Group-cohomology refinement to classify $G$-symplectic manifolds

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Abstract

“Pseudo-cohomology”, as a refinement of Lie group cohomology, is soundly studied aiming at classifying of the symplectic manifolds associated with Lie groups. In this study, the framework of symplectic cohomology provides fundamental new insight, which enriches the analysis previously developed in the setting of Cartan-Eilenberg $H^2(G,U(1))$ cohomology.

1 Introduction

From the strict mathematical point of view, the orbits of the coadjoint representation of Lie groups provide a source for symplectic manifolds on which a given Lie group acts as a group of symplectomorphisms, i.e. $G$-symplectic manifolds. Even, for finite-dimensional semi-simple groups, this mechanism essentially exhausts all models of they. These $G$-symplectic manifolds could then be considered as phase spaces of physical systems for which $G$ can be called the “basic symmetry”. From the physical point of view, however, the simplest physical systems (the free non-relativistic particle, for instance) possess a phase space endowed with a symplectic form whose associated Poisson bracket realizes the Lie algebra of a central extension of the basic “classical” symmetry. Central extensions of Lie groups by $U(1)$, associated with projective unitary representations, were classified long ago by Bargmann [1] by means of the cohomology group $H^2(G,U(1))$ [2]. Later, the momentum map from the phase space to the coalgebra $G^*$ of the basic “classical” symmetry group, constructed with the set of Noether invariants of the physical system, was used by Souriau [3] to define the symplectic cohomology group $H^1_S(G,G^*)$ characterizing equivalently the central extensions of a simply-connected group $G$.

In this paper we revisit the notion of Lie group “pseudo-cohomology” in an attempt to classify all possible (quantizable) $G$-symplectic manifolds for an arbitrary Lie group $G$, in such a way that both coadjoint orbits and phase spaces realizing central extensions can be put together into (“pseudo”-)cohomology classes. By the way, the prefix “pseudo” had its origin [4] in the fact that the corresponding central extensions are trivial from the mathematical point of view, the associated cocycle being a coboundary, although they behave as if they were non-trivial in some aspects, as we shall show.

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Our study here is made in the language of symplectic cohomology of Lie groups. The insight provided by the natural and explicit role of phase spaces in symplectic cohomology offers a more intuitive understanding of the significance of pseudo-cohomology in classifying dynamics, as well as an easier mathematical handling which allows a generalization of the mathematical results obtained in its original presentation [5, 6].

Roughly speaking, pseudo-cohomology emerges as a refinement of the equivalence classes of 2-cocycles in the cohomology group $H^2(G, U(1))$. The first clues for the need of such a refinement occurred when studying the problem of the Inönü-Wigner contraction of centrally extended Lie groups (see Saletan [7]). An example of this need appears in the contraction Poincaré → Galileo where a special kind of trivial 2-cocycles in the Poincaré group, becomes true 2-cocycles for the Galileo group in the $c \rightarrow \infty$ limit. The underlying reason is that, while the 2-cocycle is well-behaved in the limit, its generating function is not, thus occurring a generation of cohomology [4]. The second indication for the need of pseudo-cohomology appeared in the context of generalized Hopf fibrations of semi-simple Lie groups related to Čech (true, i.e. non-coboundary) cocycles of coadjoint orbits. Such pseudo-cocycles play in fact a fundamental role in the explicit construction of the local exponent associated with Lie-algebra cocycles of the corresponding Kac-Moody groups [8].

In spite of these antecedents, the importance of pseudo-cohomology is more evident in the framework of Group Approach to Quantization (GAQ), a group theoretical quantization scheme designed for obtaining the dynamics of a physical system out of a Lie group [9]. In particular, GAQ starts from a central extension $\tilde{G}$ of the symmetry group $G$ by $U(1)$ in such a way that the symplectic form of the classical phase space is derived from the 2-cocycle which defines the central extension. Nevertheless, the correspondence between central extensions and symplectic forms is not one-to-one. The most obvious illustration of this is the case of groups with trivial cohomology group $H^2(G, U(1))$ (such as the Poincaré group in 3+1 dimensions or finite-dimensional semi-simple groups). In fact, even though these groups do not admit non-trivial central extensions, genuine symplectic structures and dynamics can be derived out of them [4, 8]. The rationale for this is the existence of 2-cocycles which are coboundaries, and therefore trivial from the cohomological point of view, but which do define authentic symplectic structures. Coboundaries with this property are called pseudo-cocycles, giving rise to trivial central extension referred to as pseudo-extensions.

The study of this mechanism and the characterization of the classes of pseudo-extensions associated with non-equivalent symplectic structures, led in an explicit way to the notion of pseudo-cohomology, constituting the more systematic and clarifying approach to the problem [5, 6]. This standard view of pseudo-cohomology is described in the next section.

## 2 Pseudo-cohomology in GAQ

As commented in the introduction, GAQ is a formalism devised for obtaining the (quantum or classical) dynamics of a physical system out of a Lie group (of its symmetries). The starting point is a central extension $\tilde{G}$ of the symmetry group $G$ by $U(1)$ (or $\mathbb{R}$ to recover the classical dynamics), determined by a 2-cocycle (local exponent) $\xi : G \times G \rightarrow \mathbb{R}$. The group law then
reads:
\[ g'' = g' \ast g, \quad \zeta'' = \zeta' e^{i\xi(g',g)} \]  
where \( g'', g', g \in G \) and \( \zeta'', \zeta', \zeta \in U(1) \). On the Lie group \( \tilde{G} \) we have at our disposal left and right-invariant vector fields. If we choose a coordinate system \( \{g^i\}_{i=1}^{\dim G}, \zeta \) in \( \tilde{G} \), a basis for the vector fields is given by \( \tilde{X}_L^i \) and \( \tilde{X}_R^i \), respectively, and their dual sets of left and right invariant 1-forms are denoted by \( \theta^{L(i)} \) and \( \theta^{R(i)} \), respectively. One of the left-invariant 1-forms, \( \Theta \equiv \theta^{L(3)} \), the \( U(1) \)-component of the left-invariant canonical 1-form on the Lie group \( \tilde{G} \), is chosen as the connection 1-form of the principal bundle \( U(1) \to \tilde{G} \to G \), thus defining a notion of \textit{horizontality}.

This connection 1-form, referred to as quantization 1-form, depends directly on the 2-cocycle \( \xi \) and can be used to define a symplectic structure in a unique manner. In fact, if \( G_\Theta \) is the characteristic distribution of \( \Theta \), i.e. the intersection of the kernel of \( \Theta \) and \( d\Theta \), then \( \tilde{G} / G_\Theta \) is a quantum manifold \( P \). This means that \( \tilde{G} / G_\Theta \) is a contact manifold with contact 1-form \( \Theta|_P \). The Quantum manifold \( P \) is in turn a \( U(1) \) Principal bundle \( U(1) \to P \to S \) with base a symplectic manifold, \( S = P / U(1) \) endowed with a symplectic form \( \omega \) such that \( \pi^* \omega = d\Theta \).

