Color-flavor locking and 2SC pairing in random matrix theory

Takuya Kanazawa
Research and Development Group, Hitachi, Ltd., Kokubunji, Tokyo 185-8601, Japan

A new non-Hermitian chiral random matrix model is proposed. For three flavors, it is shown that in the large-N limit with N the matrix size, the color SU(3) and the flavor SU(3) symmetries are spontaneously broken to the diagonal SU(3) subgroup, realizing color-flavor locking. The nonlinear sigma model representation is rigorously derived and compared with the CFL chiral Lagrangian of QCD. For two flavors, the color SU(3) symmetry is shown to be spontaneously broken to SU(2) while the chiral SU(2)$_L \times$ SU(2)$_R$ symmetry remains intact, thus reproducing the symmetry breaking pattern of the 2SC phase in QCD.

I. INTRODUCTION

Random matrix theory (RMT) enjoys broad applications in mathematics, physics, statistics, biology, information theory and engineering [1–4]. In physics, RMT has long been used to describe universal energy level fluctuations of quantum-chaotic Hamiltonians [5, 6]. In nuclear and elementary particle physics, RMT having the same global symmetries as QCD was employed to understand various aspects of dynamical symmetry breaking in low-energy hadron physics [7–9]. It is well established that spontaneous breaking of chiral symmetry in QCD with massless quarks is tightly linked to the accumulation of near-zero eigenvalues of the Dirac operator [10]. The spectral fluctuations of Dirac eigenvalues on the scale $\sim 1/(V_4 \Sigma)$, with $V_4$ the spacetime volume and $\Sigma$ the chiral condensate $\langle \bar{\psi} \psi \rangle$, are universal and can be described exactly by chirally symmetric RMT [8, 11]. This limit of QCD is known as the $\varepsilon$-regime [12–14].

If one wants to investigate nuclear and quark matter present in the interior of compact stars, one has to introduce a chemical potential $\mu$; in this case the Dirac operator is no longer anti-Hermitian and the eigenvalues spread over the complex plane. An interesting question to ask is whether the powerful framework of RMT can be applied to QCD with chemical potential. In the regime $\mu \to 0$ and $V_4 \to \infty$ with $V_4 F_\pi^2 \mu^2 \sim 1$, with $F_\pi$ the pion decay constant, much work has been done [15]. RMT that explicitly includes $\mu$ was proposed [16, 17] and the microscopic spectral functions were completely worked out [17–19]. On the other hand, constructing RMT in the dense regime $\mu \neq 0$ is quite nontrivial, since just making $\mu$ large in the models of [16, 17] does not make sense. Progress has been made in [20] where RMT for two-color QCD at high density $\mu \gg \Lambda_{\text{QCD}}$ was identified. The exact microscopic spectral functions of this model were analytically derived [21–23]. Later the construction of [20] was generalized to adjoint QCD at high density and QCD with large isospin chemical potential [24]. A Banks-Casher-type relation that links the density of complex Dirac eigenvalues at the origin to the BCS gap $\Delta$ of quarks was also found for these QCD-like theories [25] and was tested in lattice simulations [26]. Furthermore, RMT for the singular-value spectrum of the Dirac operator was studied [27]. A common feature of these QCD-like theories is that the condensate at high density is color-singlet and does not induce color superconductivity. By contrast, in QCD with three colors, diquark is not color-singlet and the color symmetry is spontaneously broken at high quark chemical potential [28]. There are many phases proposed to emerge on the phase diagram of QCD [29–32]. In particular, in QCD with three degenerate flavors, the ground state at asymptotically high density and low temperatures is believed to be the so-called color-flavor-locked (CFL) phase [33] characterized by diquark condensates

$$\langle \psi_0^a \sigma_2 \psi_0^b \rangle \propto \kappa_1 \delta_{af} \delta_{bg} + \kappa_2 \delta_{ag} \delta_{bf}$$

(1)

where $a, b$ are color indices and $f, g$ are flavor indices. It breaks the $SU(3)_{\text{color}} \times SU(3)_L \times SU(3)_R$ group down to the diagonal $SU(3)_V$ group. All quarks and gluons are gapped and the low-energy dynamics is dominated by $8 + 1 + 1$ Nambu-Goldstone modes [34]. The $\varepsilon$-regime is well-defined for the CFL phase and therefrom one may derive infinitely many spectral sum rules for the complex Dirac eigenvalues on the scale $\sim 1/\sqrt{V_4 \Delta^2}$ [35]. Except for this, essentially nothing is known about statistical properties of the Dirac operator at high quark density. Because lattice simulations are beset with the notorious sign problem, it is impossible to numerically probe this regime.

