Formal Groups Arising from Formal Punctured Ribbons

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Abstract

We investigate Picard functor of a formal punctured ribbon. We prove that under some conditions this functor is representable by a formal group scheme.

1 Introduction.

First we would like to give a motivation of our present research.

Let’s consider a \( C \)-algebra \( R \) with a derivation \( \partial : R \to R \)

\[
\partial(ab) = \partial(a)b + a\partial(b), \quad a, b \in R.
\]

We construct a ring

\[
R(\!(\partial^{-1})) : \sum_{i \in \mathbb{C} + \infty} a_i \partial^i, a_i \in R
\]

\[
[\partial, a] = \partial(a), \quad \partial^{-1}a = a\partial^{-1} + C_{-1}^1\partial(a)\partial^{-2} + C_{-1}^2\partial^2(a)\partial^{-3} + \ldots,
\]

where \( C_i^k, \ i \in \mathbb{Z}, \ k \in \mathbb{N} \) is a binomial coefficient:

\[
C_i^k = \frac{i(i - 1) \ldots (i - k + 1)}{k(k - 1) \ldots 1}, \quad C_i^0 = 1.
\]

Now we consider \( R = \mathbb{C}\!(x) \) with usual derivation \( \partial(x) = 1 \). We add infinite number of ”formal times” : \( t_1, t_2, \ldots \). There is a unique decomposition in the ring \( R(\!(\partial^{-1}))[[t_1, t_2, \ldots]] \):

\[
\text{if } A \in R(\!(\partial^{-1}))[\![t_1, t_2, \ldots]\!], \quad \text{then } A = A_+ + A_-,
\]

where \( A_+ \in R[\partial][\![t_1, t_2, \ldots]\!], \quad A_- \in R[\!(\partial^{-1})] : \partial^{-1}[\![t_1, t_2, \ldots]\!]. \)

Let \( L \in R(\!(\partial^{-1}))[[t_1, t_2, \ldots]] \) be of the following type:

\[
L = \partial + a_1 \partial^{-1} + a_2 \partial^{-2} + \ldots, \quad a_i \in R[[t_1, t_2, \ldots]].
\]

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The classical KP-hierarchy is the following infinite system of equations, see [17]:

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L], \quad n \in \mathbb{N}. $$

From this system it follows

the KP equation \((4u_t - uu'' - 12uu')' = 3u_{yy}\) for \(u(t, x, y)\),

and the KdV equation \(4u_t - 7uu'' - 12uu' = 0\) for \(u(t, x)\).

Solutions of KP-hierarchy are obtained from flows on Picard varieties of algebraic curves (for example, solitons).

A.N. Parshin gave in 1999 in [14] the following generalization of KP-hierarchy. (A. Zheglov modified it later in [20].) Let \(R = \mathbb{C}[[x_1, x_2]]((\partial_1 ^{-1}))\), where the derivation

\[
\partial_1 : \mathbb{C}[[x_1, x_2]] \to \mathbb{C}[[x_1, x_2]], \quad \partial_1(x_1) = 1, \quad \partial_1(x_2) = 0.
\]

We consider a derivation \(\partial_2 : R \to R, \quad \partial_2(x_1) = 0, \quad \partial_2(x_2) = 1, \quad \partial_2(\partial_1) = 0\). As before, we construct a ring

\[
E = R((\partial_2 ^{-1})) = \mathbb{C}[[x_1, x_2]]((\partial_1 ^{-1}))((\partial_2 ^{-1})).
\]

We add "formal times" \(\{t_k\}\), \(k = (i, j) \in \mathbb{Z} \times \mathbb{Z}_+\). As before, there is a decomposition (with respect to \(\partial_2\)):

\[
E[\{t_k\}] = E_+[[\{t_k\}] \oplus E_-[[\{t_k\}]].
\]

We consider \(L, M \in E[\{t_k\}]\) such that

\[
L = \partial_1 + u_1 \partial_2^{-1} + \ldots, \quad M = \partial_2 + v_1 \partial_2^{-1} + \ldots,
\]

where \(u_i, v_i \in R[[\{t_k\}]]\).

Let \(N = (L, M) \) and \([L, M] = 0\), then hierarchy is

\[
\frac{\partial N}{\partial t_k} = V_N^k,
\]

where \(V_N^k = \left([([L^iM^j])_+, L], ([L^iM^j])_+, M]\right)\),

\(k = (i, j) \in \mathbb{Z} \times \mathbb{Z}_+, \quad i \leq \alpha j, \quad \alpha > 0\).

There is the following property. Let \(L, M \in E\) such that they satisfy conditions for Parshin’s hierarchy when all the times \(t_k = 0\). Then there is \(S \in 1 + E_- \subset E\) such that

\[
L = S^{-1}\partial_1 S, \quad M = S^{-1}\partial_2 S.
\]

Besides, the ring \(E\) acts \(\mathbb{C}\)-linearly on \(\mathbb{C}((u))((t))\) (and on the set of \(\mathbb{C}\)-vector subspaces of \(\mathbb{C}((u))((t))\)) in the following way:

\[
E/E \cdot (x_1, x_2) = \mathbb{C}((u))((t)), \quad \partial_1^{-1} \mapsto u, \quad \partial_2^{-1} \mapsto t,
\]
In classical KP-hierarchy an analogous action is an action of the ring of pseudodifferential operators on the set of Fredholm subspaces of \( C((t)) \), or more generally, on the Sato Grassmanian. This action gives the flows on generalized Jacobians of algebraic curves, which are solutions of KP-hierarchy, see [15, §1].

In article [10] we investigated new geometric objects \( \hat{X}_\infty = (C, A) \), which are ringed spaces: formal punctured ribbons with the underlying topological space \( C \) as an algebraic curve. (For simplicity we call such objects "ribbons"). Examples of ribbons come from Cartier divisors on algebraic surfaces.

We are working in formal algebraic language, therefore originally we assume that ribbons are defined over any ground field \( k \). But in many places of this article we will additionally assume that \( k \) is an algebraically closed field of characteristic zero.

We introduced the notion of a torsion free sheaf on a ribbon. An importance of such sheaves followed from theorem 1 of article [10], where we proved that torsion free sheaves on some ribbons plus some geometrical data such as formal trivialization of sheaves, local parameters at smooth points of ribbons and so on are in one-to-one correspondence with **generalized Fredholm subspaces** of two-dimensional local field \( k((u))((t)) \) (see also section 2 of this article).

In this article we investigate torsion free sheaves on ribbons \( (C, A) \) and proved that if the underlying curve \( C \) of a ribbon is a smooth curve and for any small open \( U \subset C \) there are sections \( a \in \Gamma(U, A_1) \), \( a^{-1} \in \Gamma(U, A_{-1}) \), then every torsion free sheaf on the ribbon \( (C, A) \) is a locally free sheaf on a ringed space \( (C, A) \), see proposition 1. We remark that this condition is satisfied, for example, when the ribbon \( (C, A) \) comes from a smooth curve \( C \), which is a Cartier divisor on algebraic surface.

Therefore it is important to study locally free sheaves on ribbons \( (C, A) \). We restrict ourselves to the Picard group of a ribbon. In [10] we investigated the Picard group as a set, see proposition 5 and example 8 in [10]. But it was not clear, what are the deformations (local or global) of elements of \( Pic(\hat{X}_\infty) \). We study the groups \( Pic(\hat{X}_\infty, S) \) and \( Pic(X_\infty, S) \) for an arbitrary affine scheme \( S \) as functors \( Pic_{X_\infty} \) and \( Pic_{X_\infty} \) on the category of affine schemes from the point of view of representability or formal representability of these functors by a scheme or a formal scheme, see section 4. We remark that the representability by a scheme of the sheaf associated with the presheaf given by the functor \( Pic_{X_\infty} \) follows from important Lipman's results in [11], and the functor \( Pic_{X_\infty} \) is mapped in the functor \( Pic_{\hat{X}_\infty} \).

At first, we study the tangent spaces to these Picard functors. In article [21] the "picture cohomology" \( H^0(W) \), \( H^1(W) \), \( H^2(W) \) were introduced for a generalized Fredholm subspace \( W \) of a two-dimensional local field. These cohomology groups coincide with the cohomology groups of a line bundle on an algebraic surface when a ribbon and a line bundle on it come from an algebraic surface and a line bundle on this surface. In section 3 we investigate the picture cohomology groups of generalized Fredholm subspaces \( W \) and related them with some groups which depend on cohomology groups of sheaves \( F_W \) and \( F_{W,0} \) on the curve \( C \), where \( W \leftarrow (F_W, \ldots) \) is a generalized Krichever-Parshin correspondence from [10, §5]. Due to this result we obtained that the kernel of the natural map from tangent space of functor \( Pic_{X_\infty} \) to tangent space of functor \( Pic_{\hat{X}_\infty} \) coincides with
Further, in section 4, we investigate the Picard functors on ringed spaces $X_{\infty}$ and $\tilde{X}_{\infty}$ as formal functors on Artinian rings. We prove that if the first picture cohomology group of the structure sheaf of a ribbon $\tilde{X}_{\infty}$ is finite-dimensional over the ground field $k$ and char $k = 0$, then the formal Picard functor $\widehat{Pic}_{\tilde{X}_{\infty}}$ is representable by a formal group, which can be decomposed in the product of two formal groups, where the first one is connected with the formal Picard functor $\widehat{Pic}_{X_{\infty}}$ and the second one coincides with the formal Brauer group of algebraic surface when the ribbon $\tilde{X}_{\infty}$ comes from an algebraic surface and a curve on it (see corollary 1 of proposition 4).

In section 5, under some condition, we prove the global representability of the étale sheaf associated with the presheaf (or functor) $\widehat{Pic}_{\tilde{X}_{\infty}}$ on the category of affine Noetherian schemes. This condition is equivalent to $H^1(W) = 0$, where $W \subset k((u))((t))$ correspond to the structure sheaf of the ribbon $\tilde{X}_{\infty}$ plus some local parameters via the Krichever-Parshin map. The functor $\widehat{Pic}_{\tilde{X}_{\infty}}$ classifies invertible sheaves plus morphisms of order on the ribbon $\tilde{X}_{\infty}$, see section 5.3. Then we prove in theorem 4 that (noncanonically) Picard scheme of $\tilde{X}_{\infty}$ is the product of Picard scheme of $X_{\infty}$ (see [11]) and the formal Brauer group of $\tilde{X}_{\infty}$ (see section 4.4) when the ground field $k$ is an algebraically closed field and char $k = 0$.

In the end, we remark that there are many activities in the direction of constructing of geometric objects which encode spectral properties of commutative rings of germs of differential operators in the 2-dimensional case. For an (incomplete) survey on recent activities one can consult [16].

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2 Torsion free sheaves on ribbons.

We recall the general definition of ribbon from [10].

Let $S$ be a Noetherian base scheme.

Definition 1 ([10]). A ribbon $(C, \mathcal{A})$ over $S$ is given by the following data.

1. A flat family of reduced algebraic curves $\tau : C \rightarrow S$.

2. A sheaf $\mathcal{A}$ of commutative $\tau^{-1}\mathcal{O}_S$-algebras on $C$.

3. A descending sheaf filtration $(\mathcal{A}_i)_{i \in \mathbb{Z}}$ of $\mathcal{A}$ by $\tau^{-1}\mathcal{O}_S$-submodules which satisfies the following axioms:

(a) $\mathcal{A}_i \mathcal{A}_j \subset \mathcal{A}_{i+j}$, $1 \in \mathcal{A}_0$ (thus $\mathcal{A}_0$ is a subring, and for any $i \in \mathbb{Z}$ the sheaf $\mathcal{A}_i$ is a $\mathcal{A}_0$-submodule);

(b) $\mathcal{A}_0/\mathcal{A}_1$ is the structure sheaf $\mathcal{O}_C$ of $C$;
(c) for each $i$ the sheaf $\mathcal{A}_i/\mathcal{A}_{i+1}$ (which is a $\mathcal{A}_0/\mathcal{A}_1$-module by (3a)) is a coherent sheaf on $C$, flat over $S$, and for any $s \in S$ the sheaf $\mathcal{A}_i/\mathcal{A}_{i+1} |_{C_s}$ has no coherent subsheaf with finite support, and is isomorphic to $\mathcal{O}_{C_S}$ on a dense open set.

(d) $\mathcal{A} = \lim_{\rightarrow i} \mathcal{A}_i$, and $\mathcal{A}_i = \lim_{\leftarrow j \geq 0} \mathcal{A}_i/\mathcal{A}_{i+j}$ for each $i$.

Sometimes we shall denote a ribbon $(C, \mathcal{A})$ over $\text{Spec } R$, where $R$ is a ring, as a ribbon over $R$.

There is the following example of a ribbon. If $X$ is an algebraic surface over a field $k$, and $C \subset X$ is a reduced effective Cartier divisor, we obtain a ribbon $(C, \mathcal{A})$ over $k$, where

$$\mathcal{A} := \mathcal{O}_{\hat{X}_C}(\ast C) = \lim_{\rightarrow i} \mathcal{O}_{\hat{X}_C}(-iC) = \lim_{\rightarrow i} \lim_{j \geq 0} J^i / J^{i+j}$$

and

$$\mathcal{A}_i := \mathcal{O}_{\hat{X}_C}(-iC) = \lim_{\rightarrow j \geq 0} J^i / J^{i+j}, \quad i \in \mathbb{Z},$$

where $\hat{X}_C$ is the formal scheme which is the completion of $X$ at $C$, and $J$ is the ideal sheaf of $C$ on $X$ (the sheaf $J$ is an invertible sheaf). We shall say that a ribbon which is constructed by this example is "a ribbon which comes from an algebraic surface".

We recall the definition of a torsion free sheaf on a ribbon from [10].

**Definition 2** ([10]). Let $\hat{X}_\infty = (C, \mathcal{A})$ be a ribbon over a scheme $S$. We say that $\mathcal{N}$ is a torsion free sheaf of rank $r$ on $\hat{X}_\infty$ if $\mathcal{N}$ is a sheaf of $\mathcal{A}$-modules on $C$ with a descending filtration $(\mathcal{N}_i)_{i \in \mathbb{Z}}$ of $\mathcal{N}$ by $\mathcal{A}_0$-submodules which satisfies the following axioms.

1. $\mathcal{N}_i \mathcal{A}_j \subseteq \mathcal{N}_{i+j}$ for any $i, j$.

2. For each $i$ the sheaf $\mathcal{N}_i/\mathcal{N}_{i+1}$ is a coherent sheaf on $C$, flat over $S$, and for any $s \in S$ the sheaf $\mathcal{N}_i/\mathcal{N}_{i+1} |_{C_s}$ has no coherent subsheaf with finite support, and is isomorphic to $\mathcal{O}_{C_S}$ on a dense open set.

3. $\mathcal{N} = \lim_{\rightarrow i} \mathcal{N}_i$ and $\mathcal{N}_i = \lim_{\leftarrow j \geq 0} \mathcal{N}_i/\mathcal{N}_{i+j}$ for each $i$.

**Remark 1.** Note that the sheaf $\mathcal{N}$ is flat over $S$. To show this note that all sheaves $\mathcal{N}_i/\mathcal{N}_{i+j}$ are, clearly, flat over $S$ (see, e.g. [10], prop.1). So, for any ideal sheaf $J$ on $S$ we have the embeddings $0 \to \tau^* J \otimes \mathcal{N}_i/\mathcal{N}_{i+j} \to \mathcal{N}_i/\mathcal{N}_{i+j}$ by the flatness criterium for modules. This imply that we have embeddings $0 \to \tau^* J \otimes \mathcal{N}_i \to \mathcal{N}_i$ for any $i$ and embeddings $0 \to \tau^* J \otimes \mathcal{N} \to \mathcal{N}$. Therefore, $\mathcal{N}$ is flat over $S$.

There is the following example of a torsion free sheaf of rank $r$ on a ribbon $\hat{X}_\infty = (C, \mathcal{A})$ which comes from an algebraic surface $X$. Let $E$ be a locally free sheaf of rank $r$ on the surface $X$. Then

$$\hat{E}_C := \lim_{\rightarrow i} \lim_{j} E(iC)/E(jC)$$
is a torsion free sheaf of rank $r$ on $\mathbb{X}_\infty$. We shall say that a torsion free sheaf on a ribbon constructed after this example is "a sheaf which comes from a locally free sheaf on an algebraic surface".

In [10] we defined the notion of a smooth point of a ribbon, the notion of formal local parameters at a smooth point of a ribbon, and the notion of a smooth point of a torsion free sheaf on a ribbon, see definitions 9, 10, 12 from [10]. We remark that these notions coincide with the usual notions (i.e. used in [15], [13]) when a ribbon comes from algebraic surface, a torsion free sheaf on a ribbon comes from a locally free sheaf on this surface and so on.

Let $k$ be a field. We recall (see, for example, [21]) that a $k$-subspace $W$ in $k((u))^\oplus_r$ is called a Fredholm subspace if
\[ \dim_k W \cap k[[u]]^\oplus_r < \infty \quad \text{and} \quad \dim_k \frac{k((u))^\oplus_r}{W + k[[u]]^\oplus_r} < \infty. \]

**Definition 3.** For a $k$-subspace $W$ in $k((u))((t))^\oplus_r$, for $n \in \mathbb{Z}$ let
\[ W(n) = \frac{W \cap t^n k((u))[[t]]^\oplus_r}{W \cap t^{n+1} k((u))[[t]]^\oplus_r} \]
be a $k$-subspace in $k((u))^\oplus_r = \frac{t^n k((u))[[t]]^\oplus_r}{t^{n-1} k((u))[[t]]^\oplus_r}$.

A $k$-subspace $W$ in $k((u))((t))^\oplus_r$ is called a generalized Fredholm subspace iff for any $n \in \mathbb{Z}$ the $k$-subspace $W(n)$ in $k((u))^\oplus_r$ is a Fredholm subspace.

The following definition is from [10].

