Adding dynamical generators in quantum master equations

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The quantum master equation is a widespread approach to describing open quantum system dynamics. In this approach, the effect of the environment on the system evolution is entirely captured by the dynamical generator providing a compact and versatile description. However, care needs to be taken when several noise processes act simultaneously or the Hamiltonian evolution of the system is modified. Here, we show that generators can be added at the master equation level without compromising physicality only under restrictive conditions. Moreover, even when adding generators results in legitimate dynamics, this does not generally correspond to the true evolution of the system. We establish general conditions under which direct addition of dynamical generators is justified, showing that it is ensured under weak coupling and for settings where the free system Hamiltonian and all system-environment interactions commute. In all other cases, we demonstrate by counterexamples that the exact evolution derived microscopically cannot be guaranteed to coincide with the dynamics naively obtained by adding the generators.

Introduction.—It is generally impossible to completely isolate a small system of interest from the surrounding environment. Thus, dissipative effects caused by the environment are important in almost every quantum experiment, ranging from highly controlled settings, where much effort is invested in minimising them, to areas where the dissipation is the key object of interest. In many cases, exact modelling of the environment is not practical and its effect is instead accounted for by employing effective models describing the induced noise. Different approaches exist, e.g., quantum Langevin and stochastic Schrödinger equations, quantum state diffusion models or Hilbert-space averaging methods [1–4].

Arguably, the most widely applied approach is to use the quantum master equation description [2, 3]. In this approach, the system evolution is given by a time-local differential equation, where the effect of the environment is captured by the dynamical generator. A master equation can be derived from a microscopic model of the system and environment, and their interaction, by tracing over the environment and applying appropriate approximations [2, 3]. However, master equations are also often applied directly, without explicit reference to an underlying model. In that case, care needs to be taken when several noise processes act simultaneously or the Hamiltonian evolution of the system is modified. Simultaneous coupling to multiple baths in a microscopic model does not generally correspond to simple addition of noise generators. Additivity of noise at the master equation level has been discussed for qubits [5–7] and in quantum transport [8].

In this work, we address the questions of when:

(i) Naive addition of generators yields physically valid dynamics.

(ii) The corresponding evolution coincides with the true system dynamics derived from the underlying microscopic model.

First, we show that (i) is satisfied, in addition to the case of Markovian generators, for generators which are commutative, can be interpreted as a fictitious semigroup at each time instance, and preserve commutativity of the dynamics under addition. Outside of this class, we find examples of simple qubit master equations which lead to unphysical dynamics. We find that (ii) holds in the weak coupling regime, extending previous results in this direction [8, 9], and also provide sufficient conditions for (ii) dictated by the commutativity of Hamiltonians at the microscopic level. We combine these generic considerations with a detailed study of a specific open system, namely a qubit interacting simultaneously with multiple spin baths, for which we provide examples where (ii) is not satisfied for different couplings and internal Hamiltonians. Our results are of relevance to areas of quantum physics where open systems naturally appear and master equations are routinely employed, e.g., quantum thermodynamics [10–16], quantum metrology [17–21], dissipative state engineering [22, 23], quantum transport [24], or quantum biology [25].

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FIG. 1. (a) A system $S$ interacting with a single environment $E$ (b) A system $S$ simultaneously interacting with multiple, independent environments $E_1, E_2, E_3, \ldots$
In fact, one can ensure that the interaction and Schrödinger pictures.

The system dynamics is obtained by tracing over the environmental degrees of freedom. In the Schrödinger picture, the reduced state of the system evolves as

$$\frac{d}{dt}\rho_S(t) = -i\text{Tr}_E[H_S + H_E + H_I, \rho_{SE}(t)]$$

(1)

where $\rho_{SE}$ is the total system-environment state. If the environment is initially uncorrelated with the system, $\rho_{SE}(0) = \rho_S(0) \otimes \rho_E(0)$, the state is unambiguously specified at all later times by the integral of (1).

In practice, however, it is often not viable to solve the master equation (3) only when $[V, H]$ for any initial state $\rho_S(0)$. For the dynamics to be physically valid, all $\Lambda_t$ must be completely positive and trace preserving (CPTP), see App. B. We refer to a family of dynamical generators $\mathcal{L}_t$ as physical if it generates physically valid dynamics.

Any family of dynamical generators, whether physical or not, can be uniquely decomposed as [28]

$$\mathcal{L}_t[\rho] = -i[H(t), \rho] + \sum_{i,j=1}^{d^2-1} D_{ij}(t) \left( F_i^* F_j^T - \frac{1}{2} \{F_j^T F_i, \rho\} \right),$$

(4)

where $\{F_i\}_{i=1}^{d^2}$ is any orthonormal operator basis with $\text{Tr}[F_i^T F_j] = \delta_{ij}$, and all $F_i$ traceless except $F_0 = \mathbb{1}/\sqrt{d}$. The Hamiltonian part of the generator, $\mathcal{H}_t$, is then determined by $H(t)$ in (4), while the dissipative part $D_t$ is set by the Hermitian matrix $\mathcal{D}(t)$. Although general criteria for physicality of dynamical generators are not known, two classes of physical dynamics can be identified based on the above decomposition.

First, when $\mathcal{D}(t)$ is positive semidefinite, $\mathcal{L}_t$ is said to be of Gorini-Kossakowski-SudaRshan-Lindblad (GKSL) form [28, 29]. In this case, the evolution is not only physical but also $\text{CP-divisible}$, i.e., Markovian [30, 31] (see App. B 1). When, in addition, $\mathcal{H}$ and $\mathcal{D}$ are time independent, the dynamics forms a semigroup $\Lambda_t = e^{\mathcal{L}_t}$ with $\mathcal{L} = \mathcal{L}_t$ [32].

Second, physicality is also ensured when $Z_t = \log \Lambda_t$ takes the GKSL form at all times [27]. We call such evolutions $\text{semigroup-simulable}$ (SS), since at any $t$ they can be interpreted as a fictitious semigroup $\Lambda_t = e^{\mathcal{L}_t}$, generated by $Z_t$. In general, the condition for an evolution to be SS does not take a tangible form. However, in the case of $\text{commutative}$ dynamics, for which $[\mathcal{L}_s, \mathcal{L}_t] = 0$ at all $s, t \geq 0$, it corresponds simply to $\int_0^t d\tau \mathcal{D}(\tau) \geq 0$ for any $t$ (see App. B 1).

Additivity of dynamical generators.— We define two families of physical generators, $\mathcal{L}^{(1)}_t$ and $\mathcal{L}^{(2)}_t$, to be additive if all the generators obtained from linear combinations, $\mathcal{L} = \alpha\mathcal{L}^{(1)}_t + \beta\mathcal{L}^{(2)}_t$ with $\alpha, \beta \geq 0$, are also physical. Note that, with this definition, generator families can be additive only if they are $\text{resca}	ext{leable}$—remain physical under $\Lambda_t \rightarrow \alpha\Lambda_t$ for any $\alpha > 0$. Non-rescalable examples can be found (see App. B 3) for generators that e.g. exhibit singularities at finite times [26], are derived assuming week-coupling interactions [33], or lead to physical (even commutative) but non-SS dynamics [34]. On the other hand, when $\mathcal{L}^{(1)}_t$ and $\mathcal{L}^{(2)}_t$ are of GKSL form then so is $\mathcal{L}_t$. Hence, generator families of CP-divisible dynamics must all be additive. Thus, we schematically represent them by a convex set in Fig. 2 and, as the same argumentation applies to semigroups, we depict these as a convex subset [35]. In App. B 4, we demonstrate that additivity is also respected by any pair of physical generator families which are commutative and SS, and remain commutative under addition. In particular, these form various SS commutative (SSC) subclasses that we depict in Fig. 2 as general overlapping convex sets containing both CP-divisible and non-CP-divisible evolutions [36].

Now, as indicated in Fig. 2, by adding a generator family that is SS, but not CP-divisible, and another physical
Within a microscopic model, if the system is in an environment that cares must be taken when combining generators of a non-Markovian master equation (3) can be broken (see App. B 2) [3]. Note that it follows that physicality of spin-boson and Jaynes-Cummings models, respectively demonstrate that this is also the case when both semi-group and non-Markovian contributions come from explicit microscopic derivations. In particular, as the generators (5) describe dephasing and spontaneous-emission processes, we consider non-Markovian forms derived from spin-boson and Jaynes-Cummings models, respectively (see App. B 2) [3]. Note that it follows that physicality of a non-Markovian master equation (3) can be broken even by addition of a time-invariant dissipative term.

Multiple environments. From the above, it is clear that care must be taken when combining generators corresponding to different processes. Identifying the appropriate master equation describing a system coupled to several environments may be very non-trivial [39]. Within a microscopic model, if the system is interacting with several independent environments, as in Fig. 1(b), combining multiple environments corresponds simply to adding the Hamiltonians $H_E = \sum_i H_{E_i}$, and $H_I = \sum_i H_{I_i}$. In contrast, the action of simultaneous noise processes is not generally captured by adding generators at the master equation level, leaving it ambiguous which properties the Hamiltonians must satisfy in order for generator addition to be justified [40].

For the setting in Fig. 1(b), equation (2) is replaced by

$$\frac{d}{dt} \bar{\rho}_S(t) = -\sum_{ij} \int_0^t ds \text{Tr}_E [H_{I_i}(t), [H_{I_j}(s), \bar{\rho}_{SE}(s)]]$$

(6)

where $E$ labels all the environments, and the interaction picture is generated by the total free Hamiltonian $H_S + \sum_i H_{E_i}$. Assuming a time-local master equation to exist both in the presence of each single environment and all of them, we can rewrite (6) as

$$\frac{d}{dt} \bar{\rho}_S(t) = \sum_i \mathcal{L}_i^{(t)}[\bar{\rho}_S(t)]$$

$$- \sum_{i \neq j} \int_0^t ds \text{Tr}_{E_{ij}} [H_{I_i}(t), [H_{I_j}(s), \bar{\rho}_{SE_{ij}}(s)]]$$

(7)

where $\mathcal{L}_i^{(t)}$ is the generator when only the $i$th environment is present, $\bar{\rho}_{SE_{ij}}$ denotes the joint state of the system and environments $i$ and $j$, while $\text{Tr}_{E_{ij}}$ stands for the trace over these environments. Hence, the dynamical generator corresponding to a system simultaneously interacting with multiple environments can be constructed by simple addition of the generators associated with each individual environment if and only if the cross-term in (7) vanishes. This provides a general criterion for when a microscopic derivations lead to addition of dynamical generators at the master equation level.

We first note that the cross-term in (7) vanishes under weak coupling. That this holds under the Born-Markov approximation [3] follows e.g. from Ref. [9], however, we show in App. C that it also applies to a larger (see, e.g., [39]) class of weakly interacting open systems. In particular, it holds for all master equations derived employing the time-convolutionless (TCL) approach [41] up to the second order in all the interaction parameters.

Next, we investigate the implications that commutativity of the system, environment, and interaction Hamiltonians has on the validity of generator addition. We consider the cases when all $H_{I_i}$ commute with each other (II), with $H_S$ (IS), or with all $H_{E_i}$ (IE), and summarize the results in Fig. 3. Importantly, only when the interaction Hamiltonians commute among themselves and with the system Hamiltonian—$[H_{I_i}, H_{I_j}] = 0$, and $[H_{I_i}, H_S] = 0$ for all $i, j$—dynamical generators are guaranteed to simply add in the interaction picture. In this case, the total time-evolution operator factorises

$$U_{SE}(t) = e^{-i(H_S + \sum_j H_{E_j} + H_I)t} = e^{-iH_ST \prod_j e^{-i(H_{I_j} + H_{E_j})t}},$$

(8)
Does commutativity ensure the validity of addition?

![Diagram](image)

**FIG. 3.** Connection between the commutativity of the microscopic Hamiltonians in the scenario of Fig. 1(b) and the validity of generators addition at the master equation level (in the interaction picture). Each set above indicates commutativity of the interaction Hamiltonians with: II – each other, IS – the system Hamiltonian, IE – the bath Hamiltonians.

so the system dynamics is described by a product of commuting channels \( \tilde{\rho}_S(t) = \prod_j \Lambda^{(j)}_t (\tilde{\rho}(0)) \) that may be associated with each individual environment, as (see App. D) \( \Lambda^{(j)}_t = \Tr_{E_j} \{ e^{-i[H_{E_j} + H_S]t} \} \). Note that this corresponds to adding generator families which belong to the same SSC class in Fig. 2.

In all other cases (marked ‘No’ in Fig. 3), commutativity does not ensure the generators to simply add. We prove this by establishing counterexamples with help of a concrete microscopic model, for which the evolution of the system interacting with each environment separately, as well as all simultaneously, can be explicitly solved. It is sufficient to do so for the scenarios in which either all \( H_L \) commute with all \( H_{E} \), and \( H_S \) (intersection IS \( \cap \) IE in Fig. 3), or all \( H_L \) commute with each other and all \( H_{E} \), (intersection II \( \cap \) IE in Fig. 3), since it then follows that neither II, IS, nor IE alone can ensure the validity of generator addition. It is known that \([H_I, H_S] = 0\) implies the evolution to be CP-divisible [42]. Thus, as families of CP-divisible generators are additive (c.f. Fig. 2), our counterexample for IS \( \cap \) IE (described in App. E 4) corresponds to a case where generator addition results in dynamics which is physical but does not agree with the microscopic derivation. For a setting in which \( H_L \) do not commute neither among each other nor with \( H_S \), validity of adding generators has been discussed in Ref. [6].

We consider a model of a single qubit (spin-1/2) in contact with one or more environments consisting of many identical spins. The system spin couples to the total spin (or ‘magnetisation’) of each environment, while the environment Hamiltonians are taken to vanish (all \( H_{E} = 0 \)). For a single environment with \( N \) spins, \( H_I = A \otimes \hat{m}, A \) is some system operator, and

\[
\hat{m} = \sum_{n=1}^{N} \hat{\sigma}_z^{(n)} = \sum_{k=0}^{N} m_k \Pi_k,
\]

where \( \Pi_k \) denotes a projector onto the subspace of environment states with magnetisation number \( m_k \). This corresponds to the Curie-Weiss measurement model with a null environment Hamiltonian [43, 44]. We take the initial environment state in \( \rho_{SE}(0) = \rho_S(0) \otimes \rho_E(0) \) to be a classical mixture of different magnetisations, \( \rho_E(0) = \sum_k q_k \Pi_k \), so that the probability for a given magnetisation to occur is \( p(m_k) = q_k \Tr \Pi_k = q_k(N^k) \). As a result, the joint system-environment state at all times remains a mixture of states with different magnetisations

\[
\rho_{SE}(t) = \sum_{k=0}^{N} q_k \, \rho_S^{(k)}(t) \otimes \Pi_k,
\]

where \( \rho_S^{(k)}(t) \) can be understood as the conditional state of the system for environment magnetisation \( m_k \). We determine the system dynamics by decomposing

\[
\rho_S(t) = \Tr_E \rho_{SE}(t) = \sum_{k=0}^{N} p(m_k) \rho_S^{(k)}(t),
\]

and solving the von Neumann equation (see App. E 1)

\[
\frac{d}{dt} \rho_S^{(k)}(t) = -i[H_S + m_k A, \rho_S^{(k)}(t)]
\]

independently for each \( \rho_S^{(k)}(t) \), given \( H_S \) and \( A \). In the limit of large environments \( N \rightarrow \infty \), the sum in (11) can be replaced by an integral, and \( p(m_k) \) by a continuous distribution \( p(m) \). This allows for an explicit microscopic derivation of different qubit dynamics, e.g. dephasing—see App. E 3.

