Magneto-transport signatures in periodically-driven Weyl and multi-Weyl semimetals

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We investigate the influence of a time-periodic driving (for example, by shining circularly polarized light) on three-dimensional Weyl and multi-Weyl semimetals, in the planar Hall and planar thermal Hall set-ups. We incorporate the effects of the drive by using the Floquet formalism in the large frequency limit. We evaluate the longitudinal magneto-conductivity, planar Hall conductance, longitudinal thermo-electric coefficient, and transverse thermo-electric coefficient, using the semi-classical Boltzmann transport equations. We demonstrate the explicit expressions of these transport coefficients in certain limits of the system parameters, where it is possible to perform the integrals analytically. We cross-check our analytical approximations by comparing the physical values with the numerical results, obtained directly from the numerical integration of the integrals. The answers obtained show that the topological charges of the corresponding semimetals have profound signatures in these transport properties, which can be observed in experiments.

I. Introduction

Recently, there has been an upsurge in the explorations of condensed matter systems exhibiting multiple band-crossing points in the Brillouin zone (BZ), which host gapless excitations. These include the Weyl semimetals (WSMs)\textsuperscript{1} and the multi-Weyl semimetals (mWSMs) \textsuperscript{2}, which are three-dimensional (3d) semimetals having non-trivial topological properties in the bandstructures, responsible for giving rise to various novel electrical properties (e.g., Fermi arcs). The nodal points behave as sinks and sources of the Berry flux, i.e., they are the monopoles of the Berry curvature. Since the total topological charge over the entire BZ must vanish, these nodes must come in pairs, each pair carrying positive and negative topological charges of equal magnitude. This also follows from the Nielsen–Ninomiya theorem \textsuperscript{3}. The sign of the monopole charge is often referred to as the chirality of the corresponding node. While WSMs have a linear and isotropic dispersion, and their band-crossing points show Chern numbers (i.e., values of the monopole charges) $\pm 1$, mWSMs exhibit anisotropic and non-linear dispersions, and harbour nodes with Chern numbers $\pm 2$ (double-Weyl) or $\pm 3$ (triple-Weyl). It can be proved mathematically...
that the magnitude of the Chern number in mWSMs is bounded by 3, by using symmetry arguments for crystalline structures [2, 4, 5]. Due to the non-trivial topological structure of these systems, novel optical and transport properties, such as circular photogalvanic effect [6], circular dichroism [7], negative magnetoresistance [8, 9], planar Hall effect (PHE) [10], magneto-optical conductivity [11–14], and thermopower [15], can emerge.

There has been unprecedented advancement in the experimental front, where WSMs have been realized experimentally [9, 16–19] in compounds like TaA, NbA, and TaP. These materials have been reported to have topological charges equal to ±1. Compounds like HgCr$_2$Se$_4$ and SrSi$_2$ have been predicted to harbour double-Weyl nodes [2, 4, 20]. DFT calculations have found that nodal points, in compounds of the form A(MoX)$_3$ (where A = Na, K, Rb, In, Tl, and X=S, Se, Te), have Chern numbers ±3 [21]. Dynamical / nonequilibrium topological semimetallic phases can also be designed by Floquet engineering [22–25].

When a conductor is placed in a magnetic field $B$, such that it has a nonzero component perpendicular to the electric field $E$ (which has been applied across the conductor), a current is generated perpendicular to the $E$-$B$ plane. This current is usually referred to as the Hall current, and the phenomenon is the well-known Hall effect. A generalization of this phenomenon is the PHE [10], when there is the emergence of a voltage difference perpendicular to an applied external $E$, which is in the plane along which $E$ and $B$ lie [cf. Fig. 1(a)]. The planar Hall conductivity, denoted by $\sigma_{xy}$ in this paper, is dependent on the angle between $E$ and $B$. In contrast with the canonical Hall conductivity, PHE does not require a nonzero component of $B$ perpendicular to $E$. In fact, a nonzero PHE is exhibited by ferromagnetic materials [26–30] or topological semimetals (which possess non-trivial Berry curvature and chiral anomalies), in a configuration in which the conventional Hall effect vanishes (because $E$, $B$, and the induced transverse Hall voltage, all lie in the same plane). Similar to the PHE, the planar thermal Hall effect [also referred to as the the planar Nernst effect (PNE)] is the appearance of a voltage gradient perpendicular to an applied temperature ($T$) gradient $\nabla T$ (instead of an electric field), which is co-planar with an externally applied magnetic field $B$ [cf. Fig. 1(b)].

There have been extensive theoretical [15, 31–33] and experimental [9, 34] studies of the transport coefficients in these planar Hall set-ups for various semimetals. Examples include longitudinal magneto-conductivity (LMC), planar Hall conductivity (PHC), longitudinal thermo-electric coefficient (LTEC), and transverse thermo-electric coefficient (TTEC) (also known as the Peltier coefficient). In this paper, we will compute these magneto-electric and thermo-electric transport coefficients for WSMs and mWSMs, subjected to a time-periodic drive (for example, by shining circularly polarized light with frequency $\omega$). We will use a semi-classical Boltzmann equation approach for calculating these properties.

A widely used approach to analyze periodically driven systems, where the time-independent Hamiltonian is perturbed with a periodic potential, is the application of Floquet formalism [35–40]. The approach relies on the fact that a particle can gain or lose energy in multiples of $\hbar \omega$ (quantum of a photon), where $\omega$ is the driving frequency. Since the time($t$)-dependent Hamiltonian $\hat{H}$ satisfies $\hat{H}(t + T) = \hat{H}(t)$, where $T = 2\pi/\omega$, we perform a Fourier transformation. When $\omega$ is much larger than the typical energy bandwidth of the system, we can combine
the Floquet formalism with Van Vleck perturbation theory, to obtain an effective perturbative potential of the form:

\[ V_{\text{eff}} = \sum_{n=1}^{\infty} \left( \frac{[H_{-n}, H_n]}{n \omega} + \frac{[[H_{-n}, H_0], H_n]}{n^2 \omega^2} + \cdots \right). \tag{1.1} \]

Here, \( H_n \) denotes the \( n \)th Fourier mode of the Hamiltonian.

The paper is organized as follows: In Sec. II, we show the low-energy effective Hamiltonians for the WSMs and mWSMs, and then write down the modifications needed to capture the properties of periodically driven systems. In Sec. III, we use the semi-classical Boltzmann equations to derive the magneto-electric transport coefficients for PHE. We perform similar computations in Sec. IV to determine the thermo-electric coefficients for PNE. In Sec. V, we discuss our results and their implications. Finally, we conclude with a summary and outlook in Sec. VI.

II. Model and Formalism

The low-energy effective Hamiltonian in the vicinity of a single multi-Weyl node, with topological charge \( J \), can be written as [2, 4, 5, 41, 42]:

\[ H_J (\mathbf{k}) = \alpha_J k_\perp^I \left[ \cos (J \phi_k) \sigma_x + \sin (J \phi_k) \sigma_y \right] + v_0 k_z \sigma_z = \begin{bmatrix} v_0 k_z & \alpha_J (k_x - i k_y)^J \\ \alpha_J (k_x + i k_y)^J & -v_0 k_z \end{bmatrix}, \quad J \in (1, 2, 3), \tag{2.1} \]

where \( k_\perp = \sqrt{k_x^2 + k_y^2}, \phi_k = \arctan(k_y/k_x), \) and \( \alpha_J = \frac{\hbar v_\perp}{2 J} \). Furthermore, \( v_0 \) and \( v_\perp \) are the Fermi velocities in the \( z \) direction and \( xy \)-plane, respectively, and \( k_0 \) is a system-dependent parameter with the dimension of momentum. As usual, \( \sigma \equiv (\sigma_x, \sigma_y, \sigma_z) \) is the vector of the Pauli matrices in the pseudo-spin space. Note that \( J = 1 \) represents the WSM, which has a linear and isotropic dispersion with \( \sigma \) band, and \( "1" \) refers to the conduction band.

We will use \( \hbar \) where necessary.

