of negative curvature.

The paper is organized as follows. The Klein-Gordon equation for a scalar field is first written in a generalized form valid for the curved spacetimes described in the Friedmann-Lemaître-Robertson-Walker line element of standard cosmology. Some general Green’s functions are then obtained for specific cases of static and expanding spacetimes. Finally, some comments on the results obtained are given. General solutions of the generalized Klein-Gordon equation are presented in the appendix. Throughout this work, natural units are used ($c = G = \hbar = 1$).

II. INITIAL CONSIDERATIONS

For a classical scalar field with a Lagrangian density

$$\mathcal{L} = \sqrt{-g} \left[ \frac{1}{2} \partial_{\mu} \varphi \partial^{\mu} \varphi - \left( \frac{1}{2} \xi R \varphi^2 + \frac{1}{2} m^2 \varphi^2 \right) \right],$$

(4)

one obtains the field equation

$$\Box \varphi + (\xi R + m^2) \varphi = 0,$$

(5)

where $\Box \equiv [-g]^{-1/2} \partial_{\mu} \left( \sqrt{-g} g^{\mu \nu} \partial_{\nu} \right)$ is the d’Alembertian operator, $m^2$ and $\xi$ are constants, and $R = g_{\mu \nu} R^{\mu \nu}$ is the Ricci scalar derived from the spacetime metric $g_{\mu \nu}$, with $g = \det [g_{\mu \nu}]$.

In this work the interest is focused on metrics representing cosmological solutions, in particular, those with a Friedmann-Lemaître-Robertson-Walker (FLRW) line element,

$$ds^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right],$$

(6)

which can be written as

$$ds^2 = a^2(\eta) \left[ d\eta^2 - d\chi^2 - \frac{\sin^2 \sqrt{k} \chi}{k} (d\theta^2 + \sin^2 \theta d\phi^2) \right],$$

(7)

where $k = 0, \pm 1$, with $t$ the cosmological time, $\eta$ the conformal time, and $dt = ad\eta$. Assuming Einstein’s equations with a cosmological constant $\Lambda$, and matter a perfect fluid, this line element gives the Friedmann equation

$$\frac{D^2}{a^2} \equiv \frac{1}{a^2} \left( \frac{1}{a} \frac{da}{d\eta} \right)^2 = \frac{8\pi}{3} \rho + \Lambda - \frac{k}{a^2},$$

(8)

for the evolution of the scale factor $a$, with $\rho$ the energy density of the matter, and

$$2 \frac{1}{a^3} \frac{d^2a}{d\eta^2} + \frac{k}{a^2} = \Lambda - 8\pi p,$$

(9)

where $p$ is the pressure of the matter. For the FLRW metric, the Ricci scalar is

$$R = \frac{6}{a^2} \left[ \frac{1}{a} \frac{d^2a}{d\eta^2} + k \right],$$

(10)

and

$$\Box = \frac{1}{a^2} \left[ \frac{\partial^2}{\partial \eta^2} + 2D \frac{\partial}{\partial \eta} - \nabla^2 \right].$$

(11)

Therefore, the field equation becomes

$$\frac{1}{a^2} \left[ \frac{\partial^2 G}{\partial \eta^2} + 2D \frac{\partial G}{\partial \eta} - \nabla^2 G \right] + (\xi R + m^2) G = \delta(x - x'),$$

(12)

where $\delta(x - x')$ is the Dirac delta function.

III. GREEN’S FUNCTIONS

A Green’s function can always be constructed, using the eigenfunctions and eigenvalues of the general solution of the homogeneous equation. However, this is not always an easy task, especially in non-flat spacetimes, where, for example, plane waves are replaced by generalized solutions to Helmholtz equation [9] (see appendix). Therefore, since the Green functions studied here must satisfy the more general equation,

$$\frac{1}{a^2} \left[ \frac{\partial^2 G}{\partial \eta^2} + 2D \frac{\partial G}{\partial \eta} - \nabla^2 G \right] + (\xi R + m^2) G = \frac{\delta(x - x')}{\sqrt{-g}}$$

(13)

we search for solutions to Eq. (13) in its homogeneous form, keeping in mind that such solutions must be symmetric in $x$ and $x'$. In the appendix, some general solutions of the homogeneous field Eq. (12) are presented.

