On the second moment of $S(T)$ in the theory of
the Riemann zeta function

Tsz Ho Chan

March 29, 2022

Abstract

We assume the Riemann Hypothesis and a quantitative form of the
Twin Prime Conjecture, and obtain an asymptotic formula for the second
moment of $S(T)$ with better error term.

1 Introduction

Let $\rho = \beta + i\gamma$ be the zeros of the Riemann zeta function $\zeta(s)$. For $T \neq \gamma$,

$$S(T) = \frac{1}{\pi} \arg \zeta(\frac{1}{2} + iT),$$

where the argument is obtained by continuous variation along the horizontal
line $\sigma + iT$ starting with the value zero at $\infty + iT$. For $T = \gamma$, we define

$$S(T) = \lim_{\epsilon \to 0} \frac{1}{2} \{S(T + \epsilon) + S(T - \epsilon)\}.$$ 

In [3], Goldston proved that under the Riemann Hypothesis,

$$\int_0^T |S(t)|^2 dt = \frac{T}{2\pi^2} \log \log T + \frac{T}{2\pi^2} \left[ \int_1^\infty \frac{F(\alpha,T)}{\alpha^2} + C_0 \right.$$

$$+ \sum_{m=2}^\infty \sum_p \left( \frac{-1}{m} + \frac{1}{m^2} \right) \frac{1}{p^m} \left. \right] + o(T).$$ (1)

Here $C_0$ is Euler’s constant, and

$$F(\alpha) = F(\alpha,T) = \left( \frac{T}{2\pi} \log T \right)^{-1} \sum_{\substack{0<\gamma,\gamma'\leq T \\gamma' < \gamma}} T^{i\alpha(\gamma-\gamma')} w(\gamma - \gamma')$$ (2)

with $w(u) = 4/(4 + u^2)$ is Montgomery’s pair correlation function. Here and throughout this paper, $p$ will denote a prime and sums over $p$ are over all primes.
In [1] and [2], the author assumed the Riemann Hypothesis and the following quantitative form of Twin Prime Conjecture: For any $\epsilon > 0$,

$$\sum_{n=1}^{N} \Lambda(n)\Lambda(n + d) = \mathcal{G}(d)N + O(N^{1/2+\epsilon})$$

uniformly in $|d| \leq N$. $\Lambda(n)$ is the von Mangoldt lambda function. $\mathcal{G}(d) = 2 \prod_{p>2} (1 - 1/(p-1)^2) \prod_{p|d, p>2} (p-1)/(p-2)$ if $d$ is even, and $\mathcal{G}(d) = 0$ if $d$ is odd.

He proved that, for any $\epsilon > 0$,

$$F(\alpha, T) = \begin{cases} \alpha + \frac{1}{\log T} \left[ \log T - 2 \log 2\pi - 2 \right] + O(\alpha T^{\alpha-1}) + O\left(\frac{1}{T^{(1/2-\epsilon)/\log T}}\right), & \text{if } 0 \leq \alpha \leq 1 - \frac{3 \log \log T}{\log T}, \\ \alpha + O\left(\frac{T^{\alpha-1}}{\log T}\right), & \text{if } 1 - \frac{3 \log \log T}{\log T} \leq \alpha \leq 1. \end{cases}$$

Using this and more careful calculations, we have

**Theorem 1.1.** Assume the Riemann Hypothesis and the above Twin Prime Conjecture. We have

$$\int_0^T |S(t)|^2 dt = \frac{T}{2\pi^2} \log \log T + \frac{T}{2\pi^2} \left[ \int_1^\infty \frac{F(\alpha, T)}{\alpha^2} d\alpha + C_0 ight. \\
- \sum_{m=2}^\infty \sum_p \left( \frac{1}{m} - \frac{1}{m^2} \right) \frac{1}{p^m} \left( \frac{1}{\log^2 T} \right) + O\left( \frac{T}{\log^2 T} \right).$$

So, there appears to be no $T/\log T$ term. However, one has to be careful about this because we still do not know anything very precise about

$$\int_1^\infty \frac{F(\alpha, T)}{\alpha^2} d\alpha.$$

Goldston [3] proved that under the Riemann Hypothesis, for any $\epsilon > 0$,

$$\frac{2}{3} - \epsilon < \int_1^\infty \frac{F(\alpha, T)}{\alpha^2} d\alpha < 2.$$

