Thermodynamic uncertainty relation in thermal transport

Sushant Saryal,¹ Hava Friedman,² Dvira Segal,² and Bijay Kumar Agarwalla³

¹Department of Physics, Dr. Homi Bhabha Road, Indian Institute of Science Education and Research, Pune, 411008 India
²Department of Chemistry and Centre for Quantum Information and Quantum Control, University of Toronto, 80 Saint George St., Toronto, Ontario, Canada M5S 3H6
³Department of Physics, Dr. Homi Bhabha Road, Indian Institute of Science Education and Research, Pune, 411008 India

(Dated: March 9, 2022)

We use the fundamental nonequilibrium steady state fluctuation symmetry and derive a condition on the validity of the thermodynamic uncertainty relation (TUR) in thermal transport problems, classical or quantum alike. We test this condition and study the breakdown of the TUR in different thermal transport junctions of bosonic and electronic degrees of freedom. First, we prove that the TUR is valid in harmonic oscillator junctions. In contrast, in the nonequilibrium spin-boson model, which realizes many-body effects, it is satisfied in the Markovian limit, but violations arise as we tune (reduce) the cutoff frequency of the thermal baths, thus observing non-Markovian dynamics. Finally, we consider heat transport by noninteracting electrons in a tight-binding chain model. Here we show that the TUR is feasibly violated by tuning e.g. the hybridization energy of the chain to the metal leads. These results manifest that the validity of the TUR relies on the statistics of the participating carriers, their interaction, and the nature of their couplings to the macroscopic contacts (metal electrodes, phonon baths).

I. INTRODUCTION

The thermodynamic uncertainty relation (TUR),¹ a recently discovered trade-off in the field of stochastic thermodynamics,² provides a low bound for the relative uncertainty of observables in terms of the associated entropy production. The precise statement of the TUR for a two-terminal out-of-equilibrium system in steady state reads

\[
\frac{\langle j^2 \rangle_c}{\langle j \rangle^2} \geq 2. \tag{1}
\]

Here, \( \langle j \rangle \) is the averaged current, \( \langle j^2 \rangle_c \) is the second cumulant, and \( \langle \sigma \rangle \) is the averaged entropy production rate, all evaluated in steady state; \( k_B \) is the Boltzmann constant.

The TUR, Eq. (1), was put forward in Ref.¹ for biomolecular processes in the linear response regime. A generalized version of the TUR had been proposed and tested for various systems under driving forces (e.g. Ref.¹⁶,¹⁷) that are given in terms of the total entropy production are inconsequential in the steady state limit. Therefore, for systems operating under a constant voltage or thermal bias in the steady state limit one should focus on the TUR as given by Eq. (1). The validity of this relation, so far, has been established for classical markovian dynamics. While one expects that a related fundamental bound would exist in general cases, the validity of Eq. (1) in the quantum regime,²⁸ as well as for systems beyond the markovian limit, is questionable. In our previous work²⁵, we explored the TUR in charge transport problems, which are quantum in nature, and demonstrated its violations in quantum dot junctions of noninteracting electrons. In Ref.²⁵ we further examined the validity of the TUR in thermoelectric junctions of noninteracting electrons, and analyzed the related power-efficiency-power fluctuation trade-off relation.

What is the origin of the TUR? Can we derive it from fundamental principles? Using the nonequilibrium steady state fluctuation symmetry (SSFS), we put together the relative uncertainty of heat current and the associated entropy production rate. The resulting expression recovers the structure of the TUR—albeit allowing its violation—depending on the sign of high order cumulants. This analysis, based on the SSFS, allows us to interrogate the TUR in thermal energy transport problems along several axes: classical-quantum dynamics, harmonic-anharmonic systems, bosonic-fermionic statistics for the participating particles, and Markovian-non Markovian dynamics. The role of interaction on the breakdown of the TUR is examined by contrasting a fully harmonic model to the spin-boson model. Furthermore, we study here bosonic, fermionic, and hybrid statistics (spin-boson) systems, illustrating that the particle statistics plays a central role on this bound. Quantum effects are further questioned by analyzing the TUR at high temperature for both bosonic and electronic systems. Finally, the role of non-Markovian effects on the TUR is recognized by controlling system-bath parameters such as the spectral density function of the attached (phononic or electronic) thermal baths.

Our setup includes a central system that is coupled to two heat baths \( L, R \) that are maintained at different temperatures, \( T_L > T_R \); there are no other thermodynamical forces (e.g. the chemical potential is fixed). For such a setup, the average steady-state entropy production rate is given by \( \langle \sigma \rangle = k_B \Delta \beta \langle j \rangle \), where \( k_B \Delta \beta = \frac{1}{T_R} - \frac{1}{T_L} \) is the thermodynamic affinity, driving the system out-of-equilibrium in an irreversible manner. In this setting, the
TUR (1) simplifies to
\[ \Delta \beta \frac{\langle j^2 \rangle_c}{\langle j \rangle} \geq 2. \] (2)

The objective of this work is to understand the behavior of current fluctuations in thermal transport junctions, by providing insights on the validity of the TUR, Eq. (2). First, based on the steady state fluctuation symmetry we show that TUR violations are linked to the behavior of the skewness, the third cumulant of the heat current. This derivation holds for both classical and quantum systems. We exemplify this observation and examine the TUR in three central quantum thermal transport models: (i) Chain of coupled harmonic oscillators. In this case we prove that the TUR is always satisfied. (ii) Nonequilibrium spin-boson model. Here, we demonstrate that the TUR can be violated by structuring the thermal baths, which leads to non-Markovian dynamics. (iii) Fermionic chains, where we study electronic heat transport. In the resonant transport regime we write down an analytic condition for TUR violations, which is supported by numerical simulations.

The paper is organized as follows. In Section II we employ the universal steady state fluctuation relation (Gallavotti-Cohen symmetry) for heat exchange and present a \( \Delta \beta \) perturbative expansion of the TUR. In Section III we study the non-interacting harmonic oscillator model. The nonequilibrium spin-boson (NESB) model is examined in Sec. IV. In Section V we investigate the TUR in electronic heat transport. Certain details are delegated to the Appendices. We conclude and discuss future directions in Section VI.

II. PERTURBATIVE EXPANSION FOR THE TUR FROM FLUCTUATION SYMMETRY

Invoking the steady state fluctuation symmetry for heat exchange,\(^{29-31}\) we derive here a perturbative expansion for the ratio \( \Delta \beta \frac{\langle j^3 \rangle_c}{\langle j \rangle} \) in orders of the affinity \( \Delta \beta \). In steady state, we can formally expand the averaged heat current and its higher order cumulants, \( \langle j^n \rangle_c \), \( n=1, 2, 3, \ldots \) in powers of the affinity \( \Delta \beta \) as
\[
\langle j \rangle = G_1 \Delta \beta + \frac{1}{2!} G_2 (\Delta \beta)^2 + \frac{1}{3!} G_3 (\Delta \beta)^3 + \frac{1}{4!} G_4 (\Delta \beta)^4 + \cdots
\]
\[
\langle j^2 \rangle_c = S_0 + S_1 (\Delta \beta) + \frac{1}{2!} S_2 (\Delta \beta)^2 + \frac{1}{3!} S_3 (\Delta \beta)^3 + \cdots
\]
\[
\langle j^3 \rangle_c = R_1 \Delta \beta + \frac{1}{2!} R_2 (\Delta \beta)^2 + \cdots
\]
\[
\langle j^4 \rangle_c = T_0 + T_1 (\Delta \beta) + \cdots
\]

The four cumulants are the averaged current, its variance, skewness, and the kurtosis. For the average current, \( G_1 \) is the linear transport coefficient, or the thermal conductance, \( G_2, G_3, \ldots \) are nonlinear transport coefficients. Heat current fluctuations include the equilibrium noise component \( S_0 \) and higher order nonequilibrium terms, \( S_1, S_2, \ldots \). Other coefficients appearing in the skewness and the kurtosis, such as \( R_1, R_2, T_0, \ldots \) can be similarly described.

