Time-Fractional KdV Equation for the plasma in auroral zone using Variational Methods

El-Said A. El-Wakil, Essam M. Abulwafa, Emad K. Elshewy and Aber A. Mahmoud
Theoretical Physics Research Group, Physics Department, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

Abstract
The reductive perturbation method has been employed to derive the Korteweg-de Vries (KdV) equation for small but finite amplitude electrostatic waves. The Lagrangian of the time fractional KdV equation is used in similar form to the Lagrangian of the regular KdV equation. The variation of the functional of this Lagrangian leads to the Euler-Lagrange equation that leads to the time fractional KdV equation. The Riemann-Liouville definition of the fractional derivative is used to describe the time fractional operator in the fractional KdV equation. The variational-iteration method given by He is used to solve the derived time fractional KdV equation. The calculations of the solution with initial condition $A_0 \sec h^2(c x)$ are carried out. Numerical studies have been made using plasma parameters close to those values corresponding to the dayside auroral zone. The effects of the time fractional parameter on the electrostatic solitary structures are presented.

Keywords: Euler-Lagrange equation, Riemann-Liouville fractional derivative, fractional KdV equation, He's variational-iteration method.

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1 Introduction
Because most classical processes observed in the physical world are nonconservative, it is important to be able to apply the power of variational methods to such cases. A method used a Lagrangian that leads to an Euler-Lagrange equation that is, in some sense, equivalent to the desired equation of motion. Hamilton’s equations are derived from the Lagrangian and are equivalent to the Euler-Lagrange equation. If a Lagrangian is constructed using noninteger-order derivatives, then the resulting equation of motion can be nonconservative. It was shown that such fractional derivatives in the Lagrangian describe nonconservative forces [1, 2]. Further study of the fractional Euler-Lagrange can be found in the work of Agrawal [3, 4], Baleanu and coworkers [5, 6] and Tarasov...
and Zaslavsky [7, 8]. During the last decades, Fractional Calculus has been applied to almost every field of science, engineering and mathematics. Some of the areas where Fractional Calculus has been applied include viscoelasticity and rheology, electrical engineering, electrochemistry, biology, biophysics and bioengineering, signal and image processing, mechanics, mechatronics, physics, and control theory [9].

On the other hand, electron acoustic waves (EAWs) propagation in plasmas has received a great deal attention because of its vital role in understanding different types of collective processes in laboratory devices [10, 11] as well as in space environments [12, 13]. It has been argued that when the hot to cold electron temperature ratio is greater than 10, the electron-acoustic mode may be the principal mode of the plasma in which the restoring force comes from the pressure of the hot electrons, while the inertia comes from the mass of the cold electron component [14]. The ions play the role of a neutralizing background, i.e., the ion dynamics does not influence the EAWs because its frequency is much larger than the ion plasma frequency. Several theoretical attempts have been made to explain nonlinear EAWs in plasma systems [15-17].

To the author’s knowledge, the problem of time fractional KdV equation in collisionless plasma has not been addressed in the literature before. So, our motive here is to study the effects of time fractional parameter on the electrostatic structures for a system of collisionless plasma consisting of a cold electron fluid and non-thermal hot electrons obeying a non-thermal distribution and stationary ions. Our choice of non-thermal distribution of electrons is prompted by its convenience rather than as precise fitting of the observations. We expect that the inclusion of the non-thermal electrons, time fractional parameter will change the properties as well as the regime of existence of solitons.

Several methods have been used to solve fractional differential equations such as: the Laplace transform method, the Fourier transform method, the iteration method and the operational method [18, 19]. Recently, there are some papers deal with the existence and multiplicity of solution of nonlinear fractional differential equation by the use of techniques of nonlinear analysis [20-23]. In this paper, the resultant fractional KdV equation will be solved using a variational-iteration method (VIM) firstly used by He [24, 25].

