Improved perturbative QCD formalism for $B_c$ meson decays

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We derive the $k_T$ resummation for doubly heavy-flavored $B_c$ meson decays by including the charm quark mass effect into the known formula for a heavy-light system. The resultant Sudakov factor is employed in the perturbative QCD study of the “golden channel” $B_c^+ \rightarrow J/\psi \pi^+$. With a reasonable model for the $B_c$ meson distribution amplitude, which maintains approximate on-shell conditions of both the partonic bottom and charm quarks, it is observed that the imaginary piece of the $B_c \rightarrow J/\psi$ transition form factor appears to be power suppressed, and the $B_c^+ \rightarrow J/\psi \pi^+$ branching ratio is not lower than $10^{-3}$. The above improved perturbative QCD formalism is applicable to $B_c$ meson decays to other charmonia and charmed mesons.

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I. INTRODUCTION

A $B_c$ meson is the ground state of the doubly heavy-flavored $\bar{b}c$ system in the Standard Model [1], different from the heavy-light one represented by a $B$ meson and from the heavy-heavy one represented by quarkonia $J/\psi$ and $\Upsilon$ in many aspects. Its weak transition can occur through the bottom quark decay with the spectator charm quark as displayed in Fig. 1(a), the charm quark decay with the spectator bottom quark in Fig. 1(b), and the pure weak annihilation channel in Fig. 1(c). Hence, $B_c$ meson decays contain rich heavy quark dynamics in both the perturbative and nonperturbative regimes, which is worth a thorough exploration with high precision. It is certainly a challenge to develop an appropriate theoretical framework for analyzing $B_c$ meson decays. A framework available in the literature is the perturbative QCD (PQCD) approach, which basically follows the conventional one for $B$ meson decays, with the finite charm quark mass being included in hard decay kernels but neglected in the $k_T$ resummation for meson distribution amplitudes. A rigorous resummation formalism for $B_c$ meson decays, which involve multiple scales, is expected to be more complicated than for $B$ meson decays.

In this paper, we will investigate how the charm quark mass affects the infrared structures of the $B_c$ meson and of its decay products and derive the corresponding $k_T$ resummation in the PQCD approach. The derivation depends on the power counting for the ratio $m_c/m_b$, $m_b$ ($m_c$) being the bottom (charm) quark mass. Taking the limit $m_b \rightarrow \infty$ but keeping $m_c$ finite, we treat a $B_c$ meson as a heavy-light system, the decays of which can be analyzed in the conventional PQCD approach to $B$ meson decays mentioned above. Taking the limit $m_b, m_c \rightarrow \infty$ but fixing the ratio $m_c/m_b$, we treat a $B_c$ meson as a heavy-heavy system, the decays of which may be studied in a formalism for heavy quarkonium decays. Here, we will adopt the power counting rules proposed in Ref. [2] and regard a $B_c$ meson as a multiscale system, which respects the hierarchy $m_b \gg m_c \gg \Lambda_{QCD}$, $\Lambda_{QCD}$ being the QCD scale. An intermediate impact of this power counting is that the large infrared logarithms $\ln(m_b/m_c)$, in addition to the ordinary ones $\ln(m_b/\Lambda_{QCD})$, appear in the perturbative evaluation of the $B_c$ meson distribution amplitude and need to be resummed.

The Sudakov factor from the $k_T$ resummation with the charm quark mass effect is then employed in the PQCD study of the “golden channel” $B_c^+ \rightarrow J/\psi \pi^+$. We focus on the bottom quark decay of a $B_c$ meson because the charm quark decay is believed to suffer from significant long-distance contributions, i.e., final-state interactions, though perturbative results for the $B_c \rightarrow B(s)X$ modes have been presented in the literature [3, 4]. Besides, the available models for the $B_c$ meson distribution amplitude vary dramatically from a simple $\delta$ function [5, 6] to a complicated Gaussian type [3]. We will propose a kinematic constraint on the $B_c$ meson distribution amplitude, which allows both the partonic bottom and charm quarks to be off shell only at a power-suppressed level. It is then shown, with a reasonable model for the $B_c$ meson distribution amplitude, that the
imaginary piece of the $B_c \to J/\psi$ transition form factor, supposed to be a real object [7], is indeed power suppressed. It is also found that the $B_c^+ \to J/\psi\pi^+$ branching ratio is not lower than $10^{-3}$, in agreement with those obtained in other approaches.

In Sec. II, we discuss the kinematic constraint on the charm quark momentum distribution in a $B_c$ meson. The one-loop correction to the $B_c$ meson distribution amplitude, which generates the double logarithm $\alpha_s \ln^2(m_b/m_c)$, $\alpha_s$ being the strong coupling, is calculated in Sec. III. The result hints at how the $k_T$ resummation for $B_c$ meson decays is modified from the known formula for $B$ meson decays. In Sec. IV, we predict the $B_c^+ \to J/\psi\pi^+$ branching ratio in the improved PQCD framework, including the contributions from both factorizable and nonfactorizable emission diagrams. It is then stressed in the Conclusion that the formalism developed here is ready for the extension to $B_c$ meson decays to other charmonia like $\eta_c$, $\chi_{cJ}$ ($J = 0, 1, 2, \ldots$, and charmed mesons).