For our first example, we take \( H_S = 0 \) and two environments with \( H_{I_1} = A_1 \otimes \hat{m}_1 \otimes 1_2, H_{I_2} = A_2 \otimes 1_1 \otimes \hat{m}_2 \), and \( A_1 = \frac{1}{2} g_1 \hat{\sigma}_z \), \( A_2 = \frac{1}{2} g_2 \hat{\sigma}_x \). Then the interactions commute with \( H_S \) and the \( H_{E} \) but \([H_{I_1}, H_{I_2}] \neq 0\). Generalising (12) to two magnets (see App. E 4) and associating each \( \rho_S^{(k,k')} \) to a Bloch vector \( \mathbf{r}^{(k,k')} \), one obtains the equations of motion when coupling to both environments simultaneously

\[
\frac{d \mathbf{r}^{(k,k')} (t)}{dt} \cdot \sigma = \frac{i}{2} \left[ \sigma x z (k,k'), \mathbf{r}^{(k,k')} (t) \cdot \sigma \right],
\]

where \( \sigma = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z) \) and \( \hat{\sigma}_{xz} = g_1 m_{1,k} \hat{\sigma}_z + g_2 m_{2,k} \hat{\sigma}_x \). This can be solved explicitly, and the evolutions corresponding to interaction with a single environment are obtained by setting \( g_1 = 0 \) or \( g_2 = 0 \). As in (11), we compute \( \rho_S(t) \) by taking the \( N \rightarrow \infty \) limit for both magnets and their \( p(m_i) \) to be Gaussian. As a result, we obtain the maps in \( \rho_S(t) = A_t [\rho_S(0)] \) for all \( t \) and the corresponding family of dynamical generators \( L_t = A_t A_t^{-1} \). We find that \( L_t \neq L^{(1)}_t + L^{(2)}_t \), where \( L^{(1)}_t \), \( L^{(2)}_t \), and \( L_t \) are the generators corresponding to interaction with environment 1, 2, and both, respectively (see App. E 4 for details). As both system and environment Hamiltonians vanish in this case, the Schrödinger and interaction pictures coincide. This provides the desired counterexample for the region IS \( \cap \) IE in Fig. 3.
For our second example, we take \( H_S = \frac{1}{2} \omega \sigma_x \) and \( A_1 = \frac{1}{2} g_1 \sigma_z, \ A_2 = \frac{1}{2} g_2 \sigma_z \). Then, clearly \([H_S, H_{I_1}] \neq 0\) and \([H_S, H_{I_2}] \neq 0\) while all \( H_{I_1} \) commute with each other and all \( H_{I_E} \). In this case, the Schrödinger and interaction pictures do not coincide, hence the dynamics must be solved in the interaction picture. The equations of motion become (see App. E.5)

\[
\frac{d\hat{r}_{k,k'}(t)}{dt} \cdot \sigma = -\frac{i}{2} \xi_{k,k'} [\hat{\sigma}_z(t), \hat{r}_{k,k'}(t)] \cdot \sigma, \tag{14}
\]

where \( \xi_{k,k'} = g_1 m_{1,k} + g_2 m_{2,k} \) and \( \hat{\sigma}_z(t) = \cos(\omega t) \hat{\sigma} + \sin(\omega t) \hat{\sigma}_y \). Again, this can be solved in general, for \( g_1 = 0 \), and for \( g_2 = 0 \), in order to determine respectively \( L_i, L_i^{(2)}, \) and \( L_i^{(1)} \) (in the \( N \to \infty \) limit). As before, we find that \( L_i \neq L_i^{(1)} + L_i^{(2)} \), so we conclude that it is also not generally valid to simply add generators in the region \( \Pi \cap 1E \) of Fig. 3.

Conclusions.—We have investigated under what circumstances modifications to open system dynamics can be effectively dealt with at the master equation level by adding dynamical generators. We have identified a condition (semigroup simulability and commutativity preservation) applicable beyond Markovian (CP-divisible) dynamics which guarantees generator addition to yield physical evolutions. We have also demonstrated that simple qubit generators violating this condition result in unphysical dynamics under addition. Moreover, even when physically valid, addition does not generally correspond to the real evolution derived from a microscopic model describing interactions with multiple environments. We have formulated a general criterion under which the addition of generators associated with each individual environment indeed yields the correct dynamics. In particular, it is satisfied in weak-coupling regimes in which the Born-Markov or, more generally, second-order TCL approximations hold. In parallel, we have demonstrated that, at the microscopic level, commutativity of the interaction Hamiltonians among each other and with the system Hamiltonian also ensures addition of dynamical generators (in the interaction picture) to give the correct dynamics. We believe that our results may prove useful in areas where the master equation description of open quantum systems is a common workhorse, including quantum metrology, thermodynamics, transport, and engineered dissipation.

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Appendix A: From microscopic models to master equations

For an extensive analysis of the derivations and interpretations of master equations (MEs) describing dynamics of open quantum systems, we refer the reader to the books of Breuer and Petruccione [3] and Rivas and Huelga [45]. In what follows, we present only derivations relevant to the claims and points presented in the main text.

1. Master equation in the exact form

Let us consider a time-invariant total (T) Hamiltonian consisting of Hamiltonians describing the quantum system (S) of interest, its environment (E) and their interaction (I):

\[ H_T = H_S + H_E + H_I, \]  
(A1)

and introduce the interaction picture (IP) with the notation:

\[ \bar{H}(t) := e^{i(H_S + H_E)t} H e^{-i(H_S + H_E)t}, \] \[ \bar{\rho}(t) := e^{i(H_S + H_E)t} \rho(t) e^{-i(H_S + H_E)t}, \]  
(A2)

for Hamiltonians and density matrices, respectively. Then, we may write the von Neumann equation describing the unitary evolution of the closed system-environment (SE) system in the IP as

\[ \frac{d\bar{\rho}_{SE}(t)}{dt} = -i \left[ \bar{H}_I(t), \bar{\rho}_{SE}(t) \right]. \]  
(A3)

Assuming the SE to initially be in a product state,

\[ \rho_{SE}(0) = \rho_S(0) \otimes \rho_E \]  
(A4)

with \( \rho_E \) being a stationary state of the environment that satisfies \([\bar{H}_E(t), \rho_E] = [H_E, \rho_E] = 0\), we may integrate (A3) to obtain:

\[ \bar{\rho}_{SE}(t) = \rho_S(0) \otimes \rho_E - i \int_0^t ds \left[ \bar{H}_I(s), \bar{\rho}_{SE}(s) \right]. \]  
(A5)
We then trace out the environment in (A3), so that its l.h.s. reads

$$\text{Tr}_E \left\{ \frac{d\rho_{SE}(t)}{dt} \right\} = \frac{d}{dt} \left( \text{Tr}_E \left\{ e^{i(H_S+H_E)t} \rho_{SE}(t) e^{-i(H_S+H_E)t} \right\} \right) = \frac{d}{dt} \left( e^{iH_st} \text{Tr}_E \left\{ \rho_{SE}(t) \right\} e^{-iH_st} \right) = \frac{d\tilde{\rho}_S(t)}{dt} , \quad (A6)$$

and substitute into its r.h.s. for $\tilde{\rho}_{SE}(t)$ according to (A5), in order to obtain the integro-differential equation describing the system density matrix in the IP at time $t$ that involves a convolution integral:

$$\frac{d\tilde{\rho}_S(t)}{dt} = -i \text{Tr}_E \left\{ \left[ \tilde{H}_I(t), \rho_S(0) \otimes \rho_E \right] \right\} - \int_0^t ds \text{Tr}_E \left\{ \left[ \tilde{H}_I(t), [\tilde{H}_I(s), \tilde{\rho}_{SE}(s)] \right] \right\} . \quad (A7)$$

Furthermore, we drop the first term in (A7), as without loss of generality we may assume

$$\text{Tr}_E \left\{ \tilde{H}_I(t) \rho_E \right\} = 0 . \quad (A8)$$

It is so, as we are free to shift the zero point of the energy scale. In particular, by changing $H_I$ and $H_S$ to

$$H'_I = H_I - \text{Tr}_E \{ H_I \rho_E \} \otimes \mathbb{1}_E \quad \text{and} \quad H'_S = H_S + \text{Tr}_E \{ H_I \rho_E \} , \quad (A9)$$

we do not alter the total Hamiltonian $H_T$ (A1), but can always assure that, indeed,

$$\text{Tr}_E \left\{ \tilde{H}'_I(t) \rho_E \right\} = \text{Tr}_E \left\{ e^{i(H'_S+H'_E)t} \tilde{H}_I e^{-i(H'_S+H'_E)t} \rho_E - e^{iH'_st} \text{Tr}_E \left\{ \tilde{H}_I \rho_E \right\} e^{-iH'_st} \right\} \quad (A10)$$

$$= e^{iH'_st} \text{Tr}_E \left\{ e^{iH'_st} \tilde{H}_I e^{-iH'_st} \rho_E \right\} e^{-iH'_st} - e^{iH'_st} \text{Tr}_E \left\{ \tilde{H}_I \rho_E \right\} e^{-iH'_st} \quad (A11)$$

$$= e^{iH'_st} \text{Tr}_E \left\{ \tilde{H}_I \rho_E \right\} e^{-iH'_st} - e^{iH'_st} \text{Tr}_E \left\{ \tilde{H}_I \rho_E \right\} e^{-iH'_st} = 0 , \quad (A12)$$

where in the last line we have used the steady-state property of the initial environment state, i.e., $[H_E, \rho_E] = 0$.

As a result, we obtain the master equation (ME) in the exact form describing general dynamics of a system interacting with its environment as:

$$\frac{d\tilde{\rho}_S(t)}{dt} = - \int_0^t ds \text{Tr}_E \left\{ \left[ \tilde{H}_I(t), [\tilde{H}_I(s), \tilde{\rho}_{SE}(s)] \right] \right\} , \quad (A13)$$

which does not involve any approximations and assumes only $\rho_{SE}(0) = \rho_s(0) \otimes \rho_E$ with $[H_E, \rho_E] = 0$.

**2. Master equation in the time-local form**

The ME (A13) despite being compact and exact is typically not of much use, as it involves the full SE state and a time-convoluted integral. Nevertheless, one may always formally rewrite it as a function of the system state at a given time. In order to show this, we integrate the closed von Neumann dynamics (A3) in order to obtain

$$\tilde{\rho}_{SE}(t) = \tilde{U}_{SE}(t) \left( \rho_s(0) \otimes \rho_E \right) \tilde{U}_{SE}^\dagger(t) \quad (A14)$$

with the unitary rotation being formally defined with help of a time-ordered exponential:

$$\tilde{U}_{SE}(t) := T_e \exp \left\{ - i \int_0^t ds \tilde{H}_I(s) \right\} . \quad (A15)$$

Now, as the state of the system may be obtained at any time by tracing out the environment, we may identify the resulting physical map $\tilde{\Lambda}_t$ describing the dynamics of solely the system as

$$\tilde{\rho}_S(t) = \tilde{\Lambda}_t[\rho_S(0)] := \text{Tr}_E \left\{ \tilde{U}_{SE}(t) \left( \rho_s(0) \otimes \rho_E \right) \tilde{U}_{SE}^\dagger(t) \right\} . \quad (A16)$$

Hence, if the inverse of such a map exists at $t$, i.e., $\rho_S(0) = \tilde{\Lambda}_t^{-1}[\tilde{\rho}_S(t)]$, we may equivalently rewrite the exact ME (A13) into its time-local form (in the IP) as

$$\frac{d\tilde{\rho}_S(t)}{dt} = \tilde{\mathcal{L}}_t [\tilde{\rho}_S(t)] , \quad (A17)$$
where \( \tilde{L}_t \) is time-local dynamical generator associated with the time-instance \( t \) such that [26, 27]:

\[
\tilde{L}_t := \frac{d \Lambda_t}{dt} \Lambda_t^{-1}
\]

with

\[
\frac{d \tilde{L}_t}{dt} [\bullet] := -\int_0^t ds \text{Tr}_E \left\{ \left[ \hat{H}_I(t), \left[ \hat{H}_I(s), \hat{U}_{SE}(s) (\bullet \otimes \rho_E) \hat{U}_{SE}^\dagger(s) \right] \right] \right\}.
\]

(A19)

Note that the dynamical generator (A18) may be ill-defined at time-instances for which the inverse map \( \Lambda_t^{-1} \) does not exist [26]. We return to this issue in App. B, where we discuss in more detail representations of open system dynamics by the families of quantum maps and their corresponding dynamical generators.

3. Master equation in the Schrödinger picture

One typically transforms the time-local ME (A17) into the Schrödinger picture (SP), in which it may then be written in the more familiar form:

\[
\frac{d \rho_S(t)}{dt} = -i[H_S, \rho_S(t)] + \mathcal{L}_t [\rho_S(t)]
\]

(A20)

with the SP-based dynamical generator now reading

\[
\mathcal{L}_t [\bullet] := e^{-iH_S t} \tilde{L}_t \left[ e^{iH_S t} \bullet e^{-iH_S t} \right] e^{iH_S t}.
\]

(A21)

Decomposing \( \mathcal{L}_t \) into its Hamiltonian and dissipative parts, i.e., \( \mathcal{L}_t [\bullet] = -i[H(t), \bullet] + \mathcal{D}_t [\bullet] \), one arrives at the canonical form of the ME:

\[
\frac{d \rho_S(t)}{dt} = -i[H_S + H(t), \rho_S(t)] + \mathcal{D}_t [\rho_S(t)],
\]

(A22)

where \( H(t) \) accounts for the corrections to the unitary evolution of the system due to interactions with the environment, e.g., representing the Lamb shifts when describing atom-light interactions [3], while \( \mathcal{D}_t \) may then be entirely associated with the destructive impact of the environment on the system that leads to dissipation.

Let us emphasise that, as the SP-based dynamical generator \( \mathcal{L}_t \) (A21) explicitly depends on the system Hamiltonian \( H_S \), its form (and, hence, its Hamiltonian \( H(t) \) and dissipative parts \( \mathcal{D}_t \) in (A22)) should not be generally associated with the properties of just the environment and the interactions. In particular, only in special cases the form of \( \mathcal{L}_t \) can be derived basing solely on \( H_I, H_E, \) and \( \rho_E \). For instance, this is always possible when the system and the interaction Hamiltonians commute, i.e., when \( [H_S, H_I] = 0 \). It is not hard to verify that the global unitary \( \hat{U}_{SE} \) in (A15) then also commutes with \( H_S, \hat{U}_{SE}, H_S \) = 0, and consequently the IP-based map \( \tilde{\Lambda}_t \) in (A16) becomes \( H_S \)-covariant. In particular, the map \( \tilde{\Lambda}_t \) satisfies then (for any \( s, t \geq 0 \)) the covariation relation [46]:

\[
e^{-iH_S s} \tilde{\Lambda}_t [\bullet] e^{iH_S s} = \tilde{\Lambda}_t \left[ e^{-iH_S s} \bullet e^{iH_S s} \right] \quad \Leftrightarrow \quad e^{-iH_S s} \tilde{\Lambda}_t [\bullet] e^{iH_S s} = \tilde{L}_t \left[ e^{-iH_S s} \bullet e^{iH_S s} \right],
\]

(A23)

which, as noted above, must then be generally passed onto its dynamical generators [46, 47] (as, here, assured by (A18)). Crucially, the \( H_S \)-covariance (A23) implies that there is no difference in the description of the system dynamics between the IP and SP, i.e., \( \tilde{\Lambda}_t = \Lambda_t \) and \( \tilde{\Lambda}_t = \tilde{L}_t \) (see (A21)). Thus, it becomes clear that the form of \( \mathcal{L}_t \) in (A20) (and \( H(t) \) and \( \mathcal{D}_t \) in (A22)) must then be completely independent of the system Hamiltonian and fully determined by the properties of the environment and the interactions, i.e., by \( H_E, H_I \) and \( \rho_E \).

