Deconfinement transition in a one–dimensional model

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Abstract
We present a model for quark matter with a density dependent quark–quark (confining) potential, which allows to describe a deconfinement phase transition as the system evolves from a low density assembly of bound structures to a high density free Fermi gas of quarks. A proper account of the many–body correlations induced by the medium is crucial in order to disentangle this behaviour, which does not uniquely stem from the naive density dependence of the interaction. We focus here on the effects of finite (non–zero) temperatures on the dynamical behaviour of the system. In particular we investigate the ground state energy per particle and the pair correlation function, from which one can extract the relevant information about the quarks being confined or not; the temperature dependence of the transition density is also derived.
1 Introduction

“QCD inspired” models for quark/nuclear matter have been proposed since many years, mainly within the framework of non–relativistic constituent quark models \[1, 2, 3, 4\]; the latter have proven remarkably useful for single–hadron spectroscopy, in spite of the lack of Lorentz invariance and chiral symmetry. Relativistic models, based on effective chirally invariant Lagrangians of NJL type or related to the so–called colour dielectric model, have also provided important tools for the investigation of quark matter \[5, 6, 7, 8, 9\]; however their many–body description mainly relies on the mean field approach, which does not seem the most appropriate one to deal with the problem of a quark–hadron phase transition.

The present work is based on a non–relativistic model for a one–dimensional many quark system, whose key feature is a density dependent interaction between quarks \[4\]:

\[
V(x) = \frac{1}{2} x^2 e^{-c \rho |x|},
\]

\[x \equiv x_1 - x_2\] being the relative interquark distance, \(\rho = N/L\) the (uniform) density of the system with \(N\) fermions in a length \(L\), and \(c\) a constant parameter. The potential (1.1) resembles a strong confining force in the limit of very low densities, where one expects quarks to be bound into hadrons, while it becomes negligible at large densities, where the quarks behave as a free Fermi gas (which should mimic the quark–gluon plasma phase).

Internal degrees of freedom are neglected, but antisymmetry for the global wavefunction of the system is required, as it is appropriate for an assembly of fermions. We notice that while the many–body interaction potential of \[1, 2, 3\] can only be dealt with variational Monte Carlo techniques, the present two–body potential allows to describe the system within the customary many–body schemes, which have been widely tested against nuclear matter properties.

The philosophy underlying this model is to provide a tool for a phenomenological but microscopic description of the phase transition from a plasma of (weakly interacting) quarks and gluons to an assembly of colourless quark clusters, namely hadrons. This transition should have occurred in the early stages of the Universe, at rather large temperature and density of the primordial gas of elementary constituents; as the system expanded and cooled down, hadronization took place, giving rise to the ordinary matter of
baryons and mesons, eventually bound into nuclear systems.

Relativistic heavy ion collisions are presently investigated in an attempt of reproducing the extreme density and temperature conditions at which the quark–gluon plasma phase can occur. One can thus envisage that temperature plays a crucial role in the description of the dynamical evolution of the system. Indeed, according to lattice QCD calculations, confining forces become weaker with increasing temperature, as it is suggested by the following temperature dependence of the string tension [10]:

\[
\sigma(T) = \sigma_0 \sqrt{1 - \left(\frac{T}{T_C}\right)^2} \theta(T_C - T),
\]

(1.2)

where \(\sigma_0 = \sigma(T = 0)\) and the theta function implies that confinement no longer survives above the critical temperature \(T_C\). Obviously this can only be taken as a qualitative indication of how the temperature affects the quark–quark interaction potential; moreover various uncertainties concern the precise determination of the critical temperature (as well as the order of the phase transition which shows up within this framework). Nevertheless we have assumed (1.2) as a sensible phenomenological Ansatz for the temperature dependence of the quark–quark interaction strength and employed it in our model potential (1.1) in order to describe the dynamical evolution of the system at finite temperature.

