Transport equation for the photon Wigner operator in non-commutative QED

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We derive an exact quantum equation of motion for the photon Wigner operator in non-commutative QED, which is gauge covariant. In the classical approximation, this reduces to a simple transport equation which describes the hard thermal effects in this theory. As an example of the effectiveness of this method we show that, to leading order, this equation generates in a direct way the Green amplitudes calculated perturbatively in quantum field theory at high temperature.

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I. INTRODUCTION

Classical transport equations for point particles, in the absence of collisions (scattering), have been well studied in the past. In more recent years, they have also been of considerable use in the study of hot QCD plasma \textsuperscript{1}. It is known, in particular, that in QCD the leading behavior of the $n$-point gluon functions at temperatures $T \gg p$, where $p$ represents a typical external momentum, is proportional to $T^2$ and these leading contributions to the amplitudes are all gauge independent \textsuperscript{2, 3, 4}. In order to extract the leading order contributions to the amplitude leading to gauge invariant results for physical quantities, it is necessary to perform a resummation of hard thermal loops, which are defined by

$$p \ll k \sim T, \quad (1)$$

where $k$ denotes a characteristic internal loop momentum. Such a procedure, however, is quite technical and the classical transport equation has provided a much simpler method for deriving the gauge invariant amplitudes as well as the effective action which incorporates all the effects of the hard thermal loops \textsuperscript{2, 3, 4}. In such an approach, one pictures the constituents of the plasma as classical particles carrying color charge and interacting in a self-consistent manner.

The basic idea behind the transport equation is to determine the evolution equation for the distribution function. There are basically two equivalent ways of doing this. In the first approach one pictures the thermal particles, moving in an internal loop, as classical particles in equilibrium in the hot plasma whose dynamics are governed by classical point particle equations. The transport equation can, of course, be derived in a straightforward manner once we know the dynamical equations for such a particle in the background of a gauge field. For example, let us assume that the equations of motion for a particle in the presence of a background field is given by

$$m \frac{dX^\sigma}{d\tau} = k^\sigma \quad (2a)$$

$$m \frac{dk^\sigma}{d\tau} = eX^\sigma \quad (2b)$$

where $\tau$ denotes the proper time of the particle and $X^\sigma$ represents the force it feels in the presence of a background gauge field. (In addition, for a particle carrying color charge in QCD, we have to supplement the above equations with the evolution equation for the color charge.) The explicit form of $X^\sigma$ will, of course, be different depending on the type of interaction. In general, however, the form of $X^\sigma$ must be such that $k^2 = k^\mu k^\mu$ is a constant and that the time evolution for $k^\mu$ transforms covariantly under a gauge transformation. Given these equations, the classical transport equation for the distribution function follows in a straightforward manner.

In the second approach, also known as the Wigner function approach, one starts with the Wigner distribution function for the quantum field theory of interest interacting with a background gauge field. The evolution equation for the Wigner function is, then, determined directly from the equations satisfied by the quantum fields \textsuperscript{5, 11}. In the case of self-interacting non-Abelian gauge fields, there are two particular issues that need special care. First, the Wigner function has to be defined in a gauge covariant manner and second, for a self-interacting gauge field, one has to use a self-consistent mean field approach to identify the background gauge field. We will discuss this method more in detail later. But, the advantage of the second method is that here we do not have to know the dynamical equations governing the constituent objects moving through a hot plasma. In both these approaches, once the transport equation for the distribution function is known, a current is defined in terms of it. This current, which is a functional of the background gauge fields, generates the leading hard thermal loop amplitudes for the theory of interest.

More recently, following from developments in string theory, there has been an increased interest in quantum field theories defined on a non-commutative manifold satisfying \textsuperscript{12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22}

$$[x^\mu, x^{\nu}] = i\theta^{\mu\nu} \quad (3)$$

where $\theta^{\mu\nu}$ is assumed to be a constant anti-symmetric tensor with the dimensions of length squared. The behavior of hot plasmas in such a non-commutative gauge theory and a corresponding transport equation for such
theories are interesting questions to study. The difficulty, on the other hand, lies in the fact that non-commutative field theories inherently describe particles with an extended structure \[23, 24, 25\] for which classical dynamical equations are not well understood. In an earlier paper \[26\], we tried to use the explicit forms of the leading hard thermal loop amplitudes in non-commutative QED as well as the relation between hard thermal loops and classical transport equations to derive a force law for such constituents and we had proposed a transport equation for non-commutative QED. Such a “phenomenological” equation already exhibits interesting features associated with such theories. For example, we have argued that, while the charged particles, in such theories, have the expected dipole structure, the charge neutral photon field exhibits a quadrupole nature in the hard thermal loop approximation and that to describe the correct hard thermal loop amplitudes in the leading order, the classical transport equation must necessarily involve “collision” terms arising from the extended nature of the constituent particles.

The drawback of our earlier study lies in the fact that the transport equation did not have a theoretical derivation. As we have alluded to above, this is connected with the difficulty that we do not understand well the classical dynamical equations for extended particles in a given background. As a result, in this paper, we make an attempt to try to derive theoretically a classical transport equation based on the Wigner function approach, which involves only the properties of the non-commutative QED. In Sec II, we describe briefly the properties of non-commutative QED and define the Wigner distribution function for such a theory. We study various properties of this function and derive the transport equation associated with it. We also define the current associated with such a system. In Sec III, we evaluate in a direct way the leading order amplitudes following from the current, which agree completely with the hard thermal loops calculated in perturbative quantum field theory. We also compare various features arising in this approach with those found earlier and present a brief conclusion in Sec IV. In appendix A, we give some essential technical details of the derivation of the transport equation. In appendix B, we compile some relations that are useful in understanding some other aspects of our calculations.

II. WIGNER FUNCTION APPROACH IN NON-COMMUTATIVE QED

Non-commutative QED is described by the Lagrangian density of the form

\[
\mathcal{L} = -\frac{1}{4} F_{\mu \nu} \star F^{\mu \nu}
\]

where

\[
F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{ie}{\hbar c} [A_\mu, A_\nu]_{MB}
\]

where \(e\) represents the speed of light. Here the Moyal bracket is defined to be

\[
[A, B]_{MB} = A \star B - B \star A
\]

where the Grönewold-Moyal star product has the form

\[
A(x) \star B(x) = e^{\frac{ie}{\hbar c} \int_0^1 dt \Omega(x(t))} A(x) B(x)\bigg|_{\Omega=0}
\]

We can, of course, add to this Lagrangian density charged matter fields, but since this theory is self-interacting, much like Yang-Mills theory, we will continue with this. Furthermore, since the photon is charge neutral, this case is of more interest, since we normally have transport equations for charged particles. We simply note here that the theory in \(4\) is invariant under the non-commutative \(U(1)\) gauge transformation

\[
A_\mu(x) \rightarrow \Omega^{-1}(x) \star A_\mu \star \Omega(x) + \frac{ie}{\hbar c} \frac{1}{2} \Omega^{-1}(x) \star \partial_\mu \Omega(x)
\]

