Note on the existence theory for non-induced evolution equations

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Abstract
In this note, we develop a framework which allows to prove an abstract existence result for non-linear evolution equations involving so-called non-induced operators, i.e., operators which are not prescribed by a time-dependent family of operators. Apart from this, we introduce the notion of $C_0$-Bochner pseudo-monotonicity and $C_0$-Bochner coercivity, which are appropriate adaptations of the standard notion to the framework of evolutionary problems.

KEYWORDS
evolution equation, existence result, pseudo-monotone operator

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1 | INTRODUCTION

The theory of pseudo-monotone operators proved itself as a reliable tool in the verification of the solvability of non-linear problems. At its core lies the main theorem on pseudo-monotone operators, tracing back to Brezis [5], which states the following.¹

**Theorem 1.1.** Let $(X, \| \cdot \|_X)$ be a reflexive Banach space and let $A : X \to X^*$ be a bounded, pseudo-monotone and coercive operator. Then, $R(A) = X^*$.

A remarkable number of contributions, see e.g., [1, 9, 12–19, 23], dealt with the question to what extent Brezis’ result is transferable to the framework of non-linear evolution equations. A popular time-dependent analogue of Brezis’ contribution is the following (cf. [13, 16, 17, 23]).

**Theorem 1.2.** Let $(V, H, j)$ be an evolution triple, let $I := (0, T)$ be a finite time horizon, let $y_0 \in H$ be an initial value, let $f \in L^p(I, V^*)$, $1 < p < \infty$, be a right-hand side and let $A : L^p(I, V) \to L^{p'}(I, V^*)$ be a bounded, pseudo-monotone and coercive operator. Then, there exists a solution $y \in W^{1, p, p'}(I, V, V^*)$ of the initial value problem

$$
\frac{dy}{dt} + Ay = f \quad \text{in } L^{p'}(I, V^*),
$$

$$
j(y(0)) = y_0 \quad \text{in } H.
$$

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A major drawback of Theorem 1.2 is that the number of non-monotone, but pseudo-monotone, operators \( \mathcal{A} : L^p(I, V) \to L^{p'}(I, V^*) \) is negligible and thus the scope of application of Theorem 1.2 is strictly limited. For example, consider the unsteady \( p \)-Navier–Stokes equations, which can be written as an initial value problem of type (1.1), where \( I = (0, 2\pi) \), \( V = W^{1,p}_0(\Omega) \), \( p > 3 \), \( H = L^2(\Omega) \) and \( \mathcal{A} = S + B : L^p(I, V) \to L^{p'}(I, V^*) \), given via \( \langle Sx, y \rangle_{L^p(I,V)} := \int_I \int_{\Omega} (\delta + |Dx|^p) \cdot DxDy \, dx \, dt \), for every \( x, y \in L^p(I,V) \). While \( S : L^p(I,V) \to L^{p'}(I,V^*) \) is monotone, continuous, and thus pseudo-monotone, \( B : L^p(I,V) \to L^{p'}(I,V^*) \) fails to be pseudo-monotone (cf. [12, Remark 3.6]). Therefore, Theorem 1.2 is not applicable on the unsteady \( p \)-Navier–Stokes equations.

In [13], J.-L. Lions already observed that incorporating information from the time derivative will help to overcome this restriction. To this end, he introduced the following generalization of pseudo-monotonicity.

**Definition 1.3** \( (\frac{d}{dt} \)-pseudo-monotonicity). Let \( (V,H,j) \) be an evolution triple, let \( I := (0,T) \), with \( 0 < T < \infty \), and \( 1 < p < \infty \). An operator \( \mathcal{A} : W^{1,p,p'}(I,V,V^*) \to L^{p'}(I,V^*) \) is said to be \( \frac{d}{dt} \)-pseudo-monotone, if for \( (x_n)_{n \in \mathbb{N}} \subseteq W^{1,p,p'}(I,V,V^*) \) from

\[
x_n \xrightarrow{n \to \infty} x \quad \text{in} \quad W^{1,p,p'}(I,V,V^*),
\]

\[
\limsup_{n \to \infty} \langle A x_n, x_n - x \rangle_{L^p(I,V)} \leq 0,
\]

it follows that \( \langle A x, x - y \rangle_{L^p(I,V)} \leq \liminf_{n \to \infty} \langle A x_n, x_n - y \rangle_{L^p(I,V)} \) for every \( y \in L^p(I,V) \).

With this new notion J.-L. Lions was able to extend Theorem 1.2 to \( \frac{d}{dt} \)-pseudo-monotone and coercive operators \( \mathcal{A} : W^{1,p,p'}(I,V,V^*) \to L^{p'}(I,V^*) \) satisfying a special boundedness condition, which takes the time derivative into account (cf. [13, Théorème 1.2, p. 316]). In fact, he proved that \( \mathcal{S} + \mathcal{B} : L^p(I,V) \to L^{p'}(I,V^*) \) is \( \frac{d}{dt} \)-pseudo-monotone, coercive and satisfies this special boundedness condition (cf. [13, Remarque 1.2, p. 335]). Unfortunately, [13, Théorème 1.2] is entailing an imbalance between the demanded continuity and growth conditions. To be more precise, while the required \( \frac{d}{dt} \)-pseudo-monotonicity is quite general, coercivity is a restrictive assumption, which in many applications is not fulfilled, e.g., \( \mathcal{S} - \mathcal{R} : L^p(I,V) \to L^{p'}(I,V^*) \), where \( \mathcal{S} : L^p(I,V) \to L^{p'}(I,V^*) \) is defined as above and \( \mathcal{R} : L^p(I,V) \to L^{p'}(I,V^*) \) is given via \( \langle Rx, y \rangle_{L^p(I,V)} := \int_I \int_{\Omega} x \cdot y \, dx \, dt \) for every \( x, y \in L^p(I,V) \), is \( \frac{d}{dt} \)-pseudo-monotone, but not coercive.

In [12], this restriction is overcome by introducing alternative generalizations of pseudo-monotonicity and coercivity, which, in contrast to \( \frac{d}{dt} \)-pseudo-monotonicity and coercivity, both incorporate information from the time derivative, and therefore are more in balance. The idea is to weaken the pseudo-monotonicity assumption to a bearable extend, in order to make a coercivity condition accessible, which takes the information from the time derivative into account.

**Definition 1.4** (Bochner pseudo-monotonicity and Bochner coercivity). Let \( (V,H,j) \) be an evolution triple, let \( I := (0,T) \), with \( 0 < T < \infty \), and \( 1 < p < \infty \). An operator \( \mathcal{A} : L^p(I,V) \cap L^\infty(I,H) \to L^{p'}(I,V^*) \) is said to be

\( (i) \) Bochner pseudo-monotone, if for a sequence \( (x_n)_{n \in \mathbb{N}} \subseteq L^p(I,V) \cap L^\infty(I,H) \) from

\[
x_n \xrightarrow{n \to \infty} x \quad \text{in} \quad L^p(I,V),
\]

\[
jx_n \xrightarrow{\ast} jx \quad \text{in} \quad L^\infty(I,H) \quad (n \to \infty),
\]

\[
(jx_n)(t) \xrightarrow{n \to \infty} (jx)(t) \quad \text{in} \quad H \quad \text{for a.e.} \ t \in I,
\]

\[
\limsup_{n \to \infty} \langle A x_n, x_n - x \rangle_{L^p(I,V)} \leq 0,
\]

it follows that \( \langle A x, x - y \rangle_{L^p(I,V)} \leq \liminf_{n \to \infty} \langle A x_n, x_n - y \rangle_{L^p(I,V)} \) for every \( y \in L^p(I,V) \).
(ii) Bochner coercive with respect to \( f \in L^p(I, V^*) \) and \( x_0 \in H \), if there exists a constant \( M := M(f, x_0, A) > 0 \), such that for every \( x \in L^p(I, V) \cap_j L^\infty(I, H) \) from
\[
\frac{1}{2}\|(jx)(t)\|_H^2 + \langle Ax - f, x_\chi[0,t]\rangle_{L^p(I, V)} \leq \frac{1}{2}\|x_0\|_H^2 \quad \text{for a.e. } t \in I,
\]
it follows that \( \|x\|_{L^p(I, V)\cap_j L^\infty(I, H)} \leq M \).

Bochner pseudo-monotonicity and Bochner coercivity in [12] turned out to be appropriate generalizations of pseudo-monotonicity and coercivity for evolution equations since they both take into account the additional information from the time derivative, coming from the generalized integration by parts formula (cf. Proposition 2.12). In fact, in [12] it is illustrated that (1.4)–(1.7) are natural properties of a sequence \((x_n)_{n \in \mathbb{N}} \subseteq L^p(I, V) \cap_j L^\infty(I, H)\) coming from an appropriate Galerkin approximation of (1.1). To be more precise, (1.4) and (1.5) result from the Bochner coercivity of \( A \), and therefore take into account information from both the operator and the time derivative (cf. discussion below Definition 3.5), while (1.6) and (1.7) follow directly from the Galerkin approximation. In this way, [12, Theorem 4.1] provides an existence result for the initial value problem (1.1) provided that \( A : L^p(I, V) \cap_j L^\infty(I, H) \to L^p(I, V^*) \) is Bochner pseudo-monotone, Bochner coercive and induced by a time-dependent family of operators \( A(t) : V \to V^* \), i.e., \( (Ax)(t) := A(t)x(t) \) in \( V^* \) for almost every \( t \in I \) and all \( x \in L^p(I, V) \cap_j L^\infty(I, H) \), satisfying appropriate growth conditions (cf. [12, Conditions (C.1)–(C.3)]). Note that both \( S + B : L^p(I, V) \to L^p(I, V^*) \) and \( S - R : L^p(I, V) \to L^p(I, V^*) \) are Bochner pseudo-monotone and Bochner coercive (cf. [12, Example 5.1]), and [12, Theorem 4.1] applicable.

However, there are still non-negligible disadvantages of [12, Theorem 4.1] in comparison to Theorem 1.2 and [13, Théorème 1.2], which consist in its non-applicability on non-induced operators and the needed separability of \( V \). The necessity of an induced operator can be traced back to the verification of the existence of Galerkin solutions which in turn follows from the verification of (1.6) from the Galerkin approximation method applied in [12, Théorème 1.2].

The main purpose of this paper is to remove these limitations and extend the new gap-filling concepts of [12] to the abstract level of Theorem 1.2 and [13, Théorème 1.2], to prove an existence result for non-induced, bounded, Bochner pseudo-monotone and Bochner coercive operators in the case of purely reflexive \( V \), in order to gain a proper alternative to [13, Théorème 1.2] and a generalization of [12, Theorem 4.1] and Theorem 1.2. To this end, we will combine the modi operandi of [6], [7] and [12]. To be more specific, as one fails to extract (1.6) from the Galerkin approximation method applied in [6, Theorem 1], we are forced to fall back on the usual Galerkin approach as in [12]. Therefore, we initially limit ourselves to the case of separable, reflexive \( V \) and extend this result to the case of purely reflexive \( V \) by techniques from [6, 7] afterwards.

A further intention of this paper is to point out that there is still space for generalizations of Bochner pseudo-monotonicity. Indeed, since (1.5) together with (1.6) is strictly weaker than weak convergence in \( L^\infty(I, H) \) (cf. [20, Remark 3.5] or Remark 3.2), one may be reluctant to require the latter in Definition 1.4. However, in [12] it is shown that a sequence of Galerkin approximations in \( L^p(I, V) \cap_j C^0(\bar{I}, H) \) satisfies (1.5) and (1.6) not only for almost every, but for all \( t \in \bar{I} \), which is equivalent to weak convergence in \( C^0(\bar{I}, H) \) (cf. [3, Theorem 4.3]). This suggests a generalization of Bochner pseudo-monotonicity that respects the weak sequential topology in \( C^0(\bar{I}, H) \). Therefore, we say that an operator \( A : L^p(I, V) \cap_j C^0(\bar{I}, H) \to L^p(I, V^*) \) is \( C^0 \)-Bochner pseudo-monotone, if from (1.7) and
\[
x_n \rightharpoonup x \text{ in } L^p(I, V) \cap_j C^0(\bar{I}, H),
\]
it follows that \( \langle Ax, x - y \rangle_{L^p(I, V)} \leq \liminf_{n \to \infty} \langle Ax_n, x_n - y \rangle_{L^p(I, V)} \) for every \( y \in L^p(I, V) \). We will see that Bochner pseudo-monotonicity implies \( C^0 \)-Bochner pseudo-monotonicity, but the converse is not true in general (cf. Remark 3.2).

In the same spirit, we introduce \( C^0 \)-Bochner condition (M) and \( C^0 \)-Bochner coercivity as appropriate generalizations of the condition (M) and coercivity for evolution equations, as they take the additional energy space \( C^0(\bar{I}, H) \) into account.

 Altogether, we prove an existence result for bounded, \( C^0 \)-Bochner pseudo-monotone and \( C^0 \)-Bochner coercive operators \( A : L^p(I, V) \cap_j C^0(\bar{I}, H) \to L^p(I, V^*) \), including also non-induced operators, even in the case of purely reflexive \( V \).
Note that any bounded and coercive, or Bochner coercive, operator \( A : L^p(I, V) \to L^p(I, V^*) \) is \( C^0 \)-Bochner coercive (cf. Proposition 3.6), and that Bochner pseudo-monotonicity or usual pseudo-monotonicity imply \( C^0 \)-Bochner pseudo-monotonicity (cf. Remark 3.2). We will thus gain a proper generalization of both Theorem 1.2 and [12].

**Plan of the paper.** In Section 2, we introduce the notation and some basic definitions and results concerning continuous functions, Bochner–Lebesgue spaces, Bochner–Sobolev spaces and evolution equations. In Section 3, we introduce the new notions \( C^0 \)-Bochner pseudo-monotonicity, \( C^0 \)-Bochner condition (M) and \( C^0 \)-Bochner coercivity. In Section 4, we specify the implemented Galerkin approach. In Section 5, we prove in the case of separable and reflexive \( V \) an existence result for evolution equations with not necessarily induced, bounded and \( C^0 \)-Bochner coercive operators satisfying the \( C^0 \)-Bochner condition (M). Section 6 extends the results of Section 5 to the case of purely reflexive \( V \).

The paper is an extended and modified version of parts of the thesis [11].

2  | PRELIMINARIES

2.1  | Operators

For a Banach space \( X \) with norm \( \| \cdot \|_X \), we denote by \( X^* \) its dual space equipped with the norm \( \| \cdot \|_{X^*} \), by \( \tau(X, X^*) \) the corresponding weak topology and by \( B^X_M(x) \) the closed ball with radius \( M > 0 \) and centre \( x \in X \). The duality pairing is denoted by \( \langle \cdot, \cdot \rangle_X \). All occurring Banach spaces are assumed to be real. By \( D(A) \) we denote the domain of definition of an operator \( A : D(A) \subseteq X \to Y \), and by \( R(A) := \{ Ax \mid x \in D(A) \} \) its range.

