Cubic hypersurfaces and integrable systems

Atanas Iliev, Laurent Manivel

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Abstract

Together with the cubic and quartic threefolds, the cubic fivefolds are the only hypersurfaces of odd dimension $>1$ for which the intermediate jacobian is a nonzero principally polarized abelian variety (p.p.a.v.). In this paper we show that the family of 21-dimensional intermediate jacobians of cubic fivefolds containing a given cubic fourfold $X$ is generically an algebraic integrable system. In the proof we apply an integrability criterion, introduced and used by Donagi and Markman to find a similar integrable system over the family of cubic threefolds in $X$. To enter in the conditions of this criterion, we write down explicitly the known by Beauville and Donagi symplectic structure on the family $F(X)$ of lines on the general cubic fourfold $X$, and prove that the family of planes on a cubic fivefold containing $X$ is embedded as a Lagrangian surface in $F(X)$. By a symplectic reduction we deduce that our integrable system induces on the nodal boundary another integrable system, interpreted generically as the family of 20-dimensional intermediate jacobians of Fano threefolds of genus four contained in $X$. Along the way we prove an Abel-Jacobi type isomorphism for the Fano surface of conics in the general Fano threefold of genus 4, and compute the numerical invariants of this surface.

1 Introduction

1.1 Background

For a smooth compact Kähler manifold $Y$ of dimension $n$ and any positive integer $q \leq n$, its $q$-th intermediate jacobian

$$J_q(Y) = (H^{q-1,q}(Y) \oplus \cdots \oplus H^{0,2q-1}(Y))/H^{2q-1}(Y, \mathbb{Z})$$

is a complex torus. Because of the skew-symmetry of the Riemann-Hodge bilinear relations the tori $J_q(Y)$ aren’t in general abelian varieties, except for the Picard variety $J_1(Y) = Pic^0(Y)$ and the Albanese variety $J_n(Y) = Alb(Y)$, see Ch.2 §6 in [CG]. If $n = \dim Y = 2q - 1$ is odd then $J(Y) = J_q(Y)$ is simply called the intermediate jacobian of $Y$. In the particular case when $Y \subset \mathbb{P}^{n+1}$ is a smooth hypersurface of degree $d$, the intermediate jacobian $J(Y)$ happens to be a non trivial abelian variety only when $n = 1$ and $d \geq 3$, $n = 3$ and $d = 3, 4$ (i.e. when $Y$ is a cubic or quartic 3-fold), or $n = 5$ and $d = 3$ (i.e. when $Y$ is a cubic 5-fold), see p.43 in [Col].

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In [DM1] Donagi and Markman studied certain complex algebraic analogs of integrable systems in mechanics. These analogs take as models the integrable Hamiltonian systems $f : M \to B$, where the phase space $M$ is a real symplectic $2n$-fold, and the general fibers of $f$ are compact Lagrangian $n$-folds in $M$, which by the Arnold-Liouville theorem are $n$-fold real tori. In §7 of [DM1] are given general criteria (the cubic conditions) which ensure when an algebraic family $f : M \to B$ of complex tori represents an algebraic integrable system; for more details see §2 in [DM1] or §2 in [Fr].

The main application of these criteria is the observation that the relative intermediate jacobians over the gauged moduli spaces of Calabi-Yau threefolds are in fact such algebraic integrable systems, see [DM2]. This gives rise to the question whether besides the Calabi-Yau integrable systems there exist other nontrivial examples of families of intermediate jacobians that are also algebraic integrable systems, see e.g. Question 4.6 in [AS]. It seems that until now, the only known such nontrivial example (besides the integrable systems coming from families of jacobians of curves) is given in Example 8.22 of [DM1]:

If $X \subset \mathbb{P}^5$ is a smooth cubic fourfold, and $B = \mathcal{H}_s$ is the open subset in the dual projective space parameterizing the smooth hyperplane sections $Y_b$ of $X$, then the relative intermediate jacobian $f : \mathcal{J}(Y) \to \mathcal{H}_s$ is an algebraic integrable system.

This example is based on an abstract procedure, described in §8 of [DM1], that generates algebraic integrable systems in the context of Lagrangian deformations. More precisely, for a given complex symplectic variety $W$, its Lagrangian Hilbert scheme parameterizes the Lagrangian subvarieties of $W$. By a result of Voisin and Ran the deformations of a smooth Lagrangian subvariety $F \subset W$ are unobstructed and the component $\mathcal{H}$ of the Lagrangian Hilbert scheme of $W$ containing $F$ is generically smooth, see [Voi Ran]. Consider the universal family $\mathcal{F} \to \mathcal{H}$ and its Picard bundle $\text{Pic}^o \mathcal{F} \to \mathcal{H}$. In this context, Theorem 8.1 in [DM1] states:

**Theorem** If $W$ is a smooth complex symplectic variety and $F$ is a smooth Lagrangian subvariety of $W$, then the symplectic structure on $W$ generates a natural symplectic structure on the relative Picard $\text{Pic}^o \mathcal{F} \to \mathcal{H}_s$ over the base $\mathcal{H}_s$ of smooth Lagrangian deformations of $F$ in $W$, making the relative Picard $\text{Pic}^o \mathcal{F} \to \mathcal{H}_s$ an algebraic integrable Hamiltonian system.

