PERIODIC HIGGS SUBBUNDLES IN POSITIVE AND MIXED CHARACTERISTIC

MAO SHENG AND KANG ZUO

Abstract. Let $k$ be an algebraically closed field of odd characteristic $p$ and $X$ a proper smooth scheme over the Witt ring $W(k)$. To an object $(M, Fil, \nabla, \Phi)$ in the Faltings category $\mathcal{M}F^\nabla[0,n](X), n \leq p - 2$, one associates an étale local system $V$ over the generic fiber of $X$ and a Higgs bundle $(E, \theta)$ over $X$. Our motivation is to find the analogue of the classical Simpson correspondence for the categories of subobjects of $V$ and $(E, \theta)$. Our main discovery in this paper is the notion of periodic Higgs subbundles, both in positive characteristic and in mixed characteristic. In char $p$, it relies on the inverse Cartier transform constructed by Ogus and Vologodsky in their work on the char $p$ nonabelian Hodge theory. A lifting of the inverse Cartier transform to mixed characteristic is constructed, which is used for the notion of periodicity in mixed characteristic. We show a one to one correspondence between the set of periodic Higgs subbundles of $(E, \theta)$ and the set of étale sub local systems of $V \otimes_{\mathbb{Z}} \mathbb{Z}_p^r$, where $r$ is a natural number. The notion turns out to be useful in applications. We have proven, among other results, that the reduction $(E, \theta)_0$ of $(E, \theta)$ modulo $p$ is Higgs stable, if and only if, the corresponding representation $V$ is absolutely irreducible over $k$.

Contents

1. Introduction 1
2. The category $\mathcal{M}F^\nabla$ 4
3. Periodic Higgs subbundles in positive characteristic 6
4. Periodic Higgs subbundles in mixed characteristic 12
5. Further applications 18
6. A global inverse Cartier transform over $W_2$ 25
7. Appendix: the inverse Cartier transform of Ogus and Vologodsky 29
References 31

1. Introduction

Let $V$ be a complex polarizable variation of Hodge structures (abbreviated as $\mathbb{C}$-PVHS) over a projective algebraic manifold and $(E, \theta)$ the corresponding Higgs bundle (see §1 [3], §4 [15]). Then $(E, \theta)$ is Higgs polystable of slope zero, that is, $(E, \theta) = \oplus_i (G_i, \theta_i)$ with $(G_i, \theta_i)$ Higgs stable of slope zero. Each direct factor

This work is supported by the SFB/TR 45 ‘Periods, Moduli Spaces and Arithmetic of Algebraic Varieties’ of the DFG, and partially supported by the University of Science and Technology of China.
corresponds to a sub $\mathbb{C}$-PVHS, since the Hodge metric is indeed Hermitian-Yang-Mills by the curvature formula due to P. Griffiths (see Theorem 5.2 [8]). In this paper we intend to work out a char $p$ as well as $p$-adic analogue of this result. Our guiding principle is that the relative Frobenius in the $p$-adic case is a replacement of the Hodge metric in the complex case.

**Notation 1.1.** For a nonnegative integer $m$, we denote the reduction of an object defined over $W := W(k)$ modulo $p^{m+1}$ by attaching the subscript $m$. As an example, $X_0$ means the mod $p$ reduction of $X$, i.e., the closed fiber of $X$ over $W$.

A good $p$-adic analogue of the category of $\mathbb{C}$-PVHSs is the category $\mathcal{MF}^\nabla_{[0,n]}(X)$ with $n \leq p - 2$ (abbreviated as $\mathcal{MF}^\nabla$) introduced by G. Faltings (see [4]-[5] and §2 for details). Assume $X$ is a smooth projective $W$-scheme with connected geometric generic fiber. Choose and then fix a smooth projective $W$-curve $Z \subset X$ which is a complete intersection of a very ample divisor of $X$ over $W$. The slope for a vector bundle over $X_0$ in this paper means the $\mu_{Z_0}$-slope. For an object $(M, \nabla, Fil \cdot, \Phi) \in \mathcal{MF}^\nabla$, there are two associated objects: on the one hand, Faltings loc. cit. associates it to a representation $V$ of the arithmetic fundamental group of the generic fiber $X_0 := X \times_W \text{Frac}(W)$. On the other hand, because of Griffiths transversality, one associates a Higgs bundle $(E, \theta)$ by taking grading of $(M, \nabla)$ with respect to the filtration $Fil'$. Explicitly,

$$E = \bigoplus_{i=0}^n E^{i,n-i}, \quad \theta = \bigoplus_{i=0}^n \theta^{i,n-i},$$

where $E^{i,n-i} = Fil^i/Fil^{i+1}$ and the connection $\nabla$ induces an $\mathcal{O}_X$-morphism

$$\theta^{i,n-i} : E^{i,n-i} \rightarrow E^{i-1,n-i+1} \otimes \Omega_{X|W}.$$

Borrowing a terminology in complex case (see §4 [15]), we call a Higgs bundle of the above form a system of Hodge bundles. This assumption is however not restrictive. We have taken the first step by showing the following result.

**Proposition 1.2** (Proposition 0.2 [17]). Notation as above. The Higgs bundle $(E, \theta)_0$ over $X_0$ is Higgs semistable of slope zero.

The result was first shown by Ogus and Vologodsky in the curve case under a stronger assumption on $p$ (see Proposition 4.19 [14]). In this paper we obtain a further result.

**Theorem 1.3** (Theorem 3.1). The Higgs bundle $(E, \theta)_0$ is Higgs stable if and only if the representation $\nabla \otimes k$ is irreducible.

The theorem is further generalized in §5. We obtain some results in mixed characteristic as well. Among other results, we have proven the following

**Theorem 1.4** (Theorem 5.11). Suppose that $M \in \mathcal{MF}^\nabla$ arises from geometry and is non $p$-torsion. Suppose furthermore that $X_0(k)$ contains an ordinary point. If there is a decomposition of Higgs bundles over $X$:

$$(E, \theta) = \bigoplus_{i=1}^r (G_i, \theta_i)^{\oplus m_i},$$
such that the modulo $p$ reductions $\{(G_i, \theta_i)_0\}$s are Higgs stable and pairwise non-isomorphic, then one has a corresponding decomposition of $\mathbb{Z}_{p^r}$-representations for a natural number $r$:

$$V \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r} = \bigoplus_{i=1}^r V_i$$

with the equality $\text{rank}_{\mathbb{Z}_{p^r}} V_i = m_i \text{rank}_{O_X}(G_i)$ for each $i$.

The main technical result underlying the above theorem is a $p$-adic analogue of the Higgs correspondence in the subobjects setting (see [14], [17] for char $p$ case). Combined with the result of Faltings loc. cit., we obtain a rather satisfactory $p$-adic analogue of the Simpson correspondence for crystalline representations of the arithmetic fundamental groups in the subobjects setting. Precisely, we have the following

**Theorem 1.5** (Theorem 4.2). Assume $X$ proper smooth over $W$ with connected geometric generic fiber. Suppose $M \in \mathcal{MF}^\nabla$ non $p$-torsion. Then, for each natural number $r$, there is a one to one correspondence between the set of $\mathbb{Z}_{p^r}$-subrepresentations of $V \otimes \mathbb{Z}_{p^r}$ and the set of periodic Higgs subbundles in $(E, \theta)$ whose periods are divisors of $r$.

Besides its applications, the introduction of the notion of a periodic Higgs subbundle is another contribution of the paper, which shall have its importance in the $p$-adic nonabelian Hodge theory. The first result in this field is the work of Deninger and Werner [2], which gives a $p$-adic analogue of the classical result of Narasimhan and Seshadri for stable vector bundles and unitary representations. Using the theory of almost étale extensions developed by himself, Faltings has obtained a vast generalization. In [6], he has established in the curve case a correspondence between Higgs bundles and generalized representations. One of major open problems in this field is to show semistable Higgs bundles correspond to genuine representations. In the setting of subobjects, semistability is hidden in, actually equivalent to, the degree zero condition. We show that over char $p$ this topological assumption on Higgs subbundles is equivalent to quasi-periodicity, and periodic Higgs subbundles are in one to one correspondence to subrepresentations in $V \otimes k$ of the arithmetic algebraic fundamental group. We have also considered the analogous problem over mixed characteristic and has obtained a partial result. Therefore, we believe that a proper notion of (quasi-)periodic Higgs bundles connects the notion of semistable Higgs bundles on the one hand and genuine representations of geometric fundamental groups on the other hand. In our recent joint work with Lan [10], we have developed this point of view in a general setting over positive characteristic.

**Acknowledgements.** The second named author would like to thank for the hospitality of School of Mathematical Sciences, the University of Science and Technology of China. We thank Guitang Lan for a careful reading of this paper. We thank also Xiaotao Sun for his comments to this work.
2. The category $\mathcal{MF}^\nabla$

For the convenience of the reader, we collect some results due to G. Faltings on the category $\mathcal{MF}^\nabla_{[0,n]}(X)$, $n \leq p - 2$ (see Ch. II [4], see also §3-4 [5]). Let $X$ be a smooth $W$-scheme. For $X = \text{Spec } W$, the category was introduced by Fontaine and Laffaille [7], consisting of strong $p$-divisible filtered Frobenius crystals. In this case, there is no connection involved. In Ch. II [4], Faltings generalized the category of Fontaine and Laffaille to a geometric base $X$ as well as the comparison theory, which gives an equivalence of categories between this category and a certain category of étale local systems over $X^0$. We would like also remind the reader that A. Ogus has developed the category of $F$-$T$ crystals (see [12]) from another point of view, which is however closely related to the category $\mathcal{MF}^\nabla$.

The notation $\mathcal{MF}^\nabla_{[0,n]}(X)$ in [4] means originally for $p$-torsion objects. Here we shall include into the category non $p$-torsion objects as well, whose any reduction modulo $p^{m+1}, m \geq 0$ is an object of $\mathcal{MF}^\nabla$ (see Ch. II h) [4]). To our purpose, we state here only the exact definition of a non $p$-torsion object in the category, in a form closer to §3 [5]. For a $p$-torsion object, one needs to modify the formulation below on the strong $p$-divisibility of the relative Frobenius, which shall cause problems mainly in notations.

A small affine subset $U$ of $X$ is an open affine subscheme $U \subset X$ over $W$ which is étale over $\mathbb{G}_m^d$. As $X$ is smooth over $W$, an open covering $U$ consisting of small affine subsets of $X$ exists. For each $U \in \mathcal{U}$, we choose a Frobenius lifting $F_U$ on $\hat{U}$, the $p$-adic completion of $U$. An object in $\mathcal{MF}^\nabla_{[0,n]}(X)$ is a quadruple $(M, Fil \cdot, \nabla, \Phi)$, where

i) $(M, Fil \cdot)$ is a locally filtered free $\mathcal{O}_X$-module with

$$Fil^0 M = M, \quad Fil^{n+1} M = 0.$$  

ii) $\nabla$ is an integrable connection on $M$ satisfying the Griffiths transversality:

$$\nabla(Fil^i M) \subset Fil^{i-1} M \otimes \Omega_{X|W}.$$  

iii) The valuation $\Phi_{F_U}$ of the relative Frobenius over $(U, F_U)$ is an $\mathcal{O}_U$-linear morphism $\Phi_{F_U} : F_U^* M \to M$ with the strong $p$-divisible property:

$$\Phi_{F_U}(F_U^* Fil^i M) \subset p^i M,$$

and

$$\sum_{i=0}^n \frac{\Phi_{F_U}(F_U^* Fil^i M)}{p^i} = M.$$  

iv) $\Phi_{F_U}$ is horizontal with respect to the connection $F_U^* \nabla$ on $F_U^* M$ and $\nabla$ on $M$.
The locally filtered-freeness in i) means that $M$ is locally free and the filtration $\text{Fil}^i$ on $M$ is locally split. The pull-back connection $F_U^*\nabla$ on $F_U^*M$ is the composite

$$F_U^*M = F_U^{-1}M \otimes_{F_U^{-1}\mathcal{O}_U} \mathcal{O}_U \xrightarrow{F_U^{-1}\nabla \otimes \text{id}} (F_U^{-1}M \otimes F_U^{-1}\Omega_U) \otimes_{F_U^{-1}\mathcal{O}_U} \mathcal{O}_U \xrightarrow{=} F_U^*M \otimes F_U^*\Omega_U \xrightarrow{id \otimes dF_U} F_U^*M \otimes \Omega_U.$$ 

The horizontal condition iv) is expressed by the commutativity of the diagram

$$\begin{array}{ccc}
F_U^*M & \xrightarrow{\Phi_{F_U}} & M \\
\n \downarrow \Phi_{F_U} \otimes \text{id} & & \downarrow \nabla \\
F_U^*M \otimes \Omega_U & \xrightarrow{=} & M \otimes \Omega_U.
\end{array}$$

We shall explain the Taylor formula relating two valuations of the relative Frobenius as follows. Write $\hat{\Phi}$.

Example 2.1. Let $f : Y \to X$ be a proper smooth morphism of relative dimension $n \leq p - 2$ between smooth $W$-schemes. Assume that the relative Hodge cohomologies $R^if_*\Omega^j_Y$, $i + j = n$ has no torsion. It follows from Theorem 6.2 [4] that the crystalline direct image $R^nf_*(\mathcal{O}_Y,d)$ is an object in $\mathcal{M}\mathcal{F}^\nabla_{[0,n]}(X)$.

We call an object of $\mathcal{M}\mathcal{F}^\nabla$ in the above example an object arising from geometry. The main result about this category is the following

**Theorem 2.2** (Theorem 2.6* [4]). Notations as above. There exists a fully faithful contravariant functor $D^t$ from $\mathcal{M}\mathcal{F}^\nabla$ to the category of étale local systems over $X^0$. The image is closed under subobjects and quotients.

