STANDARD MODULES OF QUANTUM AFFINE ALGEBRAS

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Abstract. We give a proof of the cyclicity conjecture of Akasaka-Kashiwara, for simply laced types, via quiver varieties. We get also an algebraic characterization of the standard modules.

1. Introduction

Let \( g \) be a simple, simply laced, complex Lie algebra. If \( g \) is of type \( A \), a geometric realization of the quantized enveloping algebra \( U \) of \( g[t, t^{-1}] \) and of its simple modules was given a few years ago in [GV], [V]. This construction involved perverse sheaves and the convolution algebra in equivariant \( K \)-theory of partial flags varieties of type \( A \). It was then observed in [N1], [N2], that these varieties should be viewed as a particular case of the quiver varieties associated to any symmetric Kac-Moody Lie algebra. This leads to a geometric realization of \( U \) via a convolution algebra in equivariant \( K \)-theory of the quiver varieties for \( g \) of (affine) type \( A^{(1)} \) in [VV] and for a general symmetric Kac-Moody algebra \( g \) in [N3]. For any symmetric Lie algebra \( g \), one gets a formula for the dimension of the finite dimensional simple modules of \( U \) in terms of intersection cohomology (see [N3]). A basic tool in this geometric approach are the standard modules. They are the geometric counterpart of the Weyl modules of \( U \) (see Remark 7.19). In this paper we give an algebraic construction of the standard modules. It answers to a question in [N3] (Corollary 7.16). An immediate corollary is a proof of the cyclicity conjecture in [AK] (Corollary 7.17) for simply laced Lie algebras.

The plan of the paper is the following: Sections 1 to 6 contain recollections on quantum affine algebras and quiver varieties. The main results are given in Section 7. The proof of Theorem 7.4 uses Lemma 8.1.

While we were preparing this paper Kashiwara mentioned to us that he has proved the conjecture in [AK] by a different approach (via canonical bases). We would like to thank the referee for numerous remarks on the first version of the paper.

2. The algebra \( U \)

Let \( g \) be a simple, simply laced, complex Lie algebra. The quantum loop algebra associated to \( g \) is the \( \mathbb{C}(q) \)-algebra \( U' \) generated by \( x_{ir}^{\pm}, k_{is}^{\pm}, k_{i0}^{\pm} \) (\( i \in I, r \in \mathbb{Z}, s \in \pm \mathbb{N} \)) modulo the following defining relations

\[
    k_{i}k_{i}^{-1} = 1 = k_{i}^{-1}k_{i}, \quad [k_{i,x_{ir}}, k_{j,s}] = 0,
\]

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\[ k_i x^\pm_{ij} k_i^{-1} = q^{\pm a_{ij}} x^\pm_{ij} , \]
\[ (w - q^{\pm a_{ij}} z) k_j^x (w) x^+_i(z) = (q^{\pm a_{ij}} w - z) x^+_i(z) k_j^x (w) , \]
\[ (z - q^{\pm a_{ij}} w) x^+_i(z) x^+_j(w) = (q^{\pm a_{ij}} z - w) x^+_j(w) x^+_i(z) , \]
\[ [x^+_i, x^-_{js}] = \delta_{ij} \frac{k_{i,r+s}^+ - k_{i,r+s}^-}{q - q^{-1}} , \]
\[ \sum_m \sum_{p=0}^m (-1)^p \begin{bmatrix} m \\ p \end{bmatrix} x^\pm_{ir(w(1))} \cdots x^\pm_{ir(w(p+1))} x^\pm_{s, r(w(m))} = 0 , \]
where \( i \neq j \), \( m = 1 - a_{ij} \), \( r_1, \ldots, r_m \in \mathbb{Z} \), and \( w \in S_m \). We have set \([n] = q^{1-n} + q^{3-n} + \ldots + q^{n-1}\) if \( n \geq 0 \), \([n]! = [n][n-1] \ldots [2]\), and
\[ \begin{bmatrix} m \\ p \end{bmatrix} = \frac{[m]!}{[p]![m-p]!} . \]
We have also set \( \varepsilon = + \) or \(-\),
\[ k^\pm_i(z) = \sum_{r \geq 0} k^\pm_{i,\pm s} z^{\mp r} , \quad x^\pm_{ij}(z) = \sum_{r \in \mathbb{Z}} x^\pm_{ij} z^{\mp r} . \]

Put \( \mathcal{A} = \mathbb{C}[q,q^{-1}] \). Consider also the \( \mathcal{A}\)-subalgebra \( U \subset U' \) generated by the quantum divided powers \( x^\pm_{ir}(n) = x^\pm_{ir} / [n]! \), the Cartan elements \( k^\pm_i \), and the elements \( h_{is} \) such that
\[ k^\pm_i(z) = k^\pm_i \exp \left( \pm (q - q^{-1}) \sum_{s \geq 1} h_{i,\pm s} z^{\mp s} \right) . \]

Let \( \Delta^0 \) be the coproduct defined in terms of the Kac-Moody generators \( e_i, f_i, k^\pm_i \), \( i \in I \cup \{0\} \), of \( U' \) as follows
\[ \Delta^0(e_i) = e_i \otimes 1 + k_i \otimes e_i , \quad \Delta^0(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i , \quad \Delta^0(k_i) = k_i \otimes k_i , \]
where \( \otimes \) is the tensor product over the field \( \mathbb{C} \), or the ring \( \mathcal{A} \). Let \( \tau \) be the antiautomorphism of \( U \) such that \( \tau(e_i) = f_i , \tau(f_i) = e_i , \tau(k_i) = k_i^{-1} \), and \( \tau(q) = q^{-1} \). It is known (see [B]) that \( \tau(x^\pm_{i,-k}) = x^\pm_{ik} \) and \( \tau(k^\pm_{i,\pm r}) = k^\pm_{i,\mp r} \). Let \( \Delta^\bullet \) be the coproduct opposit to \( \Delta^0 \). We have \((\tau \otimes \tau) \Delta^0 = \Delta^\bullet \). Hereafter \( \zeta \) is an element of \( \mathbb{C}^\times \) which is not a root of unity.

### 3. The braid group

Let \( W, P, Q \), be the Weyl group, the weight lattice, and the root lattice of \( g \). The extended affine Weyl group \( \tilde{W} = W \ltimes P \) is generated by the simple reflexions \( s_i \) and the fundamental weights \( \omega_i , i \in I \). Let \( \Gamma \) be the quotient of \( \tilde{W} \) by the normal Coxeter subgroup generated by the simple affine reflexions. The group \( \Gamma \) is a group of diagram automorphisms of the extended Dynkin diagram of \( g \). In particular \( \Gamma \) acts on \( U \) and on \( \tilde{W} \). For any \( w \in \tilde{W} \) let \( l(w) \) denote its length. The braid group \( B_{\tilde{W}} \) associated to \( \tilde{W} \) is the group on generators \( T_w , w \in \tilde{W} \), with the relation \( T_w T_{w'} = T_{ww'} \) whenever \( l(ww') = l(w) + l(w') \). The group \( B_{\tilde{W}} \) acts on \( U \) by algebra
automorphisms (see [L1], [B]). Recall that $x_{ir}^- = \nu(i)^r T_{w_i}^r (f_i)$, $x_{ir}^+ = \nu(i)^r T_{w_i}^r (e_i)$, for a fixed function $\nu : I \to \{ \pm 1 \}$ such that $\nu(i) + \nu(j) = 0$ if $a_{ij} < 0$ (see [B, Definition 4.6]). We have $T_{w_j}(k_{ir}^\pm) = k_{ir}^\pm$ for any $i, j$, and $T_{w_j}(x_{ir}^\pm) = x_{ir}^\pm$ for any $i \neq j$. For any $i \in I \cup \{0\}$ put

$$R_i = \sum_{l \geq 0} c_l T_{s_i} (f_i)^{(l)} \otimes T_{s_i} (e_i)^{(l)} \quad \text{where} \quad c_l = (-1)^l q^{-l(l-1)/2} (q - q^{-1})^l |l|!.$$  

The element $R_i$ belongs to a completed tensor product $(U \otimes U)^*$ (see [L1, §4.1.1] for instance). It is known that

$$R_i^{-1} = \sum_{l \geq 0} c^l T_{s_i} (f_i)^{(l)} \otimes T_{s_i} (e_i)^{(l)} \quad \text{where} \quad c^l = q^{l(l-1)/2} (q - q^{-1})^l |l|!.$$  

If $s_{i_1} \ldots s_{i_r}$ is a reduced expression of $w \in \tilde{W}$ with $\tau \in \Gamma$, set

$$R_w = \tau (T_{i_1}^{[2]} \ldots T_{i_r-1}^{[2]} (R_{i_r}) \ldots T_{i_1}^{[2]} (R_{i_2}) R_{i_1}),$$  

where $a^{[2]} = a \otimes a$ for any $a$. In particular if $w, w' \in \tilde{W}$ are such that $l(ww') = l(w) + l(w')$, then $R_{ww'} = T_{w'}^{[2]} (R_{w'}) R_w$. For any $w \in \tilde{W}$ set $\Delta^0_w = T_w^{[2]} \Delta^0 T_w^{-1}$. Then $R_w \cdot \Delta^0 x \cdot R_w^{-1} = \Delta^0_w x$, for all $x \in U$ (see [L1, 37.3.2] and [B, Section 5]).

4. The quiver varieties

Let $I$ (resp. $E$) be the set of vertices (resp. edges) of a finite graph $(I, E)$ with no edge loops. For $i, j \in I$ let $n_{ij}$ be the number of edges joining $i$ and $j$. Put $a_{ij} = 2 \delta_{ij} - n_{ij}$. The map $(I, E) \mapsto A = (a_{ij})_{i, j \in I}$ is a bijection from the set of finite graphs with no edge loops onto the set of symmetric generalized Cartan matrices. Let $\alpha_i$, $i \in I$, be the simple roots of the symmetric Kac-Moody algebra $\mathfrak{g}$ corresponding to $A$. Hereafter we assume that $\mathfrak{g}$ is finite dimensional, i.e. the matrix $A$ is positive definite. Let $H$ be the set of edges of $(I, E)$ together with an orientation. For $h \in H$ let $h'$ and $h''$ the incoming and the outcoming vertex of $h$. If $h \in H$ we denote by $\tilde{h} \in H$ the same edge with opposite orientation. Given two $I$-graded finite dimensional complex vector spaces $V = \bigoplus_{i \in I} V_i$, $W = \bigoplus_{i \in I} W_i$, set

$$E(V, W) = \bigoplus_{h \in H} \text{Hom} (V_{h'}, W_h), \quad L(V, W) = \bigoplus_{i \in I} \text{Hom} (V_i, W_i).$$  

Let $P^+, Q^+$ be the semi-groups $Q^+ = \bigoplus_{i \in I} \mathbb{N} \alpha_i$ and $P^+ = \bigoplus_{i \in I} \mathbb{N} \omega_i$. Let us fix once for all the following convention : the dimension of the graded vector space $V$ is identified with the element $\alpha = \sum_{i \in I} v_i \alpha_i \in Q^+$ (where $v_i$ is the dimension of $V_i$), while the dimension of $W$ is identified with the weight $\lambda = \sum_i w_i \omega_i \in P^+$ (where $w_i$ is the dimension of $W_i$). We also put $|\alpha| = \sum_i v_i$ and $|\lambda| = \sum_i w_i$. We write $\alpha \geq \alpha'$ if and only if $\alpha - \alpha' \in Q^+$. Set

$$M_{\alpha \lambda} = E(V, V) \oplus L(W, V) \oplus L(V, W).$$  

For any $(B, p, q) \in M_{\alpha \lambda}$ let $B_h$ be the component of $B$ in $\text{Hom} (V_{h'}, V_h)$ and set

$$m_{\alpha \lambda} (B, p, q) = \sum_h \varepsilon(h) B_h \frac{B}{h} + pq \in L(V, V),$$  

where $\varepsilon(h) = 1$ if $h$ is oriented, $\varepsilon(h) = -1$ if not.
where $\varepsilon$ is a function $\varepsilon : H \rightarrow \mathbb{C}^\times$ such that $\varepsilon(h) + \varepsilon(\overline{h}) = 0$. A triple $(B, p, q) \in m_{\alpha \lambda}(0)$ is stable if there is no nontrivial $B$-invariant subspace of Ker $q$. Let $m_{\alpha \lambda}(0)^\circ$ be the set of stable triples. The group $G_\alpha = \prod_i \text{GL}(V_i)$ acts on $M_{\alpha \lambda}$ by

$$g \cdot (B, p, q) = (gBg^{-1}, gp, gq^{-1}).$$

The action of $G_\alpha$ on $m_{\alpha \lambda}(0)^\circ$ is free. Put

$$Q_{\alpha \lambda} = m_{\alpha \lambda}(0)^\circ / G_\alpha \quad \text{and} \quad N_{\alpha \lambda} = m_{\alpha \lambda}(0) / G_\alpha,$$

where $\parallel$ denotes the categorical quotient. The variety $Q_{\alpha \lambda}$ is smooth and quasi-projective.

