Born–Infeld extension of Lovelock brane gravity

Miguel Cruz\(^1\) and Efraín Rojas\(^2\)

\(^1\)Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN, Apdo Postal 14-740, 07000 México DF, Mexico
\(^2\)Departamento de Física, Facultad de Física e Inteligencia Artificial, Universidad Veracruzana, 91000 Xalapa, Veracruz, Mexico

E-mail: mcruz@fis.cinvestav.mx and efrojas@uv.mx

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Abstract
We present a Born–Infeld-type theory to describe the evolution of \(p\)-branes propagating in an \(N=(p+2)\)-dimensional Minkowski spacetime. The expansion of the BI-type volume element gives rise to the \((p+1)\) Lovelock brane invariants associated with the worldvolume swept out by the brane. Contrary to the Lovelock theory in gravity, the number of Lovelock brane Lagrangians differs in this case, depending on the dimension of the worldvolume as a consequence that we consider the embedding functions, instead of the metric, as the field variables. This model depends on the intrinsic and the extrinsic geometries of the worldvolume and in consequence is a second-order theory as shown in the main text. A classically equivalent action is discussed and we comment on its Weyl invariance in any dimension which naturally requires the introduction of some auxiliary fields.

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1. Introduction

The usefulness of Born–Infeld structures has grown by leaps and bounds over the past few years because of their attractive geometric properties and capacity to implement some finite bounds in several physical theories [1–9]. These structures have found a natural place in the brane context as they may contain unknown dynamic symmetries, thus providing physical implications. Related constructions including higher order brane curvature terms but ignoring electromagnetism have also been considered [10, 11]. However, among the plethora of these approaches, no purely geometrical BI-like construction combining both the intrinsic and extrinsic geometries of a brane trajectory, leading to second-order equations, has been proposed. In this sense, it seems reasonable to ask whether there exists a connection between BI structures and the Lovelock invariants defined on the worldvolume.

The Lovelock brane invariants are second-order geometric terms constructed with the extrinsic curvature of the worldvolume swept out by the \(p\)-brane that yield second-order equations of motion. This fact is important because it ensures no propagation of extra degrees
of freedom and this renders a given system free from many of the pathologies that plague higher order derivative theories. Interest in the Lovelock brane theory has attracted much attention recently, not only for its rich structure but also for its applications to describe cosmological phenomena [12–14].

The purpose of this work is to provide a BI-type brane theory that describes the evolution of \( p \)-dimensional branes propagating in an \( N = (p + 2) \)-dimensional Minkowski spacetime, which may be written in terms of the Lovelock brane invariants associated with the co-dimension 1 worldvolume. The Lagrangian density of such a model is written close in spirit to the one developed in [3]. This contains contribution from both intrinsic and extrinsic geometries through the first and second fundamental forms \( g_{ab} \) and \( K_{ab} \), respectively, inherited by the co-dimension 1 worldvolume. We find that the expansion of this alternative volume element casts out the Lovelock brane invariants for a given dimension \( p \), which can be collected in a finite series. To explore the structure of the Lovelock brane invariants, we exploit the so-called conserved stress–tensor [15]. This will permit us to know their mechanical content in an effortless way. In fact, this tool is nothing but the Noether current associated with the translation invariance of the theory. In addition, to complement our proposal we examine a classically equivalent action and we discuss its Weyl invariance through the introduction of an auxiliary metric and a scalar field. This may be useful in the construction of a Hamiltonian formulation for the theory thus avoiding the higher nonpolynomial dependence of the field variables.

This paper is organized as follows. The aim of section 2 is to acquaint the reader with the basic facts of the Lovelock theory for branes propagating in a flat background spacetime which will be used throughout the paper. We show that the Lovelock brane invariants can all be obtained by using the antisymmetric products of the extrinsic curvature tensor and the Gauss–Codazzi integrability condition for surfaces. We obtain the general equations of motion in terms of the conserved tensors. In section 3, we introduce a BI-type action containing the induced metric and the extrinsic curvature whose curvature expansion leads to the Lovelock brane Lagrangians for a given dimension \( p \). We also discuss the Weyl invariance of a classically equivalent action. We conclude in section 4 with some comments and we discuss our results. Appendices A and B gather information about the standard Lovelock theory notation and some mathematical relations useful for expanding the BI-like structures.

2. Lovelock brane theory

Consider a spacelike brane of dimension \( p \) floating in a flat background spacetime of dimension \( N = (p + 2) \) with metric \( \eta_{\mu \nu} \) (\( \mu, \nu = 0, 1, \ldots, p + 1 \)). The brane sweeps out an oriented hypersurface manifold of dimension \( p + 1 \), known as worldvolume and denoted by \( \Sigma \), described by the embedding functions \( x^\mu = X^\mu(\xi^a) \) where \( x^\mu \) are the local coordinates in the background spacetime and \( \xi^a \) are the local coordinates for \( \Sigma \) and \( X^\nu \) are the embedding functions (\( a, b = 0, 1, \ldots, p \)). The only geometrically significant derivatives of \( X^\mu \) are encoded in the induced metric tensor \( g_{ab} = \eta_{\mu \nu} e^\mu_a e^\nu_b =: e_a \cdot e_b \) and the extrinsic curvature tensor \( K_{ab} = -n \cdot \nabla_a e_b = K_{ba} \), where \( e^\mu_a = \partial_a X^\mu \) are the tangent vectors to the worldvolume and \( n^\mu \) denotes the normal vector to \( \Sigma \) such that \( n \cdot e_a = 0 \), and \( \nabla_a \) is the covariant derivative compatible with the induced metric \( g_{ab} \).