The steady state fluctuation symmetry relates the probability of integrated heat current, \( Q(t) = \int_0^t d\tau j(\tau) \), flowing in the forward direction (from hot to cold) \( p(Q = j t) \) to that of the probability of heat flowing in the reverse direction \( p(-Q = - j t) \) (from cold to hot). The precise statement of SSFS is
\[
\lim_{t \rightarrow \infty} \frac{1}{t} \ln \left[ \frac{p(Q = j t)}{p(Q = -j t)} \right] = \Delta \beta j. \] (4)

Here, \( \sigma = k_B \Delta \beta j \) is the stochastic entropy production rate. In terms of the generating function defined as the Fourier transform of the probability distribution, \( Z(\alpha) = \int dQ e^{\alpha Q} p(Q) \), the SSFS leads to the symmetry \( Z(\alpha) = Z(-\alpha + i \Delta \beta) \). Here, \( \alpha \) is the counting parameter, which keeps track of the net amount of thermal energy flowing between a bath to the system. The cumulants for current are obtained by taking derivatives,
\[
\langle j^n \rangle_c = \frac{\partial^n \chi(\alpha)}{\partial \alpha^n} \bigg|_{\alpha = 0}, \quad \text{where } \chi(\alpha) = \lim_{t \rightarrow \infty} \frac{1}{t} \ln Z(\alpha) \text{ is the cumulant generating function (CGF) for heat transport.}
\]

As a consequence of this SSFS, it can be shown that linear and higher order transport coefficients in Eq. (3) are in fact related to each other,\(^{32}\)
\[
S_0 = 2 G_1, \quad S_1 = G_2, \quad T_0 = 2 R_1, \quad T_1 = R_2, \quad 3 S_2 - 2 G_3 = R_1, \quad 2 S_3 - G_4 = R_2, \quad \ldots
\] (5)

and so on. We substitute the expansion for the current and its fluctuations Eq. (3) into Eq. (2) and simplify it using the relationships (4). As a result, we obtain the left hand side of the TUR as a perturbative series in \( \Delta \beta \),
\[
\Delta \beta \frac{\langle j^2 \rangle_c}{\langle j \rangle} = 2 + \frac{\langle \Delta \beta \rangle^2}{6 G_1} R_1 + \frac{\langle \Delta \beta \rangle^3}{12 G_1^2} \left[ G_1 R_2 - G_2 R_1 \right] + \mathcal{O}(\Delta \beta)^4 + \cdots
\] (6)

This expression is one of the central results of the paper. Since there is no fundamental constraint on the sign of the skewness coefficient, \( R_1 \), the TUR (2) may be violated in nonequilibrium systems—within a certain range of the affinity \( \Delta \beta \).

We point out the following: (i) The expansion (6) is valid for both classical and quantum systems. (ii) The (3) linear order term, \( \Delta \beta \), does not contribute to the TUR. This is a consequence of the relation \( S_1 = G_2 \), which stems from the time-reversal symmetry of the underlying Hamiltonian. (iii) Coefficients for orders \( \langle \Delta \beta \rangle^n, n = 2, 3, \ldots \) include heat current cumulants that are greater than two. Therefore, if the probability distribution for heat exchange is Gaussian the TUR precisely saturates to the value 2, since all higher order cumulants greater
than two vanish. (iv) It is evident from this expression that for noninteracting setups (missing a diode effect) only even orders in \( \Delta \beta \) contributes since the current is an odd function of the thermal bias, while the second cumulant involves only even powers in \( \Delta \beta \). This may not be the situation for interacting junctions.

Equation (6) reveals that the TUR [2], is violated if \( R_1 > 0 \). Of course, one may observe that \( R_1 < 0 \) but the TUR is still violated due to the impact of higher cumulants. However, in this work we examine the breakdown of the TUR within the leading order of the affinity i.e., up to \( (\Delta \beta)^2 \), and we therefore focus on the sign of \( R_1 \).

In what follows we study the TUR in three representative thermal transport problem: harmonic oscillator system coupled to harmonic baths, a central spin coupled to harmonic baths, and a tight-binding electronic junction.

A recent study obtained a current fluctuation - entropy production trade-off relation based on the geometry of quantum non-equilibrium steady-state.\(^{28}\) This quantum TUR (QTUR) reads \( (j^2)_c \langle \sigma \rangle / (k_B j^2) \geq 1 \), which is two times looser than Eq. (1) or Eq. (2) for the specific case of heat transport). Of course, once a system obeys the QTUR, it satisfies as well the ‘standard’ TUR (1). We also emphasize that while we show below violations to the bound (2), our results do obey the QTUR. Nevertheless, we argue that analyzing the TUR (1) is particularly interesting for several reasons: First, at equilibrium the TUR becomes an equality, providing a clear reference point to as the role of the nonequilibrium condition on the trade-off relation. Further, the expansion (6) builds about the equilibrium value of 2, and it provides insights on violations in terms of high-order cumulants of transport and their relationships. Second, as we prove below, harmonic systems exactly satisfy the TUR (2), making it clear that its validity extends beyond Markovian dynamics to cover more general cases. Third, our simulations below satisfy the QTUR, yet we observe that this bound is quite loose, thus suggesting the existence of a tighter bound. Altogether, as we illustrate in Secs. III IV, invalidating the bound (1) allows us to recognize systems of different statistics, as well as identify the onset of interactions and non-Markovianity in the dynamics, making it profoundly useful.

\[ \chi_{\text{HO}}(\alpha) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \ln \left\{ 1 - \mathcal{T}_{\text{HO}}(\omega) \left[ n_L(\omega) - n_R(\omega) \right] \right\} \left[ e^{i\alpha \hbar \omega} - 1 \right] \left[ n_R(\omega) n_L(\omega) (e^{-i\alpha \hbar \omega} - 1) \right] , \]  

with the transmission function \( \mathcal{T}_{\text{HO}}(\omega) \), which is expressed in terms of the Green’s function of the central region and the self-energies of the baths.\(^{33,35}\) We highlight that the transmission function is derived for a harmonic system, but the oscillators within could be coupled to each other with any geometry and force constants. The baths are maintained at thermal equilibrium with the Bose-Einstein distribution function \( n_\nu(\omega) = \left[ \exp(\beta \nu \hbar \omega) - 1 \right]^{-1} \). 