**Definition 2.1.** Let \( (X, \| \cdot \|_X) \) and \( (Y, \| \cdot \|_Y) \) be Banach spaces. An operator \( A : D(A) \subseteq X \to Y \) is said to be

1. **demi-continuous**, if \( D(A) = X \) and \( x_n \to x \) in \( X \) \((n \to \infty)\) implies \( Ax_n \to Ax \) in \( Y \) \((n \to \infty)\).
2. **strongly continuous**, if \( D(A) = X \) and \( x_n \to x \) in \( X \) \((n \to \infty)\) implies \( Ax_n \to Ax \) in \( Y \) \((n \to \infty)\).
3. **compact**, if \( A : D(A) \subseteq X \to Y \) is continuous and for every bounded subset \( M \subseteq D(A) \subseteq X \), the image \( A(M) \subseteq Y \) is relatively compact.
4. **bounded**, if for every bounded subset \( M \subseteq D(A) \subseteq X \), the image \( A(M) \subseteq Y \) is bounded.
5. **locally bounded**, if for every \( x_0 \in D(A) \) there exist constants \( \varepsilon(x_0), \delta(x_0) > 0 \) such that \( \| Ax \|_Y \leq \varepsilon(x_0) \) for every \( x \in D(A) \) with \( \| x - x_0 \|_X \leq \delta(x_0) \).
6. **monotone**, if \( Y = X^* \) and \( \langle Ax - Ay, x - y \rangle_X \geq 0 \) for every \( x, y \in D(A) \).
7. **pseudo-monotone**, if \( Y = X^* \), \( D(A) = X \) and for a sequence \( (x_n)_{n \in \mathbb{N}} \subseteq X \) from \( x_n \to x \) in \( X \) \((n \to \infty)\) and \( \limsup_{n \to \infty} \langle Ax_n, x_n - x \rangle_X \leq 0 \), it follows that \( \langle Ax, x - y \rangle_X \leq \liminf_{n \to \infty} \langle Ax_n, x_n - y \rangle_X \) for every \( y \in X \).
8. **coercive**, if \( Y = X^* \), \( D(A) \) is unbounded and \( \lim \sup_{x \in D(A)} \frac{\langle Ax, x \rangle_X}{\| x \|_X} = \infty \).

The following proposition states that monotone operators satisfy certain boundedness properties and motivates to consider non-bounded operators provided that these operators are monotone.

**Proposition 2.2.** Let \( (X, \| \cdot \|_X) \) be a Banach space and let \( A : X \to X^* \) be monotone. Then, the following statements hold:

1. **(i)** \( A : X \to X^* \) is locally bounded.
2. **(ii)** Let \( S \subseteq X \), let \( h : S \to [0, 1] \) be a function and let \( M, C > 0 \) be constants such that for every \( s \in S \) it holds \( \| s \|_X \leq M \) and \( h(s)(As, s)_X \leq C \). Then, there exists a constant \( K = K(C, M, A) > 0 \) such that \( \| h(s)As \|_{X^*} \leq K \) for every \( s \in S \).

**Proof.** Concerning point (i), we refer to [9, Kapitel III, Lemma 1.2]. Point (ii) is a modification of [9, Kapitel III, Folgerung 1.2]. Being more precise, since \( A : X \to X^* \) is locally bounded due to point (i), there exist constants \( \varepsilon, \delta > 0 \) such that \( \| Ax \|_{X^*} \leq \varepsilon \) for every \( x \in X \) with \( \| x \|_X \leq \delta \). With the help of a scaled version the norm formula we finally obtain for every \( s \in S \)

\[
\| h(s)As \|_{X^*} = \sup_{\| x \|_X = \delta} h(s)(As, x)_X \leq \sup_{\| x \|_X = \delta} h(s)(As, s)_X + \frac{(Ax, x - s)_X}{\delta} \leq C + \varepsilon(\delta + M) \delta ,
\]

where we exploited the monotonicity of \( A : X \to X^* \) in the first inequality. \( \square \)
2.2 Continuous functions and Bochner–Lebesgue spaces

In this passage we collect some well-known results concerning continuous functions and Bochner–Lebesgue spaces, which will find use in the following. By \((X, \| \cdot \|_X)\) and \((Y, \| \cdot \|_Y)\) we always denote Banach spaces and by \(I := (0, T)\), with \(0 < T < \infty\), a finite time interval. The first proposition serves in parts as a motivation for \(C^0\)-Bochner pseudo-monotonicity.

**Proposition 2.3.** For a sequence \((x_n)_{n \in \mathbb{N}} \subseteq C^0(I, X)\) and a function \(x \in C^0(I, X)\), it holds \(x_n \rightharpoonup x\) in \(C^0(I, X)\) \((n \to \infty)\) if and only if \((x_n)_{n \in \mathbb{N}} \subseteq C^0(I, X)\) is bounded and \(x_n(t) \rightharpoonup x(t)\) in \(X\) \((n \to \infty)\) for every \(t \in I\).

**Proof.** See [3, Theorem 4.3]. □

**Proposition 2.4.** Let \(1 \leq p \leq \infty\) and let \(A : X \to Y\) be a linear and continuous operator. Then, the induced operator \( \mathcal{A} : L^p(I, X) \to L^p(I, Y)\), defined by \((\mathcal{A}x)(t) := A(x(t))\) in \(Y\) for almost every \(t \in I\) and all \(x \in L^p(I, X)\), is well-defined, linear and continuous. Furthermore, it holds:

(i) \(A \left( \int I x(s) \, ds \right) = \int I (\mathcal{A}x)(s) \, ds\) in \(Y\) for every \(x \in L^p(I, X)\).

(ii) If \(A : X \to Y\) is an embedding, then also \(A : L^p(I, X) \to L^p(I, Y)\) is an embedding.

(iii) If \(A : X \to Y\) is an isomorphism, then also \(A : L^p(I, X) \to L^p(I, Y)\) is an isomorphism.

**Proof.** Concerning the well-definedness, linearity and boundedness including point (i), we refer to [21, Chapter V, 5. Bochner’s Integral, Corollary 2]. The verification of assertions (ii) and (iii) is elementary and thus omitted. □

We use the in [12] proposed alternative point of view concerning intersections of Banach spaces, which is specified in the appendix. We emphasize that the standard definition of intersections of Banach spaces (cf. [2]) is equivalent to our approach and all the following assertions remain true if we use the framework in [2]. The next remark examines how the concepts of the appendix transfer to the Bochner–Lebesgue level.

**Remark 2.5.** Let \((X, Y) = (X, Y, Z, e_X, e_Y)\) be a compatible couple (cf. Definition A.2) and let \(1 \leq p, q \leq \infty\). In [2, Chapter 3, Theorem 1.3], it is proved that the sum \(R(e_X) + R(e_Y) \subseteq Z\) equipped with the norm

\[
\|z\|_{R(e_X) + R(e_Y)} := \inf_{x \in X, y \in Y} \max \{\|x\|_X, \|y\|_Y\}
\]

is a Banach space. Then, both \(e_X : X \to R(e_X) + R(e_Y)\) and \(e_Y : Y \to R(e_X) + R(e_Y)\) are embeddings (cf. Definition A.1) and therefore due to Proposition 2.4 the induced operators

\[
e_X : L^p(I, X) \to L^1(I, R(e_X) + R(e_Y)), \quad \text{given via} \quad (e_Xx)(t) := e_X(x(t)) \quad \text{for a.e. } t \in I,
\]

\[
e_Y : L^q(I, Y) \to L^1(I, R(e_X) + R(e_Y)), \quad \text{given via} \quad (e_Yy)(t) := e_Y(y(t)) \quad \text{for a.e. } t \in I
\]

are embeddings as well. Consequently, the couples

\[
(L^p(I, X), L^q(I, Y)) = \left( L^p(I, X), L^q(I, Y), L^1(I, R(e_X) + R(e_Y)), e_X, e_Y \right),
\]

\[
(L^p(I, X), C^0(I, Y)) = \left( L^p(I, X), C^0(I, Y), L^1(I, R(e_X) + R(e_Y)), e_X, e_Y \right)
\]

are compatible couples. In accordance with Definition A.3, the pull-back intersections

\[
L^p(I, X) \cap_j L^q(I, Y) \quad \text{and} \quad L^p(I, X) \cap_j C^0(I, Y),
\]

where \(j := e_Y^{-1}e_X\), and their corresponding intersection embeddings

\[
j : L^p(I, X) \cap_j L^q(I, Y) \to L^q(I, Y) \quad \text{and} \quad j : L^p(I, X) \cap_j C^0(I, Y) \to C^0(I, Y)
\]

are well-defined.
Proposition 2.6. Let \((X, Y)\) be a compatible couple, let \(X\) be a reflexive Banach space and let \(1 < p < \infty\). Then, for a sequence \((x_n)_{n \in \mathbb{N}} \subseteq L^p(I, X) \cap C^0(I, Y)\) and a function \(x \in L^p(I, X) \cap C^0(I, Y)\) it holds \(x_n \to x\) in \(L^p(I, X) \cap C^0(I, Y)\) \((n \to \infty)\) if and only if \((x_n)_{n \in \mathbb{N}} \subseteq L^p(I, X) \cap C^0(I, Y)\) is bounded and \((jx_n)(t) \to (jx)(t)\) in \(Y\) \((n \to \infty)\) for every \(t \in I\).

Proof. Immediate consequence of Proposition 2.3 and Proposition A.5 (iii).

2.3 | Bochner–Sobolev spaces

Let \((X, \| \cdot \|_X)\) and \((Y, \| \cdot \|_Y)\) be Banach spaces, let \(j : X \to Y\) be an embedding, let \(I := (0, T)\), with \(0 < T < \infty\), and \(1 \leq p, q \leq \infty\). A function \(x \in L^p(I, X)\) has a generalized time derivative with respect to \(j\) in \(L^q(I, Y)\) if there exists a function \(y \in L^q(I, Y)\) such that for every \(\varphi \in C_0^\infty(0, T)\) it holds

\[ j \left( - \int_I x(s) \varphi'(s) \, ds \right) = \int_I y(s) \varphi(s) \, ds \] in \(Y\).

As such a function \(y \in L^q(I, Y)\) is unique (cf. [22, Proposition 23.18]), \(\frac{dx}{dt} := y\) is well-defined. By

\[ W^{1,p,q}_j(I, X, Y) := \left\{ x \in L^p(I, X) \mid \exists \frac{dx}{dt} \in L^q(I, Y) \right\}, \]

we denote the Bochner–Sobolev space \(W^{1,p,q}_j(I, X, Y)\), which is equipped with norm

\[ \| \cdot \|_{W^{1,p,q}_j(I, X, Y)} := \| \cdot \|_{L^p(I, X)} + \left\| \frac{dx}{dt} \right\|_{L^q(I, Y)} \]
a Banach space (cf. [4, Lemma II.5.10]). In the case \(Y = X\) and \(j = \text{id}_X\), we define for sake of readability \(\frac{dx}{dt} := \frac{d\text{id}_X}{dt}\) and \(W^{1,p,q}(I, X) := W^{1,p,q}_{\text{id}_X}(I, X, X)\).

Proposition 2.7. Let \(j : X \to Y\) be an embedding and let \(j : L^p(I, X) \to L^p(I, Y)\) be given via \((jx)(t) := j(x(t))\) in \(Y\) for almost every \(t \in I\) and all \(x \in L^p(I, X)\). Then, it holds \(x \in W^{1,p,q}_j(I, X, Y)\) if and only if \(x \in L^p(I, X)\) and \(jx \in W^{1,p,q}(I, Y)\).

In this case, we have

\[ \frac{dx}{dt} = \frac{d(jx)}{dt} \] in \(L^q(I, Y)\). (2.1)

Proof. A straightforward application of Proposition 2.4 (i).

Proposition 2.8. Let \(A : X \to Y\) be linear and continuous. Then, the induced operator \(A : W^{1,p,q}(I, X) \to W^{1,p,q}(I, Y)\), defined by \((Ax)(t) := A(x(t))\) in \(Y\) for almost every \(t \in I\) and all \(x \in W^{1,p,q}(I, X)\), is well-defined, linear and continuous. Furthermore, it holds for every \(x \in W^{1,p,q}(I, X)\)

\[ \frac{d(jx)}{dt} = A \frac{dx}{dt} \] in \(L^q(I, Y)\). (2.2)

In addition, if \(A : X \to Y\) is an isomorphism, then \(A : W^{1,p,q}(I, X) \to W^{1,p,q}(I, Y)\) is an isomorphism as well.

Proof. Concerning the well-definedness, linearity, boundedness and (2.2) we refer to [8, Proposition 2.5.1]. The isomorphism property transfers obviously.
Proposition 2.9.

(i) First fundamental theorem of calculus for Bochner–Sobolev functions: Each \( \mathbf{x} \in W^{1, p, q}(I, X) \) (defined almost everywhere) possesses a unique representation \( \{ \mathbf{x} \}_X \in C^0(\bar{I}, X) \) with

\[
\{ \mathbf{x} \}_X(t) = \{ \mathbf{x} \}_X(t') + \int_{t'}^t \frac{d\mathbf{x}}{dt}(s) \, ds \quad \text{in } X
\]

for all \( t', t \in \bar{I} \) with \( t' \leq t \). The resulting choice function \( \{ \cdot \}_X : W^{1, p, q}(I, X) \to C^0(\bar{I}, X) \) is an embedding, which we denote by \( W^{1, p, q}(I, X) \hookrightarrow C^0(\bar{I}, X) \). In consequence, it holds \( W^{1, p, q}(I, X) = W^{1, \infty, q}(I, X) \) with norm equivalence and we thus set \( W^{1, q}(I, X) : = W^{1, \infty, q}(I, X) \).

(ii) Second fundamental theorem of calculus for Bochner–Sobolev functions: The operator \( \mathcal{V}_X : L^q(I, X) \to W^{1, q}(I, X) \), for every \( \mathbf{x} \in L^q(I, X) \) given via

\[
(\mathcal{V}_X \mathbf{x})(t) := \int_0^t \mathbf{x}(s) \, ds \quad \text{in } X \quad \text{for all } t \in \bar{I},
\]

is a continuous right inverse of \( \frac{d\mathbf{x}}{dt} : W^{1, q}(I, X) \to L^q(I, X) \).

Proof. Concerning point (i), we refer to [4, Proposition II.5.11]. Point (ii) except for the continuity one can find in [9, Kap. IV, Lemma 1.8]. The verification of the stated continuity is an elementary calculation and thus omitted. \( \square \)

The following result guarantees the compactness, which is indispensable for the applicability of Schauder’s fixed point theorem, and thus the existence of Galerkin approximations.

