Mathematical modeling of the propagation of precursors of the front edges of cracks as spatial curves on the fronts of waves of a strong discontinuity of rates and stresses

N D Verveiko\textsuperscript{1,3}, A I Shashkin\textsuperscript{1,4} and S E Krupenko\textsuperscript{2}

\textsuperscript{1}Voronezh State University, University Sq. 1, Voronezh, 394018, Russia
\textsuperscript{2}Joint-stock company Voronezh pilot plant of software products, (VOZPP) Tekstilshchikov St., Building 10a, Voronezh, 394026, Russia
E-mail: \textsuperscript{3}verveyko2017@yandex.ru, \textsuperscript{4}dean@amm.vsu.ru

Abstract. Crack propagation occurs at the macro level with finite elastic displacements, deformations and stresses far from the front edge of the crack. In the neighborhood of the front spatial edge of a crack during its development, the behavior of the material is really inelastic, when the changes take place at the micro and nano levels.

In this paper with the use of the Bingham model of an elasto viscoplastic (EVP) material, a small $\delta$-neighborhood is chosen around the front spatial edge of a crack. The neighborhood has the shape of a cylinder with a curvilinear axis. It is shown that the wave fronts of plastic deformation are waves of longitudinal or transverse deformation and bear the front edges of the precursors these are longitudinal shear cracks and transverse shear cracks or detachment cracks.

Ordinary differential equations of transfer of the intensity of tangential shear stresses and normal detachment stresses behind the precursor of the crack edge depending on the path traveled by the front edge of the crack are constructed. Exact solutions are obtained in the case of a non-stressed material. The finiteness of the length of a growing crack in an EVP material has made it possible to clarify the classical criteria for brittle fracture.

1. Introduction

The use of a mathematical model of an ideally elastic material to describe the crack dynamics requires setting boundary conditions on the moving boundary of a crack. The front edge of the crack moves at an unknown rate, that leads to the formulation of difficult mathematical problems with a boundary unknown in advance. In this statement, the evaluation of the stress state at the crack tip leads to unlimited stress values and, therefore, the incorrect application the elastic deformation model to real brittle materials having really irreversible deformations in the crack tip neighborhood [1–10].

An important point in the theory of brittle fracture is the equivalence of the energy [2–5, 11–19] and force [14, 15] criteria for the critical state of a crack, starting with which the crack propagates (develops). This allows us to consider the propagation of detachment and shear cracks taking into account the material detachment strength and shear strength criteria.

In many experimental studies a change in the structure of the material is observed in a small neighborhood of tips of detachment and shear cracks [20–22].
Various estimates [2–5, 11–15] demonstrate that the crack propagation rate is close to the rate \( c_1 \) of longitudinal wave or rate \( c_2 \) of shear wave. Basing on various models for propagation of the vertex of cut as the tip of a crack [6, 14, 15] it can be said that the influx of energy into the emerging crack surface occurs in the form of longitudinal waves, shear waves and Rayleigh waves (\( c_1 > c_2 > c_R \)).

The limit values of the propagation rates of the tips of detachment cracks and shear cracks are the propagation rate of longitudinal waves \( c_1 \) carrying longitudinal and volume deformations and the rate of shear waves \( c_2 \) carrying deformation shear in the absence of volumetric deformations [1, 16, 17], respectively. In the proposed limit variant the propagation of the tips of detachment cracks and shear cracks is local (isolated) when the propagating tip of a crack does not change the stress state ahead of itself.

Thus, it is justified to consider problems on the propagation of crack tips with limit rates \( c_1 \) and \( c_2 \) in prestressed material and to study the inelastic behavior of the material in the neighborhood of the crack tip. In this case, the brittle fracture of real solid materials is modeled more accurately.

The presence of viscosity and plasticity of the material makes it possible to take into account the energy dissipation when studying the propagation of a crack tip in real materials and to investigate in time the process of crack propagation in a stressed material. This takes into account both the loss of energy at viscous and plastic strain rates and the strains themselves, as well as the release of elastic energy at the crack tip. The model of an EVP material allows us to consider the dynamic nature of the crack tip propagation and estimate the possible propagation length of a growing crack [10, 23].

Taking into account the viscous and plastic properties of the material the finiteness of the length of growing crack makes it possible to clarify the classical criteria for brittle fracture.

### 2. The mathematical model of the spatial non-stationary motion of the EVP material

Consider the stress-strain state of a solid deformable material in the neighborhood of the tip of detachment crack. Until achieving the von Mises condition of plasticity with a limit of plasticity \( K \) the model of EVP material [16, 24] allows only elastic deformations and includes in full deformations also plastic deformations with viscous resistance at the rates of plastic deformations [10, 16, 23–25]:

\[
\sigma_{ij} = \lambda e_{kk} \delta_{ij} + 2 \mu e_{ij} - 2 \mu e_{ij}^p; \quad (1)
\]

\[
e_{ij} = e_{ij}^e + e_{ij}^p; \quad e_{ij}^p = e_{ij}^v; \quad (2)
\]

\[
e_{ij}^p = \frac{\partial e_{ij}^p}{\partial t} = \frac{(I_2 - K\sqrt{2}) \sigma'_{ij}}{I_2 \eta}. \quad (3)
\]

Here \( e_{ij} = \frac{u_{i,j} + u_{j,i}}{2} \) are total deformations according to Cauchy; \( u_{i,j} = \frac{\partial u_i}{\partial x_j} \), \( I_2 = (\sigma_{ij}' \sigma_{ij}')^{\frac{1}{2}} \); \( \sigma'_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij} \) are deviator components of the stress tensor \( \sigma_{ij} \); \( u_i \) are displacements of material particles; \( \lambda, \mu \) are the Lame elastic parameters; \( \eta \) is the viscosity coefficient; \( t \) is the time; \( K \) is the plasticity limit; indices \( p \) and \( v \) at the top are assigned respectively to plastic and viscous components.