The symplectic structure \( (S, \omega) \) is not completely determined by the cohomology class to which the 2-cocycle \( \xi \) belongs. In fact, different yet cohomologous 2-cocycles can lead to completely different symplectic structures \( (S, \omega) \) (think, for instance, of a semi-simple Lie group, with trivial cohomology but with many different kinds of symplectic structures determined by its coadjoint orbits). This phenomenon suggests, again, a refinement in the classification of 2-cocycles in such a way that a one-to-one correspondence between the refined classes and the symplectic structures could be established. This refinement will define pseudo-cohomology. Therefore, the latter is intrinsically tied to the classification of possible symplectic structures constructed out of a Lie group.

The main idea for the definition of these subclasses in \( H^2(G, U(1)) \) can be intuited from the expression of \( \Theta \) in terms of the 2-cocycle \( \xi \):

\[ \Theta = \frac{d\xi}{i\zeta} + \frac{\partial \xi (g', g)}{\partial g^i} \bigg|_{g' = g^{-1}} \, dg^i \]  

If, now, a 2-coboundary \( \xi_\lambda (g', g) = \lambda (g' \ast g) - \lambda (g') - \lambda (g) \) generated by the function \( \lambda : G \to \mathbb{R} \), is added to \( \xi \), the expression for the new quantization 1-form \( \Theta' \) (as the \( U(1) \)-component of the canonical 1-form for the centrally extended Lie group defined by \( \xi + \xi_\lambda \)) is given by:

\[ \Theta' = \Theta + \Theta_\lambda = \Theta + \lambda^0_\xi \theta^{L(i)} - d\lambda \]  

where \( \lambda^0 \equiv \frac{\partial \lambda(g)}{\partial g} |_{g=e} \). Thus, the new term \( \Theta_\lambda \) added to the connection 1-form \( \Theta \) by the inclusion of a 2-coboundary depends only, up to a total differential, on the gradient at the identity \( \lambda^0 \) of the generating function \( \lambda (g) \). In fact, if we denote \( \Theta_\lambda^0 = \lambda^0_\xi \theta^{L(i)} \), then the total differential disappears when the pre-symplectic 2-form \( d\Theta \) is considered, in such a way that \( d\Theta' = d\Theta + d\Theta_\lambda^0 \).

From these considerations two conclusions can be drawn:

1) A 2-coboundary contribute non-trivially to the connection 1-form \( \Theta \) and to the symplectic structure determined by \( d\Theta \), if and only if \( \lambda^0 \neq 0 \).
ii) This contribution depends only (up to a total differential, which does not affect the symplectic structure) on the local properties of the generating function $\lambda(g)$ at the identity of the group, through its gradient at the identity $\lambda^0$.

A 2-coboundary $\xi_\lambda$ such that $\lambda^0 \neq 0$ is named a pseudo-cocycle. The name reflects the fact that they are trivial 2-cocycles but, from the dynamical point of view, behave as if they were non-trivial. If we consider the group $G$ centrally extended by this pseudo-cocycle $\xi_\lambda$, the extended group $\tilde{G}$ is isomorphic to $G \times U(1)$. However, we will refer to this extension as a pseudo-extension, to underline the fact that, although trivial as a central extension, it can lead to a non-trivial symplectic structure and non-trivial dynamics.

The next point to explore is the conditions under which two different 2-coboundaries, $\xi_\lambda$ and $\xi_{\lambda'}$, generated by functions $\lambda$ and $\lambda'$ with different gradients at the identity $\lambda^0$ and $\lambda'^0$, determine the same symplectic structure $(S, \omega)$, up to symplectomorphisms. This condition will define a refined equivalence relation inside each cohomology class. For the sake of simplicity, we shall restrict ourselves to simply connected Lie-groups.

The clue in the definition of the new equivalence relation is given by the fact that $\lambda^0$ defines an element of $G^*$, the dual of the Lie algebra $G$ of $G$, usually named the coalgebra. This can be seen by noting that $\Theta_{\lambda^0} = \lambda^0 \theta^L(e)$ defines, at the identity of $G$, an element of $G^*$ given by $\Theta_{\lambda^0}|_{g=e} = \lambda^0$. It is also important to note that $\Theta_{\lambda}|_{g=e} = 0 \in G^*$ (due to the presence of $d\lambda$), in such a way that the quantization 1-form $\Theta$ verifies $\Theta|_{g=e} = (0, \ldots, 0, 1) \in \tilde{G}^*$, whatever the 2-cocycle $\xi$ we are considering (here $\tilde{G}^*$ is the dual of the extended algebra $\tilde{G}$ associated with the extended group $\tilde{G}$). This fact will be of relevance in the relationship between pseudo-cohomology and symplectic cohomology.

Once we have established that $\lambda^0 \in G^*$, it is natural to propose their classification in accordance with the coadjoint orbits. This will prove to be the correct ansatz, provided we use the correct coadjoint action.

A bit of notation is in order. Let us denote the equivalence class of the cocycle $\xi$, defining a certain central extension $\tilde{G}$ of $G$ by $U(1)$, by $[[\xi]] \in H^2(G, U(1))$. We are going to introduce a further partition in each class $[[\xi]]$ into equivalence subclasses $[\xi]$.

For the sake of clarity, we shall firstly define this partition for the trivial cohomology class $[[\xi]]_0$, made out of trivial cocycles, i.e. 2-coboundaries $\xi_\lambda$. This would be enough for groups with trivial cohomology $H^2(G, U(1)) = 0$ (that is, with only the trivial class), such as finite-dimensional semi-simple groups or the Poincaré group (in $3+1$ dimensions). It is also valid for fully centrally-extended groups $\tilde{G}$, for which $H^2(\tilde{G}, U(1)) = 0$. The case of groups with non-trivial cohomology or non-fully central-extended groups $\tilde{G}$, with $H^2(\tilde{G}, U(1)) \neq 0$, will be considered in section 2.2.

### 2.1 The trivial class

Given a Lie group $G$, a natural action of $G$ on $G^*$ is provided by the coadjoint action $Coad$, defined as the dual of the adjoint action of $G$ on $G$. More explicitly, with the adjoint action of $G$ on $G$ given by $Ad g(X) = (R^T_g \cdot L^T_f)(e) \cdot X$, where $g \in G$ and $X \in G$, the coadjoint action

\[\text{Throughout the paper, the differential of a given application will be denoted with a superscript T.}\]
Coad : \(G \to Aut(\mathcal{G}^*)\) has the form \(\text{Coad}(g)\mu(X) = \mu(\text{Ad} g^{-1}(X))\), where \(\mu \in \mathcal{G}^*\). It is also convenient to make explicit the infinitesimal version of this action. Linearizing on the \(g\) variable we obtain the coadjoint action of the Lie algebra on the coalgebra: \((\text{Coad})^T(e) \equiv \text{coad} : \mathcal{G} \to \text{End}(\mathcal{G}^*)\). Its explicit expression is given by \(\text{coad} X(\mu)(Y) = \mu(\text{ad} X(Y)) = \mu([X,Y])\), with \(X,Y \in \mathcal{G}\) and \(\mu \in \mathcal{G}^*\).

