Classical Extended Conformal Algebras Associated with
Constrained KP Hierarchy

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Abstract

We examine the conformal property of the second Hamiltonian structure of constrained KP hierarchy derived by Oevel and Strampp. We find that it naturally gives a family of nonlocal extended conformal algebras. We give two examples of such algebras and find that they are similar to Bilal’s V algebra. By taking a gauge transformation one can map the constrained KP hierarchy to Kupershmidt’s nonstandard Lax hierarchy. We consider the second Hamiltonian structure in this representation. We show that after mapping the Lax operator to a pure differential operator the second structure becomes the sum of the second and the third Gelfand-Dickey brackets defined by this differential operator. We show that this Hamiltonian structure defines the W-U(1)-Kac-Moody algebra by working out its conformally covariant form.

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I. Introduction

In recent years there has been a lot of interest in the connections between the classical extended conformal algebras and the Hamiltonian structures of certain classical integrable systems[1-12]. A prototype of connections of the sort is provided by the generalized KdV hierarchy which is defined as[2]

$$\frac{\partial l_n}{\partial t_k} \equiv \partial_k l_n = \left[ (l_n^{k/n})_+, l_n \right] \quad (k = 1, 2, \ldots) \quad (1.1)$$

where

$$l_n = \partial^n + u_1 \partial^{n-1} + u_2 \partial^{n-2} + \ldots + u_n \quad (1.2)$$

and \((A)_{\pm}\) stand for the differential and the integral part of the pseudodifferential operator \(A\) respectively. The second Hamiltonian structure of (1.1) is given by the second Gelfand-Dickey bracket which, in operator form, reads

$$\Theta^{GD}_2(\frac{\delta H}{\delta l_n}) \equiv \{u_1, H\} \partial^{n-1} + \{u_2, H\} \partial^{n-2} + \ldots + \{u_n, H\}$$

$$= (l_n \frac{\delta H}{\delta l_n})_+ l_n - l_n (\frac{\delta H}{\delta l_n} l_n)_+ \quad (1.3)$$

Here

$$\frac{\delta H}{\delta l_n} = \partial^{-1} \frac{\delta H}{\delta u_n} + \partial^{-2} \frac{\delta H}{\delta u_{n-1}} + \ldots + \partial^{-n} \frac{\delta H}{\delta u_1} \quad (1.4)$$

The connection of this bracket to the classical extended conformal algebra can be established as follows: First, we define

$$t(x) = u_2(x) - \frac{n-1}{2} u_1'(x) - \frac{1}{2} u_1^2(x) \quad (1.5)$$

Then it can be shown easily that \(t(x)\) satisfies the classical Virasoro algebra and that \(u_1(x)\) is a conformal spin-1 field satisfying the \(U(1)-Kac-Moody\) algebra[7]; i.e.

$$\{t(x), t(y)\} = \left[ \frac{n^3 - n}{12} \partial_x^3 + 2t(x) \partial_x + t'(x) \right] \delta(x - y)$$

$$\{u_1(x), t(y)\} = [u_1(x) \partial_x + u_1'(x)] \delta(x - y) \quad (1.6)$$

$$\{u_1(x), u_1(y)\} = -n \partial_x \delta(x - y)$$
Secondly, one can show[10] that for \( k(=3,\ldots,n) \) a spin-k field \( w_k \) can be constructed as a differential polynomial of the coefficient functions \( u_1,\ldots,u_n \) [We shall review this construction in Sec. II]. Therefore, in terms of these new fields the second Gelfand-Dickey bracket defines a classical extended conformal algebra called \( W_n-U(1)-Kac-Moody \) algebra. One usually eliminates the spin-1 field from the spectrum and the resulting algebra is the classical \( W_n \) algebra. Many generalizations of this connection can be found in the literature.

Recently there are several publications concerning the so-called “constrained KP hierarchy” from various points of view[13-24]. The constrained hierarchy is the ordinary KP hierarchy restricted to pseudodifferential operators of the form

\[
L_n = \partial^n + u_2 \partial^{n-2} + \ldots + u_n + \phi \partial^{-1} \psi
\]  

(1.7)

The evolution of the system is given by

\[
\partial_k L_n = [(L_n^{k/n})_+, L_n]
\]

\[
\partial_k \phi = (L_n^{k/n} \phi)_0, \quad \partial_k \psi = -((L_n^{k/n})^* \psi)_0
\]  

(1.8)

where \((\ )_0\) denotes the zeroth order term and \(*\) stands for the conjugate operation: \((AB)^* = B^*A^*, \partial^* = -\partial, f(x)^* = f(x)\). One should note that the second line of (1.8) is consistent with the first line. This system can be put into a different Lax representation by using the gauge transformation

\[
K_n \equiv \phi^{-1} L_n \phi
\]

\[
\equiv \partial^n + v_1 \partial^{n-1} + \ldots + v_n + \partial^{-1} v_{n+1}
\]  

(1.9)

In terms of \( K_n \) (1.8) becomes the Kuperschmidt’s nonstandard Lax hierarchy [13] (it is also called modified constrained KP hierarchy [18]):

\[
\partial_k K_n = [(K_n^{k/n})_{\geq 1}, K_n]
\]  

(1.10)

