On Brownian motion in ideal gas and related principles

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Abstract

Brownian motion of particle interacting with atoms of ideal gas is discussed as a key problem of kinetics lying at the border between “dead” systems like the Lorentz gas or formal constructs of conceptual Boltzmannian kinetics and actual “alive” systems like mere gas possessing scaleless (1/f) fluctuations in their kinetic characteristics (e.g. in diffusivity and mobility of the “Brownian particle”).

Key words: Brownian motion, diffusion, molecular random walks, BBGKY hierarchy, correlation functions, Bogolyubov functional equation, virial relations, kinetic theory of gases

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1. Introduction (once again about derivation of kinetics from dynamics)

What is simpler than ideal gas? At least, when simply gas is too hard nut to crack?

In this paper we want to consider Brownian motion of a particle interacting with infinite gas of atoms which do not interact between themselves. Our starting-point will be corresponding Bogolyubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy of equations [1,2,3] for \((n+1)\)-particle distribution functions of the “Brownian particle” and \(n\) atoms \((n = 0, 1, 2, \ldots)\) or equivalent Bogolyubov functional evolution equation (BFE) for their generating functional [1]. Our task here is not to solve these equations but only discern them and discuss their possible solution in terms of reasonably introduced \((n+1)\)-particle correlation functions, with taking in mind general invariance properties of BFE found in [4] and expressed by so-called “virial expansion of Brownian path probability distribution” and other “virial relations”. The latter were foreseen earlier in [5,6] and partly deduced in [7,8,9,10] from the generalized fluctuation-dissipation relations [11,12].

The modesty of our task is opposed to pretence of the legend existing among “nutcrackers” and stating that dilute gas (under the “Boltzmann-Grad limit” or, in other words, the “low density limit”) is so much simple object that in respect to it the BBGKY hierarchy reduces to so-called Boltzmann hierarchy and then to the single Boltzmann equation or some its derivative like the Boltzmann-Lorentz equation [13]. The legend refers to works by Lanford on gas of hard spheres, e.g. [14,15] (see also [13] and references in [3] and [13,15]). However, careful reading of work [15] shows that factually it does not suggest a proof of the proclaimed statement. Moreover, the author himself indicates that already short time evolution (five times shorter than mean free-path time) reveals definite surfaces in \(n\)-particle phase spaces \((n > 1)\) where
\textit{n}-particle distribution functions (DF) “do not converge to the desired products” of one-particle DF, thus preventing the desired proof.

The mentioned (hyper-)surfaces are $r_i - r_j = (v_i - v_j)\theta$ with $r$ and $v$ being particle coordinates and velocities and $\theta$ arbitrary time parameter. That are surfaces made of various pre- or post-collision trajectories of gas particles. Hence, a correct conclusion what suggested itself in [15] was that generally statistics of collisions is not determined by one-particle DF, and BBGKY equations can not be reduced to Boltzmann equation (BE), even under the “low density limit”.

From physical point of view, this is quite trivial conclusion. Indeed, the “low density limit” in no way removes fluctuations in density of collisions (number of pairs in pre- or post-collision states per unit volume) as well as it does not remove fluctuations in density of particles (number of particle per unit volume), sooner it strengthens all them. Therefore, even if local density of collisions was primitively thought as square function of local density of particles, an unbiased reasoning would result in the stated conclusion merely because of general inequality $\langle A * B \rangle \neq \langle A \rangle * \langle B \rangle$ (with angle brackets denoting ensemble average)!

In later attempts to derive Boltzmann hierarchy from the BBGKY one, e.g. in [13], it was postulated that values of DF for an incoming, or pre-collision, state (at $\theta < 0$ with above introduced $\theta$ ) and responding outgoing, or post-collision, state (at $\theta > 0$ ) are equal. It is good idea (although wrong in literal quantitative sense). But elementary logics requires to apply it at once in all places of the BBGKY hierarchy where differential $n$-particle Liouville operators act on DF just realizing collisional transitions from in- to out-states. Instead, the authors applied their postulate in integral terms only, while in the differential terms neglected collisions at all! Of course, results of such arbitrariness hardly can be attributed to physics.

More likely approach to kinetic theory of dilute gas (formally, in the Boltzmann-Grad limit) was suggested in [16] (or see [17]). There it was emphasized that reformulation of BBGKY hierarchy in terms of collisions of particles and “collision integrals”, in place of continuous interactions, is possible only if various consecutive stages of any collision process are presented in statistical ensemble under consideration with equal weights (probability densities). In other words, derivative of DF in respect to the “inner time of collision” $\theta$ turns to zero inside a spatial region assigned to collision. This property never realizes exactly, but it must be postulated in order to construct a correct “collisional approximation” to BBGKY hierarchy. It means [16] that \textbf{in any particular Liouville operator} the replacement

\[
\frac{p_i}{m} \frac{\partial}{\partial r_i} - \frac{p_j}{m} \frac{\partial}{\partial r_j} + \Phi'(r_i - r_j) \cdot \left( \frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_j} \right) =
\]

\[
- \frac{p_i + p_j}{2m} \cdot \left( \frac{\partial}{\partial r_i} + \frac{\partial}{\partial r_j} \right) + \frac{\partial}{\partial \theta} \to - \frac{p_i + p_j}{2m} \cdot \left( \frac{\partial}{\partial r_i} + \frac{\partial}{\partial r_j} \right)
\]

should be made within collision of \textit{i}-th and \textit{j}-th particles, where $\Phi(r)$ and $-\Phi'(r) = -\partial\Phi(r)/\partial r$ are interaction potential and interaction force, respectively, and the first equality gives formal definition of the “inner time of collision”. To be more precise, the words “within collision” mean

\[
| r_i - r_j - (v_i - v_j)\theta | \lesssim r_0 , \quad | r_i - r_j | \lesssim \lambda ,
\]

with $r_0$ being characteristic interaction radius and $\lambda$ mean free path.

Evidently, this ansatz by its nature is independent on shape of $\Phi(r)$, therefore extends to hard spheres too. It is just what was deficient in specifications of [13] \footnote{This loss demonstrates that hard spheres is treacherous model which may play a bad trick on its makers, at least when treated in the language of DF. One can see also what is principal error of [13,14,15] and similar mathematical “fantasies”: although Lebesgue measure of regions “within collision” tends to zero under the Boltzmann-Grad limit, this is not a ground to cast such regions since just they determine actual evolution of the system!}.

As the consequence, \textbf{density of collisions determined by pair DF within collision} drifts in space with velocity $(v_i + v_j)/2$. Since relative motion of colliding particles is included to collision, the latter as a whole moves with this centre of mass velocity! This is sufficient to forbid factorization of density of collisions into any quadratic functional (e.g. product) of two one-particle DF drifting with different velocities, $v_i$ and $v_j$. Hence, pair DF of colliding particles is \textbf{independent statistical characteristics} of gas evolution.
Eventually, “collisional approximation” to kinetics of dilute gas results in an infinite hierarchy of kinetic equations [16,17,18] for usual one-particle DF and infinitely many less usual mutually independent \( n \)-particle DF. At \( n > 2 \) they represent ensemble-average densities of connected clusters of \( n - 1 \) (real or virtual) collisions. This new hierarchy reduces to Boltzmann hierarchy and then to BE in case of strictly spatially uniform statistical ensemble only (in this case, possibly, \( \langle A \ast B \rangle = \langle A \rangle \ast \langle B \rangle \) since ensemble average can be identified with infinite-volume average).

