LINEAR PROPERTIES OF BANACH SPACES AND LOW DISTORTION EMBEDDINGS OF METRIC GRAPHS

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Abstract. We characterize non-reflexive Banach spaces by a low-distortion (resp. isometric) embeddability of a certain metric graph up to a renorming. Also we study non-linear sufficient conditions for $\ell^n_1$ being $(1 + \varepsilon)$-isomorphic to a subspace of a Banach space $X$.

1. Introduction

In [10], L. Sánchez and the author have shown the following characterization.

Theorem 1. There is a bounded countable metric graph $M_{\ell_1}$ with the following properties.

a) If $M_{\ell_1}$ Lipschitz-embeds into a Banach space $X$ with distortion $D < 2$ (denoted $M_{\ell_1} \rightarrow_D X$), then $X$ contains an isomorphic copy of $\ell_1$.

b) Conversely, if $X$ contains an isomorphic copy of $\ell_1$ then there is an equivalent norm $|\cdot|$ on $X$ such that $M_{\ell_1}$ embeds isometrically into $(X, |\cdot|)$.

In this article we modify the methods used to prove this theorem to obtain some further results. Namely, the aim of Section 2 is to present a similar bounded countable metric graph $M_R$ which satisfies the above theorem with the property “$X$ contains an isomorphic copy of $\ell_1$” replaced by the property “$X$ is non-reflexive” (Theorem 3). In Section 3 we briefly discuss the importance of the renorming in these theorems and answer some quantitative questions left open in [10].

In Section 4 we establish a local version of Theorem 1 a), i.e. a theorem where $M_{\ell_1}$ is replaced by a finite metric space and $\ell_1$ is replaced by a finite dimensional $\ell^n_1$. In fact, using an ultraproduct argument one can get quite immediately from Theorem 1 a) the following.

Theorem 2. Let $(M_n)$ be an increasing sequence of finite subsets of the metric space $M_{\ell_1}$ such that $M_{\ell_1} = \bigcup M_n$. Then for every $\varepsilon > 0$, $D \in [1, 2)$ and every $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that if $M_k \rightarrow_D X$ then $\ell^n_1 \subset X$ with linear distortion less than $1 + \varepsilon$.

The downside is that this theorem nor its proof do not provide any information about the dependence of $k$ on $n$. The goal of Section 4 is dispensing of the ultraproduct argument in order to study the quantitative dependence between the size of the metric space $M_k$ and the dimension of $\ell^n_1$ (Theorem 8).

Finally in Section 5 we will prove Theorem 2 and indicate some directions for possible future research.

Throughout the paper we will need the following notions and notation. A mapping $f : M \rightarrow N$ between metric spaces $(M, d)$ and $(N, \rho)$ is called Lipschitz embedding if there are constants
\( C_1, C_2 > 0 \) such that \( C_1 d(x, y) \leq \rho(f(x), f(y)) \leq C_2 d(x, y) \) for all \( x, y \in M \). The distortion \( \text{dist}(f) \) of \( f \) is defined as \( \inf \frac{C_2}{C_1} \) where the infimum is taken over all constants \( C_1, C_2 \) which satisfy the above inequality. We say that \( M \) Lipschitz embeds (embeds for brevity) into \( N \) with distortion \( D \) (in short \( M \hookrightarrow_D N \)) if there exists a Lipschitz embedding \( f : M \to N \) with \( \text{dist}(f) \leq D \). In this case, if the target space \( N \) is a Banach space, we may always assume (by taking \( C_1^{-1} f \)) that \( C_1 = 1 \).

If \( X \) is a Banach space, we will denote by \( B_X \) (resp. \( S_X \)) the closed unit ball (resp. unit sphere) of \( X \).

For integers \( m \leq n \) we denote \( \llbracket m, n \rrbracket = [m, n] \cap \mathbb{N} \) and \( \llbracket m, \infty \rrbracket = [m, \infty) \cap \mathbb{N} \). For a set \( S \) and an integer \( n \in \mathbb{N} \) we put \( \binom{S}{n} = \{A \subset S : |A| = n\} \), the \( n \)-element subsets of \( S \). If \( x \in \mathbb{R} \) we will denote by \( \lfloor x \rfloor \) the smallest integer \( n \geq x \).

