Exponential Bounds for Random Sums.
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Abstract. We construct a non-improved exponential bounds for distribution of normed sums of i.i.d. random variables with random numbers of summand.

Key words: Random sum, exponential estimation, Orlicz spaces, martingales, saddle-point method.

1. Introduction.
Let $(\Omega, F, \mu)$ be a probability space, $\{\xi(i)\}, i = 1, 2, \ldots$ be a sequence of independent identical distributed (i.i.d) centered: $E \xi(i) = 0$ random variables (r.v) with finite non-trivial variance $\sigma^2 = D \xi(i) \in (0, \infty)$, and let $\eta, \eta \geq 1$ be an integer r.v. with finite first moment $E \eta = A$, where $A \in [2, \infty)$. We assume at first that the r.v. $\eta$ and the sequence $\{\xi(i)\}$ are independent.

We will denote for arbitrary r.v. $\tau$ and $x = const \geq 0$ the tail function

$$T(\tau, x) = \max(P(\tau \geq x), P(\tau \leq -x)),$$

will write for the r.v. $\xi(1) \ R(x) = T(\xi(1), x)$, and we define the so-called normed random sum and corresponding uniform tail function

$$S = \sum_{i=1}^{\eta} \xi(i)/(\sigma \sqrt{A}),$$

$$V(x) = V(Law(\eta), Law(\xi(i)), x) = \sup_{A \geq 2} T(S, x).$$

In the case if the r.v. $\eta - 1$ has a Poisson distribution $Pois(A)$ with parameter $A : \ P(\eta - 1 = n) = A^n \exp(-A)/n!, \ n = 0, 1, 2, \ldots$ and $Law(\eta - 1) \in \cup_{A \geq 2}\{Pois(A)\}$ we will write $V(Pois; Law(\xi(i)), x) = V(\cup_{A \geq 2}\{Pois(A)\}, Law(\xi(i)), x)$; if the r.v. $\eta$ has a geometrical distribution $G(A) : \ P(\eta = n) = A^{-1}(1 - 1/A)^{n-1}, n = 1, 2, \ldots, A \geq 2$ and $Law(\eta) \in \cup_{A \geq 2}\{G(A)\}$ we will write $V(G; Law(\xi(i)), x) =$
\[ V(\cup_{A \geq 2} \{G(A)\}, \text{Law}(\xi(i)), x); \text{for the case of all distribution r.v. } \eta \text{ under the condition that } E\eta = A, \exists A \geq 2 \text{ we will use the notations correspondently } \text{Dis}(A), \text{Dis} = \cup_{A \geq 2} \{\text{Dis}(A)\} \text{ and } V(\text{Dis}, \text{Law}(\xi(i)), x) = V(\cup_{A \geq 2} \{\text{Dis}(A)\}, \text{Law}(\xi(i)), x). \]

**Our goal is the bide - side exponential estimating** \( V(x) \) **at** \( x \to \infty, x \geq 2 \text{ in the terms of distributions } \text{Law}(\xi(i)), \text{Law}(\eta).**

We have for all distribution \( \xi(i) \) under the conditions \( E\xi(i) = 0, \text{D}\xi(i) \in (0, \infty) \), since \( E\xi(i) = 0, \text{D}\xi(i) \in (0, \infty) \):

\[
V(\text{Dis}, \text{Law}(\xi(i)), x) \leq \min \left(1, x^{-2}\right)
\]

by virtue of Chebyshev inequality; this estimation is called trivial.

There are many publications about the moment estimations \( E|S|^p \) and statistical applications of those estimations (see, for example, (Gut A., 1988), (Gut A., 2003):

\[
|S|^p \leq B(p) |\eta|_{p/2}^{1/2} |\xi(1)|_p, \quad p \geq 2,
\]

where \( B(p) \) is a constant in the famous Burkholder inequality for martingales. Here and further for arbitrary r.v. \( \zeta \)

\[
|\zeta|^p = E^{1/p} |\zeta|^p = ||\zeta||_{L_p(\Omega, \mathbf{P})}.
\]

It is proved in (Hitczenko, 1990) that the best boundary for \( B(p) \) at \( p \geq 2, p \to \infty \) is \( B(p) \leq C p / \log p, C \) is an absolute constant. Note that the estimation (0) is proved in the case if r.v. \( \eta \) is the stopping time for the sequence \( \{\xi(i)\} \), for example, if \( \forall n = 1, 2, \ldots \text{ r.v.'s } \xi(1), \xi(2), \ldots, \xi(n) \) and the event \( \{\eta = n\} \) are independent.