Proposition 2.10. Let \( (X, \| \cdot \|_X) \) be a finite dimensional Banach space and let \( 1 \leq q \leq \infty \). Then, the choice function \( \{ \cdot \}_X : W^{1, q}(I, X) \to C^0(\bar{I}, X) \) in Proposition 2.9 (i) is strongly continuous. In addition, if \( 1 < q \leq \infty \), then the choice function \( \{ \cdot \}_X : W^{1, q}(I, X) \to C^0(\bar{I}, X) \) is compact.

Proof. Due to the embeddings \( W^{1, \infty}(I, X) \hookrightarrow W^{1, q}(I, X) \hookrightarrow W^{1, 1}(I, X) \) for every \( 1 < q < \infty \), it suffices to prove strong continuity for \( q = 1 \) and compactness for \( q \in (1, \infty) \). In addition, since for \( q \in (1, \infty) \), the space \( W^{1, q}(I, X) \) is reflexive and thus strong continuity of \( \{ \cdot \}_X : W^{1, q}(I, X) \to C^0(\bar{I}, X) \) implies compactness, the entire assertion already follows, if we can prove that \( \{ \cdot \}_X : W^{1, 1}(I, X) \to C^0(\bar{I}, X) \) is strongly continuous.

Due to the linearity it suffices to show the strong continuity in the origin \( 0 \in W^{1, 1}(I, X) \). To this end, we treat a sequence \( (\mathbf{x}_n)_{n \in \mathbb{N}} \subseteq W^{1, 1}(I, X) \), such that \( \mathbf{x}_n \to 0 \) in \( W^{1, 1}(I, X) \) \((n \to \infty)\). Apparently, this also implies that

\[
\frac{d\mathbf{x}_n}{dt} \xrightarrow{n \to \infty} 0 \quad \text{in } L^1(I, X).
\]  

(2.3)

On the other hand, since the choice function \( \{ \cdot \}_X : W^{1, 1}(I, X) \to C^0(\bar{I}, X) \) is an embedding (cf. Proposition 2.9 (i)), we also obtain

\[
\{ \mathbf{x}_n \}_X \xrightarrow{n \to \infty} 0 \quad \text{in } C^0(\bar{I}, X).
\]  

(2.4)

Using the characterization of weak convergence in \( C^0(\bar{I}, X) \) (cf. Proposition 2.3), we further deduce from (2.4) that

\[
\{ \mathbf{x}_n \}_X(0) \xrightarrow{n \to \infty} 0 \quad \text{in } X.
\]  

(2.5)

Thanks to the compactness of \( \bar{I} \), there exists a sequence \( (t_n)_{n \in \mathbb{N}} \subseteq \bar{I} \), which without loss of generality converges to some \( t^* \in \bar{I} \) (otherwise, we switch to a subsequence and use the standard convergence principle [9, Kap. I, Lemma 5.4] to obtain
the assertion for the entire sequence), such that for every \( n \in \mathbb{N} \) it holds

\[
\left\| \left\{ x_n \right\}_X \right\|_{C^0(I,X)} = \max_{t \in I} \left\| \left\{ x_n \right\}_X(t) \right\|_X = \left\| \left\{ x_n \right\}_X(t_n) \right\|_X.
\] (2.6)

Next, let us fix an arbitrary \( x^* \in X^* \). Using Proposition 2.4 (i), we deduce for every \( n \in \mathbb{N} \) that

\[
\left\langle x^*, \int_0^{t_n} \frac{d}{dt} x_n(s) \, ds \right\rangle_X = \left\langle x^*, \text{sgn}(t_n - t^*) \int_{\min\{t_n, t^*\}}^{\max\{t_n, t^*\}} \frac{d}{dt} x_n(s) \, ds + \int_0^{t^*} \frac{d}{dt} x_n(s) \, ds \right\rangle_X \leq \left\| x^* \right\|_{X^*} \int_{\min\{t_n, t^*\}}^{\max\{t_n, t^*\}} \left\| \frac{d}{dt} x_n(s) \right\|_X \, ds + \left\langle x^* \chi_{[0,t^*]}, \int_0^{t_n} \frac{d}{dt} x_n(s) \, ds \right\rangle_{L^q(I,X)} \to 0,
\] (2.7)

where we used that \( \left( \frac{d}{dt} x_n \right)_{n \in \mathbb{N}} \subseteq L^1(I,X) \) is uniformly integrable, due to (2.3) (cf. [3, Theorem 4.2]). As a result, (2.7), (2.5) and the integral representation in Proposition 2.9 (i) yield

\[
\left\{ x_n \right\}_X(t_n) = \left\{ x_n \right\}_X(0) + \int_0^{t_n} \frac{d}{dt} x_n(s) \, ds \to 0 \quad \text{in } X.
\] (2.8)

As \( X \) is finite dimensional, (2.8) implies that \( \left\{ x_n \right\}_X(t_n) \to 0 \) in \( X \) \((n \to \infty)\). Using the latter in (2.6), we finally conclude

\[
\left\| \left\{ x_n \right\}_X \right\|_{C^0(I,X)} = \left\| \left\{ x_n \right\}_X(t_n) \right\|_X \to 0, \quad n \to \infty,
\]
i.e., \( \{ \cdot \}_X : W^{1,1}(I,X) \to C^0(\overline{I},X) \) is strongly continuous.

\[
\square
\]

### 2.4 Evolution equations

Let \((V, \| \cdot \|_V)\) be a reflexive Banach space, let \((H, \langle \cdot, \cdot \rangle_H)\) be a Hilbert space and let \( j : V \to H \) be an embedding such that \( R(j) \) is dense in \( H \). Then, the triple \((V, H, j)\) is said to be an evolution triple.

Denote by \( R : H \to H^* \) the Riesz isomorphism with respect to \( \langle \cdot, \cdot \rangle_H \). As \( j \) is a dense embedding the adjoint \( j^* : H^* \to V^* \) and therefore \( e := j^* R j : V \to V^* \) are embeddings as well. We call \( e \) the canonical embedding of \((V, H, j)\). Note that

\[
\langle ev, w \rangle_V = \langle jv, jw \rangle_H \quad \text{for all } v, w \in V.
\] (2.9)

For an evolution triple \((V, H, j), I := (0, T)\), with \( 0 < T < \infty \), and \( 1 < p < \infty \), we set

\[
\mathcal{X} := L^p(I,V), \quad \mathcal{W} := W^{1,p,p'}_e(I,V,V^*), \quad \mathcal{Y} := C^0(\overline{I},H).
\]

**Proposition 2.11.** Let \((V, H, j)\) be an evolution triple and let \( 1 < p < \infty \). Then, it holds \( x \in \mathcal{W} \) if and only if \( x \in \mathcal{X} \) and there exists \( x^* \in \mathcal{X}^* \) such that

\[
- \int_I (j(x(s)), jv)_H \varphi'(s) \, ds = \int_I \langle x^*(s), v \rangle_{V'} \varphi(s) \, ds.
\]

for every \( v \in V \) and \( \varphi \in C^\infty_0(I) \). In this case, we have \( \frac{d}{dt} x = x^* \) in \( \mathcal{X}^* \).

**Proof.** If \( V \) is additionally separable, a proof can be found in [22, Proposition 23.20]. As the argumentation remains true if we omit the separability of \( V \), we however refer to this proof. \( \square \)
Proposition 2.12. Let \((V, H, j)\) be an evolution triple and let \(1 < p < \infty\). Then, it holds:

(i) Given \(x \in \mathcal{W}\), the function \(jx \in L^p(I, H)\), given via
\[
(jx)(t) := j(x(t)) \in H
\]
for almost every \(t \in I\), possesses a unique representation in \(\mathcal{Y}\) and the resulting mapping \(j : \mathcal{W} \to \mathcal{Y}\) is an embedding. In particular, since also \(\mathcal{W} \hookrightarrow \mathcal{X}\), we gain the embedding \(\mathcal{W} \hookrightarrow \mathcal{X} \cap j \mathcal{Y}\).

(ii) Generalized integration by parts formula: It holds
\[
\int_{t'}^t \left( \frac{dx}{dt}(s), y(s) \right)_V ds = \left[ \langle (jx)(s), (jy)(s) \rangle_H \right]_{s=t'}^{s=t} - \int_{t'}^t \left( \frac{dy}{dt}(s), x(s) \right)_V ds
\]
for every \(x, y \in \mathcal{W}\) and \(t, t' \in \bar{I}\) with \(t' \leq t\).

Proof. See [19, Chapter III.1, Proposition 1.2].

Definition 2.13 (Evolution equation). Let \((V, H, j)\) be an evolution triple and let \(1 < p < \infty\). Furthermore, let \(y_0 \in H\) be an initial value, let \(f \in \mathcal{X}^*\) be a right-hand side and let \(A : \mathcal{X} \cap j \mathcal{Y} \to \mathcal{X}^*\) be an operator. Then, the initial value problem
\[
\frac{dy}{dt} + Ay = f \quad \text{in } \mathcal{X}^*,
\]
\[
(jy)(0) = y_0 \quad \text{in } H
\]
(2.10)
is said to be an evolution equation. The initial condition has to be understood in the sense of the unique continuous representation \(jy \in \mathcal{X}^*\) (cf. Proposition 2.12 (i)).

3 NOTIONS OF CONTINUITY AND GROWTH FOR EVOLUTION EQUATIONS

In [12], Bochner pseudo-monotonicity and Bochner coercivity has been introduced as appropriate notions of continuity and growth for evolution equations, as they operate on the same energy space \(\mathcal{X} \cap j L^\infty(I, H)\) and thus are more in balance than \(\frac{d}{dt}\)-pseudo-monotonicity and coercivity. We emphasize that Bochner pseudo-monotonicity was not directly defined with respect to the weak sequential topology of \(\mathcal{X} \cap j L^\infty(I, H)\). Indeed, [20, Remark 3.5] (see also Remark 3.2 below) gives an example of a sequence \((x_n)_{n \in \mathbb{N}} \subseteq \mathcal{X} \cap j L^\infty(I, H)\), which satisfies (1.5) and (1.6), but does not weakly converge in \(L^\infty(I, H)\), wherefore one cannot simply replace (1.5) and (1.6) by weak convergence in \(L^\infty(I, H)\) in the definition of Bochner pseudo-monotonicity (cf. Definition 1.4). However, according to Proposition 2.3, weak convergence in \(\mathcal{Y}\) is equivalent to (1.5) together with (1.6) valid, not just for almost every, but for all \(t \in \bar{I}\). This motivates generalizations of Bochner pseudo-monotonicity, which incorporate the weak sequential topology of \(\mathcal{X} \cap j \mathcal{Y}\).

Definition 3.1 (\(C^0\)-Bochner pseudo-monotonicity and \(C^0\)-Bochner condition (M)). Let \((V, H, j)\) be an evolution triple and let \(1 < p < \infty\). An operator \(A : \mathcal{X} \cap j \mathcal{Y} \to \mathcal{X}^*\) is said to be

(i) \(C^0\)-Bochner pseudo-monotone, if for a sequence \((x_n)_{n \in \mathbb{N}} \subseteq \mathcal{X} \cap j \mathcal{Y}\) from
\[
x_n \to x \quad \text{in } \mathcal{X} \cap j \mathcal{Y}, \tag{3.1}
\]
\[
\limsup_{n \to \infty} \langle Ax_n, x_n - x \rangle_{\mathcal{X}} \leq 0, \tag{3.2}
\]
it follows that \(\langle Ax, x - y \rangle_{\mathcal{X}} \leq \liminf_{n \to \infty} \langle Ax_n, x_n - y \rangle_{\mathcal{X}}\) for every \(y \in \mathcal{X}\).

(ii) satisfying the \(C^0\)-Bochner condition (M), if for a sequence \((x_n)_{n \in \mathbb{N}} \subseteq \mathcal{X} \cap j \mathcal{Y}\) from (3.1),
\[
Ax_n \to \xi \quad \text{in } \mathcal{X}^*, \tag{3.3}
\]
\[
\limsup_{n \to \infty} \langle Ax_n, x_n \rangle_{\mathcal{X}} \leq \langle \xi, x \rangle_{\mathcal{X}}, \tag{3.4}
\]
it follows that \(Ax = \xi\) in \(\mathcal{X}^*\).
Remark 3.2. \(C^0\)-Bochner pseudo-monotonicity \(\Rightarrow\) Bochner pseudo-monotonicity) Clearly, Bochner pseudo-monotonicity implies \(C^0\)-Bochner pseudo-monotonicity. This is an immediate consequence of Proposition 2.3. Note that the converse is not true in general. In fact, there exist \(C^0\)-Bochner pseudo-monotone operators which are not Bochner pseudo-monotone. This can be seen by the following example (cf. [20, Remark 3.5]).

Let \(I := (-1, 1)\), \(p \in (1, \infty)\), \(V := H = \mathbb{R}\) and \(A : L^\infty(I, \mathbb{R}) \to L^p(I, \mathbb{R})\) given via \(Ax := \langle \omega, x \rangle_{L^\infty(I, \mathbb{R})} = \int_I x(s) d\omega(s)\) for every \(x \in L^\infty(I, \mathbb{R})\), where \(\omega \in (L^\infty(I, \mathbb{R}))^*\) is a finitely additive measure with \(\omega \left( \left( -\frac{1}{2n}, 0 \right) \cup \left( 0, \frac{1}{2n} \right) \right) = 1\) for all \(n \in \mathbb{N}\), whose existence is guaranteed in [20, Theorem 2.9]. We define \((x_n)_{n \in \mathbb{N}} \subseteq L^\infty(I, \mathbb{R})\) by \(x_n(0) := 0\), \(x_n(t) := 0\) if \(|t| \geq 2/n\), \(x_n(t) := 1\) if \(|t| < 1/n\), and \(x_n(t) := -n|t| + 2\) if \(1/n < |t| < 2/n\). One easily sees, that \((x_n)_{n \in \mathbb{N}} \subseteq L^\infty(I, \mathbb{R})\) with \(\sup_{n \in \mathbb{N}} \|x_n\|_{L^\infty(I, \mathbb{R})} \leq 1\) and \(x_n(t) \to 0\) \((n \to \infty)\) for every \(t \in I\), which immediately implies that \(x_n \to 0\) in \(L^\infty(I, \mathbb{R})\) \((n \to \infty)\) and \(x_n \to 0\) in \(L^p(I, \mathbb{R})\) \((n \to \infty)\). Apart from that, according to [20, Theorem 2.8], we have \(Ax_n = (\omega, x_n)_{L^\infty(I, \mathbb{R})} = 1\) for all \(n \in \mathbb{N}\), which let us exclude that \(x_n \neq 0\) in \(L^\infty(I, \mathbb{R})\) \((n \to \infty)\) and provides that \(\limsup_{n \to \infty} \langle Ax_n, x_n - 0 \rangle_{L^p(I, \mathbb{R})} = \lim_{n \to \infty} \int_I x_n(s) ds = 0\). Overall, \((x_n)_{n \in \mathbb{N}} \subseteq L^\infty(I, \mathbb{R})\) is a sequence satisfying (3.1) and (3.2), then we have \(Ax_n \to Ax\) in \(\mathbb{R}\) \((n \to \infty)\), as also \(\omega \in (C^0(I, \mathbb{R}))^*\) and therefore \(\langle Ax, x - y \rangle_{L^p(I, \mathbb{R})} \leq \liminf_{n \to \infty} \langle Ax_n, x_n - y \rangle_{L^p(I, \mathbb{R})}\) for every \(y \in L^p(I, \mathbb{R})\). In other words, \(A : L^\infty(I, \mathbb{R}) \to L^p(I, \mathbb{R})\) is \(C^0\)-Bochner pseudo-monotone.

