Weakly Compact “Matrices”, Fubini-Like Property and Extension of Densely Defined Semigroups of Operators

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Abstract

Taking matrix as a synonym for a numerical function on the Cartesian product of two (in general, infinite) sets, a simple purely algebraic “reciprocity property” says that the set of rows spans a finite-dimensional space iff the set of columns does so. Similar topological reciprocity properties serve to define strongly compact and weakly compact matrices, featured in the well-known basic facts about almost periodic functions on groups and about compact operators. Some properties, especially for the weak compact case, are investigated, such as the connection with the matrix having a Fubini-like property for general finitely additive means. These are applied to prove possibility of extension to the entire semigroup of bounded densely defined semigroups of operators in a Banach space with weak continuity properties.

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Michael Cwikel has drawn my attention to [B] and [M] where similar issues are dealt with from a somewhat different point of view.

1 Preface: Reciprocity Between Rows and Columns, Strongly Compact “Matrices”.

Taking matrix as a synonym for a numerical (i.e. real or complex) function on the Cartesian product of two (in general, infinite) sets, a simple purely algebraic “reciprocity property” says that the set of rows spans a finite-dimensional space iff the set of columns does so. It might be illuminating to sketch a proof: If the rows span a finite-dim space, there is a finite set $F$ of them so that all others are linear combinations of them. Otherwise put: all entries of a column are fixed (i.e. the same for all columns) linear combinations of the entries at $F$. But
since $F$ is finite, the entries at $F$ of all the columns surely span a finite-dim space, i.e. all depend linearly on a finite number of them, which carries over to the entire columns by the above fixed linear combinations.

Assuming the matrix is bounded and viewing the rows and columns as vectors in the $\ell^\infty$ spaces, well-known analogous functional-analytic reciprocity properties obtain similarly. Thus we have: the rows form a strongly relatively compact (i.e. precompact) set in the sup-norm iff the columns do so (an assertion which specializes, of course, to well-known basic facts about almost periodic functions on groups and about compact operators). To give an argument similar to the above algebraic case, suppose the rows form a precompact set. For any $\varepsilon > 0$, there is a finite set $F$ of rows which is an $\varepsilon$-net (i.e. any row has an $\ell^\infty$ distance no more than $\varepsilon$ from some one of them). This means that for the columns, two columns that differ no more than some $\delta > 0$ at the $F$-entries differ no more than $\delta + 2\varepsilon$ at all entries. The $F$ pieces of all the columns are a bounded set in a finite-dim space, hence precompact, i.e. contains a $\delta$-net for any $\delta > 0$ which by the above will be a $\delta + 2\varepsilon$-net for the entire columns.

Let us call a matrix where the rows, equivalently the columns, form a strongly relatively compact set in the sup-norm a strongly compact matrix. One easily obtains some properties of them (which give well-known properties for the case of almost-periodic functions).

**Proposition 1** The set of strongly compact matrices is a closed subspace of the $\ell^\infty$ space on the Cartesian product.

**Proposition 2** Let $I$, $J$ and $L$ be sets and let $f$ be a bounded complex function on $I \times J \times L$ so that

- $f$ is strongly compact as a matrix on $L$ and $I \times J$
- for any fixed $a \in L$, the $a$-section of $f$ $f(a, \cdot, \cdot)$ is strongly compact as a matrix on $I$ and $J$

then $f$ is strongly compact as a matrix on $J$ and $I \times L$ and as a matrix on $I$ and $J \times L$.

Consequently, if a bounded complex function on $I \times J \times L$ is strongly compact both as a matrix on $L$ and $I \times J$ and as a matrix on $J$ and $I \times L$, then it is so also as a matrix on $I$ and $J \times L$. ("associativity property").

**Proposition 3** Suppose $I$ and $J$ are topological spaces. Then if a bounded complex separately continuous function on $I \times J$ is strongly compact as a matrix, then it is jointly continuous. If $I$ and $J$ are compact, the converse holds: if $f$ is jointly continuous it is strongly compact as a matrix.