2. Low-Distortion Characterization of Reflexivity

Let \( M_R = \{0\} \cup \mathbb{N} \cup F \) where \( F = \{\llbracket 1, n \rrbracket : n \in \mathbb{N}\} \cup \{\llbracket n, \infty \rrbracket : n \in \mathbb{N}\} \). We introduce on \( M_R \) a graph structure: the edges are couples of the form \( \{0, a\}, a \in \mathbb{N} \), or \( \{a, A\}, a \in \mathbb{N}, A \in F \) and \( a \in F \). Finally, we equip \( M_R \) with the shortest path distance.

Theorem 3. a) Let \( X \) be a Banach space and \( D \in [1, 2) \). If \( M_R \hookrightarrow_D X \) then \( X \) is non-reflexive.

b) Conversely, if \( X \) is non-reflexive then there is an equivalent norm \( \|\cdot\| \) on \( X \) such that \( M_R \) embeds isometrically into \((X, \|\cdot\|)\).

Proof. a) Assume that \( f : M \to X \) satisfies \( f(0) = 0 \) and
\[
d(x, y) \leq \|f(x) - f(y)\| \leq D d(x, y)
\]
for some \( D < 2 \) and all \( x, y \in M_R \). Then, each \( x^* \in B_X \) that norms \( f(\llbracket 1, n \rrbracket) - f(\llbracket n + 1, \infty \rrbracket) \) satisfies
\[
\inf_{k \leq n} \langle x^*, f(k) \rangle - \sup_{k \geq n} \langle x^*, f(k) \rangle \geq 4 - 2D
\]
Hence \( (f(n))_{n \in \mathbb{N}} \subset \text{DB}_X \) satisfies \( \text{dist}(\overline{\{f(i)\}_{i=1}^{n}}) \geq 4 - 2D \) for every \( n \in \mathbb{N} \). By a well known lemma of James [3, page 51], \( X \) is not reflexive.

b) Let us first observe that \( M_R \) embeds isometrically into \((c, \|\cdot\|_\infty)\), the space of convergent sequences. We define \( \Phi : M_R \to c \) by
\[
\Phi(0) = 0
\]
\[
\Phi(\llbracket 1, n \rrbracket) = 21_{[n+1, \infty[}
\]
\[
\Phi(\llbracket n, \infty \rrbracket) = -21_{[1,n[}
\]
\[
\Phi(n) = -1_{[1,n[} + 1_{[n+1, \infty[}
\]
Then \( \Phi \) is an isometric embedding. Indeed, \( \llbracket 1, n \rrbracket \cap \llbracket m, \infty \rrbracket = \emptyset \) iff \( m \geq n + 1 \). In this case the supports of \( \Phi(\llbracket 1, n \rrbracket) \) and \( \Phi(\llbracket m, \infty \rrbracket) \) intersect and we have \( \|\Phi(\llbracket 1, n \rrbracket) - \Phi(\llbracket m, \infty \rrbracket)\|_\infty = 4 \). Otherwise the supports do not intersect and we have \( \|\Phi(I) - \Phi(J)\|_\infty = 2 \). For the possible distances between \( \Phi(\llbracket 1, n \rrbracket) \) and \( \Phi(m) \), resp. \( \Phi(\llbracket n, \infty \rrbracket) \) and \( \Phi(m) \), consult Figure 1.

Now, let \( Y \) be a one-codimensional subspace of \( X \). Clearly, \( Y \) is not reflexive. Let \( \theta \in (0, 1) \). By the proof of James lemma (see [3, page 52]), there exist \( F \in \text{B}_Y^{\ast\ast} \) and sequences \( (x_n) \subset S_Y \),
Figure 1. The purple bars correspond to $\Phi(m)$, the orange bars correspond to $\Phi([1, n])$ in the first line and to $\Phi([n, \infty[)$ in the second line.

$(x^*_n) \subset S_{Y^*}$ such that

\[ F(x^*_n) = \theta \text{ for all } n \in \mathbb{N}, \]
\[ x^*_n(x_k) = \theta \text{ for all } n \leq k, \]
\[ x^*_n(x_k) = 0 \text{ for all } n > k. \]

Observe that $X$ is isomorphic to $Z := Y \oplus \text{span}\{F\} \subset Y^{**}$. We are going to renorm $Z$ and embed $M$ isometrically into this renorming.