In this paper we construct a new RMT that describes dynamical locking of color and flavor symmetries [36] due to diquark condensates of the form (1). We rigorously take the microscopic large-N limit and determine the quark-mass dependence of the partition function. The effective action starts at second order in quark masses, implying that the chiral condensate vanishes. We show that the quark-mass dependence is controlled by fluctuations of a soft mode that resides in $U(3)$, which can be interpreted as a color-singlet four-quark field $\bar{\psi}_L \gamma_\mu \psi_R \bar{\psi}_R + \text{h.c.}$ arising from the coupling between a left-handed diquark and a right-handed diquark. This agrees with the physical picture of pions in the CFL phase [37]. One salient feature of our RMT is that there is no explicit parameter that corresponds to the chemical potential, in stark contrast to previous works [7, 16, 17, 38, 39]. The random matrix for the left-handed sector is statistically independent of that for the right-handed sector, making the Dirac matrix maximally
non-Hermitian. The decoupling of left and right quarks in the chiral limit indeed occurs in high-density QCD [30] and it is natural that our RMT reflects it. We note that RMT for dense two-color QCD also had this feature [20].

This paper is organized as follows. In section II the new matrix model is defined and basic properties are stated. In section III the model for \( N_f = 3 \) is rewritten with Hubbard-Stratonovich fields and the large-\( N \) limit is taken. The color-flavor locking is demonstrated and the sigma-model representation is derived. In section IV the model for \( N_f = 2 \) is analyzed. The color-flavor locking is demonstrated and the relevance to the 2SC phase of QCD is discussed. We conclude in section V.

II. THE MATRIX MODEL

The random matrix ensemble we propose is defined by the partition function

\[
Z = \int dV_A \int dW_A \int dX \int dY \times \prod_{f=1}^{N_f} \text{det}(\mathcal{D} + m_f \mathbb{1}_{12N}) \times \exp \left[ -2N \text{Tr}(V^{A \dagger}V^A) - 2N \text{Tr}(W^{A \dagger}W^A) - N \text{Tr}(X^\dagger X) - N \text{Tr}(Y^\dagger Y) \right] \]

where the “Dirac operator” \( \mathcal{D} \) is a \( 12N \times 12N \) matrix defined as

\[
\mathcal{D} = \begin{pmatrix} 0 & D_R \\ D_L & 0 \end{pmatrix} \tag{3}
\]

with

\[
D_L \equiv (CV^A + V^{A \dagger}C) \otimes \lambda^A + i(CX + X^\dagger C) \otimes \mathbb{1}_3 \tag{4} \\
D_R \equiv (CW^A + W^{A \dagger}C) \otimes \lambda^A + i(CY + Y^\dagger C) \otimes \mathbb{1}_3 \tag{5}
\]

Here, \( V^A, W^A (A = 0, 1, \cdots, 8) \) and \( X \) and \( Y \) are \( 2N \times 2N \) complex matrices with independently and identically distributed matrix elements, \( C = i\sigma_2 \otimes \mathbb{1}_N \) is a \( 2N \times 2N \) skew-symmetric matrix, \( \lambda^a (a = 1, \cdots, 8) \) are \( 3 \times 3 \) Gell-Mann matrices normalized as \( \text{Tr}(\lambda^a \lambda^b) = 2\delta_{ab} \), and

\[
\lambda^0 = \sqrt{\frac{2}{3}} \mathbb{1}_3. \quad dV^A, dW_A, dX \text{ and } dY \text{ are flat Cartesian measures.}
\]

Since \( D_L \) and \( D_R \) are statistically independent, \( \mathcal{D} \) is maximally non-Hermitian. In the chiral limit the model is evidently invariant under \( U(N_f)_L \times U(N_f)_R \). In addition, there are two “color” \( SU(3)_C \) symmetries that act on the left and right sectors separately. They are locked to the diagonal \( SU(3) \) subgroup by quark masses. There is also a symplectic symmetry

\[
V^A \to gV^A h^\dagger \quad X \to gXh^\dagger \tag{6} \\
W^A \to h^*W^Ag^T \quad Y \to h^*Yg^T \tag{7}
\]

where \( g, h \in \text{USp}(2N) \), i.e.,

\[
g^T Cg = h^T Ch = C, \quad g^\dagger g = h^\dagger h = \mathbb{1}_{2N}. \tag{8}
\]

