STOCHASTIC SOLUTIONS FOR SPACE-TIME FRACTIONAL EVOLUTION EQUATIONS ON BOUNDED DOMAIN

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Abstract. Space-time fractional evolution equations are a powerful tool to model diffusion displaying space-time heterogeneity. We prove existence, uniqueness and stochastic representation of classical solutions for an extension of Caputo evolution equations featuring nonlocal initial conditions. We discuss the interpretation of the new stochastic representation. As part of the proof a new result about inhomogeneous Caputo evolution equations is proven.

1. Introduction

It is a classical result that the solution to the standard heat equation $\partial_t u = \Delta u$, $u(0) = \phi_0$ allows the stochastic representation $u(t,x) = \mathbb{E}[\phi_0(X^{x,2}(t))]$, where $X^{x,2}$ is a Brownian motion started at $x \in \mathbb{R}^d$. Space-time fractional evolution equations (EEs) extend the heat equation by introducing space-time heterogeneity. This often is done by considering the Caputo EE $D^\beta_0 u = -(-\Delta)^{\frac{\alpha}{2}} u$, where one substitutes the local operators $\partial_t$ and $\Delta$ with fractional analogues. Respectively, the Caputo derivative $D^\beta_0 u(t) = c_\beta \int_0^t u'(r)(t-r)^{-\beta} dr$ and the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}} u(x) = \mathcal{F}^{-1}(|\xi|^{\alpha} \mathcal{F} u(\xi))(x)$, where $\beta \in (0,1)$, $\alpha \in (0,2)$, $c_\beta = \Gamma(1-\beta)^{-1}$ and $\mathcal{F}$ is the Fourier transform (for standard references see [19, 12]). It is well known that the fundamental solution to the Caputo EE is given by the law of the non-Markovian anomalous diffusion $Y^x(t) = X^{x,\alpha}(\tau_0(t))$ (see, e.g., [37]). Here $X^{x,\alpha}$ is the rotationally symmetric $\alpha$-stable Lévy process started at $x \in \mathbb{R}^d$ and $\tau_0(t)$ is the inverse process of the $\beta$-stable subordinator $-X^\beta(t)$. This beautiful formula was first observed in [41], and then proved for more general spatial operators in [3] (to be precise the time-change interpretation appeared later in [33, 36]). The process $Y^x$ displays space-heterogeneity due to the jump nature of $X^{x,\alpha}$. Also time-heterogeneity features in $Y^x$, as the time change $t \mapsto \tau_0(t)$ is constant precisely when the subordinator $t \mapsto -X^\beta(t)$ jumps, so that $t \mapsto Y^x(t)$ is trapped on such time intervals. This interesting trapping phenomenon leads to the process $Y^x$ spreading at a slower rate than $X^{x,\alpha}$. Indeed, in the physics literature
the anomalous diffusion $Y^x$ is often referred to as a sub-diffusion when $\alpha = 2$ (see, e.g., [46, 40, 31]). See [36] for a characterisation of $Y^x$ as the scaling limit of continuous time random walks with heavy-tailed waiting times. See [5] for a characterisation of $Y^x$ as the scaling limit of random conductance models or asymmetric Bouchaud’s trap models ($\alpha = 2$). See [30, 32] for sample path properties of $Y^x$, and [18, 16] for heat kernel asymptotic formulas. Existence of classical solutions for Caputo EEs is generally a subtle problem. The works [22] and [4] tackle classical solutions on unbounded domains. Meanwhile the works [17], [34], [35] and [29] consider bounded domains, and all their proofs rely on the spectral decomposition of the spatial operator. Stochastic representations for solutions to time-nonlocal equations is an active area of theoretical research (see [2, 14, 26, 16]). Partly because they provide formulas in the general absence of closed forms along with suggesting probabilistic proof methods. Moreover, such representations can be useful for particle tracking codes (see, e.g., [47]). Let us remark that Caputo EEs are applied in a variety of fields, such as physics, finance, economics, biology and hydrogeology (see, e.g., [41, 43, 44, 6, 25]).

In this work we focus on the following extension of the Caputo EE: the inhomogeneous space-time fractional EE on bounded domain with Dirichlet boundary conditions and non-local initial condition

$$
\begin{align*}
\begin{cases}
D^\beta_\infty \tilde{u}(t, x) &= \Delta_{\Omega}^{\alpha/2} \tilde{u}(t, x) + g(t, x), & \text{in } (0, T] \times \Omega, \\
\tilde{u}(t, x) &= 0, & \text{in } [0, T] \times \partial \Omega, \\
\tilde{u}(t, x) &= \phi(t, x), & \text{in } (-\infty, 0] \times \Omega,
\end{cases}
\end{align*}
$$

(1.1)

where $\Omega \subset \mathbb{R}^d$ is a regular domain, $\Delta_{\Omega}^{\alpha/2}$ is the restricted fractional Laplacian,\(^1\) and the time operator $-D^\beta_\infty$ is the generator of the inverted $\beta$-stable subordinator\(^2\)

$$
D^\beta_\infty f(t) = \int_0^\infty (f(t-r) - f(t)) \Gamma(-\beta)^{-1} dr \frac{1}{r^{1+\beta}}, \quad t \in \mathbb{R}.
$$

(1.2)

As the main result of this work we prove existence and uniqueness of classical solutions to problem (1.1) along with the stochastic representation for the solution

$$
\begin{align*}
\tilde{u}(t, x) &= \mathbb{E} \left[ \phi \left( X^{t, \beta}(\tau_0(t)), X^{x, \alpha}(\tau_0(t)) \right) \mathbf{1}_{\{\tau_0(t) < \tau_0(x)\}} \right] \\
&+ \mathbb{E} \left[ \int_0^{\tau_0(t) \wedge \tau_0(x)} g \left( X^{t, \beta}(s), X^{x, \alpha}(s) \right) ds \right],
\end{align*}
$$

(1.3)

where the processes $X^{x, \alpha}$ and $X^{t, \beta} = t + X^{\beta}$ are independent, and $\tau_0(x)$ is the first exit time of $X^{x, \alpha}$ from $\Omega$. To see why problem (1.1) extends the Caputo EE, let $\phi(t) = \phi(0)$

\(^1\)We define $\Delta_{\Omega}^{\alpha/2}$ on functions on $\Omega$, so that the Euclidean boundary $\partial \Omega$ makes sense in (1.1). In the literature the operator $\Delta_{\Omega}^{\alpha/2}$ is often defined through the application of the singular integral definition of $-(-\Delta)^{\alpha/2}$ to functions vanishing outside $\Omega$ (see, e.g., [11]).

\(^2\)The operator $D^\beta_\infty$ is often referred to as the Marchaud derivative in the fractional calculus literature (see, e.g., [22]).
for every \( t \in (-\infty, 0) \) and \( g = 0 \) in both (1.1) and (1.3). Then
\[
D_0^\beta \tilde{u}(t) = \int_0^t (\tilde{u}(t-r) - \tilde{u}(t)) \frac{\Gamma(-\beta)^{-1} dr}{r^{1+\beta}} - \frac{\phi(0) - \tilde{u}(t)}{\Gamma(1-\beta)} t^{-\beta} = D_0^\beta u(t),
\]
(1.4)
where \( u \) is the restriction of \( \tilde{u} \) to \( t \geq 0 \), and one obtains the homogeneous Caputo EE and its solution, respectively. The recent works [15] and [20] introduced a class of EEs that includes (1.1). They are motivated by the success of related nonlocal EEs arising in image processing, peridynamics and heat conduction (see, e.g., [23, 10, 45, 24]), and the general lack of alternatives to Caputo-type time-nonlocal models. Part of their intent is to introduce initial conditions on the ‘past’ \( \phi \) on \((-\infty, 0) \times \Omega \). Our stochastic solution (1.3) appears to be new, and it provides an interesting interpretation for the nonlocal initial condition \( \phi \). This is because the overshoot \( X^{t,\beta}(\tau_0(t)) \) is the holding/trapping time of the anomalous diffusion \( X^{x,\alpha}(\tau_0(t)) \). We discuss an interpretation where the values of \( \phi \) on \((-\infty, 0) \times \Omega \) describe the initial condition at time 0 with respect to the ‘depth’ of \( \Omega \), rather than the ‘past’ of \( \Omega \). To the best of our knowledge, the wellposedness of the EE (1.1) has only been tackled in a weak sense in [15] (for more general Lévy kernels in (1.2)), and in [38] for abstract Markovian generators. To see why the stochastic representation (1.3) is natural, one can formally apply the classical probabilistic intuition for elliptic boundary value problems (see, e.g., [21, Introduction, chapter 3]) to problem (1.1) rewritten as
\[
\left\{ \begin{array}{ll}
L \tilde{u} = -g, & \text{in } \Gamma, \\
\tilde{u} = \phi, & \text{in } \partial \Gamma,
\end{array} \right.
\]
(1.5)
where \( L = (-D_0^\beta + \Delta^\alpha_\Omega) \) is the generator of the process \( \{(X^{t,\beta}(s), X^{x,\alpha}(s))_{s \geq 0} \}_{s \geq 0} \) taking values in \((-\infty, T] \times \Omega, \Gamma = (0, T] \times \Omega, \text{ and } \partial \Gamma := (-\infty, 0] \times \Omega \cup [0, T] \times \partial \Omega \), with \( \phi = 0 \) on \((0, T] \times \partial \Omega \).

To prove our main result, Theorem 5.6, we derive two results of independent interest. Namely:

- **Theorem 4.6**: the stochastic representation
\[
\tilde{u}(t,x) = \mathbb{E} \left[ \phi_0(X^{x,\alpha}(\tau_0(t))) \mathbf{1}_{\{\tau_0(t) < \tau_\Omega(x)\}} \right] \\
+ \mathbb{E} \left[ \int_0^{\tau_0(t) \wedge \tau_\Omega(x)} f(X^{t,\beta}(s), X^{x,\alpha}(s)) \, ds \right],
\]
(1.6)
is the unique classical solution to the inhomogeneous Caputo EE on bounded domain
\[
\left\{ \begin{array}{ll}
D_0^\beta u(t, x) = \Delta^\alpha_\Omega u(t, x) + f(t, x), & \text{in } (0, T] \times \Omega, \\
u(t, x) = 0, & \text{in } [0, T] \times \partial \Omega, \\
u(t, x) = \phi_0(x), & \text{in } \{0\} \times \Omega;
\end{array} \right.
\]
(1.7)

- **Theorem 3.9**: the stochastic representation (1.6) is a weak solution to problem (1.7).
Let us outline our proof strategy for Theorem 5.6. By plugging the values of \( \phi \) in \( \tilde{u} \), it is not hard to show the equivalence of classical solutions to problem (1.1) and to problem (1.7) with forcing term \( f = g - D^\beta_\infty \phi \) and initial condition \( \phi_0 = \phi(0) \) (see Lemma 5.5). Moreover, a Dynkin formula argument proves that the respective stochastic representations (1.3) and (1.6) agree (see Lemma 5.1). Hence, it is enough to prove Theorem 4.6. We do so by proving Theorem 3.9 and then showing the required regularity of the candidate solution (1.6). The main feature of our regularity assumption on the data \( \phi \) and \( g \) is the differentiability in time. This is a consequence of the regularity assumption on \( f \) in Theorem 4.6, which we discuss now. Theorem 4.6 extends the proof of [17, Theorem 5.1], where problem (1.7) is treated for \( f = 0 \). This proof uses separation of variables combing eigenfunction expansions of \( \Delta_\Omega \) with Mittag-Leffler solutions to the Caputo initial value problem. Our separation of variables formula for the second term in (1.6) reads

\[
\sum_{n=1}^\infty \psi_n(x)u_n(t) = \sum_{n=1}^\infty \psi_n(x) \int_0^t \langle f(s), \psi_n \rangle(t - s)^\beta - 1 \beta E'_\beta(-\lambda_n(t - s)^\beta)ds,
\]

where \( E_\beta(t) = \sum_{k=0}^\infty t^k \Gamma(k\beta + 1)^{-1} \) is a Mittag-Leffler function, \( \{\lambda_n, \psi_n\}_{n \in \mathbb{N}} \) is the system of eigenvalues-eigenfunctions of \( \Delta_{\Omega}^\frac{\alpha}{2} \) and \( \langle \cdot, \cdot \rangle \) is the inner product on \( \Omega \). Unsurprisingly, each \( u_n \) is the solution to the inhomogeneous Caputo initial value problem \( D^\beta_0 u_n(t) = -\lambda_n u_n(t) + \langle f(t), \psi_n \rangle, u_n(0) = 0 \) (see [19, Theorem 7.2]). As \( D^\beta_0 u \) requires differentiability of \( t \mapsto u(t) \), we want to differentiate each \( t \mapsto u_n(t) \). To compensate for the singularity of the Mittag-Leffler kernel \( t^{\beta - 1} E'_\beta(-\lambda t^\beta) \) we require differentiability of \( t \mapsto f(t) \) to integrate by parts. Note that for the space fractional heat equation (\( \beta = 1 \)) the Mittag-Leffler kernel is an exponential, and so continuity of \( f \) is enough to differentiate the \( u_n \)'s. Briefly, the arguments for Theorem 3.9 reduce the Caputo EE (1.7) to a Poisson equation with zero boundary conditions on \( \{0\} \times \Omega \cup [0, T] \times \partial \Omega \) by constructing space-time sub-Feller semigroups. We rely on the fact that the generator \( -D^\beta_0 \) only requires boundary conditions on the trivial set \( \{0\} \). These arguments are an extension of the ideas in [26], and they appear versatile. For example, they can be used to prove stochastic weak solutions for problem (1.1) with general nonlocal operators in both space and time (ongoing work with the authors in [20]). As far as we know, stochastic representations for solutions such as (1.6) for time-nonlocal EEs appear in [26], meanwhile in [4] the solution is given a representation via the superposition principle. Possibly worth mentioning that we do not invoke [3, Theorem 3.1] and all our methods work for the standard Laplacian case \( \alpha = 2 \).