For a \((p + 1)\)-dimensional worldvolume whose field variables are the embedding functions, the action

\[
S[X] = \int d^{p+1}\xi \sqrt{-g} \sum_{n=0}^{p+1} \alpha_n L_n(g_{ab}, K_{ab}),
\]

(1)
where

\[ L_0 (g_{ab}, K) = \delta^{a_1 a_2 \ldots a_n}_{b_1 b_2 \ldots b_n} K^{a_1}_{a_2} K^{a_3}_{a_4} \ldots K^{a_n}_{a_n} \]  

(2)

ensures that the field equations of motion are of second order, as we will see below.

Here, \( \delta^{a_1 a_2 \ldots a_n}_{b_1 b_2 \ldots b_n} \) denotes the generalized Kronecker delta (gKd) (see appendix A for details), \( g = \det(g_{ab}) \) and \( \alpha_a \) are the generic coefficients with appropriate dimensions. Furthermore, we set \( L_0 = 1 \). The action (1) is a second-order derivative theory which is invariant under reparameterizations of the worldvolume. Since the independent variables are the embedding functions instead of the metric, we have a greater number of the Lovelock brane terms contrary to the standard gravity case [16]. By expanding out equation (2) in terms of minors we have

\[ L_n = \left[ \delta^{a_1 a_2 \ldots a_n}_{b_1 b_2 \ldots b_n} - \delta^{a_1 a_2 \ldots a_n}_{c_1 c_2 \ldots c_n} + \delta^{a_1 a_2 \ldots a_n}_{d_1 d_2 \ldots d_n} + \ldots + (-1)^{n-1} \delta^{a_1 a_2 \ldots a_n}_{b_1 b_2 \ldots b_{n-1}} \right] 
\times K^{a_1}_{a_2} K^{a_3}_{a_4} \ldots K^{a_n}_{a_n}, \]

(3)

where we have relabeled some indexes besides using the antisymmetric properties of the gKd. The iterative expansion of the remaining gKd in the latter equation yields

\[ L_n = \sum_{r=1}^{n} \frac{(-1)^{r-1}(n-1)!}{(n-r)!} K^{a_1}_{a_2} L_{n-r}, \]

(4)

where \( n = 1, 2, 3, \ldots, p + 1 \), and we have introduced the useful short-hand notation

\[ K_{ab}^1 := K^{a}_a K^b_b, K_{ab}^2 := K^{a}_a K^b_c K^c_b, \ldots, K_{ab}^p := K^{a}_a K^b_i K^i_j K^j_a \]

and so on. From this recursion formula, we can compute the first Lovelock brane Lagrangians

\[ L_0 = 1, \]

(5a)

\[ L_1 = K, \]

(5b)

\[ L_2 = K^2 - K_{ab}^2 = R, \]

(5c)

\[ L_3 = K^3 - 3K K_{ab}^2 + 2K_{ab}^3 = -K^3 + \frac{3}{2} K^2 - \frac{1}{2} K^3 - 3G_{ab} K_{ab}, \]

(5d)

\[ L_4 = K^4 - 6K^2 K_{ab}^2 + 8K_{ab}^3 + 3(K_{ab}^2)^2 - 6K_{ab}, \]

\[ = R^2 - 4R_{ab} R^{ab} + R_{abcd} R^{abcd}, \]

(5e)

\[ L_5 = K^5 - 10K^3 K_{ab}^2 + 20K^2 K_{ab}^3 - 30K_{ab}^4 + 15K(K_{ab}^2)^2 - 20K_{ab}^3 K_{ab}^2 + 24K_{ab}^5, \]

(5f)

\[ L_6 = K^6 - 15K^4 K_{ab}^2 + 40K^3 K_{ab}^3 - 90K^2 K_{ab}^4 + 45K^2 (K_{ab}^2)^2 \]

\[ - 120K_{ab}^3 K_{cd}^3 + 144K_{ab}^5 + 90K_{ab}^2 K_{cd}^2 - 15(K_{ab}^2)^3 + 40(K_{ab}^3)^2 - 120K_{ab}, \]

\[ = R^3 - 12R_{ab} R_{cd} R^{ab} + 16R_{ab} R^{a} R^{c}_{b} R^{d}_{c} + 24R_{ab} R_{cd} R^{abcd} \]

\[ + 3R R_{abcd} R_{ef}^{abcd} - 20R_{ab} R^{a} R^{c}_{d} R^{b}_{c} - 2R_{abcd} R_{e}^{abcd} \]

\[ - 8R R_{abcd} R^{a} R^{c}_{d} R^{b}_{c}, \]

(5g)

Here, we have used repeatedly the contracted Gauss–Codazzi integrability condition in a flat spacetime background, \( R_{abcd} = K_{ab} K_{cd} - K_{ad} K_{bc} \), where \( R_{abcd} \) denotes the worldvolume Riemann tensor, and \( K = g_{ab} K_{ab} \), and \( G_{ab} \) denotes the worldvolume Einstein tensor. Note that for even \( n \), we can recognize the form of the Gauss–Bonnet (GB) terms (see A.4–A.7) but expressed now in terms of the worldvolume extrinsic curvature. For example, for \( n = 0 \) we have the Dirac–Nambu–Goto (DNG) Lagrangian, for \( n = 2 \) we have the Ricci scalar Lagrangian also known as Regge–Teitelboim model [17–19] and for \( n = 4 \) we have the GB Lagrangian which for \( p > 3 \) produces non-vanishing equations of motion with ghost-free
contribution; in fact, for \( p = 3 \) the GB combination is a total divergence and it is a topological invariant. On the other hand, for odd \( n \), the Lagrangians are seen as the Gibbons–Hawking–York boundary terms which may exist if we have the presence of bulk Lovelock invariants (see section A.1).

