Equivalent Characterizations for Boundedness of Maximal Singular Integrals on $ax+b$–Groups

Liguang Liu, Maria Vallarino, and Dachun Yang

Abstract  Let $(S, d, \rho)$ be the affine group $\mathbb{R}^n \ltimes \mathbb{R}^+$ endowed with the left-invariant Riemannian metric $d$ and the right Haar measure $\rho$, which is of exponential growth at infinity. In this paper, for any linear operator $T$ on $(S, d, \rho)$ associated with a kernel $K$ satisfying certain integral size condition and Hörmander’s condition, the authors prove that the following four statements regarding the corresponding maximal singular integral $T^*$ are equivalent: $T^*$ is bounded from $L^\infty_c$ to $\text{BMO}$, $T^*$ is bounded on $L^p$ for all $p \in (1, \infty)$, $T^*$ is bounded on $L^p$ for some $p \in (1, \infty)$ and $T^*$ is bounded from $L^1$ to $L^{1, \infty}$. As applications of these results, for spectral multipliers of a distinguished Laplacian on $(S, d, \rho)$ satisfying certain Mihlin-Hörmander type condition, the authors obtain that their maximal singular integrals are bounded from $L^\infty_c$ to $\text{BMO}$, from $L^1$ to $L^{1, \infty}$, and on $L^p$ for all $p \in (1, \infty)$.

Keywords  Exponential growth group · Maximal singular integral · Multiplier · Dyadic cube · (Local) Sharp maximal function · Fefferman–Stein type inequality

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1 Introduction

Let $S$ be the affine group $\mathbb{R}^n \ltimes \mathbb{R}^+$ endowed with the product

$$(x, a) \cdot (x', a') = (x + ax', aa') \quad \forall (x, a), (x', a') \in S.$$ 

We call $S$ an $ax+b$–group. Clearly, $e \equiv (0, 1)$ is the unit of $S$. The inverse of any $(x, a) \in S$, denoted by $(x, a)^{-1}$, is equal to $(-x/a, 1/a)$. We endow $S$ with the left-invariant Riemannian metric $ds^2 \equiv a^{-2}(dx_1^2 + \cdots + dx_n^2 + da^2)$, which induces the following distance function $d$ on $S \times S$:

$$d((x, a), (x', a')) = \cosh^{-1}\left(\frac{a^2 + a'^2 + |x - x'|^2}{2aa'}\right) \quad \forall (x, a), (x', a') \in S; \quad (1.1)$$

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*Corresponding author.
see, for example, \[1, \text{formula (2.18)}\] or \[24, \text{pp. 119–120}\]. The left Haar measure induced by the above left-invariant Riemannian metric is \(d\lambda(x, a) = a^{-n-1}dxda\). A standard calculation yields that the group modular function \(\delta(x, a)\) is equal to \(a^{-n}\), and hence the right Haar measure is
\[
d\rho(x, a) = \delta(x, a)^{-1}d\lambda(x, a) = a^{-1}dxda. \tag{1.2}
\]

Throughout the whole paper, we work on the triple \((S, d, \rho)\), namely, the group \(S\) endowed with the left-invariant Riemannian metric \(d\) and the right Haar measure \(\rho\). For all \((x, a) \in S\) and \(r > 0\), we define the ball \(B((x, a), r)\) on \(S\) by
\[
B((x, a), r) = \{(x', a') \in S : d((x, a), (x', a')) < r\}.
\]

For any \(p \in (0, \infty]\), let \(L^p\) be the space of all measurable functions \(f\) on \((S, d, \rho)\) satisfying
\[
\|f\|_{L^p} \equiv \left\{ \int_{S} |f(x)|^p d\rho(x) \right\}^{1/p} < \infty,
\]
with a usual modification made when \(p = \infty\), and let \(L^{p, \infty}\) be the space of all measurable functions \(f\) on \((S, d, \rho)\) satisfying
\[
\|f\|_{L^{p, \infty}} \equiv \sup_{\alpha > 0} \left\{ \alpha \|\{x \in S : |f(x)| > \alpha\}\|_p \right\}^{1/p} < \infty.
\]

Denote by \(L^\infty_c\) the set of functions in \(L^\infty\) with compact support; and by \(L^\infty_{c, 0}\) the set of functions \(f\) in \(L^\infty_c\) such that \(\int_S f d\rho = 0\). Let \(L^1_{\text{loc}}\) be the set of locally integrable functions on \(S\) (with respect to the measure \(\rho\)).

It is well known that the space \((S, d, \rho)\) is of exponential growth at infinity, namely,
\[
\rho(B((0, 1), r)) \approx \begin{cases} r^{n+1} & \text{if } r < 1, \\ e^{nr} & \text{if } r \geq 1; \end{cases}
\]
see, for example, \[24, 32\]. Harmonic analysis on exponential growth groups including the space \((S, d, \rho)\) currently attracts a lot of attention. In particular, efforts have been made to study the Hardy–Littlewood maximal function (see \[11, 8\]), Riesz transforms (see \[27, 10, 9, 26\]), spectral multipliers related to a distinguished Laplacian (see \[3, 5, 16, 15, 14, 24, 32, 34\]).

In 2003, Hebisch and Steger \[16\] introduced the notion of Calderón–Zygmund spaces, and developed a variant of the Calderón–Zygmund theory of singular integrals on Calderón–Zygmund spaces, which can be applied to the \(ax + b\)-groups. Precisely, a space \(M\) endowed with a metric \(d\) and a Borel measure \(\mu\) is said to have the Calderón–Zygmund property if there exists a positive constant \(C\) such that for every \(f \in L^1(\mu)\) and each \(\alpha > C\|f\|_{L^1(\mu)}/\mu(M)\), there exist a decomposition \(f = g + \sum b_i\), sets \(\{Q_i\}_i \subseteq M\), positive numbers \(\{r_i\}_i\) and points \(\{x_i\}_i \subseteq M\) such that for all \(i\),

(i) \(\text{supp } b_i \subseteq Q_i\) and \(\int_M b_i \, d\mu = 0\),

(ii) \(\|g\|_{L^\infty(\mu)} \leq C\alpha\) and \(\sum_i \|b_i\|_{L^1(\mu)} \leq C\|f\|_{L^1(\mu)}\),

(iii) \(Q_i \subseteq B(x_i, Cr_i)\) and \(\sum_i \mu(Q_i^*) \leq C\|f\|_{L^1(\mu)}/\alpha\), where \(Q_i^* \equiv \{x \in M : d(x, Q_i) < r_i\}\);
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see [16, Definition 1.1]. Hebisch and Steger [16, Theorem 1.2] further proved that, for any Calderón–Zygmund space $(M, d, \mu)$, if $T \equiv \sum_{j \in \mathbb{Z}} K_j$ is bounded on $L^2(\mu)$ and if there exist positive constants $c \in (0, 1)$, $C$, $a$ and $b$ such that

$$\int_M |K_j(x, y)| (1 + c^j d(x, y))^a \, d\mu(x) \leq C \quad \forall y \in M,$$

(1.3)

and

$$\int_M |K_j(x, y) - K_j(x, z)| \, d\mu(x) \leq C(c^j d(y, z))^b \quad \forall y, z \in M,$$

(1.4)

then $T$ is of weak type 1 and bounded on $L^p(\mu)$ for all $p \in (1, 2]$. Obviously, spaces of homogeneous type in the sense of Coifman and Weiss [4] enjoy the Calderón–Zygmund property. An example of Calderón–Zygmund spaces, which is not a space of homogeneous type, is $(S, d, \rho)$ (see [16, Lemma 5.1]). In fact, each integrable function $f$ on $(S, \rho)$ at any level $\alpha > 0$ can be decomposed into a sum $g + \sum_i b_i$ as above, where every $b_i$ is supported on a set $R_i$ which belongs to a suitable family $R$ of sets in $S$: these sets are not balls because of the exponential growth of the space, but suitable “rectangles” in $\mathbb{R}^n \times \mathbb{R}^+$. More precisely, for any $R \in R$, there exists a positive $r_R$ such that $R$ is contained in a ball of radius comparable to $r_R$ and the measures of $R$ and its dilated set $R^* \equiv \{ x \in S : d(x, R) < r_R \}$ are comparable. The elements in $R$ are called Calderón–Zygmund sets on $(S, d, \rho)$. As an application of the aforementioned Calderón–Zygmund theory of singular integrals, spectral multipliers associated with a distinguished Laplacian $\Delta$ on $(S, d, \rho)$ which satisfy certain Mihlin-Hörmander type condition are of weak type 1 and bounded on $L^p$ for all $p \in (1, \infty)$ [16, Theorem 2.4]. Müller and Thiele [24] re-obtained the multiplier results of [16] by considering estimates of the wave propagator associated with $\Delta$.

A reformulation of Hebisch–Steger [16, Theorem 1.2] on $(S, d, \rho)$, using a condition of Hörmander’s type, is as follows: if $T$ is a linear operator which is bounded on $L^2$ and admits a locally integrable kernel $K$ off the diagonal satisfying that

$$\sup_{R \in R, y, z \in R} \int_{S \setminus R} |K(x, y) - K(x, z)| \, d\rho(x) < \infty,$$

(1.5)

then $T$ is bounded from $L^1$ to $L^{1, \infty}$ and on $L^p$ for all $p \in (1, 2]$; see [34] for the details and see also [29, 13] for the Euclidean case. For the endpoint case, Vallarino [32] developed an $H^1 – BMO$ theory on $(S, d, \rho)$, and proved that singular integrals whose kernels satisfy the condition (1.5) are bounded from $H^1$ to $L^1$ and from $L^\infty$ to BMO. As an application, spectral multipliers associated with a distinguished Laplacian $\Delta$ which satisfy certain Mihlin-Hörmander type condition are bounded from $H^1$ to $L^1$ and from $L^\infty$ to BMO (see [32, Proposition 4.2]). Moreover, Sjögren and Vallarino [27] considered $H^1 – L^1$ boundedness of various Riesz transforms associated with $\Delta$. In [35], the Calderón–Zygmund theory of [16] is generalized to Damek–Ricci spaces.

In this paper, we study the boundedness of maximal singular integrals on $(S, d, \rho)$. The importance of results in this direction is well known, and comes from the fact that they imply pointwise convergence results (see, for example, [13] and, in particular, [13, Theorem...
Recall that in the Euclidean setting, the maximal singular integral $T^*$ associated with a kernel $K$ is defined, for all suitable functions $f$ and all $x$ in $\mathbb{R}^n$, by

$$T^* f(x) = \sup_{\epsilon > 0} \left| \int_{|x-y| > \epsilon} K(x,y) f(y) \, dy \right|.$$ 

An alternative but equivalent way of expressing this operator $T^*$ is

$$T^* f(x) = \sup_{B \subset \mathbb{R}^n, B \ni x} \left| \int_{\mathbb{R}^n \setminus 2B} K(x,y) f(y) \, dy \right|,$n
where the supremum is taken over all Euclidean balls in $\mathbb{R}^n$ containing $x$.

In view of this observation, in the space $(S, d, \rho)$, we define the maximal singular integral $T^*$ associated with a kernel $K$ as

$$T^* f(x) \equiv \sup_{R \in \mathcal{R}, R \ni x} \left| \int_{S \setminus R} K(x,y) f(y) \, d\rho(y) \right| \quad \forall x \in S, \quad (1.6)$$

where $f \in L^\infty_c$ and the supremum is taken over all Calderón–Zygmund sets $R \in \mathcal{R}$ containing $x$; see Section 4 below for more details.

The main aim of this paper is to prove that, for $T^*$, defined as in (1.6), and associated with a kernel $K$ that satisfies an integral size condition and Hörmander’s condition (see (4.1) and (4.2) below), the following four statements are equivalent:

(i) $T^*$ is bounded from $L^\infty_c$ to BMO;

(ii) $T^*$ is bounded on $L^p$ for all $p \in (1, \infty)$;

(iii) $T^*$ is bounded on $L^p$ for some $p \in (1, \infty)$;

(iv) $T^*$ is bounded from $L^1$ to $L^{1, \infty}$;

see Theorem 4.1 below. Moreover, if $T$ is further assumed to be bounded on $L^2$, then the above boundedness (i) through (iv) hold for $T^*$; see Theorem 4.2 below.

The proof of the main results of the paper, namely, Theorems 4.1 and 4.2, are presented in Section 4. The main ingredients used in the proof are the Calderón–Zygmund property of $(S, d, \rho)$ and certain Fefferman–Stein weak type inequalities related to the local sharp maximal functions in the sense of John [20], Strömberg [31] and Jawerth–Torchinsky [19]; see Propositions 3.1 and 3.2 below. The proof of the aforementioned Fefferman–Stein type inequalities relies on the existence of certain “dyadic” sets on $(S, d, \rho)$, which are an analogue of the Euclidean dyadic cubes and were constructed in [22]; see Lemma 2.2 below. We remark that the proof of Theorem 4.1 invokes some ideas of [18] and Grafakos [12].

Some applications are given in Section 5. Precisely, for certain class of spectral multipliers for the distinguished Laplacian $\Delta$ we prove in Theorem 5.1 below that the corresponding maximal singular integral operators are bounded from $L^\infty_c$ to BMO, from $L^1$ to $L^{1, \infty}$ and on $L^p$ for all $p \in (1, \infty)$. The main difficulty in proving Theorem 5.1 is to show that the kernels of such spectral multipliers satisfy the integral size condition (4.1), which requires very delicate estimates (see Proposition 5.1 below).

Our paper is organized as follows. A brief recall of the geometric properties of $S$ and the Calderón–Zygmund property is presented in Section 2. In Section 3, we establish the
Fefferman–Stein (weak) type inequalities related to the local sharp maximal functions on $S$. The whole Section 4 is devoted to the proof of Theorems 4.1 and 4.2, which are the main results of the paper. Finally, an application to the spectral multipliers for the Laplacian $\Delta$ is studied in Section 5.

We make some conventions on notation. Let $\mathbb{N} \equiv \{0, 1, 2, \cdots \}$ and $\mathbb{R}^+ \equiv (0, \infty)$. For any space $X$ and any subset $E$ of $X$, set $E^c \equiv X \setminus E$ and let $\chi_E$ denote the characteristic function of $E$. Denote by $C$ a positive constant independent of the main parameters involved, which may vary at different occurrences. Constants with subscripts do not change through the whole paper. We use $f \lesssim g$ to denote $f \leq Cg$. If $f \lesssim g \lesssim f$, we write $f \approx g$.

For an operator $T$ defined on the Banach space $A$ and taking values in the Banach space $B$, we use $\|T\|_{A \to B}$ to denote its operator norm.

## 2 Preliminaries

We recall the notion of Calderón–Zygmund sets, which appears in [16] and implicitly in [11]. Let $Q$ be the collection of dyadic cubes in $\mathbb{R}^n$.

**Definition 2.1.** A Calderón–Zygmund set is a set $R = Q \times [ae^{-r}, ae^r)$, where $Q \in Q$ with side length $L$, $a \in \mathbb{R}^+$, $r > 0$ and

$$e^2ar \leq L < e^8ar \quad \text{if} \quad r < 1; \quad ae^{2r} \leq L < ae^{8r} \quad \text{if} \quad r \geq 1.$$  

Set $a_R \equiv a$, $r_R \equiv r$ and $x_R \equiv (c_Q, a)$, where $c_Q$ is the center of $Q$. Denote by $R$ the family of all Calderón–Zygmund sets on $S$. For any $x \in S$, let $R(x)$ be the collection of all $R \in R$ containing $x$.

The following lemma presents some properties of the Calderón–Zygmund sets (see [16, 32]).

