ON THE VANISHING COHOMOLOGY PROBLEM
FOR COCYCLE ACTIONS OF GROUPS ON II$_1$ FACTORS

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Abstract. We prove that any free cocycle action of a countable amenable group $\Gamma$ on any II$_1$ factor $N$ can be perturbed by inner automorphisms to a genuine action. Besides containing all amenable groups, this vanishing cohomology property, that we call $\mathcal{VC}$, is also closed to free products with amalgamation over finite groups. While no other examples of $\mathcal{VC}$-groups are known, by considering special cocycle actions $\Gamma \curvearrowright N$ in the case $N$ is the hyperfinite II$_1$ factor $R$, respectively the free group factor $N = L(F_\infty)$, we exclude many groups from being $\mathcal{VC}$. We also show that any free action $\Gamma \curvearrowright R$ gives rise to a free cocycle $\Gamma$-action on the II$_1$ factor $R' \cap R''$ whose vanishing cohomology is equivalent to Connes’ Approximate Embedding property for the II$_1$ factor $R \rtimes \Gamma$.

0. Introduction

A cocycle action of a group $\Gamma$ on a II$_1$ factor $N$ is a map $\sigma : \Gamma \to \text{Aut}(N)$ which is multiplicative modulo inner automorphisms of $N$, $\sigma_g \sigma_h = \text{Ad}(v_{g,h}) \sigma_{gh}$, $\forall g, h \in \Gamma$, with the unitary elements $v_{g,h} \in \mathcal{U}(N)$ satisfying the cocycle relation $v_{g,h}v_{gh,k} = \sigma_g(v_{h,k})v_{g,hk}$, $\forall g, h, k \in \Gamma$.

If $\Gamma$ is a free group $F_n$, for some $1 \leq n \leq \infty$, then any cocycle $\Gamma$-action on any II$_1$ factor $N$ can obviously be perturbed by inner automorphisms $\{\text{Ad}(w_g)\}_g$ of $N$ so that to become a “genuine” action, i.e., such that $\sigma'_g = \text{Ad}(w_g) \sigma_g$ is a group morphism, in fact so that the stronger condition $v_{g,h} = w_g \sigma_g(w_h)w^*_{gh}$, $\forall g, h$, holds true. We obtain in this paper several results towards identifying the class $\mathcal{VC}$ of all countable groups $\Gamma$ that satisfy this vanishing cohomology property. Thus, we first prove that any free product of amenable groups amalgamated over a common finite subgroup is in the class $\mathcal{VC}$. Then we show that if a group $\Gamma$ has an infinite...
subgroup which either has relative property (T), or has non-amenable centralizer, then $\Gamma$ is not in $\mathcal{VC}$. To prove that all amenable groups lie in $\mathcal{VC}$ we use subfactor techniques to reduce the problem to the case $N$ is the hyperfinite $\text{II}_1$ factor $R$, where vanishing cohomology holds due to results in ([Oc85]). To exclude groups from being in $\mathcal{VC}$ we apply $W^*$-rigidity results to two types of cocycle actions that are “hard to untwist”: the ones arising from $t$-amplifications of Bernoulli actions on $N = R$ introduced in [P01]; and the ones considered in [CJ84], arising from normal inclusions $\mathbb{F}_\infty \hookrightarrow \mathbb{F}_n$ with $\mathbb{F}_n/\mathbb{F}_\infty = \Gamma$, which give cocycle $\Gamma$-actions on $N = L(\mathbb{F}_\infty)$.

Untwisting cocycle actions on $\text{II}_1$ factors is a basic question in non-commutative ergodic theory and very specific to this area. Besides its intrinsic interest, the problem occurs in the classification of group actions on $\text{II}_1$ factors ([C74], [J80], [Oc85], [P01a]) and, closely related to it, in the classification of factors through unique crossed-product decomposition (as in [C74], [C75] for amenable factors, or [P01a], [P03], [P06a], [IPeP05], [PV12] for non-amenable $\text{II}_1$ factors). Another aspect, which goes back to ([CJ84]) and is important in $W^*$-rigidity, relates non-vanishing cohomology for certain cocycle $\Gamma$-actions on $L(\mathbb{F}_\infty)$ to non-embeddability of $L(\Gamma)$ into $L(\mathbb{F}_n)$. From an opposite angle, which offers a new point of view much emphasized here, vanishing cohomology results for cocycle actions are relevant to embedding problems, such as finding unusual group factors that embed into $L(\mathbb{F}_2)$ and Connes Approximate Embedding conjecture.

To describe the results in this paper in more details we need some background and notations. Let us first note that cocycle actions are more restrictive than outer actions, which are maps $\sigma : \Gamma \rightarrow \text{Aut}(N)$ that only require $\sigma_g\sigma_h\sigma_g^{-1} \in \text{Int}(N)$, $\forall g, h \in \Gamma$. It has in fact been shown in ([NT59]) that there is a scalar 3-cocycle $\mu_\sigma \in H^3(\Gamma)$ associated to an outer action $\sigma$. If $\sigma$ is free, i.e., $\sigma_g \not\in \text{Int}(N)$, $\forall g \neq e$, then $\mu_\sigma$ is trivial if and only if $\sigma$ is a cocycle action. Thus, if we view the vanishing cohomology problem as a question about lifting a 1 to 1 group morphism $\sigma : \Gamma \rightarrow \text{Out}(N)$ to a group morphism into $\text{Aut}(N)$, then the problem is not well posed unless one requires $\mu_\sigma \equiv 1$, i.e., that $\sigma$ defines a cocycle action.

Like for genuine actions, one can associate to a cocycle action $\Gamma \actson^\sigma N$ a tracial crossed product von Neumann algebra $N \rtimes \Gamma$, with the freeness of $\sigma$ equivalent to the condition $N' \cap N \rtimes \Gamma = \mathbb{C}1$. Thus, if $\sigma$ is free then $N \subset M = N \rtimes \Gamma$ is an irreducible inclusion of $\text{II}_1$ factors with the normalizer of $N$ in $M$ generating $M$ as a von Neumann algebra ($N$ is regular in $M$). Conversely, any irreducible regular inclusion of $\text{II}_1$ factors $N \subset M$ arises this way, from a crossed product construction involving a free cocycle action (cf. [J80]).

The crossed product framework allows an alternative formulation of vanishing cohomology. Thus, if $M = N \rtimes \Gamma$ denotes the crossed product $\text{II}_1$ factor associated
with the free cocycle action \((\sigma, v)\) of \(\Gamma\) on \(N\), and we let \(\{U_g\}_g \subset M\) denote the canonical unitaries implementing \(\sigma\) on \(N\), then the existence of \(w_g \in U(N)\) such that \(v_{g,h} = w_g \sigma_g(w_h)w_{gh}^*\), \(\forall g, h\) (i.e., vanishing cohomology for \(v\)) amounts to \(U'_g = w_g U_g\) being a \(\Gamma\)-representation. While the condition that \(\sigma'_g = \text{Ad}(U'_g)\), \(g \in \Gamma\), is a genuine action (i.e., weak vanishing cohomology for \(v\)) amounts to the weaker condition that \(\{U'_g\}_g\) is a projective \(\Gamma\)-representation.

Given a \(\Pi_1\) factor \(N\), we denote by \(\mathcal{VC}(N)\) (respectively \(\mathcal{VC}_w(N)\)) the class of countable groups \(\Gamma\) with the property that any free cocycle action of \(\Gamma\) on \(N\) satisfies the strong form (respectively weak form) of the vanishing cohomology. Also, we denote by \(\mathcal{VC}\) (respectively \(\mathcal{VC}_w\)) the class of countable groups \(\Gamma\) with the property that any free cocycle action of \(\Gamma\) on any \(\Pi_1\) factor \(N\) satisfies the strong form (respectively weak form) of the vanishing cohomology.

The class \(\mathcal{VC}\) contains all finite groups by ([J80], [Su80]) and all groups with polynomial growth by ([P89]). The first main result in this paper, which we prove in Section 2, shows that in fact \(\mathcal{VC}\) contains all countable amenable groups. Since by [J80] all 1-cocycles for actions of finite groups are co-boundary, this allows to deduce that, more than just containing the free groups, all amalgamated free products of amenable groups over finite groups belong to \(\mathcal{VC}\).

0.1. Theorem. The class \(\mathcal{VC}\) contains all countable amenable groups. Also, if \(\{\Gamma_n\}_n\) is a sequence of groups in \(\mathcal{VC}\) and \(K \subset \Gamma_n\) is a common finite subgroup, \(n \geq 1\), then \(\Gamma_1 *_K \Gamma_2 *_K \ldots \in \mathcal{VC}\).

To prove this result we show that any cocycle action \(\sigma\) of a countable amenable group \(\Gamma\) on a separable \(\Pi_1\) factor \(N\) can be perturbed by inner automorphisms to a cocycle action \(\sigma'\) that leaves invariant an irreducible hyperfinite subfactor \(R \subset N\) with the additional property that \(\sigma'_g \sigma'_h \sigma'^{-1}_{gh}\) are implemented by unitaries in \(R\), \(\forall g, h\), with \(\sigma'\) still free when restricted to \(R\) (see Theorem 2.1). This reduces the vanishing cohomology problem to the case \(N = R\), where one can apply the vanishing cohomology result in ([Oc85]) to finish the proof.

To prove the existence of a “large” \(R \subset N\) that’s normalized by inner perturbations of \(\sigma\) we use an idea introduced in [P89], of translating the problem into the question of whether there exists a sub-inclusion of hyperfinite factors inside the “diagonal subfactor” \(N \subset M_\sigma\) associated with \(\sigma\), so that to have a non-degenerate commuting square satisfying a strong smoothness condition on higher relative commutants. This subfactor problem was solved in [P89] in the case \(\Gamma\) is finitely generated with trivial Poisson boundary (e.g., with polynomial growth; see [KV83]), by constructing \(R\) as a limit of relative commutants \(N'_n \cap N\) of factors in an appropriately chosen Jones’ tunnel \(M \supset N \supset N_1\ldots\) (in the spirit of [P91], [P93]).

However, that construction depends crucially on the trivial Poisson boundary
condition on \( \Gamma \). We use here the amenability of \( \Gamma \) alone to construct a more elaborate decreasing sequence of subfactors \( P_n \subset N \) with \( P'_n \cap N \not
rightarrow R \) “large” in \( N \), obtained through reduction/induction in Jones tunnels. In fact, this method allows us to obtain the existence of strongly smooth embedding of hyperfinite subfactors into any finite index subfactor \( N \subset M \) with standard invariant \( G_{N \subset M} \) amenable (in the sense of [P91], i.e., with its graph \( \Gamma_{N \subset M} \) satisfying the Kesten-type condition \( \|\Gamma_{N \subset M}\|_2 = [M : N] \); see also [P93], [P94a], [P97] for other equivalent definitions).

We also show that given any countable amenable rigid \( C^* \)-tensor category of bimodules (or of endomorphisms) \( \mathcal{G} \) acting freely on a \( \text{II}_\infty \) factor \( N^\infty \), there exists a “large” hyperfinite \( \text{II}_\infty \) subfactor \( R^\infty \subset N^\infty \) that’s normalized by \( \mathcal{G} \) (modulo inner automorphisms).

In Section 5 we use the strong solidity of free group factors ([OP07]) to prove that in order for a group \( \Gamma \) to satisfy the property that any of its actions on \( \text{II}_1 \) factors normalizes a hyperfinite subfactor (modulo inner automorphisms), \( \Gamma \) must necessarily be amenable. The problem of whether this dichotomy still holds for subfactor standard invariants and rigid \( C^* \)-tensor categories, remains open.

In turn, in Sections 3 and 4 we obtain a series of obstruction criteria for groups to belong to the classes \( \mathcal{VC}(R) \), \( \mathcal{VC}(L(F_\infty)) \), \( \mathcal{VC} \), summarized in the following:

**0.2. Theorem.** 1° If a countable group \( \Gamma \) has an infinite subgroup which either has the relative property (T), or has non-amenable centralizer in \( \Gamma \), then \( \Gamma \not\in \mathcal{VC}(R) \).

2° Assume a countable group \( \Gamma \) satisfies one of the following: (a) \( \Gamma \) does not have Haagerup property (e.g., it contains an infinite subset with relative property (T)); (b) The Cowling-Haagerup invariant \( \Lambda(\Gamma) \) is larger than 1; (c) \( \Gamma \) has an infinite subgroup with non-amenable centralizer; (d) \( \Gamma \) has an infinite amenable subgroup with non-amenable normalizer. Then \( \Gamma \not\in \mathcal{VC}_w(L(F_\infty)), \) in particular \( \Gamma \not\in \mathcal{VC}_w \).

To prove the restrictions on \( \mathcal{VC}(R) \) we use the \( t \)-amplifications of Bernoulli actions on \( R \) introduced in [P01a] and results obtained there and in ([P06a]) through deformation-rigidity arguments. In turn, to get restrictions on \( \mathcal{VC}_w(L(F_\infty)) \), we use the Connes-Jones (CJ) cocycles associated with surjective group morphisms \( \pi : F_S \rightarrow (\Gamma, S) \), extending the map assigning the free generators of \( F_S \) to a set of generators \( S \subset \Gamma \). As shown in [CJ84], if \( \Gamma \) is infinite, non-free, then \( \ker \pi \simeq F_\infty \) and the inclusion \( L(F_\infty) = N \subset M = L(F_S) \) is of the form \( N \subset N \rtimes_{(\sigma_\pi, v_\pi)} \Gamma \), for a free cocycle action \( (\sigma_\pi, v_\pi) \). The vanishing of the cocycle \( v_\pi \) implies that \( L(\Gamma) \) embeds into \( L(F_S) \), hence 2° above follows from results in ([CJ84], [P01b], [O03], [P06b], [OP07]).

The CJ-cocycles seem the “most difficult to untwist”, in the sense that if all such cocycle actions of \( \Gamma \) on \( L(F_\infty) \) untwist, then \( \Gamma \) ought to be in \( \mathcal{VC} \). In particular, this would show that \( \mathcal{VC} = \mathcal{VC}(L(F_\infty)) \). Since untwisting a CJ-cocycle for \( \Gamma \) implies
that \( \Gamma \) is in the class \( \mathcal{W}^e_{\text{eq}}(\mathbb{F}_2) \) of groups whose von Neumann algebra embeds into \( L(\mathbb{F}_2) \), one has \( \mathcal{V} \subset \mathcal{V}(L(\mathbb{F}_\infty)) \subset \mathcal{W}^e_{\text{eq}}(\mathbb{F}_2) \). Very little is actually known about the class \( \mathcal{W}^e_{\text{eq}}(\mathbb{F}_2) \), which is extremely interesting by itself. Any success in proving \( \mathcal{V} \) property for some “exotic” group \( \Gamma \), would provide embeddings \( L(\Gamma) \hookrightarrow L(\mathbb{F}_2) \).

In the final part of the paper we discuss a connection between vanishing cohomology phenomena and Connes Approximate Embedding conjecture, on whether any separable II\(_1\) factor \( M \) embeds in the ultrapower \( R^\omega \) of the hyperfinite II\(_1\) factor \( R \). Thus, we notice that any free action of a group \( \Gamma \) on \( R \) (such as the “non-commutative” Bernoulli action \( \Gamma \acts \mathbb{R}^\omega \)), gives rise to a free cocycle action of \( \Gamma \) on the centralizer \( R^\omega = R' \cap R^\omega \) of \( R \) in \( R^\omega \). We deduce that this cocycle untwists if and only \( M \equiv R \rtimes_\sigma \Gamma \) satisfies the conjecture.

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1. Preliminaries and notations.

For general background on II\(_1\) factors we refer the reader to ([AP17]; also [T79], [BrO08] for general theory of operator algebras and von Neumann algebras).