Assuming isotropy of the universe, the use of the coordinates $(\chi, \theta, \phi)$ allows for a choice of orientation where the two points $x$ and $x'$ are radially aligned, i.e., $\theta = \theta'$, $\phi = \phi'$, and $\chi' = 0$, so that for the FLRW line element,

$$\nabla^2 = \frac{\partial^2}{\partial \chi^2} + 2\sqrt{k} \cot \sqrt{k} \chi \frac{\partial}{\partial \chi},$$

(14)

where $\chi$ is the spatial distance between the points $x$ and $x'$. Thus, the homogeneous equation to be solved is

$$\frac{\partial^2 G}{\partial \eta^2} + 2D \frac{\partial G}{\partial \eta} - \frac{\partial^2 G}{\partial \chi^2} - 2\sqrt{k} \cot \sqrt{k} \chi \frac{\partial}{\partial \chi} (\xi R + m^2) a^2 G = 0.$$

(15)

Several different Green’s functions obey this equation since there are several boundary conditions that can be defined. For example, according to different boundary conditions, we have the causal Green function (or Feynman propagator) $G_F(x, x')$, the Wightman’s functions...
$G^+ (x, x')$ and $G^- (x, x')$ for positive and negative frequency, respectively, or the elementary Hadamard function $G^{(1)} (x, x')$. In quantum field theory, these different functions are associated with mean values of some field functions [6].

A. Static universes

For flat spaces, the trivial solution of Friedmann’s equation is Minkowski space, which is a vacuum static solution for $\Lambda = 0$. In this simple case, $a = 1, R = 0$, and

$$\frac{\partial^2 G}{\partial \eta^2} + \frac{\partial^2 G}{\partial \xi^2} = 2 \frac{\partial G}{\partial \xi} + m^2 G = 0. \quad (16)$$

Using the invariant

$$s = \sqrt{\Delta \eta^2 - \xi^2}, \quad (17)$$

where $\Delta \eta \equiv \eta - \eta'$, Eq. (16) becomes

$$\frac{d^2 G}{ds^2} + \frac{3dG}{s ds} + m^2 G = 0, \quad (18)$$

with the general solution

$$G (x, x') = \frac{1}{s} \left[ c_1 H^{(1)}_1 (ms) + c_2 H^{(2)}_1 (ms) \right], \quad (19)$$

where $H^{(1)}_n$ and $H^{(2)}_n$ are Hankel’s functions of order $n$ of the first and second kinds, respectively [10]. Another invariant, the world function $\sigma$ [11, 12, 13], which in the Minkowski space is given as $\sigma = s^2/2$, could have been used to give a similar general solution. Values for the coefficients $c_1$ and $c_2$ are set in accordance with the particular boundary conditions used in different Green’s functions. For example, in the case where $k = m = 0$, Eq. (13) is a wave equation with the massless positive frequency Wightman function,

$$D^+ (x, x') = -\frac{1}{4 \pi^2 s^2}. \quad (20)$$

Another type of Green’s function, the massless Hadamard elementary function,

$$D^{(1)} (x, x') = -\frac{1}{2 \pi^2 s^2}, \quad (21)$$

is also a solution to Eq. (13) for this case. Since near the origin [10],

$$H^{(1)}_1 (x) \simeq -\frac{2i}{\pi x}, \quad H^{(2)}_1 (x) \simeq \frac{2i}{\pi x}, \quad (22)$$

we have

$$G^+ (x, x') = \frac{im}{8\pi s} H^{(2)}_1 (ms). \quad (23)$$

This solution can be used as a limit that any other positive frequency solution for curved spaces must reach.

