Centroaffine Duality and Loewner’s Conjecture

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Abstract. Consider a transversal vector field $\xi$ along a hypersurface $f$. Then a lifting of $(f, \xi)$ is a centroaffine codimension 2 immersion together with a transversal vector field, and its dual centroaffine immersion $G$ does not depend on the lifting.

We prove in this paper that principal lines of $(f, \xi)$ correspond to asymptotic lines of $G$. As an application, we prove that Loewner’s conjecture for asymptotic lines at inflection points of surfaces in 4-space is equivalent to Loewner’s conjecture for principal lines at umbilical points of surfaces in 3-space with an equiaffine transversal vector field.

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1. Introduction

A centroaffine codimension 2 immersion is an immersion $F : M \to \mathbb{R}^{n+2} \setminus \{0\}$ whose radial vector field does not belong to tangent space of $F$. In this context, we can generate a unique conical hypersurface with vertex at the origin and containing $F(M)$. A vector field $\Phi$ along $M$ will be called transversal if it is transversal to this conical hypersurface. We say that a transversal vector field $\Phi$ along a centroaffine immersion $F : M \to \mathbb{R}^{n+2}$ is equiaffine (or parallel, or exact, see [2], [11], [12]), if the derivative of $\Phi$ in any direction tangent to $M$ is tangent to the above conical hypersurface. A centroaffine codimension 2 immersion $F$ together with an equiaffine transversal vector field is called an equiaffine pair.

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For a codimension 2 immersion $F$ together with a transversal vector field $\Phi$, the centroaffine dual pair $(G, \Psi) : M^n \rightarrow \mathbb{R}_{n+2}$, where $\mathbb{R}_k$ denotes the dual space of $\mathbb{R}_k$, is defined by the following equations:

\begin{align}
G \cdot \Phi &= 1, & G \cdot F_s X &= 0, & G \cdot F &= 0, \quad \text{(1.1)} \\
\Psi \cdot \Phi &= 0, & \Psi \cdot F_s X &= 0, & \Psi \cdot F &= 1. \quad \text{(1.2)}
\end{align}

(see [10]). When $(F, \Phi)$ is equiaffine, the same holds for the dual pair $(G, \Psi)$. One can show that if we start with a codimension 1 immersion $f$ together with an equiaffine transversal vector field $\xi$ and lift it to a codimension 2 equiaffine pair $(F, \Phi)$, the dual immersion $G$ does not depend on the lifting ([9], note 9). We call $G$ the projective pedal of the pair $(f, \xi)$. It is proved in [3] that the projective pedal of a Blaschke pair $(f, \xi)$ is umbilical, and conversely, any umbilical centroaffine immersion is the projective pedal of a Blaschke pair.

In this paper we prove that principal lines (resp. umbilical points) of $(F, \Phi)$ correspond to asymptotic lines (resp. inflection points) of the centroaffine dual $(G, \Psi)$, and vice-versa. As a consequence, principal lines (resp. umbilical points) of an equiaffine pair $(f, \xi)$ correspond to asymptotic lines (resp. inflection points) of its projective pedal $G$. Loewner’s conjecture for isolated inflection points of a surface in 4-space states that the index of the asymptotic lines is at most 1 ([1], [5], [6], [11]). Since any immersion $G : M^2 \rightarrow \mathbb{R}^4$ is locally the affine pedal of some equiaffine immersion $(f, \xi)$, we shall verify that the above conjecture is equivalent to the conjecture that the index of curvature lines of an equiaffine pair $(f, \xi)$ at an isolated umbilical point is at most 1. This latter conjecture was proved in [4] under the additional hypothesis of semi-homogeneity of the umbilical point.

The paper is organized as follows: In section 2 we recall some basic definitions. In section 3 we consider the duality between centroaffine codimension 2 immersions and prove that principal lines of the original correspond to asymptotic lines of the dual. In section 4 we describe the main properties of the projective pedal. In section 5 we apply our results to prove the equivalence between Loewner’s conjectures for principal lines of equiaffine transversal vector fields in 3-space and for asymptotic lines of surfaces in 4-space.