**Remark 1** A continuous complex function $f$ on a topological group $G$ is called (strongly) almost periodic if the matrix $x, y \mapsto f(xy), x, y \in G$ is strongly compact. This clearly implies that the matrices $x, (y, z) \mapsto f(xyz)$ and $(x, y), z \mapsto f(xyz)$ are strongly compact, hence, by Prop. 2 that $(x, z), y \mapsto f(xyz)$ is strongly compact, which in turn implies that the original matrix $x, y \mapsto f(xy)$ is strongly compact. Hence $(x, z), y \mapsto f(xyz)$ strongly compact is an equivalent definition for almost-periodicity.
Let $X, Y$ be a Banach spaces and let $Y^*$ be the dual of $Y$. Let $A : X \to Y$ be a bounded linear operator. Consider the matrix $x, y^* \mapsto \langle y^*, Ax \rangle$, $x, y^*$ in the closed unit balls of $X$ and $Y^*$ resp. Then its rows form a relatively compact set in the $\ell^\infty$ space iff $A$ is a compact operator, and its columns do so iff $A^*$ is such. Thus the matrix is strongly compact $\iff A$ is a compact operator $\iff A^*$ is a compact operator.

2 Weakly Compact Matrices and Fubini-Like Property

Now replace strong (relative) compactness by weak (relative) compactness w.r.t. the $\ell^\infty$ spaces. We shall need the following simple

Proposition 4 Let $I$ and $J$ be Hausdorff topological spaces and let $S \subset I$ be dense. Suppose $f$ is a bounded complex matrix on $I \times J$ so that all columns are continuous functions on $I$ and all rows $f(s, \bullet)$ with $s \in S$ are continuous functions on $J$. Suppose further that the set of these rows $f(s, \bullet)$, $s \in S$ is relatively weakly compact in $\ell^\infty(J)$ (or, equivalently, in its closed subspace $C_b(J)$—bounded continuous functions). Then $f$ is separately continuous (i.e. all rows $f(s, \bullet)$, $s \in I$ are continuous).

Proof The set of rows $f(s, \bullet)$, $s \in S$, is relatively weakly compact in the space $C_b(J)$. Thus, for any $r \in I$, the rows $f(s, \bullet)$ for $s \to r$ have a weak cluster point in $C_b(J)$ (to which they will weakly converge if we take a finer filter, alternatively take a subnet). Anyhow, these rows tend pointwise to $f(r, \bullet)$ since the columns are continuous. Therefore the only possible weak cluster point is $f(r, \bullet)$, hence $f(r, \bullet)$ must be in $C_b(J)$; i.e. continuous, and we are done.

QED

Theorem 1 Let $f$ be a bounded complex matrix on the sets $I$ and $J$. TFAE:

(i) Fubini-like property: if $M$ is any mean on $\ell^\infty(I)$ (recall this means a positive linear functional mapping the constant 1 to 1, equivalently a finitely additive “measure” on $I$. $I$ is embedded into the set of means as $\delta$-“measures”) and $N$ is any mean on $\ell^\infty(J)$, then taking an “iterated integral” of $f$ w.r.t. these means does not depend on the “order of integration”.

(ii) $f$ can be extended to a separately continuous affine function (=matrix) by embedding $I$ and $J$ (as sets) in some $\tilde{I}$ and $\tilde{J}$, compact convex subsets of Hausdorff locally convex topological linear spaces.

(iii) The set of rows $f(i, \bullet)$ is weakly relatively compact in $\ell^\infty(J)$

(iv) The set of columns $f(\bullet, j)$ is weakly relatively compact in $\ell^\infty(I)$

In this case, we naturally say that the matrix is weakly compact.

Proof (i) $\Rightarrow$ (ii): Having the Fubini-like property, the “iterated integral” defines a matrix $\tilde{f}(M, N)$, which is clearly a separately continuous affine function on the sets of means, these being convex compact w.r.t. the $w^*$ topology from the $\ell^\infty$ spaces. $\tilde{f}$ is an extension of the original $f$, when $I$ and $J$ are embedded in the sets of means as $\delta$-“measures”.
(ii) ⇒ (iii) and (iv): It clearly suffices to prove the weak relatively compactness when \( I \) and \( J \) themselves are convex compact and the matrix \( f \) separately continuous affine. But note that for the sup-norm space \( A \) of continuous affine functions on a compact convex set any element of the dual is a linear combination of two evaluation functionals, hence the weak topology is the pointwise topology. Now each \( i \in I \) maps to the row \( f(i, \bullet) \), thus mapping a compact set continuously w.r.t. to the pointwise = weak topology, so the set of rows is weakly compact in the \( A \)-space, hence in the \( \ell^\infty \) space. Similarly for the columns.

