Arnold’s potentials and quantum catastrophes II.

Miloslav Znojil
The Czech Academy of Sciences, Nuclear Physics Institute,
Hlavní 130, 250 68 Řež, Czech Republic

and
Department of Physics, Faculty of Science, University of Hradec Králové,
Rokitanského 62, 50003 Hradec Králové, Czech Republic

e-mail: znojil@ujf.cas.cz

and
Denis I. Borisov
Institute of Mathematics, Ufa Federal Research Center, RAS, Chernyshevskii str. 112, 450008
Ufa, Russia

and
Bashkir State University, Zaki Validi str. 32, 450076 Ufa, Russia

and
Department of Physics, Faculty of Science, University of Hradec Králové,
Rokitanského 62, 50003 Hradec Králové, Czech Republic

e-mail: BorisovDI@yandex.ru

Keywords:
Schrödinger equation; multi-barrier polynomial potentials; avoided energy-level crossings; abrupt wavefunction re-localizations; quantum theory of catastrophes;

PACS number:
PACS 03.65.Ge - Solutions of wave equations: bound states
Abstract

In paper I (M. Znojil, Ann. Phys. 413 (2020) 168050) it has been demonstrated that besides the well known use of the Arnold’s one-dimensional polynomial potentials $V_{(k)}(x) = x^{k+1} + c_1 x^{k-1} + \ldots$ in the classical Thom’s catastrophe theory, some of these potentials (viz., the confining ones, with $k = 2N+1$) could also play an analogous role of genuine benchmark models in quantum mechanics, especially in the dynamical regime in which $N+1$ valleys are separated by $N$ barriers. For technical reasons, just the ground states in the spatially symmetric subset of $V_{(k)}(x) = V_{(k)}(-x)$ have been considered. In the present paper II we will show that and how both of these constraints can be relaxed. Thus, even the knowledge of the trivial leading-order form of the excited states will be shown sufficient to provide a new, truly rich level-avoiding spectral pattern. Secondly, the fully general asymmetric-potential scenarios will be shown tractable perturbatively.
1 Introduction

In many realistic applications of quantum mechanics (say, in molecular or nuclear physics) the structure of the bound-state spectrum is often found fairly sensitive to the variation of parameters. In particular, after a small change of these parameters some of the neighboring energy levels appear to merge and cross (cf., e.g., our preceding paper I \[1\] for references). Although a more detailed inspection of the spectrum reveals that the crossings are in fact avoided \[2\], the phenomenon permanently attracts interest of theoreticians \[3, 4, 5\] as well as experimentalists \[6\].

One of the reasons is that after an analytic continuation of the Hamiltonian in the complex plane of a parameter, the avoided energy-level crossing (ALC) phenomenon proves intimately related to the complex singularities called (Kato’s) exceptional points (EPs, \[7\]). For example, in the one-dimensional Schrödinger equation

\[
H(a, b, \ldots) \psi_n(x) = E_n(a, b, \ldots) \psi_n(x), \quad n = 0, 1, \ldots ,
\]

\[
H(a, b, \ldots) = -\frac{\hbar^2}{2\mu} \frac{d^2}{dx^2} + V(x, a, b, \ldots), \quad \psi_n(x) \in L^2(\mathbb{R}),
\]

an explicit form of the ALC-EP correspondence emerges even after the most elementary choice of the exactly solvable harmonic-oscillator potential \[8\]. A message deduced from this observation is that whenever one studies the quantum ALC phenomenon, it makes sense to start the analysis from analytic potentials of polynomial form.

In paper I, for this reason, we paid attention to the study of Schrödinger equation (1) for a specific sequence of spatially symmetric polynomial potentials

\[
V(x, c_1) = x^4 + c_1 x^2, \quad V(x, c_1, c_2) = x^6 + c_1 x^4 + c_2 x^2, \quad \ldots .
\]

In a way summarized in section 2 below this enabled us to find another connection between the quantum ALC and the phenomenon of bifurcation of equilibria in certain classical dynamical systems \[9, 10\]. Indeed, sequence (2) is just an even-parity subset of family

\[
V^{(Arnold)}_{(k)}(x, a, b, \ldots) = x^{k+1} + a x^{k-1} + b x^{k-2} + \ldots , \quad k = 1, 2, \ldots
\]

which has been proposed by Arnold \[11\]. In an extended version of the Thom’s theory Arnold emphasized that the use of the general polynomial potentials \[3\] offers a geometric picture of the branching of evolutions in the generic one-dimensional classical dynamical systems.

In paper I it has been pointed out that at the odd subscripts \(k = 2N + 1\) the classical catastrophes could be also paralleled, in consistent manner, by their quantized analogues. Under the assumption of spatial symmetry \[2\] such a project proved technically feasible. Via a suitable
ad hoc reparametrization of the couplings in (3), a systematic classification of the sudden ALC-related changes of the state of the systems has been obtained. Naturally, due to the characteristic mathematical fact of the tunneling, not all of the classical types of the catastrophe found their fully analogous quantum counterpart. Some of them (like, e.g., cusp) got smeared out and smoothed after quantization. What still survived were the abrupt, catastrophe-like changes of the topological properties of probability densities. They remained measurable and, hence, phenomenologically meaningful. For this reason it has been suggested to call these specific types of the ALC-related bifurcations of equilibria “quantum relocalization catastrophes” (QRC).

In our present paper we intend to complement these results by a few vital and important addenda. First of all we will turn attention to the role of asymmetry in section 3 and to its possible influence upon the quantum-catastrophic scenarios (section 4). Besides such an extension of physics we will reconsider also the mathematical problem of the growth of the technical obstacles encountered after one removes the parity-symmetry constraint $V(x) = V(-x)$. Resolution of the challenge will be shown provided by perturbation theory. We will decompose the general Arnold’s confining quantum potential (1) into a dominant, even-parity unperturbed component $V^{(even)}(x)$ complemented by an odd-parity perturbation $V^{(odd)}(x)$.

The basic idea will lie in a combination of the standard assumption of the smallness of perturbations, $V^{(odd)}(x) = O(\epsilon)$, with an innovative “re-use” of the above-mentioned technical user-friendliness of the reparametrizations of the even-parity polynomials. Such a reparametrization of coefficients proved essential, in paper I, during the localizations of ALCs for the even Arnold’s polynomial potentials $V^{(even)}(x) = V^{(even)}_{(N)}(x)$ of degree $2N + 2$. Here, the trick will be re-applied also to the products $V^{(odd)}(x) = \epsilon x V^{(even)}_{(N)}(x)$. The control of the shape of a perturbation in $V^{(odd)}(x)$ will be transferred to the control of the shape of the even-parity auxiliary polynomial of a smaller degree $2M + 2 < 2N + 2$. The repeated application of the same reparametrization trick to perturbations of an independently tunable shape will be illustrated in section 5.

