A simple 3-parameter random walk model for monthly fluctuations $\Delta T$ of a temperature $T$ is introduced. Applied to a time range of 170 years, temperature fluctuations of the model produce for about 14% of the runs warming that exceeds the observed global warming of the earth surface temperature from 1850 to 2019. On the other hand, there is a 50% likelihood for runs of our model resulting in cooling. If a similar random walk process can be used as an effective model for fluctuations of the global earth surface temperature, effects due to internal and external forcing could be considerably over- or underestimated.

Global warming has become a subject of major research efforts [1]. Figure 1 relies on a data set of the UK Met Office Hadley Center [2] and depicts monthly estimates for the global earth surface temperature from 1850/01 to 2019/09. From about 1975 on a global warming trend is clearly visible and consistent with satellite based monthly temperature estimates, which are available from 1979 on [3]. The zero reference line adapted in Fig. 1 is close to the mean value of all shown data, $T = -0.088 \, ^\circ C$. In this paper all temperature references (in Celsius) are with respect to the zero line and not with respect to the Celsius scale.

The global temperature is chosen for our considerations, because one expects significantly less seasonal variations than, for instance, for the temperatures of the northern or southern hemisphere. Though some asymmetry due to the seasons remains, it is not immediately visible from the graph of Fig. 1. Instead, one realizes already at a first glance that the temperature curve is not smooth, but fluctuates heavily from month to month. How does this happen? The temperatures of Fig. 1 are weighted averages over measurements in a narrow band close to the surface of the earth. Energy exchanges in the horizontal directions are balanced to zero by energy conservation, while there can be mismatches of incoming and outgoing energies in the vertical directions.

Heating comes mainly from the radiation of the sun. About 30% of the sun’s radiation gets immediately reflected back into outer space. What is left over heats the ground, the oceans and the atmosphere. Ultimately the heat escapes in the form of mostly infrared radiation. To avoid continuous heating or cooling, the energy of the incoming radiation has to agree in average with that of the outgoing radiation. Due to statistical fluctuations this balance does not hold at every instance. For example, if there are clouds at daytime, more sunlight will immediately be reflected back into space than on a clear day. The effect exceeds the trapping of infrared radiation by the clouds and it will be cooler than on a sunny day under otherwise similar conditions. At night the opposite is true, because only the trapping effect remains. Heat exchange with the oceans and the earth surface has also random components, and so on.

It is shown here that accidental fluctuations of a random walk process with mean $\langle \Delta T \rangle = 0$, i.e., a process that has no preference for increasing or decreasing the temperature, can exhibit similar temperature drifts as shown in Fig. 1.

Let us label the monthly temperatures of Fig. 1 by $T(i), i = 1, \ldots, 2037$. The corresponding dates are ap-
Monthly temperature fluctuations are then defined by

$$\Delta T(i) = T(i) - T(i-1), \quad i = 2, \ldots, 2037,$$

and their time series is depicted in Fig. 2. On closer inspection of the data one finds that the 13 largest monthly temperature fluctuations all fall into the time period before 1900 although far more data points exist from 1900 on. This may be a real effect or due to the larger uncertainties of the older data. In either case the older data enlarge substantially the standard deviation of the empirical distribution

$$\sigma_e = \sqrt{\frac{1}{2035} \sum_{i=2}^{2037} (\Delta T(i) - \overline{\Delta T})^2} = 0.138 \, ^\circ\text{C}. \quad (3)$$

Here $\overline{\Delta T} = 0.000695 \, ^\circ\text{C}$ is the mean value of the monthly temperature fluctuations:

$$\overline{\Delta T} = \frac{\sum_{i=2}^{2037} \Delta T(i)}{2036} = \frac{T(2037) - T(1)}{2036} = 1.415 \, ^\circ\text{C} \quad (4)$$

where $T(1) = -0.7 \, ^\circ\text{C}$ and $T(2037) = 0.715 \, ^\circ\text{C}$ are the temperatures for the first and last month considered.

Gaussian creates temperature fluctuations in the more recently observed range of natural variability. Note that the mean value of the Gaussian is chosen to be $\overline{x} = 0$, whereas $\Delta T$ is non-zero (4) for the empirical $\Delta T$ distribution. Further, Gaussian fluctuations are statistically independent, whereas this is not expected for the observed fluctuations. The Gaussian form is chosen for simplicity. Similar arguments could be made using the empirical histogram directly to generate uncorrelated or correlated random walk updates in some kind of bootstrap approach.