In the cited Example 8.22, the above criterion (DM) is applied in the case where $W = F(X)$ is the family of lines on a smooth cubic hypersurface $X \subset \mathbb{P}^5$ and $F = F(Y) \subset F(X)$ is the Fano surface of lines on a smooth hyperplane section $Y$ of $X$. To enter in the conditions of (DM), the following results are used:

1. The family $F(X)$ is a symplectic 4-fold, see [BI].
2. $F(Y) \subset F(X)$ is a Lagrangian surface in $F(X)$, see Ex.7 in §3 of [Voi].
3. For a smooth cubic 3-fold $Y$, its intermediate jacobian $J(Y)$ is a principally polarized abelian variety of dimension 5, the family $F(Y)$ of lines on $Y$ is a smooth surface of irregularity 5, and the Abel-Jacobi map defines an isomorphism between $\text{Alb} F(Y)$ and $J(Y)$, see [CG].

The most common situation when (DM) can be applied is the case when the symplectic variety $W = S$ is a K3 surface and $F = C$ is a smooth curve of genus $g$ on $S$; notice that any curve on a K3 surface $S$ is a Lagrangian subvariety of $S$. In this lowest dimensional case, the base $\mathcal{H}_s$ of the smooth Lagrangian deformation of $C$ in $S$ is an open subset of the complete linear system $|O_S(C)| \cong \mathbb{P}^g$, and (DM) yields that the relative jacobian $\mathcal{J}$ is a Lagrangian fibration over $\mathcal{H}_s$. When $S = S_{2g-2}$ is a K3 surface with a primitive polarization of genus $g = g(C)$ this fibration can be extended to a Lagrangian fibration $\mathcal{J} \to \mathbb{P}^g$ over the compactified
relative jacobian $\mathcal{J}$, which is a smooth compact complex symplectic variety birational to the $g$-th punctual Hilbert scheme $\text{Hilb}_g S$, see §2 in [B2]. In this trend, J. Sawon, in a collaboration with K. Yoshioka, has recently shown that for any $g, m \geq 2$ the $g$-th Hilbert power $\text{Hilb}_g T$ of the general primitive K3 surface $T$ of degree $m^2(2g - 2)$ can be represented as a torsor over the compactified relative jacobian $\mathcal{J} \rightarrow \mathbb{P}^g$ of a primitive K3 surface $S = S(T)$ of degree $2g - 2$, thus proving the existence of a regular Lagrangian fibration over $\mathbb{P}^g$ on the smooth compact complex symplectic variety $\text{Hilb}_g T$, see [Saw].

The above examples give rise to the question when the relative Picard fibrations from (DM) can be extended to Lagrangian fibrations including fibers over singular Lagrangian deformations of $F \subset W$. For general integrable systems of type relative Picard as in (DM), a partial compactification is described in Theorem 8.18 of [DM1] but the proof remains unpublished.

At the end of the same Example 8.22 of [DM1] it is shown that the symplectic structure on $\text{Pic}^0(F) \rightarrow H_s$ can be extended to a still non-degenerate symplectic structure over the set $H_n \supset H_s$ of hyperplane sections $Y$ of $X$ that are allowed to be singular but with at most one node. For a general nodal cubic 3-fold $Y_b \in \partial H_n := H_n - H_s$ the family of lines $\ell \subset Y_b$ that pass through the node of $Y$ is parameterized by a smooth curve $C_b$ of genus 4, and the generalized intermediate jacobian $J(Y) \rightarrow H_n$ induces over the nodal boundary $\partial H_n$ a boundary fibration $J(C) \rightarrow \partial H_n$ whose fibers are the Jacobians $J(C_b)$ of the genus 4 curves $C_b$. All this makes it possible to conclude that the boundary abelian fibration $J(C) \rightarrow \partial H_n$ of the algebraic integrable system $J(Y) \rightarrow H_n$ is also an algebraic integrable system, at least over an open subset of $\partial H_n$. Donagi and Markman call this system the boundary integrable system for $J(Y)$, see the end of §8 in [DM1]. As communicated to us by Ron Donagi, this is an instance of the algebraic symplectic reduction, described in §2 of his later paper [DP] with E. Previato.

1.2 Summary of the results in the paper

In this paper we describe a new example of an algebraic integrable family of intermediate Jacobians in the context of the integrability conditions (DM). It is an analog of the Example 8.22 from [DM1], where the fibers of the integrable system are the 5-dimensional intermediate Jacobians of smooth cubic 3-folds contained in a fixed cubic fourfold as hyperplane sections. In our case the fibers of the integrable system are the 21-dimensional intermediate Jacobians of the general cubic fivefolds containing the same cubic fourfold $X$ as a hyperplane section. By Theorem 13, the first main conclusion in our paper:

The relative intermediate jacobian is an algebraic integrable system over an open subset of the family of cubic fivefolds containing a fixed general cubic fourfold $X$ as a hyperplane section.

Notice once again that the cubic fivefolds are the unique hypersurfaces of odd dimension $> 3$ for which the intermediate jacobian is a non trivial abelian variety. Next, we study the degeneration of this integrable system on the boundary parameterizing the nodal cubic fivefolds through $X$, and prove that the induced boundary abelian fibration is also an algebraic integrable system.