III. COUPLED HARMONIC OSCILLATORS: PROOF FOR THE VALIDITY OF THE TUR

We consider a noninteracting thermal transport model consisting of coupled harmonic oscillators. We assume that the first and last oscillators are coupled to independent heat baths, \( L \) and \( R \), which are maintained at different temperatures, \( T_L \) and \( T_R \), respectively. The baths include a collection of harmonic oscillators, which are bilinearly coupled to the system. The model is fully harmonic and the total Hamiltonian for this setup can be written as

\[ \hat{H} = \sum_{\nu=L,R} \left( \frac{1}{2} \hat{p}_\nu^T \hat{p}_\nu + \frac{1}{2} \hat{u}_\nu^T K^\nu \hat{u}_\nu \right) + \frac{1}{2} \hat{p}_C^T \hat{p}_C + \frac{1}{2} \hat{u}_C^T K^C \hat{u}_C \]

\[ + \hat{u}_L^T V^{LC} \hat{u}_C + \hat{u}_R^T V^{RC} \hat{u}_C . \]  

Here \( \hat{u}_\nu \) and \( \hat{p}_\nu \) are column vectors of mass weighted coordinate operators and momenta for regions \( \nu = L, R \). \( K^\nu \) is the corresponding force constant matrix. Similar definitions hold for the central (C) region. \( V^{LC} \) and \( V^{RC} \) are the force constants matrices between the central system and the left and right regions; recall that the left bath is coupled to a single (‘first’) oscillator in the system, and similarly the right bath is connected to a specific (‘last’) oscillator. \( T \) stands for the transpose operation.

The integrated thermal energy current is defined as the net change of thermal energy in the reservoir, say the left one, \( Q \equiv \hat{H}_L(0) - \hat{H}_L(t) \). An exact expression for the steady state CGF, \( \chi_{\text{HO}}(\alpha) \) for the integrated thermal energy current can be obtained by following the two-time measurement protocol and employing the Keldysh non-equilibrium Green’s function approach.\(^{33,35}\) The CGF is given by

\[ \chi_{\text{HO}}(\alpha) = \int_{-\infty}^{\infty} d\omega \ln \left\{ 1 - \mathcal{T}_{\text{HO}}(\omega) \left[ n_L(\omega) - n_R(\omega) \right] \right\} \left[ e^{i\alpha \hbar \omega} - 1 \right] \left[ n_R(\omega) n_L(\omega) (e^{-i\alpha \hbar \omega} - 1) \right] , \]  

with \( \mathcal{T}_{\text{HO}}(\omega) \), which is expressed in terms of the Green’s function of the central region and the self-energies of the baths.\(^{33,35}\) We highlight that the transmission function is derived for a harmonic system, but the oscillators within could be coupled to each other with any geometry and force constants. The baths are maintained at thermal equilibrium with the Bose-Einstein distribution function \( n_\nu(\omega) = \left[ \exp(\beta \nu \hbar \omega) - 1 \right]^{-1} \). 

\[ \chi_{\text{HO}}(\alpha; T_L, T_R) = \chi_{\text{HO}}(-\alpha; T_R, T_L) , \]  

which ensures that the energy current and the associated noise are odd and even functions of the affinity \( \Delta \beta \), respectively. Therefore, for a fully harmonic model only even powers in \( \Delta \beta \) survive in the TUR expression, Equation (6).

The analytical forms for the first and second cumulants can be readily obtained from Eq. (8), and are given by

\[ \langle j \rangle = \int_{0}^{\infty} \frac{d\omega}{2\pi} \hbar \omega \mathcal{T}_{\text{HO}}(\omega) \left[ n_L(\omega) - n_R(\omega) \right] , \]  

\[ \langle j^2 \rangle_c = \int_{0}^{\infty} \frac{d\omega}{2\pi} \hbar^2 \omega^2 \left\{ \mathcal{T}_{\text{HO}}(\omega) \left[ n_L(\omega) - n_R(\omega) \right] \right\} \left[ n_R(\omega) n_L(\omega) (e^{-i\alpha \hbar \omega} - 1) \right] \left[ n_R(\omega) n_L(\omega) (e^{-i\alpha \hbar \omega} - 1) \right] , \]  

\[ + \mathcal{T}_{\text{HO}}(\omega) \left[ n_L(\omega) - n_R(\omega) \right] \left[ n_R(\omega) n_L(\omega) (e^{-i\alpha \hbar \omega} - 1) \right] . \]  

(11)
Expanding the current and the variance in powers of $\Delta \beta$, we construct the coefficient $R_1 = 3S_2 - 2G_3$, which is always positive,

$$R_1 = \int_0^\infty \frac{d\omega}{2\pi} (\hbar \omega)^4 T_{HO}(\omega)n(\omega)\bar{n}(\omega) \left[ 1 + 6T(\omega)n(\omega)\bar{n}(\omega) \right] > 0.$$  

(12)

Here $n(\omega)$ is the Bose-Einstein distribution function evaluated at the average temperature $T = (T_L + T_R)/2$. Since the transmission function and the thermal distribution functions are positive, the coefficient $R_1$ is always positive. Therefore, up to $O(\Delta \beta)^2$ the TUR is satisfied for quantum harmonic networks consisting of an arbitrary number of oscillators with general connectivity and arbitrary coupling strengths to the baths.

Moreover, we now prove that for harmonic systems the TUR is valid arbitrarily far from equilibrium. From Eq. (11), we note that the second cumulant obeys the inequality,

$$\langle j^2 \rangle_v \geq \int_0^\infty \frac{d\omega}{2\pi} (\hbar \omega)^2 T_{HO}(\omega) \left[n_L(\omega)\bar{n}(\omega) + n_R(\omega)\bar{n}(\omega) \right].$$

(13)

An equality is satisfied at thermal equilibrium, $\beta_L = \beta_R$. Interestingly, one can show the following inequality for $\forall \omega > 0$,

$$\left[n_L(\omega)\bar{n}(\omega) + n_R(\omega)\bar{n}(\omega) \right] \geq \frac{2}{\Delta \beta} \hbar \omega \left[n_L(\omega) - n_R(\omega) \right].$$

(14)

Using this inequality in the noise expression immediately implies that

$$\langle j^2 \rangle_v \geq \frac{2}{\Delta \beta} \int_0^\infty \frac{d\omega}{2\pi} \hbar \omega T_{HO}(\omega) \left[n_L(\omega) - n_R(\omega) \right]$$

$$= \frac{2}{\Delta \beta} \langle j \rangle,$$

(15)

which is the TUR, Eq. (2). This derivation is entirely independent of the details of the transmission function. The proof only emerges from the formal structure of the CGF in Eq. (1). We conclude that for harmonic junctions the TUR is satisfied in the quantum and classical (high temperature) limits irrespective of the underlying dynamics, which could be Markovian or non-Markovian. While this proof holds for classical and quantum systems alike, in Appendix A we separately study classical harmonic systems by directly studying the classical CGF.

IV. NON-EQUILIBRIUM SPIN-BOSON MODEL: TUR VIOLATION FORSTRUCTURED BATHS

In the previous section, we proved the TUR, Eq. (2) in harmonic systems. It is intriguing to depart from the harmonic limit, which describes heat transport in a system of independent normal modes, and examine the role of anharmonicity (many-body interaction) for invalidating the TUR. The nonequilibrium spin-boson model (NESB) serves as a toy model for exploring the role of anharmonicity in quantum thermal transport. Its transport characteristics have been extensively examined with different theoretical techniques; partial list includes[36–43]

The NESB model comprises a two-level system (spin) interacting with two bosonic environments, $\nu = L, R$, which are maintained at different temperatures. The Hamiltonian for the NESB model reads

$$\hat{H} = \frac{\hbar \omega_0}{2} \hat{\sigma}_z + \frac{\hbar \Delta}{2} \hat{\sigma}_x + \sum_{\nu,j} \omega_{\nu,j} \hat{b}_{\nu,j}^\dagger \hat{b}_{\nu,j} + \hat{\sigma}_z \sum_{\nu,j} \gamma_{\nu,j} (\hat{b}_{\nu,j}^\dagger + \hat{b}_{\nu,j}).$$

(16)

Here, $\hat{b}_{\nu,j}$ ($\hat{b}_{\nu,j}$) are bosonic creation (annihilation) operators of the $j$th mode in the $\nu$ thermal bath. $\omega_0$ is the spin splitting, $\Delta$ the tunneling element, $\gamma_{\nu,j}$ the coupling energy of the spin polarization to the displacements of baths’ oscillators. Given the quantum nature of the spin system and the varied applicability of the model, understanding the validity of the thermodynamic uncertainty relation in the NESB is of a significant interest.

