Regularity of weak solutions to the model Venttsel problem for solutions of linear parabolic systems with nonsmooth in time principal matrix. A(t)-caloric method

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We consider a model Venttsel type problem for linear parabolic systems of equations. The Venttsel type boundary condition is fixed on the flat part of the lateral surface of a given cylinder. It is defined by parabolic operator (with respect to the tangential derivatives) and the conormal derivative. The Hölder continuity of a weak solution of the problem is proved under optimal assumptions on the data. In particular, only boundedness in the time variable of the principal matrices of the system and the boundary operator is assumed. All results are obtained by so-called A(t)-caloric method [1].

1 Introduction

In this paper we study regularity of weak solutions of the linear parabolic systems under the Venttsel boundary condition on the flat part of the lateral surface of a given cylinder.

Let $B_1(0) = \{x \in \mathbb{R}^n : |x| < 1\}$, $B_1^+(0) = B_1(0) \cap \{x_n > 0\}$, $Q_1(0) = B_1(0) \times (-1,0)$, and $Q_1^+(0) = Q_1(0) \cap \{x_n > 0\}$.

We consider a solution $u : Q_1^+(0) \rightarrow \mathbb{R}^N$, $N > 1$, of the problem

$$u_t - \text{div}(a(z) \nabla u) = f(z), \quad z \in Q_1^+(0),$$

$$u_t - \text{div}'(b(z') \nabla' u) + \frac{\partial u}{\partial n} = \psi(z'), \quad z' \in \Gamma_1(0),$$

where $\Gamma_1(0) = Q_1(0) \cap \{x_n = 0\}$, $x = (x', x_n)$, $z' = (x', 0, t)$, $\text{div}'(\cdot) = \sum_{i=1}^{n-1} \frac{\partial u}{\partial x_i}$, $\nabla' u = (u_{x_1}, ..., u_{x_{n-1}})$. We assume that $a(z)$ and $b(z')$ are bounded positive definite nondiagonal $[nN \times nN]$ and $[(n-1)N \times (n-1)N]$ matrices, $\frac{\partial u}{\partial n} = (a(z) \nabla u, n)$- vector of the conormal derivative. Let $f \in L^2(Q_1^+(0))$, $\psi \in L^2(\Gamma_1(0))$, more exact assumptions on the data will be formulated below in conditions $\text{H1} - \text{H5}$.

Definition 1.1 A function $u \in H := L^2(\Lambda_1; W^1_2(B_1^+)) \cap L^2(\Lambda_1; W^1_2(\gamma_1))$ is a weak solution to problem (1), (2) if it satisfies the identity

$$\int_{Q_1^+} [-u \eta_t + (a(z) \nabla u, \nabla \eta)] dz + \int_{\Gamma_1} [-u \eta_t + b(z') \nabla' u, \nabla' \eta] d\Gamma = \int_{Q_1^+} f \eta dz + \int_{\Gamma_1} \psi \eta d\Gamma,$$

with any function $\eta \in C^\infty(\bar{Q}_1^+)$, $\text{supp} \eta \subset Q_1^+ \cup \Gamma_1$. 
Here and below we denote $\Lambda_1 = (-1, 0)$, $\gamma_1 = B_1(0) \cap \{x_n = 0\}$. Certainly, we can assume that the test-functions $\eta$ belong to the space $W^1_2(Q^+_1)$, $\eta|_{\partial Q^+_1 \setminus \Gamma_1} = 0$.

The boundary condition (2) includes both the conormal derivative of $u$ and parabolic second order operator with an elliptic operator relative to the tangential derivatives. Such boundary condition is referred to the Venttsel condition. A specific of the problem under consideration is that the boundary operator is very strong one, and integral identity (3) is not homogeneous one with respect to similarity transformation of the variables.

To our days the \textit{scalar} ($N = 1$) Venttsel problem (considered in [23] under more general boundary condition) was studied for elliptic and parabolic nonlinear operators of different classes (see [22] and references therein).

In the case of vector-functions ($N > 1$), regularity of weak solutions to the stationary Venttsel problem was studied in [3] and [4]. Elliptic operators with constant coefficients were considered in [3] and the Campanato integral estimates were obtained for smooth solutions of the Venttsel problem. Regularity of weak solutions of the \textit{linear} elliptic Venttsel problem was studied in the scale of the Morrey-Campanato spaces in [3]. In particular, the Hölder continuity of solutions and their first and second derivatives were proved in [4], $N > 1$.

In [5], the author considered the Venttsel problem for \textit{quasilinear} elliptic operators, $N > 1$. It is well known that one can expect only \textit{partial} regularity of weak solutions of systems with quasilinear operators ([16], [13], [14], [17] and references therein). Moreover, it was proved that singularities may be concentrated near the boundary even under trivial Dirichlet condition [15]. In [5], the author proved partial regularity of weak solutions of the Venttsel problem for quasilinear elliptic operators with matrices $a(x, u)$ and $b(x', u)$ ($f, \psi = 0$) and estimated the Hausdorff measure of admissible singularities in the vicinity of the boundary. To study regularity, it was applied the so-called A-harmonic method [20]. The method allowed to relax continuity assumptions on $a(x, u)$ and $b(x', u)$ in variables $x$ and and $x'$ respectively to the integral (VMO) continuity conditions.

Here we study regularity of weak solutions (3) of the Venttsel problem for linear parabolic operators in the model setting (1), (2) and apply so-called $A(t)$-caloric method.

The method of $A$-caloric approximation was proposed by F.Duzaar, G. Mingione in [18] to study regularity of weak solutions to a wide class of nonlinear parabolic systems (see also [19]). According to this method, one can estimate locally the difference in $L^2$-norm between a smooth solution of the simplest parabolic system with the constant matrix $A$ and a solution of the nonlinear system under consideration. The main $A$-caloric lemma was later modified in [1] to $A(t)$-caloric lemma. Such modification allowed to compare well enough a solution of quasilinear parabolic system with the principal non smooth in time matrix and a solution of the model system with the principal matrix $A(t)$ where $A(t)$ is only bounded in $t$. The application of $A(t)$-caloric lemma allowed to prove partial regularity of weak solutions of quasilinear parabolic systems with matrices $a(x, t, u)$ which are VMO-smooth in $x$ and only bounded in time variable $t$. Then it was proved partial regularity of a solution of the Cauchy-Dirichlet problem for quasilinear parabolic systems under the same assumptions on the principal matrix [2].

It should be mentioned works [6] - [11] where different classes of nonlinear parabolic \textit{scalar} equations and linear parabolic systems were studied with non smooth in time principal matrices.

In this paper we formulate $A(t)$-caloric lemmas in an appropriate form (Section 3) and prove Hölder continuity of weak solutions defined in (3) under relax conditions on the matrices $a(x, t)$ and $b(x', t)$. More exactly, we assume only integral continuity of these matrices in the space variables and boundedness in the time variable $t$. 
We introduce the following notation

\[ B_R(x^0) = \{ x \in \mathbb{R}^n : |x - x^0| < R \}, \quad B^+_R(x^0) = B_R(x^0) \cap \{ x_n > x_n^0 \}, \quad S_R(x^0) = \partial B_R(x^0), \]
\[ S^+_R(x^0) = S_R(x^0) \cap \{ x_n > x_n^0 \}, \quad \gamma_R(x^0) = B_R(x^0) \cap \{ x_n = x_n^0 \}, \quad \Lambda_R(t^0) = (t^0 - R^2, t^0), \]
\[ Q_R(z^0) = B_R(x^0) \times (\Lambda_R(t^0)), \quad Q^+_R(z^0) = B^+_R(x^0) \times \Lambda_R(t^0), \quad \Gamma_R(z^0) = \gamma_R(x^0) \times \Lambda_R(t^0); \]

we write \( Q_R, Q^+_R, \Gamma_R, B^+_R, S^+_R \) if \( z^0 = 0 \); \( |D|_m = \text{mes}_m D \) - is the Lebesgue measure of \( D \) in \( \mathbb{R}^m \), \( \omega_n = \text{mes}_n B_1 \);

\[ \int_D f \, dz = \frac{1}{|D|^{n+1}} \int_D f \, dz, \quad v_{r,z^0} = \int_{Q^+_1 \cap Q^+_r(z^0)} v \, dz, \quad v_{r,\alpha}(t) = \int_{B^+_1 \cap B_r(x^0)} v(x, t) \, dx, \quad \frac{\partial v}{\partial x_\alpha}; \]

\[ \|v\|_{p,Q} \] is the norm in \( L^p(Q) \); \( W^1_2(Q) \) is the Sobolev space of functions \( v \) in \( L^2(Q) \) possessing weak derivatives \( v_{x_1}, v_{t} \in L^2(Q), i = 1, \ldots, n \).

We denote by \( L^{2,\lambda}(Q^+_1; \delta) \) and \( L^{2,\lambda}(Q^+_1; \delta) \) the Morrey and Campanato spaces in the \textit{parabolic} metric \( \delta(z^1, z^2) = \max \{|x^1 - x^2|, |t^1 - t^2|^{1/2}\} \) \cite{[12]}:

\[ L^{2,\lambda}(Q^+_1; \delta) = \{ v \in L^2(Q^+_1) : \|v\|_{2,\lambda,Q^+_1} < \infty \}, \quad \lambda \in (0, n + 2], \]

where

\[ \|v\|_{2,\lambda,Q^+_1} = \left( \sup_{\rho \leq 1, z^0 \in Q^+_1} \frac{1}{\rho^\lambda} \int_{Q^+_\rho(z^0) \cap Q^+_1} |v|^2 \, dz \right)^{1/2}; \]

the space \( L^{2,\lambda}(Q^+_1; \delta), \lambda \in (0, n + 4], \) is the space of functions from \( L^2(Q^+_1) \) with the finite norm

\[ \|v\|_{2,\lambda,Q^+_1} = \|v\|_{2,\lambda,Q^+_1} + [v]_{2,\lambda,Q^+_1}; \]

where the seminorm

\[ [v]_{2,\lambda,Q^+_1} = \sup_{\rho \leq 1, z^0 \in Q^+_1} \frac{1}{\rho^\lambda} \int_{Q^+_\rho(z^0) \cap Q^+_1} \|v(z) - v_{r,z^0}\|^2 \, dz < \infty. \]

We put

\[ W^{1,0}(Q^+_R(z^0)) = L^2(\Lambda_R(t^0); W^1_2(B^+_R(x^0))), \quad W^{1,0}(\Gamma_R(z^0)) = L^2(\Lambda_R(t^0); W^1_2(\gamma_R(x^0))); \]
\[ H(Q^+_R(z^0)) = W^{1,0}(Q^+_R(z^0)) \cap W^{1,0}(\Gamma_R(z^0)), \]

and

\[ V^{(1)}(Q^+_R(z^0)) = C(\Lambda_R(t^0); L^2(B^+_R(x^0))) \cap L^2(\Lambda_R(t^0); W^1_2(B^+_R(x^0))), \]
\[ V^{(2)}(\Gamma_R(z^0)) = C(\Lambda_R(t^0); L^2(\gamma_R(x^0))) \cap L^2(\Lambda_R(t^0); W^1_2(\gamma_R(x^0))), \]
\[ V(Q^+_R(z^0)) = V^{(1)}(Q^+_R(z^0)) \cap V^{(2)}(\Gamma_R(z^0)). \]

We write \( v \in B(Q) \) instead of \( v \in B(Q; \mathbb{R}^N) \) for the sake of brevity. Different constants depending on the data of the problem are denoted by \( c, c_i \).
2 The main results

We formulate the basic assumptions:

- **H1** The matrix $a$ is defined in $Q_1^+(0)$ and has measurable bounded entries. There are positive constants $\nu, \mu$ such that
  \[
  (a(z) \xi, \xi) = a^\alpha(z) \xi_i \xi_j \geq \nu |\xi|^2, \quad \xi \in \mathbb{R}^{nN},
  \]
  \[
  |(a(z)p, q)| \leq \mu |p||q|, \quad p, q \in \mathbb{R}^{nN},
  \]
  for almost all $z \in Q_1^+$.

