Large time behavior of temperature in two-phase heat conductors

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Abstract

We consider the Cauchy problem for the heat diffusion equation in the whole Euclidean space consisting of two media with different constant conductivities. The large time behavior of temperature, the solution of the problem, is studied when initially temperature is assigned to be 0 on one medium and 1 on the other. We show that under a certain geometric condition of the configuration of the media, temperature is stabilized to a constant as time tends to infinity. We also show by examples that temperature in general oscillates and is not stabilized.

Key words. heat diffusion equation, two-phase heat conductors, Cauchy problem, self-similar solutions, stabilization, oscillation.

AMS subject classifications. Primary 35K05; Secondary 35K10, 35K15, 35B40, 35C06

1 Introduction

This paper concerns the Cauchy problem for the heat diffusion equation in the whole Euclidean space which is occupied by two heat conducting media with different constant conductivities. It deals with the question of stabilization of temperature (the solution of the problem) as time tends to infinity when the initial data is given by the characteristic function of a medium. The reason why we introduce such a specific initial data is to clarify geometry of the composite media. We find a condition on the configuration of the media under which temperature converges to a constant as time tends to infinity. We also show by examples that temperature in general oscillates and does not converge.

†This research was partially supported by the Grant-in-Aid for Scientific Research (B) (♯18H01126) of Japan Society for the Promotion of Science and NRF (of S. Korea) grants No. 2019R1A2B5B01069967.
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The stabilization problem of the solution of linear parabolic equations as time tends to infinity has been studied by many authors. For example, Kamin considered in [K] the Cauchy problem for the uniformly parabolic diffusion equations with bounded initial data. Under the assumption that the diffusion coefficients converge to the Kronecker symbol at infinity (the condition (9) in [K]), that is, the equation converges to the exact heat equation at infinity, it is shown that the solution converges to a constant as time tends to infinity if and only if the averages of the initial data over balls converge to a constant as the radii of the balls tend to infinity. A generalization of this result to some degenerate parabolic equation was given in [EKT, Theorem 1.1]. We refer to [E, Z] and references therein for further work on stabilization of the parabolic equations. We also mention that the oscillatory behavior of solutions of the Cauchy problem for the exact heat equation with unbounded initial data was studied in [CT].

To present the results of this paper precisely, let Ω be a domain in \( \mathbb{R}^N \) with \( N \geq 2 \) so that \( \Omega \) and \( \mathbb{R}^N \setminus \Omega \) constitute the two media. We suppose that \( \partial \Omega \neq \emptyset \) and \( \partial \Omega \) is connected. Denote by \( \sigma = \sigma(x) \ (x \in \mathbb{R}^N) \) the heat conductivity distribution of the whole medium given by

\[
\sigma = \begin{cases}
\sigma_+ & \text{in } \Omega, \\
\sigma_- & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

where \( \sigma_-, \sigma_+ \) are positive constants such that \( \sigma_- \neq \sigma_+ \). We consider the Cauchy problem for the heat diffusion equation:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \text{div}(\sigma \nabla u) \quad \text{in } \mathbb{R}^N \times (0, +\infty), \\
u &= \chi_\Omega \quad \text{on } \mathbb{R}^N \times \{0\},
\end{align*}
\]

(1.2)

where \( \chi_\Omega \) denotes the characteristic function of the set \( \Omega \). We look for a geometric condition on \( \Omega \) such that the unique bounded solution \( u = u(x, t) \) to (1.2) tends to a constant as \( t \to \infty \). As far as we are aware of, the question of stabilization for the multi-phase heat conductors has not been considered before. Recently, a geometric question related to the diffusion over such multi-phase heat conductors has been dealt with in [CSU, S].

