RATIONAL RULED SURFACES AS SYMPLECTIC DIVISORS

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Abstract. We study embeddability of rational ruled surfaces as symplectic divisors into closed integral symplectic manifolds. From this we obtain results on Stein fillability of Boothby–Wang bundles over rational ruled surfaces.

1. Introduction

The aim of this paper is to study the symplectic topology of certain symplectic hypersurfaces of closed integral symplectic manifolds, called symplectic divisors. By an integral symplectic manifold, we mean a symplectic manifold $(M, \omega)$ with $[\omega] \in H^2(M; \mathbb{Z})$. Motivated by the notion of ample divisors in complex geometry, we define a symplectic divisor of degree $k > 0$ on an integral symplectic manifold $(M, \omega)$ to be a symplectic submanifold $\Sigma \subset M$ of codimension 2 such that the homology class $[\Sigma] \in H^2(M; \mathbb{Z})$ is Poincaré dual to $k[\omega]$ and the complement $M \setminus \Sigma$ admits a Stein structure. A good source of symplectic divisors is Donaldson’s construction [Don96], which guarantees the existence of a symplectic divisor of sufficiently large degree on a given closed integral symplectic manifold.

In complex geometry, where divisors have played an important role, rigidity aspects of complex manifolds are often captured by the existence of ample divisors. For example, carrying an ample divisor forces the ambient space to have projectivity via Kodaira embedding. The Lefschetz hyperplane theorem gives a strong restriction for a projective manifold to be an ample divisor. Sommese [Som76] provided projective manifolds that cannot be embedded into any projective manifolds as ample divisors.

Inspired by complex geometry it is interesting to study rigidity and flexibility aspects of symplectic manifolds in terms of the existence of symplectic divisors. We can show that every closed Riemann surface $(\Sigma, \omega)$ with symplectic volume $\int_{\Sigma} \omega \in \mathbb{Z}_{>0}$ can be embedded into a closed symplectic 4-manifold as a symplectic divisor. While the projective space $(\mathbb{C}P^2, \omega_{FS})$ is a symplectic divisor on $(\mathbb{C}P^3, \omega_{FS})$ where $\omega_{FS}$ denotes the respective Fubini–Study form, $(\mathbb{C}P^2, k\omega_{FS})$ with $k \geq 2$ cannot be a symplectic divisor on any integral symplectic 6-manifold. This is a consequence of Stein non-fillability of a contact structure on a certain $S^1$-bundle over $(\mathbb{C}P^2, k\omega_{FS})$ (see [PP08] for example).

In this paper we study the next simplest symplectic 4-manifolds, namely rational ruled surfaces, as symplectic divisors. There are two diffeomorphism types of them: the product bundle $S^2 \times S^2$ and the non-trivial bundle $S^2 \times S^2$. We equip them with the

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symplectic forms $\omega_{a,b}$ and $\tilde{\omega}_{a,b}$, respectively, for each $a, b \in \mathbb{Z}_{>0}$, characterized by the equations (2.4) and (2.5) in Section 2.1.2.

**Theorem 1.1.** Let $(S^2 \times S^2, \omega_{a,b})$ and $(S^2 \times S^2, \tilde{\omega}_{a,b})$ be symplectic manifolds as above.

1. For any $a \geq 1$ (resp. $a \geq 2$), the symplectic manifold $(S^2 \times S^2, \omega_{a,1})$ (resp. $(S^2 \times S^2, \tilde{\omega}_{a,1})$) can be embedded into a closed integral symplectic 6-manifold as a symplectic divisor of degree 1.

2. For any odd $a \geq 5$ (resp. $a \geq 7$), the symplectic manifold $(S^2 \times S^2, \omega_{a,2})$ (resp. $(S^2 \times S^2, \tilde{\omega}_{a,2})$) cannot be embedded into any closed integral symplectic 6-manifold as a symplectic divisor of degree 1.

We construct embeddings in Theorem 1.1 from complex geometry. On the other hand the non-embeddability result comes from a holomorphic curve technique in symplectic geometry. We derive topological information of the complement of a symplectic divisor from analyzing a moduli space of holomorphic spheres in the ambient closed symplectic manifold. This approach was studied by McDuff, Floer and Eliashberg [McD91] and has developed in several directions; see for example [McD90], [Hin03], [OO05], [Wen10], [BGZ19] and the references therein. Our proof is mainly inspired by Hind [Hin06].

We would like to point out that Bădescu [Băd81, Băd82] classified complex projective 3-manifolds which contain rational ruled surfaces as ample divisors. In principle, his result gives the list of symplectic forms on rational ruled surfaces with which they cannot be symplectic divisors on any projective 3-manifolds. Our non-embeddability result can be seen as an extension of this to symplectic divisors on symplectic 6-manifolds.

We apply Theorem 1.1 to address the fillability problem of contact manifolds. Let $\Sigma$ be a symplectic divisor on an integral symplectic manifold $(M, \omega)$. Then the boundary of the complement $W$ of a tubular neighborhood of $\Sigma$ in $M$ carries a canonical principal $S^1$-bundle structure over $\Sigma$ with a contact structure. We call this bundle the **Boothby–Wang bundle** over $\Sigma$ and its total space the **Boothby–Wang manifold** (see Section 3.1). Applying Theorem 1.1 we prove the following fillability results.

**Theorem 1.2.** Let $(P_{a,b}, \xi_{a,b})$ (resp. $(\tilde{P}_{a,b}, \tilde{\xi}_{a,b})$) be the Boothby–Wang manifold over $(S^2 \times S^2, \omega_{a,b})$ (resp. $(S^2 \times S^2, \tilde{\omega}_{a,b})$).

1. The contact structure $\xi_{1,1}$ is critically Stein fillable.

2. The contact structures $\xi_{a,1}$ and $\tilde{\xi}_{a,1}$ for $a \geq 2$ are subcritically Stein fillable.

3. The contact structure $\xi_{a,2}$ for any odd $a \geq 5$ (resp. $\tilde{\xi}_{a,2}$ for any odd $a \geq 7$) is not Stein fillable.

In the theorem, a Stein fillable contact structure $\xi$ on $P$ is said to be **subcritically Stein fillable** if $(P, \xi)$ admits a subcritical Stein filling; otherwise, we call it **critically Stein fillable**. See Remarks 3.3 and 3.5 for further statements on fillability.

It is known that the contact structures $\xi_{a,b}$ and $\xi_{a',b'}$ (resp. $\tilde{\xi}_{a,b}$ and $\tilde{\xi}_{a',b'}$) are equivalent as almost contact structures if and only if $a - b = a' - b'$ (resp. $2a - 3b = 2a' - 3b'$).
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b′) (see Corollary 3.2). Lerman [Ler03, Question 1] asked whether they are actually contactomorphic or not when their almost contact structures are equivalent. Theorem 1.2 gives a partial negative answer to his question from the fillability viewpoint. We would like to refer to the result in Boyer–Pati [BP14] which proves using contact homology that if (a, b) ≠ (a′, b′), they are not contactomorphic.

It is worth emphasizing that our non-fillability result is symplectic in nature. It is known that there are Stein non-fillable contact structures on 5-manifolds (see [BCS14], [EKP06] and [PP08] for example): In fact, these known 5-manifolds cannot admit Stein fillable contact structures for topological reasons. In contrast, our 5-manifolds $P_{a,b}$ and $\tilde{P}_{a,b}$, which are diffeomorphic to the trivial $S^3$-bundle $S^2 \times S^3$ and the non-trivial one $S^2 \times S^3$ over $S^2$ respectively (see Proposition 3.1), carry Stein fillable contact structures. This means in particular that there are no topological obstructions to Stein fillability. Moreover, each Stein non-fillable contact structure in the theorem is equivalent to a subcritically Stein fillable contact structure as almost contact structures.

This paper is organized as follows: Section 2 deals with the embedding problem of symplectic rational ruled surfaces. We first describe rational ruled surfaces as complex surfaces and construct symplectic forms on them in Section 2.1. After this, we give proofs of Theorem 1.1 in Section 2.2 and 2.3, respectively. Applying Theorem 1.1, we discuss the fillability problem in Section 3. Section 3.1 is devoted to reviewing the topology of Boothby–Wang bundles over rational ruled surfaces, and Section 3.2 explains a property of almost contact structures on them. Finally, we conclude this paper by proving Theorem 1.2 in Section 3.3 and 3.4.

Convention 1.3. Since we are primarily interested in symplectic divisors of degree 1, symplectic divisors in this paper are assumed to be of degree 1 unless otherwise noted.

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2. Embedding problem of rational ruled surfaces

2.1. Rational ruled surfaces. We consider smooth 4-manifolds which fiber over the 2-sphere $S^2$ with fibers diffeomorphic to $S^2$. Motivated by complex geometry, we call such 4-manifolds rational ruled surfaces. It is known that there are only two $S^2$-bundle over $S^2$ (e.g. see [MS17, Lemma 6.2.3]). In this subsection, we describe symplectic structures on them parametrized by pairs of two integers.