- **H2** $a(\cdot, t) \in VMO(B_1^+)$ for almost all $t \in \Lambda_1$ and
  \[
  \sup_{\rho \leq r, z^0 \in Q_1^+} \int_{\Lambda_\rho(t^0)} \int_{B_\rho(x^0) \cap B_1^+} |a(y, t) - a_{\rho, x^0}(t)|^2 \, dy \, dt =: q_a^2(r) \to 0, \quad r \to 0,
  \]
  where
  \[
  a_{\rho, x^0}(t) = \int_{B_1^+ \cap B_\rho(x^0)} a(y, t) \, dy.
  \]

- **H3** The matrix $b$ is defined on $\Gamma_1$ and has measurable bounded entries. There are positive constants $\nu_0$ and $\mu_0$ such that
  \[
  (b(z') \xi, \xi) = b_i^\alpha(z') \xi_i \xi_j \geq \nu_0 |\xi|^2, \quad \xi \in \mathbb{R}^{(n-1)N},
  \]
  \[
  |b(z')p, q)| \leq \mu_0 |\xi|^2, \quad p, q \in \mathbb{R}^{(n-1)N},
  \]
  for almost all $z' \in \Gamma_1$.

- **H4** $b(\cdot, t) \in VMO(\gamma_1)$ for almost all $t \in \Lambda_1$ and
  \[
  \sup_{\rho \leq r, z^0 \in \gamma_1} \int_{\Lambda_\rho(t^0)} \int_{\gamma_1 \cap \gamma_\rho(t^0)} |b(y', t) - b_{\rho, x^0}(t)|^2 \, dy' \, dt =: q_b^2(r) \to 0, \quad z' = (\xi', t'), \quad r \to 0.
  \]

- **H5** The function $f \in L^{2,\lambda}(Q_1^+)$, $\psi \in L^{2,\lambda}(\Gamma_1)$ where $\lambda = n - 3 + 2\alpha$, $\alpha \in (0, 1)$, and $n \geq 3$.

- **H5'** The function $f \in L^{2,n-2+2\alpha}(Q_1^+; \delta)$, and $\psi \in L^{2,n-3+2\alpha}(\Gamma_1; \delta)$, $\alpha \in (0, 1)$, $n \geq 3$.

**Theorem 2.1** Let assumptions **H1-H5** hold, $n \geq 3$, and $u \in H$ be a weak solution to problem (7), (8). Then
1) $u^G \in C^\alpha(\bar{\Gamma}_R; \delta)$, $\nabla' u^G \in L^{2,n-1+2\alpha}(\Gamma_R; \delta)$ for any $R < 1$, $u^G(z') = u(x', 0, t)$ and
  \[
  ||u^G||_{L^{2,n-1+2\alpha}(\Gamma_R; \delta)} + ||\nabla' u^G||_{L^{2,n-1+2\alpha}(\Gamma_R; \delta)} \leq c \mathcal{M}_0
  \]
  where
  \[
  \mathcal{M}_0 = ||u||_{L^{2,1,0}(Q_1^+)} + ||u||_{L^{2,1,0}(\Gamma_1)} + ||f||_{L^{2,n-2+2\alpha}(Q_1^+; \delta)} + ||\psi||_{L^{2,n-3+2\alpha}(\Gamma_1; \delta)}.
  \]
Theorem 2.2 Let the assumptions H1-H4 and H5’ hold, $n \geq 3$. Then additionally to the assertion of Theorem 2.1 $u \in C^0(Q_R^+(0); \delta)$, $\nabla u \in L^{2,n+2\alpha}(Q_R^+(0); \delta)$ for any $R < 1$ and
\[
\|u\|_{C^0(Q_R^+(0); \delta)} + \|\nabla u\|_{L^{2,n+2\alpha}(Q_R^+(0); \delta)} \leq c \mathbb{L}_0
\]
where \[
\mathbb{L}_0 = \|u\|_{W^{1,0}_2(Q_1^+(0))} + \|u^G\|_{W^{1,0}_2(Q_1(0))} + \|f\|_{L^{2,n-2+2\alpha}(Q_1^+; \delta)} + \|\psi\|_{L^{2,n-3+2\alpha}(\Gamma_1; \delta)}.
\]

Theorem 2.3 Let $n = 2$, the assumptions H1-H4 hold and $u$ be a weak solution to problem (1), (2).
I. Let $f \in L^{2,\lambda_0}(\Gamma_1; \delta)$, $\psi \in L^{2,\lambda_0}(\Gamma_1; \delta)$ with some $\lambda_0 \in [0,1)$. Then
i) $u^G \in C^{1/2}(\Gamma_1-q_0; \delta)$, $\nabla u^G \in L^{2,2}(\Gamma_1-q_0; \delta)$ if $\lambda_0 = 0$;
ii) $u^G \in C^{\alpha_0}(\Gamma_1-q_0; \delta)$, $\nabla u^G \in L^{2,2+\lambda_0}(\Gamma_1-q_0; \delta)$, $\alpha_0 = \frac{\lambda_0+1}{2}$ if $\lambda_0 \in (0,1)$.
II. Let $f \in L^{2,\lambda_0+1}(Q_1^+; \delta)$, $\psi \in L^{2,\lambda_0}(\Gamma_1; \lambda_0 \in (0,1)$. Then
\[
u \in C^{\alpha_0}(\overline{Q_1^+}_1; \delta), \nabla u \in L^{2,2+\lambda_0}(Q_1^+; \delta), \alpha_0 = \frac{\lambda_0+1}{2}.
\]
In all the assertions number $q \in (0,1)$ is fixed arbitrarily.

3 Auxiliary results

In this section we introduce versions of the Caccioppoli and Poincaré inequalities for weak solutions of the problem and formulate $A(t)$-caloric lemmas in an appropriate form.

Lemma 3.1(The Caccioppoli’s inequality) Assume that the conditions H1 and H3 hold, $f \in L^2(Q_1^+)$, $\psi \in L^2(\Gamma_1)$. Then for all $z^0 \in \Gamma_1$, $Q_{R^2}(z^0) \subset Q_1^+$, $l \in \mathbb{R}^N$, a solution $u$ of (3) satisfies the inequality
\[
\int_{Q_{R/2}(z^0)} |\nabla u|^2 \, dz + \int_{\Gamma_{R/2}(z^0)} |\nabla' u|^2 \, d\Gamma \leq \frac{c}{R^2} \int_{Q_{R^2}(z^0)} |u-l|^2 \, dz + \int_{\Gamma_R(z^0)} |u-l|^2 \, d\Gamma + c R^2 \int_{Q_{R^2}(z^0)} |f|^2 \, dz + c R^2 \int_{\Gamma_R(z^0)} |\psi|^2 \, d\Gamma.
\]
The constants $c$ depend on the parameters from conditions H1 and H3.

Proof It is not difficult to check that any solution of (3) belongs the class $V$ (see the notation in Section 2). We omit smoothing in time the Steklov average procedure and put formally in (3) $\eta = (u-l)\xi^2(x)\theta(t)$ where $\xi(x)$ is a cut-off function for $B_R(x^0)$, $\xi = 1$ in $B_{R/2}(x^0)$, $\theta \in C^0([\Lambda_R(t^0)])$, $\theta = 1$ in $\Lambda_{R/2}(t^0)$, $x^0 \in \Gamma_1$. Inequality (10) follows by the standard way.
Remark 3.1 Let here and below $u^G(x', t) := u(x', 0, t)$, $u^0 = u(z) - u^G(z')$ and $l = (u^G)_{R,z^0}$ in (10). Then it follows from (10) that

$$
\int_{Q_{R/2}^+(z^0)} |\nabla u|^2 dz + \int_{\Gamma_{R/2}(z^0)} |\nabla' u|^2 d\Gamma \leq \frac{c}{R^2} \int_{Q_R^+(z^0)} |u^0|^2 dz + \frac{c}{R^2} \int_{\Gamma_R(z^0)} |u^G - u^G_{R,z^0}|^2 d\Gamma \\
+ cR^2 \int_{Q_R^+(z^0)} |f|^2 dz + cR^2 \int_{\Gamma_R(z^0)} |\psi|^2 d\Gamma \quad (11)
$$

Lemma 3.2 (The Poincaré inequality) Let the assumptions $H1$ and $H3$ hold, $z^0 \in \Gamma_1$, $Q_R^+(z^0) \subset Q_1^+$. Then for a weak solution $u$ of problem (1), (2) the following inequality hold:

$$
\int_{Q_{R/2}^+(z^0)} |u(z) - u_{R/2,z^0}|^2 dz + \int_{\Gamma_{R/2}(z^0)} |u^G(z') - u^G_{R/2,z^0}|^2 d\Gamma \\
\leq c R^2 \left( \int_{Q_R^+(z^0)} |\nabla u|^2 dz + \int_{\Gamma_R(z^0)} |\nabla' u^G|^2 d\Gamma \right) + c R^4 \left( \int_{Q_R^+(z^0)} |f|^2 dz + \int_{\Gamma_R(z^0)} |\psi|^2 d\Gamma \right) \quad (12)
$$

The constants $c$ in (12) depend on the parameters from assumptions $H1$ and $H3$.

Proof. As was noted in Lemma 3.1, weak solution of (3) is a function from the class $V$. For a point $z^0 \in \Gamma_1$ and a cylinder $Q_R^+(z^0)$ we fix $s \in \Lambda_R(t^0) \setminus \Lambda_{R/2}(t^0)$ and $\tau \in (s, t^0)$. We fix the same cut-off function $\xi(x)$ as in the proof of Lemma 3.1. Let $\chi_\varepsilon(t)$ be a piecewise linear function, $\chi_\varepsilon = 1$ for $t \in [s, \tau]$ and $\chi_\varepsilon = 0$ when $t < s - \varepsilon$ and $t > s + \varepsilon$. To simplify the explanation, we omit the Steklov average procedure and put in (3) $\eta = (u(z) - l)\xi^2(x)\chi_\varepsilon(t)$ with any constant vector $l \in \mathbb{R}^N$. After simple calculations we turn $\varepsilon$ to zero, and obtain the inequality

$$
\int_{B_R^+(x^0)} |u(x, t) - l|^2 \xi^2 dx \big|_{t=s}^{t=\tau} + \int_{\Gamma_R^+(x^0)} |u^G(x', t) - l|^2 \xi^2 d\gamma \big|_{t=s}^{t=\tau} + v \int_{s}^{\tau} \int_{B_R(x^0)} |\nabla u|^2 \xi^2 dx dt \\
+ u_0 \int_{s}^{\tau} \int_{\gamma_R(x^0)} |\nabla' u^G|^2 \xi^2 dx' dt \leq \mu \int_{s}^{\tau} \int_{B_R(x^0)} |\nabla u| |u - l| |\nabla \xi| dx dt \\
+ \mu_0 \int_{s}^{\tau} \int_{\gamma_R(x^0)} |\nabla' u^G| |u^G - l| |\nabla ^2 \xi| d\gamma dt + \mu_0 \int_{s}^{\tau} \int_{B_R(x^0)} |f| |u - l|^2 \xi^2 dx dt \\
+ \mu_0 \int_{s}^{\tau} \int_{\gamma_R(x^0)} |\psi| |u^G - l|^2 \xi^2 d\gamma dt \quad (13)
$$

Now we put $l = u^G_{R,x^0}(s)$ and estimate integrals with the constants $\mu$ and $\mu_0$ in the way:

$$
\mu \int_{s}^{\tau} \int_{B_R^+(x^0)} |\nabla u| |u - u^G_{R,x^0}(s)| |\nabla \xi| dx dt \leq \delta \sup_{t \in (s, t^0)} \int_{B_R^+(x^0)} |u(x, t) - u^G_{R,x^0}(s)|^2 \xi^2 dx \\
+ c/\delta \int_{s}^{\tau} \int_{B_R^+(x^0)} |\nabla u|^2 dx dt,
$$

$$
\mu_0 \int_{s}^{\tau} \int_{\gamma_R(x^0)} |\nabla u| |u - u^G_{R,x^0}(s)| |\nabla \xi| d\gamma dt \leq \delta \sup_{t \in (s, t^0)} \int_{\gamma_R(x^0)} |u^G - u^G_{R,x^0}(s)|^2 \xi^2 d\gamma \\
+ c/\delta \int_{s}^{\tau} \int_{\gamma_R(x^0)} |\nabla' u^G|^2 dx dt.
$$
Furthermore,

$$\int_s^\tau \int_{B_R^+(x^0)} |f| |u - u_{R,x^0}^G(s)| \xi^2 \, dx \, dt \leq \delta \sup_{t \in (s, \tau)} \int_{B_R^+(x^0)} |u - u_{R,x^0}^G(s)|^2 \xi^2 \, dx + \frac{c}{\delta} R^2 \int_{Q_R^+(x^0)} |f|^2 \, dz,$$

and the integral with the function $\psi$ is estimated in the same way.