For non-flat spaces ($k = \pm 1$), Friedmann’s equation with a non-zero cosmological constant has the vacuum ($\rho = 0$) static solution,

$$a = (3k)^{1/2} \Lambda^{-1/2}, \quad (24)$$

with $k\Lambda^{-1} > 0$. In this case, we have, from Eqs. (10) and (12),

$$\frac{\partial^2 G}{\partial \eta^2} - \frac{\partial^2 G}{\partial \xi^2} - 2\sqrt{k} \cot \sqrt{k} \frac{\partial G}{\partial \xi} + \left( 6\xi + \frac{3m^2}{\Lambda} \right) kG = 0, \quad (25)$$

where $k = \pm 1$. If

$$G (x, x') = \frac{\sqrt{k} \chi}{\sin \sqrt{k} \chi} A (x, x'), \quad (26)$$

Eq. (25) becomes similar to (18), and thus

$$G (x, x') = \frac{\sqrt{k} \chi}{\sin \sqrt{k} \chi} \frac{1}{s} \left[ c_1 H^{(1)}_1 (ms) + c_2 H^{(2)}_1 (ms) \right], \quad (27)$$

and

$$G^+ (x, x') = \frac{i\mu}{8\pi} \frac{\sqrt{k}}{\sin \sqrt{k} \chi} H^{(2)}_1 (ms), \quad (28)$$

where $s$ is given by Eq. (17), and $\mu^2 \equiv 3m^2 \Lambda^{-1} + (6\xi - 1)k$. Therefore, when $6\xi = 1, m = 0$, and $\chi \simeq 0$, we again obtain Eq. (23), as expected.

The spherical case, $k = 1$, deserves special consideration since the universe represented is finite spatially, so that there are several equivalent spatial paths between the points $x$ and $x'$, all of which must be represented by the Green’s function. Therefore, using the internal symmetry of the space $S^3$ in Eq. (28), we have

$$G^+ (x, x') = \frac{i\mu}{8\pi} \sum_{n=-\infty}^{\infty} \frac{\chi_n}{\sin \chi} \frac{H^{(2)}_1 (ms) \sqrt{\Delta \eta^2 - \chi_n^2}}{\sqrt{\Delta \eta^2 - \chi_n^2}}, \quad (29)$$

where $\sin \chi_n \equiv \sin (\chi + 2\pi n) = \sin \chi$ was used. For any other spatially finite universe one uses an equivalent procedure [7, 8].

B. Expanding universes

When $\Lambda = 0$, a group of solutions of the Friedmann equation (8) can be found by means of the ansatz

$$\rho = C a^{-n}, \quad (30)$$

where $C$ and $n$ are constants. This general group of solutions is then

$$a (\eta) = a^{-2\pi C/3} k \csc \frac{2\pi}{3} \left[ \sqrt{k} \left( \frac{n}{2} - 1 \right) \eta \right], \quad (31)$$

where $a^2 \equiv 8\pi C/3$ and $n \neq 2$. Thus we can write

$$D = \sqrt{k} \cot \left[ \sqrt{k} \left( \frac{n}{2} - 1 \right) \eta \right], \quad (32)$$
and
\[ Ra^2 = (4 - n) 3k \csc^2 \left[ \sqrt{k} \left( \frac{n}{2} - 1 \right) \eta \right]. \tag{33} \]

Using the ansatz given by Eq. (30) we can solve Friedmann’s equation for \( a \) and then substitute the expression found into Eq. (9), to obtain the linear equation of state
\[ 3p = (\alpha^2 - k a^2). \tag{34} \]

with
\[ D = (\alpha^2 - k)^{1/2}; \quad Ra^2 = 6\alpha^2. \tag{35} \]

Table I shows, for various integral values of the index \( n \), the expression for the scale factor \( a \) as a function of the cosmological time \( t \).

Although Eqs. (31) and (34) give a group of solutions which are valid for any value of the parameter \( n \), not all values of \( n \) give easily solvable field equations. Here we consider only the cases where \( n = 0, 2 \) and 4. For \( n = 2 \) and \( n = 4 \), only the massless \((m = 0)\) equations are considered.

1. Solution with \( n = 0 \)

Since in this case \( \rho \) is a constant, one has then a situation completely equivalent to the case of a vacuum with the presence of a cosmological constant \( \Lambda = 3\alpha^2 \). Therefore, the solution with \( n = 0 \) is equivalent to the de Sitter solution, the curved spacetime most studied by quantum field theorists [6], and which is to cosmology what a hydrogen atom is to atomic physics [14].

