CLIFFORD ALGEBRA, GEOMETRY AND PHYSICS

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ABSTRACT

The geometric calculus based on Clifford algebra is a very useful tool for geometry and physics. It describes a geometric structure which is much richer than the ordinary geometry of spacetime. A Clifford manifold (C-space) consists not only of points, but also of 1-loops, 2-loops, etc. They are associated with multivectors which are the wedge product of the basis vectors, the generators of Clifford algebra. Within C-space we can perform the so called polydimensional rotations which reshuffle the multivectors, e.g., a bivector into a vector, etc. A consequence of such a polydimensional rotation is that the signature can change: it is relative to a chosen set of basis vectors. Another important consequence is that the well known unconstrained Stueckelberg theory is embedded within the constrained theory based on C-space. The essence of the Stueckelberg theory is the existence of an evolution parameter which is invariant under the Lorentz transformations. The latter parameter is interpreted as being the true time - associated with our perception of the passage of time.

1 Introduction

In the usual theory of relativity there is no evolution. Worldlines are fixed, everything is frozen once for all in a 4-dimensional “block universe” \( V_4 \). This is in contradiction with our subjective experience of the passage of time. It is in contradiction with what we actually observe.

Therefore we always introduce into the theory of relativity more or less explicitly an extra postulate: that a 3-dimensional hypersurface of simultaneity moves in spacetime. We are talking about point particles, strings, etc.. Those objects exist

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in 3-dimensional space $V_3$. From the point of view of $V_4$ there are worldlines, worldsheets, etc. Relativity does not contain point particles that *evolve* in $V_4$. Something is missing in the ordinary relativity. And yet, we all: (i) assume the validity of the theory of relativity, and (ii) talk about point particles which —when moving— describe worldlines in spacetime.

The above two positions are incompatible. In the following I am going to point out how we can have both, (i) and (ii), by suitably modifying the theory of relativity. The first modification is the well known Stueckelberg theory \[1\], based on the unconstrained, Lorentz invariant action. Such a theory has been considered by a number of authors \[2\]–\[4\] and it actually describes *evolution* of a point particle (“event”) in spacetime. The essence of the Stueckelberg theory is the introduction of a *Lorentz invariant parameter* $\tau$ along which evolution (“relativistic dynamics”) takes place. This is the true *time*, whilst $X^0 \equiv t$ is just one of the spacetime coordinates, called “coordinate time”.

In search of a deeper understanding and description of geometry it has been found that Clifford algebra is such a tool. It describes a geometric structure which is much richer than the ordinary geometry of spacetime. Clifford space (shortly $C$-space) consists not only of points, but also of lines or 1-loops, 2-loops, etc. The later geometric objects are associated with multivectors. Multivectors of different grades can be superposed into the geometric objects —the so called *polyvectors* which are generic *Clifford numbers* (called also *Clifford aggregates*). Following Pezzaglia we assume that physical quantities are polyvectors and that the true space in which physics stakes place is $C$-space \[5\]–\[7\].

We formulate the action in $C$-space, which is a straightforward although not trivial generalization of the minimal length (point particle) action in ordinary spacetime. Such *constrained* action contains as a particular case the well known *unconstrained* Stueckelberg action which encompasses an invariant *evolution parameter*. From the point of view of $C$-space, the above evolution parameter is given by 4-vector part of the polyvector describing position of the “particle”\(^3\) in $C$-space.

The theory of relativity is thus shifted from the ordinary spacetime into the $C$-space. Everything that we know about relativity is now true in $C$-space: the constrained minimal length action, invariance under rotations (Lorentz transformations) in $C$-space, “block universe”, etc.. But in spacetime, a subspace of $C$-space, particles (and also extended objects) are actually *moving* as suggested by the unconstrained Stueckelberg action—which is just a reduced $C$-space action.