2. Centroaffine Immersions, Principal and Asymptotic Directions

2.1. Centroaffine immersions and equiaffine transversal vector fields

A centroaffine codimension 2 immersion is an immersion $F : M^n \rightarrow \mathbb{R}_{n+2} \setminus \{0\}$ such that the radial vector field $\eta(F(x))$ does not belong to the tangent space $F_s(T_x M)$, for any $x \in M$. A vector field $\Phi$ along $F$ is called transversal if
\(\Phi(x)\) does not belong to \(F_*(T_xM) \oplus \mathcal{S}(\eta(F(x)))\), where \(\mathcal{S}(w)\) denotes the space generated by \(w\). In order to keep notations shorter, we shall use \(F(x)\) instead of \(\eta(F(x))\). We denote by \(\mathfrak{X}(M)\) the space of smooth vector fields tangent to \(M\).

Let \(D\) be the canonical flat affine connection of \(\mathbb{R}^{n+2}\). For \(X, Y \in \mathfrak{X}(M)\), write

\[
D_XF_*Y = T(X,Y)F + F_*(\nabla_XY) + H(X,Y)\Phi,
\]

where \(H\) and \(T\) are bilinear forms and \(\nabla\) a torsion-free connection on \(M\). The bilinear form \(H\) is called the affine metric with respect to \(\Phi\). The conformal class of \(H\) does not depend on \(\Phi\) \([10]\), and thus non-degeneracy and also positiveness of \(H\) are independent of the choice of \(\Phi\). We shall assume non-degeneracy of \(H\) throughout the paper. We also write

\[
D_X\Phi = \rho(X)F - F_*(S)X + \tau(X)\Phi,
\]

where \(\rho\) and \(\tau\) are 1-forms and \(S\) a \((1,1)\)-tensor on \(M\). The \((1,1)\)-tensor \(S\) is called shape operator. We say that the pair \((F, \Phi)\) is equiaffine if \(\tau = 0\).

### 2.2. Principal directions and umbilical points

A principal direction for the pair \((F, \Phi)\) at \(x_0 \in M\) is an eigenvector of the shape operator \(S\). If the shape operator is a multiple of the identity, then we say that the point \(x_0\) is umbilical.

We say that a pair \((F, \Phi)\) is umbilical if all points are umbilical. The following proposition is a version of Theorem 4.3 of \([10]\):

**Proposition 2.1.** Consider an umbilical equiaffine pair \((F, \Phi)\). Then the planes generated by \(\{F, \Phi\}\) contain a fixed line through the origin.

### 2.3. Asymptotic directions and inflection points

Consider a hyperplane passing through \(F(x_0)\) defined by a linear functional \(w \in \mathbb{R}_{n+2}\). The contact function of the immersion with the hyperplane is given by

\[
k(x) = w \cdot (F(x) - F(x_0)),
\]

\(x \in M\). It is easy to see that \(x_0\) is a critical point of \(k\) if and only if the hyperplane defined by \(w\) contains the tangent space at \(x_0\).

Consider a vector field \(X \in \mathfrak{X}(M)\). We say that \(X(x_0) \in T_{x_0}M\) is an asymptotic direction if there exists an hyperplane \(w\) such that \(x_0\) is a degenerate critical point of the contact function \(k\) and \(X\) belongs to the kernel of its hessian, i.e. \(D_Xk_\*Y = 0\), for any \(Y \in \mathfrak{X}(M)\). Thus \(X\) is asymptotic at \(x_0\) if and only if there exists a functional \(w\) such that

\[
w \cdot (D_XF_*Y)(x_0) = 0,
\]
for any $Y \in \mathfrak{X}(M)$ (see [8, p.223] for the case $n = 2$). We may assume that 
$w \cdot F = 1$ and write $w \cdot \Phi = \mu$, for certain scalar function $\mu$. We conclude 
that $X$ is asymptotic at $x_0$ if and only if there exists $\mu = \mu(x_0)$ such that 

$$T(X, Y) - \mu H(X, Y) = 0, \quad (2.3)$$

for any $Y \in \mathfrak{X}(M)$.

If $T - \mu H = 0$ at $x_0$, then we say that $x_0$ is an inflection point. The following 
proposition is proved in [9, Prop.N9.3]

**Proposition 2.2.** The image of the immersion $F$ is contained in an affine 
hyperplane if and only if all $x \in M$ are inflection points.

### 3. Centroaffine Duality

#### 3.1. Basic equations

Assume that $F: M^n \to \mathbb{R}^{n+2}$ is a non-degenerate immersion together with 
an equiaffine transversal vector field $\Phi$. The dual pair $(G, \Psi): M^n \to \mathbb{R}^{n+2}$ of $(F, \Phi)$ is defined by Equations (1.1) and (1.2).