The Hamiltonian in Eq. (2.1) can be written in a compact form as

\[ H_J = \mathbf{n}_k \cdot \sigma, \quad \mathbf{n}_k = (\alpha_J k_\perp \cos (J \phi_k), \alpha_J k_\perp \sin (J \phi_k), v_0 k_z). \tag{2.2} \]

The energy eigenvalues are given by \( \pm \varepsilon^0_k \), where

\[ \varepsilon^0_k = \sqrt{\alpha_J^2 k_\perp^2 \varepsilon^0_0 k_z^2}, \tag{2.3} \]

and the + and − signs represent the conduction and valence bands, respectively. The quasi-particle velocity \( \mathbf{v} \) (or group-velocity) vectors, associated with the conduction and valence bands, are given by \( \pm \mathbf{v} \), where

\[ \mathbf{v}^0 \equiv (v_x^0, v_y^0, v_z^0) = \nabla_k \varepsilon^0_k = \frac{1}{\varepsilon^0_k} \left( J k_x \alpha_J^2 k_\perp^{2J-1}, J k_y \alpha_J^2 k_\perp^{2J-1}, v_0^2 k_z \right). \tag{2.4} \]

The \( \alpha^\text{th} \)-component of the Berry curvature for the \( m \)th band [with \( m \in (0, 1) \), where “0” refers to the valence band, and “1” refers to the conduction band] is given by [43]:

\[ (\Omega_\alpha^m) = \frac{(-1)^m}{4 |\mathbf{n}_k|^3} \varepsilon_{\alpha\beta\gamma} \mathbf{n}_k \cdot \left( \frac{\partial \mathbf{n}_k}{\partial k_\beta} \times \frac{\partial \mathbf{n}_k}{\partial k_\gamma} \right), \quad \text{with } \alpha, \beta, \gamma \in (x, y, z). \tag{2.5} \]

The Berry curvature of the valence band, associated with a positive-chirality WSM or mWSM node, takes the explicit form:

\[ \Omega_k^0 = \frac{J v_0 \alpha_J^2 k_\perp^{2J-2}}{2 \left( \varepsilon_k^0 \right)^3} (k_x, k_y, J k_z). \tag{2.6} \]

Clearly, the components of \( \Omega_k^0 \) are unequal (anisotropic) for \( J > 1 \).
A. Boltzmann Formalism

Let us first review the semi-classical Boltzmann formalism [44, 45], which is used to calculate the transport coefficients. The Boltzmann equation describes the evolution of the distribution function $f(k, r, t)$ of the fermionic quasiparticles as follows:

$$
\frac{\partial}{\partial t} f(k, r, t) + \dot{r} \cdot \nabla_r + \dot{k} \cdot \nabla_k f(k, r, t) = J_{\text{coll}}[f(k, r, t)].
$$

(2.7)

The term on the right-hand side represents the collision integral, and arises due to scattering of electrons (e.g., scattering from lattice or from impurities). As we are looking for a steady-state solution, for which $\partial f(k, r, t)/\partial t = 0$, we can use the simpler form:

$$
\left( \dot{r} \cdot \nabla_r + \dot{k} \cdot \nabla_k \right) f(k, r) = \frac{f_0 - f(k, r)}{\tau},
$$

(2.8)

where $\tau$ is the average time between two successive collisions of a quasiparticle (under the relaxation time approximation), and $f_0$ is the equilibrium value of $f$ (and hence, given by the Fermi-Dirac distribution function).

For the PHE, we consider the set-up consisting of an external electric field $E$ along the $x$-axis (viz., $E = (E_0, 0, 0)$), and an external magnetic field $B$ along the $xy$-plane (viz., $B = B \cos(\theta, \sin(\theta, 0)$). The semi-classical transport equations for a system with Berry curvature $\Omega_F$ are now given by [46, 47]:

$$
\dot{r} = D [v + e (E \times \Omega_F) + e (v \cdot \Omega_F) B], \quad \dot{k} = D \left[ e E + e (v \times B) + e^2 (E \cdot B) \Omega_F \right],
$$

(2.9)

where $v$ is the quasiparticle group velocity, and $D^{-1} = 1 + e [B \cdot \Omega_F(k)]$ is the modified phase-space function. The latter arises from the fact that $\Omega_F(k)$ modifies the phase-space volume element as $d^3 k d^3 x \rightarrow d^3 k d^3 x D$.

In the presence of external perturbations (e.g., an external electric field or a thermal gradient), the charge current and the thermal current are given by:

$$
J_\alpha = L_{1\beta}^{11} E_\beta + L_{1\beta}^{12} (-\nabla_\beta T), \quad Q_\alpha = L_{2\beta}^{11} E_\beta + L_{2\beta}^{12} (-\nabla_\beta T),
$$

(2.10)

respectively, which are obtained from the linear response theory. Here, $L_{1\beta}^{11}$ and $L_{1\beta}^{12}$ define the components of the charge conductivity and the thermo-electric tensors, respectively. Onsager’s relation connects $L_{1\beta}^{12}$ and $L_{2\beta}^{11}$ via $L_{1\beta}^{12} = T L_{2\beta}^{11}$.

B. Time-periodic Driving

For a periodic optical driving with frequency $\omega$, the electric field vector for this light wave can be written as $E(t) = E_0 (\cos(\omega t), \sin(\omega t), 0)$, and the vector potential can be written as $A(t) = \frac{E_0 \omega}{2} (\sin(\omega t), \cos(\omega t), 0)$ in the Landau gauge. The effect of the periodic electromagnetic field on the Hamiltonian can be obtained via the Peierls substitution $k \rightarrow k - eA$, where $e$ is the electric charge. Let us define $A_0 = \frac{E_0 \omega}{2}$. Then the gauge-dependent momentum components are found to be $k_x \rightarrow k'_x = k_x - A_0 \sin(\omega t)$, $k_y \rightarrow k'_y = k_y - A_0 \cos(\omega t)$, and $k_z \rightarrow k'_z = k_z$.

Using the binomial expansion $(k'_x + i k'_y)^J = \sum_{m=0}^{J} (k_\perp e^{\pm i \phi_x})^J m \left(-A_0\right)^m e^{\pm i (\frac{J}{2} - \omega t)} J C_m$, where $J C_m = \frac{1}{(J-m)!m!}$ represents the combinatorial factor, the time-dependent Hamiltonian takes the form:

$$
H_J(k, A) = \alpha_J \left(k'_x - i k'_y\right)^J \sigma_+ + \alpha_J \left(k'_x + i k'_y\right)^J \sigma_- + v_0 k_z \sigma_z, \quad \sigma_{\pm} = \sigma_x \pm i \sigma_y 2.
$$

(2.11)

We consider the limit where Floquet’s theorem can be applied, and extract the leading order correction terms from the high-frequency Van Vleck expansion. In this limit, one can describe the dynamics of the driven system over a time-period $T = \frac{2\pi}{\omega}$, in terms of the effective Floquet Hamiltonian $H_J^F(k) \approx H_J(k) + V(k)$, where

$$
V_k = \sum_{p=1}^{\infty} \left[ V_p, V_p \right] / p \omega, \quad V_p = \frac{1}{T} \int_0^T dt \ H_J(k, A) e^{ip\omega t}
$$

(2.12)

represents the perturbative driving term.
We restrict to corrections with leading order in $1/\omega$ throughout the paper. Using

$$V_p = \alpha J \sum_{m=1}^{J} (k_\perp)^{J-m}(-A_0)^m \mathcal{J} m \left( e^{i[(J-m)\phi_k + m \varpi]} \delta_{p,m} - e^{-i[(J-m)\phi_k + m \varpi]} \delta_{p,-m} \right),$$  

we get

$$H^F_j(k) = n_k \cdot \sigma_\perp + \sum_{p=1}^{J} \left( \frac{n C_p A_0^p}{\omega} \right)^2 \frac{t_{2J-2p}}{p} \sigma_z \equiv \alpha J \mathbf{n}_k \cdot \sigma_\perp,$$

$$n_k = (k_\perp^J \cos (J \phi_k), k_\perp^J \sin (J \phi_k), T_k/\alpha J), \quad T_k = v_0 k_z + \frac{\alpha_J^2}{\omega} \sum_{p=1}^{J} \beta_p k_\perp^{2(J-p)} \equiv \Delta_J + T'_k,$$

$$\Delta_J = \frac{\alpha_J^2 A_0^J}{J \omega}, \quad \beta_p = (\mathcal{J} C_p A_0^p)^2 / p, \quad T'_k \equiv v_0 k_z + \frac{\alpha_J^2}{\omega} \sum_{p=1}^{J-1} \beta_p k_\perp^{2(J-p)}.$$

The effective quasi-energies are $\pm \varepsilon_k$, with

$$\varepsilon_k = \sqrt{\varepsilon_k^2 k_\perp^2 + T_k^2}.$$

As before, the $+$ ($-$) sign refers to the conduction (valence) band.