4. Externally modifying the system Hamiltonian

One may ask how does the form of the ME (A20), and in particular its canonical decomposition (A22), change when the system Hamiltonian, \( H_S \), is altered. In general, as mentioned above, such modification may in principle affect both the Hamiltonian and the dissipative parts of the dynamical generator \( \mathcal{L}_t \) in (A20). Let us consider a transformation

\[ H_S \rightarrow \tilde{H}_S(t) := H_S + V(t) \]

with the Hermitian operator \( V(t) \) representing a general (potentially time-dependent) perturbation of the system Hamiltonian and discuss particular cases:
a. $[H_S, H_I] = 0$, $\forall t \geq 0$: $[V(t), H_I] = 0$. If the system Hamiltonian commutes with the interaction Hamiltonian (so that the dynamics is $H_S$-covariant—see above) and so does the perturbation $V(t)$ at all times, then the modified dynamics must exhibit covariance w.r.t. the perturbed $\tilde{H}_S$ as $[\tilde{H}_S(t), H_I] = 0$ for all $t$. Hence, the form of $\mathcal{L}_t$ in (A20) is unaffected by the perturbation remaining fully determined by $H_E, H_I$ and $\rho_E$. Moreover, the new dynamics is then correctly described by simply replacing $H_S$ with $\tilde{H}_S(t)$ in (A20) or (A22).

b. $[H_S, H_I] \neq 0$, $\forall t \geq 0$: $[V(t), H_I] = [V(t), H_S] = 0$. The above conclusion also holds when dealing with non-$H_S$-covariant dynamics, but when requiring the perturbation to commute with not only the interaction but also the system Hamiltonian. Now, as $[H_S, H_I] \neq 0$, the dynamical generator $\mathcal{L}_t$ generally depends on $H_S$. However, without affecting the total Hamiltonian $H_T$ in (A1) and hence the dynamics, we may redefine the interaction Hamiltonian as $H'_I := H_I + H_S$ with the system Hamiltonian being absent. In such a fictitious picture, the ME (A20) possesses just the second term with $\mathcal{L}_t$ now being derived basing on $H'_I$. As importantly the perturbation then fulfils $[V(t), H']_I = 0$ at all times, it becomes clear that the dynamics must be $V(t)$-covariant. Hence, the perturbation must lead to a ME that may be equivalently obtained by simply adding $V(t)$ to the Hamiltonian part of the ME (A22), which now contains $H(t)$ and $\mathcal{D}_t$ that non-trivially depend on the original system Hamiltonian $H_S$ (but not $V(t)$).

c. $[H_S, H_I] = 0$, $\forall t \geq 0$: $[V(t), H_S] = 0$. Returning to $H_S$-covariant dynamics, we may play a similar trick in order to deal with the case of $[V(t), H_S] = 0$. In contrast to the first paragraph, as now potentially $[V(t), H_I] \neq 0$, the modified dynamics is no longer guaranteed to be $\tilde{H}_S$-covariant. However, let us redefine the interaction Hamiltonian this time as $H'_I := V(t) + H_I$ that still importantly commutes with $H_S$. Hence, it becomes clear that the $H_S$-covariance is preserved and the perturbation $V(t)$ affects only the dynamical generator $\mathcal{L}_t$ in (A20) (being now derived basing on $H'_I$), whose form remains to be independent of $H_S$ and is determined solely by $H_E, H_I, \rho_E$ (and now also $V(t)$).

Finally, let us note that a prominent physical interpretation of $V(t)$ may be given when dealing with coherently driven systems, for which the perturbation then represents the externally applied force. In such a case (if the above conditions $a$, $b$ are not fulfilled), when accounting for the presence of driving one must redefine the ME, which then typically exhibits extra terms (both Hamiltonian (Lamb shifts) and dissipative) in (A22) due to the external force $[3, 45]$. Moreover, even when considering weak-coupling regimes of weak environmental interactions (also discussed below) such discrepancies persist, as, e.g., the time-dependence of the driving force may result in time-dependence of dynamical generators even in the “weakest”-coupling regime in which the Born-Markov approximation holds— for explicit discussion see Section 6.4.2 of [45]. A concrete example is provided by the “driven damped harmonic oscillator” model analysed in Sec. 4 of [39], where the driving-induced discrepancies between various approaches to weak-coupling have been explicitly studied.

Appendix B: Structure of time-local generators

1. Quantum dynamics as families of generators

a. Physical quantum dynamics

A particular evolution of an open quantum system is formally represented by a family of density matrices $\{\rho_t\}_{t \geq 0}$ that describe the system state at each time $t$. The system dynamics is then defined—indeed of the system initial state $\rho_0$—as the family of quantum channels (linear maps) associated with the evolution, $\{\Lambda_t\}_{t \geq 0}$ with $\Lambda_0 = I$, such that for any $\rho_0$:

$$\forall t \geq 0: \quad \rho_t = \Lambda_t[\rho_0]. \tag{B1}$$

Importantly, for a given dynamics to be physical the family $\{\Lambda_t\}_{t \geq 0}$ must consist of completely-positive trace preserving (CPTP) maps. In practice, any linear map $\Lambda : \mathcal{B}(\mathcal{H}_d) \rightarrow \mathcal{B}(\mathcal{H}_{d'})$ may be verified to be CPTP by constructing its corresponding Choi-Jamiolkowski (CJ) matrix $\Omega_{\Lambda} \in \mathcal{B}(\mathcal{H}_{d'} \otimes \mathcal{H}_d)$ defined as $[48, 49]$:

$$\Omega_{\Lambda} := \Lambda \otimes I \begin{pmatrix} |\psi\rangle \langle \psi| \end{pmatrix}, \tag{B2}$$

where $|\psi\rangle = \sum_{i=1}^d |i\rangle |i\rangle$ with $\{|i\rangle\}_{i=1}^d$ being some orthonormal basis spanning $\mathcal{H}_d$. In particular, a map $\Lambda$ is CPTP if its CJ matrix is positive semi-definite, i.e., $\Omega_{\Lambda} \succeq 0$, and satisfies $\text{Tr}_{\mathcal{H}_{d'}}(\Omega_{\Lambda}) = 1_{d'}$. 


b. Dynamical generators

With any quantum dynamics (B1), one may associate its corresponding family of time-local dynamical generators, \( \{L_t\}_{t \geq 0} \), which specify the master equation (ME) describing the evolution \([26, 50]\):

\[
\forall t \geq 0: \dot{\Lambda}_t = L_t \Lambda_t \Leftrightarrow \dot{\rho}_t = L_t[\rho]. \tag{B3}
\]

Here, \( \dot{\bullet} \equiv \frac{d}{dt} \bullet \). The generator can formally be defined for a given \( t \) as

\[
L_t := \dot{\Lambda}_t \Lambda_t^{-1}, \tag{B4}
\]

where \( \Lambda_t^{-1} \) is the inverse of the quantum channel at time \( t \), and is not restricted to be a CPTP map. However, \( \Lambda_t^{-1} \) is not guaranteed to exist at all time instances, even when the channel family is perfectly smooth. In such a case, \( \{L_t\}_{t \geq 0} \) can be unambiguously defined only for times at which \( \Lambda_t \) is invertible with the non-invertability manifesting itself by the emergent singularities of the resulting generator. Nevertheless, under particular conditions \([26]\), the resulting ME B3 can still be then integrated and yield correctly the original dynamics (B1).

In the other direction, given a family of dynamical generators \( \{L_t\}_t \), one may write the corresponding quantum channel at any time \( t \) with help of a time-ordered exponential (expressible in the Dyson-series form) \([26, 50]\):

\[
\Lambda_t = T_\leftarrow \exp \left\{ \int_0^t L_\tau d\tau \right\} = \sum_{i=0}^{\infty} S^{(i)}_t [L], \tag{B5}
\]

where \( S^{(0)}_t [\bullet] = I \) and for \( i \geq 1 \):

\[
S^{(i)}_t [L] : = \frac{1}{i!} \int_0^t \int_0^t \cdots \int_0^t T_\leftarrow L_t, L_{t_2} \cdots L_{t_i} dt_1 dt_2 \cdots dt_i = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{i-1}} L_{t_1} L_{t_2} \cdots L_{t_i} dt_1 \cdots dt_i. \tag{B6}
\]

c. Instantaneous generators

Given a family of quantum channels \( \{\Lambda_t\}_t \) defining the dynamics in (B1), one can also construct the corresponding family of time-local instantaneous generators \( \{X_t\}_t \) given by \([50]\)

\[
\forall t \geq 0: X_t := \frac{d}{dt} \log \Lambda_t \tag{B7}
\]

and define the instantaneous exponent \( Z_t := \int_0^t d\tau X_\tau \), so that \( X_t = \dot{Z}_t \). Crucially, one can then specify the channel family \( \{\Lambda_t\}_t \) in terms of \( \{X_t\}_t \) as follows:

\[
\Lambda_t = \exp Z_t = \exp \left\{ \int_0^t d\tau X_\tau \right\}, \tag{B8}
\]

where the expression, in contrast to (B5)), does not involve the time-ordering operation.

Although the instantaneous generators \( X_t \) cannot be associated with the ME (B3) describing the evolution, after noting that

\[
\dot{\Lambda}_t = \frac{d}{dt} \exp Z_t = \int_0^1 ds e^{sZ_t} X_t e^{(1-s)Z_t} \tag{B9}
\]

and simply \( \Lambda_t^{-1} = e^{-Z_t} \), one can, in principle, always construct the ME-based dynamical generator (B4) at time \( t \) for given \( Z_t \) and \( X_t (= \dot{Z}_t) \) by computing:

\[
L_t = \int_0^1 ds e^{sZ_t} X_t e^{-sZ_t}. \tag{B10}
\]
d. Commutative dynamics

In case of commutative dynamics the channels describing the evolution in (B1), or equivalently (by (B5)) their corresponding family of dynamical generators (B4), fulfill

\[ \forall s,t \geq 0: \quad [\Lambda_s, \Lambda_t] = 0 \quad \Leftrightarrow \quad [\mathcal{L}_s, \mathcal{L}_t] = 0. \tag{B11} \]

Moreover, it naturally follows from (B8) that the commutativity assures also the instantaneous exponents to commute, i.e., \( \forall s,t \geq 0 : [\mathcal{Z}_s, \mathcal{Z}_t] = 0 \). However, as this also implies that \( 0 = \partial_s [\mathcal{Z}_s, \mathcal{Z}_t] = [\mathcal{X}_s, \mathcal{Z}_t] \), one observes that then

\[ \mathcal{L}_t = \int_0^1 ds \, e^{s\mathcal{Z}_t} \mathcal{X}_t e^{-s\mathcal{Z}_t} = \int_0^1 ds \, \mathcal{X}_t = \mathcal{X}_t. \tag{B12} \]

Hence, for commutative dynamics, dynamical and instantaneous generators coincide at all times and

\[ \Lambda_t = \exp \left\{ \int_0^t d\tau \mathcal{X}_\tau \right\} = \exp \left\{ \int_0^t d\tau \mathcal{L}_\tau \right\} \tag{B13} \]

with the time-ordering operator becoming then irrelevant also in the master equation picture.

e. CP-divisible dynamics

The dynamics is said to be divisible into CPTP maps, i.e., CP-divisible (or Markovian [30]), if its corresponding channel family \( \{\Lambda_t\}_{t \geq 0} \) satisfies for all \( 0 \leq s \leq t \):

\[ \Lambda_t = \hat{\Lambda}_t \Lambda_s, \tag{B14} \]

where \( \hat{\Lambda}_t \) is always guaranteed to be a CPTP map. At the level of dynamical generators, such a condition is equivalent to the statement that all \( \{\mathcal{L}_t\}_{t \geq 0} \) are of the so-called Gorini-Kosakowski-Sudarshan-Linblad (GKSL) form [28, 29]:

\[ \forall t \geq 0 : \quad \mathcal{L}_t[\bullet] = -i[H_t, \bullet] + \Phi_t[\bullet] - \frac{1}{2} \{\Phi_t^*[1], \bullet\}, \tag{B15} \]

where \( H_t = H_t^\dagger \) is a time-dependent Hermitian operator (representing the Hamiltonian part of the ME (B3)), while \( \Phi_t \) a completely-positive (CP) map (representing the dissipative part of the ME (B3)) with \( \Phi_t^* \) denoting its dual. As any CP map may be represented in its Kraus form as \( \Phi_t[\bullet] = \sum_i V_i(t) \bullet V_i(t)^\dagger \) with some operators \( \{V_i(t)\}_i \) [51], one can also rewrite (B15) as [3]:

\[ \mathcal{L}_t[\bullet] = -i[H_t, \bullet] + \sum_i V_i(t) \bullet V_i(t)^\dagger - \frac{1}{2} \{V_i(t)^\dagger V_i(t), \bullet\}. \tag{B16} \]

Furthermore, fixing a particular orthonormal basis of matrices \( \{F_j\}_j \) with \( \forall i,j : \text{Tr}\{F_i^\dagger F_j\} = \delta_{ij} \), so that \( V_i(t) = \sum_j v_{ij}(t)F_j \) for all \( i \), one obtains

\[ \mathcal{L}_t[\bullet] = -i[H_t, \bullet] + \sum_{i,j} D_{ij}(t) \left( F_j \bullet F_i^\dagger - \frac{1}{2} \{F_i^\dagger F_j, \bullet\} \right) \tag{B17} \]

with now the time-dependence of the dissipative part being fully contained within the matrix \( D(t) := v(t)^\dagger v(t) \). Crucially, it then follows that the CP-divisibility of particular dynamics is assured iff one may at all times decompose the corresponding dynamical generators according to (B17) with some positive semi-definite \( D(t) \geq 0 \).

f. Semigroup dynamics

An important subclass of commutative and CP-divisible dynamics are the semigroups, in which case the whole evolution is determined by a single fixed generator \( \mathcal{L} \),

\[ \{\mathcal{L}_t = \mathcal{L}\}_{t \geq 0} \quad \Rightarrow \quad \{\Lambda_t = \exp \{\mathcal{L}t\}\}_{t \geq 0}, \tag{B18} \]

that then, in order to describe physical dynamics, must be of the GKSL form (B17) with both the Hamiltonian \( H \) and the positive semi-definite matrix \( D \geq 0 \) being now time-independent.
g. Semigroup-simulable dynamics

Definition 1. We define a map \( \Lambda_t = e^{Z_t} \) at time \( t \) to be (instantaneously) semigroup-simulable if its corresponding instantaneous exponent \( Z_t \) in (B8) is of the GKSL form, i.e.,

\[
Z_t[\bullet] = -i \left[ \hat{H}_t, \bullet \right] + \frac{1}{2} \left\{ \Phi_t^{\dagger} [\mathbb{1}] , \bullet \right\},
\]

(B19)

where, similarly to (B15), \( \hat{H}_t \) is a general Hermitian operator while \( \Phi_t \) is some CP map. If (B19) is assured to hold for all \( t \geq 0 \), we term the whole dynamics to be semigroup-simulable.

