THE BESSEL PERIOD OF U(3) AND U(2) INVOLVING A NON-TEMPERED REPRESENTATION

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Abstract. In [7], Neal Harris has given a refined Gross-Prasad conjecture for unitary group as an analogue of Ichino and Ikeda’s paper [9] concerning special orthogonal groups. In his paper, he stated a conjecture under the assumption that the pair of given representations should be tempered. In this paper, we consider a specific pair involving a non-tempered one. In this case, an analogous formula still exists but the central critical $L$-value is slightly different with the one in the conjecture. As a corollary, this verifies that the tempered condition is indispensable in formulating the conjecture.

1. THE BESSEL PERIOD OF U(3) AND U(2) INVOLVING A NON-TEMPERED REPRESENTATION

We first recall the Refined Gross-Prasad Conjecture stated in [7]. Let $E/F$ be a quadratic extension of number fields and $A_F, A_E$ are their adele rings respectively. Let $V_n \subset V_{n+1}$ be hermitian spaces of dimensions $n$ and $n+1$ over $E$, respectively. Consider the unitary groups $U(V_n) \subset U(V_{n+1})$ defined over $F$. Write $G_i := U(V_i)$. Let $\pi_n$ and $\pi_{n+1}$ be irreducible tempered cuspidal automorphic representations of $G_n(A_F)$ and $G_{n+1}(A_F)$ respectively, and we fix isomorphisms $\pi_n \cong \otimes_v \pi_{n,v}$ and $\pi_{n+1} \cong \otimes_v \pi_{n+1,v}$. We suppose that $\text{Hom}_{G_n}(\pi_{n+1,v} \otimes \pi_{n,v}, \mathbb{C}) \neq 0$ for every place $v$ of $F$.

We consider the following $G_n(A_F) \times G_n(A_F)$-invariant functional

$$P : (V_{\pi_{n+1}} \otimes V_{\pi_{n+1}}) \otimes (V_{\pi_n} \otimes V_{\pi_n}) \to \mathbb{C}$$

defined by

$$P(\phi_1, \phi_2; f_1, f_2) := \left( \int_{[G_n]} \phi_1(g) f_1(g) dg \right) \cdot \left( \int_{[G_n]} \overline{\phi_2(g)} f_2(g) dg \right)$$

for $\phi_i \in V_{\pi_{n+1}}, f_i \in V_{\pi_n}$ and $[G_n] = G_n(F) \backslash G_n(A_F)$. If $\phi_1 = \phi_2 = \phi$ and $f_1 = f_2 = f$, we simply write $P(\phi, f) := P(\phi, \phi; f, f)$ and we call $P$ the global period.

On the other hand, there is another $G_n(A_F) \times G_n(A_F)$-invariant functional constructed from the local integral of matrix coefficients. To define matrix coefficients, for each place $v$ of $F$, let $F_v$ be its completion of $F$ at $v$ and denote $G_{i,v} := G_i(F_v)$. Fix the local pairings

$$B_{\pi_i,v} : \pi_{i,v} \otimes \bar{\pi}_{i,v} \to \mathbb{C}$$

so that

$$B_{\pi_i} = \prod_v B_{\pi_i,v}$$

where $B_{\pi_i}$ is the Petersson pairing

$$B_{\pi_i}(f_1, f_2) := \int_{[G_i]} f_1(g_i) \overline{f_2(g_i)} dg_i$$

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and the $dg_i$ is Tamagawa measures on $G_i(\mathbb{A}_F)$. For each place $v$, we define a $G_{n,v} \times G_{n,v}$ invariant functional

$$\mathcal{P}_v^2 : (\pi_{n+1,v} \boxtimes \pi_{n+1,v}) \otimes (\pi_{n,v} \boxtimes \pi_{n,v})$$

by

$$\mathcal{P}_v^2(\phi_{1,v}, \phi_{2,v}; f_{1,v}, f_{2,v}) := \int_{G_{n,v}} B_{\pi_{n+1,v}}(\pi_{n+1,v}(g_{1,v}) \phi_{1,v}, \phi_{2,v}) B_{\pi_{n,v}}(\pi_{n,v}(g_{2,v}) f_{1,v}, f_{2,v}) dg_{n,v}.$$  

Here, the $dg_{n,v}$ are local Haar measures such that $\prod_v dg_{n,v} = dg_n$.

Write $\mathcal{P}_v^2(\phi_v, f_v) := \mathcal{P}_v^2(\phi_v, f_v)$ and we set

$$\Delta_{G_{n,1}} := L(M_1^\vee(1), 0)$$
$$\Delta_{G_{n,v}} := L_v(M_1^\vee(1), 0)$$

where $M_i^\vee(1)$ is the twisted dual of the motive $M_i$ associated to $G_i$ by Gross in [5]. It is known in [7, Prop. 2.1] that $\mathcal{P}_v^2$ converges absolutely if the $\pi_{n,v}$ is tempered. Furthermore, it is also known that for unramified data $\phi_v, f_v$ satisfying conditions (1) – (7) in [7, p.6], we have

$$\mathcal{P}_v^2(\phi_v, f_v) = \Delta_{G_{n+1,v}} L_{E_v}(1/2, BC(\pi_{n,v}) \boxtimes BC(\pi_{n+1,v}))$$
$$\Delta_{G_{n,v}} L_v(1, \pi_{n,v}, Ad)L_v(1, \pi_{n+1,v}, Ad)$$

(Here, $BC(\pi_i)$ is the quadratic base-change of $\pi_i$ to a representation of $GL_i(\mathbb{A}_F)$).

From this observation, we can normalize $\mathcal{P}_v^2$ as

$$\mathcal{P}_v := \Delta_{G_{n+1,v}}^{-1} L_{E_v}(1/2, BC(\pi_{n,v}) \boxtimes BC(\pi_{n+1,v})) \mathcal{P}_v^2$$

and call this the local period.

Then

$$\prod_v \mathcal{P}_v : (V_{\pi_{n+1}} \otimes \bar{V}_{\pi_{n+1}}) \otimes (V_{\pi_n} \otimes \bar{V}_{\pi_n}) \to \mathbb{C}.$$

is also another $G_n(\mathbb{A}_F) \times G_n(\mathbb{A}_F)$-invariant functional.

The Refined Gross-Prasad Conjecture predicts that these two global $G_n(\mathbb{A}_F) \times G_n(\mathbb{A}_F)$-functionals $\mathcal{P}$ and $\prod_v \mathcal{P}_v$ differs by only a certain constant, that is the central $L$-value of the product $L$-function. The precise conjecture is as follows:

**Conjecture 1.1** (Refined Gross-Prasad Conjecture for Unitary groups),

$$\mathcal{P}(\phi, f) = \frac{\Delta_{G_{n+1}}}{2^\beta} L(1/2, BC(\pi_{n}) \boxtimes BC(\pi_{n+1})) \prod_v \mathcal{P}_v(\phi_v, f_v).$$

(Here $\psi_i$ is the conjectural $L$-parameter for $\pi_i$ and $\beta$ is an integer such that $2^\beta = |S_{\psi_{n+1}}| \cdot |S_{\psi_n}|$ and $S_{\psi_i} := \text{Cent}_{G_i}(\text{Im}(\psi_i))$ is the associated component group.)

In [7], N.Harris proved this conjecture unconditionally for $n = 1$ using Waldspurger formula, and conditionally for $n = 2$ assuming $\pi_3$ is a $\Theta$-lift of a representation on $U(2)$. Recently, Wei Zhang proved for general case using relative trace formula under some local conditions [25].

Our goal is to provide an analog of this conjecture for $n = 2$ and $\pi_3$ is a theta lift of $U(1)$. Note that in this case, $\pi_3$ is no longer tempered and so the above local periods may diverge. So we first regularize the local period using the function appearing in the doubling method. Once this is done, we can define a regularized local period and this enable us to establish the following formula which can be seen as an analogue of Refined Gross-Prasad conjecture.
Theorem 1.2. Let $F$ be a totally real field and $E$ a totally imaginary quadratic extension of $F$ such that all the finite places of $F$ dividing 2 do not split in $E$. The unitary groups we are considering here are all associated to this extension. Let $\sigma$ be an automorphic characters of $U(1)(\mathbb{A}_F)$ and $\pi_3 = \Theta(\sigma), \pi_2 = \Theta(\bar{1})$ be irreducible tempered cuspidal automorphic representations of $U(2)(\mathbb{A}_F)$ which comes from a theta lift of $\sigma$ and trivial character $1$, respectively. We assume that these two theta lifts are nonvanishing and cuspidal. Then for $\phi = \otimes \phi_v \in \pi_3$ and $f = \otimes f_v \in \pi_2$,
\[
P(\phi, f) = -\frac{1}{2^3} \frac{L(3, \chi)}{L^2(1, \chi)} \frac{L_E(\frac{1}{2}, BC(\omega_2^{-1}, \omega_2^{-1}) \otimes \gamma) \cdot \text{Res}_{s=0}(L_E(s, BC(\pi_2) \otimes \gamma))}{L_E(\frac{1}{2}, BC(\sigma) \otimes \gamma^3)} \prod_v P_v(\phi_v, f_v).
\]
where $\gamma$ is a character of $\mathbb{A}_E^\times/E^\times$ such that $\gamma|_{\mathbb{A}_E^\times} = \chi_{E/F}$ and for $i = 1, 2, \omega_i$, is the central character of $\pi_i$. The normalized local periods $P_v$’s are defined by
\[
P_v(\phi_v, f_v) := c_v \cdot \lim_{s \to 0} \frac{\zeta_v(2s)}{L_v(s, BC(\pi_2, \bar{\sigma}_v) \otimes \gamma_v)} \int_{U(2)_v} B_{\pi_2, s}(g_v \cdot \phi_v, \phi_v) \cdot B_{\pi_2, s}(g_v \cdot f_v, f_v) \cdot \Delta(g_v)^s \text{d}g_v.
\]
(here, $c_v$ is a constant for each $v$ defined by
\[
c_v := \frac{L_v^3(1, \chi_{E_v}/F_v) \cdot L_E(\frac{1}{2}, BC(\sigma_v) \otimes \gamma_v^3)}{L_v(3, \chi_{E_v}/F_v) \cdot L_E(\frac{1}{2}, BC(\omega_2^{-1}, \omega_2^{-1}) \otimes \gamma_v)}
\]
and $B_{\pi_2, v}$’s are the fixed local pairings of $\theta(\bar{\sigma})_v$ s.t. $B_{\pi_2, v} = \prod_v B_{\pi_2, v}$ and $\Delta(g_v)$ is some function we will define in Section 3.)

Remark 1.3. Since $\pi_2$ comes from the theta lift of $U(1)$, its theta lift back to $U(1)$ is also nonvanishing. From [7], we see that $\text{Res}_{s=0}(L_E(s, BC(\pi_2) \otimes \gamma))$ is nonzero and so Theorem 1.2 shows that the nonvanishing of the global period is equivalent to that of $L_E(\frac{1}{2}, BC(\omega_2^{-1}, \omega_2^{-1}) \otimes \gamma)$.

Remark 1.4. In the $SO(n)$ version of the conjecture, Ichino was the first who considered the non-tempered case in [8] and, recently, Yannan Qiu has brought his result into adelic setting including the former [10]. This article can be considered as an analogue of [10] concerning unitary group.

The rest of the paper is organized as follows: in Section 2, we introduce the theta correspondence for unitary groups, as well as the Weil representation. In Section 3, we give several versions of the Rallis Inner Product Formula. With all these things put together, we prove Theorem 1.2 in Section 4 under the assumption of a lemma which we prove in Section 5.

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2. The $\Theta$-correspondence for Unitary Groups

We review the Weil Representation and $\Theta$-correspondence. Most of this section are excerpts from [7].
2.1. The Weil Representation for Unitary Groups. In this subsection, we introduce the Weil representation. Since the constructions of global and local Weil representation are similar, we will treat both of them simultaneously. For an algebraic group $G$, if the same statement can be applied to both the local and global cases, we will not use the distinguished notation $G(F_v)$ and $G(A_F)$, but just refer them to $G$.

Let $(V, \langle \cdot, \cdot \rangle_V)$ and $(W, \langle \cdot, \cdot \rangle_W)$ be two hermitian and skew-hermitian spaces of dimension $m, n$ respectively. Denote $G := U(V)$ and $H := U(W)$ and we regard them as an algebraic group over $F_v$.

Define the symplectic space

$$\mathbb{W} := \text{Res}_{E/F} V \otimes_E W$$

with the symplectic form

$$\langle v \otimes w, v' \otimes w' \rangle_{\mathbb{W}} := \frac{1}{2} \text{tr}_{E/F} (\langle v, v' \rangle_V \otimes \langle w, w' \rangle_W).$$

We also consider the associated symplectic group $Sp(\mathbb{W})$ preserving $\langle \cdot, \cdot \rangle_{\mathbb{W}}$ and the metaplectic group $\tilde{Sp}(\mathbb{W})$ satisfying the following short exact sequence:

$$1 \rightarrow \mathbb{C}^\times \rightarrow \tilde{Sp}(\mathbb{W}) \rightarrow Sp(\mathbb{W}) \rightarrow 1.$$

Let $X$ be a Lagrangian subspace of $W$ and we fix an additive character $\psi : A_F/F \rightarrow \mathbb{C}^\times$ (globally) or $\psi : F_v \rightarrow \mathbb{C}^\times$ (locally). Then we have a Schrödinger model of the Weil Representation $\omega_\psi$ of $\tilde{Sp}(\mathbb{W})$ on $S(X)$, where $S$ is the Schwartz-Bruhat function space.