The orbits of this action are specially relevant in our study. Given a point \(\mu \in \mathcal{G}^*\), the orbit through this point by the action of the whole group \(G\) is \(\text{Orb}(\mu) = \{\text{Coad}(g)\mu / g \in G\}\), diffeomorphic to \(G/G_\mu\) where \(G_\mu\) is the isotropy group of \(\mu\). The coadjoint action determines a foliation of \(\mathcal{G}^*\) in orbits, in such a way that any point belongs to one (and just one) orbit (by the definition, the point \(\mu\) belongs to \(\text{Orb}(\mu)\)), and two points in the same orbit are always connected by the coadjoint action.

Coadjoint orbits of Lie groups are interesting from the physical point of view since they possess a natural symplectic structure \((\text{Orb}(\mu), \omega)\) with the symplectic form given by

\[
\omega_\nu(X_\nu, Y_\nu) = \nu([X,Y]), X_\nu, Y_\nu \in T_\nu(\text{Orb}(\mu))
\]

where \(\nu \in \text{Orb}(\mu) \subset \mathcal{G}^*, X_\nu, Y_\nu \in T_\nu(\text{Orb}(\mu))\) and \(X \in \mathcal{G}\) is related to \(X_\nu \in T_\nu \text{Orb}(\mu)\) by \(X_\nu = \text{coad}(X) \nu\), and analogously for \(Y_\nu\) and \(Y\) (note we are using the fact that \(\mathcal{G}^*\) is a linear space in order to identify its points with tangent vectors).

There is a close relationship between pseudo-extensions and coadjoint orbits, that can be stated as follows. A pseudo-extension characterized by the generating function \(\lambda(g)\) with gradient at the identity \(\lambda^0 \neq 0\) defines a presymplectic form \(d\Theta_\lambda = d\Theta_{\lambda^0}\) depending only on \(\lambda^0\). In the trivial case we are discussing in this section, the quotient of \(\tilde{G}\) by the characteristic subalgebra \(G_{\Theta_{\lambda^0}} \equiv \ker \Theta_\lambda \cap \ker d\Theta_{\lambda^0}\), defines a quantum manifold \(P\), and the quotient \(S = \tilde{G}/(G_{\Theta_{\lambda^0}} \times U(1)) \sim G/G_{\Theta_{\lambda^0}}\) is a symplectic manifold with symplectic form \(\omega_{\lambda^0}\) given by \(\pi^*\omega_{\lambda^0} = d\Theta_{\lambda^0}\), where \(\pi : P \to S\) is the canonical projection and \(G_{\Theta_{\lambda^0}}\) is the (connected) subgroup associated with \(G_{\Theta_{\lambda^0}}\). \(G/G_{\Theta_{\lambda^0}}\) is in fact locally diffeomorphic to a coadjoint orbit (the one passing through \(\lambda^0\)). This can be seen by noting:

a) The pre-symplectic form adopts the expression:

\[
d\Theta_{\lambda^0} = \frac{1}{2} \lambda^0_k C^k_{ij} \theta^L(i) \wedge \theta^L(j)
\]

when using the Maurer-Cartan equations, and therefore,

\[
d\Theta_{\lambda^0}(X^L_i, X^L_j) = \lambda^0_k C^k_{ij} = \lambda^0([X^L_i, X^L_j])
\]

where \(\{X^L_i\}\) is a basis for \(\mathcal{G}\) and \(\lambda^0 \in \mathcal{G}^*\), thus reproducing (before falling down to the quotient).

b) The characteristic group \(G_{\Theta_{\lambda^0}}\) coincides with (the connected component of) the isotropy group of \(\lambda^0\), \(G_{\lambda^0}\) under the coadjoint action, thus defining (locally) the same quotient space. At the infinitesimal level, a vector \(Y = Y^i X^L_i\) belongs to \(G_{\Theta_{\lambda^0}}\) iff \(Y^i \lambda^0_k C^k_{ij} = 0, \forall j\), which is the same condition for \(Y\) to belong to \(G_{\lambda^0}\).
Using the transformation properties of left-invariant one-forms under translation by the group, it is easy to check that:

$$Ad(g)^*(\Theta_{\lambda^0}) = \Theta_{\text{Coad}(g)\lambda^0}$$

(7)

where $Ad(g)^*$ denotes the pull-back of the adjoint action of the group on itself (conjugation), acting on $\theta^L(i)$, and on the right hand side $\text{Coad}(g)$ acts on $\lambda^0$.

Although the connection 1-form is given by $\Theta_{\lambda}$ rather than $\Theta_{\lambda^0}$, the symplectic form is determined by just $\lambda^0$, and it transforms in a similar way:

$$Ad(g)^*(d\Theta_{\lambda^0}) = d\Theta_{\text{Coad}(g)\lambda^0}$$

(8)

These results can be summarized in the following proposition:

**Proposition 1.** Let $G$ be a Lie group and consider two coboundaries $\xi_{\lambda_1}$ and $\xi_{\lambda_2}$ with generating functions $\lambda_1(g)$ and $\lambda_2(g)$, defining the (trivial) central extensions $\tilde{G}_1$ and $\tilde{G}_2$, respectively. If $\Theta_{\lambda_1}$ and $\Theta_{\lambda_2}$ are the Quantization one-forms associated with each group, with $G_{\Theta_{\lambda_1}^0}$ and $G_{\Theta_{\lambda_2}^0}$ as their respective characteristic subgroups, the two symplectic spaces $\tilde{G}_1/(G_{\Theta_{\lambda_1}^0} \times U(1))$ and $\tilde{G}_2/(G_{\Theta_{\lambda_2}^0} \times U(1))$, with symplectic forms given by $\omega_{\lambda_1}^0$ and $\omega_{\lambda_2}^0$ such that $d\Theta_{\lambda_1}^0 = \pi^*\omega_{\lambda_1}^0$ and $d\Theta_{\lambda_2}^0 = \pi^*\omega_{\lambda_2}^0$ respectively, are symplectomorphic if there exists $h \in G$ such that:

$$\lambda_1^0 = \text{Coad}(h)\lambda_2^0,$$

(9)

the symplectomorphism being given by $Ad(h)$:

$$d\Theta_{\lambda_1}^0 = Ad(h)^*d\Theta_{\lambda_2}^0$$

(10)

**Proof:** It simply remains to proof that the two spaces $\tilde{G}_1/(G_{\Theta_{\lambda_1}^0} \times U(1))$ and $\tilde{G}_2/(G_{\Theta_{\lambda_2}^0} \times U(1))$ are diffeomorphic. Since the extensions are trivial, $\tilde{G}_i$, $i = 1, 2$ are isomorphic to $G \times U(1)$, therefore $G_i/(G_{\Theta_{\lambda_i}^0} \times U(1)) \approx G/G_{\Theta_{\lambda_i}^0}, i = 1, 2$. If $\lambda_i^0 = \text{Coad}(h)\lambda_i^0$, then $G_{\Theta_{\lambda_i}^0}$ and $G_{\Theta_{\lambda_i}^0}$ are conjugated subgroups by the adjoint action and this implies that the two spaces $G/G_{\Theta_{\lambda_i}^0}, i = 1, 2$ are diffeomorphic.

