\[ \tau \rightarrow \pi K \nu \text{ DECA}Y \text{ AND } \pi K \text{ SCATTERING} \]

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ABSTRACT

Using chiral low energy theorems and elastic unitarity assumption, the \( \tau \rightarrow \pi K \nu \) decay is investigated. The vector and scalar \( \pi K \) form factors are calculated. It is found that the \( \pi K \) spectrum is dominated by the \( K^* \) resonance. By measuring the forward-backward asymmetry, it is shown that the S wave \( \pi K \) phase shift can be determined near the \( K^* \) resonance region. The calculated branching ratio and resonance parameters are in good agreement with experiments.

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INTRODUCTION

Current Algebra was invented in the 60’s to study phenomena involving emission of soft pions (and kaons) and also the chiral symmetry breaking effects. Its first success was the study of the renormalization effect of the axial vector nucleon coupling constant $g_A$ due to the strong interaction by the celebrated Adler-Weisberger relation which relates $g_A$ to the $\pi N$ total cross sections[1]. Subsequent calculations by Weinberg [2, 3] and others [4] on the soft pion phenomena, such as the pion nucleon scattering lengths the relation between $K_{l2}, K_{l3}$ and $K_{l4}$ etc. confirmed the success of current algebra as an useful tool to study the low energy pion physics. It was later realized that the pions emitted in these processes are not really soft and methods were invented to take into account of the correction to the soft pion current algebra theorems. This development was known as the hard pion current algebra which consists in supplementing the low energy current algebra theorems with unitarity corrections in the form of the pole dominance for the hadronic matrix elements.

One of the ambitious program vigorously pursued in the late 60’s was the $\pi \rho A_1$ system [5] which ended up in failure due to the wrong prediction of the $A_1$ width. We shall deal with this problem in a future publication [6]; we discuss in this article a simpler problem $\tau \to \pi K \nu$ decay. Unlike in the study of the pion electromagnetic form factor, where the relevant current is exactly conserved and where chiral symmetry does not play a role in deriving the low energy theorems, we deal here with the $\pi K$ vector form factors whose current is not exactly conserved due to the approximate SU(3) symmetry; the physics is therefore richer. Chiral symmetry does play an important role here which enables us to derive the SU(3) breaking relation of the ratio $f_K/f_\pi$ in terms of the two form factors of the $K_{l3}$ decay [10]. These form factors are, on general grounds, analytic in the momentum transfer s plane with a cut from $(m_\pi + m_K)^2$ to infinity. The measured $K_{l3}$ form factors give us only information on the form factors below the cut. In contrast, the $\tau \to \pi K \nu$ decay form factors are measured on the cut. They are therefore the analytic continuation of the $K_{l3}$ form factors to the time like region. The role of the square root threshold singularity in the scalar form factor was emphasized by one of us and provided a semi quantitative understanding why the soft pion theorems are valid in some reactions but not in others [7].

The $\tau \to \pi K \nu$ decay amplitudes satisfy the same low energy theorems as those
of the $Kl_3$ decay because they are the analytic continuation of each other. The practical problem is how to carry out the analytic continuation.

This problem was addressed along time ago by using the boundary conditions of the low energy current algebra theorems together with analyticity and elastic unitarity relation[15]. A singular integral equation of the Muskeshvili Omnès (MO) type can be written [12] and whose exact solution can be written in terms of the $I = 1/2$ S and P wave $\pi K$ phase shifts.

One can either use experimental data or theoretical calculation of the $\pi K$ phase shifts in the solution of these integral equations in order to calculate the $\pi K$ form factors. This was done in the reference [7] where both $Kl_3$ form factors were calculated.

An alternative method consists in writing an integral equation for the inverse of the form factor, similarly to the study of the pion vector and scalar form factors [9], we then get an approximate solution by solving it perturbatively using the $\pi K$ rms radius as input. The final solution satisfies the elastic unitarity relation and can take into account of the resonant or non resonant interactions which were well demonstrated in the pion form factor calculation. This result is equivalent to applying the Padé approximant method to the one loop chiral perturbation theory (CPTh) which in the P wave case leads to a $\rho$ resonance [9]. The once iterated solution of the inverse integral equation or the Padé method are in fact the bubble summation of the $\pi\pi$ interaction of the form factor problem. This approximation is now known in the litterature as the large $N_f$ method [8], where $N_f$ is the number of Nambu-Goldstone bosons. It can be straightforwardly shown that if the strong partial wave amplitude could be represented by the bubble summation, the Padé method for the form factor would be the exact solution of the MO integral equation. We want to emphasize that the exact solution of the MO integral in terms of the phase shift is more general.