1.1. Cocycle actions and crossed products. Given a II\(_1\) factor \( N \), we denote by \( \text{Aut}(N) \) the group of automorphisms of \( N \). An automorphism \( \theta \) of \( N \) is inner if there exists \( u \) in the unitary group of \( N \), \( U(N) \), such that \( \theta(x) = \text{Ad}u(x) = uxu^*, \forall x \in N \). We denote by \( \text{Inn}(N) \subset \text{Aut}(N) \) the group of all such inner automorphisms and by \( \text{Out}(N) \) the quotient group \( \text{Aut}(N)/\text{Int}(N) \).

Given a discrete group \( \Gamma \), an action of \( \Gamma \) on \( N \) is a group morphism \( \sigma : \Gamma \to \text{Aut}(N) \). We will use the notation \( \Gamma \acts \sigma N \) to emphasize such an action.

More generally, a cocycle action \( \sigma \) of \( \Gamma \) on \( N \) is a map \( \sigma : \Gamma \to \text{Aut}(N) \) with the property that there exists \( v : \Gamma \times \Gamma \to U(N) \) such that:

\[
\begin{align*}
\sigma_e &= \text{id} \quad \text{and} \quad \sigma_g \sigma_h = \text{Ad} \ v_{g,h} \sigma_{gh}, \ \forall g, h \in G \\
v_{g,h}v_{gh,k} &= \sigma_g(v_{h,k})v_{g,hk}, \ \forall g, h, k \in \Gamma.
\end{align*}
\]

The cocycle action \( \sigma \) is free if \( \sigma_g \) cannot be implemented by unitary elements in \( N \), \( \forall g \neq \epsilon \), in other words if the factoring of \( \sigma \) through the quotient map \( \text{Aut}(N) \to \text{Out}(N) \) is 1 to 1. All cocycle actions (in particular all actions) that we will consider in this paper are assumed to be free.

Following ([KT02]), a map \( \sigma : \Gamma \to \text{Aut}(N) \) that’s a 1 to 1 group morphism when factored through the quotient map \( \text{Aut}(N) \to \text{Out}(N) \) is called an outer \( \Gamma \)-action. Thus, an outer action satisfies (1.1.1) above, but not necessarily (1.1.2). As shown in ([NT59]), if \( \sigma \) is an outer \( \Gamma \)-action, then one can associate to it a scalar 3-cocycle \( \mu \in H^3(\Gamma) \), with the property that \( \mu \equiv 1 \) if and only if \( \sigma \) is a cocycle action.
If \( \sigma \) is a cocycle action, then a map \( v \) satisfying (1.1.2) is called a 2-cocycle for \( \sigma \). The 2-cocycle is normalized if \( v_{g,e} = v_{e,g} = 1 \), \( \forall g \in G \). Note that any 2-cocycle satisfies \( v_{e,e} \in \mathbb{C} \). Thus any 2-cocycle \( v \) can be normalized by replacing it, if necessary, by \( v'_{g,h} = v^*_{e,e} v_{g,h} \), \( g, h \in \Gamma \). All 2-cocycles considered from now on will be normalized.

Also, when given a cocycle action \( \sigma \), we will sometimes specify from the beginning the 2-cocycle it comes with, thus considering it as a pair \((\sigma, v)\).

Note that the 2-cocycle \( v \) is unique modulo perturbation by a scalar 2-cocycle \( \mu \). More precisely, \( v'_{g,h} : \Gamma \times \Gamma \to U(\mathbb{N}) \), with \( v'_{e,e} = 1 \), satisfies conditions (1.1.1), (1.1.2) if and only if \( v'_{g,h} = \mu v_{g,h} \), \( g, h \in \Gamma \). All 2-cocycles considered from now on will be normalized.

The crossed product algebra associated to a cocycle action \((\sigma, v)\) on \( N \) denoted \( N \rtimes_{(\sigma, v)} \Gamma \) (or simply \( N \rtimes \sigma \) \Gamma if \( \sigma \) is a genuine action), is the von Neumann algebra generated inside \( \mathcal{B}(\ell^2(\Gamma, L^2(\mathbb{N}))) \) by the unitaries \( u_g \in \mathcal{B}(\ell^2(\Gamma, L^2(\mathbb{N}))), g \in \Gamma \), where

\[
U_g(f)(h) = f(g^{-1}h), \quad \text{for } f \in \ell^2(\Gamma, L^2(\mathbb{N})),
\]

and by a copy of the algebra \( N \) represented by

\[
(b \cdot f)(g) = \sigma_g^{-1}(b)f(g), \quad \text{for } b \in N, \quad f \in \ell^2(\Gamma, L^2(\mathbb{N}, \varphi)), g \in \Gamma.
\]

together with the vector state \( \tau(X) = \langle X \xi_e, \xi_e \rangle \), where \( \xi_e \in \ell^2(\Gamma, L^2(\mathbb{N})) \) is the function on \( \Gamma \) that takes the value 1 at \( e \) and 0 elsewhere.

The crossed product algebra \( N \rtimes_{(\sigma, v)} \Gamma \) can alternatively be viewed as the completion (on bounded sequences) of the Hilbert algebra of finite formal sums \( \sum_{g \in \Gamma} U_g b_g, b_g \in N \), with multiplication rules \( U_g U_h = v_{g,h} U_{gh}, b U_g = U_g \sigma_g^{-1}(b), b = U_e b = 1b \), for \( g, h \in G, b \in N \), and \( \ast \)-operation \( (U_g b)^* = U_{g^{-1}} \sigma_g(b^*) \), and with \( N \)-valued expectation \( E\left( \sum_g U_g b_g \right) \text{def} = b_e \) and scalar expectation

\[
\tau\left( \sum_g U_g b_g \right) \text{def} = \tau\left( E \left( \sum_g U_g b_g \right) \right) = \varphi(b_e).
\]

The crossed product algebra \((N \rtimes_{(\sigma, v)} \Gamma, \tau)\) this way defined is itself a finite von Neumann algebra, with \( \tau \) a faithful normal tracial state.

The action \( \sigma \) is free i.e., if \( \sigma_g \) is an outer automorphism, \( \forall g \neq e \), if and only if \( N' \cap N \rtimes_{(\sigma, v)} \Gamma = \mathbb{C}1 \), so in particular \( M = N \rtimes \sigma \Gamma \) is a \( \Pi_1 \) factor with the
normalize $\mathcal{N}_M(N) = \{u \in \mathcal{U}(M) \mid uNu^* = N\}$ of $N$ in $M$ generating $M$ (i.e., with $N$ regular in $M$).

Conversely, if $N \subset M$ is a regular, irreducible inclusion of II$_1$ factors and we denote $\Gamma = \mathcal{N}_M(N)/\mathcal{U}(N)$, with $U_g \in \mathcal{N}_M(N), g \in \Gamma$, a lifting of $\Gamma$, $U_e = 1$, and we denote $\sigma_g = \text{Ad}(U_g)_{|N}, v_{g,h} = U_{gh}U^*_hU^*_g$, then $(\sigma, v)$ is a free cocycle action of $\Gamma$ on $N$, with $N \subset N \rtimes (\sigma, v) \Gamma$ naturally isomorphic to $N \subset M$.

1.2. Cocycle conjugacy of cocycle actions. The cocycle actions $(\sigma_i, v_i)$ of $\Gamma$ on $N_i, i = 1, 2$, are cocycle conjugate if there exists an isomorphism $\theta : N_1 \simeq N_2$ and a map $w : \Gamma \to \mathcal{U}(N_2)$ such that the following conditions are satisfied:

\begin{equation}
\theta \sigma_1(g) \theta^{-1} = \text{Ad} w_g \sigma_2(g), \quad \forall g.
\end{equation}

\begin{equation}
\theta(v_1(g, h)) = w_g \sigma_2(g)(w_h)v_2(g, h)w^*_g h, \quad \forall g, h.
\end{equation}

The cocycle actions $\sigma_1, \sigma_2$ are outer conjugate (or weakly cocycle conjugate) if condition (1.2.1) is satisfied. Note that outer conjugacy is equivalent to the image morphisms of $\sigma_1, \sigma_2$ in Out($N$) being conjugate in Out($N$) by an automorphism of $N$.

The (cocycle) actions $\sigma_1, \sigma_2$ are conjugate if there exists an isomorphism $\theta : N_1 \simeq N_2$ such that conditions (1.2.1) is satisfied with $w = 1$. We then write $\sigma_1 \sim \sigma_2$.

We recall here the following well known observation (see e.g., [J80]), which translates cocycle conjugacy of free cocycle actions into the isomorphism of the associated crossed-product inclusions of factors.

Proposition. Let $(\sigma_i, v_i)$ be a cocycle action of the discrete group $\Gamma_i$ on the II$_1$ factor $N_i, i = 1, 2$. If there exists a $\ast$-isomorphism $\Phi : N_1 \rtimes (\sigma_1, v_1) \Gamma_1 \simeq N_2 \rtimes (\sigma_2, v_2) \Gamma_2$ such that $\Phi(N_1) = N_2$, then $\sigma_1$ and $\sigma_2$ are cocycle conjugate. More precisely, there exists a group isomorphism $\gamma : \Gamma_1 \to \Gamma_2$, and unitaries $w_g \in \mathcal{U}(N_2)$, for all $g \in \Gamma_1$, such that:

(i) $\Phi \sigma_1(g) \Phi^{-1} = \text{Ad} w_g \sigma_2(\gamma(g)), \quad \forall g \in \Gamma_1$,

(ii) $\Phi(v_1(g, h)) = w_g \sigma_2(\gamma(g))(w_h)v_2(\gamma(g), \gamma(h))w^*_g h, \quad \forall g, h \in \Gamma_1$.

Conversely, if $\Phi : N_1 \simeq N_2$ is a $\ast$-isomorphism, $\gamma : \Gamma_1 \simeq \Gamma_2$ is a group isomorphism, and there exist unitaries $w_g \in \mathcal{U}(N_2)$ for all $g \in G_1$ such that (i), (ii) are satisfied, then $\Phi$ can be extended to an isomorphism $N_1 \rtimes (\sigma_1, v_1) \Gamma_1 \simeq N_2 \rtimes (\sigma_2, v_2) \Gamma_2$ (hence, to an isomorphism of the associated inclusions).

1.3. 1-cocycles for actions. Assume $\sigma$ is a genuine action of $\Gamma$ on the II$_1$ factor $N$. A map $w : \Gamma \to \mathcal{U}(N)$ satisfying condition

\begin{equation}
w_g \sigma_g(w_h) = w_{gh}, \quad \forall g, h
\end{equation}
is called a 1-cocycle for $\sigma$. Such a 1-cocycle for $\sigma$ is a coboundary (or it is trivial) if there exists a unitary element $v \in \mathcal{U}(N)$ such that $w_g = v^* \sigma_g(v)$, $\forall g$. (Clearly, such maps $w_g$ do satisfy the 1-cocycle condition (1.3.1)).

The map $w$ is called a weak 1-cocycle if it satisfies the relation (1.2.1) modulo the scalars, i.e.,

$$w_0 \sigma(g) w_0^* \in \mathbb{T}1, \forall g, h \in \Gamma$$

(1.3.1')

Note that this is equivalent to $\text{Ad}(w_g) \sigma_g$ being an action. Note also that if $w$ is a weak 1-cocycle then $\mu_{g,h} = w_g \sigma_g(w_h) w_h^*$ is a scalar 2-cocycle for $\Gamma$, i.e., $\mu \in H^2(\Gamma)$. Also, cocycle conjugacy of two (genuine) actions $\sigma_i : \Gamma \to \text{Aut}(N_i)$, $i = 1, 2$, amounts to conjugacy of $\sigma_1$ and $\sigma_2'$, where $\sigma_2'(g) = \text{Ad}(w_g) \sigma_2(g)$, $g \in \Gamma$, for some 1-cocycle $w$ for $\sigma_2$.

A (weak) 1-cocycle $w$ is weakly trivial (or weak coboundary) if there exists a unitary element $v \in \mathcal{U}(N)$ such that $vw_g \sigma_g(v)^* \in \mathbb{T}1, \forall g$.

Two (weak) 1-cocycles $w, w'$ of the action $\sigma$ are equivalent if there exists a unitary element $v \in N$ such that $w'_g = vw_g \sigma_g(v)^*, \forall g \in \Gamma$ (resp. modulo scalars). Thus, a weak 1-cocycle is weakly trivial iff it is equivalent to a scalar valued weak 1-cocycle (N.B.: these are just plain scalar functions on $\Gamma$). Note that the scalar valued genuine 1-cocycles are just characters of $\Gamma$.

Two free actions $\sigma_1, \sigma_2$ of $\Gamma$ on $N$ are cocycle conjugate iff $\sigma_1$ is conjugate to $\sigma_2'$, where $\sigma_2'(g) = \text{Ad}(w_g) \sigma_2(g)$, $\forall g \in \Gamma$, for some 1-cocycle $w$ for $\sigma_2$.

We also mention here a well known result from [J80], showing that any 1-cocycle of an action of a finite group $\Gamma$ is co-boundary. This property is actually specific to finite groups: we use a result in [P01a] to deduce that if $\Gamma$ is infinite, then there exist free ergodic actions $\Gamma \curvearrowright R$ which admit non-trivial 1-cocycles. (N.B. In the particular case when $\Gamma$ is amenable, this fact can be derived from [Oc] as well).

**Proposition.** 1° Let $\Gamma \curvearrowright^\sigma N$ be a free action of a finite group $\Gamma$ on a $\Pi_1$ factor $N$. If $w$ is a 1-cocycle for $\sigma$, then there exists $u \in \mathcal{U}(N)$ such that $w_g = u \sigma_g(u^*)$, $\forall g \in \Gamma$.

2° Let $(N_0, \varphi_0)$ be a copy of the 2 by 2 matrix algebra with the state given by weights $\{\frac{1}{1+\lambda}, \frac{\lambda}{1-\lambda}\}$, for some $0 < \lambda < 1$, and $\Gamma$ be an infinite group. Let $\Gamma \curvearrowleft (\mathcal{N}, \varphi) = \bigotimes_g (N_0, \varphi_0)_g$ be the Bernoulli $\Gamma$-action with base $(N_0, \varphi_0)$. Let $\Gamma \curvearrowleft^\sigma N = N_\varphi \simeq R$ be the corresponding Connes-Størmer Bernoulli action. Let $B \subset N$ be an atomic von Neumann subalgebra of the form $\bigoplus_n B_n$, with $B_n \simeq M_{k_n \times k_n}(\mathbb{C})$ having minimal projections of trace $\lambda^{m_n}$, with $m_1 < m_2 < \ldots$. Then there exists a 1-cocycle $w$ for $\sigma$ such that $\sigma'_g = \text{Ad}(w_g) \sigma_g$, $g \in \Gamma$, has $B$ as its fixed point algebra. If $B \neq \mathbb{C}$, then any such 1-cocycle is not a co-boundary.
\textbf{Proof.} 1° This is part of (1.4.8 in [J80]), but we include here a proof for completeness, using Connes well known “2 by 2 matrix trick” (N.B. this argument has been used several times by Connes in his work, in particular to prove this statement in the case $\Gamma = \mathbb{Z}/n\mathbb{Z}$ in [C76]).

