KOSZULITY OF SPLITTING ALGEBRAS ASSOCIATED WITH CELL COMPLEXES

VLADIMIR RETAKH, SHIRLEI SERCONEK, AND ROBERT LEE WILSON

Abstract. We associate to a good cell decomposition of a manifold \( M \) a quadratic algebra and show that the Koszulity of the algebra implies a restriction on the Euler characteristic of \( M \). For a two-dimensional manifold \( M \) the algebra is Koszul if and only if the Euler characteristic of \( M \) is two.

0. INTRODUCTION

Let \( \Gamma = (V, E) \) be a layered graph (where \( V \) is the set of vertices and \( E \) is the set of edges). One may define an associated algebra, \( A(\Gamma) \), to be the algebra generated by \( E \) subject to the relations which state that

\[
(t - e_1)(t - e_2)\ldots(t - e_k) = (t - f_1)(t - f_2)\ldots(t - f_k)
\]

whenever the sequences of edges \( e_1, e_2, \ldots, e_k \) and \( f_1, f_2, \ldots, f_k \) define directed paths with the same origin and the same end and \( t \) is an independent central variable. As this algebra records information about the factorization of polynomials associated to \( \Gamma \) into linear factors, we call \( A(\Gamma) \) the splitting algebra associated to \( \Gamma \).

These algebras have been introduced and studied in \([5, 3, 8, 9, 10, 11]\). The algebra \( A(\Gamma) \) is defined by a set of homogeneous relations. For certain graphs \( \Gamma \) (i.e., uniform graphs as defined in \([8]\)) these relations are consequences of a family of quadratic relations and so the splitting algebra \( A(\Gamma) \) possesses a quadratic dual algebra \( A(\Gamma)' \). In \([8]\) we claimed that any algebra \( A(\Gamma) \) defined by a uniform layered graph is Koszul. Later, T. Cassidy and B. Shelton constructed a counterexample to this statement and this example forced us to rethink the situation. As a result, we discovered a large class of non-Koszul algebras \( A(\Gamma) \) where \( \Gamma \) is a graph defined by a regular cell complex \( C \). Indeed, in Sections 4 and 5 we show that, if \( \Gamma \) is the layered graph defined by a “good” cell decomposition of a two-dimensional manifold \( M \), then \( A(\Gamma) \) is Koszul if and only if the Euler characteristic of \( M \) is two.

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The analysis of [8] does provide a sufficient condition for \( A(\Gamma) \) to be Koszul. We discuss this in Section 6 and use this sufficient condition to show that if \( \Gamma \) is either a complete layered graph or the graph associated to an abstract simplicial complex then \( A(\Gamma) \) is Koszul.

Actually, any cell subdivision \( D \) of a manifold \( M \) defines two graphs: \( \Gamma_D \) with vertices \( \sigma \in D \cup \{\emptyset, M\} \) and the subgraph \( \Gamma'_D \) with vertices \( \sigma \in D \cup \{\emptyset\} \).

The properties of \( A(\Gamma_D) \) and \( A(\Gamma'_D) \) are quite different. We prove that \( A(\Gamma'_D) \) is Koszul for any simplicial complex \( D \) but as mentioned before the Koszulity of \( A(\Gamma_D) \) imposes a strong topological condition on \( M \).

We also consider a condition which is weaker than Koszulity. We call a quadratic algebra \( A \) numerically Koszul if the Hilbert series of \( H(A, z) \) and \( H(A^!, z) \) are related by the famous formula \( H(A, z)H(A^!, -z) = 1 \). This is not a common terminology; sometimes algebras with similar properties are called quasi-Koszul (see [6]). Koszul algebras are numerically Koszul but the converse is not true (see [12, 7]). We prove (Section 3), however, that if the height of \( \Gamma \) is less or equal to 4 then the numerical Koszulity of \( A(\Gamma) \) implies its Koszulity. We use this in proving the results of Section 5.

In our discussions of numerical Koszulity we need efficient techniques for computing the Hilbert series of \( A(\Gamma) \) and \( A(\Gamma^!) \). We give such a technique (Lemma 1.3) for \( A(\Gamma) \) in Section 1 and use this result to show that the example of Cassidy and Shelton is not numerically Koszul. Section 2 is devoted to results about the Hilbert series of \( A(\Gamma^!) \).

We are grateful to T. Cassidy and B. Shelton for providing us with their counter-example.

1. SPLITTING ALGEBRAS ASSOCIATED WITH LAYERED GRAPHS

Recall the definition of a layered graph. Let \( \Gamma = (V, E) \) be a directed graph. That is, \( V \) is a set (of vertices), \( E \) is a set (of edges), and \( t : E \rightarrow V \) and \( h : E \rightarrow V \) are functions. \( t(e) \) is the tail of \( e \) and \( h(e) \) is the head of \( e \).

A vertex \( u \in V \) is called maximal if there is no \( e \in E \) such that \( h(e) = u \). A vertex \( v \in V \) is called minimal if there is no \( e \in E \) such that \( t(e) = v \).

We say that \( \Gamma \) is layered if \( V \) is the disjoint union of \( V_i, 0 \leq i \leq n, E \) is the disjoint union of \( E_i, 1 \leq i \leq n, t : E_i \rightarrow V_i, h : E_i \rightarrow V_{i-1} \). We will write \( |v| = i \) if \( v \in V_i \). In this case the number \( i \) is called the level or the rank of \( v \). Note that a layered graph has no loops. There is an
obvious bijection between layered graphs and ranked partially ordered sets.

If \( v, w \in V \), a path of the length \( k \) from \( v \) to \( w \) is a sequence of edges \( \pi = \{e_1, e_2, \ldots, e_k\} \) with \( t(e_1) = v, h(e_k) = w \) and \( t(e_{i+1}) = h(e_i) \) for \( 1 \leq i < k \). We write \( v = t(\pi), w = h(\pi) \) and call \( v \) the tail of the path and \( w \) the head of the path. We also write \( v > w \) if there is a path from \( v \) to \( w \).

Recall the definition of a uniform layered graph. Let \( \Gamma \) be a layered graph. For \( v \in V \) define \( S_-(v) \) to be the set of all vertices \( w \in V \) covered by \( v \), i.e. such that there exists an edge with the tail \( v \) and the head \( w \). For \( v \in V_j, j \geq 2 \), let \( \sim \) denote the equivalence relation on \( S_-(v) \) generated by \( u \sim w \) if \( S_-(u) \cap S_-(w) \neq \emptyset \).

**Definition 1.1.** The layered graph \( \Gamma = (V, E) \) is said to be uniform if, for every \( v \in V_j, j \geq 2 \), all elements of \( S_-(v) \) are equivalent under \( \sim \).

