The Symplectic Egg

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Abstract

We invite the reader (presumably an upper level undergraduate student) to a journey leading from the continent of Classical Mechanics to the new territories of Quantum Mechanics. We’ll be riding the symplectic camel and have William of Occam as travel companion, so no excess baggage is allowed. The first part of our trip takes us from the symplectic egg to Gromov’s non-squeezing theorem and its dynamical interpretation. The second part leads us to a symplectic formulation of the quantum uncertainty principle, which opens the way to new discoveries.

Prologue

What’s in a name? That which we call a rose by any other name would smell as sweet.

Romeo and Juliet, Act 2, Scene 2 (W. Shakespeare)

Take an egg –preferably a hard boiled one, and cut it in half along its middle using a very sharp knife. The surface of section will be roughly circular and have area $\pi r^2$. Next, take a new egg of same size, and cut it this time along a line joining the egg’s tops, again as shown in Fig1. This time we get an elliptic surface of section with area $\pi R^2$ larger than that of the disk we got previously. So far, so good. But if you now take two symplectic eggs, and do the same thing, then both sections will have exactly same area! Even “worse”, no matter along which plane passing through

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the center of the egg you cut, you will always get sections having the same
area! This is admittedly a very strange property, which you probably never
have experienced (at least in a direct way) in everyday life. But what is a
symplectic egg? The eggs we are cutting are metaphors for ellipsoids; an
ellipsoid is a round ball that has been deformed by a linear transformation
of space, \emph{i.e.} a transformation preserving the alignment of three, or more,
points. In mathematics such transformations are represented by matrices.
Thus the datum of an ellipsoid is the same thing as the datum of a ball and
of a matrix. What we call a symplectic egg is an ellipsoid corresponding
to the case where the matrix is symplectic (we’ll define the concept in a
moment). The reason for which the only symplectic egg you have seen on
your breakfast table is flat –a fried egg!– is that the number of rows and
columns of a symplectic matrix must always be even. Since we are unable
to visualize things in dimension three or more, the only symplectic eggs
that are accessible to our perception are two dimensional. But what is a
symplectic matrix? In the case of smallest dimension two, a matrix

\[
S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

is symplectic if it has determinant one:

\[
ad - bc = 1.
\]

In higher dimensions, 4, 6, 8, etc. there are many more conditions: for
instance 10 if the dimension is 4, 21 if it is 6, and \(n(2n + 1)\) if it is \(2n\). We
will write these conditions explicitly in section 2.1.

So far, so good. But where do symplectic eggs come from, and what are
they good for? Let me first tell you where symplectic matrices come from.
They initially come from the study of motion of celestial bodies, which is
really rich in mathematical concepts, some of these going back to the obser-
vations of Tycho Brahe, and the work of Galileo Galilei and Johannes Kepler
(these were the “Giants” on the shoulder’s of which Isaac Newton stood). But
the notion of symplectic matrix, or more generally that of symplectic
transformation, did really have a long time to wait until it appeared explicit-
ly and was recognized as a fundamental concept. It was implicit in the
work of Hamilton and Lagrange on classical and celestial mechanics, until
the word “symplectic” was finally coined by the mathematician Hermann
Weyl in his book \textit{The Classical Groups}, edited in 1939, just before World
War II. But still then, as Ian Stewart \cite{30} reminds us, it was a rather baffling
oddity which presumably existed for some purpose –but which? It was only
later agreed that the purpose of symplectic transformations is dynamics, that is the study of motion. Let me explain this a little bit more in detail: if we have a physical system consisting of “particles” (sand corns, planets, spacecraft, or quarks) it is economical from both a notational and computational point of view to describe their motion (that is, their instantaneous location and velocity) by specifying a phase space vector, which is a matrix having only one column. For instance, if we are dealing with one single particle with coordinates \((x, y, z)\) and momentum \((p_x, p_y, p_z)\) (the momentum of a particle is just its velocity multiplied by its mass \(m\)) the phase space vector will be the column vector whose entries are \((x, y, z, p_x, p_y, p_z)\) if we have a large number \(N\) of particles with coordinates \((x_1, y_1, z_1)\) and momenta \((p_{x1}, p_{y1}, p_{z1})\) the phase space vector will be obtained by first writing all the position coordinates and thereafter the momentum coordinates in corresponding order, their momenta. These vectors form the phase space of our system of particles. It turns out that the knowledge of a certain function, the Hamiltonian (or energy) function, allows us to both predict and retrodict the motion of our particles; this is done by solving (exactly, or numerically) the Hamilton equations of motion, which are in the case \(n = 1\) given by

\[
\frac{dx}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial x}.
\]

Mathematically these equations are just a fancy way to write Newton’s second law \(F = ma\). That is, knowing exactly the positions and the momenta at some initial time, we are able to know what these are going to be at any future time (we can actually also calculate what they were in the past). The surprising, and for us very welcome fact is that the transformation which takes the initial configuration to the final configuration is always a symplectic transformation! These act on the phase vectors, and once this action is known, we can determine the future of the whole system of particles, and this at any moment (mathematicians would say we are in presence of a “phase space flow”). The relation between symplectic transformations and symplectic matrices is that we can associate a symplectic matrix to every symplectic transformation: it is just the Jacobian matrix of that transformation. In the simplest cases, for instance when no external forces act on the particles, these matrices are themselves the symplectic transformations.

The symplectic egg is a special case a deep mathematical theorem discovered in 1985 by the mathematician Gromov [17], who won the Abel Prize in 2010 for his discovery (the Abel Prize is the “true” substitute for the Nobel Prize in mathematics, as opposed to the Fields medal, which is intended to mathematicians under 40). Gromov’s theorem is nicknamed the “prin-
principle of the symplectic camel” [3] [9] [30], and it tells us that it impossible
to squeeze a symplectic egg through a hole in a plane of “conjugate coordi-
nates” if its radius is larger than that of the hole. That one can do that with
an ordinary (uncooked) egg is easy to demonstrate in your kitchen: put it
into a cup of vinegar (Coca Cola will do as well) during 24 hours. You will
then be able to squeeze that egg through the neck of a bottle without any
effort!