A united stress-strain curve for the model of this material does not exist, and the strain rate plays the role of the dynamic hardening parameter. This is confirmed by dynamic experiments, when the dynamic plasticity limit increases with strain rate increasing [1, 10, 23, 24].
The von Mises plasticity condition plays the role of a criterion for the occurrence of plastic strain rates, so this condition is formulated as the following inequalities

\[
\varepsilon_{ij}^p = 0 \quad \text{when} \quad \sigma_{ij}^p \sigma_{ij}^p - 2K^2 < 0;
\]

\[
\varepsilon_{ij}^p \neq 0 \quad \text{when} \quad \sigma_{ij}^p \sigma_{ij}^p - 2K^2 \geq 0.
\]

(4)

A mathematical model of the dynamic deformation of an EVP material in the neighborhood of the front edge of a detachment crack presented as a closed system of equations is given by rheological equations (1)–(4), equations of motion in stresses

\[
\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} + b_i;
\]

and the continuity equation

\[
\rho = \rho_0 = \text{const} \quad (i, j = 1, 2, 3).
\]

(6)

Here \( \rho \) is the density of the material, \( b_i \) is the vector of mass forces.

3. The experimentally identified types of stationary cracks and kinematics models of motion of front edges of spatial cracks

Basing on numerous experiments the following types of cracks with a specific kinematics of material particles motion near the crack tip have been defined and adopted in mathematical models in solid deformable material [4, 5, 20–22]: normal detachment cracks, longitudinal shear cracks and transverse shear cracks (figure 1).

The surface \( \Pi_T \) supported by spatial line \( L_{T\delta} \) is shown on figure 2. The surface \( \Pi_T \) is the surface of discontinuity of the material where the material displacement and stress interrupt.

The front edge of a crack is usually considered as a crack tip, referring to the concept of a "tip" to the case of a cross section of a spatial crack. The dynamic theory of elasticity allows the propagation of two types of discontinuities of rates and stresses in an unlimited medium: longitudinal and transverse deformation waves with respect to the direction of propagation of the wave front.

On figure 2 an image of the instant state of a material with elastic properties is shown. In the material the perturbation area is determined by the following factors: the initial position \( L_0 \) of the crack edge; the surface \( \Sigma_0^0 \) of \( \delta \)-neighborhood of the initial crack, inside of which the material behaviour is inelastic; the surface \( \Sigma_0 \) (\( \Sigma_1 \) or \( \Sigma_2 \)) resulting from the propagation of a disturbance with a rate of longitudinal or shear disturbances from \( \Sigma_0^0 \). One of the main curvatures of the external surface of this \( \delta \)-neighborhood is constant \( k_1 = \frac{1}{\delta} \); the second main curvature \( k_2 = \frac{1}{R} \) coincides with the curvature of the elastic precursor generated by the shape of the perturbation at the crack initiation.

The process of initiation and motion of cracks, consisting in the inelastic deformation of the material in the small \( \delta \)-neighborhood and the propagation of this \( \delta \)-neighborhood around the front edge \( L_T \) of the crack, will be considered [1, 16] as a phenomenon of motion of isolated surfaces \( \Sigma_1 \) or \( \Sigma_2 \) with the curve \( L_{T\delta} \) belonging to them, which is a precursor of the front edge of the crack surface \( \Pi_T \) (figure 3).

We assume that the inelastic deformation of the material in \( \delta \)-neighborhood of the front edge \( L_T \) does not interrupt the continuity of the material and allows discontinuities of the rates of displacements \( v_n \) or \( v_r \) [16]. These discontinuities behind the edges of the \( \delta \)-neighborhood lead to the discontinuity of displacements \( [u] = [v] \cdot \delta \) on the edge of the crack.
Figure 1. The image of the instant state of the material at the moment of passage from right to left of the front edge $L_T$ of its trace (i.e. the point $M$) in the orthogonal cross section to the edge $L_T$: a) cracks of normal detachment running left with the rate $c_2$ of elastic shear waves; b) longitudinal shear cracks running left with the rate $c_1$ of elastic longitudinal waves; c) cracks of transverse shear running left with the rate $c_2$ of elastic shear waves. Here $\tau$ is the rate of displacement of the crack boundaries; the sign “$\land$” and sign “$\lor$” are used to denote the value of the functions behind the crack precursor $\Sigma_\delta$ at points above (sign “$\land$”) or below (sign “$\lor$”) of the crack trace $L_{T\delta}$ on $\Sigma_\delta$.

Figure 2. The image of the initial area of plastic stress-strain state of the material in the neighborhood of the front edge of the crack $L_0$ bounded by the surface $\Sigma_\delta^0$ and the current position of the deformation wave on the precursor of the front edge $L_{T\delta}$ of the crack generated by the front edge of the crack $L_0$; surface $\Pi_T$ of crack.
Figure 3. The image of the curve $L_T$ of the spatial front edge of a crack; an element of the spatial surface $\Pi_T$ of a crack with a propagating front edge $L_T$; the surface $\Pi_\delta$ bounding $\delta$-neighborhood of the front edge $L_T$ of the crack; the vector of the normal $n$ to the front edge $L_T$ of the crack at the current point $M$ belonging to the straightening plane $\Pi_\Lambda$ to the curve $L_T$ at the point $M$; the radius $R$ of curvature for the curve $L_T$ at the point $M$; the radius $\delta$ of $\delta$-neighborhood of a curve $L_T$; the center $O$ of curvature of the curve $L_T$ at the point $M$.