In Section 2 we shall shortly review the many–body formalism employed to deal with the system at zero temperature, introducing the relevant quantities which might signal, as a function of the density, the occurrence of the above mentioned phase transition. Numerical results are reported at \(T = 0\) in order to better appreciate the modifications introduced by finite values of the temperature. The latter are explored in Section 3, where a few schematic details of finite temperature field theory precede the description of the many–quark dynamics at finite temperature. The results for the ground state energy per particle and for the pair correlation function are presented at various (increasing) temperatures within the framework of the independent pair approximation. Finally Section 4 summarizes the virtues and limitations of the present approach.
2 Results in the $T = 0$ limit

Before introducing the finite temperature description, we shall report here an outline of the results obtained at zero temperature. We have mainly investigated the ground state energy (per particle) and the pair correlation function at different values of the density.

The medium induced correlations have been taken into account by solving the Bethe–Goldstone equation for the wavefunction of an interacting pair, within the independent pair approximation. For this purpose we expand the (antisymmetrized) two–particle wavefunction as follows:

\[
\Psi_{k_1k_2}(x_1, x_2) = \varphi_C^{CM}(X)\psi_{nk}(x)
\]

\[
= N e^{iKX} \sqrt{L} \left\{ c_{n0} \sin(k_j x) + \sum_{k_{1j}, k_{2j} > F} c_{nj} \sin(k_{j} x) \right\},
\]

where $X = (x_1 + x_2)/2$, $x = x_1 - x_2$, $k_j = (k_{1j} - k_{2j})/2$ and the total momentum $K = k_{1j} + k_{2j}$ is conserved in all terms in the expansion. The restriction on $k_{1j}$ and $k_{2j}$ to be above the Fermi momentum $k_F$ accounts for the Pauli blocking of the states available to the considered pair, which is due to the presence of the medium. In the present one–dimensional calculation it can be rewritten as:

\[
k_j > k_F + |K|/2.
\]

The trial wavefunction (2.1) is then inserted into the Schrödinger equation for the relative motion with the potential (1.1):

\[
\left[ -\frac{d^2}{dx^2} + V(x) \right] \psi_{nk}(x) = E_{nk}\psi_{nk}(x),
\]

which, by exploiting the orthogonality of the unperturbed wavefunctions in (2.1), reduces to a system of linear equations for the coefficients $c_{nj}$ of the Bethe–Goldstone solution we are looking for. One should also keep in mind that we are interested in bound (or quasi–bound) states of the interacting pair, at least for those (low) densities where the confining potential (1.1) is strong enough to actually provide confinement. In order to disentangle the few discrete eigenvalues of (2.3) out of its continuum spectrum, we impose on $\psi_{nk}(x)$ the boundary condition

\[
\psi_{nk}(R) = 0,
\]

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$R$ being an arbitrary length, of the order of (or larger than) the peak position of the potential barrier developed by (1.1) at intermediate distances. Eq. (2.4) is easily satisfied by choosing the wavenumbers in the expansion (2.1) to be integer multiples of $\pi/R$. If $\psi_{nk}$ corresponds to a truly bound state, then its vanishing at $|x| = R$ will not be accidental and we expect such a solution (and the corresponding energy eigenvalues) to be fairly stable with respect to broad variations of $R$.

At zero temperature and very low densities ($\rho \leq 0.1$) we have found [11] that the Bethe–Goldstone wavefunctions are close to the bound states one can obtain in the pure Schrödinger equation (namely in the absence of Pauli blocking), although they develop some small components outside the “confinement” region. The importance of these components rapidly increases with the density, thus showing that the effect of the medium on the interacting pair loosens its binding and produces a gradual transition (at intermediate densities) to a phase of non–interacting particles (Fermi gas).

By summing over the energy eigenvalues of particles in occupied states, namely those with relative momentum $k \leq k_F - |K|/2$, one can evaluate the ground state energy of the system: its evolution as a function of the density is shown in Fig. 1, where it is compared with the purely kinetic energy of a Fermi gas. The two coincide for large densities (say $\rho > 1.0$) but in the low density range the effect of correlations allows to disentangle the existence of dynamically bound pairs, whose energy is fairly independent upon the density of the system.