In the final part of the paper we will turn attention to excited states. We will oppose the popular opinion (advocated even in paper I) that the only meaningful form of quantization would be a transition from the study of minima of the potentials (representing the stable long-term equilibria of classical systems) to the analogous analysis of the quantum ground states. We will remind the readers that such an attitude has recently been criticized, by Smilga [12], as over-restrictive. He proposed that in any phenomenologically responsible analysis of quantum systems the excited states may often be treated as long-term stable structures. We only have to distinguish between the “malign” (i.e., the de-excitation supporting) and “benign” (i.e., the de-excitation not supporting) nature of the states with respect to the relevant physical perturbations. This means that the stability status of the quantum excited states is model-dependent. The conventional
emphasis upon the destructive role of perturbations should not be exaggerated, especially in the most common case of the low-lying spectra of bound states which are often, in real world, all long-time stable. In this spirit we will demonstrate, in section 6, the technical feasibility of an extension of the ALC-related constructions from the exclusive ground-state scenarios of Ref. [1] to the whole low-lying spectra of the excited bound states.

In a slightly broader perspective some open questions and qualitative aspects of our present results will be discussed in Appendices A and B, with a final summary presented in section 7.

2 The smearing caused by the tunneling

2.1 Multiple-well potentials

In phenomenological applications the choice of the potential in Schrödinger equation (11) is usually dictated by an expected correspondence between quantum system and its suitable classical-physics analogue. Such a purely intuitive idea played an important role not only during the birth of quantum mechanics but also in the various related methodical considerations. In paper I, in particular, the classical-quantum correspondence has been found inspiring due to its relevance in dynamical systems. On the classical-physics side, the key role has been known to be played by the singular evolution scenarios called catastrophes [10] and/or bifurcations [13]. On the quantum theory side, unfortunately, the singularities of such a type are usually assumed “smeared out” by the quantization [14]. For this reason the progress in this field is far from rapid [15].

Incidentally, the range of applicability of the Thom’s qualitative picture of evolution patterns was by far not restricted to physics. In fact, the basic motivation of the development of the catastrophe-related mathematics lied, originally, in biology [9]. Moreover, its intuitive geometric form found, paradoxically, various formal and descriptive applications not only in disciplines like theoretical chemistry [16] but even in quantum physics itself [17, 18, 19]. Obviously, the deep conceptual differences between the classical and quantum pictures of reality are reflected also in the underlying mathematics. The necessity of replacement of the observed numbers (representing, say, an energy or position of a classical point particle) by the operators in Hilbert space implies that the traditional variational and/or geometric tractability of these observables is lost, and the tools of the functional analysis have to be used [15]. Even the Arnold’s polynomial potentials (3) will lose their quantum-mechanical applicability at the odd subscripts $k = 2N + 1$. Thus, we have to restrict our attention to the subset of polynomials

$$V(x) = x^{2N+2} + a x^{2N} + b x^{2N-1} + \ldots + z x$$

where an elementary shift of the origin on the real line of $x$ enabled us to eliminate the subdominant
term. Without any loss of generality we were also able to fix $V(0)$ via a suitable shift of the energy scale.

2.2 Double-well potentials ($N = 1$)

In the framework of the classical theory of catastrophes our confining $2N$–parametric potentials (4) can be characterized, according to Arnold [20], by the Dynkin diagrams $A_k$ with odd $k = 2N + 1$. After quantization, i.e., after insertion in Schrödinger equation (1) one could have a tendency to start from the two-parametric $N = 1$ model representing a classical catastrophic scenario called cusp,

$$V(x) = V^{(\text{cusp})}(a, b, x) = x^4 + a x^2 + b x.$$  (5)

One reveals that also such a choice is to be excluded because, in contrast to the classical case, the solutions change smoothly with the parameters. The related classical cusp catastrophe proves really smeared out by the quantization [1]. Due to the quantum tunneling the wave function of the system will reside, even in double-well case, in both of the valleys. Hence, the change of the sign of $a$ will not induce any abrupt changes of the observable features of the system.

After we incorporate the second variable coupling $b$, the left-right symmetry becomes broken, but the smoothness of the dynamics remains unchanged. The change of $b$ merely moves one of the minima downwards. Locally, the well in its vicinity gets broadened so that irrespectively of the sign of $b$ the other minimum moves up and, from the point of view of the low-lying spectrum, it loses its relevance. The ground-state wave function $\psi_0(x)$ will tunnel out of the upper well. The process remains smooth. In a search for abrupt changes our attention has to be redirected to the less elementary shapes of potentials (4) with $N \geq 2$.

2.3 Reparametrizations ($N = 3$)

At the larger $N$, the reparametrization-based localization of the ALC catastrophes is vital but it could already be perceived as a challenging task. This has been emphasized in paper I. Still, we intend to demonstrate that in a way sampled at $N = 3$, an exhaustive study of the menu of catastrophes generated by the asymmetric, six-parametric Arnold’s potential

$$V^{(N=3)}(x) = x^8 + a x^6 + b x^5 + c x^4 + d x^3 + f x^2 + g x.$$  (6)

can be simplified. What is only needed is a reinterpretation and restriction of the general asymmetric potential to a weakly asymmetric model defined as a perturbation of the symmetric three-parametric unperturbed polynomial

$$V_0(x) = x^8 + a x^6 + c x^4 + f x^2.$$  (7)
The latter model can be reparametrized: we first evaluate its derivative and factorize it in a way which is slightly different from the recipe of Ref. [1],

\[ V'_0(x) \sim (x^2 - \alpha^2) (x^2 - \alpha^2 - \beta^2) (x^2 - \alpha^2 - \beta^2 - \gamma^2) . \]

This enables us to arrive at the marginally simpler reparametrizations of the couplings,

\[ a = -4 \alpha^2 - 8/3 \beta^2 - 4/3 \gamma^2, \]
\[ c = 8 \alpha^2 \beta^2 + 4 \alpha^2 \gamma^2 + 2 \beta^4 + 6 \alpha^4 + 2 \beta^2 \gamma^2, \]
\[ f = -4 \alpha^2 \beta^2 \gamma^2 - 4 \alpha^6 - 8 \alpha^4 \beta^2 - 4 \alpha^4 \gamma^2 - 4 \alpha^2 \beta^4 . \]

In a subinterval of coordinates which lie closer to the origin the shape of the potential reaches the two symmetric minimal values

\[ V_{(\text{inner minimum})} = -\alpha^8 - 8/3 \alpha^6 \beta^2 - 4/3 \alpha^6 \gamma^2 - 2 \alpha^4 \beta^4 - 2 \alpha^4 \beta^2 \gamma^2. \]

At these coordinates, by definition, the first derivative of the potential vanishes while

\[ V'_{(\text{inner minimum})} = 16 \alpha^2 \beta^4 + 16 \alpha^2 \beta^2 \gamma^2 \]

is positive (and large) so that the harmonic-oscillator approximation is validated.