**Definition 4 ([10]).** Let a $k$-subalgebra $A$ in $k((u))((t))$ be a generalized Fredholm subspace. Let a $k$-subspace $W$ in $k((u))((t))^\oplus_r$ be a generalized Fredholm subspace. We say that $(A, W)$ is a Schur pair if $A \cdot W \subset W$.

Now we recall the main theorem of [10].

**Theorem 1 ([10]).** Schur pairs $(A, W)$ from $k((u))((t)) \oplus k((u))((t))^\oplus_r$ are in one-to-one correspondence with the data $(C, A, \mathcal{N}, P, u, t, e_P)$ up to an isomorphism, where $C$ is a projective irreducible curve over a field $k$, $(C, A)$ is a ribbon, $\mathcal{N}$ is a torsion free sheaf of rank $r$ on this ribbon, $P$ is a point of $C$ which is a smooth point of $\mathcal{N}$, $u, t$ are formal local parameters of this ribbon at $P$, $e_P$ is a formal local trivialization of $\mathcal{N}$ at $P$.

The goal of this section is to show that in "good" cases a torsion free sheaf on a ribbon $(C, A)$ is indeed a locally free sheaf on this ribbon, that is an element of the set $\hat{H}^1(C, GL_r(A))$.

We recall the following condition from [10] (see lemma 4 of [10]).

**Definition 5.** The sheaf $\mathcal{A}$ of a ribbon $(C, A)$ satisfies (**) if the following condition holds: there is an affine open cover $\{U_\alpha\}_{\alpha \in I}$ of $C$ such that for any $\alpha \in I$ there is an invertible section $a \in \mathcal{A}_1(U_\alpha) \subset \mathcal{A}(U_\alpha)$ such that $a^{-1} \in \mathcal{A}_{-1}(U_\alpha)$.
Remark 2. The condition on the sheaf $\mathcal{N}$ of rank $r$ on the ribbon $(C, \mathcal{A})$ is satisfied if for any $\alpha \in I$ there exists a surjective map, therefore such elements $\bar{c}_1, \ldots, \bar{c}_r$ exist.

We consider a map of $\mathcal{A}$ $\mathcal{N}$-modules:

$$\phi : \mathcal{A}^\oplus \mathcal{N} \mathcal{N} |_{\mathcal{V}} \rightarrow \mathcal{N} |_{\mathcal{V}}$$

where $a_i \in \mathcal{A}(U)$ for $1 \leq i \leq r$, an open $U \subset V$.

At first, we show that the map $\phi$ is a surjective map of sheaves. Let an element $b \in \mathcal{N}(U)$, $b \neq 0$ for an open $U \subset V$. Then $b \in \mathcal{N}_{i_1}(U) \setminus \mathcal{N}_{i_{l_1+1}}(U)$ for some $l_1 \in \mathbb{Z}$. Therefore an element $b_1 = a^{-l_1} \cdot b \in \mathcal{N}_0(U) \setminus \mathcal{N}_1(U)$, where $a \in \mathcal{A}_1(V) \setminus \mathcal{A}_2(V)$ such that $a^{-1} \in \mathcal{A}^{-1}(V)$. Let $\bar{b}_1 \in \mathcal{N}_0/\mathcal{N}_1(U)$ be the image of the element $b_1$. We have

$$\bar{b}_1 = \sum_{1 \leq i \leq r} \bar{e}_{1,i} \cdot \bar{c}_i,$$

where $\bar{e}_{1,i} \in \mathcal{O}_C(U)$, $1 \leq i \leq r$. We choose some elements $e_{1,i} \in \mathcal{A}_0(U)$ such that for any $1 \leq i \leq r$ the image of the element $e_{1,i}$ in $\mathcal{A}_0(U)/\mathcal{A}_1(U) = \mathcal{O}_C(U)$ coincides with the element $\bar{e}_{1,i}$ (see also proposition 3 of [10]). Now if $b_1 \neq \sum_{1 \leq i \leq r} e_{1,i} \cdot c_i$, then an element

$$(b_1 - \sum_{1 \leq i \leq r} e_{1,i} \cdot c_i) \in \mathcal{N}_{l_2}(U) \setminus \mathcal{N}_{l_2+1}(U)$$

for some $l_2 \in \mathbb{N}$, where $l_2 \geq 1$. Therefore an element

$$b_2 = a^{-l_2} \cdot (b_1 - \sum_{1 \leq i \leq r} e_{1,i} \cdot c_i) \in \mathcal{N}_0(U) \setminus \mathcal{N}_1(U).$$
And we can repeat the same procedure with $b_2$ as with $b_1$ before, and so on. Now an element
\[ d = a^{l_1} \cdot \left( \bigoplus_{1 \leq i \leq r} e_{1,i} + a^{l_2} \cdot \left( \bigoplus_{1 \leq i \leq r} e_{2,i} + \ldots \right) \right) \]
is well defined in $A(U)^{\oplus r}$ as a convergent infinite series, because $A(U)$ is a complete space and $l_n \geq 1$ for $n > 1$. And, by construction, $\phi(d) = b$, because $A(V)^{\oplus r}$ is a Hausdorff space. Therefore $\phi$ is a surjective map.

Second, we show that $\phi$ is an injective map of $A \oplus r$-modules. Let the sheaf $K$ be a kernel of the map $\phi$. Let $g \in K(U), \ g \neq 0$ for some open $U \subset V$. We have $g \in A_i(U)^{\oplus r} \setminus A_{i+1}(U)^{\oplus r}$ for some $l \in \mathbb{Z}$, then $a^{-l} \cdot g \in A_0(U)^{\oplus r} \setminus A_1(U)^{\oplus r}$. Let $e = (e_1, \ldots, e_r) \in \mathcal{O}_C(U)^{\oplus r}$ be the image of $a^{-l} \cdot g$ under the natural map. Since $a^{-l} \cdot g \in K$, we have
\[ \sum_{1 \leq i \leq r} e_i \cdot \bar{c}_i = 0. \]
Therefore for any $1 \leq i \leq r \ e_i = 0$, because $\bar{c}_1, \ldots, \bar{c}_r$ is a basis over $\mathcal{O}_C(U)$. Hence, $a^{-l} \cdot g \in A_1(U)^{\oplus r}$. We have a contradiction.

\[ \square \]

3 "Picture cohomology".

Let $k$ be a field. Let $W$ be a $k$-subspace in $k((u))(t))^{\oplus r}$. Let
\[ \mathcal{O}_1 = k((u))[t], \quad \mathcal{O}_2 = k[[u]][t), \]
be $k$-subspaces in $k((u))(t))$. We consider the following complex.
\[ (W \cap \mathcal{O}_2^{\oplus r}) \oplus (W \cap \mathcal{O}_1^{\oplus r}) \oplus (\mathcal{O}_1^{\oplus r} \cap \mathcal{O}_2^{\oplus r}) \rightarrow W \oplus \mathcal{O}_2^{\oplus r} \oplus \mathcal{O}_1^{\oplus r} \rightarrow k((u))(t))^{\oplus r} \quad (1) \]
where the first map is given by
\[ (a_0, a_1, a_2) \mapsto (a_1 - a_0, a_2 - a_0, a_2 - a_1) \]
and the second by
\[ (a_{01}, a_{02}, a_{12}) \mapsto a_{01} - a_{02} + a_{12}. \]

Remark 3. We suppose that a $k$-subspace $W \subset k((u))(t))^{\oplus r}$ is a part of a Schur pair
\[ (A, W) \subset k((u))(t)) \oplus k((u))(t))^{\oplus r}. \]

Let, by theorem 1, the pair $(A, W)$ correspond to the data $(C, A, N, P, u, t, e_P)$. We suppose that the ribbon $(C, A)$ comes from an algebraic projective surface $X$, the torsion free sheaf $N$ comes from a locally free sheaf $\mathcal{F}$ on $X$ and so on. It means that the data $(C, A, N, P, u, t, e_P)$ comes from the data $(X, C, \mathcal{F}, P, u, t, e_P)$, where $X$ is an algebraic
projective surface, $C$ is a reduced effective Cartier divisor, $\mathcal{F}$ is a locally free sheaf of rank $r$ on $X$, $P \in C$ is a point which is a smooth point on $X$ and $C$, $u, t$ are formal local parameters of $X$ at $P$ such that $t = 0$ gives the curve $C$ on $X$ in a formal neighbourhood of $P$ on $X$, $e_P$ is a formal trivialization of $\mathcal{F}$ at $P$. We suppose also that $X$ is a Cohen-Macaulay surface and $C$ is an ample divisor on $X$. Then it was proved in [13, 15] that the cohomology groups of complex (1) coincide with the cohomology groups $H^*(X, \mathcal{F})$.

The goal of this section is to relate in general situation the cohomology groups of complex (1) with the cohomology groups of sheaves $\mathcal{N}_i$, where $\mathcal{N}$ is a torsion free sheaf on the ribbon $(C, A)$ when, for example, this ribbon does not come from an algebraic surface.

**Lemma 1.** Let $W$ be a $k$-subspace in $k((u))(\langle t \rangle)^{\oplus r}$. Then the cohomology groups of complex (1) coincide with the following $k$-vector spaces:

\[
\begin{align*}
\mathcal{H}^0(W) &= W \cap \mathcal{O}_1^{\oplus r} \cap \mathcal{O}_2^{\oplus r}, \\
\mathcal{H}^1(W) &= \frac{W \cap (\mathcal{O}_1^{\oplus r} + \mathcal{O}_2^{\oplus r})}{W \cap \mathcal{O}_1^{\oplus r} + W \cap \mathcal{O}_2^{\oplus r}}, \\
\mathcal{H}^2(W) &= \frac{k((u))(\langle t \rangle)^{\oplus r}}{W + \mathcal{O}_1^{\oplus r} + \mathcal{O}_2^{\oplus r}}.
\end{align*}
\]

**Proof.** We have the following exact sequence:

\[
0 \longrightarrow \mathcal{O}_1^{\oplus r} \cap \mathcal{O}_2^{\oplus r} \longrightarrow \mathcal{O}_1^{\oplus r} \oplus \mathcal{O}_2^{\oplus r} \longrightarrow \mathcal{O}_1^{\oplus r} + \mathcal{O}_2^{\oplus r} \longrightarrow 0,
\]

where $\mathcal{O}_1^{\oplus r} + \mathcal{O}_2^{\oplus r}$ is considered as a $k$-subspace in $k((u))(\langle t \rangle)^{\oplus r}$. Now we take the factor-complex of complex (1) by the following acyclic complex:

\[
\mathcal{O}_1^{\oplus r} \cap \mathcal{O}_2^{\oplus r} \longrightarrow \mathcal{O}_1^{\oplus r} \cap \mathcal{O}_2^{\oplus r} \longrightarrow 0.
\]

We obtain the following complex:

\[
(W \cap \mathcal{O}_2^{\oplus r}) \oplus (W \cap \mathcal{O}_1^{\oplus r}) \longrightarrow W \oplus (\mathcal{O}_2^{\oplus r} + \mathcal{O}_1^{\oplus r}) \longrightarrow k((u))(\langle t \rangle)^{\oplus r}.
\]

The cohomology groups of the last complex coincide with the cohomology groups of complex (1). Therefore the statement of this lemma is evident now.

\[\square\]

**Definition 6.** The $k$-vector spaces $\mathcal{H}^i(W)$, $0 \leq i \leq 2$ are called "the picture cohomology" of $W$.

We have the following theorem.
Theorem 2. Let $W$ be a $k$-subspace in $k((u))(t))^{\oplus r}$ such that the $k$-space $W$ is a part of a Schur pair 
\[(A, W) \subset k((u))(t)) \oplus k((u))(t))^{\oplus r}.
\]
Let, by theorem 1, the pair $(A, W)$ correspond to the data $(C, A, N, P, u, t, e_P)$ (see the formulation of theorem 1). Then

\[
\mathcal{H}^0(W) = H^0(C, N_0),
\]

\[
\mathcal{H}^1(W) = \frac{H^0(C, N/N_0)}{H^0(C, N_0/N_0)},
\]

\[
\mathcal{H}^2(W) = H^1(C, N/N_0).
\]

Remark 4. Here and further in the article we consider cohomology in Zariski topology, if another topology is not specified.

Proof. By definition of a torsion free sheaf on a ribbon, we have $N_0 = \lim_{i \to 0} N_0/N_i$.

Therefore

\[
H^0(C, N_0) = \lim_{i \to 0} H^0(C, N_0/N_i) = \lim_{i \to 0} (W(0, i) \cap \mathcal{O}_2^{\oplus r}) = W \cap \mathcal{O}_1^{\oplus r} \cap \mathcal{O}_2^{\oplus r},
\]

where $W(l, i) \overset{\text{def}}{=} \frac{W \cap \mathcal{O}_2^{\oplus r}}{W \cap \mathcal{O}_2^{\oplus r}}$, $l < i \in \mathbb{Z}$. Here we used theorem 2 of [13], where the complex was constructed which calculates in our case the cohomology groups of the coherent sheaf $N_0/N_i$ of $A_0/A_i$-modules on the 1-dimensional scheme $X_{i-1} = (C, A_0/A_i)$. Formula (2) is proved.

Now we will prove formula (3). We have

\[
\mathcal{H}^1(W) = \frac{W \cap (\mathcal{O}_1^{\oplus r} + \mathcal{O}_2^{\oplus r})}{W \cap \mathcal{O}_1^{\oplus r} + W \cap \mathcal{O}_2^{\oplus r}} = \frac{W \cap (\mathcal{O}_1^{\oplus r} + \mathcal{O}_2^{\oplus r})}{W \cap \mathcal{O}_1^{\oplus r} + W \cap \mathcal{O}_2^{\oplus r}} = \frac{W \cap (\mathcal{O}_1^{\oplus r} + \mathcal{O}_2^{\oplus r})}{W \cap \mathcal{O}_1^{\oplus r} + W \cap \mathcal{O}_2^{\oplus r}}.
\]

We note that we have

\[
W \cap \mathcal{O}_2^{\oplus r} = \lim_{i \to j} W(i, j) \cap \mathcal{O}_2^{\oplus r} = \lim_{i \to j} H^0(C, N_i/N_j) = H^0(C, N).
\]

Here we used theorem 2 of [13] for the coherent sheaf $N_i/N_j$ of $A_0/A_{j-i}$-modules on the 1-dimensional scheme $X_{j-i-1} = (C, A_0/A_{j-i})$. Therefore, using it and formula (2), we have

\[
\frac{W \cap \mathcal{O}_2^{\oplus r}}{W \cap \mathcal{O}_2^{\oplus r} \cap \mathcal{O}_1^{\oplus r}} = \frac{H^0(C, N)}{H^0(C, N_0)}.
\]

Hence, to prove formula (3), we have to check that

\[
\frac{W \cap (\mathcal{O}_1^{\oplus r} + \mathcal{O}_2^{\oplus r})}{W \cap \mathcal{O}_1^{\oplus r}} = H^0(C, N/N_0).
\]
By proposition 3 of [10] we have that \( H^0(C \setminus p, \mathcal{N}) = 0. \) Therefore from the exact triple of sheaves on the curve \( C \)
\[
0 \longrightarrow \mathcal{N}_0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{N}/\mathcal{N}_0 \longrightarrow 0
\]
we have
\[
H^0(C \setminus p, \mathcal{N}/\mathcal{N}_0) = \frac{H^0(C \setminus p, \mathcal{N})}{W \cap \mathcal{O}^\oplus} = \frac{W + \mathcal{O}^\oplus}{\mathcal{O}^\oplus}.
\]
Now, as an inductive limit of complexes from theorem 2 of [13], we obtain that
\[
H^0(C, \mathcal{N}/\mathcal{N}_0) = \frac{W + \mathcal{O}^\oplus}{\mathcal{O}^\oplus} \cap \frac{\mathcal{O}^\oplus}{\mathcal{O}^\oplus}.
\]
where the intersection is considered in the \( k \)-vector space \( \frac{k((u)(t))^{\oplus}}{\mathcal{O}^\oplus} \).

There is a natural isomorphism of \( k \)-subspaces in the \( k \)-vector space \( \frac{k((u)(t))^{\oplus}}{\mathcal{O}^\oplus} \):
\[
\frac{W \cap (\mathcal{O}^\oplus + \mathcal{O}^\oplus)}{W \cap \mathcal{O}^\oplus} = \frac{W \cap \mathcal{O}^\oplus}{W \cap \mathcal{O}^\oplus} = \frac{\mathcal{O}^\oplus}{\mathcal{O}^\oplus}.
\]
Therefore we checked formula (5). Hence, we proved formula (3).