II. KINEMATIC CONSTRAINT ON $B_c$ MESON DISTRIBUTION AMPLITUDE

Consider the $B_c(P_1) \to J/\psi(P_2)$ transition at the maximal recoil, where

$$P_1 = \frac{m_{B_c}}{\sqrt{2}}(1, 1, 0_T), \quad P_2 = \frac{m_{B_c}}{\sqrt{2}}(1, r_{J/\psi}, 0_T)$$

in the light-cone coordinates label the $B_c$ and $J/\psi$ meson momenta, respectively, with $r_{J/\psi} = m_{J/\psi}/m_{B_c}$ and $m_{B_c}$ ($m_{J/\psi}$) being the $B_c$ ($J/\psi$) meson mass. This transition involves multiple scales the same as in the $B \to D^*$ transition, which has been studied in Ref. [2]: $m_b$ from the initial-state $B$ meson, $m_c$ from the final-state $D^*$ meson, and both the $B$ and $D^*$ bound states contain the nonperturbative dynamics characterized by a low hadronic scale $\Lambda$. Following the argument in Ref. [2], the scaling of the energetic $J/\psi$ momentum $P_2 \sim (m_b/m_c, r_{J/\psi}, 0_T) \sim m_c(m_b/m_c, 0_T)$ hints that the components of a collinear gluon momentum in such a multiscale system also obeys the power counting

$$l^\mu \sim \left(\frac{m_b}{m_c}, \frac{m_c}{m_b}, \Lambda, \Lambda\right),$$

with a tiny invariant mass squared $l^2 \sim O(\Lambda^2)$. A valence charm quark in the $J/\psi$ meson, after emitting such a collinear gluon, can acquire the virtuality of order $P_2 \cdot l \sim m_c\Lambda$. The momentum parametrizations for the two valence charm quarks participating in the hard subprocess should be symmetric under their exchange. Denote the spectator charm quark momentum as $k_2 = x_2 P_2$ and another as $P_2 - k_2 = (1 - x_2) P_2$ with the momentum fraction $x_2$, and assume both of them to be off-shell at most by $O(m_c\Lambda)$: $k_2^2 - m_c^2 = O(m_c\Lambda)$ and $(P_2 - k_2)^2 - m_c^2 = O(m_c\Lambda)$. To satisfy these two conditions simultaneously, we choose a charm quark mass $m_c \approx m_{J/\psi}/2 \sim 1.5$ GeV for $m_{J/\psi} = 3.097$ GeV, and the momentum fraction $x_2 = 1/2 \pm \delta$ can deviate from its central value by $\delta \sim O(\Lambda/m_c)$. That is, the $J/\psi$ distribution amplitude takes a substantial value in the above range of $x_2$ with $\delta \sim 0.3$ for $\Lambda \sim 0.5$ GeV, due to the effect of collinear gluon emissions. The model for the $J/\psi$ meson distribution amplitude, proposed in Ref. [8] and widely employed in the PQCD analyses, does exhibit these features.

Next, we discuss the kinematic constraint on the shape of the $B_c$ meson distribution amplitude. Label the momentum of the spectator charm quark in the $B_c$ meson by $k_1$ and that of the bottom quark by $P_1 - k_1$. The approximate on-shell-ness of the partons, $k_1^2 \sim m_c^2$ and $(P_1 - k_1)^2 \sim m_b^2$, implies that the zeroth component of $k_1$ is of order $k_1^0 \sim m_c$. A $B_c$ meson at rest is dominated by soft dynamics, for which the momentum of a soft gluon is characterized by the power counting [2]

$$l^\mu \sim (\Lambda, \Lambda, \Lambda),$$

with a tiny invariant mass squared $l^2 \sim O(\Lambda^2)$. The spectator charm quark, after emitting such a soft gluon, then reaches the virtuality of order $k_1 \cdot l \sim m_c\Lambda$. Parametrize the charm quark momentum by $k_1 = x_1 P_1$, $x_1$ being a momentum fraction, and
require the virtuality $k_1^2 - m_v^2 = O(m_c \Lambda)$. Given the bottom quark mass $m_b \approx m_{B_c} - m_c \sim 4.8$ GeV for $m_{B_c} = 6.276$ GeV, we find that the $B_c$ meson distribution amplitude takes a substantial value around the momentum fraction $x_1 \sim m_c/m_b \sim 0.3$ within the width of about $\Lambda/m_b \sim 0.1$. It can be verified, following the above discussion, that the bottom quark in the $B_c$ meson acquires the virtuality of $(P_1 - k_1)^2 - m_b^2 \sim O(m_b \Lambda)$, consistent with the soft gluon emission effect.

We then investigate the virtuality of the hard particles in the kinematic regions specified for the partonic bottom and charm quarks. First, the invariant mass of the hard gluon emitted by the spectator quark is written as

$$(k_1 - k_2)^2 \approx -\frac{m_b m_c}{2} + O(m_b \Lambda),$$

(4)

with the insertion of $m_{B_c} \approx m_b + m_c$ and $m_{J/\psi} \approx 2m_c$ up to the first powers in $m_c$ and in $\Lambda$. The first term on the right-hand side of Eq. (4), being $O(m_b m_c)$, indicates that the hard gluon tends to be spacelike for the chosen mass scales $m_b$, $m_c$, and $\Lambda$. The hard bottom quark, to which the hard gluon attaches, remains spacelike with the virtuality

$$(P_1 - k_1)^2 - m_b^2 \approx -\frac{m_b^2}{2}.$$  

(5)

The hard charm quark, to which the hard gluon attaches, is also spacelike with the virtuality

$$(P_2 - k_1)^2 - m_c^2 \approx -m_b m_c + O(m_b \Lambda).$$  

(6)

We conclude that, as both the partonic bottom and charm quarks are only off-shell a bit, the imaginary piece in the $B_c \to J/\psi$ transition form factor appears to be power suppressed. This observation is easily understood: the $J/\psi$ meson mass is below the $DD$ threshold, so the $B_c \to J/\psi$ transition hardly occurs through an intermediate state.

