A field theoretic approach to the energy momentum tensor for theories coupled with gravity

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We provide a field-theoretic algorithm of obtaining energy momentum tensor (EMT) for gravitationally coupled scalar field theories. The method is equally applicable to both minimal and non-minimal coupling. The algorithm illuminates the connection between the EMT, obtained by functional variation of the metric, and local balance of energy and momentum.

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the curvature of the background spacetime shows up in the nonvanishing second derivatives of the metric. The crux of our method is to view the corresponding action as an auxiliary field theory in a flat Minkowski spacetime which has an infinitesimal overlap with the tangent space at P. The whole process can be summarised in the following algorithm:

1. From the original theory subtract the dynamic part for pure gravity (i.e. Einstein–Hilbert action). This will give the scalar field theory interacting with external gravity.

2. Express the resulting action in the adapted coordinates, at a point P, which have the properties

\[ g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \]
\[ \partial_\lambda g_{\mu\nu} = 0 \]
\[ \partial_\lambda \partial_\rho g_{\mu\nu} \neq 0 \]

Note carefully that the conditions (2, 3 and 4) hold in the curved space-time within the infinitesimal patch around point P, in locally inertial frame. Uptill now \( g_{\mu\nu} \) refers to the metric of the curved space-time.

3. The action obtained in step (2) will now be considered as a new field theory in the Minkowskian space which has an infinitesimal overlap with the tangent space at the point P, with metric \( \eta_{\alpha\beta}. \phi \) and \( g_{\mu\nu} \) are now respectively scalar and second rank tensor fields in this space. This will be henceforth referred to as the auxiliary field theory. \( \phi \) and \( g_{\mu\nu} \) are now respectively scalar and second rank tensor fields in this space. This will be henceforth referred to as the auxiliary field theory. To avoid confusion we will denote the second rank tensor field by \( H_{\mu\nu} \) instead of \( g_{\mu\nu} \). Obviously, only second derivatives of \( H_{\mu\nu} \) will appear in this theory. Note that the dynamics of this newly defined second rank tensor field \( H_{\mu\nu} \) is not subject to the conditions (2, 3, 4). One should always remember that within the perview of the auxiliary field theory \( H_{\mu\nu} \) has nothing to do with the metric either of the curved space-time or of the Minkowskian space-time.

4. Now compute the EMT by applying Noether’s theorem using the translation symmetry of the new theory. Since the energy-momentum four vector is a generator of translation symmetry the EMT actually corresponds to the balance of energy and momentum all over the flat space including the tangent space at point P.

5. The auxiliary field theory becomes identical with that obtained in (2) in the overlap if the fields \( H_{\mu\nu} \) are replaced by \( g_{\mu\nu} \) and the constraints (2, 3, 4) are imposed. From steps (1), (2) and (3) it can be understood that the EMT obtained in step (4) is same as the EMT of the original scalar field theory (5) interacting with gravity locally at a point P in the region of overlap.

6. The final task is to express the EMT in step (5) in terms of general coordinates in curved space-time. Care should be taken in this step so that all the terms that appear in this EMT has unambiguous geometric meaning. What we mean by the phrase ‘unambiguous geometric meaning’ is clarified in the following, see below equation (20).

7. The EMT thus obtained should serve as the source for the gravitational field in the original theory.

From the above description of the proposed method it is apparent that the procedure is applicable for a generic scalar field theory coupled to gravity. For definiteness we take a non-minimally coupled quintessence model to illustrate our method though the same algorithm is applicable in principle for different dynamics of the scalar field, such as k-essence.

FIG. 1: Field theory defined on the tangent space
We start with the following action:

\[
S = \int d^4 x \sqrt{-g} \left[ \frac{M_{pl}^2}{2} (1 - F(\phi)) R - \frac{1}{2} g^{\mu \nu} (\nabla_\mu \phi)(\nabla_\nu \phi) - V(\phi) \right]
\]  

(5)

Note that a non-zero \(F(\phi)\) signifies nonminimal coupling. In (5), \(M_{pl}^2\) is given by \((8\pi G)^{-1}\).

As the first step of our procedure we abstract from (5) the form of the theory coupled with curved space-time when the metric is external. This leads to the Lagrangian

\[
\mathcal{L}_\phi = -\sqrt{-g} \left[ \frac{M_{pl}^2}{2} F(\phi) R + \frac{1}{2} g^{\mu \nu} (\nabla_\mu \phi)(\nabla_\nu \phi) + V(\phi) \right]
\]  

(6)

Note that metric in (6) is a background field and influences the motion through the coupling with the scalar field through curvature.