It is more convenient to use the covariant functor $D^t$, which for a non $p$-torsion object is defined by

$$D^t(M) = \text{Hom}(D(M), \mathbb{Z}_p) \otimes \mathbb{Z}_p(n),$$
where $\mathbb{Z}_p(n)$ is the Tate twist (see Ch. II h) \[4\] for $p$-torsion objects). The image of the functor $D^t$ is called the category of crystalline sheaves over $X^0$. It has an

adjoint functor $E^t$ from the category of crystalline sheaves to the category $\mathcal{M}\mathcal{F}^\nabla$ as given in Ch. II f)-g) loc. cit..

For a non $p$-torsion object $M \in \mathcal{M}\mathcal{F}^\nabla$ and each $U \in \mathcal{U}$ with chosen Frobenius lifting $F_U$, one defines a local operator $\tilde{\Phi}_{F_U}$ on the set of subbundles of $M_U$ by

$$\tilde{\Phi}_{F_U}(M') = \sum_{i=0}^{n} \frac{\Phi_{F_U}^i F_U^* Fil^i M'}{p^i},$$

where $M' \subset M_U$ and $Fil^i M' = Fil^i M \cap M'$. A de Rham subbundle $(M', \nabla)$ of $(M, \nabla)$ is said to be $\tilde{\Phi}$-stable if

(i) the induced filtration $Fil^i M'$ is filtered free,

(ii) it holds for each $U \in \mathcal{U}$ that

$$\sum_{i=0}^{n} \frac{\Phi_{F_U}^i F_U^* Fil^i M'}{p^i} = M'.$$

It follows from the above Taylor formula that the condition (ii) is independent of the choices of $F_U$s. Also, it is tedious but straightforward to formulate the notion of $\tilde{\Phi}$-stable de Rham subbundles for a $p$-torsion object of $\mathcal{M}\mathcal{F}^\nabla$. A subobject of $M \in \mathcal{M}\mathcal{F}^\nabla$ is just a $\tilde{\Phi}$-stable de Rham subbundle of $(M, \nabla)$, which by Theorem 2.2 corresponds to a subrepresentation of $\nabla$.

### 3. Periodic Higgs subbundles in positive characteristic

Assume $X$ smooth projective over $W$ with connected geometric generic fiber. Let $M$ be an object in the category $\mathcal{M}\mathcal{F}^\nabla$, $\nabla$ the corresponding representation and $(E, \theta) = Gr_{Fil} (M, \nabla)$ the associated Higgs bundle. We have shown previously that $(E, \theta)_0$ is Higgs semistable. In this section we shall show further that

**Theorem 3.1.** The Higgs bundle $(E, \theta)_0$ is Higgs stable iff the representation $\nabla \otimes k$ is irreducible.

The analogous result over $\mathbb{C}$ is a highly nontrivial result in the theory of Simpson correspondence between complex local systems and Higgs bundles \[15\]. Motivated by this correspondence, one asks further a refined version of the above theorem.

**Question 3.2.** Is there a one to one correspondence between the set of subrepresentations of $\nabla \otimes k$ and the set of Higgs subbundles of $(E, \theta)$ with trivial Chern classes?

The aim of this section is to give an answer of this question which yields the above theorem as a direct consequence. Our answer relies on the Simpson correspondence in positive characteristic established by Ogus and Vologodsky \[14\]. For simplicity, we assume from now on that $pM = 0$ and $X$ is only smooth proper over $W$, so that the de Rham bundle $(M, Fil^i, \nabla)$ as well as its associated Higgs bundle are defined over $X_0/k$. Let $X'_0 = X_0 \times_{Spec k,F_k Spec k}$, where
$F_k : \text{Spec } k \to \text{Spec } k$ is the absolute Frobenius. The pull-back of a Higgs module over $X_0$ via the natural map $X'_0 \to X_0$ is a Higgs module over $X'_0$. One identifies the two categories of Higgs modules over $X_0$ and over $X'_0$. There is a convenient $W_2$-lifting $X'_1$ of $X'_0$, obtained from $X_1$ via the base change. Then, by setting $(\mathcal{X}, \mathcal{S}) = (X_0/k, X'_1/W_2(k))$, the inverse Cartier transform $C_{\mathcal{X}/\mathcal{S}}^{-1}$ in loc. cit. associates any Higgs subbundle of $(E, \theta)$ to a flat subbundle of $(M, \nabla)^{\mathbb{L}}$. For simplicity, we denote the inverse Cartier transform in our context by $C_0^{-1}$.

Our basic observation is that the operator $Gr_{Fil} \circ C_0^{-1}$ acts on the set of Higgs subbundles, and the action is not trivial in general. Here is an example.

**Example 3.3** (Section 7 [16]). Let $F$ be a totally real field and $D$ a quaternion division algebra over $F$ which is split at one unique real place $\tau$ of $F$. Let $K$ be an imaginary quadratic field and $L$ a totally imaginary quadratic extension of $F$ contained in $D$. Put $\Phi = \text{Hom}(L, \overline{\mathbb{Q}})$. Fix an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$. Assume $F$ is unramified at $p$ and each prime of $F$ over $p$ stays prime in $L$. Let $p$ be the prime of $F$ over $p$ given by $\tau$. One formulates a moduli functor of PEL type over $\mathcal{O}_{F_p}$ (see §5.2 [1]). Under suitable conditions, it is fine represented by a smooth projective curve $\mathcal{M}$ over $\mathcal{O}_{F_p}$, together with an universal abelian scheme $f : \mathcal{X} \to \mathcal{M}$. Let $(M, Fil', \nabla)$ be the relative de Rham bundle of $f$ and $(E, \theta)$ the associated Higgs bundle. Then one has an eigen-decomposition under the action of the commutative subalgebra $\mathcal{O}_{L\otimes K}$ in the endomorphism algebra of $f$:

$$(E, \theta)_0 = \bigoplus_{\phi \in \Phi} (E_\phi, \theta_\phi) \oplus (E_{\overline{\phi}}, \theta_{\overline{\phi}}).$$

Theorem 7.3 (1) [16] shows that

$$Gr_{Fil} \circ C_0^{-1}(E_\phi, \theta_\phi) = (E_{\sigma \phi}, \theta_{\sigma \phi}),$$

where $\sigma$ is the Frobenius element in the Galois group. Thus if the orbit of a $\phi$ under the $\sigma$-action has more than one element, it holds that

$$Gr_{Fil} \circ C_0^{-1}(E_\phi, \theta_\phi) \neq (E_\phi, \theta_\phi).$$

This example leads us to introduce the following

**Definition 3.4** (Periodic Higgs subbundles over $k$). Let $(E, \theta)$ be as above. A Higgs subbundle $(G, \theta)$ in $(E, \theta)$ is said to be periodic if there is a natural number $r$ such that the equality

$$(Gr_{Fil} \circ C_0^{-1})^r(G, \theta) = (G, \theta)$$

holds. The least natural number satisfying the equality is called the period of $(G, \theta)$.

Clearly, a periodic Higgs subbundle is a subsystem of Hodge bundles, that is,

$$G = \bigoplus_i G^{i,n-i}, \quad \text{with} \quad G^{i,n-i} = G \cap E^{i,n-i}.\footnote{This statement requires clarification. To avoid interrupting the main idea, we postpone this task to the appendix.}$$
An \((E_\phi, \theta_\phi)\) in the above example, as well as its bar counterpart, is a periodic Higgs subbundle. As the algebraic cycles given by elements of \(\mathcal{O}_{L\otimes K}\) are Tate cycles, it decomposes the étale local system \(V \otimes k\) accordingly. It is not difficult to arrange a natural one to one correspondence between eigen-components in \(V \otimes k\) and those in \((E, \theta)_0\). Naturally, one may wonder if one could deduce the correspondence without reference to the action of algebraic cycles but rather relying only on the notion of periodic Higgs subbundles. Our result shows that it is indeed the case and thereby gives an answer to the original question.

**Theorem 3.5.** Notation as above. Then there exists a one to one correspondence between the set of \(\mathbb{F}_p\)-subrepresentations in \(V \otimes \mathbb{F}_p\) and the set of periodic Higgs subbundles in \((E, \theta)\) whose periods are divisors of \(r\). Moreover, the \(\sigma\)-conjugation on the representation side corresponds to the operator \(Gr_{Fil} \circ C_0^{-1}\) on the Higgs side.

**Remark 3.6.** A Higgs subbundle \((G, \theta)\) is said to be quasi-periodic if the following equality for a pair \((r, s)\) of integers with \(r > s \geq 0\) holds:

\[
(Gr_{Fil} \circ C_0^{-1})^r(G, \theta) = (Gr_{Fil} \circ C_0^{-1})^s(G, \theta).
\]

It is shown in the proof of Theorem 4.17 (4) in [14] that for any nilpotent Higgs bundle \(G\) with level less than or equal to \(p - 1\),

\[
[C_0^{-1}(G)] = [F^*_X(G)],
\]

where \([\ ]\) denotes the class of a coherent \(\mathcal{O}_X\)-module in \(K_0(X)\). The equality implies that a quasi-periodic Higgs subbundle has trivial Chern classes. Conversely, a Higgs subbundle \((G, \theta)\) with trivial Chern classes ought to be quasi-periodic. The reason for it is given in the proof of the next theorem. Clearly, a periodic Higgs subbundle is quasi-periodic. However, we are lack of a good criterion to guarantee the injectivity of the operator \(Gr_{Fil} \circ C_0^{-1}\) which is equivalent to that any quasi-periodic Higgs subbundle is indeed periodic.

Now we proceed to deduce Theorem 3.1 from Theorem 3.5.

**Proof.** First of all, we draw a simple property for a periodic Higgs subbundle.

**Proposition 3.7.** A periodic Higgs subbundle is Higgs semistable of slope zero.

**Proof.** For a Higgs subbundle \((G, \theta) \subset (E, \theta)\), it follows from Lemma 3.2 [17] that

\[
\mu(Gr_{Fil} \circ C_0^{-1}(G, \theta)) = p\mu(G).
\]

Therefore, \(\mu(G) = 0\) because of the periodicity of \(G\). By Proposition 0.2 loc. cit., \((E, \theta)\) is Higgs semistable of slope zero. Then the statement follows by noting that a Higgs subbundle of \((G, \theta)\) is also a Higgs subbundle of \((E, \theta)\).

Now assume first that \((E, \theta)\) is stable. If \(V \otimes k\) was not irreducible, then \(V \otimes \mathbb{F}_p\) is reducible for some \(r \in \mathbb{N}\). It follows from Theorem 3.5 that there is a nontrivial proper periodic Higgs subbundle in \((E, \theta)\), which contradicts the assumption by the last proposition. Thus \(V \otimes k\) is irreducible. Conversely, assume \(V \otimes k\) is irreducible. If \((E, \theta)\) was not stable, then there is a nontrivial proper Higgs subbundle \((G, \theta) \subset (E, \theta)\) of slope zero. Note that the operator \(Gr_{Fil} \circ C_0^{-1}\)
does not change the slope, rank and definition field of \((G, \theta)\). Since there are only finitely many Higgs subbundles of \((E, \theta)\) with the same slope, rank and definition field as \((G, \theta)\), \((G, \theta)\) ought to be quasi-periodic. Take a pair \((r, s)\) for \((G, \theta)\). Then the Higgs subbundle \((Gr_{Fil} \circ C_0^{-1})^s(G, \theta)\) is periodic. By Theorem \[3.5\] it follows that \(V \otimes F_{r'}\) is reducible which contradicts the assumption. This completes the proof. □

In §5 one finds more generalizations of the above result obtained as consequences of Theorem \[3.5\]. In the remaining paragraph we shall concentrate on the proof of Theorem \[3.5\]. The key is to notice a basic property of the inverse Cartier transform in the current setup.

**Proposition 3.8.** Let \((G, \theta)\) be a Higgs subbundle of \((E, \theta)\). If

\[
Gr_{Fil} \circ C_0^{-1}(G, \theta) = (G, \theta)
\]

is satisfied, then \(C_0^{-1}(G, \theta)\) is \(\tilde{\Phi}\)-stable, and hence corresponds to a subrepresentation of \(V\) by Theorem \[2.2\].

*Proof.* The question is local. For each small affine \(U \subset X\), one can express \(C_0^{-1}(G, \theta)\) locally as follows: take a local basis \(\{g^{i,n-i}\}\) of \(G^{i,n-i}\) over \(U_0\), and then a set of elements \(\{\tilde{g}^{i,n-i}\}\) in \(M\) with \(\tilde{g}^{i,n-i} \in Fil^i H\) satisfying the condition

\[
\tilde{g}^{i,n-i} \mod Fil^i+1 M = g^{i,n-i}, \quad 0 \leq i \leq n.
\]

Then it holds that

\[
C_0^{-1}(G, \theta)_{U_0} = \text{Span}\left[\frac{\tilde{\Phi}^i}{p^i} \left( F_i \tilde{g}^{i,n-i} \right), \quad 0 \leq i \leq n \right].
\]

The period one property for \((G, \theta)\) implies that we can take \(\{\tilde{g}^{i,n-i}\}\) to be a local basis of \(C_0^{-1}(G, \theta)\). Thus the \(\tilde{\Phi}\)-stability of \(C_0^{-1}(G, \theta)\) is just the local expression of \(C_0^{-1}\) as above. □

A direct consequence of the previous proposition is the special case \(r = 1\) of Theorem \[3.5\].