\(^3\)From the point of view of spacetime, of course, this is not *particle*, but an aggregate of $r$-loops, that is a polydimensional extended object (see \[5\]–\[7\]).
2 Relativistic Point Particle and Evolution

We will now briefly review the Stueckelberg theory. Let us start from the following action:

\[ I = \frac{1}{2} \int d\tau \left( \frac{\dot{x}^{\mu} \dot{x}_{\mu}}{\Lambda} + \Lambda \kappa^2 \right) \]  

(1)

where \( \kappa \) is a constant. Let us consider two distinct procedures:

a) In the standard procedure \( \Lambda \) is taken as a Lagrange multiplier whose “equations of motion” give \( \Lambda^2 = \dot{x}^{\mu} \dot{x}_{\mu}/\kappa^2 \) which is equivalent to the constraint \( p^{\mu} p_{\mu} - \kappa^2 = 0 \), where \( p_{\mu} = \partial L / \partial \dot{x}^{\mu} = \dot{x}_{\mu}/\Lambda \) is the canonical momentum. The action (1) is then equivalent to the minimal length action \( I = m \int d\tau (\dot{x}^{\mu} \dot{x}_{\mu})^{1/2} \). Fixing \( \Lambda \) in (1) means fixing a “gauge”, i.e., a choice of parametrization.

b) In the non standard procedure \( \Lambda \) in (1) is taken to be a constant with a physical meaning. Here \( \Lambda \) has nothing to do with choice of parametrization. Then (1) is an unconstrained action and all \( x^{\mu} \) are independent variables. They satisfy the following equations of motion: \( (d/d\tau) \dot{x}^{\mu}/\Lambda = 0 \) where all \( p_{\mu} = \dot{x}^{\mu}/\Lambda \) are constants of motion, and so it is the square \( p^{\mu} p_{\mu} = M^2 \).

A particle’s trajectory is given by \( x^{\mu}(\tau) \). Here \( x^0 \) is one of the coordinates, called coordinate time\(^4\), whilst \( \tau \) is the evolution parameter or historical time. The variables \( x^i(\tau), \ i = 1, 2, 3 \) describe the usual spatial motion of the particle, \( \dot{x}^i(\tau) \equiv dx^i/d\tau \) being the spatial velocity. The variable \( x^0(\tau) \) describes the progression of particle’s coordinate \( x^0 \) with increasing evolution parameter \( \tau \); \( \dot{x}^0(\tau) \) is the speed of the coordinate time with respect to the evolution parameter \( \tau \). The latter parameter we interpret as being related to the time perceived by consciousness when experiencing the passage of time. A given value of \( \tau \) denotes “now”, whilst \( x^0(\tau) \) (together with \( x^i(\tau) \)) denotes position in spacetime. The Stueckelberg theory as interpreted in \([\text{4, 6}]\) thus describes progression of “now”, the concept which is not present in the ordinary theory of relativity. This is even more transparent in the quantized theory.

From (1) it is straightforward to derive the Hamiltonian

\[ H = p_{\mu} \dot{x}^{\mu} - L = \frac{\Lambda}{2} (p^{\mu} p_{\mu} - \kappa^2). \]  

(2)

Since \( \kappa \) is an arbitrary constant, it can be taken \( \kappa = 0 \) (as it is in the usual formulation of the Stueckelberg theory).

In the quantized theory \( x^{\mu}, p_{\mu} \) become operators, satisfying \([x^{\mu}, p_{\nu}] = i\delta^{\mu}_{\nu} \) (\( \hbar = c = 1 \)). In the representation in which \( x^{\mu} \) are diagonal, momenta are \( p_{\mu} = -i\partial_{x^{\mu}} \). A state can be represented by a wave function \( \psi(\tau, x^{\mu}) \) satisfying the Schrödinger equation \( i\partial \psi / \partial \tau = H \psi \).

The wave function is normalized in spacetime according to \( \int \psi^{*} \psi d^4x = 1 \). The latter relation holds at any value of \( \tau \). Therefore the evolution operator \( U \) which

\[ ^4 \text{It is called “clock time” by Franck \([\text{8}]\).} \]
sends $\psi(\tau)$ into $\psi(\tau') = U\psi(\tau)$ is unitary. In other words, because of the above normalization unitarity is satisfied even if wave functions are localized in the coordinate time $x^0$.

A generic wave function—a wave packet localized in spacetime—is a superposition of the wave functions with definite 4-momentum:

$$
\psi(\tau, x) = \int d^4p c(p) \exp \left[ ip_{\mu} x^\mu - i \frac{\Lambda}{2} (p^2 - \kappa^2) \tau \right].
$$

The function $c(p)$ determines the profile of the wave packet. Here both $p_\mu$ and its square $p^\mu p_\mu = M^2$ are indefinite.