4.2. Let $\pi : Q_{\alpha \lambda} \rightarrow N_{\alpha \lambda}$ be the affinization map. Let $L_{\alpha \lambda} = \pi^{-1}(0) \subset Q_{\alpha \lambda}$ be the zero fiber. It is known that $\dim Q_{\alpha \lambda} = 2 \dim L_{\alpha \lambda}$. If $\alpha, \alpha' \in Q^+$ are such that $\alpha \geq \alpha'$, then the extension by zero of representations of the quiver gives an injection $N_{\alpha \lambda} \hookrightarrow N_{\alpha' \lambda}$ (see [N3, Lemma 2.5.3]). For any $\alpha, \alpha'$, we consider the fiber product

$$Z_{\alpha \alpha' \lambda} = Q_{\alpha \lambda} \times_\pi Q_{\alpha' \lambda}.$$

It is known that $\dim Z_{\alpha \alpha' \lambda} = (\dim Q_{\alpha \lambda} + \dim Q_{\alpha' \lambda})/2$. If $\alpha' = \alpha + \alpha_i$ and $V \subseteq V'$ have dimension $\alpha, \alpha'$, respectively, let $C_{\alpha \alpha' \lambda}^{\pm} \subset Z_{\alpha \alpha' \lambda}$ be the set of pairs of triples $(B, p, q), (B', p', q')$, such that $B_{\lambda V'} = B, p' = p, q'_{\lambda V'} = q$. If $\alpha' = \alpha - \alpha_i$, put $C_{\alpha \alpha' \lambda}^{-} = \phi(C_{\alpha \alpha' \lambda}^{+}) \subset Z_{\alpha \alpha' \lambda}$, where $\phi$ flips the components. The variety $C_{\alpha \alpha' \lambda}^{\pm}$ is an irreducible component of $Z_{\alpha \alpha' \lambda}$. Consider the following varieties

$$N_{\lambda} = \bigcup_{\alpha} N_{\alpha \lambda}, \quad Q_{\lambda} = \bigcup_{\alpha} Q_{\alpha \lambda}, \quad Z_{\lambda} = \bigcup_{\alpha, \alpha'} Z_{\alpha \alpha' \lambda}, \quad C_{\lambda}^{\pm} = \bigcup_{\alpha} C_{\alpha \alpha' \lambda}^{\pm}, \quad L_{\lambda} = \bigcup_{\alpha} L_{\alpha \lambda},$$

where $\alpha, \alpha'$ take all the possible values in $Q^+$. Observe that for a fixed $\lambda$, the set $Q_{\alpha \lambda}$ is empty except for a finite number of $\alpha$’s.

4.3. Put $\tilde{G}_\lambda = G_\lambda \times \mathbb{C}^\times$. The group $\tilde{G}_\lambda$ acts on $M_{\alpha \lambda}$ by

$$(g, z) \cdot (B, p, q) = (zB, zpg^{-1}, zgg).$$

This action descends to $Q_{\alpha \lambda}$ and $N_{\alpha \lambda}$. For any element $s = (t, \zeta) \in \tilde{G}_\lambda$ with $t$ semisimple, let $\langle s \rangle \subset \tilde{G}_\lambda$ be the Zariski closure of $s^\mathbb{Z}$. For any group homomorphism $\rho \in \text{Hom}((s), G_\lambda)$, let $Q(\rho) \subset Q_{\alpha \lambda}$ be the subset of the classes of the triples $(B, p, q)$ such that $s \cdot (B, p, q) = \rho(s) \cdot (B, p, q)$. The fixpoint set $Q_{\alpha \lambda}^s$ is the disjoint union of the subvarieties $Q(\rho)$. It is proved in [N3, Theorem 5.5.6] that $Q(\rho)$ is either empty or a connected component of $Q_{\alpha \lambda}^s$.

**Lemma 4.4.** (i) Fix $\lambda_1, \lambda_2 \in P^+$, such that $\lambda = \lambda_1 + \lambda_2$. The direct sum $M_{\lambda_1} \times M_{\lambda_2} \rightarrow M_\lambda$ gives a closed embedding $\kappa : Q_{\lambda_1} \times Q_{\lambda_2} \hookrightarrow Q_{\lambda}$. 
(ii) Fix a semi-simple element $t = t_1 \oplus t_2 \in G_{\lambda_1} \times G_{\lambda_2}$. Set $s = (t, \zeta) \in \tilde{G}_\lambda$, $s_1 = (t, \zeta) \in \tilde{G}_{\lambda_1}$, $s_2 = (t, \zeta) \in \tilde{G}_{\lambda_2}$. If $(\zeta^2 \text{spec} t_1) \cap \text{spec} t_2 = \emptyset$, then $Q_{\alpha \lambda}^{s_1} = \kappa(Q_{\lambda_1}^{s_1} \times Q_{\lambda_2}^{s_2}).$

**Proof of 4.4.** Fix $I$-graded vector spaces $W, W^1, W^2, V, V^1, V^2$, such that $W = W^1 \oplus W^2$, $V = V^1 \oplus V^2$, $\dim W^1 = \lambda_1$, $\dim W^2 = \lambda_2$, $\dim V^1 = \alpha_1$, and $\dim V^2 = \alpha_2$. Fix triples $x^1, x^1 \in m_{\alpha_1 \lambda_1}(0)^\circ$, $x^2, x^2 \in m_{\alpha_2 \lambda_2}(0)^\circ$. The triple $x = x^1 \oplus x^2$ is stable.
since if $V' \subseteq \ker q$ is a $B$-stable subspace, then $V^1 \cap V' = \{0\}$ by the stability of $x^1$, and then $V'$ embeds in $V/V^1$. Thus it is zero by the stability of $x^2$. Assume that $g \in G_{\alpha_1 + \alpha_2}$ maps $x$ to $x^1 + x^2$. Then,

$$q(g^{-1}(V^2) \cap V^1) \subseteq W^1 \cap W^2 = \{0\} \quad \text{and} \quad B(g^{-1}(V^2) \cap V^1) \subseteq g^{-1}(V^2) \cap V^1.$$ 

Thus the stability of $x$ gives $g^{-1}(V^2) \cap V^1 = \{0\}$. In the same way we get $g^{-1}(V^1) \cap V^2 = \{0\}$. Thus $g \in G_{\alpha_1} \times G_{\alpha_2}$. Claim (i) is proved.

Fix $\rho$ such that $Q(\rho)$ is non empty. For any $z \in \mathbb{C}^\times$ put

$$V(z) = \ker (\rho(s) - z^{-1}id_V), \quad W(z) = \ker (t - z id_W).$$

If $x \in Q(\rho)$ and $(B, p, q)$ is a representative of $x$, then

$$B(V(z)) \subset V(z/\zeta), \quad q(V(z)) \subset W(z/\zeta), \quad p(W(z)) \subset V(z/\zeta).$$

Moreover the stability of $(B, p, q)$ implies that $V = \bigoplus_{\zeta \in \zeta_{\spec}} V(z)$. Claim (ii) follows.

4.5. Let $\mathcal{V} = m_{\alpha_1, \alpha_2}^{-1}(0) \times G_{\alpha} V$ and $W$ be respectively the tautological bundle and the trivial $W$-bundle on $Q_{G\alpha}$. The $i$-th component of $\mathcal{V}, \mathcal{W}$, is denoted by $\mathcal{V}_i, \mathcal{W}_i$. The bundles $\mathcal{V}, \mathcal{W}$, are $G_{\lambda}$-equivariant. Let $q$ be the trivial line bundle on $Q_{\alpha_\lambda}$ with the degree one action of $\mathbb{C}^\times$. We consider the classes

$$\mathcal{F}_{\alpha_\lambda}^i = q^{-1}\mathcal{W}_i - (1 + q^{-2}) \mathcal{V}_i + q^{-1} \sum_{h' = 1} q^{h''} \mathcal{V}_{h''},$$

$$\mathcal{H}_{\alpha_\lambda} = qE(\mathcal{V}, \mathcal{V}) + q^2 L(qW - \mathcal{V}, \mathcal{V}) + L(\mathcal{V}, qW - \mathcal{V})$$

in $K^{G_\lambda}(Q_{\alpha_\lambda})$, and the classes

$$\mathcal{T}_{\alpha_\lambda}^{i+} = qE(\mathcal{V}, \mathcal{V}') + q^2 L(qW - \mathcal{V}, \mathcal{V}') + L(\mathcal{V}, qW - \mathcal{V}') - q^2$$

$$\mathcal{T}_{\alpha_\lambda}^{i-} = qE(\mathcal{V}', \mathcal{V}) + q^2 L(qW - \mathcal{V}', \mathcal{V}) + L(\mathcal{V}', qW - \mathcal{V}) - q^2$$

in $K^{G_\lambda}(Q_{\alpha_\lambda} \times Q_{\alpha_\lambda}^*)$ (where $\alpha' = \alpha \pm \alpha_i$). It is known that $\mathcal{T}_{\alpha_\lambda}$ is the class of the tangent sheaf to $Q_{\alpha_\lambda}$, and that $\mathcal{T}_{\alpha_\lambda}^{i+} |_{Q_{\alpha_\lambda}^{\alpha'} \times Q_{\alpha_\lambda}^{\alpha'}}$ is the class of the normal sheaf of $Q_{\alpha_\lambda}^{\alpha'}$ in $Q_{\alpha_\lambda} \times Q_{\alpha_\lambda}$ (see [N2]).

5. The convolution product

5.1. For any complex algebraic linear group $G$, and any quasi-projective $G$-variety $X$ let $K^G(X)$, $K_G(X)$ be the complexified Grothendieck groups of $G$-equivariant coherent sheaves, and locally free sheaves respectively, on $X$. We put $R(G) = K^G(point)$. For simplicity let $f_*, f^*, \otimes$ denote the derived functors $Rf_*, Lf^*, \otimes^L$ (where $\otimes$ is the tensor product of sheaves of $O_X$-modules). We use the same notation for a sheaf and its class in the Grothendieck group. Hereafter, elements of $K_G(X)$ may be identified with their image in $K^G(X)$. The class of the structural sheaf is simply denoted by $1$. Given smooth quasi-projective $G$-varieties $X_1, X_2, X_3$, consider the projection $p_{ab} : X_1 \times X_2 \times X_3 \rightarrow X_a \times X_b$ for all $1 \leq a < b \leq 3$. Consider closed subvarieties $Z_{ab} \subset X_a \times X_b$ such that the restriction of $p_{13}$ to $p_{12}^{-1}Z_{12} \cap p_{23}^{-1}Z_{23}$ is proper and maps to $Z_{13}$. The convolution product is the map

$$*: K^G(Z_{12}) \boxtimes K^G(Z_{23}) \rightarrow K^G(Z_{13}), \quad \mathcal{E} \boxtimes \mathcal{F} \mapsto p_{13\ast}(p_{12}^\ast \mathcal{E} \otimes (p_{23}^\ast \mathcal{F})).$$

See [CG] for more details. The flip $\phi : X_a \times X_b \rightarrow X_b \times X_a$ gives a map $\phi_* : K^G(Z_{ab}) \rightarrow K^G(Z_{ba})$. This maps anti-commutes with $\ast$, i.e.

$$\phi_*(x_{12} \ast x_{23}) = \phi_*(x_{23}) \ast \phi_*(x_{12}), \quad \forall x_{12}, x_{23}.$$
5.2. Let $\mathbb{D}_X$ be the Serre-Grothendieck duality operator on $K^G(X)$ (see [L2, Section 6.10] for instance). Assume that $X$ is the disjoint union of smooth connected subvarieties $X^{(i)}$. Assume also that we have fixed a particular invertible element $q \in K^G(X)$. Put $d_X^{(i)} = \dim X^{(i)}$, $D_X^{(i)} = q^{d_X^{(i)}} \mathbb{D}_X^{(i)}$, $D_X = \sum_i D_X^{(i)}$. Let $\Omega_X$ be the determinant of the cotangent bundle to $X$. Then $\mathbb{D}_X(E) = (-1)^{\dim X} \mathcal{E}^{\ast} \otimes \Omega_X$ for any $G$-equivariant locally free sheaf $\mathcal{E}$. Put $Z_{ab}^{(ij)} = Z_{ab} \cap (X_a^{(i)} \times X_b^{(j)})$. Assume that $\Omega_{X_a^{(i)}} = q^{-d_X^{(i)}}$ for all $a, i$. The operators

$$D_{Z_{ab}} = \sum_{i,j} q_{ab}^{d_{ij}} \mathbb{D}_{Z_{ab}^ij},$$

where $d_{ab}^{(ij)} = (d_{X_a^{(i)}} + d_{X_b^{(j)}})/2$, are compatible with the convolution product $\ast$, i.e.

$$D_{Z_{12}}(x_{12}) \ast D_{Z_{23}}(x_{23}) = D_{Z_{13}}(x_{12} \ast x_{23}), \quad \forall x_{12}, x_{23}$$

(see [L2, Lemma 9.5] for more details).