We show in this paper, using the rms radii of the vector and scalar $\pi K$ form factors which are either given by the experimental data or by the Callan Treiman relation [10], the main features of the $\tau \to \pi K \nu$ decay are completely determined. We wish to emphasize that the CPTh which was invented to study the physics near the $\pi K$ threshold cannot handle the main feature of the vector $\pi K$ form factor because it cannot take into account of the $K^*$ resonance. As we show below, our calculation for this decay mode yields a correct $\pi K$ spectrum and a branching ratio of $1.0\%$ which is in agreement with the experimental data of $1.4 \pm 0.2\%$. 


We then improve the above calculations with a more accurate calculation of the form factors where the t and u channels contributions to the $\pi K \rightarrow \pi K$ amplitudes are taken into account as a correction.

This paper is organized as follows: in section 1 we give the kinematic of the problem and a general phenomenological method to determine the S wave phase shift by measuring the Forward-Backward asymmetry of the $\pi K$ system. We also include, for completeness, a short review of the analysis $K \rightarrow \pi e \nu$ decay, together with the current algebra results for the form factors.

In section 2, the one loop correction to current algebra result is given and then this result is modified to take into account of the elastic unitarity condition in the approximation where the left hand cut contribution to the $\pi K$ scattering amplitude is neglected. In section 3, a more exact calculation is presented where the left hand cut contribution to the $\pi K$ scattering amplitude is taken into account. A comparison between the two methods will be made.

I) Notations and kinematical preliminaries

The most general $\tau \rightarrow \pi K \nu$ decay amplitudes are given in terms of two form factors:

$$\langle \pi^0 K^- | V_\mu^{4-i5} (0) | 0 \rangle = f_1(s)(p_2 - p_1)_\mu + f_2(s)(p_1 + p_2)_\mu$$

where $p_1$, and $p_2$ are, respectively, the pion and kaon momenta, and $s = (p_1 + p_2)^2$ is the time-like momentum transfer and $V_\mu^{4-i5}$ is the vector current operator with the superscript indices referring to the SU(3) octet currents. $f_1(s)$ is the P wave $\pi K$ form factor, $f_2(s)$ is a linear combination of S and P states as can be seen by taking the divergence of Eq (1):

$$g(s) = -i \langle \pi^0 K^- | \partial_\mu V_\mu^{4-i5} (0) | 0 \rangle = (m_K^2 - m_\pi^2)f_1(s) + sf_2(s)$$

$g(s)$ is therefore a pure scalar which describes the S wave $\pi K$ form factor. $g(s)$ measures the SU(3) violating effect because, in the exact SU(3) limit, the vector current is conserved. We expect therefore in the $\tau \rightarrow \pi K \nu$ decay, the P wave form factor $f_1(s)$ dominates.

Because of the octet current hypothesis the two channels $\pi^0 K^-$ and $\pi^- K^0$ matrix elements are related by the Clebsh Gordon coefficient

$$\langle \pi^- K^0 | V_\mu^{4-i5} (0) | 0 \rangle = \sqrt{2} \langle \pi^0 K^- | V_\mu^{4-i5} (0) | 0 \rangle$$

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In terms of form factors $f_1(s)$ and $g(s)$, and the angle $\theta$, defined as angle between $\vec p_\pi$ and $\vec p_\nu$ in the hadronic rest frame, the decay spectrum and forward-backward asymmetry are given by:

$$\frac{d\Gamma}{dsdcos\theta} = \frac{G_F^2s\sin^2\theta_c(s-m^2\pi^2)^{1/2}}{2^{9/2}\pi^3sm^2_f}\{\frac{\lambda}{s}f_1(s)\}^2 \sin^2\theta$$

$$+ \frac{m^2_\pi}{s^2}|g_0(s) + \lambda^{1/2}f_1(s)\cos\theta|^2 \}$$

where $\theta_c$ is the Cabibbo angle with $\cos \theta_c = 0.97$, and $\lambda(s,m^2_\pi,m^2_K) = (s - (m_\pi + m_K)^2)(s - (m_\pi - m_K)^2)$. For simplicity we denote it by $\lambda$.