The marvelous thing with the symplectic egg is that it contains quantum
mechanics in a nutshell... er ... an eggshell! Choose for radius the square
root of Planck’s constant $h$ divided by $2\pi$. Then each surface of section will
have radius of $h/2$. In [6] [7] [8] [14] I have called such a tiny symplectic egg
a quantum blob. It is possible –and in fact quite easy if you know the rules
of the game– to show that this is equivalent to the uncertainty principle of
quantum mechanics. The thing to remember here is that a classical property
(i.e. a property involving usual motions, as that of planets for instance),
here symbolized by the symplectic egg, contains as an imprint quantum
mechanics! The analogy between “classical” and “quantum” can actually
be pushed much further, as I have shown with Basil Hiley [15]. But this,
together with the notion of emergence [11], is another story.

Some of the ideas presented here are found in our Physics Reports paper
[16] with F. Luef; they are developed and completed here in a different way
more accessible to a general audience.

1 Notation and terminology

Position and moment vectors will be written as column vectors

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$$

and the corresponding phase vector is thus

$$\begin{pmatrix} x \\ p \end{pmatrix} = (x, p)^T = (x_1, \ldots, x_n; p_1, \ldots, p_n)^T$$

where the superscript $^T$ indicates transposition. The integer $n$ is unspec-
ified; we will call it the number of degrees of freedom. If the vector $(x, p)^T$
denotes the phase vector of a system of $N$ particles, then $n = 3N$ and the
numbers $x_1, x_2, x_3$, (resp. $p_1, p_2, p_3$) can be identified with the positions
x, y, z (resp. the momenta $p_x, p_y, p_z$) of the first particle, $x_4, x_5, x_6$, (resp. $p_4, p_5, p_6$) with those of the second particle, and so on. This is not the only possible convention, but our choice has the advantage of making formulas involving symplectic matrices particularly simple and tractable. For instance, the “standard symplectic matrix” is here $J = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}$ where $I_d$ is the $n \times n$ identity matrix and 0 the $n \times n$ zero matrix. Note that
\begin{align}
J^2 &= -I_d, \\
J^T &= J^{-1} = -J.
\end{align}
(4)

## 2 The Symplectic Egg

### 2.1 Symplectic matrices

Let $S$ be a (real) matrix of size $2n$. We say that $S$ is a symplectic matrix if it satisfies the condition
\begin{equation}
S^T JS = J
\end{equation}
(5)

Clearly the standard symplectic matrix $J$ is itself a symplectic matrix.

Assume that we write the matrix $S$ in block form
\begin{equation}
S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\end{equation}
(6)

where $A, B, C, D$ are matrices of size $n$. It is a simple exercise in matrix algebra to show that condition (5) is equivalent to the following constraints on the blocks $A, B, C, D$
\begin{equation}
A^T C = C^T A, \quad B^T D = D^T B, \quad \text{and} \quad A^T D - C^T B = I_d.
\end{equation}
(7)

Notice that the two first conditions mean that both $A^T C$ and $B^T D$ are symmetric. Observe that these conditions collapse to (2) when $n = 1$: in this case $A, B, C, D$ are the numbers $a, b, c, d$ so that $A^T C = ac$ and $B^T D = bd$ are automatically symmetric; the condition $A^T D - C^T B = I_d$ reduces to $ad - bc = 1$.

The product of two symplectic matrices is a symplectic matrix: if $S$ and $S'$ satisfy (5) then $(SS')^T JS'S' = S'^T (S^T JS)S' = S'^T JS' = J$. Also, symplectic matrices are invertible, and their inverses are symplectic as well: first, take the determinant of both sides of $S^T JS = J$ we get $\det(S^T JS) = \det J$; since $\det J = 1$ this is $(\det S)^2 = 1$ hence $S$ is indeed invertible. Knowing this, we rewrite $S^T JS = J$ as $JS = (S^{-1})^T J$, from which follows that $(S^{-1})^T JS^{-1} = JSS^{-1} = J$ hence $S^{-1}$ is symplectic. The symplectic
matrices of same size thus form a group, called the symplectic group and
denoted by $\text{Sp}(2n)$. An interesting property is that the symplectic group is
closed under transposition: if $S$ is a symplectic matrix, then so is $S^T$ (to
see this, just take the inverse of the equality $(S^{-1})^TJS^{-1} = J$). Since this
means that a matrix is symplectic if and only if its transpose is, inserting
$S^T$ in (8) and noting that $(S^T)^T = S$ we get the condition

$$SJS^T = J. \quad (8)$$

Replacing $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $S^T = \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix}$ the conditions (7) are thus
equivalent to the set of conditions:

$$AB^T = BA^T, \quad CD^T = DC^T, \quad AD^T - BC^T = I_d. \quad (9)$$

One can obtain other equivalent sets of conditions by using the fact that
$S^{-1}$ and $(S^{-1})^T$ are symplectic (see [8]).

It is very interesting to note that the inverse of a symplectic matrix is

$$S^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}. \quad (10)$$

It is interesting because this formula is very similar to that giving the inverse
$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of a $2 \times 2$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant one. The inversion
formula (10) suggests that in a sense symplectic matrices try very hard to
mimic the behavior of $2 \times 2$ matrices. We will see that this is actually
the essence of symplectic geometry, and at the origin of the symplectic egg
property!

A last property of symplectic matrices: recall that when we wanted to
show that a symplectic matrix always is invertible, we established the iden-
tity $(\det S)^2 = 1$. From this follows that the determinant of a symplectic
matrix is a priori either 1 or $-1$. It turns out –but there is no really elemen-
tary proof of this– that we always have $\det S = 1$ (see for instance §2.1.1
in [8] where I give one proof of this property; Mackey and Mackey’s online
paper [20] give a nice discussion of several distinct methods for proving that
symplectic matrices have determinant one.

Conversely, it is not true that any $2n \times 2n$ matrix with determinant one
is symplectic when $n > 1$. Consider for instance

$$M = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1/a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 1/a \end{pmatrix}. \quad (11)$$
where \( a \neq 0 \); this matrix trivially has determinant one, but the condition \( AD^T - BC^T = I_d \) in (9) is clearly violated unless \( a = \pm 1 \). Another simple example is provided by
\[
M = \begin{pmatrix}
R(\alpha) & 0 \\
0 & R(\beta)
\end{pmatrix}
\]
where \( R(\alpha) \) and \( R(\beta) \) are rotation matrices with angles \( \alpha \neq \beta \) (this counterexample generalizes to an arbitrary number \( 2n \) of phase space dimensions).