To examine the possible violation of the TUR in the NESB model, we need to study it beyond the weak-coupling Markovian dynamics[44]. The nonequilibrium Green’s function method in combination with the Majorana fermion representation for the spin[45] extends beyond the Markovian weak coupling limit, thus it may allow to observe TUR violation. Using the Majorana Green’s function approach, we derived in Ref[45] the CGF for the NESB model, where for simplicity, we focus on the unbiased case, $\omega_0 = 0$,

$$\chi_{SB}(\alpha) = \int_{-\infty}^{\infty} \frac{d\omega}{4\pi} \frac{\Delta^2}{\omega^2} \ln \left\{ 1 + T_{SB}(\omega; T_L, T_R) \left[n_L(\omega)\bar{n}(\omega) - n_R(\omega)\bar{n}(\omega) \right] \right\} (e^{\omega \hbar \omega} - 1) + n_R(\omega)\bar{n}(\omega)(e^{-\omega \hbar \omega} - 1) \right\}.$$  

(17)

The transmission function of the NESB model is given by[45]

$$T_{SB}(\omega; T_L, T_R) = \frac{4\Gamma_L(\omega)\Gamma_R(\omega)\omega^2}{(\omega^2 - \Delta^2)^2 + \omega^2 \left[ \sum_{\nu} \Gamma_\nu(\omega)(1+2n_\nu(\omega)) \right]^2}.$$  

(18)

Note the crucial sign difference between the CGF for the spin-boson model in comparison to the harmonic case, Eq. (8). Here, $\Gamma_\nu(\omega) = 2\pi \sum_j \gamma_{\nu,j}^2 (\delta(\omega - \omega_{\nu,j})$ is the spectral function of the $\nu$ bath. As a reflection of the model’s anharmonicity, the transmission function depends on temperature, which results in

$$\chi_{SB}(\alpha; T_L, T_R) \neq \chi_{SB}(-\alpha; T_R, T_L).$$

(19)

Therefore, odd powers of $\Delta \beta$ contribute to the TUR expression [5]. From Equation (17), we obtain an ana-
lytical expression for the coefficient $R_1$,
\[ R_1 = \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\hbar^3} f(\Delta^2) \mathcal{T}_{SB}(\omega; T) \left( -\frac{\partial n(\omega)}{\partial \beta} \right) \times \left[ 1 - 6 T_{SB}(\omega; T) n(\omega)(1 + n(\omega)) \right]. \] (20)

Here, $\mathcal{T}_{SB}(\omega; T)$ is evaluated at the average temperature $T = (T_L + T_R)/2$. Once again note the sign difference in $R_1$ in comparison to the harmonic case, reflecting the hybrid nature (spin-boson) of the system. The sign of $R_1$ can therefore be negative, which could result in the violation of the TUR, Eq. (2).

In Fig. 1 we present numerical simulations of the combination $\Delta \beta^2 / 4Q^2$ in the NESB model using the Majorana NEGF method. We find that the TUR can be violated by structuring the spectral density function of the thermal reservoirs. Specifically, we used an Ohmic function with either a soft cutoff, $\Gamma_\nu(\omega) = \kappa \omega e^{-|\omega|/\omega_c}$, or a hard cutoff, $\Gamma_\nu(\omega) = \kappa \omega e^{-|\omega|/\omega_c} + \Theta(\omega_c - |\omega|)$; the exponential function plays a minor role in the latter expression, but we keep it so as to maintain a qualitative comparison between the two models. In both cases, we used $\omega_c = 10\Delta$ as the cutoff frequency. This value seems to be high, thus at first sight it suggests the Markovian limit. However, a closer look (inset) reveals that in fact the transmission function comprises significant weight beyond $\omega_c$ in a region where $n_\nu(\omega)$ is still substantial, given the high temperature employed.

Figure 1 shows that the TUR is obeyed in the case of a soft-cutoff spectral function. In contrast, we observe a weak but noticeable violation of the TUR once we structure the baths and filter high frequencies using a hard cutoff. This effect can be attributed to the fact that transport is non-Markovian in this limit since the baths comprise modes only at or below the averaged temperature. Nevertheless, a fundamental understanding of the TUR violation in the NESB model in the language of the underlying transport mechanism is still missing.

It is instructive to analyze the TUR in the NESB model in two special cases:

**Weak coupling limit.** When the system-bath coupling is weak, the transmission function \[ \mathcal{T}_{SB}(\omega; T) \] evaluated at $T = T_L = T_R$, can be approximated by the Dirac delta function,
\[ \mathcal{T}_{SB}(\omega; T) = \frac{\Gamma(\Delta) \pi}{1 + 2 n(\Delta)} [\delta(\omega + \Delta) + \delta(\omega - \Delta)]. \] (21)

The spectral function is evaluated at the tunneling energy $\Delta$, and for simplicity we assume that it is identical at the two contacts, $\Gamma(\Delta) = \Gamma_\nu(\Delta)$. Similarly, the Bose-Einstein distribution is evaluated at the spin splitting $\Delta$, $n(\Delta) = [e^{\Delta/k_B T} - 1]^{-1}$. The weak coupling limit is also known as the sequential (or resonant) tunneling limit. We further calculate, in the weak coupling limit, the following function,
\[ \mathcal{T}_{SB}^2(\omega; T) = \frac{\Gamma(\Delta) \pi}{2(1 + 2 n(\Delta))} [\delta(\omega + \Delta) + \delta(\omega - \Delta)]. \] (22)

Substituting these expressions into $R_1$, Eq. (20), we get
\[ R_1 = \frac{h^4 \Delta^4 \Gamma(\Delta) n(\Delta)[1 + n(\Delta)]}{2(1 + 2 n(\Delta))^2} [n(\Delta)^2 + n(\Delta) + 1], \] (23)
which is always positive. The same conclusion is reached by following the CGF obtained using the Markovian Redfield quantum master equation approach, see Appendix B. Overall, this proves that in the weak coupling limit, there is no violation to the TUR in the quadratic order, $\langle \Delta \beta \rangle^2$. It remains a challenge to prove (or disprove) the validity of the TUR arbitrarily far from equilibrium in the weak coupling limit.