2.1.1. Construction of rational ruled surfaces. A construction of rational ruled surfaces as complex manifolds is the following: Let $\mathcal{O}(m) \to \mathbb{C}P^1$ for $m \in \mathbb{Z}_{\geq 0}$ denote the holomorphic line bundle over $\mathbb{C}P^1$ with $c_1(\mathcal{O}(m)) = m$. We write $\mathcal{O}(-m) \to \mathbb{C}P^1$ for its
dual line bundle, and set \( O := \mathcal{O}(0) \). We denote the projective bundle associated to the holomorphic vector bundle \( E_m := \mathcal{O}(-m) \oplus O \to \mathbb{CP}^1 \) by 
\[
\pi_m : \mathbb{P}(E_m) \to \mathbb{CP}^1.
\]
In this paper the projectivization \( \mathbb{P}(E_m) \) is defined so that the fiber over \( x \in \mathbb{CP}^1 \) is the space of complex lines in the fiber \( (E_m)_x \). The total space \( \mathbb{P}(E_m) \) is a complex manifold that fibers over \( \mathbb{CP}^1 \) with fibers biholomorphic to \( \mathbb{CP}^1 \) and hence it is a rational ruled surface. Its diffeomorphism type is determined by the parity of \( m \) (e.g. see [GS95, Theorem 3.4.8]): \( \mathbb{P}(E_m) \) is diffeomorphic to the total space of the trivial bundle, i.e., \( S^2 \times S^2 \) if \( m \) is even; otherwise, it is diffeomorphic to the total space of the non-trivial bundle, denoted by \( S^2 \times \tilde{S}^2 \). Taking this distinction into account, we assume that \( m \) is 0 or 1 in the rest of this section.

2.1.2. Symplectic structures on \( \mathbb{P}(E_m) \). To describe symplectic structures on \( \mathbb{P}(E_m) \), we make the following observations. Let \( F \) be a fiber of \( \mathbb{P}(E_m) \), \( C_0 \) the curve \( \mathbb{P}(0 \oplus O) \) and \( C_\infty \) the curve \( \mathbb{P}(O(-m) \oplus 0) \). Set \( \alpha = \pi_m^* c_1(\mathcal{O}(1)) \) and \( \beta = c_1(\mathcal{O}(\mathbb{P}(E_m))(1)) \in H^2(\mathbb{P}(E_m); \mathbb{Z}) \), where \( \mathcal{O}(\mathbb{P}(E_m))(1) \to \mathbb{P}(E_m) \) denotes the hyperplane line bundle. It is easy to check that
\[
\alpha([F]) = 0, \quad \alpha([C_0]) = 1, \quad \alpha([C_\infty]) = 1,
\]
\[
\beta([F]) = 1, \quad \beta([C_0]) = 0, \quad \beta([C_\infty]) = m,
\]
where the last equality is obtained by the fact that \( \mathcal{O}(\mathbb{P}(E_m))(1)|_{C_\infty} \cong \mathcal{O}_{C_\infty}(m) \). Using equalities (2.1) and (2.2), we have \([C_\infty] = [C_0] + m[F] \in H_2(\mathbb{P}(E_m); \mathbb{Z})\).

Now we equip \( \mathbb{P}(E_m) \) with symplectic structures as follows. Take two positive integers \( a \) and \( b \) with \( a - mb > 0 \). Thanks to positivity of \( \mathcal{O}(a - mb) \) and \( \mathcal{O}(b) \), the bundle \( \pi_m^* \mathcal{O}(a - mb) \otimes \mathcal{O}(\mathbb{P}(E_m))(b) \to \mathbb{P}(E_m) \) is positive, and \( \mathbb{P}(E_m) \) admits a Kähler form \( \Omega_{a,b} \) whose cohomology class \([\Omega_{a,b}]\) is given by
\[
[\Omega_{a,b}] = (a - mb)\alpha + b\beta \in H^2(\mathbb{P}(E_m); \mathbb{Z})
\]
(see [BC01, Section 2.4]). Observe that
\[
\Omega_{a,b}([C_\infty]) = a, \quad \Omega_{a,b}([F]) = b.
\]
Let \( \omega_{a,b} \) be a symplectic form on \( S^2 \times S^2 \) such that
\[
\omega_{a,b}([S^2 \times \{pt\}]) = a, \quad \omega_{a,b}([\{pt\} \times S^2]) = b,
\]
and let \( \tilde{\omega}_{a,b} \) be one on \( \tilde{S}^2 \times S^2 \) such that
\[
\tilde{\omega}_{a,b}([S_1]) = a, \quad \tilde{\omega}_{a,b}([S_2]) = b.
\]
Here \( S_1 \) (resp. \( S_2 \)) is an embedded sphere in \( \tilde{S}^2 \times S^2 \) with self-intersection 1 (resp. 0). Then the symplectomorphism type of \( (\mathbb{P}(E_m), \Omega_{a,b}) \) is given as follows.

**Lemma 2.1.** The symplectic manifold \( (\mathbb{P}(E_m), \Omega_{a,b}) \) is symplectomorphic to one of the following:

1. \( (S^2 \times S^2, \omega_{a,b}) \) if \( m = 0 \);
2. \( (S^2 \times \tilde{S}^2, \tilde{\omega}_{a,b}) \) if \( m = 1 \).
2.2. Embeddings of rational ruled surfaces.

In view of the characterizing equations (2.3), (2.4), (2.5), to complete the proof, it is therefore enough to determine the diffeomorphism type of \( \mathbb{P}(E_m) \). This depends only on the parity of \( m \) as noted in Section 2.1.1.

2.2.1. Case of \( a = 1 \).

First, let us consider the symplectic manifold \((S^2 \times S^2, \omega_{a,b})\). Let \( \omega_{FS} \) be the Fubini–Study form on \( \mathbb{CP}^3 \). Consider the projective hypersurface \( M \subset \mathbb{CP}^4 \) defined by
\[
M := \{(z_0 : z_1 : z_2 : z_3 : z_4) \in \mathbb{CP}^4 \mid z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\},
\]
and we set \( \Sigma = M \cap \{z_4 = 0\} \). Let \( \omega \) be the symplectic form on \( M \) defined to be the restriction of \( \omega_{FS} \) to \( M \). Let \( \iota: \Sigma \hookrightarrow M \) be the inclusion. Comparing the cohomology class \([\iota^*\omega]\) to \([\omega_{1,1}]\) in light of [LM96, Theorem 1.1], one can show that \((\Sigma, \iota^*\omega)\) is symplectomorphic to \((S^2 \times S^2, \omega_{1,1})\).

Proposition 2.3. The symplectic manifold \((\Sigma, \iota^*\omega)\) is a symplectic divisor on \((M, \omega)\). In particular, \((S^2 \times S^2, \omega_{1,1})\) is embedded into \((M, \omega)\) as a symplectic divisor.

Proof. By definition, \((\Sigma, \iota^*\omega)\) is a symplectic submanifold of \((M, \omega)\). Since \( \Sigma \) is a hyperplane section of \( M \), we have \([\omega] = PD[\Sigma] \). Furthermore, since \( \Sigma \) is an ample divisor on \((M, \omega)\), it follows that the complement \( M \setminus \Sigma \) is a Stein manifold. This completes the proof.
2.2.2. Case of $a > 1$. In the case of $a > 1$, we embed the rational ruled surfaces into certain projective bundles. Set $F_m := \mathcal{O}(-m) \oplus \mathcal{O} \rightarrow \mathbb{CP}^1$, where $m \in \mathbb{Z}_{>0}$. Let $\mathbb{P}(F_m)$ denote the projectivization of the vector bundle $F_m$ and $\pi : \mathbb{P}(F_m) \rightarrow \mathbb{CP}^1$ denote the bundle projection. Take two holomorphic sections $\sigma_i$ ($i = 1, 2$) of the hyperplane line bundle $\mathcal{O}_F(1) \rightarrow \mathbb{P}(F_m)$ induced by the projection $F_m \rightarrow \mathcal{O}$ from $F_m$ to the $i$-th trivial summand. We also take holomorphic sections $s_1$ and $s_2$ of the line bundle $\mathcal{O}(n) \rightarrow \mathbb{CP}^1$ ($n > 0$) such that $s_1$ is transverse to the zero-section, and $s_2$ is transverse to the intersection of the zero-section and the zero set $Z(s_1)$ of $s_1$, where we identify the base space $\mathbb{CP}^1$ with the zero-section. Define the smooth hypersurface $\Sigma_{m,n}$ of $\mathbb{P}(F_m)$ by
\[
\Sigma_{m,n} = Z(\sigma_1 \otimes \pi^* s_1 + \epsilon \sigma_2 \otimes \pi^* s_2)
\]
for a small $\epsilon > 0$. We refer the reader to [BC01, Section 2.4] for a detailed construction of $\Sigma_{m,n}$.

Lemma 2.4. The hypersurface $\Sigma_{m,n}$ is diffeomorphic to $S^2 \times S^2$ if $m - n$ is even; otherwise, it is diffeomorphic to $S^2 \times S^2$.