The Berry curvature of the valence band, associated with a positive-chirality WSM or mWSM node, is now given by:

$$\Omega_F(k) = \frac{\alpha_J^2 k_\perp^{2J-2} J}{2 \varepsilon_k^2} \left( v_0 k_x, v_0 k_y, J T_k - \beta_k k_\perp^2 \right), \quad \beta_k = \frac{2 \alpha_J^2}{\omega} \sum_{p=1}^{J} (J-p) \beta_p k_\perp^{2J-2p-2}.$$

Comparing this with Eq. (2.6), we note that the z-component of Berry curvature has drastically changed – this will produce a large effect, as the term $(\mathbf{v} \cdot \Omega_F)$ appears in the expressions for the transport coefficients. On the contrary, $\mathbf{B}$ does not have a z-component in our set-up, and therefore, the changes observed via the $(\mathbf{B} \cdot \Omega_F)$ term appearing in the expressions for transport coefficients is only via the change in the form of the energies [compare Eqs. (2.3) with (2.15)].

Employing a change of variables via $k \sin \gamma = \alpha J k_\perp^J$ and $k \cos \gamma = v_0 k_z$, Eq. (2.15) for the quasi-energies takes the form

$$\varepsilon_k = \sqrt{k^2 + 2 k \cos \gamma \chi_J(k, \gamma) + \chi_J^2(k, \gamma)}, \quad \chi_J(k, \gamma) = \frac{(k \sin \gamma)^2}{\omega} \sum_{p=1}^{J} \beta_p^J \left( \frac{\alpha_J}{k \sin \gamma} \right)^{2p}.$$

Considering the fact that the quasiparticles contributing to the transport properties lie near the Fermi surface, we have $k \approx \mu$. Hence, the expansion in large $\omega$ (i.e., about $\frac{k}{\omega} = \infty$) is valid when $|\chi_J(\mu, \gamma)/\mu| \ll 1$. Combining this with Eq. (2.17),

$$\left( e E_0 \alpha J^{1/J} \mu^{J+2} \right)^{2/3} \ll \omega$$

is the limit which we will use in our analytical approximations. This leads to

$$\varepsilon_k = k + \chi_J(k, \gamma) \cos \gamma + \mathcal{O} \left( \frac{1}{\omega^2} \right).$$

### III. Longitudinal and Transverse Magneto-Conductivities

In this section, we will consider the PHE set-up [cf. Fig. 1(a)], and evaluate the LMC and PHC. Using Eqs. (2.8) and (2.9), the expressions for the LMC and PHC are given by [32, 50–52]:

$$\sigma_{xx} = L_{xx}^{11} \approx e^2 \int \frac{d^3 k}{(2\pi)^3} \mathcal{J} r \left( \frac{\partial f_0}{\partial \varepsilon} \right) \left[ v_x + e B \cos \theta (\mathbf{v} \cdot \Omega_F) \right]^2,$$

$$\sigma_{yx} = L_{yx}^{11} \approx e^2 \int \frac{d^3 k}{(2\pi)^3} \mathcal{J} r \left( -\frac{\partial f_0}{\partial \varepsilon} \right) \left[ v_y + e B \sin \theta (\mathbf{v} \cdot \Omega_F) \right] \left[ v_x + e B \cos \theta (\mathbf{v} \cdot \Omega_F) \right].$$

(3.1)
We consider the zero-temperature limit (\( T = 0 \)). Here, \( \alpha_0 = \frac{v_0}{k_0} \), \( \alpha_1 = v_\perp \), \( \alpha_2 = \frac{v_\perp}{v_0} \); this implies that \( \alpha_3 = \frac{v_\perp}{v_0} = \frac{\alpha_2^2}{\alpha_1^2} \). In natural units, \( \hbar = c = k_B = 1 \), and \( 4 \pi \varepsilon_0 = 137 \). In our plots, we have used \( v_\perp = v_0 \) (from the table entry), leading to \( \alpha_2 = 3.9 \times 10^{-5} \text{ eV}^{-1} \), \( \alpha_3 = 2.298 \times 10^{-6} \text{ eV}^{-2} \). For \( J = 2 \) and \( J = 3 \), \( v_\perp \) has been set equal to \( v_0 \) for the sake of simplicity, while for \( J = 1 \), we have the constraint \( v_\perp = v_0 \) anyway (due to isotropy). For all the plots, except Figs. 5 and 9, we have set \( \theta = \pi/6 \).

TABLE I. The values of the various parameters which we have used in plotting the transport coefficients are tabulated here.

| Parameter        | SI Units          | Natural Units    |
|------------------|-------------------|------------------|
| \( v_0 \) from Ref. [48] | \( 1.5 \times 10^7 \text{ m s}^{-1} \) | 0.005            |
| \( \tau \) from Ref. [48] | \( 10^{-13} \text{ s} \) | 151.72 eV\(^{-1} \) |
| \( E_0 \) from Ref. [49] | \( 2.5 \times 10^3 \text{ V m}^{-1} \) | 4.95 eV\(^2 \) |
| \( T \)          | 40 K              | 3.4 \times 10^{-3} \text{ eV} |
| \( B \) from Ref. [50] | 0.168 – 0.421 Tesla | 10 – 25 eV\(^2 \) |
| \( \omega \)     | 7.6 \times 10^10 \text{ Hz} | 5 eV             |
| \( \mu \) from Ref. [32, 37] | 0.1 eV | 0.1 eV |

We have used an "approximately equal to" (\( \approx \)) sign, because we have ignored the contribution from the correction factors arising from external magnetic field. This is justified because in the semi-classical regime, it is reasonable to retain only the leading order terms in \( f \), as the correction factors are several orders of magnitude smaller than the leading order terms [50]. It is important to note that the factor \( B \cos \theta \) is associated with the signature of the chiral anomaly, which connected with the term \( \mathbf{E} \cdot \mathbf{B} \).

For the convenience of computations, \( \sigma_{yx} \) and \( \sigma_{xx} \) are decomposed as:

\[
\sigma_{yx} = \sigma_{yx}^{(1)} + \sigma_{yx}^{(2)} + \sigma_{yx}^{(3)}, \quad \sigma_{xx} = \sigma_{xx}^{(1)} + \sigma_{xx}^{(2)} + \sigma_{xx}^{(3)} + \sigma_{xx}^{(4)},
\]

where

\[
\sigma_{xx}^{(1)} = \tau \varepsilon^2 \int \frac{d^3k}{(2 \pi)^3} D v_x^2 \left( -\frac{\partial f_0}{\partial \varepsilon_k} \right), \quad \sigma_{xx}^{(2)} = \tau \varepsilon^4 B^2 \cos^2 \theta \int \frac{d^3k}{(2 \pi)^3} D (\mathbf{v} \cdot \mathbf{\Omega}_F)^2 \left(-\frac{\partial f_0}{\partial \varepsilon_k} \right),
\]

\[
\sigma_{xx}^{(3)} = 2 \tau \varepsilon^3 B \cos \theta \int \frac{d^3k}{(2 \pi)^3} D v_x (\mathbf{v} \cdot \mathbf{\Omega}_F) \left(-\frac{\partial f_0}{\partial \varepsilon_k} \right),
\]

and

\[
\sigma_{yx}^{(1)} = \tau \varepsilon^2 \int \frac{d^3k}{(2 \pi)^3} D v_y v_x \left( -\frac{\partial f_0}{\partial \varepsilon_k} \right), \quad \sigma_{yx}^{(2)} = \tau \varepsilon^4 B^2 \sin \theta \cos \theta \int \frac{d^3k}{(2 \pi)^3} D (\mathbf{v} \cdot \mathbf{\Omega}_F)^2 \left(-\frac{\partial f_0}{\partial \varepsilon_k} \right),
\]

\[
\sigma_{yx}^{(3)} = \tau \varepsilon^3 B \cos \theta \int \frac{d^3k}{(2 \pi)^3} D v_y (\mathbf{v} \cdot \mathbf{\Omega}_F) \left(-\frac{\partial f_0}{\partial \varepsilon_k} \right), \quad \sigma_{yx}^{(4)} = \tau \varepsilon^3 B \sin \theta \int \frac{d^3k}{(2 \pi)^3} D v_x (\mathbf{v} \cdot \mathbf{\Omega}_F) \left(-\frac{\partial f_0}{\partial \varepsilon_k} \right).
\]

