EXPLICIT SENTENCES DISTINGUISHING MCDUFF’S $\text{II}_1$ FACTORS

ISAAC GOLDBRING, BRADD HART, AND HENRY TOWSNER

ABSTRACT. Recently, Boutonnet, Chifan, and Ioana proved that McDuff’s examples of continuum many pairwise non-isomorphic separable $\text{II}_1$ factors are in fact pairwise non-elementarily equivalent. Their proof proceeded by showing that any ultrapowers of any two distinct McDuff examples are not isomorphic. In a paper by the first two authors of this paper, Ehrenfeucht-Fraïssé games were used to find an upper bound on the quantifier complexity of sentences distinguishing the McDuff examples, leaving it as an open question to find concrete sentences distinguishing the McDuff factors. In this paper, we answer this question by providing such concrete sentences.

1. INTRODUCTION

The first examples of continuum many nonisomorphic separable $\text{II}_1$ factors were given by McDuff in [4]. These same examples were shown to be non-elementarily equivalent (in the sense of continuous logic) by Boutonnet, Chifan, and Ioana in [1]. The way they proved that the McDuff factors were not elementarily equivalent was by showing, for any two distinct McDuff examples $\mathcal{M}$ and $\mathcal{N}$ and any two ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$, that the ultrapowers $\mathcal{M}^\mathcal{U}$ and $\mathcal{N}^\mathcal{V}$ were not isomorphic; by standard model-theoretic results, it then follows that $\mathcal{M}$ and $\mathcal{N}$ are not elementarily equivalent.

In [3], the techniques in [1] were dissected in order to give some information about the sentences distinguishing the McDuff examples. Indeed, if we enumerate the McDuff examples by $\mathcal{M}_\alpha$ for $\alpha \in 2^\omega$ and $k \in \omega$ is the least digit such that $\alpha(k) \neq \beta(k)$, then it was shown that there must be a sentence $\theta$ with at most $5k + 3$ alternations of quantifiers such that $\theta^{\mathcal{M}_\alpha} \neq \theta^{\mathcal{M}_\beta}$. The proof there used Ehrenfeucht-Fraïssé games. The game-theoretic techniques also hinted at a possible strategy of providing concrete sentences distinguishing the McDuff examples if concrete sentences distinguishing examples that differed at the first digit could be obtained. In [3, Section 4.1], such sentences were obtained, but they lacked the uniformity needed to carry out the strategy outlined there.

In this paper, an even finer analysis of the work in [1] is carried out in order to obtain concrete sentences that distinguish McDuff examples that differ at the first digit; this analysis appears in Section 3. In Section 4, the details of the plan outlined in [3, Section 4.2] are given and the inductive construction of sentences

Goldbring’s work was partially supported by NSF CAREER grant DMS-1349399.
Hart’s work was partially supported by NSERC.
Towsner’s work was partially supported by NSF grant DMS-1600263.
distinguishing all of the McDuff examples is elucidated. We note that the con-
crete sentences given here that distinguish examples at “level” $k$ also have $5k + 3$
alternations of quantifiers, agreeing with the game-theoretic bounds predicted in
[3].

We list here some conventions used throughout the paper. First, we follow set
theoretic notation and view $k \in \omega$ as the set of natural numbers less than $k$: $k =\{0, 1, \ldots, k - 1\}$. In particular, $2^k$ denotes the set of functions $\{0, 1, \ldots, k - 1\} \rightarrow \{0, 1\}$. If $\alpha \in 2^k$, then we set $\alpha_i := \alpha(i)$ for $i = 0, 1, \ldots, k - 1$ and we let $\alpha^# \in 2^{k-1}$ be such that $\alpha$ is the concatenation of $(\alpha_0)$ and $\alpha^#$. If $\alpha \in 2^\omega$, then $\alpha|k$ denotes the restriction of $\alpha$ to $\{0, 1, \ldots, k - 1\}$.

Whenever we write a tuple $\vec{x}$, it will be understood that the length of the tuple is
countable (that is, finite or countably infinite).

We will use uppercase letters to denote variables in formulae while their lower-
case counterparts will be elements from algebras. We will use $U$’s and $V$’s (some-
times with subscripts) for variables ranging over the set of unitaries; since unitaries
are quantifier-free definable relative to the theory of $C^*$-algebras, this convention
is harmless. Of course, we will then use $u$’s and $v$’s for unitaries from specific
algebras.

Given a group $\Gamma$ and $a \in \Gamma$, we let $u_a \in L(\Gamma)$ be the canonical unitary associ-
ated to $a$.

Fix a von Neumann algebra $\mathcal{M}$. For $x, y \in \mathcal{M}$, the commutator of $x$ and $y$ is the
element $[x, y] := xy - yx$. If $A$ is a subalgebra of $\mathcal{M}$, then the relative commutant
of $A$ in $\mathcal{M}$ is the set

$$A' \cap \mathcal{M} := \{x \in \mathcal{M} \mid [x, a] = 0 \text{ for all } a \in A\}.$$ 

In particular, the center of $\mathcal{M}$ is $Z(\mathcal{M}) := \mathcal{M}' \cap \mathcal{M}$. For a tuple $\vec{a}$ from $\mathcal{M}$,
we write $C(\vec{a})$ to denote $A' \cap \mathcal{M}$, where $A$ is the subalgebra of $\mathcal{M}$ generated by
the coordinates of $\vec{a}$. Technically, this notation should also mention $\mathcal{M}$, but the
ambient algebra will always be clear from context, whence we omit any mention
of it in the notation.

