Values of $L$-Series of Hecke Eigenforms

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Abstract

We determine a formula for the average values of $L$-series associated to eigenforms at complex values.

1 Introduction

Let $S_k$ denote the space of cusp forms of integer weight $k$ on the full modular group $SL_2(\mathbb{Z})$. We define the $n$-th period of $f$ in $S_k$ by

$$r_n(f) = \int_0^{\infty} f(it)t^ndt, \quad 0 \leq n \leq k-2.$$  

It is well known that $S_k$ has a rational structure given by the rationality of its periods [6]. Periods of cusp forms are actually the critical values of their corresponding $L$-series, and by definition also are the coefficients of the period polynomial associated to the modular form of degree $k - 2$. Petersson gives an average trace formula for the product of Fourier coefficients of cusp forms in terms of Kloosterman sums (see [3]). In this paper, we give an analogue for Petersson’s average formula where the Fourier coefficients are replaced by $L$-values of Hecke eigenforms at arbitrary values with certain restrictions. Restricting to integers though, we obtain an average result of the explicit formulas proved by Kohnen and Zagier in [6] for the periods of the kernel function for the special $L$-values.

Let us be more precise now. As mentioned above, the periods $r_n(f)$ give rise to a rational structure for the space of cusp forms via the Eichler-Shimura theory and those periods can be determined by taking the Petersson’s product of $R_n$ with $f$ given by $r_n(f) = < f, R_n >$ for every $f \in S_k$. In [6], the authors determine the periods of the kernel functions $R_n$ in terms of Bernoulli numbers and show certain symmetric
properties of these periods. As a result, one can determine average values of \( L \)-series associated to Hecke eigenforms at integer values. From what mentioned, one can notice the importance of critical values of \( L \)-series at integer values. In this paper, we generalize the result from \([6]\) and determine a formula for the average values of the \( L \)-functions associated to Hecke eigenforms at complex values. We would like to mention that a similar result was obtained in \([1]\) at complex values using the theory of Eisenstein series.

Let \( k \in 2\mathbb{Z} \). If \( z \in \mathbb{C} - \{0\} \) and \( w \in \mathbb{C} \), let \( z^w = e^{w \log z} \) where \(-\pi < \arg z \leq \pi\). If \( f(z) = \sum_{n \geq 1} a(n)e^{2\pi inz} \in S_k \), we define the normalized \( L \)-series associated to \( f \) by

\[
L^*(f, s) := (2\pi)^{-s} \Gamma(s) L(f, s). \tag{1}
\]

\( L^*(f, s) \) has analytic continuation to \( \mathbb{C} \) and satisfies the functional equation

\[
L^*(f, k - s) = (-1)^{k/2} L^*(f, s). \tag{2}
\]

Let \( \{f_{k,1}, f_{k,2}, \ldots, f_{k,g_k}\} \) be the basis of normalized Hecke eigenforms of \( S_k \).

We define the kernel function on \( SL_2(\mathbb{Z}) \) of integer weight \( k \) as given in \([6]\) given by

\[
R_{s,k} = \gamma_k(s) \sum_V (cz + d)^{-k} \left( \frac{az + b}{cz + d} \right)^{-s} \tag{3}
\]

for \( s \in \mathbb{C}, 1 < \Re s < k - 1 \) and where and where the sum runs over all matrices \( V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) in \( PSL_2(\mathbb{Z}) \) and

\[
\gamma_k(s) = e^{\pi is/2} \Gamma(s) \Gamma(k - s).
\]

In Lemma 1 in \([5]\), it was shown that under the usual Petersson inner product and for any \( f \in S_k \), one has

\[
< f, R_{s,k} > = c_k L^*(f, s) \tag{4}
\]

where

\[
c_k = \frac{(-1)^{k/2} \pi (k - 2)!}{2^{k-2}}.
\]

Moreover, and as a consequence of Lemma 1 in \([5]\), we have

\[
R_{s,k} = c_k \sum_{\nu=1}^{g_k} \frac{L^*(f_{k,\nu}, s)}{< f_{k,\nu}, f_{k,\nu} >} f_{k,\nu}. \tag{5}
\]

Now, let \( s' \in \mathbb{C} \) such that \( s + s' \in 2\mathbb{Z} + 1 \) and define the following integral by
\[ \int_0^\infty R_{s,k}(it)t^{s'-1}dt := \Gamma(s')(2\pi)^{-s'} L(R_{s,k}, s') = L^*(R_{s,k}, s'). \quad (6) \]

Using (5) and (6), it is easy to see that
\[ \int_0^\infty R_{s,k}(it)t^{s'-1}dt := L^*(R_{s,k}, s') = c_k \sum_{\nu=1}^{q_k} \frac{L^*(f_{k,\nu}, \bar{s})L^*(f_{k,\nu}, s')}{<f_{k,\nu}, f_{k,\nu}>}. \quad (7) \]

In what follows, we calculate the integral of the kernel function in order to determine the value of the \( L \)-function of the kernel function at complex values from (6). It is worth mentioning that the result of the following theorem is a generalization of a partial result of [6] due to the conditions on \( s \) and \( s' \).