In [4] for each layered graph \( \Gamma = (V, E) \) we constructed an associative algebra generated by the edges of \( \Gamma \). Let \( T(E) \) denote the free associative algebra on \( E \) over a field \( K \). We will do this by equating coefficients of polynomials associated with pairs of paths with the same origin and the same end. For a path \( \pi = \{e_1, e_2, \ldots, e_m\} \) define

\[
P_{\pi}(t) = (t - e_1) \ldots (t - e_m) \in T(E)[t].
\]

Note that \( P_{\pi_1 \pi_2}(t) = P_{\pi_1}(t)P_{\pi_2}(t) \) if \( h(\pi_1) = t(\pi_2) \). Write

\[
P_{\pi}(t) = \sum_{j=0}^{m} (-1)^{m-j} e(\pi, j) t^j.
\]

**Definition 1.2.** Let \( R_E \) be the linear space in \( T(E) \) spanned by

\[
\{e(\pi_1, k) - e(\pi_2, k) \mid k \geq 1, t(\pi_1) = t(\pi_2), h(\pi_1) = h(\pi_2)\}.
\]

Denote by \( < R_E > \) the ideal generated by \( R_E \) and set

\[
A(\Gamma) = T(E)/ < R_E >.
\]

Suppose that a layered graph \( \Gamma = (V, E) \) of the height \( n \) has a unique maximal vertex \( M \in V_n \) and a unique minimal vertex \( * \in V_0 \). To any path \( \pi \) from \( M \) to \( * \) there corresponds a monic polynomial \( P_{\pi} \) of degree \( n \). Then the image \( P_{\Gamma}(t) \in A(\Gamma)[t] \) of the polynomial \( P_{\pi}(t) \) does not depend on the choice of \( \pi \) and any path from \( M \) to \( * \) corresponds a factorization of \( P_{\Gamma}(t) \) into a product of linear terms.
Therefore, we call \( A(\Gamma) \) a splitting algebra of \( P_1(t) \). There is an injection of the group of level preserving automorphisms of \( \Gamma \) into the group of automorphisms of \( A(\Gamma) \).

From now on we suppose that \( \Gamma = (V,E) \) has a unique minimal vertex \( \ast \), i.e. \( V_0 = \{ \ast \} \). It was shown in [8] that if \( \Gamma \) is uniform then \( A(\Gamma) \) is quadratic. In fact, this algebra \( A(\Gamma) \) can be defined as the algebra generated by the linear space \( KV \) subject to certain relations \( R_V \subset V \otimes V \) (see [?] ). So the dual algebra \( A(\Gamma) ! \) is defined as a quadratic algebra with the space of generators \( V' \) and the space of relations \( R_V^\perp \subset V' \otimes V' \) where \( V' \) is dual to \( V \) and \( R_V^\perp \) is the annihilator of \( R_V \).

Recall that for any graded algebra \( B = \oplus_{n \geq 0} B_n \) with finite dimensional \( B_n \) one can define its Hilbert series (graded dimension) as \( H(B,z) = \sum \dim B_n z^n \). Therefore, Hilbert series can be defined for a quadratic algebra \( A = T(V)/ < R_V > \) and for its dual algebra \( A' \). The Hilbert series for splitting algebras associated with layered graphs were constructed in [10] (see [11] for a more general result).

Let \( \Gamma \) be a layered graph. We will write \( H(A(\Gamma),z) \) in a form which is convenient for computation. Let

\[
s_{a,b} = \sum_{v_1 > \ldots > v_l > \ast, |v_1| = a, |v_l| = b} (-1)^l.
\]

Lemma 1.3. \( H(A(\Gamma), z)^{-1} = 1 + \sum_{i \geq 0} (\sum_{a \geq i} a - b + 1) s_{a,b} z^i \).

Proof.

By [5] we have

\[
(H(A(\Gamma), z))^{-1} = \frac{1 - z + \sum_{v_1 > \ldots > v_l > \ast} (-1)^l \left(z^{|v_1| - |v_l| + 1} - z^{|v_1| + 1}\right)}{1 - z}.
\]

As \( |\ast| = 0 \), this is equal to

\[
1 + \sum_{v_1 > \ldots > v_l > \ast} (-1)^l z^{|v_1| - |v_l| + 1} \frac{1 - z^{|v_1|}}{1 - z} = 1 + \sum_{v_1 > \ldots > v_l > \ast} (-1)^l z^{|v_1| - |v_l| + 1} (1 + \ldots + z^{|v_l| - 1}) = 1 + \sum_{v_1 > \ldots > v_l > \ast} (-1)^l \left(z^{|v_1| - |v_l| + 1} + \ldots + z^{|v_l|}\right) = 1 + \sum_{i \geq 0} \sum_{a \geq i} a - b + 1 s_{a,b} z^i.
\]
We may apply this result to compute $H(A(\Gamma), z)^{-1}$ where $\Gamma$ is the graph occurring in Cassidy and Shelton’s example (Fig 1). The graph can be described as $\Gamma = (V, E)$ where $V = \bigcup_{i=0}^{4} V_i$ with $V_4 = \{u\}$, $V_3 = \{w_1, w_2, w_3\}$, $V_2 = \{x_1, x_2, x_3\}$, $V_1 = \{y_1, y_2, y_3\}$, $V_0 = \{\ast\}$ and there are edges from $u$ to $w_i$ for $i = 1, 2, 3$, from $w_i$ to $x_j$ if $i \neq j$, from $x_i$ to $y_j$ if $i \neq j$, and from $y_i$ to $\ast$ for $i = 1, 2, 3$.

Then one sees that:

- $s_{4,4} = -1, s_{4,3} = 3, s_{4,2} = -3, s_{4,1} = 0$,
- $s_{3,3} = -3, s_{3,2} = 6, s_{3,1} = -3$,
- $s_{2,2} = -3, s_{2,1} = 6$

and

- $s_{1,1} = -3$.

Thus

$H(A(\Gamma), z)^{-1} = -z^4 - z^3 + 8z^2 - 10z + 1$. 
Since $H(A(\Gamma), -z)^{-1}$ has a negative coefficient, it cannot be a Hilbert polynomial. Thus we recover Cassidy and Shelton’s result (which they proved by homological methods) that $A(\Gamma)$ is not Koszul.

2. HILBERT SERIES OF $A(\Gamma)^!$

We begin by noting the following description, which is immediate from the definition (cf [13]) of the Hilbert series of the dual $A^!$ of the quadratic algebra $A = T(V)/(< R_V >), R_V \subseteq V \otimes V$. To simplify notation we will write $R$ in place of $R_V$. Write $V^k$ for the $k$-th tensor power of $V$ and define

$$R^{(0)} = K, R^{(1)} = V, R^{(2)} = R$$

and

$$R^{(k+1)} = (V \otimes R^{(k)}) \cap (R \otimes V^{k-1})$$

for $k \geq 2$.

**Proposition 2.1.** If $A = T(V)/(< R >)$, then $A^j$ and $R^{(j)}$ are isomorphic vector spaces. Consequently

$$H(A^!, z) = \sum_{j=0}^{\infty} \dim(R^{(j)}) z^j.$$

We will now investigate the $R^{(j)}$ that occur for the algebras $A(\Gamma)$.