Montgomery [4] conjectured that

$$F(\alpha, T) = 1 + o(1)$$

uniformly for $1 \leq \alpha \leq M$.

for any fixed $M$. This implies

$$\int_1^\infty \frac{F(\alpha, T)}{\alpha^2} d\alpha = 1 + o(1)$$

which gives an error term $o(T)$ for our Theorem 1.1. In order to get better error terms, one has to understand $F(\alpha, T)$ better for larger range of $\alpha$, say $1 \leq \alpha \leq \log T$. This would be a great challenge.
2 Preparations

Essentially, our proof follows that of Goldston [3]. We shall recall some of his results.

Lemma 2.1. Assume the Riemann Hypothesis. For \( t \geq 1, t \neq \gamma, x \geq 4 \), we have

\[
S(t) = -\frac{1}{\pi} \sum_{n \leq x} \Lambda(n) \sin (t \log n) \frac{f(\log n/\log x)}{n^{1/2}} + \frac{1}{\pi} \sum_\gamma \sin ((t-\gamma) \log x) \int_0^\infty \frac{u}{u^2 + ((t-\gamma) \log x)^2} \sinh u \, du \tag{4}
\]

\[
+ O\left(\frac{1}{t \log x}\right) + O\left(\frac{\pi^4}{t^2 \log x}\right) + O\left(1/t \log x\right),
\]

where

\[
f(u) = \frac{\pi}{2} u \cot \left(\frac{\pi}{2} u\right),
\]

and \( \Lambda(n) = \log p \) if \( n = p^m \), for \( p \) a prime and \( m \geq 1 \), and \( \Lambda(n) = 0 \) otherwise.

Proof: This is Lemma 1 in [3].

Lemma 2.2. For \( x \geq 4 \) and \( T \geq 2 \),

\[
R = \int_1^T \left| \frac{1}{\pi} \sum_\gamma \sin ((t-\gamma) \log x) \int_0^\infty \frac{u}{u^2 + ((t-\gamma) \log x)^2} \sinh u \, du \right|^2 \, dt
\]

\[
= \frac{1}{\pi^2 \log x} \sum_{\gamma, \gamma'} k((\gamma - \gamma') \log x) + O(\log^3 T),
\]

where

\[
k(u) = \begin{cases} \left(\frac{1}{2\pi} - \frac{\pi^2}{2} \cot (\pi^2 u)\right)^2, & \text{if } |u| \leq \frac{1}{2\pi} \\ \frac{1}{2\pi}, & \text{if } |u| > \frac{1}{2\pi}. \end{cases}
\]

Proof: This is Lemma 2 in [3].

Lemma 2.3.

\[
k'(0) = 0, \quad k''(0) = \frac{\pi^8}{18},
\]

\[
k'(\frac{1}{2\pi}) = -4\pi^3, \quad k'\left(\frac{1-}{2\pi}\right) = -4\pi^3 + \pi^5,
\]

\[
k''\left(\frac{1}{2\pi}\right) = 24\pi^4, \quad k''\left(\frac{1-}{2\pi}\right) = \frac{\pi^8}{2} - 4\pi^6 + 24\pi^4.
\]

Proof: Straightforward calculations.
Lemma 2.4.

\[ \hat{k}(y) = -\frac{1}{(2\pi y)^2} \int_{-\infty}^{\infty} k''(u)e(-uy)du + \frac{\pi^3}{2y^2} \cos y, \]

where \( \hat{f} \) is the Fourier transform of \( f \),

\[ \hat{f}(y) = \int_{-\infty}^{\infty} f(u)e(-uy)du, \quad e(u) = e^{2\pi iu}. \]

