LOWER BOUNDS ESTIMATES OF THE LAPLACIAN SPECTRUM ON COMPLETE SUBMANIFOLDS

M. P. CAVALCANTE AND F. MANFIO

ABSTRACT. In this paper we obtain a lower bound estimate of the spectrum of Laplacian operator on complete submanifolds with bounded mean curvature, whose ambient space admits a Riemannian submersion over a complete Riemannian manifold with bounded negative sectional curvature. This estimate generalizes many previous known estimates.

1. Introduction

Let \( M^m \) be a complete non compact \( m \)-dimensional Riemannian manifold. Fixed a point \( p \in M \), let \( B(p; r) \) denote the open geodesic ball of radius \( r \) centered at \( p \). Let \( \lambda_1(r) > 0 \) denote the first eigenvalue of the Dirichlet value problem

\[
\begin{align*}
\Delta \phi + \lambda \phi &= 0 \quad \text{in } B(p; r), \\
\phi &= 0 \quad \text{in } \partial B(p; r),
\end{align*}
\]

where \( \Delta \) denotes the Laplace-Beltrami operator on \( M \). The fundamental tone of \( M \) is defined by

\[
\lambda_1(M) = \lim_{r \to \infty} \lambda_1(r).
\]

Of course it does not depend on the choice of the point \( p \) and coincides with the first non zero eigenvalue when \( M \) is compact. Following some authors we still call \( \lambda_1(M) \) the first eigenvalue of \( M \) when \( M \) is not compact. Moreover, \( \lambda_1(M) \) can be characterized variationally as following:

\[
\lambda_1(M) = \inf \left\{ -\frac{\int_M \varphi \Delta \varphi}{\int_M \varphi^2} : \forall \varphi \in C_0^\infty(M) \right\}.
\]

In particular, \( \lambda_1(M) \geq 0 \) and it is the bottom of the spectrum of \( -\Delta \) on \( M \). Above, and along this paper, we omit the volume element in the integrals for the sake of simplicity.

The problem to estimate the first eigenvalue \( \lambda_1(M) \) has been extensively studied. Of course, it is much harder to give a lower bound for \( \lambda_1(M) \) than...
an upper bound. According to Schoen and Yau (see §III.4 in [11]), it is an important question to find conditions on $M$ which will imply $\lambda_1(M) > 0$.

In this direction, McKean [9] showed that if $M$ is simply connected and its sectional curvature satisfies $K_M \leq -1$, then

$$\lambda_1(M) \geq \frac{(m-1)^2}{4} = \lambda_1(\mathbb{H}^m),$$

where $\mathbb{H}^m$ denotes the $m$-dimensional hyperbolic space of sectional curvature $-1$. This estimate was extended by Veeravalli [12] for a quite general class of manifolds.

In the context of submanifolds, Castillon (see Théorème 2.3 in [5]) considered a complete submanifold $M^m$ immersed in a Hadamard manifold $\tilde{M}^n$ with bounded sectional curvature $K \leq -b^2 < 0$ and bounded mean curvature vector $|H| \leq \alpha < b$ and he were able to proved that

$$\lambda_1(M) \geq \frac{(m-1)^2(b-\alpha)^2}{4}.$$

Latter, Cheung and Leung [6] found lower bounds estimates when $M$ is complete and isometrically immersed in the hyperbolic space $\mathbb{H}^n$ with bounded mean curvature vector field $|H| \leq \alpha < m-1$ (see also Theorem 4.3.(4) of [3]). Namely,

$$\lambda_1(M) \geq \frac{(m-1-\alpha)^2}{4}.$$

Recently, Bérard, Castillon and the first author [1], using a different approach, obtained a sharp lower bound estimate for $\lambda_1(M)$, when $M$ is a hypersurface immersed into $\mathbb{H}^n \times \mathbb{R}$ with constant mean curvature. We point out that Bessa and co-authors, using other techniques, obtained in [2] eigenvalue estimates for minimal submanifolds of warped product spaces.