In our case the symplectic variety $W$ from (DM) is still the same as in the example of Donagi and Markman: the 4-fold family $W = F(X)$ of lines on a fixed smooth cubic fourfold $X$. The difference is in the choice of the Lagrangian subvariety of departure $F \subset W$. At this
point we should mention that discovering principally new or nontrivial examples of Lagrangian subvarieties of symplectic varieties, especially in a projective-geometric context, looks like a happy occurrence. In our case this occurrence is realized as the Fano surface $F_2(Z)$ of planes on the general cubic fivefold $Z$ that contains $X$ as a hyperplane section; the surface $F_2(Z)$ is regarded as a subvariety of $F(X)$ by the embedding given by the intersection-map $j_Z : F_2(Y) \to F(X)$, $\mathbb{P}^2 \to \mathbb{P}^2 \cap X$. We prove:

The intersection image $j_Z(F_2(Z))$ of the Fano surface $F_2(Z)$ of planes on the general cubic fivefold $Z$ containing the cubic fourfold $X$ as a hyperplane section is a smooth Lagrangian surface inside the 4-fold family $F(X)$ of lines in $X$, see Proposition 4.

This relies on an explicit description of the symplectic form on $F(X)$, which is provided in §2.1. Recall that in [BD] the symplectic form on $F(X)$ is described only for the general Pfaffian cubic 4-folds $X$, which form a divisor in the space of all cubic fourfolds $X \subset \mathbb{P}^5$. For a general Pfaffian cubic fourfold $X$, the family $F(X)$ is known to be isomorphic to the Hilbert square $Hilb_2 S$ of a K3 surface $S$ of genus 8, and the symplectic form on $F(X)$ is the symplectic form on $Hilb_2 S$ described earlier by Fujiki and Beauville, see [HI]. The existence of a symplectic form on $F(X)$ for the general cubic fourfold $X$ then follows by a deformation argument, see [BD].

By Proposition 4 we enter in the conditions of (DM), this time with the symplectic fourfold $F(X)$ and its Lagrangian surface $j_Z(F_2(Z))$. This gives rise to a Lagrangian fibration on the relative Picard $Pic^\sigma F \to H_s$ over the scheme $H_o$ of smooth deformations of $j_Z(F_2(Z))$ in $F(X)$, see the beginning of Section 5. In this situation, the analog of 3 above is a result of Collino:

The family $F_2(Z)$ of lines on the general cubic 5-fold $Z$ is a smooth surface of irregularity 21, and the Abel-Jacobi map induces an isomorphism between the Albanese variety $Alb F_2(Z)$ and the intermediate Jacobian $J(Z)$, see [Co].

In fact $h^{5,0}(Z)$ and $h^{4,1}(Z)$ vanish, while $h^{2,1}(Z) = 21$, which explains why $J(Z)$ is a p.p.a.v. of dimension 21. By duality, $J(Z)$ is also isomorphic with the Picard variety $Pic^\sigma F_2(Z)$. This implies Theorem 13.

Next, we study a partial compactification of the relative jacobian fibration from Theorem 13, by including its base the codimension one boundary $\partial H_n$ of cubic fivefolds with one node. For a general nodal cubic 5-fold $Z_b$, the family of lines $\ell \subset Z_b$ which pass through the node of $Z_b$ is parameterized by a general smooth prime Fano threefold $Y_b$ of genus 4. This threefold $Y_b$ is the analog of the genus 4 curve $C_b$ from the boundary system in Example 8.22 of [DM]. Since in this situation the generalized intermediate jacobian $J(Z_b)$ is a $\mathbb{C}^*$-extension of $J(Y_b)$, the boundary fibration $J(Y)$ has for base the set of nodal cubic 5-folds $Z_b$ containing $X$ and as fibers the 20-dimensional intermediate jacobians of their associated Fano 3-folds $Y_b$ of genus 4. Such a Fano 3-fold $Y_b$ is a complete intersection of a quadric and a cubic in $\mathbb{P}^5$. While the cubic can be chosen to be $X$, the quadric (identified with the base of the projective tangent cone to the node of $Z_b$) can move together with $b$, see §4.2.

In §2 we find the analogs of 2 and 3 (i.e. boundary versions of Proposition 4 and Collino’s Abel-Jacobi isomorphism) over the boundary $\partial H_n$ of nodal cubic fivefolds. To find these analogs we need for a given general nodal cubic fivefold $Z_b$ to express the Fano surface $F_2(Z_b)$ with its embedding $j_Z$ in the family of lines $F(X)$ of the fixed cubic 4-fold $X$. As shown in §2.2, the Fano surface $F_2(Z_b)$ of planes on $Z_b$ is almost the same as the Fano surface $F(Y_b)$ of conics on $Y_b$. More precisely, there is a natural map $j : F(Y_b) \to F(X)$, such that the image $j(F(Y_b))$
coincides with the isomorphic intersection image of $F_2(Z_b)$ in $F(X)$. However while $F(Y_b)$ is smooth, the Fano surface $F_2(Z_b)$ has a double curve $\Gamma$, isomorphic to the family of lines on $Y_b$, see Proposition 7. Nevertheless, we conclude that $j(F(Y_b)) = j_Z(F_2(Z_b))$ is a singular Lagrangian surface in $F(X)$, see Proposition 8.

In §3.1 we study in more detail the Fano surface $F(Y)$ of the general prime Fano 3-fold $Y$ of genus 4. In particular we find the invariants of $F(Y)$, see Corollary 10. Next, in §3.2 we prove the following analog of Collino’s Abel-Jacobi isomorphism for planes on cubic fivefolds:

The family $F(Y)$ of conics on the general prime Fano 3-fold $Y$ of genus 4 is a smooth surface of irregularity 20, and the Abel-Jacobi map induces an isomorphism between $\text{Alb} F(Y)$ and the intermediate jacobian $J(Y)$, see Theorem 12.