The word “kinetic” here means that interactions are represented by a ready-made “collision integrals” instead of potential \( \Phi(r) \). But, strictly speaking, this only gently simplifies the theory, since full list of arguments and parameters of \( n \)-particle DF (\( n > 1 \)) decreases by one only (namely, by the \( \theta \)) [17]. Therefore, to derive benefit from the new hierarchy, some of its further reasonable posterization and/or approximation is necessary. Two different approaches to this task were considered in [16,17] and [18,19].

Most principal result of [16,17] was presence of long-living statistical self-correlations in random (“Brownian”) motion of any gas particle and corresponding low-frequency scaleless \( 1/f \)-type fluctuations (“\( 1/f \)-noise”) in its self-diffusivity (and mobility). In [18,19] this result was confirmed and seriously extended by showing that probability density distribution of diffusive path of a test particle possesses power-law long tails, naturally cut off at distances of ballistic flight (see also [9]). All this qualitatively justifies early phenomenological theory of \( 1/f \)-noise under realistic molecular Brownian motion [20,21] and, in turn, recently was justified in [7,8,9,10] and then in [4] basing on exact invariance properties of Liouville operator and BBGKY hierarchy.

And all this is minimum of what is neglected by the Boltzmann hierarchy and Boltzmannian kinetics at all. Unfortunately, its modern admirers think on pioneer level with Boltzmann, as if they are unacquainted with Krylov’s warning [22] that physical independence of particles or events on concrete phase trajectory (in real life) does not imply statistical independence of their images at statistical ensemble of trajectories (“ensemble of lives”). May be, from here the secret wish arises to equalize in rights “dilute gas” and “Lorentz gas” (non-interacting particles in a lattice of fixed elastic scatterers).

In fact, results of [16,17,18,19], as well as [7,8,9,10,4], say that “dilute gas” differs from “Lorentz gas” like a living matter differs from dead one. In the first, contrary to the second, random walk of a test particle never can be divided into statistically independent constituent parts, regardless of their durations and total observation time. In other words, from the point of view of time averaging, every trajectory of this random walk remains unique at arbitrary growing spatial-temporal scales, while in “Lorentz gas” almost all trajectories become identical (use of terms “ergodic” or “non-ergodic” I leave to mathematicians). A simple heuristic interpretation of this property was suggested in [18] and continued in preambles of [7,8].

Additional explanations can be found in [16,17,21] and in preambles or resumes of [9,19,20,23,24,25].

By the way, notice that the mentioned difference long ago is known as experimental fact concerning charge carriers in semiconductor crystals: the greater is concentration of hard immovable impurity atoms, the stronger is damping of relative \( 1/f \) fluctuations in carrier’s diffusivity (and mobility) due to phonons (see e.g. [21] and references therein).

The aforesaid makes clear urgency of Brownian motion of a particle interacting with ideal gas. Evidently, this problem mediates between “dilute gas” and “Lorentz gas”, especially under the Boltzmann-Grad limit when radius of the Brownian particle is arbitrary small as compared with its (fixed) mean free path. We expect that Brownian motion in ideal gas also is alive, contrary to dead random motion among static scatterers. Let us consider this expectation starting from [4] and taking in mind experience of [16,18].

2. Functions and equations of the model

As in [4], in the meantime we confine ourselves by Brownian motion in equilibrium fluid (to be ideal gas here), for concreteness, in the framework of canonical Gibbs ensemble of initial conditions for fluid atoms. What is for the Brownian particle (BP), again initially it is certainly located at given point \( \mathbf{R}_0 \). But its initial momentum is random and obeys equilibrium Maxwell distribution (with the same temperature \( T \)).

Let \( \Phi(r) \) denotes potential of interaction between BP and atoms. Since now atoms do not interact one with another, the relations between DF \( F_n(t, \mathbf{R}, \mathbf{r}^{(n)}, \mathbf{P}, \mathbf{p}^{(n)}|\mathbf{R}_0; \nu) \) and correlation functions (CF) \( V_n(t, \mathbf{R}, \mathbf{r}^{(n)}, \mathbf{P}, \mathbf{p}^{(n)}|\mathbf{R}_0; \nu), \) with \( \mathbf{r}^{(n)} = \{\mathbf{r}_1...\mathbf{r}_n\}, \mathbf{p}^{(n)} = \{\mathbf{p}_1...\mathbf{p}_n\} \) and \( \nu \) standing for mean density of
gas (density at infinity), the BBGKY equations and the Bogolyubov functional evolution equation (BFE) all strongly simplify. The BBGKY equations take form

\[
\frac{\partial F_0}{\partial t} = -\frac{P}{M} \frac{\partial F_0}{\partial R} + \nu \frac{\partial}{\partial P} \int \Phi'(R - r_1) F_1 ,
\]

\[
\frac{\partial F_n}{\partial t} = \tilde{L}_n F_n + \nu \frac{\partial}{\partial P} \int \Phi'(R - r_{n+1}) F_{n+1}
\]

at \( n > 1 \), with \( \int_k \ldots \equiv \int \ldots dP_k dr_k \),

\[
\tilde{L}_n = -\frac{P}{M} \cdot \frac{\partial}{\partial R} - \sum_{j=1}^n \frac{P_j}{m} \cdot \frac{\partial}{\partial r_j} + \sum_{j=1}^n \Phi'(R - r_j) \cdot \left( \frac{\partial}{\partial P} - \frac{\partial}{\partial p_j} \right)
\]

and obvious initial conditions:

\[
F_n(t = 0, \ldots) = \delta(R - R_0) G_M(P) \prod_{j=1}^n G_m(p_j) E(r_j) ,
\]

\[
G_m(p) \equiv (2\pi T m)^{-3/2} \exp(-p^2/2T m), \quad E(r) \equiv \exp[-\Phi(r)/T]
\]