This suggests us to define the equivalence relation in $[[\xi]]_0$ in the following way:

**Definition 1:** Two coboundaries $\xi_\lambda$ and $\xi_{\lambda'}$ with generating functions $\lambda$ and $\lambda'$, respectively, belong to the same equivalence subclass $[\xi]$ of $[[\xi]]_0$ if and only if the gradients at the identity of the generating functions are related by:

$$\lambda'' = \text{Coad}(g)\lambda^0,$$

(11)

for some $g \in G$.

We shall denote by $[\xi]_{\lambda^0}$ the equivalence class of coboundaries “passing through” $\lambda^0$. The equivalence relation introduced in this way will be named **pseudo-cohomology**\(^2\), and the

\(^2\)This equivalence relation does not define, in general, a cohomology.
subclass of (trivial) central extensions defined by all \( \xi_\lambda \in [\xi]_{\lambda^0} \) will be called the central **pseudo-extension** associated with \( [\xi]_{\lambda^0} \).

The condition \( \lambda^{0'} = Coad(g)\lambda^0 \) means that \( \lambda^0 \) and \( \lambda^{0'} \) are related by the coadjoint action of the group \( G \). Therefore, \( \lambda^0 \) and \( \lambda^{0'} \) lie in the same coadjoint orbit of \( G \) in \( G^* \). A pseudo-cohomology class is therefore directly associated with a coadjoint orbit in \( G^* \).

If \( \lambda^{0'} = \lambda^0 \), then \( \xi_\lambda, \xi_{\lambda^0} \in [\xi]_{\lambda^0} \). Thus, we can always choose a representative element in each subclass "linear" in the local coordinate system, \( \xi_{\lambda^0}(g) = \lambda^0_i g^i \). If the local coordinates \( \{g^i\} \) are canonical, and if we restrict ourselves to canonical 2-cocycles (see [1]), then two cohomologous 2-cocycles differ in a 2-coboundary \( \xi_\lambda \) with \( \lambda(g) \) linear in the canonical coordinates. Then pseudo-cohomology is a further partition of "linear" coboundaries into equivalence classes through the coadjoint action of the group \( G \) on \( G^* \) (for the trivial class, at the moment).

However, the correspondence between pseudo-cohomology classes and coadjoint orbits for a Lie group \( G \) is not onto. The relation is established in the following theorem:

**Proposition 2:** Pseudo-cohomology classes are associated with coadjoint orbits which satisfy an integrality condition: the symplectic 2-form \( \omega \) naturally defined on the coadjoint orbit by (4) has to be of integer class.

In fact, this integrality condition is required for \( \xi_{\lambda^0} \) to define a global coboundary on \( G \); non-integral coadjoint orbits of \( G \) cannot be related to central pseudo-extensions of \( G \), since they do not define a proper (global) Lie group.

**Proof:** A (pseudo-) centrally extended Lie group gives rise to a Quantum manifold in the sense of Geometric Quantization (see Sec. 2) when taking quotient by the characteristic subalgebra [9]. Therefore, as a consequence of the necessary and sufficient condition for the existence of a quantization of a given symplectic manifold (see [10, 9]), the closed 2-form on the coadjoint orbit is of integer class.

Let us see another way of looking at the integrality condition. The vector \( \lambda^0 \) is an element of \( G^* \) and, therefore, it is a linear mapping from \( G \) to \( \mathbb{R} \). It is easy to check that when restricted to \( G_{\lambda^0} \) (the Lie algebra of the isotropy group \( G_{\lambda^0} \) of the coadjoint orbit passing through \( \lambda^0 \)), \( \lambda^0 \) defines one dimensional representation of the latter. Then, the integrality condition on the coadjoint orbit parallels the requirement for \( \lambda^0 \) of being exponentiable (integrable) to a unitary character of the group \( G_{\lambda^0} \) (note however that this remark resorts to the level of representation theory of the group, whereas the above-stated theorem involves only the Lie group structure).

This relationship between integrality condition of the coadjoint orbit and "integrality" of the character defined by \( \lambda^0 \) reveals, in passing, that the Coadjoint Orbit Method of Kostant-Kirillov [11, 12], intended to obtain unitary irreducible representations of Lie groups using (what in Physics is now known as) Geometric Quantisation [10] on coadjoint orbits of Lie groups, is a particular case of the induced representation technique of Mackey [13].

Let us denote by \( \hat{G} \) the central pseudo-extension of \( G \), characterised by \( \xi_{\lambda^0} \). It defines a central pseudo-extension of \( G \):

\[
[X^L_i, X^L_j] = C^k_{ij}(X^L_k + \lambda^0_k X_0),
\]

where \( X_0 \) is the (central) generator associated with \( U(1) \) (our convention is to take \( X_0 = iI \) in any faithful unirrep of \( \hat{G} \)). The left-invariant vector fields of \( \hat{G} \) (denoted with check) are
related to those of $G$ by:
$$
\hat{X}_i^L = X_i^L + (X_i^L \lambda - \lambda_0^i)X_0,
$$
with a similar relation for the right-invariant vector fields.

From this point of view, central pseudo-extensions are on the same footing as true (non-trivial) central extensions, and we can employ with them the same techniques for obtaining (projective) unirreps of $G$ (specially for semisimple Lie groups). Once a projective representation of $G$ (which is a true representation of $\tilde{G}$, the pseudo-extended group) has been obtained in this way, in order to obtain the true (non-projective) representations of $G$ associated with it we simply redefine the generators in the following way
$$
\hat{X}_i^L \rightarrow \hat{X}_i^L + \lambda_0^i X_0 = X_i^L + (X_i^L \lambda)X_0.
$$

2.2 Non-trivial classes

Let us consider now non-trivial cohomology classes $[[\xi]] \neq [[\xi]]_0$, in the case of groups $G$ with non-trivial cohomology, or extended groups $\tilde{G}$ which still admit further central extensions, that is, with $H^2(\tilde{G}, U(1)) \neq 0$.