The forward-backward asymmetry is defined as:

$$A_{FB} = \frac{d\Gamma[\cos \theta] - d\Gamma[-\cos \theta]}{d\Gamma[\cos \theta] + d\Gamma[-\cos \theta]}$$

$$= \frac{\lambda^{1/2}m^2_sRe[g*(s)f_1(s)]}{s[\lambda(2/3 + m^2_s/3s)]f_1(s)^2 + m^2_s|g(s)|^2/s}$$

The forward-backward asymmetry is a useful phenomenological quantity, because it allows us to measure experimentally the relative phase between S and P wave amplitudes of the $\pi K$ scattering. Furthermore, because the forward-backward asymmetry vanishes in the limit of the exact SU(3) symmetry, the presence of this term allows us to measure the SU(3) breaking effect.

For completeness, we now add a short review of the $K_{\ell 3}$ decay. The total amplitude is a product of two parts, the leptonic and the hadronic ones. The hadronic matrix element is given by:

$$\langle \pi^0(p_1)|V^{4+i5}_\mu(0)|K^-(p_2)\rangle = f_+(t)(p_1 + p_2)_\mu + f_-(t)(p_2 - p_1)_\mu$$

$$f_0(t) = i\langle \pi|\partial^\mu V^{4+i5}_\mu(0)|K\rangle = (m^2_K - m^2_\pi)f_+ + tf_-(t)$$

$f_+$ and $f_-$ are dimensionless form factors depending on the momentum transfer $t = (p_2 - p_1)^2$, they are respectively the analytic continuation of $f_1$ and $f_2$. $K_{\mu 3}$ experiments give information on $f_+$ and $f_-$, while $K_{e 3}$ experiments are sensitive only to $f_+$ because of the small electron mass.

Using the Ademollo-Gatto theorem $f_+(0) = 1/\sqrt{2}$ and hence $f_0(0) = (m^2_K - m^2_\pi)/\sqrt{2}$ for $\pi^0K^-$ system.

Using the standard current algebra technique and the $SU(2)_L \times SU(2)_R$ commutation relation by taking the pion momentum $p_1$ soft we have the well known
Callan-Treiman relation [10]:

$$f_+ (m_K^2) + f_- (m_K^2) = \frac{f_K}{f_\pi \sqrt{2}} \quad (7)$$

where $f_K$ and $f_\pi$ are, respectively, the K and $\pi$ decay constants $f_K/f_\pi = 1.22$.

By evaluating Eq(2) at $t = m_K^2$ and noting that $f_- (m_K^2)$ is proportional to $m_\pi^2/m_K^2$ we have: $f_0 (m_K^2) \approx f_0 (0) f_K/f_\pi$

II) Unitarity correction to current algebra results

Before giving the details of the one loop calculation, we outline some general properties of the S matrix.

The elastic unitarity condition, which should be valid in the physical region of the $\tau \to \pi K \nu$ decay gives:

$$Im f_1 (s) = f_1 (s) \exp -i \delta_p^{1/2} \sin \delta_p^{1/2}$$
$$Im g(s) = g(s) \exp -i \delta_s^{1/2} \sin \delta_s^{1/2} \quad (8)$$

where $\delta_s^{1/2}$ and $\delta_p^{1/2}$ are respectively the phase of S and P wave $I=1/2 \pi K$ scattering amplitude. One can decompose $\pi K$ elastic amplitude into the partial waves using $T^l(s, \theta) = 16\pi \sum_l (2l+1) t^l(s) P_l(\cos \theta)$, where $l$ stands for the angular momentum and $\theta$ the angle in c.m system, and $t^l(s) = \exp i \delta^l \sin \delta^l/\rho(s)$ where $\rho(s) = \sqrt{\lambda(s, m_\pi^2, m_K^2)}/s$ is the phase space factor. From Eq(8) in order to satisfy the elastic unitarity condition the form factors $f_1(s)$ and $g(s)$ must have, respectively, the phase $\delta_p^{1/2}$ and $\delta_s^{1/2}$.