The Wigner distribution function is an operator density function in the mixed phase space. For a quantum scalar field theory in four space-time dimensions, for example, it is conventionally defined as

\[
W(x, k) = \int \frac{d^4y}{(2\pi \hbar)^4} e^{ik \cdot \phi(\frac{y}{2})} \phi(x + \frac{y}{2}) \phi(x - \frac{y}{2})
\]

For later convenience, let us define

\[
x_\pm = x \pm \frac{y}{2}
\]

The definition in \(9\) can be easily generalized to the non-commutative scalar field theory, by simply introducing star products in the product of fields (and treating \(y\) as a parameter not subject to the star product). In defining the Wigner function for the non-commutative photon fields, we have to worry about the gauge covariance properties. Since the non-commutative photon fields are self-interacting much like the usual gluon fields in QCD, we define an analogous gauge covariant Wigner function as

\[
W_{\mu \nu}(x, k) = \int \frac{d^4y}{(2\pi \hbar)^4} e^{ik \cdot \phi(\frac{y}{2})} G^{\mu \lambda}_{\nu}(x) \star G^{\nu \lambda}_{\nu}(x)
\]

where we have defined

\[
G^{\mu \nu}_{\nu}(x) = U(x, x_\pm) \star F_{\mu \nu}(x_\pm) \star U(x_\pm, x)
\]

and the \(U\)’s represent the link operators in the non-commutative theory defined along a straight path, namely

\[
U(x, x_\pm) = P(e^{\frac{ie}{\hbar c} \int_0^1 dt \gamma(x(t))})
\]

Here “\(P\)” stands for path ordering from left to right and it is straightforward to check that under a gauge transformation, \(5\), the link operators in \(13\) transform covariantly as

\[
U(x, x_\pm) \rightarrow \Omega^{-1}(x) \star U(x, x_\pm) \star \Omega(x_\pm)
\]
much like in the usual Yang-Mills theory. As a result, the Wigner function, \( (11) \), transforms covariantly in the adjoint representation under a gauge transformation,

\[
W_{\mu\nu}(x, k) \rightarrow \Omega^{-1}(x) \ast W_{\mu\nu}(x, k) \ast \Omega(x) \quad (15)
\]

We note here that, for a link operator defined along a straight path, we can also write

\[
\mathcal{G}^{(\pm)}_{\mu\nu}(x) = \left( e^{\pm \frac{i}{\hbar c} D \Phi_{\mu\nu}(x)} \right) \quad (16)
\]

where the covariant derivative is defined to be in the adjoint representation under a gauge transformation. Furthermore, from the definition of the Wigner function in \( (11) \), it is easy to check that

\[
W_{\mu\nu}^\dagger(x, k) = W_{\nu\mu}(x, k) \quad (17)
\]

In non-commutative QED, charge conjugation is defined as the simultaneous transformation \( (27) \)

\[
A_\mu(x) \rightarrow -A_\mu(x), \quad \vartheta^{\mu\nu} \rightarrow -\vartheta^{\mu\nu} \quad (18)
\]

Under charge conjugation, it is seen, using various identities for star products, that

\[
W_{\mu\nu}(x, k) \rightarrow W_{\mu\nu}^c(x, k) = W_{\nu\mu}(x, -k) \quad (19)
\]

Note from the definition in \( (11) \) that in the limit \( \theta \rightarrow 0 \), the Wigner function for usual QED is charge conjugation invariant, while it transforms non-trivially in the non-commutative case.

The derivation of the transport equation for the Wigner function, from its definition in \( (11) \), follows in a straightforward manner. We discuss the details of such a derivation in appendix A. Here, we simply note that the crucial relations that play an important role in this derivation are given by

\[
D_{(x)}^\mu U(x, x_{\pm}) = \frac{\theta^{(x)}_\mu}{\hbar c} U(x, x_{\pm}) - \frac{ie}{\hbar c} (A_\mu(x) \ast U(x, x_{\pm}) - U(x, x_{\pm}) \ast A_\mu(x_{\pm}))
\]

\[
= \frac{\theta^{(x)}_\mu}{2} \left( \int_0^1 dt \left( e^{\pm \frac{i}{\hbar c} D \Phi_{\mu\nu}(x)} \right) \ast U(x, x_{\pm}) \right)
\]

\[
D_{(x)}^\mu U(x_{\pm}, x) = \frac{\theta^{(x)}_\mu}{\hbar c} U(x_{\pm}, x) - \frac{ie}{\hbar c} (A_\mu(x_{\pm}) \ast U(x_{\pm}, x) - U(x_{\pm}, x) \ast A_\mu(x))
\]

\[
= \frac{\theta^{(x)}_\mu}{2} \left( \int_0^1 dt \left( e^{\pm \frac{i}{\hbar c} D \Phi_{\mu\nu}(x)} \right) \ast U(x, x_{\pm}) \right) \quad (20)
\]

With the relations in \( (20) \), it is straightforward to show that the Wigner function for the non-commutative photon satisfies

\[
k \cdot D W_{\mu\nu}(x, k) = \frac{e}{2\epsilon} \partial_k \kappa^\mu \left[ \left( \int_0^1 dt \left( e^{\pm \frac{i}{\hbar c} D \Phi_{\mu\nu}(x)} \right) \ast W_{\mu\nu}(x, k) + W_{\mu\nu}(x, k) \ast \left( \int_0^1 dt \left( e^{\pm \frac{i}{\hbar c} D \Phi_{\mu\nu}(x)} \right) \right) \right) \right]
\]

\[
- \frac{i}{2} \int \frac{d^4y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar c} y \cdot k} \left[ \left( \int_0^1 dt \left( \left( e^{\pm \frac{i}{\hbar c} D \Phi_{\mu\nu}(x)} \right) \ast U(x, x_{\pm}) \right) \right) \right] \ast G^{(+)}_{\mu\lambda}(x) \ast \left( \int_0^1 dt \left( \left( e^{\pm \frac{i}{\hbar c} D \Phi_{\mu\nu}(x)} \right) \ast U(x, x_{\pm}) \right) \right) \ast G^{(-)}_{\nu\lambda}(x)
\]

\[
+ k^\mu \int \frac{d^4y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar c} y \cdot k} \left[ U(x, x_{\pm}) \ast \left( D_{(x)}^{(x_{\pm})} F_{\mu\lambda}(x_{\pm}) \ast U(x_{\pm}, x) \ast G^{(+)}_{\mu\lambda}(x) \right) \right.
\]

\[
\left. \ast \left( D_{(x)}^{(x_{\pm})} F_{\nu\lambda}(x_{\pm}) \ast U(x_{\pm}, x) \ast G^{(-)}_{\nu\lambda}(x) \right) \right] \quad (21)
\]

Here \( \partial_k \) represents the derivative with respect to \( k \) and we have used the compact notation introduced in \( (16) \) to write the equation in a simpler form.