Now we will prove formula (4). We have
\[
H^1(C, \mathcal{N}/\mathcal{N}_0) = \lim_{\substack{i < 0}} H^1(C, \mathcal{N}_i/\mathcal{N}_0) = \lim_{\substack{i < 0}} \frac{t^* \mathcal{O}^\oplus}{\mathcal{O}^\oplus} = \frac{k((u)(t))^{\oplus}}{W + \mathcal{O}^\oplus + \mathcal{O}^\oplus}.
\]
Here we used the complex from theorem 2 of [13] to calculate the first cohomology group of coherent sheaf \( \mathcal{N}_i/\mathcal{N}_0 \) on the scheme \( X_{-i-1} = (C, \mathcal{A}_0/\mathcal{A}_{-i}), i < 0 \). Formula (4) is proved.

\[\square\]

4 Formal Picard group and formal Brauer group.

4.1 Picard functor

Let \( Y \longrightarrow Z \) and \( X \longrightarrow Z \) be morphisms of schemes. We consider \( W = Y \times_Z X \) with the natural projection maps \( p : W \longrightarrow Y \) and \( q : W \longrightarrow Z \). Let \( \mathcal{F} \) be an \( \mathcal{O}_Y \)-module sheaf on \( Y \), and \( \mathcal{G} \) be an \( \mathcal{O}_X \)-module sheaf on \( X \). We recall the definition of \( \mathcal{O}_W \)-module sheaf \( \mathcal{F} \boxtimes_{\mathcal{O}_Z} \mathcal{G} \) on \( W \):
\[
\mathcal{F} \boxtimes_{\mathcal{O}_Z} \mathcal{G} \overset{\text{def}}{=} p^*(\mathcal{F}) \otimes_{\mathcal{O}_W} q^*(\mathcal{G}).
\]

Now we recall the definition of a base change for a ribbon from section 2.2 of [10].
For a ribbon \( X_\infty = (C, \mathcal{A}) \) over \( S \), and a morphism \( \alpha : S \to S' \) of Noetherian schemes we define a base change ribbon \( \check{X}_\infty, S' = (C_{S'}, \mathcal{A}_{S'}) \) over \( S' \) in the following way:

\[
C_{S'} := C \times_S S',
\]

\[
\mathcal{A}_{S', j} := \lim_{i \geq 1} (\mathcal{A}_{j}/\mathcal{A}_{j+i}) \boxtimes_{\mathcal{O}_S} \mathcal{O}_{S'}, \quad \mathcal{A}_{S'} := \lim_{j} \mathcal{A}_{S', j}
\]

for any \( j \in \mathbb{Z} \). Sometimes we will denote \( \mathcal{A}_{S', j} \) by \( \mathcal{A}_{j} \boxtimes_{\mathcal{O}_S} \mathcal{O}_{S'} \), and \( \mathcal{A}_{S'} \) by \( \mathcal{A} \boxtimes_{\mathcal{O}_S} \mathcal{O}_{S'} \).

By the ribbon \( \check{X}_\infty = (C, \mathcal{A}) \) over \( S \) we construct a locally ringed space \( \check{X}_\infty = (C, \mathcal{A}) \) over \( S \). And also we define a base change locally ringed space \( \check{X}_\infty, S' = (C_{S'}, \mathcal{A}_{S', r}) \) for the morphism \( \alpha : S' \to S \) of Noetherian schemes.

**Remark 5.** Let \( \mathcal{M} \) be an \( \mathcal{O}_{S'} \)-module sheaf on \( S' \). Then we construct the sheaf of \( \mathcal{A}_{S', j} \)-modules \( \mathcal{A}_{S'} \boxtimes_{\mathcal{O}_{S'}} \mathcal{M} := \lim_{i \geq 1} (\mathcal{A}_{j}/\mathcal{A}_{j+i}) \boxtimes_{\mathcal{O}_S} \mathcal{M} \) on \( C_{S'} \) for any \( j \in \mathbb{Z} \), and the sheaf of \( \mathcal{A}_{S'} \)-modules \( \mathcal{A} \boxtimes_{\mathcal{O}_S} \mathcal{M} := \lim_{j} \mathcal{A}_{S'} \boxtimes_{\mathcal{O}_{S'}} \mathcal{M} \) on \( C_{S'} \). Now let \( \mathcal{N} \) be a coherent \( \mathcal{O}_{S'} \)-module sheaf on \( S' \). Then we have that \( \mathcal{A}_{S'} \boxtimes_{\mathcal{O}_{S'}} \mathcal{N} = \mathcal{A}_{S'} \boxtimes_{\mathcal{O}_{S'}} \mathcal{N} \) and \( \mathcal{A} \boxtimes_{\mathcal{O}_S} \mathcal{N} = \mathcal{A} \boxtimes_{\mathcal{O}_S} \mathcal{N} \).

Indeed, the second fact follows from the first one, because a tensor product commute with an inductive limit. To prove the first fact we note that it is evident when \( \mathcal{N} = \mathcal{O}_{S'}' \), and that the functor \( \mathcal{A}_{S'} \boxtimes_{\mathcal{O}_{S'}} (\cdot) \) is an exact functor on the category of coherent sheaves on \( S' \). Now using the arguments, which are similar to the proof of proposition 10.13 from [2], we obtain that the natural map \( \mathcal{A}_{S'} \boxtimes_{\mathcal{O}_{S'}} \mathcal{N} \to \mathcal{A} \boxtimes_{\mathcal{O}_S} \mathcal{N} \) is an isomorphism.

Let \( k \) be a field, and \( \check{X}_\infty = (C, \mathcal{A}) \) be a ribbon over \( k \). Let \( \mathcal{B} \) be a category of affine Noetherian \( k \)-schemes. Then we define the following contravariant functors \( Pic_{\check{X}_\infty} \) and \( Pic_{X_\infty} \) from \( \mathcal{B} \) to the category of Abelian groups.

**Definition 7.** Let \( \mathcal{B} \) be a category of affine Noetherian \( k \)-schemes. Then we define the following contravariant functors \( Pic_{\check{X}_\infty} \) and \( Pic_{X_\infty} \) from \( \mathcal{B} \) to the category of Abelian groups:

1. \( Pic_{\check{X}_\infty}(S) \overset{\text{def}}{=} Pic(\check{X}_\infty, S) = H^1(C_S, \mathcal{A}^*_S) \);
2. \( Pic_{X_\infty}(S) \overset{\text{def}}{=} Pic(X_\infty, S) = H^1(C_S, \mathcal{A}^*_{S, 0}) \).

**Remark 6.** If the locally ringed space \( X_\infty = (C, \mathcal{A}) \) is defined over a field \( k \) and \( C \) is proper, then \( X_\infty \) is a weakly Noetherian formal scheme in the sense of [11]. Therefore we have from the theorem of section 2.5 of loc.cit. that if \( \text{char} k = 0 \), then the sheaf associated on the large étale site of \( \text{Spec} k \) with the functor \( S \mapsto Pic_{X_\infty}(S) \) is a \( k \)-group scheme \( Pic_{X_\infty} \). (In other words, for any Noetherian affine scheme \( S \) over \( k \) we consider all étale covers of \( S \) and we take the sheaf associated with the presheaf \( S' \mapsto Pic_{X_\infty}(S') \) with respect to these covers. Then this sheaf is represented by a \( k \)-group scheme \( Pic_{X_\infty} \).

If \( \text{char} k = p > 0 \), then it was proved in [11] that the fpqc sheaf associated with the modified Picard functor of \( X_\infty \) is also a \( k \)-group scheme.

If \( \text{char} k = 0 \), one can obtain the structure of this \( k \)-group scheme \( Pic_{X_\infty} \) enough easily, see remark 20 below.

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4.2 Zariski tangent space.

We recall the definition of the Zariski tangent space to a functor at 0. Let $\hat{X}_\infty = (C, \mathcal{A})$ be a ribbon over a field $k$. Let $E = k \oplus k \cdot \epsilon$, where $\epsilon^2 = 0$, be a $k$-algebra.

$$T_{\text{Pic}\hat{X}_\infty}(0) \overset{\text{def}}{=} \text{Ker}(\text{Pic}\hat{X}_\infty(\text{Spec} E) \longrightarrow \text{Pic}\hat{X}_\infty(\text{Spec} k)).$$

Analogously,

$$T_{\text{Pic}X_\infty}(0) \overset{\text{def}}{=} \text{Ker}(\text{Pic}X_\infty(\text{Spec} E) \longrightarrow \text{Pic}X_\infty(\text{Spec} k)).$$

We have the following proposition.

**Proposition 2.** Let $\hat{X}_\infty = (C, \mathcal{A})$ be a ribbon over a field $k$.

1. We have $T_{\text{Pic}\hat{X}_\infty}(0) = H^1(C, \mathcal{A})$ and $T_{\text{Pic}X_\infty}(0) = H^1(C, \mathcal{A}_0)$.

2. Let the ribbon $\hat{X}_\infty$ correspond to some generalized Fredholm $k$-subalgebra $\mathcal{A}$ in $\text{k}((u))((t))$ (after a choice of a smooth point $P \in C$ of the ribbon, formal local parameters $u, t$, see theorem 1). Then we have the following exact sequence of $k$-vector spaces:

$$0 \longrightarrow H^1(\mathcal{A}) \longrightarrow T_{\text{Pic}\hat{X}_\infty}(0) \longrightarrow T_{\text{Pic}X_\infty}(0) \longrightarrow H^2(\mathcal{A}) \longrightarrow 0. \quad (6)$$

**Proof.** Let $R = \text{Spec} E$. We denote the base change sheaves 

$$\mathcal{A}' = \mathcal{A}_R = \mathcal{A} \oplus \epsilon \cdot \mathcal{A}, \quad \mathcal{A}'_0 = \mathcal{A}_{R,0} = \mathcal{A} \oplus \epsilon \cdot \mathcal{A}_0,$$

where $\epsilon^2 = 0$. Then we have canonically the following decompositions:

$$\mathcal{A}'^* = \mathcal{A}^* \times (1 + \epsilon \cdot \mathcal{A}) = \mathcal{A}^* \times \mathcal{A};$$

$$\mathcal{A}'_0^* = \mathcal{A}_0^* \times (1 + \epsilon \cdot \mathcal{A}_0) = \mathcal{A}_0^* \times \mathcal{A}_0.$$ 

Therefore we have canonically:

$$H^1(C, \mathcal{A}'^*) = H^1(C, \mathcal{A}^*) \times H^1(C, \mathcal{A});$$

$$H^1(C, \mathcal{A}'_0^*) = H^1(C, \mathcal{A}_0^*) \times H^1(C, \mathcal{A}_0).$$

Hence, we obtain

$$T_{\text{Pic}\hat{X}_\infty}(0) = H^1(C, \mathcal{A}), \quad T_{\text{Pic}X_\infty}(0) = H^1(C, \mathcal{A}_0). \quad (7)$$

We have the following exact sequence of sheaves on $C$:

$$0 \longrightarrow \mathcal{A}_0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{A}_0 \longrightarrow 0.$$ 

Hence we have the following long exact sequence

$$0 \longrightarrow \frac{H^0(C, \mathcal{A}/\mathcal{A}_0)}{H^0(C, \mathcal{A})} \longrightarrow H^1(C, \mathcal{A}_0) \longrightarrow H^1(C, \mathcal{A}) \longrightarrow H^1(C, \mathcal{A}/\mathcal{A}_0) \longrightarrow 0. \quad (8)$$

Now, using it, formulas (7) and theorem 2 we obtain exact sequence (6).
Remark 7. According to remark 3 and lemma 1, we have that if a ribbon $\tilde{X}_\infty = (C, A)$ comes from an algebraic projective Cohen-Macaulay surface $X$ and an ample Cartier divisor $C$ on $X$, then exact sequence (6) transforms to the following exact sequence:

$$0 \rightarrow H^1(X, \mathcal{O}_X) \rightarrow TPic_{X_\infty}(0) \rightarrow TPic_{\tilde{X}_\infty}(0) \rightarrow H^2(X, \mathcal{O}_X) \rightarrow 0.$$ 

4.3 Formal Brauer group of an algebraic surface.

We suppose in this subsection that a field $k$ has char $k = 0$.

Let $X$ be a projective algebraic surface over the field $k$. We recall the following definition of the formal Brauer group of the surface $X$ from [1].

**Definition 8.** Let $\mathcal{C}$ be the category of affine Artinian local $k$-schemes with residue field $k$ (i.e. the full subcategory of affine $k$-schemes such that $S \in \text{Ob}(\mathcal{C})$ iff $S = \text{Spec } B$ for an Artinian local $k$-algebra $B$ with residue field $k$). The formal Brauer group $\hat{Br}_X$ of $X$ is a contravariant functor from $\mathcal{C}$ to the category of Abelian groups which is given by the following rule:

$$\hat{Br}_X(S) \stackrel{\text{def}}{=} \text{Ker}(H^2(X \times_k S, \mathcal{O}_{X \times_k S}^*)) \rightarrow H^2(X, \mathcal{O}_X^*),$$

where $S \in \text{Ob}(\mathcal{C})$.

We used the Zariski topology for the definition of the functor $\hat{Br}_X$. But, as it was noticed in [1, ch. II] (because of the filtration with factors being coherent sheaves), we can use, for example, the étale topology, i.e. we have the following equality:

$$\hat{Br}_X(S) \stackrel{\text{def}}{=} \text{Ker}(H^2_{\text{ét}}(X \times_k S, \mathcal{O}_{X \times_k S}^*)) \rightarrow H^2_{\text{ét}}(X, \mathcal{O}_X^*),$$

where $S \in \text{Ob}(\mathcal{C})$. It explains the name “formal Brauer group”.

In [1, corollary 4.1] it was proved that under some conditions on $X$ the functor $\hat{Br}_X$ is pro-representable by the formal group scheme $\hat{Br}_X$ which is a formal group (for fields $k$ of any characteristic). It means that for any $S \in \text{Ob}(\mathcal{C})$:

$$\hat{Br}_X(S) = \text{Hom}_{\text{form.sch.}}(S, \hat{Br}_X),$$

where $\text{Hom}_{\text{form.sch.}}$ is considered in the category of formal schemes.

Since we supposed that char $k = 0$, we will give an easy proof that the functor $\hat{Br}_X$ is always pro-representable in the following lemma.

**Lemma 2.** The functor $\hat{Br}_X$ from the category $\mathcal{C}$ to the category of Abelian groups is pro-representable by the formal group scheme $\hat{Br}_X = \text{Spf } \hat{\text{Sym}}_k(H^2(X, \mathcal{O}_X)^*)$, where the group law in the formal group $\hat{Br}_X$ is given by $v \mapsto v \otimes 1 + 1 \otimes v$, $v \in H^2(X, \mathcal{O}_X)^*$. 

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Proof. By definition, we have for \( k \)-algebra
\[
\widehat{\text{Sym}}_k(H^2(X, \mathcal{O}_X)^*) = \prod_{i=0}^{\infty} S^i(H^2(X, \mathcal{O}_X)^*),
\]
where \( S^i(\cdot) \) is the \( k \)-vector space of symmetric \( i \)-th tensors over the field \( k \), \( S^0(\cdot) = k \). \( \widehat{\text{Sym}}_k(H^2(X, \mathcal{O}_X)^*) \) is a topological local \( k \)-algebra over a discrete field \( k \). This topology is given by the infinite product topology of discrete spaces.

For any \( S = \text{Spec } B \in \text{Ob}(\mathcal{C}) \) we have \( B = k \oplus I \), where \( I \) is the maximal ideal in the ring \( B \), \( \dim_k I < \infty \) and \( I^n = 0 \) for some \( n \geq 0 \). We consider the discrete topology on the ring \( B \). Therefore we have
\[
\text{Hom}_{\text{form.sch.}}(S, \text{Spf } \widehat{\text{Sym}}_k(H^2(X, \mathcal{O}_X)^*)) = \text{Hom}_{k-\text{alg.,cont.}}(\widehat{\text{Sym}}_k(H^2(X, \mathcal{O}_X)^*), B) =
\]
\[
\text{Hom}_k(H^2(X, \mathcal{O}_X)^*, I) = H^2(X, \mathcal{O}_X) \otimes_k I,
\]
where \( \text{Hom}_{k-\text{alg.,cont.}} \) is considered in the category of topological \( k \)-algebras. We used also that \( H^2(X, \mathcal{O}_X)^* = H^2(X, \mathcal{O}_X) \), because \( \dim_k H^2(X, \mathcal{O}_X) < \infty \).

On the other hand, for base change sheaf \( \mathcal{O}_X' = \mathcal{O}_{X \times_k S} \) on \( X \) we have
\[
\mathcal{O}_X' = \mathcal{O}_X \oplus (\mathcal{O}_X \otimes_k I). \text{ Hence } \mathcal{O}_X^* = \mathcal{O}_X^* \times (1 + \mathcal{O}_X \otimes_k I). \text{ The exponential map gives an isomorphism of sheaves of Abelian groups:}
\]
\[
exp : \mathcal{O}_X \otimes_k I \longrightarrow 1 + \mathcal{O}_X \otimes_k I.
\]

Therefore \( \mathcal{O}_X^* = \mathcal{O}_X^* \times (\mathcal{O}_X \otimes_k I) \). Hence
\[
H^2(X, \mathcal{O}_X^*) = H^2(X, \mathcal{O}_X') \times H^2(X, \mathcal{O}_X \otimes_k I) = H^2(X, \mathcal{O}_X^*) \times (H^2(X, \mathcal{O}_X) \otimes_k I).
\]

Therefore we have
\[
\widehat{\text{Br}}_X(S) = H^2(X, \mathcal{O}_X) \otimes_k I.
\]

Moreover, we have the universal object
\[
\tau \in \text{Ker}(H^2(X \times_k \widehat{\text{Br}}_X, \mathcal{O}_X^*_{X \times_k \widehat{\text{Br}}_X}) \rightarrow H^2(X, \mathcal{O}_X^*)�.,
\]
which is constructed in the following way. We have
\[
\text{Ker}(H^2(X \times_k \widehat{\text{Br}}_X, \mathcal{O}_X^*_{X \times_k \widehat{\text{Br}}_X}) \rightarrow H^2(X, \mathcal{O}_X')) = H^2(X, 1 + J \otimes_k \mathcal{O}_X) = H^2(X, \mathcal{O}_X) \otimes_k J,
\]
where the ideal \( J = \prod_{i=1}^{\infty} S^i(H^2(X, \mathcal{O}_X)) \). Besides, the sheaf of Abelian groups \( J \otimes_k \mathcal{O}_X \) is isomorphic to the sheaf \( 1 + J \otimes_k \mathcal{O}_X \) via the exponential map.

Now \( \tau = Id \), where \( Id \) is the identity map from
\[
\text{End}_k(H^2(X, \mathcal{O}_X)) = H^2(X, \mathcal{O}_X) \otimes_k H^2(X, \mathcal{O}_X)^*.
\]

And there is a canonical embedding of \( k \)-vector spaces:
\[
H^2(X, \mathcal{O}_X) \otimes_k H^2(X, \mathcal{O}_X)^* \subset H^2(X, \mathcal{O}_X) \otimes_k J.
\]
4.4 Formal Brauer group of a ribbon.

