Bifurcation diagram of stationary solutions of the 2D Kuramoto-Sivashinsky equation in periodic domains.

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Abstract. The solution tree of the 2D stationary Kuramoto-Sivashinsky equations in the periodic domain is analyzed using the analytical and numerical methods. The evolution of stationary solutions is considered by constructing the bifurcation diagram. Some bifurcation points on the main solution are found analytically, where secondary bifurcations are analyzed numerically. The bifurcation diagram is constructed using the deflated pseudo arc-length continuation process that allows one to find both connected and disconnected branches of solutions. The resulting bifurcation diagram is analyzed and subdivided according to characteristic properties of the solution. Different solution branches are visualized in the physical space.

1. Introduction
The Kuramoto-Sivashinsky equations (KSE) for scalar function \( u = u(x) : \Omega \to \mathbb{R} \) is considered in this paper in the following form:

\[
\alpha (2uu_x + 2uu_y + \Delta u) + 4 \Delta^2 u = 0,
\]

where \( \Omega = (\mathbb{R}/2L\pi \mathbb{Z}) \times (\mathbb{R}/2\pi \mathbb{Z}) \) is two dimensional torus, \( L > 0 \) is the integer stretch factor, \( x = (x,y)^T \) is independent variables vector, \( \alpha \) is the bifurcation parameter, \( (\cdot)_{j} \) is the derivative in the \( j \)-direction and \( \Delta \) is the Laplace operator. The integer constants in the equations are widely used in other papers \cite{1} and the zero mean is assumed in \( \Omega \), i.e. \( \int_{\Omega} u \, dx \, dy = 0 \). This equation is physically relevant in terms of model equations for turbulence as well as chaotic dynamical systems. Originally, the KSE system was derived independently by G.I. Sivashinsky \cite{2} in 1977 and Y. Kuramoto \cite{3} in 1978. Physical origin of these equations is the description of the laminar flame propagation and the description of chaos in the Belousov–Zabotinskii reaction in three dimensions. These equations also describe dynamics of thin liquid film flows down inclined plane \cite{4}. It was later used as one of the model equations for chaotic behaviour in PDEs subject to super-diffusivity. The mathematical study of these equations as well as the analysis of their inertial manifold are considered in \cite{5,6}. The analysis of these equations for the 1D spatial case was studied extensively in many papers, including mathematical and numerical nonlinear analysis, e.g. \cite{7,8,9,10} and other papers cited within. It was proven that these equations have a unique smooth solution that has a continuous dependence on the initial conditions. A global reduction to the approximate inertial manifold for the 1D KSE in periodic domain is considered in \cite{1}, where bifurcations of stationary solutions are analysed. The same results
were obtained in [11] using computer assisted rigorous computations. It is shown that multiple nontrivial stationary solutions bifurcate from the main stationary trivial solution at certain integer points in the parameter space and the kernel dimension of the linearized operator at these points is greater than unity (higher order degeneration). This results in the formation of multiple solutions at these bifurcation points by pitchfork bifurcations. The stability loss of these stationary solutions leads to the occurrence of a two-dimensional local attractor. This attractor contains all periodic functions of time. Spatially inhomogeneous case gives rise to more complicated attractors. The bifurcation analysis of periodic solutions indicated the existence of Feigenbaum and Sharkovskii sequences of bifurcations leading to the chaotic regimes.

The 2D case is still in the field of intense study. It was shown in [12] that only locally integrable stationary solutions of the 2D KSE are constant values on infinite domain. The analytical results include [13] the boundness of the solution in the \( L_2(\Omega) \cap L_\infty(t) \) norm for unstationary equations (the ones that include the term \( u_t \) in [11]), where \( \Omega = [0, L\pi] \times [0, \epsilon\pi] \) with \( L > 1 \) and \( 0 < \epsilon < 1 \). The analyticity for the 2D KSE on the torus \( \Omega \) for \( L = 1 \) is considered in [14]. It is shown that small solutions exist for all time if there are no linearly growing modes, proving also in this case that the radius of analyticity of solutions grows linearly in time. For the general case the estimates are obtained in terms of mild solutions in \( L_2(\Omega) \) and \( H^s, 0 \leq s \leq 5/3 \) norms for some finite time \( T \) that depends on the regularity of the initial conditions and the number of linearly growing modes. Authors know of no improvements of this result.