**Lemma 2.1.** There exists $\kappa_0 \in [1, \infty)$ such that for all $R \in R$, the following hold:

(i) $B(x_R, r_R) \subset R \subset B(x_R, \kappa_0 r_R)$;
(ii) $\rho(R^*) \leq \kappa_0 \rho(R)$, where $R^* \equiv \{x \in S : d(x, R) < r_R\}$ is the dilated set of $R \in R$;
(iii) every $R \in R$ can be decomposed into mutually disjoint sets $\{R_i\}_{i=1}^k \subset R$ with $k = 2$ or $k = 2^n$ such that $R = \bigcup_{i=1}^k R_i$ and $\rho(R_i) = \rho(R)/k$ for all $i \in \{1, \cdots, k\}$.

Using the geometric properties of the Calderón–Zygmund sets, the authors, in [22], constructed certain “dyadic” sets on $(S, d, \rho)$, which are analogues of the Euclidean.

**Lemma 2.2.** There exists a sequence $\{D_j\}_{j \in \mathbb{Z}}$ such that each $D_j$ consists of pairwise disjoint Calderón–Zygmund sets, and

(i) for any given $j \in \mathbb{Z}$, $S = \bigcup_{R \in D_j} R$;
(ii) if $\ell \leq k$, $R \in D_\ell$ and $R' \in D_k$, then either $R \subset R'$ or $R \cap R' = \emptyset$;
(iii) for any given $j \in \mathbb{Z}$ and $R \in D_j$, there exists a unique $R' \in D_{j+1}$ such that $R \subset R'$ and $\rho(R') \leq \max\{2^n, 3\} \rho(R)$;
Thus, the operator 
\( M \) and

The maximal function 
\( M \)

Proposition 2.1. 

\[ D ≡ \{ D_i \}_{i=1} \subset \mathcal{D}_{j-1} \] with \( k = 2 \) or \( k = 2^n \) such that \( R = \bigcup_{i=1}^k R_i \) and \( \rho(R)/2^n \leq \rho(R_i) \leq 2\rho(R)/3 \) for all \( i \in \{ 1, \cdots, k \} \).

From now on, we set \( \mathcal{D} \equiv \{ \mathcal{D}_j \}_{j \in \mathbb{Z}} \).

Hardy–Littlewood maximal functions on groups with exponential growth have been investigated in a series of works; see, for example, [8, 11, 33]. For any \( f \in L^1_{\text{loc}} \), we define the Hardy–Littlewood maximal function \( \mathcal{M}f \) and the dyadic Hardy–Littlewood maximal function \( \mathcal{M}_D f \) respectively by the formulae

\[ \mathcal{M}f(x) ≡ \sup_{R \in \mathcal{R}(x)} \frac{1}{\rho(R)} \int_R |f| \, d\rho \quad \forall x \in S, \] (2.1)

and

\[ \mathcal{M}_D f(x) ≡ \sup_{R \in \mathcal{R}(x), R \in \mathcal{D}} \frac{1}{\rho(R)} \int_R |f| \, d\rho \quad \forall x \in S. \] (2.2)

The maximal function \( \mathcal{M} \) has the following boundedness properties [33].

**Proposition 2.1.** \( \mathcal{M} \) is bounded from \( L^1 \) to \( L^{1, \infty} \), and on \( L^p \) for all \( p \in (1, \infty] \).

From (2.1), (2.2) and the differentiation theorem for integrals, it follows that

\[ f(x) \leq \mathcal{M}_D f(x) \leq \mathcal{M} f(x) \quad \forall f \in L^1_{\text{loc}}, \text{ almost every } x \in S. \]

Thus, the operator \( \mathcal{M}_D \) is bounded from \( L^1 \) to \( L^{1, \infty} \) and for any \( p \in (1, \infty] \),

\[ \| \mathcal{M}_D(f) \|_{L^p} \approx \| f \|_{L^p} \approx \| \mathcal{M}(f) \|_{L^p} \quad \forall f \in L^p. \]

It was proved in [16, Lemma 5.1] that \( (S, d, \rho) \) possesses the Calderón–Zygmund property. Indeed, the boundedness properties of \( \mathcal{M}_D \) and the differentiation theorem for integrals, together with Lemma 2.2 and a standard stopping-time argument, imply the following dyadic version of the Calderón–Zygmund property; see also [32, Proposition 2.4].

**Proposition 2.2.** Let \( f \in L^p \) with \( p \in [1, \infty) \). For any \( \alpha > 0 \), there exists a sequence of disjoint sets, \( \{ R_i \}_i \in \mathcal{D} \), such that \( f \) can be decomposed into \( f = g + \sum_i b_i \), where

(i) \( |g(x)| \leq C_1 \alpha \) for almost every \( x \in S \);

(ii) for all \( i \), \( \text{supp} b_i \subset R_i \subset \mathcal{D} \) and \( \int_S b_i \, d\rho = 0 \);

(iii) for all \( i \), \( \alpha \leq \frac{1}{\rho(R_i)} \int_{R_i} |f|^p \, d\rho \leq C_1 \alpha \);

(iv) for all \( i \), \( \| b_i \|_{L^p} \leq C_1 \alpha [\rho(R_i)]^{1/p} \),

where \( C_1 \) is a positive constant independent of \( \alpha \) and \( f \).

**Remark 2.1.** If \( f \in L^{\infty} \cap L^p \) for some \( p \in [1, \infty) \) and \( f = g + \sum_i b_i \) is a Calderón–Zygmund decomposition of \( f \) obtained as in Proposition 2.2, then \( \| g \|_{L^\infty} \leq \| f \|_{L^\infty} \). Moreover, if \( \int_S f \, d\rho = 0 \), then \( \int_S g \, d\rho = 0 \).
3 Fefferman–Stein Type Inequalities

The local maximal functions in Euclidean spaces were introduced by John [20] and later investigated by Strömberg [31], Jawerth–Torchinsky [19] and Lerner [21]; see also [18] for the setting of spaces of homogeneous type. Following this pioneering work, we introduce the local maximal functions on \((S, d, \rho)\).

**Definition 3.1.** (i) The non-increasing rearrangement of a measurable function \(f\) on \((S, d, \rho)\) is defined by

\[
    f^*(\xi) = \inf\{t > 0 : \rho(\{x \in S : |f(x)| > t\}) < \xi\} \quad \forall \, \xi \in (0, \infty).
\]

(ii) Let \(s \in (0, 1)\) and \(D\) be the family of dyadic sets. For any \(f \in L^1_{\text{loc}}\), its local maximal function \(M_{0,s}f\) is defined by

\[
    M_{0,s}f(x) = \sup_{R \in \mathcal{R}(x)} (f\chi_R)^* (s\rho(R)) \quad \forall \, x \in S,
\]

and its dyadic local maximal function \(M^D_{0,s}f\) is defined by

\[
    M^D_{0,s}f(x) = \sup_{R \in \mathcal{R}(x) \cap D} (f\chi_R)^* (s\rho(R)) \quad \forall \, x \in S.
\]

Some properties of these local maximal operators are presented in the following lemma.

**Lemma 3.1.** For any \(f \in L^1_{\text{loc}}\), the following hold:

(i) for all \(s \in (0,1)\) and \(x \in S\), \(M^D_{0,s}f(x) \leq M_{0,s}f(x)\);

(ii) if \(0 < s_1 < s_2 < 1\), then \(M^D_{0,s_2}f(x) \leq M^D_{0,s_1}f(x)\) for all \(x \in S\);

(iii) for all \(s \in (0,1)\) and \(x \in S\), \(M^D_{0,s}f(x) \leq s^{-1} M_Df(x)\);

(iv) for all \(s \in (0,1)\) and all measurable functions \(f_1\) and \(f_2\),
\[
    M^D_{0,s}(f_1 + f_2)(x) \leq M^D_{0,s/2}(f_1)(x) + M^D_{0,s/2}(f_2)(x) \quad \forall \, x \in S.
\]

(v) Properties (ii)–(iv) hold with \(M^D_{0,s}\) and \(M_D\) replaced by \(M_{0,s}\) and \(M\), respectively.

(vi) For all \(s \in (0,1)\), \(\lambda \in (0, \infty)\) and \(f \in L^1_{\text{loc}}\),
\[
    \{x \in S : M_D(\chi_{\{|f| > \lambda\}})(x) > s\} \subset \{x \in S : M^D_{0,s}f(x) > \lambda\} \\
    \subset \{x \in S : M_D(\chi_{\{|f| > \lambda\}})(x) \geq s\}.
\]

Consequently,
\[
    \rho(\{x \in S : |f(x)| > \lambda\}) \leq \rho(\{x \in S : M^D_{0,s}f(x) > \lambda\}) \\
    \leq \|M_D\|_{L^1 \to L^{1,\infty}} s^{-1} \rho(\{x \in S : |f(x)| > \lambda\}).
\]
(vii) Similarly,

$$\{ x \in S : M(\chi_{\{|f|>\lambda\}})(x) > s \} \subset \{ x \in S : M_{0,s}f(x) > \lambda \} \subset \{ x \in S : M(\chi_{\{|f|>\lambda\}})(x) \geq s \}$$

and

$$\rho(\{ x \in S : |f(x)| > \lambda \}) \leq \rho(\{ x \in S : M_{0,s}f(x) > \lambda \}) \leq \|M\|_{L^1 \rightarrow L^{1,\infty}} s^{-1} \rho(\{ x \in S : |f(x)| > \lambda \}).$$

Proof. Properties (i) and (ii) follow from the definition of $M_{0,s}$. For all $R \in \mathcal{R}(x) \cap \mathcal{D}$,

$$\rho(\{ y \in R : |f(y)| > s^{-1} M_{D}f(x) \}) < \int_{R} \frac{|f(y)|}{s^{-1} M_{D}f(x)} d\rho(y) \leq s \rho(R),$$

therefore, (iii) holds. Property (iv) follows from an argument similar to the one used in [18, Lemmas 2.2], while (v) is proven as (i)-(iv). Finally, proceeding as in the proof of [18, Lemmas 2.3] yields (vi) and (vii). \qed

Next, we recall the notion of the median value; see [31] for the Euclidean setting.

**Definition 3.2.** Suppose that $f$ is a real function in $L^1_{\text{loc}}$ and $R \in \mathcal{R}$. A median value $m_f(R)$ of $f$ over $R$ is defined to be one of the real numbers satisfying

$$\rho(\{ x \in R : f(x) > m_f(R) \}) \leq \rho(R)/2,$$

and $\rho(\{ x \in R : f(x) < m_f(R) \}) \leq \rho(R)/2$. In the case when $f$ is complex, define

$$m_f(R) \equiv m_{\Re(f)}(R) + im_{\Im(f)}(R),$$

where $\Re(f)$ and $\Im(f)$ denote, respectively, the real part and the imaginary part of $f$.

In the following lemma we show an analogue of the inequality proved by Jawerth and Torchinsky in [19, p. 238], which will be used in the proof of “good-$\lambda$” inequalities in Lemma 3.5 below.

**Lemma 3.2.** Let $f \in L^1_{\text{loc}}$ and $R \in \mathcal{R}$. Then

$$|m_f(R)| \leq \sqrt{2} \inf \{ t > 0 : \rho(\{ y \in R : |f(y)| > t \}) < \rho(R)/2 \}. \quad (3.1)$$

Consequently, for all $s \in (0,1/2]$ and $R \in \mathcal{D}$,

$$|m_f(R)| \leq \sqrt{2} \inf_{x \in R} M_{0,1/2}^{D}f(x) \leq \sqrt{2} \inf_{x \in R} M_{0,s}^{D}f(x). \quad (3.2)$$
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Proof. We first show (3.1). If \( f \) is real and \( m_f(R) > 0 \), then for all \( \epsilon \in (0, m_f(R)) \),
\[
\rho(\{ y \in R : |f(y)| > m_f(R) - \epsilon \}) \geq \rho(\{ y \in R : f(y) \geq m_f(R) \}) \geq \rho(R)/2,
\]
which implies that
\[
\inf \{ t > 0 : \rho(\{ y \in R : |f(y)| > t \}) < \rho(R)/2 \} \geq m_f(R) - \epsilon.
\]
Letting \( \epsilon \to 0 \) yields
\[
\inf \{ t > 0 : \rho(\{ y \in R : |f(y)| > t \}) < \rho(R)/2 \} \geq m_f(R).
\] (3.3)

If \( f \) is real and \( m_f(R) < 0 \), applying the above argument to \(-f\) and \(-m_f(R)\), we also obtain (3.3).

For any complex function \( f \), we have
\[
|m_f(R)|^2 = |m_Rf(R)|^2 + |m_\Im f(R)|^2 \leq 2 \max\{|m_Rf(R)|^2, |m_\Im f(R)|^2\}.
\]
Without loss of generality, we may assume that \( |m_Rf(R)| \leq |m_\Im f(R)| \). Then, by (3.3),
\[
|m_f(R)| \leq \sqrt{2}|m_\Im f(R)| \leq \sqrt{2}\inf \{ t > 0 : \rho(\{ y \in R : |\Im f(y)| > t \}) < \rho(R)/2 \}
\leq \sqrt{2}\inf \{ t > 0 : \rho(\{ y \in R : |f(y)| > t \}) < \rho(R)/2 \}.
\]
This proves (3.1). The inequalities (3.2) easily follow from the definition of \( M^D_{0,1/2} \) and Lemma 3.1(ii).

Vallarino [32] introduced the space \( \text{BMO} \) of functions with bounded mean oscillation on \((S,d,\rho)\) as follows. For any \( f \in L^1_{\text{loc}} \) and \( R \in \mathcal{R} \), set \( f_R \equiv \frac{1}{\rho(R)} \int_R f d\rho \); the function \( f \) is said to be in \( \text{BMO} \) if
\[
\|f\|_{\text{BMO}} \equiv \sup_{R \in \mathcal{R}} \frac{1}{\rho(R)} \int_R |f - f_R| d\rho < \infty.
\]
For any \( q \in (0, \infty) \) and \( f \in L^1_{\text{loc}} \), set
\[
\|f\|_{\text{BMO}_q} \equiv \sup_{R \in \mathcal{R}} \left\{ \frac{1}{\rho(R)} \int_R |f - f_R|^q d\rho \right\}^{1/q}.
\]
It was proved in [32, Section 3] that for any \( q \in (1, \infty) \), there exists a positive constant \( C_q \) such that for all \( f \in L^1_{\text{loc}} \),
\[
\frac{1}{C_q} \|f\|_{\text{BMO}_q} \leq \|f\|_{\text{BMO}} \leq C_q \|f\|_{\text{BMO}_q}.
\] (3.4)
It turns out that (3.4) also holds for \( q \in (0,1) \). This is proved in the following lemma, using some ideas of [23].
Lemma 3.3. For any $\sigma \in (0, 1)$ there exists a positive constant $C_\sigma$, which depends only on $\sigma$, such that for all $f \in L^1_{\text{loc}}$,
\[
C_\sigma \| f \|_{\text{BMO}} \leq \| f \|_{*, \sigma} \leq \| f \|_{\text{BMO}, \sigma} \leq \| f \|_{\text{BMO}},
\]
where
\[
\| f \|_{*, \sigma} = \sup_{R \in \mathcal{R}} \inf_{c \in \mathbb{C}} \left\{ \frac{1}{\rho(R)} \int_R |f - c|^\sigma \, d\rho(x) \right\}^{1/\sigma}.
\]

Proof. The third inequality of (3.5) follows from Hölder’s inequality, while the second inequality of (3.5) follows from the definitions of $\| \cdot \|_{*, \sigma}$ and $\| \cdot \|_{\text{BMO}, \sigma}$.