**Corollary 3.9.** Let \(\mathcal{V}, \mathcal{H}\) and \(\mathcal{E}\) be the set of subrepresentations of \(\mathcal{V}\), \(\tilde{\Phi}\)-stable de Rham subbundles of \(M\) and periodic Higgs subbundles of \((E, \theta)\) of period one respectively. Then there are one to one correspondences between the sets:

\[
\mathcal{V} \overset{D^1}{\leftrightarrow} \mathcal{H} \overset{C_0^{-1}}{\leftrightarrow} \mathcal{E}.
\]

*Proof.* Theorem \[2.2\] settles the correspondence between \(\mathcal{V}\) and \(\mathcal{H}\). It is to show the correspondence between \(\mathcal{H}\) and \(\mathcal{E}\). Note first that the locally filtered freeness of \(M' \subset M\) is equivalent to the locally freeness of the grading \(Gr_{Fil} M'\). If a de Rham subbundle \(M'\) is further \(\tilde{\Phi}\)-stable, its grading \(Gr_{Fil} M'\) satisfies the period one condition:

\[
Gr_{Fil} \circ C_0^{-1}(Gr_{Fil} M') = Gr_{Fil} (M'),
\]

which follows by taking the grading of the \(\tilde{\Phi}\)-stability condition. The converse direction is just Proposition \[3.8\]. □
The proof of Theorem 3.5 for a general \( r \) will be reduced to the above case. The main idea is as follows: for a periodic Higgs subbundle \((G, \theta)\) of period \( r \), we embed the Higgs subbundles \( \{(Gr_{Fil} \circ C^{-1}_0)(G, \theta)\}_{0 \leq i \leq r-1} \) into \((E, \theta)^{\oplus r}\) in a suitable way such that the image is periodic of period one. The above corollary gives us then an \( \mathbb{F}_p \)-subrepresentation \( W \subset V^{\oplus r} \). Considering \( \mathbb{F}_p \)-algebra structure, we label the \( r \) copies of \((M, \nabla, Fil^i, \Phi)\) as \( \{(iH, i\nabla, Fil^i, i\Phi)\}_{0 \leq i \leq r-1} \).

We observe that the map \( s \) on \( V \otimes \mathbb{F}_p \), induced by multiplication with \( \xi \) on the tensor factor \( \mathbb{F}_p \), is an endomorphism in the category of crystalline sheaves, and hence by the equivalence of categories corresponds to an endomorphism \( s_{MF} \) on \( M^{\oplus r} \) in the category \( \mathcal{MF}^\nabla \) with the minimal polynomial \( P(t) \). As \( \mathbb{F}_p \subset k \subset \mathcal{O}_{X_0} \), the endomorphism \( s_{MF} \) decomposes \( \oplus_{i=0}^{r-1}(iH) \) into a direct sum of eigenspaces. We need to describe the eigen-decomposition in an explicit way which will be applied in our reduction step. For sake of completeness, the proof of the following simple lemma in linear algebra is supplied.

**Lemma 3.10.** Let \( A \in \text{GL}_r(\mathbb{F}_p) \) be the representation matrix of the \( \mathbb{F}_p \)-linear map \( m_\xi : \mathbb{F}_p \to \mathbb{F}_p \) under the basis \( \{1, \xi, \cdots, \xi^{r-1}\} \). Then there is an invertible matrix

\[
S = (S_1, S_1^\sigma, \cdots, S_1^{\sigma^{r-1}}) \in \text{GL}_r(\mathbb{F}_p)
\]

such that

\[
S^{-1}AS = \text{diag}\{\xi, \xi^\sigma, \cdots, \xi^{\sigma^{r-1}}\},
\]

where \( S_1 \) is the first column vector of \( S \) and \( \sigma \in \text{Gal}(\mathbb{Q}_p|\mathbb{Q}) \) is the Frobenius element.

**Proof.** As \( P(t) \in \mathbb{F}_p[t] \) splits over \( \mathbb{F}_p \) into the product \( \prod_{i=0}^{r-1}(t - \xi^\sigma^i) \) of linear factors, one has a basis of eigenvectors of

\[
m_\xi \otimes \text{id} : \mathbb{F}_p \otimes_{\mathbb{F}_p} \mathbb{F}_p \to \mathbb{F}_p \otimes_{\mathbb{F}_p} \mathbb{F}_p.
\]
Pick an eigenvector $S_1$ to the eigenvalue $\xi$, which namely satisfies the equality $AS_1 = \xi S_1$ holds. Applying $\sigma^i$ on both sides, one obtains then

$$AS_1^{\sigma^i} = \xi \sigma^i S_1^{\sigma^i}.$$ 

So the matrix $S = (S_1, \cdots, S_1^{\sigma^{r-1}})$ satisfies

$$AS = S\text{diag}\{\xi, \xi^\sigma, \cdots, \xi^{\sigma^{r-1}}\}.$$ 

Note that $\{S_1^{\sigma^i}\}_{0 \leq i \leq r-1}$ makes a basis of eigenvectors of $m_\xi \otimes 1$ and hence $S$ is invertible. $\square$

**Lemma 3.11.** The endomorphism $s_{MF}$ decomposes $\bigoplus_{i=0}^{r-1} iH$ into direct sum $\bigoplus_{i=0}^{r-1} M^i$ of eigenspaces such that

(i) one has an explicit isomorphism

$$(M, \nabla, Fil^i) \cong (M^i, \nabla^i := \nabla^{\otimes r}|_{M^i}, Fil^i := Fil^{\otimes r}|_{M^i}),$$

(ii) $\phi := \Phi^{\otimes r}$ permutes $\{M^i\}_{0 \leq i \leq r-1}$ cyclically.

**Proof.** Let $\Delta : M \to \bigoplus_{i=0}^{r-1} iH$ be the diagonal embedding with $i$-th component $\Delta^i$, and

$$S_1 = \begin{pmatrix} a_0 \\ \vdots \\ a_{r-1} \end{pmatrix}.$$ 

It follows from the last lemma that for $0 \leq i \leq r-1$,

$$\sum_{j=0}^{r-1} a_j^{\sigma^i} \Delta^j(M)$$

is the eigenspace of $s_{MF}$ with eigenvalue $\xi^{\sigma^i}$. Clearly, the isomorphism of vector bundles

$$\alpha_i = \sum_{j=0}^{r-1} a_j^{\sigma^i} \Delta^j : M \to M^i$$

respects also the connection and the filtration. Hence (i) follows. We have (ii) immediately because of the semilinearity of $\phi$. $\square$

Set $(E^i, \theta^i) = Gr_{Fil^i}(M^i, \nabla^i)$. Then $\alpha_i$ in the above lemma induces the isomorphism of Higgs bundles

$$\beta_i : (E, \theta) \cong (E^i, \theta^i).$$

The following easy lemma follows immediately from the semilinearity of $C_0^{-1}$ and will be applied below.

**Lemma 3.12.** It holds for a Higgs subbundle $(G, \theta) \subset (E, \theta)$,

$$Gr_{Fil} \circ C_0^{-1}(\beta_i(G, \theta)) = \beta_{i+1} \mod r(Gr_{Fil} \circ C_0^{-1}(G, \theta)), \ 0 \leq i \leq r-1$$

Now we proceed to the proof of Theorem 3.5.
Proof. Let $\mathcal{E}^r$ be the set of periodic Higgs subbundles of $(E, \theta)$ whose periods divide $r$, and $\mathcal{V}^r$ the set of $F_{p^r}$-subrepresentations of $V \otimes F_{p^r}$. Now we consider $M^{\oplus r} \in \mathcal{MF}^\Sigma$ together with the endomorphism $s_{MF}$ described as above. Let $\mathcal{H}^r$ be the set of $s_{MF}$-invariant $\tilde{\phi}$-stable de Rham subbundles of $(M, \nabla, Fil.)^{\oplus r}$. We shall show the correspondences of Corollary 3.9 for $M^{\oplus r}$ (instead of $M$) induce the claimed correspondence between $\mathcal{V}^r$ and $\mathcal{E}^r$, using $\mathcal{H}^r$ as a bridge. First of all, an $F_{p^r}$-subrepresentation of $V \otimes F_{p^r}$ is nothing but a $F_{p^r}$-subrepresentation of

$$V \otimes F_{p^r}\{1\} \oplus \cdots \oplus V \otimes F_{p^r}\{\xi^{r-1}\} = V \otimes F_{p^r},$$

which is invariant under the endomorphism $s$. Thus the functors $D^*$ and $E^*$ restricts to a one to one correspondence between $\mathcal{V}^r$ and $\mathcal{H}^r$. By Lemma 3.11 on the eigen-decomposition of $s_{MF}$, an element of $\mathcal{H}^r$ is given by a direct sum $\oplus_{i=0}^{r-1}(M^i)$ which is $\tilde{\phi}$-stable, where $M^i \subset M^i$ is $\nabla^i$-invariant for each $i$, and since $\tilde{\phi}$ permutes the factors $\{M^i\}$s cyclically, $M^i$ is just $\tilde{\phi}^i(M^0)$ for $1 \leq i \leq r - 1$. Thus an element of $\mathcal{H}^r$ can be represented by

$$M^0 \oplus \tilde{\phi}(M^0) \oplus \cdots \oplus \tilde{\phi}^{r-1}(M^0),$$

where $M^0 \subset M^0$ is a de Rham subbundle satisfying $\tilde{\phi}(M^0) = M^0$. It is clear that the functors $Gr_{Fil}$ and $C_0^{-1}$ induce a one to one correspondence between $\mathcal{H}^r$ and a set of Higgs subbundles of $(E, \theta)^{\oplus r} = \oplus_{i=0}^{r-1}(E^i, \theta^i)$ of the following form:

$$(G, \theta) \oplus Gr_{Fil} \circ C_0^{-1}(G, \theta) \oplus \cdots \oplus (Gr_{Fil} \circ C_0^{-1})^{-1}(G, \theta)$$

for a Higgs subbundle $(G, \theta) \subset (E^0, \theta^0)$ with the property

$$(Gr_{Fil} \circ C_0^{-1})^{-1}(G, \theta) = (G, \theta).$$

By Lemma 3.12, the isomorphism $\beta_0^{-1}: (E^0, \theta^0) \cong (E, \theta)$ induces an identification between the previous set of Higgs subbundles and $\mathcal{E}^0$. Therefore, we have established a one to one correspondence between $\mathcal{V}^r$ and $\mathcal{E}^r$. Finally, let $W \subset V \otimes F_{p^r}$ be an element of $\mathcal{V}^r$. Its $\sigma$-conjugation $W^{\sigma} := W \otimes F_{p^r, \sigma} \otimes F_{p^r}$ is considered as an $F_{p^r}$-subrepresentation of $V \otimes F_{p^r}$ via the natural isomorphism

$$V \otimes F_{p^r} \otimes F_{p^r, \sigma} \otimes F_{p^r} \cong V \otimes F_{p^r} \otimes F_{p^r}, \quad v \otimes \lambda \otimes \mu \mapsto v \otimes \lambda \mu^{\sigma}.$$

So one considers the endomorphism $s^\sigma$ on $V \otimes F_{p^r}$ induced by multiplication with $\xi^{r-1}$ and its corresponding endomorphism $s_{MF}^\sigma$ on $(M, \nabla, Fil., \Phi)^{\oplus r}$. Chasing the proof of Lemma 3.11 one sees that the $i$-th eigenspace of $s_{MF}^\sigma$ is the $i + 1$ mod $r$-th eigenspace of $s_{MF}$. This means that under the above correspondence between $\mathcal{V}^r$ and $\mathcal{H}^r$, if $M^0 \oplus \cdots \oplus M^{r-1}$ corresponds to $W$, then $M^1 \oplus \cdots \oplus M^{r-1} \oplus M^0$ corresponds to $W^{\sigma}$. Therefore if $(G, \theta) \in \mathcal{E}^r$ corresponds to $W$, then $Gr_{Fil} \circ C_0^{-1}(G, \theta)$ corresponds to $W^{\sigma}$. □

4. Periodic Higgs subbundles in mixed characteristic

We would like to extend our previous results in char $p$ to mixed characteristic. For a non $p$-torsion object $M \in \mathcal{MF}^\Sigma$ we ask the following question, which is parallel to Question 3.2.
**Question 4.1.** Is there a one to one correspondence between the set of subrepresentations of $V \otimes \hat{\mathbb{Z}}_p$ and the set of Higgs subbundles of $(E, \theta) \otimes O_X$ with trivial Chern classes, where $X := X \times_W \text{Spec } \hat{\mathbb{Z}}_p$?

This question seems to be much more difficult than the question in char $p$ case. What we have obtained in this section is a partial result. The following theorem is the lifted version of Theorem 3.5.

**Theorem 4.2.** For each $r \in \mathbb{N}$, there is a one to one correspondence between the set of $\mathbb{Z}_p[r]$-subrepresentations of $V \otimes \mathbb{Z}_p/r\mathbb{Z}_p$ and the set of periodic Higgs subbundles of $(E, \theta)$ whose periods divide $r$.

The meaning of a periodic Higgs subbundle in mixed characteristic will be explained below. The key to the above theorem is the construction of a lifting of the inverse Cartier transform to mixed characteristic. The construction is done in an inductive way. So we shall consider first the lifting of the inverse Cartier transform to $W_2$. While the inverse Cartier transform associates to any Higgs subbundle of $(E, \theta)_0$ a de Rham subbundle, the objects of our lifted inverse Cartier transform over $W_2$ are not all Higgs subbundles of $(E, \theta)_1$, rather those subject to the periodic condition in char $p$. See Proposition 4.9 for the precise statement.

Let $(G, \theta) \subset (E, \theta)_1$ be a subsystem of Hodge bundles. By abuse of notation, we denote the image of $(G, \theta)_0$ in $(E, \theta)_0$ again by $(G, \theta)_0$. A similar abuse applies to the modulo $p$ reduction of a de Rham subbundle in $(M, \nabla)_1$.

**Theorem 4.3.** If $(G, \theta)_0 \subset (E, \theta)_0$ is a periodic Higgs subbundle of period one, then one constructs a de Rham subbundle $C_1^{-1}(G, \theta) \subset (M, \nabla)_1$ with the same rank as $G$ satisfying the equality

$$(C_1^{-1}(G, \theta))_0 = C_0^{-1}(G, \theta)_0.$$

Furthermore, if the equality

$$\text{Gr}_{\text{Fil}} \circ C_1^{-1}(G, \theta) = (G, \theta)$$

is satisfied, then $C_1^{-1}(G, \theta) \subset M_1$ is $\bar{\Phi}$-stable, hence corresponds to a subrepresentation of $V \otimes \mathbb{Z}_p/p^2$ by Theorem 2.2.

