Kaluza-Klein gravitons are negative energy dust in brane cosmology

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We discuss the effect of Kaluza-Klein (KK) modes of bulk metric perturbations on the second Randall-Sundrum (RS II) type brane cosmology, taking the possible backreaction in the bulk and on the brane into account. KK gravitons may be produced via quantum fluctuations during a de Sitter (dS) inflating phase of our brane universe. In an effective 4-dimensional theory in which one integrates out the extra-dimensional dependence in the action, KK gravitons are equivalent to massive gravitons on the brane with masses $m > 3H/2$, where $H$ represents the expansion rate of a dS brane. Thus production of even a tiny amount of KK gravitons may eventually have a significant impact on the late-time brane cosmology. As a first step to quantify the effect of KK gravitons on the brane, we calculate the effective energy density and pressure for a single KK mode. Surprisingly, we find that a KK mode behaves as cosmic dust with a negative energy density on the brane. We note that the bulk energy density of a KK mode is positive definite and there occurs no singular phenomenon in the bulk.

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I. INTRODUCTION

The idea that our Universe might be a brane embedded in a higher dimensional bulk spacetime has attracted tremendous attention in the last few years. A particularly attractive framework, especially from a gravitational point of view, is the so-called second Randall-Sundrum (RS II) scenario, where our brane-universe is embedded in an Anti-de Sitter (AdS) five-dimensional bulk spacetime [1]. If the brane is endowed with a (positive) tension, tuned with respect to the (negative) bulk cosmological constant $\Lambda_5 \equiv -6/\ell^2$, then, as shown by Randall and Sundrum, the geometry on the brane is Minkowski and the gravity felt on the brane is similar to standard 4D gravity on scales $r \gg \ell$ [2, 3].

A considerable amount of work has also been devoted to the cosmological extension of the RSII model, in order to describe the cosmological behavior of a brane in this framework (see [4, 5, 6, 7] for reviews). A significant result in this perspective has been the realization that the Friedmann equations must be modified when the cosmological energy density becomes of the order of the brane tension. Another difference with the standard Friedmann equation is the presence of an additional term, usually called dark radiation, which represents the influence of the bulk geometry on the brane cosmological evolution. The dark radiation term can be related, from the bulk point of view, to a five-dimensional gravitational mass.

All the fundamental results just discussed have been obtained by assuming from the start an exact cosmological symmetry, by which we mean that the bulk spacetime is supposed to be foliated by homogeneous and isotropic three-surfaces. At all instants in its history the brane is supposed to coincide with one of these symmetric three-surfaces.

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However, things are more complicated if one considers a more realistic framework where perturbative deviations from homogeneity and isotropy are allowed. Such perturbations must be taken into account, for example, if one wishes to confront the predictions of brane cosmology with the high-precision measurements of the CMB anisotropies.

Even if one is interested only in homogeneous and isotropic cosmology, the existence of perturbations can affect the homogeneous evolution. An interesting example is the question of dark radiation. The presence of matter fluctuations on the brane generates perturbations of the spacetime metric in the bulk. This process may be regarded as the emission of bulk gravitons from the brane. These bulk gravitons will on average contribute to the gravitational mass in the bulk, and hence modify the evolution of the dark radiation on the brane. That is, it no longer behaves like a free, conserved radiation as in the strictly symmetric case. Detailed analytical and numerical calculations of the effect of the bulk graviton generation were performed for a radiation-dominated brane universe \([8, 9, 10, 11, 12, 13, 14]\).

In the present work, we consider a similar, but slightly different problem. We study the impact of the bulk metric perturbations which are generated in the bulk (or present from the beginning) on the homogeneous evolution of the brane. The bulk metric perturbations are naturally produced via quantum fluctuations during brane inflation \([15]\), and we will concentrate in this work mainly on a de Sitter (dS) brane. From the 4-dimensional point of view, the bulk metric perturbations can be decomposed into a massless zero mode and an infinite number of Kaluza-Klein (KK) modes with effective mass \(m > 3H/2\), where \(H\) is the expansion rate of the dS brane. The time evolution of the zero mode is similar to the standard four dimensional perturbations although its amplitude, as determined by the vacuum quantum fluctuations, depends on the energy scale of the dS brane expansion. In contrast, the squared-amplitude of KK modes on the brane decays as \(a^{-3}\) and thus becomes rapidly negligible during brane inflation. However, after brane inflation the background energy density in a radiation-dominated Friedmann-Lemaître-Robertson-Walker (FLRW) era decays as \(a^{-4}\), hence the massive modes of gravitons may affect the late-time cosmological evolution of the brane. Here it should be noted that the concept of KK modes, which assumes the separation of variables with respect to the fifth dimensional coordinate, is only approximately defined in general. It is an important but longstanding problem to quantify the effect of these approximately defined KK modes on the brane evolution.

In this paper, in order to discuss the cosmological impact of KK gravitons on the brane cosmology, we derive the effective stress-energy of a KK mode on a separable (e.g., a dS brane) background, taking possible backreaction in the bulk and on the brane into account. Then we extrapolate our result to a FLRW cosmological background on which a KK mode can be approximately defined. We show that a sufficiently massive KK mode, which may constitute a non-negligible, if not dominant, fraction of the contribution of all the KK mode, when they are summed up, behaves as cosmic dust, which is consistent with the linear perturbations, but the effective energy density is negative.

This paper is organized as follows. In Section II, we discuss the case of a massless, minimally coupled scalar field because the situation is similar to the tensor case but simpler. We find that a massive KK mode behaves like dust with negative energy density. In Section III, we turn to the main topic of this paper, namely, the backreaction of the KK gravitons on the cosmology of the brane. We find again that a KK graviton mode behaves as negative energy dust. In Section IV, we consider our results from the bulk point of view, and discuss its impact on the cosmological evolution. In Section V, we summarize our results. Some useful formulae are given in two appendices. In Appendix A, the components of the bulk curvature tensor up to second order in the metric perturbations are given. In Appendix B, the computational rules for averaging tensor components that are quadratic in the metric perturbation are given.

II. THE CASE OF A BULK SCALAR FIELD

Before tackling the main subject of this paper, that is the backreaction of KK gravitons, it is instructive to discuss the case of a massless, minimally coupled scalar field, because the behavior of its perturbations is quite similar to the KK gravitons but it is much simpler to analyze \([16]\).

We thus consider a homogeneous scalar field and assume its amplitude \(\phi\) to be small so that its effect can be treated perturbatively: in particular, the backreaction of the scalar field on the metric will be of order \(O(\phi^2)\).

Eventually, we would like to discuss the backreaction for a general cosmological background. However, for the general case, it turns out that the field equation (either for the scalar field or, later, for the gravitons) is not separable and the notion of a KK mode cannot be well defined. The separability property is satisfied only for two limiting cases. One is the case of a de Sitter brane. In this case, the brane is exponentially expanding with a constant Hubble rate \(H\) and one finds a mass gap \(\Delta m = 3H/2\) between the zero mode and KK modes. Thus the continuum of KK modes starts above the mass \(3H/2\). The other case is a low energy cosmological brane, in which case the dependence on the extra dimension can be approximated by the profile obtained for a static brane, i.e. the RS brane.
A. Einstein scalar theory in the bulk

We start from the five-dimensional action which consists of the Einstein-Hilbert term, a cosmological constant $\Lambda_5$ and a bulk scalar field, complemented by the four-dimensional action for the brane:

$$S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \left( \mathcal{R} - 2\Lambda_5 \right) + \int d^5x \sqrt{-g} \left( -\frac{1}{2} g^{ab} \partial_\alpha \phi \partial_\beta \phi - V(\phi) \right) + \int dt x \sqrt{-q} \left( -\sigma + \mathcal{L}_m \right), \quad (2.1)$$

where $q$ is the determinant of the induced metric on the brane, which we denote by $q_{\alpha\beta}$, and $\mathcal{L}_m$ is the Lagrangian density of the matter confined on the brane. The Latin indices $\{a, b, \cdots\}$ and the Greek indices $\{\alpha, \beta, \cdots\}$ are used for tensors defined in the bulk and on the brane, respectively. We will assume that the brane tension on the brane is tuned to its RS value so that $\kappa_5^2 \sigma^2 = -6\Lambda_5$. We also take a constant bulk potential

$$V(\phi) = V_0 > 0, \quad (2.2)$$

so that the scalar field is effectively massless.