### 2.2 The first Poincaré invariant

In what follows \( \gamma(t), 0 \leq t \leq 2\pi, \) is a loop in phase space: we have \( \gamma(t) = (x(t), p(t)) \) where \( x(0) = x(2\pi), p(0) = p(2\pi) \); the functions \( x(t) \) and \( p(t) \) are supposed to be continuously differentiable. By definition, the first Poincaré invariant associated to \( \gamma(t) \) is the integral
\[
I(\gamma) = \oint_\gamma p \, dx = \int_0^{2\pi} p(t)^T \dot{x}(t) \, dt.
\]
(12)

The fundamental property –from which almost everything else in this paper stems– is that \( I(\gamma) \) is a symplectic invariant. By this we mean that if we replace the loop \( \gamma(t) \) by the a new loop \( S\gamma(t) \) where \( S \) is a symplectic matrix, the first Poincaré invariant will keep the same value: \( I(S\gamma) = I(\gamma) \), that is
\[
\oint_\gamma p \, dx = \oint_{S\gamma} p \, dx.
\]
(13)

The proof is not very difficult if we carefully use the relations characterizing symplectic matrices (see Arnol’d [2], §44, p.239, for a shorter but more abstract proof). We will first need a differentiation rule for vector-valued functions, generalizing the product formula \( d(uv)/dt = u(dv/dt) + v(du/dt) \) from elementary calculus. Suppose that
\[
u(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}, \quad v(t) = \begin{pmatrix} v_1(t) \\ \vdots \\ v_n(t) \end{pmatrix}
\]
are vectors depending on the variable \( t \) and such that each component \( u_j(t) \), \( v_j(t) \) is differentiable. Let \( M \) be a symmetric matrix of size \( n \) and consider
the real-valued function $u(t)^TMv(t)$. That function is differentiable as well and its derivative is given by the formula
\[
\frac{d}{dt} [u(t)^TMv(t)] = \dot{u}(t)^TMv(t) + u(t)^TM\dot{v}(t) \tag{14}
\] (we are writing $\dot{u}, \dot{v}$ for $du/dt, dv/dt$ as is customary in mechanics); for a proof I refer you to your favorite calculus book.

Let us now go back to the proof of the symplectic invariance of the first Poincaré invariant. We write as usual the symplectic matrix $S$ in block form
$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
$$
so that the loop $S\gamma(t)$ is parametrized by
$$
S\gamma(t) = \begin{pmatrix} Ax(t) + Bp(t) \\ Cx(t) + Dp(t) \end{pmatrix}, \quad 0 \leq t \leq 2\pi.
$$
We thus have, by definition of the Poincaré invariant,
$$
I(S\gamma) = \int_0^{2\pi} (Cx(t) + Dp(t))^T(A\dot{x}(t) + B\dot{p}(t))dt;
$$
expanding the product in the integrand, we have $I(S\gamma) = I_1 + I_2$ where
\[
I_1 = \int_0^{2\pi} x(t)^T C^T A\dot{x}(t)dt + \int_0^{2\pi} p(t)^T D^T B\dot{p}(t)dt
\]
\[
I_2 = \int_0^{2\pi} x(t)^T C^T B\dot{p}(t)dt + \int_0^{2\pi} p(t)^T D^T A\dot{x}(t)dt.
\]
We claim that $I_1 = 0$. Recall that $C^T A$ and $C^T B$ are symmetric in view of the two first equalities in (7); applying the differentiation formula (14) with $u = v = x$ we have
\[
\int_0^{2\pi} x(t)^T C^T A\dot{x}(t)dt = \frac{1}{2} \int_0^{2\pi} \frac{d}{dt} (x(t)^T C^T Ax(t))dt
\]
\[
= \frac{1}{2} \left[ x(2\pi)^T C^T Ax(2\pi) - x(0)^T C^T Ax(0) \right]
\]
\[
= 0
\]
because $x(0) = x(2\pi)$. Likewise, applying (14) with $u = v = p$ we get
\[
\int_0^{2\pi} p(t)^T D^T B\dot{p}(t)dt = 0
\]
hence \( I_1 = 0 \) as claimed. We next consider the term \( I_2 \). Rewriting the integrand of the second integral as

\[
x(t)^T C^T B \dot{p}(t) = \dot{p}(t)^T B^T C x(t)^T
\]

(because it is a number, and hence equal to its own transpose!) we have

\[
I_2 = \int_0^{2\pi} \dot{p}(t)^T B^T C x(t)^T \, dt + \int_0^{2\pi} p(t)^T D^T A \dot{x}(t) \, dt
\]

that is, since \( D^T A = I_d + B^T C \) by transposition of the third equality in (7),

\[
I_2 = \int_0^{2\pi} p(t)^T \dot{x}(t) \, dt + \int_0^{2\pi} [p(t)^T B^T C A \dot{x}(t) + \dot{p}(t)^T B^T C A x(t)] \, dt.
\]

Using again the rule (14) and noting that the first integral is precisely \( I(\gamma) \) we get, \( D^T A \) being symmetric,

\[
I_2 = I(\gamma) + \int_0^{2\pi} \frac{d}{dt} [p(t)^T B^T C A x(t)] \, dt.
\]

The equality \( I(S\gamma) = I(\gamma) \) follows noting that the integral in the right-hand side is

\[
p(2\pi)^T B^T C A x(2\pi) - p(0)^T B^T C A x(0) = 0
\]

since \((x(2\pi), p(2\pi)) = (x(0), p(0))\).

The observant reader will have observed that we really needed all of the properties of a symplectic matrix contained in the set of conditions (7); this shows that the symplectic invariance of the first Poincaré invariant is a characteristic property of symplectic matrices.

### 2.3 Proof of the symplectic egg property

Let us denote by \( B_R \) the phase space ball centered at the origin and having radius \( R \). It is the set of all points \( z = (x, p) \) such that \( |z|^2 = |x|^2 + |p|^2 \leq R^2 \). What we call a “symplectic egg” is the image \( S(B_R) \) of \( B_R \) by a symplectic matrix \( S \). It is thus an ellipsoid in phase space, consisting of all points \( z \) such that \( S^{-1} z \) is in the ball \( B_R \), that is \( |S^{-1} z|^2 \leq R^2 \). Using formula (10) giving the inverse of \( S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) together with the relations \( A^T C = C^T A \), \( B^T D = D^T B \) in (7) we get the following explicit expression after some easy calculations:

\[
x^T (CC^T + DD^T) x - 2x^T (DB^T + CA^T) p + p^T (AA^T + BB^T) p \leq R^2
\]
(don’t worry: we will not have to use this cumbersome inequality in what follows!).