Here,

\[
\mathbf{v} \cdot \mathbf{\Omega}_F \approx \frac{v_0 \alpha_0^2 J^2 k_0^{2J-2}}{2 \varepsilon_k^2}, \quad \mathbf{B} \cdot \mathbf{\Omega}_F = \frac{J \alpha_0^2 k_0^{2J-1} v_0 B \cos (\theta - \phi)}{2 \varepsilon_k^2}.
\]

We consider the zero-temperature limit (\( T = 0 \)), and hence \( \frac{\partial f_0(\varepsilon_k)}{\partial \varepsilon_k} = -\delta(\varepsilon_k - \mu) \). Due to the presence of the Jacobian factor \( D \), we will need to assume the small \( B \) limit to get analytical expressions.

After a straightforward but tedious algebra (see Appendix A for the details on the intermediate steps), we get
the final expressions for LMC as:

\[
\sigma_{xx}^{(1)} = \begin{cases} 
\frac{e^2 \tau}{6 \pi^2 v_0} \left( \mu^2 + \frac{A_1^2 \alpha_1^4}{\omega^2} \right) + \frac{B^2 e^4 v_0 \tau [2 + \cos(2\theta)]}{840 \pi^3 \mu^2} \left( 7 \mu^2 + \frac{12 A_1^4 \alpha_1^4}{\omega^2} \right) & \text{for } J = 1 \\
\frac{e^2 \tau}{\pi^2 v_0} \left( \frac{2^2}{3} + \frac{92\mu^2 A_1^2 \alpha_2^2}{35 \omega^2} + \frac{7\pi \mu A_1^2 \alpha_2^3}{16 \omega^2} + \frac{A_1^6 \alpha_2^4}{12 \omega^2} + \frac{5 \pi \mu^2}{208} + \frac{63 \pi \mu^2 A_1^2 \alpha_2^2}{64 \omega^2} + \frac{32 \mu A_1^6 \alpha_2^3}{40 \omega^2} + \frac{25 \pi A_1^8 \alpha_2^4}{2048 \omega^2} \right) & \text{for } J = 2 \\
\frac{e^2 \tau}{2 \pi^2 v_0} \left[ \mu^2 + \frac{7533 \sqrt{\pi} \alpha_3^4 \mu^8/3 \Gamma(\frac{1}{3}) A_1^4}{80 \omega^2 \Gamma(\frac{2}{3}) \Gamma(\frac{4}{3}) \Gamma(\frac{2}{3})} + \frac{2916 \alpha_3^2 \mu^2 A_1^6}{35 \omega^2} + \frac{27945 \sqrt{\pi} \alpha_3^8/3 \mu^{4/3} \Gamma(\frac{5}{3}) \Gamma(\frac{2}{3}) A_1^4}{494 \omega^2 \Gamma(\frac{2}{3}) \Gamma(\frac{4}{3})} \\
+ \frac{171 \sqrt{\pi} \alpha_3^{10/3} \mu^{2/3} \Gamma(\frac{10}{3}) A_1^6}{56 \omega^2 \Gamma(\frac{4}{3}) \Gamma(\frac{2}{3})} + \frac{\alpha_3^4 A_1^{12/3}}{9 \omega^2} \right] & \text{for } J = 3 \\
+ \frac{3 B^2 e^4 v_0 \alpha_3^{2/3} [2 + \cos(2\theta)]}{\pi^2 \mu^{3/2}} \left[ 729 \sqrt{\pi} \mu^2 \Gamma(\frac{4}{3}) + \frac{648 \alpha_3^{4/3} \mu^{8/3} A_1^8}{35 \omega^2} + \frac{21141 \sqrt{\pi} \alpha_3^{8/3} \mu^{2/3} \Gamma(\frac{2}{3}) \Gamma(\frac{4}{3}) A_1^4}{896 \omega^2 \Gamma(\frac{2}{3}) \Gamma(\frac{4}{3})} \right. \\
\left. + \frac{13832 \sqrt{\pi} \alpha_3^{10/3} \mu^{2/3} \Gamma(\frac{10}{3}) A_1^6}{496 \omega^2 \Gamma(\frac{4}{3}) \Gamma(\frac{2}{3})} + \frac{17 \sqrt{\pi} \alpha_3^4 \Gamma(\frac{4}{3}) A_1^{12/3}}{704 \omega^2 \Gamma(\frac{2}{3}) \Gamma(\frac{4}{3})} \right] & \text{for } J = 3 
\end{cases}
\]

(3.6)
FIG. 3. Plots for $\sigma_{xx}$ (LMC) and $\sigma_{yx}$ (PHC) as functions of the frequency $\omega$, for $B=25$ eV$^2$. The values of the remaining parameters are taken from Table I. Subfigures (a) and (d) correspond to $J=1$; (b) and (e) correspond to $J=2$; and (c) and (f) correspond to $J=3$.

\[
\sigma_{xx}^{(2)} = \frac{\tau \cos^2 \theta (12 B^2 e^4 v_0 \mu^4 \alpha_1^2 + 3 B^4 e^6 v_0^2 \alpha_1^4)}{96 \pi^2 \mu^2} - \frac{\tau \cos^2 \theta A_0^4 (8 B^2 e^4 v_0 \mu^4 \alpha_1^2 + 3 B^4 e^6 v_0^2 \alpha_1^4)}{96 \pi^2 \mu^2 \omega^2}
\]

for $J = 1$.

\[
\sigma_{xx}^{(3)} = \frac{e^2 \tau (e v_0 B \cos \theta \alpha_1)^2}{16 \pi v_0 \mu} \left( \frac{4}{3} \pi \mu + \frac{4 A_6^4 \alpha_1^4}{\pi \mu^2 \omega^2} \right)
\]

for $J = 1$.

\[
\sigma_{xx}^{(3)} = \frac{3 e^2 \tau (\alpha_2^{1/2} B e v_0 \cos \theta)^2}{16 \pi v_0 \mu} - \frac{B^2 e^4 v_0 \tau \cos^2 \theta \alpha_2^3}{\mu^2 \omega^2} \left( \frac{105 \mu^2 A_6^4}{16 \pi} + \frac{32 \mu A_6^2 \alpha_2}{128 \pi} + \frac{9 A_6^4 \alpha_2^2}{128 \pi} \right)
\]

for $J = 2$.

\[
\sigma_{xx}^{(3)} = \frac{729 \alpha_3^{2/3} B^2 e^4 \tau v_0 \Gamma \left( \frac{1}{2} \right) \cos^2 \theta}{128 \pi^{5/2} \mu^2 \Gamma \left( \frac{2}{3} \right)} - \frac{3 \alpha_3^{5/3} B^2 e^4 \tau v_0 \Gamma \left( \frac{1}{2} \right) \cos^2 \theta}{64 \pi^2 \mu^{5/2} \omega^2} \left[ -\frac{1117290 \alpha_3^{5/3} \mu^{4/3} \Gamma \left( \frac{-4}{3} \right) \Gamma \left( \frac{-1}{2} \right) A_0^4}{4301 \sqrt{3}} + \frac{1728 \alpha_3^{1/3} \mu^{2/3} \left( 5 \alpha_3^2 A_0^6 + 162 \mu^2 \right) + 5.60111 \times 10^{-15} \left( \alpha_3 \Gamma \left( \frac{-4}{3} \right) A_0^4 \left( 270 \alpha_3^2 A_0^6 + 26872 \mu^2 \right) \right)}{\Gamma \left( \frac{-4}{3} \right) \omega^2} \right]
\]

for $J = 3$.  

