MATRIX TRANSPOSITION AND BRAID REVERSION

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Abstract. Matrix transposition induces an involution \( \tau \) on the isomorphism classes of semi-simple \( n \)-dimensional representations of the three string braid group \( B_3 \). We show that a connected component of this variety can detect braid-reversion or that \( \tau \) acts as the identity on it. We classify the fixed-point components.

1. Introduction

If \( \phi = (X_1, X_2) \) is an \( n \)-dimensional representation of the three string braid group \( B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle \), then so is the pair of transposed matrices \( \tau(\phi) = (X_{tr}^1, X_{tr}^2) \). In this paper we investigate when \( \phi \simeq \tau(\phi) \).

This problem is relevant to detect braid- and knot-reversion. Recall that a knot is said to be invertible if it can be deformed continuously to itself, but with the orientation reversed. There do exist non-invertible knots, the unique one with a minimal number of crossings is knot \( 8_{17} \), see the Knot Atlas \([4]\), which is the closure of the three string braid \( b = \sigma_{-2}^2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1^{-2} \). Proving knot-vertibility of \( 8_{17} \) comes down to separating the conjugacy class of the braid \( b \) from that of its reversed braid \( b' = \sigma_2^2 \sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_2^2 \). Now, observe that a \( B_3 \)-representation \( \phi \) can separate \( b \) from \( b' \), via \( Tr_\phi(b) \neq Tr_\phi(b') \), only if \( \phi \) is not isomorphic to the transposed representation \( \tau(\phi) \).

The involution \( \tau \) on the affine variety \( \text{rep}_n B_3 \) of all \( n \)-dimensional representations passes to an involution \( \tau \) on the quotient variety \( \text{rep}_n B_3 / \text{PGL}_n = \text{iss}_n B_3 \), classifying isomorphism classes of semi-simple \( n \)-dimensional representations. Recall from \([7]\) and \([9]\) that \( \text{iss}_n B_3 \) decomposes as a disjoint union of its irreducible components

\[ \text{iss}_n B_3 = \sqcup_\alpha \text{iss}_n B_3 \]

and the components containing a Zariski open subset of simple \( B_3 \)-representations are classified by \( \alpha = (a, b; x, y, z) \in \mathbb{N}^5 \) satisfying \( n = a + b = x + y + z, a \geq b \) and \( x = \max(x, y, z) \leq b \).

Theorem 1. If the component \( \text{iss}_n B_3 \) contains simple representations, then \( \tau \) acts as the identity on it if and only if \( \alpha \) is equal to

- \( (1, 0; 1, 0, 0) \), or \( (4, 2; 2, 2, 2) \), or
- \( (k, k; k, k - 1, 1) \), or \( (k, k; k, 1, k - 1) \), or
- \( (k + 1, k; k, k, 1) \), or \( (k + 1, k; k, 1, k) \)

for some \( k \geq 1 \). In all these cases, \( \dim \text{iss}_n B_3 = n \). In all other cases, \( \text{iss}_n B_3 \) contains simple representations \( \phi \) such that \( Tr_\phi(b) \neq Tr_\phi(b') \), that is, \( \text{iss}_n B_3 \) can detect braid-reversion.
Hence, the modular group $\Gamma = \text{PSL}_2$, which is the free product of cyclic groups of order two and three (and thus isomorphic to the involution $\tau$ representing to specific stable quiver-representations, and we will describe the element $c$ of the eigenspace $V$ such that

$$\phi^n = V$$

for every proper sub-representations $\rho$ of $\phi$. Choose a vector-space basis for $V$, then the corresponding base-change matrix

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix} \in GL_n(C)$$

determines the quiver-representation $V_B$ with dimension vector $\alpha = (a, b; x, y, z)$.