To prove the first inequality of (3.5), we fix $f \in L^1_{\text{loc}}$. Suppose that $\| f \|_{*, \sigma} < \infty$; otherwise there is nothing to prove. For any $R \in \mathcal{R}$, by (3.6) and the fact that $\| f \|_{*, \sigma} < \infty$, we have
\[
\inf_{c \in \mathbb{C}} \left\{ \frac{1}{\rho(R)} \int_R |f - c|^\sigma \, d\rho(x) \right\}^{1/\sigma} = \left\{ \frac{1}{\rho(R)} \int_R |f - A_R|^\sigma \, d\rho(x) \right\}^{1/\sigma} \leq \| f \|_{*, \sigma}.
\]

For all $R' \in \mathcal{R}$, by (3.6) and the fact that $|a^\sigma - b^\sigma| \leq |a - b|^\sigma$ for all $a, b \in (0, \infty)$, we have
\[
\frac{1}{\rho(R')} \int_{R'} |f - A_R|^\sigma - |A_R - A_{R'}|^\sigma \, d\rho \leq \frac{1}{\rho(R')} \int_{R'} |f - A_{R'}|^\sigma \, d\rho \leq \| f \|_{*, \sigma}.
\]
which implies that
\[
\| |f - A_R|^\sigma \|_{\text{BMO}} \leq 2 \| f \|_{*, \sigma}.
\]
For all $a, b \in (0, \infty)$, we have $(a + b)^{1/\sigma} \leq 2^{1/\sigma - 1} \left[ a^{1/\sigma} + b^{1/\sigma} \right]$, which together with (3.6) yields that for all $R \in \mathcal{R}$,
\[
\frac{1}{\rho(R)} \int_R |f - A_R| \, d\rho = \frac{1}{\rho(R)} \int_R \{|f - A_R|^\sigma\}^{1/\sigma} \, d\rho(x)
\]
\[
\leq 2^{1/\sigma - 1} \left\{ \frac{1}{\rho(R)} \int_R |f(x) - A_R|^\sigma - \frac{1}{\rho(R)} \int_R |f(y) - A_R|^\sigma \, d\rho(y) \right\}^{1/\sigma} \, d\rho(x)
\]
\[
+ \left[ \frac{1}{\rho(R)} \int_R |f(y) - A_R|^\sigma \, d\rho(y) \right]^{1/\sigma}
\]
\[
\leq 2^{1/\sigma - 1} \left\{ \left( \| |f - A_R|^\sigma \|_{\text{BMO}, 1/\sigma} \right)^{1/\sigma} + \| f \|_{*, \sigma} \right\}.
\]

Observe that (3.4) and (3.7) imply that
\[
\left( \| |f - A_R|^\sigma \|_{\text{BMO}, 1/\sigma} \right)^{1/\sigma} \leq (C_{1/\sigma} \| f - A_R|^\sigma \|_{\text{BMO}})^{1/\sigma} \leq (2C_{1/\sigma})^{1/\sigma} \| f \|_{*, \sigma},
\]
where $C_{1/\sigma}$ is as in (3.4). Applying this estimate, we see that
\[
\frac{1}{\rho(R)} \int_R |f - f_R| \, d\rho \leq \frac{2}{\rho(R)} \int_R |f - A_R| \, d\rho \leq 2^{1/\sigma} (2C_{1/\sigma})^{1/\sigma} + 1 \| f \|_{*, \sigma}.
\]
Lemma 3.5. Indeed, the converses of (3.8) and (3.9) hold for small $s$ of a certain kind of Lemma 3.4(iv) and Lemma 3.1(vii) also imply that for all $f \in L^1$ the proof.

Now we introduce the (local) sharp maximal functions on $(S,d,\rho)$; see [13, 29, 31] for their definitions in the Euclidean setting and [18] for their definitions in spaces of homogeneous type.

**Definition 3.3.** (i) For any $f \in L^1_{loc}$, its *sharp maximal function* $f^\#$ is defined by

$$f^\#(x) \equiv \sup_{R \in \mathcal{R}(x)} \frac{1}{\rho(R)} \int_R |f(y) - f_R| \, d\rho(y) \quad \forall x \in S.$$ 

(ii) Let $s \in (0,1)$. For any $f \in L^1_{loc}$, its *local sharp maximal function* $M^\#_{0,s}f$ is defined by

$$M^\#_{0,s}f(x) \equiv \sup_{R \in \mathcal{R}(x)} \inf_{c \in \mathbb{C}} \left( (f - c) \chi_R \right)^*(s\rho(R)) \quad \forall x \in S.$$ 

The following lemma summarizes some properties of the (local) sharp maximal functions on $(S,d,\rho)$. The proofs are easy and hence omitted; see [13, 31, 18, 19].

**Lemma 3.4.** Let $f, g \in L^1_{loc}$. Then, for all $x \in S$,

(i) $f^\#(x) \leq 2Mf(x)$, where $M$ is defined in (2.1);

(ii) $|f|^\#(x) \leq 2f^\#(x)$ and $(f + g)^\#(x) \leq f^\#(x) + g^\#(x)$;

(iii) $f^\#(x)/2 \leq \sup_{R \in \mathcal{R}(x)} \inf_{a \in \mathbb{C}} \frac{1}{\rho(R)} \int_R |f(y) - a| \, d\rho(y) \leq f^\#(x)$;

(iv) when $s \in (0,1)$, $M^\#_{0,s}f(x) \leq M_{0,s}f(x)$ and $M^\#_{0,s}f(x) \leq s^{-1}f^\#(x)$, where $M_{0,s}$ is defined in (3.1);

(v) if $s \in (0,1/2)$ and $R \in \mathcal{R}$, then

$$\rho\{y \in R : |f(y) - m_f(R)| > 2\sqrt{2} \inf_{x \in R} M^\#_{0,s}f(x)\} \leq s\rho(R).$$

For any $p \in (1, \infty)$ and $s \in (0,1)$, an immediate consequence of Lemma 3.4 (i) and (iv) and the $L^p$-boundedness of $M$ is that for all $f \in L^p$,

$$\|M^\#_{0,s}f\|_{L^p} \leq 2s^{-1}\|M\|_{L^p \to L^p}\|f\|_{L^p}; \quad (3.8)$$

**Lemma 3.4(iv) and Lemma 3.1(vii) also imply that for all $f \in L^{p,\infty}$,**

$$\|M^\#_{0,s}f\|_{L^{p,\infty}} \leq \|M\|_{L^1 \to L^{1,\infty}}^{1/p} s^{-1/p}\|f\|_{L^{p,\infty}}. \quad (3.9)$$

Indeed, the converses of (3.8) and (3.9) hold for small $s$. To see this, we need the following certain kind of “good-$\lambda$” inequalities involving the dyadic local maximal function and the local sharp maximal function.

**Lemma 3.5.** Let $0 < s_1, s_2 \leq \frac{1}{2}$. Then, there exists a positive constant $C_2$ depending only on $S$ such that for all $f \in L^1_{loc}$ and all $\lambda > 0$,

$$\rho\{x \in S : M^D_{0,s_1}f(x) > 3\lambda, M^\#_{0,s_2}f(x) \leq \lambda/4\}$$

$$\leq C_2s_1^{-1}s_2\rho\{x \in S : M^D_{0,s_1}f(x) > \lambda\}.$$
Proof. Take \( \lambda > 0 \) and set \( \Omega_\lambda \equiv \{ x \in S : M_{D_{0,s_1}}^\mathcal{D}(x) > \lambda \} \). We may assume that \( \rho(\Omega_\lambda) < \infty \), otherwise there is nothing to prove. For any fixed \( x \in \Omega_\lambda \), there exists a “maximal dyadic cube” \( R_x \in \mathcal{D} \) containing \( x \) such that

\[
\inf \{ t > 0 : \rho(\{ y \in R_x : |f(y)| > t \}) < s_1\rho(R_x) \} > \lambda. \tag{3.10}
\]

Here “maximal dyadic cube” means that if \( R' \in \mathcal{D} \cap \mathcal{R}(x) \) satisfies (3.10), then \( R' \subset R_x \). Such a “maximal dyadic cube” exists since \( R_x \subset \Omega_\lambda \) and \( \rho(\Omega_\lambda) < \infty \). Let \( \{ R_j \}_{j \in I} \) be the collection of all such “maximal dyadic cubes” obtained by running \( x \) over \( \Omega_\lambda \). From the maximality, it follows that any two “maximal dyadic cubes” are disjoint. Moreover, \( \Omega_\lambda = \cup_{j \in I} R_j \). Therefore, to show Lemma 3.5, it suffices to prove that there exists a positive constant \( C_2 \) such that for all \( j \in I \),

\[
\rho \left( \left\{ x \in R_j : M_{D_{0,s_1}}^\mathcal{D}(x) > 3\lambda, M_{D_{0,s_2}}^\mathcal{D}(x) \leq \lambda/4 \right\} \right) \leq C_2 s_1^{-1} s_2 \rho(R_j). \tag{3.11}
\]

Fix \( j \in I \). We may assume that there exists \( x_0 \in R_j \) such that \( M_{D_{0,s_2}}^\mathcal{D}(x_0) \leq \lambda/4 \); otherwise (3.11) holds trivially. Suppose that \( R_j \in \mathcal{D}_{j_0} \) for some \( j_0 \in \mathbb{Z} \). Using Lemma 2.2(iii), we take \( \tilde{R}_j \) to be the unique Calderón–Zygmund set in \( \mathcal{D}_{j_0+1} \) that contains \( R_j \). Then, \( \rho(\tilde{R}_j) \leq \max\{2^n, 3\} \rho(R_j) \). By the maximality of \( R_j \) and (3.10), we have

\[
\inf \{ t > 0 : \rho(\{ y \in \tilde{R}_j : |f(y)| > t \}) < s_1\rho(\tilde{R}_j) \} \leq \lambda. \tag{3.12}
\]

From this and Lemma 3.2 together with the hypothesis \( s_1 \leq 1/2 \), it follows that

\[
|f_{\tilde{R}_j}(x)| \leq \sqrt{2} \inf \{ t > 0 : \rho(\{ y \in \tilde{R}_j : |f(y)| > t \}) < \rho(\tilde{R}_j)/2 \} \leq \sqrt{2}\lambda.
\]

Thus,

\[
M_{D_{0,s_1}}^\mathcal{D}(m_{\tilde{R}_j}x_{\tilde{R}_j})(x) \leq |m_{\tilde{R}_j}(x)| \leq \sqrt{2}\lambda.
\]

If \( M_{D_{0,s_1}}^\mathcal{D}(x_0) > 3\lambda \), then by (3.12) and the definition of \( \tilde{R}_j \), we have

\[
M_{D_{0,s_1}}^\mathcal{D}(x_{\tilde{R}_j})(x) = M_{D_{0,s_1}}^\mathcal{D}(x_0) > 3\lambda.
\]

So applying Lemma 3.1(iv) yields that for all \( x \in R_j \) satisfying \( M_{D_{0,s_1}}^\mathcal{D}(x) > 3\lambda \),

\[
M_{D_{0,s_1}}^\mathcal{D}(f_{\tilde{R}_j}x_{\tilde{R}_j})(x) \geq M_{D_{0,s_1}}^\mathcal{D}(f_{\tilde{R}_j})(x) - M_{D_{0,s_1}}^\mathcal{D}(m_{\tilde{R}_j}x_{\tilde{R}_j})(x) > \lambda. \tag{3.13}
\]

By using (3.13), Lemma 3.1(vi), \( M_{D_{0,s_2}}^\mathcal{D}(x_0) \leq \lambda/4 \) and Lemma 3.4(v), we obtain

\[
\rho \left( \left\{ x \in R_j : M_{D_{0,s_1}}^\mathcal{D}(x) > 3\lambda, M_{D_{0,s_2}}^\mathcal{D}(x) \leq \lambda/4 \right\} \right) \leq \rho \left( \left\{ x \in R_j : M_{D_{0,s_1}}^\mathcal{D}(f_{\tilde{R}_j}x_{\tilde{R}_j})(x) > \lambda \right\} \right)
\]
\[ \leq 2 \|M\|_{L^1\to L^{1,\infty}} s \int_1^{N} \rho \left( \left\{ x \in \tilde{R}_j : |f(x) - m f(\tilde{R}_j)| > \lambda \right\} \right) \]
\[ \leq 2 \|M\|_{L^1\to L^{1,\infty}} s \int_1^{N} \rho \left( \left\{ x \in \tilde{R}_j : |f(x) - m f(\tilde{R}_j)| > 2\sqrt{2} \inf_{z \in R_j} M_{0,s_2}^{\#} f(z) \right\} \right) \]
\[ \leq 2 \|M\|_{L^1\to L^{1,\infty}} s \int_1^{N} \rho(\tilde{R}_j). \]

Hence, (3.11) holds with \( C_2 \equiv 2 \max\{2^n, 3\} \|M\|_{L^1\to L^{1,\infty}}. \) This finishes the proof. \( \square \)

Applying the previous “good-\( \lambda \)” inequality, we prove the following Fefferman–Stein type inequality.

**Proposition 3.1.** Let \( p_0 \in (0, \infty) \) and \( C_2 \) be as in Lemma 3.5. Then, for any \( p \in (p_0, \infty) \) and \( s \in (0, 1/2] \) satisfying \( s < (2^2 3^p C_2)^{-1} \), there exists a positive constant \( C \) such that

\[ \|f\|_{L^p} \leq C \|M_{0,s_2}^{\#} f\|_{L^p} \quad \forall \; f \in L^{p_0, \infty}. \]

**Proof.** Fix \( f \in L^{p_0, \infty} \). For any \( N \in \mathbb{N} \), set

\[ I_N \equiv \int_0^{3N} \rho \left( \left\{ x \in S : M_{0,1/2}^p f(x) > \lambda \right\} \right) \rho \lambda d\lambda. \]

Then, Lemma 3.1(vi) and \( p > p_0 \) imply that \( I_N < \infty \). Applying Lemma 3.5 yields that

\[ I_N = 3^p \int_0^{N} \rho \lambda^{-1} \left( \left\{ x \in S : M_{0,1/2}^p f(x) > 3\lambda \right\} \right) d\lambda \]
\[ \leq 3^p \int_0^{N} \rho \lambda^{-1} \left( \left\{ x \in S : M_{0,1/2}^p f(x) > 3\lambda, M_{0,s}^p f(x) \leq \lambda/4 \right\} \right) d\lambda \]
\[ + \int_0^{N} \rho \lambda^{-1} \left( \left\{ x \in S : M_{0,s}^p f(x) > \lambda/4 \right\} \right) d\lambda \]
\[ \leq 3^p 2 C_2 s \int_0^{N} \rho \lambda^{-1} \left( \left\{ x \in S : M_{0,1/2}^p f(x) > \lambda \right\} \right) d\lambda + 3^p 4^p \|M_{0,s}^p f\|_{L^p} \]
\[ \leq s 3^p 2 C_2 I_N + 3^p 4^p \|M_{0,s}^p f\|_{L^p}. \]

Since \( s < (3^p 2 C_2)^{-1} \), we have \( I_N \lesssim \|M_{0,s}^p f\|_{L^p} \). Letting \( N \to \infty \) and using Lemma 3.1(vi), we obtain \( \|f\|_{L^p} \leq \|M_{0,1/2}^p f\|_{L^p} \lesssim \|M_{0,s}^p f\|_{L^p} \), which completes the proof. \( \square \)

The corresponding weak-type Fefferman–Stein inequality is the following.