This paper is organized as follows: Section 2 is devoted to describe the formulation of the time-fractional KdV (FKdV) equation using the variational Euler-Lagrange method. In section 3, the resultant time-FKdV equation is solved approximately using VIM. Section 4 contains the results of calculations and discussion of these results.

2 Basic Equations and Time-fractional KdV equation

We consider a homogeneous system of unmagnetized collisionless plasma consisting of cold electrons fluid, non-thermal hot electrons obeying a non-thermal
distribution and stationary ions. Such system is governed by the following normalized equations in one dimension [26]:

\[
\frac{\partial}{\partial t} u_c(x, t) + \frac{\partial}{\partial x} \left[ u_c(x, t) n_c(x, t) \right] = 0, \tag{1a}
\]

\[
\frac{\partial}{\partial t} u_c(x, t) + u_c(x, t) \frac{\partial}{\partial t} u_c(x, t) - \gamma \phi(x, t) = 0, \tag{1b}
\]

with Poisson’s equation

\[
\frac{\partial^2}{\partial x^2} \phi(x, t) - \frac{1}{\gamma} n_c(x, t) - n_h(x, t) + \left(1 + \frac{1}{\gamma} \right) = 0. \tag{1c}
\]

The non-thermal hot electrons density \( n_h(x, t) \) is given by:

\[
n_h(x, t) = [1 - \beta \phi(x, t) + \beta \phi(x, t)^2] \exp[\phi(x, t)], \quad \beta = 4\delta/(1 + 3\delta). \tag{2}
\]

In these equations, \( n_c(x, t)[n_h(x, t)] \) is the cold (non-thermal hot) electrons density normalized by equilibrium value \( n_{c0}[n_{h0}] \), \( u_c(x, t) \) is the cold electrons fluid velocity normalized by hot electron acoustic speed \( C_e = \sqrt{\frac{k_B T_h}{m_e}} \), \( \phi(x, t) \) is the electric potential normalized by \( \frac{k_B T_h}{\gamma m_e} \), \( \gamma = \frac{n_h}{n_{c0}} \) is the hot to cold electron equilibrium densities ratio, \( m_e \) is the electron mass, \( \delta \) is a parameter which determines the population of energetic non-thermal hot electrons, \( e \) is the electron charge, \( x \) is the space co-ordinate and \( t \) is the time variable. The distance is normalized to the hot electron Debye length \( \lambda_{Dh} \), the time is normalized by the inverse of the cold electron plasma frequency \( \omega_{ce}^{-1} \) and \( k_B \) is the Boltzmann’s constant. Equations (1a) and (1b) represent the inertia of cold electron and (1c) is the Poisson’s equation needed to make the self consistent. The hot electrons are described by non-thermal distribution given by (2).

According to the general method of reductive perturbation theory, we introduce the slow stretched co-ordinates [27]:

\[
\tau = e^{2/3}t \quad \text{and} \quad \xi = e^{1/2}(x - \lambda t), \quad \tag{3}
\]

where \( \epsilon \) is a small dimensionless expansion parameter and \( \lambda \) is the wave speed normalized by \( C_e \). All physical quantities appearing in (1) are expanded as power series in \( \epsilon \) about their equilibrium values as:

\[
n_c(\xi, \tau) = 1 + \epsilon n_1(\xi, \tau) + \epsilon^2 n_2(\xi, \tau) + \epsilon^3 n_3(\xi, \tau) + ..., \tag{4a}
\]

\[
u_c(\xi, \tau) = \epsilon u_1(\xi, \tau) + \epsilon^2 u_2(\xi, \tau) + \epsilon^3 u_3(\xi, \tau) + ..., \tag{4b}
\]

\[
\phi(\xi, \tau) = \epsilon \phi_1(\xi, \tau) + \epsilon^2 \phi_2(\xi, \tau) + \epsilon^3 \phi_3(\xi, \tau) + ..., \tag{4c}
\]