Another quantity which provides a clear signature for the existence (if any) of bound clusters in the system is the so–called pair correlation function ($r = |x|$)

$$g(r) = \frac{N(N - 1)}{\rho^2} \langle \Psi | \rho_2(x_1 - x_2) | \Psi \rangle,$$

which is obtained as the ground state expectation value of the two–body density operator:

$$\rho_2(x_1 - x_2) = \frac{1}{N(N - 1)} \sum_{i \neq j} \delta(x_i - x_1) \delta(x_j - x_2).$$

(2.6)

Within the independent pair approximation $g(r)$ turns out to be:

$$g_{BG}(r) = \frac{1}{\rho^2 L} \sum_{|k_1|, |k_2| \leq k_F} |\psi_{0k}(x)|^2 [k = (k_1 - k_2)/2]$$

(2.7)
Figure 1: The ground state energy per particle at $T = 0$ calculated in the independent pair approximation (black dots) is shown as a function of density, together with the one of a free Fermi gas (dashed line).

and can thus be evaluated using the ground state Bethe–Goldstone wavefunctions ([24]). In Fig. 2 we display a few $g_{BG}(r)$ at $T = 0$ and for different densities: one can appreciate the evolution of the pair correlation function from the low–density regime, where confinement is evident up to $\rho \simeq 0.15$ (lower densities, not displayed here, show even higher peaks at small distances), to the transition density ($\rho \simeq 0.19$), where the Fermi gas component starts developing at large distances, up to densities of the order of 0.5 and higher, where the pair correlation function practically coincides with the one of an uncorrelated Fermi gas. The correlation function obtained within the present model compares fairly well with the one of ref. [3], thus showing the equivalence of the two approaches for what concerns the physical properties of the system.
Figure 2: The pair correlation function $g(r)$ derived from the Bethe–Goldstone wave functions at $T = 0$ is displayed as a function of the relative interquark distance: the various curves are labelled by the value of the density. The free Fermi gas correlation function is shown only for $\rho = 0.5$ (dotted line), where it practically coincides with the one of the correlated system.

It is worth noticing that, in the presence of strong short range interactions, the importance of the medium grows with the density (which is proportional to $k_F$); in the present model, however, the interparticle potential exponentially decreases with the density and thus the largest modifications on the relative motion of a pair of particles only occur within an intermediate, limited range of densities. This is an interesting interplay between Pauli and dynamical correlations.
3 Finite temperatures

In order to deal consistently with the hadron–quark plasma phase transition one should take into account the behaviour of the system with the temperature, thus extending the microscopic description within the formalism of finite temperature field theory. In this case it is convenient to statistically treat the system within the grand canonical ensemble, by defining

\[ K = H - \mu \hat{N}, \]  

(3.1)

\( \hat{N} \) being the number operator and \( \mu \) the chemical potential.

The expectation value of any operator will then be evaluated by implementing the ensemble average

\[ \langle O \rangle = \text{Tr}(\hat{\rho}_G O), \]  

(3.2)

\( \hat{\rho}_G \) being the statistical density matrix \( (\beta = 1/k_BT) \)

\[ \hat{\rho}_G = \frac{1}{Z_G} e^{-\beta K} \]  

(3.3)

and \( Z_G \) the grand partition function

\[ Z_G \equiv e^{-\beta \Omega} = \text{Tr}e^{-\beta K}. \]  

(3.4)

In the above the trace (Tr) implies a sum over a complete set of states in the Hilbert space with any number of particles.

Moreover one can introduce the so–called temperature Green’s functions: for example the single particle propagator is defined as

\[ G_{\alpha\beta}(x\tau, x'\tau') = -\text{Tr} \left\{ \hat{\rho}_G T_\tau \left[ \hat{\psi}^\dagger_{K\alpha}(x\tau) \hat{\psi}_{K\beta}(x'\tau') \right] \right\}, \]  

(3.5)

where \( \hat{\psi}(x\tau) \) is the imaginary–time field operator in the (modified) Heisenberg picture \([12]\) and \( T_\tau \) the corresponding time ordering operator.

For a non–interacting system of fermions the temperature Green’s function reads then (in momentum space):

\[ G^0_{\alpha\beta}(k, \omega; T) = \delta_{\alpha\beta} \left[ \frac{n_k^0(T)}{\omega - (\varepsilon_k^0 + i\eta)} + \frac{1 - n_k^0(T)}{\omega - (\varepsilon_k^0 - i\eta)} \right] \]  

(3.6)
where the occupation probability

$$n_k^0(T) = \frac{1}{1 + e^{\beta(\varepsilon_k^0 - \mu)}}$$  \hspace{1cm} (3.7)

reduces to the sharp theta function momentum distribution of the Fermi gas in the limit of zero temperature:

$$n_k^0(T) \xrightarrow{T \to 0} \theta(\varepsilon_F^0 - \varepsilon_k^0).$$  \hspace{1cm} (3.8)

These ingredients can be used to evaluate, at $T \neq 0$, the polarization (particle–hole) propagator within the customary perturbation theory and, from it, to extract the two–body correlation function according to the procedure described in ref. [11].