Figure 4. The scheme of the precursor $\Sigma_\delta$ of the crack and the trace $L_{T\delta}$ of the front edge of the crack on the precursor. The sign “$\wedge$” and sign “$\vee$” are used to denote the value of the functions behind $\Sigma_\delta$ at points above (sign “$\wedge$”) or below (sign “$\vee$”) of the crack trace $L_T$ on $\Sigma_\delta$ for the case of transverse or longitudinal shear detachment crack.

4. The model of the motion of the front edge of the detachment crack behind the front of the shear wave $\Sigma_2$ and the surface of material fracture by pressing behind the front of the longitudinal wave $\Sigma_1$

The detachment crack develops from the initial position of the front edge (figure 2), which can be defined by a spatial curve $L_0$, in the rectifying plane of which limit conditions of the material rupture are fulfilled. Such conditions are the obtaining of a limit of normal to an element of a crack near its tip of stress $\sigma_n$ [11–16] or an equivalent to it condition of equality of the increase of rate of surface elastic energy at the crack tip and the rate of decrease of the elastic energy of the material in the neighborhood of the crack tip.

Analytical studies of the elastic plane stress state show that approaching the crack tip leads to an unlimited increase of stresses [1, 2, 11, 12]. Taking into account that there are no unlimited stresses in real materials, select a $\delta$-neighborhood at the crack tip, where the stress-strain state is described by the model of an EVP material. The stress state on the surface of the $\delta$-neighborhood
can be approximated by normal pressure [1] leading to rupture of the material. In the case of a
high-speed impact by a solid body [16, 23, 24] on an EVP material, the initial stress state on the
surface of δ-neighborhood of the tip is determined by the impact rate $V_0 < c_1$ and the conditions
of contact of the body surface with the material: sticking, friction or slipping.

The instant application of stresses on a surface of δ-neighborhood of the tip of a detachment
crack leads to the appearance of a discontinuity surface of tangential to $\Sigma_2$ stresses and rates
propagating on $\Sigma_2$ with rate $c_2$ (figure 5).

5. The model of the motion of the front edge of the crack of the longitudinal shear
behind the front of the longitudinal wave $\Sigma_1$ and the crack of the transverse shear
behind the front of the shear wave $\Sigma_2$

The rate normal to $\Sigma_1$ has a jump when crossing the line $LT_{\delta}$. This jump leads to a rupture of
the longitudinal along $n$ rate $[v_n]$ at the front edge of the crack $LT$. The front edge of the shear wave $\Sigma_2$ (figure 7) allowing the jump of tangent to line $LT_{\delta}$ rate $v_\tau$ when crossing front edge $LT_{\delta}$ can be considered as front edge of the precursor of transverse
shear crack $LT$ at the edge of which a rupture of longitudinal displacement $[u_r] = [v_r] \cdot \frac{\delta}{c_2}$ arises.

The kinematics of rate jumps on $\Sigma_1$ and $\Sigma_2$ leads to the conclusion that the front edge of
a longitudinal shear crack moves with the normal rate $c_1$ of longitudinal deformation waves $\Sigma_1$
(figure 6) and the front edge $LT$ of transverse shear and detachment cracks propagates with the
rate $c_2$ of shear wave $\Sigma_2$.

6. Identified patterns of propagation of front edges of spatial cracks in unlimited
space

6.1. Statement of the Goursat problem on the stress-strain state behind the precursor front of
the front edge of a crack

The initial parameters of the surface of the front edge of a crack are the main radii of curvature
$R_0 + \delta$ of the curve $LT_{\delta}$ in its rectifying plane $\Pi_A$ and the radius $\delta$ of surface of δ-neighborhood
of the inelastic behavior of material in the neighborhood of the crack edge $LT$ (figure 3).
Figure 6. The image of the surface $\Sigma_1$ moving at a rate $c_1$ in the direction of the vector $n$ normal to the surface of $\delta$-neighborhood at the point $M$ having main radii of curvature $\rho_1 = R + \delta$ and $\rho_2 = \delta$; the rate vectors $\vec{v}_n^1$ and $\vec{v}_n^\alpha$ behind $\Sigma_1$ discontinuous when crossing a crack edge in the neighborhood of the point $M$; the surface $\Pi_{TP}$ of longitudinal shear crack; the symbols “+” and “−” denoting the material in front of $\Sigma_1$ and behind its front; the center $O$ of curvature of the curve $L_{T\delta}$.

Figure 7. The image of the surface $\Sigma_2$ moving at a rate $c_2$ in the direction of the normal $\vec{n}$ at the current point $M$ of the front edge $L_T$ of the shear crack; the rate vectors $\vec{v}_n^2$ and $\vec{v}_n^\alpha$ tangent to $L_{T\delta}$ and $\Sigma_2$ and discontinuous when crossing the trace $L_{T\delta}$ of the edge of a crack; the surface $\Pi_{Ta}$ of a crack of antiplane strain.

