Instantons and scattering in $\mathcal{N} = 4$ SYM in 4D

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ABSTRACT: We study classical solutions (ic–instantons) in $\mathcal{N} = 4$ SYM in 4D which, in the strong coupling limit, correspond to complex two–dimensional manifolds. Asymptotically in time the latter have boundaries represented by compact real three–manifolds. Therefore they lend themselves to an interpretation in terms of 3–brane scattering. We suggest that these solutions may represent scattering of D3–branes of type IIB theory in 10D. In particular we show that the world–volume theory on complex two–dimensional manifolds is the correct one for D3–branes.

KEYWORDS: ic–instantons, $\mathcal{N} = 4$ supersymmetry, D3–branes, brane scattering.
1. Introduction

In this paper we discuss a new type of classical solutions in $\mathcal{N} = 4$ SYM theory in 4D with $U(N)$ gauge group. This theory is well-known for its self-duality properties $[1]$ and its duality properties with type IIB supergravity (superstring) theory via AdS/CFT correspondence, $[2]$, are still under intensive study. Here we show that there is a (still unexplored) nonperturbative sector of the theory based on a new type of instantons. This is partly parallel to what happens in $\mathcal{N} = (8, 8)$ 2D theory with gauge group $U(N)$, which has been called Matrix String Theory (MST), $[3]$. In MST one finds classical solutions that, in the strong coupling limit, become Riemann surfaces with punctures, which are natural candidates to represent scatterings of closed strings, $[4, 5, 6]$. This idea was confirmed by the subsequent analysis in $[7, 8, 9, 10]$. This led to the identification of the strong coupling limit of MST with perturbative type IIA theory. The solutions in question were called stringy or Riemannian instantons.
Similar classical solutions can be found in other dimensions. In this paper we deal with 4D. In view of these generalizations the world instanton may sound misleading, therefore we will use for these new kind of solutions the term interaction–carrying instantons or simply ic–instantons. The reason we keep calling them generically instantons is due to the analogy with ordinary instantons: just as the latter are thought to represent interpolating solutions between different vacua, we think of ic–instantons as interpolating solutions between given initial and final asymptotic states.

In this paper we construct such ic–instantons in 4D SYM and we suggest that the ic–instantons of \( \mathcal{N} = 4 \) SYM theory with gauge group \( U(N) \), in the strong YM coupling limit, may represent scattering processes involving 3–branes, which we identify as D3–branes of type IIB theory in 10 dimensions. In support of this suggestion we show that ic–instantons at strong coupling describe branched coverings of the 4 dimensional base manifold which we assume to have a complex structure. These branched coverings are complex 2 dimensional surfaces\(^1\) with boundaries which have the correct geometry to describe scatterings of 3–branes. Moreover we show that the world–volume theory on the surface is the correct one for D3–branes. Finally we show that the sum over ic–instantons gives rise to a series weighted by powers of the inverse YM coupling constant, with an exponent given by the Euler characteristics of the corresponding surfaces. The analysis of this last part is largely incomplete and what we report in this paper can only be considered as a preliminary exploration on this subject.

The paper is organized as follows. In section 2 we introduce the notation and derive in various ways the equations for ic–instantons. In section 3 we describe some general properties for ic–instantons. Section 4 is devoted to the explicit construction of such classical solutions; in particular we illustrate the factorization of ic–instantons in a group theoretical factor and a branched covering factor. In the strong coupling limit only the second factor is relevant. In section 5 we discuss some general properties and give a few explicit examples of branched coverings. In section 6 we expand the SYM action about an ic–instanton solution in the strong coupling limit: we find that the dominant part of the action is lifted to the covering surface, say \( \Sigma \), and becomes the action of \( \mathcal{N} = 4 \) free supersymmetric Maxwell theory. We show that the amplitude induced by an ic–instanton corresponding to \( \Sigma \) is proportional to a power of the inverse YM coupling constant whose exponent is the Euler characteristics of \( \Sigma \). This result comes from a counting of zero modes over \( \Sigma \). In section 7 we discuss the relation with Matrix Theory and other problematic or unresolved questions.

2. Interaction–carrying classical solutions

The Minkowski action of \( \mathcal{N} = 4 \) SYM theory in 4D is

\[
S = \int_X d^4x \, \text{Tr} \left( -\frac{1}{4g^2} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2} D_\mu X^i D^\mu X_i + \frac{g^2}{4} [X^i, X^j]^2 + \frac{i}{2} \bar{\lambda} \gamma^\mu D_\mu \lambda \right)
\]

\(^1\)Throughout the paper, by surface without qualifier we mean a complex two–dimensional surface. Whenever we want to indicate a real two–dimensional surface we use the term 'Riemann surface'.

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where \( i = 1, \ldots, 6 \). \( \mathcal{X} \) is a four dimensional manifold of the type \( \mathcal{X} = \mathbb{R} \times M_3 \), where \( M_3 \) is a three–dimensional compact manifold and \( \mathbb{R} \) is the line \(-\infty < x^0 < \infty\). Although the action (2.1) can be studied on more general manifolds, we will consider in the following essentially two examples: \( M_3 = S^3 \), the 3–sphere, and \( M_3 = T^3 \), the 3–torus defined by periodic \( x^1, x^2, x^3 \). We always suppose that \( \mathcal{X} \) admit complex structures.

\( F^{\mu\nu} \) is the field strength of the gauge field \( A_\mu \), the \( X^i \) are \( N \times N \) hermitean matrices in the adjoint of \( U(4) \); from a geometrical point of view, we understand the existence of a vector bundle \( E \), with structure group \( U(4) \), so that \( X^i \) are sections of \( \text{End}E \). \( \lambda \) is an \( N \times N \) matrix whose entries are both Weyl spinors of \( SO(1,3) \) and vectors in the fundamental of \( SU(4) \): namely the \( \gamma^\mu \)'s will act on the \( SO(1,3) \) spinorial indices, while the \( \gamma^i \)'s on the \( SU(4) \) ones. Since we make explicit use of them in the following, we write down our definitions for the gamma matrices:

\[
\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \Gamma^i = \begin{pmatrix} 0 & \gamma^i \\ \gamma^{i\dagger} & 0 \end{pmatrix}
\]

where \( \sigma^0 = -\bar{\sigma}^0 = 1 \), \( \sigma^i = \bar{\sigma}^i \) are the Pauli matrices; the \( \Gamma^i \) are the 8×8 6D gamma matrices as in [11] and \( C \) is the 4D charge conjugation matrix; they satisfy the usual anticommutation relations:

\[
\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \quad \{\Gamma^i, \Gamma^j\} = 2\delta^{ij}.
\]

The supersymmetric transformations are

\[
\delta X^i = \frac{i}{g} \left( \epsilon^T C \gamma^{i\dagger} \lambda - \epsilon^{\dagger} C \gamma^i \lambda^* \right)
\]

\[
\delta A_\mu = -i \left( \bar{\epsilon} \gamma^\mu \lambda - \bar{\lambda} \gamma^\mu \epsilon \right)
\]

\[
\delta \lambda = -\frac{1}{g^2} F^{\mu\nu} \gamma^\mu \gamma^\nu \epsilon - i \left[ X_i, X_j \right] \gamma^{ij} \epsilon^* + \frac{2}{g} D_\mu X_i \gamma^\mu \gamma^0 C \gamma^i \epsilon^*.
\]
of the latter if Weyl/Majorana fermions are involved, which is the case here. There are several recipes to deal with this problem, see [12] and references therein. We will follow [12]: such an approach amounts to an effective doubling of the degrees of freedom of the Euclidean version with respect to the Minkowski one.

With some abuse of language we will call the above solutions BPS solutions, in the sense of supersymmetry preserving solutions. This is substantially motivated by the fact that the final theory on the covering space at strong coupling will turn out to be supersymmetric (see below).

2.1 Ic–instanton equations as BPS solutions

We write first the Euclidean action in terms of the complex coordinates $v = \frac{1}{2}(x^1 + ix^2)$, $w = \frac{1}{2}(x^3 + ix^4)$,

$$S = \int X d^2v d^2w \ Tr \left( D_v X^i D \bar{v} X^i + D_w X^i D \bar{w} X^i - \frac{g^2}{2} [X^i, X^j]^2 \right)$$

$$- \frac{1}{4g^2} (F_{v \bar{v}}^2 + F_{w \bar{w}}^2 - 2F_{vw} F_{\bar{v}\bar{w}} - 2F_{v \bar{w}} F_{\bar{v}w} )$$

$$- 2(\lambda_1 D_{\bar{v}} \lambda_1 + \lambda_2 D_w \lambda_2) - 2(\lambda_1 D_{\bar{w}} \lambda_2 - \lambda_2 D_w \lambda_1 )$$

$$- \frac{g}{2} \left( \lambda^T C \gamma^i \lambda \right)$$

(2.4)

Next we write the Euclidean version of the $\mathcal{N} = 4$ supersymmetric transformations

$$\delta X^i = \frac{i}{g} \left( \epsilon^T C \gamma^i \lambda - \epsilon^\dagger C \gamma^i \lambda^* \right)$$

$$\delta A_\mu = - \left( \epsilon^\dagger \gamma^\mu \lambda - \lambda^\dagger \gamma^\mu \epsilon \right)$$

$$\delta \lambda = - \frac{1}{g^2} F_{\mu \nu} \gamma^{\mu \nu} \epsilon - i [X_i, X_\lambda] \gamma^i \epsilon - \frac{2i}{g} D_\mu X_i \gamma^\mu C \gamma^i \epsilon^*$$

(2.5)

where, according to [12], we consider the variables $\lambda^*$ and $\epsilon^*$ as independent from $\lambda$ and $\epsilon$, respectively. The superscript $^T$ represents the transpose matrix and $^\dagger$ stands for $^{*T}$.