We consider backgrounds given by a fixed value of the scalar field which we choose $\phi = 0$. For a non-zero $V_0$, one has a de Sitter brane background, which will be discussed in subsection B below. For $V_0 = 0$, one has a low energy cosmological brane, discussed in subsection C.

The field equation for the bulk scalar field is linear and given by

$$\Box_5 \phi = 0. \quad (2.3)$$

Since we consider a background configuration with $\phi = 0$, the solution of the above equation can be seen as a perturbation. This perturbation will induce a bulk energy-momentum tensor, of order $O(\phi^2)$, which embodies the backreaction of the scalar field on the metric. This is the effect we wish to calculate explicitly.

The variation of the action (2.1) yields the five-dimensional Einstein equations

$$(5) G_{ab} + \Lambda_5 g_{ab} = -\kappa_5^2 V_0 g_{ab} + \kappa_5^2 T_{ab} + (-\sigma q_{ab} + \tau_{ab}) \delta(y - y_0) \quad (2.4)$$

where we have implicitly assumed a coordinate system in which the brane stays at a fixed location $y = y_0$ and where

$$\tau_{\alpha\beta} = \frac{2}{\sqrt{-q}} \frac{\delta}{\delta q^{\alpha\beta}} \left( \sqrt{-q} \mathcal{L}_m \right) \quad (2.5)$$

represents the energy-momentum tensor of matter confined on the brane. The stress energy tensor of the bulk scalar field, not including the constant potential $V_0$, is given by

$$T_{ab} = \phi_{,a} \phi_{,b} - \frac{1}{2} g_{ab} q^{cd} \phi_{,c} \phi_{,d}. \quad (2.6)$$

It is useful to consider the projection of the gravitational equations on the brane$[17]$. Taking into account the bulk energy-momentum tensor, one finds

$$(4) G^\alpha_{\beta} = -\frac{1}{2} \kappa_5^2 V_0 \delta^\alpha_{\beta} + \frac{1}{6} \kappa_5^2 \sigma T^\alpha_{\beta} + \kappa_5^2 T^{(b)\alpha}_{\beta} - E^\alpha_{\beta}, \quad (2.7)$$

where

$$T^{(b)\alpha}_{\beta} = \frac{2}{3} \left[ \phi_{,\alpha} \phi_{,\beta} + \delta_{,\beta} \left( \frac{3}{8} \phi_{,y}^2 - \frac{5}{8} q^{\rho\sigma} \phi_{,\rho} \phi_{,\sigma} \right) \right], \quad (2.8)$$

and $E_{\alpha\beta}$ is the projection on the brane of the bulk Weyl tensor and is traceless by construction. If, in addition, one assumes the brane geometry to be homogeneous and isotropic then the components of $E_{\alpha\beta}$ (in an appropriate coordinate system) reduce to $E^i_t$ and

$$E^i_j = -\frac{1}{3} \delta^i_j E^t_t. \quad (2.9)$$

By using the four-dimensional Bianchi identities, and assuming that the brane matter content is conserved, one is able to express the component $E^i_t$ in terms of the values on the brane of the bulk scalar field and its derivatives$[10]$:

$$E^i_t = \frac{\kappa_5^2}{a^4} \int_{t_0}^t dt' a^4 \left( \partial_i T^{(b)\alpha}_{\alpha} t + \frac{3}{a} \partial T^{(b)\alpha}_{\alpha} t - \frac{a}{a} T^{(b)\alpha}_{\alpha} t \right), \quad (2.10)$$
B. KK mode on a de Sitter brane

First, we consider the case of a de Sitter brane. The bulk metric around a de Sitter brane can be expressed as

\[ ds^2 = dy^2 + b^2(y) \gamma_{\mu\nu} dx^\mu dx^\nu, \]

where the warp factor \( b(y) \) is given by

\[ b(y) = H \ell \sinh(y/\ell), \]

and \( \gamma_{\mu\nu} \) is the 4-dimensional de Sitter metric, which may be expressed by using a flat slicing for simplicity:

\[ \gamma_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j, \]

\[ a(t) = e^{Ht}, \quad H^2 = \frac{1}{6} \kappa_5^2 V_0. \]

The brane is located at \( y = y_0 \) such that \( b(y_0) = 1 \), that is,

\[ \sinh(y_0/\ell) = \frac{1}{H\ell}. \]

In this geometry, the equation of motion for the scalar field is

\[ \frac{1}{b^4} \partial_y \left( b^4 \partial_y \phi \right) - \frac{1}{b^2} \left( \ddot{\phi} + 3H \dot{\phi} - \frac{1}{a^2} \Delta \phi \right) = 0. \]

This equation is separable and one can solve it by looking for a solution of the form \( \phi = f(y) \varphi(t, x^i) \), with

\[ \frac{1}{b^2} \partial_y \left( b^4 \partial_y f \right) + m^2 f = 0, \]

\[ \ddot{\varphi} + 3H \dot{\varphi} - \frac{1}{a^2} \Delta \varphi + m^2 \varphi = 0. \]

The separation constant \( m^2 \) corresponds to the square of the KK mass, as measured by an observer on the brane.

Since there is no coupling between the brane and the bulk scalar field, the boundary condition for the scalar field at the brane location is simply \( \partial_y \phi = 0 \), and therefore \( \partial_y f = 0 \). The equation along the \( y \)-direction implies that the mass spectrum is characterized by a mass gap \( 3H/2 \) \(^1\text{[13]}\). The corresponding eigenfunctions \( f \) can be written in terms of the associated Legendre functions.

Let us now focus on a single KK mode, which is spatially homogeneous and sufficiently massive: \( m \gg H \). One finds from (2.15) that

\[ \varphi(t) = \frac{1}{a^{3/2}} \cos(mt). \]

If we take a time average over a time scale much longer than the period of oscillation \( m^{-1} \), we can ignore the oscillatory behavior and use

\[ \langle \sin^2(mt) \rangle = \langle \cos^2(mt) \rangle = \frac{1}{2}, \quad \text{etc.} \]

From Eq. (2.18), we thus find

\[ T^{(b)}_{1t} = -\frac{5}{8} |f_m|^2 m^2 \frac{1}{a^3}, \]

\[ T^{(b)}_{ij} = \frac{5}{24} |f_m|^2 m^2 \frac{1}{a^3} \delta_{ij}, \]

where \( f_m \) is the value of \( f(y) \) on the brane for the eigenvalue \( m^2 \). From Eqs. (2.11) and (2.16), and from the fact that \( \partial_y^2 \phi = -m^2 \phi \) on the brane, we can evaluate \( E_{\mu\nu} \) as

\[ -E^t_t = \frac{5}{8} \kappa_5^2 |f_m|^2 m^2 \frac{1}{a^3}, \]

\[ -E^i_j = -\frac{5}{24} \kappa_5^2 |f_m|^2 m^2 \frac{1}{a^3} \delta^i_j, \]

(2.19)
where we have neglected the terms that depend on the initial data, which behave as \( a^{-4} \) and thus become negligible at late times.

The above results show that the Weyl term \( E_{\mu\nu} \) contributes negatively to the effective energy density and pressure on the brane for a massive mode. Moreover, if one computes the total effective contribution of the bulk, i.e., the sum of \( T^{(b)}_{\alpha\beta} \) and of the Weyl term \( E_{\alpha\beta} \), one finds for the effective energy density and pressure on the brane

\[
\kappa_4^2 \rho_{(\text{eff})} = - \left( \kappa_5^2 T^{(b)}_{tt} - E_{tt} \right) = - \frac{1}{2} \kappa_5^2 |f_m|^2 m^2 \frac{1}{a^2},
\]

\[
\kappa_4^2 p_{(\text{eff})} = \frac{1}{3} \left( \kappa_5^2 T^{(b)}_{ii} - E_{ii} \right) = 0.
\]  

(2.20)

This represents the backreaction effects of the bulk scalar field, which are of order \( \mathcal{O}(\phi^2) \). Whereas the effective pressure due to the KK mode vanishes, because the bulk component and the Weyl component exactly cancel each other, the effective energy, remarkably, is negative.