Thus, the front edge $L_T$ of a spatial crack generates ahead of itself a $\delta$-neighborhood of the inelastic behavior of the material bounded by a moving surface $\Sigma_{\delta}$ with main curvatures $\delta$ and $R_0 + \delta + ct$. The stress-strain state of a material in the $\delta$-neighborhood of inelastic behavior is considered satisfying the system of partial differential equations of the mathematical dynamic model [10, 16, 23, 24] of an EVP material (1)–(6) in the Goursat statement for rates $v_i$ and stresses $\sigma_{ij}$ as functions of spatial coordinates $x_i$ ($i = 1, 2, 3$) and of time $t$.

The Goursat problem in this case is:

1) to specify a moving surface $\Sigma_{\delta}$ limiting $\delta$-neighborhood of the crack $L_T$ and the initial values of the rates $v_i|_{L_{T0}}$ and $\sigma_{ij}|_{L_{T0}}$

$$v_i|_{T0} = v_i(y_1, y_2); \quad \sigma_{ij}|_{T0} n_i = \sigma_{in}^0(y_1, y_2)$$  \hspace{1cm} (7)

in the initial position $L_{T0}$ or $\Sigma_0$;

2) to determine the rates field $v_i(x, t)$ and stresses field $\sigma_{ij}(x, t)$ in the area behind $\Sigma_{\delta}$.

The dynamic development of inelastic deformation is of interest up to that point in time until
the von Mises plasticity condition (4) or the condition of detachment across the crack [1] are satisfied.

One of the acceptable methods for solving the stated Goursat spatial problem is the splitting method by selecting the direction along the normal \( \vec{n} \) to the direction of the leading front \( \Sigma_0 \) of a crack, which is taken as the wave front of a discontinuity of rates and stresses. Taking into account the smallness of the distance of the \( \delta \)-neighborhood along the direction \( \vec{n} \) consider the finite number of members of the Taylor series in the decomposition of rates and stresses on \( \vec{n} \) according to [17].

\[
f(x_1, x_2, x_3, t) = f(n, y_1, y_2, t) = \sum_{k=1}^{\infty} \frac{1}{k!} \left. \frac{\partial^k f(n, y_1, y_2, t)}{\partial n^k} \right|_{\Sigma_{\delta}^-} \cdot n^k,
\]

where the points of space located on the negative (back) side of the surface \( \Sigma_{\delta} \) are denoted by \( \Sigma_{\delta}^- \); \( x_i = x_i(n, y_1, y_2, t) \) is the coordinate transformation from fixed ones \( x_i \) to moving ones \( (n, y_1, y_2, t) \) associated with the surface \( \Sigma_{\delta} \) (figure 8).

**Figure 8.** The image of the current position of the surface \( \Sigma_{\delta} \) bounding the \( \delta \)-neighborhood of the front edge of a spatial crack \( L_T \). The curvilinear coordinate \( y_1 \) coincides with the curve \( L_T \); i.e. it is a precursor of a crack; the curvilinear coordinate \( y_2 \) is an arc of a circle of radius \( \delta \). The values of rates and stresses are supplied with a plus sign in front of the surface \( \Sigma_{\delta} \) and with a minus sign behind it, i.e. \((v_i^+, \sigma_{ij}^+)\) and \((v_i^-, \sigma_{ij}^-)\), respectively.

It is convenient to count the coefficients \( \frac{\partial^k f(n, y_1, y_2, t)}{\partial n^k} \) of the Taylor series from their value before \( \Sigma_{\delta}^- \) introducing into consideration the jumps of the corresponding quantities

\[
\left. \left[ \frac{\partial^k f}{\partial n^k} \right] \right|_{\Sigma_{\delta}^-} = \left. \frac{\partial^k f}{\partial n^k} \right|_{\Sigma_{\delta}^+} - \left. \frac{\partial^k f}{\partial n^k} \right|_{\Sigma_{\delta}^-}.
\]

The expansion (8) of the function in a Taylor series behind \( \Sigma_{\delta} \) in the area of the Goursat problem solution is presented as

\[
f(n, y_1, y_2, t) = \sum_{k=1}^{\infty} \frac{1}{k!} \left( \left. \frac{\partial^k f}{\partial n^k} \right|_{\Sigma_{\delta}^+} - \left. \frac{\partial^k f}{\partial n^k} \right|_{\Sigma_{\delta}^-} \right) \cdot n^k.
\]

Jumps of the members of the Taylor series (9) must be determined from the system of equations (1)–(6) with the initial data of the Goursat problem (7) with the use of the ray method [16].
6.2. The ray method to construct a solution for rates and stresses behind the front of the precursor \( \Sigma_\delta \) of the front edge of a crack \( LT \)

6.2.1. Possible types of wave precursors of spatial cracks Let’s introduce a curvilinear coordinate system \((n, y_1, y_2)\) associated with a moving surface \( \Sigma_0 \) (figure 8)

\[
x_i = x_i(n, y_1, y_2, t) \quad (i = 1, 2, 3).
\]

The equation of surface \( \Sigma_0 \) is determined by law (11) at \( n = 0 \)

\[
x_i = x_i(0, y_1, y_2, t) \quad (i = 1, 2, 3).
\]

The system of equations (1)–(6) containing partial derivatives with respect to \( x_i \) and with respect to time \( t \) can be brought to the form containing the derivatives with respect to coordinates \((n, y_1, y_2)\) and with respect to time \( t \) in the moving coordinate system by replacing [1, 26]

\[
\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial n} n_i + g^{\alpha\beta} \frac{\partial x_i}{\partial y_\alpha} \frac{\partial f}{\partial y_\beta},
\]

\[
\frac{\partial f}{\partial t} = \frac{\delta f}{\delta t} - \frac{c}{\partial n} \frac{\partial f}{\partial n}, \quad (i = 1, 2, 3; \quad \alpha, \beta = 1, 2).
\]

Here \( \frac{\delta f}{\delta t} \) is the local derivative with respect to time \( t \) of the function given on the moving surface \( \Sigma \); \( g^{\alpha\beta} = \frac{\partial x_i}{\partial y_\alpha} \frac{\partial x_i}{\partial y_\beta} \) is metric tensor.