At last, by the Friedrichs and Poincare inequalities

$$\int_{B_R^+(x^0)} |u(x, s) - u_{R,x^0}^G(s)|^2 \, dx \leq 2 \int_{B_R^+(x^0)} |u^0(x, s)|^2 \, dx + 2 \int_{B_R^+(x^0)} |u_{R,x^0}^G(x', s) - u_{R,x^0}^G(s)|^2 \, dx \leq c R^2 \int_{B_R^+(x^0)} |\nabla u^0(x, s)|^2 \, dx + c R^3 \int_{\gamma_R(x^0)} |\nabla' u(x', s)|^2 \, d\gamma.$$  

Now we fix $\delta = 1/8$ and derive from (13) that

$$\int_{B_R^+(x^0)} |u(x, \tau) - u_{R,x^0}^G(s)|^2 \xi^2 \, dx + \int_{\gamma_R(x^0)} |u_{R,x^0}^G(x', \tau) - u_{R,x^0}^G(s)|^2 \xi^2 \, d\gamma \leq 1/2 \sup_{t \in (s, \tau)} \int_{B_R^+(x^0)} |u(x, t) - u_{R,x^0}^G(s)|^2 \xi^2 \, dx + 1/2 \sup_{t \in (s, \tau)} \int_{\gamma_R(x^0)} |u_{R,x^0}^G(x', t) - u_{R,x^0}^G(s)|^2 \xi^2 \, d\gamma + c R^2 \int_{B_R^+(x^0)} |\nabla u^0|^2 \, dx + c R^3 \int_{\gamma_R(x^0)} |\nabla' u^G(x', s)|^2 \, d\gamma + c \int_{\Gamma_R(x^0)} |\nabla u|^2 \, d\Gamma + c R^2 \int_{P_R^+(x^0)} |f|^2 \, dz + c R^2 \int_{\Gamma_R(x^0)} |\psi|^2 \, d\Gamma.$$  

Taking supremum in $\tau \in (s, \ell^0)$ in the left hand side of (14) we obtain the inequality

$$\sup_{\tau \in (s, \ell^0)} \int_{B_R^+(x^0)} |u(x, \tau) - u_{R,x^0}^G(s)|^2 \xi^2 \, dx + \sup_{\tau \in (s, \ell^0)} \int_{\gamma_R(x^0)} |u_{R,x^0}^G(x', \tau) - u_{R,x^0}^G(s)|^2 \xi^2 \, d\gamma \leq c R^2 \int_{B_R^+(x^0)} |\nabla u^0(x, s)|^2 \, dx + c R^3 \int_{\gamma_R(x^0)} |\nabla' u(x', s)|^2 \, d\gamma + c \int_{\Gamma_R(x^0)} |\nabla u|^2 \, d\Gamma + c R^2 \int_{\Gamma_R(x^0)} |f|^2 \, dz + c R^2 \int_{\Gamma_R(x^0)} |\psi|^2 \, d\Gamma.$$  

To estimate the left hand side of (15), we use the fact that any function $\Phi(c) = \int_D |v - c|^2 \, dD$ takes its minimum in $c = (v)_D$. Then the inequality follows:

$$\sup_{t \in \Lambda_{R/2}(\ell^0)} \int_{B_{R/2}^+(x^0)} |u(x, t) - u_{R/2,x^0}(t)|^2 \, dx + \sup_{t \in \Lambda_{R/2}(\ell^0)} \int_{\gamma_{R/2}(x^0)} |u_{R/2,x^0}(x', t) - u_{R,x^0}^G(t)|^2 \xi^2 \, d\gamma \leq c R^2 \int_{B_{R/2}^+(x^0)} |\nabla u^0(x, s)|^2 \, dx + c R^3 \int_{\gamma_{R/2}(x^0)} |\nabla' u^G(x', s)|^2 \, d\gamma + c \int_{\Gamma_{R}(x^0)} |\nabla u|^2 \, d\Gamma + c R^2 \int_{\Gamma_{R}(x^0)} |f|^2 \, dz + c R^2 \int_{\Gamma_{R}(x^0)} |\psi|^2 \, d\Gamma.$$  

The last inequality we integrate over the interval $\Lambda_{R}(\ell^0) \setminus \Lambda_{R/2}(\ell^0)$ and divide by the measure of this interval. In a result we have

$$\sup_{t \in \Lambda_{R/2}(\ell^0)} \int_{B_{R/2}^+(x^0)} |u(x, t) - u_{R/2,x^0}(t)|^2 \, dx + \sup_{\Lambda_{R/2}(\ell^0)} \int_{\gamma_{R/2}(x^0)} |u_{R/2,x^0}(x', t) - u_{R,x^0}^G(t)|^2 \xi^2 \, d\gamma \leq c \left( \int_{Q_R^+(x^0)} |\nabla u|^2 \, dz + \int_{\Gamma_R(x^0)} |\nabla' u|^2 \, d\Gamma \right) + c R^2 \left( \int_{Q_R^+(x^0)} |f|^2 \, dz + \int_{\Gamma_R(x^0)} |\psi|^2 \, d\Gamma \right).$$  

(17)
Note that for a fixed \( s \in \Lambda_R(t^0) \setminus \Lambda_{R/2}(t^0) \)

\[
\int_{Q_{R/2}(z^0)} |u(z) - u_{R/2, z^0}|^2 \, dz \leq \int_{Q_{R/2}(z^0)} |u(z) - u_{R, z^0}(s)|^2 \, dx \leq \frac{R^2}{4} \sup_{t \in \Lambda_{R/2}(t^0)} \int_{B_{R/2}(z^0)} |u(x, t) - u_{R, z^0}(s)|^2 \, dx
\]

and integral

\[
\int_{\Gamma_{R/2}(z^0)} |u^G(z') - u_{R/2, z^0}^G|^2 \, d\Gamma
\]

we estimate in the same way.

Now estimate (12) follows from (15).

As was said in the Introduction, we apply in this work \( A(t) \)-caloric method. Here we introduce two assertions in an appropriate form [1], [2].

**Lemma 3.3** Let positive numbers \( \nu_0 < \mu_0 \) be fixed. Suppose that \( m = (n-1) N \) and \( A(t) \) is \([m \times m]\) matrix, \( A \in L^\infty(\Lambda_R(z^0)) \), satisfying for almost all \( t \in \Lambda_R(t^0) \) the conditions

\[
(A(t) \xi, \xi) \geq \nu_0 \| \xi \|^2, \quad |(A(t) \xi, \eta)| \leq \mu_0 \| \xi \| \| \eta \|, \quad \forall \xi, \eta \in \mathbb{R}^m.
\]

Let a function \( u^\Gamma \in W^{1,0}(\Gamma_R(z^0)) \) be fixed. For any \( \varepsilon > 0 \) there exist a constant \( C_\varepsilon = C(\varepsilon, \nu_0, \mu_0, n, N) > 0 \), an \( A(t) \)-caloric function \( h^G \) in \( W^{1,0}(\Gamma_{R/2}(z^0)) \), and a function \( \phi \in C^1_0(\Gamma_R(z^0)) \), \( \sup_{R,\phi} \| \nabla' \phi \| \leq 1 \) such that

\[
\int_{\Gamma_{R/2}(z^0)} (|h^G(z') - h_{R/2, z^0}^G|^2 + R^2 |\nabla' h^G|^2) \, d\Gamma \leq 2^{n+2} \int_{\Gamma_{R}(z^0)} (|u^G(z') - u_{R, z^0}^G|^2 + R^2 |\nabla' u^G|^2) \, d\Gamma; \quad (18)
\]

\[
\int_{\Gamma_{R/2}(z^0)} |u^G - h^G|^2 \, d\Gamma \leq \varepsilon \left( \int_{\Gamma_{R}(z^0)} (|u^G - u_{R, z^0}^G|^2 + R^2 |\nabla' u^G|^2) \, d\Gamma \right) + C_\varepsilon R^2 \mathcal{L}_b^2(R, \phi) \quad (19)
\]

where

\[
\mathcal{L}_b(R, \phi) = \int_{\Gamma_{R}(z^0)} (-u^G \phi_t + (A(t) \nabla' u^G, \nabla' \phi) \, d\Gamma).
\]

**Lemma 3.4** Let positive numbers \( \nu < \mu \) be fixed, \( m = n N \). Let an \([m \times m]\) matrix \( A(t) \) satisfy the conditions of Lemma 3.3 with the parameters \( \nu, \mu \) and \( m = n N \). Suppose that a function \( u \in W^{1,0}(Q_{R}^+(z^0)) \), \( u|_{\Gamma_R(z^0)} = 0 \). For any \( \varepsilon > 0 \) there exist a constant \( C_\varepsilon = C(\varepsilon, \nu, \mu, m) \), an \( A(t) \)-caloric function \( h \) in \( Q_{R/2}^+(z^0) \), \( h|_{\Gamma_{R/2}(z^0)} = 0 \), and a function \( \phi \in C^1(\Gamma_{R}(z^0)) \), \( \sup_{\Gamma_{R}} \| \nabla \phi \| \leq 1 \) such that

\[
\int_{Q_{R/2}^+(z^0)} (|h|^2 + R^2 |\nabla h|^2) \, dz \leq 2^{n+2} \int_{Q_{R}^+(z^0)} (|u|^2 + R^2 |\nabla u|^2) \, dz; \quad (21)
\]

\[
\int_{Q_{R/2}^+(z^0)} |u - h|^2 \, dz \leq \varepsilon \int_{Q_{R}^+(z^0)} (|u|^2 + R^2 |\nabla u|^2) \, dz + C_\varepsilon R^2 \mathcal{L}_b^2(R, \phi), \quad (22)
\]
where

$$L_a(R, \phi) = \left| \int_{Q_R^+(z_0)} (-u \phi_t + (A(t) \nabla u, \nabla \phi)) \, dz \right|. \quad (23)$$