The bifurcation analysis of the trivial solution for the 2D KSE for \( L = 1 \) is performed in [15]. It is shown that for wave numbers \((j, k) = \{(1, 1); (1, 2); (2, 1)\}\) of the Fourier representation of [1] there exist nontrivial branches and the null-space of the linearized operator has dimensions two and four. The bifurcated nontrivial solutions are stable at least in the infinitesimal positive parameter segment at the bifurcation point. The detailed numerical analysis of the non-stationary 2D KSE is performed in [16] for three different cases: \( L = 1, L = 10 \) and somewhat \( L >> 1 \). The statistics of chaotic solutions is considered and the solutions are classified where the trivial solution is unstable and the long-time dynamics is completely two-dimensional. Various paths to chaos are observed that are not through period doubling, unlike in the 1D KSE case. However, a detailed numerical analysis of the stationary solutions of the 2D KSE is not considered up until now. The authors used [1] as benchmark model problem to tune the deflation pseudo arc-length continuation method but didn’t find obtained analysis of stationary solutions. This paper is an attempt to present such analysis.

The paper is laid out as follows. First, the analytical analysis of stationary bifurcations of the trivial solution is considered. The numerical results are summarized to the bifurcation diagram on different solution representation functions and various branches are analysed. This follows by the conclusion. The obtained results also demonstrate the capabilities of the developed software regarding deflated continuation process.

2. Bifurcation points on the main solution

For periodic domain the equation [1] is transferred to the Fourier domain with the ansatz \( u(x) = \sum_{(j,k) \in \mathbb{Z}^2} \hat{u}_{j,k} e^{i(jx/L + ky)} \), \( \hat{u}_{0,0} = 0 \) and \( \hat{u}_{-j,-k} = (\hat{u}_{j,k})^* \), due to reality condition that results in the following discrete infinite dimensional operator:

\[
F(\hat{u}, \alpha) = -\alpha \left[ 2 \sum_{(l,m) \in \mathbb{Z}^2} \hat{u}_{l,m} \hat{u}_{l-j,m-k} + m \hat{u}_{l,m} \hat{u}_{l-j,m-k} \right] - \left( \frac{1}{L^2} j^2 + k^2 \right) \hat{u}_{j,k} - 4 \left( \frac{1}{L^4} j^4 + \frac{2}{L^2} j^2 k^2 + k^4 \right) \hat{u}_{j,k} = 0, \forall\{j, k\} \in \mathbb{Z}^2, \]

with \( \hat{u}_0 = \hat{u}_{j,k} = 0 \) as a trivial solution.
poses a symmetry w.r.t. the main branch. Plugging the Fourier coefficients of these solutions rather interested in verifying obtained results with the numerical solutions. These solutions where

\[ \beta \]

In this case the eigenvectors \( e_l \) are \( \sin(jx/L + ky), \sin(jx/L - ky) \) and all solutions lies in the Hilbert space with the zero mean value, we designate \( \mathbb{H}^0 \).

2.1. Bifurcations for \( L = 1, \alpha = 4 \)

In this case \( j = \pm 1, k = 0 \) or \( j = 0, k = \pm 1 \). Consider an infinitesimal perturbation with the magnitude \( \varepsilon \) and a positive constant \( \beta(\varepsilon) \) being applied to the parameter and a solution function:

\[ \alpha = \alpha_1 + \beta(\varepsilon)\varepsilon^2, \]

\[ u = \sum_{l=1}^{2} \varepsilon a_l e_l + \varepsilon^2 v(x, y, \varepsilon), \]

such that:

\[ v \in \text{Sp}(e_1, e_2) \perp \mathbb{H}^0, \]

and \( \sum_{l=1}^{2} a_l^2 = 1 \), due to the Lyapunov-Schmidt reduction [17]. Plugging these values into the equation [1] and using the fact [2] one finds the following eight solutions:

\[ u_{1,2} = \varepsilon(x - 2\beta(0)) \sin(x) + O(\varepsilon^2), \text{ for } a_1 = 1, a_2 = 0, \]

\[ u_{3,4} = \varepsilon(y - 2\beta(0)) \sin(y) + O(\varepsilon^2), \text{ for } a_1 = 0, a_2 = 1, \]

\[ u_{5,6,7,8} = \varepsilon^2(4 + \beta(0))a_1 \sin(x) + a_2 \sin(y) + O(\varepsilon^3), \text{ for } a_1 = \pm 1/\sqrt{2}, a_2 = \mp 1/\sqrt{2}, \]