Proposition 3.3. Let \((V, H, j)\) be an evolution triple and let \(1 < p < \infty\). Then, it holds:

(i) \(A : \mathcal{X} \cap J \to X^*\) is \(C^0\)-Bochner pseudo-monotone, then it satisfies the \(C^0\)-Bochner condition \((M)\).

(ii) \(A : \mathcal{X} \cap J \to X^*\) is locally bounded and satisfies the \(C^0\)-Bochner condition \((M)\), then it is semi-continuous.

Proof. (i) Let \((x_n)_{n \in \mathbb{N}} \subseteq \mathcal{X} \cap J\) be a sequence satisfying (3.1), (3.3) and (3.4). In particular, (3.3) and (3.4) imply (3.2). The \(C^0\)-Bochner pseudo-monotonicity of \(A : \mathcal{X} \cap J \to X^*, (3.2)\) and (3.3) thus yield for every \(y \in \mathcal{X}\)

\[
\langle Ax, x - y \rangle_{X^*} \leq \liminf_{n \to \infty} \langle Ax_n, x_n - y \rangle_{X^*}
\]

\[
\leq \limsup_{n \to \infty} \langle Ax_n, x_n - x \rangle_{X^*} + \limsup_{n \to \infty} \langle Ax_n, x - y \rangle_{X^*}
\]

\[
\leq \langle \xi, x - y \rangle_{X^*}.
\]

Inserting \(y = x - z \in \mathcal{X}\) for arbitrary \(z \in \mathcal{X}\) in (3.5), we conclude that \(Ax = \xi \in X^*\).

(ii) Let \((x_n)_{n \in \mathbb{N}} \subseteq \mathcal{X} \cap J\) be a sequence such that \(x_n \rightharpoonup x\) in \(\mathcal{X} \cap J\) \((n \to \infty)\). From the locally boundedness of \(A : \mathcal{X} \cap J \to X^*\) and reflexivity of \(X^*\) we obtain a subsequence \((Ax_n)_{n \in \Lambda} \subseteq X^*, \) with \(\Lambda \subseteq \mathbb{N}\) and \(\xi \in X^*\) such that \(Ax_n \rightharpoonup \xi\) \((\Lambda \ni n \to \infty)\). Hence, it holds \(\langle Ax_n, x_n \rangle_X \to \langle \xi, x \rangle_X\) \((\Lambda \ni n \to \infty)\), i.e., (3.4) with respect to \(\Lambda\). From the \(C^0\)-Bochner condition \((M)\) we conclude that \(Ax = \xi \in X^*\). As this argumentation stays valid for each subsequence of \((x_n)_{n \in \mathbb{N}} \subseteq \mathcal{X} \cap J\), \(Ax \in X^*\) is weak accumulation point of each subsequence of \((Ax_n)_{n \in \mathbb{N}} \subseteq X^*\). The standard convergence principle (cf. [9, Kap. I, Lemma 5.4]) finally yields \(Ax_n \rightharpoonup Ax\) in \(X^*\) \((n \to \infty)\).\]

Remark 3.4 \((C^0\)-Bochner condition \((M) \Rightarrow C^0\)-pseudo-monotonicity) According to Proposition 3.3 (i), \(C^0\)-Bochner pseudo-monotonicity implies the \(C^0\)-Bochner condition \((M)\). But note that there exist operators satisfying the \(C^0\)-Bochner condition \((M)\) without being \(C^0\)-Bochner pseudo-monotone.

For example, let \(I := (0, T)\), with \(0 < T < \infty\), \(p \in (1, \infty)\), let \(V = H\) be a separable Hilbert space with orthonormal basis \((e_n)_{n \in \mathbb{N}} \subseteq H\) and Riesz isomorphism \(R : H \to H^*\). Moreover, let \(A : C^0(\bar{T}, H) \to L^p(I, H^*)\) be given via \(\langle Ax(t) \rangle := -R(x(t))\) in \(H^*\) for almost every \(t \in I\) and all \(x \in C^0(\bar{T}, H)\). Then, \(A : C^0(\bar{T}, H) \to L^p(I, H^*)\) satisfies the \(C^0\)-Bochner condition \((M)\), which is weakly continuous, but is not \(C^0\)-Bochner pseudo-monotone. In fact, the sequence \((x_n)_{n \in \mathbb{N}} \subseteq C^0(\bar{T}, H)\), given via \(x_n(t) := e_n\) in \(H\) for every \(t \in \bar{T}\) and \(n \in \mathbb{N}\), satisfies \(x_n \rightharpoonup 0\) in \(C^0(\bar{T}, H)\) \((n \to \infty)\) and \(\limsup_{n \to \infty} \langle Ax_n, x_n - 0 \rangle_{L^p(I, H)} = -T < 0\), but \(\liminf_{n \to \infty} \langle Ax_n, x_n - y \rangle_{L^p(I, H)} = -T < 0 = \langle A0, 0 - y \rangle_{L^p(I, H)}\) for every \(y \in L^p(I, H)\).
**Definition 3.5 (C^0-Bochner coercivity).** Let \((V, H, j)\) be an evolution triple and let \(1 < p < \infty\). An operator \(A : \mathcal{X} \cap j \mathcal{Y} \to \mathcal{X}^*\) is said to be

(i) \(C^0\)-Bochner coercive with respect to \(f \in \mathcal{X}^*\) and \(x_0 \in H\), if there exists a constant \(M := M(f, x_0, A) > 0\) such that for every \(x \in \mathcal{X} \cap j \mathcal{Y}\) from

\[
\frac{1}{2} \|jx(t)\|_H^2 + \langle Ax - f, x_{\chi(0,t)} \rangle_{\mathcal{X}} \leq \frac{1}{2} \|x_0\|_H^2 \quad \text{for all } t \in \bar{T},
\]

it follows that \(\|x\|_{\mathcal{X} \cap j \mathcal{Y}} \leq M\).

(ii) \(C^0\)-Bochner coercive, if it is \(C^0\)-Bochner coercive with respect to all \(f \in \mathcal{X}^*\) and \(x_0 \in H\).

Note that \(C^0\)-Bochner coercivity, similar to semi-coercivity (cf. [16]) in conjunction with Gronwall’s inequality, takes into account the information from the operator and the time derivative. In fact, \(C^0\)-Bochner coercivity is a more general property. In the context of the main theorem on pseudo-monotone perturbations of maximal monotone mappings (cf. [23, §32.4]), which implies Theorem 1.2, \(C^0\)-Bochner coercivity is phrased in the spirit of a local coercivity type condition of the form\(\frac{dx}{dt} + A : \mathcal{W} \subseteq \mathcal{X} \to \mathcal{X}^*\). Being more precise, if \(A : \mathcal{X} \cap j \mathcal{Y} \to \mathcal{X}^*\) is \(C^0\)-Bochner coercive with respect to \(f \in \mathcal{X}^*\) and \(x_0 \in H\), then for \(x \in \mathcal{W}\) from \(\|jx(0)\|_H \leq \|x_0\|_H\), i.e., \(\langle \frac{dx}{dt}, x \rangle_{\mathcal{X}} \geq -\frac{1}{2} \|x_0\|_H^2\), and

\[
\langle \frac{dx}{dt} + Ax, x_{\chi(0,t)} \rangle_{\mathcal{X}} \leq \langle f, x_{\chi(0,t)} \rangle_{\mathcal{X}} \quad \text{for all } t \in \bar{T},
\]

it follows that \(\|x\|_{\mathcal{X} \cap j \mathcal{Y}} \leq M\), since (3.7) is just (3.6). In other words, if the image of \(x \in \mathcal{W}\) with respect to \(\frac{dx}{dt}\) and \(A\) is bounded by the data \(x_0, f\) in this weak sense, then \(x\) is contained in a fixed ball in \(\mathcal{X} \cap j \mathcal{Y}\). We chose (3.6) instead of (3.7) in Definition 3.5, since \(x \in \mathcal{X} \cap j \mathcal{Y}\) is not admissible in (3.7).

We emphasize that there is a relation between \(C^0\)-Bochner coercivity and coercivity in the sense of Definition 2.1. In fact, in the case of bounded operators \(A : \mathcal{X} \to \mathcal{X}^*\), \(C^0\)-Bochner coercivity extends the standard concept of coercivity.

**Proposition 3.6.** Let \((V, H, j)\) be an evolution triple and let \(1 < p < \infty\). If \(A : D(A) \subseteq \mathcal{X} \to \mathcal{X}^*\) with \(D(A) = \mathcal{X} \cap j \mathcal{Y}\) is bounded and coercive (in the sense of Definition 2.1), then \(A : \mathcal{X} \cap j \mathcal{Y} \to \mathcal{X}^*\) is \(C^0\)-Bochner coercive.

**Proof.** A straightforward adaptation of [12, Lemma 3.11].

**Lemma 3.7 (Induced Bochner pseudo-monotonicity and Bochner coercivity).** Let \((V, H, j)\) be an evolution triple, let \(1 < p < \infty\) and let \(A(t) : V \to V^*\), \(t \in I\), be a family of operators with the following properties:

- (C.1) \(A(t) : V \to V^*\) is pseudo-monotone for almost every \(t \in I\).
- (C.2) \(A(\cdot)v : I \to V^*\) is Bochner measurable for all \(v \in V\).
- (C.3) There exist non-negative functions \(\alpha, \gamma \in L^p(I, \mathbb{R}), \beta \in L^\infty(I, \mathbb{R})\) and a non-decreasing function \(R : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\) such that

\[
\|A(t)v\|_{V^*} \leq R(\|ju\|_H^p)(\alpha(t) + \beta(t)\|v\|_{V^*}^{p-1}) + \gamma(t)
\]

for almost every \(t \in I\) and all \(v \in V\).

- (C.4) There exist a constant \(c_0 > 0\) and non-negative functions \(c_1, c_2 \in L^1(I, \mathbb{R})\) such that

\[
\langle A(t)v, v \rangle_{V^*} \geq c_0\|v\|_{V^*}^p - c_1(t)\|ju\|_H^2 - c_2(t)
\]

for almost every \(t \in I\) and all \(v \in V\).

Then, the induced operator \(\mathcal{A} : \mathcal{X} \cap j L^\infty(I, H) \to \mathcal{X}^*\), given via \((\mathcal{A}x)(t) := A(t)(x(t))\) in \(V^*\) for almost every \(t \in I\) and all \(x \in \mathcal{X} \cap j L^\infty(I, H)\), is well-defined, bounded, Bochner pseudo-monotone and Bochner coercive.
If $V$ is additionally separable, a proof can be found in [12, Proposition 3.13]. As the argumentation remains true if we omit the separability of $V$, we however refer to this proof. 

\section{Abstract Galerkin Approach}

In this section, we specify the exact framework of the implemented Galerkin approach.

\begin{remark}[Galerkin approximation] Let $(V, H, j)$ be an evolution triple and let $1 < p < \infty$. Furthermore, let $y_0 \in H$, $f \in X^*$ and $A : X \cap J \rightarrow X^*$.

(i) \textit{Galerkin-Basis:} Let $U$ be a system of subspaces $U \subseteq V$, such that $(U, \| \cdot \|_V)$ is a Banach space and

$$\bigcup_{U \in U'} U = V.$$ 

Moreover, for $U \in U'$ we set $H_U := j(U)$ and by $y_0^U \in H_U$ we denote a net of approximative initial values such that $y_0^U \to y_0$ in $H$ and $\sup_{U \in U'} \| y_0^U \|_H \leq \| y_0 \|_H$.

where the notion of convergence initially has to be understood in the sense of nets but will later be realized by sequential convergence. Such a family $(U, y_0^U)_{U \in U'}$ is called a \textit{Galerkin basis} of $(V, y_0)$.

(ii) \textit{Restriction of evolution triple structure:} For $U \in U'$, the space $(H_U, (\cdot, \cdot)_H)$ is a Hilbert space and the restricted operator $j_U := j \mid U : U \to H_U$ is a dense embedding, i.e., $(U, H_U, j_U)$ forms an evolution triple. In particular, the corresponding canonical embedding $e_U : U \to U^*$ satisfies

$$\langle e_U u, \tilde{u} \rangle_U = (ju, j \tilde{u})_H = \langle e_u, \tilde{u} \rangle_V \quad \text{for all } u, \tilde{u} \in U. \quad (4.1)$$

(iii) \textit{Restriction of energy spaces:} For $U \in U'$ and $I := (0, T)$, with $0 < T < \infty$, we set

$$X_U := L^p(I, U), \quad W_U := W^{1, p, p'}(I, U, U^*), \quad Y_U := C^0(I, H_U).$$

Due to Remark 2.5, the couple $(X_U, Y_U) := (X_U, Y_U, L^1(I, H_U), j_U, id y_0^U)$ forms a compatible couple, where $j_U : X_U \cap J_U \to Y_U$ is given via $(j_U x)(t) := j_U(x(t)) = (j x)(t)$ for almost every $t \in I$ and all $x \in X_U \cap J_U$.

(iv) \textit{Restriction of operators:} For $U \in U'$, we define the restricted operator and right-hand side by

$$A_U := (\text{id}_{X_U})^* A : X_U \cap J_U \to X_U^* \quad \text{and} \quad f_U := (\text{id}_{X_U})^* f \in X_U^*.$$ 

Then, it holds for every $x \in X_U \cap J_U$ and $\tilde{x} \in X_U$

$$\langle A_U x, \tilde{x} \rangle_{X_U} = \langle Ax, \tilde{x} \rangle_X \quad \text{and} \quad \langle f_U, \tilde{x} \rangle_{X_U} = \langle f, \tilde{x} \rangle_X. \quad (4.3)$$
(v) **Galerkin system:** Given a Galerkin basis \((U, y^U_0)\) \(U \subseteq U^*\), we obtain the well-posedness of the system of evolution equations

\[
\frac{d_{eU}y_U}{dt} + Af_U = f_U \quad \text{in } U^*,
\]

\[
(f_U y_U)(0) = y^U_0 \quad \text{in } H_U, \quad U \in U^*.
\]

Such a system is called **Galerkin system with respect to\( (U, y^U_0)\) \(U \subseteq U^*\).**

The next lemma examines to what extent the properties of the global operator, especially those developed in Section 3, transfer to its restriction as per Remark 4.1 (iv).