Let $y_n \in 2B_Z$ such that $x^*_n(y_n) = 1$. Let $C = \frac{\theta}{12}$. We define, for every $n \in \mathbb{N}$ and every $x \in Z$,

\[ \|x\|_n := \max \{|x^*_n(x)|, C \|x - x^*_n(x)y_n\|\}. \]

Then

\[ \frac{C}{4} \|x\| \leq \|x\|_n \leq \|x\| \]

for every $n \in \mathbb{N}$ and so $|x| := \sup_n \|x\|_n$ defines an equivalent norm on $Z$. We define an embedding $f : M \to (Z, |\cdot|)$ as follows

\[ f(0) = 0 \]
\[ f([1,n]) = 2(F - x_n) \]
\[ f([n, \infty[) = -2x_n \]
\[ f(n) = F - 2x_n. \]

When evaluated against the functionals $(x^*_k)$ this embedding reminds the embedding $\Phi : M \to c$. Indeed, one can check easily that for every $a \in M$ and each $k \in \mathbb{N}$ we have $(x^*_k, f(a)) = \theta \Phi(a)(k)$. Moreover $C \|x - x^*_n(x)y_n\| \leq \theta$ for all $n \in \mathbb{N}$ whenever $x = f(a) - f(b)$. It follows that

\[ |f(a) - f(b)| = \sup_{n \in \mathbb{N}} |x^*_n(f(a) - f(b))| = \theta \|\Phi(a) - \Phi(b)\|_\infty = \theta d(a, b) \]

for every $a, b \in M$. Thus $g = f/\theta$ is the desired embedding. \hfill \square
3. The role of stability

We recall that a metric space \((M, d)\) is stable if for all bounded sequences \((x_n), (y_n) \subset M\) and for all non-principal ultrafilters \(\mathcal{U}, \mathcal{V}\) we have

\[
\lim \lim_{n,\mathcal{U}, m,\mathcal{V}} d(x_n, y_m) = \lim \lim_{m,\mathcal{V}, n,\mathcal{U}} d(x_n, y_m).
\]

The space \(\ell_1\) is stable, see [4, page 212].

The next proposition shows at once that in general we need to pass to a renorming in both theorems 1 b) and 3 b).

**Proposition 4.** Let \(C > 1\). Let \(M\) be uniformly discrete and bounded metric space with the property that \(M \hookrightarrow X, D < C\) only if \(X\) is non-reflexive. Then \(M \hookrightarrow \ell_1 \Rightarrow D \geq C\).

**Proof.** Let \(\Phi : M \hookrightarrow \ell_1, D < C\). Then \(\Phi(M)\) is uniformly discrete, bounded and stable (see [4]). Baudier and Lancien [2] have shown that stable metric spaces nearly isometrically embed into the class of reflexive spaces. In the uniformly discrete and bounded case that means that there exists a reflexive space \(X\) such that \(\Phi(M) \hookrightarrow X\), thus \(M \hookrightarrow X\) which is impossible. \(\Box\)

**Remark 5.** If \((x_n) \subset B_X\) is a 1-separated sequence, then any bijection \(f : M_R \to \{x_n\}\) satisfies \(\text{dist}(f) \leq 8\). Thus \(M_R\) embeds with distortion 8 into any infinite-dimensional Banach space. We do not know whether \(M_R\) embeds into some reflexive space \(X\) with distortion 2. In other words, we do not know whether the constant 2 in Theorem 3 a) is optimal.

On the other hand, let \(\rho : M_R \times M_R \to \mathbb{R}\) be defined by \(\rho(0, n) = \frac{2}{3}, \rho(0, A) = \frac{4}{3}, \rho(n, A) = 2, \rho(n, m) = \frac{4}{3}\) and \(\rho(A, B) = \frac{8}{3}\) for all \(n \neq m \in \mathbb{N}\) and all \(A \neq B \in F \subset M_R\). Then \(\rho\) is a stable metric on \(M_R\) and \(\text{dist}(\text{Id}) = 3\) for the identity \(\text{Id} : (M_R, d) \to (M_R, \rho)\). To see the stability consider the following isometric embedding \(g : (M_R, \rho) \to \ell_1(M_R)\) defined by \(g(0) = 0, g(n) = \frac{2}{3}e_n\), for every \(n \in \mathbb{N}\), and \(g(A) = \frac{4}{3}e_A\), for every \(A \in F\). It follows, using again [2], that there is a reflexive Banach space \(X\) such that \(M_R \hookrightarrow X\).

Finally let us mention that the method using [2] does not work for distortions less than 3 as it follows from the next lemma that \(M_R\) does not embed into a stable metric space with distortion less than 3.

**Lemma 6.** Let \((M, d)\) be a metric space containing sequences \((x_n)\) and \((y_n)\) such that

\[
\lim \lim_{n, m} d(x_n, y_m) \geq C \lim \lim_{m, n} d(x_n, y_m).
\]

Then \(M\) does not embed into any stable metric space with distortion \(D < C\).