5.3. If $\mathcal{E}$ is a $G$-bundle on $X$, we have the element $\bigwedge_z (\mathcal{E}) = \sum_{i=1}^{rk\mathcal{E}} \bigwedge_z \bigwedge^i \mathcal{E} \cdot z_i \in K_G(X)[z]$, where $\bigwedge^i \mathcal{E}$ is the $i$-th wedge product. Clearly $\bigwedge_z (\mathcal{E} + \mathcal{F}) = \bigwedge_z (\mathcal{E}) \otimes \bigwedge_z (\mathcal{F})$ for any $\mathcal{E}, \mathcal{F}$, and

$$(5.4) \quad \bigwedge_{-1} (\mathcal{E}) = (-1)^{rk\mathcal{E}} \text{Det} (\mathcal{E}) \otimes \bigwedge_{-1} (\mathcal{E}^{\ast}).$$

Observe also that $\bigwedge_z (\mathcal{E})$ admits an inverse in $K_G(X)[[z]]$. Let $\mathbf{R}(G)$ be the fraction field of $\mathbf{R}(G)$ and set $\tilde{K}_G(X) = K_G(X) \otimes_{\mathbf{R}(G)} \mathbf{R}(G)$, $K^G_{\text{et}}(X) = K^G(X) \otimes_{R(G)} \mathbf{R}(G)$. Assume that $G$ is a diagonalizable group and that the set of $G$-fixed points of the restriction $\mathcal{E}|_{X^G}$ is $X^G$. Then $\bigwedge_{-1} (\mathcal{E})$ is invertible in $K_G(X)$ by the localization theorem and [CG, Proposition 5.10.3]. In particular the element $\bigwedge_{-1} (\mathcal{F} - \mathcal{E}) = \bigwedge_{-1} (\mathcal{F}) \otimes \bigwedge_{-1} (\mathcal{E})^{-1}$ is well-defined in $K_G(X)$ for any $\mathcal{F}$. If $G$ is a product of $\text{GL}_n$’s and $H \subset G$, then $\bigwedge_{-1} (\mathcal{E})$ is still invertible in $\tilde{K}_G(X)$ if the set of $H$-fixed points of $\mathcal{E}|_{X^H}$ is $X^H$ (use [CG, Theorem 6.1.22] which holds, although the group $G$ is not simply-connected). In the sequel we may identify a $G$-bundle and its class in the Grothendieck group.

5.5. Assume that the group $G$ is Abelian. For any $\mathbf{R}(G)$-module $M$ and any $s \in G$, let $M_s$ be the specialisation of $M$ at the maximal ideal in $\mathbf{R}(G)$ associated to $s$. The localization theorem gives isomorphisms of modules

$$\iota_s : K^G(X^s)_s \to K^G(X)_s, \quad \iota_s : K^G(X^s) \to \tilde{K}^G(X),$$

where $\iota : X^s \to X$ is the closed embedding.

6. Nakajima’s theorem

6.1. We fix a subset $H^+ \subset H$ such that $H^+ \cap \check{H}^+ = \emptyset$ and $H^+ \cup \check{H}^+ = H$. For any $i, j \in I$ let $n_{ij}^+$ be the number of arrows in $H^+$ from $i$ to $j$. Put $n_{ij}^- = n_{ij} - n_{ij}^+$. Observe that $n_{ij}^+ = n_{ji}^-$. Put

$$F_{\alpha \lambda}^{ij} = -V_i + q^{-1} \sum_j n_{ij}^+ V_j, \quad F_{\alpha \lambda}^{ij} = q^{-1} V_i - q^{-2} V_i + q^{-1} \sum_j n_{ij}^- V_j.$$
Let \((\mid) : Q \times P \to \mathbb{Z}\) be the pairing such that \((\alpha_i|\omega_j) = \delta_{ij}\) for all \(i, j \in I\). The rank of \(F_{\alpha\lambda}\) is \((\alpha_i|\lambda - \alpha)\). Put \(F_{\alpha\lambda}^+ = \bigoplus_{\alpha} F_{\alpha\lambda}^{i+}\) and \(F_{\alpha\lambda}^- = \bigoplus_{\alpha} F_{\alpha\lambda}^{i-}\). Let \(f_{\alpha\lambda}^i, f_{\alpha\lambda}^j\), be the diagonal operators acting on \(K^{G_\lambda}(Q_{\alpha\lambda})\) by the scalars \(f_{\alpha\lambda}^i \in \text{rk} F_{\alpha\lambda}^i\) and \(f_{\alpha\lambda}^j \in \text{rk} F_{\alpha\lambda}^j\). Let \(p, p' : (Q_\lambda)^2 \to Q_\lambda\) be the first and the second projection. We denote by \(\mathcal{V}, \mathcal{V}' \in K^{G_\lambda}((Q_\lambda)^2)\) the pull-back of the tautological sheaf (i.e. \(\mathcal{V} = p^*\mathcal{V}\) and \(\mathcal{V}' = p'^*\mathcal{V}\)). Set \(L = q^{-1}(\mathcal{V}' - \mathcal{V})\). For any \(r \in \mathbb{Z}\) set

\[
(6.2) \quad x_{ir}^\pm = \sum_{\alpha'}(\pm L)^{\otimes r} f_{\alpha'\lambda}^{i\pm} \otimes \delta_{\pm} x_{i\alpha'\lambda}, \quad k_r^\pm(z) = \delta_{\alpha} q^{f_{\alpha\lambda}^i} \Lambda_{-1/2} ((q^{1} - q) F_{\alpha\lambda}^i)^\pm,
\]

where \(x_{i\alpha'\lambda} = (-1)^{f_{\alpha'\lambda}^i} \det (F_{\alpha'\lambda}^{i\pm})\), the map \(\delta\) is the diagonal embedding \(Q_\lambda \hookrightarrow (Q_\lambda)^2\), and \(\pm\) is the expansion at \(z = \infty\) or \(0\). Hereafter we may omit \(\delta\), hoping that it makes no confusion. Let \(U_{\lambda}\) be the quotient of \(K^{G_\lambda}(Z_\lambda)\) by its torsion \(R(\hat{G}_\lambda)\)-submodule. The space \(U_{\lambda}\) is an associative algebra for the convolution product \(*\) (with \(Z_{12} = Z_{23} = Z_\lambda\)). It is proved in [N3, Theorems 9.4.1 and 12.2.1] that the map \(x_{ir}^\pm \mapsto x_{ir}^\pm, k_r^\pm \mapsto k_r^\pm\), extends uniquely to an algebra homomorphism \(\Phi_\lambda : U \to U_{\lambda}\).

**Remark 6.3.** The morphism \(\Phi_\lambda\) is not the one used by Nakajima, although the operators \(h_{ir}\) in (6.2) and in [N3, §9.2] are the same. The proof of Nakajima still works in our case : the relations [N3, (1.2.8) and (1.2.10)] are checked in the appendix, the relations involving only one vertex of the graph are proved as in [N3, §11], the Serre relations are proved as in [N3, §10.4].

**6.4.** Recall that \(\dim Q_{\alpha\lambda} = (\alpha|2\lambda - \alpha)\). From the formula for \(\mathcal{I}_{\alpha\lambda}\) in Section 4.5 we get \(\Omega_{\alpha\lambda} = q^{-d_{\alpha\lambda}}\). Thus the hypothesis in Section 5.2 are satisfied. Consider the anti-automorphism \(\gamma_U = \phi_\ast D_{Z_\lambda}\) of \(U_{\lambda}\).

**Lemma 6.5.** We have \(\Phi_\lambda^\tau = \gamma_U \Phi_\lambda\).

**Proof of 6.5.** For any \(\alpha' \in Q^+\) the Hecke correspondence \(C_{\alpha'\lambda}^{i\pm}\) is smooth and

\[
\Omega_{C_{\alpha'\lambda}^{i\pm}} = q^{f_{\alpha'\lambda}^i} p^* \Omega_{Q_{\alpha\lambda}} \otimes p^* \det (F_{\alpha\lambda}^{i\pm}) \otimes (\pm L)^{f_{\alpha\lambda}^i},
\]

where \(\alpha = \alpha' \mp \alpha_i\) (see the proof of 7.4). Using the identities

\[
p^* F_{\alpha'\lambda}^{i\pm} - p^* F_{\alpha\lambda}^{i\pm} = -q^{1} L \quad \text{and} \quad \dim Z_{\alpha'\lambda} = \mp f_{\alpha'\lambda}^i + d_{\alpha'\lambda} + 1,
\]

and the commutation of the Serre-Grothendieck duality with closed embeddings we get

\[
\gamma_U(x_{ir}^\pm) = \sum_{\alpha'} q^{-1} x_{i\alpha'\lambda}^\mp (\mp L)^{\otimes f_{\alpha'\lambda}^i - r} C_{\alpha'\lambda}^{i\mp} \quad \text{and} \quad x_{i\lambda}^\mp = x_{i\lambda}^\mp.
\]

\(\Box\)
6.6. Put $W_\lambda = K^{\tilde{G}_\lambda}(L_\lambda)$, $W'_\lambda = K^{\tilde{G}_\lambda}(Q_\lambda)$, and $R_\lambda = R(\tilde{G}_\lambda)$. The $R_\lambda$-modules $W'_\lambda$, $W_\lambda$ are free. Thus $W_\lambda$, resp. $W'_\lambda$, may be viewed as $U$-module via the algebra homomorphism $U \to U_\lambda \to \text{End } W_\lambda$, resp. $U \to \text{End } W'_\lambda$, which is composed of $\Phi_\lambda$ and of the convolution product $\star : U_\lambda \otimes W_\lambda \to W_\lambda$, resp. $\star : U_\lambda \otimes W'_\lambda \to W'_\lambda$ (for $Z_{12} = Z_\lambda$ and $Z_{23} = Q_\lambda$, resp. $Z_{23} = L_\lambda$). The varieties $L_{0\lambda}$ and $Q_{0\lambda}$ are reduced to a point. Let $[0]$ be their fundamental class in $K$-theory. By [N3, Propositions 12.3.2, 13.3.1] the $U$-module $W_\lambda$ is cyclic generated by $[0]$, and we have

$$x_t^\pm(z) \ast [0] = 0, \quad k^\pm_\lambda(z) \ast [0] = q^{(\lambda|\alpha_i)} \Lambda_{-1/z}((q^{-2} - 1)W^\pm_i) \otimes [0].$$

6.7. Fix a semi-simple element $s = (t, \zeta)$ in $\tilde{G}_\lambda$. Let $\langle s \rangle \subset \tilde{G}_\lambda$ be the Zariski closed subgroup generated by $s$. Put

$$W_s = K^{(s)}(L_\lambda)_s, \quad W'_s = K^{(s)}(Q_\lambda)_s, \quad U_s = K^{(s)}(Z_\lambda)_s.$$

Let $\Phi_s : U \to U_s$ be the composition of $\Phi_\lambda$ and the specialization at $s$. Consider the $C$-algebra

$$U|_{q=\zeta} = U \otimes_A (\mathbb{A}/(q - \zeta)).$$

The spaces $W_s$, $W'_s$, are $U|_{q=\zeta}$-modules. The $U|_{q=\zeta}$-module $W_s$ is called a standard module. It is cyclic generated by $[0]$.