(ii) ⇒ (i): in (ii), one may assume that \( \tilde{I} \) is the closed convex hull of the embedded \( I \) and similarly for \( \tilde{J} \). This makes the norm in \( \ell^\infty(I) \) identical with that of \( A(\tilde{I}) \) (the sup-norm space of continuous affine functions), so \( A(\tilde{I}) \) may be viewed as a subspace of \( \ell^\infty(I) \), similarly for \( J \). Any mean on \( \ell^\infty(I) \) “collapses” on \( A(\tilde{I}) \) to an evaluation functional at a point of \( \tilde{I} \), similarly for \( \tilde{J} \). For evaluation functionals Fubini is immediate.

(iii) ⇒ (ii): We wish to invoke Prop. 4. Indeed, taking “iterated integral” first on the first argument then on the second, one extends \( f \) to \( \tilde{f}(M, N) \) on the compact convex sets of means, continuous (and affine) in \( N \) (i.e. with continuous rows), and also continuous (and affine) in \( M \) if \( N = \delta_j \), \( j \in J \) hence if \( N \) is a finite convex combination of \( \delta_j \)'s, these being dense in the set of means on \( J \). Also the sup-norm for continuous affine functions on the set of means on \( J \), such as the rows of \( \tilde{f} \), coincides with that of their restriction to \( J \), making \( \ell^\infty(J) \) isometric to the sup-norm space of these continuous affine functions, and our assumption that the rows form a weakly relatively compact set translates accordingly. Therefore by Prop. 4 \( \tilde{f} \) is separately continuous (and affine) and we have (ii).

QED

Thus the matrix being weakly compact is determined by equations – the equality of the “iterated integrals”, and one concludes

Corollary 5 The set of weakly compact matrices is a closed subspace of the \( \ell^\infty \) space on the Cartesian product.

While jointly continuous matrices on two compact spaces are strongly compact, separately continuous matrices on two compact spaces are weakly compact: (Thm. 2 is not needed in §4)

Theorem 2 Let \( I \) and \( J \) be compact Hausdorff spaces, and let \( f \) be a bounded complex matrix on \( I \times J \), separately continuous. Then the matrix \( f \) is weakly compact.

Proof

• Assume \( J \) is convex compact and every \( f(i, \bullet) \) is affine. Then for the sup-norm space \( A(J) \) of continuous affine functions on \( J \) every element of the dual is a linear combination of two evaluation functionals, hence the weak topology is the pointwise topology. \( i \mapsto f(i, \bullet) \) maps the compact \( I \) continuously to \( A(J) \) with the weak = pointwise topology, hence the image is weakly compact so the rows form a weakly compact set.

• Assume \( I \) metrizable. Let \( \text{Pr}(J) \) be the compact convex space of the probability measures on \( J \) (with the \( w^* \)-topology from the continuous functions). Extend \( f \) to a matrix \( \tilde{f} \) on \( I \times \text{Pr}(J) \) by \( \tilde{f}(i, \mu) := f f(i, j) \, d\mu(j) \). \( \tilde{f} \) is continuous in \( \mu \), but also in \( i \) – on the metrizable \( I \) continuity of functions is determined by sequences and one uses Lebesgue’s convergence theorem. So by the previous \( \bullet \), \( \tilde{f} \) is weakly compact, hence so is \( f \).
• $I$ and $J$ general. It is a well-known fact (due to Eberlein and Šmulian) that a subset $A$ of a Banach space, such as $\ell^\infty$, is weakly relatively compact iff every countable subset of $A$ is so. Therefore to prove that the set of rows $f(i, \bullet)$ is weakly relatively compact it suffices to prove that every sequence of rows $f(i_n, \bullet)$ is such. But such a sequence may be viewed as a continuous mapping from $J$ with metrizable compact image $Q \subset C^N$.

Consider $\beta N$ – the Stone-Čech compactification of $N$, i.e. the compact space of the ultrafilters in $N$. One can define a matrix $m$ on $\beta N \times Q$ by $m(\tau, q) := \lim_{n \to \tau} f(i_n, j)$. $m$ is continuous in $\tau$, but also in $q$, because substituting $q(j) = (f(i_n, j))_{n=1}^{\infty}$ gives

$$m(\tau, q) = \lim_{n \to \tau} f(i_n, j) = f \left( \lim_{n \to \tau} i_n, j \right),$$

where one uses the continuity of $f$ in $i$ and the compactness of $I$, ensuring the existence of $\lim_{n \to \tau} i_n$.

