Abstract

We investigate the origin of order in the low-lying spectra of many-body systems with random two-body interactions. Our study based both on analytical as well as on numerical arguments shows that except for the most $J$-stretched states, the ground states in the higher $J$-sectors are more orderly and develop larger energy gaps than the ones in the $J = 0$-sector. Due to different characteristic energy scales in different $J$-sectors the $J = 0$ ground states may predominate only when all the states are taken together.

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I. INTRODUCTION

Nature of the mechanism generating order out of randomness constitutes one of the most fundamental issues of the contemporary physics. Theories based on various versions of ensembles of the random matrices provide one possible theoretical frame for studying such effects. The Gaussian orthogonal ensemble (GOE) constitutes the most common reference. The related concepts originate \cite{1,2} from nuclear physics and prove very useful in the area of strongly interacting Fermi systems or in quantum chaos \cite{3,4}. At present they find even broader applications in such diverse fields like the brain research \cite{5}, econophysics \cite{6–8} and most recently in the "Real-World" networks or graphs \cite{9}. Utility of the standard random matrix theory (RMT) results form the fact that a potential agreement reflects the generic properties of a system and thus in many cases it provides an appropriate null hypothesis. From this perspective the deviations are even more interesting as they can be used to quantify some properties which are nonrandom and thus system specific.

In this context the recently identified \cite{10,11} preponderance of the $J = 0$ ground states in strongly interacting Fermi systems, such as atomic nuclei, arising from random two-body interactions seems to indicate the effect reflecting a 'sparser connectivity' than just pure random. Several closely related issues have also been addressed in the context of mesoscopic \cite{12} and randomly interacting many-spin systems \cite{13}. One purpose of the present investigation is to identify the origin of the related physically relevant deviations from standard RMT and to quantify their character. Since it was nuclear physics which gave birth to RMT we believe that the present example, even though addressed in the nuclear context, may also stimulate much broader activity and understanding of similar effects in other areas.
II. STATISTICS OF MATRIX ELEMENTS

Our theoretical framework is thus analogous to this of ref. [10]. Then schematically, indicating nevertheless all the relevant ingredients, the interaction matrix elements $v_{\alpha,\alpha'}^J$ of good total angular momentum $J$ in the shell-model basis $|\alpha\rangle$ can be expressed as follows [14]:

$$v_{\alpha,\alpha'}^J = \sum_J \sum_{ii'} c_{J\alpha\alpha'}^{J'ii'} g_{ii'}^{J'}. \tag{1}$$

The summation runs over all combinations of the two-particle states $|i\rangle$ coupled to the angular momentum $J'$ and connected by the two-body interaction $g$. $g_{ii'}^{J'}$ denote the radial parts of the corresponding two-body matrix elements while $c_{J\alpha\alpha'}^{J'ii'}$ globally represent elements of the angular momentum recoupling geometry. Structures analogous to eq. (1) can be identified in various other areas. The quantum open systems [16] or the neural nets [13] provide immediate examples.

In statistical ensembles of matrices the crucial factor determining the structure of eigen-spectrum is the probability distribution $P_V(v)$ of matrix elements [17]. Especially relevant are the tails of such distributions since they prescribe the probability of appearance of the large matrix elements. From the point of view of the mechanism producing the energy gaps they are most effective in generating a local reduction of dimensionality responsible for such effects. In principle, the probability distribution of the shell model matrix elements is prescribed by their general structure expressed by the eq. (1), provided the probability distributions of both $g_{ii'}^{J'}$ and $c_{J\alpha\alpha'}^{J'ii'}$ are known. In general terms this structure can be considered to be of the form

$$V = \sum_{i=1}^N V_i \tag{2}$$

and each $V_i$ to be a product of another two variables denoted as $C_i$ and $G_i$. By making use of the convolution theorem [18] the probability distribution $P_V(v)$ that $V$ assumes a value equal to $v$ can be expressed as:

$$P_V(v) = F^{-1}[F(P_{V_1}(v_1)) \cdot F(P_{V_2}(v_2)) \cdot \ldots \cdot F(P_{V_N}(v_N))], \tag{3}$$
where $F$ denotes a Fourier transform, $F^{-1}$ its inverse and $P_{v_i}(v_i)$ the probability distributions of individual terms. Taking in addition into account the fact that

$$P_{v_i}(v_i) = \int dg_i P_{G_i}(g_i) P_{C_i}(\frac{v_i}{g_i}) \frac{1}{|g_i|}$$

