A NOTE ON BRAID GROUP ACTIONS ON SEMIORTHONORMAL BASES OF MUKAI LATTICES

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Abstract

We shed some light on the problem of determining the orbits of the braid group action on semiorthonormal bases of Mukai lattices as considered in [7] and [8]. We show that there is an algebraic (and in particular algorithmic) equivalence between this problem and the Hurwitz problem for integer matrix groups finitely generated by involutions. In particular we consider the case of $K_0(P^n)$ $n \geq 4$ which was considered in [8] and show that the only obstruction for showing the transitivity of the braid group action on its semiorthonormal bases is the determination of the relations of particular finitely generated integer matrix groups. Although we prove transitivity for an infinite set of Mukai lattices, our work, however, indicates quite strongly that the question of transitivity of semiorthonormal bases of Mukai lattices under the braid group action cannot be answered in general and can, at most, be resolved only in particular cases.

Key words: Braid Group, Helix Theory, Exceptional Bundles, Mukai Lattice, Monodromy Action

1 Background and motivation

The bounded derived category of coherent sheaves of an algebraic variety $X$, $\mathcal{D}^b(X)$, has been the focus of recent studies in various fields related with algebraic geometry (see for example [5], [6]). Although $\mathcal{D}^b(X)$ in general contains less information than $\text{Coh}(X)$ (the abelian category of coherent sheaves on $X$), it contains enough structure for the reconstruction of important invariants of $X$ such as higher Chow groups, K-theory and cohomology, as well as birational geometric invariants of $X$. Furthermore, Bondal and Orlov [6] have shown that projective varieties with ample canonical or anticanonical bundle are uniquely determined by their derived categories.

The main idea of doing computations with $\mathcal{D}^b(X)$ consists of reducing the computations in the derived category essentially to linear algebra using the Grothendieck
$K_0$ functor. This was first conceived by Beilinson in [3]. We thus have a notion of a base, a so called **exceptional collection**, which consists of exceptional objects which generate $D^b(X)$ and, by using $K_0$ becomes a vector space basis which is called a **semiorthonormal base**. For example, the exceptional collection of $K_0(\mathbb{P}^n)$ is induced by $\{\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n)\}$ (see [3, 16]). The exceptional collections of del-Pezzo surfaces are also well studied [12]. We also have the **Euler form**, an (unimodular non-symmetric) inner product:

$$\chi(E, F) = \sum (-1)^k \text{dim} \text{Ext}^k(E, F)$$

defined on coherent sheaves and is respected by the $K_0$ functor.

A semiorthonormal base equipped with such an inner product is called a **Mukai lattice**. Furthermore, there exists a non-trivial action of the braid group on such bases which is called **mutation** (we give the precise definitions in the next section). This action is transitive, for example, in the case of del-Pezzo surfaces [12]. Thus, in order to study semiorthonormal bases of Mukai lattices it is quite useful to study the orbits under the braid group action. In this paper we address this problem which, as far as we know, was only considered in [7] and in particular cases in [8] and [16]. For further details on this theory, which is known as **helix theory** one can consult [7] and [16].

Our motivation in this paper was to try to understand the algebraic constraints in the constructions appearing in [1] and [2]. In these works, Auroux Kuleshov and Orlov, have constructed categorical equivalences between bounded derived categories of certain algebraic varieties (or their deformations) and the so called **Fukaya-Seidel categories** of their corresponding dual symplectic manifolds according to the celebrated homological mirror symmetry conjecture of Kontsevich [13]. Apart from their physical significance (see for example [11]), these constructions can also be viewed as a means for extracting effective algebraic invariants of symplectic manifolds. The Mukai lattices can thus be considered as particular cases of **lagrangian vanishing cycles** appearing in Seidel’s work [17, 18] on symplectic manifolds. Although these constructions cannot be applied for general symplectic manifolds, they might give insights for the construction of analytic invariants (as opposed to synthetic empirical invariants) of symplectic manifolds. We plan to pursue this point of view in future works.

### 2 Definitions and main results

We start off with the necessary definitions.

Recall that the **braid group**, $B_n$, is a group with generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } i = 1, \ldots, n-2$$

\(^{1}\)strictly speaking it is a free $\mathbb{Z}$-module basis which becomes a vector space basis after tensoring with $\mathbb{C}$ for example
\[
\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i - j| > 1
\]

In what follows, \( G \) is a group and \( g_1, \ldots, g_n \in G \).