**Lemma 4.2.** Let \((V, H, j)\) be an evolution triple and let \(1 < p < \infty\). Furthermore, let \(y_0 \in H, f \in X^*\) and \(A : X \cap J Y \to X^*\). If \((U, y^U_0)_{U \subseteq U^*}\) is a Galerkin basis of \((V, y_0)\), \(A_U := \left( id_{X_U} \right)^* A : X_U \cap J U Y_U \to X^*_U\) and \(f_U := \left( id_{X_U} \right)^* f \in X^*_U\), then it holds:

(i) If \(A : X \cap J Y \to X^*\) is bounded, semi-continuous or \(C^0\)-Bochner pseudo-monotone, then also \(A_U : X_U \cap J U Y_U \to X^*_U\).

(ii) If \(A : X \cap J Y \to X^*\) is \(C^0\)-Bochner coercive with respect to \(f \in X^*\), then \(A_U : X_U \cap J U Y_U \to X^*_U\) is \(C^0\)-Bochner coercive with respect to \(f_U \in X^*_U\).

(iii) If \(A_0 : X \to X^*\) is monotone, \(B : X \cap J Y \to X^*\) is bounded and \(A := A_0 + B : X \cap J Y \to X^*\) satisfies the \(C^0\)-Bochner condition (M), then \(A_U : X_U \cap J U Y_U \to X^*_U\) satisfies the \(C^0\)-Bochner condition (M).

**Proof.** \(\text{ad (i) \& (ii) (i)}\) follows from the embedding \(X_U \cap J U Y_U \hookrightarrow X \cap J Y\), the weak continuity of \(\left( id_{X_U} \right)^* : X^* \to X^*_U\) and the identities (4.3). (ii) follows from (4.3), \(|y^U_0|^2_H \leq \|y_0\|^2_H\), \(\|A U n \cap J U Y U \|= \|A X n \cap J Y\|\), \(\|y^n_0\|^2_H\) on \(X \cap J Y\) and \(\|\cdot\|_{H U} = \|\cdot\|_{H}\) on \(H_U\).

\(\text{ad (iii)}\) Let \((x_n)_{n \in N} \subseteq X_U \cap J U Y_U\) be a sequence satisfying (3.1)–(3.4) with respect to \(X_U\) and \(Y_U\), i.e.,

\[
x_n \xrightarrow{n \to \infty} x \quad \text{in } X_U \cap J U Y_U,
\]

\[
A_U x_n \xrightarrow{n \to \infty} \xi_U \quad \text{in } X^*_U,
\]

\[
\limsup_{n \to \infty} \langle A_U x_n, x_n \rangle_{X^*_U} \leq \langle \xi_U, x \rangle_{X^*_U}.
\]

The embedding \(X_U \cap J U Y_U \hookrightarrow X \cap J Y\) and (4.4), immediately imply \(x_n \xrightarrow{n \to \infty} \xi \) in \(X \cap J Y\) \((n \to \infty)\). In addition, due to (4.4)_1, 2, there exist constants \(M, M' > 0\), such that \(|x_n|^2_{X \cap J Y} = \|x_n\|^2_{X U \cap J U Y U} \leq M\) and \(\|A_U x_n\|^2_{X^*_U} \leq M'\) for every \(n \in N\).

Since \(B : X \cap J Y \to X^*\) is bounded, we obtain a further constant \(C > 0\) such that \(|B x_n|^2_{X^*} \leq C\) for every \(n \in N\). From this and (4.3) we deduce for every \(n \in N\) that

\[
\langle A_0 x_n, x_n \rangle \leq \|A_U x_n\|^2_{X^*_U} \cdot \|x_n\|^2_{X^*_U} + \|B x_n\|^2_{X^*} \cdot \|x_n\|^2_{X} \leq (M' + C)M.
\]

Due to (4.5) and the monotonicity of \(A_0 : X \to X^*\), Proposition 2.2 with \(S = (x_n)_{n \in N} \subseteq X\) and \(h \equiv 1\) provides a constant \(K > 0\) such that \(|A_0 x_n|^2_{X^*} \leq K\) for every \(n \in N\). Thus, \((Ax_n)_{n \in N} \subseteq X^*\) is bounded and, in virtue of the reflexivity of \(X^*\), we extract a subsequence \((Ax_n)_{n \in A} \subseteq X^*\), with \(A \subseteq N\), and an element \(\xi \in X^*\) such that

\[
Ax_n \xrightarrow{n \to \infty} \xi \quad \text{in } X^* \quad (n \in A).
\]
We infer $A_U x_n = (\text{id}_{X_U}^*)^* A x_n \to (\text{id}_{X_U}^*)^* \xi$ in $X_U^*$ ($\forall n \to \infty$), i.e., $(\text{id}_{X_U}^*)^* \xi = \xi_U$ in $X_U^*$, and the weak continuity of $(\text{id}_{X_U}^*)^* : X^* \to X_U^*$. Finally, we use (4.3) once more and (4.4) to obtain

$$
\limsup_{n \to \infty} \langle A x_n, x_n \rangle_{X_U^*} = \limsup_{n \to \infty} \langle (\text{id}_{X_U}^*)^* \xi, x \rangle_{X_U^*} = \langle \xi_U, x \rangle_{X_U^*}.
$$

Altogether, $(x_n)_{n \in \mathbb{N}} \subseteq X \cap j_Y$ satisfies (3.1)–(3.4) with respect to $\mathbb{X}$ and $J$, and the $C^0$-Bochner condition (M) of $A : X \cap j_Y \to X^*$ finally yields $A x = \xi$ in $X^*$, and therefore $A_U x = (\text{id}_{X_U}^*)^* A x = (\text{id}_{X_U}^*)^* \xi = \xi_U$ in $X_U^*.

5 | MAIN THEOREM: (SEPARABLE CASE)

**Theorem 5.1.** Let $(V, H, j)$ be an evolution triple, let $V$ be separable and let $1 < p < \infty$. Furthermore, we require the following conditions:

(i) $A_0 : X \to X^*$ is monotone.
(ii) $B : X \cap j_Y \to X^*$ is bounded.
(iii) $A := A_0 + B : X \cap j_Y \to X^*$ satisfies the $C^0$-Bochner condition (M) and is $C^0$-Bochner coercive with respect to $f \in X^*$ and $y_0 \in H$.

Then, there exists a solution $y \in W$ of the evolution equation

$$
\frac{d_e y}{dt} + A y = f \quad \text{in } X^*,
$$

$(jy)(0) = y_0 \quad \text{in } H$.

**Proof.**

**0. Reduction of assumptions:** It suffices to treat the special case of trivial right-hand side $f = 0$ in $X^*$. Otherwise, we switch to $\hat{A} := A_0 + \hat{B} : X \cap j_Y \to X^*$ with the shifted bounded part $\hat{B} := B - f : X \cap j_Y \to X^*$. It is straightforward to check that $\hat{A}$ still satisfies the $C^0$-Bochner condition (M) and is $C^0$-Bochner coercive with respect to $0 \in X^*$ and $y_0 \in H$.

1. **Galerkin approximation:** We apply the Galerkin approach of Section 4. As Galerkin basis of $(V, y_0)$ will serve a sequence $(V_n, ja_n)_{n \in \mathbb{N}}$ with the following properties:

- $V_n \subseteq V_{n+1} \subseteq V$, dim $V_n < \infty$ and $\bigcup_{n \in \mathbb{N}} V_n = V$. 
- $a_n \in V_n, ja_n \to y_0$ in $H$ ($n \to \infty$) and $\sup_{n \in \mathbb{N}} \|ja_n\|_H \leq \|y_0\|_H$.

The existence of such a sequence is a consequence of the separability of $V$ in conjunction with the given evolution triple structure. The well-posedness of the Galerkin system with respect to $(V_n, ja_n)_{n \in \mathbb{N}}$ follows as in Remark 4.1. We denote for $n \in \mathbb{N}$ by $y_n \in W_{V_n}$ the $n$th Galerkin solution, if

$$
\frac{d_{V_n} y_n}{dt} + A_{V_n} y_n = 0 \quad \text{in } X_{V_n}^*,
$$

$(j_{V_n} y_n)(0) = ja_n \quad \text{in } H_{V_n}$.

2. **Existence of Galerkin solutions:** As the operator $A$ is not necessarily induced, Carathéodory's theorem is not available. However, we will prove the existence of Galerkin solutions similarly to Carathéodory's theorem by translating (5.1) into an equivalent fixed point problem and then exploiting an appropriate version of Schauder's fixed point theorem.
To this end, we first translate (5.1) into an equivalent differential equation with values in $V_n$ instead of $V^*_n$, to have a chance to meet the in Schauder’s fixed point theorem demanded self map property.

2.1 Equivalent differential equation: As $\mathcal{e}_V : V_n \rightarrow V^*_n$ is an isomorphism, Proposition 2.4 ensures that the induced operator $\mathcal{e}_n : L^p(I, V_n) \rightarrow \mathcal{X}^*_V$, given via $(\mathcal{e}_n x)(t) := \mathcal{e}_V(x(t))$ in $V^*_n$ for almost every $t \in I$ and all $x \in L^p(I, V_n)$, is also an isomorphism. Apart from this, Proposition 2.8 additionally yields that $\mathcal{e}_n : W^{1,p'}(I, V_n) \rightarrow W^{1,p'}(I, V^*_n)$ is an isomorphism and that for every $x \in W^{1,p'}(I, V_n)$, it holds

$$\frac{dV_n}{dt} \mathcal{e}_n x = e_n \frac{dV_n}{dt} x \quad \text{in } \mathcal{X}^*_V. \quad (5.2)$$

Using Proposition 2.7 and (2.1), we see that $y_n \in \mathcal{W}_V$ satisfies (5.1) if and only if $y_n \in \mathcal{X}_V$ and $\mathcal{e}_n y_n \in W^{1,p'}(I, V^*_n)$ with

$$\frac{dV_n}{dt} y_n = \mathcal{e}_n^{-1} \frac{dV_n}{dt} \mathcal{e}_n y_n = -\mathcal{A}_V y_n \quad \text{in } L^{p'}(I, V_n). \quad (5.3)$$

By exploiting Proposition 2.8 and (5.2), we further deduce the equivalence of (5.3) and $y_n = \mathcal{e}_n^{-1} \mathcal{e}_n y_n \in W^{1,p'}(I, V_n)$ with

$$\frac{dV_n}{dt} y_n = \mathcal{e}_n^{-1} \frac{dV_n}{dt} \mathcal{e}_n y_n = -\mathcal{e}_n^{-1} \mathcal{A}_{V^*_n} y_n \quad \text{in } L^{p'}(I, V_n). \quad (5.4)$$

Proposition 2.9 (i) provides the choice function $\{ \cdot \}_{V_n} : W^{1,p'}(I, V_n) \rightarrow C^0(I, V_n)$. Thus, $y_n \in \mathcal{W}_V$ satisfies (5.1) if and only if $\{ y_n \}_{V_n}(0) = a_n$ in $V_n$ in the sense of the unique continuous representation $\{ y_n \}_{V_n} \in C^0(I, V_n)$. Altogether, $y_n \in \mathcal{W}_V$ is a solution of (5.1) if and only if $\{ y_n \}_{V_n} \in C^0(I, V_n)$. By exploiting Proposition 2.8 and (5.2), we further deduce the equivalence of (5.3) and $y_n = \mathcal{e}_n^{-1} \mathcal{e}_n y_n \in W^{1,p'}(I, V_n)$ with

$$\frac{dV_n}{dt} y_n = \mathcal{e}_n^{-1} \frac{dV_n}{dt} \mathcal{e}_n y_n = -\mathcal{e}_n^{-1} \mathcal{A}_{V^*_n} y_n \quad \text{in } L^{p'}(I, V_n). \quad (5.4)$$

2.2 Equivalent fixed point problem: Clearly, we have $\mathcal{C}^0(\bar{I}, V_n) \cong \mathcal{X}_V \cap j_{V_n} \mathcal{Y}_{V^*_n}$. In addition, since $j_{V_n} : V_n \rightarrow H_{V^*_n}$ is an isomorphism, the induced operator $j_{V_n} : C^0(\bar{I}, V_n) \rightarrow \mathcal{X}_{V^*_n}$, given via $(j_{V_n} x)(t) := j_{V_n}(x(t))$ for every $t \in \bar{I}$ and all $x \in C^0(\bar{I}, V_n)$, is an isomorphism as well. Thus, each function $x \in \mathcal{X}_{V^*_n} \cap j_{V_n} \mathcal{Y}_{V^*_n}$, i.e., $j_{V_n} x \in \mathcal{Y}_{V_n}$, also satisfies $x = j_{V_n}^{-1} j_{V_n} x \in C^0(\bar{I}, V_n)$ with $\|x\|_{C^0(\bar{I}, V_n)} \leq \|j_{V_n}^{-1}\|_{C^0(\bar{I}, V_n)} \|j_{V_n} x\|_{\mathcal{Y}_{V^*_n}}$. As a result, we have $C^0(\bar{I}, V_n) = \mathcal{X}_{V^*_n} \cap j_{V_n} \mathcal{Y}_{V^*_n}$ with $n$-dependent norm equivalence. From this and Proposition 2.9 (ii) we deduce the well-definedness of the fixed point operator $\mathcal{F}_n : C^0(\bar{I}, V_n) \rightarrow W^{1,p'}(I, V_n)$, for every $x \in C^0(\bar{I}, V_n)$ defined by

$$(\mathcal{F}_n x)(t) := a_n - (V_n e_n^{-1} \mathcal{A}_{V^*_n}) x(t) = a_n - \int_0^t (e_n^{-1} \mathcal{A}_{V^*_n} x)(s) ds \quad \text{in } V_n \quad \text{for all } t \in \bar{I}. \quad (5.5)$$

In addition, the embedding $W^{1,p'}(I, V_n) \hookrightarrow C^0(\bar{I}, V_n)$ (cf. Proposition 2.9 (i)) provides the well-definedness of $\mathcal{F}_n : W^{1,p'}(I, V_n) \subseteq C^0(\bar{I}, V_n) \rightarrow W^{1,p'}(I, V_n)$. Analogously to the theory of ordinary differential equations we conclude under the renewed application of Proposition 2.9 the equivalence of (5.4) and the existence of a fixed point of $\mathcal{F}_n : W^{1,p'}(I, V_n) \subseteq C^0(\bar{I}, V_n) \rightarrow W^{1,p'}(I, V_n)$.