We impose the boundary conditions as \( \xi \to \infty, \quad n_c = n_h = 1, \quad u_c = 0 \) and \( \phi = 0 \).
Substituting (3) and (4) into (1) and equating coefficients of like powers of \( \epsilon \), the lowest-order equations in \( \epsilon \) lead to the following results:

\[
\begin{align*}
n_1(\xi, \tau) &= \frac{\gamma}{\lambda^2} \phi_1(\xi, \tau) \quad \text{and} \quad u_1(\xi, \tau) = \frac{\gamma}{\lambda} \phi_1(\xi, \tau). \\

\end{align*}
\]  

Poisson’s equation gives the linear dispersion relation

\[
\lambda = \sqrt{\frac{1}{1 - \beta}} = \sqrt{\frac{1 + 3\delta}{1 - \delta}}.
\]

Considering the coefficients of \( O(\epsilon^2) \) and eliminating the second order-perturbed quantities \( n_2, u_2 \) and \( \phi_2 \) lead to the following KdV equation for the first-order perturbed potential:

\[
\frac{\partial}{\partial \tau} \phi_1(\xi, \tau) + A \phi_1(\xi, \tau) \frac{\partial}{\partial \xi} \phi_1(\xi, \tau) + B \frac{\partial^3}{\partial \xi^3} \phi_1(\xi, \tau) = 0,
\]

where

\[
A = -\frac{3\gamma + \lambda^4}{2\lambda}, \quad B = \frac{\lambda^3}{2}.
\]

\( \phi_1(\xi, \tau) \) is a field variable, \( \xi \) is a space coordinate in the propagation direction of the field and \( \tau \in T(= [0, T_0]) \) is the time. The resultant KdV equation (7a) can be converted into time-fractional KdV equation as follows:

Using a potential function \( \psi(\xi, \tau) \) where \( \phi_1(\xi, \tau) = \psi(\xi, \tau) \) gives the potential equation of the regular KdV equation (1) in the form

\[
v_\xi(\xi, \tau) + A v_\xi(\xi, \tau) v_\xi(\xi, \tau) + B v_{\xi\xi\xi}(\xi, \tau) = 0,
\]

where the subscripts denote the partial differentiation of the function with respect to the parameter. The Lagrangian of this regular KdV equation (7) can be defined using the semi-inverse method [28, 29] as follows.

The functional of the potential equation (8) can be represented by

\[
J(v) = \int_R d\xi \int_T d\tau \{v_\xi(\xi, \tau)[c_1 v_\xi(\xi, \tau) + c_2 A v_\xi(\xi, \tau) v_\xi(\xi, \tau) + c_3 B v_{\xi\xi}(\xi, \tau)]\},
\]

where \( c_1, c_2 \) and \( c_3 \) are constants to be determined. Integrating by parts and taking \( v_\tau|_R = v_\xi|_R = v_\xi|_T = 0 \) lead to

\[
J(v) = \int_R d\xi \int_T d\tau \{v(\xi, \tau)[-c_1 v_\xi(\xi, \tau) v_\tau(\xi, \tau) - \frac{1}{2} c_2 A v_\xi^2(\xi, \tau) + c_3 B v_{\xi\xi}^2(\xi, \tau)]\}.
\]

The unknown constants \( c_i \) \( (i = 1, 2, 3) \) can be determined by taking the variation of the functional (10) to make it optimal. Taking the variation of this
functional, integrating each term by parts and make the variation optimum give
the following relation

\[ 2c_1 v_{\xi \tau} (\xi, \tau) + 3c_2 A v_\xi (\xi, \tau) v_{\xi \xi} (\xi, \tau) + 2c_3 B v_{\xi \xi \xi \xi} (\xi, \tau) = 0. \quad (11) \]

As this equation must be equal to equation (8), the unknown constants are
given as

\[ c_1 = \frac{1}{2}, \quad c_2 = \frac{1}{3} \text{ and } c_3 = \frac{1}{2}. \quad (12) \]

