QUOTIENT SINGULARITIES IN THE GROTHENDIECK RING OF VARIETIES

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ABSTRACT. Let $G$ be a finite group, $X$ be a smooth complex projective variety with a faithful $G$-action, and $Y$ be a resolution of singularities of $X/G$. Larsen and Lunts asked whether $[X/G] - [Y]$ is divisible by $[\mathbb{A}^1]$ in the Grothendieck ring of varieties. We show that the answer is negative if $BG$ is not stably rational and affirmative if $G$ is abelian. The case when $X = \mathbb{Z}^n$ for some smooth projective variety $Z$ and $G = S_n$ acts by permutation of the factors is of particular interest. We make progress on it by showing that $[\mathbb{Z}^n/S_n] - [\mathbb{Z} \langle n \rangle / S_n]$ is divisible by $[\mathbb{A}^1]$, where $\mathbb{Z} \langle n \rangle$ is Ulyanov’s polydiagonal compactification of the $n$-th configuration space of $Z$.

1. INTRODUCTION

For a field $k$, the Grothendieck ring of varieties $K_0(\text{Var}_k)$ is generated as an abelian group by isomorphism classes $[X]$ of separated $k$-schemes $X$ of finite type, subject to the relations $[X \setminus Y] = [X] - [Y]$, where $Y$ is a closed subscheme of $X$. The product of classes is induced by the fiber product of varieties over $k$.

The class $[X]$ of a variety $X$ in $K_0(\text{Var}_k)$ retains a great deal of geometric information about $X$. This information is often expressed by motivic measures, i.e., ring homomorphisms $K_0(\text{Var}_k) \to A$. When $k$ has characteristic zero, one important example is the homomorphism

$$sb: K_0(\text{Var}_k) \to \mathbb{Z}[\text{SB}_k].$$

Here $\text{SB}_k$ is the monoid of stable birational equivalence classes of smooth projective varieties over $k$, and $sb$ is the homomorphism that sends a smooth projective connected variety to its stable birational equivalence class. By a result of Larsen and Lunts [13], $sb$ induces an isomorphism $K_0(\text{Var}_k)/\langle \mathbb{L} \rangle \sim \mathbb{Z}[\text{SB}_k]$, where $\mathbb{L} = [\mathbb{A}^1_k]$. Therefore, it’s natural to consider classes of varieties $X$ in the quotient $K_0(\text{Var}_k)/\langle \mathbb{L} \rangle$, but the interpretation of such classes is less clear when $X$ is not smooth.

Let $Z$ be a smooth projective variety. By definition, the $n$th symmetric power $\text{Sym}^n(Z)$ is the quotient of $Z^n$ by the action of the symmetric group $S_n$ which permutes the factors of $Z^n$. When $\dim(Z) > 1$ and $n > 1$, $\text{Sym}^n(Z)$ is singular. The classes of symmetric powers in the Grothendieck ring are of particular interest because they appear in the motivic zeta function of $Z$:

$$\zeta_Z(t) := \sum_{n=0}^{\infty} [\text{Sym}^n(Z)] t^n.$$
When $Z$ is defined over a finite field, $\zeta_Z(t)$ specializes to the ordinary Weil zeta function under the point-counting measure. However, unlike the Weil zeta function, the motivic zeta function $\zeta_Z(t)$ is not always rational [14, Theorem 1.1]. Not much is known in general about the motivic zeta function or the classes of symmetric powers. In particular, the following basic question of Larsen and Lunts is open.

**Question 1.1.** (Larsen and Lunts [14, Question 6.7]) (i) Let $Z$ be a smooth projective complex variety, $n \geq 1$ be an integer, and $Y \to \text{Sym}^n(Z)$ be a resolution of singularities. Does the equality

$$[Y] = [\text{Sym}^n(Z)]$$

hold in the ring $K_0(\text{Var}_\mathbb{C})/(L)$?

(ii) Let $X$ be a smooth projective complex variety with the action of a finite group $G$. If $Y \to X/G$ is a resolution of singularities, do we have

$$[Y] = [X/G]$$

in $K_0(\text{Var}_\mathbb{C})/(L)$?

Question (i) is a special case of (ii).

Question 1.1 arose as part of the study of the rationality of the motivic zeta function in [14]. Let $\mathbb{Z}[s]$ be a polynomial ring in one variable $s$ and $M \subset \mathbb{Z}[s]$ be the sub-monoid consisting of polynomials with constant term 1. Larsen and Lunts considered a motivic measure

$$\mu: K_0(\text{Var}_\mathbb{C}) \to \mathbb{Z}[M]$$

which sends the class of a $d$-dimensional smooth projective irreducible complex variety $X$ to $\sum_{i=0}^{d} h^{1,0}(X)s^i$. This homomorphism factors through $K_0(\text{Var}_k)/(L)$. Both $K_0(\text{Var}_\mathbb{C})$ and $\mathbb{Z}[M]$ have natural structures as $\lambda$-rings, and it is natural to expect that $\mu$ is a $\lambda$-homomorphism. As observed by Larsen and Lunts, an affirmative answer to part (i) implies that $\mu$ is a $\lambda$-homomorphism.

Our first main result shows that the answer to Question 1.1(ii) is negative. Let $G$ be a finite group and let $k$ be a field. We say that the classifying space $BG$ is not stably rational over $k$ if the $k$-variety $V/G$ is not stably rational for a faithful $G$-representation $V$. This definition is closely related to the famous Noether’s problem on rationality of fields of invariants. It is independent of the choice of $V$ because the stable birational equivalence class of $V/G$ does not depend on the faithful representation by Bogomolov-Katsylo’s double fibration method (see, e.g., the proof of [19, Theorem 2.5]).

There are many known examples of groups $G$ with $BG$ not stably rational. Swan [18] and Voskresenskiĭ [22] independently showed that $B(\mathbb{Z}/47\mathbb{Z})$ is not stably rational over $\mathbb{Q}$. Lenstra [11] later classified the finite abelian groups $A$ such that $BA$ is stably rational over $k$; in particular, he showed that $B(\mathbb{Z}/8\mathbb{Z})$ is not stably rational over $\mathbb{Q}$. When $k$ is algebraically closed, the first examples of groups with this property were constructed by Saltman [16] and Bogomolov [4]. For example, if $n$ is any positive integer such that $2^6|n$ or $p^3|n$ for some odd prime $p$, then there exists a group $G$ of order $n$ such that $BG$ is not stably rational over $k$ [9, Theorem 1.13].
For any $G$ such that $BG$ is not stably rational over $k$, we produce an example answering Question 1.1(ii) in the negative.

**Theorem 1.2.** Let $k$ be a field of characteristic zero, $G$ be a finite $k$-group, $V$ be a faithful $k$-linear representation of $G$ containing at least one copy of the trivial representation, and $Y \to \mathbb{P}(V)/G$ be a resolution of singularities. Then $[Y] = [\mathbb{P}(V)/G]$ in $K_0(\text{Var}_k)/\mathbb{L}$ if and only if $BG$ is stably rational over $k$.

Conversely, we show that Question 1.1(ii) has a positive answer when $G$ is abelian. This means that the motivic measure $sb$ can be used directly to detect the stable birational equivalence class of $X/G$ when $X$ is a smooth complex projective variety and $G$ is abelian. We refer the reader to Definition 2.7 for the definition of $\mathbb{L}$-rational singularities.

**Theorem 1.3.** Let $G$ be a finite discrete abelian group of exponent $e$, $k$ be a field of characteristic zero containing a primitive $e$th root of unity, and $X$ be a smooth projective $k$-variety with a faithful $G$-action. Then $X/G$ has $\mathbb{L}$-rational singularities. In particular, if $Y$ is a resolution of singularities of $X/G$, then $[Y] = [X/G]$ in $K_0(\text{Var}_k)/\mathbb{L}$.

Finally, we make progress toward answering Question 1.1(i). For every $n \geq 1$, let $X(n)$ be the polydiagonal compactification of the configuration space of $n$ distinct points on $X$, first constructed by Ulyanov [20]. This model, which is “larger” than the Fulton-Macpherson compactification $X[n]$ of [6], is obtained by successively blowing up certain diagonals in $X^n$ in an $S_n$-equivariant way. In particular, $X(n)$ admits an $S_n$-action and the natural morphism $X(n) \to X^n$ is $S_n$-equivariant. This map factors as $X(n) \to X[n] \to X^n$.

We refer the reader to Definition 2.7 for the definition of $\mathbb{L}$-rational fibers.