**Hurwitz action** : This is an action of the braid group \( B_n \) on \( n \)-tuples of group elements, defined on the generators of \( B_n \) as follows (denote \( g^h = hgh^{-1} \))

\[
\sigma_i(g_1, \ldots, g_n) = (g_1, \ldots, g_{i-1}, g_i^{g_{i+1}}, g_i, g_{i+2}, \ldots, g_n)
\]

\[
\sigma_i^{-1}(g_1, \ldots, g_n) = (g_1, \ldots, g_{i-1}, g_{i+1}, g_i^{g_{i-1}}, g_i, g_{i+2}, \ldots, g_n)
\]

(That this is a well defined action see [7]).

**Remark 2.1.** We shall sometimes call the Hurwitz action a **mutation** in accord with proposition [2.1] and the following definitions.

**Hurwitz problem for groups** : Let \( \{g_1, \ldots, g_n\} \) be an \( n \)-tuple of elements in the group \( G \). Then the Hurwitz problem for \( G \) and \( \{g_1, \ldots, g_n\} \) is to decide whether the tuple \( (g_1, \ldots, g_n) \) and a given tuple \( (h_1, \ldots, h_n) \) (where each \( h_i \) is expressed in terms of the \( g_j \)-s or their inverses) are in the same \( B_n \)-orbit, where the action of \( B_n \) is the Hurwitz action defined above.

**Remark 2.2.** Note that our definition of the Hurwitz problem is more restrictive than [20] where the input is any two tuples of elements from the group. And so, the undecidability result, as well as the different constructions in [20] do not apply in our work.

**Definition 2.1.** Mukai lattice : This is a free \( \mathbb{Z} \)-module, \( M \), of finite rank, equipped with a bilinear map \( M \times M \to \mathbb{Z} \), denoted \( \langle *, * \rangle \), which we assume to be unimodular; this means that the natural map of \( M \) into \( \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \), sending \( m \) to \( \langle m, * \rangle \) is a \( \mathbb{Z} \)-module isomorphism. A semi-orthonormal (or exceptional) basis for a Mukai lattice is a basis \( \{E_1, \ldots, E_n\} \) of \( M \) such that its Gram matrix \( \chi_{ij} = \langle E_i, E_j \rangle \) satisfies \( \chi_{ii} = 1 \) and \( \chi_{ij} = 0 \) for \( i > j \) (We note that such a basis does not always exist but we shall work with Mukai lattices which admit such bases). In this paper we shall use the term semi-orthonormal basis to mean a tuple \( (E_1, \ldots, E_n) \), where \( \{E_1, \ldots, E_n\} \) is a semi-orthonormal basis of a Mukai lattice. Note that since \( M \) is a free \( \mathbb{Z} \)-module, it can be faithfully represented as the free \( \mathbb{Z} \)-module \( \mathbb{Z}^n \).

**Mutations of semi-orthonormal bases** : This is an action of the braid group \( B_n \) on semi-orthonormal bases of a Mukai lattice of rank \( n \) defined as follows

\[
\sigma_i(E_1, \ldots, E_n) = (E_1, \ldots, E_{i-1}, L_{E_i}E_{i+1}, E_i, E_{i+2}, \ldots, E_n)
\]

\[
\sigma_i^{-1}(E_1, \ldots, E_n) = (E_1, \ldots, E_{i-1}, E_{i+1}, R_{E_{i+1}}E_i, E_{i+2}, \ldots, E_n)
\]
where

\[ L_{E_i}E_{i+1} \overset{\text{def}}{=} E_{i+1} - \langle E_i, E_{i+1} \rangle E_i \]
\[ R_{E_{i+1}}E_i \overset{\text{def}}{=} E_i - \langle E_i, E_{i+1} \rangle E_{i+1} \]

(that this is a well defined action on semi-orthonormal bases see [7]).

the actions of \( \sigma_i \) and \( \sigma_i^{-1} \) are called left and right mutations respectively. Thus, for example, \( L_{E_i}E_{i+1} \) is called the left mutation of \( E_{i+1} \) by \( E_i \).

Analogously to the Hurwitz problem on groups, we have

**Hurwitz problem for Mukai lattices** : Given a Mukai lattice \( M \) of rank \( n \) and its semi-orthonormal basis \( \{E_1, \ldots, E_n\} \), the Hurwitz problem for \( M \) is to decide whether a given semi-orthonormal basis \( \{F_1, \ldots, F_n\} \) is in the same \( B_n \)-orbit as \( \{E_1, \ldots, E_n\} \) (for each \( 1 \leq i \leq n \), \( F_i \) is given as a composition of left or right mutations of \( E_{\tau(i)} \) by the \( E_i \)-s where \( \tau \) is a permutation of \( 1, \ldots, n \)).

As will be shown in the next proposition, the following set could be considered as equivalent, in some sense, to the set of all semi-orthonormal bases.