Proof: Note that \( k(u) \ll \min(1, 1/u^2) \), \( k'(u) \ll \min(1, 1/u^3) \) and \( k''(u) \ll \min(1, 1/u^4) \) except at \( u = 1/2\pi \). Integrating by parts twice,

\[ \hat{k}(y) = \int_{-\infty}^{\infty} k(u)e(-uy)du \]

\[ = -\frac{1}{2\pi iy} \int_{-\infty}^{\infty} k(u)de(-uy) = \frac{1}{2\pi iy} \int_{-\infty}^{\infty} e(-uy)k'(u)du \]

\[ = -\frac{1}{(2\pi iy)^2} \int_{-\infty}^{\infty} k'(u)de(-uy) = \frac{1}{(2\pi iy)^2} \int_{-\infty}^{\infty} e(-uy)dk'(u) \]

\[ = \frac{1}{(2\pi iy)^2} \int_{-\infty}^{\infty} k''(u)e(-uy)du \]

\[ + \left(k'\left(\frac{1}{2\pi}\right) - k'\left(-\frac{1}{2\pi}\right)\right)e\left(\frac{y}{2\pi}\right) + \left(k'\left(-\frac{1}{2\pi}\right) - k'\left(-\frac{1}{2\pi}\right)\right)e\left(-\frac{y}{2\pi}\right) \]

\[ = -\frac{1}{(2\pi y)^2} \int_{-\infty}^{\infty} k''(u)e(-uy)du + \frac{\pi^3}{2y^2} \cos y \]

by Lemma 2.3 and the fact that \( k(u) \) is even.

Our key improvement is the following

Lemma 2.5. Let \( x = T^\beta \). For any \( \beta > 0 \),

\[ \sum_{0 < \gamma, \gamma' \leq T} \hat{k}((\gamma - \gamma') \log x) \frac{(\gamma - \gamma')^2}{4 + (\gamma - \gamma')^2} \]

\[ = \frac{\pi^2 T}{16 \log T} \frac{F(\beta)}{\beta^2} - \frac{T}{64 \pi^4 \log T \beta^3} \int_{-\infty}^{\infty} F(\alpha)k''\left(\frac{\alpha}{2\pi \beta}\right)d\alpha, \]

where \( F(\alpha) \) is as defined in 4.
Proof: By Lemma 2.4, the above sum
\[
\sum_{0<\gamma,\gamma'\leq T} \cos \left( (\gamma - \gamma') \log x \right) \frac{1}{4 + (\gamma - \gamma')^2}
\]
\[
= \frac{\pi^3}{2(\log x)^2} \sum_{0<\gamma,\gamma'\leq T} \cos \left( (\gamma - \gamma') \log x \right) \frac{1}{4 + (\gamma - \gamma')^2}
\]
\[
= \frac{\pi^3}{8(\log x)^2} \sum_{0<\gamma,\gamma'\leq T} x^{(\gamma-\gamma')} w(\gamma - \gamma')
\]
\[
- \frac{1}{(4\pi \log x)^2} \int_{-\infty}^{\infty} \left( \sum_{0<\gamma,\gamma'\leq T} e(-u(\gamma - \gamma') \log x) w(\gamma - \gamma') \right) k''(u) du
\]
which gives the lemma by substituting \( \alpha = 2\pi u \beta \) in the integral.

Using Lemma 2.2 and Lemma 2.5, we have the following improvement of Lemma 3 in [3]:

**Lemma 2.6.** Let \( x = T^\beta \). Then for \( \beta > 0 \),
\[
R = \frac{T}{(2\pi^2 \beta)} \int_{-\infty}^{\infty} F(\alpha) k\left( \frac{\alpha}{2\pi \beta} \right) d\alpha + \frac{T}{6 \log T} \int_{-\infty}^{\infty} F(\alpha) k''\left( \frac{\alpha}{2\pi \beta} \right) d\alpha + O(\log^3 T).
\]

Proof: We simply note that
\[
\sum_{0<\gamma,\gamma'\leq T} \hat{k}(\gamma - \gamma') \log x = \sum_{0<\gamma,\gamma'\leq T} \hat{k}(\gamma - \gamma') \log x w(\gamma - \gamma')
\]
\[
+ \sum_{0<\gamma,\gamma'\leq T} \hat{k}(\gamma - \gamma') \log x \frac{(\gamma - \gamma')^2}{4 + (\gamma - \gamma')^2}
\]
Using Lemma 2.2 and Lemma 2.3, we get the second and third terms. Meanwhile,
\[
\sum_{0<\gamma,\gamma'\leq T} \hat{k}(\gamma - \gamma') \log x w(\gamma - \gamma')
\]
\[
= \int_{-\infty}^{\infty} k(u) \sum_{0<\gamma,\gamma'\leq T} e(-u(\gamma - \gamma') \log x) w(\gamma - \gamma') du
\]
\[
= \frac{T}{2\pi} \log T \int_{-\infty}^{\infty} F(2\pi u \beta) k(u) du
\]
\[
= \frac{T}{(2\pi)^2 \beta} \int_{-\infty}^{\infty} F(\alpha) k\left( \frac{\alpha}{2\pi \beta} \right) d\alpha.
\]
which accounts for the first term.
Lemma 2.7. For any $\beta > 0$,
\[ \int_0^\beta T^{-2\alpha} k''\left(\frac{\alpha}{2\pi \beta}\right) d\alpha = 16\pi^2 \beta^2 (\log T)^2 \int_0^\beta T^{-2\alpha} k\left(\frac{\alpha}{2\pi \beta}\right) d\alpha. \]