**Co-tunneling limit.** At low temperatures, $\Gamma < T_c < \Delta$, sequential tunneling is exponentially suppressed since incoming phonons fall short of the spin splitting $\Delta$. The residual off-resonant transmission probability is therefore small, $\mathcal{T}_{SB}(\omega) \ll 1$. From the CGF \[ \mathcal{T}_{CO}(\omega) \] we find the current and its noise in this co-tunneling limit
\[ \langle j \rangle = \frac{2}{\pi} \int_0^{\omega_h} d\omega \omega \mathcal{T}_{CO}(\omega) [n_L(\omega) - n_R(\omega)], \]
\[ \langle j^2 \rangle_c = \frac{2}{\pi} \int_0^{\omega_h} d\omega (\omega)^2 \mathcal{T}_{CO}(\omega) [n_L(\omega) n_R(\omega) + n_R(\omega) n_L(\omega)]. \] (24)

Here, the upper limit of the integral $\omega_h$ is determined by the smallest of two energy scale: temperature or the cutoff frequency of the baths. $\mathcal{T}_{CO}(\omega)$ stands for the transmission function in the co-tunneling limit
\[ \mathcal{T}_{CO}(\omega) = \frac{\Gamma_L(\omega) \Gamma_R(\omega)}{\Delta^2}, \] (25)
V. THERMAL TRANSPORT OF NONINTERACTING ELECTRONS: TUR VIOLATION IN THE RESONANT TRANSPORT REGIME

After analyzing the validity and breakdown of the TUR for bosonic heat transport and for the NESB model, in this Section we study electronic heat transport. For simplicity, we focus on a one-dimensional noninteracting tight-binding chain model for fermions with the Hamiltonian

\[ \hat{H} = \sum_{\nu=L,C,R} \tilde{c}^\dagger_{\nu} h^\nu \tilde{c}_{\nu} + \sum_{\nu=L,R} \left( \tilde{c}^\dagger_{\nu} V_{eC} \tilde{c}_{C} + \text{h.c.} \right). \] (26)

Here, \( \tilde{c}^\dagger_{\nu}(\tilde{c}_{\nu}) \) is the row (column) vector consisting of electronic creation (annihilation) operators in the \( \nu \) region, with \( h^\nu \) the single-particle Hamiltonian matrix in that domain. \( V_{eC} \) is the coupling matrix between the metals and the central system. The baths (\( \nu = L, R \)) and the central system are initially decoupled and are prepared at their respective grand canonical equilibrium state with temperature \( T_\nu \) and chemical potential \( \mu_\nu \).

The steady state expression for the joint CGF corresponding to integrated particle and energy current can be obtained exactly for this model. It was first derived by Levitov and Lesovik following a wave scattering approach. It was later derived by employing different rigorous approaches. The joint CGF is given as

\[ \chi_{el}(\alpha_p, \alpha_e) = \int_{-\infty}^{\infty} \frac{d \epsilon}{2\pi \hbar} \ln \left\{ 1 + T_{el}(\epsilon) \left[ f_L(\epsilon) f_R(\epsilon) (e^{i(\alpha_p + \epsilon \alpha_e)} - 1) \right. \right. \]
\[ \left. \left. + f_R(\epsilon) f_L(\epsilon) (e^{-i(\alpha_p + \epsilon \alpha_e)} - 1) \right] \right\}. \] (27)

Here, \( \alpha_p(\alpha_e) \) is the counting parameter keeping track of the net particle (energy) exchange between the system and the left bath in steady state. \( f_\nu(\epsilon) = 1/[e^{\beta_\nu(\epsilon - \mu_\nu)} + 1] \) is the Fermi-Dirac distribution function for the leads, \( \nu = L, R \), \( f_\nu(\epsilon) = 1 - f_\nu(\epsilon) \), \( \mu_\nu \) is the corresponding chemical potential and \( T_{el}(\epsilon) \) is the transmission function, containing the structural information of the electrodes and the system. The joint CGF satisfies the steady state fluctuation symmetry, \( \chi_{el}(\alpha_p, \alpha_e) = \chi_{el}(-\alpha_p + i(\beta_L \mu_L - \beta_R \mu_R), -\alpha_e + i\Delta \beta) \).

Following Eq. (27), the CGF for heat exchange can be obtained by simply replacing \( \alpha_p = -\mu_L \alpha \) and \( \alpha_e = \alpha \), where \( \alpha \) keeps track of net amount heat transfer at the left contact. We then receive

\[ \chi_{el}(\alpha) = \int_{-\infty}^{\infty} \frac{d \epsilon}{2\pi \hbar} \ln \left\{ 1 + T_{el}(\epsilon) \left[ f_L(\epsilon) f_R(\epsilon) (e^{i\alpha(\epsilon - \mu_L)} - 1) \right. \right. \]
\[ \left. \left. + f_R(\epsilon) f_L(\epsilon) (e^{-i\alpha(\epsilon - \mu_L)} - 1) \right] \right\}. \] (28)

Since our focus here is on the heat current, we consider identical chemical potentials \( (\mu = \mu_L = \mu_R) \) but different temperatures \( (T_L \neq T_R) \) for the leads. In this case, the CGF follows the symmetry \( \chi_{el}(\alpha) = \chi_{el}(-\alpha + i\Delta \beta) \). Note that, similarly to the noninteracting bosonic case, the CGF here also satisfies

\[ \chi_{el}(\alpha; \mu, T_L, T_R) = \chi_{el}(-\alpha; \mu, T_R, T_L), \] (29)

which implies that the TUR expression in Eq. (6) contains only even powers in \( \Delta \beta \). Using Eq. (28), the heat current and the associated noise are given by

\[ \langle j \rangle = \int_{-\infty}^{\infty} \frac{d \epsilon}{2\pi \hbar} (\epsilon - \mu) T_{el}(\epsilon) [f_L(\epsilon) - f_R(\epsilon)], \] (30)
\[ \langle j^2 \rangle_c = \int_{-\infty}^{\infty} \frac{d \epsilon}{2\pi \hbar} (\epsilon - \mu)^2 \left\{ T_{el}(\epsilon) [f_L(\epsilon) f_R(\epsilon) \right. \]
\[ \left. + f_R(\epsilon) f_L(\epsilon)] - T_{el}^2(\epsilon) (f_L(\epsilon) - f_R(\epsilon))^2 \right\}. \] (31)

As before, we compute \( R_1 \), which is given by

\[ R_1 = \int_{-\infty}^{\infty} \frac{d \epsilon}{2\pi \hbar} (\epsilon - \mu)^4 T_{el}(\epsilon) f(\epsilon) [1 - f(\epsilon)] \times \]
\[ \left[ 1 - 6T_{el}(\epsilon) f(\epsilon) [1 - f(\epsilon)] \right]. \] (32)

The Fermi function \( f(\epsilon) \) is evaluated at the averaged temperature. It is important to note that the above obtained expressions are valid for arbitrary temperatures, bias voltage, system-bath coupling and the details of the chain, encapsulated within the transmission function. In what follows, we once again consider different limiting cases to analyze the TUR.

Low-transmission. In the limit when \( T_{el}(\epsilon) \ll 1 \) for all values of \( \epsilon \), one can discard the term \( T_{el}^2(\epsilon) \) in Eq. (32) relative to \( T_{el}(\epsilon) \). This immediately implies that the TUR is satisfied.