The rheological equations (1)–(3) and the equations of motion in stresses (5) take the following form

\[
\sigma_{ij} = \lambda \epsilon_{kk}\delta_{ij} + 2\mu \left( \epsilon_{ij} - \epsilon_{ij}^p \right),
\]

\[
\epsilon_{ij}^p = \frac{\partial \epsilon_{ij}^p}{\partial t} = \frac{\delta \epsilon_{ij}^p}{\delta t} - c \frac{\partial \epsilon_{ij}^p}{\partial n} = \frac{I_2 - K\sqrt{2}}{I_2\eta} \sigma_{ij},
\]

\[
\epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial n} n_j + \frac{\partial u_j}{\partial n} n_i + g^{\alpha\beta} \frac{\partial x_j}{\partial y_\alpha} \frac{\partial u_i}{\partial y_\beta} + g^{\alpha\beta} \frac{\partial x_i}{\partial y_\alpha} \frac{\partial u_j}{\partial y_\beta} \right),
\]

\[
v_i = \frac{\partial u_i}{\partial t} = \frac{\delta u_i}{\delta t} - c \frac{\partial u_i}{\partial n},
\]

\[
\rho \frac{\partial v_i}{\partial t} = \frac{\rho \delta v_i}{\delta t} - c \frac{\partial v_i}{\partial n} = \frac{\partial \sigma_{ij}}{\partial n} n_j + g^{\alpha\beta} \frac{\partial x_i}{\partial y_\alpha} \frac{\partial \sigma_{ij}}{\partial y_\beta} + b_i.\]

The integration of partial differential equations (13), (14) with respect to \( n \) ranging from \(-\varepsilon\) to \(+\varepsilon\) and the transition to the limit on \( \varepsilon \to 0 \) leads to the following two equations

\[
-\rho c [v_i] = [\sigma_{ij}] n_j,
\]

\[
[e_{ij}^p] = 0, \quad \left[ \frac{\partial \epsilon_{ij}^p}{\partial n} \right] \neq 0.
\]

Equation (16) reflects the physical fact that plastic deformations \( \epsilon_{ij}^p \) are continuous on the crack precursor surface \( \Sigma_\delta \) and only a plastic deformation gradient arises [16].
Rheological equations (12) and expressions $e_{ij}$ for deformations and rates $v_i$, taken as the difference in the values of the incoming values to the right and left of the fracture precursor surface lead to the following expressions:

$$[e_{ij}] = \frac{1}{2} \left( [u_{i;j}] + [u_{j;i}] + g_{x;j} \left[ \frac{\partial u_i}{\partial y} \right] + g_{x;i} \left[ \frac{\partial u_j}{\partial y} \right] \right).$$

(19)

The system of equations (15)–(18) is a linear homogeneous system of algebraic equations for rate jumps $[v_i]$, displacement gradients $\left[ \frac{\partial u_i}{\partial n} \right]$, stresses $[\sigma_{ij}]$, displacements $[u_i]$ and displacement gradients $\left[ \frac{\partial u_i}{\partial y_\alpha} \right]$ being tangent along $y_\alpha$.

The condition of continuity of displacements along the surface $\Sigma_\delta$ in the direction $y_\alpha$ and the material continuity condition, the absence of discontinuities of displacements in the precursor result in equalities

$$\frac{\partial [u_i]}{\partial t} = 0; \quad \left[ \frac{\partial u_i}{\partial y_\alpha} \right] = \left[ \frac{\partial [u_i]}{\partial y_\alpha} \right] = 0.$$

(20)

The system of equations (17)–(20) by eliminating $[\sigma_{ij}]$, $[e_{ij}]$, $[u_{i;j}]$ is reduced to the form

$$\rho c^2 [v_i] = (\lambda + \mu) [v_j] n_j n_i + \mu [v_i].$$

There are only two non-zero solutions for $[v_i]$: $[v_i] n_i = \omega_n$

(21)

when

$$\rho c^2_1 = \lambda + 2\mu$$

(22)

and

$$[v_i] \tau_i = \omega_\tau$$

(23)

when

$$\rho c^2_2 = \mu.$$

(24)

Conditions (21)–(24) show that the front $\Sigma_\delta$ of the precursor of a spatial crack can be a longitudinal strain wave for a longitudinal crack propagating with rate $c_1$ ($\rho c^2_1 = \lambda + 2\mu$) and a shear strain wave for detachment cracks or transverse shear cracks propagating with rate $c_2$ ($\rho c^2_2 = \mu$).

The jumps of stresses on $\Sigma_\delta$ can be expressed through jumps of rates as follows

$$[\sigma_{ij}] = -\frac{1}{c} \lambda [v_n] \delta_{ij} - \frac{1}{c} \mu \left( [v_i] n_j + [v_j] n_i \right).$$

(25)
6.2.2. Intensity of precursors of spatial front edges of cracks Consider the trace of the front edge of a crack at the front $\Sigma_\delta$ of a wave precursor, which is represented by a spatial curve $L_{T\delta}$ belonging to the surface $\Sigma_\delta$ (figure 4).

