Interactions for a collection of spin-two fields intermediated by a massless $p$-form

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Abstract

Under the general hypotheses of locality, smoothness of interactions in the coupling constant, Poincaré invariance, Lorentz covariance, and preservation of the number of derivatives on each field, we investigate the cross-couplings of one or several spin-two fields to a massless $p$-form. Two complementary cases arise. The first case is related to the standard interactions from General Relativity, but the second case describes a new, special type of couplings in $D = p + 2$ space-time dimensions, which break the PT-invariance. Nevertheless, no consistent, indirect cross-interactions among different gravitons with a positively defined metric in internal space can be constructed.

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1 Introduction

Theories involving one or several spin-two fields have raised a constant interest over the last thirty years, especially at the level of direct or intermediated graviton interactions \[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\]. In this context more results on the impossibility of consistent cross-couplings among different gravitons have been obtained, either without other fields \[14\] or in the presence of a scalar field \[14\], a Dirac spinor \[15\], or respectively of a massive Rarita-Schwinger field \[16\]. All these no-go results have been deduced under some specific hypotheses, always including the preservation of the derivative order of each field equation with respect to its free limit (derivative order assumption). Through their implications, these findings support the common belief that the only consistent interactions in graviton theories require a single spin-two field and are subject to the standard prescriptions of General Relativity (meaning diffeomorphisms for the gauge transformations of the graviton and diffeomorphism algebra for the gauge algebra of the interacting theory). This idea is also strengthened by the confirmation of the uniqueness of Einstein-Hilbert action \[14\] having the Pauli-Fierz model as its free limit or the uniqueness of \(N = 1, D = 4\) SUGRA action \[17\] allowing for a Pauli-Fierz field and a massless Rarita-Schwinger spinor as the corresponding uncoupled limit. Indirect arguments are thus presented in favour of ruling out \(N > 8\) extended supergravity theories since they require more than one spin-two field. It is nevertheless known that the relaxation of the derivative order condition may lead to exotic couplings for one or a collection of spin-two fields \[18\], which are no longer mastered by General Relativity.

Our paper submits to the same topic, of constructing spin-two field(s) couplings, initially in the presence of a massless vector field and then of a \(p\)-form, with \(p > 1\). We employ a systematic approach to the construction of interactions in gauge theories \[19, 20, 21\], based on the cohomological reformulation of Lagrangian BRST symmetry \[22, 23, 24, 25, 26\]. In this approach interactions result from the analysis of consistent deformations of the generator of the Lagrangian BRST symmetry (known as the solution of the master equation) by means of specific cohomological techniques, relying on local BRST cohomology \[27, 28\]. The emerging deformations, and hence also the interactions, are constructed under the general hypotheses of locality, smoothness in the coupling constant, Poincaré invariance, Lorentz covariance, and derivative order assumption. In this specific situation the derivative order assumption requires that the interaction vertices contain at
most two spacetime derivatives of the fields, but does not restrict the polynomial order in the undifferentiated fields either in the Lagrangian or in the gauge symmetries. Our analysis envisages three steps, which introduce gradually the situations under investigation, according to the complexity of their cohomological content.

Initially, we consider the case of couplings between a single Pauli-Fierz field [29, 30] and a massless vector field. In this setting we compute the coupling terms to order two in the coupling constant $k$ and find two distinct solutions. The first solution leads to the full cross-coupling Lagrangian in all $D > 2$

$$\mathcal{L}_1^{(\text{int})} = -\frac{1}{4} \sqrt{-g} g^\mu_\nu g^\rho_\lambda \bar{F}_\mu_\rho F_\nu_\lambda + k \left( q_1 \delta^D_3 \varepsilon^{\mu_1\mu_2\mu_3} V_\mu_1 \bar{F}_\mu_2 F_\mu_3 
+ q_2 \delta^D_5 \varepsilon^{\mu_1\mu_2\mu_3\mu_4\mu_5} V_\mu_1 F_{\mu_2 \mu_3} F_{\mu_4 \mu_5} \right),$$

which respects the standard rules of General Relativity. The second solution is more unusual: it ‘lives’ only in $D = 3$, produces polynomials of order two in the coupling constant (and not series, like in the first case), and the couplings are mixing-component terms that can be written in terms of a deformed field strength (of the massless vector-field) as

$$\mathcal{L}_1^{(\text{int})} = -\frac{1}{4} F^\mu_\nu F'^\nu_\mu, \quad F'^\nu_\mu = F^\mu_\nu + 2k \varepsilon_{\mu_\nu\rho_\theta} \partial[\theta, \rho_\theta].$$

By contrast to General Relativity, where all the gauge symmetries are deformed, here only those of the vector field are modified by terms of order one in the coupling constant that involve the Pauli-Fierz gauge parameters, while the spin-two field keeps its original gauge symmetries, namely the linearized version of diffeomorphisms. To our knowledge, this is the first situation where the linearized version of the spin-two field allows for non-trivial couplings, other than those subject to General Relativity, which fulfill all the working hypotheses, including that on the derivative order.

Next, we focus on the investigation of cross-interactions among different gravitons intermediated by a massless vector field. In view of this, we start from a finite sum of Pauli-Fierz actions with a positively defined metric in internal space and a massless vector field. The cohomological analysis reveals again two cases. The former is related to the standard graviton-vector field interactions from General Relativity and exhibits no consistent cross-interactions among different gravitons (with a positively defined metric in internal space) in the presence of a massless vector field. At most one
The graviton can be coupled to the vector field via a Lagrangian similar to \( \mathcal{L}_{\text{int}} \), while each of the other spin-two fields may interact only with itself through an Einstein-Hilbert action with a cosmological term. The latter case seems to describe some new type of couplings in \( D = 3 \), which appear to allow for cross-couplings among different gravitons. The coupled Lagrangian is, like in the case of a single graviton, a polynomial of order two in the coupling constant, obtained by deforming the vector field strength

\[
\mathcal{L}_{\text{int}}^{(\text{int})} = -\frac{1}{4} \hat{F}^{\mu\nu} \hat{F}_{\mu\nu}, \quad \hat{F}^{\mu\nu} = F^{\mu\nu} + 2k \varepsilon^{\mu\nu\rho} \sum_{A=1}^{n} (y_{3}^{A} \partial_{\rho} h_{\rho}^{A} \theta),
\]

where \( y_{3}^{A} \) are some arbitrary, nonvanishing real constants. Nevertheless, these cross-couplings can be decoupled through an orthogonal, linear transformation of the spin-two fields, in terms of which \( \mathcal{L}_{\text{int}}^{(\text{int})} \) becomes nothing but \( \mathcal{L}_{\text{int}}^{(\text{int})} \), with \( h_{\mu\nu} \) replaced for instance by the first transformed spin-two field from the collection. In consequence, these case also leads to no indirect cross-couplings between different gravitons.

Then, we show that all the new results obtained in the case a massless vector field can be generalized to an arbitrary \( p \)-form. More precisely, if one starts from a free action describing an Abelian \( p \)-form and a single Pauli-Fierz field, then it is possible to construct some new deformations in \( D = p + 2 \) that are consistent to all orders in the coupling constant and are not subject to the rules of General Relativity. It is important to remark that all the working hypotheses, including the derivative order assumption, are fulfilled. There are several physical consequences of these couplings, such as the appearance of a constant linearized scalar curvature if one allows for a cosmological term or the modification of the initial \((p + 1)\)-order conservation law for the \( p \)-form by terms containing the spin-two field. Regarding a collection of spin-two fields, we find that the deformed Lagrangian does not allow for cross-couplings between different gravitons intermediated by a \( p \)-form, either in the setting of General Relativity or in the special, \((p + 2)\)-dimensional situation.

This paper is organized in seven sections. In Section 2 we construct the BRST symmetry of a free model with a single Pauli-Fierz field and one massless vector field. Section 3 briefly addresses the deformation procedure based on the BRST symmetry. In Sections 4 and 5 we compute the deformations corresponding to a vector field and one or respectively several spin-two fields, and emphasize the Lagrangian formulation of the resulting theories. Section 6 discusses the generalization of the previous results to the case of couplings...
between one or several gravitons and an arbitrary $p$-form gauge field. Section 7 ends the paper with the main conclusions.

2 BRST symmetry of the free model

Our starting point is represented by a free Lagrangian action, written as the sum between the linearized Hilbert-Einstein action (also known as the Pauli-Fierz action) and Maxwell’s action in $D > 2$ spacetime dimensions

$$S_{0}[h_{\mu\nu}, V_{\mu}] = \int d^{D}x \left[ -\frac{1}{2} (\partial_{\mu} h_{\nu\rho}) \partial^{\nu} h^{\mu\rho} + (\partial_{\mu} h^{\mu\rho}) \partial^{\nu} h_{\nu\rho} - (\partial_{\mu} h) \partial_{\nu} h^{\mu\nu} + \frac{1}{2} (\partial_{\mu} h) \partial^{\mu} h - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right]$$

$$\equiv \int d^{D}x \left( L_{0}^{\text{PF}} + L_{0}^{\text{vect}} \right).$$

The restriction $D > 2$ is required by the spin-two field action, which is known to reduce to a total derivative in $D = 2$. Throughout the paper we work with the flat metric of ‘mostly plus’ signature, $\sigma_{\mu\nu} = (- + \ldots +)$. In the above $h$ denotes the trace of the Pauli-Fierz field, $h = \sigma_{\mu\nu} h^{\mu\nu}$, and $F_{\mu\nu}$ represents the Abelian field-strength of the massless vector field ($F_{\mu\nu} \equiv \partial_{[\mu} V_{\nu]}$). The theory described by action (1) possesses an Abelian and irreducible generating set of gauge transformations

$$\delta_{\epsilon} h_{\mu\nu} = \partial_{(\mu} \epsilon_{\nu)}, \quad \delta_{\epsilon} V_{\mu} = \partial_{\mu} \epsilon,$$

with $\epsilon_{\mu}$ and $\epsilon$ bosonic gauge parameters. The notation $[\mu \ldots \nu]$ (or $(\mu \ldots \nu)$) signifies antisymmetry (or symmetry) with respect to all indices between brackets without normalization factors (i.e., the independent terms appear only once and are not multiplied by overall numerical factors).

In order to construct the BRST symmetry for action (1), it is necessary to introduce the field/ghost and antifield spectra

$$\Phi^{\alpha\dot{\alpha}} = (h_{\mu\nu}, V_{\mu}), \quad \Phi^{\ast\alpha\dot{\alpha}} = (h^{*\mu\nu}, V^{*\mu}),$$

$$\eta_{\alpha} = (\eta_{\mu}, \eta), \quad \eta^{\ast\alpha} = (\eta^{*\mu}, \eta^{*}).$$

The fermionic ghosts $\eta_{\alpha}$ are associated with the gauge parameters $\epsilon_{\alpha} = \{\epsilon_{\mu}, \epsilon\}$ respectively and the star variables represent the antifields of the corresponding fields/ghosts. (According to the standard rule of the BRST method,
the Grassmann parity of a given antifield is opposite to that of the corresponding field/ghost.) Since the gauge generators are field-independent and irreducible, it follows that the BRST differential decomposes into

\[ s = \delta + \gamma, \]

(5)

where \( \delta \) is the Koszul-Tate differential and \( \gamma \) denotes the exterior longitudinal derivative. The Koszul-Tate differential is graded in terms of the antighost number (agh, \( \text{agh} (\delta) = -1 \), \( \text{agh} (\gamma) = 0 \)) and enforces a resolution of the algebra of smooth functions defined on the stationary surface of field equations for action \( (\Sigma), C^\infty (\Sigma), \Sigma: \delta S_0 / \delta \Phi^a = 0 \). The exterior longitudinal derivative is graded in terms of the pure ghost number (pgh, \( \text{pgh} (\gamma) = 1 \), \( \text{pgh} (\delta) = 0 \)) and is correlated with the original gauge symmetry via its cohomology in pure ghost number zero computed in \( C^\infty (\Sigma) \), which is isomorphic to the algebra of physical observables for this free theory. These two degrees of the BRST generators are valued as

\[ \text{agh} (\Phi^a_0) = \text{agh} (\eta_{\alpha_1}) = 0, \quad \text{agh} (\Phi^*_{a_0}) = 1, \quad \text{agh} (\eta^{*\alpha_1}) = 2, \]

(6)

\[ \text{pgh} (\Phi^a_0) = 0, \quad \text{pgh} (\eta_{a_1}) = 1, \quad \text{pgh} (\Phi^*_{a_0}) = \text{pgh} (\eta^{*\alpha_1}) = 0. \]

(7)

The overall degree that grades the BRST complex is named ghost number (gh) and is defined like the difference between the pure ghost number and the antighost number, such that \( \text{gh} (s) = \text{gh} (\delta) = \text{gh} (\gamma) = 1 \). The actions of the operators \( \delta \) and \( \gamma \) (taken to act as right differentials) on the BRST generators read as

\[ \delta h^{*\mu\nu} = 2H^{\mu\nu}, \quad \delta V^{*\mu} = -\partial_{\nu} F^{\nu\mu}, \]

(8)

\[ \delta \eta^{*\mu} = -2\partial_{\nu} h^{*\mu\nu}, \quad \delta \eta^{*} = -\partial_{\mu} V^{*\mu}, \]

(9)

\[ \delta \Phi_{a_0} = 0, \quad \delta \eta_{a_1} = 0, \]

(10)

\[ \gamma \Phi^{*}_{a_0} = 0, \quad \gamma \eta^{*\alpha_1} = 0, \]

(11)

\[ \gamma h_{\mu\nu} = \partial_{(\mu} \eta_{\nu)}, \quad \gamma V_{\mu} = \partial_{\mu} \eta, \]

(12)

\[ \gamma \eta_{\mu} = 0, \quad \gamma \eta = 0. \]

(13)

In the above \( H^{\mu\nu} \) is the linearized Einstein tensor

\[ H^{\mu\nu} = K^{\mu\nu} - \frac{1}{2} \sigma^{\mu\nu} K, \]

(14)

with \( K^{\mu\nu} \) and \( K \) the linearized Ricci tensor and the linearized scalar curvature respectively, both obtained from the linearized Riemann tensor

\[ K_{\mu\nu;\alpha\beta} = -\frac{1}{2} (\partial_{\nu} \partial_{\alpha} h_{\mu\beta} + \partial_{\beta} \partial_{\nu} h_{\mu\alpha} - \partial_{\nu} \partial_{\alpha} h_{\mu\beta} - \partial_{\beta} \partial_{\nu} h_{\mu\alpha}), \]

(15)
from its trace and double trace respectively

\[ K_{\mu\alpha} = \sigma^{\nu\beta} K_{\mu\nu|\alpha\beta}, \quad K = \sigma^{\mu\alpha} \sigma^{\nu\beta} K_{\mu\nu|\alpha\beta}. \] (16)

The BRST differential is known to have a canonical action in a structure named antibracket and denoted by the symbol \( (\cdot, \cdot) \) \( (s \cdot = (\cdot, \bar{S})) \), which is obtained by considering the fields/ghosts conjugated respectively to the corresponding antifields. The generator of the BRST symmetry is a bosonic functional of ghost number zero, which is solution to the classical master equation \( (\bar{S}, \bar{S}) = 0 \). The full solution to the master equation for the free model under study reads as

\[ \bar{S} = S^L_0[h_{\mu\nu}, V_{\mu}] + \int d^D x \left( h^{*\mu\nu} \partial_{(\mu} \eta_{\nu)} + V^{*\mu} \partial_{\mu} \eta \right) \] (17)

and encodes all the information on the gauge structure of the theory (1)–(2).

3 Brief review of the deformation procedure

We begin with a “free” gauge theory, described by a Lagrangian action \( S^L_0[\Phi^0] \), invariant under some gauge transformations \( \delta \Phi^0 = \tilde{Z}^\alpha_0 \epsilon^\alpha_1 \), i.e. \( \frac{\delta S^L_0}{\delta \Phi^0} \tilde{Z}^\alpha_0 = 0 \), and consider the problem of constructing consistent interactions among the fields \( \Phi^0 \) such that the couplings preserve the field spectrum and the original number of gauge symmetries. This matter is addressed by means of reformulating the problem of constructing consistent interactions as a deformation problem of the solution to the master equation corresponding to the “free” theory [19, 20, 21]. Such a reformulation is possible due to the fact that the solution to the master equation contains all the information on the gauge structure of the theory. If an interacting gauge theory can be consistently constructed, then the solution \( \bar{S} \) to the master equation associated with the “free” theory, \( (\bar{S}, \bar{S}) = 0 \), can be deformed into a solution \( S \)

\[ S \rightarrow S = \bar{S} + kS_1 + k^2S_2 + \cdots = \bar{S} + k \int d^D x a + k^2 \int d^D x b + \cdots \] (18)

of the master equation for the deformed theory

\[ (S, S) = 0, \] (19)
such that both the ghost and antifield spectra of the initial theory are preserved. The projection of equation (19) on the various orders in the coupling constant \( k \) leads to the equivalent tower of equations

\[
\begin{align*}
\langle \bar{S}, \bar{S} \rangle &= 0, \\
2 \langle S_1, \bar{S} \rangle &= 0, \\
2 \langle S_2, S \rangle + \langle S_1, S_1 \rangle &= 0,
\end{align*}
\]

Equation (20) is fulfilled by hypothesis. The next equation requires that the first-order deformation of the solution to the master equation, \( S_1 \), is a co-cycle of the “free” BRST differential \( s \), \( ss_1 = 0 \). However, only cohomologically nontrivial solutions to (21) should be taken into account, since the BRST-exact ones can be eliminated by some (in general nonlinear) field redefinitions. This means that \( S_1 \) pertains to the ghost number zero cohomological space of \( s \), \( H^0(s) \), which is nonempty because it is isomorphic to the space of physical observables of the “free” theory. It has been shown (by of the triviality of the antibracket map in the cohomology of the BRST differential) that there are no obstructions in finding solutions to the remaining equations, namely (22), etc. However, the resulting interactions may be nonlocal and there might even appear obstructions if one insists on their locality. The analysis of these obstructions can be done with the help of cohomological techniques.

4 Consistent interactions between the spin-two field and a massless vector field

4.1 Standard material: basic cohomologies

The aim of this section is to investigate the cross-couplings that can be introduced between the spin-two field and a massless vector field. This matter is addressed in the context of the antifield-BRST deformation procedure described in the above and relies on computing the solutions to equations (21)–(22), etc., with the help of the BRST cohomology of the free theory. The deformations are obtained under the following (reasonable) assumptions: smoothness in the deformation parameter, locality, Lorentz covariance, Poincaré invariance, and the presence of at most two derivatives in the coupled Lagrangian. ‘Smoothness in the deformation parameter’ refers to the
fact that the deformed solution to the master equation, (18), is smooth in
the coupling constant $k$ and reduces to the original solution, (17), in the free
limit $k = 0$. The hypothesis on the deformed theory to be Poincaré invariant
means that one does not allow an explicit dependence on the spacetime
coordinates into the deformed solution to the master equation. The require-
ment concerning the maximum number of derivatives allowed to enter the
deformed Lagrangian is frequently imposed in the literature; for instance,
see the case of couplings between the Pauli-Fierz and the massless Rarita-
Schwinger fields [17] or of cross-interactions for a collection of Pauli-Fierz
fields [14]. If we make the notation $S_1 = \int d^D x \ a$, then equation (21), which
controls the first-order deformation, takes the local form

$$sa = \partial_\mu m^\mu, \quad \text{gh} (a) = 0, \quad \varepsilon (a) = 0,$$  \hspace{1cm} (23)

for some local current $m^\mu$. It shows that the nonintegrated density of the
first-order deformation pertains to the local cohomology of the free BRST
differential in ghost number zero, $a \in H^0 (s|d)$, where $d$ denotes the exterior
spacetime differential. The solution to (23) is unique up to $s$-exact pieces
plus divergences

$$a \rightarrow a + sb + \partial_\mu n^\mu,$$  \hspace{1cm} (24)

with $\text{gh} (b) = -1, \ \varepsilon (b) = 1, \ \text{gh} (n^\mu) = 0, \ \text{and} \ \varepsilon (n^\mu) = 0$. At the same time, if
the general solution of (23) is found to be completely trivial, $a = sb + \partial_\mu n^\mu$,
then it can be made to vanish, $a = 0$.

In order to analyze equation (23) we develop $a$ according to the antighost
number

$$a = \sum_{i=0}^{I} a_i, \quad \text{agh} (a_i) = i, \quad \text{gh} (a_i) = 0, \quad \varepsilon (a_i) = 0,$$  \hspace{1cm} (25)

and assume, without loss of generality, that decomposition (25) stops at some
finite value of $I$. This can be shown for instance like in Appendix A of [14].
Replacing decomposition (25) into (23) and projecting it on the various values
of the antighost number by means of (5), we obtain that (23) is equivalent
with the tower of equations

$$\gamma a_I = \partial_\mu m_I^\mu,$$  \hspace{1cm} (26)
$$\delta a_I + \gamma a_{I-1} = \partial_\mu m_{I-1}^\mu,$$  \hspace{1cm} (27)
$$\delta a_i + \gamma a_{i-1} = \partial_\mu m_{i-1}^\mu, \quad 1 \leq i \leq I - 1,$$  \hspace{1cm} (28)
where \((m_i^\mu)\) are some local currents, with \(\text{agh} (m_i^\mu) = \mu\). Moreover, according to the general result from \cite{14} in the absence of collection indices, equation (26) can be replaced in strictly positive antighost numbers by

\[
\gamma a_I = 0, \quad I > 0. \tag{29}
\]

Due to the second-order nilpotency of \(\gamma (\gamma^2 = 0)\), the solution to (29) is unique up to \(\gamma\)-exact contributions

\[
a_I \rightarrow a_I + \gamma b_I, \quad \text{agh} (b_I) = I, \quad \text{pgh} (b_I) = I - 1, \quad \varepsilon (b_I) = 1. \tag{30}
\]

Meanwhile, if it turns out that \(a_I\) reduces to \(\gamma\)-exact terms, \(a_I = \gamma b_I\), then it can be made to vanish, \(a_I = 0\). In other words, the nontriviality of the first-order deformation \(a\) is translated at its highest antighost number component into the requirement that \(a_I \in H^I (\gamma)\), where \(H^I (\gamma)\) denotes the cohomology of the exterior longitudinal derivative \(\gamma\) in pure ghost number equal to \(I\). So, in order to solve equation (23) (equivalent with (29) and (27)–(28)), we need to compute the cohomology of \(\gamma\), \(H (\gamma)\), and, as it will be made clear below, also the local cohomology of \(\delta\), \(H (\delta | d)\).

Using the results on the cohomology of \(\gamma\) in the Pauli-Fierz sector \cite{14} as well as definitions (11)–(13), we can state that \(H (\gamma)\) is generated on the one hand by \(\Phi^*_{\alpha_0}, \eta^{* \alpha_1}, F_{\mu \nu},\) and \(K_{\mu \nu \alpha \beta}\), together with their spacetime derivatives and, on the other hand, by the undifferentiated ghosts \(\eta\) and \(\eta_\mu\) as well as by their antisymmetric first-order derivatives \(\partial [\mu \eta_\nu]\). (The spacetime derivatives of \(\eta\) are \(\gamma\)-exact, in agreement with the latter definition from (12), and the same is valid for the derivatives of \(\eta_\mu\) of order two and higher.) So, the most general (and nontrivial) solution to (29) can be written, up to \(\gamma\)-exact contributions, as

\[
a_I = \alpha_I ([F_{\mu \nu}], [K_{\mu \nu \rho \lambda}], [\Phi^*_{\alpha_0}], [\eta^{* \alpha_1}]) e^I (\eta, \eta_\mu, \partial [\mu \eta_\nu]), \tag{31}
\]

where the notation \(f ([q])\) means that \(f\) depends on \(q\) and its derivatives up to a finite order, while \(e^I\) denotes the elements of a basis in the space of polynomials with pure ghost number \(I\) in \(\eta, \eta_\mu,\) and \(\partial [\mu \eta_\nu]\). The objects \(\alpha_I\) (obviously nontrivial in \(H^0 (\gamma)\)) were taken to have a finite antighost number and a bounded number of derivatives, and therefore they are polynomials in the antifields, in the linearized Riemann tensor \(K_{\mu \nu \alpha \beta}\), and in the field-strength \(F_{\mu \nu}\) as well as in their subsequent derivatives. They are required to fulfill the property \(\text{agh} (\alpha_I) = I\) in order to ensure that the ghost number of
$a_I$ is equal to zero. Due to their $\gamma$-closeness, $\gamma a_I = 0$, and to their polynomial character, $\alpha_I$ will be called invariant polynomials. In antighost number zero the invariant polynomials are polynomials in the linearized Riemann tensor, in the field-strength of the Abelian field, and in their derivatives. The result that one can replace equation (26) with (29) is a consequence of the triviality of the cohomology of the exterior spacetime differential in the space of invariant polynomials in strictly positive antighost numbers. For more details, see subsection A.1 from [14].

Inserting (31) in (27), we obtain that a necessary (but not sufficient) condition for the existence of (nontrivial) solutions $a_{I-1}$ is that the invariant polynomials $\alpha_I$ are (nontrivial) objects from the local cohomology of the Koszul-Tate differential $H(\delta|d)$ in antighost number $I > 0$ and in pure ghost number zero

$$\delta \alpha_I = \partial \mu j_I^{\mu}, \quad \text{agh} \left( j_I^{\mu} \right) = I - 1, \quad \text{pgh} \left( j_I^{\mu} \right) = 0. \quad (32)$$

We recall that the local cohomology $H(\delta|d)$ is completely trivial in both strictly positive antighost and pure ghost numbers (for instance, see Theorem 5.4 from [27] and also [28]). Using the fact that the Cauchy order of the free theory under study is equal to two, the general results from [27] and [28], according to which the local cohomology of the Koszul-Tate differential in pure ghost number zero is trivial in antighost numbers strictly greater than its Cauchy order, ensure that

$$H_J(\delta|d) = 0, \quad J > 2, \quad (33)$$

where $H_J(\delta|d)$ denotes the local cohomology of the Koszul-Tate differential in antighost number $J$ and in pure ghost number zero. It can be shown that any invariant polynomial that is trivial in $H_J(\delta|d)$ with $J \geq 2$ can be taken to be trivial also in $H_J^{\text{inv}}(\delta|d)$. ($H_J^{\text{inv}}(\delta|d)$ denotes the invariant characteristic cohomology in antighost number $J$ — the local cohomology of the Koszul-Tate differential in the space of invariant polynomials.) Thus:

$$(\alpha_J = \delta b_{J+1} + \partial \mu e_I^{\mu}, \text{agh} (\alpha_J) = J \geq 2) \Rightarrow \alpha_J = \delta \beta_{J+1} + \partial \mu \gamma_I^{\mu}, \quad (34)$$

with both $\beta_{J+1}$ and $\gamma_I^{\mu}$ invariant polynomials. Results (34) and (33) yield the conclusion that the invariant characteristic cohomology is trivial in antighost numbers strictly greater than two

$$H_J^{\text{inv}}(\delta|d) = 0, \quad J > 2. \quad (35)$$
By proceeding in the same manner like in [14] and [31], it can be proved that the spaces $H_2(\delta|d)$ and $H_2^{\text{inv}}(\delta|d)$ are spanned by
\[ H_2(\delta|d), H_2^{\text{inv}}(\delta|d) : (\eta^*, \eta^*\mu). \] (36)

In contrast to the groups $(H_J(\delta|d))_{J \geq 2}$ and $(H_J^{\text{inv}}(\delta|d))_{J \geq 2}$, which are finite-dimensional, the cohomology $H_1(\delta|d)$ in pure ghost number zero, known to be related to global symmetries and ordinary conservation laws, is infinite-dimensional since the theory is free. Fortunately, it will not be needed in the sequel.