Proof: Integrating by parts twice.

Lemma 2.8. Assume the Riemann Hypothesis and Twin Prime Conjecture. For any $\epsilon > 0$ and $0 < \beta < 1$,
\[ \int_{-\infty}^\infty F(\alpha) k\left(\frac{\alpha}{2\pi \beta}\right) d\alpha = 2\pi^2 \beta^2 \left[ 1 - \frac{\pi^2}{8} + \log \frac{\pi}{2} + \int_1^\infty \frac{F(\alpha)}{\alpha^2} d\alpha - \log \beta \right] \]
\[ + 2(\log T + C) \int_0^\beta T^{-2\alpha} k\left(\frac{\alpha}{2\pi \beta}\right) d\alpha \]
\[ + O\left(\frac{1}{\beta^2 \log^4 T}\right) + O\left(\frac{\log T}{T^{(1/2-\epsilon)\beta}}\right) + O\left(\beta^2 \log^2 T\right) \]
with $C = -2 \log 2\pi - 2$.

Proof: Let $\epsilon_T = 3 \log \log T / \log T$. Since $F$ and $k$ are even,
\[ \int_{-\infty}^\infty F(\alpha) k\left(\frac{\alpha}{2\pi \beta}\right) d\alpha = 2 \left( \int_0^\beta + \int_0^{1-\epsilon_T} + \int_{1-\epsilon_T}^1 + \int_1^\infty \right) F(\alpha) k\left(\frac{\alpha}{2\pi \beta}\right) d\alpha \]
\[ = 2(I_1 + I_2 + I_3 + I_4) \]
From (3), with $C = -2 \log 2\pi - 2$,
\[ I_1 = \int_0^\beta \left[ \alpha + \frac{\log T + C}{T^{2\alpha}} \right] \left[ \frac{\pi \beta}{\alpha} - \frac{\pi^2}{2} \cot \left(\frac{\pi \alpha}{2\beta}\right) \right]^2 d\alpha \]
\[ + O\left( \int_0^\beta \alpha T^{\alpha-1} \alpha^2 d\alpha \right) + O\left( \int_0^\beta \frac{T^{-\epsilon_T} \alpha^2}{\beta^2} d\alpha \right) \]
\[ = \int_0^\beta \alpha \left[ \frac{\pi \beta}{\alpha} - \frac{\pi^2}{2} \cot \left(\frac{\pi \alpha}{2\beta}\right) \right]^2 d\alpha + (\log T + C) \int_0^\beta T^{-2\alpha} k\left(\frac{\alpha}{2\pi \beta}\right) d\alpha \]
\[ + O\left( \frac{T^{\beta-1}}{\beta^2 \log^4 T} \right) + O\left( \frac{1}{\beta^2 \log^4 T} \right) \]
because $\cot(x) = 1/x + O(x)$ when $0 \leq x \leq \pi/2$. The first integral is elementary to evaluate.
\[ I_1 = \pi^2 \beta^2 \left[ 1 - \frac{\pi^2}{8} + \log \frac{\pi}{2} \right] + (\log T + C) \int_0^\beta T^{-2\alpha} k\left(\frac{\alpha}{2\pi \beta}\right) d\alpha + O\left( \frac{1}{\beta^2 \log^4 T} \right). \]
By (3) again, we have,

\[
I_2 = \int_{\beta}^{1-\epsilon_T} \left[ \alpha + O(\alpha^\alpha - 1) + O\left(\frac{\log T}{T^{(1/2-\epsilon)\alpha}}\right) \right] \left(\frac{\pi \beta}{\alpha}\right)^2 \, d\alpha \\
= \pi^2 \beta^2 [\log (1 - \epsilon_T) - \log \beta] + O\left(\frac{1}{\log^4 T}\right) + O\left(\frac{\log T}{T^{(1/2-\epsilon)\beta}}\right)
\]

\[
I_3 = \int_{1-\epsilon_T}^{1} \left[ \alpha + O\left(\frac{\alpha^\alpha - 1}{\log T}\right) \right] \left(\frac{\pi \beta}{\alpha}\right)^2 \, d\alpha \\
= -\pi^2 \beta^2 \log (1 - \epsilon_T) + O\left(\frac{\beta^2}{\log^2 T}\right)
\]