The elastic unitarity condition is a good approximation to describe $\tau \to \pi K \nu$ decay owing to the experimental fact that the inelastic effects are not large. Using the analytic properties of the form factors, and the elastic unitarity condition, we have:

$$f_1 (s) = f_1 (0) + f_1 (0) \frac{\langle r_v^2 \rangle}{6} s + \frac{s^2}{\pi} \int _{m_\pi + m_K} ^{+\infty} f_1 (z) \exp -i \delta_p^{1/2} \sin \delta_p^{1/2} dz$$
$$g(s) = g(0) + g(0) \frac{\langle r_s^2 \rangle}{6} s + \frac{s^2}{\pi} \int _{m_\pi + m_K} ^{+\infty} g(z) \exp -i \delta_s^{1/2} \sin \delta_s^{1/2} dz \quad (9)$$
where \( \langle r_v^2 \rangle \) and \( \langle r_s^2 \rangle \) are, respectively, vector and scalar radii of \( \pi K \) system.

The solutions to these integral equations are well known \[12\]

\[
f_1(s) = f_1(0) \exp\left(\frac{s}{\pi} \int_{(m_\pi+m_K)^2}^{+\infty} \frac{\delta^{1/2}_p dz}{z(z-s-i\epsilon)}\right) \\
g(s) = g(0) \exp\left(\frac{s}{\pi} \int_{(m_\pi+m_K)^2}^{+\infty} \frac{\delta^{1/2}_s dz}{z(z-s-i\epsilon)}\right)
\tag{10}
\]

These solutions can also be derived by an infinite iteration of the integral equations Eq[9].

From the Ademollo-Gatto theorem as discussed above, we have \( f_1(0) = 1/\sqrt{2} \) and hence \( g(0) = (m_K^2 - m_\pi^2)/\sqrt{2} \). We ignore the so called polynomial ambiguity which is obtained by multiplying the rhs of Eq [10] by a polynomial. They correspond to higher energy contributions which are assumed to be small. We can either use the experimental or theoretical phase shifts in Eq [10] to calculate the form factors \( f_1(s) \) and \( g(s) \).

A simplest approximation for \( f_1(s) \) and \( g(s) \) can be obtained by modifying the one loop CPTh as was done in the reference [9] for the pion form factor. This can be done by calculating the strong \( \pi K I = 1/2 \) amplitude. The tree amplitudes are:

\[
t_1^{\text{tree}}(s) = \frac{\lambda(s, m_\pi^2, m_K^2)}{128\pi s f_\pi^2} \\
t_0^{\text{tree}}(s) = \frac{(2s - 3\lambda(s, m_\pi^2, m_K^2)/(4s) - 2m_\pi^2 - 2m_K^2)}{32\pi f_\pi^2}
\tag{11}
\]

where the subscript refers to the \( l \) partial wave.

Using these expressions and replacing \( f_1(z) \) by \( f_1(0) \) and \( g(z) \) by \( g(0) \) in Eq[9], we have the one loop perturbative results for the form factor \( f_1 \) and \( g \):

\[
f_1^{\text{pert.}}(s) = f_1(0) + f_1(0) \frac{\langle r_v^2 \rangle}{6} s + \frac{f_1(0)}{128\pi f_\pi^2} (-I_1(s) + 2(m_\pi^2 + m_K^2)I_2(s) - (m_\pi^2 - m_K^2)^2 I_3(s)) \tag{12 - a}
\]

\[
g^{\text{pert.}}(s) = g(0) + g(0) \frac{\langle r_s^2 \rangle}{6} s + \frac{g(0)}{32\pi f_\pi^2} (-\frac{5}{4} I_1(s) + \frac{1}{2} (m_\pi^2 + m_K^2)I_2(s) + \frac{3}{4}(m_\pi^2 - m_K^2)^2 I_3(s)) \tag{12 - b}
\]
where $I_1(s)$, $I_2(s)$ and $I_3(s)$ are given in the Appendix A. As was explained in reference [9], these expressions only satisfy perturbatively the unitarity relation. We can resum the perturbative results, Eq[12-a, 12-b], to implement the elastic unitarity relation. For this purpose, following ref[9], we write $f_1$ and $g$ as:

$$f = \frac{f_{\text{tree}}}{1 - f_{\text{loop}}/f_{\text{tree}}}$$  

which is just the diagonal [1,1] Padé approximant of the form factors, hence:

$$f_1(s) = \frac{f_1(0)}{1 - s\langle r_v^2 \rangle / 6 - \frac{1}{128\pi f_0^2}(-I_1(s) + 2(m_\pi^2 + m_K^2)I_2(s) - (m_\pi^2 - m_K^2)^2I_3(s))}$$  

$$g(s) = \frac{g(0)}{1 - s\langle r_s^2 \rangle / 6 - \frac{1}{32\pi f_0^2}(-\frac{5}{4}I_1(s) + \frac{1}{2}(m_\pi^2 + m_K^2)I_2(s) + \frac{3}{4}(m_\pi^2 - m_K^2)^2I_3(s))}$$  

(13-a)

(13-b)

We show below that these results can also be directly obtained by the $N/D$ method in the approximation where the t and u one loop graphs are represented by an adjustable polynomial in the D function [11] or by the once iterated solution of the integral equation for the inverse of the form factor. It is obvious that Eq(13) can also be obtained by the infinite bubble summation of the $\pi K S$ and P wave interactions.

Because the partial wave amplitude has both right and left hand cut, we can always write it as a product of two cuts; $t_l^I(s) = N_l^I(s)/D_l^I(s)$, we normalize $D_l^I$ such that $D_l^I(0) = 1$. The elastic unitarity implies $\text{Im}(t_l^I(s)) = \rho(s)|t_l^I(s)|^2$ and hence $\text{Im}(D_l^I(s)) = -\rho(s)N_l^I(s)$. Using analyticity and unitarity, we can write the following dispersion relation for the partial wave amplitude

$$t_l^I(s) = \frac{N_l^I(s)}{1 + sD'(0) - \frac{s^2}{2\pi} \int_{m_\pi + m_K}^{\infty} \frac{\rho(z)N_l^I(z)dz}{z^2(z-s-i\epsilon)}}$$  

(14)

$D'(0)$ is an adjustable phenomenological parameter. We shall approximate $N_l^I(s)$ by $t_l^{\text{tree}}(s)$. Because $1/D_l^I(s)$ has the following phase representation: $1/D_l^I(s) = \text{exp}(\frac{2}{\pi} \int_{m_\pi + m_K}^{\infty} \frac{\delta_l^I dz}{z(z-s-i\epsilon)})$, hence $1/D_l^{1/2}$ and $1/D_l^{0/2}$ are proportional to $f_1(s)$ and $g(s)$.
given by Eq[10], and hence we have:

\[
f(s) = \frac{f(0)}{1 + sD'(0) - \frac{s^2}{\pi} \int \frac{\rho(z)}{(m_n+m_K)^2} \frac{t^{tree}(z)dz}{z^2(z-s-i\epsilon)} + \infty}
\]  \hspace{1cm} (15)

This expression is equivalent to (13-a) and (13-b) if we identify \(D'(0)\) with the rms radius.

From the expression for \(f_1(s)\), the phase of the form factor which is identical to the P wave phase shifts of \(\pi K\) scattering amplitude, can be calculated using the experimental value of \(\langle r^2 \rangle = 0.34 \pm 0.03\) fm\(^2\). Using this value we have \(m_{K^*} = 810 \pm 30\) MeV and agrees with the experimental data \(m_{K^*} = 892\) MeV. Its width satisfies the following modified KSRF relation [13]:

\[
\Gamma_{K^*} = \frac{\lambda^{3/2}(m_{K^*}^2, m_{\pi^*}^2, m_{K^*}^2)}{128\pi m_{K^*} f_{\pi}^2}
\]  \hspace{1cm} (16)

Using the experimental value \(m_{K^*} = 892\) MeV the numerical result of the right hand side of Eq(16) is 55 MeV, compared to the experimental value of 49.8 \(\pm\) 0.8 MeV.