Proof. It is easy to see that $\Sigma_{m,n}$ is a rational ruled surface. As in Section 2.1.1 its diffeomorphic type is determined by the parity of the self-intersection number of a holomorphic section of the fibration $\Sigma_{m,n} \rightarrow \mathbb{CP}^1$. Hence it suffices to compute the one of $D_\infty := \mathbb{P}(\mathcal{O}(-m) \oplus 0 \oplus 0)$ in $\Sigma_{m,n}$; this agrees with $c_1(N_{D_\infty}/\Sigma_{m,n}) \langle [D_\infty] \rangle$. The splitting $N_{D_\infty}/\mathbb{P}(F_m) \cong N_{D_\infty}/\Sigma_{m,n} \oplus N_{\Sigma_{m,n}}/\mathbb{P}(F_m)|_{D_\infty}$ yields
\[
c_1(N_{D_\infty}/\Sigma_{m,n}) = c_1(N_{D_\infty}/\mathbb{P}(F_m)) - c_1(N_{\Sigma_{m,n}}/\mathbb{P}(F_m)|_{D_\infty}).
\]
Thus, we shall focus on the right-hand side in the rest of the proof.

Before the computation, we introduce some notations. Let $F'$ be a general line in a fiber of $\mathbb{P}(F_m)$ and $D_0$ the curve $\mathbb{P}(0 \oplus 0 \oplus \mathcal{O})$. Set $\alpha' = \pi^* c_1(\mathcal{O}(1))$ and $\beta' = c_1(\mathcal{O}_F(1)) \in H^2(\mathbb{P}(F_m); \mathbb{Z})$. Similarly to Section 2.1.2, we have
\[
\alpha'([F']) = 0, \quad \alpha'([D_0]) = 1, \quad \alpha'([D_\infty]) = 1,
\]
\[
\beta'([F']) = 1, \quad \beta'([D_0]) = 0, \quad \beta'([D_\infty]) = m,
\]
and this leads to $[D_\infty] = [D_0] + m[F']$.

Let us compute $c_1(N_{D_\infty}/\mathbb{P}(F_m)) \langle [D_\infty] \rangle$. By definition,
\[
Z(\sigma_1) = \mathbb{P}(\mathcal{O}(-m) \oplus 0 \oplus \mathcal{O}) \text{ and } Z(\sigma_1) \cap Z(\sigma_2) = D_\infty,
\]
and hence coupling the adjunction formula with the fact that $\mathcal{O}_F(1)|_{D_\infty} \cong \pi^* \mathcal{O}(m)|_{D_\infty}$, we have
\[
N_{D_\infty}/\mathbb{P}(F_m) \cong N_{\mathbb{P}(\mathcal{O}(-m) \oplus 0 \oplus \mathcal{O})}/\mathbb{P}(F_m)|_{D_\infty} \cong \mathcal{O}_{\mathbb{P}(\mathcal{O}(-m) \oplus 0 \oplus \mathcal{O})}(1)|_{D_\infty} \cong (\pi^* \mathcal{O}(m) \oplus \pi^* \mathcal{O}(m))|_{D_\infty}.
\]
Thus,
\[
c_1(N_{D_\infty}/\mathbb{P}(F_m)) \langle [D_\infty] \rangle = 2m.
\]
Next, compute $c_1(N_{\Sigma_{m,n}/\mathbb{P}(F_m)}|D_{\infty})([D_{\infty}])$. By the adjunction formula, we have

$$N_{\Sigma_{m,n}/\mathbb{P}(F_m)} \cong (\mathcal{O}_{\mathbb{P}(F_m)}(1) \otimes \pi^*\mathcal{O}(n))|_{\Sigma_{m,n}}.$$ 

Thus, $c_1(N_{\Sigma_{m,n}/\mathbb{P}(F_m)}|D_{\infty}) = \nu^*(n\alpha' + \beta')$ and

$$c_1(N_{\Sigma_{m,n}/\mathbb{P}(F_m)}|D_{\infty})([D_{\infty}]) = n + m,$$

where $\nu: \Sigma_{m,n} \hookrightarrow \mathbb{P}(E_m)$ denotes the inclusion. Combining (2.8) with (2.9), we conclude that

$$c_1(N_{D_{\infty}/\mathbb{P}(F_m)}|D_{\infty}) = 2m - (n + m) = m - n.$$ 

Since $[F']^2 = 0$, the parity of $[D_{\infty}]^2 = m - n$ determines the diffeomorphism type of $\Sigma_{m,n}$.

Now we discuss symplectic aspects of $\mathbb{P}(F_m)$ and $\Sigma_{m,n}$. Thanks to semi-positivity of $\mathcal{O}(m)$ and $\mathcal{O}(n)$, the bundle $\mathcal{O}_{\mathbb{P}(F_m)}(1) \otimes \pi^*\mathcal{O}(n) \to \mathbb{P}(F_m)$ is positive, and $\mathbb{P}(F_m)$ admits a Kähler form $\Omega_m$ with the cohomology class $[\Omega_m] = n\alpha' + \beta'$ (see [BC01 Section 2.4]); of course $\Omega_m$ depends on $n$ although we omit this from the notation for the sake of simplicity. Let $\eta_{m,n}$ denote the restriction of $\Omega_m$ to $\Sigma_{m,n}$.

**Lemma 2.5.** The symplectic manifold $(\Sigma_{m,n}, \eta_{m,n})$ is symplectomorphic to one of the following:

1. $(S^2 \times S^2, \omega_{2m-3\ell,1})$ if $m - n = 2\ell$ is even;

2. $(S^2 \times S^2, \tilde{\omega}_{2m-3\ell-1,1})$ if $m - n = 2\ell + 1$ is odd.

**Proof.** The following argument is similar to [Bor12 Proposition 2.2]. As we have already seen, $\Sigma_{m,n}$ is a rational ruled surface. Hence, by a result in [LM96 Theorem 1.1] combined with Lemma 2.3, the symplectomorphism type of $(\Sigma_{m,n}, \eta_{m,n})$ is determined by the parity of $m - n$ and the cohomology class of $[\eta_{m,n}] \in H^2(\Sigma_{m,n}; \mathbb{Z})$. Recall from the proof of Lemma 2.4 that the homology classes $[D_{\infty}]$ and $[F']$ form a basis for $H_2(\Sigma_{m,n}; \mathbb{Z})$, and their intersections are given by $[D_{\infty}]^2 = m - n, [F']^2 = 0$ and $[D_{\infty}] \cdot [F'] = 1$. We also have $\eta_{m,n}([D_{\infty}]) = m + n$ and $\eta_{m,n}([F']) = 1$.

When $m - n = 2\ell$ is even, $(\Sigma_{m,n}, \eta_{m,n})$ is symplectomorphic to $(S^2 \times S^2, \omega_{a,b})$ for some $a$ and $b$. The homology classes of the spheres of the latter space are identified with

$$[S^2 \times pt] = [D_{\infty}] - \ell[F'] \quad \text{and} \quad \{|pt\} \times S^2 = [F'].$$

This shows that $\omega_{a,b}([S^2 \times pt]) = 2m - 3\ell$ and $\omega_{a,b}(|\{pt\} \times S^2|) = 1$. This proves the first assertion.

When $m - n = 2\ell + 1$ is odd, $(\Sigma_{m,n}, \eta_{m,n})$ is symplectomorphic to $(S^2 \times S^2, \tilde{\omega}_{a,b})$ for some $a$ and $b$. The homology classes $[S_1]$ and $[S_2]$ defined after the equation (2.5) are identified with

$$[S_1] = [D_{\infty}] - \ell[F'] \quad \text{and} \quad [S_2] = [F'],$$

and this shows that $\omega_{a,b}([S_1]) = 2m - 3\ell - 1$ and $\omega_{a,b}([S_2]) = 1$. This proves the second assertion. \(\square\)
Proof of Theorem 1.1 (1). In view of Proposition 2.3, it suffices to deal with the case when \( a \geq 2 \). It follows from Lemma 2.5 that for any \( a \geq 2 \), there exist \( m \) and \( n \) such that \((S^2 \times S^2, \omega_{a,1})\) is symplectomorphic to \((\Sigma_{m,n}, \eta_{m,n})\); this also holds for \((S^2 \times S^2, \omega_{a,2})\). Now, suppose that \((\Sigma_{m,n}, \eta_{m,n})\) is symplectomorphic to either \((S^2 \times S^2, \omega_{a,1})\) or \((S^2 \times S^2, \omega_{a,2})\). The hypersurface \( \Sigma_{m,n} \) is the smooth ample divisor on \( \mathbb{P}(F_m) \) given by

\[
Z(\sigma_1 \otimes \pi^* s_1 + \sigma_2 \otimes \pi^* s_2),
\]

and the Kähler form \( \Omega_m \) satisfies \([\Omega_m] = c_1(\mathcal{O}_{\mathbb{P}(F_m)}(1) \otimes \pi^*(\mathcal{O}(n)))\). This shows that \( \Sigma_{m,n} \) is a symplectic divisor on \((\mathbb{P}(F_m), \Omega_m)\). □

2.3. Non-embeddability of rational ruled surfaces.

2.3.1. Almost complex structures on symplectic manifolds. For non-embeddability results, we first define the set of almost complex structures on a symplectic manifold satisfying certain conditions.

