Corwin-Greenleaf multiplicity function for compact extensions of the Heisenberg group

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Abstract

Let $H_n$ be the $(2n+1)$-dimensional Heisenberg group and $K$ a closed subgroup of $U(n)$ acting on $H_n$ by automorphisms such that $(K, H_n)$ is a Gelfand pair. Let $G = K \ltimes H_n$ be the semidirect product of $K$ and $H_n$. Let $\mathfrak{g} \supset \mathfrak{k}$ be the respective Lie algebras of $G$ and $K$, and $\text{pr} : \mathfrak{g}^* \to \mathfrak{k}^*$ the natural projection. For coadjoint orbits $O^G \subset \mathfrak{g}^*$ and $O^K \subset \mathfrak{k}^*$, we denote by $n(O^G, O^K)$ the number of $K$-orbits in $O^G \cap \text{pr}^{-1}(O^K)$, which is called the Corwin-Greenleaf multiplicity function. In this paper, we give two sufficient conditions on $O^G$ in order that $n(O^G, O^K) \leq 1$ for any $K$-coadjoint orbit $O^K \subset \mathfrak{k}^*$.

For $K = U(n)$, assuming furthermore that $O^G$ and $O^K$ are admissible and denoting respectively by $\pi$ and $\tau$ their corresponding irreducible unitary representations, we also discuss the relationship between $n(O^G, O^K)$ and the multiplicity $m(\pi, \tau)$ of $\tau$ in the restriction of $\pi$ to $K$. Especially, we study in Theorem 4 the case where $n(O^G, O^K) \neq m(\pi, \tau)$. This inequality is interesting because we expect the equality as the naming of the Corwin-Greenleaf multiplicity function suggests.

Keywords. Heisenberg motion group, generic unitary representation, generic coadjoint orbit, Corwin-Greenleaf multiplicity function.

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1 Introduction

Let $G$ be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and $\hat{G}$ the unitary dual of $G$, i.e. the set of all equivalence classes of irreducible unitary representations of $G$. Then Kirillov proved that the unitary dual $\hat{G}$ of $G$ is parametrized by $\mathfrak{g}^*/G$, the set of coadjoint orbits. The bijection

$$\hat{G} \simeq \mathfrak{g}^*/G$$

is called the Kirillov correspondence (see [7]). Let $\pi$ be the unitary representation corresponding to a given coadjoint orbit $O^G \subset \mathfrak{g}^*$. Let $K$ be a subgroup of $G$. 

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Then the restriction $\pi|_K$ is decomposed into a direct integral of irreducible unitary representations of $K$:

$$
\pi|_K \simeq \int_K m(\pi, \tau) d\mu(\tau) \quad \text{(branching rule)}
$$

where $d\mu$ is a Borel measure on the unitary dual $\hat{K}$. Then Corwin and Greenleaf proved that the above multiplicity $m(\pi, \tau)$ coincides almost everywhere with the “mod $K$” intersection number $n(O^G, O^K)$ defined as follows:

$$
n(O^G, O^K) := \sharp \bigg( \bigg( O^G \cap \text{pr}^{-1}(O^K) \bigg)/K \bigg)
$$

(see [4]). Here, $O^G \subset g^*$ and $O^K \subset \mathfrak{k}^*$ are the coadjoint orbits corresponding to $\pi \in \hat{G}$ and $\tau \in \hat{K}$, respectively, under the Kirillov correspondence $\hat{G} \simeq g^*/G$ and $\hat{K} \simeq \mathfrak{k}^*/K$, and

$$
\text{pr} : g^* \longrightarrow \mathfrak{k}^*
$$

is the natural projection. The function

$$
n : g^*/G \times \mathfrak{k}^*/K \longrightarrow \mathbb{N} \cup \{\infty\}, \quad (O^G, O^K) \longmapsto n(O^G, O^K)
$$

is sometimes referred as the Corwin-Greenleaf multiplicity function. In the special case that $\tau = 1_K$, the formula for the multiplicity function $n(O^G, \{0\})$ is

$$
n(O^G, \{0\}) := \sharp \bigg( \bigg( O^G \cap \mathfrak{k}^\perp \bigg)/K \bigg),
$$

where $\mathfrak{k}^\perp := \text{pr}^{-1}(\{0\}) = \{\ell \in g^*; \ell(\mathfrak{k}) = 0\}$.

In the spirit of the orbit method due to Kirillov, R. Lipsman established a bijection between a class of coadjoint orbits of $G$ and the unitary dual $\hat{G}$ (see [13]). Given a linear form $\psi \in g^*$, we denote by $G(\psi)$ its stabilizer in $G$. Then $\psi$ is called admissible if there exists a unitary character $\chi$ of the identity component of $G(\psi)$ such that $d\chi = i\psi|_{g(\psi)}$. Let $g^\dagger$ be the set of all admissible linear forms on $g$. For $\psi \in g^\dagger$, one can construct an irreducible unitary representation $\pi_\psi$ by holomorphic induction. According to Lipsman [13], every irreducible unitary representation of $G$ arises in this manner. By observing that $\pi_\psi$ is equivalent to $\pi_{\psi'}$ if and only if $\psi$ and $\psi'$ lie in the same $G$-orbit, we get finally a bijection between the space $g^\dagger/G$ of admissible coadjoint orbits and $\widehat{G}$.