Finally,

\[
I_4 = \pi^2 \beta^2 \int_{1}^{\infty} \frac{F(\alpha)}{\alpha^2} \, d\alpha.
\]

Combining the results for \(I_1, I_2, I_3\) and \(I_4\), we have the lemma.

**Lemma 2.9.** Assume the Riemann Hypothesis and Twin Prime Conjecture. For any \(\epsilon > 0\) and \(0 < \beta < 1\), where \(\beta = \log x/\log T\),

\[
\int_{-\infty}^{\infty} F(\alpha) k''\left(\frac{\alpha}{2\pi \beta}\right) d\alpha = 4\pi^6 \beta^2 - 24\pi^4 \beta^4 + 48\pi^4 \beta^4 \int_{1}^{\infty} F(\alpha) \alpha^4 \, d\alpha \\
+ 32\pi^2 \beta^2 (\log T)^2 (\log T + C) \int_{0}^{\beta} T^{-2\alpha} k\left(\frac{\alpha}{2\pi \beta}\right) \, d\alpha \\
+ O\left(\frac{1}{\log^2 T}\right) + O\left(\frac{\log T}{T^{(1/2-\epsilon)\beta}}\right)
\]

Proof: Let \(\epsilon_T = 3 \log \log T/\log T\). Again, since \(F\) and \(k\) are even,

\[
\int_{-\infty}^{\infty} F(\alpha) k''\left(\frac{\alpha}{2\pi \beta}\right) d\alpha = 2\left(\int_{0}^{\beta} + \int_{\beta}^{1-\epsilon_T} + \int_{1-\epsilon_T}^{1} + \int_{1}^{\infty}\right) F(\alpha) k''\left(\frac{\alpha}{2\pi \beta}\right) d\alpha \\
= 2(J_1 + J_2 + J_3 + J_4)
\]

By (3),

\[
J_1 = \int_{0}^{\beta} \alpha k''\left(\frac{\alpha}{2\pi \beta}\right) d\alpha + 16\pi^2 \beta^2 (\log T)^2 (\log T + C) \int_{0}^{\beta} T^{-\alpha} k\left(\frac{\alpha}{2\pi \beta}\right) \, d\alpha \\
+ O\left(\int_{0}^{\beta} \alpha T^{-\alpha - 1} \, d\alpha\right) + O\left(\int_{0}^{\beta} T^{-\left(1/2-\epsilon\right)\alpha} \, d\alpha\right) \\
= 4\pi^2 \beta^2 \int_{0}^{1/2\pi} u k''(u) \, du + 16\pi^2 \beta^2 (\log T)^2 (\log T + C) \int_{0}^{\beta} T^{-2\alpha} k\left(\frac{\alpha}{2\pi \beta}\right) \, d\alpha \\
+ O\left(\frac{T^{\beta-1}}{\log^2 T}\right) + O\left(\frac{1}{\log^2 T}\right)
\]
since $k''(x) \ll 1$ when $0 \leq x \leq 1/2\pi$. Using integration by parts twice and Lemma 2.3 to compute the first integral, we have
\[
J_1 = 2\pi^4(\pi^2 - 6)\beta^2 + 16\pi^2\beta^2(\log T)^2(\log T + C) \int_0^\beta T^{-2\alpha}k\left(\frac{\alpha}{2\pi\beta}\right) d\alpha
\]
\[+ O\left(\frac{1}{\log^2 T}\right).
\]
By (3) again, we have
\[
J_2 = \int_{\beta}^{1-\epsilon_T} \left[\alpha + O(\alpha T^{\alpha-1}) + O\left(\frac{\log T}{T^{(1/2-\epsilon)\alpha}}\right)\right] \left(\frac{24\pi^4\beta^4}{\alpha^4}\right) d\alpha
\]
\[= 12\pi^4\beta^4 \left[\frac{1}{\beta^2} - \frac{1}{(1 - \epsilon T)^2}\right] + O\left(\frac{1}{\log T}\right) + O\left(\frac{\log T}{T^{(1/2-\epsilon)\beta}}\right).
\]
\[
J_3 = \int_{1-\epsilon_T}^1 \left[\alpha + O\left(\frac{\log T}{T}\right)\right] \left(\frac{24\pi^4\beta^4}{\alpha^4}\right) d\alpha
\]
\[= 12\pi^4\beta^4 \left[\frac{1}{(1 - \epsilon T)^2} - 1\right] + O\left(\frac{\beta^4}{\log^2 T}\right) \int_{1-\epsilon T}^1 T^{\alpha-1} d\alpha
\]
\[= 12\pi^4\beta^4 \left[\frac{1}{(1 - \epsilon T)^2} - 1\right] + O\left(\frac{\beta^4}{\log^2 T}\right).
\]
Finally,
\[
J_4 = 24\pi^4\beta^4 \int_1^\infty \frac{F(\alpha)}{\alpha^4} d\alpha.
\]
Combining the results for $J_1$, $J_2$, $J_3$ and $J_4$, we have the lemma.

