Research Article

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On split regular BiHom-Poisson color algebras

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Abstract: The purpose of this paper is to introduce the class of split regular BiHom-Poisson color algebras, which can be considered as the natural extension of split regular BiHom-Poisson algebras and of split regular Poisson color algebras. Using the property of connections of roots for this kind of algebras, we prove that such a split regular BiHom-Poisson color algebra $L$ is of the form $L = \bigoplus_{[\alpha] \in \Lambda} I_{[\alpha]}$ with $I_{[\alpha]}$ a well described (graded) ideal of $L$, satisfying $[I_{[\alpha]}, I_{[\beta]}] + I_{[\alpha]} I_{[\beta]} = 0$ if $[\alpha] \neq [\beta]$. In particular, a necessary and sufficient condition for the simplicity of this algebra is determined, and it is shown that $L$ is the direct sum of the family of its simple (graded) ideals.

Keywords: BiHom-Lie color algebra, BiHom-Poisson algebra, root space, root system

MSC 2020: 17B75, 17A60, 17B22, 17B65

1 Introduction

The interest in Poisson algebras has grown in the last few years, motivated especially by their applications in geometry and mathematical physics. For example, Poisson algebras play a fundamental role in deformation of commutative associative algebras [1]. Moreover, the cohomology group, deformation, tensor product and $\Gamma$-graded of Poisson algebras have been studied by many authors in [2–5]. A Hom-algebra is an algebra such that a linear homomorphism appears in the identities satisfied by its multiplication. This class of algebras appeared in the study of quasi-deformations of vector fields, in particular quasi-deformations of Witt and Virasoro algebras in [6]. So far, many authors have studied Hom-type algebras [7–13]. In particular, the notion of Hom-Lie color algebra was introduced in [8] and presented the methods to construct this color algebra, which can be viewed as an extension of a Hom-Lie algebra to a $\Gamma$-graded algebra, where $\Gamma$ is any abelian group. Furthermore, a BiHom-algebra is an algebra in such a way that the identities defining the structure are twisted by two homomorphisms $\phi$, $\psi$. A BiHom-Poisson color algebra has simultaneously a BiHom-Lie algebra structure and a BiHom-associative algebra structure, satisfying the BiHom-Leibniz identity.

The class of the split algebras is especially related to addition quantum numbers, graded contractions and deformations. For instance, for a physical system which displays a symmetry of $L$, it is interesting to know in detail the structure of the split decomposition because its roots can be seen as certain eigenvalues which are the additive quantum numbers characterizing the state of such system. Recently, the structure of different classes of split algebras such as split regular Hom-Poisson algebras, split regular Hom-Poisson color algebras, split regular BiHom-Lie superalgebras, split BiHom-Leibniz superalgebras and split Leibniz triple systems have been studied by using techniques of connections of roots (see for instance [14–26]).

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Later, these techniques of connections become powerful to study not only split algebras but also graded algebras and algebras having multiplicative bases [27–30]. The purpose of this paper is to consider the decomposition and simplicity of split regular BiHom-Poisson color algebras by the techniques of connections of roots.

2 Preliminaries

First we recall the definitions of Lie color algebra, Poisson color algebra, Hom-Lie color algebra and Hom-Poisson color algebra. The following definition is well known from the theory of graded algebra.

**Definition 2.1.** [8] Let $\Gamma$ be an abelian group. A bi-character on $\Gamma$ is a map $\varepsilon : \Gamma \times \Gamma \to \mathbb{K}\setminus\{0\}$ satisfying

1. $\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = 1$,
2. $\varepsilon(\alpha, \beta + \gamma) = \varepsilon(\alpha, \beta)\varepsilon(\alpha, \gamma)$,
3. $\varepsilon(\alpha + \beta, \gamma) = \varepsilon(\alpha, \gamma)\varepsilon(\beta, \gamma)$,

for all $\alpha, \beta, \gamma \in \Gamma$.

It is clear that $\varepsilon(\alpha, 0) = \varepsilon(0, \alpha) = 1$ for any $\alpha \in \Gamma$, where 0 denotes the identity element of $\Gamma$.

**Definition 2.2.** [14] Let $L = \oplus_{\gamma \in \Gamma} L_{\gamma}$ be a $\Gamma$-graded $\mathbb{K}$-vector space. For a nonzero homogeneous element $v \in L$, denote by $\bar{v}$ the unique group element in $\Gamma$ such that $v \in L_{\bar{v}}$, which will be called the homogeneous degree of $v$. We shall say that $L$ is a **Lie color algebra** if it is endowed with a $\mathbb{K}$-bilinear map $[\cdot, \cdot] : L \times L \to L$ satisfying

$$\left[[v, w], t\right] + \varepsilon(v, w)[v, [w, t]] = \varepsilon(v, w)[v, t] + [\varepsilon(v, w)v, [w, t]]$$ (Jacobi identity),

for all homogeneous elements $v, w, t \in L$.

Lie superalgebras are examples of Lie color algebras with $\mathbb{K} = \mathbb{Z}_2$ and $\varepsilon(i, j) = (-1)^{ij}$, for any $i, j \in \mathbb{Z}_2$. We also note that $L_0$ is a Lie algebra.

**Definition 2.3.** [5] A **Poisson color algebra** is a $\Gamma$-graded vector space $L = \oplus_{\gamma \in \Gamma} L_{\gamma}$, an even bilinear mapping $[\cdot, \cdot] : L \times L \to L$, and a bi-character $\varepsilon$ on $\Gamma$ satisfying the following conditions:

1. $(L, \varepsilon)$ is an associative color algebra,
2. $(L, [\cdot, \cdot])$ is a Lie color algebra,
3. Leibniz color identity $[xy, z] = x[y, z] + \varepsilon(y, z)[x, z]y$,

for any $x, y, z \in L$, $\bar{y}, \bar{z}$ denote the homogeneous degree of $y, z$.

**Definition 2.4.** [8] A **Hom-Lie color algebra** $L$ is a quadruple $(L, [\cdot, \cdot], \phi, \varepsilon)$ consisting of a $\Gamma$-graded space $L$, an even bilinear mapping $[\cdot, \cdot] : L \times L \to L$, a homomorphism $\phi : L \to L$ and a bi-character $\varepsilon$ on $\Gamma$ satisfying

1. $[x, y] = -\varepsilon(\bar{x}, \bar{y})[y, x]$,
2. $\varepsilon(x, \bar{y})[\phi(x), [y, z]] + \varepsilon(x, y)[\phi(y), [z, x]] + \varepsilon(y, \bar{z})[\phi(z), [x, y]] = 0$,

for any $x, y, z \in L$, $\bar{y}, \bar{z}$ denote the homogeneous degree of $y, z$. When $\phi$ is an algebra automorphism it is said that $L$ is a regular Hom-Lie color algebra.

**Definition 2.5.** [17] A **Hom-Poisson color algebra** is a Hom-Lie color algebra $(L, [\cdot, \cdot], \phi, \varepsilon)$ endowed with a Hom-associative color product, that is, a bilinear product denoted by juxtaposition such that

$$\phi(x)(yz) = (xy)\phi(z),$$

for all $x, y, z \in L$, and such that the Hom-Leibniz color identity

$$[xy, \phi(z)] = \phi(x)[y, z] + \varepsilon(\bar{y}, \bar{z})[x, z]\phi(y)$$

holds for any $x, y, z \in L$, $\bar{y}, \bar{z}$ denote the homogeneous degree of $y, z$. 


If \( \phi \) is furthermore a Poisson automorphism, that is, a linear bijective on such that 
\[
\phi([x, y]) = [\phi(x), \phi(y)]
\]
and \( \phi(xy) = \phi(x)\phi(y) \) for any \( x, y \in L \), then \( L \) is called a regular Hom-Poisson color algebra.