2. Preliminaries

In this section, we gather most of the background material needed in the rest of
the paper. First, we recall McDuff’s examples. Let $\Gamma$ be a countable group. For
$i \geq 1$, let $\Gamma_i$ denote an isomorphic copy of $\Gamma$ and let $\Lambda_i$ denote an isomorphic copy
of $\mathbb{Z}$. Let $\tilde{\Gamma} := \bigoplus_{i \geq 1} \Gamma_i$. If $S_\omega$ denotes the group of permutations of $\mathbb{N}$ with finite
support, then there is a natural action of $S_\omega$ on $\bigoplus_{i \geq 1} \Gamma_i$ (given by permutation of
indices), whence we may consider the semidirect product $\tilde{\Gamma} \rtimes S_\omega$. Given these
conventions, we can now define two new groups:

$$T_0(\Gamma) := \langle \tilde{\Gamma}, (\Lambda_i)_{i \geq 1} \mid [\Gamma_i, \Lambda_j] = 0 \text{ for } i \geq j \rangle$$

and

$$T_1(\Gamma) := \langle \tilde{\Gamma} \rtimes S_\omega, (\Lambda_i)_{i \geq 1} \mid [\Gamma_i, \Lambda_j] = 0 \text{ for } i \geq j \rangle.$$
Note that if $\Delta$ is a subgroup of $\Gamma$ and $\alpha \in \{0, 1\}$, then $T_{\alpha}(\Delta)$ is a subgroup of $T_{\alpha}(\Gamma)$. Given a sequence $\alpha \in 2^{<\omega}$, we define a group $K_\alpha(\Gamma)$ as follows:

1. $K_\alpha(\Gamma) := \Gamma$ if $\alpha = \emptyset$;
2. $K_\alpha(\Gamma) := (T_{\alpha_0} \circ T_{\alpha_1} \circ \cdots T_{\alpha_{n-1}})(\Gamma)$ if $\alpha \in 2^n$;
3. $K_\alpha$ is the inductive limit of $(K_\alpha(n))_n$ if $\alpha \in 2^\omega$.

We then set $M_\alpha(\Gamma) := L(T_{\alpha}(\Gamma))$. When $\Gamma = F_2$, we simply write $M_\alpha$ instead of $M_\alpha(F_2)$; these are the McDuff examples referred to the introduction.

Given $n \geq 1$, we let $\tilde{\Gamma}_{\alpha,n}$ denote the subgroup of $T_{\alpha_0}(K_{\alpha}(\Gamma))$ given by the direct sum of the copies of $K_{\alpha}(\Gamma)$ indexed by those $i \geq n$ and we let $P_{\alpha,n} := L(\tilde{\Gamma}_{\alpha,n})$. When $\alpha$ has length 1, we simply refer to $\tilde{\Gamma}_{\emptyset,n}$ as $\tilde{\Gamma}_n$ and $P_{\emptyset,n}$ as $P_n$; if, in addition, $n = 1$, then we simply refer to $\tilde{\Gamma}_1$ as $\tilde{\Gamma}$. As introduced in [3], we define a generalized McDuff ultraproduct corresponding to $\alpha$ and $\Gamma$ to be an ultraproduct of the form $\prod_{\mathcal{U}} M_\alpha(\Gamma)^{\otimes t_s}$, where $(t_s)$ is a sequence of natural numbers and $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$.

The following definition, implicit in [1] and made explicitly in [3], is central to our work in this paper.

**Definition 2.1.** We say that a pair of unitaries $u, v$ in a $\Pi_1$ factor $\mathcal{M}$ are good unitaries if, for all $\zeta \in \mathcal{M}$,

$$
\inf_{\eta \in C(u,v)} \|\zeta - \eta\|^2 \leq 100(\|\zeta, u\|^2 + \|\zeta, v\|^2).
$$

In the terminology of [1], this says that $C(u, v)$ is a $(2, 100)$-residual subalgebra of $\mathcal{M}$.

We will need the following key facts, whose proofs are outlined in [3] Facts 2.6).

**Facts 2.2.** Suppose that $\alpha \in 2^{<\omega}$ is nonempty, $\Gamma$ is a countable group, and $(t_s)$ is a sequence of natural numbers.