The connection between generating functional of DF,

\[
\mathcal{F}\{t, R, P, \psi | R_0; \nu\} = F_0(t, R, P | R_0; \nu) + \sum_{n=1}^\infty \frac{\nu^n}{n!} \int \ldots \int F_n(t, R, r^{(n)}, P, p^{(n)} | R_0; \nu) \prod_{j=1}^n \psi(r_j, p_j) ,
\]

and quite similarly defined generating functional of CF [4], \( \mathcal{V}\{t, R, P, \psi | R_0; \nu\} \), becomes

\[
\mathcal{F}\{t, R, P, \psi | R_0; \nu\} = \exp \left[ \nu \int G_m(p) E(r - R) \psi(r, p) dP dR \right] \mathcal{V}\{t, R, P, \psi | R_0; \nu\} = \exp \left[ \nu \int E(r - R) \phi(r) dR \right] \mathcal{V}\{t, R, P, \psi | R_0; \nu\} ,
\]

where \( \phi(r) \equiv \int \psi(r, p) G_m(p) dp \). Recall that in fact this is definition of CF. According to it,

\[
F_0(t, R, P | R_0; \nu) = V_0(t, R, P | R_0; \nu) ,
\]

\[
F_1(t, R, r_1, P, p_1 | R_0; \nu) = V_0(t, R, P | R_0; \nu) E(r_1 - R) G_m(p_1) + V_1(t, R, r_1, P, p_1 | R_0; \nu) ,
\]

and so on. The BFE, that is compact functional form of BBGKY hierarchy, now reads

\[
\frac{\partial \mathcal{F}}{\partial t} + \frac{P}{M} \cdot \frac{\partial \mathcal{F}}{\partial R} = -\int \psi(x) \cdot \frac{P}{m} \cdot \frac{\partial}{\partial \psi(x)} \Phi'(R - r) \cdot \left( \frac{\partial}{\partial P} - \frac{\partial}{\partial \psi(x)} \right) \frac{\delta \mathcal{F}}{\delta \psi(x)}
\]

with \( x = \{r, p\} \) and \( \int \ldots = \int \ldots dx = \int \ldots dP dR \), thus turning into first-order differential equation in respect to \( \psi(x) \). Substitution of (5) to (7) yields equivalent BFE in terms of CF:

\[
\frac{\partial \psi}{\partial t} + \frac{P}{M} \cdot \frac{\partial \psi}{\partial R} = -\int \psi(x) \cdot \frac{P}{m} \cdot \frac{\partial}{\partial \psi(x)} \psi(x) \cdot \left( \frac{\partial}{\partial P} - \frac{\partial}{\partial \psi(x)} \right) \frac{\delta \mathcal{F}}{\delta \psi(x)} + \nu T \left[ \int G_m(p) E'(r - R) \psi(x) \right] \cdot \left( \frac{P}{MT} + \frac{\partial}{\partial P} \right) \mathcal{V} ,
\]

with \( E'(r) = dE(r)/dr = -\Phi'(r)E(r)/T \).
Initial conditions and boundary conditions to these equations and (8) are very simple:

\[ V_n(t = 0, \ldots) = \delta_{n, 0} \delta(R - R_0) G_M(P), \quad \forall \{t = 0, R, P, \psi|R_0; \nu\} = \delta(R - R_0) G_M(P) \]  

(I have to detect misprint in similar formula (13) in [4]: the factor \( G_M(P) \) was missed!),

\[ V_{n>0}(\ldots r_k \ldots) \to 0 \quad \text{at} \quad r_k - R \to \infty \]  

At last, consider the invariance group found in [4] (a group of such transformations of arguments of generating functional determined by (8)-(13) which do not change value of the functional). In case of ideal gas it simplifies to

\[ \forall\{t, R, P, \sigma + \psi|R_0; \nu\} = \forall\{t, R, P, \psi/(1 + \sigma)|R_0; (1 + \sigma)\nu\} \],

where \( \sigma(r, p) = \sigma = \text{const} \) is arbitrary constant from interval \(-1 < \sigma < \infty\). This functional identity implies exact “virial expansions” of CF:

\[ \forall\{t, R, r^{(n)}(n), P, p^{(n)}|R_0; \nu + \sigma\nu\} = \forall\{t, R, r^{(n)}(n), P, p^{(n)}|R_0; \nu\} + \sum_{k=1}^{\infty} \frac{(\nu\sigma)^k}{k!} \int_{n+1} V_{n+1}(t, R, r^{(n+k)}(n+k), P, p^{(n+k)}|R_0; \nu) \]  

Their infinitesimal form yields exact “virial relations”

\[ \frac{\partial V_{n}(t, R, r^{(n)}(n), P, p^{(n)}|R_0; \nu)}{\partial \nu} = \int_{n+1} V_{n+1}(t, R, r^{(n+1)}(n+1), P, p^{(n+1)}|R_0; \nu) \]  

At \( n = 0 \), in particular, we obtain connection between density derivative of the probability distribution of path, \( \Delta R = R - R_0 \), of the Brownian particle and integrated pair CF:

\[ \frac{\partial V_0(t, R, P|R_0; \nu)}{\partial \nu} = \int V_1(t, R, r, p|R_0; \nu) \, dr \, dp \]  

where, of course, both sides depend on current position of BP, \( R \), and it starting position, \( R_0 \), through their difference \( \Delta R = R - R_0 \) only.

Relations like (17), as combined with identity (6) and trivial positivity of DF \( F_1 \), lead to principal consequence [4,7,8,9,10] mentioned in Introduction: asymptotic of BP’s path distribution,

\[ V_0(t, \Delta R; \nu) = \int V_0(t, R, P|R_0; \nu) \, dP \]  

at \( t \gg \tau \), with \( \tau \) being mean free-path time of BP, as a function of \( \Delta R \) has power-law long tails lasting up to \( |\Delta R| \sim t^{\sqrt{T/M}} \).
Such statement strikingly contradicts Boltzmannian kinetics which always gravitates towards the “law of large numbers” and Gaussian asymptotic characterized by short exponential tails. To avoid new repeating myself, I ask dear reader to search for proper comments in [8,9,10,16,17,18]. But one not popular truism deserves repetition: in statistical physics, unlike probability theory, one should not rely on the “law of large numbers” since practically independent events or quantities may be nevertheless statistically dependent. It would be a good thing to understand in detail how all this realize in case of Brownian motion in ideal gas.

Notice that due to simplicity of BBGKY equations (1)-(2) or (9)-(11), in comparison with general case [4,16,18], one can easy verify virial relations (16) by deriving them directly from (9)-(11).

3. Relative coordinates and characteristic function

Eventually, we would like to obtain the probability distribution (18) of BP’s path or its characteristic function, that is Fourier transform

$$V_0(t, ik; \nu) \equiv \int \exp (ik \cdot \Delta R) V_0(t, \Delta R; \nu) d\Delta R$$

Therefore, first, let us consider all the DF and CF as functions of $$\Delta R = R - R_0$$ and relative distances $$\rho_j = r_j - R$$. Such change of spatial variables implies operator changes

$$\frac{\partial}{\partial R} \rightarrow \frac{\partial}{\partial \Delta R} - \sum_j \frac{\partial}{\partial \rho_j}, \quad \frac{\partial}{\partial r_j} \rightarrow \frac{\partial}{\partial \rho_j}$$

everywhere in (1)-(2), (9)-(11). Second, make Fourier transform in respect to $$\Delta R$$ and go to functions

$$V_n(t, ik, \rho^{(n)}, P, p^{(n)}; \nu) \equiv \int \exp [ik \cdot (R - R_0)] V_n(t, R, R + \rho^{(n)}, P, p^{(n)} | R_0; \nu) \, dR$$