(3.7)  

and

(3.8)
FIG. 4. Plots for (a) $\sigma_{xx}$ (LMC) and (b) $\sigma_{yx}$ (PHC), as functions of the magnetic field magnitude $B$, for $\omega = 5$ eV. The values of the various other physical quantities used are shown in Table I.

Similarly, the expressions for PHC (Appendix A explains the intermediate steps) are given by:

\[
\sigma_{yx}^{(J)} = \begin{cases}
\frac{e^2 \tau (e v_0 B \alpha_1)^2 \sin(2\theta)}{840 \pi^2 \mu^2 v_0} \left( 7 \mu^2 + \frac{12 A_1^2}{\omega^2} \right) & \text{for } J = 1 \\
\frac{5 \alpha_2 B^2 e^4 \tau v_0 \sin(2\theta)}{128 \pi \mu} + \frac{B^2 e^4 \tau v_0 \sin(2\theta) \alpha_2^2}{\mu^3 \omega^2} \left( \frac{63 \mu^2 A_4^4}{32 \pi} + \frac{64 \mu A_0^4}{45 \pi^2} + \frac{25 A_0^2 \alpha_2^2}{1024 \pi^2} \right) & \text{for } J = 2 \\
\frac{27 B^2 e^4 \tau v_0 \sin(2\theta) \alpha_2^2}{64 \pi^2 \mu^3} \left[ \frac{1296 \sqrt{\pi} \mu^2 \Gamma\left(\frac{3}{2}\right)}{1729 \pi^2} + \frac{9216 \alpha_2^4}{35 \omega^2} \right] & \text{for } J = 3
\end{cases}
\]

Finally,

\[
\sigma_{yx}^{(2)} = \sigma_{xx}^{(2)} \tan \theta \quad \text{and} \quad \sigma_{yx}^{(3)} = \sigma_{xx}^{(3)} \frac{\tan \theta}{2} \quad \forall J \in (1, 2, 3).
\]

We note that the above expansions provide analytical approximations for the integrals in Eqs. (3.3) and (3.4). In order to show how closely these expressions match with the actual values, we plot the transport coefficients [cf. Figs. 2 and 3] for some realistic parameter regimes, as shown in Table I. The curves demarcated as “Numerics” show the results obtained from numerically evaluating the integrals in Eqs. (3.3) and (3.4), while the curves labelled as “Analytics” show the values obtained from our results obtained in the large $\omega$ limit. We also show the dependence of these transport coefficients on the magnitude ($B$) and direction (via the angle $\theta$) of the magnetic field, in Figs. 4 and 5, respectively.

IV. Longitudinal and Transverse Thermo-Electric Coefficients

In this section, we will consider the PNE set-up [cf. Fig. 1(b)], and evaluate the LTEC and TTEC. For the sake of completeness, we review the generic derivation for these transport coefficients in Appendix B.

Using the modifications of Eqs. (2.8) and (2.9) for a temperature gradient along the $x$-axis [viz., $\nabla T = (\frac{dT}{dx}, 0, 0)$], instead of an electric field, the LTEC and the TTEC are given by:

\[
\alpha_{xx} \approx e \int \frac{d^3k}{(2\pi)^3} \frac{(\mu - \epsilon_k) T}{T} \left( -\frac{\partial f_0}{\partial \epsilon_k} \right) \left\{ v_x + e B \cos \theta (\mathbf{v} \cdot \mathbf{\Omega}_F) \right\}^2 \quad \text{and}
\alpha_{yx} \approx e \int \frac{d^3k}{(2\pi)^3} \frac{(\mu - \epsilon_k) T}{T} \left( -\frac{\partial f_0}{\partial \epsilon_k} \right) \left\{ v_y + e B \sin \theta (\mathbf{v} \cdot \mathbf{\Omega}_F) \right\} \left\{ v_x + e B \cos \theta (\mathbf{v} \cdot \mathbf{\Omega}_F) \right\},
\]

respectively.
For the ease of the computations, we split the integrals as:

\[
\alpha_{xx} = \alpha_{xx}^{(1)} + \alpha_{xx}^{(2)} + \alpha_{xx}^{(3)}, \quad \alpha_{yx} = \alpha_{yx}^{(1)} + \alpha_{yx}^{(2)} + \alpha_{yx}^{(3)} \tag{4.2}
\]

where

\[
\begin{align*}
\alpha_{xx}^{(1)} &= \tau e \int \frac{d^3k}{(2\pi)^3} \beta (\mu - \varepsilon_k) v_x^2 \left( - \frac{\partial f_0}{\partial \varepsilon_k} \right), \\
\alpha_{xx}^{(2)} &= \tau e (e B \cos \theta)^2 \int \frac{d^3k}{(2\pi)^3} \beta (\mu - \varepsilon_k) (\mathbf{v} \cdot \mathbf{\Omega}_F)^2 \left( - \frac{\partial f_0}{\partial \varepsilon_k} \right), \\
\alpha_{xx}^{(3)} &= \tau (2 e B \cos \theta) \int \frac{d^3k}{(2\pi)^3} \beta (\mu - \varepsilon_k) v_x (\mathbf{v} \cdot \mathbf{\Omega}_F) \left( - \frac{\partial f_0}{\partial \varepsilon_k} \right),
\end{align*}
\]

(4.3)

and

\[
\begin{align*}
\alpha_{yx}^{(1)} &= \tau e \int \frac{d^3k}{(2\pi)^3} \beta (\mu - \varepsilon_k) v_y v_x \left( - \frac{\partial f_0}{\partial \varepsilon_k} \right), \\
\alpha_{yx}^{(2)} &= \tau e^3 B^2 \sin \theta \cos \theta \int \frac{d^3k}{(2\pi)^3} \beta (\mu - \varepsilon_k) (\mathbf{v} \cdot \mathbf{\Omega}_F)^2 \left( - \frac{\partial f_0}{\partial \varepsilon_k} \right), \\
\alpha_{yx}^{(3)} &= \tau e^2 B \cos \theta \int \frac{d^3k}{(2\pi)^3} \beta (\mu - \varepsilon_k) v_y (\mathbf{v} \cdot \mathbf{\Omega}_F) \left( - \frac{\partial f_0}{\partial \varepsilon_k} \right), \\
\alpha_{yx}^{(4)} &= \tau e^2 B \sin \theta \int \frac{d^3k}{(2\pi)^3} \beta (\mu - \varepsilon_k) v_x (\mathbf{v} \cdot \mathbf{\Omega}_F) \left( - \frac{\partial f_0}{\partial \varepsilon_k} \right),
\end{align*}
\]

(4.4)

with \( \beta = 1/T \). Unlike the PHE set-up, there is no Jacobian factor \( D \), and hence we can consider generic values of \( B \).

We will assume a small temperature \( T \ll 1 \), and apply Sommerfeld expansion to facilitate analytic evaluations. Using the large \( \omega \) approximation, in addition with the Sommerfeld expansion, alters the regime of Eq. (2.18). In this case, the large \( \omega \) limit is captured by:

\[
\left[ \frac{e E_0 \alpha_{ij}^{1/2} \mu_j^{1/2}}{(\beta \mu)^{1/4} \Gamma^{1/4}} \right]^2 \ll \omega. \tag{4.5}
\]

Appendix C gives the details of the intermediate steps for calculating the final expressions of the transport coefficients.