To gain a better understanding of the condition for the stabilization given later in (1.7), let us first consider conic regions. Let \( S^{N-1} \) be the unit sphere in \( \mathbb{R}^N \). For a domain \( A \) of \( S^{N-1} \) such that \( \partial A \neq \emptyset \), we set \( \Omega_A \) to be the cone over \( A \), namely,

\[
\Omega_A = \{x \in \mathbb{R}^N : x = r\omega, \ r > 0, \ \omega \in A\}.
\]

(1.3)

Denote by \( \sigma_A = \sigma_A(x) \ (x \in \mathbb{R}^N) \) the conductivity distribution of the whole medium given...
by
\[
\sigma_A = \begin{cases} 
\sigma_+ & \text{in } \Omega_A, \\
\sigma_- & \text{in } \mathbb{R}^N \setminus \Omega_A.
\end{cases}
\] (1.4)

The following proposition can be proved using the self-similarity of the solution (see the proof in the next section).

**Proposition 1.1** Let \( u_A = u_A(x, t) \) be the unique bounded solution of problem (1.2) with \( \Omega, \sigma \) replaced by \( \Omega_A, \sigma_A \), respectively. It holds that \( 0 < u_A(0, 1) < 1 \) and
\[
\lim_{t \to \infty} u_A(x, t) = u_A(0, 1)
\] (1.5)
uniformly in \( x \) belonging to any fixed compact set in \( \mathbb{R}^N \).

We now present a condition on the shape of the domain which is sufficient for stabilization of the solution. Let \( A \) be a domain of \( S^{N-1} \) such that there is a point \( p \in A \) with \(-p \notin A\). For \( A \subset S^{N-1} \) and such a point \( p \), we say \( A \) is *starshaped* with respect to \( p \) if for every point \( \omega \in A \) the shortest geodesic connecting \( \omega \) and \( p \) in \( S^{N-1} \) is contained in \( A \), or equivalently, if \( t\omega + sp \in A \) for any \( \omega \in A \) and a pair of nonnegative numbers \( t, s \) such that \( t\omega + sp \in S^{N-1} \). If \( A \subset S^{N-1} \) is starshaped with respect to \( p \in A \), then
\[
\Omega_A \subset \Omega_A - sp
\] (1.6)
for all \( s > 0 \). Here and throughout this paper \( \Omega_A - y \) denotes the translate of \( \Omega_A \) by \( y \), that is,
\[
\Omega_A - y = \{ x - y : x \in \Omega_A \}.
\]
In fact, if \( x \in \Omega_A \), then \( x = t\omega \) for some \( t > 0 \). Since \( A \) is starshaped with respect to \( p \), \( x + sp \in \Omega_A \) for all \( s > 0 \), and hence (1.6) follows.

The sufficient condition for the stabilization of the solution to (1.2) to hold on the domain \( \Omega \) is as follows:
\[
\Omega_A \subset \Omega \subset \Omega_A - hp
\] (1.7)
for some \( h > 0 \). Roughly speaking, this condition means \( \partial \Omega \) lies in \( (\Omega_A - hp) \setminus \Omega \), but is of arbitrary shape. For such domains we have the following theorem.

**Theorem 1.2** If the domain \( \Omega \) in \( \mathbb{R}^N \) satisfies (1.7) for some \( h > 0 \), \( A \subset S^{N-1} \) and \( p \in A \) with \(-p \notin A\), where \( A \) is starshaped with respect to \( p \), then the unique bounded solution \( u \) of problem (1.2) satisfies that
\[
\lim_{t \to \infty} u(x, t) = u_A(0, 1)
\] (1.8)
uniformly in \( x \) belonging to each compact set in \( \mathbb{R}^N \).
The second theorem gives an example of oscillatory behavior of the solutions of problem (1.2). For that we deal with general conductivity distributions \( \sigma = \sigma(x) \), not necessarily two-phase conductivity distribution.