Note that, as a result, a valid physical semigroup with generator \( L = Z_t \) is guaranteed to exist:

\[
\left\{ \tilde{\Lambda}(t)_{\tau \geq 0} : \tilde{\Lambda}(t)_{\tau = 1} = \Lambda_t \right\} := e^{Z_t \tau},
\]

(B20)

such that it coincides with the original map at \( \tau = 1 \), \( \tilde{\Lambda}(t)_{\tau = 1} = \Lambda_t \) or, in other words, “simulates” its action at this instance of “fictitious time” \( \tau \).

Fact 1. Semigroup-simulability provides a sufficient but not necessary condition for physicality of dynamics.

The fact that a channel family \( \{ \Lambda_t = e^{Z_t} \}_{t \geq 0} \) possesses the exponents of the form (B19) assures the dynamics to consist of maps which coincide with various semigroup evolutions (B20) at all times, i.e., \( \Lambda_t = \tilde{\Lambda}(t)_{\tau = 1} \) for all \( t \). Hence, as it must then trivially consist of CPTP maps, the family must yield physical dynamics. In the other direction, however, there exist dynamics that are not semigroup-simulable but nonetheless define a physical family \( \{ \Lambda_t \}_{t \geq 0} \) consisting of CPTP maps. An example is provided by the eternal non-Markovianity model introduced in [34] for random unitary qubit dynamics, as shown in [52] and also discussed below.

In case of commutative dynamics \( Z_t = \int_0^t d\tau L_\tau \), so that parametrising the dynamical generator as in (B17), one may write

\[
Z_t[\bullet] = -i \int_0^t d\tau H_\tau, \bullet \] + \sum_{i,j} \int_0^t d\tau D_{ij}(\tau) \left( F_j^\dagger \bullet F_i^\dagger - \frac{1}{2} \left\{ F_i^\dagger F_j^\dagger, \bullet \right\} \right).
\]

(B21)

As \( \int_0^t d\tau H_\tau \) above can always be interpreted as \( \hat{H}_t \) in (B19), the GKSL form of \( Z_t \) is then fully assured by the condition \( \int_0^t d\tau D(\tau) \geq 0 \). Hence, one may generally conclude that

Fact 2. Any commutative dynamics is semigroup-simulable (and, hence, physical) if and only if for all \( t \):

\[
\Gamma(t) := \int_0^t d\tau D(\tau) \geq 0,
\]

(B22)

where \( D(t) \) is the matrix (potentially negative at some time-instances) defined by the dynamical generators via (B17).

In various previous works [53, 54], the notion of semigroup-simulability has been identified as Markovianity of the dynamics. However, let us emphasise that such a notion is non-trivially related to the concept of CP-divisibility introduced in Sec. B 1 e more commonly associated with the notion of Markovianity [30]. The CP-divisibility assures \( D(t) \geq 0 \) in (B17) at all times, so that in the case of commutative dynamics the semigroup-simulability condition (2) is trivially fulfilled. However, let us emphasise that there exist (also commutative) evolutions that are semigroup-simulable but not CP-divisible, e.g., instances of random unitary qubit dynamics [26, 37] discussed below. Such a fact can be intuitively understood by inspecting (B10), as the GKSL form (B19) of the instantaneous exponent \( Z_t \) (and \( \mathcal{X}_t = \hat{Z}_t \)) does not necessary assure corresponding dynamical generator \( L_t \) to also be of the GKSL form (B15).

2. Example: Qubit dynamics

In order to elaborate on the notions introduced in the previous sections, we consider in more detail the qubit dynamics and, in particular, discuss its two important commutative subclasses, i.e., the random unitary and phase-covariant qubit evolutions, as well as their exemplary microscopic derivations for the cases of dephasing and amplitude damping models.
a. Random unitary qubit dynamics

Random unitary (RU) qubit dynamical model is abstractly defined at the level of dynamical generators as [26, 37]:

\[ \mathcal{L}_t[\rho] = \sum_{k=1}^{3} \gamma_k(t) (\hat{\sigma}_k \rho \hat{\sigma}_k - \rho), \quad (B23) \]

so that the family \( \{ \mathcal{L}_t \}_{t \geq 0} \) is then fully specified by the three rates \( \gamma_k(t) \) that correspond to a diagonal form of the general D-matrix in (B17) with Pauli operators \( \{ \mathbb{1}, \{ \hat{\sigma}_i \}_{i=x,y,z} \} \) constituting a basis of two-dimensional Hermitian matrices. Hence, it directly follows that the GKSL form of dynamical generators (B23), and hence the CP-divisibility (B14) of the dynamics is assured iff at all times all \( \gamma_k(t) \geq 0 \) are non-negative.

Importantly, one may straightforwardly verify (thanks to Pauli operators properties yielding \( \forall_{x,y,z} : [\mathcal{L}_x, \mathcal{L}_y] = 0 \)) that the corresponding dynamics is commutative. As a result, owing to (B12), the family of dynamical generators (B23) coincides with the instantaneous ones, i.e., \( \mathcal{X}_t = \mathcal{L}_t \), while

\[ Z_t[\rho] = \sum_k \Gamma_k(t) (\hat{\sigma}_k \rho \hat{\sigma}_k - \rho), \quad (B24) \]

where \( \Gamma_k := \int_0^t d\tau \gamma_k(\tau) \). Hence, it directly follows from the condition (B22) that any RU qubit dynamics is semigroup-simulable iff

\[ \forall_{t \geq 0, k=x,y,z} : \Gamma_k(t) \geq 0, \quad (B25) \]

which assures instantaneous exponent (B24) to be of the GKSL form (B19)) at all times. Note that one may easily construct families of dynamical generators (B23) that satisfy (B25) without requiring \( \gamma_k(t) \geq 0 \) for all \( t \). Hence, indeed, there exist random unitary qubit dynamics that are semigroup-simulable but not CP-divisible.

On the other hand, one may explicitly verify that RU dynamics is physical, i.e., leads to a family of CPTP qubit maps in (B1), iff for all cyclic permutations of \( i, j, k \in \{1, 2, 3\} \) (i.e., such that \( \epsilon_{ijk} = 1 \) [26, 52]

\[ \mu_i(t) + \mu_j(t) \leq 1 + \mu_k(t), \quad (B26) \]

where \( \mu_i(t) := \exp[-2(\Gamma_j(t) + \Gamma_k(t))] \). As the physicality conditions (B26) are even less restrictive than (B25), it follows that there must also exist RU dynamics that still are physical but cannot be simulated with semigroups at every instance of time.

Eternally non-Markovian model. An example is provided by the eternally non-Markovian model introduced in Ref. [34] that corresponds to the choice of the rates:

\[ \gamma_1(t) = \gamma_2(t) = \frac{1}{2}, \quad \gamma_3(t) = -\frac{1}{2} \tanh(t) \quad (B27) \]

in (B23). Importantly, it fulfils the condition (B26) at all times, yet leads to \( \gamma_3(t) < 0 \), and hence \( \Gamma_3(t) < 0 \), for all \( t \). Thus, the choice (B27) yields dynamics that is physical but neither CP-divisible nor semigroup-simulable.

Dephasing dynamics. On the other hand, the simplest case of the random unitary dynamics (B23) is the dephasing model:

\[ \mathcal{L}_t[\rho] = \gamma(t) (\hat{\sigma}_n \rho \hat{\sigma}_n - \rho), \quad (B28) \]

where \( \hat{\sigma}_n = n \cdot \sigma = \sum_i n_i \hat{\sigma}_i \), and \( \hat{\sigma}_n^2 = \mathbb{1} \) implies \( \|n\| = 1 \). The vector \( n \) should be interpreted as a choice (a passive rotation in the Bloch-ball picture) of the Pauli operator basis, in which then (B28) corresponds to the (rank-one Pauli) dynamics (B23) with only a single term present in the sum. One may easily verify that for the dephasing model to be physical \( \Gamma(t) = \int_0^t d\tau \gamma(\tau) \geq 0 \) with the notions of physicality and semigroup-simulability then trivially coinciding.

The dephasing dynamics (B28) can be explicitly obtained by considering various macroscopic derivations, in which the qubit is coupled to a large environment via some \( H_{\text{int}} \propto \hat{\sigma}_n \otimes \mathcal{O}_{\text{env}} \). In App. E 3, we provide a compact example by using a toy-model of a qubit coupled to a large magnet. However, the most common derivation typically corresponds to a qubit coupled to a large, thermal bosonic bath \( \mathcal{O}_{\text{env}} \). In particular, the interaction is then modelled by \( H_{\text{int}} \propto \hat{\sigma}_n \otimes \left( \sum_k g_k a_k + g_k^* a_k^\dagger \right) \) coupling the qubit to a bosonic reservoir with an Ohmic-like spectral density [55]:

\[ J(\omega) := \sum_k g_k^2 \delta(\omega - \omega_k) = \frac{\omega^s}{\omega_c^s - i} e^{-\frac{\omega}{\omega_c} i}, \quad (B29) \]
where \( \omega_c \) represents then the reservoir cutoff frequency while \( s \geq 0 \) is the so-called Ohmicity parameter. Assuming further the reservoir to be at zero temperature, the dynamical generators describing the qubit evolution take then exactly the form (B28) with the dephasing rate reading [56]:

\[
\gamma(t) = \omega_c \left[ 1 - (\omega_c t)^2 \right]^{-\frac{1}{2}} \Gamma[s] \sin[s \arctan(\omega_c t)],
\]

(B30)

where \( \Gamma[s] \) above represents the Euler gamma function. Moreover, one may show that the dephasing rate temporarily takes negative values iff \( s > 2 \), so that the dynamics ceases to be CP-divisible in such a regime [56].

b. Phase covariant dynamics

Another widely used class of commutative qubit dynamics are the so-called phase covariant (PC) evolutions [38], i.e., families of qubit CPTP maps possessing azimuthal symmetry with respect to rotations about the z axis in the Bloch ball representation. Importantly, PC dynamics possess a direct physical interpretation at the master equation level—they correspond to the family of dynamical generators:

\[
\mathcal{L}_\gamma[\rho] = \gamma_-(t) \left( \hat{\sigma}_- \rho \hat{\sigma}_+ - \frac{1}{2} \{ \hat{\sigma}_+ \hat{\sigma}_-, \rho \} \right) + \gamma_+(t) \left( \hat{\sigma}_+ \rho \hat{\sigma}_- - \frac{1}{2} \{ \hat{\sigma}_- \hat{\sigma}_+, \rho \} \right) + \gamma_z(t) \left( \hat{\sigma}_z \rho \hat{\sigma}_z - \rho \right),
\]

(B31)

which represents a combination of the relaxation, excitation and coherence-loss processes occurring with rates: \( \gamma_-(t) \), \( \gamma_+(t) \) and \( \gamma_z(t) \), respectively; while \( \hat{\sigma}_\pm := \frac{1}{\sqrt{2}} (\hat{\sigma}_x \pm i \hat{\sigma}_y) \) constitute the transition operators. In general, the dynamical generators of RU (B23) and PC (B31) dynamics do not commute unless either of the two corresponds to pure dephasing (with \( n = \{0, 1\} \) and \( \sigma_n = \hat{\sigma}_z \) in (B28)) aligned along the \( z \) direction, i.e., either \( \gamma_z(t) = \gamma_0(t) = 0 \) in (B23) or \( \gamma_+(t) = \gamma_-(t) = 0 \) in (B31).

As \( \{1, \hat{\sigma}_+, \hat{\sigma}_-, \hat{\sigma}_z\} \) equivalently constitute a basis of two-dimensional Hermitian matrices, the properties of PC dynamics can be determined analogously to the RU case. In particular, considering now \( k = \{+, -, z\} \), a given PC evolution is CP-divisible iff all \( \gamma_k(t) \geq 0 \) at all times, while it is semigroup-simulable iff all \( \Gamma_k(t) = \int_0^t \gamma_k(\tau) d\tau \geq 0 \) for any \( t \). Only the physicality condition for PC dynamics cannot be straightforwardly inferred, yet it is not hard to verify that the family of generators (B31) yields physical dynamics iff for all \( t \) [38]:

\[
\eta_\perp(t) \pm \kappa(t) \leq 1 \quad \text{and} \quad (1 + \eta_\perp(t))^2 \geq 4 \eta_\perp(t)^2 + \kappa(t)^2,
\]

(B32)

where \( \eta_\parallel(t) := e^{-\delta(t)}, \eta_\perp(t) := e^{-\frac{1}{2}(\delta(t) - 4 \Gamma_\parallel(t))}, \kappa(t) := e^{-\delta(t)} \int_0^t d\tau e^{\delta(\tau)} [\gamma_+(\tau) - \gamma_-(\tau)] \) and \( \delta(t) := \Gamma_+(t) + \Gamma_-(t) \).

Amplitude damping dynamics. The most common example of PC dynamics (that is not RU) is the amplitude damping evolution that represents spontaneous emission, i.e., a pure relaxation process of a two-level system [3] corresponding to the choice \( \gamma_+(t) = \gamma_z(t) = 0 \) in (B31). The canonical microscopic derivation of such an evolution corresponds then to the Jaynes-Cummings model describing coupling (\( \text{Hint} \propto \hat{\sigma}_+ \otimes \sum_k g_k n_k + \text{h.c.} \)) to a cavity with a Lorentzian spectral density [3]:

\[
J(\omega) = \frac{1}{2\pi} \frac{\gamma_0 \lambda^2}{(\omega_0 - \Delta - \omega)^2 + \lambda^2},
\]

(B33)

where \( \Delta \) is the detuning parameter from the transition frequency \( \omega_0 \) of the two-level system, while \( \lambda \) represents the spectral width of the coupling. In such a case, one obtains the PC dynamics described by (B31) with only the relaxation rate that is non-zero [57],

\[
\gamma_-(t) = \Re \left\{ \frac{2\gamma_0 \lambda}{\lambda - i\Delta + d \coth \left( \frac{d t}{2} \right)} \right\},
\]

(B34)

and does not possess any singularities thanks to the complex parameter \( d := \sqrt{(\lambda - i\Delta)^2 - 2\gamma_0 \lambda} \) with \( \Re\{d\} > 0 \). An exception is the on-resonance (\( \Delta = 0 \)) strong-coupling (\( \gamma_0 \gg \frac{\lambda}{2} \)) regime, for which the parameter becomes purely imaginary, i.e., \( d = i |d| \) with \( |d| = \sqrt{2\gamma_0 \lambda - \lambda^2} \), so that the rate reads

\[
\gamma(t) = \frac{2\gamma_0 \lambda}{\lambda + |d| \cot \left( \frac{|d| t}{2} \right)}
\]

(B35)

and diverges for all \( t = \frac{2 n \pi}{|d|} \left( \arccot \left( \frac{\lambda}{|d|} \right) + n\pi \right) \) with \( n \in \mathbb{N}^+ \) [3].
3. Rescalability of dynamical generators

**Definition 2.** We define a physical family of dynamical generator \( \{L_t\}_{t \geq 0} \) to be rescalable if by multiplying its elements by any positive constant, \( \alpha > 0 \), one obtains a generator family

\[
L'_t := \alpha L_t
\]

(B36)

that importantly also yields physical dynamics.