where \( \beta(0) \) is the appropriate constant. We are not interested in the particular constant value, rather interested in verifying obtained results with the numerical solutions. These solutions poses a symmetry w.r.t. the main branch. Plugging the Fourier coefficients of these solutions into the linear operator [3] one sees that solutions \( u_{1,2} \) and \( u_{3,4} \) are stable and \( u_{5,6} \) and \( u_{7,8} \) are unstable. The representation of these solutions is shown in figure [1]. The same solutions at \( \alpha - 4 \leq 0.0025 \) for \( L = 1 \) that were obtained numerically during the continuation process are presented in figure [2]. The stability of these solutions is only valid in the vicinity of the bifurcation point for infinitesimal \( \varepsilon > 0 \).

2.2. Bifurcations for \( L = 2, \alpha = 1 \)

In this case \( j = \pm 1, k = 0 \). Using the same approach as before one can obtain two solutions bifurcating from the trivial one:

\[ u_{1,2} = \frac{\varepsilon}{2} \left(- \sin \left( \frac{x}{2} \right) \pm \beta(0) \sin \left( \frac{x}{2} \right) \right) + O(\varepsilon^2), \]

where one is stable, and the other is not for \( \beta(0) > 1 \).
2.3. Bifurcations for $L \to \infty$
In this case $\alpha \to 0$ and for any finite $|j| > 0, k = 0$ one can observe that a bifurcation occurring near zero and there are infinitely many bifurcations. These solutions have a form:

$$u = -\varepsilon \frac{2 \sin(x/L)}{L^2} - 4 \varepsilon \frac{2 \beta(0) \sin(x/L)}{L^4} + O(\varepsilon^2).$$

(9)

Their stability depends on the constant $\beta(0)$ which should be found for any particular solution. However this will involve considering an expansion with parts, having higher degree of $\varepsilon$.

2.4. Bifurcations for $L = 1$, $\alpha = 8$
In this case $j^2 + k^2 = 2$, and $j = \pm 1, k = \pm 1$. This will result in two solutions bifurcating at this point (not taking symmetric ones into account) of the form:

$$u_{1,2} = \varepsilon 32 \beta 0 \cos(x \pm y) \sin(x \pm y) + O(\varepsilon^2).$$

(10)

These solutions are unstable and correspond to $a_{1,2}$ being 1 or 0. Other weights combinations don’t respect the reduced Lyapunov–Schmidt form. This result can be compared with the results from [15] where the full set of eigenvalues was considered without symmetry reduction resulting in four solutions at this point (the case in the paper corresponds to constant in the ($1$ being 1 and 1 instead of 2 and 4). All solutions are linearly unstable.

2.5. Bifurcations for $L > 1$, $\alpha = 4 \left(1 + L^{-2}\right)$
In this case $j = \pm 1, k = \pm 1$ and the bifurcation point $\alpha$ approaches 4 as the value of integer $L$ approaches infinity. The solutions have a form:

$$u = \mp \varepsilon (4 + 4/L^2) 1/L (2\cos(x/L - y) \mp L \cos(x/L - y) \pm \cos(x/L + y)$$

$$\pm L \cos(x/L + y))(\sin(x/L - y) + \sin(x/L + y))) + O(\varepsilon^2),$$

(11)
and all these solutions are linearly unstable.

For the sake of numerical verification we take $L = 1$ and observe, that the points $\alpha$, where $j^2 + k^2 = \alpha/4$ and $\{j, k\} \in \mathbb{Z}^2$, involve bifurcations of the primary solution, i.e. $\alpha = \{4, 8, 16, 20, 32, 36, \ldots\}$.

3. Numerical continuation analysis

For the implementation of the numerical analysis we use the earlier developed methods of the deflated pseudo arc-length continuation being implemented on the multiple graphics processing units computational architecture [18]. The modified implicitly restarted Arnoldi method (IRAM) is used [19] with the inexact exponent and inexact exponent shift and inverse matrix transformations [20]. The analysis is performed according to the following scheme:

- The set of deflation knots $D := \{\alpha_j\}_{j=1}^K$ and parameter interval $[\alpha_s; \alpha_e]$ are provided, such, that $\forall j : \alpha_s < \alpha_j < \alpha_e$.
- Parameters of the numerical schemes and methods are prescribed.
- The branches and solutions database is initialized.
- Perform deflation-continuation process according to [18].
- Use found solutions to perform linear stability analysis using the IRAM.
- Generate results for each branch and each bifurcation point.