**Theorem 1.4.** Let $k$ be a field of characteristic zero. Then the morphism $X(n)/S_n \to X^n/S_n$ has $\mathbb{L}$-rational fibers. In particular, $[X(n)/S_n] = [X^n/S_n]$ in $K_0(\text{Var}_k)/\mathbb{L}$.

This does not answer Question 1.1(i) because $X(n)/S_n$ is typically singular. However, the stabilizers of the $S_n$-action on $X(n)$ are simpler than on $X^n$, in fact even simpler than on $X[n]$: they are abelian [20, Theorem 3.11]. In particular, the singularities of $X(n)/S_n$ may be amenable to study by toroidal methods.

**Notation.** We work over a field $k$, with algebraic closure $\overline{k}$. A $k$-variety is a separated integral $k$-scheme of finite type. If $X$ is a $k$-scheme and $x$ is a point of $X$, we write $k(x)$ for the residue field of $x$.

A $k$-group is a smooth affine group scheme of finite type over $k$. If $G$ is a finite $k$-group and $X$ is a quasi-projective $k$-scheme with a $G$-action, we denote by $X/G$ the quotient $k$-scheme. We let $\mathbb{G}_{m,X} := \mathbb{G}_{m,k} \times_k X$ denote the multiplicative group scheme over $X$.

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2. Preliminaries

2.1. Quotients in the Grothendieck Ring. We first prove some properties of quotients that will be used throughout the paper. We also use these results to prove Theorem 1.2. We do not assume that $\text{char}(k) = 0$ until it is necessary.

**Lemma 2.1.** Let $k$ be a field, $G$ be a finite $k$-group, $X = \coprod_{i=1}^{r} X_i$ be a disjoint union of quasi-projective $k$-schemes such that $G$ acts on $X$ and the $G$-action transitively permutes the $X_i$. Let $G_1 \subset G$ be the stabilizer of $X_1$. Then the open and closed embedding $X_1 \to X$ induces an isomorphism $X_1/G_1 \sim \to X/G$. Furthermore, if $f : Y \to X$ is a $G$-equivariant morphism, with $Y_i := f^{-1}(X_i)$ and $f_i := f|_{Y_i}$, then $X_1 \to X$ and $Y_1 \to Y$ induce a commutative diagram

$$
\begin{array}{ccc}
Y_1/G_1 & \sim \to & Y/G \\
\downarrow & & \downarrow \\
X_1/G_1 & \sim \to & X/G,
\end{array}
$$

where the vertical maps are induced by $f_1$ and $f$, respectively.

**Proof.** The commutativity of the diagram is clear, so it suffices to show that the horizontal maps are isomorphisms. For this, we are allowed to base change to $\bar{k}$, hence we may assume that $k$ is algebraically closed; in particular, $G$ is a finite discrete group. We may also suppose that $X$ and $Y$ are affine. The conclusion is implied by the following simple observation: if a finite group $G$ acts on a $k$-algebra $A$ via $k$-algebra automorphisms, $A = \prod_{i=1}^{r} A_i$, the $A_i$ are transitively permuted by the $G$-action, and $G_1 \subset G$ is the stabilizer of $A_1$, then $A^G = (A_1)^{G_1}$. \hfill \Box

Let $G$ be a finite $k$-group, $X$ be a $k$-scheme with a $G$-action, and $\pi : P \to X$ be a $\mathbb{G}_{m,X}$-torsor. We say that $\pi$ is a $G$-equivariant $\mathbb{G}_{m,X}$-torsor if $G$ acts on $P$ and the $G$-action on $P$ commutes with the $\mathbb{G}_{m,X}$-action, that is, the action map $\mathbb{G}_{m,X} \times X P \to P$ is $G$-equivariant, where $G$ acts on $\mathbb{G}_{m,X} = \mathbb{G}_{m,k} \times_k X$ via its action on $X$. By restricting the action map to $\{1\} \times P$, this implies that $\pi$ is $G$-equivariant.

**Proposition 2.2.** Let $k$ be a field, $G$ be a finite $k$-group, $X$ be a quasi-projective $k$-scheme of finite type, and $\pi : P \to X$ be a $G$-equivariant $\mathbb{G}_{m,X}$-torsor. Then

$$[P/G] = (\mathbb{L} - 1)[X/G]$$

in $K_0(\text{Var}_k)$.

**Proof.** The conclusion of Proposition 2.2 is clear if $G$ is the trivial group because $\mathbb{G}_{m,X}$-torsors are Zariski-locally trivial. When $G$ is arbitrary, by induction we may suppose that the conclusion of Proposition 2.2 is true for all proper quotients of $G$. The proposition also holds for the group $G$ and torsors over quasi-projective schemes of dimension zero, since we can use Lemma 2.1 to reduce to a single point. Therefore, by induction, we can also assume that the conclusion of the proposition holds for the restriction of $\pi : P \to X$ to any $G$-invariant closed subscheme $Y \subset X$ of strictly smaller dimension. The scheme $X$ may consist of many irreducible components that intersect one another, but by the scissor
In particular, $G$ acts freely on the open subscheme $X$. We may assume that Claim 2.3.

We may assume that $G$ acts faithfully on $P$.

Let $K \subset G$ be the kernel of the $G$-action on $P$. Then $K$ is contained in the kernel of the $G$-action on $X$, hence we may regard $\pi$ as a $G/K$-equivariant $\mathbb{G}_{m,X}$-torsor. If $K$ is a non-trivial subgroup of $G$, then $G/K$ is a proper quotient of $G$ and the conclusion follows from the inductive assumption. This means that we may assume that $K$ is trivial, thus proving Claim 2.3.

Claim 2.4. We may assume that all $\bar{k}$-points of $X$ have the same $G$-stabilizer $H$ and that $G$ acts freely on $P$.

Let $H \subset G$ be the kernel of the $G$-action on $X$, so that $H$ is a normal subgroup. Since $G$ is finite and $X$ is irreducible, there exists a dense open subscheme $U \subset X$ such that for all geometric points $x \in U(\bar{k})$, the $G$-stabilizer of $x$ is equal to $H$. The complement $X \setminus U$ is a union of subvarieties of lower dimension, hence we may replace $X$ by $U$ and $P$ by $P|_U$, so that the $G$-stabilizer of every $x \in X(\bar{k})$ is equal to $H$. In particular, every $G$-stabilizer in $P$ is a subgroup of $H$, and so coincides with the $H$-stabilizer.

By Claim 2.3, we may assume that $G$ acts faithfully on $P$. Since $G$ is finite, this implies the existence of a dense open subscheme $V$ of $P$ such that $G$ acts freely on $V$.

Since $G$ acts on $P$ via $\mathbb{G}_{m,X}$-torsor automorphisms and $H$ acts trivially on $X$, for all $x \in X(\bar{k})$, $p \in P_x(\bar{k})$, $\lambda \in \mathbb{G}_{m,k}(\bar{k})$, and $h \in H$, we have $h \cdot p = P_x(\bar{k})$ and

$$h \cdot p = p \quad \Rightarrow \quad h \cdot (\lambda p) = \lambda(h \cdot p) = \lambda p.$$ 

This shows that the $H$-stabilizer of $p$ is contained the $H$-stabilizer of $\lambda p$. Replacing $p$ by $\lambda p$ and $\lambda$ by $\lambda^{-1}$ then shows that the $H$-stabilizer of $p$ is equal to the $H$-stabilizer of $\lambda p$. In other words, the $G$-stabilizers of geometric points of $P$ (which as we said in the previous paragraph coincide with their $H$-stabilizers) are constant along each fiber of $\pi$. In particular, $G$ acts freely on the open subscheme $\mathbb{G}_{m,k} \cdot V = \pi^{-1}(\pi(V)) \subset P$. By the inductive assumption, we are allowed to replace $X$ by the open subscheme $\pi(V)$ and $P$ by $P|_{\pi(V)}$. This proves Claim 2.4.