We have for the function \( T(x) \) under the conditions:

\[
T(0) = 1, T(\infty) = 0, \text{monotonically non - increasing and right continuous with finite second moment } |\int_0^\infty x^2 dT(x)| < \infty \text{ the operator }
\]

\[
W[T](x) = \min \left(1, 4 \inf_{z>0} \left[ \exp(-x^2/(8z^2)) - \int_z^\infty x^2 dT(x) \right] \right).
\]
Further, put
\[ \varphi(\lambda) = \max_{\pm} \log \mathbb{E} \exp(\pm \lambda \xi(i)) . \]

This definition is non-trivial only if the variable \( \xi(1) \) satisfies the so-called Kramer condition:
\[ \exists \lambda_0 \in (0, \infty), \forall \lambda \in (-\lambda_0, \lambda_0) \Rightarrow \varphi(\lambda) < \infty ; \]
in other case we set \( \varphi(\lambda) = \infty \forall \lambda \neq 0. \)

Let us introduce the function \( \chi(\lambda) = \sup_{n=1,2,...} n \varphi(\lambda/\sqrt{n}) , \chi^*(x) = \sup_{\lambda} (\lambda x - \chi(\lambda)) , \)
\[ Q[R](x) = \min \{ W[R](x) , \exp(-\chi^*(x)) \} . \]

**Lemma 1.** (Buldygin at al., 1992), (Lesign at al., 2001).
\[ \sup_n T \left( n^{-1/2} \frac{1}{n} \sum_{i=1}^{n} \xi(i) , x \right) \leq Q[R](x) , \quad x \geq 0 . \] (1)

Let us introduce the Orlicz spaces \( G(m,r) \) (in order to describe the examples) of random variables as the set of all r.v. \( \{ \tau \} \) with finite norm
\[ ||\tau||_{m,r} = \sup_{p \geq 2} |\tau|_p p^{-1/m} \log^r/m p . \]

Here \( m = \text{const} > 0, \ r = \text{const} \in R^1 . \) It is easy to verify that \( G(m,r) \) is isomorphic to the Orlicz space with \( N- \) function
\[ N(u) = \exp(|u|^m \log^r |u|) , \quad |u| \geq 2 , \]
and that \( \tau \in G(m,r) \) if and only if
\[ T(\tau, x) \leq \exp (-C_1(m,r)x^m \log^r (C_2(m,r) + x)) . \] (2)

See (Buldygin at al, 1992, p.351).

For example, assume that the r.v. \( \tau \) has Poisson distribution with parameter \( A; \ A \geq 2 . \) Then for some non-trivial positive absolute constants \( C_1, C_2 \) and all \( p, x \geq 2 \)
\[ |\tau - A|_p \leq C_1 \sqrt{A} p/ \log p , \]
or
\[ \mathbf{P}(|\tau - A|/\sqrt{A} > x) \leq \exp(-C_2 x \log x) . \]
Let us suppose, for example, that for some $m = \text{const} > 0, r \in \mathbb{R}^1$ $\xi(1) \in G(m, r)$. We define the following functions $M = M(m, r), L = L(m, r)$: at $m \in (0, 1)$ or $m = 1, r < 0 \Rightarrow M = 2m/(m + 2), L = 2r/(m + 2)$; at $m = 1, r \geq 0$ or $m \in (1, 2), r < 0 \Rightarrow M = m, L = r$; at $m = 2, r \geq 0$ or $m > 2 \Rightarrow M = 2, L = 0$. We can define formally in the case $m = +\infty, r \in \mathbb{R} \Rightarrow M = 2, L = 0$.