In order to proceed, a representative element $\xi \in [[\xi]]$ must be chosen. We can add to $\xi$ a coboundary $\xi_\lambda$ generated by a function $\lambda$, with non-trivial gradient at the identity of $G$. The resulting cocycle $\xi' = \xi + \xi_\lambda$ defines a new central extension $\tilde{G}'$ of $G$ isomorphic, from the group-theoretical point of view, to $\tilde{G}$. The question is whether these pseudo-extensions can be classified into equivalence classes leading to the same symplectic structures, as in the case of the trivial class of Sec. 2.1. The naive classification in coadjoint orbits of the group $G$ does not work in this case (since there is a mixture of true cohomology and pseudo-cohomology), and there is no clue, at this level, of how the classification should be done.

The direct relation between pseudo-cohomology and coadjoint orbits obtained for the trivial class, allows us to resort to symplectic cohomology, as a tool for classifying symplectic structures (see Sec. 3), to come in our help. In this framework, it will be shown that a classification of pseudo-cocycles in the non-trivial classes is possible and entails a slight generalization with respect to that of the trivial class, in the sense that the classification should be done using the deformed coadjoint action (associated with the central extension determined by the non-trivial class we are considering).

3 Symplectic Cohomology

In the previous section we have seen how GAQ can be used to define symplectic structures out of a Lie group, naturally leading to the notion of pseudo-cohomology. In this section we review a different approach to the discussion of the symplectic structures defined in terms of a Lie group $G$. Firstly, we briefly recall the fundamentals of the so-called symplectic cohomology of $G$. The rationale for this structure can be found in the context of momentum mapping ([3] and

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3 Or, equivalently, the functions of the Hilbert space carrying the representation should be redefined multiplying them by an appropriate factor.
below). Secondly, we use this mathematical structure to classify a family of symplectic spaces which generalize the ones obtained by the coadjoint action of a group $G$.

### 3.1 Lie group cohomology. Symplectic cohomology

Given a Lie group $G$, an abelian Lie group $A$ and a (left) action $L$ of $G$ on $A$, we define the $n$-cochains $\gamma_n$ as mappings

$$\gamma_n : G \times \ldots \times G \to A ,$$

in such a way that the standard sum of mappings

$$(\gamma_n + \gamma'_n)(g_1, \ldots, g_n) = \gamma_n(g_1, \ldots, g_n) + \gamma'_n(g_1, \ldots, g_n) .$$

ends the space of $n$-cochains, denoted as $C^n_L(G, A)$, with the structure of an abelian group. The coboundary operators $\delta : C^n_L(G, A) \to C^{n+1}_L(G, A)$ are defined by

$$(\delta \gamma_n)(g_1, \ldots, g_n, g_{n+1}) \equiv L(g_1)(\gamma_n(g_2, \ldots, g_n, g_{n+1}) +$$

$$+ \sum_{i=1}^{n} (-1)^i \gamma_n(g_1, \ldots, g_ig_{i+1}, g_{i+2}, \ldots, g_{n+1}) +$$

$$+ (-1)^{n+1}\gamma_n(g_1, \ldots, g_n) ,$$

satisfying the nilpotency condition $\delta \circ \delta = 0$. We can define the subspaces of $n$-cochains $Z^n \equiv Ker(\delta) \subset C^n_L(G, A)$, whose elements are closed $n$-cochains and $B^n \equiv Im(\delta) \subset C^n_L(G, A)$, whose elements are $n$-coboundaries. Two exact $n$-cochains are equivalent if their difference is a coboundary. Cohomology groups are defined by this equivalence

$$H^n_L(G, A) = \frac{Z^n}{B^n}$$

and their elements are called $n$-cocycles. For the first cohomology groups the expression of $\delta$ takes the form,

$$\begin{align*}
(\delta \gamma_0)(g) & = L(g_1)\gamma_0 - \gamma_0 \\
(\delta \gamma_1)(g_1, g_2) & = L(g_1)\gamma_1(g_2) - \gamma_1(g_1g_2) + \gamma_1(g_1) \\
(\delta \gamma_2)(g_1, g_2, g_3) & = L(g_1)\gamma_2(g_2, g_3) + \gamma_2(g_1, g_2g_3) - \gamma_2(g_1g_2, g_3) - \gamma_2(g_1, g_2)
\end{align*}$$

As we will see below, the generalization of the coadjoint action and its associated orbits naturally involves a cohomological structure. In order to address this point, we consider the general elements above and choose $A = G^*$ and $L = Coad$, i.e. the coadjoint action of $G$ on $G^*$. This choice leads in particular to the cohomology group $H^1_{Coad}(G, G^*)$, where a 1-cocycle $\gamma : G \to G^*$ is characterized by ($\delta \gamma \equiv 0$)

$$\gamma(g'g) = Coad(g')\gamma(g) + \gamma(g') ,$$

\(9\)
meanwhile a 1-coboundary has the form \((\Delta_\mu \equiv \delta \mu)\)

\[
\Delta_\mu = \text{Coad}(g)\mu - \mu , \quad g \in G, \mu \in G^*.
\] (21)

Symplectic cohomology \(H_S(G, G^*)\) is defined out of this cohomology group by restricting the 1-cocycles to functions \(\gamma\) which satisfy the following antisymmetry condition on its differential \(\gamma^T\):

\[
\begin{align*}
\gamma^T(e)(X,Y) &\equiv \gamma^T(e) \cdot X(Y) \\
\gamma^T(e)(X,Y) &= -\gamma^T(e)(Y,X) \quad \forall X,Y \in G.
\end{align*}
\] (22)

The reason for this condition will be apparent in the next subsection.

For the sake of completeness, we mention that the cohomology group \(H^2(G, U(1))\) we found in the previous section and which classifies the central extensions of the Lie group \(G\), is obtained by setting \(A = U(1)\) and \(L\) as the trivial representation in the general construction above of Lie group cohomology. Second and third lines in (19) then define the expression of a coboundary and the cocycle condition.

### 3.2 Deformed coadjoint orbits

As we have seen in Section 2.1, orbits of the coadjoint action of a group \(G\) on its coalgebra \(G^*\) constitute a class of symplectic manifolds characterized in terms of group-theoretical structures. Symplectic cohomology provides a way of introducing a notion of affine-deformations of coadjoint actions which allow us to generalize the notion of coadjoint orbit. Defining the mapping \(g \mapsto \text{Coad}_\gamma(g)\)

\[
\text{Coad}_\gamma(g)\mu_0 \equiv \text{Coad}(g)\mu_0 + \gamma(g) \quad \mu_0 \in G^* ,
\] (23)

the condition for this expression to actually define a (left) action of \(G\) on \(G^*\), i.e. \(\text{Coad}_\gamma(g^*g)\mu = \text{Coad}_\gamma(g')(\text{Coad}_\gamma(g)\mu)\), reduces to expression (20), which is simply the cocycle condition in \(H_S(G, G^*)\).