**Proof.** Let \(D < C\) and assume that

\[
\text{sd}(x, y) \leq d(f(x), f(y)) \leq sDd(x, y)
\]

for some \(f : M \to N\) and some stable metric space \(N\). Then

\[
\text{sd} \lim_{n, m} d(x_n, y_m) \leq \lim \lim_{n, m} d(f(x_n), f(y_m)) = \lim \lim_{m, n} d(f(x_n), f(y_m))
\]

\[
\leq sD \lim \lim_{m, n} d(x_n, y_m) < sC \lim \lim_{m, n} d(x_n, y_m) \leq s \lim \lim_{m, n} d(x_n, y_m)
\]

which is impossible. \(\Box\)
 Remark 7. We recall the definition of the space $M_{\ell_1}$. The vertex set is $M_{\ell_1} = \{0\} \cup \mathbb{N} \cup F$ with $F = \{A \subseteq \mathbb{N} : 1 \leq |A| < \infty\}$. A pair $\{a, b\}$ is an edge either if $a = 0$ and $b \in \mathbb{N}$, or if $a \in \mathbb{N}$, $b \in F$ and $a \notin b$. The space $M_{\ell_1}$ is equipped with the shortest path metric.

In [10], we asked for the best constant $D$ such that $M_{\ell_1} \hookrightarrow_{D} \ell_1$ and also for the value of $d_{BM}(\ell_1, F(M_{\ell_1}))$ where $F(M_{\ell_1})$ is the Lipschitz free space of $M_{\ell_1}$.

First, using the above lemma, one can easily see that $M_{\ell_1}$ does not embed with distortion less than $3$ into any stable space. Second, if we define $\rho : M_{\ell_1} \times M_{\ell_1} \rightarrow \mathbb{R}$ as in Remark 5, we have that $\text{dist}(Id) = 3$ for the identity map $Id : (M_{\ell_1}, d) \rightarrow (M_{\ell_1}, \rho)$. Defining the isometric embedding $g : (M_{\ell_1}, \rho) \rightarrow \ell_1$ as in Remark 5, this already gives that $D = 3$.

But it also follows from the theory of Lipschitz free spaces that $\text{dist}(\hat{Id}) = 3$ where $\hat{Id} : F(M_{\ell_1}, d), F(M_{\ell_1}, \rho))$ is the unique linear extension of $Id$. Moreover, $F(M_{\ell_1}, \rho) \equiv \ell_1$. This can be seen by noticing that $F(M_{\ell_1}, \rho)$ is isometric to a negligible subset of an $\mathbb{R}$-tree which contains all the branching points, and applying [6]. Thus $d_{BM}(\ell_1, F(M_{\ell_1})) = 3$.

4. Low-distortion representation of $\ell_1^n$

In this section we state and prove a quantitative version of Theorem 2 for a particular choice of spaces $M_n \subset M_{\ell_1}$. Having in mind the definition of the space $M_{\ell_1}$ (see Remark 7), the most natural choice of the spaces $M_n$ seems to be the following. We put $M_n = \{0\} \cup [1,n] \cup F_n$ where $F_n = 2^{[1,n]} \setminus \{0\}$. The graph structure and the metric are induced by the space $M_{\ell_1}$.

The main result of this section follows.

Theorem 8. a) Let $D \in [1, \frac{4}{3})$ and $n \in \mathbb{N}$. Then $M_n \hookrightarrow_{D} X$ implies that $\ell_1^n$ is $\frac{D}{4 - 3D}$-isomorphic to a subspace of $X$.

b) Let $D \in [1,2)$. For every $\alpha \in (0,1)$ there exists $\eta = \eta(\alpha, D) \in (0,1)$ such that $M_k \hookrightarrow_{D} X$ implies that $\ell_1^{[\eta k]}$ is $\frac{2D}{2 - D}$-isomorphic to a subspace of $X$ whenever $k > \frac{\log_2(\frac{2D}{2 - D})}{1 - \alpha}$.

The isomorphism constant can be arbitrarily reduced, at the cost of augmenting the size of the metric space, by virtue of the following finite version of James’s $\ell_1$-distortion theorem, see Proposition 30.5 in [12]: If $X$ contains a $b^2$-isomorphic copy of $\ell_1^{n^2}$, then $X$ contains a $b$-isomorphic copy of $\ell_1^n$.

