Entanglement in bosonic systems

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We present a technique to resolve a Gaussian density matrix and its time evolution through known expectation values in position and momentum. Further we find the full spectrum of this density matrix and apply the technique to a chain of harmonic oscillators to find agreement with conformal field theory in this domain. We also observe that a nonconformal state has a divergent entanglement entropy.

I. INTRODUCTION

Entanglement is today considered a fundamental resource in nature when it comes to quantum computation and information, and measures of entanglement has become a major field of research. In particular, entanglement in condensed-matter systems and the entanglement’s critical behavior is well investigated. In these terms, the entanglement entropy is an analytically well suited tool for investigating the properties of ground states in condensed-matter systems. In this paper, we focus on bosonic states with Gaussian wave functions, and the entanglement properties of the ground state of a simple harmonic chain, which belongs to this class.

We consider the notion of entanglement entropy, that is, considering a quantum system denoted $C$ in a pure state with wave function $|\psi\rangle$, we trace out some portion $B$ to obtain the density matrix of the remaining space $A$ as $\rho_A = \text{Tr}_B |\psi\rangle\langle\psi|$. Then the entanglement of $A$ with respect to $B$ is well defined by the entropy (measured in ebits) of the reduced density matrix $\rho_A$:

$$S_A = -\text{Tr} \rho_A \log_2 \rho_A.$$  

(1)

The measure of entropy in units of ebits is customary and we will use logarithms base two throughout the paper. This procedure is well established, and works well for all cases where the entire strip $C$ is in a pure state, though entanglement measures for mixed states are still incomplete. Most work has been focused on this entanglement in spin chains, but we will focus on a one-dimensional bosonic strip.

At critical points in a parameter space we have scale and translational invariance, and thus expect the theory to be conformally invariant, and one can use this fact to efficiently detect critical systems. As was computed by Holzhey, Larsen, and Wilczek, conformally invariant systems in 1+1 dimension can be considered as a string of length $\Lambda$ of which we trace out some fraction $1 - \sigma \in [0, 1]$, and then the entanglement entropy of the remaining space with respect to the rest is

$$S(\sigma) = \frac{c + \bar{c}}{6} \log \left( \frac{\Lambda}{\pi \epsilon \sin(\pi \sigma)} \right).$$  

(2)

Here $c$ and $\bar{c}$ are the holomorphic and anti holomorphic charges respectively. $\epsilon$ is some cutoff parameter that we will consider arbitrary. When considering the limit $\sigma \ll 1$, the formula reduces simply to $S \sim \log \sigma \Lambda$, which has been a matter of keen interest. However, we will focus on any $\sigma$, in particular, when keeping the $\Lambda$ constant provides a very specific signature of a conformally invariant system. Thus this formula presents two independent (as long as $\epsilon$ is held constant) signatures of a finite conformal system. First the logarithmic divergence of the entropy as $\sigma$ is held constant while $\Lambda$ increases, and second the characteristic $\log \sin$ signature when $\Lambda$ is constant and $\sigma$ varies.

Another, more trivial, measure of the entanglement of a reduced density matrix is the product state identification

$$E_M = 1 - \text{Tr} \rho_A^M, \quad M \geq 2,$$  

(3)

which is zero for a product state, and unity for a maximally entangled state. This measure is equivalent to the R"enyi entropy and is not well suited for much more than to single out a pure state, as with increasing $M$ any entangled state will converge to zero in this measure,

$$\lim_{M \to \infty} E_M = \begin{cases} 1 & \text{entangled state} \\ 0 & \text{product state.} \end{cases}$$

The density matrix contains all information of any system in a mixed or pure state, and computing vital physical information on any such system is mostly determined by the eigenvalues of the density matrix. Therefore there is a great need to compute these in an efficient way. In particular, considering a bosonic system in one spatial dimension in its ground state, it can be modeled as a harmonic chain. The quantum correlations as measured by the entanglement are nonzero, as will be shown. Thus the vacuum itself is entangled, which is a highly intriguing result. Our main purpose in this paper is twofold, first we wish to demonstrate how to compute this entanglement from the vacuum ground state, and second we

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wish to see how and if the conformal invariance identified by Eq. (2) arises in this case. The free boson is known to have a central charge $c = \bar{c} = 1$, so the prefactor in $S(\sigma)$ reduces to simply $1/3$. It turns out that we indeed have a conformally invariant ground state, as identified by the two conformal signatures provided. As we will demonstrate in Sec. III, the entropy diverges in the massless limit, but it nevertheless seems conclusive that the theory is conformally invariant below a certain threshold mass.