We shall call $C_1^{-1}$ in the theorem an inverse Cartier transform over $W_2$. Its construction is based on our previous work [17] aiming at a 'physical' understanding of the inverse Cartier transform of Ogus and Vologodsky. Recently, we have generalized the construction to a certain category of nilpotent Higgs bundles which have no bearing with the category $\mathcal{M}\mathcal{H}^{-}$. For clarity, we shall carry out only the local version in this section and complete the global construction in §6.

A local inverse Cartier transform over $W_2$. In the following paragraph, $X$ is assumed to be affine with a Frobenius lifting $F_X : \hat{X} \to \hat{X}$. Let $M \in \mathcal{M}\mathcal{F}^{-}$ with $pM \neq 0$ and $p^2M = 0$. Let $(G, \theta) \subset (E, \theta)$ be a Higgs subbundle satisfying

$$\text{Gr}_{\text{Fil}} \circ C_0^{-1}(G, \theta)_0 = (G, \theta)_0.$$
Our starting point of the construction is to observe the existence of special liftings for a basis of \( G \).

**Lemma 4.4.** Let \( \{g^{i,n-i}\}_{0 \leq i \leq n} \) be a basis of \( G = \bigoplus_{i=0}^{n} G^{i,n-i} \). Then there exists a set of elements \( \{\tilde{g}^{i,n-i}\}_{0 \leq i \leq n} \) in \( M \) with \( \tilde{g}^{i,n-i} \in \text{Fil}^{i+1}M \) satisfying two conditions:

(i) \( \tilde{g}^{i,n-i} \mod \text{Fil}^{i+1}M = g^{i,n-i} \),
(ii) \( \tilde{g}^{i,n-i} \mod pM \in C^{-1}_0(G,\theta)_0 \).

**Proof.** Without loss of generality we can assume in the argument that each set \( g^{i,n-i} \) consists of only one element if nonempty. Put \( \tilde{g}^{n,0} = g^{n,0} \).

For each \( 0 \leq i \leq n-1 \), we take any \( \tilde{g}^{i,n-i} \in M \) satisfying (i). We shall modify it as follows: consider its modulo \( p \) reduction \( \tilde{g}^{i,n-i}_0 \in M_0 \), which satisfies
\[
\tilde{g}^{i,n-i}_0 \mod \text{Fil}^{i+1}M_0 = g^{i,n-i}_0 \in G^{i,n-i}_0.
\]
Since \( Gr_{\text{Fil}} [C^{-1}_0(G,\theta)_0] = (G,\theta)_0 \) holds by assumption, there exists a \( \tilde{g}^{i,n-i}_0 \in \text{Fil}^{i+1}C^{-1}_0(G,\theta)_0 \) satisfying
\[
\tilde{g}^{i,n-i}_0 \mod \text{Fil}^{i+1}M_0 = g^{i,n-i}_0.
\]
In other words, it holds that
\[
\omega^{i+1,n-i-1}_0 := \tilde{g}^{i,n-i}_0 - g^{i,n-i}_0 \in \text{Fil}^{i+1}M_0.
\]
Now we pick an \( \omega^{i+1,n-i-1} \in \text{Fil}^{i+1}M \) lifting \( \omega^{i+1,n-i-1}_0 \) and set
\[
\tilde{g}^{i,n-i} = \tilde{g}^{i,n-i}_0 - \omega^{i+1,n-i-1}.
\]
Then this modified element satisfies both conditions. \( \square \)

**Proposition and Definition 4.5.** Notation as above. Then the \( O_{X_1} \)-submodule
\[
\text{Span}\left[\frac{\Phi_{F_X}}{p^i}(F_X^{\ast}\tilde{g}^{i,n-i}), \ 0 \leq i \leq n\right]
\]
is a well-defined de Rham submodule of \((M,\nabla)\). More precisely, the span is independent of the choice of basis elements \( \{g^{i,n-i}\} \) of \( G \) and the choice of liftings \( \{\tilde{g}^{i,n-i}\} \) in Lemma 4.4. We call it \( C^{-1}_1(G,\theta) \).

**Remark 4.6.** In \([36]\) we show further that \( C^{-1}_1(G,\theta) \) is also independent of the choice of Frobenius lifting \( F_X \).

**Proof.** For simplicity, we omit the subscript of \( \Phi_{F_X} \) and denote \( e \otimes 1 \) for the pullback of an element \( e \in M \) via \( F_X \).

**Claim 4.7.** The span is independent of the choice of elements \( \{\tilde{g}^{i,n-i}\} \) in Lemma 4.4 as well as the choice of basis elements \( \{g^{i,n-i}\} \) of \( G \). Hence \( C^{-1}_1(G,\theta) \subset M \) is well defined.
Proof. Let \( \{ \tilde{g}^{i,n-i} \} \) be another set of elements in Lemma 4.4. Then by condition (i) we can write
\[
\tilde{g}^{i,n-i} = g^{i,n-i} + \omega^{i+1,n-i-1},
\]
for an \( \omega^{i+1,n-i-1} \in Fil^{i+1}M \). By condition (ii),
\[
\omega_0^{i+1,n-i-1} \in Fil^{i+1}C_{0}^{-1}(G, \theta)_0.
\]
Note that the two conditions of Lemma 4.4 imply the equality:
\[
Fil^{i+1}C_{0}^{-1}(G, \theta)_0 = \text{Span}[g_0^{n,0}, \ldots, \tilde{g}_0^{i+1,n-i-1}].
\]
It follows that
\[
\omega^{i+1,n-i-1} \in pFil^{i+1}M + \text{Span}[g_0^{n,0}, \ldots, \tilde{g}_0^{i+1,n-i-1}].
\]
Now we use the induction on \( i \) to show the claim: for \( i = n \), there is nothing to prove. The induction hypothesis for \( i + 1 \) means that
\[
\text{Span}\left[ \frac{\Phi}{p^i}(\tilde{g}^{i,n-j} \otimes 1), \ i + 1 \leq j \leq n \right] = \text{Span}\left[ \frac{\Phi}{p^i}(g^{j,n-j} \otimes 1), \ i + 1 \leq j \leq n \right].
\]
It follows from the above discussion that
\[
\frac{\Phi}{p^i}(\tilde{g}^{i,n-i} \otimes 1) - \frac{\Phi}{p^i}(g^{i,n-i} \otimes 1) = \frac{\Phi}{p^i}(\omega^{i+1,n-i-1} \otimes 1)
\]
\[
\in \frac{\Phi}{p^i}F_{\tilde{X}}[pFil^{i+1}M + \text{Span}[g_0^{n,0}, \ldots, \tilde{g}_0^{i+1,n-i-1}]]
\]
\[
= \frac{\Phi}{p^i}F_{\tilde{X}}[\text{Span}[g^{n,0}, \ldots, \tilde{g}^{i+1,n-i-1}]].
\]
The last equality follows from the fact \( \frac{\Phi}{p^i}(pFil^{i+1}M) = 0 \). Since clearly
\[
\frac{\Phi}{p^i}F_{\tilde{X}}[\text{Span}[g^{n,0}, \ldots, \tilde{g}^{i+1,n-i-1}]] \subset \text{Span}\left[ \frac{\Phi}{p^i}(\tilde{g}^{j,n-j} \otimes 1), \ i + 1 \leq j \leq n \right],
\]
the case for \( i \) then follows. The \( i = 0 \) case is the first statement of the claim. Note that two different choice of bases of \( G \) are interrelated through an invertible matrix. It relates also special liftings in Lemma 4.4 for these two bases. Clearly, they define the same span. Thus \( C_1^{-1}(G, \theta) \) is a well defined submodule of \( M \). □

It remains to show the following

Claim 4.8. The \( \mathcal{O}_X \)-submodule \( C_1^{-1}(G, \theta) \subset M \) is invariant under the action of \( \nabla \). Hence \( C_1^{-1}(G, \theta) \), equipped with the induced connection, is a de Rham submodule of \( (M, \nabla) \).

Proof. Without loss of generality, we assume \( X \) to be a curve. Take a local coordinate \( t \) of \( X \), i.e. \( \Omega_{X|M} = \mathcal{O}_X\{dt\} \), and set \( \partial = \frac{d}{dt} \). It suffices to show that for each \( 0 \leq i \leq n \),
\[
\nabla_{\partial}(\frac{\Phi}{p^i}(\tilde{g}^{i,n-i} \otimes 1)) \in C_1^{-1}(G, \theta).
\]
The horizontal property of \( \Phi \) can be explicitly expressed by
\[
\nabla_{\partial}\left[ \frac{\Phi}{p^i}(\tilde{g}^{i,n-i} \otimes 1) \right] = \frac{\Phi}{p^{i-1}}[\nabla_{\partial}(\tilde{g}^{i,n-i} \otimes a)],
\]
for an \( a \in \mathcal{O}_{X_1} \). We shall show that
\[
\Phi \frac{p^i-1}[\nabla_\partial(\tilde{g}^{i,n-i}) \otimes 1] \in C_1^{-1}(G, \theta).
\]
The assumption that \( G \subset E \) is a Higgs subbundle means that \( G \) is invariant under the action of \( \theta_\partial = Gr_{Fil} \nabla_\partial \). Thus one finds a unique \( b \in \mathcal{O}_{X_1} \) such that
\[
\nabla_\partial(\tilde{g}^{i,n-i}) \mod Fil^i M = bg^{i-1,n-i+1}.
\]
It follows then
\[
\omega^{i,n-i} := \nabla_\partial(\tilde{g}^{i,n-i}) - bg^{i-1,n-i+1} \in Fil^i M.
\]
As clearly
\[
\Phi \frac{p^i-1}(bg^{i-1,n-i+1} \otimes 1) \in C_1^{-1}(G, \theta),
\]
it suffices to show
\[
\Phi \frac{p^i-1}(\omega^{i,n-i} \otimes 1) \in C_1^{-1}(G, \theta).
\]
As \( \Phi \frac{p^i-1}(\omega^{i,n-i} \otimes 1) \in pM \), it is equivalent to show
\[
\Phi \frac{p^i-1}(\omega^{i,n-i} \otimes 1) \in [C_1^{-1}(G, \theta)]_0 = C_0^{-1}(G, \theta)_0.
\]
The equivalence is clear from Lemma \[6.1\], which is elementary. Now that
\[
\omega^{i,n-i}_0 = \nabla_\partial(\tilde{g}^{i,n-i}_0) - b_0g^{i-1,n-i+1}_0,
\]
with \( \tilde{g}^{i,n-i}_0 \in C_0^{-1}(G, \theta)_0 \) and \( C_0^{-1}(G, \theta)_0 \) is \( \nabla \)-invariant, one has
\[
\omega^{i,n-i}_0 \in C_0^{-1}(G, \theta)_0.
\]
Finally, because \((G, \theta)_0 \) is periodic of period one,
\[
\Phi \frac{p^i-1}(\omega^{i,n-i} \otimes 1) \in C_0^{-1}(G, \theta)_0
\]
by Proposition \[3.8\]. This shows the claim. \( \square \)

Now we are going to show Theorem \[4.3\] by assuming the global existence of \( C_1^{-1} \).

**Proof.** It follows from the strong \( p \)-divisibility of \( \Phi \) that \( C_1^{-1}(G, \theta) \) is free \( \mathcal{O}_{X_1} \)-module of the same rank as \( G \). Note that the set of elements \( \{g^{i,n-i}_0\} \) is a basis of \( G_0 \). Then we have
\[
(C_1^{-1}(G, \theta))_0 = \text{Span}_{\mathcal{O}_{X_0}}[\Phi \frac{p^i-1}{p^i}(g^{i,n-i}_0 \otimes 1), \ 0 \leq i \leq n],
\]
which is exactly \( C_0^{-1}(G, \theta)_0 \) by its very construction. Now assume furthermore
\[
Gr_{Fil} \circ C_1^{-1}(G, \theta) = (G, \theta).
\]
Then we can take \( \{\tilde{g}^{i,n-i}\} \) of Lemma \[4.4\] to be a basis of \( C_1^{-1}(G, \theta) \), and the \( \Phi \)-stability of \( C_1^{-1}(G, \theta) \) is just the definition of \( C_1^{-1} \). \( \square \)
The assumption of \((G, \theta)\) for the existence of \(C_1^{-1}(G, \theta)\) can be relaxed. In fact, using the same technique in the reduction step from a general period to the period one case, we can show the following

**Proposition 4.9.** Let \((G, \theta) \subset (E, \theta)_1\) be a Higgs subbundle whose modulo \(p\) reduction is a periodic Higgs subbundle in \((E, \theta)_0\). Then there exists \(C_1^{-1}(G, \theta) \subset (M, \nabla)_1\) with the same rank as \(G\) satisfying the equality

\[
(C_1^{-1}(G, \theta))_0 = C_0^{-1}(G, \theta)_0.
\]

**Remark 4.10.** In the above proposition as well as Theorem 4.3, we have assumed that \((G, \theta) \subset (E, \theta)_1\) to be a subsystem of Hodge bundles. However, this assumption is not necessary. In fact, \(C_1^{-1}(G, \theta)\) exists for any Higgs subbundle of \((E, \theta)_1\) with the periodic condition in char \(p\).