Although the standard deviation $\sigma_g = 0.1 \, ^\circ\text{C}$ is chosen smaller than $\sigma_e$ (3) of the observed fluctuations, Gaussian temperature fluctuations with $\sigma_g = 0.1 \, ^\circ\text{C}$ are too large to comply with the observed temperature drift over the last 170 years. This is shown next by investigating a sample of $n_{\text{rpt}} = 1001$ Gaussian random walks, each of 2040 steps generated with the probability density (5). For all simulations reported here we use Marsaglia random numbers and other software from [4].

![Probability density of global monthly temperature fluctuations.](image)

The probability density of $\Delta T$ is depicted in Fig. 3 in form of a histogram together with the Gaussian probability density

$$H = \frac{1}{\sigma_g \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{x}{\sigma_g} \right)^2 \right]. \quad (5)$$

of standard deviation $\sigma_g = 0.1 \, ^\circ\text{C}$. This Gaussian fits the peak of the histogram quite well, while it ignores the outliers. For our purposes it is sufficient that the

![Typical temperature random walk for Gaussian fluctuations with $\sigma_g = 0.1 \, ^\circ\text{C}$.](image)

For each of the random walks the minimum $T_{\text{min}}(i_{\text{rpt}})$ and the maximum $T_{\text{max}}(i_{\text{rpt}}), \quad i_{\text{rpt}} = 1, \ldots, 1001$, temperature were recorded. The differences $T_{\text{max}}(i_{\text{rpt}}) - T_{\text{min}}(i_{\text{rpt}})$ are then found in the range 2.7 $^\circ\text{C}$ to 18.3 $^\circ\text{C}$ with a median of 6.7 $^\circ\text{C}$. An example with $T_{\text{max}} - T_{\text{min}} = 5.0^\circ\text{C}$ is given in Fig. 4. Over the considered time period temperature excursions were far too large to resemble those of Fig. 1. They increase proportional to $\sigma_g \sqrt{n} (n = 2040)$, and $\sigma_g = 0.1 \, ^\circ\text{C}$ is too large to allow the random walk to stay within the range given by the observations of Fig. 1. On the other hand, we cannot change $\sigma_g$ much because of the $\Delta T$ time series of Fig. 2. The requirement $\overline{x} = 0$ is not sufficient to prevent a runaway to very hot or cool temperatures. Some kind of “thermostat” is needed, which drives the random walk back to the neighborhood of $\overline{x} = 0$ without changing its variance.

We achieve this by turning a proposed update $\Delta T = x$ into $\Delta T = -x$ with a suitable likelihood. Due to $\overline{x} = 0$
the variance \( \sigma^2 = \langle x^2 \rangle \) is invariant under \( x \to -x \). For the construction of a suitable likelihood we introduce the probabilities

\[
p = \int_{-\infty}^{\sigma} g(x) \, dx \quad \text{and} \quad q = \int_{\sigma}^{\infty} g(x) \, dx ,
\]

where \( g(x) \) is the Gaussian probability density (5) with \( \sigma_g = 1 \) and \( a > 0 \) a free parameter. The update will now be \( \Delta T(i) = \pm |x| \) where the sign is determined as follows:

\[
\Delta T(i) = \begin{cases} 
-|x| & \text{with probability } p, \\
+|x| & \text{with probability } q .
\end{cases}
\]

In each case the larger of the probabilities \( p \) and \( q \) drives the random walk defined by

\[
T(i - 1) \to T(i) = T(i - 1) + \Delta T(i)
\]

closer to zero without changing the variance. These fluctuations are no longer statistically independent. In the following they are called “modified” Gaussian fluctuations. The large fluctuations around zero of Fig. 5 exhibit a typical example of a random walk for the temperatures \( T(i) \) obtained from our modified Gaussian fluctuations. It starts with \( T(1) = 0 \), and \( a = 4 \) is used for the free parameter.