In this work we apply the ideas of [1] to find a general lower bound for $\lambda_1(M)$ on complete submanifolds with bounded mean curvature, whose ambient space only admits a Riemannian submersion over a complete Riemannian manifold with bounded negative sectional curvature or over the class of Riemannian manifolds considered in [12], (see Section 5). In particular, when the base manifold of the submersion is the hyperbolic space $\mathbb{H}^k$ we obtain:

**Theorem 1.1.** Let $f : M^m \to \tilde{M}^n$ be an isometric immersion of a complete Riemannian manifold $M^m$ into a Riemannian manifold $\tilde{M}^n$, which admits a Riemannian submersion $\pi : \tilde{M} \to \mathbb{H}^k$. Let $H$ be the mean curvature of $M$, $\alpha^F$ the second fundamental form of the fibers of $\tilde{M}$, $H^F$ its mean curvature and $A$ the O’Neill tensor of $\tilde{M}$. If

$$c = \inf\{k - 1 - \|H\| - \|H^F\| - (n-m)(2\|A\| + \|\alpha^F\| + 1)\} > 0,$$

then $\lambda_1(M) \geq \frac{(m-1-\alpha)^2}{4}$.
then
\[ \lambda_1(M) \geq \frac{c^2}{4}. \]

This paper is organized as follows. In Sections 2 we present the preliminaries results for latter use. In particular we recall an useful condition on a Riemannian manifold which implies a positive lower bound estimate for the first eigenvalue. In Sections 3 and 4 we present some results on Riemannian submersions and on Busemann functions in order to make the paper more clear and self-contained. A main step in the our approach is to use a comparison theorem for the Hessian of Busemann functions. In Section 5 we state and prove the main theorems which generalizes Theorem 1.1 in two cases, namely, when the base manifold has bounded negative sectional curvature and when the base manifold is a Riemannian warped-product of a complete manifold by the real line (see Theorem 5.1). We finish this Section describing new examples of submersions where the constant in the main theorem is positive.

The authors are grateful to Professors P. Piccione, H. Rosenberg and D. Zhou for helpful comments about this work.

2. Preliminaries

In this section we present two general lemmas, which will be used latter. The first result is known in the literature, but we present here its proof for the sake of completeness.

**Lemma 2.1.** Let \( M^m \) be a complete Riemannian manifold that carries a smooth function \( F : M \to \mathbb{R} \) satisfying
\[ \|\text{grad } F\| \leq 1 \quad \text{and} \quad |\Delta F| \geq c, \]
for some constant \( c > 0 \). Then, for any smooth and relativelycompact domain \( \Omega \subset M \) we have
\[ \lambda_1(\Omega) \geq \frac{c^2}{4}, \]
where \( \lambda_1(\Omega) \) is the first eigenvalue of the Laplacian operator \( \Delta \) in \( \Omega \), with Dirichlet boundary condition.

**Proof.** Let \( \varphi \in C_0^\infty(\Omega) \), where \( \Omega \subset M \) is a pre-compact domain. Multiplying the inequality \( c \leq \Delta F \) by \( \varphi^2 \), integrating over \( \Omega \) and applying the Green’s first identity, we get:
\[ c \int_\Omega \varphi^2 \leq \int_\Omega \varphi^2 \cdot \Delta F = -\int_\Omega \langle \text{grad } \varphi^2, \text{grad } F \rangle. \]
Using the Cauchy-Schwarz geometric-inequality and the fact that \( \|\text{grad } F\| \leq 1 \), we have:
\[ |\langle \text{grad } \varphi^2, \text{grad } F \rangle| \leq 2|\varphi| : \|\text{grad } \varphi\| \leq \epsilon |\varphi|^2 + \frac{1}{\epsilon} \|\text{grad } \varphi\|^2, \]
for any $\epsilon > 0$. It follows that
\[ \epsilon (c - \epsilon) \int_{\Omega} \varphi^2 \leq \int_{\Omega} \|\nabla \varphi\|^2. \]
We can maximize the constant in the left-hand side by choosing $\epsilon$ to be $\hat{\epsilon}$ and the conclusion follows from the min-max principle. \qed

Given an isometric immersion $f : M^m \to \tilde{M}^n$ between Riemannian manifolds $M$ and $\tilde{M}$, let $\alpha$ denote its second fundamental form. Then, the mean curvature vector (not normalized) $H$ of $M$ is defined by $H = \text{tr} \alpha$.

The following well-known result relates the Laplacian of a function on $\tilde{M}$ and its restriction to $M$.