To prove Theorem 12, we follow the same program as Clemens and Griffiths in their proof of the Abel-Jacobi isomorphism for the cubic threefold. This program was subsequently applied in [Lett] to prove a similar Abel-Jacobi isomorphism for the Fano surface of conics on the general quartic hypersurface in $\mathbb{P}^4$, then in [Col] for the Fano surface of planes on the general cubic 5-fold. In brief, this program consists in verifying that in a general Leftschetz pencil $\{Y_t : t \in \mathbb{P}^1\}$ of Fano 3-folds of genus 4, for a finite number of values of $t$ either the Fano surface $F(Y_t)$ acquires isolated singular points but $Y_t$ remains smooth, or when $Y_t$ becomes singular then its singularity is a simple node and in this case $F(Y_t)$ has a smooth double curve for singular locus, see [Lett] and §3.2. As above, the Abel-Jacobi isomorphism makes it possible to identify $\text{Pic}^0 F(Y)$ with $J(Y)$.

In §4.2 we use these results together with the algebraic symplectic reduction procedure from [DP] to get Theorem 17, the second main conclusion of this paper:

The algebraic integrable system from Theorem 13 induces on the nodal boundary $\partial \mathcal{H}_n$ a fibration by intermediate jacobians of Fano 3-folds of genus 4 which is generically an algebraic integrable system.

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2 Revisiting the family of lines on the cubic fourfold

2.1 The symplectic form

Let $X \subset \mathbb{P}V = \mathbb{P}^5$ be a general cubic hypersurface, with equation $P = 0$ for some general polynomial $P \in S^3 V^*$. The Fano variety $F(X)$ of lines contained in $X$ is a subvariety of the Grassmannian $G(2, V)$. It can be defined as the zero locus of the global section $s_P$ of the vector bundle $S^3 T^*$, naturally defined by $P$, where $T$ denotes the rank two tautological bundle on the Grassmannian. Since $s_P$ is a general section of the globally generated vector bundle $S^3 T^*$, $F(X)$ is a smooth four dimensional variety.

It was proved by Beauville and Donagi that $F(X)$ has a symplectic structure. Indeed they showed that for $X$ a Pfaffian cubic hypersurface there exists a K3 surface $S$ such that $F(X) = \text{Hilb}_2 S$, which is well known to inherit a symplectic structure from that of $S$. Then the general
case follows by a deformation argument, see [BD]. Nevertheless the use of deformations makes that the symplectic form on \( F(X) \) is not explicit. Markushevich and Tikhomirov [MT] showed how to deduce it from the Voisin’s observation that the Fano varieties of lines in the hyperplane sections of \( X \) are Lagrangian surfaces in \( F(X) \) (see section 3). We shall give below a completely explicit and down-to-earth description of this form. In fact it suffices to exhibit a non zero holomorphic two form on \( F(X) \) – indeed since \( h^{2,0}(F(X)) = 1 \) we know that there exists a unique such form, so it must define the symplectic structure, that it, it will automatically be a non degenerate closed form. Consider the tangent exact sequence

\[
0 \to TF(X) \to TG(2,V)_{|F(X)} \to S^3T^*_{|F(X)} \to 0.
\]

Remember that \( TG(2,V) \simeq Hom(T,Q) \), where \( Q = V/T \) denote the quotient bundle on the Grassmannian. Therefore, the tangent space to \( TF(X) \) at a line \( \ell = \mathbb{P}T \) is

\[
T_tF(X) = \{ u \in Hom(T,Q), \ P(x,x,u(x)) = 0 \ \forall x \in T \},
\]

where \( P(x,y,z) \) is the polarization of \( P(x) \) and \( u(x) \in V \) is any representative of \( u(x) \in Q \).

Note that \( P(x,x,x) = 0 \) is an equation of \( T_xX \). If \( \ell \) is a general line on \( X \) and \( x \) varies in \( \ell \), we get a quadratic pencil of hyperplanes in \( \mathbb{P}V \), whose intersection is a plane \( \Pi = \mathbb{P}S \supset \ell = \mathbb{P}T \). In particular, \( T_tF(X) \supset Hom(T,S/T) \).

There is a natural skew-symmetric form on \( T_tF(X) \), defined up to scalar as follows. Choose some vectors \( e, f, g, h \) of \( T \) and let, for \( u, v \in T_tF(X) \),

\[
\Omega(u,v;e,f,g,h) = P(e,u(u(g)))P(f,f,v(h)) - P(e,e,v(g))P(f,f,u(h)).
\]

This can be seen as defining a skew-symmetric bilinear map on \( S^2T \otimes T \) with values in \( \wedge^2 T_tF(X)^* \). Note that \( S^2T \otimes T = S^3T \oplus T \otimes \wedge^2 T \). We get the component \( S^3T \) by letting \( g = e \) in the formula above, which gives zero on \( T_tF(X)^* \). So we need only consider the other component, which gives a map \( (\wedge^2 T^*)^3 \to \wedge^2 T_tF(X)^* \) defined by the formula

\[
\omega(u,v;e,f) = P(e,u(u(f)))P(f,f,v(e)) - P(e,e,v(f))P(f,f,u(e)) + 2P(e,f,u(f))P(e,e,v(f)) - 2P(e,e,u(f))P(e,f,v(f)) + 2P(f,f,u(e))P(e,e,v(e)) - 2P(f,f,u(e))P(f,f,v(e)).
\]

We can see \( \omega \) as a form with values in the line bundle \( (\wedge^2 T^*)^\otimes^3 = \mathcal{O}(3) \). This form has rank two at the generic point of \( F(X) \) and its radical is precisely \( Hom(T,S/T) \).