The bihamiltonian structures of (1.8) and (1.10) have been constructed by Oevel and Strampp[18]. The second Hamiltonian structures are somewhat different from the second
Gelfand-Dickey bracket defined by a pure differential operator. It is a natural question to examine whether or not they still have the nice conformal property owned by the bracket defined by (1.2) and (1.3). If yes, what are the corresponding conformal algebras? In this paper, we study these two questions in some details. After a brief review on the conformal property of the second Gelfand-Dickey bracket defined by a pure differential operator in Sec.II, we show in Sec.III. that the second Hamiltonian structure defined by (1.7) can be easily put into a conformally covariant form and gives a nonlocal extended conformal algebra. We discuss in details two of such nonlocal algebras. We find that they are very similar to the V algebra discovered by Bilal[25]. In Sec. IV. we consider the second Hamiltonian structure defined by (1.9). After working out two examples explicitly, we show that in terms of the differential operator $l_{n+1} \equiv \partial K_n$ the second structure is noting but the sum of the second and the third Gelfand-Dickey brackets defined by $l_{n+1}$. Based on this result we show how to write $K_n$ in a conformally covariant form. Some concluding remarks are presented in Sec. V.

II. Conformally Covariant Differential Operators

In this section we review briefly the method to construct the conformally covariant form of a differential operator[10,11], which will be useful for later discussions. First, let us recall a few basic definitions. A function $f(x)$ is called is spin-k field if it transform under coordinate change $x \to t(x)$ as $f(t) = (dx/dt)^k f(x)$. The space of all spin-k fields is denoted by $F_k$. An operator $\Delta$ is called a covariant operator if it maps from $F_h$ to $F_l$ for some $h$ and $l$. Symbolically, we denote $\Delta : F_h \to F_l$. In other words, under $x \to t(x)$

$$\Delta(t) = (dx/dt)^l \Delta(x)(dx/dt)^{-h}$$

(2.1)

Usually, $\Delta$ is a pseudodifferential operator and we can easily write down the infinitesimal form of (2.1). Taking $t(x) = x - \epsilon(x)$ we can derive easily the infinitesimal change of $\Delta$[11]:

$$\delta_\epsilon \Delta(x) = [\epsilon(x) \partial_x + l\epsilon'(x)] \Delta(x) - \Delta(x) [\epsilon(x) \partial_x + h\epsilon'(x)]$$

(2.2)
The key step to construct the conformally covariant differential operators is to recognize that the flow (defined by the second Gelfand-Dickey bracket) generated by the Virasoro generator $t(x)$ given by (1.5) takes the form of (2.2). That is, for a suitable choice of $h$ and $l$ a differential operator $l_n$ can be regarded as a covariant operator mapping from $F_h$ to $F_l$. It is not hard to work out the values of $h$ and $l$. First, since $l_n$ is of order $n$, $l-h$ must be $n$. Secondly, the function $u_1$ transforms like a spin-1 field under the Virasoro flow (flow generated by $\int dx t(x)\epsilon(x)$). One can show easily that $h = -\frac{n-1}{2}$. In short, we have

$$l_n : F_{-\frac{n-1}{2}} \rightarrow F_{\frac{n-1}{2}}$$

(2.3)

It is also easy to derive the infinitesimal form of (2.3) by using (1.3), (1.5), (2.1) and (2.3).

The next step is to construct a family of covariant operators such that each of them depends on a spin field and the Virasoro generator $t$. To this end, we introduce a “anomalous” spin-1 field $b(x)$ which obeys the transformation law: $b(t) = (\frac{dt}{dx})b(x) + \frac{d^2x}{dt^2}(\frac{dt}{dx})^{-1}$. The Virasoro generator is then represented by

$$t(x) = \frac{n^3 - n}{12} \left( b'(x) - \frac{1}{2}b(x)^2 \right)$$

(2.4)

It is a simple matter to check that the $t$ represented by (2.4) has the correct transformation law. The main use of $b$ is to define a sequence of covariant operators:

$$D^l_k \equiv [\partial_x - (k + l - 1)b(x)][\partial_x - (k + l - 2)b(x)]\ldots[\partial_x - kb(x)] \quad (l \geq 0)$$

(2.5)

One may verify that $D^l_k$ maps from $F_k$ to $F_{k+l}$. Now given a spin-$k(=1,2\ldots)$ field $w_k$ the covariant operator[10]

$$\Delta_k(w_k,t) \equiv \sum_{l=0}^{n-k} \alpha^{(n)}_{k,l} [D^l_k w_k] D_{-\frac{n-k-l}{2}}$$

(2.6)

where

$$\alpha^{(n)}_{k,l} = \binom{k + l - 1}{l} \binom{n-k}{l} \binom{2k+l-1}{l}$$

(2.7)
maps from $F_{-\frac{n-1}{2}}$ to $F_{\frac{n+1}{2}}$. The numerical coefficients $\alpha^{(n)}_{k,l}$'s are determined by requiring
the right hand side of (2.6) depends on $b$ only through $t$ defined by (2.4). With (1.5) and
$\alpha^{(n)}_{1,1} = \frac{n-1}{2}$ in mind we now can write down the covariant form of $l_n$ as

$$l_n = D^{n}_{-\frac{n-1}{2}} + \Delta_1(u_1, t) + \frac{1}{2}\Delta_2(u_1^2, t) + \sum_{k=3}^{n} \Delta_k(w_k, t)$$

(2.8)

Eq.(2.8) decomposes the coefficient functions $u_k$'s into spin fields and the Virasoro generator. Inverting these relations then gives the expressions for spin fields as differential polynomials of $u_k$'s. The second Gelfand-Dickey bracket (1.3), when expressed in terms of $t$ and the spin fields $u_1, w_3, \ldots w_n$, defines the $W_n$-$U(1)$-Kac-Moody algebra mentioned in Introduction.