Equation \( (21) \) represents the full transport equation for the photon Wigner function in non-commutative QED. We note that by using various identities (which we discuss in the appendix A) the second term in \( (21) \) can be written as
\[ \frac{ie}{4c} \frac{\partial}{\partial k_{\sigma}} \int \frac{d^4y}{(2\pi)^4} \ e^{-i\frac{\pi}{2} y \cdot k} \int dt \left \{ \left [ \left( e^{\frac{i\pi}{2} D^\rho} F^\rho(x) \right) , U(x, x+) \ast (D_\mu F_{\mu\lambda}(x_+)) \ast U(x_+, x) \right]_{MB} \ast G^{\lambda(-)}(x) \right \} \]

\[ + U(x_+, x) \ast (D_\mu F_{\mu\lambda}(x_+)) \ast U(x_+, x) \ast \left [ \left( e^{-\frac{i\pi}{2} D^\rho} F^\rho(x) \right) , G_{\lambda}^{\nu(-)}(x) \right]_{MB} \]

\[ + \left \{ \left( e^{\frac{i\pi}{2} D^\rho} F^\rho(x) \right) , U(x_+, x) \ast (D_\mu F_{\mu\lambda}(x_+)) \ast U(x_+, x) \ast G_{\lambda}^{\nu(-)}(x) \right \} \]

\[ + \frac{e}{c} \int \frac{d^4y}{(2\pi)^4} e^{-i\frac{\pi}{2} y \cdot k} \left \{ U(x_+, x) \ast \left [ F_{\mu\nu}(x_+), F_{\lambda}^{\nu}(x_+) \right]_{MB} \ast U(x_+, x) \ast G_{\lambda}^{\nu(-)}(x) \right \} \]

\[ - G_{\mu\lambda}^{(+)}(x) \ast U(x_+, x) \ast \left [ F_{\lambda}^{\nu}(x_+), F_{\mu\nu}(x_+) \right]_{MB} \ast U(x_+, x) \]  \tag{22}

In the hard thermal loop approximation, as we have mentioned earlier, it is assumed that the gradients in the system are small compared with \( k/\hbar \). Moreover, for the semi-classical picture to hold, one also assumes that the ensemble average of the covariant derivative, \( \langle D W \rangle \), may be considered as being sufficiently small compared with \( \langle k W \rangle /\hbar \). In this approximation, it may be verified that the ensemble average of the term (22), namely, of the second term in (21) is small compared to that of the first term. Under the above conditions, the exponentials \( \exp(\pm i\frac{\pi}{2} y \cdot D) \sim \exp(\pm \frac{i\pi}{2} \hbar \cdot D) \) which appear in the first term of (21) may also be approximated by 1. Therefore, if we are only interested in the leading order behavior in the hard thermal loop approximation, we can neglect the second term in (21) and the classical transport equation for the Wigner function takes the simple form

\[ k \cdot DW_{\mu\nu}(x, k) = \frac{e}{2c} \frac{\partial}{\partial k_\sigma} k_\rho \left [ F_{\rho\sigma}(x) \ast W_{\mu\nu}(x, k) + W_{\mu\nu}(x, k) \ast F_{\rho\sigma}(x) \right ] - 2 \int \frac{d^4y}{(2\pi)^4} e^{-i\frac{\pi}{2} y \cdot k} G_{\mu}^{(+)}(x) \ast F_{\rho\sigma}(x) \ast G_{\lambda}^{\nu(-)}(x) \]  \tag{23}

Since non-commutative QED is a self-interacting theory, very much like the conventional QCD, one has to consistently separate the gauge field into a background part and a quantum part. Normally, this is done by assuming a mean field decomposition of the form

\[ A_\mu(x) = \tilde{A}_\mu(x) + a_\mu(x) \]  \tag{24}

where it is assumed that, in this mean field approximation,

\[ \langle A_\mu(x) \rangle = \tilde{A}_\mu(x), \quad \langle a_\mu(x) \rangle = 0 \]  \tag{25}

In making such a decomposition, it is assumed, as is the case in the usual background field method, that under a gauge transformation, \( \tilde{A}_\mu \),

\[ \tilde{A}_\mu(x) \rightarrow \Omega^{-1}(x) \ast \tilde{A}_\mu(x) \ast \Omega(x) \]

\[ + \frac{i\hbar c}{e} \Omega^{-1}(x) \ast \partial_\mu \Omega(x), \]  \tag{26a}

\[ a_\mu(x) \rightarrow \Omega^{-1}(x) \ast a_\mu(x) \ast \Omega(x) \]  \tag{26b}

This is very important in understanding the gauge transformation properties of the quantities resulting from the transport equation. We note that, under such a decomposition,

\[ F_{\mu\nu}(x) = \tilde{F}_{\mu\nu}(x) + \partial_\mu a_\nu(x) - \partial_\nu a_\mu(x) - \frac{ie}{\hbar c} [a_\mu(x), a_\nu(x)]_{MB} \]  \tag{27}

where

\[ \tilde{D}_\mu a_\nu = \partial_\mu a_\nu - \frac{ie}{\hbar c} [\tilde{A}_\mu, a_\nu]_{MB}, \]  \tag{28a}

\[ \tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu - \frac{ie}{\hbar c} [\tilde{A}_\mu, \tilde{A}_\nu]_{MB} \]  \tag{28b}

It is clear that every term in (27) transforms covariantly under the gauge transformation (26).

Let us also define

\[ G_{\mu\nu}(x, k) = \langle W_{\mu\nu}(x, k) \rangle - \tilde{W}_{\mu\nu}(x, k) \]  \tag{29}
where $\tilde{W}_{\mu\nu}(x,k)$ represents the Wigner function associated with the background field and it is important to recognize that

\[ \langle W_{\mu\nu}(x,k) \rangle \neq \tilde{W}_{\mu\nu}(x,k) \] (30)

We note that $\tilde{W}_{\mu\nu}(x,k)$ satisfies the same equation as

\[ k \cdot D G_{\mu\nu}(x,k) = \frac{e}{2c} \frac{\partial}{\partial k} \left[ \bar{F}_{\rho\sigma}(x) \ast G_{\mu\nu}(x,k) + G_{\mu\nu}(x,k) \ast \bar{F}_{\rho\sigma}(x) \right] - 2 \int \frac{d^4y}{(2\pi\hbar)^4} e^{-iy \cdot k} \left\{ \langle \left( G^{(+)}_{\mu\lambda}(x) \ast \bar{F}_{\rho\sigma}(x) \ast G^{(-)}_{\lambda\rho}(x) \right) \rangle - \left( G^{(+)}(x) \rightarrow \tilde{G}^{(+)}(x) \right) \right\} \] (31)

It also follows from this, that

\[ F(x,k) = \frac{1}{k^2} \eta^{\mu\nu} G_{\mu\nu}(x,k) \] (32)

This equation is manifestly covariant under the gauge transformation and can be solved order by order in powers of $\hbar$ to give the ensemble average of the Wigner function.