We use in this subsection the same notations as in subsection 4.3. In particularly, a field $k$ has char $k = 0$, $C$ is the category of affine Artinian local $k$-schemes with residue field $k$. We introduce the following definition.

**Definition 9.** Let $\hat{X}_\infty = (C, \mathcal{A})$ be a ribbon over a field $k$. The formal Brauer group $\hat{\text{Br}}_{\hat{X}_\infty}$ of $\hat{X}_\infty$ is a contravariant functor from $C$ to the category of Abelian groups which is given by the following rule:

$$\hat{\text{Br}}_{\hat{X}_\infty}(S) \overset{\text{def}}{=} \ker(H^1(C_S, \mathcal{A}_S^*/\mathcal{A}_{S,0}^*)) \to H^1(C, \mathcal{A}^*/\mathcal{A}_0^*)),$$

where $S \in \text{Ob}(C)$.

Below, in remark 9, we will explain, why we use the name "formal Brauer group" for a ribbon. Now we have the following proposition.

**Proposition 3.** Assume $C$ is a projective curve. Then the functor $\hat{\text{Br}}_{\hat{X}_\infty}$ from the category $C$ to the category of Abelian groups is pro-representable by a formal group scheme $\hat{\text{Br}}_{\hat{X}_\infty}$.

**Proof.** We denote a $k$-vector space $V = H^1(C, \mathcal{A}/\mathcal{A}_0)$. (We note that, by theorem 2, $V = H^2(A)$ when $A$ is a generalized Fredholm subspace in $k((u))(t))$ which correspond to the ribbon $X_\infty = (C, \mathcal{A})$ with some smooth point $P \in C$ of the ribbon, formal local parameters $u, t$, see theorem 1). We note that we have canonically:

$$V = \lim_{i \geq 0} V_i,$$

where $V_i = H^1(C, \mathcal{A}_{-i}/\mathcal{A}_0), \ i \geq 0$. And dim $V_i < \infty$, since $C$ is a projective curve and $\mathcal{A}_{-i}/\mathcal{A}_0$ are coherent sheaves on the scheme $X_{i-1}$ by [10, prop. 1]. Therefore we have

$$V^* = \lim_{i \geq 0} V_i^*.$$

The $k$-vector space $V^*$ has a natural linearly compact topology, which is given by topology of this projective limit, where every $V_i^*$ has a discrete topology, see [4, ch.III, §2, ex. 15].

For any $l \geq 0$ we define

$$S^l_{cont}(V^*) \overset{\text{def}}{=} \lim_{i \geq 0} S^l(V_i^*).$$

These spaces has also a linearly compact topology, which is given by the projective limit. We define

$$T = \text{Sym}_{k, cont}(V^*) \overset{\text{def}}{=} \prod_{l=0}^{\infty} S^l_{cont}(V^*).$$

By construction, we have $S^0_{cont}(V^*) = k$. The $k$-vector space $T$ has the product topology. Therefore $T$ is a linearly compact space as the product of linearly compact spaces. Hence $T$ is Hausdorff and complete, see [4, ch.III, §2, ex. 16].

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For any $l_1 \geq 0, l_2 \geq 0$ we have canonical continuous bilinear map over $k$:

$$S^{l_1}_{cont}(V^*) \times S^{l_2}_{cont}(V^*) \longrightarrow S^{l_1+l_2}_{cont}(V^*).$$

Therefore $T$ is a topological local $k$-algebra (over the discrete field $k$). The maximal ideal $J$ in $T$ is given as

$$J = \prod_{l=1}^{\infty} S^l_{cont}(V^*).$$

By construction, for any open $k$-subspace $U \subset T$ there is $m > 0$ such that $U \supset J^m$. Using these properties of topological $k$-algebra $T$ we obtain that the following formal scheme is well-defined (see [6, ch. I, §10]):

$$\hat{\text{Br}}_{X,\infty} \overset{\text{def}}{=} \text{Spf}(T).$$

Moreover, $\hat{\text{Br}}_{X,\infty}$ is a formal group with the group law $v \mapsto v \otimes 1 + 1 \otimes v$, $v \in V^* = S^1_{cont}(V^*)$. ($V^*$ topologically generates the $k$-algebra $T$.)

Now we have to check that the formal group scheme $\hat{\text{Br}}_{X,\infty}$ pro-represents the functor

$$\text{Hom}_{\text{form.sch.}}(S, \hat{\text{Br}}_{X,\infty}) = \text{Hom}_{k-\text{alg}}(T, B) = \text{Hom}_{k,\text{cont}}(V^*, I),$$

where $\text{Hom}_{k,\text{cont}}$ is considered in the category of topological $k$-vector spaces. Since $\dim_k(V_i) < \infty$, $i \geq 0$, we have

$$\text{Hom}_{k,\text{cont}}(V^*, k) = V.$$  \hspace{1cm} (9)

(We note also that $V^* = \text{Hom}_{k,\text{cont}}(V, k)$, where $V$ has a discrete topology.) Since $\dim_k I < \infty$ and $I$ has a discrete topology, we obtain the following formula from formula (9):

$$\text{Hom}_{k,\text{cont}}(V^*, I) = V \otimes_k I.$$  \hspace{1cm} (10)

Therefore

$$\text{Hom}_{\text{form.sch.}}(S, \hat{\text{Br}}_{X,\infty}) = V \otimes_k I.$$ \hspace{1cm} (10)

On the other hand, we have the following split exact sequence:

$$1 \to 1 + I \otimes_k \mathcal{A}/\mathcal{A}_0 \to \mathcal{A}^*_S/\mathcal{A}^*_{S,0} \to \mathcal{A}^*/\mathcal{A}^*_0 \to 0,$$

which is the factor of the exact sequence

$$1 \to 1 + I \otimes_k \mathcal{A} \to \mathcal{A}^*_S \to \mathcal{A}^* \to 0$$ \hspace{1cm} (11)

by the following exact sequence

$$1 \to 1 + I \otimes_k \mathcal{A}_0 \to \mathcal{A}^*_{S,0} \to \mathcal{A}^*_0 \to 0.$$ \hspace{1cm} (12)

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The sheaf of Abelian groups $I \otimes_k \mathcal{A}/\mathcal{A}_0$ is isomorphic to the sheaf $1 + I \otimes_k \mathcal{A}/\mathcal{A}_0$ via the exponential map. Therefore, we have

$$\widehat{Br}_{\mathcal{X}_\infty}(S) = H^1(C, 1 + I \otimes_k \mathcal{A}/\mathcal{A}_0) \simeq H^1(C, I \otimes_k \mathcal{A}/\mathcal{A}_0) = I \otimes_k H^1(C \otimes_k \mathcal{A}/\mathcal{A}_0).$$

So, $\widehat{Br}_{\mathcal{X}_\infty}(S) = \text{Hom}_{\text{form.sch.}}(S, \widehat{Br}_{\mathcal{X}_\infty}).$

\[\square\]

**Remark 8.** Let $A$ be any commutative $k$-algebra. Then we have an analog of formula (10):

$$\text{Hom}_{\text{form.sch.}}(\text{Spec } A, \widehat{Br}_{\mathcal{X}_\infty}) = V \otimes_k N_A,$$

where $N_A$ is the nilradical of the ring $A$. Indeed, following the proof of formula (10), we see that it is enough to prove the following formula

$$\text{Hom}_{k,\text{cont}}(V^*, N_A) = V \otimes_k N_A, \quad (13)$$

where $N_A$ has a discrete topology. But we have that $V^*$ is a linearly compact $k$-vector space. Therefore for any $\phi \in \text{Hom}_{k,\text{cont}}(V^*, N_A)$ we have that $\phi(V^*)$ is a linearly compact $k$-vector subspace in a discrete $k$-vector space $N_A$. Hence $\dim_k \phi(V^*) < \infty$. Now, using formula (9), we obtain formula (13).

**Remark 9.** By remark 3 and theorem 2 we have that if a ribbon $\mathcal{X}_\infty = (C, \mathcal{A})$ comes from an algebraic projective Cohen-Macaulay surface $X$ and an ample Cartier divisor $C$ on $X$, then $H^2(X, \mathcal{O}_X) = H^1(C, \mathcal{A}/\mathcal{A}_0)$. Therefore we have

$$\widehat{Br}_X = \text{Spf } \widehat{\text{Sym}}_k(H^2(X, \mathcal{O}_X)^*) = \widehat{Br}_{\mathcal{X}_\infty}.$$

**4.5 Formal Picard group of a ribbon.**

We use in this subsection the same notations as in subsection 4.4.

**Definition 10.** Let $\mathcal{X}_\infty = (C, \mathcal{A})$ be a ribbon over a field $k$. The formal Picard group $\widehat{\text{Pic}}_{\mathcal{X}_\infty}$ of $\mathcal{X}_\infty$ is a contravariant functor from $\mathcal{C}$ to the category of Abelian groups which is given by the following rule:

$$\widehat{\text{Pic}}_{\mathcal{X}_\infty}(S) \overset{\text{def}}{=} \text{Ker}(H^1(C_S, \mathcal{A}_S^*) \longrightarrow H^1(C, \mathcal{A}^*)),$$

where $S \in \text{Ob}(\mathcal{C})$.

Analogously, the formal Picard group $\widehat{\text{Pic}}_{X_\infty}$ of $X_\infty$ is a functor from $\mathcal{C}$ to the category of Abelian groups given by

$$\widehat{\text{Pic}}_{X_\infty}(S) \overset{\text{def}}{=} \text{Ker}(H^1(C_S, \mathcal{A}_{S,0}^*) \longrightarrow H^1(C, \mathcal{A}_0^*)).$$
Let $S = \text{Spec} \, B \in \text{Ob}(C)$, where $B = k \oplus I$, $I$ is the maximal ideal of the ring $B$, $\dim_k I < \infty$, and $I^n = 0$ for some $n \geq 0$.

As it follows from split exact sequences (11), (12), using exponential and logarithmic maps, we have

$$\widehat{\text{Pic}}_{X_\infty}(S) = I \otimes_k H^1(C, A), \quad \widehat{\text{Pic}}_{X_\infty}(S) = I \otimes_k H^1(C, A_0).$$

So, if we define the contravariant functor $P$ from $C$ to the category of Abelian groups by the rule

$$P(S) \overset{\text{def}}{=} I \otimes_k H,$$

where $H \overset{\text{def}}{=} \frac{H^0(C, A/A_0)}{H^0(C, A)}$, we get the following exact sequence of groups, which is functorial on $C$ (compare with sequence (8)):

$$0 \longrightarrow P(S) \longrightarrow \widehat{\text{Pic}}_{X_\infty}(S) \longrightarrow \widehat{\text{Pic}}_{X_\infty}(S) \longrightarrow \widehat{\text{Br}}_{X_\infty}(S) \longrightarrow 0 \quad (14)$$

We define another functor $\overline{\text{Pic}}$ from $C$ to the category of Abelian groups by the rule

$$\overline{\text{Pic}}(S) \overset{\text{def}}{=} I \otimes_k \frac{H^1(C, A_0)}{H}.$$ (If $H = 0$, then $\overline{\text{Pic}} = \widehat{\text{Pic}}_{X_\infty}$.) Then from sequence (14) we obtain another exact sequence:

$$0 \longrightarrow \overline{\text{Pic}}(S) \longrightarrow \overline{\text{Pic}}_{X_\infty}(S) \longrightarrow \overline{\text{Pic}}_{X_\infty}(S) \longrightarrow \widehat{\text{Br}}_{X_\infty}(S) \longrightarrow 0 \quad (15)$$

**Proposition 4.** Assume $C$ is a projective curve. Then we have:

1. There is a noncanonical functorial (with $S$) splitting of sequence (15).

2. If $\dim_k H < \infty$, then the functor $\overline{\text{Pic}}$ from the category $C$ to the category of Abelian groups is pro-representable by the formal group scheme $\overline{\text{Pic}}$.

**Remark 10.** The condition $\dim_k H < \infty$ of this proposition is satisfied, for example, if the ribbon comes from an algebraic projective surface $X$ as in remark 7, since in this case $H = H^1(X, \mathcal{O}_X)$. Another example is a ribbon coming from the Schur pair $(A, W)$ as in theorem 1, where $A$ is chosen so that $\dim_k H^1(A) < \infty$. Due to lemma 1 one can easily construct a lot of examples of such spaces.

**Proof.** The first claim is clear, because we can fix any $k$-linear section of the map $H^1(C, A) \rightarrow H^1(C, A/A_0)$ and then extend it for any $S$ in (15) by tensor product with identity map on $I$ over $k$.

The proof of the second assertion is similar to the proof of proposition 3. Namely, let $\dim_k H < \infty$. We have $H^1(C, A_0) = \lim_{j \geq 0} H^1(C, A_j/A_0)$ by [10, corollary 1], and $\dim_k H^1(C, A_0/A_j) < \infty$. The homomorphism $i : H \rightarrow H^1(C, A_0)$ gives a system of compatible homomorphisms $i_j : H \rightarrow H^1(C, A_0/A_j)$. Denote by $K_j$ the kernel of $i_j$ and
by $C_j$ the cokernel of $i_j$. Then we obtain the exact sequence of projective systems of $k$-vector spaces:

$$0 \longrightarrow (K_j) \longrightarrow (H_j) \longrightarrow H^1(C, \mathcal{A}_0/\mathcal{A}_j) \longrightarrow (C_j) \longrightarrow 0,$$

where $H_j = H$ for all $j$. Since $\dim_k K_j < \infty$ for all $j$, the systems $(K_j), (H_j)$ satisfy the Mittag-Leffler condition. Then by [10, lemma 1] we obtain that

$$H^1(C, \mathcal{A}_0)/H \simeq \lim_{\longleftarrow} j \in \mathbb{N} C_j,$$

where $\dim_k C_j < \infty$ for all $j$. Denote by $V$ the $k$-vector space $H^1(C, \mathcal{A}_0)/H$.

We have $V^*_{\text{cont}} := \text{Hom}_{k, \text{cont}}(V, k) = \lim_{\longleftarrow} j C_j^*$ is a $k$-vector space with a discrete topology. Now we define

$$T := \widehat{\text{Sym}}_k(V^*_{\text{cont}}) = \prod_{l=0}^\infty S^l(V^*_{\text{cont}}).$$

$T$ is a topological local $k$-algebra with the maximal ideal $J = \prod_{l=1}^\infty S^l(V^*_{\text{cont}})$. The topology on $T$ is a linear product topology. It is clear that $J$ is the maximal ideal of definition and that $T$ is an admissible ring (and moreover, adic) in the sense of [6, 0.7.1]. Therefore we can define

$$\widehat{\text{Pic}} \overset{\text{def}}{=} \text{Spf}(T).$$

Again, as in proposition 3, $\widehat{\text{Pic}}$ is a formal group with the group law $v \mapsto v \otimes 1 + 1 \otimes v, v \in V^*_{\text{cont}}$.

For any $S = \text{Spec} B \in \text{Ob}(\mathcal{C})$ we have

$$\text{Hom}_{\text{form.sch.}}(S, \widehat{\text{Pic}}) = \text{Hom}_{k-\text{alg,cont}}(T, B) = \text{Hom}_{k, \text{cont}}(V^*_{\text{cont}}, I) = V \otimes_k I = \widehat{\text{Pic}}(S).$$

Now we have the following obvious corollary of proposition 4.

**Corollary 1.** Let $C$ be a projective curve, and $\dim_k H < \infty$. Then the functor $\widehat{\text{Pic}}_{X_{\infty}}$ is (noncanonically) pro-representable by the formal group scheme $\widehat{\text{Pic}} \times_k \widehat{\text{Br}}_{X_{\infty}}$. Such decompositions are in one-to-one correspondence with functorial (with $S$) splittings of sequence (15).

## 5 Picard functor of a ribbon.

### 5.1 The order function

In this subsection we will give an appropriate generalization of the order function used in [10, §4]. It will be used later.

For a topological space $U$ let $W_U(\mathbb{Z})$ be a sheaf of functions on $U$ with values in $\mathbb{Z}$.
Let $\mathcal{X}_\infty$ be a ribbon over a Noetherian scheme $S$.

In the following we assume that for every $s \in S$ there exists a point $P_s \in \mathcal{X}_\infty$ such that $(\mathcal{A}_{s,1})_{P_s} (\mathcal{A}_{s,-1})_{P_s} = (\mathcal{A}_{s,0})_{P_s}$ and that the underlying topological space of $\mathcal{X}_{\infty,s}$ is irreducible. We also assume that the morphism $\tau : C \to S$ from definition 1 is locally of finite type.

Note that by [10, prop.9] the function of order $\text{ord}$ defined in [10, def.6] is a morphism of sheaves of groups on any $\mathcal{X}_{\infty,s}$.

**Remark 11.** For example, this assumption is satisfied in the case of the ribbon $\mathcal{X}_{\infty,S}$, where $S \to \text{Spec} \ k$ is a base change, and $\mathcal{X}_\infty$ is a ribbon over an algebraically closed field $k$ with irreducible underlying topological space and either with a smooth point in the sense of [10, def. 9], or with condition $(\ast\ast)$ from definition 5.

Indeed, in this case for every $s \in S$ we have that the underlying topological space of $\mathcal{X}_\infty$ is an irreducible curve by [19, vol.I, ch.III, §15, th.40, cor.1] (see also [9, ch. II, ex.3.20]), and $\tau$ is of finite type. If $P$ is a smooth point of the ribbon $\mathcal{X}_\infty$, and $P_s$ is the closed point that maps to $P$, then $P_s$ is a smooth point of the ribbon $\mathcal{X}_{\infty,s}$. The reason is that we can lift the elements $t \in \mathcal{A}_{1,P}$, $t' \in \mathcal{A}_{-1,P}$ with $tt' = 1$ to analogous elements $t_s \in \mathcal{A}_{1,P_s}$, $t'_s \in \mathcal{A}_{-1,P_s}$. Then, for example, the arguments from the proof of [10, prop.7] show that $P_s$ is smooth. The same arguments work in the case of condition $(\ast\ast)$.