The de Sitter solution is, in fact, a group of solutions, each one with a different \( k \), but all related to portions of the 4-dimensional surface \( \eta_{\mu\nu} x^\mu x^\nu = -\alpha^{-2} \), which is an hyperboloid embedded in a 5-dimensional Minkowski space of metric \( \eta_{\mu\nu} = \text{diag} \{ 1, -1, -1, -1, -1 \} \). Different sections of this surface represent the various de Sitter spacetimes, each one with a distinct curvature. There also exists a static representation of de Sitter spacetime, which will not be treated here [6, 15, 16, 17, 18]. It is, however, worthwhile to note that despite much investigation of the de Sitter spacetime, it is rare to find descriptions of the hyperbolic de Sitter solution, which is not even cited in some standard texts [6, 18]), perhaps due to an ‘aesthetic’ prejudice: “the open model has the unattractive feature that in horospherical coordinates it

\[ ds^2 = \alpha^{-2} \eta^{-2} \left[ dy^2 - d\chi^2 - \chi^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \tag{36} \]

with the corresponding parametrization
\[
\begin{align*}
x^0 &= 2^{-1} \eta - (\alpha^{-2} + \chi^2) (2\eta)^{-1} \\
x^1 &= -2^{-1} \eta - (\alpha^{-2} - \chi^2) (2\eta)^{-1} \\
x^2 &= - (\alpha\eta)^{-1} \chi \sin \theta \cos \phi \\
x^3 &= - (\alpha\eta)^{-1} \chi \sin \theta \sin \phi \\
x^4 &= - (\alpha\eta)^{-1} \chi \cos \theta
\end{align*}
\tag{37}
\]

where \( 0 \leq \phi \leq 2\pi, 0 \leq \theta \leq \pi, 0 \leq r < \infty, -\infty < \eta < 0 \).

In this case, it is easy to see that the parametrization of the hyperboloid is incomplete since \( x^0 + x^1 \geq 0 \).

In the spherical case, the line element is
\[ ds^2 = \alpha^{-2} \csc^2 \eta \left[ dy^2 - d\chi^2 - \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2) \right], \tag{38} \]

with the relations
\[
\begin{align*}
x^0 &= \alpha^{-1} \cot \eta \\
x^1 &= \alpha^{-1} \csc \eta \cos \chi \\
x^2 &= \alpha^{-1} \csc \eta \sin \chi \sin \theta \cos \phi \\
x^3 &= \alpha^{-1} \csc \eta \sin \chi \sin \theta \sin \phi \\
x^4 &= \alpha^{-1} \csc \eta \sin \chi \cos \theta
\end{align*}
\tag{39}\]

where \( 0 \leq \phi \leq 2\pi, 0 \leq \theta \leq \pi, 0 \leq \chi \leq \pi, 0 \leq \eta < \pi \).

Finally, for \( k = -1 \), we have
\[ ds^2 = \frac{\csc^2 \eta}{\alpha^2} \left[ dy^2 - d\chi^2 - \sinh^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2) \right], \tag{40} \]

with
\[
\begin{align*}
x^0 &= \alpha^{-1} \cosh \eta \cosh \chi \\
x^1 &= \alpha^{-1} \coth \eta \\
x^2 &= \alpha^{-1} \cosh \eta \sinh \chi \sin \theta \cos \phi \\
x^3 &= \alpha^{-1} \cosh \eta \sinh \chi \sin \theta \sin \phi \\
x^4 &= \alpha^{-1} \cosh \eta \sinh \chi \cos \theta
\end{align*}
\tag{41}\]

where \( 0 \leq \phi \leq 2\pi, 0 \leq \theta \leq \pi, 0 \leq \chi < \infty, -\infty < \eta < \infty \).

The invariant quantity to be considered in this case is not simply the interval \( s \), as defined in Eq. (17), but
rather the geodesic interval $\sigma (x, x')$ in the hyperboloid [20, 21], a geometric quantity which is another example of a world function [11, 12, 13], given in this case as [22, 23, 24, 25]

$$\sigma (x, x') \equiv \alpha^{-1} \text{arccosh} |p(x, x')|, \quad (42)$$

where $p(x, x') \equiv -\alpha^2 \eta_{\mu \nu} x^\mu (x')^\nu$ is an auxiliary quantity. Using Eqs. (37), (39), and (41) and the radial alignment between the points $x$ and $x'$, we can write