So far, the coordinate system was general, charting the curved spacetime. The next step of our method is to concentrate at the neighbourhood of the point P (see fig.1) and use the adapted coordinates. The Riemann tensor is expressed, using (2), (3), in the form

\[
R_{\alpha \beta \gamma \delta} = \frac{1}{2} (g_{\beta \gamma,\alpha \delta} - g_{\alpha \gamma,\beta \delta} - g_{\beta \delta,\alpha \gamma} + g_{\alpha \delta,\beta \gamma})
\]  

(7)

The Ricci tensor is

\[
R_{\beta \delta} = \eta^{\alpha \gamma} R_{\alpha \beta \gamma \delta}
\]  

(8)

The Ricci scalar is obtained on another contraction as,

\[
R = \eta^{\alpha \gamma} \eta^{\beta \delta} R_{\alpha \beta \gamma \delta} = - \left( \eta^{\alpha \gamma} \partial_\lambda \partial^\lambda g_{\alpha \beta} - \partial^\alpha \partial^\beta g_{\alpha \beta} \right)
\]  

(9)

The Lagrangian (6) can now be written in terms of the adapted coordinates, using (9) as

\[
\mathcal{L} = \frac{M_{pl}^2}{2} F(\phi) \left\{ \eta^{\alpha \gamma} \partial_\lambda \partial^\lambda g_{\alpha \beta} - \partial^\alpha \partial^\beta g_{\alpha \beta} \right\} - \frac{1}{2} \eta^{\mu \nu} (\partial_\mu \phi)(\partial_\nu \phi) - V(\phi)
\]  

(10)

Note that due to the condition (2), \(\sqrt{-g} = 1\). The region in which the adapted coordinates are defined is the tangent space at the point P. The range of these can however, be extended to the whole flat (Minkowski) space that has a region of overlap with the tangent space at P.

As the next step of our algorithm we will construct the Lagrangian of the auxiliary field theory in these coordinates as

\[
\mathcal{L} = \frac{M_{pl}^2}{2} F(\phi) \left\{ \eta^{\alpha \gamma} \partial_\lambda \partial^\lambda H_{\alpha \beta} - \partial^\alpha \partial^\beta H_{\alpha \beta} \right\} - \frac{1}{2} \eta^{\mu \nu} (\partial_\mu \phi)(\partial_\nu \phi) - V(\phi)
\]  

(11)

where \(H_{\mu \nu}\) is a second rank tensor field (see the discussion in (3)). These fields, along with the scalar field \(\phi\) form a closed system. The theory has translation symmetry (as a part of the more general Poincare symmetry). Note that no constraint is imposed on \(H_{\mu \nu}\).

The equation of motion for the \(\phi\) field is obtained as usual,

\[
\Box \phi - \frac{M_{pl}^2}{2} F'(R_H) - V' = 0
\]  

(12)

In the above, \(A'\) denotes differentiation of \(A(\phi)\) with respect to \(\phi\). \(R_H\) is given by

\[
R_H = - \left( \eta^{\alpha \gamma} \partial_\lambda \partial^\lambda H_{\alpha \beta} - \partial^\alpha \partial^\beta H_{\alpha \beta} \right)
\]  

(13)

A characteristic feature in the Lagrangian (11) is the presence of second derivatives of the tensor field \(H_{\mu \nu}\). Thus it is a higher derivative field theory [12]. The equation of motion for the \(H_{\mu \nu}\) field is [13],

\[
\partial_\mu \partial_\nu \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \partial_\nu H_{\alpha \beta})} \right] - \partial_\mu \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu H_{\alpha \beta})} \right] + \frac{\partial \mathcal{L}}{\partial H_{\alpha \beta}} = 0
\]  

(14)
which gives

\[ \frac{\partial^\nu}{\partial x^\nu} \{ \eta^{\alpha\beta} \eta^{\mu\nu} - \eta^{\mu\alpha} \eta^{\nu\beta} \} = 0 \]  

(15)

Apprently this equation makes no reference to the field \( H_{\mu\nu} \). However, eliminating \( F(\phi) \) from equation (15) and (12), the explicit equation of motion for the fields \( H_{\mu\nu} \) can be obtained. Also one can alternatively write equation (15) as,

\[ \partial^\alpha \partial^\beta F - \eta^{\alpha\beta} \square F = 0 \]

(16)