For example, in the case a), we get: If $D < \frac{4}{3}$ and $w \geq -\log_2 \left( \frac{\log_2(1 + \varepsilon)}{\log_2(\frac{2D}{2 - 3D})} \right)$, then $M_n^{w} \hookrightarrow_{D} X$ implies that $\ell_1^n$ is $(1 + \varepsilon)$-isomorphic to a subspace of $X$.

In order to prove Theorem 8 we will need the following lemma, which is a finite-dimensional version of Proposition 4 in [11].

Lemma 9. Let $S$ be a set, $K > 0$ and let $(f_i)_{i=1}^n \subset K B_{\ell_\infty}(S)$. Assume that there are $\delta > 0$ and $r \in \mathbb{R}$ such that these functions satisfy that for every $A \subseteq [1,n]$ there is $s \in S$ for which $f_j(s) \leq r < r + \delta \leq f_i(s)$ for all $i \in A$ and for all $j \in [1,n] \setminus A$. Then $(x_i)_{i=1}^n$ is $\frac{2K}{\delta}$-equivalent to the unit vector basis of $\ell_1^n$. 


Lemma 11 (Sauer, Shelah, and Vapnik and Červonenkis). Let \( S \subset 2^{[1,k]} \) such that \(|S| > \sum_{i=0}^{m-1} \binom{k}{i}\) for some \( m \leq k \). Then there is \( H \subset \binom{[1,k]}{\eta_k} \) such that \( \{A \cap H : A \in S\} \geq 2^H \).
Lemma 12. For every $1 \leq m \leq k$ one has
\[ \sum_{i=0}^{m} \binom{k}{i} \leq \left( \frac{ek}{m} \right)^m. \]

Proof of Lemma 10. Let $\eta > 0$ satisfy $2^\alpha > \left( \frac{\varepsilon}{2} \right) ^\eta$. And $m = \lceil \eta k \rceil$. Then, using $\eta < 1$ and Lemma 12,
\[ |S| \geq 2^{\alpha k} > \left( \frac{ek}{\eta k} \right)^\eta \geq \left( \frac{ek}{m-1} \right)^{m-1} \geq \sum_{i=0}^{m-1} \binom{k}{i}. \]
So we get the existence of $H$ by Lemma 11.

Proof of Theorem 8 b). Given $D < 2$, let $c = \lceil \frac{2D}{2-D} \rceil - 1$. Let $k > \frac{\log(c-1)}{1-\alpha}$. Assume that $f : M_k \to X$ satisfies $f(0) = 0$ and
\[ d(x,y) \leq \|f(x) - f(y)\| \leq Dd(x,y) \]
for all $x,y \in M_k$. To simplify notation we will denote $x' := f(x)$ for each $x \in M_k$. Let $r_j := -D + j(D - 2)$ for $j \in \{1,c\}$. Then any closed interval of length $4 - 2D$ which is contained in $[-D,D]$ contains at least two different points $r_j$. We claim that for every $A \in 2^{[1,k]}$ there are $j \in [1,c-1]$ and $x^* \in B_{Y^*}$ such that
\[ \langle x^*, b' \rangle \leq r_j < r_{j+1} \leq \langle x^*, a' \rangle \]
for all $a \in A$ and all $b \in B := [1, k] \setminus A$. Indeed, we take $x^* \in B_{Y^*}$ such that
\[ \langle x^*, A' - B' \rangle = \|A' - B'\| \text{ if } A \neq \emptyset \neq B \]
\[ \langle x^*, A' \rangle = \|A'\| \text{ if } B = \emptyset \]
\[ \langle x^*, -B' \rangle = \|B'\| \text{ if } A = \emptyset \]
In the first case we get for all $a \in A$ and all $b \in B$ that $\langle x^*, a' - b' \rangle \geq 4 - 2D$. Moreover $\langle x^*, a' \rangle, \langle x^*, b' \rangle \in [-D,D]$ so the claim follows by the choice of $(r_j)$. In the second case we get for all $a \in A$ that $\langle x^*, a' \rangle \geq 2-D$, and in the last case we get for all $b \in B$ that $\langle x^*, b' \rangle \leq D - 2$. In both cases we use that the interval $[D - 2, 2-D]$ contains at least two different $r_j$ to finish the proof of the claim.