**Theorem 1. (The Main Theorem)** For \( s + s' \in 2\mathbb{Z} + 1, \) \( \text{Re } s > \text{Re } s' + 1, 1 < s + s' < k - 1, 1 < \text{Re } s < k - 1, \) and \( 1 < \text{Re } s' < k - 1 \) we have
\[
\gamma_k(s)^{-1} \int_0^\infty R_{s,k}(it)t^{s'-1}dt = i^{-k} \frac{(2\pi i)^{k-s}}{(2\pi)^{k-s'}} \zeta(s - s' + 1) \frac{\Gamma(k - s')}{\Gamma(k - s)} + i^{-k} \frac{(-2\pi i)^s}{(2\pi)^{k-s'}} \zeta(k - s' - s + 1) \frac{\Gamma(k - s')}{\Gamma(s)} + 2\pi e^{\pi i(s'-1)/2} \frac{(-s'+1)_{k-1}}{(k-1)!} \sum_{a,b,c,d>0 \atop ad-bc=1} a^{-s} c^{s'-s} d^{s'-k} \frac{2F1(s,-s'+k;\bar{k};1/ad)}{\Gamma(s)\Gamma(s')}\zeta(s - s') \left( e^{-\pi is'}/2 - e^{3\pi is'}/2 \right) \left( \frac{\Gamma(s')\Gamma(s - s')}{\Gamma(s)} \right). \quad (8) \]

where \( 2F1 \) is the hypergeometric series.

Note that \( \text{Re } s > \text{Re } s' + 1 \) will be used in a crucial way in e.g. equation (28) and therefore this condition cannot be replaced by the symmetric condition \( \text{Re } (s - s') \notin [-1, 1] \). It is also worth noting that the condition that \( s + s' \) is an odd integer will play a significant role in equation (16) where we were able to recombine the integrals and get an integral over the real line. However, that condition could be removed but one has to re-evaluate additional integrals and the strategy used to evaluate \( S_\varepsilon \) by considering equation (18) has to be completely changed. However, removing the condition on \( s \) and \( s' \) will reveal symmetry in the right side of equation (8).

Using (7), we deduce the following corollary about the average values of the \( L \)-series.

**Corollary 1.** For \( s + s' \in 2\mathbb{Z} + 1, 1 < s + s' < k - 1, 1 < \text{Re } s < k - 1, 1 < \text{Re } s' < k - 1 \) and
For $s > Re\ s' + 1$, we have
\[
\sum_{\nu=1}^{g_k} \frac{L^* (f_{k, \nu}, \bar{s}) L^* (f_{k, \nu}, s')}{< f_{k, \nu}, \tilde{f}_{k, \nu}>} = \sum_{\nu=1}^{g_k} \frac{\gamma_k (s)}{c_k} i^{-k} \frac{(2\pi i)^{k-s}}{(2\pi)^{k-s'}} \zeta (s-s'+1) \frac{\Gamma (k-s')}{\Gamma (k-s)} \\
+ \frac{\gamma_k (s)}{c_k} i^{-k} \frac{(-2\pi i)^s}{(2\pi)^{k-s'}} \zeta (k-s' - s + 1) \frac{\Gamma (k-s')}{\Gamma (s)}
\]
\[
+ 2\pi \gamma_k (s) e^{\pi i (s'-1)/2} \frac{(-s'+1)k-1}{(k-1)!} \sum_{a,b,c,d > 0 \atop ad-bc = 1} a^{-s} c^{s-s'} d^{s'-k} \zeta (s-s') \left( e^{-\pi is'/2} - e^{\pi is'/2} \right) \frac{\Gamma (s') \Gamma (s-s')}{\Gamma (s)}
\]
\[
+ \frac{\gamma_k (s)}{c_k} \zeta (s-s') \left( e^{-\pi is'/2} - e^{\pi is'/2} \right) \frac{\Gamma (s') \Gamma (s-s')}{\Gamma (s)}
\]
\[
(9)
\]

For $k = 8, 10, 14$ one has $S_k = \{0\}$ and we deduce a relation between $\zeta (s)$ and the hypergeometric function $2 F_1$.