On the other hand, and denoting the orbit of \(\text{Coad}_\gamma\) through the point \(\mu_0 \in G^*\) by \(\text{Orb}_\gamma(\mu_0)\), we note that \(\gamma\) functions which differ by a coboundary, (21), define the same set of orbits. In fact, since

\[
\begin{align*}
\text{Coad}_{\gamma+\Delta_\mu}(g)\mu_0 &= \text{Coad}_\gamma(g)\mu_0 + \text{Coad}(g)\mu - \mu \\
&= \text{Coad}_\gamma(g)(\mu_0 + \mu_0) - \mu \quad \forall g \in G, \forall \mu_0 \in G^*,
\end{align*}
\] (24)

we realize that \(\text{Orb}_{\gamma+\Delta_\mu}(\mu_0)\) and \(\text{Orb}_\gamma(\mu_0 + \mu)\) coincide modulo a translation \(\mu\). Therefore, if we allow \(\mu\) to vary on \(G^*\), each element in \(H^1_{\text{Coad}}(G, G^*)\) characterizes a family of orbits (modulo translations) obtained from the deformed coadjoint action on \(G^*\).

Finally, the antisymmetry condition on \(\gamma^T(e)\) is necessary in order to define a symplectic structure on \(\text{Orb}_\gamma(\mu)\). If we define

\[
\Gamma(X, Y) \equiv \gamma^T(e) \cdot X(Y)
\] (25)
the following theorem follows (33).

**Theorem.** The orbit $Orb_\gamma(\mu) \subset \mathcal{G}^*$ admits a symplectic form $\omega$ which is pointwise given by

$$\omega_\nu(X, Y) = \nu([X_\mathcal{G}, Y_\mathcal{G}]) + \Gamma(X_\mathcal{G}, Y_\mathcal{G}), \quad \nu \in Orb_\gamma(\mu), X, Y \in T_\nu Orb_\gamma(\mu),$$

(26)

where $\nu \in Orb_\gamma(\mu)$, $X, Y \in T_\mu Orb_\gamma(\mu)$ and $X_\mathcal{G} \in \mathcal{G}$ is related to $X \in T_\mu Orb_\gamma(\mu)$ by $X = coad_\gamma X_\mathcal{G}(\nu)$ and analogously for $Y$ and $Y_\mathcal{G}$ (where $coad_\gamma \equiv (Coad_\gamma)^T(e)$).

3.3 Convergence with the problem of central extensions

In Section 2 the techniques of GAQ were used in order to define a specific symplectic structure that could be used as the support for the Hamiltonian description of a classical system. The algorithm started from a $U(1)$-centrally extended Lie group $\tilde{G}$, where the 2-cocycle which defines the central extension permits the identification of the set of variables building the sought phase space (for concreteness, those coordinates associated with non-vertical vector fields which are absent from the characteristic module of $\Theta \equiv \mathfrak{g}^{L(C)}$). However, the object classifying the non-isomorphic $U(1)$-central extensions of $G$, $H^2(G, U(1))$, is not fine enough in order to classify the specific symplectic spaces, since some ambiguity still remains linked to the choice of the particular coboundary for the 2-cocycle.

In an analogous manner, the approach followed in this Section, based on deformed coadjoint actions, permits the classification of the different classes of deformed coadjoint orbits by the elements of $H_S(G, \mathcal{G}^*)$, but not the characterization of individual symplectic spaces.

Therefore, the crucial mathematical structures of both approaches, $H^2(G, U(1))$ and $H_S(G, \mathcal{G}^*)$ respectively, need to be refined in order to account for such specific symplectic manifolds.

Even at this intermediate step, a non-trivial convergence occurs between the conceptually different problems of classifying the central extensions of a given Lie group $G$ by $U(1)$, on the one hand, and the affine deformations of the coadjoint actions on $\mathcal{G}^*$, on the other hand. In fact, the same object classifies the solutions to both problems, since $H^2(G, U(1)) \approx H_S(G, \mathcal{G}^*)$.

Although we shall dwell on this point in subsection 3.3.2., we can outline this equivalence by noting that, for simply connected groups, the isomorphism $H^2(\mathcal{G}, U(1)) \approx H^2(G, U(1))$ is satisfied and therefore it is enough to discuss the equivalence at the infinitesimal level. In fact, the cocycle condition (20) implies the following condition on its differential $\Gamma(X, Y)$

$$\Gamma([X, Y], Z) + \Gamma([Y, Z], X) + \Gamma([Z, X], Y) = 0,$$

(27)

which, together with the antisymmetry condition (22), $\Gamma(X, Y) = -\Gamma(Y, X)$ defines a 2-cocycle in $H^2(\mathcal{G}, U(1))$ (from the point of view of the central extensions of the Lie algebra $\mathcal{G}$) is simply the Jacobi identity for the central generator in the Lie algebra; see [14]). Likewise the infinitesimal expression of the coboundary condition (21) implies,

$$\Gamma_{cob}(X, Y) = \mu([X, Y]) \quad \text{for some} \quad \mu \in \mathcal{G}^*,$$

(28)

On behalf of concision, we avoid a presentation of Lie algebra cohomology and refer the reader to standard references like [14]. We simply note that for simply connected groups, Lie algebra cohomology emerges as an infinitesimal version of Lie group cohomology.
which is the coboundary condition in $H^2(\mathcal{G}, U(1))$.

In section 2 pseudo-extensions have been introduced as the element necessary to account for the specific symplectic manifolds, that we can construct out of a Lie group $G$ via a central extension of it. However the discussion was carried out only for the trivial class of $H^2(G, U(1))$. For the non-trivial cohomology classes the analysis was not so straightforward. However the convergence with the approach based on symplectic cohomology, and which aims directly at the problem of defining symplectic structures completely in terms of a Lie group, sheds a new light on the problem. From this perspective, the characterization of a specific symplectic structure for a (in general non-trivial) cohomology class $\gamma$ of $H^2(\mathcal{G}, U(1)) \approx H^2(G, U(1))$, simply parallels the characterization of a particular orbit in the family of orbits defined by $\text{Coad}_\gamma$.

### 3.3.1 Singularization of coadjoint orbits in symplectic cohomology

In order to singularize a specific symplectic manifold out of the family defined by a cocycle in $H_S(G, \mathcal{G}^*)$, i.e. in order to characterize a particular deformed coadjoint orbit, we have two options:

1. We can fix a pair $(\gamma, \mu_0)$, where $\gamma$ specifies the cocycle which defines the deformation of the action and $\mu_0$ precises a point in the orbit. In this case, varying the second entry we scan all the possible orbits.

2. Alternatively, we can fix the point $\mu_0$ in the coalgebra and vary instead the representative of the cocycle $\gamma$ by modifying the coboundary, $\Delta_\mu$. Since the coalgebra is a linear space, there is a canonical choice for the fixed point $\mu_0$: the zero vector. We can see from expression (24) that the set of spaces constructed this way is the same that the one derived with option i), although translated with respect to them in such a way that all these spaces share the zero vector in $\mathcal{G}^*$.