(4)

one can explicitly derive the form of $P_{v}(v)$ in several cases. Assuming for instance that all the above constituents are identically Gaussian distributed (then, according to eq. (4), $P_{v_i}(v_i) = K_0(|v_i|)/\pi$ and thus $F(P_{v_i}(v_i)) = 1/\sqrt{1 + \omega^2}$) one arrives at

$$P_{v}(v) = \frac{|v|^{(N-1)/2}K_{(N-1)/2}(|v|)}{2^{(N-1)/2}\Gamma(N/2)\sqrt{\pi}},$$

(5)

where $K$ stands for the modified Bessel function. Asymptotically, for large $v$, this leads to

$$P_{v}(v) \sim \exp(-|v|) \ |v|^{N/2-1}.$$  

(6)

For such a global estimate the identical Gaussian distribution of $g_{ii'}^\alpha$ is consistent both with the Two-Body Random Ensemble (TBRE) [19] and with the Random Quasiparticle Ensemble (RQE) [10]. The only anticipated difference originates from the fact that in the second case the variance of the distribution drops down with $J'$ like the inverse of $2J' + 1$ which is expected to result in a smaller effective $N$ as compared to TBRE. By contrast, in both versions of the above random ensembles the geometry expressed by $c_{J\alpha\alpha'}^{J_{ii'}}$ enters explicitly. However, the complicated quasi-random coupling of individual spins is believed [20] to result in the so-called geometric chaoticity [3]. For the extreme values of $J$ the underlying selection rules may however impose severe constraints in achieving such a limit. Below we therefore explicitly verify its range of applicability.

III. THE MODEL AND RESULTS

The model to be quantitatively explored here consists, similarly as in ref. [10], of 6 identical particles (all single-particle energies are set to zero) operating in the sd shell. Fig. 1 shows distributions of the corresponding geometrical factors $c_{J\alpha\alpha'}^{J_{ii'}}$ for $\alpha \neq \alpha'$ and for several relevant values of $J$ and $J'$. 

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As one can see, the Gaussian may be considered quite a reasonable representation of
the distribution of such factors for all combinations of $J$ and $J'$ shown, with one exception,
for those which involve $J = 0$. In this later case the distribution of $c_{J'ii}^{\alpha\alpha'}$ resembles more a
uniform distribution over a finite interval located symmetrically with respect to zero. One
principal reason for this fact is that the $6j$ symbols which enter $c_{J'ii}^{\alpha\alpha'}$ are here more selective.
These empirical facts justify well the estimates of $P_V(v)$ based on eq. (5) for $J \neq 0$ and not so
well for $J = 0$. More appropriate in this particular case is to assume a uniform distribution
of $c_{J'ii}^{\alpha\alpha'}$ over an interval confined by say $-c_0$ and $c_0$, i.e., $P_{C_i}(c_i) = 1/2c_0$, retaining $P_{G_i}(g_i)$
in its original Gaussian form of course. By making use of eqs. (3) and (4) one then obtains
\[
P_V(v) = \frac{1}{\pi} \int_0^{\infty} \left[ \sqrt{\frac{\pi}{2}} \text{erf}(c_0\omega/\sqrt{2}) \right]^N \cos(\omega v) d\omega
\]
which for large $v$ behaves like
\[
P_V(v) \sim \exp(-|v|^2).
\]

An explicit calculation of the distribution of the shell model off-diagonal matrix elements
for the various $J$-values based on the present model with two-body matrix elements drawn
from RQE (TBRE results in similar relations among different $J$-sectors though the distribu-
tions are somewhat broader as compared to RQE) confirms the above analytical estimates
as is illustrated in Fig. 2.