The previous results on $H(\delta|d)$ and $H^{\text{inv}}(\delta|d)$ in strictly positive antighost numbers are important because they control the obstructions of removing the antifields from the first-order deformation. Based on formulas (33)–(35), one can eliminate all the pieces of antighost number strictly greater than two from the nonintegrated density of the first-order deformation by adding only trivial terms. Consequently, one can take (without loss of nontrivial objects) $I \leq 2$ into the decomposition (25). (The proof of this statement can be realized like in subsection A.3 from [14].) In addition, the last representative reads as in (31), where the invariant polynomial is necessarily a nontrivial object from $H_2^{\text{inv}}(\delta|d)$ if $I = 2$ and from $H_1(\delta|d)$ if $I = 1$ respectively.

4.2 Computation of first-order deformations

Assuming $I = 2$, the nonintegrated density of the first-order deformation (25) becomes
\[ a = a_0 + a_1 + a_2. \] (37)

We can further decompose $a$ in a natural manner as
\[ a = a^{(\text{PF})} + a^{(\text{int})} + a^{(\text{vect})}, \] (38)
where $a^{(\text{PF})}$ contains only fields/ghosts/antifields from the Pauli-Fierz sector, $a^{(\text{int})}$ mixes both fields, and $a^{(\text{vect})}$ involves only the vector field sector. The component $a^{(\text{PF})}$ is completely known [14] and satisfies by itself an equation of the type (23). It admits a decomposition similar to (37)
\[ a^{(\text{PF})} = a_0^{(\text{PF})} + a_1^{(\text{PF})} + a_2^{(\text{PF})}, \] (39)
where
\[ a_2^{(\text{PF})} = \frac{f}{2} \eta^* \eta^* \partial(\mu \eta_\nu), \] (40)
\[ a_{1}^{(PF)} = f h^{\mu \rho} \left( (\partial_{\rho} \eta^\nu) h_{\mu \nu} - \eta^\nu \partial_{[\mu} h_{\nu] \rho} \right), \]  
and \( a_{0}^{(PF)} \) is the cubic vertex of the Einstein-Hilbert Lagrangian multiplied by a real constant \( f \) plus a cosmological term \[ a_{0}^{(PF)} = f a_{0}^{(EH-cubic)} - 2 \Lambda h, \] with \( \Lambda \) the cosmological constant. Due to the fact that \( a^{(int)} \) and \( a^{(vect)} \) contain different sorts of fields, it follows that they are subject to two separate equations

\[ sa^{(vect)} = \partial_{\mu} m^{(vect)\mu}, \]  
\[ sa^{(int)} = \partial_{\mu} m^{(int)\mu}, \]  
for some local \( m^{\mu} \)s. It is known (for instance, see [32]) that the general solution to (43) reduces to its component of antighost number zero and reads as

\[ a^{(vect)} = a_{0}^{(vect)} = \sum_{j>0} q_{j} \delta^{D}_{2j+1} \epsilon^{\mu_{1}\mu_{2}\mu_{3}...\mu_{2j+1}} V_{\mu_{1} F_{\mu_{2}} F_{\mu_{3}} ... F_{\mu_{2j+1}}}, \]  
with \( q_{j} \) some real constants. Selecting from (45) only the terms with maximum two spacetime derivatives, we conclude that we must ask \( q_{j} = 0 \) for all \( j > 2 \), so

\[ a^{(vect)} = a_{0}^{(vect)} = q_{1} \delta_{3}^{D} \epsilon^{\mu \lambda} V_{\mu F_{\nu}} + q_{2} \delta_{5}^{D} \epsilon^{\mu \nu \lambda \alpha \beta} V_{\mu F_{\nu} F_{\alpha} F_{\beta}}. \]  
The notation \( \delta_{m}^{D} \) signifies the Kronecker symbol. In the sequel we analyze the general solution to equation (44).

Due to (37) we should consider that the general solution to (44) stops at antighost number two, \( a^{(int)} = a_{0}^{(int)} + a_{1}^{(int)} + a_{2}^{(int)} \). Equation (44) is equivalent to the fact that the components of \( a^{(int)} \) are subject to equations (29) and (27)–(28) with \( I = 2 \) and \( a \) replaced by \( a^{(int)} \). It can be shown that  

\[ ^{1}\text{The terms } a_{2}^{(PF)} \text{ and } a_{1}^{(PF)} \text{ given in (40) and (41) differ from those present in (14) (in the absence of collection indices) by a } \gamma \text{-exact and respectively a } \delta \text{-exact contribution. However, the difference between our } a_{2}^{(PF)} + a_{1}^{(PF)} \text{ and that from (14) is a } s \text{-exact modulo } d \text{ quantity. The associated } a_{0}^{(PF)} \text{ is nevertheless the same in both formulations. As a consequence, } a^{(PF)} \text{ and the first-order deformation from (14) belong to the same cohomological class from } H_{0}^{0}(s|d). \]
there exist no such solutions ending at antighost number two. For the sake of simplicity, we omit the proof of this result, which is mainly based on showing that there is no nontrivial $a_{2}^{(\text{int})}$ yielding a consistent $a_{0}^{(\text{int})}$. In view of this finding, we approach the next situation, where the solution to (44) stops at antighost number one

$$a^{(\text{int})} = a_{0}^{(\text{int})} + a_{1}^{(\text{int})},$$

(47)
such that the components on the right-hand side of (47) are subject to the equations

$$\gamma a_{1}^{(\text{int})} = 0,$$

(48)

$$\delta a_{1}^{(\text{int})} + \gamma a_{0}^{(\text{int})} = \partial_{\mu}a_{0}^{(\text{int})}\mu,$$

(49)

In agreement with (31) for $I = 1$ and the discussion from the end of subsection 4.1, the general solution to (48) is (up to trivial, $\gamma$-exact contributions)

$$a_{1}^{(\text{int})} = \alpha_{1}\eta + \alpha_{1\mu}\eta^{\mu} + \alpha_{1\mu\nu}\partial^{\mu}\eta^{\nu},$$

(50)

where $\alpha_{1}$, $\alpha_{1\mu}$, and $\alpha_{1\mu\nu}$ are nontrivial invariant polynomials from $H_{1}((\delta | d))$ (but not necessarily from $H_{1}^{\text{inv}}((\delta | d))$) in order to produce a consistent $a_{0}^{(\text{int})}$. Because they are nontrivial invariant polynomials of antighost number one, we can always assume that they are linear in the undifferentiated antifields $V^{*\mu}$ and $h^{*\mu\nu}$, such that (50) becomes

$$a_{1}^{(\text{int})} = V^{*\mu} \left( M_{\mu\nu}\eta + M_{\mu\nu\rho}\partial^{\nu}\eta^{\rho} \right) + h^{*\mu\nu} \left( N_{\mu\nu}\eta + N_{\mu\nu\rho}\eta^{\rho} + N_{\mu\nu\rho\lambda}\partial^{\rho}\eta^{\lambda} \right),$$

(51)

where all the coefficients, denoted by $M$ or $N$, must be $\gamma$-closed quantities, and therefore they may depend on $F_{\mu\nu}$, $K_{\mu\alpha|\nu\beta}$, and their derivatives. In addition, these tensors are subject to the symmetry/antisymmetry properties

$$M_{\mu\rho} = -M_{\mu\rho}, \quad N_{\mu\nu} = N_{\nu\mu},$$

(52)

$$N_{\mu\rho} = N_{\nu\rho}, \quad N_{\mu\rho\lambda} = N_{\nu\rho\lambda} = -N_{\nu\lambda\rho}.$$}

(53)

At this point we recall the hypothesis on the derivative order of the deformed Lagrangian, which imposes that $a_{0}^{(\text{int})}$ as solution to (49) contains at most two spacetime derivatives of the fields. Then, relation (51), equation (49), and definitions (8)–(13) yield the following results: A. none of the $M$- or $N$-type tensors entering (51) are allowed to depend on $K_{\mu\alpha|\nu\beta}$ or its derivatives; B.
$M_{\mu\nu\rho}$ and $N_{\mu\nu\rho\lambda}$ cannot involve either $F_{\mu\nu}$ or its derivatives, and therefore they are nonderivative, constant tensors; C. the tensors $M_\mu$, $M_{\mu\nu}$, $N_{\mu\nu}$, and $N_{\mu\nu\rho}$ may depend on $F_{\mu\nu}$ (and not on its derivatives), but only in a linear manner. These results are synthesized by the formulas

\[ M_\mu = C_\mu + C_{\mu\rho}F^{\nu\rho}, \quad M_{\mu\nu} = C_{\mu\nu} + C_{\mu\nu\lambda}F^{\rho\lambda}, \]  

\[ N_{\mu\nu} = D_{\mu\nu} + D_{\mu\nu\lambda\rho}F^{\mu\lambda\rho}, \quad N_{\mu\nu\rho} = D_{\mu\nu\rho} + D_{\mu\nu\rho\lambda\sigma}F^{\mu\lambda\rho\sigma}, \]  

\[ M_{\mu\nu\rho} = \bar{C}_{\mu\nu\rho}, \quad N_{\mu\nu\rho\lambda} = \bar{D}_{\mu\nu\rho\lambda}, \]  

where the quantities denoted by $C$, $\bar{C}$, $D$, or $\bar{D}$ are nonderivative, constant tensors, subject to some symmetry/antisymmetry properties such that (52) and (53) are fulfilled. Since we work in $D > 2$ spacetime dimensions, the only choice that complies with the above mentioned properties and leads to consistent cross-couplings between the Pauli-Fierz field and the vector field is:

\[ C_\mu = 0, \quad C_{\mu\rho} = 0, \quad C_{\mu\nu} = y_1\sigma_{\mu\nu}, \]  

\[ C_{\mu\rho\lambda} = \frac{p}{2} (\sigma_{\mu\rho}\sigma_{\nu\lambda} - \sigma_{\mu\lambda}\sigma_{\nu\rho}), \]  

\[ D_{\mu\nu} = y_2\sigma_{\mu\nu}, \quad D_{\mu\nu\rho} = D_{\nu\rho\mu} = 0, \]  

\[ D_{\mu\nu\rho\lambda\sigma} = 0, \quad \bar{C}_{\mu\nu\rho} = y_3\delta_3^{\mu\nu\rho}. \]  

Substituting (57)–(60) in (54)–(56) and the resulting expressions in (51), we obtain

\[ a_1^{(\text{int})} = y_1V^{*\lambda}\eta_{\lambda} + y_2h^{*}\eta + y_3\delta_3^{\mu\nu\rho}V^{*\mu}\partial^{[\nu}[\eta^{\rho]} + pV^{*\mu}F_{\mu\nu}\eta^{\nu}, \]  

where $h^{*} = h^{*\mu\nu\sigma}_{\mu\nu\sigma}$. Acting with $\delta$ on (61), we infer

\[ \delta a_1^{(\text{int})} = \gamma \left[-(D - 2)y_2V^{*\lambda}\partial_{[\mu}h^{\lambda}_{\nu]} + y_3\delta_3^{\mu\nu\rho}F^{\lambda\mu}\partial^{[\nu}[h^{\rho]}_{\lambda}\right]. \]  

\[ ^2\text{Strictly speaking, there is a nonvanishing solution } C_{\mu\nu\rho} = z\delta_3^{\mu\nu\rho}, \text{ which adds to } a_1^{(\text{int})} \text{ the term } z\delta_3^{\mu\nu\rho}V^{*\mu}F^{\nu\rho}\eta. \text{ Even if consistent, this term would lead to selfinteractions in the Maxwell sector. However, } a_1^{(\text{int})} \text{ is restricted by hypothesis to provide only cross-couplings between the Pauli-Fierz field and the electromagnetic field, so this term must be removed from this context by setting } z = 0. \text{ Apparently, there are two more possibilities, } C_{\mu\nu\rho\lambda} = z'\delta_4^{\mu\nu\rho\lambda} \text{ and } D_{\mu\nu\rho\lambda\sigma} = z''\delta_3^{\mu\nu\rho\sigma}, \text{ which add to } a_1^{(\text{int})} \text{ the terms } (z'\delta_4^{\mu\nu\rho\lambda}h^{*\mu\nu\lambda} - z'\delta_4^{\mu\nu\rho\sigma}V^{*\mu\nu\sigma}\eta). \text{ They are not eligible to enter } a_1^{(\text{int})} \text{ since the corresponding invariant polynomial, } z''\delta_3^{\mu\nu\rho}h^{*\mu\nu\rho} - z'\delta_4^{\mu\nu\rho\lambda}V^{*\mu\nu\lambda}\eta, \text{ does not belong to } H^1(\delta|d), \text{ such that they cannot lead to consistent pieces in } a_0^{(\text{int})} \text{ unless } z' = 0 = z''. \]
Comparing (62) with (49) and observing that (61) already contains a term of the type $V^{\lambda \eta \lambda}$, it follows that $a^{(\text{int})}_{1}$ is consistent at antighost number zero if and only if

$$y_{1} + (D - 2) y_{2} = 0.$$  \hspace{1cm} (63)

Replacing (63) into (61) and (62), we get finally

$$a^{(\text{int})}_{1} = y_{2} \left[ h^{*} \eta - (D - 2) V^{\lambda \eta \lambda} \right] + y_{3} \delta_{D}^{D} \varepsilon_{\mu \nu \rho} V^{\lambda \eta \lambda} \partial^{[\lambda \mu \nu \rho]} + p V^{\mu \nu} F_{\mu \nu} \eta^{\lambda},$$  \hspace{1cm} (64)

$$a^{(\text{int})}_{0} = (D - 2) y_{2} V^{\lambda \eta \lambda} \partial^{\mu}_{\lambda} h_{\mu}^{\lambda 
mu} + y_{3} \delta_{D}^{D} \varepsilon_{\mu \nu \rho} V^{\lambda \eta \lambda} \partial^{[\lambda \mu \nu \rho]} + \frac{p}{2} \left( F^{\alpha \mu} F_{\mu \nu} h^{\alpha \nu} + \frac{1}{4} F^{\alpha \mu} F_{\alpha \mu} h \right) + \tilde{a}^{(\text{int})}_{0},$$  \hspace{1cm} (65)

where $\tilde{a}^{(\text{int})}_{0}$ is the general solution to the homogeneous equation

$$\gamma \tilde{a}^{(\text{int})}_{0} = \partial^{\mu} \tilde{m}^{(\text{int})}_{\mu}.$$  \hspace{1cm} (66)

Such solutions correspond to $\tilde{a}^{(\text{int})}_{1} = 0$ and thus they cannot deform either the gauge algebra or the gauge transformations, but only the Lagrangian at order one in the coupling constant. There are two main types of solutions to (66). The first one corresponds to $\tilde{m}^{(\text{int})}_{\mu} = 0$ and is given by gauge-invariant, nonintegrated densities constructed from the original fields and their space-time derivatives. According to (31) for both pure ghost and antighost numbers equal to zero, they are given by $a^{(\text{int})}_{0} = a^{(\text{int})}_{0} \left( [F_{\mu \nu}] , [K_{\mu \alpha \nu \beta}] \right)$, up to the conditions that they describe true cross-couplings between the two types of fields and cannot be written in a divergence-like form. Unfortunately, this type of solutions must depend simultaneously at least on the linearized Riemann tensor and on the Abelian field strength in order to provide cross-couplings, so they would lead to terms with at least three derivatives in the deformed Lagrangian. By virtue of the derivative order assumption, they must be discarded by setting $\tilde{a}^{(\text{int})}_{0} = 0$. The second kind of solutions is associated with $\tilde{m}^{(\text{int})}_{\mu} \neq 0$ in (66), being understood that we maintain the requirements on $a^{(\text{int})}_{0}$ to contain maximum two derivatives of the fields and to
describe cross-couplings. In order to simplify the presentation, we omit the technical aspects regarding the analysis of these solutions. The main result is that, without loss of generality, we can take \( \bar{a}^{(\text{int})}_0 = 0 \) in (65). Very briefly, we mention that the procedure used for obtaining this result relies on decomposing \( \bar{a}^{(\text{int})}_0 \) along the number of derivatives, \( \bar{a}^{(\text{int})}_0 = \omega_0 + \omega_1 + \omega_2 \), where \( \omega_i \) contains exactly \( i \) derivatives of the fields. As a consequence, equation (66) becomes equivalent to three independent equations, one for each component. The terms \( \omega_0 \) and respectively \( \omega_1 \) are ruled out because they cannot produce cross-couplings. As for \( \omega_2 \), it requires the existence of a nonderivative, real, constant tensor \( C^{\mu\nu\beta\sigma} \), which displays the generalized symmetry properties of the Riemann tensor with respect to its first four indices and is simultaneously antisymmetric in its last three indices. Since there are no such tensors in any \( D \geq 3 \) spacetime dimension, we must discard \( \omega_2 \), which finally leaves us with \( \bar{a}^{(\text{int})}_0 = 0 \).

Replacing (64), (65), and \( \bar{a}^{(\text{int})}_0 = 0 \) in (47), we obtain the concrete form of the general solution \( a^{(\text{int})} \) to (44). We can still remove certain trivial, s-exact modulo \( d \) terms from the resulting \( a^{(\text{int})} \). Indeed, we have that

\[
a^{(\text{int})} = a^{(\text{int})} + s \left[ -p \left( \eta^* V^\mu \eta_\mu + \frac{1}{2} V^* V^\nu h_{\mu\nu} \right) \right] + \partial_\mu t^\mu,
\]

such that, in agreement with the discussion made in the beginning of this section, we can work with

\[
a^{(\text{int})} = a^{(\text{int})} + s \left[ p \left( \eta^* V^\mu \eta_\mu + \frac{1}{2} V^* V^\nu h_{\mu\nu} \right) \right] - \partial_\mu t^\mu
\]

\[
\equiv y_2 \left[ h^* \eta + (D - 2) \left( -V^* \eta^\lambda \eta_\lambda + V^\lambda \partial_\mu h^\mu \right) \right] 
\]

\[
+ y_3 \delta^D \xi_{\mu\rho} \left( V^* \partial^\nu \eta^\rho \eta_\nu + F^\lambda \partial^\mu h^\rho \right) + p \left[ \eta^* \eta_\mu \partial^\mu \eta \right] - \frac{1}{2} V^* \partial_\mu \left( h^\eta \eta_\mu \right) + 2 (\partial_\nu V^\mu) \eta^\nu - h_{\mu\nu} \partial^\mu \eta
\]

\[
+ \frac{1}{8} F^\mu_\nu \left( 2 \partial_\mu \left( h^\rho \rho_\nu \right) + F_{\mu\nu} h - 4 F_{\mu\rho} h^\rho \right)
\]

instead of \( a^{(\text{int})} \).

In view of the results (39), (46), and (68) we conclude that the most general, nontrivial first-order deformation of the solution to the master equation...
corresponding to action $(\Pi)$ and to its gauge transformations $(\mathcal{Z})$, which complies with all the working hypotheses, is expressed by

\[ S_1 = S_1^{(PF)} + S_1^{(int)}, \quad (69) \]

where

\[ S_1^{(PF)} \equiv \int d^D x a^{(PF)} = \int d^D x \left( a_2^{(PF)} + a_1^{(PF)} + a_0^{(PF)} \right), \quad (70) \]

and

\[ S_1^{(int)} = \int d^D x \left( a^{(int)} + a^{(vect)} \right) \]

\[ = \int d^D x \left\{ y_2 \left[ h^* \eta + (D - 2) \left( -V^* \lambda \eta + V^\lambda \partial_{[\mu} h_{\nu]} \right) \right] \]

\[ + y_3 \delta_3^{D} \varepsilon_{\mu\nu\rho} \left( V^{*\mu} \partial_{[\nu} \eta^{\rho]} + F^{\lambda\nu} \partial_{[\nu} h_{\rho]} \right) + p [\eta^{*} \eta_{\mu} \partial^{\mu} \eta] \]

\[ - \frac{1}{2} V^{*\mu} \left( V^{\nu} \partial_{[\nu} \eta_{\rho]} + 2 \left( \partial_{\nu} V_{\mu} \right) \eta^{\mu} - h_{\mu\nu} \partial^{\nu} \eta \right) \]

\[ + \frac{1}{8} F^{\mu\nu} \left( 2 \partial_{[\mu} \left( h_{\nu]} \right) V^{\rho} + F_{\mu\nu} h - 4 F_{\mu\rho} h^{\rho} \right) \]

\[ + q_1 \delta_3^{D} \varepsilon_{\mu\nu\lambda} V_{\mu} F_{\nu\lambda} + q_2 \delta_3^{D} \varepsilon_{\mu\nu\lambda\beta} V_{\mu} F_{\nu\lambda} F_{\alpha\beta} \}. \quad (71) \]

Thus, the first-order deformation of the solution to the master equation for the model under study is parameterized by seven independent, real constants, namely $f$, and $\Lambda$ corresponding to $S_1^{(PF)}$ (see $(40)$, $(41)$, and $(42)$) together with $p$, $y_2$, $y_3 \delta_3^{D}$, $q_1 \delta_3^{D}$, and $q_2 \delta_3^{D}$ associated with $S_1^{(int)}$.

### 4.3 Computation of second-order deformations

Here, we approach the construction of the second-order deformation of the solution to the master equation, governed by equation $(22)$. Replacing $(69)$ into $(22)$ we find that it becomes equivalent to the equations

\[ \begin{aligned}
    \left( S_1^{(PF)}, S_1^{(PF)} \right) + \left( S_1^{(int)}, S_1^{(int)} \right)^{(PF)} + 2 s S_2^{(PF)} &= 0, \\
    2 \left( S_1^{(PF)}, S_1^{(int)} \right) + \left( S_1^{(int)}, S_1^{(int)} \right)^{(int)} + 2 s S_2^{(int)} &= 0,
\end{aligned} \quad (72),(73) \]

where $\left( S_1^{(int)}, S_1^{(int)} \right)^{(PF)}$ comprises only BRST generators from the Pauli-Fierz sector and each term from $\left( S_1^{(int)}, S_1^{(int)} \right)^{(int)}$ contains at least one BRST
generator from the one-form sector. By writing down (72) and (73), it is understood that the second-order deformation decomposes as

$$S_2 = S_2^{(PF)} + S_2^{(int)},$$  \hspace{1cm} (74)$$

where $S_2^{(PF)}$ represents the component from the Pauli-Fierz sector and $S_2^{(int)}$ signifies the complementary part.

Initially, we analyze equation (72). It is known from the literature (for instance, see [14] in the absence of collection indices) that there exists $S_2^{(PF)}(f^2, f \Lambda)$ such that

$$\left( S_1^{(PF)}, S_1^{(PF)} \right) + 2s S_2^{(PF)}(f^2, f \Lambda) = 0,$$  \hspace{1cm} (75)$$

where

$$S_2^{(PF)}(f^2, f \Lambda) = f^2 S_2^{(EH-quartic)} + f \Lambda \int d^Dx \left( \varepsilon_{\mu\nu} h_{\mu\nu} - \frac{1}{2} h^2 \right),$$  \hspace{1cm} (76)$$

with $S_2^{(EH-quartic)}$ the second-order Einstein-Hilbert deformation, including the quartic vertex of the Einstein-Hilbert Lagrangian. On the other hand, direct computation based on (71) leads to

$$\left( S_1^{(int)}, S_1^{(int)} \right)^{(PF)} = -2s \int d^Dx \left[ y_2^2 \frac{(D-2)^2}{4} \left( h^2 - \varepsilon_{\mu\nu} h_{\mu\nu} \right) + y_2 y_3 \left( D - 2 \right) \delta_3 \epsilon_{\mu\nu} \left( \partial^{[\nu} h^{\rho]}_{\mu\lambda} \right) h^{\mu}_{\lambda} + y_3^2 \delta_3 \epsilon_{\mu\nu} \left( \partial^{[\nu} h^{\rho]}_{\mu\lambda} \right) h_{[\nu} h_{\rho]} \right] \equiv -2s \left( S_2^{(PF)}(y_2^2) + S_2^{(PF)}(y_2 y_3) + S_2^{(PF)}(y_3^2) \right),$$  \hspace{1cm} (77)$$

where we used the obvious notations

$$S_2^{(PF)}(y_2^2) = y_2^2 \frac{(D-2)^2}{4} \int d^Dx \left( h^2 - \varepsilon_{\mu\nu} h_{\mu\nu} \right),$$  \hspace{1cm} (78)$$

$$S_2^{(PF)}(y_2 y_3) = y_2 y_3 \left( D - 2 \right) \delta_3 \epsilon_{\mu\nu} \int d^Dx \left( \partial_{[\nu} h^{\rho]}_{\mu\lambda} \right) h^{\mu}_{\lambda},$$  \hspace{1cm} (79)$$

$$S_2^{(PF)}(y_3^2) = y_3^2 \delta_3 \epsilon_{\mu\nu} \int d^Dx \left( \partial^{[\nu} h^{\rho]}_{\mu\lambda} \right) h_{[\nu} h_{\rho]} \lambda.$$  \hspace{1cm} (80)$$

Taking into account relations (75)–(77) it follows that (72) becomes equivalent with

$$s \left[ S_2^{(PF)} - \left( S_2^{(PF)}(f^2, f \Lambda) + S_2^{(PF)}(y_2^2) + S_2^{(PF)}(y_2 y_3) + S_2^{(PF)}(y_3^2) \right) \right] = 0,$$  \hspace{1cm} (81)$$
which allows us to determine the component \( S_2^{(PF)} \) from the second-order deformation (74), up to trivial, \( s \)-exact contributions 3, in the form
\[
S_2^{(PF)} = S_2^{(PF)} (f^2, fA) + S_2^{(PF)} (y^2_2) + S_2^{(PF)} (y^2_2y_3) + S_2^{(PF)} (y^2_3). \tag{82}
\]