$$p(x, x') = \begin{cases} 1 + (\Delta \eta^2 - \chi^2) (2\eta')^{-1} \\ 1 - \csc \eta \csc \eta' (\cos \Delta \eta - \cos \chi) \\ 1 + \csc \eta \csc h \eta' (\cosh \Delta \eta - \cosh \chi) \end{cases}, \quad (43)$$

for $k = 0, 1,$ and $-1,$ respectively, so that, for any value of $k,$ we obtain

$$(p^2 - 1) \frac{d^2 G}{dp^2} + 4p \frac{dG}{dp} + (12\xi + m^2\alpha^{-2}) G = 0, \quad (44)$$

with the solution

$$G(x, x') = c_1 F \left( \frac{3}{2} - \nu, \frac{3}{2} + \nu; 2; \frac{1 - p}{2} \right) + c_2 F \left( \frac{3}{2} - \nu, \frac{3}{2} + \nu; 2; \frac{1 + p}{2} \right), \quad (45)$$

where $\nu^2 \equiv 2^{-2} 3^2 - (12\xi + m^2\alpha^{-2}).$ An analysis of the conditions that must be imposed on this solution to obtain the values of $c_1$ and $c_2$ can be found in the literature [25] and will not be reproduced here.

We note that Eq. (44) has also solutions in terms of the associated Legendre functions [22, 23],

$$G(x, x') = (p^2 - 1)^{-1/2} \left[c_1 P_{-\frac{1}{2} + \nu} (p) + c_2 Q_{-\frac{1}{2} + \nu} (p)\right], \quad (46)$$

or

$$G(x, x') = \frac{d}{dp} \left[c_1 P_{\frac{1}{2} + \nu} (p) + c_2 Q_{\frac{1}{2} + \nu} (p)\right], \quad (47)$$

valid since $p$ assumes only real values.

In the specific case of a massless field ($m = 0$) with conformal coupling ($\xi = 1/6$), we get

$$D_{\xi = \frac{1}{6}} (x, x') = c_1 p + c_2 \left(2p^2 - 1\right), \quad (48)$$

which gives

$$D_{\xi = \frac{1}{6}}^+ (x, x') = -\frac{\alpha^2}{8\pi^2 (p - 1)} = -\left[\frac{16\pi^2 \sinh^2 \frac{\alpha \sigma}{2}}{\alpha^2} \right]^{-1}, \quad (49)$$

a result formally similar to the one obtained for an uniformly accelerated observer in Minkowski space (see also Eq. (20)) [6].

2. Solution with $n = 2$

For $n = 2$ we have the line element

$$ds^2 = e^{2\eta \sqrt{\alpha^2 - k}} \left[\frac{d\eta^2}{1 - kr^2} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)\right], \quad (50)$$

which, for $k = -1$ and $\alpha = 0,$ can be transformed in Minkowski space by the change of coordinates

$$\tau = e^\nu r, \quad t = e^\nu \sqrt{1 + \tau^2}. \quad (51)$$

This peculiar parametrization of the flat Minkowski space is known as the Milne universe [26, 27].

In the general case where $n = 2,$ the homogeneous Eq. (15) becomes simpler in the flat massless case, i.e., with $k = m = 0,$ and we have

$$\frac{\partial^2 G}{\partial \eta^2} + 2\alpha \frac{\partial G}{\partial \eta} - \frac{\partial^2 G}{\partial \chi^2} - \frac{2 \partial G}{\chi} + 6\xi \alpha^2 G = 0. \quad (52)$$

This damped wave equation [28], has the solution

$$G(x, x') = e^{-\alpha \Delta \eta} G_0 (x, x'), \quad (53)$$

where $G_0 (x, x')$ obeys the equation

$$\frac{\partial^2 G_0}{\partial \eta^2} - \frac{\partial^2 G_0}{\partial \chi^2} - \frac{2 \partial G_0}{\chi} + (6\xi - 2) \alpha^2 G_0 = 0, \quad (54)$$

which is Eq. (16), for Minkowski space. Therefore,

$$G(x, x') = e^{-\alpha \Delta \eta} \frac{1}{s} \left[c_1 H_1^{(1)} (\beta s) + c_2 H_1^{(2)} (\beta s)\right], \quad (55)$$

where $\beta^2 \equiv (6\xi - 2) \alpha^2$ and $s$ is given by Eq. (17). The non-flat massless cases can be obtained from this solution in a similar way to that used for static non-flat cases.