Third, introduce velocities $$V \equiv P/M$$ and $$v_j \equiv p_j/m$$ of BP and gas atoms, and new operator

$$\hat{\Lambda}(j) \equiv - (V_j - V) \cdot \frac{\partial}{\partial \rho_j} + \Phi'(\rho_j) \cdot \left( \frac{\partial}{\partial P_j} - \frac{\partial}{\partial \rho_j} \right)$$

Then BBGKY equations (9)-(11) take the form

$$\frac{\partial V_0}{\partial t} = i(k \cdot V) V_0 - \nu \frac{\partial}{\partial P} \int_1 \Phi'(\rho_1) V_1,$$  \hspace{1cm} (22)

$$\frac{\partial V_1}{\partial t} = i(k \cdot V) V_1 + \hat{\Lambda}(1) V_1 - \nu \frac{\partial}{\partial P} \int_2 \Phi'(\rho_2) V_2 + G_m(P_1) E'(\rho_1) \left( V + T \frac{\partial}{\partial P} \right) V_0,$$ \hspace{1cm} (23)

$$\frac{\partial V_n}{\partial t} = i(k \cdot V) V_n + \sum_{j=1}^{n} \hat{\Lambda}(j) V_n - \nu \frac{\partial}{\partial P} \int_{n+1} \Phi'(\rho_{n+1}) V_{n+1} +$$

$$+ \sum_{j=1}^{n} G_m(P_j) E'(\rho_j) \cdot \left( V + T \frac{\partial}{\partial P} \right) V_{n-1}(\ldots \rho_{j-1}, \rho_j + \ldots P_{j-1}, P_j + \ldots)$$ \hspace{1cm} (24)

with initial and boundary conditions as follow,

$$V_n(t = 0, \ldots) = \delta_{n,0} G_M(P), \quad V_{n>0}(\ldots \rho_k \ldots) \rightarrow 0 \quad \text{at} \quad \rho_k \rightarrow \infty,$$ \hspace{1cm} (25)

and BP’s path distribution (18) under interest presented by

$$V_0(t, \Delta R; \nu) = \int \exp (-ik \cdot \Delta R) \int V_0(t, ik, P; \nu) \, dP \frac{dk}{(2\pi)^d},$$ \hspace{1cm} (26)

where $$d$$ is space dimension ($$d = 3$$ by default).
Taking into account quite obvious identity \((V + T \partial / \partial P) G_M(P) = 0\) it is easy to see that at \(k = 0\) the system (22)-(24) always stays in its initial state:

\[
\begin{align*}
V_0(t, ik = 0, P; \nu) &= G_M(P), \\
V_n(t, ik = 0, \rho^{(n)}, P, p^{(n)}; \nu) &= \int V_n(t, R, R + \rho^{(n)}, P, p^{(n)} | R_0; \nu) \, dR = 0 \quad (n > 0)
\end{align*}
\]

First of these expresses mere normalization of BP’s path distribution and besides says that (unconditional) probability distribution of BP’s velocity in equilibrium gas always stays equilibrium.

Collecting all \(V\)'s \(V_n(t, ik, \rho^{(n)}, P, p^{(n)}; \nu)\) into generating functional

\[
\mathcal{V}(t, ik, P, \psi; \nu) = V_0(t, ik, P, \psi; 0) + \sum_{n=1}^{\infty} \frac{\nu^n}{n!} \int \ldots \int V_n(t, ik, \rho^{(n)}, P, p^{(n)}; \nu) \prod_{j=1}^{n} \psi(\rho_j, p_j),
\]

one may replace all equations (22)-(24) by single functional equation:

\[
\frac{\partial \mathcal{V}}{\partial t} = \hat{\mathcal{L}} \mathcal{V}
\]

with evolution operator

\[
\hat{\mathcal{L}} = i(k \cdot V) + \int \psi(x) \left[ (V - v) \cdot \frac{\partial}{\partial \rho} + \Phi'(|\rho|) \cdot \left( \frac{\partial}{\partial \rho} - \frac{\partial}{\partial P} \right) \right] \frac{\delta}{\delta \psi(x)} - \frac{\partial}{\partial P} \int \Phi'(\rho) \frac{\delta}{\delta \psi(x)} + \nu \int G_m(p) E'(\rho) \psi(x) \cdot (V + T \frac{\partial}{\partial P})
\]

and initial condition

\[
\mathcal{V}(t = 0, ik, P, \psi; \nu) = G_M(P)
\]

In essence, of course, this is full equivalent of (8) following from (8) under change \(\psi(r, p) \rightarrow \psi(R + \rho, p)\) and correspondingly

\[
\frac{\partial}{\partial R} \rightarrow \frac{\partial}{\partial R} + \int dP \int d\rho \frac{\partial \psi(\rho, p)}{\partial \rho} \frac{\delta}{\delta \psi(\rho, p)}
\]

4. Boson representation and path integrals

1. According to formulas of previous section, direct formal exact solution of BBGKY equations (22)-(24) in respect to the characteristic function (19) of BP’s path distribution can be written as

\[
V_0(t, ik; \nu) = \int \mathcal{V}(t, ik, P, \psi; 0; \nu) \, dP = \left[ \int dP \, e^{t \hat{L}} G_M(P) \right]_{\psi = 0},
\]

where the evolution operator \(\hat{\mathcal{L}}\) represents, in respect to BP’s momentum, a linear combination of two vector operators, \(V + T \partial / \partial P\) and \(-T \partial / \partial P\) (factor \(V\) in first row of (29) is their sum). Let us normalize them so that their components commute one with another exactly as boson birth and annihilation operators:

\[
A^\dagger \equiv -\sqrt{TM} \frac{\partial}{\partial P}, \quad A \equiv \sqrt{\frac{M}{T}} \left( V + T \frac{\partial}{\partial P} \right), \quad A_\alpha A^\dagger_\beta - A^\dagger_\beta A_\alpha = \delta_{\alpha \beta}
\]

Then notice that \(A G_M(P) = 0\) and \(\int dP \, A^\dagger \ldots = 0\). Therefore the Maxwell distribution, \(G_M(P) = 0\), can be treated as ket-vector of ground state of “quantum harmonic oscillators” represented by \(A\) and \(A^\dagger\), while integration over momentum, \(\int dP \ldots\), acts as multiplication by bra-vector of the ground state.
Besides, notice or recall that operators of multiplication by $\psi(x)$ and differentiation $\delta/\delta\psi(x)$ also behave as boson birth and annihilation operators, and we can write
\[ a^\dagger(x) \equiv c(x) \psi(x) , \quad a(x) \equiv c^{-1}(x) \frac{\delta}{\delta\psi(x)} , \quad a(x)a^\dagger(y) - a^\dagger(y)a(x) = \delta(x-y) , \tag{33} \]
with arbitrary fixed $c(x) \neq 0$. At that, unit from the right of them and putting on $\psi = 0$ after their action again represent ket- and bra-vectors of vacuum state, respectively. A proper choice of $c(x)$ is
\[ c(x) = \sqrt{\nu G_m(p) E(\rho)} \]
For further let us introduce also $v_0 \equiv \sqrt{T/m}$, $u_0 \equiv \sqrt{T/M}$.