Let $\Gamma = (V, E)$ be a layered graph, $V = \sum_{i=0}^{n} V_i$. Assume $v \in V_j, 1 \leq j \leq n$ and let $l$ satisfy $1 \leq l \leq j$. Denote by $C(v, l)$ the set

$$\{v \otimes v(1) \otimes ... \otimes v(l-1)|v(m) \in V_{j-m}, v(m) > v(m+1)\} \subseteq T(V).$$

We call the elements of $C(v, l)$ linked monomials of length $l$ starting at $v$. Also, if $w \in V_{j-l}$ define

$$C(v, w) = \{v \otimes v(1) \otimes ... \otimes v(l-1) \otimes w \in C(v, l+1)\}$$

and call elements of $C(v, w)$ linked monomials from $v$ to $w$.

We say that a linked monomial $v \otimes v(1) \otimes ... \otimes v(l-1) \in C(v, l)$ is an admissible monomial if, for each $p, 1 \leq p \leq l - 1$ there is a linked monomial

$$v \otimes w(1) \otimes ... \otimes w(p-1) \otimes w(p) \otimes w(p+1) ... \otimes w(l-1) \in C(v, l)$$

such that $w(m) = v(m)$ for all $m, 1 \leq m \leq l - 1, m \neq p$ and $v(p) \neq w(p)$.

We denote the set of all admissible monomials in $C(v, l)$ by $A(v, l)$ and, if $w \in V_{j-l}$, define $A(v, w) = A(v, l+1) \cap C(v, w)$.

Clearly $R^{(k)} = V^{k-2}R \cap V^{k-3}RV \cap ... \cap RV^{k-2}$. For $v \in V_j$ set

$$R^{(k)}_v = vV^{k-1} \cap R^{(k)}.$$
Lemma 2.2. $R^{(k)}_v \subseteq \text{span } A(v, k)$.

Proof. For $w \in V$ define a linear map
\[ \phi_w : KV \to K \]
by
\[ \phi_w(u) = \delta_{w,u} \quad \forall \ u \in V. \]

Let
\[ D = \{(w_1, w_2) \in V \times V | w_1 \neq w_2 \text{ or } |w_1| - |w_2| \neq 1\}. \]
Then if $(w_1, w_2) \in D$ we have
\[ \phi_{w_1} \otimes \phi_{w_2} R = 0. \]
Thus
\[ R^{(k)}_v \subseteq \bigcap_{l=0}^{k-2} \bigcap_{(w_1, w_2) \in D} \ker(id^l \otimes \phi_{w_1} \otimes \phi_{w_2} \otimes id^{k-2-l}). \]

Clearly
\[ vV^{k-1} \cap \bigcap_{l=0}^{k-2} \bigcap_{(w_1, w_2) \in D} \ker(id^l \otimes \phi_{w_1} \otimes \phi_{w_2} \otimes id^{k-2-l}) = \text{span } C(v, k) \]
and so
\[ R^{(k)}_v \subseteq \text{span } C(v, k). \]

Now define $\psi = \sum_{w \in V} \phi_w$ so that
\[ \psi(u) = 1 \quad \forall \ u \in V. \]
Then $R^{(2)}_v = C(v, 2) \cap \ker(id \otimes \psi)$ and so
\[ R^{(k)}_v = C(v, k) \cap \left( \bigcap_{l=1}^{k-1} \ker(id^l \otimes \psi \otimes id^{k-l-1}) \right). \]

But
\[ \text{span } C(v, k) \cap \bigcap_{l=1}^{k-1} \ker(id^l \otimes \psi \otimes id^{k-l-1}) \subseteq \text{span } A(v, k) \]
and the lemma is proved. $\square$

Assume $v \in V_j$, $w \in V_{j-l}$, $2 \leq l \leq j$. Define $R^{(l)}_{v,w} = R^{(l)} \cap A(v, w)$. We say that the graph $\Gamma$ is oriented if for any $v \in V_j$, $w \in V_{j-l}$, $1 \leq k \leq l-1$ there exists a function
\[ o_{v,w} : A(v, w) \to \{\pm 1\} \]
such that if $m = v \otimes v_1 \otimes \ldots \otimes v_{l-1}$ and $m' = v \otimes w_1 \otimes \ldots \otimes w_{l-1} \in \mathcal{A}(v, w)$ satisfy $v_q = w_q$ for $1 \leq q \leq l-1, q \neq k, \text{and} v_k \neq w_k$ then

$$o(m) + o(m') = 0.$$  

(This is similar to the definition of an oriented combinatorial cell complex given by Basak [1].) Clearly there is at most one such $m'$. However, $m \in \mathcal{A}(v, w)$ implies there is at least one such $m'$. Thus $m'$ is unique. We call it the $k$-conjugate of $m$.

Assume $\Gamma$ is oriented and $v \in V_j, w \in V_{j-1}, 1 \leq k \leq l - 1$. Define the permutation

$$\tau_{v,w,k} : \mathcal{A}(v, w) \to \mathcal{A}(v, w)$$

to be the map that takes any $m \in \mathcal{A}(v, w)$ to its $k$-conjugate. Let $\mathcal{P}(v, w)$ denote the group of permutations of $\mathcal{A}(v, w)$ generated by $\{\tau_{v,w,k} | 1 \leq k \leq l - 1\}$. Clearly

$$o_{v,w} \tau_{v,w} = -\tau_{v,w}.$$ Extend elements of $\mathcal{P}(v, w)$ to endomorphisms of $\text{span} \ \mathcal{A}(v, w)$.

**Lemma 2.3.** Let $\Gamma$ be oriented, $v, w \in V, v > w, |v| - |w| = l$, and $1 \leq k \leq l - 1$. Let $u \in \text{span} \ \mathcal{A}(v, w)$. Then $(\text{id}^{k-1} \otimes \psi \otimes \text{id}^{l-k}) u = 0$ if and only if $\tau_{v,w,k} u = -u$.

**Proof.** For any linked monomial $m = y_{(1)} \otimes \ldots \otimes y_{(r)}$ define

$$\psi_m = \phi_{y_{(1)}} \otimes \ldots \otimes \phi_{y_{(r)}} : V^r \to K.$$ 

For $m = v \otimes v_1 \otimes \ldots \otimes v_{(j)} \otimes \ldots \otimes v_{(l-1)} \in \mathcal{C}(v, w), |v| - |w| = l$ define $m^{[j]} = (\text{id}^j \otimes \psi^{l-j}) m = v \otimes v_1 \otimes \ldots \otimes v_{(j-1)}, m^{[j]} = v_{(j)},$ and $m^{[j]} = (\psi^{j+1} \otimes \text{id}^{l-j-1}) m = v_1 \otimes \ldots \otimes v_{(l-1)}$. Thus $m = m^{[j]} \otimes m^{[j]} \otimes m^{[j]}$. Let $m_1, m_2 \in \mathcal{A}(v, w)$. Then $m_1^{[k]} \otimes m_2^{[k]} = m_2^{[k]} \otimes m_1^{[k]}$ if and only if $m_1 = m_2$ or $m_1 = \tau_{v,w,k} m_2$. Since the set of distinct monomials in $V^{l-1}$ is linearly independent, $\ker (\text{id}^k \otimes \psi \otimes \text{id}^{l-k-1})$ is spanned by $\{m - \tau_{v,w,k} m | m \in \mathcal{A}(v, w)\}$. Thus $\ker (\text{id}^k \otimes \psi \otimes \text{id}^{l-k-1}) = \ker (\text{id} + \tau_{v,w,k}).$

Define

$$m_{v,w} = \sum_{m \in \mathcal{A}(v, w)} o_{v,w}(m)m.$$ 

Let $X(v, w)$ denote a complete set of representatives for the orbits of $\mathcal{A}(v, w)$ under $\mathcal{P}(v, w)$.