At the two outer minima the value of the potential has the slightly more complicated form

\[ V_{(\text{outer minimum})} = -\alpha^8 - 2 \alpha^4 \beta^2 \gamma^2 + 1/3 \beta^8 - 2 \alpha^4 \beta^2 \gamma^2 - 4/3 \alpha^6 \gamma^2 - 2 \alpha^4 \beta^4 + 2/3 \beta^6 \gamma^2 - 1/3 \gamma^8 \]

while the value of the second derivative is positive as it should be,

\[ V'_{(\text{outer minimum})} = 16 \beta^4 \gamma^2 + 16 \alpha^2 \beta^2 \gamma^2 + 32 \beta^2 \gamma^4 + 16 \gamma^6 + 16 \alpha^2 \gamma^4 \]

A comparison of these formulae with their rescaled analogues in paper I reveals that the absence of the rescaling keeps in fact the transparency of the formulae practically unchanged. This means that also the evaluation of the locally supported low-lying spectra as well as the inclusion of asymmetric perturbations remains to be an entirely routine exercise which can be left to the readers.
3 The role of the asymmetry of $V(x)$

Let us consider the general confining Arnold’s potential

\[ V^{(\text{Arnold})}_{(2N+1)}(x, a, b, \ldots, q) = x^{2N+2} + ax^{2N} + bx^{2N-1} + \ldots + px^2 + qx \]  

(8)

and let us simplify the discussion by the special choice of $q = 0$ and $b > 0$. In any nontrivial case this enables us to introduce the even and odd components of the potential,

\[ V^{(\text{even})}_{(N)}(x) = \frac{1}{2} \left[ V^{(\text{Arnold})}_{(2N+1)}(x) + V^{(\text{Arnold})}_{(2N+1)}(-x) \right] = x^{2N+2} + ax^{2N} + cx^{2N-2} + \ldots + px^2, \]  

(9)

\[ V^{(\text{odd})}_{(N)}(x) = \frac{1}{2} \left[ V^{(\text{Arnold})}_{(2N+1)}(x) - V^{(\text{Arnold})}_{(2N+1)}(-x) \right] = bx \left[ x^{2M+2} + a' x^{2M} + c' x^{2M-2} + \ldots + p' x^2 \right] \]  

(10)

where $M = M(N) = N - 2$ and $a' = d/b$ and $c' = f/b$, etc.

3.1 Spatially symmetric special cases

Our analysis of the related ALC/QRC phenomena will be separated in two halves. In its first half we will study just the models in which the original potential is even, $V^{(\text{Arnold})}_{(2N+1)}(x) = V^{(\text{even})}_{(N)}(x)$.

Along the lines indicated above we will reparametrize the $N-$plet of its coupling constants $a, c, \ldots, p$ in terms of another $N-$plet of parameters $\alpha, \beta, \ldots, \omega$ which specify the spatially symmetric $N-$plet of zeros $\xi_n$ of the first derivative of the potential,

\[ \left[ V^{(\text{even})}_{(N)} \right]'(x) = (2N + 2) \left( x^2 - \alpha^2 \right) \left( x^2 - \alpha^2 - \beta^2 \right) \ldots \left( x^2 - \alpha^2 - \beta^2 - \ldots - \omega^2 \right). \]  

(11)

We will assume that all of the new parameters $\alpha, \beta, \ldots, \omega$ are large, $O(\lambda)$, $\lambda \gg 1$, making our spatially symmetric potential composed of an $(N+1)-$plet of the well separated deep wells. Near the local minima, these wells may be represented, with good precision, by the exactly solvable harmonic-oscillator potentials,

\[ V(x) \sim F_n + G_n(x - X_n)^2 + \text{corrections}, \quad x - X_n = \text{small}, \quad G_n > 0. \]  

(12)

This observation, thoroughly analyzed in paper I, can be given the more explicit form.

**Lemma 1** [1] In the dynamical regime with the large parameters $\alpha = O(\lambda)$, $\beta = O(\lambda), \ldots$ and positions of minima $X_n = O(\lambda)$, $\lambda \gg 1$, the coefficients in formula (12) will be large,

\[ F_n = F_n(\alpha, \beta, \ldots) = O(\lambda^{2N+2}), \quad G_n = G_n(\alpha, \beta, \ldots) = O(\lambda^{2N}), \quad n = 0, 1, \ldots, N. \]  

(13)

8
This is an important result because it excludes the possibility of the loss of the applicability of the present ubiquitous leading-order harmonic-oscillator approximations. In other words, the Lemma guarantees that the local minima of the Arnold’s potential remain “pronounced” and will not “flatten” even in the small vicinity of any Kato’s complex EP degeneracy, i.e., in our present terminology, directly in the “catastrophic” real-coupling ALC/QRC dynamical regime. In parallel, the barriers between the separate minima remain, for the same reason, high and thick. In the light of paper I, this in fact leads to the most important consequence concerning the wave functions: during their “passage through the barrier”, their decrease is exponentially quick.

Out of the multi-well ALC-supporting “bifurcation-admitting” parametric domain (the size of which is exponentially small), the numerical value of the wave function becomes negligible near any non-dominant minimum of the potential. Interested readers are advised to simulate and check the phenomenon (and, in particular, the speed of the exponential suppression of the size of the wave function inside a high and thick barrier) via an exactly solvable square-well model, with the typical closed-form wave functions sampled via example Nr. 12 on page 42 in monograph [21]).

In an immediate consequence of Lemma 1, also the values of the lowermost ground-state energies

\[ E_0^{(n)}(\alpha, \beta, \ldots) = F_n(\alpha, \beta, \ldots) + \sqrt{G_n(\alpha, \beta, \ldots)} + \text{corrections}, \quad n = 0, 1, \ldots, N \]  

will be large. This observation is certainly useful for a generalization of the \( N = 2 \) ALC/QRC condition (see Eq. (26) below) to any \( N \).

Thirdly, in the dynamical regime with the large parameters \( \alpha = O(\lambda), \beta = O(\lambda), \ldots \) the following set of the ALC/QRC conditions

\[ E_0^{(m)}(\alpha, \beta, \ldots) = E_0^{(n)}(\alpha, \beta, \ldots), \quad m > n = 0, 1, \ldots, N \]  

is, due to the symmetry of the potential, over-complete. This is an apparent puzzle which has the following remedy.

**Corollary 2** For both even \( N = 2J \) and odd \( N = 2J + 1 \) the “thin” sub-domain \( D^{(\text{QRC})}_{(\text{maximal})} \) of the parameters \( \alpha, \beta, \ldots \) which would be compatible with a complete ALC degeneracy of the ground-state energies (14) will be determined, up to small uncertainties, by any subset of \( J \) independent constraints (15).

By constraints (15), in general, the initial \( N \)-plet \( \alpha, \beta, \ldots, \omega \) becomes reduced, roughly, to one half (more precisely, to \( N = \text{entier}[(N + 1)/2] \) parameters). An approximate ALC coincidence of all of the local-well ground-state energies will be achieved. For illustration we may recall the most elementary \( N = 2 \) sample \( D^{(\text{QRC})}_{(\text{maximal})} = \{ (\alpha, \beta) | \beta = \sqrt{2}\alpha \pm \text{small corrections} \} \) of the “thin” domain of Corollary 2 above.
At any \( N \) the ALC-related measurable probability density \( \varrho(x) = \psi^*(x)\psi(x) \) will be spread over all of the vicinities of the minima \( X_n \). The corresponding QRC equilibrium characterized by the almost fully degenerate spectrum is highly unstable of course. Even the smallest perturbation will move the parameters out of the equilibrium domain \( \mathcal{D}^{(QRC)} \), causing a collapse and opening a transition, presumably, to a new, stable (or at least less unstable) equilibrium.