Since the evolution of $u_1$ determined by the Lax equation (1.1) is trivial, it is often to set $u_1 = 0$. Under this constraint (2.3) becomes even more natural in the sense that it is the only choice which preserves this constraint. As usual, imposing a constraint causes a modification of its Hamiltonian structure. For (1.3) it is easy to show following the Dirac procedure that the modified bracket reads

$$\Theta_{GD}^{2b} \left( \frac{\delta H}{\delta l_n} \right) = (l_n \frac{\delta H}{\delta l_n})_+ l_n - l_n (\frac{\delta H}{\delta l_n})_+ + \frac{1}{n} [l_n, \int^{x} (res[l_n, \frac{\delta H}{\delta l_n}])]$$

(2.9)

Here, $u_1$ in $l_n$ and $\frac{\delta H}{\delta u_1}$ in $\frac{\delta H}{\delta l_n}$ are both set to zero and $res(\sum_i a_i \partial^i) \equiv a_{-1}$. Now the Virasoro generator $t$ is simply $u_2$ and the covariant form is still given by (2.8) except that the $u_1$ dependent terms must be removed. The algebra defined by (2.9) is the $W_n$ algebra.

Before ending this section we like to remark that the decomposition of a coefficient function into spin fields given by (2.8) is by no means unique. Redefinitions like $w_3 \rightarrow w_3 + u_1^3$ and $w_4 \rightarrow w_4 + w_3u_1$ are certainly allowed. In other words, the definition of a higher spin field is not unique.
III. Nonlocal Extended Conformal Algebras From Constrained KP Hierarchy

We now consider the Lax operator \( L_n \), given by (1.7), for the constrained KP hierarchy. Here we have set \( u_1 = 0 \). For \( u_1 \neq 0 \) a bihamiltonian structure associated with \( L_n \) has been worked out by Oevel and Strampp[18]. Using their result and the Dirac procedure, we can easily write down the second Hamiltonian structure defined by \( L_n \):

\[
\Theta_{2}^{KP}\left( \frac{\delta H}{\delta L_n}, \frac{\delta H}{\delta \phi}, \frac{\delta H}{\delta \psi} \right) = (L_n \frac{\delta H}{\delta L_n})_+ L_n - L_n (\frac{\delta H}{\delta L_n} L_n)_+ + \frac{1}{n} [L_n, \int_{x} (\operatorname{res}[L_n, \frac{\delta H}{\delta L_n}])]
\]

\[
- \phi \partial^{-1} \frac{\delta H}{\delta \phi} L_n + L_n \frac{\delta H}{\delta \psi} \partial^{-1} \psi + \frac{1}{n} [L_n, \int_{x} (\frac{\delta H}{\delta \phi} - \psi \frac{\delta H}{\delta \psi})]
\]

(3.1)

Here

\[
\frac{\delta H}{\delta L_n} = \partial^{-1} \frac{\delta H}{\delta u_n} + \ldots + \partial^{-n+1} \frac{\delta H}{\delta u_2}
\]

(3.2)

The bracket (3.1) is obviously nonlocal. In fact,

\[
\{ \phi(x), \phi(y) \} = -\frac{n+1}{n} \phi(x) \epsilon(x - y) \phi(y)
\]

\[
\{ \psi(x), \psi(y) \} = -\frac{n+1}{n} \psi(x) \epsilon(x - y) \psi(y)
\]

\[
\{ \phi(x), \psi(y) \} = \frac{n+1}{n} \phi(x) \epsilon(x - y) \psi(y) + (L_n)_+ \delta(x - y)
\]

(3.3)

where \( \epsilon(x - y) \equiv \partial_x^{-1} \delta(x - y) \) is the antisymmetric step function.

For \( n \geq 2 \) it is not hard to check that \( t = u_2 \) still satisfies the first of (1.6). Therefore, (3.1) defines an extended conformal algebra if the higher spin fields can be constructed. To see this is true, let us try to interpret \( L_n \) as a covariant operator as we did for \( l_n \) in Section II. Since the positive part and the negative part of \( L_n \) transform independently under changes of coordinate, the constraint \( u_1 = 0 \) forces the positive part \( (L_n)_+ \) maps from \( F_{\frac{n-1}{2}} \) to \( F_{\frac{n+1}{2}} \). As a result, its negative part does the same. Hence, we have

\[
L_n : F_{\frac{n-1}{2}} \rightarrow F_{\frac{n+1}{2}}
\]

(3.4)

and, in particular,

\[
\phi(t) \partial_t^{-1} \psi(t) = (\frac{dx}{dt})^{\frac{n+1}{2}} \phi(x) \partial_x^{-1} \psi(x) (\frac{dx}{dt})^{\frac{n-1}{2}}
\]

(3.5)
Using $\partial_t^{-1} = \partial_x^{-1}(\frac{dx}{dt})^{-1}$ we obtain immediately

$$\phi(t) = (\frac{dx}{dt})^{\frac{n+1}{2}} \phi(x) \quad \psi(t) = (\frac{dx}{dt})^{\frac{n+1}{2}} \psi(x)$$

(3.6)

that is, $\phi$ and $\psi$ are both spin-$\frac{n+1}{2}$ fields. Thus the spectrum contains two half integer spin fields if the order $n$ is even. To complete the discussion we must show that the flow, defined by (3.1), generated by the functional $\int dx u_2(x) \epsilon(x)$ gives the infinitesimal form of (3.4). The verification is completely identical to that for the pure differential operator $l_n$ with $u_1 = 0$ and hence we shall not spell out. From the above discussions it should be clear that the covariant form of $L_n$ is given by

$$L_n = D_{t-1}^{-n} + \sum_{k=3}^{n} \Delta_k(w_k, t) + v_{n+1}^+ \partial^{-1} v_{n+1}^-$$

(3.7)

where $v_{n+1}^\pm$ denote two fields of spin-$\frac{n+1}{2}$.