**Proposition 3.2.** Let \( p_0 \in (0, \infty) \) and \( C_2 \) be as in Lemma 3.5. Then, for any \( p \in [p_0, \infty) \) and \( s \in (0, 1/2] \) satisfying \( s < (2^2 3^p C_2)^{-1} \), there exists a positive constant \( C \) such that

\[ \|f\|_{L^{p, \infty}} \leq C \|M_{0,s}^p f\|_{L^{p, \infty}} \quad \forall \; f \in L^{p_0, \infty}. \]
Proof. For any $N \in \mathbb{N}$, set $I_N \equiv \sup_{0<\lambda<3N} \lambda^p \rho(\{x \in S : \mathcal{M}_{0,1/2}^p f(x) > \lambda\})$. Combining Lemma 3.1(vi) with $f \in L^{p_0,\infty}$ and $p \geq p_0$ implies that $I_N < \infty$. Thus, by Lemma 3.5,

$$I_N = 3^p \sup_{0<\lambda<N} \lambda^p \rho(\{x \in S : \mathcal{M}_{0,1/2}^p f(x) > 3\lambda\}) \leq 3^p \sup_{0<\lambda<N} \lambda^p \rho(\{x \in S : \mathcal{M}_{0,1/2}^p f(x) > 3\lambda, \mathcal{M}_{0,s}^p f(x) \leq \lambda/4\}) + 3^p \sup_{0<\lambda<N} \lambda^p \rho(\{x \in S : \mathcal{M}_{0,1/2}^p f(x) > \lambda/4\}) \leq 2sC_23^p \sup_{0<\lambda<N} \lambda^p \rho(\{x \in S : \mathcal{M}_{0,1/2}^p f(x) > \lambda\}) + 3^p 4^p \|\mathcal{M}_{0,s}^p f\|_{L^{p,\infty}} \leq s23^p C_2 I_N + 3^p 4^p \|\mathcal{M}_{0,s}^p f\|_{L^{p,\infty}}.$$

Since $I_N$ is finite, the assumption $s < (2^2 3^p C_2)^{-1}$ implies that $I_N \lesssim \|\mathcal{M}_{0,s}^p f\|_{L^{p,\infty}}$. Letting $N \to \infty$ and using Lemma 3.1(vi) yield the desired conclusion.

**Remark 3.1.** A Fefferman–Stein inequality involving the sharp maximal function also holds. More precisely, let $p_0$ be in $(1, \infty)$. Then for any $p \in (p_0, \infty)$, by Lemma 3.4(iv) and Proposition 3.1,

$$\|f\|_{L^p} \leq C \|f\|_{L^{p_0,\infty}} \quad \forall f \in L^{p_0,\infty}. \quad (3.14)$$

As we shall see, such a Fefferman–Stein inequality is not enough for the proof of Theorem 4.1 below. This explains why we studied Fefferman–Stein type inequalities related to the local sharp maximal function as in Proposition 3.2, which is crucial in the proof of Theorem 4.1. However, (3.14) would be enough to give a direct proof of Theorem 4.2 below.

## 4 Maximal Singular Integrals

In this section, we consider the boundedness of maximal singular integrals whose kernels satisfy some integral size condition and Hörmander’s condition.

Assume that $K$ is a locally integrable function on $(S \times S) \setminus \{(x, x) : x \in S\}$ such that

$$\sup_{y \in S} \sup_{r>0} \int_{r<d(x,y) \leq 2r} |K(x,y)| + |K(y,x)| \, d\rho(x) \equiv \nu_1 < \infty, \quad (4.1)$$

and

$$\sup_{R \in \mathcal{R}} \sup_{y, y' \in R \setminus (R^*)^c} \int_{d(x,y') \leq 2} |K(x,y) - K(x,y')| + |K(y,x) - K(y',x)| \, d\rho(x) \equiv \nu_2 < \infty, \quad (4.2)$$

where, for any $R$ in $\mathcal{R}$, $R^* \equiv \{x \in S : d(x,R) < r_R\}$ and $(R^*)^c \equiv S \setminus R^*$.

Let $T$ be the **linear operator associated with a kernel $K$** satisfying (4.1) and (4.2); in particular, for all $f \in L_c^{\infty}$ and $x \notin \text{supp } f$,

$$Tf(x) = \int_S K(x,y) f(y) \, d\rho(y).$$
We define the maximal singular integral operator $T^*$ by

$$T^*f(x) \equiv \sup_{R \in \mathcal{R}(x)} |T_Rf(x)| \quad \forall \, f \in L^\infty_c, \, \forall \, x \in S, \quad (4.3)$$

where $T_R$ is the truncated operator defined by

$$T_Rf(x) \equiv \int_{(R^*)^c} K(x,y)f(y) \, d\rho(y) \quad \forall \, x \in S, \, \forall \, R \in \mathcal{R}(x).$$

The main result concerning such maximal singular integrals is the following.

**Theorem 4.1.** Suppose that $T^*$ is the maximal singular integral operator as in (4.3) associated with a kernel $K$ satisfying (4.1) and (4.2). The following statements are equivalent:

(i) $T^*$ is bounded from $L^\infty_c$ to BMO;
(ii) $T^*$ is bounded on $L^p$ for all $p \in (1,\infty)$;
(iii) $T^*$ is bounded on $L^p$ for some $p \in (1,\infty)$;
(iv) $T^*$ is bounded from $L^1$ to $L^{1,\infty}$.

To prove Theorem 4.1, we first establish the following lemma by following some ideas used by Grafakos [12, Lemma 1].

**Lemma 4.1.** Let $T^*$ and $K$ be as in Theorem 4.1. Suppose that $p \in [1,\infty)$, $\lambda > 0$, $b \equiv \sum_{i \in I} b_i$ and $\{R_i\}_{i \in I} \subset \mathcal{R}$ are such that for a fixed positive constant $C_3$ and all $i \in I$, $\text{supp} b_i \subset R_i$, $\int_S b_i \, d\rho = 0$, $\|b_i\|_{L^p} \leq C_3 \lambda [\rho(R_i)]^{1/p}$, and $\{R_i\}_{i \in I}$ are pairwise disjoint. Then

$$\rho \left( \left\{ x \notin \bigcup_{i \in I} R_i^* : T^*b(x) > (C_{\kappa_0} C_3 \nu_1 + 3)\lambda \right\} \right) \leq C_3 \left[ \nu_1 \tilde{C}_{\kappa_0} + 3 \nu_2 \right] \sum_{i \in I} \rho(R_i), \quad (4.4)$$

where $\kappa_0$ is the constant which appears in Lemma 2.1, $C_{\kappa_0} \equiv 3 + \log_2(\kappa_0 + 1)$ and $\tilde{C}_{\kappa_0} \equiv 2 + \log_2[(4\kappa_0 + 3)\kappa_0]$.

**Proof.** For any fixed $x \notin \bigcup_{i \in I} R_i^*$ and $R \in \mathcal{R}(x)$, we set $I_1(x,R) \equiv \{ i \in I : R_i \subset R^* \}$, $I_2(x,R) \equiv \{ i \in I : R_i \cap R^* = \emptyset \}$ and $I_3(x,R) \equiv \{ i \in I : R_i \cap R^* \neq \emptyset, R_i \cap (R^*)^c \neq \emptyset \}$. Then, by the definition of $T_R$,

$$|T_Rb(x)| \leq |T_R \left( \sum_{i \in I_2(x,R)} b_i \right)(x)| + |T_R \left( \sum_{i \in I_3(x,R)} b_i \right)(x)|. \quad (4.5)$$

Denote by $x_i$ the center of $R_i$. By $\int_S b_i \, d\rho = 0$ and supp $b_i \subset R_i$, we obtain

$$\left| T_R \left( \sum_{i \in I_2(x,R)} b_i \right)(x) \right| = \left| \sum_{i \in I_2(x,R)} \int_{R_i} [K(x,y) - K(x,x_i)] b_i(y) \, d\rho(y) \right|$$
\[
\leq \sum_{i \in I} \int_{R_i} |K(x, y) - K(x, x_i)| |b_i(y)|\, d\rho(y) \equiv Z_1(x).
\]

To estimate the second term in the right-hand side of (4.5), for any \(i \in I_3(x, R)\), we set

\[
c_i(R) \equiv \frac{1}{\rho(R_i)} \int_{R_i} b_i(y) \chi_{(R^*)^c}(y)\, d\rho(y).
\]

Notice that

\[
\left| T_R \left( \sum_{i \in I_3(x, R)} b_i \right)(x) \right| = \left| \sum_{i \in I_3(x, R)} \int_{R_i} K(x, y)b_i(y)\chi_{(R^*)^c}(y)\, d\rho(y) \right|
\leq \left| \sum_{i \in I_3(x, R)} \int_{R_i} K(x, y) \left[ b_i(y)\chi_{(R^*)^c}(y) - c_i(R) \right]\, d\rho(y) \right|
+ \left| \sum_{i \in I_3(x, R)} \int_{R_i} K(x, y)c_i(R)\, d\rho(y) \right| \equiv J_1(x) + J_2(x). \tag{4.6}
\]

For all \(i \in I_3(x, R)\), by the hypothesis \(\|b_i\|_{L^p} \leq C_3\lambda \rho(R_i)^{1/p}\), we have \(|c_i(R)| \leq C_3\lambda\).

From this, it follows that

\[
J_1(x) = \left| \sum_{i \in I_3(x, R)} \int_{R_i} [K(x, y) - K(x, x_i)] \left[ b_i(y)\chi_{(R^*)^c}(y) - c_i(R) \right] d\rho(y) \right|
\leq Z_1(x) + C_3\lambda \sum_{i \in I} |K(x, y) - K(x, x_i)| d\rho(y) \equiv Z_1(x) + Z_2(x).
\]

To estimate \(J_2(x)\), set \(I^{(1)}_3(x, R) \equiv \{i \in I_3(x, R) : r_R \geq 4\kappa_0 r_{R_i}\}\) and

\(I^{(2)}_3(x, R) \equiv \{i \in I_3(x, R) : r_R < 4\kappa_0 r_{R_i}\}\).

For any \(i \in I_3(x, R)\), assume that \(w_i \in R_i \cap R^*\) and \(u_i \in R_i \cap (R^*)^c\). If \(i \in I^{(1)}_3(x, R)\), by the assumption that \(x \in R\) and Lemma 2.1(i), we obtain that for all \(y \in R_i\),

\[
d(x, y) \leq d(x, w_i) + d(w_i, y) < (2\kappa_0 + 1)r_R + 2\kappa_0 r_{R_i} < 2(\kappa_0 + 1)r_R
\]

and

\[
d(x, y) \geq d(x, u_i) - d(u_i, y) > r_R - 2\kappa_0 r_{R_i} \geq r_R/2.
\]

These facts together with the pairwise disjointness of \(\{R_i\}_{i \in I}\) yield

\[
\left| \sum_{i \in I^{(1)}_3(x, R)} \int_{R_i} K(x, y)c_i(R)\, d\rho(y) \right| \leq C_3\lambda \sum_{i \in I^{(1)}_3(x, R)} \int_{R_i} |K(x, y)|\, d\rho(y)
\]
\[ \leq C_3 \lambda \int_{r_R/2 < d(x,y) \leq 2(\kappa_0 + 1)r_R} |K(x,y)|\,d\rho(y) \]
\[ < [3 + \log_2(\kappa_0 + 1)]C_3\nu_1 \lambda. \]

If \( i \in I_3^{(2)}(x, R) \) and since \( x \in R \setminus \bigcup_{i \in I} R_i^*, \) for all \( y \in R_i, \)

\[ r_{R_i} < d(x, y) \leq d(x, w_i) + d(w_i, y) \leq (2\kappa_0 + 1)r_R + 2\kappa_0r_{R_i} < 2(4\kappa_0 + 3)\kappa_0r_{R_i}, \]

which implies that

\[ \left| \sum_{i \in I_3^{(2)}(x, R)} \int_{R_i} K(x,y)c_i(R)\,d\rho(y) \right| \leq C_3 \lambda \sum_{i \in I} \int_{r_{R_i} < d(x,y) \leq 2(4\kappa_0 + 3)\kappa_0r_{R_i}} |K(x,y)|\chi_{R_i}(y)\,d\rho(y) \]
\[ \equiv Z_3(x). \]

Therefore, \( J_2(x) \leq [3 + \log_2(\kappa_0 + 1)]C_3\nu_1 \lambda + Z_3(x). \)

Set \( C_{\kappa_0} \equiv 3 + \log_2(\kappa_0 + 1). \) The estimates of \( J_1(x) \) and \( J_2(x) \) together with (4.5) and (4.6) imply that for all \( x \not\in \bigcup_{i \in I} R_i^*, \)

\[ T^*b(x) \leq 2Z_1(x) + Z_2(x) + C_{\kappa_0}C_3\nu_1 \lambda + Z_3(x). \] (4.7)

By this, (4.1) and (4.2), we obtain

\[ \rho \left( \left\{ \frac{1}{2} \sum_{i \in I} \int_{S \setminus R_i^*} \int_{R_i} |K(x,y) - K(x,x_i)||b_i(y)|\,d\rho(y)\,d\rho(x) \right\} \right) \]
\[ \leq \rho \left( \left\{ x \not\in \bigcup_{i \in I} R_i^* : Z_1(x) > \lambda/2 \right\} \right) + \rho \left( \left\{ y \not\in \bigcup_{i \in I} R_i^* : Z_1(x) > \lambda \right\} \right) \]
\[ + \rho \left( \left\{ x \not\in \bigcup_{i \in I} R_i^* : Z_3(x) > \lambda \right\} \right) \]
\[ \leq \frac{2}{\lambda} \sum_{i \in I} \int_{S \setminus R_i^*} \int_{R_i} |K(x,y) - K(x,x_i)||b_i(y)|\,d\rho(y)\,d\rho(x) \]
\[ + C_3 \sum_{i \in I} \int_{S \setminus R_i^*} \int_{R_i} |K(x,y) - K(x,x_i)|\,d\rho(y)\,d\rho(x) \]
\[ + C_3 \sum_{i \in I} \int_{S \setminus R_i^*} \int_{r_{R_i} < d(x,y) \leq 2(4\kappa_0 + 3)\kappa_0r_{R_i}} |K(x,y)|\chi_{R_i}(y)\,d\rho(y)\,d\rho(x) \]
\[ \leq \frac{2\nu_2}{\lambda} \sum_{i \in I} ||b_i||_{L^1} + C_3\nu_2 \sum_{i \in I} \rho(R_i) + C_3\nu_1 \left( 2 + \log_2[(4\kappa_0 + 3)\kappa_0] \right) \sum_{i \in I} \rho(R_i). \] (4.8)

The hypothesis \( ||b_i||_{L^p} \leq C_3\lambda[\rho(R_i)]^{1/p} \) implies that

\[ \sum_{i \in I} ||b_i||_{L^1} \leq \sum_{i \in I} ||b_i||_{L^p} [\rho(R_i)]^{1-1/p} \leq C_3\lambda \sum_{i \in I} \rho(R_i). \]

Combining all these facts yields (4.4). This finishes the proof. \( \square \)
Lemma 4.2. Let $T^*$ and $K$ be as in Theorem 4.1. If $T^*$ is bounded from $L^\infty_c$ to BMO, then $T^*f \in L^{1,\infty}$ for all $f \in L^\infty_{c,0}$.