Therefore, the functional given by (10) gives the Lagrangian of the regular
KdV equation as

\[ L(v_\tau, v_\xi, v_{\xi \xi}) = -\frac{1}{2} v_\xi (\xi, \tau) v_\tau (\xi, \tau) - \frac{1}{6} A v_\xi^3 (\xi, \tau) + \frac{1}{2} B v_{\xi \xi}^2 (\xi, \tau). \quad (13) \]

Similar to this form, the Lagrangian of the time-fractional version of the
KdV equation can be written in the form

\[ F(0 D_\tau^\alpha v, v_\xi, v_{\xi \xi}) = -\frac{1}{2} 0 D_\tau^\alpha v (\xi, \tau) v_\xi (\xi, \tau) - \frac{1}{6} A v_\xi^3 (\xi, \tau) + \frac{1}{2} B v_{\xi \xi}^2 (\xi, \tau), \quad 0 \leq \alpha < 1, \quad (14) \]

where the fractional derivative is represented, using the left Riemann-Liouville
fractional derivative definition as [18, 19]

\[ \frac{1}{\Gamma(k - \alpha)} \int_a^t dt \frac{d^k}{dt^k} [0 D_\tau^\alpha f(t)] = \int_a^t dt \frac{d^k}{dt^k} [0 D_\tau^\alpha g(t)] = \frac{1}{\Gamma(k - \alpha)} \int_a^t dt \frac{d^k}{dt^k} [0 D_\tau^\alpha f(t)], \quad k - 1 < \alpha < 1, \quad t \in [a, b]. \quad (15) \]

The functional of the time-FKdV equation can be represented in the form

\[ J(v) = \int_R d\xi \int_T d\tau F(0 D_\tau^\alpha v, v_\xi, v_{\xi \xi}), \quad (16) \]

where the time-fractional Lagrangian \( F(0 D_\tau^\alpha v, v_\xi, v_{\xi \xi}) \) is defined by (14).

Following Agrawal’s method [3, 4], the variation of functional (16) with
respect to \( v(\xi, \tau) \) leads to

\[ \delta J(v) = \int_R d\xi \int_T d\tau \left\{ \frac{\partial F}{\partial 0 D_\tau^\alpha v} \delta 0 D_\tau^\alpha v + \frac{\partial F}{\partial v_\xi} \delta v_\xi + \frac{\partial F}{\partial v_{\xi \xi}} \delta v_{\xi \xi} \right\}. \quad (17) \]

The formula for fractional integration by parts reads [3, 18, 19]

\[ \int_a^b dt f(t) 0 D_\tau^\alpha g(t) = \int_a^t dt g(t) 0 D_\tau^\alpha f(t), \quad f(t), g(t) \in [a, b]. \quad (18) \]
\[ \tau D^\alpha_0 f(t) = \frac{(-1)^k}{\Gamma(k - \alpha)} \frac{d^k}{dt^k} \left[ \int_t^b d\tau (\tau-t)^{k-\alpha-1} f(\tau) \right], \quad k-1 \leq \alpha \leq 1, \quad t \in [a, b]. \quad (19) \]

Integrating the right-hand side of (17) by parts using formula (18) leads to

\[ \delta J(v) = \int_R d\xi \int_T d\tau \left[ \tau D^\alpha_0 \left( \frac{\partial F}{\partial \xi} \right) - \frac{\partial}{\partial \xi} \left( \frac{\partial F}{\partial v_\xi} \right) + \frac{\partial^2}{\partial \xi^2} \left( \frac{\partial F}{\partial v_\xi v_\xi} \right) \right] \delta v, \quad (20) \]

where it is assumed that \( \delta v|_T = \delta v|_R = \delta v|_\xi|_R = 0. \)