Given the ensemble average of the Wigner function, we can now define a covariantly conserved current as follows. First, we note that the current is a four vector which is odd under charge conjugation. It transforms covariantly under the gauge transformation and furthermore, using (33), is easily seen to be covariantly conserved,

\[ J_\mu(x) = \frac{e}{2} \int d^4k \theta(k^0) \frac{k_\mu}{k^2} \eta^{\lambda\rho} (G_{\lambda\rho}(x,k) - G_{\rho\lambda}(x,-k)) \]

\[ = \frac{e}{2} \int d^4k \theta(k^0) k_\mu (F(x,k) - F(x,-k)) \] (34)

where it is assumed that we sum over the helicity states. We note that this current is manifestly odd under charge conjugation. It transforms covariantly under the gauge transformation and furthermore, using (33) is easily seen to be covariantly conserved,

\[ \bar{D}_\mu J^\mu(x) = \partial_\mu J^\mu - \frac{ie}{\hbar c} [\bar{A}_\mu, J^\mu]_{MB} = 0 \] (35)

This can, therefore, be defined as the current associated with our system and can be determined to any order in powers of $\hbar$ once $F(x,k)$ is determined. This is a functional of $A_\mu(x)$ and would generate $n$-point amplitudes through functional derivation. These can then be compared with the leading order calculations from perturbation theory in the hard thermal loop approximation.

### III. LEADING ORDER AMPLITUDES

Given the equation for $F(x,k)$, we can now determine it order by order in $\hbar$. In this section, we will calculate this quantity explicitly to second order which would give us photon amplitudes up to the three point function for which we have explicit results from the calculations in perturbation theory. For simplicity of notation, we shall use the following natural units $c = \hbar = 1$.

To zeroth order, $F^{(0)}$ can be calculated from its definition in (32) using Eqs. (11) and (29),

\[ F^{(0)}(x,k) = \int \frac{d^4y}{(2\pi)^4} e^{-iy \cdot k} \frac{2}{k^2} \langle (\partial_\mu a_\lambda(x+) - \partial_\lambda a_\mu(x+)) \ast \partial^\lambda a^\mu(x-) \rangle \] (36)

The thermal ensemble average in (36) can be calculated in a standard manner using the field decomposition for
the quantum field \( a_\mu \). The only important thing to remember is that we want manifest gauge covariance preserved in the calculation. This suggests, as in the background field method, that the proper gauge condition on the quantum fields should maintain this invariance and, in particular, a transverse gauge has to be generalized to the form

\[
D_\mu a^\mu(x) = 0 \tag{37}
\]

This would suggest that the polarization tensors, in such a case, need not be transverse to the momentum four vector and, in fact, can have a longitudinal component depending on the background field. Fortunately, up to the leading three point amplitudes, this modification does not give rise to any contribution and, for all practical purposes of our calculations in this paper, we can take the polarization to be transverse to the momentum four vector. In higher order calculations or in the sub-leading terms, however, one has to include such contributions carefully.

With this observation, we note that Eq. \( \text{(36)} \) can be evaluated in a straightforward manner and the thermal contribution has the form

\[
\mathcal{F}^{(0)}(x, k) = \int \frac{d^4y}{(2\pi)^4} e^{-iy \cdot k} \int \frac{d^4\tilde{k}}{(2\pi)^4} \theta(\tilde{k}^0)\delta(\tilde{k}^2)n_B(\tilde{k}^0) \left( -2 \sum_s \epsilon_\lambda(\tilde{k}, s)e^\lambda(\tilde{k}, s) \right) \left( -e^{i\hat{k} \cdot x} + e^{-i\hat{k} \cdot x} \right)
\]

where \( n_B(\tilde{k}_0) = (\exp(\tilde{k}_0/T) - 1)^{-1} \).

We can now substitute \( \mathcal{F}^{(0)} \) into \( \text{(36)} \) to determine \( \mathcal{F}^{(1)} \). However, the calculation is not quite as iterative in the Wigner function approach, as it normally is in the other way of doing. We still have to evaluate some terms on the right hand side of \( \text{(36)} \) that do not involve \( \mathcal{F} \) directly. In trying to evaluate such terms, we note that, to the leading order, we can set the link operators to unity in these terms. This is easily seen from the fact that if we expand the link operators to any order in \( \epsilon^\nu \), they will involve powers of \( y \) which can be thought of as \( \frac{\partial}{\partial \epsilon^\nu} \) acting on the integral. Each power of \( y \), therefore, gives a contribution that is more and more sub-leading (since \( k \) is large) and, consequently, if we are interested only in the leading contributions, we can approximately set the link operators to unity in the second group of terms on the right hand side of \( \text{(33)} \). With this, it follows that to lowest order in \( \epsilon^\nu \), these terms give a contribution of the form

\[
-\epsilon^\nu \frac{\partial}{\partial \epsilon^\nu} \int \frac{d^4y}{(2\pi)^4} e^{-iy \cdot k} \int \frac{d^4\tilde{k}}{(2\pi)^4} \theta(\tilde{k}^0)\delta(\tilde{k}^2)n_B(\tilde{k}^0) \left( -2 \sum_s \epsilon_\lambda(\tilde{k}, s)e^\lambda(\tilde{k}, s) \right) \left( -e^{i\hat{k} \cdot x} + e^{-i\hat{k} \cdot x} \right)
\]

Here, we have identified the Abelian part of the field strength tensor as

\[
\tilde{f}_{\rho\sigma}(x) = \partial_{[\rho}A_{\sigma]}(x) - \partial_{\sigma}A_{\rho}(x) \tag{40}
\]

The thermal ensemble average in \( \text{(39)} \) can be evaluated in a straightforward manner and the temperature dependent part has the form

\[
-\epsilon^\nu \frac{\partial}{\partial \epsilon^\nu} \int \frac{d^4y}{(2\pi)^4} e^{-iy \cdot k} \int \frac{d^4\tilde{k}}{(2\pi)^4} \theta(\tilde{k}^0)\delta(\tilde{k}^2)n_B(\tilde{k}^0) \left( -2 \sum_s \epsilon_\lambda(\tilde{k}, s)e^\lambda(\tilde{k}, s) \right) \times \left( e^{i\hat{k} \cdot x} \tilde{f}_{\rho\sigma}(x) e^{-i\hat{k} \cdot x} + e^{-i\hat{k} \cdot x} \tilde{f}_{\rho\sigma}(x) e^{i\hat{k} \cdot x} \right)
\]

where the star product identity

\[
e^{i\hat{k} \cdot x} \star f(x) \star e^{-i\hat{k} \cdot x} = f(x + \theta k) \tag{42}
\]

with the identification

\[
(\theta k)^\mu = \theta^\mu_\nu k_\nu \tag{43}
\]