1. Suppose that $(n_s)$ and $(m_s)$ are two sequences of natural numbers such that $n_s < m_s$ for all $s$. Further suppose that $\Gamma$ is an ICC group. Then $(\prod_{\mathcal{U}} P_{\alpha,m_s}^{\otimes t_s})' \cap (\prod_{\mathcal{U}} P_{\alpha,n_s}^{\otimes t_s})$ is a generalized McDuff ultraproduct corresponding to $\alpha$ and $\Gamma$.
2. For any sequence $(n_s)$, there is a pair of good unitaries $\tilde{u}$ from $\prod_{\mathcal{U}} M_\alpha(\Gamma)^{\otimes t_s}$ such that $\prod_{\mathcal{U}} P_{\alpha,n_s}^{\otimes t_s} = C(\tilde{u})$.
3. Given any separable subalgebra $A$ of $\prod_{\mathcal{U}} M_\alpha(\Gamma)^{\otimes t_s}$, there is a sequence $(n_s)$ such that $\prod_{\mathcal{U}} P_{\alpha,n_s}^{\otimes t_s} \subset A' \cap \prod_{\mathcal{U}} M_\alpha(\Gamma)^{\otimes t_s}$.

3. **Distinguishing examples at level one**

In this section, we will find sentences that distinguish $L(T_{0}(\Gamma))$ and $L(T_{1}(\Gamma))$ for nonamenable groups $\Gamma$. For the purposes of the next section, where the main theorem of the paper is proved, we will actually need to prove a bit more.

In the rest of this paper, we set $\chi(X, U_1, U_2) := 100(\|[X, U_1]\|_2^2 + \|[X, U_2]\|_2^2)$.

**Lemma 3.1.** Let $\Gamma$ be a countable group and $\alpha \in \{0, 1\}$. For any $t, n \in \mathbb{N}$ with $t \geq 1$, there are $a, b \in \bigoplus_{\mathcal{U}} T_{\alpha}(\Gamma)$ such that, for any $\zeta \in L(\bigoplus_{\mathcal{U}} T_{\alpha}(\Gamma))$, we have

$$
\|\zeta - \mathbb{E}_{L(\bigoplus_{\mathcal{U}} T_{\alpha}(\Gamma))}(\zeta)\|^2_2 \leq \chi(\zeta, a, b) L(\bigoplus_{\mathcal{U}} T_{\alpha}(\Gamma)).
$$
Proof. This follows from [1] Lemmas 2.6-2.10. \qed

**Definition 3.2.** We set $\psi_m(V_a, V_b)$ to be the formula

$$
\sup_{X,Y}(\inf_{U} \max_{1 \leq i,j \leq m} \| UX_i U^* Y_j \|_2) - 2 \max_{1 \leq i \leq m} \sqrt{\chi(x_i, V_a, V_b)}
$$

and set $\tau_m := \inf_{V_a, V_b} \psi_m$.

**Proposition 3.3.** Suppose that $\Gamma$ is a countable group and that $t \geq 1$. Then for any $m \geq 1$, we have

$$
\tau_m^{L(\bigoplus_i T_1(\Gamma))} = 0.
$$

**Proof.** Apply Lemma 3.1 with $n = 1$, obtaining $a, b \in \bigoplus_i T_1(\Gamma)$. Let $V_a := u_a$ and $V_b := u_b$. Fix $m$-tuples $\vec{x}, \vec{y} \in L(\bigoplus_i T_1(\Gamma))$ and $\epsilon > 0$. For each $i = 1, \ldots, m$, we have that

$$
\|x_i - \mathbb{E}_{L(\bigoplus_i \Gamma_j)}(x_i)\|_2 \leq \chi(x_i, u_a, u_b)^{L(\bigoplus_i T_1(\Gamma))} + \epsilon.
$$

In particular, there is $k > 0$ such that

$$
\|x_i - \mathbb{E}_{L(\bigoplus_i \Gamma_j)}(x_i)\|_2 \leq \sqrt{\chi(x_i, u_a, u_b)^{L(\bigoplus_i T_1(\Gamma))}} + \epsilon.
$$

Set $x_i^+ := \mathbb{E}_{L(\bigoplus_i \Gamma_j)}(x_i)$ and $x_i^- := x_i - x_i^+$. Let $H_p$ be the subgroup of $T_1(\Gamma)$ generated by $\bigoplus_{j \leq p} \Gamma_j \rtimes S_p$ and $\Lambda_1, \ldots, \Lambda_p$. For $p > 0$ sufficiently large, setting $y_i^+ := \mathbb{E}_{L(\bigoplus_i H_j)}(y_i)$ and $y_i^- := y_i - y_i^+$, we have $\|y_i^-\|_2 \leq \epsilon$.