This work is structured as follows: in Section 2 we provide general notation and basic results about several stochastic processes obtained from \( X^{t,\beta} \) and \( X^{x,\alpha} \), with a focus on semigroup results. In Section 3 we prove Theorem 3.9. In Section 4 we prove Theorem 4.6. In Section 5 we prove that the stochastic representation (1.3) is the unique classical solution to the EE (1.1). In Section 6 we discuss an interpretation of the stochastic representation (1.3).
2. Preliminaries

2.1. General notation. We denote by \( \mathbb{N}, \mathbb{R}, \Gamma(\cdot), 1_A(\cdot), a \wedge b, a.e., \) lhs, rhs, the set of natural numbers, the \( d \)-dimensional Euclidean space, the gamma function, the indicator function of the set \( A \), the minimum between \( a, b \in \mathbb{R} \), the statements almost everywhere with respect to Lebesgue measure, left hand side, right hand side, respectively. We define the one parameter Mittag-Leffler function for \( \beta \in (0, 1) \) as \( E_\beta(t) = \sum_{k=0}^{\infty} t^k \Gamma(k\beta + 1)^{-1}, t \geq 0 \). We define the Banach spaces

\[ B(A) = \{ f : A \to \mathbb{R} \text{ is bounded and measurable} \}, \]

\[ C(K) = \{ f \in B(K) : f \text{ is continuous} \}, \]

\[ C_{\partial\Omega}(\Omega) = \{ f \in C(\overline{\Omega}) : f = 0 \text{ on } \partial\Omega \}, \]

\[ C_0([0, T]) = \{ f \in C([0, T]) : f(0) = 0 \}, \]

\[ C_c(\Omega) = \{ f \in C(\Omega) : f \text{ is continuous and vanishes at infinity} \}, \]

\[ C_c^k(\Omega) = \{ f \in C^k(\Omega) : f \text{ is } k\text{-times continuously differentiable} \}, \]

\[ C_c^\infty(\Omega) = \{ f \in C(\Omega) : f \text{ is smooth and compactly supported} \}, \]

\[ C_1([0, T]) = \{ f \in C([0, T]) : f' \in C([0, T]) \}, \]

\[ C_0^1([0, T]) = \{ f \in C_0([0, T]) : f' \in C_0([0, T]) \}, \]

\[ C_c^1((-\infty, T)) = \{ f \in C_c((-\infty, T)) : f' \in C_c((-\infty, T)) \}, \]

\[ C_c^{1,k}((0, T) \times \Omega) = \{ f \in C^k((0, T) \times \Omega) : f \text{ is } k\text{-times continuously differentiable in time and space} \}, \]

\[ C_1^{1,k}((0, T) \times \Omega) = \{ f \in C_1^{1,k}((0, T) \times \Omega) : f' \in C_1^{1,k}((0, T) \times \Omega) \text{ and } f' \text{ is compactly supported} \}, \]

\[ C_0^{1,k}([0, T] \times \Omega) = \{ f \in C_0([0, T] \times \Omega) : f \in C^{1,k}([0, T] \times \Omega) \text{ and } f' \in C_{\partial\Omega}([0, T] \times \Omega) \}, \]

\[ C_c^{1,k}((-\infty, T) \times \Omega) = \{ f \in C_c((-\infty, T) \times \Omega) : f \text{ is } k\text{-times continuously differentiable in time and space} \}, \]

all equipped with the supremum norm, where \( A \) is any subset of \( \mathbb{R}^d \), the set \( K \subset \mathbb{R}^d \) is compact, the set \( \Omega \subset \mathbb{R}^d \) is bounded and open, \( T \geq 0 \). For a function \( f : A \to \mathbb{R} \) we denote its supremum norm by either \( \|f\|_\infty \) or \( \|f\|_{C(A)} \). We define the spaces

\[ C(O) = \{ f : O \to \mathbb{R} \text{ is continuous} \}, \]

\[ C^k(O) = \{ f \in C(O) : f \text{ is } k\text{-times continuously differentiable} \}, \]

\[ C_c^k(O) = \{ f \in C(O) : f \in C^k(O) \text{ and compactly supported} \}, \]

\[ C_c^\infty(O) = \{ f \in C(O) : f \text{ is smooth and compactly supported} \}, \]

\[ C_1^1(O) = \{ f \in C_1(O) : f' \in C_1(O) \}, \]

\[ C_0^1(O) = \{ f \in C_0(O) : f' \in C_0(O) \}, \]

\[ C_c^1((-\infty, T)) = \{ f \in C_c((-\infty, T)) : f' \in C_c((-\infty, T)) \}, \]

\[ C_c^{1,k}((0, T) \times \Omega) = \{ f \in C_c((0, T) \times \Omega) : f \text{ is } k\text{-times continuously } \]

\[ \text{differentiable in time and space, respectively} \}, \]

\[ C_1^{1,k}((0, T) \times \Omega) = \{ f \in C_1((0, T) \times \Omega) : f' \in C_1((0, T) \times \Omega) \text{ and } f' \text{ is compactly supported} \}, \]

\[ C_c^{1,k}((-\infty, T) \times \Omega) = \{ f \in C_c((-\infty, T) \times \Omega) : f \text{ is } k\text{-times continuously } \]

\[ \text{differentiable in time and space} \}, \]

\[ C_1^{1,k}((-\infty, T) \times \Omega) = \{ f \in C_1((-\infty, T) \times \Omega) : f' \in C_1((-\infty, T) \times \Omega) \text{ and } f' \text{ is compactly supported} \}, \]

\[ C_c^{1,k}((0, T) \times \Omega) = \{ f \in C_c((0, T) \times \Omega) : f \text{ is } k\text{-times continuously differentiable in time and space} \}, \]

\[ C_1^{1,k}((0, T) \times \Omega) = \{ f \in C_1((0, T) \times \Omega) : f' \in C_1((0, T) \times \Omega) \text{ and } f' \text{ is compactly supported} \}, \]

\[ C_c^{1,k}((-\infty, T) \times \Omega) = \{ f \in C_c((-\infty, T) \times \Omega) : f \text{ is } k\text{-times continuously differentiable in time and space} \}, \]

\[ C_1^{1,k}((-\infty, T) \times \Omega) = \{ f \in C_1((-\infty, T) \times \Omega) : f' \in C_1((-\infty, T) \times \Omega) \text{ and } f' \text{ is compactly supported} \}, \]
where the set \( O \subset \mathbb{R}^d \) is open. We write \( C^1_\infty(O)((-\infty,T] \times \Omega) = C^1_\infty,\partial\Omega((-\infty,T] \times \Omega) \). By \((L^1(O), \| \cdot \|_{L^1(O)}), (L^2(O), \| \cdot \|_{L^2(O)})\) and \((L^\infty(O), \| \cdot \|_{L^\infty(O)})\) we mean the standard Banach spaces of real-valued Lebesgue integrable, square-integrable and essentially bounded functions on \( O \), respectively. Without risk of confusion we write \( \| \cdot \|_{L^\infty(O)} = \| \cdot \|_\infty \). For a bounded linear operator \( L : B \to \tilde{B} \), where \( B \) and \( \tilde{B} \) are Banach spaces, we denote the operator norm by \( \|L\| \). Given two sets of real-valued functions \( F \) and \( \tilde{F} \), we define \( F \cdot \tilde{F} := \{ ff : f \in F, \tilde{f} \in \tilde{F} \} \), and by \( \text{Span}\{F\} \) we mean the set of all linear combinations of functions in \( F \). The notation we use for an \( E \)-valued stochastic process started at \( x \in E \) is \( X^x = \{X^x(s)\}_{s \geq 0} \). Note that the symbol \( t \) will often be used to denote the starting point of a stochastic process with state space \( E \subset \mathbb{R} \). By a strongly continuous contraction semigroup \( P \) we mean a collection of linear operators \( P_s : B \to B \), \( s \geq 0 \), where \( B \) is a Banach space, such that \( P_{s+r} = P_s P_r \), for every \( s, r \geq 0 \), \( P_0 \) is the identity operator, \( \lim_{s \downarrow 0} P_s f = f \) in \( B \), for every \( f \in B \), and \( \sup_s \|P_s\| \leq 1 \). The generator of the semigroup \( P \) is defined as the pair \((\mathcal{L}, \text{Dom}(\mathcal{L}))\), where \( \text{Dom}(\mathcal{L}) := \{ f \in B : \mathcal{L}f := \lim_{s \downarrow 0} s^{-1}(P_s f - f) \text{ exists in } B \} \). We say that a set \( C \subset \text{Dom}(\mathcal{L}) \) is a core for \((\mathcal{L}, \text{Dom}(\mathcal{L}))\) if the generator equals the closure of the restriction of \( \mathcal{L} \) to \( C \). We say that a set \( C \subset B \) is invariant under \( P \) if \( P_s C \subset C \) for every \( s \geq 0 \). If a set \( C \) is invariant under \( P \) and a core for \((\mathcal{L}, \text{Dom}(\mathcal{L}))\), then we say that \( C \) is an invariant core for \((\mathcal{L}, \text{Dom}(\mathcal{L}))\). For a given \( \lambda \geq 0 \) we define the resolvent of \( P \) by \( (\lambda - \mathcal{L})^{-1} := \int_0^\infty e^{-\lambda s} P_s ds \), and recall that for \( \lambda > 0 \), \( (\lambda - \mathcal{L})^{-1} : B \to \text{Dom}(\mathcal{L}) \) is a bijection and it solves the abstract resolvent equation

\[ \mathcal{L}(\lambda - \mathcal{L})^{-1} f = \lambda(\lambda - \mathcal{L})^{-1} f - f, \quad f \in B, \]

see for example [21, Theorem 1.1]. By a sub-Feller semigroup we mean a strongly continuous contraction semigroup on any of the Banach spaces of continuous functions defined above such that \( P \) preserves non-negative functions. A Feller semigroup is a sub-Feller semigroup such that its extension to bounded measurable functions preserves constants.

### 2.2. Fractional derivatives, stable processes and related space-time semigroups.

**Definition 2.1.** For parameters \( \beta \in (0,1) \) and \( \alpha \in (0,2) \), we define: the \( \beta \)-stable generator \(-D^\beta_\infty\) by formula (1.2); the Caputo operator \( D_0^\beta \) by

\[ D_0^\beta f(t) = \int_0^t (f(t - r) - f(t)) \frac{\Gamma(-\beta)^{-1}dr}{r^{1+\beta}} + (f(0) - f(t)) \int_t^\infty \frac{\Gamma(-\beta)^{-1}dr}{r^{1+\beta}}, \quad t > 0, \]

and \( D_0^\beta f(0) = \lim_{t \downarrow 0} D_0^\beta f(t) \); the restricted fractional Laplacian \( \Delta_\Omega^\alpha,\partial \Omega \) by

\[ \Delta_\Omega^\alpha f(x) = \lim_{\varepsilon \downarrow 0} \int_{\Omega \setminus B_\varepsilon(x)} (f(y) - f(x)) \frac{c_{\alpha,d}dy}{|x - y|^{d+\alpha}} - f(x) \int_{\mathbb{R}^d \setminus \Omega} \frac{c_{\alpha,d}dy}{|x - y|^{d+\alpha}}, \quad x \in \Omega, \]

and \( \Delta_\Omega^\alpha f(z) = \lim_{x \to z} \Delta_\Omega^\alpha f(x) \) for \( z \in \partial \Omega \), where \( c_{\alpha,d}^{-1} = \int_{\mathbb{R}^d} \frac{1 - \cos y}{|y|^{d+\alpha}} dy \), \( \| \cdot \| \) denotes the Euclidean norm on \( \mathbb{R}^d \) and \( B_\varepsilon(x) \) denotes the Euclidean ball of radius \( \varepsilon > 0 \) around \( x \in \Omega \).
We now define several sub-Feller semigroups that relate to the operators in Definition 2.1 and collect some results relevant for us.

**Definition 2.2.** For \( \beta \in (0, 1) \), we denote by \( X^{t,\beta} = \{X^{t,\beta}(s)\}_{s \geq 0} \) the inverted \( \beta \)-stable subordinator started at \( t \in \mathbb{R} \), characterised by the Laplace transforms \( \mathbb{E}[e^{kX^{0,\beta}(s)}] = e^{-k^\beta s} \), \( k, s \geq 0 \). We denote the law of \( X^{t,\beta}(s) \) by \( \mathbb{P}[X^{t,\beta}(s) \in dr] = p^{\beta}_0(t-r)dr, \ s > 0 \), recalling that the smooth density \( p^{\beta}_0 \) is supported on \( (0, \infty) \). We define the exit/first passage times \( \tau_0(t) = \inf\{s > 0 : -X^{0,\beta}(s) \geq t\}, t \in \mathbb{R} \).

**Definition 2.3.** For \( \alpha \in (0, 2) \), we denote by \( X^{x,\alpha} = \{X^{x,\alpha}(s)\}_{s \geq 0} \) the rotationally symmetric \( \alpha \)-stable Lévy process with values in \( \mathbb{R}^d \), started at \( x \in \mathbb{R}^d \), with characteristic functions \( \mathbb{E}[e^{ikX^{0,\alpha}(s)}] = e^{-s|k|^{\alpha}}, k \in \mathbb{R}^d, s \geq 0 \). Recall that the law of \( X^{x,\alpha}(s) \) is smooth for each \( s > 0 \) (see for example [12, page 10]). We define the exit times \( \tau_\Omega(x) = \inf\{s > 0 : X^{x,\alpha}(s) \notin \Omega\}, x \in \mathbb{R}^d \).

**Proposition 2.4.** Fix \( T > 0 \). For the the inverted \( \beta \)-stable subordinator \( X^{t,\beta} \), define the Feller semigroup \( P^{\beta,\infty} = \{P^{\beta}_s\}_{s \geq 0} \) on \( C^\infty((\infty, T)) \), by \( P^{\beta,\infty}_s f(t) := \mathbb{E}[f(X^{t,\beta}(s))] \), \( s \geq 0 \), denote by \( (\mathcal{L}_\beta^{\infty}, \text{Dom}(\mathcal{L}_\beta^{\infty})) \) the generator of \( P^{\beta,\infty} \), and recall that \( C^\infty_1((\infty, T)) \) is an invariant core for \((\mathcal{L}_\beta^{\infty}, \text{Dom}(\mathcal{L}_\beta^{\infty})) \) with \( \mathcal{L}^{\infty}_\beta = -D_0^\beta \) on \( C^1_\infty((\infty, T)) \).

(i) Define the absorbed process \( X^{t,\beta}_0 \) by

\[
X^{t,\beta}_0(s) := \begin{cases} X^{t,\beta}(s), & \text{if } s < \tau_0(t), \\ 0, & \text{if } s \geq \tau_0(t). \end{cases} \tag{2.1}
\]

Then the process \( X^{t,\beta}_0 \) induces a Feller semigroup on \( C([0, T]) \), denoted by \( P^\beta = \{P^\beta_s\}_{s \geq 0} \) with generator \( (\mathcal{L}_\beta, \text{Dom}(\mathcal{L}_\beta)) \). Moreover, \( C^\infty_1([0, T]) \) is an invariant core for \( (\mathcal{L}_\beta, \text{Dom}(\mathcal{L}_\beta)) \) and

\[
\mathcal{L}_\beta = -D_0^\beta \quad \text{on} \quad C^\infty_1([0, T]).
\]

(ii) The sub-Feller semigroup \( P^{\beta,\text{kill}} := P^\beta \) on \( C_0([0, T]) \) is the the sub-Feller semigroup induced by the killed version of the process \( X^{t,\beta} \), and its generator is \( (\mathcal{L}_\beta^{\text{kill}}, \text{Dom}(\mathcal{L}_\beta^{\text{kill}})) = (\mathcal{L}_\beta, \text{Dom}(\mathcal{L}_\beta) \cap \{f(0) = 0\}) \). Moreover, \( C^1_0([0, T]) \) is an invariant core for \( (\mathcal{L}_\beta^{\text{kill}}, \text{Dom}(\mathcal{L}_\beta^{\text{kill}})) \) and

\[
\mathcal{L}_\beta^{\text{kill}} = -D_0^\beta \quad \text{on} \quad C^1_0([0, T]).
\]

(iii) The following three identities hold

\[
\mathbb{E}[	au_0(t)] = \frac{t^\beta}{\Gamma(\beta + 1)}, \quad \mathbb{E}[e^{-\lambda \tau_0(t)}] = E_\beta(-\lambda t^\beta), \ t, \lambda \geq 0, \ \text{and} \tag{2.2}
\]

\[
\int_0^\infty p^\beta_0(t-r)dr = \frac{(t-r)^{\beta-1}}{\Gamma(\beta)}, \ t > r. \tag{2.3}
\]
(iv) The alternative representation of Caputo derivatives

$$D_0^\beta u(t) = \int_0^t u'(r) \frac{(t-r)^{-\beta} dr}{\Gamma(1-\beta)}, \quad \text{for } 0 < t < T,$$

holds if \( u \in C([0, T]) \cap C^1((0, T)) \) and \( u' \in L^1((0, T)) \).