Let \((M, \omega)\) be a symplectic manifold and \( A \in H_2(M; \mathbb{Z}) \) a spherical class. Given a decomposition \( A = A_1 + \cdots + A_N \) with \( N \geq 1 \) consisting of non-trivial spherical classes \( A_i \)'s, we define the set \( \mathcal{J}_{\text{reg}}(\{A_i\}) \) of \( \omega \)-compatible almost complex structures \( J \) on \( M \) for which every simple \( J \)-holomorphic sphere \( u_i : \mathbb{C}P^1 \rightarrow M \) representing \( A_i \) is Fredholm regular. Considering all possible decompositions \( \{A_i\}\) of \( A \), set

\[
\mathcal{J}_{\text{reg}}(A) := \bigcap_{\{A_i\}} \mathcal{J}_{\text{reg}}(\{A_i\}).
\]

It is known that \( \mathcal{J}_{\text{reg}}(A) \) is residual in the space of \( \omega \)-compatible almost complex structures on \( M \) with respect to the \( C^\infty \)-topology (cf. [MS04, Section 6.2]).

2.3.2. Proof of non-embeddability. For simplicity, we often use the notation \((\mathbb{P}(E_m), \Omega_{a,2})\) instead of \((S^2 \times S^2, \omega_{a,2})\) and \((S^2 \times S^2, \omega_{a,2})\). Recall that \( F \) denotes a fiber of the fibration \( \mathbb{P}(E_m) \rightarrow \mathbb{C}P^1 \), and \( C_0 \) and \( C_\infty \) denote \( \mathbb{P}(0 \oplus \mathcal{O}) \) and \( \mathbb{P}(\mathcal{O}(-m) \oplus 0) \), respectively. The following is a key lemma for non-embeddability results.

Lemma 2.6. Let \( \Sigma \) be a symplectic divisor on a closed integral symplectic 6-manifold \((M, \omega)\). Suppose that \((\Sigma, \omega|_\Sigma)\) is symplectomorphic to \((\mathbb{P}(E_m), \Omega_{a,2})\) with \( a > m+2 \) where \( m = 0, 1 \), and the fiber class \( B = \iota_*[F] \in H_2(M; \mathbb{Z}) \) is \( J \)-indecomposable for \( J \in \mathcal{J}_{\text{reg}}(B) \), where \( \iota : \mathbb{P}(E_m) \rightarrow M \) is a symplectic embedding with \( \iota(\mathbb{P}(E_m)) = \Sigma \). Then, \( \pi_1(M \setminus \Sigma) \) is isomorphic to \( \mathbb{Z}/(a-2m)\mathbb{Z} \).

In the lemma, a \( J \)-indecomposable homology class \( A \in H_2(M; \mathbb{Z}) \) is a spherical class admitting no decomposition \( A = A_1 + \cdots + A_N \) of \( A \) with \( N \geq 2 \) such that each \( A_i \) can be represented by a nonconstant \( J \)-holomorphic sphere.

The following lemma will be used in the proof of Lemma 2.6.

Lemma 2.7. Let \( \iota : (\mathbb{P}(E_m), \Omega_{a,2}) \rightarrow (M, \omega) \) be a symplectic embedding as in Lemma 2.6. If \( a > m+2 \), the induced homomorphism \( \iota_* : H_2(\mathbb{P}(E_m); \mathbb{Z}) \rightarrow H_2(M; \mathbb{Z}) \) is an isomorphism.
Proof. The surjectivity of \( \iota_* \) directly follows from the long exact sequence in homology for the pair \((M, \iota(\mathbb{P}(E_m)))\). To see injectivity, suppose that there exist integers \( k_1, k_2 \in \mathbb{Z} \) such that \( \iota_*(k_1[C_\infty] + k_2[F]) = 0 \). By definition of symplectic divisors, we have

\[
 c_1(N_{\Sigma/M})(\iota_*[C_\infty]) = a \quad \text{and} \quad c_1(N_{\Sigma/M})(\iota_*[F]) = 2.
\]

We obtain

\[
 (2.10) \quad 0 = c_1(N_{\Sigma/M})(0) = c_1(N_{\Sigma/M})(\iota_*(k_1[C_\infty] + k_2[F])) = ak_1 + 2k_2.
\]

Moreover, using Lemma 2.2, we have

\[
 c_1(TM)(\iota_*[C_\infty]) = c_1(T\mathbb{P}(E_m))([C_\infty]) + c_1(N_{\Sigma/M})(\iota_*[C_\infty]) = a + m + 2,
\]

and

\[
 c_1(TM)(\iota_*[F]) = c_1(T\mathbb{P}(E_m))([F]) + c_1(N_{\Sigma/M})(\iota_*[F]) = 4.
\]

We deduce

\[
 (2.11) \quad 0 = c_1(TM)(0) = c_1(TM)(\iota_*(k_1[C_\infty] + k_2[F])) = (a + m + 2)k_1 + 4k_2.
\]

Combining the equations (2.10) and (2.11), we conclude that \( k_1 = 0 \) and \( k_2 = 0 \), and this completes the proof. \( \square \)

Now we prove Lemma 2.6.

Notation 2.8. In the proof below, we will consider fundamental groups of several topological spaces. To avoid confusion, let us denote a homotopy class of a loop \( f : [0, 1] \to X \) based at a point \( p \) in a topological space \( X \) by \([f]_X \) or \([f]_{(X, p)}\).

Proof of Lemma 2.6. The following proof is in spirit contained in [Hin06, Theorem 1.1(ii)] and has five steps.

Step 1. We define a moduli space of holomorphic spheres in \( M \).

Choose an \( \omega \)-compatible almost complex structure \( J_0 \) on \( M \) so that a tubular neighborhood of \( \Sigma \) in \( M \) is biholomorphic to the total space of the holomorphic line bundle \( \mathcal{O}(a, 2) := \pi_m^*\mathcal{O}(a - 2m) \otimes \mathcal{O}_{\mathbb{P}(E_m)}(2) \to \mathbb{P}(E_m) \). Here, \( \pi_m : \mathbb{P}(E_m) \to \mathbb{C}P^1 \) denotes the bundle projection. Recall that \( c_1(N_{\Sigma/M})(B) = [\Omega_{a, 2}(B) = 2 \) and \( c_1(N_{\iota(F)/\Sigma})(B) = 0 \). In light of [MS04, Lemma 3.3.1], these evaluations of the first Chern classes together with integrability of \( J_0 \) near \( \Sigma \) show that every simple \( J_0 \)-holomorphic sphere near \( \Sigma \) homologous to \( B \) is Fredholm regular.

Define the (unparametrized) \( J_0 \)-holomorphic spheres \( Q_0 \) and \( Q_\infty \) in \( \Sigma \) to be \( \iota(C_0) \) and \( \iota(C_\infty) \), respectively. Note that \( Q_0 \) and \( Q_\infty \) are disjoint. Consider the moduli space \( \mathcal{M}(B; J_0; Q_0 \times Q_\infty) \) consisting of all \( J_0 \)-holomorphic spheres \( u : \mathbb{C}P^1 \to M \) and points \( z_0, z_\infty \) of \( \mathbb{C}P^1 \) such that \([u(\mathbb{C}P^1)] = B \in H_2(M; \mathbb{Z}) \) and \( u(z_0) \in Q_0 \) and \( u(z_\infty) \in Q_\infty \):

\[
 \mathcal{M}(B; J_0; Q_0 \times Q_\infty) = \left\{ (u, z_0, z_\infty) \in W^{1,p}(\mathbb{C}P^1, M) \times \mathbb{C}P^1 \times \mathbb{C}P^1 \mid \begin{array}{l}
 u \text{ is } J_0\text{-holomorphic}, [u(\mathbb{C}P^1)] = B \text{ and } u(z_i) \in Q_i \text{ for } i = 0, \infty
\end{array}\right\}.
\]

For the sake of simplicity we abbreviate \( \mathcal{M} := \mathcal{M}(B; J_0; Q_0 \times Q_\infty) \).

We can actually choose \( J_0 \) to be in \( \mathcal{J}_{reg}(B) \) making a neighborhood of \( \Sigma \) biholomorphic to \( \mathcal{O}(a, 2) \) and all simple \( J_0 \)-holomorphic spheres in \( \mathcal{M} \) Fredholm regular (see [Wen15].
Step 3. We construct a map \( \xi : D \to \mathbb{C}P^1 \) by \( \hat{\psi} (z_0) = \xi (z) \) for \( (u, z_0, z_\infty) \in D, z \in \mathbb{C}P^1 \) and \( g \in G \).

By the assumption that \( B \) is \( J_0 \)-indecomposable, the space \( M \times_G \mathbb{C}P^1 \) is compact (cf. the proof of [MS04, Lemma 7.1.8]), and hence \( \psi \) is proper.

