Cutting rules on a cylinder
and simplified diagrammatic approach to CP violation in quantum kinetic theory

Tomáš Blažek & Peter Maták
Based on 2102.05914, 2104.06395

DISCRETE 2020-2021
29 November to 3 December 2021, Bergen
Unitarity in the Boltzmann equation and $CP$ asymmetric reaction rates from zero-temperature Feynman rules. [Phys. Rev. D 103, L091302]

Quantum thermal corrections to the lepton number source-term and their novel diagrammatic representation. [Eur. Phys. J. C (2021) 81:1050]

Mass-derivative relations for leptogenesis. [2111.03419]
\[ S^\dagger S = 1 \quad \rightarrow \quad i T^\dagger = i T - i T i T^\dagger \quad \text{for} \quad i T = S - 1 \]
Truncated unitarity expansion

\[ S^\dagger S = 1 \quad \rightarrow \quad i T^\dagger = i T - iT i T^\dagger \quad \text{for} \quad i T = S - 1 \]  

\[ |T_{fi}|^2 = -iT_{if}iT_{fi} = -iT_{if}iT_{fi} + \sum_n iT_{in}iT_{nf}iT_{fi} - \sum_{mn} iT_{im}iT_{mn}iT_{nf}iT_{fi} + \ldots \]  

\[ 1 - iT^\dagger = (1 + iT)^{-1} \quad \rightarrow \quad iT^\dagger = iT - (iT)^2 + (iT)^3 - \ldots \]
CP violation and unitarity cancellations in a CPT symmetric theory

\[ T_{fi} = C_{fi}^{\text{tree}} K_{fi}^{\text{tree}} + C_{fi}^{\text{loop}} K_{fi}^{\text{loop}} \rightarrow \Delta |T_{fi}|^2 \propto \text{Im} \left[ C_{fi}^{\text{tree}} C_{fi}^{\text{loop*}} \right] \text{Im} \left[ K_{fi}^{\text{tree}} K_{fi}^{\text{loop*}} \right] \] (6)

\[ \Delta |T_{fi}|^2 = |T_{fi}|^2 - |T_{if}|^2 \] (7)

\[ = \sum_n \left( i T_{in} i T_{nf} i T_{fi} - i T_{if} i T_{fn} i T_{ni} \right) \] [Covi, Roulet, Vissani '98]

\[ - \sum_{mn} \left( i T_{im} i T_{mn} i T_{nf} i T_{fi} - i T_{if} i T_{fn} i T_{nm} i T_{mi} \right) \]

\[ + \ldots \]
\[ T_{fi} = C_{fi}^{\text{tree}} K_{fi}^{\text{tree}} + C_{fi}^{\text{loop}} K_{fi}^{\text{loop}} \to \Delta |T_{fi}|^2 \propto \text{Im} \left[ C_{fi}^{\text{tree}} C_{fi}^{\text{loop} *} \right] \text{Im} \left[ K_{fi}^{\text{tree}} K_{fi}^{\text{loop} *} \right] \] (6)

\[ \Delta |T_{fi}|^2 = |T_{fi}|^2 - |T_{if}|^2 \] (7)

\[ \sum_{n} \left( i T_{in} i T_{nf} i T_{fi} - i T_{if} i T_{fn} i T_{ni} \right) \] [Covi, Roulet, Vissani ’98]

\[ - \sum_{mn} \left( i T_{im} i T_{mn} i T_{nf} i T_{fi} - i T_{if} i T_{fn} i T_{nm} i T_{mi} \right) \]

\[ + \ldots \to \sum_{f} \Delta |T_{fi}|^2 = 0 \] [Dolgov ’79; Kolb, Wolfram ’80]
Unitarity in classical kinetic theory

\[ \dot{n}_{i_1} + 3Hn_{i_1} = -\dot{\gamma}_{fi} + \dot{\gamma}_{if} + \ldots \] (8)

The classical \( i \rightarrow f \) reaction rates \( \dot{\gamma}_{fi} \) receive contributions from

\[ \int [dp_{i_k}] \dot{\gamma}_{i_k} (p_{i_k}) \] for each initial-state particle,

\[ \int [dp_{f_1}] \ldots [dp_{f_q}] (2\pi)^4 \delta^{(4)} (p_{i_1} + \ldots + p_{i_p} - p_{f_1} - \ldots - p_{f_q}) \] for each dotted line.