**Proof.**

(i) It is easy to prove that \( P_s^\beta f(t) := \int_0^t f(r)p_s^\beta(t-r)dr + f(0)\int_{-\infty}^0 p_s^\beta(t-r)dr \) is a Feller semigroup on \( C([0, T]) \), and the corresponding process is indeed \( X_t^\beta \). By using the proof\(^3\) of \([7, \text{Proposition 14}]\) with \( c_+ = \Gamma(-\alpha)^{-1} \) and \( c_- = 0 \), it holds that \( C^1([0, T]) \subset \text{Dom}(\mathcal{L}_\beta) \), and that \( \mathcal{L}_\beta = -D_0^\beta \) on \( C^1([0, T]) \). To prove that \( C^1([0, T]) \) is invariant under \( P_s^\beta \), we directly compute for \( g \in C^1([0, T]), t \in (0, T) \) and \( s > 0 \),

\[
\partial_t P_s^\beta g(t) = \partial_t \left( \int_0^t g(t-r)p_s^\beta(r)dr + g(0)\int_{-\infty}^t p_s^\beta(-r)dr \right)
\]

\[
= \int_0^t g'(t-r)p_s^\beta(r)dr \pm g(0)p_s^\beta(t).
\]

Then \( C^1([0, T]) \) is a dense subspace of \( \text{Dom}(\mathcal{L}_\beta) \) which is invariant under \( P_s^\beta \), and so it is a core for \((\mathcal{L}_\beta, \text{Dom}(\mathcal{L}_\beta))\) by \([13, \text{Lemma 1.34}]\).

(ii) Similarly to part (i), it can be shown that \( P_{s, \text{kill}}^\beta f(t) = \int_0^t f(r)p_{s, \text{kill}}^\beta(t-r)dr \). To show \( \text{Dom}(\mathcal{L}_\beta) \cap \{f(0) = 0\} \subset \text{Dom}(\mathcal{L}_{\beta, \text{kill}}) \), let \( f \in \text{Dom}(\mathcal{L}_\beta) \cap \{f(0) = 0\} \), then for some \( \lambda > 0 \), let \( g \in C([0, T]) \) such that

\[
f(t) = \int_0^\infty e^{-\lambda s} P_s^\beta g(t)ds, \quad \text{and} \quad g(0)\frac{1}{\lambda} = \int_0^\infty e^{-\lambda s} P_s^\beta g(0)ds = f(0) = 0,
\]

and so \( g \in C_0([0, T]) \). As \( P_s^\beta = P_{s, \text{kill}}^\beta \) on \( C_0([0, T]) \), it follows that \( f \in \text{Dom}(\mathcal{L}_{\beta, \text{kill}}) \). The inclusion \( \text{Dom}(\mathcal{L}_\beta) \cap \{f(0) = 0\} \supset \text{Dom}(\mathcal{L}_{\beta, \text{kill}}) \) is immediate using \( P_{s, \text{kill}}^\beta = P_{s, \text{kill}}^{\beta, \text{kill}} \) on \( C_0([0, T]) \). By equating a resolvent equation, it follows that \( \mathcal{L}_{\beta, \text{kill}} = \mathcal{L}_\beta \) on \( \text{Dom}(\mathcal{L}_{\beta, \text{kill}}) \). Invariance of \( C_0^1([0, T]) \) can be proven as in part (i). The last statement now follows from part (i).

(iii) The first identity follows from the third identity \([2.3]\). The second identity follows by \([18, \text{Theorem 2.10.2}]\). To prove the third identity \([2.3]\), recall that

\[
p_s^\beta(t-r) = s^{-1/\beta} p_1^\beta(s^{-1/\beta}(t-r)), \quad t > r,
\]

and then compute

\[
\int_0^\infty p_s^\beta(t, r)ds = (t-r)^{\beta - 1} \int_0^\infty u^{-1/\beta} p_1^\beta(u^{-1/\beta})du = (t-r)^{\beta - 1} \frac{1}{\Gamma(\beta)},
\]

\(^3\) In the statement of \([7, \text{Proposition 14}]\) it is required that \( F \in C^2([0, \infty)) \), but \( F \in C^1([0, \infty)) \) is enough.
using the Mellin transform of the $\beta$-stable density $p^\beta_1$ for the last equality (see for example [43] Theorem 2.6.3).

(iv) This is a standard computation and we omit it.

We say that a bounded open set $\Omega \subset \mathbb{R}^d$ is a regular set if $\Omega$ satisfies the exterior cone condition at every point $\partial \Omega$, i.e. for each $x \in \partial \Omega$ there exists a finite right circular open cone $V_x$ with vertex $x$, such that $V_x \subset \Omega^c$ (see [17] end of Section 4). From now on $\Omega$ is always a regular set.

**Proposition 2.5.** Define the sub-process $X_{\Omega}^{x,\alpha}$ started at $x \in \Omega$ by

$$X_{\Omega}^{x,\alpha}(s) := \begin{cases} X^{x,\alpha}(s), & s < \tau_{\Omega}(x), \\ \text{cemetery}, & s \geq \tau_{\Omega}(x), \end{cases}$$

(i) Then $X_{\Omega}^{x,\alpha}$ induces a sub-Feller semigroup on $C_{\partial \Omega}(\Omega)$, which we denote by $P_{\Omega} = \{P_{\Omega}^s\}_{s \geq 0}$, and we denote its generator by $(\mathcal{L}_{\Omega}, \text{Dom}(\mathcal{L}_{\Omega}))$. Moreover if $u \in \text{Dom}(\mathcal{L}_{\Omega})$ then there exists a sequence $u_n \in C_{\partial \Omega}(\Omega) \cap C^2(\Omega)$ such that $u_n \to u$ uniformly and $\Delta_{\Omega}^{\alpha} u_n \to \mathcal{L}_{\Omega} u$ uniformly on compact subsets of $\Omega$. The transition density of $X_{\Omega}^{x,\alpha}(s)$, denoted by $p_{\Omega}^{x,\alpha}(s)$, is jointly continuous in $x$ and $y$, for every $s > 0$.

(ii) For every $u \in \text{Dom}(\mathcal{L}_{\Omega})$ and $\varphi \in C^2(\Omega)$ it holds

$$\int_{\Omega} \mathcal{L}_{\Omega} u \varphi dx = \int_{\Omega} u \Delta_{\Omega}^{\alpha} \varphi dx. \quad (2.4)$$

(iii) The semigroup $P_{\Omega}$ induces a strongly continuous contraction semigroup on $L^2(\Omega)$, and we denote its generator by $(\mathcal{L}_{\Omega,2}, \text{Dom}(\mathcal{L}_{\Omega,2}))$. Moreover there exists a sequence of positive numbers $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$, and an orthonormal basis $\{\psi_n\}_{n \in \mathbb{N}}$ of $L^2(\Omega)$, so that $P_{\Omega}^s \psi_n = e^{-\lambda_n s} \psi_n$ in $L^2(\Omega)$, for every $n \in \mathbb{N}$, $s > 0$. For $k \geq 1$, we denote by $\text{Dom}(\mathcal{L}_{\Omega,2}^k)$ the subset of $L^2(\Omega)$ such that $\|f\|_{\mathcal{L}_{\Omega,2}^k} := \left( \sum_{n=1}^{\infty} \lambda_n^{2k} \langle f, \psi_n \rangle^2 \right)^{1/2} < \infty$. Moreover, $P_{\Omega}$ on $C_{\partial \Omega}(\Omega)$ has the same set of eigenvalues and eigenfunctions as $P_{\Omega}$ on $L^2(\Omega)$.

**Proof.** (i) The first two statements are a consequence of [2] Lemma 2.2 and Theorem 2.7]. The last statement follows by the strong Markov property along with joint continuity of the transition densities of $X^{x,\alpha}$ (see for example [17] Section 4).

(ii) The operator $\Delta_{\Omega}^{\alpha}$ is self-adjoint in the sense that

$$\int_{\Omega} \Delta_{\Omega}^{\alpha} u \varphi dx = \int_{\Omega} u \Delta_{\Omega}^{\alpha} \varphi dx, \quad (2.5)$$

if $\varphi \in C^2(\Omega)$ and $u \in C_{\partial \Omega}(\Omega) \cap C^2(\Omega)$. Now use the approximating sequence from part (i) of the current proposition to conclude.
(iii) These results can be found in [17], Section 4 and references therein.

□

In the next lemma we construct three sub-Feller semigroups by combining in space-time the
sub-Feller semigroups defined so far. We combine them in a way that allows us to describe
the newly constructed space-time generator as the closure of the sum of the time and space
generators. This is how we give meaning to the boundary value problem viewpoint formally
presented in [15].

Lemma 2.6. Consider the four tuples

\((P^{\beta,\infty}, C_\infty((-\infty, T]), L_\beta^\infty, \text{Dom}(L_\beta^\infty)), \quad (P^\beta, C([0, T]), L_\beta, \text{Dom}(L_\beta)),\)

\((P^{\beta,\text{kill}}, C_0([0, T]), L_\beta^\text{kill}, \text{Dom}(L_\beta^\text{kill})), \quad (P^\Omega, C_{\partial \Omega}(\Omega), L_\Omega, \text{Dom}(L_\Omega)),\)

defined in Proposition 2.4, Proposition 2.4-(i), Proposition 2.4-(ii) and Proposition 2.5-(i),
respectively. Let \(L_\beta^\infty, C_\beta, L_\beta^\text{kill}, \text{Dom}(L_\beta^\text{kill}), C_{\partial \Omega}(\Omega), L_\Omega, \text{Dom}(L_\Omega)\) be invariant cores for \((L_\beta^\infty, \text{Dom}(L_\beta^\infty)), (L_\beta, \text{Dom}(L_\beta)), (L_\beta^\text{kill}, \text{Dom}(L_\beta^\text{kill})), (L_\Omega, \text{Dom}(L_\Omega))\), respectively.

(i) Then \(P^{\beta,\Omega} = \{P_s^\beta P_s^\Omega\}_{s \geq 0}\) is a sub-Feller semigroup on \(C_{\partial \Omega}([0, T] \times \Omega)\). The generator \((L_{\beta,\Omega}, \text{Dom}(L_{\beta,\Omega}))) of \(P^{\beta,\Omega}\) is the closure of

\((L_\beta + L_\Omega, \text{Span}\{C_\beta \cdot C_\Omega\})\) in \(C_{\partial \Omega}([0, T] \times \Omega),\)

(where \(P^\beta\) and \(L_\beta\) act on the \([0, T]\)-variable, and \(P^\Omega\) and \(L_\Omega\) act on the \(\Omega\)-variable).

(ii) Then \(P^{\beta,\Omega,\text{kill}} = \{P_s^{\beta,\text{kill}} P_s^\Omega\}_{s \geq 0}\) is a sub-Feller semigroup on \(C_{0,\partial \Omega}([0, T] \times \Omega)\). The generator \((L_{\beta,\Omega}^\text{kill}, \text{Dom}(L_{\beta,\Omega}^\text{kill}))) of \(P^{\beta,\Omega,\text{kill}}\) is the closure of

\((L_{\beta}^\text{kill} + L_\Omega, \text{Span}\{C_{\beta}^\text{kill} \cdot C_\Omega\})\) in \(C_{0,\partial \Omega}([0, T] \times \Omega),\)

(where \(P^{\beta,\text{kill}}\) and \(L_{\beta}^\text{kill}\) act on the \([0, T]\)-variable, and \(P^\Omega\) and \(L_\Omega\) act on the \(\Omega\)-variable).

(iii) Then \(P^{\beta,\infty,\Omega} = \{P_s^{\beta,\infty} P_s^\Omega\}_{s \geq 0}\) is a sub-Feller semigroup on \(C_{\infty,\partial \Omega}((-\infty, T] \times \Omega)\). The generator \((L_{\beta,\Omega}^\infty, \text{Dom}(L_{\beta,\Omega}^\infty))) of \(P^{\beta,\infty,\Omega}\) is the closure of

\((L_{\beta}^\infty + L_\Omega, \text{Span}\{C_{\beta}^\infty \cdot C_\Omega\})\) in \(C_{\infty,\partial \Omega}((-\infty, T] \times \Omega),\)

(where \(P^{\beta,\infty}\) and \(L_{\beta}^\infty\) act on the \((-\infty, T]\)-variable, and \(P^\Omega\) and \(L_\Omega\) act on the \(\Omega\)-variable).

(iv) It holds that \(P_s^{\beta,\Omega} = P_s^{\beta,\text{kill}} P_s^\Omega\) on \(C_{0,\partial \Omega}([0, T] \times \Omega)\), \(L_{\beta,\Omega} = L_{\beta,\Omega}^\text{kill}\) on \(\text{Dom}(L_{\beta,\Omega}^\text{kill})\), and \(\text{Dom}(L_{\beta,\Omega}^\text{kill}) = \text{Dom}(L_{\beta,\Omega}) \cap \{f(0) = 0\}\).

Proof. The proofs of (i), (ii) and (iii) can be found in Appendix A.II.
(iv) The first claim is an immediate consequence of \( P^{\beta, \text{kill}} = P^\beta \) on \( C_0([0, T]) \). The second claim follows from the third by considering a resolvent equation. To prove the third claim, we show the equivalent statement

\[
\text{Dom}(\mathcal{L}^{\text{kill}}_{\beta, \Omega}) \subset \text{Dom}(\mathcal{L}_{\beta, \Omega}), \quad \text{and if } u \in \text{Dom}(\mathcal{L}_{\beta, \Omega}), \text{ then } u - u(0) \in \text{Dom}(\mathcal{L}^{\text{kill}}_{\beta, \Omega}).
\]

The first inclusion is immediate using \( P^\beta_{s, \Omega} = P^\beta_{s, \text{kill}, \Omega} \), on \( C_0([0, T] \times \Omega) \). For the second part, let \( u \in \text{Dom}(\mathcal{L}_{\beta, \Omega}) \) and consider its resolvent representation for some \( \lambda > 0 \) and \( g \in C_{\partial \Omega}([0, T] \times \Omega) \). Then

\[
u(0, x) = \int_0^\infty e^{-\lambda s} P^\beta_s P^\Omega(g(0), x) ds = \int_0^\infty e^{-\lambda s} P^\beta_s P^\Omega(g(0))(t, x) ds,
\]
as \( P^\beta g(0, x) = P^\beta(g(0))(t, x) \). Now consider

\[
u(t, x) - u(0, x) = \int_0^\infty e^{-\lambda s} P^\beta_s P^\Omega(g(g(0)))(t, x) ds = \int_0^\infty e^{-\lambda s} P^\beta_s P^\beta_{s, \text{kill}}(g(0))(t, x) ds \in \text{Dom}(\mathcal{L}^{\text{kill}}_{\beta, \Omega}),
\]
where we use the fact that \( P^\beta_{s, \text{kill}} = P^\beta \) on \( C_{\partial \Omega}([0, T] \times \Omega) \) and that \( g - g(0) \in C_{\partial \Omega}([0, T] \times \Omega) \). \( \square \)

**Remark 2.7.** Note that

\[
(\mathcal{L}^{\text{kill}}_{\beta, \Omega})^{-1} g(t, x) = \int_0^\infty P^\beta_{s, \Omega} g(t, x) ds = E \left[ \int_0^{\tau_0(t) \wedge \tau_\Omega(x)} g(X^{t, \beta}(s), X^{x, \alpha}(s)) ds \right],
\]
for \( g \in C_{\partial \Omega}([0, T] \times \Omega) \). Also, from now on we might write \( \tau_{t, x} \) for \( \tau_0(t) \wedge \tau_\Omega(x) \).