**Remark 3.2** It was proved in [2] that any $A(t)$-caloric function $h$ in $Q_R^+(\xi), h|_{R(\xi)} = 0, \xi \in \Gamma_1$, satisfies Campanato type integral inequalities. We introduce here the following estimate from Lemma 4 [2]:

$$\int_{Q^+_R(\xi)} |h|^2 \, dz \leq \left( \frac{P}{r} \right)^2 \int_{Q^+_R(\xi)} |h|^2 \, dz, \quad \rho \leq r \leq R; \quad (24)$$

If $h^G$ is $A(t)$-caloric function in $\Gamma_R(z^0)$ then (see [1])

$$\int_{\Gamma_{\rho}(z^0)} |h - h_{\rho,z^0}|^2 \, d\Gamma \leq c \left( \frac{\rho}{r} \right)^2 \int_{Q^+_R(z^0)} |h - h_{R,z^0}|^2 \, d\Gamma, \quad \rho \leq r \leq R. \quad (25)$$

### 4 H"{o}lder continuity of $u$ on $\Gamma_1$. Proof of Theorem 2.1

First, we apply Lemma 3.3 to estimate the function $u^G(x', t) = u(x', 0, t) \in W^{1,0}(\Gamma_R(z^0))$. Here $z^0 = (x^0, z^0) \in \Gamma_1$ and $\Gamma_{2R}(z^0) \subset \Gamma_1$ are fixed arbitrarily. We put $A^G(t) = \int_{\gamma_R(z^0)} b(x', t) \, d\gamma$. The matrix $A^G(t)$ satisfies the conditions of Lemma 3.3 with the parameters $\nu_0 < \mu_0$. Now we fix an $\varepsilon > 0$. By Lemma 3.3, there exist a constant $C_\varepsilon > 0$, an $A^G(t)$ caloric function $h^G$ on $\Gamma_{R/2}(z^0)$, and a function $\phi \in C^0((\Gamma_R(z^0))), \sup_{\Gamma_R(z^0)} |\nabla^0 \phi| \leq 1$ such that relations (18) and (19) are valid. We put $\eta(z) = \phi(z') m(x_n)$ in [3] where $m \in C^1[0, R], m(0) = 1, m(R) = 0$. Now we estimate the expression $L_b(R, \phi)$ defined in (20):

$$L_b(R, \phi) \leq \int_{\Gamma_R(z_0)} |\Delta b| |\nabla u^G| |\nabla \phi| \, d\Gamma + R \int_{Q_R^+(z_0)} |u \eta_t - (a(z) \nabla u, \nabla \eta)| \, dz$$

$$+ R \int_{Q_R^+(z_0)} |f| \, dz + \int_{\Gamma_R(z_0)} |\psi| \, d\Gamma. \quad (26)$$

where

$$\Delta b = b_{R,x^0}(t) - b(z').$$

Taking into account that $|\phi(z')| \leq c R, |\phi_t| \leq c/R, |\eta_t| \leq c/R$, we estimate the right hand side of (26) and derive the inequality

$$L^2_b(R, \phi) \leq c \int_{Q_R^+(z)} |u^0|^2 \, dz + c \int_{\Gamma_R(z_0)} |u^G - u_{R,x^0}^G|^2 \, d\Gamma + c R^2 \int_{Q_R^+(z_0)} |f|^2 \, dz + c R^4 \int_{Q_R^+(z_0)} |\psi|^2 \, d\Gamma$$

$$+ c R^2 \int_{Q_R^+(z_0)} |\nabla u|^2 \, dz + c q_0^2(R) \int_{\Gamma_R(z_0)} |\nabla u^G|^2 \, d\Gamma, \quad u^0(z) = u(z) - u^G(z'). \quad (27)$$
We estimate two last terms in relation (27) by (11). Then we multiply new relation by \( R^2 \) and obtain the inequality

\[
R^2 \mathcal{L}_b^2(R, \phi) \leq c_q^2(R) R \int_{Q_{2R}(z^0)} |u^0|^2 \, dz + c (q_b^2(R) + R) \int_{\Gamma_{2R}(z^0)} |u^G - u_{2R,z^0}^G|^2 \, d\Gamma
+ c R^5 \int_{Q_{2R}(z^0)} |f|^2 \, dz + c R^4 \int_{\Gamma_{2R}(z^0)} |\psi|^2 \, d\Gamma.
\]

(28)

Now we introduce the function

\[
\Phi(r, z^0) = r \int_{Q_r^+(z^0)} |u^0|^2 \, dz + \int_{\Gamma_r(z^0)} |u^G(z') - u_{r,z^0}^G|^2 \, d\Gamma, \quad r \leq 2R.
\]

(29)

Then (28) can be written in the form

\[
R^2 \mathcal{L}_b^2(R, \phi) \leq c (q_b^2(R) + R) \Phi(2R, z^0) + c K(2R, z^0)
\]

(30)

where

\[
K(2R, z^0) = R^5 \int_{Q_{2R}(z^0)} |f|^2 \, dz + R^4 \int_{\Gamma_{2R}(z^0)} |\psi|^2 \, d\Gamma.
\]

(31)

It follows from inequalities (18), (19), (30) and (11) that

\[
\int_{\Gamma_{R/2}(z^0)} (|h^G - h_{R/2,z^0}^G|^2 + R^2 |\nabla h^G|^2) \, d\Gamma \leq c \Phi(2R, z^0) + c K(2R, z^0),
\]

(32)

\[
\int_{\Gamma_{R/2}(z^0)} |u^G - h^G|^2 \, d\Gamma \leq (\varepsilon + C\varepsilon(R + q_b^2(R))) \Phi(2R, z^0) + C \varepsilon c K(2R, z^0).
\]

(33)

The next step of the proof is to apply Lemma 3.4 to the function \( u^0 \) in \( Q_{R}^+(z^0) \), \( u^0|_{\Gamma_{R}(z^0)} = 0 \). We put the matrix \( A(t) = \int_{Q_{R}^+(z^0)} a(x, t) \, dx \). For the fixed earlier \( \varepsilon \), there exist a constant \( C\varepsilon \), an \( A(t) \) caloric function \( h \) defined in \( Q_{R/2}^+(z^0) \), \( h|_{\Gamma_{R/2}(z^0)} = 0 \), and a function \( \phi^* \in C_0^1(Q_{R}^+(z^0)) \), \( \sup_{Q_{R}^+(z^0)} |\nabla \phi^*| \leq 1 \) such that inequalities (21), (22) hold with \( u^0 \) instead of \( u \). Using identity (3), we estimate the expression \( \mathcal{L}_a(R, \phi^*) \) in the way

\[
\mathcal{L}_a(R, \phi^*) := \int_{Q_{R}^+(z^0)} \left( -u^0 \phi^*_t + (A(t) \nabla u^0, \nabla \phi^*) \right) \, dz \leq \int_{Q_{R}^+(z^0)} |\Delta a| |\nabla u^0| |\nabla \phi^*| \, dz
+ \int_{Q_{R}^+(z^0)} |u^G \phi^*_t - (a(z) \nabla u^G, \nabla \phi^*)| \, dz + \int_{Q_{R}^+(z^0)} |f \phi^*| \, dz.
\]

(34)

Recall that \( |\phi^*| \leq c R \) and \( |\phi^*_t| \leq c/R \). Thus,

\[
\int_{Q_{R}^+(z^0)} u^G \phi^*_t \, dz = \int_{Q_{R}^+(z^0)} (u^G - u_{R,z^0}^G) \phi^*_t \, dz \leq c R^{-1} \int_{\Gamma_{R}(z^0)} |u^G - u_{R,z^0}^G| \, d\Gamma,
\]

\[
\int_{Q_{R}^+(z^0)} |f| \phi^* \, dz \leq c R \int_{Q_{R}^+(z^0)} |f| \, dz, \quad \int_{Q_{R}^+(z^0)} |a \nabla u^G, \nabla \phi^*| \, dz \leq c \int_{\Gamma_{R}(z^0)} |\nabla u^G| \, d\Gamma.
\]
Now the estimate follows
\[
\mathcal{L}_a^2(R, \phi^*) \leq \int_{Q_R^+(z^0)} |\Delta a|^2 \, dz + \int_{Q_R^+(z^0)} |\nabla u^0|^2 \, dz + \frac{c}{R^2} \int_{\Gamma_R(z^0)} |u^G - u_{R,\phi^0}^G|^2 \, d\Gamma \\
+ c R^2 \int_{Q_R^+(z^0)} |f|^2 \, dz + \int_{\Gamma_R(z^0)} |\nabla' u^G|^2 \, d\Gamma. \tag{35}
\]

Then we can apply condition \textbf{H2} and inequality (11) to estimate the first and the last integrals in the right hand side of (35).

In a result, we obtain the inequality
\[
R^2 \mathcal{L}_a^2(R, z^0) \leq c (q_a^2(R) R^{-1} + 1) [\Phi(2R, z^0) + K(2R, z^0)]. \tag{36}
\]

Now it follows from (22) and (36) that
\[
\int_{Q_R^+(z^0)} |u^0 - h|^2 \, dz \leq \varepsilon \int_{Q_R^+(z^0)} (|u^0|^2 + r^2 |\nabla u^0|^2) \, dz + C^* R^2 \mathcal{L}_a^2(R, \phi^*) \\
\leq c \{\varepsilon R^{-1} + C^* (q_a^2(R) R^{-1} + 1)\} \Phi(2R, z^0) + c C^* (q_a^2(R) R^{-1} + 1) K(2R, z^0). \tag{37}
\]

On the next step we will use known estimates for $A(t)$ caloric functions $h^G$ and $h$ (see Remark 3.2) and estimates (33) and (37). The following chain of the inequalities are valid for $\rho \leq R/2$:

\[
\Phi(\rho, z^0) = \rho \int_{Q_{R/2}^+(z^0)} |u^0|^2 \, dz + \int_{\Gamma_{R/2}(z^0)} |u^G - u_{R,\phi^0}^G|^2 \, d\Gamma \leq 2 \rho \int_{Q_{R/2}^+(z^0)} |u^0 - h|^2 \, dz \\
+ 2 \rho \int_{Q_{R/2}^+(z^0)} |h|^2 \, dz + 2 \int_{\Gamma_{R/2}(z^0)} |(u^G - h^G) - (u_{R,\phi^0}^G - h_{R,\phi^0}^G)|^2 \, d\Gamma + 2 \int_{\Gamma_{R/2}(z^0)} |h^G - h_{R,\phi^0}^G|^2 \, d\Gamma. \tag{38}
\]

Applying estimates (24) and (25) for $A(t)$- caloric functions $h$ and $h^G$, $\rho \leq R/2$, we obtain from (38) the inequality
\[
\Phi(\rho, z^0) \leq \rho \left(\frac{R}{\rho}\right)^{n+2} \int_{Q_{R/2}^+(z^0)} |u^0 - h|^2 \, dz + c \left(\frac{R}{\rho}\right)^{n+1} \int_{\Gamma_{R/2}(z^0)} |u^G - h^G|^2 \, d\Gamma \\
+ \rho \left(\frac{R}{\rho}\right)^2 \int_{Q_{R/2}^+(z^0)} |h|^2 \, dz + c \left(\frac{R}{\rho}\right)^2 \int_{\Gamma_{R/2}(z^0)} |h^G - h_{R/2,\phi^0}^G|^2 \, d\Gamma. \tag{39}
\]

Now we use (21) and (11) to obtain the inequality
\[
\int_{Q_{R/2}^+(z^0)} |h|^2 \, dz \leq c R^{-1} (\Phi(2R, z^0) + K(2R, z^0)). \tag{40}
\]

Further, applying relations (32), (33), (37) and (40), we estimate the right-hand side of (39) with $r = 2R$ as follows:
\[
\Phi(\rho, z^0) \leq c_0 \left\{ \left(\frac{\rho}{r}\right)^2 + \varepsilon \left(\frac{r}{\rho}\right)^{n+1} + \tilde{C}_\varepsilon \left(\frac{r}{\rho}\right)^{n+1} \left[r + q_a^2(r) + q_b^2(r)\right] \right\} \Phi(r, z^0) \\
+ \tilde{C}_\varepsilon c K(r, z^0), \quad \rho \leq r/4, \quad \tilde{C}_\varepsilon = \max\{C^*_a, C^*_\varepsilon\}. \tag{41}
\]
It follows from the assumptions $H5$ on $f$, $\psi$ and the definition (31) that

$$K(r) \leq K_0 r^{2\alpha}, \quad n \geq 3.$$  

$$K_0 = \|f\|_{L^{2,\lambda}(Q^+_0; \delta)} + \|\psi\|_{L^{2,\lambda}(\Gamma_0; \delta)}.$$  

We put now in (41) $\rho = \tau r$ with $\tau \leq 1/4$ to be chosen later.