Throughout the rest of the paper, let $\chi_{E/F}$ be the quadratic character of $A_F/F$ associated to $E/F$ by the global and local class field theory. (For split place $v$, we define $\chi_{E/F}$ the trivial character.) And we also fix some unitary character $\gamma : A_E/E \rightarrow \mathbb{C}^\times$ whose restriction to $A_F/F$ or $E_v^\times$ is $\chi_{E/F}$.

If we set

$$\gamma_V := \gamma^m, \quad \gamma_W := \gamma^n,$$

then $(\gamma_V, \gamma_W)$ gives a splitting homomorphism

$$\iota_{\gamma_V, \gamma_W} : G \times H \rightarrow \tilde{Sp}(\mathbb{W})$$

and so by composing this to $\omega_\psi$, we have a Weil representation of $G \times H$ on $S(X)$.

When the choice of $\psi$ and $(\gamma_V, \gamma_W)$ is fixed as above, we simply write

$$\omega_{W,V} := \omega_\psi \circ \iota_{\gamma_V, \gamma_W}.$$

Remark 2.1. For $n = 1$, the image of $H = U(1)$ in $\tilde{Sp}(\mathbb{W})$ coincides with the image of the center of $G$, so we can regard the Weil representation of $G \times H$ as the representation of $G$.

2.2. The Local $\Theta$-Correspondence. In this subsection, we deal with only the local case and so we suppress $v$ from the notation. (Note that if $v$ is non-split, $E$ is the quadratic extension of $F$ and in the split case, $E = F \oplus F$.) As in previous subsection, for non-split $v$, we denote $\chi_{E/F}$ the quadratic character associated to $E/F$ by local class field theory and for the split case, $\chi_{E/F}$ is trivial.
2.2.1. Howe Duality. Suppose that \((G, G')\) is a dual reductive pair of unitary groups in a symplectic group \(Sp(W)\). (Recall that a dual reductive pair \((G, G')\) in \(Sp(W)\) is a pair of reductive subgroups of \(Sp(W)\) which are mutual centralizers, i.e. \(Z_{Sp(W)}(G) = G'\) and \(Z_{Sp(W)}(G') = G\).

After fixing the characters \(\psi\) and \(\gamma\) as in subsection 2.1, we obtain a Weil representation \((\omega_{\psi, \gamma}, S)\) of \(G \times G'\). For an irreducible admissible representation \(\pi\) of \(G\), the maximal \(\pi\)-isotypic quotient of \(\omega\), say \(S(\pi)\), is of the form

\[ S(\pi) \cong \pi \otimes \Theta(\pi). \]

The Howe Duality Principle says that if \(\Theta(\pi)\) is nonzero, then

1. \(\Theta(\pi)\) is a finite-length admissible representation of \(G'\).
2. \(\Theta(\pi)\) has the unique maximal semisimple quotient \(\theta(\pi)\) and it is irreducible.
3. The correspondence \(\pi \mapsto \theta(\pi)\) gives a bijection between the irreducible admissible representations of \(G\) and \(G'\) that occur as the maximal semisimple quotients of \(S\).

The third is called the local \(\Theta\)-correspondence. The Howe duality is proved for \(v \nmid 2\) and not yet proved for \(v \mid 2\). In this paper, we assume that Howe duality holds for \(v \mid 2\).

2.3. The Explicit Local Weil representation for \(GL(3)(F_v)\). The local Weil representation of unitary groups is explicitly described in [6]. In particular, if \(v\) splits, \(U(3)(F_v) = \{(A, B) \in M_3(F_v) \mid AB = 1\}\) and so by sending \((x, x^{-1})\) to \(x\), it is identified to \(GL(3)(F_v)\). We record here the explicit local Weil representation of \(GL(3)(F_v)\) for later use.

Let \(X = F_v^3\) be a 3-dimensional vector space over \(F_v\) with a fixed basis. Then there is a Weil-representation \(\omega_{\text{GL}(3)(F_v)}\) realized on \(S(F_v^3)\), which is uniquely determined by the following formula:

\[ \omega(g)f(x) = \gamma(|\det(g)|)^{\frac{3}{2}}f(g^tx), \quad x \in F_v^3 \]

Since \(E_v = F_v \times F_v\) and \(\gamma\), we defined in 2.1, is trivial on \(F_v\), we can write \(\gamma = (\gamma_1, \gamma_1^{-1})\) for some unitary character \(\gamma_1\) of \(F_v\). Using the above isomorphism of \(U(3)\) and \(GL(3)\), we can write \(\gamma(\det(g)) = \gamma_1^2(\det(g))\). We will use this formula in Section 5.

2.4. The Global \(\Theta\)-Correspondence. The global \(\Theta\)-correspondence is realized using \(\Theta\)-series.

To do this, we first define the theta kernel as follows. For any \(\varphi \in \mathcal{S}(\mathcal{X}(A_F))\), let

\[ \theta(g, h, \varphi) := \sum_{\chi \in \mathcal{X}(F)} \omega_{W_{\gamma W}, V_{\gamma V}, \psi}(g, h)(\varphi)(\chi). \]

Note that this is slowly increasing function. Thus if \(f\) is some cusp form on \(G(A_F)\), it is rapidly decreasing and so we can define

\[ \theta(f, \varphi)(h) := \int_{[G]} \theta(g, h, \varphi)f(g) \, dg \]

where \(dg\) is the Tamagawa measure.

Then the \(\Theta\)-lift of a cuspidal representation of \(G\) as follows:

**Definition 2.2.** For a cuspidal automorphic representation \(\pi\) of \(G(A_F)\),

\[ \Theta_{V_{\gamma W}, V_{\gamma V}, \psi}(\pi) = \left\{ \theta(f, \varphi) : f \in \pi, \varphi \in \mathcal{S}(\mathcal{X}(A_F)) \right\} \]

is called the \(\Theta\)-lift of \(\pi\) with data \((\gamma_W, \gamma_V, \psi)\).

The Howe Duality Principle implies the following. ([4], proposition 1.2)

**Proposition 2.3.** If \(\Theta(\pi)\) is a cuspidal representation of \(U(V)(A_v)\), then it is irreducible and is isomorphic to the restricted tensor product \(\otimes_v \theta(\pi_v)\).
Remark 2.4. Since we integrated $\mathcal{I}$ (instead of $f$) against the theta series, $\pi$ and $\Theta(\pi)$ have the same central characters.

Remark 2.5. In the theory of theta lift, there are two main issues, that is, the cuspidality and non-vanishing of the theta lift. The cuspidality issue was treated by Rallis in terms of so-called tower property. So to make our Theorem 1.2 not vacuous, we record the criterion in 3.6 which ensures the non-vanishing of two theta lifts $\pi_3$ and $\pi_2$.

3. The Rallis Inner Product Formula

The Rallis inner product formula enables us to express the Petersson inner product of the global theta lift with respect to the source information. Since we will need three different version of Rallis inner product formulas, we record them for lifts from $U(1)$ to $U(3)$, $U(1)$ to $U(1)$ and $U(1)$ to $U(2)$. To give a uniform description, we introduce some related notions.

3.1. Global and Local zeta-integral. Let $V$ be a hermitian space over $E$ of dimension $m$, and $W$ a skew-hermitian space of dimension $n$. Let $V^-$ be the same space as $V$, but with hermitian form $-\langle \cdot, \cdot \rangle_V$. Note that $U(V) = U(V^-)$. Let $\tau$ be an irreducible cuspidal automorphic representation of $U(V)$.

Denote $\mathcal{G} := U(V) = U(V^-), H := U(W), G^\circ := U(V \oplus V^-)$ and $i : G \times G \to G^\circ$ be the inclusion map $U(V) \times U(V^-) \to U(V \oplus V^-)$. Let $v$ be a finite place of $F$ and $\mathcal{O}_v$ the ring of integer of $F_v$ and denote by $\varpi$ a generator of its maximal ideal. We fix a maximal compact subgroup $K = \prod_v K_v$ of $G$ such that $K_v := G(\mathcal{O}_v)$ for finite places and $K_\infty := G(F_\infty) \cap U(2m)$ for archimedean places. Let $P$ be a Siegel-parabolic subgroup of $G^\circ$ stabilizing $V^\Delta := \{(x, x) \in V \oplus V^-\}$ with Levi-component $GL(V^\Delta)$ and $\tilde{K}$ a maximal compact subgroup of $G^\circ$ such that $i(K \times K) \subseteq \tilde{K}$ and $G^\circ = P\tilde{K}$. Let $I(s, \gamma_W) := \text{Ind}_{P(\mathbb{Z}_p)}^{G^\circ}(\gamma_W \circ \det) \cdot |\det|^s$ be the degenerate principal series representation induced from the character $\gamma_W$ of $\mathbb{A}_F^\times$ and $|\det|^s$. (Here, we took $\gamma_W$ as the one we defined in 2.1 and the determinants are taken with respect to $GL(V^\Delta)$ which is isomorphic to the Levi of $P$.)

Then for $\Phi_s \in I(\gamma_W, s)$, we define the Eisenstein series

$$E(\Phi_s, \mathring{\mathfrak{g}}) := \sum_{x \in P(F)\backslash G^\circ(F)} \Phi_s(x \mathring{\mathfrak{g}})$$

for $\mathring{\mathfrak{g}} \in G^\circ$. Then for $f_1, f_2 \in \tau$, we can define

Definition 3.1. The Piatetski-Shapiro-Rallis zeta integral is defined as follows:

$$Z(s, f_1, f_2, \Phi_s, \gamma_W) := \int_{[G \times G]} f_1(g_1) f_2(g_2) E(\Phi_s, \varpi(g_1, g_2)) \gamma_W^{-1}(\det_U(V^-)g_2) \det g_1 \det g_2.$$

This integral converges only for $\text{Re}(s) \gg 0$. However, once the convergence is ensured, it can be factored into the product of the local-zeta integrals. So we define the local zeta-integrals. Assume that $\Phi_s = \otimes_v \Phi_{s,v}$ and $f_i = \otimes_v f_{i,v}$. Then for each place $v$, the local zeta-integral is defined by

$$Z_v(s, f_{1,v}, f_{2,v}, \Phi_{s,v}) := \int_{U(V)_v} \Phi_{s,v}(i(g_v, 1)) \langle \pi_v(g_v) f_{1,v}, f_{2,v} \rangle \pi_v d g_v$$

We note that the integral defining the $Z_v$ converges for $\text{Re}(s)$ sufficiently large. However, $Z_v$ can be extended to all of $\mathbb{C}$ by meromorphic continuation. For large $s$, there is a factorization theorem of the zeta integral. (See [13] for more detail)
Theorem 3.2. For \( \text{Re}(s) \gg 0 \),
\[ Z(s, f_1, f_2, \Phi_s, \gamma_W) = \prod_v Z_v(s, f_1, f_2, \Phi_s, \gamma_W) \]

The local-zeta integral has a simple form for unramified places. Take \( S \) to be a sufficiently large finite set of places of \( F \) such that for all \( v \notin S \), the relevant data is unramified, and the local vectors \( f_{i,v} \) are normalized spherical vectors so that \( \langle f_{1,v}, f_{2,v} \rangle_{\pi_v} = 1 \). Recall that \( m = \dim E_V, n = \dim E_W \) and set
\[ d_{m,v}(s, \gamma_W) := m - 1 \prod_{r=0}^{m-1} L(2s + m - r, \chi_{E/F}^{n+r}). \]

It is known that for \( v \notin S \), \( Z_v \) has the following simple form,
\[ Z_v(s, f_{1,v}, f_{2,v}, \Phi_{s,v}) = L_v(s + 1/2, \pi \otimes \gamma_W) \cdot d_{m,v}(s, \gamma_W), \]
and so we can normalize them defining \( Z_v^\# \) by
\[ Z_v^\#(s, f_{1,v}, f_{2,v}, \Phi_{s,v}) = \frac{d_{m,v}(s, \gamma_W)}{L_v(s + 1/2, \pi \otimes \gamma_W)} \cdot Z_v(s, f_{1,v}, f_{2,v}, \Phi_{s,v}) \]

Thus, we can rewrite Theorem 3.2 as follows:

For \( f_1, f_2 \in \tau \), we have
\[ Z(s, f_1, f_2, \Phi_s, \gamma_W) = L(s + 1/2, \pi \otimes \gamma_W) \cdot \prod_v Z_v^\#(s, f_1, f_2, \Phi_s, \gamma_W) \]

3.2. The Siegel-Weil section. The Rallis Inner Product Formula relates the Petersson inner product of the global theta lifts to the global zeta-integral for a special section \( \Phi_s \in I(s, \gamma_W) \), so called Siegel-Weil section. In this section, we give the definition of the Siegel-Weil section introducing the doubled Weil representation.

The setting for the doubled Weil representation is as follows.

We have
\[ \mathcal{W} := \text{Res}_{E/F} 2V \otimes E W \]
where \( 2V := V \oplus V^- \). We also denote
\[ V^\nabla := \{(v,-v) : v \in V\} \subset V \oplus V^- \]
Since \( V^\nabla \otimes W \) is a Lagrangian subspace of \( \mathcal{W} \) over \( F \), with some fixed choice of characters \( \psi \) and \( \gamma \), we have a Schrödinger model of the Weil representation \( \tilde{\omega} \) of \( G^\circ \times H \) realized on \( \mathcal{S}((V^\nabla \otimes W)) \).