### 3.2 Slightly asymmetric potentials

In the preceding paragraph we implicitly assumed that the perturbations causing an unfolding of the maximal ALC/QRC collapse do not violate the spatial symmetry of the potential. In such an arrangement, the control of the relocalizations may be expected to proceed, at arbitrary \( N > 2 \), along the lines which would parallel the same pattern of explicit predictions. This means that the changes of the “outer” parameters would influence, predominantly, the shifts of the “outer” local-well minima and/or of the associated low-lying “outer” subspectra.

No similar intuitive \textit{a priori} estimate of the behavior of the subspectra exists in the full-fledged asymmetric models with \( q = 0 \). At any \( N \), the analysis can still be simplified by the decomposition

\[
V_{(2N+1)}^{(Arnold)}(x,a,b,\ldots,p) = V_{(N)}^{(even)}(x) + V_{(N)}^{(odd)}(x)
\]

of the general \( N \geq 2 \) Arnold’s potential \( \otimes \) with \( q = 0 \). Thus, let us assume that the even component \( V_{(N)}^{(even)}(x) \) of the potential becomes fixed and fine-tuned to the above-described, maximally ALC unstable equilibrium considered, for the sake of definiteness, just in the ground-state regime. Then we may expect that a rich menu of the QRC transitions to an alternative stable equilibrium will be realizable via the odd component of the potential with a sufficiently small \( b = \epsilon \),

\[
V_{(N)}^{(odd)}(x) = \epsilon x V_{(N-2)}^{(even)}(x)
\]

or with \( b = 0 \) and with a sufficiently small \( d = \epsilon \),

\[
V_{(N)}^{(odd)}(x) = \epsilon x V_{(N-4)}^{(even)}(x)
\]

etc. Along these lines, the trick is that we may ask for the presence of pronounced minima and maxima not only in the dominant even-parity potential \( V_{(N)}^{(even)}(x) \) but also in the small, \( \mathcal{O}(\epsilon) \) contributions coming from the odd function correction \( V_{(N)}^{(odd)}(x) = \epsilon x V_{(M)}^{(even)}(x) \) with \( M = N - 2 \) or \( M = N - 4 \), etc. Once such a small \( \mathcal{O}(\epsilon) \) term is added to the dominant but sensitive, ALC-fine-tuned \( V_{(N)}^{(even)}(x) \) of Eq. (12), it may cause enhancements and/or suppressions of the dominant-potential wells via the new, movable \( \mathcal{O}(\epsilon) \) wells and/or barriers in \( V_{(M)}^{(odd)}(x) \). This might facilitate the selection and control of the ultimate QRC equilibria as long as the coordinates
One can summarize that our task is reduced to the analysis of the adaptability of the properties of the spatially antisymmetric perturbation component $\epsilon x V^{(\text{even})}(x)$ of the full potential. It is important that in such a component the maximal power $x^{2M+3}$ can be much smaller than the maximal power $x^{2N+2}$ characterizing the full, perturbed Arnold’s potential. Hence, the generic multi-well form of the perturbation can be kept as simple as possible. In particular, its freely variable local extremes could be more easily matched, added to, or subtracted from, their unperturbed partners. Decisively, such a matching can be facilitated by the following observation.

**Lemma 3** In a vicinity of the minimum $\lambda X > 0$ of $V(x) = \lambda^{2M+2} F + \lambda^2 M (x - \lambda X)^2$ with $G > 0$, the minimum of the third-order polynomial $x V(x)$ in $x$ lies at $x_0 = \lambda (1 + \delta) X$ where the shift is $\lambda$-independent,

$$\delta = -\frac{F}{G X^2 + X \sqrt{G^2 X^2 - 3 FG}}.$$  

**Proof.** The odd function $x V(x)$ of $x$ has two local extremes (viz., a local maximum and a local minimum), with the coordinates given as roots of quadratic equation. □

This result shows that in a small vicinity of a preselected minimum of a dominant symmetric part of a slightly asymmetric Arnold’s potential, an enhancement or suppression of this minimum can be achieved via an antisymmetric *ad hoc* perturbation $V^{(\text{odd})}(x)$, the global shape of which can be controlled by an $M$–plet of independent parameters where $M$ can be chosen much smaller than $N$.

## 4 Quantum catastrophes

### 4.1 Symmetric triple-well potentials

Schrödinger equation

$$\left[ -\Lambda^2 \frac{d^2}{dx^2} + V^{(\text{butterfly})}(x) \right] \psi_n(x) = E_n \psi_n(x), \quad \Lambda^2 = \hbar^2/(2\mu), \quad n = 0, 1, \ldots$$  

with the four-parametric potential

$$V^{(\text{butterfly})}(x) = x^6 + a x^4 + b x^3 + c x^2 + d x$$  

is simpler to solve in its spatially symmetric version where $b = d = 0$. The basic technical aspects of such a special case were discussed in paper I. In a slightly different notation let us now
summarize these results briefly. First, in the spirit of loc. cit., potential

\[ V(x) = x^6 + ax^4 + cx^2 \]  

will only be considered here in its most interesting deep-triple-well dynamical regime. Under this assumption, the first derivative of the potential can be factorized in terms of the coordinates of the extremes of \( V(x) \) or, in other words, in terms of suitable real \( \alpha \) and \( \beta \),

\[ V'(x) = 6x^5 + 4ax^3 + 2cx = 6x(x^4 + \frac{4a}{6}x^2 + \frac{2c}{6}x) = 6x(x^2 - \alpha^2)(x^2 - \alpha^2 - \beta^2), \]  

This induces a reparametrization of the original couplings,

\[ a = a(\alpha, \beta) = -3(\alpha^2 + \beta^2/2), \quad c = c(\alpha, \beta) = 3\alpha^2(\alpha^2 + \beta^2). \]  

The pronounced, deeply triple-well shape of the potential possessing two high and thick inner barriers will be achieved when and only when the coordinates of the extremes are chosen sufficiently large, \( \alpha^2 \gg \Lambda^2 \) and \( \beta^2 \gg \Lambda^2 \). Thus, in units such that \( \Lambda^2 = 1 \) we have to have \( \alpha \gg 1 \) and \( \beta \gg 1 \).

### 4.2 Ground states and the avoided level crossing phenomenon

The latter, phenomenologically motivated assumption simplifies the approximate construction of bound states. In the case of the dominance of the central attraction the approximate low-lying spectrum will acquire the elementary leading-order form

\[ E_n^{\text{(central)}} = (2n + 1) \sqrt{c(\alpha, \beta)} + \text{higher order corrections}, \quad n = 0, 1, \ldots \]  

The alternative assumption of the dominance of the off-central attraction yields the almost degenerate energy-level doublets

\[ E_m^{\text{(even/odd)}} = V(\sqrt{\alpha^2 + \beta^2}) + (2m + 1)\Omega + \text{higher order corrections}, \quad m = 0, 1, \ldots \]  

\[ \Omega = \sqrt{V''(\sqrt{\alpha^2 + \beta^2})/2}, \quad V''(\sqrt{\alpha^2 + \beta^2}) = 12\alpha^2\beta^2 + 12\beta^4 \]

which are, naturally, never strictly degenerate.