The construction of the EMT, \( T^{\mu\nu} \) will now be detailed. Following Noether’s prescription,

\[ T^{\mu\nu} = \frac{\partial L_H}{\partial (\partial_\mu \phi)} (\partial^\nu \phi) - \eta^{\mu\nu} L_H + \frac{\partial L_H}{\partial (\partial_\lambda \partial_\mu H_{\alpha\beta})} (\partial_\lambda \partial^\nu H_{\alpha\beta}) - \partial_\lambda \left( \frac{\partial L_H}{\partial (\partial_\lambda \partial_\mu H_{\alpha\beta})} \right) (\partial^\nu H_{\alpha\beta}) \]

(17)

which gives

\[ T^{\mu\nu} = - (\partial^\nu \phi) (\partial_\mu \phi) + \eta^{\mu\nu} \left[ \frac{M^2_{pl}}{2} F(\phi) R_H + \frac{1}{2} \eta^{\alpha\beta} (\partial_\alpha \phi) (\partial_\beta \phi) + V(\phi) \right] - \frac{M^2_{pl}}{2} \left( \eta^{\lambda\mu} \eta^{\sigma\beta} - \eta^{\mu\sigma} \eta^{\lambda\beta} \right) (\partial_\nu \partial_\sigma H_{\lambda\beta}) F(\phi) + \partial_\sigma \left[ \frac{M^2_{pl}}{2} \left( \eta^{\lambda\mu} \eta^{\sigma\beta} - \eta^{\mu\sigma} \eta^{\lambda\beta} \right) \partial_\nu \{ H_{\lambda\beta} \partial_\sigma F(\phi) \} \right] - \frac{M^2_{pl}}{2} \left[ \eta^{\lambda\mu} (\partial^\beta \partial_\nu F) H_{\lambda\beta} - \eta^{\lambda\beta} (\partial^\mu \partial_\nu F) H_{\lambda\beta} \right] \]

(18)

An explicit check of the conservation of \( T^{\mu\nu} \) is due. A straight forward calculation shows,

\[ \partial_\mu T^{\mu\nu} = 0 \]

(19)

In arriving at the above conservation law we have used the equations of motion (12) and (15). Now that we have obtained a conserved EMT for the field theory (11). For reasons that will be clear in the following, we rewrite (18) such that first derivative of \( H_{\mu\nu} \) is not explicit in the EMT. It follows as,

\[ T^{\mu\nu} = - (\partial^\nu \phi) (\partial_\mu \phi) + \eta^{\mu\nu} \left[ \frac{M^2_{pl}}{2} F(\phi) R_H + \frac{1}{2} \eta^{\alpha\beta} (\partial_\alpha \phi) (\partial_\beta \phi) + V(\phi) \right] - \frac{M^2_{pl}}{2} \left( \eta^{\lambda\mu} \eta^{\sigma\beta} - \eta^{\mu\sigma} \eta^{\lambda\beta} \right) (\partial_\nu \partial_\sigma H_{\lambda\beta}) F(\phi) + \partial_\sigma \left[ \frac{M^2_{pl}}{2} \left( \eta^{\lambda\mu} \eta^{\sigma\beta} - \eta^{\mu\sigma} \eta^{\lambda\beta} \right) \partial_\nu \{ H_{\lambda\beta} \partial_\sigma F(\phi) \} \right] - \frac{M^2_{pl}}{2} \left[ \eta^{\lambda\mu} (\partial^\beta \partial_\nu F) H_{\lambda\beta} - \eta^{\lambda\beta} (\partial^\mu \partial_\nu F) H_{\lambda\beta} \right] \]

(20)

The program now is to import the EMT (20) to the local patch at \( P \) with substitution of \( H_{\mu\nu} \) by \( g_{\mu\nu} \) and express it in terms of general coordinates. But for the latter, we have to ensure that all the terms have unambiguous correspondence with geometric objects. For example, the first and second term of (20) (on substitution of \( H \) by \( g \), are already in a form that has such a correspondence, but not the third term, owing to the presence of the factor \( (\partial^\nu \partial_\sigma g_{\lambda\beta}) \). We have to improve the EMT (20) so as to get rid of such ambiguity.