Now, fix polarizations
\[ V = X^+ \oplus Y^+ \]
\[ V^- = X^- \oplus Y^- \]
and denote
\[ X := X^+ \oplus X^- \]
\[ Y := Y^+ \oplus Y^- \]

Then
\[ 2V = X \oplus Y \]
and so we have another Lagrangian \( X \otimes W \) of \( \mathcal{W} \).
If we set
\[ X := X \otimes W \]
\[ X^+ := X^+ \otimes W \]
\[ X^- := X^- \otimes W, \]
then there is a \( U(V)(\mathbb{A}_F) \times U(V^-)(\mathbb{A}_F) \)-intertwining map
\[ \rho_{m,n} : S(X^+)(\mathbb{A}_F) \otimes S(X^-)(\mathbb{A}_F) \to S(X)(\mathbb{A}_F) \to S(V^\vee \otimes W)(\mathbb{A}_F) \]
where the first map is the obvious one, and the second map is given by the Fourier transform. Furthermore, it satisfies \( \rho_{m,n}(\varphi_1 \otimes \varphi_2)(0) = \langle \varphi_1, \varphi_2 \rangle \) and so \( \langle \hat{\omega}(i(g,1)) \cdot \rho_{m,n}(\varphi_1 \otimes \varphi_2)(0) = \langle \omega_{W,V}(g) \cdot \varphi_1, \varphi_2 \rangle \). By the explicit formula for \( \hat{\omega} \) described in [11], there is an intertwining map \([ \cdot ] : S(V^\vee \otimes W) \to I(s_m, \gamma_W)\) given by \( \Phi \to f_\rho^\Phi(g) = \hat{\omega}(\Phi)(0). \)
We can also extend \( f_\rho^\Phi \) to \( f_\rho^\Phi \in I(s, \gamma_W) \) for all \( s \in \mathbb{C} \) by defining \( f_\rho^\Phi := f_\rho^\Phi \cdot |\det|^\sigma \) and call this the Siegel-Weil section in \( I(s, \gamma_W) \). (Here the determinant map was taken as in [11]) Then we can define the function \( \Delta_m \) of \( G \) as \( \Delta_m(g) := |\det(i(g,1))| \) and using \( \Delta_m \), we can write the Siegel-Weil section as
\[ (3.3) \quad f_{[\rho_{m,n}(\varphi_1 \otimes \varphi_2)]}(g) = \langle \omega_{W,V}(g) \cdot \varphi_1, \varphi_2 \rangle \cdot \Delta_m(g)^{s-m}. \]
Note that \( \Delta_m(g) \) is \( K \times K \) invariant and \( \Delta_m(1) = 1 \). (For \( k_1, k_2 \in K \), \( (k_1, k_1) \cdot (g, 1) \cdot (k_2, k_2^{-1}) \) and \( (k_1, k_1) \in P \), \( (k_2, k_2^{-1}) \in K \)) Using the similar argument of Prop.6.4 in [15], Yamana [22, Lemma A.4.] computed \( \Delta_m(g_v) \) explicitly for split place \( v \) of \( F \). We record his computation for the non-archimedean split places not dividing 2.
Let \( v \) be a finite place of \( F \) which splits in \( E \) and not divide 2. Let \( \mathcal{O}_v \) be the ring of integer of \( F_v \) and \( \varpi \) a generator of its maximal ideal. Since \( v \) splits, \( U(m)(F_v) \simeq GL(m)(F_v) \) and by Cartan decomposition, \( GL(m)(F_v) = K_v D^+_m K_v \) where \( K_v = GL(m)(\mathcal{O}_v) \) and \( D^+_m = \text{diag}[^{\varpi^a}, \ldots, ^{\varpi^m}] \). Then,
\[ (3.4) \quad \Delta_m(g_v) = |\varpi|^{\sum_{i=1}^{m}|a_i|} \]

**Remark 3.3.** Since \( |a + b| \neq |a| + |b| \), we cannot expect \( \Delta_m(g_v l_v) \neq \Delta_m(g_v) \Delta_m(l_v) \) for central diagonal matrix \( l_v = \text{diag}[^{\varpi^e}, \ldots, ^{\varpi^m}] \in GL(m)(F_v). \)

Now, we are ready to state the three versions of Rallis Inner Product formula. The first one is as follows:

**3.3. Lifting from \( U(1) \) to \( U(3) \).** Here, \( \dim V = 1, \dim W = 3 \) and \( \tau \) is an irreducible automorphic representation of \( U(1)(\mathbb{A}_F) \). Suppose that \( f_1 = \otimes_v f_{1,v} \in \tau, \varphi_1 = \otimes_v \varphi_{1,v} \in S(X^+)(\mathbb{A}_F) \) and \( \varphi_2 = \otimes_v \varphi_{2,v} \in S(X^-)(\mathbb{A}_F) \). Let \( \Phi_{s,v} \in I(s, \gamma^{3}) \) is a holomorphic Siegel-Weil section given by \([\rho_{1,3}(\varphi_1 \otimes \varphi_2)]\). Then,

**Theorem 3.4.**
\[ (\theta(f_1, \varphi_1), \theta(f_2, \varphi_2))_{\theta(\tau)} = \frac{L_E(\frac{3}{2}, BC(\tau) \otimes \gamma^3)}{L(3, \chi_{E/F})} \prod_v Z^s_v(1, f_{1,v}, f_{2,v}, \Phi_{1,v}) \]
where
\[ Z^s_v := \frac{L_v(3, \chi_{E/F})}{L_v(\frac{3}{2}, BC(\tau_v) \otimes \gamma_v^3)} \cdot Z_v \]

**Proof.** This follows immediately from Theorem 2.1 in [12] and [3.2] the normalization of the local-zeta integral. □

The next two versions of Rallis Inner product formula come from Lemma 10.1 in [21].
3.4. Lifting from \( U(1) \) to \( U(1) \). Here, \( \dim V = \dim W = 1 \) and \( \tau \) is an irreducible automorphic representation of \( U(1)\( \mathbb{A}_F \). Suppose that \( f_i = \otimes_v f_{i,v} \in \tau, \ \varphi_1 = \otimes_v \varphi_{1,v} \in S(X^+(\mathbb{A}_F)) \) and \( \varphi_2 = \otimes_v \varphi_{2,v} \in S(X^-(\mathbb{A}_F)) \). Let \( \Phi_{s,v} \in I(s,\gamma) \) is a holomorphic Siegel-Weil section given by \([\rho_{p,1}(\varphi_1 \otimes \varphi_2)]\). By \[21\] Theorem 4.1 and \[3.2\] we have

**Theorem 3.5.**

\[
\langle \theta(f_1, \varphi_1), \theta(f_2, \varphi_2) \rangle_{\Theta(\tau)} = \frac{1}{2} \frac{L_E(\frac{s}{2}, BC(\tau) \otimes \gamma)}{L(1, \chi_{E/F})} \prod_v \mathcal{Z}^2_v(0, f_{1,v}, f_{2,v}, \Phi_{0,v})
\]

where

\[
\mathcal{Z}^2_v = \frac{L_v(1, \chi_{E/F} + 1)}{L_v(1, \chi_{E/F})} \cdot Z_v
\]

3.5. Lifting from \( U(2) \) to \( U(1) \). Here, \( \dim V = 2 \), \( \dim W = 1 \) and \( \tau \) is an irreducible automorphic representation of \( U(2)\( \mathbb{A}_F \). Suppose that \( f_i = \otimes_v f_{i,v} \in \tau, \ \varphi_1 = \otimes_v \varphi_{1,v} \in S(X^+(\mathbb{A}_F)) \) and \( \varphi_2 = \otimes_v \varphi_{2,v} \in S(X^-(\mathbb{A}_F)) \). Let \( \Phi_{s,v} \in I(s,\gamma) \) be a holomorphic Siegel-Weil section given by \([\rho_{p,2}(\varphi_1 \otimes \varphi_2)]\). Then,

**Theorem 3.6.**

\[
\langle \theta(f_1, \varphi_1), \theta(f_2, \varphi_2) \rangle_{\Theta(\tau)} = \frac{-\text{Res}_{s=0}(L_E(s, BC(\tau) \otimes \gamma))}{L(1, \chi_{E/F})} \prod_v \mathcal{Z}^2_v(s, f_{1,v}, f_{2,v}, \Phi_{s,v})
\]

where

\[
\mathcal{Z}^2_v(s, f_{1,v}, f_{2,v}, \Phi_{s,v}) = \lim_{s \to 0} \frac{L_v(2s + 1, \chi_{E/F} \cdot \gamma_v)}{L_v(2s, BC(\tau_v) \otimes \gamma_v)} \cdot Z_v(s) = \frac{1}{2} \frac{L(1, \chi_{E/F}) \cdot Z(0)}{L(1, \chi_{E/F})}
\]

Proof. By Lemma 10.1 (2) in \[21\] and \[3.2\],

\[
\langle \theta(f_1, \varphi_1), \theta(f_2, \varphi_2) \rangle_{\Theta(\tau)} = \frac{1}{2} \lim_{s \to 0} \frac{L_E(s, BC(\tau) \otimes \gamma)}{L(2s + 1, \chi_{E/F})} \cdot Z_v(s) = \frac{1}{2} \frac{L(1, \chi_{E/F}) \cdot Z(0)}{L(1, \chi_{E/F})}
\]

For each \( v \), \( d_v(s - \frac{1}{2}, \gamma \tau_v) \cdot \Phi_{s-\frac{1}{2},v}(g) \) is not holomorphic but good section (see, \[21\]), so by Theorem 5.2 in \[21\], the quotient of \( L_v(2s + 1, \chi_{E/F} \cdot \gamma_v) \cdot Z_v(s - \frac{1}{2}, f_{1,v}, f_{2,v}, \Phi_{s-\frac{1}{2},v}) \) by \( L_v(s, BC(\tau_v) \otimes \gamma_v) \) is holomorphic. Thus each \( \mathcal{Z}^2_v(s, f_{1,v}, f_{2,v}, \Phi_{s,v}) \) exists and it proves theorem when \( \Theta(\tau) \) is nonvanishing. When \( \Theta(\tau) \) is zero, then \( L_E(s, BC(\tau) \otimes \gamma) \) is holomorphic by Lemma 10.2 in \[21\], and so \( \text{Res}_{s=0}(L_E(s, BC(\tau) \otimes \gamma)) \) is zero. So the theorem also holds in this case. \( \square \)

3.6. The local-to-global criterion for the non-vanishing of the theta lifts. Since we will assume \( \pi_3 \) and \( \pi_2 \) are non-vanishing, we give two non-vanishing criterion of the theta lifts \( \pi_1, \pi_2 \).

3.6.1. Theta lift from \( U(1) \) to \( U(3) \). Let \( \tau \) be a character of \( U(1) \). Note that \( L_E(\frac{1}{2}, BC(\tau) \otimes \gamma^3) \neq 0 \). Then by \[3.4\], we see that \( \pi_3 = \Theta(\tau) \) non-vanishes when the local theta integral

\[ Z_v(1, \cdot) \in \text{Hom}(I(1, \gamma_v) \otimes \tau_v \otimes \tau_v) \]

is nonzero for all the places \( v \).

3.6.2. Theta lift from \( U(1) \) to \( U(2) \). Let \( \tau \) be a character of \( U(1) \). Then by [Theorem 5.10, \[7\]], the theta lift \( \pi_3 = \Theta(\tau) \) does not vanish when \( L_E(1, BC(\tau) \otimes \gamma^2) \neq 0 \) and local theta lift \( \theta_v(\tau_v) \neq 0 \) for all the places \( v \).
4. Proof of Theorem 1.2

We remind the reader of our setting.

4.1. The Setup. $F$ is a totally real number field and $E$ a totally imaginary quadratic extension of $F$. We consider the following seesaw diagram:

\[
\begin{array}{ccc}
U(V \oplus L) & U(W) \times U(W) \\
\downarrow & \downarrow \\
U(V) \times U(L) & U(W)
\end{array}
\]

(Here, $V$ is a 2-dimensional hermitian space over $E/F$ and $W$ is a 1-dimensional skew-hermitian space over $E/F$ and $L$ is a hermitian line over $E/F$.

Using the seesaw duality, we will relate the period integral in Theorem to the triple product integral over $U(W)$.

We first fix the following:

- $\pi_2 = \otimes \pi_{2,v}$ is an irreducible, cuspidal, tempered, automorphic representation of $U(V)(A_F)$.
- $\sigma = \otimes \sigma_v$ is an automorphic character of $U(W)(A_F)$.
- $\mu := w_{\pi_2}^{-1} \cdot \sigma$ is an automorphic character of $U(L)(A_F)$, where $\omega_{\pi_2}$ is the central character of $\pi_2$ and $\mu = \otimes \mu_v$ where $\mu_v = w_{\pi_2,v}^{-1} \cdot \sigma_v$.
- $(\omega_{W \otimes L, \psi})$ is a Weil representation of $\widetilde{Sp}(W)(A_F)$. (See Chapter 2 for notation.)

We also fix local pairings $B_{\pi_2,v}, B_{\sigma_v}, B_{\mu_v}$ such that $\prod_v B_{\pi_2,v}, \prod_v B_{\sigma_v}, \prod_v B_{\mu_v}$ give the respective Petersson inner products on the global representation and $B_{\mu_v}(\mu_v, \mu_v) = B_{\sigma_v}(\sigma_v, \sigma_v)$ for all places $v$. (Since $B_{\sigma}(\sigma, \sigma) = B_{\mu}(\mu, \mu) = \text{Vol}([U(1)])$, these choices can stand with no conflict.)