**Definition 2.6.** A BiHom-Lie color algebra \( L \) is a quintuple \( (L, [, , ], \phi, \psi, \epsilon) \) consisting of a \( \Gamma \)-graded space \( L \), an even bilinear mapping \([, , ] : L \times L \rightarrow L \) two homomorphisms \( \phi, \psi \) and a bi-character \( \epsilon \) on \( \Gamma \) satisfying

1. \( \phi \circ \psi = \psi \circ \phi \),
2. \( [\psi(x), \phi(y)] = \epsilon(x, y)[\psi(y), \phi(x)] \) (BiHom-skew-symmetry),
3. \( \epsilon(z, x)[\psi(z), \phi(y)] + \epsilon(x, z)[\psi(y), \phi(z)] + \epsilon(z, y)[\psi(y), \phi(z)] = 0 \) (BiHom-Jacobi identity),

for any \( x, y, z \in L, \tilde{x}, \tilde{y}, \tilde{z} \) denote the homogeneous degree of \( x, y, z \).

When \( \phi, \psi \) furthermore are algebra automorphisms, it is said that \( L \) is a regular BiHom-Lie color algebra.

**Definition 2.7.** A BiHom-Poisson color algebra is a BiHom-Lie color algebra endowed with a BiHom-associative color product, that is, a bilinear product denoted by juxtaposition such that

\[
\phi(x)(yz) = (xy)\psi(z),
\]

for all \( x, y, z \in L \), and such that the BiHom-Leibniz color identity

\[
\epsilon(x, y)[\psi(y), \phi(z)] + \epsilon(z, y)[\psi(y), \phi(z)] + \epsilon(z, y)[\psi(y), \phi(z)] = 0
\]

(BiHom-Jacobi identity),

Poison color algebras are examples of BiHom-Poisson color algebras by taking \( \phi = \psi = \text{Id} \). Hom-Poisson color algebras are also examples of BiHom-Poisson color algebras by considering \( \psi = \phi \).

**Example 2.8.** Let \( (L, [, , ], \epsilon) \) be a Poisson color algebra, \( \phi, \psi : L \rightarrow L \) two automorphisms and \( \phi \circ \psi = \psi \circ \phi \). If we endow the underlying linear space \( L \) with a new product \([, , ]' : L \times L \rightarrow L \) defined by \( [x, y]' = [\phi(x)\psi(y)] \), \( (xy)' = \phi(x)\psi(y) \) for any \( x, y, z \in L \), we have that \( (L, [, , ]', \phi, \psi, \epsilon) \) becomes a regular BiHom-Poisson color algebra.

**Proof.** First we check that \( (\phi(x)(yz))' = ((xy)\psi(z))' \).

\[
(\phi(x)(yz))' = ((\psi(y))'(\phi(z)))' = \phi^2(x)\psi(y)\psi(z) = \phi^2(x)\psi(y)\psi(z)
\]

and

\[
((xy)'\psi(z))' = ((\phi(x)\psi(y))\psi(z))' = \phi^2(x)\psi(y)\psi^2(z) = \phi^2(x)\psi(y)\psi^2(z).
\]

That is, \( (\phi(x)(yz))' = ((xy)'\psi(z))' \).

Next check that \( [(xy)', \phi\psi(z)]' = (\phi(x)\psi(y))'(\phi(z))' + \epsilon(\tilde{y}, \tilde{z})[(x, \psi(z))' \phi(y)]' \), by Leibniz color identity and \([x, y]' = [\phi(x), \psi(y)]\), we get

\[
[(xy)', \phi\psi(z)]' = [\phi(xy)', \phi\psi(z)]' = [\phi(\phi(x)\psi(y))\psi(z)]' = [\phi^2(x)\psi(y)\psi^2(z)]
\]

\[
= \phi^2(x)[\psi(y), \psi^2(z)] + \epsilon(\tilde{y}, \tilde{z})[\phi^2(x), \psi^2(z)]\psi(y),
\]

and

\[
\epsilon(\tilde{y}, \tilde{z})[(x, \psi(z))' \phi(y)]' = \epsilon(\tilde{y}, \tilde{z})[(x, \psi(z))' \phi(y)]' = \epsilon(\tilde{y}, \tilde{z})[\phi^2, \psi^2(z)]\psi(y).
\]

Hence, \([xy, \phi\psi(z)] = \phi(x)[y, \phi(z)] + \epsilon(\tilde{y}, \tilde{z})[x, \psi(z)]\phi(y)\). This gives the conclusion. \( \square \)
Throughout this paper we will consider a regular BiHom-Poisson color algebra $L$ being of arbitrary dimension and over an arbitrary base field $K$. $\mathbb{N}$ denotes the set of all non-negative integers and $\mathbb{Z}$ denotes the set of all integers. A subalgebra $A$ of $L$ is a graded subspace such that $[A, A] + AA \subset A$ and $\phi(A) = \psi(A) = A$. A subalgebra $I$ of $L$ is called an ideal if $[I, L] + IL + LI \subset I$ and $\phi(I) = \psi(I) = I$. A BiHom-Poisson color algebra $L$ will be called simple if $[L, L] + LL \not\subset 0$ and its only ideals are $\{0\}$ and $L$.

Let us introduce the class of split algebras in the framework of regular Hom-Poisson color algebras $L$. First, we recall that a Hom-Poisson color algebra $(L, [\cdot, \cdot], \phi, \epsilon)$, over a base field $K$, is called split with respect to a maximal Abelian subalgebra $H$ of $L$, if $L$ can be written as the direct sum

$$L = H \oplus \left( \bigoplus_{a \in \Lambda} L_a \right),$$

where

$$L_a = \{ v_a \in L : [h_0, \phi(v_a)] = \alpha(h_0) \phi(v_a) \text{ for any } h_0 \in H_0 \},$$

being any $\alpha : H_0 \to K$, $\alpha \in \Gamma$, a nonzero linear functional on $H_0$ such that $L_a \neq 0$.

**Definition 2.9.** Denote by $H = \oplus_{a \in \Gamma} H_a$ a maximal Abelian (graded) subalgebra, of a regular BiHom-Poisson color algebra $L$. For a linear functional $\alpha : H_0 \to K$, we define the root space of $L$ (with respect to $H$) associated with $\alpha$ as the subspace

$$L_{\alpha} = \{ v_a \in L : [h_0, \phi(v_a)] = \alpha(h_0) \phi\psi(v_a) \text{ for any } h_0 \in H_0 \}.$$

The elements $\alpha : H_0 \to K$ satisfying $L_{\alpha} \neq 0$ are called roots of $L$ with respect to $H$. We denote $\Lambda = \{ \alpha \in (H_0)^\ast \setminus \{ 0 \} : L_{\alpha} \neq 0 \}$. We say that $L$ is a **split regular BiHom-Poisson color algebra**, with respect to $H$, if

$$L = H \oplus \left( \bigoplus_{\alpha \in \Lambda} L_{\alpha} \right).$$

We also say that $\Lambda$ is the root system of $L$.

Note that when $\phi = \psi = \text{Id}$, the split Poisson color algebras become examples of split regular BiHom-Poisson color algebras and when $\phi = \psi$, the split regular Hom-Poisson color algebra become examples of split regular BiHom-Poisson color algebras. Hence, the present paper extends the results in [15].

From now on $L = H \oplus \left( \oplus_{a \in \Lambda} L_a \right)$ denotes a split regular BiHom-Poisson color algebra. Also, and for an easier notation, the mappings $\phi|_{L_{\alpha}}, \psi|_{L_{\alpha}}, \phi|_{L_{\alpha}}, \psi|_{L_{\alpha}} : H \to H$ will be denoted by $\phi, \psi, \phi^{-1}, \psi^{-1}$, respectively.