### C. KK mode for a low energy cosmological brane

We now calculate the effective energy density and pressure of a KK mode for a low energy cosmological brane. The bulk geometry around a brane, with a flat FLRW geometry and located at \( y = 0 \) is given by the metric

\[
ds^2 = -N^2(t, y)dt^2 + Q^2(t, y) a^2(t) dx^2 + dy^2,
\]

(2.21)

where

\[Q(t, y) = \cosh(y/\ell) - \eta \sinh(|y|/\ell)\]

\[N(t, y) = \cosh(y/\ell) - \left( \eta + \frac{\dot{\eta}}{H} \right) \sinh(|y|/\ell),\]

(2.22)

with

\[\eta = \sqrt{H^2 \ell^2 + 1}.
\]

(2.23)

We have assumed that there is no dark radiation, i.e., that the bulk geometry is strictly AdS and not Schwarzschild-AdS. In general, this metric is non-separable. However, in the low energy limit characterized by \( H \ell \ll 1 \) and \( \dot{H} \ell^2 \ll 1 \), we have \( \eta \simeq 1 \) and \( \dot{\eta}/H \ll 1 \) so that the metric can be approximated by

\[
ds^2 = dy^2 + e^{-2|y|/\ell} (-dt^2 + a^2(t) dx^2),
\]

(2.24)

which is now separable. If one considers the evolution of a massless, minimally coupled scalar field in the above background metric, one finds that the field equation is separable and thus admits a solution of the form \( \phi(t, y) = f(y) \varphi(t) \) with

\[\partial_y^2 f - \frac{4}{\ell} \partial_y f + m^2 e^{2y/\ell} f = 0,
\]

\[\ddot{\varphi} + 3H \dot{\varphi} + m^2 \varphi = 0,
\]

(2.25)

where the function \( f(y) \) is assumed to be \( Z_2 \)-symmetric.

The solution for \( f(y) \) with the appropriate Neumann boundary condition on the brane, \( f'(0) = 0 \) is given in terms of the Hankel functions. There is a zero mode corresponding to \( m = 0 \) as well as a continuum of KK modes with \( m > 0 \). For a massive KK mode \( m \gg H \), the four-dimensional part evolves as

\[\varphi = \frac{1}{a^{3/2}} \cos(mt),
\]

(2.26)

Similarly to the de Sitter brane case, one can compute the projection of the bulk energy-momentum tensor on the brane and one finds for its components:

\[T^{(b)}_{tt} = - \frac{1}{4a^3(t)} |f_m|^2 m^2 \sin^2(mt) = - \frac{1}{8a^3(t)} |f_m|^2 m^2,
\]

\[T^{(b)}_{ii} = \frac{5}{4a^3(t)} |f_m|^2 m^2 \sin^2(mt) = \frac{5}{8a^3(t)} |f_m|^2 m^2.
\]

(2.27)
This gives
\[ \kappa_5^{-2} E^t_t = \frac{1}{a^4} \int_0^t dt' a^4 \left( \partial_t T^{(b)t} + 3 \frac{\dot{a}}{a} T^{(b)t} - \frac{\dot{a}}{a} T^{(b)i} \right) = - \frac{5}{8 a^3(t)} |f_m|^2 m^2 \left( 1 - \frac{a(t_0)}{a(t)} \right), \] (2.28)
and \( E^t_t = -E^i_i \). Thus we obtain
\[ T^{(b)t} - \kappa_5^{-2} E^t_t = \frac{1}{2a^3(t)} |f_m|^2 m^2, \]
\[ T^{(b)i} - \kappa_5^{-2} E^i_i = 0, \] (2.29)
at late times. Therefore, the effective energy density and pressure for a KK mode becomes
\[ \kappa_4^2 \rho_{\text{eff}} = - \left( \kappa_5^2 T^{(b)t} - E^t_t \right) = - \frac{\kappa_5^2}{2a^3(t)} |f_m|^2 m^2, \]
\[ \kappa_4^2 p_{\text{eff}} = \frac{1}{3} \left( \kappa_5^2 T^{(b)i} - E^i_i \right) = 0. \] (2.30)
This means that, also for a low energy cosmological brane, a massive KK mode behaves as cosmic dust with negative energy density.

The analyses given above imply that the result is independent of the existence of a mass gap and the essential factor is the background expansion of thebrane. A KK mode can be approximately defined only for a cosmological brane which slightly deviates from the dS geometry and for a low energy brane, thus we expect that our result can be applied at least for these cases. However, for intermediate energy scales a KK mode is not well-defined in general and it is not clear how our result might be applied.

Finally, we note that the bulk energy density of a KK mode on the brane remains positive as
\[ \kappa_5^2 \rho_{\text{bulk}} := -\kappa_5^2 T^t_t = \frac{1}{4} \kappa_5^2 |f_m|^2 m^2 \frac{1}{a^3} > 0, \] (2.31)
for both de Sitter and low energy branes (with the understanding that the time average over scales greater than \( m^{-1} \) is taken). It shows that there is no singular effect in the bulk in contrast to the peculiar behavior on the brane.

### III. EFFECTIVE THEORY IN THE BULK AND ON THE BRANE INCLUDING THE GRAVITATIONAL BACKREACTION

After having studied the backreaction of the KK modes of a bulk scalar field, we now turn to the main subject of this paper, which is to study the backreaction of the gravitational perturbations of the metric itself on the cosmology of the brane. In this section, we adopt a more general perspective by considering a \((d-1)\)-brane embedded in a \((d+1)\)-dimensional bulk spacetime, although we remain primarily interested by the case \( d = 4 \). This allows us to investigate the dependence on the number of dimensions of various quantities introduced in this section.

#### A. Effective theory in the bulk

We now consider only pure gravity in the bulk. The action of the system is given by
\[ S[g] = \frac{1}{2 \kappa_{d+1}^2} \int d^{d+1}x \sqrt{-g} \left( R^{(d+1)} - 2 \Lambda_{d+1} \right) - \int d^d x \sqrt{-\sigma} \sigma, \] (3.1)
where \( \Lambda_{d+1} \) is the bulk cosmological constant and \( \sigma \) is the brane tension. We mainly consider a dS brane background in this section and assume that its tension is larger than that of the corresponding RS value \( 2(d-1)/(\kappa_{d+1}^2 \ell) \), where \( \ell = (-d(d-1)/(2 \Lambda_{d+1}))^{1/2} \) is the bulk AdS curvature radius.

We start from an unperturbed metric \( g^{(0)} \), which is a solution of Einstein’s equations and thus satisfies
\[ \frac{\delta S}{\delta g} \bigg|_{g^{(0)}} = 0, \] (3.2)
where and in what follows the notation, $Q[a + g]_{f}$, means that a functional $Q[a + g]$ of $g$ is evaluated for a function $f$, i.e.,

$$Q[a + g]_{f} = Q[a + f].$$  \hfill (3.3)

We then consider (small) linear perturbations of this metric, which we write $\epsilon \, g^{(1)}$ and such that its average vanishes i.e.,

$$\langle \epsilon \, g^{(1)} \rangle = 0.$$  \hfill (3.4)

Here we should specify our definition of averaging. We assume that the perturbation $g^{(1)}$ has a typical wavelength $\lambda$ which is much smaller than the characteristic curvature radius $L$ of the background $g^{(0)}$, $\lambda \ll L$. Then we take the average over a length scale much larger than $\lambda$ but much smaller than $L$. In our case, we can take this average in the spacetime dimensions parallel to the brane. However, the situation is dramatically different in the direction of the extra spatial dimension because the brane is infinitesimally thin, which implies that the curvature radius along the extra dimension is infinitely small. Therefore one cannot take an average in that direction at or around the brane. Thus our averaging will include only the average over the $1 + (d - 1)$ spacetime dimensions. (For spatially homogeneous perturbations, we take only the time average.)

What we are interested in is the correction to the original metric due to the backreaction of the metric perturbations. The total metric we consider can thus be written as

$$g_{\text{tot}} = g^{(0)} + \epsilon \, g^{(1)} + \epsilon^{2} \, g^{(2)},$$  \hfill (3.5)

where the quantity $g^{(2)}$ represents the backreaction due to the metric perturbations, so that the effective background (homogeneous) metric, after averaging, is given by

$$\bar{g} = g^{(0)} + \epsilon^{2} \, g^{(2)}.$$  \hfill (3.6)

For convenience, the parameter $\epsilon$ is introduced as an expansion parameter, which is to be set to unity at the end of the calculation.