In this paper we will in Sec. III compute how a Gaussian density matrix can be recovered from expectation values in position and momentum, and how this can be used to compute the time evolution of the density matrix. This can also be computed in terms of Weyl representation, though we apply a more explicit representation here. In Sec. III we compute the two entanglement measures written in this section from the Gaussian density matrix, and finally in Sec. IV we apply the formulas to the ground state of a harmonic chain.

II. GAUSSIAN DENSITY MATRICES

A. Recovery of the density matrix from expectation values

It is known that a Gaussian state in a quantum harmonic oscillator potential will evolve in such a way that the Gaussian shape is preserved at all times. However, the different parameters of the Gaussian state may also evolve in time so that the exact appearance of the density matrix may be difficult to predict. Consider a Gaussian density matrix of $N$ particles with positions $q_i$,

$$
\rho(q, q') = \frac{\sqrt{\det (A' - C'')}}{\pi^{N/2}} \exp \left[ -\frac{1}{2} (q, A' q) + \frac{1}{2} (q', C'' q') - d' q + d q_i \right]
$$

summing over repeated indices. Here $A$, a positive, symmetric $N \times N$ matrix, while $C$ is a Hermitian $N \times N$ matrix, and $d$ is an $N$-dimensional vector. We use $A'$, $C'$, and $d'$ to denote the real parts of $A$, $C$, and $d$, respectively. We further denote the imaginary parts such that $A = A' + i A''$, etc. Positions and momenta are real valued. The matrix $A' - C'$ must be invertible and positive in order to have an orthonormalizable density matrix. Also, in order to keep $\rho$ positive, that is that $\langle \psi | \rho | \psi \rangle > 0$ for any $| \psi \rangle$, one must have $A' > 0$. Hence follows that even $C$ must be positive. Next we define three matrices of variances in position and momentum,

$$
Q_{ij} = \langle q_i q_j \rangle - \langle q_i \rangle \langle q_j \rangle,
$$

$$
P_{ij} = \langle p_i p_j \rangle - \langle p_i \rangle \langle p_j \rangle,
$$

$$
S_{ij} = \frac{1}{2} (q_i p_j + p_j q_i) - \langle q_i \rangle \langle p_j \rangle.
$$

(5)

$Q$ and $P$ are symmetric, while all are real. Furthermore, since the system in translationally invariant we can assume $Q_{ij} = f(|i - j|)$ for some function $f$, and similarly with the other matrices. Thus $Q$, $P$, and $S$ are Toeplitz matrices, potentially simplifying numerical computations. Toeplitz matrices are known to be central also in the study of quantum spin chains [12]. These three matrices can be assumed known in a given model of a bosonic quantum system. We refer to the set of variables in the density matrix as $\Theta = \{ A, C, d \}$, and the expectation value matrices as $\Xi = \{ Q, P, S, \langle q \rangle, \langle p \rangle \}$. A simple count shows that the two sets both have $2D^2 + 3D$ degrees of freedom, and thus the two sets may contain equal amounts of information. The expectation value matrices can be computed for a given density matrix through a straightforward calculation that computes expectation values of an operator $\hat{O}$ as

$$
\langle \hat{O} \rangle = \text{Tr} \hat{O} \rho(q, q') = \int dq dq' \delta(q - q') \hat{O} \rho(q, q').
$$

This means that the expectation value matrices $\Xi[\Theta]$ can be computed in terms of the parameters in the density matrix,

$$
Q = \frac{1}{2} (A' - C')^{-1},
$$

$$
P = A - (A - C)Q(A - C)^T,
$$

$$
S = -Q(A'' + C''),
$$

$$
\langle q \rangle = 2Qd',
$$

$$
\langle p \rangle = -2(A'' - C'')Qd' + d''.
$$

(6)