![Figure 2](image)

FIG. 2. (Color online) Behavior of $\phi_{B_c}(x)$ for the different shape parameters $\beta_{B_c} = 0.8$ GeV (black-solid curve), 1.0 GeV (red-dashed curve), and 1.2 GeV (blue-dotted curve).

The $B_c$ meson wave function with an intrinsic $k_T$ dependence is parametrized in a Gaussian form as [9]

$$\phi_{B_c}(x,k_T) = \frac{f_{B_c}}{2\sqrt{2}N_c \, 2\beta_{B_c}^2/\Lambda} \exp \left[ -\frac{1}{8\beta_{B_c}^2} \left( \frac{|k_T|^2 + m_c^2}{x} + \frac{|-k_T|^2 + m_b^2}{1-x} \right) \right],$$

(7)

in which $k_T$ ($-k_T$) is the transverse momentum carried by the charm (bottom) quark, $N_c$ is the number of colors, $\beta_{B_c}$ is the shape parameter, and $N_{B_c}$ is the normalization constant. The $B_c$ meson distribution amplitude is given by

$$\phi_{B_c}(x,b) = \frac{f_{B_c}}{2\sqrt{2}N_c} \, N_{B_c} x (1-x) \exp \left[ -\frac{(1-x)m_c^2 + x m_b^2}{8\beta_{B_c}^2} \frac{1}{x(1-x)} \right] \exp \left[ -2\beta_{B_c}^2 x(1-x)b^2 \right],$$

(8)

with the impact parameter $b$ being conjugate to $k_T$. The normalization constant $N_{B_c}$ is fixed by the relation

$$\int_0^1 \phi_{B_c}(x,b=0) dx \equiv \int_0^1 \phi_{B_c}(x) dx = \frac{f_{B_c}}{2\sqrt{2}N_c},$$

(9)

where the decay constant $f_{B_c} = 0.489 \pm 0.005$ GeV has been obtained in lattice QCD by the TWQCD Collaboration [10]. Figure 2, in which the behavior of $\phi_{B_c}(x)$ is plotted for the different shape parameters $\beta_{B_c}$, indicates that the peak of $\phi_{B_c}(x)$ shifts toward larger $x$ and becomes broader with the increase of $\beta_{B_c}$. Note that data for $B_c$ meson decay branching ratios are not yet available, so it is difficult to determine $\beta_{B_c}$ unambiguously. However, the kinematic constraint derived above hints that $\beta_{B_c} = 1.0$ GeV seems to be a reasonable choice. On the other hand, the existent models [3, 11] of the $B_c$ meson distribution amplitude roughly correspond to the range [0.6, 1.0] GeV of the parameter $\beta_{B_c}$.
A theoretical challenge from the $B_c \to J/\psi$ transition is to derive the $k_T$ resummation for energetic charm quarks with a finite mass. To proceed, we construct a transverse momentum-dependent $J/\psi$ meson wave function in the $k_T$ factorization theorem [12, 13] and then perform the perturbative evaluation according to the wave-function definition as a hadronic matrix element of a nonlocal operator. The double logarithms attributed to the overlap of the collinear and soft radiative corrections are expected to differ from those in $B$ meson decays into light mesons, which have been elaborated in Ref. [14]. According to the one-loop analysis in Ref. [15], the only source of the double logarithms is the correction to the quark-Wilson-line vertex as displayed in Fig. 3(a), in which the loop momentum does not flow into a hard subprocess. When the gluon in Fig. 3(a) attaches to the lower piece of the Wilson lines, the loop momentum flows through a hard subprocess. Since the region with small parton momenta dominates in the $k_T$ factorization, the large collinear gluon momentum induces power suppression on the hard kernel [13], such that this one-loop diagram does not generate the double logarithm. The similar vertex diagram with the gluon being radiated by the spectator charm quark either in the $J/\psi$ meson [Fig. 3(b)] or in the $B_c$ meson [Fig. 3(c)] may produce the double logarithms. Nevertheless, their effects ought to be weaker, due to the lack of phase space for collinear gluons from less energetic quarks.