Then after some algebra formulas (29) and (31) can be rewritten as
\[ V_0(t, ik; \nu) = \langle 0| e^{it\hat{L}} |0 \rangle , \]
\[ \hat{L} = \hat{L}_1 + \hat{L}_2 + \hat{L}_3 , \tag{34} \]
where $|0\rangle$ is common ground state ("vacuum state"), and three parts of the evolution operator are linear, quadratic and cubic forms of the birth and annihilation operators, respectively:
\[ \hat{L}_1 = i(k \cdot (A + A^\dagger)) u_0 , \tag{35} \]
\[ \hat{L}_2 = \int a^\dagger(x) \left[ -v \cdot \frac{\partial}{\partial p} + \Phi'(\rho) \cdot \frac{\partial}{\partial p} \right] a(x) + u_0 \int c(x) \frac{\Phi'(\rho)}{T} \cdot [a(x)A^\dagger - a^\dagger(x)A] , \tag{36} \]
\[ \hat{L}_3 = u_0 \int a^\dagger(x) \left[ (A^\dagger + A) \cdot \frac{\partial}{\partial p} + (A^\dagger - A) \cdot \frac{\Phi'(\rho)}{2T} \right] a(x) \tag{37} \]
In such way calculation of exact characteristic function of BP’s path transforms to calculation of the "vacuum-vacuum transition" amplitude, $\langle 0| e^{-it\hat{H}} |0 \rangle$, for excited system of interacting bosons, or quantum oscillators, with cubic Hamiltonian $\hat{H} = i\hat{L}$. Due to the boundary conditions (13),(25) one can think that $a^\dagger(x)a(x) \to 0$ at $\rho \to \infty$ in those sense that far enough oscillators almost surely stay in their ground states. Therefore $\hat{H}$ can be treated as Hermitian operator.

2. On this way, next possible step is use of so-called holomorphic form of path integrals (see e.g. [26,27,28]). According to it, we can replace (34) by path integral
\[ V_0(t, ik; \nu) = \langle 0| e^{it\hat{L}} |0 \rangle = \int \exp \left\{ \int_0^t \left[ \frac{1}{2} (\dot{A}^* \cdot A - A^* \cdot \dot{A}) + \hat{L}(A^*, A) \right] d\xi \right\} \prod_\xi \frac{dA^* dA}{2\pi i} , \tag{38} \]
supplemented with edge conditions $A^*(\xi = t) = 0$, $A(\xi = 0) = 0$, where $A^* = A^* (\xi) = \{a^*(x, \xi), A^* (\xi)\}$ and $A = A(\xi) = \{a(x, \xi), A(\xi)\}$ are holomorphic images of all above introduced birth and annihilation operators in the form of complex-valued time functions, and the dot means derivative in respect to time $\xi$. Integrating (38) first over variables $a^*(x, \xi), a(x, \xi)$ deputed by $a^\dagger(x)$ and $a(x)$ one obtains
\[ V_0(t, ik; \nu) = \int \exp \left\{ \int_0^t \left[ \frac{1}{2} (\dot{A}^* \cdot A - A^* \cdot \dot{A}) + i(k \cdot (A + A^*)) u_0 \right] d\xi \right\} \times \]
\[ \times \exp \left\{- \int_{t \leq \xi_1 \leq \xi_2 > 0} A^*_\alpha(\xi_1) \ G_{\alpha\beta}(\xi_1, \xi_2, A^*, A) \ A_\beta(\xi_2) \ d\xi_2 d\xi_1 \right\} \prod_\xi \frac{dA^* dA}{(2\pi i)^2} \tag{39} \]
with edge conditions $A^*(t) = 0$, $A(0) = 0$, where repeated indices imply summation, the kernel $G_{\alpha, \beta}$ is functional of $A^*(\xi) = 0$ and $A(\xi) = 0$ expressed by
\[ \mathcal{G}_{\alpha \beta} \{ \xi_1, \xi_2, \mathbf{A}^*, \mathbf{A} \} = \frac{u_0^2}{T^2} \int c(x) \Phi_\alpha' (\rho) \exp \left[ \int_{\xi_2}^{\xi_1} \tilde{\Lambda}(\mathbf{A}^*(\xi), \mathbf{A}(\xi)) \, d\xi \right] \Phi_\beta'(\rho) \, c(x) , \]  

(40)

and \( \tilde{\Lambda}(\mathbf{A}^*, \mathbf{A}) \) is sum of evolution operators from (36) and (37),

\[ \tilde{\Lambda}(\mathbf{A}^*, \mathbf{A}) = - \mathbf{v} \cdot \frac{\partial}{\partial \rho} + \Phi(\rho) \cdot \frac{\partial}{\partial \mathbf{p}} + u_0 \left[ (\mathbf{A}^* + \mathbf{A}) \cdot \frac{\partial}{\partial \rho} + (\mathbf{A}^* - \mathbf{A}) \cdot \frac{\Phi'(\rho)}{2T} \right] \]  

(41)

(clearly, this is image of the Liouville operator (21)). Here and below \( \exp \) designates chronologically ordered exponential and, as before, \( \int \ldots = \int \ldots \, d\mathbf{p} \, d\rho \). With use of obvious identities

\[ \tilde{\Lambda}(\mathbf{A}^*, \mathbf{A}) \, c(x) = - \frac{u_0 \Phi'(\rho) \cdot \mathbf{A} \, c(x)}{T} , \quad \tilde{\Lambda}^\top(\mathbf{A}^*, \mathbf{A}) \, c(x) = \frac{u_0 \Phi'(\rho) \cdot \mathbf{A}^* \, c(x)}{T} \]  

(where \( \top \) symbolizes transposition) one can transform (39) into

\[ V_0(t, i\mathbf{k}; \nu) = \int \exp \left\{ \int_0^t \left[ \frac{1}{2} \left( \mathbf{A}^* - \mathbf{A}^* \cdot \mathbf{A} \right) + i (\mathbf{k} \cdot (\mathbf{A}^* + \mathbf{A})) \, u_0 \right] \, d\xi \right\} \times \]  

\[ \times \exp \left\{ \int c(x) \left( \exp \left[ \int_0^t \tilde{\Lambda}(\mathbf{A}^*, \mathbf{A}) \, d\xi \right] - 1 \right) \, c(x) \right\} \prod_{\xi} \frac{d\mathbf{A}^* \, d\mathbf{A}}{(2\pi i)^d} \]  

(42)

Exact analogue of formula (42) was obtained in [29], in slightly different notation, by means of so-called “stochastic representation of deterministic interactions” (see also references from [29]).