We look for solutions that preserve $\frac{1}{2}$ supersymmetry, by setting all fermions and all $X^i$, with $i = 3, \ldots, 6$, to zero, and defining $X = X^1 + iX^2$ and $\bar{X} = X^{\dagger}$. The equations that define such solutions are

$$F_{v \bar{v}} + F_{w \bar{w}} - ig^2 [X, \bar{X}] = 0$$

(2.6)

$$F_{vw} = 0, \quad F_{\bar{v}\bar{w}} = 0,$$

(2.7)

$$D_v X = 0 = D_{\bar{v}} \bar{X}, \quad D_w \bar{X} = 0 = D_{\bar{w}} X$$

(2.8)

We will refer to the solutions of these equations as ic–instantons. Analogous equations for ic–anti–instantons can be obtained by an anti–holomorphic involution. Similar equations were previously discussed, in the context of $\mathcal{N} = 4$ theory, for compact manifolds by [13].

2.2 Ic–instanton equations from self–duality in 8D

Self–dual YM solutions in 4D are the well–known instantons. Self–duality in 8D for the YM curvature is less known and was studied a few years ago more as a curiosity than with
the real aim at applying it in physical problems, [14, 15, 16, 17, 18]. In components of the curvature the self–duality condition reads:

\[
\begin{align*}
F_{12} + F_{34} + F_{56} + F_{78} &= 0 \\
F_{13} + F_{42} + F_{57} + F_{86} &= 0 \\
F_{14} + F_{23} + F_{76} + F_{85} &= 0 \\
F_{15} + F_{62} + F_{73} + F_{48} &= 0 \\
F_{16} + F_{25} + F_{38} + F_{47} &= 0 \\
F_{17} + F_{82} + F_{35} + F_{64} &= 0 \\
F_{18} + F_{27} + F_{63} + F_{54} &= 0
\end{align*}
\] (2.9)

The anti–self–duality equations are obtained from these by changing the sign of the first entry of each one.

Let us reduce this system to 4D by keeping the dependence on \(x^1, \ldots, x^4\) and dropping the dependence on the remaining coordinates. Let us introduce the complex coordinates \(v = \frac{1}{2}(x^1 + ix^2)\), \(w = \frac{1}{2}(x^3 + ix^4)\), set \(A_7 = A_8 = 0\) and call \(X = A_5 - iA_6\). Then the system (2.9) becomes:

\[
\begin{align*}
F_{v\bar{v}} + F_{w\bar{w}} + i[X, \bar{X}] &= 0 \\
F_{vw} &= 0, \quad F_{\bar{v}\bar{w}} = 0, \quad D_vX = 0 = D_v\bar{X}, \quad D_{\bar{w}}X = 0 = D_{\bar{w}}\bar{X}
\end{align*}
\] (2.10)

After an obvious rescaling, this is the system of equations found above as BPS equations, (2.6,2.7,2.8). The connection of these sets of equations with integrability is under study\(^2\).

In the following section we would like to discuss solutions of (2.10) for which \(X \neq 0\), i.e. ic–instantons.

**2.3 Ic–instanton equations: other derivations and covariant form**

It is worth spending a few more words on the equations (2.6–2.8). We want to show here other ways in which they can be derived. This gives us in particular the opportunity to write them in covariant form. First we notice that they are dimensional reduction of a single equation in 6D Kähler manifold. The latter can be cast in covariant form using the Kähler form \(\omega\):

\[
*F = \omega \wedge F.
\] (2.11)

To see this, just use complex coordinates and write down its components; they can be reexpressed as

\[
\omega \cdot F = 0, \quad F^{(2,0)} = F^{(0,2)} = 0.
\] (2.12)

Now it is very easy to see that the dimensional reduction of this is nothing but equations (2.6–2.8). One can compare this with the situation in MST [6], where the instanton equations (Hitchin’s equations) are dimensional reduction of the single self-duality equation in 4d, which decomposes just as in (2.12).

\(^2\)We acknowledge useful discussions with C.Constantinidis and L.Ferreira on this point
One can take an even more general point of view and look for solutions which preserve some fraction of supersymmetry in ten–dimensional SYM. This yields more general instanton equations. Writing it explicitly in components, in 10d complex coordinates \((z_1 \ldots z_5)\), they look

\[
\sum_{i=1}^{5} F_{z_i \bar{z}_i} = 0; \quad F_{z_i \bar{z}_j} = 0 \quad \forall i < j,
\]

which can be, again, rewritten in covariant form as \(\ast F = \omega^3 \wedge F\). As particular cases, taking some \(A_{z_i}\) vanishing, we find the 8d equation \(\ast F = \omega^2 \wedge F\) and the above mentioned 6d one.

If one dimensional reduces these to the dimension of interest, in this case 4, one has also cases with more than one active scalar:

\[
F_{\bar{v} \bar{w}} + F_{w \bar{w}} - ig^2 \sum_{a=1}^{3} [X_a, X_{\bar{a}}] = 0
\]

\[
F_{v v} = 0, \quad F_{\bar{v} \bar{w}} = 0, \quad [X_a, X_b] = 0, \quad \forall 1 \leq a < b \leq 3
\]

\[
D_{\bar{v}} X_a = 0 = D_v X_{\bar{a}}, \quad D_w X_{\bar{a}} = 0 = D_w X_{\bar{a}}.
\]

These, however, do not represent new solutions as far as the problem we study in this paper is concerned. In fact, anticipating the discussion of the subsequent section, \([X_a, X_{\bar{b}}] = 0\) implies that \(X_a = Y S \tilde{X}_a S^{-1} Y^{-1}\), i.e., all \(X_a\) are diagonalized by the same matrix \(YS\). This entails in particular that they all have the same monodromy. Now, as we will see later, each of these matrices \(\tilde{X}_a\) defines a covering of the base space, and lifts to a holomorphic section of the trivial line bundle over the covering. It follows that the \(X_a\)’s have to be multiple of one another. Therefore we can make a complex linear transformation and go back to the situation with just one active complex scalar.

Apart from this, the above equations open the way to interesting considerations, which are however outside the mainstream of this paper. For this reason we limit ourselves here to some concise remarks. Let us stick in particular to the 6d case. If \(F\) is the curvature of a connection on a vector bundle, the quantity \(\int F \cdot \omega \nu\) (where \(\nu\) is the volume form) is a topological invariant, the degree, which can be thought of as the intersection \([c_1],[\omega^2]\) in homology and naturally generalizes the degree in two dimensions (= \(c_1\)); one may easily show that a holomorphic line bundle admits holomorphic sections iff its degree vanishes. The second equation in (2.12) just means that our connection defines a holomorphic structure on the vector bundle which is integrable; the first one means that the bundle has degree zero. This condition fits into a more general framework. A connection is called Hermitian-Yang-Mills if, for some \(\mu, \omega \cdot F = \mu I_d\). A theorem \([21]\) gives necessary and sufficient conditions for solutions to these equations to exist in terms of a condition of stability. This generalizes the results known in 4d for ASD equations \([19]\), in 2d (Narasimhan-Seshadri) and for Hitchin equations.

2.4 General properties of ic–instantons

The system of equations (2.6,2.7,2.8) has various types of solutions. Notice that, if we set \(X = 0\), (2.10) becomes the usual self–duality condition in 4D. Therefore the set of solutions
of (2.10) will include in particular all the ordinary instants of YM in 4D, compatible with
the topology of the base manifold. In principle we could consider solutions with \( X \neq 0 \)
and nonvanishing instanton number. However the vector bundle \( E \) is such that \( c_2(E) \) is
trivial, therefore we only consider solutions with vanishing instanton number.

Let us now compute the action of a configuration that satisfies (2.10). Starting from
(2.4) we get

\[
S_{\text{inst}} = \frac{1}{2} \int_X d^2v d^2w \, \text{Tr} \left( D_v X D_{\bar{v}} X + D_w X D_{\bar{w}} X + \frac{g^2}{2} [X, \bar{X}]^2 
- \frac{1}{2g^2} (F_{v\bar{v}}^2 + F_{w\bar{w}}^2 - 2 F_{vw} F_{\bar{v}\bar{w}} - 2 F_{v\bar{w}} F_{\bar{v}w}) \right) \quad (2.15)
\]

This can be rewritten as

\[
S_{\text{inst}} = S_{\text{bulk}} + S_{\text{boundary}}
\]

\[
S_{\text{bulk}} = \int d^2v d^2w \, \text{Tr} \left( D_v X D_{\bar{v}} X + D_w X D_{\bar{w}} X + \frac{1}{g^2} F_{vw} F_{\bar{v}\bar{w}}
- \frac{1}{4g^2} (F_{v\bar{v}} + F_{w\bar{w}} - ig^2[X, \bar{X}])^2 \right)
\]

\[
S_{\text{boundary}} = \int d^2v d^2w \, \text{Tr} \left( D_v (X D_{\bar{v}} X) + D_w (X D_{\bar{w}} X) + \frac{1}{4g^2} dK(A_v, A_w) \right) \quad (2.16)
\]

where \( K \) is the Chern–Simons term corresponding to \( F \wedge F \). More explicitly

\[
(F_{v\bar{w}} F_{\bar{v}w} - F_{vw} F_{v\bar{w}} + F_{v\bar{v}} F_{w\bar{w}}) d^2v d^2w = dK(A_v, A_w)
\]

where \( d \) is the exterior derivative in 4D. One sees immediately that for an ic–instanton
\( S_{\text{bulk}} = 0 \). It is well–known that the Chern–Simons term in \( S_{\text{boundary}} \) is equal to the
instanton number. Therefore, since in this paper we only consider solutions with instanton
number 0, the Chern–Simons term does not contribute. The other term in \( S_{\text{boundary}} \) is
usually divergent for the ic–instantons solutions and apparently one cannot attach any
geometric meaning to it. On the other hand it is very easy to get rid of it by simply
saying that our starting action is (2.4), in which the first two terms have been modified to

\[-\frac{1}{2} \left( X^i \{ D_v, D_{\bar{v}} \} X^i + X^i \{ D_w, D_{\bar{w}} \} X^i \right) \]

With these provisos the action of the ic–instantons considered in this paper vanishes. A similar conclusion holds for ic–anti–instantons.