It follows from (1)

$$\sup_n T \left( \sum_{i=1}^{n} \xi(i)/\sqrt{n}, \ x \right) \leq \exp \left( -C_3 \ x^M \ \log^L(C_2 + x) \right), \quad (3)$$

$C_{2,3} = C_{2,3}(m, r)$, or, equally,

$$\sup_{n \geq 1} \left| \sum_{i=1}^{n} \xi(i)/\sqrt{n} \right|_{M,L} \leq C(m, r) \left| \xi(1) \right|_{m,r}.$$ 

It is proved in (Ostrovsky, 1999, p. 34) that in the case $m > 1$ the estimation (2) is exact at $x \to \infty$.

In this paper, the letter $C, C_j(\cdot)$ will denote positive finite various non-essentially constant which may differ from one formula to the next and which does not depend upon $x, n$. We make no attempt to obtain the best values for these constants.

2. Main result. Upper bound. Examples.

Theorem 1.

$$V(Law(\eta), Law(\xi(i)), \eta, x) \leq E \ Q[R](\sigma \ x/\sqrt{A/\eta}). \quad (4)$$

Proof. We will assume without loss of generality $\sigma = 1$. We receive from (1), using the formula of full probability and denoting $q_n = q_n(A) = P(\eta = n) : V(x) =$

$$P \left( \sum_{i=1}^{n} \xi(i) > x\sqrt{A} \right) = \sum_{n=1}^{\infty} q_n P \left( \sum_{i=1}^{n} \xi(i)/\sqrt{n} > x\sqrt{A/n} \right) \leq$$

$$\sum_{n} q_n \ Q[R](x\sqrt{A/n}) = E \ Q[R](x\sqrt{A/\eta}).$$

Example 1. Let us suppose here that the r.v. $\eta$ has a geometric distribution $G(A)$ with parameter $A, A \geq 2$, and that $\exists m > 0, \exists r \in \mathbb{R} L(\xi(1)) \in G(m, r)$. We assert that at $x \geq 2 \Rightarrow \sup_{A \geq 2} V(G(A), G(m, r), x) \leq$

$$C_3(m, r) \exp \left( -C(m, r)x^{2M/(M+2)} (\log x)^{2L/(M+2)} \right)$$
\[
J(M, L; x) = J(x).
\]

**Proof.** We have using theorem 1:

\[
V(G(A), G(m, r), x) \leq \sum_{n=1}^{\infty} (A - 1)^{-1} \exp(n[\log(1 - 1/A)]) - \\
C x^M n^{-M/2} A^{M/2} n^{-M/2}(\log^L(C_2 + x\sqrt{A/n})) \leq (A - 1)^{-1} \times \\
C_1 \sum_{n=2}^{\infty} \exp \left(-C(n/A + x^M(A/n))^{M/2}(\log^L(C_2 + x\sqrt{A/n})\right) \overset{\text{def}}{=} \\
\sum_{n=2}^{\infty} a(n; A, x).
\]

We denote

\[
N_0 = N_0(A, x) = \arg\max_{n \geq 2} a(n; A, x) = A \, n_0(x),
\]

where at \( x \to \infty \Rightarrow \)

\[
N_0/A \sim C_2(M, L) \, x^{2M/(M+2)} \, (\log x)^{2L/(M+2)}.
\]

Since the function \( n \to a(n; A, x) \) is monotonic in the intervals \([1, N_0]\) and \([N_0, \infty)\), we can estimate

\[
C_2 \, (A - 1) \, V(G, G(m, r), x) \leq \\
\int_1^{\infty} \exp \left(-C \left((y/A) + x^M(\sqrt{A/y})^M \, \log^L(C_2 + x \sqrt{A/y})\right)\right) \, dy = \\
C_3 A \, \int_{1/A}^{\infty} \exp \left(-C \left(z - x^M z^{-M/2} \, \log^L(C_2 + x / \sqrt{z})\right)\right) \, dz \leq \\
C_3 \, A \, \int_0^{\infty} \exp \left(-C(z - x^M z^{-M/2} \, \log^L(C_2 + x / \sqrt{z})\right) \, dz.
\]