Let us now consider two simplest examples. Using (3.7) with $n = 2$ we have

$$L_2 = \partial^2 + t + v_2^+ \partial^{-1} v_2^-$$

(3.8)

In this parametrization (3.1) gives [18]

$$\{ t(x), t(y) \} = \frac{1}{2} \partial_x^3 + 2t(x) \partial_x + t'(x) \delta(x - y)$$

$$\{ v_2^\pm(x), t(y) \} = \frac{3}{2} v_2^\pm(x) \partial_x + v_2^\pm(y) \delta(x - y)$$

$$\{ v_2^\pm(x), v_4^\pm(y) \} = -\frac{3}{2} v_2^\pm(x) \epsilon(x - y) v_4^\pm(y)$$

$$\{ v_2^\pm(x), v_4^\mp(y) \} = \frac{3}{2} v_2^\pm(x) \epsilon(x - y) v_4^\mp(y) \pm [\partial_x^3 + t(x)] \delta(x - y)$$

(3.9)

This is a nonlocal extension of Virasoro algebra by two spin-$\frac{3}{2}$ fields. (3.9) is quite similar to Bilal’s V algebra which is a nonlocal extension of Virasoro algebra by two spin-2 fields $v_2^\pm$ [25]:

$$\{ t(x), t(y) \} = \frac{1}{2} \partial_x^3 + 2t(x) \partial_x + t'(x) \delta(x - y)$$

$$\{ v_2^\pm(x), t(y) \} = [2v_2^\pm(x) + v_2^\pm'(x)] \delta(x - y)$$

$$\{ v_2^\pm(x), v_2^\pm(y) \} = -2v_2^\pm(x) \epsilon(x - y) v_2^\pm(y)$$

$$\{ v_2^\pm(x), v_2^\mp(y) \} = 2v_2^\pm(x) \epsilon(x - y) v_2^\mp(y) + [\frac{1}{2} \partial_x^3 + 2t(x) \partial_x + t'(x)] \delta(x - y)$$

(3.10)
Next, we consider \( n = 3 \). In this case, the covariant Lax operator reads

\[
L_3 = \partial^2 + t \partial + w_3 + \frac{1}{2} t' + v_2^\pm \partial^{-1} v_2^\mp
\]  

(3.11)

and we have

\[
\{ t(x), t(y) \} = [2 \partial_x^2 + 2 t(x) \partial_x + t'(x)] \delta(x - y)
\]

\[
\{ w_3(x), t(y) \} = [3 w_3(x) \partial_x + w_3'(x)] \delta(x - y)
\]

\[
\{ v_2^\pm(x), t(y) \} = [2 v_2^\pm(x) + v_2^\pm'(x)] \delta(x - y)
\]

\[
\{ w_3(x), w_3(y) \} = -\left[\frac{1}{6} \partial_x^3 + \frac{5}{6} t(x) \partial_x^2 + \frac{5}{4} t'(x) \partial_x + \left(\frac{3}{4} t''(x) + \frac{2}{3} t(x)^2\right) \partial_x + \frac{1}{6} t'''(x)\right] \delta(x - y)
\]

\[
\{ v_2^\pm(x), w_3(y) \} = \pm\left[\frac{5}{6} v_2^\pm \partial_x^2 + \frac{5}{4} v_2^\pm'(x) \partial_x + v_2^\pm''(x) + \frac{2}{3} t(x) v_2^\pm(x)\right] \delta(x - y)
\]

\[
\{ v_2^\pm(x), v_2^\mp(y) \} = -\frac{4}{3} v_2^\pm(x) \epsilon(x - y) v_2^\mp(y)
\]

\[
\{ v_2^\pm(x), v_2^\mp(y) \} = \frac{4}{3} v_2^\pm(x) \epsilon(x - y) v_2^\mp(y) + [\partial_x^3 + t(x) \partial_x + \frac{1}{2} t'(x) \pm w_3(x)] \delta(x - y)
\]  

(3.12)

Eqs.(3.12) gives some sort of spin-3 extension of V algebra (3.11). However, one should note that the elimination of the spin-3 field \( w_3 \) from (3.12) is not quite well-defined even though the eliminations of \( v_2^\pm \) is quite trivial. It is perhaps better to regard (3.12) as a nonlocal extension of \( W_3 \) algebra by two spin-2 fields \( v_2^\pm \).