The system of equations (16)–(19) can be reduced to three equations for the values of the displacement gradients

$$\rho c^2 [u_{i,n}] = (\lambda + \mu) [u_{n,n}] n_i + \mu [u_{i,n}] + \mu [u_{n,\alpha}] g^{\alpha\beta} x_{i,\beta}; \ (i = 1, 2, 3; \ \alpha, \beta = 1, 2). \quad (26)$$

In the projection on the direction of the normal $\vec{n}$ to $\Sigma_\delta$ these equations give a degenerate equation

$$\rho c^2 [u_n] = (\lambda + \mu) [u_{n,n}], \quad (27)$$

from which follows the known fact of the propagation of longitudinal strain waves $[u_{n,n}] = \frac{\omega}{c_1} \neq 0$ with rate $c_1 = \rho c^2 = \lambda + 2\mu$.

In the projection on the tangent to the normal $\vec{n}$ direction $y_\alpha$ equation (26) takes the form

$$\rho c^2 [u_{i,n}] x_{i,\alpha} g^{\alpha\beta} = \mu [u_{i,n}] x_{i,\beta} g^{\alpha\beta} + \mu [u_{j,\beta}] n_j g^{\alpha\beta}$$

or

$$\rho c^2 [u_{r,n}] = \mu [u_{r,n}] + \mu [u_{n,n}]. \quad (28)$$

The known fact of the propagation of shear waves $[u_{r,n}] = -\frac{[u_r]}{c_2} = -\frac{\omega_r}{c_2} \neq 0$ with rate $c_2 = \rho c^2 = \mu$ follows from (28), at the same time

$$[u_{n,\tau}] = [u_{n,n}] = 0. \quad (29)$$

Equations (21)–(28) are satisfied on the fronts of longitudinal and shear waves at all points of the continuous behavior of the jumps $[v_n]$, $[v_r]$, $[u_{n,n}]$, $[u_{r,n}]$.

Cases of discontinuous behavior of rate jumps $[v_i]$ or displacement gradients $[u_{i,n}]$ on the line $L_{T\delta}$ of the precursor front edge of a crack can be considered as the limit of a sequence of smooth functions with large but continuous derivatives in directions $y_n$ and $y_\tau$ being normal or tangent to $L_{T\delta}$ in a small $\varepsilon$-neighborhood of a point on $L_{T\delta}$ and investigate jumps of “jumps” crossing through $L_{T\delta}$

$$[\vec{f}] = [f'] - [f'],$$

where $[\vec{f}] = [f']|_{L_\delta} - [f']|_{L_\delta}$ can be used as the intensity of the corresponding value of rate or stress on the precursor $L_{T\delta}$ of the crack. Carrying out the operations of subtracting equations (25)–(29) given from below and above of $L_{T\delta}$ we obtain relations for the intensities of rates jumps and stresses jumps at the front edges of the spatial precursors of the crack.

6.2.3. Equation of transfer of the intensity of wave precursors of spatial cracks To construct the equations of transfer of intensities $\omega_n$ and $\omega_r$ of jumps of normal and tangential rates on the precursor $\Sigma_\delta$ of a crack we can consider a system of partial differential equations of the first order these are equations of motion and rheological equations in jumps on $\Sigma_\delta$

$$[\sigma_{ij,ij}] = \rho \left[ \frac{\partial v_i}{\partial t} \right];$$

$$\left[ \frac{\partial \sigma_{ij}}{\partial t} \right] = \lambda [v_{k,k}] \delta_{ij} + \mu ([v_{i,i}] + [v_{j,j}]) - 2\mu \left[ \frac{\sigma'}{\sigma_{ij}} \right].$$
In this system of equations we pass from differentiation with respect to Cartesian coordinates \( x_i \) to derivatives with respect to normal \( \vec{t} \) to \( \Sigma_0 \) and curvilinear coordinates \((y_1, y_2)\), and also select the local time derivative with respect to time \( \frac{\delta}{\delta t} \) [19]:

\[
[\sigma_{ij,n}] n_j + g^{\alpha\beta} [\sigma_{ij}]_{,\alpha} x_{j,\beta} = -\rho c [\bar{v}_{i,n}] + \rho \frac{\delta[\bar{v}_i]}{\delta t};
\]

\[
c [\sigma_{ij,n}] + \delta [\sigma_{ij}] = \lambda \left([\bar{v}_{k,n}] n_k + g^{\alpha\beta} [\bar{v}_{\alpha}]_{,\alpha} x_{k,\beta}\right) \delta_{ij} +
\]

\[
+ \mu \left([\bar{v}_{i,n}] n_j + [\bar{v}_{j,n}] n_i + g^{\alpha\beta} \left([\bar{v}_{\alpha}]_{,\alpha} x_{j,\beta} + [\bar{v}_{\beta}]_{,\beta} x_{i,\alpha}\right)\right) - 2\mu \left[\bar{v}_{ij}\right].
\]

The system of linear algebraic equations for \([\sigma_{ij,n}]\) and \([\bar{v}_{i,n}]\) is simplified to three equations for \([\bar{v}_{i,n}]\) by eliminating \([\sigma_{ij,n}]\)

\[
(\lambda + \mu) [\bar{v}_{k,n}] n_k n_i + (\mu - \rho c^2) [\bar{v}_{i,n}] + \rho c \frac{\delta[\bar{v}_i]}{\delta t} -
\]

\[
- \frac{\delta [\sigma_{ij}]}{\delta t} n_j - 2\mu \left[\bar{v}_{ij}\right] n_j - cg^{\alpha\beta} [\sigma_{ij}]_{,\alpha} x_{j,\beta} -
\]

\[
- \lambda g^{\alpha\beta} [\bar{v}_{\alpha}]_{,\alpha} x_{k,\beta} n_i + \mu g^{\alpha\beta} [\bar{v}_{\beta}]_{,\beta} x_{i,\alpha} n_j = 0, \quad i = 1, 2, 3,
\]

where \([\sigma_{ij}]\) is expressed through \([v_k]\) from (25).