3. **Stochastic weak solution for problem** (1.7)

3.1. **Definition of weak solution.** Define the operator

\[
-D^{\beta, \ast}_0 \varphi(s) := \partial_s I^{1-\beta}_T \varphi(s) + \delta_0(ds) I^{1-\beta}_T \varphi(0),
\]
where \( \delta_0 \) is the delta-measure at 0, and the Riemann-Liouville integral \( I^{1-\beta}_T \) is defined as

\[
I^{1-\beta}_T f(s) := \int_s^T f(t) (t-s)^{-\beta} dt \frac{1}{\Gamma(1-\beta)}, \quad s < T.
\]

In the current section only the pairing \( \langle \cdot, \cdot \rangle \) is defined as

\[
\langle f, g \rangle := \int_0^T \int_\Omega f(t, x) g(t, x) dx dt.
\]

**Definition 3.1.** Let \( f \in L^\infty((0, T) \times \Omega) \) and \( \phi_0 \in C_{\partial \Omega}(\Omega) \). A function \( u \in L^2((0, T) \times \Omega) \) is said to be a **weak solution to problem** (1.7) if

\[
\langle u, (-D^{\beta, \ast}_0 + \Delta^\beta_0) \varphi \rangle = \langle -f, \varphi \rangle, \quad \text{for every } \varphi \in C^{1,2}_c((0, T) \times \Omega),
\]

(3.1)
and \( u(t) \to \phi_0 \) a.e. as \( t \downarrow 0 \).

The next proposition motivates Definition 3.1.

**Proposition 3.2.** Let \( \varphi \in C^1_c((0, T)) \) and \( u \in C([0, T]) \cap C^1((0, T)) \) such that \( u' \in L^1((0, T)) \). Then

\[
\int_0^T D^\beta_0 u(t) \varphi(t) dt = -\int_0^T u(t) \left( \partial_t I^{1-\beta}_T \varphi(t) \right) dt - u(0) \partial_t I^{1-\beta}_T \varphi(0).
\]

**Proof.** Using Proposition 2.4-(iv), Fubini’s Theorem and integration by parts, compute

\[
\int_0^T D^\beta_0 u(t) \varphi(t) dt = \int_\mathbb{R} \int_\mathbb{R} u'(s) \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)} \varphi(t) 1_{\{0 \leq t \leq T\}} 1_{\{0 \leq s \leq t\}} ds dt
\]

\[
= \int_\mathbb{R} u'(s) 1_{\{0 \leq s \leq T\}} \left( \int_s^T \frac{(t-s)^{-\beta}}{\Gamma(1-\beta)} \varphi(t) dt \right) ds
\]

\[
= \int_0^T u'(s) I^{1-\beta}_T \varphi(s) ds
\]

\[
= -\int_0^T u(s) \partial_s I^{1-\beta}_T \varphi(s) ds - u(0) I^{1-\beta}_T \varphi(0).
\]

\( \square \)

From Proposition 3.2 and the identity in (2.5), it is straightforward to prove the following lemma.

**Lemma 3.3.** Let \( \varphi \in C^1_c((0, T) \times \Omega) \) and \( u \in C(\partial \Omega([0, T] \times \Omega) \cap C^1((0, T) \times \Omega) \) such that \( \partial_t u \in L^1((0, T) \times \Omega) \). Then

\[
\langle u, (-D^\beta_0 + \Delta^\frac{\alpha}{2}_\Omega) \varphi \rangle = \langle (-D^\beta_0 + \Delta^\frac{\alpha}{2}_\Omega) u, \varphi \rangle.
\]

### 3.2. Existence of a weak solution

Following [26], we define two auxiliary notions of solution for problem (1.7), starting from the abstract evolution equation

\[
\mathcal{L}_{\beta, \Omega} u = -f \text{ on } (0, T) \times \Omega, \quad u = \phi_0 \text{ on } \{0\} \times \Omega, \quad u \in \text{Dom}(\mathcal{L}_{\beta, \Omega}). \tag{3.2}
\]

**Definition 3.4.** Let \( f \in C(\partial \Omega([0, T] \times \Omega) \) and \( \phi_0 \in \text{Dom}(\mathcal{L}_{\Omega}) \) such that \( f(0) = -\mathcal{L}_{\Omega} \phi_0 \). We say that a function \( u \in C(\partial \Omega([0, T] \times \Omega) \) is a solution in the domain of the generator to problem (1.7) if \( u \) satisfies (3.2).

The next solution concept for problem (1.7) is defined as a pointwise approximation of solutions in the domain of the generator \( \{u_n\}_{n \in \mathbb{N}} \) such that the approximating forcing term \( \{f_n\}_{n \in \mathbb{N}} \) satisfies a dominated convergence type of condition.
Definition 3.5. Let \( f \in B([0,T] \times \overline{\Omega}) \) and \( \phi_0 \in \text{Dom}(L_\Omega) \). We say that a function \( u \in B([0,T] \times \overline{\Omega}) \) is a generalised solution to problem (1.7) if

\[
    u = \lim_{n \to \infty} u_n \quad \text{pointwise},
\]

where each \( u_n \) is the solution in the domain of the generator for a corresponding forcing term \( f_n \in C_{\partial \Omega}([0,T] \times \Omega) \) such that

\[
    f_n \to f \quad \text{a.e. on} \quad (0,T] \times \Omega, \quad \sup_n \| f_n \|_\infty < \infty, \quad \text{and} \quad f_n(0) = -L_\Omega \phi_0 \quad \text{for each} \quad n \in \mathbb{N}.
\]

Remark 3.6. Any generalised solution must satisfy the boundary conditions \( u = 0 \) on \([0,T] \times \partial \Omega\) and \( u = \phi_0 \) on \( \{0\} \times \overline{\Omega} \).

Lemma 3.7. Let \( \phi_0 \in \text{Dom}(L_\Omega) \). Then

(i) If \( f + L_\Omega \phi_0 \in C_{0,\partial \Omega}([0,T] \times \Omega) \), then there exists a unique solution in the domain of the generator to problem (1.7).

(ii) If \( f \in B([0,T] \times \overline{\Omega}) \), then there exists a unique generalised solution to problem (1.7).

(iii) Both solutions in part (i) and (ii) allow the stochastic representation (1.6).

Proof. (i) Observe that the potential \((-L_{\beta,\Omega}^{\text{kill}})^{-1}\) maps \( C_{0,\partial \Omega}([0,T] \times \Omega) \) to itself. This follows from \( P_s^{\beta,\Omega,\text{kill}} g \in C_{0,\partial \Omega}([0,T] \times \Omega) \) for \( g \in C_{0,\partial \Omega}([0,T] \times \Omega) \), \( s \geq 0 \), and Dominated Convergence Theorem (DCT) with dominating function \( G(s) := \| g \|_\infty P [ s < \tau_0(T) ] \). Note that we use the first identity in (2.2) to prove that \( G \in L^1((0,\infty)) \). The potential \((-L_{\beta,\Omega}^{\text{kill}})^{-1}\) is also bounded by the inequality

\[
    \left| (-L_{\beta,\Omega}^{\text{kill}})^{-1} g(t,x) \right| \leq \| g \|_\infty \mathbb{E}[\tau_0(T)], \quad g \in C_{0,\partial \Omega}([0,T] \times \Omega).
\]

It then follows by [21, Theorem 1.1] that \( \bar{u} := (-L_{\beta,\Omega}^{\text{kill}})^{-1} (f + L_\Omega \phi_0) \) is the unique solution to the abstract evolution equation

\[
    L_{\beta,\Omega} \bar{u} = -(f + L_\Omega \phi_0) \quad \text{on} \quad (0,T] \times \overline{\Omega}, \quad \bar{u} = 0 \quad \text{on} \quad \{0\} \times \overline{\Omega}, \quad \text{and} \quad \bar{u} \in \text{Dom}(L_{\beta,\Omega}^{\text{kill}}) \quad (3.3)
\]

It is now enough to show that \( \bar{u} \) satisfies (3.3) if and only if \( u = \bar{u} + \phi_0 \) satisfies (3.2). For the ‘if’ direction, let \( u \in \text{Dom}(L_{\beta,\Omega}) \) satisfy (3.2). Note that \( u(0) = \phi_0 \). Then \( \tilde{u} := u - \phi_0 \in \text{Dom}(L_{\beta,\Omega}^{\text{kill}}) \), and \( L_{\beta,\Omega} \tilde{u} = L_{\beta,\Omega}^{\text{kill}} \tilde{u} \), by Lemma 2.6(iv). So we can compute

\[
    L_{\beta,\Omega}^{\text{kill}} \tilde{u} = L_{\beta,\Omega}(u - \phi_0) = L_{\beta,\Omega} u - L_{\Omega} \phi_0 = -f - L_{\Omega} \phi_0,
\]

where we use

\[
    L_{\beta,\Omega} \phi_0 = (L_{\beta} + L_{\Omega}) \phi_0 = L_{\Omega} \phi_0,
\]

from Lemma 2.6(i) taking the invariant cores \( C_\beta = \text{Dom}(L_{\beta}) \) and \( C_\Omega = \text{Dom}(L_{\Omega}) \) (recalling that \( L_{\beta} 1 = 0 \)). For the ‘only if’ direction, let \( \bar{u} \) satisfy (3.3), and define \( u := \bar{u} + \phi_0 \). Then with the same justifications as just above, compute

\[
    L_{\beta,\Omega} u = L_{\beta,\Omega}^{\text{kill}} \bar{u} + L_{\beta,\Omega} \phi_0 = -(f + L_\Omega \phi_0) + L_\Omega \phi_0 = -f.
\]
It follows that
\[ u = (-\mathcal{L}_{\beta,\Omega}^{\text{kill}})^{-1}(f + \mathcal{L}_{\Omega}\phi_0) + \phi_0. \]

(ii) Let \( f \in B([0, T] \times \Omega) \). Then \( f + \mathcal{L}_{\Omega}\phi_0 \in B([0, T] \times \Omega) \). Now take a sequence \( \{\tilde{f}_n\}_{n \in \mathbb{N}} \in C_{0,\partial}\Omega([0, T] \times \Omega) \) such that \( \tilde{f}_n \to f + \mathcal{L}_{\Omega}\phi_0 \) a.e., and \( \sup_n \|\tilde{f}_n\|_\infty < \infty \). Define \( f_n := \tilde{f}_n - \mathcal{L}_{\Omega}\phi_0 \) for each \( n \in \mathbb{N} \) and note that \( f_n \to f \) a.e., \( \sup_n \|f_n\|_\infty < \infty \) and \( f_n(0) = -\mathcal{L}_{\Omega}\phi_0 \), as required by Definition 3.5. Now, for each \( f_n \) consider the stochastic representation of the respective solution in the domain of the generator
\[ u_n(t, x) = \mathbb{E} \left[ \int_0^{\tau_{t,x}} f_n \left( X^{t,\beta}_{s}(s), X^{x,\alpha}_{s}(s) \right) ds \right] + \mathbb{E} \left[ \int_0^{\tau_{t,x}} \mathcal{L}_{\Omega}\phi_0 \left( X^{x,\alpha}_{s}(s) \right) ds \right] + \phi_0(x). \]

Fix \((t, x) \in (0, T) \times \Omega\). Using absolute continuity with respect of Lebesgue measure of the laws of \( X^{t,\beta}_{s}(s) \) and \( X^{x,\alpha}_{s}(s) \) for each \( s > 0 \), and the bound \( \mathbb{E} [\tau_{t,x}] \leq \mathbb{E} [\tau_0(t)] < \infty \), we can apply DCT twice to obtain as \( n \to \infty \)
\[ \mathbb{E} \left[ \int_0^{\tau_{t,x}} f_n \left( X^{t,\beta}_{s}(s), X^{x,\alpha}_{s}(s) \right) ds \right] = \int_0^\infty P_{s}^{\beta,\text{kill}} P_{s}^{\Omega} f_n(t, x) ds \]
\[ = \int_0^\infty P_{s}^{\beta,\text{kill}} P_{s}^{\Omega} f(t, x) ds \]
\[ = \mathbb{E} \left[ \int_0^{\tau_{t,x}} f \left( X^{t,\beta}_{s}(s), X^{x,\alpha}_{s}(s) \right) ds \right], \]
using as a dominating function \( G := \sup_n \|f_n\|_\infty \) to show that for each \( s > 0 \)
\[ F_n(s) := P_{s}^{\beta,\text{kill}} P_{s}^{\Omega} f_n(t, x) \to P_{s}^{\beta,\text{kill}} P_{s}^{\Omega} f(t, x) =: F(s), \]
and the dominating function \( G(s) := \sup_n \|f_n\|_\infty \mathbb{P}[s < \tau_{t,x}] \) to show that
\[ \int_0^\infty F_n(s) ds \to \int_0^\infty F(s) ds. \]

The convergence on \([0, T] \times \partial\Omega \cup \{0\} \times \Omega\) is trivial. It follows that a generalised solution \( u \) exists and it is given by
\[ u = (-\mathcal{L}_{\beta,\Omega}^{\text{kill}})^{-1}(f + \mathcal{L}_{\Omega}\phi_0) + \phi_0. \]

Finally, independence of the approximating sequence proves uniqueness.

(iii) This is a standard application of Dynkin formula (21 Theorem 5.1) using the finite stopping times \( \tau_{t,x}, (t, x) \in (0, T) \times \Omega \), namely
\[ (-\mathcal{L}_{\beta,\Omega}^{\text{kill}})^{-1}(\mathcal{L}_{\Omega}\phi_0)(t, x) = \mathbb{E} \left[ \int_0^{\tau_{t,x}} \mathcal{L}_{\beta,\Omega}\phi_0 \left( X^{x,\alpha}_{s}(s) \right) ds \right] = \mathbb{E} [\phi_0(X^{x,\alpha}_{\tau_{t,x}})] - \phi_0(x). \]
\[ \square \]

We now show that the dual of \( \mathcal{L}_{\beta,\Omega} \) is \((-D_{0+}^{\beta,\alpha} + \Delta_{\Omega}^{\alpha}).\)
Lemma 3.8. Let $u \in \text{Dom}(\mathcal{L}_\beta, \Omega)$. Then
\[
\langle \mathcal{L}_\beta, \Omega u, \varphi \rangle = \langle u, (-D_0^{\beta, \ast} + \Delta^\frac{\alpha}{2}) \varphi \rangle, \quad \text{for every } \varphi \in C^{1,2}_c((0, T) \times \Omega).
\]

**Proof.** By Lemma 2.6 (i) and Proposition 2.4 (i) we can pick a sequence
\[
\{u_n\}_{n \in \mathbb{N}} \subset \text{Span} \{C^1([0, T]) \cdot \text{Dom}(\mathcal{L}_\Omega)\},
\]
such that $u_n \to u$ and $\mathcal{L}_\beta, \Omega u_n \to \mathcal{L}_\beta, \Omega u$ in $C_{\partial \Omega}([0, T] \times \Omega)$, with the additional property
\[
\mathcal{L}_\beta, \Omega u_n = (-D_0^{\beta} + \mathcal{L}_\Omega) u_n, \quad \text{for every } n \in \mathbb{N}.
\]
Hence, for every $\varphi \in C^{1,2}_c((0, T) \times \Omega)$, as $n \to \infty$
\[
\langle \mathcal{L}_\beta, \Omega u, \varphi \rangle = \langle \mathcal{L}_\beta, \Omega u_n, \varphi \rangle = \langle u_n, (-D_0^{\beta, \ast} + \Delta^\frac{\alpha}{2}) \varphi \rangle \to \langle u, (-D_0^{\beta, \ast} + \Delta^\frac{\alpha}{2}) \varphi \rangle,
\]
where we use DCT for both limits, and for the equality we use the identity (3.4) along with Proposition 3.2 and the dual identity in Proposition 2.5 (ii). \qed

We now combine Lemma 3.8 with the notion of generalised solution to obtain the main theorem of this section

**Theorem 3.9.** Let $f \in L^\infty((0, T) \times \Omega)$ and $\phi_0 \in C_{\partial \Omega}(\Omega)$. Then the function $u \in B([0, T] \times \Omega)$ defined in (1.6) is a weak solution to problem (1.7).