**Proposition 2.4.** Assume that $\Gamma$ is oriented. Then $\{m_{v,w} | m \in X(v, w)\}$ is a basis for $\mathcal{R}^{(l)}$. 

Proof. Since
\[(\text{id} + \tau_{v,w,k})m_{v,w} = \sum_{m \in A(v,w)} o_{v,w}(m)(m + \tau_{v,w,k}m) = \sum_{m \in A(v,w)} (o_{v,w}(m) + o_{v,w}(\tau_{v,w,k}m)m = 0),\]
the lemma shows that each $m_{v,w}$ is an element of $R_{v,w}$. Moreover, the $m_{v,w}$ for $m \in X_{v,w}$ are sums over disjoint subsets of $A(v,w)$ and so $\{m_{v,w}| m \in X(v,w)\}$ is linearly independent.

Now let $u = \sum_{m \in A(v,w)} a(m)m \in R_{v,w}$. Then
\[u' = u - \sum_{m \in X(v,w)} a(m)m_{v,w} \in R_{v,w}\]
and if we write $u' = \sum_{m \in A(v,w)} a'(m)m$, we have $a'(m) = 0$ for all $m \in X(v,w)$. Now by the lemma we see that $a'(m) = 0$ implies $a'(\tau_{v,w,k}m) = 0$ for all $k, 1 \leq k \leq l - 1$. Thus $a'(\tau m) = 0$ for all $\tau \in P(v,w)$ and $m \in X(v,w)$, so $u' = 0$ and the proposition is proved.

\[\square\]

3. SMALL DIFFERENCE IN LEVELS

Let $V$ be a finite-dimensional vector space over a field $K$. Recall that we write $V^k$ for the $k$-th tensor power $V \otimes ... \otimes V \subseteq T(V)$. Let $R$ be a subspace of $V^2$, let $A = \sum_{n \geq 0} A_n$ be the (graded) quadratic algebra $T(V)/ < R >$ and let $A^! = \sum_{n \geq 0} A^!_n$ be the quadratic dual of $A$.

Proposition 3.1. Write $H(A,t)H(A^!, -t) = \sum_{n \geq 0} b_n t^n$. If $b_4 = 0$, then the lattice in $V^4$ generated by $V^2 R, VRV$ and $RV^2$ is distributive.

Proof. The lattice is distributive if and only if $V^2 R \cap (VRV + RV^2) = V^2 R \cap V RV + V^2 R \cap RV^2$. But
\[V^2 R \cap (VRV + RV^2) \supseteq V^2 R \cap V RV + V^2 R \cap RV^2\]
and so it is sufficient to prove that
\[\dim(V^2 R \cap (VRV + RV^2)) = \dim(V^2 R \cap VRV + V^2 R \cap RV^2).\]

Write $R^{(j)} = V^{j-2} R \cap V^{j-3} RV \cap ... \cap RV^{j-2}$ and recall (Proposition 2.1) the well-known fact that $\dim(A^!_j) = \dim R^{(j)}$.

Now
\[b_4 = \dim(A_4) - \dim(A_3)\dim(A^!_1) + \dim(A_2)\dim(A^!_2) - \dim(A_1)\dim(A^!_3) + \dim(A^!_4) = \]
\[ \dim(V^4) - \dim(V^2R + VRV + RV^2) - (\dim(V^3) - \dim(VR + RV)) \dim(V) + \\
(\dim(V^2) - \dim(R)) \dim(R) - \dim(V) \dim(R^{(3)}) + \dim(R^{(4)}). \]

Thus if \( b_4 = 0 \) we have
\[ (\dim(V) \dim(R^{(3)}) - \dim(R^{(4)})) = \\
-\dim(V^2R + VRV + RV^2) + \dim(VR + RV)) \dim(V) + \dim(V^2) \dim(R). \]

Now the left-hand side may be written as
\[ \dim(V^2R \cap RV^2) + \dim(V^2R \cap VRV) - \dim((V^2R \cap RV^2) \cap (V^2R \cap VRV)) = \\
\dim((V^2R \cap RV^2) + (V^2R \cap VRV)). \]

Similarly, the right-hand side may be written as
\[ -\dim(V^2R + VRV + RV^2) + \dim(VRV + RV^2) + \dim(V^2R) = \\
\dim((V^2R \cap VRV + RV^2)). \]

Thus the proposition is proved. \( \Box \)

4. CELL COMPLEXES

Let \( C \) be a finite dimensional cell complex (see [14], Example 3.8.7) and \( M \) be the underlying manifold. Denote by \( P \) the ranked poset of cells of \( C \), ordered by defining \( \sigma_i \leq \sigma_j \) if \( \bar{\sigma}_i \subseteq \bar{\sigma}_j \) with \( rk(\sigma) = \dim(\sigma) + 1 \). Let \( \hat{P} \) be the ranked poset obtained from \( P \) by adding to \( P \) the underlying space \( M \) and the empty set where \( rk(M) = \dim(M) + 1 \) and \( rk(\emptyset) = 0 \). Recall ([14], Proposition 3.8.8) that
\[ \mu(M, \emptyset) = \chi(M) - 1 \]
where \( \mu(M, \emptyset) \) is the Möbius function and \( \chi(M) \) the Euler characteristic of \( M \). In the notations of [14] \( \mu(M, \emptyset) = \mu_{\hat{P}}(\emptyset, \hat{1}) \).

Let \( \Gamma \) be the layered graph corresponding to the ranked poset \( \hat{P} \) and let \( n = \dim(M) + 2 \). Since the cells of \( C \) are connected, the layered graph \( \Gamma \) is uniform. Therefore \( A(\Gamma) \) is a quadratic algebra. Recall that \( R \) denotes the space of defining relations.

**Theorem 4.1.** If the algebra \( A(\Gamma) \) is numerically Koszul then
\[ (-1)^n \dim(R^{(n)}) = \chi(M) - 1. \]

**Proof.** According to our computation [8, 11] (see also Lemma 1.3),
\[ H(A(\Gamma), z)^{-1} = \mu(M, \emptyset) z^n + \cdots + 1. \]

Since the Hilbert polynomial \( H(A(\Gamma), z) = \sum_{j=0}^{n} \dim(R^{(j)}) z^j \), if \( A(\Gamma) \) is numerically Koszul, then \( \mu(M, \emptyset) = (-1)^n \dim(R^{(n)}) \) and the theorem follows from Proposition 3.8.8 of [14].
5. 2-DIMENSIONAL MANIFOLDS

To any subdivision of a two-dimensional connected, oriented manifold $M$ into faces, edges and vertices there corresponds a layered graph $\Gamma = (V, E)$, where $V_0 = \{\emptyset\}$, $V_1$ is the set of vertices, $V_2$ is the set of edges, $V_3$ is the set of faces, and $V_4 = \{M\}$. An edge $x \in E$ goes from $a \in V_i$ to $b \in V_{i-1}$, $i = 4, 3, 2, 1$ if and only if $b$ belongs to the border of $a$.