Choose $\sigma \in S_\infty$ with $\sigma(j) > p$ for all $j \leq m$. Let $\sigma_1 := (\sigma, \sigma, \ldots, \sigma) \in \bigoplus_i L(T_1(\Gamma))$. Note that $\sigma_1(\bigoplus_{j \leq m} \Gamma_j)\sigma_1^{-1}$ commutes with $L(\bigoplus_i H_p)$. Let $u \in U(L(\bigoplus_i T_1(\Gamma)))$ be the unitary corresponding to $\sigma_1$. It follows, for $1 \leq i, j \leq m$, that $\|ux_i^+ u^*, y_j^+\| = 0$, so

$$
\|ux_i^+ u^*, y_j^+\| \leq \|ux_i^+ u^* y_j^-\| + \|ux_i^- u^* y_j^+\| + \|ux_i^- u^* y_j^-\|.
$$

Now

$$
\|ux_i^+ u^*, y_j^-\| \leq \|ux_i^+ u^* y_j^-\| + \|y_j^- ux_i^+ u^*\| \leq 2\|y_j^-\|_2 \leq 2\epsilon.
$$

Here we use that conditional expectation is a contractive map, so $\|x_i^+\| \leq \|x_i\| \leq 1$. Since $\|x_i^-\| \leq 2$, one shows that $\|ux_i^- u^* y_j^-\|_2 \leq 4\epsilon$ in a similar fashion. Finally, we have

$$
\|ux_i^- u^* y_j^+\| \leq 2\|x_i^-\|_2 \leq 2\sqrt{\chi(x_i, u_a, u_b)^{L(\bigoplus_i T_1(\Gamma))}} + \epsilon).
$$

Letting $\epsilon$ go to 0, we get the desired result. \qed

The following is probably obvious and/or well-known, but in any event:

**Lemma 3.4.** There is a function $v: \mathbb{R}^* \to \mathbb{R}^*$ such that, for every $\epsilon > 0$ and an inclusion $N \subseteq M$ of II$_1$ factors, if $x \in N$ is such that $d(x, U(M)) < v(\epsilon)$, then $d(x, U(N)) < \epsilon$. 
Proposition 3.6. Let \( \psi(x) := \max(d(x^*x, 1), d(xx^*, 1)) \). Then \( \psi \) is weakly stable, so there is \( \eta > 0 \) such that if \( N \) is any II_1 factor and \( \psi(x)^N < \eta \), then \( d(x, U(N)) < \epsilon \). Let \( v(\epsilon) := \Delta_\psi(\eta) \), where \( \Delta_\psi \) is the modulus of uniform continuity for the formula \( \psi \). Now suppose that \( N \subseteq M \) are II_1 factors and \( x \in N \) is such that \( d(x, U(M)) < v(\epsilon) \). Then \( \psi(x)^N = \psi(x)^M < \eta \), whence \( d(x, U(N)) < \epsilon \). \( \square \)

The following result, which is Lemma 4.6 in \([1]\), will be very important to us. In what follows, \( \pi_n : \Gamma \to \hat{\Gamma} \) is the canonical embedding with \( \pi_n(\Gamma) = \Gamma_n \).

Fact 3.5. Suppose that \( \Gamma \) is a countable non-amenable group and \( Q \) is a tracial von Neumann algebra. Then there are \( g_1, \ldots, g_m \in \Gamma \) and a constant \( C > 0 \) such that, for any \( n \geq 1 \), unitaries \( v_1, \ldots, v_m \in U(L(\hat{\Gamma}_{n+1} \otimes Q)) \), and \( \zeta \in L(T_0(\Gamma)) \otimes Q \), we have that

\[
\|\zeta\|_2 \leq C \sum_{k=1}^{m} \|u_{\pi_n(g_k)}\zeta - \zeta v_k\|_2.
\]

Note that in the version of \([1]\) currently available, the lemma only allows for unitaries in \( L(\hat{\Gamma}_{n+1}) \) rather than \( L(\hat{\Gamma}_{n+1} \otimes Q) \). However, the proof readily adapts to this more general situation and, indeed, the lemma is used in this more general form in the proof of \([1]\) Lemma 4.4.

For a nonamenable group \( \Gamma \), let \( C(\Gamma) \) and \( m(\Gamma) \) be as in Fact 3.5.

Proposition 3.6. Suppose that \( \Gamma \) is a nonamenable group. Let \( m = m(\Gamma) \), \( C = C(\Gamma) \), and \( \delta := \sqrt{\frac{1}{200(30)^2}} \). Then whenever \( M \) is an intermediate subalgebra \( L(T_0(\Gamma)) \subseteq M \subseteq L(T_0(\Gamma)) \otimes Q \), it follows that \( \tau_m^M \geq \delta \).

Proof. Suppose, towards a contradiction, that \( v_a, v_b \in U(M) \) are such that \( \psi_m(v_a, v_b)^M < \delta \). For each \( n \), let \( \rho_n : \Gamma \to U(P_n) \) be given by \( \rho_n(g) := u_{\pi_n(g)} \). Since \( \bigcup_n (P_n \otimes Q) \) is dense in \( L(T_0(\Gamma)) \otimes Q \), there is \( n \) sufficiently large so that

\[
\max(\|\rho_n(g, v_a)\|_2, \|\rho_n(g, v_b)\|_2) < \delta
\]

for all \( g \in \Gamma \). Fix such an \( n \) and set \( \rho := \rho_n \). It follows that \( \chi(\rho(g), v_a, v_b)^M \leq 200\delta^2 \) for all \( g \in \Gamma \).