The loop integral corresponding to Fig. 3(a) is written as
\begin{equation}
\phi^{(1)} = -\frac{i}{4g^2 C_F \mu_T^2} \int \frac{d^4 l}{(2\pi)^4} l \cdot \left[ \gamma_5 \not{l} + \gamma_\nu \cdot \not{l} + m_c \gamma_\nu \not{n} - \gamma_5 \right] \frac{1}{l^2/n \cdot l} \tag{10}
\end{equation}
with $\not{l} \equiv P_2 - k_2$, the eikonal vertex $n_\nu$, and the eikonal propagator $1/n \cdot l$. The dimensionless vector $n$ with $n^\perp > 0$ represents the direction of the Wilson lines, which is allowed to be away from the light cone [15]. The projectors $\gamma_5 n_+ \not{n} - \gamma_5 n_-$, arising from the insertion of the Fierz identity for factorizing the fermion flow, work for the selection of the logarithm $\ln(m_b/m_c)$ up to corrections in powers of $m_c/m_b$. A straightforward calculation leads to
\begin{equation}
\phi^{(1)} = \frac{\alpha_s}{4\pi} C_F \left[ \frac{1}{e} + \ln \frac{4\pi \mu_T^2}{m_c^2} - \ln^2 \frac{\zeta^2}{k_T^2} + \ln \frac{m_c^2}{k_T^2} + \ln \frac{\zeta^2}{m_c^2} + 2 - \frac{2\pi^2}{3} \right], \tag{11}
\end{equation}
with the factorization scale $\mu_T$, the Euler constant $\gamma_E$, and the variable $\zeta^2 \equiv 4(n \cdot \not{k})^2/n^2$. It is found that the infrared logarithms in the above expression reproduce those in the pion case [16], as $m_c$ is replaced by $k_T$. The double logarithms can be understood in the way that the soft divergence is regularized by the quark virtuality $k_T$, and the collinear divergence is regularized by the charm quark mass $m_c$, giving
\begin{equation}
-\ln^2 \frac{\zeta^2}{k_T^2} + \ln^2 \frac{m_c^2}{k_T^2} = -\ln \frac{\zeta^2 m_c^2}{k_T^4} \ln \frac{\zeta^2}{m_c^2}. \tag{12}
\end{equation}

The partial cancellation between the two double logarithms implies that the resummation effect in the case of energetic massive quarks is smaller than in the case of light quarks [17].

The aforementioned lack of phase space for the collinear gluons in Figs. 3(b) and 3(c) can be understood by means of the contour integration. Take Fig. 3(b), the loop integrand of which contains a denominator $(k_2 - l^\perp)^2 - m_c^2$ from the anticharm quark propagator, as an example. To get a nonvanishing contribution from the contour integration over the minus component $l^\perp$ of the loop momentum, some poles of $l^\perp$ have to be located in the upper half-plane, and some have to be located in the lower half-plane. This is possible only when the coefficients of $l^\perp$ in the denominators of the corresponding loop integrand are not of the same sign. Hence, the plus component $l^\perp$ must take a value in the range $0 < l^\perp < k_T^\perp$ for our gauge choice $n^\perp > 0$ as stated below Eq. (10). In the dominant region with small parton momenta, i.e., with small $k_T^\perp$, the phase space for $l^\perp$ is then limited, implying a weaker double logarithmic effect.

We will not attempt a complete one-loop computation and an exact next-to-leading-logarithm resummation associated with an energetic massive quark in the present work. Instead, we will infer an approximate Sudakov exponent from the implication
of Eq. (11). It has been known that the $k_T$ resummation for an energetic light quark yields the Sudakov exponent in the $b$ space [14, 18],

$$s(Q, b) = \int_{1/b}^{Q} \frac{d\mu}{\mu} \left[ \int_{1/b}^{\mu} \frac{d\bar{\mu}}{\bar{\mu}} A(\alpha_s(\bar{\mu})) + B(\alpha_s(\mu)) \right],$$

(13)

at the next-to-leading-logarithm accuracy, where the universal anomalous dimension $A(\alpha_s)$ given to two loops is responsible for the collection of the double logarithms, the factor $B(\alpha_s)$ given to one loop is for the collection of single logarithms, and $Q$ is related to the major light-cone component of the quark momentum through the variable $\zeta$. The $\mu_f$-independent logarithms in Eq. (11) can be cast into two pieces,

$$- \left( \ln^2 \frac{m^2}{k_T^2} - \ln \frac{Q^2}{k_T^2} \right) + \left( \ln^2 \frac{m^2}{k_T^2} - \ln \frac{m^2}{k_T^2} \right),$$

(14)

which are of the same form. This hints that the above infrared logarithms may be organized into the Sudakov exponents with the different upper bounds $Q$ and $m_c$; namely, the Sudakov exponent $s_c(Q, b)$ for an energetic charm quark up to next-to-leading-logarithm might be expressed as the difference

$$s_c(Q, b) = s(Q, b) - s(m_c, b),$$

$$= \int_{m_c}^{Q} \frac{d\mu}{\mu} \left[ \int_{1/b}^{\mu} \frac{d\bar{\mu}}{\bar{\mu}} A(\alpha_s(\bar{\mu})) + B(\alpha_s(\mu)) \right].$$

(15)

This observation applies to the organization of the double logarithms in Figs. 3(b) and 3(c).

At last, the $\mu_f$-dependent logarithm $\ln(\mu_f^2/m_c^2)$ in Eq. (11) means that the $J/\psi$ (as well as $B_c$) meson distribution amplitude is defined at the scale $m_c$ and that the renormalization-group evolution for the $B_c \to J/\psi$ transition runs from $\mu_f = m_c$ to the hard scale of the process. We summarize the exponents of the total evolution factors for the $B_c$ and $J/\psi$ meson distribution amplitudes as

$$S_{B_c} = s_c \left( x_1 P_1^-, b_1 \right) + \frac{5}{3} \int_{m_c}^{t} \frac{d\bar{\mu}}{\bar{\mu}} \gamma_q(\alpha_s(\bar{\mu})),$$

$$S_{J/\psi} = s_c \left( x_2 P_2^+, b_2 \right) + s_c \left( (1 - x_2) P_2^+, b_2 \right) + 2 \int_{m_c}^{t} \frac{d\bar{\mu}}{\bar{\mu}} \gamma_q(\alpha_s(\bar{\mu})),$$

(16)

with the hard scale $t$, and the quark anomalous dimension $\gamma_q = -\alpha_s/\pi$, that governs the aforementioned renormalization-group evolution. The coefficient $5/3$ in the first line of Eq. (16) differs from the coefficient 2 in the second line, since we have employed the effective heavy quark field for the bottom quark in the definition of the $B_c$ meson distribution amplitude, as exhibited by the horizontal double line in Fig. 3(c). For the numerical analysis below, we insert the one-loop running coupling constant $\alpha_s$ into Eq. (16) in order to match the expected next-to-leading-logarithm accuracy of our resummation formula.