In general, a state $\psi(\tau, x)$ has indefinite mass; the wave packet is localized in spacetime. The region of localization depends on the evolution parameter $\tau$. The centre of the wave packet describes a classical world line (see figures in [4, 7]).

It is now natural to interpret the wave function localized in spacetime as being related to our perception of “now” [4, 7]. When the wave packet evolves with $\tau$, its region of localization (center of the wave packet) moves in spacetime along a time-like direction. This is then a physical description of the “passage of time”.

At this point let me mention that the wave function for a localized point particle (“event”) in spacetime is just a first step. Instead of localized point particles we can consider localized extended objects (strings, membranes) in spacetime whose dynamics is given in terms of wave functionals satisfying the unconstrained Schrödinger functional equation [9].

An example is a string extended along a time-like direction. Such time-like strings, if charged, yield the correct electromagnetic interaction with the Coulomb law [5].

Another example is a 4-dimensional membrane $V_4$ in an $N$-dimensional embedding space. According the “brane world” scenario such a membrane $V_4$ could be our world [6]. Quantum mechanically motion of $V_4$ is described by a wave functional which can be sharply localized within a certain 4-region $\Omega$ on $V_4$. Such region could correspond to “here” and “now”. With the passage of $\tau$ the wave functional evolves so that the region of sharp localization $\Omega$ changes and so also “here” and “now” change. In short, I assume the interpretation that such localized wave functional provides a physical description of our perception of “here” and “now”. Much more on this topics is to be found in a recent book [7] and in [10].

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5Charged point particles localized in spacetime (charged “events”) do not lead to the Coulomb law.

6In [11] it was shown that self-intersections of $V_4$ (or the intersections of $V_4$ with other branes) give rise to localized matter on $V_4$. 
3 Geometric Calculus Based on Clifford Algebra

I am going to provide a brief, simplified, introduction into the calculus with vectors and their generalizations. Geometrically, a vector is an oriented line element. Mathematically, it can be elegantly described as a Clifford number [12].

How to multiply vectors? There are two possibilities:

1. *The inner product*

   \[ a \cdot b = b \cdot a \]  

   of vectors \(a\) and \(b\). The quantity \(a \cdot b\) is a scalar.

2. *The outer product*

   \[ a \wedge b = -b \wedge a \]  

   which is an oriented element of a plane.

The products 1 and 2 can be considered as the symmetric and the anti symmetric parts of the Clifford product, called also geometric product

\[ ab = a \cdot b + a \wedge b \]  

where

\[ a \cdot b \equiv \frac{1}{2} (ab + ba), \quad a \wedge b \equiv \frac{1}{2} (ab - ba). \]  

This suggests a generalization to trivectors, quadivectors, etc. It is convenient to introduce the name r-vector and call \(r\) its degree: \(A_r = a_1 \wedge a_2 \wedge ... \wedge a_r\). Another name for a generic r-vector is multivector. The highest possible multivector in \(V_n\) is \(n\)-vector, since \((n + 1)\)-vector is identically zero.

Let \(e_1, e_2, ..., e_n\) be linearly independent vectors, and \(\alpha, \alpha^i, \alpha^{i_1i_2}, ...\) scalar coefficients. A generic Clifford number can then be written as

\[ A = \alpha + \alpha^\mu e_\mu + \frac{1}{2!} \alpha^{\mu_1\mu_2} e_{\mu_1} \wedge e_{\mu_2} + ... + \frac{1}{n!} \alpha^{\mu_1...\mu_n} e_{\mu_1} \wedge ... \wedge e_{\mu_n}. \]  

Since it is a superposition of multivectors of all possible grades it will be called polyvector [8]. Another name, also often used in the literature, is Clifford aggregate. These mathematical objects have far reaching geometrical and physical implications that will be discussed and explored to some extent in the rest of the paper.