Next, we pass to equation (73). If we denote by \( \Delta^{(\text{int})} \) and \( b^{(\text{int})} \) the nonintegrated densities of \( 2 \left( S_1^{(PF)}, S_1^{(\text{int})} \right) + \left( S_1^{(\text{int})}, S_1^{(\text{int})} \right) \) and \( S_2^{(\text{int})} \) respectively,
\[
2 \left( S_1^{(PF)}, S_1^{(\text{int})} \right) + \left( S_1^{(\text{int})}, S_1^{(\text{int})} \right) \equiv \int d^D x \Delta^{(\text{int})}, \tag{83}
\]
\[
S_2^{(\text{int})} \equiv \int d^D x b^{(\text{int})}, \tag{84}
\]
then the local form of equation (73) reads as
\[
\Delta^{(\text{int})} = -2sb^{(\text{int})} + \partial \mu n^\mu, \tag{85}
\]
where
\[
gh (\Delta^{(\text{int})}) = 1, \quad gh (b^{(\text{int})}) = 0, \quad gh (n^\mu) = 1, \tag{86}
\]
for some local currents \( n^\mu \). By direct computation, from (70) and (71) we deduce that \( \Delta^{(\text{int})} \) decomposes as
\[
\Delta^{(\text{int})} = \sum_{I=0}^2 \Delta_I^{(\text{int})}, \quad \text{agh} \left( \Delta_I^{(\text{int})} \right) = I, \tag{87}
\]
where
\[
\Delta_2^{(\text{int})} = \gamma \left[ p\eta^* \left( p \left( \partial^\mu \eta \right) \eta^\nu h_{\mu\nu} - (p + f) V^\mu \eta^\nu \partial_{[\mu} \eta_{\nu]} \right) \right] + \partial_\mu w_2^\mu, \tag{88}
\]
\[
\Delta_1^{(\text{int})} = \delta \left[ p\eta^* \left( p \left( \partial^\mu \eta \right) \eta^\nu h_{\mu\nu} - (p + f) V^\mu \eta^\nu \partial_{[\mu} \eta_{\nu]} \right) \right]
+ \gamma \left\{ p^2 V^* \left[ \partial_{[\nu} V_{\mu]} h^\nu \rho \eta^\rho + \frac{1}{2} \left( \partial_{[\mu} h_{\nu]} \rho \right) V^\nu \eta^\rho \right\},
\]
\[\]
\[3\text{Strictly speaking, we must add to (52) the nontrivial solution } F \text{ to the homogeneous equation } sF = 0. \text{ However, this solution brings nothing new and can always be absorbed into the full deformed solution to the master equation } S \text{ (actually in } S_1^{(PF)} \text{) through a convenient redefinition of the coupling constant and of the other constants that parameterize } S_1^{(PF)}. \text{ For instance, see Section 7 from [14].} \]
\[-\frac{1}{4} V^\nu h_{\mu}^\rho (\partial_\nu \eta_\rho) - \frac{1}{4} V^\nu (\partial_\rho \eta_{\mu}) h_{\nu}^\rho - \frac{3}{4} h_{\mu}^\nu h_{\nu}^\rho \partial_\rho \eta \]

\[
+ \frac{1}{2} p(p + f) V^{*\mu} V^\nu \left[(\partial_\mu h_{\rho \nu}) + \partial_{(\rho \nu} h_{\mu)\nu}\right) \eta^\rho - h_{\mu}^\rho \partial_\nu \eta_\rho \\
- h_{\nu}^\rho \partial_\mu \eta_\rho - y_3 \delta_3^D \varepsilon_{\mu \nu \rho} V^{*\mu} \left[f h^\nu \chi \iota^{\rho \eta \lambda} + (2 p + f) \eta_\lambda \iota^{\nu \rho \lambda}\right] \\
+p y_2 V^{*\nu} \left[(D - 2) h_{\mu \eta}^\nu - V_\mu \eta\right] - y_2 h^{*\nu \mu} \left[f (h_{\mu \eta} + 2 V_{\mu \eta})\right] \\
- 2(p + f) \left[\sigma_{\mu \nu} V^\rho \eta_\rho\right] - p(p + f) \frac{p - f}{p} F_{\mu \nu}^\rho \eta^\rho \partial_\rho \eta_\lambda \\
+ (2p + f) V^{*\nu} \left[y_3 \delta_3^D \varepsilon_{\mu \nu \rho} \left(\partial_\nu \eta \lambda\right) \iota^{\rho \eta \lambda} \sigma_\lambda \tau\right] \\
+y_2 (D - 2) (\partial_\mu \eta_\rho) \eta_\nu] + \partial_\mu \eta_{\nu}, \quad (89) \]

\[
\Delta_0^{(\text{int})} = \delta \left\{ p^2 V^{*\mu} \left[(\partial_\mu V_\nu) h^\nu \rho \eta_\rho + \frac{1}{2} (\partial_\mu h_{\nu \rho}) V^\nu \rho \eta^\rho - \frac{1}{4} V^\nu h_{\mu}^\rho (\partial_\nu \eta_\rho) \right] \\
- \frac{1}{4} V^\nu (\partial_\rho \eta_\mu) h_{\nu}^\rho - \frac{3}{4} h_{\mu}^\nu h_{\nu}^\rho \partial_\rho \eta_\rho] \right. \\
+ \frac{1}{2} p(p + f) V^{*\mu} V^\nu \left[(\partial_\mu h_{\rho \nu}) + \partial_{(\rho \nu} h_{\mu)\nu}\right) \eta^\rho - h_{\mu}^\rho \partial_\nu \eta_\rho - h_{\nu}^\rho \partial_\mu \eta_\rho \right\} \\
+ \gamma \left\{ \frac{p^2}{8} \left[V_{\mu} ((\partial_\mu h_{\nu \rho}) (\partial_\nu h_{\rho \lambda}) \right. \left. V_{\lambda} - 2 (\partial_\mu h_{\nu \rho}) h_{\lambda \mu} (\partial_\nu V_{\lambda}) \right] \\
+ h_{\mu}^\rho \left( \partial_\rho V_{\nu} \right) h_{\lambda \mu} \left( \partial_\nu V_{\lambda} \right) + F_{\mu \nu} h_{\rho}^\lambda \left( h_{\lambda \mu} \left( \partial_\nu V_{\rho} \right) - \partial_{(\mu h_{\rho \lambda})} V_{\rho} \right) \\
+ F_{\mu \nu} h_{\rho}^\lambda \left( \partial_\nu h_{\rho \lambda} \right) V_{\lambda} \right\} + p^2 F_{\mu \nu} \left[F_{\mu \rho} h_{\nu}^\lambda h_{\lambda \rho} + \frac{1}{16} F_{\mu \nu} (h^2 - 2 h_{\rho \lambda} h_{\rho \lambda}) \right] \\
- h_{\nu}^\rho \left( \left( \partial_\mu h_{\rho \lambda} \right) V_{\lambda} - h_{\rho \nu} \left( \partial_\mu V_{\lambda} \right) \right) + \frac{1}{2} \left( F_{\rho \nu} h_{\mu \rho} h_{\nu \lambda} - F_{\mu \rho} h_{\nu \rho} h_{\nu \lambda} \right) \\
+ \frac{1}{4} \left( \left( \partial_\mu h_{\nu \rho} \right) V_{\rho} - h_{\mu}^\rho \left( \partial_\nu V_{\rho} \right) \right) h_{\nu}^\rho + \frac{1}{4} p(p + f) (F_{\mu \nu}^\rho F_{\nu \rho} \right\} \\
+ \frac{1}{4} \delta_3^D F_{\mu \nu} \left[h_{\alpha \nu} h_{\rho}^\sigma + q_1 \delta_3^D \varepsilon_{\mu \nu \rho \lambda} (h V_{\mu} F_{\nu \lambda} - 2 h_{\lambda}^\alpha V_{\mu} F_{\nu \alpha}) \right] \\
+ h_{\mu}^\alpha V_{\nu} F_{\nu \alpha} + q_2 \delta_3^D \varepsilon_{\mu \nu \rho \lambda} (h V_{\mu} F_{\nu \lambda} F_{\nu \alpha} - 4 h_{\nu}^\alpha V_{\mu} F_{\nu \lambda} F_{\nu \alpha} \right\} \\
+ 2 h_{\mu}^\alpha V_{\nu} F_{\nu \lambda} F_{\nu \alpha} - 16 q_1 \delta_3^D V_{\nu} \left[\partial_\nu h_{\rho \lambda} \right. \left. - (D - 2) (D - 1) y_3^2 V_{\nu} V_{\mu} \right] \\
- 4 q_1 y_2 \delta_3^D (D - 2) \varepsilon_{\mu \nu \rho \lambda} F_{\mu \nu}^\rho \eta_\rho - 6 q_2 y_3 \delta_3^D \varepsilon_{\mu \nu \rho \lambda} F_{\mu \nu}^\rho \eta_\rho \right\} \\
+ \frac{1}{2} p(p + f) \left(F_{\mu \nu}^\rho F_{\nu \rho} + \frac{1}{4} \delta_3^D F_{\mu \nu} \right) \left[h_{\rho}^\sigma \partial_\mu \eta_\nu - 2 \partial_\mu h_{\rho \sigma} \eta_\nu \right] \\
+ y_2 \left[ f A_0^{(\text{int})} \left( \partial \partial_\Phi \Phi \partial_\Phi \right) \right. \left. + p B_0^{(\text{int})} \left( \partial \partial_\Phi \Phi \partial_\Phi \right) - 4 D \Lambda \eta \right] \]
\[ + y_0 \delta^D \left[ f C_0^{(\text{int})} \left( \partial \partial \Phi^{\alpha_0} \Phi^{\delta_0} \eta_{\alpha_1} \right) + p D_0^{(\text{int})} \left( \partial \partial \Phi^{\alpha_0} \Phi^{\delta_0} \eta_{\alpha_1} \right) \right] + \partial_{\mu} w_{\mu}^{(90)} \] 

In (90) \( A_0^{(\text{int})}, B_0^{(\text{int})}, C_0^{(\text{int})}, \) and \( D_0^{(\text{int})} \) are linear in their arguments; for instance the notation \( A_0^{(\text{int})} \left( \partial \partial \Phi^{\alpha_0} \Phi^{\delta_0} \eta_{\alpha_1} \right) \) means that each term from \( A_0^{(\text{int})} \) contains two spacetime derivatives and is simultaneously quadratic in the fields \( \Phi^{\alpha_0} \) from (3) and linear in the ghosts \( \eta_{\alpha_1} \) from (4).

Replacing decomposition (87) into equation (85) and using (5), one can assume, without loss of generality, that \( b^{(\text{int})} \) and \( n^{\mu} \) stop at antighost number three: \( b^{(\text{int})} = \sum_{I=0}^{3} b_I^{(\text{int})} \), \( n^{\mu} = \sum_{I=0}^{3} n_I^{\mu} \). However, it can be shown in a direct manner (based on the result \( H_{3}^{\text{inv}} ( \delta | d ) = 0 \)) that one can take \( b_3^{(\text{int})} = 0 \), so we can work with

\[ b^{(\text{int})} = \sum_{I=0}^{2} b_I^{(\text{int})}, \quad \text{agh} (b_I^{(\text{int})}) = I, \quad (91) \]

\[ n^{\mu} = \sum_{I=0}^{2} n_I^{\mu}, \quad \text{agh} (n_I^{\mu}) = I. \quad (92) \]

The above expansions inserted into equation (85) produce the equivalent equations

\[ \Delta_2^{(\text{int})} = -2 \gamma b_2^{(\text{int})} + \partial_{\mu} n_2^{\mu}, \quad (93) \]

\[ \Delta_1^{(\text{int})} = -2 \left( \delta b_2^{(\text{int})} + \gamma b_1^{(\text{int})} \right) + \partial_{\mu} n_1^{\mu}, \quad (94) \]

\[ \Delta_0^{(\text{int})} = -2 \left( \delta b_1^{(\text{int})} + \gamma b_0^{(\text{int})} \right) + \partial_{\mu} n_0^{\mu}. \quad (95) \]

At this stage it is useful to make the notations

\[ b_2^{(\text{int})} = -\frac{1}{2} p \eta^\nu \left[ p ( \partial^\mu \eta ) h_{\mu \nu} - ( p + f ) V^\nu \eta^\nu \partial_{\mu} \eta_\alpha \right] + \bar{b}_2^{(\text{int})}, \quad (96) \]

\[ b_1^{(\text{int})} = -\frac{1}{2} p^2 \left( V^{\nu} \partial_{\mu} V_\nu \right) \eta^\rho + \frac{1}{2} \left( \partial_{[\mu} h_{\nu \rho]} \right) V^\nu \eta^\rho \]

\[ - \frac{1}{4} V^\nu h_{[\mu}^\rho \partial_{\nu] \eta_\rho} - \frac{1}{4} V^\nu \left( \partial_{\rho} \eta_{[\mu} \right) h_{\nu]}^\rho - \frac{3}{4} h_{\nu}^\rho V^\nu \partial_{\rho} \eta \]

\[ - \frac{1}{4} p ( p + f ) V^{\nu} V_\nu \left[ ( \partial_{[\mu} h_{\rho]}^\nu + \partial_{[\nu} h_{\rho] \mu} ) \eta^\rho - h_{\mu}^\rho \partial_{\nu} \eta_\rho \right] \]

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\[
-b_{\mu}^\rho \partial_{\mu} \eta_\rho + \frac{1}{2} y_3 \delta^D \varepsilon_{\mu \rho} V^{\nu \mu} \left[ f h^{\nu} \lambda \partial^\rho \eta^\lambda \right] + (2 p + f) \eta_\lambda \partial^{[\nu} h^{\rho] \lambda} \\
- \frac{1}{2} p y_2 V^{\nu \mu} \left[ (D - 2) h_{\mu \nu} \eta_\rho - V_\mu \eta_\rho \right] + \frac{1}{2} y_2 h^{\nu \mu} \left[ f (h_{\mu \eta} + 2 V_\mu \eta_\nu) \right] \\
- 2 (p + f) \sigma_{\mu \nu} V^{\rho \eta_\rho} - \frac{8}{D - 2} y_3 q_1 \delta^D h^{* \rho} + \tilde{b}_1^{(\text{int})},
\]

(97)

\[
b_0^{(\text{int})} = -\frac{p^2}{16} \left[ \left( \partial^{[\nu} h^{\rho]} \right) \left( \partial_{[\mu} h_{\nu]} \right) \right] V^\lambda - 2 \left( \partial^{[\nu} h^{\rho]} \right) h_{\lambda \mu} \left( \partial_{[\nu} V^\lambda \right) \\
+ h_{\rho} \left[ \partial^{[\nu} V^\rho \right] h_{\lambda \mu} \left( \partial_{[\nu} V^\lambda \right) + F^{\mu \nu} h_{\rho} \left( h^\lambda \left[ \mu \right] \left( \partial_{[\nu} V^\lambda \right) - \left( \partial_{[\nu} h^{\lambda \nu} \right) V_{\rho} \right) \\
+ F^{\mu \nu} h_{\rho} \left( \partial_{[\nu} V_{\rho} \right) V_{\lambda} \right] - \frac{1}{2} p^2 F^{\mu \nu} \left[ F_{\mu \rho} h_{\nu} \left( \partial_{\lambda} h_{\rho} \right) \right] + \frac{16}{16} F^{\mu \nu} \left( h^2 - 2 h^{\rho \lambda} h_{\rho \lambda} \right) \\
- h_{\nu} \left( \left( \partial_{[\nu} h_{\rho]} \right) \right) V_{\lambda} - h_{\nu} \left( \partial_{[\nu} V^\lambda \right) \right] + \frac{1}{2} \left( \partial^{[\nu} h_{\rho]} \right) \left( \partial_{[\nu} V^\lambda \right) \right] - \frac{1}{8} (p + f) \left( F^{\mu \nu} F_{\nu \rho} \right) \\
+ \frac{1}{4} \left( \left( \partial_{[\nu} h_{\rho]} \right) \right) \left( \partial_{[\nu} V_{\rho} \right) \left( \partial_{[\nu} V^\lambda \right) \right] - \frac{1}{8} (p + f) \left( F^{\mu \nu} F_{\nu \rho} \right) \\
+ \frac{1}{4} \delta_{\rho}^\mu F^{\nu \lambda} V_{\nu} \left( \partial_{[\nu} h_{\rho]} \right) h_{\mu \nu} h^{\sigma \rho} - \frac{1}{2} q_1 p \delta^D \varepsilon_{\mu \nu \lambda} \left( h V_{\mu} F_{\nu \lambda} - 2 h_{\lambda} \alpha V_{\mu} F_{\nu \alpha} \right) \\
+ h_{\mu} \alpha V_{\alpha} F_{\nu \lambda} \right] - \frac{1}{2} q_2 p \delta^D \varepsilon_{\mu \nu \lambda} \left( h V_{\mu} F_{\nu \lambda} F_{\alpha \beta} - 4 h_{\beta} \alpha V_{\mu} F_{\nu \lambda} F_{\alpha \beta} \right) \\
+ 2 h_{\mu} \alpha V_{\alpha} F_{\nu \lambda} F_{\alpha \beta} \right) + 8 y_3 q_1 \delta^D V_{\nu} \partial^{[\nu} h_{[\rho]} \rho + \frac{1}{2} \left( D - 2 \right) \left( D - 1 \right) y_2^2 V_{\nu} V^\mu \\
+ \tilde{b}_0^{(\text{int})}.
\]

(98)

Using the above notations and recalling the expressions (88)–(90) of $\Delta^{(\text{int})}_{\mu}$, equations (93)–(95) (equivalent to (85)) become

\[
\tilde{b}_2^{(\text{int})} = \partial_{\mu} \rho_2^\mu, \\
\delta \tilde{b}_2^{(\text{int})} + \gamma \tilde{b}_1^{(\text{int})} = \partial_{\mu} \rho_1^\mu + \frac{1}{2} \chi_1, \\
\delta \tilde{b}_1^{(\text{int})} + \gamma \tilde{b}_0^{(\text{int})} = \partial_{\mu} \rho_0^\mu + \frac{1}{2} \chi_0,
\]

(99) \hspace{1cm} (100) \hspace{1cm} (101)

where

\[
\rho_1^\mu = \frac{1}{2} \left( w_1^\mu - n_1^\mu \right), \quad I = 0, 2,
\]

\[
\chi_1 = V^{\nu \mu} \left( - p (p + f) F_{\mu \nu} \eta_\rho \partial^{[\rho \eta} \eta_{\nu]} \right)
\]

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\[
\chi_0 = \delta \left\{ y_3 \delta_3^D \varepsilon_{\mu\nu\rho} V_{\ast\mu} \left( \partial^{[\nu} \eta^{\lambda]} \right) \partial^{[\rho} \eta^{\gamma]} \sigma_{\lambda\tau} + y_2 (D - 2) \left( \partial_{[\mu} \eta_{\nu]} \right) \eta^{\nu} \right\},
\]
\[
\chi_1 = \delta (y_3 \delta_3^D \varepsilon_{\mu\nu\rho} V_{\ast\rho} F_{\nu\sigma} \eta^{\alpha} + \frac{1}{2} \eta (p + f) (F_{\mu\sigma} F_{\nu}) + \frac{1}{4} \delta_3^\alpha \delta_3^\beta \delta_3^\gamma \partial_{[\alpha} \eta_{\beta]} - 2 \partial_{[\alpha} \eta_{\beta]} \partial_{[\gamma]} \eta_{\sigma]} \sigma^\sigma\rho \sigma) + y_2 \left( f A_{0}^{\text{int}} \left( \partial \partial \Phi^{\alpha} \Phi^\beta \eta_{\alpha} \right) + p B_{0}^{\text{int}} \left( \partial \partial \Phi^{\alpha} \Phi^\beta \eta_{\alpha} \right) - 4 D \Lambda \eta \right) + y_3 \delta_3^D \left( f C_{0}^{\text{int}} \left( \partial \partial \Phi^{\alpha} \Phi^\beta \eta_{\alpha} \right) + p D_{0}^{\text{int}} \left( \partial \partial \Phi^{\alpha} \Phi^\beta \eta_{\alpha} \right) \right)\right].
\] (104)

One can replace (99) with
\[
\gamma \tilde{b}_2^{\text{int}} = 0,
\] (105)
such that (85) is in fact equivalent to (105) and (100)–(101). So far, we have shown that the second-order deformation of the solution to the master equation, (74), is completely known once we manage to solve equations (105) and (100)–(101). This is our next concern.

From (100) we obtain a necessary condition for the existence of \( \tilde{b}_2^{\text{int}} \) and \( \tilde{b}_1^{\text{int}} \), namely
\[
\chi_1 = \delta \phi_2 + \gamma \omega_1 + \partial \mu l^\mu_1,
\] (106)
where \( \text{agh} (\phi_2) = 2 = \text{pgh} (\phi_2), \text{agh} (\omega_1) = 1 = \text{pgh} (\omega_1), \text{agh} (l^\mu) = 1, \text{pgh} (l^\mu) = 2 \). It is essential to remark that all the functions \( \phi_2, \omega_1, \) and \( l^\mu \) must be local since otherwise we cannot obtain local second-order deformations from (100). Assuming (106) holds, we act with \( \delta \) on it and use its nilpotency and its anticommutation with \( \gamma \), which yields
\[
\delta \chi_1 = \gamma (-\delta \omega_1) + \partial \mu (\delta l^\mu_1).
\] (107)
Without entering technical details, we mention that the validity of (107) can be checked by means of standard cohomological arguments. In fact, after direct manipulations of (103), it can be shown that (107) (and thus also (106)) requires that the following conditions are simultaneously satisfied:
\[
F_{\mu\sigma} F_{\nu} = \frac{1}{4} \delta_3^\alpha \delta_3^\beta \delta_3^\gamma \partial_{[\alpha} \eta_{\beta]} - 2 \partial_{[\alpha} \eta_{\beta]} \partial_{[\gamma]} \eta_{\sigma]} \sigma^\sigma\rho \sigma) + \delta \Lambda \eta \right) + y_3 \delta_3^D \left( f C_{0}^{\text{int}} \left( \partial \partial \Phi^{\alpha} \Phi^\beta \eta_{\alpha} \right) + p D_{0}^{\text{int}} \left( \partial \partial \Phi^{\alpha} \Phi^\beta \eta_{\alpha} \right) \right)\right].
\] (104)

One can replace (99) with
\[
\gamma \tilde{b}_2^{\text{int}} = 0,
\] (105)
such that (85) is in fact equivalent to (105) and (100)–(101). So far, we have shown that the second-order deformation of the solution to the master equation, (74), is completely known once we manage to solve equations (105) and (100)–(101). This is our next concern.

From (100) we obtain a necessary condition for the existence of \( \tilde{b}_2^{\text{int}} \) and \( \tilde{b}_1^{\text{int}} \), namely
\[
\chi_1 = \delta \phi_2 + \gamma \omega_1 + \partial \mu l^\mu_1,
\] (106)
where \( \text{agh} (\phi_2) = 2 = \text{pgh} (\phi_2), \text{agh} (\omega_1) = 1 = \text{pgh} (\omega_1), \text{agh} (l^\mu) = 1, \text{pgh} (l^\mu) = 2 \). It is essential to remark that all the functions \( \phi_2, \omega_1, \) and \( l^\mu \) must be local since otherwise we cannot obtain local second-order deformations from (100). Assuming (106) holds, we act with \( \delta \) on it and use its nilpotency and its anticommutation with \( \gamma \), which yields
\[
\delta \chi_1 = \gamma (-\delta \omega_1) + \partial \mu (\delta l^\mu_1).
\] (107)
Without entering technical details, we mention that the validity of (107) can be checked by means of standard cohomological arguments. In fact, after direct manipulations of (103), it can be shown that (107) (and thus also (106)) requires that the following conditions are simultaneously satisfied:
\[
F_{\mu\sigma} F_{\nu} + \frac{1}{4} \delta_3^\alpha \delta_3^\beta \delta_3^\gamma \partial_{[\alpha} \eta_{\beta]} - 2 \partial_{[\alpha} \eta_{\beta]} \partial_{[\gamma]} \eta_{\sigma]} \sigma^\sigma\rho \sigma) + \delta \Lambda \eta \right) + y_3 \delta_3^D \left( f C_{0}^{\text{int}} \left( \partial \partial \Phi^{\alpha} \Phi^\beta \eta_{\alpha} \right) + p D_{0}^{\text{int}} \left( \partial \partial \Phi^{\alpha} \Phi^\beta \eta_{\alpha} \right) \right)\right].
\] (104)

One can replace (99) with
\[
\gamma \tilde{b}_2^{\text{int}} = 0,
\] (105)
\[ F^{\theta \mu} = \delta \bar{\Omega}^{\theta \mu}, \quad (109) \]
\[ \partial_{[\mu} h_{\lambda]}^{\theta} = \delta \Omega^{\theta}_{\mu \lambda}, \quad (110) \]
\[ (\partial_{[\theta h_{\nu]}^{\theta]} \partial^{[\mu h^{\nu\rho]}} - (\partial_{\nu h^{\theta \rho]}^{[\mu} \partial_{\theta h_{\rho]}}^{\mu} = \delta \Omega. \quad (111) \]

All the quantities denoted by \( \Omega \) or \( \bar{\Omega} \) must be local; their locality is essential in obtaining local deformations, which is one of the main working hypotheses of our paper. One can explicitly reveal locality obstructions to each of these conditions. For instance, assuming that equation (108) is satisfied for some local \( \Omega^{\mu \rho}_{\theta} \) and taking its divergence, it follows that the relation
\[ \partial_{\mu} \left( F^{\mu \nu} F_{\nu \rho} + \frac{1}{4} \delta^{\mu \rho}_{F^\nu F^{\nu \lambda}} \right) = \delta \left( \partial_{\mu} \Omega^{\mu \rho}_{\theta} \right) \quad (112) \]
should also take place. On the other hand, it is easy to see that
\[ \partial_{\mu} \left( F^{\mu \nu} F_{\nu \rho} + \frac{1}{4} \delta^{\mu \rho}_{F^\nu F^{\nu \lambda}} \right) = \delta \left( -V^{\nu \nu} F_{\nu \rho} \right). \quad (113) \]
Since \(-V^{\nu \nu} F_{\nu \rho}\) obviously is not a divergence of a local function, equation (112) cannot hold for some local \( \Omega^{\mu \rho}_{\theta} \), so neither does (108). Acting in a similar manner with respect to equation (109), we infer \( \partial_{\mu} F^{\theta \mu} = \delta V^{* \theta} \neq \delta \left( \partial_{\mu} \Omega^{\theta \mu}_{\theta} \right) \), such that (109) cannot be satisfied for some local \( \bar{\Omega}^{\theta \mu} \). Related to (110), if we apply \( \partial^{\mu} \) on it and then take its trace, we obtain
\[ \partial^{\mu} \partial_{\mu h_{\lambda]}^{\lambda} = \delta \left( \frac{h^{*}}{2} \right) \neq \delta \left( \partial^{\mu} \Omega^{\lambda}_{\mu \lambda} \right), \]
and hence (110) is not valid for some local \( \Omega^{\theta \lambda}_{\mu \lambda} \). Concerning equation (111), it can be shown directly that its left-hand side reads as
\[ \delta \left( -h_{\mu \nu} h^{* \nu \mu} \right) + \partial_{\mu} u^{\mu} \right), \] with \( \partial_{\mu} u^{\mu} \neq 0 \) and \( u^{\mu} \neq \delta u^{\mu}_{1} \) for some local \( u^{\mu}_{1} \), so (111) also fails to be true. Combining these last results, it follows that (107) (and hence also (106)) cannot hold locally unless \( \chi_{1} = 0 \), which yields
\[ p (p + f) = 0, \quad (114) \]
\[ (2p + f) y_{3} \delta_{D}^{1} = 0, \quad (115) \]
\[ (2p + f) y_{2} = 0. \quad (116) \]

There are three relevant solutions to the above equations:\(^4\)

\[ \text{Case I : } p = -f \neq 0, \quad y_{2} = 0 = y_{3} \delta_{D}^{1}, \quad D > 2, \quad (117) \]

\(^4\)By ‘relevant solution’ we mean that the resulting deformations lead to a maximum number of consistent couplings and gauge symmetries. For instance, another possible solution to (117)(119) is \( p = 0, f \neq 0, y_{2} = 0, y_{3} \delta_{D}^{1} = 0 \). This case is not relevant since it would mean to allow the Einstein-Hilbert selfinteractions of the graviton, but forbid: (i) the standard couplings graviton-photon and (ii) the diffeomorphism sector of the vector field gauge symmetries prescribed by General Relativity.
Case II: \( p = f = 0, \quad D = 3 \) \hspace{1cm} (118)

Case III: \( p = f = 0, \quad D > 3 \) \hspace{1cm} (119)

which require an individual treatment.