Then

$$\Phi(\tau r, z^0) \leq c_0 \{\tau^2 + \varepsilon \tau^{-(n+1)} + \hat{C}_\varepsilon \tau^{-(n+1)}[r + q_0^2(r) + q_1^2(r)]\} \Phi(r, z^0) + c \hat{C}_\varepsilon K_0 r^{2\alpha}.$$  

(42)

Now we fix $\beta = \frac{\alpha + 1}{2}$, $\beta > \alpha$, and choose $\tau$ to satisfy the relation

$$c_0 \tau^{2\beta} \leq \frac{\tau^{2\beta}}{3}.$$  

(43)

Further, we fix $\varepsilon < 1$ in the way

$$c_0 \varepsilon \tau^{-(n+1)} \leq \frac{\tau^{2\beta}}{3}.$$  

(44)

The parameters $\tau$ and $\varepsilon$ are fixed by the data of the problem and do not depend on $z^0$.

At last, we can specify the choice of $r_0 = 2R_0$ by requiring that

$$c \hat{C}_\varepsilon \tau^{-(n+1)}[r_0 + q_0^2(r_0) + q_1^2(r_0)] \leq \frac{\tau^{2\beta}}{3}.$$  

(45)

In a result,

$$\Phi(\tau r, z^0) \leq \tau^{2\beta} \Phi(r, z^0) + c K_0 r^{2\alpha}.$$  

(46)

Proceeding by induction in relation (40) for $r_j = \tau^j r$, $j \in \mathbb{N}$, we obtain the inequality

$$\Phi(r^j, z^0) \leq \tau^{2\alpha j}(\Phi(r, z^0) + c_1 K_0 r^{2\alpha}).$$  

(47)

We can assert now that

$$\Phi(\rho, z^0) \leq c \left(\frac{\rho}{r}\right)^{2\alpha} [\Phi(r, z^0) + K_0 r^{2\alpha}], \quad \forall \rho \leq r \leq r_0.$$  

(48)

Thus,

$$\sup_{\rho \leq r_0} \rho^{-(n+1+2\alpha)} \left(\int_{Q^+_\rho(z^0)} |u^0|^2 dz + \int_{\Gamma_\rho(z^0)} |u^G - u^{G, \rho, \psi}_{\tau^0}|^2 d\Gamma\right) \leq c(r_0^{-1})(\|\nabla u\|_{L^2(Q^+_0)} + \|\nabla' u^G\|_{L^2(\Gamma, \rho)} + K_0) \leq c(r_0^{-1}) M_0,$$  

(49)

where $M_0$ is defined in (8). Taking supremum in $z^0 \in \Gamma_{1-q}(0)$ in the left-hand side of (49), $(q \in (0, 1)$ is any fixed number, $r_0$ satisfies (45) and $r_0 \leq q)$, we obtain the estimate of the seminorm of $u^G$ in $L^{2,n+1+2\alpha}(\Gamma_{1-q}(0); \delta)$:

$$[u^G]_{L^{2,n+1+2\alpha}(\Gamma_{1-q}(0); \delta)} \leq c(r_0^{-1}, q^{-1}) (\|\nabla u\|_{L^2(Q^+_0)} + \|\nabla' u^G\|_{L^2(\Gamma, \rho)} + K_0) \leq c M_0.$$  

(50)

Due to the isomorphism between $L^{2,n+1+2\alpha}(\Gamma_{1-q}(0); \delta)$ and $C^\alpha(\overline{\Gamma}_{1-q}(0); \delta)$ in the parabolic metric, we obtain estimate of $C^\alpha$-norm of $u^G$ in $\Gamma_{1-q}(0)$. Moreover, estimate (41) provides that

$$\|\nabla' u^G\|_{L^{2,n+1+2\alpha}(\Gamma_{1-q}(0); \delta)} \leq c M_0,$$  

and estimate (8) follows. •
5 Proofs of Theorem 2.2 and 2.3

Here we consider problem (11), (2) in the form

\[ u_t - \text{div}(a(z) \nabla u) = f(z), \quad z \in Q^+_1, \quad (51) \]
\[ u|_{\Gamma_1} = \phi(z), \quad (52) \]

where \( \phi(z) = u^G(x', t) \) and \( f \in L^{2,n-2+2\alpha}(Q^+_1; \delta) \). If all assumptions of Theorem 2.2 hold then \( u^G(x', t) \in C^\alpha(\Gamma_{1-q}(0); \delta), \forall q \in (0, 1) \) by Theorem 2.1.

Here we prove further smoothness results for \( u \). First, we want to recall some known results on the regularity of solutions (51), (52).

**Proposition 5.1.** Let \( N = 1, n \geq 2, \) the assumptions H1–H4 hold, \( f \in L^p(Q^+_1 R_0), \) \( p > \frac{n+2}{2}, n \geq 2, \) and \( \psi \) satisfies the condition H5 with \( \alpha = 2 - \frac{n+2}{p} > 0. \) Then \( u \in C^\beta(Q^+_R; \delta) \) with some \( \beta \leq \alpha \) and \( R < R_0. \)

This result is a consequence of a weak form of the maximum principle (see, for example [21], Chapter III, §10). Indeed, if \( f \in L^p(Q^+_1) \) with \( p > \frac{n+2}{2} \) then it belongs to the space \( L^{2,n-2+2\alpha}(Q^+_1; \delta), \) \( \alpha = 2 - \frac{n+2}{p} > 0. \) Using Theorem 2.1, we can assert that \( u^G \in C^\alpha(\Gamma_{1-q}(0); \delta). \) Thus, by the mentioned integral form of the maximum principle \( u \in C^\beta(Q^+_{1-q}(0); \delta), \beta \leq \alpha \) in the scalar case.

Let now \( N > 1 \) and the assumptions of Theorem 2.2 hold. Analyzing the proof of Theorem 1 in [1] where regularity problem for quasilinear systems was studied, one can assert local smoothness of weak solutions of systems (51). More exactly, the following proposition is valid.

**Proposition 5.2.** Let the matrix \( a \) satisfy the conditions H1, H2, \( n \geq 2, \ f \in L^{2,n-2+2\alpha}(Q^+_1; \delta), \) \( \alpha \in (0, 1), \) and \( u \) be a weak solution of (51) from \( H_1 = W^{1,0}_2(Q^+_1). \) Then \( u \in C^\alpha(Q^+_1; \delta), \forall Q' \subset Q^+_1, \) and

\[ \|u\|_{C^\alpha(Q^+_1; \delta)} + \|\nabla u\|_{L^{2,n-2+2\alpha}(Q^+_1; \delta)} \leq c_1 \|u\|_{H_1} + c_2 \|f\|_{L^{2,n-2+2\alpha}(Q^+_1; \delta)}. \quad (53) \]

Moreover, for any \( \xi \in Q' \) and \( \rho \leq r \leq \delta(\xi, \partial_p(Q^+_1)) \)

\[ \Phi(\rho, \xi) := \frac{1}{\rho^{n+2+2\alpha}} \int_{Q^+_\rho(\xi)} |u - u^0|\alpha dz \leq c_3 \{ \Phi(\rho, \xi) + \|f\|_{L^{2,n-2+2\alpha}(Q^+_1; \delta)} \}. \quad (54) \]

The constants \( c_1 - c_3 \) depend on the parameters from conditions H1, H2, \( \alpha, \) and the constant \( c_1 \) also depends on \( \delta(Q', \partial_p(Q^+_1)) > 0. \)

As a consequence of Proposition 5.2 and Theorem 2.1, we obtain the following assertion.

**Proposition 5.3.** Let the assumptions H1–H4 hold. If \( 2\alpha - 1 > 0 \) in the assumption H5' then \( u \in C^\beta(Q^+_1; \delta) \) with \( \beta = \alpha - 1/2 > 0 \) and any fixed \( q \in (0, 1). \)

Indeed, if \( z^0 \in \Gamma_{1-q}(0) \) then by Theorem 2.1 for \( \rho \leq r_0 \leq q \)

\[ \Psi(\rho, z^0) := \frac{1}{\rho^{n+2+2\alpha}} \int_{Q^+_\rho(z^0)} |u - u^{G^0}|\alpha dz \leq \frac{1}{\rho^{n+2+2\alpha}} \int_{Q^+_\rho(z^0)} |u - u^{G^0}|\alpha dz \]
\[ \leq \frac{2}{\rho^{n+2+2\alpha}} \int_{Q^+_\rho(z^0)} |u^0|^2 dz + \frac{2\rho}{\rho^{n+2+2\alpha}} \int_{\Gamma_{1-q}(z^0)} |u^G - u^{G^0}|^2 d\Gamma, \quad u^0(z) = u(z) - u^G(z'). \quad (55) \]
The right hand side of (55) can be estimated by Theorem 2.1. Thus,
\[ \Psi(\rho, z^0) \leq c \mathbb{L}_0 \]
for any point \( z^0 \in \Gamma_{1-q}(0) \), here \( r_0 \) does not depend on \( z^0 \) and \( \mathbb{L}_0 \) is defined by (9).
The standard procedure of ”sewing” together local inner and boundary estimates for \( \Psi(\rho, \cdot) \) provides estimate of this function for all \( \xi \in \overline{Q_{1-q}^+(0)} \). We remark that \( n + 1 + 2\alpha = n + 2 + 2\beta, \ \beta = \alpha - 1/2, \) and the Hölder continuity of \( u \) in \( \overline{Q_{1-q}^+(0)} \) follows with the exponent \( \beta = \alpha - 1/2 \). We do not explain in details the proof of Proposition 5.3 because below we prove the more strong assertion of Theorem 2.2.

**Proof of Theorem 2.2.**

We start with the transformation of problem (51), (52) to the homogeneous one.

We put \( u^0(z) = u(z) - u^G(z') \), \( u^0|_{\Gamma_1} = 0 \), and define a weak solution to the problem
\[ u_t - \text{div}(a(z)\nabla u^0) = -u_t^G + \text{div}(a(z)\nabla' u^G) + f(z), \quad z \in Q_1^+(0), \]
\[ u^0|_{\Gamma_1(0)} = 0. \]

**Definition 5.1.** A function \( u^0 \in W_2^{1,0}(Q_1^+(0)), \ u^0|_{\Gamma_1(0)} = 0 \), is a weak solution to problem (56) if it satisfies the identity
\[ \int_{Q_1^+(0)} [-u^0 \eta_t + (a(z)\nabla u^0, \nabla \eta)] \, dz = \int_{Q_1^+(0)} [u^G \eta_t - (a(z)\nabla' u^G, \nabla \eta) + f \eta] \, dz \]
for any \( \eta \in W_2^{1,1}(Q_1^+(0)) \).