Note that since $A' - C'$ is invertible, $Q$ is well defined invertible. This set of matrix equations can then be inverted to yield $\Theta[\Xi]$,

$$
A' = P + \frac{1}{4} Q^{-1} - S^T Q^{-1} S,
$$

$$
A'' = \frac{1}{2} (S^T Q^{-1} + Q^{-1} S),
$$

$$
C' = P - \frac{1}{4} Q^{-1} - S^T Q^{-1} S,
$$

$$
C'' = \frac{1}{2} (S^T Q^{-1} - Q^{-1} S),
$$

$$
d' = \frac{1}{2} Q^{-1} \langle q \rangle,
$$

$$
d'' = (A' + C'') \langle q \rangle + \langle p \rangle.
$$

(7)

We note that $S = 0$ implies both $A$ and $C$ to be real, and that in a state where $\langle \dot{q} \rangle = \langle \dot{p} \rangle = 0$, $d = 0$. This way it is simple to recover the density matrix given $\Xi$. A strategy to find the nonlinear time evolution of the density matrix is sketched in Fig. 4, one would use the fact that the time evolution of $\Xi$ is much simpler to find than that of $\Theta$. And since the formulas above enables us to switch between the two sets in an easy manner, we can take the time evolution via $\Xi$ instead of taking it directly on $\Theta$. 




we will show in this section. To this end, consider what
in the range $[0$ $t$ $\mathbf{m}$]
three-vector describing external forces. The equations of
motion become $\ddot{\mathbf{r}} = \mathbf{F}$, which implies four
coupled differential equations for the time evolution of the
expectation value matrices

$$
\dot{Q} = \mathcal{T}(S), \quad \dot{P} = -\mathcal{T}(\Omega S), \quad \dot{S} = P - \frac{1}{2}Q\Omega,
$$

where we have defined the symmetrizing operator $\mathcal{T}(A) = A + A^T$. These relations specify the time evolu-
tion of the system, given some initial condition. Also,
combining the first and third equation gives also

$$
\dot{Q} + 2\dot{P} - \frac{1}{2}\{Q, \Omega\} = 0,
$$
or, in the case of a diagonal $\Omega, \Omega_{ij} = \delta_{ij}\omega_i$, this reads

$$
\dot{Q}_{ij} + 2\dot{P}_{ij} - \frac{1}{2}(\omega_i + \omega_j)Q_{ij} = 0.
$$

III. ENTANGLEMENT MEASURES FOR GAUSSIAN DENSITY MATRICES

The eigenvalues of Eq. (4) can be found explicitly as
we will show in this section. To this end, consider what
we will refer to as the single-particle density matrix

$$
\rho^0(x, x'; \eta, d) = \frac{e^{-d^2/2}}{\sqrt{\pi}} \delta^2 \sqrt{1 - \eta} \sqrt{1 - \eta}
\times \exp\left[-\frac{1}{2} (x^2 + x'^2) + \eta xx' + dx + dx'ight].
$$

Here $d = 1 + d_2/2$ is a complex number, while $\eta$ is real
in the range $[0, 1]$. The latter constraint is consistent
with the mentioned requirement on the density matrix
that $A' - \mathbf{C}$ is positive. We will omit the two latter
parameters in the argument list of $\rho^0$ when there is no
risk of confusion. The eigenvalue equation of this density
matrix is

$$
\int dx' \rho^0(x, x')\Psi_n(x') = \lambda_n \Psi_n(x).
$$

Also, the density matrix obeys

$$
\rho^0(x + x_0, x' + x_0) = e^{id'x} \rho^0(x, x'; \eta, 0)e^{-id'x'},
$$

where $x_0 = d'/1 - \eta$. This means that the scaled eigen-
function $\tilde{\Psi}_n(x) = e^{id'(x-x_0)}\Psi_n(x - x_0)$ also is an eigen-
vector of $\rho^0(x, x')$ with the same eigenvalue as $\Psi_n(x)$, and
hence the eigenvalues are independent of the displacement $d$.
Furthermore, Eq. (11) shows that any traces
$\text{Tr} \rho^0$ are invariant under $d$.