For $B$ invertible, the representation $V_B$ is semi-stable in the sense of Westbury, meaning that for every proper sub-representations $W$, with dimension vector $\beta = (a', b'; x', y', z')$ we have $x' + y' + z' \geq a' + b'$. If all these inequalities are strict, we call $V_B$ a stable representation, which is equivalent to the $\Gamma$-representation $V = C_{\rho'}^n$, being simple. Westbury showed that two $\Gamma$-representations $V \simeq V'$ if and only if $V_B \simeq V'_B$, as quiver-representations, that is, there exist base-changes in the eigenspaces

$$(M_1, M_2; N_1, N_2, N_3) \in GL_a \times GL_b \times GL_x \times GL_y \times GL_z$$
such that
\[
\begin{pmatrix}
N_1 & 0 & 0 \\
0 & N_2 & 0 \\
0 & 0 & N_3
\end{pmatrix}
\begin{bmatrix}
B_{11}' & B_{12}' \\
B_{21}' & B_{22}' \\
B_{31}' & B_{32}'
\end{bmatrix}
\begin{bmatrix}
M_{1}^{-1} & 0 \\
0 & M_{2}^{-1}
\end{bmatrix}
= 
\begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22} \\
B_{31} & B_{32}
\end{bmatrix}
\]

Working backwards, we recover the $B_3$-representation $\phi = (X_1, X_2)$ from the invertible matrix $B$ via
\[
\begin{align*}
X_1 &= \lambda^{1/6} B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho \end{bmatrix} B \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \\ 0 & 0 & \rho \end{bmatrix} \\
X_2 &= \lambda^{1/6} B^{-1} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho \end{bmatrix} B \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \\ 0 & 0 & \rho \end{bmatrix}
\end{align*}
\]

**Proposition 1.** If the $n$-dimensional simple $B_3$-representation $\phi = (X_1, X_2)$ is determined by $\lambda \in \mathbb{C}^*$ and the stable quiver-representation $V_B$, then $\tau(\phi) = (X_1^r, X_2^r)$ is isomorphic to the representation determined by $\lambda$ and the stable quiver-representation $V_{(B^{-1})^r}$.

**Proof.** Taking transposes of the formulas (*) for the $X_i$ we get
\[
\begin{align*}
X_1^r &= \lambda^{1/6} B^{tr} \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \\ 0 & 0 & \rho \end{bmatrix} (B^{-1})^{tr} \\
X_2^r &= \lambda^{1/6} B^{tr} \begin{bmatrix} 1_x & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho \end{bmatrix} (B^{-1})^{tr} \begin{bmatrix} 1_a & 0 \\ 0 & -1_b \\ 0 & 0 & \rho \end{bmatrix}
\end{align*}
\]

Conjugating these with the matrix $\begin{bmatrix} 1_a & 0 \\ 0 & -1_b \\ 0 & 0 & \rho \end{bmatrix}$ (which is also a base-change action in $GL(\alpha)$) we obtain again a matrix-pair in standard-form (*), this time replacing the matrix $B$ by the matrix $(B^{-1})^r$. \qed

That is, we have reduced the original problem of verifying whether or not $\phi \simeq \tau(\phi)$ as $B_3$-representations to the problem of verifying whether or not the two stable representations $V_B$ and $V_{(B^{-1})^r}$ lie in the same $GL(\alpha)$-orbit.

**Example 1.** The two components $\mathfrak{iss}_\alpha B_3$ containing simple 2-dimensional $B_3$-representations for $\alpha = (1, 1; 1, 1, 0) \text{ or } (1, 1; 1, 0, 1)$ are fixed-point components for the involution $\tau$. A general stable $\alpha = (1, 1; 1, 1, 0)$ dimensional representation $V_B$ is isomorphic to one of the form

\[
\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (1,1) {1};
  \node (3) at (1,-1) {1};
  \node (4) at (2,1) {1};
  \node (5) at (2,-1) {1};
  \draw (1) -- (2);
  \draw (1) -- (3);
  \draw (2) -- (4);
  \draw (3) -- (5);
\end{tikzpicture}
\]
with $a \neq 1$. Hence, we can take $B = \begin{bmatrix} 1 & 1 \\ a & 1 \end{bmatrix}$. But then, $V_B$ and $V_{(B^{-1})^{tr}}$ lie in the same $GL(\alpha) = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$-orbit because

$$(B^{-1})^{tr} = \frac{1}{1-a} \begin{bmatrix} 1 & -a \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{a}{1-a} \end{bmatrix} B \begin{bmatrix} \frac{1}{1-a} & 0 \\ 0 & \frac{-a}{1-a} \end{bmatrix}$$

3. The stratification and potential fixed-point components

In this section we will show that a component $\text{iss}_\alpha B_3$ containing $n$-dimensional simple $B_3$-representations is a fixed-point component for the involution $\tau$ only if $\alpha$ is among the list of theorem 1.