By Lemma 5.1, we may find \( a', b' \in T_0(\Gamma) \) such that, for all \( \zeta \in L(T_0(\Gamma)) \), we have

\[
\|\zeta - E_L(\hat{\Gamma}_{n+1}^{\Gamma \otimes Q}) (\zeta)\|_2 \leq \chi(\zeta, u_{a'}, u_{b'})^{L(T_0(\Gamma))}.\]

For simplicity, write \( E \) instead of \( E_L(\hat{\Gamma}_{n+1}^{\Gamma \otimes Q}) \). It then follows that, for all \( \zeta \in L(T_0(\Gamma)) \otimes Q \), we have

\[
\|\zeta - E(\zeta)\|_2 \leq \chi(\zeta, u_{a'}, u_{b'})^{L(T_0(\Gamma)) \otimes Q}.
\]

Let \( g_1, \ldots, g_m \in \Gamma \) be as in Fact 3.5. Since \( \psi_m(v_a, v_b)^M < \delta \), we may find \( u \in U(M) \) such that, for all \( 1 \leq k \leq m \), we have

\[
\max(\|u\rho(g_k)u^*, u_{a'}\|_2, \|u\rho(g_k)u^*, u_{b'}\|_2) < 20\sqrt{2}\delta + \delta \leq 30\delta.
\]
Let \( v_k := up(g_k)u^* \in U(L(T_0(\Gamma)) \otimes Q) \) and let \( v'_k := E(v_k) \). It follows that 
\[
\|v_k - v'_k\|^2 \leq \chi(v_k, u_{a'}, u_{b'})L(T_0(\Gamma)) \otimes Q \leq 200(30\delta)^2.
\]
By the choice of \( \delta \), there is \( v''_k \in U(L(\Gamma_{n+1}) \otimes Q) \) such that \( \|v'_k - v''_k\|^2 < \frac{1}{2\epsilon_0m} \). By Fact [3.5] we have that
\[
\|u\|^2 \leq C \sum_k \|\rho(g_k) - uv''_k\|^2 \leq C \sum_k uv'_{k} - uv''_k\|^2 < \frac{1}{2},
\]
yielding the desired contradiction. \( \square \)

4. The Inductive Construction

In this section, we describe an inductive construction of sentences that allows us to carry out the argument hinted at in [3, Section 4.2]. By [3, Section 4.2], we know that centralizers of good unitaries and relative commutants between centralizers of good unitaries are definable sets, whence we can quantify over them. We actually need to know that we can do this in a uniform manner that does not depend on the ambient II_1 factor nor the good unitaries at hand. Such uniformity is the content of the next lemma. Note that if \( \mathcal{M} \) is a II_1 factor, \( u_1, u_2 \in \mathcal{M} \) are good unitaries and \( x \in \mathcal{M} \), then:
- \( d(x, C(u_1, u_2)) \leq \sqrt{\chi(x, u_1, u_2)^\mathcal{M}} \)
- if \( x \in C(u_1, u_2) \), then \( \chi(x, u_1, u_2)^\mathcal{M} = 0. \)

**Lemma 4.1** (Quantification Lemma).

1. For every formula \( \psi(X, \vec{Y}, \vec{U}) \), there are formulae \( \hat{\psi}_s(\vec{Y}, \vec{U}) \) and \( \hat{\psi}_r(\vec{Y}, \vec{U}) \) such that, for any II_1 factor \( \mathcal{M} \), any pair of good unitaries \( \vec{u} \in \mathcal{M} \), and any tuple \( \vec{y} \in \mathcal{M} \), we have
   \[
   \hat{\psi}_s(\vec{y}, \vec{u})^\mathcal{M} = \sup\{\psi(x, \vec{y}, \vec{u})^\mathcal{M} : x \in C(\vec{u})\}
   \]
   and
   \[
   \hat{\psi}_r(\vec{y}, \vec{u})^\mathcal{M} = \inf\{\psi(x, \vec{y}, \vec{u})^\mathcal{M} : x \in C(\vec{u})\}.
   \]
2. For every formula \( \rho(X, \vec{Y}, \vec{U}_1, \vec{U}_2) \), there are formulae \( \overline{\rho}_s(\vec{Y}, \vec{U}_1, \vec{U}_2) \) and \( \overline{\rho}_r(\vec{Y}, \vec{U}_1, \vec{U}_2) \) such that, for any II_1 factor \( \mathcal{M} \) and any two pairs of good unitaries \( \vec{u}_1, \vec{u}_2 \in \mathcal{M} \) with \( C(\vec{u}_2) \subseteq C(\vec{u}_1) \) and any tuple \( \vec{y} \in \mathcal{M} \), we have
   \[
   \overline{\rho}_s(\vec{y}, \vec{u}_1, \vec{u}_2)^\mathcal{M} = \sup\{\rho(x, \vec{y}, \vec{u}_1, \vec{u}_2)^\mathcal{M} : x \in C(\vec{u}_2) \cap C(\vec{u}_1)\}
   \]
   and
   \[
   \overline{\rho}_r(\vec{y}, \vec{u}_1, \vec{u}_2)^\mathcal{M} = \inf\{\rho(x, \vec{y}, \vec{u}_1, \vec{u}_2)^\mathcal{M} : x \in C(\vec{u}_2) \cap C(\vec{u}_1)\}.
   \]