**Theorem 1.3** Let \( m \leq M \) be positive constants. There exists a domain \( \Omega \) in \( \mathbb{R}^N \) such that for any conductivity \( \sigma = \sigma(x) \) (\( x \in \mathbb{R}^N \)) satisfying

\[
0 < m \leq \sigma(x) \leq M \quad \text{for every } x \in \mathbb{R}^N,
\]

the unique bounded solution \( u \) of problem (1.2) satisfies that

\[
0 < \liminf_{t \to \infty} u(0, t) < \limsup_{t \to \infty} u(0, t) < 1.
\]

We remark that since Theorem 1.3 is proved only using the Gaussian bounds for the fundamental solutions of diffusion equations (see (3.1)), it holds also for the diffusion equations of the form

\[
u_t = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left( a_{ij}(x, t) \frac{\partial u}{\partial x_j} \right)
\]

where the coefficients satisfy the following for some positive constants \( m, M \)

\[
m|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x, t) \xi_i \xi_j \leq M|\xi|^2 \quad \text{for } x, \xi \in \mathbb{R}^N \text{ and } t > 0.
\]

Such diffusion equations have been dealt with in [K].

This paper is organized as follows. In section 2 we prove Proposition 1.1 and Theorem 1.2 by introducing the one-parameter families of solutions \( \{u^k_A\}, \{u^k\} \) as in [K]. Section 3 is to prove Theorem 1.3.

### 2 Proofs of Proposition 1.1 and Theorem 1.2

**Proof of Proposition 1.1** Note that the function \( u_A^k \) defined by \( u_A^k(x, t) = u_A(kx, k^2t) \) for \( k > 0 \) also solves problem (1.2) with \( \Omega, \sigma \) replaced by \( \Omega_A, \sigma_A \), respectively. Thus it follows from the uniqueness of the solution of (1.2) that the solution \( u_A \) is self-similar, namely,

\[
u_A(kx, k^2t) = u_A(x, t)
\]

for every \((x, t) \in \mathbb{R}^N \times (0, +\infty)\) and every \( k > 0 \), from which we infer that

\[
u_A(0, t) = \nu_A(0, 1) \quad \text{for every } t > 0.
\]

Note that \( 0 < \nu_A < 1 \) in \( \mathbb{R}^N \times (0, +\infty) \) by the maximum principle.
By the Hölder estimate for $u_A$ as in [Z, p. 526], there exist constants $C > 0$ and $0 < \theta < 1$ such that
\[ |u_A(x, 1) - u_A(0, 1)| \leq C|x|^\theta \quad \text{for every } x \in \mathbb{R}^N. \tag{2.3} \]

It then follows from (2.1) that
\[ |u_A(x, t) - u_A(0, t)| \leq C|x|^\theta t^{-\frac{\theta}{2}} \quad \text{for every } (x, t) \in \mathbb{R}^N \times (0, +\infty). \tag{2.4} \]

Combining (2.2) with this yields (1.5).

**Proof of Theorem 1.2.** Let $u = u(x, t)$ be the unique bounded solution of problem (1.2). As in [EKT, K], we introduce the one-parameter family of functions \( \{u^k\} \) by
\[ u^k(x, t) = u(kx, k^2t) \quad \text{for } (x, t) \in \mathbb{R}^N \times (0, +\infty) \quad \text{and } k > 0. \]

Let us show that $u^k$ converges to the self-similar solution $u_A$ as $k \to \infty$ uniformly in each compact set in $\mathbb{R}^N \times (0, +\infty)$. By the maximum principle we have
\[ 0 < u^k(x, t) < 1 \quad \text{for every } (x, t) \in \mathbb{R}^N \times (0, +\infty) \quad \text{and every } k > 0. \tag{2.5} \]

Moreover, each $u^k$ solves the problem (1.2) with $\Omega, \sigma$ replaced by $\Omega^k, \sigma^k$, respectively, where
\[ \Omega^k = \{x \in \mathbb{R}^N : kx \in \Omega\} \quad \text{and} \quad \sigma^k = \begin{cases} \sigma_+ & \text{in } \Omega^k, \\ \sigma_- & \text{in } \mathbb{R}^N \setminus \Omega^k. \end{cases} \]

As in [K, Lemmas 1 and 2], we have the following two lemmas which come from (2.5) together with the Hölder and energy estimates for solutions of parabolic equations of second order with discontinuous coefficients (see, for example, [LSU, Chapter III]).