To compute the degree of \( \psi \), we claim that every point \( p_0 \in \Sigma \setminus (Q_0 \cup Q_\infty) \) is a regular value of \( \psi \), and that \( \psi^{-1}(p_0) \) consists of only one point. This implies \( \text{deg}(\psi) = 1 \). By positivity of intersections, the unique \( J_0 \)-holomorphic sphere in \( \Sigma \) passing through \( p_0 \) in the class \( B \) intersects \( Q_0 \) and \( Q_\infty \). Note that

\[
B \cdot [\Sigma] = \int_B \omega = \int_F \Omega_{u,2} = 2,
\]

that is, the intersection number of \( J_0 \)-holomorphic spheres in \( M \) representing \( B \) with \( \Sigma \) equals 2. It follows from positivity of intersections that any \( J_0 \)-holomorphic sphere passing through \( Q_0, Q_\infty \) and \( p_0 \) must lie in \( \Sigma \), and hence it is unique and \( \psi^{-1}(p_0) \) is a one-point set.

We next prove that \( p_0 \) is a regular value of \( \psi \). It is easy to see that for a given vector \( Y \in T_{p_0} u(\mathbb{C}P^1) \subset T_{p_0} M \), there exists a vector \( X \in T_{[u(0),0,\infty,1]} (M \times_G \mathbb{C}P^1) \) such that \( d_{[u(0),0,\infty,1]} \psi(X) = Y \). To deal with normal directions, consider the pull-back bundle of the normal bundle \( N_u(\mathbb{C}P^1)/M \) to \( u(\mathbb{C}P^1) \) in \( M \) under the unique \( J_0 \)-holomorphic sphere \( u : \mathbb{C}P^1 \to M \) with \( u(0) \in Q_0, u(\infty) \in Q_\infty \) and \( u(1) = p_0 \). The bundle \( u^* N_u(\mathbb{C}P^1)/M \to \mathbb{C}P^1 \) is isomorphic to \( O(\mathbb{C}P^1) \to \mathbb{C}P^1 \) as holomorphic vector bundles. Hence, for a given vector \( Y \) of \( (N_u(\mathbb{C}P^1)/M)_{p_0} \), holomorphic sections of the latter bundle give a curve \( (-\epsilon, \epsilon) \ni s \mapsto [u_s,0,\infty,1] \in M \times_G \mathbb{C}P^1 \) with \( u_0 = u \) such that

\[
d_{[u_s,0,\infty,1]} \psi \left( \frac{d[u_s,0,\infty,1]}{ds} \right) \bigg|_{s=0} = \frac{du_s}{ds} \bigg|_{s=0} = Y \in T_{p_0} M.
\]

We conclude that the differential of \( \psi \) at \( p_0 \) is surjective, and hence \( p_0 \) is a regular value.

Step 3. We construct a map \( \varphi : \pi_1(X) \to \pi_1(D_0(\xi)) \), where \( X = M \setminus \Sigma \) and the space \( D_0(\xi) \) is defined below.

Let \( \xi \) be the total space of the normal bundle \( N_{\Sigma/M} \) to \( \Sigma \) in \( M \) restricted to \( Q_0 \). Denoting by \( \nu_{\Sigma}(Q_0) \) a tubular neighborhood of \( Q_0 \) in \( \Sigma \), we write \( \hat{\xi} \) for the total space of the restricted normal bundle \( N_{\Sigma/M} |_{\nu_{\Sigma}(Q_0)} \). Equip \( N_{\Sigma/M} \) with a bundle metric and consider the total space \( D(\xi) \) of the disk bundle associated to \( N_{\Sigma/M} |_{\nu_{\Sigma}(Q_0)} \), and \( D(\xi) \subset D(\hat{\xi}) \) is defined in the same manner. We will identify \( D(\xi) \) with a tubular neighborhood of \( \nu_{\Sigma}(Q_0) \) in \( M \) and regard \( \nu_{\Sigma}(Q_0) \) as a submanifold of this tubular neighborhood (see
Figure 1 for schematic pictures). The space $D_0(\hat{\xi})$ is defined to be the complement of the zero-section in $D(\hat{\xi})$.

![Figure 1](image)

**Figure 1.** (A) The neighborhoods $\nu_M(\Sigma)|Q_0$ (green) and $\nu_\Sigma(Q_0)$ (yellow). (B) The neighborhood $D(\hat{\xi})$ (green).

Now we construct a map $\varphi: \pi_1(X) \to \pi_1(D_0(\hat{\xi}))$. Note that $\pi_1(D_0(\hat{\xi})) \cong \mathbb{Z}/(a - 2m)\mathbb{Z}$, and $\varphi$ shall provide the desired isomorphism for the assertion of the lemma. Take a point $p \in X$ sufficiently close to a point on $\Sigma \setminus (Q_0 \cup Q_\infty)$ so that it is a regular value of the map $\text{ev}$. We can perturb any loop $f: [0, 1] \to X$ based at $p$ so that its image is an embedded 1-dimensional submanifold and it is transverse to the evaluation map $\text{ev}$.

Then, thanks to transversality, $\text{ev}^{-1}(f([0, 1]))$ is a 1-dimensional submanifold in $M \times_G \mathbb{C}P^1$ parametrized by $t \in [0, 1]$. In fact, it is a circle since $\text{ev}^{-1}(p)$ is a one-point set. Write $u_1^f: \mathbb{C}P^1 \to M$ for a $J_0$-holomorphic sphere such that $u_1^f(0) \in Q_0$, $u_1^f(\infty) \in Q_\infty$ and $u_1^f(1) = f(t)$, i.e.,

$$\text{ev}([[u_1^f, 0, \infty), 1]) = f(t).$$

Note that $u_0^f = u_1^f$, and $u_0^f$ depends only on $p$ (not $f$); this allows us to set $u_0 := u_0^f$. Take a sufficiently small real number $\epsilon_0 > 0$ such that the image $u_0(D^2(\epsilon_0))$ of the closed disk $D^2(\epsilon_0) = \{ z \in \mathbb{C} \mid |z| \leq \epsilon_0 \}$ is contained in $D(\hat{\xi})$. We set $p' := u_0(\epsilon_0)$ to be a base point for loops on $D_0(\hat{\xi})$. Depending on the circle $u_1^f$ of $J_0$-holomorphic spheres, we can also take $\epsilon > 0$ such that $u_1^f(D^2(\epsilon))$ is contained in $D(\hat{\xi})$ for all $t$. This provides the loop $g(f)$ in $D_0(\hat{\xi})$ given by $g(f)(t) := u_1^f(\epsilon)$. We may assume that $\epsilon < \epsilon_0$, and since $D_0(\hat{\xi})$ is path-connected, we can regard $g(f)(t)$ as an element of $\pi_1(D_0(\hat{\xi}), p')$ up to base point change. Note that the choice of $\epsilon$ does not affect the homotopy class $[g(f)(t)]_{D_0(\hat{\xi})} \in \pi_1(D_0(\hat{\xi}), p')$. Assuming that $[g(f)(t)]_{D_0(\hat{\xi})}$ actually depends only on the homotopy class $[f]_{X} \in \pi_1(X, p)$, which will be shown in the next step, we define a map $\varphi: \pi_1(X, p) \to \pi_1(D_0(\hat{\xi}), p')$ by

$$\varphi([f]_{X}) = [g(f)]_{D_0(\hat{\xi})}.$$  

**Step 4.** We prove that $\varphi$ is a well-defined homomorphism.

To see its well-definedness as a map, take two homotopic smooth loops $f_1$ and $f_2$ based at $p$ whose images are 1-dimensional submanifolds of $X$ and which are transverse to $\text{ev}$. Then, one can check that the lifts $\tilde{f}_1$ and $\tilde{f}_2$ in $M \times_G \mathbb{C}P^1$ of $f_1$ and $f_2$, respectively, are homologous. This shows that the corresponding loops $g(f_1)$ and $g(f_2)$ in $D_0(\hat{\xi})$ are homologous. As any two homologous loops in the latter space are homotopic, $g(f_1)$ and $g(f_2)$ are homotopic, and $\varphi$ is a well-defined map.
To show that $\varphi$ is a homomorphism, take two smooth loops $f_1$ and $f_2$ based at $p$ whose images are 1-dimensional submanifolds of $X$ and which are transverse to $ev$. All smoothings of the composition $f_1 \cdot f_2$ give homotopic loops in $D_0(\hat{\xi})$ via $\varphi$ as discussed above. Moreover, an argument similar to the above shows that respective lifted loops $\tilde{f}_1 \cdot \tilde{f}_2$ and $\tilde{f}_{1,2}$ of $f_1 \cdot f_2$ and its smoothing $f_{1,2}$ are homologous, which implies that $g(f_1) \cdot g(f_2)$ and $g(f_{1,2})$ are homotopic in $D_0(\hat{\xi})$.

**Step 5.** We prove that $\varphi$ is an isomorphism, and hence $\pi_1(X)$ is isomorphic to $\mathbb{Z}/(a - 2m)\mathbb{Z}$.