\( T_{fi} \) computed using zero-temperature quantum field theory.
Unitarity in classical kinetic theory

\[ \dot{n}_{i_1} + 3Hn_{i_1} = -\dot{\gamma}_{f_i} + \dot{\gamma}_{i_f} + \ldots \]  

(8)

The classical \( i \to f \) reaction rates \( \dot{\gamma}_{f_i} \) receive contributions from

In equilibrium, \( \dot{\gamma}_{f_i} \) acts like a trace.

\[ f_{i_k}^\text{eq}(p_{i_k}) \to \exp\left\{ -E_{i_k}/T \right\} \]  

implies

\[ f_{i_1}^\text{eq}(p_{i_1}) \cdots f_{i_p}^\text{eq}(p_{i_p}) = f_{f_1}^\text{eq}(p_{f_1}) \cdots f_{f_q}^\text{eq}(p_{f_q}) \]  

(9)

[Phys. Rev. D 103, L091302]
LO asymmetries in seesaw type-I leptogenesis

\[ \dot{n}_{\Delta L} + 3Hn_{\Delta L} = \Delta \dot{\gamma}_{N_i \rightarrow lH} - \Delta \dot{\gamma}_{lH \rightarrow N_i} - 2\Delta \dot{\gamma}_{lH \rightarrow \bar{\nu}H} + \ldots \] (10)

[Flanz, Paschos '98]

\[ \Delta \dot{\gamma}_{N_i \rightarrow lH} \supset \]

\[ \Delta \dot{\gamma}_{lH \rightarrow \bar{\nu}H} \supset \]

\[ \Delta \dot{\gamma}_{lH \rightarrow N_i} \supset \]

\[ \Delta \dot{\gamma}^{eq}_{N_i \rightarrow lH} = \Delta \dot{\gamma}^{eq}_{lH \rightarrow \bar{\nu}H} = -\Delta \dot{\gamma}^{eq}_{lH \rightarrow N_i} \] (12)
NLO asymmetries and top-Yukawa corrections

\[ \Delta \gamma_{N_i Q \rightarrow lt}^{eq} + \Delta \gamma_{N_i Q \rightarrow lHQ}^{eq} + \Delta \gamma_{N_i Q \rightarrow \bar{t}H}^{eq} + \Delta \gamma_{N_i Q \rightarrow \bar{l}Q\bar{t}}^{eq} = 0 \]  

[Pilaftsis, Underwood ’05; Abada, et al. ’06; Nardi, Racker, Roulet ’07; Racker ’19]
Completing the classical picture

At the lowest order, the $N_i$ number density evolution is given by

$$\dot{n}_{N_i} + 3Hn_{N_i} = -\dot{\gamma}_{N_i\rightarrow lH} - \dot{\gamma}_{N_i\rightarrow \bar{l}\bar{H}} + \dot{\gamma}_{lH\rightarrow N_i} + \dot{\gamma}_{\bar{l}\bar{H}\rightarrow N_i}$$

(14)

where

$$\dot{\gamma}_{N_i\rightarrow lH} = \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \Quad
Completing the classical picture

At the lowest order, the $N_i$ number density evolution is given by

\[ \dot{n}_{N_i} + 3Hn_{N_i} = -\gamma_{N_i \rightarrow lH} - \gamma_{N_i \rightarrow \bar{l}H} + \gamma_{lH \rightarrow N_i} + \gamma_{\bar{l}H \rightarrow N_i} \]  

(14)

where

\[ \gamma_{N_i \rightarrow lH} = \ldots \]  

(15)

To obtain a complete density evolution at the given order in the coupling, we should include all respective forward diagrams!

\[ \gamma_{N_i H \rightarrow lHH} = \ldots \]  

(16)
Completing the classical picture

\[ \sum_{w=0}^{\infty} \left( f_{eq}^{H} \right)^w = \frac{1}{1 - \exp\{-E_H/T\}} = 1 + f_{eq}^{H} \quad (17) \]

\[ \gamma_{N_i \to lH} = \int \cdots f_{N_i} (1 - f_{l}) (1 + f_{eq}^{H}) \quad (18) \]