Set $g = |V_1|$, $h = |V_2|$, $f = |V_3|$ and $u = g + f$. Note that $u - h$ is the Euler characteristic of $M$.

**Proposition 5.1.** Suppose that in a subdivision of $M$ every edge separates two distinct faces and the boundary of every edge consists of two points. Then
\[
H(A(\Gamma), z)^{-1} = 1 - (1 + u + h)z + (3h - 1)z^2 - (h + 1)z^3 + (u - h - 1)z^4,
\]
\[
H(A(\Gamma)^{\dagger}, z) = 1 + (1 + u + h)z + (3h - 1)z^2 + (u - 1)z^3 + z^4.
\]

The proof of this proposition will depend on a sequence of lemmas.

**Lemma 5.2.** $\Gamma$ may be given the structure of an oriented graph.

**Proof.** For each edge $e$ we arbitrarily label the two endpoints as $h(e)$ and $t(e)$. Let $F$ be any face of $M$ and $e$ be an edge of $F$. Then the orientation of $M$ induces an orientation of $F$ and this orientation of $F$ induces an orientation of the edge $e$. Then we define the orientation of the linked monomial $m_{F,e} = M \otimes F \otimes e$ by $o(m_{F,e}) = 1$ if the oriented edge $e$ is directed from $t(e)$ to $h(e)$ and $o(m_{F,e}) = -1$ if the oriented edge $e$ is directed from $h(e)$ to $t(e)$. Let $v$ be an endpoint of $e$. We define the orientation of the linked monomial $m_{F,e,v} = F \otimes e \otimes v$ by $o(m_{F,e,v}) = 1$ if the oriented edge $e$ is directed towards $v$ and define $o(m_{F,e,v}) = -1$ if the oriented edge $e$ is directed away from $v$.

Now for any edge $e$ there are exactly two faces incident to $e$ and the orientations induced on $e$ from $F_1$ and $F_2$ are opposite. Also, if $F$ is any face and $v$ is any endpoint occurring in $F$, there are exactly two edges of $F$ incident to $v$ and one of these edges is directed towards $v$ and one is directed away from $v$. Thus the function $o$ we have defined gives $\Gamma$ the structure of an oriented graph.

□
Lemma 5.3. (a) Let $F$ be a face of $M$. Then $\mathcal{P}(F,*)$ acts transitively on $\mathcal{A}(F,*)$.

(b) Let $w$ be a vertex of $M$. Then $\mathcal{P}(M,w)$ acts transitively on $\mathcal{A}(M,w)$.

(c) $\mathcal{P}(M,*)$ acts transitively on $\mathcal{A}(M,*)$.

Proof.

(a) Let the face $F$ have vertices $v_1, \ldots, v_l$ and edges $e_1, \ldots, e_l$ with $e_i$ incident to $v_i$ and $v_{i+1}$ where the subscripts are taken modulo $l$. Then $\mathcal{A}(F,*) = \{ F \otimes e_i \otimes v_i, F \otimes e_i \otimes v_{i+1} | 1 \leq i \leq l \}, \tau_{F,*1}$ interchanges $F \otimes e_i \otimes v_i$ and $F \otimes e_{i-1} \otimes v_i$ and $\tau_{F,*2}$ interchanges $F \otimes e_i \otimes v_i$ and $F \otimes e_{i+1} \otimes v_i$. Then $\tau_{F,*1} \tau_{F,*1}$ maps $F \otimes e_i \otimes v_i$ to $F \otimes e_{i-1} \otimes v_{i-1}$ and so the transitivity of $\mathcal{P}(F,*)$ acting on $\mathcal{A}(F,*)$ is clear.

(b) Let the vertex $w$ be incident to edges $e_1, \ldots, e_l$ and faces $F_1, \ldots, F_l$ with $F_i$ incident to $e_i$ and $e_{i+1}$ where the subscripts are taken modulo $l$. Then $\mathcal{A}(M,w) = \{ M \otimes F_i \otimes e_i, M \otimes F_i \otimes e_{i+1} | 1 \leq i \leq l \}, \tau_{M,w,1}$ interchanges $M \otimes F_i \otimes e_i$ and $M \otimes F_{i-1} \otimes e_i$ and $\tau_{M,w,2}$ interchanges $M \otimes F_i \otimes e_i$ and $M \otimes F_{i+1} \otimes e_{i+1}$. Then $\tau_{M,w,1} \tau_{M,w,1}$ maps $M \otimes F_i \otimes e_i$ to $M \otimes F_{i-1} \otimes e_{i-1}$ and so the transitivity of $\mathcal{P}(M,w)$ acting on $\mathcal{A}(M,w)$ is clear.

(c) Let $M \otimes F_1 \otimes e_1 \otimes v_1$ and $M \otimes F_2 \otimes e_2 \otimes v_2$ be two elements of $\mathcal{A}(M,*)$. Then since $M$ is connected, there is a path $\pi$ in $M$ from $v_1$ to $v_2$. We may assume that this path does not contain any endpoint other than $v_1$ and $v_2$. Let $n(\pi)$ denote the number of faces that $\pi$ traverses.

If $n(\pi) = 1$, then $F_1 = F_2$ and the result follows from the transitivity of $\mathcal{P}(F_1,*)$ on $\mathcal{A}(F_1,*)$.

Now assume $n(\pi) > 1$. Then the path $\pi$ remains in $F_1$ until it crosses some edge $e$ separating $F_1$ from some face $F'$. Let $v$ be an endpoint of $e$. Then by the transitivity of $\mathcal{P}(F_1,*)$ on $\mathcal{A}(F_1,*)$ there is some $\tau \in \mathcal{P}(M,*)$ such that $\tau(M \otimes F_1 \otimes e_1 \otimes v_1) = M \otimes F_1 \otimes e \otimes v$. But there is a path $\pi'$ from $v$ to $v_2$ with $n(\pi') = n(\pi) - 1$. By induction we may assume that there is $\tau' \in \mathcal{P}(M,*)$ such that $\tau'(M \otimes F_1 \otimes e \otimes v) = M \otimes F_2 \otimes e_2 \otimes v_2$. Then $\tau' \tau(M \otimes F_1 \otimes e_1 \otimes v_1) = M \otimes F_2 \otimes e_2 \otimes v_2$ and the proof is complete.