We are now ready to complete the proof of Proposition 2.2. We may assume that Claim 2.4 holds. We may also replace $X$ by a dense open subscheme over which $\pi$ is trivial, that is, we may identify $\pi$ with the second projection $\mathbb{G}_{m,X} \to X$. The $H$-action on the fibers of $\pi$ induces an $X$-group embedding $H_X \hookrightarrow \mathbb{G}_{m,X}$. By the classification of isotrivial groups of multiplicative type [8, Exposé X, Corollaire 1.2], the proper isotrivial $X$-subgroups of $\mathbb{G}_{m,X}$ are all of the form $\mu_{n,X}$ for some integer $n \geq 1$. Therefore, $H_X$ is $X$-isomorphic to $\mu_{n,X}$ for some integer $n \geq 1$, and this isomorphism identifies the action of $H_X$ on $P$ with the $\mu_{n,X}$-action obtained by restricting the $\mathbb{G}_{m,X}$-torsor action. (If $X(k)$ is empty, $H_X \cong \mu_{n,X}$ does not necessarily imply that $H \cong \mu_n$ over $k$.)
Therefore, the morphism \( P/G \to X/G \) can be obtained by first taking the quotient of \( P \) by the action of \( H_X \cong \mu_{n,X}, \) and then taking the quotient by the action of \((G/H)_X\). Said otherwise, \( P/G \to X/G \) is the same as \((P/\mu_{n,X})/(G/H) \to X/(G/H), \) that is, the quotient of \( P/\mu_{n,X} \to X \) by the induced \( G/H \)-action. Note that \( \mathbb{G}_{m,X} \)-action of scalar multiplication on \( P \) descends to a \( \mathbb{G}_{m,X} \)-action on \( P/\mu_{n,X} \to X, \) with action map fitting into the commutative square

\[
\begin{array}{ccc}
\mathbb{G}_{m,X} \times_X P & \longrightarrow & P \\
\downarrow & & \downarrow \\
\mathbb{G}_{m,X} \times_X (P/\mu_{n,X}) & \longrightarrow & P/\mu_{n,X},
\end{array}
\]

where the vertical map on the left sends \((t,p) \mapsto (t^n, [p])\), and the map on the right sends \( p \mapsto [p]. \) (Equivalently, we quotient both \( \mathbb{G}_{m,X} \) and \( P \) by \( \mu_{n,X} \) and then identify \( \mathbb{G}_{m,X}/\mu_{n,X} \) with \( \mathbb{G}_{m,X}. \))

If \( H \) is a non-trivial subgroup of \( G, \) then the conclusion follows from the inductive assumption. If \( H \) is the trivial subgroup of \( G, \) then by Claim 2.4 the group \( G \) acts freely on \( X \) and \( P. \) In this case, by descent along étale-torsors [21, Theorem 4.46] the quotient \( P/G \to X/G \) is a \( \mathbb{G}_{m,X}/G \)-torsor, hence a Zariski-locally trivial fibration with fibers isomorphic to \( \mathbb{A}^1 \setminus \{0\}, \) and we conclude that \([P/G] = (\mathbb{L} - 1)[X/G]). \)

**Corollary 2.5.** Let \( G \) be a finite \( k \)-group, \( W \) be a finite \( k \)-scheme with a \( G \)-action such that \( W/G = \text{Spec}(k), \) and \( V \) be a \( G \)-equivariant vector bundle on \( W. \) Then

\[ [V/G] = (\mathbb{L} - 1)[\mathbb{P}(V)/G] + 1 \]

in \( K_0(\text{Var}_k). \) In particular, the following holds in \( K_0(\text{Var}_k)/(\mathbb{L}): \) \([\mathbb{P}(V)/G] = 1 \) if and only if \([V/G] = 0). \)

**Proof.** Consider the \( \mathbb{G}_m \)-torsor \( \pi: V \setminus s(W) \to \mathbb{P}(V), \) where \( s: W \to V \) is the zero section of the vector bundle. Since \( G \) acts linearly on \( V \) over \( W, \) the \( G \)-action commutes with the \( \mathbb{G}_m \)-action, that is, \( \pi \) is a \( G \)-equivariant \( \mathbb{G}_m \)-torsor. Therefore, the assumptions of Proposition 2.2 are satisfied. By Proposition 2.2, we have

\[ [V/G] = [(V \setminus s(W))/G] + [s(W)/G] = (\mathbb{L} - 1)[\mathbb{P}(V)/G] + 1 \]

Corollary 2.5 is an important step towards our proof of Theorem 1.4. It also allows us to prove Theorem 1.2.

**Proof of Theorem 1.2.** By assumption, \( V = V' \oplus k \) for some \( k \)-linear representation \( V' \) of \( G. \) Note that \( V' \) is faithful because \( V \) is faithful and \( G \) acts trivially on \( k. \) The variety \( \mathbb{P}(V) \) is a union of the \( G \)-invariant affine chart isomorphic to \( V' \) and its complement \( \mathbb{P}(V'), \) both with the natural \( G \)-action. Therefore, in \( K_0(\text{Var}_k) \) we have

\[ [\mathbb{P}(V)/G] = [V'/G] + [\mathbb{P}(V')/G] = (\mathbb{L} - 1)[\mathbb{P}(V')/G] + 1 + [\mathbb{P}(V')/G] = \mathbb{L}[\mathbb{P}(V')/G] + 1, \]

where the second equality follows from Corollary 2.5. This implies that \([\mathbb{P}(V)/G] = 1 \) in \( K_0(\text{Var}_k)/(\mathbb{L}). \) However, \( \mathbb{P}(V)/G \) contains the open subvariety \( V'/G \) which is stably rational if and only if \( BG \) is. Therefore, if \( Y \to \mathbb{P}(V)/G \) is any resolution, \([Y] \equiv 1 \) in
We conclude this section with the observation that Proposition 2.2 allows one to compute the class of the quotient of a \(G\)-equivariant line bundle.

**Remark 2.6.** Let \(k\) be a field, \(G\) be a finite group, \(X\) be a quasi-projective \(k\)-scheme of finite type, and \(L \to X\) be a \(G\)-equivariant line bundle. Let \(s: X \to L\) be the zero section of \(L\), and let \(P := L \setminus s(X)\). Then \(L/G\) is the union of a closed subscheme isomorphic to \(X/G\) and with complement \(P/G\). (This is automatic when \(\text{char}(k) = 0\), but is true in arbitrary characteristic: the \(G\)-equivariant surjection \(O_P \to O_X\) induced by \(s\) is \(G\)-equivariantly split, and so the induced map \(O_P^G \to O_X^G\) is also surjective.) Since \(P\) is a \(G\)-equivariant \(\mathbb{G}_m\)-torsor, Proposition 2.2 implies

\[
[L/G] = [P/G] + [X/G] = \mathbb{L}[X/G]
\]

in \(K_0(\text{Var}_k)\).

### 2.2. Toroidal embeddings and \(\mathbb{L}\)-rational singularities.

In this section, we’ll introduce \(\mathbb{L}\)-rational singularities and some of the language of toroidal embeddings. These notions will be used to prove the equalities of classes in \(K_0(\text{Var}_k)/(\mathbb{L})\) given in Theorem 1.3 and Theorem 1.4.

**Definition 2.7.** (cf. [15, Definition 4.2.4]) Let \(k\) be a field and \(X\) and \(Y\) be \(k\)-varieties. We say that a proper morphism of schemes \(h: Y \to X\) has \(\mathbb{L}\)-rational fibers if for each point \(x\) of the scheme \(X\), \([h^{-1}(x)] = 1\) in \(K_0(\text{Var}_k(x))/(\mathbb{L})\). We say that \(X\) has \(\mathbb{L}\)-rational singularities if there exists a resolution of singularities \(h: Y \to X\) with \(\mathbb{L}\)-rational fibers.

These definitions are local on \(X\) in the Zariski topology. As observed after [15, Definition 4.2.4], the weak factorization theorem in characteristic zero [2] implies that the definition of \(\mathbb{L}\)-rational singularities is independent of the choice of resolution of \(X\). The following lemma relates this local definition to the global equality of classes:

**Lemma 2.8.** [15, Lemma 4.2.5] Let \(k\) be a field and \(h: Y \to X\) be a proper morphism of \(k\)-varieties. If \(h\) has \(\mathbb{L}\)-rational fibers, then \([Y] = [X]\) in \(K_0(\text{Var}_k)/(\mathbb{L})\).

In particular, this implies that if \(X\) has \(\mathbb{L}\)-rational singularities, the class of \(X\) equals the class of its resolution modulo \(\mathbb{L}\).