If we expand the action with respect to $g^{(1)}$, we have

$$S[\bar{g} + \epsilon \, g^{(1)}] = S[\bar{g}] + \frac{\delta S}{\delta g}[\bar{g}] \big|_{\bar{g}} \left( \epsilon \, g^{(1)} \right) + \frac{1}{2} \frac{\delta^{2} S}{\delta g^{2}}[\bar{g}] \big|_{\bar{g}} \left( \epsilon \, g^{(1)} \right)^{2} + O(\epsilon^{3}).$$  \hfill (3.7)

Hence the variation of the above expression with respect to $g^{(1)}$ yields

$$\epsilon \frac{\delta S}{\delta g}[\bar{g} + \epsilon \, g^{(1)}] \big|_{\epsilon \, g^{(1)}} = \epsilon \frac{\delta S}{\delta g}[\bar{g}] \big|_{\bar{g}} + O(\epsilon^{2}) = \epsilon \frac{\delta S}{\delta g}[\bar{g}] \big|_{\bar{g}} + O(\epsilon^{2}) = O(\epsilon^{2}),$$  \hfill (3.8)

where we have used Eq. (3.8) in the final equality. This implies that, up to $O(\epsilon)$, the equation of motion for the perturbation $g^{(1)}$ is given by

$$\frac{\delta S}{\delta g}[\bar{g} + g^{(1)}] \big|_{\epsilon \, g^{(1)}} = 0.$$  \hfill (3.9)

On the other hand, the variation of the action with respect to $g_{\text{tot}}$ gives

$$0 = \frac{\delta S}{\delta g}[\bar{g}] \big|_{g_{\text{tot}}} = \frac{\delta S}{\delta g}[\bar{g}] \big|_{\bar{g} + \epsilon \, g^{(1)}} = \frac{\delta S}{\delta g}[\bar{g}] \big|_{\bar{g}} + \frac{\delta^{2} S}{\delta g^{2}}[\bar{g}] \big|_{\bar{g}} \left( \epsilon \, g^{(1)} \right) + \frac{1}{2} \frac{\delta^{3} S}{\delta g^{3}}[\bar{g}] \big|_{\bar{g}} \left( \epsilon \, g^{(1)} \right)^{2} + O(\epsilon^{3})$$  \hfill (3.10)
where, to get the last expression, the argument of the coefficient of the third term, \( g \), has been replaced by \( g^{(0)} \), which is justified within the accuracy of \( O(\epsilon^2) \). If one averages the above expression, the second term on the right-hand side vanishes and we obtain the equation that determines the backreaction-corrected background metric \( g \), in the form

\[
\frac{\delta S}{\delta g} \bigg|_g = -\frac{1}{2} \epsilon^2 \left\langle \frac{1}{g} \frac{\delta S}{\delta g} \bigg|_g \right\rangle^{(1)}_g \bigg|^{(1)}_g .
\]  \( (3.11) \)

Substituting the explicit form for the braneworld action, we find that Eq. \((3.11)\) yields

\[
(d+1)\tilde{G}^a_{(d+1) b} + \Lambda d+1 \tilde{G}^a_{(d+1) b} = \kappa_{d+1}^2 \mathcal{T}^a_{(d+1) b} \ + \bar{\ell}_{(\text{brane})}^a b + \delta t_{(\text{brane})}^a b ,
\]  \( (3.12) \)

where \((d+1)\tilde{G}\) is the background bulk Einstein tensor including the backreaction effects, i.e., for the metric \( \bar{g} \). And the stress-energy tensor due to the backreaction in the bulk is given by

\[
\kappa_{d+1}^2 \mathcal{T}^a_{(d+1) b} = -\left\langle (d+1)^{(2)} \mathcal{G}^a_{(d+1) b} \right\rangle ,
\]  \( (3.13) \)

where \((d+1)^{(2)} \mathcal{G}^a_{(d+1) b}\) is the bulk Einstein tensor at quadratic order. Here it may be worth noting that averaging is necessary for this effective energy-momentum tensor to be physically meaningful, since there exists no locally covariant gravitational energy-momentum tensor due to the equivalence principle. The tensor \( \bar{\ell}_{(\text{brane})}^a b \) corresponds to the brane energy-momentum tensor in the background configuration defined by the metric \( \bar{g} \) and thus comes from the variation of the brane action in the left-hand side of \((3.11)\). Finally, \( \delta t_{(\text{brane})}^a b \), which comes from the brane-dependent part in the right-hand side of \((3.11)\), denotes the backreaction due to the brane fluctuations and will be discussed in Section III. C. The existence of this term is the most important difference when compared to the case of the scalar field, in which case the backreaction originates purely from the bulk.

Hereafter, we write \((d+1)\mathcal{G}^a_{(d+1) b}\) as \((d+1)\mathcal{G}^a_{d+1} b\) for simplicity. For the moment, we concentrate on the effective theory in the bulk,

\[
(d+1)\mathcal{G}^a_{d+1} b + \Lambda_d \delta^a_{d+1} b = \kappa_{d+1}^2 \mathcal{T}^a_{d+1} b .
\]  \( (3.14) \)

Our first task is to evaluate the effective bulk energy-momentum tensor \( \mathcal{T}^a_{d+1} b \), which is quadratic in the metric perturbations. Then we will take the limit to the brane.

We now identify the background metric \( g^{(0)} \) with the separable metric of AdS\(_{d+1}\) bulk-dS brane spacetime and \( g^{(1)} \) as the linear perturbation of this system. Namely,

\[
ds^2 = dy^2 + b^2(y)(\gamma_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu , \quad h^a_{\alpha \beta} = h^{-1}_{\alpha \beta} = 0 ,
\]  \( (3.15) \)

where \( b(y) \) is the warp factor defined in Eq. \((2.12)\) and \( \gamma_{\mu\nu} \) is the metric of a \( d \)-dimensional dS spacetime which is an extension of Eq. \((2.11)\). Note that we have adopted the so-called RS gauge for the perturbations \( 1 \ 20 \). The equation of motion for the perturbations in the bulk reads

\[
\left[ \frac{1}{b^2} \partial_y \left( b^2 \partial_y \right) + \frac{1}{b^2} \left( \Box_d - 2H^2 \right) \right] h_{\alpha \beta} = 0 .
\]  \( (3.16) \)

This equation is separable and one considers solutions of the form \( h_{\alpha \beta} = f(y) \varphi_{\alpha \beta}(x^\mu) \), where \( f(y) \) is the generalization of the solution of Eq. \((2.15)\) to the case of a \( d \)-dimensional brane with boundary condition \( \partial_y f(y) = 0 \) at \( y = y_0 \) because \( \partial_y h_{\alpha \beta} = 0 \) on the brane. Similarly to the scalar case, the separation constant \( m \) represents the effective mass of a KK graviton mode and satisfies \( m > (d-1)H/2 \). The \( d \)-dimensional part \( \varphi_{\alpha \beta} \) satisfies

\[
\left[ \Box_d - 2H^2 \right] \varphi_{\alpha \beta} = m^2 \varphi_{\alpha \beta} .
\]  \( (3.17) \)

We focus on a KK mode with \( m^2 \gg H^2 \). Furthermore, we focus on perturbations of the tensor-type with respect to the spatial \((d-1)\)-geometry, namely on those with \( h^i_{\ i} = h^{i}_{\ i} = 0 \). Taking the slicing of the de Sitter space with the flat spatial \((d-1)\)-geometry, they will have the form,

\[
h^i_{\ j} = \frac{f_m}{\alpha^{(d-1)/2}} \cos(mt) Q^i_{\ j} ,
\]  \( (3.18) \)

where \( f_m \) is the amplitude of the KK mode and \( Q^i_{\ j} \) is the polarization tensor on the flat \((d-1)\)-space. The amplitude \( f_m \) can be determined, for instance, by the normalization condition if one considers a quantized perturbation theory.
As mentioned earlier, in order to obtain the stress-energy tensor that embodies the backreaction due to the metric perturbations, one needs to “average” the Einstein tensor at quadratic order, according to Eq. (3.14). The components of the bulk curvature tensors, up to quadratic order in the perturbations are listed in Appendix A. As explained after Eq. (3.14), we take the spacetime average in the 1 + (d − 1) dimensions parallel to the brane, but not along the extra dimension. In particular, because of the cosmological symmetry, we can take the average in the (d − 1) dimensions over the complete space. The derivatives along the extra dimension are replaced by using the field equation (3.16) and the boundary conditions on the brane. Our procedure is detailed in Appendix B.