The detail of the proof is postponed to [6] because it is more urgent to note that we are already in an inductive situation:

For a non-\(p\)-torsion \(M \in \mathcal{MF}^\nabla\), we define inductively the set of periodic Higgs subbundles of \((E, \theta)_m\) and a lifting of the inverse Cartier transform \(C_m^{-1}\) over \(W_{m+1} := W_{m+1}(k)\), where \(m\) runs from zero to infinity. Based on the inverse Cartier transform \(C_0^{-1}\) of Ogus and Vologodsky, we have defined previously the set of periodic Higgs subbundles in \((E, \theta)_0\). The last proposition asserts that the inverse Cartier transform lifts to an operator over \(W_2\) for those Higgs subbundles of \((E, \theta)_1\) whose modulo \(p\) reduction are periodic. Thus we make the following

**Definition 4.11** (Periodic Higgs subbundles over \(W_2\)). Notation as above. A Higgs subbundle \((G, \theta)\) of \((E, \theta)_1\) is periodic if there are two natural numbers \(r_0, r_1\) such that

1. \((Gr_{Fil} \circ C_0^{-1})^{r_0}(G, \theta)_0 = (G, \theta)_0\) and then,
2. \((Gr_{Fil} \circ C_1^{-1})^{r_1}(G, \theta) = (G, \theta)\) hold.

Via a direct generalization of the construction of \(C_1^{-1}\), one defines a further lifting \(C_2^{-1}\) over \(W_3\) for the set of Higgs subbundles of \((E, \theta)_2\) whose modulo \(p^2\) reduction are periodic in the above sense, and then the set of periodic Higgs subbundles of \((E, \theta)_2\), and so on. This process culminates with the following

**Definition 4.12** (Periodic Higgs subbundles over \(W_{m+1}\) and \(W\)). A Higgs subbundle \((G, \theta) \subset (E, \theta)_m\) is periodic if there are a sequence of natural numbers \(\{r_i\}_{0 \leq i \leq m}\) such that inductively from \(i = 0\) to \(i = m\) the equality

\[
(Gr_{Fil} \circ C_i^{-1})^{r_i}(G, \theta)_i = (G, \theta)_i
\]

holds. A Higgs subbundle of \((E, \theta)\) is periodic if its reduction in \((E, \theta)_m\) is periodic for all \(m \geq 0\).

For a periodic Higgs subbundle \((G, \theta)\), we list the periods of \((G, \theta)_m\) into a sequence of natural numbers \(r_0, r_1, \ldots\). Clearly, \(r_i\) divides \(r_j\) for \(i > j\). Since the numbers of Higgs subbundles in \((E, \theta)_m\) are bounded by a constant independent of \(m\), the above sequence is stable after finitely many terms. Thus a Higgs subbundle \((G, \theta)\) is periodic if there exists a natural number \(r\) such that

\[
(Gr_{Fil} \circ C_m^{-1})^r(G, \theta)_m = (G, \theta)_m, \ \forall m \geq 0.
\]
Now we come to the proof of Theorem 4.2.

Proof. Our lifting \(C_{m+1}^{-1}\) of the inverse Cartier transform lifts the basic property of \(C_0^{-1}\) as given in Proposition 3.8. For \(m = 0\), this is a part of statements in Theorem 4.3 and its proof generalizes directly to a general \(m\). Using the same argument as in Corollary 3.9, one shows the one to one correspondence between the set of subrepresentations of \(\mathcal{V}\) and the set of periodic Higgs subbundles of \((E, \theta)\) of period one by identifying both with the set of \(\tilde{\Phi}\)-stable de Rham subbundles of \(M\). To show the general case, we note first that Lemma 3.10 and its consequent lemmas hold over \(\mathbb{Z}_{p^r}\). So the reduction step to the period one case as carried in the proof of Theorem 3.5 can be applied to the mixed characteristic situation as well. This shows the theorem. □

5. Further applications

In this section, \(X\) is assumed to be smooth projective over \(W\) throughout. Notations as before. We start with a direct consequence of Theorem 3.5.

Proposition 5.1. If \(\mathcal{V} \otimes k\) is semisimple, then \((E, \theta)_0\) is polystable.

Proof. The assumption implies the decomposition

\[
\mathcal{V} \otimes \mathbb{F}_{p^r} = \bigoplus_i \mathcal{V}_i
\]

into direct sum of \(\mathbb{F}_{p^r}\)-representations whose direct factors are all absolutely irreducible. It is clear that the correspondence in Theorem 3.5 respects direct sum. So we obtain a corresponding decomposition

\[
(E, \theta)_0 = \bigoplus_i (G_i, \theta_i)
\]

into direct sum of periodic Higgs subbundles. Each factor ought to be stable, since the corresponding factor is otherwise not absolutely irreducible by a similar argument in Theorem 3.1. □

We would like to have the converse statement of the above proposition. What we have obtained is a conditional result. The proof of the following lemma is identical to that for a semistable bundle of degree zero, which is standard.

Lemma 5.2. The following statements hold for \((E, \theta)_0\):

(i) there is a filtration \(0 = (G_0, \theta_0) \subset (G_1, \theta_1) \subset \cdots \subset (G_r, \theta_r) = (E, \theta)_0\) by Higgs subbundles such that the quotient \((G_i, \theta_i)/(G_{i-1}, \theta_{i-1})\) is Higgs stable and \(\text{deg}(G_i) = 0\) for \(1 \leq i \leq r\),

(ii) if \(0 = (G'_0, \theta'_0) \subset (G'_1, \theta'_1) \subset \cdots \subset (G'_s, \theta'_s) = (E, \theta)_0\) is another filtration enjoying the properties of (i), then \(r = s\) and there exists a permutation \(\tau\) of \(\{1, \cdots, r\}\) such that \((G_i, \theta_i)/(G_{i-1}, \theta_{i-1})\) is isomorphic to \((G'_{\tau(i)}, \theta'_{\tau(i)}/(G'_{\tau(i)-1}, \theta'_{\tau(i)-1})\).

A filtration in (i) is called a Jordan-Hölder (abbreviated as JH) filtration of \((E, \theta)_0\). Put

\[
gr(E, \theta)_0 = \bigoplus_{i=1}^r (G_i, \theta_i)/(G_{i-1}, \theta_{i-1}),
\]

which is independent of the choice of a JH filtration. The number \(r\) in the above expression is said to be the length of a JH filtration on \((E, \theta)_0\).
Assumption 5.3. Assume the factors in $\text{gr}(E, \theta)_0$ are non-isomorphic to each other.

Proposition 5.4. Assume 5.3. If $(E, \theta)_0$ is polystable, then $V \otimes k$ is semisimple.

Note that the operator $\text{Gr F il} \circ C^{-1}_0$ does not commute with direct sum in general, although $C^{-1}_0$ does. This makes the problem subtle. We observe however the following property of the operator, which implies the result under the assumption. If we know that the operator is injective, then we can even remove the assumption.

Proposition 5.5. Let $(G, \theta) \subset (E, \theta)_0$ be a Higgs subbundle of degree zero. Then $\text{Gr F il} \circ C^{-1}_0(G, \theta)$ is Higgs stable iff $(G, \theta)$ is Higgs stable.

We derive Proposition 5.4 first.

Proof. As $(E, \theta)_0$ is polystable, we write

$$(E, \theta)_0 = \bigoplus_i (G_i, \theta_i)$$

into a direct sum of stable factors. Because of Proposition 5.5, $\text{Gr F il} \circ C^{-1}_0(G_i, \theta_i)$ is again stable of degree zero for each $i$. For a chosen $i_0$, we consider the composite

$$\text{Gr F il} \circ C^{-1}_0(G_{i_0}, \theta_{i_0}) \subset (E, \theta)_0 echt (G_i, \theta_i).$$

It is either zero or an isomorphism because of stability. By the assumption, there is a unique $j_0$ such that the composite onto $(G_{j_0}, \theta_{j_0})$ is an isomorphism. It follows that

$$\text{Gr F il} \circ C^{-1}_0(G_{i_0}, \theta_{i_0}) = (G_{j_0}, \theta_{j_0}),$$

and that the operator induces a map on the set of indices of direct factors. This map must be injective and hence bijective: assume the contrary and say

$$\text{Gr F il} \circ C^{-1}_0(G_1, \theta_1) = \text{Gr F il} \circ C^{-1}_0(G_2, \theta_2).$$

For

$$M_i := C^{-1}_0(G_i, \theta_i), \quad \text{Fil}_i := \text{Fil} \cdot M_i, \quad i = 1, 2,$$

it holds that $M_1 \cap M_2 = 0$, and then the previous equality implies inductively

$$\text{Fil}_1^n = \text{Fil}_2^n = 0, \quad \text{Fil}_1^{n-1} = \text{Fil}_2^{n-1} = 0, \ldots, \text{Fil}_1^0 = \text{Fil}_2^0 = 0,$$

which is absurd. Therefore, each direct stable factor is periodic and by Theorem 3.5, $V \otimes k$ is a direct sum of irreducible representations, i.e, semisimple. \[\square\]

One direction of Proposition 5.5 is clear. Namely, if $\text{Gr F il} \circ C^{-1}_0(G, \theta)$ is stable, then $(G, \theta)$ must be stable. To show the converse direction, we need a lemma.

Lemma 5.6. Let $0 = (G_0, \theta_0) \subset (G_1, \theta_1) \subset \cdots \subset (G_r, \theta_r) = (E, \theta)_0$ be a JH filtration of $(E, \theta)_0$. Then

$$0 = \text{Gr F il} \circ C^{-1}_0(G_0, \theta_0) \subset \text{Gr F il} \circ C^{-1}_0(G_1, \theta_1) \subset \cdots \subset \text{Gr F il} \circ C^{-1}_0(G_r, \theta_r)$$

is again a JH filtration of $(E, \theta)_0$. 

Proof. Note first that $Gr_{Fil} \circ C_0^{-1}(G_i, \theta_i)$ is Higgs semistable of degree zero. So the grading $Gr_{Fil} \circ C_0^{-1}(G_i, \theta_i)/Gr_{Fil} \circ C_0^{-1}(G_{i-1}, \theta_{i-1})$ for each $i$ is Higgs semistable of degree zero. It is to show that each grading is in fact Higgs stable. For $1 \leq i \leq r$ let

$$\pi_i : Gr_{Fil} \circ C_0^{-1}(G_i, \theta_i) \to Gr_{Fil} \circ C_0^{-1}(G_i, \theta_i)/Gr_{Fil} \circ C_0^{-1}(G_{i-1}, \theta_{i-1})$$

be the natural projection. If this grading was not Higgs stable, then by Lemma 5.2 (i), there is a nontrivial JH filtration of this grading. Then the preimage of this JH filtration in $Gr_{Fil} \circ C_0^{-1}(G_i, \theta_i)$ via $\pi_i$ is a nontrivial refinement of the inclusion $Gr_{Fil} \circ C_0^{-1}(G_{i-1}, \theta_{i-1}) \subset Gr_{Fil} \circ C_0^{-1}(G_i, \theta_i)$ whose gradings are by construction Higgs stable of degree zero. Therefore we will obtain a new JH filtration of $(E, \theta)_0$ with strictly greater length, which contradicts Lemma 5.2 (ii).

Then Proposition 5.3 is shown as follows.

Proof. One can refine the inclusion $(G, \theta) \subset (E, \theta)_0$ into a JH filtration

$$0 \subset (G, \theta) = (G_1, \theta_1) \subset \cdots \subset (E, \theta)_0.$$

Then the previous lemma shows that

$$0 \subset Gr_{Fil} \circ C_0^{-1}(G, \theta) \subset \cdots \subset (E, \theta)_0$$

is again a JH filtration. In particular, $Gr_{Fil} \circ C_0^{-1}(G, \theta)$ is Higgs stable of degree zero. \qed

A composition series for $V \otimes k$ is a filtration of subrepresentations whose gradings are irreducible. The length of a composition series is the number of the irreducibles in its grading. It follows from Schur’s lemma that two composition series have the same length. A natural question is to compare this length on the representation side with that on this Higgs side. The next result generalizes Theorem 3.1.

**Proposition 5.7.** The length of a composition series of $V \otimes k$ is less than or equal to the length of a JH filtration of $(E, \theta)_0$. Assume 5.3. Then they are equal.

Proof. By Theorem 3.5, a composition series on $V \otimes k$ gives rise to a filtration on $(E, \theta)_0$ whose constituents are periodic Higgs subbundles, which are Higgs semistable of degree zero by Proposition 3.7. Thus the first statement is clear. To get the second statement, it suffices to produce a JH filtration on $(E, \theta)_0$ consisting of periodic Higgs subbundles. We start with an arbitrary JH filtration

$$0 = (G_{0}', \theta_0') \subset (G_1', \theta_1') \subset \cdots \subset (G_r', \theta_r') = (E, \theta)_0.$$

For each $l \in \mathbb{N}$ and $1 \leq i \leq r$ we write the grading

$$(Gr_{Fil} \circ C_0^{-1})^l(G_i', \theta_i')/(Gr_{Fil} \circ C_0^{-1})^l(G_{i-1}', \theta_{i-1}')$$

by $gr_i^l$ for short. It follows from Lemma 5.2 (ii) and Lemma 5.6 that $gr_i^l$ has its isomorphism class in the set $\{gr_1, \cdots, gr_r\}$. So for $i = r - 1$, there exist nonnegative integers $r \geq n_{r-1} > m_{r-1} \geq 0$ such that $gr_{r-1}^{n_{r-1}} \cong gr_{r-1}^{m_{r-1}}$. Thus, because of the assumption, the composite

$$(Gr_{Fil} \circ C_0^{-1})^{n_{r-1}}(G_{r-1}', \theta_{r-1}') \subset E_0 \to E_0/(Gr_{Fil} \circ C_0^{-1})^{m_{r-1}}(G_{r-1}', \theta_{r-1}') = gr_{r-1}^{m_{r-1}}.$$
is zero. Hence the above inclusion factorizes through
\[(Gr_{Fil} \circ C_0^{-1})^{n_{r-1}}(G_{r-1}^r, \theta_{r-1}^r) \subset (Gr_{Fil} \circ C_0^{-1})^{m_{r-1}}(G_{r-1}^r, \theta_{r-1}^r)\].

As they have the same rank and both are of degree zero, the inclusion is actually an equality
\[(Gr_{Fil} \circ C_0^{-1})^{n_{r-1}}(G_{r-1}^r, \theta_{r-1}^r) = (Gr_{Fil} \circ C_0^{-1})^{m_{r-1}}(G_{r-1}^r, \theta_{r-1}^r)\].