Let $G_\Gamma$ denote the set of all linear functionals $\alpha : KE \to K$ such that for every face $F \in V_3$ we have

$$\sum o(F,e)\alpha(e) = 0$$

where the sum is taken over all edges $e$ incident to $F$. Let $F' \in V_3$ be a face with more than 3 vertices and let $v,v'$ be vertices of $F'$ that are not connected by an edge of $F'$. Then we may create a new subdivision of $M$ by dividing the face $F'$ into two faces $F''$ and $F'''$ by adding a
new edge \( e' \) connecting \( v \) and \( v' \). Let \( \Gamma' \) be the graph corresponding to this new subdivision and let \( E' = E \cup \{ e' \} \), the set of edges of the new subdivision. Then any \( \alpha \in G_\Gamma \) has a unique extension to an element \( \alpha' \in G_{\Gamma'} \) since we must have \( \alpha'(e') = -\sum o(F, e)\alpha(e) \) where the sum is taken either over all edges \( e \) of \( F'' \) different from \( e' \) or, equivalently, over all edges \( e \) of \( F''' \) different from \( e' \). Define

\[
\theta : (KE')^* \to (KE)^*
\]

by

\[
\theta : \beta \mapsto \beta|_{KE}
\]

Then \( \theta(G_{\Gamma'}) = G_\Gamma \).

Let \( w \in V_1 \). Define \( q_{w,\Gamma} \in (KE)^* \) by

\[
q_{w,\Gamma} = \sum_{t(e)=w} e^* - \sum_{h(e)=w} e^*.
\]

Clearly \( q_{w,\Gamma} \in G_\Gamma \) and \( \theta(q_{w,\Gamma'}) = q_{w,\Gamma} \).

**Lemma 5.4.** \( G_\Gamma \subseteq \sum_{w \in V_1} Kq_{w,\Gamma} \).

**Proof.** By the previous remarks, if

\[
G_{\Gamma'} \subseteq \sum_{w \in V_1} Kq_{w,\Gamma'}
\]

then

\[
G_\Gamma \subseteq \sum_{w \in V_1} Kq_{w,\Gamma}.
\]

Therefore, it is sufficient to prove the lemma under the additional assumption that every face is a triangle.

Let \( \alpha \in G_\Gamma \) and suppose that \( S \subseteq V_1 \) has the property that \( \alpha(e) = 0 \) whenever both endpoints of \( e \) are contained in \( S \). Clearly \( S = \emptyset \) has this property and if \( S = V_1 \) then \( \alpha \in \sum_{w \in V_1} Kq_{w,\Gamma} \). Thus if, whenever \( S \neq V_1 \), we can find \( S' \subseteq V_1 \) properly containing \( S \) and \( \alpha' \in \alpha + \sum_{w \in V_1} Kq_{w,\Gamma} \) such that \( \alpha'(e) = 0 \) whenever both endpoints of \( e \) are contained in \( S' \), the lemma will follow by induction.

Now assume \( S \neq V_1 \). Since \( M \) is connected, there is some \( u \in V_1 \) and some edge \( e' \in V_2 \) connecting \( u \) to an element \( s \in S \). If \( u = t(e') \) set \( \alpha' = \alpha - \alpha(e')q_{u,\Gamma} \) and if \( u = h(e') \) set \( \alpha' = \alpha + \alpha(e')q_{u,\Gamma} \). Then \( \alpha'(e') = 0 \) and \( \alpha'(e) = 0 \) for any edge \( e \) connecting two vertices in \( S \). Now, if there is an edge \( e'' \) from \( u \) to some vertex \( s' \in S, s' \neq s \) then \( u, s, s' \) are the vertices of some face and consequently \( \alpha'(e'') = 0 \). Then we may take \( S' = S \cup \{ u \} \).

Next suppose there is no edge from \( u \) to any element of \( S \) other than \( s \). Then, since \( u \) is the endpoint of at least two edges, there is an edge
e' from u to some u' \notin S. Then u, u', s are the three vertices of some face F' and so there is an edge e'' from u' to s. If there is also an edge from u' to some other element of S, then the previous argument with u replaced by u' shows that we may find the required S'. If there is no edge from u' to a vertex of S other than s, then setting S' = S \cup \{u, u'\} gives the result. \qed

We can now prove the proposition.

**Proof.** Because \(V_4 = \{M\}\) and \(V_0 = \{\star\}\) we have
\[
R^{(4)} = R^{(4)}_{M,\star}.
\]
Thus by Lemma 5.3 and Proposition 2.4 we have
\[
R^{(4)} = Km_{M,\star}
\]
and so \(\text{dim } R^{(4)} = 1\).

Next, observe that \(R^{(3)} = R^{(3)}_M \oplus \sum_{F \in V_3} R^{(3)}_{F,\star}\). Again by Lemma 5.3 and Proposition 2.4, \(R^{(3)}_{F,\star} = Km_{F,\star}\) and since \(R^{(3)}_{F,\star} \subseteq FV^2\) and \(\sum_{F \in V_3} FV^2\) is direct, we have
\[
\text{dim}(\sum_{F \in V_3} R^{(3)}_{F,\star}) = |V_3| = f.
\]
Now \(\text{dim}(\sum_{w \in V_1} K(m_{M,w} \otimes w)) = |V_1|\). By Lemma 5.4 \(R^{(3)}_M = \sum_{w \in V_1} R^{(3)}_{M,w}\). Then, by Lemma 5.3 and Proposition 2.4, the map
\[
id^{3} \otimes \psi : \sum K(m_{M,w} \otimes w) \to R^{(3)}_M
\]
is surjective. The kernel is
\[
K.A(M,4) \cap (\cap_{i=1}^{3} \ker(id^{i} \otimes \psi \otimes id^{3-i})) = R^{(4)}_M.
\]
Thus
\[
0 \to R^{(4)}_M \to \sum K(m_{M,w} \otimes w) \to R^{(3)}_M \to 0
\]
is exact and so \(\text{dim}(R^{(3)}_M) = |V_1| - 1 = g - 1\). Thus
\[
\text{dim}(R^{(3)}) = f + g - 1 = u - 1.
\]
Now
\[
R^{(2)} = R^{(2)}_M \oplus (\oplus_{F \in V_3} R^{(2)}_F) \oplus (\oplus_{e \in V_2} R^{(2)}_e).
\]
Clearly \(\text{dim}(R^{(2)}_M) = f - 1\). Since \(\text{dim } R^{(2)}_F\) is equal to the number of edges of \(F\) minus 1 and since every edge is incident to two faces we have \(\text{dim}(\oplus_{F \in V_3} R^{(2)}_F) = 2h - f\). Finally, \(\text{dim}(R^{(2)}_e) = 1\) and so \(\text{dim}(\oplus_{e \in V_2} R^{(2)}_e) = h\). Thus \(\text{dim}(R^{(2)}) = f - 1 + 2h - f + h = 3h - 1\). As the number of generators of \(A(\Gamma)^1\) is \(|V_1| + |V_2| + |V_3| + |V_4| = g + h + f + 1 = u + h + 1\), the expression for \(H(A(\Gamma),z)^{-1}\) is proved.
The expression for \( H(A(\Gamma)^!, z) \) follows from Lemma 1.3 by noting that
\[
\begin{align*}
s_{4,4} &= -1, s_{4,3} = f, s_{4,2} = -h, s_{4,1} = g, \\
s_{3,3} &= -f, s_{3,2} = 2h, s_{3,1} = -2h, \\
s_{2,2} &= -h, s_{2,1} = 2h,
\end{align*}
\]
and
\[s_{1,1} = -g.\]

\[\square\]

**Corollary 5.5.** For \( \Gamma \) as above, the following statements are equivalent:

i) \( A(\Gamma) \) is numerically Koszul;

ii) \( A(\Gamma) \) is Koszul;

iii) the Euler characteristic of \( M \) is 2, i.e. \( M \) is a topological sphere.