3. Ic–instantons

Our purpose is to find solutions \( (A, X) \) of (2.6,2.7,2.8). For definiteness let us consider a
concrete case, say \( X = \mathbb{R} \times T^3 \). The construction is parallel to the one carried out in [6, 8].
In the following we stick to the complex structure of the punctured sphere \( \mathbb{P}^1 \times T^2 \),
with local coordinates \( v \) and \( w \). At times it is convenient to use the coordinate \( z = e^v \). We
start from the (simple) ansatz

\[
A_v = i \partial_v Y^\dagger (Y^{-1})^\dagger, \quad A_w = i \partial_w Y^\dagger (Y^{-1})^\dagger, \quad X = Y^{-1} MY \quad (3.1)
\]
where $Y$ is a generic element in the complex group $SL(N, \mathbb{C})$ and $M$ specifies a branched covering of the base manifold. A more general ansatz will be considered later on. As a consequence of (3.1) the equations $D_{\bar{v}}X = 0 = D_v \bar{X}$ are equivalent to

$$\partial_v M = 0 = \partial_{\bar{v}} M$$

(3.2)

which means that the matrix $M$ is holomorphic in $v, w$. Eq. (3.2) guarantees that eqs. (2.8) are satisfied.

The ansatz (3.1) is given in terms of two matrices, $Y$ and $M$. $Y$ will be called the group theoretical factor, while $M$ defines a general branched covering of the base manifold, i.e. a two dimensional complex manifold. The factor $Y$ will be discussed below, while branched coverings will be discussed later on. For the time being let us give some essential information. Let us consider the polynomial

$$P_X(y) = \det(y - X) = y^N + \sum_{i=0}^{N-1} y^i a_i,$$

where $y$ is a complex indeterminate. The equation

$$P_X(y) = 0$$

(3.3)

can also be written as the matrix equation

$$X^N + a_{N-1}X^{N-1} + \cdots + a_0 = 0.$$  

(3.4)

A diagonalizable matrix, which is solution of eq. (3.4), can always be cast in the canonical form

$$M = \begin{pmatrix}
-a_{N-1} & -a_{N-2} & \ldots & \ldots & -a_0 \\
1 & 0 & \ldots & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 1 & 0
\end{pmatrix}.$$  

(3.5)

Due to (3.3), we have $\partial_1 a_i = 0 = \partial_{\bar{w}} a_i$, which means that the set of functions $\{a_i\}$ are holomorphic in $v, w$, although they are allowed to have poles at $z = 0$ and $z = \infty$. The point is that, as we shall see in many examples, Eq. (3.3) identifies in the $(y, z, w)$ space a complex 2–manifold (a surface) $\Sigma$, which is an $N$–sheeted branched covering of the base manifold. The explicit form of the covering is given by the set $\{x^{(1)}(z, w), \ldots, x^{(N)}(z, w)\}$ of eigenvalues of $X$. Each eigenvalue spans a sheet. The projection map to the base cylinder $\mathcal{X}$ will be denoted $\pi : \Sigma \to \mathcal{X}$. The divisor (complex 1–submanifold) where two eigenvalues coincide is the branch locus. We can also define branch cuts: they are 3d manifolds that connect disconnected components of the branch locus.

We stress that the covering is independent of the coupling $g$. 

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3.1 Explicit construction of ic–instantons

The aim of the present subsection is to construct the group theoretical factor corresponding to the most general covering. The construction is close to the one in [8], so we will be brief.

Let us recall our ansatz (3.1). The group theoretical factor $Y$ takes values in the complex group $SL(N, \mathbb{C})$, while the matrix $M$ determines the branched covering. The dependence on the Yang-Mills coupling constant $g$ is contained in the $Y$ factor, while $M$ does not depend on $g$. We set $Y = KL$ where $L$, the dressing factor, is expected to tend to 1 in the strong coupling limit outside the branch locus, while $K$ is a special matrix, independent of $g$, endowed with the property that $K^{-1}MK$ and $K^\dagger M^\dagger (K^\dagger)^{-1}$ are simultaneously diagonalizable.

It is well-known, [4], that the matrix $M$ can be diagonalized

$$M = S M S^{-1}, \quad \hat{M} = \text{Diag}(\lambda_1, \ldots, \lambda_N)$$

by means of the following matrix $S \in SL(N, \mathbb{C})$:

$$S = \Delta^{-\frac{1}{N}} \begin{pmatrix} \lambda_1^{N-1} & \lambda_2^{N-1} & \ldots & \lambda_N^{N-1} \\ \lambda_1^{N-2} & \lambda_2^{N-2} & \ldots & \lambda_N^{N-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{pmatrix},$$

where

$$\Delta = \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j).$$

$\Delta$ vanishes whenever two eigenvalues coincide. Two coincident eigenvalues define a component of the branch locus of the covering. Going around a branch locus and crossing a branch cut in the $v, w$–plane, produces a reshuffling of the eigenvalues that can be represented via a monodromy matrix $\Lambda$: $\hat{M} \to \Lambda \hat{M} \Lambda^{-1}$. Correspondingly we have $S \to S \Lambda^{-1}$, so that the single-valuedness of $M$ is preserved.

The explicit construction of $K$ and $L$ in the general case is given in [8] and will not be reported here. The qualitative features are as follows. First one introduces a monodromy–invariant $K$ such that $K^{-1}S = U$ be unitary. To this end one sets $K = \sqrt{SS^\dagger}$ and easily verifies that $U$ is unitary. As it turns out, $K$ may have singularities at the points of $X$ where any two eigenvalues of $M$ coincide, i.e. at the branch locus of the spectral covering (the elements of $K$ contains as factors fractional powers of $|\Delta|$). Therefore $K^{-1}MK$ is in general singular at these points. That is why we must introduce into the game a new monodromy invariant matrix $L$, with the purpose of canceling the singularities of $K^{-1}MK$ in such a way that $L^{-1}K^{-1}MKL$ be smooth and satisfy (2.6, 2.7). Let us denote again by $\phi$ the generic entry of $L$. For (2.8) to be satisfied $\phi$ must satisfy, [8], an equation of the WZNW type with the following general structure

$$(\partial_v \partial_{\bar{v}} + \partial_w \partial_{\bar{w}})\phi + \ldots \sim (\partial_v \partial_{\bar{v}} + \partial_w \partial_{\bar{w}}) \ln |\Delta| = \pi \left( \partial_v \Delta \partial_{\bar{v}} \Delta + \partial_w \Delta \partial_{\bar{w}} \Delta \right) \delta(\Delta),$$

where dots represent all the other terms, which are irrelevant in the cancellation of singularities. In some equations (but not in all) the coefficients in front of the delta–function
terms may vanish. This term has support at the zeroes of \( \Delta \), i.e. at the branch locus. The equation \( F_{vw} = 0 \) in (2.7), on the other hand, does not give rise to delta function terms:
\[
\partial_v \phi + ... = 0 \tag{3.10}
\]

Let us refer to the above equations collectively as the ‘dressing equations’.

By construction \( K \) is independent of \( g \) while \( L \) does depend on \( g \). One can show that in fact \( L \to 1 \) as \( g \to \infty \), outside the zeroes of the discriminant. Let us present a simple argument in this sense.

The solution \( X \) exists with the required properties only if the ‘dressing equations’ admit solutions that vanish at \( v = \pm \infty \) (to this end, of course, we have to exclude possible branch locus components at \( t = \pm \infty \) from the right hand side of eq.(3.9). To our best knowledge, not much is known in the literature concerning the existence of such solutions. Based on the analysis of [7], we assume that the ‘dressing equations’ do admit solutions that vanish at \( v = \pm \infty \). Once one assumes this, it is rather easy to argue, on a completely general ground, that in the strong coupling limit, \( g \to \infty \), such solutions vanish outside the zeroes of the discriminant. The argument goes as follows. Consider a candidate solution of (2.6,2.7) in which \( \phi = 0 \) outside the zeroes of the discriminant, for all the \( \phi \)’s. Then, there, \( L = 1 \), and \( X = K^{-1}MK \). As noted previously, in such a situation \( [X,\bar{X}] = 0 \), since both \( X \) and \( \bar{X} \) are simultaneously diagonalized by the matrix \( U = K^{-1}S \). Now we have to show that also \( F_{\bar{v}v} \) and \( F_{\bar{w}w} \) vanish outside the zeroes of the discriminant if \( L = 1 \). In fact when \( L = 1 \),
\[
A_{\bar{v}} = -iK^{-1}\partial_{\bar{v}}K = -i(K^{-1}SS^{-1})\partial_{\bar{v}}(SS^{-1}K) = -iU(\partial_{\bar{v}} + \bar{A}_{\bar{v}})U^{-1},
\]
where \( \bar{A}_{\bar{v}} = S^{-1}\partial_{\bar{v}}S \). But \( \partial_{\bar{v}}S \equiv 0 \) due to holomorphicity of the eigenvalues of \( M \). Therefore \( F_{\bar{v}v} = 0 \). The same can be done for \( A_{\bar{w}} \), therefore \( F_{\bar{w}\bar{w}} = F_{\bar{v}v} = 0 \). In conclusion (2.6,2.7) is identically satisfied by the ansatz \( L = 1 \) outside the zeroes of the discriminant.