**Definition 11** (order map). Define a morphism of sheaves of sets

$$\text{ord} : \mathcal{A}^* \to W_C(\mathbb{Z}), \quad \text{ord}(a)(x) \overset{\text{def}}{=} \max\{j \mid a|_{U_s} \in \mathcal{A}_{s,j}(U_s)\},$$

where $a \in \mathcal{A}^*(U)$ for an open $U \subset C$, $x \in U$, $s = \tau(x)$.

On any $\mathcal{X}_{\infty,s}$, $\text{ord}$ coinside with the order function from [10, def. 6]. By [10, prop. 9] $\text{ord}$ is compatible with restriction homomorphisms on any $\mathcal{X}_{\infty,s}$. So, our definition is correct.

We want to give a condition when $\text{ord}$ is a morphism of sheaves of groups, and when it factors through the sheaf $\mathbb{Z}_C \subset W_C(\mathbb{Z})$ of locally constant functions. We also want to describe in this case the kernel of the order map.

If $S = \text{Spec} \ K$, where $K$ is a field, then this definition coinside with the definition 6 of [10]. In [10, prop. 8, prop. 9] we gave certain sufficient conditions for the order function to be a homomorphism (obviously, in this case it is locally constant), see also counter-example 7 in loc.cit.

**Lemma 3.** In our assumptions we have: for every $P_s$ there exists its affine neigbourhood $U_{P_s} \subset C$ such that all $\mathcal{A}_j|_{U_{P_s}}$ are invertible sheaves of $\mathcal{A}_0|_{U_{P_s}}$-modules and $\mathcal{A}_{-j}|_{U_{P_s}} = \mathcal{A}_{j}^{-1}|_{U_{P_s}}$.

**Proof.** First, let’s prove that the natural homomorphism of $\mathcal{O}_{C,P_s}$-modules

$$\mathcal{A}_{-1}/\mathcal{A}_{0} \otimes_{\mathcal{O}_{C,P_s}} (\mathcal{A}_{1}/\mathcal{A}_{2})_{P_s} \to (\mathcal{A}_{0}/\mathcal{A}_{1})_{P_s} = \mathcal{O}_{C,P_s}$$

is an isomorphism.
the homomorphism (16) is surjective by Nakayama’s lemma. Let $K$ be the kernel of this map. Then, since $\mathcal{O}_{C,P_s}$ is a flat $\mathcal{O}_{S,s}$-module, the following sequence is exact (by [2, ch.2, ex.26]):

$$0 \rightarrow K \otimes_{\mathcal{O}_{S,s}} k(s) \rightarrow ((\mathcal{A}_1/\mathcal{A}_0)_{P_s} \otimes_{\mathcal{O}_{C,P_s}} (\mathcal{A}_1/\mathcal{A}_2)_{P_s}) \otimes_{\mathcal{O}_{S,s}} k(s) \rightarrow \mathcal{O}_{C,P_s} \rightarrow 0.$$ 

Since

$$(\mathcal{A}_1/\mathcal{A}_0)_{P_s} \otimes_{\mathcal{O}_{C,P_s}} (\mathcal{A}_1/\mathcal{A}_2)_{P_s} \otimes_{\mathcal{O}_{S,s}} k(s) \cong [(\mathcal{A}_1/\mathcal{A}_0)_{P_s} \otimes_{\mathcal{O}_{S,s}} k(s)] \otimes_{\mathcal{O}_{C,P_s}} [(\mathcal{A}_1/\mathcal{A}_2)_{P_s} \otimes_{\mathcal{O}_{S,s}} k(s)],$$

we get $0 = K \otimes_{\mathcal{O}_{S,s}} k(s) = K/\mathcal{M}_sK$. Therefore, $K = 0$ by Nakayama’s lemma.

Now let $U_{P_s}$ be an affine neighbourhood, where there exist $\tilde{t} \in \mathcal{A}_1/\mathcal{A}_0(U_{P_s})$, $\tilde{t} \in \mathcal{A}_1/\mathcal{A}_2(U_{P_s})$ such that $\tilde{t}\tilde{t}' = 1$. Then, by [10, prop.3], we can lift the elements $\tilde{t}, \tilde{t}'$ and find $t' \in \mathcal{A}_1^{-1}(U_{P_s})$, $t \in \mathcal{A}_1(U_{P_s})$ such that $tt' = 1$. Then for any $j$ we have $\mathcal{A}_j|_{U_{P_s}} = t^j(\mathcal{A}_0|_{U_{P_s}})$ (compare with the arguments in the proof of prop. 7 from [10]).

\[\square\]

**Remark 12.** If for any $s \in S$ the ribbon $\tilde{X}_{\infty,s}$ satisfies the condition (***) from definition 5, then the statements of lemma are valid for the sheaves $\mathcal{A}_j$ on the whole space $C$ (not only on $U_{P_s}$). The proof is the same.

Let’s consider several cases.

**Case 1.** Let $S$ be an integral scheme. We claim that the order map on $\mathcal{A}^*|_{U_{P_s}}$ factors through $\mathbb{Z}_C|_{U_{P_s}}$ and is a morphism of sheaves of Abelian groups. Moreover, $(\mathcal{A}^*/\mathcal{A}_0^*)|_{U_{P_s}} \cong \mathbb{Z}_C|_{U_{P_s}}$.

Let $a \in \mathcal{A}^*(U_{P_s})$ and $j$ be the biggest integer with $a \in \mathcal{A}_j(U_{P_s})$. Then $j = \text{ord}(a)(x)$ for any $x \in U_{P_s}$. Indeed, by lemma 3 there exists an invertible element $t \in \mathcal{A}_1(U_{P_s})$. So, $a = a_0t^j$ with $a_0 \in \mathcal{A}_0(U_{P_s}) \setminus \mathcal{A}_1(U_{P_s})$. If $a^{-1} \in \mathcal{A}_k(U_{P_s}) \setminus \mathcal{A}_{k+1}(U_{P_s})$, then $a^{-1} = b_0t^k$, $b_0 \in \mathcal{A}_0(U_{P_s}) \setminus \mathcal{A}_1(U_{P_s})$. Then $1 = a^{-1}a = a_0b_0t^{j+k}$, hence $j + k \leq 0$ and $a_0b_0 = t^{-j-k} \notin \mathcal{A}_1(U_{P_s})$, because $\mathcal{A}_0(U_{P_s})/\mathcal{A}_1(U_{P_s}) \cong \mathcal{O}_C(U_{P_s})$ has no zero divisors, since $C$ is irreducible and reduced, what follows from our assumptions (see remark 13 below).

Therefore, $j + k = 0$, $b_0 = a_0^{-1}$ and $a_0 \in \mathcal{A}_0^*(U_{P_s})$, $a = a_0t^j$, $b^{-1} = a_0^{-1}t^{-j}$. Clearly, this is preserved under base change $s \rightarrow S$, therefore $j = \text{ord}(a)(x)$ for any $x \in U_{P_s}$. So, the degree map factors through $\mathbb{Z}_C|_{U_{P_s}}$ and is, obviously, a morphism of sheaves of Abelian groups with $(\mathcal{A}^*/\mathcal{A}_0^*)|_{U_{P_s}} \cong \mathbb{Z}_C|_{U_{P_s}}$, since $\text{ord}(t)|_{U_{P_s}} = 1$.

**Remark 13.** $C$ is irreducible, because $S$ is irreducible. Indeed, assume the converse. Then there are two open subsets $U_1 \subset C$, $U_2 \subset C$ with $U_1 \cap U_2 = \emptyset$. Since $\tau : C \rightarrow S$ is flat and locally of finite type, it is open and therefore $\tau(U_1) \cap \tau(U_2) \neq \emptyset$. So, if $s \in \tau(U_1) \cap \tau(U_2)$, then $C_s \cap U_1 \neq \emptyset$, $C_s \cap U_2 \neq \emptyset$ and therefore $C_s$ is reducible, a contradiction with our assumption.
To prove that \( C \) is reduced, let’s assume the converse. We can assume \( S \) is affine and the nilradicals \( \mathcal{N}il(\mathcal{O}_C(U)) \neq 0 \) for any open \( U \). Let \( S' = \mathcal{N}il(\mathcal{O}_C(U \times_S S')) \neq 0 \) for any affine \( U \subset C \), because we have the embedding \( \mathcal{N}il(\mathcal{O}_C(U)) \hookrightarrow \mathcal{N}il(\mathcal{O}_C(U \times_S S')) \). For any point \( s \in S' \) of codimension 1 let \( T \subset C' \) with \( \tau'(T) = s \). Then we have \( \mathcal{N}il(\mathcal{O}_{C',T}) \neq 0 \). But \( \mathcal{O}_{S',T} \) is a flat \( \mathcal{O}_{S',s} \)-module and \( \mathcal{O}_{S',s} \) is a regular local ring of dimension 1. Moreover, \( \mathcal{O}_{C',T} \otimes k(s) \simeq \mathcal{O}_{C,T}/u\mathcal{O}_{C,T} \simeq \mathcal{O}_{C,T} \), where \( u \) is a generator of the maximal ideal of \( \mathcal{O}_{S',s} \), has no zero divisor, because by our assumptions \( C_s \) is irreducible curve. Therefore, \( \mathcal{O}_{C,T}/u\mathcal{O}_{C,T} \simeq \mathcal{O}_{C,T,\text{red}}/u\mathcal{O}_{C,T,\text{red}} \), where \( \mathcal{O}_{C,T,\text{red}} = \mathcal{O}_{C,T}/\mathcal{N}il(\mathcal{O}_{C,T}) \). Note also that \( u \) is not a zero divisor in \( \mathcal{O}_{C,T,\text{red}} \), since it is not a zero divisor in \( \mathcal{O}_{C,T} \) by the local flatness criterium ([4, ch.III, §5, th.1] or [9, ch.III, lemma 10.3.1]). Therefore, \( \mathcal{O}_{C,T,\text{red}} \) is a flat \( \mathcal{O}_{S',s} \)-module by this criterium. Hence, \( \mathcal{N}il(\mathcal{O}_{C,T}) \otimes k(s) = 0 \) and by the Nakayama lemma \( \mathcal{N}il(\mathcal{O}_{C',T}) = 0 \), a contradiction.

**Remark 14.** In situation of remark 12 the statements of our case are valid for the whole space \( C \) (not only on \( U_{P_s} \)). The proof is the same.

Now we claim that the order map on \( \mathcal{A}_x \) factors through \( \mathbb{Z}_C \) on the whole space \( C \) (although may be \( (\mathcal{A}_x/\mathcal{A}_x^s) \neq \mathbb{Z}_C \)).

Indeed, let \( U \) be a neighbourhood of a point \( x \in C \), \( a \in \mathcal{A}_x(U) \). Then by [10, prop.9] and by definition, for all points \( y \in U_s \), where \( s = \tau(x) \), we have \( \mathcal{O}(a)(y) = \mathcal{O}(a)(x) \), because \( C_s \) is irreducible. Since \( C \) is irreducible, we have \( U \cap U_s \neq \emptyset \) for any \( q \in S \). Then, by the arguments above, we have \( \mathcal{O}(a)(x) = \mathcal{O}(a)(y) \) for any \( y \in U \cap U_{P_s} \).

Analogously, for \( x' \in U \), \( s' = \tau(x') \) we have \( \mathcal{O}(a)(x') = \mathcal{O}(a)(y) \) for any \( y \in U \cap U_{P_{s'}} \). Since \( U \cap U_{P_s} \cap U_{P_{s'}} \neq \emptyset \), we obtain \( \mathcal{O}(a)(x) = \mathcal{O}(a)(x') \) for any \( x' \in U \).

**Remark 15.** We can define in our case the map \( \mathcal{O}(a) \) as in [10]:

\[
\mathcal{O}(a) \equiv \max\{ j \mid a \in \mathcal{A}_j(U) \};
\]

where \( a \in \mathcal{A}_x(U) \). Then we claim that \( \mathcal{O}(a) = \mathcal{O}(a)(x) \) for any \( x \in U \).

Indeed, we have proved above that \( \mathcal{O}(a)(x) = \mathcal{O}(a)(x') \) for any \( x' \in U \) and

\[
\mathcal{O}(a|_{U_{P_s} \cap U})(y) = \mathcal{O}(a|_{U_{P_s} \cap U})(y) \geq \mathcal{O}(a)
\]

for any \( y \in U_{P_s} \cap U \). If \( \mathcal{O}(a|_{U_{P_s} \cap U}) > \mathcal{O}(a) \), then this would mean that that the image of the element \( \bar{a} \in \mathcal{A}_{\mathcal{O}(a)} / \mathcal{A}_{\mathcal{O}(a)+1}(U) \) under the map \( \varphi : \mathcal{A}_{\mathcal{O}(a)}/\mathcal{A}_{\mathcal{O}(a)+1}(U) \to (\mathcal{A}_{\mathcal{O}(a)}/\mathcal{A}_{\mathcal{O}(a)+1})_\eta(U_\eta) \), where \( \eta \) is a general point on \( S \), is zero. But \( \varphi \) is an injective map, because \( \mathcal{A}_{\mathcal{O}(a)}/\mathcal{A}_{\mathcal{O}(a)+1}(U) \) is a flat \( \mathcal{O}_S(\tau(U)) \)-module, \( \mathcal{A}_{\mathcal{O}(a)}/\mathcal{A}_{\mathcal{O}(a)+1} \) is a coherent sheaf, and the map \( \mathcal{O}_S(\tau(U)) \to \mathcal{O}_{S,\eta} \) is an embedding. So, \( \mathcal{O}(a|_{U_{P_s} \cap U}) = \mathcal{O}(a) \) and \( \mathcal{O}(a) = \mathcal{O}(a)(x) \) for any \( x \in U \).

**Case 2.** Let \( S \) be a reduced scheme. We claim that the same assertions as in Case 1 hold.

Let \( a \in \mathcal{A}_x(U_{P_s}) \) and \( j = \mathcal{O}(a)(x) \), where we can assume \( x \in C \) to be a point such that \( s = \tau(x) \) belong to several irreducible components \( S_1, \ldots S_k \) (without loss of
Since \( A \) and \( N \) are nilradical. Since \( A \) and \( N \) are nilradical.

Summarized we get \( \mathcal{A}^*_{S,0}(U_{P_s,s_i}) \). For any \( l, m \in \mathbb{Z} \) we have the exact sequences of sheaves of filtered \( \mathcal{A}_0 \)-modules

\[
0 \longrightarrow \mathcal{A}_{m,S_1 \cup (S_2 \cup \ldots \cup S_l)} \longrightarrow \mathcal{A}_{m,S_1} \times \mathcal{A}_{m,(S_2 \cup \ldots \cup S_l)} \longrightarrow \mathcal{A}_{m,S_1 \cap (S_2 \cup \ldots \cup S_l)} \longrightarrow 0.
\]

Therefore, by obvious induction arguments, using the exact sequences, we obtain \( a_0 \in \mathcal{A}_0(U_{P_s}) \). Similarly, \( b_0 = a^{-1}t^j \in \mathcal{A}_0(U_{P_s}) \) and \( a_0b_0 = 1 \). So, \( a_0, b_0 \in \mathcal{A}^*_0(U_{P_s}) \).

**Remark 16.** In situation of remark 12 the statements of our case are valid for the whole space \( C \) (not only on \( U_{P_s} \)). The proof is the same.

To show that \( \text{ord}(a) \) is locally constant on any \( U \subset C \), we can repeat the arguments from the end of Case 1, because \( C_s \) is irreducible (so, \( U \cap U_{P_s} \neq \emptyset \), and \( U \cap U_{P_s} \cap S_i \times S C \neq \emptyset \) for any \( i = 1, \ldots k \) (so, \( U \cap U_{P_s} \cap U_{P_s} \neq \emptyset \) for \( s' \in S_i, k \in \{1, \ldots, k\} \)).

Note that, using remark 15 and above arguments, we obtain that an element \( a \in \mathcal{A}^*(U) \) with \( \text{ord}(a) \equiv 0 \) must belong to \( \mathcal{A}^*_0(U) \).

**Case 3.** Let \( S \) be an arbitrary Noetherian scheme. Let \( \mathcal{N}_S \) be the nilradical. Since \( \mathcal{N}_S \) is a coherent sheaf on \( S \), we have by the arguments of remark 5

\[
\mathcal{N}_S \mathcal{A} \overset{\text{def}}{=} \mathcal{N}_S^* \otimes_{\mathcal{O}_S} \mathcal{A} = \text{Ker}(A \to \mathcal{A}_{s_{red}}), \quad \mathcal{N}_S \mathcal{A}_0 \overset{\text{def}}{=} \tau^*(\mathcal{N}_S^* \otimes_{\mathcal{O}_S} \mathcal{A}_0) = \text{Ker}(A_0 \to \mathcal{A}_{s_{red},0})
\]

and

\[
\mathcal{A}_{s_{red}}^* = \mathcal{A}^*/(1 + \mathcal{N}_S \mathcal{A}), \quad \mathcal{A}_{s_{red},0}^* = \mathcal{A}_0^*/(1 + \mathcal{N}_S \mathcal{A}_0).
\]

Since \( A/A_0 \) is flat over \( S \), we obtain, by comparing the exact sequences

\[
0 \to \mathcal{N}_S \mathcal{A}_0 \to A_0 \to A_{s_{red},0} \to 0, \quad 0 \to \mathcal{N}_S \mathcal{A} \to A \to A_{s_{red}} \to 0,
\]

that \( \mathcal{N}_S \mathcal{A} \cap A_0 = \mathcal{N}_S \mathcal{A}_0 \) and therefore \( 1 + \mathcal{N}_S \mathcal{A}_0 = (1 + \mathcal{N}_S \mathcal{A}) \cap \mathcal{A}_0^* \).