**Corollary 2.** For $k = 8, 10, 14$ and for $s + s' \in 2\mathbb{Z} + 1, 1 < s + s' < k - 1, 1 < Re\ s < k - 1, 1 < Re\ s' < k - 1$ and $Re\ s > Re\ s' + 1$, we have
\[
i^{-k} \frac{(2\pi i)^{k-s}}{(2\pi)^{k-s'}} \frac{\Gamma (k-s')}{\Gamma (k-s)} \zeta (s-s'+1) + i^{-k} \frac{(-2\pi i)^s}{(2\pi)^{k-s'}} \frac{\Gamma (k-s')}{\Gamma (s)} \zeta (k-s' - s + 1)
\]
\[
+ \zeta (s-s') \left( e^{-\pi is'/2} - e^{\pi is'/2} \right) \frac{\Gamma (s') \Gamma (s-s')}{\Gamma (s)}
\]
\[
= -2\pi e^{\pi i (s'-1)/2} \frac{(-s'+1)k-1}{(k-1)!} \sum_{a,b,c,d > 0 \atop ad-bc = 1} a^{-s} c^{s-s'} d^{s'-k} 2 F_1 (s, -s' + k; k; 1/ad).
\]
\[
(10)
\]

## 2 Proof of the Main Theorem

**Proof:** We divide the sum of $R_{s,k}$ defined in (3) inside the integral on the left side of equation (8) into three with terms where; $bd = 0, bd > 0$ and $bd < 0$. We start with the first case when $bd = 0$.

For $bd = 0$, we have elements of the form $\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ and $\begin{pmatrix} n & -1 \\ 1 & 0 \end{pmatrix}$.

Thus we get
\[
A := \int_0^\infty \sum_{bd=0} \sum_{n \in \mathbb{Z}} \frac{\gamma_k (s)}{c_k} i^{-k} \frac{(2\pi i)^{k-s}}{(2\pi)^{k-s'}} t^{s'-1} dt
\]
\[
= \int_0^\infty \sum_{n \in \mathbb{Z}} (n-i/t)^{s-k} (it)^{-k} t^{s'-1} dt + \int_0^\infty \sum_{n \in \mathbb{Z}} (n+i/t)^{-s} (it)^{-k} t^{s'-1} dt
\]
\[
= i^{-k} \int_0^\infty \sum_{n \in \mathbb{Z}} (n-i/t)^{s-k} t^{s'-k-1} dt + i^{-k} \int_0^\infty \sum_{n \in \mathbb{Z}} (n+i/t)^{-s} t^{s'-k-1} dt.
\]

(11)
We now replace $t$ by $1/t$ and we get

$$A = i^{-k} \int_0^\infty \sum_{n \in \mathbb{Z}} (n-it)^{s-k} t^{k-s'-1} dt + i^{-k} \int_0^\infty \sum_{n \in \mathbb{Z}} (n+it)^{-s} t^{k-s'-1} dt. \quad (12)$$

Lipschitz summation formula tells us for $s \in \mathbb{C}$, $Re \ s > 0$ that

$$\sum_{n \in \mathbb{Z}} (z+n)^{-s-1} = \frac{(-2\pi i)^{s+1}}{\Gamma(s+1)} \sum_{n \geq 1} n^s e^{2\pi i nz}, \quad \Im z > 0$$

Letting $z = it$ in Lipchitz summation formula for real $s$ and then using the complex conjugate, we can analytically continue this formula for complex $s$ and we get

$$A = i^{-k} \frac{(2\pi i)^{k-s}}{\Gamma(k-s)} \int_0^\infty \sum_{n \geq 1} n^{k-s-1} e^{-2\pi nt} t^{k-s'-1} dt + i^{-k} \frac{(-2\pi i)^s}{\Gamma(s)} \int_0^\infty \sum_{n \geq 1} n^{s-1} e^{-2\pi nt} t^{k-s'-1} dt$$

$$= i^{-k} \frac{(2\pi i)^{k-s}}{(2\pi)^{k-s'}} \zeta(s-s'+1) \frac{\Gamma(k-s')}{\Gamma(k-s)} + i^{-k} \frac{(-2\pi i)^s}{(2\pi)^{k-s'}} \zeta(k-s'-s) \frac{\Gamma(k-s')}{\Gamma(s)} \quad (13)$$

where $s + s' < k - 2$ and $Res > Res'$ are satisfied.