We take $\gamma_L, \gamma_W = \gamma$ and $\gamma_V = \gamma^2$, where $\gamma$ is a unitary character of $\mathbb{A}_E^\times/E^\times$ such that $\gamma|_{A_F^\times} = \chi_{E/F}$ and fix additive character $\psi : A_F \to \mathbb{C}$. After fixing these splitting data $(\gamma_V, \gamma_L, \gamma_W, \psi)$, we can define the relevant theta lifts and denote them $\Theta(\pi_2) := \Theta_{W,V,\gamma_L,\gamma_W,\psi}(\pi_2)$ on $U(W)(A_F)$, $\Theta(\sigma) := \Theta_{W,V \otimes L, \omega_L, \gamma_V, \gamma_W, \gamma_L, \psi}(\sigma)$ on $U(V \oplus L)(A_F)$, and $\Theta(\mu) := \Theta_{W,L,\gamma_W,\gamma_L, \psi}(\mu)$ on $U(W)(A_F)$. We assume that all $\Theta$-lifts we consider here are non-vanishing and cuspidal.

4.2. Proof of Theorem 1.2. In the course of the proof, we will regard $\mu$ and $\sigma$ as automorphic forms in the 1-dimension representations of $\mu$ and $\sigma$ and take $f_\mu = \mu$, and $f_\sigma = \sigma$. Since $\omega_{W,V \otimes L} = \omega_W V \otimes \omega_W L$, we prove the theorem assuming $\varphi = \varphi_1 \otimes \varphi_2$ for $\varphi_1 \in \omega_W V$ and $\varphi_2 \in \omega_W L$.

Step 1. First, we consider another global period

$$\mathcal{P}' : V_{\Theta(\sigma)} \otimes V_{\pi_2} \otimes V_{\mu} \to \mathbb{C}$$

defined by

$$\mathcal{P}'(f_{\Theta(\sigma)}, f_{\pi_2}, f_{\mu}) := \left| \int_{[U(V) \times U(L)]} f_{\Theta(\sigma)}(i(g,l))f_{\pi_2}(g)f_{\mu}(l)dgdl \right|^2.$$

(Here, $i$ is the natural embedding $i : U(V) \times U(L) \hookrightarrow U(V \oplus L).$)

By making a change of variables $g \to gl$, we see that

$$\int_{[U(V) \times U(L)]} f_{\Theta(\sigma)}(i(g,l))f_{\pi_2}(g)f_{\mu}(l)dgdl = \int_{[U(V) \times U(L)]} f_{\Theta(\sigma)}(i(gl,l))f_{\pi_2}(gl)f_{\mu}(l)dgdl.$$
By Remark 2.4, the central character of $\Theta(\sigma)$ is $\omega_{\sigma}^{-1} = \sigma^{-1}$. So, after observing that $(l, l)$ is in the center of $U(V \oplus L)$ and $l$ is in the center of $U(V)$, we have

$$
\int_{[U(V) \times U(L)]} f_{\Theta(\sigma)}(i(g(l), l)) f_{\pi_2}(g(l)) \mu(l) dgdll
= \int_{[U(V) \times U(L)]} \omega_{\Theta(\sigma)}(l) \omega_{\pi_2}(l) \mu(l) f_{\Theta(\sigma)}|_{U(V)}(g) f_{\pi_2}(g) dgdll
= \int_{[U(V) \times U(L)]} f_{\Theta(\sigma)}|_{U(V)}(g) f_{\pi_2}(g) dgdll
= \text{Vol}([U(L)]) \int_{[U(V)]} f_{\Theta(\sigma)}|_{U(V)}(g) f_{\pi_2}(g) dg
= 2 \int_{[U(V)]} f_{\Theta(\sigma)}|_{U(V)}(g) f_{\pi_2}(g) dg. \quad \text{(note that Vol}([U(1)]) = 2)\]

Thus, we get $P(f_{\Theta(\sigma)}, f_{\pi_2}) = \frac{1}{2} P'(f_{\Theta(\sigma)}, f_{\pi_2}, f_{\mu})$.

**Step 2.** By the global seesaw duality, we see that

$$
\int_{[U(V) \times U(L)]} \theta(\bar{\sigma}, \varphi)(i(g(l), l)) f_{\pi_2}(g(l)) \mu(l) dgdll = \int_{[U(W)]} \theta(f_{\pi_2}, \varphi_1)(l) \theta(\bar{\mu}, \varphi_2)(l) \sigma(h) dh
$$

(The order change of integration is justified by the rapidly decreasing property of cusp forms and the moderate growth of the theta series.)

Since $\Theta(\pi_2)$ and $\Theta(\mu)$ have central characters $\omega_{\pi_2}^{-1}$ and $\mu^{-1}$ respectively, we see that

$$
P'(\theta(f_{\pi_2}, \varphi_1), \theta(\bar{f}_\mu, \varphi_2), f_\sigma) = |\theta(f_{\pi_2}, \varphi_1)(1) \theta(\bar{\mu}, \varphi_2)(1) \sigma(1)|^2 \cdot \text{Vol}([U(W)])^2.
$$

For $\tau = \pi_2$ or $\mu$ and $i = 1, 2$,

$$
\mathcal{B}_{\Theta(\tau)}(\theta(f_\tau, \varphi_1), \theta(\bar{f}_\tau, \varphi_i)) = |\theta(f_\tau, \varphi_1)(1)|^2 \cdot \text{Vol}([U(W)]) \text{ and } \sigma(1) = 1.
$$

Thus we can write

$$
P'(\theta(f_{\pi_2}, \varphi_1), \theta(\bar{f}_\mu, \varphi_2), f_\sigma) = \mathcal{B}_{\Theta(\pi_2)}(\theta(f_{\pi_2}, \varphi_1), \theta(\bar{f}_\pi, \varphi_1)) \cdot \mathcal{B}_{\Theta(\mu)}(\theta(f_\mu, \varphi_2), \theta(\bar{f}_\mu, \varphi_2)).
$$

By theorem 3.5 and 3.6 we see that

$$
P'(\theta(f_{\pi_2}, \varphi_1), \theta(\bar{f}_\mu, \varphi_2), f_\sigma) = -\frac{1}{2} \cdot \frac{L_E(\frac{1}{2}, BC(\mu) \otimes \gamma)}{L(1, \chi_{E/F})} \cdot \prod_{s, \gamma \neq 0} \frac{L_E(s, BC(\pi_2) \otimes \gamma)}{L(1, \chi_{E/F})} \cdot Z^2_{\psi}(f_{\mu}, f_{\pi_2}, \varphi_1, \varphi_2, v)
$$

where $Z^2_{\psi}(f_{\pi_2}, f_{\mu_0}, \varphi_1, \varphi_2, v) = Z^2_{\psi}(s, f_{\pi_2}, f_{\pi_2}, \Phi_0, v) \cdot Z^2_{\psi}(0, f_{\mu_0}, f_{\mu_0}, \Phi_0, v)$ and

$$
\Phi_0 = [p_{2,1}(\varphi_1 \otimes \bar{\varphi}_1)] \in I(s, \gamma), \Phi_0, v = [p_{1,1}(\varphi_2 \otimes \bar{\varphi}_2)] \in I(0, \gamma).
$$

(Note that $Z^2_{\psi}(f_{\pi_2}, f_{\mu_0}, \varphi_0, \varphi_2, v) = 1$ for unramified data)

**Step 3.** Recall the abbreviations for various matrix coefficients made in Theorem 1.2

$$
\mathcal{B}_{\omega_{\psi}}^{\pi_2}(g_v) := \mathcal{B}_{\omega_{W}}^{\pi_2}(\omega_{W}, V(g_v), \varphi_1, \varphi_1, \varphi_1), \mathcal{B}_{\omega_{\psi}}^{\pi_2}(l_v) := \mathcal{B}_{\omega_{W}}^{\pi_2}(\omega_{W}, L(l_v), \varphi_2, \varphi_2),
\mathcal{B}_{\omega_{\psi} V, \varphi_1}(g_v, l_v) := \mathcal{B}_{\omega_{W}, \varphi_1}(\omega_{W}, V(l_v), \varphi_1, \varphi_2, \varphi_2)
$$

and

$$
\mathcal{B}_{\pi_2, \varphi}(g_v) := \mathcal{B}_{\pi_2, \varphi}(g_v, f_{\pi_2}, f_{\pi_2}, \varphi_1), \mathcal{B}_{\pi_2, \varphi}(l_v) := \mathcal{B}_{\pi_2, \varphi}(l_v, f_{\pi_2}, f_{\pi_2}) \text{ for } \tau = \sigma \text{ or } \mu.
If we unfold $Z_{s,v}^t(f_{s_1,s_2},f_{s_2,v},\Phi_{s,v})$ in $Z_{t}^s(f_{s_2,v},f_{s,v},\Phi_{s,v},\varphi_{s,v})$, we can write

$$Z_{t}^s(f_{s_2,v},f_{s,v},\Phi_{s,v},\varphi_{s,v}) = \lim_{s \to 0^+} \frac{L_{t}(2s + 1, \chi_{E_{2}}/\Lambda_{t})}{L_{t}(s, BC(\pi_{2,v}) \otimes \gamma_{t})} \cdot \int_{U(V)} Z_{s,v}^t(0, f_{s,v}, f_{s,v}, \Phi_{s,v}, \varphi_{s,v}) \Delta_{2}(g_{v}) \cdot dg_{v}$$

$$= \lim_{s \to 0^+} \frac{L_{t}(1, \chi_{E_{2}}/\Lambda_{t})}{L_{t}(s, BC(\pi_{2,v}) \otimes \gamma_{t})} \cdot \int_{U(V)} \zeta_{t}(2s) \cdot \mathcal{I}_{v}(s, \varphi_{s,v}, f_{s_2,v}, f_{s,v})$$

where

$$\mathcal{I}_{v}(s, \varphi_{s,v}, f_{s_2,v}, f_{s,v}) := \int_{U(V)} \mathcal{B}_{\omega_{W,L}}^{\varphi_{s,v}}(g_{v}) \cdot B_{\mu_{s,v}}^{f_{s,v}}(l_{v}) \cdot \mathcal{B}_{\omega_{W,L} \otimes \Lambda_{t}}^{\varphi_{s,v}}(g_{v}) \cdot B_{\mu_{s,v}}^{f_{s,v}}(l_{v}) \cdot \Delta_{2}(g_{v}) \cdot dg_{v}.$$

Set $J(s, g_{v}, l_{v}, \varphi_{s,v}, f_{s_2,v}, f_{s,v}) := \mathcal{B}_{\omega_{W,L}}^{\varphi_{s,v}}(g_{v}) \cdot B_{\mu_{s,v}}^{f_{s,v}}(l_{v}) \cdot \mathcal{B}_{\omega_{W,L} \otimes \Lambda_{t}}^{\varphi_{s,v}}(g_{v}) \cdot B_{\mu_{s,v}}^{f_{s,v}}(l_{v}) \cdot \Delta_{2}(g_{v})$. Then we can write $\mathcal{I}_{v}(s, \varphi_{s,v}, f_{s_2,v}, f_{s,v})$ as a double integral,

$$\mathcal{I}_{v}(s, \varphi_{s,v}, f_{s_2,v}, f_{s,v}) = \int_{U(V)} \int_{U(L)} J(s, g_{v}, l_{v}, \varphi_{s,v}, f_{s_2,v}, f_{s,v}) \cdot dg_{v} \cdot dl_{v}.$$

Since $\pi_{2}$ is tempered, by Lemma 7.2 in [21], $Z_{s}(f_{s_2,v}, f_{s,v}, [\rho(\varphi_{s,v} \otimes \varphi_{s,v})])$ absolutely converge for $\Re(s) > 0$ and so $Z_{s}(0, f_{s,v}, f_{s,v}, [\rho(\varphi_{s,v} \otimes \varphi_{s,v})])$ does. For $\Re(s) > 0$, $I_{v}(s)$ is just the product of $Z_{s}(f_{s_2,v}, f_{s_2,v}, [\rho(\varphi_{s_2,v} \otimes \varphi_{s_2,v})])$ and $Z_{s}(0, f_{s,v}, f_{s,v}, [\rho(\varphi_{s,v} \otimes \varphi_{s,v})])$, the above double integral for $I_{v}(s)$ absolutely converges for $\Re(s) > 0$.