Some remarks are in order. To avoid ambiguities for the possible gauge invariance for the case of odd \( n \) Lagrangians, it is necessary to make a choice in the direction for the normal vector to \( \Sigma \) in order to have a theory defined on the right-hand side of the worldvolume; hence, we assume that \( n^p \) is such that it is pointing outward of the region of interest. In addition, we point out that the Lovelock brane invariants have a linear dependence on the acceleration of the brane. This remarkable fact leads to explore the alternative Hamiltonian constructions since, as happens in some braneworld scenarios, when one tries to study the symmetries or quantize canonically, we cannot obtain quadratic constraints in the momenta in a straightforward way [17–20].

2.1. Lovelock brane tensors and equations of motion

By virtue of the properties of the gKd function one can define the brane-conserved tensors as follows:

\[
J^{ab}_{(n)} := \delta^{a b}_{b_{1}b_{2}...b_{n}} K^{b_{1}a_{1}} K^{b_{2}a_{2}} K^{b_{3}a_{3}} \cdots K^{b_{n}a_{n}},
\]

(6)

They are symmetric and obey that \( \nabla_{\sigma} J_{(n)}^{ab} = 0 \). This fact is shown by using the properties of the gKd and the Codazzi–Mainardi integrability condition in a flat background spacetime, \( \nabla_{\sigma} K_{bc} = \nabla_{b} K_{bc} \). Note that for a \((p + 1)\)-dimensional \( \Sigma \) there are at most an equal number of conserved tensors \( J^{ab} \). As developed above, by expanding out the determinant in (6) in terms of minors, we obtain a recursion relation

\[
J^{ab}_{(n)} = \delta^{a b}_{b_{1}b_{2}...b_{n}} - \delta^{a b}_{b_{1}b_{2}b_{3}...b_{n}} + \cdots + (-1)^{n} \delta^{a b}{_{b_{1}b_{2}...b_{n+1}}} + (-1)^{n+1} \delta^{a b}{_{b_{1}b_{2}...b_{n}}} \]

(7)

\[
+ (-1)^{n} b_{1} n K^{a b}_{(n-1)b}.
\]

The recurrent use of this identity allows us to have an expression for the conserved tensors

\[
J^{ab}_{(n)} = \sum_{j=0}^{n} \frac{(-1)^{n}}{(n-s)!} K^{a b}_{(s)} L_{n-s}, \quad n = 0, 1, 2, 3, 4, \ldots, p,
\]

(8)

where we have adopted the notation: \( K^{a b}_{(0)} \equiv g^{a b}, K^{a b}_{(3)} = K^{a b}, K^{a b}_{(2)} = K^{a c} K^{c b} \) and so forth. From equation (8), we may compute the first conserved tensors

\[
J^{ab}_{(0)} = g^{ab} = -2G^{ab}_{(0)},
\]

(9a)

\[
J^{ab}_{(1)} = g^{ab} L_{1} - K^{ab},
\]

(9b)

\[
J^{ab}_{(2)} = -2G^{ab}_{(1)},
\]

(9c)

\[
J^{ab}_{(3)} = g^{ab} L_{3} - 3 \mathcal{R} K^{ab} + 6 K^{c a} K^{c b} - 6 K^{c a} K^{c b} K^{d b},
\]

(9d)

\[
J^{ab}_{(4)} = -2G^{ab}_{(2)},
\]

(9e)

\[
J^{ab}_{(5)} = g^{ab} L_{3} - 5 L_{3} K^{ab} + 20 L_{3} K^{c a} K^{c b} - 60 L_{3} K^{c a} K^{c b} K^{d b} + 120 L_{4} K^{c a} K^{c d} K^{c b} - 120 L_{4} K^{c a} K^{c d} K^{c b} K^{f b},
\]

(9f)

\[
J^{ab}_{(6)} = -2G^{ab}_{(3)},
\]

(9g)

3 Relation (7) was introduced by mathematicians under the name of Newton transformation [21]. In fact, the framework given in (2) was outlined in [21, 22] but, from our perspective, it lacks a physical insight.
where $G^{(n)}_{ab}$ are defined in appendix A. In view of these facts, $J^{(n)}_{ab}$ are to be referred to as the Lovelock brane tensors. As a consequence of equation (2), the contraction of (6) with the extrinsic curvature provides an identity among the Lovelock brane tensors and the Lovelock brane Lagrangians

$$J^{(n)}_{ab} = L_n + 1.$$  

It follows immediately from equations (7) and (10) that $J^{(n)}_{ab} = (p + 1 - n)L_n = (N - n - 1)L_n$.