Let us now cut \( S(B_R) \) by a plane \( \Pi_j \) of conjugate coordinates \( x_j, p_j \). We get an elliptic surface \( \Gamma_j \), whose boundary is an ellipse denoted by \( \gamma_j \). Since that ellipse lies in the plane \( \Pi_j \) we can parametrize it by only specifying coordinates \( x_j(t), p_j(t) \) all the other being identically zero; relabeling if necessary the coordinates we may as well assume that \( j = 1 \) so that the curve \( \gamma_j \) can be parametrized as follows:

\[
\gamma_j(t) = (x_1(t), 0, \ldots, 0; p_1(t), 0, \ldots, 0)^T
\]

for \( 0 \leq t \leq 2\pi \) with \( x_1(0) = x_1(2\pi) \) and \( p_1(0) = p_1(2\pi) \). Since \( x_k(t) = 0 \) and \( p_k(t) = 0 \) for \( k > 1 \) the area of the ellipse is given by the formula

\[
\text{Area}(\Gamma_1) = \int_0^{2\pi} p_1(t) \dot{x}_1(t) dt = \sum_{k=1}^n \int_0^{2\pi} p_k(t) \dot{x}_k(t) dt = \oint_{\gamma_1} p dx
\]

hence \( \text{Area}(\Gamma_1) = I(\gamma_1) \). Since the inverse matrix \( S^{-1} \) is symplectic, we have \( I(\gamma_1) = I(S^{-1}\gamma_1) \). But the loop \( S^{-1}\gamma_1 \) bounds a section of the ball \( B_R \) by a plane (the plane \( S^{-1}\Pi_j \)) passing through its center. This loop is thus a great circle of \( B_R \) and the area of the surface \( S^{-1}\Gamma_1 \) is thus exactly \( \pi R^2 \), which was to be proven.

We urge the reader to notice that the assumption that we are cutting \( S(B_R) \) with a plane of conjugate coordinates is essential, because it is this assumption that allowed us to identify the area of the section with action. Here is a counterexample which shows that the property does not hold for arbitrary sections of \( S(B_R) \). Take, for instance

\[
S = \begin{pmatrix}
\lambda_1 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 \\
0 & 0 & 1/\lambda_1 & 0 \\
0 & 0 & 0 & 1/\lambda_2
\end{pmatrix}, \quad \lambda_1 > 0, \lambda_2 > 0, \text{ and } \lambda_1 \neq \lambda_2
\]  

(15)

so that \( S(B_R) \) is defined by the inequality

\[
\frac{1}{\lambda_1} x_1^2 + \frac{1}{\lambda_2} x_2^2 + \lambda_1 p_1^2 + \lambda_2 p_2^2 \leq R^2.
\]
The section of $S(B_R)$ by the $x_2, p_2$ plane is the ellipse

$$\frac{1}{\lambda_1} x_1^2 + \lambda_1 p_1^2 \leq R^2$$

which has area $\pi(R^2\sqrt{\lambda_1/\lambda_1}) = \pi R^2$ as predicted, but its section with the $x_2, p_1$ plane is the ellipse

$$\frac{1}{\lambda_2} x_1^2 + \lambda_2 p_1^2 \leq R^2$$

which has area $\pi(R^2\sqrt{\lambda_1/\lambda_2})$ which is different from $\pi R^2$ since $\lambda_1 \neq \lambda_2$.

The assumption that $S$ is symplectic is also essential. Assume that we scramble the diagonal entries of the matrix $S$ above in the following way:

$$S' = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 1/\lambda_2 & 0 \\ 0 & 0 & 0 & 1/\lambda_1 \end{pmatrix}.$$ 

The matrix $S'$ still has determinant one, but it is not symplectic (cf. (11)). The section $S'(B_R)$ by the $x_2, p_2$ plane is now the ellipse

$$\frac{1}{\lambda_1} x_1^2 + \lambda_2 p_1^2 \leq R^2$$

with area $\pi R^2\sqrt{\lambda_1/\lambda_2} \neq \pi R^2$.

3 The Symplectic Camel

The property of the symplectic camel is a generalization of the property of the symplectic egg for arbitrary canonical transformations; it reduces to the latter in the linear case.

3.1 Gromov’s non-squeezing theorem: static formulation

As we mentioned in the Prologue, the property of the symplectic egg is related to the “non-squeezing theorem” of Gromov [17] in 1985. To understand it fully we have to introduce the notion of canonical transformation [2, 4]. A canonical transformation is an invertible infinitely differentiable mapping

$$f : \begin{pmatrix} x \\ p \end{pmatrix} \mapsto \begin{pmatrix} x' \\ p' \end{pmatrix}$$
of phase space on itself whose inverse $f^{-1}$ is also infinitely differentiable and such that its Jacobian matrix

$$f'(x,p) = \frac{\partial (x', p')}{\partial (x, p)}$$

is symplectic at every point $(x, p)$. A symplectic matrix $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ automatically generates a linear canonical transformation by letting it act on phase space vectors $\begin{pmatrix} x \\ p \end{pmatrix} \rightarrow S \begin{pmatrix} x \\ p \end{pmatrix}$; it is an invertible transformation (because symplectic matrices are invertible), trivially infinitely differentiable, and the Jacobian matrix is the matrix $S$ itself. Phase space translations, that is mappings $\begin{pmatrix} x \\ p \end{pmatrix} \rightarrow \begin{pmatrix} x + x_0 \\ p + p_0 \end{pmatrix}$ are also canonical: their Jacobian matrix is just the identity $\begin{pmatrix} I_d & 0 \\ 0 & I_d \end{pmatrix}$. By composing linear canonical transformations and translations one obtains the class of all affine canonical transformations.