III. THREE FLAVORS

Let us cast the model for \( N_f = 3 \) into a form that is more amenable to the large-\( N \) analysis. First we introduce quarks \( \psi_{fa}^a \) and anti-quarks \( \overline{\psi}_{fa}^a \). Here and in the following we label colors of quarks by \( a, b, c, d, e \in \{1, 2, 3\} \) and flavors by \( f, g, h, i, j \in \{1, 2, 3\} \). The Greek indices \( \alpha, \beta, \gamma, \delta \) run from 1 to 2\( N \). The quark mass matrix is defined as \( M = \text{diag}(m_1, m_2, m_3) \). Then the determinant can be expressed as a Grassmann integral

\[
Z = \int d\overline{\psi}_R d\psi_L d\psi_R d\psi_L \int dV_A \int dW_A \int dX \int dY \times \prod_{f=1}^{N_f} \text{det}(\mathcal{D} + m_f \mathbb{1}_{12N}) \times \exp \left[ -2N \text{Tr}(V^{A \dagger}V^A) - 2N \text{Tr}(W^{A \dagger}W^A) - N \text{Tr}(X^\dagger X) - N \text{Tr}(Y^\dagger Y) \right] \times \exp \left[ \psi_R^a M_{fa}^a + \overline{\psi}_L^a M_{La}^a \right] \times \exp \left[ -2N V^A_{af} V^A_{fa} + \overline{\psi}_{fa}^a (C_{af} V^A_{af} + V^A_{Af} C_{bf}) \right] \lambda^A_{a\beta} \psi_{f\beta}^b + iN X^a_{af} X_{fa}^a + \overline{\psi}_{fa}^a (C_{af} X^a_{af} + X^a_{f\gamma} C_{bf} \psi_{f\beta}^b - NY^a_{af} Y_{fa}^a + i\overline{\psi}_{fa}^a (C_{af} Y^a_{af} + Y^a_{f\gamma} C_{bf} \psi_{f\beta}^b) \right]. \tag{9}
\]

It is tedious but straightforward to integrate out the Gaussian random matrices \( V^A, W^A, X \) and \( Y \). The result reads

\[
Z \propto \int d\overline{\psi}_R d\psi_L d\psi_R d\psi_L \exp \left[ \overline{\psi}_{fa}^a M_{fa}^a \right].
\]
\[ Z \propto \int \text{d} \bar{\psi} \text{d} \psi \exp \left[ -\frac{1}{2N} \left( \bar{\psi} L L \gamma C \gamma L L \gamma \right) \left( \bar{\psi} L L \gamma L L \gamma \right) + \left( L \leftrightarrow R \right) \right] \]

Next we use the relation \( \lambda^a \lambda^b = 2 \delta^a \delta^b \) to obtain

\[ Z \propto \int \text{d} \bar{\psi} \text{d} \psi \exp \left[ -\frac{1}{N} \left( \bar{\psi} L L \gamma C \gamma L L \gamma \right) \left( \bar{\psi} L L \gamma L L \gamma \right) + \left( L \leftrightarrow R \right) \right] \]

To bilinearize the quartic interaction we insert the constant factor

\[ \int \text{d} \Delta \exp \left[ -2N \left\{ (\Delta L)_j - \frac{1}{2N} \varepsilon_{cde} \varepsilon_{hij} \bar{\psi}_{L L \gamma} C \bar{\psi}_{L L \gamma} \right\} \right] \]

where \( \Delta_{LR} \) are 3 \times 3 complex matrices. They transform as triplet under both color SU(3) and flavor SU(3). Then

Now one can integrate out quarks and obtain a product of Pfaffians:

\[ Z \propto \int \text{d} \Delta \int \text{d} \Delta \int \text{d} \Delta \int \text{d} \Delta \exp \left[ -2N \left( \Delta_{L L} \Delta_{R L} - 2N \left( \Delta_{R L} \Delta_{R L} \right) \right) \right] \]