Indeed, in the context of charge transport, low transmission probability is associated with a Poisson process, which further implies uncorrelated electron transport through the junction. In this limit, one simplifies Eq. (28) by approximating \( \ln(1 + x) \approx x \) and writes the CGF as

\[ \chi_{el}(\alpha) = \int_{-\infty}^{\infty} \frac{d \epsilon}{2\pi \hbar} T_{el}(\epsilon) \left\{ f_L(\epsilon) f_R(\epsilon) (e^{i\alpha(\epsilon - \mu_L)} - 1) \right. \]
\[ \left. + f_R(\epsilon) f_L(\epsilon) (e^{-i\alpha(\epsilon - \mu_L)} - 1) \right\}. \] (33)

In Appendix C we further prove that the TUR is valid arbitrarily far from equilibrium in this low transmission transport regime.
Constant transmission. If the transmission is a constant i.e., \( T_d(\epsilon) = \tau \), the integration in Eq. (32) can be performed analytically and the expression for TUR up to \( O(\Delta \beta)^2 \) simplifies to
\[
\Delta \beta \frac{\langle j^2 \rangle_c}{\langle j \rangle} = 2 + \frac{(\Delta \beta)^2}{\beta^2} \left[ \frac{7}{5} \pi^2 (1 - \tau) + 12 \tau \right] \geq 0, \tag{34}
\]
which once again validates the TUR. For perfect transmission \( \tau = 1 \), the above expression simplifies to
\[
\Delta \beta \frac{\langle j^2 \rangle_c}{\langle j \rangle} = 2 + 12 \frac{(\Delta \beta)^2}{\beta^2} \geq 0.
\]

Weak coupling—Resonant tunneling limit. Another experimentally realizable case is the weak system-bath coupling limit, also known as the sequential tunneling or the resonant tunneling regime. In this case, the transmission function is sharply peaked about resonance frequencies (corresponding to molecular/quantum dot electronic levels), while the Fermi functions are relatively broad (constant) in the range where the transmission function is non-zero i.e., \( k_B T_v > \Gamma, \epsilon_d \), where \( \Gamma \) is the hybridization energy and \( \epsilon_d \) the characteristic energy level of the quantum dots. As a precaution we note that while for charge transport we assess the width of the transmission function itself—relative to the Fermi function—for thermal energy transport we need instead to confirm that the combined function \( (\epsilon - \mu)^4 T_d(\epsilon) \) is sufficiently narrow relative to the alteration of the Fermi functions.

Assuming for simplicity that the central region includes electronic resonances clustered around the energy \( \epsilon_d \), Eq. (32) simplifies to
\[
R_1 = f(\epsilon_d)[1 - f(\epsilon_d)]T_1 - 6f(\epsilon_d)^2(1 - f(\epsilon_d))^2T_2, \tag{35}
\]
where we define
\[
\tau_\alpha = \int_{-\infty}^\infty \frac{d\epsilon}{2\pi\hbar} (\epsilon - \mu)^4 T_d(\epsilon). \tag{36}
\]
These integrals converge if the transmission \( T_d(\epsilon) \) decays faster than \( 1/\epsilon^4 \). Based on Eq. (35) and the inequality \( 0 \leq f(1 - f) \leq 1/4 \), violation of the TUR \( (R_1 < 0) \) occurs when
\[
\frac{T_2}{T_1} > \frac{2}{3}. \tag{37}
\]
This inequality can be satisfied by tuning the electronic parameters of the chain and its hybridization to the metal, as we show below.

In Figs. 2(a), we present numerical results for the TUR in electronic heat transport following Eq. (32). The model consists a junction with its central part including quantum dots in a serial configuration, and we set the spectral function for the baths to be constant and identical, \( \Gamma = \Gamma_L = \Gamma_R \). The \( N \) serial quantum dots are described by a tight-binding model with onsite energy \( \epsilon_d \) and hopping parameter \( \Omega \). The transmission function is calculated in a standard way from the Green’s function of the system and self energy of the baths. [31, 32]

In Fig. 2(a), we display the ratio \( R_1/6G_1 \) as a function of \( \Gamma \) and demonstrate the violation \( (R_1/6G_1 < 0) \) of the TUR within a certain range of this coupling. We present results for \( N = 3 \) and \( N = 4 \). Interestingly, the violation weakly depends on the number of quantum dots in the setup, and in fact for the present parameters, results
saturate beyond $N = 4$. We further show in Fig. 2(b) that the condition of Eq. (37), which was derived under the assumption of resonant tunneling, very well captures the violation region. Note that for $N = 3$ and $N = 4$ the transmission function $T_n(\epsilon)$ for large $\epsilon$ decays as $1/\epsilon^6$ and $1/\epsilon^8$ respectively and therefore the integrals in Eq. (36) converge.

Figure 2 displays a map of $R/LG_1$, while extending beyond the resonant tunneling limit, with $\epsilon_d$ exceeding $T$. Negative values correspond to the breakdown of the TUR, and we identify a significant basin of TUR violation when $\Gamma \ll \epsilon_d < T$.

VI. SUMMARY

We used the steady state fluctuation symmetry to explore the validity of the TUR in thermal transport problems. From the SSFS, we wrote down relationships between transport coefficients and organized an expression for the TUR, which was perturbative in the affinity $\Delta \beta$, and given in terms of nonlinear transport coefficients. The first central result of this work is that negative skewness (to the lowest order in the perturbative expansion around equilibrium) reveals TUR violations. This result is consistent with our previous work on charge transport. Our expansion, building the TUR, is universally valid for quantum and classical problems, as well as for arbitrary interactions in the conducting system.

The second important result of our work is the proof that the TUR is satisfied in harmonic junctions, classical and quantum, irrespective of the underlying stochastic dynamics. Furthermore, to understand the impact of interaction on the violation of the TUR we studied the nonequilibrium spin-boson model. For this model we showed that the TUR can be violated if the phonon baths are structured by employing a hard frequency cut-off for their spectral density function, thus eliminating a significant portion of the transmission function.

Finally, we studied the TUR in fermionic chains. Here, we focused on the resonant transport limit where an analytical condition for TUR violation was derived, demonstrated to be in a quantitative agreement with simulations. The TUR was satisfied when electron transmission probability was small (Poissonian statistics). Violations were identified within a certain range of the system-bath hybridization energy.

The thermodynamic uncertainty relation attracts significant interest given its impact on the performance of thermal machines. Unlike our previous study, which was focused on quantum charge transport and assumed non-interacting electrons, the present work deals with heat exchange in classical and quantum systems, and it exposes that the validity of the TUR depends on the statistics of the participating particles as well as on their interaction. For bosonic systems, interactions are necessary to invalidate the TUR, while fermionic chains can show violations even in the case of noninteracting electrons. Future work will focus on the behavior of the TUR in classical anharmonic chains to understand the impact of quantum effects and many-body interaction on fluctuation-entropy production trade-off relations.

ACKNOWLEDGMENTS

The work of HMF was supported by the NSERC PGSD program. DS acknowledges the NSERC Discovery Grant and the Canada Chair Program. BKA would like to thank G. Schaller, G. Landi and G. Guarnieri for useful discussions. BKA gratefully acknowledges the start-up funding from IISeR Pune, the Max Planck-India mobility grant and the hospitality of the Department of Chemistry at the University of Toronto.