In particular, given the family of physical maps \( \{\Lambda_t\}_{t \geq 0} \) related to the original generators \( \{L_t\}_{t \geq 0} \) via (B5), the rescalability of dynamical generators assures that for any \( \alpha > 0 \) the new Dyson series

\[
\Lambda'_t = T_\alpha \exp \left\{ \int_0^t L'_\tau \, d\tau \right\} = \sum_{i=0}^{\infty} S_i^{(i)} [L'_t] = \sum_{i=0}^{\infty} \alpha^i S_i^{(i)} [L_t]
\]

(B37)

also defines a valid family of CPTP maps \( \{\Lambda'_t\}_{t \geq 0} \).

Let us emphasise that—even in the case of commutative dynamics for which the time ordering in (B37) can be dropped—the Dyson series (B37) for \( \alpha \neq 1 \) cannot be generally guaranteed to also yield a valid family of CPTP maps. However, in some special cases the physicality of dynamics is clearly preserved when rescaling the dynamical generators. For instance, all CP-divisible dynamics must be rescalable, as the condition (B15) assures also \( L'_t \) to be of the GKSL form and, hence, also yield physical CP-divisible dynamics. Moreover, in case of commutative dynamics, the dynamical generators must also be rescalable in case of the larger class of semigroup-simulable evolutions. The instantaneous exponents must transform then similarly to (B36) as \( Z'_t := \alpha Z_t \), so that their GKSL form (B19)) is trivially preserved for any \( \alpha > 0 \).

In what follows, we discuss three important situations in which non-rescalable families of generators may emerge.

**a. Dynamical generators with singularities**

Firstly, let us note that dynamical generators are generally not rescalable when dealing with dynamics that contain non-invertible CPTP maps and, hence, lead to \( \{L_t\}_{t \geq 0} \) in (B4) that are temporarily singular. In particular, as in such a case the dynamics can be unambiguously recovered from the dynamical generators only for times smaller than \( T \) denoting occurrence of the (first) singularity [26], the dynamical generators typically cease to be rescalable (\( \{\alpha L_{t \geq 0}\}_t \) does not yield physical dynamics for any \( \alpha > 0 \) for \( t \geq T \), even if for \( \alpha = 1 \) the dynamics satisfies the sufficient conditions (see [26]) for the fully recoverable despite singular behaviour of \( \{L_t\}_{t \geq 0} \).

A simple example is provided by the Jaynes-Cummings model of the spontaneous decay described above, in which case when considering the on-resonance regime with strong coupling the damping rate (B35) becomes divergent periodically in \( t \)—manifesting instantaneous non-invertibility of the corresponding dynamical maps \( \Lambda_t \) in (B4). Nevertheless, considering then a rescaled version (B36), \( \alpha L_t \), of the relevant amplitude damping generator (B31) discussed above, one may, in principle, integrate (ignoring potential complex solutions) the resulting ME to obtain the family of maps describing the dynamics. For instance, setting \( \gamma_0 = 3/2 \) and \( \lambda = 1 \) in (B35), one may naively derive \( \{\Lambda_t\}_{t \geq 0} \) describing the dynamics and, hence, their corresponding Choi-Jamiolkowski (CJ) matrices (B2), whose eigenvalues then read

\[
\lambda_{\text{evals}} (\Omega_{\Lambda_t}) = 1 \pm 2^{-\alpha} e^{-\alpha t} \left[ \sqrt{2} \cos \left( \frac{t}{\sqrt{2}} \right) + \sin \left( \frac{t}{\sqrt{2}} \right) \right]^{2\alpha}.
\]

(B38)

Thus, although (B38) proves that the original dynamics (case of \( \alpha = 1 \)) is physical for all \( t \) with eigenvalues being real and positive, it becomes clear that only for times before the occurrence of the first singularity, i.e., \( t \leq T = \frac{2}{|\lambda|} \left( \arccot \left( \frac{\lambda}{|\lambda|} \right) + \pi \right) = \sqrt{2} \left[ \pi - \arctan (\sqrt{2}) \right] \) the eigenvalues are guaranteed to remain so for any \( \alpha \geq 0 \). For \( t \geq T \) they are not unambiguously defined, as the corresponding ME when integrated starts to exhibit complex (and, hence, unphysical) solutions.

**b. Weak-coupling-based generators**

On the other hand, such a phenomenon also occurs when dealing with families of dynamical generators derived with a particular timescale \( T (0 \leq t \leq T) \) in mind. A canonical example are the MEs (B3) derived only approximately—assuming the weak-coupling regime of interactions (as in Sec. C). Consider the family \( \{\lambda^2 L_t\}_{0 \leq t \leq T} \) of generators
derived by employing a microscopic model and making a weak-coupling approximation to $O(\lambda^2)$ [33]—so that $L_{\text{real}}^t \approx \lambda^2 L_t$ (e.g., by assuming the Redfield form of the ME [58]). One may then simply interpret the rescaling constant as the square of the coupling constant, i.e., $\alpha^2 = \lambda^2$. Importantly, such a generator family is guaranteed to yield physical dynamics—a family of CPTP maps—only on timescales in which the weak-approximation approximation is valid, i.e., for $T \ll \lambda^{-2}$ [59]. Now, as the rescaling by $\alpha > 1$ corresponds then to increasing the interaction strength $\lambda$, by rescaling the generators or, in other words, by choosing strong enough coupling, one eventually must invalidate the weak-coupling approximation enough to make the dynamics unphysical.

\section*{c. Commutative but not semigroup-simulable dynamics}

Finally, let us remark that, although all families of dynamical generators that lead to commutative and semigroup-simulable (SSC) dynamics must be rescalable, it is enough to lose the semigroup-simulability (SS) property for the rescalability to be also invalidated. In particular, general dynamical generators of commutative, but not semigroup-simulable dynamics may not be rescalable. A direct example is provided by the \textit{eternally non-Markovian model} discussed above in App. B 2 corresponds to random unitary (RU) qubit dynamics (B23) with rates specified in (B27), which is commutative but non-SS. Considering now the family of generators $\{L^{(1)}_t = \alpha L_t^t\}_{t \geq 0}$, it has been shown that the physicality conditions (B26) are only fulfilled iff $\alpha \geq 1$ [60], so that the model is indeed non-rescalable.

\section*{4. Additivity of dynamical generators}

\textbf{Definition 3.} Let us consider two families of dynamical generators $\{L^{(1)}_t\}_{t \geq 0}$ and $\{L^{(2)}_t\}_{t \geq 0}$ that are guaranteed to yield physical evolutions and the family $\{L'_t\}_{t \geq 0}$ formed by a linear combination of the two:

$$L'_t := \alpha L^{(1)}_t + \beta L^{(2)}_t$$

(B39)

with some $\alpha, \beta \geq 0$. We define the families $\{L^{(1)}_t\}_{t \geq 0}$ and $\{L^{(2)}_t\}_{t \geq 0}$ to be additive (with respect to one another), if for any $\alpha, \beta \geq 0$ the family $\{L'_t\}_{t \geq 0}$ yields physical dynamics.

First, note that if either of the generator families, e.g., $\{L^{(1)}_t\}_{t \geq 0}$, is not rescalable, then it trivially follows that the pair cannot be additive as there must exist $\alpha > 0$ and $\beta = 0$ that give $\{L'_t\}_{t \geq 0}$, yielding non-physical dynamics. Hence, from the beginning, when discussing additivity of various dynamical classes according to Def. 3 one must exclude the cases of non-rescalable generators.

On the other hand, it is also straightforward to observe that (as depicted in Fig. 2 of the main text) all pairs of CP-divisible dynamics must be additive. It is so as any combination (B39) of CP-divisible dynamical generators, i.e., of the GKSL form (B17) with $D(t) \geq 0$, must yield for any $\alpha, \beta \geq 0$ a generator $L'_t$ that is also of the GKSL form and, hence, not only physical but also CP-divisible.

\textbf{Definition 4.} We define a pair of families of dynamical generators $\{L^{(1)}_t\}_{t \geq 0}$ and $\{L^{(2)}_t\}_{t \geq 0}$ to be mutually commutative if for all $\alpha, \beta \geq 0$ their combinations defined via (B39) yield commutative dynamics, i.e., $\forall_{s,t \geq 0}: [L'_s, L'_t] = 0$.

It then trivially follows that any mutually commutative pair $\{L^{(1)}_t\}_{t \geq 0}$ and $\{L^{(2)}_t\}_{t \geq 0}$ must always consist of commutative families of dynamical generators as by considering the choices $\alpha > 0$, $\beta = 0$ and $\alpha = 0$, $\beta > 0$; the Def. 4 of mutual commutativity assures that for all $s,t \geq 0$ $[L^{(1)}_s, L^{(1)}_t] = 0$ and $[L^{(2)}_s, L^{(2)}_t] = 0$, respectively.

\textbf{Theorem 1.} If the families $\{L^{(1)}_t\}_{t \geq 0}$ and $\{L^{(2)}_t\}_{t \geq 0}$ of dynamical generators are mutually commutative and yield semigroup-simulable dynamics, then they are also additive.

\textbf{Proof.} As the families are mutually commutative, their instantaneous exponents also add according to (B39), i.e., $Z'_t = \alpha Z^{(1)}_t + \beta Z^{(2)}_t$. Moreover, as each of them is also semigroup-simulable (and commutative), they both satisfy $\Gamma^{(1)}_{ij}(t) \geq 0$ and $\Gamma^{(2)}_{ij}(t) \geq 0$ in (B22). Hence, the resulting family $\{L'_t\}_{t \geq 0}$ is also semigroup-simulable (and commutative) as it must lead to positive semi-definite

$$\Gamma'_{ij}(t) = \alpha \Gamma^{(1)}_{ij}(t) + \beta \Gamma^{(2)}_{ij}(t) \geq 0$$

(B40)
FIG. 4. CJ eigenvalues as functions of time for linear combinations of generators, $\alpha \mathcal{L}^{(1)}_t + \beta \mathcal{L}^{(2)}_t$, when addition leads to unphysical dynamics. In all cases: $\alpha = \beta = 1$, $\mathcal{L}^{(1)}_t$ is the dephasing along $x$, $\mathcal{L}^{(2)}_t$ is the amplitude damping in $z$; while the rate functions are chosen differently: (a) $\gamma_1(t) = \sin(2t)$, while $\gamma_2(t) = 1$; (b) $\gamma_1(t) = 1/2$, while $\gamma_2(t) = \sin(t)$; (c) $\gamma_1(t)$ is super-Ohmic with cut-off frequency $\omega_c = 1$, and Ohmicity parameter $s = 4.5$, while $\gamma_2(t) = 1$; (d) $\gamma_1(t) = 1$, while $\gamma_2(t)$ is Lorentzian, off-resonant, with detuning $\Delta = 3$, spectral width $\lambda = 0.05$ and excited-state decay rate $\gamma_0 = 150$.

Note that in all cases negative eigenvalues occur, indicating that the evolution ceases to be CP at some point in time.

in (B22) for any $\alpha, \beta \geq 0$. Finally, as it then follows from Fact. 1 that $\{\mathcal{L}'_t\}_{t \geq 0}$ is physical the starting families of dynamical generators must be additive.

Note that Thm. 1 importantly does not assure physicality when adding dynamical generators that despite being semigroup-simulable and commutative (SSC) alone (and, hence, rescalable) are not mutually commutative. We demonstrate this below in App. B5 by considering the two canonical SSC classes of qubit evolution: random unitary and phase covariant dynamics introduced in App. B2 above.

5. Additivity of dynamical generators: Counterexamples based on qubit dynamics

In order to construct counterexamples to additivity, we consider the dynamical generators given in (5) of the main text and make specific choices for their corresponding rates $\gamma_1(t)$ and $\gamma_2(t)$. We then solve the master equation (ME):

$$\frac{d}{dt}\rho(t) = \mathcal{L}'_t[\rho(t)] = \alpha \mathcal{L}^{(1)}_t[\rho(t)] + \beta \mathcal{L}^{(2)}_t[\rho(t)],$$

for some $\alpha, \beta \geq 0$ to explicitly obtain the channel family $\{\Lambda_t\}_{t \geq 0}$. Finally, we show that such a family contains maps that cease to be completely positive (CP) for some time, what is manifested by some eigenvalues of the corresponding Choi-Jamiołkowski (CJ) defined in (B2)) matrices $\{\Omega_{\Lambda_t}\}_{t \geq 0}$ becoming negative.