With the determination of the second term on the right hand side of \( \text{(36)} \) to the lowest order, we can now determine \( \mathcal{F}^{(1)} \) from \( \text{(38)} \) as
Since $\mathcal{F}^{(0)}$ is independent of $x$, its Moyal bracket with $k \cdot A(x)$ vanishes and using the form for $\mathcal{F}^{(0)}$ in (35), we determine

$$\mathcal{F}^{(1)}(x, k) = \frac{4e}{(2\pi)^3} \frac{1}{k} \frac{\partial}{\partial k} k^\rho \left( \delta(k^2)n_B(|k^0|)k^\rho (\bar{f}_{\rho \sigma}(x) - \tilde{f}_{\rho \sigma}(x + \theta k)) \right)$$

(45)

Before going onto calculate $\mathcal{F}^{(2)}$, let us discuss some of the features of the results in (35) and (44). We note that $\mathcal{F}^{(0)}(x, k)$ is manifestly covariant under a gauge transformation. However, even though we start from a gauge covariant equation, we see signs of violation of gauge covariance in the calculation of $\mathcal{F}^{(1)}(x, k)$. This is manifest more clearly in the right hand side of (41). Namely, we note that $e^{ik \cdot x} \bar{f}_{\rho \sigma}(x) e^{-ik \cdot x}$

(46)

does not transform under a gauge transformation, as we would expect. In a non-commutative gauge theory, space-time translations form a subgroup of the gauge group and, in particular, because of relations like (42), gauge covariance appears to be violated. We want to emphasize that gauge covariance is manifest before taking the ensemble average. However, the naive mean field ensemble average seems to be incompatible with gauge covariance. It is worth pointing out that this is not a problem in the usual QCD where factors such as $e^{\pm i k \cdot x}$ are ordinary functions. This, therefore, is a very special feature of the non-commutative nature of the theory and implies that, in such theories, the naive mean field ensemble average must be modified.

For lack of a more fundamental understanding of the mean field average method in such theories, we proceed as follows. We note from (42) that the ordinary plane wave function in non-commutative theories leads to a translation which does not commute with gauge transformation. The simplest way to covariantize such an expression would be to replace the coordinate in the exponent of the plane wave by a covariant coordinate which would generate a covariant translation. Such a covariant coordinate can, in fact, be uniquely determined to the leading order, from a few general conditions that we discuss in detail in appendix B. For the present, however, we simply note that we can uniquely identify the covariant coordinate with

$$X^\mu = x^\mu + e\theta^{\mu \nu} \tilde{A}_\nu(x)$$

(47)

where we identify (we will discuss this point more in detail in appendix B)

$$\tilde{A}_\mu(x) = \bar{A}_\mu(x) + \frac{1}{k \cdot D} \bar{F}_{\mu \nu}(x)k^\nu$$

(48)

Thus, we see that the simplest way to covariantize the mean field calculations in non-commutative theories is to replace

$$e^{\pm i k \cdot x} \rightarrow e^{\pm i(k \cdot x + c k \cdot \tilde{A}(x))}$$

(49)

where we have used the standard notation of non-commutative theories,

$$A \times B = \theta^{\mu \nu} A_\mu B_\nu$$

(50)

We note that such a covariantization vanishes in the usual Yang-Mills theories simply because $\theta^{\mu \nu} = 0$, but is crucial for the covariantization of the results in a non-commutative theory.

The covariantization in (10) contributes only at higher orders in $\theta^{\mu \nu}$. Therefore, $\mathcal{F}^{(1)}(x, k)$ is unaffected by this. However, in calculating $\mathcal{F}^{(2)}(x, k)$, we have to take into account contributions coming from this in order to maintain covariance under a gauge transformation. At order $\theta^{\mu \nu}$, the leading order contributions from the second term on the right hand side of (35) have the forms

\begin{align*}
2e & \frac{\partial}{\partial k^\rho} k^\rho \int \frac{d^4 y}{(2\pi)^4} e^{-i y \cdot k} \frac{1}{k^2} \\
& \times \left\{ \langle \partial_{\mu} a_\lambda(x_+) - \partial_\lambda a_{\mu}(x_+) \rangle \star \tilde{f}_{\rho \sigma}(x) \star \partial^\lambda a^\mu(x_-) \rangle \right\}^{\text{cov}} \\
& - ie \langle \langle \partial_{\mu} a_\lambda(x_+) - \partial_\lambda a_{\mu}(x_+) \rangle \star [\tilde{A}_\rho(x), \tilde{A}_\sigma(x)]_{MB} \star \partial^\lambda a^\mu(x_-) \rangle \\
& - ie \langle \langle [\tilde{A}_\mu, a_\lambda(x_+)]_{MB} - [\tilde{A}_\lambda, a_\mu(x_+)]_{MB} \rangle \star \tilde{f}_{\rho \sigma}(x) \star \partial^\lambda a^\mu(x_-) \rangle \\
& - ie \langle \langle \partial_{\mu} a_\lambda(x_+) - \partial_\lambda a_{\mu}(x_+) \rangle \star \tilde{f}_{\rho \sigma}(x) \star [\tilde{A}_\lambda, a_\mu(x_-)]_{MB} \rangle \right\}
\end{align*}

(51)
Here, \( \langle \cdots \rangle_{\text{cov}} \) stands for the linear terms in \( \mathcal{A} \) coming from the covariantization discussed in [19]. The other terms are already of order \( \sim e^2 \) so that the covariantization does not contribute in such terms at this order. Equation (51) can be evaluated in a straightforward manner to give

\[
\frac{4ie^2}{(2\pi)^3} \frac{\partial}{\partial k_\sigma} k^\rho \left( \delta(k^2)n_B(|k^0|) \right) \left\{ \left[ \frac{1}{k \cdot \partial} k \cdot (\mathcal{A}(x + \theta k) - \mathcal{A}(x)), \tilde{f}_{\rho\sigma}(x + \theta k) \right]_{\text{MB}} + \left[ \mathcal{A}_\rho(x + \theta k), \mathcal{A}_\sigma(x + \theta k) \right]_{\text{MB}} \right\} + \cdots
\]

where \( \cdots \) represent terms that are of sub-leading order. Substituting this into (33), we can now determine to leading order,

\[
\mathcal{F}^{(2)}(x, k) = ie \frac{1}{k \cdot \partial} \left[ k \cdot \mathcal{A}(x), \mathcal{F}^{(1)}(x, k) \right]_{\text{MB}} - \frac{4ie^2}{(2\pi)^3} \frac{1}{k \cdot \partial} k^\rho \left\{ \delta(k^2)n_B(|k^0|) \left[ \left[ \mathcal{A}_\rho(x), \mathcal{A}_\sigma(x) \right]_{\text{MB}} - \left[ \mathcal{A}_\rho(x + \theta k), \mathcal{A}_\sigma(x + \theta k) \right]_{\text{MB}} \right. \right. \\
+ \left. \left. \left[ \frac{1}{k \cdot \partial} k \cdot (\mathcal{A}(x) - \mathcal{A}(x + \theta k)), \tilde{f}_{\rho\sigma}(x + \theta k) \right]_{\text{MB}} \right\} \right\}
\]

(52)

This is all we need to determine the leading amplitudes up to the three point function in the hard thermal loop approximation. However, let us first note some of the basic features of these results. We note from \( \mathcal{F}^{(1)} \) and \( \mathcal{F}^{(2)} \) that these Wigner functions have the dipole structure characteristic of non-commutative theories. As we have alluded to in the introduction, the constituents of non-commutative theories can be thought of as extended particles and have a dipole structure in the charged sector. This is basically reflected in these calculations of the Wigner functions.