Now, to find the eigenvalues of the single-particle den-
sity matrix when $d = 0$, consider the Green’s function
$G(z, z'; \tau) = \langle z| e^{-\tau \mathcal{H}} | z' \rangle$, with $\mathcal{H}$ being the single har-
monic oscillator Hamiltonian $H = -\frac{p^2}{2} + \frac{1}{2}m^2$, and the states $|z\rangle$ the position eigenstates of the quantum
harmonic oscillator. The eigenvalue result for this matrix
is, with $\langle z|\Psi\rangle_n = \Psi_n(z)$ and $|\Psi\rangle_n$, the energy eigenstates

$$
\int dz' G(z, z'; \tau)\Psi_n(z') = e^{-\omega\tau(\eta+1/2)}\Psi_n(z).
$$

Furthermore, $G$ must solve the initial value problem

$$
\frac{-\partial}{\partial \tau} G(z, z'; \tau) = 0,
$$

lim $\tau \rightarrow 0$ $G(z, z'; \tau) = \delta(z - z'),$

which, through a scaling of variables, $x = z\sqrt{\omega\coth(\omega\tau)}$, has the solution

$$
G(x, x'; \tau) = \frac{\omega}{2\pi \sinh(\omega\tau)} \times \exp\left[-\frac{1}{2} (x^2 + x'^2) + \frac{xx'}{\cosh(\omega\tau)}\right].
$$

There also exists a complex solution, which we disregard
as unphysical. This solution means that if we iden-
tify $\eta = 1/\cosh(\omega\tau)$, we can conclude that $\rho^0$ has the
infinite set of eigenvalues

$$
\lambda_n = \sqrt{\omega\coth(\omega\tau)} e^{-\omega\tau(\eta+1/2)} = \lambda_0 \xi^n
$$

for $n \in \{0, 1, 2, \ldots\}$,

$$
\xi = e^{-\omega\tau} = \eta/(1 + \sqrt{1 - \eta^2}).
$$

Hence we have identified the eigenvalues and eigen-
functions of our density matrix in the case $d = 0$, which
is sufficient to know the eigenvalues also in the $d \neq 0$
case.

Now, let us again turn to our general density matrix
and show that this can be transformed into the sim-
ple form of Eq. (10). In general, the matrix $A''$
which means that the density matrix \( \rho \) may be rewritten as
\[
\tilde{\rho} = \tilde{O} \sqrt{a^{-1} O^T C O a^{-1}} \tilde{O}.
\] (14)

Note that both \( O \) and \( \tilde{O} \) are real, since \( C \) is hermitian and \( A \) is symmetric. The coordinates and vectors \( d \) transform as
\[
\tilde{q} = \tilde{O} \sqrt{a} O^T q, \quad \tilde{d} = \tilde{O} \sqrt{a^{-1}} O^T d,
\]
which means that the density matrix \( \tilde{\rho} \) may be rewritten as
\[
\tilde{\rho}(q,q') = \prod_{i=1}^{N} \rho^{0}(q_{i}, q'_{i}; \eta_{i}, \tilde{d}_{i}),
\] (15)
\( \eta_{i} \) being the diagonal elements of \( \tilde{\eta} \) and \( \tilde{\rho}(q,q') = \rho(\tilde{q}, \tilde{q}') \). In other words, the density matrix \( \tilde{\rho} \) may be expressed as a product of single-particle density matrices [15]. Thus each of the product terms in Eq. (15) gives the infinite set of eigenvalues (13), and we may write the eigenvalues for the total matrix as
\[
\lambda_{n_{1}, n_{2}, \ldots, n_{N}} = \prod_{i=1}^{N} \lambda_{n_{i}} = \Lambda_{0} \prod_{i=1}^{N} \xi_{i}^{n_{i}}.
\] (16)

Here we have defined \( \Lambda_{0} = \prod_{i} \lambda_{n_{i}}^{(i)} \). Here we have let \( \lambda_{0} \rightarrow \lambda_{n_{i}}^{(i)}, \eta_{i} \rightarrow \eta_{i}, \text{ and } \xi_{i} \rightarrow \xi_{i} \) from Eq. (15), where the index \( i \) refers to particle number. \( \Lambda_{0} \) can be calculated from the normalization condition
\[
\text{Tr} \rho = \sum_{n_{1}=1}^{\infty} \cdots \sum_{n_{D}=1}^{\infty} \lambda_{n_{1}, \ldots, n_{D}} = 1
\]
giving \( \Lambda_{0} = \prod_{i=1}^{N} (1 - \xi_{i})/\xi_{i} \). This sets the stage for the calculation of entropy and traces over the density matrix.