Note that, by definition, the order map on \( \mathcal{A}^* \) coincide with the order map on \( \mathcal{A}_{s_{red}}^* \). Summarized we get

**Proposition 5.** 1. If \( X_\infty \) is a ribbon over a Noetherian scheme \( S \) satisfying the assumptions in the beginning of this section, then

(a) The order map

\[
\text{ord} : \mathcal{A}^* \longrightarrow \mathbb{Z}_C
\]

is a morphism of sheaves of groups.

(b) There exist neighbourhoods \( U_{P_s} \subset C \) such that for each \( U_{P_s} \) the map \( \text{ord} |_{U_{P_s}} \) is a surjective morphism.

(c) We have the equality of sheaves

\[
\text{Ker}(\text{ord}) = \mathcal{A}_0^* \cdot (1 + \mathcal{N}_S \mathcal{A}) \simeq \mathcal{A}_0^* \prod_{1 + \mathcal{N}_S \mathcal{A}_0} (1 + \mathcal{N}_S \mathcal{A}),
\]

where on the right hand side we consider the amalgamated sum.
2. If $\tilde{\mathcal{X}}_\infty$ is a ribbon over a Noetherian scheme $S$ satisfying the assumptions from Remark 12, then the statement (1b) of this proposition holds for $U_p = C$.

3. If $\tilde{\mathcal{X}}_\infty$ is a ribbon obtained by the base change from a ribbon over a field $k$ of characteristic zero that satisfies the assumptions from Remark 12, then

(a) Using exponential and logarithmic maps we can write the equality of sheaves

$$\text{Ker}(\text{ord}) = \mathcal{A}_0^* \prod_{N_S \otimes_{\mathcal{O}_S} \mathcal{A}} N_S \otimes_{\mathcal{O}_S} \mathcal{A},$$

where on the right hand side we consider the amalgamated sum.

(b) We have the following exact sequences of sheaves

$$\begin{align*}
1 & \to \text{Ker}(\text{ord}) \to \mathcal{A}^* \to \mathbb{Z}_C \to 0, \\
0 & \to N_S \otimes_{\mathcal{O}_S} \mathcal{A}/\mathcal{A}_0 \to \mathcal{A}^*/\mathcal{A}_0^* \to \mathbb{Z}_C \to 0, \\
1 & \to \mathcal{A}_0^* \to \text{Ker}(\text{ord}) \to N_S \otimes_{\mathcal{O}_S} \mathcal{A}/\mathcal{A}_0 \to 0.
\end{align*}$$

Proof. The proof of statements 1, 2 was given above. The proof of statement 3 follows when we use the power series for $\log(1 + z)$ to identify: $1 + N_S \mathcal{A} \simeq N_S \mathcal{A}$, $1 + N_S \mathcal{A}_0 \simeq N_S \mathcal{A}_0$, and $1 + N_S \mathcal{A}/((1 + N_S \mathcal{A}) \cap \mathcal{A}_0^*) = 1 + N_S \mathcal{A}/1 + N_S \mathcal{A}_0$ with $N_S \mathcal{A}/N_S \mathcal{A}_0 = N_S \otimes_{\mathcal{O}_S} \mathcal{A}/\mathcal{A}_0$.

5.2 Vanishing theorems

In this subsection we will prove one fact which we will use later and which may be of independent interest.

**Proposition 6.** Let $C$ be a normal variety over an algebraically closed field $k$ of characteristic zero, $S$ be a $k$-scheme, $X := C \times_k S$, $\pi : X \to S$ be the projection morphism. Then $R^1\pi_* (\mathbb{Z}_{X\text{et}}) = 0$.

Proof. Our arguments will be similar to the arguments in the proof of theorem 2.5 in [18].

It is enough to proof that for any geometric point $\bar{s}$ of $S$, $R^1\pi_* (\mathbb{Z}_{X\text{et}})_{\bar{s}} = 0$. By [12, ch. III, th.1.15] we have $R^1\pi_* (\mathbb{Z}_{X\text{et}})_{\bar{s}} \simeq H^1_{\text{et}} (X \times_S \text{Spec} \mathcal{O}^\text{h}_{\text{sh}}_{\bar{s}}, \mathbb{Z})$. So, we can assume $S$ is a spectrum of a strict hensel ring $R$. Since every hensel ring is the union of henselizations of its finitely generated subrings, we can apply [12, ch. III, lemma 1.16] and assume that $R$ is a strict henselization of a finitely generated ring. By [8, IV.18.7.3], $R$ is a pseudogeometric ring (or a universally japanese ring).

We are going to use induction on dimension of $R$, where $R$ is a strict hensel pseudogeometric ring. If $\dim R = 0$, then we can assume $R$ is a field, because by [18, th. 7.6, corol. 7.6.1] $H^1_{\text{et}} (X, \mathbb{Z}) = H^1_{\text{et}} (X_{\text{red}}, \mathbb{Z})$. Then $H^1_{\text{et}} (X, \mathbb{Z}) = 0$ by prop. 7.4, th.7.6 in [18], because $C \times_k R$ is a normal scheme by [4, Ch.V, §1, prop. 19] and [19, vol.I, ch.III, §15, th.40, cor.1].
Now let \( \dim R > 0 \). Let’s consider two cases.

**Case 1.** \( R \) is a domain. Let \( \tilde{R} \) be the normalization of \( R \). Since \( R \) is a pseudo-geometric and strict hensel ring, the ring \( \tilde{R} \) is also a strict hensel domain.

The scheme \( \tilde{X} := X \otimes_R \tilde{R} = C \times_k \tilde{R} \) is normal by \([4, \text{Ch.V, §1, prop. 19, cor. 1}]\) and \([19, \text{vol.I, ch.III, §15, th.40, cor.1}]\). Let \( I = \text{Ann}_R(\tilde{R}/R) \) be the conductor ideal in \( R \). Then we have an isomorphism \( \phi : I \to \text{Hom}_R(\tilde{R}, R) \) by the following rule: \( \phi(i)(r) = ir \), \( i \in I \), \( r \in \tilde{R} \). Since \( R \) is pseudo-geometric, \( \tilde{R} \) is a finite \( R \)-module. Therefore, we have \( I \neq 0 \) and \( R/I \) is a strict hensel ring with \( \dim R/I < \dim R \). By \([8, \text{IV.7.7.2}]\), \( R/I \) is also a pseudo-geometric ring. Denote by \( Y \) the subscheme \( X \times_R (R/I) \subset X \), and by \( \bar{Y} \) the subscheme \( X \times_R (\tilde{R}/I) \). Now we are in the situation of §7, prop.7.8 of \([18]\). So, we have the following long exact sequence

\[
0 \to H^0(X, \mathbb{Z}) \to H^0(\tilde{X}, \mathbb{Z}) \times H^0(Y, \mathbb{Z}) \to H^0(\bar{Y}, \mathbb{Z}) \to H^1_{et}(X, \mathbb{Z}) \to H^1_{et}(\tilde{X}, \mathbb{Z}) \times H^1_{et}(Y, \mathbb{Z}).
\]

By induction on dimension of the ring \( R \) we have \( H^1(\tilde{Y}, \mathbb{Z}) = 0 \). \( H^1_{et}(\tilde{X}, \mathbb{Z}) = 0 \) by prop. 7.4, th.7.6 in \([18]\), because \( \tilde{X} \) is a normal scheme. Since \( \tilde{X} \), \( \bar{Y} \) are connected schemes, the map \( H^0(\tilde{X}, \mathbb{Z}) \to H^0(\bar{Y}, \mathbb{Z}) \) is surjective. Therefore, \( H^1_{et}(X, \mathbb{Z}) = 0 \).

**Case 2.** In general case, we can assume that \( R = R_{\text{red}} \) by \([18, \text{corol. 7.6.1}]\). Let \( (0) = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_n \) be a primary decomposition in \( R \). Set \( \tilde{R} = R/\mathfrak{p}_1 \times \ldots \times R/\mathfrak{p}_n \). Set \( I = \bigoplus_i \mathfrak{p}_j \) \( \cap \mathfrak{p}_j \). We have

\[
I = \bigoplus_{i \neq j} \mathfrak{p}_j = \bigoplus_i \text{Hom}_R(R/\mathfrak{p}_i, R)
\]

is a conductor ideal in \( R \), and it contains a nonzerodivisor. Now in the notations of case 1 we have \( \tilde{X} = \prod_{i=1}^n (X \times_R R/\mathfrak{p}_i) \), \( \tilde{Y} = \prod_{i=1}^n (X \times_R R/(\mathfrak{p}_i + \bigcap_{j \neq i} \mathfrak{p}_j)) \). So, in the sequence (17) we have \( H^1_{et}(\tilde{Y}, \mathbb{Z}) = 0 \) by induction on dimension of the ring \( R \), since \( \dim R/I < \dim R \); \( H^1_{et}(\tilde{X}, \mathbb{Z}) = \prod_{i=1}^n H^1_{et}(X \times_R R/\mathfrak{p}_i, \mathbb{Z}) = 0 \) by case 1; and \( H^0(\tilde{X}, \mathbb{Z}) \to H^0(\bar{Y}, \mathbb{Z}) \) is a surjective map. So, again \( H^1_{et}(X, \mathbb{Z}) = 0 \).

Now we can prove the same result in the flat topology, where under the flat topology we understand the fppf or fpqc topology on \( X \).

**Theorem 3.** Let \( C \) be a normal variety over an algebraically closed field \( k \) of characteristic zero, \( S \) be a \( k \)-scheme, \( X := C \times_k S \), \( \pi : X \to S \) be the projection morphism.

Then \( R^1\pi_* (\mathbb{Z}_{X_{fg}}) = 0 \).

**Proof.** By definition of the sheaf \( R^1\pi_* (\mathbb{Z}_{X_{fg}}) \), we need to prove that for any element \( x \in H^1_{fl}(C \times_k U, \mathbb{Z}) \), where \( U \) is flat over \( S \), there exists a cover \( (U_i) \) of \( U \) in the flat topology such that \( \text{res}_{U_i}(x) = 0 \) for any \( i \).

For any scheme \( Z \) set \( Y = Z[t, t^{-1}] = Z \times_Z Z[t, t^{-1}] \), and let \( p : Y \to Z \) be the structure map. Similarly, let \( p^+ \) and \( p^- \) denote the structure maps from \( Y^+ = Z[t] = Z \times_Z Z[t] \) and \( Y^- = Z[t^{-1}] = Z \times_Z Z[t^{-1}] \) to \( Z \).
Recall that for a covariant functor $F$ from the category of commutative rings (or for a contravariant functor from the category of schemes) to some Abelian category the following functors are defined (see [18], §1 or [3], ch.XII):

$$NF(R) = N_t F(R) = \text{Ker}[F(t = 1) : F(R[t]) \to F(R)] \simeq \text{Coker}[F(i_+) : F(R) \to F(R[t])].$$

$$LF(R) = \text{Coker}[F(R[t]) \oplus F(R[t^{-1}]) \stackrel{add}{\to} F(R[t, t^{-1}])].$$

Clearly, $F(R[t]) \simeq F(R) \oplus NF(R)$.

Consider now the following sequence of sheaves on $Z$ (in the flat topology) from [18], proof of prop. 7.2:

$$0 \to G_{m, Z} \to p^+_* (G_{m, Y^+}) \times p^-_* (G_{m, Y^-}) \to p_* (G_{m, Y}) \to \mathbb{Z} \to 0.$$  \hspace{1cm} (18)

By (1.1) in [18] this sequence is exact and there is a splitting $\mathbb{Z} \to p_* (G_{m, Y})$ given by multiplication by $t$ ($n \mapsto t^n$) and the splitting $p_* (G_{m, Y}) \to G_{m, Z}$ given by evaluation at $t = 1$. So, we have

$$p^+_* (G_{m, Y^+}) \simeq G_{m, Z} \times \mathcal{N}_t G_{m, Z}, \quad p^-_* (G_{m, Y^-}) \simeq G_{m, Z} \times \mathcal{N}_{t^{-1}} G_{m, Z}$$  \hspace{1cm} (19)

and

$$p_* (G_{m, Y}) \simeq G_{m, Z} \times \mathcal{N}_t G_{m, Z} \times \mathcal{N}_{t^{-1}} G_{m, Z} \times \mathbb{Z},$$

where $\mathcal{N}_t G_{m, Z}$ is the sheaf, associated to the presheaf $U \mapsto \text{Coker}(G_{m, Z}(U) \to p^+_* (G_{m, Y^+})(U))$, in the flat topology.

Now, comparing the Leray spectral sequences for $p^+$, $p^-$ and $p$ (in the flat topology), we obtain the following exact diagram (compare the diagram in the proof of th.7.6 in [18]):

$$
\begin{array}{cccc}
0 & 0 & \\
\downarrow & \downarrow & \\
0 \to Pic(Z) & H^1_{fl}(Z, p^+_* (G_{m, Y^+})) \oplus H^1_{fl}(Z, p^-_* (G_{m, Y^-})) & H^1_{fl}(Z, Z) & 0 \\
\| & \downarrow & \\
0 \to Pic(Z) & Pic(Y^+) \oplus Pic(Y^-) & Pic(Y) & LPic(Z) \\
\downarrow & \downarrow & \\
0 \to H^0(Z, R^1 p^+_* (G_{m, Y^+})) \oplus H^0(Z, R^1 p^-_* (G_{m, Y^-})) & H^0(Z, R^1 p_* (G_{m, Y})) & 0 \\
\| & \\
0 \to H^0(Z, R^1 p_* (G_{m, Y})) & \\
\end{array}
$$

Here the first row is exact by (18) and (19). In the second row we use Hilbert’s 90 theorem: $H^1_{fl}(T, G_{m, T}) = Pic(T) = H^1_{et}(T, G_{m, T})$, where $T$ is a scheme. So, the second row is exact by [18, th. 7.6]. Let’s show that the third row is also exact.

Assume the converse, and let $x \in H^0(Z, R^1 p^+_* (G_{m, Y^+}))$ be an element from the kernel of the map from the third row. This means that there exists a cover $(U_i)$ of $Z$ in the flat topology such that

$$res_{Z, U_i}(x) \in \text{Ker}(H^1_{fl}(U_i \times Z \mathbb{Z}[t], G_m) \to H^1_{fl}(U_i \times Z \mathbb{Z}[t, t^{-1}], G_m))$$

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for all $i$. By Hilbert’s 90 theorem we have

$$\text{res}_{Z,U_i}(x) \in \ker(H^1_{\text{et}}(U_i \times_k Z[t], \mathbb{G}_m) \rightarrow H^1_{\text{et}}(U_i \times_k Z[t, t^{-1}], \mathbb{G}_m)).$$

By (5.1) in [18] (see also the proof of th. 7.6 there) we have $R^1p^*_x(\mathbb{G}_{m,Y^+}) \oplus R^1p^*_x(\mathbb{G}_{m,Y^-}) \simeq R^1p^*_x(\mathbb{G}_{m,Y})$ in the étale topology on $Z$ for any scheme $Z$. So, our conditions mean that for every $i$ there exists a cover $(V_{i\alpha})$ of $U_i$ such that $0 = \text{res}_{U\times V_{i\alpha}}(\text{res}_{Z,U_i}(x)) \in H^1_{\text{et}}(V_{i\alpha} \times_k Z[t], \mathbb{G}_m)$). Again by Hilbert’s 90 theorem this means $\text{res}_{Z,V_{i\alpha}}(x) = 0$ for the flat cover $(V_{i\alpha})$ of $Z$, i.e. $x = 0$. The same arguments work for $x \in H^0(Z, R^1p^*_x(\mathbb{G}_{m,Y^-})).$

A diagram chase now shows that $H^1_{\text{et}}(Z, \mathbb{Z}) \rightarrow \text{LPic}(Z) \simeq H^1_{\text{et}}(Z, \mathbb{Z})$ is an injective map. Note also that for an open étale $U \rightarrow Z$ the diagram

$$
\begin{array}{ccc}
H^1_{\text{et}}(Z, \mathbb{Z}) & \rightarrow & \text{LPic}(Z) \\
\downarrow & & \downarrow \\
H^1_{\text{et}}(U, \mathbb{Z}) & \rightarrow & \text{LPic}(U)
\end{array}
$$

is commutative.

Now let $x \in H^1_{\text{et}}(C \times_k U, \mathbb{Z})$, where $U$ is flat over $S$. By the arguments above, the element $x$ is embedded in $H^1_{\text{et}}(C \times_k U, \mathbb{Z})$. By proposition 6 there exists an étale cover $(U_i)$ of $U$ such that $0 = \text{res}_{U\times U_i}(x) \in H^1_{\text{et}}(C \times_k U_i, \mathbb{Z})$. Then by the arguments above this imply that for the cover $(U_i)$ in the flat topology we also have $0 = \text{res}_{U\times U_i}(x) \in H^1_{\text{et}}(C \times_k U_i, \mathbb{Z})$. So, $R^1\pi^*_x(\mathbb{Z}_{\underline{X}}) = 0$.

\[ \square \]

### 5.3 Representability of the Picard functor

In this subsection we will show that under certain conditions on a ribbon the étale sheaf, associated with the Picard functor, is representable by a formal scheme.

Let $C$ be an irreducible projective curve over an algebraically closed field $k$ of characteristic zero, and $\underline{X}_\infty$ be a ribbon over $k$ with underlying topological space $C$ and either with a smooth point in the sense of definition 9 from [10], or satisfying the condition $(\ast\ast)$ from definition 5. We consider the ribbon $\underline{X}_\infty,S$ for some base change $S \rightarrow \text{Spec } k$.

Let $\mathcal{F} \in \text{Pic}(\underline{X}_\infty,S)$. We define a sheaf of generating sections $\mathcal{B}(\mathcal{F})$ (which is a sheaf of sets) by the rule

$$\mathcal{B}(\mathcal{F})(U) = \{ \text{sections } \lambda \in \mathcal{F}(U) \text{ with } \mathcal{F}|_U = \mathcal{A}|_U \cdot \lambda \},$$

where $U$ is open in $C \times_k S$. We have $\mathcal{B}(\mathcal{F})(U) = \emptyset$ or (after choice of one generator) $\mathcal{B}(\mathcal{F})|_U \simeq \mathcal{A}^*_U$. Thus, $\mathcal{B}(\mathcal{F})$ is a torsor over the sheaf of groups $\mathcal{A}^*$.