IV. $B_c^+ \to J/\psi\pi^+$ DECAY

![Leading-order diagrams for the $B_c^+ \to J/\psi\pi^+$ decay in the PQCD approach.](image)

After the pioneering paper on $B_c$ meson decays by Bjorken in 1986 [19], numerous investigations in different formalisms have been devoted to this subject, but the predictions vary in a wide range. For example, the $B_c^+ \to J/\psi\pi^+$ branching ratio was predicted to be between orders of $10^{-3}$ and $10^{-2}$ [20–24]. In particular, it takes the values $1.2 \times 10^{-3}$ in the QCD factorization approach [21] and $(1.4 \sim 2.5) \times 10^{-3}$ [22], $2.32^{+0.64-0.38}_{-0.04-0.2} \times 10^{-3}$ [23], and $2.6^{+0.2} \times 10^{-3}$ in the conventional
PQCD approach. These results manifest the sensitivity to the hadronic inputs in the theoretical frameworks for $B_c$ meson decays. However, the current data, appearing only as the ratios of the decay rates because of experimentally complicated background, such as
\[
R_{K/\pi}^{J/\psi} = \frac{Br(B_c \to J/\psi K^+)}{Br(B_c \to J/\psi \pi^+)},
\]
(17)
cannot be used to discriminate the branching-ratio predictions. The factorizable emission diagrams in Figs. 4(a) and 4(b) dominate the $B_c^+ \to J/\psi K^+$ and $B_c^+ \to J/\psi \pi^+$ modes, so the associated uncertain $B_c \to J/\psi$ transition form factor cancels in the ratio. This explains why the various formalisms lead to similar $R_{K/\pi}^{J/\psi}$ in agreement with the latest measurement [25], although they give quite distinct values for the individual branching ratios.

In this section, we calculate the $B_c \to J/\psi$ transition form factor and the $B_c^+ \to J/\psi \pi^+$ branching ratio in the improved PQCD approach developed in Sec. III. The relevant weak effective Hamiltonian $H_{\text{eff}}$ is written as [26]
\[
H_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{cb}^* V_{ud}[C_1(\mu)O_1(\mu) + C_2(\mu)O_2(\mu)] + \text{H.c.},
\]
(18)
where $C_{1,2}(\mu)$ are the Wilson coefficients evaluated at the renormalization scale $\mu$ and the local four-quark operators are
\[
O_1 = \bar{d}_\alpha \gamma_\mu (1 - \gamma_5) u_\beta \bar{c}_\beta \gamma_\mu (1 - \gamma_5) b_\alpha, \quad O_2 = \bar{d}_\alpha \gamma_\mu (1 - \gamma_5) u_\alpha \bar{c}_\beta \gamma_\mu (1 - \gamma_5) b_\beta,
\]
(19)
with the color indices $\alpha$ and $\beta$ and the Fermi constant $G_F = 1.16639 \times 10^{-5}$ GeV$^{-2}$. For the Cabibbo-Kobayashi-Maskawa matrix elements $V_{cb}$ and $V_{ud}$, we employ the Wolfenstein parametrization at leading order with the parameters $A = 0.811$ and $\lambda = 0.22506$ [27]. The momenta of the $B_c$ and $J/\psi$ mesons have been chosen in Eq. (1), from which the pion momentum is given by $P_\pi = m_{B_c}/\sqrt{2}(0, 1 - r_{J/\psi}^J, 0_T)$, for the vanishing pion mass. The momenta of the spectator quarks in the involved hadrons are parametrized as
\[
k_1 = (x_1 P_{t1}^+, x_1 P_{t1}^-, k_{1T}), \quad k_2 = (x_2 P_{t2}^+, x_2 P_{t2}^-, k_{2T}), \quad k_3 = (x_3 P_{t3}^+, x_3 P_{t3}^-, k_{3T}).
\]
(20)
The $B_c$, $J/\psi$, and $\pi$ meson distribution amplitudes have the structures
\[
\Phi_{B_c}(x, b) \equiv \frac{i}{\sqrt{2N_c}} (P_{B_c} + m_{B_c}) \gamma_5 \phi_{B_c}(x, b),
\]
(21)
\[
\Phi_\pi(x) \equiv \frac{i}{\sqrt{2N_c}} \gamma_\mu \left[ P_\pi \phi_\pi^A(x) + m_\pi \phi_\pi^P(x) + m_\pi (\not{Q}_\mu - 1) \phi_\pi^T(x) \right],
\]
(22)
\[
\Phi_{J/\psi}^L(x) \equiv \frac{1}{\sqrt{2N_c}} \left[ m_{J/\psi} \epsilon_{J/\psi}^L \phi_{J/\psi}^L(x) + \epsilon_{J/\psi}^L P_{J/\psi} \phi_{J/\psi}^T(x) \right],
\]
(23)
with the dimensionless vectors $n = (0, 1, 0_T)$ and $v = (1, 0, 0_T)$ and the longitudinal polarization vector for the $J/\psi$ meson
\[
\epsilon_{J/\psi}^L = \frac{1}{\sqrt{2r_{J/\psi}^J}}(1, -r_{J/\psi}^J, 0_T).
\]
(24)
Owing to the experimental status stated before, we adopt the shape parameter $\beta_{B_c} = 1$ GeV for the $B_c$ meson distribution amplitude inferred from the kinematic constraint. The light-cone pion distribution amplitudes $\phi_\pi^A$ (twist 2), and $\phi_\pi^P$ and $\phi_\pi^T$ (twist 3) have been parametrized as [28–30]
\[
\phi_\pi^A(x) = \frac{f_\pi}{2\sqrt{2N_c}} 6x(1-x) \left[ 1 + a^2_\pi C_2^{3/2}(2x - 1) + a^2_\pi C_4^{3/2}(2x - 1) \right],
\]
(25)
\[
\phi_\pi^P(x) = \frac{f_\pi}{2\sqrt{2N_c}} \left[ 1 + \left( 30 \eta_3 - \frac{5}{2} \rho_\pi^2 \right) C_2^{1/2}(2x - 1) \right.
\]
\[
- 3 \left( \eta_3 \omega_3 + \frac{9}{20} \rho_\pi^2 (1 + 6a^2_\pi) \right) C_4^{1/2}(2x - 1) \right],
\]
(26)
\[
\phi_\pi^T(x) = \frac{f_\pi}{2\sqrt{2N_c}} (1 - 2x) \left[ 1 + 6 \left( 5 \eta_3 - \frac{1}{2} \eta_3 \omega_3 - \frac{7}{20} \rho_\pi^2 - \frac{3}{5} \rho_\pi^2 a^2_\pi \right) (1 - 10x + 10x^2) \right],
\]
(27)
with the decay constant $f_\pi = 0.130$ GeV; the Gegenbauer moments $a^2_\pi = 0.115 \pm 0.115$ and $a^2_\pi = -0.015$; the parameters $\eta_3 = 0.015$ and $\omega_3 = -3$ [28, 29]; the mass ratio $\rho_\pi = m_\pi/m_0$, $m_0 = 1.4$ GeV being the pion chiral mass; and the Gegenbauer polynomials $C_\alpha^n(t)$,
\[
C_2^{1/2}(t) = \frac{1}{2} (3t^2 - 1), \quad C_4^{1/2}(t) = \frac{1}{8} (3 - 10t^2 + 35t^4),
\]
\[
C_2^{3/2}(t) = \frac{3}{2} (5t^2 - 1), \quad C_4^{3/2}(t) = \frac{15}{8} (1 - 14t^2 + 21t^4).
\]
(28)
The \( J/\psi \) meson distribution amplitudes \( \phi_{J/\psi}^L \) (twist 2) and \( \phi_{J/\psi}^T \) (twist 3) have been derived as [8]