To solve the ME (B41), we choose an operator basis allowing us to use matrix and vector representations for the generators and states, respectively. Specifically, we take the basis:

$$\mu_0 = 1/\sqrt{2}, \quad \mu_1 = \hat{\sigma}_x/\sqrt{2}, \quad \mu_2 = \hat{\sigma}_y/\sqrt{2}, \quad \mu_3 = \hat{\sigma}_z/\sqrt{2},$$

such that $\text{Tr}[\mu_i \mu_j] = \delta_{ij}$, in which any generator $\mathcal{L}$ may be represented by a matrix $M_{ij} = \text{Tr}[\mu_i \mathcal{L} \mu_j]$, while any state $\rho$ by a vector $x_i = \text{Tr}[\rho \mu_i]$. The ME (B41) is then equivalent to the set of linear, coupled differential equations:

$$\frac{d}{dt}x(t) = M'_i x(t),$$

where $M'_i$ is the matrix representation of $\mathcal{L}'_t$. 
a. Simple non-Markovian dephasing

For our first example, we take
\[ \gamma_1(t) = \sin(\omega t), \quad (B44) \]
\[ \gamma_2(t) = \gamma, \quad (B45) \]
with \( \omega, \gamma > 0 \) constants. Since \( \int_0^t \gamma_1(s) ds = 1 - \cos(\omega t)/\omega \geq 0 \) for all \( t \), both generators are physical (and rescalable). They give:

\[
M'_t = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -\frac{\beta \gamma}{2} - 2\alpha \sin(t\omega) & 0 \\
\beta \gamma & 0 & 0 & -\beta \gamma - 2\alpha \sin(t\omega)
\end{pmatrix}. \quad (B46)
\]

Solving (B43), one finds

\[
x(t) = \begin{pmatrix}
x_0(0) \\
\frac{e^{-\frac{\beta \gamma t}{2}} x_1(0)}{\gamma} \\
\frac{e^{-\frac{2\alpha}{\gamma} + \frac{2\alpha \cos(t\omega)}{\omega}} - \frac{2\alpha t}{\omega}}{-\beta \gamma t} [x_3(0) + \beta e^{\frac{\beta t}{\omega}} I(t) x_0(0)] \\
\frac{e^{-\frac{2\alpha}{\gamma} + \frac{2\alpha \cos(t\omega)}{\omega}} - \frac{2\alpha t}{\omega}}{-\beta \gamma t} [x_3(0) + \beta e^{\frac{\beta t}{\omega}} I(t) x_0(0)] \\
\end{pmatrix}, \quad (B47)
\]

where
\[
I(t) = \int_0^t e^{\beta \gamma s - \frac{2\alpha \cos(s\omega)}{\omega}} ds. \quad (B48)
\]

The CJ-matrix eigenvalues are then given by

\[
\frac{1}{2} e^{-\frac{2\alpha}{\gamma} - \beta \gamma t} \left( e^{\frac{2\alpha}{\gamma} + \beta \gamma t} + e^{\frac{2\alpha \cos(t\omega)}{\omega}} \right) \pm \sqrt{\beta^2 \gamma^2 e^{\frac{4\alpha \cos(t\omega)}{\omega} + 1} I(t)^2 + e^{\beta \gamma t} \left( e^{\frac{2\alpha}{\gamma} + \frac{2\alpha \cos(t\omega)}{\omega}} \right)^2}, \quad (B49)
\]

\[
\frac{1}{2} e^{-\frac{2\alpha}{\gamma} - \beta \gamma t} \left( e^{\frac{2\alpha}{\gamma} + \beta \gamma t} - e^{\frac{2\alpha \cos(t\omega)}{\omega}} \right) \pm \sqrt{\beta^2 \gamma^2 e^{\frac{4\alpha \cos(t\omega)}{\omega} + 1} I(t)^2 + e^{\beta \gamma t} \left( e^{\frac{2\alpha}{\gamma} - \frac{2\alpha \cos(t\omega)}{\omega}} \right)^2}, \quad (B50)
\]

and plotted in Fig. 4(a) for \( \alpha = \beta = 1, \omega = 2, \) and \( \gamma = 1. \) For \( t = \pi, \) we have \( I \approx 23.36, \) and it is easy to check that two of the eigenvalues are negative. Hence, the evolution is clearly unphysical.

b. Simple non-Markovian amplitude damping

For our second example, we take
\[ \gamma_1(t) = \gamma, \quad (B51) \]
\[ \gamma_2(t) = \sin(\omega t), \quad (B52) \]
Again, both generators are physical and rescalable. This time we get

\[
M'_t = \begin{pmatrix}
0 & -\frac{1}{2} \beta \sin(t\omega) & 0 & 0 \\
0 & 0 & 0 & 0 \\
\beta \sin(t\omega) & 0 & -2\alpha \gamma - \frac{1}{2} \beta \sin(t\omega) & 0 \\
0 & 0 & 0 & -2\alpha \gamma - \beta \sin(t\omega)
\end{pmatrix}. \quad (B53)
\]

and solving (B43) we obtain

\[
x(t) = \begin{pmatrix}
x_0(0) \\
\frac{e^{\beta \gamma t} x_1(0)}{\gamma} \\
\frac{e^{-\frac{\beta \gamma t}{2}} x_2(0)}{\gamma} - \frac{2\alpha \gamma}{\omega} [x_3(0) + \frac{\beta}{\omega} e^{\frac{\beta t}{\omega}} I(t) x_0(0)] \\
\frac{e^{-\frac{\beta \gamma t}{2}} x_2(0)}{\gamma} - \frac{2\alpha \gamma}{\omega} [x_3(0) + \frac{\beta}{\omega} e^{\frac{\beta t}{\omega}} I(t) x_0(0)] \\
\end{pmatrix}, \quad (B54)
\]
where now

$$I(t) = \int_0^t \sin(s\omega)e^{2\alpha\gamma s - \beta \tan^{-1}(s\omega)} ds.$$  \hspace{1cm} (B55)

The CJ eigenvalues in this case are

$$\frac{1}{2}e^{-\frac{1}{2}2\alpha\gamma t}\left(e^{\frac{1}{2}2\alpha\gamma t} + e^{\frac{\beta \tan^{-1}(t\omega)}{\omega}}\right) \pm \sqrt{e^{\frac{\beta \tan^{-1}(t\omega)}{\omega}}\left(\beta e^{\frac{\beta \tan^{-1}(t\omega)}{\omega}}I(t)^2 + (e^{2\alpha\gamma t} - 1)^2\right)}, \hspace{1cm} (B56)$$

$$\frac{1}{2}e^{-\frac{1}{2}2\alpha\gamma t}\left(e^{\frac{1}{2}2\alpha\gamma t} - e^{\frac{\beta \tan^{-1}(t\omega)}{\omega}}\right) \pm \sqrt{e^{\frac{\beta \tan^{-1}(t\omega)}{\omega}}\left(\beta e^{\frac{\beta \tan^{-1}(t\omega)}{\omega}}I(t)^2 + (e^{2\alpha\gamma t} + 1)^2\right)}. \hspace{1cm} (B57)$$

They are plotted in Fig. 4(b) for $\alpha = \beta = 1$, $\omega = 1$, and $\gamma = 1/2$. For $t = 2\pi$ the integral is $I \approx -204.81$, and again two of the eigenvalues are negative and the evolution is unphysical.

c. Non-Markovian dephasing from a spin-boson model

To get a more physical example with an underlying microscopic model, we consider the dephasing generator to arise from a single spin coupled to a bosonic reservoir with an Ohmic-like spectrum, described in App. B 2. In that case, the rate functions are (c.f. (B30))

$$\gamma_1(t) = \omega_c \left[1 - (\omega_c t)^2\right]^{-\frac{3}{2}} \Gamma[s \sin[s \arctan(\omega_c t)]], \hspace{1cm} (B58)$$

$$\gamma_2(t) = \gamma, \hspace{1cm} (B59)$$

with $\omega_c$ the cut-off frequency and $s$ the Ohmicity parameter.

In this case, we cannot solve the master equation analytically. However, we can solve it numerically for given parameter settings at each $t$ and obtain the corresponding CJ eigenvalues. These are plotted in Fig. 4(c) for $\alpha = \beta = 1$, $\omega_c = 1$, $s = 4.5$, which corresponds to a super-Ohmic spectrum, and $\gamma = 1$. We see that the evolution becomes unphysical around $t = \pi/2$.

d. Non-Markovian amplitude damping from a Jaynes-Cummings model

We get a second example with an underlying microscopic model by taking the amplitude damping to come from a Jaynes-Cummings model with a Lorentzian spectrum described in App. B 2. The rate functions are (c.f. (B34))

$$\gamma_1(t) = \gamma, \hspace{1cm} (B60)$$

$$\gamma_2(t) = \text{Re}\left\{\frac{2\gamma_0\lambda}{\lambda - i\Delta + d \coth \left(\frac{\gamma_0}{2}\right)}\right\}. \hspace{1cm} (B61)$$

Again, we find the CJ eigenvalues by solving the master equation numerically for fixed parameter values, and they are plotted in Fig. 4(d) for $\alpha = \beta = 1$, $\Delta = 3$, $\lambda = 0.05$, $\gamma_0 = 150$, and $\gamma = 1$. We see that the evolution becomes unphysical a bit before $t = \pi/2$.

Appendix C: Validity of adding dynamical generators in the weak-coupling regime

We demonstrate that in the weak-coupling regime interactions with multiple environments are adequately described by adding the corresponding dynamical generators, as then the cross-term in (7) must vanish. The proof for the regime in which the Born-Markov approximation is valid can be found in Ref. [9]. In this section, we extend this result to master equations derived for broader regimes in which only the time-convolutionless (TCL) approximation to second order holds.

We decompose each of the interaction Hamiltonians into system and environment parts, so that for a single environment the interaction Hamiltonian reads:

$$H_I = \sum_k A_k \otimes B_k. \hspace{1cm} (C1)$$
On the other hand, we can rewrite the system operators in the basis of the system Hamiltonian, \( A_k = \sum \omega A_k(\omega) \), where \( A_k(\omega) \) are projections onto the eigenspaces of \( H_S \). In the interaction picture with respect to the free Hamiltonian of the total system, \( H_S + H_E \), the interaction Hamiltonian takes the form \( \tilde{H}_I(t) = \sum_k \tilde{A}_k(t) \otimes \tilde{B}_k(t) \), with \( \tilde{A}_k(t) = \sum \omega e^{-i\omega t} A_k(\omega) \). In this notation the ME (A13) reads

\[
\frac{d\tilde{\rho}(t)}{dt} = -\sum_{k,l} \int_0^t ds \text{ Tr}_E \left\{ \left[ \tilde{A}_k(t) \otimes \tilde{B}_k(t), \left[ \tilde{A}_l(s) \otimes \tilde{B}_l(s), \tilde{\rho}(s) \right] \right] \right\}
\]

(2)

\[
= -\sum_{k,l} \int_0^t ds \left[ \tilde{A}_k(t) \tilde{A}_l(s) \text{ Tr}_E \left\{ \tilde{\rho}_SE(s) \tilde{B}_k(t) \tilde{B}_l(t) \right\} - \tilde{A}_k(t) \text{ Tr}_E \left\{ \tilde{\rho}_SE(s) \tilde{B}_l(t) \tilde{B}_k(t) \right\} \tilde{A}_l(s) + \right.
\]

(3)

\[
- \tilde{A}_l(s) \text{ Tr}_E \left\{ \tilde{\rho}_SE(s) \tilde{B}_k(t) \tilde{B}_l(t) \right\} \tilde{A}_k(t) + \text{ Tr}_E \left\{ \tilde{\rho}_SE(s) \tilde{B}_l(s) \tilde{B}_k(t) \right\} \tilde{A}_l(s) \tilde{A}_k(t) \right]\n\]

(4)

with condition (A8) now corresponding to the requirement that \( \text{ Tr}_E \left\{ \tilde{B}_k(t) \rho_E \right\} = 0 \) for all \( k \) (or equivalently \( \text{ Tr}_E \left\{ \tilde{B}_k \rho_E \right\} = 0 \) as \( H_E \rho_E = 0 \).

For multiple environments one can analogously write

\[
\tilde{H}_I = \sum_k A_{i,k} \otimes \tilde{B}_k^{E_i},
\]

(C5)

and \( \tilde{H}_I(t) = \sum_k \tilde{A}_{i,k}(t) \otimes \tilde{B}_k^{E_i}(t) \) in the interaction picture w.r.t. \( H_S + \sum_i H_{E_i} \). The key assumption that allows for the equivalence between coupling to independent reservoirs and adding their corresponding generators at the ME level is the weak-coupling assumption. Specifically, the cross-term in the ME (7) always vanishes under the TCL weak-coupling approximation to second order, as we now show.

Pulling out an explicit coupling strength parameter \( \alpha \) from each interaction Hamiltonian (we can take it to be the same for all of them without loss of generality, as we do not restrict their norms), we can write the exact ME (6) as

\[
\frac{d}{dt} \tilde{\rho}(t) = -\alpha^2 \sum_{ij} \int_0^t ds \text{ Tr}_E \left\{ \left[ \tilde{H}_{ij}(t), \left[ \tilde{H}_{ij}(s), \tilde{\rho}(s) \otimes \rho_E \right] \right] \right\}.
\]

(C6)

Assuming the reservoirs to be weakly coupled \( (\alpha \ll 1) \) with the system, we may employ the TCL projective methods and derive the ME valid up to \( O(\alpha^2) \) (TCL2):

\[
\frac{d}{dt} \tilde{\rho}(t) = -\alpha^2 \sum_{ij} \int_0^t ds \text{ Tr}_E \left\{ \left[ \tilde{H}_{ij}(t), \left[ \tilde{H}_{ij}(s), \tilde{\rho}(s) \otimes \rho_E \right] \right] \right\}.
\]

(C7)

Using the ”A-B decomposition” of the interaction Hamiltonian (C5), the above ME reads

\[
\frac{d}{dt} \tilde{\rho}(t) = -\alpha^2 \sum_{ij} \int_0^t ds \text{ Tr}_E \left\{ \left[ \sum_k \tilde{A}_{i,k}(t) \otimes \tilde{B}_k^{E_i}(t), \sum_l \tilde{A}_{j,l}(s) \otimes \tilde{B}_l^{E_j}(s), \tilde{\rho}(s) \otimes \rho_E \right] \right\}.
\]

(C8)

\[
= \sum_{i,j} \sum_{k,l} C_{i,k,j,l}(t,s) \left( \tilde{A}_{i,k}(t) \tilde{A}_{j,l}(s) \rho_S(t) - \tilde{A}_{j,l}(s) \rho_S(t) \tilde{A}_{i,k}(t) \right) +
\]

(C9)

\[
+ C_{i,l,j,k}(t,s) \left( \rho_S(t) \tilde{A}_{j,l}(s) \tilde{A}_{i,k}(t) - \tilde{A}_{i,k}(t) \rho_S(t) \tilde{A}_{j,l}(s) \right),
\]

(C10)

where in the last line we have introduced the second-order bath correlation function

\[
C_{i,k,j,l}(t,s) := \text{ Tr}_{E_{i,j}} \left\{ \tilde{B}_k^{E_i}(t) \otimes \tilde{B}_l^{E_j}(s) \rho_{E_{i,j}} \right\}.
\]

(C11)

In the TCL method, the state \( \rho_E \) is a fixed state of the environment and is usually taken to be the initial state. Since we are investigating the reduced dynamics of an open system under the interaction with independent reservoirs, it will remain in a product state of the initial states of the respective reservoirs, in particular \( \rho_{E_{i,j}} = \rho_{E_i} \otimes \rho_{E_j} \). Therefore, the second-order bath correlation function corresponding to different reservoirs reduces to a product of first-order bath correlation functions for each of the baths, \( C_{i,k,j,l}(t,s) = \text{ Tr}_{E_i} \left\{ \tilde{B}_k^{E_i}(t) \rho_{E_i} \right\} \text{ Tr}_{E_j} \left\{ \tilde{B}_l^{E_j}(s) \rho_{E_j} \right\} \). This observation together with the previously taken assumption that the single coupling operator expectation value is zero (A8), assures the cross-term in the ME (6) to accordingly vanish in the weak-coupling regime. Note that the Born-Markov approximation, within which generator addition has been proved to be valid in Ref. [9], corresponds to the special case of time-independent correlation functions \( C_{i,k,j,l} \) in (C11).
Appendix D: Validity of adding dynamical generators ensured by commutativity of the microscopic Hamiltonians

We provide the proof for the ‘Yes’ region in the Venn diagram depicted in Fig. 3 of the main text. In particular, we show that one can simply add (in the interaction picture) the dynamical generators associated with individual environments at the master equation level when the interaction Hamiltonians commute among themselves and also with the system Hamiltonian ($H \cap IS$) within the microscopic description of the dynamics.