Thus, let $\tilde{\sigma}$ be the action of $\Gamma$ on $\tilde{N} = M_{2 \times 2}(N) = N \otimes M_{2 \times 2}(\mathbb{C})$ given by $\tilde{\sigma}_g = \sigma_g \otimes \text{id}$. If $\{e_{ij} \mid 1 \leq i, j \leq 2\}$ is a matrix unit for $M_{2 \times 2}(\mathbb{C}) \subset \tilde{N}$, then $\tilde{w}_g = e_{11} + w_g e_{22}$ is a cocycle for $\tilde{\sigma}$. If $Q \subset \tilde{N}$ denotes the fixed point algebra of the action $\tilde{\sigma}_g = \text{Ad}(\tilde{w}_g)\tilde{\sigma}$, then $e_{11}, e_{22} \in Q$ and the existence of a unitary element $u \in N$ satisfying $w_g = u\sigma_g(u^*)$, $\forall g$, is equivalent to the fact that $e_{11}, e_{22}$ are equivalent projections in $Q$. But the fixed point algebra of any action of a finite group on a $\text{II}_1$ factor is a $\text{II}_1$ factor. Thus, $e_{11}, e_{22}$ are equivalent in $Q$ and $w$ follows co-boundary.

2° For each $n \geq 1$, let $\{V_n^j\}_{1 \leq j \leq k_n} \subset N$ be isometries such that $V_n^j V_n^j \in N$, $\tau(V_n^j V_n^j \ast) = \lambda_n^{mn}$, $V_n^j N V_n^j \ast = N$ and $\{V_n^j V_n^j \ast \mid 1 \leq i, j \leq k_n\}$ be the matrix units of $B_n$. Let also $\pi_n$ be the trivial representation of $\Gamma$ of multiplicity $k_n$. Then by (Theorem 3.2 in [P01a]), $w_g = \Sigma_n \Sigma_{i,j} V_n^j \sigma_g(V_n^j)^\ast$, $g \in \Gamma$, defines a 1-cocycle for $\sigma$ and $\sigma_g' = \text{Ad}(w_g)\sigma_g$ has $B$ as its fixed point algebra.

Since the fixed point algebra of an action is a conjugacy invariant of the action and $\sigma$ is mixing (thus ergodic), it follows that $\sigma, \sigma'$ are not conjugate, in particular there exists no $u \in \mathcal{U}(N)$ such that $\sigma_g' = \text{Ad}(u)\sigma_g\text{Ad}(u^*)$, $\forall g$, a relation that amounts to $w_g = u\sigma_g(u^*)$ modulo scars, $\forall g$. \hfill \Box

\subsection*{1.4. Vanishing cohomology and property V\text{C}} The 2-cocycle $v$ for the cocycle action $\sigma$ vanishes (or it is a coboundary) if there exists a map $w : \Gamma \to \mathcal{U}(N)$ such that $w_e = 1$ and $v = \partial w$, i.e.:

\begin{equation}
(1.4.1) \quad v_{g,h} = (\partial w)_{g,h} \overset{\text{def}}{=} \sigma_g(w_h^*)w_g^*w_{gh}, \forall g, h \in \Gamma.
\end{equation}

The 2-cocycle $v$ weakly-vanishes (or it is a weak coboundary) if there exists $w : \Gamma \to \mathcal{U}(N)$ such that $w_e = 1$ and $v = \partial w$ modulo scalars, i.e.:

\begin{equation}
(1.4.2) \quad w_g \sigma_g(w_h)w_{gh}^*w_{gh} \in \mathbb{C}1, \forall g, h \in \Gamma.
\end{equation}

Note that this is equivalent to

\begin{equation}
(1.4.2') \quad (\text{Ad } w_g \sigma_g)(\text{Ad } w_h \sigma_h) = \text{Ad } w_{gh} \sigma_{gh}, \forall g, h
\end{equation}

i.e., to $\sigma_g' \overset{\text{def}}{=} \text{Ad } w_g \sigma_g$ being a “genuine” action.

In turn, the vanishing of $v$ is equivalent to the existence of unitary elements $\{w_g\}_g \subset N$ such that $U_g' = w_g U_g \in M = N \rtimes_{(\sigma,v)} \Gamma$ give a representation of $\Gamma$ (i.e., $U_g' U_h' = U_{gh}', \forall g, h \in \Gamma$).
Given a II$_1$ factor $N$, we denote by $\mathcal{VC}(N)$ the class of countable groups $\Gamma$ for which any free cocycle action $(\sigma, v)$ of $\Gamma$ on $N$ has the property that the 2-cocycle $v$ vanishes (or is coboundary) and by $\mathcal{VC}_w(N)$ the class of groups $\Gamma$ for which any free cocycle action $(\sigma, v)$ of $\Gamma$ on $N$ has the property that $v$ is a weak-coboundary.

We denote by $\mathcal{VC}$ (respectively $\mathcal{VC}_w$) the class of countable groups $\Gamma$ with the property that $\Gamma \in \mathcal{VC}(N)$ (resp. $\Gamma \in \mathcal{VC}_w(N)$) for any II$_1$ factor $N$. If $\Gamma \in \mathcal{VC}$ then we also say that $\Gamma$ has property $\mathcal{VC}$ or that it is a $\mathcal{VC}$-group.

We are especially interested in identifying the $\mathcal{VC}$ and $\mathcal{VC}_w$ groups, i.e., groups that have the most “universal” vanishing cohomology property. Other classes of interest will be $\mathcal{VC}(N)$ for $N$ equal to the hyperfinite II$_1$ factor $R$ and for $N$ equal to the free group factor $L(\mathbb{F}_\infty)$. This is because $R$ and $L(\mathbb{F}_\infty)$ are the most interesting “non-commutative probability spaces”. Also, any countable group $\Gamma$ has “many” free actions on these factors, in fact both of them have a lot of generalized symmetries (notably $L(\mathbb{F}_\infty)$, on which by [PS01] any “group-like” object admits free actions). Also, both factors admit many cocycle actions that are “hard to untwist” (cf. [CJ83], [P01a] and Section 3 and 4 below). (N.B. It should be noticed that by the way we have defined $\mathcal{VC}(N)$, if a factor $N$ has only inner automorphisms, i.e., $\text{Out}(N) = \{1\}$, like the examples in [IPeP05], then any $\Gamma$ belongs to $\mathcal{VC}(N)$!)

We’ll now show that the class $\mathcal{VC}$ is close to amalgamated free products over finite subgroups and that vanishing cohomology for cocycle actions of countable groups is essentially a “separability” property:

**1.5. Proposition.** 1° if $\{\Gamma_n\}_{n \geq 0} \subset \mathcal{VC}(N)$ (respectively $\mathcal{VC}_w(N)$) for some II$_1$ factor $N$ and $K \subset \Gamma_n$ is a common finite subgroup, $n \geq 0$, then $\Gamma_0 *_K \Gamma_1 *_K \Gamma_2 *_K ... \in \mathcal{VC}(N)$ (respectively $\mathcal{VC}_w(N)$). Also, if $\{\Gamma_n\} \subset \mathcal{VC}$ (resp. $\mathcal{VC}_w$), then $\Gamma_0 *_K \Gamma_1 *_K ... \in \mathcal{VC}$ (resp. $\mathcal{VC}_w$).

2° Let $N$ be a II$_1$ factor, $\Gamma \subset \text{Out}(N)$ a countable group with a lifting $\{\sigma_g\}_{g \in \Gamma} \subset \text{Aut}(N)$ and denote $v_{g,h} \in \mathcal{U}(N)$ a set of unitaries satisfying $\text{Ad}(v_{g,h}) = \sigma_g \sigma_h \sigma_{gh}^{-1}$, $g,h \in \Gamma$. There exists a separable II$_1$ subfactor $Q \subset N$ that contains the countable set $\{v_{g,h} \mid g,h \in \Gamma\}$ and is normalized by $\sigma$, with $\sigma_g$ outer, $\forall g \in \Gamma$.

3° $\mathcal{VC}$ (respectively $\mathcal{VC}_w$) coincides with the class of countable groups $\Gamma$ with the property that $\Gamma \in \mathcal{VC}(N)$ (resp. $\Gamma \in \mathcal{VC}_w(N)$) for any separable II$_1$ factor $N$.

**Proof.** 1° Assume $\Gamma_n \in \mathcal{VC}(N)$. Let $(\sigma, v)$ be a free cocycle action of $G = *_K \Gamma_n$ on $N$ and denote $M = N \rtimes_\sigma G$ with $U_g, g \in G$ the corresponding canonical unitaries. Since $\Gamma_n \in \mathcal{VC}(N)$, there exist unitaries $\{w^n_g \mid g \in \Gamma_n\}$ in $N$ such that $U^n_g = w^n_g U_g, g \in \Gamma_n$, give left regular representations of $\Gamma_n$. Replacing $U_g$ by $w^n_g U^n_0$, $g \in \Gamma_0$, we may assume $w^n_g = 1, \forall g \in \Gamma_0$.

But then for each $n \geq 1, U^n_k = w^n_k U_k, k \in K$, for some 1-cocycles $w^n : K \to \mathcal{U}(N)$ for the restriction to $K$ of the $\Gamma_n$-action $\sigma_n$ implemented by $U^n_g, g \in \Gamma_n$. By
(\cite{J80}; see Proposition 1.3 above) any 1-cocycle of a free action of a finite group vanishes. Hence, there exists \( v_n \in \mathcal{U}(N) \) (with \( v_0 = 1 \)) such that \( w_k^n = v_n \sigma_n(k)(v_n^*) \), equivalently \( U_k^n = v_n U_k v_n^* \), \( k \in K \). But then the unitaries \( \{ v_n^* U_g^m \sigma_g(v_n) \mid g \in \Gamma, n \geq 0 \} \) generate inside \( M \) a copy of the left regular representation of \( G = \ast_{k} \Gamma_n \) implementing a \( G \)-action on \( N \) that gives an inner perturbation of the initial cocycle \( G \)-action \( \sigma \).

2° We construct recursively an increasing sequence of separable von Neumann subalgebras \( Q_n, n \geq 0 \), such that \( Q_0 \supset \{ v_{g,h} \mid g, h \in \Gamma \} \) and for each \( m \geq 1 \) we have

\[ (a) \quad E_{Q_m \cap N}(x) = \tau(x)1, \forall x \in (Q_{m-1})_1; \]
\[ (b) \quad E_{Q_m \cap N \ltimes \Gamma}(U_g) = 0, \forall g \neq e; \]
\[ (c) \quad Q_m \supset \cup_g \sigma_g(Q_{m-1}), \]

where \( U_g \in N \ltimes \Gamma \) are the canonical unitaries implementing \( \sigma \).

Assume we have constructed these algebras up to \( m = n \). Since \( N' \cap N \ltimes \Gamma = \mathbb{C} \), by using (Theorem 0.1 in \cite{P13}) we can get a Haar unitary \( v = (v_k)_{k} \in N' \) that’s free independent to \( Q_{n-1} \cup \{ U_g \}_{g} \). Thus, if we take \( Q_n^0 \) to be the von Neumann algebra generated by \( Q_{n-1} \) and \( \{ v_k \}_{k} \), then we already have (a) and (b) satisfied for \( Q_n = Q_n^0 \), and then we can replace this “initial” \( Q_n^0 \) by the von Neumann algebra generated by \( Q_n = \cup_g \sigma_g(Q_n^0) \), to have (c) satisfied as well.

Finally, if we define \( Q = \overline{\cup_n Q_n^w} \), then \( Q \) is clearly separable, condition (c) insures that \( Q \) is normalized by \( \sigma \), condition (a) shows that \( E_{Q' \cap N}(\cup_n Q_n) \in \mathbb{C}1 \), implying that \( Q \) is a factor, while condition (b) shows that \( \sigma_g \) is outer on \( Q \), \( \forall g \neq e \).

3° This part is now trivial by 2°. \( \square \)

**1.6. Remarks**

1° As we will see in Sections 3 and 4, it is in general not true that if \( \Gamma_i \) are in \( \mathcal{V} \mathcal{C} \) then their amalgamated free product over a common (infinite) amenable subgroup \( H \subset \Gamma_i, \Gamma = \Gamma_1 \ast_H \Gamma_2 \) is in \( \mathcal{V} \mathcal{C} \). For instance, \( \mathbb{Z}^2 \rtimes SL(2, \mathbb{Z}) \) does not even belong to \( \mathcal{V} \mathcal{C}_w(R) \) (see Theorem 3.2).

2° The classes \( \mathcal{V} \mathcal{C} \) may satisfy other general permanence properties. For instance, it may be true that \( \Gamma \in \mathcal{V} \mathcal{C} \) implies \( \Gamma_0 \in \mathcal{V} \mathcal{C} \) for any subgroup \( \Gamma_0 \subset \Gamma \) (or at least when \( [\Gamma : \Gamma_0] < \infty \)). However, the obvious idea for a proof, which is to “co-induce” a given cocycle action \( \Gamma_0 \rtimes \sigma_0 N_0 \) to a set of automorphisms \( \{ \sigma_g \mid g \in \Gamma \} \) on \( N = N_0 \otimes^{\Gamma/\Gamma_0} \) doesn’t work when \( [\Gamma : \Gamma_0] = \infty \), because an infinite tensor product of inner automorphisms may become outer, so the \( \sigma_g \)’s may in fact not give a cocycle action of \( \Gamma \). When the index of \( \Gamma_0 \) in \( \Gamma \) is finite, then \( \sigma \) defined this way does give a cocycle action of \( \Gamma \) on \( N \), but it is not immediate of how to “bring down to \( N_0 \)” the vanishing of the cohomology for \( \sigma \) to the vanishing of the cohomology for the initial \( \Gamma_0 \rtimes \sigma_0 N_0 \).
2. Groups with the property \( \mathcal{VC} \)

In this section we prove a vanishing cohomology result for arbitrary free cocycle actions of countable amenable groups on arbitrary II\(_1\) factors.

To do this, we’ll first show that any amenable subgroup \( \Gamma \subset \text{Aut}(N)/\text{Int}(N) \) can be lifted to a set \( \{\sigma_g \mid g \in \Gamma\} \subset \text{Aut}(N) \) normalizing a “large” hyperfinite subfactor of \( N \) (see Theorem 2.1). As it happens, this property, which is interesting by itself, characterizes the amenability of the group \( \Gamma \). Indeed, we will show in Section 5 that any non-amenable group admits a free action on \( N = L(\mathbb{F}_\infty) \) that cannot be perturbed to a cocycle action that normalizes a hyperfinite subfactor of \( N \).

Once we prove that any cocycle action of an amenable group \( \Gamma \overset{\sigma}{\curvearrowright} N \) normalizes (modulo inner perturbation) a hyperfinite II\(_1\) factor \( R \subset N \), we reduce the vanishing cohomology problem to the case \( N = R \), where by a well known result of Ocneanu [Oc85] free cocycle actions of amenable groups can indeed be “untwisted” to genuine actions. The fact that \( R \) is “large in \( N \)” assures that by untwisting \( \sigma \) on \( R \) we have untwisted it as an action on \( N \) as well.

The result about amenable subgroups \( \Gamma \subset \text{Out}(N) \) normalizing large hyperfinite factors \( R \subset N \) will be derived from a much more general phenomenon in subfactor theory, showing that any finite index inclusion of separable II\(_1\) factors \( N \subset M \) with amenable standard invariant \( G_{N\subset M} \) contains an inclusion of hyperfinite factors \( (Q \subset R) \subset (N \subset M) \), that makes a non-degenerate commuting square with \( N \subset M \) and has identical higher relative commutants in the associated Jones tower (in particular same standard invariant), in fact even satisfies the strong smoothness condition \( Q' \cap R_n = Q' \cap M_n = N' \cap M_n, \forall n \) (see Theorem 2.8 below).

2.1. **Theorem.** Let \( N \) be a II\(_1\) factor and \( \sigma : \Gamma \rightarrow \text{Aut}(N) \) an outer action of a countable amenable group \( \Gamma \) on \( N \) (i.e., a map that’s a 1 to 1 group morphism when factoring to \( \text{Out}(N) \)). Then there exist inner perturbations \( \{\sigma'_g\}_g \subset \text{Aut}(N) \) of \( \sigma \) that leave invariant a hyperfinite subfactor \( R \subset N \), such that:

1. The automorphisms \( \sigma'_g \sigma'_h \sigma^{-1}_g \in \text{Int}(N) \) are implemented by unitaries in \( R \), \( \forall g, h \in \Gamma \).
2. \( \sigma'_g \) is outer on \( R \), \( \forall g \neq e \).