**Definition 2.2. Reflect** : Given a Mukai lattice \( M \), the set \( \text{Reflect}(M) \) consists of the tuples \( (\psi_{E_1}, \ldots, \psi_{E_n}) \) of elements in \( \text{Hom}_\mathbb{Z}(M) \) where \( \{E_1, \ldots, E_n\} \) is a semi-orthonormal base of \( M \) and

\[ \psi_{E_i}(E_j) = E_j - (\langle E_i, E_j \rangle + \langle E_j, E_i \rangle) : E_i \]

we shall see below that in fact each \( \psi_{E_i} \) is an automorphism of \( M \) of order 2.

**Definition 2.3. M-Bases** Given a Mukai lattice \( M \), the set \( \text{M-Bases}(M) \) consists of the tuples \( \{E_1, \ldots, E_n\} \) where \( \{E_1, \ldots, E_n\} \) is a semi-orthonormal base of \( M \).

Mutations of semi-orthonormal bases of a Mukai lattice can be considered in terms of Hurwitz action on group tuples as shown in the following

**Proposition 2.1** (Bijective Equivariance). Given a Mukai lattice \( M \) of rank \( n \), there exists a bijective equivariant mapping between the set \( \text{M-Bases}(M) \) and the set \( \text{Reflect}(M) \) w.r.t the \( B_n \) action up to the sign of the vectors in the semi-orthonormal bases.

**Proof.** Given a Mukai lattice \( M \) of rank \( n \) with a semiorthonormal base \( \{E_1, \ldots, E_n\} \) and an inner product \( \langle \cdot, \cdot \rangle \), first let us denote its symmetrized inner product by \( B \), that is

\[ B(E, F) \overset{\text{def}}{=} \langle E, F \rangle + \langle F, E \rangle \]
Consider the following map from $M - \text{Bases}(M)$ into $\text{Reflect}(M)$

$$\Psi(E_1, \ldots, E_n) \overset{\text{def}}{=} (\psi_{E_1}, \ldots, \psi_{E_n})$$

where we define $\psi_{E_i} \in \text{Hom}_Z(M)$ as follows

$$\psi_{E_i}(F) = F - B(E_i, F)E_i \quad F, E_i \in M$$

First, let us show that $\psi_{E_i}$ is a map of order 2 for all $i = 1 \ldots n$:

$$\psi_{E_i}(\psi_{E_i}(A)) = \psi_{E_i}(A - B(E_i, A)E_i) = A - B(E_i, A)E_i - B(A - B(E_i, A)E_i, E_i)E_i =$$

$$= A - B(E_i, A)E_i - B(A, E_i)E_i + B(E_i, A) \cdot B(E_i, E_i)E_i = A$$

Secondly, let us show that $\psi_{E_i}$ is an isometry w.r.t $B$:

$$B(\psi_{E_i}(F), \psi_{E_i}(G)) = B(F - B(F, E_i)E_i, G - B(G, E_i)E_i) =$$

$$= B(F, G) - B(G, E_i) \cdot B(F, E_i) - B(F, E_i) \cdot B(E_i, G) + B(F, E_i) \cdot B(G, E_i) \cdot B(E_i, E_i) =$$

$$= B(F, G) - 2B(G, E_i) \cdot B(F, E_i) + 2B(F, E_i) \cdot B(G, E_i) =$$

$$= B(F, G)$$

(2.1)

Now, that $\Psi$ is surjective is clear from definition 2.2. To show it is injective up to the signs of the $E_i$-s note first that

$$\psi_{-E_i}(F) = F - B(-E_i, F) \cdot (-E_i) = F - B(E_i, F)E_i = \psi_{E_i}(F)$$

on the other hand, suppose that $(E_1, \ldots, E_n)$ and $(F_1, \ldots, F_n)$ are two semi-orthonormal bases of $M$ and suppose that

$$(\psi_{E_1}, \ldots, \psi_{E_n}) = (\psi_{F_1}, \ldots, \psi_{F_n})$$

then for every $i$ we have that

$$\psi_{E_i}(E_i) = \psi_{F_i}(E_i) \implies -E_i = E_i - B(F_i, E_i)F_i \implies 2E_i = B(F_i, E_i)F_i$$

$$\psi_{E_i}(F_i) = \psi_{F_i}(F_i) \implies F_i - B(F_i, E_i)E_i = -F_i \implies 2F_i = B(F_i, E_i)E_i$$

the last two equations imply that $4E_i = (B(E_i, F_i))^2 E_i$ but this must mean that $B(E_i, F_i) = \pm 2$ and hence $E_i = \pm F_i$.