**Proof.** We only prove the infimum statements. We first prove (1). Let \( \alpha \) be a continuous, nondecreasing function such that \( \alpha(0) = 0 \) and
\[
|\psi(x, \vec{y}, \vec{u}) - \psi(x', \vec{y}, \vec{u})| \leq \alpha(d(x, x'))
\]
for all \( x, x', \vec{y}, \vec{u} \). We claim that
\[
\hat{\psi}_r(\vec{Y}, U_1, U_2) := \inf_X (\psi(X, \vec{Y}, U_1, U_2) + \alpha(\sqrt{\chi(X, U_1, U_2)}))
\]
works. Fix a II\textsubscript{1} factor \(\mathcal{M}\), a pair of good unitaries \(u_1, u_2 \in \mathcal{M}\), and a tuple \(\vec{y} \in \mathcal{M}\). It is clear that
\[
\hat{\psi}(\vec{y}, u_1, u_2)^{\mathcal{M}} \leq \inf \{\psi(x, \vec{y}, u_1, u_2)^{\mathcal{M}} : x \in C(u_1, u_2)\}.
\]
To see the other direction, fix \(x, x' \in \mathcal{M}\) and note that
\[
\psi(x, \vec{y}, u_1, u_2)^{\mathcal{M}} \leq \psi(x', \vec{y}, u_1, u_2)^{\mathcal{M}} + \alpha(d(x, x')),
\]
whence, taking the infimum over \(x \in C(u_1, u_2)\), we have
\[
\inf \{\psi(x, \vec{y}, u_1, u_2)^{\mathcal{M}} : x \in C(u_1, u_2)\} \leq \psi(x', \vec{y}, u_1, u_2)^{\mathcal{M}} + \alpha(\sqrt{\chi(x', u_1, u_2)^{\mathcal{M}}}),
\]
whence the desired result follows from taking the infimum over \(x'\).

The proof of part (2) proceeds in the same way once we find a formula \(\zeta(X, \vec{U}_1, \vec{U}_2)\) such that, for any \(\Pi_1\) factor \(\mathcal{M}\), any two pairs of good unitaries \(\vec{u}_1, \vec{u}_2 \in \mathcal{M}\) such that \(C(\vec{u}_2) \subseteq C(\vec{u}_1)\), and any \(x \in \mathcal{M}\), we have that \(d(x, C(\vec{u}_2)' \cap C(\vec{u}_1)) \leq \zeta(x, \vec{u}_1, \vec{u}_2)^{\mathcal{M}}\). Let
\[
\mathbb{E} : \mathcal{M} \to C(\vec{u}_2)' \cap C(\vec{u}_1), \quad \mathbb{E}_1 : \mathcal{M} \to C(\vec{u}_1), \quad \text{and} \quad \mathbb{E}_2 : C(\vec{u}_1) \to C(\vec{u}_2)' \cap C(\vec{u}_1)
\]
be the usual conditional expections, so \(\mathbb{E} = \mathbb{E}_2 \circ \mathbb{E}_1\) and \(d(x, C(\vec{u}_2)' \cap C(\vec{u}_1)) = \|x - \mathbb{E}(x)\|_2\). Note that
\[
\|x - \mathbb{E}(x)\|_2 \leq \|x - \mathbb{E}_1(x)\|_2 + \|\mathbb{E}_1(x) - \mathbb{E}_2(\mathbb{E}_1(x))\|_2.
\]
Now \(\|x - \mathbb{E}_1(x)\|_2 \leq \sqrt{\chi(x, u_{11}, u_{12})^{\mathcal{M}}}\). As proved in [3, Section 4.2],
\[
\|\mathbb{E}_1(x) - \mathbb{E}_2(\mathbb{E}_1(x))\|_2 \leq \sqrt{\sup_{y \in C(\vec{u}_2)} \|[y, \mathbb{E}_1(x)]\|_2}.
\]
Now notice that
\[
\|[y, \mathbb{E}_1(x)]\|_2 \leq \|\mathbb{E}_1(x)y - xy\|_2 + \|yx - yx\|_2 + \|yx - y\mathbb{E}_1(x)\|_2.
\]
Let \(\psi(X, Y, \vec{U}_2) := 2\chi(X, \vec{U}_2) + \|XY - YX\|_2\). It follows that
\[
\sup_{y \in C(\vec{u}_2)} \|[y, \mathbb{E}_1(x)]\|_2 \leq \hat{\psi}(x, \vec{u}_2)^{\mathcal{M}}.
\]
Letting
\[
\zeta(X, \vec{U}_1, \vec{U}_2) := \sqrt{\chi(X, \vec{U}_1) + \sqrt{\hat{\psi}(X, \vec{U}_2)}}
\]
yields the desired formula. \(\square\)

Repeatedly applying the Quantification Lemma yields:

**Theorem 4.2** (Relativization Theorem). For any sentence \(\theta\) in prenex normal form, there is a formula \(\tilde{\theta}(\vec{U}_1, \vec{U}_2)\) such that, for any \(\Pi_1\) factor \(\mathcal{M}\) and any two pairs of good unitaries \(\vec{u}_1, \vec{u}_2 \in \mathcal{M}\) with \(C(\vec{u}_2) \subseteq C(\vec{u}_1)\), we have
\[
\tilde{\theta}(\vec{u}_1, \vec{u}_2)^{\mathcal{M}} = \theta^{C(\vec{u}_2)' \cap C(\vec{u}_1)}.
\]

Moreover, \(\tilde{\theta}\) is also in prenex normal form and has the same number of alternations of quantifiers as \(\theta\).
We now introduce the formulae

\[ \varphi_{\text{good}}(U_1, U_2) := \sup_X \inf_{\tilde{Y}} \max_{i=1,2} \| [Y, U_i] \|_2, d(X, Y) = \sqrt{\chi(X, U_1, U_2)} \]

and

\[ \varphi^0_{\theta}(\tilde{Y}; \tilde{U}) := \sup_{X \in C(\tilde{U})} \max_{i=1,\ldots,n} \| [X, Y_i] \|_2. \]

In the definition of \( \varphi_{\leq} \), we are abusing notation and really mean the formula one obtains from Lemma \ref{lem:4.1}. In what follows, we will only need to consider \( \varphi^3_{\leq} \) and denote this formula simply by \( \varphi_{\leq} \).

Note that:
\begin{itemize}
  \item If \( M \) is an \( \aleph_1 \)-saturated \( \Pi_1 \) factor, then \( \varphi_{\text{good}}(u_1, u_2)^{\mathcal{M}} = 0 \) if and only if \( u_1, u_2 \) is a pair of good unitaries.
  \item If \( M \) is any \( \Pi_1 \) factor, \( \tilde{u} \in \mathcal{M} \) is a pair of good unitaries, and \( \tilde{y} \in \mathcal{M}^n \) is arbitrary, then \( \varphi_{\leq}(\tilde{y}, \tilde{u})^{\mathcal{M}} = 0 \) if and only if \( \tilde{y} \leq \tilde{u} \).
\end{itemize}

**Definition 4.3.** Given a sentence \( \theta \), we recursively define a sequence of sentences \( \theta_n \) as follows: Set \( \theta_1 := \theta \). Supposing that \( \theta_n \) has been defined, we set \( \theta_{n+1} \) to be the sentence

\[ \inf_{\tilde{u}_1} \max_{\tilde{u}_2} (\varphi_{\text{good}}(\tilde{U}_1), \sup_{A} \max_{\tilde{V}_1} (\varphi_{\text{good}}(\tilde{U}_2), \varphi_{\leq}(A, \tilde{U}_1; \tilde{U}_2), \tilde{\theta}_n(\tilde{U}_1, \tilde{U}_2))). \]

When \( \theta = \tau_m \), we write \( \theta_{m,n} \) for \((\tau_m)_{n}\). Here is the main result of this paper:

**Theorem 4.4.** For each nonamenable group \( \Gamma \), there is a sequence \((r_n(\Gamma))\) of positive real numbers such that, for any \( n, t \in \mathbb{N} \) with \( t \geq 1 \) and any \( \alpha \in 2^n \), we have:

\[ \theta_{m,n}^{L(T_{\alpha}(\Gamma))^\otimes t} = 0 \quad \text{for all } m \geq 1 \quad \text{if } \alpha(n-1) = 1; \]

\[ \theta_{m,n}^{L(T_{\alpha}(\Gamma))^\otimes t} \geq r_n(\Gamma) \quad \text{if } \alpha(n-1) = 0. \]

**Proof:** We prove the theorem by induction on \( n \). When \( n = 1 \), the theorem holds by Propositions \ref{prop:3.3} and \ref{prop:3.6}.

Inductively suppose that the theorem is true for \( n \). Fix a non-amenable group \( \Gamma \). First suppose that \( \alpha \in 2^{n+1} \) is such that \( \alpha(n) = 1 \). Fix also \( m, t \geq 1 \). Let \( \mathcal{M} \) be the ultrapower of \( L(T_{\alpha}(\Gamma))^\otimes t \); by Łos’ theorem, it suffices to show that \( \theta_{m,n}^{\mathcal{M}} = 0 \). Fix a pair of good unitaries \( \tilde{u}_1 \). Given \( a \in \mathcal{M} \), we can find a pair of good unitaries \( \tilde{u}_2 \in \mathcal{M} \) such that \( \tilde{u}_2 > \{a, \tilde{u}_1\} \). We then have that \( C(\tilde{u}_2)^{C(a)} \cap C(\tilde{u}_1) \) is a generalized McDuff ultraproduct corresponding to \( \alpha^{\#} \) and \( \Gamma \), whence, by the inductive hypothesis, we have that \( \tilde{\theta}_{m,n}(\tilde{u}_1, \tilde{u}_2)^{\mathcal{M}} = \theta_{C(\tilde{u}_2)^{C(a)} \cap C(\tilde{u}_1)} = 0 \). It follows that \( \theta_{m,n}^{\mathcal{M}} = 0 \).