Using Eq. (3.17) of Appendix A and the computational rules detailed in Appendix B, we obtain in the limit y → +0 the expressions

\[
\left\langle (d+1)^{(2)} G_{\alpha\beta} \right\rangle = -\frac{1}{8} \left\langle h^{\rho\sigma} \square_d h_{\rho\sigma} \right\rangle,
\]

\[
\left\langle (d+1)^{(2)} G_{y\nu} \right\rangle = -\frac{1}{2} \left\langle h^{\rho\sigma} \square_d h_{y\rho\sigma} \right\rangle - \frac{d - 3}{8d} \delta^{\alpha}_\beta \left\langle h^{\rho\sigma} \square_d h_{\rho\sigma} \right\rangle - \frac{1}{4} \left\langle h^{\rho\sigma} [h_{\rho\sigma}]_\beta \right\rangle.
\]

(3.19)

A priori, the effective energy-momentum tensor includes an anisotropic stress, to which each mode will contribute with a factor \(O(m^2)\). However, if the perturbations are described by a random field which is statistically homogeneous and isotropic, the average over all modes of the anisotropic part must cancel. What remains is thus to justify the randomness of the perturbations. In this respect, the quantum fluctuations are indeed expected to have this property. Also, \(\left\langle (d+1)^{(2)} G_{y\nu} \right\rangle\) vanishes on the brane by using the boundary conditions \(\partial_y h_{\alpha\beta} = 0\) on it.

### B. Backreaction on the brane

Let us now discuss the effect of the backreaction onto the brane. The projected gravitational equation on the brane reads

\[
^{(d)}G_{\alpha\beta} = -\Lambda_{\text{eff}} \delta_{\alpha\beta} + \kappa_{d+1}^2 \tau_{\alpha\beta} - E_{\alpha\beta};
\]

(3.20)

where

\[
\Lambda_{\text{eff}} = \frac{d - 2}{d} \Lambda_{d+1} + \frac{d - 2}{8(d - 1)} \kappa_{d+1}^2 \sigma^2,
\]

(3.21)

is the effective cosmological constant on the brane, and

\[
\kappa_{d+1}^2 T_{\alpha\beta}^{(b)} = \frac{d - 2}{d - 1} \left\langle T_{\alpha\beta}^{(b)} \right\rangle + \frac{d - 2}{8(d - 1)} \left\langle T_{y\nu}^{(b)} - \frac{1}{d} T_{a}^{(b)} \right\rangle
\]

\[
= \frac{d - 2}{2(d - 1)} \left\langle h^{\rho\sigma} \square_d h_{\rho\sigma} \right\rangle + \frac{(d - 2)(d - 3)}{8d(d - 1)} \delta^{\alpha}_\beta \left\langle h^{\rho\sigma} \square_d h_{\rho\sigma} \right\rangle + \frac{d - 2}{4(d - 1)} \left\langle h^{\rho\sigma} \rho\sigma \right\rangle,
\]

(3.22)

is the projection of the effective energy-momentum tensor of the bulk gravitons. The tensor \(\tau_{\alpha\beta}\), corresponding to \(\delta_{\text{t}(\text{brane})}\) of the previous subsection, describes the brane perturbation induced by the bulk perturbation. We will show in the next subsection that, for our purposes, this term can be neglected. We now concentrate on the effect of the effective energy-momentum of the bulk gravitons projected on the brane, i.e., the terms \(T_{\alpha\beta}^{(b)}\) and \(E_{\alpha\beta}\).

Let us first consider \(T_{\alpha\beta}^{(b)}\). Because of the assumed symmetries, i.e., the spatial homogeneity and isotropy, this gives in the brane an effective perfect fluid with some energy density and pressure. Decomposing the metric perturbations into KK modes, one finds that the contribution of a sufficiently massive mode to the energy density and pressure is given by

\[
\kappa_{d+1}^2 T_{\alpha\beta}^{(b)t} = \frac{(d + 3)(d - 2)}{16d(d - 1)} \frac{1}{a^{d-1}} m^2 |f_m|^2 \left\langle Q^{k\ell} Q_{k\ell}^* \right\rangle,
\]

\[
\kappa_{d+1}^2 T_{\alpha\beta}^{(b)i} = \frac{(d^2 + 3)(d - 2)}{16d(d - 1)^2} \frac{1}{a^{d-1}} m^2 |f_m|^2 \left\langle Q^{k\ell} Q_{k\ell}^* \right\rangle \delta_i^j.
\]

(3.23)

We must also take into account the projection of the Weyl tensor on the brane, \(E_{\alpha\beta}\). Although this term is not included in the “bulk energy-momentum” tensor because it is a part of the bulk Weyl tensor, it contributes nevertheless...
to the projected gravitational equations as an “energy-momentum” tensor. Although its direct evaluation is rather delicate, this term can be computed by resorting once more to the cosmological symmetry. From Eq. (3.20), the contracted Bianchi identities \( D^\alpha \varepsilon^{(d)}_\alpha = 0 \), together with the conservation of \( \tau_{\alpha \beta} \), give

\[
D^\mu E_{\mu \nu} = \kappa_{d+1}^2 D(t)T^{(b)\mu \nu}.
\]  

(3.24)

Because of the cosmological symmetry, the only non-trivial component of the above equation is the time component, which reads

\[
\partial_t E^t_t + \frac{d}{a} \dot{a} E^t_t = \kappa_{d+1}^2 \left( \partial_t T^{(b)\mu \nu} + (d-1) \frac{\dot{a}}{a} T^{(b)\mu \nu} - \frac{\dot{a}}{a} T^{(b)\mu \nu} \right),
\]

(3.25)

where, on the left-hand side, we have used the property that \( E_{\mu \nu} \) is traceless and thus \( E^{t} = -E^{t} \). The integration then yields

\[
E^t_t = \kappa_{d+1}^2 \int_{t_0}^{t} dt' a^d \left( \partial_t T^{(b)\mu \nu} + (d-1) \frac{\dot{a}}{a} T^{(b)\mu \nu} - \frac{\dot{a}}{a} T^{(b)\mu \nu} \right).
\]

(3.26)

As before, we neglect the contribution from the initial condition, which is valid at late times. Substituting a KK graviton mode given by Eq. (3.18) into the integrand on the right-hand side of Eq. (3.20), and taking the time average, one finds

\[
k_{d+1}^2 \left( \partial_t T^{(b)\mu \nu} + (d-1) \frac{\dot{a}}{a} T^{(b)\mu \nu} - \frac{\dot{a}}{a} T^{(b)\mu \nu} \right) = -\left( \frac{d^2 + 3}{16(d-1)} \right) \frac{H}{a^{d-1}} |f_m|^2 m^2 \left( Q^{it} Q_{it} \right).
\]

(3.27)

This gives, at late times,

\[
E^t_t = -\left( \frac{d^2 + 3}{16(d-1)} \right) \frac{1}{a^{d-1}} m^2 |f_m|^2 \left( Q^{it} Q_{it} \right).
\]

(3.28)

Because of the traceless nature of this tensor, we then obtain \( E^{i}j = -(1/(d-1)) E^{i}j \).

The total contribution of the two tensors is therefore

\[
k_{d+1}^2 T^{(b)\mu \nu} - E^{\mu \nu} = \frac{d-2}{16} \frac{1}{a^{d-1}} m^2 |f_m|^2 \left( Q^{it} Q_{it} \right).
\]

(3.29)

for the temporal part and

\[
k_{d+1}^2 T^{(b)i} - E^{i} = 0
\]

(3.30)

for the spatial part. This means that the contributions of a KK mode to the total effective energy density and pressure are respectively given by

\[
k_{d}^2 \rho_{(\text{eff})} = -\frac{d-2}{16} \frac{1}{a^{d-1}} m^2 |f_m|^2 \left( Q^{it} Q_{it} \right),
\]

\[
k_{d}^2 p_{(\text{eff})} = 0.
\]

(3.31)

For instance, for \( d = 4 \), we obtain

\[
k_{4}^2 \rho_{(\text{eff})} = -\frac{1}{8 a^{4}} m^2 |f_m|^2 \left( Q^{it} Q_{it} \right),
\]

\[
k_{4}^2 p_{(\text{eff})} = 0.
\]

(3.32)

The effective isotropic pressure vanishes and the effective energy density is negative. This is the same as in the case of the scalar field discussed in the previous section.

We note that the bulk energy density of a KK mode on the brane remains positive

\[
k_{d+1}^2 \rho_{(\text{bulk})} := -k_{d+1}^2 T^t_t = \frac{d+3}{16d} \frac{1}{a^{d-1}} m^2 |f_m|^2 \left( Q^{it} Q_{it} \right) > 0,
\]

(3.33)

as in the scalar case, Eq. (2.31). It shows again that there is no singular effect in the bulk. The negativity of the effective energy density on the brane originates from the projected Weyl tensor \( E_{\mu \nu} \).
C. Brane intrinsic contributions

We now consider the brane intrinsic contributions. In order to discuss the gravitational perturbations in the brane world, it is not sufficient to consider the contribution from the bulk. The brane perturbations must be taken into account as well. We take an approach in which we derive the second order boundary action and regard it as the action for an effective matter on the brane.