General out-of-equilibrium densities? Technically, we may define

\[ \delta f(p) = \left( \frac{f(p)}{1 \pm f(p)} \right)^{\pm 1} \quad \text{such that} \quad f(p) = \sum_{w=0} (\pm 1)^w \delta f^{w+1}(p). \quad (19) \]
Thermal propagators

\[
\frac{i}{p_H^2 + i\epsilon} + 2\pi \sum_{w=1}^{\infty} \delta(p_H^0) \delta(p_H^2) \rightarrow \frac{i}{p_H^2 + i\epsilon} + 2\pi f(p_H) \theta(p_H^0) \delta(p_H^2)
\]  

(Eur. Phys. J. C (2021) 81:1050)
Lepton number source-term with quantum statistical factors

\[ \Delta \gamma_{N_i \rightarrow lH} = \int \ldots f_{N_i} \left( 1 - f_{eq}^l \right) \left( 1 + f_{eq}^H \right) \left( 1 - f_{eq}^l \right) \left( 1 + f_{eq}^{\bar{H}} \right) \]  

(21a)

\[ \Delta \gamma_{lH \rightarrow \bar{l}H} = \int \ldots f_{eq}^l f_{eq}^H \left( 1 - f_{eq}^l \right) \left( 1 + f_{eq}^H \right) \left( 1 - f_{eq}^l \right) \left( 1 - f_{N_i} \right) \]  

(21b)

\[ \dot{n}_{\Delta L} + 3Hn_{\Delta L} = \Delta \gamma_{N_i \rightarrow lH} - \Delta \gamma_{lH \rightarrow \bar{l}H} + \ldots \]  

(22)

\[ = \int \ldots \delta f_{N_i} \left( 1 - f_{eq}^l \right) \left( 1 + f_{eq}^H \right) \left( 1 - f_{eq}^l + f_{eq}^{\bar{H}} \right) + \ldots \]

[Garny, Hohenegger, Kartavtsev, Lindner ’09; ’10; Garny, Hohenegger, Kartavtsev ’10; Beneke, Garbrecht, Herranen, Schwaller ’10]
Thermal mass effects and NLO leptogenesis

\[ \gamma_{N_i Q \rightarrow l t} = \gamma_{N_i Q \rightarrow l H Q} \]

is IR finite by the Kinoshita-Lee-Nauenberg theorem [Racker '19]
Thermal mass effects and NLO leptogenesis

\[ \frac{i}{k^2 + i \epsilon} = \text{P.V.} \frac{i}{k^2} + \pi \delta(k^2) \rightarrow \]

\[ \frac{1}{2} \left( k_0^2 - |k| \right)^2 \frac{\partial \delta(k_0^0 - |k|)}{\partial k_0^0} \]

\[ \gamma_{N_i Q \rightarrow lHQ} = -2\text{P.V.} \left( k_0^0 \right) \delta(k^2) \text{P.V.} \frac{1}{k^2} = -\frac{1}{(k_0^0 + |k|)^2} \left( k_0^0 - |k| \right)^2 \frac{\partial \delta(k_0^0 - |k|)}{\partial k_0^0} \]

[Racker '19]
Thermal mass effects and NLO leptogenesis

\[-i\tilde{\Pi}_T(k^0, k) = \begin{array}{c}
Q \\
H
\end{array} + \begin{array}{c}
\bar{Q} \\
H
\end{array} + \begin{array}{c}
\bar{Q} \\
H
\end{array} + \begin{array}{c}
\bar{Q} \\
H
\end{array} \tag{25}\]

For $Q$ and $t$ in equilibrium

\[\dot{m}_H^2(T) = \tilde{\Pi}_T(k^0, k)_{k^0=|k|} = 12Y_t^2 \int [dp_t] \exp \{-E_t/T\} \tag{26}\]

and using dimensional regularization of IR divergences

\[\gamma_{N_i Q \rightarrow lHQ} + \gamma_{N_i \bar{Q} \rightarrow lH \bar{Q}} + \gamma_{N_i t \rightarrow lHt} + \gamma_{N_i \bar{t} \rightarrow lH \bar{t}} = \dot{m}_H^2(T) \frac{\partial}{\partial m_H^2} \bigg|_{m_H^2=0} \gamma_{N_i \rightarrow lH} \tag{27}\]