**Lemma 2.2.** Let $f : M^m \to \tilde{M}^n$ be an isometric immersion with mean curvature vector $H$. Let $\tilde{F} : \tilde{M} \to \mathbb{R}$ be a smooth function and let $F = \tilde{F}|_M$ be its restriction to $M$. Then, on $M$, we have:
\[
\Delta \tilde{F} = \Delta F + \sum_{i=1}^{n-m} \text{Hess} \tilde{F}(N_i, N_i) - H(\tilde{F}),
\]
where $\{N_1, \ldots, N_{n-m}\}$ is an orthonormal frame of $TM^\perp$.

**Proof.** See, for example, [7, Lemma 2]. \qed

### 3. Riemannian Submersions

Let $\tilde{M}$ and $B$ be differentiable manifolds of dimensions $n$ and $k$ respectively. A smooth map $\pi : \tilde{M} \to B$ is a **submersion** if it is surjective and its differential $d\pi(p)$ has maximal rank at every point $p \in \tilde{M}$. It follows that for all $x \in B$ the fiber $\mathcal{F}_x = \pi^{-1}(x)$ is a $(n-k)$-dimensional embedded smooth submanifold of $\tilde{M}$. Moreover, for every point $p \in \mathcal{F}_x$, the tangent space of $\mathcal{F}_x$ at point $p$ coincides with the kernel of $d\pi(p)$, i.e., $T_p\mathcal{F}_x = \ker d\pi(p)$.

If $\tilde{M}$ and $B$ are Riemannian manifolds, then a submersion $\pi : \tilde{M} \to B$ is called a **Riemannian submersion** if for all $x \in B$ and for all $p \in \mathcal{F}_x$, the differential map $d\pi$ restricted to the orthogonal subspace $T_p\mathcal{F}_x^\perp$ is an isometry onto $T_xB$. A vector field on $\tilde{M}$ is called **vertical** if it is always tangent to fibers, and it is called **horizontal** if it is always orthogonal to fibers. Let $\mathcal{V}$ denote the vertical distribution consisting of vertical vectors and $\mathcal{H}$ denote the horizontal distribution consisting of horizontal vectors on $M$. The corresponding projections from $T\tilde{M}$ to $\mathcal{V}$ and $\mathcal{H}$ are denoted by the same symbols.

For any given vector field $X \in \mathfrak{X}(B)$, there exists a unique horizontal vector field $\tilde{X} \in \mathfrak{X}(\tilde{M})$ which is $\pi$-related to $X$, that is, $d\pi(p) \cdot \tilde{X}(p) = X(x)$ for all $x \in B$ and all $p \in \mathcal{F}_x$, called *horizontal lifting* of $X$. A horizontal vector field $\tilde{X} \in \mathfrak{X}(\tilde{M})$ is called *basic* if it is $\pi$-related to some vector field $X \in \mathfrak{X}(B)$. 

The following proposition, which can be found in [10] Lemma 1, summarize some basic proprieties about $\pi$-related fields.

**Proposition 3.1.** Let $\pi : \tilde{M} \to B$ be a Riemannian submersion and let $\tilde{X}$ and $\tilde{Y}$ be basic vector fields, $\pi$-related to $X$ and $Y$, respectively. Then:

(a) $\langle \tilde{X}, \tilde{Y} \rangle = \langle X, Y \rangle \circ \pi$,

(b) $[\tilde{X}, \tilde{Y}]^H$ is basic and it is $\pi$-related to $[X, Y]$,

(c) $(\tilde{\nabla}_{\tilde{X}} \tilde{Y})^H$ is basic and it is $\pi$-related to $\nabla_X Y$,

(d) $\tilde{\nabla}_{\tilde{X}} \tilde{Y} = \nabla_X Y + \frac{1}{2}[\tilde{X}, \tilde{Y}]^V$,

where $\nabla$ and $\tilde{\nabla}$ are the Levi-Civita connections of $B$ and $\tilde{M}$, respectively.