To show the injectivity of $\varphi$, suppose that $\varphi([f]_X) = [g(f)]_{D_0(\hat{\xi})}$ is the identity element. Set

$$h_0(t) := u_0(1 + t(\epsilon_0 - 1)),$$

which is a path connecting $p$ to $p'$. By definition of $g(f)$, the loop $h_0 \cdot g(f) \cdot h_0^{-1}$ is homotopic to $f$ in $X$. This proves that $[f]_X$ is trivial and $\varphi$ is injective.

Next, to see its surjectivity, consider the loop $f_0(t)$ in $X$ defined by

$$f_0(t) = u_0(e^{2\pi it}).$$

We can see that each point $u_0(e^{2\pi it})$ on this loop is a regular value of $ev$ as follows: Identify $\nu_M(\Sigma)$ with a neighborhood of the zero-section of $\mathcal{O}(a, 2) \to \mathbb{P}(E_m)$. Then, one can take a holomorphic section $\tau: \mathbb{P}(E_m) \to \mathcal{O}(a, 2)$ such that $u_0$ lies on the image of $\tau$. In particular, $u_0(e^{2\pi it})$ can be seen as a point of $\tau(\mathbb{P}(E_m))$. By replacing $\mathbb{P}(E_m)$ by $\tau(\mathbb{P}(E_m))$, an argument similar to that for regularity in Step 2 concludes that $u_0(e^{2\pi it})$ is a regular value of $ev$.

Now we claim that the loop $g_0(t)$ in $D_0(\hat{\xi})$ defined by

$$g_0(t) := g(f_0)(t) = u_0(\epsilon_0 e^{2\pi it})$$

gives rise to a generator of $\pi_1(D_0(\hat{\xi}), p')$. This loop bounds the disk $u_0(D^2(\epsilon_0))$, which intersects $\Sigma$ only at one point, namely $u_0(0) \in Q_0$. Let $S(\xi)$ and $S(\hat{\xi})$ denote the total spaces of the circle bundles associated to the restricted normal bundles $\xi \to Q_0$ and $\xi \to \nu_{\Sigma}(Q_0)$, respectively. The disk $u_0(D^2(\epsilon_0))$ may be assumed to give an element of not only $H_2(D(\xi), S(\hat{\xi}); \mathbb{Z})$ but also $H_2(D(\hat{\xi}), S(\xi); \mathbb{Z})$. Letting $I: H_2(D(\xi), S(\xi); \mathbb{Z}) \times H_2(D(\hat{\xi}); \mathbb{Z}) \to H_0(D(\xi); \mathbb{Z}) \cong \mathbb{Z}$ be the intersection pairing, we have

$$I([u_0(D^2(\epsilon_0))], [Q_0]) = 1,$$

where $[Q_0]$ is the class of the image of the zero-section of $D(\xi) \to Q_0$. Coupled with the fact that $H_2(D(\xi), S(\xi); \mathbb{Z}) \cong H^2(D(\xi); \mathbb{Z}) \cong \mathbb{Z}$, this shows that $[u_0(D^2(\epsilon_0))]$ is nontrivial, and especially it is a generator of $H_2(D(\xi), S(\xi); \mathbb{Z})$. The group $H_1(D(\xi); \mathbb{Z})$ is trivial, and hence the homomorphism $\partial_*: H_2(D(\xi), S(\xi); \mathbb{Z}) \to H_1(S(\xi); \mathbb{Z})$ appearing in the homology long exact sequence for the pair $(D(\xi), S(\xi))$ is surjective. Therefore, $\partial_*$ maps $[u_0(D^2(\epsilon_0))]$ to a generator of $H_1(S(\xi); \mathbb{Z}) \cong \mathbb{Z}/(a - 2m)\mathbb{Z}$, and by definition we have

$$\partial_*([u_0(D^2(\epsilon_0))] = [g_0].$$

Since the three groups $\pi_1(D_0(\hat{\xi}), p')$, $\pi_1(D_0(\xi))$ and $H_1(S(\xi); \mathbb{Z})$ can be canonically identified, $[g_0]$ is a generator of $\pi_1(D_0(\hat{\xi}), p')$. This completes the proof. □
The \( J \)-indecomposability assumption in Lemma \[2.6\] is satisfied if \( a \geq m + 5 \):

**Lemma 2.9.** Let \((M, \omega)\) be a closed integral symplectic 6-manifold admitting a symplectic divisor symplectomorphic to \((\mathbb{P}(E_m), \Omega_{a,b})\) for \(m = 0, 1\). If \(a \geq m + 5\) and \(b = 2\), then the homology class \(B\) corresponding to \([F]\) is \( J \)-indecomposable for \(J \in \mathcal{J}_{\text{reg}}(B)\).

**Proof.** Let \( \iota: \mathbb{P}(E_m) \hookrightarrow M \) denote a symplectic embedding with \( \iota(\mathbb{P}(E_m)) = \Sigma \). Set

\[
A = \iota_*[C_0] \quad \text{and} \quad B = \iota_*[F] \in H_2(M; \mathbb{Z}).
\]

Suppose on the contrary that \(B\) is not \( J \)-indecomposable, that is, there exist homology classes \(C_1, \ldots, C_N \in H_2(M; \mathbb{Z})\) \((N \geq 2)\) such that \(B = C_1 + \cdots + C_N\) and each \(C_i\) is represented by a nonconstant \( J \)-holomorphic sphere. Since \([\omega]\) has an integral lift, the minimal symplectic energy of nonconstant \( J \)-holomorphic spheres in \(M\) must be 1, and this implies that \(N = 2\). Moreover, by Lemma \[2.7\] we can represent each \(C_i\) as a linear combination of \(A\) and \(B\); namely \(C_i = k_iA + \ell_iB\) for some \(k_i, \ell_i \in \mathbb{Z}\) with

\[
(2.12) \quad k_1 + k_2 = 0 \quad \text{and} \quad \ell_1 + \ell_2 = 1.
\]

In general, the dimension of the moduli space of (unparametrized) \( J \)-holomorphic spheres \(u: \mathbb{CP}^1 \to M\) with \([u(\mathbb{CP}^1)] = kA + \ell B\) is given by

\[
\chi(\mathbb{CP}^1) \cdot \frac{\dim M}{2} + 2c_1(TM)([u(\mathbb{CP}^1)]) - \dim \text{Aut}(\mathbb{CP}^1) = 2((a + 2 - 3m)k + 4\ell),
\]

where \(\chi(\mathbb{CP}^1)\) is the Euler characteristic of \(\mathbb{CP}^1\). Suppose that the symplectic energy of such \( J \)-holomorphic curves is equal to 1, i.e.,

\[
E(u) := \int_{kA + \ell B} \iota^*\omega = (a - 2m)k + 2\ell = 1.
\]

Then, for the moduli space to be nonempty, \(k\) and \(\ell\) need to satisfy the following inequality and equation:

\[
2((a + 2 - 3m)k + 4\ell) \geq 0 \quad \text{and} \quad (a - 2m)k + 2\ell = 1.
\]

From these we have

\[
(2.13) \quad k \leq \frac{2}{a - m - 2} < 1,
\]

where the last inequality follows from the assumption that \(a \geq m + 5\).

Let us return to our case. By the inequality \[2.13\] combined with the equation for \(k_i\) in \[2.12\], we obtain \(k_1 = k_2 = 0\). However, note that there is no \( J \)-holomorphic sphere \(u: \mathbb{CP}^1 \to M\) with \([u(\mathbb{CP}^1)] = \ell_iB\) and \(E(u) = 1\). This leads to a contradiction. \( \square \)

We are ready to prove the second assertion in Theorem \[1.1\].

**Proof of Theorem \[1.1\]**. In the following proof, we will use the same notations as in the proof of Lemma \[2.6\]. Suppose on the contrary that there exists a symplectic embedding \( \iota: \mathbb{P}(E_m) \to M \) of \((\mathbb{P}(E_m), \omega_{a,2})\) into a closed integral symplectic 6-manifold \((M, \omega)\) such that the image \(\Sigma = \iota(\mathbb{P}(E_m))\) is a symplectic divisor on \((M, \omega)\). By Lemma \[2.9\] the homology class \(B = \iota_*[F]\) is \( J \)-indecomposable for any \(J \in \mathcal{J}_{\text{reg}}(B)\), and \(\Sigma\) and \(B\)
satisfy the assumption of Lemma 2.6. Therefore the fundamental group \( \pi_1(X) \) of the complement \( X = M \setminus \Sigma \) is isomorphic to \( \mathbb{Z}/(a - 2m)\mathbb{Z} \).