[2111.03419; See also: Fujimoto, et al. ’84; Salvio, Lodone, Strumia ’11]
Thermal mass effects and NLO leptogenesis

\[ \gamma_{N_i H \rightarrow lHH} + \gamma_{N_i \bar{H} \rightarrow lH\bar{H}} = m^2_{H,\lambda}(T) \frac{\partial}{\partial m^2_H} \bigg|_{m^2_H = 0} \gamma_{N_i \rightarrow lH} \] (28)

\[ \mathcal{F}(k^0, k) = \frac{1}{4\pi^2} (\mathcal{Y}^\dagger \mathcal{Y})_{ii} (M_i^2 - k^2) \delta_+ [(p_{N_i} - k)^2][1 + f_H(k^0)][1 - f_l(E_{N_i} - k^0)] \] (29)

\[ \frac{\partial}{\partial k^0} \bigg|_{k^0 = |k|} \frac{\mathcal{F}(k^0, k)}{(k^0 + |k|)^2} = \frac{\partial}{\partial m^2_H} \bigg|_{m_H = 0} \frac{\mathcal{F}(E_k, k)}{2E_k} \] (30)
General one-particle densities

The hermiticity and positive definiteness of $\hat{\rho}$ allows us to write

$$\hat{\rho} = \frac{1}{Z} \exp \left\{ -\hat{F} \right\}, \quad Z = \text{Tr} \exp \left\{ -\hat{F} \right\},$$

assuming

$$\hat{F} = \sum_p \mathcal{F}_p a_p^\dagger a_p.$$  \hspace{1cm} (31)

where

$$Z = \sum_{\{i\}} \exp \{ -\mathcal{F}_1 i_1 - \mathcal{F}_2 i_2 - \ldots \} = \prod_p Z_p \quad \text{where} \quad Z_p = \frac{\exp\{\mathcal{F}_p\}}{\exp\{\mathcal{F}_p\} - 1} \hspace{1cm} (33)$$

$$f_p = \text{Tr} \left[ \hat{\rho} a_p^\dagger a_p \right] = \frac{1}{\exp\{\mathcal{F}_p\} - 1} \quad \rightarrow \quad \hat{f}_p \overset{\text{def.}}{=} \exp \left\{ -\mathcal{F}_p \right\}$$
General one-particle densities

\[ \hat{\rho}' = S\hat{\rho}S^\dagger \implies \hat{\rho}' - \hat{\rho} = T\hat{\rho}T^\dagger - \frac{1}{2} TT^\dagger \hat{\rho} - \frac{1}{2} \hat{\rho}TT^\dagger + \ldots \]  

(McKellar, Thomson '94)

Tracing with \( a_p^\dagger a_p \) over \(|i_1, i_2, \ldots \rangle\) we get

\[ f_p' - f_p = \text{Tr} \left[ a_p^\dagger a_p \left( T\hat{\rho}T^\dagger - \hat{\rho}TT^\dagger \right) \right] = \ldots = \]

\[ = \frac{1}{Z} \sum_{k=1}^{\infty} (-1)^k \sum_{\{i\}} \sum_{\{n\}} (n_p - i_p) \hat{o}_1^{i_1} \hat{o}_2^{i_2} \ldots (i T)^k_{in} i T_{ni} \]

leading to the same statistical factors as seen in equilibrium case. [Eur. Phys. J. C (2021) 81:1050]
Direct use of unitarity and $CPT$ invariance makes $CP$ asymmetry cancellations easy to track at any perturbative order. [Phys. Rev. D 103, L091302]

Thermal corrected rates in the quantum Boltzmann equations can be represented by diagrams with internal lines wound on a cylindrical surface and cut into as many pieces as kinematically allowed. [Eur. Phys. J. C (2021) 81:1050; 2111.03419]

Thank You!