The main fact behind the Lagrangians (2) is that their associated equations of motion are of second order in the derivatives of the embedding functions. To prove this, we shall use the so-called conserved stress tensor associated with each term in (1) defined as follows [15, 23]:

$$f^{a \mu}_{(n)} = \left( L_n g^{ab} - L^a_c K^b_c \right) \delta^{(n)}_{\mu b} + \left( \nabla_a K_i^b \right) \mu,$$

where $L^n_a := \partial L_n / \partial K_{ab}$. It is conserved in the sense that $\nabla_a f^{a \mu}_{(n)} = 0$. This geometrical object is a powerful tool to study the mechanical content of branes [15]. In our case, from equations (2) and (6), we find that $J^{(n)}_{ab} = nJ^{(n-1)}_{ab}$. Note that in some sense, $J^{(n-1)}_{ab}$ is the derivative of $L_n$. Thus, we obtain

$$f^{a \mu}_{(n)} = J^{(n)}_{ab} \delta^{(n)}_{\mu b},$$

where we have considered the conservation property of $J^{(n)}_{ab}$ as well as identity (7). Note further that, $f^{a \mu}_{(n)}$ is only tangential to $\Sigma$. On the physical grounds, (12) is merely the linear momentum density of the brane whose dynamics is governed by the action (1) which is invariant under Poincaré transformations in the bulk. Following [15], the vanishing of the normal projection of the conservation law, $n \cdot \nabla_a f^{a \mu}_{(n)} = 0$, yields the equations of motion for the Lagrangian (2):

$$J^{(n)}_{ab} K_{ab} = L_n + 1 = 0,$$

whereas the tangential projection, $e_a \cdot \nabla_b f^{b \mu}_{(n)} = 0$, results in a geometrical identity being the conservation of the tensors (6). Obviously, we have only one equation of motion which is of second order in the field variables. This fact, in particular, means that we have only one physical degree of freedom for these types of branes.

By virtue of the definition of the extrinsic curvature $K_{ab} = -n \cdot \nabla_a e_b$, we can observe that the equation of motion (13) may also be written as a set of conservation laws:

$$\nabla_a \left( J^{(n)}_{ab} \delta^{(n)}_{\mu b} \right) = 0,$$

where we identify the linear momentum density (12) as a conserved current. It should be pointed out that equations (13) and (14) resemble the constraints that certain higher co-dimension self-gravitating branes in the Lovelock gravity obey, where, for instance, the Ricci scalar plays the role of correction to the DNG branes (see, for example, [24, 25]).

### 3. Born–Infeld extension of Lovelock brane gravity

Let us consider the dynamical evolution of a $p$-dimensional brane propagating in a Minkowski spacetime of dimension $N = p + 2$ that follows from the local BI-type action:

$$S[X] = \Lambda \int d^{p+1}x \sqrt{-\det(g_{ab} + X_{ab})},$$

where

$$X_{ab} = 2\alpha K_{ab} + \alpha^2 K_{ac} K^{cb},$$

and $\Lambda$ is a constant with dimensions $[L]^{N-p-1}$ and $\alpha$ is a concomitant constant with dimensions $[L]$ characterizing the relative weight of the nonlinear terms in the model. At first glance,
this second-order action in the embedding variables seems to lead to fourth-order equations of motion, but this appearance is however deceptive. Note that $X_{ab} = X_{ba}$, contrary to the ordinary Dirac–Born–Infeld theory where in such a case $X_{ab} = F_{ab}$ is the electromagnetic field strength. The action (15) is invariant under reparameterizations of the worldvolume. It is expected that for the small values of $X_{ab}$ the action (15) will reproduce small correction terms to the DNG model as we will uncover later on.

This effective theory has a natural geometric interpretation. Indeed, if we consider the embedding

$$x^\mu = Y^\mu (\xi^a) = X^\mu + \alpha n^\mu,$$

(17)

which is anchored to the former embedding $X^\mu$, then the tangent vectors are $E^\mu_a = e^{\mu}_a + \alpha K^a_b e^{\mu}_b$, where we have used the Gauss–Weingarten equation $\nabla_{an} = K^a_b e^{\mu}_b$. The corresponding induced metric $M_{ab} := \eta_{\mu\nu} E^{\mu}_a E^{\nu}_b = g_{ab} + X_{ab}$ leads to the volume element form in (15). The embedding (17) is equivalent to foliate the background spacetime by timelike leaves along the transverse deformations of the worldvolume [13, 14, 26].

The BI-type volume element form may be written in terms of the Lovelock brane Lagrangians for a given dimension $p$. The key to show this fact is to note that $M_{ab} = g_{ab} (g_{d}^{c} + \alpha K_{db}) (g_{b}^{c} + \alpha K_{ab})$. This entails that the action (15) can be rewritten as

$$S[X] = \Lambda \int d^{p+1} \xi \sqrt{-g} \left| \text{det} (g_{ab} + \alpha K_{ab}) \right|.$$

(18)

Now we turn to expand the characteristic polynomial of $K_{ab}$ by using equation (B.3). It therefore follows, from equations (B.4–B.7) specialized to $f_{ab} = \alpha K_{ab}$, that

$$f_{(1)} = \alpha K,$$

(19a)

$$f_{(2)} = \frac{\alpha^2}{2} R,$$

(19b)

$$f_{(3)} = \frac{\alpha^3}{6} \left(-K_{ab} + 3 K_{ab}^2 - \frac{1}{2} K^3 - 3 G^{(1)} K_{ab} \right),$$