The system of three equations (30) for \([\bar{v}_{i,n}]\) is degenerated in two cases:

1) \(\rho c_1^2 = \lambda + 2\mu\) at \([\bar{v}_{k,n}] n_k = [\bar{v}_{n,n}] \neq 0\);

2) \(\rho c_2^2 = \mu\) at \([\bar{v}_{n,n}] \neq 0\).

The compatibility conditions of equations (30) give the equations of transfer of rate jumps \([v_k] n_k = \omega_n\) and \([v_k] \tau_k = \omega_\tau\) on the fronts of longitudinal and shear waves, respectively. To do this we obtain the projection of equations (30) on the normal \(\vec{t}\) to \(\Sigma_0\) and on the tangent direction \(\vec{t}'\) to \(\Sigma_0\) multiplying equation (30) by the normal vector and the tangent vector, respectively:

\[
\rho c_1 \frac{\delta \bar{v}_n}{\delta t} = (\lambda + 2\mu) \Omega \bar{v}_n + \mu \left[\bar{v}_{ij}\right] n_i n_j;
\]

\[
\rho c_2 \frac{\delta \bar{v}_\tau}{\delta t} = \mu \Omega \bar{v}_\tau + \mu \left[\bar{v}_{ij}\right] n_i \tau_j .
\]

Here \(\Omega = \frac{1}{2} \left( \frac{1}{\rho_1} + \frac{1}{\rho_2} \right)\) is the average curvature of the surface \(\Sigma_0\); \(\rho_1\) and \(\rho_2\) are main curvatures of \(\Sigma_0\) at the current time.

These equations can be represented as follows for projections \([v_i]\).

On \(\Sigma_1\):

\[
[v_i] = \bar{v}_n n_i \quad \text{and} \quad \frac{\delta [v_i]}{\delta t} = c_1 \Omega [v_i] + \frac{c_2}{c_1} [\bar{v}_{ij}] n_i .
\]

On \(\Sigma_2\):

\[
[v_i] = \bar{v}_\tau \tau_i \quad \text{and} \quad \frac{\delta [v_i]}{\delta t} = c_2 \Omega [v_i] + c_2 \left( [\bar{v}_{ij}] n_j - [\bar{v}_{pq}] n_p n_q n_i \right) ,
\]

and also in the form of an ordinary differential equation for the traveled distance \(s\) \((s = c_2t)\):

\[
\frac{\delta [v_i]}{\delta s} = \Omega [v_i] + [\bar{v}_{ij}] n_j - [\bar{v}_{kj}] n_k n_q n_i .
\]
6.2.4. Propagation of front edges of elastic precursors after stopping cracks

The propagation of elastic precursors after a crack has stopped is determined by equations (31)–(32) in the absence of plastic deformations on the leading front of the wave $\Sigma_\delta$:

$$\frac{\delta [\tau_n]}{\delta t} - c_1 \Omega [\tau_n] = 0; \quad \frac{\delta [\tau_r]}{\delta t} - c_2 \Omega [\tau_r] = 0.$$ 

These equations have an exact solution

$$\Lambda_n = \left[ \frac{\tau_n}{\tau_n|_0} \right] = \frac{1}{\sqrt{1 + 2\Omega_0 c_1 t + K_0 c_1^2 t^2}}; \quad \Lambda_r = \left[ \frac{\tau_r}{\tau_r|_0} \right] = \frac{1}{\sqrt{1 + 2\Omega_0 c_2 t + K_0 c_2^2 t^2}},$$ (33)

here $\Omega_0 = \frac{1}{2} \left( \frac{1}{\delta_0} + \frac{1}{\rho_L} \right); \quad K_0 = \frac{1}{\delta_0 \rho L}$.

Stresses $[\sigma_{nn}]$ and $[\sigma_{nr}]$ are determined behind $\Sigma_1$ and $\Sigma_2$:

$$[\sigma_{nn}] = - \frac{1}{c_1} (\lambda + 2\mu) [\tau_n] = - \rho c_1 [\tau_n]; \quad [\sigma_{nr}] = - \frac{1}{c_2} \mu [\tau_r] = - \rho c_2 [\tau_r].$$

Expressions (33) for the relative intensity of the precursor of a crack can be represented through the initial arithmetic mean radius of curvature for $\Sigma_\delta$ and the geometric mean radius

$$\bar{R} = \frac{1}{2} (\delta_0 + \rho L); \quad R^* = \sqrt{\delta_0 \rho L}; \quad \Lambda = \left[ \frac{\tau}{\tau|_0} \right] = \frac{R^*}{\sqrt{R^* + 2R^* c t + c^2 t^2}},$$ (34)

where $[\tau] = [\tau_n]$ on the longitudinal wave and $[\tau] = [\tau_r]$ on the shear wave.