In this paper, the brane is treated as a thin wall. In the thin wall approximation, the second order action on the boundary has been derived in the Appendix of [20]. When there is no ordinary matter on the brane and thus no brane bending mode, the second order boundary action is given by [20]

$$\delta^2 S = \frac{1}{2\kappa_{d+1}^2} \int_{\partial M} d^d x \sqrt{-q} \left[ -\Delta k^{\rho\sigma} \tilde{h}_{\rho\sigma} + \frac{\sigma \kappa_{d+1}^2}{2(d-1)} \tilde{h}^{\rho\sigma} \tilde{h}_{\rho\sigma} \right],$$  \hspace{1cm} (3.34)

where $\tilde{h}_{\mu\nu} = b^2 h_{\mu\nu}$, $k_{\rho\sigma} = \partial_{\rho} \tilde{h}_{\sigma\rho}$, and $\Delta Q = Q^{(+)} - Q^{(-)}$. For an AdS-bulk configuration and with the assumption of $Z_2$ symmetry about the brane, this reduces to

$$\delta^2 S = \frac{3}{4(d-1)} \sigma \int_{\partial M} d^d x \sqrt{-q} \tilde{h}_{\rho\sigma} h^{\rho\sigma}.$$  \hspace{1cm} (3.35)

The second order action can be regarded as an action for some effective matter induced on the brane

$$\int_{\partial M} d^d x \sqrt{-q} \mathcal{L}_m, \quad \mathcal{L}_m := \frac{3}{2(d-1)} \sigma h_{\rho\sigma} h^{\rho\sigma}.$$  \hspace{1cm} (3.36)

Its variation with respect to the background metric $\tilde{q}_{\mu\nu}$ yields the induced matter energy-momentum tensor on the brane

$$\tau_{\alpha\beta} = \frac{2}{\sqrt{-q}} \frac{\delta}{\delta q^{\alpha\beta}} \left( \sqrt{-q} \mathcal{L}_m \right) = \frac{6}{d-1} \sigma \left( h_{\alpha\rho} h^{\rho\beta} - \frac{4}{3} \tilde{q}_{\alpha\beta} h^{\rho\sigma} h_{\rho\sigma} \right).$$  \hspace{1cm} (3.37)

Note that, strictly speaking, $\tilde{q}_{\mu\nu}$ does not include the backreaction. However, as discussed in Section III. A, for perturbations with small amplitude, the linear perturbation equations are identical to those for the background metric in which the backreaction is taken into account. Thus we can add this term as a part of the (effective) matter contribution in the effective equation on the brane.

We can readily calculate the effective energy density and pressure of this contribution. One finds

$$\kappa_{d-1}^2 \rho_{(\text{brane})} = \frac{3(d-2)}{16(d-1)^2} \kappa_{d+1}^4 \sigma^2 |f_m|^2 \frac{1}{a^{d-1}} \langle Q^{kl} Q_{*kl} \rangle,$$

$$\kappa_{d-1}^2 p_{(\text{brane})} = \frac{3(d-2)(-d+5)}{16(d-1)^3} \kappa_{d+1}^4 \sigma^2 |f_m|^2 \frac{1}{a^{d-1}} \langle Q^{kl} Q_{*kl} \rangle.$$  \hspace{1cm} (3.38)

This gives the equation of state

$$w_{(\text{brane})} = -\frac{d-5}{d-1}.$$  \hspace{1cm} (3.39)

For $d = 4$ the boundary contribution thus behaves as radiation, for $d = 5$ as dust and for $d > 6$ as matter with negative pressure. However, its contribution to $^{(d)}G^{\alpha\beta}$ is of order

$$\kappa_{d+1}^4 \sigma \delta^{\alpha\beta} \sim \frac{1}{\ell^2} \left( 1 + \langle H \ell \rangle^2 \right) h^{\rho\sigma} h_{\rho\sigma},$$  \hspace{1cm} (3.40)

where we use [20]

$$\sigma = \frac{2(d-1)}{\kappa_{d+1}^2 \ell} \left( 1 + \langle H \ell \rangle^2 \right)^{1/2}.$$  \hspace{1cm} (3.41)

For the cases $H \ell \ll 1$ and $H \ell \gg 1$, the right-hand side of Eq. (3.40) is $\mathcal{O}(\ell^{-2} h^2)$ and $\mathcal{O}(H^2 h^2)$, respectively. Thus as long as we consider sufficiently massive KK modes, with $m \gg \max\{\ell^{-1}, H\}$, the brane perturbations can be safely neglected and only the projected bulk contributions are relevant for the effective theory on the brane.
IV. NEGATIVE ENERGY DENSITY FROM THE BULK POINT OF VIEW

Intuitively, the “negative energy density” of a KK mode is rather puzzling. However, we can understand its cause by regarding the KK modes as a part of the dark component, such as the dark radiation. The energy density of the dark component evolves as

$$\rho_{(D)} + 4H\rho_{(D)} = -2(1 + \frac{\rho}{\sigma})T_{ab}u^an^b - 2(H\ell)T_{ab}n^an^b, \quad (4.1)$$

where $u^a$ and $n^a$ are the tangent and normal vectors to the brane, respectively. Since there is no matter on the brane and no brane-bulk energy exchange, the first term on the right-hand side of Eq. (4.1) vanishes and only the second one, related to the pressure transverse to the brane, survives. In terms of the energy conservation law in the bulk, this has the simple interpretation that the work done by the pressure on the brane to move it outward in the direction of the AdS infinity reduces the energy in the bulk. As a result, the dark energy density decreases, since the dark energy density on the brane is proportional to the total mass (energy) in the bulk. For a massless scalar field,

$$T_{ab}n^an^b = \frac{1}{2}\phi^2 > 0. \quad (4.2)$$

Thus, the dark component decays faster than ordinary radiation. For the KK modes, after time averaging, we have

$$T_{ab}n^an^b = \frac{1}{4a^3}|f_m|^2 m^2. \quad (4.3)$$

The formal solution of Eq. (4.1) is

$$\rho_{(D)} = -\frac{2}{a^4}\int_{t_0}^t dt a^4(H\ell)T_{ab}n^an^b + \frac{C}{a^4}, \quad (4.4)$$

where $C$ denotes the initial mass in the bulk. For the KK modes,

$$\int_{t_0}^t dt a^4(H\ell)T_{ab}n^an^b = \frac{\ell}{4}|f_m|^2 m^2 \int_{t_0}^t dt \dot{a} = \frac{\ell}{4}(a(t) - a(t_0))|f_m|^2 m^2. \quad (4.5)$$

Hence

$$\rho_{(D)} = -\frac{\ell}{2a^4}|f_m|^2 m^2 + \frac{C}{a^4}, \quad (4.6)$$

where we have redefined the mass parameter $C$ by absorbing into it the initial data dependent term of the integral.

Anyway, in the case of a dS brane (or a cosmological brane which slightly deviates from the dS geometry), the effective cosmological constant dominates the cosmological evolution and the KK effect does not have a significant impact on the brane. For a low energy brane, especially for a radiation-dominated brane, naively one might worry that this result would imply the appearance of a negative energy density within a finite time. However from Eq. (4.1), the bulk pressure term is proportional to $H$. Hence if $\dot{H} < 0$ at $H = 0$, the energy density will remain positive at the expense of rendering the universe to recollapse.

For simplicity, we consider the case where the cosmological evolution of the brane is determined solely by the dark component. Note that this discussion can be generalized when one considers ordinary dust or radiation in addition to the dark component. The Hubble parameter on the brane is obtained from

$$H^2 = \frac{\kappa^2}{3\ell}\rho_{(D)} = -\frac{\kappa^2}{6}\frac{1}{a^3}|f_m|^2 m^2 + \frac{\kappa^2}{2}\frac{C}{3\ell a^3} = -\frac{2\kappa^2}{3}\frac{C}{3\ell a^3}. \quad (4.7)$$

Taking the time derivative of this equation, we obtain

$$\dot{H} = \frac{\kappa^2}{4}\frac{1}{a^3}|f_m|^2 m^2 - \frac{2\kappa^2}{3\ell a^4} = \kappa^2 T_{ab}n^an^b - \frac{2\kappa^2}{3\ell a^4} C. \quad (4.8)$$

Therefore, at $H = 0$, we have

$$\dot{H} = -\frac{1}{3}T_{ab}n^an^b < 0, \quad (4.9)$$
and the universe begins to collapse. Thus, the backreaction of the KK modes leads to a collapsing universe.