There is also a natural quadratic form on \( \wedge^2 T_tG(2,V) \), defined again up to scalar by the following formula: if \( u, v, u', v' \in Hom(T,Q) \) and \( e, f \) is a basis of \( T \), let

\[
K(u\wedge v, u'\wedge v') = u(e)\wedge u'(f)\wedge v(e)\wedge v'(f) - u(f)\wedge u'(e)\wedge v(e)\wedge v'(e) + u(f)\wedge u'(e)\wedge v(f)\wedge v'(e) - u(e)\wedge u'(e)\wedge v(f)\wedge v'(e),
\]

seen as an element of the line \( \wedge^4 Q \). More precisely, \( K \) is a quadratic form on \( \wedge^2 T_tG(2,V) \) with values in the line bundle \( Hom((\wedge^2 T),\wedge^4 Q) = \mathcal{O}(3) \otimes \det(V) \).

Observe that this form can be described in terms of the natural decomposition \( \wedge^2 T_tG(2,V) = \wedge^2 T^* \otimes S^2Q \oplus S^2T^* \otimes \wedge^2 Q \), as the composition of the natural morphisms

\[
S^2(\wedge^2 T_tG(2,V)) \to S^2(S^2T^* \otimes \wedge^2 Q) \to S^2(S^2T^*) \otimes S^2(\wedge^2 Q) \to (\wedge^2 T^*)^\otimes^2 \otimes \wedge^4 Q.
\]
Now we restrict the quadratic form $K$ to $\bigwedge^2 T_\ell F(X)$. We claim that the restriction has rank at most two. Indeed, it is clear that $K$ vanishes on $Hom(T, S/T) \wedge T_\ell F(X)$, a hyperplane in $\bigwedge^2 T_\ell F(X)$. In particular we can write

$$K(u \wedge v, u \wedge v) = \omega(u, v)\Omega(u, v)$$

for $u, v \in T_\ell F(X)$, where $\Omega$ is now a well-defined skew-symmetric form on $T_\ell F(X)$ (with values in $\det(V)$, to be precise), since $K$ and $\omega$ both take values in $O(3)$.

**Theorem 1** The form $\Omega$ defines a symplectic structure on the Fano variety $F(X)$.

**Proof.** As explained above we just need to prove that $\Omega$ is non trivial at some point $\ell = \mathbb{P}T$ of $F(X)$. This reduces to an easy computation in coordinates: we chose a basis $e_1, \ldots, e_6$ of $V$ such that $T = \langle e_1, e_2 \rangle$. If $z_1, \ldots, z_6$ are the corresponding coordinates on $V$, we may suppose that $P(e_1, e_1, z) = z_4$, $P(e_1, e_2, z) = z_5$, $P(e_2, e_2, z) = z_6$. In particular the intersection of the tangent spaces of $X$ along $\ell$ is $S = \langle e_1, e_2, e_3 \rangle$.

An element of the tangent space $T_\ell F(X)$ is a homomorphism $u \in Hom(T, Q)$ such that

$$u(e_1) = ae_3 + be_5 - 2ce_6,$$
$$u(e_2) = de_3 - 2be_4 + ce_5$$

for some scalars $a, b, c, d$. Take another $u' \in T_\ell F(X)$ defined by the scalars $a', b', c', d'$. Then $K(u \wedge u', u \wedge u')$ is proportional to $u(e_1) \wedge u(e_2) \wedge u'(e_1) \wedge u'(e_2)$, that is, to the determinant

$$\begin{vmatrix}
a & d & a' & d' \\
0 & -2b & 0 & -2b' \\
b & c & b' & c' \\
-2c & 0 & -2c' & 0
\end{vmatrix} = 4(bc' - b'c)(a'c - ac' + bd' - b'd).$$

The factor $bc' - b'c$ is proportional to $\omega(u, u')$. We conclude that $\Omega(u, u')$ is proportional to $a'c - ac' + bd' - b'd$, so that $\Omega$ has maximal rank. $\square$

Note that $\omega(u, u') = 0$ if and only if the tangent plane generated by $u$ and $u'$ contains an element $u'' \in Hom(T, Q)$ whose image is contained in $S/T$. If it is not the case, then this tangent plane is isotropic if and only if

$$u(T) + u'(T) \neq Q,$$

that is, there is a hyperplane $H \subset Q$ containing the image of any $u''$ in the tangent plane.

**2.2 Lagrangian surfaces in $F(X)$**

Our very explicit description of the symplectic structure of $F(X)$ will help us to identify Lagrangian surfaces.
2.2.1 Voisin’s example

First we recover the following observation of Claire Voisin (see [Voi]).

**Proposition 2**  Let \( Y = X \cap H \) be a generic hyperplane section. Then the Fano surface \( F(Y) \) of lines in \( Y \) is a Lagrangian surface in \( F(X) \).