7. The coproduct

7.1. Fix $\lambda^1, \lambda^2 \in P^+$, such that $\lambda = \lambda^1 + \lambda^2$. Put $G_{\lambda^1, \lambda^2} = G_{\lambda^1} \times G_{\lambda^2} \times C^\times$, $R_{\lambda^1, \lambda^2} = R(G_{\lambda^1, \lambda^2})$, and let $R_{\lambda^1, \lambda^2}$ be the fraction field of $R_{\lambda^1, \lambda^2}$. Set

$$W'_\lambda = W'_\lambda \otimes_{\mathbb{R}_\lambda} R_{\lambda^1, \lambda^2}, \quad W'_{\lambda^1, \lambda^2} = K_{G_{\lambda^1, \lambda^2}}(Q_{\lambda^1} \times Q_{\lambda^2}), \quad W'_{\lambda^1, \lambda^2} = W'_{\lambda^1, \lambda^2} \otimes_{\mathbb{R}_{\lambda^1, \lambda^2}} R_{\lambda^1, \lambda^2}.$$

Similarly we define $W_\lambda$, $W_{\lambda^1, \lambda^2}$, $W_{\lambda^1, \lambda^2}$ (using $L_\lambda$ instead of $Q_\lambda$), and $\tilde{U}_\lambda$, $\tilde{U}_{\lambda^1, \lambda^2}$ (using $Z_\lambda$ instead of $Q_\lambda$). By [N3, §7], [CG, §5.6] we have two Kunneth isomorphisms

$$W'_{\lambda^1} \otimes W'_{\lambda^2} \cong W'_{\lambda^1, \lambda^2}, \quad W_{\lambda^1} \otimes W_{\lambda^2} \cong W_{\lambda^1, \lambda^2}.$$

Let them by $\theta$. Set $\tilde{U}_{\lambda^1, \lambda^2} = (U_{\lambda^1} \otimes U_{\lambda^2}) \otimes_{\mathbb{R}_{\lambda^1, \lambda^2}} R_{\lambda^1, \lambda^2}$. We do not know if there is a Kunneth isomorphism

$$K_{G_{\lambda^1}}(Z_{\lambda^1}) \otimes K_{G_{\lambda^2}}(Z_{\lambda^2}) \simeq K_{G_{\lambda^1, \lambda^2}}(Z_{\lambda^1} \times Z_{\lambda^2}).$$

However, it is easy to see that the map $\theta : \tilde{U}_{\lambda^1, \lambda^2} \to \tilde{U}_{\lambda^1, \lambda^2}$ induced by the external tensor product is invertible: if $H = H_{\lambda^1} \times H_{\lambda^2} \times C^\times \subseteq G_{\lambda^1, \lambda^2}$ is a maximal torus, then

$$\tilde{R}(H) \otimes_{\mathbb{R}_{\lambda^1, \lambda^2}} \tilde{U}_{\lambda^1, \lambda^2} \simeq \tilde{K}^{H_{\lambda^1} \times C^\times}(Z_{\lambda^1}) \otimes \tilde{K}^{H_{\lambda^2} \times C^\times}(Z_{\lambda^2})$$

$$\simeq \tilde{K}^{H_{\lambda^1} \times C^\times}(Q_{\lambda^1} \times Q_{\lambda^2}) \otimes \tilde{K}^{H_{\lambda^2} \times C^\times}(Q_{\lambda^2} \times Q_{\lambda^2})$$

$$\simeq \tilde{K}^{H}(Q_{\lambda^1} \times Q_{\lambda^2})$$

$$\simeq \tilde{K}^{H}(Z_{\lambda^1} \times Z_{\lambda^2})$$

$$\simeq \tilde{R}(H) \otimes_{\mathbb{R}_{\lambda^1, \lambda^2}} \tilde{U}_{\lambda^1, \lambda^2}.$$
Here we have used the localization theorem, the identity
\[(Z_{\lambda b})^{H_{\lambda b} \times \mathbb{C}^X} = (Q_{\lambda b} \times Q_{\lambda b})^{H_{\lambda b} \times \mathbb{C}^X} \quad \text{for} \quad b = 1, 2,\]
the Kunneth formula for $Q_{\lambda 1} \times Q_{\lambda 2}$, and [CG, Theorem 6.1.22] (which is valid, although $G_{\lambda 1 \lambda 2}$ is not simply connected). Taking the invariants under the Weyl group we get the required invertibility. This invertibility is not needed in the sequel.

Set also
\[T_+ = qE_+(V^2, V^1) + L(V^1, qV^2 - V^2) + q^{-1}E_+(V^1, V^2) + q^{-2}L(V^2, qV^1 - V^1)\]
\[T_+^\prime = qE(V^1, V^2) + L(V^1, qV^2 - V^2) + q^2L(qV^1 - V^1, V^2)\]
in $W_{\lambda 1 \lambda 2}^\prime$, where $E_{\pm}(V^1, V^2) = \bigoplus_{i,j} n_{ij}^\pm V_i^\ast \otimes V_j^2$.

**Lemma 7.2.** The class of the normal bundle of $Q_{\lambda 1} \times Q_{\lambda 2}$ in $Q_{\lambda}$ is $T_+ + q^2T_+^\ast$. In particular, the class $\Omega^\prime = \Lambda_{-1}(T_+^\ast + q^2T_+^\ast)$ is well-defined and is invertible in $W_{\lambda 1 \lambda 2}^\prime$.

**Proof of 7.2.** Follows from Lemma 4.4 and Sections 4.5, 5.3. \qed

Let $\Delta'_{W'} : W_{\lambda} \to W_{\lambda 1 \lambda 2}^\prime$ be the map induced by the pull-back $\kappa^* : W_{\lambda} \to W_{\lambda 1 \lambda 2}^\prime$ (which is well defined, since $Q_{\lambda 1} \times Q_{\lambda 2}$ and $Q_{\lambda}$ are smooth). Since $Z_\lambda$ is a closed subvariety of $(Q_{\lambda 2})^2$, the restriction with support with respect to the embedding $(Q_{\lambda 2})^2 \to (Q_{\lambda})^2$ gives a map

\[(k \times k)^* : K^{G_{\lambda} \times \mathbb{C}^X}(Z_\lambda) \to K^{G_{\lambda 1} \times G_{\lambda 2} \times \mathbb{C}^X}(Z_{\lambda 1} \times Z_{\lambda 2}).\]

Put $\Delta'_{W'} = 1 \otimes \Omega^{-1} \ast (k \times k)^* : U_{\lambda} \to U_{\lambda 1 \lambda 2}$, where $1 \otimes \Omega^{-1}$ is the pull-back of $\Omega^{-1}$ by the second projection $Z_{\lambda 1} \times Z_{\lambda 2} \to Q_{\lambda 1} \times Q_{\lambda 2}$. There is a unique linear map $Q \to P$, $\alpha \mapsto \alpha^+$, such that $f^{\alpha_{\lambda 1} + \alpha_{\lambda 2}} = (\alpha_1 \lambda - \alpha^+)$. By (5.4) and Lemma 7.2, the class $\Lambda_{-1}(T_+|_{Q_{\alpha 1 \lambda 1} \times Q_{\alpha 2 \lambda 2}})$ is well-defined and is invertible in the ring $K^{G_{\lambda 1} \times G_{\lambda 2} \times \mathbb{C}^X}(Q_{\alpha 1 \lambda 1} \times Q_{\alpha 2 \lambda 2})$. Set

\[\Omega = \sum_{\alpha 1, \alpha 2} q^{(\alpha 1, \alpha 2) \ast - \lambda^2} \Lambda_{-1}(T_+|_{Q_{\alpha 1 \lambda 1} \times Q_{\alpha 2 \lambda 2}}) \in W_{\lambda 1 \lambda 2}^\prime.\]

The class $\Omega$ is invertible. We put $\Delta'_{W'} = \Omega^{-1} \otimes \Delta'_{W'}$ (here $\otimes$ is the tensor product on $Q_{\lambda 1} \times Q_{\lambda 2}$). Let $\delta_* \Omega^{\pm 1}$ be the image of $\Omega^{\pm 1}$ in $U_{\lambda 1 \lambda 2}$. We put $\Delta'_{U} = \delta_* \Omega^{-1} \ast \Delta'_{U} \ast \delta_* \Omega$. Hereafter $\delta_*$ may be omitted.

**7.3.** Let $R \in U_{\lambda 1 \lambda 2}$ be the element defined in Lemma 8.1.(iv), and set $\tilde{R} = \theta(\tilde{R})$. We put $\Delta_W = R^{-1} \ast \Delta_W$, and $\Delta = R^{-1} \ast \Delta^{\circ} \ast \tilde{R}$. Recall that we have anti-involutions $\gamma_U$ of $U_{\lambda}$ and $U_{\lambda 1 \lambda 2}$ (see §6.4). We set $\Delta^\gamma_{U} = \gamma_U \Delta U \gamma_U$.

**Theorem 7.4.** The map $\Delta_U : U_{\lambda} \to U_{\lambda 1 \lambda 2}$ satisfies

\[\Delta_U \Phi_{\lambda} = \theta(\Phi_{\lambda 1} \otimes \Phi_{\lambda 2}) \Delta^{\circ} \quad \text{and} \quad \Delta_U^\gamma \Phi_{\lambda} = \theta(\Phi_{\lambda 1} \otimes \Phi_{\lambda 2}) \Delta^\bullet.\]
Lemma 7.5. Assume that $M, M'$, are smooth quasi-projective $G$-varieties. Let $p : M \times M' \to M$ be the projection. Fix a semisimple element $s \in G$ and a smooth closed $G$-subvariety $X \subset M \times M'$. Put $N = TX - (p^*TM)|_X$, and $N^s = TX^s - (p^*TM^s)|_X$.

(i) The element $\bigwedge_{-1}(-N^s|_X + N^{ss}) \in K(s)(X^s)$ is well-defined. Its image in $K(X^s)$ under the evaluation map is still denoted by $\bigwedge_{-1}(-N^s|_X + N^{ss})$.

(ii) For any $G$-bundle $E$ on $X$, the bivariant localization morphism $r : K(s)(X)_s \to K(X^s)$ defined in [CG, §5.11] maps $E$ to $E|_X \otimes \bigwedge_{-1}(-N^s|_X + N^{ss})$ (here $\otimes$ is the tensor product on $X^s$).

Remark 7.6. Fix $s = (t, \zeta) \in G_\lambda$, $s^1 = (t^1, \zeta) \in G_{\lambda^1}$, $s^2 = (t^2, \zeta) \in G_{\lambda^2}$, such that $t = t^1 \oplus t^2$ and $t$ is semi-simple. Put $U_{s^1,s^2} = K(s)(Z_{\lambda^1} \times Z_{\lambda^2})_s$, and let

$$r_s : U_s \to K(Z^s_\lambda), \quad r_{s^1,s^2} : U_{s^1,s^2} \to K(Z^s_{\lambda^1} \times Z^s_{\lambda^2})$$

be the bivariant localization maps. These maps are invertible and commutes to the convolution product $\star$. If $(c^s\text{spec}^1) \cap \text{spec}^2 = \emptyset$, then $Q_s^\alpha \simeq Q_{\lambda^1}^{s^1} \times Q_{\lambda^2}^{s^2}$, $Z^s_\lambda \simeq Z^{s^1}_{\lambda^1} \times Z^{s^2}_{\lambda^2}$, and the specialization of $\Delta_U'$ at $s$ is well-defined and it coincides with the map $r_{s^1,s^2} \cdot r_s$.