The situation is the same for the case of KK gravitons as long as the brane fluctuations are negligible, because we have

\[
\kappa_5^2 \mathcal{T}_{ab} n^a n^b = - \langle \frac{1}{2} G_{y y} \rangle = \frac{1}{8} m^2 \langle h^{\alpha \sigma} h_{\alpha \sigma} \rangle = \frac{1}{16} f_m^2 m^2 \frac{1}{a^3} \langle Q^{k\ell} Q^{*}_{k\ell} \rangle > 0. \tag{4.10}
\]

Thus, provided that the brane fluctuations can be neglected, the brane universe will start to collapse within a finite time. For more realistic situations in cosmology, our result suggests that for a low energy brane the brane universe will eventually collapse unless the contribution of the true (normal) dust matter is larger than that of KK modes.

V. SUMMARY AND DISCUSSION

We have investigated the effect of a Kaluza-Klein (KK) mode on brane cosmology, focusing on the second Randall-Sundrum (RSII) type model, i.e., for a $Z_2$-symmetric brane embedded in the anti-de Sitter bulk.

The KK gravitons, which are just the metric perturbations in the bulk, are produced during a de Sitter (dS) brane inflation phase via vacuum fluctuations. From the four-dimensional point of view they are effectively equivalent to massive gravitons with masses $m > 3H/2$, where $H$ represents the dS expansion rate of the brane. The theory of linear perturbations reveals that the squared amplitude of a KK mode decays as $a^{-3}$ and its contribution rapidly becomes negligible during the brane inflation. However, after brane inflation in the radiation-dominated Friedmann-Lemaître-Robertson-Walker (FLRW) era the background radiation energy density decays as $a^{-3}$, which implies that the contribution of KK gravitons may have a significant impact on the brane cosmology at late times.

Before discussing the case of the KK gravitons, we have considered the case of a massless, minimally coupled bulk scalar field as an exercise. This is because a massless, minimally coupled scalar field has many properties in common with the gravitational perturbation, but it is much simpler to deal with the former than with the latter. We have considered two limiting cases of the background spacetime in which the equations of motion become separable, namely, the case of a de Sitter brane and the case of a low energy cosmological brane. In both cases, we found that a sufficiently massive KK mode, with mass much greater than the Hubble expansion rate, $m \gg H$, indeed behaves like dust but, rather surprisingly, its energy density is negative.

Then we have turned to the case of the bulk gravitational perturbations. We have first derived the effective energy-momentum tensor of a KK mode in the bulk, and investigated the effect of the KK mode on the brane cosmology by projecting the effective energy-momentum tensor on the brane. We have found exactly the same behavior as that of the scalar, i.e., a massive KK mode behaves as a negative energy density dust.

The negative energy density of a KK mode may sound rather puzzling. But, from the bulk point of view, we have shown that this result can be regarded as a natural consequence of the energy conservation law in the bulk. Here the essence is to recall that the so-called dark radiation term, which behaves like radiation on the brane, describes the total mass in the bulk. Then, a very massive KK mode corresponds to a particle with a high momentum in the direction of the extra dimension, which exerts a pressure on the brane and pushes it outward in the direction of the AdS infinity. As a result, the energy in the bulk decreases, leading to the decrease of the dark energy term. Thus, a massive KK mode gives a negative contribution to the dark radiation term. This is why a KK mode behaves like a negative energy density dust.

Note that the negative energy of a KK mode emerges only from the effective four-dimensional point of view on the brane. The bulk energy density for a KK mode still remains positive and thus there is no singular effect in the bulk.

We have studied the two cases (de Sitter and low energy branes) in which the bulk equations are separable, hence the KK modes are well defined. However, for a general cosmological brane, one cannot define a KK mode since its very definition depends on the separability of the equations in the bulk. Nevertheless, considering the discussion from the bulk point of view given in the previous paragraph, it seems reasonable to expect that this backreaction effect of the bulk metric perturbations persists for a general cosmological brane. Thus we conclude that the effect of very massive KK modes is to reduce the energy density on the brane, hence the expansion rate, and for a low energy brane the universe will recollapse unless the contribution of a normal (true) dust matter is larger than that of the KK modes.

To quantify this effect in realistic cosmological models, there are some additional issues that remain to be resolved. In this paper, we have considered only a single KK mode and calculated the effective energy density and pressure. In reality, one should integrate over all the KK modes that contribute to the cosmology of the brane. This requires first the knowledge of the whole spectrum of the KK modes, which will presumably be determined by vacuum fluctuations in the bulk \[21\] \[22\] \[23\]. However, knowing the whole spectrum may not be enough, because a naive integral of the KK spectrum is expected to diverge. One would then need an appropriate regularization scheme. In connection with...
this, it may be important to take into account the thickness of a brane, either classically as in the case of a classical domain wall or quantum mechanically by considering the wavefunction of a brane.

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APPENDIX A: SECOND ORDER CURVATURE TENSORS IN THE BULK

Here we spell out the components of the curvature tensors up to quadratic order in the bulk metric perturbation. We consider the \((d + 1)\)-dimensional perturbed metric in the form,

\[ ds^2 = dy^2 + b^2(y)\left(\gamma_{\mu\nu} + h_{\mu\nu}\right)dx^\mu dx^\nu, \]

where \(\gamma_{\mu\nu}\) is the metric of the background \(d\)-dimensional spacetime section. In the text, we identify \(\gamma_{\mu\nu}\) with the metric of a de Sitter spacetime. We impose the following gauge conditions on the perturbation:

\[ h^\gamma_\alpha = h^\beta_\alpha = 0, \]

where the vertical bar (\(\cdot\)) denotes the covariant derivative associated with the \(d\)-dimensional metric \(\gamma_{\mu\nu}\), and the tensor indices of \(h_{\mu\nu}\) are raised or lowered by the metric \(\gamma_{\mu\nu}\) (not by the 5-dimensional metric).

The non-trivial components of the connection are given by

\[
(d+1)\Gamma^\mu_{\nu\rho} = \frac{b'}{b}\delta^\mu_\nu + \frac{1}{2}h^\mu_\nu - \frac{1}{2}h^{\mu\rho}h^\nu_\rho, \\
(d+1)\Gamma^\mu_{\rho\nu} = -bb'\gamma_{\mu\nu} - \frac{1}{2}b^2h^\mu_\nu - bb'h_{\mu\nu}, \\
(d+1)\Gamma^\rho_{\alpha\beta} = (d)\Gamma^\rho_{\alpha\beta} + \frac{1}{2}\gamma^{\rho\mu}(h_{\alpha\beta\mu} + h_{\beta\mu\alpha} - h_{\alpha\mu\beta} - h_{\alpha\beta\mu}) - \frac{1}{2}h^{\mu\rho}(h_{\alpha\rho\beta} + h_{\beta\rho\alpha} - h_{\alpha\beta\rho}),
\]

where the prime (\(\cdot'\)) denotes the \(y\)-derivative.