**Proof.** By Proposition 5.1, \( H(A(\Gamma), Z)^{-1} = H(A(\Gamma)^!, -z) = (u - h - 2)(z^3 + z^4) \). Thus (i) and (iii) are equivalent. Also (i) implies (ii) by Proposition 3.1 and (iii) is well-known to imply (i). \[\square\]

6. KOSZULITY OF CERTAIN \( A(\Gamma) \)

As noted in the introduction, we are indebted to Cassidy and Shelton for pointing out errors in our previous paper [8]. First of all, Lemma 1.4 in that paper and the line immediately preceding it should be replaced by:

For every \( v \in V^+ \), set \( \tilde{v} = v^{(0)} + \cdots + v^{(|v|-1)} \in FV^+ \). then \( \tau(e_i) = e(v_{j-1}, 1) - e(v_i, 1) \) and hence \( \nu(e_i) = \tilde{v}_{j-1} - \tilde{v}_i \). Since \( \theta \) is induced by \( \nu \) we have

\[
\theta(e(\pi, k)) = (-1)^k \sum_{1 \leq i_1 < \ldots < i_k \leq m} (\tilde{v}_{i_{k-1}} - \tilde{v}_{i_1}) \cdots (\tilde{v}_{i_{k-1}} - \tilde{v}_{i_k}).
\]

A more serious error is that we have incorrectly asserted (in Theorems 4.6 and 5.2) that \( gr A(\Gamma) \) and \( A(\Gamma) \) are Koszul algebras for any uniform layered graph \( \Gamma \) with a unique minimal vertex. The argument given for these assertions depends on a subsidiary result, Lemma 4.4, which asserts that if \( l \geq 0, j \geq 1, v \in \bigcup_{j=2}^n V_j \) then

\[
P_j(v) V^{l+1} \cap R^{(l+2)} = g_{l+3}(S_{j-1}(v) V^{l+2} \cap R^{(l+3)}).
\]

Here \( g_j : V^j \to V^j-1 \) denotes \( \psi \otimes id^{j-1} \) (recall that \( \psi(v) = 1 \) for all \( v \in V \)), \( S_j(v) \) denotes \( \text{span}\{w \in V|v > w, |v| - |w| = j\} \), and \( P_j(v) = \ker g_1|_{S_j(v)} \).
The Cassidy-Shelton example provides a counter-example to this assertion. Let \( r = (x_1 - x_2)(y_1 - y_3) - (x_1 - x_3)(y_1 - y_2) = x_1(y_2 - y_3) - x_2(y_1 - y_3) + x_3(y_1 - y_2) \), where the notation is as in Section 1. Then \( r \in P_2(v)V \cap R^{(2)} \), while \( S_1(u)V^3 \cap R^{(3)} \subseteq \sum_{i=1}^3 R_{w_i}^{(3)} = (0) \).

The proof in [8] shows:

**Proposition 6.1.** If 
\[
P_j(v)V^{l+1} \cap R^{(l+2)} = g_{l+3}(S_{j-1}(v)V^{l+2} \cap R^{(l+3)})
\]
whenever \( l \geq 0, j \geq 1, v \in \cup_{j=2}^n V_j \) then \( gr A(\Gamma) \) and \( A(\Gamma) \) are Koszul algebras.

We use this sufficient condition to show that \( A(\Gamma) \) is Koszul for certain classes of graphs \( \Gamma \).

We begin by recalling (from [9]) that a complete layered graph is a layered graph \( \Gamma = (V, E) \) with \( V = \cup_{i=0}^n V_i \) such that if \( 1 \leq i \leq n, v \in V_i, w \in V_{i-1} \) then there is an edge from \( v \) to \( w \).

**Theorem 6.2.** Let \( V = (V, E) \) be a complete layered graph where \( V = \cup_{i=0}^n V_i \) and \( V_0 = \{\ast\} \). Then \( gr A(\Gamma) \) and \( A(\Gamma) \) are Koszul algebras.

**Proof.** It is sufficient to prove the result for \( gr A(\Gamma) \). For simplicity of notation we will write \( R^{(j)} \) for \( (gr R)^{(j)} \). Let \( v \in V_k, j \leq k \). We first note that 
\[
vV^{j-1} \cap R^{(j)} = \text{span}\{v(v_{k-1,1} - v_{k-1,2}) \ldots (v_{k-j+1,1} - v_{k-j+1,2})|v_{i,1}, v_{i,2} \in V_i, 0 \leq i \leq k-1\}.
\]

To see this note that it is clear that \( vV^{j-1} \cap R^{(j)} \) contains the right-hand side. Also, any element \( q \) of \( vV^{j-1} \cap R^{(j)} \) is a sum of elements in \( vwV^{(j-2)} \cap vV^{(j-1)} \) where \( w \in V_{k-j} \). Equality now follows by applying induction to \( V^{(j-1)} \) and using \( q \in RV^{j-2} \).

Using this equality we have 
\[
P_j(v)V^{l+1} \cap R^{(l+2)} = P_j(v)\text{span}\{(v_{k-j,1} - v_{k-j,2}) \ldots (v_{k-j-l+1,1} - v_{k-j-l+1,2})|v_{i,1}, v_{i,2} \in V_i, k - j - l \leq i \leq k - j\} = g_{l+3}(S_{j-1}(v)V^{l+2} \cap R^{(l+3)}),
\]
proving the theorem. \( \square \)

Next we will prove the Koszulity of splitting algebras associated to abstract simplicial complexes. We recall that an abstract simplicial complex on a set \( X \) is a subset \( \Delta \) of the power set \( \mathcal{P}(X) \) of \( X \) such that \( \{x\} \in \Delta \) for all \( x \in X \) and if \( S \in \Delta \) and \( T \subseteq S \), then \( T \in \Delta \).
Let $X$ be a finite set and $\Delta$ be an abstract simplicial complex on $X$. Define a layered graph $\Gamma_\Delta = (V_\Delta, E_\Delta)$ by $V_{\Delta,i} = \{S \in \Delta \mid |S| = i\}$ and $E_\Delta = \{e_{S,s} | S \in \Delta, s \in S\}$ where $t(e_{S,s}) = S, h(e_{S,s}) = S \setminus \{s\}$.

Note that $\Delta_n = \mathcal{P}\{\{1, ..., n\}\}$ is an abstract simplicial complex in $\{1, ..., n\}$. The algebra $A(\Gamma_{\Delta_n})$ is the algebra studied in [5]. This algebra is denoted by $Q_n$ there and we will continue to denote it by $Q_n$.