**Lemma 2.9.** The class of morphisms with \(\mathbb{L}\)-rational fibers is closed under base change and composition.

**Proof.** First, we prove stability under base change. Suppose that \(h: X \to Y\) is a proper morphism with \(\mathbb{L}\)-rational fibers and that \(g: S \to Y\) is an arbitrary morphism of \(k\)-varieties. Let \(f: X \times_Y S \to S\) be the base change, \(s \in S\) be a point of the scheme \(S\), and \(y = g(s)\) the image in \(Y\). We have a commutative diagram
Since the two inner squares are pullbacks, the outer rectangle is as well. The morphism \( \text{Spec}(k(s)) \to Y \) factors as \( \text{Spec}(k(s)) \to \text{Spec}(k(y)) \to Y \), where \( k(s)/k(y) \) is an extension of fields. Thus, in the new diagram

\[
\begin{array}{ccc}
\text{Spec}(k(s)) & \longrightarrow & \text{Spec}(k(y)) \\
\downarrow & & \downarrow \\
S & \longrightarrow & Y
\end{array}
\]

the outer and right rectangles are pullbacks, so the left is too. Since \([h^{-1}(y)] = 1\) in \( K_0(\text{Var}_k(y))/(\mathbb{L}) \) by assumption, we have \([f^{-1}(s)] = 1\) in \( K_0(\text{Var}_k(s))/(\mathbb{L}) \) by functoriality of the Grothendieck ring.

Now we’ll prove stability under composition. Suppose that \( g: X \to Y \) and \( h: Y \to Z \) are proper with \( \mathbb{L} \)-rational fibers and let \( z \in Z \) be a point. We have \([h^{-1}(z)] = 1\) in \( K_0(\text{Var}_k(z))/(\mathbb{L}) \) by assumption. Further, \( g^{-1}(h^{-1}(z)) \to h^{-1}(z) \) has \( \mathbb{L} \)-rational fibers by stability under base change. Then by Lemma 2.8, \([g^{-1}(h^{-1}(z))] = [h^{-1}(z)] = 1\) in \( K_0(\text{Var}_k(z))/(\mathbb{L}) \), completing the proof.

As a consequence, we also obtain some useful properties of \( \mathbb{L} \)-rational singularities.

**Lemma 2.10.** Let \( g: X' \to X \) be a proper birational morphism of \( k \)-varieties with \( \mathbb{L} \)-rational fibers. Then, if \( X' \) has \( \mathbb{L} \)-rational singularities, so does \( X \).

**Proof.** Let \( h: Y \to X' \) be a resolution of singularities, which has \( \mathbb{L} \)-rational fibers by assumption. Since \( g \) is birational, \( g \circ h: Y \to X \) is also a resolution of singularities. The morphism \( g \circ h \) is also proper with \( \mathbb{L} \)-rational fibers by Lemma 2.9, completing the proof.

**Lemma 2.11.** If a \( k \)-variety \( X \) has \( \mathbb{L} \)-rational singularities and \( f: X' \to X \) is an \( \acute{e} \text{tale} \) morphism, then \( X' \) has \( \mathbb{L} \)-rational singularities as well.

**Proof.** If \( Y \to X \) is a resolution of singularities, then the pullback \( Y'' = Y \times_X X' \to X' \) is also a resolution of singularities. This is because the properties of being proper and birational are preserved under pullback by an \( \acute{e} \text{tale} \) morphism, while \( Y'' \) is an \( \acute{e} \text{tale} \) cover of a smooth variety \( Y \) and hence is smooth. By Lemma 2.9, \( Y' \to X' \) has \( \mathbb{L} \)-rational fibers, completing the proof.

Next, we’ll introduce some of the language of toroidal embeddings. This machinery is used for the proof of Theorem 1.3 in Section 3, but is not required for Theorem 1.4. The datum of a toroidal embedding consists of a normal variety \( X \) over a field \( k \) and a non-empty open subscheme \( U \subset X \) with the property that \( (X, U) \) looks \( \acute{e} \text{tale} \)-locally like the embedding of the open torus orbit in a toric variety. That is, for every point \( x \) of the
scheme $X$, there is an étale neighborhood $f: V \to X$ of $x$ and a toric variety $(X_\sigma, T_\sigma)$ such that $V$ admits an étale morphism $g: V \to X_\sigma$ and $g^{-1}(T_\sigma) = f^{-1}(U)$. A toroidal embedding is strict if we instead require that for each $x \in X$ there is a Zariski open neighborhood $V$ of $x$ satisfying the same property.

There are equivalent definitions of toroidal and strictly toroidal originally due to Mumford ([10, p. 195, footnote]; see also [1, Section 2] or [23, Section 4] for more on these definitions).

**Lemma 2.12.** [10, p. 195, footnote], [23, Lemma 4.8.3] Let $k$ be a field of characteristic zero, $\bar{k}$ be an algebraic closure of $k$, $X$ be a normal $k$-variety, and $U$ be a non-empty open subscheme of $X$.

1. The pair $(X, U)$ is a toroidal embedding if and only if for every closed point $x$ in $X_{\bar{k}}$, there exists a toric variety $(X_\sigma, T_\sigma)$ over $\bar{k}$, a point $t \in X_\sigma$ and an isomorphism
   \[ \varphi: \hat{O}_{X_{\bar{k}}, x} \to \hat{O}_{X_\sigma, t} \]
   of complete local algebras over $\bar{k}$ with the property that $\varphi$ takes the ideal of $X_{\bar{k}} \setminus U_{\bar{k}}$ to the ideal of $X_\sigma \setminus T_\sigma$.
2. A toroidal embedding $(X, U)$ is strict if and only if each irreducible component of $X \setminus U$ is normal.

Strict toroidal embeddings enjoy many of the properties of toric varieties. In particular, the singularities of strictly toroidal embeddings are $\mathbb{L}$-rational [15, Example 4.2.6(3)]. This is because toric singularities are $\mathbb{L}$-rational and strictly toroidal embeddings Zariski-locally admit étale morphisms to toric varieties, so we may apply Lemma 2.11.

Let $(X, U)$ be a toroidal embedding, and let $G$ be a finite discrete group acting on $X$ and leaving $U$ invariant. Following [1, Section 2.3], we say that the action of $G$ on $(X, U)$ is toroidal if for each closed point $x$ in $X_{\bar{k}}$ the following holds: there exists a local model $(X_\sigma, T_\sigma)$ as in Lemma 2.12(1) and a homomorphism $\alpha: G_x \to T_\sigma$ from the stabilizer $G_x$ of $x$ under the induced action to the torus such that the action of $G_x$ on $\hat{O}_{X_{\bar{k}}, t}$ via the isomorphism $\varphi$ agrees with the action on the toric variety $X_\sigma$ via $\alpha$. We say the $G$-action on $(X, U)$ is strict if $X \setminus U = D$ has the property that $\bigcup_{g \in G} g \cdot Z$ is normal for every irreducible component $Z$ of $D$. Equivalently, $\bigcup_{g \in G} g \cdot Z$ is a disjoint union of normal components, so that either $g \cdot Z = Z$ or $g \cdot Z \cap Z = \emptyset$.

In the event that we have a strict toroidal action on a strict toroidal embedding, the quotient also inherits a strict toroidal structure (see [1, Section 2.3], [3, Proposition 2.5]):

**Proposition 2.13.** If $(X, U)$ is a strict toroidal embedding on which $G$ acts strictly and toroidally, then $(X/G, U/G)$ is also a strict toroidal embedding.

### 3. Diagonalizable Groups

Recall that a $k$-group $G$ is diagonalizable if it embeds in $\mathbb{G}_{m,k}^n$ for some $n \geq 1$. If $G$ is an abstract finite group, then $G$ is diagonalizable if and only if $G$ is abelian and $k$ contains a primitive root of unity of order equal to the exponent of $G$.

Let $X$ be a smooth projective variety. In this section, we prove Theorem 1.3, namely that the singularities of a quotient $X/G$ are $\mathbb{L}$-rational in the special case that the abstract
finite group $G$ is diagonalizable. As a first step, we see that an equivariant blowup of $X$ in a smooth subvariety leaves the class of the quotient unchanged. This result holds for any (not necessarily discrete) diagonalizable $k$-group $G$.

**Proposition 3.1.** Suppose that $G$ is a diagonalizable $k$-group acting on a smooth quasi-projective variety $X$ and $f: Y \to X$ is an equivariant blowup of $X$ in a smooth proper subvariety $Z$. Then the induced morphism $h: Y/G \to X/G$ has $\mathbb{L}$-rational fibers. In particular,

$$[Y/G] = [X/G]$$

holds in $K_0(\text{Var}_k)/\langle \mathbb{L} \rangle$.