Proof of 7.5. Claim (i) is well-known, see [CG, Proposition 5.10.3] for instance. Claim (ii) is immediate from the Koszul resolution of $O_X$ by sheaves of locally free $O_{M \times M'}$-modules in a neighborhood, in $M \times M'$, of each point of $X^s$.

Proof of 7.4. Assume that $s$ is generic. Then $\kappa(Q_{\lambda^1}^{s^1} \times Q_{\lambda^2}^{s^2}) = Q^s_\lambda$ and $\Delta_U'$ specializes to the map $r_{s^1,s^2} \cdot r_s$ by Remark 7.6. Assume that $\alpha' = \alpha \pm \alpha_i \in Q^+$.

Observe that

$$TC^{s^1}_{\alpha^i \lambda} - TCQ_{\alpha \lambda} \boxtimes 1 = 1 \boxtimes T_{\alpha^i \lambda} - T_{\alpha \lambda}^{s^1} = E(L, V') + L(L, W) - (q + q^{-1})L(L, V') + q^2$$

$$= q^2 + q\mathcal{E}^s \otimes p^*\mathcal{F}_{\alpha^i \lambda}.$$
Lemma 7.9. The following lemma is proved as in [N3, Proposition 12.3.2 and Theorem 7.3.5].

Let \( \Omega \in \mathcal{W}_{\lambda,\lambda} \) specialize to a class in \( \mathcal{K}^{(s)}(Q_{\lambda_1} \times Q_{\lambda_2}) \).

Let \( \Omega_{s_1,s_2} \) be this class. A direct computation gives the following identities in \( U_{s_1,s_2} \):

\[
\Omega_{s_1,s_2}^{-1} \otimes (x_{ir}^+ \otimes 1) \otimes \Omega_{s_1,s_2} = q^{\mp f_{\lambda_1}^\pm} \bigwedge_{-1} (q^{-1} F_{\lambda_1}^+ \otimes \mathcal{L}^*) \otimes (1 \otimes x_{ir}^+) \]

Using (5.4) we get

\[
\Delta_{U}^\circ(x_{ir}^+) = \theta(x_{ir}^+ \otimes 1 + q^{f_{\lambda_1}^\pm} \bigwedge_{-1}((q^{-1} - q) F_{\lambda_1}^+ \otimes \mathcal{L}^*) \otimes (1 \otimes x_{ir}^+))
\]

(7.7)

\[
\Delta_{U}^\circ(x_{ir}^-) = \theta(x_{ir}^- \otimes 1 + q^{f_{\lambda_1}^\pm} \bigwedge_{-1}((q^{-1} - q) (-\mathcal{L})^* \otimes F_{\lambda_2}) + 1 \otimes x_{ir}^-) \]

Using (7.7) it is proved in Lemma 8.1.(iii) that

\[
\bar{R}_{s_1,s_2}^{-1} \ast (\Delta_{U}^\circ \Phi_s) \ast \bar{R}_{s_1,s_2} = \theta(\Phi_{s_1} \otimes \Phi_{s_2}) \Delta^\circ.
\]

We are done. The second identity follows from the first one and Lemma 6.5 since \( \Delta^{\circ,\tau} = \Delta^* \).

\[
\square
\]

7.8. Let \( H \subset G_{\lambda_1} \times G_{\lambda_2} \times \mathbb{C}_x \) be the maximal torus of diagonal matrices. Assume that the elements \( s, s_1, s_2 \in G_{\lambda_1, \lambda_2} \) belong to \( H \). Put

\[
\mathcal{W}'_{\lambda,H} = \mathcal{H}^H(Q_{\lambda_1}), \quad \mathcal{W}'_{\lambda,\lambda_2,H} = \mathcal{H}^H(Q_{\lambda_1} \times Q_{\lambda_2}),
\]

and idem for \( \mathcal{W}_{\lambda,H}, \mathcal{W}_{\lambda_1,\lambda_2,H}. \) The corresponding \( \mathcal{R}(H) \)-vector spaces are overlined (i.e. we set \( \mathcal{W}_{\lambda,H} = \mathcal{K}^H(Q_{\lambda_1}) \), etc). Let \( \theta \) denote the Kunneth isomorphisms

\[
\mathcal{W}_{\lambda_1,H} \otimes_{\mathcal{R}(H)} \mathcal{W}_{\lambda_2,H} \simeq \mathcal{W}_{\lambda_1,\lambda_2,H}, \quad \mathcal{W}'_{\lambda_1,H} \otimes_{\mathcal{R}(H)} \mathcal{W}'_{\lambda_2,H} \simeq \mathcal{W}'_{\lambda_1,\lambda_2,H}.
\]

Consider the bilinear pairing

\[
(\ ) : \mathcal{W}'_{\lambda,H} \otimes_{\mathcal{R}(H)} \mathcal{W}_{\lambda,H} \to \mathcal{R}(H), \quad \mathcal{E} \otimes \mathcal{F} \mapsto q_*(\mathcal{E} \otimes \mathcal{F}),
\]

where \( q \) is the projection to a point and \( \otimes \) is the tensor product of sheaves on \( Q_{\lambda_1}^s \).

The following lemma is proved as in [N3, Proposition 12.3.2 and Theorem 7.3.5].

Lemma 7.9. (i) The \( U \)-module \( \mathcal{W}_{\lambda,H} \) is generated by \( \mathcal{R}(H) \otimes [0] \).

(ii) The \( \mathcal{R}(H) \)-modules \( \mathcal{W}'_{\lambda,H}, \mathcal{W}_{\lambda,H} \) are free and the pairing \( (\ ) \) is perfect.

Let \( (\ ) \) denote also the pairing with the scalars extended to the field \( \mathcal{R}(H) \), and the pairing between \( \mathcal{W}'_{\lambda_1,\lambda_2,H} \) and \( \mathcal{W}_{\lambda_1,\lambda_2,H} \).

The automorphism \( D_{L_\lambda} \otimes id_{\mathcal{R}(H)} : \mathcal{W}_{\lambda,H} \to \mathcal{W}_{\lambda,H} \) is still denoted by \( D_{L_\lambda} \). By §5.2 we have

\[
D_{Z_\lambda}(u) \ast D_{L_\lambda}(m) = D_{L_\lambda}(u \ast m), \quad \forall u \in U_\lambda, \forall m \in \mathcal{W}_{\lambda,H}.
\]

Given a \( U_\lambda \)-module \( M \), let \( M^\circ \) be its contragredient module, that is \( M^\circ = M^* \) as a vector space and \( (uf)(m) = f(\phi(u)m) \) for all \( f \in M^\circ, m \in M, u \in U_\lambda \). The symbols \( \circ \) and \( \bullet \) denote the tensor product of \( U \)-modules relative to the coproduct \( \Delta^\circ \) and \( \Delta^* \) respectively. To simplify let \( \Delta_{W^}\circ \) denote also the map \( \Delta_{W^}\circ \otimes id_{\mathcal{R}(H)} : \mathcal{W}'_{\lambda,H} \to \mathcal{W}'_{\lambda_1,\lambda_2,H}. \)
Proposition 7.10. (i) The map $\Delta_W : \hat{W}_{\lambda,H}^s \to \hat{W}_{\lambda^1 \lambda^2,H}^s$ is invertible.
(ii) The map $\Delta_U$ is an algebra homomorphism and
\[ \Delta_U(u) \ast \Delta_W(m) = \Delta_W(u \ast m'), \quad \forall u, m'. \]
(iii) The pairing identifies $W_{\lambda,H}^r$ with the contragredient module $W_{\lambda,H}^s$.
(iv) Set $\Delta_W = D_{L_\lambda}(\Delta_{W'})D_{L_\lambda}$, where the transpose is relative to the pairing (\lvert \rvert).
We have
\[ \Delta_W^\ast(u) \ast \Delta_W(m) = \Delta_W(u \ast m), \quad \forall u, m. \]
(v) The map $\Delta_W$ is an embedding of $U$-modules $W_{\lambda,H} \hookrightarrow W_{\lambda^1 \lambda^2,H}$.

Proof of 7.10. Claim (i) follows from the localization theorem since the fixpoint sets $Q^H_\lambda$ and $Q^H_{\lambda^1 \lambda^2}$ are equal. It suffices to check Claim (ii) on a dense subset of $\text{spec} R_{\lambda^1 \lambda^2}$. If $s$ is generic then $\Delta_U^\ast = r^{-1}_{s,\lambda^1 \lambda^2}r_s$ (see Remark 7.6). Thus $\Delta_U$ is an algebra homomorphism, since the map $r_s$ commutes to the convolution product.

The case of $\Delta_W$ is similar. Claim (iii) means that for any $m \in W_{\lambda,H}$, $m' \in W_{\lambda,H}^r$, $u \in U_\lambda$, we have
\[ (m' | u \ast m) = (\phi_s(u) \ast m'|m). \]

It suffices to check the two identities below:
\[ (m' | u \ast m) = (m' \ast u|m) \quad \text{and} \quad m' \ast u = \phi_s(u) \ast m'. \]

These identities are standard. The first one is the associativity of the convolution product, the second one is essentially the fact that $\phi_s$ is an anti-homomorphism. Claim (iv) is a direct computation: for any $u, m, m'$ as above, we have
\[
(\Delta_{W'}(m')|D_{L_\lambda} \Delta_W(u \ast m)) = (m'|D_{L_\lambda}(u \ast m)) = (m'|D_{Z_\lambda}(u) \ast D_{L_\lambda}(m)) = (\gamma_U(u) \ast m'|D_{L_\lambda}(m)) = (\Delta_{W'}(\gamma_U(u) \ast m'|D_{L_\lambda} \Delta_W(m)) = (\Delta_\lambda \gamma_U(u) \ast \Delta_{W'}(m'|D_{L_\lambda} \Delta_W(m)) = (\Delta_{W'}(m'|D_{L_\lambda} \Delta_W(m)) = (\Delta_{W'}(m'|D_{L_\lambda}(\Delta_U^\ast(u) \ast \Delta_W(m))).
\]

Since $W_{\lambda,H}$ is a free $R(H)$-module the restriction of $\Delta_W$ to $W_{\lambda,H}$ is injective. The $U$-module $W_{\lambda,H}$ is generated by $R(H) \otimes [0]$ and $\Delta_W([0]) = [0] \otimes [0]$. Thus $\Delta_W(W_{\lambda,H}) \subseteq W_{\lambda^1 \lambda^2,H}$. \(\square\)

7.11. We can now state the main result of this paper. Using Theorem 7.4, Proposition 7.10, and the Kunneth isomorphisms
\[ \theta^{-1} : W_{\lambda^1 \lambda^2,H} \overset{\sim}{\to} W_{\lambda^1,H} \boxtimes_{R(H)} W_{\lambda^2,H}, \quad \theta^{-1} : W_{\lambda^1 \lambda^2,H} \overset{\sim}{\to} W_{\lambda^1,H} \boxtimes_{R(H)} W_{\lambda^2,H}, \]
we get morphisms of $U|_{q=\zeta}$-modules
\[ \theta^{-1} \Delta_W : W_s \to W_{s_1} \ast W_{s_2} \quad \text{and} \quad \theta^{-1} \Delta_{W'} : W'_s \to W'_{s_1} \circ W'_{s_2}. \]
Theorem 7.12. Assume that $(\zeta^{-1}N_{\text{spec} t}) \cap \text{spec} t^2 = \emptyset$. Then the maps

$$\theta^{-1}\Delta_W : W_s \to W_{s_1} \oplus W_{s_2} \quad \text{and} \quad \theta^{-1}\Delta_{W'} : W_s' \to W_{s_1}' \circ W_{s_2}'$$

are isomorphisms of $U|_{q=\zeta}$-modules.