Proof. Without loss of generality, we may assume that $\text{supp } f \subset R_0 \subset R$. Since $f \in L^\infty_{c,0}$, we have that $T^*f \in \text{BMO}$, which implies the local integrability of $T^*f$ and

$$\sup_{\alpha > 0} \alpha \rho \{x \in (R_0)^* : T^*f(x) > \alpha\} \leq \|T^*f \chi_{(R_0)^*}\|_{L^1} < \infty.$$ 

Therefore, it suffices to prove that

$$\sup_{\alpha > 0} \alpha \rho \{x \notin (R_0)^* : T^*f(x) > \alpha\} < \infty. \quad (4.9)$$

To this end, fix $x \notin (R_0)^*$ and $R \in \mathcal{R}(x)$. If $R_0 \subset R^*$, then $T_R f(x) = 0$. If $R_0 \cap R^* = \emptyset$, then we denote by $x_0$ the center of $R_0$ and use the fact that $\int_S f \, d\rho = 0$ to obtain

$$|T_R f(x)| \leq \int_{R_0} |K(x, y) - K(x, x_0)||f(y)||d\rho(y)| \equiv Y_1(x).$$

If $R_0 \cap R^* \neq \emptyset$ and $R_0 \cap (R^*)^C \neq \emptyset$, we take a Calderón–Zygmund decomposition of $f$ at level $\alpha$ and write $f = g^\alpha + b^\alpha$ with $b^\alpha \equiv \sum_{i \in I_\alpha} b_i^\alpha$, $\|g^\alpha\|_{L^\infty} \leq C_1 \alpha$, supp $b_1^\alpha \subset R_i^\alpha$, $\{R_i^\alpha\}_{i \in I_\alpha} \subset \mathcal{R}$ are mutually disjoint, $\int_S b_i^\alpha \, d\rho = 0$, $\alpha \leq \frac{1}{\mu(R_0^\alpha)} \int_{R_i^\alpha} |f(x)| \, d\rho \leq C_1 \alpha$ and $\|b_i^\alpha\|_{L^1} \leq C_1 \alpha \rho(R_i^\alpha)$, where $I_\alpha$ is a certain index set and $C_1$ is the constant which appears in Proposition 2.2. Then

$$\begin{align*}
|T_R f(x)| &= \left| \int_{R_0 \cap (R^*)^C} K(x, y) f(y) \, d\rho(y) \right| \\
&\leq \left| \int_{R_0 \cap (R^*)^C} K(x, y) g^\alpha(y) \, d\rho(y) \right| + \sum_{i \in I_\alpha} \left| \int_{R_0 \cap (R^*)^C} K(x, y) b_i^\alpha(y) \, d\rho(y) \right| \\
&\equiv Y_2(x) + Y_3(x).
\end{align*}$$

Observe that if $R_0 \cap R^* \neq \emptyset$ and $R_0 \cap (R^*)^C \neq \emptyset$, then for all $y \in R_0 \cap (R^*)^C$,

$$\frac{r_R + r_{R_0}}{2} \leq d(x, y) \leq d(x, y_1) + d(y_1, y) \leq (1 + 2\kappa_0)r_R + 2\kappa_0 r_{R_0} < (2\kappa_0 + 1)(r_R + r_{R_0}),$$

where $y_1$ is some fixed point in $R_0 \cap R^*$. Thus, for all $x \notin (R_0)^*$,

$$Y_2(x) \leq \int_{\frac{r_R + r_{R_0}}{2} \leq d(x, y) < (2\kappa_0 + 1)(r_R + r_{R_0})} |K(x, y)||g^\alpha(y)||d\rho(y)| \leq \nu_1 C_1 \alpha [3 + \log_2(2\kappa_0 + 1)].$$

By proceeding as in the proof of Lemma 4.1, we obtain that for all $x \notin \cup_{i \in I_\alpha} (R_i^\alpha)^*$,

$$Y_3(x) \leq 2Z_1'(x) + Z_2'(x) + C_{\kappa_0} C_1 \nu_1 \alpha + Z_3'(x),$$

where $C_{\kappa_0}$ is as in Lemma 4.1, and

$$Z_i'(x) \equiv \sum_{i \in I_\alpha} \int_{R_i^\alpha} |K(x, y) - K(x, x_i)||b_i^\alpha(y)||d\rho(y)|.$$
By Lemma 2.1(ii), the condition $\alpha > 0$ and hence for all $\alpha > 0$,

$$Z'_2(x) \equiv C_1\alpha \sum_{i \in I_a} \int_{R_i^0} |K(x, y) - K(x, x_i)| \, d\rho(y),$$

Combining all these estimates, we obtain that for all $\alpha > 0$,

$$T^* f(x) \leq Y_1(x) + \nu_1 C_1 [3 + \log_2(2\kappa_0 + 1)] + 2Z'_1(x) + 2Z'_2(x) + C_{\kappa_0} C_1 \nu_1 \alpha + Z'_3(x),$$

and hence for all $\alpha > 0$,

$$\rho(\{ x \notin (R_0)^* : T^* f(x) > (C_{\kappa_0} C_1 + 4 + \nu_1 C_1 [3 + \log_2(2\kappa_0 + 1)]) \alpha \}) \leq \rho(\cup_{i \in I_a} (R_i^0)^*) + \rho(\{ x \notin (R_0)^* : Y_1(x) > \alpha \}) + \rho(\{ x \notin \cup_{i \in I_a} (R_i^0)^* : 2Z'_1(x) + 2Z'_2(x) + Z'_3(x) > 3\alpha \}).$$

By Lemma 2.1(ii), the condition $\alpha \leq \frac{1}{\rho(R_i^0)} \int_{R_i^0} |f| \, d\rho$ and the pairwise disjointness of $\{ R_i^0 \}_{i \in I_a}$, we have $\rho(\cup_{i \in I_a} (R_i^0)^*) \leq \kappa_0 \sum_{i \in I_a} \rho(R_i^0) \leq \kappa_0 \alpha^{-1} \| f \|_{L^1}$, where $\kappa_0$ is the constant which appears in Lemma 2.1. For the second term, applying (4.2) yields

$$\rho(\{ x \notin (R_0)^* : Y_1(x) > \alpha \}) \leq \alpha^{-1} \int_{x \notin (R_0)^*} \int_{(R_0)^*} |K(x, y) - K(x, x_0)| |f(y)| \, d\rho(y) \, d\rho(x) \leq \alpha^{-1} \nu_2 \| f \|_{L^1}.$$

Finally, an argument similar to (4.8) yields that

$$\rho(\{ x \notin \cup_{i \in I_a} (R_i^0)^* : 2Z'_1(x) + 2Z'_2(x) + Z'_3(x) > 3\alpha \}) \leq C_1 \nu_1 \tilde{C}_{\kappa_0} + 3\nu_2 \| f \|_{L^1},$$

where $\tilde{C}_{\kappa_0}$ is the constant which appears in Lemma 4.1. Combining all these estimates gives (4.9). Hence, $T^* f \in L^{1, \infty}$. \qed

**Proof of Theorem 4.1.** To show that (i) implies (ii), by the Marcinkiewicz interpolation theorem (see [13, Theorem 1.4.19]) and the fact that $L^\infty_{c, 0}$ is dense in $L^q$ when $q \in (1, \infty)$ (see [32, Lemma 5.3]), it suffices to show that for any $p \in (1, \infty)$ and all $f \in L^\infty_{c, 0}$,

$$\| T^* f \|_{L^{p, \infty}} \lesssim \| f \|_{L^p}. \quad (4.10)$$

To prove (4.10), we fix $p \in (1, \infty)$ and $f \in L^\infty_{c, 0}$. By applying Proposition 2.2 and Remark 2.1, we know that for any given $\alpha > 0$, there exist a positive constant $C_1$ and a sequence of pairwise disjoint sets $\{ R_i^0 \}_{i \in I_a} \subset \mathcal{R}$ such that $f$ is decomposed into $f = g^\alpha + b^\alpha = g^\alpha + \sum_{i \in I_a} b_i^\alpha$ such that

(a) $|g^\alpha(x)| \leq C_1 \alpha$ for almost all $x \in S$ and $\| g^\alpha \|_{L^\infty} \leq \| f \|_{L^\infty}$;
(b) for all $i \in I_a$, $\supp b_i^\alpha \subset R_i^0 \subset \mathcal{R}$ and $\int_S b_i^\alpha \, d\rho = 0$;
(c) for all $i \in I_a$, $\alpha \leq \frac{1}{\rho(R_i^0)} \int_{R_i^0} |f|^p \, d\rho)^{1/p} \leq C_1 \alpha$;
(d) for all $i \in I_a$, $\| b_i^\alpha \|_{L^p} \leq C_1 \alpha [\rho(R_i^0)]^{1/p}$,
where \( I_\alpha \) is an index set and \( C_1 \) is the constant which appears in Proposition 2.2. By proceeding as in the proof of \([18, (3.5)]\), we obtain that for all \( s \in (0, 1) \), \( \alpha > 0 \), and \( x \in S \),
\[
\mathcal{M}_{0,s}^\alpha(T^*f)(x) = \mathcal{M}_{0,s}^\alpha(T^*(g^\alpha + b^\alpha))(x) \leq \mathcal{M}_{0,s/2}^\alpha(T^*g^\alpha)(x) + \mathcal{M}_{0,s/2}(T^*b^\alpha)(x);
\]
therefore, with \( C_{\kappa_0} \) as in Lemma 4.1 and \( C_4 \equiv \|T^*\|_{L^\infty \to \text{BMO}} \),
\[
\rho(\{x \in S : \mathcal{M}_{0,s}^\alpha(T^*f)(x) > (2C_4C_1s^{-1} + C_{\kappa_0}C_1\nu_1 + 3\alpha)\}) \\
\leq \rho(\{x \in S : \mathcal{M}_{0,s/2}^\alpha(T^*g^\alpha)(x) > 2C_4C_1s^{-1}\alpha\}) \\
+ \rho(\{x \in S : \mathcal{M}_{0,s/2}(T^*b^\alpha)(x) > (C_{\kappa_0}C_1\nu_1 + 3\alpha)\}) = I + II.
\]
By (i), Lemma 3.4(iv) and Property (a), we obtain
\[
\|\mathcal{M}_{0,s/2}^\alpha(T^*g^\alpha)\|_{L^\infty} \leq 2s^{-1}\|T^*g^\alpha\|_{L^\infty} \leq 2s^{-1}\|T^*\|_{L^\infty \to \text{BMO}}\|g^\alpha\|_{L^\infty} \leq 2C_4C_1s^{-1}\alpha,
\]
which implies that \( I = 0 \). By Lemma 3.1(vii), the term \( II \) can be estimated by
\[
II \leq 2\|\mathcal{M}\|_{L^1 \to L^{1,\infty}s^{-1}}\rho(\{x \in S : T^*b^\alpha(x) > (C_{\kappa_0}C_1\nu_1 + 3\alpha)\}),
\]
then, applying Lemma 4.1, Lemma 2.1(ii) and Property (c) yields that
\[
II \lesssim s^{-1}\left[\rho\left(\bigcup_{i \in I_\alpha} (R_i^\alpha)^*\right) + \rho\left(\{x \notin \bigcup_{i \in I_\alpha} (R_i^\alpha)^* : T^*b^\alpha(x) > (C_{\kappa_0}C_3\nu_1 + 3\alpha)\}\right)\right] \\
\lesssim s^{-1}\sum_{i \in I_\alpha} \rho(R_i^\alpha) \lesssim s^{-1}\alpha^{-p}\|f\|_{L^p}.
\]
The estimates of \( I \) and \( II \) above imply that for any given \( s \in (0, 1) \),
\[
\|\mathcal{M}_{0,s}^\alpha(T^*f)\|_{L^{1,\infty}} \lesssim s^{-1/p}\|f\|_{L^p}. \tag{4.11}
\]
By the assumption \( f \in L^{\infty}_{\text{c}} \) and Lemma 4.2, \( T^*f \in L^{1,\infty} \). Then by applying Proposition 3.2 to the function \( T^*f \) and by (4.11), we obtain that for \( s \in (0, 1/2) \) such that \( s < (2^3pC_2)^{-1} \),
\[
\|T^*f\|_{L^{1,\infty}} \leq C\|\mathcal{M}_{0,s}^\alpha(T^*f)\|_{L^{1,\infty}} \leq C\|f\|_{L^p}.
\]
This proves (4.10). Thus, (ii) holds.

It is obvious that (ii) implies (iii). Now we assume that (iii) holds for an index \( p \in (1, \infty) \) and show that (iv) holds. To this end, for any given \( f \in L^1 \) and \( \alpha > 0 \), we use Proposition 2.2 to obtain a sequence of mutually disjoint sets, \( \{R_i^\alpha\}_{i \in I_\alpha} \subset \mathcal{R} \), and a decomposition of \( f \) as \( f = g^\alpha + b^\alpha = g^\alpha + \sum_{i \in I_\alpha} b_i^\alpha \) where \( \|g^\alpha\|_{L^\infty} \lesssim \alpha \), every \( b_i^\alpha \) is supported on \( R_i^\alpha \) and has integral 0, \( \frac{1}{\rho(R_i^\alpha)} \int_{R_i^\alpha} |f| \, dp \approx \alpha \) and \( \|b_i^\alpha\|_{L^1} \lesssim \alpha \rho(R_i^\alpha) \). By Lemma 4.1 and Lemma 2.1(ii), there exists a sufficiently large positive constant \( C \) such that for all \( \alpha > 0 \),
\[
\rho\left(\{x \in S : T^*f(x) > (C+1)\alpha\}\right)
\]
Lemma 2.1(ii), we obtain

\[
\leq \rho(\{x \in S : T^* g^\alpha(x) > \alpha\}) + \rho \left( \bigcup_{i \in I_\alpha} (R_i^\alpha)^* \right) + \rho \left( \left\{ x \notin \bigcup_{i \in I_\alpha} (R_i^\alpha)^* : T^* b^\alpha(x) > C\alpha \right\} \right) \leq \alpha^{-p}\|T^* g^\alpha\|_{L^p}^p + \sum_{i \in I_\alpha} \rho(R_i^\alpha).
\]

Notice that \(\sum_{i \in I_\alpha} \rho(R_i^\alpha) \lesssim \alpha^{-1}\|f\|_{L^1}\). Using the \(L^p\)-boundedness of \(T^*\) by (iii) and the properties of \(g^\alpha\), we have

\[
\alpha^{-p}\|T^* g^\alpha\|_{L^p}^p \lesssim \alpha^{-p}\|T^*\|_{L^p \rightarrow L^p}^p\|g^\alpha\|_{L^p}^p \lesssim \alpha^{-1}\|g^\alpha\|_{L^1} \lesssim \alpha^{-1}\|f\|_{L^1}.
\]

Combining all the above estimates yields that \(\|T^* f\|_{L^1,\infty} \lesssim \|f\|_{L^1}\). Hence, (iv) holds.