Let $\mathcal{D} \subset T\tilde{M}$ denote the smooth distribution on $\tilde{M}$ consisting of vertical vectors; $\mathcal{D}$ is clearly integrable, the fibers of the submersion being its maximal integral leaves. The orthogonal distribution $\mathcal{D}^\perp$ is the smooth rank $k$ distribution on $\tilde{M}$ consisting of horizontal vectors. The second fundamental form of the fibers is a symmetric tensor $\alpha^F : \mathcal{D} \times \mathcal{D} \to \mathcal{D}^\perp$, defined by

$$\alpha^F(v, w) = (\tilde{\nabla}_v W)^H,$$

where $W$ is a vertical extension of $w$. The mean curvature vector of the fiber is the horizontal vector field $H^F$ defined by $H^F = tr \alpha^F$. In terms of an orthonormal frame, we have

$$H^F(p) = \sum_{i=1}^{n-k} \alpha^F(e_i, e_i) = \sum_{i=1}^{n-k} (\tilde{\nabla}_{e_i} e_i)^H,$$

where $\{e_1, \ldots, e_{n-k}\}$ is a local orthonormal frame to the fiber at $p$. The fibers are minimal submanifolds of $\tilde{M}$ when $H^F \equiv 0$, and are totally geodesic when $\alpha^F \equiv 0$.

We need some formulas relating the derivatives of $\pi$-related objects in $\tilde{M}$ and $B$. Let us start with the divergence of vector fields.

**Lemma 3.2.** Let $\tilde{X} \in \mathfrak{X}(\tilde{M})$ be a basic vector field, $\pi$-related to $X \in \mathfrak{X}(B)$. The following relation holds between the divergence of $\tilde{X}$ and $X$ at $x \in N$ and $p \in F_x$:

$$\text{div} \tilde{X}(p) = \text{div} X(x) - \langle \tilde{X}(p), H^F(p) \rangle.$$

**Proof.** Let $\tilde{X}_1, \ldots, \tilde{X}_k, \tilde{X}_{k+1}, \ldots, \tilde{X}_n$ be a local orthonormal frame of $T\tilde{M}$, where $\tilde{X}_1, \ldots, \tilde{X}_k$ are basic fields. The equality follows from identities (a) and (c) in Proposition 3.1 and formula (3.1) using this frame. \qed

 Giving a smooth function $F : B \to \mathbb{R}$, denote by $\tilde{F} = F \circ \pi : \tilde{M} \to \mathbb{R}$ its lifting to $\tilde{M}$. It is easy to see that the gradient of $\tilde{F}$ is the horizontal lifting of the gradient of $F$, i.e.,

$$\text{grad} \tilde{F} = \tilde{\text{grad}} F.$$
The Laplace operator in $B$ of a smooth function $F : B \to \mathbb{R}$ and the Laplace operator in $\tilde{M}$ of its lifting $\tilde{F} = F \circ \pi$ are related by the following formula.

**Lemma 3.3.** Let $F : B \to \mathbb{R}$ be a smooth function and set $\tilde{F} = F \circ \pi$. Then, for all $x \in B$ and all $p \in F_x$:

$$\tilde{\Delta} \tilde{F}(p) = \Delta F(x) + \langle \text{grad} \tilde{F}(p), H^F(p) \rangle.$$

**Proof.** It follows easily from (3.2) and Lemma 3.2 applied to the vector fields $\tilde{X} = \text{grad} \tilde{F}$ and $X = \text{grad} F$. □

Associated with a Riemannian submersion $\pi : \tilde{M} \to B$, there are two natural $(1,2)$–tensors $T$ and $A$ on $\tilde{M}$, introduced by O’Neill in [10], and defined as follows: for vector fields $X, Y$ tangent to $\tilde{M}$, the tensor $T$ is defined by

$$T_{XY} = \left( \nabla_X Y^\mathcal{V} \right)^\mathcal{H} + \left( \nabla_X Y^\mathcal{H} \right)^\mathcal{V}.$$

Note that $\pi : \tilde{M} \to B$ has totally geodesic fibers if and only if $T$ vanishes identically. The tensor $A$, known as the integrability tensor, is defined by

$$A_{XY} = \left( \nabla_X Y^\mathcal{H} \right)^\mathcal{V} + \left( \nabla_X Y^\mathcal{V} \right)^\mathcal{H}.$$

The tensor $A$ measures the obstruction to integrability of the horizontal distribution $\mathcal{H}$. In particular, for any horizontal vector field $X$ and any vertical vector field $V$, we have:

$$A_X V = \left( \nabla_X V \right)^\mathcal{H}.$$

The following lemma gives useful expressions for the Hessian of the lifting $\tilde{F} : \tilde{M} \to \mathbb{R}$ of a smooth function $F : B \to \mathbb{R}$, when we consider horizontal and vertical vector fields.