Now suppose, towards a contradiction, that there is no constant \( r_{n+1}(\Gamma) \). Then for each \( l > 1 \), there is \( \alpha_l \in 2^{n+1} \) and \( t_l \in \mathbb{N} \) with \( t_l \geq 1 \) such that \( \theta_{m(n),n}^{L(T_{\alpha}(\Gamma))^\otimes t_l} < \frac{1}{l} \). Without loss of generality, each \( \alpha_l = \alpha \) for some fixed \( \alpha \in 2^{n+1} \). Let \( \mathcal{M} := \prod_{l} L(T_{\alpha}(\Gamma))^\otimes t_l \), a generalized McDuff ultraproduct corresponding to \( \alpha \) and \( \Gamma \).
We then have that $\bar{\theta}_{m,n+1}^{M,\Gamma} = 0$. Let $\vec{u}_1$ be a pair of good unitaries witnessing the infimum. Take any $a > \vec{u}_1$ and then take a pair of good unitaries $\vec{u}_2 > a$ witnessing the infimum for that $a$. We then have that $C(\vec{u}_2)^* \cap C(\vec{u}_1)$ is a McDuff ultraproduct corresponding to $\alpha^*\Gamma$ and $\Gamma$, whence $\bar{\theta}_{m,n}(\vec{u}_1, \vec{u}_2)^{\Gamma} = \bar{\theta}_n^{C(\vec{u}_2)^* \cap C(\vec{u}_1)} \geq \tau_n(\Gamma)$, contradicting the fact that $\bar{\theta}_{m,n}(\vec{u}_1, \vec{u}_2)^{\Gamma} = 0$. \hfill \Box

**Remark 4.5.** Note that each $\tau_m$ is equivalent to a formula in prenex normal form that begins with an inf and has three alternations of quantifiers. By the construction, it is easy to check, by induction on $n$, that each $\theta_{m,n}$ is equivalent to a formula in prenex normal form that begins with an inf and has $5n + 3$ alternations of quantifiers. This agrees with the theoretical bounds given in [3].

**Corollary 4.6.** Suppose that $\Gamma$ is any countable group and $\alpha, \beta \in 2^n$ are such that $\alpha|n-1 = \beta |n-1$, $\alpha(n) = 1$, and $\beta(n) = 0$. Write $\bar{\beta} = (\beta | n+1)^{\alpha \beta}$. Set $m := m(T_{\beta^{\Gamma}}(\Gamma))$ and $r := r_{n+1}(T_{\beta^{\Gamma}}(\Gamma))$. Then $\bar{\theta}_{m,n+1}^{M} = 0$ and $\theta_{m,n+1}^{M,\beta(\Gamma)} \geq r$.

**Remark 4.7.** As pointed out in [1], the results there also show, for any countable group $\Gamma$ and any distinct $\alpha, \beta \in 2^n$, that $C^*_r(T_{\alpha}(\Gamma))$ and $C^*_r(T_{\beta}(\Gamma))$ are not elementarily equivalent. Our results here do indeed yield concrete sentences distinguishing these algebras. As mentioned in [1], the groups $T_{\alpha}(\Gamma)$ are increasing unions of Powers groups, whence, by the proof of [2] Proposition 7.2.3, the unique trace on $C^*_r(T_{\alpha}(\Gamma))$ is definable, and uniformly so over all $\alpha \in 2^n$. Consequently, the $\theta_{m,n}$’s can be construed as formulae in the language of $C^*$-algebras with imaginary sorts added and, since the completion of $C^*_r(T_{\alpha}(\Gamma))$ with respect to the GNS representation induced by the unique trace is $\mathcal{M}_{\alpha}(\Gamma)$, we have that the $\theta_{m,n}$’s distinguish the $C^*_r(T_{\alpha}(\Gamma))$’s as well.

[1] R. Boutonnet, I. Chifan, and A. Ioana, $II_1$ factors with non-isomorphic ultrapowers, to appear in Duke Math. J. arXiv 1507.06340.

[2] I. Farah, B. Hart, M. Lupini, L. Robert, A.P. Tikuisis, A. Vignati, and W. Winter, Model theory of $C^*$-algebras, arXiv 1602.08072.

[3] I. Goldbring and B. Hart, On the theories of McDuff’s $II_1$ factors, International Math Research Notices, to appear.

[4] D. McDuff, Uncountably many $II_1$ factors, Ann. of Math. 90 (1969) 372-377.

REFERENCES

E-mail address: isaac@math.uci.edu
URL: http://www.math.uci.edu/~isaac
DEPARTMENT OF MATHEMATICS AND STATISTICS, McMaster University, 1280 Main St., Hamilton ON, Canada L8S 4K1
E-mail address: hartb@mcmaster.ca
URL: http://ms.mcmaster.ca/~bradd/

DEPARTMENT OF MATHEMATICS, University of Pennsylvania, 209 South 33rd Street
Philadelphia, PA 19104-6395.
E-mail address: htowsner@math.upenn.edu
URL: http://www.sas.upenn.edu/~htowsner/