Since the solutions are uniquely determined by their boundary conditions, we can conclude that, as \( g \to \infty \), the only solution of the dressing equations outside the zeroes of the discriminant, is the identically vanishing solution. We infer from this argument that the solutions of the dressing equations for large \( g \) are concentrated around the branch locus and become more and more spiky as \( g \) grows larger and larger. Therefore the matrix \( L \) has the properties we expect.

The previous argument hinges on the occurrence that, as \( g = \infty \), we have both \( [X^\infty,\bar{X}^\infty] = 0 \) and \( F_{\bar{v}v}^\infty = F_{\bar{w}\bar{w}}^\infty = 0 \) (the superscript \( \infty \) obviously represents the strong coupling value of a field). Other types of solutions can be envisaged, see below and [8].

### 3.2 Generalized ic–instantons

In this paper we will have to take into consideration more general solutions than those just studied. Instead of (3.1) let us start from
\[
A_v = i\mathcal{D}_v Y^{\dagger}(Y^{-1})^{\dagger}, \quad A_w = i\mathcal{D}_w Y^{\dagger}(Y^{-1})^{\dagger}, \quad X = Y^{-1}MY \tag{3.11}
\]

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where \( Y \) is as before and the covariant derivative \( D \) is relative to a connection \( A \) which commutes with \( M \). As a consequence of this, we have again that the equations \( D_v X = 0 = D_v \tilde{X} \) imply (3.2). Moreover, the connection \( A \) is diagonalized by \( S \) and

\[
A = S \hat{A} S^{-1}
\]

Since any solution \( A, X \) must be smooth, the monodromy of \( \hat{A} \) must be the same as the monodromy of \( \hat{M} \), i.e. going around a branch locus produces a reshuffling of the eigenvalues that can be represented via the same monodromy matrix \( \Lambda: \hat{A} \rightarrow \Lambda \hat{A} \Lambda^{-1} \).

The construction of such ic-instantons carries through as before. The only remarkable difference is that in the strong coupling limit the connection \( A^\infty = U \hat{A} U^{-1} \) does not evaporate into a pure gauge as before. Not only do we have a covering described by \( \tilde{X} \), but also a connection \( \hat{A} \) valued in the Cartan subalgebra.

In the strong coupling limit, instead of (2.6), in this case we find

\[
\left[ X^\infty, \tilde{X}^\infty \right] = 0
\]

\[
F^\infty_{\bar{v}v} + F^\infty_{\bar{w}w} = 0
\]  

(3.12)

i.e. we obtain a non-trivial self-dual connection. Of course we can do the same with anti-self-dual ic-instantons and obtain strong coupling antiself-dual connections.

An important proviso: a basic condition for us to call all the above solutions ic-instantons is that \( [X, \tilde{X}] \neq 0 \) for finite \( g \): only in this case do they represent interpolating solutions between genuine initial and final brane configurations (see below).

4. Spectral coverings and scattering

In the previous section we have seen that in the strong coupling limit any ic-instanton reduces to a branched covering of the base manifold \( \mathcal{X} \). In this section we analyze a few general facts and examples of branched coverings of 4-manifolds (without any illusion of completeness). We will see that, in parallel to what happens in MST, such coverings may describe scatterings of D3-branes. The idea is simple. The ic-instantons present in our theory describe four-manifolds of various topologies, which cover various base spaces; the latter are topologically of the form \( \mathbb{R} \times M_3 \), and so we may define slices of the covering at constant time. In general our ic-instantons at \( t = -\infty \) are represented by a disjoint union of 3-manifolds of various topology and at \( t = +\infty \) by another (in general different) disjoint union. Now, we interpret the \( t = -\infty \) configuration as a set of incoming 3-branes and the \( t = \infty \) one as a set of outgoing 3-branes. Any ic-instanton interpolates between two such asymptotic configurations. If we want to describe a given scattering process we will choose, among all the ic-instantons, those with the given asymptotic structure, i.e. whose slices at \( t \to \pm\infty \) correspond to the assigned unions of 3-manifolds.

We are therefore faced with two classification problems: 1) given a base manifold \( \mathcal{X} \), classifying all possible branched coverings; 2) analyzing the effect of a change of base manifold.

From a path-integral point of view it is clear from the example of MST that we have to sum over all the branched coverings of 1), which means a discrete sum over topologies.
and an integration over moduli. We will argue later on that we have perhaps to sum also over different base manifolds.

In this paper we will actually limit ourselves to analyzing two special cases of scattering topology, with the purpose of illustrating these two problems. Let us, however, point out first a general result. We will denote by $\Sigma$ the complex surface associated to a given instanton and by $\pi : \Sigma \to \mathcal{X}$. As we will see later on any instanton relevant for a given process will contribute to the path integral a term proportional to $g^{-\chi}$, where $\chi$ is the Euler characteristic of the 4-manifold $\Sigma$. To compute $\chi$, we can triangulate the manifold in such a way that it gives a triangulation of the branch locus as well. Doing so we obtain the result

$$\chi_{\Sigma} = N\chi_{\mathcal{X}} - \sum (r_i - 1)\chi_{R_i}, \quad (4.1)$$

where $r_i$ are the ramification orders at the branch loci $R_i$, $N$ is the order of the covering and $\mathcal{X}$ is the base. We recall that in our case $\chi_{\mathcal{X}} = 0$.

### 4.1 Scattering of $S^3$–branes

The natural choice of base space is in this case $\mathbb{R} \times S^3$. In this case, the complex structure we can take is obviously $\mathbb{C}^2 - \{(0,0)\}$. In terms of the complex coordinates $(v, w)$ introduced above the time is given by $e^t = \sqrt{|v|^2 + |w|^2}$. The characteristic polynomial depends on both coordinates; the branch locus is a curve in $\mathbb{C}^2 - \{(0,0)\}$, given by some equation $\Delta(v, w) = 0$, and its slices at constant time are generically unions of $S^1$.

Let us consider first the asymptotics at $t = -\infty$. In a generic situation we expect the inverse image of $S^3$ under $\pi$ to be a disjoint union of $N$ copies of $S^3$. But of course we are interested in less trivial asymptotic configurations. This is so if the point at $t = -\infty$ belongs to the branch locus. In such a situation we can use, for instance, the well-understood theory that relates germs of plane curves (i.e. local forms of equations in $\mathbb{C}^2$) to knots and links embedded in a small $S^3$ around the origin [22]. Let us review some of those results. First of all, it is obvious that if the equation $\Delta = 0$ has a constant term, a small sphere does not intersect the branch locus. Apart from this trivial case, the rule is that each component of $\Delta$ as a polynomial corresponds to an $S^1$, and the multiplicity of intersection of two components is exactly the linking number of the corresponding $S^1$. As for each single component, its local Puiseux expansion encodes the knot type of the corresponding $S^1$.

Once we have understood the structure of the branch locus $R$, given that the number of sheets of the covering is $N$, we need to know the action of the first homotopy group $\pi_1(S^3 - R)$ on the discrete fiber (the monodromy). From these data we can reconstruct topologically the covering space. Suppose, for definiteness, that the covering is totally branched along a knot, and the monodromy is the generator of the cyclic group $\mathbb{Z}_N$ (cyclic covering). An easy case we may analyze is that in which the knot is trivial; think indeed the base $S^3$ as $\mathbb{R}^3 \cup \infty$, choose a line in it as branch (from the point of view of $S^3$ it is a circle) and construct the branched covering as usual, attaching in sequence $N$ copies of $\mathbb{R}^3$ along the $S^1$ (this is the same as the usual picture of a 2d branched covering, translated along one more spatial direction). The covering space is again an $\mathbb{R}^3$ with a point at infinity, so it is an $S^3$ as well.
To describe more complicated cases, there is a beautiful theory relating branched coverings, knots and surgeries \[20\]. Surgery is a technique that allows one to obtain any 3–manifold from a sphere, cutting a solid 2–torus constructed along a knot, and regluing it in a different way. By applying it to the simple branched covering \(S^3 \to S^3\) we just described, one may induce other branched coverings \(M_3 \to S^3\), branched along knots. In fact, one may show that any \(M_3\) can be obtained as a covering of \(S^3\), with branch along a knot. The problem is, however, that the knots obtainable as branches from complex geometry are not of general type; they are called itorated torus knots. Therefore we see that the base \(\mathbb{R} \times S^3\) may give rise to many different topologies at \(t = -\infty\), but we have no guarantee that it gives rise to any desired topology for the incoming branes.

In some cases, there is a further method to understand the topological structure of the scattering surface. If the equation of the covering \(p(y, v, w) = 0\) is homogeneous in \((y, v, w)\), and if the coefficient of \(y^N\) is 1, we may map homeomorphically the solutions \(
abla |v|^2 + |w|^2 = c, p = 0\) above the 3–sphere to solutions \(
abla |y|^2 + |v|^2 + |w|^2 = c, p = 0\), exploiting the fact that \(p(y, v, w) = 0 \Leftrightarrow p(\lambda y, \lambda v, \lambda w) = 0\). Think of this as the stereographic projection from a cylinder to the sphere inscribed in it. In this case it is simpler to understand the topology: it is the intersection of a homogeneous equation in \(\mathbb{C}^3\) with the sphere \(S^5\). The homogeneous equation can be read as an equation in \(\mathbb{P}^2\); \(S^5\) can be thought of as the \(U(1)\) bundle inside \(\mathcal{O}_{\mathbb{P}^2}(-1)\), and so we get in this case that our brane has the topology of a \(S^1\) bundle over a Riemann surface (it is, in fact, just the \(S^1\) inside the line bundle which embeds the Riemann surface in \(\mathbb{P}^2\)).

For quasi–homogeneous equations, a similar projection can be done; an analysis in terms of weighted projective spaces is however less straightforward, and one has to resort to other methods, \[27\].