Now as for the other elements where $bd < 0$ and $bd > 0$, we set

$$B := \int_0^\infty \sum_{bd < 0} (cit + d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} t^{s'-1} dt$$

and

$$B' := \int_0^\infty \sum_{bd > 0} (cit + d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} t^{s'-1} dt.$$

We will do certain substitutions in order to combine $B$ with $B'$. We have

$$\sum_{bd < 0} (cit + d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} = \sum_{bd < 0} ((-c)(-it) + d)^{-k} \left( \frac{(-a)(-it) + b}{(-c)(-it) + d} \right)^{-s}.$$

Replace $t$ by $-t$ and we get

$$B = \int_{-\infty}^0 \sum_{bd < 0} (-cit + d)^{-k} \left( \frac{-ait + b}{-cit + d} \right)^{-s} (-t)^{s'-1} dt. \quad (14)$$

Replace $(a, d)$ by $(-a, -d)$ and we get

$$B = \int_{-\infty}^0 \sum_{bd > 0} (-cit - d)^{-k} \left( \frac{ait + b}{-cit - d} \right)^{-s} (-t)^{s'-1} dt. \quad (15)$$

As a result, we have

$$B = e^{-\pi i (s+s'-1)} \int_{-\infty}^0 \sum_{bd > 0} (cit + d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} t^{s'-1} dt = \int_{-\infty}^0 \sum_{bd > 0} (cit + d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} t^{s'-1} dt \quad (16)$$
since \( s + s' \in 2\mathbb{Z} + 1 \). We combine the cases \( bd > 0 \) and \( bd < 0 \) to get the following sum

\[
B + B' = \int_0^\infty \sum_{bd > 0} (cit + d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} t^{s' - 1} dt + \int_0^0 \sum_{bd > 0} (cit + d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} t^{s' - 1} dt. \tag{17}
\]

To be able to interchange summation and integration, we define the following integrals. Let

\[
S_\epsilon = \sum_{bd > 0} \int_0^{\epsilon} (cit + d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} t^{s' - 1} dt + \sum_{bd > 0} \int_{-\epsilon}^{-1/\epsilon} (cit + d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} t^{s' - 1} dt.
\]

Notice that \( \lim_{\epsilon \to 0} S_\epsilon = B + B' \). We already determined \( A \) and thus when calculating \( \lim_{\epsilon \to 0} S_\epsilon \), we prove Theorem \( \Box \) and we get

\[
\gamma_k(s)^{-1} \int_0^\infty R_{s,k}(it)t^{s' - 1} dt = A + B + B'.
\]

2.1 The Value of \( \lim_{\epsilon \to 0} S_\epsilon \)

In what follows, we calculate the \( \lim_{\epsilon \to 0} S_\epsilon \). We have

\[
\sum_{bd > 0} \left( \int_0^{1/\epsilon} + \int_{-1/\epsilon}^{-\epsilon} \right) = \sum_{bd > 0} \left( \int_{-\epsilon}^{-1/\epsilon} + \int_{-\epsilon}^0 - \int_{-\epsilon}^{-1/\epsilon} - \int_{-1/\epsilon}^0 \right) = \sum_{bd > 0} \left( \int_{-\epsilon}^{-1/\epsilon} - \int_{-\epsilon}^0 \right) + S_0^0, \tag{18}
\]

where

\[
S_0^0 = \sum_{bd > 0} \left( - \int_{-\epsilon}^{-1/\epsilon} - \int_{-1/\epsilon}^0 \right).
\]

We now try to simplify \( S_0^0 \). Substitute \( t \) by \( 1/t \) in \( S_0^0 \) and we get

\[
S_0^0 = \sum_{bd > 0} \left( \int_0^\epsilon (ci + dt)^{-k} \left( \frac{ai + bt}{ci + dt} \right)^{-s} t^{k-s' - 1} dt + \int_{-\epsilon}^{\epsilon} (ci + dt)^{-k} \left( \frac{ai + bt}{ci + dt} \right)^{-s} t^{k-s' - 1} dt \right) \tag{19}
\]

\[
= - \sum_{bd > 0} \int_{-\epsilon}^{\epsilon} (ci + dt)^{-k} \left( \frac{ai + bt}{ci + dt} \right)^{-s} t^{k-s' - 1} dt.
\]

We now replace \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) by \( \begin{pmatrix} b & -a \\ d & -c \end{pmatrix} \), we get

\[
S_0^0 = -i^{-k} \sum_{ac > 0} \int_{-\epsilon}^{\epsilon} (cit + d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} t^{k-s' - 1} dt. \tag{20}
\]
As a result and using (18), we have
\[
S_\epsilon = \sum_{bd > 0} \int_{-\infty}^{\infty} (cit + d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} t^{s'-1} dt - \sum_{bd > 0} \int_{-\epsilon}^{\epsilon} (cit + d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} t^{s'-1} dt
\]
\[- i^{-k} \sum_{ac > 0} \int_{-\epsilon}^{\epsilon} (cit + d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} t^{k-s'-1} dt.
\]