To prove Theorem 2.2 it is enough to state Hölder continuity of \( u^0 \) in \( \overline{Q_{1-q}^+(0)} \), \( q \in (0, 1) \).

As a first step, we prove that there exist half derivatives in \( t \) of the functions \( u \) and \( u^G \).

We fix \( z^0 \in \Gamma_1(0) \) and \( Q_{2R}^+(z^0) \subset Q_1^+(0) \). Let \( \omega(t) \in C_0^0(\Lambda_{2R}(t^0)) \), \( \omega(t) = 1 \) in \( \Lambda_R(t^0) \); let \( d(x) \) be a cut-off function for \( B_{2R}(x^0), \ d(x) = 1 \) in \( B_R(x^0) \). Note that \( |\omega'(t)| \leq \frac{c}{R^2}, \ |\nabla d(x)| \leq \frac{c}{R} \). We put
\[ v(z) = (u(z) - u_{2R}^G) \omega(t) d(x), \quad v|_{\partial Q_{2R}(z^0) \\cap \Gamma_{2R}(z^0)} = 0, \]
and prove the following proposition.

**Proposition 5.4.** Let assumptions H1, H3 hold and \( u \) be a weak solution to problem (51), (52) in \( Q_{2R}^+(z^0) \subset Q_1^+(0), \ z^0 \in \Gamma_1(0) \). Then
\[ v \in H^{1/2}(\Lambda_{2R}(t^0); L^2(B_{2R}(x^0))), \ v^G = v|_{x_n=0} \in H^{1/2}(\Lambda_{2R}(t^0); L^2(\gamma_{2R}(x^0))) \]
where \( v \) defined by (58), and
\[ \|v\|^2_{H^{1/2}(\Lambda_{2R}(t^0); L^2(B_{2R}(x^0)))} + \|v^G\|^2_{H^{1/2}(\Lambda_{2R}(t^0); L^2(\gamma_{2R}(x^0)))} \]
\[ \leq c_1 \left\{ \|\nabla v\|^2_{2, Q_{2R}(z^0)} + \|\nabla v^G\|^2_{2, \Gamma_{2R}(z^0)} + R^{-2}(\|u^0\|^2_{2, Q_{2R}(z^0)} + \|u^G - u_{2R}^G\|^2_{1, \Gamma_{2R}}) \right. \]
\[ \left. + R^2(\|f\|^2_{2, Q_{2R}} + \|\psi\|^2_{2, \Gamma_{2R}}) \right\}; \]
2) \( u \in H^{1/2}(\Lambda_R(t^0); L^2(B^+_R(x^0))) \), \( u^G \in H^{1/2}(\Lambda_R(t^0); L^2(\gamma_R(x^0))) \) and

\[
\|u\|_{H^{1/2}(\Lambda_R(t^0); L^2(B^+_R(x^0)))}^2 + \|u^G\|_{H^{1/2}(\Lambda_R(t^0); L^2(\gamma_R(x^0)))}^2 \leq c_2 \left\{ \|\nabla u\|_{2, Q^+_R}^2 + \|\nabla' u^G\|_{2, \Gamma_R}^2 + R^{-2}(\|u^0\|_{2, Q^+_R}^2 + \|u^G - u^G_{2R}\|_{2, \Gamma_R}^2) + R^2(\|f\|_{2, Q^+_R}^2 + \|\psi\|_{2, \Gamma_R}^2) \right\}. \tag{60}
\]

The constants \( c_1 \) and \( c_2 \) depend on the parameters from the conditions \( H1, H3, \) \( n, N, \) and do not depend on \( x^0 \) and \( R. \)

**Proof.** We consider identity (53) with \( \eta(z) = \omega(t)d(x)\xi(z) \) where \( \omega(t) \) and \( d(x) \) are the same as in (53). The function \( \xi \in W^{1,1}_2(\hat{Q}) \cap W^{1,1}_2(\hat{\Gamma}) \), here and below we denote

\( \hat{Q} = B^+_R(x^0) \times \mathbb{R}^1, \quad \hat{\Gamma} = \gamma_R(x^0) \times \mathbb{R}^1 \).

The identity (53) with the fixed \( \eta \) we rewrite in the form

\[
\int_{Q^+_R(z^0)} -v \xi_t \, dz + \int_{\Gamma_R(z^0)} -v^G \xi_t \, d\Gamma = \int_{Q^+_R(z^0)} [(\Phi, \nabla \xi) + F \xi] \, dz + \int_{\Gamma_R(z^0)} [(\Phi^G, \nabla' \xi) + F^G \xi] \, d\Gamma \tag{61}
\]

where

\[
\Phi(z) = (a(z)\nabla u) \omega(t) d(x), \quad \Phi^G(z') = (b(z')\nabla' u^G) \omega(t) d(x', 0),
\]

\[
F(z) = (u(z) - u^G_{R, \omega}) \omega(t) d(x) + (a(z)\nabla u, \nabla d) \omega(t) + f(z) \omega(t) d(x),
\]

\[
F^G(z') = (u^G - u^G_{R, \omega}) \omega(t) d(x', 0) + (b(z')\nabla' u^G, \nabla' d(x', 0)) \omega(t) + \psi(z') \omega(t) d(x', 0).
\]

We put \( f(z) \) and \( \psi(z') = 0 \) for \( t \in \mathbb{R}^1 \setminus \Lambda_R(t^0) \) and remark that the functions \( v, v^G, \Phi, \Phi^G, F, \) and \( F^G \) vanish for \( t \in \mathbb{R}^1 \setminus \Lambda_R(t^0). \) The identity (61) can be written in the form

\[
\int_{\hat{Q}} -v \xi_t \, dz + \int_{\hat{\Gamma}} -v^G \xi_t \, d\Gamma = \int_{\hat{Q}} [(\Phi, \nabla \xi) + F \xi] \, dz
\]

\[
+ \int_{\hat{\Gamma}} [(\Phi^G, \nabla' \xi) + F^G \xi] \, d\Gamma, \quad \forall \xi \in W^{1,1}_2(\hat{Q}) \cap W^{1,1}_2(\hat{\Gamma}).
\]

For any \( w(t) \in L^1(\mathbb{R}^1) \) we define the Steklov averages

\[
w_h(t) = \frac{1}{h} \int_t^{t+h} w(\tau) \, d\tau, \quad w_{-h}(t) = \frac{1}{h} \int_{t-h}^t w(\tau) \, d\tau.
\]

We put \( \xi(z) = g^\tau(x, t) \) in (63) for any \( g \in W^{1,1}_2(\hat{Q}) \cap W^{1,1}_2(\hat{\Gamma}). \)

It allows us to transform (63) in the way:

\[
\int_{\hat{Q}} -v_h \gamma_t \, dz + \int_{\hat{\Gamma}} -v^G_h \gamma_t \, d\Gamma = \int_{\hat{Q}} [(\Phi_h, \nabla g) + F_h g] \, dz + \int_{\hat{\Gamma}} [(\Phi^G_h, \nabla' g) + F^G_h g] \, d\Gamma. \tag{64}
\]

If to fix \( g(z) = \chi(t) \theta(x), \chi \in C^\infty_0(\mathbb{R}^1) \) and \( \theta \in W^{1,1}_2(B^+_R(x^0)) \cap W^{1,1}_2(\gamma_R(x^0)), \theta|_{S^+_R(x^0)} = 0, \) then

\[
-\int_{\mathbb{R}^1} \chi(t)[(v_h, \theta)_{2, B^+_R} + (v^G_h, \theta)_{2, \gamma_R}] \, dt \tag{65}
\]

\[
= \int_{\mathbb{R}^1} \chi(t)[(\Phi_h, \nabla \theta)_{2, B^+_R} + (F_h, \theta)_{2, B^+_R} + (\Phi^G_h, \nabla' \theta)_{2, \gamma_R} + (F^G_h, \theta)_{2, \gamma_R}] \, dt
\]
for any $\chi \in C^\infty_0(\mathbb{R})$.
By the definition of the weak derivative,
\[
\frac{d}{dt} \left[ (v_h, \theta)_{2,B_{2R}^+} + (v_h^G, \theta)_{2,\gamma_{2R}} \right] = K_h(\theta)
\]  
(66)
for almost all $t \in \mathbb{R}$, here
\[
K_h(\theta) := (\Phi_h, \nabla \theta)_{2,B_{2R}^+} + (F_h, \theta)_{2,B_{2R}^+} + (\Phi_h^G, \nabla' \theta)_{2,\gamma_{2R}} + (F_h^G, \theta)_{2,\gamma_{2R}}.
\]
If we fix $\theta \in W_2^1(B_{2R}^+(x^0))$ in (65) then
\[
\frac{d}{dt} (v_h, \theta)_{2,B_{2R}^+} = (\Phi_h, \nabla \theta)_{2,B_{2R}^+} + (F_h, \theta)_{2,B_{2R}^+}
\]  
(67)
for almost all $t \in \mathbb{R}$.
Now using well-known properties of the Steklov averages (see, for example, [21], Ch.2, Lemma 4.7, Ch.3, Lemma 4.1) we obtain the relation
\[
(\frac{d}{dt}v_h, \theta)_{2,B_{2R}^+} + (\frac{d}{dt}v_h^G, \theta)_{2,\gamma_{2R}} = K_h(\theta), \quad a.a. \ t \in \mathbb{R}.
\]  
(68)
By $\tilde{p}(\alpha)$ we denote the Fourier transformation of a function $p \in L^1(\mathbb{R})$:
\[
\tilde{p}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} p(t) \exp^{-i\alpha t} \, dt.
\]
We apply the Fourier transformation in $t$ to equality (68) and get
\[
-i \alpha (\tilde{v}_h(x, \alpha), \theta)_{2,B_{2R}^+} - i \alpha (\tilde{v}_h(x, \alpha), \theta)_{2,\gamma_{2R}} = \tilde{K}_h(\theta).
\]
Multiplying the last relation by $i \text{sign} \alpha$ and putting in it $\theta = \tilde{v}_h(x, \alpha)$ we obtain the relation
\[
|\alpha| \|\tilde{v}_h(x, \alpha)\|_{2,B_{2R}^+}^2 + |\alpha| \|\tilde{v}_h^G(x', \alpha)\|_{2,\gamma_{2R}}^2 = i \text{sign} \alpha (\tilde{\Phi}_h, \nabla \tilde{v}_h)_{2,B_{2R}^+} + (\tilde{F}_h, \tilde{v}_h)_{2,B_{2R}^+}
\]
\[
+ (\tilde{\Phi}_h^G, \nabla' \tilde{v}_h^G)_{2,\gamma_{2R}} + (\tilde{F}_h^G, \tilde{v}_h^G)_{2,\gamma_{2R}}.
\]
Now we integrate the last relation in $\alpha \in \mathbb{R}$ and derive the following equality:
\[
J_h := \int_{\mathbb{R}} |\alpha| \|\tilde{v}_h(x, \alpha)\|_{2,B_{2R}^+}^2 \, d\alpha + \int_{\mathbb{R}} |\alpha| \|\tilde{v}_h^G(x', \alpha)\|_{2,\gamma_{2R}}^2 \, d\alpha
\]  
(69)
\[
= \int_{\mathbb{R}} i \text{sign} \alpha \{ (\tilde{\Phi}_h, \nabla \tilde{v}_h)_{2,B_{2R}^+} + (\tilde{F}_h, \tilde{v}_h)_{2,B_{2R}^+} + (\tilde{\Phi}_h^G, \nabla' \tilde{v}_h^G)_{2,\gamma_{2R}} + (\tilde{F}_h^G, \tilde{v}_h^G)_{2,\gamma_{2R}} \} d\alpha.
\]
The Parseval’s equality provides the estimate
\[
J_h \leq \|\tilde{\Phi}_h\|_{2,\tilde{Q}} \|\nabla \tilde{v}_h\|_{2,\tilde{Q}} + \|\tilde{\Phi}_h^G\|_{2,\tilde{f}} \|\nabla' \tilde{v}_h^G\|_{2,\tilde{f}}
\]
\[+ R^2(\|F_h\|_{2,\tilde{Q}}^2 + \|F_h^G\|_{2,\tilde{f}}^2) + R^{-2}(\|v_h\|_{2,\tilde{Q}}^2 + \|v_h^G\|_{2,\tilde{f}}^2).\]
(70)
Let now \( h \to 0 \) in (70). Then
\[
J = \lim_{h \to 0} J_h = \|v\|^2_{H^{1/2}(\mathbb{R}^4; L^2(\mathcal{B}_2^+(\mathbb{R})))} + \|v^G\|^2_{H^{1/2}(\mathbb{R}^4; L^2(\gamma_2^+(\mathbb{R})))}.
\]
Thus,
\[
J \leq \|\Phi\|_{2,q} \|\nabla v\|_{2,q} + \|\Phi^G\|_{2,\mathcal{F}} \|\nabla v^G\|_{2,\mathcal{F}} + R^2(\|F\|_{2,q} + \|F^G\|_{2,\mathcal{F}}) + R^{-2}(\|\nabla v\|_{2,q}^2 + \|v^G\|_{2,\mathcal{F}}^2).
\]
Using definitions (58) and (52) of the functions \( v, v^G, \Phi, \Phi^G, F \) and \( F^G \), we derive estimates (59) and (60).