Using a combination of the Sommerfeld expansion and the large \( \omega \) expansion, in the limit outlined above, we
obtain the expressions for the LTEC as:

\[
\alpha_{xx}^{(1)} = \begin{cases} 
-\frac{\varepsilon \mu \tau}{9 \beta v_0} + \frac{\varepsilon \tau A_0^6 \alpha_2}{25200 \v_0^3 \beta^2 \mu^8 \omega^4} \left[ 60 \beta \mu \left( 4 \beta^4 \mu^4 - 120 \beta^3 \mu^3 + 428 \beta^2 \mu^2 + 25 \beta \mu - 1020 \right) 
+ 7 \pi^2 \left( 120 \beta^4 \mu^4 - 140 \beta^3 \mu^3 - 1633 \beta^2 \mu^2 + 4764 \beta \mu - 3276 \right) \right] & \text{for } J = 1 \\
-\frac{2 \varepsilon \mu \tau}{9 \beta v_0} + \frac{\varepsilon \tau A_0^4 \alpha_2^2}{60480 \v_0^5 \beta^2 \mu^6 \omega^2} \left[ 3675 \pi \left( \pi^2 - 12 \right) \alpha_2 \beta A_0^2 + 2048 \left\{ \pi^2 \left( 70 \beta \mu + 7 \right) + 60 - 780 \beta \mu \right\} \right] & \text{for } J = 2, \\
-\frac{\varepsilon \mu \tau}{3 \beta v_0} + \frac{56 A_0^4 \varepsilon \pi^3 / 10 \alpha_2 \beta}{\sqrt{3} \v_0^4 \beta^4 \mu^{3/2} \omega^2 \Gamma(\frac{2}{3}) \Gamma(\frac{1}{3})} \left( \frac{14 \pi^2}{8415} + \frac{280 \beta \mu}{4301} - \frac{518 \pi^2 \beta \mu}{17595} + \frac{100 \beta^2 \mu^2}{391} + \frac{21 \pi^2 \beta^2 \mu}{391} 
- \frac{11688 \beta^3 \mu^3}{4301} + \frac{98 \pi^2 \beta^3 \mu^3}{391^2} \right) & \text{for } J = 3
\end{cases}
\]
FIG. 7. Plots for $\alpha_{xx}$ (LTEC) and $\alpha_{xx}$ (TTEC) as functions of the frequency $\omega$, for $B = 25 \text{eV}^2$. The values of the remaining parameters are taken from Table I. Subfigures (a) and (d) correspond to $J = 1$; (b) and (e) correspond to $J = 2$; and (c) and (f) correspond to $J = 3$.

$$\alpha^{(2)}_{xx} = \begin{cases} 
\frac{B^2 \pi^3 v_0^2 \cos^2 \theta \alpha_1^2}{6 \beta \mu^2} \left( \frac{7 \pi^2 \beta}{6 \beta \mu} + \frac{1}{2} \right) - \frac{B^2 e^2 v_0 \tau \cos^2 \theta A_0^0 \alpha_1^6}{\beta^3 \mu^3 \omega^2} \left[ \frac{1519 \pi^2}{108} + \frac{325 \beta \mu}{9} - \frac{385 \beta^2 \mu}{3} + \frac{25 \beta^2 \mu^2}{4} \right] 
& \text{for } J = 1 \\
\frac{B^2 \pi^3 v_0^2 \tau \cos^2 \theta A_0^0}{12 \beta \mu^2} \left( 1 + \frac{7 \pi^2 \beta}{6 \beta \mu} \right) + \frac{B^2 e^2 v_0 \tau \cos^2 \theta A_0^0 \alpha_3^3}{\beta^3 \mu^3 \omega^2} \left[ \frac{77 \pi^3 \beta \mu}{18} - \frac{35 \pi^3 \mu}{9} - \frac{50 \pi \beta \mu^3}{3} + \frac{5 \pi \beta^3 \mu}{3} \right] 
& + \frac{414 \pi^4 \beta^2 \mu^2 A_0^0 \alpha_2^2}{228} + \frac{112 \pi^3 \beta \mu A_0^0 \alpha_2^2}{96} + \frac{25 \pi \beta \mu A_0^0 \alpha_2^2}{192} + \frac{225 \pi \beta \mu A_0^0 \alpha_2^2}{192} + \frac{259 \pi \beta \mu A_0^0 \alpha_2^2}{192} + \frac{85 \pi \beta^2 \mu A_0^0 \alpha_2^2}{96} + \frac{133 \pi^3 \beta^2 \mu^2 A_0^0 \alpha_2^2}{1152} + \frac{33 \pi^3 \beta^3 \mu A_0^0 \alpha_2^2}{1152} 
& + \frac{17 \pi \beta \mu A_0^0 \alpha_2^2}{96} - \frac{35 \pi^3 \beta \mu A_0^0 \alpha_2^2}{2004} 
& \text{for } J = 2 \\
\frac{B^2 e^2 \pi v_0^2 \left( 27 \beta^2 \mu^2 + 2 \pi^2 \right) \cos^2 \theta A_0^0 \alpha_3^3}{21 \beta^5 \mu^{1/3} \Gamma \left( \frac{2}{3} \right)} 
+ \frac{B^2 e^2 \pi v_0^2 \alpha_3^3 \tau A_0^0 \cos^2 \theta A_0^0 \alpha_3^3}{512 \beta^4 \mu^{1/3} \omega^2 \Gamma \left( \frac{1}{2} \right)} \left[ 7 \pi^2 (90 \beta^3 \mu^3 + 135 \beta^2 \mu^2 - 7880 \beta \mu + 6424) 
& - 36 \beta \mu (1086 \beta^2 \mu^2 + 275 \beta \mu - 5800) \right] 
& \text{for } J = 3 \\
\end{cases}
$$

(4.7)

and

$$\alpha^{(3)}_{xx} = 0 \quad \forall J \in \{1, 2, 3\}.$$

(4.8)
In this section, we will discuss the results obtained for the various transport coefficients. With the choice of parameter values listed in Table I, the condition \((\beta \mu) \gg 1\) is always satisfied, and hence the validity of the semi-classical Boltzmann transport theory is justified. While the LMC and PHC expressions have been evaluated at \(T = 0\), and in the small \(B\) and large \(\omega\) limit, the TTEC and LTEC have been computed in the large \((\beta \mu)\) and large \(\omega\) limit. We also show the dependence of these transport coefficients on the magnitude \((B)\) and direction (via the angle \(\theta\)) of the magnetic field in Figs. 8 and 9, respectively.

V. Discussions and Physical Interpretation of the Results

In this section, we will discuss the results obtained for the various transport coefficients. With the choice of parameter values listed in Table I, the condition \((\beta \mu) \gg 1\) is always satisfied, and hence the validity of the semi-classical Boltzmann transport theory is justified. While the LMC and PHC expressions have been evaluated at \(T = 0\), and in the small \(B\) and large \(\omega\) limit, the TTEC and LTEC have been computed in the large \((\beta \mu)\) and large \(\omega\) regime.

**LMC and PHC** — For large \(\omega\), both the LMC and the PHC, regardless of the topological charge \(J\), gravitate towards the static case values (cf. Ref. [32]) very fast as \(\omega\) is increased, as seen in Figs. 2 and 3. This can be traced to the fact that the leading order correction falls off as \(A_0^2/\omega^2\) (with \(A_0 = eE_0/\omega\)), and even for \(\omega > 2\ eV\), the values of the conductivities start to saturate to the static values. In fact, this the reason why the dependencies on parameters like \(\mu\), \(B\), and \(\theta\) in the static case continue to dictate the dominant behaviour, even after including the time periodic drive. For all the cases of WSM and mWSMs, the Fermi energy dependence of LMC has a dominant \(\mu^2\) dependence. In contrast, PHC shows a more diverse behaviour, since PHC scales as (1) \(\mu^{-2}\) for \(J = 1\), (2) \(\mu^{-1}\) for \(J = 2\), and (3) \(\mu^{-2/3}\) for \(J = 3\), which was also observed in the static case [32].

For small \(B\) and large \(\omega\), LMC and PHC increase with \(B\) quadratically, as observed in the analytical expressions, as well as in Fig. 4. LMC has of course a piece which is independent of \(B\), while PHC is directly proportional to \(B^2\), which means it vanishes for \(B = 0\). The zeroth order term, which is independent of \(1/\omega\) and \(B\), is directly proportional to the topological charge \(J\), as is evident from the analytical expressions [as well as Fig. 4(a)]. The PHC, on the other hand, has a complicated dependence on \(J\), via \(\mu\) and the polynomial function \(\alpha_J\). It is worth pointing out that these dependencies continue to remain dominant even after the application of the periodic driving.