**Lemma 2.1** For every $\tau > 0$, there exist two constants $C_1 > 0$ and $0 < \theta < 1$ such that
\[ |u^k(x, t) - u^k(y, s)| \leq C_1(|x - y|^\theta + |t - s|^\frac{\theta}{2}) \]
for every $(x, t), (y, s) \in \mathbb{R}^N \times [\tau, +\infty)$ and for every $k > 0$.

**Lemma 2.2** For every $T > 0$ and $\rho > 0$ there exists a constant $C_2 > 0$ such that
\[ \int_0^T \int_{B_\rho(0)} |\nabla u^k|^2 dx \, dt \leq C_2 \]
for every $k > 0$. Here, $B_\rho(0) = \{x \in \mathbb{R}^N : |x| < \rho\}$.
Since the assumption (1.7) means that $\partial \Omega$ lies between the two conical surfaces, $\partial \Omega_A$ and its translate $\partial (\Omega_A - hp)$, this geometric condition, combined with the homothety of $\Omega^k$ and $\Omega$ with ratio $k$, implies directly that $X_{\Omega^k}(x), \sigma^k(x)$ converge to $X_{\Omega_A}(x), \sigma_A(x)$, respectively, as $k \to \infty$, for every $x \in \mathbb{R}^N$. Therefore, by using the diagonal process with the aid of the estimates (2.5), Lemma 2.1 and Lemma 2.2, we infer from the uniqueness of the solution of problem (1.2) that $u^k$ converges to $u_A$ as $k \to \infty$ uniformly in each compact set in $\mathbb{R}^N \times (0, +\infty)$. Hence in particular $u(0, t)$ converges to $u_A(0, 1)$ as $t \to \infty$.

Moreover it follow from Lemma 2.1 that

$$|u(x, t) - u(0, t)| \leq C_1 |x|^{\theta} t^{-\frac{\theta}{2}}$$

for every $(x, t) \in \mathbb{R}^N \times (0, +\infty)$, which yields the desired conclusion (1.8).

### 3 Proof of Theorem 1.3

We utilize the Gaussian bounds for the fundamental solutions of diffusion equations due to Aronson [A, Theorem 1, p. 891] (see also [FS, p. 328]). Let $g = g(x, \xi, t)$ be the fundamental solution of $u_t = \text{div}(\sigma \nabla u)$. Then there exist two positive constants $\lambda < \Lambda$ such that

$$\lambda t^{-\frac{N}{2}} e^{-\frac{|x-\xi|^2}{4t}} \leq g(x, \xi, t) \leq \Lambda t^{-\frac{N}{2}} e^{-\frac{|x-\xi|^2}{4t}} \quad (3.1)$$

for all $(x, t), (\xi, t) \in \mathbb{R}^N \times (0, +\infty)$, where the constants $\lambda, \Lambda$ depend only on $N$ and the bounds $m, M$ of $\sigma$.

To construct the domain $\Omega$ with the desired property, we first choose two domains $A, B$ in $S^{N-1}$ such that $A \subset B \subset S^{N-1}$ and

$$\mathcal{H}^{N-1}(B) \lambda^{\frac{N+2}{2}} > \mathcal{H}^{N-1}(A) \Lambda^{\frac{N+2}{2}}, \quad (3.2)$$

where $\mathcal{H}^{N-1}$ is the standard $(N-1)$-dimensional Hausdorff measure. Let

$$\alpha := \mathcal{H}^{N-1}(A) \quad \text{and} \quad \beta := \mathcal{H}^{N-1}(B) \quad (3.3)$$

for simplicity of expression. According to (3.2), we may choose a small number $\varepsilon$ with $0 < \varepsilon < 1$ so that

$$[(1 - \varepsilon) \beta + \varepsilon \alpha] \lambda^{\frac{N+2}{2}} > [(1 - \varepsilon) \alpha + \varepsilon \beta] \Lambda^{\frac{N+2}{2}}. \quad (3.4)$$