Here is an example of a nonlinear canonical transformation: assume that $n = 1$ and denote the phase space variables by $r$ and $\varphi$ instead of $x$ and $p$; the transformation defined by $(r, \varphi) \rightarrow (x, p)$ with

$$x = \sqrt{2r} \cos \varphi, \quad p = \sqrt{2r} \sin \varphi, \quad 0 \leq \varphi < 2\pi,$$

has Jacobian matrix

$$f'(r, \varphi) = \begin{pmatrix} \frac{1}{\sqrt{2r}} \cos \varphi & \frac{1}{\sqrt{2r}} \sin \varphi \\ -\frac{1}{\sqrt{2r}} \sin \varphi & \frac{1}{\sqrt{2r}} \cos \varphi \end{pmatrix}$$

which has determinant one for every choice of $r$ and $\varphi$. The transformation $f$ is thus canonical, and can be extended without difficulty to the multi-dimensional case by associating a similar transformation to each pair $(x_j, p_j)$. It is in fact a symplectic version of the usual passage to polar coordinates (the reader can verify that the latter is not canonical by calculating its Jacobian matrix); it can also be viewed as the simplest example of action-angle variable [2, 4]; for instance it reduces the isotropic harmonic oscillator Hamiltonian $H = \frac{1}{2}(p^2 + x^2)$ to $K = r$.

We will see in a moment why canonical transformations play such an important role in Physics (and especially in classical mechanics), but let us first state Gromov’s theorem:

**Gromov’s theorem:** No canonical transformation can squeeze a ball $B_R$ through a circular hole in a plane $\Pi_j$ of conjugate coordinates $x_j, p_j$ with smaller radius $r < R$. 

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This statement is surprisingly simple, and one can wonder why it took so long time to discover it. There are many possible answers. The most obvious is that all known proofs Gromov’s theorem are extremely difficult, and make use of highly non-trivial techniques from various parts of pure mathematics, so the result cannot be easily derived from elementary principles. Another reason is that it seems, as we will discuss below, to contradict the common conception of Liouville’s theorem, and was therefore unsuspected!

So, what is the relation of Gromov’s theorem with our symplectic eggs, and where does its nickname “principle of the symplectic camel” come from? The denomination apparently appeared for the first time in Arnol’d’s paper [3]. Recalling that in [21] it is stated that

‘...Then Jesus said to his disciples, ‘Amen, I say to you, it will be hard for one who is rich to enter the kingdom of heaven. Again I say to you, it is easier for a camel to pass through the eye of a needle than for one who is rich to enter the kingdom of God’.

The biblical camel is here the ball $B_R$, and the eye of the needle is the hole in the $x_j,p_j$ plane! (For alternative interpretations of the word “camel”; see the reader’s comments following E. Samuel Reich’s New Scientist paper [20] about [9].)

Let us next show that the section property of the symplectic egg is indeed a linear (or affine) version of Gromov’s theorem. It is equivalent to prove that no symplectic egg $S(B_R)$ with radius $R$ larger than that, $r$, of the hole in the $x_j,p_j$ plane can be threaded through that hole. Passing $S(B_R)$ through the hole means that the section of the symplectic egg by the $x_j,p_j$ plane, which has area $\pi R^2$, is smaller than the area $\pi r^2$ of the hole; hence we must have $R \leq r$.

### 3.2 Dynamical interpretation

The reason for which canonical transformations play an essential role in Physics comes from the fact that Hamiltonian phase flows precisely consist of canonical transformations. Consider a particle with mass $m$ moving along the $x$-axis under the action of a scalar potential $V$. The particle is subject to a force $F = -\frac{d}{dx}V(x)$. Since $F = m\frac{dv}{dt} = \frac{dp}{dt}$ (Newton’s second law), the equations of motion can be written

$$m\frac{dx}{dt} = p, \quad \frac{dp}{dt} = -\frac{dV}{dx}.$$  

(16)
In terms of the Hamilton function

\[ H(x, p) = \frac{1}{2m}p^2 + V(x) \]

this system of differential equations is equivalent to Hamilton’s equations of motion

\[ \frac{dx}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial x}. \]  

(17)

We will more generally consider the \( n \)-dimensional version of (17) which reads

\[ \frac{dx_j}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial x_j}, \quad 1 \leq j \leq n. \]  

(18)

(In mathematical treatments of Hamilton’s equations \[2, 4, 8\] the function \( H \) can be of a very general type, and even depend on time \( t \). In either case, these equations determine—as any system of differential equations does—a flow. By definition, the Hamiltonian flow is the infinite set of mappings \( \phi^H_t \) defined as follows: suppose we solve the system (18) after having chosen initial conditions \( x_1(0), ..., x_n(0) \) and \( p_1(0), ..., p_n(0) \) at time \( t = 0 \) for the position and momentum coordinates. Denote the initial vector thus defined \((x(0), p(0))\). Assuming that the solution to Hamilton’s equations at time \( t \) exists (and is unique), we denote it by \((x(t), p(t))\). By definition, \( \phi^H_t \) is just the mapping that takes the initial vector to the final vector:

\[ \left( \begin{array}{c} x(t) \\ p(t) \end{array} \right) = \phi^H_t \left( \begin{array}{c} x(0) \\ p(0) \end{array} \right). \]  

(19)

As time varies, the initial point describes a curve in phase space; it is called a “flow curve” or a “Hamiltonian trajectory”.

The essential property to remember is that each mapping \( \phi^H_t \) is a canonical transformation; Hamiltonian flows are therefore volume preserving: this is Liouville’s theorem \[2, 4\]. This easily follows from the fact that symplectic matrices have determinant one. Since it is not true that every matrix with determinant one is symplectic, as soon as \( n > 1 \) volume preservation also holds for other transformations, and is therefore not a characteristic property of Hamiltonian flows; see Arnold \[2\], Ch.3, §16 for a discussion of this fact. The thing to observe is that volume preservation does not imply conservation of shape, and one could therefore imagine that under the action of a Hamiltonian flow a subset of phase space can be stretched in all directions, and eventually get very thinly spread out over huge regions of phase space.
space, so that the projections on any plane could \textit{a priori} become arbitrary small after some (admittedly, very long) time $t$. In addition, one may very well envisage that the larger the number $n$ of degrees of freedom, the more that spreading will occur since there are more directions in which the ball is likely to spread! This possibility, which is ruled out by the symplectic camel as we will explain below, has led to many philosophical speculations about Hamiltonian systems. For instance, in his 1989 book Roger Penrose ([23], p.174–184) comes to the conclusion that phase space spreading suggests that “\textit{classical mechanics cannot actually be true of our world}” (p.183, l.–3). In fact, our discussion of Gromov’s theorem shows that Hamiltonian evolution is much less disorderly than Penrose thought. Consider again our phase space ball $B_R$. Its orthogonal projection (or “shadow”) on any two-dimensional subspace $\Pi$ of phase space is a circular surface with area $\pi R^2$.