Let $\pi \in \hat{G}$ and $\tau \in \hat{K}$ correspond to admissible coadjoint orbits $O^G$ and $O^K$ respectively and let $\text{pr} : g^* \longrightarrow \mathfrak{k}^*$ be the restriction map. One expects that the multiplicity of $\tau$ in $\pi|_K$ is given by $\sharp \bigg( \bigg( O^G \cap \text{pr}^{-1}(O^K) \bigg)/K \bigg)$. Results in this direction have been established for compact extensions of $\mathbb{R}^n$ (see [1]). In this setting the Corwin-Greenleaf multiplicity function $n(O^G, O^K)$ may become greater than one, or even worse, may take infinity. For example, if $(K, \mathbb{H}_n)$ is a Gelfand pair then $n(O^G, \{0\}) = 1$, i.e., $O^G \cap \mathfrak{k}^\perp$ is a single $K$-orbit (see [3]).
**Question.** Give a sufficient condition on the admissible coadjoint orbit $O^G$ in $\mathfrak{g}^*$ in order that

$$n(O^G, O^K) \leq 1$$

for any admissible coadjoint orbit $O^K \subset \mathfrak{k}^*$. 

Our interest for this question is motivated by the formulation and the results by Kobayashi-Nasrin [9,16] which may be interpreted as the “classical limit” of the multiplicity-free theorems in the branching laws of semisimple Lie groups that were established in [10,11,12] by three different methods, explicit branching laws [10], the theory of visible actions [11], and Verma modules [12].

Let $H_n = \mathbb{C}^n \times \mathbb{R}, n \geq 1$, be the standard Heisenberg group of real dimension $2n+1$. The maximal compact subgroup of $Aut(H_n)$ is the unitary group $U(n)$, and it acts by $k.(z,t) = (kz,t)$. In this paper we consider the Lie group $G = K \ltimes H_n$, the semidirect product of the $K$ and $H_n$, where $K$ stands for a closed subgroup of $U(n)$ acting on $H_n$ as above. Our group $G$ is obviously a subgroup of the so-called Heisenberg motion group, which is the semidirect product $U(n) \ltimes H_n$.

The group $K$ acts on the unitary dual $\widehat{H_n}$ of $H_n$ via

$$k.\sigma = \sigma \circ k^{-1}$$

for $k \in K$ and $\sigma \in \widehat{H_n}$. Let $K_\sigma$ denote the stabilizer of $\sigma$ (up to unitary equivalence). Let $\pi$ be an irreducible unitary representation of $G$ associated to a given admissible coadjoint orbit $O$ in $\mathfrak{g}^*/G$. Mackey’s little group theory [14,15] tells us that $\pi$ is determined by a pair $(\sigma, \tau)$ where $\sigma \in \widehat{H_n}$ and $\tau \in K_\sigma$. We consider here the case where the representation $\pi$ is generic, i.e., $\pi$ has Mackey parameters $(\sigma, \tau)$ such that the stabilizer $K_\sigma$ is all of $K$. In this case we have

$$\pi(k,z,t) = \tau(k) \otimes \sigma(z,t) \circ W_\sigma(k),$$

$(k, z, t) \in G$, with $W_\sigma$ being a (non-projective) unitary representation of $K$ in the Hilbert space $H_\sigma$ of $\sigma$ that intertwines $k.\sigma$ with $\sigma$:

$$(k,\sigma)(z,t) = W_\sigma(k)^{-1} \circ \sigma(z,t) \circ W_\sigma(k)$$

for all $k \in K, (z,t) \in \mathbb{H}_n$. The main results of the present work are

**Theorem 1.** If $(K, \mathbb{H}_n)$ is a Gelfand pair and $U$ is a central element of $\mathfrak{t}$, then

$$n(O^G_{(U,0,z)}, O^K_{\lambda}) \leq 1$$

for any coadjoint orbit $O^K_{\lambda}$ in $\mathfrak{t}^*$.

**Theorem 2.** We have

$$m(\pi_{(\lambda,\alpha)}, \tau_\mu) \neq 0 \Rightarrow n(O^G_{(\lambda,\alpha)}, O^K_\mu) \neq 0.$$
The matrix $B_{\lambda,\mu}$ is defined in Section 4.3 p 11.

**Theorem 4.** Let $n \geq 2$. If the dominant weight $\lambda = (\lambda_1, ..., \lambda_n)$ of $K$ satisfies $\lambda_1 = ... = \lambda_n = a$ for some $a \in \mathbb{Z}$, then for any dominant weight $\mu$ of $K$ with $\mu \neq \lambda$ we have

$$n(O_{(\lambda,\alpha)}^G, O_{\mu}^K) \leq 1$$

Moreover, $n(O_{(\lambda,\alpha)}^G, O_{\mu}^K) \neq 0$ if and only if $\mu$ is of the form

**Case 1:** if $\alpha > 0$ then $\mu = (b, ..., b, a, ..., a) \in \mathbb{Z}^n$, $p + q = n$, $b \in \mathbb{Z}$ with $b > a$.

**Case 2:** if $\alpha < 0$ then $\mu = (a, ..., a, b, ..., b) \in \mathbb{Z}^n$, $p + q = n$, $b \in \mathbb{Z}$ with $a > b$.

Consequently, if $\mu_{n-1} \neq a$ and $n(O_{(\lambda,\alpha)}^G, O_{\mu}^K) \neq 0$ then $m(\pi_{(\lambda,\alpha)}, \tau_\mu) \neq n(O_{(\lambda,\alpha)}^G, O_{\mu}^K)$.