For this calculation we set $L = 1$,

$$D = \{3.5, 4.5, 5.0, 6.0, 7.5, 8.5, 9.0, 10.5, 12.0, 13.5, 15.0, 16.5, 18.0, 19.5, 21.0, 22.5, 24.0, 24.5, 25.0\},$$

(12)

$\alpha_s = 3$, $\alpha_e = 30$. Hence the continuation is terminated for each branch as soon as the parameter value reaches 3 or 30. New solution branches, including those that are disconnected from the main branch, are being found only in deflation knot points according to the methodology in [18].

The discrete problem is formulated according to (2), where the sums are taken on the finite ring of integers: $2048 \times 2048$ Fourier harmonics were used, resulting in $N = 2099200$ active degrees of freedom (DOF), the nonlinearity is accounted for using pseudo-spectral approach with the two thirds de-aliased approach.

All numerical methods in the deflation-continuation and eigensolver processes execute the solution of the linear system $Ax = b$, where linear operator $A$ is $F_u(\hat{u})$ from [3] with finite sums. The system is solved using BiCGStab(L) with the left preconditioner. On each iteration of the BiCGStab(L) a residual vector $r$ is formed as:

$$z = b - Ax,$$

$$r = Pz,$$

(13)

where the preconditioner $P \sim A^{-1}$ is formed as the inverse of the Laplace and biharmonic operators. It is done easily, since these operators are diagonal matrices in the Fourier space. The iterations continue, while $\|r\| > \epsilon_L$.

All methods for the deflation-continuation are used with Newton-Krylov solvers. The linear systems are solved up to a small residual reduction followed by the Newton method updates with the given $\epsilon_N$. For all deflation-continuation process we set $\epsilon_L = 1.0 \cdot 10^{-2}$ and $\epsilon_N = 1.0 \cdot 10^{-9}$. The IRAM can also be used with relaxed tolerance for the linear solver, so we set $\epsilon_L = 1.0 \cdot 10^{-4}$ for the IRAM, and $1.0 \cdot 10^{-9}$ for the Ritz vectors norms, see [19]. The points of bifurcation are found using the bisection method with the parameter value accuracy up to the third significant digit in mantissa. The parallel performance on multiple GPUs is discussed in [20, 18] and is not elaborated here.
The solution database is constructed using adaptive strategy for saving intermediate results on the hard drive, since each solution vector takes roughly 16MB. So only each 50-th solution is saved on the branch with additional saving solutions when the curve intersects knots. We also save each 20-th solution when the parameter gradient w.r.t. the arc-length has inclination greater than 1 e.g. at saddle-node bifurcations, and the solution is also saved for each bifurcation point. Filled database consumes 60GB of hard disk space in binary data file format.

4. Results

From the literature overview and results of the section one can expect a complicated behaviour of the solutions branches. It is difficult to represent the solutions behaviour adequately in the 2D or 3D space, hence we needed to split the results into different sections. First we present the full bifurcation diagram. Next we discuss separate branches and bifurcation points. The following solution representation functions are used for the solution vector to plot the bifurcation diagrams:

\[ \|u\|_2 := N^{-1/2} \sum_{j=1}^{N} |\tilde{u}_j|^2, \]

\[ s_1(u) := u(2\pi/3, 2\pi/3), \]

\[ s_2(u) := u(2\pi/5, 2\pi/3), \]

\[ s_3(u) := u(2\pi/4, 2\pi/5) \]

and

\[ s_4(u) := u(\pi, 2\pi/3). \]

4.1. The whole bifurcation diagram for \( \alpha \in [4, 30] \)

The whole bifurcation diagram is presented in two figures.

![Figure 3.](image)

**Figure 3.** The whole bifurcation diagram for \( 4 \leq \alpha \leq 30 \) using \( s_1, s_2 \) functions, colors represent different branches, totalling 94 curves.