The following proposition can be found in [10, Lemmas 3.2 and 3.3]:

**Proposition 3.1.** Consider a non-degenerate centroaffine immersion $F$ and 
an equiaffine transversal vector field $\Phi$.

1. $G$ defined by Equation (1.1) is a non-degenerate centroaffine immersion 
   and we can write
   
   $$D_X G \ast Y = -H(SX, Y)G + G \ast (\nabla^\ast_X Y) + H(X, Y)\Psi, \quad (3.1)$$
   
   which means that $H^\ast = H$ and $T^\ast(X, Y) = -H(SX, Y)$. Moreover, the 
dual connection $\nabla^\ast$ is conjugate to $\nabla$, i.e.,
   
   $$Z(H(X, Y)) = H(\nabla_Z X, Y) + H(X, \nabla^\ast_Z Y).$$

2. $\Psi$ defined by Equation (1.2) is an equiaffine vector field and
   
   $$D_X \Psi = -\rho(X)G - G \ast (S^\ast X),$$

   where
   
   $$H(S^\ast X, Y) = -T(X, Y),$$

   which implies that $S^\ast$ is $H$-self-adjoint.

#### 3.2. Projective invariance

Assume that $(F_0, \Phi_0)$ and $(G_0, \Psi_0)$ are dual equiaffine pairs. Next lemma 
computes the dual of the equiaffine pair

$$F = \lambda F_0, \quad \Phi = \Phi_0 + \mu F_0, \quad (3.2)$$

where $\lambda$ and $\mu$ are arbitrary smooth real functions on $M$, $\lambda(x) \neq 0$ for any 
$x \in M$. Denote by $H_0$ the metric associated with $(F_0, \Phi_0)$. 
Lemma 3.2. The centroaffine dual of the equiaffine pair \((F, \Phi)\) given by Equations (3.2) is \((G, \Psi)\), where \(G = G_0\) and
\[
\Psi = \lambda^{-1} \Psi_0 - \mu \lambda^{-1} G_0 + (G_0)_* Z, \tag{3.3}
\]
where \(Z\) is the vector field on \(M\) defined by the condition \(H_0(X, Z) = d(\log \lambda)(X)\), for any \(X \in \mathfrak{X}(M)\).

Proof. We first verify conditions (1.1) with \(G = G_0\). We have
\[
G_0 \cdot (\Phi_0 + \mu F_0) = 1, \quad G_0 \cdot (\lambda F_0) = 0,
\]
and
\[
G_0 \cdot (d\lambda(X) F_0 + \lambda(F_0)_* X) = 0,
\]
which proves that in fact \(G = G_0\). Now we must verify conditions (1.2) for \(\Psi\) given by Equation (3.3). We have
\[
(\lambda^{-1} \Psi_0 - \mu \lambda^{-1} G_0 + (G_0)_* Z) \cdot (\Phi_0 + \mu F_0) = \lambda^{-1} \mu - \mu \lambda^{-1} = 0,
\]
where we have used that \((G_0)_* Z \cdot F_0 = -G_0 \cdot (F_0)_* Z = 0\) and \((G_0)_* Z \cdot \Phi_0 = -G_0 \cdot (\Phi_0)_* Z = 0\). Moreover
\[
(\lambda^{-1} \Psi_0 - \mu \lambda^{-1} G_0 + (G_0)_* Z) \cdot (\lambda F_0) = 0,
\]
since \((G_0)_* Z \cdot F_0 = -G_0 \cdot (F_0)_* Z = 0\). Finally
\[
(\lambda^{-1} \Psi_0 - \mu \lambda^{-1} G_0 + (G_0)_* Z) \cdot (d\lambda(X) F_0 + \lambda(F_0)_* X) = \lambda^{-1} d\lambda(X) - \lambda H_0(X, Z), \tag{3.4}
\]
where we have used that
\[
(G_0)_* Z \cdot (F_0)_* X = -H_0(X, Z).
\]
Now since \(H_0(X, Z) = d(\log \lambda)(X)\), we conclude that the second member of Equation (3.4) equals zero, thus proving the lemma.