4.3.1 Case I — General Relativity

According to (117), the first-order deformation (69) is parameterized in this situation by four real constants, namely, \( f, \Lambda, q_1, \delta_3^D \), and \( q_2 \delta_5^D \). For the sake of simplicity we set \( f = 1 \), so \( p = -1 \), such that the \( S_1 \) (see (70) with the components (40), (41), and (42) plus (71)) takes the concrete form

\[
S_1^{(d)} = S_1^{(PF)} + S_1^{(int)} = \int d^D x \left\{ \frac{1}{2} \eta^\mu \eta^\nu \partial_\mu \eta_\nu \right\} + h^{\mu\rho} \left[ (\partial_\rho \eta^\nu) h_{\mu\nu} - \eta^\nu \partial_\nu h_{\nu\rho} \right] + a_0^{(EH-\text{cubic})} - 2\Lambda h \right\} + \int d^D x \left\{ -\eta^\mu \partial^\mu \eta \right. \\
\left. + \frac{1}{2} V^\mu \left[ V^\nu \partial_\mu \eta_\nu \right] + 2 \left( \partial_\nu V^\mu \right) \eta^\nu - h_{\mu\nu} \partial^\nu \eta \right\} - \frac{1}{8} F^{\mu\nu} \left[ 2 \partial_\mu \left( h_{\nu\rho} V^\rho \right) + F_{\mu\nu} h - 4 F_{\mu\rho} h_{\nu} \right] + q_1 \delta_3^D \varepsilon_{\mu\nu\lambda} V_{\mu} F_{\nu\lambda} + q_2 \delta_5^D \varepsilon_{\mu\nu\lambda\beta} V_{\mu} F_{\nu\lambda} F_{\alpha\beta} \right\}. \hspace{1cm} (120)
\]

Replacing (117) into (103) and (104), we find that

\( \chi_1 = 0, \quad \chi_0 = 0 \), \hspace{1cm} (121)

such that equations (105) and (100)–(101) become

\[
\gamma \bar{b}_2^{(\text{int})} = 0, \hspace{1cm} (122)
\]

\[
\delta \bar{b}_2^{(\text{int})} + \gamma \bar{b}_1^{(\text{int})} = \partial_\mu \rho_1^\mu, \hspace{1cm} (123)
\]

\[
\delta \bar{b}_1^{(\text{int})} + \gamma \bar{b}_0^{(\text{int})} = \partial_\mu \rho_0^\mu. \hspace{1cm} (124)
\]

These equations have already been considered in Section 4.2 at the construction of the first-order deformation, so their solutions can be absorbed into \( S_1^{(\text{int})} \) from (120) by a suitable redefinition of the constants \( p, q_1, \) and \( q_2 \). In conclusion, we can work with

\[
\bar{b}_2^{(\text{int})} = 0, \quad \bar{b}_1^{(\text{int})} = 0, \quad \bar{b}_0^{(\text{int})} = 0. \hspace{1cm} (125)
\]
Inserting the previous results together with (117) for \( f = 1 \) in (96), (97), and (88) and then the resulting expressions in (91), we complete the interacting component \( S_{2}^{(\text{int})} \) from the second-order deformation of the solution to the master equation, in agreement with notation (84). Particularizing (76) and (78)–(80) to the case (117) for \( f = 1 \), we also infer \( S_{2}^{(\text{PF})} \) with the help of relation (76). Putting together these expressions of \( S_{2}^{(\text{int})} \) and \( S_{2}^{(\text{PF})} \) via formula (74), we can state that the full second-order deformation to the master equation in case I reads as

\[
S_{2}^{(\text{I})} = S_{2}^{(\text{PF})} + S_{2}^{(\text{int})}
\]

\[
\equiv \left[ S_{2}^{(\text{EH--quartic})} + \Lambda \int d^{D}x \left( h_{\mu\nu} h_{\mu\nu} - \frac{1}{2} h^{2} \right) \right] - \frac{1}{2} \int d^{D}x \{ \eta^{*} (\partial^{\mu} \eta) \eta^{\nu} h_{\mu\nu}
\]

\[
+ V^{*\mu} \left[ (\partial_{\nu} V_{\mu}) h^{\nu} \rho \eta^{\rho} + \frac{1}{2} (\partial_{\rho} h_{\nu} \rho) V^{\nu} \eta^{\rho} - \frac{1}{4} V^{\nu} h_{\nu}^{\rho} (\partial_{\nu} \eta_{\rho})
\]

\[
- \frac{1}{4} V^{\nu} (\partial_{\rho} \eta_{\mu}) h_{\nu}^{\rho} - \frac{3}{4} h_{\mu}^{\nu} h_{\nu}^{\rho} \partial_{\nu} \eta_{\rho} \right] + \frac{1}{8} \left[ F_{\mu\nu} h_{\rho}^{\lambda} (h^{\lambda} \mu \eta_{\nu}) \right.
\]

\[
- (\partial_{\nu} h_{\rho}^{\lambda}) V_{\nu} + F_{\mu\nu} h_{\rho}^{\lambda \nu} (\partial_{\rho} h_{\nu}^{\lambda}) V_{\lambda} + F_{\mu\nu} (\partial_{\nu} h_{\rho}^{\lambda}) (h_{\rho}^{\lambda \nu} \eta_{\rho})
\]

\[
- 2 (\partial_{\nu} h_{\rho}^{\lambda}) V_{\lambda} + F_{\mu\nu} \left[ F_{\rho}^{\lambda \mu} (\partial_{\lambda} V_{\mu}) + h_{\rho}^{\mu \nu} (\partial_{\nu} V_{\rho}) \right] h_{\lambda \nu} \eta_{\nu} \right]
\]

\[
+ F_{\mu\nu} \left[ F_{\nu}^{\lambda \rho} h_{\lambda}^{\rho} + \frac{1}{2} (F_{\rho}^{\lambda \mu} h_{\nu}^{\lambda} - F_{\nu}^{\mu} h_{\nu}^{\rho}) h_{\rho}^{\lambda} \right.
\]

\[
+ \frac{1}{16} F_{\mu\nu} (h^{2} - 2 h^{\rho \lambda} h_{\rho \lambda}) - h_{\nu}^{\rho} \left( (\partial_{\rho} h_{\nu}^{\lambda}) V_{\lambda} - h_{\nu}^{\rho} \right) V_{\rho} \right]
\]

\[
+ \frac{1}{4} \left( (\partial_{\rho} h_{\nu}^{\lambda}) V_{\rho} - h_{\nu}^{\rho} (\partial_{\rho} V_{\rho}) \right) h \right] - q_{1} \delta_{3}^{D} \varepsilon^{\mu \nu \lambda} (h V_{\mu} F_{\nu} \lambda)
\]

\[
- 2 h_{\nu}^{\lambda \rho} V_{\mu} F_{\nu \alpha} + h_{\mu}^{\alpha \nu} V_{\lambda} F_{\alpha \lambda} - q_{2} \delta_{5}^{D} \varepsilon^{\mu \nu \lambda \alpha \beta} (h V_{\mu} F_{\nu \lambda} F_{\alpha \beta}
\]

\[
- 4 h_{\beta}^{\nu \lambda} V_{\mu} F_{\nu \lambda} F_{\alpha \rho} + 2 h_{\mu}^{\nu \lambda} V_{\rho} F_{\nu \lambda} F_{\alpha \beta}) \right). \tag{126}
\]

The deformation procedure goes on indefinitely in the sense that it produces an infinite number of nontrivial higher-order components of the deformed solution to the master equation

\[
S^{(n)} \neq 0, \quad \text{for all} \quad n > 0. \tag{127}
\]

Nevertheless, we will see in Section 4.4.1 that the first two deformations derived so far for case I are enough in order to describe the overall deformed theory at all orders in the coupling constant, which turns out to describe
nothing but the standard graviton-vector interactions from General Relativity.

4.3.2 Case II — new solutions in $D = 3$

In this situation we substitute (118) into (103) and (104) and obtain that

$$\chi_1 = 0, \quad \chi_0 = -4y_2 (q_1 \varepsilon_{\mu\nu\rho} F^{\mu\nu\eta^\rho} + 3\Lambda \eta). \quad (128)$$

Thus, from (101) we obtain a necessary condition for the existence of $\bar{b}_1^{(\text{int})}$ and $\bar{b}_0^{(\text{int})}$, namely

$$\chi_0 = \delta \varphi_1 + \gamma \omega_0 + \partial_\mu l_0^\mu, \quad (129)$$

where $\text{agh} (\varphi_1) = 1 = \text{pgh} (\omega_0)$, $\text{agh} (\omega_0) = 0 = \text{pgh} (l_0^\mu)$, $\text{pgh} (l_0^\mu) = 1$. We insist that all the quantities $\varphi_1, \omega_0$, and $l_0^\mu$ from (129) must be local in order to render a local second-order deformation via (101). This is the second place where we analyze the possible obstructions in finding local deformations. It is clear from (128) that $\chi_0$ is a nontrivial element from $H^1 (\gamma)$ of antighost number zero, $\gamma \chi_0 = 0$, since it is written as $\chi_0 = \alpha_{0M} (F_{\mu\nu} \epsilon^{1M})$, where $\alpha_{0M}$ are invariant polynomials not depending on the antifields and $\epsilon^{1M}$ are the elements of a basis in the space of polynomials with pure ghost number one in $\eta_\mu$ and $\eta$. The latter term from the right-hand side of (128) is derivative-free while the non-vanishing actions of $\delta$ and $\gamma$ contain at least one derivative, so it cannot be written as in (129), and, as a consequence, we must require $y_2 \Lambda = 0$. (From the latter definition in (12) we have that $\gamma (\partial^\mu V_\mu) = \Box \eta$, so we can indeed write $\eta = \gamma (\Box^{-1} \partial^\mu V_\mu)$. But $\Box^{-1} \partial^\mu V_\mu$ is not local, so this solution must be discarded.)

Regarding the former term, proportional with $\varepsilon_{\mu\nu\rho} F^{\mu\nu\eta^\rho}$, since $\text{agh} (\varphi_1) = 1$, it follows that $\varphi_1$ is linear in the antifields $\Phi^*_{a_0} = (h^{*\mu\nu}, V^{*\mu})$. On behalf of definitions (8), it would produce in (129) terms with two spacetime derivatives. But $\varepsilon_{\mu\nu\rho} F^{\mu\nu\eta^\rho}$ contains only pieces with at most one derivative, so the locality assumption requires $\varphi_1 = 0$ in (129), such that this becomes

$$-4y_2 q_1 \varepsilon_{\mu\nu\rho} F^{\mu\nu\eta^\rho} = \gamma \omega_0 + \partial_\mu l_0^\mu. \quad (130)$$

From definitions (12) it is clear now that (130) cannot hold for some local $\omega_0$ and $l_0^\mu$. By virtue of the above discussion we must impose $\chi_0 = 0$, which is equivalent with the supplementary conditions

$$y_2 q_1 = 0, \quad y_2 \Lambda = 0, \quad (131)$$

---

5Note that in $D = 3$ we have $q_2 \delta_5^D = 0$. 

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displaying two relevant solutions

\[ \begin{align*}
  y_2 &= 0, \\
  q_1 &= 0 = \Lambda.
\end{align*} \tag{132} \tag{133} \]

Thus, the second case admits two subcases, deserving separate analyses.

**Subcase II.1** results from (118) and (132), so it corresponds to the choice

\[ D = 3, \quad p = f = q_2 \delta^D = y_2 = 0. \tag{134} \]

We observe that the deformations lie in three spacetime dimensions and are parameterized by three constants, namely \( \Lambda, y_3, \) and \( q_1 \). Under these circumstances, the first-order deformation \( S_1 \) (see (70) with the components (10), (11), and (12) plus (71), all particularized to (134)) is expressed by

\[
S_1^{(II.1)} = S_1^{(PF)} + S_1^{(int)} = -2\Lambda \int d^3x h + \int d^3x \varepsilon_{\mu \nu \rho} \left[ y_3 \left( V^* \partial^\nu h^\rho \right) + F^{\lambda \mu} \partial^{[\nu} h^{\rho]} \lambda \right] + q_1 V^\mu F^{\nu \rho}. \tag{135} \]

Substituting relations (134) into (103) and (104), we find that

\[ \chi_1 = 0, \quad \chi_0 = 0, \tag{136} \]

so the discussion from subsection 4.3.1 applies here as well and we can take

\[ \bar{b}_2^{(int)} = 0, \quad \bar{b}_1^{(int)} = 0, \quad \bar{b}_0^{(int)} = 0 \tag{137} \]

in (96), (97), and (98). Consequently, with the help of formulas (74), (82) (with the components 78–80), (84), and (91) (with the components 96–98) written in the presence of conditions (134) and (137) we determine the second-order deformation in the form

\[
S_2^{(II.1)} = S_2^{(PF)} + S_2^{(int)} = y_3^2 \int d^3x \left( \partial^{[\nu} h^{\rho]} \lambda \right) \partial_{[\nu} h^{\rho]} \lambda + 8y_3q_1 \int d^3x \left( h^* \eta + V_\nu \partial^{[\nu} h^{\rho]} \lambda \right). \tag{138} \]

Next, we approach the consistency of \( S_2^{(II.1)} \), i.e. we solve the equation introducing the third-order deformation of the solution to the master equation

\[
\left( S_1^{(II.1)}, S_2^{(II.1)} \right) + sS_3^{(II.1)} = 0. \tag{139} \]

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By direct computation we obtain

\[
\left( S_1^{(II.1)}, S_2^{(II.1)} \right) = s \left( 4y_3^2q_1 \int d^3x \varepsilon_{\mu\nu\rho} h_\mu^\chi \partial^{[\nu} h^{\rho]}_\chi \right) + 48\Lambda y_3 q_1 \int d^3x \eta. \tag{140}
\]

Substituting the last result into (139) we arrive at

\[
s \left( S_3^{(II.1)} + 4y_3^2q_1 \int d^3x \varepsilon_{\mu\nu\rho} h_\mu^\chi \partial^{[\nu} h^{\rho]}_\chi \right) + 48\Lambda y_3 q_1 \int d^3x \eta = 0. \tag{141}
\]

The last equation possesses local solutions if and only if the integrand of the last term from the left-hand side of (141) is written in a \(s\)-exact modulo \(d\) form from local functions. We discussed a similar term in the beginning of Section 4.3.2 (see the second term on the right-hand side of (128) and equation (129)) and concluded that it cannot be written in a \(s\)-exact modulo \(d\) form from local functions until its coefficient vanishes. Then, we can state that (141) holds if and only if

\[
\Lambda y_3 q_1 = 0. \tag{142}
\]

The relevant solutions to the above equation are

\[
y_3 \neq 0, \quad \Lambda \neq 0, \quad q_1 = 0, \tag{143}
\]

\[
y_3 \neq 0, \quad q_1 \neq 0, \quad \Lambda = 0. \tag{144}
\]

Thus, the first subcase from case II splits again into two complementary situations.

In **subcase II.1.1**, where (134) and (143) hold simultaneously,

\[
D = 3, \quad p = f = q_2 \delta_5^D = y_2 = q_1 = 0, \quad y_3 \neq 0, \quad \Lambda \neq 0, \tag{145}
\]

we have that the deformed solution to the master equation is parameterized by two constants, \(y_3\) and \(\Lambda\). Its first two components result from (135) and (138) where we set \(q_1 = 0\) and read as

\[
S_1^{(II.1.1)} = \int d^3x \left[ -2\Lambda h + y_3 \varepsilon_{\mu\nu\rho} \left( V^{*\mu} \partial^{[\nu} \eta^{\rho]} + F^{\lambda\mu} \partial^{[\nu} h^{\rho]}_\lambda \right) \right]. \tag{146}
\]

---

\(6\)The solution \(y_3 = 0\) and \(\Lambda q_1 \neq 0\) yields no couplings: the original gauge transformations [2] are maintained and two gauge-invariant terms are added to the starting Lagrangian [1]: \(-2k\Lambda h\) and \(kq_1 \delta_3^D \varepsilon^{\mu\nu\rho} V_\mu F_{\nu\rho}\).
\[ S_{2}^{(II.1.1)} = y_{3}^{2} \int d^{3}x \left( \partial^{\nu} h^{\rho}{\lambda} \right) \partial_{\nu} h_{\rho}{\lambda}. \] (147)

Consequently, \((S_{1}^{(II.1.1)}, S_{2}^{(II.1.1)}) = 0\), so (139) becomes
\[ sS_{3}^{(II.1.1)} = 0, \] (148)
whose solution can be taken to be trivial
\[ S_{3}^{(II.1.1)} = 0 \] (149)
(the solution to the homogeneous equation (148) can be absorbed into (146) by a suitable redefinition of the involved constants). Inserting (149) into the next deformation equation
\[ \frac{1}{2} \left( S_{2}^{(II.1.1)}, S_{2}^{(II.1.1)} \right) + \left( S_{1}^{(II.1.1)}, S_{3}^{(II.1.1)} \right) + sS_{4}^{(II.1.1)} = 0 \] (150)
and observing that \((S_{2}^{(II.1.1)}, S_{2}^{(II.1.1)}) = 0\), we can again take
\[ S_{4}^{(II.1.1)} = 0. \] (151)
It is easy to see that in fact we can set
\[ S_{n}^{(II.1.1)} = 0, \quad \text{for all} \quad n > 2. \] (152)

We can therefore conclude that in subcase II.1.1, described by conditions (145), the deformation procedure stops nontrivially at a finite step \((n = 2)\) and the deformed solution to the master equation, consistent to all orders in the deformation parameter, takes the form
\[ S^{(II.1.1)} = \bar{S} + kS_{1}^{(II.1.1)} + k^{2}S_{2}^{(II.1.1)} \equiv \int d^{3}x \left[ \mathcal{L}_{0}^{(PF)} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right. \\
+ h^{* \mu\nu} \partial_{(\mu} \eta_{\nu)} + V^{* \mu} \partial_{\mu} \eta - 2k \Lambda h \\
+ k y_{3} \varepsilon_{\mu\nu\rho} \left( V^{* \mu} \partial^{\nu} h^{\rho}{\lambda} + F^{\lambda\mu} \partial^{\nu} h^{\rho}{\lambda} \right) + k^{2} y_{3}^{2} \left( \partial^{\nu} h^{\rho}{\lambda} \right) \partial_{\nu} h_{\rho}{\lambda} \right], \] (153)
where \(\mathcal{L}_{0}^{(PF)}\) is the Pauli-Fierz Lagrangian.

We choose not to expose in detail the remaining possibilities, whose investigation is merely technical, but simply state their main conclusions. Thus,
in subcase II.1.2, where (134) and (144) are assumed to take place concurrently, the deformed solution to the master equation is parameterized by \( y_3 \) and \( q_1 \) and starts like in (135) and (138) where we set \( \Lambda = 0 \). There appear no obstructions in solving the higher-order deformation equations, of order three and four, while that of order five requires the supplementary condition \( y_3^3 q_1^2 = 0 \). Its relevant solution is \( q_1 = 0 \) since in the opposite situation, \( y_3 = 0 \), there are no cross-couplings at all between the graviton and the vector field: the original gauge transformations are not affected and the Lagrangian is modified by an Abelian Chern-Simons term \( kq_1 \varepsilon_{\mu
u\rho} V^\mu F^{\nu\rho} \).

Based on \( q_1 = 0 \), it can be shown that all the deformations of order three or higher can be made to vanish, such that the resulting deformed solution to the master equation precisely reduces to a particular solution of subcase II.1.1: it is expressed by (153) for \( \Lambda = 0 \). Regarding subcase II.2, it is pictured by conditions (118) and (133), so the deformations ‘live’ again in a three-dimensional spacetime, being parameterized by \( y_2 \) and \( y_3 \). The first-order deformation reduces to (71) where we set \( D = 3 \) and \( p = 0 = q_1 \). There are no obstructions in finding the deformation of order two in the coupling constant, but the existence of the third-order deformation imposes the additional condition \( y_2 = 0 \), which further implies that all the deformations of order three or higher are trivial. Therefore, the fully deformed solution to the master equation is nothing but the same particular solution from subcase II.1.1, being equal to (153) for \( \Lambda = 0 \).

Combining all the results exposed so far, we can state that the most general solution of the deformation procedure in case II is provided by a three-dimensional, consistent solution to the master equation that stops at the second order in the deformation parameter, is parameterized by \( y_3 \) and \( \Lambda \), and reads as in (153). We will argue in Section 4.4.2 that this solution describes a new mechanism for coupling a spin-two field to a massless vector field in \( D = 3 \), which is completely different from the standard one, based on General Relativity prescriptions.

### 4.3.3 Case III — nothing new

Case III is subject to conditions (119), so it is valid only in \( D > 3 \) spacetime dimensions\(^7\), being parameterized in the first instance by \( y_2 \), \( q_2 \delta_5^D \), and \( \Lambda \).

\(^7\)Note that \( D > 3 \) implies automatically \( y_3 \delta_5^D = 0 = q_1 \delta_5^D \).
agreement with (119), formulas (103) and (104) will be

\[ \chi_1 = 0, \quad \chi_0 = -2y_2 \left( 3q_2^D \varepsilon_{\mu\rho\alpha\beta} F^{\mu\nu} F^{\rho\alpha} \eta^\beta + 2DA\eta \right), \]

such that (101) yields the same necessary condition for the existence of \( \bar{b}_1^{(\text{int})} \) and \( \bar{b}_0^{(\text{int})} \) like in case II

\[ \chi_0 = \delta \varphi_1 + \gamma \omega_0 + \partial_\mu l_0^\mu, \]

where \( \text{agh} (\varphi_1) = 1 = \text{pgh} (\varphi_1), \text{agh} (\omega_0) = 0 = \text{pgh} (\omega_0), \text{agh} (l_0^\mu) = 0, \) \( \text{pgh} (l_0^\mu) = 1. \) The locality of the second-order deformation requires that all \( \varphi_1, \omega_0, \) and \( l_0^\mu \) are local functions. From (154) and definitions (8) and (12) it is obvious that (155) cannot be satisfied for some local \( \varphi_1, \omega_0, \) and \( l_0^\mu \) until we set \( \chi_0 = 0, \) which further demands

\[ y_2q_2^D = 0, \quad y_2\Lambda = 0. \]

There are obviously two complementary solutions to these equations

\[ q_2^D = 0, \quad \Lambda = 0, \]

\[ y_2 = 0. \]

Once more, we try to simplify the presentation by avoiding the technical details involved and mentioning only the key points. **Subcase III.1**, described by (119) and (157), is parameterized by a single constant, \( y_2, \) such that the first-order deformation reduces to the component of (71) proportional with this parameter

\[ S_{1}^{(\text{III.1})} = y_2 \int d^Dx \left[ h^*\eta + (D - 2) \left( -V^{*\lambda}\eta_\lambda + V^\lambda \partial_\mu h_\mu^\lambda \right) \right]. \]

The second-order deformation, \( S_2^{(\text{III.1})}, \) is then easily obtained from the observation that \( \chi_1 = 0 = \chi_0, \) so equations (103) and (100)–(101) reduce, like in case I, to (122)–(124), whose solution can be taken to vanish, like in (125). Consequently, the nonintegrated density of \( S_2^{(\text{III.1})} \) contains only terms of antighost number zero and reduces to the integrand of (78) plus the terms proportional with \( y_2^2 \) from (98). It is easy to show that the existence of a local third-order deformation requires \( y_2 = 0, \) so subcase III.1 leads to no nontrivial deformations, \( S_n^{(\text{III.1})} = 0, \) for all \( n \geq 1. \) **Subcase III.2**, pictured
by (119) and (158), is parameterized by $q_2 \delta^D_5$ and $\Lambda$ (see also footnote [7]). Only the first-order deformation is found non-trivial, being equal to

$$S^{(III.2)}_1 = \int d^D x \left( -2\Lambda h + q_2 \delta^D_5 \varepsilon^{\mu\nu\lambda\alpha\beta} V_\mu F_{\nu\lambda} F_{\alpha\beta} \right). \quad (160)$$

Analyzing (160), we can state that subcase III.2 is not interesting since the deformation procedure does not modify the original gauge transformations (2), but mainly adds to the original action (11) two gauge-invariant terms: a cosmological one and a generalized Abelian Chern-Simons action. In conclusion, case III brings no new information on the possible couplings between a spin-two field and a massless one-form.

4.4 Analysis of the deformed theory

The main aim of this section is to give an appropriate interpretation of the Lagrangian formulation of the deformed theories obtained previously from the deformation of the solution to the master equation. We will analyze the first two cases separately since we have seen that the third one gives nothing interesting. It is useful to recall the relationship between some quantities appearing in the deformed solution of the master equation, $S$, and the associated gauge theory: the component of antighost number zero from the former is nothing but the Lagrangian action of the coupled model, the piece of antighost number one provides the gauge transformations of the deformed theory, and the terms of antighost number two contain the structure functions defined by the commutators among the deformed gauge transformations. More precisely, the gauge transformations of the coupled theory result from the terms of antighost number one present in $S$ (generically written as $\Phi^*_a Z^{a_0}_{a_1} \eta^{a_1}$) by replacing the ghosts with the gauge parameters $e^{a_1}$, $\delta \epsilon^a = Z^{a_0}_{a_1} e^{a_1}$. The functions

$$Z^{a_0}_{a_1} = Z^{a_0}_{a_1} + k Z^{a_0}_{a_1} + k^2 Z^{a_0}_{a_1} + \cdots \quad (161)$$

define the gauge generators of the coupled model, where the components $Z^{a_0}_{a_1}$ are responsible for the original gauge transformations.

4.4.1 Case I: standard couplings from General Relativity

We discussed in detail in Section 4.3.1 a first case of obtaining consistent interactions between a Pauli-Fierz field and a vector field. This is defined
by conditions (117), in which situation the deformed solution to the master equation starts like

\[
S^{(I)} = \bar{S} + k S_1^{(I)} + k^2 S_2^{(I)} + \cdots \\
= \bar{S} + k \left( S_1^{(PF)} + S_1^{(int)} \right) + k^2 \left( S_2^{(PF)} + S_2^{(int)} \right) + \cdots ,
\]

(162)

where \( \bar{S}, S_1^{(I)}, \) and \( S_2^{(I)} \) read as in (17), (120), and (126) respectively.