So by the previous $\bullet$, $m$ is weakly compact, hence so is the sub-matrix of $f$ formed by the sequence of rows, which is a substitution in $m$.

QED

Theorem 3 Let $A$ be compact convex (subset of a Hausdorff locally convex topological linear space), let $I$ be a Hausdorff topological space and let $S \subset I$ be dense. Suppose $f(\alpha, s)$ is a complex function on $A \times I$ satisfying:

(i) $f$ is continuous affine in $\alpha$ on $A$ for each fixed $s \in S$.

(ii) $f$ is continuous in $s$ on $I$ for each fixed $\alpha \in A$.

Then $f$ is continuous in $\alpha$ on $A$ also for each fixed $r \in I$, i.e. it is separately continuous on $A \times I$.

Proof By Prop. 4 it is enough to prove that the matrix $f$ on $A \times S$ is weakly compact. By Thm. 1 one just needs to establish the Fubini-like property for $f$ w.r.t. any means on $\ell^\infty(A)$ and $\ell^\infty(S)$. But any mean on $\ell^\infty(A)$ “collapses”, on the sections $f(\bullet, s)$ $s \in S$ parallel to $A$ (the columns), to an evaluation at some point $\alpha \in A$. Thus, if $M$ is a mean on $\ell^\infty(S)$, then both iterated integrals give $M(f(\alpha, \bullet))$. So Fubini holds and we are done.

QED

Remark 2 The analogue of the “associativity property” Prop. 2 does not hold for weakly compact matrices. An example is $B =$ the unit ball of a Hilbert space $H$, $L =$ the set of operators in $H$ of norm $\leq 1$ and $f$ defined on $B \times B \times L$ by $f(x, y, A) := \langle Ax, y \rangle$. It is weakly compact as a matrix on $(x, A)$ and $y$ and as a matrix on $(y, A)$ and $x$ but not as a matrix on $(x, y)$ and $A$. 
3 A Digression: Upper- and Lower- Semicontinuous Envelopes

Let $S$ be a dense subset of a completely regular topological space $I$. For any bounded real function $f$ defined on $S$ let $f^-$ be defined (on the whole $I$) as the pointwise infimum of all continuous functions (on $I$) that majorize $f$ on the $S$. $f^-$ is upper semicontinuous (an infimum of continuous, even of upper semicontinuous functions is always upper semicontinuous, and any upper semicontinuous function is the infimum of all continuous functions which majorize it) and is the smallest upper semicontinuous function majorizing $f$ on $S$.

Note that $f^-$ at a point $r \in I$ is, in fact, defined locally: is the same for two functions $f$ coinciding on $S$ in a neighborhood of $r$.

Equivalently, since for any function $f$ on $S$ the function of $t$ in $I \limsup_{s \to t} f(s)$ is upper semicontinuous, we have $f^-(t) = \limsup_{s \to t} f(s)$. So this is another way to define $f^-$. Clearly $f^- \geq f_-$ on $I$.

Surely, $f$ can be extended to a continuous function on $I$ iff $f^- = f_-$ everywhere on $I$.

4 Densely Defined Semigroups of Operators

Theorem 4 Let $\Sigma$ be a topological semigroup which is an open sub-semigroup of a topological group which is a Baire space (e.g. a locally compact group or a Polish group). Let $\mathcal{F}$ be a filter in $\Sigma$ having a basis consisting of open sets and converging to the unit element of the group (think of $\Sigma = (0, \infty)$ with addition and $\mathcal{F}$ tending to $0^+$). Let $S$ be a dense sub-semigroup of $\Sigma$. Let $X$ be a Banach space, taken as real. (Elements of $X$ denote by $x$ and elements of its dual $X^*$ by $\rho$). Let $T_s, s \in S$ be a uniformly bounded semigroup of linear operators on $X$ defined on $S$. Suppose $T_s \to 1$ weakly as $s$ tends to $\mathcal{F}$ in $S$. Then $T_s, s \in S$ can be extended to a weakly continuous (in $r$) semigroup $T_r, r \in \Sigma$.