**Proof.** Assume for the moment that $\phi_0 \in \text{Dom}(\mathcal{L}_\Omega)$. By the definition of a generalised solution we can take an approximating sequence of forcing terms $\{f_n\}_{n \in \mathbb{N}} \subset C_{\partial \Omega}([0, T] \times \Omega)$ such that $f_n \to f$ a.e., $\sup_n \|f_n\|_\infty < \infty$, and the respective solutions in the domain of the generator $\{u_n\}_{n \in \mathbb{N}}$ satisfy
\[
u_n(0) = \phi_0 \quad \text{for all } n \in \mathbb{N}, \quad u_n \to u \text{ pointwise on } [0, T] \times \Omega, \quad \sup_n \|u_n\|_\infty < \infty,
\]
where the last property is an immediate consequence of the stochastic representation (1.6). Hence, we obtain for every $\varphi \in C^{1,2}_c((0, T) \times \Omega)$, as $n \to \infty$
\[
\langle -f, \varphi \rangle = \langle -f_n, \varphi \rangle = \langle \mathcal{L}_\beta, \Omega u_n, \varphi \rangle = \langle u_n, (-D_0^{\beta, \ast} + \Delta^\frac{\alpha}{2}) \varphi \rangle \to \langle u, (-D_0^{\beta, \ast} + \Delta^\frac{\alpha}{2}) \varphi \rangle,
\]
where we applied DCT for both limits, the first equality is due to the $u_n$’s being solutions in the domain of the generator, and the second equality holds as a consequence of Lemma 3.8.

Now, for $\phi_0 \in C_{\partial \Omega}(\Omega)$, let $\{\phi_{0,n}\}_{n \in \mathbb{N}} \subset \text{Dom}(\mathcal{L}_\Omega)$ such that $\phi_{0,n} \to \phi_0$ in $C_{\partial \Omega}(\Omega)$. Let $u_n$ be the generalised solution to problem (1.1) for $f \in B([0, T] \times \Omega)$ and $\phi_n \in \text{Dom}(\mathcal{L}_\Omega)$, and $u$ defined as in (1.6). Then $u_n \to u$ pointwise and $\sup_n \|u_n\|_\infty < \infty$, which in turn implies by DCT
\[
\langle -f, \varphi \rangle = \lim_{n \to \infty} \langle u_n, (-D_0^{\beta, \ast} + \Delta^\frac{\alpha}{2}) \varphi \rangle = \langle u, (-D_0^{\beta, \ast} + \Delta^\frac{\alpha}{2}) \varphi \rangle.
\]
It is clear that we the result holds for $f \in L^\infty((0, T) \times \Omega)$. Finally, the required convergence of $u$ to the initial condition $\phi_0$ follows by the argument in Remark 5.3 using the stochastic representation (1.6). \qed
4. Stochastic classical solution for problem (1.7)

**Definition 4.1.** Let $f \in C((0, T] \times \Omega)$ and $\phi_0 \in C(\Omega)$. A function $u \in C_{\partial \Omega}([0, T] \times \Omega) \cap C^{1,2}((0, T) \times \Omega)$, such that $|\partial_t u(t, x)| \leq C t^{-\gamma}$, for every $(t, x) \in (0, T] \times \Omega$, for some $\gamma \in (0, 1), C > 0$, is said to be a classical solution to problem (1.7) if $u$ satisfies the identities in (1.7), and for every $x \in \Omega$

$$\lim_{t \downarrow 0} |u(t, x) - \phi_0(x)| = 0.$$ 

In this section the pairing $\langle \cdot, \cdot \rangle$ is defined as

$$\langle f, g \rangle := \int_{\Omega} f(x)g(x)dx.$$ 

The proof of the main theorem of this section (Theorem 4.6), extends the eigenfunction expansion argument in [17, Theorem 5.1], using the next lemma as the key extra ingredient.

**Lemma 4.2.** Let $\lambda > 0$ and $f \in C([0, T])$. Then

(i) $E \left[ \int_0^{\tau_0(t)} e^{-\lambda s} f(X_t^{t, \beta}(s))ds \right] = F_{\lambda} [f] (t), \quad t > 0.$

(ii) The bound

$$|F_{\lambda} [f] (t)| \leq \frac{c}{\lambda} \|f\|_\infty, \quad t > 0, \quad (4.1)$$

holds, and if $f \in C^1([0, T])$ then

$$|\partial_t F_{\lambda} [f] (t)| \leq \frac{c}{\lambda} \left( \|f'\|_\infty + f(0) \frac{t^{\beta-1}}{1 + \lambda t^{\beta}} \right), \quad t > 0, \quad (4.2)$$

for some positive constant $c$.

**Proof.** (i) Given the second identity in (2.2), it is enough to prove the equivalent identity

$$E \left[ \int_0^{\tau_0(t)} e^{-\lambda s} f(X_t^{t, \beta}(s))ds \right] + u_0 E \left[ e^{-\lambda \tau_0(t)} \right] = F_{\lambda} [f] (t) + u_0 E_\beta (-\lambda t^{\beta}), \quad (4.3)$$

where $u_0$ is some constant. We show that the lhs of (4.3) is the unique continuous solution to the Caputo initial value problem solved by the rhs of (4.3). Let $w \in C_0([0, T])$ such that $w' \in C([0, T])$. Then $u(t) := (\lambda - \mathcal{L}_\beta)^{-1} w(t) = E[ \int_0^{\tau_0(t)} e^{-\lambda s} w(X_t^{t, \beta}(s))ds ]$ solves the resolvent equation

$$\mathcal{L}_\beta u = \lambda u - w, \quad u(0) = 0,$$
and \( u \in \text{Dom}(\mathcal{L}_\beta) \), by Proposition 2.4-(i). By the following computation

\[
\partial_t u(t) = \partial_t \int_0^t w(t - y) \int_0^\infty e^{-\lambda s} p_{s}^{\beta}(y) ds dy
\]

\[
= w(0) \int_0^\infty e^{-\lambda s} p_{s}^{\beta}(t) ds + \int_0^t w'(t - y) \int_0^\infty e^{-\lambda s} p_{s}^{\beta}(y) ds dy,
\]

it follows that \( u \in C_0^1([0, T]) \), and so \( \mathcal{L}_\beta u = -D_0^\beta u \) by Proposition 2.4-(i). Let \( u_0 \in \mathbb{R} \). Then \( \bar{u} := u + u_0 \) is a continuous solution to the Caputo initial value problem

\[
-D_0^\beta \bar{u} = \mathcal{L}_\beta u - D_0^\beta u_0 = \lambda u - w = \lambda \bar{u} - (w + \lambda u_0),
\]

with initial value \( \bar{u}(0) = u_0 \). By [19, Theorem 6.5 and Theorem 7.2] we obtain \( \bar{u} = \text{rhs of (4.3)} \) for \( f = w + \lambda u_0 \). Now compute

\[
\bar{u}(t) = E \left[ \int_0^{\tau_0(t)} e^{-\lambda s} \left( w(X_{t, \beta}(s)) \pm \lambda u_0 \right) ds \right] + u_0
\]

\[
= E \left[ \int_0^{\tau_0(t)} e^{-\lambda s} \left( w(X_{t, \beta}(s)) + \lambda u_0 \right) ds \right] - \lambda u_0 \frac{E \left[ e^{-\lambda \tau_0(t)} \right] - 1}{-\lambda} + u_0
\]

\[
= E \left[ \int_0^{\tau_0(t)} e^{-\lambda s} \left( w(X_{t, \beta}(s)) + \lambda u_0 \right) ds \right] + u_0 E \left[ e^{-\lambda \tau_0(t)} \right].
\]

For \( f \in C^1([0, T]) \), by picking \( w \equiv f - f(0) \) and \( u_0 \equiv f(0) \lambda^{-1} \), we obtain the equality (4.3). A straightforward application of DCT proves the claim for \( f \in C([0, T]) \).

(ii) Recall that there exists a constant \( c > 0 \) such that \( 0 \leq -\partial_t E_\beta(-\lambda t^\beta) \leq c \frac{\lambda t^{\beta-1}}{1 + \lambda \beta} \) by [19, Theorem 7.3] and [28, Equation (17)], and \( E_\beta(-\lambda t^\beta) \leq \frac{c}{1 + \lambda \beta} \). Then

\[
(-\lambda)^{-1} \int_0^t f(r) \frac{d}{dr} E_\beta(-\lambda(t - r)^\beta) dr \leq \|f\|_\infty \frac{1 - E_\beta(-\lambda t^\beta)}{\lambda} \leq \|f\|_\infty \frac{1 + c}{\lambda}.
\]

For the second inequality we exploit the smoothness of \( f \), computing for \( t > 0 \)

\[
\partial_t F_\lambda[f](t) = (-\lambda)^{-1} \partial_t \left( -\int_0^t f(r) \partial_r E_\beta(-\lambda(t - r)^\beta) dr \right)
\]

\[
= (-\lambda)^{-1} \partial_t \left( \int_0^t f'(r) E_\beta(-\lambda(t - r)^\beta) dr - f(t) + f(0) E_\beta(-\lambda t^\beta) \right)
\]

\[
= (-\lambda)^{-1} \left( \int_0^t f'(r) \frac{d}{dt} E_\beta(-\lambda(t - r)^\beta) dr \pm f'(t) + f(0) \partial_t E_\beta(-\lambda t^\beta) \right)
\]

\[
= F_\lambda[f](t) - \lambda^{-1} f(0) \partial_t E_\beta(-\lambda t^\beta).
\]
Then
\[ |\partial_t F_\lambda[f](t)| \leq \|f'\|_\infty \frac{1+c}{\lambda} + f(0) \frac{c t^{\beta-1}}{\lambda 1 + \lambda t^{\beta}}. \]

\[ \square \]

From the proof of [17, Theorem 5.1], we infer the following lemma.

**Lemma 4.3.** Working with the notation of Proposition 2.3 (iii):

(i) the system of eigenvectors \( \{ \psi_n \}_{n \in \mathbb{N}} \) forms an orthonormal basis of \( \text{Dom}(L_{\Omega,2}^k) \subset L^2(\Omega) \). The corresponding eigenvalues can be ordered so that \( \lambda_n \leq \lambda_{n+1} \), and also there are constants \( c_1 \leq \lambda_n \leq c_1 n^{\alpha/d} \) for some constant \( c_1 > 0 \). Also, for any compact subset \( K \) of \( \Omega \), \( j = 0, 1, 2 \), there are constants \( c_1 = c_1(K, j, d, \alpha) \) such that
\[ |\nabla^j \psi_n(x)| \leq c_1 \lambda_n^{(d+2j)/(2\alpha)}, \quad (4.4) \]

where \( c_1(K, 0, d, \alpha) \) is independent of \( K \).

(ii) Suppose \( \phi_0 \in \text{Dom}(L_{\Omega,2}^k) \) for \( k > -1+ (3d+4)/(2\alpha) \). Then \( N := \sum_{n=1}^{\infty} \lambda_n^{2k} (\phi_0, \psi_n)^2 < \infty \), and the series
\[ \sum_{n=1}^{\infty} E_\beta(-\lambda_n t^\beta) (\phi_0, \psi_n) \psi_n(x) = E_0 \left[ \phi_0(X^{x,\alpha}(\tau_0(t))) |_{\tau_0(t) < \tau_0(x)} \right], \]
defines a function in \( C_{\partial \Omega}([0, T] \times \Omega) \cap C^{1,2}((0, T) \times \Omega) \), with bounds for \( j = 1, 2 \),
\[ \sum_{n=1}^{\infty} \left| E_\beta(-\lambda_n t^\beta) (\phi_0, \psi_n) \nabla^j \psi_n(x) \right| \leq c_2 t^{-\frac{\beta}{2}} \sqrt{\beta} \sum_{n=1}^{\infty} \lambda_n^{(d+4)/(2\alpha)-1-k} < \infty, \quad t > 0, \]
\[ \sum_{n=1}^{\infty} \left| \partial_t E_\beta(-\lambda_n t^\beta) (\phi_0, \psi_n) \psi_n(x) \right| \leq c_3 t^{-\frac{\beta}{2}-1}, \quad x \in \Omega, \]

where \( c_2 = c_2(K, j, d, \alpha), c_3 = c_3(\Omega, \alpha) \), and \( 0 \leq \gamma \leq 1 \land \left( \frac{4}{2\alpha} - 1 \right) \).

We will assume that the forcing term \( f \) in (1.7) belongs to the space of functions
\[ C^1([0, T]; \text{Dom}(L_{\Omega,2}^k)) := \left\{ f \in C_{\partial \Omega}^1([0, T] \times \Omega) : \sup_t \| f(t) \|_{L_{\Omega,2}^k} + \sup_t \| \partial_t f(t) \|_{L_{\Omega,2}^k} < \infty \right\}. \quad (4.5) \]

Note that if \( f \in C^1([0, T]; \text{Dom}(L_{\Omega,2}^k)) \), then there exists \( M > 0 \) such that for every \( n \in \mathbb{N} \)
\[ \sup_{t \in [0,T]} |\langle f(t), \psi_n \rangle| \leq M \lambda_n^{-k}, \quad \text{and} \quad \sup_{t \in [0,T]} |\langle \partial_t f(t), \psi_n \rangle| \leq M \lambda_n^{-k}. \quad (4.6) \]

**Remark 4.4.** The inclusion \( \text{Span}\{C^1([0, T]); \text{Dom}(L_{\Omega,2}^k)\} \subset C^1([0, T]; \text{Dom}(L_{\Omega,2}^k)) \) is clear. Moreover, if \( k \in \mathbb{N} \), then the inclusion \( C^1([0, T] \times \Omega) \subset C^1([0, T]; \text{Dom}(L_{\Omega,2}^k)) \) holds. \(^4\) To \(^4\)We define \( C_{c}^{1,2k}([0, T] \times \Omega) = C^{1,2k}((0, T) \times \Omega) \cap \{ f, \partial_t f \in C([0, T] \times \Omega), \text{supp}\{ f \} \subset [0, T] \times \Omega \text{ is compact} \} \).
see this, let \( f \in C^{1,2k}_c([0,T] \times \Omega) \), and compute for each \( t \in [0,T] \)
\[
\sum_{n=1}^{\infty} \lambda_n^{2k} \langle f(t), \psi_n \rangle^2 = \sum_{n=1}^{\infty} \langle f(t), \mathcal{L}_{\Omega}^k \psi_n \rangle^2 = \sum_{n=1}^{\infty} \langle (\Delta_{\Omega}^2)^k f(t), \psi_n \rangle^2 = \| (\Delta_{\Omega}^2)^k f(t) \|_{L^2(\Omega)}^2 < \infty,
\]
where the second equality holds by the same argument at the end the proof of Theorem 4.6 using \((\Delta_{\Omega}^2)^m f(t) \in L^2(\Omega)\) for each \( t \in [0,T] \) and \( m \leq k \). Now observe that by DCT the function \( t \mapsto \| (\Delta_{\Omega}^2)^k f(t) \|_{L^2(\Omega)} \) is continuous on \([0,T]\), because \((\Delta_{\Omega}^2)^k f \in C([0,T] \times \Omega)\). Repeat the argument for \( \partial_t f \) to conclude.