Let us denote \( \beta = 2M/(2 + M) \),

\[
R(x, v) = v + (2/M) \, v^{-M/2} \, \log^L \left(C_2 + x^{2/(M+2)} \, v^{-1/2}\right),
\]
\[ U(x, v) = v + (2/M)v^{-M/2} \log^L x, \ x \geq 2, v > 0; \]

\[ S(x, v) = x^\beta \left( v + (2/M)v^{-M/2} \log^L \left( C_2 + x^{2/(M+2)}v^{-1/2} \right) \right). \]

Let \( \Delta = \Delta(M, L) \) be the arbitrary function on \( M, L \) so that \( \Delta > 2|L|/(M + 2) \). After the substitution \( z = x^\beta v \) we receive:

\[ \sup_{A \geq 2} \| S \|_{2M/(M+2), 2L/(M+2)} \leq C_6 \| \xi(1) \|_{m, r}. \]
Example 2. Let us now suppose again that $L\{\xi(i)\} \in G(m, r)$ for some $m > 0, r \in R$ and assume that the r.v. $\eta - 1$ has a Poisson distribution with parameter $B = A - 1$; $A \geq 2$. It follows from theorem 1 and Stirling’s formula that $V(Pois(A), G(m, r), x) \leq$

$$C \sup_{B \geq 1} \sum_{n=1}^{\infty} \exp(-B + n \log B - \log n! -$$

$$C x^M B^{M/2} n^{-M/2} (\log^L (C_2 + x\sqrt{B/n})) \leq$$

$$C \sup_{B \geq 1} \sum_{n=1}^{\infty} \exp(-B - n \log n +$$

$$n - C x^M B^{M/2} n^{-M/2} (\log^L (C_2 + x\sqrt{B/n})) \leq$$

$$C \sup_{n=1}^{\infty} \sup_{B \geq 1} \exp(-B - n \log n + n +$$

$$C x^M B^{M/2} n^{-M/2} (\log^L (C_2 + x\sqrt{B/n})).$$  \hfill (6)

It is easy to verify that the maximum of arbitrary member of the right-hand side (6) over $B \geq 1$ for sufficiently greater values $x \geq x_0$, $x_0 = const \geq 2$ is achieved at $B = 1$. Therefore $V(Pois(A), G(m, r), x)/C \leq$

$$\leq \sum_{n=2}^{\infty} \exp(-1 - n \log n + n - C x^M n^{-M/2} \log^L (C_2 + x/\sqrt{n})).$$

Since for $x \geq 2$

$$\sum_{n \geq x^2} \exp(-n \log n + n) \leq \exp(-C x^2 \log x),$$

we have:

$$V(Pois(A), G(m, r), x)/C \leq \exp(-C x^2 \log x) +$$

$$\sum_{n \in [1, x^2]} \exp(n - n \log n - C x^M n^{-M/2} \log^L (C_2 + x/\sqrt{n})).$$

Let us denote $N_1 = N_1(x) =$

$$\argmax_{n \in [1, x^2]} (n - n \log n - C x^M n^{-M/2} \log^L (C_2 + x/\sqrt{n}));$$
it is easy to calculate:

\[ N_1(x) \asymp C(M, L)x^{2M/(M+2)}(\log x)^{(2L-2)/(M+2)}, \quad x \to \infty. \]

We obtain for values \( x \geq 2 \):

\[
V(Pois, G(m, r), x)/C(m, r) \leq \exp \left(-Cx^2 \log x\right) + \\
(x^2 + 1) \exp \left(N_1 - N_1 \log N_1 - Cx^M N_1^{-M/2} \log^L(C_2 + x/\sqrt{N_1})\right) \leq \\
C_3 \exp \left(-C(m, r)x^{2M/(M+2)}(\log x)^{(2L+M)/(M+2)}\right). \tag{7}
\]

In the case \( m = \infty \), i.e. if the variable \( \xi(1) \) is bounded (mod P), then \( 2M/(M + 2) = 1, (2L + M)/(M + 2) = 1/2 \).