Consider a sphere \( R = \iota(F') \subset \Sigma \) passing through \( u_0(0) \), where \( F' \) is a fiber of \( \pi : \mathbb{P}(E_m) \to \mathbb{CP}^1 \). Note that \( [R] = B \in H_2(M; \mathbb{Z}) \). Let \( \eta \) be the total space of the restricted normal bundle \( N_{\Sigma/M}\mid R \) and \( \hat{\eta} \) the total space of the normal bundle \( N_{\Sigma/M}\mid B(R) \). Write \( D(\eta), S(\hat{\eta}) \) and \( S(\eta) \) for the total spaces of the disk and circle bundles associated to \( \eta \) and \( \hat{\eta} \), respectively, with respect to some bundle metric on \( N_{\Sigma/M} \). Consider the loop \( g_0 : [0, 1] \to D_0(\hat{\xi}) \) defined by

\[
g_0(t) = u_0(\epsilon_0e^{2\pi it}).
\]

Taking \( \epsilon_0 \) sufficiently small if necessary, we may assume that \( g_0([0, 1]) \) lies in \( D_0(\hat{\xi}) \cap D_0(\hat{\eta}) \), where \( D_0(\hat{\eta}) \) is the complement of the image of the zero-section in \( D(\hat{\eta}) \). A homological argument similar to that in the proof of Lemma 2.6 shows that \( g_0 \) serves as a generator of the group \( \pi_1(D_0(\hat{\eta}), p') \cong \mathbb{Z}/2\mathbb{Z} \).

Write \( j : D_0(\hat{\eta}) \hookrightarrow X \) for the inclusion map. Define the base change map \( \phi_{\hat{h}_0} : \pi_1(X, p') \to \pi_1(X, p) \) by

\[
\phi_{\hat{h}_0}([f]_{(X, p')}) = [h_0 \cdot f : h_0^{-1}]_{(X, p)}.
\]

Then, by definition we have \( (\phi_{\hat{h}_0} \circ j_\ast)([g_0]_{D_0(\hat{\eta})}) = [f_0]_X \), which is a generator of \( \pi_1(X, p) \), and in particular it is a nontrivial element. Hence, it follows that \( \phi_{\hat{h}_0} \circ j_\ast : \pi_1(D_0(\hat{\eta}), p') \to \pi_1(X, p) \) is a nontrivial homomorphism. However, \( \pi_1(D_0(\hat{\eta}), p') \cong \mathbb{Z}/2\mathbb{Z} \not\cong \mathbb{Z}/(a - 2m)\mathbb{Z} \cong \pi_1(X, p) \), and \( a \) and \( 2 \) are coprime. This contradicts the existence of the above nontrivial homomorphism.

\[ \square \]

3. Fillability of Boothby–Wang bundles over rational ruled surfaces

3.1. Boothby–Wang bundles over rational ruled surfaces. Let \((M, \omega)\) be a closed integral symplectic manifold with a fixed integral lift \([\omega] \in H^2(M; \mathbb{Z})\). There is a unique (up to isomorphism) principal \( S^1 \)-bundle \( p : P \to M \) with Euler class \( e(P) = -[\omega] \).

We can take a connection 1-form \( A \in \Omega^1(P) \) on \( P \) with \( dA = 2\pi p^*\omega \). Throughout this paper, we regard connection 1-forms on principal \( S^1 \)-bundles as usual \( \mathbb{R} \)-valued differential forms on them. By definition, the connection 1-form \( A \) serves as a contact form on \( P \) whose Reeb orbits are fibers of \( p \). We call the manifold \( P \) the Boothby–Wang manifold over \((M, \omega)\) and refer to the contact manifold \((P, \ker(\pi))\) as the Boothby–Wang contact manifold over \((M, \omega)\).

Let \((P_{a,b}, \xi_{a,b})\) and \((\tilde{P}_{a,b}, \tilde{\xi}_{a,b})\) be the Boothby–Wang contact manifolds over the (integral) symplectic rational ruled surfaces \((S^2 \times S^2, \omega_{a,b})\) and \((S^2 \tilde{\times} S^3, \tilde{\omega}_{a,b})\), respectively.

Proposition 3.1. Suppose that \( a \) and \( b \) are coprime. Then,

1. \( P_{a,b} \) is diffeomorphic to \( S^2 \times S^3 \); \( \tilde{P}_{a,b} \) is diffeomorphic to \( S^2 \tilde{\times} S^3 \), where \( S^2 \tilde{\times} S^3 \) denotes the non-trivial \( S^3 \)-bundle over \( S^2 \);

2. \( c_1(\xi_{a,b}) = (2a - 2b)\gamma \) and \( c_1(\tilde{\xi}_{a,b}) = (2a - 3b)\tilde{\gamma} \), where \( \gamma \) and \( \tilde{\gamma} \) are generators of \( H^2(P_{a,b}; \mathbb{Z}) \) and \( H^2(\tilde{P}_{a,b}; \mathbb{Z}) \), respectively.
Proof. In this proof, we only deal with \((P_{a,b}, \xi_{a,b})\). A similar argument also works for \((\tilde{P}_{a,b}, \tilde{\xi}_{a,b})\).

To examine the diffeomorphism type of \(P_{a,b}\), we first claim that \(H_1(P_{a,b}; \mathbb{Z}) \cong \mathbb{Z}\). According to \cite[Lemma 17]{Ham13}, the cohomology group \(H^2(P_{a,b}; \mathbb{Z})\) is isomorphic to \(H^2(S^2 \times S^2; \mathbb{Z})/\mathbb{Z}([\omega_{a,b}])\). To see that the latter is isomorphic to \(\mathbb{Z}\), consider the homomorphism defined by

\[
\Phi: H^2(S^2 \times S^2; \mathbb{Z}) \cong \mathbb{Z}(\alpha) \oplus \mathbb{Z}(\beta) \rightarrow \mathbb{Z}, \quad (r, s) \mapsto as - br,
\]

where \(\alpha = \pi^*\omega_1(\mathcal{O}(1))\) and \(\beta = c_1(O_E(m))\) are the cohomology classes as in Section 2.1. It is easy to check that its kernel is \(\mathbb{Z}([\omega_{a,b}])\). Moreover, since \(a\) and \(b\) are coprime, \(\Phi\) is surjective. Hence, the first isomorphism theorem concludes the claim.

Next, since \(S^2 \times S^2\) is simply connected and carries a spin structure, and \([\omega_{a,b}] \in H^2(S^2 \times S^2; \mathbb{Z})\) is indivisible, Lemmata 16 and 19 in \cite{Ham13} show that the Boothby–Wang manifold \(P_{a,b}\) is simply connected and spinable. Moreover, one can easily check that the map \(H_2(P_{a,b}; \mathbb{Z}) \rightarrow H_2(S^2 \times S^2; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}\) induced by the bundle projection \(P_{a,b} \rightarrow S^2 \times S^2\) is injective, which implies that \(H_2(P_{a,b}; \mathbb{Z}) \cong H^3(P_{a,b}; \mathbb{Z})\) is torsion free. Thus, the universal coefficient theorem combined with the above claim concludes that

\[
H_2(P_{a,b}; \mathbb{Z}) \cong \text{Hom}(H^2(P_{a,b}; \mathbb{Z}), \mathbb{Z}) \oplus \text{Ext}(H^3(P_{a,b}; \mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}.
\]

Now the first assertion follows from \cite[Proposition 8.2.1]{Gei08}.

According to \cite[Section 6]{Ham13}, we have

\[
c_1(\xi_{a,b}) = \Phi(c_1(T(S^2 \times S^2)))
\]

under the identification \(H^2(P_{a,b}; \mathbb{Z}) \cong H^2(S^2 \times S^2; \mathbb{Z})/\mathbb{Z}([\omega_{a,b}])\) via \(\Phi\). It follows from Lemma 2.2 that

\[
c_1(\xi_{a,b}) = \Phi(2, 2) = (2a - 2b)\gamma,
\]

where \(\gamma\) is a generator of \(H^2(P_{a,b}; \mathbb{Z})\).

3.2. Almost contact structures on Boothby–Wang bundles. Recall that an almost contact structure on an oriented manifold \(P\) is a cooriented codimension 1 subbundle \(\xi \subset TP\) together with a complex bundle structure on it. We say that two almost contact structures \(\xi\) and \(\xi'\) on \(P\) are equivalent if they can be connected by a combination of smooth homotopies of almost contact structures on \(P\) and orientation-preserving diffeomorphisms on \(P\).

Corollary 3.2. Let \(\xi_{a,b}\) (resp. \(\tilde{\xi}_{a,b}\)) be the contact structure on \(P_{a,b}\) (resp. \(\tilde{P}_{a,b}\)) given as in Section 3.1. Suppose that each pair of \((a, b)\) and \((a', b')\) is coprime.