![Plots for (a) \(\alpha_{xx}\) (LTEC) and (b) \(\alpha_{yx}\) (TTEC), as functions of the magnetic field magnitude \(B\), for \(\omega = 5\ eV\). The values of the various other physical quantities used are shown in Table I.](image-url)
FIG. 9. Plots for (a) $\alpha_{xx}/(\alpha_{xx}|_{\theta=0})$ and (b) $\alpha_{yx}$ (TTEC), as functions of the angle $\theta$ between the external magnetic and electric fields, for $\omega = 10$ eV and $B = 10$ eV$^2$. The values of the various other physical quantities used are shown in Table I.

Hence, the topological charge etches a unique signature on both the LMC and PHC, and these can indeed be used as the identifiers of $J$ of the system.

From Eqs. (3.6), (3.7), and (3.8), we find that the LMC has a part which is independent of $\theta$ (where $\theta$ is the angle between $E$ and $B$), and a part which is proportional to $\cos(2\theta)$. This gives it a $\pi$-periodicity, as shown in Fig. 5(a). From Eq. (3.10), we find that the PHC is proportional to $\sin(2\theta)$, which gives it a $\pi$-periodicity as well [cf. Fig. 5(b)].

**LTEC and TTEC** — The LTEC for large $\omega$, small $T$, and small $B$, have similar dependencies on $\mu$ and $\beta$ for all values of $J$. The zeroth order term, which is independent of $1/\omega$, $B$, and $\beta$, is directly proportional to the topological charge $J$, as can be seen from the analytical expressions as well as the plots [cf. Fig. 8(a)]. The TTEC, on the other hand, is inversely proportional to $\beta\mu^{1+\frac{3}{2}}$. We also observe that the first nonzero term of the LTEC is independent of $\alpha_J$, while that of the TTEC is proportional to $\alpha_J^{2/J}$. The $\alpha_J$-dependence in LTEC shows up only in the leading order corrections. We again emphasize on the fact that these imprints of the topological charge in LTEC and TTEC are dominant, and can be used for experimental detection. The absence of the Jacobian factor $D$ from the integral expressions of LTEC and TTEC eliminates the need for small $B$ approximation. As shown in Fig. 8, the results from analytical approximations overlap with the numerical results even for large $B$, and show a quadratic dependence.

In Fig. 6(b), we find the LTEC goes through a minimum (and a sharp dip) near $\omega \sim 0.4$ eV. For the $J = 1$ case, this feature appears at a larger value of $B$, and a lower value of $\omega$. For $J = 3$, this occurs at even smaller values of $\omega$. Also, we note that the change in $B$ can lead to one of the terms completely eclipsing the other. This is the reason why there is no sharp dip visible in Fig. 7(b) – the sharpness observed in Fig. 6(b) has smoothed out here. In fact, as $B$ increases, the size of the dip decreases. We also point out that the extreme negative values, observed for the analytical results in these plots, appear for very small $\omega$ values, where our approximations start failing. Hence, these can be remedied to some extent by the inclusion of higher order corrections.

In Fig. 9(a), which shows the variations of LTEC with the change in the angle $\theta$ between the temperature gradient $\nabla T$ and $B$, we observe that the LTEC for $J = 2$ is the lowest in the entire range of $\theta$. This suggests that the leading order correction term for $J = 2$ is relatively small in comparison with that for $J = 1$ or $J = 3$. This is because $\alpha_{xx} = \alpha_{xx}^{(1)} + \alpha_{xx}^{(2)} + \alpha_{xx}^{(3)}$ (with $\alpha_{xx}^{(3)} = 0$) [cf. Eqs. (4.6), (4.7), and (4.8)], and $\alpha_{xx}^{(2)}/\alpha_{xx}^{(1)}$ is the smallest for $J = 2$. We also note that $\alpha_{xx}^{(1)}$ is $\theta$-independent, while $\alpha_{xx}^{(2)}$ is proportional to $\cos^2 \theta$, which makes $\alpha_{xx}$ periodic in $\pi$. From Eq. (4.9), we note that the TTEC is proportional to $\sin(2\theta)$, which again indicates a $\pi$-periodicity [cf. Fig. 9(b)].

**VI. Summary and Outlook**

In this paper, we have evaluated various transport coefficients for WSMs and mWSMs in the planar Hall and planar thermal Hall set-ups, when the system is perturbed by a periodic drive. We have used the low-energy effective Hamiltonian for a single node, and using the Floquet theorem, we have obtained the leading order corrections in the high frequency limit. This serves as a complementary signature for these semimetal systems, in addition to studies
of other transport properties (see, for example, Refs. [7, 32, 37, 42, 50, 53–55]). In other words, the periodic drive provides another control knob, in terms of the frequency dependence of the drive, making the transport coefficients depend on an additional parameter \( \omega \).

For evaluating the analytical expressions for the magneto-transport coefficients, we have used the large frequency limit captured by Eq. (2.18), and shown only the leading order corrections. Using the systematic scheme we have followed, higher order corrections can be incorporated into the expressions. Our results show that for large but finite \( \omega \), the values of the concerned transport coefficients are significantly different compared to the static case (corresponding to \( \omega \to \infty \)). As is evident from Eqs. (3.6)-(3.9) and Eqs. (4.6)-(4.9), the transport coefficients are complicated functions of the chemical potential \( \mu \) and the parameter \( \alpha J \). Such dependence is absent in the static counterpart, as can be verified by substituting \( \omega = \infty \). The difference from the static values amplifies especially for small values of the chemical potential, as some of the terms in the analytical expressions contain inverse powers of \( \mu \).

The thermopower in the presence of a quantizing magnetic field was computed for the 2d double-Weyl case in Ref. [15]. It will be interesting to compute the effect of Floquet driving for such cases. Yet another direction is to consider the Magnus Hall effect [55] in presence of a periodic drive. A challenging avenue is to carry out the conductivity calculations in the presence of interactions (such that when interactions affect the quantized physical observables in the topological phases [56–58], or when non-Fermi liquids emerge [33, 59, 60]), and / or disorder [61–64].

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A. Computational details for LMC and PHC

In this section, we will outline the computation of the terms shown in Eq. (3.2). Using \( \mathbf{v} \cdot \mathbf{\Omega}_F = \frac{v_0 \alpha_j^2 j^2 k_{\perp}^{2j-2}}{2 \varepsilon_k^j} \) and \( \mathbf{B} \cdot \mathbf{\Omega}_F = \frac{J \alpha_j^2 k_{\perp}^{2j-1} v_0 B \cos(\theta - \phi)}{2 \varepsilon_k^j} \) [cf. Eq. (3.5)], we expand the integrals in Eqs. (3.3) and (3.4) in small \( \varepsilon_k \) and take care of the terms in the Jacobian \( \mathcal{D} \), and keep terms upto \( \mathcal{O}(B^2) \). This leads to:

\[
\sigma_{xx}^{(1)} = \frac{e^2 \tau J}{v_0} \int_{\phi=0}^{2\pi} \int_{\gamma=0}^{\pi} \int_{k=0}^{\infty} \frac{d\phi \, d\gamma \, dk}{(2\pi)^3} \frac{k \cos^2 \phi}{\varepsilon_k^2} \delta(\mu - \varepsilon_k) \left[ \frac{(k \sin \gamma)^3}{\varepsilon_k^2} - \frac{(k \sin \gamma)^{5-2\theta} \cos(\theta - \phi) e v_0 B J \alpha_0^j}{2 \varepsilon_k^7} \right],
\]

\[
\sigma_{xx}^{(2)} = \frac{e^2 \tau J^2}{4 v_0} \frac{e v_0 B \cos \theta \alpha_0^j}{\varepsilon_k^4},
\]

\[
\sigma_{xx}^{(3)} = \frac{e^2 \tau J^2}{v_0} \frac{e v_0 B \cos \theta \alpha_0^j}{\varepsilon_k^4},
\]

\[
\sigma_{yx}^{(1)} = \frac{e^2 \tau J}{2 v_0} \int_{\phi=0}^{2\pi} \int_{\gamma=0}^{\pi} \int_{k=0}^{\infty} \frac{d\phi \, d\gamma \, dk}{(2\pi)^3} \frac{k \sin(2\phi)}{\varepsilon_k^2} \delta(\mu - \varepsilon_k) \left[ \frac{(k \sin \gamma)^3}{\varepsilon_k^2} - \frac{(k \sin \gamma)^{5-2\theta} \cos(\theta - \phi) e v_0 B J \alpha_0^j}{2 \varepsilon_k^7} \right].
\]