The paper is organized as follows. Section 2 introduces the coadjoint orbits of $K \ltimes \mathbb{H}_n$. In Sec. 3, we give two sufficient conditions on $O^G$ in order that $n(O^G, O^K) \leq 1$ for any $K$-coadjoint orbit $O^K \subset \mathfrak{t}^*$. Section 4.1 deals with the description of the generic unitary dual $U(n) \ltimes \mathbb{H}_n$ of $U(n) \ltimes \mathbb{H}_n$. Section 4.2 is devoted to the description of the subspace of generic admissible coadjoint orbits of $U(n) \ltimes \mathbb{H}_n$ and to the branching rules from $U(n) \ltimes \mathbb{H}_n$ to $U(n)$. In Sec. 4.3, the Corwin-Greenleaf multiplicity function for $U(n) \ltimes \mathbb{H}_n$ is studied in some situations and the main results of this work are derived.

## 2 Coadjoint orbits of $K \ltimes \mathbb{H}_n$

On the $n$-dimensional complex vector space $\mathbb{C}^n$, we fix the usual scalar product $(\cdot, \cdot)$. Let $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$ with group law

$$(z, t)(z', t') := (z + z', t + t' - \frac{1}{2}Im(z, z'))$$

denote the $(2n + 1)$-dimensional Heisenberg group. Let $K$ be a closed subgroup of $U(n)$. The group $K$ acts naturally on $\mathbb{H}_n$ by automorphisms, and then one can form the semidirect product $G = K \ltimes \mathbb{H}_n$. Let us denote by $(k, z, t)$ the elements of $G$ where $k \in K$ and $(z, t) \in \mathbb{H}_n$. The group law of $G$ is given by

$$(k, z, t) \cdot (k', z', t') = (kk', z + kz', t + t' - \frac{1}{2}Im(z, kz')).$$

We identify the Lie algebra $\mathfrak{h}_n$ of $\mathbb{H}_n$ with $\mathbb{H}_n$ via the exponential map. We also identify the Lie algebra $\mathfrak{k}$ of $K$ with its vector dual space $\mathfrak{t}^*$ through the $K$-invariant inner product

$$(A, B) = tr(AB).$$

For $z \in \mathbb{C}^n$ define the $\mathbb{R}$-linear form $z^*$ in $(\mathbb{C}^n)^*$ by

$$z^*(w) := Im(z, w).$$
One defines a map \( \times : \mathbb{C}^n \times \mathbb{C}^n \to \mathfrak{k}, (z, w) \mapsto z \times w \) by
\[
(z \times w, B) = z \times w(B) := w^*(Bz)
\]
with \( B \in \mathfrak{k} \). It is easy to verify that for \( k \in K \), one has
\[
\text{Ad}_K(k)(z \times w) = (kz \times kw).
\]
Each element \( \nu \in \mathfrak{g}^* = (\mathfrak{f} \ltimes \mathfrak{h}_n)^* \) can be identified with an element \((U, u, x) \in \mathfrak{f} \times \mathbb{C}^n \times \mathbb{R}\) such that
\[
\langle (U, u, x), (B, w, s) \rangle = \langle U, B \rangle + u^*(w) + xs,
\]
where \((B, w, s) \in \mathfrak{g} \). By a direct computation, one obtains that the coadjoint action of \( G \) is
\[
\text{Ad}^*_G(k, z, t)(U, u, x) = (\text{Ad}_K(k)U + z \times (ku) + \frac{x}{2} z \times z, ku + xz, x).
\]
Letting \( k \) and \( z \) vary over \( K \) and \( \mathbb{C}^n \) respectively, the coadjoint orbit \( O^G_{(U, u, x)} \) through the linear form \((U, u, x)\) can be written
\[
O^G_{(U, u, x)} = \left\{ (\text{Ad}_K(k)U + z \times (ku) + \frac{x}{2} z \times z, ku + xz, x); k \in K, z \in \mathbb{C}^n \right\}
\]
or equivalently, replacing \( z \) by \( kz \),
\[
O^G_{(U, u, x)} = \left\{ k \cdot (U + z \times u + \frac{x}{2} z \times z, u + xz, x); k \in K, z \in \mathbb{C}^n \right\}.
\]
**Remark** Here we regard \( z \) as a column vector \( z = (z_1, \ldots, z_n)^T \) and \( z^* := \overline{z}^T \). Then \( z \times u \in \mathfrak{u}^*(n) \cong \mathfrak{u}(n) \) is the \( n \) by \( n \) skew Hermitian matrix \( \frac{1}{2}(uz^* + zu^*) \). Indeed, for all \( B \in \mathfrak{u}(n) \) we compute
\[
(uz^* + zu^*, B) = \text{tr}((uz^* + zu^*)B) = \sum_{1 \leq i, j \leq n} B_{ij}z_i\overline{u}_j - \sum_{1 \leq i, j \leq n} u_i\overline{B}_{ij}\overline{z}_j = -2iz \times u(B).
\]
In particular, \( z \times z \) is the skew Hermitian matrix \( izz^* \) whose entries are determined by \((izz^*)_ij = iz_i\overline{z}_j \).