The analysis carried out so far only concerns asymptotic branes (the \(t = +\infty\) case is analogous to the \(t = -\infty\) one). At finite time it is still true that, given the structure of the branch locus, we can single out the intermediate configurations but the analysis is in general more difficult. In general, what happens is that the initial branes will join and split in branes of the same or different topologies. As an example, consider the polynomial \(y^N = v^2 - w^2 - e^{t_0}\). The branch locus is absent for \(t < t_0\), and an unknotted \(S^1\) for \(t > t_0\). So, by the above discussion, this describes \(N\) spheres which join to form one.

Up to now, we tried to describe a scattering of spheres by considering the most natural base \(\mathbb{R} \times S^3\), and we found scattering states of very general topology. But to be complete, we should describe other bases, and see whether there are instantons with spheres as asymptotic structure. As we said, this means that there should be a covering \(S^3 \to M_3\) coming from the restriction at fixed time of a complex branched covering. Just as the base \(\mathbb{R} \times S^3\) contributes to scattering of all manifolds, other bases may contribute to the scattering of \(S^3\)'s. One would have to extend the theory we cited above \[20\] to base 3–manifolds different from \(S^3\). We will not try this here.

4.2 Scattering of \(T^3\)–branes.

Also in this case we begin with a base space \(\mathbb{R} \times T^3\). There are indeed complex structures on it: think of this base as \(\mathbb{R}^4/\Lambda\), where \(\Lambda\) is a 3–lattice spanned, say, by \(v_1, v_2, v_3\). Now
take the standard complex structures on $\mathbb{R}^4 = \mathbb{C}^2$: we obtain, varying $v_i$, different complex structures. To be more precise, they are really different only modulo the action of $GL(2, \mathbb{C})$ and modular transformations. This general setting, however, yields results not different from those obtained by taking the simplest choice among them: thinking of the base space as $(\mathbb{P}^1 - \{0, \infty\})$ times an elliptic curve $C$. We call $z$ the coordinate on the first factor (time is given by $e^t = |z|$) and $w$ the one on $C$.

The coverings are defined by the characteristic polynomial $P_X$, whose coefficients $a_i$ are holomorphic in $z$ and $w$ by 3.2. As functions of $z$, they are just meromorphic functions on $\mathbb{P}^1$, with poles in the excluded points 0 and $\infty$; as functions of $w$, they are constant. The resulting coverings are very simple: for each fixed $z$, the covering space is nothing but a disjoint union of 2-tori (the eigenvalues are constant in $w$); so the process is of the type $(\text{scattering of strings}) \times T^2$, where the first factor is exactly what was already examined in MST [6, 7, 8]. This simply means that 3-tori are really scattering just along one of their dimensions. Since in this case the branches $B_i$ are tori, by (4.1) these processes all have $\chi = 0$, and therefore contribute only to the zero-th order term in $1/g$ in the path integral.

This is the simplest possibility; but again we have to consider contribution from other bases, with branched or unbranched coverings $T^3 \to M_3$. The first idea is to use $\mathbb{R} \times S^3$, which we have considered above. We are not sure that there is actually a covering yielding $T^3$, the technique to construct it would be to analyze coverings along iterated torus knots, coming from equations $\Delta(v, w) = 0$, and then construct a holomorphic covering having $\Delta$ a discriminant. Similar analysis should be done for other bases.

5. Expansion about a classical solution

Our purpose in this section is to expand the action about a classical ic–instanton solution. For definiteness we choose an instanton rather than an anti–instanton, but everything can be repeated for the latter. The analysis is along the lines of [7], but there are important differences which we will try to emphasize while going rapidly through the repetitive aspects.

As a first step let us analyze the background part. The dependence on the coupling is entirely contained in the factor $L$. We have seen that in the strong coupling limit $L \to 1$ outside the branch locus of the covering. Since here we are interested in expanding the action (2.4) in inverse powers of $1/g$, and actually in singling out the dominant term in this expansion (see below), we will consider the action (2.4) around a given classical solution stripped of the above dressing factor, and exclude from the integration region the branch locus on the base manifold, $\mathcal{X}$. In other words we will consider from now on the action (2.4) in which the relevant $Y$ is replaced by $K$ and the integral extends over $\mathcal{X}_0$ which is the initial $\mathcal{X}$ from which small tubular neighborhoods have been cut out around the branch locus. Said otherwise, we introduce in our integrated action a regulator (which will eventually be removed).

After getting rid of the dressing factor, the classical background configuration is specified by $X^{\infty}$ and $A^{\infty}$ (see section 3.2). As expected, this configuration is singular exactly at the branch locus. We have seen that $M = \tilde{M} S^{-1}$. $\tilde{M}$ is the matrix of eigenvalues of $M$
and of $X$, so we denote it equivalently by $\hat{X}$. In the strong coupling limit $X \to U \hat{X} U^{-1}$, where $U = K^{-1} S$ is a unitary matrix and therefore simultaneously diagonalizes $X$ and $\hat{X}$. Corresponding to $\hat{X}$ we have $\hat{A}_v, \hat{A}_w$.

$U$ is finite in $\mathcal{X}_0$. Therefore, with a gauge transformation, we can remove it from the action defined in $\mathcal{X}_0$. This leads us to

- $\hat{X}$ and $\hat{A}$ diagonal,

for the classical background in the strong coupling limit.

Let us return now to the bosonic action (2.4) (the fermionic part will be analyzed later on) with the above understanding of the background part. To extract the strong coupling effective theory, we first rewrite the action in the following useful form

$$S^{(b)} = \int d^2 \nu d^2 w \text{Tr} \left( D_v X^I D_v X^I + D_w X^I D_{\bar{w}} X^I - \frac{g^2}{2} [X^I, X^J]^2 - g^2 [X^I, X] [X^I, \bar{X}] \right)$$

$$+ D_v \bar{X} D_v X + D_w \bar{X} D_{\bar{w}} X + \frac{1}{g^2} F_{\nu \bar{w}} F_{\nu \bar{w}} - \frac{1}{4 g^2} (F_{\nu \bar{w}} + F_{w \bar{w}} - i g^2 [X, \bar{X}])^2, $$

where $I = 3, \ldots, 6$. We now expand the action around a generic ic–configuration as follows

$$\Phi = \Phi^{(b)} + \phi^i + \phi^n \equiv \Phi^{(b)} + \phi \equiv \Phi^0 + \phi^n, \quad (5.1)$$

where $\Phi^{(b)}$ is the background value of the field at infinite coupling, $\phi^i$ are the fluctuations along the Cartan directions and $\phi^n$ are the fluctuations along the complementary directions in the Lie algebra $u(N)$. In the following we suppose we have carried out the operation described above and by background value we refer to the diagonal representation.

The expansion of the action starts with quadratic terms in the fluctuations and $\hat{A}$ drops out from all the terms, except from the kinetic energy term of the (diagonal) Yang–Mills field. To simplify the subsequent formulas we will drop $\hat{A}$ for the time being and resume it later on.

To proceed further let us fix the gauge. We use, in the strong coupling limit, the following gauge–fixing term

$$S_{gf} = \frac{1}{4 \pi g^2} \int d^2 \nu d^2 w \text{Tr} \mathcal{G}^2 \quad (5.2)$$

where

$$\mathcal{G} = D^\circ_\nu a_{\bar{v}} + D^\circ_{\bar{v}} a_\nu + D^\circ_\nu a_{\bar{w}} + D^\circ_{\bar{w}} a_\nu + ig^2 ([X^\circ, \bar{x}] + [\bar{X}^\circ, x]) + 2ig^2 [X^{\circ I}, x^I], \quad (5.3)$$

and $D^\circ$ is the covariant derivative with respect to $A^\circ$. Next we introduce the Faddeev–Popov ghost and antighost fields $c$ and $\bar{c}$ and expand them like all the other fields and add to the action the corresponding Faddeev–Popov ghost term

$$S_{\text{ghost}} = -\frac{1}{2 \pi g^2} \int d^2 \nu d^2 w \text{Tr} \left( c \frac{\delta \mathcal{G}}{\delta c} \right), \quad (5.4)$$

where $\delta$ represents the gauge transformation with parameter $c$. 

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At this point, to single out the strong coupling limit of the action, we rescale the fields in appropriate manner. Precisely, we redefine our fields as follows

\[ A_v = g a_v^l + \alpha_v^n, \quad A_w = g a_w^l + \alpha_w^n, \quad X = \hat{X} + x^l + \frac{1}{g} x^n, \quad \tilde{X} = x^l + \frac{1}{g} x^n \]

and likewise for the conjugate variables. For the ghosts we set

\[ c = gc^l + \sqrt{g} c^n, \quad \tilde{c} = gc^l + \frac{1}{\sqrt{g}} c^n. \]

After these rescalings the action becomes

\[ S^{(b)} = S^{(b)}_{sc} + S^{(b)}_n + o \left( \frac{1}{\sqrt{g}} \right), \]

where

\[
S^{(b)}_{sc} = \int_{\chi_0} d^2 v d^2 w \, \text{Tr} \left[ \partial_v x^l \partial_v x^l + \partial_w x^l \partial_w x^l + \partial_v x^l \partial_v \bar{x}^l + \partial_w x^l \partial_w \bar{x}^l + \partial_v c^l \partial_v c^l + \partial_w c^l \partial_w c^l + \partial_v a^l \partial_v a^l + \partial_w a^l \partial_w a^l + \partial_v a^l \partial_v a^l + \partial_w a^l \partial_w a^l \right] \quad (5.5)
\]

\[ S^{(b)}_n \] is the purely quadratic term in the \( \phi^n \) fluctuations. Let us see this in detail. \( S^{(b)}_n \) has the form

\[ S_n = \int d^2 v d^2 w \, \text{Tr} \left[ \bar{x}^n Q x^n + x^n Q x^n + a_v^n Q a_v^n + a_w^n Q a_w^n + \bar{c}^n Q c^n \right], \quad (5.6) \]

where

\[ Q = \text{ad} \chi^* \cdot \text{ad} \chi^* + \text{ad} a_v^l \cdot \text{ad} a_v^l + \text{ad} a_w^l \cdot \text{ad} a_w^l + \text{ad} x^l \cdot \text{ad} x^l \]