2.3 Existence of a fixed point of $\mathcal{F}_n$: The verification of the existence of a fixed point is based on the following version of Schauder’s fixed point theorem:

**Theorem 5.2.** Let $(\mathcal{X}, \| \cdot \|_\mathcal{X})$ be a Banach space, let $\mathcal{F} : K \subseteq \mathcal{X} \rightarrow K$ be a continuous operator and let $\mathcal{K} \subseteq \mathcal{X}$ be a non-empty, convex and compact set. Then, there exists $x \in \mathcal{K}$ such that

$$\mathcal{F} x = x \quad \text{in } \mathcal{X}.$$
Proof. See [17, Kapitel 1, Satz 2.46].

It remains to verify the assumptions of Schauder’s fixed point theorem.

(i) **Continuity of** $F_n$: Lemma 4.2 (i) in conjunction with Proposition 3.3 (ii) yields the demi-continuity of the operator $A_{V_n} : \mathcal{X}_{V_n \cap j_{V_n}} \rightarrow \mathcal{X}_{V_n}$. Thus, as there holds $C^0(\bar{T}, V_n) = \mathcal{X}_{V_n \cap j_{V_n}}$, with norm equivalence, and both $e^{-1} : \mathcal{X}_{V_n} \rightarrow L^p(I, V_n) \text{ and } V_n : L^p(I, V_n) \rightarrow W^{1,p}(I, V_n)$ are weakly continuous (cf. Proposition 2.4 and 2.9 (ii)), $F_n : C^0(\bar{T}, V_n) \rightarrow W^{1,p}(I, V_n)$ is demi-continuous. Proposition 2.10 eventually provides the strong continuity of the embedding $W^{1,p}(I, V_n) \hookrightarrow C^0(\bar{T}, V_n)$ and consequently the continuity of $F_n : C^0(\bar{T}, V_n) \rightarrow C^0(\bar{T}, V_n)$.

(ii) **Self-map property of the compressed fixed point operator**: Since $A_{V_n}$, in general, fails to comply with the in Schauder’s fixed point theorem demanded self-map property, we construct a compression operator $\tau_n \cdot C^0(\bar{T}, V_n) \rightarrow (0, 1]$ such that the compressed operator $\tau_n A_{V_n}$ meets the self-map property and has coinciding fixed point set with $A_{V_n}$. Then, it suffices to prove the existence of a fixed point of the compressed fixed point operator.

As we are not aware of how to construct the desired compression operator $\tau_n$, we first consider $\tau_n A_{V_n}$ with an arbitrary operator $\tau_n : C^0(\bar{T}, V_n) \rightarrow (0, 1]$ and demonstrate the existence of a-priori estimates which are independent of $\tau_n$.

(a) **Invariance of the a priori estimates with respect to compressions**: We fix an arbitrary operator $\tau_n : C^0(\bar{T}, V_n) \rightarrow (0, 1]$ and assume there exists a fixed point $y_n \in W^{1,p}(I, V_n)$ of $\tau_n A_{V_n} : W^{1,p}(I, V_n) \subseteq C^0(\bar{T}, V_n) \rightarrow W^{1,p}(I, V_n)$. Then, we deduce analogously to the discussion in Step 2.1 and 2.2 that $y_n \in \mathcal{X}_{V_n}$ satisfies

$$
\frac{de_{V_n} y_n}{dt} + \tau_n(y_n) A_{V_n} y_n = 0 \quad \text{in } \mathcal{X}_{V_n},
$$

$$(j_{V_n} y_n)(0) = \tau_n(y_n) j_{V_n} y_0 \quad \text{in } H_{V_n}.
$$

Testing (5.5) by $y_n \chi_{[0,t]} \in \mathcal{X}_{V_n}$, where $t \in (0,T]$ is arbitrary, and a subsequent application of the generalized integration by parts formula (4.2) and identity (4.3) with $U = V_n$, yield

$$
\tau_n(y_n) \left( A y_n, y_n \chi_{[0,t]} \right)_{\mathcal{X}} = -\left( \frac{de_{V_n} y_n}{dt}, y_n \chi_{[0,t]} \right)_{\mathcal{X}_{V_n}}
$$

$$
= -\frac{1}{2} \left\| (j_{V_n} y_n)(t) \right\|_H^2 + \frac{\tau_n(y_n)}{2} \left\| j_{V_n} y_n \right\|_H^2.
$$

From dividing (5.6) by $0 < \tau_n(y_n) \leq 1$ and using $\| j_{V_n} y_n \|_H \leq \| y_0 \|_H$, we further obtain

$$
\frac{1}{2\tau_n(y_n)} \left\| (j_{V_n} y_n)(t) \right\|_H^2 + \left( A y_n, y_n \chi_{[0,t]} \right)_{\mathcal{X}} \leq \frac{\tau_n(y_n)}{2} \left\| y_0 \right\|_H^2 \leq \frac{1}{2} \left\| y_0 \right\|_H^2.
$$

As $t \in \bar{T}$ was arbitrary, $1/2 \leq 1/(2\tau_n(y_n))$ and $A : \mathcal{X}_{\cap j} \mathcal{Y} \rightarrow \mathcal{X}^*$ is $C^0$-Bochner coercive with respect to $0 \in \mathcal{X}^*$ and $y_0 \in H$, there exists an $n$-independent constant $M > 0$, such that

$$
\| y_n \|_{\mathcal{X}_{\cap j} \mathcal{Y}} \leq M.
$$

The boundedness of $B : \mathcal{X}_{\cap j} \mathcal{Y} \rightarrow \mathcal{X}^*$ and (5.8) further yield an $n$-independent constant $C > 0$ such that $\| By_n \|_{\mathcal{X}^*} \leq C$. From this, (5.7) in the case $t = T$, and (5.8) we obtain

$$
\left( A_0 y_n, y_n \right)_{\mathcal{X}} = \left( A y_n, y_n \right)_{\mathcal{X}} - \left( By_n, y_n \right)_{\mathcal{X}} \leq \frac{1}{2} \left\| y_0 \right\|_H^2 + CM.
$$
Finally, due to (5.9) and the monotonicity of \( \mathcal{A}_0 : \mathcal{X} \to \mathcal{X}^* \), Proposition 2.2 with \( S = (y_n)_{n \in \mathbb{N}} \subseteq \mathcal{X} \) and \( h \equiv 1 \), provides an \( n \)-independent constant \( M' > 0 \) such that

\[
\| \mathcal{A} y_n \|_{\mathcal{X}^*} \leq M'.
\]

(b) **Construction of the compression operator:** The demi-continuity of \( \mathcal{A} : \mathcal{X} \cap \mathcal{Y} \to \mathcal{X}^* \) (cf. Propositions 2.2 (i) and 3.3 (ii)) and the embedding \( C^0(\overline{I}, V_n) \hookrightarrow \mathcal{X} \cap \mathcal{Y} \) imply the continuity of

\[
(x \mapsto \langle \mathcal{A} x, x \rangle_{\mathcal{X}}) : C^0(\overline{I}, V_n) \to \mathbb{R}_{\geq 0}, \quad \text{and} \quad \| \cdot \|_{\mathcal{X} \cap \mathcal{Y}} : C^0(\overline{I}, V_n) \to \mathbb{R}_{\geq 0}.
\]

From this, we deduce the continuity of \( g, h : C^0(\overline{I}, V_n) \to \mathbb{R}_{\geq 0} \), for every \( x \in C^0(\overline{I}, V_n) \) defined by

\[
g(x) := \begin{cases} 1 & \text{if } \|x\|_{\mathcal{X} \cap \mathcal{Y}} \leq 2M, \\ \frac{2M}{\|x\|_{\mathcal{X} \cap \mathcal{Y}}} & \text{else.} \end{cases}
\]

\[
h(x) := \begin{cases} 1 & \text{if } |\langle \mathcal{A} x, x \rangle_{\mathcal{X}}| \leq M'M, \\ \frac{M'M}{|\langle \mathcal{A} x, x \rangle_{\mathcal{X}}|} & \text{else.} \end{cases}
\]

Finally, the compression operator

\[
\tau_n := (x \mapsto g(h(x) \mathcal{F}_n x) h(x)) : C^0(\overline{I}, V_n) \to (0, 1],
\]

and therefore the compressed fixed point operator \( \tau_n \mathcal{F}_n : C^0(\overline{I}, V_n) \to C^0(\overline{I}, V_n) \) are continuous.

(c) **Equivalence of the fixed point problems:** Since \( \tau_n : C^0(\overline{I}, V_n) \to (0, 1] \) was still an arbitrary operator in (5.5), the a priori estimates (5.8) and (5.10) hold true for both \( \tau_n \equiv 1 \) and the compression operator defined in (5.13).

Being more precise, a fixed point \( y_n \in W^{1,p}(I, V_n) \) of \( \tau_n \mathcal{F}_n \) as well as a fixed point of \( \mathcal{F}_n \) satisfies the estimates

\[
\|y_n\|_{\mathcal{X} \cap \mathcal{Y}} \leq M \quad \text{and} \quad \|\mathcal{A} y_n\|_{\mathcal{X}^*} \leq M',
\]

with the same \( n \)-independent constants \( M, M' > 0 \). These imply \( |\langle \mathcal{A} y_n, y_n \rangle_{\mathcal{X}}| \leq M'M \) and therefore \( h(y_n) = 1 \) due to the definition of \( h \) (cf. (5.12)). From this, we are able to derive the equivalence of the fixed point problems. In fact, there holds:

If \( y_n \in W^{1,p}(I, V_n) \) is a fixed point of \( \mathcal{F}_n \), then, having regard to \( \|y_n\|_{\mathcal{X} \cap \mathcal{Y}} \leq M < 2M \) and the definition of \( g \) (cf. (5.11)), we deduce

\[
g(h(y_n) \mathcal{F}_n y_n) = g(\mathcal{F}_n y_n) = g(y_n) = 1.
\]

From this, we obtain \( \tau_n(y_n) = 1 \) and thus \( \tau_n(y_n) \mathcal{F}_n y_n = \mathcal{F}_n y_n = y_n \) in \( W^{1,p}(I, V_n) \).

On the other hand, if \( y_n \in W^{1,p}(I, V_n) \) is a fixed point of the compressed operator \( \tau_n \mathcal{F}_n \), then

\[
\tau_n(y_n) = g(h(y_n) \mathcal{F}_n y_n) h(y_n) = 1
\]

has to be valid. Otherwise, taking into account \( h(y_n) = 1 \) and the definition of \( g \) (cf. (5.11)), \( \|\mathcal{F}_n y_n\|_{\mathcal{X} \cap \mathcal{Y}} > 2M \) would hold true. But this yields the contradiction

\[
M \geq \|y_n\|_{\mathcal{X} \cap \mathcal{Y}} = \|\tau_n(y_n) \mathcal{F}_n y_n\|_{\mathcal{X} \cap \mathcal{Y}} = \|g(h(y_n) \mathcal{F}_n y_n) h(y_n) \mathcal{F}_n y_n\|_{\mathcal{X} \cap \mathcal{Y}} = \frac{2M}{\|\mathcal{F}_n y_n\|_{\mathcal{X} \cap \mathcal{Y}}} \|\mathcal{F}_n y_n\|_{\mathcal{X} \cap \mathcal{Y}} = 2M > M.
\]

As a consequence, it holds \( \mathcal{F}_n y_n = \tau_n(y_n) \mathcal{F}_n y_n = y_n \) in \( W^{1,p}(I, V_n) \).
(d) **Existence of a fixed point of the equivalent compressed problem:** We define $\mu_n := \|e_n^{-1}\|_{L(\mathcal{X}_{V_n}, L^p(I, V_n))}$ for $n \in \mathbb{N}$. Then, for arbitrary $x \in C^0(I, V_n)$ there holds:

$$
\|\tau_n(x)F_n x\|_{\mathcal{X}_{V_n}} = \begin{cases} 
\|h(x)F_n x\|_{\mathcal{X}_{V_n}} \quad &\text{if } \|h(x)F_n x\|_{\mathcal{X}_{V_n}} \leq 2M, \\
2M \|h(x)F_n x\|_{\mathcal{X}_{V_n}} \quad &\text{else}
\end{cases} \leq 2M.
$$

(5.14)

$$
\left\| \frac{dV_n}{dt} \tau_n(x)F_n x \right\|_{L^p(I, V_n)} = \left\| \frac{dV_n}{dt} (\tau_n(x)V_n e_n^{-1} A_{V_n} x) \right\|_{L^p(I, V_n)} 
\leq \left\| h(x) \frac{dV_n}{dt} (V_n e_n^{-1} A_{V_n} x) \right\|_{L^p(I, V_n)} 
\leq \left\| (id_{\mathcal{X}_{V_n}})^* h(x)A x \right\|_{\mathcal{X}^*_{V_n}} 
\leq \mu_n \left\| h(x)A x \right\|_{\mathcal{X}^*}.
$$

(5.15)

In the first inequality in (5.15) we made use of $|g(h \cdot F_n)| \leq 1$. The subsequent equal sign and inequality stem from the identity $\frac{dV_n}{dt} V_n = \text{id}_{L^p(I, V_n)}$ (cf. Proposition 2.9 (ii)), the definition of $A_{V_n}$ (cf. Remark 4.1) and

$$
\left\| (id_{\mathcal{X}_{V_n}})^* \right\|_{L(\mathcal{X}^*, \mathcal{X}_{V_n})} = \left\| id_{\mathcal{X}_{V_n}} \right\|_{L(\mathcal{X}_{V_n}, \mathcal{X})} \leq 1.
$$

Next, we fix the closed ball

$$
S := B_{2M}^{\mathcal{X}_{V_n}} (0) \subseteq \mathcal{X} \cap \mathcal{Y}.
$$

From the boundedness of $B : \mathcal{X} \cap \mathcal{Y} \to \mathcal{X}^*$ we obtain a constant $K_0 > 0$, such that $\|Bx\|_{\mathcal{X}^*} \leq K_0$ for all $x \in S$. Due to the definition of $h$ (cf. (5.12)), we further deduce for every $x \in S$

$$
h(x)(A_0 x, x)_{\mathcal{X}} = h(x)(A x, x)_{\mathcal{X}} - h(x)(B x, x)_{\mathcal{X}} \leq M'M + 2K_0 M.
$$

Proposition 2.2, with $S = S$ and $h$ as in (5.12), yields a constant $K_1 > 0$, such that $\|h(x)A x\|_{\mathcal{X}^*} \leq K_1$ for all $x \in S$. This and (5.15) imply for every $x \in S \cap C^0(I, V_n)$

$$
\left\| \frac{dV_n}{dt} \tau_n(x)F_n x \right\|_{L^p(I, V_n)} \leq \mu_n K_1.
$$

(5.16)

Finally, we define

$$
B_n := \left\{ x \in W^{1, p}(I, V_n) \cap S \mid \left\| \frac{dV_n}{dt} x \right\|_{L^p(I, V_n)} \leq \mu_n K_1 \right\} \subseteq W^{1, p}(I, V_n),
$$

$$
\mathcal{K}_n := \overline{B_n}_{\mathcal{C}^0(I, V_n)} \subseteq C^0(I, V_n).
$$

From (5.14) and (5.16) we derive the self map property for $\tau_nF_n$ on $\mathcal{K}_n$. In particular,

$$
\tau_nF_n : \mathcal{K}_n \subseteq C^0(I, V_n) \to \mathcal{K}_n.
$$
is well-defined and continuous. Thus, in view of Schauder’s fixed point theorem it remains to inquire into the properties of \( \mathcal{K}_n \). To this end, let us focus on \( \mathcal{B}_n \), which is obviously non-empty, bounded and convex in \( W^{1,p}(I, V_n) \). Proposition 2.10 provides the compact embedding \( W^{1,p}(I, V_n) \hookrightarrow C^0(\overline{I}, V_n) \). In consequence, \( \mathcal{K}_n \) is non-empty, convex and compact in \( C^0(\overline{I}, V_n) \) and therefore Schauder’s fixed point theorem applicable. Due to Step 2.3 (ii) (c), the existing fixed point is also a fixed point of \( \mathcal{F}_n \) and, looking back to Step 2.2 and 2.1, a solution of both (5.4) and (5.1).