### Entanglement measures

Having obtained the density matrix eigenvalues, the entropy \( S = -\text{Tr} \rho \log \rho \) is easily calculated,
\[
S = - \sum_{n_{1}=1}^{\infty} \cdots \sum_{n_{N}=1}^{\infty} \lambda_{n_{1}, \ldots, n_{N}} \ln \lambda_{n_{1}, \ldots, n_{N}}
= - \sum_{i=1}^{N} \left[ \log \left( 1 - \xi_{i} \right) + \frac{\xi_{i} \log \xi_{i}}{1 - \xi_{i}} \right],
\] (17)
a result that agrees exactly with that found by Srednicki [12].

We can proceed to find exact formulas for the entanglement measures \( E_{M} \equiv 1 - \text{Tr} \rho^{M} \) where \( M \geq 2 \), for which we find that
\[
E_{M} = 1 - \prod_{i=1}^{N} \frac{(1 - \xi_{i})^{M}}{1 - \xi_{i}^{M}}.
\] (18)

These two entanglement measures are now quite straightforward to compute. Since \( Q \) is a Toeplitz matrix, the inversion involved can be done efficiently with existing linear algebra packages. Also the computation involves two diagonalizations of real, symmetric \( N \times N \) matrices, which is also numerically straightforward and efficient. We will focus primarily on the entropy due to its analytical usability.

### IV. APPLICATION

We now apply the above formalism to the quantum Klein-Gordon field \( \phi_{n} \), defined by the Lagrangian with \( \xi = 0 \) and \( \Omega_{ij} = \delta_{ij}/\kappa \) on \( N \) lattice points with periodic boundary conditions [16]. The lattice constant is denoted \( a \), and the system size is thus \( \Lambda = a N \). This field has the Fourier expansion in bosonic creation and annihilation operators \( a_{k} \) and \( a_{k}^{\dagger} \) respectively;
\[
\phi_{n} = \sum_{k} \frac{1}{2 \Lambda \omega_{k}} \left( a_{k} e^{-i(k \omega_{k} - \omega_{k} t)} + a_{k}^{\dagger} e^{i(k \omega_{k} + \omega_{k} t)} \right)
\] (19)
and conjugate field
\[
\pi_{n} = -i \sum_{k} \sqrt{\frac{\omega_{k}}{2 \Lambda}} \left( a_{k} e^{-i(k \omega_{k} - \omega_{k} t)} - a_{k}^{\dagger} e^{i(k \omega_{k} + \omega_{k} t)} \right).
\] (20)
In this discrete field theory the dispersion relation is
\[
\omega_{k}^{2} = \frac{4}{a^{2}} \sin^{2}(k/2) + \kappa^{2},
\]
and the sum is over all allowed wave vectors \( k \) in this space. We have the nonzero commutation rules (at equal time)
\[
[i \phi(x), i \pi(x')] = i \delta(x - x'), \quad [a_{k}, a_{k'}^{\dagger}] = \delta_{k, k'},
\]
indicating the harmonic oscillator nature of the system, with positions \( \phi_{n} \) and momenta \( \pi_{n} \). When rescaling this theory it is imperative to keep \( \Lambda \) constant in order to maintain any scaling invariance and hence any conformal invariance in the model. The \( \Xi \)-matrices of the ground state are the Toeplitz matrices
\[
Q_{mn} = \langle \phi_{n} \phi_{m} \rangle = \frac{1}{2N} \sum_{k} \frac{1}{\omega_{k}} e^{ik(m-n)},
\]
\[
P_{mn} = \frac{1}{2N} \sum_{k} \omega_{k} e^{ik(m-n)},
\] (21)
\[
S_{mn} = 0,
\]
with the sums running over all lattice points numbered \(i\), and \(k = 2\pi i/N\). \(Q\) and \(P\) are thus essentially finite Fourier transforms of \(\omega_k^{\pm 1}\). Now, since \(S\) is vanishing, we can conclude that both \(A\) and \(C\) are real. As shown this is all we need to recover the density matrix, and calculating the traces.

In the limit of a massless theory, \(\kappa \to 0\), \(Q_n\) will diverge due to the \(k = 0\) (zero mode) term in the sum (21). We can exclude this zero mode from the sum by summing over lattice point \(m\) using \(k = \pi(2m + \alpha)/N\) and choose \(\alpha = 0\) (1) to compute with (without) the zero mode. This corresponds to using (anti)periodic boundary conditions on the field \(\phi\).