\[
\begin{align*}
\phi_{J/\psi}^L(x) &= 9.58 \frac{f_{J/\psi}}{2\sqrt{2}N_c} x(1-x) \left[ \frac{x(1-x)}{1-2.8x(1-x)} \right]^{0.7}, \quad (29) \\
\phi_{J/\psi}^T(x) &= 10.94 \frac{f_{J/\psi}}{2\sqrt{2}N_c} (1-2x)^2 \left[ \frac{x(1-x)}{1-2.8x(1-x)} \right]^{0.7}, \quad (30)
\end{align*}
\]

with the decay constant \( f_{J/\psi} = 0.405 \pm 0.014 \) GeV.

The \( B_c^+ \rightarrow J/\psi \pi^+ \) decay amplitude is decomposed into

\[
A(B_c \rightarrow J/\psi \pi) = V_{ub}V_{ud}^*(f_2 F + M). \quad (31)
\]

The factorizable emission diagrams, i.e., Figs. 4(a) and 4(b), give the factorization formula

\[
F = 8\pi C_F m_{B_c}^2 \int_0^1 dx_1 dx_2 \int_0^\infty b_1 b_2 \phi_{B_c}(x_1, b_1)(r_{J/\psi}^2 - 1) \cdot \left\{ \right.
\begin{array}{c}
[r_{J/\psi}(r_b + 2x_2 - 2) \phi_{J/\psi}^L(x_2) - (r_b + x_2 - 1) \phi_{J/\psi}^T(x_2)] h_a(x_1, x_2, b_1, b_2) E_f(t_a) \\
[r_{J/\psi}(x_1 - 1) - r_c \phi_{J/\psi}^L(x_2) h_b(x_1, x_2, b_1, b_2) E_f(t_a)]
\end{array}
\right\}, \quad (32)
\]

where the ratios \( r_b = m_u/m_{B_c} \) and \( r_c = m_c/m_{B_c} \) and \( b_i \) are the impact parameters conjugate to the transverse momenta \( k_T \).