3. Passage to the limit:

3.1 Convergence of Galerkin solutions: Due to Step 2.3 (ii) (a), the verified solution \( y_n \in W_{V_n} \) of (5.1) satisfies the estimates (5.8) and (5.10). In virtue of the reflexivity of \( \mathcal{X} \) and \( \mathcal{X}^* \) together with the existing separable pre-dual \( L^1(I, H^*) \) of \( L^\infty(I, H) \cong (L^1(I, H^*))^* \) (cf. [3, Theorem 3.3]), we obtain a not relabelled subsequence \( (y_n)_{n\in\mathbb{N}} \subseteq \mathcal{X} \cap j \mathcal{Y} \) as well as elements \( y \in \mathcal{X} \cap j L^\infty(I, H) \) and \( \xi \in \mathcal{X}^* \) such that

\[
\begin{align*}
\langle y_n \rangle_{n\to\infty} &\to y \quad \text{in} \ \mathcal{X}, \\
\langle jy_n \rangle_{n\to\infty} &\to jy \quad \text{in} \ L^\infty(I, H) \quad (n \to \infty), \\
\langle Ay_n \rangle_{n\to\infty} &\to \xi \quad \text{in} \ \mathcal{X}^*.
\end{align*}
\]

(5.17)

3.2 Regularity and trace of the weak limit: Let \( v \in V_k \), where \( k \in \mathbb{N} \) is arbitrary, and \( \varphi \in C^\infty_0(I) \) with \( \varphi(T) = 0 \). Testing (5.1) for every \( n \in \mathbb{N} \) with \( n \geq k \) by \( v \varphi \in \mathcal{X}_{V_k} \subseteq \mathcal{X}_{V_n} \) and a subsequent application of the generalized integration by parts formula (4.2) with \( U = V_n \), yield for every \( n \in \mathbb{N} \) with \( n \geq k \)

\[
\langle A_{V_n} y_n, v \varphi \rangle_{\mathcal{X}_{V_n}} = -\left( \frac{d e_{V_n} y_n}{dt}, v \varphi \right)_{\mathcal{X}_{V_n}} + \langle e_{V_n}(v) \varphi', y_n \rangle_{\mathcal{X}_{V_n}} + \langle ja_n, jv \rangle_H \varphi(0). \tag{5.18}
\]

Together with (4.1) and (4.3) in the case \( U = V_n \), (5.18) reads for every \( n \in \mathbb{N} \) with \( n \geq k \)

\[
\langle A y_n, v \varphi \rangle_{\mathcal{X}} = \langle e(v) \varphi', y_n \rangle_{\mathcal{X}} + \langle ja_n, jv \rangle_H \varphi(0). \tag{5.19}
\]

By passing for \( n \to \infty \) in (5.19), using (5.17) and \( ja_n \to y_0 \) in \( H \ (n \to \infty) \), we obtain

\[
\langle \xi, v \varphi \rangle_{\mathcal{X}} = \langle e(v) \varphi', y \rangle_{\mathcal{X}} + \langle y_0, jv \rangle_H \varphi(0) \tag{5.20}
\]

for every \( v \in \bigcup_{k \in \mathbb{N}} V_k \) and \( \varphi \in C^\infty(I) \) with \( \varphi(T) = 0 \). In the case \( \varphi \in C^\infty_0(I) \), (5.20) reads

\[
\langle e(v) \varphi', y \rangle_{\mathcal{X}} = \langle \xi, v \varphi \rangle_{\mathcal{X}} \tag{5.21}
\]

for every \( v \in \bigcup_{k \in \mathbb{N}} V_k \) and Proposition 2.11 thus proves that \( y \in \mathcal{W} \) with a continuous representation \( jy \in \mathcal{Y} \) and

\[
\frac{d_jy}{dt} = -\xi \quad \text{in} \ \mathcal{X}^*. \tag{5.21}
\]

In addition, we are allowed to apply the generalized integration by parts formula (cf. Proposition 2.12) in (5.20) in the case \( \varphi \in C^\infty(I) \) with \( \varphi(T) = 0 \) and \( \varphi(0) = 1 \). In so doing, we further deduce for every \( v \in \bigcup_{k \in \mathbb{N}} V_k \) that

\[
\langle (jy)(0) - y_0, jv \rangle_H = 0. \tag{5.22}
\]

Since \( \bigcup_{k \in \mathbb{N}} V_k \) is dense in \( V \) and \( R(j) \) is dense in \( H \), we conclude from (5.22) that

\[
(jy)(0) = y_0 \quad \text{in} \ H. \tag{5.23}
\]
3.3 Weak convergence in $\mathcal{X} \cap J \mathcal{Y}$: The objective of this passage is to exploit the characterization of weak convergence in $\mathcal{X} \cap J \mathcal{Y}$ (cf. Proposition 2.6). To be more precise, it remains to verify that $(j y_n) (t) \to (j y)(t)$ in $H (\text{for } n \to \infty)$ for every $t \in T$.

To this end, let us fix an arbitrary $t \in (0, T]$. From the a-priori estimate $\sup_{n \in \mathbb{N}} \|j y_n\|_Y \leq M$ we obtain the existence of a subsequence $((j y_n) (t))_{n \in \Lambda_t} \subseteq H$ with $\Lambda_t \subseteq \mathbb{N}$, initially depending on this fixed $t$, and an element $y_{\Lambda_t} \in H$ such that

$$\lim_{n \to \infty} (j y_n) (t) = y_{\Lambda_t} \quad \text{in } H \quad \left( n \in \Lambda_t \right). \quad (5.24)$$

For $\nu \in V_k$, where $k \in \Lambda_t$ is arbitrary, and $\varphi \in C^\infty([0, T])$ with $\varphi(0) = 0$ and $\varphi(t) = 1$, we test (5.1) for every $n \in \Lambda_t$ with $n \geq k$ by $\nu \varphi \chi_{[0, t]} \in \mathcal{X}_k \subseteq \mathcal{X}_n$, use the generalized integration by parts formula (4.2), (4.3) with $U = V_n$, to obtain for every $n \in \Lambda_t$ with $n \geq k$

$$\langle A y_n, \nu \varphi \chi_{[0, t]} \rangle_{\mathcal{X}} = \langle e (\nu) \varphi' \chi_{[0, t]}, y_n \rangle_{\mathcal{X}} - \langle (j y_n) (t), j \nu \rangle_H. \quad (5.25)$$

By passing for $\Lambda_t \ni n \to \infty$ in (5.25), using (5.17) and (5.24), we obtain for every $\nu \in \bigcup_{k \in \Lambda_t} V_k$

$$\langle \xi, \nu \varphi \chi_{[0, t]} \rangle_{\mathcal{X}} = \langle e (\nu) \varphi' \chi_{[0, t]}, y_{\Lambda_t} \rangle_{\mathcal{X}} - \langle y_{\Lambda_t}, j \nu \rangle_H. \quad (5.26)$$

The generalized integration by parts formula (cf. Proposition 2.12) applied on (5.26) and (5.21) yield for every $\nu \in \bigcup_{k \in \Lambda_t} V_k$

$$(j y)(t) - y_{\Lambda_t}, j \nu \rangle_H = 0. \quad (5.27)$$

Thanks to $V_k \subseteq V_{k+1}$ for every $k \in \mathbb{N}$, there holds $\bigcup_{k \in \Lambda_t} V_k = \bigcup_{k \in \mathbb{N}} V_k$. Thus, $j \left( \bigcup_{k \in \Lambda_t} V_k \right)$ is dense in $H$ and we obtain from (5.27) that $(j y)(t) = y_{\Lambda_t}$ in $H$, i.e.,

$$\lim_{n \to \infty} (j y_n) (t) = (j y)(t) \quad \text{in } H \quad \left( n \in \Lambda_t \right). \quad (5.28)$$

As this argumentation stays valid for each subsequence of $((j y_n) (t))_{n \in \mathbb{N}} \subseteq H$, $(j y)(t) \in H$ is a weak accumulation point of each subsequence of $((j y_n) (t))_{n \in \mathbb{N}} \subseteq H$. The standard convergence principle (cf. [9, Kap. I, Lemma 5.4]) yields $\Lambda_t = \mathbb{N}$ in (5.28). Since $t \in (0, T]$, was arbitrary in (5.28) and $(y_n)_{n \in \mathbb{N}} \subseteq \mathcal{X} \cap J \mathcal{Y}$ is bounded (cf. (5.8)), we conclude using Proposition 2.6 that

$$y_n \rightharpoonup y \quad \text{in } \mathcal{X} \cap J \mathcal{Y}. \quad (5.29)$$

3.4 Identification of $Ay$ and $\xi$: Since $\tau_n (y_n) = 1$ for every $n \in \mathbb{N}$, estimate (5.7) with $t = T$ reads for every $n \in \mathbb{N}$

$$\langle A y_n, y_n \rangle_{\mathcal{X}} \leq -\frac{1}{2} \| (j y_n) (T) \|_H^2 + \frac{1}{2} \| y_0 \|_H^2. \quad (5.30)$$

The limit superior with respect to $n \to \infty$ on both sides in (5.30), (5.23), (5.28) with $\Lambda_t = \mathbb{N}$ in the case $t = T$, the weak lower semi-continuity of $\| \cdot \|_H$, the generalized integration by parts formula (cf. Proposition 2.12) and (5.21) yield

$$\limsup_{n \to \infty} \langle A y_n, y_n \rangle_{\mathcal{X}} \leq -\frac{1}{2} \| (j y) (T) \|_H^2 + \frac{1}{2} \| (j y)(0) \|_H^2 \quad (5.31)$$

In view of (5.17), (5.29) and (5.31), we conclude from the $C^0$-Bochner condition (M) of $A : \mathcal{X} \cap J \mathcal{Y} \to \mathcal{X}^*$ that $Ay = \xi$ in $\mathcal{X}^*$. All things considered, we proved

$$\frac{d}{dt} y + Ay = 0 \quad \text{in } \mathcal{X}^*, \quad (j y)(0) = y_0 \quad \text{in } H.$$ 

This completes the proof of Theorem 5.1. \[\square\]
6 | MAIN THEOREM: (PURELY REFLEXIVE CASE)

This section is concerned with the extension of Theorem 5.1 to the case of purely reflexive \( V \).

**Theorem 6.1.** Theorem 5.1 stays valid if we omit the separability of \( V \).

A lack of separability of \( V \) results in a non-existence of an increasing sequence of finite dimensional subspaces which approximates \( V \) up to density. We circumvent this problem by regarding a probably uncountable system of separable subspaces. But this system might be orderless, such that the increasing structure, which was indispensable for the proof of Theorem 5.1, has to be generated locally. The latter will be guaranteed by the subsequent lemma. Then, we perform the passage to limit as in Theorem 5.1, Step 3, first locally and assemble the extracted local information to the desired global assertion afterwards.

**Lemma 6.2.** Let \((V, H, j)\) be an evolution triple, let \( 1 < p < \infty \), \( M > 0 \), \( y_0 \in H \) and

\[
U_{y_0} := \left\{ U \subseteq V \mid (U, \| \cdot \|_V) \text{ is a separable Banach space, } y_0 \in H_U := j(U) \right\}.
\]

Then, it holds:

(i) \( (U, y_0)_{U \in U_{y_0}} \) is a Galerkin basis of \( (V, y_0) \) in the sense of Remark 4.1 (i).

(ii) Suppose for a mapping \( \Psi : U_{y_0} \to 2^{B_X(0)} \setminus \{ \emptyset \} \) with

\[
\Psi(U) \subseteq \mathcal{X}_U := L^p(I, U) \quad \text{and} \quad L_U := \bigcup_{Z \in U_{y_0}} \Psi(Z) \subseteq 2^{B_X(0)} \text{ for all } U \in U_{y_0},
\]

there exists \( y \in \bigcap_{U \in U_{y_0}} \overline{\cup_{Z \subseteq U} (\mathcal{X}_Z \cap \mathcal{X}^*)} \). Then, for every \( U \in U_{y_0} \), there exist sequences \( (U_n)_{n \in \mathbb{N}} \subseteq U_{y_0} \) and \( (y_n)_{n \in \mathbb{N}} \subseteq \mathcal{X} \), with \( U \subseteq U_n \subseteq U_{n+1} \) and \( y_n \in \Psi(U_n) \) for every \( n \in \mathbb{N} \) such that

\[
y_n \overset{n \to \infty}{\rightharpoonup} y \text{ in } \mathcal{X}.
\]

**Proof.** Point (i) follows right from the definition in Remark 4.1 (i). The verification of (ii) is a straightforward modification of [7, Proposition 11]. For a detailed proof we refer to [11, Lemma 7.1].

**Proof of Theorem 6.1.** It suffices anew to treat the special case \( f = 0 \) in \( \mathcal{X}^* \).

1. **Galerkin approximation:** Let \( U_{y_0} \) be defined as in Lemma 6.2. In line with Remark 4.1, we see the well-posedness of the Galerkin system with respect to \( (U, y_0)_{U \in U_{y_0}} \). For \( U \in U_{y_0} \) we denote by \( y_U \in \mathcal{W}_U \) the **Galerkin solution** with respect to \( U \), if

\[
\frac{d}{dt} y_U + A_U y_U = 0 \quad \text{in } \mathcal{X}_U^*,
\]

\[
(j_U y_U)(0) = y_0 \quad \text{in } H_U. \tag{6.1}
\]

2. **Existence of Galerkin solutions:** Each \( U \in U_{y_0} \) is separable and reflexive. Thus, it remains to inquire into the properties of the restricted operators \( A_U := (id_{\mathcal{X}_U})^* A : \mathcal{X}_U \cap j_U \mathcal{Y}_U \to \mathcal{X}_U^*, U \in U_{y_0} \). Lemma 4.2 immediately provides for every \( U \in U_{y_0} \):

(i) \( (A_0)_U := (id_{\mathcal{X}_U})^* A_0 : \mathcal{X}_U \to \mathcal{X}_U^* \) is monotone.