Combining Lemma 2.6, Lemma 2.8 and Lemma 2.9, we have Lemma 2.10.

**Lemma 2.10.** Assume the Riemann Hypothesis and Twin Prime Conjecture, for fixed $0 < \beta < 1$, where $\beta = \log x / \log T$,
\[
R = \frac{T}{2\pi^2} \left[1 - \frac{\pi^2}{8} + \log \frac{\pi}{2} + \int_0^\infty \frac{F(\alpha)}{\alpha^2} d\alpha - \log \beta\right] + \frac{3T}{8\pi^2 \log^2 T}
\]
\[\hspace{1cm} - \frac{3T}{4\pi^2 \log^2 T} \int_1^\infty \frac{F(\alpha)}{\alpha^4} d\alpha + O\left(\frac{T}{\log^2 T}\right) + O\left(\frac{T}{\beta^4 \log^4 T}\right).
\]

Note: This is a more precise version of Lemma 4 of [3]. Also, we keep some of the $T / \log^2 T$ terms explicit because one can actually make the $O(T / \log^2 T)$ error term $= C_1 T / \log^2 T + O(T \log \log T / \log^3 T)$ for some constant $C_1$ by using Theorem 1.1 in [2].

Following [3], we need to compute the mean value of the Dirichlet series in Lemma 2.11 and the cross term obtained from multiplying $S(t)$ with this series. Let
\[
G(T) = \int_1^T \left|\frac{1}{\pi} \sum_{n \leq x} \Lambda(n) \frac{\sin (t \log n)}{\log n} f\left(\frac{\log n}{\log x}\right)\right|^2 dt
\]
and
\[ H(T) = \frac{2}{\pi} \int_1^T S(t) \sum_{n \leq x} \frac{\Lambda(n) \sin(t \log n)}{n^{1/2} \log n} f\left(\frac{\log n}{\log x}\right) dt, \]
where \( f \) is defined as in (5). We need a lemma.

**Lemma 2.11.** For \( C \geq 2 \) and \( k \geq 1 \),
\[ \sum_{n=1}^{\infty} \frac{n^k}{C^n} \ll_k \frac{1}{C}. \]

Proof: First, we note that \( u^k C^{-u} \) is decreasing when \( u > \frac{k}{\log C} \). So,
\[ \sum_{n=1}^{\infty} \frac{n^k}{C^n} = \sum_{n=1}^{k/\log C} \frac{n^k}{C^n} + \sum_{n>k/\log C} \frac{n^k}{C^n} \leq \left(\frac{k}{\log 2}\right)^k \frac{1}{C-1} + \int_1^{\infty} u^k C^{-u} du \ll_k \frac{1}{C} + \frac{1}{\log C} C^{-1} \]
by integration by parts. This gives the lemma.