However both characterizations are redundant since different pairs $(\gamma, \mu_0)$, or alternatively different specific representatives $\gamma + \Delta_\mu$, give rise essentially to the same orbits. Therefore it is necessary to establish an equivalence relationship in order to eliminate this ambiguity. The analysis of section 2.1. establishing the relationship between specific symplectic structures and pseudo-cohomology understood as a refinement of a true cohomology, suggest us to choose the characterization ii) for the deformed orbits. In fact, it directly leads to a refinement of symplectic cohomology, intrinsically tied to group cohomology.

In this sense we have to determine under which conditions two coboundaries $\Delta_\mu$ and $\Delta_\mu'$ generate the same orbit. A direct computation shows that if there exists an element $h \in G$ such that $\mu = \text{Coad}_\gamma(h)\mu'$ (that is, if $\mu$ and $\mu' \in \mathcal{G}^*$ belong to the same $\gamma$-orbit) then

$$\text{Coad}_{\gamma + \Delta_\mu}(g)0 = \text{Coad}_{\gamma + \Delta_\mu'}(gh)0 + \mu' - \mu \quad \forall g \in G.$$  \hspace{1cm} (29)

Since $\mu' - \mu$ is independent of $g$, spaces spanned by the action of $\text{Coad}_{\gamma + \Delta_\mu}$ and $\text{Coad}_{\gamma + \Delta_\mu'}$ through the zero in $\mathcal{G}^*$ coincide, modulo a rigid translation. These orbits are trivially symplectomorphic, the symplectomorphism being this translation in $\mathcal{G}^*$.
Summarizing with the language of symplectic cohomology, individual symplectic spaces associated with deformed coadjoints actions are classified by refinement of symplectic cohomology in such a way that two coboundaries $\Delta_\mu$ and $\Delta_{\mu'}$ are equivalent if $\mu$ and $\mu'$ belong to the same $\gamma-$orbit. In other words, these individual symplectic spaces are classified by elements $\mu \in \mathcal{G}^*$ modulo the corresponding $\gamma$-deformed coadjoint action. Note the similarity with Definition 1, to which it directly generalizes in the context of deformed coadjoint orbits.

### 3.3.2 Pseudo-cohomology from symplectic cohomology

In this section we see in a more systematic way the close relation between pseudo-cohomology and symplectic cohomology for the non-trivial classes ($H^2(G, U(1)) \neq 0$). The idea is to investigate how the coadjoint action $\text{Coad}$ of $G$ on $\mathcal{G}^*$ is modified by a central extension. The result is that when $G$ is centrally extended by a 2-cocycle $\xi$, the coadjoint action of the extended group $\widetilde{G}$, denoted by $\widetilde{\text{Coad}}$, acting on $\widetilde{\mathcal{G}}^* = \mathcal{G}^* \times \mathbb{R}$, turns out to be

$$\widetilde{\text{Coad}}(\tilde{g})\tilde{\mu} = (\text{Coad}(g)\mu + \mu\zeta F(g), \mu\zeta)$$

(30)

where $\tilde{g} = (g, \zeta) \in \widetilde{G}$, $\zeta \in U(1)$ and $\tilde{\mu} = (\mu, \mu\zeta) \in \widetilde{\mathcal{G}}^*$. Here $F(g) \in \mathcal{G}^*$, and it is related to the 2-cocycle $\xi$ through the quantization 1-form $\Theta$, by $F_i(g) = i\tilde{\chi}_i^\Theta$. These functions are nothing other than the Noether invariants of the classical theory [9]. Observe that $\widetilde{\text{Coad}}(\tilde{g})$ does not depend on $\zeta$, and that $\mu\zeta$ does not change by this extended action (these two facts are related to the central character of $U(1)$). Since the case $\mu\zeta = 0$ reproduces the original coadjoint action $\text{Coad}$ of $G$, let us suppose $\mu\zeta \neq 0$.

From (30) it can be derived that $\widetilde{\text{Coad}}(g)$ can be restricted to the foliations of $\widetilde{\mathcal{G}}^*$ of constant $\mu\zeta$, which can be identified with $\mathcal{G}^*$. Since $\widetilde{\text{Coad}}$ is an action, so it is its restriction, and this implies that $F(g)$ must verify the condition (this relation can also be checked by direct computation):

$$F(g'g) = \text{Coad}(g')F(g) + F(g')$$

(31)

Therefore Noether invariants are nothing other than 1-cocycles for the coadjoint action $\text{Coad}$ of $G$. Even more, they are symplectic, since its differential at the identity is precisely the Lie algebra 2-cocycle. Therefore, $\widetilde{\text{Coad}}$ can be identified with a deformed coadjoint action $\text{Coad}_\gamma$, with $\gamma(g) = \mu\zeta F(g)$. Without losing generality, we can take $\mu\zeta = 1$. Let us see what happens to $\text{Coad}_\gamma$ when we add to $\xi$ a coboundary $\xi_\lambda$ generated by $\lambda(g)$. A simple calculation shows that $\gamma$ changes to $\gamma' = \gamma + \gamma_\lambda$, where $\gamma_\lambda$ is given by:

$$\gamma_\lambda(g) = \text{Coad}(g)\lambda^0 - \lambda^0$$

(32)

Surprisingly, $\gamma_\lambda$ is a symplectic coboundary, associated with $\lambda^0 \in \mathcal{G}^*$, and, what is more important, it depends just on $\lambda^0$, not on the particular choice of $\lambda$. This simple relation has deep consequences since it provides the close relation between pseudo-cocycles and symplectic cohomology. It also guides us in the correct definition of subclasses of pseudo-cocycles for the
non-trivial case, using the characterization of single coadjoint orbits found in the symplectic cohomology setting (see Sec. 3.3.1).

According to this, and since the quantization 1-form $\Theta$ for any central extension $\tilde{G}$ characterized by the 2-cocycle $\xi$ always verifies $\Theta|_e = (0, 0, \ldots, 1)$ (that is, $\mu = 0$ and $\mu_\xi = 1$), we can singularize a deformed orbit in $G^*$ by considering

$$\tilde{\text{Coad}}(g) \Theta|_e = (\text{Coad}_\gamma(g) 0, 1) = (\text{Coad}(g) 0 + F(g), 1) = (F(g), 1)$$

(33)

That is, this orbit is the image of the Noether invariants. This fact simply affirms that Noether invariants parameterize classical phase spaces.

The question now is that if we add to the 2-cocycle $\xi$ a pseudo-cocycle $\xi^\lambda$ generated by $\lambda(g)$ with gradient at the identity $\lambda^0$, does it define a new deformed coadjoint orbit? Can we define an equivalence relation among pseudo-cocycles as for the case of the trivial class?