This is an exact rewriting of the original partition function and so far no approximation has been made. We are now ready to take the microscopic large-\( N \) limit in which \( M \to 0 \) and \( N \to \infty \) such that \( NM^2 \sim 1 \). The first step is to determine the saddle point in the chiral limit. Setting \( M = 0 \) we find

\[ Z \propto \int \text{d} \Delta \int \text{d} \Delta \int \text{d} \Delta \int \text{d} \Delta \exp \left[ -2N \left( \Delta_{L L} \Delta_{R L} - 2N \left( \Delta_{R L} \Delta_{R L} \right) \right) \right] \]

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Note that \( \varepsilon_{abc} \varepsilon_{fgh} \Delta_{ch} \) is regarded here as a 9 \times 9 symmetric matrix with a left index \((a, f)\) and a right index \((b, g)\). It holds that [40]

\[
\det(\varepsilon_{abc} \varepsilon_{fgh} \Delta_{ch}) = -2(\det \Delta)^3.
\] (18)

Hence

\[
Z \propto \int d\Delta_L \int d\Delta_R \exp \left[ -2N \text{Tr}(\Delta_L^\dagger \Delta_L) - 2N \text{Tr}(\Delta_R^\dagger \Delta_R) \right] \times \det^{3N}(\Delta_L^\dagger \Delta_L) \det^{3N}(\Delta_R^\dagger \Delta_R).
\] (19)

A singular value decomposition yields \( \Delta_L = u \Lambda_L v \) with \( \Lambda_L \) a diagonal matrix with non-negative entries and \( u, v \in U(3) \), and likewise for \( \Delta_R \). Then it follows that the saddle point at large \( N \) is located at

\[
\Delta_L = \Lambda_L = \sqrt{\frac{3}{2}} 1_3, \quad \Delta_R = \sqrt{\frac{3}{2}} V.
\] (20)

This means that the SU(3) color symmetry and the SU(3) flavor symmetry are spontaneously broken to the diagonal SU(3) subgroup, i.e., color-flavor locking. The soft mode around the saddle point can be parametrized in terms of unitary matrices \( U \) and \( V \) as

\[
\Delta_L = \sqrt{\frac{3}{2}} U, \quad \Delta_R = \sqrt{\frac{3}{2}} V.
\] (21)

(This \( V \) should not be confused with the random matrix \( V^A \) appearing in the Dirac operator \( \mathcal{D} \).) Now we reinstate the quark masses and find

\[
Z \cong \int_{U(3)} dU \int_{U(3)} dV \det^N \left( \varepsilon_{abc} \varepsilon_{fgh} U^\ast_{ch} \right) \frac{(M^\dagger)^{fg} \delta_{ab}/\sqrt{6}}{-M^\dagger_{fg} \delta_{ab}/\sqrt{6} - \varepsilon_{abc} \varepsilon_{fgh} V_{ch}} \times \det^N \left( \varepsilon_{abc} \varepsilon_{fgh} V_{ch} \right) \frac{M_{fg} \delta_{ab}/\sqrt{6}}{(M^\dagger)^{fg} \delta_{ab}/\sqrt{6} - \varepsilon_{abc} \varepsilon_{fgh} U_{ch}}.
\] (22)

The evaluation of these determinants needs some preparation. Let us define

\[
U_{af, bg} \equiv \varepsilon_{abc} \varepsilon_{fgh} U_{ch},
\]

(23)

\[
V_{af, bg} \equiv \varepsilon_{abc} \varepsilon_{fgh} V_{ch},
\]

(24)

\[
\tilde{U}_{af, bg} \equiv U_{ab} \delta_{fg},
\]

(25)

\[
\tilde{V}_{af, bg} \equiv V_{ab} \delta_{fg},
\]

(26)

\[
\mathcal{M}_{af, bg} \equiv M_{fg} \delta_{ab},
\]

(27)

\[
\mathcal{Z}_{af, bg} \equiv \varepsilon_{abc} \varepsilon_{fgc},
\]

(28)

\[
e^{i\phi_V} \equiv \det U,
\]

(29)

\[
e^{i\phi_V} \equiv \det V.
\] (30)

Then it holds that

\[
\tilde{V}^\dagger \tilde{U}^\ast \equiv e^{-i\phi_V} Z,
\]

(31)

\[
\tilde{V}^T V \equiv e^{i\phi_V} Z,
\]

(32)

\[
\tilde{V}^\dagger \tilde{M} \tilde{U} = (V^\dagger U) \otimes M.
\] (33)