In what follows, we shall write $\psi_{E_i}$ instead of $\psi_{E_i}^{-1}$, since, as mentioned above $\psi_{E_i}$ is a map of order 2.
To show equivariance, let us first show that for each \( i, 1 \leq i < n \), we have

\[
\psi_{L_i E_i E_{i+1}} = \psi_{E_{i+1}} = \psi_{E_i} \psi_{E_{i+1}} \psi_i E_i
\]  

(2.2)

To prove this, we shall show that both sides of equation 2.2 agree on a basis of \( M \).

Consider then the following set of vectors \( \{F^j_k\}_{k=1}^n \) which is defined for each \( j \), \( 1 \leq j \leq n \), as follows:

\[
F^j_k = \begin{cases} 
\psi_{E_i} E_k + E_k & k \neq j \\
E_j & k = j
\end{cases}
\]  

(2.3)

that this is indeed a basis for \( M \) is easy to see by considering the faithful representation of \( M \) in \( \mathbb{Z}^n \) as mentioned in definition 2.1 where the \( E_i \)-s are mapped to the standard basis vectors : After this representation is applied it is easy to see that the matrix whose columns are \( F^j_k \) has determinant 1 and hence it is inverted by an integer matrix which means that its columns span \( \mathbb{Z}^n \) over \( \mathbb{Z} \). Now, for a fixed \( i \), we have that \( \{\psi_{E_i} (F^j_k)\}_{k=1}^n \) is also a basis of \( M \) (as each \( \psi_{E_i} \) is an automorphism of \( M \)). We shall thus show that both sides of equation 2.2 agree on the basis \( \{\psi_{E_i} (F^j_k)\}_{k=1}^n \) of \( M \).

Note first that

\[
\forall A \in M, \forall i = 1 \ldots n: \quad B(A, E_i) = 0 \iff \psi_{E_i}(A) = A
\]  

(2.4)

and that for all \( k, k \neq i \):

\[
\psi_{E_i} (F^j_k) = \psi_{E_i} (\psi_{E_i} E_k + E_k) = \psi_{E_i} (\psi_{E_i} (E_k)) + \psi_{E_i} (E_k) = E_k + \psi_{E_i} (E_k) = F^j_k
\]  

(2.5)

Now, on the one hand, using equation 2.5, we have that

\[
\psi_{E_i} (F^j_k) = \psi_{E_i} (\psi_{E_i} (F^j_{k+1})) = \psi_{E_i} (\psi_{E_i} (F^j_k)) = \psi_{E_i} (F^j_k) \quad k \neq i + 1
\]

\[
\psi_{E_i} (F^j_{i+1}) = \psi_{E_i} (\psi_{E_i} (F^j_{i+1})) = \psi_{E_i} (\psi_{E_i} (E_{i+1})) = -\psi_{E_i} (E_{i+1}) = -\psi_{E_i} (F^j_{i+1})
\]

and on the other hand we have that

\[
B(L_i E_i E_{i+1}, \psi_{E_i} (F^j_k)) = B(\psi_{E_i} (E_{i+1}), \psi_{E_i} (F^j_k)) = B(E_{i+1}, F^j_k) = 0
\]

the second equality follows the fact that \( \psi_{E_i} \) is an isometry and the third equality follows 2.5 and 2.4 now, using 2.4 we have that this equality is equivalent to

\[
\psi_{L_i E_i E_{i+1}} (\psi_{E_i} (F^j_k)) = \psi_{E_i} (F^j_k) \quad k \neq i + 1
\]
and furthermore we have

\[ \psi_{LE_i}E_{i+1}(\psi_{E_i}(F_{i+1}^i)) = \psi_{LE_i}E_{i+1}(\psi_{E_i}(E_{i+1})) = \psi_{E_i}(E_{i+1})(\psi_{E_i}(E_{i+1})) = \psi_{E_i}(E_{i+1}) = -\psi_{E_i}(F_{i+1}^i) \]

Thus, we have shown that both sides of equation 2.2 agree on the basis \( \{\psi_{E_i}(F_{k+1}^i)\}_{k=1,\ldots,n} \), and so

\[ \psi_{LE_i}E_{i+1} = \psi_{E_i}^{E_{i+1}} \quad (2.6) \]

as \( R_{E_{i+1}}(E_i) = \psi_{E_{i+1}}(E_i) \), a completely analogous computation shows that

\[ \psi_{RE_{i+1}}E_i = \psi_{E_i}^{E_{i+1}} \quad (2.7) \]

Now, (2.6) and (2.7) show that for every braid group generator \( \sigma_i \) we have

\[ \Psi(\sigma_i(E_1, \ldots, E_n)) = \sigma_i(\Psi(E_1, \ldots, E_n)) \]

proceeding with induction on the number of generators this establishes equivariance.