**Step 4.** By making a change of variables $g_{v} \to g_{v} l_{v}$,

$$\mathcal{I}_{v}(s, \varphi_{s,v}, f_{s_2,v}, f_{s,v}) = \int_{U(V) \times U(L)} J(s, g_{v} l_{v}, \varphi_{s,v}, f_{s_2,v}, f_{s,v}) \cdot dg_{v} \cdot dl_{v}$$

$$= \int_{U(V) \times U(L)} \mathcal{B}_{\omega_{W,L}}^{\varphi_{s,v}}(g_{v} l_{v}) \cdot B_{\mu_{s,v}}^{f_{s,v}}(l_{v}) \cdot \mathcal{B}_{\omega_{W,L}}^{\varphi_{s,v}}(g_{v} l_{v}) \cdot B_{\mu_{s,v}}^{f_{s,v}}(l_{v}) \cdot \Delta_{2}(g_{v} l_{v}) \cdot dg_{v} \cdot dl_{v}$$

$$= \int_{U(V) \times U(L)} \mathcal{B}_{\omega_{W,L} \otimes \Lambda_{t}}^{\varphi_{s,v}}(g_{v} l_{v}) \cdot B_{\mu_{s,v}}^{f_{s,v}}(l_{v}) \cdot \mathcal{B}_{\omega_{W,L} \otimes \Lambda_{t}}^{\varphi_{s,v}}(g_{v} l_{v}) \cdot B_{\mu_{s,v}}^{f_{s,v}}(l_{v}) \cdot \Delta_{2}(g_{v} l_{v}) \cdot dg_{v} \cdot dl_{v}$$

$$= \int_{U(V) \times U(L)} \mathcal{B}_{\omega_{W,L} \otimes \Lambda_{t}}^{\varphi_{s,v}}(g_{v} l_{v}) \cdot B_{\mu_{s,v}}^{f_{s,v}}(l_{v}) \cdot \mathcal{B}_{\omega_{W,L} \otimes \Lambda_{t}}^{\varphi_{s,v}}(g_{v} l_{v}) \cdot B_{\mu_{s,v}}^{f_{s,v}}(l_{v}) \cdot \Delta_{2}(g_{v} l_{v}) \cdot dg_{v} \cdot dl_{v}$$

(The last equality follows from $\mathcal{B}_{\lambda_{v}}(f_{s,v}, f_{s,v}) = B_{\mu_{s,v}}(f_{s,v}, f_{s,v})$).

**Step 5.** By the lemma 5.1 in the next section, we see that

$$\lim_{s \to 0^+} \frac{\zeta_{t}(2s)}{L_{t}(s, BC(\pi_{2,v}) \otimes \gamma_{t})} \int_{U(V) \times U(L)} \mathcal{B}_{\omega_{W,L} \otimes \Lambda_{t}}^{\varphi_{s,v}}(g_{v} l_{v}) \cdot B_{\mu_{s,v}}^{f_{s,v}}(l_{v}) \cdot \Delta_{2}(g_{v} l_{v}) \cdot dg_{v} \cdot dl_{v} =$$

$$\lim_{s \to 0^+} \frac{\zeta_{t}(2s)}{L_{t}(s, BC(\pi_{2,v}) \otimes \gamma_{t})} \int_{U(V) \times U(L)} \mathcal{B}_{\omega_{W,L} \otimes \Lambda_{t}}^{\varphi_{s,v}}(g_{v} l_{v}) \cdot B_{\mu_{s,v}}^{f_{s,v}}(l_{v}) \cdot \Delta_{2}(g_{v} l_{v}) \cdot dg_{v} \cdot dl_{v} =$$

$$\lim_{s \to 0^+} \frac{\zeta_{t}(2s)}{L_{t}(s, BC(\pi_{2,v}) \otimes \gamma_{t})} \int_{U(V)} Z(1, \varphi_{s,v}, f_{s,v}, [\rho(g_{v} \circ \varphi_{s,v})]) \cdot B_{\mu_{s,v}}^{f_{s,v}}(g_{v}) \cdot \Delta_{2}(g_{v}) \cdot dg_{v}.$$
We normalize $Z_v(1, f_{\sigma_v}, f_{\varphi_v}, [\rho(\varphi_v \otimes \varphi_v)])$ by
$$Z_v^2(1, f_{\sigma_v}, f_{\varphi_v}, [\rho(\varphi_v \otimes \varphi_v)]) := \frac{L_v(3, \chi_{E/F})}{L_v(3/2, BC(\sigma_v) \otimes \gamma_v^3)} \cdot Z_v(1, f_{\sigma_v}, f_{\varphi_v}, [\rho(\varphi_v \otimes \varphi_v)]).$$

We define the local inner product $B_{\Theta(\sigma_v)}$ on $\theta_0(\sigma_v)$ as follows:
$$\mathcal{B}_{\theta(\sigma_v)}(\theta_v(f_{\sigma_v}, \varphi_v), \theta_v(f_{\sigma_v}, \varphi_v)) := \begin{cases} \frac{L_v(3/2, BC(\sigma) \otimes \gamma_v^3)}{L_v(3, \chi_{E/F})} \cdot Z_v^2(1, f_{\sigma_v}, f_{\varphi_v}, [\rho(\varphi_v \otimes \varphi_v)]) & \text{for some place } v \\ Z_v^2(1, f_{\sigma_v}, f_{\varphi_v}, [\rho(\varphi_v \otimes \varphi_v)]) & \text{for the remaining places} \end{cases}$$

Then we see that
$$\mathcal{B}_{\theta(\sigma_v)}(\theta_v(f_{\sigma_v}, \varphi_v), \theta_v(f_{\sigma_v}, \varphi_v)) = \prod_v \mathcal{B}_{\theta(\sigma_v)}(\theta_v(f_{\sigma_v}, \varphi_v), \theta_v(f_{\sigma_v}, \varphi_v))$$
and $\mathcal{B}_{\theta(\sigma_v)}(\theta_v(f_{\sigma_v}, \varphi_v), \theta_v(f_{\sigma_v}, \varphi_v)) = 1$ for unramified data $(f_{\sigma_v}, \varphi_v)$. (Note that the 'small' local theta-lift is the maximal semisimple quotient of the 'big' theta-lift, and so we should check whether these pairings are well-defined. But since we are assuming $\Theta(\sigma)$ is cuspidal, it is semisimple and so $\mathcal{B}_{\theta(\sigma_v)}(\theta(f_{\sigma_v}, \varphi_v))$ factors as a map $\sigma_v \otimes \sigma_v \otimes \varphi_{w_{\sigma_v \cdot 0}} \otimes \varphi_{w_{\sigma_v \cdot 0}} \rightarrow \Theta(\sigma) \otimes \Theta(\sigma)$. Thus theorem (3.4) shows that $B_{\Theta(\sigma)}$ descends to $B_{\Theta(\sigma)}$.)

**Step 6.** With the things we developed so far, we see that
$$\mathcal{P}(\theta(f_{\sigma_v}, f_{\varphi_v})) = \frac{1}{4} \mathcal{P}(\theta(f_{\sigma_v}, f_{\varphi_v})) = \frac{1}{4} \mathcal{P}(\theta(f_{\varphi_v}, f_{\varphi_v})) = \frac{1}{4} \mathcal{P}(\theta(f_{\varphi_v}, f_{\varphi_v})) - \frac{1}{4} \mathcal{P}(\theta(f_{\varphi_v}, f_{\varphi_v})) \cdot Z_v(f_{\sigma_v}, f_{\varphi_v}, \varphi_v, \varphi_v)$$

This proves the theorem.

**Remark 4.1.** Since $Z_v(0, f_{\mu_v}, f_{\mu_v}, \Phi_{\mu_v}) \cdot Z_{v, s, n = -\frac{1}{2}}(s, f_{\sigma_v}, f_{\sigma_2}, \Phi_{\frac{1}{2}}) = 1$ for unramified vectors, our local periods $P_v$'s are also 1 at infinitely many places and so the above product is indeed a finite product.

5. **Proof of Lemma 5.1**

In this section, we prove the lemma upon which we developed Step 5 in the proof of [12]. We retain the same notations as in the previous section and since everything occurs in local case, we suppress $v$ from the notation. We remind the reader that $\varphi_v$ is given by the theta lift of the trivial character $I$ of $U(1)$.

**Lemma 5.1.** Let $t$ be the order of $\frac{2}{\Delta_v^{(2s)}(s, BC(\sigma_v) \otimes \gamma_v)}$ at $s = 0$. Then,

$$\lim_{R(s) \rightarrow 0^+} \int_{U(V) \times U(L)} \beta_{\omega_{w_{\sigma_v \cdot 0}}}(g, l) \cdot B_{\beta_{\sigma_v}}(g) \cdot B_{\beta_{\gamma_v}}(l) \cdot (\Delta_2(g)^s - \Delta_2(g)\gamma) \, dg \, dl = 0$$

**Proof.** When $E$ is quadratic field extension of $F$, $U(L)$ is the centralizer of $U(V)$ and compact and so it is included in every maximal compact subgroup of $U(V)$. Then $\Delta_2(g)^s - \Delta_2(g)\gamma = 0$ and so the lemma is immediate in this case. So we assume $E = F \times F$ and by our hypothesis, all archimedean places do not split, and so we consider only p-adic only.
Since $E = F \times F$, $U(n) \simeq GL_n(F)$ and by Cartan decomposition, $GL_1(F) = \bigcup_{l \in \mathbb{Z}} \mathbb{W}^l K_1$, $GL_2(F) = \bigcup_{n \in \mathbb{Z}, m \geq 0} K_2 \left( \mathbb{W}^{n+m} \mathbb{W}^n \right) K_2$. (Here, $\mathcal{O}$ is the ring of integers of $F$ and $\mathbb{W}$ is a uniformizer of $\mathcal{O}$ and $K_1 = GL_2(\mathcal{O})$.)

Since the theta lift preserves the central character, $\omega_{\mathcal{O}}(\mathbb{W}) = 1$ and let $\alpha = \sigma(\mathbb{W})$. For $i = 1, 2$ and diagonal matrix $m \in GL_2(F)$, let $\mu_i(m) := \frac{Vol(K_1 m K_1)}{Vol(K_1)^2}$. Since $GL_1(F)$ is abelian, $\mu_1(m) = 1$ and by the Lemma 2.1 in [17], $\mu_2(\text{diag}(a, b)) = C \cdot \frac{1}{|a|}$ for some constant $C \in \mathbb{R}_{>0}$.

Then the measure decomposition formula turns (5.1) to

$$\prod_{(s) \to \infty} \sum_{n, l \in \mathbb{Z}, m \geq 0} \alpha^l \cdot |\mathbb{W}|^{-m} \cdot \left( |\mathbb{W}|^{s(n+m+l)+|n+l|} - |\mathbb{W}|^{s(n+m)+|n|} \right) \cdot I(s, \varphi, f_{\pi_2}, m, n, l) = 0$$

where $I(s, \varphi, f_{\pi_2}, m, n, l) = \int_{K_2 \times K_2} B_{\omega, \nu, \varphi, l} \left( k_2 \text{diag}(\mathbb{W}^{n+m}, \mathbb{W}^n) k_2, \mathbb{W}^l k_1 \right) \cdot B_{\pi_2}^{l} (k_2 \text{diag}(\mathbb{W}^m, 1) k_2^l) dk_1 dk_2 dk_2^l$.

Since $\varphi$ and $f_{\pi_2}$ are $K \times K$-finite functions, we are sufficient to show

$$\prod_{(s) \to \infty} \sum_{n, l \in \mathbb{Z}, m \geq 0} \alpha^l \cdot |\mathbb{W}|^{-m} \cdot \left( |\mathbb{W}|^{s(n+m+l)+|n+l|} - |\mathbb{W}|^{s(n+m)+|n|} \right) \cdot c_{n, m, l} \cdot d_m = 0$$

where $c_{n, m, l} = B_{\omega, \nu, \varphi, l} (\text{diag}(\mathbb{W}^{n+m}, \mathbb{W}^n), \mathbb{W}^l)$ and $d_m = B_{\pi_2}^{l} (\text{diag}(\mathbb{W}^m, 1))$.

Now we invoke the asymptotic formulas of $c_{n, m, l}$ and $d_m$. Recall (2.1) in Section 2.2 and write $c = c_1^2(\mathbb{W})$. (Note that $|c| = 1$.) Since $\varphi$ is locally constant and has compact support, there is $l_1 \in \mathbb{N}$ such that for $X, Y \in F^3$, if $|X - Y| \leq |\mathbb{W}|^{1/2}$, $\text{Supp}(|X| \cdot X \in \text{supp}(\varphi) \subset F^3)$, then $\varphi(X) = \varphi(Y)$. Thus

$c_{n, m, l} = \begin{cases} 
\frac{c^{2n+m+l}}{2} \cdot |\mathbb{W}|^{n+m+l} \cdot \int_{F^3} \varphi(\mathbb{W}^{n+m} x_1, \mathbb{W}^n x_2, 0) \cdot \varphi(x_1, x_2, x_3) dX, & \text{if } l \geq l_1 \\
\frac{c^{2n+m+l}}{2} \cdot |\mathbb{W}|^{n+m+l} \cdot \int_{F^3} \varphi(\mathbb{W}^{n+m} x_1, \mathbb{W}^n x_2, x_3) \cdot \varphi(x_1, x_2, 0) dX, & \text{if } l \leq l_1.
\end{cases}$

Write $a_{n, m} = \int_{F^3} \varphi(\mathbb{W}^{n+m} x_1, \mathbb{W}^n x_2, 0) \cdot \varphi(x_1, x_2, x_3) dX$, $b_{n, m} = \int_{F^3} \varphi(\mathbb{W}^{n+m} x_1, \mathbb{W}^n x_2, x_3) \cdot \varphi(x_1, x_2, 0) dX$. Then $a_{n, m} = \begin{cases} 
a_{n, m}^1, & \text{if } n \geq l_1 \\
a_{n, m}^2, & \text{if } n \leq l_1.
\end{cases}$

Write $a_{n, m}^1 = \int_{F^3} \varphi(\mathbb{W}^{n+m} x_1, 0, 0) \cdot \varphi(x_1, x_2, x_3) dX$, $a_{n, m}^2 = \int_{F^3} \varphi(\mathbb{W}^{n+m} x_1, x_2, 0) \cdot \varphi(x_1, 0, x_3) dX$ and $b_{n, m} = \begin{cases} 
b_{n, m}^1, & \text{if } n \geq l_1 \\
b_{n, m}^2, & \text{if } n \leq l_1.
\end{cases}$

Write $a_{n, m}^1 = \int_{F^3} \varphi(\mathbb{W}^{n+m} x_1, 0, 0) \cdot \varphi(x_1, x_2, x_3) dX$, $a_{n, m}^2 = \int_{F^3} \varphi(\mathbb{W}^{n+m} x_1, x_2, 0) \cdot \varphi(x_1, 0, x_3) dX$ and $b_{n, m} = \begin{cases} 
b_{n, m}^1, & \text{if } n \geq l_1 \\
b_{n, m}^2, & \text{if } n \leq l_1.
\end{cases}$

Again $a_{n, m}^1 = \begin{cases} 
k_1, & \text{if } n + m \geq l_1 \\
k_2, & \text{if } n + m \leq -l_1
\end{cases}$ and $b_{n, m} = \begin{cases} 
k_1, & \text{if } n + m \geq l_1 \\
k_2, & \text{if } n + m \leq -l_1.
\end{cases}$

for some constants $k_1, k_2, k_3, k_4$.