When finding the entropy we define some of the points as “inside” (region \(A\)) and some as “outside” (region \(B\)), and trace over the outside to find the geometric entropy of the inside region. Formally, this amounts to calculating the \(\Xi\) matrices for the \(\sigma N\) inside oscillators, \(\sigma\) being the fraction of the entire system that \(A\) constitutes. When the \(\Xi\)s are known, we can compute the density matrix and the entropy for the state. Clearly, the entropy is symmetric with respect to interchange of \(A\) and \(B\), so any entanglement measure will be symmetric around \(\sigma = 1/2\). Hence we expect a maximal entanglement at this half size, and we can consider the half size entanglement with respect to \(\kappa\) and \(N\). Results for \(E_2\) at half size is shown in Fig. 2. Hence we conclude that the massive \(\kappa \to \infty\) system in nonentangled, while we have a transition to a maximally entangled case in the massless system \(\kappa \to 0\). Also, the entanglement is larger for larger systems, and the transition occurs at a larger mass in a larger system. The correlations are greater in a large system, and the inertia of the mass \(\kappa\) must be larger to prevent them.

We now consider the entanglement entropy. The half size measure for both \(\alpha = 0\) and \(\alpha = 1\) are shown in Fig. 3. We see that when the zero mode is included, the entropy diverges in the massless limit, while without the zero mode it converges to some system dependent value. In Fig. 4 we see how the half size entropy diverges logarithmically with system size for small \(\kappa\). Most remarkably, this divergence occurs regardless of the zero mode, and the correct scaling factor \(c/3\) is reproduced.

Finally, we look at the second feature of the conformal signature, namely the log sin shape of a finite system. For the \(\alpha = 0\) case this is shown in Fig. 4 and we again see a good characteristic of the conformal system in a massless system. For the massive system, the entanglement saturates, and fits the signature only at small \(\sigma\), which is also observed earlier in noncritical quantum spin chains.
FIG. 5: $S(\sigma)$ for some values of $\kappa$ with $\alpha = 0$. Here, $N = 100(\pm)$ and $N = 10(\bigcirc)$. From top to bottom, $\kappa = 10^{-4}$, $\kappa = 10^{-2}$, $\kappa = 1$, and $\kappa = 10^2$. The full lines are $f(\sigma) = \frac{1}{\pi} \log_2(\sin \pi \sigma) + a$, with $a$ chosen to fit the lines at the ends. For large $\kappa$, the entanglement saturates and does not obey the conformal signature. In the massless limit the conformal signature is obeyed, even at $N = 10$.

FIG. 6: The largest entropy contributions in the sum $S = \sum_n s_n$, with ordered terms. The graph shows the terms for system size $N = 100$, $\alpha = 0$ (\pm), and $\alpha = 1$ (\times) and also system size $N = 25$, $\alpha = 0$ (\star), and $\alpha = 1$ (\square). All values are for an essentially massless theory with $\kappa = 10^{-3}$.

When the zero mode is omitted, however, the log sin signature is not present, as the system is identical to the $\alpha = 0$ case for some massive $\kappa$, a state that is not conformally invariant. But then it is nevertheless notable that the logarithmic divergence with system size still fits the conformal theory although the fixed-size signature does not, and the state is not conformally invariant.

It is valuable to note that the conformal signature in the entanglement entropy is present even in small systems, such as $N = 10$. This has been seen in quantum spin chains earlier, there enabling efficient identification of criticality.

As a final feature, we investigate which modes, or terms, contribute to the entanglement entropy in the expansion $S = \sum_n s_n$ where the terms are defined by the sum (17). The individual terms are shown in Fig. 6 where we see that for the conformally invariant case, the eigenvalues fall of faster than exponentially, which indicates that only very few terms in the sum are needed to compute the entropy. Indeed, to compute the entropy to within an error $\pm 10^{-6}$ one needs only eight out of a possible 50 terms in the expansion. Also, the figure shows that for the nonconformal state without the zero mode, the eigenvalues are paired since in a not scale invariant, though translationally invariant state, any mode with a nonzero impulse will be degenerate with another state of opposite impulse.

V. CONCLUSION

We have shown how to compute the entanglement entropy of a chain of bosonic harmonic oscillators, using scalings and rotations of the density matrix to put it in a single-particle form. The results show what we call conformal signatures in the massless limit. Moreover, the results show that even a nonconformal state can show a logarithmic divergence as predicted by conformal field theory, but not the log sin signature that the author believes to be a uniquely conformal feature.

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