(19c)

$$f_{(4)} = \frac{\alpha^4}{24} (R^2 - 4 R_{ab} R_{ab} + R_{abcd} R^{abcd}),$$

(19d)

$$\vdots$$

$$f_{(s)} = \frac{\alpha^s}{s!} L_s,$$

(19e)

where $L_s$ is given by (2). Now, if we set $f_{(0)} = 1$ in equation (B.3), we may therefore write the action (18) in the form

$$S[X] = \Lambda \int d^{p+1} \xi \sqrt{-g} \sum_{n=0}^{p+1} \left( \frac{\alpha^n}{n!} \right) L_n,$$

(20)

where expressions (2) have been invoked. This expansion clearly exhibits that the action (15) leads to second-order equations of motion since each one of the emerging terms does. Moreover, from equation (17), we can infer that when the spacetime coordinates suffer a deviation along the direction of the normal $n^\mu$ to the worldvolume, the ordinary DNG volume element undergoes a deformation becoming in another one that can be expressed as a finite series which involves to the Lovelock brane invariants. In a similar manner, there is an

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4 In this BI spirit, a closed approach for strings with $X_{ab} = K_{ab}$, where $i$ keeps the track of the number of normal vectors to the worldvolume.
analogous construction where the strategy of deforming a determinant results to be useful to obtain ghost-free nonlinear massive gravity actions [27].

The BI-like structure outlined above is a particular case of the action (1). As discussed extensively in [28] for the gravitational case, one may relate this type of BI-like actions to a Chern–Simons limit of the Lovelock theory by a particular choice of the $\alpha_n$'s in the Lovelock series (1). On the other hand, this choice for the bulk coordinates (17) induces, for small deformations of the brane, a certain type of brane bending mode in connection with the so-called Galileons, i.e. scalar fields that are claimed to explain the cosmic acceleration, inflation and dark energy [13, 14, 26, 29–32].

The combination $X_{ab}$ is not constrained to have the form (16) only but it seems reasonable to think of some other possible geometric choices for $X_{ab}$ maintaining this BI-type structure. Some of them are not so attractive due to their complexity and limitation, but others still underlie the interesting geometrical information. For example, we can choose $X_{ab} = \alpha K_{ab} + \beta K K_{ab}$. In such a case, by using (B.9), the cubic expansion in the small curvature of (15) casts out only the first four terms of the Lovelock brane invariants by considering $\beta = \alpha^2 / 4$ but not beyond this. In this sense, the action (15) can also be considered as a minimal Born–Infeld extension of the Lovelock brane theory. A special concern is provided by the unitarity and stability of these types of BI-like brane theories, since there are many possibilities that one can write for them. This issue is highly non-trivial to be resolved, unless further constraints are imposed as the particular choices that can be made for the parameter $\alpha$ [33]. We will discuss this case elsewhere.

3.1. Weyl invariance

There are some classically equivalent actions to (15)\(^5\) that might play a similar role for this model, but they result again in higher nonpolynomial actions. One of the most promising is the one that contains a scalar field as an auxiliary field which might help to the construction of the associated canonical formalism as well as the study of symmetries [36].

Consider the modified action

$$S[X^\mu, h_{ab}, \phi] = \Lambda \frac{1}{2} \int d^{p+1}\xi \sqrt{-h}[\phi^{1-\frac{p}{2}d}\, h^{ab}(g_{ab} + X_{ab}) - \phi(p - 1)].$$

(21)

where $h_{ab}$ is an auxiliary intrinsic metric on the worldvolume, not to be confused with the induced metric, and $h = \det(h_{ab})$. Also here, $\phi$ is an auxiliary scalar field. The above action, for any $p$, is invariant under the Weyl symmetry

$$h_{ab} \rightarrow h'_{ab} = e^{2\omega(\xi)} h_{ab}, \quad \phi \rightarrow \phi' = e^{-(p+1)\omega(\xi)} \phi.$$  

(22)

In addition, the action (21) is classically equivalent to the action (15) upon the elimination of the auxiliary fields $h_{ab}$ and $\phi$ by using its own equations of motion. To see this, we first obtain the variation of the action (21) with respect to $h_{ab}$:

$$\frac{1}{2}[h^{cd}(g_{cd} + X_{cd}) - \phi^{\frac{p}{2}d}(p - 1)] h_{ab} = g_{ab} + X_{ab}. \quad \text{(23)}$$

We contract now with the inverse metric $h^{ab}$ to obtain $h^{ab}(g_{ab} + X_{ab}) = (p + 1)\phi^{\frac{p}{2}d}$. Thus, the determinant of (23) is given by $\det(g_{ab} + X_{ab}) = (h/2p+1)[h^{cd}(g_{cd} + X_{cd}) - \phi^{\frac{p}{2}d}(p - 1)]^{p+1}$. On the other hand, the variation of the action (21) with respect to $\phi$ is

$$h^{ab}(g_{ab} + X_{ab}) = (p + 1)\phi^{\frac{p}{2}d}. \quad \text{(24)}$$

When we plug back (24) and the determinant $h$ into the action (21), we recover the action (15).

We believe that the action (21) deserves a closer examination as it seems reasonable to think

\(^5\) See [34–37] for related developments for DNG extended objects and also for $D_p$-branes.
that the presence of the scalar field will allow us to make contact with the Galileon theory. It will be considered in a forthcoming development.