Given \( \mathcal{F}^{(i)}(x, k), i = 0, 1, 2 \), we can now construct the current up to third order in the coupling constant from the definition in \( \mathcal{F}^{(3)} \). In momentum space, they have the explicit forms

\[
J^{(0)}_\mu(-p_1) = 0
\]

(54a)

\[
J^{(1)}_\mu(-p_1) = -8e^2 \int dK \, n_B(k^0) \left( 1 - \cos(p_1 \times k) \right) L_{\mu\sigma}(p_1, k) L^\sigma_{\nu\rho}(p_1, k) \mathcal{A}_\nu(-p_1)
\]

(54b)

\[
J^{(2)}_\mu(-p_1) = 16ie^3 \int d^4p_2 d^4p_3 dK \, n_B(k^0) \delta(p_1 + p_2 + p_3) \sin \left( \frac{p_1 \times p_2}{2} \right) \times \left( 1 - \cos(p_3 \times k) \right) \left( L_{\mu\sigma}(p_1, k) \frac{k_\rho}{p_1 \cdot k} + L_{\nu\sigma}(p_3, k) \frac{k_\rho}{p_3 \cdot k} \right) \tilde{A}_\nu(p_2) L^\lambda_{\sigma\lambda}(p_3, k) \mathcal{A}_\lambda(p_3)
\]

(54c)

\[
\begin{align*}
&+ \left( 1 - \cos(p_1 \times k) \right) \frac{k \cdot \tilde{A}(p_3)}{p_1 \cdot k} L_{\mu\sigma}(p_1, k) L^\sigma_{\nu\rho}(p_1, k) \mathcal{A}_\nu(p_2) \\
&+ \left( \cos(p_1 \times k) - \cos(p_3 \times k) \right) \frac{p_1 \cdot k \cdot \tilde{A}(p_2)}{p_1 \cdot k \cdot p_2 \cdot k} L_{\mu\sigma}(p_1, k) L^\lambda_{\sigma\lambda}(p_3, k) \mathcal{A}_\lambda(p_3)
\end{align*}
\]

where we have defined

\[
dK = \frac{d^4k}{(2\pi)^3} \theta(k^0) \delta(k^2)
\]

(55a)

\[
L_{\mu\nu}(p, k) = \eta_{\mu\nu} - \frac{k_\mu p_\nu}{p \cdot k}
\]

(55b)
Similarly, the leading order three point amplitude is obtained to be

\[ \Pi_{\mu \nu}(p) = \frac{\delta J_\mu(-p)}{\delta A^\nu(p)} \bigg|_{A=0} = \frac{\delta J_\mu^{(1)}(-p)}{\delta A^\nu(p)} = -\frac{8 \epsilon^2}{(2\pi)^3} \int dK n_B(k^0) \left(1 - \cos(p \times k)\right) G_{\mu \nu}(p; k) \]  

(56)

where we have defined

\[ G_{\mu \nu}(p; k) = \eta_{\mu \nu} - \frac{k_\mu p_\nu + k_\nu p_\mu}{(k \cdot p)} + \frac{p^2 k_\mu k_\nu}{(k \cdot p)^2} = L_{\mu \sigma}(p, k) L_{\nu}^\sigma(p, k) \]  

(57)

Similarly, the leading order three point amplitude is obtained to be

\[
\Gamma_{\mu \nu \lambda}(p_1, p_2, p_3) = \frac{\delta^2 J_\mu(-p_1)}{\delta A^\nu(p_2) \delta A^\lambda(p_3)} \bigg|_{A=0} = \frac{\delta^2 J_\mu^{(2)}(-p_1)}{\delta A^\nu(p_2) \delta A^\lambda(p_3)} \\
= \frac{16 i \epsilon^3}{(2\pi)^3} \sin \left(\frac{p_1 \times p_2}{2}\right) \int dK n_B(k^0) \frac{1}{k \cdot p_1} \left\{ [1 - \cos(p_1 \times k)] k_\lambda G_{\mu \nu}(p_1; k) \\
+ [1 - \cos(p_3 \times k)] [k_\mu G_{\lambda \nu}(p_3; k) + k_\nu G_{\lambda \mu}(p_3; k)] G^\lambda_{\sigma \mu}(p_3; k) \\
+ [\cos(p_1 \times k) - \cos(p_3 \times k)] \frac{k_\lambda}{k \cdot p_2} k_\nu G_{\mu \sigma}(p_1; k) G^\lambda_{\sigma \nu}(p_3; k) - (p_2 \leftrightarrow p_3; \nu \leftrightarrow \lambda) \right\} . 
\]  

(58)

These amplitudes agree completely with the leading order perturbative results, arising from the gauge and the ghost loops, in the hard thermal loop approximation discussed in [26]. They are gauge independent and satisfy simple Ward identities. For example, from the structure in [57], one can easily verify the transversality of the photon self-energy

\[ p^\mu \Pi_{\mu \nu}(p) = 0, \]  

(59)

Similarly, the identity relating the two and the three point functions

\[ p_3^\lambda \Gamma_{\mu \nu \lambda}(p_1, p_2, p_3) = 2 i \epsilon \sin \left(\frac{p_1 \times p_2}{2}\right) \left[ \Pi_{\mu \nu}(p_1) - \Pi_{\nu \mu}(p_2) \right] . \]  

(60)

can also be seen to hold.

**IV. SUMMARY AND DISCUSSIONS**

In this paper, we have studied the Wigner function approach to the transport equation for photon in noncommutative QED. This is complementary to our earlier work [27] in that here we have tried to give a theoretical derivation for the transport equation. While noncommutative QED has many similarities with the conventional QCD, some differences arise in this method. In particular, we have shown that because the gauge symmetry in non-commutative QED is intermixed with space-time translations, the naive mean field ensemble average breaks down and needs modification. While we have proposed a covariantization with an interest in the leading order hard thermal loop calculations, a systematic study of this question remains to be carried out. We have shown that the leading order amplitudes resulting from the current in the Wigner function transport equation approach in non-commutative QED agree completely with the explicit perturbation calculations up to the three point amplitudes.

We can also derive the force law for a neutral noncommutative particle from our calculations. Let us recall that in the usual description of the transport equation (using dynamical equations for the particles), the current...
We see that each of the above force terms indeed has a dipole structure. However, the charge conjugation odd nature of the current converts the dipoles into a quadrupole which effectively has the structure of a pair of dipoles back to back.