Optimizing this variation of the functional \( J(v) \), i.e.; \( \delta J(v) = 0 \), gives the Euler-Lagrange equation for the time-FKdV equation in the form

\[ \tau D^\alpha_0 \left( \frac{\partial F}{\partial \xi} \right) - \frac{\partial}{\partial \xi} \left( \frac{\partial F}{\partial v_\xi} \right) + \frac{\partial^2}{\partial \xi^2} \left( \frac{\partial F}{\partial v_\xi v_\xi} \right) = 0. \quad (21) \]

Substituting the Lagrangian of the time-FKdV equation (14) into this Euler-Lagrange formula (21) gives

\[ - \frac{1}{2} \tau D^\alpha_0 v_\xi(\xi, \tau) + \frac{1}{2} 0 D^\alpha_0 v_\xi(\xi, \tau) + A v_\xi(\xi, \tau) v_\xi(\xi, \tau) + B v_\xi v_\xi(\xi, \tau) = 0. \quad (22) \]

Substituting for the potential function, \( v_\xi(\xi, \tau) = \phi_1(\xi, \tau) = \Phi(\xi, \tau) \), gives the time-FKdV equation for the state function \( \Phi(\xi, \tau) \) in the form

\[ \frac{1}{2} \left[ 0 D^\alpha_0 \Phi(\xi, \tau) - \tau D^\alpha_0 \Phi(\xi, \tau) \right] + A \Phi(\xi, \tau) \Phi(\xi, \tau) + B \Phi \Phi(\xi, \tau) = 0, \quad (23) \]

where the fractional derivatives \( 0 D^\alpha_0 \) and \( \tau D^\alpha_0 \), are, respectively the left and right Riemann-Liouville fractional derivatives and are defined by (15) and (19).

The time-FKdV equation represented in (14) can be rewritten by the formula

\[ \frac{1}{2} R D^\alpha_0 \Phi(\xi, \tau) + A \Phi(\xi, \tau) \Phi(\xi, \tau) + B \Phi \Phi(\xi, \tau) = 0, \quad (24) \]

where the fractional operator \( R D^\alpha_0 \) is called Riesz fractional derivative and can be represented by \([4, 18, 19]\)

\[ R D^\alpha_0 f(t) = \frac{1}{2} \left[ 0 D^\alpha_0 f(t) + (-1)^k t D^\alpha_0 f(t) \right] = \frac{1}{2} \frac{1}{\Gamma(k - \alpha)} \frac{d^k}{dt^k} \int_a^t d\tau [t - \tau]^{k-\alpha-1} f(\tau)], \]

\[ k-1 \leq \alpha \leq 1, \quad t \in [a, b]. \quad (25) \]

The nonlinear fractional differential equations have been solved using different techniques \([18-23]\). In this paper, a variational-iteration method (VIM) \([24, 25]\) has been used to solve the time-FKdV equation that formulated using Euler-Lagrange variational technique.
3 Variational-Iteration Method

Variational-iteration method (VIM) [24, 25] has been used successfully to solve different types of integer nonlinear differential equations [30, 31]. Also, VIM is used to solve linear and nonlinear fractional differential equations [32, 33]. This VIM has been used in this paper to solve the formulated time-FKdV equation.

A general Lagrange multiplier method is constructed to solve nonlinear problems, which was first proposed to solve problems in quantum mechanics [24]. The VIM is a modification of this Lagrange multiplier method [25]. The basic features of the VIM are as follows. The solution of a linear mathematical problem or the initial (boundary) condition of the nonlinear problem is used as initial approximation or trail function. A more highly precise approximation can be obtained using iteration correction functional. Considering a nonlinear partial differential equation consists of a linear part \( \hat{L}U(x, t) \), nonlinear part \( \hat{N}U(x, t) \) and a free term \( f(x, t) \) represented as

\[
\hat{L}U(x, t) + \hat{N}U(x, t) = f(x, t),
\]