In order to identify the main ingredients of the coupled model in the first case we use the result proved in Section 5 of [15], according to which the local BRST cohomologies of the Pauli-Fierz model and of the linearized version of vielbein formulation of spin-two field theory are isomorphic. Because the local BRST cohomology (in ghost numbers zero and one) controls the deformation procedure, it results that this isomorphism allows one to pass in a consistent manner from the Pauli-Fierz model to the linearized version of the vielbein formulation and conversely during the deformation procedure. Nevertheless, the linearized vielbein formulation possesses more fields (the antisymmetric part of the linearized vielbein) and more gauge parameters (Lorentz parameters) than the Pauli-Fierz model. The switch from the former version to the latter is realized via the above mentioned isomorphism by imposing some partial gauge-fixing conditions, chosen to annihilate the antisymmetric components of the vielbein. An appropriate interpretation of the Lagrangian description of the interacting theory in case I requires the generalized expression of these partial gauge-fixing conditions [33]

\[
\sigma_{\mu[a} e^{\mu}_{b]} = 0
\]

(163)

and the development of the vielbein \( e^a_{\mu} \) and of its inverse \( e^a_{\mu} \) up to the second order in the coupling constant in terms of the Pauli-Fierz field

\[
e^a_{\mu} = (0)^{\mu}_a + k (1)^{\mu}_a + k^2 (2)^{\mu}_a + \cdots = \delta^a_{\mu} - \frac{k}{2} h^a_{\mu} + \frac{3k^2}{8} h^a_{\mu} h^\nu_{\nu} + \cdots ,
\]

(164)

\[
e^a_{\mu} = (0)^{\mu}_a + k (1)^{\mu}_a + k^2 (2)^{\mu}_a + \cdots = \delta^a_{\mu} + \frac{k}{2} h^a_{\mu} - \frac{k^2}{8} h^a_{\mu} h^\nu_{\nu} + \cdots .
\]

(165)

The expansion of the inverse of the metric tensor \( g^{\mu\nu} \) and of the square root from the minus determinant of the metric tensor \( \sqrt{-g} = \sqrt{-\det g_{\mu\nu}} \) in terms of the Pauli-Fierz field,

\[
g^{\mu\nu} = (0)^{\mu\nu} + k (1)^{\mu\nu} + k^2 (2)^{\mu\nu} + \cdots = \sigma^{\mu\nu} - k h^{\mu\nu} + k^2 h^{\mu}_{\mu} h^{\nu}_{\nu} + \cdots ,
\]

(166)
\[
\sqrt{-g} = \sum_{n=0}^{\infty} \left( k^{n} \sqrt{-g} \right) + \cdots
\]

will also be necessary in what follows. We note that the metric tensor is

\[
g_{\mu\nu} = \sigma_{\mu\nu} + kh_{\mu\nu}.
\] (168)

The interacting Lagrangian at order one in the coupling constant, \( L^{(\text{int})}_{1} \), is the nonintegrated density of the piece of antighost number zero from the first-order deformation in the interacting sector, \( S^{(\text{int})}_{1} \). Using (120) and expansions (164)–(167), we can write

\[
L^{(\text{int})}_{1} \equiv -\frac{1}{4} F_{\mu\nu} \partial_{\mu} \left( h_{\nu}\right) V_{\rho} - \frac{1}{8} F_{\mu\nu} F_{\mu\nu} h_{\rho} + \frac{1}{2} F_{\mu\nu} F_{\mu\rho} h_{\nu} + q_{1} \delta^{D} \varepsilon^{\mu\nu\lambda\alpha\beta} V_{\lambda} F_{\alpha\beta} + q_{2} \delta^{D} \varepsilon^{\mu\nu\lambda\alpha\beta\gamma\delta} V_{\lambda} F_{\alpha\beta} F_{\gamma\delta}
\]

\[
= -\frac{1}{4} \left( \sqrt{-g} g_{\mu\nu} \partial_{\mu} \left( h_{\nu}\right) V_{\rho} + \frac{1}{8} \left( \sqrt{-g} g_{\mu\nu} g_{\rho\lambda} \right) + \frac{1}{2} \left( \sqrt{-g} g_{\mu\nu} g_{\rho\lambda} \right) + \frac{1}{2} \left( \sqrt{-g} g_{\mu\nu} g_{\rho\lambda} \right) \right) F_{\mu\nu} F_{\nu\lambda}
\]

\[
+ \sqrt{-g} g_{\mu\nu} g_{\rho\lambda} \left( F_{\mu\rho} F_{\nu\lambda} + F_{\mu\lambda} F_{\nu\rho} \right)
\]

\[
+ q_{1} \delta^{D} \varepsilon^{\mu\nu\lambda\alpha\beta} V_{\lambda} F_{\alpha\beta} + q_{2} \delta^{D} \varepsilon^{\mu\nu\lambda\alpha\beta\gamma\delta} V_{\lambda} F_{\alpha\beta} F_{\gamma\delta},
\] (169)

where

\[
\begin{align*}
^{(0)} V_{\mu} &= e_{\mu}^{a} V_{a}, & ^{(0)} F_{\mu\nu} &= \partial_{\mu} \left( e_{\nu}^{a} V_{a} \right), & ^{(1)} F_{\mu\nu} &= \partial_{\mu} \left( e_{\nu}^{a} V_{a} \right),
\end{align*}
\] (170)

Along the same line, the interacting Lagrangian at order two, \( L^{(\text{int})}_{2} \), results from \( S^{(\text{int})}_{2} \) at antighost number zero. Taking into account formula (126) and expansions (164)–(167), we have that

\[
L^{(\text{int})}_{2} \equiv -\frac{1}{4} \left[ \sqrt{-g} g_{\mu\nu} g_{\rho\lambda} \right] F_{\mu\rho} F_{\nu\lambda} + F_{\mu\lambda} F_{\nu\rho},
\]
From the expressions of $\mathcal{L}_1^{(\text{int})}$ and $\mathcal{L}_2^{(\text{int})}$, we observe that the first three terms from the full interacting Lagrangian in case I

$$
\mathcal{L}_1^{(\text{int})} = \mathcal{L}_0^{(\text{vect})} + k\mathcal{L}_1^{(\text{int})} + k^2\mathcal{L}_2^{(\text{int})} + \cdots
$$

(173)
coincide with the first orders of the Lagrangian describing the standard vector field-graviton cross-couplings from General Relativity

\[
\mathcal{L}^{(\text{vector-graviton})} = -\frac{1}{4}\sqrt{-g}g^{\mu\nu}g^{\rho\lambda}F_{\mu\rho}F_{\nu\lambda} + k \left( q_1 \delta^D_5 \varepsilon^{\mu_1\mu_2\mu_3}V_{\mu_1}F_{\mu_2\mu_3} + q_2 \delta^D_5 \varepsilon^{\mu_1\mu_2\mu_3\mu_4\mu_5}V_{\mu_1}F_{\mu_2\mu_3}F_{\mu_4\mu_5} \right),
\]

where the fully deformed field strength \( \tilde{F}_{\mu\nu} \) and the Levi-Civita symbol with curved indices \( \varepsilon^{\mu_1\ldots\mu_D} \) are given by

\[
\tilde{F}_{\mu\nu} = \partial_\mu (e^{0}_{\nu}V_a) \equiv F_{\mu\nu} + k F_{\mu\nu} + k^2 F_{\mu\nu} + \cdots
\]

\[
\varepsilon^{\mu_1\ldots\mu_D} = \sqrt{g}e^{\mu_1}_{a_1} \cdots e^{\mu_D}_{a_D} \varepsilon^{a_1\ldots a_D}.
\]

The self-interactions of the Pauli-Fierz field at orders one and two in the coupling constant, \( \mathcal{L}_{1,2}^{(PF)} \), result from the terms of antighost number zero present in \( S^{(PF)}_1 \) and \( S^{(PF)}_2 \) (see (120) and (126)), so the full Lagrangian describing the self-interactions of the graviton in case I starts like

\[
\tilde{\mathcal{L}}_{I}^{(PF)} = \mathcal{L}_{0}^{(PF)} + k \mathcal{L}_{1}^{(PF)} + k^2 \mathcal{L}_{2}^{(PF)} + \cdots,
\]

where \( \mathcal{L}_{0}^{(PF)} \) is the Pauli-Fierz Lagrangian. Using (160)–(168), one finds that the first three terms from \( \tilde{\mathcal{L}}_{I}^{(PF)} \) are nothing but the first orders of the Einstein-Hilbert Lagrangian with a cosmological term [14]

\[
\mathcal{L}^{(EH)} = \frac{2}{k^2}\sqrt{-g} \left( R + 2k^2\Lambda \right),
\]

where \( R \) is the full scalar curvature.

As explained in the beginning of this section, the terms present in (162) (see (17), (120), and (126)) that are linear in the antifields \( V^*_{\mu} \) provide the deformed gauge transformations of the vector field

\[
\delta^{(1)}(I) V_\alpha = \left( \delta^\beta_\alpha - \frac{k}{2} h^\beta_{\alpha} + \frac{3k^2}{8} h^\mu_\nu h^\nu_{\alpha} h^\nu_{\beta} + \cdots \right) \partial_\nu \varepsilon + \left[ \frac{k}{2} \partial_\alpha \varepsilon_{\beta \gamma} \right]
\]

\[
+ k^2 \left( \frac{1}{4} \left( \partial_\alpha h_{\beta \gamma} \right) \varepsilon^\gamma + \frac{1}{8} h_{\alpha \beta \gamma} \partial_\gamma h^\gamma_{\beta} + \frac{1}{8} \left( \partial_\gamma h^\gamma_{\alpha} \right) h^\gamma_{\beta} \right) V^\beta + \cdots
\]

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\[ + (\partial_\mu V_\alpha) \left( k\delta_\mu^\alpha - \frac{k^2}{2} h_\beta^\mu + \frac{3k^3}{8} h_\nu^\mu h_\beta^\nu + \cdots \right) \epsilon^\beta. \]  

(179)

In the last formula the indices of the one-form, even if written in Latin letters, are flat. In standard, Latin notation the above gauge transformations can be written as

\[ \delta^{(1)}_\epsilon V_\alpha = (0)^{\mu}_\epsilon V_\alpha + (1)^{\mu}_\epsilon V_\alpha + k^{(2)}\delta_\epsilon V_\alpha + \cdots, \]

where the first orders of the gauge transformations read as

\[ (0)^{\mu}_\epsilon V_\alpha = \epsilon^{a}_\mu \partial_\mu \epsilon_a, \]

(180)

\[ (1)^{\mu}_\epsilon V_\alpha = (1)^{\mu}_\epsilon \partial_\mu \epsilon + (0)^{\mu}_\epsilon \epsilon \epsilon^a b V_\beta + (\partial_\mu V_\alpha)^{(0)^{\mu}_\epsilon}, \]

(181)

\[ (2)^{\mu}_\epsilon V_\alpha = (2)^{\mu}_\epsilon \partial_\mu \epsilon + (1)^{\mu}_\epsilon \epsilon \epsilon^a b V_\beta + (\partial_\mu V_\alpha)^{(1)^{\mu}_\epsilon}, \]

(182)

and the various orders of the gauge parameters are expressed by

\[ (0)^{\mu}_\epsilon = \epsilon^{a}_\mu \equiv \epsilon^{a}_\mu h_a^\mu, \]

(183)

\[ (0)^{\mu}_\epsilon ab = \frac{1}{2} \partial_{[a} \epsilon_{b]}, \]

(184)

\[ (1)^{\mu}_\epsilon ab = -\frac{1}{4} \epsilon \epsilon_{[a} b |c| + \frac{1}{8} h_a^\rho |a| b |c| + \frac{1}{8} (\partial_\rho \epsilon_{[a}) h^c b]. \]

(185)

Based on the above notations, we can re-write the gauge transformations of the vector field with a flat index as

\[ \delta^{(1)}_\epsilon V_\alpha = \left( (0)^{\mu}_\epsilon + k^{(1)^{\mu}_\epsilon} \partial_\mu \epsilon + k \left( (0)^{\mu}_\epsilon \epsilon + k (1)^{\mu}_\epsilon \epsilon \epsilon^a b V_\beta \right) + \left( (0)^{\mu}_\epsilon \epsilon + k (1)^{\mu}_\epsilon \epsilon \right) \right) V_\beta. \]

(186)

The gauge parameters \( (0)^{\mu}_\epsilon \epsilon_a b \) and \( (1)^{\mu}_\epsilon \epsilon_a b \) are precisely the first two terms from the Lorentz parameters expressed in terms of the flat parameters \( \epsilon^a \) via the partial gauge-fixing (163). Indeed, (163) leads to

\[ \delta_{\epsilon} \left( \sigma_{\mu[a} e^\mu_{b]} \right) = 0. \]

(187)

Using

\[ \delta_{\epsilon} e^\mu_a = \bar{\epsilon}^a \partial_\mu e^\mu_a - e^a \partial_\mu \bar{\epsilon} + \epsilon^b a e^\mu_a, \]

(188)
and inserting (164) together with the expansions

\[ \bar{\epsilon}^{\mu} = (0)^{\mu} + k \bar{\epsilon}^{(1)\mu} + \cdots = \left( \delta^{\mu}_{a} - \frac{k}{2} h^{\mu}_{a} + \cdots \right) \epsilon^{a}, \]  

(189)

\[ \epsilon_{ab} = (0)_{ab} + k (1)_{ab} + \cdots \]  

(190)
in (187), we arrive precisely at (184) and (185). At this point it is easy to see that the first orders of the gauge transformations (186) coincide with those arising from the perturbative expansion of the formula

\[ \delta^{(1)}_{\epsilon} V_{a} = \epsilon^{a}_{\mu} \partial_{\mu} \epsilon + k \epsilon_{ab} V^{b} + k (\partial_{\mu} V_{a}) \bar{\epsilon}^{\mu}. \]  

(191)

Concerning the vector field with a curved index \( \bar{V}_{\mu} \)

\[ \bar{V}_{\mu} = \epsilon^{a}_{\mu} V_{a}, \]  

(192)
it its gauge transformations will be correctly described at the level of the first orders in the coupling constant by the well-known gauge transformations

\[ \delta^{(1)}_{\epsilon} \bar{V}_{\mu} = \partial_{\mu} \epsilon + k (\partial_{\mu} \epsilon) \bar{V}_{\mu} + k (\partial_{\nu} \bar{V}_{\mu}) \bar{\epsilon}^{\nu} \]  

(193)
of the vector field (in interaction with the Einstein-Hilbert graviton) from General Relativity. Finally, from the terms present in (162) linear in the Pauli-Fierz antifields \( h^{*\mu\nu} \) (see (17), (120), and (126)) one infers that the deformed gauge transformations of the metric tensor (168) reproduce the first orders of diffeomorphisms

\[ \delta^{(1)}_{\epsilon} g_{\mu\nu} = k \epsilon_{(\mu;\nu)}, \]  

(194)

where \( \epsilon_{\mu;\nu} \) is the (full) covariant derivative of \( \epsilon_{\mu} \).

So far, we argued that in the first case the consistent interactions between a graviton and a vector field are described in all \( D > 2 \) dimensions by the first orders of the Lagrangian and gauge transformations prescribed by the standard rules of General Relativity (see (174), (178), (193), and (194)). Our result follows as a consequence of applying a cohomological procedure based on the “free” BRST symmetry in the presence of a few natural assumptions: locality, smoothness in the coupling constant, Poincaré invariance, Lorentz covariance, and preservation of the number of derivatives on each field. General covariance was not imposed a priori, but was gained in a natural way.
from the cohomological setting developed here under the previously mentioned hypotheses. It can be shown that formulas (174), (178), (193), and (194) apply in fact to all orders in the coupling constant, so we can state that the fully interacting Lagrangian action in case I reads as

\[ S_{L(I)}[g_{\mu\nu}, \bar{V}_\mu] = \int d^3 x \left[ \frac{2}{k^2} \sqrt{-g} \left( R - 2k^2 \Lambda \right) - \frac{1}{4} \sqrt{-g} g^{\mu\nu} g^{\rho\lambda} \bar{F}_{\mu\rho} \bar{F}_{\nu\lambda} \right. \]

\[ + k \left( q_1 \delta^D_3 \varepsilon^{\mu_1 \mu_2 \mu_3} \bar{V}_{\mu_1} \bar{F}_{\mu_2 \mu_3} + q_2 \delta^D_5 \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5} \bar{V}_{\mu_1} \bar{F}_{\mu_2 \mu_3} \bar{F}_{\mu_4 \mu_5} \right) \] (195)

and is invariant under the deformed gauge transformations

\[ \delta^{(I)}_\epsilon g_{\mu\nu} = k\epsilon_{(\mu; \nu)}, \quad \delta^{(I)}_\epsilon \bar{V}_\mu = \partial_\mu \epsilon + k (\partial_\mu \bar{\epsilon}_\nu) \bar{V}_\nu + k (\partial_\nu \bar{V}_\mu) \epsilon_\nu. \] (196)

The validity of (195) and (196) to all orders in the coupling constant can be done by developing the same technique used in Section 7 of [14].

### 4.4.2 Case II: new couplings in \( D = 3 \)

As discussed in Section 4.3.2, the second case of interest allowing for non-trivial, consistent couplings between a Pauli-Fierz field and a vector field is pictured by the deformed solution to the master equation given in (153). We can re-write the deformation in a more convenient way by adding to (153) some s-exact terms, since we know that this does not affect the physical content of the coupled model (see (24)). Because the most general couplings in case II are obtained in subcase II.1.1, described by conditions (145), we will denote the deformed solution (153) to which we add the previously mentioned s-exact terms and where we set \( y_3 = 1 \) by \( S^{(II)} \)

\[ S^{(II)} \equiv S^{(II.1.1)}|_{y_3=1} - s \left[ 2k^2 \int d^3 x \left( h^{*\nu} h_{\nu} + \eta^{*\mu} \eta_\mu \right) \right] \]

\[ = \int d^3 x \left[ \mathcal{L}^{(PF)}_0 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - 2k \Lambda h \right. \]

\[ - k F^{\mu\nu} \varepsilon_{\mu\rho\sigma} \partial^{[\theta \rho]} h^{\sigma]}_\theta + 2k^2 \left( \partial^{[\nu} h^{\rho]}_\nu \right) \partial_{[\rho} h^{\sigma]}_\sigma \]

\[ + h^{*\mu} \partial_{(\mu} \eta_{\nu)} + V^{*\mu} \left( \partial_\mu \eta + k \varepsilon_{\mu\rho\sigma} \partial^{[\nu} \eta^{\rho]} \right) \]. (197)

Essentially, it is not trivial and is consistent to all orders in the coupling constant, namely

\[ (S^{(II)}, S^{(II)}) = 0. \] (198)
From the terms of antighost number zero we deduce the Lagrangian action of the coupled model

\[ S_{L(II)}[h_{\mu\nu}, V_\mu] = \int d^3 x \left[ \mathcal{L}_0^{(PF)} - \frac{1}{4} F_{\mu\nu}^\prime F^{\mu\nu} - 2k \Lambda h - k F_{\mu\nu}^\prime \epsilon_{\mu\nu\rho} \partial^\theta h_\theta^\rho + 2k^2 (\partial^{[\mu} h_\rho^\nu)] \partial_{[\nu} h_{\rho]}^\mu \right], \tag{199} \]

where \( \mathcal{L}_0^{(PF)} \) is the Pauli-Fierz Lagrangian and \( \Lambda \) is the cosmological constant. The component of antighost number one provides the gauge symmetries of \( \delta_{\xi} h_{\mu\nu} = \partial_{(\mu} \epsilon_{\nu)}, \quad \delta_{\xi} V_\mu = \partial_\mu \epsilon + k \epsilon_{\mu\rho\sigma} \partial^{[\nu} \epsilon^{\rho]} \). \tag{200} \]

The absence of terms of antighost number strictly greater than one shows that the above gauge transformations are independent (irreducible) and their algebra remains Abelian, like the original one. Action \( \mathcal{L}_0 \) can be set in a more suggestive form by introducing a deformed field strength

\[ F_{\mu\nu}^\prime = F_{\mu\nu} + 2k \epsilon_{\mu\nu\rho} \partial^\theta h_\theta^\rho, \tag{201} \]

in terms of which we can write

\[ S_{L(II)}[h_{\mu\nu}, V_\mu] = \int d^3 x \left( \mathcal{L}_0^{(PF)} - 2k \Lambda h - \frac{1}{4} F_{\mu\nu}^{\prime} F^{\prime\mu\nu} \right). \tag{202} \]

Under this form, action \( \mathcal{L}_0 \) is manifestly invariant under the gauge transformations \( \delta_{\xi} F_{\mu\nu}^\prime = 0 \). \tag{203} \]

This result is new and will be generalized in Section 6 to the case of couplings between a graviton and an arbitrary p-form. In conclusion, this case yields another possibility to establish nontrivial couplings between the Pauli-Fierz field and a vector field. It is complementary to case I (General Relativity) and is valid only in \( D = 3 \). The resulting Lagrangian action and gauge transformations are not series in the coupling constant. The Lagrangian contains pieces of maximum order two in the coupling constant,
which are mixing-component terms (there is no interaction vertex at least cubic in the fields) and emphasize the deformation of the standard Abelian field strength of the vector field like in (201). Concerning the new gauge transformations, only those of the massless vector field are modified at order one in the coupling constant by adding to the original $U(1)$ gauge symmetry a term linear in the antisymmetric first-order derivatives of the Pauli-Fierz gauge parameters. As a consequence, the gauge algebra, defined by the commutators among the deformed gauge transformations, remains Abelian, just like for the free theory. We cannot stress enough that these two cases (I and II) cannot coexist, even in $D = 3$, due to the consistency conditions (114)–(116).

5 No cross-couplings in multi-graviton theories intermediated by a vector field

As it has been proved in [14], there are no direct cross-couplings that can be introduced among a finite collection of gravitons and also no cross-couplings among different gravitons intermediated by a scalar field. Similar conclusions have been drawn in [15, 16] related to the couplings between a finite collection of spin-two fields and a Dirac or a massive Rarita-Schwinger field. In this section, under the same hypotheses like before, namely, locality, smoothness in the coupling constant, Poincaré invariance, Lorentz covariance, and preservation of the number of derivatives on each field, we investigate the existence of cross-couplings among different gravitons intermediated by a massless vector field. The Greek field indices are (Lorentz) flat: they are lowered and raised with a flat metric of ‘mostly plus’ signature, $\sigma_{\mu\nu} = (-+++)$. 

5.1 First- and second-order deformations. Consistency conditions

5.1.1 Generalities

We start now from a finite sum of Pauli-Fierz actions and a single Maxwell action in $D > 2$

\[
S_0^{L} \left[ h_{\mu\nu}^A, V_{\mu} \right] = \int d^{D}x \left[ -\frac{1}{2} \left( \partial_{\mu} h_{\nu\rho}^A \right) \partial^{\mu} h_{\rho\nu}^A + \left( \partial_{\mu} h_{\nu\rho}^{\mu A} \right) \partial^{\nu} h_{\rho\nu}^A \right]
\]
\[ - (\partial_\mu h^A) \partial_\nu h^\nu_A + \frac{1}{2} (\partial_\mu h^A) \partial^\mu h_A - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \], \quad (204) \]

where \( h_A \) is the trace of the Pauli-Fierz field \( h^\mu_\nu \) \((h_A = \sigma_\mu^\nu h^\mu_\nu)\), with \( A = 1, n \) and \( n > 1 \). The collection indices \( A, B, \) etc., are raised and lowered with a quadratic form \( k_{AB} \) that determines a positively-defined metric in the internal space. It can always be normalized to \( \delta_{AB} \) by a simple linear field redefinition, so from now on we take \( k_{AB} = \delta_{AB} \) and re-write (204) as

\[ S_0^L [h^A_\mu\nu, V_\mu] = \int d^Dx \left[ \sum_{A=1}^n L_0^{(PF)} (h^A_\mu\nu, \partial_\lambda h^A_\mu\nu) + L_0^{(\text{vect})} \right], \quad (205) \]

where \( L_0^{(PF)} (h^A_\mu\nu, \partial_\lambda h^A_\mu\nu) \) is the Pauli-Fierz Lagrangian for the graviton \( A \). Action (204) is invariant under the gauge transformations

\[ \delta h^A_\mu\nu = \partial_\mu \epsilon^A_\lambda, \quad \delta V_\mu = \partial_\mu \epsilon. \quad (206) \]

The BRST complex comprises the fields, ghosts, and antifields

\[ \hat{\Phi}^{\alpha_0} = (h^A_\mu\nu, V_\mu), \quad \hat{\eta}^{\alpha_1} = (\eta^A_\mu, \eta), \quad (207) \]
\[ \hat{\Phi}^{*\alpha} = (h^{*A}_\mu\nu, V^*\mu), \quad \hat{\eta}^{*\alpha_1} = (\eta^{*A}_\mu, \eta^*), \quad (208) \]

whose degrees are the same like in the case of a single Pauli-Fierz field. The BRST differential decomposes exactly like in (5) and its components act on the BRST generators via the relations

\[ \delta h^{*A}_\mu\nu = 2 H^A_\mu\nu, \quad \delta V^{*\mu} = - \partial_\nu F^{*\mu}, \quad (209) \]
\[ \delta \eta^{*A}_\mu = -2 \partial_\nu h^{*A}_\nu\mu, \quad \delta \eta^* = - \partial_\mu V^{*\mu}, \quad (210) \]
\[ \delta \hat{\Phi}^{\alpha_0} = 0, \quad \delta \hat{\eta}^{\alpha_1} = 0, \quad (211) \]
\[ \gamma \hat{\Phi}^{*\alpha_0} = 0, \quad \gamma \hat{\eta}^{*\alpha_1} = 0, \quad (212) \]
\[ \gamma h^{A}_\mu\nu = \partial_\mu \eta^A_\nu, \quad \gamma V_\mu = \partial_\mu \eta, \quad (213) \]
\[ \gamma \eta^{A}_\mu = 0, \quad \gamma \eta = 0, \quad (214) \]

where \( H^A_\mu\nu = K^A_\mu\nu - \frac{1}{2} \sigma^{\mu\nu} K_A \) is the linearized Einstein tensor of the Pauli-Fierz field \( h^A_\mu\nu \). The solution to the master equation for this free model takes the simple form

\[ \bar{S}' = S_0^L [h^A_\mu\nu, V_\mu] + \int d^Dx \left( h^{*A}_\mu\nu \partial_\mu \eta^A_\nu + V^{*\mu} \partial_\mu \eta \right). \quad (215) \]
5.1.2 First-order deformation

The first-order deformation of the solution to the master equation decomposes like in the case of a single graviton in a sum of three independent components

\[ \hat{a} = \hat{a}^{(PF)} + \hat{a}^{(int)} + \hat{a}^{(vect)}. \]  

The first-order deformation in the Pauli-Fierz sector, \( \hat{a}^{(PF)} \), can be shown to expand as

\[ \hat{a}^{(PF)} = \hat{a}^{(PF)}_2 + \hat{a}^{(PF)}_1 + \hat{a}^{(PF)}_0, \]

where

\[ \hat{a}^{(PF)}_2 = \frac{1}{2} f^A_{BC} \eta^A_\mu \eta^B_\nu \partial_\mu \eta^C_\nu, \]

with \( f^A_{BC} \) some real constants. The requirement that \( \hat{a}^{(PF)}_2 \) produces a consistent \( \hat{a}^{(PF)}_1 \) as solution to the equation

\[ \delta \hat{a}^{(PF)}_2 + \gamma \hat{a}^{(PF)}_1 = \partial_\mu \hat{m}^{(PF)\mu} \]

restricts the coefficients \( f^A_{BC} \) to be symmetric with respect to their lower indices (commutativity of the algebra defined by \( f^A_{BC} \)) \[14\]

\[ f^A_{BC} = f^A_{CB}. \]

Based on (219), it follows that

\[ \hat{a}^{(PF)}_1 = f^A_{BC} h_A^{\nu\rho} \left( (\partial_\rho \eta^B_\nu) h^C_\mu - \eta^B_\nu \partial_\mu h^C_\rho \right). \]