**Proof.** For \( u, u' \) in \( T_1F(Y) \subset \text{Hom}(T, Q) \), the images \( u(T) \) and \( u'(T) \) are contained in the hyperplane of \( Q \) defined by \( H \). So clearly \( K(u \wedge u', u \wedge u') = 0 \). But \( \omega(u, u') \) is non zero, at least generically, so \( \Omega(u, u') = 0 \). \( \square \)

2.2.2 Planes in cubic fivefolds

We now turn to the inverse situation where \( X \) is a hyperplane section of a cubic fivefold \( Z \subset \mathbb{P}^6 = \mathbb{P}W \), say \( X = Z \cap H \) with \( H = \mathbb{P}V \). Generically, such a cubic fivefold contains projective planes, and the Fano variety \( F_2(Z) \) of planes in \( Z \) is a smooth surface in the Grassmannian \( G(3, W) \). Cutting such a plane \( \Pi \) with the hyperplane \( H \) we get a line in \( X \), because \( X \) contains no plane. Hence a map

\[
i_Z : F_2(Z) \to F(X).
\]

**Lemma 3**  For general \( X \) and \( Z \), the map \( i_Z \) is a closed embedding.

**Proof.** Two planes in a general cubic fivefold cannot meet along a line, so \( i_Z \) is injective. We check it is also immersive.

Let \( \Pi = \mathbb{P}S \) be a plane in \( Z \), such that \( \ell = P \cap H \) is a line in \( X \). If \( R = 0 \) is an equation of \( Z \), we have

\[
T_{\Pi}F_2(Z) = \{ u \in \text{Hom}(S, W/S), \quad R(x, x, u(x)) = 0 \quad \forall x \in S \}.
\]

The differential of \( i_Z \) at \( \Pi \) maps \( u \in \text{Hom}(S, W/S) \) to its restriction \( u|_T \) from \( T \) to \( V/T \cong W/S \). If this restriction is zero, then \( u = e^* \otimes f \) has rank one and \( R(x, x, f) = 0 \) for all \( x \in S \). This means that the intersection of the tangent hyperplanes to \( Z \) along \( \Pi \) intersect along a linear space strictly larger that \( \Pi \). Call such a plane special. We claim that a general cubic fivefold contains no special plane. Indeed, choose linear coordinates \( x_0, \ldots, x_6 \) on \( \mathbb{P}^6 \), such that the plane \( \Pi \) is defined by \( x_3 = x_4 = x_5 = x_6 = 0 \). If \( Z \) contains \( \Pi \), its equation is of the form \( x_3Q_3 + x_4Q_4 + x_5Q_5 + x_6Q_6 \) for some quadrics \( Q_3, \ldots, Q_6 \). The intersection of the tangent hyperplanes to \( Z \) along \( \Pi \) contains \( \Pi^+ \) of equations \( x_4 = x_5 = x_6 = 0 \) if and only if the quadric \( Q_3 = 0 \) contains \( \Pi \). So \( \Pi \) is special if and only if it is special in the sense of [Col]. Collino proves that if \( \Pi \) is contained in the smooth locus of \( Z \), then it defines a smooth point of \( F_2(Z) \) if and only if it is non special. In particular a smooth \( Z \) such that \( F_2(Z) \) is smooth contains no special plane, and \( i_Z \) is immersive. \( \square \)

**Proposition 4**  The map \( i_Z \) embeds \( F_2(Z) \) in \( F(X) \) as a Lagrangian surface.

**Proof.** If \( \omega \) vanishes on \( (i_Z)_*T_{\Pi}F_2(Z) \), then \( T_{\Pi}F_2(Z) \) must contain a morphism \( u \in \text{Hom}(S, W/S) \) whose restriction to \( T \) has rank one. But this cannot happen generically, so we just need to prove that our quadratic form \( K \) on \( \wedge^2T_1F(X) \) vanishes on the line \( \wedge^2(i_Z)_*T_{\Pi}F_2(Z) \).

By definition of \( K \) this means that two morphisms in \( T_{\Pi}F_2(Z) \) cannot send \( T \) to two independent subspaces in \( W/S \). So our claim follows from the following observation:
Lemma 5 There exists a hyperplane \( h_\Pi \) in \( W/S \) such that for all \( u \in T_\Pi F_2(Z) \subset Hom(S,W/S) \), we have \( u(S) \subset h_\Pi \).

Proof. Again we just need to prove this for a general cubic fivefold \( Z \) and a general plane \( \Pi \in F_2(Z) \). The cubic equation \( R \) of \( Z \) defines a linear map \( \rho : W/S \to Sym^2 S^* \), mapping \( \bar{w} \) to the quadratic form \( x \mapsto R(x,x,w) \) on \( S \), for any representative \( w \) of \( \bar{w} \) in \( W \). Consider the diagram

\[
\begin{array}{ccc}
\wedge^2 S^* \otimes S^* & \downarrow & S^* \otimes Sym^2 S^* \\
T_\Pi F_2(Z) \subset Hom(S,W/S) & \overset{id \otimes \rho}{\longrightarrow} & Sym^3 S^*
\end{array}
\]

where the vertical strand is part of a Koszul complex. Since \( T_\Pi F_2(Z) \) maps to zero in \( Sym^3 S^* \), its image by \( id \otimes \rho \) lies in the image of \( \wedge^2 S^* \otimes S^* \). Note that since \( S \) has dimension three, \( \wedge^2 S^* \otimes S^* = det S \otimes End(S) \). Once we chose a generator \( \alpha \) of \( det S \), we thus conclude that for any \( u \in T_\Pi F_2(Z) \), there exists some \( \theta_u \in End(S) \) such that

\[
R(x,x,u(y))\alpha = \theta_u(x) \wedge x \wedge y \quad \forall x, y \in S.
\]

Generically, the endomorphism \( \theta_u \) is semisimple, and for any \( z \in u(S) \) the quadratic form \( x \mapsto R(x,x,z) \) vanishes along the three eigenlines of \( \theta_u \).