Finally, we show that (iv) implies (i). Fix \(\sigma \in (0,1)\). By Lemma 3.3, it suffices to prove that for all \(f \in L^\infty_c\), \(\|T^* f\|_{L^\infty,\sigma} \lesssim \|f\|_{L^\infty}\). To this end, for any given \(f \in L^\infty_c\) and \(R \in \mathcal{R}\), we decompose \(f\) into \(f = f \chi_{R^*} + f \chi_{R \setminus R^*} \equiv f_1 + f_2\). Notice that for all \(c \in \mathbb{C}\) and \(x \in S\), \(|T^* f(x) - c| \leq T^* f_1(x) + |T^* f_2(x) - c|\). Then, for all \(R \in \mathcal{R}(x)\),

\[
\inf_{c \in \mathbb{C}} \frac{1}{\rho(R)} \int_R |T^* f(x) - c|^\sigma \, d\rho(x) \leq \frac{1}{\rho(R)} \int_R |T^* f_1(x)|^\sigma \, d\rho(x) + \inf_{c \in \mathbb{C}} \frac{1}{\rho(R)} \int_R |T^* f_2(x) - c|^\sigma \, d\rho(x) \equiv Z_1 + Z_2. \quad (4.12)
\]

Using \(\sigma \in (0,1)\) and the hypothesis that \(T^*\) is bounded from \(L^1\) to \(L^{1,\infty}\) together with Lemma 2.1(ii), we obtain

\[
Z_1 = \frac{1}{\rho(R)} \int_0^\infty \sigma t^{\sigma-1} \rho(\{x \in R : T^* f_1(x) > t\}) \, dt \\
\leq \frac{1}{\rho(R)} \left\{ \int_0^{\|f\|_{L^\infty}} \sigma t^{\sigma-1} \rho(R) \, dt + \int_0^\infty \sigma t^{\sigma-1} \frac{\|T^*\|_{L^1 \rightarrow L^{1,\infty}}}{t} \frac{\|f\|_{L^1}}{t} \, dt \right\} \lesssim \|f\|_{L^\infty}^\sigma.
\]

By this and (4.12), the proof of (i) is reduced to the estimate \(Z_2 \lesssim \|f\|_{L^\infty}^\sigma\). Since \(f_2 \in L^\infty_c\), an argument similar to the one used for \(Z_1\) yields \(|T^* f_2|^\sigma \in L^1_{\text{loc}}\); thus there exists some \(z_R \in R\) such that \(T^* f_2(z_R) < \infty\). Notice that for all \(x \in R\) and \(\hat{R} \in \mathcal{R}(x)\), Lemma 2.1(i) implies that \(\{y \in S : d(y, x) > r_{\hat{R}}\} = (\hat{R}^*)^c \cup \{y \in \hat{R}^* : d(y, x) > r_{\hat{R}}\}\). From this, it follows that for all \(x \in R\), we can write \(T^* f_2(x)\) as follows:

\[
T^* f_2(x) = \sup_{\hat{R} \in \mathcal{R}(x)} \left| \int_{d(y, x) > r_{\hat{R}}} K(x, y) f_2(y) \, d\rho(y) \right| - \int_{d(y, x) > r_{\hat{R}}} K(x, y) f_2(y) \, d\rho(y) \right|.
\]

(4.13)
To see this, by symmetry, it suffices to show that 

\[ T^*f_2(x) - T^*f_2(z_R) \]

In particular, the equality (4.13) holds for 

\[ T^*f_2(z_R) \]. Thus, for all \( x \in R \),

\[
\begin{align*}
|T^*f_2(x) - T^*f_2(z_R)| & \leq \sup_{\tilde{R} \in \mathcal{R}(x)} \left| \int_{d(y,x) > r_{\tilde{R}}} K(x,y) f_2(y) \, d\rho(y) \right| - \sup_{\tilde{R} \in \mathcal{R}(z_R)} \left| \int_{d(y,z_R) > r_{\tilde{R}}} K(z_R,y) f_2(y) \, d\rho(y) \right| \\
& + \sup_{\tilde{R} \in \mathcal{R}(x)} \left| \int_{d(y,x) > r_{\tilde{R}}} K(x,y) f_2(y) \, d\rho(y) \right| + \sup_{\tilde{R} \in \mathcal{R}(z_R)} \left| \int_{d(y,z_R) > r_{\tilde{R}}} K(z_R,y) f_2(y) \, d\rho(y) \right| \\
& \equiv L_1 + L_2 + L_3.
\end{align*}
\]

To see this, by symmetry, it suffices to show that 

\[ T^*f_2(x) - T^*f_2(z_R) \leq L_1 + L_2 + L_3, \]

which follows by first writing \( T^*f_2(x) \) and \( T^*f_2(z_R) \) as in (4.13), then applying

\[
\sup_{i \in \Lambda} |a_i - b_i| \leq \sup_{i \in \Lambda} |a_i| + \sup_{i \in \Lambda} |b_i|
\]

to the expression of \( T^*f_2(x) \), and \( \sup_{i \in \Lambda} |a_i - b_i| \geq \sup_{i \in \Lambda} |a_i| - \sup_{i \in \Lambda} |b_i| \) in the expression of \( T^*f_2(z_R) \), where \( \Lambda \) denotes an index set which might be uncountable.

When \( d(y,x) > r_{\tilde{R}} \) and \( y \in \tilde{R}^* \), by \( x \in R \cap \tilde{R} \) and Lemma 2.1(i), we have \( r_{\tilde{R}} < d(y,x) \leq (2\kappa_0 + 1)r_{\tilde{R}} \), which combined with (4.1) implies that \( L_2 \lesssim \nu_1 \|f\|_{L^\infty} \). Similarly, \( L_3 \lesssim \nu_1 \|f\|_{L^\infty} \). To estimate \( L_1 \), by the properties of Calderón–Zygmund sets, we obtain

\[
\sup_{\tilde{R} \in \mathcal{R}(x)} \left| \int_{d(y,x) > r_{\tilde{R}}} K(x,y) f_2(y) \, d\rho(y) \right| = \sup_{\epsilon > 0} \left| \int_{d(y,x) > \epsilon} K(x,y) f_2(y) \, d\rho(y) \right|.
\]

By this and the inequality \( \sup_{i \in \Lambda} |a_i - b_i| \geq \sup_{i \in \Lambda} |a_i| - \sup_{i \in \Lambda} |b_i| \), we obtain that

\[
L_1 \leq \sup_{\epsilon > 0} \left| \int_{d(y,x) > \epsilon} K(x,y) f_2(y) \, d\rho(y) - \int_{d(y,z_R) > \epsilon} K(z_R,y) f_2(y) \, d\rho(y) \right| \\
\leq \sup_{\epsilon > 0} \left| \int_{d(y,x) > \epsilon, d(z_R,y) > \epsilon} K(x,y) - K(z_R,y) |f_2(y)| \, d\rho(y) \right| \\
+ \sup_{\epsilon > 0} \left| \int_{d(y,x) > \epsilon, d(z_R,y) > \epsilon} |K(x,y) f_2(y)| \, d\rho(y) \right| \\
+ \sup_{\epsilon > 0} \left| \int_{d(y,x) > \epsilon, d(z_R,y) > \epsilon} |K(z_R,y) f_2(y)| \, d\rho(y) \right| = J_1 + J_2 + J_3.
\]

From (4.2) and \( \text{supp} f_2 \subset (R^*)^0 \), it follows that \( J_1 \lesssim \nu_2 \|f\|_{L^\infty} \). If \( x \in R \), \( y \notin R^* \) and \( d(x,y) > \epsilon \geq d(z_R,y) \), by Lemma 2.1(i), we have \( r_R < d(y,R) \leq d(y,z_R) \leq \epsilon \) and \( d(x,y) \leq d(x,z_R) + d(z_R,y) < 2\kappa_0 r_R + \epsilon < (2\kappa_0 + 1)\epsilon \),

which together with (4.1) implies that \( J_2 \lesssim \nu_1 \|f\|_{L^\infty} \). Similarly, \( J_3 \lesssim \nu_1 \|f\|_{L^\infty} \). Thus, \( L_1 = \sum_{i=1}^3 J_i \lesssim \|f\|_{L^\infty} \). Combining the estimates of \( L_1, L_2 \) and \( L_3 \) yields \( Z_2 \lesssim \|f\|_{L^\infty}^2 \).

This finishes the proof of (iv) implies (i), and hence the proof of Theorem 4.1. \(\square\)
Applying Theorem 4.1 and the Calderón–Zygmund decomposition, we obtain the following result.

**Theorem 4.2.** Let $T$ be the integral operator associated with a kernel $K$ satisfying (4.1) and (4.2). If $T$ is bounded on $L^2$, then the maximal singular integral $T^*$ defined as in (4.3) is bounded from $L^1$ to $L^{1,\infty}$, from $L^p$ to $L^p$ for all $p \in (1, \infty)$, and from $L^\infty$ to $\text{BMO}$.

To prove Theorem 4.2, we need the following Cotlar-type inequality.

**Lemma 4.3.** Under the assumptions of Theorem 4.2, for all $g \in L^\infty_c$ and $x \in S$,

$$T^* g(x) \leq M(T g)(x) + \left[ \nu_2 + \kappa_0^{1/2} \|T\|_{L^2 \to L^2} \right] \|g\|_{L^\infty},$$  \hspace{1cm} (4.14)

where $M$ is the Hardy–Littlewood maximal operator defined in (2.1).

**Proof.** We only give an outline of the proof because of its similarity to the argument used in [12, Theorem 1]; see also [13, p. 295]. Indeed, for all $x \in S$, $R \in \mathcal{R}(x)$, $z \in R$ and all $g \in L^\infty_c$, we use Hörmander’s condition (4.2) to obtain

$$|T_R g(x)| \leq |T(g \chi_{(R \cap \mathbb{Z})})(x)| - |T(g \chi_{(R \cap \mathbb{Z})})(z)| + |T g(z)| + |T(g \chi_{R^*})(z)|$$

Taking the integral average over $R$ with respect to the variable $z$ in both sides of this inequality yields that

$$|T_R g(x)| \leq \nu_2 \|g\|_{L^\infty} + \frac{1}{\rho(R)} \int_R |T g(z)| \, d\rho(z) + \frac{1}{\rho(R)} \int_R |T(g \chi_{R^*})(z)| \, d\rho(z).$$

By (2.1), Hölder’s inequality, the $L^2$-boundedness of $T$ and Lemma 2.1(ii), we obtain

$$|T_R g(x)| \leq \nu_2 \|g\|_{L^\infty} + M(T g)(x) + \left( \frac{1}{\rho(R)} \int_R |T(g \chi_{R^*})(z)|^2 \, d\rho(z) \right)^{1/2}$$

$$\leq M(T g)(x) + \left[ \nu_2 + \kappa_0^{1/2} \|T\|_{L^2 \to L^2} \right] \|g\|_{L^\infty},$$

which completes the proof. \hfill $\square$

**Proof of Theorem 4.2.** By Theorem 4.1, it suffices to show that $T^*$ is bounded from $L^1$ to $L^{1,\infty}$. To this end, for any $f \in L^1$ with bounded support and $\alpha > 0$, we decompose $f$ at level $\alpha$ into $f = g^\alpha + b^\alpha = g^\alpha + \sum_{i \in I_\alpha} b_i^\alpha$, where $I_\alpha$ is a certain index set, $\|g^\alpha\|_{L^\infty} \leq C_1 \alpha$, $\text{supp} b_i^\alpha \subset R_i^\alpha$, $\{R_i^\alpha\}_{i \in I_\alpha} \subset \mathcal{R}$ are mutually disjoint, $\int_S b_i^\alpha \, d\rho = 0$, $\alpha \leq \frac{1}{\rho(R^\alpha_i)} \int_{R^\alpha_i} |f| \, d\rho \leq C_1 \alpha$ and $\|b_i^\alpha\|_{L^1} \leq C_1 \alpha \rho(R^\alpha_i)$. Here $C_1$ is the constant which appears in Proposition 2.2. For $C_0 > C_{\kappa_0} C_1 \nu_1 + 3$ sufficiently large, which will be determined later, we have

$$\rho \left( \{ x \in S : T^* f(x) > C_0 \alpha \} \right) \leq \rho \left( \{ x \in S : T^* g^\alpha(x) > (C_0 - C_{\kappa_0} C_1 \nu_1 - 3)\alpha \} \right) + \rho \left( \{ x \in S : T^* b_i^\alpha(x) > (C_{\kappa_0} C_1 \nu_1 + 3)\alpha \} \right) \equiv Z_1 + Z_2.$$
The aim of this section is to apply the results in Section 4 to the multipliers of a distinguished Laplacian $\Delta$ on $(S, d, \rho)$. Let us begin with some known facts related to the integration formulae and spherical analysis on $S$; for details we refer the reader to [1, 2, 6, 7, 17].

A radial function on $S$ is a function that depends only on the distance from the identity. If $f$ is radial and $f \in C_c^\infty(S)$, then we have the following integration formula:

$$
\int_S f(x) \, d\lambda(x) = C \int_0^\infty f(r) A(r) \, dr,
$$

where $C$ is a positive constant depending only on $S$, $A(r) = 4^n \sinh^n \left( \frac{r}{2} \right) \cosh^n \left( \frac{r}{2} \right)$ for all $r > 0$ and $\lambda$ denotes the left Haar measure. One easily checks that

$$
A(r) \lesssim \left( \frac{r}{1 + r} \right)^n e^{nr} \quad \forall \ r > 0.
$$

A radial function $\phi$ is spherical if it is an eigenfunction of the Laplace-Beltrami operator $\mathcal{L} = -\text{div} \cdot \text{grad}$ and $\phi(0) = 1$. For $s \in \mathbb{C}$, let $\phi_s$ be the spherical function with eigenvalue $s^2 + n^2/4$. It is known ([2]) that the spherical function $\phi_0$ satisfies the estimate

$$
0 < \phi_0(r) \lesssim (1 + r)e^{-nr/2} \quad \forall \ r > 0,
$$

and that for every radial function $f \in C_c^\infty(S)$,

$$
\int_S \delta^{1/2}(x) f(x) \, d\rho(x) = \int_0^\infty \phi_0(r) f(r) A(r) \, dr.
$$

The spherical Fourier transform of a radial function $f$ in $L^1(\lambda)$ is defined by the formula

$$
\mathcal{H}f(s) = \int_S \phi_s(x) f(x) \, d\lambda(x) \quad \forall \ s \in \mathbb{C}.
$$

For radial functions $f \in C_c^\infty(S)$, a Plancherel formula holds:

$$
\int_S |f(x)|^2 \, d\lambda(x) = C \int_0^\infty |\mathcal{H}f(s)|^2 |c(s)|^{-2} \, ds,
$$

where $c(s)$ is the spherical function with eigenvalue $s$.
where $C$ is a positive constant depending only on $S$, and $|c(s)|^{-2} ds$ denotes the Plancherel measure which satisfies the following estimates (see Chapter IV of [17]):

$$
|c(s)|^{-2} \leq \begin{cases} 
|C|s|^2 & \text{if } |s| \leq 1, \\
|C|s^n & \text{if } |s| > 1,
\end{cases}
$$

(5.6)

where $C$ is a positive constant independent of $s$. In particular, when $n = 1$, the estimate (5.6) becomes

$$
|c(s)|^{-2} \leq C \min\{|s|^2, |s|\} \quad \forall s \in \mathbb{R}^+,
$$

(5.7)

where $C$ is a positive constant independent of $s$.

Denote by $\mathcal{A}$ the Abel transform and by $\mathcal{A}^{-1}$ the inverse Abel transform. If $n$ is even, then

$$
\mathcal{A}^{-1} f(r) = (2\pi)^{-n/2} \left( -\frac{1}{\sinh r} \frac{\partial}{\partial r} \right)^{n/2} f(r) \quad \forall r > 0,
$$

(5.8)

and if $n$ is odd, then for all $r > 0$,

$$
\mathcal{A}^{-1} f(r) = (2\pi)^{-n/2} \int_r^\infty \left[ \left( -\frac{1}{\sinh s} \frac{\partial}{\partial s} \right)^{(n+1)/2} f(s) \right] (\cosh s - \cosh r)^{-1/2} \sinh s ds.
$$

(5.9)

Denote by $\mathcal{F}(g)$ or $\hat{g}$ the Fourier transform of $g$ on $\mathbb{R}$, namely, $\mathcal{F}g(s) = \int_{\mathbb{R}} g(r) e^{-irs} dr$. It is known that $\mathcal{H} = \mathcal{F} \circ \mathcal{A}$, and hence $\mathcal{H}^{-1} = \mathcal{A}^{-1} \circ \mathcal{F}^{-1}$.