Asking that \( \hat{a}^{(PF)}_1 \) provides a consistent \( \hat{a}^{(PF)}_0 \) as solution to the equation

\[ \delta \hat{a}^{(PF)}_1 + \gamma \hat{a}^{(PF)}_0 = \partial_\mu \hat{m}^{(PF)\mu} \]

further constrains the coefficients with lowered indices, \( f^A_{ABC} = k_{AD} f^D_{BC} \equiv \delta_{AD} f^D_{BC} \), to be fully symmetric \[14\]

\[ f^A_{ABC} = \frac{1}{3} f^A_{(ABC)}. \]

From (221) we obtain that \( \hat{a}^{(PF)}_0 \) coincides with that from \[14\] (where it is denoted by \( a_0 \) and the coefficients \( f^A_{ABC} \) by \( a_{abc} \))

\[ \hat{a}^{(PF)}_0 = f^A_{ABC} \hat{a}^{(cubic)}_0 ABC - 2 \Lambda_A h^A, \]

\[ \text{The term (215) differs from that corresponding to [14] through a } \gamma \text{-exact term, which does not affect (219).} \]

\[ \text{The piece (220) differs from that corresponding to [14] through a } \delta \text{-exact term, which does not change (221).} \]
where \( \hat{a}_0^{(\text{cubic})ABC} \) contains only vertices that are cubic in the Pauli-Fierz fields and reduce to the cubic Einstein-Hilbert vertex in the absence of collection indices. \( \Lambda_A \) play the role of cosmological constants. Employing exactly the same line like in 4.2, we find that the first-order deformation giving the cross-couplings between the gravitons and the vector fields ends at antighost number one

\[
\hat{a}^{(\text{int})} = \hat{a}_1^{(\text{int})} + \hat{a}_0^{(\text{int})},
\]

where

\[
\hat{a}_1^{(\text{int})} = y_{2A} \left[ h^{*A} \eta - (D - 2) V^{*A} \eta_A \right] \\
+ y_3^A \varepsilon_{\mu \nu \rho} V^{*A} \partial^{[\nu} h_A^{\rho]} + p_A V^{*A} F_{\mu \nu} \eta^{A \nu},
\]

\[
\hat{a}_0^{(\text{int})} = (D - 2) y_{2A} V^A \partial_\mu h_A^A + y_3^A \varepsilon_{\mu \nu \rho} F^{\lambda \nu} \partial^{[\nu} h_A^{\rho]} + \frac{p_A}{2} \left( F^{\alpha \mu} F_{\mu \nu} h_A^A + \frac{1}{4} F^{\alpha \mu} F_{\alpha \mu} h_A^A \right).
\]

and \( y_{2A}, y_3^A \) together with \( p_A \) are some arbitrary, real constants. Like in Section 4.2, we eliminate some \( s \)-exact modulo \( d \) terms from \( \hat{a}^{(\text{int})} \) and work with

\[
\hat{a}^{(\text{int})} = \hat{a}^{(\text{int})} + s \left[ p_A \left( \eta^* V^{\mu} h_A^A + \frac{1}{2} V^{*A} V^{\nu} h_A^{A \nu} \right) \right] - \partial_\mu \hat{t}^\mu.
\]

The component \( \hat{a}^{(\text{vect})} \) coincides with that from Section 4.2 (see 46)

\[
\hat{a}^{(\text{vect})} = a^{(\text{vect})} = q_1 \varepsilon_{\mu \nu \lambda} V_\nu F_{\mu \lambda} + q_2 \varepsilon_{\mu \nu \lambda} V_\nu F_{\mu \lambda} F_{\alpha \beta}.
\]

Putting together (217) and (223)–(227) with the help of (216), we can write the first-order deformation of the solution to the master for a single vector field and a collection of Pauli-Fierz fields like

\[
\hat{S}_1 = \hat{S}_1^{(\text{PF})} + \hat{S}_1^{(\text{int})},
\]

where

\[
\hat{S}_1^{(\text{PF})} = \int d^D x \left( \hat{a}_2^{(\text{PF})} + \hat{a}_1^{(\text{PF})} + \hat{a}_0^{(\text{PF})} \right)
\]

\[
= \int d^D x \left\{ \frac{1}{2} f_{BC}^{A} \eta^{*A} \eta^{B \nu} \partial_\mu h_A^{C \nu} + f_{BC}^{A} h_A^{*\mu \rho} \left[ \partial_\rho \eta^{B \nu} \right] h_C^{\mu \nu} \right\}
\]

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\[-\eta^{B\nu}\partial_{[\mu}h^{C]}_{\nu\rho]} + f_{ABC}\hat{a}_0^{(\text{cubic})ABC} - 2\Lambda_A h^A \}\), \hspace{1cm} (229)

\[
\hat{S}_{1}^{(\text{int})} \equiv \int d^D x \left( \hat{a}_{\text{int}}^{(\text{int})} + \hat{a}_{\text{vect}}^{(\text{int})} \right) \\
= \int d^D x \left\{ y_{2A} \left[ h^{*A}\eta + (D - 2) \left( -V^{*\lambda}h^{A\mu} + V^\lambda \partial_{[\mu}h^{A\rho]} \right) \right] \\
+ y_3^{A}\delta^D_3 \varepsilon_{\mu\nu} \left( V^{*\mu}h^{A\rho]} + F^{\lambda\rho} \partial_{[\nu}h^{A\lambda]} \right) + p_A \left[ \eta^{*A}\partial_{[\mu}\eta_A \right] \\
- \frac{1}{2} V^{*\mu} \left( V^\nu \partial_{[\mu}\eta_A + 2 (\partial_{\nu}V^\mu) \eta_A^{A\nu} - h^{A\mu}\partial^{\nu}\eta \right) \\
+ \frac{1}{8} F^{\mu\nu} \left( 2\partial_{[\mu} \left( h^{A\rho]}V^{\nu} \right) + F_{\mu\nu}h^{A\rho] - 4F_{\mu\rho}h^{A\nu]} \right) \\
+ 2\delta^D_3 \varepsilon_{\mu\nu\lambda}\eta_A \partial_{\nu\lambda} + q_1\delta^D_3 \varepsilon_{\mu\nu\lambda\rho}V_{\mu}F_{\nu\lambda} + q_2\delta^D_5 \varepsilon_{\mu\nu\lambda\rho\sigma}V_{\mu}F_{\nu\lambda}F_{\rho\sigma} \right\}. \hspace{1cm} (230)
\]

It is parameterized by seven types of real, constant coefficients, namely \( f_{BC}^A \), \( \Lambda_A \), \( y_{2A} \), \( y_3^{A}\delta^D_3 \), \( p_A \), \( q_1\delta^D_3 \), and \( q_2\delta^D_5 \), with \( f_{BC}^A \) fully symmetric (see (221)).

5.1.3 Consistency of the first-order deformation

Next, we investigate the consistency of the first-order deformation, expressed by equation (22), with \( S_{1,2} \) replaced by \( \hat{S}_{1,2} \):

\[
\left( \hat{S}_1, \hat{S}_1 \right) + 2s\hat{S}_2 = 0. \hspace{1cm} (231)
\]

We decompose the second-order deformation as

\[
\hat{S}_2 = \hat{S}_{2}^{(\text{PF})} + \hat{S}_{2}^{(\text{int})}, \hspace{1cm} (232)
\]

where \( \hat{S}_{2}^{(\text{PF})} \) is responsible only for the self-interactions of the Pauli-Fierz fields and \( \hat{S}_{2}^{(\text{int})} \) for the cross-couplings between the gravitons and the vector field. Using (228), we find that (231) becomes equivalent with two independent equations

\[
\left( \hat{S}_{1}^{(\text{PF})}, \hat{S}_1^{(\text{PF})} \right) + \left( \hat{S}_{1}^{(\text{int})}, \hat{S}_1^{(\text{int})} \right)^{(PF)} + 2s\hat{S}_2^{(PF)} = 0, \hspace{1cm} (233)
\]

\[
2 \left( \hat{S}_1^{(\text{PF})}, \hat{S}_1^{(\text{int})} \right) + \left( \hat{S}_{1}^{(\text{int})}, \hat{S}_1^{(\text{int})} \right)^{(int)} + 2s\hat{S}_2^{(int)} = 0, \hspace{1cm} (234)
\]
where \( \left( \hat{S}_1^{\text{(int)}} , \hat{S}_1^{\text{(int)}} \right)^{\text{(PF)}} \) contains only Pauli-Fierz BRST generators and each term of \( \left( \hat{S}_1^{\text{(int)}} , \hat{S}_1^{\text{(int)}} \right)^{\text{(int)}} \) includes at least one BRST generator from the Maxwell sector.

Initially, we analyze the existence of \( \hat{S}_2^{\text{(PF)}} \), governed by equation (233). By direct computation we find

\[
\left( \hat{S}_1^{\text{(int)}} , \hat{S}_1^{\text{(int)}} \right)^{\text{(PF)}} = -2s \int d^D x \left[ y_2 Ay_2 B \frac{(D - 2)^2}{4} \left( \hbar^A h^B - h^{A\mu \nu} h^B_{\mu \nu} \right) + y_2 Ay_3 B \delta^D_3 (D - 2) \varepsilon_{\mu \nu \rho} h^A_{\mu \lambda} \left( \partial^{[\nu} h_{B\rho]}^{\rho]} \lambda \right) \partial_{[\nu} h_{B\rho]}^{\rho]} \lambda \right] 
\]

where

\[
\hat{S}_2^{\text{(PF)}} (y_2 Ay_2 B) = y_2 Ay_2 B \frac{(D - 2)^2}{4} \int d^D x \left( \hbar^A h^B - h^{A\mu \nu} h^B_{\mu \nu} \right),
\]

\[
\hat{S}_2^{\text{(PF)}} (y_2 Ay_3 B) = y_2 Ay_3 B \delta^D_3 (D - 2) \varepsilon_{\mu \nu \rho} \int d^D x h^A_{\mu \lambda} \left( \partial^{[\nu} h_{B\rho]}^{\rho]} \lambda \right) \partial_{[\nu} h_{B\rho]}^{\rho]} \lambda,
\]

\[
\hat{S}_2^{\text{(PF)}} (y_3 Ay_3 B) = y_3 Ay_3 B \delta^D_3 \int d^D x \left( \partial^{[\nu} h_{B\rho]}^{\rho]} \lambda \right) \partial_{[\nu} h_{B\rho]}^{\rho]} \lambda.
\]

Replacing (235) into (233), it becomes equivalent to

\[
\left( \hat{S}_1^{\text{(PF)}}, \hat{S}_1^{\text{(PF)}} \right) + 2s \left[ \hat{S}_2^{\text{(PF)}} - \hat{S}_2^{\text{(PF)}} (y_2 Ay_2 B) \right.

\left. - \hat{S}_2^{\text{(PF)}} (y_2 Ay_3 B) - \hat{S}_2^{\text{(PF)}} (y_3 Ay_3 B) \right] = 0,
\]

so the existence of \( \hat{S}_2^{\text{(PF)}} \) requires that \( \left( \hat{S}_1^{\text{(PF)}}, \hat{S}_1^{\text{(PF)}} \right) \) is s-exact, where \( \hat{S}_1^{\text{(PF)}} \) reads as in (229). It has been shown in [14] (Section 5.4) that this requirement restricts the coefficients \( f_{ABC} \) to satisfy the supplementary conditions

\[
f_{ABC} f_{CDE} = 0.
\]

Combining (219), (221), and (240), we conclude that the coefficients \( f_{ABC} \) define the structure constants of a real, commutative, symmetric, and associative (finite-dimensional) algebra. The analysis realized in [14] (Section 6)
shows that such an algebra has a trivial structure: it is a direct sum of one-dimensional ideals. Therefore, \( f_{AB}^C = 0 \) whenever two indices are different

\[
f_{AB} = 0, \quad \text{if} \quad (A \neq B \quad \text{or} \quad B \neq C \quad \text{or} \quad C \neq A).
\] (241)

For notational simplicity, we denote \( f_{ABC} \) for \( A = B = C \) by

\[
f_{AAA} \equiv f_A \quad \text{without summation over} \ A.
\] (242)

Using (241), it follows that \( \hat{S}_1^{(PF)} \) cannot couple different gravitons: it will be written as a sum of \( s \)-exact terms, each term involving a single graviton

\[
\left( \hat{S}_1^{(PF)}, \hat{S}_1^{(PF)} \right) = -2s \left\{ \sum_{A=1}^{n} f_A \left[ f_A S_2^{(\text{EH-quartic})A} + \Lambda_A \int d^Dx \left( h^{A\mu\nu}h^A_{\mu\nu} - \frac{1}{2}(h^A)^2 \right) \right] \right\} \equiv -2s \sum_{A=1}^{n} \hat{S}_2^{(PF)} \left( f_A^2, f_A \Lambda_A \right).
\] (243)

Each \( S_2^{(\text{EH-quartic})A} \) is the second-order Einstein-Hilbert deformation in the sector of the graviton \( A \). It includes the quartic Einstein-Hilbert Lagrangian for the field \( h^A_{\mu\nu} \) and is written only in terms of the BRST generators from the \( A \) sector, namely \( h^A_{\mu\nu}, \eta^A_{\mu} \), and their antifields. Also, it is important to note that (241) restricts \( \hat{S}_1^{(PF)} \) to have the same property (see (229)) of being written as a sum of individual components, each component involving a single graviton sector

\[
\hat{S}_1^{(PF)} = \sum_{A=1}^{n} \left\{ f_A \int d^Dx \left[ \frac{1}{2} \eta^{*A}_{\mu} \eta^{A\nu} \partial_{[\mu} h^A_{\nu]} + h^{*A\mu\nu} \left( \partial_{\mu} \eta^{A\nu} \right) h^A_{\mu\nu} \right. \right. \\
\left. \left. - \eta^{A\nu} \partial_{[\mu} h^A_{\nu]} \right] + a_0^{(\text{EH-cubic})A} \right\} - 2 \sum_{A=1}^{n} \left( \Lambda_A \int d^Dx h^A \right).
\] (244)

Now, \( a_0^{(\text{EH-cubic})A} \) is nothing but the cubic Einstein-Hilbert Lagrangian involving only the graviton field \( h^A_{\mu\nu} \). Substituting (243) into (239) we find the equation

\[
s \left[ \hat{S}_2^{(PF)} - \hat{S}_2^{(PF)} (y_2A y_2B) - \hat{S}_2^{(PF)} (y_2A y_3B) \right]
\]
\[-\hat{S}_2^{(PF)} (y_3^A y_3^B) - \sum_{A=1}^{n} \hat{S}_2^{(PF)} (f_A^2, f_A \Lambda_A) \right] = 0, \quad (245)\]

whose solution reads as (up to the solution of the homogeneous equation, \(s \hat{S}_2^{(PF)} = 0\), which can be incorporated into (244) by a suitable redefinition of the constants involved)

\[
\hat{S}_2^{(PF)} = \hat{S}_2^{(PF)} (y_2 A y_2 B) + \hat{S}_2^{(PF)} (y_2 A y_2 B) + \hat{S}_2^{(PF)} (y_2 A y_2 B) + \sum_{A=1}^{n} \hat{S}_2^{(PF)} (f_A^2, f_A \Lambda_A). \quad (246)\]

Inspecting (244) and (246), we observe that the latter component contains at this stage three pieces that mix different graviton sectors, namely those proportional with \(y_i A y_j B\) for \(i, j = 2, 3\) and \(A \neq B\).

Next, we approach the solution \(\hat{S}_2^{(int)}\) to equation (234). We act like in Section 4.3. If we make the notations

\[
2 \left(\hat{S}_1^{(PF)}, \hat{S}_1^{(int)}\right) + \left(\hat{S}_1^{(int)}, \hat{S}_1^{(int)}\right)^{(int)} = \int d^D x \hat{\Delta}^{(int)}, \quad (247)\]

\[
\hat{S}_2^{(int)} = \int d^D x \hat{b}^{(int)}, \quad (248)\]

then equation (234) takes the local form

\[
\hat{\Delta}^{(int)} = -2 s \hat{b}^{(int)} + \partial_{\mu} \hat{n}^{\mu}. \quad (249)\]

Developing \(\hat{\Delta}^{(int)}\) according to the antighost number, we obtain that

\[
\hat{\Delta}^{(int)} = \sum_{I=0}^{2} \hat{\Delta}_I^{(int)}, \quad \text{agh} \left(\hat{\Delta}_I^{(int)}\right) = I, \quad I = 0, 2, \quad (250)\]

with

\[
\hat{\Delta}_2^{(int)} = \gamma \left[ \eta^* \left( p_{APB} (\partial^{\mu} \eta) \eta^{Av} h_{\mu \nu}^B \right. \left. - \left(f_{CAB}^B + p_{APB} \right) V^{\mu} \eta^{Av} \partial_{[\mu} \eta_{\nu]} \right) \right] + \partial_{\nu} \hat{n}_2^{\mu}, \quad (251)\]

\[
\hat{\Delta}_1^{(int)} = \delta \left[ \eta^* \left( p_{APB} (\partial^{\mu} \eta) \eta^{Av} h_{\mu \nu}^B \right. \left. - \left(f_{CAB}^B + p_{APB} \right) V^{\mu} \eta^{Av} \partial_{[\mu} \eta_{\nu]} \right) \right]
\]

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\[
+ \gamma \left\{ p_{ABP} V^{\mu} \left[ (\partial_{\nu} V_{\mu}) h^{A}_{\nu \rho} \eta^{B}_{\rho} + \frac{1}{2} (\partial_{\nu} h^{A}_{\nu \rho}) V^{\nu} \eta^{B}_{\rho} \right]
- \frac{1}{4} V^{\nu} h^{A}_{\mu \rho} (\partial_{\nu} \eta^{B}_{\rho}) - \frac{1}{4} V^{\nu} (\partial_{\nu} h^{A}_{\nu \rho}) h^{B}_{\rho} - \frac{3}{4} h^{A}_{\mu} h^{B}_{\rho} \rho \partial_{\nu} \eta^{B}_{\rho} \right] + \frac{1}{2} \left( f^{C}_{ABPC} + p_{ABP} \right) V^{\mu} V^{\nu} \left[ (\partial_{\nu} h^{A}_{\nu \rho} + \partial_{\nu} h^{A}_{\nu \rho}) \eta^{B}_{\rho} \right]
- h^{A}_{\mu} \rho \partial_{\nu} \eta^{B}_{\rho} - h^{A}_{\mu} \rho \partial_{\nu} \eta^{B}_{\rho} \right] - \delta^{D}_{3} \varepsilon^{\mu \nu \rho} V^{\ast_{1}} \left[ y_{3C} f^{C}_{AB} h^{A}_{\nu} \lambda \partial_{\nu} \eta^{B}_{\rho} \right] + (2 y_{3BP} + y_{3C} f^{C}_{AB}) \eta^{\lambda} \lambda \partial_{\nu} h^{B}_{\rho} \right] + (2 y_{3BP} + y_{3C} f^{C}_{AB}) (D - 2) \sigma^{\mu \nu} \left[ (\partial_{\nu} h^{A}_{\nu \rho}) \eta^{B}_{\rho} \right] + \partial_{\nu} W_{\nu}
\right), \tag{252}
\]

\[
\Delta_{0}^{(\text{int})} = \delta \left\{ p_{ABP} V^{\mu} \left[ (\partial_{\nu} V_{\mu}) h^{A}_{\nu \rho} \eta^{B}_{\rho} + \frac{1}{2} (\partial_{\nu} h^{A}_{\nu \rho}) V^{\nu} \eta^{B}_{\rho} \right]
- \frac{1}{4} V^{\nu} h^{A}_{\mu \rho} (\partial_{\nu} \eta^{B}_{\rho}) - \frac{1}{4} V^{\nu} (\partial_{\nu} h^{A}_{\nu \rho}) h^{B}_{\rho} - \frac{3}{4} h^{A}_{\mu} h^{B}_{\rho} \rho \partial_{\nu} \eta^{B}_{\rho} \right] + \frac{1}{2} \left( f^{C}_{ABPC} + p_{ABP} \right) V^{\mu} V^{\nu} \left[ (\partial_{\nu} h^{A}_{\nu \rho} + \partial_{\nu} h^{A}_{\nu \rho}) \eta^{B}_{\rho} \right]
- h^{A}_{\mu} \rho \partial_{\nu} \eta^{B}_{\rho} - h^{A}_{\mu} \rho \partial_{\nu} \eta^{B}_{\rho} \right] + \frac{16}{D - 2} \left( y_{3A} q_{1} \delta^{D}_{3} h^{A} \eta \right)
\]

\[
+ \gamma \left\{ \frac{p_{ABP}}{8} \left[ V_{\nu} \left( (\partial_{\nu} h^{A}_{\nu \rho}) (\partial_{\nu} h^{B}_{\nu \lambda}) V^{\lambda} - 2 \left( \partial_{\nu} h^{A}_{\nu \rho} \right) h^{B}_{\lambda} \left( \partial_{\nu} V^{\lambda} \right) \right) + h^{A}_{\nu} \rho \partial_{\nu} V^{\lambda} \right] + \frac{1}{2} \left( f^{C}_{ABPC} + p_{ABP} \right) F^{\mu \nu} h^{A}_{\nu} \lambda \partial_{\nu} h^{B}_{\rho} \right] + \frac{1}{16} F_{\mu \nu} \left( h^{A} h^{B} \right)
- 2 h^{A}_{\nu} \rho h^{B}_{\nu \lambda} \left( \partial_{\nu} h^{B}_{\nu \rho} \right) V^{\lambda} - h^{B}_{\mu} \lambda \left( \partial_{\nu} V^{\lambda} \right) \right] + \frac{1}{2} \left( f^{C}_{ABPC} + p_{ABP} \right) \left[ F^{\mu \nu} F_{\nu \rho} + \frac{1}{4} \delta^{\mu}_{\rho} F_{\nu \lambda} F^{\nu \lambda} \right] h^{A}_{\mu} h^{B}_{\rho}

+ q_{1} \delta^{D}_{3} p_{A} \varepsilon^{\mu \nu \lambda \rho} \left( h^{A} V_{\mu} F_{\nu \lambda} - 2 h^{A}_{\lambda} V_{\mu} F_{\nu \lambda} + h^{A}_{\mu} V_{\lambda} F_{\nu \lambda} \right)
\]

51
If we make the notations

\[
\begin{align*}
\hat{b}^{(\text{int})} &= \sum_{i=0}^{2} \hat{b}^{(\text{int})}_i, \quad \text{aggh.} \left(\hat{b}^{(\text{int})}_i\right) = I, \quad I = 0, 2, \\
\hat{\eta}^{\mu} &= \sum_{i=0}^{1} \hat{\eta}^{\mu}_i, \quad \text{aggh.} \left(\hat{\eta}^{\mu}_i\right) = I, \quad I = 0, 1.
\end{align*}
\]

In (253) the functions \(\hat{A}^{(\text{int})BC}, \hat{B}^{(\text{int})AB}, \hat{C}^{(\text{int})BC}, \) and \(\hat{D}^{(\text{int})AB}\) are linear in their arguments, just like in (90).

Acting exactly like in the case of a single graviton (see Section 4.3), we deduce that \(\hat{b}^{(\text{int})}\) and \(\hat{\eta}^{\mu}\) from (249) can be taken to stop at antighost number two and one respectively

\[
\hat{b}^{(\text{int})} = \sum_{i=0}^{2} \hat{b}^{(\text{int})}_i, \quad \text{aggh.} \left(\hat{b}^{(\text{int})}_i\right) = I, \quad I = 0, 2, \\
\hat{\eta}^{\mu} = \sum_{i=0}^{1} \hat{\eta}^{\mu}_i, \quad \text{aggh.} \left(\hat{\eta}^{\mu}_i\right) = I, \quad I = 0, 1.
\]

If we make the notations

\[
\hat{b}^{(\text{int})}_2 = -\frac{1}{2} \eta^* \left[ p_{A\sigma} \left( \partial^{\mu} \eta \right) \eta^{\rho} \eta^{B\mu} - \left( f_{A\sigma B}^{C} + p_{A\sigma B} \right) V^{\mu} \left( \partial^{\nu} \eta \right) \eta^{B\nu} \right] + \hat{b}^{(\text{int})}_2,
\]

\[
\hat{b}^{(\text{int})}_1 = \frac{-p_{A\sigma} V^{\mu}}{2} \left[ \left( \partial_{\rho} \eta \right) h^{A \mu \rho} \eta^{B \rho} + \frac{1}{2} \left( \partial_{\mu} h^{A \rho \sigma} \right) V^{\rho} \eta^{B \sigma} \right] - \frac{1}{4} \left( \partial_{\nu} \eta \right) h^{A \mu \rho} \eta^{B \rho} - \frac{1}{4} h^{A \nu \rho} \partial_{\rho} \eta^{B \mu} - \frac{1}{4} \left( f_{A\sigma B}^{C} + p_{A\sigma B} \right) V^{\mu} V^{\nu} \left[ \left( \partial_{\mu} h^{A \rho \sigma} \right) + \partial_{\sigma} h^{A \rho \mu} \right] \eta^{B \rho} - h^{A \rho \sigma} \partial_{\nu} \eta^{B \rho} - h^{A \rho \sigma} \partial_{\sigma} \eta^{B \rho} + \frac{1}{2} \frac{\delta_{D}^{3}}{\epsilon_{\mu \rho \nu}} V_{\mu} \left[ y_{3C} f_{A\sigma B}^{C} h^{A \lambda \rho} \partial_{\rho} \eta^{B \lambda} \right]
\]

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becomes equivalent with the tower of equations

\[\begin{align*}
+ (2y_{3BPA} + y_{3CF_{AB}}) \eta^{\alpha\lambda} \partial_{\mu} h_{\rho\lambda}^B \\
- \frac{1}{2} y_{2ABP} V^{*\mu} [(D - 2) h^{A \mu}_{\nu} \eta^{B \nu} - \delta^{AB} V \eta] \\
+ \frac{1}{2} h^{*A\mu} [y_{2CF_{AB}} (h_{\mu\nu}^B \eta + 2V_{\mu} \eta_{\nu}) - 2(y_{2ABP}) \\
+ y_{2CF_{AB}}] \sigma_{\mu
u} V^\rho \eta_{\rho}^B] - \frac{8}{D - 2} y_{3A}\frac{\delta^{D}_3}{\delta^{D}_3} h^{*A} \eta + \hat{\eta}_1^{\text{(int)}},
\end{align*}\]

(257)

\[
\hat{b}_0^{(\text{int})} = -\frac{p_{ABP}}{16} \left[ F_{\rho\mu} h^{A\lambda B}_{\nu} h^{B}_{\rho\lambda} + \frac{1}{16} F_{\mu\nu} (h^{A B} - 2h^{A\rho\lambda} h^{B \lambda}_{\rho}) \\
- h^{A \mu}_\nu \left( (\partial_{\mu} h^{B \lambda}_{\rho}) V_{\lambda} - h^{B \lambda}_{\nu} (\partial_{\nu} V_{\lambda}) \right) + \frac{1}{2} \left( F_{\rho\lambda} h^{A \mu}_{\mu \rho} h^{B \lambda}_{\nu} \right) \\
- F_{\mu\nu} h^{A \rho \nu} h^{B}_{\mu} \right] + \frac{1}{4} \left( \left( \partial_{\mu} h^{A \rho}_{\nu} \right) V_{\rho} - h^{A \rho}_{\mu} (\partial_{\nu} V_{\rho}) \right) h^{B}_{\nu}
\]