Moreover, if \( N \) is separable, then one can choose \( (\sigma', R) \) so that \( R' \cap N = \mathbb{C} \).

As mentioned above, Theorem 2.1 will be a particular case of a general result in subfactor theory (Theorem 2.8 below). In turn, by combining Theorem 2.1 with Ocneanu’s Theorem in [Oc85], we can already derive the vanishing cohomology result for cocycle actions of arbitrary amenable groups:

2.2. **Theorem.** Let \( N \) be a II\(_1\) factor, \( \Gamma \) a countable amenable group and \( \Gamma \overset{\sigma}{\curvearrowright} N \)
a free cocycle action of $\Gamma$ on $N$. Then there exist unitary elements $\{w_g \in \mathcal{U}(N) \mid g \in \Gamma\}$ such that $\sigma'_g = \text{Ad}(w_g) \circ \sigma_g$, $g \in \Gamma$, is a genuine action of $\Gamma$ on $N$. Moreover, if $M = N \rtimes_{\sigma} \Gamma$ denotes the corresponding crossed product $\Pi_1$ factor with $\{U_g\}_g \subset M$ the canonical unitaries implementing the cocycle $\Gamma$-action $\sigma$, then $w_g$ can be chosen so that $U'_g = w_g U_g$, $g \in \Gamma$, satisfy $U'_g U'_h = U'_g h$, $\forall g, h \in \Gamma$.

**Proof of 2.2.** By Theorem 2.1, there exist unitary elements $w^0 \in N$, $g \in \Gamma$, and a hyperfinite $\Pi_1$ subfactor $R \subset N$ such that:

1. $U^0_g = w^0_g U_g$ normalize $R$;
2. $U^0_g U^0_h = v^0_{g,h} U^0_{gh}$, $\forall g, h \in \Gamma$, where $v^0 : \Gamma \times \Gamma \to \mathcal{U}(R) \subset \mathcal{U}(N)$ is the 2-cocycle for the cocycle $\Gamma$-action $\sigma^0$ of $\Gamma$ on $N$ given by $\sigma^0_g = \text{Ad}(U^0_g)$, $g \in \Gamma$;
3. $\sigma^0_{g|R}$ is outer, $\forall g \neq e$.

By Ocneanu’s Theorem [Oc85], the 2-cocycle $v^0$ vanishes on $R$. In other words, there exist unitary elements $w^1_g \in R$ such that $U'_g = w^1_g U^0_g$, $g \in \Gamma$, give a copy of the left regular representation of $\Gamma$. This shows that $w_g = w^1_g w^0_g$ satisfy the required condition. \hfill \Box

**2.3. Corollary.** The class $\mathcal{VC}$ contains all countable amenable groups and is closed to free products with amalgamation over finite subgroups, i.e., if $\{\Gamma_n\}_n \subset \mathcal{VC}$ and $K \subset \Gamma_n$ is a common finite subgroup, then $\Gamma_0 \ast_K \Gamma_1 \ast_K \ldots \in \mathcal{VC}$.

For the rest of this section, we will use concepts and notations from [J83] (such as the basic construction, the Jones tower of factors, etc), as well as from ([PiP84], [P91], [P93], [94a], [94b], [P97]). In particular, we will often use as framework the symmetric enveloping (SE) inclusion $M \boxtimes_{e_N} M^{op} \subset M \boxtimes M^{op}$ of $N \subset M$, introduced in [P94b] (cf. also the extended version of the paper, [P97]).

We begin by recalling some properties relating Jones tower/tunnel of a subfactor with its symmetric enveloping inclusion.

**2.4. Lemma.** Let $N \subset M$ be an extremal inclusion of $\Pi_1$ factors of index $[M : N] = \lambda^{-1} < \infty$, $T = M \boxtimes_{e_N} M^{op} \subset M \boxtimes M^{op} = S$ its symmetric enveloping (SE) inclusion of $\Pi_1$ factors and ... $\subset N_m \subset \ldots \subset N \subset e^{-1} M \subset e^{e_N=e_0} M_1 \subset e^1 \ldots M_m \subset e^m$ .... a tunnel-tower for $N \subset M$ inside $S$.

1. If $e^{-n}_n \in M_{n+1} \subset S$ denotes the projection of trace $[M : N]^{-n-1}$ obtained as a scalar multiple of the word of maximal length in $e^{-n}, \ldots, e_0, \ldots, e_n$, then $e^{-n}_n$ implements $E^{M}_{N_n}$, $E^{M^{op}}_{N^{op}_n}$ and one has $vN(T, e^{-n}_n) = S$. Thus, the map that acts as the identity on $M \vee M^{op}$ and takes $e^{-n}_n$ to $e^{-n}_n$ gives a natural identification between $M \vee M^{op} \subset M \boxtimes M^{op}$ and $T \subset S$. 

\[\]
2° Let $p \in \mathcal{P}(N'_n \cap M)$ and $p^{op} \in M'_1 \cap M_{n+1} \subset S$ its image under the antiisomorphism $^{op}$ of $S$. Then $N_{n+1} p^{op} \subset pMp^{op} \subset pp^{op}M_{n+1}p^{op}$ is a basic construction for $(V \subset U) = (N_{n+1} p^{op} \subset pMp^{op})$, with Jones projection $f = \tau(p)^{-1} p^{op} e_p n p^{op} = \tau(p)^{-1} p e_p n p = \tau(p)^{-1} p^{op} f p^{op}$. Moreover $U \otimes U^{op}$ is naturally embedded into $S$ as the weakly closed *-subalgebra generated by $pMp^{op}$, $pp^{op}M^{op}$, $p^{op}$ and $f$. If in addition $p \in N'_n \cap N$, then this latter algebra is actually equal to $p^{op} S(p^{op})$, thus giving a natural identification between $(U \otimes U^{op})_{ev} \subset U \otimes U^{op}$ and the amplification by $\tau(p)^2$ of $(T \subset S)$, with $e_{V} \mapsto f$.

3° Let $p \in \mathcal{P}(N'_n \cap N)$ be as in the last part of 2°. Let $P \subset N$ be a subfactor such that $P \subset M$ is a downward basic construction for $M \simeq M^{op}_p \subset pp^{op}M_{n+1}p^{op}$ and denote $M \otimes M^{op} = T \subset S_1 = M \boxtimes M^{op}$ its SE inclusion. If $\{m_j\}_{j \in N}$ is an orthonormal basis of $N$ over $P$, then $e = \Sigma_j m_j e_P m_j^{*} = \Sigma_j m_j^{op} e m_j^{op*}$ is a projection of trace $\lambda = [M : N]^{-1}$ in $S_1$ that implements both $E_N^{M}$ and $E_{N^{op}}^{M^{op}}$ and satisfies $\sigma N(T, e) = S_1$, thus giving an identification between $T \subset S$ and $T \subset S_1$, with $e_N \mapsto e$.

Proof. Part 1° is (Proposition 2.9(a) in [P97]), the first part of 2° is (Propositions 2.8.(c) and 2.9(c) in [P97]), while the first part of 3° is (Proposition 2.10 in [P97]).

To prove the last part of 2°, note that with the notation $U = pMp^{op}$ and $S_0 = U \otimes U^{op} \subset pp^{op}S(p^{op})$ we have $UL^2(S_0)_{ev} \subset UL^2(pp^{op}S(p^{op}))_{ev}$ as Hilbert bimodules. Then notice that by (Theorem 4.5 in [P97]), $UL^2(pp^{op}S(p^{op}))_{ev} = \bigoplus_{k \in K} H_k^{op} \otimes H_k^{op}$, where $\{H_k^{op}\}_{k \in K}$ denotes the reduction by $pp^{op}$ of all distinct irreducible Hilbert $M$-bimodules in $\bigcup_n L^2(M_n)$. Since the list of irreducible $U$-bimodules in the Jones tower for $N_p \subset pMp$ contains all $\{H_k^{op}\}_{k \in K}$ (because $p \in N'_n \cap N$), it follows that $UL^2(pp^{op}S(p^{op}))_{ev} \subset UL^2(S_0)_{ev}$ as well. Thus, the inclusion $UL^2(S_0)_{ev} \subset UL^2(pp^{op}S(p^{op}))_{ev}$ is an equality, forcing $S_0 = S$ as well.

The last part of 3° follows by a similar “exhaustion by bimodules” argument. □

2.5. Definition. Let $N \subset M$ be an inclusion of II$_1$ factors with finite index and denote by $N < M < M_1 < M_2 < ...$ its Jones tower. A subfactor $P \subset N$ is said to be $(N < M)$-compatible if there exist $n \geq 1$ and a non-zero projection $p^{op} \in M'_1 \cap M_n$ such that $PP^{op} \subset MP^{op} \subset p^{op} \subset p^{op} \subset M_nP^{op}$ is a basic construction.

For the reader’s convenience, we recall here two of the equivalent definitions of amenability for “group-like” objects arising from subfactors, that we have introduced and studied in ([P91], [P93], [P94b], [P97]), and that we need hereafter.

The standard invariant $G_{N \subset M}$ of an extremal inclusion of factors with finite Jones index $N \subset M$ is amenable if its principal graph $\Gamma_{N \subset M}$ satisfies the Kesten-
type condition $\|\Gamma_{N \subset M}\|^2 = [M : N]$.

This very first definition of amenability was proposed in [P91], and we will also take it to be the definition of amenability for the various abstractizations of standard invariants: a standard $\lambda$-lattice $\mathcal{G}$ as in [P04b] (or a planar algebra as in [J99]) is amenable if its graph $\Gamma_\mathcal{G}$ satisfies the condition $\|\Gamma_\mathcal{G}\|^2 = \lambda^{-1}$.

An alternative notion of amenability was introduced in [P93], by requiring the following Følner-type condition on $\mathcal{G} = \mathcal{G}_{N \subset M}$: let $\{v_k\}_{k \in K}$ denote the standard weights of $\Gamma_{N \subset M}$ (resp. $\Gamma_\mathcal{G}$), obtained for instance as $\text{dim}(M \mathcal{H}_kM)^{1/2}$, where $\{\mathcal{H}_k\}_{k \in K}$ is the list of irreducible $M - M$ Hilbert bimodules appearing at even levels in $\mathcal{G}$, indexed by the set $K$ of left vertices of the bipartite graph $\Gamma_\mathcal{G} = \Gamma_{N \subset M}$; $\mathcal{G}$ satisfies the Følner condition if for any $\varepsilon > 0$ there exists a finite set $F \subset K$ such that if one denotes by $\partial F = \{k \in K \setminus F \mid \exists k_0 \in F \text{ with } (\Gamma_\mathcal{G})_{k_0} \neq 0\}$ (the boundary of $F$) then $\sum_{k \in \partial F} v_k^2 < \varepsilon \sum_{k \in F} v_k^2$.

These two conditions were shown equivalent in [P97] (the result had already been announced in [P93] and [P94b]). Several other equivalent amenability conditions were in fact introduced and studied in [P97], notably a local finite dimensional approximation property which will be crucial for the proof of Theorem 2.8 below.

The above Kesten-type condition provides in particular a definition of amenability for finitely generated rigid $C^*$-tensor categories $\mathcal{G}$ (as defined for instance in [PV14]), having the bimodules $\{\mathcal{H}_k\}_{k \in K}$ as irreducible objects. Indeed, such $\mathcal{G}$ are equivalent to standard $\lambda$-lattices (or standard invariants) of the form $\mathcal{G}_{M_1 \subset M_2}$, where $N \subset M \subset M_1 \subset M_2 \subset \ldots$ if the Jones tower of an extremal subfactor $N \subset M$. $\mathcal{G}$ can also be viewed as the group-like object generated by the corresponding endomorphisms $\theta_k = \theta_{\mathcal{H}_k}$ of $M^\infty = M \overline{\otimes} \mathcal{B}(l^2 \mathbb{N})$, via Connes’ well known correspondence between the Hilbert $M^\infty$-bimodules $\mathcal{H}_k$ and $M^\infty$-endomorphisms ([C80]).

In turn, since it involves only finite subsets of $K$, the Følner-type condition in ([P03]) makes sense for any (not necessarily finitely generated) rigid $C^*$-tensor category. Thus, we’ll say that a (arbitrary) rigid $C^*$-tensor category $\mathcal{G}$ is amenable, if it satisfies the above Følner-type condition. This condition is of course equivalent to the condition that any finitely generated sub $C^*$-tensor category of $\mathcal{G}$ is amenable (for which in turn one has the alternative Kesten-type characterization).

2.6. Lemma. Let $N \subset M$ be a finite index extremal inclusion of $II_1$ factors with amenable standard invariant and SE factor $S = M \bigotimes_{\varepsilon N} M^{\varepsilon \nu}$. Given any $(N \subset M)$-compatible subfactor $P \subset N$ and any $\varepsilon > 0$, there exists a $(P \subset M)$-compatible subfactor $Q \subset P$ such that $\|E_{(Q \cap P) \cap S}(x) - E_{P \cap S}(x)\|_2 \leq \varepsilon \|x\|$, $\forall x \in P' \cap S$.

Proof. Let us first note a few Facts needed in the proof.

Fact 1. It is sufficient to show that there exists a compatible subfactor $Q \subset P$
with the property that \( \|E_{Q' \cap M \cap S}(f) - \tau(f)1\|_2 \leq \varepsilon/([M : P] + 1)^2 \), where \( f \in S \) is the Jones projection for \( P \subset M \subset \langle M, P \rangle \), viewed inside \( S \) (cf. 2.4.2°, 2.4.3° above).

Indeed, because if \( \{m_j\}_j \subset M \) is an orthonormal basis of \( M \) over \( P \) with \([M : P] + 1 \) many elements of norm \( \leq [M : P]^{1/2} \) (cf. [PiP84]), then \( \{[M : P]^{1/2}m_j^{op} f\}_j \) is an orthonormal basis of \( P' \cap S \) over \( M' \cap S = M^{op} \) and any \( x \in P' \cap S = \langle M^{op}, f \rangle \) is of the form \( x = \sum_j [M : P]^{1/2}m_j^{op} f y_j^{op} \), where \( y_j^{op} = [M : P]^{1/2}E_{M^{op}}(f m_j^{op} x) \in M^{op} \) has operator norm majorized by \([M : P]^{1/2}\|x\|\|m_j f\| = [M : P]^{1/2}\|x\|\), thus giving the estimates

\[
\|E_{(Q' \cap M) \cap S}(x) - E_{M' \cap S}(x)\|_2 = \|\sum_j [M : P]^{1/2}m_j^{op} (E_{(Q' \cap M) \cap S}(f) - \tau(f)1)y_j^{op}\|_2 \\
\leq [M : P]^{1/2} \sum_j \|m_j\|\|y_j^{op}\| \\
\leq [M : P]([M : P] + 1)\|x\|\varepsilon/([M : P] + 1)^2 < \varepsilon\|x\|.
\]

**Fact 2.** By (Corollary 6.4 in [P97]), \( G_{N \subset M} \) amenable implies \( G_{P \subset M} \) amenable.

**Fact 3.** By (Theorem 6.1 in [P97]), if \( P \subset M \) is an extremal inclusion of factors with amenable standard invariant then for any \( \delta > 0 \) there exists \( n \geq 1 \) and a projection \( p \in P_n' \cap P \) such that \( \|E_{p_n'(\cap M) \cap P}\cap P_{0}\cap (M, P)_{0}(f_0p) - \tau(f_0)p\|_2 < \delta\tau(p) \), where \( ... \subset P_n \subset P_{n-1} \subset ... P \subset M \subset f_{0} = e_{P} \langle M, P \rangle \) denotes a Jones tunnel and basic construction for \( P \subset M \).