We have seen that the bracket (3.1) indeed defines a family of nonlocal extended algebra when \( n \geq 2 \). Let us go back to the \( n = 1 \) case:

\[
L_1 = \partial + \phi \partial^{-1} \psi
\]  

(3.13)

From (3.3) we have the following:

\[
\{ \phi(x), \phi(y) \} = -2 \phi(x) \epsilon(x - y) \phi(y)
\]

\[
\{ \psi(x), \psi(y) \} = -2 \psi(x) \epsilon(x - y) \psi(y)
\]  

(3.14)

\[
\{ \phi(x), \psi(y) \} = \partial_x \delta(x - y) + 2 \phi(x) \epsilon(x - y) \psi(y)
\]
Here we do not have a natural candidate for the Virasoro generator \( t \). Based on the dimension consideration one expects that a function of the form \( a\phi^2 + b\phi\psi + c\psi^2 + e\phi' + f\psi' \) could do the job. Since \( \{t(x),t(y)\} \) contains only local terms, a little thinking tells us that \( a, c, e \) and \( f \) must be all zero. Moreover, simple calculations give

\[
\{ \phi(x)\psi(x),\phi(y)\psi(y) \} = [2\phi(x)\psi(x)\partial_x + (\phi(x)\psi(x))]\delta(x-y)
\]

Hence, \( \phi \) and \( \psi \) are spin-1 fields with respect to the Virasoro generator which is simply their product \( \phi\psi \). This is fairly interesting situation. We shall come back to this example.

\section*{IV. \( W_n \) Algebras From Nonstandard Lax Operators}

We now consider the Kuperschmidt’s nonstandard Lax hierarchy defined by (1.9) and (1.10). The second Hamiltonian structure for this system is\(^\text{[18]}\)

\[
\Theta_2^{NS} \left( \frac{\delta H}{\delta K_n} \right) = (K_n \frac{\delta H}{\delta K_n})_+ - K_n \left( \frac{\delta H}{\delta K_n} K_n \right)_+ + [K_n, (K_n \frac{\delta H}{\delta K_n})_0] + \partial^{-1} \text{res} [K_n, \frac{\delta H}{\delta K_n}] K_n + [K_n, \int^x \text{res} [K_n, \frac{\delta H}{\delta K_n}]]
\]

(4.1)

Here

\[
\frac{\delta H}{\delta K_n} = \frac{\delta H}{\delta v_{n+1}} + \partial^{-1} \frac{\delta H}{\delta v_n} + \ldots + \partial^{-n} \frac{\delta H}{\delta v_1}
\]

(4.2)

As in Section III, we like to understand the conformal property of \( K_n \) and the algebra defined by the bracket (4.1). In view of the gauge transformation

\[
K_n = \phi^{-1} L_n \phi = \partial^n + n \frac{\phi'}{\phi} \partial^{n-1} + \left( u_2 + \frac{n(n-1)}{2} \frac{\phi''}{\phi} \right) \partial^{n-2} + \ldots
\]

(4.3)

and of (3.4) and (3.6) we deduce that

\[
K_n : \quad F_{-n} \longrightarrow F_0
\]

(4.4)

and that

\[
t \equiv u_2 = v_2 - \frac{n-1}{2} v_1' - \frac{n-1}{2n} v_1^2
\]

(4.5)
is the corresponding Virasoro generator. However, (4.4) and (4.5) are not what we are interested in since with respect to this $t$ the first coefficient function $v_1$ is not a spin-1 field.

[Recall: It is a spin-1 field only when $K_n$ maps from $F_{-\frac{n-1}{2}}$ to $F_{\frac{n+1}{2}}$.] The last statement is actually quite obvious since $v_1 = n\frac{\phi'}{\phi}$ which is surely not a spin field with respect to $t = u_2$. As a result (4.1) does not define an extended conformal algebra with the choice of Virasoro generator given by (4.5) even though it has a Virasoro subalgebra. So if we want to know whether (4.1) defines an extended conformal algebra or not then we have to give up (4.4), which is the only way $K_n$ can be a covariant operator, and search for a new Virasoro generator of the form: $t = v_2 + av_1 + bv_1^2$. Let us start with two explicit examples. First, we look at

$$K_1 = \partial + v_1 + \partial^{-1}v_2$$

(4.6)

The second Hamiltonian structure associated with $K_1$ has been discussed in details in Ref.[23]. Here we have

$$\{v_1(x),v_1(y)\} = 2\partial_x \delta(x - y)$$

$$\{v_1(x),v_2(y)\} = [\partial_x^2 + v_1(x)\partial_x + v_1'(x)]\delta(x - y)$$

(4.7)

$$\{v_2(x),v_2(y)\} = [2v_2(x)\partial_x + v_2'(x)]\delta(x - y)$$

We see that $v_2$ satisfies the Virasoro algebra with zero "anomalous term", $\partial_x^2 \delta(x - y)$. If we take $v_2$ as the Virasoro algebra then $v_1$ is not a spin-1 field due to the presence of an anomalous term, $\partial_x^2 \delta(x - y)$ in the bracket $\{v_1(x),v_2(y)\}$. However, it was observed that[23] if we define $t = v_2 + \frac{1}{2}v_1'$ then

$$\{v_1(x),t(y)\} = [v_1(x)\partial_x + v_1'(x)]\delta(x - y)$$

$$\{t(x),t(y)\} = [\frac{1}{2}\partial_x^3 + 2t(x)\partial_x + t'(x)]\delta(x - y)$$

(4.8)

In other words, with the new choice of Virasoro generator $v_1$ become a genuine spin-1 field and the second Hamiltonian structure associated with $K_1$ is nothing but the Virasoro-$U(1)$-Kac-Moody algebra.
Next, we consider
\[ K_2 = \partial^2 + v_1 \partial + v_2 + \partial^{-1} v_3 \]  
(4.9)