Suppose now that we move the ball $B_R$ using a Hamiltonian flow $\phi^H_t$ and choose for $\Pi$ the plane $\Pi_j$ of conjugate coordinates $x_j, p_j$. The ball will slowly get deformed, while keeping same volume. But, as a consequence of the principle of the symplectic camel, its “shadow” on any plane $\Pi_j$ will never decrease below its original value $\pi R^2$! Why is it so? First, it is clear that if the area of the projection of $f(B_R)$ on a plane $x_j, p_j$ (under a canonical transformation) will never be smaller than $\pi R^2$, then we cannot expect that $f(B_R)$ lies inside a cylinder $(p_j - a_j)^2 + (x_j - b_j)^2 = r^2$ if $r < R$. So is the “principle of the symplectic camel” stronger than Gromov’s theorem? Not at all, it is equivalent to it! Let us prove this. We assume as in section 2.3 that $j = 1$; this does not restrict the generality of the argument. Let $\gamma_1$ be the boundary of the projection of $f(B_R)$ on the $x_1, p_1$ plane; it is a loop encircling a surface $\Gamma_1$ with area at least $\pi R^2$. That surface $\Gamma_1$ can be deformed into a circle with same area using an area-preserving mapping of the $x_1, p_1$ plane; call that mapping $f_1$ and define a global phase space transformation $f$ by the formula

$$f(x_1, p_1, x_2, p_2, \ldots, x_n, p_n) = (f_1(x_1, p_1), x_2, p_2, \ldots, x_n, p_n).$$

Calculating the Jacobian matrix it is easy to check that the matrix $f$ is a canonical transformation, hence our claim. For a more detailed discussion of this and related topics see [6] [16].

4 Quantum Blobs

By definition, a quantum blob is a symplectic egg with radius $R = \sqrt{\hbar}$. The section of quantum blob by a plane of conjugate coordinates is thus
\( \pi \hbar = \frac{1}{2} \hbar \). We will see that quantum blobs qualify as the smallest units of phase space allowed by the uncertainty principle of quantum mechanics. We begin with a very simple example illustrating the basic idea, which is that a closed (phase space) trajectory cannot be carried by an energy shell smaller (in a sense to be made precise) than a quantum blob. As simple as this example is, it allows us to recover the ground energy of the anisotropic quantum harmonic oscillator.

### 4.1 The harmonic oscillator

The fact that the ground energy level of a one-dimensional harmonic oscillator

\[
H = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2x^2
\]

is different from zero is heuristically justified in the physical literature by the following observation: since Heisenberg’s uncertainty relation \( \Delta p_x \Delta x \geq \frac{1}{2} \hbar \) prevent us from assigning simultaneously a precise value to both position and momentum, the oscillator cannot be at rest. To show that the lowest energy has the value \( \frac{1}{2} \hbar \omega \) predicted by quantum mechanics one can then argue as follows: since we cannot distinguish the origin \( (x = 0, p = 0) \) of phase space from a phase plane trajectory lying inside the double hyperbola \( p_x x < \frac{1}{2} \hbar \), we must require that the points \( (x, p) \) of that trajectory are such that \( |p_x x| \geq \frac{1}{2} \hbar \); multiplying both sides of the trivial inequality

\[
\frac{p_x^2}{m\omega} + m\omega^2x^2 \geq 2|p_x| \geq \hbar
\]

by \( \omega/2 \) we then get

\[
E = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2x^2 \geq \frac{1}{2} \hbar \omega
\]

which is the correct lower bound for the quantum energy. This argument can be reversed: since the lowest energy of an oscillator with frequency \( \omega \) and mass \( m \) is \( \frac{1}{2} \hbar \omega \), the minimal phase space trajectory will be the ellipse

\[
\frac{p_x^2}{m\hbar \omega} + \frac{x^2}{(\hbar/m\omega)} = 1
\]

which encloses a surface with area \( \frac{1}{2} \hbar \). Everything in this discussion immediately extends to the generalized anisotropic \( n \)-dimensional oscillator

\[
H = \sum_{j=1}^{n} \frac{p_j^2}{2m_j} + \frac{1}{2}m_j\omega_j^2x^2
\]
and one concludes that the smallest possible trajectories in $x_j, p_j$ space are the ellipses

$$\frac{p_j^2}{m_j \hbar \omega_j} + \frac{x_j^2}{(h/m_j \omega_j)} = 1.$$ 

By the same argument as above, using each of the Heisenberg uncertainty relations

$$\Delta p_j \Delta x_j \geq \frac{1}{2} \hbar$$  \hspace{1cm} (20)

we recover the correct ground energy level

$$E = \frac{1}{2} h \omega_1 + \frac{1}{2} h \omega_2 + \cdots + \frac{1}{2} h \omega_n$$

as predicted by standard quantum theory [22]. In addition, one finds that, the projection of the motion on any plane of conjugate variables $x_j, p_j$ will always enclose a surface having an area at least equal to $\frac{1}{2} \hbar$. In other words, the motions corresponding to the lowest possible energy must lie on a quantum blob!

4.2 Quantum blobs and uncertainty

The Heisenberg inequalities (20) are a weak form of the quantum uncertainty principle; they are a particular case of the more accurate Robertson–Schrödinger [25, 27] inequalities

$$(\Delta p_j)^2 (\Delta x_j)^2 \geq \Delta(x_j, p_j)^2 + \frac{1}{4} \hbar^2$$  \hspace{1cm} (21)

(see Messiah [22] for a simple derivation). Here, in addition to the standard deviations $\Delta x_j, \Delta p_j$ we have the covariances $\Delta(x_j, p_j)$ which are a measurement of how much the two variables $x_j, p_j$ change together. (We take the opportunity to note that the interpretation of quantum uncertainty in terms of standard deviations goes back to Kennard [19]; Heisenberg’s [18] own interpretation was much more heuristic). Contrarily to what is often believed the Heisenberg inequalities (20) and the Robertson–Schrödinger inequalities (21) are not statements about the accuracy of our measurements; their derivation assumes on the contrary perfect instruments; see the discussion in Peres [24], p.93. Their meaning is that if the same preparation procedure is repeated a large number of times on an ensemble of systems, and is followed by either by a measurement of $x_j$, or by a measurement of $p_j$, then the results obtained have standard deviations $\Delta x_j, \Delta p_j$; in addition these measurements need not be independent: this is expressed by the statistical covariances $\Delta(x_j, p_j)$ appearing in the inequalities (21).
It turns out that quantum blobs can be used to give a purely geometric and intuitive idea of quantum uncertainty. Let us first consider the case $n = 1$, and define the covariance matrix by

$$
\Sigma = \begin{pmatrix}
\Delta x^2 & \Delta(x, p) \\
\Delta(p, x) & \Delta p^2
\end{pmatrix}.
$$