Remark 7.13. Let first recall the following standard facts (see [N3, Section 7.1] for instance). Let $X$ be a smooth quasi-projective $G$-variety ($G$ a linear group) with a finite partition into $G$-stable locally closed subsets $X_i$, $i \in I$. Fix an order on $I$ such that the subset $\bigcup_{j \leq i} X_j \subset X$ is closed for all $i$. Let $\kappa : Y \to X$ be the embedding of a smooth closed $G$-stable subvariety such that the intersections $Y_i = X_i \cap Y$, are the connected components of $Y$ (in particular, $Y_i$ is smooth). Assume that there is a $G$-invariant vector bundle map $\pi_i : X_i \to Y_i$ for each $i$. Let $K_{i, \text{top}}^G$ be the complexified equivariant topological $K$-group of degree $r$ ($r = 0, 1$). Assume also that $K_{i, \text{top}}^G(Y) = \{0\}$, $K_{0, \text{top}}^G(Y) = K^G(Y)$ and $K^G(Y)$ is a free $R(G)$-module. Let consider the vector bundle $N_i = (TX|_{Y_i})/(TX|_{Y_1})$ on $Y_i$. For each $i$ fix a basis $(\xi_{ij} : j \in J_i)$ of the space $K^G(Y_i)$. Fix also an element $\tilde{\xi}_{ij} \in K^G(X_i)$ whose restriction to $X_i$ is $\pi_i^*\xi_{ij}$. Then $K^G_{i, \text{top}}(X) = \{0\}$, $K^G_{0, \text{top}}(X) = K^G(X)$, and the $\tilde{\xi}_{ij}$ form a basis of $K^G(X)$. Moreover $\kappa^*(\tilde{\xi}_{ij}) = \Lambda_{-1}(N^*_{ij}) \otimes E_{ij}$ modulo $K^G(U_{i < j}, Y_i)$ (here $\otimes$ is the tensor product on $Y$).

Proof of 7.12. Consider the co-character

$$\gamma : \mathbb{C}^\times \to \hat{G}_\lambda, \ z \mapsto (z\text{id}_{|\lambda|_1} \oplus \text{id}_{|\lambda|_2}, 1).$$

Let $\langle s, \gamma \rangle$ be the Zariski closed subgroup generated by $s$ and $\gamma(\mathbb{C}^\times)$. We have $\kappa(Q^{s_1}_{\lambda_1} \times Q^{s_2}_{\lambda_2}) = Q^{(s, \gamma)}_{\lambda}$ (see Remark 7.6). We claim that $\gamma$ gives a Byalnicki-Birula partition of the variety $Q^\times_{\lambda}$ such that each piece is a $H$-equivariant vector bundle over a connected component of $Q^{(s, \gamma)}_{\lambda}$. Fix $I$-graded vector spaces $V, W^1, W^2$, of dimension $\alpha, \lambda_1, \lambda_2$. Given a triple $(B, p, q)$ representing a point $x \in Q^\times_{\lambda}$, let $V^1$ be the largest $B$-stable subspace contained in $q^{-1}(W^1)$. Assume that $x$ is fixed by $s$. Let $Q(p) \subset Q^\times_{\lambda}$ be the connected component containing $x$. For any $z \in \mathbb{C}^\times$ let $V(z)$ and $W(z)$ be defined as in Lemma 4.4. Assume that $(\zeta^{-1}N_{\text{spec} t}) \cap \text{spec} t^2 = \emptyset$. Then $V(z) \subset V^1$ for any $z \in \zeta^{-1}\text{spec}(t^1)$. In particular $p(W^1) \subset V^1$. Thus, the subspace $V^1 \oplus W^1 \subset V \oplus W$ is stable by $B, p, q$. The restriction of $(B, p, q)$ to $V^1 \oplus W^1$ is a stable triple. The projection of $(B, p, q)$ to $V^1 \oplus W/W^1$ is stable either by the maximality of $V^1$. Let $x^1 \in Q^{s_1}_{\lambda_1}$, $x^2 \in Q^{s_2}_{\lambda_2}$, be the classes of those triples. We have

$$\kappa(x^1, x^2) = \lim_{z \to 0} \gamma(z)x$$

(set $g(z) = z\text{id}_{V^1} \oplus \text{id}_S$ where $S$ is a $I$-graded vector space such that $V = S \oplus V^1$ and write the triple $\gamma(z)g(z)(B, p, q)$, which represents $\gamma(z)x$, in a basis adapted to the splitting $V = S \oplus V^1$. Then, do the limit $z \to 0$). The claim is proved.

Consider the piece

$$Q^+_{\rho^1 \rho^2} = \{x \in Q^s_{\lambda} \mid \lim_{z \to 0} \gamma(z)x \in Q^s_{\rho^1 \rho^2}\},$$

where $Q^s_{\rho^1 \rho^2}$ is the connected component $\kappa(Q^1_{\rho^1} \times Q^2_{\rho^2}) \subseteq Q^{(s, \gamma)}_{\lambda}$. By definition of $Q^s_{\rho^1 \rho^2}$, the one-parameter subgroup $\gamma$ acts on $\mathcal{T}Q^s_{\rho^1 \rho^2}|_{Q^s_{\rho^1 \rho^2}}$ with non-negative
weights. By Lemma 7.2 the class of the normal bundle of $Q_{\lambda_1} \times Q_{\lambda_2}$ in $Q_{\lambda}$ is the sum of $T_+^\gamma$ and $q^2 T_+^\gamma$. It is easy to see that $\gamma$ acts on $T_+^\gamma$ with negative weights, and on $q^2 T_+^\gamma$ with positive weights. We get the following equality of classes in $K^H(Q_{\rho_1, \rho_2})$

$$\mathcal{T} Q^\gamma_{\lambda_1, \lambda_2} - \mathcal{T} Q^\gamma_{\rho_1^i, \rho_2^i} = (\mathcal{T}^\gamma Q_{\eta_1, \eta_2})^s. \tag{7.14}$$

Here we use the notation: if $E$ is the class of a virtual $H$-bundle on $Q_{\rho_1, \rho_2}$, then $E^s$ is the class of the $s$-invariant part of the virtual bundle. Recall that

$$Q_{\lambda}^{(s, \gamma)} = \kappa(Q_{\lambda_1}^{s_1} \times Q_{\lambda_2}^{s_2}) = \bigcup_{\rho_1^i, \rho_2^i} Q_{\rho_1^i, \rho_2^i}.$$

We can apply Remark 7.13 to the following situation

$$X = Q_{\lambda}, \quad Y = \kappa(Q_{\lambda_1}^{s_1} \times Q_{\lambda_2}^{s_2}), \quad I = \{(\rho_1, \rho_2^i)\}, \quad \{X_i\} = \{Q_{\rho_1^i, \rho_2^i}\}, \quad \{Y_i\} = \{Q_{\rho_1^i, \rho_2^i}\}$$

(see also [N3, Theorem 7.3.5]). We get particular bases $B_\lambda$ of $K^H(Q_{\lambda})$, and $B_{\lambda_1, \lambda_2}$ of $K^H(Q_{\lambda_1} \times Q_{\lambda_2})$. In these bases, the $\mathbb{R}(H)$-linear map

$$\Lambda_{-1}((T_+^\gamma Q_{\eta_1, \eta_2})^s)^{-1} \otimes_1 \kappa_{1}^s : \mathbb{K}^H(Q_{\lambda}) \to \mathbb{K}^H(Q_{\lambda_1}^{s_1} \times Q_{\lambda_2}^{s_2})$$

(where $\kappa_1$ is the restriction of $\kappa$ to $Q_{\lambda_1}^{s_1} \times Q_{\lambda_2}^{s_2}$, and $\otimes_1$ is the tensor product on $\kappa(Q_{\lambda_1}^{s_1} \times Q_{\lambda_2}^{s_2})$) is triangular unipotent by Remark 7.13 and (7.14). Since $E_-(V^1, V^2)^s = E_+(V^2, V^1)$, the elements

$$\Lambda_{-1}((T_+^\gamma Q_{\eta_1, \eta_2})^s), \quad \Lambda_{-1}((T_+^\gamma Q_{\eta_1, \eta_2})^s) \in \mathbb{R}^H(Q_{\rho_1, \rho_2})$$

coincide up to the product by the class in $K^H(Q_{\rho_1, \rho_2})$ of an invertible sheaf (see §7.1 and (5.4)). Thus, the product by

$$\Lambda_{-1}((T_+^\gamma Q_{\eta_1, \eta_2})^s)^{-1} \otimes_1 \Lambda_{-1}((T_+^\gamma Q_{\eta_1, \eta_2})^s)$$

belongs to $GL(K^H(Q_{\rho_1, \rho_2})) \subset GL(K^H(Q_{\rho_1, \rho_2}))$. In particular, the determinant of the $\mathbb{R}(H)$-linear map

$$\Lambda_{-1}((T_+^\gamma Q_{\eta_1, \eta_2})^s)^{-1} \otimes_1 \kappa_{1}^s : \mathbb{K}^H(Q_{\lambda}) \to \mathbb{K}^H(Q_{\lambda_1}^{s_1} \times Q_{\lambda_2}^{s_2})$$

respectively to $B_\lambda, B_{\lambda_1, \lambda_2}$, is an invertible element of $\mathbb{R}(H)$. Let

$$\iota_\lambda : Q_{\lambda} \to Q_{\lambda}, \quad \iota_{\lambda_1, \lambda_2} : Q_{\lambda_1}^{s_1} \times Q_{\lambda_2}^{s_2} \to Q_{\lambda_1} \times Q_{\lambda_2},$$

be the closed embeddings. The localization theorem gives isomorphisms of $\mathbb{R}(H)$-modules

$$\iota_{\lambda^s} : \mathbb{K}^H(Q_{\lambda}^s) \simeq \mathbb{W}_{\lambda, H}, \quad \iota_{\lambda_1, \lambda_2, s} : \mathbb{K}^H(Q_{\lambda_1}^{s_1} \times Q_{\lambda_2}^{s_2}) \simeq \mathbb{W}_{\lambda_1, \lambda_2, H}.$$
Moreover, $\iota_{\lambda_*}(\mathcal{K}^H(Q^s_{\lambda})) \subset W'_{\lambda, H}$ are free $R(H)$-submodules of $\tilde{W}'_{\lambda, H}$ of maximal rank such that $\iota_{\lambda_*}(\mathcal{K}^H(Q^s_{\lambda}))_s = W'_s$. Thus, the basis $\iota_{\lambda_*}(B_\lambda)$ differs from any basis of $W'_{\lambda, H}$ by the action of an $R(H)$-linear operator whose determinant is regular and non-zero at $s$. Idem for $\iota_{\lambda_1\lambda_2*}(B_{\lambda_1\lambda_2}) \subset W'_{\lambda_1\lambda_2, H}$. Set

$$\bar{f} = \Omega^{-1} \otimes \kappa^* : \tilde{W}'_{\lambda, H} \rightarrow \tilde{W}'_{\lambda_1\lambda_2, H}.$$ 

By §7.1 we have

$$\iota_{\lambda_1\lambda_2}^{-1} \circ \bar{f} \circ \iota_{\lambda_*} = \sum_{\rho_1, \rho_2} A_{\rho_1, \rho_2} \otimes_1 (T_+|_{Q_{\rho_1, \rho_2}})^{-1} \otimes_1 \kappa^*_1,$$

for some element $A_{\rho_1, \rho_2} \in \tilde{K}^H(Q_{\rho_1, \rho_2})$. Moreover, the product by

$$A_{\rho_1, \rho_2} \otimes_1 (T_+|_{Q_{\rho_1, \rho_2}})^{-1} \otimes_1 (T_+|_{Q_{\rho_1, \rho_2}})^{-1}$$

is an invertible operator in $\text{GL}(\tilde{K}^H(Q_{\rho_1, \rho_2}))$ whose determinant, in $R(H)$, is regular and non-zero at $s$. The element $\tilde{R} \in U_{\lambda_1\lambda_2}$ is unipotent by Lemma 8.1.(v). Thus the determinant of the $R(H)$-linear map

$$\Delta_{W'} : \tilde{W}'_{\lambda, H} \rightarrow \tilde{W}'_{\lambda_1\lambda_2, H},$$

with respect to $\iota_{\lambda_*}(B_\lambda)$, $\iota_{\lambda_1\lambda_2*}(B_{\lambda_1\lambda_2})$ is regular and non zero at $s$. By Proposition 7.10.(iv), (v) we have $\Delta_{W'}(W'_{\lambda, H}) \subset W'_{\lambda_1\lambda_2, H}$. Thus, the map $\theta^{-1} \Delta_{W'} : W'_{\lambda, H} \rightarrow W'_{\lambda_1, H} \otimes R(H) W'_{\lambda_2, H}$ specializes to an isomorphism $W'_s \rightarrow W'_{s_1} \otimes W'_{s_2}$ of $U_{\lambda = \zeta}$-modules. The other claim is due to the following easy fact. Consider the tensor product of $U_\lambda$-modules $M_1 \boxtimes M_2$ relative to the coproduct $\Delta_\gamma$. If the map $\Delta_M : M \rightarrow M_1 \otimes M_2$ is an isomorphism of $U_\lambda$-modules, then the map $^t\Delta_{M}^{-1} : M^2 \rightarrow M_1 \boxtimes M_2$ is an isomorphism of $U_\lambda$-modules, either, where $M_1 \boxtimes M_2$ is the tensor product relative to the coproduct $\phi_{\lambda} \Delta_\gamma \phi_{\lambda}$. \hfill \Box

7.15. For any $\alpha \in \mathbb{C}^\times$ and any $k \in I$, let $V_\zeta(\omega_k)\alpha$ be the simple finite dimensional $U_{|q = \zeta}$-module with the $j$-th Drinfeld polynomial $(z - \zeta^{-1}\alpha)^{\delta_{jk}}$. By [N3, Theorem 14.1.2] we have $V_\zeta(\omega_k)\alpha = W_\lambda$ if $\lambda = \omega_k$ and $s = (\alpha, \zeta) \in \tilde{G}_\lambda$. Theorem 7.12 has the following corollaries which were conjectured in [N3] (in a less precise form) and in [AK] (for all types respectively).