It should be pointed out that, in principle, the chemical potential has to be self–consistently determined in order to satisfy the relation:

$$\frac{N}{L} \equiv \rho = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{d\varepsilon}{\varepsilon \sqrt{1 + \exp\{\beta(\varepsilon - \mu)\}}}.$$  \hspace{1cm} (3.9)

which, in the limit of very small temperatures leads, for a free Fermi gas, to the expansion:

$$\mu \simeq \varepsilon_F \left[1 + \frac{1}{12} \left(\frac{\pi}{\beta\mu}\right) + \ldots\right].$$  \hspace{1cm} (3.10)

The latter, however, becomes unreliable for temperatures of the order of the Fermi energy, which are typical values we are interested in; thus we have utilized the chemical potential provided by equation (3.9).

Concerning the evaluation of the pair correlation function, the alternative method we have employed here (in analogy with the previous Section) consists in solving the Bethe–Goldstone equation for the relative motion of two interacting quarks. At $T \neq 0$ the Pauli operator, which limits the available states in the formal development (2.1) of the correlated wavefunction, will be modified by employing the smooth distribution function (3.7):

$$\theta(k_\alpha - k_F)\theta(k_\beta - k_F) \xrightarrow{T \neq 0} \left[1 - n_{k_\alpha}^0(T)\right]\left[1 - n_{k_\beta}^0(T)\right].$$  \hspace{1cm} (3.11)

As a result the effect of Pauli correlations is somewhat weakened and the influence of the medium can be expected to be, at a fixed density, less important at finite temperature than at $T = 0$. 

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Figure 3: The ground state energy per particle at various (finite) temperatures, calculated in the independent pair approximation is shown as a function of the density: black dots correspond to $T/T_C = 0.1$, stars to $T/T_C = 0.5$ and crosses to $T/T_C = 0.995$. The dashed lines represent the corresponding free kinetic energy.

Moreover, as already anticipated in the Introduction, as the temperature increases, one should also take into account some temperature dependence of the effective coupling, which we have assumed in the form suggested by (1.2). This potential has been employed in the evaluation of the two–body correlation function derived from the solution of the Bethe–Goldstone equation, where, as stated above, we also account for the temperature dependence of the occupation probability, according to (3.11).

In addition, from the corresponding energy eigenvalues we have evaluated the ground state energy per particle as a function of the density, to be compared with the pure kinetic energy of the Fermi gas, and followed its evolution with increasing temperature (see Fig. 3). At high densities the sys-
tem is clearly a gas of non-interacting quarks; at low densities, instead, for small (and zero) temperatures the effect of pair correlations is clearly visible, but tends to fade away, as it is obvious from the temperature dependence of the quark–quark interaction, as the temperature approaches the critical value (here arbitrarily fixed “a priori” to coincide with twice the Fermi energy at $\rho = 0.5$, $T_C = \pi^2/4$). Notably the discontinuity in the derivative of the energy per particle, which is seen at $T = 0$ and $\rho \simeq 0.2$, is smoothed out at $T \neq 0$.

Although, in the present treatment, we cannot identify a specific order parameter, which would allow to consider the phase transition from a thermodynamical point of view, we have interpreted as a transition density, $\rho_C$, the one where the medium induced correlations vanish: this quantity can be defined as the average value (with respect to the relative and total momentum of a pair) of the difference between the matrix elements of the bare potential (1.1) and the G-matrix which one obtains from the solution of the Bethe–Goldstone equation:

$$\Delta U(\rho) = <k, K|V|k, K> - <k, K|G|k, K>.$$  \hspace{1cm} (3.12)

In the above, according to the usual definition,

$$<k, K|G|k, K> = \frac{1}{R} \int_0^R dx \sin(kx)V(x)\psi_{0k}(x),$$  \hspace{1cm} (3.13)

where both the unperturbed and the correlated wavefunctions are normalized over the above mentioned distance $R$ and the integral over the center of mass coordinate is unity; the dependence upon $K$ of the right hand side of (3.13) is implicit in the Bethe–Goldstone wavefunction through the action of the Pauli operator.