It is clear that the root space associated with the zero root $L_0$ satisfies $H \subset L_0$. Conversely, given any $v_0 \in L_0$ we can write $v_0 = h \oplus \left( a_{\epsilon_a} v_a \right)$, where $h \in H$ and $v_a \in L_a$, for $i = 1, \ldots, n$, with $a_i \neq a_j$ if $i \neq j$. Since for any $h_0 \in H_0$ we have $[h_0, v_0] = 0$, then $0 = [h_0, h \oplus \left( a_{\epsilon_a} \phi^{-1}(v_a) \right)] = a_{\epsilon_a} h \phi(h_0) \psi(v_a)$. From here, taking into account the direct character of the sum with the fact $a_i \neq 0$ gives us that any $v_a = 0$. Hence, $v_0 = h \in H$. Consequently,

$$H = L_0.$$ (2.1)

**Lemma 2.10.** Let $L$ be a split regular BiHom-Poisson color algebra. Then, for any $\alpha, \beta \in \Lambda \cup \{0\}$, the following assertions hold.

1. $\phi(L_\alpha) = L_{\alpha \phi^{-1}}$ and $\phi^{-1}(L_\alpha) = L_{\alpha \phi}$.
2. $\psi(L_\alpha) = L_{\alpha \psi^{-1}}$ and $\psi^{-1}(L_\alpha) = L_{\alpha \psi}$.
3. $[L_\alpha, L_\beta] \subset L_{\alpha \phi^{-1} \beta \phi^{-1}}$.
4. $L_\alpha L_\beta \subset L_{\alpha \phi^{-1} \beta \phi^{-1}}$.
5. If $\alpha \in \Lambda$, then $\alpha \phi^{-1} \psi^{-1} \in \Lambda$ for any $z_1, z_2 \in \mathbb{Z}$.

**Proof.** 1. For any $h_0 \in H_0$ and $v_a \in L_a$, since

$$[h_0, \phi(v_a)] = \alpha(h_0) \phi\psi(v_a),$$ (2.2)
we have that by writing $h_0' = \phi(h_0)$ then
\[
[h_0', \phi^2(v_a)] = \phi([h_0, \phi(v_a)]) = a(h_0) \phi^2\psi(v_a) = \alpha \phi^{-1}([h_0', \phi^2(v_a)]) = \alpha \phi^{-1}(h_0') \phi \psi(\phi(v_a)).
\]
Therefore, we get $\phi(v_a) \in L_{\phi^{-1}}$, and so
\[
\phi(L_\alpha) \subset L_{\phi^{-1}}.
\] (2.3)

Now, let us show
\[
L_{\phi^{-1}} \subset \phi(L_\alpha).
\]
Indeed, for any $h_0 \in H_0$ and $v_a \in L_\alpha$, equation (2.2) shows $[\phi^{-1}(h_0), v_a] = a(h_0) \psi(v_a)$. From here, we get $[\phi(h_0), v_a] = \alpha \phi^2(\phi(v_a))$ and conclude
\[
\phi^{-1}(L_\alpha) \subset L_{\phi^{-1}}.
\] (2.4)

Hence, since for any $x \in L_{\phi^{-1}}$, we can write $x = \phi(\phi^{-1}(x))$ and by equation (2.4) we have $\phi^{-1}(x) \in L_\alpha$, we conclude $L_{\phi^{-1}} \subset \phi(L_\alpha)$. This fact together with (2.3) show $\phi(L_\alpha) = L_{\phi^{-1}}$.

To show
\[
\phi^{-1}(L_\alpha) = L_{\phi^{-1}}.
\]
By equation (2.4), while the fact $L_{\phi^{-1}} \subset \phi(L_\alpha)$ is a consequence of writing any element $x \in L_{\phi^{-1}}$ of the form $x = \phi^{-1}(\phi(x))$ and apply equation (2.3).

2. To verify
\[
\psi(L_\alpha) \subset L_{\psi^{-1}},
\] (2.5)
observe that equation (2.2) gives us $\psi(h_0, \phi^2(v_a)) = a(h_0) \psi^2(v_a)$, and so $\psi(h_0, \phi(v_a)) = \alpha \psi^{-1}(\psi(h_0)) \phi \psi(\phi(v_a))$. Since equation (2.2) and the identity $\psi^{-1} \psi = \phi \psi^{-1}$ also give us
\[
\psi^{-1}(L_\alpha) \subset L_{\psi^{-1}},
\] (2.6)
we conclude as above that $\psi(L_\alpha) = L_{\phi^{-1}}$. We can argue similarly with equations (2.5) and (2.6) to get $\psi^{-1}(L_\alpha) = L_{\phi^{-1}}$.

3. For each $h_0 \in H_0$, $v_a \in L_{\alpha,i}$, $v_b \in L_{\beta,i}$, we can write
\[
[h_0, \phi([v_a, v_b])] = [\psi^2(h_0), \phi([v_a, v_b])].
\]
So, by denoting $h_0' = \psi^{-2}(h_0)$, we can apply BiHom-Jacobi identity and BiHom-skew-symmetry to get
\[
[\psi^2(h_0'), \phi([v_a, v_b])] = [\psi^2(h_0'), \phi^2(v_a), \phi(v_b)]
\]
\[
= -\varepsilon(\tilde{h}_0^\alpha, \beta + \tilde{h}_0^\beta) [\psi(v_a), [\psi(v_b), [\psi(h_0'), \phi^2(v_a)]]] - \varepsilon(\tilde{h}_0^\alpha, \beta) [\psi(v_b), [\psi(h_0'), \phi^2(v_a)]] - \varepsilon(\tilde{h}_0^\alpha, \beta) [\psi^2(v_a), [\psi(h_0'), \phi^2(v_a)]] + \varepsilon(\tilde{h}_0^\alpha, \beta) [\psi(h_0'), \phi^2(v_a)]
\]
\[
= \varepsilon(\tilde{h}_0^\alpha, \beta) [\psi^2(h_0'), [\psi^2(v_a), [\psi(h_0'), \phi^2(v_a)]] + [\psi^2\psi^{-1}(h_0'), \phi^2(v_a), \phi(v_b)]
\]
\[
= \varepsilon(\tilde{h}_0^\alpha, \beta) [\psi(h_0'), \phi(v_b)] + \varepsilon(\tilde{h}_0^\alpha, \beta) [\psi^2\psi^{-1}(h_0'), [\psi(v_a), \phi^2(v_a)], \phi(v_b)]
\]
\[
= \varepsilon(\tilde{h}_0^\alpha, \beta) [\psi^2\psi^{-1}(h_0'), [\psi^2\psi^2(v_a), [\psi(h_0'), \phi^2(v_a)]]
\]
\[
= \varepsilon(\tilde{h}_0^\alpha, \beta) [\psi^2\psi^{-1}(h_0'), [\psi(v_a), \phi^2(v_a)]
\]
\[
= (\beta \psi + \alpha \psi^2\phi^{-1})(h_0') \phi \psi([v_a, v_b]).
\]
Taking into account $h_0' = \psi^{-2}(h_0)$ we have shown
\[
[h_0, \phi([v_a, v_b])] = (\beta \psi + \alpha \psi^2\phi^{-1})(h_0') \phi \psi([v_a, v_b]).
\]
From here, $[L_\alpha, L_\beta] \subset L_{\phi^{-1}}, \phi^{-1}$.
From Lemma 2.10-3, we can assert that 

\[ [L_{a,g}, L_{b,g}] \subset L_{\alpha^\phi^{-1}, \beta^\psi^{-1}; g_1, g_2}, \]

for any \( g_1, g_2 \in \Gamma \).