First, these results represent high complexity of the stationary solutions as the parameter variates. Figure demonstrates, that the unstable manifold dimension is limited to 13. Although it is of relatively low complexity (compared, i.e. to the Navier-Stokes equations), the need of high DOF scheme ensures the fact of the complete scheme resolution, since the unstable manifold dimension has no link to the approximating linear space DOF. For example a one dimensional periodic orbit can be expanded into multiple frequencies that would require high DOF scheme. The analysis of this data as a whole is imposible, so it is conducted as follows: first we analize...
all branches that bifurcate from the main zero solution that we call primary. Next, we analyze branches bifurcating from primary branches and disconnected branches, we call these branches secondary. A trivial zero solution is not traced. We consider \( \| \cdot \|_2 \) norms for bifurcation diagram display and \( s_1, s_2, s_3, s_4 \) representation functions for solution visualization on branches. Some curves overlap in the \( \| \cdot \|_2 \) norm, but this allows a more clear view of a general picture. All bifurcation points marked in figures correspond to the Hopf bifurcations that are not analyzed in this paper, except saddle-node, transcritical and pitchfork bifurcations (both sub- and supercritical) that result in the change of the unstable manifold dimension. Those are explicitly discussed.

4.2. Primary branches for \( \alpha \in [4,8] \)

The results are presented in figure 5. The first bifurcation corresponds to the analytically obtained results \( \text{(7)} \) at \( \alpha = 4 \). We classify it as symmetry-breaking bifurcation with \( \mathbb{D}_4 \) symmetry group according to the Equivariant Branching Lemma [21, p.82]. As a result of this bifurcation one has: four steady state solutions denoted by \( u_{1,2,3,4} \) which are stable near the bifurcation point (these solutions labeled 1 and 2 in the figure); four steady state solutions denoted by \( u_{5,6,7,8} \) which are unstable near the bifurcation point (these solutions labeled 3 in the figure). The number of solutions is half of the expected for \( \mathbb{D}_4 \) group because we only consider solutions symmetric relative to the central point which corresponds to the quotient group \( \mathbb{D}_4/\mathbb{Z}_2 \). All bifurcation points marked in figure 5 correspond to the Hopf bifurcations, except one pitchfork bifurcation point near the beginning of the curve 4 and one unidentified bifurcation near its ending. This unidentified type of bifurcation will be discussed later in section 4.4.

The evolution of four bifurcating solutions is shown in figure 6. Those solutions lose stability at \( |\alpha - 8| < 0.005 \) with the Hopf bifurcation. These branches suffer multiple Hopf bifurcations resulting in 4 linear operator eigenvalues in the right hand side as \( \alpha \to 16 \) and two pitchfork
Figure 5. Bifurcation diagram and stability of the primary branches for $\alpha \in [4, 8]$ shown in bold. Curve colors indicate unstable manifold dimension.

Figure 6. Evolution of stable solutions (ones marked with physical space pics) bifurcated at $\alpha = 4$ using $s3$. Colors represent unstable manifold dimension.

bifurcations on each curve that create secondary connecting branches shown on both curves in figure 5. Bifurcation near $|\alpha - 14.4405| < 0.005$ is supercritical and near $|\alpha - 15.9121| < 0.005$ is subcritical.

The evolution of other four bifurcating solutions is shown in figure 6. These solutions are initially unstable, also suffer Hopf bifurcations at $|\alpha - 8| < 0.005$ and end with pitchfork bifurcations which connect them with primary branches emerging at $\alpha = 8$.

The latter solutions, described by (10) bifurcate at $\alpha = 8$ and are labeled 4 in the figure 5. They suffer multiple Hopf bifurcations, already mentioned pitchfork bifurcations with curves labeled 3 and six other pitchfork bifurcations - four supercritical and two subcritical. The branch
Figure 7. Evolution of unstable solutions (ones marked with physical space pics) at $\alpha = 4$ in $s4$ norm. Colors represent unstable manifold dimension.

Figure 8. Evolution of the pitchfork bifurcated solutions at $\alpha = 8$ in $s4$ norm and four solutions bifurcated at $\alpha = 16$ with unstable manifold dimension 2 in $s1$ norm. Colors represent unstable manifold dimension.

reaches the termination value of the parameter and is not continued beyond $\alpha = 30$. Detailed diagram for these solutions is presented on the left side of the figure 8.

4.3. Primary branches at $\alpha = 16$
At $\alpha = 16$ we have $j^2 + k^2 = 4$ and, hence $j = \pm 2, k = 0$ and $k = \pm 2, j = 0$, which repeats the situation for $\alpha = 4$ but with higher frequency solutions; all bifurcating solutions are unstable.
Figure 9. Bifurcation diagram and stability of the primary branches for $\alpha = 16$ shown in bold. Curve colors indicate unstable manifold dimension.

and have greater unstable manifold dimension (two and five) compared to the one at $\alpha = 4$. The main branch curves are shown in figure 9 in bold lines labeled as 5, 6 (corresponding to the four solutions with unstable manifold dimension 2) and 7 (corresponding to the four solutions with unstable manifold dimension 5) totalling eight curves.