We recall that for any topological space $Y$ we denoted by $\mathbb{Z}_Y$ the sheaf of locally constant functions on $Y$ with values in $\mathbb{Z}$.

**Definition 12.** Let $\mathcal{B}$ be a category of affine Noetherian $k$-schemes. We define the contravariant functor $\widehat{\text{Pic}}_{\underline{X}_\infty}^\prime$ from $\mathcal{B}$ to the category of Abelian groups:

$$\widehat{\text{Pic}}_{\underline{X}_\infty}^\prime(S) \overset{\text{def}}{=} \{ \text{the group of isomorphism classes of pairs } (\mathcal{F}, d) \},$$

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where \( F \in \text{Pic}(\mathcal{X}_\infty, S) \), and \( d : \mathcal{B}(F) \to \mathcal{L}_{C \times_k S} \) is a morphism of sheaves of sets such that
\[
d(a\lambda)(x) = \text{ord}(a)(x) + d(\lambda)(x)
\]
for any \( a \in \mathcal{A}_S^*(U) \), \( \lambda \in \mathcal{B}(F)(U) \), \( U \subset C \times_k S \) is open, and \( x \in U \). Two pairs \((\mathcal{F}, d)\) and \((\mathcal{F}', d')\) are isomorphic, if there is an isomorphism of sheaves of \( \mathcal{A}_S \)-modules \( \mathcal{F} \) and \( \mathcal{F}' \) compatible with \( d, d' \). Besides
\[
(\mathcal{F}_1, d_1) \otimes (\mathcal{F}_2, d_2) = (\mathcal{F}_1 \otimes_{\mathcal{A}_S} \mathcal{F}_2, d),
\]
where \( d(\lambda_1 \otimes \lambda_2) = d_1(\lambda_1) + d_2(\lambda_2) \).

**Example 1.** Any locally free sheaf of rank 1 on \( \hat{\mathcal{X}}_\infty \) has a filtration (see example 11 in [10]), i.e., this sheaf is a torsion free sheaf on \( \hat{\mathcal{X}}_\infty \) in the sense of definition 2. Its base change give some sheaf \( F \in \text{Pic}(\mathcal{X}_\infty, S) \), which also has a filtration. We define a morphism of order \( d : \mathcal{B}(F) \to \mathcal{L}_{C \times_k S}(\mathbb{Z}) \) (where \( \mathcal{L}_{C \times_k S} \) is the sheaf of all functions on \( C \times_k S \) with values in \( \mathbb{Z} \)) by the rule
\[
d(\lambda)(x) = \max\{j \mid \lambda|_{U_s} \in (\mathcal{F}_s)_{x,j}\},
\]
where \( \lambda \in \mathcal{B}(F)(U) \) for an open \( U \subset C \times_k S \), \( x \in U \), \( s = \tau(x) \), \( U_s \) is an open set in \( C_s \), which is obtained from \( U \) by the base change \( s \to S \), \( \mathcal{F}_s \) is a sheaf after the base change. One can easily check that \( d \) is a morphism of sheaves of sets. Moreover, by section 5.1 it factors through the subsheaf \( \mathcal{L}_{C \times_k S} \subset \mathcal{L}_{C \times_k S} \). Besides, we have
\[
d(a\lambda)(x) = \text{ord}(a)(x) + d(\lambda)(x)
\]
for any \( a \in \mathcal{A}_S^*(U) \), because \( \text{ord} \) is a morphism of sheaves of groups by proposition 5. Thus, \((\mathcal{F}, d) \in \text{Pic}(\mathcal{X}_\infty, S)\).

We define the following sheaf on \( C \times_k S \):
\[
\mathcal{Y}_S \overset{\text{def}}{=} \ker(\text{ord} : \mathcal{A}_S^* \to \mathcal{L}_{C \times_k S}) = \mathcal{A}_S^* \prod_{\mathbb{N}_S \otimes_k \mathcal{A}} \mathcal{N}_S \otimes_k \mathcal{A}.
\]
where the last equality follows from proposition 5.

**Definition 13.** Let \( \mathcal{B} \) be a category of affine Noetherian \( k \)-schemes. We define the contravariant functor \( \tilde{\text{Pic}}_{\mathcal{X}_\infty} \) from \( \mathcal{B} \) to the category of Abelian groups:
\[
\tilde{\text{Pic}}_{\mathcal{X}_\infty}(S) \overset{\text{def}}{=} H^1(C \times_k S, \mathcal{Y}_S),
\]
where the restriction maps of this functor are compositions of the natural maps
\[
H^1(C \times_k S, \mathcal{Y}_S) \to H^1(C \times_k S, (id \times f)_*(\mathcal{Y}_S')) \to H^1(C \times_k S', \mathcal{Y}_S'),
\]
and the second map is the embedding from the Cartan-Leray spectral sequence for a morphism \( f : S' \to S \).
We have always an evident morphism of functors:

\[
\widetilde{\text{Pic}}_{\hat{X}_\infty} \longrightarrow \widetilde{\text{Pic}}'_{\hat{X}_\infty}
\]

such that for any \( S \in \text{Ob}(\mathcal{B}) \) we have an embedding of Abelian groups:

\[
\widetilde{\text{Pic}}_{\hat{X}_\infty}(S) \hookrightarrow \widetilde{\text{Pic}}'_{\hat{X}_\infty}(S).
\]

**Proposition 7.** Let a ribbon \( \hat{X}_\infty \) satisfies the condition \( (**\) \) from definition 5. Then we have the natural isomorphism of functors:

\[
\widetilde{\text{Pic}}_{\hat{X}_\infty} \simeq \widetilde{\text{Pic}}'_{\hat{X}_\infty}.
\]

**Proof.** Let \( S \in \text{Ob}(\mathcal{B}) \). The sheaf of automorphisms of a pair \( (\mathcal{F}, d) \in \widetilde{\text{Pic}}'_{\hat{X}_\infty}(S) \) (i.e., the sheaf of automorphisms of \( \mathcal{A} \)-module \( \mathcal{F} \), which preserve the function \( d \)) is equal to the sheaf \( \mathcal{I}_S \). Besides, by remark 12, the pair \( (\mathcal{F}, d) \) is isomorphic to the pair \( (\mathcal{A}_S, \text{ord}) \) locally on \( C \times_k S \). Therefore, by standard arguments with twisted forms (see, e.g. [12], ch.III, §4) we obtain the statement of the proposition. 

\( \square \)

**Definition 14.** We denote by \( \widetilde{\text{Pic}}_{\hat{X}_\infty} \) the sheaf in the étale topology over \( \text{Spec} \, k \), associated with the presheaf \( S \mapsto \widetilde{\text{Pic}}_{\hat{X}_\infty}(S) \) (i.e., the sheaf on the large étale site of \( \text{Spec} \, k \), compare with remark 6).

Analogously, we denote by \( \text{Pic}_{\hat{X}_\infty} \) the sheaf in the étale topology over \( \text{Spec} \, k \), associated with the presheaf \( S \mapsto \text{Pic}_{\hat{X}_\infty}(S) \).

In view of theorem 1, propositions 1 and 7, it is important to obtain that the sheaf \( \widetilde{\text{Pic}}_{\hat{X}_\infty} \) is a \( k \)-group scheme. Our first aim is to prove this under some conditions, and then we will compare the sheaf \( \widetilde{\text{Pic}}_{\hat{X}_\infty} \) with the sheaf \( \text{Pic}_{\hat{X}_\infty} \).

**Lemma 4.** Let \( S \) be an affine scheme over \( k \). We have

\[
H^h(C \times_k S, \mathcal{A}_i \hat{\boxtimes}_k \mathcal{N}_S) \simeq \lim_{\leftarrow j>i} H^h(C \times_k S, (\mathcal{A}_i/\mathcal{A}_j) \hat{\boxtimes}_k \mathcal{N}_S) \simeq \lim_{\leftarrow j>i} (H^h(C, \mathcal{A}_i/\mathcal{A}_j) \otimes_k H^0(S, \mathcal{N}_S)),
\]

\[
H^q(C \times_k S, \mathcal{A}_i \hat{\boxtimes}_k \mathcal{N}_S) = 0
\]

for any \( i \in \mathbb{Z}, \ h \leq 1, \ q > 1 \).

**Proof.** We have the analog of the Küneth formula:

\[
p_*((\mathcal{A}_i/\mathcal{A}_{i+h}) \hat{\boxtimes}_k \mathcal{N}_S) \simeq (\mathcal{A}_i/\mathcal{A}_{i+h}) \otimes_k H^0(S, \mathcal{N}_S),
\]

(21)

where \( p : C \times_k S \to C \) is the projection. Indeed, if \( U \) is an affine open set on \( C \), and \( \tau_S : C \times_k S \to S \) is a projection, we have the natural isomorphisms

\[
p_*((\mathcal{A}_i/\mathcal{A}_{i+h}) \hat{\boxtimes}_k \mathcal{N}_S)(U) \simeq p^*(\mathcal{A}_i/\mathcal{A}_{i+h})(U \times_k S) \otimes \mathcal{O}_{U \times_k S} \tau_S^* \mathcal{N}_S(U \times_k S) \simeq
\]
Corollary 2. For an affine Noetherian scheme $S$ we have

$$H^q(C \times_k S, (\mathcal{A}_i/\mathcal{A}_{i+h}) \boxtimes_k \mathcal{N}_S) \simeq (\mathcal{A}_i/\mathcal{A}_{i+h})(U) \otimes_k H^q(S, \mathcal{N}_S),$$

since $(\mathcal{A}_i/\mathcal{A}_{i+h}), \mathcal{N}_S$ are coherent sheaves of modules on $X_{h-1}, S$ correspondingly (see prop. 1 in [10]). These isomorphisms are obviously compatible with the restriction homomorphisms corresponding to the embedding of affine sets $U' \subset U$ for both sheaves $p_*((\mathcal{A}_i/\mathcal{A}_{i+h}) \boxtimes_k \mathcal{N}_S)$ and $(\mathcal{A}_i/\mathcal{A}_{i+h}) \otimes_k H^q(S, \mathcal{N}_S)$. Therefore, the sheaves from formula (21) are isomorphic.

Since $p$ is an affine morphism, we have then

$$H^q(C \times_k S, (\mathcal{A}_i/\mathcal{A}_{i+h}) \boxtimes_k \mathcal{N}_S) \simeq H^q(C, \mathcal{A}_i/\mathcal{A}_{i+h}) \otimes_k H^q(S, \mathcal{N}_S)$$

for all $q$ (see [9, ch.III, ex.8.2]).

For all $i, h, k$ with $h \leq k$ we have surjective morphism of sheaves

$$(\mathcal{A}_i/\mathcal{A}_{i+k}) \boxtimes_k \mathcal{N}_S \rightarrow (\mathcal{A}_i/\mathcal{A}_{i+h}) \boxtimes_k \mathcal{N}_S,$$

because $(\mathcal{A}_i/\mathcal{A}_{i+k}) \rightarrow (\mathcal{A}_i/\mathcal{A}_{i+h})$ is a surjective morphism of sheaves on $C$.

For any affine $U$ in $C \times_k S$ the maps

$$\Gamma(U, (\mathcal{A}_i/\mathcal{A}_{i+k}) \boxtimes_k \mathcal{N}_S) \rightarrow \Gamma(U, (\mathcal{A}_i/\mathcal{A}_{i+h}) \boxtimes_k \mathcal{N}_S)$$

are surjective, since $(\mathcal{A}_l/\mathcal{A}_m) \boxtimes_k \mathcal{N}_S$ are coherent sheaves of modules for all $l < m$ on $X_{m-l-1} \times_k S$. By the same reason we have $H^q(U, (\mathcal{A}_l/\mathcal{A}_m) \boxtimes_k \mathcal{N}_S) = 0$ for all $q > 0$.

At last, since $C$ is a projective curve, the projective systems

$$\{H^q(C, \mathcal{A}_i/\mathcal{A}_{i+h}) \otimes_k H^0(S, \mathcal{N}_S)\}_{h \in \mathbb{N}}, \quad q \geq 0$$

satisfy the ML-condition. So, by [7, ch. 0, prop.13.3.1] we have

$$H^q(C \times_k S, \mathcal{A}_i \boxtimes_k \mathcal{N}_S) \simeq \varprojlim_{j>i} H^q(C \times_k S, (\mathcal{A}_i/\mathcal{A}_j) \boxtimes_k \mathcal{N}_S) \simeq \varprojlim_{j>i} (H^q(C, \mathcal{A}_i/\mathcal{A}_j) \otimes_k H^0(S, \mathcal{N}_S))$$

for $q \geq 1$.

For $q = 0$ it follows from the definition of the sheaf $\mathcal{A}_i \boxtimes_k \mathcal{N}_S$.

$\square$

**Corollary 2.** For an affine Noetherian scheme $S$ we have

$$H^1(C \times_k S, (\mathcal{A}/\mathcal{A}_0) \boxtimes_k \mathcal{N}_S) \simeq H^1(C, \mathcal{A}/\mathcal{A}_0) \otimes_k H^0(S, \mathcal{N}_S)$$

and

$$H^q(C \times_k S, \mathcal{A} \boxtimes_k \mathcal{N}_S) = H^q(C \times_k S, (\mathcal{A}/\mathcal{A}_0) \boxtimes_k \mathcal{N}_S) = 0$$

for $q \geq 2$.

**Proof.** The proof is clear, since cohomology commute with $\varprojlim$ on Noetherian schemes.

$\square$
Theorem 4. Let $C$ be an irreducible projective curve over an algebraically closed field $k$ of characteristic zero, and $X_\infty$ be a ribbon with underlying topological space $C$, which satisfies conditions from the beginning of section 5.3. Assume that

$$\text{Coker}(H^0(C, A) \to H^0(C, A/A_0)) = 0.$$  \hspace{1cm} (22)$$

Then the sheaf $\widetilde{\text{Pic}}_{X_\infty}$ is a formal group scheme, which is isomorphic (non-canonically) to the product $\text{Pic}_{X_\infty} \times_k \tilde{\text{Br}}_{X_\infty}$, where $\text{Pic}_{X_\infty}$ is the Picard scheme of $X_\infty$ (see remark 6), and $\tilde{\text{Br}}_{X_\infty}$ is the formal Brauer group of $X_\infty$ (see section 4.4).

Remark 17. Compare assumption formula (22) with theorem 2.

Proof. It is enough to prove that for any affine Noetherian scheme $S$ over $k$ the following sequence is split exact (see corollary 2 for the last term):

$$0 \to H^1(C \times_k S, A_{S,0}^*) \to H^1(C \times_k S, \mathcal{Z}_S) \to H^1(C, A/A_0) \otimes_k H^0(S, \mathcal{N}_S) \to 0,$$  \hspace{1cm} (23)$$

and the splitting is functorial with $S$. Indeed, by [10, prop. 5], [11, §2] the étale sheaf, associated to the presheaf $S \mapsto H^1(C \times_k S, A_{S,0}^*)$, is a scheme $\text{Pic}_{X_\infty}$. By remark 8 we have

$$\text{Hom}_{\text{form.sch.}}(S, \tilde{\text{Br}}_{X_\infty}) = H^1(C, A/A_0) \otimes_k H^0(S, \mathcal{N}_S).$$

On the other hand, the presheaf $S \mapsto \text{Hom}_{\text{form.sch.}}(S, \tilde{\text{Br}}_{X_\infty})$ is a sheaf in the étale topology. For, by [12, ch II, prop.1.5] it is enough to show that the restriction of the presheaf to a scheme $U$ is a sheaf in the Zariski topology, and for an étale cover $U' \to U$ with affine $U', U$ the sequence

$$\text{Hom}_{\text{form.sch.}}(U, \tilde{\text{Br}}_{X_\infty}) \to \text{Hom}_{\text{form.sch.}}(U', \tilde{\text{Br}}_{X_\infty}) \to \text{Hom}_{\text{form.sch.}}(U' \times_U U', \tilde{\text{Br}}_{X_\infty})$$

is exact. The first fact follows from [6, 1.10.4.6] and the sheaf properties of $\mathcal{O}_U$. The second fact follows from the following fact:

$$\text{Hom}_{\text{form.sch.}}(U, \tilde{\text{Br}}_{X_\infty}) \simeq \lim_{\leftarrow k} \text{Hom}_{\text{sch.}}(U, \text{Spec}(T/J^k)),$$

(here $T$ and $J$ are from the proof of proposition 3), see [6, 1.10.6.9], and from [12, ch I, th.2.17] (see also [12, ch II, cor.1.7]).

Now let's prove that (23) is exact. This sequence is a part of the long exact cohomology sequence that comes from the short exact sequence

$$0 \to A_{S,0}^* \to \mathcal{Z}_S \to (A/A_0) \otimes_k \mathcal{N}_S \to 0.$$  

Let's show that sequence (23) is left exact due to our assumption formula (22). Suffice it to show that the map

$$H^0(C \times_k S, A \otimes_k \mathcal{N}_S) \to H^0(C \times_k S, (A/A_0) \otimes_k \mathcal{N}_S)$$

is exact.
is surjective (see formula (20), or that the maps 
\[ H^0(C \times_k S, A_i \otimes_k \mathcal{N}_S) \rightarrow H^0(C \times_k S, (A_i/A_0) \otimes_k \mathcal{N}_S) \]
are surjective for all \( i < 0 \).