**Proof.** Since $f$ is equivariant, we must have that the action of $G$ on $X$ restricts to an action of $G$ on $Z$. Over $Z$, the morphism $f$ is a projective bundle $\mathbb{P}(N_{Z/X})$, where $N_{Z/X}$ is the normal bundle to $Z$ in $X$. Denote by $p: X \to X/G$ the quotient morphism. We’ll show that $[h^{-1}(p(x))] = 1$ in $K_0(\text{Var}_{k(p(x))})/\langle \mathbb{L} \rangle$ for any point $x \in X$. The quotient map $h: Y/G \to X/G$ is an isomorphism away from $Z/G$, so we need only consider the case that $x \in Z$. Suppose that $W$ is the orbit of the point $x$ under the $G$ action. It is a finite disjoint union of points.

The quotient $W/G$ is the point $\text{Spec}(k(p(x)))$ and the fiber of $h$ over $p(x)$ is the quotient of the projectivization $\mathbb{P}N_W$ by $G$, where $N_W$ is the pullback of the normal bundle to the scheme $W$ (here we use that projectivization commutes with base change). Therefore, we may apply Corollary 2.5 to conclude that $[h^{-1}(p(x))] = [\mathbb{P}(N_W)/G] = 1$ if and only if $[N_W/G] = 0$ in the ring $K_0(\text{Var}_{k(p(x))})/\langle \mathbb{L} \rangle$. The total space $N_W$ is a disjoint union of vector spaces over each point in $W$, which are transitively permuted by the $G$-action. Therefore, by Lemma 2.1, $N_W/G \cong N_x/G_1$, where $G_1 \subset G$ is the subgroup that stabilizes the point $x \in W$. It will suffice to prove that the class $[N_x/G_1]$ is trivial in $K_0(\text{Var}_{k(p(x))})/\langle \mathbb{L} \rangle$.

The fiber $N_x$ is a finite-dimensional vector space over $k(x)$, but $G_1$ may not act by $k(x)$-automorphisms because it also acts on the field $k(x)$ non-trivially. However, $k(x)$ is a finite extension of the subfield $k(p(x)) = k(x)^{G_1}$ and $G_1$ does act by $k(p(x))$-automorphisms. Since $G_1 \subset G$ is diagonalizable over $k$, it is certainly diagonalizable over $k(p(x))$, so $N_x/G_1$ is the quotient of a finite-dimensional vector space by a diagonalizable group action. This has trivial class in $K_0(\text{Var}_{k(p(x))})/\langle \mathbb{L} \rangle$ by the following lemma.

**Lemma 3.2.** Let $V$ be a vector space over a field $k$ with a diagonalizable action by a $k$-group $H$. Then $[V/H] = 0$ in $K_0(\text{Var}_k)/\langle \mathbb{L} \rangle$.

**Proof.** Using the scissor relations, we can decompose $V$ into coordinate strata, each of which is a split torus $(\mathbb{G}_{m,k})^d$ with a diagonal action of $H$. The quotient of this stratum by this action is again isomorphic to $(\mathbb{G}_{m,k})^d$, so the class in the Grothendieck ring is unchanged. Reassembling the pieces, we have $[V/H] = [V] = 0$ in $K_0(\text{Var}_k)/\langle \mathbb{L} \rangle$. □

This shows that the fiber of $h: Y/G \to X/G$ over the point $p(x)$ has class 1 in $K_0(\text{Var}_{k(p(x))})/\langle \mathbb{L} \rangle$ for any point $x$ of the scheme $X$. Since the quotient $X \to X/G$ is surjective, the image $p(x)$ ranges over every point of the scheme $X/G$ and the proof is complete. □
The proof of Theorem 1.3 relies on the existence of a $G$-equivariant blowup of $X$ on which the action of $G$ is well-behaved. We’ll apply the main theorem of [3]:

**Theorem 3.3.** (cf. [3, Theorem 0.1]) Let $X$ be a projective variety over a field $k$ of characteristic zero with an action by a finite (discrete) group $G$. Then there is a $G$-equivariant proper birational morphism $r : X_1 \to X$ and an open set $U \subset X_1$ such that $X_1$ is a nonsingular projective variety, $(X_1, U)$ has the structure of a strictly toroidal embedding, and $G$ acts strictly and toroidally on $(X_1, U)$.

The statement of the theorem differs slightly here from [3], but this is what their proof shows. That paper also assumes that $k$ is algebraically closed, but this assumption can be removed.

**Proof of Theorem 1.3.** Let $G$ be any discrete, diagonalizable $k$-group. Given a smooth $G$-variety $X$, find a birational morphism $X_1 \to X$ satisfying the conditions of Theorem 3.3. Then $X_1/G$ is strictly toroidal by Proposition 2.13. If $Y \to X_1/G$ is a resolution of singularities, we therefore have $[Y] = [X_1/G]$ in $K_0(\text{Var}_k)/((\mathbb{L})$. But $Y$ is also a resolution of $X/G$. By the $G$-equivariant weak factorization theorem, $X_1$ can be constructed from $X$ via some series of equivariant blowups and blowdowns in smooth centers. Using Proposition 3.1 at each step, we have $[X_1/G] = [X/G]$ in $K_0(\text{Var}_k)/((\mathbb{L})$. This completes the proof.

4. Symmetric Powers

In this section, we’ll prove Theorem 1.4, which shows that the class of a symmetric power $\text{Sym}^n(X)$ in $K_0(\text{Var}_k)/((\mathbb{L})$ equals the class of a certain birational modification with simpler singularities. This modification is the quotient of the polydiagonal compactification $X\langle n \rangle$ of the configuration space of $n$ distinct points on $X$ by $S_n$. In Section 4.1, we’ll prove some general facts about quotients of symmetric group representations in the Grothendieck ring. Section 4.2 will introduce the polydiagonal compactification and complete the proof of Theorem 1.4.

4.1. Stratification of Symmetric Group Representations. We’ll introduce some notation for partitions that will be used throughout the remainder of this section. For a positive integer $n$, let $P(n)$ denote the set of partitions of the set $[n] = \{1, \ldots, n\}$. Suppose $\pi$ is a partition in $P(n)$. We say that $\pi$ is of type $\mathbf{a} = (1^{m_1}, \ldots, n^{m_n})$ if it contains exactly $m_h$ blocks of size $h$, $h = 1, \ldots, n$. We’ll write $\pi_1, \ldots, \pi_i$ for the blocks of $\pi$, where $i = m_1 + \cdots + m_n$ is the total number of blocks. For each block $\pi_j$, $|\pi_j|$ denotes the number of elements in the block.

The group $S_n$ naturally acts on $[n] = \{1, \ldots, n\}$, hence on the set of partitions $P(n)$. The orbits of this action are precisely the sets of partitions of a fixed type $\mathbf{a}$. Let $S_\pi \subset S_n$ be the stabilizer of $\pi$. We can naturally write $S_\pi$ as a semi-direct product

$$S_\pi \cong S'_\pi \rtimes S_\pi.$$

Here $S'_\pi \cong S_{\pi_1} \times \cdots \times S_{\pi_i}$, where $S_{\pi_l}$ is the symmetric group on the elements of $\pi_l$. The elements of the subgroup $S'_\pi \subset S_n$ therefore preserve each block individually. The group $S_\pi$ is defined as $S_{m_1} \times \cdots \times S_{m_n}$, where $S_{m_h}$ permutes the $m_h$ factors $S_{\pi_l}$ such that $|\pi_l| = m_h$. 

Definition 4.1. We say that a morphism of \(k\)-varieties \(Y \to Z\) is stratified by vector bundles (with height \(N\)) if it can be written as a composition

\[ Y = Y_N \to Y_{N-1} \to \cdots \to Y_1 \to Y_0 = Z \]

such that for all \(i = 1, \ldots, N\) there exists a locally closed stratification of \(Y_{i-1}\) where the restriction of \(Y_i \to Y_{i-1}\) to each stratum is a vector bundle.

It follows from the definition that the composition of two morphisms stratified by vector bundles is also stratified by vector bundles. If a morphism \(f : Y \to Z\) is stratified by vector bundles, then \([f^{-1}(z)] = 0\) in \(K_0(\text{Var}_k(z))/(\mathbb{L})\) for every point \(z\) in \(Z\).

Now let \(X\) be a \(k\)-variety, \(E \to X\) be a vector bundle of rank \(r\), and \(d\) be a positive integer. We denote by \((E^{\oplus d})_0 \to X\) the \(S_d\)-equivariant vector subbundle of \(E^{\oplus d}\) given by the kernel of the addition map \(E^{\oplus d} \to E\). The symmetric group \(S_d\) acts on \(E^{\oplus d}\) over \(X\) by permutation of the \(d\) summands \(E\), and this action leaves \((E^{\oplus d})_0\) invariant.