**Lemma 3.4.** If $X$ and $Y$ are basic, and $V$ and $W$ are vertical vector fields, we have the following expressions for the Hessian of the lifting $\tilde{F} = F \circ \pi$ of $F$ to $\tilde{M}$:

(a) $\text{Hess} \tilde{F}(X,Y) = \text{Hess} F(\pi_* X, \pi_* Y) \circ \pi$,

(b) $\text{Hess} \tilde{F}(V,W) = - \left\langle \alpha^F(V,W), \text{grad} \tilde{F} \right\rangle$,

(c) $\text{Hess} \tilde{F}(X,V) = - \left\langle A_X V, \text{grad} \tilde{F} \right\rangle$.

**Proof.** The first assertion follows from (3.2) and item (c) in the Proposition 3.1. The second one is a straightforward calculation, and the third assertion follows directly from (3.3). □
4. Busemann Functions

In this section we describe comparison results for the Hessian of Busemann functions on two classes of Riemannian manifolds, both are generalization of the hyperbolic space. These classes of manifolds will be used as the base space of the Riemannian submersions we will consider in our main theorem.

4.1. Busemann functions on manifolds with bounded negative sectional curvature. Given $a > 0$, let $\mathbb{H}^k(-a^2)$ denote de $k$-dimensional hyperbolic space with constant sectional curvature $-a^2$. We consider the warped-product model, that is

$$\mathbb{H}^k(-a^2) = (\mathbb{R}^{k-1} \times \mathbb{R}, h),$$

where

$$h = e^{-2as} dx^2 + ds^2.$$  

In this model, the curve $\gamma: \mathbb{R} \to \mathbb{H}^k(-a^2)$, given by $\gamma(s) = (x_0, s)$, is a geodesic for any $x_0 \in \mathbb{R}^{k-1}$, and the function $F: \mathbb{H}^k(-a^2) \to \mathbb{R}$, given by

$$F(x, s) = s,$$

is its associated Busemann function. By a direct computation we get

$$\left\{ \begin{align*} 
\text{Hess}F &= e^{-2as} dx^2, \\
\Delta F &= (k-1)a.
\end{align*} \right.$$  

Now we will estimate the Hessian of the Busemann function $F$ defined in a complete Riemannian manifold $B^k$ with sectional curvature between two negative constants. In order to obtain the Hessian of $F$, one takes a point $p$ on a geodesic sphere of radius $r$, and let the center of the sphere go to infinity. In this case, the sphere converges to a horosphere, and the Hessian of the distance function will converge to the Hessian of the Busemann function. So, a comparison theorem for the hessian of a Busemann function follows from the comparison theorem for the Hessian of the distance function. See [4] for a proof.

**Lemma 4.1.** Let $B^k$ be a complete Riemannian manifold with sectional curvature $K$ satisfying $-a^2 \leq K \leq -b^2$, for some constants $a, b > 0$. If $F: B \to \mathbb{R}$ is a Busemann function, then

$$b\|X\|^2 \leq \text{Hess}F(X, X) \leq a\|X\|^2,$$

for any vector $X$ orthogonal to $\text{grad} F$. 
4.2. Busemann functions on manifolds with warped product structure. Let $(N^{k-1}, g)$ be a complete Riemannian manifold and let $w : \mathbb{R} \to \mathbb{R}$ be a smooth function. Inspired in the hyperbolic space, we consider the Riemannian warped-product manifold

$$B = (N \times \mathbb{R}, h),$$

where

$$h = e^{2w(s)}g + ds^2.$$

Consider now the Busemann function $\overline{F} : B \to \mathbb{R}$ defined by $\overline{F}(x, s) = s$. As above, a direct computation gives

$$\begin{aligned}
\text{Hess } \overline{F} &= w'(s)e^{2w(s)}g, \\
\Delta \overline{F} &= w'(s)(k - 1).
\end{aligned}$$

In particular we have the following lemma:

**Lemma 4.2.** Let $B^k$ be a Riemannian manifold as in (4.2) and assume that the function $w$ satisfies $b \leq w' \leq a$, for some constants $a, b > 0$. If $\overline{F} : B \to \mathbb{R}$ is the Busemann function defined as above, then

$$b\|X\|^2 \leq \text{Hess } \overline{F}(X, X) \leq a\|X\|^2$$

for any vector $X$ orthogonal to $\text{grad } \overline{F}$.