The \( G \)-coadjoint orbit arising from the initial point \((U, 0, x)(x \neq 0)\) is said to be generic. Notice that the space of generic coadjoint orbits of \( G \) is parametrized by the set \((\mathfrak{t}/K) \times (\mathbb{R} \setminus \{0\}) \). Concluding this section, let us underline that the union of all generic coadjoint orbits of \( G \) is dense in \( \mathfrak{g}^* \).
3 Corwin-Greenleaf multiplicity function for $K \ltimes \mathbb{H}_n$

We keep the notation of Sec. 2. Consider the generic coadjoint orbit $O^G(U,0,x)$ through the element $(U,0,x) \in g^*$. For $X \in \mathfrak{k}$, we introduce the set

$$F_X := \{ z \in \mathbb{C}^n; U + \frac{x}{2} z \times z \in O^K_X \}.$$ 

Here $O^K_X$ is the $K$-coadjoint orbit in $\mathfrak{k}^* \simeq \mathfrak{k}$ through $X$. Letting $H$ be the stabilizer of $U$ in $K$, we define an equivalence relation in $F_X$ by

$$z \sim w \iff \exists h \in H; w =hz.$$ 

The set of equivalence classes is denoted by $F_X/H$.

**Proposition 1** For any $X \in \mathfrak{k}$, we have

$$n(O^G(U,0,x),O^K_X) = \#(F_X/H).$$

**Proof.** Fix an element $X$ in $\mathfrak{k}$. For $z \in \mathbb{C}^n$, let us set

$$E_z := \{ k \cdot (U + \frac{x}{2} z \times z, xz, x); k \in K \}.$$ 

Observe that

$$E_z = E_w \iff z \sim w.$$ 

Since

$$O^G(U,0,x) \cap \text{pr}^{-1}(O^K_X) = \bigcup_{z \in F_X} E_z,$$

it follows that

$$n(O^G(U,0,x),O^K_X) = \#(O^G(U,0,x) \cap \text{pr}^{-1}(O^K_X))/K = \#(F_X/H).$$

This completes the proof of the proposition. \qed

Following [2], we define the moment map $\tau : \mathbb{C}^n \to \mathfrak{k}^*$ for the natural action of $K$ on $\mathbb{C}^n$ by

$$\tau(z)(A) = z^*(Az)$$

for $A \in \mathfrak{k}$. Since $\langle z, Az \rangle$ is pure imaginary, one can also write $\tau(z)(A) = \frac{1}{i} \langle z, Az \rangle$. The map $\tau$ is a key ingredient in the proof of the following result.

**Theorem 1** If $(K,\mathbb{H}_n)$ is a Gelfand pair and $U$ is a central element of $\mathfrak{k}$, then

$$n(O^G(U,0,x),O^K_X) \leq 1$$

for any coadjoint orbit $O^K_X$ in $\mathfrak{k}^*$. 

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Proof. Let $U$ be a central element of $\mathfrak{k}$. Then for any $X \in \mathfrak{k}$,
\[ n(\mathcal{O}^G_{(U,0,x)}, \mathcal{O}^K_X) = \sharp \left( \mathcal{F}_X / K \right). \]

Fix a non-zero element $X \in \mathfrak{k}$ and assume that the set $\mathcal{F}_X$ is not empty. It is clear that $\mathcal{F}_X$ is stable under the natural action of $K$ on $\mathbb{C}^n$. If $z$ and $w$ are two elements in $\mathcal{F}_X$, then there exists $k \in K$ such that
\[ w \times w = \text{Ad}_K(k)(z \times z). \]

Thus we get the equality $\mathcal{O}_X^K = \mathcal{O}_{w(z)}^K$. Since $(K, H^*_n)$ is a Gelfand pair, the moment map $\tau : \mathbb{C}^n \to \mathfrak{k}^*$ is injective on $K$-orbits [2]. That is, if $\mathcal{O}_X^K = \mathcal{O}_{w(z)}^K$, then $Kz = Kw$. We conclude that the $K$-action on $\mathcal{F}_X$ is transitive and hence
\[ n(\mathcal{O}^G_{(U,0,x)}, \mathcal{O}^K_X) = 1. \]

\[ \square \]

4 Corwin-Greenleaf multiplicity function for $U(n) \ltimes \mathbb{H}_n$

and branching rules

4.1 Generic unitary dual of $U(n) \ltimes \mathbb{H}_n$

In the sequel, we fix $K = U(n)$ with $n \geq 2$. Then $G = K \ltimes \mathbb{H}_n$ is the so-called Heisenberg motion group. The description of the unitary dual $\hat{G}$ of $G$ is based on the Mackey little group theory. In the present paper we consider only the generic irreducible unitary representation of $G$.

Let us recall a useful fact from the representation theory of the Heisenberg group (see, e.g., [5] for details). The infinite dimensional irreducible representations of $\mathbb{H}_n$ are parametrized by $\mathbb{R}^*$. For each $\alpha \in \mathbb{R}^*$, the Kirillov orbit $\mathcal{O}^n_{\alpha}$ of the irreducible representation $\sigma_\alpha$ is the hyperplane $\mathcal{O}^n_{\alpha} = \{(z, \alpha), z \in \mathbb{C}^n \}$. It is clear that for every $\alpha$ the coadjoint orbit $\mathcal{O}_\alpha$ is invariant under the $K$-action. Therefore $K$ preserves the equivalence class of $\sigma_\alpha$. The representation $\sigma_\alpha$ can be realized in the Fock space

\[ \mathcal{F}_\alpha(n) = \left\{ f : \mathbb{C}^n \to \mathbb{C} \text{ holomorphic} \mid \int_{\mathbb{C}^n} |f(w)|^2 e^{-\frac{1}{2}|w|^2} dw < \infty \right\} \]

as

\[ \sigma_\alpha(z, t)f(w) = e^{iat - \frac{1}{2}|z|^2 - \frac{1}{2}\langle w, z \rangle} f(w + z) \]