Because the group algebra $\mathbb{C}[\Gamma] = \mathbb{C}C_2 \ast \mathbb{C}C_3$ is a formally smooth algebra, we have a Luna stratification of $\text{iss}_\alpha \Gamma$ by representation types, see [5, §5.1]. A point $p$ in $\text{iss}_\alpha \Gamma$ determines the isomorphism class of a semi-simple $\Gamma$-representation

$$V_p = S_{1e_1}^{\beta_1} \oplus \ldots \oplus S_{k}^{\beta_k}$$

with all $S_i$ distinct simple $\Gamma$-representations with corresponding dimension vectors $\beta_i = (a_i, b_i; x_i, y_i, z_i)$. We say that $p$ (or $V_p$) is of representation type

$$\tau = (e_1, \beta_1; \ldots; e_k, \beta_k) \quad \text{and clearly} \quad \alpha = \sum_i e_i \beta_i$$

With $\text{iss}_\alpha \Gamma(\tau)$ we denote the subset of all points of representation type $\tau$. Recall that $\beta_i$ is the dimension vector of a simple $\Gamma$-representation if and only if $a_i + b_i = x_i + y_i + z_i$ and $\max(x_i, y_i, z_i) \leq \min(a_i, b_i)$ if $x_i y_i z_i \neq 0$ (the remaining cases being the 1- and 2-dimensional components). It follows from Luna’s results [8] that every $\text{iss}_\alpha \Gamma(\tau)$ is a locally closed smooth irreducible subvariety of $\text{iss}_\alpha \Gamma$ of dimension $\sum_i (1 + 2a_i b_i - (x_i^2 + y_i^2 + z_i^2))$ and that

$$\text{iss}_\alpha \Gamma = \bigcup_{\tau} \text{iss}_\alpha \Gamma(\tau)$$

is a finite smooth stratification of $\text{iss}_\alpha \Gamma$. Degeneration of representation types, see [6, p. 247], defines an ordering $\leq$ on representation types and by [5, Prop. 5.3] we have that $\text{iss}_\alpha \Gamma(\tau')$ lies in the Zariski closure of $\text{iss}_\alpha \Gamma(\tau)$ if and only if $\tau' \leq \tau$.

Observe that the involution $\tau$ on $\text{iss}_\alpha \Gamma$ induced by $\tau(V_B) = V_{(B^{-1})^{tr}}$ preserves the strata and its restriction to $\text{iss}_\alpha \Gamma(\tau)$ is induced by the involutions $\tau$ on the components $\text{iss}_\beta \Gamma$. As the fixed-point set of $\tau$ is a closed subvariety of $\text{iss}_\alpha \Gamma$ we deduce immediately:

**Lemma 1.** If $\tau$ is the identity on a Zariski open subset of $\text{iss}_\alpha \Gamma(\tau')$, then $\tau = \text{id}$ on all strata $\text{iss}_\alpha \Gamma(\tau')$ with $\tau' \leq \tau$. Conversely, if $\tau = (e_1, \beta_1; \ldots; e_k, \beta_k)$ and $\tau \neq \text{id}$ on one of the components $\text{iss}_\beta \Gamma$, then $\tau \neq \text{id}$ on all strata $\text{iss}_\alpha \Gamma(\tau')$ with $\tau \leq \tau'$.

In [7] we have shown that for $\beta = (3, 3; 2, 2, 2)$ there are simple $B_3$-representations able to separate the braid $b$ from the introduction from its reversed braid $b'$. In particular, $\tau$ does not act as the identity on $\text{iss}_\beta \Gamma$. We proved this by parametrizing
the matrices $B$ for a dense open subset of $\text{iss}_3 \Gamma$ by

$$B = \begin{bmatrix}
1 & 0 & 0 & a & 0 & f \\
0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & d & e \\
0 & 1 & 0 & b & c & 0 \\
g & 0 & 1 & 0 & 0 & 1
\end{bmatrix}$$

for free parameters $a, \ldots, g$. We then computed the matrix-pair $\phi = (X_1, X_2)$ from (*) with generic values of the parameters in $\mathbb{Z}[\rho]$ and checked that $\text{Tr}_\phi(b) \neq \text{Tr}_\phi(b')$.