The proof is straightforward. For instance,

\[
(V^T \tilde{V} V)_{af, bg} = (\tilde{V}^T)_{af, a'} V_{a' f', b' g'} \tilde{V}_{b' g'} \delta_{bg},
\]

(34)

\[
= \varepsilon_{af} \varepsilon_{fg} V_{a' a} V_{f' b'} V_{b' g'} \delta_{bg},
\]

(35)

\[
= \varepsilon_{fg} \varepsilon_{ab} \delta_{f' f} \delta_{b' b} V_{ch} V_{b' b} \delta_{bg},
\]

(36)

\[
= \varepsilon_{fg} \varepsilon_{ab} \delta_{f' f} \delta_{b' b} V_{ch} \delta_{bg},
\]

(37)

\[
= (\det V) Z_{af, bg}.
\] (38)

Using (31), (32) and (33) we find

\[
\det \left( \begin{array}{cc}
U^\ast & M^\dagger/\sqrt{6} \\ -M^\dagger/\sqrt{6} & -V
\end{array} \right) \det \left( \begin{array}{cc}
V^\ast & M/\sqrt{6} \\ -M/\sqrt{6} & -U
\end{array} \right)
\]

(39)

\[
= \det \left( \begin{array}{cc}
\tilde{U}^\dagger U \tilde{V}^\ast & \tilde{M}^\dagger/\sqrt{6} \\ -\tilde{M}^\dagger/\sqrt{6} & -V
\end{array} \right) \det \left( \begin{array}{cc}
\tilde{V}^\dagger V & M/\sqrt{6} \\ -V/\sqrt{6} & -U
\end{array} \right)
\]

= \det \left( \begin{array}{cc}
e^{i\phi_V} Z & (U^\dagger V) \otimes M^\dagger/\sqrt{6} \\ -(V^\dagger U) \otimes M/\sqrt{6}
\end{array} \right)
\]

(40)

\[
= \det (I_9 - e^{-i(\phi_V - \phi_V)}) \times Z^{-1}[(U^\dagger U^\ast) \otimes M^\dagger/\sqrt{6}]
\]

(41)

\[
= \det (I_9 - e^{-i(\phi_V - \phi_V)}) \times Z^{-1}[(U^\dagger U^\ast) \otimes M^\dagger/\sqrt{6}]
\]

(42)

Now we take the scaling limit \( N \rightarrow \infty \) with \( \tilde{M} \equiv \sqrt{N} M \) fixed. The result is

\[
Z \sim \int_{U(3)} d\tilde{U} \exp \left( -\frac{1}{6} \text{Tr} \left\{ Z^{-1}(\tilde{U}^T \otimes \tilde{M}^T) Z^{-1}(\tilde{U} \otimes \tilde{M}) \right\} + \text{c.c.} \right)
\] (43)

where we have defined \( \tilde{U} \equiv e^{-i(\phi_V - \phi_V)/2} V^\dagger U \). Using a software [41] we find

\[
\text{Tr} \left\{ Z^{-1}(\tilde{U}^T \otimes \tilde{M}^T) Z^{-1}(\tilde{U} \otimes \tilde{M}) \right\} = \frac{5}{4} \text{Tr}(\tilde{M} \tilde{U})^2 - \text{Tr}(\tilde{M} \tilde{U})^2.
\] (44)
Thus
\[
Z \sim \int \frac{dU}{U(3)} \exp \left( -\frac{1}{6} \left\{ \frac{5}{4} \text{Tr}(\hat{M}U)^2 - \text{Tr}[(\hat{M}U)^2] \right\} \right) + \text{c.c.} \right). \tag{45}
\]

Notably the linear term \(\text{Tr}(\hat{M}U)\) is absent, indicating that the chiral condensate is vanishing in this model and symmetry breaking is instead triggered by diquark condensates. As \(U\) \((V)\) represents fluctuations of the left-handed (right-handed) diquark, respectively, the variable \(U \sim V^\dagger U\) can be interpreted as a color-singlet four-quark state, \(\bar{\psi}_R \bar{\psi}_L \psi_L \psi_L + \text{h.c.}\).