Integration of (38) at first over \( \mathbf{A} \) and \( \mathbf{A}^* \) results in another kind of holomorphic path integrals:

\[ V_0(t, i\mathbf{k}; \nu) = \int \exp \left\{ \int_0^t \int \left[ \frac{1}{2} \left( \mathbf{a}^* \, \mathbf{a} - \mathbf{a}^* \, \mathbf{a} \right) + \mathbf{a}^* \tilde{\Lambda}_0 \, \mathbf{a} \right] \, dx \, d\xi \right\} \times \]  

\[ \times \exp \left\{ u_0^2 \int_0^t \int_{\xi_1}^{\xi_2} \left[ i \mathbf{k} \cdot \mathbf{K}^*(\mathbf{a}^*(\xi_1), \mathbf{a}(\xi_1)) \right] \cdot \left[ i \mathbf{k} \cdot \mathbf{K}(\mathbf{a}^*(\xi_2), \mathbf{a}(\xi_2)) \right] \right\} \prod_{x, \xi} \frac{da^* \, da}{2\pi i} , \]  

(43)

where edge conditions \( a^*(x, \xi = t) = 0, a(x, \xi = 0) = 0 \) must be satisfied,

\[ \tilde{\Lambda}_0 = - \mathbf{v} \cdot \frac{\partial}{\partial \rho} + \Phi'(\rho) \cdot \frac{\partial}{\partial \mathbf{p}} \]  

is Liouville operator of atom interacting with BP (fixed at coordinate origin), and

\[ \mathbf{K}^*(\mathbf{a}^*, \mathbf{a}) = - \int c(x) \frac{\Phi'(\rho)}{T} \, a^*(x) \, dx + \int a^*(x) \left[ \frac{\partial}{\partial \rho} - \frac{\Phi'(\rho)}{2T} \right] \, a(x) \, dx , \]  

\[ \mathbf{K}(\mathbf{a}^*, \mathbf{a}) = \int c(x) \frac{\Phi'(\rho)}{T} \, a(x) \, dx + \int a^*(x) \left[ \frac{\partial}{\partial \rho} + \frac{\Phi'(\rho)}{2T} \right] \, a(x) \, dx , \]  

where \( \int \ldots = \int \ldots \, d\mathbf{p} \, d\rho \).

Unfortunately, a correct non-perturbation treatment of such strongly non-Gaussian path integrals as (39) or (42) or even (43) is in itself non-trivial problem [27]. To make use of them, one has to construct some perturbation expansion (e.g. over \( m/M \) in the limit of hard BP).
5. Generalized boson representation and continued fractions

A different boson-like representation is prompted by characteristic triple-diagonal form of the system of equations (22)-(24). Considering $V_n$ as components of vector in the Fock space, let us define in this space birth and annihilation operators, $\hat{A}^\dagger$ and $\hat{A}$, $\hat{a}$ and $\hat{a}^\dagger$, as follows:

$$\hat{A}^\dagger V_n = \left( \mathbf{V} + T \frac{\partial}{\partial \mathbf{P}} \right) \cdot \left[ G_m(p_{n+1}) E'(\rho_{n+1}) V_n + \sum_{j=1}^{n} G_m(p_j) E'(\rho_j) V_n (...) x_j \rightarrow x_{n+1} \ldots \right] \equiv$$

$$= \left( \mathbf{P} + TM \frac{\partial}{\partial \mathbf{P}} \right) \cdot \hat{a}^\dagger V_n ,$$

$$\hat{A} V_{n>0} = -\nu \frac{\partial}{\partial \mathbf{P}} \int \Phi'(\rho_n) V_n \equiv \frac{\partial}{\partial \mathbf{P}} \cdot \hat{a} V_n , \quad \hat{A} V_0 = 0 , \quad \hat{a} V_0 = 0 ,$$

where symmetry of all the CF $V_n$ is taken in mind. According to this definition,

$$\left( \hat{a}_\alpha \hat{a}^\dagger_\beta - \hat{a}^\dagger_\beta \hat{a}_\alpha \right) = a^2 \delta_{\alpha\beta} ,$$

$$\left( \hat{A} \hat{A}^\dagger - \hat{A}^\dagger \hat{A} \right) V_n = a^2 \left[ \frac{\partial}{\partial \mathbf{P}} \cdot \left( \mathbf{P} + TM \frac{\partial}{\partial \mathbf{P}} \right) + \sum_{j=1}^{n} \hat{\Pi}(j) \right] V_n ,$$

$$a^2 \equiv -\frac{\nu}{Md} \int \Phi'(\rho) \cdot E'(\rho) d\rho ,$$

if we assume that $\Phi(\rho)$ is spherically symmetric potential and define projection operator $\hat{\Pi}$ as

$$\hat{\Pi}(j) V_n = d G_m(p_j) E'(\rho_j) \cdot \int \Phi'(\rho_j) V_n d\rho_j d\mathbf{p}_j \left[ \int \Phi'(\rho) \cdot E'(\rho) d\rho \right]^{-1}$$

Besides, introduce operator $\hat{\Lambda}_k$ by equalities $\hat{\Lambda}_k V_0 \equiv i(\mathbf{k} \cdot \mathbf{V}) V_0 ,$

$$\hat{\Lambda}_k V_n \equiv \left[ i(\mathbf{k} \cdot \mathbf{V}) + \sum_{j=1}^{n} \hat{\Lambda}(j) \right] V_n$$

Then formal solution of equations (22)-(24), under initial conditions (25), in respect to Laplace transform of $V_0(t; i\mathbf{k} ; \nu) , can be represented by continued fraction:

$$\int_0^\infty e^{-zt} V_0(t; i\mathbf{k} ; \nu) dt =$$

$$= \int d\mathbf{P} \left[ z - \hat{\Lambda}_k - \hat{A} \left[ z - \hat{\Lambda}_k - \hat{A} \left[ z - \hat{\Lambda}_k - \ldots \right]^{-1} \hat{A}^\dagger \right]^{-1} \hat{A}^\dagger \right] \cdot G_M(\mathbf{P}) =$$

$$= \int d\mathbf{P} \left[ z - i\mathbf{k} \cdot \mathbf{V} - \frac{\partial}{\partial P_\alpha} \tilde{\Gamma}_\alpha^{(1)}(z,i\mathbf{k}) \left( P_\beta + TM \frac{\partial}{\partial P_\beta} \right) \right]^{-1} G_M(\mathbf{P}) ,$$

where $\tilde{\Gamma}_\alpha^{(1)}$ begins recursive chain of operators

$$\tilde{\Gamma}_\alpha^{(n)}(z,i\mathbf{k}) = \tilde{\alpha}_\alpha \left[ z - i\mathbf{k} \cdot \mathbf{V} - \sum_{j=1}^{n} \hat{\Lambda}(j) - \frac{\partial}{\partial P_\gamma} \tilde{\Gamma}_\gamma^{(n+1)}(z,i\mathbf{k}) \left( P_\delta + TM \frac{\partial}{\partial P_\delta} \right) \right]^{-1} \hat{a}^\dagger_\beta$$
6. Stochastic form of the boson representation