4. Let \( h_0 \in H_0, v_a \in L_{a,i}, v_b \in L_{b,i} \), we can write

\[ [h_0, \phi(v_a v_b)] = [\psi^{-1}(h_0), \phi(v_a v_b)], \]

and denote \( h_0' = \psi^{-1}(h_0) \). By applying the BiHom-Leibniz color identity and BiHom-skew-symmetry, we get

\[
\begin{align*}
[\psi(h_0'), \phi(v_a v_b)] &= -\varepsilon(h_0', (v_a v_b))\psi(v_a) \phi(v_b) \\
&= -\varepsilon(h_0', (v_a v_b))\varepsilon(v_a, h_0)\phi\psi(v_b) - \varepsilon(h_0', (v_a v_b))\phi\psi(v_a)\varepsilon(v_b) \phi(v_b) \\
&= -[\psi(v_a), \phi\psi^{-1}(h_0')]\phi\psi(v_b) - \phi\psi(v_a)[\psi(v_b), \phi\psi^{-1}(h_0')] \\
&= -[\psi(v_a), \phi\psi^{-1}(h_0')]\phi\psi(v_b) - (\varepsilon(v_a, h_0)\phi\psi(v_b)) \\
&= [\phi\psi^{-1}(h_0), \phi\psi(v_b)] + [\phi\psi(v_a), h_0] \phi\psi(v_b) \\
&= [\phi\psi^{-1}(h_0), \phi\psi(v_b)] + \phi\psi(v_a)[\psi(v_b), \phi\psi^{-1}(h_0)] \\
&= [\phi\psi^{-1}(h_0), \phi\psi(v_b)] + \phi\psi(v_a)[\psi(v_b), \phi\psi^{-1}(h_0)] \\
&= (\phi\psi^{-1} + \beta\psi^{-1})(h_0) \phi\psi(v_a) \phi\psi(v_b) \\
&= (\phi\psi^{-1} + \beta\psi^{-1})(h_0) \phi(v_a v_b).
\end{align*}
\]

From here, \( L_{a}L_{b} \subset L_{a^\phi^{-1}, b^\psi^{-1}} \).

From Lemma 2.10-4 we can assert that

\[ L_{a,g}L_{b,g} \subset L_{a^\phi^{-1}, b^\psi^{-1}; g_1, g_2}, \]

for any \( g_1, g_2 \in \Gamma \).

5. This is a consequence of Lemma 2.10-1,2.

\[ \square \]

**Definition 2.11.** A root system \( \Lambda \) of a split BiHom-Poisson color algebra is called **symmetric** if it satisfies that \( \alpha \in \Lambda \) implies \(-\alpha \in \Lambda \).

**3 Decompositions**

In the following, \( L \) denotes a split regular BiHom-Poisson color algebra with a symmetric root system \( \Lambda \) and \( L = H \oplus (a_{\alpha \in \Lambda} L_{a}) \) the corresponding root decomposition. We begin by developing the techniques of connections of roots in this section.

**Definition 3.1.** Let \( \alpha \) and \( \beta \) be two nonzero roots. We shall say that \( \alpha \) is **connected** to \( \beta \) if there exists \( \alpha_1, \ldots, \alpha_k \in \Lambda \) such that

If \( k = 1 \), then

1. \( \alpha_i \in \{ \alpha \phi^{-\nu} \psi^{-r} : n, r \in \mathbb{N} \} \cap \{ \pm \beta \phi^{-m} \psi^{-s} : m, s \in \mathbb{N} \} \).

If \( k \geq 2 \), then

1. \( \alpha_i \in \{ \alpha \phi^{-\nu} \psi^{-r} : n, r \in \mathbb{N} \} \).
2. \( \alpha_1 \phi^{-1} + \alpha_2 \psi^{-1} \in \Lambda \),
3. \( \alpha_1 \phi^{-2} + \alpha_2 \phi^{-1} \psi^{-1} + \alpha_3 \psi^{-1} \in \Lambda \),
4. ...
We shall also say that \( \{a_i, \ldots, a_k\} \) is a connection from \( \alpha \) to \( \beta \).

Our next goal is to show that the connection is an equivalence relation on \( \Lambda \).

**Proposition 3.2.** The relation \( \sim \) in \( \Lambda \), defined by \( \alpha \sim \beta \) if and only if \( \alpha \) is connected to \( \beta \), is of equivalence.

**Proof.** This can be proved completely analogously to [15, Corollary 2.1]. \( \Box \)

For any \( \alpha \in \Lambda \), we denote by

\[
\Lambda_\alpha := \{ \beta \in \Lambda : \beta \sim \alpha \}.
\]

Clearly if \( \beta \in \Lambda_\alpha \), then \( -\beta \in \Lambda_\alpha \) and, by Proposition 3.2, if \( \gamma \notin \Lambda_\alpha \), then \( \Lambda_\alpha \cap \Lambda_\gamma = \emptyset \).

Our next goal is to associate an adequate ideal \( L_{\Lambda_\alpha} \) of \( L \) with any \( \Lambda_\alpha \). For \( \Lambda_\alpha \), \( \alpha \in \Lambda \), we define

\[
H_{\Lambda_\alpha} := \text{span}_K \left\{ L_{\beta \phi^1}, L_{-\beta \phi} : \beta \in \Lambda_\alpha \right\}
\]

and

\[
V_{\Lambda_\alpha} := \bigoplus_{\beta \in \Lambda_\alpha} L_{\beta},
\]

We denote by \( L_{\Lambda_\alpha} \) the following graded subspace of \( L \),

\[
L_{\Lambda_\alpha} = H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}.
\]

**Proposition 3.3.** For any \( \alpha \in \Lambda \), we have \( [L_{\Lambda_\alpha}, L_{\Lambda_\alpha}] + L_{\Lambda_\alpha}L_{\Lambda_\alpha} \subset L_{\Lambda_\alpha} \).

**Proof.** First we have to check that \( L_{\Lambda_\alpha} \) satisfies \( [L_{\Lambda_\alpha}, L_{\Lambda_\alpha}] \subset L_{\Lambda_\alpha} \). Taking into account \( H = L_0 \), then \( [H_{\Lambda_\alpha}, H_{\Lambda_\alpha}] = 0 \) and we have

\[
[L_{\Lambda_\alpha}, L_{\Lambda_\alpha}] = [H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}, H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}] \subset [H_{\Lambda_\alpha}, V_{\Lambda_\alpha}] + [V_{\Lambda_\alpha}, H_{\Lambda_\alpha}] + \Sigma_{\beta, \gamma \in \Lambda_\alpha} [L_{\beta}, L_{\gamma}].
\]  

(3.7)

Let us consider the first summand \( [H_{\Lambda_\alpha}, V_{\Lambda_\alpha}] \) in equation (3.7). Given \( \beta \in \Lambda_\alpha \), we have \( [H_{\Lambda_\alpha}, L_{\beta}] \subset [L_0, L_{\beta}] \subset L_{\beta \phi^1} \), being \( \beta \phi^1 \in \Lambda_\alpha \) by Lemma 2.10. Hence,

\[
[H_{\Lambda_\alpha}, V_{\Lambda_\alpha}] \subset V_{\Lambda_\alpha}.
\]  

(3.8)

Similarly, we can also get

\[
[V_{\Lambda_\alpha}, H_{\Lambda_\alpha}] \subset V_{\Lambda_\alpha}.
\]  

(3.9)