There are no zero modes involved; therefore the integration gives a certain power of the determinant of \( Q \). This has to be compared with the fermionic part of the action. So let us look at the latter. After the rescaling \( \lambda = \lambda^l + \frac{1}{\sqrt{g}} \lambda^n \), we have analogously

\[ S^{(f)} = S^{(f)}_{sc} + S^{(f)}_n + o \left( \frac{1}{\sqrt{g}} \right) \]

where

\[
S^{(f)}_{sc} = \int_{\chi_0} d^2 v d^2 w \left[ -2 \left( \lambda_1^* \partial_v \lambda_1^1 + \lambda_2^* \partial_v \lambda_2^1 \right) - 2 \left( \lambda_1^* \partial_w \lambda_2^1 - \lambda_2^* \partial_w \lambda_1^1 \right) \right] \quad (5.7)
\]

The fermionic off–diagonal fluctuations contribute quadratically in the following way. We arrange the \( \lambda_\alpha^n \) and \( \lambda_\alpha^{*n} \) in a unique “spinor” \( \psi^{nT} = (\lambda_1^n, \lambda_2^n, \lambda_1^{*n}, \lambda_2^{*n}) \),

\[ S^{(f)}_n = \int d^2 v d^2 w \, \psi^{nT} A \psi^n, \quad (5.8) \]
where

\[
\mathcal{A} = \begin{pmatrix}
0 & \gamma^i \text{ad}_{X_i^0} & -i \text{ad}_{a^t_{b}} & -i \text{ad}_{\bar{a}^t_{w}} \\
-\gamma^i \text{ad}_{X_i^0} & 0 & i \text{ad}_{a^t_{b}} & -i \text{ad}_{a^t_{t}} \\
-i \text{ad}_{a^t_{b}} & i \text{ad}_{a^t_{b}} & 0 & -\gamma^i \text{ad}_{X_i^0} \\
-i \text{ad}_{\bar{a}^t_{w}} & -i \text{ad}_{\bar{a}^t_{w}} & \gamma^i \text{ad}_{X_i^0} & 0
\end{pmatrix}
\]

(5.9)

Now let us observe that the components of this matrix commute with respect to the action of the adjoint, and to the $SU(4)$ indices, so that we can directly compute the determinant looking at it as a $4 \times 4$ matrix. Taking into account the $SU(4)$ and Lorentz indices, we get

\[\text{Det}\mathcal{A} = (\text{Det}Q)^8\]

(5.10)

As this is precisely the determinant provided by the path integration on fermions, we now have to compare it with the bosonic one. This last turns out to be $(\text{Det}Q)^{-8}$, obtained counting 6 scalars plus 4 gauge bosons minus 2 ghosts, and taking into account that the number of bosons too has been doubled, as an effect of the Wick rotation. So the final net contribution of the $n$ fields to the partition function is 1.

As it was pointed out in [7], each separate entry of the diagonal matrix fields appearing in (5.5) is not a true free field, as it is not single–valued. However each diagonal matrix field defines a unique (single–valued) field on the covering surface $\Sigma$ of $X$ (see Appendix). For example the matrices $x^I$ represent scalar fields $x^I$, the matrix $a^t$ represents a one–form field $a$ on $\Sigma$ and so on. A boldface letter will be henceforth the hallmark of a well–defined bosonic field on $\Sigma$. As for $\lambda$ its global existence on $\Sigma$ understands that the latter is a spin manifold.

In conclusion the strong coupling theory represents a free $U(1)$ gauge theory with matter on $\Sigma$:

\[S = \frac{1}{2} \int_{\Sigma} d^4\xi \left( \frac{1}{2} \partial_{\mu} x^i \partial^{\mu} x^i + \frac{1}{2} \partial_{\mu} a^t_{b} \partial^{\mu} a^t_{b} + \frac{1}{2} \partial_{\mu} \bar{c} \partial^{\mu} c - \frac{1}{2} \lambda^I \gamma_{\mu} \partial_{\mu} \lambda \right)\]

(5.11)

where $\xi$ are local coordinates on $\Sigma$ (for example, $z$ and $w$). The expression of the strong coupling (5.11) is only symbolic. It is in fact strictly valid only if $\Sigma$ is a flat manifold, in which case we recover full $N = 4$ supersymmetry. But of course in general $\Sigma$ will not be flat. In the non–flat cases (5.11) will only hold outside a neighborhood of the ramification locus, in which the curvature is concentrated. The problem of course is not how to extend the action (5.11) in such a way as to incorporate a non–trivial metric, which is straightforward, but rather how to do it in a supersymmetric way, so as to obtain an $\mathcal{N} = 4$ supersymmetric theory. This problem is analogous to the covariant formulation of Green–Schwarz superstring theory on a generic Riemann surface, met in MST. The difficulty of such problems stems from the fact that, at first sight, it would seem inevitable to introduce supergravity on the world–volume in order to guarantee supersymmetry. However this is not necessary. In fact both these problems, as well as other similar problems concerning D–brane actions embedded in space–time, have been solved using the superembedding principle. An essential role is played by $\kappa$ symmetry, and the above mentioned difficulty
is overcome by pulling back the (possibly trivial) metrics and gravitinos from the ambient space, which are therefore non-dynamical. All this fits very well in our approach, and we limit ourselves to relying on the literature: the action will be an extension of (5.11) to include the branch locus – possibly substituting the SYM action with the corresponding DBI one. In our specific case we have in mind [28] (for a review of this and related problems, see [29]).

We remark that (5.11) contains the fields which are expected to live on a D3–brane and it is itself the low energy and low curvature action for a D3–brane. We will further comment on it later.

In (5.11) the gauge coupling constant is 1. However, as shown in [7], in the path integral there is a non–trivial dependence on the original gauge coupling \(g\) which is due to the integration over the zero modes. For our previous rescaling of the various fields by powers of \(g\) involves, in particular, a rescaling of both the gauge and ghost diagonal degrees of freedom. When defining the path integral we have to take this fact into account, which amounts to rescaling it by an overall factor for any given instanton. This factor is a power of \(g\), the exponent being the number of zero modes for each rescaled field with the appropriate sign. It would seem therefore that we have to count the number of ghost and gauge zero modes. However this would lead us to a wrong result for the reason explained below.

### 5.1 Summing over line bundles

Eq. (5.11) does not tell the whole story. In fact in the previous subsection we have dropped the diagonal connection \(\hat{A}\). Reintroducing now this connection amounts to replacing \(a\) with \(A + a\), where \(A\) is a non–trivial self–dual or anti–self–dual connection. Since self–dual and anti–self–dual instantons lead to the same coverings, when selecting a definite interpolating surface \(\Sigma\) (to represent a given scattering process) we have to allow for (i.e. to sum over) all the ie–instanton solutions that contain such a surface as a covering, both self–dual and anti–self–dual, and with all the possible non–trivial connections \(A\). These are line–bundle connections (it is useful to clarify that the fluctuation \(a\) is a 1–form: added to a line bundle connection it supplies another connection; in the treatment of the previous subsection it was supposed to be added to the 0 connection, i.e. to represent a connection in the trivial line bundle over \(\Sigma\); in turn the fluctuations \(x^i\) as well as all the other fluctuating fields are section of trivial line bundles). In conclusion we have to sum over all line bundles on \(\Sigma\) and integrate over all the connections in each line bundle.

There is another way one can view the same problem: we have to admit on \(\Sigma\) any line bundle whose direct image under \(\pi\) coincides with the initial vector bundle \(E\) on \(\mathcal{X}\). The construction is, roughly speaking, as follows. In a covering with \(N\) sheets, any line bundle \(L\) generates an \(N\)–component ‘vector’ on \(\mathcal{X}\): its components are just the \(N\) lines that lie over the same point of \(\mathcal{X}\). Therefore to any line bundle over \(\Sigma\) there corresponds a vector bundle \(E\) over \(\mathcal{X}\). The Chern classes \(c_1(E), c_2(E)\) are connected to the Chern class of \(L\) via the Grothendieck–Riemann–Roch theorem, but, in the case of a noncompact manifold like \(\mathcal{X}\), these constraints may become irrelevant.
A clarification is in order concerning $c_1(E)$ ($c_2(E)$ is trivial, therefore $c_2(E)$ does not need a comment). A non–trivial first Chern class on a brane world–volume is usually interpreted as the signal of the presence of a membrane. According to our interpretation membranes are not present in the theory, and $c_1(E)$ is a pure geometrical feature of the base manifold, which must be considered on the same footing as the complex structure and the like. It is only if we sum over all non–trivial $c_1(E)$’s on the base that we are allowed to sum over all the line bundles on the covering.

The correspondence we have just outlined is described in more detail and with more appropriate language in Appendix. Summarizing, the spirit of our approach implies that we have to allow for anything in $\Sigma$ can be lifted from $\mathcal{X}$, or, equivalently, for anything in $\Sigma$ can be projected down to something that lives in $\mathcal{X}$. Therefore, on $\Sigma$, we have to allow for all possible non–trivial line bundles. The path integral must include the sum over such line bundles over $\Sigma$, as well as the path integration over all the connections on such line bundles.

A path integration with sum over all line bundles in a Maxwell theory has already been carried out in [30], see also [31], and we follow this calculation. For the sake of clarity we partially reproduce it here.