**Corollary 7.16.** The standard modules are the tensor products of the modules

$$V_\zeta(\omega_{i_1})\alpha_{j_1} \circ \cdots \circ V_\zeta(\omega_{i_n})\alpha_{j_n}$$

such that $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_n$, $j = 1, 2, \ldots, r$, and the complex numbers $\alpha_{j}$ are distincts modulo $\zeta^2$. \hfill \Box

**Corollary 7.17.** The $U$-module $V_\zeta(\omega_{i_1})\zeta^{\tau_1} \circ \cdots \circ V_\zeta(\omega_{i_n})\zeta^{\tau_n}$ is cyclic if $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_n$ and is acyclic if $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_n$. Moreover it is generated (resp. cogenerated) by the tensor product of highest weight vectors.

**Remark 7.18.** Corollary 7.16 is false if $\zeta$ is a root of unity.

**Remark 7.19.** The module $W_\lambda$ is presumably isomorphic to the Weyl module introduced in [K], and studied in [CP]. This statement is related to the flatness of the Weyl modules over the ring $R_\lambda$. This is essentially equivalent to the conjecture in [CP].
8. The $R$-matrix

Fix $\lambda^1, \lambda^2 \in P^+$, and put $\lambda = \lambda^1 + \lambda^2$. For any subset $T \subset \mathbb{C}$ we put $|T| = \{|z| : z \in T\}$. If $T, T' \subset \mathbb{C}$ we write $|T| < |T'|$ if and only if $t < t'$ for all $(t, t') \in |T| \times |T'|$.

Let $S$ be the set of pairs of semi-simple elements $(s^1, s^2) \in G_{\lambda^1} \times G_{\lambda^2}$ such that $s^1 = (t^1, \zeta)$, $s^2 = (t^2, \zeta)$ and $|\text{spec } \rho^1(s^1)| - 1 < 1 < |\text{spec } \rho^2(s^2)|$ for all $\rho^1, \rho^2$ such that $Q_{\rho^1} \otimes Q_{\rho^2}$ is a non-empty connected component of $Q_{\lambda^1} \times Q_{\lambda^2}$. The projection of $S$ to $\text{spec } R_{\lambda^1 \lambda^2}$ is a Zariski-dense subset. For any $s^1, s^2$, we put $s = (t^1 \oplus t^2, \zeta)$.

As usual, we put $\rho = \sum_{i \in I} \omega_i \in P^+$.

**Lemma 8.1.** Assume that $(s^1, s^2) \in S$.

(i) If $m^+ = x^+_{i_1 r_1} \cdots x^+_{i_k r_k}, m^- = x^-_{j_1 s_1} \cdots x^-_{j_s s_k} \in U$, then $(\Phi_{s^1} \otimes \Phi_{s^2})(T_{n \rho}^{|2|}(m^- \boxtimes m^+))$ goes to zero when $n \to \infty$.

(ii) The sequence $(\Phi_{s^1} \otimes \Phi_{s^2})(R_{2n \rho})$ admits a limit in $U_{s^1} \otimes U_{s^2}$ when $n \to \infty$. This limit, denoted by $\tilde{R}_{s^1 s^2}$, is an invertible element.

(iii) Put $\tilde{R}_{s^1 s^2} = \theta(\tilde{R}_{s^1 s^2})$. Then $\tilde{R}_{s^1 s^2} \circ \theta(\Phi_{s^1} \otimes \Phi_{s^2})\Delta^o \star \tilde{R}_{s^1 s^2}^{-1} = \Delta^o \otimes \Phi_{s^1} \otimes \Phi_{s^2}$. (iv) There is a unique invertible element $\tilde{R} \in \tilde{U}_{\lambda^1 \lambda^2}$ which specializes to $\tilde{R}_{s^1 s^2}$ for any $(s^1, s^2) \in S$.

(v) The element $\tilde{R} \in \tilde{U}_{\lambda^1 \lambda^2}$ is unipotent.

**Proof of 8.1.** In Part (i) we can assume that $k = 1$, i.e. $m^- = x^-_{j \alpha}$ and $m^+ = x^+_{i \beta}$.

Let $\mathcal{L}$ be the virtual bundle on $Q_{\lambda} \times Q_{\lambda}$ introduced in §6.1. By (6.2) we have

$$(\Phi_{\lambda^1} \otimes \Phi_{\lambda^2})(T_{n \rho}^{|2|}(m^- \boxtimes m^+)) = \pm (\mathcal{L}^{|s+n|} \boxtimes \mathcal{L}^{|r-n|}) \otimes (x^+_{j_0} \boxtimes x^+_{j_0})$$

where $\otimes$ is the tensor product on $Q_{\lambda} \times Q_{\lambda}$. For any $n > 0$, let $\mathcal{L}_{s^1}^n, b = 1, 2$, be the image in $U_{s^b}$ of the restriction of $\mathcal{L}^n$ to $Z_{\lambda}$. We want to prove that $\lim_{n \to \infty} \mathcal{L}_{s^1}^n \boxtimes \mathcal{L}_{s^2}^n = 0$. Fix $\rho^1, \rho^2$, such that the subset $Q_{\rho^1} \otimes Q_{\rho^2} \subset Q_{\lambda^1} \times Q_{\lambda^2}$ is non empty. The set of the eigenvalues of $s^b$ acting on the bundle $V|Q(\rho^b)$ is the spectrum of the semi-simple element $\rho^b(s^b)|^{-1}$. For any $\alpha \in \text{spec } \rho^b(s^b)|^{-1}$, let $V_{\rho^b}^\alpha \subseteq V|Q(\rho^b)$ be the corresponding eigen-sub-bundle. The image of $\mathcal{L}_{s^1}^n \boxtimes \mathcal{L}_{s^2}^n$ by $r_{s^1} \boxtimes r_{s^2}$ is the restriction to $Z_{\lambda^1}^s \times Z_{\lambda^2}^s$ of the product of

$$\sum_{\rho^1, \rho^2} \sum_{\alpha, \beta} (\alpha/\beta)^n (V_{\rho^1}^{\alpha} - V_{\rho^2}^{\alpha}) \otimes (V_{\rho^1}^{\beta} - V_{\rho^2}^{\beta})$$

by a constant which does not depend on $n$. Let us recall that if $E$ is a line bundle on a smooth variety $X$, then the element $E - 1$ is a nilpotent element in the ring $K(X)$ (see [CG, Proposition 5.9.4] for instance). Since $(s^1, s^2) \in S$, we are done.

Claim (ii) follows from Claim (i), from the formula

$$R_{n \rho} = T_{(n-1) \rho}^{|2|}(R_{\rho}) T_{(n-2) \rho}^{|2|}(R_{\rho}) \cdots R_{\rho}$$

and from [D, Theorem 4.4(2)], applied to the partial $R$-matrix $R_{\rho}$.

We will prove Claim (iii) as in [KT, Appendix B]. Fix $r \in \mathbb{Z}$. Since $R_{2n \rho} \cdot \Delta^o(\mathbf{x}^r_{i \beta}) \cdot R_{2n \rho}^{-1} = \Delta^o_{2n \rho}(\mathbf{x}^r_{i \beta})$, see §3, the limit $\lim_{n \to \infty} (\Phi_{s^1} \otimes \Phi_{s^2})\Delta^o_{2n \rho}(\mathbf{x}^r_{i \beta})$ is well-defined by Claim (ii). Let us prove that the series $x^r_{i \beta} \boxplus 1 + \sum_{s \geq 0} k^+_{i s} \boxtimes x^r_{i \beta} \boxtimes x^r_{i \beta}$ converges, and coincides with this limit. We will assume that $r \geq 0$, the case $r < 0$
being very similar. The element $\Delta^\circ(x_i^+ - x_i^+ \otimes 1 - k_i \otimes x_i^-)$ is a linear combination of monomials of the form

$$
(\prod_{u=1}^{a} x_{j_u, s_u}^+) \left( \prod_{u=1}^{b} k_{h_u, q_u}^+ \right) \otimes \left( \prod_{u=0}^{c} x_{a_u, r_u}^+ \right),
$$

where $\prod_{u=1}^{a}$ is the ordered product (the term corresponding to $u = 1$ is on the left), $1 \leq s_u \leq r \geq r_u \geq 0$, $\sum_u \alpha_{i-u} - \sum_u a_{j_u} = \alpha_i$, and $\sum_u s_u + \sum_u r_u + \sum_u q_u = r$, see [D, Theorem 4.4(3)] and its proof in [DD, §3.5]. Since $\Delta_{2n \rho}^\circ (x_i^+) = T_{2n \rho}^{[2]} \Delta^\circ (x_{i+2n}^+)$, the element $\Delta_{2n \rho}^\circ (x_i^+) - x_i^+ \otimes 1 - k_i \otimes x_i^+$ is, thus, a linear combination of monomials of the form

$$
(8.2) \quad \left( \prod_{u=1}^{a} x_{j_u, s_u+2n}^+ \right) \left( \prod_{u=1}^{b} k_{h_u, q_u}^+ \right) \otimes \left( \prod_{u=0}^{c} x_{a_u, r_u-2n}^+ \right),
$$

where $0 \leq s_u \leq r + 2n \geq r_u \geq 0$, $\sum_u \alpha_{i-u} - \sum_u a_{j_u} = \alpha_i$, and $\sum_u s_u + \sum_u r_u + \sum_u q_u = r + 2n$. By Claim (i) the image of the monomials (8.2) by $\Phi_{s1} \otimes \Phi_{s2}$ cannot contribute to $\lim_{r \to \infty}$ unless $a = 0$ (use the relation $k_i^+ = [x_i^+, x_i^-]$), and the inequalities $\sum_u (s_u + 2n) + \sum_u q_u \geq 2na$, $\sum_u (r_u - 2n) \leq r - 2na$. Then, $i_0 = i$. Let us consider the monomials

$$
[(h_u), (q_u), r_0] = \left( \prod_{u=1}^{a} k_{h_u, q_u}^+ \right) \otimes x_{i, r_0}^+, \quad \text{where } r_0, q_0 \geq 0 \text{ and } r_0 + \sum u q_u = r.
$$