(22)

Its determinant is $\det \Sigma = (\Delta p)^2 (\Delta x)^2 - \Delta(x, p)^2$, so in this case the Robertson–Schrödinger inequality is the same thing as $\det \Sigma \geq \frac{1}{4} \hbar^2$. Now to the geometric interpretation. In statistics it is customary to associate to $\Sigma$ the so-called covariance ellipse: it is the set of $\Omega_\Sigma$ points $(x, p)$ in the phase plane satisfying

$$
\frac{1}{2} (x, p) \Sigma^{-1} \begin{pmatrix}x \\ p \end{pmatrix} \leq 1.
$$

(23)

Its area is $2\pi \sqrt{\det \Sigma}$, that is

$$
\text{Area}(\Omega_\Sigma) = 2\pi \left[ (\Delta p)^2 (\Delta x)^2 - \Delta(x, p)^2 \right]^{1/2}
$$

and the inequality $\det \Sigma \geq \frac{1}{4} \hbar^2$ is thus equivalent to $\text{Area}(\Omega_\Sigma) \geq \pi \hbar = \frac{1}{2} \hbar$.

We have thus succeeded in expressing the rather complicated Robertson–Schrödinger inequalities (21) in terms of the area of a certain ellipse. In higher dimensions the same argument applies, but contrarily to what common intuition suggests, the Robertson–Schrödinger inequalities are not expressed in terms of volume (which is the generalization of area to higher dimensions), but again in terms of areas–namely those of the intersections of the conjugate planes $x_j, p_j$ with the covariance ellipsoid

$$
\Sigma = \begin{pmatrix}
\Delta(x, x) & \Delta(x, p) \\
\Delta(p, x) & \Delta(p, p)
\end{pmatrix}.
$$

(24)

Here $\Delta(x, x), \Delta(x, p)$, etc. are the $n \times n$ block-matrices $(\Delta(x, x))_{1 \leq i, j \leq n}$, $(\Delta(x, p))_{1 \leq i, j \leq n}$ etc. Notice that the diagonal terms of $\Sigma$ are just the variances $\Delta x_1^2, ..., \Delta x_n^2; \Delta p_1^2, ..., \Delta p_n^2$ so that (24) reduces to (22) for $n = 1$.

Defining the covariance ellipsoid $\Omega_\Sigma$ as above, one then proves that the inequalities (21) are equivalent to the property that the intersection of $\Omega_\Sigma$ with the planes $x_j, p_j$ is at least $\frac{1}{2} \hbar$. These inequalities are saturated (i.e. they become equalities) if and only if these intersections have exactly area $\frac{1}{2} \hbar$, that is, if and only if $\Omega_\Sigma$ is a quantum blob! The proof goes as follows (for a detailed argument see [9, 16]): one first remarks, using a simple algebraic argument that the Robertson–Schrödinger inequalities are equivalent to the following condition of the covariance matrix, well-known in quantum optics, see e.g. [28, 29] and the references therein:
The eigenvalues of the Hermitian matrix $\Sigma + \frac{i\hbar}{2} J$ are non-negative: $\Sigma + \frac{i\hbar}{2} J \geq 0$.

The next step consists in noting that in view of Sylvester’s theorem from linear algebra that the leading principal minors of

$$\Sigma + \frac{i\hbar}{2} J = \begin{pmatrix} \Delta(x,x) & \Delta(x,p) + \frac{i\hbar}{2} I \\ \Delta(p,x) - \frac{i\hbar}{2} I & \Delta(p,p) \end{pmatrix}$$

are non-negative. This applies in particular to the minors of order 2 so that we must have

$$\begin{vmatrix} \Delta x_j^2 & \Delta(x_j,p_j) + \frac{i\hbar}{2} \\ -\frac{i\hbar}{2} & \Delta p_j^2 \end{vmatrix} \geq 0$$

and this condition is precisely the Robertson–Schrödinger inequality \[21\].

As we have seen, the fact that the covariance ellipsoid is cut by the conjugate coordinate planes along ellipsoids with areas $\geq \frac{1}{2} \hbar$ implies the Robertson–Schrödinger inequalities. This is thus a geometric statement—and a strong one—of the quantum uncertainty principle, which can be rephrased as follows:

*Every quantum covariance ellipsoid contains a quantum blob, i.e. a symplectic egg with radius $\sqrt{\hbar}$. When this ellipsoid is a quantum blob, the Robertson–Schrödinger inequalities are saturated.*

This statement can be extended in various ways; in a very recent paper \[13\] we have applied this geometric approach to the quantum uncertainty principle to the study of partial saturation of the Robertson–Schrödinger inequalities for mixed quantum states. We show, in particular, that partial saturation corresponds to the case where some (but not all) planes of conjugate coordinates cut the covariance ellipsoid along an ellipse with exactly area $\frac{1}{2} \hbar$; this allows us to characterize those states for which this occurs (they are generalized Gaussians).

Another important thing we will unfortunately not be able to discuss in detail because of length limitations, is the following: everything we have said above still holds true if we replace the sentence “planes of conjugate coordinates $x_j, p_j$” with the sentence “symplectic planes”. A symplectic plane is a two-dimensional subspace of phase space which has the property that if we restrict the symplectic form to it, then we obtain a new symplectic form, defined on this two-dimensional space. For instance, it is easy to check that the $x_j, p_j$ are symplectic planes (but those of coordinates $x_j, p_k, j \neq k$, or
$x_j, x_k, p_j, p_k$ are not). One proves \[2, 8\] that every symplectic plane can be obtained from any of the $x_j, p_j$ planes using a symplectic transformation. This property implies, in particular, that the Robertson–Schrödinger inequalities \([21]\) are covariant under symplectic transformations: if one defines new coordinates $x', p'$ by $(x', p')^T = S(x, p)^T$, $S$ a symplectic matrix, then

$$\left(\Delta p_j\right)^2 \left(\Delta x_j\right)^2 \geq \Delta(x_j, p_j)^2 + \frac{1}{4} \hbar^2$$

we also have

$$\left(\Delta p'_j\right)^2 \left(\Delta x'_j\right)^2 \geq \Delta(x'_j, p'_j)^2 + \frac{1}{4} \hbar^2.$$

Also, there are possible non-trivial generalizations of the uncertainty principle, using new results in symplectic topology, for instance \([1]\) which extends Gromov’s theorem (in the linear case) to projections on symplectic subspaces with dimension greater than 2. In \([12]\) we have shown how this result leads to “quantum universal invariants”.