4. Conclusions

In this work, we have first reviewed the Lovelock framework for branes. We have derived the general equations of motion which were written in terms of the conserved brane tensors. We then constructed a Born–Infeld-type action which may be written in terms of the Lovelock brane Lagrangians for a given dimension $p$. In particular, the action (15) becomes the action (1) where the constants $\alpha_n$ acquire a certain form in terms of the concomitant constant $\alpha$. We observe further that this model exhibits a natural geometric interpretation. Indeed, we may think of this BI-type action as a modification of the ordinary DNG action in the sense that it is proportional to the volume as measured now by using a peculiar metric consisting of the induced metric $g_{ab}$ modified by some terms constructed from the extrinsic curvature $K_{ab}$. We may include matter fields in our approach, like the electromagnetic interaction via $X_{ab} \rightarrow X_{ab} + F_{ab}$, extending our framework to a DBI-type approach for branes. We are also interested in the Ostrogradski–Hamiltonian development of the Lovelock brane invariants, since it may help to understand the internal symmetries of the theory as well as to canonically quantize the braneworld models that include the Lovelock brane invariants. This topic will be reported elsewhere. In addition, a classically equivalent action to (15) was provided and we analyzed its Weyl invariance through the introduction of the auxiliary fields. We hope that this BI-type effective theory may lead us to an entirely new geometrical standpoint for the understanding of the mechanics of the extended objects.

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Appendix A. Lovelock gravity review

We give an overview of the Lovelock gravity theory. This is the most general theory of gravity satisfying the following three conditions. (1) The field equations are written in terms of a symmetric rank-2 tensor. (2) The theory is consistent with the conservation law of the energy–momentum tensor. (3) The theory does not include higher than third-order derivatives of the metric. Consider an $N$-dimensional spacetime manifold $M$ with metric $g_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, \ldots, N - 1$). In this spacetime, the most general action which maintains the field equations of motion for the metric of second order, as the Einstein–Hilbert action, is the Lovelock gravity action [16] given by

$$
S[g_{\mu\nu}] = \frac{1}{2\kappa^2} \int_M d^N x \sqrt{-g} \sum_{n=0}^{p} \alpha_n \mathcal{L}_n(g_{\mu\nu}, R_{\mu\nu\alpha\beta}),
$$

(A.1)

where

$$
\mathcal{L}_n(g_{\mu\nu}, R_{\mu\nu\alpha\beta}) = \frac{1}{2^n} g^{\alpha_1\beta_1 \alpha_2\beta_2 \ldots \alpha_n\beta_n} \prod_{\tau=1}^{n} R^{\mu_\tau \nu_\tau}_{\alpha_\tau \beta_\tau},
$$

(A.2)
and $R_{\mu \nu \alpha \beta}$ is the spacetime Riemann tensor. Here $p := \lceil \frac{N}{2} \rceil$ represents the integer part of $\frac{N}{2}$, $a_n$ and $a_n^2$ are the constant values. In addition, we have used the generalized Kronecker delta function defined by [38]

$$\delta_{\mu_1 \nu_1 \mu_2 \nu_2 \ldots \mu_n \nu_n} = \begin{cases}
\delta^{a_1}_{\mu_1} \delta^{a_1}_{\nu_1} \ldots \delta^{a_1}_{\mu_n} \delta^{a_1}_{\nu_n} \\
\delta^{b_1}_{\mu_1} \delta^{b_1}_{\nu_1} \ldots \delta^{b_1}_{\mu_n} \delta^{b_1}_{\nu_n} \\
\vdots \quad \vdots \quad \vdots \quad \vdots \\
\delta^{a_n}_{\mu_1} \delta^{a_n}_{\nu_1} \ldots \delta^{a_n}_{\mu_n} \delta^{a_n}_{\nu_n} \\
\delta^{b_n}_{\mu_1} \delta^{b_n}_{\nu_1} \ldots \delta^{b_n}_{\mu_n} \delta^{b_n}_{\nu_n}
\end{cases}.$$  \hfill (A.3)

It must be noted that in $N$ dimensions, all terms for which $n > \lceil N/2 \rceil$ are the total derivatives and the term $n = N/2$ is the Euler density [16, 28]. Thus, only terms for which $n < N/2$ contribute to the field equations. The first four Lovelock Lagrangians are given by

$$L_0 = 1,$$  \hfill (A.4)

$$L_1 = R,$$  \hfill (A.5)

$$L_2 = R^2 - 4R_{\mu \nu} R^{\mu \nu} + R_{\mu \nu \alpha \beta} R^{\mu \nu \alpha \beta},$$  \hfill (A.6)

$$L_3 = R^3 - 12R_{\mu \nu} R^{\mu \nu} + 16 R_{\mu \nu} R_{\alpha \beta} R^{\mu \nu \alpha \beta} + 24 R_{\mu \nu \alpha \beta} R_{\gamma \delta} R^{\mu \nu \alpha \beta \gamma \delta} - 8 R_{\mu \nu \alpha \beta} R_{\gamma \delta} R^{\mu \nu \alpha \beta \gamma \delta}.$$  \hfill (A.7)