As above, it is sufficient to prove the result for $\text{gr} A(\Gamma)$ and we will write $R^{(j)}$ for $(\text{gr} R)^{(j)}$. Set

$$R_\Delta = \text{span}\{S((S \setminus \{s\}) - (S \setminus \{t\})) | S \in \Delta, s, t \in S\},$$

$$\mathcal{R}_\Delta = T(V)R_\Delta T(V),$$

and

$$\mathcal{R}_\Delta^{(k)} = \bigcap_{j=0}^{k-2} V_j^2 R_\Delta V^{k-2-j}.$$ 

For $S \in \Delta, |S| \geq k$, define

$$\mathcal{R}_\Delta^{(k)}|_S = SV_{\Delta}^{k-1} \cap \mathcal{R}_\Delta^{(k)}.$$ 

Then, since $\mathcal{R}_\Delta = \bigoplus_{S \in \Delta} SV_{\Delta} \cap \mathcal{R}_\Delta$, we have

$$\mathcal{R}_\Delta^{(k)} = \bigoplus_{S \in \Delta} \mathcal{R}_\Delta^{(k)}|_S.$$ 

The following result is immediate.

**Lemma 6.3.** Let $S = \{s_1, ..., s_i\} \in \Delta$. The map

$$j \mapsto s_j, 1 \leq j \leq i$$

extends to an injection

$$T(V_{\Delta,i}) \rightarrow T(V_{\Delta})$$

and this map restricts to an isomorphism of vector spaces

$$\mathcal{R}_{\Delta,i\{1, ..., i\}} \rightarrow \mathcal{R}_{\Delta,S}.$$ 

**Theorem 6.4.** Let $\Delta$ be an abstract simplicial complex. Then $A(\Gamma_\Delta)$ is a Koszul algebra.

**Proof.** Let $S = \{s_1, ..., s_i\} \in \Delta$. Let

$$\phi : \mathcal{R}_{\Delta,i\{1, ..., i\}} \rightarrow \mathcal{R}_{\Delta,S}$$

be the isomorphism constructed in the lemma.

Now let $A \subseteq \{1, ..., n\}$. Then Proposition 6.4 of [2] together with the well known identification of $\mathcal{R}_\Delta^{(k)}$ with the space of elements of
degree $k$ in $A(\Gamma_\Delta)_k$ (proposition 2.1) allows us to describe a basis for $AV^{k-1}_\Delta \cap R^k_\Delta$. For $B = \{b_1, ..., b_k\} \subseteq A$ with $b_1 < ... < b_k$ define

$$S(A : B) = \sum_{\sigma \in S_k} sgn(\sigma)A(A \setminus \{b_{\sigma(1)}\})...(A \setminus \{b_{\sigma(1)}, ..., b_{\sigma(k-1)}\}).$$

Then

$$\{\phi(S(A, B))|B \subseteq A \subseteq \{1, ..., n\}, \min A \notin B, |B| = k\}$$

is a basis for $AV^{k-1}_\Delta \cap R^k_\Delta$.

Now if $B \subseteq A \subseteq \{1, ..., n\}, \min A \notin B, |B| = k$ then

$$g_k S(A : B) = \sum_{\sigma \in S_k} sgn(\sigma)(A \setminus \{b_{\sigma(1)}\})...(A \setminus \{b_{\sigma(1)}, ..., b_{\sigma(k-1)}\}).$$

Then writing $\sigma' = \sigma \circ (12)$ and taking all sums over all $\sigma \in S_k$ such that $\sigma(1) < \sigma(2)$, we have

$$g_k S(A : B) = \sum sgn(\sigma)(A \setminus \{b_{\sigma(1)}\})...(A \setminus \{b_{\sigma(1)}, ..., b_{\sigma(k-1)}\}) +$$

$$\sum sgn(\sigma')(A \setminus \{b_{\sigma'(1)}\})...(A \setminus \{b_{\sigma'(1)}, ..., b_{\sigma'(k-1)}\}) =$$

$$\sum sgn(\sigma)((A \setminus \{b_{\sigma(1)}\}) - (A \setminus \{b_{\sigma(2)}\}))(A \setminus \{b_{\sigma(1)}, b_{\sigma(2)}\})...(A \setminus \{b_{\sigma(1)}, ..., b_{\sigma(k-1)}\}).$$

Thus

$$g_k S(A : B) \in P_1(A) V^{k-2} \cap R^{(k)}_\Delta,$$

and so the hypothesis of Proposition 6.1 is satisfied and the theorem is proved.

Since any Koszul algebra is numerically Koszul, we have the following corollary.

**Corollary 6.5.** Let $\Delta$ be an abstract simplicial complex. Then $A(\Gamma_\Delta)$ is a numerically Koszul algebra.

We remark that this result may be established directly without the use of Theorem 6.4.

To see this, recall that

$$H(A(\Gamma_\Delta), z) = (1 - z)(1 + \sum_{v_1 > ... > v_{\ast}} (-1)^{v_{\ast}} z^{|v_1| - |v_{\ast}| + 1})^{-1}$$

and that

$$H(A(\Gamma_\Delta)^\dagger, z) = \sum_{k \geq 0} dim(R^{(k)}_\Delta) z^k.$$
Thus $A(\Gamma_\Delta)$ is numerically Koszul if and only if
\[
(1 - z) \sum_{k \geq 0} (-1)^k \dim(\mathcal{R}^{(k)}) z^k = 1 + \sum_{v_1 > \ldots > v_i \geq *}(1 - z)^{|v_1| - |v_i| + 1}.
\]

By Proposition 6.4 of [2] we have that, if $i \geq k$,
\[\dim\mathcal{R}^{(k)}_{\Delta, (1, \ldots, i)} = \binom{i - 1}{k - 1}.
\]

Thus, in view of Lemma 6.1,
\[
(1 - z) \sum_{k \geq 0} (-1)^k \dim(\mathcal{R}^{(k)}) z^k = (1 - z) \sum_{k \geq 0} (-1)^k \sum_{i \geq k} |V_{\Delta, i}| \binom{i - 1}{k - 1} z^k = \sum_{k \geq 0} (-1)^k \sum_{i \geq k - 1} |V_{\Delta, i}| \binom{i}{k - 1} z^k.
\]

Now consider $1 + \sum_{v_1 > \ldots > v_i \geq *}(1 - z)^{|v_1| - |v_i| + 1}$. By Example 3.8.3 of [14] we have
\[
\sum_{v_1 > \ldots > v_i \geq *}(-1)^l z^{|v_1| - |v_i| + 1} = \sum_{v_1 > v_i \geq *}(-z)^{|v_1| - |v_i| + 1} = \sum_{i \geq k} |V_{\Delta, i}| \binom{i}{k - 1} (-z)^k,
\]
establishing the numerical Koszulity of $A(\Gamma_\Delta)$.

**Remark 6.6.** The layered graph $A(\Gamma)$ corresponding to an abstract simplicial complex will not, in general, have a unique maximal vertex. Adding such vertex will destroy numerical Koszulity of the corresponding splitting algebra.

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Department of Mathematics, Rutgers University, Piscataway, NJ 08854-8019, USA
*E-mail address*: vretakh@math.rutgers.edu

IME-UFG, CX Postal 131, Goiania - GO, CEP 74001-970, Brazil
*E-mail address*: serconek@math.rutgers.edu

Department of Mathematics, Rutgers University, Piscataway, NJ 08854-8019, USA
*E-mail address*: rwilson@math.rutgers.edu