for $\alpha > 0$ and

\[ \sigma_\alpha(z, t)f(\overline{w}) = e^{iat + \frac{1}{2}|z|^2 + \frac{1}{2}\langle w, \overline{z} \rangle} f(\overline{w} + \overline{z}) \]

for $\alpha < 0$. We refer the reader to [5] or [6] for a discussion of the Fock space. For each $A \in K$, the operator $W_\alpha(A) : \mathcal{F}_\alpha(n) \to \mathcal{F}_\alpha(n)$ defined by

\[ W_\alpha(A)f(w) = f(A^{-1}w) \]
intertwines $\sigma_\alpha$ and $(\sigma_\alpha)_A$ given by $(\sigma_\alpha)_A(z,t) := \sigma_\alpha(Az,t)$. Observe that $W_\alpha$ is a unitary representation of $K$ in the Fock space $F_\alpha(n)$.

As usual, the dominant weights of $K = U(n)$ are parametrized by sequences $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Denote by $(\tau_{\lambda}, H_{\lambda})$ an irreducible unitary representation of $K$ with highest weight $\lambda$. Then by Mackey \cite{15}, for each nonzero $\alpha \in \mathbb{R}$

$$
\pi_{(\lambda,\alpha)}(A,z,t) := \tau_{\lambda}(A) \otimes \sigma_\alpha(z,t) \circ W_\alpha(A), \quad (A,z,t) \in G,
$$

is an irreducible unitary representation of $G$ realized in $H_{\lambda} \otimes F_\alpha(n)$. This representation $\pi_{(\lambda,\alpha)}$ is said to be generic. The set of all equivalence classes of generic irreducible unitary representations of $G$, denoted by $\hat{G}_{gen}$, is called the generic unitary dual of $G$. Notice that $\hat{G}_{gen}$ has full Plancherel measure in the unitary dual $\hat{G}$ (see \cite{8}).

### 4.2 Generic admissible coadjoint orbits of $U(n) \ltimes \mathbb{H}_n$ and Branching rules

We shall freely use the notation of the previous subsection. Given a dominant weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ of $K$, we associate to $\pi_{(\lambda,\alpha)}$ the linear form $\ell_{\lambda,\alpha} = (U_\lambda, 0, \alpha)$ in $\mathfrak{g}^*$ where

$$
U_\lambda = \begin{pmatrix}
  i\lambda_1 & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & i\lambda_n
\end{pmatrix}.
$$

Observe that $\ell_{\lambda,\alpha}$ is an admissible linear form on $\mathfrak{g}$. Denote by $G(\ell_{\lambda,\alpha})$, $K(\ell_{\lambda,\alpha})$ and $\mathbb{H}_n(\ell_{\lambda,\alpha})$ the stabilizers of $\ell_{\lambda,\alpha}$ respectively in $G$, $K$ and $\mathbb{H}_n$. We have

$$
G(\ell_{\lambda,\alpha}) = \{ (A,z,t) \in G; (AU_\lambda A^* + \frac{\alpha}{2} z \times z, \alpha z, \alpha) = (U_\lambda, 0, \alpha) \}
$$

$$
= \{ (A,0,0) \in G; AU_\lambda A^* = U_\lambda \},
$$

$$
K(\ell_{\lambda,\alpha}) = \{ A \in K; (AU_\lambda A^* + \frac{\alpha}{2} z \times z, \alpha z, \alpha) = (U_\lambda, 0, \alpha) \}
$$

$$
= \{ A \in K; AU_\lambda A^* = U_\lambda \},
$$

$$
\mathbb{H}_n(\ell_{\lambda,\alpha}) = \{ (z,t) \in \mathbb{H}_n; (U_\lambda + \frac{\alpha}{2} z \times z, \alpha z, \alpha) = (U_\lambda, 0, \alpha) \}
$$

$$
= \{ 0 \} \times \mathbb{R}.
$$

It follows that $G(\ell_{\lambda,\alpha}) = K(\ell_{\lambda,\alpha}) \ltimes \mathbb{H}_n(\ell_{\lambda,\alpha})$. According to Lipsman \cite{13}, the representation $\pi_{(\lambda,\alpha)}$ is equivalent to the representation of $G$ obtained by holomorphic induction from the linear form $\ell_{\lambda,\alpha}$. Now, for an irreducible unitary representation $\tau_\mu$ of $K$ with highest weight $\mu$, we take the linear functional $\ell_\mu := (U_\mu, 0, 0)$ of $\mathfrak{g}^*$ where

$$
U_\mu = \begin{pmatrix}
  i\mu_1 & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & i\mu_n
\end{pmatrix}.
$$
which is clearly aligned and admissible. Hence, the representation of $G$ obtained by holomorphic induction from the linear functional $\ell_\mu$ is equivalent to the representation $\tau_\mu$. We denote by $O^{G}_\mu$ the coadjoint orbit of $\ell_\mu$ and by $O^{G}_{(\lambda, \alpha)}$ the coadjoint orbit associated to the linear form $\ell_{\lambda, \alpha}$. Let $g^+$ be the set of all admissible linear forms of $G$. The orbit space $g^+/G$ is called the space of admissible coadjoint orbits of $G$. The set of all coadjoint orbits $O^{G}_{\lambda, \alpha}$ turns out to be the subspace of generic admissible coadjoint orbits of $G$.