We have \(ac.bd = ad.bc = (1 + bc)bc \geq 0\) and this implies that \(bd \geq 0\) in the third sum in (21) and thus we get
\[
S_\epsilon = \sum_{bd > 0} \int_{-\infty}^{\infty} (cit + d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} t^{s'-1} dt - i^{-k} \sum_{ac > 0} \int_{-\epsilon}^{\epsilon} (cit + d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} t^{k-s'-1} dt
\]
\[+ i^{-k} \sum_{ac > 0, bd > 0} \int_{-\epsilon}^{\epsilon} (cit + d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} t^{k-s'-1} dt
\]
\[- \sum_{bd > 0} \left( \int_{-\epsilon}^{\epsilon} (cit + d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} t^{s'-1} dt + i^{-k} \sum_{bd > 0} \int_{-\epsilon}^{\epsilon} (cit + d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} t^{k-s'-1} dt \right).
\]

(22)

Now to determine \(S_\epsilon\), we divide our integral as follows and we put
\[
S_\epsilon = S + S'_\epsilon + S''_\epsilon + S'''_\epsilon
\]
where
\[
S = \sum_{bd > 0} \int_{-\infty}^{\infty} (cit + d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} t^{s'-1} dt,
\]
\[
S'_\epsilon = - i^{-k} \sum_{ac > 0, bd = 0} \int_{-\epsilon}^{\epsilon} (cit + d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} t^{k-s'-1} dt,
\]
\[
S''_\epsilon = i^{-k} \sum_{ac > 0, bd > 0} \int_{-\epsilon}^{\epsilon} (cit + d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} t^{k-s'-1} dt
\]
(24)
and
\[
S'''_\epsilon = - \sum_{bd > 0} \left( \int_{-\epsilon}^{\epsilon} (cit + d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} t^{s'-1} dt + i^{-k} \sum_{bd > 0} \int_{-\epsilon}^{\epsilon} (cit + d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} t^{k-s'-1} dt \right).
\]
(25)

The evaluation of \(S\) is given in the appendix as a last section of this paper. We get
\[
S = 2\pi e^{\pi i(s'-1)/2} \frac{(-s' + 1)_{k-1}}{(k-1)!} \sum_{bd > 0, ad - bc = 1} a^{-s} e^{s-s'} d^{s'-k} F_1(s, -s' + k; k; 1/ad)
\]
\[
+ i\zeta(s - s') \left( e^{3\pi i(s'-1)/2} - e^{-\pi i(s'-1)/2} \right) \left( \frac{\Gamma(s')\Gamma(s - s')}{\Gamma(s)} \right),
\]
(26)
We now evaluate \( \lim_{\epsilon \to 0} S_{\epsilon} \) by evaluating \( \lim_{\epsilon \to 0} S'_{\epsilon}, \lim_{\epsilon \to 0} S''_{\epsilon} \) and \( \lim_{\epsilon \to 0} S'''_{\epsilon} \). We start with \( \lim_{\epsilon \to 0} S'_{\epsilon} \) defined in (23) by separating the sum into two sums where \( b = 0 \) and where \( d = 0 \). Thus we have

\[
S'_{\epsilon} = -i^{-k} \sum_{n>0} \int_{\eta - \epsilon}^{\eta} (nit + 1)^{-k} \left( \frac{it}{nit + 1} \right)^{-s} t^{k-s'-1} dt - i^{-k} \sum_{n>0} \int_{\eta - \epsilon}^{\eta} (it)^{-k} \left( \frac{nit - 1}{it} \right)^{-s} t^{k-s'-1} dt.
\]

Substituting \( t \) by \( \epsilon t \) and \( t \) by \( -\epsilon t \) respectively in the above sums, we get

\[
S'_{\epsilon} = -i^{-k} \sum_{n=1}^{\infty} \int_{-1}^{1} (n\epsilon t + 1)^{-k} \left( \frac{it}{n\epsilon t + 1} \right)^{-s} t^{k-s'-1} dt - i^{-k} \sum_{n=1}^{\infty} \int_{-1}^{1} (it)^{-k} \left( \frac{n\epsilon t - 1}{it} \right)^{-s} t^{k-s'-1} dt.
\]

(27)

Note that for \( n = 0 \), we have \( s + s' \in 2\mathbb{Z} + 1 \) and the integrals are 0. Since \( s + s' < k - 1 \) and \( Re s - Re s' > 1 \), we now take the limit as \( \epsilon \to 0 \) of the two summands. Observe that the two sums in the brackets are the Riemann sums of the integrals

\[
\int_{0}^{\infty} \int_{-1}^{1} (ixt + 1)^{-k} \left( \frac{it}{ixt + 1} \right)^{-s} t^{k-s'-1} dt dx \quad \text{and} \quad \int_{0}^{\infty} \int_{-1}^{1} (it)^{-k} \left( \frac{ixt - 1}{it} \right)^{-s} t^{k-s'-1} dt dx.
\]