**Proposition 2.** If $\alpha$ is the dimension vector of a simple $\Gamma$-representation such that $\alpha \geq \beta = (3, 3; 2, 2, 2)$, then $\tau \neq \text{id}$ on $\text{iss}_3 \Gamma$ and there are simple representations $\phi \in \text{iss}_3 \Gamma$ such that $\text{Tr}_\phi(b) \neq \text{Tr}_\phi(b')$.

**Proof.** The unique open stratum of $\text{iss}_3 \Gamma$ corresponds to the unique maximal representation type $\tau_{\text{gen}} = (1, \alpha)$, that is, $\text{iss}_3 \Gamma(\tau_{\text{gen}})$ is the open set of simple $\Gamma$-representations.

If $\alpha - \beta$ is the dimension vector of a simple $\Gamma$-representation, then we have a representation type $\tau = (1, \beta; 1, \alpha - \beta)$ such that $\tau \neq \text{id}$ and $\text{Tr}(b) \neq \text{Tr}(b')$ on $\text{iss}_3 \Gamma(\tau)$. But then, by the previous lemma, these facts also hold for $\text{iss}_3 \Gamma(\tau_{\text{gen}})$.

If $\alpha - \beta$ is not the dimension vector of a simple $\Gamma$-representation, we consider the generic (maximal) representation type $\tau' = (e_1, \beta_1; \ldots; e_k, \beta_k)$ in $\text{iss}_{3-\beta} \Gamma$. But then, $\tau = (1, \beta; e_1, \beta_1; \ldots; e_k, \beta_k)$ is a representation type for $\text{iss}_3 \Gamma$ and we can repeat the foregoing argument. \qed

**Proposition 3.** If $\alpha = (a, b; x, y, z)$ is a simple dimension vector such that $\tau$ acts trivially on $\text{iss}_3 B_3$, then

$$\dim \text{iss}_3 B_3 = n = a + b = x + y + z$$

**Proof.** By the previous result we must have $\beta \preceq \alpha$ and hence either $n \leq 5$ or $\min(x, y, z) = 1$. For a simple $B_3$-dimension vector we may assume that $a \geq b$ and $x = \max(x, y, z)$, which leaves us with the following list of potential fixed-point components

| $n$ | $\alpha$ | $\dim \text{iss}_3 B_3$ |
|-----|-----------|-------------------|
| 1   | (1, 0; 1, 0, 0) | 1 |
| 2   | (1, 1; 1, 1, 0) | 2 |
|     | (1, 1; 1, 0, 1) | 2 |
| 3   | (2, 1; 1, 1, 1) | 3 |
| 4   | (2, 2; 2, 1, 1) | 4 |
| 5   | (3, 2; 2, 2, 1) | 5 |
|     | (3, 2; 2, 1, 2) | 5 |
| 6   | (3, 3; 3, 2, 1) | 6 |
|     | (3, 3; 3, 1, 2) | 6 |
|     | (4, 2; 2, 2, 2) | 6 |
| $2k$ | $(k, k; k, k - 1, 1)$ | $2k$ |
|     | $(k, k; k, 1, k - 1)$ | $2k$ |
| $2k + 1$ | $(k + 1, k; k, k, 1)$ | $2k + 1$ |
|     | $(k + 1, k; k, 1, k)$ | $2k + 1$ |
By example [3] we know that the 1- and 2-dimensional components are fixed-point components. All other potential fixed-point components belong to the infinite families, with one exception: \((4, 2; 2, 2, 2)\). In the following sections we will prove that all of these are indeed fixed-point components.

4. The infinite families

In this section we will prove that for \(\alpha = (k, k; k, k - 1, 1)\) (the even case) and \(\alpha = (k + 1, k; k, 1)\) (the odd case), \(\text{iss}_\alpha B_3\) is a fixed-point component. We will prove the even case by direct matrix calculations and deduce the odd case from it by a degeneration argument.

Proposition 4. For all \(k \in \mathbb{N}_+\) and \(\alpha = (k, k; k, k - 1, 1)\), \(\text{iss}_\alpha B_3\) is a fixed-point component.