Let us now proceed with the proof of the lemma. By Fact 2, \( P \subset M \) is amenable so we can apply Fact 3 to \( (P \subset M \subset f_{0}\langle M, P \rangle) \), to get an \( n \geq 1 \) and a projection \( p \in P_n' \cap P \) such that

\[
(2.6.1) \quad \|E_{p(P_n'(\cap M) \cap P)}(f_0p) - \tau(f_0)p\|_2 < \varepsilon\|p\|_2/([M : P] + 1)^2
\]

By amplifying by \( \alpha = \tau(p)^{-1} \) the inclusions of factors

\[
(2.6.2) \quad P_n p \subset p P p \subset p M p \subset f_{0} p \langle M, P \rangle p
\]

using partial isometries in \( P \), we obtain inclusions of factors

\[
(2.6.3) \quad (P_n p)^{\alpha} = Q \subset P \subset M \subset f_{0}\langle M, P \rangle
\]
having same relative commutants as (2.6.2). Thus, by (2.6.1), it follows that $Q$ satisfies
\begin{equation}
\|E_{Q' \cap (M, P)}(f_0) - \tau(f_0)\|_2 < \varepsilon / ([M : P] + 1)^2,
\end{equation}

By the way it is defined, $Q \subset P$ is an $(P \subset M)$-compatible subfactor, and thus $(N \subset M)$-compatible as well, while by Fact 1 and (2.6.4) we have \[\|E_{(Q' \cap M) \cap S}(x) - E_{M' \cap S}(x)\|_2 \leq \varepsilon / \|x\|,\] for all $x \in P' \cap S$.

\section*{2.7. Lemma.} \textit{Let $N \subset M$ be a finite index extremal inclusion of $\Pi_1$ factors with $N \subset M \subset M_1, \ldots \not\subset M_\infty$ its Jones tower of factors and $S$ its SE factor. If $B \subset M$ is a diffuse von Neumann subalgebra, then $B \not\subset_{M_\infty} M' \cap M_\infty$ and $B \not\subset_{S} M'^{\text{op}}$.}

\textbf{Proof.} By [P03], in order to prove $B \not\subset_{M_\infty} M' \cap M_\infty$, it is sufficient to prove that for any finite set $F$ in a given total subset $X$ of $M_\infty$, there exist $u_n \in \mathcal{U}(B)$ such that $\lim_n \|E_{M' \cap M_\infty}(y^*u_n x)\|_2 = 0$, $\forall x, y \in F$. Taking $X = \cup_n M_m$, it is sufficient to show this for any $m$ and any finite $F \subset M_m$. But by [P03] this amounts to $M \not\subset M' \cap M_m$, which is trivial since $B$ is diffuse and $M' \cap M_m$ is finite dimensional.

To prove $B \not\subset_{S} M'^{\text{op}}$ we use the same criterion, but with $X = \cup_n M_m M'^{\text{op}}$, which is total in $S$ by [P97]. Thus, if $F \subset X$ is finite then we may assume $F \subset M_m M'^{\text{op}}$. Moreover, by [P97], thus, if $F \subset X$ is finite then we may assume $F \subset M_m M'^{\text{op}}$ for some large $m$ so if $x = x_1 x_2^\text{op}, y = y_1 y_2^\text{op} \in F$, with $x_1, y_1 \in M_m, x_2, y_2 \in M$, and we take $u_n \in \mathcal{U}(B)$, then we get the estimate
\[\|E_{M'^{\text{op}}}(y_2^\text{op} y_1^\text{op} u_n x_1 x_2)\|_2 \leq \|x_2\| \|y_2\| \|E_{M' \cap M_m}(y_1^\text{op} u_n x_1)\|_2.\]

This shows that it is actually sufficient to check the criterion for $F \subset M_m$, which amounts again to $M \not\subset M' \cap M_m$ as before. \hfill \Box

\section*{2.8. Theorem.} 1° \textit{Let $N \subset M$ be a finite index extremal inclusion of separable $\Pi_1$ factors with amenable standard invariant. There exists a sub-inclusion of hyperfinite factors $(Q \subset R) \subset (N \subset M)$, that makes a non-degenerate commuting square with $N \subset M$, such that $Q' \cap R_n = Q' \cap M_n = N' \cap M_n$ and $R' \cap R_n = R' \cap M_n = M' \cap M_n$, $\forall n$, where $N \subset M \subset M_0 \subset \ldots$ is the Jones tower for $N \subset M$ and $R_n = vN(R, e_0, \ldots, e_{n-1})$, $n \geq 1$, the tower for $Q \subset R$. Moreover, $Q \subset R$ can be obtained as $Q = \cup_n P'_n \cap N \subset \cup_n P'_n \cap M = R$, for some decreasing sequence of $(N \subset M)$-compatible subfactors $M \supset N \supset P_0 \supset P_1, \ldots$.

2° \textit{Let $M_\infty$ be a separable $\Pi_\infty$ factor and $G$ a countable rigid $C^*$-tensor category acting freely on $M_\infty$ by endomorphisms. Fix a set $\{\theta_k\}_{k \in K}$ of endomorphisms of $M_\infty$ representing the irreducible objects of $G$, with $\text{Tr} \circ \theta_k = [M_\infty : \theta_k(M_\infty)]^{1/2}$ for all $k \in K$. If $G$ is amenable, then there exists an irreducible $\Pi_\infty$ subfactor}
$R^\infty \subset M^\infty$ with normal conditional expectation such that for each $k \in K$ there exists $W_k \in \mathcal{U}(M^\infty)$ with the property that $\theta'_k = \text{Ad}(W_k) \circ \theta_k$ leaves $R^\infty$ invariant and such that $\{\theta'_k|_{R^\infty} \mid k \in K\}$ defines a free action of $\mathcal{G}$ on $R^\infty$.

**Proof.** We split the proof of $1^\circ$ into several parts.

**Fact 1.** There exists a sequence of $(N \subset M)$-compatible subfactors .... $\subset P_{n-1} \subset ... P_0 \subset N \subset M$ such that if we define $Q = \bigcup_n P'_n \cap N \subset \bigcup_n P'_n \cap M = R$ and $R_n = \bigcup_n P'_n \cap M_n$, $n \geq 1$, then $(Q \subset R) \cap (N \subset M)$ is a non degenerate commuting square of $\Pi_1$ factors, $Q \subset R \subset e N_0$ $R_1 \subset e N_1 ...$ is its Jones tower and $Q' \cap R_n = N' \cap M_n$, $R' \cap R_n = M' \cap M_n$, $\forall n$.

To see this, let $M \vee M^{\text{op}} \subset M$ $\otimes M^{\text{op}} = S$ be the SE inclusion of factors associated with $N \subset M$. By applying recursively Lemma 2.6, we obtain a sequence of subfactors $M \supset N = P_0 \supset P_1 ...$ such that for each $n \geq 1$ we have

(a) $P_n \subset P_{n-1}$ is $(P_{n-1} \subset M)$-compatible (thus also $(N \subset M)$-compatible).

(b) $\|E_{P'_n \cap M} \cap S(x) - E_{M'} \cap S(x)\|_2 \leq 2^{-n}\|x\|$, $\forall x \in P'_{n-1} \cap S$.

Let $Q = \bigcup_n P'_n \cap N$, $R = \bigcup_n P'_n \cap M$. If we denote $S_0 = \bigcup_n P'_n \cap S$, then by property (b) above, it follows that $R' \cap S_0 = M^{\text{op}}$. In particular, $R$ is a $\Pi_1$ factor. By the definitions of $Q, R$, it follows that $(Q \subset R) \cap (N \subset M)$ is a commuting square, with $e_0 = e_N$ implementing the conditional expectation of $R$ onto $Q$ and $Q = \{e_0\}' \cap R$. From (b), one also gets $E_{Q \cap S}(e_0) = \lambda 1$. This implies that the algebra $R_1^0 := \text{sp}Re_0R$ has support 1 and thus any orthonormal basis $\{m_j\}_j$ of $R$ over $Q$ must “fill up the identity”, i.e., $\sum_j m_j e_0 m_j^* = 1$. Hence, $(Q \subset R) \subset (N \subset M)$ is in fact a non-degenerate commuting square. Moreover, $R_1^0 \cap S_0 = \{e_0\}' \cap M^{\text{op}} = N^{\text{op}}$, implying that $R_1^0$ is a $\Pi_1$ factor. Thus, $Q \simeq Qe_0 = e_0 R_1^0 e_0$ is a $\Pi_1$ factor as well, and $R_0 := vN(R, e_0, ..., e_{n-1})$, $n \geq 1$, is the Jones tower for $Q \subset R$.

At the same time, if for each $n \geq 1$ we define $R_n = \bigcup_m P'_m \cap M_n$, then both this sequence and the sequence $Q \subset R \subset R_1^0 \subset ...$ make (non-degenerate) commuting squares with $N \subset M \subset M_1 \subset ...$, with $R_0 \subset R_n$. This forces $R_0 = R_n$ and $R_n$, $N_0^{op} \subset R_n$, be the higher relative commutants, we have the equalities

$$Q' \cap R_n = (Q' \cap S_0) \cap R_n = (M_1^{op} \cap M_n) \cap R_n$$

$$= (N' \cap M_n) \cap R_n \subset Q' \cap R_n = M_1^{op} \cap M_n \cap (N_1^{op})' = N' \cap M_n,$$

finishing the proof of Fact 1.

**Fact 2.** Assume $... \subset P_1 \subset P_0 \subset N \subset M$ are as in Fact 1. If $u_n \in \mathcal{U}(P_n)$, $n \geq 0$, and we define $P_n = u_0...u_n P_n u^*...u^*_0$, then $P_n$ is an $(N \subset M)$-compatible subfactor
in $P_{n-1}^n$ and $P_n^m \subset P_{n-1}^m \subset ... P_1^1 \subset P_0^0 \subset N \subset M$ is a sequence of factors still satisfying the conditions in the statement of Fact 1.

Indeed, for each $k$ the systems of commuting squares of algebras $\{P_n^m \cap N \subset P_n^m \cap M \subset ... P_n^m \cap M_k\}_m$, $\{P_n^m \cap N \subset P_n^m \cap M \subset ... P_n^m \cap M_k\}_m$ (standard $\lambda$-lattices in the sense of [P94a]) are isomorphic via the map $\Phi(x) = \lim_{n} u_0 ... u_n x u_n^* ... u_0^*$, $x \in \bigcup_{n} P_n^m \cap M_k$. Thus, $\Phi$ implements an isomorphism between $R' \cap R_i \subset Q' \cap R_i$ and $R^0 \cap R_i \subset Q^0' \cap R_i$, where $R_i = \bigcup_{n} P_n^m \cap M_i$, $i \leq n$. Since the isomorphism $\Phi$ leaves $N' \cap M_i = Q' \cap R_i$ and $M' \cap M_i = R' \cap R_i$ fixed, by equality of dimensions via $\Phi$ it follows that $N' \cap M_i = Q' \cap R_i$, $M' \cap M_i = R' \cap R_i$, $\forall i$.

Fact 3. Assume $... \subset P_1 \subset P_0 \subset N \subset M$ are as in Fact 1. Then there exist integers $k_0 = 0 < k_1 < ...$ and unitaries $v_n \in \mathcal{U}(P_{k_n-1}^m)$, $n \geq 0$, such that if we define $P_{k_n}^m = v_1 ... v_n P_{k_n} v_n^* ... v_1^*$ and let $Q = \bigcup_{n} P_n^m \cap N \subset \bigcup_{n} P_n^m \cap M = R$, then $Q' \cap M_m = Q' \cap R_m$, $M' \cap M_m = R' \cap R_m$, $\forall m$.

To show this, let $\{b_k\}_k \subset (\bigcup_{n} M_n)_1$ be a $\| \|$-2-dense sequence. We choose recursively $k_m > k_{m-1}$, $v_m \in \mathcal{U}(P_{k_m-1}^m)$ such that

(F3) $\| E_{(P_{k_n}^m \cap P_{k_{m-1}^m}^m) \cap M_m} (b_j) - E_{P_{k_{m-1}^m}^m} (b_j) \|_2 \leq 2^{-m}, \forall j \leq m.$

Assume we made this construction up to $m = n$. By (Theorem 0.1 (a) in [P13]), if $A_0 \subset P_{k_n}^m$ is any finite dimensional abelian von Neumann algebra with all minimal projections of sufficiently small trace, then there exists $u \in \mathcal{U}(P_{k_n}^m)$ such that $\| E_{u A_0 u' \cap M_{n+1}} (b_j) - E_{P_{k_n}^m} (b_j) \|_2 < 2^{-n-1}, \forall j \leq n + 1$. For each $m \geq k_n$ denote $P_{k_n}^m = v_0 ... v_m P_{k_n} v_m^* ... v_0^*$. Since $B = \bigcup_{m} P_{k_m}^m \cap P_{k_n}^m$ is diffuse, we may assume $A_0 \subset B$, and hence

$$\| E_{u B u' \cap M_{n+1}} (x) - E_{P_{k_n}^m} (b_j) \|_2$$

$$< \| E_{u A_0 u' \cap M_{n+1}} (b_j) - E_{P_{k_n}^m} (b_j) \|_2 < 2^{-n-1}.$$

Since $B$ is a “limit” of $P_{k_m}^m \cap P_{k_n}^m$, for $m$ sufficiently large we’ll still have

$$\| E_{(u P_{k_m}^m u' \cap P_{k_n}^m) \cap M_{n+1}} (b_j) - E_{P_{k_n}^m} (b_j) \|_2 < 2^{-n-1}, \forall j \leq n + 1.$$

We choose such a large $m$ and put $k_{n+1} = m$. Letting $v_{n+1} = u^* ... u^* u v_1 ... v_n \in P_{k_n}^m$, $P_{k_{n+1}}^m = v_1 ... v_{n+1} P_{n+1} v_{n+1}^* ... v_1^* = u P_{k_{n+1}}^m u^*$, we see that (F3) is satisfied for $m = n + 1$. 

If we now define $Q = \bigcup_n P_n^* \cap N \subset \bigcup_n P_n^* \cap M = R$, then condition (F3) implies $Q' \cap M_n \subset Q' \cap R_n$, $\forall n$, while Fact 2 implies we have $Q' \cap R_n = N' \cap M_n$. The calculations for the relative commutants of $R$ are similar, thus finishing the proof of part 1°.

To prove part 2°, we need some notations. Let $\{p_k \mid k \in K\}$ be a partition of 1 with finite projections in $M = M^\infty$ of trace $Tr(p_k) = [M : \theta_k(M)]^{1/2}$. By perturbing if necessary $\theta_k$ by inner automorphisms of $M$, we may assume $\theta_k(p_1) = p_k$, $\forall k \in K$, where $* \in K$ is the trivial element in $K$, corresponding to the endomorphism $id_M$ (or bimodule $L^2(M)$). Denote by $M \subset M$ the II$_1$ factor $\simeq p_* M p_*$ embedded diagonally into $M$ as $M = \{ \oplus_k \theta_k(x) \mid x \in p_* M p_* \} \subset M$. Let also $F_n \subset K$ be an increasing sequence of finite sets exhausting $K$ and denote $q_n = \Sigma_{k \in F_n} p_k$. The embedding $M \subset M$ implements (by reduction) a sequence of embeddings of II$_1$ factors $M \simeq M q_n \subset q_n M q_n \simeq M^{Tr(q_n)}$, $n \geq 1$.