As a result, we get that the limit of the two summands to be 0 except when \( k - 1 = s + s' \) in the first summand (which cannot happen because \( s + s' < k - 1 \)) and when \( s - s' = 1 \) in the second summand (which also cannot happen because \( s - s' > 1 \)) and thus we get

\[
\lim_{\epsilon \to 0} S'_{\epsilon} = 0.
\]

(29)

As for \( S''_{\epsilon} \) as defined in (24), we follow similar evaluation as in \( \lim_{\epsilon \to 0} S'_{\epsilon} \) as above to show that the \( \lim_{\epsilon \to 0} S''_{\epsilon} = 0 \). By directly estimating the integrals that emerge from the cases one takes \( a = 0 \) and \( c = 0 \), one can easily check when taking the limit as \( \epsilon \) goes to 0, we get

\[
\lim_{\epsilon \to 0} S''_{\epsilon} = 0.
\]

(30)

To clarify, we display the summand when taking \( a = 0 \) and replacing \( t \) by \( \epsilon t \), we get

\[
\epsilon^{-k}(s') \sum_{n>0} \int_{-1}^{1} \left( \frac{1}{n\epsilon t + n} \right)^{-s} (-\epsilon t + n)^{-k} t^{k-s'-1} dt.
\]

Notice that the integral converges because we have \( k - s' - 1 > -1 \). We do the same thing for the case when \( c = 0 \).

What left is to determine the \( \lim_{\epsilon \to 0} S'''_{\epsilon} \). Recall that we our group is \( PSL_2(\mathbb{Z}) \) and that the sums below run
over all \(bd > 0\). It is important to note that in this case, and without loss of generality, we can take \(b > 0\) and \(d > 0\). Thus there exists \(a_0\) and \(c_0\) such that \(a_0d - bc_0 = 1\). All the other elements with the same \((b, d)\) are given as \(\begin{pmatrix} a_0 + bn & b \\ c_0 + dn & d \end{pmatrix}\). Now taking \(l = \frac{c_0}{d} + n\), one can notice that the first summand in \(S'''_\epsilon\) can be written as

\[
\sum_{bd > 0} \int_{-\epsilon}^{\epsilon} (cit + d)^{-k} \left(\frac{ait + b}{cit + d}\right)^{-s} t^{s' - 1} dt
\]

We replace \(t\) by \(\epsilon t\) and we get

\[
- \sum_{bd > 0} \int_{-\epsilon}^{\epsilon} (cit + d)^{-k} \left(\frac{ait + b}{cit + d}\right)^{-s} t^{s' - 1} dt
\]

Similarly, we see that the sum of the integral expression is again a Riemann sum

\[
- \sum_{bd > 0} d^{k-s} b^{-s} \int_{-\epsilon}^{\epsilon} \sum_{l \in \mathbb{Z} + c/d} (ilt + 1)^{-k} \left(\frac{1 + it/bd + ilt}{ilt + 1}\right)^{-s} t^{s' - 1} dt.
\]

Note that \(s' > 1\) and thus the expression vanishes as \(\epsilon \to 0\).

We similarly deal with the second sum of \(S'''_\epsilon\) as defined in (25) and we also get the limit to vanish as \(\epsilon \to 0\) because \(Res' < k - 1\). As a result, we get

\[
\lim_{\epsilon \to 0} S'''_\epsilon = 0. \tag{34}
\]

Adding eqs (26), (29), (30) and (34), we get

\[
B = \lim_{\epsilon \to 0} S_\epsilon = 2\pi e^{\pi i(s'-1)/2} \frac{(-s'+1)_{k-1}}{(k-1)!} \sum_{a,b,c,d > 0 \atop ad - bc = 1} a^{-s} e^{s-s'} d^{s'-k} {}_2F_1(s, -s' + k; k; 1/ad)
\]

\[
+ i\zeta(s - s') \left( e^{3\pi i(s'-1)/2} - e^{-\pi i(s'-1)/2} \right) \left( \frac{\Gamma(s') \Gamma(s - s')}{\Gamma(s)} \right).
\]

Finally, we get
\[
\gamma_k(s)^{-1} \int_0^\infty R_{s,k}(it) t^{s'-1} dt = i^{-k} \frac{(2\pi i)^k}{(2\pi)^{k-s'}} \zeta(s-s'+1) \frac{\Gamma(k-s')}{\Gamma(k-s)} + i^{-k} \frac{(2\pi i)^s}{(2\pi)^{k-s}} \zeta(k-s-s'+1) \frac{\Gamma(k-s')}{\Gamma(s)} \\
+ 2\pi e^{\pi i(s'-1)/2} \frac{(-s'+1)k-1}{(k-1)!} \sum_{a,b,c,d>0 \atop ab=bc-1} a^{-s} e^{s-s'} d^{s'-k} \frac{2F_1(s,-s'+k;k;1/ad)}{(\Gamma(s')\Gamma(s-s')) \Gamma(s)}.
\]