Indeed, such a distribution in the $J = 0$ sector resembles more a Gaussian and the large
$v$ tails of this distribution drop down faster as compared to the $J \neq 0$-sectors where this
asymptotics is exponential (eq. (5)). At the same time the $J \neq 0$ sectors are dominated by
very small matrix elements to a larger degree than $J = 0$. The probability of appearance of
a large off-diagonal matrix element which in magnitude overwhelms the remaining ones is
thus greater for $J \neq 0$ than for $J = 0$. Such an effective reduction of the rank in the former
case is expected to result in a stronger tendency to localization as compared to GOE [17,21].
The corresponding characteristics can be quantified in terms of the information entropy
\[
K^J_i = -\sum_{\alpha=1}^{M_J} |a_{i,\alpha}^J|^2 \ln |a_{i,\alpha}^J|^2
\]
of an eigenstate labelled by $l$ from the $J$-sector. The coefficients $a_{l,\alpha}^J$ denote the eigenvector components in the basis $|\alpha>$. Such a mean field basis offers an appropriate reference for the present purpose. Since the definition of $K^J_l$ involves the total number of states $M_J$ which differ for different $J$’s, before relating the result to the GOE we normalise $K^J_l$ to the GOE limit of this quantity

$$K^J_{\text{GOE}} = \psi(M_J/2 + 1) - \psi(3/2),$$

where $\psi$ is the digamma function. Within our model the so-calculated and RQE ensemble averaged quantity for all the states versus their corresponding energies $E^J_l$ is illustrated in Fig. 3. As anticipated, it is not $J = 0$ whose lowest eigenstate comes out most localised, i.e., most regular. The lowest states for several higher $J$ values (like 2 and especially 4) deviate much more from GOE. This thus indicates more favorable conditions for the emergence of energy gaps for larger $J$ than for $J = 0$.

Fig. 3 provides one more information which turns out helpful to properly interpret the results. The $J = 0$ states are spread over the broadest energy interval even though the number of states ($M_0 = 14$) is here significantly smaller than for several larger $J$ values ($M_1 = 19, M_2 = 33, M_3 = 29, M_4 = 26$). As a result, the average level spacing is a factor of few larger for $J = 0$ than for the remaining ones.

In Fig. 4 (dashed line) we therefore show distributions of the ground state ($E^J_1$) gaps

$$s^J = (E^J_2 - E^J_1)/D^J,$$

where similarly as in ref. [10],

$$D^J = <E^J_3 - E^J_2 >,$$

though here for each $J$ individually.

As it is clearly seen the $J = 0$-sector does not significantly distinguish from the remaining ones. In view of our investigations presented above one would however expect a reduced probability for occurance of the large ground state energy gaps in this particular sector. As
the solid lines in Fig. 4 indicate such an effect does indeed take place when $D^J$ in eq. (11) is replaced by

$$D^J = \langle E^J_{M_J} - E^J_2 \rangle / (M_J - 2).$$

(13)

In fact, it seems more appropriate and more consistent with the above global considerations to relate the ground state energy gap just to the average global level spacing among the remaining states, characteristic for a given $J$, as expressed by eq. (13).

Finally one may ask a question why this tendency does not extend to the highest $J$-values. In this connection one has to remember that the off-diagonal matrix elements is not the only relevant element. These are the diagonal matrix elements which constitute the driving term. Irrespective of the value of $J$ their distribution is always Gaussian-like. This can be observed numerically and is consistent with arguments formulated in terms of eqs. (1 - 6) since the geometrical factors $c^J_{\alpha\alpha'}$ entering the diagonal matrix elements are always nonnegative. As it is shown in Fig. 5 (dashed lines), increasing however $J$ beyond 4 results in a significant reduction of the variance of $P_V(v)$ for the diagonal matrix elements and consequently a larger fraction of the off-diagonal matrix elements becomes effective in mixing the basis states. In addition, due to a smaller number of terms entering the eq. (1), for the stretched high-$J$ states the effect of geometric chaoticity is no longer effective and in this respect the conditions become similar to those for $J = 0$. As a result, the distribution of off-diagonal matrix elements converts back towards more Gaussian-like shaped, i.e. $N_{eff}$ becomes larger (for $J = 6$ not shown in Fig. 2 $N_{eff} = 2.67$). Superposing the above two effects one thus obtains even smaller gaps and even more delocalized states at the edges of the spectra as compared to $J = 0$. In fact the spectral density (solid lines in Fig. (5)) becomes even somewhat closer to semicircular in this case.