In general \(e_\mu\) in eq.(8) are arbitrary so that their inner products form the metric tensor of arbitrary signature. In particular \(e_\mu\) can be four linearly independent vectors \(e_\mu = \gamma_\mu, \mu = 0, 1, 2, 3\), satisfying \(\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu}\), generating the Clifford algebra of spacetime, called the Dirac algebra. By using the relations \(\gamma_{\mu\nu\rho\sigma} = \gamma_5 \epsilon_{\mu\nu\rho\sigma}\) and

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7 A more elaborate discussion is in [7].
8 Following a suggestion by Pezzaglia I call a generic Clifford number *polyvector* and reserve the name *multivector* for an r-vector, since the latter name is already widely used for the corresponding object in the calculus of differential forms.
\[ \gamma_{\mu\nu\rho} = \gamma_{\mu\rho\sigma} \gamma_{\nu}^\sigma, \]

where \( \gamma_{\mu\nu...} \equiv \gamma_{\mu} \wedge \gamma_{\nu} \wedge ... \) we can rewrite \( A \) as a superposition of a scalar, vector, bivector, pseudovector and pseudoscalar:

\[ A = S + V^\mu \gamma_{\mu} + T^{\mu\nu} \gamma_{\mu\nu} + C^\mu \gamma_5 \gamma_{\mu} + P \gamma_5 \]

(9)

where \( S \equiv \alpha, \ V^\mu \equiv \alpha^\mu, \ T^{\mu\nu} \equiv (1/2) \alpha^{\mu\nu}, \ C_\sigma \equiv (1/3!) \alpha^{\mu\nu\rho} \epsilon_{\mu\nu\rho\sigma} \) and \( P \equiv (1/4!) \alpha^{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma}. \)

**Relativity of signature.** In eq.(9) we assumed the Minkowski signature of the metric tensor. We are now going to find out that within Clifford algebra the signature is a matter of which amongst the available Clifford numbers we choose as the basis vectors (i.e., as the generators of Clifford algebra).

Let us assume that the basis vectors \( e_\mu, \mu = 0, 1, 2, 3, \) satisfy

\[ e_\mu \cdot e_\nu \equiv \frac{1}{2} (e_\mu e_\nu + e_\nu e_\mu) = \delta_{\mu\nu} \]

(10)

where \( \delta_{\mu\nu} \) is the Euclidean metric.

Let us consider the set of four Clifford numbers \( (e_0, e_i e_0), \ i = 1, 2, 3 \) and denote them as

\[ e_0 \equiv \gamma_0, \quad e_i e_0 \equiv \gamma_i. \]

(11)

The Clifford numbers \( \gamma_\mu, \mu = 0, 1, 2, 3 \) satisfy

\[ \gamma_\mu \cdot \gamma_\nu = \frac{1}{2} (\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = \eta_{\mu\nu} \]

(12)

where \( \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \) is the Minkowski tensor. We see that the \( \gamma_\mu \) behave as basis vectors in a 4-dimensional space \( V_{1,3} \) with signature \((+---)\). We can form a Clifford aggregate \( \alpha = \alpha^\mu \gamma_\mu \) which has the properties of a vector in \( V_{1,3} \). From the point of view of the space \( V_4 \) the same object \( \alpha \) is a linear combination of a vector and bivector: \( \alpha = \alpha^0 e_0 + \alpha^i e_i e_0. \)

We may use \( \gamma_\mu \) as generators of the Clifford algebra \( C_{1,3} \) defined over the pseudo-Euclidean space \( V_{1,3} \). The basis elements of \( C_{1,3} \) are \( \gamma_J = (1, \gamma_\mu, \gamma_{\mu\nu}, \gamma_{\mu\nu\alpha}, \gamma_{\mu\nu\alpha\beta}), \) with \( \mu < \nu < \alpha < \beta. \) A generic Clifford aggregate in \( C_{1,3} \) is given by

\[ B = b^J \gamma_J = b^\mu \gamma_\mu + b^{\mu\nu} \gamma_\mu \gamma_\nu + b^{\mu\nu\alpha} \gamma_\mu \gamma_\nu \gamma_\alpha + b^{\mu\nu\alpha\beta} \gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta. \]

(13)

With suitable choice of the coefficients \( b^J = (b^\mu, b^{\mu\nu}, b^{\mu\nu\alpha}, b^{\mu\nu\alpha\beta}) \) we have that \( B \) of eq.(13) is equal to \( A \) of eq.(8). Thus the same number \( A \) can be described either within \( C_4 \) or within \( C_{1,3} \). The expansions (13) and (8) exhaust all possible numbers of the Clifford algebras \( C_{1,3} \) and \( C_4 \). The algebra \( C_{1,3} \) is isomorphic to the algebra \( C_4 \) and actually they are just two different representations of the same set of Clifford numbers (called also polyvectors or Clifford aggregates).
4 Extending Relativity from Spacetime to $C$-space