Let $A^\dagger$, $A$ be a set of pairs of boson birth and annihilation operators, such that $A_\alpha A_\beta^\dagger - A_\beta^\dagger A_\alpha = \delta_{\alpha\beta}$. Then any path integral like (38) or (39) or (42) or (43),

$$\int \exp \left\{ \int_0^t \frac{1}{2} (\dot{A}^* \cdot A - A^* \cdot \dot{A}) d\xi + F\{A^*(\xi), A(\xi)\} \right\} \prod_\xi \frac{dA^* dA}{2\pi i} = \int \exp \left[ \int F\{A^*(\xi), A(\xi)\} dM\{A^*, A\} \right],$$

with arbitrary functional $\exp[F\{A^*(\xi), A(\xi)\}]$ defined on interval $0 \leq \xi \leq t$, can be formally considered as averaging of this functional over Gaussian “probabilistic” measure $dM\{A^*, A\}:

$$\int \exp \left[ \int F\{A^*(\xi), A(\xi)\} dM\{A^*, A\} \right] = (\exp[F\{A^*(\xi), A(\xi)\}])$$

The measure $dM\{A^*, A\}$ is completely characterized by corresponding pair correlation functions:

$$\langle A_\alpha(t_1) A_\beta^\dagger(t_2) \rangle = \langle A_\alpha^*(t_1) A_\beta^\dagger(t_2) \rangle = 0, \quad \langle A_\alpha(t_1) A_\beta(t_2) \rangle = \delta_{\alpha\beta} \Theta(t_1 - t_2),$$

where $\Theta(t)$ is Heaviside step function. To prove these equalities, it is sufficient to calculate Gaussian integral which represents characteristic functional of the stochastic processes $A^\dagger(t)$ and $A(t),

$$\int \exp \left\{ \int_0^t \left[ b(\xi) \cdot A^*(\xi) + b^*(\xi) \cdot A(\xi) \right] d\xi \right\} dM\{A^*, A\} = \exp \left[ \int_0^t dt' \int_0^{t''} b^*(t') \cdot b(t'') \right],$$

where the mentioned edge conditions are taken in mind, and $b(t)$, $b^*(t)$ are arbitrary time functions.

In this sense, in particular,

$$V_0(t, i\mathbf{k}; \nu) = \langle 0 | e^{i\mathcal{L}(A^\dagger, A)} | 0 \rangle = (\exp \int \mathcal{L}(A^*(\xi), A(\xi)) d\xi)$$

with operator function $\mathcal{L}(A^\dagger, A)$ defined by (35)-(37), and

$$V_0(t, i\mathbf{k}; \nu) = \exp \left[ i\mathbf{w} \cdot \mathbf{k} \cdot \int_0^t \left[ A(\xi) + A^*(\xi) \right] d\xi - \int_{t_1 < \xi_2} A^*(\xi_2) \cdot \mathcal{G}\{\xi_1, \xi_2, A^*, A\} \cdot A(\xi_2) \right]$$

instead of (39), with Gaussian random processes $A^*(t)$ and $A(t)$ defined by

$$\langle A_\alpha(t_1) A_\beta(t_2) \rangle = \langle A_\alpha^*(t_1) A_\beta^\dagger(t_2) \rangle = 0, \quad \langle A_\alpha(t_1) A_\beta^\dagger(t_2) \rangle = \delta_{\alpha\beta} \Theta(t_1 - t_2)$$

Such “stochastic” point of sight at calculation of path integrals can make it more constructive. Notice that for any functional $F = F\{A^*(\xi), A(\xi)\}$ arranged like second term in (51),

$$\mathcal{G}\{A^*(\xi), A(\xi)\} = - \int_{t > \xi > \xi_2 > 0} \int A^*(\xi_1) \mathcal{G}_{\alpha\beta}\{\xi_1, \xi_2, A^*, A\} A_\beta(\xi_2) d\xi_2 d\xi_1,$$

where $\mathcal{G}_{\alpha\beta}\{\xi_1, \xi_2, A^*, A\}$ involves $A^*(\xi)$ and $A(\xi)$ from interval $\xi_2 \leq \xi \leq \xi_1$ only, the identities

$$\langle F^n\{A^*(\xi), A(\xi)\} \rangle = 0 \quad (n > 0), \quad (\exp[F\{A^*(\xi), A(\xi)\}]) = 1$$
take place, that is by itself such functional is identical to zero. In special case of quadratic (bilinear) functional, when $G_{\alpha\beta}(\xi_1, \xi_2, A^*, \mathbb{A}) = G_{\alpha\beta}(\xi_1 - \xi_2)$,

$$
\left\langle \exp \left( \int_0^t \left[ b(\xi) \cdot A^*(\xi) + b^*(\xi) \cdot A(\xi) \right] d\xi - \int_{t>\xi_1,\xi_2>0} A^*_{\alpha}(\xi_1) G_{\alpha\beta}(\xi_1 - \xi_2) A_{\beta}(\xi_2) \right) \right\rangle = \exp \left\{ \int_0^t dt' \int_0^{t''} b^*_{\alpha}(t') Q_{\alpha\beta}(t' - t'') b_{\beta}(t'') \right\},
$$

where matrix function $Q_{\alpha\beta}(t)$ is defined by

$$
Q \equiv \Theta \left[ 1 + \otimes G \otimes \Theta \right]^{-1}, \quad \int_0^\infty e^{-zt} Q(t) dt = \left[ z + \int_0^\infty e^{-zt} G(t) dt \right]^{-1},
$$

with symbol $\otimes$ standing for time convolution. The first of the latter expressions remains valid also when $G_{\alpha\beta} = G_{\alpha\beta}(\xi_1, \xi_2)$ is not a difference kernel.

7. Discussion and conclusion

At present, unfortunately, none of formally exact expressions (31), (34), (39), (42), (43), (45) or (51) can be calculated exactly or at least correctly, at least in the long-time limit $t/\tau_p \to \infty$ with $\tau_p$ denoting relaxation time of BP’s momentum. Therefore it remains only to discuss the convenient approximation of exact theory and establish discreditable invalidity of this approximation from the point of view of exact “virial relations” (15)-(17) (see [4,5,6,7,8,9] and remark at end of Sec.2). 1. The mentioned approximation follows from the chain (22)-(24) if we cut off it already at second level, i.e. neglect second-order (three-particle) correlation and thus all higher-order correlations. This is just what one always makes (knowingly or unknowingly) when creating Boltzmannian kinetics. Then

$$
V_1(t, ik, P, \rho; \nu) = \int_0^t e^{i[k \cdot V + \Lambda](t-t')} G_m(P) E'(\rho) \cdot \left( V + T \frac{\partial}{\partial P} \right) V_0(t', ik, P; \nu) dt',
$$

and the first BBGKY equation (22) turns into closed kinetic equation

$$
\frac{\partial V_0(t, ik, P; \nu)}{\partial t} = i(k \cdot V) V_0(t) + \frac{\partial}{\partial P_\alpha} \int_0^t \tilde{G}_{\alpha\beta}(t-t', ik) \left( P_\beta + TM \frac{\partial}{\partial P_\beta} \right) V_0(t') dt',
$$

with $V_0(t) \equiv V_0(t, ik, P; \nu)$ on the right and operator-valued kernel

$$
\tilde{G}_{\alpha\beta}(\theta, ik) \equiv \frac{\nu}{TM} \int d\rho \int dP \Theta'(\rho) e^{i[k \cdot V + \Lambda] \theta} \Phi'_{\beta}(\rho) G_m(P) E(\rho)
$$

From the point of view of continued fraction (45), this is “one-loop approximation” when one substitutes zero for $\tilde{F}^{(2)}_{\alpha\beta}$ and thus leaves two floors of the fraction only.