Let $\tau_\lambda$ be an irreducible unitary representation of the unitary group $K = U(n)$ with highest weight $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{Z}^n$. Recall that the irreducible representations of $G = K \ltimes \mathbb{H}_n$ that come from an infinite dimensional irreducible representation $\sigma_\alpha \in \widehat{\mathbb{H}_n}$, $\alpha \in \mathbb{R}^*$, are of the form $\pi(\lambda, \alpha)$ with

$$
\pi(\lambda, \alpha)(A, z, t) = \tau_\lambda(A) \otimes \sigma_\alpha(z, t) \circ W_\alpha(A)
$$

for $(A, z, t) \in G$. Here $W_\alpha$ denotes the natural representation of $K$ on the ring $\mathbb{C}[z_1, ..., z_n]$ of holomorphic polynomials on $\mathbb{C}^n$, given by

$$(A.p)((z_1, ..., z_n)^T) = p(A^{-1}(z_1, ..., z_n)^T).$$

The space $\mathbb{C}[z_1, ..., z_n]$ decomposes under the action of $K$ as

$$
\mathbb{C}[z_1, ..., z_n] = \bigoplus_{k=0}^{\infty} \mathbb{C}_k[z_1, ..., z_n]
$$

where $\mathbb{C}_k[z_1, ..., z_n]$ denotes the space of homogeneous polynomials of degree $k$, thus we have $W_\alpha = \bigoplus_{k \in \mathbb{N}} \tau_{\alpha, k}$ where $\tau_{\alpha, k}$ is the representation of $K$ on $\mathbb{C}_k[z_1, ..., z_n]$.

Consider now an irreducible unitary representation $\tau_\mu$ of $K$ with highest weight $\mu$. The multiplicity of $\tau_\mu$ in the representation $\pi(\lambda, \alpha)$ is given by

$$
m(\pi(\lambda, \alpha), \tau_\mu) = \text{mult}(\pi(\lambda, \alpha)|_{K}, \tau_\mu) = \text{mult}(\tau_\lambda \otimes W_\alpha, \tau_\mu) = \text{mult}(\bigoplus_{k \in \mathbb{N}} \tau_\lambda \otimes \tau_{\alpha, k}, \tau_\mu).
$$

### 4.3 Corwin-Greenleaf multiplicity function for $U(n) \ltimes \mathbb{H}_n$

We continue to use the notation of the previous sections. Fix $\alpha$ a nonzero real. Let $\pi(\lambda, \alpha) \in \hat{G}$ and $\tau_\mu \in \hat{K}$ be as before. To these unitary representations, we attach respectively the generic coadjoint orbit $O^{G}_{(\lambda, \alpha)}$ and the coadjoint orbit $O^{K}_{\mu}$. Here $O^{K}_{\mu}$ is the orbit in $\mathfrak{k}^*$ through $U_\mu$, i.e., $O^{K}_{\mu} = \text{Ad}^*_K(K)U_\mu$. Now, we turn our attention to the multiplicity $m(\pi(\lambda, \alpha), \tau_\mu)$ of $\tau_\mu$ in the restriction of $\pi(\lambda, \alpha)$ to $K$, we shall prove the following result:
Theorem 2. We have

\[ m(\pi_{(\lambda,\alpha)}, \tau_{\mu}) \neq 0 \Rightarrow n(O^G_{(\lambda,\alpha)}, O^K_{\mu}) \neq 0. \]

Proof. Denote by \( \tau_{\alpha,k} = \tau_{(0,\ldots,0,-k)} \) the irreducible representation of \( K \) on \( \mathbb{C}_k[z_1,\ldots,z_n] \) with highest weight \( (0,\ldots,0,-k) \in \mathbb{Z}^n \). Then, we have

\[
\pi_{(\lambda,\alpha)}|_K = \tau_{\lambda} \otimes W_{\alpha} = \tau_{\lambda} \otimes \bigoplus_{k \in \mathbb{N}} \tau_{(0,\ldots,0,-k)} = \bigoplus_{k \in \mathbb{N}} \tau_{\lambda} \otimes \tau_{(0,\ldots,0,-k)}.
\]

Consider again the set \( F_{\mu} = \{ z \in \mathbb{C}^n; U_{\lambda} + \frac{\alpha}{2} z \times z \in O^K_{\mu} \} \). Now, assume that \( m(\pi_{(\lambda,\alpha)}, \tau_{\mu}) \neq 0 \). Then there exists \( k \in \mathbb{N} \) such that

\[ \tau_{\mu} \subset \tau_{\lambda} \otimes \tau_{(0,\ldots,0,-k)} \]

hence

\[ O_{\mu} \subset O_{\lambda} + O_{(0,\ldots,0,-k)} \]

So, there exists \( C \in U(n) \) such that

\[ U_{\lambda} + C U_{(0,\ldots,0,-k)} C^{-1} \in O_{\mu} \]

Let \( z = C(0,\ldots,0,r)^t \) with

\[
r = \begin{cases} 
    i \sqrt{\frac{2k}{\alpha}} & \text{if } \alpha > 0, \\
    \sqrt{\frac{-2k}{\alpha}} & \text{if } \alpha < 0.
\end{cases}
\]

Therefore, we have \( \frac{\alpha}{2} z \times z = C U_{(0,\ldots,0,-k)} C^{-1} \). It follows that \( F_{\mu} \neq \emptyset \), and then \( n(O^G_{(\lambda,\alpha)}, O^K_{\mu}) \neq 0 \). \( \Box \)