3. Solution with $n = 4$

Although for $n = 4,$ we have $Ra^2 = 0$ from Eq. (33), it is not easy to find a Green’s function in this case. However, it is interesting to see that in the homogeneous massless ($m = 0$) equation there exists a symmetry between the coordinates $\eta$ and $\chi,$ i.e.,

$$\frac{\partial^2 G}{\partial \eta^2} + 2\sqrt{k} \cot \sqrt{k} \eta \frac{\partial G}{\partial \eta} = \frac{\partial^2 G}{\partial \chi^2} + 2\sqrt{k} \cot \sqrt{k} \chi \frac{\partial G}{\partial \chi}. \quad (56)$$

IV. FINAL REMARKS

We presented the Green’s functions for a number of interesting cosmological solutions. Not all plausible possibilities for scalar fields have been covered and there remains a number of physically possible cosmological solutions, for example, the expanding solutions with $n = 2$ or $n = 4,$ for which there is not yet a complete, easy
available, analytical treatment. Even where many studies have been made, there are lapses, such as the almost complete absence of references to the hyperbolic de Sitter solution. It is the intention of this article to stimulate more work in this field.

In the context of cosmic topology, the task of writing Green’s functions is a preparatory step, and can be done analytically. The bulk of the work, the use of these functions in cosmic topology, is mainly numerical and, for this reason, is not presented here.

The importance of studies linked to the field of cosmic topology must not be underestimated. Despite some recent claims about the beginning of a new era of ‘precision cosmology’, we are still unable to determine whether the universe has a really null curvature or just a very small one, which would be a remnant from a pre-inflation era. In this context, finding topological properties of the universe is very important. In the theoretical inflation era. In this context, finding topological properties of the universe is very important. In the theoretical search for these properties, the study of a scalar cosmological Casimir effect in finite spaces, while not providing the final answer about the shape of the universe, would certainly supply important information.

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APPENDIX: SOLUTIONS FOR THE FIELD EQUATION

A Green’s function can always be written using the eigenfunctions and eigenvalues of the homogeneous equation it obeys [28, 29]. However, to obtain Green’s functions by this process for the finite spaces of non-trivial topology is not an easy task, since the eigenvalues in general are not easy to find. Therefore, it is more convenient to write a Green’s function for infinite space and force it to satisfy the symmetries of the finite space numerically. Nevertheless, even the functions for the infinite space are not easy to obtain in this fashion because the eigenfunctions are, in general, very complicated. In this appendix, eigenfunctions for some specific cosmological solutions are presented.

The field equation (12) has general solutions of the kind

$$\varphi (\eta, x) = u (\eta) v (x),$$  \hspace{1cm} (A.1)

where $u$ and $v$ are functions satisfying the equations

$$\frac{d^2 u}{d\eta^2} + 2D \frac{du}{d\eta} + [\ell^2 + (\xi R + m^2) a^2] u = 0,$$  \hspace{1cm} (A.2)

and

$$\nabla^2 v = -\ell^2 v,$$  \hspace{1cm} (A.3)

the Helmholtz equation.

1. Spatial modes

To obtain information about the size and shape of the universe, we must study the spatial modes of the general solutions of the field equation. A general form for the solutions of the Helmholtz equation, without taking into account the finiteness of the universe, is [6]

$$v (x) = \begin{cases} \exp \eta \ell x^j, & \ell^2 = \eta \ell \ell \ell \ell (k = 0) \\ \Pi^{(\pm)}_ {\ell, J} (\chi) Y^M_{\ell} (\theta, \phi), & \ell = (\ell, J, M) (k = \pm 1) \end{cases},$$  \hspace{1cm} (A.4)

where $Y^M_{\ell}$ are the usual spherical harmonics (with limitations on the possible values of $\ell, J, M$), and

$$\Pi^{(-)}_{\ell, J} (\chi) = c_{\ell, J} \sinh^j \chi \left( \frac{d}{d \cosh \chi} \right)^{1+j} \cos \ell \chi,$$  \hspace{1cm} (A.5)

with $\Pi^{(+)}_{\ell, J} (\chi)$ obtained by replacing $\ell$ with $-i \ell$ and $\chi$ with $-i \chi$ [6].