It is of interest to compare our RMT partition function (45) with the mass term of the chiral effective theory of the CFL phase of QCD [42-44],
\[
\mathcal{L} = -\frac{3\Delta^2}{4\pi^2} \left\{ \text{Tr}(MU)^2 - \text{Tr}[(MU)^2] \right\} + \text{c.c.} \tag{46}
\]
which neglects the chiral condensate and the color-sextet condensate for simplicity. (As for the validity of this approximation, see [30] and references therein.) The numerical factors in (45) and (46) are slightly different [45]. Where does this discrepancy come from? In the CFL phase of QCD, the diquark condensate of the form (1) with \(\kappa_1 \approx -\kappa_2\) develops. In contrast, our RMT hosts a diquark condensate
\[
\langle \psi_f^a \psi_f^b \rangle \propto \frac{1}{2} \delta_{af} \delta_{bg} - \delta_{ag} \delta_{bf}. \tag{47}
\]
This can be verified with a quick calculation. For a squared quark mass term, we have
\[
(\bar{\psi}_R M \psi_L)^2 \sim \bar{\psi}_R^a \bar{\psi}_f \bar{\psi}_f^b \psi_L^b M_{fg} M_{f'g'} \tag{48}
\]
\[
\sim \left( \frac{1}{2} \delta_{af} \delta_{bg} - \delta_{ag} \delta_{bf} \right) \left( \frac{1}{2} \delta_{af} \delta_{bg} - \delta_{ag} \delta_{bf} \right) M_{fg} M_{f'g'} \tag{49}
\]
\[
= \frac{5}{4} \delta_{fg} \delta_{f'g'} - \delta_{fg} \delta_{f'g'} \tag{50}
\]
\[
= \frac{5}{4} \text{Tr}(M)^2 - \text{Tr}(M^2). \tag{51}
\]
Thus the mass dependence of RMT (45) is reproduced. The bottom line is that the requirement that color and flavor be locked alone does not uniquely fix the form of the chiral Lagrangian. Our RMT obeys exactly the same pattern of symmetry breaking as the CFL phase of QCD, and the difference is a purely quantitative one.

\section*{IV. Two Flavors}

We now discuss the case of two flavors. We shall begin with (10) which is valid for an arbitrary number of flavors:
\[
Z \propto \int d\bar{\psi}_R d\bar{\psi}_L d\psi_R d\psi_L \exp \left[ \bar{\psi}_R^a M \psi_L^a + \bar{\psi}_L^a M^\dagger \psi_R^a + \frac{1}{2N} \delta_{abc} \delta_{cde} (\delta_{fhi} \delta_{gj} - \delta_{fh} \delta_{gj}) \right. \tag{52}
\]
\[
\left. \times (\bar{\psi}_f \gamma_i C \gamma_j \psi_L^e)(\bar{\psi}_L^i C \gamma_j \psi_R^e) + (L \leftrightarrow R) \right].
\]
For two flavors there is an identity
\[
\varepsilon_{fg} \varepsilon_{hi} = \delta_{fh} \delta_{gi} - \delta_{fi} \delta_{gh}, \tag{53}
\]
thus
\[
Z \propto \int d\bar{\psi}_R d\bar{\psi}_L d\psi_R d\psi_L \exp \left[ \bar{\psi}_R^a M \psi_L^a + \bar{\psi}_L^a M^\dagger \psi_R^a \right. \tag{54}
\]
\[
\left. + \frac{1}{2N} \delta_{abc} \delta_{cde} \varepsilon_{fg} \varepsilon_{hi} (\bar{\psi}_f \gamma_i C \gamma_j \psi_L^e)(\bar{\psi}_L^i C \gamma_j \psi_R^e) + (L \leftrightarrow R) \right].
\]
To bilinearize the quartic interaction we insert the constant factor
\[
\int_{C_3} d\Omega_L \exp \left[ -2N \left( \Omega_{Le} - \frac{1}{2N} \varepsilon_{abc} \varepsilon_{fg} \varepsilon_{hi} \psi_L^e C \delta_{fg} \psi_R^h \right) \right. \tag{55}
\]
\[
\times \left( \Omega_{Le} - \frac{1}{2N} \varepsilon_{abc} \varepsilon_{fg} \varepsilon_{hi} \psi_L^e C \delta_{fg} \psi_R^h \right) \right],
\]
where \(\Omega_{L,R}\) are complex three-component vectors that transform as triplet under color SU(3). They are also chaged under \(U(1)_B\) and \(U(1)_A\). Then
\[
Z \propto \int_{C_3} d\Omega_L \int_{C_3} d\Omega_R \int d\bar{\psi}_R d\bar{\psi}_L d\psi_R d\psi_L \exp \left[ -2N(|\Omega_L|^2 + |\Omega_R|^2) + \bar{\psi}_R^a M \psi_L^a + \bar{\psi}_L^a M^\dagger \psi_R^a \right. \tag{56}
\]
\[
+ \Omega_{Le} \varepsilon_{abc} \varepsilon_{fg} \bar{\psi}_L^e C \delta_{fg} \psi_R^h + \Omega_{Le} \varepsilon_{abc} \varepsilon_{fg} \bar{\psi}_L^e C \delta_{fg} \psi_R^h + (L \leftrightarrow R) \right]\]
The quarks can be readily integrated out and yield a product of Pfaffians:

\[
Z \propto \int d\Omega_L \int d\Omega_R \, e^{-2N(\|\Omega_L\|^2 + \|\Omega_R\|^2)}
\]

\[
\times \text{Pf} \left( \begin{array}{cc}
\Omega_{Rc}^{\varepsilon \alpha \beta} f g C_{\alpha \beta} & \frac{M f g \delta_{\alpha \beta}}{2} \\
-(M^T) f g \delta_{\alpha \beta}/2 & \Omega_{Le}^{\varepsilon \alpha \beta} f g C_{\alpha \beta}
\end{array} \right)
\]

\[
\times \text{Pf} \left( \begin{array}{cc}
\Omega_{Lc}^{\varepsilon \alpha \beta} f g C_{\alpha \beta} & \frac{M f g \delta_{\alpha \beta}}{2} \\
-(M^T) f g \delta_{\alpha \beta}/2 & \Omega_{Rc}^{\varepsilon \alpha \beta} f g C_{\alpha \beta}
\end{array} \right)
\]

where

\[
\frac{2}{\sqrt{N}} \left( \frac{m u m d}{4} \Omega_{Lc}^{\varepsilon \alpha \beta} f g C_{\alpha \beta} - \frac{m d}{2} \Omega_{Le}^{\varepsilon \alpha \beta} f g C_{\alpha \beta} \right)^2 \frac{2}{\sqrt{N}} \left( \frac{m u m d}{4} \Omega_{Rc}^{\varepsilon \alpha \beta} f g C_{\alpha \beta} - \frac{m d}{2} \Omega_{Rc}^{\varepsilon \alpha \beta} f g C_{\alpha \beta} \right)^2
\]

This is an exact rewriting of the original theory and no approximation has been made yet. The fact that the partition function vanishes with high powers of \(m\) in the chiral limit indicates that there are a macroscopic number of exact zero modes in this system.

Now we shall take the large-\(N\) limit. In the chiral limit with masses factored out, we have

\[
\frac{Z}{\|m u m d\|^{4N}} \sim \int d\Omega_L \int d\Omega_R e^{-2N(\|\Omega_L\|^2 + \|\Omega_R\|^2)}
\]

\[
\times \left[ \frac{1}{\|\Omega_{Lc}\|^8 N} \right]
\]

Let us define

\[
\Omega_L = \xi v_L, \quad \Omega_R = \eta v_R,
\]

\[
\xi \geq 0, \quad \eta \geq 0, \quad v_L^+ v_L = v_R^+ v_R = 1.
\]

Then

\[
\frac{Z}{\|m u m d\|^{4N}} \sim \int_0^\infty d\xi \int_0^\infty d\eta \int dv_L \int dv_R e^{-2N(\xi^2 + \eta^2)} \|v_R v_L\|^{8N}.
\]

Extremization of the integrand yields

\[
\xi = \eta = \sqrt{\frac{2}{N}}, \quad v_L = e^{i \varphi} v_R
\]

where \(e^{i \varphi}\) is an arbitrary \(U(1)\) phase. With a suitable rotation in \(U(3)/U(2)\), \(v_L\) can be brought to \((1,0,0)^T\), hence the saddle point is given by

\[
\Omega_L = (\sqrt{2},0,0)^T
\]

and likewise for \(\Omega_R\). Obviously this breaks the color symmetry spontaneously as

\[
SU(3)_C \to SU(2)_C
\]

but leaves the chiral symmetry \(SU(2)_L \times SU(2)_R\) intact. It is very pleasing to see that this breaking pattern is identical to the 2SC phase of two-flavor QCD at high density [28, 46, 47].