Let $\delta, R$ be two numbers such that $0 < \delta < 1 < R$ and

$$\int_{\delta}^{R} e^{-s^2} s^{N-1} ds = (1 - \varepsilon) \int_{0}^{\infty} e^{-s^2} s^{N-1} ds. \quad (3.5)$$
We then define a sequence of numbers \( \{r_n\} \) by \( r_0 = 0 \) and
\[
r_n = \delta R^{n-1}, \quad n \in \mathbb{N},
\]
and a sequence of sets \( \{E_n\} \) in \( \mathbb{R}^N \) by
\[
E_{2(k-1)} = \{ x \in \mathbb{R}^N : x = r\omega, \; r_{2(k-1)} \leq r \leq r_{2k-1}, \; \omega \in A \},
\]
\[
E_{2k-1} = \{ x \in \mathbb{R}^N : x = r\omega, \; r_{2k-1} \leq r \leq r_{2k}, \; \omega \in B \}
\]
for \( k \in \mathbb{N} \). At last, we define the domain \( \Omega \) to be the interior of the set \( \bigcup_{n=0}^{\infty} E_n \).

Since the initial condition of the problem \( (1.2) \) is the characteristic function of \( \Omega \), the solution \( u(x,t) \) is given by
\[
u(x,t) = \int_{\Omega} g(x,\xi,t) d\xi.
\]
It follows from \( (3.1) \) that for every \( t > 0 \)
\[
\lambda t^{-\frac{N}{2}} \int_\Omega e^{-\frac{|\xi|^2}{\lambda t}} d\xi \leq u(0,t) \leq \Lambda t^{-\frac{N}{2}} \int_\Omega e^{-\frac{|\xi|^2}{\lambda t}} d\xi.
\]
Let us calculate the both sides of \( (3.7) \). Using the notation \( (3.3) \), we have
\[
\lambda t^{-\frac{N}{2}} \int_\Omega e^{-\frac{|\xi|^2}{\lambda t}} d\xi = \lambda t^{-\frac{N}{2}} \sum_{k=1}^{\infty} \left[ \int_{E_{2(k-1)}} e^{-\frac{|\xi|^2}{\lambda t}} d\xi + \int_{E_{2k-1}} e^{-\frac{|\xi|^2}{\lambda t}} d\xi \right]
\]
\[
= \lambda t^{-\frac{N}{2}} \sum_{k=1}^{\infty} \left[ \alpha \int_{r_{2(k-1)}}^{r_{2k-1}} e^{-\frac{s^2}{\lambda t}} s^{N-1} ds + \beta \int_{r_{2k-1}}^{r_{2k}} e^{-\frac{s^2}{\lambda t}} s^{N-1} ds \right]
\]
Replacing \( \lambda \) by \( \Lambda \) yields that
\[
\Lambda t^{-\frac{N}{2}} \int_\Omega e^{-\frac{|\xi|^2}{\lambda t}} d\xi = \Lambda t^{-\frac{N}{2}} \sum_{k=1}^{\infty} \left[ \alpha \int_{\frac{r_{2(k-1)}}{\sqrt{\lambda t}}}^{\frac{r_{2k-1}}{\sqrt{\lambda t}}} e^{-\frac{s^2}{\lambda t}} s^{N-1} ds + \beta \int_{\frac{r_{2k-1}}{\sqrt{\lambda t}}}^{\frac{r_{2k}}{\sqrt{\lambda t}}} e^{-\frac{s^2}{\lambda t}} s^{N-1} ds \right].
\]
Let us consider the sequence of times \( \{t_n\} \) defined by
\[
t_n = \frac{1}{\lambda} R^{4(n-1)}, \quad n \in \mathbb{N}.
\]
According to \( (3.7) \) and \( (3.8) \), we have
\[
u(0,t_n) \geq \lambda t^{-\frac{N}{2}} \sum_{k=1}^{\infty} \left[ \alpha \int_{0}^{\infty} e^{-s^2} s^{N-1} ds + (\beta - \alpha) \int_{\frac{r_{2k-1}}{\sqrt{\lambda t_n}}}^{\frac{r_{2k}}{\sqrt{\lambda t_n}}} e^{-s^2} s^{N-1} ds \right].
\]
Since \( \beta > \alpha \), we have
\[
u(0,t_n) \geq \lambda t^{-\frac{N}{2}} \sum_{k=1}^{\infty} \left[ \alpha \int_{0}^{\infty} e^{-s^2} s^{N-1} ds + (\beta - \alpha) \int_{\frac{r_{2k-1}}{\sqrt{\lambda t_n}}}^{\frac{r_{2k}}{\sqrt{\lambda t_n}}} e^{-s^2} s^{N-1} ds \right].
By the definition \( r_{\ell} \) of \( r_{\ell} \), \( r_{\ell} = \frac{\delta}{\sqrt{M_n}} = \delta R \), and hence we have
\[
\int_{r_{\ell} - \sqrt{\lambda n}}^{r_{\ell} + \sqrt{\lambda n}} e^{-s^2} s^{N-1} ds = \int_\delta^{\delta R} e^{-s^2} s^{N-1} ds = (1 - \varepsilon) \int_0^\infty e^{-s^2} s^{N-1} ds,
\]
where the second identity follows from (3.5). Thus we have
\[
u(0, t_n) \geq \lambda \frac{N + 2}{2} ((1 - \varepsilon) \beta + \varepsilon \alpha) \int_0^\infty e^{-s^2} s^{N-1} ds,
\]
for every \( n \in \mathbb{N} \), and hence
\[
\limsup_{t \to \infty} u(0, t) \geq \lambda \frac{N + 2}{2} ((1 - \varepsilon) \beta + \varepsilon \alpha) \int_0^\infty e^{-s^2} s^{N-1} ds. \tag{3.11}
\]