Next consider the third summand \( \Sigma_{\beta, \gamma \in \Lambda_\alpha} [L_{\beta}, L_{\gamma}] \). Given \( \beta, \gamma \in \Lambda_\alpha \) such that \( [L_{\beta}, L_{\gamma}] \neq 0 \), if \( \beta \phi^1 + \gamma \psi^1 = 0 \), then clearly \( [L_{\beta}, L_{\gamma}] \subset H_{\Lambda_\alpha} \). Suppose that \( \beta \phi^1 + \gamma \psi^1 \neq 0 \), since \( [L_{\beta}, L_{\gamma}] \neq 0 \) together with Lemma 2.10 ensures that \( \beta \phi^1 + \gamma \psi^1 \in \Lambda_\alpha \), we have that \( [\beta, \gamma] \) is a connection from \( \beta \) to \( \beta \phi^1 + \gamma \psi^1 \). The transitivity of \( \sim \) gives now that \( \beta \phi^1 + \gamma \psi^1 \in \Lambda_\alpha \) and so

\[
[L_{\beta}, L_{\gamma}] \subset L_{\beta \phi^1 + \gamma \psi^1} \subset V_{\Lambda_\alpha}.
\]  

(3.10)

From equations (3.7)–(3.10), we conclude that \( [L_{\Lambda_\alpha}, L_{\Lambda_\alpha}] \subset L_{\Lambda_\alpha} \).

Second, we will verify that \( L_{\Lambda_\alpha} \cap L_{\Lambda_\alpha} \subset L_{\Lambda_\alpha} \). We have

\[
L_{\Lambda_\alpha}L_{\Lambda_\alpha} = (H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha})(H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}) \subset H_{\Lambda_\alpha}H_{\Lambda_\alpha} + H_{\Lambda_\alpha}V_{\Lambda_\alpha} + V_{\Lambda_\alpha}H_{\Lambda_\alpha} + V_{\Lambda_\alpha}V_{\Lambda_\alpha} + \Sigma_{\beta, \gamma \in \Lambda_\alpha} (L_{\beta}L_{\gamma}).
\]
By arguing as above, we have
\[ H_{\Lambda_a} V_{\Lambda_a} + V_{\Lambda_a} H_{\Lambda_a} + \Sigma_{\beta, \gamma \in \Lambda_a} (L_{\beta} L_{\gamma}) \subset H_{\Lambda_a}. \]

Hence, it just remains to check that \( H_{\Lambda_a} H_{\Lambda_a} \), observe that
\[ H_{\Lambda_a} H_{\Lambda_a} \subset (\Sigma_{\beta, \gamma \in \Lambda_a} (L_{\beta}^{-1} L_{\gamma}^{-1})) H \]
\[ \subset (\Sigma_{\beta, \gamma \in \Lambda_a} (L_{\beta}^{-1} L_{\gamma}^{-1})) H + (\Sigma_{\beta, \gamma \in \Lambda_a} (L_{\beta}^{-1} L_{\gamma}^{-1})) H. \]

From the above, by BiHom-Leibniz color identity, we have
\[ [L_{\beta}^{-1} L_{\gamma}^{-1}] H = [L_{\beta}^{-1} L_{\gamma}^{-1}] \phi (\psi^{-1} (H)) \]
\[ \subset [L_{\beta}^{-1} \psi^{-1} (L_{\beta}^{-1}) + L_{\gamma}^{-1} \psi^{-1} (L_{\gamma}^{-1})] H \]
\[ \subset [L_{\beta}^{-1} \psi^{-1} (L_{\beta}^{-1}) + L_{\gamma}^{-1} \psi^{-1} (L_{\gamma}^{-1})] H \subset H_{\Lambda_a}. \]

By BiHom-associativity, we have
\[ (L_{\beta}^{-1} L_{\gamma}^{-1}) H = (L_{\beta}^{-1} L_{\gamma}^{-1}) \psi (\psi^{-1} (H)) \subset (L_{\beta}^{-1} L_{\gamma}^{-1}) \psi (\psi^{-1} (H)) \subset (L_{\beta}^{-1} L_{\gamma}^{-1}) \psi (\psi^{-1} (H)) \subset H_{\Lambda_a}. \]

Proposition 3.4. For any \( \alpha \in \Lambda \), we have \( \phi (L_{\Lambda_a}) = L_{\Lambda_a} \) and \( \psi (L_{\Lambda_a}) = L_{\Lambda_a} \).

Proof. This is a direct consequence of Lemma 2.10-1,2.

Proposition 3.5. If \( \gamma \notin \Lambda_a \), then \( [L_{\Lambda_a}, L_{\Lambda_a}] + L_{\Lambda_a} L_{\Lambda_a} = 0. \)

Proof. We have
\[ [L_{\Lambda_a}, L_{\Lambda_a}] = [H_{\Lambda_a} \oplus V_{\Lambda_a}, H_{\Lambda_a} \oplus V_{\Lambda_a}] \subset [H_{\Lambda_a}, V_{\Lambda_a}] + [V_{\Lambda_a}, H_{\Lambda_a}] + [V_{\Lambda_a}, V_{\Lambda_a}] \]
(3.11)
and
\[ L_{\Lambda_a} L_{\Lambda_a} = (H_{\Lambda_a} \oplus V_{\Lambda_a} (H_{\Lambda_a} \oplus V_{\Lambda_a}) \subset H_{\Lambda_a} H_{\Lambda_a} + H_{\Lambda_a} V_{\Lambda_a} + V_{\Lambda_a} H_{\Lambda_a} + V_{\Lambda_a} V_{\Lambda_a}. \]
(3.12)

First, we consider \( [V_{\Lambda_a}, V_{\Lambda_a}] + V_{\Lambda_a} V_{\Lambda_a} \) and suppose that there exist \( \beta \in \Lambda_a \) and \( \eta \in \Lambda_a \) such that \( [L_{\beta}, L_{\eta}] + L_{\beta} L_{\eta} \neq 0. \) As necessarily \( \beta \phi^{-1} \neq -\eta \psi^{-1} \), then \( \beta \phi^{-1} \eta \psi^{-1} \in \Lambda \). So \( \beta \eta, -\beta \eta \) is a connection between \( \beta \) and \( \eta \). By the transitivity of the connection relation we have \( \gamma \in \Lambda_a \), a contradiction. Hence, \( [L_{\beta}, L_{\eta}] + L_{\beta} L_{\eta} = 0 \) and so
\[ [V_{\Lambda_a}, V_{\Lambda_a}] + V_{\Lambda_a} V_{\Lambda_a} = 0. \]
(3.13)

Second, we consider \([H_{\Lambda_a}, V_{\Lambda_a}] + H_{\Lambda_a} V_{\Lambda_a} \), and suppose there exist \( \beta \in \Lambda_a \) and \( \eta \in \Lambda_a \) such that
\[ [[L_{\beta}, L_{\beta}], L_{\eta}] + [L_{\beta} L_{\beta}, L_{\eta}] + [L_{\beta}, L_{\beta}, L_{\eta}] + (L_{\beta} L_{\beta}, L_{\eta}) \neq 0. \]

The following is divided into four situations to discuss.