Consider the following basis of left-invariant vector fields of the Lie algebra of $S$:

$$
X_0 \equiv a\partial_a, \quad X_i \equiv a\partial_{x_i}, \quad i = 1, 2, \ldots, n.
$$

The Laplacian $\Delta \equiv -\sum_{i=0}^{n} X_i^2$ is a left-invariant essentially selfadjoint operator on $L^2(\rho)$. The operator $\Delta$ has a special relationship with the Laplace–Beltrami operator $\mathcal{L}$ associated with the Riemannian structure of $S$. Indeed, if we denote by $\mathcal{L}_n$ the shifted operator $\mathcal{L} - n^2/4$, it is known that

$$
\delta^{-1/2} \Delta \delta^{1/2} f = \mathcal{L}_n f
$$

(5.10)

for all smooth radial functions $f$ on $S$ (see [2]), where $\delta$ denotes the modular function. The spectra of both $\Delta$ on $L^2(\rho)$ and $\mathcal{L}_n$ on $L^2(\lambda)$ are $[0, \infty)$. Let $E_\Delta$ and $E_{\mathcal{L}_n}$ be the spectral resolutions of identity for which $\Delta = \int_0^\infty t dE_\Delta(t)$ and $\mathcal{L}_n = \int_0^\infty t dE_{\mathcal{L}_n}(t)$. For each bounded measurable function $m$ on $\mathbb{R}^+$, the operators $m(\Delta)$ and $m(\mathcal{L}_n)$, spectrally defined by

$$
m(\Delta) = \int_0^\infty m(t) dE_\Delta(t) \quad \text{and} \quad m(\mathcal{L}_n) = \int_0^\infty m(t) dE_{\mathcal{L}_n}(t),
$$

are respectively bounded on $L^2(\rho)$ and $L^2(\lambda)$ by the spectral theorem. By (5.10) and the spectral theorem, we see that for all radial functions $f \in C_c^\infty(S)$,

$$
\delta^{-1/2} m(\Delta) \delta^{1/2} f = m(\mathcal{L}_n) f.
$$
Denote by \( k_{m(\Delta)} \) the convolution kernel of \( m(\Delta) \), namely, for all \( f \in C_c^\infty(S) \),

\[
m(\Delta)f(x) = \int_S f(xy^{-1})k_{m(\Delta)}(y)\,d\rho(y) \quad \forall \ x \notin \text{supp} \, f.
\]

As in (5.11), denote by \( k_{m(\mathcal{L}_n)} \) the convolution kernel of \( m(\mathcal{L}_n) \). It was proved in [1, 2] that for all bounded measurable function \( m \) on \( \mathbb{R}^+ \), the convolution kernel \( k_{m(\mathcal{L}_n)} \) is radial,

\[
k_{m(\Delta)} = \delta^{1/2}k_{m(\mathcal{L}_n)} \text{ and } \mathcal{H}k_{m(\mathcal{L}_n)}(s) = m(s^2) \quad \forall \ s \in \mathbb{R}^+.
\]

Let \( K_{m(\Delta)} \) be the integral kernel of \( m(\Delta) \), namely, for all \( f \in C_c^\infty(S) \),

\[
m(\Delta)f(x) = \int_S K_{m(\Delta)}(x, y)f(y)\,d\rho(y) \quad \forall \ x \notin \text{supp} \, f.
\]

In view of (5.11) and (5.13), by changing variables and using the left-invariant property of \( \lambda \) and the right-invariant property of \( \rho \), we obtain that for all \( x, y \in S \),

\[
K_{m(\Delta)}(x, y) = k_{m(\Delta)}(y^{-1}x)\delta(y). \quad (5.14)
\]

For any \( s \in (0, \infty) \), we denote by \( H^s(\mathbb{R}) \) the Sobolev space \( W^{s, 2}(\mathbb{R}) \) of order \( s \) on \( \mathbb{R} \). Let \( \phi \in C_c^\infty(\mathbb{R}^+) \) be a function supported in \([1/4, 4]\) such that

\[
\sum_{j \in \mathbb{Z}} \phi(2^{-j}t) = 1 \quad \forall \ t \in \mathbb{R}^+. \quad (5.15)
\]

For any given \( s_0, s_\infty \in \mathbb{R}^+ \), a bounded measurable function \( m \) on \( \mathbb{R}^+ \) is said to satisfy a mixed Mihlin-Hörmander condition of order \((s_0, s_\infty)\) if

\[
\|m\|_{s_0} \equiv \sup_{t < 1} \|m(t \cdot)\phi(\cdot)\|_{H^{s_0}(\mathbb{R})} < \infty \quad \text{and} \quad \|m\|_{s_\infty} \equiv \sup_{t \geq 1} \|m(t \cdot)\phi(\cdot)\|_{H^{s_\infty}(\mathbb{R})} < \infty.
\]

For any \( j \in \mathbb{Z} \) and any bounded measurable function \( m \) on \( \mathbb{R}^+ \), we define \( m_j \) by

\[
m_j(t) \equiv m(2^jt)\phi(t) \quad \forall \ t \in \mathbb{R}^+. \quad (5.16)
\]

Obviously, we have

\[
m(\Delta) = \sum_{j \in \mathbb{Z}} m_j(2^{-j}\Delta). \quad (5.17)
\]

As in (5.11) and (5.13), we denote by \( k_{m_j(2^{-j}\Delta)} \) and \( K_{m_j(2^{-j}\Delta)} \) the convolution kernel and the integral kernel of \( m_j(2^{-j}\Delta) \), respectively.

Assume that \( m \) satisfies a mixed Mihlin-Hörmander condition of order \((s_0, s_\infty)\) with \( s_0 > 3/2 \) and \( s_\infty > \max\{3/2, (n+1)/2\} \). Choose \( \sigma > 0 \) small enough such that \( s_0 > 3/2 + \sigma \) and \( s_\infty > \max\{3/2, (n+1)/2\} + \sigma \). Hebisch and Steger [16, Theorems 2.4 and 6.1] proved that there exists a positive constant \( C \) such that for all \( j \in \mathbb{Z} \) and \( y \in S \),

\[
\int_S |K_{m_j(2^{-j}\Delta)}(x, y)|(1 + 2^{j/2}d(x, y))^{\sigma} \,d\rho(x) \leq \begin{cases} C\|m\|_{s_0} & \text{if } j \leq 0, \\ C\|m\|_{s_\infty} & \text{if } j > 0; \end{cases} \quad (5.18)
\]
and that for all \( y, z \in S \),

\[
\int_S |K_{m_j(2^{-j} \Delta)}(x, y) - K_{m_j(2^{-j} \Delta)}(x, z)| \, d\rho(x)
\leq \begin{cases} 
C 2^{j/2} d(y, z) \|m\|_{s_0} & \text{if } j \leq 0, \\
C 2^{j/2} d(y, z) \|m\|_{s_\infty} & \text{if } j > 0.
\end{cases}
\tag{5.19}
\]

From (5.18) and (5.19), it is easy to deduce that

\[
\sup_{R \in \mathcal{R}} \sup_{y \in R} \int_{S \setminus R^*} |K_{m(\Delta)}(x, y) - K_{m(\Delta)}(x, z)| \, d\rho(x) < \infty; \tag{5.20}
\]

see [16, Remark 1.4]. From the proofs of [16, Theorems 2.4 and 6.1], it follows that (5.18) and (5.19) still hold if we interchange the two variables of \( K_{m_j(2^{-j} \Delta)} \). Thus, \( K_{m(\Delta)} \) satisfies Hörmander’s condition (4.2).

The estimates (5.18) and (5.19) imply that the operator \( m(\Delta) \) is bounded from \( L^1 \) to \( L^{1, \infty} \) and bounded on \( L^p \) for all \( p \in (1, \infty) \) [16, Theorem 2.4] and that it is also bounded from \( H^1 \) to \( L^1 \) and from \( L^{\infty} \) to BMO [32, Proposition 2.4].

We now consider the boundedness of the maximal singular integral operator \( (m(\Delta))^* \) as defined in (4.3).

**Theorem 5.1.** Let \( m \) satisfy a mixed Mihlin-Hörmander condition of order \( (s_0, s_\infty) \) with \( s_0 > 3/2 \) and \( s_\infty > \max\{3/2, (n + 1)/2 \} \). Then the maximal singular integral \( (m(\Delta))^* \) is bounded from \( L^1 \) to \( L^{1, \infty} \), from \( L^{\infty} \) to BMO and bounded on \( L^p \) for all \( p \in (1, \infty) \).

Theorem 5.1 follows immediately from Theorem 4.2 if we know that \( K_{m(\Delta)} \) satisfies (4.1) and (4.2). We already noticed that the kernel \( K_{m(\Delta)} \) satisfies Hörmander’s condition (4.2). Condition (4.1) for \( K_{m(\Delta)} \) is equivalent to the following estimate:

\[
\sup_{\epsilon > 0} \int_{\epsilon < d(y, e) \leq 2\epsilon} |k_{m(\Delta)}(y)| \, d\rho(y) < \infty. \tag{5.21}
\]

To see this, by using the right-invariant property of \( \rho \) and the left-invariant property of \( \lambda \), we have that for all \( x, y \in S \),

\[
\delta(y) d\rho(y) = d\lambda(y) = d\rho(y^{-1}) = d\rho(y^{-1} x) \quad \text{and} \quad d(x, y) = d(y^{-1} x, e).
\]

We then apply (5.14) to obtain

\[
\int_{\epsilon < d(x, y) \leq 2\epsilon} |K_{m(\Delta)}(x, y)| \, d\rho(y) = \int_{\epsilon < d(x, y) \leq 2\epsilon} |k_{m(\Delta)}(y^{-1} x)| \delta(y) \, d\rho(y)
= \int_{\epsilon < d(y^{-1} x, e) \leq 2\epsilon} |k_{m(\Delta)}(y^{-1} x)| \, d\rho(y^{-1} x)
= \int_{\epsilon < d(y, e) \leq 2\epsilon} |k_{m(\Delta)}(y)| \, d\rho(y).
\]
Similarly, by (5.14) and
\[
\delta(y)\rho(x) = \delta(y)[\delta(x)]^{-1}d\lambda(x) = [\delta(y^{-1}x)]^{-1}d\lambda(y^{-1}x) = \delta(y^{-1}x)\rho(y^{-1}x),
\]
we have
\[
\int_{\epsilon<d(x,e)\leq2\epsilon} |K_{m(\triangle)}(x,y)| \, d\rho(x) = \int_{\epsilon<d(x,y)\leq2\epsilon} |k_{m(\triangle)}(y^{-1}x)|\delta(y) \, d\rho(x)
= \int_{\epsilon<d(y^{-1}x,e)\leq2\epsilon} |k_{m(\triangle)}(y^{-1}x)| \, d\rho(y^{-1}x)
= \int_{\epsilon<d(y,e)\leq2\epsilon} |k_{m(\triangle)}(y)| \, d\rho(y).
\]
Therefore, it suffices to show (5.21). To this end, we will use some ideas from the proof of [34, Theorem 4.3]. Let us start with an integral estimate of the kernel which is more delicate than the one proved in [34, Lemma 4.5].

**Lemma 5.1.** Let \( f \) be a bounded even function on \( \mathbb{R} \) such that \( \text{supp} \hat{f} \subset [-r, r] \). Then, there exists a positive constant \( C \) independent of \( r \) such that \( k_{f(\sqrt{\triangle})} \) satisfies the following:

(i) if \( \epsilon \geq r \), then \( \int_{\epsilon<d(x,e)\leq2\epsilon} |k_{f(\sqrt{\triangle})}(x)| \, d\rho(x) = 0 \);

(ii) for all \( \epsilon \in (0, 1) \) such that \( \epsilon < r \),
\[
\int_{\epsilon<d(x,e)\leq2\epsilon} |k_{f(\sqrt{\triangle})}(x)| \, d\rho(x) \leq Ce^{(1+1)/2} \left\{ \int_{0}^{\infty} |f(s)|^2 (s^2 + s^n) \, ds \right\}^{1/2}; \quad (5.22)
\]

(iii) for all \( \epsilon \in [1, \infty) \) such that \( \epsilon < r \),
\[
\int_{\epsilon<d(x,e)\leq2\epsilon} |k_{f(\sqrt{\triangle})}(x)| \, d\rho(x) \leq Ce^{3/2} \left\{ \int_{0}^{\infty} |f(s)|^2 (s^2 + s^n) \, ds \right\}^{1/2}; \quad (5.23)
\]

(iv) when \( n = 1 \), the right-hand sides of (5.22) and (5.23) can be respectively replaced by the better estimates: \( Ce^{(1+1)/2} \left\{ \int_{0}^{\infty} |f(s)|^2 \min\{s, s^2\} \, ds \right\}^{1/2} \) and
\[
Ce^{3/2} \left[ \int_{0}^{\infty} |f(s)|^2 \min\{s, s^2\} \, ds \right]^{1/2}.
\]

**Proof.** Let \( k_{f(\sqrt{\triangle})} \) be the convolution kernel of the operator \( f(\sqrt{\triangle}) \). By (5.12), we know that \( k_{f(\sqrt{\triangle})} \) is radial on \( S \), \( k_{f(\sqrt{\triangle})} = \delta^{1/2}k_{f(\sqrt{\triangle})} \) and \( \mathcal{H}k_{f(\sqrt{\triangle})}(t) = f(t) \) for all \( t \in \mathbb{R}^+ \).

Since \( \mathcal{H}^{-1} = \mathcal{A}^{-1} \circ \mathcal{F}^{-1} \) and \( f \) is even, we obtain
\[
k_{f(\sqrt{\triangle})}(t) = \mathcal{H}^{-1}f(t) = \mathcal{A}^{-1}\mathcal{F}^{-1}f(t) = \mathcal{A}^{-1}(\hat{f})(t). \quad (5.24)
\]

Notice that \( \text{supp} \hat{f} \subset [-r, r] \) and the inverse formulae for the Abel transform (5.8) and (5.9) imply that \( \text{supp} \mathcal{A}^{-1}(\hat{f}) \subset B(e, r) \). Then, \( \text{supp} k_{f(\sqrt{\triangle})} \subset B(e, r) \), and hence
supp $k_{f(\sqrt{\Delta})} \subset B(\epsilon, r)$. So the integral of $k_{f(\sqrt{\Delta})}$ in the domain $\{x \in S : \epsilon < d(x, e) \leq 2\epsilon\}$ is 0 when $\epsilon \geq r$. This proves (i).