Note that in codimension 0, 1 case, the theta lift sends a tempered representation to a tempered one. Thus we know that $\pi_2$ is tempered and by [Prop.8.1, [2]], we see that it is the irreducible unitary induced representation $B(\gamma_1^2, \gamma_2^{-1})$ of $GL(2)(F)$. (Here, since $\gamma = (\gamma_1, \gamma_2^{-1})$, if we regard $\gamma$ as a character of $F^\times$ using the isomorphism of $U(1)$ and $GL(1)$, $\gamma(x) = \gamma_1^2(x)$. Then by (17), Lemma 3.9), if we take $l_1$ large enough, we assume that for $m \geq l_1$, $d_m = |\mathbb{W}|^{\frac{1}{2}} \cdot (c_1 \cdot c^m + c_2 \cdot c^{-m})$ where $c_1, c_2$ are constants.
Since $\pi_2$ is the theta lift of the trivial representation, by the relation of their $L$-parameters in [Prop.8.1, [1]], we have $BC(\pi_2) = BC(1)\gamma^{-1} \otimes \gamma$ and so $L_E(s, BC(\pi_2) \otimes \gamma) = (\frac{1}{1-s})^2 \cdot \frac{1}{1-\gamma_1(\varpi^{-1})} \cdot \frac{1}{1-\gamma_2(\varpi^{-1})}$. (Recall $\gamma = (\gamma_1, \gamma_2^{-1})$ for some unitary character $\gamma_1$ of $F$.) Thus if $\gamma_1(\varpi) = 1$, $L_E(s, BC(\pi_2) \otimes \gamma)$ has a quadruple pole at $s = 0$ and if $\gamma_1(\varpi) \neq 1$, then it has double pole at $s = 0$. In any cases, $t \geq 1$.

Now, we introduce two notation that we will use in this argument:

• If two meromorphic functions $f_1, f_2$ differ by a constant multiplicity, we write $f_1 \approx f_2$.

• For two meromorphic functions $f_1, f_2$ and $m \in \mathbb{N}$, if $\lim_{\Re(s) \to 0^+} s^n \cdot (f_1(s) - f_2(s)) = 0$, we write $f_1 \sim f_2$ and if $f_1 \sim f_2$, we simply write $f_1 \sim f_2$.

Since the integral in (5.1) absolutely converges on $\Re(s) > 0$, to prove the Lemma, it suffices to show that each component of (5.1)

$$\sum_{n \in \mathbb{Z}, m \geq 0} c_{n,m} \cdot \left( n^{\frac{m}{2}} \cdot \alpha\left( \frac{1}{2} - n \right) \right)$$

are all $\approx 0$.

We will first show (5.2) $\approx 0$. To do this, we decompose (5.2) into three component

$$\sum_{n \in \mathbb{Z}, m \geq 0} c_{n,m} \cdot \left( n^{\frac{m}{2}} \cdot \alpha\left( \frac{1}{2} - n \right) \right)$$

and show each component is $\approx 0$.

For fixed $m \in \mathbb{N}$ and small $\Re(s) > 0$,

$$\sum_{n \in \mathbb{Z}} c_{n,m} \cdot \left( n^{\frac{m}{2}} \cdot \alpha\left( \frac{1}{2} - n \right) \right)$$

where

$$f_1(s) = \frac{(\alpha|\varpi|^{\frac{1}{2} - 2s})\cdot \left( \varpi \right)}{1 - \alpha|\varpi|^s}$$

and note that $f_1 \sim 0$.

Since

$$\sum_{m \geq 0} c_{n,m} \cdot \left( n^{\frac{m}{2}} \cdot \alpha\left( \frac{1}{2} - n \right) \right)$$

is the theta lift of the trivial representation.
Next we will show

\[ \sum_{m \geq 0} d_m(c|\omega|^{-s - \frac{3}{2}})^m \left( \sum_{n \leq -(m+1)} (c^2|\omega|^{-1-2s})^n \right) \approx (c^{-2}|\omega|^{-1+2s})^{l+1} \sum_{m \geq 0} d_m \left( c^{-1}|\omega|^{-s - \frac{3}{2}} \right)^m = \]

\[ \frac{(c^{-2}|\omega|^{-1+2s})^{l+1}}{1 - c^{-2}|\omega|^{-1+2s}} \cdot \left( \sum_{m=0}^{l-1} d_m (c^{-1}|\omega|^{-s - \frac{3}{2}})^m \right) + c_1 \cdot \frac{|\omega|^n}{1 - |\omega|^s} + c_2 \cdot \frac{(c^{-2}|\omega|^{-s})^l}{1 - c^{-2}|\omega|^{-s}} \]

and so \((\sum_{m \geq 0} \sum_{n \leq -(m+1)} c^{2n+m} \cdot |\omega|^{n(1-2s)-m(\frac{3}{2}+s)} d_n a_{n,m}) \cdot f_1(s) \overset{!}{\sim} 0.\)

Furthermore,

\[ \sum_{m \geq 0} \sum_{n \leq -(m+1)} c^{2n+m} \cdot |\omega|^{n(1-2s)-m(\frac{3}{2}+s)} d_n a_{n,m} \cdot \left( \frac{(ca|\omega|^{-\frac{3}{2}-2s})^{-(n+m)}}{1 - ca|\omega|^{-\frac{3}{2}-2s}} - \frac{(ca|\omega|^{-\frac{3}{2}})^{-(n+m)}}{1 - ca|\omega|^{-\frac{3}{2}}} \right) \approx \]

\[ \sum_{m \geq 0} d_m a^{-m} \left( |\omega|^{-s-2m} \sum_{n \leq -(m+1)} (c|\omega|^{-\frac{3}{2}-2s})^n \right) + (|\omega|^{-s-2m} \sum_{n \leq -(m+1)} (c|\omega|^{-\frac{3}{2}+2s})^n) = \]

\[ \frac{(c^{-1}|\omega|^{-\frac{3}{2}})^{l+1}}{1 - c^{-1}|\omega|^{-\frac{3}{2}}} - \frac{(c^{-1}|\omega|^{-\frac{3}{2}})^{l+1}}{1 - c^{-1}|\omega|^{-\frac{3}{2}+2s}} \cdot \left( \sum_{m=0}^{l-1} d_m (c^{-1}|\omega|^{-s - \frac{3}{2}})^m \right) + c_1 \cdot \frac{|\omega|^n}{1 - |\omega|^s} + c_2 \cdot \frac{(c^{-2}|\omega|^{-s})^l}{1 - c^{-2}|\omega|^{-s}} \overset{!}{\sim} 0.\]

Thus we see that

\[ \sum_{n \in \mathbb{Z}, m \geq 0} c^{2n+m} |\omega|^{n-\frac{3}{2}} d_n a_{n,m} \cdot \sum_{l \geq 1, l \leq -(n+m)} ((ca|\omega|^{-\frac{3}{2}-2s})^l \cdot |\omega|^{-s-2(n+m)} - (ca|\omega|^{-\frac{3}{2}})^l \cdot |\omega|^{-s-2(n+m)}) \overset{!}{\sim} 0.\]

Next we will show

\[ \sum_{m \in \mathbb{N}} c^m d_m |\omega|^{-\frac{s}{2}} \sum_{n \in \mathbb{Z}} c^{2n} |\omega|^{n} a_{n,m} \cdot \sum_{l \geq 1, -(n+m) \leq l \leq -n} (ca|\omega|^{-\frac{3}{2}})^l \cdot (|\omega|^{-s} - |\omega|^{-s+(n+m)-n}) \overset{!}{\sim} 0.\]

Let

\[ p_{n,m}(s) = c^{2n} |\omega|^{n} a_{n,m} \cdot \sum_{l \geq 1, -(n+m) \leq l \leq -n} (ca|\omega|^{-\frac{3}{2}})^l \cdot (|\omega|^{-s} - |\omega|^{-s+(n+m)-n}).\]

Then

\[ \sum_{n \in \mathbb{Z}} p_{n,m}(s) = \sum_{n < \min(-(l-1), -m)} c^{2n} |\omega|^{n} a_{n,m} \cdot (|\omega|^{-s} - |\omega|^{-s-(2n-m)s}) \cdot \left( \frac{(ca|\omega|^{-\frac{3}{2}})^{\max(l-1, -(n+m))} - (ca|\omega|^{-\frac{3}{2}})^{-n}}{1 - ca|\omega|^{-\frac{3}{2}}} \right) \]

and so to show \(\sum_{m \in \mathbb{N}} c^m d_m |\omega|^{-\frac{s}{2}} \sum_{n \in \mathbb{Z}} p_{n,m}(s) \overset{!}{\sim} 0,\) it is sufficient to check

\[ \sum_{0 \leq m < l} c^m d_m |\omega|^{-\frac{s}{2}} \cdot \left( \sum_{-l_1 - m < n < -l_1} p_{n,m}(s) \right) \overset{!}{\sim} 0 \]

\[ \sum_{0 \leq m < l} c^m d_m |\omega|^{-\frac{s}{2}} \cdot \left( \sum_{n \leq -l_1 - m} p_{n,m}(s) \right) \overset{!}{\sim} 0 \]

\[ \sum_{m \geq l_1} c^m d_m |\omega|^{-\frac{s}{2}} \cdot \left( \sum_{-l_1 - m < n \leq -m} p_{n,m}(s) \right) \overset{!}{\sim} 0 \]

\[ \sum_{m \geq l_1} c^m d_m |\omega|^{-\frac{s}{2}} \cdot \left( \sum_{n \leq -l_1 - m} p_{n,m}(s) \right) \overset{!}{\sim} 0. \]
For each \(0 \leq m < l_1, -l_1 - m \leq n < -l_1\),
\[
e^{m d_m |w|} p_{n,m}(s) \lesssim 0
\]
and so (5.5) easily follows.
For each \(m \in \mathbb{N}\),
\[
\sum_{n \leq -l_1 - m} p_{n,m}(s) \approx \left( (c^{-2}|w|^s)^m \cdot g_1(s) - (c^{-1}a|w|^s)^m \cdot g_2(s) \right)
\]
where
\[
g_1(s) = \frac{(c^{-1}a|w|^s)^{l_1}}{1 - c^{-2}|w|^s}, \quad g_2(s) = \frac{(c^{-1}a|w|^s)^{l_1}}{1 - c^{-2}a|w|^s}
\]
and so (5.6) and (5.8) follow from this.

For each \(-l_1 < k < 0\), note that
\[
\sum_{m \geq l_1} e^{m d_m |w|} p_{k-m,m}(s) \approx \left( 1 - |w|^{-2k} \right) \sum_{m \geq l_1} (c_1 |w|^{sm} + c_2 (c^{-2}|w|^s)^{2m}) \cdot ((ca|w|^s)^{l_1} - (ca|w|^s)^{m-k})
\]
\[
\sim 0 \quad \text{and so we have (5.7).}
\]

Next we decompose
\[
\sum_{m \in \mathbb{N}} e^{m d_m |w|} \frac{d}{d} \sum_{n \geq 2} e^{2n |w|^n a_{n,m}} \sum_{l_1 \leq l \leq 2-n} (ca|w|^s)^{l_1} \cdot |w|^s(2n+m) - (ca|w|^s)^{l} \cdot |w|^s(n+m+n))
\]
into three components
\[
\sum_{m \in \mathbb{N}} e^{m d_m |w|} \frac{d}{d} \sum_{n \geq 2} e^{2n |w|^n a_{n,m}} \cdot |w|^s(2n+m) \cdot \frac{(ca|w|^s)^{l_1}}{1 - (ca|w|^s)^{2n+1}} - \frac{(ca|w|^s)^{l}}{1 - (ca|w|^s)^{2n+1}}
\]
\[
+ \sum_{m \in \mathbb{N}} e^{m d_m |w|} \frac{d}{d} \sum_{-m \leq n < 0} e^{2n |w|^n a_{n,m}} (|w|^s(2n+m) \cdot \frac{(ca|w|^s)^{l_1}}{1 - (ca|w|^s)^{2n+1}} - |w|^s m \cdot \frac{(ca|w|^s)^{l_1}}{1 - (ca|w|^s)^{2n+1}})
\]
\[
+ \sum_{m \in \mathbb{N}} e^{m d_m |w|} \frac{d}{d} \sum_{n < -m} e^{2n |w|^n a_{n,m}} (|w|^s(2n+m) \cdot \frac{(ca|w|^s)^{l_1}}{1 - (ca|w|^s)^{2n+1}} - |w|^s(2n+m) \cdot \frac{(ca|w|^s)^{l_1}}{1 - (ca|w|^s)^{2n+1}})
\]
and we will show each component is \(\lesssim 0\).