- \frac{1}{8} \left( f_{ABC} + p_{ABP} \right) \left( F^{\mu\nu} F_{\nu\rho} + \frac{1}{4} \delta^{\mu}_{\nu} F^{\nu\lambda} F_{\lambda\rho} \right) h_{\mu\alpha} h^{B\sigma\rho} \\
- \frac{p_A}{2} q_{\frac{1}{D}} \frac{\delta^{D}_3}{\delta^{D}_3} \epsilon^{\mu\nu\lambda} (h^{A \nu} V_{\mu} F_{\nu\lambda} - 2h^{A \lambda \nu} V_{\mu} F_{\nu\alpha} + h^{A \mu \nu} V_{\alpha} F_{\nu\lambda}) \\
- \frac{p_A}{2} q_{\frac{1}{D}} \frac{\delta^{D}_3}{\delta^{D}_3} \epsilon^{\mu\nu\lambda\alpha\beta} (h^{A \nu} V_{\mu} F_{\nu\lambda} F_{\alpha\beta} - 4h^{A \rho \nu} V_{\mu} F_{\nu\lambda} F_{\alpha\rho}) \\
+ 2h^{A \rho \nu} V_{\rho} F_{\nu\lambda} F_{\alpha\beta} + 8y_{3A} q_{\frac{1}{D}} \frac{\delta^{D}_3}{\delta^{D}_3} V^{*\nu} \partial_{\mu} h^{A \rho}_{\nu} \\
+ \frac{1}{2} (D - 2) (D - 1) (y_{2A} y_{\frac{1}{2}}) V_{\mu} V^\mu + \hat{\eta}_1^{(int)}
\]

(258)

and take into account expansions (254) - (255) and (5), then equation (249) becomes equivalent with the tower of equations

\[\begin{align*}
\gamma \hat{b}_2^{(\text{int})} &= 0, \\
\delta \hat{b}_2^{(\text{int})} + \gamma \hat{b}_1^{(\text{int})} &= \partial_{\mu} \hat{b}_1^\mu + \frac{1}{2} \hat{\chi}_1, \\
\delta \hat{b}_1^{(\text{int})} + \gamma \hat{b}_0^{(\text{int})} &= \partial_{\mu} \hat{b}_0^\mu + \frac{1}{2} \hat{\chi}_0,
\end{align*}\]

(259)

(260)

(261)
where \( \hat{\rho}_I^\mu = \frac{1}{2} (\hat{w}_I^\mu - \hat{n}_I^\mu) \) and

\[
\hat{x}_1 = V_\mu^* \left[ - (f_{ABPC}^C + p_{APB}) F^{\mu\nu} \eta^{A\rho} \partial_{[\mu} \eta_{B]}^B + (y_{3APB} + y_{3BP} \right.
\]

\[+ \left. y_{3C} f_{AB}^C \right] \delta_3^D \varepsilon^{\mu\rho\nu} \left( \partial_{(\nu} \eta_A^\Lambda \right) \partial_{[\rho} \eta_B^{\Lambda \sigma} + (y_{2AP} + y_{2BP} \right.
\]

\[+ \left. y_{2C} f_{AB}^C \right] (D - 2) \sigma^{\mu\nu} \left( \partial_{(\nu} \eta_A^B \right) \eta^{B\rho} \right].
\]

(262)

\[
\hat{x}_0 = \delta \left\{ \delta_3^D \varepsilon^{\mu\rho\nu} V_\mu^* \left[ y_{3C} f_{AB}^C h^{A\Lambda} \partial_{[\mu} \eta_B^{\Lambda \sigma} \right] + (2y_{3BP} \right.
\]

\[+ \left. y_{3C} f_{AB}^C \right] \eta^{A\Lambda} \partial_{[\mu} h_B^{\Lambda \rho} \right\] - y_{2AP} V_{\mu} \right. \left( \partial_{(\nu} \eta_B^B + (D - 2) h^{A\mu} \eta_B^{B\nu} \right.
\]

\[+ \left. \delta_3^A \right] V_{\mu} \right\} + h^{A\mu} \left[ y_{2C} f_{AB}^C \left( h^{A\mu} \eta_{BC} + 2V_{\eta_{\mu}} \right) - 2 (y_{2AP} \right.
\]

\[+ \left. y_{2C} f_{AB}^C \right] \sigma^{\mu\nu} V^{B\rho} \right\} - 4q_1 y_{2A} \delta_3^D (D - 2) \varepsilon_{\mu\nu\rho} F^{\mu\nu} \eta^{A\rho} \right.
\]

\[+ \left. - 6q_2 y_{2A} \delta_5^D \varepsilon_{\mu\nu\rho\beta} F^{\mu\nu} F^{\rho\beta} \eta^{A\beta} + \left( f_{ABPC}^C + p_{APB} \right) (F^{\mu\nu} F_{\nu\rho} \right.
\]

\[+ \left. + \frac{1}{4} \delta_5^D \right] V_{\mu} \right\} \left( h^{A\rho\sigma} \partial_{[\mu} \eta_B^{\sigma} - 2 \partial_{[\mu} h^{A\rho} \eta_B^{\sigma} \right.
\]

\[+ \left. + y_{2A} \left[ - 4DA \right] \right] A_{BC} \hat{A}_0^{(int)} BC \left( \partial \partial \hat{\Phi}_A^\alpha \hat{\Phi}_B^\beta \hat{\eta}_{\alpha_1} \right.
\]

\[+ \left. + p_{BP} \hat{D}_0^{(int)} AB \right] \left( \partial \partial \hat{\Phi}_A^\alpha \hat{\Phi}_B^\beta \hat{\eta}_{\alpha_1} \right)
\]

\[+ \left. + y_{3A} \delta_3^D \right] f_{BC}^A \hat{C}_0^{(int)} BC \left( \partial \partial \hat{\Phi}_A^\alpha \hat{\Phi}_B^\beta \hat{\eta}_{\alpha_1} \right)
\]

\[+ \left. + p_{BP} \hat{D}_0^{(int)} AB \right] \left( \partial \partial \hat{\Phi}_A^\alpha \hat{\Phi}_B^\beta \hat{\eta}_{\alpha_1} \right).
\]

(263)

The component \( S_2^{(int)} \), given by (248), is thus completely determined once we compute \( \hat{b}^{(int)} \), which expands as in (254). The only unknown components from \( \hat{b}^{(int)} \) are \( \left( \hat{j}^{(int)} \right)_I \) appearing in formulas (250)-(258). They are subject to equations (259)-(261). In conclusion, the final step needed in order to construct \( \hat{S}_2^{(int)} \) is to solve equations (259)-(261).

Related to equation (260), we observe that the existence of \( \hat{j}^{(int)}_2 \) and \( \hat{j}^{(int)}_1 \) requires that (262) must be written as

\[
\hat{x}_1 = \delta \hat{\varphi}_2 + \gamma \hat{\omega}_1 + \partial_{\mu} \hat{n}_1^\mu,
\]

(264)

where \( \hat{\varphi}_2, \hat{\omega}_1, \) and \( \hat{n}_1^\mu \) exhibit the same properties like the corresponding unhatted quantities from (106). We require that the second-order deformation
is local, so $\hat{\varphi}_2$, $\hat{\omega}_1$, and $\hat{l}_1^\mu$ must be local functions. Assuming (264) is fulfilled, we apply $\delta$ on it and find the necessary condition

\[
\delta \hat{\chi}_1 = \gamma (-\delta \hat{\omega}_1) + \partial_\mu \left( \delta \hat{l}_1^\mu \right).
\]  
(265)

We do not insist on the investigation of equation (265), which can be done by standard cohomological techniques, but simply state that it can be shown to hold if the following conditions are simultaneously satisfied

\[
F^{\mu\nu} F_{\nu\rho} + \frac{1}{4} \delta_\rho^\mu F_{\nu\lambda} F^{\nu\lambda} = \delta \Omega^\mu_\rho,  
\]  
(266)

\[
F^{\theta\mu} = \delta \tilde{\Omega}^{\theta\mu},  
\]  
(267)

\[
\partial_{[\mu} h_{\lambda]}^{A \theta} = \delta \gamma^{A\theta}_{\mu\lambda},  
\]  
(268)

\[
(\partial_{[\nu} h_{\lambda]}^{A \theta}) \partial^{[\mu} h_{B\nu]}^{B \theta} - (\partial^{[\nu} h^{A \theta]}_{\mu}) \partial_{\nu} h_{\theta\mu}^{B} = \delta \Omega^{AB}.  
\]  
(269)

All the quantities denoted by $\Omega$ or $\tilde{\Omega}$ must be local in order to produce local deformations. It is easy to see, by arguments similar to those exposed in the end of the preamble of Section 4.3, that none of equations (266)–(269) is fulfilled (for local functions), so (265), and therefore (264), cannot hold unless

\[
\hat{\chi}_1 = 0,  
\]  
(270)

which further implies the following equations

\[
f_{AB}^{C} p_{C} + p_{A} p_{B} = 0,  
\]  
(271)

\[
(p_{Ay3B} + p_{By3A} + f_{AB}^{C} y_{3C}) \delta_3^D = 0,  
\]  
(272)

\[
p_{Ay2B} + p_{By2A} + f_{AB}^{C} y_{2C} = 0.  
\]  
(273)

We recall that the constants $f_{AB}^{C}$ are not arbitrary. They have been restricted previously to define the structure constants of a real, commutative, symmetric, and associative (finite-dimensional) algebra, so in addition they satisfy relations (241).

Let us analyze briefly the solutions to (271)–(273). Taking into account (241) and recalling (242), equations (271)–(273) become equivalent to

\[
p_{A} p_{B} = 0, \quad \text{for all } A \neq B,  
\]  
(274)

\[
(p_{Ay3B} + p_{By3A}) \delta_3^D = 0, \quad \text{for all } A \neq B,  
\]  
(275)

\[
p_{Ay2B} + p_{By2A} = 0, \quad \text{for all } A \neq B,  
\]  
(276)
\[ p_A (f_A + p_A) = 0, \quad \text{without summation over } A, \quad (277) \]
\[ (f_A + 2p_A) y_3 A \delta_3^D = 0, \quad \text{without summation over } A, \quad (278) \]
\[ (f_A + 2p_A) y_2 A = 0, \quad \text{without summation over } A. \quad (279) \]

Unlike Section 4.3 where we searched only the solutions relevant from the point of view of deformations, here we must discuss all the solutions, since our aim is to see whether they allow or not cross-couplings among different gravitons. Inspecting (274)–(279), we observe that there appear two complementary cases related to the \( p_A \)'s: either at least one is nonvanishing, say \( p_1 \), or all the \( p_A \)'s vanish. In case I

\[ p_1 \neq 0, \quad (280) \]

so from (277) for \( A = 1 \) it follows that at least \( f_1 \) is non-vanishing

\[ f_1 = -p_1 \neq 0, \quad (281) \]

while (274) restricts all the other \( p_B \)'s to vanish

\[ p_B = 0, \quad B = 2, n. \quad (282) \]

Thus, (275) and (276) for \( A = 1 \) and \( B \neq 1 \) imply

\[ p_1 y_3 B \delta_3^D = 0, \quad p_1 y_2 B = 0, \quad B = 2, n, \quad (283) \]

while (278) and (279) for \( A = 1 \) together with (281) lead to

\[ p_1 y_3 1 \delta_3^D = 0, \quad p_1 y_2 1 = 0. \quad (284) \]

The last two sets of equations, (283) and (284), display a unique solution

\[ y_3 A \delta_3^D = 0 = y_2 A, \quad A = 1, n. \quad (285) \]

In case II

\[ p_A = 0, \quad A = 1, n, \quad (286) \]

equations (274)–(277) are identically satisfied, while the other two take the simple form

\[ f_A y_3 A \delta_3^D = 0, \quad \text{without summation over } A, \quad (287) \]
\[ f_A y_2 A = 0, \quad \text{without summation over } A. \quad (288) \]

Therefore, we have a single option, namely the set \( \{1, 2, \ldots, n\} \) is divided into two complementary subsets such that \( A = (\bar{A}, A') \) with \( \bar{A} \neq A' \) and \((f_{\bar{A}} = 0, y_3 A' \delta_3^D = 0, y_2 A' = 0)\). Re-ordering the indices we can always write

\[ f_{\bar{A}} = 0, \quad \bar{A} = 1, m, \quad y_3 A' \delta_3^D = 0 = y_2 A', \quad A' = m + 1, n. \quad (289) \]

The above solution contains two limit situations: \( m = n \) and \( m = 0 \).
5.2 Main cases. Coupled theories

5.2.1 Case I: no-go results in General Relativity

As we have discussed previously, the first case is governed by the solution

\[ p_1 = -f_1 \neq 0, \quad (p_B)_{B=2,\ldots,n} = 0, \quad (y_{3A}\delta^D_A)_{A=1,\ldots,n} = 0 = (y_{2A})_{A=1,\ldots,n}, \quad (290) \]

so the deformed solution to the master equation in all \( D > 2 \) spacetime dimensions is maximally parameterized by \( (f_A)_{A=1,\ldots,n}, p_1 = -f_1 \neq 0, (\Lambda_A)_{A=1,\ldots,n}, \]

\( q_1\delta^D_A, \) and \( q_2\delta^D_A. \) Of course, it is possible that some of \( f_B \) (for \( B \neq 1 \)), \( \Lambda_A, q_1, \)

or \( q_2 \) vanish. Inserting (290) into (263) we find

\[ \hat{\chi}_0 = 0. \quad (291) \]

Combining this result with (270) we observe that the tower of equations (259)–(261) takes the ‘homogeneous’ form

\[ \gamma \hat{b}_2^{(\text{int})} = 0, \quad (292) \]

\[ \delta \hat{b}_2^{(\text{int})} + \gamma \hat{b}_1^{(\text{int})} = \partial_\mu \hat{\rho}_1^\mu, \quad (293) \]

\[ \delta \hat{b}_1^{(\text{int})} + \gamma \hat{b}_0^{(\text{int})} = \partial_\mu \hat{\rho}_0^\mu, \quad (294) \]

so we can take

\[ \hat{b}_2^{(\text{int})} = \hat{b}_1^{(\text{int})} = \hat{b}_0^{(\text{int})} = 0 \quad (295) \]

and incorporate the ‘homogeneous’ solution into the first-order deformation \( \hat{S}_1^{(\text{int})} \) (see (230)) through a suitable redefinition of the parameterizing constants. At this point we act like in sections 4.3.1 and 4.4.1. Replacing (295) and (290) into (230), (243), (244), (246), and (256)–(258) and regrouping the terms from (228) and (232) with the help of (248) and (254), we find that there are no cross-couplings among different gravitons intermediated by the vector field. The vector field gets coupled to a single graviton (the first one in our convention) and the resulting interactions fit the rules prescribed by General Relativity.

The Lagrangian formulation of the coupled model can be completed by imposing some gauge-fixing conditions similar to (163), one for each graviton sector. If in addition we make the convention

\[ f_1 = 1 = -p_1, \quad (296) \]
then the fully deformed solution to the master equation

\[ \hat{\mathcal{S}}^{(1)} = \hat{\mathcal{S}}' + k \hat{\mathcal{S}}_1^{(1)} + k^2 \hat{\mathcal{S}}_2^{(1)} + \cdots, \]  

(297)

where \( \hat{\mathcal{S}}' \) is the “free” solution (215), leads to a Lagrangian action in which a single graviton \((A = 1)\) couples to the vector field \(V_\mu\) according to the standard coupling from General Relativity, while each of the other gravitons \((B = \overline{2, n})\) interacts only with itself according to an Einstein-Hilbert action (or possibly a Pauli-Fierz action if \(f_B = 0\)) with a cosmological term. Accordingly, in case I we obtain the Lagrangian action

\[
\hat{\mathcal{S}}^{(1)}[h_{\mu\nu}^A, V_\mu] = \int d^Dx \left[\frac{2}{k^2} \sqrt{-g^1} \left( R^1 - 2k^2 \Lambda^1 \right) \right. \\
-\frac{1}{4} \sqrt{-g^1} g^1_{\mu\nu} g^{1\rho\lambda} F^1_{\rho\mu} F^1_{\lambda\nu} + k \left( q_1 \delta^D_3 \varepsilon^{1\mu_1\mu_2\mu_3} \bar{V}^1_{\mu_1} \tilde{F}^1_{\mu_2\mu_3} \\
+ q_2 \delta^D_3 \varepsilon^{1\mu_1\mu_2\mu_3\mu_4\mu_5} \bar{V}^1_{\mu_1} F^1_{\mu_2\mu_3} \bar{F}^1_{\mu_4\mu_5} \right) \\
+ \sum_{B=2}^n \left[ \int d^Dx \frac{2}{k_B^2} \sqrt{-g^B} \left( R^B - 2k_B \Lambda^B \right) \right]
\]
\[\equiv \hat{\mathcal{S}}^{(1)}[g_{\mu\nu}^1, \bar{V}^1_\mu] + \sum_{B=2}^n \hat{\mathcal{S}}^{(E-H)}[\tilde{g}_{\mu\nu}^B], \]  

(298)

where \(\bar{V}^1_\mu\) and \(\bar{F}^1_{\mu\nu}\) are ‘curved’ with the vielbein fields from the first graviton sector

\[
\bar{V}^1_\mu = e^1_{\mu a} V_a, \quad \bar{F}^1_{\mu\nu} = \partial_{[\mu} \left( e^1_{\nu]} V_a \right), \\
\varepsilon^{1\mu_1\mu_2...\mu_D} = \sqrt{-g^1} e^1_{a_1} \cdots e_{a_D}^1 \varepsilon^{a_1...a_D}.
\]  

(300)

The notations \(R^A\) and \(g^A\) \((A = \overline{1, n})\) denote the full scalar curvature and the determinant of the metric tensor \(g^A_{\mu\nu} = \sigma_{\mu\nu} + k_A \delta^A_{\mu\nu}\) (without summation over \(A\)) from the \(A\)-th graviton sector respectively, while \(k_B = kf_B, B = \overline{2, n}\). The final conclusion is that in the first case there is no cross-interaction among different gravitons to all orders in the coupling constant.

### 5.2.2 Case II: no-go results for the new couplings in \(D = 3\)

The second case is subject to the conditions

\[
(p_A)_{\overline{A=1,n}} = 0, \quad (f_A)_{\overline{A=1,m}} = 0, \quad (y_{3A'}^D)_{\overline{A'=m+1,n}} = 0 = (y_{2A'})_{\overline{A'=m+1,n}},
\]  

(301)
so the deformed solution to the master equation is maximally parameterized by \((f_{A'})_{A' = m+1,n} (y_{3A'}^{D})_{A' = 1,n}, (y_{2A'})_{A = 1,m}, (\Lambda_{A'})_{A' = 1,m}, q_{1}\delta_{A'}^{D}, q_{2}\delta_{A'}^{D} \). Substituting (301) into (263), it follows that

\[
\hat{\chi}_0 = -4q_{1}\delta_{3A}^{D}y_{2A} (D-2) \varepsilon_{\mu\nu}\eta^{A\mu} - 6q_{2}\delta_{5A}^{D}y_{2A} \varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} \eta^{A\beta} - \left( \sum_{A=1}^{m} y_{2A}\Lambda^{A}\right) 4D\eta. \tag{302}
\]

Reasoning exactly like in the case of formulas (128) and (154), we deduce that equation (261) demands an equation of the type (129), \(\hat{\chi}_0 = \delta\hat{\varphi}_1 + \gamma\hat{\omega}_0 + \partial_{\mu}\hat{l}^{\mu}\), which cannot be satisfied for local \(\hat{\varphi}_1, \hat{\omega}_0\), and \(\hat{l}^{\mu}_0\) unless

\[
\hat{\chi}_0 = 0, \tag{303}
\]

which further requires

\[
(q_{1}\delta_{3A}^{D}y_{2A})_{A = 1,m} = 0, \quad (q_{2}\delta_{5A}^{D}y_{2A})_{A = 1,m} = 0, \quad \sum_{A=1}^{m} (y_{2A}\Lambda^{A}) = 0. \tag{304}
\]

Clearly, there are two distinct solutions to the above equations

\[
q_{1}\delta_{3A}^{D} = 0 = q_{2}\delta_{5A}^{D}, \quad \sum_{A=1}^{m} (y_{2A}\Lambda^{A}) = 0, \tag{305}
\]

\[
y_{2A} = 0, \quad A = 1,m, \tag{306}
\]

deserving separate analyses. In each subcase (270) and (303) hold, such that equations (259)–(261) take the ‘homogeneous’ form (292)–(294), whose solution can be taken of the form (295).

**Subcase II.1** From (301) and (305) we observe that the deformed solution to the master equation is maximally parameterized in this situation by \((f_{A'})_{A' = m+1,n} (y_{3A'}^{D})_{A' = 1,n}, (y_{2A'})_{A = 1,m}, (\Lambda_{A'})_{A' = 1,m}, \) where in addition the first \(m\) cosmological constants are restricted to satisfy the condition

\[
\sum_{A=1}^{m} (y_{2A}\Lambda^{A}) = 0. \tag{307}
\]
Consequently, the first- and second-order deformations of the solution to the master equation, (228) and (232), read as

\[
\hat{S}_{1}^{(I/II)} = \sum_{A'=m+1}^{n} \left\{ \int d^{D}x \left\{ f_{A'} \left[ \frac{1}{2} \eta^{*A'}_{\mu} \eta^{A'\nu} \partial_{\mu} \eta^{A'}_{\nu} + h^{*A'\mu\nu} \left( \partial_{\rho} \eta^{A'\nu} \right) h^{A'}_{\rho\nu} \right. \right. \\
- \eta^{A'\nu} \partial_{\mu} h^{A'}_{\nu\rho} \right) + \tilde{a}_{0}^{(EH-cubic)A'} \right\} - 2\Lambda_{A'} h^{A'} \right\} \right. \\
+ \sum_{A=1}^{m} \left\{ \int d^{D}x \left[ y_{2A} \left( h^{*A} \eta + (D-2) \left( -V^{*A} \eta^{A} + V^{A} \partial_{\mu} h^{A}_{\mu} \right) \right) \right. \right. \\
+ y_{3} \delta^{D}_{3} \varepsilon_{\mu\nu\rho} \left( V^{*A} \partial^{[\nu} \eta_{\lambda^{\rho]}} + F^{\lambda\mu} \partial^{[\nu} h_{\lambda^{\rho]} A} \right) - 2\Lambda_{A} h^{A} \right\}, \\
\tag{308}
\]

\[
\hat{S}_{2}^{(I/II)} = \sum_{A'=m+1}^{n} \left\{ f_{A'} \left[ f_{A'} S_{2}^{(EH-quartic)A'} + \Lambda_{A'} \int d^{D}x \left( h^{A'\mu\nu} h^{A'}_{\mu\nu} \right) \right. \right. \\
- \frac{1}{2} \left( h^{A'} \right)^{2} \right) \right\} \right. \\
+ \sum_{A,B=1}^{m} \left\{ \int d^{D}x \left[ y_{2A} y_{2B} \left( D-2 \right)^{2} \left( h^{A} h^{B} - h^{A} h^{B} \right) \right. \right. \\
+ y_{2A} y_{3} \delta^{D}_{3} \left( D-2 \right) \varepsilon_{\mu\nu\rho} h^{A}_{\mu} \left( \partial^{[\nu} h_{\lambda^{\rho]} B} \right) + y_{3} y_{3} \delta^{D}_{3} \left( \partial^{[\nu} h_{\lambda^{\rho]} A} \right) \partial_{\nu} h^{B}_{\rho\lambda} \right) \right. \\
+ \frac{1}{2} \left( D-2 \right) \left( D-1 \right) \sum_{A=1}^{m} \left( y_{2A} \right)^{2} \right\} \int d^{D}x \left( V_{\mu} V_{\mu} \right) \\
\tag{309}
\]

respectively. The third-order deformation results from the equation

\[
\left( \hat{S}_{1}^{(I/II)}, \hat{S}_{2}^{(I/II)} \right) + s \hat{S}_{3}^{(I/II)} = 0. \\
\tag{310}
\]

If we make the notations

\[
S_{1}^{(EH-\Lambda)A'} \equiv \int d^{D}x \left\{ f_{A'} \left[ \frac{1}{2} \eta^{*A'\mu} \eta^{A'\nu} \partial_{\mu} \eta^{A'}_{\nu} + h^{*A'\mu\nu} \left( \partial_{\rho} \eta^{A'\nu} \right) h^{A'}_{\rho\nu} \right. \right. \\
- \eta^{A'\nu} \partial_{\mu} h^{A'}_{\nu\rho} \right) + \tilde{a}_{0}^{(EH-cubic)A'} \right\} - 2\Lambda_{A'} h^{A'} \right\}, \\
\tag{311}
\]

\[
S_{2}^{(EH-\Lambda)A'} \equiv f_{A'} \left[ f_{A'} S_{2}^{(EH-quartic)A'} + \Lambda_{A'} \int d^{D}x \left( h^{A'\mu\nu} h^{A'}_{\mu\nu} - \frac{1}{2} \left( h^{A'} \right)^{2} \right) \right] \right\}, \\
\tag{312}
\]

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then we observe that $S_1^{(EH-\Lambda)A'}$ and $S_2^{(EH-\Lambda)A'}$ are nothing but the first- and second-order components respectively (in the coupling constant) of the solution to the master equation corresponding to the full Einstein-Hilbert theory in the presence of a cosmological constant for the graviton $A'$. Therefore,

$$
\left( \sum_{A'=m+1}^{n} S_1^{(EH-\Lambda)A'}, \sum_{B'=m+1}^{n} S_2^{(EH-\Lambda)B'} \right) = -s \left[ \sum_{A'=m+1}^{n} S_3^{(EH-\Lambda)A'} \right], \tag{313}
$$

where $S_3^{(EH-\Lambda)A'}$ is the third-order component of the solution to the master equation associated with the full Einstein-Hilbert theory with a cosmological term in the graviton sector $A'$. By direct computation we then infer that

$$
\left( \hat{S}_1^{(II,1)}, \hat{S}_2^{(II,1)} \right) = s \left[ \frac{4}{D-2} \left( \sum_{A=1}^{m} \tilde{y}_{2A}\tilde{y}_{3A} \right) \left( \sum_{B=1}^{m} \tilde{y}_{3B}\delta_3^{D,h^*B}\eta \right) - \sum_{A'=m+1}^{n} S_3^{(EH-\Lambda)A'} \right] + \left( \sum_{A=1}^{m} \tilde{y}_{2A} \right)^2 (D-2)(D-1) \times
$$

$$
\times \left\{ \sum_{B=1}^{m} \left[ \int d^{D-2} x \left( \frac{(D-2)}{2} \tilde{y}_{2B} \left( h^{\tilde{B}} \eta - 2V^{\lambda\eta}\tilde{B}\eta \right) + \tilde{y}_{3B}\delta_3^{D}\varepsilon_{\mu\nu\rho}F^{\mu\nu}\tilde{B}\eta \right) \right] \right\}, \tag{314}
$$

such that the existence of local solutions to equation (310) demands that $(h^{\tilde{B}} \eta - 2V^{\lambda\eta}\tilde{B}\eta)$ and $\varepsilon_{\mu\nu\rho}F^{\mu\nu}\tilde{B}\eta$ are $s$-exact modulo $d$ quantities from local functions for each $\tilde{B} = \overline{1, m}$. It is easy to show that none of them has this property, so we must set

$$
\left( \sum_{A=1}^{m} (y_{2A})^2 \right) y_{2\tilde{B}} = 0, \quad \tilde{B} = \overline{1, m}, \tag{315}
$$

$$
\left( \sum_{A=1}^{m} (y_{2\tilde{A}})^2 \right) y_{3\tilde{B}}\delta_3^{D} = 0, \quad \tilde{B} = \overline{1, m}. \tag{316}
$$

The solution to these equations,

$$
y_{2\tilde{B}} = 0, \quad \tilde{B} = \overline{1, m}, \tag{317}
$$
solves in addition equation (307). Substituting (317) into (314) and then in (310) we find the equivalent equation
\[
s \left( \hat{S}_3^{(\text{II.1})} - \sum_{A' = m+1}^{n} S_3^{(EH-\Lambda)A'} \right) = 0, \tag{318}
\]
whose solution can be chosen, without loss of generality, of the form
\[
\hat{S}_3^{(\text{II.1})} = \sum_{A' = m+1}^{n} S_3^{(EH-\Lambda)A'}. \tag{319}
\]

We recall that \( S_3^{(EH-\Lambda)A'} \) gathers the contributions of order three in the coupling constant from the solution of the master equation corresponding to the full Einstein-Hilbert action with a cosmological constant for the graviton \( A' \).