Lemma 4.2. Let \(r, d\) be non-negative integers. Suppose that \(S_d\) acts on \((\mathbb{A}^r)^d\) by permutation of the \(d\) factors \(\mathbb{A}^r\), and consider the induced \(S_d\)-action on \((\mathbb{A}^r)^d_0\). The projection \(p : ((\mathbb{A}^r)^d_0)/S_d \to ((\mathbb{A}^r)^d_0)/S_d\) is stratified by vector bundles with height 1.

Proof. We denote by \(\iota_r : (\mathbb{A}^r)^d_0 \hookrightarrow (\mathbb{A}^r)^d\) the canonical linear inclusion as the kernel of the addition map. We view the trivial vector bundle \((\mathbb{A}^r)^d\) as the space of \(r\) by \(d\) matrices

\[
A = \begin{bmatrix}
    a_{1,1} & \cdots & a_{1,d} \\
    \vdots & \ddots & \vdots \\
    a_{r,1} & \cdots & a_{r,d}
\end{bmatrix}.
\]

In this notation, the \(S_d\)-action on \((\mathbb{A}^r)^d\) permutes the columns of \(A\), the map \(p\) is induced by forgetting the last row of \(A\), and \((\mathbb{A}^r)^d_0\) parametrizes the matrices \(A\) such that \(a_{i,1} + \cdots + a_{i,d} = 0\) for all \(1 \leq i \leq r\). We obtain a commutative diagram of \(S_d\)-equivariant linear maps

\[
\begin{array}{ccc}
(\mathbb{A}^r)^d_0 & \xrightarrow{\iota_r} & (\mathbb{A}^r)^d \\
\downarrow{p} & & \downarrow{p} \\
(\mathbb{A}^{r-1})^d_0 & \xleftarrow{\iota_{r-1}} & (\mathbb{A}^{r-1})^d,
\end{array}
\]

where the square on the right is cartesian. Note that \(\iota_{r-1}^*((\mathbb{A}^r)^d)\) parametrizes those matrices \(A\) for which \(a_{i,1} + \cdots + a_{i,d} = 0\) for all \(1 \leq i \leq r - 1\). The inclusion of \((\mathbb{A}^r)^d_0\) in \(\iota_{r-1}^*((\mathbb{A}^r)^d)\) is defined by the extra condition \(a_{r,1} + \cdots + a_{r,d} = 0\).

Consider the \(S_d\)-equivariant locally-closed stratification \(\{Y_\pi\}_{\pi \in P(d)}\) of \((\mathbb{A}^{r-1})^d\), where \(Y_\pi\) parametrizes \(d\)-tuples \((v_1, \ldots, v_d) \in (\mathbb{A}^{r-1})^d\) such that \(v_s = v_t\) if and only if \(s\) and \(t\) are in the same block of \(\pi\). The action of \(S_d\) on the set of partitions \(P(d)\) breaks into orbits by partition type. By Lemma 2.1, \(p\) is stratified by the union of maps of the form \(p^{-1}(Y_\pi)/S_\pi \to Y_\pi/S_{\pi}\), that is,

\[(Y_\pi \times (\mathbb{A}^1)^d)/S_\pi \to Y_\pi/S_{\pi}.\]
Here there is one stratum for each partition type \( a = (1^{m_1}, \ldots, n^{m_n}) \) and \( \pi \) can be taken to be any representative of this partition type.

We rearrange the factors of \((\mathbb{A}^1)^d\) by putting together the factors \( \mathbb{A}^1 \) corresponding to elements of \( \{1, \ldots, d\} \) in the same block of \( \pi \). The previous map becomes

\[
(Y_\pi \times \prod_{j=1}^i \mathbb{A}^{[\pi_j]})/S_\pi \to Y_\pi/S_\pi.
\]

The subgroup \( S'_\pi \) acts trivially on \( Y_\pi \), and the direct factor \( S_{\pi_j} \subset S'_\pi \) acts on \( (\mathbb{A}^1)^{[\pi_j]} \) by permuting the coordinates and trivially on the other factors. By the fundamental theorem on symmetric polynomials, the quotient \((\mathbb{A}^1)^{[\pi_j]}/S_{\pi_j}\) is isomorphic to \( \mathbb{A}^{[\pi_j]} \), and the quotient map \((\mathbb{A}^1)^{[\pi_j]} \to \mathbb{A}^{[\pi_j]}\) is given by the elementary symmetric functions. We have obtained a commutative diagram

\[
\begin{array}{ccc}
Y_\pi \times (\mathbb{A}^1)^d & \xrightarrow{\sigma} & Y_\pi \times \prod_{j=1}^i \mathbb{A}^{[\pi_j]} \\
p & & \downarrow p \\
Y_\pi & \xrightarrow{p^{-1}} & Y_\pi/S_\pi,
\end{array}
\]

where the square on the right is cartesian, \( \sigma \) is given by quotienting by \( S'_\pi \), and \( p \) is the induced quotient morphism. For all \( h = 1, \ldots, d \), recall that \( m_h \) is the number of blocks of \( \pi \) of size exactly \( h \). Then the morphism \( Y_\pi \to Y_\pi/S_\pi \) in the bottom row of (2) is an \( \overline{S}_\pi \)-Galois cover. The \( \overline{S}_\pi \)-action on \( Y_\pi \times \prod_{j=1}^i \mathbb{A}^{[\pi_j]} \) is given by permutation of the factors corresponding to blocks of the same size. It follows by étale descent that the map \( \overline{p} : \overline{p}^{-1}(Y_\pi/S_\pi) \to Y_\pi/S_\pi \) is a vector bundle. (A priori this vector bundle is only étale-locally trivial, but this implies that it is Zariski-locally trivial by Grothendieck’s version of Hilbert Theorem 90.)

Define \( Z_\pi := \iota_{r-1}^{-1}(Y_\pi) \) for every \( \pi \in P(d) \), so that \( \{Z_\pi\}_{\pi \in P(d)} \) is a locally closed stratification of \((\mathbb{A}^{r-1})^d_0 \subset (\mathbb{A}^{r-1})^d \). Write \( \sigma_{1,j}, \ldots, \sigma_{[\pi_j],j}, 1 \leq j \leq i \), for the standard basis of \( \mathbb{A}^{[\pi_j]} \). By definition, \( \sigma_{h,j} \) is the degree \( h \) elementary symmetric polynomial on the coordinates of \((\mathbb{A}^1)^{[\pi_j]}\), which are the \( a_{r,l} \) such that \( l \in \pi_j \). The trivial vector bundle \( p_0 \) of (1) is the subbundle of \( \iota_{r-1}^{-1}p \) defined by the equation \( a_{d,1} + \cdots + a_{d,n} = 0 \). Over \( Z_\pi \), this is exactly the inverse image of the subbundle of \( Z_\pi \times (\prod_{j=1}^i \mathbb{A}^{[\pi_j]})_0 \) given by the equation \( \sigma_{1,1} + \sigma_{1,2} + \cdots + \sigma_{1,j} = 0 \), which we call \( Z_\pi \times (\prod_{j=1}^i \mathbb{A}^{[\pi_j]})_0 \). Therefore, we have a cartesian square

\[
\begin{array}{ccc}
Z_\pi \times (\mathbb{A}^1)^d_0 & \xrightarrow{\sigma} & Z_\pi \times (\prod_{j=1}^i \mathbb{A}^{[\pi_j]})_0 \\
\downarrow & & \downarrow \\
Z_\pi \times (\mathbb{A}^1)^d & \xrightarrow{\sigma} & Z_\pi \times (\prod_{j=1}^i \mathbb{A}^{[\pi_j]}).
\end{array}
\]

We now observe that \( Z_\pi \times (\prod_{j=1}^i \mathbb{A}^{[\pi_j]})_0 \) is a \( \prod S_{m_h} \)-equivariant vector subbundle of \( Z_\pi \times (\prod_{j=1}^i \mathbb{A}^{[\pi_j]}) \), and so descends to a subbundle of \( \overline{p}^{-1}(Z_\pi/S_\pi) \to Z_\pi/S_\pi \). In other words,

\[
(Z_\pi \times (\mathbb{A}^1)^d_0)/S_\pi = \overline{p}^{-1}(Z_\pi/S_\pi) \to Z_\pi/S_\pi.
\]
is a vector bundle, as desired.