Using the asymptotic formulae of \(d_m\) and \(a_{n,m}\), one can easily see that the first sum is \(\lesssim 0\).
Write
\[
p_{n,m}(s) = e^{m d_m |w|} \frac{d}{d} \sum_{n \geq 2} e^{2n |w|^n a_{n,m}} (|w|^s(2n+m) \cdot \frac{(ca|w|^s)^{l_1}}{1 - (ca|w|^s)^{2n+1}} - |w|^s m \cdot \frac{(ca|w|^s)^{l_1}}{1 - (ca|w|^s)^{2n+1}})
\]
and note that \(p_{n,m}(s) \sim 0\). The second sum is decomposed into
\[
\sum_{0 \leq m < l_1} \sum_{-m \leq n < 0} p_{n,m}(s) + \sum_{l_1 \leq m - l_1 n < 0} p_{n,m}(s) + \sum_{l_1 \leq m} \sum_{-l_1 n < 0} p_{n,m}(s)
\]
and since \(\sum_{0 \leq m < l_1} \sum_{-m \leq n < 0} p_{n,m}(s)\) is a finite sum, it is \(\lesssim 0\). For each \(-l_1 \leq n < 0\), one can easily check \(\sum_{l_1 \leq m} \sum_{-l_1 n < 0} p_{n,m}(s) \sim 0\) and so \(\sum_{l_1 \leq m} \sum_{-l_1 n < 0} p_{n,m}(s) \sim 0\).
If \( n < -l_1 \),
\[
|\varpi|^s |(2n+m) \cdot (\alpha_1 |\varpi|^{\frac{1}{2} + 2s})^\max(l_1,-n) \\
1 - \alpha_1 |\varpi|^{\frac{1}{2} + 2s} - |\varpi|^{s m} \cdot \frac{(\alpha_1 |\varpi|^{\frac{1}{2}})^\max(l_1,-n)}{1 - \alpha_1 |\varpi|^{\frac{1}{2}}} = 0
\]
and so \( \sum_{l_1 \leq m} \sum_{-m \leq n < -l_1} p_{n,m}^1(s) = 0 \). Thus the second sum \( \sum_{m \in \mathbb{N}} \sum_{-m < n \leq 0} p_{n,m}^1(s) = 0 \).

To show the third sum is \( 1 \sim 0 \), write
\[
p_{n,m}^2(s) = c^n d_m |\varpi|^{-2} \cdot c^{2n} |\varpi|^n a_{n,m} \left( |\varpi|^{s (2n+m)} \cdot \frac{(\alpha_1 |\varpi|^{\frac{1}{2} + 2s})^{\max(l_1, -n)}}{1 - \alpha_1 |\varpi|^{\frac{1}{2} + 2s}} - |\varpi|^{-s (2n+m)} \cdot \frac{(\alpha_1 |\varpi|^{\frac{1}{2}})^{\max(l_1, -n)}}{1 - \alpha_1 |\varpi|^{\frac{1}{2}}(1 + 2s)} \right).
\]

We decompose \( \sum_{m \in \mathbb{N}} \sum_{n < -m} p_{n,m}^2(s) = \sum_{m \in \mathbb{N}} \sum_{-m-1 < n < -m} p_{n,m}^2(s) + \sum_{m \in \mathbb{N}} \sum_{-m < n < -m-1} p_{n,m}^2(s) \).

Write \( k = m + n \) and for each \( -l_1 < k < 0 \),
\[
\sum_{m \in \mathbb{N}} p_{k-m,m}^2(s) \sim \sum_{m \geq l_1} p_{k-m,m}^2(s) = c^k (c_1 |\varpi|^s)^m + c_2 (c_1 |\varpi|^s)^m \cdot g_k(s) \sim 0
\]
where
\[
g_k(s) = \frac{(\alpha_1 |\varpi|^{\frac{1}{2}})^{-k}}{1 - \alpha_1 |\varpi|^{\frac{1}{2} + 2s}} \cdot \frac{(\alpha_1 |\varpi|^{\frac{1}{2}})^{-k}}{1 - \alpha_1 |\varpi|^{\frac{1}{2}}}.
\]
Thus \( \sum_{m \in \mathbb{N}} \sum_{n < -m} p_{n,m}^2(s) = 0 \).

Next, for each \( m \in \mathbb{N} \), some calculation shows that
\[
\sum_{n \leq -m - l_1} p_{n,m}^2(s) = c^n d_m |\varpi|^{-\frac{1}{2}} \cdot k_2 \cdot (c_1 |\varpi|^{\frac{1}{2} + s})^m \cdot g(s) \text{ where } g(s) = \frac{(c_1 |\varpi|^{\frac{1}{2}})^{l_1}}{1 - c_1 |\varpi|^{\frac{1}{2} + 2s}} - \frac{(c_1 |\varpi|^{\frac{1}{2}})^{l_1}}{1 - c_1 |\varpi|^{\frac{1}{2}}}.
\]
and so \( \sum_{m \in \mathbb{N}} \sum_{n \leq -m - l_1} p_{n,m}^2(s) \sim 0 \). Thus we have showed \( (5.2) \sim 0 \).

Now, we will show \( (5.3) \sim 0 \). To do this, for each \( -l_1 < l < l_1 \), we decompose
\[
\sum_{n \in \mathbb{N}, m \geq 0} \left( |\varpi|^{-m} d_{m} \cdot \alpha^{d(\varpi)^{s(m+n+m+n+l)}} - |\varpi|^{-n} d_{n} \right) \cdot c_{n,m,l}
\]
into three summands \( \sum_{m \in \mathbb{N}, n \geq 1} + \sum_{m \in \mathbb{N}, n < 1} + \sum_{m \in \mathbb{N}, n < -1} \) and show that each is \( \sim 0 \).

Write \( f_{n,m,l}(s) = \left( |\varpi|^{-m} d_{m} \cdot \alpha^{d(\varpi)^{s(n+n+m+n+l)}} - |\varpi|^{-n} d_{n} \right) \cdot c_{n,m,l} \) and note that for each fixed \( n, m, l \), \( f_{n,m,l} \sim 0 \).

For each \( -l_1 < l < l_1 \), we see that
\[
\sum_{m \in \mathbb{N}, n \geq 1} f_{n,m,l}(s) \approx \left( \sum_{n \geq 1} (c_2 |\varpi|^{l+2s})^{n} \right) \cdot \left( \sum_{m \in \mathbb{N}} c_1 \cdot (c_2 |\varpi|^s)^m + c_2 |\varpi|^s \cdot (||\varpi||^{2s} - 1) \right) \sim 0.
\]

For all \( -l_1 < n, l < l_1 \), there exists \( N_1 \in \mathbb{N} \) such that \( N_1 > 2l_1 \) and if \( m \geq N_1 \), then \( c_{n,m,l} = (c_2 |\varpi|^{l})^m \cdot f_{n,l} \) for some constants \( f_{n,l} \). Thus \( \sum_{m \geq 0, -l_1 < n < l_1} f_{n,m,l}(s) = \sum_{0 \leq m < N_1, -l_1 < n < l_1} f_{n,m,l}(s) + \sum_{m \geq N_1} c_1 (c_2 |\varpi|^s)^m + c_2 |\varpi|^s \cdot (||\varpi||^{2s} - 1) \sim 0 \).
and so
\[ \sum_{m \geq 0, -l_1 < n < l_1} f_{n,m,l}(s) \lesssim 0. \]

Next we decompose
\[ \sum_{m \geq 0} f_{n,m,l} \]
into
\[ \sum_{n \leq -l_1, m+n \geq \max\{-l_0, -l_i\}} f_{n,m,l} + \sum_{n \leq -l_1, -l_i \leq m+n < 0} f_{n,m,l} + \sum_{n \leq -l_1, 0 \leq m+n < -l_i} f_{n,m,l} + \sum_{n \leq -l_1, m+n < \min\{-l_0, -l_i\}} f_{n,m,l}. \]

The first sum is zero. The second sum is
\[ \sum_{-l \leq k < 0} \sum_{m \geq k+l_1} f_{k-m,m,l} \]
and for each \(-l \leq k < 0\), there exists \(N_2 \in \mathbb{N}\) such that \(N_2 \geq l_1\) and if \(m \geq N_2\), then \(f_{k-m,m,l} \approx \|\varpi\|^{\frac{k}{2}} e^{-m}\). Thus
\[ \sum_{-l \leq k < 0} \sum_{m \geq k+l_1} f_{k-m,m,l} \approx (\sum_{k+l_1 \leq m < N_2} f_{k-m,m,l}) + \left(1 - \|\varpi\|^{-2+2s}\right) \sum_{m \geq N_2} (c_1 \|\varpi\|^m + c_2 e^{-2 \|\varpi\|^m}) \lesssim 0. \]

Similarly, we can show the third sum \(\lesssim 0\).

The fourth sum is decomposed into
\[ \sum_{n \leq -l_1, -l_i < m+n < \min\{-l_0, -l_i\}} f_{n,m,l} + \sum_{n \leq -l_1, m+n \leq -l_i} f_{n,m,l} \]
and as we have done in the above, it is easy to see
\[ \sum_{n \leq -l_1, -l_i < m+n < \min\{-l_0, -l_i\}} f_{n,m,l} \lesssim 0. \]

Note
\[ \sum_{n \leq -l_1, m+n \leq -l_i} f_{n,m,l} = \sum_{0 \leq m < l_1, n \leq -l_1, m+n \leq -l_i} f_{n,m,l} + \sum_{m \geq l_1, n \leq -l_1, m+n \leq -l_i} f_{n,m,l}. \]

For each \(0 \leq m < l_1\),
\[ \sum_{n \leq -l_1-m} f_{n,m,l} \approx d_m(c_1 \|\varpi\|^{\frac{k}{2}+s})^m \cdot (\|\varpi\|^{-2ls} - 1) \sum_{n \leq -l_1-m} (c_2 \|\varpi\|^{-1+2s})^n \lesssim 0. \]

On the other hand,
\[ \sum_{m \geq l_1} \sum_{n \leq -l_1-m} f_{n,m,l} \approx (\|\varpi\|^{-2ls} - 1)(c_1 (c_2 \|\varpi\|^{1+s})^m + c_2 \|\varpi\|^{-1+2s})^n \sum_{n \leq -l_1-m} (c_2 \|\varpi\|^{1+2s})^n \]
\[ = (e^{-2 \varpi})^{l_1} \cdot (\|\varpi\|^{-2ls} - 1) \cdot \left( \sum_{m \geq l_1} c_1 \cdot \|\varpi\|^m + c_2 \cdot (e^{-2 \varpi})^m \right) \lesssim 0. \]

Thus we see that the fourth sum \(\sum_{n \leq -l_1, m+n < \min\{-l_0, -l_i\}} f_{n,m,l}\) is also \(\lesssim 0\) and we showed \(\textbf{[5.3]} \lesssim 0\).

Last, we will show \(\textbf{[5.4]} \lesssim 0\). To do this, write \(\sum_{l \geq l_1} e^{\alpha l} \|\varpi\|^{-l} (\|\varpi\|^s |(n+l+m)+(n+l)| - \|\varpi\|^s |(n+m)+|n|)\) as
\[ \sum_{l \geq l_1} \left( e^{-\alpha l} \|\varpi\|^l \cdot (\|\varpi\|^s |(n+l+m)+(n+l)| - \|\varpi\|^s |(n+m)+|n|) \right) \]
and decompose it into three summands
\[ \sum_{l \geq l_1, l > (n+m)} (e^{-\alpha l} \|\varpi\|^l \cdot (\|\varpi\|^s (2n+m) - (e^{-\alpha} \|\varpi\|^l \cdot |\varpi|^s |(n+m)+|n|))) \]
\[ + \sum_{l \geq 1, s \leq l \leq n} \left( \frac{(c^{-1} \alpha - 1 |x|^s)^l}{(1 - c^{-1} \alpha^{-1}|x|^{s+2})} \cdot (|x|^{s(n+m)+|n|}) \right) \]

\[ + \sum_{l \geq 1, l \geq n} \left( \frac{(c^{-1} \alpha - 1 |x|^s)^l}{(1 - c^{-1} \alpha^{-1}|x|^{s+2})} \cdot (|x|^{s(n+m)+|n|}) \right) \]

We write \( M_{n,m} = \max\{l_1, m + n + 1\} \). Then for fixed \( m, n \in \mathbb{N} \) and small \( \Re(s) > 0 \),

\[ \sum_{l \geq 1, l \geq (n+m)} \left( \frac{(c^{-1} \alpha - 1 |x|^s)^l}{(1 - c^{-1} \alpha^{-1}|x|^{s+2})} \cdot (|x|^{s(n+m)+|n|}) \right) = \]

\[ \frac{|x|^{-s(n+m)}(c^{-1} \alpha - 1 |x|^s)^{\frac{s}{2}+2}) M_{n,m} - |x|^{s(n+m)+|n|}(c^{-1} \alpha - 1 |x|^s)^{\frac{s}{2}+2}) M_{n,m}}{1 - c^{-1} \alpha^{-1}|x|^{s+2}} \]

Denote

\[ c^{2n+m}|x|^n = \frac{d_m g_{n,m}(s)}{2} \]

by \( g_{n,m}(s) \) and note \( g_{n,m}(s) \sim 0 \). We will show \( \sum_{m \geq 0, n \in \mathbb{N}} g_{n,m}(s) \sim 0 \).