A possible way is to add a term \( \partial_\sigma M^{\sigma\mu\nu} \) to the EMT (20), where \( M^{\sigma\mu\nu} \) is a third rank tensor antisymmetric in its first two indices such that \( \partial_\mu \partial_\sigma M^{\sigma\mu\nu} \) vanishes identically. This ensures the conservation of the improved EMT (14). An appropriate choice is

\[ M^{\sigma\mu\nu} = \eta^{\nu\beta} \left( \eta^{\mu\lambda} \eta^{\rho\sigma} - \eta^{\mu\rho} \eta^{\lambda\sigma} \right) \{ (\partial_\rho H_{\lambda\beta}) F(\phi) - 3 \{ \partial_\rho F(\phi) \} H_{\lambda\beta} \} \]

(21)

Note that \( M^{\sigma\mu\nu} \) has the required anti symmetry in \( \sigma \) and \( \mu \). Adding \( \frac{M^2_{pl}}{2} \partial_\sigma M^{\sigma\mu\nu} \) with \( T^{\mu\nu} \) to (20), we get the improved tensor,

\[ \Theta^{\mu\nu} = - (\partial^\nu \phi) (\partial_\mu \phi) + \eta^{\mu\nu} \left[ \frac{M^2_{pl}}{2} F(\phi) R_H + \frac{1}{2} \eta^{\alpha\beta} (\partial_\alpha \phi) (\partial_\beta \phi) + V(\phi) \right] - M^2_{pl} R^{\mu\nu} F(\phi) + \frac{M^2_{pl}}{2} \eta^{\nu\rho} \left( \eta^{\mu\lambda} \eta^{\sigma\beta} - \eta^{\mu\sigma} \eta^{\lambda\beta} \right) \partial_\rho \{ H_{\lambda\beta} \partial_\sigma F(\phi) \} - \frac{M^2_{pl}}{2} \left[ \eta^{\lambda\mu} (\partial^\beta \partial_\rho F) H_{\lambda\beta} - \eta^{\lambda\beta} (\partial^\mu \partial_\rho F) H_{\lambda\beta} \right] \]

(22)
Here we have used the relation
\[
(\eta^{\mu\nu} \eta^{\sigma\beta} - \eta^{\mu\sigma} \eta^{\nu\beta}) \{\partial^\mu \partial_\sigma H_{\lambda\beta}\} F(\phi) = 2R_H \mu^\nu F(\phi) + (\eta^{\mu\nu} \eta^{\rho\sigma} \square H_{\rho\sigma} - \eta^{\mu\rho} \partial^\rho \partial_\sigma H_{\lambda\beta}) F(\phi)
\]
\[
= 2R_H \mu^\nu F(\phi) + \partial_\sigma \left[\eta^{\rho\beta} \left(\eta^{\mu\lambda} \partial^\rho \partial^\nu \partial_\mu \partial_\lambda \phi\right) \right] - \eta^{\mu\beta} \left(\eta^{\rho\nu} \partial^\sigma \partial_\rho \partial_\lambda \phi\right) \partial_\sigma H_{\lambda\beta} F(\phi)
\]
(23)

obtained using the substitution of \(g_{\mu\nu} H_{\mu\nu}\) in equation (7) and (15).

The right hand side of (25) is in covariant form and passes to the left hand side when the defining properties (2), (3) thereby emphasising the deviation from the Einstein structure (27). The non-conservation has been criticised [9].

The correspondence with the Hilbert action principle must be investigated. First observe that if we put \(F = \Theta(\phi)\) in (26) we get minimally coupled scalar field action. The different terms on the right hand side of (24) are in appropriate tensorial form suitable to be lifted to general coordinates. For instance, \(M_{pl}^2 \eta^{\mu\nu} \eta^{\rho\sigma} \partial_\rho g_{\lambda\beta} \partial_\sigma F(\phi) = g_{\lambda\beta} - \eta^{\lambda\beta} (\partial^\mu \partial^\nu F) g_{\lambda\beta}\)
(24)

and identify the EMT (24) to that of the original theory (5) in the adapted coordinates at the point P and then convert it to the general coordinates. Once this identification is done, \(g_{\mu\nu}\) becomes the metric. The different terms on the right hand side of (24) are in covariant form and passes to the left hand side when the defining properties (2), (3) and (4) are used. Similar generalization of the other terms is obvious, considering that \(F\) is a scalar. One can now readily obtain the expression for the EMT in general coordinates,
\[
\Theta^{\mu\nu} = - (\partial^\mu \phi)(\partial^\nu \phi) + g^{\mu\nu} \left[ g_{\alpha\beta} \frac{1}{2} g^{\rho\sigma} (\partial_\alpha \phi)(\partial_\beta \phi) + V(\phi) \right] - M_{pl}^2 F(\phi) G^{\mu\nu} - M_{pl}^2 g^{\mu\nu} \Delta F + M_{pl}^2 \nabla^\mu \nabla^\nu F
\]
(26)

where \(\Delta F = \nabla_\mu \nabla^\mu F(\phi)\) and \(G^{\mu\nu}\) is the Einstein tensor. Note that metric compatibility is assumed.