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APPENDIX A: DERIVATION OF THE TRANSPORT EQUATION

In this appendix, we will give some details on the derivation of the transport equation in [24]. First, we note that, for a link operator along a straight path, we can write,

\[ U(x, x_{\pm}) = P \left( \epsilon^+ \int_0^1 dt \xi(t) \cdot A(x + \xi(t)) \right) \]
\[ = P \left( \epsilon^+ \int_0^1 dt \frac{\xi(t) \cdot A(x \pm (1-t)\xi)}{2} \right) \]  

where we have identified

\[ \xi^\mu(t) = \pm (1-t) \frac{y^\mu}{2} \]  

It follows now from its definition that, under a general change of the end points, the link operator will change as

\[ \delta U(x, x_{\pm}) = \frac{ie}{\hbar c} \int_0^1 dt U(x, x + \xi(t)) \ast \delta \left( \frac{\xi(t)}{2} \right) \ast U(x + \xi(t), x_{\pm}) \]  

With some algebraic manipulations involving integration by parts, the change can be rewritten in the form

\[ \delta U(x, x_{\pm}) = \frac{ie}{\hbar c} \delta x^\mu \left( A_\mu(x) \ast U(x, x_{\pm}) - U(x, x_{\pm}) \ast A_\mu(x_{\pm}) \right) + \frac{ie}{2\hbar c} \delta y^\mu U(x, x_{\pm}) \ast A_\mu(x_{\pm}) \]
\[ - \frac{ie}{\hbar c} \int_0^1 dt \delta(x + \xi(t)) \left[ \frac{dU(x, x + \xi(t))}{dt} \ast A_\mu(x + \xi(t)) \ast U(x + \xi(t), x_{\pm}) + U(x, x + \xi(t)) \ast A_\mu(x + \xi(t)) \ast \frac{dU(x + \xi(t), x_{\pm})}{dt} \right] \]
\[ \ast \left( \partial_\mu A_\nu(x + \xi(t)) - \partial_\nu A_\mu(x + \xi(t)) \right) \ast U(x + \xi(t), x_{\pm}) \]  

Let us next recall that, by definition,

\[ U(x, x + \xi(t)) = P \left( \epsilon^+ \int_0^1 dt \xi(t') \cdot A(x + \xi(t')) \right) \]
\[ U(x + \xi(t), x_{\pm}) = P \left( \epsilon^+ \int_0^1 dt \xi(t') \cdot A(x + \xi(t')) \right) \]
It follows from this that
\[
\frac{dU(x, x + \chi(t))}{dt} = -\frac{i e}{\hbar c} U(x, x + \chi(t)) \ast \dot{\chi}(t) \cdot A(x + \chi(t))
\]
and
\[
\frac{dU(x + \chi(t), x)}{dt} = \frac{ie}{\hbar c} \dot{\chi}(t) \cdot A(x + \chi(t)) \ast U(x + \chi(t), x)
\]
(A6)

Using these relations, the variation in (A4) can be written in the simpler form
\[
\delta U(x, x) = \frac{ie}{\hbar c} \delta x^\mu (A_\mu (x) \ast U(x, x) - U(x, x) \ast A_\mu (x)) \pm \frac{ie}{2\hbar c} \delta y^\mu U(x, x) \ast A_\mu (x) - \frac{ie}{\hbar c} \int_0^1 dt \dot{\chi}(t) \delta (x + \chi(t)) \ast U(x + \chi(t), x)
\]
(A7)

It now follows from this that, for \(\delta y^\mu = 0\) and an infinitesimal translation of \(x^\mu\), we have
\[
D_\mu(x)^2 U(x, x) = \pm \frac{i}{2 \hbar c} y^\nu \left( \int_0^1 dt \left( e^{\pm i F_{\mu \nu}(x)} \right) \right) \ast U(x, x)
\]
(A8)

where the covariant derivative of the link operator is defined in (20) and we have used the group properties of the link operators in writing the expression in the final form. The second relation in (20) can also be derived in an exactly similar manner. Given these relations, the transport equation (21) follows from the observation that
\[
D_\rho(x)^2 G_{\mu \nu}^{(\pm)}(x) = D_\rho(x)^2 U(x, x) \ast U(x, x) \ast U(x, x) \ast (D_\rho(x)^2 F_{\mu \nu}(x)) \ast U(x, x)
\]
(A9)

The simplification of the second term in (21) presented in (22) can be obtained as follows. First, we note that the factor \(k^\rho\) can be taken inside the integral as
\[
k^\rho \rightarrow i \hbar \frac{\partial}{\partial y^\rho}
\]
(A10)

Furthermore, the \(y\)-derivatives can be converted to covariant derivatives and, from (A11), we see that we can write
\[
D_\mu(y)^2 U(x, x) = D_\mu(U(x, x) \pm \frac{ie}{2\hbar c} y^\nu U(x, x) \ast A_\mu (x) = -\frac{ie}{4\hbar c} y^\nu \left( \int_0^1 dt \left( e^{\pm i F_{\mu \nu}(x)} \right) \right) \ast U(x, x)
\]

Using these as well as the identity (following from the Bianchi identity as well as the equations of motion),
\[
D^2 F_{\mu \nu} = 2 \frac{ie}{\hbar c} [F_{\mu \lambda}, F_{\nu}^\lambda]_{\text{MB}}
\]
(A12)

one obtains the expression given in Eq. (22).

**APPENDIX B: SOME USEFUL RELATIONS**

In this appendix, we will derive some other useful relations. First, let us note from (3) that
\[
[x^\mu, f(x)]_{\text{MB}} = i \theta^{\mu \nu} (\partial_\nu f)
\]
or,
\[
[-i \theta_{\mu \nu}^\pm x^\nu, f(x)]_{\text{MB}} = (\partial_\nu f)
\]
(B1)

where we are assuming that \(\theta^{\mu \nu}\) is invertible. Using this, we see that we can write a covariant derivative (in the adjoint representation) with a gauge connection \(\tilde{A}_\mu\) as
\[
(D_\mu f) = (\partial_\mu f) - \frac{ie}{\hbar c} [\tilde{A}_\mu, f]_{\text{MB}}
\]
\[
= \left[ -i \left( \theta_{\mu \nu}^{-1} x^\nu + \frac{e}{\hbar c} \tilde{A}_\mu \right), f \right]_{\text{MB}}
\]
\[
= \left[ -i \theta_{\mu \nu}^{-1} \left( x^\nu + \frac{e}{\hbar c} \theta^{\nu \lambda} \tilde{A}_\lambda \right), f \right]_{\text{MB}}
\]
(B2)

Here,
\[
X^\mu = x^\mu + \frac{e}{\hbar c} \theta^{\mu \nu} \tilde{A}_\nu (x)
\]
is known as the covariant coordinate, since its Moyal bracket leads to the covariant derivative. Using relation
It follows that

\[ e^{ik \cdot x} * f(x) * e^{-ik \cdot x} = (e^{-k \times \partial} f) \quad (B4) \]