Since $G$ is amenable, all subfactors $M \subset q_n M q_n$ have amenable standard invariant. Using this, we will apply part 1° to construct recursively a sequence of subfactors of finite index $M \supset P_0 \supset P_1 \supset \ldots$ such that if we denote $Q = \bigcup_n P_n \cap M$, $R = \bigcup_n P_n^* \cap M$, then $R$ is an irreducible hyperfinite II$_\infty$ subfactor in $M$ generated by finite projections in $M$ such that $Q' \cap M = \{p_g \mid g \in G\}' = M' \cap M$ and each inclusion $(Q \subset q_n R q_n) \subset (M \subset q_n M q_n)$ is a nondegenerate commuting square (the construction also gives equality of higher relative commutants of the inclusions, but this condition will follow automatically).

Note that the above conditions imply that $R \simeq R^\infty$ satisfies the conditions required in part 2°, thus finishing the proof of the theorem.

Let $X = \{b_n\}_n \subset (\bigcup_n q_n M q_n)_1$ be a countable set such that $X \cap q_n M q_n$ is $\| \cdot \|_2$-dense in the unit ball of the II$_1$ factor $q_n M q_n$, $\forall n$. For each $n$, let $e_n^0$ denote the Jones projection for the inclusion of II$_1$ factors $M \subset q_n M q_n$. We construct the decreasing sequence of subfactors $P_m \subset M$ recursively, such that

(i) $P_m \subset P_{m-1}$ is $(P_{m-1} \subset q_m M q_m)$-compatible.

(ii) $\| E(Q_n' \cap M) \cap q_m M q_m (b_j) - E(M' \cap q_m M q_m (b_j) \|_2 < 2^{-m}/Tr(q_m)$, $j \leq m$.

(iii) $\| E(Q_n' \cap q_m M q_m ) \cap q_m M q_m (b_j) - \tau(b_j) 1 \|_2 < 2^{-m}/Tr(q_m)$, $j \leq m$.

(iv) $\| E(Q_n' \cap q_m M q_m ) (e_0^j) - [q_j M q_j : M]^{-1} 1 \|_2 < 2^{-m}/Tr(q_j)$, $j \leq m$.

If we made this construction up to $m = n$, then by applying Part 1° to the inclusion of II$_1$ factors $P_n \subset q_{n+1} M q_{n+1}$ (which has amenable standard invariant) we get an $(M \subset q_{n+1} M q_{n+1})$-compatible subfactor $P_{n+1} \subset P_n$ such that (i) – (iv) above are satisfied for $m = n + 1$.

With the sequence $\{P_n\}_n$ this way constructed, define $R = \bigcup_n P_n^* \cap M$ and $R = \bigcup_n P_n^* \cap M$. Then $R$ is clearly AFD and generated by finite projections of $M$, while
condition (iii) implies $R' \cap M = \mathbb{C}$. Moreover, condition (ii) shows that $R' \cap M = M' \cap M = R' \cap R$ while condition (iv) shows that $(R \subset q_n R q_n) \subset (M \subset q_n M q_n)$ is a non-degenerate commuting, $\forall n$.

The definition shows that $R p_* = p_* R p_*$ and the above properties imply the inclusion $R \subset R$ is of a diagonal form $\{\oplus g \theta'_g(x) | x \in R p_*\}$ with each $\theta'_k$ an endomorphism of $R$ that's the restriction of an inner perturbation of $\theta_k$.

\[ \square \]

**Proof of Theorem 2.1.** By Proposition 1.5.2°, we may assume $N$ is separable. By Theorem 2.8.2°, one can perturb the $\sigma_g$'s by inner automorphisms to a set of automorphisms $\{\sigma'_g\}_g$ that normalize an irreducible hyperfinite subfactor $R \subset N$, on which each $\sigma'_g$ is outer, $\forall g \neq e$, and so that $\{\sigma'_g|_R\}_g$ is an outer $\Gamma$-action on $R$.

Since $\sigma'_g \sigma'_h \sigma'_{gh}^{-1}$ is inner on $N$ and is implemented by a unitary $v_{g,h} \in R$ when restricted to $R$, by the irreducibility of $R$ in $N$ it follows that $\sigma'_g \sigma'_h \sigma'_{gh}^{-1} = \text{Ad}(v_{g,h})$ on all $N$.

\[ \square \]

**2.9. Remark.** The first result about normalizing a “large” hyperfinite subfactor $R$ modulo inner perturbations was obtained in [P83], in the case $\Gamma = \mathbb{Z}$, with $R$ being constructed “by hand”, using Rokhlin towers and an iterative procedure. Shortly after, the question of whether any cocycle action of $\mathbb{Z}$ on an arbitrary II$^1$ factor can be untwisted was asked in [CJ84]. While we realized at that time that if a similar normalization result could be proved for $\Gamma = \mathbb{Z}$ then the problem would reduce to the case $N = R$, where vanishing cohomology was just shown in [Oc88], we could not extend the arguments in [P83] from $\mathbb{Z}$ to $\mathbb{Z}^2$, despite much effort. Several years later in [P89], we were able to solve this problem by using tools from subfactor theory. However, the argument in [P89] could only cover groups $\Gamma$ that have a finite set of generators $S \subset \Gamma$ with respect to which $\Gamma$ has trivial Poisson boundary (e.g., groups with polynomial growth, in particular $\mathbb{Z}^2$), depending crucially on this condition. In retrospect, it is quite surprising that in fact any outer action of any amenable group $\Gamma$ on any II$^1$ factor $N$ normalizes (modulo Int$(N)$) an irreducible hyperfinite subfactor, a property which turns out to characterize the amenability of $\Gamma$.

## 3. Non-vanishing cohomology for amplifications of actions

**3.1. Definition ([P01a]).** Let $\Gamma \curvearrowright^\sigma N$ be a free action of a group $\Gamma$ on a type II$^1$ factor $N$. Let $p \in N$ be a projection and for each $g \in \Gamma$ choose a partial isometry $w_g \in N$ such that $w_g w_g^* = p$, $w_g^* w_g = \sigma_g(p)$ and $w_e = p$. Define $\sigma'_g \in \text{Aut}(p N p)$ by $\sigma'_g(x) = w_g \sigma_g(x) w_g^*$, $x \in p N p$. Then $\sigma'$ is a free cocycle action of $\Gamma$ on $p N p$, with 2-cocycle $\gamma_{g,h}^{p} = w_g \sigma_g(w_h) w_h^*$, $g, h \in \Gamma$. Moreover, up to cocycle conjugacy
infinite subgroup with the relative property (T).

Proof. Assume now that \( \Gamma \) has an infinite subgroup with the relative property (T), then (Corollary 4.10 in [P01a]) shows that the support \( \sigma \) of the group \( \Gamma \) on \( N^t \), called the amplification of \( \sigma \) by \( t \).

If in addition \( \{w_g\}_g \subset N \) satisfy \( w_g\sigma_g(w_h) = w_{gh}, \forall g, h \in \Gamma \), then \( w \) is called a generalized 1-cocycle (of support \( t \)) for \( \sigma \), while if this equality holds modulo scalars then it is called a weak generalized 1-cocycle for \( \sigma \) (of support \( t \)).

Note that the vanishing (resp. weak-vanishing) of the cocycle \( \tau \) amounts to being able to chose the partial isometries \( \{w_g \mid g \in \Gamma \} \subset N \) so that \( w \) be a generalized 1-cocycle (resp. weak generalized 1-cocycle) for \( \sigma \).

3.2. Theorem. Let \( \Gamma \) be a countable group and \( \Gamma \acts^\sigma R = R^\sigma_0 \) the non-commutative Bernoulli \( \Gamma \)-action with base \( R_0 \simeq R \). Let \( 0 < t < 1 \) and denote by \( \sigma^t \) the free cocycle action obtained by amplifying \( \sigma \) by \( t \), with its 2-cocycle denoted \( \nu^t \). Assume one of the following properties holds true: (a) \( \Gamma \) contains an infinite subgroup with the relative property (T); (b) \( \Gamma \) contains an infinite subgroup with non-amenable centralizer. Then the cocycle \( \nu^t \) is not weak-vanishing.

Proof. If we assume \( \Gamma \) has an infinite group with the relative property (T), then (Corollary 4.10 in [P01a]) shows that the support \( t \) of the generalized weak 1-cocycle \( w_g \) must be 1, a contradiction.

Assume now that \( \Gamma \) has an infinite subgroup \( H \subset \Gamma \) such that its centralizer \( H' = \{g \in \Gamma \mid gh = hg, \forall h \in H\} \) is non amenable.

Let \((\alpha, \beta)\) be the \( s \)-malleable deformation of \( R \times R \) that commutes with the double action \( \tilde{\sigma} = \sigma \times \sigma : \Gamma \acts \tilde{R} = R \times R = (R_0 \times R_0) \rtimes \Gamma \), as in ([P01a]).

Denote \( M = R \times \Gamma, M_0 = R \rtimes H, \tilde{M} = R \times R \rtimes \tilde{\sigma} \Gamma \) and \( \tilde{M}_0 = R \times R \rtimes \tilde{\sigma} H \), with \( M \) identified as the subfactor \( R \rtimes 1 \vee \{U_g\}_g \) in \( \tilde{M} \), where \( \{U_g\}_g \subset \tilde{M} \) are the canonical unitaries implementing \( \tilde{\sigma} \).

Since \( H' \) is non-amenable, the action \( \text{Ad}(w_g U_g) \) of \( H' \) on \( (p \otimes 1)\tilde{M}_0(p \otimes 1) \) has spectral gap relative to \( (p \otimes 1)M_0(p \otimes 1) \). Arguing like in ([P06a]), this implies that \( w_h U_h \) and \( \alpha_s(w_h U_h) \), \( h \in H \), are uniformly \( \| \| \) 2-close, for \( s \in \mathbb{R} \) with \( |s| \) small. Equivalently, the map \( \xi \mapsto (w_g \otimes 1)\sigma_g(\xi)\alpha_s(w_g^* \otimes 1) \) gives a unitary representation \( \pi_s \) of the group \( H \) on the Hilbert space \( (p \otimes 1)\tilde{L}^2(R \times R)\alpha_s(p \otimes 1) \) which for \( |s| \) small has the vector \( \xi_0 = (p \otimes 1)\alpha_s(p \otimes 1) \) almost fixed by \( \pi_s(h) \), uniformly in \( h \in H \).

We are thus in exactly the same situation as when \( H \subset \Gamma \) is rigid. So the proof of (Corollary 4.10 in [P01a]) applies to conclude that the support \( t \) of the generalized weak 1-cocycle \( \{w_h \mid h \in H\} \) for \( \sigma \) must necessarily be equal to 1, a contradiction.

\[ \square \]

3.3. Corollary. If \( \Gamma \) is a group that contains an infinite subgroup which either has relative property (T), or has non-amenable centralizer in \( \Gamma \), then \( \Gamma \notin \mathcal{VC}_w(R) \).
3.4. Theorem. Let $\Gamma$ be a countable group and $\Gamma \curvearrowright \rho L(\mathbb{F}_\Gamma)$ the action implemented by the translation from the left by elements $h \in \Gamma$ on the set $\{a_g\}_{g \in \Gamma}$ of generators of the free group $\mathbb{F}_\Gamma$, $\rho_h(a_g) = a_{hg}$, $\forall h, g \in \Gamma$. Let $1 > t > 0$ and denote by $\rho^t$ the free cocycle action obtained by amplifying $\rho$ by $t$, with its 2-cocycle denoted $\nu^t$. Assume one of the following properties is satisfied: (a) $\Gamma$ contains an infinite subgroup with the relative property (T); (b) $\Gamma$ contains an infinite subgroup with non-amenable centralizer; (c) $\Gamma$ contains an infinite amenable group with non-amenable normalizer. Then the cocycle $\nu^t$ is not weak-vanishing.

Proof. The assumptions (a) and (b) lead to exactly the same argument as in the proof of Theorem 3.2 above, by using the “free s-malleable deformation” of $L(\mathbb{F}_\Gamma)$, like in the proof of (Theorem 6.1 in [P01a]).

So let us assume we are under the assumption (c) and let $H \subset \Gamma$ be an infinite amenable subgroup with its normalizer $G \subset \Gamma$ non-amenable. Note that $\mathbb{F}_\Gamma \rtimes \Gamma$ is naturally isomorphic to $\Gamma * \mathbb{Z}$. Denote $N = L(\mathbb{F}_\Gamma)$, $M = N \rtimes \Gamma = L(\mathbb{F}_\Gamma) \rtimes \Gamma = L(\Gamma * \mathbb{Z})$. Let $p \in \mathcal{P}(N)$ be a projection of trace $t < 1$ and assume $(\rho^p, \nu^p, pNp)$ has weak vanishing cohomology.

Thus, if $\{U_g\}_{g \in \Gamma} \subset M$ denote the canonical unitaries implementing $\rho$ on $N = L(\mathbb{F}_\Gamma)$, then there exist partial isometries $w_g \in N$ of left support $p$ such that $U'_g = w_g U_g \subset U(pNp)$ gives a projective representation of $\Gamma$ with scalar 2-cocycle $\mu$. In particular, $\{U'_g \mid g \in G\}'^\prime$ gives an embedding of $L_\mu(H) \subset L_\mu(G)$ into $pMp$.

Note that $L_\mu(H)$ is amenable diffuse, $L_\mu(G)$ has no amenable direct summand and $L_\mu(H)$ is regular in $L_\mu(G)$. Thus, by (Corollary 1.7 in [I13]) it follows that $L_\mu(G) \prec_M L(\Gamma)$.

But $L_\mu(G) \prec_M L(\Gamma)$ implies that the “free” malleable deformation $(\alpha, \beta)$ of $M \subset \tilde{M} = \mathbb{F}_\Gamma \rtimes \Gamma$ is uniform on $\{U'_g \mid g \in G\}$ (because $L(\Gamma)$ is in the fixed point algebra of the malleable deformation and $L_\mu(G)$ is subordinated to $L(\Gamma)$). Thus, for each $s \in \mathcal{R}$, the $G$-representation $\xi \mapsto (w_g * 1) \sigma_g(\xi) \alpha_s(w_g * 1)$ gives a unitary representation $\pi_s$ of the group $G$ on the Hilbert space $(p \otimes 1)L^2(N * N)\alpha_s(p \otimes 1)$ which for $|s|$ small has the vector $\xi_0 = (p * 1)\alpha_s(p * 1)$ almost fixed by $\pi_s(g)$, uniformly in $g \in G$. As in the proof of (Theorem 4.1 in [P01a]) this shows that we must necessarily have $t = 1$, a contradiction.

3.5. Remark. Let $(N_0, \tau_0)$ be an arbitrary tracial von Neumann algebra $\neq \mathbb{C}$ and $\Gamma$ a group that satisfies one of the conditions (a) or (b) in Theorem 3.2. If $\Gamma \curvearrowright N = N_0^{\otimes \Gamma}$ is the Bernoulli action with base $N_0$ and $0 < t < 1$, then by using the s-malleable deformation of $N = N_0^{\otimes \Gamma}$ in ([I06]) in combination with the argument in the proof of Theorem 3.2, one obtains that the $t$-amplification $(\sigma^t, \nu^t, N^t)$ of $\sigma$ has non-vanishing cohomology. Similarly, by using the s-malleable
deformation in [IPeP05] one can obtain a generalization of Theorem 3.4 for the free-Bernoulli action $\Gamma \curvearrowright \rho N = N_0^\Gamma$.