(1) \(\xi_{a,b}\) and \(\xi_{a',b'}\) are equivalent as almost contact structures if and only if \(a - b = a' - b'\);

(2) \(\tilde{\xi}_{a,b}\) and \(\tilde{\xi}_{a',b'}\) are equivalent as almost contact structures if and only if \(2a - 3b = 2a' - 3b'\).
Proof. According to [Ham13, Corollary 10], two almost contact structures $\xi_{a,b}$ and $\xi_{a',b'}$ are equivalent if and only if $c_1(\xi_{a,b})$ and $c_1(\xi_{a',b'})$ have the same maximal divisibility in the integral cohomology; this also applies to $\tilde{\xi}_{a,b}$ and $\tilde{\xi}_{a',b'}$. Therefore the corollary follows from Proposition 3.1 immediately. □

3.3. Stein fillable contact structures. Let us prove Theorem 1.2 (1) and (2).

Proof of Theorem 1.2 (1) and (2). Let $(M,\omega)$ be the complex quadric in $(\mathbb{CP}^4,\omega_{FS})$ and $\Sigma$ the hyperplane section in $M$ as in Section 2.2

$$M = \{(z_0 : z_1 : z_2 : z_3 : z_4) \in \mathbb{CP}^4 \mid z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\}, \quad \Sigma = M \cap \{z_4 = 0\}.$$ By Proposition 2.3, $(\Sigma,\omega|_\Sigma)$ is a symplectic divisor on $(M,\omega)$ and is symplectomorphic to $(S^2 \times S^2,\omega_{1,1})$. Hence, the contact manifold $(P_{1,1},\xi_{1,1})$ is obtained as the convex boundary of the complement of a symplectic tubular neighborhood of $(\Sigma,\omega|_\Sigma)$ in $M$. Furthermore, $M \setminus \Sigma$ admits a Stein structure and is written as the affine hypersurface $X$ given by

$$X = \{(w_0, w_1, w_2, w_3) \in \mathbb{C}^4 \mid w_0^2 + w_1^2 + w_2^2 + w_3^2 + 1 = 0\}.$$ One can see that $X$ is diffeomorphic to the cotangent bundle $T^*S^3$. According to [BGZ19, Theorem 1.2(1)], all Stein fillings of a subcritically Stein fillable contact manifold have isomorphic homology groups. In particular, their middle-dimensional ones are trivial. Thus, $H_3(X;\mathbb{Z}) \cong H_3(T^*S^3;\mathbb{Z})$ being non-trivial implies that $(P_{1,1},\xi_{1,1})$ does not admit a subcritical Stein filling, which concludes that $\xi_{1,1}$ is critically Stein fillable.

In the rest of the proof, we show the second assertion of $\xi_{a,1}$ with $a \geq 2$; the same argument also works for $\tilde{\xi}_{a,1}$. Following the proof of Theorem 1.1 (1), we have an identification

$$(S^2 \times S^2,\omega_{a,1}) \cong (\Sigma_{m,n},\eta_{m,n})$$

for some $m$ and $n$, and $(\Sigma_{m,n},\eta_{m,n})$ is a symplectic divisor on $(\mathbb{P}(F_m),\Omega_m)$. Thus, the Boothby–Wang contact manifold $(F_{a,1},\xi_{a,1})$ appears as the boundary of the complement of a symplectic tubular neighborhood of $\Sigma_{m,n}$ in $\mathbb{P}(F_m)$. Employing the result of Biran and Cieliebak in [BC04, Theorem 2.4.1], this complement is in fact a subcritical Stein domain. Therefore, $\xi_{a,1}$ is subcritically Stein fillable. □

Remark 3.3. In each homotopy class of almost contact structures on $S^2 \times S^3$ and $S^2 \times S^3$, there is a unique contact structure which is subcritically Stein fillable [DGZ18, Theorem 1.2]. In view of the result of Boyer and Pati [BP14, Proposition 3.11] and Theorem 1.2 we deduce that any contact structure $\xi_{a,b}$ (and also $\tilde{\xi}_{a,b}$) with $b \neq 1$ is not subcritically Stein fillable.

3.4. Stein non-fillable contact structures. We will prove Stein non-fillability by contradiction. To do this, let us suppose that a Boothby–Wang contact manifold over a closed integral symplectic manifold $\Sigma$ admits a Stein filling. By capping off the boundary, we can construct a closed symplectic manifold containing $\Sigma$ as a symplectic divisor. In what follows we first present this construction and then give a proof of Theorem 1.2 (3).
A similar construction of closed symplectic manifolds can be found in [CDvK14, Section 6.1].

Let \((W, d\lambda)\) be a Stein domain whose contact boundary is the total space of the Boothby–Wang bundle

\[ p : (P, \text{ker } A) \to (\Sigma, \omega_{\Sigma}) \]

over an integral symplectic manifold \((\Sigma, \omega_{\Sigma})\) with the Euler class equal to a fixed integral lift \(-[\omega_{\Sigma}] \in H^2(\Sigma; \mathbb{Z})\). Up to Liouville homotopy, we may assume that \(A = \lambda|_{\partial W}\). See [Gut17, Lemma 4.25].

Consider the complex line bundle \(p : E \to \Sigma\) with the first Chern class \(c_1(E) = [\omega_{\Sigma}]\). Take a hermitian metric on \(E\), and let \(\Theta \in \Omega^1(E \setminus \Sigma)\) be a connection 1-form, called the global angular form, with \(d\Theta = -p^*\omega_{\Sigma}\). Denote the corresponding unit disk bundle by \(N \to \Sigma\). We may assume that the circle bundle \(\partial N \to \Sigma\) coincides with the Boothby–Wang bundle \(P \to \Sigma\) and that \(\Theta|_{\partial N} = -A/2\pi\). Following [BK13, Section 2.2], we equip \(E\) with the symplectic form \(\omega_E\) given by

\[
\omega_E = -d(f(r^2)\Theta) = -df(r^2) \wedge \Theta + f(r^2)p^*\omega_{\Sigma}
\]

where \(r\) denotes the norm on \(E\) with respect to the hermitian metric, and \(f\) is a smooth function such that \(f(0) = 1\) and \(f' < 0\). Note that \(\omega_E\) is exact away from the zero section with a primitive \(-f(r^2)\Theta\). The corresponding Liouville vector field is transverse to the boundary \(\partial N\) of \(N\) pointing inwards. This concave boundary coincides with \((\partial W, \lambda|_{\partial W})\) up to positive rescaling of \(\lambda|_{\partial W}\). This allows us to glue \(W\) and \(N\) together symplectically. Let \((M, \omega)\) denote the glued symplectic manifold \(W \cup_N (N, \omega_E|_N)\).

**Proposition 3.4.** The symplectic manifold \((M, \omega)\) contains \((\Sigma, \omega_{\Sigma})\) as a symplectic divisor.

**Proof.** Since \(\Sigma\) is a symplectic submanifold of codimension 2 and its complement is a Stein manifold, it remains to show that \([\omega] = \text{PD} [\Sigma]\). Recall that the symplectic form \(\omega\) is given by the symplectic form \(d\lambda\) on the Stein domain piece \(W\) and by the symplectic form \(\omega_E|_N\) on the disk bundle piece. In particular \(d\lambda\) is exact and hence is vanishing in the cohomology \(H^2_c(\text{Int} W)\). In light of the Mayer–Vietoris sequence for the decomposition \(M = W \cup N\) which is compatible with taking Poincaré dual, it suffices to show that the Poincaré dual of the cohomology class \([\omega_E] \in H^2_c(E)\) is given by the homology class of the zero section \(\{\Sigma\} \in H_{2n-2}(E)\). In view of (3.1), we see that \([\omega_E]\) gives the Thom class of the line bundle \(E \to \Sigma\) as described e.g. in [BT82, Equation (6.40)]. See also [BK13, Section 2.2]. Therefore its Poincaré dual is the homology class of the zero section by [BT82 Proposition 6.24(b)].

Using Proposition 3.4, we prove Theorem 1.2 (3).

**Proof of Theorem 1.2 (3).** Suppose on the contrary that \((P_{a,2}, \xi_{a,2})\) admits a Stein filling \(W\). By Proposition 3.4, we can construct a closed integral symplectic 6-manifold \(M\) such that the rational ruled surface \((\mathbb{P}(E_0), \Omega_{a,2})\) is embedded in \(M\) as a symplectic divisor. This contradicts Theorem 1.1 (2) as we have assumed \(a \geq 5\). The same argument shows \(\tilde{\xi}_{a,2}\) with \(a \geq 6\) is Stein non-fillable. \(\square\)
Remark 3.5. Combining the results in Theorems 1.1 and 1.2, we obtain interesting contact structures on $S^2 \times S^3$ (and also on $S^2 \times \tilde{S}^3$) in terms of various fillability: There are infinitely many pairwise non-contactomorphic contact structures on $S^2 \times S^3$ which are strongly fillable and almost Weinstein fillable, but not Stein fillable. Those contact structures are given by the contact structures of the form $\xi_{a,2}$ on $P_{a,2}$. For, note that the underlying almost contact structure of $\xi_{a,2}$ is equivalent to the one of $\xi_{a-1,1}$ by Corollary 3.2. Since $\xi_{a-1,1}$ is Stein fillable and hence is almost Weinstein fillable by Theorem 1.2, it follows that $\xi_{a,2}$ is also almost Weinstein fillable. Since $\xi_{a,2}$ comes from a Boothby–Wang bundle, the associated disk bundle provides a strong symplectic filling. The distinguishing result in [BP14, Proposition 3.11] tells us that $\xi_{a,2}$'s are pairwise non-contactomorphic. For more information on almost Weinstein fillability, we refer the reader to [BCS14], [Laz20, Section 2.1.2] and [Zho19, Section 1.3].

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