Then we replace the starting JH filtration with the one after the \(m_{r-1}\)-iterated \(Gr_{Fil} \circ C_0^{-1}\)-action. Put
\[(G_{r-1}, \theta_{r-1}) = (Gr_{Fil} \circ C_0^{-1})^{m_{r-1}}(G_{r-1}^r, \theta_{r-1}^r),\]
which is periodic with period \(\leq n_{r-1} - m_{r-1} \leq r\), and denote the obtained JH filtration by
\[0 = (G_0^r, \theta_0^r) \subset (G_1^r, \theta_1^r) \subset \cdots \subset (G_{r-1}, \theta_{r-1}) \subset (E, \theta)_0.\]

Note that \((G_i^r, \theta_i^r)\) may differ from the original one. Next, we shall apply the same argument to the filtration
\[0 = (G_0^r, \theta_0^r) \subset (G_1^r, \theta_1^r) \subset \cdots \subset (G_{r-1}, \theta_{r-1}).\]

But we shall take the number of iterations of the operator \(Gr_{Fil} \circ C_0^{-1}\) to be a multiple of \((n_{r-1} - m_{r-1})\). It yields then nonnegative integers \(r - 1 \geq n_{r-2} \geq m_{r-2} \geq 0\) such that
\[(Gr_{Fil} \circ C_0^{-1})^{n_{r-1} - m_{r-1} n_{r-2}}(G_{r-2}^r, \theta_{r-2}^r) = (Gr_{Fil} \circ C_0^{-1})^{m_{r-2}}(G_{r-2}^r, \theta_{r-2}^r)\]
holds. Put
\[(G_{r-2}, \theta_{r-2}) = (Gr_{Fil} \circ C_0^{-1})^{n_{r-1} - m_{r-1} m_{r-2}}(G_{r-2}^r, \theta_{r-2}^r).\]

Then continue the argument. In the end, we will obtain a JH filtration whose constituents \((G_i^r, \theta_i^r), 1 \leq i \leq r - 1\) are all periodic.

The following corollary is immediate after the above discussions.

**Corollary 5.8.** Assume\[5.3\] If \((E, \theta)_0\) decomposes
\[(E, \theta)_0 = \bigoplus_{i=1}^{r} (G_i, \theta_i)\]
into a direct sum of stable factors, then
\[\mathbb{V} \otimes k = \bigoplus_{i=1}^{r} \mathbb{V}_i\]
with \(\mathbb{V}_i\) is irreducible and \(\dim_k \mathbb{V}_i = \text{rank}_{O_{X_0}} G_i\).

Our next result replaces the assumption\[5.3\] with a geometric one.

**Proposition 5.9.** Let \(M \in \mathcal{MF}^\vee\) be an object arising from geometry. Suppose that \(X_0(k)\) contains an ordinary point (over which the Newton polygon coincides with the Hodge polygon). If
\[(E, \theta)_0 = \bigoplus_{i=1}^{r} (G_i, \theta_i)^{m_i}\]
decomposes into a direct sum such that each \((G_i, \theta_i)\) is Higgs stable and
\[(G_i, \theta_i) \not\cong (G_j, \theta_j), \quad i \neq j,\]
then accordingly
\[
V \otimes k = \bigoplus_{i=1}^{r} V_i
\]
with \(\dim_k V_i = \text{rank}_{\mathcal{O}_{X_0}} G_i\).

Proof. In the argument we shall use \(F_{\text{hod}}\) to mean the Hodge filtration \(\text{Fil}^\cdot\) and \(F_{\text{con}}\) the conjugate filtration on \(M_0\). We shall prove only the weight one case, because the higher weight case is entirely the same. Forgetting the multiplicities \(\{m_i\}\), we rewrite the decomposition of \((E, \theta)_0\) into
\[
(E, \theta)_0 = \bigoplus_{i'} (G_{i'}, \theta_{i'}).
\]
As \(C_0^{-1}\) respects direct sum, it yields the decomposition
\[
(M, \nabla)_0 = \bigoplus_{i'} (M_{i'}, \nabla_{i'})
\]
with \((M_{i'}, \nabla_{i'}) = C_0^{-1}(G_{i'}, \theta_{i'})\). If follows from the Cartier-Katz descent (see e.g. Proposition 2.11 [17]) that \(F_{\text{con}}\) on \(M_0\) decomposes accordingly. That is,
\[
F_{\text{con}} = \bigoplus_{i'} M_{i'}' \cap F_{\text{con}}.
\]
Now Proposition 5.5 implies that the images of \((G_{i'}, \theta_{i'})\) and \((G_{j'}, \theta_{j'})\) under \(\text{Gr}_{\text{Fil}} \circ C_0^{-1}\) intersect trivially or coincide. Thus we can take the following argument over the generic point of \(X_0\). So we are considering vector spaces and their linear maps over the function field of \(X_0\). For simplicity, we shall not change the notations for this base change. The ordinary assumption is equivalent to that over the generic point the two filtrations \(F_{\text{hod}}\) and \(F_{\text{con}}\) on \(M_0\) are complementary. We claim that \(F_{\text{hod}}\) on \(M_0\) decomposes accordingly. For that, we consider the composite \(\pi\) of natural morphisms in the following diagram:

\[
\begin{array}{ccc}
F_{\text{con}} & \hookrightarrow & \pi \\
\downarrow & & \downarrow \\
F_{\text{hod}} & \longrightarrow & M_0 \longrightarrow M_0/F_{\text{hod}}
\end{array}
\]

It is an isomorphism by the ordinary assumption. So the decomposition of \(F_{\text{con}}\) induces via \(\pi\) the following decomposition:
\[
M_0/F_{\text{hod}} = \bigoplus_{i'} \pi(M_{i'} \cap F_{\text{con}}).
\]
Because \(\pi\) maps \(M_{i'} \cap F_{\text{con}}\) into \(M_{i'}/M_{i'} \cap F_{\text{hod}}\) and any two of \(M_{i'}/M_{i'} \cap F_{\text{hod}}\) either intersect trivially or coincide, it follows that any two of them intersect trivially and
\[
\pi(M_{i'} \cap F_{\text{con}}) = M_{i'}/M_{i'} \cap F_{\text{hod}}.
\]
It follows that
\[
M_0/F_{\text{hod}} = \bigoplus_{i'} M_{i'}/M_{i'} \cap F_{\text{hod}}.
\]
which implies that \( F_{\text{ hod}} = \bigoplus \nu \mathcal{M}' \cap F_{\text{ hod}} \) as claimed. Therefore, the following equality holds:
\[
\bigoplus \nu (G_{\nu}, \theta_{\nu}) = \text{Gr}_{\text{Fil}} \circ C_0^{-1} \bigoplus \nu (G_{\nu}, \theta_{\nu}) = \bigoplus \nu \text{Gr}_{\text{Fil}} \circ C_0^{-1}(G_{\nu}, \theta_{\nu}).
\]
In particular, we have
\[
\bigoplus \nu (G_i, \theta_i)^{\oplus m_i} = \bigoplus \nu \text{Gr}_{\text{Fil}} \circ C_0^{-1}[(G_i, \theta_i)^{\oplus m_i}].
\]
As for any given \( i \) there is a unique \( j \) such that
\[
\text{Gr}_{\text{Fil}} \circ C_0^{-1}(G_i, \theta_i) \cong (G_j, \theta_j)
\]
holds, the previous equality implies that \( m_i = m_j \) and
\[
\text{Gr}_{\text{Fil}} \circ C_0^{-1}[(G_i, \theta_i)^{\oplus m_i}] = (G_j, \theta_j)^{\oplus m_j}.
\]
So the operator \( \text{Gr}_{\text{Fil}} \circ C_0^{-1} \) permutes the factors \( \{(G_i, \theta_i)^{\oplus m_i}\} \) and therefore each \( (G_i, \theta_i)^{\oplus m_i} \) is periodic. The result follows from Theorem 3.5. \( \square \)

The following lemma allows us to lift many results in char \( p \) to mixed characteristic.

**Lemma 5.10.** Let \( (G, \theta) \) be a Higgs subbundle of \( (E, \theta)_m \) satisfying the following two conditions:

(i) the quotient \( (E, \theta)_m/(G, \theta) \) is a locally free \( \mathcal{O}_{X_m} \)-module,

(ii) the set \( \text{Hom}_{\mathcal{O}_{X_0}}((G, \theta)_0, (E, \theta)_0/(G, \theta)_0) \) of morphisms of Higgs bundles is trivial.

Then if \( (G, \theta)' \subset (E, \theta)_m \) is a Higgs subbundle with the same rank as \( G \) and its modulo \( p \) reduction \( (G, \theta)'_0 \) is equal to \( (G, \theta)_0 \), then one concludes that
\[
(G, \theta)' = (G, \theta).
\]

**Proof.** The assumption (i) implies that \( G \) has no \( p \)-torsion as a local basis element and hence the modulo \( p \) reduction map \( G \otimes \mathbb{F}_p \to E_0 \) is injective. As \( G' \) has the same rank as \( G \) and the same modulo \( p \) reduction, the modulo \( p \) reduction map \( G' \otimes \mathbb{F}_p \to E_0 \) is also injective. For \( m = 0 \), there is nothing to prove. So we begin with the modulo \( p^2 \) reductions of \( (G, \theta)' \) and \( (G, \theta) \). Denote the composite \( (G, \theta)'_1 \subset (E, \theta)_1 \to (E, \theta)_1/(G, \theta)_1 \) by \( \tau \). By the condition (ii), the modulo \( p \) reduction \( \tau_0 : (G, \theta)'_0 = (G, \theta)_0 \to (E, \theta)_0/(G, \theta)_0 \) is zero. So
\[
\tau((G, \theta)'_1) \subset p[(E, \theta)_1/(G, \theta)_1].
\]
Then we continue to consider the composite
\[
(G, \theta)'_1 \to p[(E, \theta)_1/(G, \theta)_1] \overset{1/p}{\to} (E, \theta)_0/(G, \theta)_0.
\]
It descends clearly to a morphism \( \tau/p : (G, \theta)_0 \to (E, \theta)_0/(G, \theta)_0 \) and hence is zero for the same reason. This means that \( \tau \) is zero and therefore an inclusion
\[
(G, \theta)'_1 \subset (G, \theta)_1.
\]
Since they have the same rank and the same modulo \( p \) reduction, they are equal. An easy induction on \( m \) shows the lemma. \( \square \)

The following result uses the full strength of our theory, which therefore can be regarded as our best approximation to our original question in the introduction.
Theorem 5.11. Let $M \in \mathcal{MF}^\tau$ be a non-$p$-torsion object with $\mathcal{V}$ and $(E, \vartheta)$ as before. Assume one of the following two situations:

(i) $(E, \vartheta) = \bigoplus_{i=1}^r (G_i, \theta_i)$ and the reductions mod $p$ of $(G_i, \theta_i)_{1 \leq i \leq r}$ are Higgs stable and nonisomorphic to each other.

(ii) $M$ arises from geometry and $(E, \vartheta) = \bigoplus_{i=1}^r (G_i, \theta_i)^{\oplus m_i}$, and the reductions mod $p$ of $(G_i, \theta_i)_{1 \leq i \leq r}$ are Higgs stable and nonisomorphic to each other.

Then it holds accordingly that for a natural number $r$,

$$
\mathcal{V} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r} = \bigoplus_{i=1}^r \mathcal{V}_i
$$

with $\text{rank}_{\mathbb{Z}_{p^r}} \mathcal{V}_i = \text{rank}_{\mathcal{O}_X} (G_i)$ in Case (i) and

$$
\mathcal{V} \otimes_{\mathbb{Z}_p} \mathbb{Z}_{p^r} = \bigoplus_{i=1}^r \mathcal{V}_i
$$

with $\text{rank}_{\mathbb{Z}_{p^r}} \mathcal{V}_i = m_i \text{rank}_{\mathcal{O}_X} (G_i)$ in Case (ii).

Proof. We show only Case (i) since the proof for Case (ii) is entirely similar. Each $(G_i, \theta_i)_0$ is a periodic Higgs subbundle by the assumption. Let $r_i$ be its period. Thus we can apply $C_1^{-1}$ to $(G_i, \theta_i)_1$ by Proposition 4.9 Then by construction, $(Gr_{Fil} \circ C_1^{-1})_{G_i, \theta_i, 1}$ has the same rank as $(G_i)_1$ and its reduction mod $p$ is equal to $(Gr_{Fil} \circ C_0^{-1})_{G_i, \theta_i_1, 0}$, which is equal to $(G_i, \theta_i)_0$ by periodicity. So Lemma 5.10 applies, and we get the equality

$$(Gr_{Fil} \circ C_1^{-1})_{G_i, \theta_i, 1} = (G_i, \theta_i)_1,$$

which means that $(G_i, \theta_i)_1 \subset (E, \vartheta)_1$ is periodic with period $r_i$. We continue the argument, and show inductively that $(G_i, \theta_i)_m \subset (E, \vartheta)_m$ is periodic of period $r_i$ for any $m \geq 0$. Theorem 4.12 (and its proof) implies the corresponding direct decomposition of $\mathcal{V}$ with the claimed properties after tensoring with $\mathbb{Z}_{p^r}$ with $r$ is the least common multiple of all $r_i$'s.

We conclude this section by pointing out a connection with the notion of strongly semistable vector bundles, which was introduced in [11] and has played a central role in the work of Deninger and Werner [2]. This connection becomes more evident in our recent work [10].

Proposition 5.12. Let $(G^{i,n-i}, 0) \subset (E, \vartheta)_0$ be a periodic Higgs subbundle of period $r$. If $(Gr_{Fil} \circ C_0^{-1})_{G^{i,n-i}}$ is of pure type for each $1 \leq j \leq r$, then $G^{i,n-i}$ is étale trivializable and particularly strongly semistable.