As long as we have \( V(\sqrt{\alpha^2 + \beta^2}) = \alpha^6 + 3/2 \alpha^4 \beta^2 - 1/2 \beta^6 \) the ground-state special case of formula (24) reads

\[ E_0^{\text{(even/odd)}} = \alpha^6 + 3/2 \alpha^4 \beta^2 - 1/2 \beta^6 + \sqrt{6} \alpha^2 \beta^2 + 6 \beta^4 + \text{corrections}. \]  

We may set \( \beta^2 = \mu^2 \alpha^2 \) with, say, \( \mu = O(1) \). Asymptotically (i.e., in the regime of very large parameters \( \alpha \gg 1 \) and \( \beta \gg 1 \)) we then deduce, from formula (23), that \( E_n^{\text{(central)}} = O(\alpha^2) \). In
contrast, Eq. (25) implies that \( E_m^{(\text{even/odd})} = \mathcal{O}(\alpha^6) \). This comparison reveals the clearly dominant behavior of the off-central minimum of the potential.

In the leading-order approximation the “quantum-catastrophic” instant of transition between the central and off-central dominance of the probability density of the ground states will be merely dictated by the trivial constraint \( V(\sqrt{\alpha^2 + \beta^2}) = \mathcal{O}(\alpha^2) \), i.e., up to corrections, by the elementary equation \( V(\sqrt{\alpha^2 + \beta^2}) = 0 \). This enables us to deduce that the quantum relocalization catastrophe becomes controlled by \( \beta \) and realized at \( \mu^2 \approx \mu_\infty = 2 \), i.e., along the line of \( \beta \approx \sqrt{2} \alpha \). Only in the next order approximation one has to impose the explicit ALC condition

\[
E_0^{(\text{central})}(\alpha, \beta) = E_0^{(\text{even/odd})}(\alpha, \beta) + \text{small corrections}.
\] (26)

After an appropriate change of the notation conventions, a more detailed numerical analysis of its solutions may be found described in paper I.

5 Asymmetric \( N = 2 \) illustration

The turn of attention to the spatially asymmetric general version of the butterfly [9] potential

\[
V(x) = x^6 + a x^4 + b x^3 + c x^2 + d x
\] (27)

opens a number of possibilities of reaching an asymmetric quantum catastrophe. A small change of parameters could now cause an abrupt jump of the dominant part of the observable probability density (i.e., of function \( \rho(x) = \psi^\ast(x)\psi(x) \)) between the central and strictly one of the off-central local minima.

5.1 Simplification of mathematics: \( d = 0 \)

For illustration, for the sake of brevity, let us set \( d = 0 \) and keep the asymmetry controlled by the single coupling constant \( b \neq 0 \). In the limit of \( b \to 0 \) the left-right-symmetric set of the zeros of polynomial (21)

\[
\{-\sqrt{\alpha^2 + \beta^2}, -\alpha, 0, \alpha, +\sqrt{\alpha^2 + \beta^2}\}
\] (28)

is known. Thus, for our present methodical purposes it will suffice to keep the asymmetry small. Using, without any loss of generality, a positive \( b = \epsilon > 0 \), the set of extremes (28) becomes modified, in general, as follows,

\[
\{-\sqrt{\alpha^2 + \beta^2} - \epsilon p^2(\epsilon), -\alpha + \epsilon q^2(\epsilon), 0, \alpha + \epsilon u^2(\epsilon), \sqrt{\alpha^2 + \beta^2} - \epsilon v^2(\epsilon)\}.
\]
The weakly $\epsilon -$dependent shifts $p^2(\epsilon), q^2(\epsilon), u^2(\epsilon)$ and $v^2(\epsilon)$ must be all positive. Their values can easily be determined from their definition $V'(x) = 0$, i.e., from equation

$$V'(x) = 6 x^5 + 4 a x^3 + 3 \epsilon x^2 + 2 c x = 6 x (x^2 - \alpha^2) (x^2 - \alpha^2 - \beta^2) + 3 \epsilon x^2 = 0. \quad (29)$$

Step by step, the insertion of the first-order ansatz $x = \alpha + u^2 \epsilon$ using abbreviation $u^2 = u^2(0)$ in Eq. (29) will yield the relation

$$V'(x) \sim 2 (x^2 - \alpha^2) (x^2 - \alpha^2 - \beta^2) + \epsilon x \sim -4 u^2 \beta^2 + 1 = O(\epsilon). \quad (30)$$

Its leading-order solution is $u^2 = 1/(4 \beta^2)$. Similarly, the insertion of ansatz $x = \sqrt{\alpha^2 + \beta^2} - v^2 \epsilon$ in the same equation yields an analogous solution $v^2 = 1/(4 \beta^2)$. Along the same lines we also obtain $q^2 = 1/(4 \beta^2)$ and, finally, $p^2 = 1/(4 \beta^2)$. If needed, a systematic evaluation of the next-order corrections would be lengthier but also routine, yielding

$$u^2(\epsilon) = \frac{1}{4 \beta^2} + \left(\frac{\beta^2 + 4 \alpha^2}{32 \alpha \beta^6}\right) \epsilon + O(\epsilon^2)$$

e tc.

5.2 Asymmetry treated as a perturbation

We are now prepared to analyze the above-mentioned asymmetric relocalization catastrophe as a scenario during which, after a small change of parameters, the mechanism of quantum tunneling would force the dominant part of the observable probability density $\rho(x)$ to perform an abrupt jump between the central and the leftmost local minimum of the potential. In other words, the initial localization of the dominant part of $\rho(x)$ in the center will get suppressed while the initially negligible component of $\rho(x)$ near the left minimum will become abruptly dominant.

In a way explained in paper I, the explicit demonstration that the probability density indeed goes through a sharp change at the catastrophe is based on the very initial formulation of the problem in which the wave functions of the system are well approximated by the harmonic-oscillator states near all of the local minima. The task of an approximate quantitative description of such a jump is then feasible, simplified further by the fact that the inclusion of asymmetry does not modify the central candidate (23) for the low-lying spectrum.