4. Non-vanishing cohomology for cocycle actions on $L(\mathcal{F}_\infty)$

4.1. Definition ([CJ84]). Let $\Gamma$ be an infinite countable group with a set of generators $S \subset \Gamma$ and assume $(\Gamma, S) \neq (\mathcal{F}_S, S)$. Let $\pi : \mathcal{F}_S \rightarrow \Gamma$ be the unique group morphism taking the free generators $S$ of $\mathcal{F}_S$ onto $S \subset \Gamma$ (so our assumption is equivalent to $\pi$ not being 1 to 1). Then $\ker \pi \cong \mathcal{F}_\infty$ and $\mathcal{F}_S$ has infinite conjugacy classes relative to $\ker \pi$. Thus, the inclusion of $II_1$ factors $L(ker \pi) = N \subset M = L(\mathcal{F}_S)$ is irreducible and regular, with $N_M(N)/U(N) = \Gamma$. Consequently, $N \subset M$ is a crossed product inclusion of the form $L(\mathcal{F}_\infty) \subset L(\mathcal{F}_\infty) \rtimes (\sigma_\pi, v_\pi) \Gamma$, for some free cocycle action $(\sigma_\pi, v_\pi)$ of $\Gamma$ on $L(\mathcal{F}_\infty)$, that we’ll call the Connes-Jones cocycle action associated with $\pi$, with $v_\pi$ its 2-cocycle (abbreviated CJ-cocycle).

We summarize here some straightforward consequences of results from ([CJ84], [P01a], [O03], [P06b], [OP07]), but which are stated in a manner pertaining to our vanishing cohomology problem:

4.2. Theorem. Let $(\Gamma, S)$, $\pi : \mathcal{F}_S \rightarrow \Gamma$, $\Gamma \curvearrowright (\sigma_\pi, v_\pi) L(\mathcal{F}_\infty)$ be as in 4.1. Assume $\Gamma$ satisfies one of the following conditions: (a) it does not have Haagerup property (e.g., it contains an infinite subset with relative property (T)); (b) it has Cowling-Haagerup invariant $\Lambda(\Gamma)$ larger than 1; (c) it has an infinite subgroup with non-amenable centralizer; (d) it has an infinite amenable subgroup with non-amenable normalizer. Then the CJ-cocycle $v_\pi$ is not weak-vanishing. Thus, if $\Gamma$ satisfies any of these properties then $\Gamma \not\in VV_w(\mathcal{F}_\infty)$.

Proof. If $v_\pi$ is weak-vanishing, then we can choose representatives $\{U_g \mid g \in \Gamma\}$ in $N_M(N)$ such that $U_gU_h = \mu_{g,h}U_{gh}$, $\forall g, h \in \Gamma$, for some scalar 2-cocycle $\mu \in H^2(\Gamma)$. Thus, $L_\mu(\Gamma) \subset M = L(\mathcal{F}_S)$.

If now $H \subset \Gamma$ is an infinite subgroup with the relative property (T) of Kazhdan-Margulis, then by [P01b] the inclusion of von Neumann algebras $L_\mu(H) \subset L_\mu(\Gamma)$ is rigid, so $L_\mu(H) \subset M = L(\mathcal{F}_S)$ is rigid as well. But $H$ infinite implies $L_\mu(H)$ is diffuse. Hence, $L(\mathcal{F}_S)$ contains a diffuse rigid von Neumann subalgebra, which by ([P01b]) contradicts the fact that $L(\mathcal{F}_S)$ has the Haagerup property.

If in turn $\Gamma$ has an infinite subgroup $H$ with non-amenable centralizer $H'$, then by (Remark 3° in §4 of [P06b]) $L_\mu(\Gamma)$ would contain a diffuse von Neumann subalgebra $B$ with non-amenable centralizer. But then $B \subset M = L(\mathcal{F}_S)$ has non-amenable centralizer, contradicting the solidity of free group factors ([O03]).

If $\Gamma$ has an infinite amenable subgroup $H \subset \Gamma$ with non-amenable normalizer $G \subset \Gamma$, then the normalizer of $B = L_\mu(H)$ in $M = L(\mathcal{F}_S)$ contains all $\{U_g\}_{g \in G}$. 
Since \( \{U_g\}_g'' \simeq L_\mu(G) \) is non-amenable, this contradicts the strong solidity of free group factors ([OP07]).

\[ \square \]

4.3. Notations. Given two discrete groups \( \Gamma, \Lambda \), one writes \( \Gamma \leq \Lambda \) whenever \( L(\Gamma) \) can be embedded (tracially) into \( L(\Lambda) \). Denote \( W_{\text{leq}}(\Lambda) \) the class of all groups \( \Gamma \) that can be subordinated this way to \( \Lambda \), i.e., \( W_{\text{leq}}(\Lambda) = \{ \Gamma \mid \Gamma \leq \Lambda \} \).

4.4. Corollary. One has \( \mathcal{VC} \subset \mathcal{VC}(L(F_\infty)) \subset W_{\text{leq}}(F_2) \). Thus, if \( \Gamma \in \mathcal{VC} \), then \( \Gamma \) has Haagerup property, Cowling-Haagerup constant equal to 1, any infinite subgroup of \( \Gamma \) has amenable centralizer, and any infinite amenable subgroup has amenable normalizer.

4.5. Remarks

1° The above criteria for \( \Gamma \) not to be in \( W_{\text{leq}}(F_2) \) (and thus not in \( \mathcal{VC}(L(F_\infty)) \subset \mathcal{VC} \)) are in fact not stated in their optimal form. Thus, the results in [OP07] show that in order for \( \Gamma \) not to be in \( W_{\text{leq}}(F_2) \), it is enough that \( L(\Gamma) \) is not strongly solid, i.e., that \( L(\Gamma) \) merely has a diffuse amenable von Neumann subalgebra whose normalizer in \( L(\Gamma) \) generates a non-amenable von Neumann algebra. The list in 4.2 is also not exhaustive. For instance, by [Pe07] it follows that if \( \Gamma \in W_{\text{leq}}(F_2) \), then \( L(\Gamma) \) needs to be \( L^2 \)-rigid, while a result of Ozawa (see e.g. [BrO08]) shows that \( \Gamma \) needs to be exact. One should also note that an embedding \( L_\mu(\Gamma) \hookrightarrow L(F_2) \), for some \( \mu \in H^2(\Gamma) \), gives rise to an embedding \( L(\Gamma) \hookrightarrow L(F_2 \times F_2) \), by simply doubling the canonical unitaries \( \{u_g \mid g \in \Gamma\} \subset L_\mu(\Gamma) \subset L(F_2) \), i.e., by taking \( L(\Gamma) \simeq \{u_g \otimes u_g^{op} \mid g \in \Gamma\}'' \subset L(F_2) \otimes L(F_2) = L(F_2 \times F_2) \).

2° It is reasonable to expect that \( \mathcal{VC} = \mathcal{VC}(L(F_\infty)) \), and more specifically that the CJ-cocycles are in some sense the “worse possible”, i.e., if any such cocycle vanishes for some group \( \Gamma \), then \( \Gamma \in \mathcal{VC} \). We also believe that \( \mathcal{VC} = \mathcal{VC}_w \).

3° Given a group \( \Lambda \), denote by \( \text{ME}(\Lambda) \) the class of groups \( \Gamma \) that are measure equivalent (ME) to \( \Lambda \) and by \( \text{ME}_{\text{leq}}(\Lambda) \) the class of groups \( \Gamma \) that have a free m.p. action which can be realized as a sub equivalence relation of a free ergodic m.p. \( \Lambda \)-action. It would be interesting to explore the possible correlations between the classes \( \mathcal{VC} \), \( W_{\text{leq}}(F_2) \), \( \text{ME}_{\text{leq}}(F_2) \), etc. In this respect, one should point out that while \( W_{\text{leq}}(F_2) \), \( \text{ME}_{\text{leq}}(F_2) \) are obviously “hereditary” classes (i.e., if \( \Gamma \) belongs to any of them, then all subgroups of \( \Gamma \) belong too), we could not prove such hereditarity for \( \mathcal{VC} \) (cf. Remark 1.6.2°). See also Section 7 in [PeT07] for more comments on \( \text{ME}_{\text{leq}}(F_2) \) and its relations to \( W_{\text{leq}}(F_2) \). One should also note that \( \text{ME}_{\text{leq}}(F_2) \) consists of groups \( \Gamma \) that are ME to either \( Z = F_1 \), \( F_2 \), or \( F_\infty \), i.e., \( \text{ME}_{\text{leq}}(F_2) = \text{ME}(Z) \cup \text{ME}(F_2) \cup \text{ME}(F_\infty) \) (cf. [G04], [Hj04]).
4° We do not know of any examples of groups in $\mathcal{VC}$, $W_{\text{eq}}^*(\mathbb{F}_2)$ other than amalgamated free products of amenable groups over finite groups. The approach in 4.2 indicates that these two classes may coincide (perhaps with $\text{ME}_{\text{eq}}(\mathbb{F}_2)$ as well).

An intriguing class of groups that are known from [G04] to belong to $\text{ME}_{\text{eq}}(\mathbb{F}_2)$ (in fact, even to $\text{ME}(\mathbb{F}_2)$), are the free products of finitely many copies of $\mathbb{F}_2$ with amalgamation over the subgroup $\mathbb{Z} \subset \mathbb{F}_2$ generated by the commutator $aba^{-1}b^{-1}$ (where $a, b$ are the generators of $\mathbb{F}_2$). Gaboriau conjectured that in fact any amalgamated free product of $\mathbb{F}_{k_i}, 2 \leq k_i \leq \infty, i \in I$ (with $I$ finite or $I = \mathbb{N}$), over some $\mathbb{Z} \hookrightarrow \mathbb{F}_{k_i}$ which is maximal abelian in the corresponding $\mathbb{F}_{k_i}$, $\forall i$, is in $\text{ME}_{\text{eq}}(\mathbb{F}_2)$.

Thus, according to the above speculations, the groups $\mathbb{F}_{k_1} *_{\mathbb{Z}} \mathbb{F}_{k_2} *_{\mathbb{Z}} ...$, with $\mathbb{Z}$ maximal abelian in each $\mathbb{F}_{k_i}$, should belong to $\mathcal{VC}$ as well. However, we were not able to prove this for any such example, except of course the case when any subgroup $\mathbb{Z}$ is freely complemented in $\mathbb{F}_{k_i}$. Related to this, we ask the following

**Question:** Let $\mathbb{F}_n \curvearrowleft^\sigma N$ be a free action of a free group of rank $n$ on a II$_1$ factor $N$ and let $\mathcal{W}$ denote the set of all 1-cocycles $w$ for $\sigma$ (i.e., maps $w: \mathbb{F}_n \rightarrow \mathcal{U}(N)$ satisfying $w_g\sigma_g(w_h) = w_{gh}, \forall g, h \in \mathbb{F}_n$). Let $\mathbb{Z} \subset \mathbb{F}_n$ be a maximal abelian subgroup, generated by some element $g \in \mathbb{F}_n$. Is it then true that the set $\{w_g \mid w \in \mathcal{W}\}$ coincides with the unitary group $\mathcal{U}(N)$?

Taking into account the way the 1-cocycles $w \in \mathcal{W}$ are constructed, from $n$-tuples of unitaries in $N$ that are taken as perturbations of the canonical unitaries $U_1, ..., U_n \in N \rtimes_\sigma \mathbb{F}_n$ that implement $\sigma_{a_1}, ..., \sigma_{a_n}$ (where $a_1, ..., a_n$ are the generators of $\mathbb{F}_n$), it immediately follows that this question has an affirmative answer in the case $\mathbb{Z}$ is freely complemented in $\mathbb{F}_n$. It is also trivial to see that if the statement holds true, then all of the above Gaboriau groups $\Gamma = \mathbb{F}_{k_1} *_{\mathbb{Z}} \mathbb{F}_{k_2} *_{\mathbb{Z}} ...$ lie in $\mathcal{VC}$.

It is not known whether these groups are in $W_{\text{eq}}^*(\mathbb{F}_2)$ either. In fact, deciding that a group $\Gamma$ satisfies $L(\Gamma) \hookrightarrow L(\mathbb{F}_2)$ is at least as interesting as deciding that it has property $\mathcal{VC}$. So the fact that $\mathcal{VC} \subset W_{\text{eq}}^*(\mathbb{F}_2)$ gives another strong motivation for proving the universal vanishing cohomology property for various groups.

The above question can also be stated for free measure preserving actions on the probability measure space, $\mathbb{F}_n \curvearrowleft (X, \mu)$, by simply replacing $N$ by $A = L^\infty(X, \mu)$ throughout that statement. Besides answering this question, it would be interesting to know if an affirmative answer would imply Gaboriau’s conjecture that the groups $\mathbb{F}_{k_1} *_{\mathbb{Z}} \mathbb{F}_{k_2} *_{\mathbb{Z}} ...$ belong to $\text{ME}_{\text{eq}}(\mathbb{F}_2)$.

5° It has been conjectured by Peterson and Thom (see end of Sec. 7 in [PeT10]) that if two amenable von Neumann subalgebras $B_1, B_2$ of the free group factor $L(\mathbb{F}_2)$ have diffuse intersection, then $B_1 \vee B_2$ should follow amenable. There has been an accumulation of evidence towards this fact being true (e.g., [P81], [Ju06], [Pe07]). For us here, this would imply that if $\Gamma \in W_{\text{eq}}^*(\mathbb{F}_2)$ is generated by amenable
subgroups $\Gamma_1, \Gamma_2 \subset \Gamma$ with $H = \Gamma_1 \cap \Gamma_2$ infinite, then $\Gamma$ must be amenable. Thus, if $\Gamma = \Gamma_1 \ast_H \Gamma_2$ then $\Gamma \not\in W_{\text{eq}}(F_2)$, unless either $H$ is finite, or $[\Gamma_1 : H] \leq 2, [\Gamma_2 : H] \leq 2$. In particular, $\mathcal{VC}(L(F_\infty))$ should not contain such groups either.

6° We expect that $\mathcal{VC}(R)$ is equal to $\mathcal{VC}(L(F_\infty))$. This fact suggests various new statements in deformation-rigidity for factors arising from Bernoulli actions. For instance, it should be possible to prove that if a group $\Gamma$ does not have Haagerup property, or if it has an infinite amenable subgroup with non-amenable normalizer, then some of the $W^*$-rigidity results in [P01a], [P03], [P06a] should be true. Combining this conjecture with the Peterson-Thom conjecture and remark 5° above, this also suggests that $\mathcal{VC}(R)$ doesn’t contain any non-amenable group $\Gamma$ that can be generated by amenable subgroups $\Gamma_1, \Gamma_2$ with $H = \Gamma_1 \cap \Gamma_2$ infinite (in particular $\Gamma = \Gamma_1 \ast_H \Gamma_2$). But the obstruction in this case should be of a completely different nature. The II$_1$ factors arising from Bernoulli actions of such groups (and more generally from non-amenable groups $\Gamma$ generated by $n \geq 2$ amenable groups with infinite intersection) may actually have additional $W^*$-rigidity properties, providing a new class of factors on which deformation-rigidity techniques should be tested, in the spirit of ([P01a], [P03], [P06a], [IPeP05], [PV12]).

7° Note that the $W^*$-algebra version of von Neumann’s conjecture on whether any non-amenable group $\Lambda$ contains a copy of $F_2 \leq_w \Lambda$. Note also that by the Gaboriau-Lyons result in [GL07] one indeed has $F_2 \leq_w Z \wr \Lambda$ for any non-amenable $\Lambda$, while by [OP07] it follows that if $Z \wr \Lambda \leq_w F_2$ then $\Lambda$ must be amenable.

5. A related characterization of amenability

We prove in this section that the “normalization” property for cocycle $\Gamma$-actions in Theorem 2.1 can only be true when the group $\Gamma$ is amenable. More precisely, for any non-amenable $\Gamma$ we exhibit examples of embeddings $\Gamma \subset \text{Out}(L(F_\infty))$ which admit no lifting to $\text{Aut}(L(F_\infty))$ that normalizes a hyperfinite subfactor of $L(F_\infty)$.