The evolution of solutions on curves 5 and 6 is shown in figure 8 right. These curves suffers 12 secondary pitchfork bifurcations: eight supercritical at $|\alpha - 16.18| < 0.005$, $|\alpha - 16.684| < 0.005$ and four subcritical at $|\alpha - 16.14| < 0.005$. Multiple Hopf bifurcations are observed near the origin of the primary bifurcation point and are all concentrated at $16 < \alpha \leq 16.03$.

Figure 10. Evolution of four solutions with unstable manifold dimension 5 bifurcated at $\alpha = 16$ in s3 norm. Colors represent unstable manifold dimension.
The evolution of other four solutions labeled 7 at $\alpha = 16$ is shown in figure 10. Apart from initial symmetry-breaking bifurcation at $\alpha = 16$, these curves suffer transcritical secondary bifurcations near $|\alpha - 28.8454| < 0.005$ resulting in the stability interchange with one of the secondary branches, four more subcritical pitchfork bifurcations near $|\alpha - 17.4135| < 0.005$ and eight more supercritical pitchfork bifurcations near $|\alpha - 26.2583| < 0.005$. Other points mark Hopf bifurcations.

4.4. Primary branches at $\alpha = 20$

Figure 11. Bifurcation diagram and stability of the primary branches for $\alpha = 20$ shown in bold. Secondary branches are removed for better visual representation. Curve colors indicate unstable manifold dimension.

A high order degeneration is observed at $\alpha = 20$. We have $j^2 + k^2 = 5$ and, hence $j = \pm 2, k = \pm 1$ and $j = \pm 1, k = \pm 2$. All bifurcating solutions are unstable. The main branch curves are shown in figure 11 in bold lines labeled as 8,9 for supercritical and 10 – 12 for subcritical bifurcations. There are 32 curves bifurcating at $\alpha = 20$, 16 of them, labeled 8,9 are directed in the positive direction of the parameter (supercritical bifurcations) and 16 - in the negative direction (subcritical bifurcations), labeled 10–12. We interpret this situation as imposition of four (two supercritical and two subcritical) symmetry-breaking bifurcations analogous to the ones occurring at $\alpha = 4$ and $\alpha = 16$, however, another interpretations might be possible. The visual representation of these curves is difficult, so 3D visualization is used. Curves 8, 9 are presented in figure 12.

The results indicate that there are 4 looped curves clearly visible in figure 12. The curves are formed by subcritical pitchfork bifurcations at $|\alpha - 26.0597| < 0.005$ and are terminated at $\alpha = 20$ with supercritical symmetry-breaking bifurcations discussed above. The evolution of solutions on these supercritical curves is presented in figure 13. One can observe that most of these solutions are formed by eigenvectors $\{\sin(2x \pm y), \sin(x \pm 2y)\}$.

Subcritical curves, bifurcated from $\alpha = 20$ are presented in figure 14. One can also observe four loops that are formed by symmetry-breaking and pitchfork bifurcations. A better insight into the topological structure of solutions can be view in figure 15 where all curves are projected.
Figure 12. 3D representation of 16 supercritically bifurcated solutions at $\alpha = 20$ in coordinates $s_1, s_2$. Colors represent unstable manifold dimension.

onto the $s_1\alpha$ plane that respects symmetry of solutions. The curves in the figure are colored according to the curve classes. Curves 2 are formed by the supercritical bifurcation near $|\alpha - 17.9025| < 0.005$ of curves 5. Then the curves 2 suffer three saddle-node bifurcations in $19.47 < \alpha < 19.53$, followed by the subcritical bifurcation at $\alpha = 20$. Curves 5 apart from noticed bifurcations and several Hopf bifurcations are initiated at $|\alpha - 17.02| < 0.005$ by saddle-node bifurcations and again terminated at $\alpha = 20$ by the above mentioned subcritical bifurcation.

Curves 3 and 4 are entangled in a complicated manner. Both curves bifurcate subcritically at $\alpha = 20$, followed by multiple turning points (saddle-node bifurcations) that can be observed in figures 15.