Again, from the symplectic cohomology framework (see 3.3.1), we have the answer. Firstly we define,

**Definition 2:** Two coboundaries $\xi^\lambda$ and $\xi^{\lambda'}$ with generating functions $\lambda$ and $\lambda'$, respectively, define two cocycles $\xi + \xi^\lambda$, $\xi + \xi^{\lambda'}$ belonging to the same equivalence subclass $[\xi]$ of $[[\xi]]$ if and only if the gradients at the identity of the generating functions are related by:

$$\lambda^{0'} = \text{Coad}_\gamma(g) \lambda^0,$$

(34)

for some $g \in G$, where $\text{Coad}_\gamma$ stands for the deformed coadjoint action, which is equivalent to the coadjoint action $\tilde{\text{Coad}}$ of $\tilde{G}$ on $\tilde{G}^* = G^* \times \mathbb{R}$, where $\tilde{G}$ is the central extension associated with the two-cocycle $\xi$.

According to this definition, equivalent pseudo-extensions (for the non-trivial class $[[\xi]]$) are determined by generating functions whose gradient at the identity lie in the same coadjoint orbit of $\tilde{G}$.

The ultimate justification of this definition is the following proposition:

**Proposition 3.** Given a Lie group $G$ and a 2-cocycle $\xi$ on $G$, consider the two coboundaries $\xi_{\lambda_1}$ and $\xi_{\lambda_2}$ with generating functions $\lambda_1(g)$ and $\lambda_2(g)$. Define the central extensions $\tilde{G}_1$ and $\tilde{G}_2$ characterized by the two-cocycles $\xi + \xi_{\lambda_1}$ and $\xi + \xi_{\lambda_2}$, respectively, and construct the Quantization one-forms $\Theta_1 = \Theta + \Theta_{\lambda_1}$ and $\Theta_2 = \Theta + \Theta_{\lambda_2}$, following expressions (2) and (3). The two symplectic spaces $\tilde{G}_1/(G_{\Theta_1} \times U(1))$ and $\tilde{G}_2/(G_{\Theta_2} \times U(1))$, with symplectic forms $\omega_1$ and $\omega_2$, such that $d\Theta_1 = \pi^*\omega_1$ and $d\Theta_2 = \pi^*\omega_2$, respectively, are symplectomorphic if there exists $h \in G$ such that:

$$\lambda^o_1 = \text{Coad}_\gamma(h) \lambda^o_2,$$

(35)

the symplectomorphism being given by $\tilde{\text{Ad}}(\tilde{h})$:

$$d\Theta_1 = (\tilde{\text{Ad}}(\tilde{h}))^*d\Theta_2$$

(36)

where $\tilde{h}$ is such that $p(\tilde{h}) = h$, with $p : \tilde{G} \to G$ the canonical projection.
Proof: Even though the result can be shown by direct calculation, the most straightforward derivation comes from splitting the central extension into two steps. Firstly, the central extension by \( \xi \) alone is constructed, and this group \( \tilde{G} \) is taken as the departing point for a second trivial extension by \( \xi_{\lambda_1} \) and \( \xi_{\lambda_2} \). The study of the trivial class in \( \tilde{G} \) amounts for the study of that non-trivial class in \( G \) characterized by the cocycle \( \xi \). At this point we can apply Proposition 1 to the trivial extension of \( \tilde{G} \) and then take advantage of the identification between the non-trivial part (\( G^* \)-component in \( \tilde{G}^* \)) of \( \tilde{Coad} \) in \( \tilde{G} \) and \( Coad_\gamma \) in \( G \), which follows from expression (30) and its subsequent discussion. This leads directly to the claimed result.

As in the case of the trivial class, the correspondence between pseudo-cohomology classes in \( [[\xi]] \) and “deformed” coadjoint orbits in \( G^* \) is not onto. Only when we demand these coadjoint orbits to satisfy the integrality condition (that is, to be quantizable), the correspondence with pseudo-cohomology classes is one-to-one. The proof of this statement is the same as in the trivial case, see [9].

4 Final remarks

In this paper we have established a neat characterization of the concept of pseudo-cohomology as the mathematical object classifying the single \( G \)-symplectic spaces that can be constructed out of a Lie group \( G \). The role of symplectic cohomology has been crucial in this analysis: a) on the one hand it provides a clearer setting for the problem than the one based on central extensions; b) on the other hand, it offers a straightforward bridge for the translation of the results into the language of central extensions.

This characterization is something more than an academic problem, since these symplectic spaces constitute the classical phase spaces of the quantum theories associated with a fundamental symmetry. The a \textit{priori} knowledge of the available classical structures provides a most valuable information in the study of the quantum theory. In this sense, and although this paper focuses on the discussion of classical structures, a remark on their quantum counterparts is in order. In fact, once the classification of symplectic spaces (deformed coadjoint orbits) associated with a symmetry group has been done by means of pseudo-cohomology, the question on the existence of non-equivalent quantizations corresponding to a given coadjoint orbit \( S \) naturally arises. As is well-known from Geometric Quantization [8] (see also [10]) such a variety of non-isomorphic quantum manifolds is classified by \( \pi_1^*(S) \), i.e. the dual group of the first homotopy group of the classical phase space. Although this problem goes beyond the scope of the present work, where we are interested in the classification of symplectic spaces, not in their quantization, let us remark that when considering multiply connected coadjoint orbits there exists the possibility of finding pseudo-cocycles associated with the same coadjoint orbit and which leads to non-equivalent representations (see the end of Sec. 2.1 for the relation between pseudoextensions and quantization), hence to non-equivalent quantizations. These pseudo-cocycles are generated by non-homotopic functions having the same gradient at the identity. An example of this situation can be found in the case of the \( SL(2, \mathbb{R}) \) group which admits two non-equivalent classes of unirreps associated with the multiply connected coadjoint orbits [15]. A precise analysis of this example can be seen in [16]. As a consequence, the
classification of non-equivalent representations associated with the same coadjoint orbit would require a further refinement in the characterization of pseudo-cohomology classes.

It should be stressed that although pseudo-cohomology with values on $U(1)$ classifies quantizable $G$-symplectic manifolds through the integrality condition, general classical $G$-symplectic manifolds can be regained by considering pseudo-cohomology with values on the additive group $R$, rather than $U(1)$. In fact, the Group Approach to Quantization recovers Classical Mechanics, in the Hamilton-Jacobi version, by just considering the additive group $\mathbb{R}$ instead of the multiplicative one $U(1)$, the former being a local approximation to the latter.

Finally, and coming back to the original motivation for pseudo-cohomology in terms of İnönü-Wigner contractions, let us remark that, conversely, given a pseudo-cohomology class and the corresponding quantization group $\tilde{G}$, with quantization form $\Theta$, an İnönü-Wigner contraction with respect the characteristic subgroup $G_\Theta$ of $\Theta$ automatically leads to a contracted group $\tilde{G}_c$ which proves to be a non-trivial extension by $U(1)$ of the contraction $G_c$ of $G$ by the same subgroup $G_\Theta$.

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