The converse of this theorem is false in general if we take for example \( \lambda = (-1,\ldots,-1) \) and \( \mu = (0,\ldots,0,-1) \) we will see in the last theorem that \( n(O^G_{(\lambda,\alpha)}, O^K_{\mu}) \neq 0 \) (see Theorem 4) but

\[
\tau_{\lambda} \otimes W_{\alpha} = \bigoplus_{k \in \mathbb{N}} \tau_{\lambda} \otimes \tau_{(0,\ldots,0,-k)} = \bigoplus_{k \in \mathbb{N}} \tau_{(-1,\ldots,-1,-1-k)}.
\]

Therefore \( \tau_{\mu} = \tau_{(0,\ldots,0,-1)} \notin \tau_{\lambda} \otimes W_{\alpha} \) and then \( m(\pi_{(\lambda,\alpha)}, \tau_{\mu}) = 0. \)
In the remainder of this paper, we give two situations where the Corwin-Greenleaf multiplicity function is less than one and discuss the relationship between \( n(O^G_{(\lambda,\alpha)}, O^K_\mu) \) and \( m(\pi_{(\lambda,\alpha)}, \tau_\mu) \). For some particular dominant weight \( \mu \), we shall prove in the first situation that \( m(\pi_{(\lambda,\alpha)}, \tau_\mu) \) coincides with \( n(O^G_{(\lambda,\alpha)}, O^K_\mu) \), but in the second situation we have \( m(\pi_{(\lambda,\alpha)}, \tau_\mu) \neq n(O^G_{(\lambda,\alpha)}, O^K_\mu) \).

Let us first fix some notation that we will use later. Let \( \lambda = (\lambda_1, \ldots, \lambda_n), \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n \) such that \( \lambda_1 \geq \ldots \geq \lambda_n \) and \( \mu_1 \geq \ldots \geq \mu_n \). To these dominant weights of \( K \) we attach the matrix \( B_{\lambda,\mu} \) and the vector \( V_{\lambda,\mu} \) defined as follows

\[
B_{\lambda,\mu} = \left( \prod_{k=1, k \neq j}^{n} (\mu_i - \lambda_k) \right)_{1 \leq i, j \leq n}
\quad \text{and} \quad
V_{\lambda,\mu} = \left( \prod_{k=1}^{n} (\mu_1 - \lambda_k), \ldots, \prod_{k=1}^{n} (\mu_n - \lambda_k) \right)^T.
\]

Now, we are in position to prove

**Theorem 3** Let \( n \geq 2 \). Assume that \( \lambda \) is strongly dominant weight of \( K \). Then for any dominant weight \( \mu \) of \( K \) such that \( B_{\lambda,\mu} \) is invertible we have

\[
n(O^G_{(\lambda,\alpha)}, O^K_\mu) \leq 1.
\]

**Proof.** Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \) be a strongly dominant weight of \( K \). We shall denote by \( H_\lambda \) the stabiliser of \( U_\lambda \) in \( K \). Assume that \( n(O^G_{(\lambda,\alpha)}, O^K_\mu) \neq 0 \) for some dominant weight \( \mu \) of \( K \). Then there exists \( z \in \mathbb{C}^n \) such that \( U_\lambda + \frac{\alpha}{2} z \times z = AU_\mu A^* \) for some \( A \in K \). For all \( x \in \mathbb{R} \), we have

\[
det(U_\lambda + \frac{\alpha}{2} z \times z - ixI) = (-i)^n P(x)
\]

where \( P \) is the unitary polynomial of degree \( n \) given by

\[
P(x) = \prod_{i=1}^{n} (x - \lambda_i) \quad \text{and} \quad \sum_{j=1}^{\alpha} \prod_{i=1, i \neq j}^{n} (x - \lambda_i) |z_j|^2.
\]

Therefore we have \( P(\mu_k) = 0 \) for \( k = 1, \ldots, n \). It follows that

\[
V_{\lambda,\mu} = \frac{\alpha}{2} B_{\lambda,\mu} (|z_1|^2, \ldots, |z_n|^2)^T
\]

Consider again the set \( \mathcal{F}_\mu = \{ z \in \mathbb{C}^n, U_\lambda + \frac{\alpha}{2} z \times z \in O^K_\mu \} \). Hence

\[
\mathcal{F}_\mu = \left\{ z \in \mathbb{C}^n, (|z_1|^2, \ldots, |z_n|^2)^T = \frac{2}{\alpha} B^{-1}_{\lambda,\mu} V_{\lambda,\mu} \right\}.
\]

Since \( H_\lambda = \mathbb{T}^n \) the \( n \)-dimensional torus, we conclude that \( n(O^G_{(\lambda,\alpha)}, O^K_\mu) = 1 \). \( \square \)
Corollary 1 Let $n \geq 2$. Assume that $\lambda = (\lambda_1, \ldots, \lambda_n)$ is strongly dominant weight of $K$ and $\mu = (\lambda_1, \ldots, \lambda_{n-1}, \lambda_n - k)$ for some $k \in \mathbb{N}$. Then we have

$$m(\pi(\lambda, \alpha), \tau_\mu) = n(O^{G}_{(\lambda, \alpha)}, O^{K}_{\mu})$$

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a strongly dominant weight of $K$. Suppose that $\mu = (\lambda_1, \ldots, \lambda_{n-1}, \lambda_n - k)$ for some $k \in \mathbb{N}$, then $B_{\lambda, \mu}$ is invertible, therefore $n(O^{G}_{(\lambda, \alpha)}, O^{K}_{\mu}) \leq 1$. Since $\pi(\lambda, \alpha)|_K = \bigoplus_{k \in \mathbb{N}} \tau(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n - k)$ then $m(\pi(\lambda, \alpha), \tau_\mu) = 1$ and by the theorem 2 we deduce that

$$m(\pi(\lambda, \alpha), \tau_\mu) = n(O^{G}_{(\lambda, \alpha)}, O^{K}_{\mu}).$$