(26)

where \( \hat{L} \) is the linear operator and \( \hat{N} \) is the nonlinear operator. According to the VIM, the \( (n+1) \)th approximation solution of (26) can be given by the iteration correction functional as [24, 25]

\[
U_{n+1}(x, t) = U_n(x, t) + \int_0^t d\tau \lambda(\tau) \left[ \hat{L}U_n(x, \tau) + \hat{N}U_n(x, \tau) - f(x, \tau) \right], \quad n \geq 0,
\]

(27)

where \( \lambda(\tau) \) is a Lagrangian multiplier and \( \hat{U}_n(x, \tau) \) is considered as a restricted variation function, i.e., \( \delta \hat{U}_n(x, \tau) = 0 \). Extreme the variation of the correction functional (27) leads to the Lagrangian multiplier \( \lambda(\tau) \). The initial iteration can be used as the solution of the linear part of (26) or the initial value \( U(x, 0) \). As \( n \) tends to infinity, the iteration leads to the exact solution of (26), i.e.,

\[
U(x, t) = \lim_{n \to \infty} U_n(x, t).
\]

(28)

For linear problems, the exact solution can be given using this method in only one step where its Lagrangian multiplier can be exactly identified.

4 Time-fractional KdV equation Solution

The time-FKdV equation represented by (24) can be solved using the VIM by the iteration correction functional (27) as follows:

Affecting from left by the fractional operator on (24) leads to
\[
\frac{\partial}{\partial \tau} \Phi(\xi, \tau) = R\int_0^\tau D_\tau^{\alpha - 1} \Phi(\xi, \tau) |_{\tau = 0} \frac{\tau^\alpha - 2}{\Gamma(\alpha - 1)} \\
- R\int_0^\tau D_\tau^{1-\alpha} [A \Phi(\xi, \tau) \frac{\partial}{\partial \xi} \Phi(\xi, \tau) + B \frac{\partial^3}{\partial \xi^3} \Phi(\xi, \tau)]
\]

\[
0 \leq \alpha \leq 1, \tau \in [0, T_0], \quad (29)
\]

where the following fractional derivative property is used [18, 19]

\[
R_a D_b^\alpha \left[ R_a D_b^\beta f(t) \right] = R_a D_b^{\alpha + \beta} f(t) - \sum_{j=1}^{k-1} R_a D_b^{\beta - j} f(t) |_{t=a} \frac{(t-a)^{-\alpha - j}}{\Gamma(1 - \alpha - j)},
\]

\[
k - 1 \leq \beta < k. \quad (30)
\]

As \( \alpha < 1 \), the Riesz fractional derivative \( R_0 D_\tau^{\alpha - 1} \) is considered as Riesz fractional integral \( R_0 I_\tau^{1-\alpha} \) that is defined by [18, 19]

\[
R_0 I_{\tau}^\alpha f(t) = \frac{1}{2} [0 I_{\tau}^\alpha f(t) + \tau I_{\tau}^\alpha f(t)] = \frac{1}{2 \Gamma(\alpha)} \int_a^b d\tau |t - \tau|^{\alpha - 1} f(\tau), \quad \alpha > 0. \quad (31)
\]

where \( 0 I_{\tau}^\alpha f(t) \) and \( \tau I_{\tau}^\alpha f(t) \) are the left and right Riemann-Liouville fractional integrals, respectively [18, 19].

The iterative correction functional of equation (29) is given as

\[
\Phi_{n+1}(\xi, \tau) = \Phi_n(\xi, \tau) + \int_0^\tau d\tau' \lambda(\tau') \left\{ \frac{\partial}{\partial \tau} \Phi_n(\xi, \tau') \right\} \\
- R_0 I_{\tau}^{1-\alpha} \Phi_n(\xi, \tau') |_{\tau' = 0} \frac{\tau^{\alpha - 2}}{\Gamma(\alpha - 1)} \\
+ R_0 D_\tau^{1-\alpha} [A \Phi_n(\xi, \tau') \frac{\partial}{\partial \xi} \Phi_n(\xi, \tau') + B \frac{\partial^3}{\partial \xi^3} \Phi_n(\xi, \tau')] \quad (32)
\]