The construction of its left-well alternative is also not too complicated because its dominant component is represented just by the deep and $\alpha -$sensitive $O(\alpha^6)$ minimum of the potential. Its asymmetry-dependence alias $\epsilon -$dependence can be evaluated in its closed leading-order form to read

$$V(-\sqrt{\alpha^2 + \beta^2} - \epsilon p^2(0)) = V(-\sqrt{\alpha^2 + \beta^2}) - \epsilon (\sqrt{\alpha^2 + \beta^2})^3 + O(\alpha^2) + O(\epsilon^2). \quad (31)$$
Thus, the only rather lengthy calculation is needed for the evaluation of the level-spacing parameter corresponding to the leftmost local well. One obtains the formula

\[ V''(-\sqrt{\alpha^2 + \beta^2 - \epsilon \mu^2(\epsilon)}) = V''(-\sqrt{\alpha^2 + \beta^2}) + 3 \frac{\sqrt{\alpha^2 + \beta^2} (4 \alpha^2 + 5 \beta^2)}{\beta^2} \epsilon + O(\epsilon^2) \]

which implies that the \( O(\epsilon) \) correction to the level-spacing parameter \( \Omega \) is of order \( O(\alpha) \), i.e., inessential (cf. Eq. (24) above). Only the first two terms of Eq. (31) remain relevant for the proof of the existence of the asymmetric relocalization catastrophe.

Its explicit approximate localization proceeds via relation

\[ \alpha^6 + 3/2 \alpha^4 \beta^2 - 1/2 \beta^6 = (\alpha^2 + \beta^2)^{3/2} \epsilon + \text{corrections} \quad (32) \]

i.e.,

\[ \epsilon = -\frac{1}{2} \alpha^3 \delta \sqrt{3 + \delta} + \text{corrections} \quad (33) \]

Here, the (small) value of \( \epsilon \) and the (large) value of \( \alpha \) should be interpreted as an input information about the potential (i.e., about dynamics). Once we return to the ansatz \( \beta^2 = \mu^2 \alpha^2 \) with a (small) variable \( \delta \) in \( \mu^2 = 2 + \delta \) we obtain the desirable solution of the catastrophe-determining Eq. (32) in its linearized leading-order form

\[ \delta = -\frac{2 \epsilon}{\sqrt{3} \alpha^3}. \quad (34) \]

This formula indicates that the impact of the asymmetry of the potential is suppressed by the factor of \( \alpha^{-3} \). Still, in the light of Eq. (22) where \( a = O(\alpha^2) \) and \( c = O(\alpha^4) \), an asymptotic enhancement of the magnitude of the symmetry-violating couplings up to \( b = \epsilon = O(\alpha^3) \) seems appropriate.

In such an extended dynamical regime, a return to the more precise implicit cubic-equation definition (33) of the (negative) critical shift \( \delta = O(1) \) would be necessary. The resulting, more visible asymmetric quantum relocalization catastrophe will be characterized by a substantial, non-negligible decrease of the critical value of the ratio \( \mu^2 = \beta^2 / \alpha^2 \) of the two dynamical parameters of the model.

6 **Excited states** \((N = 2)\)

The above-discussed conclusions concerning the existence of the ground-state relocalization transitions in the spatially symmetric potentials reconfirm the results of paper I. We only have to keep in mind here that in *loc. cit.* the meaning of some of the symbols was different. Irrespectively of that, in both cases one works with large \( \alpha \gg 1 \) and \( \beta \gg 1 \) so that the higher-order anharmonicities remain small. The pronounced form of the minima and maxima of the potential is
guaranteed. This opens the way towards an innovation in which the new “quantum-catastrophic” relocalization transitions involve also the excited states.

In a concise explanation of the emergence of such a new class of observable abrupt-change phenomena we have to point out, first of all, that in the deep-triple-well regime of the model with $N = 2$ the level spacings $\sqrt{c(\alpha, \beta)} \sim E_0^{(\text{central})}$ and $\Omega(\alpha, \beta) \sim E_0^{(\text{even/odd})} - V(\sqrt{\alpha^2 + \beta^2})$ are both of the order of magnitude $O(\alpha^2)$, i.e., subdominant but still large and practically excitation-independent. For this reason, also the separate excited states can still be well identified experimentally. In the language of the relocalization theory, its above-outlined ground-state version admits an immediate and meaningful upgrade, therefore. In particular, the ALC-related Eq. (26) describing the instant of bifurcation between the central and off-central localization of the probability density $\rho(x) = \psi^*(x)\psi(x)$ becomes complemented by an analogous excited-state-matching extension which involves any pair of the bound states lying in the low-lying spectrum of the system,

$$E_m^{(\text{even/odd})}(\alpha, \beta) = E_n^{(\text{central})}(\alpha, \beta) + \text{small corrections}, \quad m, n = 0, 1, \ldots, M_{\text{max}}.$$  

(35)

At every pair of the main quantum numbers $m$ and $n$ the search for the solutions remains straightforward, albeit just numerical.

Table 1: The localization of bifurcations $E_m^{(\text{even/odd})}(\alpha, \beta) = E_n^{(\text{central})}(\alpha, \beta)$ as defined by Eq. (35). The search is performed at a fixed $\alpha = 4$ and with a variable $\beta = \mu \alpha$ or rather $\mu = \sqrt{2 + \delta}$ with a small $\delta$. The critical ALC values $\delta = \delta(m, n)$ were found numerically.
Table 1 offers an illustrative sample of the corresponding search for the ALC coincidences obtained as solutions of Eq. (35) at $\alpha = 4$. In the Table we consider a variable $\beta = \mu \alpha$ where the ratio $\mu = \sqrt{2 + \delta}$ or, more precisely, the value of $\delta$ is a variable parameter. What one immediately notices is the smallness of the deviations $\delta = \delta(m, n)$ of the numerically evaluated critical ratios $\mu^2 = \beta^2/\alpha^2$ from their unique asymptotic value $\mu^2_\infty = 2$. Theoretically, this smallness is easily explained by the proportionality of the off-central energy values to the local bottom $V(\sqrt{\alpha^2 + \beta^2})$ of the potential. The decrease of this minimum is dictated by its very large component $-\beta^6/2$ so that the shift $\delta$ itself must remain small. From the point of view of experimentalists, the high sensitivity of the process of the relocalization of the probability density to a parameter should be interpreted as an abrupt change of the topology of the system near a critical $\delta = \delta(m, n)$. Hence, it makes sense to speak about the phenomenon of a quantum relocalization catastrophe even when the excited states are concerned.

The second striking feature of the excitation-dependent relocalization catastrophes which occur at the $m-$ and $n-$ dependent parameters $\mu^2 = \mu^2_\infty + \delta(m, n)$ may be seen, in Table 1, in their approximate pairwise degeneracy, with $\delta(1, 3) \approx \delta(0, 1)$, etc. Again, the theoretical explanation of the phenomenon remains straightforward. The process of the matching of the separate levels as prescribed by Eq. (35) is basically controlled by the decrease of their off-central partners which is, roughly speaking, proportional to the increase of $\mu$. Once we use the asymptotically correct parameter $\mu = \mu_\infty$ and evaluate the asymptotically correct spring constants (i.e., level-distances) $\sqrt{c_\infty} = 48$ and $\Omega_\infty = 96$, we reveal the proportionality in disguise, $\Omega_\infty = 2 \sqrt{c_\infty}$.

This would imply the exact pairwise degeneracies of the shifts. Such a phenomenon (reflecting the low-degree polynomial nature of the interaction) might deserve an experimental verification and/or simulation. In the context of pure theory it is really remarkable that in our illustrative Table 1 such a degeneracy is only very weakly broken by the nonlinearity of the equations. We can deduce that even after the consequent next-order inclusion of the nonlinearity in both of our two topologically different dynamical regimes we will still have, with good precision, the commensurate level spacings $\Omega(\alpha, \beta) \approx 2 \sqrt{c(\alpha, \beta)}$.