Concluding this section, let us prove the following result:

Theorem 4 Let $n \geq 2$. If the dominant weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ of $K$ satisfies $\lambda_1 = \ldots = \lambda_n = a$ for some $a \in \mathbb{Z}$, then for any dominant weight $\mu$ of $K$ with $\mu \neq \lambda$ we have

$$n(O^{G}_{(\lambda, \alpha)}, O^{K}_{\mu}) \leq 1$$

Moreover, $n(O^{G}_{(\lambda, \alpha)}, O^{K}_{\mu}) \neq 0$ if and only if $\mu$ is of the form

**Case 1:** if $\alpha > 0$ then $\mu = (b, \ldots, b, a, \ldots, a) \in \mathbb{Z}^n$, $p + q = n$, $b \in \mathbb{Z}$ with $b > a$.

**Case 2:** if $\alpha < 0$ then $\mu = (a, \ldots, a, b, \ldots, b) \in \mathbb{Z}^n$, $p + q = n$, $b \in \mathbb{Z}$ with $a > b$.

Consequently, if $\mu_{n-1} \neq a$ and $n(O^{G}_{(\lambda, \alpha)}, O^{K}_{\mu}) \neq 0$ then $m(\pi(\lambda, \alpha), \tau_\mu) \neq n(O^{G}_{(\lambda, \alpha)}, O^{K}_{\mu})$.

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a dominant weight of $K$ such that $\lambda_1 = \ldots = \lambda_n = a$ with $a \in \mathbb{Z}$. Assume that $n(O^{G}_{(\lambda, \alpha)}, O^{K}_{\mu}) \neq 0$ for some dominant weight $\mu$ of $K$. Then there exists $z \in \mathbb{C}^n$ such that $U_\lambda + \frac{\alpha}{2} z \times z = A U_\mu A^*$ for some $A \in K$. For all $x \in \mathbb{R}$, we have

$$det(U_\lambda + \frac{\alpha}{2} z \times z - i x I) = (-i)^n P(x)$$

with

$$P(x) = (x - a)^{n-1} \left( x - a - \frac{\alpha}{2} \sum_{j=1}^{n} |z_j|^2 \right).$$

Then we have $P(\mu_k) = 0$ for $k = 1, \ldots, n$. It follows that

$$\begin{cases} 
\mu_k = a \\
\text{or} \\
\mu_k \neq a \text{ and } \mu_k = a + \frac{\alpha}{2} \sum_{j=1}^{n} |z_j|^2.
\end{cases}$$
Since $\mu \neq \lambda$ then there exists $1 \leq k \leq n$ such that $\mu_k \neq a$.

**Case $\alpha > 0$:** Let $p = \max\{1 \leq k \leq n, \mu_k \neq a\}$ then

$$\mu_p = a + \frac{\alpha}{2} \sum_{j=1}^{n} |z_j|^2 > a.$$  

Since $\mu_1 \geq \ldots \geq \mu_p \geq \ldots \geq \mu_n$, we obtain

$$\mu = (b, \ldots, b, a, \ldots, a)$$  

with $b = a + \frac{\alpha}{2} \sum_{j=1}^{n} |z_j|^2$.

Consider again the set $\mathcal{F}_\mu = \left\{ z \in \mathbb{C}^n, U_\lambda + \frac{\alpha}{2} z \times z \in \mathcal{O}_\mu^K \right\}$ then

$$\mathcal{F}_\mu = \left\{ z \in \mathbb{C}^n, \sum_{j=1}^{n} |z_j|^2 = (b - a) \frac{2}{\alpha} \right\}.$$

Since $H_\lambda = K$ we can deduce that $n(\mathcal{O}_{(\lambda,\alpha)}^G, \mathcal{O}_\mu^K) = 1$.

**Case $\alpha < 0$:** Let $l = \min\{1 \leq k \leq n, \mu_k \neq a\}$ then

$$\mu_l = a + \frac{\alpha}{2} \sum_{j=1}^{n} |z_j|^2 < a.$$  

Hence

$$\mu = (a, \ldots, a, b, \ldots, b)$$  

with $b = a + \frac{\alpha}{2} \sum_{j=1}^{n} |z_j|^2$, $p = l - 1$

and so $n(\mathcal{O}_{(\lambda,\alpha)}^G, \mathcal{O}_\mu^K) = 1$.

Now, Suppose that $\mu_{n-1} \neq a$, if $\alpha > 0$ we get $\mu = (b, \ldots, b, a) \in \mathbb{Z}^n$ with $b > a$ and if $\alpha < 0$, $\mu = (a, \ldots, a, b, \ldots, b) \in \mathbb{Z}^n$ with $a > b$ and $q \geq 2$. Since $\pi_{(\lambda,\alpha)}|_K = \bigoplus_{k \in \mathbb{N}} \tau(a, \ldots, a, a - k)$ then $m(\pi_{(\lambda,\alpha)}, \tau_\mu) = 0$ and hence $m(\pi_{(\lambda,\alpha)}, \tau_\mu) \neq n(\mathcal{O}_{(\lambda,\alpha)}^G, \mathcal{O}_\mu^K)$.

This completes the proof of the theorem.  

□

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