It follows from expressions (33) and (34) that the relative intensity $\Lambda$ of the elastic precursors of the stopped front edges of spatial cracks fades according to the laws of geometric optics.

7. Conclusion

The proposed in this paper mathematical model for the propagation of spatial front edges of cracks describes the intensity of the precursors of cracks in the process of their propagation providing the initial impulse of forces or rate impulse on the impact surface in the neighborhood of the initiation of the front edge of a spatial crack is given.

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References

[1] Kachanov L M 1974 *Fundamentals of Fracture Mechanics* (Moscow: Nauka) 312 (in Russian)
[2] Cherepanov G P 1974 *Mechanics of Brittle Fracture* (Moscow: Nauka) 640 (in Russian)
[3] Shemyakin E I 1968 *Dynamic Problems of the Theory of Elasticity and Plasticity* (Novosibirsk: NSU Publishers) 338 (in Russian)
[4] Parton V Z and Morozov E M 1985 *Mechanics of elastoplastic fracture* (Moscow: Nauka) 504 (in Russian)
[5] Parton V Z and Borisovskiy V G 1988 *Dynamics of Brittle Fracture* (Moscow: Mashinostroenie) 240 (in Russian)
[6] Morozov N F and Petrov Yu V 1997 *Problems of the Dynamics of the Fracture of Solids* (Saint-Petersburg: Saint-Petersburg State University Publishers) 132 (in Russian)
[7] Sinclair G B, Meda G and Smallwood B S 2011 On crack-tip stresses as crack-tip radii decrease *Trans. ASME. J. Appl. Mech.* 78:1 1–8
[8] Ivants'kyi Ya L, Acmbara O V, Amiyan O D and Kovalic M 2011 Evolution of the concentration of hydrogen in the process zone near the crack tip *Mater. Sci.* 46:6 769–774
[9] Verveiko N D, Shashkin A I and Krupenko S E 2018 On impact by a hard cone on elastovisco-plastic material, leading to the generation of a conical crack *IOP Conf. Series: Mater. Sci. Eng.* 973:1 012021 pp 1–10 (Russia, Voronezh: IOP Publishing) DOI: 10.1088/1742-6596/973/1/012021
[10] Kukudzhanov V N and Kondaurov V I 1975 Numerical solutions of inhomogeneous problems of dynamic deformation of solids (in *Problems of the dynamics of visco-plastic media*) (Moscow: Mir) 38–84 (in Russian)
[11] Kostrov B V, Nikitin L V and Flitman L M 1969 Mechanics of brittle fracture *Izv. Academy of Sciences of the USSR. Solid Mechanics* 3 112–125 (in Russian)
[12] Kostrov B V, Nikitin L V and Flitman L M 1970 Crack propagation in viscoelastic bodies *Physics of the Earth* 7 20–35 (in Russian)
[13] Rice J R 1968 *Mathematical Analysis in the Mechanics of Fracture* Chapter 3 of *Fracture: An Advanced Treatise (Vol 2, Mathematical Fundamentals)* ed H Liebowitz (New York: Academic Press) pp 191–311
[14] Rice J and Druccer D 1967 Energy changes in stressed bodies due to void and crack growth *IJFM* 3:1 19–27
[15] Rice J and Rosengren G 1968 Plane strain deformation near a crack tip in a power law hardening material *JMPS* 16:1 1–12
[16] Verveiko N D 1997 *Ray Theory of Elastoviscoelastic Waves and Hydrodynamic Waves* (Voronezh: VSU Publishers) 204 (in Russian)
[17] Verveiko N D, Shashkin A I and Krupenko S E 2017 *The Origin and Motion of the Tips of Cracks behind the Fronts of Elasto Viscoplastic Waves* (Voronezh: Quarta Publishers) 124 (in Russian)
[18] Glushko V P and Savchenko Yu B 1985 Degenerate high-order elliptic equations: spaces, operators, boundary problems *Results of science and technology. Series "Mathematical analysis"* 23 125–218 (in Russian)
[19] Glushko V P and Glushko A V 2006 *Asymptotic Methods* (Voronezh: VSU Publishers) 56 (in Russian)
[20] Ievlev V M 2008 *Thin Films of Inorganic Materials: Growth Mechanisms and Structure: Study Guide* (Voronezh: VSU Publishers) 496 (in Russian)
[21] Ievlev V M 1992 *The Structure of the Interfaces in Metal Films* (Moscow: Metallurgiya) 172 (in Russian)
[22] Ievlev V M, Bugakov A V and Trofimov V I 2000 *Growth and Substructure of Condensed Films: Study Guide* (Voronezh: Voronezh State Technical University Publishers) 386 (in Russian)
[23] Kukudzhanov V N 1967 *The Propagation of Elasto Plastic Waves in the Rod, Taking into Account the Influence of the Strain Rate* (Moscow: USSR Academy of Sciences Computing Center Publishers) 48 (in Russian)
[24] Krupenko S E, Fursov A A, Al Imam and Adel Al Wahab 2017 The influence of the stress state of the medium on the direction of the development of shear cracks behind the fronts of reflected waves during a sliding impact with a wedge *Int. Conf. “Applied Mathematics, Computational Science and Mechanics: Current Problems”* (Voronezh: VSU Publishers) pp 1112–1117 (in Russian)
[25] Ilev D D 2005 *Theory of Ultimate State and Perfect Plasticity: Selected Works* (Voronezh: VSU Publishers) 357 (in Russian)
[26] Thomas T Y 1961 *Plastic Flow and Fracture in Solids* (New York-London: Academic Press) vi+267