The trick consists in passing to a dual formulation by introducing auxiliary fields and enlarging the gauge symmetry. Given a connection $A$ on a line bundle $L$, one first introduces the auxiliary field $G_{\mu \nu}$, and requires the theory to be invariant under an extended gauge symmetry whose local version is

$$A \rightarrow A + \Omega, \quad G \rightarrow G + d\Omega$$  \hspace{1cm} (5.12)

where $\Omega$ is a local one–form. In addition (global version) one requires that $G$ be defined up to closed two–forms. This is tantamount to asking that the integrals of $G$ over two–cycles be defined up to integers. Then, if $F$ is the curvature of $A$, one defines the combination $\tilde{F}_{\mu \nu} = F_{\mu \nu} - G_{\mu \nu}$. $\tilde{F}$ is clearly invariant under the generalized gauge transformation (5.12) because $F$ integrated over a two–cycle gives an integer.

Now one considers the series of dual line bundles $\tilde{L}$ with dual connection $V_{\mu}$ and curvature $W_{\mu \nu}$, and writes the action

$$I = \frac{1}{2} \int_{\Sigma} d^4 \xi \sqrt{h} \left( \frac{i}{4\pi} \epsilon^{\mu \nu \lambda \rho} W_{\mu \nu} G_{\lambda \rho} + \tilde{F}^{\mu \nu} \tilde{F}^{\mu \nu} + \tilde{F}^{\mu \nu} \tilde{F}^{\mu \nu} \right)$$  \hspace{1cm} (5.13)

where $+$ and $-$ denotes the self–dual and anti–self–dual part of a two form, respectively. There are two alternatives. On the one hand, integrating over $V$ one obtains that $dG = 0$ and $G$ has integral periods, which allows us to set $G = 0$, in view of (5.12); in this way we get back the original lagrangian for $A$. On the other hand, using (5.12) one can simply set $A = 0$, and end up with the dual formulation, where the basic fields are the connection $V$ and the two–form $G$.

5.2 Counting zero modes

The dual formulation has the virtue of transforming the discrete summation over line bundles into an integration over continuous fields. Now we can see that the relevant zero
modes are those of the one–forms \( V \) together with the corresponding ghosts (the dual of \( c \)), and the two forms \( G \). Looking at (5.13) and at the definition of \( \mathcal{F} \) one sees that \( V \) rescales inversely with respect to \( A \), while \( G \) rescales in the same way. Therefore the zero modes of \( G \) and the ghosts of \( V \) will contribute with the same sign, while the zero modes of \( V \) will contribute with the opposite sign to the overall factor in front of the path integral. We are now ready to compute the latter.

Let us recall that our surface \( \Sigma \) is a two–dimensional complex variety with \( n \) punctures (see section 4). If it were a compact surface we would say that there are \( 2b_1 \) zero modes of \( V \), two zero modes of the ghosts and \( b_2 \) zero modes of \( G \), where \( b_1, b_2 \) are Betti numbers of \( \Sigma \). In conclusion we would get an overall factor \( g^{2b_1-b_2-2} \). Now since \( 2 - 2b_1 + b_2 \) is the Euler characteristics of a compact 4D manifold, and since a puncture takes away one unit of Euler characteristics (as can be seen for example by triangulating the manifold), we are led to the conclusion that the overall factor in the presence of \( n \) puncture is \( g^{-\chi} \) where \( \chi = 2 - 2b_1 + b_2 - n \). This is the correct result and there are several ways one can convince oneself of it. The easiest one is probably by use of a doubling construction, as in [7]. Let us first make more precise the concept of puncture. The open su rface \( \Sigma \) has boundaries \( B_i \) with \( i = 1, \ldots, n \), which are 3–manifolds. For each \( B_i \) let us consider the cone \( C_i \), which is obtained from the ‘cylinder’ \( B_i \times I \), where \( I \) is a finite interval, by ‘squeezing’ to a point one of the boundaries of the cylinder. We can attach the boundaries of these cones to the corresponding boundaries \( B_i \) of \( \Sigma \) and obtain a compact surface \( \Sigma_c \). Since each \( B_i \) has Euler characteristic 1, using additivity of the latter, we get \( \chi \equiv \chi_{\Sigma} = \chi_c - n \), where \( \chi_c \) is the Euler characteristic of the compact surface, i.e. \( \chi_c = 2 - 2b_1 + b_2 \). Now let us turn to the double of \( \Sigma \): one constructs a complex surface \( \hat{\Sigma} \) endowed with an antianalytic involution (locally this is \( z \rightarrow \bar{z} \)). Roughly speaking the double is made of two copies of \( \Sigma \) attached by the boundaries to form a compact surface: each boundary of one copy is attached to the corresponding boundary of the other copy. Denoting by a hat the quantities relevant to the double and using again additivity of the Euler characteristic, we have

\[
\hat{\chi} = 2 - 2\hat{b}_1 + \hat{b}_2 = 2\chi = 4 - 4b_1 + 2b_2 - 2n
\]

Now the number \( 2 - 2\hat{b}_1 + \hat{b}_2 \) is the total alternating sum of zero modes on the double. Since it can be expressed via the Gauss–Bonnet theorem as an integral over \( \hat{\Sigma} \), we expect that the same integral over \( \Sigma \) would yield half of it, i.e. \( \chi \), thanks to the symmetry implied by the anti–involution. We expect therefore that the total number of zero modes on \( \Sigma \) with the appropriate sign be given by \( \chi \), which is the result anticipated above.

Now one can see that for all \( \Sigma \)'s considered in this paper as branched coverings of \( \mathcal{X} \), \( \chi \geq 0 \), therefore the sum over ic–instantons gives rise to a series of non–negative powers \(^3\) of \( 1/g \). This suggests that we interpret \( g_3 = 1/g \) as the D3–brane perturbative interaction coupling, analogous to the string coupling of [7].

What we have shown so far is not enough to draw definite and unambiguous conclusions, however we have seen that in the strong coupling limit of 4D SYM theory there

\(^3\)In MST the corresponding series is in terms of \( g^{\chi} \); however Riemann surfaces with at least two punctures have negative \( \chi \).
is room to describe scattering processes of D3–branes. Let us discuss a few general aspects of these processes. Instantons become four–manifolds with boundaries consisting of three–manifolds. These geometrical configurations lend themselves to an interpretation in terms of three–brane scattering. In turn this interpretation fits very well in the path integral formalism, since it gives rise to a perturbative series in $1/g$. It remains for us to specify what are the amplitudes involved in these scattering of branes. Like in the case of string scattering, we will not really mean scattering of full branes but rather scattering of particle states which represent brane excitations. Although we do not know the spectrum of states of a D3–brane, in the case at hand we know plenty of such states: all the fields $x^i, a_\mu$ as well as the fermions which appear in (5.11), together with their derivatives and products are eligible to create such states. It is clear how to proceed: whenever we want to represent scattering of 3–branes excitations with given incoming and outgoing states, we have to insert in the path integral the appropriate fields and evaluate the corresponding amplitudes. Of course this is not the end of the story, since one should then sum over all the instantons that interpolate between the same initial and final states, which means a sum over the appropriate instanton topologies and, at fixed topology, an integral over the appropriate moduli space.

The states we have just mentioned are local states, i.e. they should be associated to points of $\Sigma$, not to boundary 3–manifolds $B_i$. Suitable 4–manifolds are obtained by attaching cones $C_i$ to $B_i$, as explained above, and smoothing out the result. Alternatively, we can imagine diffeomorphisms that deform the boundaries to points, and restrict ourselves to such configurations.

To end this section let us remark another difference with MST. While in MST the Euler characteristics entirely determines the instanton topology, that is not so in the present case. In fact since $\chi = 2 - 2b_1 + b_2 - n$, there may be and in fact there are manifolds with different $b_1$ and $b_2$ but the same $\chi$. Therefore each term in the perturbative expansion in $1/g$ consist in general of a sum over different topologies. This sum is potentially infinite. However, as long as $N$ is finite, the number of topologies which is possible to realize as branched coverings will be finite. Therefore $N$ may be considered as a regulator for these sums. In this regard there is an interesting possibility: it is possible to introduce a parameter that allows us to discriminate among the various terms of a sum corresponding to a given $\chi$. This is $\theta$, the angle in front of the topological theta term, which can be introduced in the theory in the usual way (see [30] for an analogous context). We will not do it here.

6. Discussion

In the course of the paper we have set aside a few problems which we would like now to comment on. The first question we want to address is that of the interpretation of the scattering theory we have digged out in the previous sections. We have found several indications that it is a scattering theory of D3–branes. Let us further justify this claim in the light of Matrix theory.
6.1 Connection with Matrix theory

At least in the case $X = \mathbb{R} \times \mathbb{T}^3$, the theory \[\text{(2.4)}\] can be easily derived from Matrix Theory, \[\cite{32, 34, 35, 36, 37}\], via compactification on a dual 3-torus, \[\cite{38}\]. Here we recall what is essential to make this connection, following in particular \[\cite{39}\] (see also \[\cite{40, 41}\]).

Matrix Theory hinges on the idea that a system of $N$ D0–branes infinitely boosted along a fixed direction, say the 11–th, describes the essential features of M theory. Each D0–brane has the 11-th component of the momentum $p_{11} \sim 1/R_{11}$ far larger that the transverse components (where $R_{11}$ is interpreted as a large compactification radius for M theory). It is expected that the Matrix Theory description of M theory becomes more and more faithful as $N$ becomes larger and larger. Matrix Theory is represented by supersymmetric quantum mechanics of matrices (SYM theory in 0+1 dimensions with gauge group $U(N)$ and 16 supercharges). Compactification of M–theory on a circle of radius, say, $R_9$ is expected to lead to IIA theory in the $R_9 \to 0$ limit. In Matrix Theory the corresponding operation consists in compactifying the base manifold of SYM theory on the dual radius, so that one ends up with 1+1 dimensional SYM theory with $U(N)$ gauge group and $\mathcal{N} = (8,8)$ supersymmetry, i.e. MST. That this leads to type IIA theory in the strong YM coupling limit is by now a well–known result, which has been recalled in the introduction.