The choice of $\Delta U$ as an “order parameter” is arbitrary, but it is closely related to the energy gap usually considered in the microscopic description of superconductivity; in that case a non–vanishing gap signals the existence of bound electron pairs which profoundly alter the global properties of the system. Here we assume (3.12) as a discriminant between a system of “hadrons” (bound pairs of quarks) and a weakly interacting Fermi gas of quarks. For each value of the temperature, the solution of the equation $\Delta U(\rho) = 0$ provides the critical density for the deconfinement phase transition.

The behaviour of $\rho_C$ with the temperature is shown in Fig. 4, which can be interpreted as the phase diagram for the model under investigation. For
a wide range of temperatures the critical density remains fairly constant, then it rapidly drops to zero as one approaches the critical temperature. It should be noticed the slight increase of $\rho_C$ in the low temperature regime: this outcome, somewhat opposite to the intuitive expectation, stems from the fact that a non–vanishing temperature weakens the Pauli blocking on the correlated wavefunction while the interaction strength is still practically unaffected. Thus for low temperatures the dominant effect is the appearance of the Fermi gas component at densities higher than for $T = 0$; as the temperature increases, however, this tendency is balanced and overcome by the temperature dependence of the interaction itself.

Finally we have evaluated the pair correlation function, eq. (2.7), by utilizing the finite temperature Bethe–Goldstone wavefunctions for the relative
Figure 5: The pair correlation function \( g(r) \) derived from the Bethe–Goldstone wave functions at \( T/T_c = 0.1 \) is displayed as a function of the relative interquark distance: the various curves are labelled by the value of the density.

A few examples of \( g(r) \) at various densities are illustrated in Figs. 5–7 for typical values of the temperature (in units of the above defined critical temperature). For low \( T \) (\( T = 0.1T_c \)) the same behaviour as for \( T = 0 \) is observed, although, in agreement with the phase diagram of Fig. 4, the transition density appears to be somewhat larger than in the zero temperature limit. As the temperature increases the binding effects become weaker and weaker and the correlation function rapidly approaches the free Fermi gas one, since the quark–quark model interaction becomes vanishingly small. Notice that the finite temperature washes out completely the quantum oscillations of the free correlation function.
Remarks and conclusions

In this work we have investigated some specific properties of a system of fermions, strongly interacting with a density dependent force, such as to provide bound pairs in the low density limit of a free Fermi gas at large densities. As already pointed out in a previous work [11], this could bear some relevance in the investigation of the evolution of the primordial quark–gluon plasma toward the hadronic phase. However the evolution of the early Universe non only implies a decreasing local barionic density but also a rapid fall off of the temperature: the latter indeed should play a non–negligible role in the phase transition which lead to the formation of hadrons and nuclei.

Here we have thus focussed the attention on the effects of temperature, starting from the hypothesis (supported by lattice QCD calculations) that with increasing temperature the strength of the confining potential becomes

Figure 6: The same as in Fig. 5, at $T/T_c = 0.5$
weaker and finally vanishes at some critical temperature. On this basis we have analyzed the interplay of density and temperature dependences of the quark–quark interaction: the pair correlation function shows indeed that, with increasing temperature, the transition density remains fairly stable until the critical temperature is approached and then rapidly drops to zero.

The above findings show that the present approach is on the right path to achieve a microscopic description of the phase transition from hadrons to a quark–(gluon) plasma. However it can only be considered as a very preliminary stage: indeed the model for quark matter employed here suffers from many major limitations. Beside being one-dimensional and non-relativistic, it does not embody internal degrees of freedom like spin, colour, etc. Also it does not contain explicit gluonic degrees of freedom, not even in the form of a perturbative one gluon exchange.

Moreover, with respect to the present and future experiments of rela-
tivistic heavy ion collisions, one would also like to describe the dynamical evolution with time of the many–quark system, so that estimates of the probability for QGP formation and subsequent hadronization could be done. Future work is required in order to improve along these directions.

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