Case 1:

BiHom-skew-symmetry and BiHom-Jacobi identity give
\[ 0 \neq [\psi (\phi^{-1} (L_{\beta})), \phi (\psi^{-1} (L_{\beta})), \psi^{2} (\phi^{-1} (L_{\eta}))]
\[ \subset [\psi^{2} (\phi^{-1} (L_{\beta})), \phi (\psi^{-1} (L_{\beta})), \psi^{2} (\phi^{-1} (L_{\eta}))] \]
\[ \subset [\psi (\phi^{-1} (L_{\beta})), \phi (\psi^{-1} (L_{\eta})), \phi (\psi^{2} (L_{\beta}))]. \]

We get either \( [\psi (\phi^{-1} (L_{\beta})), \phi (\psi^{2} (L_{\eta}))] \neq 0 \) or \( [\psi^{2} (L_{\beta}), \phi (\psi^{-1} (L_{\eta}))] \neq 0. \) In any case, which contradicts equation (3.13). Hence, \([L_{\beta}, L_{\beta}, L_{\eta}] = 0. \)
Case 2:
\[ [L_\beta L_\beta, L_\eta] \neq 0. \]

BiHom-Leibniz color identity gives
\[
0 \neq [L_\beta L_\beta, \phi \psi (\phi^{-1} \psi^{-1}(L_\eta))] \\
\subset \phi(L_\beta) [L_\beta, \phi(\phi^{-1} \psi^{-1}(L_\eta))] + [L_\beta, \psi(\phi^{-1} \psi^{-1}(L_\eta))] \phi(L_\beta) \\
= \phi(L_\beta) [L_\beta, \phi^{-1}(L_\eta)] + [L_\beta, \phi^{-1}(L_\eta)] \phi(L_\beta).
\]

We get either \([L_\beta, \psi^{-1}(L_\eta)] \neq 0\) or \([L_\beta, \phi^{-1}(L_\eta)] \neq 0\). In any case, which contradicts equation (3.13). Hence, \([L_\beta L_\beta, L_\eta] = 0.\)

Case 3:
\[ [L_\beta, L_\beta]L_\eta \neq 0. \]

BiHom-Leibniz color identity gives
\[
0 \neq [L_\beta, \psi (\psi^{-1}(L_\beta))] \phi (\phi^{-1}(L_\eta)) \\
\subset [L_\beta (\phi^{-1}(L_\eta)), \phi (\psi^{-1}(L_\beta))] + [L_\beta, \psi(\phi^{-1}(L_\beta))][\phi(\phi^{-1}(L_\eta))] \\
= [L_\beta, \phi^{-1}(L_\eta)] + [L_\beta, \phi^{-1}(L_\eta)] \phi(L_\beta).
\]

We get either \([L_\beta \phi^{-1}(L_\eta)] \neq 0\) or \([\phi^{-1}(L_\eta), \phi \psi^{-1}(L_\beta)] \neq 0\). In any case, which contradicts equation (3.13). Hence, \([L_\beta, L_\beta]L_\eta = 0.\)

Case 4:
\[ (L_\beta L_\beta) L_\eta \neq 0. \]

BiHom-associativity gives
\[
0 \neq (L_\beta L_\beta) \psi (\psi^{-1}(L_\eta)) < \phi(L_\eta)(L_\beta (\psi^{-1}(L_\eta))),
\]
which contradicts equation (3.13). Hence, \((L_\beta L_\beta) L_\eta = 0.\) Consequently,
\[
[H_{\Lambda_\alpha}, V_{\Lambda_\alpha}] + H_{\Lambda_\alpha} V_{\Lambda_\alpha} = 0. \tag{3.14}
\]

In a similar way, we get
\[
[V_{\Lambda_\alpha}, H_{\Lambda_\alpha}] + V_{\Lambda_\alpha} H_{\Lambda_\alpha} = 0. \tag{3.15}
\]

Finally, we consider \(H_{\Lambda_\alpha} H_{\Lambda_\alpha}\), suppose there exist \(\beta \in \Lambda_\alpha\) and \(\eta \in \Lambda_\gamma\) such that
\[
[L_\beta, L_\gamma][L_\eta, L_\gamma] + [L_\beta, L_\gamma][L_\eta, L_\gamma] + (L_\beta L_\beta)[L_\eta, L_\gamma] + (L_\beta L_\beta)[L_\eta, L_\gamma] \neq 0.
\]

As above, BiHom-Leibniz color identity or BiHom-associativity identity gives us that this fact implies
\[
[H_{\Lambda_\alpha}, V_{\Lambda_\alpha}] + H_{\Lambda_\alpha} V_{\Lambda_\alpha} + [V_{\Lambda_\alpha}, H_{\Lambda_\alpha}] + V_{\Lambda_\alpha} H_{\Lambda_\alpha} \neq 0,
\]
a contradiction either with equation (3.14) or equation (3.15). From here,
\[
H_{\Lambda_\alpha} H_{\Lambda_\alpha} = 0.
\]

Hence, \([L_{\Lambda_\alpha}, L_{\Lambda_\alpha}] + L_{\Lambda_\alpha} L_{\Lambda_\alpha} = 0.\) The proof is completed. \(\square\)

**Proposition 3.6.** For any \(\alpha \in \Lambda\), we have \(H_{\Lambda_\alpha} H + HH_{\Lambda_\alpha} \subset H_{\Lambda_\alpha}.\)

**Proof.** Fix any \(\beta \in \Lambda_\alpha.\) On one hand, by BiHom-Leibniz color identity, we get
\[
[L_\beta, L_\beta] H + H[L_\beta, L_\beta] = [L_\beta, \psi (\psi^{-1}(L_\beta))] \phi (\psi^{-1}(H)) + (\phi (\phi^{-1}(H)) [L_\beta, \phi (\phi^{-1}(L_\beta))]) \\
\subset [L_\beta (\phi^{-1}(H)), \phi (\psi^{-1}(L_\beta))] + (L_\beta [\phi^{-1}(H), \phi (\psi^{-1}(L_\beta))])
\]
\[
\subset [L_\beta, \phi^{-1}(H)] + [L_\beta, \phi^{-1}(L_\beta)] \phi (\phi^{-1}(L_\beta))
\]
\[
\subset [L_\beta L_\beta, L_{\Lambda_\alpha}] + [L_\beta L_\beta, L_{\Lambda_\alpha}] + [L_{\Lambda_\alpha}, L_{\Lambda_\alpha}] + L_{\Lambda_\alpha} L_{\Lambda_\alpha} \subset H_{\Lambda_\alpha}.
\]
On the other hand, by BiHom-associativity, we get
\[(L_\beta L_\beta)H + H(L_\beta L_\beta) = (L_\beta L_\beta)\psi(\psi^{-1}(H)) + \phi(\phi^{-1}(H))(L_\beta L_\beta)
\subset \phi(L_\beta)(L_\beta L_\beta) + (\psi^{-1}(H)L_\beta)\psi(L_\beta)
\subset L_{\beta \phi^{-1}} L_{\beta \phi^{-1}} \subset L_{\Lambda_\psi}.
\]

The proof is completed.

**Theorem 3.7.** The following assertions hold.
1. For any \( a \in \Lambda \), the subalgebra
\[ L_{\Lambda_a} = H_{\Lambda_a} \oplus V_{\Lambda_a} \]
of \( L \) associated with \( \Lambda_\alpha \) is an ideal of \( L \).
2. If \( L \) is simple, then there exists a connection from \( a \) to \( \beta \) for any \( a, \beta \in \Lambda \) and \( H = \sum_{a \in \Lambda} [[L_{\alpha \phi^{-1}} L_{-\alpha \phi^{-1}}] + L_{\alpha \phi^{-1}} L_{-\alpha \phi^{-1}}] \).