The branching ratio \(B.R = \frac{\Gamma(\tau \rightarrow \pi K\nu)}{\Gamma(\tau \rightarrow all)}\) is 1.0\% and is in agreement with the experimental result of \(B.R_{exp.} = (1.4 \pm 0.2)\%\)

Because the S wave \(\pi K\) scattering length does not vanish, \(g(s)\) has a square root threshold singularity at the threshold (the derivative of \(g(s)\) is discontinuous at this point) as it can be seen in Fig(3). The scalar form factor contributes very little to the \(\pi K\) spectrum owing to the fact that it appears as a square of the amplitude. The forward backward asymmetry, being proportionnal to the amplitude, is reasonably large. It is about 10\% in the \(K^*\) resonance region where the number of event is maximum. The Forward-Backward asymmetry could be a useful quantity for studying the relative phases of the S and P waves as can be seen from Eq(5).

**III) A more exact calculation of \(\pi K \rightarrow \pi K\) scattering and form factors**

The \(\pi K\) scattering problem was calculated up to the one loop order by ref[14] and will not be repeated here. Below the inelastic thresholds, the one loop chiral perturbation theory satisfies the perturbative unitarity: \(\text{Im} t^{(1)} = \rho(s)t^{(0)}^2\) where \(t = t^{(0)} + t^{(1)}\) and the superscripts stand for the tree graph and one loop calculation, the isospin and the partial wave indices are omitted for convenience. This relation is not really satisfied in the one loop amplitude of ref[14] unless we replace \(f_K\) by
$f_\pi$. In fact, at the order $0(p^4)$, it is not clear whether $f_\pi$ or $f_K$ should be used. The difference is of the order $O(p^6)$. The way to circumvent this problem is to calculate the $\pi K$ scattering in the $SU(2) \times SU(2)$ theory which is not yet available. Using the standard current algebra technique [2] by treating the K meson as a heavy target one expect the $\pi K \rightarrow \pi K$ amplitude to be proportionnal to $1/f_\pi^2$. In what follows we replace $f_K$ by $f_\pi$ in order to get the same absorptive part in the s channel as in our calculation in section 2. It is straightforward to show that the reconstructed amplitude:

$$t(s) = \frac{t(0)}{1 - t(1)/t(0)}$$

(17)

satisfies exactly the elastic unitarity. The counterterms for the one loop amplitude are discussed in ref[16]. $L_4$, $L_5$, $L_6$ and $L_8$ measure the chiral symmetry breaking effects; their contributions to the scattering amplitude are proportional to the pion and Kaon mass squared. In this work they are taken to be the values given by the reference [16]. One linear combination of $L_1$, $L_2$ and $L_3$ is given by the $K^*$ mass. The other two constraints were considered in ref[11] in studying $\pi\pi$ scattering: $L_2 - 2L_1 - L_3$ is fixed by the $\rho$ mass, the third constraint, in order to completely determine $L_1$, $L_2$ and $L_3$, was obtained from an experimental S wave I=0 $\pi\pi$ phase shift at 500 MeV. The $K^*$ and $\rho$ mass are defined, respectively, as the energies where $I=1/2 \ l=1 \ \pi K$ and $I=1 \ l=1 \ \pi\pi$ partial wave phase shifts pass through 90 degrees. These three constraints give $L_1 = 1.23 \times 10^{-3}$, $L_2 = 1.51 \times 10^{-3}$ and $L_3 = -4.1 \times 10^{-3}$.

The predictions of the scattering lengths are: $a_{1/2}^{1/2} = 0.22m_\pi^{-1}$, and $a_{1/2}^{1/2} = 0.016m_\pi^{-3}$. They are in agreement with experiments. The P wave phase I=1/2 phase shifts are given in Fig(2). It is seen that they are in a reasonable agreement with the experimental data [17] and also with the results calculated above by N/D method where the the left hand cut discontinuity is neglected. More explicitly for the same value of the $K^*$ mass, this more complete calculation yields $\Gamma_{K^*} = 45MeV$, while the N/D method gives $\Gamma_{K^*} = 55MeV$. If we had taken the $\pi K \rightarrow \pi K$ one loop amplitude to be inversely proportional to $1/f_K^2f_\pi^2$ with $f_K = 1.22f_\pi$, we would have obtained $\Gamma_{K^*} = 38MeV$ which is too small. The corresponding form factors using the phase representation are shown in Fig(1). It is seen that the peak value of the vector form factor squared in the present calculation, where the left hand cut discontinuity is neglected, is 25% higher than that obtained from N/D method where the left hand cut is neglected. The discrepancy
is due to the difference in \( K^* \) width obtained in these two calculations. Away from the peak, the approximate solution agrees well with the more exact calculation.