Decompose \( \sum_{m \geq 0, n \in \mathbb{N}} g_{n,m}(s) \) into \( \sum_{m \geq 0, n \geq 0} g_{n,m}(s) + \sum_{m \geq 0, n < 0} g_{n,m}(s) + \sum_{m < 0, n \geq 0} g_{n,m}(s) \) and the first sum decomposes again into

\[ \sum_{0 \leq m \leq l_1, 0 \leq n \leq l_1 - m} g_{n,m}(s) + \sum_{0 \leq m \leq l_1, 0 \leq n \leq l_1 - m} g_{n,m}(s) + \sum_{0 \leq m \leq l_1, 0 \leq n \leq l_1} g_{n,m}(s) \]

Since the first term in the above is a finite sum, \( 0 \leq m \leq l_1 - 1 \sum_{0 \leq n \leq l_1 - m} g_{n,m}(s) \sim 0 \).

For each \( 0 \leq m \leq l_1 - 1, \sum_{0 \leq n \leq l_1 - m} g_{n,m}(s) \sim \sum_{l_1 \leq m} g_{n,m}(s) \approx d_m (\alpha^{-1}|x|^{s})^m \cdot \sum_{l_1 \leq n} g_{n,m}(s) \)

where

\[ g^2_m(s) = \frac{(c^{-1} \alpha^{-1}|x|^s)^{n}(c^{-1} \alpha^{-1}|x|^s)^{\frac{s}{2}+2})}{1 - c^{-1} \alpha^{-1}|x|^{s+2}} \]

and note that \( \sum_{n \geq l_1} g^2_m(s) \sim 0 \). Thus the second term \( \sum_{0 \leq m \leq l_1 - 1} \sum_{l_1 - m \leq n} g_{n,m}(s) \sim 0 \).

The third term \( \sum_{l_1 \leq m} \sum_{0 \leq n \leq l_1} g_{n,m}(s) \) is \( \sum_{l_1 \leq m} \sum_{0 \leq n \leq l_1} g_{n,m}(s) + \sum_{l_1 \leq m} \sum_{l_1 \leq n} g_{n,m}(s) \).

For each \( 0 \leq n < l_1, \sum_{l_1 \leq m} c_1 (\alpha^{-1}|x|^s)^m + c_2 (c^{-1} \alpha^{-1}|x|^s)^m \cdot g^2_m(s) \)

where

\[ g^2_m(s) = \frac{(c^{-1} \alpha^{-1}|x|^s)^{n}(c^{-1} \alpha^{-1}|x|^s)^{\frac{s}{2}+2})}{1 - c^{-1} \alpha^{-1}|x|^{s+2}} \]

Thus \( \sum_{l_1 \leq m} \sum_{0 \leq n \leq l_1} g_{n,m}(s) \sim 0 \) and

\[ \sum_{l_1 \leq m} \sum_{0 \leq n \leq l_1} g_{n,m}(s) \approx \left( \sum_{n \leq l_1} g^2_m(s) \right) \cdot \left( \sum_{l_1 \leq m} c_1 (\alpha^{-1}|x|^s)^m + c_2 (c^{-1} \alpha^{-1}|x|^s)^m \right) \sim 0 \]

Since the above three components of \( \sum_{m \geq 0, n \geq 0} g_{n,m}(s) \) are all \( \sim 0, \sum_{m \geq 0, n \geq 0} g_{n,m}(s) \sim 0 \).

Next we divide \( \sum_{m \geq 0, n \leq 0} g_{n,m}(s) \) into \( \sum_{m \geq 0, n < 0} g_{n,m}(s) + \sum_{m \geq 0, n \leq -1} g_{n,m}(s) \).

For each \( -l_1 < n < 0 \),

\[ \sum_{m \geq 0, n \leq -1} g_{n,m}(s) = \frac{(c^{-1} \alpha^{-1}|x|^s)^{\frac{s}{2}+2})}{1 - c^{-1} \alpha^{-1}|x|^{s+2}} \]
where \( k_n = \int_{\mathbb{R}^3} \varphi(0, w^n x_2, 0) \cdot \varphi(x_1, x_2, x_3) \, dX \) and
\[
 f_n^1(s) = \frac{c_1 (\alpha^{-1} |w|^{\frac{1}{2} + s})^{-n+l_1}}{1 - \alpha^{-1} |w|^{\frac{1}{2} + s}} + \frac{c_2 (e^{-2} \alpha^{-1} |w|^{\frac{1}{2} + s})^{-n+l_1}}{1 - e^{-2} \alpha^{-1} |w|^{\frac{1}{2} + s}}.
\]

Thus
\[
\sum_{m \geq n, -l_1 < n < 0} g_{n,m}(s) = \sum_{-n \leq m < -n+l_1, -l_1 < n < 0} g_{n,m}(s) + \sum_{m \geq -n+l_1, -l_1 < n < 0} g_{n,m}(s) \sim 0.
\]

Next, we divide \( \sum_{n < n, -l_1 < m < -n-l_1, n < 0} g_{n,m}(s) \) and for each \( 0 \leq m < l_1 \),
\[
\sum_{n \leq -2l_1} g_{n,m} = d_m (c |w|^{-\frac{3}{2} - s})^m \left( \frac{(e^{-1} \alpha^{-1} |w|^{\frac{1}{2} + s})^l_1}{1 - e^{-1} \alpha^{-1} |w|^{\frac{1}{2} + s}} \right) \sum_{n \leq -2l_1} \left( c^2 |w|^{-1 - 2s} \right)^n \sim 0.
\]

Note that
\[
\sum_{n \leq -2l_1, l_1 \leq m < -n-l_1} g_{n,m} = \left( \frac{(e^{-1} \alpha^{-1} |w|^{\frac{1}{2} + s})^l_1}{1 - e^{-1} \alpha^{-1} |w|^{\frac{1}{2} + s}} \right) \sum_{n \leq -2l_1} f_n^1(s) \cdot (c^2 |w|^{-1 - 2s})^n
\]
where
\[
f_n^2(s) = c_1 \cdot \left( \frac{(c^2 |w|^{-1 - s})^l_1}{1 - c_1 (c^2 |w|^{-1 - s})^{-n+l_1}} \right) + c_2 \cdot \left( \frac{(c^2 |w|^{-1 - s})^l_1 - (c^2 |w|^{-1 - s})^{-n+l_1}}{1 - c_1 (c^2 |w|^{-1 - s})^{-n+l_1}} \right).
\]

Thus \( \sum_{n \leq -2l_1, l_1 \leq m < -n-l_1, n < 0} g_{n,m}(s) \sim 0 \) and so \( \sum_{n < n, 0 \leq m < -n-l_1} g_{n,m}(s) \sim 0. \)

To show \( \sum_{n < n, -l_1 \leq m < -n, n < 0} g_{n,m}(s) \) \( \sim 0 \), let \( k = n + m \) and for each \( -l_1 \leq k < 0 \), we will check
\[
\sum_{n < 0} g_{n,k-n}(s) \sim \sum_{n < 2l_1} g_{n,k-n}(s) \approx (c_1 \cdot \sum_{n < 2l_1} |w|^{-\frac{1}{2} - s})^m + c_2 \cdot \sum_{n < 2l_1} (c^2 |w|^{-\frac{1}{2} - s})^n
\]
where
\[
f_n^3(s) = \left( \frac{(e^{-1} \alpha^{-1} |w|^{\frac{1}{2} + s})^l_1}{1 - e^{-1} \alpha^{-1} |w|^{\frac{1}{2} + s}} \right) \left( (c^2 |w|^{-1 - s})^l_1 \cdot |w|^{-s(2n+m)} - (c^2 \alpha^{-1} |w|^{\frac{1}{2}})^l_1 \cdot |w|^{s(n+m) + |n|}) \right) \sim 0.
\]

Next we turn to show
\[
\sum_{n \in \mathbb{Z}, m \geq 0} c^{2n+m}|w|^{n - \frac{3}{2}} d_m a_{n,m} \cdot \left( \sum_{l \geq l_1, n < l \leq n + m} (c^2 |w|^{-1 - s})^l \cdot (|w|^{2n+m}) \right) \sim 0.
\]

It equals \( \sum_{m \geq 0} c^m d_m |w|^{-\frac{3}{2}} \sum_{n \geq 0} f_{n,m}(s) \) where
\[
f_{n,m}(s) = c^{2n}|w|^{n} a_{n,m} \cdot \left( \sum_{l \geq l_1, n < l \leq n + m} (c^2 |w|^{-1 - s})^l \cdot (|w|^{2n+m}) \right).
\]
Then
\[ \sum_{n \geq 0} f_{n,m}(s) = \sum_{0 \leq n \leq l_1 - 1} f_{n,m}(s) + \sum_{l_1 \leq n} f_{n,m}(s) \]
and
\[ \sum_{m \geq 0} e^m d_m |w|^{-\frac{m}{2}} \left( \sum_{0 \leq n \leq l_1 - 1} f_{n,m}(s) \right) \sim \sum_{m \geq 2l_1} e^m d_m |w|^{-\frac{m}{2}} \left( \sum_{0 \leq n \leq l_1 - 1} f_{n,m}(s) \right) = \]
\[ \sum_{0 \leq n \leq l_1 - 1} e^{2n} |w|^n (1 - |w|^{2n}) \sum_{m \geq 2l_1} e^m d_m |w|^{(s-\frac{1}{2})m} a_{n,m} \cdot \frac{(c^{-1} - |w|^{\frac{3}{2}})^l 1 - (c^{-1} - |w|^{\frac{3}{2}})^n + m + 1}{1 - c^{-1} - |w|^{\frac{3}{2}}} \]
For each \( 0 \leq n \leq l_1 - 1 \),
\[ \sum_{m \geq 2l_1} e^m d_m |w|^{-\frac{m}{2}} \left( \sum_{0 \leq n \leq l_1 - 1} f_{n,m}(s) \right) \sim 0. \]
and so
\[ \sum_{m \geq 0} e^m d_m |w|^{-\frac{m}{2}} \left( \sum_{0 \leq n \leq l_1 - 1} f_{n,m}(s) \right) \sim 0. \]
For each \( m \in \mathbb{N} \),
\[ \sum_{n \geq l_1} f_{n,m} \sim |w|^m (1 - (c^{-1} - |w|^{\frac{3}{2}})^m) \cdot (\sum_{n \geq l_1} (c^{-1} - |w|^{\frac{3}{2}})^n - (c^{-1} - |w|^{\frac{3}{2} + 2s})^n). \]
Thus \( \sum_{m \geq 0} e^m d_m |w|^{-\frac{m}{2}} \sum_{n \geq l_1} f_{n,m} \sim 0 \) and so we showed
\[ \sum_{n \in \mathbb{Z}, m \geq 0} e^{2n+m} |w|^n - \frac{1}{2} \sum_{n \geq l_1} |w|^{s((n + m + |n|))} \cdot \left( \sum_{l \geq 1, n \leq l \leq n + m} (c^{-1} - |w|^{\frac{3}{2}})^l \cdot (|w|^{sm} - |w|^{s((n + m + |n|))}) \right) \sim 0. \]
Finally, we investigate the last sum
\[ \sum_{n \in \mathbb{Z}, m \geq 0} e^{2n+m} |w|^n - \frac{1}{2} \sum_{n \geq l_1} |w|^{s((n + m + |n|))} \cdot \left( \sum_{l \geq 1, n \leq l \leq n + m} (c^{-1} - |w|^{\frac{3}{2}})^l \cdot (|w|^{sm} - |w|^{s((n + m + |n|))}) \right). \]
It equals
\[ k_1 \cdot \sum_{m \geq 0} d_m (c|w|^{s-\frac{1}{2}}) \left( \sum_{n \geq l_1} (c^2|w|^{1 + 2s})^n \cdot g_n(s) \right) \text{ where} \]
\[ g_n(s) = \frac{(c^{-1} - |w|^{\frac{3}{2}})^l 1 - (c^{-1} - |w|^{\frac{3}{2}})^n + 1}{1 - c^{-1} - |w|^{\frac{3}{2}}}. \]
Thus \( \sum_{n \geq l_1} (c^2|w|^{1 + 2s})^n \cdot g_n(s) \sim 0 \) and \( \sum_{m \geq 0} d_m (c|w|^{s-\frac{1}{2}}) \sim 0 \), and so we see that
\[ \sum_{n \in \mathbb{Z}, m \geq 0} e^{2n+m} |w|^n - \frac{1}{2} \sum_{n \geq l_1} |w|^{s((n + m + |n|))} \cdot \left( \sum_{l \geq 1, n \leq l \leq n + m} (c^{-1} - |w|^{\frac{3}{2}})^l \cdot (|w|^{sm} - |w|^{s((n + m + |n|))}) \right) \sim 0. \]
We have checked (5.4) \( \sim 0 \).
Putting all these things together, we verified our claim (5.1). \( \square \)
THE BESSEL PERIOD OF U(3) AND U(2) INVOLVING A NON-TEMPERED REPRESENTATION

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