(ii) \( B_U := (id_{\mathcal{X}_U})^* B : \mathcal{X}_U \cap j_U \mathcal{Y}_U \to \mathcal{X}_U^* \) is bounded.

(iii) \( A_U := (A_0)_U + B_U : \mathcal{X}_U \cap j_U \mathcal{Y}_U \to \mathcal{X}_U^* \) satisfies the \( C^0 \)-Bochner condition (M) and is \( C^0 \)-Bochner coercive with respect to \( 0 \in \mathcal{X}_U^* \) and \( y_0 \in H_U \).
All things considered, Theorem 5.1 yields for every $U \in U'_{y_0}$ the solvability of (6.1). In addition, we obtain as in Theorem 5.1, Step 2.3 (ii) (a), $U$-independent constants $M, M' > 0$ such that for every $U \in U'_{y_0}$, there holds
\[ \|y_U\|_{\mathcal{X}} \leq M \quad \text{and} \quad \|A_U y_U\|_{\mathcal{X}^*} \leq M'. \tag{6.2} \]

Therefore, the mapping $\Psi : U'_{y_0} \to 2^{B^W_M(0)} \setminus \{\emptyset\}$, for every $U \in U'_{y_0}$ given via
\[ \Psi(U) := \{ y_U \in \mathcal{W}_U \mid y_U \text{ solves (6.1) with respect to } U \}, \tag{6.3} \]
is well-defined. Apart from this, we define $L_U := \bigcup_{Z \in U'_{y_0}} \Psi(Z) \neq \emptyset$ for every $U \in U'_{y_0}$.

3. Passage to the limit: Our next objective is to show that
\[ \bigcap_{U \in U'_{y_0}} L_U^{\tau(\mathcal{X}, \mathcal{X}^*)} \neq \emptyset. \tag{6.4} \]

Then, Lemma 6.2 is applicable and we are in the position to perform the passage to the limit as in Theorem 5.1, Step 3, locally for every $U \in U'_{y_0}$.

By construction, holds $L_Z \subseteq L_W$ for every $Z, W \in U'_{y_0}$ with $W \subseteq Z$. Since $\langle Z \cup W \rangle^{\|\cdot\|_V} \in U'_{y_0}$ for every $Z, W \in U'_{y_0}$, we thus have for every $Z, W \in U'_{y_0}$
\[ \emptyset \neq L_{\langle Z \cup W \rangle}^{\tau(\mathcal{X}, \mathcal{X}^*)} \subseteq L_Z \cap L_W. \tag{6.5} \]

By induction, we deduce from (6.5) that $\left( L_U^{\tau(\mathcal{X}, \mathcal{X}^*)} \right)_{U \in U'_{y_0}}$ satisfies the finite intersection property. As $B^W_M(0)$ is compact with respect to $\tau(\mathcal{X}, \mathcal{X}^*)$ and $L_U \subseteq B^W_M(0)$ for every $U \in U'_{y_0}$ (cf. (6.2)), we conclude (6.4) by means of the finite intersection principle (cf. [17, Appendix, Lemma 1.3]).

Now, we perform the passage to the limit locally for each $U \in U'_{y_0}$. To this end, we fix an arbitrary $U \in U'_{y_0}$. Then, Lemma 6.2 provides sequences $(U_n)_{n \in \mathbb{N}} \subseteq U'_{y_0}$ and $(y_n)_{n \in \mathbb{N}} \subseteq \mathcal{X} \cap \mathcal{J}$, with $U \subseteq U_n \subseteq U_{n+1}$ and $y_n \in \Psi(U_n) \subseteq \mathcal{X}_{U_n}$ for every $n \in \mathbb{N}$ such that for every $n \in \mathbb{N}$
\[ y_n \overset{n \to \infty}{\rightharpoonup} y \quad \text{in } \mathcal{X}, \tag{6.6} \]
\[ \frac{d}{dt} U_n y_n + A_{U_n} y_n = 0 \quad \text{in } \mathcal{X}_{U_n}^*, \tag{6.7} \]
\[ (j_{U_n} y_n)(0) = y_0 \quad \text{in } H_{U_n}. \]

For this specific $U \in U'_{y_0}$ and $(U_n)_{n \in \mathbb{N}}$ from Lemma 6.2, we set $U_\infty := \overline{\text{span}\{ \bigcup_{n \in \mathbb{N}} U_n \}^{\|\cdot\|_V}} \in U'_{y_0}$. Then, $U_\infty$ is a separable, reflexive Banach space and $A_{U_\infty} : \mathcal{X}_{U_\infty} \cap j_{U_\infty} \mathcal{Y}_{U_\infty} \to \mathcal{X}_{U_\infty}^*$ satisfies the assumptions of Theorem 5.1 with respect to $U_\infty$ in the role of $V$. As $\mathcal{X}_{U_\infty}$ is closed with respect to $\tau(\mathcal{X}, \mathcal{X}^*)$ and $(y_n)_{n \in \mathbb{N}} \subseteq \mathcal{X}_{U_\infty} \cap j_{U_\infty} \mathcal{Y}_{U_\infty}$, we deduce from (6.6) that $y \in \mathcal{X}_{U_\infty}$ and
\[ y_n \overset{n \to \infty}{\rightharpoonup} y \quad \text{in } \mathcal{X}_{U_\infty}. \tag{6.8} \]

From $\|j_{U_\infty} y_n\|_{U_\infty} \leq \|j y_n\|_{y_0} \leq M$ and $\|A_{U_\infty} y_n\|_{\mathcal{X}_{U_\infty}} \leq \|A y_n\|_{\mathcal{X}^*} \leq M'$ for every $n \in \mathbb{N}$, we additionally obtain a subsequence $(y_n)_{n \in \Lambda_{U_\infty}} \subseteq \mathcal{X}_{U_\infty} \cap j_{U_\infty} \mathcal{Y}_{U_\infty}$, with $\Lambda_{U_\infty} \subseteq \mathbb{N}$, and an element $\xi_{U_\infty} \in \mathcal{X}_{U_\infty}^*$, such that
\[ j_{U_\infty} y_n \overset{\ast}{\to} j_{U_\infty} y \quad \text{in } L^\infty(I, H_{U_\infty}) \quad (\Lambda_{U_\infty} \ni n \to \infty), \tag{6.9} \]
\[ A_{U_\infty} y_n \overset{n \to \infty}{\rightharpoonup} \xi_{U_\infty} \quad \text{in } \mathcal{X}_{U_\infty}^* \quad (n \in \Lambda_{U_\infty}). \]
All things considered, we are now in the situation of Theorem 5.1, Step 3.1, if $U_\infty$ takes the role of $V$ with Galerkin basis $(U_n, y_0)_{n \in \mathbb{N}}$. Thus, we recapitulate Step 3.1 till 3.4 in Theorem 5.1. In doing so, we need to replace (5.1) by (6.7) and (5.17) by (6.8) together with (6.9). Thus, we infer that $y \in W_\infty$ with a continuous representation $j U_\infty y \in Y_\infty$ and

$$
\begin{align*}
\frac{dU_\infty y}{dt} + A_{U_\infty} y &= 0 \quad \text{in } \mathcal{X}_{U_\infty}^*, \\
(j U_\infty y)(0) &= y_0 \quad \text{in } H_{U_\infty}.
\end{align*}
$$

(6.10)

Let $u \in U \subseteq U_\infty$ and $\varphi \in C_0^\infty(I)$ be arbitrary. Testing (6.10) by $u \varphi \in \mathcal{X}_{U_\infty}$ and a subsequent application of the generalized integration by parts formula (4.3), (4.2) and (4.1) with $U = U_\infty$, provide for every $u \in U$ and $\varphi \in C_0^\infty(I)$

$$
\langle Ay, u \varphi \rangle_{\mathcal{X}} = \langle A_{U_\infty} y, u \varphi \rangle_{\mathcal{X}_{U_\infty}} = \langle e_{U_\infty}(u) \varphi', y \rangle_{\mathcal{X}_{U_\infty}} = \langle e(u) \varphi', y \rangle_{\mathcal{X}}.
$$

(6.11)

Since $U \subseteq U_\infty$ was chosen arbitrarily, (6.11) is actually valid for every $u \in V$ and $\varphi \in C_0^\infty(I)$. Therefore, Proposition 2.11 proves that $y \in W$ with a continuous representation $j y \in Y$ and

$$
\begin{align*}
\frac{dy}{dt} + Ay &= 0 \quad \text{in } \mathcal{X}^*, \\
(j y)(0) &= y_0 \quad \text{in } H.
\end{align*}
$$

This completes the proof of Theorem 6.1.

With the help of Theorem 6.1, we are able to extend the results in [12] to the case of purely reflexive $V$.

**Corollary 6.3.** Let $(V, H, j)$ be an evolution triple, let $1 < p < \infty$ and let $A(t) : V \to V^*$, $t \in I$, be a family of operators satisfying (C.1)–(C.4). Then, for arbitrary $y_0 \in H$ and $f \in \mathcal{X}^*$ there exists a solution $y \in W$ of (2.10).

**Proof.** Immediate consequence of Theorem 6.1 and Lemma 3.7, since Bochner pseudo-monotonicity and Bochner coercivity imply $C^0$-Bochner pseudo-monotonicity and $C^0$-Bochner coercivity.

There is still some room for improvement. Indeed, the statements of Theorem 6.1 and Corollary 6.3 remain true under more general assumptions. For proofs we refer to [11].

**Remark 6.4.**

(i) Corollary 6.3 remains true if we replace the evolution triple $(V, H, j)$ by a pre-evolution triple $(V, H, j)$, i.e., not $V$ but $V \cap j H$ embeds continuously and densely into $H$ (cf. [12] or [11, Definition 8.1]).

(ii) Theorem 6.1 remains true if we replace $\mathcal{X} = L^p(I, V)$ by the intersection $\mathcal{X} = L^p(I, V) \cap L^q(I, H)$, where $1 < p \leq q < \infty$ and $(V, H, j)$ is a pre-evolution triple (cf. [11, Satz 8.7]).

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**Endnotes**

1. All notions are defined in Section 2.
2. $W^{1, p}_{0, \text{div}}(\Omega)$ is the closure of $\mathcal{V} = \{ v \in C_0^\infty(\Omega)^3 \mid \text{div } v \equiv 0 \}$ with respect to $\|\nabla \cdot \|_{L^p(\Omega)}$ and $L^2_{\text{div}}(\Omega)$ the closure of $\mathcal{V}$ with respect to $\|\cdot\|_{L^2(\Omega)}$.
3. $Dy = \frac{1}{2}(\nabla y + \nabla y^T)$ denotes the symmetric gradient.
4. $A : D(A) \subseteq X \to X^*$ is said to be coercive (cf. [23, §32.4.]) with respect to $f \in X^*$, if $D(A)$ is unbounded and there exists a constant $M > 0$ such that for $x \in X$ from $(Ax, x)_X \leq (f, x)_X$, it follows $\|x\|_X \leq M$, i.e., all elements whose images with respect to $A : D(A) \subseteq X \to X^*$ do not grow beyond the data $f \in X^*$ in this weak sense are contained in a fixed ball in $X$.
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APPENDIX: A PULL-BACK INTERSECTIONS

This passage is highly inspired by [2, Chapter 3]. For proofs we refer to [12].

**Definition A.1** (Embedding). Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be topological vector spaces. The operator $j : X \to Y$ is said to be an *embedding* if it is linear, injective and continuous. In this case, we use the notation

$$X \xrightarrow{j} Y.$$

If $X \subseteq Y$ and $j = \text{id}_X$, then we write $X \hookrightarrow Y$ instead.

**Definition A.2** (Compatible couple). Let $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ be Banach spaces such that embeddings $e_X : X \to Z$ and $e_Y : Y \to Z$ into a Hausdorff vector space $(Z, \tau_Z)$ exist. Then, we call $(X,Y) := (X,Y,Z,e_X,e_Y)$ a *compatible couple*. 
Definition A.3 (Pull-back intersection of Banach spaces). Let \((X, Y)\) be a compatible couple. Then, the operator 
\[ j := e_X^{-1}e_Y : e_X^{-1}(R(e_X) \cap R(e_Y)) \to Y \]
is well-defined and we denote by 
\[ X \cap_j Y := e_X^{-1}(R(e_X) \cap R(e_Y)) \subseteq X \]
the pull-back intersection of \(X\) and \(Y\) in \(X\) with respect to \(j\). Furthermore, \(j\) is said to be the corresponding intersection embedding. If \(X, Y \subseteq Z\) with \(e_X = \text{id}_X\) and \(e_Y = \text{id}_Y\), then we set \(X \cap Y := X \cap_j Y\).

Proposition A.4 (Completeness of \(X \cap_j Y\)). Let \((X, Y)\) be a compatible couple. Then, \(X \cap_j Y\) is a vector space and equipped with norm
\[ \|\cdot\|_{X \cap_j Y} := \|\cdot\|_X + \|j \cdot\|_Y \]
a Banach-space. Moreover, \(j : X \cap_j Y \to Y\) is an embedding.

Proposition A.5 (Properties of \(X \cap_j Y\)). Let \((X, Y)\) be a compatible couple. Then, it holds:

(i) If \(X\) and \(Y\) are reflexive or separable, then \(X \cap_j Y\) is as well.

(ii) First characterization of weak convergence in \(X \cap_j Y\): A sequence \((x_n)_{n \in \mathbb{N}} \subseteq X \cap_j Y\) and \(x \in X \cap_j Y\) satisfy \(x_n \rightharpoonup x \) in \(X \cap_j Y\) \((n \to \infty)\) if and only if \(x_n \rightharpoonup x \) in \(X\) \((n \to \infty)\) and \(j x_n \rightharpoonup j x \) in \(Y\) \((n \to \infty)\).

(iii) Second characterization of weak convergence in \(X \cap_j Y\): In addition, let \(X\) be reflexive. A sequence \((x_n)_{n \in \mathbb{N}} \subseteq X \cap_j Y\) and \(x \in X \cap_j Y\) satisfy \(x_n \rightharpoonup x \) in \(X \cap_j Y\) \((n \to \infty)\) if and only if \(\sup_{n \in \mathbb{N}} \|x_n\|_X < \infty\) and \(j x_n \rightharpoonup j x \) in \(Y\) \((n \to \infty)\).