From p.165-166 of [3], we have, assuming the Riemann Hypothesis,
\[ G(T) = \frac{T}{2\pi^2} \sum_{n \leq x} \frac{\Lambda^2(n)}{n \log^2 n} f\left(\frac{\log n}{\log x}\right) + O(x^2), \] \hspace{1cm} (6)
\[ H(T) = -\frac{T}{\pi^2} \sum_{n \leq x} \frac{\Lambda^2(n)}{n \log^2 n} f\left(\frac{\log n}{\log x}\right) + O(x^{2+\epsilon}) \] \hspace{1cm} (7)
for any \( \epsilon > 0 \). Adding (6) and (7), we have
\[ G(T) + H(T) = \frac{T}{2\pi^2} \left[ \sum_{p \leq x} \frac{1}{p} f\left(\frac{\log p}{\log x}\right) - 2 \sum_{p \leq x} \frac{1}{p} f\left(\frac{\log p}{\log x}\right) - \sum_{m=2}^{\infty} \sum_{p^m \leq x} \frac{1}{m^2 p^m} \right. \]
\[ + \left. \sum_{m=2}^{\infty} \sum_{p^m \leq x} \frac{1}{m^2 p^m} \left(f\left(\frac{m \log p}{\log x}\right) - 1\right)^2 \right] + O(x^{2+\epsilon}) \]
\[ = \frac{T}{2\pi^2} [S_1 - 2S_2 - S_3 + S_4] + O(x^{2+\epsilon}). \]

\[ S_3 = \sum_{m=2}^{\infty} \sum_{p} \frac{1}{m^2 p^m} + O\left(\sum_{m=2}^{\infty} \sum_{n \geq x^{1/m}} \frac{1}{n^m}\right) \]
\[ = \sum_{m=2}^{\infty} \sum_{p} \frac{1}{m^2 p^m} + O\left(\sum_{m=2}^{\infty} \frac{1}{m^2 (m-1)x^{1-1/m}}\right) \]
\[ = \sum_{m=2}^{\infty} \sum_{p} \frac{1}{m^2 p^m} + O\left(\frac{1}{x^{1/2}}\right). \]
By Taylor’s expansion of $\tan x$, we have $f(u) = 1 + O(u^2)$ when $0 \leq u \leq 1$. Thus,

\[
S_4 \ll \frac{1}{\log^4 x} \sum_{m=2}^{\infty} m^2 \sum_{p^m \leq x} \frac{\log^4 p}{p^m} \\
\ll \frac{1}{\log^4 x} \sum_{m=2}^{\infty} m^2 \sum_{i=1}^{\infty} \sum_{2^{i+1} \leq p \leq 2^{i+1}} \frac{i^4}{2mi} \\
\ll \frac{1}{\log^4 x} \sum_{m=2}^{\infty} m^2 \sum_{i=1}^{\infty} \frac{i^4}{(2m-1)i} \\
\ll \frac{1}{\log^4 x} \sum_{m=2}^{\infty} \frac{m^2}{2m-1} \ll \frac{1}{\log^4 x}
\]

by using Lemma 2.11 twice.

We now define

\[T(u) = \sum_{2 \leq p \leq u} \frac{1}{p},\]

and have

\[T(u) = \log \log u + C_0 + \sum_p \left( \log \left( 1 - \frac{1}{p} \right) + \frac{1}{p} \right) + r(u),\]

where $r(u) \ll \log u/\sqrt{u}$ under the Riemann Hypothesis, and the sum over primes on the right is equal to

\[- \sum_{m=2}^{\infty} \sum_p \frac{1}{mp^m}.

Then,

\[
S_1 = \int_2^x f^2 \left( \frac{\log u}{\log x} \right) dT(u) \\
= \int_2^x f^2 \left( \frac{\log u}{\log x} \right) \frac{du}{u \log u} + \int_2^x f^2 \left( \frac{\log u}{\log x} \right) dr(u) = I_1 + I_2.
\]

Similarly,

\[
S_2 = \int_2^x f \left( \frac{\log u}{\log x} \right) \frac{du}{u \log u} + \int_2^x f \left( \frac{\log u}{\log x} \right) dr(u) = J_1 + J_2.
\]