5 Conclusion

Quoting the great mathematician Hermann Weyl:

‘In these days the angel of topology and the devil of abstract algebra fight for the soul of each individual mathematical domain’

(H. Weyl, 1939)

This quotation from goes straight to the point, and applies to Physics as well: while algebra (in the large) has dominated the scene of quantum mechanics for a very long time (in fact, from its beginning), we are witnessing a slow but steady emergence of geometric ideas. Not only do these geometric ideas add clarity to many concepts, but they also lead to new insights (see e.g. \([12]\)). This is what we had in mind while writing the present paper.

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References

[1] A. Abbondandolo and S. Matveyev. How large is the shadow of a symplectic ball? Preprint arXiv:1202.3614v1[math.SG]

[2] V.I. Arnold. Mathematical Methods of Classical Mechanics, Graduate Texts in Mathematics, 2nd edition, Springer-Verlag, 1989

[3] V.I. Arnold. First steps in symplectic topology. Uspekhi Mat. Nauk. 41:6, 3–18 (1986); translation: Russian Math. Surveys 41:6, 1–21 (1986)

[4] H. Goldstein. Classical Mechanics. Addison–Wesley, 1950; 2nd edition, 1980; 3d edition 2002

[5] M. de Gosson. The Principles of Newtonian and Quantum Mechanics, Imperial College Press, London, 2001

[6] M. de Gosson. Phase space quantization and the uncertainty principle. Phys. Lett. A 317, 365–369 (2003)

[7] M. de Gosson. The optimal pure Gaussian state canonically associated to a Gaussian quantum state. Phys. Lett. A 330, 161–167 (2004)

[8] M. de Gosson. Symplectic Geometry and Quantum Mechanics, series “Operator Theory: Advances and Applications” Vol. 166, Birkhäuser, Basel, 2006

[9] M. de Gosson. The Symplectic Camel and the Uncertainty Principle: The Tip of an Iceberg? Found. Phys. 99, 194–214 (2009)

[10] M. de Gosson. On the Use of Minimum Volume Ellipsoids and Symplectic Capacities for Studying Classical Uncertainties for Joint Position-Momentum Measurements. J. Stat. Mech. (2010) P11005

[11] M. de Gosson. A Pseudo-Quantum Triad: Schrödinger’s Equation, the Uncertainty Principle, and the Heisenberg Group. J. Phys.: Conf. Ser. 361 012015 (2012)

[12] M. de Gosson. The Symplectic Camel and Quantum Universal Invariants: the Angel of Geometry vs. the Demon of Algebra.

[13] M. de Gosson. On the partial saturation of the uncertainty relations of a mixed Gaussian state. To appear in J.Phys.A:Math.Theor (2012)
[14] M. de Gosson. Quantum Blobs. To appear in Found. Phys. (2012), Preprint: [arXiv:1106.5468v1 [quant-ph]]

[15] M. de Gosson and B. Hiley. Imprints of the Quantum World in Classical Mechanics. Found Phys. 41, 1415–1436 (2011)

[16] M. de Gosson and F. Luef. Symplectic capacities and the geometry of uncertainty: The irruption of symplectic topology in classical and quantum mechanics. Phys. Rep. 484(5), 131–179 (2009)

[17] M. Gromov. Pseudoholomorphic curves in symplectic manifolds, Invent. Math. 82, 307–347 (1985)

[18] W. Heisenberg. Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik, Zeitschrift für Physik A, Hadrons and Nuclei, Volume 43, Numbers 3-4, 172–198 (1927)

[19] E.H. Kennard. Zur Quantenmechanik einfacher Bewegungstypen. Zeitschrift für Physik 44 (4–5) 326–352 (1927)

[20] D.S. Mackey and N. Mackey. On the Determinant of Symplectic Matrices. [http://math.nju.edu.cn/~guoxj/notes/detsym.pdf](http://math.nju.edu.cn/~guoxj/notes/detsym.pdf)

[21] Matthew 19(24), St Luke 18(25), and Mk 10(25)

[22] A. Messiah. Quantum Mechanics (Vol.1). North–Holland Publ. Co. (1991); translated from the French: original title Mécanique Quantique. Dunod, Paris, 1961

[23] R. Penrose. The Emperor’s New Mind, Oxford University Press, 1989

[24] A. Peres. Quantum Theory: Concepts and Methods, Kluwer Academic Publishers, 1993

[25] H. P. Robertson. The uncertainty principle, Phys. Rev. 34, 163–164 (1929)

[26] E. Samuel Reich. How camels could explain quantum uncertainty, New Scientist 2697, 12 (2009)

[27] E. Schrödinger. Zum Heisenbergschen Unschärfeprinzip. Berliner Berichte 296–303 (1930) [English translation: Angelow, A., Batoni, M.C.: About Heisenberg Uncertainty Relation. Bulg. Journal of Physics, 26, nos.5/6, 193–203 (1999), and [http://arxiv.org/abs/quant-ph/9903100](http://arxiv.org/abs/quant-ph/9903100)]
[28] R. Simon, N. Mukunda, and N. Dutta, Quantum Noise Matrix for Multimode Systems: U(n)-invariance, squeezing and normal forms. Physical Reviews A 49 (1994) 1567–1583

[29] R. Simon, E.C.G. Sudarshan, N. Mukunda. Gaussian–Wigner distributions in quantum mechanics and optics. Physical Reviews A 36(8) (1987) 3868–3880

[30] I. Stewart. The symplectic camel. Nature 329, 17–18 (1987)

[31] E.C.G. Sudarshan, Charles B. Chiu, and G. Bhamati. Generalized uncertainty relations and characteristic invariant for the multimode states. Physical Reviews A 52(1) (1995) 43–54

[32] H. Weyl. Invariants. Duke Math. J. 5(3): 489–502 (1939)