As can be checked directly, our algorithm, based on Noether theorem and dynamics of fields, leads to an EMT which is symmetric and covariantly conserved. Based as it is on the local conservation of energy-momentum, this EMT serves as the source of gravity \(a\ la\ Einstein\). Hence following the original spirit of general relativity, we write the equation of motion of non-minimally coupled quintessence model as
\[
G^{\mu\nu} = - \frac{1}{M_{pl}^2} \Theta^{\mu\nu}
\]
(27)

The correspondence with the Hilbert action principle must be investigated. First observe that if we put \(F(\phi) = 0\) in [5] we get minimally coupled scalar field action. The corresponding EMT obtained by our algorithm is (see equation-26)
\[
\Theta^{\mu\nu} = - (\nabla^\mu \phi)(\nabla^\nu \phi) + g^{\mu\nu} \left[ g_{\alpha\beta} \frac{1}{2} g^{\rho\sigma} (\nabla_\alpha \phi)(\nabla_\beta \phi) + V(\phi) \right]
\]
(28)

which is nothing but the EMT obtained by using [1]. Thus the connection of the usual EMT, obtained from (1), as source of the gravitational field with the energy-momentum balance is properly elucidated by our algorithm. This is a new result as far as we know.

The case of \(F(\phi) \neq 0\), i.e., non-minimally coupled theories, is naturally more involved. Such theories appear in different contexts, e.g., quantum corrections [13], renormalization of classical theory [14], in the string theoretic context [17] and in the Scalar-tensor theories [18]. Now a days such theories are intensly investigated in context of dark energy also. However, in the literature there are different approaches in identifying the EMT for a non-minimal theory. To illustrate this, let us first write the equation of motion obtained by varying the metric of [5]
\[
M_{pl}^2 (1-F) G_{\mu\nu} = (\nabla_\mu \phi \nabla_\nu \phi) - g_{\mu\nu} \left[ g_{\alpha\beta} \frac{1}{2} g^{\rho\sigma} (\nabla_\alpha \phi)(\nabla_\beta \phi) + V(\phi) \right] + M_{pl}^2 g_{\mu\nu} \Delta F - M_{pl}^2 \nabla_\mu \nabla_\nu F = \tilde{\Theta}_{\mu\nu}
\]
(29)

One can keep the equation as it is [10] and interprete \(\tilde{\Theta}_{\mu\nu}\) as the EMT, which is clearly not covariantly conserved, thereby emphasising the deviation from the Einstein structure (27). The non-conservation has been criticised [9].
Alternatively, conserved EMT can be identified from (29) by algebraic manipulations which may take different courses [8, 9] leading to EMTs with apparently different forms, e.g.

\[ \Theta^D_{\mu\nu} = \frac{\tilde{\Theta}_{\mu\nu}}{(1 - F)} \]  
\[ \Theta^A_{\mu\nu} = \tilde{\Theta}_{\mu\nu} + M_{pl}^2 F G_{\mu\nu} \]

Looking back at our equation (26) it is easy to see that it agrees with (31). Thus our approach predicts a covariantly conserved EMT for non-minimally coupled theories.

In this paper we have developed a novel algorithm to obtain the Energy momentum tensor (EMT) of scalar field theories coupled with gravity. The method rests on the corresponding theory where gravity is non dynamical. Adapted (Lorentzian) coordinates are chosen in the locally inertial frame where the first derivative of the metric vanishes in an infinitesimal patch. We have then defined a field theory in the extended flat space that has the patch as overlap with the tangent space at P by the same action as the one obtained in the locally inertial frame. This theory has translation symmetry and the conserved Noether current gives an EMT. Identifying it as the EMT of the original theory in the locally inertial frame, we generalized the same to curved coordinates.

When the coupling of the scalar field with gravity is minimal, the standard form of the EMT is reproduced from the expression derived here. Non trivial results follow from the non minimally coupled scalar tensor theories, where incidentally different prescriptions of obtaining the EMT are available in the literature. It is gratifying to observe that our method leads to a covariant expression for the EMT which agrees with one of the widely used forms [9]. It is also apparent that the algorithm presented in this paper has a general applicability.

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