**Lemma 4.5.** If \( f(t) \in \text{Dom}(\mathcal{L}_{\Omega,2}^k) \) for \( k > -1 + (3d + 4)/(2\alpha) \), for every \( t \in [0,T] \), and \( f \in C^{1}_{\partial\Omega}([0,T] \times \Omega) \), then
\[
E \left[ \int_0^{\tau_{t,x}} f(X^{t,\beta}(s), X^{x,\alpha}(s)) ds \right] = \sum_{n=1}^{\infty} \psi_n(x) F_{\lambda_n} \left[ \langle f(\cdot), \psi_n \rangle \right](t).
\]

If in addition \( f \in C^1([0,T];\text{Dom}(\mathcal{L}_{\Omega,2}^k)) \), then there exists a constant \( C \) such that for \( t > 0 \)
\[
\sum_{n=1}^{\infty} \psi_n(x) \partial_t F_{\lambda_n} \left[ \langle f(\cdot), \psi_n \rangle \right](t) \leq Ct^{\beta-1} \sum_{n=1}^{\infty} n^{(\alpha/d)(-k-1+d/(2\alpha))}.
\]

**Proof.** We justify the following equalities
\[
E \left[ \int_0^{\tau_{t,x}} f(X^{t,\beta}(s), X^{x,\alpha}(s)) ds \right] = \int_0^{\infty} P_{s}^{\beta,\text{kill}} \mathbf{P}_{s \uparrow}^\Omega f(t,x) ds
\]
\[
= \int_0^{\infty} P_{s}^{\beta,\text{kill}} \left( \sum_{n=1}^{\infty} \langle f(t), \psi_n \rangle \psi_n(x) e^{-s\lambda_n} \right) ds
\]
\[
= \sum_{n=1}^{\infty} \psi_n(x) \int_0^{\infty} P_{s}^{\beta,\text{kill}} \langle f(t), \psi_n \rangle e^{-s\lambda_n} ds
\]
\[
= \sum_{n=1}^{\infty} \psi_n(x) E \left[ \int_0^{\tau_{0}(t)} \langle f(X^{t,\beta}(s)), \psi_n \rangle e^{-s\lambda_n} ds \right]
\]
\[
= \sum_{n=1}^{\infty} \psi_n(x) F_{\lambda_n} \left[ \langle f(\cdot), \psi_n \rangle \right](t).
\]

We can apply Fubini’s Theorem in the third equality as
\[
\sum_{n=1}^{\infty} \| \langle f(t), \psi_n \rangle \|_{L^\infty} \leq C \sum_{n=1}^{\infty} n^{(\alpha/d)(d/(2\alpha)-k-1)} < \infty,
\]
for some constant \( C > 0 \) and each \( t \geq 0 \), by the condition on \( k \) and the bounds in Lemma 4.3(i) and in (4.6). We apply Lemma 4.2(i) in the fifth equality as \( r \mapsto \langle f(r), \psi_n \rangle \in \)

C([0,T]) for each \( n \in \mathbb{N} \). The other equalities are clear.

For the last claim we use the bounds in (4.2), (4.6) and Lemma 4.3-(i) to obtain

\[
\left| \sum_{n=1}^{\infty} \psi_n(x) \partial_t F_{\lambda_n} \left[ \langle f(t), \psi_n \rangle \right] (t) \right| \leq \sum_{n=1}^{\infty} \left| \psi_n(x) \right| \frac{c}{\lambda_n} \left( \sup_{r \in [0,T]} |\partial_r f(r), \psi_n| \right) + \frac{t^{\beta-1}}{1 + \lambda_n t^\beta} |\langle f(0), \psi_n \rangle|
\]

\[
\leq \sum_{n=1}^{\infty} \frac{cM \lambda_n}{\lambda_n} \left( 1 + \frac{t^{\beta-1}}{1 + \lambda_n t^\beta} \right)
\]

\[
\leq (c_1 c M) \sum_{n=1}^{\infty} \frac{\lambda_n^{\alpha/(2\alpha)} \lambda_n^{-k}}{\lambda_n} \left( 1 + \frac{t^{\beta-1}}{1 + \lambda_n t^\beta} \right)
\]

\[
\leq (c_1 c M) t^{\beta-1} \sum_{n=1}^{\infty} \lambda_n^{-k-1 + d/(2\alpha)}
\]

\[
\leq (\tilde{c}_1 c_1 c M) t^{\beta-1} \sum_{n=1}^{\infty} \left( \alpha/d \right)(-k-1+d/(2\alpha)),
\]

where the constants \( \tilde{c}_1, c_1, c \) and \( M \) follow the notation of the referenced inequalities.

\[\square\]

**Theorem 4.6.** Let \( \Omega \subset \mathbb{R}^d \) be a regular set. Assume that \( \phi_0 \in \text{Dom}(L^k_{\Omega,2}) \), and \( f \in C^1([0,T]; \text{Dom}(L^k_{\Omega,2})) \) for some \( k > -1 + (3d + 4)/(2\alpha) \), where \( C^1([0,T]; \text{Dom}(L^k_{\Omega,2})) \) is defined in (4.5). Then

\[ u \in C_{\partial\Omega}([0,T] \times \Omega) \cap C^{1,2}((0,T) \times \Omega), \quad \text{and} \]

\[ |\partial_t u(t,x)| \leq Ct^{-\gamma}, \text{ for every } (t,x) \in (0,T] \times \Omega, \text{ for some } \gamma \in (0,1), \ C > 0, \quad (4.8) \]

where \( u \) is defined in (1.6). Moreover, \( u \) is the unique classical solution to problem (1.7).

**Proof.** (The notation for constants is consistent with the referenced inequalities.)

By Lemma 4.3-(ii) and Lemma 4.5 we can write our candidate solution (1.6) as

\[ u(t,x) = \sum_{n=1}^{\infty} E_{\beta}(-\lambda_n t^\beta) \langle \phi_0, \psi_n \rangle \psi_n(x) + \sum_{n=1}^{\infty} F_{\lambda_n} \left[ \langle f(\cdot), \psi_n \rangle \right] (t) \psi_n(x), \]

and the first sum enjoys the regularity properties stated in (4.8). We now prove the same regularity for the second sum. Observe that \( \sum_{n=1}^{\infty} F_{\lambda_n} \left[ \langle f(\cdot), \psi_n \rangle \right] (t) \psi_n(x) \) converges
uniformly to a function in \( C_{\partial \Omega}([0, T] \times \Omega) \), since we have the uniform bound
\[
\sum_{n=1}^{\infty} |F_{\lambda_n} [(f(\cdot), \psi_n)] (t)\psi_n(x)| \leq \sum_{n=1}^{\infty} \frac{c}{\lambda_n} \| f(\cdot), \psi_n \|_{C([0,T])} C_1 \lambda_n^{d/(2\alpha)}
\]
\[
\leq (cc_1 M) \sum_{n=1}^{\infty} \lambda_n^{-k-1+d/(2\alpha)}
\]
\[
\leq (\tilde{c}_1 c_1 c M) \sum_{n=1}^{\infty} n^{(\alpha/d)(-k-1+d/(2\alpha))} < \infty,
\]
for any \( k > -1 + (3d/2\alpha) \), using the bounds in (4.4) and (4.1). Further, for \( j = 1,2 \), and for any \( x \) in a compact subset \( K \) of \( \Omega \), the term-wise space derivative of \( u \) can be bounded as follows,
\[
\sum_{n=1}^{\infty} |F_{\lambda_n} [(f(\cdot), \psi_n)] (t)\nabla^j \psi_n|_{\infty} \leq \sum_{n=1}^{\infty} \frac{c}{\lambda_n} \| f(\cdot), \psi_n \|_{C([0,T])} C_1 \lambda_n^{d+4)/2\alpha}
\]
\[
\leq (\tilde{c}_1 c_1 c M) \sum_{n=1}^{\infty} n^{(\alpha/d)((d+4)/(2\alpha)-k-1)} < \infty,
\]
as
\[
\frac{\alpha}{d} \left( \frac{d + 4}{2\alpha} - k - 1 \right) < -1 \iff \ k > \frac{3d + 4 - 2\alpha}{2\alpha},
\]
where we use the bounds in (4.4), (4.1) and Lemma 4.3-(i). Thus, Weierstrass M-test implies that for any \( t > 0 \), \( u(t) \) is a \( C^2 \) function on every \( K \subset \Omega \) compact. For the time regularity, we use the inequality in (4.7), which is again finite under the assumption \( k > (3d + 4 - 2\alpha)/(2\alpha) \).

By Theorem 3.9 \( u \) is also a weak solution to problem (1.7), and by Lemma 3.3 and standard approximation arguments, \( u \) satisfies the equalities in (1.7). Continuity at \( t = 0 \) can be proved as in Remark 5.3.

To prove uniqueness, consider two classical solutions to problem (1.7), denoted by \( u, v \). Then \( w := u - v \) is a classical solution to problem (1.7), with \( f = 0, \phi_0 = 0 \). Consider the continuous functions on \([0,T]\), \( t \mapsto \langle w(t), \psi_n \rangle, n \in \mathbb{N} \). If we can justify
\[
D_0^\beta \langle w(t), \psi_n \rangle = \langle D_0^\beta w(t), \psi_n \rangle = \langle \Delta_{\Omega}^\beta w(t), \psi_n \rangle = \langle v(t), L_{\Omega,2}\psi_n \rangle = -\lambda_n \langle w(t), \psi_n \rangle, \quad (4.10)
\]
for \( t > 0 \), it follows by [19] Theorem 6.5 and Theorem 7.2] that \( \langle w(t), \psi_n \rangle = 0 \) for every \( t \in [0, T], n \in \mathbb{N} \), and we are done. The first equality is a consequence of \( |\partial_t w(r,y)| \leq Cr^{-\gamma} \), for some \( \gamma \in (0,1) \). The second and fourth equalities in (4.10) are clear. Now, as \( \psi_n \in \text{Dom}(L_{\Omega,2}) \), there exists a sequence \( \{\psi_{n,j}\}_{j \in \mathbb{N}} \subset C_0^\infty(\Omega) \), such that as \( j \to \infty \)
\[
\psi_{n,j} \to \psi_n, \quad \text{and} \quad \Delta_{\Omega}^\beta \psi_{n,j} = L_{\Omega,2}\psi_{n,j} \to L_{\Omega,2}\psi_n, \quad \text{in} \ L^2(\Omega), \quad (4.11)
\]
\[\text{Note that if } f(0) = 0, \text{ then } \partial_t u \text{ is bounded.}\]
where the equality in (4.11) holds by [17, Lemma 4.1]. Combining (4.11) with the equality (2.5) and $\Delta^2_{\Omega}w(t) \in L^2(\Omega)$ for each $t > 0$, the third equality in (4.10) is proven. □

5. Stochastic classical solution for problem (1.1)

5.1. Stochastic representation and continuity at $t = 0$.

**Lemma 5.1.** Define the function $f_\phi : (0, T] \times \Omega \rightarrow \mathbb{R}$ as

$$f_\phi(t, x) := \int_{t}^{\infty} (\phi(t-r, x) - \phi(t, x)) \frac{-\Gamma(-\beta)^{-1}dr}{r^{1+\beta}},$$

(5.1)

assuming that $\phi \in C_{\infty, \partial\Omega}((-\infty, 0] \times \Omega)$, $\phi(0) \in \text{Dom}(\mathcal{L}_\Omega)$, and the extension of $\phi$ to $\phi(0)$ on $(0, T] \times \Omega$ is such that

$$\phi \in \text{Dom}(\mathcal{L}^\infty_{\beta, \Omega}), \quad \text{and} \quad \mathcal{L}^\infty_{\beta, \Omega} \phi = (-D_{-\infty}^{-\beta} + \mathcal{L}_\Omega) \phi.$$  

(5.2)

Then $f_\phi \in C([0, T] \times \Omega)$ and the function $u$ defined in (1.6) for $f = f_\phi$ and $\phi_0 = \phi(0)$, equals the function $\tilde{u}$ defined in (1.3) for $g = 0$, on $(0, T] \times \Omega$.

**Proof.** The first claim follows from $f_\phi = -D_{-\infty}^{-\beta} \phi \in C([0, T] \times \Omega)$, using (5.2) and $\mathcal{L}_\Omega \phi(t, x) = \mathcal{L}_\Omega \phi(0, x)$ for all $(t, x) \in [0, T] \times \Omega$. Recall that we write $\tau_{t,x} = \tau_0(t) \wedge \tau_\Omega(x)$. Fix $(t, x) \in (0, T] \times \Omega$. It is enough to justify the following equalities

$$u(t, x) = E \left[ \phi(0, X_{t,x}(\tau_0(t))1_{\{\tau_0(t) \leq \tau(x)\}} + \int_0^{\tau_{t,x}} f_\phi \left( X_{s,s}(X_{t,x}(s)), X_{s,s}(s) \right) ds \right]$$

$$= E \left[ \phi(0, x) + \int_0^{\tau_{t,x}} \mathcal{L}_\Omega \phi(0, X_{s,s}(s)) ds + \int_0^{\tau_{t,x}} f_\phi \left( X_{s,s}(s), X_{s,s}(s) \right) ds \right]$$

$$= E \left[ \int_0^{\tau_{t,x}} \mathcal{L}_\Omega \phi \left( X_{s,s}(s), X_{s,s}(s) \right) - D_{-\infty}^{-\beta} \phi \left( X_{s,s}(s), X_{s,s}(s) \right) ds \right] + \phi(0, x)$$

$$= E \left[ \int_0^{\tau_{t,x}} \mathcal{L}^\infty_{\beta, \Omega} \phi \left( X_{s,s}(s), X_{s,s}(s) \right) ds \right] + \phi(0, x)$$

$$= E \left[ \phi \left( X_{\tau_{t,x}, \tau_{t,x}}(\tau_{t,x}), X_{\tau_{t,x}, \tau_{t,x}}(\tau_{t,x}) \right) \right] \pm \phi(0, x).$$

For the second equality we use Dynkin formula with Lemma 2.6 (i) and $\phi(0) \in \text{Dom}(\mathcal{L}_\Omega)$; for the third equality, as we extended $\phi(t, x) = \phi(0, x)$ on $[0, T] \times \Omega$, we use the identities $f_\phi(t, x) = -D_{-\infty}^{-\beta} \phi(t, x)$ and $\mathcal{L}_\Omega \phi(0, x) = \mathcal{L}_\Omega \phi(t, x)$ on $(0, T] \times \Omega$; in the fourth equality we use assumption (5.2); the fifth equality is again an application of Dynkin formula with Lemma 2.6 (ii) and $\phi(t, x) = \phi(0, x)$ on $(0, T] \times \Omega$. □

**Corollary 5.2.** If $\phi \in C_{\infty, \partial\Omega}^1((-\infty, 0] \times \Omega) \cap \{\partial_i f(0) = 0\}$, then

$$E \left[ \phi(0, X_{\tau_{t,x}, \tau_{t,x}}(\tau_{t,x})) + \int_0^{\tau_{t,x}} f_\phi \left( X_{s,s}(s), X_{s,s}(s) \right) ds \right] = E \left[ \phi \left( X_{\tau_{t,x}, \tau_{t,x}}(\tau_{t,x}), X_{\tau_{t,x}, \tau_{t,x}}(\tau_{t,x}) \right) \right].$$