We interpret this as follows. The pull-back by \( \rho \) of the discriminant defines a cubic surface \( \Sigma \) in \( \mathbb{P}(W/S) \), and this surface is smooth by the genericity assumption. Since the image of the plane \( \mathbb{P}u(S) \) by \( \rho \) is the net of conics passing through three general points, \( \mathbb{P}u(S) \) is a trisecant plane to the surface \( \Sigma \). But there are only a finite number of such trisecant planes, so \( u(S) \) does not depend on \( u \) (as long as \( u \) has maximal rank). This is what we wanted to prove. \( \square \)

2.2.3 Conics on Fano threefolds of genus four

A Fano threefold of genus four is the complete intersection \( Y = Q \cap W \) of a quadric \( Q \) and a cubic hypersurface \( W \) in \( \mathbb{P}^5 \).

Proposition 6 For \( Y \) general, the set of lines in \( Y \) is a smooth curve \( \Gamma(Y) \) and the set of conics in \( Y \) is a smooth surface \( F(Y) \).

Proof. Choosing equations for \( Q \) and \( W \) we get sections \( \sigma_Q \) and \( \sigma_W \) of \( S^2 T^* \) and \( S^3 T^* \) on \( G(2,6) \), and the zero locus of \( \sigma_Q \oplus \sigma_W \), is precisely \( \Gamma(Y) \). Hence the first assertion.

The Hilbert scheme \( \mathcal{H} \) of conics in \( \mathbb{P}^5 \) is nothing else than the total space of the projective bundle \( \mathbb{P}(S^2 T^*) \xrightarrow{\pi} G(3,6) \). On \( \mathcal{H} \) we have a tautological sequence

\[
0 \to \mathcal{O}(-1) \to \pi^* S^2 T^* \to Q \to 0,
\]

and the equation of \( Q \) defines a section \( \tau_Q \) of \( Q \) whose zero-locus is the set of conics contained in \( Q \). In a similar way, the vector bundle \( \mathcal{M} \) defined by the exact sequence

\[
0 \to \pi^* T^* \otimes \mathcal{O}(-1) \to \pi^* S^3 T^* \to \mathcal{M} \to 0,
\]
has a global section \( \tau_W \) defined by the equation of \( W \) whose zero locus is the set of conics contained in \( C \). Now \( \tau_Q \oplus \tau_W \) is a general section of the globally generated vector bundle \( Q \oplus M \), whose zero locus is \( F(Y) \). So by Bertini, \( F(Y) \) is a smooth surface. \( \square \)

**Remark.** The invariants of the curve \( \Gamma(Y) \) were computed in [Mar], in particular its genus is \( g(\Gamma(Y)) = 271 \). We will compute the invariants of \( F(Y) \) in the next section. As noticed in Ch. 2, §4 of [Isk], \( F(Y) \) contains a special curve \( B(Y) \subset F(Y) \) that consists of conics that lie in the ruling planes of \( Q \). Since \( Q \) has two family of ruling planes, this curve has two connected components \( B_+(Y) \) and \( B_-(Y) \).

Now consider in \( \mathbb{P}^6 \) a general nodal cubic 5-fold, \( X \), with equation \( x_0Q + W \), where \( x_0 \) is an equation of the hyperplane \( \mathbb{P}^5 \supset Q, W \). Denote its node by \( e_o = (1 : 0 : \ldots : 0) \). The projective tangent cone \( K(Q) \) to \( X \) at \( e_o \) is the cone in \( \mathbb{P}^6 \) with vertex \( e_o \) over the quadric \( Q \) its intersection with \( X \) is the cone \( K(Y) \) over the Fano 3-fold \( Y = W \cap Q \subset \mathbb{P}^5 \).

From another point of view, the cone \( K(Y) \) is the union of all lines \( \ell \subset X \) which pass through the node of \( X \); in other words, \( Y \) is the base of the family of lines on \( X \) that pass through \( e_o \). Denote by \( F_2(X) \) the Fano variety of planes in \( X \) We shall construct a natural map

\[
f : F(Y) \rightarrow F_2(X)
\]

as follows: Let \( q \in Y \) be a conic. Then \( q \), together with the node \( e_o \) span a 3-space \( P^3_q \), which intersects \( X \) along a cubic surface \( \mathcal{S}_q = P^3_q \cap X \). This surface is not irreducible, since it contains the quadratic cone \( K(q) \) with vertex \( e_o \) and base \( q \). Therefore

\[
\mathcal{S}_q = K(q) + \mathbb{P}^2_q,
\]

where \( P^2_q \subset X \) is a plane, and we let \( q \mapsto f(q) := P^2_q \).

Note that \( \ell_q := P^2_q \cap W \) is a line that is 2-secant to \( q \). We denote by \( j \) the map \( F(Y) \rightarrow F(W) \) sending \( q \) to \( \ell_q \).