Proof Replacing the norm in $X$ by the equivalent norm

$$\|x\|_1 := \sup_s \|T_s(x)\|$$

we may and do assume that the $T_s$ are contractions.

Let $B^*$ be the unit ball of the dual $X^*$, with the $w^*$-topology. The members $x \in X$ may be identified with the functions on $B^* \ x(\rho) := \rho(x), \rho \in B^*$. (And denote that function by the same $x$.) This is an isometry of $X$ into $C^\infty(B^*)$. Moreover, the image of $X$ are precisely all functions on $B^*$ that are continuous affine and vanish at 0, i.e. extend to linear on $X^*$. (That follows from a well-known fact about Banach spaces, due to Dieudonné: any linear functional on $X^*$ which is continuous on $B^*$ w.r.t. the $w^*$ topology comes from an element of $X$.)

The action of $T_s$ on $X$ translates as follows:

$$(T_s x)(\rho) = \rho(T_s x) = (T_s^* \rho)(x) = x(T_s^* \rho).$$
Thus our semigroup translates to a semigroup induced by a semigroup of (continuous affine fixing 0) transformations $T^*_s$ of the space $B^*$.

Our task, extending the semigroup, will be accomplished if, for each $x$, we can extend the function $(\rho, s) \mapsto x(T^*_s\rho)$ to a function on $(\rho, t) \in B^* \times \Sigma$ which is separately continuous. Indeed, then by continuity it will be affine in $\rho$, vanishing at 0, hence by Dieudonné’s theorem will be of the form $(T^*_t x)(\rho)$, thus defining $T_t x$ and $T_t$, $t \in \Sigma$, which will depend weakly continuously on $t$ and will coincide with the original $T_s$ for $s \in S$, therefore, by continuity, will satisfy the semigroup identity.

But note, that if we can extend each $s \mapsto x(T^*_s\rho)$, for fixed $x$ and $\rho$, to a function continuous on $\Sigma$, then Thm. 3 (taking there $A = B^*$ and $I = \Sigma$) will insure the separate continuity.

Thus we are left with proving that $s \mapsto x(T^*_s\rho)$, for fixed $x$ and $\rho$, can be continuously extended to $\Sigma$.

For each $\rho \in B^*$ and $h \in \ell^\infty(B^*)$ there is the “orbit function” on $S$ $O_\rho(h) := s \mapsto h(T^*_s\rho)$. The map $h \mapsto O_\rho(h)$ is, of course, a bounded linear operator from $\ell^\infty(B^*)$ to $\ell^\infty(S)$. Moreover, the action $T_s h(\rho) := h(T^*_s\rho)$ translates to

$$O_\rho(T_s h)(s') = T_s h(T^*_s\rho) = h(T^*_s T^*_s\rho) = h((T_s T_s)^*\rho) = h(T^*_s T^*_s\rho) = (O_\rho(h))(s's),$$

that is, to the shift $f \mapsto (s' \mapsto f(s's)) = f(\bullet \cdot s)$.

Now, the fact that $T_s \to 1$ as $s \to \mathcal{F}$ in $S$ translates, for the orbit function $x(T^*_s\rho)$, which is the function that we have to continuously extend to $\Sigma$, to “$f(\bullet \cdot s)$ tends to $f$ weakly in $\ell^\infty(S)$ as $s \to \mathcal{F}$ in $S$”.

Hence it suffices to prove:

Lemma 6 Let $f$ be a real bounded function on $S$ with the property:

$s' \mapsto f(s's)$, i.e. $f(\bullet \cdot s)$, tends to $f$ weakly in $\ell^\infty(S)$ as $s \to \mathcal{F}$ in $S$.

Then $f$ can be extended to a continuous function on $\Sigma$.

Proof We shall use $f^-$ and $f_-$ defined in §3. Our task is to prove that $f^- = f_-$ everywhere on $\Sigma$.