We now consider the sequence of times \( \{T_n\} \) defined by
\[
T_n = \frac{1}{\Lambda} R^{2(2n-3)}, \quad n = 2, 3, \ldots.
\]
In the same way as above, one can show using (3.9) that
\[
u(0, T_n) \leq \Lambda \frac{N + 2}{2} ((1 - \varepsilon) \alpha + \varepsilon \beta) \int_0^\infty e^{-s^2} s^{N-1} ds,
\]
and hence
\[
\liminf_{t \to \infty} u(0, t) \leq \Lambda \frac{N + 2}{2} ((1 - \varepsilon) \alpha + \varepsilon \beta) \int_0^\infty e^{-s^2} s^{N-1} ds. \tag{3.12}
\]
Therefore, it follows from (3.4), (3.11) and (3.12) that
\[
\liminf_{t \to \infty} u(0, t) < \limsup_{t \to \infty} u(0, t). \tag{3.13}
\]

Since \( \alpha < \beta \), we have from (3.7), (3.8) and (3.10) that for every \( t > 0 \)
\[
u(0, t) \geq \lambda \frac{N + 2}{2} \alpha \int_0^\infty e^{-s^2} s^{N-1} ds,
\]
which shows that
\[
\liminf_{t \to \infty} u(0, t) > 0. \tag{3.15}
\]
Observe that \( 1 - u \) solves problem (1.2) where \( \Omega \) is replaced by \( \Omega^c = \mathbb{R}^N \setminus \Omega \), and the fact that \( \Omega_{B^c} \subset \Omega^c \) with \( B^c = S^{N-1} \setminus B \). Here we used the notation (1.3) for \( \Omega_{B^c} \). Hence, by the same argument as that for (3.14), we infer that for every \( t > 0 \)
\[
1 - u(0, t) \geq \lambda \frac{N + 2}{2} \mathcal{H}^{N-1}(B^c) \int_0^\infty e^{-s^2} s^{N-1} ds,
\]
which yields that
\[
\limsup_{t \to \infty} u(0, t) < 1. \tag{3.16}
\]
This completes the proof. □
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