**Proof.**
1. Since \([L_{\Lambda_a}, H] = [L_{\Lambda_a}, L_0] \subset V_{\Lambda_a} \), taking into account Propositions 3.3 and 3.5, we have
\[ [L_{\Lambda_a}, L] = \left[ L_{\Lambda_a}, H \oplus \left( \bigoplus_{\beta \in \Lambda_a} L_\beta \right) \oplus \left( \bigoplus_{\gamma \not\in \Lambda_a} L_\gamma \right) \right] \subset L_{\Lambda_a}. \]

By Propositions 3.3 and 3.6, we get
\[ L_{\Lambda_a} L + LL_{\Lambda_a} = L_{\Lambda_a} \left( H \oplus \left( \bigoplus_{\beta \in \Lambda_a} L_\beta \right) \oplus \left( \bigoplus_{\gamma \not\in \Lambda_a} L_\gamma \right) \right) + H \oplus \left( \bigoplus_{\beta \in \Lambda_a} L_\beta \right) \oplus \left( \bigoplus_{\gamma \not\in \Lambda_a} L_\gamma \right) \subset L_{\Lambda_a}. \]

And by Proposition 3.4 also have \( \phi(L_{\Lambda_a}) = L_{\Lambda_\beta}, \psi(L_{\Lambda_a}) = L_{\Lambda_\alpha} \). So we conclude that \( L_{\Lambda_a} \) is an ideal of \( L \).
2. The simplicity of \( L \) implies \( L_{\Lambda_a} = L \). From here, it is clear that \( \Lambda_\alpha = \Lambda \) and \( H = \sum_{a \in \Lambda} [[L_{\alpha \phi^{-1}} L_{-\alpha \phi^{-1}}] + L_{\alpha \phi^{-1}} L_{-\alpha \phi^{-1}}] \).

**Theorem 3.8.** For a vector space complement \( U \) of
\[ \text{span}_K[[L_{\alpha \phi^{-1}} L_{-\alpha \phi^{-1}}] + L_{\alpha \phi^{-1}} L_{-\alpha \phi^{-1}} : \alpha \in \Lambda] \]
in \( H \), we have
\[ L = U + \sum_{[\alpha] \in \Lambda^\sim} I_{[\alpha]}, \]
where any \( I_{[\alpha]} \) is one of the ideals \( L_{\Lambda_a} \) of \( L \) described in Theorem 3.7-1, satisfying \( I_{[\alpha]} I_{[\beta]} + I_{[\alpha]} I_{[\beta]} = 0 \), whenever \([\alpha] \neq [\beta]\).

**Proof.** By Proposition 3.2, we can consider the quotient set \( \Lambda^\sim = \{[\alpha] : \alpha \in \Lambda\} \). Let us denote by \( I_{[\alpha]} := L_{\Lambda_a} \).
We have \( I_{[\alpha]} \) is well defined and by Theorem 3.7-1, an ideal of \( L \). Therefore,
\[ L = U + \sum_{[\alpha] \in \Lambda^\sim} I_{[\alpha]} \]
By applying Proposition 3.5 we also obtain \( I_{[\alpha]} I_{[\beta]} + I_{[\alpha]} I_{[\beta]} = 0 \) if \([\alpha] \neq [\beta]\).

**Definition 3.9.** Let \( Z(L) \) be the center of \( L \) satisfying
\[ Z(L) = \{x \in L : \{x, L\} + xL + Lx = 0\}. \]
Theorem 3.10. If $Z(L) = 0$ and $H = \sum_{\alpha \in \Lambda}([L_{\alpha g}, L_{-\alpha f}] + L_{\alpha f} L_{-\alpha g})$, then $L$ is the direct sum of the ideals given in Theorem 3.7,

$$L = \bigoplus_{[\alpha] \in \Lambda'_{-}} L_{[\alpha]}.$$  

Furthermore, $[L_{[\alpha]} , L_{[\beta]}] + L_{[\alpha]} L_{[\beta]} = 0$, whenever $[\alpha] \neq [\beta]$.

**Proof.** Since $H = \sum_{\alpha \in \Lambda}([L_{\alpha g}, L_{-\alpha f}] + L_{\alpha f} L_{-\alpha g})$, we get $L = \bigoplus_{[\alpha] \in \Lambda'_{-}} L_{[\alpha]}$. To finish, we show the direct character of the sum. Given $x \in L_{[\alpha]} \cap \sum_{\beta \in \Lambda'_{-} \neq [\alpha]} I_{[\beta]}$, since $x \in L_{[\alpha]}$, then using again the equation $[L_{[\alpha]} , L_{[\beta]}] + L_{[\alpha]} L_{[\beta]} = 0$, for $[\alpha] \neq [\beta]$, we obtain

$$x, \sum_{\beta \in \Lambda'_{-} \neq [\alpha]} I_{[\beta]} + x \sum_{\beta \in \Lambda'_{-} \neq [\alpha]} I_{[\beta]} = 0.$$  

In a similar way, since $x \in \sum_{\beta \in \Lambda'_{-} \neq [\alpha]} I_{[\beta]}$, we have

$$[x, L_{[\alpha]}] + x L_{[\alpha]} + L_{[\alpha]} x = 0.$$  

It implies $[x, L] + xL + Lx = 0$, that is, $x \in Z(L)$. Thus $x = 0$, as desired. $\square$

4 The simple components

In this section, we study if any of the components in the decomposition given in Corollary 3.10 is simple. Under certain conditions we give an affirmative answer.

Observe the grading of $I$, we have

$$I = \bigoplus_{g \in \Gamma} I_{g} = \bigoplus_{g \in \Gamma}(I_{g} \cap H_{g}) \bigoplus \left( \bigoplus_{\alpha \in \Lambda}(I_{g} \cap L_{\alpha g}) \right). \tag{4.16}$$

**Lemma 4.1.** Let $L$ be a split regular BiHom-Poisson color algebra, suppose $$H = \sum_{\alpha \in \Lambda}([L_{\alpha g}, L_{-\alpha f}] + L_{\alpha f} L_{-\alpha g}).$$  

If $I$ is an ideal of $L$ such that $I \subset H$, then $I \subset Z(L)$.

**Proof.** Observe that $[I, H] \subset [H, H] = 0$ and

$$[I_{\alpha}, I_{\alpha}] + I \left( \bigoplus_{\alpha \in \Lambda} I_{\alpha} \right) + \left( \bigoplus_{\alpha \in \Lambda} I_{\alpha} \right) I \subset I \cap \left( \bigoplus_{\alpha \in \Lambda} I_{\alpha} \right) \subset H \cap \left( \bigoplus_{\alpha \in \Lambda} I_{\alpha} \right) = 0.$$  

Since $H = \sum_{\alpha \in \Lambda}([L_{\alpha g}, L_{-\alpha f}] + L_{\alpha f} L_{-\alpha g})$, by the BiHom-Leibniz color identity and the above observation, that $HI + IH = 0$. So $I \subset Z(L)$. $\square$

Let us introduce the concepts of root-multiplicativity and maximal length in the framework of split BiHom-Poisson color algebras, in a similar way to the ones for split BiHom-Lie algebras (see [15]). For each $g \in \Gamma$, we denote by $\Lambda_{g} = \{ \alpha \in \Lambda : L_{\alpha g} \neq 0 \}$.

**Definition 4.2.** A split regular BiHom-Poisson color algebra $L$ is **root-multiplicative** if given $\alpha \in \Lambda_{g}$ and $\beta \in \Lambda_{g}$, with $g, g \in \Gamma$, such that $\alpha + \beta \in \Lambda$, then

$$[L_{\alpha g}, L_{\beta g}] + L_{\alpha g} L_{\beta g} \neq 0.$$
Definition 4.3. A split regular BiHom-Poisson color algebra \( L \) is of **maximal length** if for any \( \alpha \in \Lambda_g, g \in \Gamma \), we have \( \dim L_{\alpha \cdot g} = 1 \) for \( \kappa \in \{\pm 1\} \).