where \( n \geq 0 \) and the function \( \Phi_n(\xi, \tau) \) is considered as a restricted variation function, i.e; \( \delta \Phi_n(\xi, \tau) = 0 \). The extreme of the variation of (32) using the restricted variation function leads to

\[
\delta \Phi_{n+1}(\xi, \tau) = \delta \Phi_n(\xi, \tau) + \int_0^\tau d\tau' \lambda(\tau') \frac{\partial}{\partial \tau} \Phi_n(\xi, \tau') \\
= \delta \Phi_n(\xi, \tau) + \lambda(\tau) \delta \Phi_n(\xi, \tau) - \int_0^\tau d\tau' \frac{\partial}{\partial \tau} \lambda(\tau') \delta \Phi_n(\xi, \tau') = 0.
\]

This relation leads to the stationary conditions \( 1 + \lambda(\tau) = 0 \) and \( \frac{\partial}{\partial \tau} \lambda(\tau') = 0 \), which leads to the Lagrangian multiplier as \( \lambda(\tau') = -1 \).
Therefore, the correction functional (32) is given by the form

\[ \Phi_{n+1}(\xi, \tau) = \Phi_n(\xi, \tau) - \int_0^\tau d\tau' \left\{ \frac{\partial}{\partial \tau'} \Phi_n(\xi, \tau') \right\} \\
- R_0 \frac{1}{\Gamma(\alpha - 1)} \left[ \partial_\tau'^{-\alpha} \Phi_n(\xi, \tau') \right]_{\tau' = 0}^{\tau' = \tau} \\
+ R_0 D_0^{1-\alpha} \left[ A \frac{\partial}{\partial \xi} \Phi_n(\xi, \tau') + B \frac{\partial^3}{\partial \xi^3} \Phi_n(\xi, \tau') \right], (33) \]

where \( n \geq 0 \).

In Physics, if \( \tau \) denotes the time-variable, the right Riemann-Liouville fractional derivative is interpreted as a future state of the process. For this reason, the right-derivative is usually neglected in applications, when the present state of the process does not depend on the results of the future development [3]. Therefore, the right-derivative is used equal to zero in the following calculations.

The zero order correction of the solution can be taken as the initial value of the state variable, which is taken in this case as

\[ \Phi_0(\xi, \tau) = \Phi(\xi, 0) = A_0 \sec h^2(c \xi). \quad (34) \]

where \( A_0 \) and \( c \) are constants.

Substituting this zero order approximation into (33) and using the definition of the fractional derivative (25) lead to the first order approximation as

\[ \Phi_1(\xi, \tau) = A_0 \sec h^2(c \xi) \\
+ 2A_0c \sinh(c \xi) \sec h^2(c \xi) [4c^2B] \\
+ \left( A_0 A - 12c^2B \right) \sec h^2(c \xi) \frac{\tau^\alpha}{\Gamma(\alpha + 1)}. \quad (35) \]

Substituting this equation into (33), using the definition (25) and the Maple package lead to the second order approximation in the form

\[ \Phi_2(\xi, \tau) = A_0 \sec h^2(c \xi) \\
+ 2A_0c \sinh(c \xi) \sec h^2(c \xi) [4c^2B + \left( A_0 A - 12c^2B \right) \sec h^2(c \xi)] \frac{\tau^\alpha}{\Gamma(\alpha + 1)}. \quad (36) \]

The higher order approximations can be calculated using the Maple or the Mathematica package to the appropriate order where the infinite approximation leads to the exact solution.
5 Results and calculations

For small amplitude electron-acoustic waves, the time fractional Korteweg-de Vries equation has been derived. The Riemann-Liouville fractional derivative \[18, 19\] is used to describe the time fractional operator in the FKdV equation. He’s variational-iteration method \[24, 25\] is used to solve the derived time-FKdV equation.

To make our result physically relevant, numerical studies have been made using plasma parameters close to those values corresponding to the dayside auroral zone \[15, 34\].