The brackets needed for our discussions are
\[
\begin{align*}
\{v_1(x), v_1(y)\} &= 6\partial_x \delta(x - y) \\
\{v_2(x), v_1(y)\} &= 4v_1(x)\partial_x \delta(x - y) \\
\{v_2(x), v_2(y)\} &= [2\partial_x^3 + 2(v_2(x) + v_1(x)^2)\partial_x + (v_2(x) + v_1(x)^2)']\delta(x - y) \\
\{v_3(x), v_1(y)\} &= [2\partial_x^3 - 2v_1(x)\partial_x^2 + 2(v_2(x) - v_1'(x))\partial_x]\delta(x - y) \\
\{v_3(x), v_2(y)\} &= [-\partial_x^4 + \partial_x^3 v_1(x) + \partial_x v_1(x)\partial_x^2 - \partial_x v_1(x)\partial_x v_1(x) \\
&\quad - v_2(x)\partial_x^3 + v_2(x)\partial_x v_1(x) + 3v_3(x)\partial_x + v_3'(x)]\delta(x - y)
\end{align*}
\]  
(4.10)

Some straightforward algebras show that if we define
\[
t \equiv v_2 - \frac{1}{4}v_1^2 \quad w_3 \equiv v_3 + \frac{1}{2}v_2' - \frac{1}{3}v_1'' - \frac{1}{3}v_1v_2 + \frac{1}{12}v_1^3
\]  
(4.11)

then
\[
\begin{align*}
\{v_1(x), t(y)\} &= [v_1(x)\partial_x + v_1'(x)]\delta(x - y) \\
\{t(x), t(y)\} &= [2\partial_x^3 + 2t(x)\partial_x + t'(x)]\delta(x - y) \\
\{w_3(x), t(y)\} &= [3w_3(x)\partial_x + w_3'(x)]\delta(x - y)
\end{align*}
\]  
(4.12)

That is, \(v_1\) and \(w_3\) are, respectively, spin-1 and spin-3 fields with respect to the Virasoro generator. In terms of \(v_1\), \(t\) and \(w_3\) we expect the second Hamiltonian structure associated with \(K_2\) defines the \(W_3\)-\(U(1)\) \(-\)Kac-Moody algebra. These two examples suggest that for any \(K_n\) a new Virasoro generator and a new set of higher spin fields can be found such that (4.1) gives the \(W_{n+1}\)-\(U(1)\)-Kac-Moody algebra.

Let us now show that the above expectation is indeed true. The key hint comes from an observation on the coefficient of the anomalous term \(\partial_x^3 \delta(x - y)\) in the bracket \(\{t(x), t(y)\}\). From (4.8) and (4.12) we see that it is \(\frac{1}{2}\) for \(K_1\) and is 2 for \(K_2\). These two values are precisely given by the formula \(\frac{n^3 - n}{12}\) [see (1.6)] with \(n = 2\) and \(n = 3\) respectively. It is then natural to suspect that the conformal algebras associated with \(K_1\) and \(K_2\) might
be related to the conformal property of the pure differential operators $l_2$ and $l_3$. We are therefore motivated to consider the following mapping:

$$l_{n+1} \equiv \partial K_n = \partial^{n+1} + v_1 \partial^{n} + (v_2 + v_2') \partial^{n-1} + \ldots + (v_{n+1} + v_{n+1}')$$

\[ (4.13) \]

From (4.13) it can be proved rigorously [see Appendix] that effectively

$$\frac{\delta H}{\delta l_{n+1}} \equiv \partial^{-1} \frac{\delta H}{\delta u_{n+1}} + \ldots + \partial^{-n-2} \frac{\delta H}{\delta u_1} = \frac{\delta H}{\delta K_n} \partial^{-1}$$

\[ (4.14) \]

A simple way to understand (4.14) is to note that it implies $tr(l_{n+1} \frac{\delta H}{\delta l_{n+1}}) = tr(K_n \frac{\delta H}{\delta K_n})$.

[Recall: $tr(A) \equiv \int dx res(A)$] Now we like to express (4.1) in terms of $l_{n+1}$ and $\frac{\delta H}{\delta l_{n+1}}$. By the use of (4.13) and (4.14) we easily derive

$$(l_{n+1} \frac{\delta H}{\delta l_{n+1}})_+ = (K_n \frac{\delta H}{\delta K_n})_+ + [\partial, (K_n \frac{\delta H}{\delta K_n})_{\geq 1}] \partial^{-1}$$

\[ (4.15) \]

$$res[l_{n+1}, \frac{\delta H}{\delta l_{n+1}}] = res[K_n, \frac{\delta H}{\delta K_n}] + (K_n \frac{\delta H}{\delta K_n})_0 K_n$$

Eqs. (4.15) then lead to

$$\partial \Theta^{NS}_2 \left( \frac{\delta H}{\delta K_n} \right) = (l_{n+1} \frac{\delta H}{\delta l_{n+1}})_+ l_{n+1} - l_{n+1}(\frac{\delta H}{\delta l_{n+1}}) l_{n+1} + [l_{n+1}, \int^x (res[l_{n+1}, \frac{\delta H}{\delta l_{n+1}}])]$$

\[ (4.16) \]

The last piece in the first line of (4.16) is called the third Gelfand-Dickey bracket [10]. We have shown that under the mapping (4.13) the Hamiltonian structure (4.1) is transformed to the sum of the second and the third Gelfand-Dickey brackets defined by the pure differential operator $l_{n+1}$ of order $n + 1$.