So far it has been assumed that the arena in which physics takes place is spacetime. The nice properties of Clifford algebra suggest to extend the arena to a larger manifold, called Clifford space or $C$-space, whose points are described by coordinate polyvectors

$$X = \frac{1}{r!} \sum_{r=0}^{n} X^{\mu_1...\mu_r} \gamma_{\mu_1} \wedge ... \wedge \gamma_{\mu_r} \equiv X^A E_A.$$  \hspace{1cm} (14)

Here $X^A$ are coordinates, and $E_A = (1, \gamma_\mu, \gamma_\mu \wedge \gamma_\nu, ...)$ basis vectors of $C$-space.

Points $X$ of $C$-space embrace, from the spacetime perspective, not only the usual points, but also 1-loops, 2-loops, etc. Thus $C$-space is a polydimensional continuum [5] in which the extended object of different dimensionalities coexist on the same footing, and can be transformed into each other by polydimensional rotations which are a generalization of Lorentz transformations [6].

The line element in $C$-space is given by the scalar product of an infinitesimal polyvector $dX = dX^A E_A$ and its reverse $dX^\dagger$:

$$|dX|^2 \equiv dX^\dagger * dX = dX^A dX^B G_{AB} = dX^A dX_A$$ \hspace{1cm} (15)

where $G_{AB} = E_A^\dagger * E_B$ is the metric of $C$-space. The scalar product of two polyvectors $A$ and $B$ is defined as the scalar part of the Clifford product $AB$, i.e., $A * B = \langle AB \rangle_0$. The symbol $A^\dagger$ denotes the reverse of $A$, that is the polyvector in which the order of all products of vectors in a decomposition of a polyvector $A$ is reverse (e.g., $(\gamma_1 \gamma_2 \gamma_3)^\dagger = \gamma_1 \gamma_2 \gamma_3$).

The reparametrization invariant action for a point particle in $C$-space is

$$I[X^A] = \kappa \int d\tau (\dot{X}^A \dot{X}_A)^{1/2}$$ \hspace{1cm} (16)

where $\kappa$ is a constant. Here $\dot{X}^A \equiv dX^A / d\tau$, where $\tau$ is an arbitrary parameter. The canonical momenta are $p_A = \kappa \dot{x}_A / (\dot{x}^B \dot{x}_B)^{1/2}$ and they satisfy the constraint $p^A p_A = \kappa^2$.

Let us assume that spacetime dimension is $n = 4$. Then the velocity polyvector is

$$\dot{X} \equiv \dot{X}^A E_A = \dot{\sigma} \mathbf{1} + \dot{x}^\mu \gamma_\mu + \frac{1}{2} \dddot{x}^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + \dddot{x}^\mu \gamma_5 \gamma_\mu + \dot{s} \gamma_5.$$ \hspace{1cm} (17)

From the action (16) it follows that $\dot{X}$ is constant when $C$-space is flat.

In particular, when the initial conditions happen to be such that $\dot{\sigma} = 0$, $\dddot{x}^{\mu\nu} = 0$, $\dddot{x}^\mu = 0$, we have $\dot{X} = \dot{x}^\mu \dot{s} \gamma_5$, $|\dot{X}|^2 \equiv \dot{X}^A \dot{X}_A = \dddot{x}^\mu \dddot{x}_\mu - \dot{s}^2$, and the action takes the form

$$I[\dot{x}^\mu, s] = \kappa \int d\tau (\dddot{x}^\mu \dddot{x}_\mu - \dot{s}^2)^{1/2}.$$ \hspace{1cm} (18)

\textsuperscript{9}Here I am considering flat $C$-space. Curved $C$-space is considered in [3], [7].
Not all the variables $X^A(\tau)$ in (16) are independent dynamical degrees of freedom. Here $x^\mu(\tau)$, $s(\tau)$ in (18) also are not all independent. But we can fix a gauge (choose a parametrization) and thus reduce the set of variables to the physical, independent, dynamical variables. A natural choice of gauge in (18) is $s = \tau$. The reduced action is then

$$I[x^\mu] = \kappa \int ds \left( \dot{x}^\mu \dot{x}_\mu - 1 \right)^{1/2}. \tag{19}$$