APPENDIX A: HARMONIC JUNCTIONS IN THE CLASSICAL LIMIT

In this Appendix, we prove the validity of TUR directly in the classical limit for harmonic systems starting from the classical generating function for heat exchange obtained by Kundu et al. One can as well reach the classical result from the exact quantum CGF in Eq. 8 by taking the high temperature limit, $\beta_L \hbar \omega \ll 1, \beta_R \hbar \omega \ll 1$. The CGF in the classical limit, $\chi_{HO}^{cl}(\alpha)$, is given by,

$$\chi_{HO}^{cl}(\alpha) = -\int_{-\infty}^{\infty} d\omega \frac{1}{4\pi} \ln \left[ 1 - \mathcal{T}_{HO}(\omega) \frac{\alpha}{\beta_L \beta_R} (i\alpha + (\beta_R - \beta_L)) \right]$$

(A1)

Of course, this classical version also satisfies the steady state fluctuation relation i.e., $\chi_{HO}^{cl}(\alpha) = \chi_{HO}^{cl}(-\alpha + i\Delta \beta)$. Using this CGF, one can immediately derive the current and its noise,

$$\langle j \rangle = \frac{\partial \chi_{HO}^{cl}(\alpha)}{\partial (i\alpha)} \bigg|_{\alpha = 0} = k_B (T_L - T_R) T_1$$

$$\langle j^2 \rangle_c = \frac{\partial^2 \chi_{HO}^{cl}(\alpha)}{\partial (i\alpha)^2} \bigg|_{\alpha = 0} = 2k_B^2 T_L T_R T_1 + k_B^2 (T_L - T_R)^2 T_2$$

(A2)

where $T_n = \int_{-\infty}^{\infty} d\omega \mathcal{T}_n^{\alpha}(\omega)$. Since the last term in the noise expression is positive, one can then write

$$\langle j^2 \rangle_c \geq 2T_L T_R T_1 = \frac{2 \langle j \rangle^2}{\Delta \beta},$$

(A3)

which recovers the TUR. The equality is reached in the equilibrium limit. The TUR is therefore satisfied for a coupled harmonic oscillator system in the classical limit. The proof is general for arbitrary spectral function of the baths and internal parameters for the system. Thus, the TUR is valid independent of the nature of the underlying stochastic dynamics of the oscillators.
We obtain here the weak coupling expression for $R_1$, Eq. (23), directly from the weak-coupling CGF. The CGF of the NESB model in the sequential tunneling limit (weak system-bath coupling) was derived in Ref. 38 using the Redfield quantum master equation approach. It was also received as a special case following the Majorana Green’s function technique.}

\begin{equation}
\chi_{SB}^{\text{redfield}}(\alpha) = -\frac{1}{2}[C(\Delta) - \sqrt{C^2(\Delta) + 4A(\Delta, \alpha)}],
\end{equation}

(B1)

where

\begin{equation}
C(\Delta) = \Gamma_L(\Delta)[1 + 2n_L(\Delta)] + \Gamma_R(\Delta)[1 + 2n_R(\Delta)]
A(\Delta, \alpha) = \Gamma_L(\Delta)\Gamma_R(\Delta)[n_L(\Delta)n_R(\Delta)(e^{i\alpha\hbar\Delta} - 1)
+ n_R(\Delta)n_L(\Delta)(e^{-i\alpha\hbar\Delta} - 1)].
\end{equation}

(B2)

A quick way to get the expression for $R_1$ is to obtain the third cumulant of the current $\langle j^3 \rangle_c = \frac{\partial^3 \chi_{SB}^{\text{redfield}}}{\partial \beta^3(\alpha)} \bigg|_{\alpha = 0}$ and perform a linear response analysis. Following that, we receive

\begin{equation}
R_1 = \frac{\hbar^4\Delta^4\Gamma(\Delta)n(\Delta)[1 + n(\Delta)]}{2[1 + 2n(\Delta)]^3}[n(\Delta)^2 + n(\Delta) + 1]
\end{equation}

(B3)

which is always positive. Here $n(\Delta) = \exp(\beta\hbar\Delta) - 1^{-1}$ is the Bose-Einstein distribution function at temperature $T = 1/k_B\beta$. This expression matches exactly with Eq. (23).

**APPENDIX C: VALIDITY OF TUR FOR ELECTRONIC HEAT TRANSPORT IN THE LOW-TRANSMISSION LIMIT**

We prove here that the TUR relation for non-interacting electronic heat transport is valid at arbitrary far from equilibrium in the low transmission limit. We start from Eqs. (30)-(31), shift the energy around $\mu$, and take the limit of $\gamma_{el}(\epsilon) \ll 1$,

\begin{equation}
\langle j \rangle = \int_0^\infty \frac{d\epsilon}{2\pi\hbar} \epsilon \left[ T_{el}(\epsilon + \mu) + T_{el}(\epsilon - \mu) \right] \left[ f_{L}(\epsilon) - f_{R}(\epsilon) \right],
\end{equation}

\begin{equation}
\langle j^2 \rangle_c = \int_0^\infty \int_0^\infty \frac{d\epsilon_1}{2\pi\hbar} \frac{d\epsilon_2}{2\pi\hbar} \epsilon_1 \epsilon_2 \left[ T_{el}(\epsilon_1 + \mu) + T_{el}(\epsilon_2 - \mu) \right] \left[ f_{L}(\epsilon_1)f_{R}(\epsilon_1) + f_{L}(\epsilon_2)f_{R}(\epsilon_2) \right].
\end{equation}

(C1)

We notice the following inequality involving the Fermi functions for $\epsilon \geq 0$.

\begin{equation}
\left[ f_{L}(\epsilon)f_{R}(\epsilon) + f_{L}(\epsilon)f_{R}(\epsilon) \right] = \coth \left( \frac{\Delta\beta\epsilon}{2} \right) \left[ f_{L}(\epsilon) - f_{R}(\epsilon) \right],
\end{equation}

\begin{equation}
\geq 2 \frac{\Delta\beta\epsilon}{\hbar} \left[ f_{L}(\epsilon) - f_{R}(\epsilon) \right].
\end{equation}

(C2)

Introducing this inequality in the above noise expression immediately implies that $\Delta\beta \frac{\langle j^2 \rangle_c}{\langle j \rangle} \geq 2$, completing our proof on the validity of the TUR in the limit of low transmission probability.