5.1. Theorem. Let $\Gamma$ be a countable group and $\Gamma \rhd^\sigma L(F_\Gamma)$ the action implemented by the translation from the left by elements $h \in \Gamma$ on the set $\{a_g\}_{g \in \Gamma}$ of generators of the free group $F_\Gamma$. Let $M = L(F_\Gamma) \rtimes \Gamma$ with $\{U_g \mid g \in \Gamma\} \subset M$ the canonical unitaries implementing $\sigma$. The following conditions are equivalent:

(a) $\Gamma$ is amenable.

(b) $L(F_\Gamma)$ contains a hyperfinite subfactor $R$ with $R' \cap M = C$ for which there exist $\{w_g\}_{g \in \Gamma} \subset U(L(F_\Gamma))$ with the property that $U'_g = w_g U_g, g \in \Gamma$, normalize $R$, implement a free action on it and satisfy $U'_g U'_h = U'_{gh}, \forall g, h \in \Gamma$. 
Proof. By Theorem 2.2 we have \((a) \Rightarrow (b)\), while \((b) \Rightarrow (c)\) is trivial.

To see that \((c) \Rightarrow (a)\), note first that one has a natural identification between
\(M = L(\mathbb{F}_\Gamma) \rtimes_\sigma \Gamma\) and \(L(\Gamma) \ast L(\mathbb{Z})\). If \(\Gamma\) would be non-amenable, then \(P = B \vee \{w_g U_g | g \in \Gamma\}\) would be non-amenable. Thus, by (Corollary 1.7 in [I13]) it would follow that \(P \prec_M L(\Gamma)\), in particular \(B \prec_M L(\Gamma)\). But by applying (Corollary 2.3 in [P03]), it is trivial to see that \(L(\mathbb{F}_\Gamma)\) (an algebra on which \(\{U_g\}_{g \in \Gamma}\) acts) has no diffuse von Neumann subalgebra that can be subordinated \(\prec_M\) to \(L(\Gamma) = \{U_g\}''\), a contradiction. \(\square\)

5.2. Theorem. Let \(\Gamma\) be a countable group with a set of generators \(S \subset \Gamma\) and corresponding surjective morphism \(\pi : \mathbb{F}_S \rightarrow \Gamma\), with kernel \(\ker \pi \simeq \mathbb{F}_\infty\). Let \(L(\mathbb{F}_\infty) = N \subset M = L(\mathbb{F}_S)\) be the associated irreducible, regular inclusion of free group factors, which satisfies \(N_M(N)/U(N) \simeq \Gamma\), and denote by \(\Gamma \bowtie^\sigma N\) the corresponding cocycle \(\Gamma\)-action. The following conditions are equivalent:

(a) \(\Gamma\) is amenable.

(b) \(N\) contains a hyperfinite subfactor \(R \subset N\) with \(R' \cap M = \mathbb{C}\) for which there exist \(\{w_g\}_g \subset U(N)\) with the property that \(U'_g = w_g U_g, g \in \Gamma\), normalize \(R\), implement a free action on it, and satisfy \(U'_g U'_h = U'_g h\), \(\forall g, h \in \Gamma\).

(c) \(N\) contains a diffuse AFD von Neumann subalgebra \(B\) such that \(\forall g \in \Gamma, \exists w_g \in U(N)\) with the property that \(\text{Ad} w_g \circ \sigma_g\) normalizes \(B\).

Proof. Theorem 2.2 shows that \((a) \Rightarrow (b)\) and \((b) \Rightarrow (c)\) is trivial. If \((c)\) holds but we assume \(\Gamma\) is non-amenable, then the von Neumann algebra generated by \(B\) and its normalizer in \(M = L(\mathbb{F}_S)\) is non-amenable, contradicting the strong solidity of the free group factors ([OP07]). \(\square\)

5.3. Remark. The dichotomy amenable/non-amenable in the above results can probably be extended to cover the converse to Theorem 2.8 as well. Thus, it should be true that if \(G\) is a non-amenable standard \(\lambda\)-lattice, then there exists an extremal inclusion of separable II\(_1\) factors \(N \subset M\) with standard invariant equal to \(G\) in which one cannot embed with non-degenerate strongly smooth commuting squares any inclusion of hyperfinite II\(_1\) factors \(Q \subset R\) (as before, strongly smooth commuting square inclusion \((Q \subset R) \subset (N \subset M)\) means that it is non-degenerate and satisfies \(Q' \cap R_n = Q' \cap M_n = N' \cap M_n, R' \cap R_n = R' \cap M_n = M' \cap M_n, \forall n\)).

Similarly, it should be true that if \(G\) is a non-amenable countable rigid \(C^*\)-tensor category, then \(G\) admits an action (by endomorphisms) on a II\(_\infty\) factor \(M^\infty\) which has
no hyperfinite II$_\infty$ subfactors $R^\infty$ with normal expectation that are left invariant by $\mathcal{G}$ (modulo inner perturbations) and on which $\mathcal{G}$ acts freely as a $C^*$-tensor category. In both statements, the obvious candidate for a proof is the canonical inclusion $N^\mathcal{G}(L(\mathcal{F}_\infty)) \subset M^\mathcal{G}(L(\mathcal{F}_\infty))$, from ([P94a]), which has $L(\mathcal{F}_\infty)$ as “initial data” and $\mathcal{G}$ as standard invariant. Note that the resulting factors $N = N^\mathcal{G}(L(\mathcal{F}_\infty))$, $M = M^\mathcal{G}(L(\mathcal{F}_\infty))$ were in fact shown to be isomorphic to $L(\mathcal{F}_\infty)$ in ([PS01]). If one assumes by contradiction that there does exist a hyperfinite inclusion $Q \subset R$ with $\mathcal{G}$ as standard invariant and which can be embedded with strongly smooth commuting square into $N \subset M$ and one takes the associated SE inclusions, then deformation/rigidity arguments in the style of the proofs of 5.1 and 5.2 above should contradict the non-amenability of $\mathcal{G}$. A study case is when $\mathcal{G}$ is the Temperley-Lieb-Jones $\lambda$-lattice $\mathcal{G}_\lambda$ of index $\lambda^{-1} > 4$. One difficulty in proving such a result is that so far (relative) strong solidity results can only say something about normalizers of diffuse AFD von Neumann subalgebras, while in the case of an acting standard $\lambda$-lattice $\mathcal{G}$ (or of an acting rigid $C^*$-tensor category) one generally has to deal with quasi-normalizers (see [BHV15] for related results).

6. Vanishing cohomology and Connes Embedding conjecture

In this section, we’ll show that Connes Approximate Embedding (CAE) conjecture for factors of the form $R \rtimes \Gamma$ can be reformulated as a vanishing cohomology problem for a certain cocycle action of $\Gamma$. Thus, let $\omega$ be an (arbitrary) non-principal ultrafilter on $\mathbb{N}$ and denote by $R^\omega$ the corresponding $\omega$-ultrapower II$_1$ factor, with $R \subset R^\omega$ viewed as constant sequences.

**6.1. Proposition.** 1$^\circ$ $R_\omega := R' \cap R^\omega$ is a II$_1$ factor whose centralizer in $R^\omega$ is equal to $R$, i.e., $R'_\omega \cap R^\omega = R$.

2$^\circ$ Given any $\theta \in \text{Aut}(R)$ there exists a unitary element $U_\theta \in N_{R^\omega}(R)$ such that $\text{Ad}(U_\theta)|_R = \theta$. If $U'_\theta \in N_{R^\omega}(R)$ is another unitary satisfying $\text{Ad}(U'_\theta)|_R = \theta$, then $U'_\theta = vU_\theta = U_\theta v'$ for some $v, v' \in \mathcal{U}(R_\omega)$. Moreover, if $U \in N_{R^\omega}(R)$ and one denotes $\theta = \text{Ad}(U)|_R \in \text{Aut}(R)$, then $U \in \mathcal{U}(R_\omega)U_\theta$.

3$^\circ$ If $\theta, U_\theta$ are as in 2$^\circ$ above, then $\text{Ad}(U_\theta)|_{R_\omega}$ implements an element $\theta_\omega \in \text{Out}(R_\omega)$ and an element $\tilde{\theta}_\omega = \text{Ad}(U_\theta)|_{R \vee R_\omega} \in \text{Out}(R \vee R_\omega)$, with $\theta \in \text{Aut}(R)$ outer iff $\theta_\omega$ outer and iff $\tilde{\theta}_\omega$ outer.

4$^\circ$ The application $\text{Out}(R) \ni \theta \mapsto \tilde{\theta}_\omega \in \text{Out}(R \vee R_\omega)$ is a 1 to 1 group morphism whose image has trivial scalar 3-cocycle, with corresponding cocycle crossed product II$_1$ factor $(R \vee R_\omega) \rtimes \text{Out}(R)$ equal to the von Neumann algebra generated in $R^\omega$ by $R \vee R_\omega$ and $\{U_\theta \mid \theta \in \text{Aut}(R)\}$ (thus equal to $N_{R^\omega}(R)^{''}$ as well).
5° Any free action $\Gamma \smallfrown^\sigma R$ gives rise to a free cocycle action $\tilde{\sigma}_\omega$ of $\Gamma$ on $R \vee R_\omega$, by
$$\tilde{\sigma}_\omega(g) = \text{Ad}(U_{\sigma(g)})|_{R \vee R_\omega}, \ g \in \Gamma,$$
with corresponding 2-cocycle $v^\omega_\omega : \Gamma \times \Gamma \to \mathcal{U}(R_\omega)$.

\begin{proof}
1° This is a particular case of (Theorem 2.1 in [P13]).

2° This is well known (see e.g., [C74]) and is due to the fact that any automorphism of $R$ is approximately inner.

3° Since $U_\theta$ normalizes $R$, it also normalizes its relative commutant $R' \cap R_\omega = R_\omega$, and therefore $R \vee R_\omega$ as well. If the automorphism $\theta_\omega$ it implements on $R_\omega$ is inner, say implemented by some $v \in \mathcal{U}(R_\omega)$, then $v^* U_\theta \in R'_\omega \cap R_\omega = R$, implying that $\text{Ad}(U_\theta)$ is inner on $R$, i.e., $\theta$ is inner. Similarly, if $\text{Ad}(U_\theta)$ is inner on $R$, then it is inner on $R_\omega$. Since $R \vee R_\omega \simeq R \overline{\otimes} R_\omega$ with $\text{Ad}(U_\theta)$ splitting as a tensor product of its restrictions to $R, R_\omega$, one also has that this automorphism is inner iff both restrictions are inner.

4° The $\text{II}_1$ factor $R \vee R_\omega$ has trivial relative commutant in $R_\omega$ and so if we denote by $\mathcal{N}$ the unitaries in its normalizer that leave $R$ (and thus also $R_\omega$) invariant, then $\mathcal{G} = \mathcal{N} / \mathcal{U}(R) \mathcal{U}(R_\omega)$ is a discrete group implementing a cocycle action on $R \vee R_\omega$, with $(R \vee R_\omega) \vee \mathcal{N} \simeq (R \vee R_\omega) \rtimes \mathcal{G}$. Also, from the construction of the map of $\text{Aut}(R) \ni \theta \mapsto U_\theta \in \mathcal{N}$ and part 3°, we see that this map implements an isomorphism $\text{Out}(R) \simeq \mathcal{G}$.

5° This part is trivial from 3° above. \hfill \Box
\end{proof}

6.2. Definition. A $\text{II}_1$ factor $M$ (respectively a group $\Gamma$) has the \textit{CAE property} if it can be embedded into $R_\omega$ (respectively into the unitary group of $R_\omega$). Note that by a result in [R06], $\Gamma$ has a faithful representation into $\mathcal{U}(R_\omega)$ iff $R_\omega$ contains a copy of the left regular representation of $\Gamma$, equivalently $L(\Gamma) \subset \in R_\omega$. Thus, $\Gamma$ has the CAE property iff $L(\Gamma)$ has the CAE property.

6.3. Theorem. Let $\Gamma \smallfrown^\sigma R$ be a free action of a countable group $\Gamma$ on the hyperfinite $\text{II}_1$ factor $R$. The $\text{II}_1$ factor $R \rtimes^\sigma \Gamma$ has the CAE property if and only if the $\mathcal{U}(R_\omega)$-valued 2-cocycle $v_\omega^\sigma$ vanishes, i.e., iff there exist unitary elements $\{U_g \mid g \in \Gamma\} \subset \mathcal{N}_{R_\omega}(R)$ that implement $\sigma$ on $R$ and satisfy $U_g U_h = U_{gh}, \ \forall g, h \in \Gamma$.

\begin{proof}
Let $M = R \rtimes^\sigma \Gamma$ with $\{U_g \mid g \in \Gamma\}$ denoting the canonical unitaries implementing $\sigma$. If $M$ is embeddable into $R_\omega$, then by using the fact that any two copies of the hyperfinite $\text{II}_1$ factor in $R_\omega$ are conjugated by a unitary element in $R_\omega$, it follows that we may assume the hyperfinite subfactor $R$ in $M = R \rtimes \Gamma$ coincides with the algebra of constant sequences in $R_\omega$, with the action $\sigma$ on it being implemented by $\{U_g\}_g \subset M \subset R_\omega$. By Proposition 6.1.5° above, this means the 2-cocycle $v_\omega^\sigma$ vanishes.

Conversely, if $v_\omega^\sigma$ vanishes, then we clearly have $R \rtimes^\sigma \Gamma \hookrightarrow R_\omega$. \hfill \Box
6.4. **Corollary.** Let $\Gamma$ be a countable group and $\Gamma \ltimes^\sigma R_{\mathbb{T}} \Gamma$ the non-commutative Bernoulli $\Gamma$-action with base $R$. Let $H$ be an ICC amenable group (such as the group $S_{\infty}$ of finitely supported permutations of $\mathbb{N}$, or the lamp-lighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$). Then $H \wr \Gamma$ is a CAE group iff $v_\omega^\sigma$ vanishes.

6.5. **Remarks.** 1° It has been shown in [HS16] that if two groups $H, \Gamma$ are sofic, then their wreath product $H \wr \Gamma$ is sofic as well, so in particular it is CAE. Taking $H$ to be an (arbitrary) amenable ICC group $H$, for which by Connes Theorem one has $L(H) \simeq R$, it follows that the crossed product $\Pi_1$ factor $R \rtimes_\sigma \Gamma = L(H \wr \Gamma)$ is CAE, where $\Gamma \ltimes^\sigma R_{\mathbb{T}} \simeq R$ is the non-commutative Bernoulli $\Gamma$-action with base $\simeq R$ as in 6.4. Thus, the corresponding cocycle $v_\omega^\sigma$ vanishes. Equivalently, one can choose $U_{\sigma(g)} \in \mathcal{N}_{R-n}(R)$ so that to be a representation of $\Gamma$. One can in fact show that these unitaries can be taken so that to also normalize the ultrapower of the Cartan subalgebra, i.e., $\{U_g\} \subset \mathcal{N}_{R-n} \cap \mathcal{N}_{R-n}(D_\omega)$, where $D = D_0 \mathbb{T}$, $D_0$ being the Cartan subalgebra of the base.

2° Given any $\Gamma \in \mathcal{VC}$, the wreath product group $S_{\infty} \wr \Gamma$ is CAE by Corollary 6.4 above, and thus it is a CAE group. However, one already knows this, since we have seen in Section 4 that $\mathcal{VC}$ is contained in $W_{\text{eq}}(F_2)$, and $L(F_2) \subset R_\omega$. But while the class $\mathcal{VC}$ has a lot of restrictions on it (cf. Sections 3 and 4 in this paper), the class of CAE groups is manifestly huge, in fact it may well be that all groups are CAE.

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