Proof. We shall show an isomorphism

$$
F^s_{X_0} G^{i,n-i} \cong G^{i,n-i},
$$

which implies that $G^{i,n-i}$ is étale trivializable by Satz 1.4 [11]. To show that, we consider first $Gr_{Fil} \circ C_0^{-1}(G^{i,n-i}, 0)$. As it is of pure type by assumption, it is isomorphic to $C_0^{-1}(F^{s}_{X_0} G^{i,n-i}, 0)$, which is isomorphic to $F^s_{X_0} G^{i,n-i}$ by Remark 2.2 [14] (see also Proposition 2.9 [17]). Note that we can continue the argument and show inductively for $1 \leq j \leq r$ an isomorphism

$$(Gr_{Fil} \circ C_0^{-1})_{j} (G^{i,n-i}, 0) \cong F^s_{X_0} G^{i,n-i}.$$
Then the claimed isomorphism follows from the periodicity and the $j = r$ case.

**Remark 5.13.** The assumption on purity of the Hodge type of 

$$(Gr_{F_{\hat{X}}} \circ C_{0}^{-1})^{j}(G^{i,n-i}), \ 1 \leq j \leq r$$

made for the strong semistability of $G^{i,n-i}$ is necessary. The example in Proposition 6.6 (ii) shows that a certain power of the operator $Gr_{F_{\hat{X}}} \circ C_{0}^{-1}$ can turn a Higgs subbundle with zero Higgs field into a Higgs subbundle with maximal Higgs field, which is not semistable but Higgs semistable.

6. A global inverse Cartier transform over $W_2$

Our aim of this section is to globalize the construction of the local inverse Cartier transform over $W_2$ in §4. In particular, the proof of Theorem 4.3 is completed in this section. The main technique underlying the construction is an extensive use of the Taylor formula (see §2). Thus, without loss of generality, we can assume $X$ to be a curve. This assumption amounts to simplify a multi-index into a usual index in the arguments. Our strategy is as follows: firstly we show that the local inverse Cartier transform over $W_2$ does not depend on the choice of Frobenius liftings. Secondly, we modify the arguments to show that the local constructions glue into a global one. Finally, we adapt the technique of §3 to show that the local inverse Cartier transform over $W_2$ extends over to those Higgs subbundles whose reductions are periodic in char $p$.

Assume $X$ affine with a Frobenius lifting $F_{\hat{X}}$, and $M \in \mathcal{M}F_{\nabla}$ satisfies $p^2 M = 0$ and $pM \neq 0$. The following simple lemma reduces certain issues over $W_2$ to char $p$.

**Lemma 6.1.** Let $M' \subset M$ be a free $O_X$-submodule, and $M'_0 \subset M_0$ the image of its modulo $p$ reduction in $M_0$. Then the isomorphism $\frac{1}{p} : pM \cong M_0$ restricts to an isomorphism $pM' \cong M'_0$.

**Proof.** Fix an $O_{X_1}$-basis $e$ (resp. $e'$) of $M$ (resp. $M'$). Under these bases, the inclusion $M' \hookrightarrow M$ is given by a matrix $A = (a_{ij})$ with $a_{ij} \in O_{X_1}$. Then the image $M'_0$ under the map $A_0 := A \mod p : M' \mod p \to M \mod p = M_0$ is generated by $A_0 \cdot e_0$. On the other hand, $pM' \subset pM$ is generated by $(pA) \cdot e$. So its image under the isomorphism $\frac{1}{p}$ is generated also by $A_0 \cdot e_0$. □

**Proposition 6.2.** Let $(G, \theta)$ be as given in Proposition and Definition 4.3. Then $C_{-1}^{-1}(G, \theta)$ is independent of the choice of Frobenius liftings over $\hat{X}$.

**Proof.** Take another Frobenius lifting $F_{\hat{X}}'$. For short, we write $\phi = \Phi_{F_{\hat{X}}}$ and $\phi' = \Phi_{F_{\hat{X}}'}$. Because of the symmetric roles of $\phi$ and $\phi'$, it suffices to show for each $0 \leq i \leq n$,

$$\frac{\phi'_{p^i}}{p^i}(\tilde{g}^{i,n-i} \otimes 1) - \frac{\phi_{p^i}}{p^i}(\tilde{g}^{i,n-i} \otimes 1) \in \text{Span}[\frac{\phi_{p^i}}{p^i}(\tilde{g}^{i,n-i} \otimes 1), \ 0 \leq i \leq n].$$
We show this by an explicit calculation via the Taylor formula. Choose a local coordinate \( t \in \mathcal{O}_X \) of \( X \). Then the Taylor formula says that

\[
(\phi' - \phi)(e \otimes 1) = \sum_{j=1}^{\infty} \phi(\nabla^j_{\partial}(e) \otimes 1,1) \otimes z^j / j!
\]

where \( \partial = \frac{d}{dt} \), \( e \) is an element in \( M \), and \( z = F_t'(t) - F_t(t) \in \mathcal{O}_X \) which is divisible by \( p \). For an \( e \in M \), the above formula then reads by modulo \( p^2 \). So in this case the right hand side of the above formula is just a finite sum. By the Griffiths transversality,

\[
\nabla_{\partial}^{i-j}(\tilde{g}^{i,n-i}) \in Fil^jM, \quad 0 \leq j \leq i - 1.
\]

As \( i \leq n \leq p - 2 \), the above formula for \( e = g^{i,n-i} \in M \) can be written into

\[
\frac{\partial^j}{p^j}(\tilde{g}^{i,n-i} \otimes 1) - \frac{\phi}{p^j}(g^{i,n-i} \otimes 1) = I + II,
\]

with \( I = \sum_{j=0}^{i-1} \frac{\phi}{p^j}(\nabla_{\partial}^{i-j}(\tilde{g}^{i,n-i}) \otimes 1) \otimes \frac{z^{i-j}}{p^j(i-j)!} \)

and

\[
II = \sum_{j \geq i+1} \phi(\nabla^j_{\partial}(g^{i,n-i}) \otimes 1) \otimes \frac{z^j}{p^j j!}.
\]

We are going to show the terms \( I \) and \( II \) belong to \( \text{Span}[\frac{\phi}{p^j}(\tilde{g}^{i,n-i} \otimes 1), 0 \leq i \leq n] \). Consider first the term \( II \). Note that \( \frac{z^j}{p^j j!} \) is divisible by \( p \) for \( j \geq i + 1 \). So \( II \in pM \). By Lemma 6.1, \( II \in p\text{Span}[\frac{\phi}{p^j}(\tilde{g}^{i,n-i} \otimes 1), 0 \leq i \leq n] \) iff

\[
II/p \in \text{Span}[\frac{\phi}{p^j}(\tilde{g}^{i,n-i} \otimes 1), 0 \leq i \leq n]_0 = C_0^{-1}(G, \theta) \]

So we consider the modulo \( p \) reduction of \( \phi(\nabla^j_{\partial}(g^{i,n-i}) \otimes 1) \). By the condition (ii) of Lemma 4.4, \( g^{i,n-i} \in C_0^{-1}(G, \theta) \). As it is \( \nabla \)-invariant, it follows that for any \( j \),

\[
\nabla_{\partial}^j(g^{i,n-i}) \in C_0^{-1}(G, \theta) \]

By Proposition 3.8, if follows further that

\[
\phi(\nabla^j_{\partial}(g^{i,n-i}) \otimes 1) \in C_0^{-1}(G, \theta) \]

Therefore,

\[
II \in p\text{Span}[\frac{\phi}{p^j}(\tilde{g}^{i,n-i} \otimes 1), 0 \leq i \leq n] \subset \text{Span}[\frac{\phi}{p^j}(\tilde{g}^{i,n-i} \otimes 1), 0 \leq i \leq n] \]

Consider next the term \( I \). As \( G \subset E \) is \( \theta \)-invariant, there exists a unique \( b_j \in \mathcal{O}_X \) such that

\[
\nabla_{\partial}^{i-j}(g^{i,n-i}) \mod Fil^{j+1}M = b_j g^{i,n-j} \in G^{j,n-j}.
\]

As clearly \( b_j \tilde{g}^{i,n-j} \mod Fil^{j+1}M = b_j g^{i,n-j} \), it follows that

\[
\omega^{i+1,n-j-1} := \nabla_{\partial}^{i-j}(\tilde{g}^{i,n-i}) - b_j \tilde{g}^{i,n-j} \in Fil^{j+1}M.
\]
Note that \( \frac{\partial}{p^j}(\omega^{j+1,n-j-1} \otimes 1) \in pM \). Again by Lemma \[6.1\] in order to show

\[
\frac{\partial}{p^j} \left( \omega^{j+1,n-j-1} \otimes 1 \right) \in p\text{Span}\left[ \frac{\partial}{p^j}(\tilde{g}^{i,n-i} \otimes 1), \ 0 \leq i \leq n \right],
\]

it suffices to show \( \frac{\partial}{p^{j+1}}(\omega_0^{j+1,n-j-1} \otimes 1) \in C_0^{-1}(G, \theta)_0 \). But this is rather clear, because

\[
\omega_0^{j+1,n-j-1} = \nabla_{i,j}^{n-i}(\tilde{g}_0^{i,n-i}) - b_{j,0}g_0^{i,n-j}
\]

belongs to \( C_0^{-1}(G, \theta)_0 \) and by Proposition \[3.8\]

\[
\frac{\partial}{p^{j+1}}(F_{i,0}^*C_0^{-1}(G, \theta)_0) \subset C_0^{-1}(G, \theta)_0.
\]

As clearly

\[
\frac{\partial}{p^j}(b_jg^{i,n-j} \otimes 1) \in \text{Span}\left[ \frac{\partial}{p^j}(\tilde{g}^{i,n-i} \otimes 1), \ 0 \leq i \leq n \right],
\]

we have also shown

\[
I \in \text{Span}\left[ \frac{\partial}{p^j}(\tilde{g}^{i,n-i} \otimes 1), \ 0 \leq i \leq n \right].
\]

This completes the proof. \( \Box \)

From now on, \( X \) is assumed to be proper smooth over \( W \). Let \( \mathcal{U} \) be a small open affine covering of \( X \), together with a choice of Frobenius lifting \( F_{\mathcal{U}} \) over \( \hat{U} \) for each \( U \in \mathcal{U} \). Thus for a Higgs subbundle \((G, \theta)\) in the situation of Theorem \[4.3\] we have constructed a set of local de Rham subbundles \( \{C_1^{-1}(G, \theta)|U\}_{U \in \mathcal{U}} \) with local properties listed ibid. In order to show \( C_1^{-1}(G, \theta) \) exists, it suffices to show the following equality of subbundles in \( M|_{U \cap V_1} \) for any \( U, V \in \mathcal{U} \):

\[
[C_1^{-1}(G, \theta)|U_1]|_{U \cap V_1} = C_1^{-1}[(G, \theta)|U \cap V_1].
\]

Its proof modifies the previous one. Take Frobenius liftings \( F_{\hat{U}}, F_{\hat{V}}, F_{\hat{U} \cap \hat{V}} \) on \( \hat{U}, \hat{V}, \hat{U} \cap \hat{V} \) respectively and write

\[
z = F_{\hat{U}} \circ \iota(t) - \iota \circ F_{\hat{U} \cap \hat{V}}(t),
\]

where \( \iota : \hat{U} \cap \hat{V} \hookrightarrow \hat{U} \) is the natural inclusion. Then the difference

\[
\iota_1^*[\Phi_{F_{\hat{U}}}^*(\tilde{g}^{i,n-i} \otimes 1)] - \Phi_{F_{\hat{U} \cap \hat{V}}}^*[\iota_1^*\tilde{g}^{i,n-i} \otimes 1]
\]

is again expressed by the Taylor formula. Thus the previous proof carries over, and it shows that

\[
[C_1^{-1}(G, \theta)|U_1]|_{U \cap V_1} \subset C_1^{-1}[(G, \theta)|U \cap V_1].
\]

In order to obtain the equality rather than an inclusion, we shall examine the proof of Proposition \[6.2\] Consider first the above difference for \( i = 0 \). One sees from the proof that the difference belongs to \( p\Phi_{F_{\hat{U} \cap \hat{V}}}^*(\iota_1^*\tilde{g}^{0,n}) \). So it holds that

\[
\iota_1^*[\Phi_{F_{\hat{U}}}^*(\tilde{g}^{0,n} \otimes 1)] = \Phi_{F_{\hat{U} \cap \hat{V}}}^*[\iota_1^*\tilde{g}^{0,n} \otimes 1].
\]
For a general \( 1 \leq i \leq n \), we shall use induction on \( i \). Assume the truth of the equality for \( i - 1 \), namely,

\[
\text{Span}[t_1^i \frac{\Phi_{F_{U_j}}}{p^i}(\tilde{g}^{i,n-i} \otimes 1)], \quad 0 \leq j \leq i - 1 = \text{Span}[t_1^i \frac{\Phi_{F_{U_j}}}{p^i}(\tilde{g}^{i,n-i} \otimes 1)], \quad 0 \leq j \leq i - 1.
\]

As one sees from the proof that the difference

\[
t_1^i [\frac{\Phi_{F_{U_j}}}{p^i}(\tilde{g}^{i,n-i} \otimes 1)] - \text{Span}[t_1^i \frac{\Phi_{F_{U_j}}}{p^i}(\tilde{g}^{i,n-j} \otimes 1)], \quad 0 \leq j \leq i - 1,
\]

belongs to

\[
p^i \frac{\Phi_{F_{U_j}}}{p^i}[t_1^i(\tilde{g}^{i,n-i} \otimes 1)] + \text{Span}[\frac{\Phi_{F_{U_j}}}{p^i}[t_1^i(\tilde{g}^{i,n-j} \otimes 1)], \quad 0 \leq j \leq i - 1,
\]

one obtains the equality also for \( i \). So the local subbundles \( \{ C_1^{-1}(G, \theta)_U \}_{U \in \mathcal{U}} \) glue into a global subbundle \( C_1^{-1}(G, \theta) \) of \((M, \nabla)\) as claimed. Now we proceed to the proof of Proposition 4.9.