Evidently, this kernel is a sharp function of $\theta$, with width nearly equal to time duration of BP-atom collision, $\tau_m = \tau_0 / v_0$. At that, principally we are interested in the long-time long-range limit only, when $t/\tau_p \to \infty$ and $k \to 0$ under $k \cdot t = const$. Therefore factor $i(k \cdot V)$ in (58) can be neglected, and equation (57) reduces to the “Boltzmann-Lorentz equation”

$$
\frac{\partial V_0}{\partial t} = i(k \cdot V) V_0 + \tilde{B} V_0, \quad \tilde{B} \equiv \frac{\partial}{\partial P} \cdot \int_0^\infty G(\theta, 0) d\theta \cdot \left( P + TM \frac{\partial}{\partial P} \right),
$$
where $\hat{B}$ plays role of linearized collision operator.

Interestingly, above “derivation” of Boltzmannian kinetics had not required the Boltzmann’s “Stosszahlansatz”. Though the latter is necessary if one wants to transform $\hat{B}$ into standard “collision integral”. This observation shows that the heart of Boltzmannian kinetics is neglect of third- and higher-order correlations (and thus, in essence, neglect of true second-order correlation).

Recall that $\hat{A}$ contains $P$ and $\partial/\partial P$, therefore in general operator $\hat{G}(\theta, 0)$ significantly depends on $P$ and $\partial/\partial P$. But under the long-range limit it effectively interchanges to $\int dP \hat{G}(\theta, 0) G_M(P)$. Therefore, regardless of details of $\hat{B}$, long-range asymptotic what follows from (57) or (59) is the Gaussian one:

$$V_0(t, i\mathbf{k}, P; \nu) \rightarrow \exp \left[ -D(\nu) k^2 t \right] G_M(P) \left( 1 + \frac{\tau_p}{M} i\mathbf{k} \cdot P + ... \right) ,$$

$$V_0(t, i\mathbf{k}; \nu) \rightarrow \exp \left[ -D(\nu) k^2 t \right] ,$$

where BP’s diffusivity $D(\nu)$ and momentum relaxation time are presented by

$$D(\nu) = u_0^2 \tau_p \propto \nu^{-1} ,$$

$$\frac{1}{\tau_p} = \frac{\nu}{TMd} \int_0^\infty d\theta \int d\mathbf{P} \int d\rho \int d\mathbf{p} \quad \Phi'(\rho) \cdot e^{\hat{N}_0} \Phi'(\rho) G_m(p) G_M(P) E(\rho) =$$

$$= \frac{2m}{M + m} \nu \int |v - V| \Sigma(|v - V|) G_m(p) G_M(P) d\mathbf{p} d\mathbf{P} \quad (61)$$

In (60) last multiplier of $V_0(t, i\mathbf{k}, P; \nu)$ is important for (56), while the dots replace unimportant terms, and in (61) $\Sigma$ is full effective cross-section of BP-atom collisions.

All that seems beautiful till one confronts equations (56) and (59) to the simplest virial relation (17). The latter requires that

$$\frac{\partial V_0(t, i\mathbf{k}, P; \nu)}{\partial \nu} = \int \int \int V_1(t, i\mathbf{k}, P, \rho, P; \nu) d\mathbf{p} d\rho d\mathbf{P} \quad (62)$$

Combining this identity from exact theory with (60) and (61) we see that if the mentioned approximation was correct then we would have

$$\int \int \int V_1(t, i\mathbf{k}, P, \rho, P; \nu) d\mathbf{p} d\rho d\mathbf{P} \rightarrow \frac{D(\nu) k^2 t}{\nu} e^{-D(\nu) k^2 t} \quad (63)$$

In fact, however, expression (56), as combined with (60) and (61), after quite standard (although rather troublesome) manipulations yields

$$\int \int \int V_1(t, i\mathbf{k}, P, \rho, P; \nu) d\mathbf{p} d\rho d\mathbf{P} \rightarrow$$

$$- \frac{u_0^2 k^2}{\nu} \int_0^t \exp \left[ -\frac{1}{2} u_0^2 k^2 (t - t')^2 - D(\nu) k^2 t' \right] (t - t') dt' \rightarrow - \frac{1}{\nu} e^{-D(\nu) k^2 t} , \quad (64)$$

since exponential in (56) corresponds to free ballistic flight of BP after its single collision with an atom.

The difference between (63) and (64) is qualitative, and it says that the conventional picture, including the Boltzmann-Lorentz equation, is far from truth!

Such strong discrepancy came from our neglect of the third-order correlations in (23), i.e. $V_2$’s contribution to (23), and thus neglect of all higher-order correlations. However, the “virial expansions” (15), in particular, first of them, as written via Fourier transforms (20),

$$V_0(t, i\mathbf{k}, P; \nu + \sigma \nu) = V_0(t, i\mathbf{k}, P; \nu) + \sum_{n=1}^{\infty} \frac{\nu \sigma}{n!} \int \int \int V_n(t, i\mathbf{k}, \rho^{(n)}, P, \rho^{(n)}; \nu) \quad (65)$$

(recall that $\sigma > -1$), help us to understand that cutting of any even high correlations is bad idea.
Indeed, gas densities on two sides of (65), $\nu + \sigma \nu$ and $\nu$, can be different in arbitrary strong proportion $0 < \sigma + 1 < \infty$. Therefore any cutoff in the infinite series in (65) would give a faulty result (like e.g. cutoff in the series representing exponential function). Hence, in practice all terms on right-hand side of (65) are equally important, regardless of value of gas density!

Consequently, all equations of the hierarchy (9)-(11) or (22)-(24) are equally important for correct analysis of the BP’s path probability distribution, even in the “low density limit” (“Boltzmann-Grad limit”). This means that Boltzmannian kinetics is not a true “zero-order approximation” of rigorous theory in respect to the density. Or, better to say, true kinetics has no literal “zero-order approximation” at all.

2. The reason for all this was explained more than once in [20,21] twenty five years ago, then in [16] and later in [17,18,19,23,24,25] and [6,7,8,9,10] (and in principal sense anticipated in [22] sixty years ago). Indifference of many-particle system to a number of happened events of definite kind (BP’s collisions with gas atoms here) leads to scaleless fluctuations (“1/f-noise”) in “mean number of events per unit time” and related quantities (e.g. BP’s diffusivity and mobility here).

In spite of this understanding, one can envy creators of conventional kinetics (see Introduction): it already resolved all its problems. But, from the other hand, this is kinetics of an invented tiresome and “dead” world. Our consideration demonstrated that even such simple world as Brownian particle interacting with ideal gas apparently is “alive” and interesting. Nobody is able to predict what “number of collisions per unit time” will meet this particle in particular life. Hence, real theory is not in the past, it is yet in the future.

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