We can rewrite (7) in the considered case \( Law(\eta - 1) = Pois, Law(\xi(i) \in G(m, r) \text{ in the terms of } G(m, r) \text{ spaces:} \\
\sup_{A \geq 2} ||S||_{2M/(M+2), (M+2L)/(M+2)} \leq C(m, r)||\xi(1)||_{m, r}. \)

In the case if \( Law\{\xi(i)\} \in \cup_{m>1}\{G(m, r)\} \) the estimation (7) improves some result of (Gine at al, 2003). For instance, if \( m \in (1, \infty), r = 0 \), then from (7) it follows the inequality: \( p \geq 2 \Rightarrow \\
\sup_{A \geq 2} |S|_p \leq C_1 p^{1/2+1/\min(m,2)}/\sqrt{\log p}, \ p \geq 2, \)

but we receive from (Gine at all, 2003):

\[
\sup_{A \geq 2} |S|_p \leq C_2 p^{1+1/m}/\log p,
\]

and we obtain from (Gut, 1988), (Gut, 2003):

\[
\sup_{A \geq 2} |S|_p \leq C_3 p^{1+1/m} \log^{-1/2} p.
\]

3. Low bounds. We will prove further that our estimations are non - improved in general case, for example, even for normal distribution of values \( \{\xi(i)\} \).

Theorem 2. We assert that for all values \( m > 1 \) and sufficiently larges \( x, x \geq x_0 = \text{const} \geq 2 \) : \( V(G, G(m, r), x) \geq \\
C_6(m, r)\exp\left(-C_7(m, r)x^{2M/(M+2)}(\log x)^{2L/(M+2)}\right), \tag{8}
\]

\[ 3. \text{Low bounds.} \text{ We will prove further that our estimations are non} \]
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\]
\[ V(\text{Pois}, G(m, r), x) \geq C_8(m, r) \exp \left( -C_9(m, r)x^{2M/(M+2)} (\log x)^{(2L+M)/(M+2)} \right). \] (9)

**Proof** is very simple. It is enough to prove the inequality (8); the proposition (9) is proved analogously. Let \( \xi(i) \) be independent symmetrically distributed r.v. with distributions

\[ P(|\xi(i)| > x) = \exp \left( -x^m \left[ \log(C(m, r) + x) \right]^r \right), \quad x \geq 0, \]

and let us introduce the even smooth convex function \( \varphi_{m,r}(\lambda) = \log \mathbb{E} \exp(\lambda \xi(i)), \lambda \in (-\infty, \infty). \) It is proved in (Buldygin at al., 1992, p.341), (Ostrovsky, 1999, p.34) that

\[ C_1 \varphi_{m,r}(\lambda) \leq \psi_{m,r}(\lambda) \leq C_2 \varphi_{m,r}(\lambda), \]

where \( \psi_{m,r}(\lambda) = \lambda^2, \lambda \in [-2, 2]; \)

\[ \psi_{m,r}(\lambda) = C_3 |\lambda|^{m/(m-1)} \left[ \log(C_4 + |\lambda|) \right]^{-r/(m-1)} \]

in the case \(|\lambda| > 2\). Since the r.v. \( \{\xi(i)\} \) are i., i.d., we have for the non-random sum:

\[ \log \mathbb{E} \exp \left( \lambda n^{-1/2} \sum_{i=1}^{n} \xi(i) \right) = n \varphi_{m,r}(\lambda/\sqrt{n}) \asymp \]

\[ n \psi_{m,r}(\lambda/\sqrt{n}), \text{ where the symbol } \asymp \text{ is understood uniformly on } n; \lambda \in R: \]

\[ 0 < C_1(m, r) \leq \inf_{n \geq 1} \inf_{\lambda \in R} \frac{n \varphi_{m,r}(\lambda/\sqrt{n})}{n \psi_{m,r}(\lambda/\sqrt{n})} \leq \]

\[ \sup_{n \geq 1} \sup_{\lambda \in R} \frac{n \varphi_{m,r}(\lambda/\sqrt{n})}{n \psi_{m,r}(\lambda/\sqrt{n})} \leq C_2(m, r) < \infty. \]

We conclude at \( x \geq 2, A \geq 2: \)

\[ P(S > x) = \sum_{n=1}^{\infty} A^{-1}(1 - 1/A)^{n-1} P \left( \sum_{i=1}^{n} \xi(i)/\sqrt{n} > x\sqrt{A/n} \right). \]