Remark 2.3. There exists another natural action on semi-orthonormal bases. Consider \( \text{Isom}(M) \): The group of isometries of \( M \) as an inner product space. Then the action of \( \varphi \in \text{Isom}(M) \) on the semiorthonormal base \( (E_1, \ldots, E_n) \) is \( (\varphi(E_1), \ldots, \varphi(E_n)) \). As can be seen by the following computation :

\[ \psi_{\varphi(E_i)}(\varphi(E_j)) = \varphi(E_j) - B(\varphi(E_i), \varphi(E_j))\varphi(E_i) = \varphi \circ \psi_{E_i}(E_j) \]

and hence

\[ \psi_{\varphi(E_i)} \circ \varphi = \varphi \circ \psi_{E_i} \]

hence

\[ \psi_{\varphi(E_i)} = \psi_{\varphi(E_i)} \quad \varphi \in \text{Isom}(M) \]

and so, in the notation of proposition 2.7 we have that

\[ \Psi(\varphi \cdot (E_1, \ldots, E_n)) = (\psi_{E_1}^\varphi, \ldots, \psi_{E_n}^\varphi) \quad \varphi \in \text{Isom}(M) \]

so the action of \( \text{Isom}(M) \) on the semiorthonormal bases corresponds to global conjugation on the corresponding group tuples. We will address this subject more generally in the last section.

(In [7] the considered action on semiorthonormal bases of Mukai lattices is actually the action of the group \( \text{Isom}(M) \rtimes B_n \), however, it is not clear what is meant by this semi-product as the stabilizer of \( B_n \) for example may be non-trivial. For this reason we decided to address the action of \( \text{Isom}(M) \) separately).
Proposition 2.1 gives a good notion of the algebraic difficulty of analysing the orbits in Mukai lattices. Indeed, although the groups generated by the tuples in \( \text{Reflect}(M) \) are finitely generated, there is no reason that they should be finitely presented (i.e. have a finite number of relations). However, as the following proposition shows, groups which have simple relations (such as free Coxeter groups) are easy to analyse in this context. Before stating the proposition, let us first say what we mean by a **transitive** orbit of the \( B_n \) action on a group tuple \((g_1, \ldots, g_n)\). Note that if \((h_1, \ldots, h_n)\) is in the same \( B_n \) orbit of \((g_1, \ldots, g_n)\) (where each \( h_i \) is expressed in terms of the \( g_i \)) then necessarily
\[
h_i = g_{\tau(i)}^{f_i}
\]
where \( \tau \) is a permutation of \( (1, \ldots, n) \) and \( f_i \in G \). Furthermore, we have
\[
h_1 \cdots h_n = g_1 \cdots g_n.
\]
Both properties are preserved under a single mutation, as can be readily checked, and in general follow by induction on the number of mutations. We shall thus say that the \( B_n \) action on \((g_1, \ldots, g_n)\) is **transitive** if every tuple of the form \((g_{\tau(1)}^{f_1}, \ldots, g_{\tau(n)}^{f_n})\) is in the orbit of \((g_1, \ldots, g_n)\).

**Proposition 2.2.** Let \( G = \{g_1, \ldots, g_n | g_i^2 = 1, \quad i = 1 \ldots n\} \) (i.e. \( G \) is a free Coxeter group), then the \( B_n \) action on \((g_1, \ldots, g_n)\) is transitive.

**Proof.** Throughout the proof we shall assume that all elements of \( G \) are presented in terms of the generating set \( \{g_1, \ldots, g_n\} \).

To prove the claim, we shall show that, given a tuple \((g_{\tau(1)}^{f_1}, \ldots, g_{\tau(n)}^{f_n})\), where \( \tau \) is a permutation of \( (1, \ldots, n) \) and \( f_i \in G \) and furthermore \( g_{\tau(1)}^{f_1} \cdots g_{\tau(n)}^{f_n} = g_1 \cdots g_n \), we have that \((g_{\tau(1)}^{f_1}, \ldots, g_{\tau(n)}^{f_n})\) is in the same \( B_n \) orbit of \((g_1, \ldots, g_n)\).

Since \( G \) is a free Coxeter group, we can use the notion of length of words and its properties (see [10] chapter 5). We denote the length of a word \( u \), as usual, by \( l(u) \).