It is known that the above formula is related to the transition form factor \( A_{0, B_c^+ \rightarrow J/\psi}(q^2 = 0) \) [31–33] with \( q = P_1 - P_2 \). As pointed out in the Introduction, the PQCD approach is applicable to the evaluation of the nonfactorizable emission diagrams, i.e., Fig. 4(c) and 4(d). The corresponding factorization formula is expressed as

\[
M = -\frac{32}{\sqrt{6}} \pi C_F m_{B_c}^2 \int_0^1 dx_1 dx_2 dx_3 \int_0^\infty b_1 b_2 b_3 \phi_{B_c}(x_1, b_1) \phi_{J/\psi}^L(x_2)(r_{J/\psi}^2 - 1) \cdot \left\{ \right.
\begin{array}{c}
[(r_{J/\psi}^2 - 1)(x_1 + x_3 - 1) \phi_{J/\psi}^L(x_2) + r_{J/\psi}(x_2 - x_1) \phi_{J/\psi}^T(x_2)] h_c(x_1, x_2, x_3, b_1, b_3) E_f(t_c) \\
[2x_1 - (x_2 + x_3) + r_{J/\psi}(x_2 - x_3) \phi_{J/\psi}^L(x_2) + r_{J/\psi}(x_2 - x_1) \phi_{J/\psi}^T(x_2)] h_d(x_1, x_2, x_3, b_1, b_3) E_f(t_d)
\end{array}
\right\}. \quad (33)
\]

In the above expressions, the hard functions \( h_{a,b,c,d} \) are defined by

\[
\begin{align*}
h_a(x_1, x_2, b_1, b_2) &= \left[ \theta(b_2 - b_1) I_0(\sqrt{\beta_a} b_1) K_0(\sqrt{\beta_a} b_2) + (b_1 \leftrightarrow b_2) \right] K_0(\sqrt{\alpha} b_1), \quad (34) \\
h_b(x_1, x_2, b_1, b_2) &= \left[ \theta(b_2 - b_1) I_0(\sqrt{\beta_b} b_1) K_0(\sqrt{\beta_b} b_2) + (b_1 \leftrightarrow b_2) \right] K_0(\sqrt{\gamma} b_2), \quad (35) \\
h_{c,d}(x_1, x_2, x_3, b_1, b_3) &= \left[ \theta(b_3 - b_1) I_0(\sqrt{\beta_c} b_1) K_0(\sqrt{\beta_c} b_3) + (b_1 \leftrightarrow b_3) \right] K_0(\sqrt{\beta_{c,d}} b_3), \quad (36)
\end{align*}
\]

with the factors \( \alpha \) and \( \beta_{a,b,c,d} \) and the hard scales \( t_{a,b,c,d} \)

\[
\begin{align*}
\alpha &= -[(x_1 - x_2)(x_1 - x_2 r_{J/\psi})] m_{B_c}^2, \quad (37) \\
\beta_a &= -[(1 - x_2)(1 - x_2 r_{J/\psi}^2) - r_{c}^2] m_{B_c}^2, \quad \beta_b = -[(1 - x_1)(r_{J/\psi}^2 - x_1) - r_{c}^2] m_{B_c}^2, \quad (38) \\
\beta_c &= -[(x_2 r_{J/\psi}^2 + (1 - x_3)(1 - r_{J/\psi}^2) - x_1)(x_2 - x_1)] m_{B_c}^2, \quad (39) \\
\beta_d &= -[(x_2 r_{J/\psi}^2 + (1 - x_3)(1 - r_{J/\psi}^2) - x_1)(x_2 - x_1)] m_{B_c}^2, \quad (40) \\
t_a &= \max(\sqrt{\alpha}, \sqrt{\beta_a}, 1/b_1, 1/b_2), \quad t_b = \max(\sqrt{\alpha}, \sqrt{\beta_b}, 1/b_1, 1/b_2), \quad (41) \\
t_c &= \max(\sqrt{\alpha}, \sqrt{\beta_c}, 1/b_1, 1/b_3), \quad t_d = \max(\sqrt{\alpha}, \sqrt{\beta_d}, 1/b_1, 1/b_3). \quad (42)
\end{align*}
\]

Note that, as \( \alpha \) and \( \beta_{a,b,c,d} \) are negative, the associated Bessel functions transform as

\[
K_0(\sqrt{\alpha}) = K_0(i\sqrt{|\alpha|}) = \frac{i\pi}{2} [J_0(\sqrt{|\alpha|}) + iN_0(\sqrt{|\alpha|})], \quad I_0(\sqrt{\alpha}) = J_0(\sqrt{|\alpha|}), \quad (43)
\]

for \( y < 0 \). The evolution functions \( E_f(t) = \alpha_s(t) C_i(t) S_i(t) \) contain the Wilson coefficients

\[
C_{ab}(t) = \frac{1}{3} C_1(t) + C_2(t), \quad C_{cd}(t) = C_1(t) \quad (44)
\]
and the Sudakov factors

\[
S_{ab}(t) = s_c \left( x_1 P_{1}^+, b_1 \right) + s_c \left( x_2 P_{2}^+, b_2 \right) + s_c \left( (1-x_2) P_{2}^-, b_2 \right) - \frac{1}{\beta_1} \left[ \frac{11}{6} \ln \frac{\ln(t/\Lambda)}{\ln(m_c/\Lambda)} \right],
\]

\[
S_{cd}(t) = s_c \left( x_1 P_{1}^+, b_1 \right) + s_c \left( x_2 P_{2}^+, b_1 \right) + s_c \left( (1-x_2) P_{2}^+, b_1 \right) + s \left( x_3 P_{3}^+, b_3 \right) + s \left( (1-x_3) P_{3}^-, b_3 \right)
- \frac{1}{\beta_1} \left[ \frac{11}{6} \ln \frac{\ln(t/\Lambda)}{\ln(m_c/\Lambda)} + \ln \frac{\ln(t/\Lambda)}{-\ln(b_3\Lambda)} \right]
\]

(45)

(46)

where the explicit expression of the Sudakov exponent \( s(Q, b) \) for an energetic light quark is referred to Refs. [31, 32].