For any \( i < 0 \), \( h \geq 0 \) we define
\[ K_{i,h} \overset{\text{def}}{=} \text{Coker}(H^0(C, A_i/A_h) \rightarrow H^0(C, A_i/A_0)). \]

We have the following exact sequences:
\[ 0 \rightarrow H^0(C, A_0/A_h) \rightarrow H^0(C, A_i/A_h) \rightarrow H^0(C, A_i/A_0) \rightarrow K_{i,h} \rightarrow 0, \]  \hspace{1em} (24)
\[ 0 \rightarrow H^0(C, A_0/A_h) \otimes_k H^0(S, \mathcal{N}_S) \rightarrow H^0(C, A_i/A_h) \otimes_k H^0(S, \mathcal{N}_S) \rightarrow H^0(C, A_i/A_0) \otimes_k H^0(S, \mathcal{N}_S) \rightarrow K_{i,h} \otimes_k H^0(S, \mathcal{N}_S) \rightarrow 0. \]  \hspace{1em} (25)

Since \( C \) is a projective curve, the projective systems
\[ \{H^0(C, A_0/A_h)\}_{h \in \mathbb{N}}, \quad \{H^0(C, A_0/A_h) \otimes_k H^0(S, \mathcal{N}_S)\}_{h \in \mathbb{N}}, \]
\[ \{H^0(C, A_i/A_h)\}_{h \in \mathbb{N}}, \quad \{H^0(C, A_i/A_h) \otimes_k H^0(S, \mathcal{N}_S)\}_{h \in \mathbb{N}} \]
satisfy the ML-condition. Therefore, by [10, lemma 1] the sequences of projective limits are also exact, i.e., using lemma 4, we obtain the exact sequences
\[ 0 \rightarrow H^0(C, A_0) \rightarrow H^0(C, A_i) \rightarrow H^0(C, A_i/A_0) \rightarrow \varprojlim_{h \in \mathbb{N}} K_{i,h} \rightarrow 0, \]  \hspace{1em} (26)
\[ 0 \rightarrow H^0(C \times_k S, A_0 \otimes_k \mathcal{N}_S) \rightarrow H^0(C \times_k S, A_i \otimes_k \mathcal{N}_S) \rightarrow H^0(C \times_k S, (A_i/A_0) \otimes_k \mathcal{N}_S) \rightarrow \varprojlim_{h \in \mathbb{N}}(K_{i,h} \otimes_k H^0(S, \mathcal{N}_S)) \rightarrow 0. \]  \hspace{1em} (27)

So, our assertion will follow from the fact that \( K_{i,h} = 0 \).

First, note that \( K_i \overset{\text{def}}{=} \varprojlim_{h \in \mathbb{N}} K_{i,h} = 0 \). Indeed, by the assumption formula (22) and sequence (26) we have that
\[ 0 = \text{Coker}(H^0(C, \mathcal{A}) \rightarrow H^0(C, \mathcal{A}/A_0) = \varprojlim_i K_i. \]

We consider the following exact diagram:
\[
\begin{array}{cccccc}
0 & 0 & & & & \\
\downarrow & & \downarrow & & \downarrow & \\
0 & H^0(C, A_0) & \rightarrow & H^0(C, A_i) & \rightarrow & H^0(C, A_i/A_0) \rightarrow & K_i \rightarrow 0 \\
\| & & \downarrow & & \downarrow & & \\
0 & H^0(C, A_0) & \rightarrow & H^0(C, A_{i-1}) & \rightarrow & H^0(C, A_{i-1}/A_0) \rightarrow & K_{i-1} \rightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & H^0(C, A_{i-1}/A_i) = H^0(C, A_{i-1}/A_i) & & & & \\
\end{array}
\]
The diagram chase shows that the map $K_i \to K_{i-1}$ is injective. Therefore, we must have $K_i = 0$ for any $i < 0$.

Now, if $K_{i,h} \neq 0$ for some $h > 0$, then this would mean that $K_i \neq 0$. For, we have $\phi_{h+1}(H^0(C, A_i/A_{h+1})) \subset \phi_h(H^0(C, A_i/A_h))$ for any $h$ (see sequence (24)). So, if $K_{i,h} \neq 0$, then $\phi_h(H^0(C, A_i/A_h)) \neq H^0(C, A_i/A_0)$, and a preimage in $H^0(C, A_i/A_0)$ of any nonzero element from $K_{i,h}$ gives a nonzero element in $K_i$. Therefore, $K_{i,h} = 0$.

Let’s show that (23) is right exact. This follows from the fact that the map

$$H^2(C \times_k S, A^*_S, 0) \to H^2(C \times_k S, \mathcal{S}_S)$$

is an isomorphism. Indeed, this map is a part of the following diagram, which is exact by lemma 4, corollary 2 of this lemma and definitions:

$$
\begin{array}{ccccccccc}
0 & \to & H^2(C \times_k S, A^*_S, 0) & \to & H^2(C \times_k S, A^*_S, 0) & \to & 0 \\
\downarrow & & \downarrow & & \parallel & & \\
0 & \to & H^2(C \times_k S, \mathcal{S}_S) & \to & H^2(C \times_k S, \mathcal{S}_S, 0) & \to & 0 \\
\end{array}
$$

(We used here exact sequences:

$$
0 \to A_0 \widehat{\otimes}_k \mathcal{N}_S \to A^*_S, 0 \to A^*_S, 0 \to 0,
$$

$$
0 \to A \widehat{\otimes}_k \mathcal{N}_S \to \mathcal{S}_S \to \mathcal{S}_S, 0 \to 0,
$$

and $A^*_S, 0 = \mathcal{S}_S, 0$.)

Now we show that (23) splits and there is a splitting, which is functorial with $S$. We consider the following diagram, which is exact by lemma 4, corollary 2 of this lemma, definitions and our assumptions:

$$
\begin{array}{ccccccccc}
0 & \to & H^1(C \times_k S, A_0 \widehat{\otimes}_k \mathcal{N}_S) & \to & H^1(C \times_k S, A^*_S, 0) & \to & H^1(C \times_k S, A^*_S, 0) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \parallel & & \\
0 & \to & H^1(C \times_k S, \mathcal{N}_S) & \to & H^1(C \times_k S, \mathcal{S}_S) & \to & H^1(C \times_k S, \mathcal{S}_S, 0) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^1(C, A_0) \otimes_k H^0(S, \mathcal{N}_S) & \to & H^1(C, A_0) \otimes_k H^0(S, \mathcal{N}_S) & \to & H^1(C, A_0) \otimes_k H^0(S, \mathcal{N}_S) \to 0 \\
\end{array}
$$

A splitting of the left vertical exact sequence is given by system of compatible $k$-linear sections of surjective maps $H^1(C, A_i/A_h) \to H^1(C, A_i/A_0), h > 0, i < 0$ and tensor multiplication (over $k$) of these sections with identity map on $H^0(S, \mathcal{N}_S)$. It gives the functorial with $S$ splitting of sequence (23). (Compare with the proof of proposition 4.) The theorem is proved.

\[\Box\]
Remark 18. The first maps in the rows of the diagram above are embeddings. To show this it suffice to prove that the map
\[ H^0(C \times_k S, A^*_S, 0) \to H^0(C \times_k S, A^*_{S, a, 0}) \]
is surjective.

If \( a \in H^0(C \times_k S, A_S, 0) \) is an invertible element, then its image in \( H^0(C \times_k S, \mathcal{O}_{C \times_k S}) \) must be also invertible. But we have
\[ H^0(C \times_k S, \mathcal{O}_{C \times_k S}) \cong H^0(C, \mathcal{O}_C) \otimes_k H^0(S, \mathcal{O}_S) \cong H^0(S, \mathcal{O}_S), \]
because \( S \) is an affine scheme (see the proof of lemma 4) and \( H^0(C, \mathcal{O}_C) \cong k \) for an irreducible projective curve. Now since we have embeddings
\[ H^0(S, \mathcal{O}_S) \hookrightarrow H^0(C, \mathcal{A}_0/A_j) \otimes_k H^0(S, \mathcal{O}_S) \]
for all \( j > 0 \), we obtain the embedding
\[ H^0(S, \mathcal{O}_S) \hookrightarrow H^0(C \times_k S, A_{S, 0}) \]
for any scheme \( S \) by lemma 4. Therefore, we can reduce the proof to the following fact: the map
\[ H^0(C \times_k S, A_{S, 1}) \to H^0(C \times_k S, A_{S, 0}) \]
is surjective. The last fact follows from the following observations:

1) using the same arguments as in the proof of lemma 4, we have
\[ H^0(C \times_k S, A_{S, 1}) \cong \lim_{j \to 1} H^0(C \times_k S, A_{S, 0} / A_j) \otimes_k H^0(S, \mathcal{O}_S), \]
\[ H^0(C \times_k S, A_{S, 0}) \cong \lim_{j \to 1} H^0(C \times_k S, A_{S, 0} / A_j) \otimes_k H^0(S, \mathcal{O}_S); \]

2) we have short exact sequences
\[ 0 \to H^0(C, \mathcal{A}_1/A_j) \otimes_k H^0(S, \mathcal{N}_S) \to H^0(C, \mathcal{A}_1/A_j) \otimes_k H^0(S, \mathcal{O}_S) \to \]
\[ H^0(C, \mathcal{A}_1/A_j) \otimes_k H^0(S, \mathcal{O}_S) \to 0 \quad (28) \]
for all \( j > 1 \), and the projective system \( \{ H^0(C, \mathcal{A}_1/A_j) \otimes_k H^0(S, \mathcal{N}_S) \}_{j \to 1} \) satisfies the ML-condition. Therefore, passing to projective limit with respect to \( j \) in sequence (28) we obtain again the short exact sequence.

Now we compare the sheaves \( \widetilde{\text{Pic}}_{X_{\infty}} \) and \( \text{Pic}_{X_{\infty}} \). Let \( \text{Pic}^0_{X_{\infty}} \) be the connected component of zero in \( \text{Pic}_{X_{\infty}} \), which is known to be a closed irreducible subgroup with
\[ \text{Pic}^0_{X_{\infty}}(k) = \lim_{i \to 0} \text{Pic}^0_{X_i} \]
(here \( X_i = (C, \mathcal{A}_0/A_{i+1}) \) is a scheme), see §3 in [11]. Besides, we have the following exact sequence of sheaves:
\[ 0 \to \text{Pic}^0_{X_{\infty}} \to \text{Pic}_{X_{\infty}} \to \mathbb{Z} \to 0. \]
Then we have the following exact sequence of étale presheaves:
\[
0 \rightarrow \mathbb{Z} \rightarrow \tilde{\text{Pic}}_{X,\infty} \rightarrow \text{Pic}_{X,\infty} \rightarrow 0,
\]
and \( \text{Pic}_{X,\infty} \) is a formal group scheme, which is non-canonically isomorphic to
\[
\left( \prod_{i=1}^{\vert(C-C) \vert} \text{Pic}_{X,\infty}^i \right) \times_k \widehat{\text{Br}}_{X,\infty}.
\]

\textbf{Proof.} By our assumption, the ribbon satisfies the condition (**). So, by definition of \( \mathcal{A} \) \((20)\) (see also proposition 5), we have the exact sequence of Zariski sheaves for each \( S \):
\[
0 \rightarrow \mathcal{A} \rightarrow \mathcal{A}^*_S \rightarrow \mathbb{Z} \rightarrow 0.
\]
Then we have the following exact sequence of étale presheaves:
\[
H^1(C \times_k S, \mathcal{A}) \rightarrow H^1(C \times_k S, \mathcal{A}^*) \rightarrow H^1(C \times_k S, \mathbb{Z}).
\]
(29)

By proposition 6, the étale sheaf, associated to the presheaf \( S \mapsto H^1(C \times_k S, \mathbb{Z}) \), is zero.

Let’s show that the kernel of the first map is \( H^0(C \times_k S, \mathcal{A}^*_{S,0}) \approx H^0(C \times_k S, \mathcal{A}^*_S) \) for a reduced and connected scheme \( S \). Indeed, if it is true, then this isomorphism holds for any scheme \( S \), because for any affine Noetherian scheme \( S \) we then have the following diagram, which is exact by definitions and remark 18:
\[
\begin{array}{cccc}
0 & \rightarrow & H^0(C \times_k S, \mathcal{A}\widehat{\otimes}_k \mathcal{N}_S) & \rightarrow & H^0(C \times_k S, \mathcal{A}^*_{S,0}) \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & H^0(C \times_k S, \mathcal{A}\widehat{\otimes}_k \mathcal{N}_S) & \rightarrow & H^0(C \times_k S, \mathcal{A}^*_S) & \rightarrow & H^0(C \times_k S, \mathcal{A}^*_{S,red})
\end{array}
\]

Now, if \( S \) is \( \text{Spec} \ k \), the isomorphism \( H^0(C \times_k S, \mathcal{A}^*_{S,0}) \approx H^0(C \times_k S, \mathcal{A}^*_S) \) for a reduced \( S \) follows from example 8, [10]. Recall that in this case we have the exact sequence
\[
0 \rightarrow \mathbb{Z} \overset{\alpha}{\rightarrow} \text{Pic}(X) \rightarrow \text{Pic}(X_{\infty}),
\]
where \( \alpha(1) = \mathcal{A}_1 \), and \( \mathcal{A}_1 \) is not a torsion element in the group \( \text{Pic}(X_{\infty}) \), because its image in \( \text{Pic}(C) \) has degree equal to \( -(C \cdot C) \neq 0 \). So, the element \( \mathcal{A}_1\widehat{\otimes}_k \mathcal{O}_S \) is not a torsion element in \( \text{Pic}(X_{\infty,S}) \), and therefore \( \mathbb{Z} \rightarrow \text{Pic}(X_{\infty,S}) \) is injective and \( H^0(C \times_k S, \mathcal{A}^*_{S,0}) \approx H^0(C \times_k S, \mathcal{A}^*_S) \) as the long exact sequence
\[
0 \rightarrow H^0(C \times_k S, \mathcal{A}^*_{S,0}) \rightarrow H^0(C \times_k S, \mathcal{A}^*_S) \rightarrow \mathbb{Z} \rightarrow \text{Pic}(X_{\infty,S})
\]
shows.
Therefore, the sequence (29) leads to the exact sequence of étale sheaves
\[ 0 \rightarrow \mathbb{Z} \rightarrow \widehat{\Pic}_{X_\infty} \rightarrow \Pic_{\tilde{X}_\infty} \rightarrow 0. \]
Immediately from the construction of the sequences above follows that the sheaf \( \Pic_{X_\infty}/\mathbb{Z} \) is representable by the scheme \( \prod_{i=1}^{[C:C]} \Pic_{X_\infty}^0 \). Therefore, using theorem 4, we obtain
\[ \Pic_{\tilde{X}_\infty} \simeq \left( \prod_{i=1}^{[C:C]} \Pic_{X_\infty}^0 \right) \times_k \Br_{\tilde{X}_\infty}. \]

\[ \square \]

**Remark 19.** If \( C \) is an ample divisor, then the condition (22) from theorem 4 is equivalent to the condition \( H^1(X, \mathcal{O}_X) = 0 \) (see section 3).

**Remark 20.** Since we supposed that \( \text{char } k = 0 \), we can use the series for \( \exp(z) \) and \( \log(1 + z) \) to construct easily the Picard scheme \( \Pic_{X_\infty} \) studied by Lipman.

For any affine Noetherian scheme \( S \) we have exact sequences of sheaves on \( C \times_k S \):
\[ 0 \rightarrow A_{S,1} \xrightarrow{\exp} A_{S,0}^* \rightarrow \mathcal{O}_{C \times_k S}^* \rightarrow 1. \]
Therefore, using lemma 4 and remark 18, we obtain the following exact sequence
\[ 0 \rightarrow H^1(C, A_1) \hat{\otimes}_k H^0(S, \mathcal{O}_S) \rightarrow \Pic_{X_\infty}(S) \rightarrow \Pic_C(S) \rightarrow 0, \]
where \( H^1(C, A_1) \hat{\otimes}_k H^0(S, \mathcal{O}_S) \overset{\text{def}}{=} \lim_{j>1} (H^1(C, A_1/A_j) \otimes_k H^0(S, \mathcal{N}_S)) \), and
\[ S \mapsto \Pic_C(S) = H^1(C \times_k S, \mathcal{O}_{C \times_k S}^* \) is the Picard functor of the curve \( C \).

We define
\[ H \overset{\text{def}}{=} \text{Hom}_{k, \text{cont}}(H^1(C, A_1), k) = \lim_{j>1} (H^1(C, A_1/A_j))^*, \]
and \( \mathbb{V} \overset{\text{def}}{=} \text{Spec}(\text{Sym}_k(H)) \) is an affine \( k \)-group scheme, where \( \text{Sym}_k(H) \overset{\text{def}}{=} \bigoplus_{i=0}^{\infty} S^i(H) \), and the group law is given by \( v \mapsto v \otimes 1 + 1 \otimes v, \ v \in H \).

We have for any affine Noetherian scheme \( S \) over \( k \)
\[ \text{Hom}_{\text{sch}}(S, \mathbb{V}) = \text{Hom}_{k, \text{alg}}(\text{Sym}_k(H), H^0(S, \mathcal{O}_S)) = \text{Hom}_k(H, H^0(S, \mathcal{O}_S)) = \lim_{j>1} (H^1(C, A_1/A_j) \otimes_k H^0(S, \mathcal{N}_S)) = H^1(C, A_1) \hat{\otimes}_k H^0(S, \mathcal{O}_S). \quad (30) \]
Thus, we have the exact sequence of groups, which is functorial with \( S \):
\[ 0 \rightarrow \text{Hom}_{\text{sch}}(S, \mathbb{V}) \rightarrow \Pic_{X_\infty}(S) \rightarrow \Pic_C(S) \rightarrow 0, \quad (31) \]
The sheaves associated with the first and the last presheaves (or functors) in sequence (31) are \( k \)-group schemes. Therefore the existence of the Picard scheme \( \Pic_{X_\infty} \) of \( X_\infty \) as a \( k \)-group scheme follows now from [5, III, § 4, 1.8] and [5, III, § 4, 1.9].

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