We proceed to the derivation of the action for the soft mode \(\varphi\), which is the Nambu-Goldstone mode associated with the \(U(1)_A\) symmetry. Plugging (67) into (62) yields

\[
Z \sim \int_0^{2\pi} d\varphi \left| \frac{m u m d}{4} \right|^{8N}
\]

For \(N \gg 1\) with \(\tilde{m}_{a,d} = \sqrt{N} m_{a,d}\) fixed, one gets

\[
Z \sim \tilde{m}_u \tilde{m}_d \int_0^{2\pi} d\varphi \exp \left( -\frac{1}{2} \tilde{m}_u \tilde{m}_d e^{-i \varphi} + \text{c.c.} \right).
\]

For real masses, a very concise expression is obtained:

\[
Z \sim \tilde{m}_u \tilde{m}_d \left| \tilde{m}_u \tilde{m}_d \right|^{4N} I_0(\tilde{m}_u \tilde{m}_d)
\]

where \(I_0\) is the modified Bessel function of the first kind. This result looks a bit pathological, because in the limit \(N \to \infty\) the partition function either blows up to infinity
or shrinks to zero depending on the magnitude of $|\tilde{m}_a\tilde{m}_d|$. This is an inevitable consequence of the fact that only quarks of two colors participate in Cooper pairing and quarks of the third color remain gapless. They do not decouple in the low-energy limit and make it hard to define the $\varepsilon$-regime for the 2SC phase. A similar situation arises in dense two-color QCD with an odd number of flavors [20, 23, 48, 49].

V. SUMMARY AND OUTLOOK

In this paper, we put forward a novel RMT and analyzed its properties in the large-$N$ limit. For three flavors, we showed that the model exhibits a striking similarity to the CFL phase of QCD in that the SU(3) color is broken and gets locked to the flavor SU(3). For two flavors, we showed that an isospin-singlet pairing of quarks occurs and breaks the SU(3) color symmetry down to SU(2), in analogy to the 2SC phase of QCD. This work opens up an entirely new avenue to investigate color superconductivity in dense QCD.

A lot of work remains to be done. As one can see from (11), in this work we have restricted ourselves to the color and flavor-antisymmetric interaction only. Extending this by incorporating the color and flavor-symmetric interaction is of much interest. This may help us identify RMT that exactly reproduces the CFL chiral Lagrangian (46). It would also be intriguing to numerically investigate the spectrum of the random matrix Dirac operator $\mathcal{D}$. This would be much easier (at least in the quenched limit) than analytically obtaining the microscopic spectral density, which seems to be a gargantuan task given the awful complexity of $\mathcal{D}$.

It is well known that the CFL phase exhibits many features that match the low-density hadronic phase of QCD, and there is even a proposal that the two phases may be continuously connected [50][51]. Therefore one may wonder if our new RMT can be continuously deformed into the conventional chiral RMT. Apparently this is not a trivial task, since our model requires $NM^2 \sim 1$ in the large-$N$ limit whereas in the conventional RMT $NM \sim N\mu^2 \sim 1$ where $\mu$ is a parameter that breaks anti-Hermiticity of the Dirac operator. An insight comes from the study of RMT for dense two-color QCD [23], where it was found that results in the large-$N$ limit of strong non-Hermiticity ($NM^2 \sim \mu \sim 1$) can be recovered from those in the limit of weak non-Hermiticity ($NM \sim N\mu^2 \sim 1$) by first taking the limit $N \to \infty$ with $N\mu^2$ fixed and then taking the second limit $N\mu^2 \to \infty$. A similar methodology may work for our RMT as well. In fact, if

$$V^A = -CW^A C$$  \hspace{1cm} (73)

$$X = CY^\dagger C$$  \hspace{1cm} (74)

holds, then $\mathcal{D}$ becomes anti-Hermitian. In this case we conjecture that the ordinary pattern of chiral symmetry breaking would be recovered, because the microscopic limit is universal and independent of a detailed structure of a random matrix [52]. If this is the case, then one may break the above relations with a parameter $\mu$ and take the limit $N \to \infty$ with $N\mu^2 \sim 1$, followed by another limit $N\mu^2 \to \infty$, to connect the two regimes. This intuitive speculation must be borne out by a more detailed calculation.

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