With the QCD scale \( \Lambda_{QCD}^{(4)} = 0.25 \) GeV and the \( B_c \) meson lifetime \( \tau_{B_c} = 0.507 \) ps, we obtain \( Br(B_c^+ \rightarrow J/\psi \pi^+) = 1.60 \times 10^{-3} \). This result is consistent with \( 1.2 \times 10^{-3} \) derived in the QCD factorization approach [21], in which the transition form factor \( A_0^{B_c^+ \rightarrow J/\psi}(0) \) was treated as an input, a bit larger value of \( A_0^{B_c^+ \rightarrow J/\psi}(0) = 0.6 \) was employed, and the one-loop correction to the \( b \rightarrow c \) decay vertex was included. Our prediction can be compared to the measured branching ratio of the corresponding mode with the replacement of the spectator charm quark by an up quark, \( Br(B^+ \rightarrow D^* \pi^+) = (5.18 \pm 0.26) \times 10^{-3} \) [27], which receives an additional color-suppressed tree contribution. The dependence of the quantities \( A_0^{B_c^+ \rightarrow J/\psi}(0) \) and \( Br(B_c^+ \rightarrow J/\psi \pi^+) \) on \( \beta_{B_c} \) in the range \([0.8, 1.2] \) GeV is shown in Table I. It is clearly seen that the imaginary piece of the \( B_c \rightarrow J/\psi \) transition form factor is greatly suppressed, being only 10%–20% of the real piece, and that the \( B_c^+ \rightarrow J/\psi \pi^+ \) branching ratio is unlikely to be lower than \( 10^{-3} \). Roughly speaking, the preferred range of \( Br(B_c^+ \rightarrow J/\psi \pi^+) \) from the PQCD approach can be preliminarily read as \([0.9, 2.8] \times 10^{-3} \). When the data are available for individual branching ratios, or for the ratios of decay rates that are more sensitive to the nonfactorizable emission contributions, it is possible to pin down the shape parameter \( \beta_{B_c} \) and to make more precise predictions in the PQCD approach. In the latter case, the emitted meson could be a scalar or tensor, such that the dominant nonfactorizable emission diagrams do not cancel in the ratios of decay rates.

| Shape Parameter | \( A_0^{B_c^+ \rightarrow J/\psi}(0) \) | \( Br(B_c^+ \rightarrow J/\psi \pi^+) \) |
|-----------------|-----------------|-----------------|
| \( \beta_{B_c} = 0.8 \) GeV | 0.488 - 0.095 | 2.80 \times 10^{-4} |
| \( \beta_{B_c} = 0.9 \) GeV | 0.434 - 0.070 | 2.10 \times 10^{-3} |
| \( \beta_{B_c} = 1.0 \) GeV | 0.384 - 0.053 | 1.60 \times 10^{-3} |
| \( \beta_{B_c} = 1.1 \) GeV | 0.341 - 0.039 | 1.23 \times 10^{-3} |
| \( \beta_{B_c} = 1.2 \) GeV | 0.306 - 0.029 | 0.94 \times 10^{-3} |

**V. CONCLUSION**

In this paper, we have deduced the shape of the \( B_c \) meson distribution amplitude \( \phi_{B_c}(x) \) resulting from the soft gluon emission effect based on the parton kinematic analysis and found that \( \phi_{B_c}(x) \) exhibits a peak around the momentum fraction \( x \sim m_c/m_b \sim 0.3 \) of the spectator charm quark with a width of order \( \Lambda/m_b \sim 0.1 \). These features were then implemented into the parametrization of \( \phi_{B_c}(x) \) in terms of a Gaussian form with the shape parameter \( \beta_{B_c} \sim 1.0 \) GeV. We have estimated the potential imaginary piece in the \( B_c \rightarrow J/\psi \) transition form factor, which should be power suppressed according to the specified parton kinematics and the argument on the absence of intermediate states. It is worth emphasizing that the resummation formula adopted in the conventional PQCD approach to \( B_c \) meson decays [22–24] is not appropriate. We have modified the \( k_T \) resummation by taking into account the finite charm quark mass, the effect of which was shown to enhance the decay rates. We point out that this modification is exact only at the leading-logarithm level, and a precise next-to-leading-logarithm resummation formalism for a hadronic process involving the multiple scales \( m_b, m_c, \) and \( \Lambda_{QCD} \) is still urged; it demands a complete one-loop calculation for determining the factor \( B(\alpha_s) \) in Eq. (13).

Given the \( B_c \) meson distribution amplitude preferred by the kinematic constraints and the newly derived Sudakov factor for the \( B_c \rightarrow J/\psi \) transition, we have calculated, at leading order in the strong coupling, the transition form factor \( A_0^{B_c^+ \rightarrow J/\psi}(0) \) and the \( B_c^+ \rightarrow J/\psi \pi^+ \) branching ratio in the range \([0.8, 1.2] \) GeV of the shape parameter \( \beta_{B_c} \). It was observed that the strong phase in \( A_0^{B_c^+ \rightarrow J/\psi}(0) \) is indeed largely suppressed and that the predicted \( Br(B_c^+ \rightarrow J/\psi \pi^+) \sim 1.60 \times 10^{-3} \) is comparable to the data \( Br(B^+ \rightarrow D^0 \pi^+) = (5.18 \pm 0.26) \times 10^{-3} \). The definite value of the shape parameter demands the input data of some individual \( B_c \) decay channels from LHCb, with which it is then possible to make more precise predictions for various modes. At last, we stress that the improved PQCD formalism developed in this work is applicable to \( B_c \) meson decays to other charmonia and charmed mesons.
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