After performing the \( \phi \)-integrals, we can express the above integrals in a generic form as shown below:

\[
I_{\gamma, k} = \int d\gamma \, dk \, f(\gamma, k) \delta(\mu - k - \chi J \cos \gamma),
\]
where we have suppressed the \( k \) and \( \gamma \) dependence of \( \chi_J(k, \gamma) \) in order to unclutter the notations. Using the integral representation of the Dirac delta function, we recast it as:

\[
I_{\gamma, k} = \frac{1}{2\pi} \int_{\xi = -\infty}^{\infty} \int_{\gamma = 0}^{\pi} d\xi \, dk \, e^{i \xi (\mu - \chi_J \cos \gamma)} \, f(\gamma, k).
\]

Next, we incorporate the large \( \omega \) approximation (i.e., expand about \( \frac{1}{\omega} = \infty \)), and retain the leading order correction in \( \frac{1}{\omega} \). This gives us:

\[
I_{\gamma, k} = \frac{1}{2\pi} \int d\gamma \, dk \, d\xi \, e^{i \xi (\mu - k)} \, f(\gamma, k) (1 - i \xi \chi_J \cos \gamma) = \int_{0}^{\pi} d\gamma \, f(\gamma, \mu) - \left[ \frac{d}{dk} \int_{0}^{\pi} d\gamma \, f(k, \mu) \, \chi_J(k, \gamma) \cos \gamma \right]_{k=\mu}.
\]

(A3)

We now use this form to evaluate the final integrals, and get the final expressions [cf. Eqs. (3.6)-(3.10)] shown in the main text.

**B. Derivation of the Conductivities for the Planar Thermal Hall Set-up**

Using Eq. (2.8), and the semi-classical transport equations

\[
\dot{r} = \mathcal{D} \{ v + e (v \cdot \Omega_F) B \}, \quad \dot{k} = \mathcal{D} \{ e (v \times B) + e^2 (E \cdot B) \Omega_F \},
\]

within the planar thermal Hall setup [i.e., with \( \mathbf{E} = 0 \) and \( \nabla T = (\frac{dT}{dx}, 0, 0) \)], we get:

\[
\frac{f_0 - f}{\mathcal{D} \tau} = \{ [v + e (v \cdot \Omega_F) B] \cdot \nabla r f + e (v \times B) \cdot \nabla k f \}.
\]

(B2)

From

\[
\nabla r f = \beta (\varepsilon_k - \mu) \nabla T \left( -\frac{\partial f_0}{\partial \varepsilon_k} \right) \quad \text{and} \quad \nabla k f = \varepsilon \frac{\partial f_0}{\partial \varepsilon_k},
\]

we note that \( \nabla T \) enters in the response by causing a variation in \( f \). These partial derivatives allow us to rewrite Eq. (B2) as:

\[
\frac{f - f_0}{\mathcal{D} \tau} = \{ [v + e (v \cdot \Omega_F) B] \cdot \nabla T \} \beta (\varepsilon_k - \mu) \frac{\partial f_0}{\partial \varepsilon_k}.
\]

Finally, the electric current for a single WSM or mWSM node is given by [51]:

\[
J_{\alpha} = -e \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \mathcal{D} \{ \dot{r} \}_\alpha f + e \beta \int \frac{d^3 \mathbf{k}}{(2\pi)^3} \{ \nabla T \cdot \Omega_F \} \left[ (\varepsilon_k - \mu) f_0 + \frac{\ln (1 + e^{-\beta (\varepsilon_k - \mu)})}{\beta} \right], \quad \alpha \in (x, y).
\]

(B3)

Comparing this with Eq. (2.10) furnishes the integrals in Eq. (4.1).

**C. Computational details for LTEC and TTEC**

In this section, we will outline the computation of the terms shown in Eq. (4.2). From Eqs. (4.3) and (4.4), after some simplifications, the nonzero and independent terms are given by:

\[
\alpha^{(1)}_{xx} = \int_{0}^{\tau} \frac{e}{v_0} \int \frac{dk \, d\gamma \, d\phi}{(2\pi)^3} \cos^2 \phi \, k \, (k \sin \gamma)^{1/2} \, \beta (\mu - \varepsilon_k) \frac{\beta e^{\beta (\varepsilon_k - \mu)}}{[1 + e^{\beta (\varepsilon_k - \mu)}]^2},
\]

\[
\alpha^{(2)}_{xx} = \frac{J^3 \tau e}{4 v_0} (e v_0 B \cos \theta) \alpha^{(1)}_{jJ} \int \frac{dk \, d\gamma \, d\phi}{(2\pi)^3} \, (k \sin \gamma)^{3/2} \, \frac{\beta (\mu - \varepsilon_k)}{\varepsilon_k^{3/2}} \frac{\beta e^{\beta (\varepsilon_k - \mu)}}{(1 + e^{\beta (\varepsilon_k - \mu)})^2}.
\]

(C1)

After performing the \( \phi \)-integrals, we get:

\[
\alpha^{(\eta)}_{xx} \propto \int \frac{dk \, d\gamma \, k \, (k \sin \gamma)^{3-2(\eta-1)} \beta (\mu - \varepsilon_k) \beta (\varepsilon_k - \mu)}{\varepsilon_k^{\eta} [1 + e^{\beta (\varepsilon_k - \mu)}]^2}, \quad \eta \in (1, 2).
\]
We proceed by expanding the integrand in the large ω limit (i.e., about $\frac{1}{\beta} = \infty$). In the next step, the terms need to be evaluated by using the Sommerfeld’s expansion. This involves integrals of the form:

$$J_{p,q} = \int dk k^m \frac{\beta e^{\beta(k-\mu)}}{(1 + e^{\beta(k-\mu)})^q},$$  \hspace{1cm} \text{(C2)}

where $p$ and $q$ are positive integers.

First, let us consider

$$J_{1,2} = \frac{dI}{d\mu}, \text{ where } I = \int_0^\infty dk k^m \frac{1}{1 + e^{\beta(k-\mu)}}.$$

With a variable change $\beta (k-\mu) = x$, we get:

$$I = \frac{1}{\beta} \int_{-\beta \mu}^{\infty} dx \left( \frac{x + \mu}{\beta} \right)^m \frac{1}{1 + e^x}$$

$$= \frac{1}{\beta} \int_0^{\beta \mu} dx \left( -\frac{x}{\beta} + \mu \right)^m \frac{1}{1 + e^{-x}} + \frac{1}{\beta} \int_0^{\infty} dx \left( \frac{x + \mu}{\beta} \right)^m \frac{1}{1 + e^x}$$

$$= \frac{\mu^{m+1}}{m+1} + \mu^{m+1} \left[ \frac{\pi^2 m}{6 (\beta \mu)^2} + \frac{7 \pi^4 m (m-1)(m-2)}{360 (\beta \mu)^4} + \cdots \right].$$

As $(\beta \mu) \gg 1$, the integrand decreases exponentially with $x$. We have used this fact to arrive at the above expression. Hence, for $(\beta \mu) \gg 1$, we get:

$$J_{1,2} = \int_0^\infty dk k^m \frac{\beta e^{\beta(m-k)}}{[1 + e^{\beta(m-k)}]^2} = \mu^m \left[ 1 + \frac{\pi^2 m (m-1)}{6 \beta^2 \mu^2} + \frac{7 \pi^4 m (m-1)(m-2)(m-3)}{360 \beta^4 \mu^4} + \cdots \right].$$ \hspace{1cm} \text{(C3)}

Finally, we can compute $J_{p+1,q+1}$ and $J_{p,p+2}$ (integrals of these forms are sufficient to evaluate the expressions for $\alpha_{xx}^{(1)}$ and $\alpha_{xx}^{(2)}$) using the following relations:

$$J_{p+1,q+1} = \frac{d}{dq} \frac{p \beta}{q \beta} J_{p,q} = J_{p,p+2} = J_{p+1} - J_{p+1,p+2}.$$

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