In particular the following consequence will be use in the main theorem.

**Corollary 4.3.** Under the conditions of Lemma 4.1 or Lemma 4.2 we have

$$\Delta \overline{F} \geq (k - 1)b.$$

**Remark 4.4.** It is important to point out that Riemannian manifolds given by (4.2) form a wide class. In particular, we may choose the manifold $N$ in such way that $B$ has positive sectional curvature in some directions (see [12]).

5. Main result and examples

In this section we use all the above previous results to prove a lower bound estimates for the first eigenvalue of the Laplace operator on manifolds isometrically immersed on Riemannian manifolds which carries a Riemannian submersion on the two classes of manifolds described above. In particular, using Lemmas 4.1 and 4.2 and its corollary above, we are able to present a unified proof to both cases. Bellow, we use a bar for geometric objects related with the metric of $B$ and a tilde for geometric objects related with the metric of $\tilde{M}$. 
Theorem 5.1. Let $B^k$ be a complete Riemannian manifold as in Lemma 4.1 or as in Lemma 4.2 and let $\pi: \tilde{M}^n \to B^k$ be a Riemannian submersion. Let $M^m$ be a complete Riemannian manifold and $f: M^m \to \tilde{M}^n$ be an isometric immersion. Assume that $F: B \to \mathbb{R}$ is a Busemann function and consider its lifting $\tilde{F}: \tilde{M} \to \mathbb{R}$. If $F = \tilde{F}|_M$ is its restriction to $M$, then

$$\Delta F \geq (k-1)b + H^F(\tilde{F}) - (n-m)(a + 2\|A\| + \|\alpha^F\| + H(\tilde{F})).$$

In particular, if

$$c = \inf\{(k-1)b - \|H^F\| - (n-m)(a + 2\|A\| + \|\alpha^F\|) - \|H\|\} > 0,$$

then

$$\lambda_1(M) \geq \frac{c^2}{4}.$$

Proof. From Lemma 3.3 and Corollary 4.3 we have:

$$\Delta \tilde{F} = \Delta F + \langle \text{grad} \ F, H^F \rangle \geq (k-1)b + H^F(\tilde{F}).$$

On the other hand, from Lemma 2.2,

$$\Delta \tilde{F} = \Delta F + \sum_{i=1}^{n-m} \text{Hess} \ F(N_i, N_i) - H(\tilde{F}),$$

where $\{N_1, \ldots, N_{n-m}\}$ is an orthonormal frame of $TM^\perp$. For each $1 \leq i \leq n-m$, we write

$$N_i = N_i^H + N_i^V,$$

where $N_i^H$ and $N_i^V$ denote the horizontal and vertical projection of $N_i$ onto $T\tilde{M}$, respectively. Moreover, since (5.2) is a tensorial equation, we may assume that each $N_i^H$ is basic. Thus, using Lemmas 3.4, 4.1 and 4.2 we get

$$\Delta \tilde{F} \leq \Delta F + (n-m)(a + 2\|A\| + \|\alpha^F\|) - H(\tilde{F}).$$

So, plugging this in (5.1) we obtain

$$\Delta F \geq (k-1)a + H^F(\tilde{F}) - (n-m)(b + 2\|A\| + \|\alpha^F\|) + H(\tilde{F}).$$

The result follows from Lemma 2.1. \qed

5.1. Lower bounds in warped products. Suppose that the ambient space $\tilde{M}^n = \mathbb{H}^k \times_\rho F^{n-k}$ admits a warped product structure, where the warped function $\rho$ satisfies $\|\text{grad} \ \rho\|/\rho \leq 1$.

By considering the projection on the first factor $\pi: \mathbb{H}^k \times_\rho F^{n-k} \to \mathbb{H}^k$ as a Riemannian submersion, we have that the tensor $A$ is identically zero, $\|\alpha^F\| \leq 1$, and in particular $\|H^F\| \leq n-k$.