The next step is to compactify the base manifold of SYM theory along some additional dimensions. Let us denote by $\tilde{R}_i$ the field theory compactification radii and by $R_i$ the corresponding M–theory radii. They are related by, see \[\cite{39}\],

$$\tilde{R}_i = \frac{\ell_{11}^3}{R_i R_{11}} , \quad i = 9, 8, ... \tag{6.1}$$

where $\ell_{11}$ is the 11–th dimensional length scale. For example, if one compactifies on a two–torus ($i = 9, 8$) and takes the limit $R_9, R_8 \to 0$, one can convince oneself that one series of massless states is produced, which is interpreted as a new dimension that opens up. This new dimension plus the one implicit in the large $N$ limit lead us back to 10 dimensions in a type IIB framework. If, instead, we compactify on a three–torus ($i = 9, 8, 7$) and take the limit $R_9, R_8, R_7 \to 0$ we can see that three new dimensions open up: in this case dimensions and context are those of M–theory. The latter however is not the limit we are interested in in this paper. We will rather consider the limit

$$R_7, R_8 \to 0, \quad R_9 \to \infty \tag{6.2}$$

It is easy to see that in this case only one series of massless states is produced, i.e. only one new dimension opens up (instead of three). Naturally we have to add the decompactification related to $R_9$ being very large and the new dimension implicit in the large $N$ limit. Therefore the context of \[\text{(6.2)}\] is that of a 10 dimensional type IIB theory. That it describes D3–branes can be seen by starting from Matrix Theory, which is theory of D0–branes, compactifying along the 7,8,9–th directions and t–dualizing the three circles. The D0–branes become D3–branes wrapped around the three–torus. Finally one takes the limit \[\text{(6.2)}\] or the corresponding in the dual variables according to \[\text{(6.1)}\].
The YM theory we obtain is exactly (2.1). In fact the dependence on the compactification radii can be entirely collected in the dimensionless coupling constant

\[ g^2 = \frac{\ell_1^{11}}{R_7 R_8 R_9}. \]  

(6.3)

The action can be brought to the form (2.1) via a sequence of rescalings.

In conclusion, the connection with Matrix Theory tells us that the scattering theory that looms through the previous sections, if confirmed by further analysis, can be interpreted as a scattering theory of D3–branes in type IIB theory in 10D.

6.2 Other questions

In the previous subsection we have considered the case of \( \mathbb{R} \times \mathbb{T}^3 \). Toroidal compactifications are the most well–known cases of compactifications of Matrix theory, [37]. Our point of view about Matrix Theory, however, is that its content is revealed and the information stored in it can be retrieved by considering all possible compactifications. This means that we should analyze other compactifications beside \( \mathbb{R} \times \mathbb{T}^3 \) and the ensuing ic–instantons in order to capture the full content of Matrix Theory. On the other hand, when studying a given D3–brane scattering process, we saw that ic–instantons with the same in and out configurations can come from different base manifolds. This creates a potential problem: which is the right base manifold? One possible answer suggested by the previous considerations is that we should perhaps sum over all instantons that interpolate between the relevant initial and final configurations, regardless of what base manifolds these instantons are originated from.

One final comment concerns the comparison with the scattering of macroscopic D–branes mediated by open strings stuck on them, which is the way scattering of macroscopic D–branes has been described in the literature up to now, [42]. This is an open problem, however the following remark might be helpful. In our approach a string mediated interaction would imply manifolds of real dimensions two being exchanged among the interacting strings, instead of manifolds of complex dimensions two, as in our case. We are clearly in the presence of a limiting (singular) case of the scattering described in this paper. This can be rephrased by saying that string–mediated D3–brane scattering amplitudes may be a limiting case of the general scheme presented here: they may become the leading contributions under particular kinematical conditions.

Appendix. Mathematical description of the lifting.

To really describe the process we called “lifting”, it is convenient to start from the opposite, i.e. how to push down a line bundle. Given a covering \( \pi : \tilde{M} \to M \), one has an easy way to get a line bundle on \( \tilde{M} \) from one on \( M \), i.e. the pullback \( \pi^* \); to “push forward” a line bundle on \( \tilde{M} \) we have to resort to the machinery of sheaves, in a way that may, however, be easily translated in simple terms.

Let \( L \) the line bundle on \( \tilde{M} \), and denote with the same symbol the sheaf of its holomorphic sections. Now we define the direct image sheaf \( \pi_* L \) on \( M \) through the formula

\[ \pi_* L (U) = L(\pi^{-1}(U)), \]  

(6.4)
where $U$ are open sets on the base $M$; this, as we will discuss in a moment, is the sheaf of sections of a vector bundle. The correspondence is this: the holomorphic sections of the vector bundle over $U \subset M$ are given by the holomorphic sections of the line bundle over the open set $\pi^{-1}(U) \subset \tilde{M}$.

If $U$ is a disc that does not intersect the branch locus, $\pi^{-1}(U)$ consists of $N$ (the order of the covering) distinct discs. The sections of $\mathcal{L}$ over these $N$ discs are simply $N$-uples of functions; these are interpreted as the local sections of the vector bundle on the base, which therefore has rank $N$. On a neighborhood of a branch point, the situation is different, since there are less than $N$ discs. So it would seem that the rank changes, and that the sheaf we defined does not correspond to a vector bundle. Let us analyze more closely what happens in a situation of total branching, with a map from a disc $\tilde{U}$ with coordinate $z$ to a disc $U$ with coordinate $w$, branched $k$ times: $z \mapsto z^k = w$. Any function on $\tilde{U}$ can be written, by the Weierstrass preparation theorem, as $s(z) = s_1(z^k) + z s_2(z^k) + \ldots + z^{k-1} s_k(z^k)$; the $s_i$ are functions of $w$, so any function on $U_z$ gives $k$ functions on $U$. This shows that the rank is constant: the local sections of our direct image sheaf over a neighborhood of every point are $N$ holomorphic functions.

Consider now a section $y$ of the trivial line bundle over $\tilde{M}$ (remember it is non compact, in our case; on a compact manifold, we would have to take a non trivial line bundle and slightly modify the whole construction, making another line bundle appear also on the base). Multiplication by it gives a map $y : H^0(\pi^{-1}(U), \mathcal{L}) \to H^0(\pi^{-1}(U), \mathcal{L})$ and hence, by definition of $\pi_* \mathcal{L} \equiv E$, a map that we call $X : H^0(U, E) \to H^0(U, E)$. In a neighborhood of a non branching point, a basis for the space of our local sections is given by local sections $s_i$ of $\mathcal{L}$ over each of the discs $U_i$, such that $s_i|_{U_j} = 0$ for $i \neq j$. On this basis, $X$ acts as follows: $X s_i = y_i s_i$. This means that the $y_i s_i$ are the eigenvalues of $X$, and $s_i$ the eigenvectors. On neighborhoods of the branch points, we would have to choose a different basis of sections, as described above; however, by continuity it is still true that $y$ is given by the spectrum of $X$.

Suppose now that we have a connection on $\mathcal{L}$: this means, for each tangent vector $v$, an endomorphism $D_v$ of the space of sections of $\mathcal{L}$. To obtain a connection on $E$, consider again the situation locally around a non branching point $p \in M$; given a tangent vector $v_p$, $(D_{v_p} s_i)$ is the $N$-uple having as components $D_{v_i} s_i$, where $v_i$ are the $N$ counterimages of $v_p$. This means that we do the same analysis we did for $y$ for each component of $a$, where $d + a$ is a local expression for the connection.

In a disc containing a branch point, the analysis is different due to the different basis; we don’t describe explicitly the computations here, but let us briefly sketch the result. Pushing down the connection gives, in general, a connection on the base with poles; this gives relations between the Chern classes of $\mathcal{L}$ and of $E$, which agree with the results one can find by the Grothendieck-Riemann-Roch theorem. On our non compact manifolds, however, most of these conditions are uneffective.

Now that we have described what happens going downstairs, we can try to invert the process. In our situation, we simply have a section $X$ of a bundle $\text{End}(E)$ over the base manifold $M$, a connection over this bundle, and fluctuations. We have to reconstruct the covering manifold, the line bundle $\mathcal{L}$, the connection over it, and the map $y$. 

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The manifold is described by the spectrum of $X$: in $M \times \mathbb{C}$, it is given by the equation $\det(X - y) = 0$, and $y$ is a well-defined function over it (it is one of the coordinates of the ambient space $M \times \mathbb{C}$). To obtain the line bundle, consider the pullback $\pi^*X$, section of $\pi^*\text{End}(E)$, which is a rank $N$ vector bundle over $\tilde{M}$. The eigenspace $\ker(X - Y) \subset \pi^*E$ defines, completing by continuity also over the branching points, a line bundle $\mathcal{L}$; this is clearly the inverse of the construction we gave above, since we observed that the $s_i$ were indeed eigenvectors of $X$ – we just put on $M$ the eigenspace corresponding to its eigenvalue.

What remains to be lifted is the connection, and all the fluctuations of the fields. The process is similar, and we describe it for the connection. Take the connection on $M$: it is a connection on $\text{End}(E)$, but we may consider it as a connection on $E$ (it is a matter of choosing the representation on which it acts). Pull it back to $\tilde{M}$: it defines a connection on $\pi^*E$. Since we know that it commutes with $X$, it preserves eigenvalues, and so it defines a connection on the line bundle. This is done with the eigenvalues of $A$, and so what we described is nothing but the formalization of the process described in the text. On the branches, the process is different, as we mentioned above; but its analysis is beyond our scope now, since we know the connection outside branches by what we described, and over the branches we know that the lifting gives the right number of delta functions to make the relations between the Chern classes of $\mathcal{L}$ and $E$ match.

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