To reproduce the temperature increase seen in Fig. 1 one may now add a smooth curve representing systematic causes like internal and external forcings. However, they are not the subject of this paper. Here we investigate whether a purely statistical random walk model can create similar temperature drifts. The idea is to achieve this by adding to the large modified Gaussian fluctuations ordinary Gaussian fluctuations with a standard deviation \( \sigma_g' \ll 0.1 \, ^\circ C \) scaled so that a temperature difference like the one depicted in Fig. 1 is within reach over the given time range. A good choice for this are Gaussian random walks with \( \sigma_g' = 0.02 \, ^\circ C \). An example is given by the “small fluctuations” of Fig. 5. This assumes that in the range of presently relevant temperatures there is no effective backdriving mechanism for these small temperature fluctuations.

\[
T_1 = -0.2780 \, ^\circ C , \quad (T^2 - T_1^2) = 0.9904 \, ^\circ C .
\]
For our model we sorted the \( n_{\text{rpt}} = 10001 \) generated temperature random walks in increasing order with respect to \((T_2 - T_1)(i_{\text{rpt}})\). Of those 1372 exceeded the value of 0.99 [\(^{\circ}\)C].

![Graph showing cumulative distribution function (F).

FIG. 7. Cumulative distribution function \( F(T_2 - T_1) \) of the random walk model temperature increase \( T_2 - T_1 \).

![Graph showing examples of random walks from our model.

FIG. 8. Examples of random walks from our model: Largest temperature increase \( T_2 - T_1 > 0 \), smallest absolute value for its maximum minus its minimum temperature, largest temperature decrease \( T_2 - T_1 < 0 \).

Figure 7 shows the cumulative distribution function \( F(T_2 - T_1) \) from our simulation. In the upper right the 13.7\% q-tile is indicated for which the warming is greater than the observed warming. In the lower left we have the 50\% range of values for which there is cooling by the random process. Figure 8 depicts three extreme cases from our sample of 10001 random walks.

In conclusion, the simple stochastic model of this paper exhibits the variability needed to describe the global temperature changes for the time period over which global temperature records exist.

The large monthly temperature fluctuations of Fig. 2 are an observed fact of the real world. In Fig. 3 the central part of their empirical probability density is approximated by a Gaussian with a variance suggested by the empirical data. However, Gaussian fluctuations with this variance lead to temperature excursions which are far too large to comply with observations. An example is shown in Fig. 4. This problem is overcome by correlating moves in a suitable statistical way (7), where the particular mechanism used is not really of importance for our present discussion.

The challenge remaining is to exclude the existence of subleading small fluctuations which are effectively uncorrelated over a temperature range larger or equal to that of Fig. 1. In our illustration subleading Gaussian fluctuations with a variance of 2.5\% of the variance (20\% of the standard deviation) of the leading fluctuations are chosen. That is about the maximum allowed. Substantially larger subleading Gaussian fluctuations are excluded by the observations.

The assumed subleading fluctuations make it impossible to identify the causes for the temperature increase in Fig. 1. With a probability of almost 14\% one would have a natural temperature increase that is larger than the observed one, while with a probability of 50\% causal reasons like external forcing by an increase of the \( \text{CO}_2 \) contents of the atmosphere would have to be even larger than indicated by the observations. Phenomena like the medieval warm period and the little ice age would need no causal explanations anymore.

These problems would be demagnified for subleading Gaussian fluctuations with a variance smaller than the value assumed here. But at their core the problems would not go away. Observations can only be used to support internal or external forcing mechanisms when the stochastic climate noise is under control. Otherwise one has to rely on theoretical calculations alone.

[1] Intergovernmental Panel on Climate Change (IPCC), Climate Change 2013: The Physical Science Basis. Available at www.ipcc.ch/report/ar5/wg1 (an update is announced for 2021).
[2] Met Office Hadley Centre observations dataset, Nov. 8, 2019: Global (NH+SH)/2 monthly downloaded from https://www.metoffice.gov.uk/hadobs/hadcrut4/data/current/time_series/HadCRUT.4.6.0.0.monthly_ns_avg.txt
[3] J. Christy and R. Spencer dataset, Nov. 8, 2019: https://www.nsstc.uah.edu/data/msu/v6.0/tlt/uahncdc_v6.0.txt
[4] B.A. Berg, Markov Chain Monte Carlo Simulations and Their Statistical Analysis (With Web-Base Fortran Code), World Scientific, 2004.