This more exact calculation yields a branching ratio of 1.15\% for \( \tau \to \pi K \nu \) decay which is in a better agreement with the experimental value 1.4 \( \pm \) 0.2\%.

The S wave phase shift calculated using the unitarized \( \pi K \) amplitude eq(17), agrees also better with the experimental data \([17, 18]\). The approximate S wave phase shift eq(13-b) where the left hand cut of \( \pi K \) is neglected, differs from the experimental results at high energies.

Our result shows that the calculated S and P wave form factors, using the usual rule of neglecting the left hand cut for the \( \pi K \) scattering are not always accurate. When the left hand cut of the \( \pi K \) scattering is taken into account, a better agreement with the experimental data is obtained. In other words, there are some sizable corrections to the large \( N_f \) expansion.

We can also calculate the vector and scalar \( \pi K \) rms radii using:

\[
\langle r^2_V \rangle_{\pi K} = \frac{6}{\pi} \int_0^{+\infty} \frac{\delta_1^{1/2} dz}{z^2} \quad (m_\pi + m_K)^2
\]

\[
\langle r^2_S \rangle_{\pi K} = \frac{6}{\pi} \int_0^{+\infty} \frac{\delta_0^{1/2} dz}{z^2} \quad (m_\pi + m_K)^2
\]

(18)

Numerical calculation using the \( \pi K \) phase shift from the exact calculation gives \( \langle r^2_V \rangle_{\pi K} = 0.27 \text{ fm}^2 \) and \( \langle r^2_S \rangle_{\pi K} = 0.13 \text{ fm}^2 \). The experimental data from \( K_{\mu 3} \) decay are \( \langle r^2_V \rangle_{\pi K} = 0.34 \pm 0.03 \text{ fm}^2 \). The experimental situation of the \( \langle r^2_S \rangle_{\pi K} [19] \) is unsatisfactory since the values obtained by the expriments[20,21] are quite dispersed. The best result is given by Donaldson et al. [21], \( \langle r^2_S \rangle_{\pi K} = 0.23 \pm 0.05 \text{ fm}^2 \) which is larger than our theoretical prediction \( \langle r^2_S \rangle_{\pi K} = 0.13 \text{ fm}^2 \). The reason for this discrepancy is due to the assumption of the linear dependence in \( s \) of the form factor in the analysis of the experimental data of \( K_{\mu 3} \). Our scalar form factor calculation disagrees with this assumption as can be seen from Fig(3).

In this article we have calculated the \( \pi K \) S and P wave form factors by two different methods. In the first method, using the input as the \( \pi K \) r.m.s radius and the bubble summation for the form factor, the calculated form factor moduli and phases are in a rough agreement with the experimental data.

In the second method, we calculate first the CPTh for the \( \pi K \) scattering and then we unitarize this amplitude (where both left and right cuts are included)
by the Padé Approximant method; the strong $\pi K$ elastic amplitudes are in good agreement with the experimental data (e.g $\pi K$ phase shifts, width and mass of the $K^*$ resonance). We then calculate the $\pi K$ S and P wave form factors using the Omnes representation. This method yields a better agreement with data than the first one.

**APPENDIX A**

In section 2 we have given the form factors in terms of $I_1(s), I_2(s)$ and $I_3(s)$ which are easily expressed in terms of a generating function;

$$\psi(s) = -\frac{\lambda(s, m_{\pi}^2, m_K^2)}{2} \int_{(m_{\pi} + m_K)^2}^{\infty} \frac{dz}{\sqrt{\lambda(z, m_{\pi}^2, m_K^2)(z - s - i\epsilon)}}$$

For convenience we give the analytic continuation of this function to all regions.