However, since one of our motivations was to study effects of time fractional order \(\alpha\), the energetic population parameter \(\delta\) and the hot to cold electron equilibrium densities ratio \(\gamma\) on the existence of solitary waves. The electrostatic potential as a function of space variable \(\xi\) and time variable \(\tau\) is represented in Figure (1) that has a single rarefactive soliton shape. In Figure (2), the relation between the hot to cold electron equilibrium densities ratio \(\gamma\) and the amplitude of the electrostatic potential solitary wave \(|\Phi(0, \tau)|\) is obtained at different values of the time variable \(\tau\). It is seen that the soliton amplitude decreases with the increase of \(\gamma\). In addition the soliton amplitude \(|\Phi(0, \tau)|\) against the fractional order \(\alpha\) is represented in Figure (3) at different time variable values. It is seen that \(|\Phi(0, \tau)|\) increases with the increase of \(\alpha\). Figure (4) shows the relation between \(|\Phi(0, \tau)|\) and the energetic population parameter \(\delta\). The effect of the fractional order \(\alpha\) on the electrostatic soliton amplitude \(|\Phi(0, \tau)|\) for different values of the energetic population parameter \(\delta\) is given in Figure (5). To compare our result (the amplitude of the electrostatic potential \(|\Phi(0, \tau)|\)) with that observed in the auroral zone, we choose a set of available parameters corresponding to the dayside auroral zone where an electric field amplitude \(E_0 = 100\) mV/m has been observed \[15, 34\] with \(T_c = 5\) eV, \(T_h = 250\) eV, \(n_{c0} = 0.5\) m\(^{-3}\) and \(n_{h0} = 2.5\) m\(^{-3}\). These parameters correspond to \(\lambda_{Dh} \approx 7430\) cm and the normalized electrostatic wave potential amplitude \(\Phi_0 = \frac{E_0 \lambda_{Dh} c}{k_B T_h} \approx 0.03\) V, which is obtained for different values of \(\alpha\) and \(\delta\) \[cf. Figure (5)\]. This value of the electrostatic potential amplitude is given at values of \(\alpha\) and \(\delta\) as: \((\alpha = 0.61, \delta = 0.01), (\alpha = 0.69, \delta = 0.05), (\alpha = 0.78, \delta = 0.1)\) and \((\alpha = 0.9, \delta = 0.15)\) at velocity \(v = 0.04, \gamma = 5\) and time \(\tau = 10\).

In summery, it has been found that the amplitude of the electron acoustic solitary waves as well as the parametric regime where the solitons can exist are sensitive to the time fractional order.

We have stressed out that it is necessary to include the space fractional parameter. This is beyond the scope of the present paper and it will be include in a further work in electron-acoustic solitary wave. The application of our model might be particularly interesting in the auroral region.
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Figure Captions:

Fig. (1): The electrostatic potential $\Phi(\xi, \tau)$ against $\xi$ and $\tau$ at $\gamma = 5$, $v = 0.04$, $\alpha = 0.5$ and $\delta = 0.1$.

Fig. (2): The amplitude of the electrostatic potential $|\Phi(0, \tau)|$ against $\gamma$ at different values of time for $v = 0.04$, $\alpha = 0.5$ and $\delta = 0.1$.

Fig. (3): The amplitude of the electrostatic potential $|\Phi(0, \tau)|$ against $\alpha$ at different values of time for $\gamma = 5$, $v = 0.04$ and $\delta = 0.1$.

Fig. (4): The amplitude of the electrostatic potential $|\Phi(0, \tau)|$ against $\delta$ at different values of time for $\gamma = 5$, $v = 0.04$ and $\alpha = 0.5$.

Fig. (5): The amplitude of the electrostatic potential $|\Phi(0, \tau)|$ against $\alpha$ at different values of $\delta$ for $v = 0.04$, $\tau = 10$ and $\gamma = 5$. 