7 Summary

In technical terms, our present message has two parts. In one, we turned attention to the excited states and we strengthened the claim of paper I that the mathematical complications introduced by the use of quantum dynamics still remain surmountable. Secondly, we amended and completed the results of paper I by an outline of a perturbative inclusion of the “missing”, parity-violating components in the interactions. In this manner we extended the class of the admissible and
tractable models to the whole set of the slightly asymmetric confining Arnold’s potentials. One can say that in spite of the existence of multiple questions which are still open (see Appendices A and B for their sample), the overall picture of the ALC phenomenon formulated in the language of the low-lying quantum bound states supported by Arnold’s potentials is mathematically consistent.

It is worth adding that the latter result was based on a tacit assumption of the smallness of the antisymmetric part of the potential. Such an assumption has two aspects. On positive side it has been shown useful and efficient in the vicinity of the (approximate) complete, \((N+1)\)-tuple ALC degeneracy where the branching of the unfolding scenarios becomes maximally sensitive to the tiniest changes of the parameters. On negative side, the perturbative tractability of the other, less extreme ALC degeneracies were not discussed and their analysis remains an open question. Far from the exceptional dynamical regime of a maximal ALC degeneracy, unfortunately, our present perturbation-approximation technique might fail and would require an independent study. This is a serious uncertainty which could lead to some \textit{a posteriori} limitations of our constructive perturbation approach in applications.

The currently missing exhaustive classification of all of the alternative QRC re-arrangement scenarios might, indeed, require the use of some alternative, non-perturbative methods in the future. At present we must admit that in such a case it may happen that one would have to sacrifice the simplicity of the picture. Perhaps, an entirely different treatment of the ALC phenomena will be needed for the quantum Arnold’s potentials characterized by an extreme spatial asymmetry.
References

[1] M. Znojil, Ann. Phys. 413 (2020) 168050.

[2] J. von Neumann and E. P. Wigner, Physikalische Zeitschrift 30 (1929) 465 - 467.

[3] M. V. Berry, Czech. J. Phys. 54 (2004) 1039 - 1047;
   D. I. Borisov, Acta Polytechnica 54 (2014) 93.

[4] F. Gesztesy, D. Gurarie1, H. Holden, M. Klaus, L. Sadun, B. Simon and P. Vogl, Commun.
   Math. Phys. 118 (1988) 597;
   G. V. Dunne, T. Sulejmanpasic and M. Ünsal, Bions and Instantons in Triple-well and
   Multi-well Potentials. in A. Niemi, T. Tomboulis and K. K. Phua, eds, Roman Jackiw 80th
   Festschrift. World Scientific, Singapore, 2020.

[5] D. I. Borisov and M. Znojil, Mathematical and physical meaning of the crossings of energy
   levels in PT-symmetric systems, in F. Bagarello, R. Passante and C. Trapani, eds, Non-
   Hermitian Hamiltonians in Quantum Physics. Springer, Cham, 2016.

[6] http://www.nithep.ac.za/2g6.htm (accessed on September 22nd, 2019).

[7] T. Kato, Perturbation theory for linear operators. Springer, Berlin, 1966.

[8] M. Znojil, Phys. Lett. A 259 (1999) 220 - 223.

[9] R. Thom, Structural Stability and Morphogenesis: An Outline of a General Theory of Models.
   Addison-Wesley, Reading, 1989.

[10] E.C. Zeeman, Catastrophe Theory-Selected Papers 1972-1977. Addison-Wesley, Reading,
    1977.

[11] J. Poston and I. Stewart, Catastrophe Theory and Its Applications. Pitnam, London, 1978.

[12] A. Smilga, Int. J. Mod. Phys. A 32 (2017) 1730025;
    A. Smilga, videorecorded seminar on July 23, 2020 (https://vphhq.com).

[13] V. I. Arnold, Dynamical Systems V: Bifurcation Theory and Catastrophe Theory. Springer-
    Verlag, Berlin, 1994.

[14] V. Mukhanov, Physical Foundations of Cosmology. CUP, Cambridge, 2005;
    A. Ashtekar, A. Corichi and P. Singh, Phys. Rev. D 77 (2008) 024046.
[15] M. Znojil, J. Phys. A: Math. Theor. 45 (2012) 444036;
    D. I. Borisov, F. Ruzicka and M. Znojil, Int. J. Theor. Phys. 54 (2015) 4293.

[16] X. Krokidis, S. Noury and B. Silvi, J. Phys. Chem 101 (1997) 7277.

[17] D. S. Lohr-Robles, E. Lopez-Moreno and P. O. Hess, Nucl. Phys. B 992 (2019) UNSP 121629.

[18] R. Gilmore, S. Kais, and R. D. Levine, Phys. Rev. A 34 (1986) 2442;
    C. Emary, N. Lambert, and T. Brandes, Phys. Rev. A 71 (2005) 062302.

[19] D. H. J. O’Dell, Phys. Rev. Lett. 109 (2012) 150406;
    A. Z. Goldberg, A. Al-Qasimi, J. Mumford, and D. H. J. O’Dell, Phys. Rev. A 100 (2019)
    063628.

[20] V. I. Arnold, Catastrophe Theory. Berlin, Springer-Verlag, 1992.

[21] F. Constantinescu and E. Magyari, Problems in quantum mechanics. Pergamon Press, Oxford,
    1971.

[22] M. Znojil, Mod. Phys. Lett. B 34 (2020) 2050378.

[23] M. Znojil, Ann. Phys. (NY) 416 (2020) 168161.

[24] F. G. Scholtz, H. B. Geyer and F. J. W. Hahne, Ann. Phys. (NY) 213 (1992) 74.

[25] A. Mostafazadeh, Int. J. Geom. Meth. Mod. Phys. 7 (2010) 1191.

[26] N. Moisyev, Non-Hermitian quantum mechanics. Cambridge, CUP, 2011.

[27] C. M. Bender, Rep. Prog. Phys. 70 (2007) 947.

[28] F. Bagarello, J.-P. Gazeau, F. Szafraniec and M. Znojil, Eds., Non-Selfadjoint Operators in
    Quantum Physics: Mathematical Aspects. Wiley, Hoboken, 2015.

[29] A. Turbiner, J. Carlos del Valle, Anharmonic Oscillator: a solution. arXiv:2011.14451

[30] B. Simon, Ann. Math. 120 (1984) 89 - 118.
Appendix A: Physics behind the Arnold’s potentials: Open questions and further research

Naturally, the complexity of the implementation of the basic ideas of our present paper will increase with the growth of $N$. Via several tentative preliminary approximative calculations we revealed that, fortunately, the rate of this increase remains reasonable. First of all, the proofs of the existence of the topology-changing quantum catastrophes as well as of the localization of the corresponding ALC/QRC instants do not change too much with the growth of $N$. Secondly, the use of any commercially available symbolic manipulation software appears remarkably suitable for the purpose. Last but not least, the results of the computer-assisted algebraic manipulations still remain comparatively compact and transparent, in spite of being less suitable for the presentation in print, mainly due to their increasing length. A modified strategy of their analysis as well as presentation will be needed, in particular, after the present turn of attention to the asymmetric potentials $V(x) \neq V(-x)$ and to the low-lying excited states.