The equations of motion derived from (19) are

$$\frac{dp_\mu}{ds} = 0, \quad p_\mu = \frac{\kappa \dot{x}_\mu}{\left( \dot{x}^\nu \dot{x}_\nu - 1 \right)^{1/2}} = \text{constant}. \tag{20}$$

The square $p^\mu p_\mu = M^2 = \kappa^2 \dot{x}^\mu \dot{x}_\mu / \left( \dot{x}^\nu \dot{x}_\nu - 1 \right)^{1/2}$ is also a constant of motion. Inserting the latter relation between $M$ and $\kappa$ into the expression (20) we obtain

$$p_\mu = \frac{M \dot{x}_\mu}{\left( \dot{x}^\nu \dot{x}_\nu \right)^{1/2}}, \quad M = \kappa \left( \dot{x}^\mu \dot{x}_\mu \right)^{1/2} \left( \dot{x}^\nu \dot{x}_\nu - 1 \right)^{1/2}. \tag{21}$$

The latter expression for 4-momentum has formally the same form as the momentum of the usual 4-dimensional relativity. The difference is in that $M$ is not a fixed constant entering the action, but a constant of motion. But the latter property is just typical for the Stueckelberg unconstrained theory described by the action (1).

We shall now directly demonstrate that the constrained action (18) is equivalent to the Stueckelberg action (1). First we observe that instead of (18) we can use the Howe-Tucker type action in which there occurs a Lagrange multiplier $\lambda$:

$$I[x^\mu, s, \lambda] = \kappa \int d\tau \left( \frac{\dot{x}^\mu \dot{x}_\mu - \dot{s}^2}{\lambda} + \lambda \right) \tag{22}$$

which is classically equivalent to (18). Variation of (22) with respect to $x^\mu$, $s$, $\lambda$ gives

$$\frac{d}{d\tau} \left( \frac{\kappa \dot{x}^\mu}{\lambda} \right) = 0, \quad \frac{d}{d\tau} \left( \frac{\kappa \dot{s}}{\lambda} \right) = 0, \quad \lambda = (\dot{x}^\mu \dot{x}_\mu - \dot{s}^2)^{1/2}. \tag{23}$$

The second equation in (23) gives $\left( d/d\tau \right)(\kappa \dot{s}/\lambda) = \kappa \dot{s}^2/\lambda$. Using the latter equation we can rewrite eq.(22) in the form

$$I[x^\mu, s, \lambda] = \frac{\kappa}{2} \int d\tau \left[ \frac{\dot{x}^\mu \dot{x}_\mu}{\lambda} + \lambda - \frac{d}{d\tau} \left( \frac{\kappa \dot{s}}{\lambda} \right) \right]. \tag{24}$$

The Lagrange multiplier $\lambda$ can be chosen arbitrarily: this determines a choice of parametrization. Let us choose $\lambda = \Lambda \kappa$, i.e., $(\dot{x}^\mu \dot{x}_\mu - \dot{s}^2)^{1/2} = \Lambda \kappa$, where $\lambda$ is a fixed constant. Omitting the total derivative, eq.(24) becomes just the Stueckelberg action (1)! The equations of motion derived from the unconstrained action (1) are the same as the $x^\mu$ equations (23) derived from the constrained action (22).
5 Conclusion

The formulation of relativity in $C$-space leads to the point particle with an extra variable $s$ along which the evolution in spacetime takes place. The extra variable $s$ does not come from an extra dimension of spacetime $V_4$, but from the Clifford algebra of $V_4$. In $C$-space we have “block universe”, no evolution, everything frozen. But in Minkowski space $V_4$ we have evolution. All the elegance of the theory of relativity is preserved, not in $V_4$, but in $C$-space. All the nice features of the Stueckelberg unconstrained theory are also present, not in $C$-space, but in its subspace $V_4$.

It is often claimed that the passage of time is just an illusion of the observer. Well, but good physics has always been capable of explaining certain illusions. Physics did not raise hands at why we see a “lake” in a desert, a colored arc in the rainy and sunny sky, or why far away objects appear smaller than the nearby ones. Now it is time to explain why we experience the passage of time. In the present paper I have presented a theoretical framework in which such a problem could be tackled.

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