The next step is to derive the energy estimate for a weak solution \( u^0 \) of problem (56).

**Proposition 5.5.** Let the assumptions H1–H4 and H5' hold, \( u^0 \) be a weak solution to problem (56). For any fixed \( Q_{2R}^+(z^0) \), \( z^0 \in \Gamma_{1-q}(0) \), \( 2R < q \), where \( q \in (0, 1) \), the following estimate
\[
\int_{Q_{2R}^+(z^0)} |\nabla u^0|^2 \, dz \leq \frac{c}{R^2} \int_{Q_{2R}^+(z^0)} |u^0|^2 \, dz + c\mathbb{L}_0 R^{n+2\alpha},
\]
is valid where \( \mathbb{L}_0 \) is defined in (72).

**Proof.** We fix a number \( q \in (0, 1) \), a cylinder \( Q_{2R}^+(z^0) \), \( z^0 \in \Gamma_{1-q}(0) \), \( 2R < q \), and put in (57)
\[
\eta = u^0(z) \omega^2(t) \, d^2(x) \in \mathcal{W}_2 (Q_{2R}^+(z^0)),
\]
the functions \( d(x) \) and \( \omega(t) \) such as in (58). Let \( v \) be the function defined by (58), we put
\[
v^0(z) = u^0(z) \omega(t) \, d(x), \quad v^G(z') = u^G(z') \omega(t) \, d(x'), \quad \hat{u}^G = u^G(z') - u_{2R,z^0}^G,
\]
\[
\hat{u}^G(z) = u^G(z') \omega(t) \, d(x), \quad v(z) = v^0(z) + u^G(z).
\]
After trivial calculations in (57) with the fixed \( \eta \) we use estimate (8) and obtain the inequalities
\[
\int_{Q_{2R}^+(z^0)} |\nabla v^0|^2 \, dz \leq |J_{2R}| + \frac{c}{R^2} \int_{Q_{2R}^+(z^0)} |u^0|^2 \, dz + \frac{c}{R} \int_{\Gamma_{2R}} |u^G - u_{2R}|^2 \, d\Gamma \quad \text{(74)}
\]
\[
+ c R \int_{\Gamma_{2R}} |\nabla' u^G|^2 \, d\Gamma + c R^2 \int_{Q_{2R}^+(z^0)} |f|^2 \, dz \leq |J_{2R}| + \frac{c}{R^2} \int_{Q_{2R}^+(z^0)} |u^0|^2 \, dz + c\mathbb{L}_0 R^{n+2\alpha}
\]
where the integral
\[
J_{2R} = \int_{Q_{2R}^+(z^0)} u^G(z') \omega(t) \, d(x) \, v^0(z) \, dz
\]
we estimate below.

As \( u^G \in C^\alpha(\Gamma_{1-q}(0)) \),
\[
\left| \int_{Q_{2R}^+(z^0)} \hat{u}^G(z') v^0(z) \omega'(t) \, d(x) \, dz \right| \leq \frac{c}{R^2} \int_{Q_{2R}^+(z^0)} |u^0|^2 \, dz + c\mathbb{L}_0 R^{n+2\alpha},
\]
and it follows from (74) that
\[
\int_{Q_{2R}^+(z^0)} |\nabla v^0|^2 \, dz \leq c |J_{2R}| + \frac{c}{R^2} \int_{Q_{2R}^+(z^0)} |u^0|^2 \, dz + c\mathbb{L}_0 R^{n+2\alpha},
\]
(75)
where

$$|I_{2R}| = \left| \int_{Q_{2R}(z^0)} w^G(z) v^0(z) \, dz \right|. $$

To estimate $|I_{2R}|$ we go back to the proof of the Proposition 5.4 and apply the Fourier transformation (with respect to the variable $t$) to the relation (77). Then

$$-i \alpha(v_h(x, \alpha), \theta(x))_{2, B_{2R}^+} = (\Phi_h(x, \alpha), \nabla \theta)_{2, B_{2R}^+} + (F_h(x, \alpha), \theta)_{2, B_{2R}^+},$$

for any $\theta \in \mathbb{W}_2(B_{2R}^+(x^0))$.

We multiply the last relation by $i\text{sign} \alpha$ and put $\theta = \tilde{v}_h(x, \alpha)$. It follows that

$$|\alpha| (v_h(x, \alpha), v_h^0(x, \alpha))_{2, B_{2R}^+} = i \text{sign} \alpha [(\Phi_h, \nabla \tilde{v}_h^0)_{2, B_{2R}^+} + (F_h, \tilde{v}_h^0)_{2, B_{2R}^+}]$$

As $\tilde{v}_h = \tilde{v}_h^0 + w_h^G$, we obtain (after integrating in $\alpha \in \mathbb{R}^1$ the last relation) that

$$\int_{\mathbb{R}^1} |\alpha| \|v_h^0(x, \alpha)\|^2_{2, B_{2R}^+} d\alpha + \int_{\mathbb{R}^1} |\alpha| \|w_h^G, v_h^0\|_{2, B_{2R}^+} d\alpha \leq \|\Phi_h\|_{2, Q_{2R}^+} \|\nabla \tilde{v}_h^0\|_{2, Q_{2R}} + \|F_h\|_{2, Q_{2R}} \|\tilde{v}_h^0\|_{2, Q_{2R}}. \tag{76}$$

Taking into account the inequality

$$\left| \int_{\mathbb{R}^1} |\alpha| \|\tilde{w}_h^G, \tilde{v}_h^0\|_{2, B_{2R}^+} d\alpha \right| \leq \frac{1}{2} \int_{\mathbb{R}^1} |\alpha| \|\tilde{w}_h^G\|^2_{2, B_{2R}^+} d\alpha + \frac{1}{2} \int_{\mathbb{R}^1} |\alpha| \|\tilde{v}_h^0\|^2_{2, B_{2R}^+} d\alpha,$$

we derive from (76) that

$$\int_{\mathbb{R}^1} |\alpha| \|\tilde{v}_h^0\|^2_{2, B_{2R}^+} d\alpha \leq \int_{\mathbb{R}^1} |\alpha| \|\tilde{w}_h^G\|^2_{2, B_{2R}^+} d\alpha + 2 \|\tilde{\Phi_h}\|_{2, Q_{2R}^+} \|\nabla \tilde{v}_h^0\|_{2, Q_{2R}} + 2 \|\tilde{F_h}\|_{2, Q_{2R}} \|\tilde{v}_h^0\|_{2, Q_{2R}}.$$

If $h \to 0$ in the last inequality then

$$\|v^0\|^2_{H^{1/2}(\mathbb{R}^1, L^2(B_{2R}^+(x^0)))} \leq 2 \|\Phi\|_{2, Q_{2R}^+(x^0)} \|\nabla v^0\|_{2, Q_{2R}^+} + 2 \|F\|_{2, Q_{2R}^+} \|v^0\|_{2, Q_{2R}^+} \tag{77}$$

By the definition of the function $w^G$ and due to the Hölder continuity of the function $u^G(z')$ along $\Gamma_{1-q}(0)$, we obtain from (77) that

$$\|w^G\|^2_{H^{1/2}(\mathbb{R}^1, L^2(B_{2R}^+(x^0)))} = \int_{\mathbb{R}^1} \int_{\mathbb{R}^1} \int_{B_{2R}^+} d^2(x) |\tilde{u}^G(x', t) \omega(t) - \tilde{u}^G(x', \tau) \omega(\tau)|^2 \frac{dx \, dt \, d\tau}{|t - \tau|^2} \tag{78}$$

$$\leq c \mathbb{L}_0 R^{n+2\alpha}.$$

Using the definition (62) of the functions $\Phi$ and $F$, we derive from (77) and (78) the inequality

$$\|v^0\|^2_{H^{1/2}(\mathbb{R}^1, L^2(B_{2R}^+(x^0)))} \leq c \|\nabla v^0\|^2_{2, Q_{2R}^+} + \frac{c \|u^0\|^2_{2, Q_{2R}^+}}{R^2} + c \mathbb{L}_0 R^{n+2\alpha}. \tag{79}$$

Now estimates (78) and (79) allows us to estimate the expression $|I_{2R}|$:

$$|I_{2R}| \leq \|w^G\|_{H^{1/2}(\mathbb{R}^1, L^2(B_{2R}^+(x^0)))} \|v^0\|_{H^{1/2}(\mathbb{R}^1, L^2(B_{2R}^+(x^0)))}.$$
\[
\leq \varepsilon \|v^0\|_{H^{1/2}(\Lambda_{2R}(\nu), L^2(B_{2R}^+))}^2 + \varepsilon^{-1}\|w^G\|_{H^{1/2}(\Lambda_{2R}(\nu), L^2(B_{2R}^+))}^2 \\
\leq \varepsilon |\nabla v^0|_{2, Q_{2R}^+}^2 + c \mathbb{L}_0 (\varepsilon^{-1} + 1) R^{n+2\alpha} + \frac{c\|u_0\|_{2, Q_{2R}^+}^2}{R^2}, \quad \forall \varepsilon > 0.
\]

We put \(\varepsilon = 1/2\) in the last inequality and apply it to estimate the right hand side of (75). Thus,
\[
\int_{Q_{2R}^+} |\nabla v^0|^2 \, dz \leq c \mathbb{L}_0 R^{n+2\alpha} + c R^{-2} \int_{Q_{2R}^+} |u_0|^2 \, dz.
\]  
(80)

As \(v^0 = u_0\) in \(Q_{R}(z^0)\), estimate (72) follows from (80).

**Proof of Theorem 2.2.** Let now \(z^0 \in \Gamma_{1-q}(0)\) where the number \(q \in (0, 1)\) is fixed arbitrarily, and a cylinder \(Q_{2R}^+(z^0) \subset Q_{1}^+(0)\).