We note that the solutions given above are not the only possible options: for 3-D flat spaces, there are 11 coordinate systems in which the Helmholtz equation is separable [29, 30], while for hyperbolic spaces, there are at least 12 such systems [31]. Such diversity is particularly important in universes with compact spatial sections. In these universes the coordinate system $(\chi, \theta, \phi)$ is not always a good choice since the characteristic symmetries of the spatial sections of such universes generally do not fit in these coordinates, especially in the hyperbolic case. An example where this occurs is in the calculation of the volumes and other related functions of the polyhedra that serve as the fundamental cells of compact hyperbolic spaces and the use of the coordinate set $(\chi, \theta, \phi)$ makes the problem almost intractable [32].

2. Non-spatial modes

Trying a solution of the kind

$$u (\eta) = a^p U (\eta),$$  \hspace{1cm} (A.6)

where $p$ is a constant, in Eq. (A.2), we obtain

$$\frac{d^2 U}{d\eta^2} + 2 (p + 1) D \frac{dU}{d\eta} + [\ell^2 + (\xi R + m^2) a^2] U$$

$$+ p \left[ (p + 1) D^2 + \frac{1}{a} \frac{d^2 a}{d\eta^2} \right] U = 0.$$  \hspace{1cm} (A.7)
In the case where \( p = -1 \), we obtain, using Eq. (10),
\[
\frac{d^2 U}{d\eta^2} + \left\{ \ell^2 + k + \left[ \left( \xi - \frac{1}{6} \right) R + m^2 \right] \alpha^2 \right\} U = 0. \tag{A.8}
\]
This equation, which clearly becomes simpler for the case of a conformal coupling, \( \xi = 1/6 \), can be solved analytically for several cases of cosmological interest.

### a. Static universes

For Minkowski space, \( a = 1, k = R = 0 \) and
\[
U = c_1 \sin \left( \ell^2 + m^2 \right)^{1/2} \eta + c_2 \cos \left( \ell^2 + m^2 \right)^{1/2} \eta. \tag{A.9}
\]
For \( k \neq 0 \), we have from Eqs. (10) and (A.8),
\[
\frac{d^2 U}{d\eta^2} = - \left( \ell^2 + 6\xi + m^2 \frac{2k}{\alpha} \right) U \equiv -\lambda^2 U. \tag{A.10}
\]

where the expression for \( a \) given in Eq. (31) was used.

**Solutions for** \( n = 0 \)

When \( n = 0 \), Eq. (A.13) becomes
\[
\frac{d^2 U_{k,0}}{d\eta^2} + \left\{ k + \ell^2 + \frac{1}{\sin^2 \sqrt{k} \eta} \right\} U_{k,0} = 0, \tag{A.14}
\]
where \( \nu^2 \equiv 2 - 3 - (12\xi + m^2 \alpha^{-2}) \). The solutions to this equation are
\[
U_{0,0} = \sqrt{\eta} \left[ c_1 J_\nu (\ell \eta) + c_2 N_\nu (\ell \eta) \right], \tag{A.15}
\]
for \( k = 0 \) and
\[
U_{1,0} = \sin \frac{\eta}{2} \left[ c_1 P_\nu^{-\frac{\eta}{2} + \sqrt{\alpha^{-1} - k}} (\cos \eta) + c_2 Q_\nu^{-\frac{\eta}{2} + \sqrt{\alpha^{-1} - k}} (\cos \eta) \right], \tag{A.16}
\]
and
\[
U_{-1,0} = \sinh \frac{\eta}{2} \left[ c_1 P_\nu^{-\frac{\eta}{2} + \sqrt{\alpha^{-1} - k}} (\cosh \eta) + c_2 Q_\nu^{-\frac{\eta}{2} + \sqrt{\alpha^{-1} - k}} (\cosh \eta) \right], \tag{A.17}
\]
for \( k = 1 \) and \( k = -1 \), respectively.