In particular, assuming that for all $i$ and $j$

$$[H_I, H_J] = 0 \quad \text{and} \quad [H_I, H_S] = 0, \quad (D1)$$

where $H_I := \sum_i H_{I_i}$ is the full interaction Hamiltonian, the unitary operator describing evolution of the total system can be decomposed as follows

$$U_{SE}(t) = e^{-i(H_S+H_E+H_I)t} = e^{-iH_ST} \prod_i e^{-i(H_{E_i}+H_{I_i})t} =: U_S(t) \prod_i U_{SE_i}(t). \quad (D2)$$

To simplify the notation, for any unitary transformation we write $U_t[\rho] = U(t) \rho U(t)$. The transformation defined by (D2) then reads $U_t^{SE} = U_t^S \prod_i U_t^{SE_i}$, where $U_t^{SE_i}$ commute among each other. Consequently, the reduced dynamics of the open system is described by

$$\rho_S(t) = \text{Tr}_E \{U_t^{SE}[\rho_S(0) \otimes \rho_E]\} = \text{Tr}_E \left\{ U_t^S \prod_i U_t^{SE_i} \left[ \rho_S(0) \otimes \bigotimes_{k \neq 1} \rho_{E_k} \right] \right\}$$

$$= U_t^S \left( \text{Tr}_{E_i \neq 1} \prod_{i \neq 1} U_t^{SE_i} \left( \text{Tr}_{E_i} \{U_t^{SE_i}[\rho_S(0) \otimes \rho_{E_i}]\} \otimes \bigotimes_{k \neq 1} \rho_{E_k} \right) \right)$$

$$= U_t^S \left( \text{Tr}_{E_i \neq 1} \prod_{i \neq 1} U_t^{SE_i} \left( \Lambda_t^{(i)}[\rho_S(0)] \otimes \bigotimes_{k \neq 1} \rho_{E_k} \right) \right)$$

$$= \cdots$$

$$= U_t^S \left( \prod_i \Lambda_t^{(i)}[\rho_S(0)] \right), \quad (D7)$$

where $\Lambda_t^{(i)}[\rho] := \text{Tr}_{E_i} \{U_t^{SE_i}[\rho \otimes \rho_{E_i}]\}$ is the dynamical map describing the dissipative part of evolution coming solely from the interaction with the $i$th environment. In the interaction picture with respect to the free Hamiltonian $H_S + \sum_i H_{E_i}$, the system evolution then reads

$$\bar{\rho}_S(t) = e^{iH_St} \rho_S(t) e^{-iH_ST} = \prod_i \Lambda_t^{(i)}[\rho_S(0)] = \prod_i \Lambda_t^{(i)}[\bar{\rho}_S(0)]. \quad (D8)$$

Note that the commutativity conditions imposed on the system and interaction Hamiltonians (D1) give rise to a dynamics for which all maps originating from interactions with different reservoirs commute with each other, i.e., $\forall_{i,j} : [\Lambda_t^{(i)}, \Lambda_t^{(j)}] = 0$. It then follows that the dynamics at the level of generators is additive in the interaction picture. Specifically, denoting the full channel (D8) by $\Lambda_t$, the corresponding generator is given by

$$\bar{\mathcal{L}}_t = \Lambda_t \Lambda_t^{-1} = \left( \sum_i \Lambda_t^{(i)} \prod_{j \neq i} \Lambda_t^{(j)} \right) \prod_i \left( \Lambda_t^{(i)} \right)^{-1} = \sum_i \Lambda_t^{(i)} \left( \Lambda_t^{(i)} \right)^{-1} = \sum_i \bar{\mathcal{L}}_t^{(i)}, \quad (D9)$$

where $\bar{\mathcal{L}}_t^{(i)}$ is the generator corresponding to the interaction with the $i$th environment alone, and we have used the commutativity of the channels.

Appendix E: Validity of adding dynamical generators: Counterexamples based on the spin-magnet model

We develop a model of a single spin in contact with one or more baths consisting of many spins. Within this model, not only the dynamics of the total system with both one or multiple baths can be solved, but also when tracing out
various reservoirs it allows to obtain the exact dynamics of the open single-spin system. As a result, we can then compare the corresponding master-equation generators for coupling with one or multiple baths. In particular, we use the model to explicitly construct counterexamples that allow us to prove invalidity of adding generators at the master equation level for the relevant cases summarised in Fig. 3 in the main text.

1. Spin-magnet model

We refer to the single spin-1/2 (a qubit) as the ‘system’ and the reservoirs as ‘magnets’. The system Hamiltonian $H_S$ can be arbitrary, while we always take the bath Hamiltonians to vanish, $H_E = 0$. We take all Hamiltonians, including the system-bath interaction, to be time independent.

One magnet consists of $N$ spin-1/2 particles, and we introduce the magnetisation operator

$$\hat{m} = \sum_{n=1}^{N} \hat{\sigma}_z^{(n)} = \sum_{k=0}^{N} m_k \Pi_k,$$

where $\Pi_k$ are the projectors on the subspaces with magnetisation $m_k$ (i.e. with $k$ spins pointing up). The magnetisation values, $m_k$, takes $N + 1$ equally spaced values between $-N$ and $N$, i.e.,

$$m_k = -N + 2k \quad \text{for} \quad k = 0, ..., N.$$  \hspace{1cm} (E2)

We will always assume that the initial state (at $t = 0$) of system and bath factorises, $\rho_{SE}(0) = \rho_S(0) \otimes \rho_E(0)$, and we will take the initial state of the bath to be a classical mixture of different magnetisations. That is, the state can be written

$$\rho_E(0) = \sum_{k=0}^{N} q_k \Pi_k.$$  \hspace{1cm} (E3)

The probability for an observation of the magnetisation to yield $m_k$ is

$$p(m_k) = \text{Tr}[\rho_E(0) \Pi_k] = q_k \text{Tr} \Pi_k.$$  \hspace{1cm} (E4)

In the limit of large $N$, this discrete distribution approaches a continuous distribution, and its moments can be computed as

$$\sum_k (m_k)^s p(m_k) \xrightarrow{N \to \infty} \int_{-\infty}^{\infty} dm m^s p(m).$$  \hspace{1cm} (E5)

We will further consider only interaction Hamiltonians which couple the system spin to the magnetisation of the bath, i.e. $H_I = A \otimes \hat{m}$, for some system observable $A$. With such an interaction, and an initial state as above, the global state of system and bath at any time is a mixture of states for different bath magnetisation (i.e. there is no coherence between states of different magnetisation). We can then write

$$\rho_{SE}(t) = \sum_k q_k \rho_S^{(k)}(t) \otimes \Pi_k,$$  \hspace{1cm} (E6)

where we can understand $\rho_S^{(k)}$ as the conditional (not necessarily normalised) state of the system for bath magnetisation $m_k$. The idea is now that we will be able to solve the global dynamics of system and bath exactly, i.e., we will be able to find $\rho_S^{(k)}(t)$. Once we have this, we can then compute the reduced system dynamics as

$$\rho_S(t) = \text{Tr}_E \rho_{SE}(t) = \sum_k p(m_k) \rho_S^{(k)}(t) \xrightarrow{N \to \infty} \int_{-\infty}^{\infty} dm p(m) \rho_S(m, t).$$  \hspace{1cm} (E7)

To see how this is going to work, consider an arbitrary system Hamiltonian $H_S$ and insert (E6) and the Hamiltonians
into the von Neumann equation:

\[ \dot{\rho}_{SE}(t) = \sum_{k} q_k \rho_S^{(k)}(t) \otimes \Pi_k = -i[H_S + H_I, \rho_{SE}(t)] \]  

(E8)

\[ = -i[H_S \otimes \mathbb{1} + A \otimes \hat{m}, \rho_{SE}(t)] \]  

(E9)

\[ = -i[H_S \otimes \mathbb{1} + \sum_k m_k A \otimes \Pi_k, \sum_k q_k \rho_S^{(k)}(t) \otimes \Pi_k] \]  

(E10)

\[ = -i \sum_k q_k [H_S, \rho_S^{(k)}(t)] \otimes \Pi_k + \sum_k q_k [m_k A, \rho_S^{(k)}(t)] \otimes \Pi_k \]  

(E11)

\[ = -i \sum_k q_k [H_S + m_k A, \rho_S^{(k)}(t)] \otimes \Pi_k. \]  

(E12)

We see that there is no coupling between different magnetisations (e.g. between \( k \) and \( k + 1 \)) – we get a set of identical differential equations for each \( k \). We also see that

\[ \text{Tr}[\rho_S^{(k)}(t)] = p(m_k)^{-1} \text{Tr}[\hat{\rho}_{SE}(t)(\mathbb{1} \otimes \Pi_k)] = p(m_k)^{-1} \text{Tr}[H_S + m_k A, \rho_S^{(k)}(t)] \text{Tr} \Pi_k = 0. \]  

(E13)

It follows that the distribution over magnetisations does not change, and that since initially \( \rho_S^{(k)}(t) = \rho_S(0) \) for all \( k \), the \( \rho_S^{(k)}(t) \) remain normalised at all times. We can therefore write them in Bloch representation

\[ \rho_S^{(k)}(t) = \frac{1}{2}(\mathbb{1} + \mathbf{r}^{(k)}(t) \cdot \mathbf{\sigma}), \]  

(E14)

and from the von Neumann equation above we get that

\[ \mathbf{r}^{(k)}(t) \cdot \mathbf{\sigma} = -i[H_S + m_k A, \mathbf{r}^{(k)}(t) \cdot \mathbf{\sigma}]. \]  

(E15)

This defines a set of three coupled, linear differential equations whose solutions gives us \( \mathbf{r}^{(k)}(t) \) in terms of \( \mathbf{r}^{(k)}(0) \). Note that, decomposing \( H_S \) and \( A \) in terms of the Pauli operators and the identity, the commutators of Pauli operators will always yield other Pauli operators while the identity does not contribute to the commutator. Consequently, the general solution can always be written in the matrix form without any inhomogeneous shift-terms, i.e.,

\[ \mathbf{r}^{(k)}(t) = R^{(k)}(t) \mathbf{r}^{(k)}(0), \]  

(E16)

where \( R^{(k)}(t) \) specifies the transformation of the Bloch vector for each magnetisation index \( k \). The reduced state of the system then evolves according to

\[ \rho_S(t) = \frac{1}{2}(\mathbb{1} + \mathbf{r}(t) \cdot \mathbf{\sigma}), \]  

(E17)

where

\[ \mathbf{r}(t) = R(t) \mathbf{r}(0) = \sum_k p(m_k) \mathbf{r}^{(k)}(t) = \sum_k p(m_k) R^{(k)}(t) \]  

(E18)

As the dynamical map is completely defined by the Bloch vector transformation \( R(t) \), we can work directly with this matrix and define its corresponding generator analogously to the definition of the dynamical generator (B4):

\[ K_t := \hat{R}(t) R^{-1}(t). \]  

(E19)

Now, crucially, note that, due to linearity of transformations switching between the pictures, adding any two dynamical generators \( K_t^{(1)} + K_t^{(2)} \) is completely equivalent to adding their corresponding generators in the Bloch vector representation, \( K_t^{(1)} + K_t^{(2)} \).

Initial state of the bath: Microcanonical distribution. For some of the examples below, we will take the initial magnet state to be a micro-canonical distribution, i.e. each spin configuration is equally probable, then \( q_k = 1/2^N \), and,

\[ p(m_k) = \text{Tr}[\rho_E(0) \Pi_k] = \frac{1}{2^N} \binom{N}{k}. \]  

(E20)

Note that \( p(m_k) \) is a binomial distribution of magnetisation number with zero mean (\( \mu = 0 \)) and variance equal to the number of spins (\( \sigma^2 = N \)) Using the central limit theorem, for large \( N \), this approaches a continuous normal distribution, i.e., \( m_k \sim p(m_k) \rightarrow m \sim \mathcal{N}(\mu, \sigma^2) \),

\[ p(m) = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{m^2}{2\sigma^2}}. \]  

(E21)
Initial state of the bath: Lorentzian distribution. We will also consider a Lorentzian distribution for the initial reservoir state, which in the continuous limit is given by
\[
p(m) = \frac{\lambda}{\pi(\lambda^2 + m^2)},
\]
where \(\lambda\) is the scale parameter (half width at half maximum).

2. The Schrödinger and interaction pictures

For the magnet model, since \(H_E = 0\), the Schrödinger and interaction pictures are related by
\[
\tilde{\rho}_S(t) = e^{iH_st} \rho(t) e^{-iH_st} = \sum_k q_k \rho^{(k)}(t) \otimes \Pi_k,
\]
\[
\hat{H}_I(t) = e^{iH_st} H_I e^{-iH_st} = (e^{iH_st} \hat{A} e^{-iH_st}) \otimes \hat{m} = \hat{A}(t) \otimes \hat{m}.
\]
The von Neumann equation of motion in the interaction picture becomes
\[
\dot{\tilde{\rho}}_S(t) = -i[\hat{H}_I(t), \tilde{\rho}_S(t)],
\]
which results in
\[
\dot{\tilde{r}}^{(k)}(t) = -im_k \tilde{A}(t), \tilde{r}^{(k)}(t) \cdot \sigma
\]
where \(\tilde{r}^{(k)}\) are the Bloch vectors in the interaction picture
\[
\tilde{\rho}^{(k)}_S(t) = \frac{1}{2}(1 + \tilde{r}^{(k)}(t) \cdot \sigma).
\]

Again, these differential equations can be solved. For a solution on matrix form
\[
\tilde{r}^{(k)}(t) = \tilde{R}^{(k)}(t) \tilde{r}^{(k)}(0),
\]
one can derive the corresponding interaction picture dynamical generator as before
\[
\hat{K}_t = \hat{\dot{R}}(t) \hat{R}^{-1}(t).
\]

3. Coupling to a single bath: Recovering dephasing qubit dynamics

To illustrate the model, we first consider interaction with a single magnet only. We take
\[
H_E = 0, \quad H_S = 0, \quad H_I = \frac{1}{2} g \hat{\sigma}_z \otimes \hat{m}.
\]
We note that as both bath and system Hamiltonians vanish, in this case the Schrödinger and interaction pictures coincide. From (E15) we get
\[
\dot{\tilde{r}}^{(k)}(t) \cdot \sigma = \frac{i}{2} gm_k [\hat{\sigma}_z, \tilde{r}^{(k)}(t) \cdot \sigma],
\]
which gives the coupled equations (dropping the index \(k\) and the explicit time dependence for simplicity)
\[
\dot{r}_x = -gmr_y, \quad \dot{r}_y = gmr_x, \quad \dot{r}_z = 0.
\]
This set of equations is easily solved, giving a solution of the form
\[
\tilde{r}(m, t) = R(m, t) \tilde{r}(0),
\]
with
\[
R(m, t) = \begin{pmatrix}
\cos (tg_m) & -\sin (tg_m) & 0 \\
\sin (tg_m) & \cos (tg_m) & 0 \\
0 & 0 & 1
\end{pmatrix}.
\] (E36)

The reduced state of the system is obtained by averaging over the initial magnetisation distribution
\[
R(t) = \int_{-\infty}^{\infty} dm \, p(m) R(m, t). \tag{E37}
\]

If we take a Gaussian (micro-canonical) initial state with \( p(m) \) given according to (E21), we find
\[
R(t) = \begin{pmatrix}
e^{-\frac{1}{2}g^2 \sigma^2 t^2} & 0 & 0 \\
0 & e^{-\frac{1}{2}g^2 \sigma^2 t^2} & 0 \\
0 & 0 & 1
\end{pmatrix}, \tag{E38}
\]
while for a Lorentzian initial state, with \( p(m) \) given in (E22), we have
\[
R(t) = \begin{pmatrix}
e^{-\lambda g t} & 0 & 0 \\
0 & e^{-\lambda g t} & 0 \\
0 & 0 & 1
\end{pmatrix}. \tag{E39}
\]

The corresponding generators (c.f. (E19)) are, for the Gaussian
\[
K_t = \begin{pmatrix}
-\sigma^2 g^2 t & 0 & 0 \\
0 & -\sigma^2 g^2 t & 0 \\
0 & 0 & 0
\end{pmatrix},
\] (E40)
and for the Lorentzian
\[
K_t = \begin{pmatrix}
-\lambda g & 0 & 0 \\
0 & -\lambda g & 0 \\
0 & 0 & 0
\end{pmatrix}. \tag{E41}
\]

Each of these generators correspond to a simple ME of the GKSL form (B17). In particular, for the Gaussian:
\[
L_t(\rho) = \sigma^2 g^2 t (\hat{\sigma}_z \rho \hat{\sigma}_z - \rho), \tag{E42}
\]
which corresponds to dephasing with a time-dependent rate \( \gamma(t) = \sigma^2 g^2 t \) in (B28), while for the Lorentzian:
\[
L_t(\rho) = \lambda g (\hat{\sigma}_z \rho \hat{\sigma}_z - \rho) \tag{E43}
\]
that is the usual semigroup dephasing generator (B28) with a constant rate \( \gamma(t) = \lambda g \).