Proof. A general representation in \(\text{iss}_\alpha \Gamma\) corresponds to an invertible \(2m \times 2m\) matrix \(B\) and quiver-representation

\[
\begin{array}{c}
\bullet_1 & \rightarrow & \bullet_2 \\
B_{11} & & B_{12} \\
B_{21} & & B_{22} \\
B_{31} & & B_{32} \\
\end{array}
\]

After a base-change in the lower-left hand vertex, we may assume that the modified matrix-blocks become

\[
\begin{bmatrix}
B'_{22} \\
B'_{32}
\end{bmatrix} = I_k
\]

The block \(B_{12}\) is modified to an invertible \(k \times k\) matrix \(B'_{12}\) which becomes the identity matrix \(I_k\) after performing a base-change in the top-right hand vertex. This changes the block \(B_{11}\) to an invertible matrix \(B'_{11}\) which becomes the identity matrix \(I_k\) after a base-change in the top-left hand vertex. Hence, we may assume that, up-to isomorphism, the matrix \(B\) has the following block-form

\[
B = \begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22} \\
B_{31} & B_{32}
\end{bmatrix} = \begin{bmatrix}
I_k & I_k \\
A & I_k
\end{bmatrix}
\]

with all blocks of sizes \(k \times k\) and \(A\) an invertible matrix such that \(B\) is invertible. One verifies that

\[
(B^{-1})^r = \begin{bmatrix}
-C & I_k + C \\
C & -C
\end{bmatrix} \quad \text{with} \quad C = (A - I_k)^{-1}
\]

and performing the base-change

\[
(AC^{-1}, -C; -A^{-1}, I_{k-1}, I_1) \in GL_k \times GL_k \times GL_k \times GL_{k-1} \times GL_1
\]

we obtain

\[
B = \begin{bmatrix}
I_k & I_k \\
A & I_k
\end{bmatrix} = \begin{bmatrix}
-A^{-1} & 0 \\
0 & I_k
\end{bmatrix} \begin{bmatrix}
-C & I_k + C \\
C & -C
\end{bmatrix} \begin{bmatrix}
C^{-1}A & 0 \\
0 & -C^{-1}
\end{bmatrix}
\]
Therefore, the $\Gamma$-representations determined by the matrices $B$ and $(B^{-1})^{tr}$ are isomorphic and hence the involution $\tau$ is the identity map on the component $\text{iss}_\alpha B_3$. □

**Proposition 5.** For all $k \in \mathbb{N}_+$ and $\alpha = (k + 1, k; k, k, 1)$, $\text{iss}_\alpha B_3$ is a fixed-point component.

**Proof.** Let $\alpha_+ = (k + 1, k + 1; k + 1, k, 1)$, then the stratum $\tau = (1, \alpha; 1, (0, 1; 1, 0, 0))$ lies in the closure of the generic stratum $\tau_{\text{gen}} = (1, \alpha_+)$ in $\text{iss}_{\alpha_+} \Gamma$. The result follows from the foregoing proposition and lemma 1. □

5. The exceptional component and vectorbundles on $\mathbb{P}_2$

To finish the proof of theorem 1 it suffices to show that $\text{iss}_\beta B_3$ is a fixed-point component for $\beta = (4, 2; 2, 2, 2)$. In [7] we have given a parametrization of the matrices $B$ for a dense open subset of $\text{iss}_\beta \Gamma$

$$B = \begin{bmatrix}
1 & 0 & 0 & 0 & a & 0 \\
0 & 1 & e & 1 & 0 & 1 \\
1 & c & d & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & b \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}$$

One can attempt to show that $B$ and $(B^{-1})^{tr}$ belong to the same $GL(\beta)$-orbit by explicit computation. We follow a different approach, allowing us to connect this component to the study of stable vectorbundles on $\mathbb{P}_2$.

**Proposition 6.** For $\alpha = (2n, n; n, n, n)$, the component $\text{iss}_\alpha \Gamma$ is birational to $M_{\mathbb{P}_2}(n; 0, n, n)$, the moduli space of semi-stable rank $n$ bundles on $\mathbb{P}_2$ with Chern classes $c_1 = 0$ and $c_2 = n$.