We deduce, choosing again in this sum only the member with \( n = N_0 \) (recall that \( N_0 = N_0(x) \)) and using the main result of paper (Bagdasarov at al, 1995):

\[ P \left( n^{-1/2} \sum_{i=1}^{n} \xi(i) > u \right) \geq \exp \left( -C_1 u^M \log^L(C_2 + u) \right), \]
where $C_1, C_2 = C_1(m,r), C_2(m,r); \ u = u(n) \geq 2$:

$$
P(S > x) \geq (A - 1)^{-1}(1 - 1/A)^N \mathbf{P} \left( \sum_{i=1}^{N_0} \xi(i)/\sqrt{N_0} > x\sqrt{A/N_0} \right) \geq C_6(m) \exp \left( -C_7(m)x^{2M/(M+2)} \left[ \log x \right]^{2L/(M+2)} \right).
$$

**Theorem 3.** For all values $x \geq 3$

$$
V(\text{Dis}, N(0,1), x) \geq C \ x^{-2}.
$$

Here $N(0,1)$ denotes the normal distribution with parameters $0,1$; and $C$ is an absolute constant.

**Proof.** We suppose now $L(\xi(i)) = N(0,1)$, i.e.

$$
P(\xi(i) > x) = \Psi(x) = (2\pi)^{-1/2} \int_{x}^{\infty} \exp(-y^2/2)dy.
$$

We define $\alpha = 1/\text{Ent}[x^2], \text{Ent}[z]$ denotes the integer part of $z$, for $x \geq 3$ and choose the r.v. $\eta$ by the following way: $P(\eta = 2) = 1 - 1/\alpha$, $P(\eta = 1/\alpha) = \alpha$. Then $A = E\eta = 3 - 2\alpha \geq 25/9 > 2$;

$$
P(S > x) > \alpha \mathbf{P} \left( \sum_{i=1}^{1/\alpha} \xi(i) > x\sqrt{A} \right) = (1/[x^2]) \Psi \left( x\sqrt{3 - 2/[x^2]\sqrt{1/[x^2]} \right) \geq x^{-2} \Psi \left( 3\sqrt{3}/8 \right) = Cx^{-2}.
$$

4. **Upper exponential bound for stopping time.** In this section we will obtain the exponential bounds for the tails of distribution r.v. $S$ in the case if $\eta$ is the stopping time for the sequence $\{\xi(i)\}$, in addition to the moment estimations of $S$ in (Gut, 1988), (Gut, 2003). Recall that again $E S = 0$ and $D S = 1$.

**Theorem 4.** Assume that the r.v. $\eta$ belong to the space $G(m,r)$ for some $m > 0, r \in R: \ \eta \in G(m,r)$ and is the stopping time for the sequence $\{\xi(i)\}$, where $\xi(1) \in G(a,b), a = \text{const} > 0, b = \text{const}: \ \forall x \geq 2$

$$
T(\eta, x) \leq \exp \left( -C \ x^m \log^r x \right), \ T(\eta, x) \leq \exp \left( -C \ x^a \log^b x \right).
$$
Denote
\[
q = \frac{2am}{2am + 2a + m}, \quad w = \frac{2am + mb + 2am}{2am + 2a + m}.
\]
We assert that at \( x \geq 2 \)
\[
\sup_{A \geq 2} \sup_{\eta : E\eta = A} T(S, x) \leq \exp \left( -C(a, b, m, r)x^q \log w \right).
\]
(11)

Proof of theorem 4. It follows from our conditions and the theory of \( G \) – spaces (2) that for all \( p \geq 2 \)
\[
|\xi(1)|_p \leq C_1 p^{1/m} \log^{-r/m} p, \quad |\eta|_p \leq C_2 p^{1/a} \log^{-b/a} p.
\]
We obtain using the inequality (0) with optimal constant \( B(p) \):
\[
|S|_p \leq C_3 p^{1+(1/2a)+1/m} \log^{-1-(b/(2a)-(r/m))} p = C_3 p^{1/q} \log^{-w/q} p.
\]
The last inequality is equivalent to (11), see (2).
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