A codimension 1 equiaffine pair \((f, \xi)\) in \(\mathbb{R}^{n+1}\) is represented in homogeneous coordinates by a lifting, which is a codimension equiaffine pair \((F, \Phi)\) in \(\mathbb{R}^{n+2}\). More explicitly, let \((F_0, \Phi_0) : M^n \to \mathbb{R}^{n+2}\) be given by
\[
F_0(x) = (f(x), 1), \quad \Phi_0(x) = (\xi(x), 0). \tag{3.5}
\]
Then any lifting of \((f, \xi)\) is given by
\[
F = \lambda F_0, \quad \Phi = \Phi_0 + \mu F_0, \tag{3.6}
\]
where \(\lambda\) and \(\mu\) are arbitrary smooth functions on \(M\), with \(\lambda(x) \neq 0, x \in M\). The following corollary can be found in [9, Note 9]:

Corollary 3.3. The centroaffine dual surface \(G\) of the lifting of a codimension 1 equiaffine pair \((f, \xi)\) is independent of the lifting.
3.3. Duality between asymptotic and principal directions

The main tool of this paper is the following:

**Proposition 3.4.** Consider non-degenerate dual equiaffine pairs \((F, \Phi)\) and \((G, \Psi)\). Fix \(x_0 \in M\) and \(X \in T_{x_0} M\). Then \(X\) is a principal direction for \(F\) at \(x_0\) if and only if \(X\) is an asymptotic direction for \(G\) at \(x_0\).

**Proof.** From Equations (2.3) and (3.1), \(X\) is an asymptotic direction for \(G\) at \(x_0\) if

\[ H(SX,Y) + \mu H(X,Y) = 0, \]

at \(x_0\), for any \(Y \in \mathfrak{X}(M)\). Thus \(X\) is an asymptotic direction for \(G\) if and only if

\[ H((S + \mu I)X,Y) = 0, \]

for any \(Y\). Since \(H\) is non-degenerate, this is equivalent to \((S + \mu I)X = 0\), which is the condition for \(X\) to be a principal direction for \(F\) at \(x_0\). \(\square\)

4. The Projective Pedal

We call the surface \(G\) in \(\mathbb{R}^{n+2}\) given by Corollary 3.3 the projective pedal of the equiaffine pair \((f, \xi)\).

4.1. Co-normal vector field

For a non-degenerate immersion \(f : M^n \to \mathbb{R}^{n+1}\) with an equiaffine transversal vector field \(\xi\), denote by \(\nu : M^n \to \mathbb{R}_{n+1}\) the co-normal map associated to \(\xi\), i.e.,

\[ \nu(x) \cdot \xi(x) = 1, \quad \nu(x) \cdot f_*X = 0, \quad X \in TM. \quad (4.1) \]

**Lemma 4.1.** The projective pedal \(G : M \to \mathbb{R}_{n+2}\) is given by

\[ G(x) = (\nu(x), -\nu(x) \cdot f(x)). \quad (4.2) \]

**Proof.** Let \((F_0, \Phi_0)\) be given by Equation (3.5). Then one can directly check that

\[ G \cdot F_0 = 0, \quad G \cdot (F_0)_*, X = 0, \quad G \cdot \Phi_0 = 1, \]

for any \(X \in \mathfrak{X}(M)\), thus proving the lemma. \(\square\)

4.2. Centroaffine dual of the pedal

Next proposition is a type of converse of Corollary 3.3

**Proposition 4.2.** Let \(G\) be the projective pedal of \((f, \xi)\) and \(\Psi\) be any equiaffine vector field along \(G\). Then the centroaffine dual of \((G, \Psi)\) is a lifting of \((f, \xi)\).
Proof. Denote by $H$ the metric induced by the equiaffine vector field $\Psi$ along $G$. Writing

$$\Psi = \alpha G + G^*_Z + \beta \Psi_0,$$

the equiaffine condition implies that $d(\log \beta) = -H(X, Z)$. Now take $\lambda = \beta^{-1}$ and $\mu = -\alpha \beta^{-1}$, so $H(X, Z) = d(\log \lambda)(X)$, which implies that $\Psi$ is given by Equation (3.3). Thus, by Lemma 3.2, the centroaffine dual of $(G, \Psi)$ is a lifting of $(f, \xi)$, thus proving the proposition. □

4.3. Projective invariance

Denote by $L$ a invertible linear transformation of $\mathbb{R}^{n+2}$ and by $\tilde{L}$ the corresponding projective transformation of $P\mathbb{R}^{n+1}$. It is proved in [12] that if $(f, \xi)$ is an equiaffine pair, then $(\tilde{L} \cdot f, d\tilde{L} \cdot \xi)$ is also an equiaffine pair.