**Proof.** A representation in $\text{rep}_\alpha \Gamma$ in general position

is such that $\psi : \mathbb{C}^{2n} \xrightarrow{B_{11} \oplus B_{21} \oplus B_{31}} \mathbb{C}^n \oplus \mathbb{C}^n \oplus \mathbb{C}^n$ in injective, whence its cokernel defines maps $\text{Cok}(\psi) : \mathbb{C}^n \oplus \mathbb{C}^n \oplus \mathbb{C}^n \xrightarrow{(C_{12}, C_{22}, C_{32})} \mathbb{C}^n$ and therefore a representation for the quiver-setting
By the general theory of reflection functors, isomorphism classes of representations are preserved under this construction. By the fundamental theorem of \(GL_n\)-invariants \([5, \text{Thm. II.4.1}]\) we can eliminate the base-change action in the middle vertices and obtain a representation of the quiver-setting

\[
\begin{array}{c}
C_{12}B_{12} \\
\downarrow \\
C_{22}B_{22} \\
\downarrow \\
C_{32}B_{32}
\end{array}
\]

By results of Klaus Hulek \([2]\), the corresponding moduli space of semi-stable quiver-representations (as in \([3]\) for the stability structure \((-1,1))\) is birational to \(M_{\mathbb{P}^2}(n;0,n)\).

**Proposition 7.** \(\text{iss}_3 B_3\) is a fixed point component.

**Proof.** By results of Wolf Barth \([1]\), we know that a stable rank 2 bundle \(E\) on the projective plane with Chern-classes \(c_1 = 0\) and \(c_2 = 2\) is determined up to isomorphism by its curve of jumping lines, that is the collection of those lines \(L \subset \mathbb{P}^2\) such that \(E|_L \not\cong O_{\mathbb{P}^2}^2\). If \(E\) is determined by the quiver-representation as in the previous proposition and if \(x,y,z\) are projective coordinates of the dual plane \(\mathbb{P}^2^*\), then the equation of this curve of jumping lines is given by

\[
det(C_{12}B_{12}x + C_{22}B_{22}y + C_{32}B_{32}z) = 0
\]

In terms of the matrix \(B\) and its inverse \(B^{-1}\) these \(2 \times 2\) matrices are given as

\[
\begin{pmatrix}
* & * & * \\
* & * & * \\
C_{12} & C_{22} & C_{32}
\end{pmatrix}
\begin{pmatrix}
* & * \\
* & * \\
B_{12} & B_{22} & B_{32}
\end{pmatrix}
= 
\begin{pmatrix}
I_2 & 0 & 0 \\
0 & I_2 & 0 \\
0 & 0 & I_2
\end{pmatrix}
\]

But then, the bundle \(F\) corresponding to the matrix \((B^{-1})^tr\) is determined by the \(2 \times 2\) matrices

\[
\begin{pmatrix}
* & * & * \\
* & * & * \\
B_{12}^tr & B_{22}^tr & B_{32}^tr
\end{pmatrix}
\begin{pmatrix}
* & * \\
* & * \\
C_{12}^tr & C_{22}^tr & C_{32}^tr
\end{pmatrix}
= 
\begin{pmatrix}
I_2 & 0 & 0 \\
0 & I_2 & 0 \\
0 & 0 & I_2
\end{pmatrix}
\]

and hence its curve of jumping lines

\[
det(B_{12}^trC_{12}^trx + B_{22}^trC_{22}^try + B_{32}^trC_{32}^trz)
\]

is the same as that for \(E\) and hence by Barth’s result \(E \cong F\). \(\square\)

**Remark 1.** One can repeat the above argument verbatim for \(\alpha = (2n,n;n,n,n)\). However, if \(n > 2\), the bundle \(E\) corresponding to the matrix \(B\) is determined by its curve of jumping lines (defined as above by the \(n \times n\) matrices \(B_{ij}\) and \(C_{ij}\)) together with a half-canonical divisor on it, see \([2]\). Whereas the curve of jumping lines \(Y\) of the bundle \(F\) corresponding to the matrix \((B^{-1})^tr\) coincides with that of \(E\), the involution \(\tau\) acts non-trivially on the Jacobian \(\text{Pic}_d^F\) where \(d = \frac{1}{2}n(n - 1)\).
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