With (4.16) the remained tasks are quite easy. We like to find a Virasoro generator, which is a differential polynomial in the coefficients of $l_n$, such that the corresponding flow defined by (4.16) gives the infinitesimal form of (3.4); i.e.

$$\delta_{\epsilon} l_n = [\epsilon \partial + \frac{n+1}{2} \epsilon'] l_n - l_n[\epsilon \partial - \frac{n-1}{2} \epsilon']$$

\[ (4.17) \]
We find that
\[ t = u_2 - \frac{n - 1}{2} u_1' - \frac{n - 2}{2(n - 1)} u_1'^2 \] (4.18)
does the job. \( u_1 \) and \( t \) together satisfy (1.6) except that the last bracket there is now replaced by
\[ \{ u_1(x), u_1(y) \} = n(n - 1) \partial_x \delta(x - y) \] (4.19)
Since \( v_1 = u_1 \) according to (4.13), (4.19) agrees with (4.7) and (4.10). Eq.(4.18) enables us the write down the covariant form of \( l_n \) with respect to the bracket defined by (4.16):
\[ l_n = D^n - \frac{n - 1}{2} + \Delta_1(u_1,t) + \frac{n - 2}{2(n - 1)} \Delta_2(u_1^2,t) + \sum_{k=3}^n \Delta_k(w_k,t) \] (4.20)
The formula (4.20) differs from (2.8) only by the coefficient of the third term. Let us compare the decompositions given by (4.20) with the previous explicit results for \( K_1 \) and \( K_2 \). In terms of \( t \) and spin fields we have
\[ l_2 \equiv \partial K_1 = \partial^2 + v_1 \partial + t + \frac{1}{2} v_1' \]
\[ l_3 \equiv \partial K_2 = \partial^3 + v_1 \partial^2 + (t + v_1' + \frac{1}{4} v_1'^2) \partial \]
\[ + w_3 + \frac{1}{2} t' + \frac{1}{3} v_1'' + \frac{1}{3} v_1 t + \frac{1}{4} v_1 v_1' \] (4.21)
It is easy to check that (4.21) completely agrees with (4.20) once \( v_1 = u_1 \) is considered.

Finally, we like to impose the constraint \( u_1 = 0 \) on (4.16). By the virtue of (4.13) this is essentially equivalent to imposing the constraint \( v_1 = 0 \) on (4.1). We leave the discussion of this equivalence to the Appendix. Following the Dirac procedure the modified form of \( \varphi \) defined by (4.16) reads
\[ \bar{\varphi}(\frac{\delta H}{\delta l_{n+1}}) = (l_{n+1} \frac{\delta H}{\delta l_{n+1}}) + l_{n+1}(\frac{\delta H}{\delta l_{n+1}}) l_{n+1} + \frac{1}{n + 1}[l_{n+1}, \int^x (res[l_{n+1}, \frac{\delta H}{\delta l_{n+1}}])] \] (4.22)
\( \bar{\varphi} \) is exactly equal to the modified second Gelfand-Dickey bracket given by (2.9). This shows that when we remove the spin-1 field \( u_1 \) the resulting algebra is simply the \( W_{n+1} \) algebra. This result together with the fact that \( u_1 \) and \( t \) together define the \( U(1) \)-Kac-Moody
algebra confirm our expectation that (4.1) [or equivalently (4.16)] defines the $W_{n+1}\cdot U(1)$-Kac-Moody algebra. However, it is not equivalent to the corresponding one defined by (1.3) and (2.8) in the sense that no redefinitions, of differential polynomial type, of the Virasoro generator and of spin fields makes two algebras identical (We assume that $u_i$’s are real-valued functions). This inequivalence actually can be easily seen by comparing the last of (1.6) and (4.19): no redefinitions of $u_1$ and of $t$ can make two Virasoro – $U(1)$-Kac-Moody subalgebras identical.

V. Concluding Remarks

In this paper we have studied the conformal property of the Lax operator of the constrained KP hierarchy and the associated second Hamiltonian structure. We have seen that the analysis for the second Gelfand-Dickey bracket defined by a pure differential operator can be straightforwardly carried over to the present case. The conformal decomposition defined by (3.7) is a simple extension of (2.8). The extended conformal algebras defined by (3.1) are nonlocal and contain two half integer spin fields when the leading order is even. We have given two examples of such nonlocal algebras. They are very similar to Bilal’s V algebra.

We also study the constrained KP hierarchy in Kuperschmidt’s nonstandard Lax representation. Here we have found that the corresponding second Hamiltonian structure defines the $W_{n+1}\cdot U(1)$-Kac-Moody algebra just like the second Gelfand-Dickey bracket of the differential operator of order $n + 1$ does. However, the Virasoro generator in this algebra does not make its Lax operator $K_n$, a covariant operator. The natural covariant operator now is actually the pure differential operator $l_{n+1} \equiv \partial K_n$ of order $n + 1$. More unexpectedly, in terms of $l_{n+1}$, the Hamiltonian structure (4.1) becomes the sum of the second and the third Gelfand-Dickey brackets defined by $l_{n+1}$. The conformally covariant form is then worked out. Since the two brackets (3.1) and (4.1) are connected by the gauge
transformation (1.9), what is interesting here is that a nonlocal extended conformal algebra defined by (3.1) is nothing but a local $W-U(1)$-Kac-Moody algebra in disguise. Of course, the transformations between the two sets of Virasoro generator and spin fields are not of differential polynomial type. For instance, connecting (3.13) to (4.6) by $K_1 = \phi^{-1}L_1\phi$ we have

$$v_1 = \frac{\phi'}{\phi}$$
$$t = \phi\psi + \frac{1}{2}(\frac{\phi'}{\phi})'$$

which maps the nonlocal algebra (3.14) to the local one given by (4.7) and (4.8).