A variational procedure applied to the action (A.1) casts out the equations of motion

$$G^{\mu \nu}_{\alpha \beta} = \frac{\partial^2 S_{\text{matter}}}{\partial R^{\alpha \beta} \partial R_{\mu \nu}},$$  \hfill (A.8)

with $T_{\mu \nu}$ being the energy–momentum tensor for matter fields coming from a possible matter action $S_{\text{matter}}$ appearing in (A.1), and $G_{\mu \nu} = \sum_{n=0}^{p} a_n G^{(n)}_{\mu \nu}$ where the so-called Lovelock tensors $G^{(n)}_{\mu \nu}$ are defined as

$$G^{(0)}_{\mu \nu} = \frac{1}{2n!} g^{\mu \nu} \cdot R^{\mu \nu} - \frac{1}{2} g_{\mu \nu} L_0,$$  \hfill (A.10)

$$G^{(1)}_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} L_1,$$  \hfill (A.11)

$$G^{(2)}_{\mu \nu} = 2 (R R_{\mu \nu} - 2 R_{\rho \sigma} R^{\rho \sigma} + 2 R_{\rho \sigma \lambda \sigma} R^{\rho \sigma \lambda \sigma}) - \frac{1}{2} g_{\mu \nu} L_2,$$  \hfill (A.12)

$$G^{(3)}_{\mu \nu} = 3 (R^2 R_{\mu \nu} - 4 R_{\rho \mu \sigma} R^{\rho \sigma} R_{\nu \rho} + 8 R_{\rho \mu \sigma \lambda} R^{\rho \sigma \lambda} R_{\nu \rho}) + 8 R_{\rho \mu} R_{\lambda \sigma} R^{\rho \sigma} + 8 R_{\rho \mu \sigma} R_{\lambda \sigma} R^{\rho \sigma} + 2 R R_{\rho \mu \sigma \lambda} R^{\rho \sigma \lambda}.$$  \hfill (A.13)
A.1. Gibbons–Hawking–York–Myers boundary terms

We review briefly the so-called Gibbons–Hawking–York–Myers (GHYM) boundary terms. In order to have a well-posed variational principle in the Lovelock gravity we must consider the appropriate surface terms given by [39–42]

\[ S_b = -2c_n \int_{\partial M} d^{N-1}x \sqrt{h} L_b^{(a)}, \quad \text{(A.14)} \]

where the Lagrangian surface terms may be written in a tensorial form as follows [41, 42]:

\[ L_b^{(a)} \sim \int_0^1 dt \delta_{b_1b_2 \ldots b_{2m-1}} \delta_{a_1a_2 \ldots a_{2m-1}} \mathcal{K}_{a_1}^{b_1} \mathcal{K}_{a_2}^{b_2} \mathcal{K}_{a_3}^{b_3} \mathcal{K}_{a_{2m-1}}^{b_{2m-1}} \mathcal{K}_{a_1}^{b_1b_2} \mathcal{K}_{a_2}^{b_2b_3} \mathcal{K}_{a_3}^{b_3b_4} \cdots \mathcal{K}_{a_{2m-1}}^{b_{2m-2}b_{2m-1}}. \quad \text{(A.15)} \]

Here \( \mathcal{K}_{\mu\nu} \) is the extrinsic curvature in the spacelike surface manifold \( \partial M \), and we have suppressed the coupling constants. Hence, \( \mathcal{K} = g^{\mu\nu} \mathcal{K}_{\mu\nu} \). The first three counterterms to the Lovelock Lagrangians take the compact form:

\[ L_b^{(1)} = K \quad \text{(A.16)} \]
\[ L_b^{(2)} = -\frac{1}{3} K^3 + KK_{\mu\nu}^2 - \frac{2}{3} K_{\mu\nu}^3 - 2G_{\mu\nu}K_{\mu\nu} \quad \text{(A.17)} \]
\[ L_b^{(3)} = -\frac{1}{3} K^3 + 15K_{\mu\nu}^2 + 20K^2K_{\mu\nu} + 50K^3K_{\mu\nu} - 30KK_{\mu\nu}^2 - 20K_{\mu\nu}^2K_{\mu\nu} + 24K_{\mu\nu}^3 - 2G_{\mu\nu}^2K_{\mu\nu}. \quad \text{(A.18)} \]

Appendix B. Expansion of the BI-type action

The determinant of an \( n \times n \) matrix \( A_{ab} \) in terms of the gKd is

\[ A := \det(A_{ab}) = \frac{1}{n!} \delta_{a_1b_1 \ldots a_nb_n} A_{a_1b_1} A_{a_2b_2} \cdots A_{a_nb_n}. \quad \text{(B.1)} \]

In addition, the associated inverse matrix can be computed as follows:

\[ (A^{-1})_{ab} = \frac{1}{(n-1)!} \delta_{a_1b_1 \ldots a_nb_n} A_{a_2b_2} A_{a_3b_3} \cdots A_{a_nb_n}. \quad \text{(B.2)} \]