Let \( Y \) be a \( k \)-variety, \( d \) be a positive integer, and let \( S_d \) act on \((\mathbb{A}^1 \times_k Y)^d\) by permutation of the \( d \) factors. In [7, Lemma 4.4], Totaro used a stratification argument to show that \([((\mathbb{A}^1 \times_k Y)^d)/S_d] = [\mathbb{A}^d \times_k (Y^d/S_d)]\) in \( K_0(\text{Var}_k)\). His arguments prove the following stronger statement.

**Lemma 4.3.** [7, Lemma 4.4] Let \( Y \) be a \( k \)-variety, \( d \) be a positive integer, and let \( S_d \) act on \((\mathbb{A}^1 \times_k Y)^d\) and \( Y^d \) by permutation of the \( d \) factors \( \mathbb{A}^1 \times_k Y \) and \( Y^d \), respectively. Then the projection \((\mathbb{A}^1 \times_k Y)^d/S_d \to Y^d/S_d\) can be stratified by vector bundles (with height 1).

We are ready to prove the main result of this subsection. In what follows, \( S^m_d \times S_m \) denotes the semidirect product, where \( S_m \) acts on \( S^m_d \) by permuting the factors.

**Proposition 4.4.** Let \( X \) be a quasi-projective \( k \)-variety, \( d, m \) be non-negative integers, \( S_m \) act on \( X^m \) by permutation of factors, and \( \text{pr}_i : X^m \to X \) be the \( i \)-th projection, for \( i = 1, \ldots, m \). Let \( E \to X \) be a vector bundle, and consider the \((S^m_d \times S_m)\)-equivariant vector bundle \( E := \bigoplus_{i=1}^m \text{pr}_i^*(E^{\otimes d})_0 \) over \( X^m \). Then the morphism \( E/(S^m_d \times S_m) \to X^m/S_m \) can be stratified by vector bundles. In particular, for every \( x \in X^m/S_m \) we have \([(E/(S^m_d \times S_m))_{k(x)}] = 0 \text{ in } K_0(\text{Var}_{k(x)})/(L)\).

**Proof.** Define \( \Sigma_{d,m} := S^m_d \times S_m \). For the purpose of proving that the morphism \( E/\Sigma_{d,m} \to X^m/S_m \) can be stratified by vector bundles, we are allowed to pass to a Zariski open cover of \( X \). Indeed, let \( x \) be a point of \( X^m/S_m \), let \( Z_x \) be the set-theoretic fiber of \( X^m \to X^m/S_m \) at \( x \), and set \( Z'_x := \cup_{i=1}^m \text{pr}_i(Z_x) \subset X \). Since \( X \) is quasi-projective and \( Z'_x \) is a finite subset of \( X \), by [17, 00DS] there exists an affine open subscheme \( U \subset X \) such that \( Z_x \subset U \), which implies that \( Z_x \) is contained in \( U^m \). Shrinking \( U \) if necessary, we may also assume that \( E|_U \) is trivial. Therefore, we may suppose that \( X = U \), that is, \( E \cong \mathbb{A}^d_X \) is trivial, so that we have an isomorphism

\[ E \cong ((\mathbb{A}^d)_0 \times_k X)^m \]

of \( \Sigma_{d,m} \)-equivariant vector bundles on \( X^m \), where \( r \) is the rank of \( E \) and \([-]^m \) denotes \( m \)-fold fibered product over \( \text{Spec}(k) \).

The projection maps \( \mathbb{A}^r \to \mathbb{A}^{r-1} \to \cdots \to \mathbb{A}^1 \to \text{Spec}(k) \) given by forgetting the last component induce the maps

\[
((\mathbb{A}^d)_0 \times_k X)^m/\Sigma_{d,m} \xrightarrow{p_r} ((\mathbb{A}^{r-1})_0 \times_k X)^m/\Sigma_{d,m} \xrightarrow{p_{r-1}} \cdots \xrightarrow{p_1} (\mathbb{A}^1)_0 \times_k X)^m/\Sigma_{d,m} \xrightarrow{p_0} X^m/S_m.
\]

In order to conclude, it suffices to show that \( p_i \) can be stratified by vector bundles for all \( i = 1, \ldots, d \).

By Lemma 4.2, there exist a finite set \( J \) and an \( S_d \)-equivariant locally closed stratification \( \{U_j\}_{j \in J} \) of \((\mathbb{A}^1)_{\mathbb{F}}^d/S_d \) such that the restriction of \(((\mathbb{A}^1)_{\mathbb{F}}^d)/S_d \to ((\mathbb{A}^1)_{\mathbb{F}}^d)/S_d \) to each \( U_j \) is a trivial vector bundle. Let \( X_j := U_j \times X \). Since \( S_d \) acts trivially on \( X \), the restriction of \(((\mathbb{A}^1)_{\mathbb{F}}^d \times_k X)/S_d \to ((\mathbb{A}^1)_{\mathbb{F}}^d \times_k X)/S_d \) to each \( X_j \) is a trivial vector bundle. It follows that \( \{X_{j_1} \times \cdots \times X_{j_m}\}_{(j_1, \ldots, j_m) \in J^m} \) is a \( \Sigma_{d,m} \)-equivariant stratification of \(((\mathbb{A}^1)_{\mathbb{F}}^d \times_k X)^m/S_d \) such that the restriction of \(((\mathbb{A}^1)_{\mathbb{F}}^d \times_k X)^m/S_d \to ((\mathbb{A}^1)_{\mathbb{F}}^d \times_k X)^m/S_d \) to each stratum is a trivial vector bundle:

\[ \mathbb{A}^d_{X_{j_1}} \times \cdots \times \mathbb{A}^d_{X_{j_m}} \to X_{j_1} \times \cdots \times X_{j_m}. \]
It remains to take the quotient by $S_m$. For every $j = (j_1, \ldots, j_m) \in J^m$, let $\pi(j)$ be the partition of $\{1, \ldots, m\}$ such that $s$ and $t$ are in the same block of $\pi$ if and only if $j_s = j_t$. The symmetric group $S_m$ acts on the stratification $\{X_{j_1} \times \cdots \times X_{j_m}\}_{j \in J^m}$ via its componentwise permutation action on $J^m$. The stabilizer of $X_{j_1} \times \cdots \times X_{j_m}$ is the direct product of the symmetric groups of each block of $\pi(j)$. By Lemma 2.1, it is enough to show that each

$$(A^{d-1}_{X_{j_1}} \times \cdots \times A^{d-1}_{X_{j_m}})/S'_\pi(j) \rightarrow (X_{j_1} \times \cdots \times X_{j_m})/S'_\pi(j)$$

can be stratified by vector bundles. The latter map factors as the composition

$$(A^{d-1}_{X_{j_1}} \times \cdots \times A^{d-1}_{X_{j_m}})/S'_\pi(j) \rightarrow (A^{d-2}_{X_{j_1}} \times \cdots \times A^{d-2}_{X_{j_m}})/S'_\pi(j) \rightarrow \cdots \rightarrow (X_{j_1} \times \cdots \times X_{j_m})/S'_\pi(j),$$

where each map forgets the last component of each of the $A^h_{X_{j_s}}$. Following the decomposition of $\{1, \ldots, m\}$ into blocks of $\pi(j)$, each of these maps is a product of maps of the form

$$((A^h_{X_{j_s}})^{m(j_s)})/S_m(j_s) \rightarrow ((A^{h-1}_{X_{j_s}})^{m(j_s)})/S_m(j_s).$$

Here $m(j_s)$ is the number of $t \in \{1, \ldots, m\}$ such that $j_s = j_t$, the notation $(-)^{m(j_s)}$ indicates $m(j_s)$-fold direct product over $\text{Spec}(k)$, and $S_m(j_s)$ acts by permuting the factors $A^h_{X_{j_s}}$. Setting $Y = A^{h-1}_{X_{j_s}}$ and $m = m(j_s)$ puts this map in the form of Lemma 4.3, which gives the conclusion.

4.2. The Polydiagonal Compactification. Let $X$ be a smooth quasi-projective variety over a field $k$ of characteristic zero, and let $n \geq 1$ be an integer. The symmetric power $\text{Sym}^n(X)$ is given by the quotient $X^n/S_n$, which does not act with diagonalizable stabilizers for $n \geq 3$. However, there is a canonical equivariant blowup of $X^n$ such that the action does have diagonalizable stabilizers; this is the polydiagonal compactification of configuration space due to Ulyanov. We’ll review some facts about this construction below and refer to [20] for more details.

We’ll continue to use the notation for partitions from the beginning of this section. For any partition $\pi \in P(n)$, define the polydiagonal $\Delta^\pi \subset X^n$ to be the closed subscheme parametrizing $n$-tuples $(x_1, \ldots, x_n)$ such that $x_k = x_l$ whenever $k$ and $l$ belong to the same block of the partition $\pi$. Note that $\Delta^\pi \cong X^i$, where $i$ is the number of blocks of $\pi$.