3 Appendix

In this section, we evaluate \( S \) defined in subsection 2.1. We start with the terms when \( ac \neq 0 \). Note that since \( bd > 0 \), we can assume that \( b \) and \( d \) are both positive. As a result and using the same argument as on page 7 between equations (22) and (23), we have that \( ac > 0 \). If \( a \) and \( c \) are both negative while \( b \) and \( d \) are both positive, we replace \( \int_{-\infty}^\infty \) by the standard contour in the upper half plane that joins some small \( \delta > 0 \) to some large \( R > 0 \) on the real positive axis, the semi-circle in the upper half plane and the segment above the negative real axis that joins \(-R\) to \(-\delta\) to skip the cut and then joins \(-\delta\) to \(\delta\) in the upper half plane. The singularities of

\[
\int_{-\infty}^\infty (cit + d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} t^{s'-1} dt
\]

are at \( t = ib/a \) and \( t = id/c \) and in this case, they both lie in the lower half plane. Moreover, the integrand on the semi-circle is \( O(R^{Re s'-k}) \) since \( Re s' < k - 1 \). As a result, the original integral is 0.

Now in the case where \( a, b, c, d \) are all positive, we have the poles \( t = ib/a \) and \( t = id/c \) in the upper half plane. Note that the integral is single valued in the upper half plane minus the cut joining \( ib/a \) and \( id/c \). In this case, we take the same contour as the previous case when \( a, c < 0 \), which can be deformed to the contour around the cut joining \( t = ib/a \) and \( t = id/c \). We call \( C \) the contour around the segment cut joining \( t = ib/a \) and \( t = id/c \). As a result, we have

\[
\int_{-\infty}^\infty (cit + d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} t^{s'-1} dt = \int_{C} (cit + d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} t^{s'-1} dt.
\]

In order to evaluate the integral at \( C \), we make a change of variables by taking \( x = a(cit + d) \) and we get
\[
\int_{C'} (cit+d)^{-k} \left( \frac{ait + b}{cit + d} \right)^{-s} t^{s'-1} dt = \frac{1}{i} \int_{C'} x^{-k} \left( \frac{1 - x}{x} \right)^{-s} \left( 1 - \frac{x}{ad} \right)^{s'-1} dx
\]

where \( C' \) is the contour around the cut going around 0 counterclockwise, connecting below the x-axis going around 1, and then back to 0. (we will call such a contour, the dumbbell contour about \([0,1]\)). Note that \( ad = 1 + bc \geq 2 \) and as a result, we get As a result, we get

\[
\frac{1}{i} a^{k-s-1} c^{s-s'} d^{s'-1} e^{\pi i (s'-1)/2} \int_{C'} x^{-k} \left( \frac{1 - x}{x} \right)^{-s} \left( 1 - \frac{x}{ad} \right)^{s'-1} dx
\]

\[
= \frac{1}{i} a^{k-s-1} c^{s-s'} d^{s'-1} e^{\pi i (s'-1)/2} \sum_{m \geq 0} \int_{C'} x^{m-k} \left( \frac{1 - x}{x} \right)^{-s} \left( \frac{-s' + 1}{a^m d^m m!} \right) dx
\]

where \((.)_m\) is the rising Pochhammer symbol. Now deform \( C' \) to come close to \( \infty \). With the only pole at \( \infty \), we use the residue theorem by also taking \( w = 1/x \) to get

\[
\int_{C'} x^{m-k} \left( \frac{x-1}{x} \right)^{-s} dx = 2\pi i \text{Res}_{w=0} \left( w^{-m+k-2}(1-w)^{-s} \right) = \frac{(s)_l}{l!}
\]

where \( l = m - k + 1 \). As a result, we get

\[
\frac{1}{i} a^{k-s-1} c^{s-s'} d^{s'-1} e^{\pi i (s'-1)/2} \sum_{m \geq 0} \int_{C'} x^{m-k} \left( \frac{1 - x}{x} \right)^{-s} \left( \frac{-s' + 1}{a^m d^m m!} \right) dx
\]

\[
= 2\pi a^{k-s-1} c^{s-s'} d^{s'-1} e^{\pi i (s'-1)/2} \sum_{l \geq 0} \frac{(s)_l (-s' + 1)_{k+1+l}}{l!(k+1+l)!} \left( \frac{1}{ad} \right)^{k+1+l}
\]

\[
= 2\pi a^{-s} c^{s-s'} d^{s'-k} e^{\pi i (s'-1)/2} (-s' + 1)_{k-1} \frac{2F_1(s,-s'+k;k;1/ad)}{(k-1)!}
\]

Now if \( ac = 0 \), we start with the case \( a = 0 \). Recall that \( bd > 0 \) and hence we have to evaluate

\[
\sum_{n \geq 0} \int_{-\infty}^{\infty} (-it + n)^{s-k} t^{s'-1} dt
\]

Notice that one can consider the standard semi-circle contour in the upper half plane to evaluate this integral.