Proposition 4.3. Let $(f, \xi)$ be an equiaffine pair with affine pedal $G$. Then the affine pedal of $(\tilde{L} \cdot f, d\tilde{L} \cdot \xi)$ is $G \cdot L^{-1}$.

Proof. The projective transformation $\tilde{L}$ of $(f, \xi)$ corresponds to the linear transformation $L$ of its lifting $(F, \Phi)$. Moreover, the centroaffine dual of $(L \cdot F, L \cdot \Phi)$ is $(GL^{-1}, \Psi \cdot L^{-1})$, thus proving the proposition. □

4.4. Principal and asymptotic directions

Let $(f, \xi)$ be a codimension 1 equiaffine immersion and write

$$D_X \xi = -f_\ast(SX),$$

where $S$ is the shape operator of $(f, \xi)$. Let $(F_0, \Phi_0)$ be given by Equation (3.5). Then

$$D_X \Phi_0 = (D_X \xi, 0) = -(f_\ast(SX), 0) = -(F_0)_\ast(SX),$$

which implies that the shape operator of $(F_0, \Phi_0)$ is also $S$.

Proposition 4.4. The principal lines of $(f, \xi)$ correspond to the asymptotic lines of $G$. Moreover, umbilical points of $(f, \xi)$ correspond to inflection points of $G$.

Proof. As we have seen above, the principal lines, resp. umbilical points, of $(f, \xi)$ correspond to principal lines, resp. umbilical points, of the lifting $(F_0, \Phi_0)$. Then the proposition follows from Proposition 3.4 □

4.5. A useful remark

Next proposition will be useful in what follows:

Proposition 4.5. Consider a non-degenerate immersion $G : M^n \to R_{n+2}$. Then $G$ is locally the projective pedal of an equiaffine pair $(f, \xi)$. 
Proof. Fix a point \( x_0 \in M \) and an origin of \( \mathbb{R}^{n+2} \) such that \( G \) becomes a centroaffine immersion at \( x_0 \). Then choose a vector \( \Psi(x_0) \) which is transversal to the centroaffine immersion \( G \) at \( x_0 \). In a neighborhood of \( x_0 \), the constant vector field \( \Psi(x_0) \) is equiaffine and transversal to \( G \). By Proposition 4.2, \( G \) is the affine pedal of some equiaffine pair \((f, \xi)\). □

5. Loewner’s Type Conjectures

Consider an isolated inflection point \( x_0 \) of an immersion \( G : M^2 \to \mathbb{R}^4 \). It is conjectured that the index of the asymptotic line foliation is at most 1 ([7], [6]). For later reference, we shall call it Conjecture 1. For generic immersions, it is well-known that this conjecture holds: It is proved in [7] that for a generic immersion \( G : M^2 \to \mathbb{R}^4 \), the index of an inflection point \(-1/2 \) or \( 1/2 \).

Conjecture 1 is equivalent to the following two conjectures:

Conjecture 1a. Consider an equiaffine pair \((f, \xi)\), where \( f : M^2 \to \mathbb{R}^3 \) is an immersion with positive affine metric. Then the index of the curvature lines of \((f, \xi)\) at \( x_0 \) is at most 1.

Conjecture 1b. Consider an equiaffine pair \((F, \Phi)\), where \( F : M^2 \to \mathbb{R}^4 \) is a centroaffine immersion with positive bilinear form \( H \). Then the index of the curvature lines of \((F, \Phi)\) at \( x_0 \) is at most 1.

Proposition 5.1. Conjectures 1, 1a and 1b are equivalent.

Proof. If Conjecture 1 holds, then Conjecture 1a also holds by taking the affine pedal. Conversely, assume Conjecture 1a holds and let \( G \) is an immersion. By Proposition 4.5, \( G \) is the affine pedal of some equiaffine pair \((f, \xi)\), and by Proposition 3.4, \( x_0 \) is an umbilical point for this pair. Moreover, the curvature lines of \((f, \xi)\) correspond to asymptotic lines of \( G \), which proves that Conjecture 1 holds.

If Conjecture 1 holds, then Conjecture 1b holds by taking the dual. Conversely, assume Conjecture 1b holds. Then by taking the lifting, Conjecture 1a also holds, thus proving the claim. □

Remark 5.2. Under a semi-homogeneity hypothesis, it is proved in [4] that Conjecture 1a holds. Thus, under the semi-homogeneity hypothesis, Conjecture 1 and Conjecture 1b also hold.

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