We shall prove that given a tuple \((g_{\tau(1)}^{f_1}, \ldots, g_{\tau(n)}^{f_n})\) there always exists a mutation (at least one such) that reduces the total length of words in \((g_{\tau(1)}^{f_1}, \ldots, g_{\tau(n)}^{f_n})\). This shall be proved by induction on the total length of the words in \((g_{\tau(1)}^{f_1}, \ldots, g_{\tau(n)}^{f_n})\). We assume that each word in the given tuple is reduced.

The base of the induction is simply the tuple \((g_1, \ldots, g_n)\) and the claim follows trivially. To prove the induction step, we will show that given the tuple
\[
(g_{\tau(1)}^{f_1}, \ldots, g_{\tau(n)}^{f_n})
\]
there exists a mutation for this tuple which reduces the total length of the words (and hence the claim will follow by the induction hypothesis).

In the following equations we shall use the fact that for every word \( u \) in a Coxeter group \( l(u) = l(u^{-1}) \). Furthermore, we shall use the following computation (recall that we denote \( g^h = hgh^{-1} \))
\[
(g^h)^f = fhgh^{-1}f^{-1} = g^{fh}
\] (2.8)
Now, applying a left mutation on the given tuple \((g^f_{T(1)}, \ldots, g^f_{T(n)})\) gives
\[
(g^f_{T(1)}, \ldots, g^f_{T(i-1)}, g^f_{T(i+1)}, g^f_{T(i)})^{(g^f_{T(i+1)})^{-1}} = (g^f_{T(1)}, \ldots, g^f_{T(i-1)}, g^f_{T(i+1)}, g^f_{T(i)})\]
(2.9)
And, similarly, applying a right mutation on the given tuple \((g^f_{T(1)}, \ldots, g^f_{T(n)})\) gives
\[
(g^f_{T(1)}, \ldots, g^f_{T(i-1)}, g^f_{T(i)}, g^f_{T(i+2)}, \ldots, g^f_{T(n)})^{g^f_{T(i)}} = (g^f_{T(1)}, \ldots, g^f_{T(i-1)}, g^f_{T(i)}, g^f_{T(i+2)}, \ldots, g^f_{T(n)})
\]
(2.10)
Note that the only new element after applying a left mutation in the \(i\)-th place on the given tuple is
\[
(g^f_{T(i)})^{(g^f_{T(i+1)})^{-1}} = (g^f_{T(i)})^{g^f_{T(i+1)}} = (g^f_{T(i)})^{f_i g^f_{T(i+1)} f_i^{-1}} = (g^f_{T(i)})^{f_{i+1} g^f_{T(i+1)} f_{i+1}^{-1}}
\]
(2.11)
Similarly, the only new element after applying a right mutation in the \(i\)-th place is
\[
(g^f_{T(i+1)})^{g^f_{T(i)}} = (g^f_{T(i)})^{g^f_{T(i+1)} f_i} = (g^f_{T(i)})^{f_i g^f_{T(i+1)} f_i^{-1}}
\]
(2.12)
The change of the total length of the words comprising the tuple in case of the left mutation is
\[
l((g^f_{T(i)})^{f_{i+1} g^f_{T(i+1)} f_i}) - l((g^f_{T(i)}))
\]
(2.13)
and in the case of a right mutation it is
\[
l((g^f_{T(i+1)})^{g^f_{T(i)} f_{i+1}}) - l((g^f_{T(i+1)}))
\]
(2.14)
We would like to show that either of these quantities is negative for some \(i\), hence it is enough to show that there exists at least one \(i\) for which either
\[
l(f_i^{-1} g^f_{T(i+1)}) < l(f_i)
\]
(2.15)
or
\[
l(g^f_{T(i)} f_{i+1}) < l(f_{i+1})
\]
(2.16)
Consider the following equation which holds by assumption
\[
g^f_{T(1)} \cdots g^f_{T(n)} = g_1 \cdots g_n
\]
or more explicitly
\[
f_1 g^f_{T(1)} f_1^{-1} f_2 g^f_{T(2)} f_2^{-1} \cdots f_n g^f_{T(n)} f_n^{-1} = g_1 \cdots g_n
\]
If none of the \(g^f_{T(i)}\) appearing on the left hand side of this equality is cancelled in order for the equation to hold then necessarily for all \(i\) we have that \(l(f_i) = 0\) (i.e. we are in the induction base). So we can assume that for some \(i\), \(g^f_{T(i)}\) is cancelled during the cancellation process. We can assume, without lose of generality, that the subword
which contains the letter that cancels $g_{r(i)}$ is on its right. Consider then the following subword appearing on the left hand side of the above equation

$$u = g_{r(i)} \cdot f_i^{-1} \cdot f_{i+1} \cdots f_{k-1}^{-1} \cdot f_k \cdot g_{r(k)} \cdot f_k^{-1} \cdot f_{k+1} \cdots f_{j-1}^{-1} \cdot f_j \cdot g_{r(j)} f_j^{-1}\)$$