The poles of \((-it + n)^{-k+s}\) are at \( t = -in \) and thus we have

\[
\sum_{n \geq 0} \int_{-\infty}^{\infty} (-it + n)^{s-k} t^{s'-1} dt = 0.
\]

Now if \( c = 0 \), we take \( t = n \tau \) and we get

\[
\sum_{n \geq 0} \int_{-\infty}^{\infty} (it + n)^{-s} t^{s'-1} dt = \zeta(s-s') \sum_{n \geq 0} \int_{-\infty}^{\infty} (i\tau + 1)^{-s} \tau^{s'-1} d\tau
\]
We divide the integral now and we substitute \( \tau = t \) and \( \tau = -v \) respectively in the following integrals and we get

\[
\zeta(s - s') \int_{-\infty}^{\infty} (it + 1)^{-s} t^{s'-1} dt = \zeta(s - s') \left( \int_{0}^{\infty} (it + 1)^{-s} t^{s'-1} dt + \int_{0}^{\infty} (-iv + 1)^{-s} (-v)^{s'-1} dv \right)
\]

\[
= \zeta(s - s') \left( \int_{0}^{\infty} (it + 1)^{-s} t^{s'-1} dt + e^{\pi i (s'-1)} \int_{0}^{\infty} (-iv + 1)^{-s} v^{s'-1} dv \right)
\]

We now evaluate each of the above integrals separately. We start with \( \int_{0}^{\infty} (-iv + 1)^{-s} v^{s'-1} dv \). We start with the change of variables \( v = iw \), we get

\[
\int_{0}^{\infty} (-iv + 1)^{-s} v^{s'-1} dv = ie^{\pi s/2} \int_{0}^{\infty} (w + 1)^{-s} w^{s'-1} dw.
\]

One can easily show by estimating the integral over the contour in the fourth quadrant consisting of the quarter circle connecting \(-i\infty \) to \(+\infty \) that

\[
ie^{\pi s/2} \int_{0}^{\infty} (w + 1)^{-s} w^{s'-1} dw = ie^{\pi s/2} \int_{0}^{\infty} (w + 1)^{-s} w^{s'-1} dw
\]

Recall that for \( \text{Re } p > 0 \) and \( \text{Re } q > 0 \), we have the beta-integral given by

\[
B(p, q) = \int_{0}^{1} x^{p-1} (1 - x)^{q-1} dx = \frac{\Gamma(p) \Gamma(q)}{\Gamma(p + q)}
\]

where \( \Gamma(x) \) is the usual Gamma function. Notice that for \( \text{Re } p, \text{Re } q > 0 \), we get

\[
ie^{\pi s/2} \int_{0}^{\infty} (w + 1)^{-s} w^{s'-1} dw = ie^{\pi (s'-1)/2} \frac{\Gamma(s') \Gamma(s - s')}{\Gamma(s)}.
\]

We now evaluate \( \int_{0}^{\infty} (it + 1)^{-s} t^{s'-1} dt \) using the same method we used to evaluate the above integral but now taking \( w = it \) and hence the quarter circle contour will be in the first quadrant. So we get

\[
\int_{0}^{\infty} (it + 1)^{-s} t^{s'-1} dt = -ie^{-\pi i (s'-1)/2} \int_{0}^{\infty} (v + 1)^{-s} v^{s'-1} dv = -ie^{-\pi i (s'-1)/2} \frac{\Gamma(s') \Gamma(s - s')}{\Gamma(s)}.
\]

Substitute (45) and (46) in (43), we get

\[
\zeta(s - s') \int_{-\infty}^{\infty} (i\tau + 1)^{-s} \tau^{s'-1} d\tau = i\zeta(s - s') \left( e^{3\pi i (s'-1)/2} - e^{-\pi i (s'-1)/2} \right) \frac{\Gamma(s') \Gamma(s - s')}{\Gamma(s)}.
\]

As a result, we have

\[
S = 2\pi e^{\pi i (s'-1)/2} \frac{(-s' + 1)_{k-1}}{(k - 1)!} \sum_{ad - bc = 1, ac \neq 0} a^{-s} c^{s-s'} d^{s'-k} 2F_1(s, -s' + k; k; 1/ad)
\]

\[
+ i\zeta(s - s') \left( e^{3\pi i (s'-1)/2} - e^{-\pi i (s'-1)/2} \right) \frac{\Gamma(s') \Gamma(s - s')}{\Gamma(s)}.
\]
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