(5.3)
Proof. It is enough to prove (5.3) for \( \phi \in C^1_{0,\partial \Omega}((\infty, 0] \times \Omega) \cap \{ \partial_t f(0) = 0 \} \) with compact support in \((-\infty, 0] \times \Omega\). For such \( \phi \), let \( K > 0 \) such that \( \phi \) is supported in \((-K, 0] \times \Omega\). By the same arguments as in the proof of Lemma 2.6(ii), it follows that \( \text{Span}\{C([-K, 0]) \cap \{ f(-K) = f(0) = 0 \} \cdot C_{\partial \Omega}(\Omega) \} \) is dense in \( C_{\partial \Omega}([-K, 0] \times \Omega) \cap \{ f(-K) = f(0) = 0 \} \) with respect to the supremum norm. We can use this fact to construct a sequence \( \{ \phi_n \}_{n \in \mathbb{N}} \in \text{Span}\{C^1_{0,\partial \Omega}((\infty, 0]) \cap \{ f'(0) = 0 \} \cdot C_{\partial \Omega}(\Omega) \} \) such that
\[
\| \phi_n - \phi \|_{C((-\infty, 0] \times \Omega)} + \| \partial_t (\phi_n - \phi) \|_{C((-\infty, 0] \times \Omega)} \to 0, \quad \text{as } n \to \infty.
\]
Moreover, it follows that (5.3) holds for functions in \( \text{Span}\{C^1_{\infty}((\infty, 0]) \cap \{ f'(0) = 0 \} \cdot C_{\partial \Omega}(\Omega) \} \), as DCT applied to the sequences above yields the result. By Lemma 2.6(iii) with \( C_{\beta}^\infty = C_{\infty}^\beta((-\infty, T]), \) Proposition 2.4 and Lemma 5.1, equality (5.3) holds for \( \phi \in \text{Span}\{C^1_{\infty}((\infty, 0]) \cap \{ f'(0) = 0 \} \cdot \text{Dom}(\mathcal{L}_\Omega) \} \). As \( \text{Dom}(\mathcal{L}_\Omega) \) is dense in \( C_{\partial \Omega}(\Omega) \), equality (5.3) holds for \( \phi \in \text{Span}\{C^1_{\infty}((\infty, 0]) \cap \{ f'(0) = 0 \} \cdot C_{\partial \Omega}(\Omega) \} \) by DCT. \( \square \)

Remark 5.3. If we can apply Corollary 5.2 then we can prove continuity at \( t = 0 \) for the solution (1.3) via the following argument
\[
|\text{Formula (1.6)} - \phi_0(x)| \leq |E[\phi_0(X^{x,\alpha}(\tau_0(t) \wedge \tau_\Omega(x))) - \phi_0(x)]| + |f|_\infty E[\tau_0(t)]
\]
\[
= o_{t \downarrow 0}(1) + |f|_\infty \frac{t^\beta}{\Gamma(\beta + 1)},
\]
for each \( x \in \Omega \), using stochastic continuity of the process \( t \mapsto X^{x,\alpha}(\tau_0(t)) \) at \( t = 0 \). One could also use stochastic continuity at \( t = 0 \) of \( X^{t,\beta}(\tau_0(t)) = t + X^{0,\beta}(\tau_0(t)) \), bypassing Corollary 5.2. In Proposition A.2 in the Appendix we prove continuity at \( t = 0 \) by proving a bound on big overshootings \( X^{t,\beta}(\tau_0(t)) \) for small times.

5.2. Equivalence of the classical solutions to problems (1.1) and (1.7).

Definition 5.4. Let \( \phi \in C((\infty, 0] \times \Omega) \) and \( g \in C((0, T] \times \Omega) \). A function \( \tilde{u} \in C_{\infty,\partial \Omega}((\infty, T] \times \Omega) \cap C^{1,2}((0, T] \times \Omega) \) such that \( |\partial_t \tilde{u}(t, x)| \leq Ct^{-\gamma} \), for every \( (t, x) \in (0, T] \times \Omega \), for some \( \gamma \in (0, 1) \), \( C > 0 \), is said to be a classical solution to problem (1.1) if \( \tilde{u} \) satisfies the equations in (1.1), and for every \( x \in \Omega \)
\[
\lim_{t \downarrow 0} |\tilde{u}(t, x) - \phi(0, x)| = 0.
\]

Lemma 5.5. Let \( \phi \) be a continuous function on \((-\infty, 0] \times \Omega \), define \( f_\phi \) as in (5.1), and let \( g \) be a continuous function on \((0, T] \times \Omega \). Then, if \( u \) is a classical solution to problem (1.7) with \( f = f_\phi + g \) and \( \phi_0 = \phi(0) \), then the extension
\[
\tilde{u} := \begin{cases} u, & \text{in } (0, T] \times \Omega, \\ \phi, & \text{in } (\infty, 0] \times \Omega, \end{cases}
\]

\footnote{This follows as \( X^{x,\alpha}(s) \) is right continuous and \( \tau_0(t) \) is right continuous, increasing with \( \tau_0(0) = 0 \).}
is a classical solution to problem (1.1). Conversely, if $\tilde{u}$ is a classical solution to the problem (1.1), then the restriction of $\tilde{u}$ to $[0, T] \times \bar{\Omega}$ is a classical solution to problem (1.7) with $f = f_\phi + g$ and $\phi_0 = \phi(0)$.

**Proof.** The equivalence of convergence to initial data and the required regularities are clear. Now observe that $\Delta^\beta_{\Omega}\tilde{u} = \Delta^\beta_{\bar{\Omega}}\tilde{u}$ on $(0, T] \times \Omega$. Write $\nu(r) = -\Gamma(-\beta)^{-1}r^{-1-\beta}$. On $(0, T] \times \Omega$ we have the equality

$$-D^\beta_\infty\tilde{u}(t, x) = \int_0^\infty (\tilde{u}(t-r, x) - \tilde{u}(t, x))\nu(r)dr$$

$$= \int_0^t (\tilde{u}(t-r, x) - \tilde{u}(t, x))\nu(r)dr + \int_t^\infty \phi(t-r, x)\nu(r)dr$$

$$- \tilde{u}(t, x)) \int_t^\infty \nu(r)dr \pm \phi(0, x) \int_t^\infty \nu(r)dr$$

$$= -D^\beta_0\tilde{u}(t, x) + f_\phi(t, x).$$

This is enough to prove both directions. \qed

### 5.3. Main result.

**Theorem 5.6.** Let $\Omega \subset \mathbb{R}^d$ be a regular set. Assume that $\phi \in C^1_{\infty, \partial \Omega}((-\infty, 0] \times \Omega) \cap \{\partial_t f(0) = 0\}$ with $\phi(0) \in \text{Dom}(\mathcal{L}^k_{\Omega,2})$ and $f_\phi, g \in C^1([0, T]; \text{Dom}(\mathcal{L}^k_{\Omega,2}))$, for some $k > -1 + (3d + 4)/(2\alpha)$, where $f_\phi$ is defined in (5.1) and $C^1([0, T]; \text{Dom}(\mathcal{L}^k_{\Omega,2}))$ is defined in (4.5). Then

$$\tilde{u} \in C^1_{\infty, \partial \Omega}((-\infty, T] \times \Omega) \cap C^{1,2}((0, T) \times \Omega),$$

and

$$|\partial_t \tilde{u}(t, x)| \leq C t^{-\gamma}, \text{ for every } (t, x) \in (0, T] \times \Omega, \text{ for some } \gamma \in (0, 1), \ C > 0,$$

where $\tilde{u}$ is defined as in (1.3). Moreover, $\tilde{u}$ is the unique classical solution to problem (1.1).

**Proof.** By the assumptions on $\phi$ and $g$, and Lemma 5.5, existence and uniqueness of classical solutions follows by Theorem 4.6 with $\phi_0 = \phi(0)$ and $f = f_\phi + g$. Now apply Corollary 5.2 to obtain the stochastic representation (1.3) from the stochastic representation (1.6). \qed

**Remark 5.7.** Using Corollary 5.2 (or [27] Theorem 1 for $\lambda = 0$), $\mathbb{P}[X^t(\tau_0(t)) \in \{0\}] = 0$ for every $t > 0$ (see [8] III, Theorem 4) and the independence of $X^{x,\alpha}$ and $X^{t,\beta}$, one can show that for $(t, x) \in (0, T] \times \Omega$

$$\mathbb{E}[\phi \left( X^{t,\beta}(\tau_0(t)), X^{x,\alpha}(\tau_0(t)) \right) 1_{\{\tau_0(t) < \tau_0(x)\}}] = \int_0^0 \int_\Omega \phi(r, y)\Phi_{\beta,\alpha}^x(r, y)drdy,$$

where

$$\Phi_{\beta,\alpha}^x(r, y) = \int_0^t -\Gamma(-\beta)^{-1} (z-r)^1+\beta \left( \int_0^\infty p_\alpha^0(x, y)p_\beta^0(t-z)dz \right)dz.$$
It is straightforward to compute for \((t, x) \in (0, T] \times \Omega\)
\[
E \left[ \int_0^{\tau_0(t) \wedge \tau_1(x)} g \left( X^{t, \beta}(s), X^{x, \alpha}(s) \right) ds \right] = \int_0^t \int_{\Omega} g(z, y) \left( \int_0^{\infty} p_{\alpha}^\Omega(x, y) p_{\beta}^\delta(t - z) ds \right) dy dz.
\]

**Remark 5.8.** Notice that the value \(\phi(0)\) does not contribute to the solution \((1.3)\) because \(P[X^t(\tau_0(t)) \in \{0\}] = 0\) for all \(t > 0\). However, \(u(t) \to \phi(0)\) as \(t \to 0\). We discuss the continuity of the solution at \(t = 0\) in more detail in Appendix A.1.

**Remark 5.9.** We could drop the condition \(\partial_t \phi(0) = 0\) in Theorem 5.6 by weakening Corollary 5.2, for example to \(\phi\) being \(\beta\)-Hölder continuous at \(t = 0\), for some \(\beta > \beta\) and \(\phi \in L^\infty((-\infty, 0) \times \Omega)\). This is essentially because \(\lim_{t \to 0} f_\phi(t)\) remains well-defined. However, in order to apply Theorem 4.6 in the proof of Theorem 5.6 we need to assume \(f_\phi \in C^1([0, T]; \text{Dom}(L^k_{\Omega, 2}))\). Hence, a minimal requirement is that \(\phi\) is continuously differentiable in time and both \(\phi\) and \(\partial_t \phi\) are \(\mathcal{O}(|r|^{2\beta})\) at \(-\infty\) and \(\beta\)-Hölder continuous at 0, for some \(\beta < \beta < \beta\), as we need \(f_\phi\) and \(\partial_t f_\phi\) to be continuous on \([0, T] \times \Omega\).

**Remark 5.10.** Suppose that \(\phi \in C^{2.2k}_{\infty, \partial\Omega}((-\infty, 0] \times \Omega)\) and \(\phi(t)\) along with its partial derivatives in space are compactly supported in \(\Omega\), for each \(t \in (-\infty, 0]\), where \(k \in \mathbb{N}\) and \(k > -1 + (3d + 4)/(2\alpha)\). Then, an application of Remark 4.4 implies that \(f_\phi \in C^1([0, T]; \text{Dom}(L^k_{\Omega, 2}))\).

### 6. Intuition for the stochastic solution \((1.3)\)

We give an intuition for the stochastic representation \((1.3)\) as the solution to the EE \((1.1)\). Let us write \(L(t) := t + X^{0, \beta}(\tau_0(t)) = X^{t, \beta}(\tau_0(t))\). Then \(-L(t)\) is the overshoot of the subordinator \(-X^{0, \beta}(s)\) with respect to the barrier \(t\), recalling that the hitting time-inverse subordinator is given by \(\tau_0(t) = \inf\{s > 0 : t \leq -X^{0, \beta}(s)\}\). To ease notation we write \(Y^x := \{X^{x, \alpha}(\tau_0(t)) \mathbf{1}_{\{\tau_0(t) < \tau_1(x)\}}\}_{t \geq 0}\). Let us start from the intuition of Caputo EE as if \(\phi(t, x) = \phi(0, x) := \phi_0(x)\) for every \(t \in (-\infty, 0] \times \Omega\), then the solution \((1.3)\) reads
\[
u(t, x) = E[\phi_0(Y^x(t))],
\]
and the EE \((1.1)\) equals the Caputo EE \((1.7)\) (for \(g = f = 0\)). The probabilistic object defining the solution \((6.1)\) is the time-changed anomalous diffusion \(Y^x\). Recall that the particle \(Y^x\) is either trapped or diffusing.

**Key observation:** reasoning path-wise, for some \(\bar{x} \in \Omega\)

the interval \((t_1, t_2)\) is the maximal open interval so that \(t \mapsto Y^x(t) = \bar{x}\) is constant
\[
\iff
\]
the interval \((t_1, t_2)\) is the maximal open interval so that \(t \mapsto \tau_0(t)\) is constant
\[
\iff
\]
the interval \((t_1, t_2)\) is the maximal open interval so that \(t \mapsto -X^{0, \beta}(\tau_0(t))\) is constant
\[
\iff
\]
The last statement implies that
\[ L(t) = -X^{0, \beta}(\tau_0(t)) - t = t_2 - t \in (0, t_2 - t_1) \text{ for every } t \in (t_1, t_2), \]
which is the leftover trapping time or holding time of \( Y^x(t) \). In words: the event of the diffusion \( Y^x \) being trapped at a point \( \bar{x} \in \Omega \) at time \( t \) until time \( t + s \) happens precisely when the overshoot \( L(t) = -s \). Hence the law of \( L(t) \) provides a weighting of the initial condition \( \phi(\bar{x}) \) depending on the holding time of \( Y^x(t) \). Notice that the process \( t \mapsto L(t) \) is self-similar with index 1 and it is composed by right continuous 45 degrees increasing slopes with 0 leftmost limit (see Figure 1).

**Figure 1.** A typical path of the overshoot \( t \mapsto L(t) = X^{t, \beta}(\tau_0(t)) \).

### 6.1. A non-memory interpretation.

It is possibly appealing to think about the values \((-\infty, 0) \times \Omega\) for the initial condition \( \phi \) as the ‘thickness’ underneath the surface \( \{0\} \times \Omega \) where the particle \( Y^x \) moves. Then one can think about the particle \( Y^x(t) \) as falling instantaneously at the bottom of a hole/trap of depth \( |t_2 - t_1| \), and then taking time \( |t_2 - t_1| \) to climb back up to the surface. Then, at time \( t \) one can observe the particle being \( |t_2 - t| \)-depth-units down in the hole. From this viewpoint, once the particle is in the hole it just drifts upward with unit speed. As a quick example, consider the variable separable initial condition \( \phi(t, x) = p(t)q(x) \) where \( p(t) = 1_{\{t \leq -1\}} \). Then the solution reads
\[ u(t, x) = E \left[ q(Y^x(t))1_{\{L(t) \leq -1\}} \right], \quad (t, x) \in (-\infty, T] \times \Omega. \]

Notice that in this example the diffusive particle \( Y^x \) will have to be at least a unit deep in a hole (trapped for at least a unit time) for the values at its trapping point at its depth (in the past) to contribute to the solution.
Proposition A.1. For every $p, \varepsilon > 0$, the following bound on small overshootings holds,

$$P[-X^{t,\beta}(\tau_0(t)) \leq \varepsilon] \geq (1 - p), \text{ for every } t \leq \varepsilon p^{\frac{1}{\beta}}.$$ 