in this subword we choose $k$ to be maximal such that $v = g_{r(i)} \cdot f_i^{-1} \cdot f_{i+1} \cdots f_{k-1}^{-1}$ is reduced. Note that the last letter of the reduced subword $v$ in $u$ must be the last letter of $f_i^{-1}$ for some $l$ as all the subwords of the form $f_l \cdot g_{r(l)} \cdot f_l^{-1}$ are reduced by assumption. Furthermore, we choose $j$ to be the minimal index such that $g_{r(j)}$ does not appear in $v^{-1}$, or we choose $j = n$ if all $g_{r(j)}$ appear in $v^{-1}$ for all $j > i$. In any case, as $\tau$ is a permutation of $(1 \ldots n)$, we must have that the last word of $v^{-1}$, which is $g_{r(i)}$, appears in $f_{j-1}^{-1}$ or in $f_j$ or in $f_n^{-1}$. Consider the following subword in $u$ (for the same index $k$ as above)

$$g_{r(k-1)} \cdot f_{k-1}^{-1} \cdot f_k \cdot g_{r(k)}$$

(2.17)

clearly, either of $f_{k-1}^{-1}$ or $f_k$ is not trivial (for otherwise $g_{r(k-1)}$ and $g_{r(k)}$ could not be cancelled). Now, first we address the case where both $g_{r(k-1)}$ and $g_{r(k)}$ are cancelled. Suppose first that $l(f_k) > l(f_{k-1}^{-1})$ ($l(f_k) \neq l(f_{k-1}^{-1})$ since $g_{r(k-1)}$ cannot cancel $g_{r(k)}$). Then necessarily $f_k = (g_{r(k-1)} \cdot f_{k-1}^{-1})^{-1} \cdot w$ for some reduced (possibly trivial) word $w$. For otherwise $g_{r(k-1)}$ would remain uncanned as opposed to the assumption. Similarly, if $l(f_k) < l(f_{k-1}^{-1})$ then $f_{k-1}^{-1} = y \cdot (f_k \cdot g_{r(k)})^{-1}$ for some reduced (possibly trivial) word $y$. In the first case we have that

$$l(f_{k-1} g_{r(k-1)} f_{k-1}^{-1} f_k) \leq l(f_k) - l(f_{k-1}^{-1}) - l(g_{r(k-1)}) + l(f_{k-1}) =
\leq l(f_k) - l(f_{k-1}) - 1 + l(f_{k-1}) < l(f_k)$$

Thus we have that inequality (2.16) holds for $i = k - 1$. Similarly, in the second case we have that inequality (2.15) holds for $i = k$.

Now, while $g_{r(k-1)}$ always cancels in the expression (2.17) according to our choice of indices, $g_{r(k)}$ may not cancel, however, we must have in this case that $f_k = v^{-1} \cdot w$ for some reduced (possibly trivial) word $w$. Hence this must be the first case considered in the above argument (i.e. that $f_k = (g_{r(k-1)} \cdot f_{k-1}^{-1})^{-1} \cdot w$ for some reduced (possibly trivial) word $w$) and so the claim follows.

\[\square\]

**Example 2.1.** In light of proposition 2.3 let us consider the case of $K_0(\mathbb{P}^4)$ which is yet to be determined (see [5]). Its Gram matrix is easily seen to be

$$
\begin{pmatrix}
1 & 5 & 15 & 35 & 70 \\
0 & 1 & 5 & 15 & 35 \\
0 & 0 & 1 & 5 & 15 \\
0 & 0 & 0 & 1 & 5 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

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Let us write $\text{Reflect}(K_0(\mathbb{P}^4))$ explicitly. It is generated by the following five matrices:

\[
\begin{align*}
    s_1 &= \begin{pmatrix}
        -1 & -5 & -15 & -35 & -70 \\
        0 & 1 & 0 & 0 & 0 \\
        0 & 0 & 1 & 0 & 0 \\
        0 & 0 & 0 & 1 & 0 \\
        0 & 0 & 0 & 0 & 1
    \end{pmatrix}, \\
    s_2 &= \begin{pmatrix}
        1 & 0 & 0 & 0 & 0 \\
        -5 & -1 & -5 & -15 & -35 \\
        0 & 0 & 1 & 0 & 0 \\
        0 & 0 & 0 & 1 & 0 \\
        0 & 0 & 0 & 0 & 1
    \end{pmatrix}, \\
    s_3 &= \begin{pmatrix}
        1 & 0 & 0 & 0 & 0 \\
        0 & 1 & 0 & 0 & 0 \\
        -15 & -5 & -1 & -5 & -15 \\
        0 & 0 & 0 & 1 & 0 \\
        0 & 0 & 0 & 0 & 1
    \end{pmatrix}, \\
    s_4 &= \begin{pmatrix}
        1 & 0 & 0 & 0 & 0 \\
        0 & 1 & 0 & 0 & 0 \\
        -35 & -15 & -5 & -1 & -5 \\
        0 & 0 & 0 & 0 & 1 \\
        0 & 0 & 0 & 0 & 1
    \end{pmatrix}, \\
    s_5 &= \begin{pmatrix}
        1 & 0 & 0 & 0 & 0 \\
        0 & 1 & 0 & 0 & 0 \\
        0 & 0 & 1 & 0 & 0 \\
        0 & 0 & 0 & 1 & 0 \\
        -70 & -35 & -15 & -5 & -1
    \end{pmatrix}.
\end{align*}
\]

According to proposition 2.2, the action of the braid group on semiorhonormal bases of $K_0(\mathbb{P}^4)$ is transitive if the group generated by \{s_1, s_2, s_3, s_4, s_5\} is a free Coxeter group. It is not hard to check that this group contains $\mathbb{F}_2$ (the free group with 2 generators) according to the Tits alternative [4], however it seems quite difficult to determine the relations in this group [3].

Finally let us shortly discuss the algorithmic aspect of the problem. Recall that an algorithmic problem $A_1$ is said to be (Turing-) Reducible to an algorithmic problem $A_2$ if, given an algorithm that solves $A_2$, the problem $A_1$ can also be solved. Furthermore $A_1$ is (Turing-)equivalent to $A_2$ if each of them is reducible to the other. Hence, proposition 2.1 immediately gives

**Corollary 2.1.** For a given Mukai lattice $M$, the Hurwitz problem for $\text{Reflect}(M)$ is equivalent to the Hurwitz problem for $M$.

This observation is useful especially on the negative side: it is sometimes easier to see that an algorithmic problem is undecidable rather than analysing it algebraically. In our specific case it might be easier to determine such undecidability results for the Hurwitz problem on $\text{Reflect}(M)$ rather than trying to disprove the transitivity of the action on the Mukai lattice.

---

\[2\] Stefan Kohl who helped with the analysis of this group determined that there are strong 'empirical' evidence that this group is indeed a free Coxeter group, but a proof (for example using the table-tennis lemma) seems hard.
3 Conclusions and further remarks

The bijective equivariance between Mukai lattices and group tuples of reflections proved in proposition 2.1 indicates that the main difficulty in analysing the braid group orbits lies in the analysis of the relations in those groups of reflections. These groups are matrix integer groups finitely generated by matrices of order 2. It seems that there exists no theory that could determine in general the relations in these kind of groups. Indeed, as noted above, there is no guarantee that these groups are even finitely presented (i.e. have a finite number of relations). As can be seen in example 2.1, even particular cases seem quite hard to analyse. Although we do not have a specific example, it is quite reasonable to assume that there exists a particular Mukai lattice for which the Hurwitz problem is undecidable (see for example the diagram on page 31 in [15] for the different undecidability results of finitely generated linear groups). Furthermore, it would also be quite reasonable to assume that the question of transitivity is not algorithmic in general.

According to remark 2.3, the conjugacy problem for a group of the form Reflect(M) is algorithmically reducible to the problem of determining whether two semiorthonormal bases are in the same Isom(M) orbit. Thus, the undecidability of the conjugacy problem for Reflect(M) would imply the undecidability for the Isom(M) orbit problem. This fact, combined with the different undecidability results for finitely generated linear groups in [14] and [15] might imply that it would be even easier to construct such an undecidability result in this case.

A natural extension of proposition 2.2 (apart from the obvious extension to general Coxeter groups) might be to prove the same claim for biautomatic groups (a free Coxeter group is an example of a biautomatic group), intuitively these groups have the property that relations in them have a computationally simple description which allows for the solution of different decision problems defined on them (see section 7 in [15]). However, as noted above, it is not reasonable to assume that these kind of results could be strengthened to provide a sufficient condition for the transitivity of the braid action.

Finally, although our work implies that the analysis of braid group actions on (semiorthonormal bases of) general Mukai lattices is not algorithmic in general, it is not clear whether the same holds for Mukai lattices induced from algebraic varieties. It would be interesting to understand exactly which Mukai lattices can or cannot be induced from algebraic varieties. Although this problem also seems quite difficult (if not altogether impossible in general), such an understanding could imply at least which kind of Mukai lattices are worth focusing our attention on.

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