Thus,

\[S_1 - 2S_2 = I_1 - 2J_1 + (J_2 - 2J_2).\]
By integration by parts,
\[
I_2 - 2J_2 = -r(2^-) \left[ f^2 \left( \frac{\log 2}{\log x} \right) - 2f \left( \frac{\log 2}{\log x} \right) \right] - \int_2^x r(u) \\
\left[ 2f \left( \frac{\log u}{\log x} \right) f' \left( \frac{\log u}{\log x} \right) - f' \left( \frac{\log u}{\log x} \right) \right] \frac{du}{u \log x}
\]
\[
= r(2^-) - r(2^-) \left[ f \left( \frac{\log 2}{\log x} \right) - 1 \right]^2 \\
- \frac{2}{\log x} \int_2^x r(u) f' \left( \frac{\log u}{\log x} \right) \left[ f \left( \frac{\log u}{\log x} \right) - 1 \right] \frac{du}{u}
\]
\[
= -\log \log 2 - C_0 + \sum_{m=2}^{\infty} \sum_p \frac{1}{mp^n} + O \left( \frac{1}{\log^4 x} \right) \\
+ O \left( \frac{1}{\log x} \int_2^x \frac{\log u}{\sqrt{n}} \left( \frac{\log u}{\log x} \right) \frac{2du}{u} \right) \\
= -\log \log 2 - C_0 + \sum_{m=2}^{\infty} \sum_p \frac{1}{mp^n} + O \left( \frac{1}{\log^4 x} \right)
\]

because \( f(u) = 1 + O(u^2) \) and \( f'(u) \ll u \) when \( 0 \leq u \leq 1 \). The integrals in \( I_1 \) and \( J_1 \) are elementary to evaluate. Using \( u \cot u = 1 - u^2/3 + O(u^4) \) and \( \sin u = u - u^3/6 + O(u^5) \), one has

\[
I_1 = \log \log x - \log \log 2 - \frac{\pi^2}{8} + 1 - \log \frac{\pi}{2} + \frac{\pi^2(\log 2)^2}{12 \log^2 x} + O \left( \frac{1}{\log^4 x} \right),
\]
\[
J_1 = \log \log x - \log \log 2 - \log \frac{\pi}{2} + \frac{\pi^2(\log 2)^2}{24 \log^2 x} + O \left( \frac{1}{\log^4 x} \right).
\]

Hence,

\[
S_1 - 2S_2 = -\log \log x + \log \frac{\pi}{2} - \frac{\pi^2}{8} + 1 - C_0 + \sum_{m=2}^{\infty} \sum_p \frac{1}{mp^n} + O \left( \frac{1}{\log^4 x} \right),
\]

Therefore,

\[
G(T) + H(T) = \frac{T}{2\pi^2} \left[ -\log \log x + \log \frac{\pi}{2} - \frac{\pi^2}{8} + 1 - C_0 \\
+ \sum_{m=2}^{\infty} \sum_p \left( \frac{1}{m} - \frac{1}{m^2} \right) \frac{1}{p^m} \right] + O \left( \frac{T}{\log^4 x} \right). \tag{8}
\]

3 Proof of Theorem 1.1

Suppose \( x = T^3 \) and \( \beta \) is a fixed positive number less than 1/2. We have, by Lemma 2.1, that (3) holds except on a countable set of points. Hence, on
squaring both sides of (4) and integrating from 1 to $T$,

$$\int_1^T (S(t))^2\,dt + H(T) + G(T) = R + O(T^{1/2}x^{1/2}),$$

where the error term is obtained by Cauchy-Schwarz inequality since $R \ll T$. The lower limit of integration may be replaced by zero since $\int_0^1 (S(t))^2\,dt \ll 1$. Then, Lemma 2.10 and (8) give the theorem. The author would like to thank Professor Daniel Goldston for suggestion and discussion on this problem.

References

[1] T.H. Chan, *On a conjecture of Liu and Ye*, submitted.

[2] T.H. Chan, *More precise Pair Correlation Conjecture on the zeros of the Riemann zeta function*, submitted.

[3] D.A. Goldston, *On the Function $S(T)$ in the Theory of the Riemann Zeta-Function*, J. Number Theory 27 (1987), 149-177.

[4] H.L. Montgomery, *The pair correlation of zeros of the zeta function*, Analytic Number Theory (St. Louis Univ., 1972), Proc. Sympos. Pure Math. 24, Amer. Math. Soc., Providence, 1973, pp. 181-193.

Tsz Ho Chan
Case Western Reserve University
Mathematics Department, Yost Hall 220
10900 Euclid Avenue
Cleveland, OH 44106-7058
USA
txc50@po.cwru.edu