With
\[
U = c_1 \sin \lambda \eta + c_2 \cos \lambda \eta. \tag{A.11}
\]

**b. Expanding universes**

When \( \Lambda = 0 \) and the scale factor \( a \) is given by Eq. (31), the function \( u \) satisfies the equation
\[
\frac{d^2 u}{d\eta^2} + 2D \frac{du}{d\eta} + \frac{3k \xi (4 - n) u}{\sin^2 \sqrt{k} (\frac{\eta}{2} - 1) \eta} + (\ell^2 + m^2 a^2) u = 0. \tag{A.12}
\]

Then,
\[
\frac{d^2 U_{k,n}}{d\eta^2} + \left\{ k + \ell^2 + \frac{k (4 - n) (6\xi - 1)}{2 \sin^2 \sqrt{k} (\frac{\eta}{2} - 1) \eta} + \frac{k^2 \alpha^{-2} \eta^2 k \frac{6\pi}{\alpha}}{\sin^2 \sqrt{k} (\frac{\eta}{2} - 1) \eta} \right\} U_{k,n} = 0, \tag{A.13}
\]

**Solutions for** \( n = 2 \)

When \( n = 2 \), Eq. (A.8) is
\[
\frac{d^2 U_{k,2}}{d\eta^2} + \left\{ \ell^2 + k + m^2 e^{2\eta \sqrt{\alpha^{-1} - k}} + (6\xi - 1) \alpha^2 \right\} U_{k,2} = 0, \tag{A.18}
\]
which has the solution
\[
U_{k,2} = c_1 J_\gamma \left[ \frac{m e^{(\alpha^2 - k)^{1/2} \eta}}{(\alpha^2 - k)^{1/2}} \right] + c_2 N_\gamma \left[ \frac{m e^{(\alpha^2 - k)^{1/2} \eta}}{(\alpha^2 - k)^{1/2}} \right], \tag{A.19}
\]
where
\[
\gamma^2 \equiv - (\alpha^2 - k)^{-1} [\ell^2 + k + (6\xi - 1) \alpha^2]. \tag{A.20}
\]
This solution reduces to
\[
U_{k,2} = c_1 \sin \left( \gamma \eta \sqrt{\alpha^2 - k} \right) + c_2 \cos \left( \gamma \eta \sqrt{\alpha^2 - k} \right), \tag{A.21}
\]
when \( m = 0 \).

**Solutions for** \( n = 4 \)

When \( n = 4 \), Eq. (A.13) becomes
\[
\frac{d^2 U_{k,4}}{d\eta^2} + \left\{ k + \ell^2 + \frac{m^2 \alpha^2}{k} \sin^2 \sqrt{k} \eta \right\} U_{k,4} = 0, \tag{A.22}
\]
whose solution for flat space is

\[
U_{0,4} = \frac{1}{\sqrt{\eta}} \left[ c_1 M_{\ell_2(4i\alpha)}^{-1,4-1,\sqrt{7}}(i\alpha \eta^2) + c_2 M_{\ell_2(4i\alpha)}^{-1,4-1,\sqrt{7}}(i\alpha \eta^2) \right], \quad (A.23)
\]

where

\[
M_{\lambda\mu}(z) = z^{\mu+1/2} e^{-z/2} \Phi \left( \mu - \lambda + \frac{1}{2}, 2\mu + 1; z \right)
\]

is the Whittaker \( M \) function [10]. This solution can also be written using the parabolic cylinder function \( D_p(z) \) [10].

For \( k = 1 \) and \( k = -1 \), Eq. (A.22) yields a Mathieu equation,

\[
\frac{d^2 U_{1,4}}{d\eta^2} + \left[ \ell^2 + 1 + \frac{m^2\alpha^2}{2} - \frac{m^2\alpha^2}{2} \cos 2\eta \right] U_{1,4} = 0,
\]

(A.25)

and a modified Mathieu equation,

\[
\frac{d^2 U_{-1,4}}{d\eta^2} + \left[ \ell^2 - 1 - \frac{m^2\alpha^2}{2} + \frac{m^2\alpha^2}{2} \cosh 2\eta \right] U_{-1,4} = 0,
\]

(A.26)

respectively.

In the massless case, Eq. (A.22) has a single solution for all \( k \)'s,

\[
U_{k,4}^{m=0} = c_1 \left[ (k + \ell^2)^{1/2} \eta \right] + c_2 \cos \left[ (k + \ell^2)^{1/2} \eta \right].
\]

(A.27)

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