Let us choose for every $A \subset [1,k]$ one such $j$ which we will denote $j_A$. By the pigeonhole principle, there is some $j \in [1,c-1]$ such that $|S| \geq \frac{2^k}{c-1}$ for $S = \{ A \in 2^{[1,k]} : j_A = j \}$. By the choice of $k$ we have $\frac{2^k}{c-1} \geq 2^{ak}$. Let $\eta \in (0,1)$ and $H \in \left( \frac{[1,k]}{\lceil m \rceil} \right)$ be as in Lemma 10. Applying Lemma 9 we can see that $(f(i))_{i \in H}$ is $\frac{2D}{2-D}$-equivalent to the unit vector basis of $\ell_1^{\lceil \eta k \rceil}$.

5. ULTRAPRODUCT TECHNIQUES FOR LOW DISTORTION REPRESENTATION

Let us first see the proof of Theorem 2.

Proof of Theorem 2. Suppose that the assertion is not true for some $\varepsilon > 0$, $D \in [1,2]$ and $n \in \mathbb{N}$. Then for every $k$ there is $X_k$ such that $M_k \xrightarrow{D} X_k$ and $\ell_1^n$ is not $(1+\varepsilon)$-isomorphic to a subspace of $X_k$. Let $X = \prod X_k$ be an ultraproduct along some free ultrafilter on $\mathbb{N}$. Then $M_{\ell_1} \xrightarrow{D} X$ and so, by Theorem 1 a), $\ell_1$ embeds into $X$ linearly and does so arbitrarily well (by
James’s $\ell_1$-distortion theorem, see e.g. [1]) . Therefore $\ell_1^n$ embeds into $X$ arbitrarily well and in particular it must be $(1 + \frac{\epsilon}{2})$-isomorphic to a subspace of some $X_k$. Contradiction. \hfill \Box

Let us mention the following folklore result which according to G. Lancien goes back to G. Schechtman.

**Theorem 13.** Let $X$ be a Banach space such that $\dim X < \infty$. Then for every $D \geq 1$ and $\epsilon > 0$ there is a finite set $F \subset X$ such that for any given Banach space $Y$ the fact $F \hookrightarrow_D Y$ implies that $X$ is $(D + \epsilon)$-isomorphic to a subspace of $Y$.

**Proof.** Let $(F_n) \subset 2^X$ be any increasing family of finite sets such that $\bigcup F_n = X$. Let us assume that there are $D > 1$ and $\epsilon > 0$ such that for every $k \in \mathbb{N}$ there is a Banach space $Y_k$ and an embedding $f_k : F_k \hookrightarrow_D Y_k$ but $X$ is not $(D + \epsilon)$-isomorphic to any subspace of $Y_k$. Then $\Phi(x) := \left( (f_n(x))_n \right)$ for $x \in \bigcup F_n$ and extended by continuity to the whole $X$ is an embedding of $X$ into $\prod Y_k$ with distortion $D$. By the theorem of Heinrich and Mankiewicz [4, Theorem 7.9] $X$ linearly embeds into $(\prod Y_k)^*$ with distortion $D$. By local reflexivity $X$ embeds linearly into $\prod Y_k$ with distortion $D + \epsilon/3$ and therefore $X$ embeds into some $Y_k$ linearly with distortion $D + 2\epsilon/3$ which is a contradiction. \hfill \Box

**Remark 14.** 1) Clearly the above theorem is qualitatively better than Theorem 2 as it works for any space $X$ and withouth any restriction on the distortion. Also the set $F$ is isometrically in $X$. On the other hand, any other information on the nature of $F$ is completely inaccessible. It could be interesting to give a concrete example of such sets for a given finite dimensional space $X$.

2) When $X = \ell^n_p$ with $1 \leq p \leq 2$, one can say more. Let $C^n_p$ be the “$n$-cube” equipped with the $\ell_p^n$-norm. We recall a result of Bourgain, Milman and Wolffson [5, page 297] which says: Let $1 \leq p \leq 2$. Assume that there exists $D > 0$ such that $C^n_p \hookrightarrow_D Y$ for every $n$. Then for every $\epsilon > 0$ and $n \in \mathbb{N}$ there is a subspace of $Y$ which is $(1 + \epsilon)$-isomorphic to $\ell^n_p$.

Essentially the same proof as the one of Theorem 2 gives therefore the following. Let $1 \leq p \leq 2$. Then for every $\epsilon > 0$, $D > 0$ and $n \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that for every Banach space $Y$ we have that $C^n_p \hookrightarrow_D Y$ implies that $\ell^n_p$ is $(1 + \epsilon)$-isomorphic to a subspace of $Y$.

The dependence of $k$ on $n$ does not seem to follow from the proof in [5].

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