The characteristic determinant of the matrix \( M_{ab} = \delta_{ab} + f_{ab} \) may be expressed in the form [38]

\[ \det(M_{ab}) = 1 + \sum_{s=1}^n \frac{1}{s!} \delta_{a_1b_1 \ldots a_rb_r} f_{a_1b_1} f_{a_2b_2} \cdots f_{a_rb_r}, \]

\[ = 1 + \sum_{s=1}^n s! f_s(a), \quad \text{(B.3)} \]

where \( s!f_s(a) = \delta_{a_1a_2 \ldots a_s} f_{a_1b_1} f_{a_2b_2} \cdots f_{a_sb_s} \) denotes the determinant of the s-rowed minor. These minors can be expressed in terms of the traces of the \( f_{ab} \) matrix

\[ f_1 = \delta_{a_1b_1} f_{a_1b_1} = f_{a} = \text{Tr}(f), \quad \text{(B.4)} \]

\[ f_2 = \frac{1}{2!} \delta_{a_1b_1} \delta_{a_2b_2} f_{a_1b_1} f_{a_2b_2} = \frac{1}{2} (\delta_{a_1b_1} \delta_{a_2b_2} - \delta_{a_1b_2} \delta_{a_2b_1}) f_{a_1b_1} f_{a_2b_2}, \]

\[ = \frac{1}{2} (f_{a} f_{b} - f_{a} f_{b}) = \frac{1}{2} [\text{Tr}(f)^2 - \text{Tr}(f^2)], \quad \text{(B.5)} \]
\[ f(3) = \frac{1}{3!} \delta^a_{b_1 b_2 b_3} f^{b_1}_{a_1} f^{b_2}_{a_2} f^{b_3}_{a_3}, \]
\[ = \frac{1}{6} \left( \delta^a_{b_1} \delta^a_{b_2} \delta^a_{b_3} - \delta^a_{b_2} \delta^a_{b_1} \delta^a_{b_3} + \delta^a_{b_3} \delta^a_{b_1} \delta^a_{b_2} \right) f^{b_1}_{a_1} f^{b_2}_{a_2} f^{b_3}_{a_3}, \]
\[ = \frac{1}{6} \left[ b_0 \left( \delta^a_{b_1} \delta^a_{b_2} - \delta^a_{b_2} \delta^a_{b_1} \right) \right] f^{b_1}_{a_1} f^{b_2}_{a_2} f^{b_3}_{a_3}, \]
\[ = \frac{1}{6} \left[ 3 f^a f^b f^c - 3 f^a f^b f^c + 2 f^a f^b f^c \right], \]
\[ = \frac{1}{6} \left[ \text{Tr}(f)^3 - 3 \text{Tr}(f^2) \text{Tr}(f) + 2 \text{Tr}(f^3) \right]. \quad (B.6) \]

\[ f(4) = \frac{1}{4!} \delta^a_{b_1 b_2 b_3 b_4} f^{b_1}_{a_1} f^{b_2}_{a_2} f^{b_3}_{a_3} f^{b_4}_{a_4}, \]
\[ = \frac{1}{24} \delta^a_{b_1} \left( \delta^a_{b_2} \delta^a_{b_3} \delta^a_{b_4} - \delta^a_{b_2} \delta^a_{b_3} \delta^a_{b_4} + \delta^a_{b_2} \delta^a_{b_3} \delta^a_{b_4} \right) \]
\[ = \frac{1}{24} \left[ f^a f^b f^c f^d - 8 f^a f^b f^c f^d + 6 f^a f^b f^c f^d \right] \]
\[ = \frac{1}{24} \left[ 3 f^a f^b f^c f^d + 8 f^a f^b f^c f^d - 6 f^a f^b f^c f^d \right], \]
\[ = \frac{1}{24} \left[ \text{Tr}(f)^4 + 8 \text{Tr}(f^3) \text{Tr}(f) - 6 \text{Tr}(f^2) \text{Tr}(f)^2 + 3 \text{Tr}(f)^3 \right] - 6 \text{Tr}(f^4). \quad (B.7) \]

In some cases it will be useful to obtain the Taylor expansion of the square root of the characteristic determinant \((B.3)\) which may be obtained by using the well-known expansion \((1+x)^{1/2} = 1 + \frac{1}{2} x - \frac{1}{8} x^2 + \frac{1}{16} x^3 - \cdots \) for \( |x| \leq 1 \). Hence,
\[ \left[ \text{det}(\delta^a_b + f^a_b) \right]^{1/2} = 1 + \frac{1}{2} \sum_{s=1}^{n} f(s) - \frac{1}{8} \left( \sum_{s=1}^{n} f(s) \right)^2 + \frac{1}{16} \left( \sum_{s=1}^{n} f(s) \right)^3 - \cdots. \quad (B.8) \]

Thus, up to \( O(f^6) \) we have
\[ \left[ \text{det}(\delta^a_b + f^a_b) \right]^{1/2} = 1 + \frac{1}{2} \text{Tr}(f) - \frac{1}{8} [2 \text{Tr}(f^2) - \text{Tr}(f)^2] + \frac{1}{16} [8 \text{Tr}(f^3) - 6 \text{Tr}(f^2) \text{Tr}(f)] \]
\[ - \frac{1}{8} [2 \text{Tr}(f^2) - \text{Tr}(f)^2] + \frac{1}{32} [48 \text{Tr}(f^4) - 32 \text{Tr}(f^3) \text{Tr}(f)] \]
\[ + 12 \text{Tr}(f^2) \text{Tr}(f^2) - 12 \text{Tr}(f^2)^2 - \text{Tr}(f)^4] + \frac{1}{32} [384 \text{Tr}(f^5) - 240 \text{Tr}(f^4) \text{Tr}(f) + 80 \text{Tr}(f^3)^2 - 20 \text{Tr}(f^3) \text{Tr}(f)^3] \]
\[ + 60 \text{Tr}(f) \text{Tr}(f^2)^2 - 160 \text{Tr}(f^2) \text{Tr}(f^3) + \text{Tr}(f)^5] - O(f^6). \quad (B.9) \]

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