We have $\Delta^\circ(x_i^+) = A(x_i^+ \otimes 1 - k_i^+ \otimes x_i^-)A^{-1}$, where $A = T_{r \rho}^{[2]} (R_{r \rho})$. By (3.1) the partial $R$-matrix $R_{r \rho}$ is a sum of monomials in the elements $x_{-s}^+ \otimes x_{j-s}^+$ with $s \geq 1$. Moreover, the coefficients of $x_{i,-r_0}^+ \otimes x_{i,r_0}^+$ in $A$, $A^{-1}$ are respectively $c_1, \tilde{c}_1$ (because the coefficients of $x_{i,-r_0}^+ \otimes x_{i,r_0-r}^+$ in $R_{r \rho}$ are respectively $c_1, \tilde{c}_1$, see §3). Thus, the coefficient of $[(h_u), (q_u), r_0]$ in $\Delta^\circ(x_i^+) - x_i^+ \otimes 1 - k_i^+ \otimes x_i^-$ and in

$$
(c_1 x_{i,-r_0}^+ x_i^+ + \tilde{c}_1 x_i^+ x_{i,-r_0}^-) \otimes x_{i,r_0}^+ = (q - q^{-1})[x_{i,i}^+, x_{i,-r_0}^-] \otimes x_{i,r_0}^+ = k_{j,i}^+ \otimes x_i^+
$$

coincide. We are done. A similar argument gives the equality (and the convergence of both sides)

$$
\lim_{n \to \infty} \Phi_{s1} \otimes \Phi_{s2} \Delta_{n \rho}^\circ (x_i^+) = 1 \otimes x_i^+ + \sum_{s \geq 0} x_{i,r+s}^- \otimes k_{i,-s}^+,
$$

for all $r \in \mathbb{Z}$. By (6.2) the $r$-th Fourier coefficient in

$$
x_i^+ (z) \otimes 1 + q^{L_1} \Lambda_{-1} \left( (q^{-1} - q) z^{-1} F_{\lambda_1}^i \right) \otimes x_i^+ (z)
$$

is $x_i^+ \otimes 1 + \sum_{s \geq 0} k_{i,s}^+ \otimes x_{i+r-s}^-$. Similarly, $1 \otimes x_i^+ + \sum_{s \geq 0} x_{i,r+s}^- \otimes k_{i,-s}^-$ is the $r$-th Fourier coefficient in

$$
(x_i^- (z) \otimes 1) \otimes q^{L_2} \Lambda_{-1} \left( (q^{-1} - q) z^{-1} F_{\lambda_2}^i \right) \otimes 1 \otimes x_i^- (z).
$$

Since $(s_1, s_2) \in S$ the class $\Lambda_{-1} ((q^{-1} - q) F_{\lambda_1}^i \otimes L^*_{s_2})$ is well-defined in $U_{s_1} \otimes U_{s_2}$. Recall that, by (7.7), the element $\Delta_{\theta}^\circ(x_i^+)$ is the image by $\theta$ of the $r$-th Fourier coefficient in

$$
(x_i^+ (z) \otimes 1 + q^{L_1} \Lambda_{-1} \left( (q^{-1} - q) F_{\lambda_1}^i \otimes L^* \right) \otimes (1 \otimes x_i^+ (z)),
$$

$$
(x_i^- (z) \otimes 1) \otimes q^{L_2} \Lambda_{-1} \left( (q^{-1} - q) (-L)^* \otimes F_{\lambda_2}^i \right) + 1 \otimes x_i^- (z).
$$
Since $\mathcal{L} \otimes x^+_i(z) = \pm z x^+_i(z)$ by (6.2), we get

$$\lim_{n \to \infty} \theta(\Phi_{s^A} \otimes \Phi_{s^B}) \Delta^{\infty}_{2n\rho}(x^+_i) = \Delta^{\infty}_{U}(x^+_i).$$

We are done because the algebra $U$ is generated by the elements $x^+_i$.

Let us prove Part (iv). The map $\Delta^\prime_U : \tilde{W}'_{\lambda} \to \tilde{W}'_{\lambda_1 \lambda_2}$ is the bivariant localization map (associated to the embedding $(Q_{\lambda_1} \times Q_{\lambda_2}) \times pt \hookrightarrow Q_{\lambda_3} \times pt$). In particular, it is invertible. The map $\Delta^\prime_W$, is invertible either. Thus $\tilde{W}'_{\lambda_1 \lambda_2}$ can be viewed as a $U$-module in two different ways: via $\Delta^\prime_W$, or via the coproduct $\Delta^{\infty}$ and the Kunneth isomorphism $\tilde{W}'_{\lambda_1 \lambda_2} \simeq \tilde{W}'_{\lambda_1} \otimes_{R_{\lambda_1 \lambda_2}} \tilde{W}'_{\lambda_2}$. These representations of $U$ are denoted by $\tilde{W}'_{\lambda_1 \lambda_2}$ and $\tilde{W}'_{\lambda_1 \lambda_2}$. Both $U$-modules are finite dimensional $R_{\lambda_1 \lambda_2}$-vector spaces. They are simple by [N3, Theorem 14.1.2], since the tensor product of two generic simple finite-dimensional $U$-modules is still simple. They have also the same Drinfeld polynomials (see §6.6). Thus they are isomorphic. Obviously,

$$\text{Hom}_{R_{\lambda_1 \lambda_2}}(\tilde{W}'_{\lambda_1 \lambda_2}, \tilde{W}'_{\lambda_1 \lambda_2}) = \text{End}_{R_{\lambda_1 \lambda_2}}(\tilde{W}'_{\lambda_1 \lambda_2}).$$

Let $H_{\lambda^b} \subset G_{\lambda^b}$ be a maximal torus, $b = 1, 2$. Put $H = H_{\lambda_1} \times H_{\lambda_2} \times \mathbb{C}^\times$. Then

$$(Z_{\lambda^b})^{H_{\lambda^b} \times \mathbb{C}^\times} = (Q_{\lambda^b} \times Q_{\lambda^b})^{H_{\lambda^b} \times \mathbb{C}^\times}.$$  

The localization theorem and the Kunneth formula, give an isomorphism

$$\tilde{R}^{H_{\lambda^b} \times \mathbb{C}^\times}(Z_{\lambda^b}) \simeq \text{End}_{\tilde{R}(H_{\lambda^b} \times \mathbb{C}^\times)}(\tilde{R}^{H_{\lambda^b} \times \mathbb{C}^\times}(Q_{\lambda^b})).$$

Moreover, [CG, Theorem 6.1.22] gives

$$\tilde{R}(H) \otimes_{R_{\lambda_1 \lambda_2}} \tilde{W}'_{\lambda_1 \lambda_2} \simeq \tilde{K}^{H_{\lambda_1} \times \mathbb{C}^\times}(Q_{\lambda_1}) \otimes \tilde{K}^{H_{\lambda_2} \times \mathbb{C}^\times}(Q_{\lambda_2})$$

$$\tilde{R}(H) \otimes_{R_{\lambda_1 \lambda_2}} \tilde{U}_{\lambda_1 \lambda_2} \simeq \tilde{K}^{H_{\lambda_1} \times \mathbb{C}^\times}(Z_{\lambda_1}) \otimes \tilde{K}^{H_{\lambda_2} \times \mathbb{C}^\times}(Z_{\lambda_2}).$$

Thus,

$$\tilde{R}(H) \otimes_{R_{\lambda_1 \lambda_2}} \text{Hom}_{R_{\lambda_1 \lambda_2}}(\tilde{W}'_{\lambda_1 \lambda_2}, \tilde{W}'_{\lambda_1 \lambda_2}) \simeq \tilde{R}(H) \otimes_{R_{\lambda_1 \lambda_2}} \tilde{U}_{\lambda_1 \lambda_2}.$$  

Taking the invariants by the Weyl group of $G_{\lambda_1} \times G_{\lambda_2}$, we get an isomorphism

$$\text{Hom}_{R_{\lambda_1 \lambda_2}}(\tilde{W}'_{\lambda_1 \lambda_2}, \tilde{W}'_{\lambda_1 \lambda_2}) \simeq \tilde{U}_{\lambda_1 \lambda_2}.$$  

Let $\tilde{R} \in \tilde{U}_{\lambda_1 \lambda_2}$ be the element corresponding to the isomorphism of $U$-modules $\tilde{W}'_{\lambda_1 \lambda_2} \cong \tilde{W}'_{\lambda_1 \lambda_2}$ which is the identity on the highest weight vectors. Claim (iv) follows from Claim (iii), since $R_{\lambda_1 \lambda_2}$ intertwines the specialized modules, whenever it is defined and invertible.

Claim (v) is immediate, since $R_{2n\rho}$ is a sum of monomials in the elements $x^+_i \otimes x^-_{i-r}$, $i \in I$, $r \geq 1$. \qed
A. Appendix

Let us check that the operators introduced in Section 6 still satisfy the Drinfeld relations. As indicated in Remark 6.3 it is sufficient to check [N3, (1.2.8) and (1.2.10)]. By [N3, Section 10.2] the proof of the first relation is reduced to the equality

\[(q^{-1}V_l/V_l)^{l^-} \otimes (q^{-1}V_k/V_k)^{l^-} \otimes x_{\alpha^3\lambda}^{l^-} \otimes x_{\alpha^3\lambda}^{k^+} =
= (q^{-1}V_l/V_l)^{k^+} \otimes (q^{-1}V_k/V_k)^{k^+} \otimes x_{\alpha^3\lambda}^{k^+} \otimes x_{\alpha^3\lambda}^{l^-}, \]

where \(\alpha^2, \alpha^3, \alpha^4 \in \mathbb{Q}^+\), are such that \(\alpha^4 = \alpha^3 - \alpha_k, \alpha^2 = \alpha^3 + \alpha_l\), and \(k \neq l\). Then (A.1) follows from

\[\begin{align*}
 x_{\alpha^3\lambda}^{l^-} &= (-1)^n_{ik} x_{\alpha^3\lambda}^{l^-} \otimes (q^{-1}V_k/V_k)^{n_{ik}}, \\
 x_{\alpha^3\lambda}^{k^+} &= (-1)^n_{ik} x_{\alpha^3\lambda}^{k^+} \otimes (q^{-1}V_l/V_l)^{n_{ik}}, \\
 f_{\alpha^3\lambda}^{l^-} &= f_{\alpha^3\lambda}^{l^-} - n_{ik}, \\
 f_{\alpha^3\lambda}^{k^+} &= f_{\alpha^3\lambda}^{k^+} + n_{ik},
\end{align*}\]

and the identity \(n_{ik}^+ = n_{ik}^-\). By [N3, Section 10.3] the proof of the second relation is reduced to the equality

\[(q^{-1}V_l/V_l)^{a_{ki}} \otimes (q^{-1}V_k/V_k)^{a_{ki}} \otimes (q^{-1}V_k/V_k)^{f_{\alpha^3\lambda}^{l^+}} \otimes x_{\alpha^3\lambda}^{l^+} \otimes x_{\alpha^3\lambda}^{k^+} =
= (-1)^{a_{ki}} (q^{-1}V_l/V_l)^{a_{ki}} \otimes (q^{-1}V_l/V_l)^{f_{\alpha^3\lambda}^{l^+}} \otimes (q^{-1}V_k/V_k)^{f_{\alpha^3\lambda}^{l^+}} \otimes x_{\alpha^3\lambda}^{l^+} \otimes x_{\alpha^3\lambda}^{k^+}, \]

where \(\alpha^2, \alpha^3, \alpha^4 \in \mathbb{Q}^+\), are such that \(\alpha^2 = \alpha^3 - \alpha_l, \alpha^4 = \alpha^3 - \alpha_k\), and \(k \neq l\). Then (A.2) follows from

\[\begin{align*}
 x_{\alpha^3\lambda}^{l^+} &= (-1)^n_{ik} x_{\alpha^3\lambda}^{l^+} \otimes (q^{-1}V_k/V_k)^{n_{ik}}, \\
 x_{\alpha^3\lambda}^{k^+} &= (-1)^n_{ik} x_{\alpha^3\lambda}^{k^+} \otimes (q^{-1}V_l/V_l)^{n_{ik}}, \\
 f_{\alpha^3\lambda}^{l^+} &= f_{\alpha^3\lambda}^{l^+} - n_{ik}, \\
 f_{\alpha^3\lambda}^{k^+} &= f_{\alpha^3\lambda}^{k^+} - n_{ik},
\end{align*}\]

and the identity \(n_{ik}^+ + n_{ik}^- = -a_{ik}\).

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