IV. SUMMARY

The present investigation based both on theoretical as well as on numerical arguments clearly shows that the many-body problems described in terms of various variants of the
two-body random ensembles (like RQE or TBRE) develop quantitatively well identified deviations from the GOE. These deviations can be linked to differences in the distribution of matrix elements and quantified in terms of the localisation or of energy gaps in eigenspectra. Contrary to the common belief they point to the intermediate total angular momenta as those $J$-sectors whose ground states are ordered most. From this perspective, a predominance of the $J = 0$ ground states [10] can be viewed as a result of mixing states with different characteristic energy scales from different $J$-sectors. It seems also appropriate to notice here that the arguments formulated in terms of eqs. (1-6) provide a more adequate approach towards understanding the distribution of matrix elements in realistic nuclear shell-model calculations than the ones based on multipole expansion [3]. Finally, similar parallels between the distribution of matrix elements and the structure of eigenspectra relative to the GOE can be set in the recent econophysics [24] applications.

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FIGURE CAPTIONS

Fig. 1 The normalised distribution of geometrical factors $c^{J_{\nu\alpha'}_{\nu\alpha'}}_{J_{\alpha\beta}J_{\alpha'}}$ entering the off-diagonal matrix elements (eq. (4)) for the model of 6 particles in the $sd$-shell.

Fig. 2 The probability distributions of nonzero many-body off-diagonal matrix elements in different $J$-sectors drawn from one thousand of RQE samples of two-body matrix elements. The energy scale is set by $\bar{v}$, where $w_{J'} = \bar{v}^2/(2J' + 1)$ and $w_{J'}$ determines the RQE mean square variance. These distributions are fitted (solid lines) in terms of eq. (5) with $N$ treated as a fitting parameter. The corresponding best $N$’s ($N_{eff}$) for each $J$ are listed. By increasing $N$ the distribution prescribed by eq. (5) quickly approaches (as a consequence of the central limit theorem) the Gaussian distribution. In this way the $J = 0$ distribution is demonstrated to be much closer to the Gaussian than the remaining ones whose asymptotic behaviour is consistent with a slower, exponential fall-off.

Fig. 3 The information entropy normalised to its GOE limit ($K_{Jl}/K_{J_{\text{GOE}}}$) for all the states $l$ from various $J$-sectors (all positive parity) versus energies ($E_{Jl}$) of those states. All the quantities are ensemble averaged. The energy units are the same as in Fig. 2.

Fig. 4 Distributions of ground state energy gaps $s^J$ as defined by the eq. (11) for successive $J$’s. The dashed line uses $D^J$ defined by eq. (12) while the solid line the one defined by eq. (13).

Fig. 5 The probability distributions of the diagonal matrix elements in different $J$-sectors drawn from one thousand of RQE samples of two-body matrix elements (dashed lines) and the corresponding spectral densities (solid lines). The energy units are the same as in Fig. 2.
$P_V(v)$

- $J=0$, $N_{\text{eff}}=8.08$
- $J=1$, $N_{\text{eff}}=2.05$
- $J=2$, $N_{\text{eff}}=1.85$
- $J=3$, $N_{\text{eff}}=1.98$
- $J=4$, $N_{\text{eff}}=2.14$
- $J=5$, $N_{\text{eff}}=2.47$

$v \rightarrow \bar{v}$