We put
\[
\Psi(\rho, z^0) = \int_{Q_{\rho}(z^0)} |u^0(z)|^2 \, dz, \quad \rho \leq 2R, \quad u^0(z) = u(z) - u^G(z').
\]

For a fixed \(\varepsilon > 0\) and the matrix
\[
A(t) = \int_{B_R(z^0)} a(x, t) \, dx
\]
we apply Lemma 3.4 to the function \(u^0 \in W_2^{1,0}(Q_{R}^+(z^0), u^0|_{\Gamma_{R}(z^0)} = 0\). By the lemma, there exist a constant \(C_\varepsilon\), an \(A(t)\)-caloric function \(h \in W_2^{1,0}(Q_{R/2}^+(z^0)), h|_{\Gamma_{R/2}(z^0)} = 0\), and a function \(\phi_0 \in C_0^1(Q_R(z^0)), \sup_{Q_R(z^0)} |\nabla \phi_0| \leq 1, (|\phi_0| \leq cR, |(\phi_0)| \leq \frac{2}{R})\) such that
\[
\int_{Q_{R/2}^+(z^0)} (|h|^2 + R^2 |\nabla h|^2) \, dz \leq 2^{n+2} \int_{Q_{R}^+(z^0)} (|u^0|^2 + R^2 |\nabla u^0|^2) \, dz,
\]  
(81)

\[
\int_{Q_{R/2}^+(z^0)} |u^0 - h|^2 \, dz \leq \varepsilon \int_{Q_{R}^+(z^0)} (|u^0|^2 + R^2 |\nabla u^0|^2) \, dz + c R^2 \mathcal{L}_a^2(R, \phi_0),
\]  
(82)

where
\[
\mathcal{L}_a^2(R, \phi_0) = \left| \int_{Q_R(z^0)} (-u^0 (\phi_0), + (A(t) \nabla u^0, \nabla \phi_0)) \, dz \right|^2.
\]

We estimate the function \(\Psi(\rho) = \Psi(\rho, z^0), \rho \leq R/2\), with the help of the Campanato estimate (24) for the \(A(t)\)-caloric function \(h\), inequalities (81), (82), and the Friedrichs inequality:
\[
\Psi(\rho) \leq 2 \int_{Q_R(z^0)} |u^0 - h|^2 \, dz + 2 \int_{Q_R(z^0)} |h|^2 \, dz \leq c \left( \frac{R}{\rho} \right)^{n+2} \int_{Q_{R/2}^+(z^0)} |u^0 - h|^2 \, dz
\]  
+ c \left( \frac{\rho}{R} \right)^2 \int_{Q_{R/2}^+(z^0)} |h|^2 \, dz \leq c \left( \frac{R}{\rho} \right) \int_{Q_{R/2}^+(z^0)} |\nabla u^0|^2 \, dz + C_\varepsilon R^2 \mathcal{L}_a^2(R, \phi_0) \left( \frac{R}{\rho} \right)^{n+2}
\]  
+ c \left( \frac{\rho}{R} \right)^2 R^2 \int_{Q_{R}^+(z^0)} |\nabla u^0|^2 \, dz.
\]  
(83)

By estimate (72),
\[
R^2 \int_{Q_R(z^0)} |\nabla u^0|^2 \, dz \leq c \int_{Q_{2R}^+(z^0)} |u_0|^2 \, dz + c \mathbb{L}_0 R^{2\alpha}.
\]  
(84)
It follows from (83), (84) that
\[
\Psi(\rho) \leq c \left[ \left( \frac{\rho}{R} \right)^2 + \varepsilon \left( \frac{R}{\rho} \right)^{n+2} \right] \Psi(2R) + cL_0 R^{2\alpha} + C_\varepsilon R^2 L^2_a(R, \phi_0) \left( \frac{R}{\rho} \right)^{n+2}.
\] (85)

Now we address to identity (57) to estimate $L^2_a(R, \phi_0)$. We put
\[
\Delta a = a(x, t) - A(t)
\]
and attract conditions $H^2, H^5'$ to derive the inequalities:
\[
L^2_a(R, \phi_0) = \left| \int_{Q^+_R} (\Delta a \nabla u^0, \nabla \phi_0) \, dz + \int_{Q^+_R} [\hat{u}^G (\phi_0)_t - (a \nabla' u^G, \nabla \phi_0) + f \phi_0] \, dz \right|^2
\]
\[
\leq \int_{Q^+_R} |\Delta a|^2 \, dz \int_{Q^+_R} |\nabla u^0|^2 \, dz + \int_{Q^+_R} |\hat{u}^G|^2 \, dz \int_{Q^+_R} |(\phi_0)_t|^2 \, dz + c \int_{\Gamma_{2R}} |\nabla' u^G|^2 \, d\Gamma
\]
\[
+ cR^2 \int_{Q^+_R} |f|^2 \, dz \leq q^2_a(R) \int_{Q^+_R} |\nabla u^0|^2 \, dz + cL_0 R^{-2+2\alpha}.
\]

Certainly, we have used estimate (8) from Theorem 2.1. Now it follows from (85) and (72) that
\[
\Psi(\rho) \leq c \left[ \left( \frac{\rho}{R} \right)^2 + \varepsilon \left( \frac{R}{\rho} \right)^{n+2} + C_\varepsilon q^2_a(R) \left( \frac{R}{\rho} \right)^{n+2} \right] \Psi(2R) + cL_0 R^{2\alpha}.
\] (86)

We put $r = 2R$ in the last inequality and obtain the relation
\[
\Psi(\rho) \leq c_0 \left[ \left( \frac{\rho}{r} \right)^2 + \varepsilon \frac{r}{\rho} \right]^{n+2} + C_\varepsilon \left( \frac{r}{\rho} \right)^{n+2} q^2_a(r) \left( r \right) \Psi(r) + c_1 r^{2\alpha} L_0, \quad \forall \rho \leq r/4.
\] (87)

Now we fix $\beta = \frac{1+\alpha}{2} > \alpha$ and put $\rho = \tau r$, $\tau \leq 1/4$ in (87):
\[
\Psi(\tau r) \leq c_0 [\tau^2 + \varepsilon \tau^{-(n+2)} + C_\varepsilon \tau^{-(n+2)} q^2_a(r)] \Psi(r) + c_1 L_0 r^{2\alpha}.
\] (88)

Further we fix $\tau \leq 1/4$ such that
\[
c_0 \tau^2 < \frac{\tau^{2\beta}}{4}.
\] (89)

Then we put $\varepsilon > 0$ to satisfy the inequality
\[
c_0 \varepsilon \tau^{-(n+2)} < \frac{\tau^{2\beta}}{4}.
\] (90)

At last, we fix $r_0$:
\[
c_0 C_\varepsilon \tau^{(n+2)} q^2_a(r_0) < \frac{\tau^{2\beta}}{4}.
\] (91)
Under conditions (89) - (91) we have the inequality
\[ \Psi(\tau r) < \tau^{2\beta} \Psi(r) + c_1 r^{2\alpha} L_0. \] (92)

For the fixed \( \tau, \varepsilon, r_0 \), we can change \( r \) by \( \tau^j r, j \in \mathbb{N} \), and repeat all considerations. In a result, we get
\[ \Psi(\tau^{j+1} r) \leq \tau^{2\beta j} \Psi(\tau^j r) + c_1 \tau^{2\alpha j} L_0 r^{2\alpha}. \] (93)

The iterating process provides that
\[ \Psi(\tau^{j+1} r) \leq c_1 \tau^{2\beta j} \Psi(r) + c_2 \tau^{2\alpha j} L_0 r^{2\alpha}. \] (94)

The constants \( c_1 \) and \( c_2 \) in (93) and (94) do not depend on \( z^0 \) and \( r \).

It follows from (94) that
\[ \Psi(\rho) \leq c_2 \rho^{2\alpha} \left( \frac{\Psi(r)}{r^{2\alpha}} + L_0 \right), \quad \forall \rho \leq r \leq r_0. \] (95)

Thus, for any \( z^0 \in \Gamma_{1-q}(0) \)
\[ \sup_{\rho \leq r_0} \frac{1}{\rho^2 + 2 + 2\alpha} \int_{Q_0^+(z^0)} |u|^2 dz \leq c \{ \int_{\Gamma_\rho(z^0)} |u|^2 dz + L_0 \}. \] (96)

Taking into account that \( u^G \in C^\alpha(\Gamma_{1-q}(0); \delta) \), we derive from (96) that
\[ \Phi(\rho, z^0) := \frac{1}{\rho^{2 + 2\alpha}} \int_{Q_\rho(z^0)} |u - u_{\rho,z^0}|^2 dz \leq \frac{1}{\rho^{2 + 2\alpha}} \int_{Q_\rho(z^0)} |u - u_{\rho,z^0}|^2 dz \]
\[ \leq \frac{2}{\rho^{2 + 2\alpha}} \int_{Q_\rho(z^0)} |u|^2 dz + \frac{2\rho}{\rho^{2 + 2\alpha}} \int_{\Gamma_\rho(z^0)} |u^G - u_{\rho,z^0}|^2 d\Gamma \]
\[ \leq c(r_0^{-1}) \{ \| u \|^2_{W^{1,0}_2(Q_1^+)} + \| u^G \|^2_{W^{2,0}_2(\Gamma_1)} \} + c(\| f \|^2_{L^{2,n-2+2\alpha}(Q_1^+; \delta)} + \| \psi \|^2_{L^{2,n-3+2\alpha}(\Gamma_1; \delta)}). \]

It follows that
\[ \sup_{z^0 \in \Gamma_{1-q}, \rho \leq r_0} \Phi(\rho, z^0) \leq c(r_0^{-1}) \{ \| u \|^2_{W^{1,0}_2(Q_1^+)} + \| u^G \|^2_{W^{2,0}_2(\Gamma_1)} \} + c(\| f \|^2_{L^{2,n-2+2\alpha}(Q_1^+; \delta)} + \| \psi \|^2_{L^{2,n-3+2\alpha}(\Gamma_1; \delta)}). \] (97)

We recall that in estimate (97) the number \( r_0 \leq q \) depends on the data only.

At the same time, it follows from Proposition 5.2 that \( \Phi(\rho, \xi) \) is estimated for \( \xi \in Q_{1-q}^+(0) \) and \( \rho < \delta(\xi, \Gamma_1) \) (see (54)). Usual "sewing" procedure allows us to derive the estimate
\[ \sup_{\rho \leq r_0, z^0 \in Q_{1-q}^+(0)} \Phi(\rho, z^0) \leq c L_0. \] (98)

It means that the seminorm of \( u \) in \( L^{2,n+2+2\alpha}(Q_{1-q}^+(0); \delta) \) is estimated. By the isomorphism of this space to \( C^\alpha(Q_{1-q}^+(0); \delta) \) we have got estimate of the Hölder norm of \( u \). Further, using estimate (72), we derive (9).
Proof of Theorem 2.3  To prove the assertion I we should repeat proof of Theorem 2.1 up to the relation (41). In this case

\[ K(r) \leq K_0 r^{2\alpha_0}, \quad K_0 = \|f\|^2_{L^2,\lambda_0(Q_1^+;\delta)} + \|\psi\|^2_{L^2,\lambda_0(\Gamma_1;\delta)}, \]  

(99)

and inequality (50) is valid with \(\alpha_0 = \frac{\lambda_0+1}{2}\) and \(K_0\) defined by (99). Indeed, taking into account the definition (31) of the expression \(K(r)\) and the assumptions on \(f\) and \(\psi\), we obtain validity of (99) and the assertions i) and ii) of Theorem 2.3.

To prove the assertion II of Theorem 2.3 we repeat all steps of the proof of Theorem 2.2 with \(f \in L^{2,\alpha_0}(Q_1^+;\delta)\) and \(\psi \in L^{2,\lambda_0}(\Gamma_1;\delta)\) where \(\lambda_0 \in (0,1)\) and \(\alpha_0 = \frac{\lambda_0+1}{2}\). •

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