Similarly, using relation (B2) repeatedly, it follows that

\[ e^{i(k \cdot x + \theta \times k \times \hat{A})} * f * e^{-i(k \cdot x + \theta \times k \times \hat{A})} = (e^{\theta \cdot \hat{D} f(x)}) = (e^{\hat{k} \cdot \hat{D} f(x)}) \quad (B5) \]

where we have used the conventional definition (in non-commutative field theories)

\[ \hat{k}^\mu = \theta^{\mu \nu} k_\nu \quad (B6) \]

Let us next present the arguments that determine the form of \( \hat{A}_\mu \) uniquely. First, it must transform like a gauge field and since the component of \( \tilde{\hat{A}} \) parallel to \( k^\mu \) does not enter into (B5), without loss of generality, we can identify

\[ k \cdot \tilde{\hat{A}} = k \cdot \hat{A} \quad (B7) \]

Thus, if we write

\[ \hat{A}_\mu = \tilde{\hat{A}}_\mu + Y_\mu \quad (B8) \]

it follows from (B7) that

\[ k \cdot Y = 0 \quad (B9) \]

Since both \( \tilde{\hat{A}}_\mu \) and \( \hat{A}_\mu \) transform like gauge connections, it follows that \( Y_\mu \) must transform covariantly in the adjoint representation under a gauge transformation. If we now assume that \( Y_\mu \) has no explicit \( \theta \) dependence except for the ones coming from the star products (in order to be compatible with charge conjugation), the form of \( Y_\mu \) is determined to be

\[ Y_\mu = H(k, \hat{D}, \hat{F}) \tilde{\hat{F}}_{\mu \nu} k^{\nu} \quad (B10) \]

where \( H \) is a gauge covariant function depending on \( k^\mu, \hat{D}_\mu \) and \( \tilde{\hat{F}}_{\mu \nu} \).

Since \( k^2 = 0 \), on dimensional grounds, we can write

\[ H(k, \hat{D}, \hat{F}) = h(k, \hat{D}, \hat{F}) \frac{1}{k \cdot \hat{D}} + g(k, \hat{D}, \hat{F}) \frac{1}{(k \cdot \hat{D})^2} \quad (B11) \]

where \( h, g \) are dimensionless functions. We next make an important observation following from the structure of thermal theories in the hard thermal loop expansion. It is well known that in such a case only singularities of the type \( \frac{1}{(k \cdot \hat{D})^n} \) arise. Thus, if we are interested in making contact with thermal field theories in the hard thermal loop approximation, the second structure in (B11) cannot be present. Furthermore, if we write

\[ h(k, \hat{D}, \hat{F}) = \sum_{n=0} C_n (k, \hat{D}, \hat{F}) \quad (B12) \]

with \( C_0 = \text{constant} \), it is clear that the higher order terms in the series will be more and more sub-leading by power counting. If we are only interested in the leading order terms, we can restrict ourselves to the \( n = 0 \) term in which case we can write

\[ Y_\mu = C_0 \frac{1}{k \cdot \hat{D}} \tilde{\hat{F}}_{\mu \nu} k^{\nu} \quad (B13) \]

Finally, the constant \( C_0 \) is determined by requiring that the current following from using such a \( Y_\mu \) must be integrable (so that it can be interpreted as coming from an effective action),

\[ \frac{\delta J_\mu}{\delta \hat{A}_\nu} = \frac{\delta^2 \Gamma}{\delta \hat{A}^\rho \delta \hat{A}^\nu} = \frac{\delta J_\nu}{\delta \hat{A}^\rho} \quad (B14) \]

This uniquely determines the constant \( C_0 = 1 \). It is interesting that \( C_0 = 0 \) is ruled out by the integrability of the current. Thus, we determine uniquely that

\[ \hat{A}_\mu = \tilde{\hat{A}}_\mu + \frac{1}{k \cdot \hat{D}} \tilde{\hat{F}}_{\mu \nu} k^{\nu} \quad (B15) \]

which is the result used in (45).

Finally, let us indicate how one obtains terms linear in \( \hat{A}_\mu \) from the exponents involving covariant coordinates in (51). Using (B3) and setting \( c = \hbar = 1 \), we note that to linear order in the gauge field, we can write

\[ e^{i(k \cdot x + \epsilon k \times \hat{A})} * \tilde{\hat{F}}_{\rho \sigma} (x) * e^{-i(k \cdot x + \epsilon k \times \hat{A})} \bigg|_{\text{lin}} = \left( e^{\epsilon k \cdot \hat{D} \tilde{\hat{F}}_{\rho \sigma} (x)} \right) \bigg|_{\text{lin}} = -i e^{\sum_{n=1} \frac{1}{n!} \left[ (k \cdot \partial)^{n-1} (k \cdot \hat{A}) + (k \cdot \partial)^{n-2} (k \cdot \hat{A}) (k \cdot \partial) + \cdots + (k \cdot \hat{A}) (k \cdot \partial)^{n-1} \right] \tilde{\hat{F}}_{\rho \sigma} } \quad (B16) \]
This expression is best evaluated in momentum space, where the right hand side takes the form

\[
\begin{align*}
  &= -ie \int d^4p_2 d^4p_3 \delta(p_1 + p_2 + p_3) \left( -2i \sin \left( \frac{p_2 \times p_3}{2} \right) \right) \left( \vec{k} \cdot \vec{A}(p_2) \right) \bar{f}_{\rho \sigma}(k_3) \\
  &\quad \times \sum_{n=1}^{\infty} \frac{1}{n!} \left\{ (i \vec{k} \cdot p_1)^n + (i \vec{k} \cdot p_1)^{n-1} \right. \\
  &= -2e \int d^4p_2 d^4p_3 \delta(p_1 + p_2 + p_3) \sin \left( \frac{p_1 \times p_2}{2} \right) \left( \vec{k} \cdot \vec{A}(p_2) \right) \bar{f}_{\rho \sigma}(p_3) \\
  &\quad \times \sum_{n=1}^{\infty} \frac{1}{n!} \frac{(i \vec{k} \cdot p_1)^n - (-i \vec{k} \cdot p_3)^n}{(i \vec{k} \cdot p_1) - (-i \vec{k} \cdot p_3)} \\
  &= -2ie \int d^4p_2 d^4p_3 \delta(p_1 + p_2 + p_3) \sin \left( \frac{p_1 \times p_2}{2} \right) \frac{\vec{k} \cdot \vec{A}(p_2)}{\vec{k} \cdot p_2} \bar{f}_{\rho \sigma}(p_3) \left( e^{i \vec{k} \cdot p_1} - e^{-i \vec{k} \cdot p_3} \right) \\
\end{align*}
\]  

(B17)

where

\[
\bar{f}_{\rho \sigma}(p_3) = -i \left( p_3 \rho A_\sigma(p_3) - p_3 \sigma A_\rho(p_3) \right) 
\]

(B18)

When charge conjugation is implemented in the current, the last parenthesis in (B17) becomes a difference of two cosines, as is manifested in the last term of \( J^{(2)}_\mu \) in (B11).