Proof. With the first equality holding by \cite{27} Theorem 1 for $\lambda = 0$ along with the identity \cite{23}, compute

$$P[-X^{t,\beta}(\tau_0(t)) \leq \varepsilon] = \int_{-\varepsilon}^{0} \frac{1}{\Gamma(\beta)} \int_{0}^{t} (-\partial_y (y - r)^{-\beta})(t - y)^{\beta - 1} \frac{dy}{\Gamma(1 - \beta)} dr$$

$$= \int_{-\varepsilon}^{0} \frac{\beta}{\Gamma(\beta)\Gamma(1 - \beta)} \int_{0}^{t} (y - r)^{-\beta - 1}(t - y)^{\beta - 1} dy dr$$

$$= -\frac{\Gamma(\beta)^{-1}}{\Gamma(-\beta)} \int_{0}^{t} (t - y)^{-\beta - 1} \left(\int_{-\varepsilon}^{0} (y - r)^{-\beta - 1} dr\right) dy$$

$$= -\frac{\Gamma(\beta)^{-1}\beta^{-1}}{\Gamma(-\beta)}(a - a_\varepsilon(t)),$$

where $a_\varepsilon(t) := \int_{0}^{t}(t - y)^{\beta - 1}(y + \varepsilon)^{-\beta} dy$ and $a := \int_{0}^{t}(t - y)^{\beta - 1} y^{-\beta} dy = \Gamma(\beta)\Gamma(1 - \beta)$ for every $t > 0$. Now pick $\tilde{t} = \varepsilon p^{1/\beta}$. Then for every $0 \leq y \leq \tilde{t}$

$$(y + \varepsilon)^{-\beta} = (y + p^{-1/\beta}\tilde{t})^{-\beta} \leq p^{\tilde{t}^{-\beta}} \leq py^{-\beta},$$

hence for every $t \leq \tilde{t}$

$$\frac{a_\varepsilon(t)}{a} = \frac{\int_{0}^{t}(t - y)^{\beta - 1}(y + \varepsilon)^{-\beta} dy}{\int_{0}^{t}(t - y)^{\beta - 1} y^{-\beta} dy} \leq p.$$ 

Then $a_\varepsilon(t) \leq pa$ for every $t \leq \tilde{t}$, which is equivalent to $a - a_\varepsilon(t) \geq (1 - p)a$ for every $t \leq \tilde{t}$. And so we obtain

$$P[-X^{t,\beta}(\tau_0(t)) \leq \varepsilon] \geq (1 - p)\frac{-\Gamma(\beta)^{-1}}{\Gamma(-\beta)}\beta^{-1}\Gamma(1 - \beta)\Gamma(1 - \beta) = (1 - p).$$

We now use the bound in Proposition A.1 to prove the following continuity result

Proposition A.2. Consider the function $\tilde{u}$ defined in \cite{13}, with an arbitrary $\Omega$-valued stochastic (sub-)process $X^x$ in place of $X^{x,\alpha}$, such that $t \mapsto X^x(\tau_0(t))$ is stochastically continuous at $t = 0$. Also assume $\phi \in B((-\infty, 0] \times \Omega)$ and $\phi$ is continuous at every point in $\{0\} \times \Omega$. Then for every $x \in \Omega$

$$\lim_{t \downarrow 0} |\tilde{u}(t, x) - \phi(0, x)| = 0.$$
Proof. Let $x \in \Omega$. Let $\delta > 0$ be arbitrary. Pick $\varepsilon, \varepsilon' > 0$ such that

$$\sup_{(s,y) \in (-\varepsilon,0) \times B_\varepsilon(x)} |\phi(s,y) - \phi(0,x)| \leq \delta.$$ 

Then

$$|\tilde{u}(t,x) - \phi(0,x)| \leq E\left[(\phi(X^{t,\beta}(\tau_0(t)), X^x(\tau_0(t)) - \phi(0,x))1_{\{-X^{t,\beta}(\tau_0(t)) > \varepsilon\}}\right]$$

$$+ E\left[(\phi(X^{t,\beta}(\tau_0(t)), X^x(\tau_0(t)) - \phi(0,x))1_{\{-X^{t,\beta}(\tau_0(t)) \leq \varepsilon\}}\right]$$

$$\leq 2\|\phi\|_\infty P[-X^{t,\beta}(\tau_0(t)) > \varepsilon]$$

$$+ E\left[|\phi(X^{t,\beta}(\tau_0(t)), X^x(\tau_0(t))) - \phi(0,x)|1_{\{-X^{t,\beta}(\tau_0(t)) \leq \varepsilon, X^x(\tau_0(t))-x| > \varepsilon'\}}\right]$$

$$+ E\left[|\phi(X^{t,\beta}(\tau_0(t)), X^x(\tau_0(t))) - \phi(0,x)|1_{\{-X^{t,\beta}(\tau_0(t)) \leq \varepsilon, X^x(\tau_0(t))-x| > \varepsilon'\}}\right]$$

$$\leq 2\|\phi\|_\infty P[-X^{t,\beta}(\tau_0(t)) > \varepsilon] + \delta + 2\|\phi\|_\infty P[|X^x(\tau_0(t)) - x| > \varepsilon'], \quad \text{for every } t \leq \frac{1}{\delta^2} \varepsilon.$$ 

To conclude, by stochastic continuity, pick a possibly smaller threshold $\bar{t}$ to obtain

$$P[|X^x(\tau_0(t)) - x| > \varepsilon'] \leq \delta \quad \text{for every } t \leq \bar{t}. \quad \square$$

Remark A.3. The continuity at $t = 0$ of Proposition A.2 is not obvious. For example it is clear the Proposition A.2 fails if we replace $X^{t,\beta}$ with a decreasing Poisson process. In fact Proposition A.2 fails in general if we replace $X^{t,\beta}$ with a decreasing compound Poisson process $Y^t(s)$ with generator

$$-D^{(\nu)} f(t) := \int_0^\infty (f(t-r) - f(t)) \nu(dr), \quad \text{where} \quad 0 < \lambda := \int_0^\infty \nu(dr) < \infty.$$ 

To see this, observe that for every $\varepsilon, t > 0$

$$P[-Y^t(\tau_0(t)) > \varepsilon] \geq P[\text{first jump of } Y^t \text{ is greater than } t + \varepsilon] = \int_{t+\varepsilon}^\infty \frac{\nu(dr)}{\lambda},$$

and note that the right hand side is non-decreasing as $t \downarrow 0$, where $\tau_0$ is the right continuous inverse of $-Y^0$. As $\int_0^\infty \nu(dr) > 0$ we can choose $\varepsilon_0 > 0$ and $\bar{t} > 0$ so that

$$\inf_{t \leq \bar{t}} P[-Y^t(\tau_0(t)) > \varepsilon_0] \geq \int_{t+\varepsilon_0}^\infty \frac{\nu(dr)}{\lambda} =: c > 0.$$
Now, consider a continuous non-negative \( \phi \) with \( \phi(0) = 0 \), such that \( \inf_{r \in (-\infty, -\varepsilon_0]} \phi(r) > 0 \). Then for every \( t \leq \bar{t} \)
\[
\| \tilde{u}(t) - \phi(0) \| = E \left[ \phi(Y^{\bar{t}}(\tau_0(t))) \left( 1_{\{ -Y^{\bar{t}}(\tau_0(t)) > \varepsilon_0 \}} + 1_{\{ -Y^{\bar{t}}(\tau_0(t)) \leq \varepsilon_0 \}} \right) \right] \\
\geq E \left[ \phi(Y^{\bar{t}}(\tau_0(t)))1_{\{ -Y^{\bar{t}}(\tau_0(t)) > \varepsilon_0 \}} \right] \\
\geq \inf_{r \in (-\infty, -\varepsilon_0]} \phi(r) P \left[ -Y^{\bar{t}}(\tau_0(t)) > \varepsilon_0 \right] \\
\geq \inf_{r \in (-\infty, -\varepsilon_0]} \phi(r)c > 0.
\]

A.II. Proof of Lemma 2.6 (i)-(ii)-(iii). The three proofs are essentially the same, hence we prove only (ii).

Note that \( P_s^\beta \cdot P_r^\Omega = P_r^\Omega P_s^\beta \) for every \( s, r \geq 0 \), and that
\[
\| P_s^\beta \|_{C([0, \bar{t}] \times \Omega)} \quad \| P_s^\beta \cdot P_r^\Omega \|_{C([0, \bar{t}] \times \Omega)} \leq \| f \|_{C([0, \bar{t}] \times \Omega)},
\]
for every \( f \in C_{0, \partial \Omega}([0, \bar{t}] \times \Omega), s \geq 0 \). It is then easy to prove that \( P_s^\beta \cdot P_r^\Omega \) is sub-Feller semigroup on \( C_{0, \partial \Omega}([0, \bar{t}] \times \Omega) \). We denote the generator of \( P_s^\beta \cdot P_r^\Omega \) by \( \mathcal{L}_{\beta, \Omega}^\beta \cdot \mathcal{L}_{\beta, \Omega}^\Omega \).

Let \( f = pq \), where \( p \in C_{0, \partial \Omega}^\beta \) and \( q \in C_\Omega \). Then, by a standard triangle inequality argument, we obtain
\[
\left| \frac{P_h^\beta \cdot P_r^\Omega f(t, x) - f(t, x)}{h} - (\mathcal{L}_{\beta, \Omega}^\beta + \mathcal{L}_{\beta, \Omega}^\Omega)f(t, x) \right| \leq \| f \|_{C([0, \bar{t}])} \left( \left| \frac{P_h^\beta \cdot P_r^\Omega q - q}{h} \right|_{C(\Omega)} + \| \mathcal{L}_{\beta, \Omega}^\Omega \|_{C(\Omega)} \right) \| \frac{P_h^\beta \cdot P_r^\Omega p - p}{h} \|_{C([0, \bar{t}])} \\
+ \| q \|_{C(\Omega)} \left( \left| \frac{P_h^\beta \cdot P_r^\Omega p - p}{h} - \mathcal{L}_{\beta, \Omega}^\beta \right|_{C([0, \bar{t}])} \right) \to 0,
\]
as \( h \downarrow 0 \). An induction argument proves that \( \text{Span}\{C_{0, \partial \Omega}^\beta \cdot C_\Omega\} \subset \text{Dom}(\mathcal{L}_{\beta, \Omega}^\beta) \) and \( \mathcal{L}_{\beta, \Omega}^\beta \cdot \mathcal{L}_{\beta, \Omega}^\Omega = (\mathcal{L}_{\beta, \Omega}^\beta + \mathcal{L}_{\beta, \Omega}^\Omega) \) on \( \text{Span}\{C_{0, \partial \Omega}^\beta \cdot C_\Omega\} \). Observing that \( \text{Span}\{C_{0, \partial \Omega}^\beta \cdot C_\Omega\} \) is invariant under \( P_s^\beta \cdot P_r^\Omega \) and it is a subspace of \( \text{Dom}(\mathcal{L}_{\beta, \Omega}^\beta \cdot \mathcal{L}_{\beta, \Omega}^\Omega) \), if we can prove that \( \text{Span}\{C_{0, \partial \Omega}^\beta \cdot C_\Omega\} \) is dense in \( C_{0, \partial \Omega}([0, \bar{t}] \times \Omega) \), we are done by [13] Lemma 1.34. So proceed by noting that set \( \text{Span}\{C^\infty([0, \bar{t}] \times \Omega) \cdot C^\infty(\Omega)\} \) is a sub-algebra of \( C([0, \bar{t}] \times \Omega) \) that contains constant functions and separates points. Hence \( \text{Span}\{C^\infty([0, \bar{t}] \times \Omega) \cdot C^\infty(\Omega)\} \) is dense in \( C([0, \bar{t}] \times \Omega) \) by Stone-Weierstrass Theorem for compact Hausdorff spaces. We now prove density of the following set
\[
\text{Span}\{C^\infty_c([0, \bar{t}] \times \Omega) \cdot C^\infty_c(\Omega)\} \subset C_{0, \partial \Omega}([0, \bar{t}] \times \Omega).
\]

For \( f \in C_{0, \partial \Omega}([0, \bar{t}] \times \Omega) \) we take a sequence \( \{f_n\}_{n \in \mathbb{N}} \subset \text{Span}\{C^\infty([0, \bar{t}] \times \Omega) \cdot C^\infty(\Omega)\} \) such that \( f_n \to f \), where \( f_n(t, x) = \sum_{i=1}^{N_n} p_{i, n}(t)q_{i, n}(x) \), for some \( N_n \in \mathbb{N} \) depending on \( n \in \mathbb{N} \). Let \( 1_{T, n} \in C^\infty_c((0, \bar{t}) \times \Omega) \) and \( 1_{\Omega, n} \in C^\infty_c(\Omega) \) be smooth functions for each \( n \in \mathbb{N} \), such that
\footnote{In the case of unbounded domains (part (iii) of the current lemma) use the Stone-Weierstrass Theorem for locally compact Hausdorff spaces.}
Then, as $n \to \infty$

$$
\|\tilde{f}_n - f\|_{C([0,T] \times \Omega)} \leq \|f_n - f\|_{C((\frac{1}{n+1}, 1] \times K_n)} + \|\tilde{f}_n - f\|_{C((\frac{1}{n+1}, 1] \times \Omega)} \to 0.
$$

As $C_c^\infty(\Omega) \not\subset \text{Dom}(\mathcal{L}_\Omega)$ we need to work a bit more. For any $u \in C_{0,\partial \Omega}([0,T] \times \Omega)$ we can now take a uniformly approximating sequence $\{u_n\}_{n \in \mathbb{N}} \subset \text{Span}\{C_c^\infty((0,T]) \cdot C_c^\infty(\Omega)\}$. Denote $u_n(t, x) = \sum_{i=1}^{N_n} p_{i,n}(t)q_{i,n}(x)$, for some $N_n \in \mathbb{N}$ depending on $n \in \mathbb{N}$, where $p_{i,n} \in C_c^\infty(\Omega)$ are non-zero, for each $i \in \{1, \ldots, N_n\}$, $n \in \mathbb{N}$. As $C_\beta$ and $C_\Omega$ are dense in $C_0([0,T]) \supset C_c^\infty((0,T])$ and $C_{0,\partial \Omega}(\Omega) \supset C_c^\infty(\Omega)$, respectively, we can pick $\{(\tilde{p}_{i,n}, \tilde{q}_{i,n}) : i \in \{1, \ldots, N_n\}, n \in \mathbb{N}\} \subset C_\beta \times C_\Omega$, in the following fashion: for each triplet $(N_n, p_{i,n}, q_{i,n})$, first pick $\tilde{p}_{i,n}$ so that

$$
\|p_{i,n} - \tilde{p}_{i,n}\|_{C[0,T]} \leq \frac{1}{nN_n\|q_{i,n}\|_{C[0,T]}},
$$

secondly pick $\tilde{q}_{i,n}$ so that

$$
\|q_{i,n} - \tilde{q}_{i,n}\|_{C[0,T]} \leq \frac{1}{nN_n\|\tilde{p}_{i,n}\|_{C[0,T]}}.
$$

Then, after defining $\tilde{u}_n(t, x) := \sum_{i=1}^{N_n} \tilde{p}_{i,n}(t)\tilde{q}_{i,n}(x)$, we obtain

$$
\|u - \tilde{u}_n\|_{\infty} \leq \|u - u_n\|_{\infty} + \|u_n - \tilde{u}_n\|_{\infty}
$$

$$
\leq \|u - u_n\|_{\infty} + \sum_{i=1}^{N_n} \|p_{i,n}q_{i,n} - \tilde{p}_{i,n}\tilde{q}_{i,n}\|_{\infty}
$$

$$
\leq \|u - u_n\|_{\infty} + \sum_{i=1}^{N_n} (\|q_{i,n}\|_{\infty}\|p_{i,n}\|_{\infty} + \|\tilde{p}_{i,n}\|_{\infty}\|q_{i,n} - \tilde{q}_{i,n}\|_{\infty})
$$

$$
\leq \|u - u_n\|_{\infty} + \sum_{i=1}^{N_n} \left( \frac{\|q_{i,n}\|_{\infty}}{nN_n\|\tilde{q}_{i,n}\|_{C[0,T]}} + \frac{\|\tilde{p}_{i,n}\|_{\infty}}{nN_n\|\tilde{p}_{i,n}\|_{C[0,T]}} \right)
$$

$$
= \|u - u_n\|_{\infty} + \sum_{i=1}^{N_n} \frac{2}{nN_n}
$$

$$
\leq \|u - u_n\|_{\infty} + \frac{2}{n} \to 0, \quad \text{as } n \to \infty.
$$
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