1 A. C. Barato and U. Seifert, Thermodynamic uncertainty relation for biomolecular processes, Phys. Rev. Lett. 114, 158101 (2015).
2 T. R. Gingrich, J. M. Horowitz, N. Perunov, and J. L. England, Dissipation bounds all steady state current fluctuations, Phys. Rev. Lett. 116, 120601 (2016).
3 M. Polettini, A. Lazarescu, and M. Esposito, Tightening the uncertainty principle for stochastic currents, Phys. Rev. E 94, 052104 (2016).
4 P. Pietzonka, A. C. Barato, and U. Seifert, Universal bounds on current fluctuations, Phys. Rev. E 93, 052145 (2016).
5 C. Hyeon and W. Hwang, Physical insight into the thermodynamic uncertainty relation using Brownian motion in tilted periodic potentials, Phys. Rev. E 96, 012156 (2017).
6 J. M. Horowitz and T. R. Gingrich, Proof of the finite-time thermodynamic uncertainty relation for steady-state currents, Phys. Rev. E 96, 020103(R) (2017).
7 K. Proesmans and C. V. den Broeck, Discrete-time thermodynamic uncertainty relation, EPL 119, 20001 (2017).
8 J. P. Garrahan, Simple bounds on fluctuations and uncertainty relations for first-passage times of counting observables, Phys. Rev. E 95, 032134 (2017).
9 A. Dechant, Multidimensional thermodynamic uncertainty relations, J. Phys. A: Math. Theor. 52, 035001 (2019).
10 P. Pietzonka, F. Ritort, and U. Seifert, Finite-time generalization of the thermodynamic uncertainty relation, Phys. Rev. E 96, 012101 (2017).
11 G. Falasco, M. Esposito, and J.-C. Delvenne, Unifying thermodynamic uncertainty relations, Phys. Rev. E 96, 012101 (2017).
12 P. P. Potts and P. Samuelsson, Thermodynamic uncertainty relations including measurement and feedback, arXiv:1906.11360.
13 T. Koyuk, U. Seifert, and P. Pietzonka, A generalization of the thermodynamic uncertainty relation to periodically driven systems, J. Phys. A: Math. Theor. 52, 02LT02 (2018).
14 K. Macieszczak, K. Brandner, and J. P. Garrahan, Unified thermodynamic uncertainty relations in linear response, Phys. Rev. Lett. 121, 130601 (2018).
15 C. Barato, R. Chetrite, A. Faggionato, and D. Gabrielli, Bounds on current fluctuations in periodically driven systems, New J. Phys. 20, 103023 (2018).
16 Y. Hasegawa and T. Van Vu, Generalized thermodynamic uncertainty relation via the fluctuation theorem, arXiv:1902.06376v3.
17 A. M. Timpanaro, G. Guarnieri, J. Goold, and G. T. Landi, Thermodynamic uncertainty relations from exchange fluctuation theorems, arXiv 1904.07574.
18 Y. Hasegawa and T. V. Vu, Thermodynamic uncertainty relation for time-delayed Langevin systems, arXiv:1809.03292.
19 M. L. Rosinberg and G. Tarjus, Comment on Thermodynamic uncertainty relation for time-delayed Langevin systems, arXiv:1810.12467.
20 K. Brandner, T. Hanazato, and K. Saito, Thermodynamic Bounds on Precision in Ballistic Multiterminal Transport, Phys. Rev. Lett. 120, 090601 (2018).
21 D. Gupta and A. Maritan, Thermodynamic uncertainty relations via second law of thermodynamics, https://arxiv.org/abs/1905.08854.
22 K. Ptaszynski, Coherence-enhanced constancy of a quantum thermo-electric generator, Phys. Rev. B 98, 085425 (2018).
23 B. K. Agarwalla and D. Segal, Assessing the validity of the thermodynamic uncertainty relation in quantum systems, Phys. Rev. B 98, 155436 (2018).
24 J. Liu and D. Segal, Thermodynamic uncertainty relation in quantum thermo-electric junctions Phys. Rev. E 99, 062141 (2019).
25 S. Kheradshou, N. Dashti, M. Misiorny, P. P. Potts, J. Splettstoesser, and P. Samuelsson, Power, efficiency and fluctuations in a quantum point contact as steady-state thermo-electric heat engine, arXiv:1904.03912.
26 U. Seifert, Stochastic thermodynamics, fluctuation theorems, and molecular machines, Rep. Prog. Phys. 75, 126001 (2012).
27 U. Seifert, Stochastic thermodynamics: Principles and perspective, Eur. Phys. J. B. 64, 423 (2008).
28 G. Guarnieri, G. T. Landi, S. R. Clark, and J. Goold, Thermodynamics of precision in quantum non-equilibrium steady states, arXiv:1901.10428.
29 M. Esposito, U. Harbola, and S. Mukamel, Nonequilibrium fluctuations, fluctuation theorems, and counting statistics in quantum systems, Rev. Mod. Phys. 81, 1665 (2009).
30 M. Campisi, P. Hänggi, and P. Talkner, Colloquium: Quantum fluctuation relations: Foundations and applications, Rev. Mod. Phys. 83, 771 (2011).
31 S. Pal, T. S. Mahesh, and B. K. Agarwalla, Experimental verification of quantum heat exchange fluctuation relation, arXiv:1811.07291.
32 K. Saito and Y. Utsumi, Symmetry in full counting statistics, fluctuation theorem, and relations among nonlinear transport coefficients in the presence of a magnetic field, Phys. Rev. B 78, 115429 (2008).
33 K. Saito and A. Dhar, Fluctuation theorem in quantum heat conduction, Phys. Rev. Lett. 99, 180601 (2007).
34 K. Saito and A. Dhar, Generating function formula of heat transfer in harmonic networks, Phys. Rev. E 83, 041121 (2011).
35 B. K. Agarwalla, B. Li, and J.-S. Wang, Full-counting statistics of heat transport in harmonic junctions: Transient, steady states, and fluctuation theorems, Phys. Rev. E 85, 051142 (2012).
36 D. Segal and A. Nitzan, Spin-boson thermal rectifier, Phys. Rev. Lett. 94, 034301 (2005).
37 K. A. Velizhanin, H. Wang, and M. Thoss, Heat transport through model molecular junctions: A multilayer multiconfiguration time-dependent Hartree approach, Chem. Phys. Lett. 460, 325 (2008).
38 L. Nicolin and D. Segal, Non-equilibrium spin-boson model: Counting statistics and the heat exchange fluctuation theorem, J. Chem. Phys. 135, 164106 (2011).
39 C. Wang, J. Ren, and J. Cao, Nonequilibrium energy transfer at nanoscale: A unified theory from weak to strong coupling, Sci. Rep. 5, 11787 (2015).
40 T. Yamamoto, M. Kato, T. Kato, and K. Saito, Heat transport via a local two-state system near thermal equilibrium, New J. Phys. 20, 093014 (2018).
41 J. Liu, C.-Y. Hsieh, D. Segal, and G. Hanna, Heat transfer statistics in mixed quantum-classical systems, J. Chem. Phys. 149, 224104 (2018).
42 M. Kilgour, B. K. Agarwalla, and D. Segal, Path-integral methodology and simulations of quantum thermal transport: Full counting statistics approach, J. Chem. Phys. 150, 084111 (2019).
43 A. Kelly, Mean field theory of thermal energy transport in molecular junctions, J. Chem. Phys. 150, 204107 (2019).
44 H. M. Friedman, B. K. Agarwalla, and D. Segal, Quantum energy exchange and refrigeration: A full-counting statistics approach, New J. Phys. 20, 083026 (2018).
45 B. K. Agarwalla and D. Segal, Energy current and its statistics in the nonequilibrium spin-boson model: Majorana fermion representation, New J. Phys. 19 043030 (2017).
46 L. S. Levitov and G. B. Lesovik, Charge distribution in quantum shot noise, JETP Lett. 58, 230 (1993).
47 L. S. Levitov, H.-W. Lee, and G. B. Lesovik, Electron counting statistics and coherent states of electric current, J. Math. Phys. 37, 4845 (1996).
48 K. Schmieder, Full counting statistics for noninteracting fermions: Exact results and the Levitov-Lesovik formula, Phys. Rev. B 75, 205329 (2007).
49 K. Schmieder, Full counting statistics for noninteracting fermions: exact finite-temperature results and generalized long-time approximation, J. Phys.: Condens. Matter 21, 495306 (2009).
50 I. Klich, in Quantum Noise in Mesoscopic Physics, NATO Science Series II, Vol. 97, edited by Yu. V. Nazarov (Kluwer, Dordrecht, 2003).
51 D. Bernard and B. Doyon, Full Counting Statistics in the resonant-level model, J. Math. Phys. 53, 122302 (2012).
52 A. Kundu, S. Sabhapandit, and A. Dhar, Large deviations of heat flow in harmonic chains, J. Stat. Mech P03007 (2011).
53 A. Nitzan, Chemical Dynamics in Condensed Phases: Relaxation, Transfer and Reactions in Condensed Molecular Systems, (Oxford University Press, UK 2006).
54 M. Di Ventra, Electrical Transport in Nanoscale Systems, (Cambridge University Press, Cambridge, U.K., 2008).