4. **Counterexample: Interactions commute with system and bath but not each other (II \( \cap \) IS \( \cap \) IE)**

We now consider the regions marked ‘No’ in the Venn diagram of Fig. 3 in the main text, in order to prove by counterexamples that the commutation relations they specify are not sufficient for the dynamical generators to simply add at the master equation level. We start by the case where the interaction Hamiltonians commute with the system and bath Hamiltonians, but not among each other. We take the magnet model with two baths and set:
\[
H_{E_1} = 0, \quad H_{E_2} = 0, \quad H_S = 0, \\
H_{I_1} = \frac{1}{2} g_1 \hat{\sigma}_z \otimes \hat{m}_1, \quad H_{I_2} = \frac{1}{2} g_2 \hat{\sigma}_z \otimes \hat{m}_2. \tag{E44}
\]
Note that, as the system and bath Hamiltonians are absent, the Schrödinger and interaction pictures coincide, while the interaction Hamiltonians trivially commute with \( H_S \) and \( H_{E_i} \), yet not with each other.
In the case of simultaneous coupling to two magnets, (E6) naturally generalises to
\[
\rho_{SE,E_1}(t) = \sum_{k,k'} q_{k,k'} \rho_S^{(k,k')}(t) \otimes \Pi_k \otimes \Pi_{k'},
\]
where \(q_{k,k'}\) now represents the joint probability of finding the magnets in magnetisations \(m_k\) and \(m_{k'}\), respectively, and \(\rho_S^{(k,k')}(t)\) stands for the corresponding conditional reduced state of the system.

Consequently, the von Neumann equation (E15) describing the dynamics for each such conditional state in the Bloch representation is also now parametrised by the two indices, \(k\) and \(k'\), and reads
\[
\dot{r}^{(k,k')}(t) \cdot \sigma = -\frac{1}{2} [g_{1m_{1,k}} \sigma_z + g_{2m_{2,k}} \sigma_x, r^{(k,k')}(t) \cdot \sigma].
\]
Hence, it leads to coupled equations (dropping the indices \(k, k'\) and the explicit time dependence for simplicity)
\[
\begin{align*}
\dot{r}_x &= -g_{1m_1} \tau_y, \\
\dot{r}_y &= g_{1m_1} \tau_x - g_{2m_2} \tau_z, \\
\dot{r}_z &= g_{2m_2} \tau_y,
\end{align*}
\]
which can be solved analytically, giving a solution of the form
\[
R(m_1, m_2, t) = R(m_1, m_2, t)R(0),
\]
with
\[
R(m_1, m_2, t) = \begin{pmatrix}
\cos(t \sqrt{g_{1m_1}^2 + g_{2m_2}^2}) g_{1m_1}^2 + g_{2m_2}^2 & -\sin(t \sqrt{g_{1m_1}^2 + g_{2m_2}^2}) g_{1m_1} g_{2m_2} \\
\sin(t \sqrt{g_{1m_1}^2 + g_{2m_2}^2}) g_{1m_1} g_{2m_2} & \cos(t \sqrt{g_{1m_1}^2 + g_{2m_2}^2}) g_{1m_1}^2 + g_{2m_2}^2 \end{pmatrix}
\]
\[
\begin{pmatrix}
\cos(t g_{1m_1}) & -\sin(t g_{1m_1}) \\
\sin(t g_{1m_1}) & \cos(t g_{1m_1})
\end{pmatrix}
\]
\[
\begin{pmatrix}
1 & 0 \\
0 & \cos(t g_{2m_2}) - \sin(t g_{2m_2})
\end{pmatrix}
\]
\[
\begin{pmatrix}
1 & 0 \\
0 & \cos(t g_{2m_2}) \sin(t g_{2m_2})
\end{pmatrix}
\]
Consistently, when only the first magnet is present \((g_2 = 0)\), this reduces to the expression (E36), i.e.,
\[
R(m_1, t) = \begin{pmatrix}
\cos(t g_{1m_1}) & -\sin(t g_{1m_1}) \\
\sin(t g_{1m_1}) & \cos(t g_{1m_1})
\end{pmatrix}
\]
while with only the second magnet \((g_1 = 0)\) it becomes
\[
R(m_2, t) = \begin{pmatrix}
1 & 0 \\
0 & \cos(t g_{2m_2}) - \sin(t g_{2m_2})
\end{pmatrix}
\]
\[
\begin{pmatrix}
1 & 0 \\
0 & \cos(t g_{2m_2}) \sin(t g_{2m_2})
\end{pmatrix}
\]
We then average over the magnetisations of both magnets, taking the \(N \to \infty\) limit for each of them independently and assuming both \(p(m_1)\) and \(p(m_2)\) to be normally distributed according to (E21). The effective transition matrix \(R(t)\) can then be evaluated as in (E18), but it now generally includes a double integral in both \(m_1\) and \(m_2\). For one magnet we recover (E37), as
\[
R(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dm e^{-\frac{m^2}{2\sigma^2}} R(m, t),
\]
while for the two
\[
R(t) = \frac{1}{2\pi \sigma_1 \sigma_2} \int_{-\infty}^{\infty} dm_1 dm_2 e^{-\frac{m_1^2}{2\sigma_1^2}} e^{-\frac{m_2^2}{2\sigma_2^2}} R(m_1, m_2, t).
\]
Not all the integrals can easily be performed analytically, but if not, we resort to numerics. Our final goal is to compute the generator \(K_t = \dot{R}(t)R^{-1}(t)\). We note that we do not need to take the derivative numerically, since \(\dot{R}(m, t)\) can be computed analytically and
\[
\dot{R}(t) = \frac{1}{2\pi \sigma_1 \sigma_2} \int_{-\infty}^{\infty} dm_1 dm_2 e^{-\frac{m_1^2}{2\sigma_1^2}} e^{-\frac{m_2^2}{2\sigma_2^2}} \dot{R}(m_1, m_2, t).
\]
We compute the generators $K^{(1)}_j$, $K^{(2)}_j$, and $K_t$ corresponding to only magnet 1 or 2 present, or both respectively, in order to check whether $\mathcal{K} = K^{(1)} + K^{(2)}$. Since we can only compute the integrals numerically, we fix the parameters $g_1, g_2, \sigma_1, \sigma_2$, and the time $t$. We take
\[
g_1 = 2, \quad g_2 = 2, \quad \sigma_1 = 1, \quad \sigma_2 = 1, \quad t = 0.5.
\] (E57)
For these values, we find
\[
K^{(1)} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix},
\] (E58)
and
\[
\mathcal{K} = \begin{pmatrix} -1.56835 & 0 & 0 \\ 0 & -7.26687 & 0 \\ 0 & 0 & -1.56835 \end{pmatrix}.
\] (E59)
Clearly, $\mathcal{K} \neq K^{(1)} + K^{(2)}$. Hence, we conclude that commutativity of the interaction Hamiltonians with the system and bath Hamiltonians is not sufficient for the generators to simply add in the interaction picture (which in this particular case coincides with the Schrödinger picture) at the master equation level.

5. **Counterexample: Interactions commute with the bath and each other but not the system ($II \cap IS \cap IE$)**

Again, we consider a single spin interacting with two magnets. This time we take
\[
H_{E_1} = 0, \quad H_{E_2} = 0, \quad H_S = \frac{1}{2}\omega \hat{\sigma}_z, \quad H_1 = \frac{1}{2}g_1 \hat{\sigma}_z \otimes \hat{m}_1, \quad H_2 = \frac{1}{2}g_2 \hat{\sigma}_z \otimes \hat{m}_2.
\] (E60)
Note that the interaction Hamiltonians trivially commute with bath Hamiltonians, and also with each other, however they do not commute with the system Hamiltonian. Also note that since the system Hamiltonian is non-zero, in this case, the Schrödinger and interaction pictures do not coincide. We therefore need to compute the interaction Hamiltonians in the interaction picture, $H_{I_1}(t), H_{I_2}(t)$. They are found by replacing $\hat{\sigma}_z$ in the above expressions by
\[
\hat{\sigma}_z(t) := e^{iH_{S}t}\hat{\sigma}_z e^{-iH_{S}t} = \cos(\omega t)\hat{\sigma}_z + \sin(\omega t)\hat{\sigma}_y.
\] (E61)
For this case, the equations of motion (E26) in the interaction picture, with two baths, become
\[
\hat{r}^{(k,k')}(t) \cdot \sigma = -\frac{i}{2} \left( g_1 m_{1,k} + g_2 m_{2,k'} \right) [\cos(\omega t)\hat{\sigma}_z + \sin(\omega t)\hat{\sigma}_y, \hat{r}^{(k,k')}(t) \cdot \sigma],
\] (E62)
which gives the coupled equations (again dropping the indices $k,k'$ and the explicit time dependence for simplicity)
\[
\dot{\hat{r}}_x = (g_1 m_1 + g_2 m_2)(\sin(\omega t)\hat{r}_z - \cos(\omega t)\hat{r}_y), \quad \dot{\hat{r}}_y = (g_1 m_1 + g_2 m_2)\cos(\omega t)\hat{r}_x, \quad \dot{\hat{r}}_z = -(g_1 m_1 + g_2 m_2)\sin(\omega t)\hat{r}_x.
\] (E63-65)
These differential equations have time dependent coefficients. To solve them, we first transform to a rotating frame
\[
\tilde{r} = M_t \hat{r}, \quad \text{such that} \quad \begin{pmatrix} \tilde{r}_x \\ \tilde{r}_y \\ \tilde{r}_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\cos(\omega t) & \sin(\omega t) \\ 0 & \sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} \hat{r}_x \\ \hat{r}_y \\ \hat{r}_z \end{pmatrix}.
\] (E66)
In the new variables we have (defining $\gamma = g_1 m_1 + g_2 m_2$)
\[
\dot{\tilde{r}}_x = \gamma \tilde{r}_y, \quad \dot{\tilde{r}}_y = \omega \tilde{r}_x - \gamma \tilde{r}_x, \quad \dot{\tilde{r}}_z = -\omega \tilde{r}_y.
\] (E67-69)
This set of equations has then solution of the form
\[ \mathbf{r}(m_1, m_2, t) = \mathbf{R}(m_1, m_2, t)\mathbf{r}(0), \] (E70)
with
\[ \mathbf{R}(m_1, m_2, t) = \begin{pmatrix}
\cos\left(\frac{t\sqrt{\gamma^2 + \omega^2}}{\gamma + \omega^2}\right) & \gamma \sin\left(\frac{t\sqrt{\gamma^2 + \omega^2}}{\gamma + \omega^2}\right) & -\gamma \omega \left(\cos\left(\frac{t\sqrt{\gamma^2 + \omega^2}}{\gamma + \omega^2}\right) - 1\right) \\
-\gamma \sin\left(\frac{t\sqrt{\gamma^2 + \omega^2}}{\gamma + \omega^2}\right) & \cos\left(\frac{t\sqrt{\gamma^2 + \omega^2}}{\gamma + \omega^2}\right) & \omega \sin\left(\frac{t\sqrt{\gamma^2 + \omega^2}}{\gamma + \omega^2}\right) \\
-\gamma \omega \left(\cos\left(\frac{t\sqrt{\gamma^2 + \omega^2}}{\gamma + \omega^2}\right) - 1\right) & -\omega \sin\left(\frac{t\sqrt{\gamma^2 + \omega^2}}{\gamma + \omega^2}\right) & \gamma^2 + \omega^2 \cos\left(\frac{t\sqrt{\gamma^2 + \omega^2}}{\gamma + \omega^2}\right)
\end{pmatrix}. \] (E71)

The Bloch vector in the non-rotating frame then fulfils
\[ \mathbf{r}(m_1, m_2, t) = \mathbf{R}(m_1, m_2, t)\mathbf{r}(0), \] (E72)
with
\[ \mathbf{R}(m_1, m_2, t) = M_1^{-1} \mathbf{R}(m_1, m_2, t)M_0. \] (E73)

As in the previous example, we now need to average over the magnetisations accounting for the coupling to the two magnets, i.e.,
\[ R(t) = \frac{1}{2\pi\sigma_1\sigma_2} \int_{-\infty}^{\infty} dm_1 dm_2 \ e^{-\frac{m_1^2}{2\sigma_1^2}} e^{-\frac{m_2^2}{2\sigma_2^2}} R(m_1, m_2, t), \] (E74)
where, again, we can evaluate the integrals only numerically.

As before, we compute the generators \( \mathcal{K}^{(1)} \), \( \mathcal{K}^{(2)} \), and \( \mathcal{K} \), corresponding to only magnet 1 or 2 present, or both respectively, in order to check whether \( \mathcal{K} = \mathcal{K}^{(1)} + \mathcal{K}^{(2)} \). Since we can only compute the integrals numerically, we fix the parameters \( g_1, g_2, \sigma_1, \sigma_2, \omega \), and the time \( t \). We take
\[ g_1 = 2, \quad g_2 = 2, \quad \sigma_1 = 1, \quad \sigma_2 = 1, \quad \omega = 2, \quad t = 0.5. \] (E75)

For these values, we obtain
\[ \mathcal{K}^{(1)} = \mathcal{K}^{(2)} = \begin{pmatrix}
0.379798 & 0. & 0. \\
0. & 0.779093 & -1.40007 \\
0. & 1.70235 & -3.05922
\end{pmatrix}, \] (E76)
giving
\[ \mathcal{K}^{(1)} + \mathcal{K}^{(2)} = \begin{pmatrix}
0.759597 & 0. & 0. \\
0. & 1.55819 & -2.80015 \\
0. & 3.4047 & -6.11843
\end{pmatrix}. \] (E77)

On the other hand, we find
\[ \mathcal{K} = \begin{pmatrix}
1.28248 & 0. & 0. \\
0. & 13.0326 & -16.6865 \\
0. & 28.4767 & -36.4608
\end{pmatrix}, \] (E78)
so that, again, clearly \( \mathcal{K} \neq \mathcal{K}^{(1)} + \mathcal{K}^{(2)} \). Hence, we conclude that the commutativity of interaction Hamiltonians with each other and the bath Hamiltonians is not sufficient for the generators to simply add in the interaction picture at the master equation level.