Observe that if \( L \) is of maximal length, then equation (4.16) let us assert that given any nonzero ideal \( I \) of \( L \) then

\[
I = \bigoplus_{g \in \Gamma} \left( (I_g \cap H_g) \oplus \left( \bigoplus_{\alpha \in \Lambda'_g} L_{\alpha \cdot g} \right) \right),
\]

where \( \Lambda'_g = \{ \alpha \in \Lambda : I_g \cap L_{\alpha \cdot g} \neq 0 \} \) for each \( g \in \Gamma \).

Theorem 4.4. Let \( L \) be a split regular BiHom-Poisson color algebra of maximal length, root multiplicative and with \( Z(L) = 0 \). Then \( L \) is simple if and only if it has all its nonzero roots connected and \( H = \sum \alpha \in \Lambda ([L_{\alpha \cdot g}^{-1}, L_{-\alpha \cdot g}^{-1}]+L_{a \cdot g}^{-1}L_{-a \cdot g}^{-1}) \).

**Proof.** The first implication is Theorem 3.7-2. To prove the converse, consider \( I \) a nonzero ideal of \( L \). By Lemma 4.1 and equation (4.17), we can write

\[
I = \bigoplus_{g \in \Gamma} \left( (I_g \cap H_g) \oplus \left( \bigoplus_{\alpha \in \Lambda'_g} L_{\alpha \cdot g} \right) \right),
\]

with \( \Lambda'_g \subset \Lambda_g \) for any \( g \in \Gamma \) and some \( \Lambda'_g \neq \emptyset \). Hence, we may choose \( \alpha_0 \in \Lambda'_g \) being so \( 0 \neq L_{\alpha_0 \cdot g} \subset I \).

Since \( \phi(I) = I \) and \( \psi(I) = I \) and by making use of Lemma 2.10 we can assert that

\[
\text{if } \alpha \in \Lambda_1, \text{ then } \{ [a \alpha \cdot \phi + \beta \cdot \psi : z_1, z_2 \in Z] \} \subset \Lambda_1.
\]

In particular,

\[
\{ L_{\alpha_0 \cdot \phi \cdot \psi g}^{-1} : z_1, z_2 \in Z \} \subset I.
\]

Now, let us take any \( \beta \in \Lambda \) satisfying \( \beta \neq \{ \pm a_0 \alpha \cdot \phi + \beta \cdot \psi : z_1, z_2 \in Z \} \). Since \( a_0 \) and \( \beta \) are connected, we have a connection \( \{ a_0, \ldots, a_k \}, k \geq 2 \), from \( a_0 \) to \( \beta \) satisfying:

\[
\begin{align*}
a_1 &= a_0 \phi^n \psi^r, \text{ for some } n, r \in \mathbb{N}, \\
a_2 \phi^1 + a_2 \phi^1 \psi^1 &\in \Lambda, \\
a_2 \phi^2 + a_2 \phi^2 \psi^1 + a_3 \phi^1 \psi^1 &\in \Lambda, \\
\vdots \\
a_k \phi^{k + 1} + a_k \phi^{k + 1} \psi^1 &\in \Lambda, \\
a_k \phi^{k + 2} + a_k \phi^{k + 2} \psi^1 &\in \Lambda, \\
\vdots \\
a_k \phi^{k + 3} + a_k \phi^{k + 3} \psi^1 &\in \Lambda, \\
a_k \phi^{k + 4} + a_k \phi^{k + 4} \psi^1 &\in \Lambda, \\
\vdots \\
e \beta \phi^m \psi^s &\text{ for some } m, s \in \mathbb{N} \text{ and } e \in \{\pm 1\}.
\end{align*}
\]

Consider \( a_1, a_2 \) and \( a_k \phi^{k + 1} + a_k \phi^1 \psi^1 \). Since \( a_2 \in \Lambda \), there exists \( g_1 \in \Gamma \) such that \( L_{a_2 \cdot g_1} \neq 0 \) and so \( a_2 \in \Lambda_{g_1} \). From here, we have \( a_1 \in \Lambda_{g_1} \) and \( a_2 \in \Lambda_{g_1} \), such that \( a_k \phi^{k + 1} + a_k \phi^1 \psi^1 \in \Lambda_{g_1, g_1} \). The root-multiplicativity and maximal length of \( L \) show \( 0 \neq [L_{a_2 \cdot g_1}, L_{a_2 \cdot g_1}] = L_{a_2 \cdot g_1, a_2 \cdot g_1, g_1, g_1} = L_{a_2 \cdot g_1, a_2 \cdot g_1, g_1, g_1} \). Since \( 0 \neq L_{a_2 \cdot g_1} \subset I \) as a consequence of equation (4.18) we get

\[
0 \neq L_{a_2 \cdot g_1, a_2 \cdot g_1, g_1, g_1} \subset I.
\]

We can argue in a similar way from \( a_k \phi^{k + 1} + a_k \phi^1 \psi^1, a_3 \) and \( a_k \phi^{k + 2} + a_k \phi^{k + 1} \psi^1 + a_k \psi^1 \) to get

\[
0 \neq L_{a_k \cdot g_1, a_k \cdot g_1, a_k \cdot g_1, g_1, g_1} \subset I,
\]

We refer to the proof of Theorem 3.7-2 for details.
for some $g_z \in \Gamma$. Following this process with the connection $\{a_1, \ldots, a_d\}$, we obtain that
\[ 0 \neq L_{a_{i\Phi^{r_{i+1} \cdots r_1} + \cdots + a_1\Phi^{r_1}}} \subset I, \]
and so either $0 \neq L_{a_{i\Phi^{r_{i+1} \cdots r_1} + \cdots + a_1\Phi^{r_1}}} \subset I$ or $0 \neq L_{-a_{i\Phi^{r_{i+1} \cdots r_1} + \cdots + a_1\Phi^{r_1}}} \subset I$ for some $g_z \in \Gamma$. That is,
\[ 0 \neq L_{\varepsilon \Phi^{r_{i+1} \cdots r_1} + \cdots + a_1\Phi^{r_1}} \subset I, \]
for some $\varepsilon \in \{\pm 1\}$, some $g_z \in \Gamma$.

From equations (4.19)–(4.20), we get either $0 \neq L_{\alpha \Phi^{r_{i+1} \cdots r_1} + \cdots + a_1\Phi^{r_1}} \subset I$ or $0 \neq L_{-\alpha \Phi^{r_{i+1} \cdots r_1} + \cdots + a_1\Phi^{r_1}} \subset I$ for some $g_z \in \Gamma$. That is,
\[ 0 \neq L_{\alpha \Phi^{r_{i+1} \cdots r_1} + \cdots + a_1\Phi^{r_1}} \subset I, \]
for any $\alpha \in \Lambda$, some $\varepsilon \in \{\pm 1\}$, some $g_z \in \Gamma$.

This can be reformulated by saying that for any $\alpha \in \Lambda$ either $\{\alpha \Phi^{r_{i+1} \cdots r_1} + \cdots + a_1\Phi^{r_1}\}$ or $\{-\alpha \Phi^{r_{i+1} \cdots r_1} + \cdots + a_1\Phi^{r_1}\}$ is contained in $\Lambda_{\varepsilon}^I$.

Taking into account $H = \sum_{\varepsilon \in \Lambda} (L_{\varepsilon \Phi^{r_{i+1} \cdots r_1} + \cdots + a_1\Phi^{r_1}} + L_{\varepsilon \Phi^{r_{i+1} \cdots r_1} + \cdots + a_1\Phi^{r_1}})$, we have
\[ H \subset I. \]  
(4.21)
Now for any $\alpha \in \Lambda$, since $L_\alpha = [H, L_{\alpha \Phi}]$ by the maximal length of $I$, equation (4.21) gives $L_\alpha \subset I$, and so $I = L$. Consequently, $L$ is simple. \hfill \Box

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