To prove (ii) and (iii), we set $w(x) \equiv \delta^{-1/2}(x)e^{nd(x, e)/2}$ for all $x \in S$. Applying Hölder’s inequality yields that

$$
\int_{\epsilon < d(x, e) \leq 2\epsilon} |k_{f(\sqrt{\Delta})}(x)| \, d\rho(x)
\leq \left\{ \int_{\epsilon < d(x, e) \leq 2\epsilon} w(x)^{-1} \, d\rho(x) \right\}^{1/2} \left\{ \int_{\epsilon < d(x, e) \leq 2\epsilon} |k_{f(\sqrt{\Delta})}(x)|^2 w(x) \, d\rho(x) \right\}^{1/2}
= 1^{1/2} \cdot J^{1/2},
$$

where we denoted by I and J respectively the integral in the first and second bracket. Recall that if $x = (y, a) \in S$ with $y \in \mathbb{R}^n$ and $a \in \mathbb{R}^+$, then $\delta(x) = \delta(y, a) = a^{-n}$. When $\epsilon \in (0, 1]$, for all $d(x, e) \leq 2\epsilon$, we have $|\delta(x)| \lesssim \epsilon \lesssim 1$, and hence

$$
I = \int_{\epsilon < d(x, e) \leq 2\epsilon} \delta^{1/2}(x)e^{-nd(x, e)/2} \, d\rho(x) \lesssim \rho(B(e, 2\epsilon)) \lesssim \epsilon^{n+1}.
$$

When $\epsilon > 1$, by (5.4) together with the estimates (5.3) and (5.2) of $\phi_0$ and of the density function $A$, we obtain

$$
I \leq \int_{d(x, e) \leq 2\epsilon} \delta^{1/2}(x)e^{-nd(x, e)/2} \, d\rho(x)
= \int_0^{2\epsilon} \phi_0(t)e^{-nt/2}A(t) \, dt \lesssim \int_0^{2\epsilon} (1 + t)^n dt \lesssim \epsilon^2.
$$

To estimate J, since $k_{f(\sqrt{\Delta})} = \delta^{1/2}k_{f(\sqrt{\Delta})}$, we have

$$
J = \int_{\epsilon < d(x, e) \leq 2\epsilon} \delta(x)|k_{f(\sqrt{\Delta})}(x)|^2 \delta^{-1/2}(x)e^{nd(x, e)/2} \, d\rho(x).
$$

Again, using (5.4) and the estimates (5.3) and (5.2), we estimate J by

$$
J = \int_{\epsilon}^{2\epsilon} \phi_0(t)|k_{f(\sqrt{\Delta})}(t)|^2 e^{nt/2}A(t) \, dt
\lesssim (1 + \epsilon) \int_{\epsilon}^{2\epsilon} |k_{f(\sqrt{\Delta})}(t)|^2 A(t) \, dt \lesssim (1 + \epsilon) \int_S |k_{f(\sqrt{\Delta})}(x)|^2 \, d\lambda(x),
$$

where the last inequality is due to (5.1). Applying the Plancherel formula (5.5) and the estimate for the Plancherel measure (5.6) (when $n = 1$ we use (5.7) instead) yields that

$$
\int_S |k_{f(\sqrt{\Delta})}(x)|^2 \, d\lambda(x) \approx \int_0^\infty |\mathcal{H}k_{f(\sqrt{\Delta})}(t)|^2 |c(t)|^{-2} \, dt
\approx \int_0^\infty |f(t)|^2 |c(t)|^{-2} \, dt \lesssim \int_0^\infty |f(t)|^2 (t^2 + t^n) \, dt,
$$

(5.25)
which implies that $J \lesssim (1 + \epsilon) \int_0^\infty |f(t)|^2(t^2 + t^n) dt$. Combining the estimate of $I$ and $J$ yields (ii) and (iii).

The proof for (iv) follows from the same argument except that in (5.25) we use (5.7) instead of (5.6).

The following decomposition of functions with compact support was proved in [15, Lemma 1.3]; see also [34, Lemma 4.6].

**Lemma 5.2.** Let $q, Q \in [0, \infty)$. Suppose that $f \in H^s(\mathbb{R})$ and supp $f \subset [1/2, 2]$. Then there exist even functions $\{f_{\ell}\}_{\ell=0}^\infty$ and a positive constant $C$, independent of $f$ and $\ell$, such that for all $\tau \in \mathbb{R}^+$,

(i) $f(\tau) = \sum_{\ell=0}^{\infty} f_{\ell, \tau}(\cdot)$ on $\mathbb{R}^+$, where $f_{\ell, \tau}(\cdot) \equiv f_{\ell}(\cdot)$ and supp $\hat{f_{\ell, \tau}} \subset [-2^{\ell+1} \tau, 2^{\ell+1} \tau]$;

(ii) for all $\ell \geq 0$ and $\tau \in [1, \infty),

\begin{equation}
\int_0^\infty |f_{\ell, \tau}(\xi)|^2 (\xi^2q + \xi^{2Q}) d\xi \leq C \tau^{-2\min\{q, Q\}-1} 2^{-2\ell} \|f\|^2_{H^s(\mathbb{R})};
\end{equation}

(iii) for all $\ell \geq 0$ and $\tau \in (0, 1),

\begin{equation}
\int_0^\infty |f_{\ell, \tau}(\xi)|^2 (\xi^2q + \xi^{2Q}) d\xi \leq C \tau^{-2\max\{q, Q\}-1} 2^{-2\ell} \|f\|^2_{H^s(\mathbb{R})}.
\end{equation}

**Proposition 5.1.** Let $m$ satisfy a mixed Mihlin-Hörmander condition of order $(s_0, s_\infty)$ with $s_0 > 3/2$ and $s_\infty > \max\{3/2, (n + 1)/2\}$. Then $k_{m(\Delta)}$ satisfies (5.21).

**Proof.** Let $m_j$ be as in (5.16). By (5.17), we obtain that for all $x \in S$ and $\epsilon > 0$,

\begin{equation}
\int_{\epsilon < d(x, e) \leq 2\epsilon} |k_{m(\Delta)}(x)| \, dp(x)
\end{equation}

\begin{equation}
= \sum_{\{j \in \mathbb{Z}; 2^{j+1} \epsilon \geq 1\}} \int_{\epsilon < d(x, e) \leq 2\epsilon} \left| k_{m_j(2^{-j} \Delta)}(x) \right| \, dp(x)
\end{equation}

\begin{equation}
+ \sum_{\{j \in \mathbb{Z}; 2^{j+1} \epsilon < 1\}} \int_{\epsilon < d(x, e) \leq 2\epsilon} \left| k_{m_j(2^{-j} \Delta)}(x) \right| \, dp(x) \equiv I + J.
\end{equation}

Observe that, by (5.14), $K_{m_j(2^{-j} \Delta)}(x, e) = k_{m_j(2^{-j} \Delta)}(x)$ for all $x \in S$. From this and (5.18), it follows that

\begin{equation}
I = \sum_{\{j \in \mathbb{Z}; 2^{j+1} \epsilon \geq 1\}} \int_{\epsilon < d(x, e) \leq 2\epsilon} \left| K_{m_j(2^{-j} \Delta)}(x, e) \right| \, dp(x)
\end{equation}

\begin{equation}
\lesssim \sum_{\{j \in \mathbb{Z}; 2^{j+1} \epsilon \geq 1\}} \frac{1}{(1 + 2^j \epsilon)^\sigma} \int_{\epsilon < d(x, e) \leq 2\epsilon} \left| K_{m_j(2^{-j} \Delta)}(x, e) \right| (1 + 2^j \epsilon \|d(x, e)\|^\sigma) \, dp(x) \lesssim 1.
\end{equation}

To estimate $J$, for each $j \in \mathbb{Z}$, set $f^{(j)}(t) \equiv m_j(t^2)$ for all $t \in \mathbb{R}^+$. Since supp $f^{(j)} \subset [1/2, 2]$, we use Lemma 5.1 to decompose each $f^{(j)}$. Hence, there exists a sequence of even functions
\{f^{(j)}_{\ell}\}_{\ell=0}^{\infty}$ such that $f^{(j)}(2^{-j/2}) = \sum_{\ell=0}^{\infty} f^{(j)}_{\ell, 2^{-j/2}}(\cdot)$ on $\mathbb{R}^+$ and the Fourier transform of each $f^{(j)}_{\ell, 2^{-j/2}}$ is supported in $[-2^{-\ell-j/2}, 2^{-\ell-j/2})$. For simplicity, in the sequel, we denote $f^{(j)}_{\ell, 2^{-j/2}}$ by $h_{\ell, j}$. Notice that $m_j(2^{-j} \Delta) = f^{(j)}(2^{-j/2} \sqrt{\Delta})$. Summarizing all these facts, we obtain that for all $j \in \mathbb{Z}$,

$$
\int_{\epsilon<d(x, e)\leq 2\epsilon} \left| k_{m_j(2^{-j}\Delta)}(x) \right| \, d\rho(x) = \sum_{\ell=0}^{\infty} \int_{\epsilon<d(x, e)\leq 2\epsilon} \left| k_{h_{\ell, j}(\sqrt{\Delta})}(x) \right| \, d\rho(x). \quad (5.28)
$$

By the support condition of $h_{\ell, j}$, we have $\text{supp} k_{h_{\ell, j}(\sqrt{\Delta})} \subset B(e, 2^{\ell-j/2})$. By this and Lemma 5.2(i), the sum in (5.28) reduces to the sum for $\ell$ satisfying $2^{\ell-j/2} > \epsilon$, therefore,

$$
J = \sum_{\{j \in \mathbb{Z}: 2^{j/2} \epsilon < 1\}} \sum_{\{\ell \in \mathbb{N}: 2^{\ell-j} > \epsilon\}} \int_{\epsilon<d(x, e)\leq 2\epsilon} \left| k_{h_{\ell, j}(\sqrt{\Delta})}(x) \right| \, d\rho(x).
$$

Set

$$
W_1 \equiv \{(j, \ell) : 2^{j/2} \epsilon < 1, 2^{\ell-j} > \epsilon, \ell \geq 0, j > 0, 2^{\ell-j} \geq 1\},
$$

$$
W_2 \equiv \{(j, \ell) : 2^{j/2} \epsilon < 1, 2^{\ell-j} > \epsilon, \ell \geq 0, j > 0, 2^{\ell-j} < 1\},
$$

$$
W_3 \equiv \{(j, \ell) : 2^{j/2} \epsilon < 1, 2^{\ell-j} > \epsilon, \ell \geq 0, j \leq 0, 2^{\ell-j} \geq 1\}
$$

and

$$
W_4 \equiv \{(j, \ell) : 2^{j/2} \epsilon < 1, 2^{\ell-j} > \epsilon, \ell \geq 0, j \leq 0, 2^{\ell-j} < 1\}.
$$

Observe that $W_4 = \emptyset$. Correspondingly, for all $k = 1, 2, 3$, we set

$$
J_k \equiv \sum_{(j, \ell) \in W_k} \int_{\epsilon<d(x, e)\leq 2\epsilon} \left| k_{h_{\ell, j}(\sqrt{\Delta})}(x) \right| \, d\rho(x).
$$

Thus, $J = \sum_{k=1}^{3} J_k$.

If $(j, \ell) \in W_1$, then $j > 0$ and hence $\epsilon < 2^{-j/2} < 1$. By this, (5.22), (5.27) and the assumption $s_{\infty} > \max\{3/2, (n+1)/2\}$, we obtain

$$
J_1 \lesssim \sum_{j>0} \sum_{\{\ell \in \mathbb{N}: 2^{\ell-j} \geq 1\}} \epsilon^{n+1/2} \left\{ \int_{0}^{\infty} \left| h_{j, \ell}(t) \right|^2 (t^2 + t^n) \, dt \right\}^{1/2} \lesssim \sum_{j>0} \sum_{\ell \geq j/2} 2^{\frac{j}{2}} \max\{3/2, (n+1)/2\} 2^{-s_{\infty}} \|f^{(j)}\|_{H^{s_{\infty}}(\mathbb{R})} \lesssim \sum_{j>0} \sum_{\ell \geq j/2} 2^{\frac{j}{2}} \max\{3/2, (n+1)/2\} 2^{-s_{\infty} (\ell-j/2)} \|m\|_{s_{\infty}} \lesssim \|m\|_{s_{\infty}}.
$$

If $(\ell, j) \in W_2$, then $\epsilon \in (0, 1)$. Applying (5.22) and (5.27) yields that when $n \geq 2$,

$$
J_2 \lesssim \sum_{\ell< j/2} \sum_{j=1}^{2^{\log_2} \epsilon} \epsilon^{n+1/2} \left\{ \int_{0}^{\infty} \left| h_{j, \ell}(t) \right|^2 (t^2 + t^n) \, dt \right\}^{1/2} \lesssim \sum_{\ell< j/2} \sum_{j=1}^{2^{\log_2} \epsilon} \epsilon^{n+1/2} \left\{ \int_{0}^{\infty} \left| h_{j, \ell}(t) \right|^2 (t^2 + t^n) \, dt \right\}^{1/2}
$$
\[ \leq \sum_{j=1}^{\infty} \sum_{\ell=0}^{\infty} \epsilon^{(n+1)/2} 2^j \max\{3/2,(n+1)/2\} 2^{-s_\infty} \|m\|_{s_\infty} \lesssim \|m\|_{s_\infty}; \]

when \( n = 1 \), by Lemma 5.1(iv), we replace \( \int_0^\infty |h_{\ell,j}(t)|^2(t^2 + t^n) dt \) in the above estimate by \( \int_0^\infty |h_{\ell,j}(t)|^2 t dt \) and then a similar argument also implies that \( J_2 \lesssim \|m\|_{s_\infty} \).

When we assume \( n \geq 2 \) and estimate \( J_3 \). In this case, when \( \epsilon \in [1, \infty) \), we use (5.23), (5.26) and \( 2^{j/2} \epsilon < 1 \) to obtain

\[ J_3 \lesssim \sum_{\{j \leq 0; 2^{j/2} \epsilon < 1\}} \sum_{\ell=0}^{\infty} \epsilon^{3/2} \left( \int_0^\infty |h_{\ell,j}(t)|^2(t^2 + t^n) dt \right)^{1/2} \lesssim \sum_{\{j \leq 0; 2^{j/2} \epsilon < 1\}} \sum_{\ell=0}^{\infty} \epsilon^{3/2} 2^{3j/4} 2^{-s_\infty} \|m\|_{s_\infty} \lesssim \|m\|_{s_\infty}. \]  

When \( \epsilon \in (0, 1) \), applying (5.23) and (5.26) yields that

\[ J_3 \lesssim \sum_{j=-\infty}^{0} \sum_{\ell=0}^{\infty} \epsilon^{(n+1)/2} \left( \int_0^\infty |h_{\ell,j}(t)|^2(t^2 + t^n) dt \right)^{1/2} \lesssim \sum_{j=-\infty}^{0} \sum_{\ell=0}^{\infty} 2^{3j/4} 2^{-s_\infty} \|m\|_{s_\infty} \lesssim \|m\|_{s_\infty}. \]  

If \( n = 1 \), Lemma 5.1(iv) implies that we can replace \( \int_0^\infty |h_{\ell,j}(t)|^2(t^2 + t^n) dt \) in (5.29) and (5.30) with \( \int_0^\infty |h_{\ell,j}(t)|^2 t dt \) and then a similar argument also yields \( J_3 \lesssim \|m\|_{s_\infty} \).

Combining the estimates of \( J_1 \) through \( J_3 \) yields \( J \lesssim 1 \). From this and the estimate of \( I \), we deduce that \( k_{m(\Delta)} \) satisfies (5.21), which completes the proof. \( \square \)

**Proof of Theorem 5.1.** By Proposition 5.1, the \( L^2 \)-boundedness of \( m(\Delta) \) and the fact that the kernel of \( m(\Delta) \) satisfies Hörmander’s condition (4.2), the operator \( m(\Delta) \) satisfies all the assumptions of Theorem 4.2. Then \( (m(\Delta))^* \) satisfies the desired properties. \( \square \)

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LIGUANG LIU
School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, China
(Present address) Department of Mathematics, School of Information, Renmin University of China, Beijing 100872, People’s Republic of China
E-mail liuliguang@ruc.edu.cn

MARIA VALLARINO
Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy
E-mail maria.vallarino@polito.it

DACHUN YANG (Corresponding author)
School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, China
E-mail dcyang@bnu.edu.cn