**Proof.** Let \((G, \theta) \subset (E, \theta)_1\) be a Higgs subbundle with the equality in \( \text{char } p \):

\[(Gr_{\text{Fil}} \circ C_0^{-1})^r(G, \theta)_0 = (G, \theta)_0.
\]

As remarked in the proof of Theorem 4.2, Lemma 3.10 and its consequent lemmas extend to \( W_2 \). So we have the eigen-decomposition

\[
(E, \theta)_{0r} = \bigoplus_{i=0}^{r-1} (E^i, \theta^i),
\]

and isomorphisms of Higgs bundles

\[
\beta_i : (E, \theta)_1 \cong (E^i, \theta^i).
\]

By Lemma 3.12, the Higgs subbundle

\[
\bigoplus_{i=0}^{r-1} \beta_i[(Gr_{\text{Fil}} \circ C_0^{-1})^i(G, \theta)_0] \subset \bigoplus_{i=0}^{r-1} (E^i, \theta^i)_0
\]

is periodic of period one. So one might be able to reduce the construction to Theorem 4.3. But this does not quite succeed. This is because the existence of a Higgs subbundle in \( \bigoplus_{i=0}^{r-1} (E^i, \theta^i) \), whose reduction modulo \( p \) is \( \bigoplus_{i=0}^{r-1} \beta_i[(Gr_{\text{Fil}} \circ C_0^{-1})^i(G, \theta)_0] \), is not part of our assumption. Instead, we shall modify our original local construction suitably so that the previous arguments carries over. We can assume in the following argument that \( r = 2 \). This assumption does not affect much the proof for a general \( r \), but will simplify the notations greatly.

Firstly, the proof of Lemma 4.4 shows that for each \( U \in \mathcal{U} \), there exists a set of elements \( \{ \tilde{g}^{i,n-i} \} \subset M_1|_{U_1} \) such that

(i) \( \tilde{g}^{i,n-i} \mod Fil^{i+1}M_1 = g^{i,n-i} \),

(ii) \( \tilde{g}^{i,n-i} \mod p \in C_0^{-1}[Gr_{Fil} \circ C_0^{-1}(G, \theta)_0] \).
Then we define as before
\[ C_1^{-1}(G, \theta)|_{U_1} = \text{Span}[\Phi_{F_i}^0(g^{i,n-i} \otimes 1), \ 0 \leq i \leq n]. \]

It is direct to check the following equalites:
\[ [C_1^{-1}(G, \theta)|_{U_1}]_0 = [C_0^{-1}(G, \theta)|_{U_1}], \]
\[ \text{and} \]
\[ [\tilde{\Phi}(C_1^{-1}(G, \theta)|_{U_1})]_0 = [C_0^{-1}(Gr_{Fil} \circ C_0^{-1}(G, \theta)_0)|_{U_1}]. \]

We use now the eigen-decomposition and isomorphisms in the lifted Lemma 3.11 over \( W_2 \):
\[
(M, \nabla)^{e2} = (M^0, \nabla^0) \oplus (M^1, \nabla^1), \ \alpha_i : (M, \nabla) \cong (M^i, \nabla^i), \ i = 0, 1.
\]

We claim that the local subbundles
\[
\{\alpha_0[\tilde{\Phi}(C_1^{-1}(G, \theta)|_{U_1})] \oplus \alpha_1[C_1^{-1}(G, \theta)|_{U_1}]\}_{U \in U}
\]
of \( (M^0, \nabla^0) \oplus (M^1, \nabla^1) \) are well defined, i.e. independent of the choices of elements \( \{\hat{g}^{i,n-i}\} \) given as above, and \( \nabla \)-invariant, and they glue. Our old proofs for \( C_1^{-1} \) go through, because the mod \( p \) reductions of these local bundles are simply
\[
\{\alpha_0[C_0^{-1}(Gr_{Fil} \circ C_0^{-1}(G, \theta)_0)] \oplus \alpha_1[C_0^{-1}(G, \theta)_0]\}_{U_0},
\]
and obviously they glue into
\[
\alpha_0[C_0^{-1}(Gr_{Fil} \circ C_0^{-1}(G, \theta)_0)] \oplus \alpha_1[C_0^{-1}(G, \theta)_0],
\]
which is just
\[
C_0^{-1}[\beta_0(G, \theta)_0 \oplus \beta_1(Gr_{Fil} \circ C_0^{-1}(G, \theta)_0),
\]
and the Higgs bundle \( \beta_0(G, \theta)_0 \oplus \beta_1(Gr_{Fil} \circ C_0^{-1}(G, \theta)_0) \) is periodic of period one. The gluing implies that \( \{\alpha_1[C_1^{-1}(G, \theta)|_{U_1}]\}_{U \in U} \) glue into a de Rham subbundle of \( M^1 \). Therefore, \( \{C_1^{-1}(G, \theta)|_{U_1}\}_{U \in U} \) glue into a de Rham subbundle \( C_1^{-1}(G, \theta) \) of \( (M, \nabla) \) whose modulo \( p \) reduction is \( C_0^{-1}(G, \theta)_0 \), as claimed. \( \square \)

7. APPENDIX: THE INVERSE CARTIER TRANSFORM OF OGUS AND VOLGOLODSKY

The appendix explains the equivalence (up to sign) of the inverse Cartier transform of Ogus and Vologodsky \([14]\) and the association defined in \([17]\) in the subobjects setting. We thank heartily Arthur Ogus for pointing out the equivalence follows from Remark 2.10 \([14]\) in \([13]\). Our exposition is based on his remark. In the following we shall quote the notations and results in \([14]\, [17]\ and \([9]\) freely. In his forthcoming doctor thesis \([18]\), H. Xin shall explain the equivalence as well as that in the logarithmic case in detail. We divide the proof into two steps:

**Step 1.** Let \((E, \theta)\) be a nilpotent Higgs bundle of exponent \( \leq p - 1 \). The inverse Cartier transform of \((E, \theta)\) after Ogus and Vologodsky is defined by
\[
(M, \nabla)_{(E, \theta)} := B_{X/S} \otimes_{F, T^*_X/S} i^* \pi^*(E, \theta).
\]
The construction is global. In [9], we construct a flat bundle \((M, \nabla, \theta)\) by gluing the local flat subbundles
\[
\{ F_{U_0}^* E, \nabla_{can} + (id \otimes \frac{dF_U^*}{p}) \circ F_{U_0}^* \theta \}_{U \in \mathcal{U}}
\]
via an exponential function, where \(F_{U_0}\) is the absolute Frobenius over \(U_0\).

**Claim 7.1.** There is a functorial isomorphism
\[
(M, \nabla) \cong (M, \nabla)_{(E, \theta)}.
\]

**Proof.** Recall that we have chosen an affine covering \(\mathcal{U}\) of \(X\), together with a choice of Frobenius liftings for each \(U \in \mathcal{U}\). Thus over each \(U\), the lifting \(F_U\) defines an isomorphism of \(\hat{T}_{U_0}^*/k\)-modules:
\[
\mathcal{B}_{X/S}|_{U_0} \cong F_{U_0/k}^* \hat{T}_{U_0}^*/k,
\]
where \(F_{U_0/k} : U_0 \to U_0'\) is the relative Frobenius. Therefore, one has a natural isomorphism
\[
[(M, \nabla)]|_{U_0} \cong \mathcal{B}_{X/S}|_{U_0} \otimes \hat{T}_{U_0'}^*/k \quad \tau^* \pi^* E|_{U_0}
\]

This gives actually an isomorphism of flat bundles
\[
[(M, \nabla)]|_{U_0} \cong [(M, \nabla)]|_{U_0},
\]
followed from the description in Formula (2.11.2) [3]. The sign comes from the involution \(\tau\). Now Remark 2.10 loc. cit. tells how the above isomorphisms change when we choose another Frobenius lifting. Precisely, let \(F_U'\) be another choice, then their difference defines an element \(\xi \in F_{U_0/k}^* \hat{T}_{U_0}^*/k\). As \(\mathcal{B}_{X/S}\) is a \(F_{X_0/k}^* \hat{T}_{X_0/k}\)-tortor, different local trivializations are related via the Taylor formula or equivalently the exponential \(\exp \cdot D\xi\). As \(E\) is nilpotent with exponent \(\leq p - 1\) by assumption, the action on the Higgs field becomes a twist via the usual exponential function. When we interpret the change of isomorphisms into the gluing data, this is exactly the form given in [9]. Therefore, there is a natural isomorphism between \((M, \nabla)\) and \((M, \nabla)_{(E, \theta)}\). \(\square\)

**Step 2.** Let \(M \in \mathcal{M} F_{\Delta}^\nabla(X), n \leq p - 2\) with \(pM = 0\). We remark that Faltings category exists also for \(n \leq p - 1\) (see Theorem 2.3 [4]). But in several places of [17] invoking the Taylor formula, the assumption \(n \leq p - 2\) has been explicitly used. So we have to keep the assumption \(n \leq p - 2\) here. Let \((G, \theta)\) be a Higgs subbundle of \((E, \theta)\) which is nilpotent of exponent \(\leq p - 2\). In [17], we associate \((G, \theta)\) a de Rham subbundle \((M(G, \theta), \nabla)\) of \((M, \nabla)\) by a local lifting and gluing process.

**Claim 7.2.** There is an isomorphism \(\phi : (M, \nabla)_{(E, \theta)} \cong (M, \nabla)\) such that for any Higgs subbundles \(G \subset E\),
\[
\phi[(M, \nabla)_{(G, \theta)}] = (M(G, \theta), \nabla).
\]
Proof. This follows from Proposition 5 [9] and its proof. □

We summarize the previous discussions into the following statement. Recall the notations in §3: \((\mathcal{X}, \mathcal{S}) = \left(\frac{X_0}{k}, \frac{X'_1}{W_2(k)}\right)\) and \(\pi : X'_0 \rightarrow X_0\) is the natural map setting in the Cartesian diagram of the base change.

**Proposition 7.3.** Let \(C_{\mathcal{X}/S}^{-1}\) be the inverse Cartier transform of Ogus and Vologodsky [14]. For an \(M \in \mathcal{MF}^\nabla\) with \(pM = 0\) let \((E, \theta) = \text{Gr}_{Ful}(M, \nabla)\) the associated Higgs bundle over \(X_0\). Then there is an isomorphism of flat bundles

\[
\psi : C_{\mathcal{X}/S}^{-1}\pi^*(E, -\theta) \cong (M, \nabla)
\]

such that for any Higgs subbundle \((G, \theta) \subset (E, \theta)\),

\[
\psi[C_{\mathcal{X}/S}^{-1}\pi^*(G, -\theta)] = (M_{(G, \theta)}, \nabla),
\]

where \((M_{(G, \theta)}, \nabla)\) is the associated de Rham subbundle of \((M, \nabla)\) constructed in [17].

**References**

[1] H. Carayol, Sur la mauvaise r´eduction des courbes de Shimura, Comp. Math. 59 (1986), no. 2, 151-230.
[2] C. Deninger, A. Werner, Vector bundles on p-adic curves and parallel transport, Ann. Scient. Éc. Norm. Sup. 38 (2005), 553-597.
[3] P. Deligne, Un théorème de finitude pour la monodromie, Discrete Groups in Geometry and Analysis, Birkhauser 1987, 1-19.
[4] G. Faltings, Crystalline cohomology and p-adic Galois-representations, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), 25-80, Johns Hopkins Univ. Press, Baltimore, MD, 1989.
[5] G. Faltings, Integral crystalline cohomology over very ramified valuation rings, Journal of the AMS, Vol. 12, no. 1, 117-144, 1999.
[6] G. Faltings, A p-adic Simpson correspondence, Advances in Mathematics 198 (2005), 847-862.
[7] J.-M. Fontaine, G. Laffaille, Construction de représentation p-adiques, Ann. Sci. Ec. Norm. Sup. 15 (1982), 547-608.
[8] P. Griffiths, Periods of integrals on algebraic manifolds III, Publ. Math. I.H.E.S., 38 (1970) 125-180.
[9] G.-T. Lan, M. Sheng, K. Zuo, An inverse Cartier transform via exponential in positive characteristic, arXiv: 1205.6599, 2012.
[10] G.-T. Lan, M. Sheng, K. Zuo, Semistable Higgs bundles and representations of algebraic fundamental groups: Positive characteristic case, arXiv: 1210.8280, 2012.
[11] H. Lange, U. Stuhler, Vektorbündel auf Kurven und Darstellungen der algebraischen Fundamentalgruppe, Math. Z. 156 (1977), 73-83.
[12] A. Ogus, F-crystals, Griffiths transversality, and the Hodge decomposition, Astérisque No. 221 (1994).
[13] A. Ogus, Private communication dated on August 12th, 2012.
[14] A. Ogus, V. Vologodsky, Nonabelian Hodge theory in characteristic p, Publ. Math. Inst. Hautes études Sci. 106 (2007), 1-138.
[15] C. Simpson, Higgs bundles and local systems, Publ. Math. Inst. Hautes étud. Sci. 75 (1992), 5-95.
[16] M. Sheng, J.-J. Zhang, K. Zuo, Higgs bundles over the good reduction of a quaternionic Shimura curve, J. reine angew. Math., DOI 10.1515, 2011.
[17] M. Sheng, H. Xin, K. Zuo, A note on the characteristic $p$ nonabelian Hodge theory in the geometric case, arXiv: 1202.3942, 2012.
[18] H. Xin, On Fontaine modules and $F$-$T$ crystals, Doctor Thesis, University of Mainz, in preparation.

E-mail address: msheng@ustc.edu.cn

School of Mathematical Sciences, University of Science and Technology of China, Hefei, 230026, China

E-mail address: zuok@uni-mainz.de

Institut für Mathematik, Universität Mainz, Mainz, 55099, Germany