The polydiagonal compactification $X\langle n \rangle$ is constructed by blowing up $X^n$ as follows:

**Definition 4.5.** [20, Definition-Theorem 2.1] The polydiagonal compactification $X\langle n \rangle$ of the configuration space $F(X, n)$ of ordered points is a sequence of $n-1$ blowups of $X^n$:

$$X\langle n \rangle = Y_{n-1} \xrightarrow{\alpha_{n-1}} Y_{n-2} \xrightarrow{\alpha_{n-2}} \cdots \xrightarrow{\alpha_2} Y_1 \xrightarrow{\alpha_1} Y_0 = X^n.$$

Here $Y_i \rightarrow Y_{i-1}$ is the blowup of the disjoint union of proper transforms $\Delta^\pi_{i-1}$ of polydiagonals $\Delta^\pi$ in $X^n$ corresponding to partitions $\pi$ with exactly $i$ blocks.

Note that the polydiagonals $\Delta^\pi$ for $\pi$ of a particular type with exactly $i \geq 1$ blocks are not disjoint in $X^n$, but their proper transforms become disjoint by the $i-1$ step.

The action of $S_n$ on $X\langle n \rangle$ has abelian isotropy subgroups by [20, Theorem 3.11]. In fact, the proof shows even more: the isotropy subgroup is contained in a direct sum of copies of $k^\times$, hence is diagonalizable.
We are now ready to prove Theorem 1.4.

**Proof of Theorem 1.4.** We’ll break the morphism $X(n)/S_n \to X^n/S_n$ into a composition of $Y_i/S_n \to Y_{i-1}/S_n$ for all $i = 1, \ldots, n - 1$. We can subdivide this process further and consider the blowup of the disjoint union of all partitions of a particular type one at a time. Choose a type $a$ with $i$ blocks, $1 \leq i < n$; then $\mathbb{P}(\mathcal{N}_{Z_a}/Y_{i-1})$ is the exceptional divisor in $Y_i$ obtained by blowing up the disjoint union $Z_a$ of all proper transforms $\Delta_{i-1}^\pi$ in $Y_{i-1}$ of polydiagonals of type $a$. Then both $Z_a$ and $\mathbb{P}(\mathcal{N}_{Z_a}/Y_{i-1})$ are $S_i$-invariant.

By Lemma 2.9, the proof will be complete if we can show that the morphism

$$\mathbb{P}(\mathcal{N}_{Z_a}/Y_{i-1})/S_n \to Z_a/S_n$$

has $\mathbb{L}$-rational fibers. Using Lemma 2.1, we can further reduce to showing that

$$\mathbb{P}(\mathcal{N}_{\Delta_{i-1}^\pi}/Y_{i-1})/S_n \to \Delta_{i-1}^\pi/S_\pi$$

has $\mathbb{L}$-rational fibers for each partition $\pi$. By the blowup formula for the normal bundle [5, Proposition B.6.10], there exists an $S_\pi$-equivariant line bundle $L_{\pi,i-1}$ on $\Delta_{i-1}^\pi$ and an $S_\pi$-equivariant isomorphism

$$\mathcal{N}_{\Delta_{i-1}^\pi}/Y_{i-1} \cong (\alpha_1 \circ \cdots \circ \alpha_{i-1})^*(\mathcal{N}_{\Delta_i^\pi/X^n}) \otimes L_{\pi,i-1}.$$ 

Therefore, the composition $\alpha_1 \circ \cdots \circ \alpha_{i-1}$ induces a commutative $S_\pi$-equivariant square

$$\begin{array}{ccc}
\mathbb{P}(\mathcal{N}_{\Delta_{i-1}^\pi}/Y_{i-1}) & \longrightarrow & \mathbb{P}(\mathcal{N}_{\Delta_i^\pi/X^n}) \\
\downarrow & & \downarrow \\
\Delta_{i-1}^\pi & \longrightarrow & \Delta_i^\pi.
\end{array}$$

The fiber of a pullback bundle over a point is isomorphic to the base change of the fiber over the image point, and the projectivization (with its $S_\pi$-action) is insensitive to twisting by $L_{\pi,i-1}$. Therefore, it suffices to show instead that $\mathbb{P}(\mathcal{N}_{\Delta_i^\pi/X^n})/S_\pi \to \Delta_i^\pi/S_\pi$ has $\mathbb{L}$-rational fibers.

Since $k$ is of characteristic zero, by [12, Proposition 1.9], finite group quotients commute with arbitrary base change. Moreover, projectivization commutes with arbitrary base change. Choose a point $x$ of $\Delta_i^\pi/S_\pi$ and let $W_x \subset \Delta_i^\pi$ be the fiber of $x$, so that $W_x/S_\pi = \text{Spec}(k(x))$. We deduce that

$$\tag{3} (\mathbb{P}(\mathcal{N}_{\Delta_i^\pi/X^n})/S_\pi)_k(x) \cong (\mathbb{P}(\mathcal{N}_{\Delta_i^\pi/X^n})_{W_x})/S_\pi \cong \mathbb{P}(\mathcal{N}_{\Delta_i^\pi/X^n})_{W_x}/S_\pi.$$ 

For the first isomorphism, we have applied [12, Proposition 1.9] to $\mathbb{P}(\mathcal{N}_{\Delta_i^\pi/X^n}) \to \Delta_i^\pi/S_\pi$. For the second isomorphism, we have used the commutativity of projectivization and base change.

From (3) and Corollary 2.5, we deduce that $[(\mathbb{P}(\mathcal{N}_{\Delta_i^\pi/X^n})/S_\pi)_k(x)] = 1$ in $K_0(\text{Var}_{k(x)})/\langle \mathbb{L} \rangle$ is equivalent to $[(\mathcal{N}_{\Delta_i^\pi/X^n})_{W_x}/S_\pi] = 0$ in $K_0(\text{Var}_{k(x)})/\langle \mathbb{L} \rangle$. In order to conclude, it now suffices to show that $[(\mathcal{N}_{\Delta_i^\pi/X^n}/S_\pi)_k(x)] = 0$ in $K_0(\text{Var}_{k(x)})/\langle \mathbb{L} \rangle$.

The closed embedding $\Delta_i^\pi \subset X^n$ determines an $S_\pi$-equivariant short exact sequence of vector bundles on $\Delta_i^\pi$:

$$0 \to \bigoplus_{j=1}^i p_j^* \mathcal{T}_X \to \mathbb{P} \subseteq \bigoplus_{l=1}^n q_l^* \mathcal{T}_X |_{\Delta_i^\pi} \to \mathcal{N}_{\Delta_i^\pi/X^n} \to 0.$$
a retraction given by the vector bundles \( \iota \) map \( j \) the \( (\text{Recall that } i \) where on the right side, each \( S_{\pi} \) acts on the corresponding summand \( T^\pi_X(\{\pi_j\}^{-1}) \) as \( (T^\pi_X(\{\pi_j\}^0))_0 \), that is, as the kernel of the addition map \( T^\pi_X(\{\pi_j\}) \rightarrow T_X \), where \( T^\pi_X(\{\pi_j\}) \) is the standard permutation representation of \( S_{\pi_j} \), and trivially on the other summands, and each \( S_m \) acts by permuting the factors corresponding to the blocks \( \pi_j \) which satisfy \( |\pi_j| = h \). (Recall that \( m_h \) is defined as the number of blocks of \( \pi \) of size exactly \( h \).) We can write the map in question
\[
\mathcal{N}_{\Delta^\pi/X^n} \cong \text{ker } \rho_n \cong \oplus_{j=1}^n p_j^*(T^\pi_X(\{\pi_j\}^{-1}))/S_{\pi_j},
\]
as a direct product of maps of the form
\[
(\oplus_{j:|\pi_j|=h} p_j^*(T^h_X)_0)/(S_{m_h} \times S_{m_h}) \rightarrow X^{m_h}/S_{m_h},
\]
where \( h \in \{1, \ldots, n\} \) satisfies \( m_h \geq 1 \). (The condition \( m_h \geq 1 \) appears because we considered the groups \( S_m \) for \( m \geq 1 \) only.) The conclusion then follows from Proposition 4.4.

By Lemma 2.10, to answer Question 1.1(i) it would therefore be enough to determine if \( X(n)/S_n \) has \( \mathbb{L} \)-rational singularities.

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