Putting together the results expressed by formulas (301), (305), and (317) we conclude that in subcase II.1 the consistency of the deformed solution to the master equation requires the conditions
\[
\begin{align*}
(p_A)_{A=1,n} &= 0 = (y_{2A})_{A=1,n}, \quad (f_A)_{A=1,n} = 0, \tag{320} \\
(y_{3A}A_3^D)_{A'=m+1,n} &= 0, \quad q_1 \delta_3 = 0 = q_2 \delta_5. \tag{321}
\end{align*}
\]

The full deformed solution to the master equation \( \hat{S}^{(\text{II.1})} \) reads as
\[
\hat{S}^{(\text{II.1})} = S' + k \hat{S}_1^{(\text{II.1})} + k^2 \hat{S}_2^{(\text{II.1})} + k^3 \hat{S}_3^{(\text{II.1})} + \cdots, \tag{322}
\]
(with \( S' \) the solution of the master equation for the free model, (215)) and it is maximally parameterized by \( (f_A')_{A'=m+1,n} \), \( (y_{3A}A_3^D)_{A=1,n} \), and the cosmological constants \( (\Lambda_A)_{A=1,n} \). Taking into account relations (215), (308), (309), (319) and notations (311)–(312), we can decompose \( \hat{S}^{(\text{II.1})} \) as a sum between two basic parts
\[
\hat{S}^{(\text{II.1})} = \left( \sum_{A'=m+1}^{n} S^{(EH-\Lambda)A'} \right) + \hat{S}^{(\text{special})} \tag{323}
\]
that are independent one of the other. The first part decomposes into \((n - m)\) components that are all series in the constant coupling \( k \)
\[
S^{(EH-\Lambda)A'} = S' + k S_1^{(EH-\Lambda)A'} + k^2 S_2^{(EH-\Lambda)A'} + k^3 S_3^{(EH-\Lambda)A'} + \cdots,
\]

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with
\[ \bar{S}^{A'} = \int d^D x \left[ \mathcal{L}_0^{(PF)} \left( h_{\mu\nu}^{A'}, \partial_{\lambda} h_{\mu\nu}^{A'} \right) + h^{*A'\mu\nu} \partial_{(\mu} \eta_{\nu)}^{A'} \right] \] (324)
and \( \mathcal{L}_0^{(PF)} \left( h_{\mu\nu}^{A'}, \partial_{\lambda} h_{\mu\nu}^{A'} \right) \) the Pauli-Fierz Lagrangian for the graviton \( A' \). Each \( S^{(EH-\Lambda,A')} \) represents a copy of the solution to the master equation for the full Einstein-Hilbert theory with a cosmological constant associated with the graviton field \( h_{\mu\nu}^{A'} (A' = m + 1, n) \), so they cannot produce couplings among different gravitons. We emphasize that none of the \( (n - m) \) gravitons gets coupled to the vector field \( V_{\mu} \). Let us analyze in more detail the second part. It stops at order two in the coupling constant.

In order to focus in more detail on (325) we take the limit situation \( m = n \) (so \( \bar{A} \rightarrow A \)) in the conditions (320)–(321) and work in \( D = 3 \), such that the entire deformed solution to the master equation, \( \bar{S}^{(II,1)} \), consistent to all orders in the coupling constant, reduces to (325). We can express \( \bar{S}^{(\text{special})} \) in a nicer form by acting in a manner similar to that followed in Section 4.4.2. Based on the observation that the deformed solution to the master equation is unique up to addition of \( s \)-exact terms, in the sequel we work with

\[ \bar{S}^{(\text{special})} \bigg|_{m=n}^{D=3} - s \left\{ 2k^2 \sum_{A,B=1}^{n} \left[ \int d^3 x y_{3}^{A} y_{3}^{B} \left( h^{*A\mu\nu} h_{\mu\nu}^{B} + \eta^{*A\mu} \eta_{\mu}^{B} \right) \right] \right\} 
= \int d^3 x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + V_{\mu}^{*} \partial_{\mu} \eta + \sum_{A=1}^{n} \left[ \mathcal{L}_0^{(PF)} \left( h_{\mu\nu}^{A}, \partial_{\lambda} h_{\mu\nu}^{A} \right) \right. \right.
- 2k \Lambda h_{\mu\nu}^{A} h^{*A\mu\nu} \partial_{(\mu} \eta_{\nu)}^{A} + k y_{3}^{A} \varepsilon_{\mu\nu\rho} \left( V_{\mu}^{*} \partial_{(\nu} \eta_{\rho)}^{A} - F_{\mu\nu} \partial_{(\mu} h_{\rho)}^{A} \right) \} \]
\[ +2k^2 \sum_{A,B=1}^{n} \left[ y_3^A y_3^B \left( \partial_{[\mu} h_{\rho]}^A \mu \right) \partial_{[\nu} h_{\lambda]}^B \nu \sigma^{\rho\lambda} \right] \]. (326)

The part of antighost number zero gives the Lagrangian action of the coupled model

\[ \hat{S}^{(II,1)}[h_{\mu\nu}, V^\mu] = \int d^3x \left\{ -\frac{1}{4} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} + \sum_{A=1}^{n} \left[ \mathcal{L}^{(PF)}_0 \left( h_{\mu\nu}^A, \partial_{[\mu} h_{\rho]}^A \right) 
-2k\Lambda_A h^A - ky_3^A \epsilon^{\mu\nu\rho} F_{\mu\nu} \partial_{[\theta} h_{\rho]}^A \theta \right] 
+2k^2 \sum_{A,B=1}^{n} \left[ y_3^A y_3^B \left( \partial_{[\mu} h_{\rho]}^A \mu \right) \partial_{[\nu} h_{\lambda]}^B \nu \sigma^{\rho\lambda} \right] \right\} \] (327)

and the terms of antighost one provide its gauge symmetries

\[ \delta^{(II,1)}_\xi h^A_{\mu\nu} = \partial_{(\mu} \xi^A_{\nu)}, \quad \delta^{(II,1)}_\xi V^\mu = \partial^\mu \xi + k \sum_{A=1}^{n} \left( y_3^A \epsilon^{\mu\nu\rho} \partial_{[\mu} h_{\rho]}^A \theta \right). \] (328)

This Lagrangian action can be brought to a simpler form by redefining the field strength of the vector field as

\[ \hat{F}^{\mu\nu} = F^{\mu\nu} + 2k \sum_{A=1}^{n} \left( y_3^A \epsilon^{\mu\nu\rho} \partial_{[\mu} h_{\rho]}^A \theta \right), \] (329)

in terms of which

\[ \hat{S}^{(II,1)}[h_{\mu\nu}, V^\mu] = \int d^3x \left[ \sum_{A=1}^{n} \left( \mathcal{L}^{(PF)}_0 \left( h_{\mu\nu}^A, \partial_{[\mu} h_{\rho]}^A \right) -2k\Lambda_A h^A - \frac{1}{4} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} \right) \right]. \] (330)

The absence of terms of antighost number strictly greater than one indicates that the deformed gauge symmetries (328) are independent and Abelian (their commutators close everywhere in the space of field histories). We remark that this case corresponds to the situation from Section 4.4.2 (in the absence of internal Pauli-Fierz indices), where we obtained a result complementary to the usual couplings prescribed by General Relativity. The gauge symmetries of the vector field are modified by terms proportional with the antisymmetric first-order derivatives of the Pauli-Fierz gauge parameters, while the gravitons keep their original gauge symmetries. The invariance of
$\hat{S}^{L(II.1)}$ under (328) is ensured by the gauge invariance of the deformed field strength, $\delta^{(II.1)}_\epsilon \hat{F}_{\mu\nu} = 0$.

Unfortunately, action (330) does not describe in fact cross-couplings between different spin-two fields. In order to make this observation clear, let us denote by $Y$ the matrix of elements $y^A_3y^B_3$. It is simple to see that the rank of $Y$ is equal to one. By an orthogonal transformation $M$ we can always find a matrix $\hat{Y}$ of the form

$$\hat{Y} = M^T Y M,$$

(331)

with $M^T$ the transposed of $M$, such that

$$\hat{Y}^{11} = \sum_{A=1}^n (y^A_3)^2 \equiv \lambda^2, \quad \hat{Y}^{1A'} = \hat{Y}^{B'1} = \hat{Y}^{A'B'} = 0, \quad A', B' = 2, n.$$

(332)

If we make the notation

$$\hat{y}^A = M^{AC} y^C_3,$$

(333)

then relation (332) implies

$$\hat{y}^A = \lambda \delta_1^A.$$

(334)

Now, we make the field redefinition

$$h^A_{\mu\nu} = M^{AC} \hat{h}^C_{\mu\nu},$$

(335)

with $M^{AC}$ the elements of $M$. This transformation of the spin-two fields leaves $\sum_{A=1}^n L_0^{(PF)} (\hat{h}^A_{\mu\nu}, \partial_\lambda \hat{h}^A_{\mu\nu})$ invariant and, moreover, based on the above results, we obtain

$$\sum_{A,B=1}^n \left[ y^A_3 y^B_3 \left( \partial_{[\mu} \hat{h}^A_{\nu]} \right) \partial_{[\rho} \hat{h}^B_{\lambda]} \nu \sigma^{\rho\lambda} \right] = \lambda^2 \left( \partial_{[\mu} \hat{h}^1_{\nu]} \right) \partial_{[\rho} \hat{h}^1_{\lambda]} \nu \sigma^{\rho\lambda},$$

(336)

$$\sum_{A=1}^n (y^A_3 \varepsilon^{\mu\nu\rho} F_{\mu\nu\rho} \partial_\theta h^A_{\theta \nu}) = \lambda \varepsilon^{\mu\nu\rho} F_{\mu\nu\rho} \partial_\theta \hat{h}^1_{\theta \nu},$$

(337)

such that (330) becomes

$$\hat{S}^{L(II.1)}[\hat{h}^A_{\mu\nu}, V^\mu] = \int d^3x \left[ \sum_{A=1}^n \left( L_0^{(PF)} \left( \hat{h}^A_{\mu\nu}, \partial_\lambda \hat{h}^A_{\mu\nu} \right) - 2k \Lambda_A \hat{h}^A \right) - \frac{1}{4} \hat{F}^\nu_{\mu\nu} \hat{F}^{\eta\mu} \right],$$

(338)
where
\[
\hat{\Lambda}_A = \Lambda_B M^{BA}, \quad \hat{F}^{\mu\nu} = F^{\mu\nu} + 2k\lambda\epsilon^{\mu\nu\rho\sigma}\partial_\rho\hat{h}_1^{\theta}. \quad (339)
\]
Action (338) is invariant under the gauge transformations
\[
\delta^{(III)}(\hat{\eta}^A_{\mu\nu}) = \partial_\mu \hat{\epsilon}^A_\nu, \quad \delta^{(III)}(V^\mu) = \partial^\mu \epsilon + k\lambda\epsilon^{\mu\nu\rho\sigma}\partial_\nu \hat{\epsilon}^1_\rho, \quad (340)
\]
where
\[
\hat{\epsilon}^A_\mu = \epsilon^B_{\mu} M^{BA}. \quad (341)
\]
We observe that action (338) decouples into action (199) (derived in Section 4.4.2) for the first spin-two field \((A = 1)\) and a sum of Pauli-Fierz actions with cosmological terms for the remaining \((n - 1)\) spin-two fields. In conclusion, we cannot couple different spin-two fields even outside the framework of General Relativity.

Subcase II.2 Now, we start from conditions (301) and (306), such that the deformed solution to the master equation is maximally parameterized in this situation by \((f_A')_{A'=m+1, n, \ldots}, (y_3\delta_D^A)_{A=1, m}, (\Lambda A)_{A=1, m}, q_1\delta_D^1, \text{ and } q_2\delta_D^5\). Without entering unnecessary details, we only mention that this case is similar to subcase II.1.2 in the absence of Pauli-Fierz internal indices, briefly discussed in the final part of Section 4.3.2. The consistency of the deformed solution to the master equation goes on unobstructed up to order five in the coupling constant, where the existence of a local \(\hat{S}^{(II.1.2)}_5\) requires a condition of the type \(y_3^2 q_1^2 = 0\), namely
\[
q_1^2 \left( \sum_{A=1}^{m} (y_3 A)^2 \right) y_3 B \delta_D^1 = 0, \quad \bar{B} = \bar{1}, m. \quad (342)
\]
There are two main possibilities, none of them leading to cross-couplings between different spin-two fields. Thus, if we take \(D \neq 3\), then no couplings among different gravitons are allowed since the Lagrangian of the interacting model is a sum of independent Einstein-Hilbert Lagrangians with cosmological terms for the last \((n - m)\) gravitons (none of them coupled to the vector field), a sum of Pauli-Fierz Lagrangians plus simple cosmological terms \(-2k\Lambda A h^A\) for the first \(m\) gravitons and the Maxwell Lagrangian supplemented by the generalized Abelian Chern-Simons density \(kq_2\delta_D^5 \epsilon^{\mu\nu\lambda\alpha\beta} V_\mu F_{\nu\lambda} F_{\alpha\beta}\). If \(D = 3\), then either \(q_1 = 0\), in which situation we re-obtain the case from the previous section, described by formula (323), where we have shown that there
are no cross-couplings between different gravitons, or \((y_3 A) \tilde{\omega} = 0\), such that again no cross-couplings are permitted and the resulting Lagrangian is like in the above for \(D \neq 3\) (after formula (342)) up to replacing the density \(k q_\omega \delta Y \varepsilon^{\mu \nu \lambda \alpha} V_\mu F_{\nu \lambda} F_{\alpha \beta}\) with the standard Abelian Chern-Simons term \(k q_\omega \varepsilon^{\mu \nu \lambda} V_\mu F_{\nu \lambda}\).

6 Generalization to an arbitrary \(p\)-form

The results obtained so far in the presence of a massless vector field can be generalized to the case of deformations for one or several gravitons and an arbitrary \(p\)-form gauge field.

In the case of a single graviton the starting point is the sum between the Pauli-Fierz action and the Lagrangian action of an Abelian \(p\)-form with \(p > 1\)

\[
S^L_0 [h_{\mu \nu}, V_{\mu_1 ... \mu_p}] = \int d^D x \left( \mathcal{L}_{(PF)}^0 - \frac{1}{2 \cdot (p + 1)!} F_{\mu_1 ... \mu_{p+1}} F^{\mu_1 ... \mu_{p+1}} \right),
\]

in \(D \geq p + 1\) spacetime dimensions, with \(F_{\mu_1 ... \mu_{p+1}}\) the Abelian field strength of the \(p\)-form gauge field \(V_{\mu_1 ... \mu_p}\)

\[
F_{\mu_1 ... \mu_{p+1}} = \partial_{[\mu_1} V_{\mu_2 ... \mu_{p+1}]}.
\]

This action is known to be invariant under the gauge transformations

\[
\delta e h_{\mu \nu} = \partial (\mu e_\nu), \quad \delta e V_{\mu_1 ... \mu_p} = \partial_{[\mu_1} e^{(p)}_{\mu_2 ... \mu_p]}.
\]

Unlike the Maxwell field \((p = 1)\), the gauge transformations of the \(p\)-form for \(p > 1\) are off-shell reducible of order \((p - 1)\). This property has strong implications at the level of the BRST complex and of the BRST cohomology in the form sector: a whole tower of ghosts of ghosts and of antifields will be required in order to incorporate the reducibility, only the ghost of maximum pure ghost number, \(p\), will enter \(H(\gamma)\), and the local characteristic cohomology will be richer in the sense that (33) and (35) become

\[
H_J (\delta |d) = 0 = H_j^{\mu \nu} (\delta |d), \quad J > p + 1.
\]

In spite of these new cohomological ingredients, which complicate the analysis of deformations, the results from Sections 4.4.1 and 4.4.2 can still be generalized.
Thus, two complementary cases are revealed. One describes the standard graviton-$p$-form interactions from General Relativity and leads to a Lagrangian action similar to (195) up to replacing $(1/4) g^{\mu\nu} g^{\rho\lambda} \bar{F}_{\mu\rho} \bar{F}_{\nu\lambda}$ with the expression $((2 \cdot (p + 1))^{-1} g^{\mu_1\nu_1} \cdots g^{\mu_{p+1}\nu_{p+1}} \bar{F}_{\mu_1\cdots\mu_{p+1}} \bar{F}_{\nu_1\cdots\nu_{p+1}}$ and, if $p$ is odd, also the terms containing $\delta^D_{3} \varepsilon_{\mu_1\mu_2\mu_3}$ and $\delta^D_{5} \varepsilon_{\mu_1\mu_2\mu_3\mu_4\mu_5}$ with some densities involving $\delta^D_{2p+1} \varepsilon_{\mu_1\cdots\mu_{2p+1}}$ and $\delta^D_{3p+2} \varepsilon_{\mu_1\cdots\mu_{3p+2}}$ respectively (if $p$ is even, the terms proportional with either $q_1$ or $q_2$ must be suppressed). The other case emphasizes that it is possible to construct some new deformations in $D = p + 2$, describing a spin two-field coupled to a $p$-form and having (343) and (345) as a free limit, which are consistent to all orders in the coupling constant and are not subject to the rules of General Relativity. Their source is a generalization of the terms proportional with $y_3$ from the first-order deformation (71)

$$S^{(\text{int})}_1(y_3) = y_3 \varepsilon_{\mu_1\cdots\mu_p\nu} \int d^{p+2} x \left( V_{\nu} \varepsilon_{\mu_1\cdots\mu_p} \partial^2 h^\rho \right) + \frac{1}{p!} F_{\mu_1\cdots\mu_p} \bar{F}_{\nu_1\cdots\nu_p} \bar{F}_{\lambda\mu_1\cdots\mu_p} \partial \left[ \nu \rho \partial^2 h^\lambda \right].$$

Performing the necessary computations, we find the Lagrangian action

$$S^L[h_{\mu\nu}, V_{\mu_1\cdots\mu_p}] = \int d^{p+2} x \left( \mathcal{L}^{(PF)}_0 - 2k \Lambda h - \frac{1}{2 \cdot (p + 1)!} F'_{\mu_1\cdots\mu_{p+1}} F''_{\nu_1\cdots\nu_{p+1}} \right),$$

where the field strength of the $p$-form is deformed as

$$F'_{\mu_1\cdots\mu_{p+1}} = F_{\mu_1\cdots\mu_{p+1}} + 2 (-)^{p+1} k y_3 \varepsilon_{\mu_1\cdots\mu_{p+1}\nu} \partial^2 h^\rho.$$  

This action is fully invariant under the original Pauli-Fierz gauge transformations and

$$\bar{\delta}_e V_{\mu_1\cdots\mu_p} = \partial_{[\mu_1} e_{\mu_2\cdots\mu_p]} + k y_3 \varepsilon_{\mu_1\cdots\mu_{p+1}} \partial^2 h^\rho.$$  

The gauge algebra remains Abelian and the reducibility of (350) is not affected by these couplings: the associated functions and relations are the initial ones.

It is important to notice that all the standard hypotheses imposed to consistent deformations are fulfilled. Indeed, in the free limit ($k = 0$) the field strength (349) is restored to its original form (344), the cosmological term $-2k \Lambda h$ is destroyed, and the Pauli-Fierz gauge parameters $e^\rho$ are discarded from the gauge transformations $\bar{\delta}_e V_{\mu_1\cdots\mu_p}$, leaving us with the original action
and initial gauge transformations (345). Also, the spacetime locality, Lorentz covariance, and Poincaré invariance of action (348) are obvious. Likewise, the smoothness of the deformed theory in the coupling constant is ensured by the polynomial behaviour of (348) and (350) with respect to $k$: the action is a polynomial of order two and the gauge transformations are polynomials of order one. Furthermore, the differential order of the coupled field equations is preserved with respect to that of the free equations (derivative order assumption), being equal to two, as it can be observed from the concrete form of the Euler-Lagrange derivatives of action (348):

$$\frac{\delta S_L}{\delta h_{\mu\nu}} = \frac{\delta S_L^0[h_{\mu\nu}, V_{\mu_1...\mu_p}]}{\delta h_{\mu\nu}} - 2k\Lambda\sigma^{\mu\nu}$$

$$- \frac{k y_3}{(p + 1)!} \left[ (-)^p \varepsilon^{\mu_1...\mu_{p+1}(\mu} \partial^{\nu)} F'_{\mu_1...\mu_{p+1}} 
+ 2\sigma^{\mu\nu} \varepsilon^{\mu_1...\mu_{p+2}} \partial_{\mu_1} F'_{\mu_2...\mu_{p+2}} \right]$$

$$\equiv \Box h^{\mu\nu} + \left(1 + 4k^2 y_3^2\right) \partial^\mu \partial^\nu h - \left(1 + 2k^2 y_3^2\right) \partial^{(\mu} \partial_{\nu)} h^{\rho)} - 2k\Lambda\sigma^{\mu\nu} + \left(-\right)^{p+1} \frac{k y_3}{(p + 1)!} \varepsilon^{\mu_1...\mu_{p+1}(\mu} \partial^{\nu)} F'_{\mu_1...\mu_{p+1}}, \quad (351)$$

$$\frac{\delta S_L}{\delta V_{\mu_1...\mu_p}} = \frac{1}{p!} \partial_\nu F'^{\nu\mu_1...\mu_p}$$

$$= \frac{\delta S_L^0[h_{\mu\nu}, V_{\mu_1...\mu_p}]}{\delta V_{\mu_1...\mu_p}} - \frac{k y_3}{p!} \varepsilon^{\mu_1...\mu_{p}\nu\rho} \partial_\nu \partial^\rho h^{\rho}_{\theta}$$

$$\equiv \frac{1}{p!} \left( \partial_\nu F'^{\nu\mu_1...\mu_p} - k y_3 \varepsilon^{\mu_1...\mu_{p}\nu\rho} \partial_\nu \partial^\rho h^{\rho}_{\theta} \right). \quad (352)$$

It is truly remarkable that these new couplings comply with the derivative order assumption.

Let us analyze the main physical consequences of these new couplings. First, we investigate some direct outcomes of the field equations, obtained by equating (351) and (352) to zero. By taking the trace of the field equations for the graviton, $\sigma_{\mu\nu} \delta S_L/\delta h_{\mu\nu} = 0$, we infer the equivalent equation

$$K = \frac{2k\Lambda (p + 2)}{p + (p + 1) 4k^2 y_3^2}, \quad (353)$$

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where $K$ is the linearized scalar curvature, $K = \partial_\rho \partial_\theta h^{\rho \theta} - \Box h$. Due to the presence of the cosmological constant, the linearized scalar curvature is a non-vanishing constant. The field equation of the $p$-form, $\delta S^L/\delta V_{\mu_1...\mu_p} = 0$, is nothing but a nontrivial conservation law of order $(p + 1)$, $\partial_\nu F^{\nu \mu_1...\mu_p} = 0$, where the associated current is precisely the deformed field strength (349). It is not a usual conservation law because it results from some rigid symmetries of the solution to the master equation for the coupled theory. The main difference between the free theory (343) and the coupled one is that the $(p + 1)$-order conservation law of the latter contains a nontrivial component from the Pauli-Fierz sector, $2 (-)^{p+1} k y_3 \varepsilon^{\nu \mu_1...\mu_p \rho} \partial_\theta h_\rho^{\theta}$. Another interesting observation is that, unlike the free limit (343), the field equations of the coupled model admit to be written in a compact form. Indeed, it can be shown that both field equations, $\delta S^L/\delta h_{\mu \nu} = 0$ and $\delta S^L/\delta V_{\mu_1...\mu_p} = 0$, are completely equivalent with the following expression of the first-order derivatives of the Abelian field strength (344)

$$\partial^\nu F_{\mu_1...\mu_{p+1}} = \left(-\right)^p \varepsilon_{\mu_1...\mu_{p+1} \rho} \left\{ 2 k y_3 \partial^\nu \partial_\theta h^{\rho \theta} + \frac{1}{k y_3} \left[ 2k\Lambda \sigma^{\nu \rho} - \Box h^{\nu \rho} \right. \right.$$

$$- (1 + 4 k^2 y_3^2) \partial^\nu \partial^\rho h + \left. (1 + 2 k^2 y_3^2) \partial^{(\nu} \partial_\theta h^{\rho)} \theta \right. \right.$$

$$- (1 + 4 k^2 y_3^2) \sigma^{\mu \rho} \left( \partial_\lambda \partial_\theta h^{\lambda \theta} - \Box h \right) \right\}. \tag{354}$$

The direct as well as the converse implication results from simple algebraic manipulations of the coupled field equations or respectively of (354) and also by means of the identity

$$\varepsilon^{\nu \mu_1...\mu_p \rho} \partial^\rho F_{\mu_1...\mu_p} = \left(-\right)^{p+1} \frac{1}{p+1} \varepsilon^{\mu_1...\mu_{p+1} [\mu \partial^\nu]} F_{\mu_1...\mu_{p+1}], \tag{355}$$

valid in $D = p + 2$.

Regarding a collection of spin-two fields and a $p$-form, it can be used a line similar to that employed in Section 5. Thus, it can be shown that two complementary cases are again unfolded. One is similar to the situation discussed in Section 5.2.1 and the other with the result from Section 5.2.2. In both cases there are no cross-couplings among different spin-two fields intermediated by a $p$-form gauge field: the $p$-form couples to a single spin-two field.
7 Conclusion

To conclude with, in this paper we have investigated the couplings between a single spin-two field or a collection of such fields (described in the free limit by a sum of Pauli-Fierz actions) and a massless $p$-form using the powerful setting based on local BRST cohomology. Under the hypotheses of locality, smoothness in the coupling constant, Poincaré invariance, Lorentz covariance, and preservation of the number of derivatives on each field (plus positivity of the metric in the internal space in the case of a collection of spin-two fields), we found two complementary situations. One submits to the well-known prescriptions of General Relativity, but the other situation discloses some new type of couplings in $(p + 2)$ spacetime dimensions, which only modify the gauge symmetries of the $p$-form. It is remarkable that these $(p + 2)$-dimensional cross-couplings comply with the derivative order assumption, unlike other situations from the literature. Unfortunately, in the case of a collection of spin-two fields none of these coupled theories allows for (indirect) cross-couplings between different gravitons.

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