The non-trivial components of the Riemann tensor are given by

\[
(d+1)R^\mu_{\nu\rho\gamma} = -\frac{b'}{b}\delta^\mu_\nu - \frac{1}{2}h^\mu_\nu - \frac{b'}{b}h^\nu_\rho + \frac{1}{4}h^{\mu\rho}h^\nu_\rho + \frac{1}{4}h^{\mu\rho}h^\nu_\rho + \frac{1}{2}h^{\mu\rho}h^\nu_\rho, \\
(d+1)R^\mu_{\rho\nu\gamma} = -bb'\gamma_{\mu\nu} - bb'h_{\mu\nu} - \frac{1}{2}b^2h^\mu_\nu - bb'h_{\mu\nu} + \frac{1}{4}h^{\mu\rho}h^\nu_\rho, \\
(d+1)R^\alpha_{\nu\rho\mu} = \left(\frac{1}{4}h_{\rho\mu\nu} + h_{\rho\mu\nu} - h_{\rho\mu\nu}\right)_{\mu
u} - \left(\frac{1}{2}h^\rho_{\mu\nu} - \frac{1}{2}h^\nu_{\mu\rho}\right)_{\rho
u}, \\
(d+1)R^\alpha_{\nu\rho\mu} = -\frac{1}{2}b^2(h_{\alpha\nu\mu} - h_{\alpha\mu\nu}) - \frac{b^2}{8}h^\mu_\nu(h_{\alpha\nu\mu} + h_{\alpha\mu\nu} - h_{\alpha\nu\mu}), \\
(d+1)R^\mu_{\alpha\beta\rho} = -(d)R^\mu_{\alpha\beta\rho} - b^2(\delta^\mu_{\nu}\gamma_{\alpha\beta} - \delta^\mu_{\beta}\gamma_{\alpha\nu}) - \frac{1}{2}(h_{\alpha\beta}|_{\nu} + h_{\beta\alpha}|_{\nu} - h_{\alpha\beta}|_{\nu} - h_{\alpha\beta}|_{\nu} - h_{\alpha\beta}|_{\nu} + h_{\alpha\beta}|_{\nu} - h_{\alpha\beta}|_{\nu} + h_{\alpha\beta}|_{\nu}), \\
\]

-\frac{1}{2}b^2(h_{\alpha\rho}|_{\nu} + h_{\beta\rho}|_{\nu} - h_{\alpha\rho}|_{\nu} - h_{\alpha\rho}|_{\nu} - h_{\alpha\rho}|_{\nu} + h_{\alpha\rho}|_{\nu} - h_{\alpha\rho}|_{\nu} + h_{\alpha\rho}|_{\nu}), \\
-\frac{1}{2}b^2(h_{\alpha\beta}|_{\nu} - h_{\alpha\beta}|_{\nu} - h_{\alpha\beta}|_{\nu} - h_{\alpha\beta}|_{\nu} - h_{\alpha\beta}|_{\nu} + h_{\alpha\beta}|_{\nu} - h_{\alpha\beta}|_{\nu} + h_{\alpha\beta}|_{\nu}).
Using these results, the components of the Einstein tensor are given by

\[ R_{\mu\nu} = \frac{1}{2} b' \left( \frac{\gamma_{\alpha\beta}}{b} \right) h^\rho_\mu h^\sigma_\nu - \frac{1}{2} h^\rho_\mu h^\sigma_\nu + \frac{1}{2} b' \left( \frac{\gamma_{\alpha\beta}}{b} \right) h^\rho_\mu h^\sigma_\nu \]

The mixed components of the Ricci tensor are given by

\[ R^\alpha_\beta = \frac{b'}{b} \delta^\alpha_\beta + \frac{1}{b^2} (d') R^\alpha_\beta - (d-1) \left( \frac{b'}{b} \right)^2 \delta^\alpha_\beta \]

The Ricci scalar is given by

\[ R = -2d \frac{b'}{b} + \frac{1}{b^2} (d') R - d(d-1) \left( \frac{b'}{b} \right)^2 - \frac{1}{b^2} (d') R_{\rho\sigma} h^{\rho\sigma} \]

Using these results, the components of the Einstein tensor are given by

\[ G^\alpha_\beta = \frac{1}{2} b^2 (d') R + \frac{1}{2} d(d-1) \left( \frac{b'}{b} \right)^2 + \frac{1}{2} \frac{b'}{b} (d') R_{\rho\sigma} h^{\rho\sigma} \]

The mixed components of the Ricci tensor are given by

\[ R^\alpha_\beta = \frac{b'}{b} \delta^\alpha_\beta + \frac{1}{b^2} (d') R^\alpha_\beta - (d-1) \left( \frac{b'}{b} \right)^2 \delta^\alpha_\beta \]
\[
\frac{1}{2} \delta^{\alpha \beta} \left[ \frac{3}{4} h^{\rho \sigma} h^\prime_{\rho \sigma} + d_{\beta} h^{\rho \sigma} h^\prime_{\rho \sigma} + h^{\rho \sigma} h^{\prime \prime}_{\rho \sigma} + \frac{1}{b^2} h^{\rho \sigma} \Box d h_{\rho \sigma} \\
+ \frac{1}{b^2} \left( \frac{3}{4} h^{\rho \sigma \mu} h_{\rho \sigma \mu} - \frac{1}{2} h^{\rho \sigma \mu} h_{\rho \sigma | \mu} \right) - \frac{1}{b^2} h^{\mu | \rho \sigma \mu} h^{\rho \sigma} + \frac{1}{b^2} (d R)_{\rho \sigma} h^{\rho \mu} h^{\mu \sigma} \right].
\]

(A7)

APPENDIX B: COMPUTATIONAL RULES FOR AVERAGING

In this appendix, we describe the computational rules for averaging the components of the second order part of the curvature tensors listed in Appendix A. As we have noted in the main text, the notation \((A)\) includes both the averaging along the ordinary spatial dimensions which are assumed to be homogeneous and isotropic, and the small-scale time averaging as defined in (2.17). In both cases, the computational rules are similar (See e.g., Ref. [24]). However, we do not apply the same rules for terms with derivatives in the bulk direction, because we are dealing with a braneworld and the averaging along the bulk direction is ill-defined.

First, we note that we are interested in massive KK modes. So, we can neglect terms coupled to the background curvature tensor as

\[
\left\langle (d) R_{\rho \sigma} h^{\rho \mu} h^{\sigma \nu} \right\rangle,
\]

(B1)

which are of order \(O(h^2/L^2)\), where \(L\) is the \(d\)-dimensional characteristic background curvature radius, in comparison with terms as

\[
\left\langle h^{\rho \sigma} |_{\rho \sigma | \nu} \right\rangle, \quad \left\langle h^{\rho \sigma} h_{\rho \sigma | \mu} \right\rangle,
\]

\[
\left\langle h^{\rho \mu} h^\prime_{\rho \mu} \right\rangle, \quad \left\langle h^{\rho \mu} h^{\prime \prime}_{\rho \mu} \right\rangle, \ldots,
\]

(B2)

which are of order \(O(m^2 h^2)\). For instance for a cosmological brane with expansion rate \(H\), we have \(L = O(1/H)\). Thus \(H \gg m\) implies \(m \gg L^{-1}\), and we can safely neglect corrections of the form \((B1)\).

As a consequence, when taking the average, we are allowed to freely interchange the order of the covariant derivatives. For example,

\[
\left\langle h^{\rho \sigma} |_{\rho \sigma | \mu} h_{\rho \sigma} \right\rangle \simeq \left\langle h^{\rho \sigma} |_{\nu \mu} h_{\rho \sigma} \right\rangle,
\]

(B3)

where corrections of order \(O(h^2/L^2)\) are neglected. From now on, as in the main text we will use \(\sim\) instead of \(\simeq\) by neglecting the corrections.

Another computation rule is that total derivative terms can be neglected. For example,

\[
\left\langle h^{\rho \sigma} |_{\rho \sigma | \mu} h_{\rho \sigma} \right\rangle = \left\langle h^{\rho \sigma} h_{\rho \sigma | \mu} \right\rangle - \left\langle h^{\rho \sigma} h_{\rho \sigma | \nu} \right\rangle = - \left\langle h^{\rho \sigma} h_{\rho \sigma | \nu} \right\rangle.
\]

(B4)

This is because the total derivative term can be cast into the surface integral which is smaller in magnitude than the volume term by a factor \(mR(\gg 1)\), where \(R\) is the length scale of the averaging volume which is taken to satisfy \(R \gg m^{-1}\).

As mentioned above, we do not apply the same rules for terms with derivatives with respect to the bulk coordinate \(y\). However, when one considers projections onto the brane, some simplifications occur. On the brane, we have the boundary condition \(h^\prime_{\alpha \beta}|_{\text{brane}} = 0\), which enables us to neglect all the first derivative terms, e.g.,

\[
\left\langle h^{\alpha \mu} h^{\rho \beta}_\prime \right\rangle|_{\text{brane}} = \frac{b}{b} \left\langle h^{\alpha \rho} h^{\rho \beta}_\prime \right\rangle|_{\text{brane}} = \left\langle h^{\alpha \rho | \mu} h^{\rho \beta}_\prime \right\rangle|_{\text{brane}} = 0.
\]

(B5)

In addition, using the bulk equation of motion (3.16), we have

\[
\left\langle h^{\rho \sigma} h^{\prime \prime}_{\rho \sigma} \right\rangle|_{\text{brane}} = - \left\langle \hat{h}^{\rho \sigma} \Box d h_{\rho \sigma} \right\rangle|_{\text{brane}}.
\]

(B6)

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