As $s \to \mathcal{F}$ in $S$, $f(\bullet \cdot s)$ tends to $f$ weakly in $\ell^\infty(S)$, hence, in particular, $f(s's) \to f(s')$ pointwise (i.e. for each $s' \in S$). But this implies that $f^-(s's) \to f(s')$ and $f_-(s's) \to f(s')$ when $s \in \Sigma$ tends to $\mathcal{F}$ in $\Sigma$. Indeed, fix $s'$ and $\varepsilon > 0$. Choosing $U$ small enough in an open basis of $\mathcal{F}$ one has $f(s') - \varepsilon \leq f(s's) \leq f(s') + \varepsilon$ for $s \in U \cap S$, so the upper semicontinuous function $g$, defined as the constant $f(s') + \varepsilon$ in the open $s'U$ and as $+\infty$ elsewhere, majorizes $f$ on $S$, hence majorizes $f^-$ everywhere, making $f^-(s's) \leq f(s') + \varepsilon$ for $s \in U$. In a similar manner one bounds $f_-$ below using a lower semicontinuous function equal to $f(t) - \varepsilon$ in $s'U$ and to $-\infty$ elsewhere.

Consider means $M$ on $\ell^\infty(S)$ (i.e. positive linear functionals mapping the constant 1 to 1) such that for restrictions of functions continuous on $\Sigma$, $M$ evaluates the function at some fixed $r \in \Sigma$. They are characterized by

$$g^-(r) \geq M(g) \geq g_-(r) \quad \text{for } g \in \ell^\infty(S).$$

(1)
(A particular case is evaluation functional at some \( r \in S \).)

We have \( f(\bullet \cdot s) \to f \) weakly in \( \ell^\infty(S) \) as \( s \to F \) in \( S \), therefore

\[
M(f(\bullet \cdot s)) \to M(f) \quad \text{as } s \to F \in S
\]  

(2)

Plug in (1) \( g = f(\bullet \cdot s) \). Note that then \( g^- = f^-(\bullet \cdot s) \), \( g_+ = f_-(\bullet \cdot s) \) (this follows from \( \Sigma \) being an open part of a topological group where shifts are, of course, homeomorphisms). One gets

\[
f^-(rs) \geq M(f(\bullet \cdot s)) \geq f_-(rs),
\]

(3)

and (2) and (3) imply:

\[
\liminf_{s\to F \in S} f^-(rs) \geq M(f) \geq \limsup_{s\to F \in S} f_-(rs).
\]

(4)

Now, for any \( f^-(r) \geq a \geq f_-(r) \), using Hahn-Banach one gets a mean as above so that \( M(f) = a \). (Just extend to a positive linear functional the linear functional on the space generated by \( f \) and the restrictions of functions bounded and continuous on \( \Sigma \), which evaluates the latter at \( r \) and gives value \( a \) for \( f \).) Thus (4) reads:

\[
\liminf_{s\to F \in S} f^-(rs) \geq f^-(r), \quad f_-(r) \geq \limsup_{s\to F \in S} f_-(rs).
\]

By the semicontinuity of \( f^- \) and \( f_- \) one can replace here \( s \to F \) in \( S \) by \( s \to F \) in \( \Sigma \). Indeed, if \( \lambda < f^-(r) \) we know that the set \( \{ s \mid f^-(rs) \geq \lambda \} \) is closed and contains \( S \cap U \) for some open \( U \) in \( F \), therefore contains \( U \). Similarly for \( f_- \). Also, since \( f^- \) is upper semicontinuous,

\[
f^-(r) \geq \limsup_{s\to F \in \Sigma} f^-(rs),
\]

making \( f^-(rs) \to f^-(r) \) and similarly \( f_-(rs) \to f_-(r) \) as \( s \to F \) in \( \Sigma \).

Since, by what we found, for \( r = s' \in S \) these limits are \( f(s') \), one concludes that \( f^- \) and \( f_- \) coincide with \( f \) on \( S \).

Now, \( f^- - f_- \) is upper semicontinuous and its value at \( s' \)'s tends to 0 as \( s \to F \) in \( \Sigma \) for each \( s' \in S \). Hence for any \( \varepsilon > 0 \), the open set \( \{ r \in \Sigma \mid f^-(r) - f_-(r) < \varepsilon \} \) is dense in \( \Sigma \). Thus by Baire category the set \( \{ r \in \Sigma \mid f^-(r) = f_-(r) \} \) is dense \( G_\delta \). With \( f^-(rs) \to f^-(r) \) and \( f_-(rs) \to f_-(r) \) as \( s \to F \) in \( \Sigma \), this makes \( f^- = f_- \) everywhere, which concludes the proof of the lemma and the theorem.

QED

References

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