Technically, the analysis and predictions proved perceivably less difficult in the context of paper I where it has been conjectured and demonstrated that the Thom’s classification of catastrophes (meaning the abrupt changes of a system’s equilibria after a very small change of its parameters) could consistently be paralleled in quantum world. The basic idea was of a topological nature, and the construction task has been facilitated by the assumption that the parities of the benchmark potentials were even [cf. their sample in Eq. (2)]. Furthermore, attention was restricted to a subfamily of $V(x)$s with the shapes characterized by the multiple pronounced and deep minima separated by the multiple high and thick barriers.

A persuasive and systematic qualitative classification has been then obtained thanks to the acceptance of several auxiliary technical constraints. The main one can be seen in the attention paid, more or less exclusively, to the “strong-coupling” dynamical regime. In its framework, both the necessary spectral analysis and the subsequent predictions of the observable topological effects characterized, first of all, by the abrupt relocalizations of the probability densities appeared simplified, feasible and, in principle, detectable in the laboratory.

In paper I the main observation was that what remains uninfluenced by the ubiquitous tunneling are the topologically non-equivalent probability densities $\rho(x)$. The first nontrivial sample of such a QRC scenario proved provided by the butterfly potential model with $N = 2$ barriers. In this system strictly two topologically non-equivalent equilibria were identified, with either the centrally dominated or the off-centrally dominated probability density $\rho(x)$. On this background it has been claimed that at least some of the subsequent studies of quantum ALC-related phenomena might still be based on Schrödinger equations with analytic and polynomial interactions. Soon,
the latter expectations were confirmed by an extension of the analysis to several more or less realistic two-dimensional \cite{22} and three-dimensional \cite{23} descendants of the Thom’s one-dimensional cusp and butterfly potentials of Eq. (2). In this context, our present extension of the scope of the analysis to the spatially asymmetric Arnold’s potentials and to the identification of the ALC phenomena in the excited quantum states appeared well phenomenologically motivated, indeed.

In the earlier attempts at the continuation and extension of the project we were, originally, not too successful. We tried to use the non-perturbative methods and we wished to extend the theory to the general asymmetric Arnold’s potentials, but we failed. The crisis had only been overcome when we imagined that the bifurcation of the maximally degenerate ALC equilibrium could rather straightforwardly be still described via a restricted, controllably small modification of the potential. This proved to be one of the key ideas which inspired our present study and which is also to be pursued in our future research.

In a more specific preliminary remark let us mention that the most important open question and theoretical challenge is twofold. In both of its forms one will encounter a generalization of the theory based on a relaxation of our present “traditional” assumption of having the Arnold’s potentials real, i.e., of having the corresponding Hamiltonians self-adjoint. The two forms of such a relaxation of constraints may be seen as motivated by the recent growth of popularity of the so called pseudo-Hermitian reformulation of the model-building strategy in quantum mechanics (see, e.g., reviews \cite{24} or \cite{25}).

From the point of view of our future “quantum-catastrophic” studies and projects, we will have to find a contact with both of the two conceptually different implementations of the latter reformulation of quantum theory dealing, respectively, with the so called open quantum systems (OQS, a monograph \cite{26} might be cited in this case) and with the (conceptually very different) closed quantum systems (CQS) as considered in reviews \cite{24} and \cite{25} or \cite{27, 28}. Indeed, from the different perspectives, both of these OQS and CQS approaches share the interest and emphasis put upon the concept of the Kato’s exceptional points. At the same time, we may only repeat that the ALC-EP correspondence will always play a key role in any future complete quantum theory of catastrophes.

Appendix B: Several open mathematical questions behind the Arnold’s potentials: Semiclassical approximation, etc

One of the explanations of the success and popularity of the Thom’s classification of elementary non-quantum catastrophes lies in its simplicity. Indeed, the emphasis upon the qualitative aspects of the classical evolution patterns made his theory universal. In parallel, the underlying intuitive
perception of the concept of the equilibrium rendered many of its applications straightforward. A priori one would expect that both of these merits of the Thom’s theory (viz., its simplicity and a straightforward applicability) must necessarily be lost after quantization.

In this sense, our present paper can be read as advocating an opposite opinion. In a way co-supported by the results of Ref. [1] we may claim that up to the trivial single-well and single-barrier exceptions (where any observable change remains smooth, due to the tunneling in the second case), all of the multi-parametric confining Arnold’s potentials prove able to serve as nontrivial benchmark realizations of the ALC-related quantum relocalization processes. In a way caused by a tiny change of the parameters these processes were shown to exhibit all of the characteristic features (like, in particular, the abruptness) of the popular classical catastrophes.

After the latter conclusion one has to ask the natural question concerning the dependence of the ALC phenomenon on a hypothetical strengthening \( V(x) \to g V(x) \) of the Arnold’s potential. The question is legal because in Eq. (3) which defines the potential the leading-power coefficient is equal to one. Indeed, once we fixed the units (such that \( \hbar^2/(2 \mu) = 1 \)) we lost the variability of the parameter which could control the semiclassical limit of the model. In other words, we fixed the arrow of our inspiration (from classical to quantized) and we lost the opportunity of studying the quantum-classical correspondence of the systems under consideration. In the opposite direction of moving from quantum to classical, the idea of reduction of the quantum picture to its classical limit could find its implementation, in the future research, in a way based on the variational [29], semiclassical [30] or even some brutally numerical treatments of the initial quantum system.

Technically, the goal could be achieved when one abbreviates \( \hbar^2/(2 \mu) = \Lambda^2 \) and keeps the latter parameter variable. One of the consequences would be that Schrödinger equation

\[
\left[ -\Lambda^2 \frac{d^2}{dx^2} + V(x) \right] \psi_n(x) = E_n \psi_n(x), \quad n = 0, 1, \ldots
\]

becomes more flexible. What remains unchanged is that near one of the deep minima the potential can still be replaced by its harmonic-oscillator approximation. This would yield the approximate low-lying spectrum in which the \( \Lambda = 1 \) energies \( E_n \sim V(x_{\text{min}}) + (2n + 1)\omega \) would be replaced by their rescaled \( \Lambda \neq 1 \) descendants \( E_n(\Lambda) \sim V(x_{\text{min}}) + \Lambda (2n + 1)\omega \). The variable \( \Lambda \) interconnects the large-mass and semi-classical limits as well as, alternatively the small-mass and ultra-quantum dynamical regimes (cf. also [29] in this respect).

The classical catastrophe theory reemerges in the semiclassical limit in which all of the low-lying levels would converge to the minimum of the potential [30]. Vice versa, the quantum effects are enhanced at large \( \Lambda \) while becoming divergent in the vanishing-mass limit \( \mu \to 0 \).