$$\psi(s) = \begin{cases} 
\sqrt{\lambda(s, m_{\pi}^2, m_K^2)} \log\left(\frac{\sqrt{s - (m_\pi + m_K)^2} + \sqrt{s - (m_\pi - m_K)^2}}{2\sqrt{m_\pi m_K}}\right) & \text{if } s \geq s_t \\
-\frac{i\pi}{2} \sqrt{\lambda(s, m_{\pi}^2, m_K^2)} & \text{if } s \leq s_t \\
-\sqrt{\lambda(s, m_{\pi}^2, m_K^2)} \log\left(\frac{\sqrt{-s + (m_\pi + m_K)^2} + \sqrt{-s + (m_\pi - m_K)^2}}{2\sqrt{m_\pi m_K}}\right) & \text{if not} \\
\sqrt{\lambda(s, m_{\pi}^2, m_K^2)} \text{arctan}\left(\sqrt{\frac{s - (m_\pi - m_K)^2}{-s + (m_\pi + m_K)^2}}\right) & \text{if not}
\end{cases}$$

where $s_t = (m_\pi + m_K)^2$

$$I_1(s) = -\frac{s^2}{\pi} \int_{(m_{\pi} + m_K)^2}^{\infty} \frac{\sqrt{\lambda(z, m_{\pi}^2, m_K^2)}}{z^2(z - s - i\epsilon)} dz$$

$$I_2(s) = -\frac{s^2}{\pi} \int_{(m_{\pi} + m_K)^2}^{\infty} \frac{\sqrt{\lambda(z, m_{\pi}^2, m_K^2)}}{z^3(z - s - i\epsilon)} dz$$

$$I_3(s) = -\frac{s^2}{\pi} \int_{(m_{\pi} + m_K)^2}^{\infty} \frac{\sqrt{\lambda(z, m_{\pi}^2, m_K^2)}}{z^4(z - s - i\epsilon)} dz$$
\[ I_1(s) = \frac{2}{\pi} (\psi(s) - \psi(0) - s\psi'(0)) \]
\[ I_2(s) = \frac{2}{\pi s} (\psi(s) - \psi(0) - s\psi'(0) - \frac{s^2}{2} \psi''(0)) \]

and
\[ I_3(s) = \frac{2}{\pi s^2} (\psi(s) - \psi(0) - s\psi'(0) - \frac{s^2}{2} \psi''(0) - \frac{s^3}{6} \psi'''(0)) \]

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FIGURE CAPTIONS

Figure 1: The calculated $\pi K$ P wave form factors (solid/dashed/dot-dashed curves) corresponding respectively to, the Omnés representation using the $\pi K$ phase shift calculated from the unitarized one loop CPTh as defined in Eq(10), the [1,1] Padé approximant as given in Eq(13-a), CPTh prediction as given in Eq(12-a).

Figure 2: The solid line represents the I=1/2, l=1 $\pi K$ scattering phase shift calculated from the unitarized CPTh Eq(17). The dashed line corresponds to the similar phase shift when the left hand cut is neglected as given by eq(13-a). The dot-dashed line is the CPTh prediction phase Eq(12-a) (which is not the same as the phase shift due to the violation of the full elastic unitarity relation in this method). The experimental results are those of ref. [17].

Figure 3: The calculated $\pi K$ S wave form factors (solid/dashed/dot-dashed curves) corresponding respectively to, the Omnés representation using the $\pi K$ phase shift calculated from unitarized one loop CPTh as defined in Eq(10), the [1,1] Padé approximant as given in eq(13-b), CPTh prediction as given by Eq(12-b).

Figure 4: The solid line represents the I=1/2, l=0 $\pi K$ scattering phase shift calculated from the unitarized CPTh Eq(17). The dashed line corresponds to the similar phase shift when the left hand cut is neglected as given by eq(13-b). The dot-dashed line is the CPTh prediction phase Eq(12-b) (which is not the same as the phase shift due to the violation of the full elastic unitarity relation in this method). The experimental results are those of ref. [17, 18].

Figure 5: Calculation of the $\pi K$ invariant mass squared spectrum of $\tau \to \pi K \nu$ decay. The dashed/solid curves correspond respectively to the calculations with/without the left hand cut of $\pi K$ scattering amplitude.

Figure 6: Prediction for the Forward-Backward asymmetry $A_{FB}$ defined in eq(5) as a function of the $\pi K$ invariant mass squared. The dashed/solid curves correspond respectively to the calculations with/without the left hand cut of $\pi K$ scattering amplitude.
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