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Appendix

In this appendix we give a proof of (4.14). From (4.13) we have

$$v_1 = u_1$$
$$v_2 = u_2 - u_1'$$
$$v_3 = u_3 - u_2' + u_1''$$
$$\ldots$$
$$\ldots$$
$$\ldots$$
$$v_{n+1} = u_{n+1} - u_n' + \ldots + (-1)^n u_1^{(n)}$$

(A.1)
and hence
\[
\frac{\delta H}{\delta u_1} = \frac{\delta H}{\delta v_1} + \left(\frac{\delta H}{\delta v_2}\right)' + \ldots + \left(\frac{\delta H}{\delta v_{n+1}}\right)^{(n)}
\]

\[
\ldots
\]

\[
\frac{\delta H}{\delta u_n} = \frac{\delta H}{\delta v_n} + \left(\frac{\delta H}{\delta v_{n+1}}\right)'
\]

\[
\frac{\delta H}{\delta u_{n+1}} = \frac{\delta H}{\delta v_{n+1}}
\]

Using
\[
f \partial^{-1} = \partial^{-1} f + \partial^{-2} f' + \partial^{-3} f'' + \ldots
\]

we obtain
\[
\frac{\delta H}{\delta K_n} \partial^{-1} = \left(\frac{\delta H}{\delta v_{n+1}} + \partial^{-1} \frac{\delta H}{\delta v_n} + \ldots + \partial^{-n} \frac{\delta H}{\delta v_1}\right) \partial^{-1}
\]

\[
= \partial^{-1} \frac{\delta H}{\delta v_{n+1}} + \partial^{-2} \left[\frac{\delta H}{\delta v_n} + \left(\frac{\delta H}{\delta v_{n+1}}\right)\right] + \ldots
\]

\[
+ \partial^{-n-1} \left[\frac{\delta H}{\delta v_1} + \left(\frac{\delta H}{\delta v_2}\right)' + \ldots + \left(\frac{\delta H}{\delta v_{n+1}}\right)^{(n)}\right] + O(\partial^{-n-2})
\]

\[
= \partial^{-1} \frac{\delta H}{\delta u_{n+1}} + \partial^{-2} \frac{\delta H}{\delta u_n} + \ldots + \partial^{-n-1} \frac{\delta H}{\delta u_1} + O(\partial^{-n-2})
\]

\[
= \frac{\delta H}{\delta l_{n+1}} + O(\partial^{-n-2})
\]

Since terms in \(O(\partial^{-n-2})\) do not contribute to (4.15) or (4.16), we can simply drop them. This completes the proof for (4.14).

Now let us imposing constraint \(v_1 = 0\) on (4.1). The Dirac procedure now gives the modified bracket:
\[
\Theta_2^{NS} \left(\frac{\delta H}{\delta K_n}\right) = (K_n \frac{\delta H}{\delta K_n}) + K_n - K_n \frac{\delta H}{\delta K_n} + \frac{1}{n+1} \{[K_n, (K_n \frac{\delta H}{\delta K_n})_0] + K_n, \int^x (res[K_n, \frac{\delta H}{\delta K_n}]) \} + \partial^{-1} res[K_n, \frac{\delta H}{\delta K_n}] K_n - n \partial^{-1} \{K_n \frac{\delta H}{\delta K_n}\}_0 K_n\]

As in Section IV. we like to express (A.5) in terms of \(l_{n+1}\) and \(\frac{\delta H}{\delta u_1}\). Here some care must be taken. From (A.2) we see that \(\frac{\delta H}{\delta v_1} = 0\) does not imply \(\frac{\delta H}{\delta u_1} = 0\). As a consequence,
instead of (A.4) we have
\[ \frac{\delta H}{\delta K_n} \partial^{-1} = \partial^{-1} \frac{\delta H}{\delta u_{n+1}} + \ldots + \partial^{-n} \frac{\delta H}{\delta u_2} + O(-n-2) \]
\[ \equiv \frac{\delta H}{\delta l_{n+1}} + O(-n-1) \]  
(A.6)

where $O(-n-1)$ denotes terms of order $-n - 1$. Since the term $O(-n-1)$ could affect (4.15), in order to use (4.15) the symbol $\frac{\delta H}{\delta l_{n+1}}$ there must be regarded as the sum of $\frac{\delta H}{\delta l_{n+1}}$ and $O(-n-1)$ defined in (A.6). A derivation similar to that for (4.16) then gives
\[ \partial \Theta_2^{NS} = (l_{n+1} \frac{\delta H}{\delta l_{n+1}}) + l_{n+1} - l_{n+1}(\frac{\delta H}{\delta l_{n+1}}) + + \frac{1}{n+1} [l_{n+1}, \int_x^x \text{res}[l_{n+1}, \frac{\delta H}{\delta l_{n+1}}]] \]  
(A.7)

We observe that the term $O(-n-1)$, which was previously put into $\frac{\delta H}{\delta l_{n+1}}$, does not contribute to (A.7). Hence we can drop it and the equality (4.14) is again correct. As promised, (A.7) is identical to (4.22). We thus have shown that imposing the constraint $v_1 = 0$ on (4.1) is completely equivalent to imposing the constraint $u_1 = 0$ on (4.16).

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