Curves 3 are formed by two unidentified bifurcations from the curves 4 near $|\alpha - 17.9025| < 0.005$ in such manner, that the curves 4 increase the unstable manifold dimension by one, which can be seen in figure 14. The problem with classification of these bifurcations is that there are seemingly no second (paired) branches starting from bifurcation points. This situation is somewhat similar to the one depicted in [22, p.437] on figure 4.5 and called “mixed-mode”, connecting different primary branches. However, this analogy could be just visual coincidence and some further investigation is required. As the result, all curves, but one, entering the knot at $\alpha = 20$ are returning trajectories. However, these trajectories form an unstable manifold of higher dimension near the knot. Part of these bifurcations are due to the secondary branches connecting the primary, that will be discussed later. The visualization of all subcritical trajectories that bifurcate at $\alpha = 20$ is shown in figure 16.

4.5. Secondary branches at $4 < \alpha \leq 30$

Secondary branches of solutions are considered in this subsection that are either formed by bifurcations on the primary branches, or by disconnected from the primary branches separate bifurcations. All secondary bifurcations are presented in figure 17 in bold lines. The
detail analysis of all secondary branches is next to impossible but one can introduce general classification and describe these branches per class.

The first class contains branches that connect at least one primary branch with other branches. Such curves are usually formed by above mentioned unidentified unpaired bifurcations on primary branches. These curves are not closed meaning that these curves contain at least two such bifurcations at each end. Examples of such curves are presented in figure 18.

The second class contains branches that connect at least one primary branch with other branch and are closed. These branches are usually formed by a pair of pitchfork bifurcations - one subcritical and the other is supercritical. Examples of these curves are presented in figure 19.

The third class is a subclass of the first two. It contains curves form class 1 or class 2 that were interrupted during the continuations process and cannot be classified. Those curves are presented in figure 20.

The final, forth class contains four curves that were found during the deflation process and have no direct connection to other branches. Curves of this type are formed by saddle-node bifurcations at $|\alpha - 18.2725| < 0.005$ and $|\alpha - 19.5075| < 0.005$. Other bifurcations on these solutions are Hopf bifurcations, thus these solutions have no connection to the main branch. The unstable manifold dimension on these solutions changes from 4 to 7 due to these Hopf bifurcations.

The evolution of all solutions on secondary branches is rich and cannot be demonstrated in the paper. We can refer the reader to the youtube movie at https://www.youtube.com/watch?v=xoG9NhSqqfo that contains evolution of all solutions glued together, both primary and secondary.

5. Conclusion
We present results of the bifurcation diagram for the 2D stationary Kuramoto-Sivashinsky equations for $0 < \alpha \leq 30$. It is shown that the stationary solutions undergo process of bifurcations that turns out to be very complex as the parameter value increases. We defined two sets of solution curves: primary, that bifurcated from the main solution and secondary that bifurcated from other solutions, connected to the primary branch, or disconnected. We found four disconnected solutions that can be formed by high dimensional saddle-node bifurcations. The next step in the analysis is the generalization of the deflated pseudo arc-length continuation method to the periodic orbits.

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**Figure 13.** Evolution of supercritical solutions, bifurcated at $\alpha = 20$ using $s2$ representation function.
Figure 14. 3D representation of 16 subcritically bifurcated solutions at $\alpha = 20$ in coordinates $s_1, s_2$. Colors represent unstable manifold dimension.

Figure 15. Projection to the $s_1\alpha$ plane of all curves that bifurcated subcritically at $\alpha = 20$ (left) and zoom-in near the (0, 20) plane point. Colors represent different curve classes, bifurcation points are represented by colored dots.
Figure 16. Evolution of subcritical solutions, bifurcated at $\alpha = 20$ using $s1$ representation function.
Figure 17. Bifurcation diagram and stability of the secondary branches shown in bold curves. Curve colors indicate unstable manifold dimension.

Figure 18. Secondary branches of the first class in s1, s2 coordinates. Colors indicate unstable manifold dimension on each curve.
Figure 19. Secondary branches of the second class in $s_1, s_2$ coordinates. Colors indicate unstable manifold dimension on each curve.

Figure 20. Secondary branches of the third class in $s_1, s_2$ coordinates. Colors indicate unstable manifold dimension on each curve.
Figure 21. Disconnected solutions (curves of the forth class) location in the bifurcation diagram in $s_1, s_2$ coordinates (left) and evolution of solutions (right).