Let $M^m$ be a complete Riemannian manifold and $f: M^m \to \tilde{M}^n$ be an isometric immersion such that its mean curvature vector $H$ satisfies $\|H\| \leq \alpha$, where $\alpha$ is a positive constant to be determined. If $F: \mathbb{H}^k \to \mathbb{R}$ is the
Busemann function given in (4.1), a lower bound estimates for the infimum in (5.1) goes as follows:

\[
c = \inf \{ k - 1 - \| H^F \| - (n - m)(1 + \| \alpha^F \|) - \| H \| \} \\
\geq \inf \{ k - 1 - n + k - 2(n - m) - \| H \| \} \\
= 2(k + m) - 3n - 1 - \alpha.
\]

In particular, \( \lambda_1(M) > 0 \) if we take \( 0 < \alpha < 2(k + m) - 3n - 1 \).

### 5.2. Lower bounds in submersions with totally geodesic fibbers.

Let \( \tilde{M}^n \) be a Riemannian manifold with nonpositive sectional curvature and \( \pi: \tilde{M}^n \to \mathbb{H}^k \) be a Riemannian submersion with totally geodesic fibers. This means that \( \alpha^F = 0 \), and thus \( H^F = 0 \). Furthermore, the submersion \( \pi \) is integrable in the sense that the horizontal distribution is integrable (cf. [8, Proposition 3.1]). Thus, if \( f: M^m \to \tilde{M}^n \) is an isometric immersion, whose mean curvature vector \( H \) satisfies \( \| H \| \leq \alpha \), for some positive constant \( \alpha < k + m - n - 1 \), we have

\[
c \geq k - 1 - (n - m) - \| H \| \\
\geq k + m - n - 1 - \alpha > 0,
\]

and thus \( \lambda_1(M) > 0 \).

### References

1. P. Bérard, P. Castillon, M. Cavalcante, Eigenvalue estimates for hypersurfaces in \( \mathbb{H}^m \times \mathbb{R} \) and applications, Pacific J. Math. 253 no. 1, 19–35, (2011).
2. G. P. Bessa, S.C. García-Martínez, L. Mari, H.F. Ramírez-Ospina, Eigenvalue estimates for submanifolds of warped product spaces, to appear in Math. Proc. Cambridge Philos. Soc.
3. G. P. Bessa, J. F. Montenegro, Eigenvalue estimates for submanifolds with locally bounded mean curvature, Ann. Global Anal. Geom. 24, 279–290, (2003).
4. G. P. Bessa, J. H. de Lira, S. Pigola, A. Setti, Curvature Estimates for Submanifolds in Horocylinder, [arXiv:1308.5926 [math.DG]].
5. P. Castillon, Sur l’opérateur de stabilité des sous-variétés à courbure moyenne constante dans l’espace hyperbolique, Manuscripta Math., 94, 385–400, (1997).
6. L.-F. Cheung, P.-F. Leung, Eigenvalue estimates for submanifolds with boundary mean curvature in the hyperbolic space, Math. Z. 236, 525–530, (2001).
7. J. Choe, R. Gulliver, Isoparametric inequalities on minimal submanifolds of space forms, Manuscripta Math. 77: 2-3, 169–189, (1992).
8. R. H. Escobales, Jr., Riemannian submersions with totally geodesic fibers, J. Diff. Geom., 10, 253–276, (1975).
9. H. P. McKean, An upper bound to the spectrum of \( \Delta \) on a manifold of negative curvature, J. Diff. Geom., 4, 359–366, (1970).
10. B. O’Neill, The fundamental equations of a submersion, Mich. Math. J. 13, 459–469, (1966).
11. R. Schoen, S.T. Yau, Lectures on Differential Geometry, 414 p., International Press, Cambridge, MA (2010).
12. A. R. Veeravalli, Une remarque sur l’inégalité de McKean, Comment. Math. Helv. 78, 884–888, (2003).
IM, Universidade Federal de Alagoas, Maceió, AL, CEP 57072-970, Brazil
E-mail address: marcos@pos.mat.ufal.br

ICMC, Universidade de São Paulo, São Carlos, SP, CEP 13561-060, Brasil
E-mail address: manfio@icmc.usp.br