Next, we shall try to find where is defined the inverse of \( f : F(Y) \rightarrow F_2(X) \).

First note that any plane \( \Pi \) in \( X \) that passes through \( e_o \) evidently lies in \( K(Y) \) and intersects on \( Y \) a line \( \ell = \Pi \cap Y \). Conversely, \( \Pi \) is just the span of \( \ell \cup e_o \). In other words, the curve of lines \( \Gamma(Y) \) is the base of the family \( \Gamma(X) \) of planes on \( X \) that pass through \( e_o \). in particular \( \Gamma(X) \cong \Gamma(Y) \).

Let \( \Pi \in F_2(X) - \Gamma(X) \), and let \( \ell = \Pi \cap Y \in \Gamma(Y) \) be its corresponding line on \( Y \). Since \( e_o \notin \Pi \), the span of \( \Pi \cup e_o \) is a 3-space, that intersects \( X \) along a cubic surface \( \mathcal{S}_\Pi = \Pi \cup Q_\Pi \), where \( Q_\Pi \) is a quadric surface in \( X \).

Since \( \mathcal{S}_\Pi \) clearly passes through \( e_o = \text{Sing}(X) \), but \( \Pi \) does not, the quadric \( Q_\Pi \) must be singular at \( e_o \). Therefore \( Q_\Pi \) must be a quadratic cone with vertex \( e_o \), meeting \( \mathbb{P}^5 \) along a conic \( q \) on \( Y = W \cap Q \). Clearly \( q = f^{-1}(\Pi) \).

Thus \( f \) is invertible outside \( \Gamma(X) \cong \Gamma(Y) \). Their remains to find the pre-images on \( F(Y) \) of the planes \( \Pi \in \Gamma(X) \subset F_2(X) \). Let \( \ell = \Pi \cap Y \in \Gamma(Y) \).

Let \( q \in f^{-1}(\Pi) \). Then with the previous notations \( \mathcal{S}_q = P^3_q \cap X \) is a cone with vertex \( e_o \), which can be defined as the set of points \( (x_0, y) \), with \( y \) in the span \( \mathbb{P}^2_q \) of \( q \), such that \( x_0Q(y) + W(y) = 0 \). So \( Q \) must contain the plane \( \mathbb{P}^2_q \), which itself contains \( \ell \). But a line in a four dimensional quadric is contained in exactly one plane in each of the two rulings of \( Q \). Denote
the two planes in $Q$ that contain $\ell$ by $P_+$ and $P_-$. The intersection of $W$ with $P_{\pm}$ is the union of $\ell$ with a conic $q_{\pm}$, and we conclude that

$$f^{-1}(\Pi) = \{q_+, q_-\}.$$ We summarize our discussion.

**Proposition 7** The map $f : F(Y) \to F_2(X)$, $q \mapsto P^2_q$ is an isomorphism outside the curve $B(Y) = B_+(Y) \cup B_-(Y)$ subset $F(Y)$, and the restriction of $f$ to $B_{\pm}(Y)$ is an isomorphism with $\Gamma(Y)$.

In particular $\Gamma(Y)$ is the singular locus of $F_2(X)$, whose normalization is $f$.

Now, since the condition of being Lagrangian is closed, we can deduce from Propositions 4 and 7 the following result:

**Proposition 8** For any general cubic fourfold $W \subset \mathbb{P}^5$ and any general quadric $Q \subset \mathbb{P}^5$, the map sending a conic $q \subset Y = Q \cap W$ to the residual line $\ell_q$ in the intersection of the linear span $P^2_q$ of $q$, with $W$, maps the surface $F(Y)$ of conics in $Y$ to a singular Lagrangian surface in $F(W)$.

3 The family of conics on the Fano threefold of genus four

3.1 Invariants of the Fano surface $F(Y)$

Since $F(Y) \subset \mathcal{H}$ is defined as the zero-locus of a general section of the vector bundle $\mathcal{Q} \oplus \mathcal{M}$, its fundamental class is given by the Thom-Porteous formula, that is

$$[F(Y)] = c_{12}(\mathcal{Q} \oplus \mathcal{M}) = c_5(\mathcal{Q})c_7(\mathcal{M}) \in A^{12}(\mathcal{H}).$$

Recall that the Chow ring $A(\mathcal{H})$ is generated over $A(G)$ by the hyperplane class $h = c_1(\mathcal{O}(1))$, modulo the relation $c_6(\mathcal{Q}) = 0$. In particular $A(\mathcal{H})$ is a free $A(G)$-module with basis $1, \ldots, h^5$.

We say a class in $A(\mathcal{H})$ is written in normal form when it is expressed in that basis. We have

$$c(\mathcal{Q}) = \frac{c(S^2T^*)}{1 - h}, \quad c(\mathcal{M}) = \frac{c(S^3T^*)}{c(T^*(1))}.$$ In particular the first identity gives the relation $hc_5(\mathcal{Q}) = -c_6(S^2T^*)$.

The Chern classes of $S^2E$ and $S^3E$, for $E$ a vector bundle of rank three, are given by universal formulas in terms of the Chern classes of $E$, that we denote by $c_1, c_2, c_3$. The splitting principle easily gives

$$c(S^2E) = (1 + 2c_1 + 4c_2 + 8c_3)(1 + 2c_1 + c_1^2 + c_2 + c_1c_2 - c_3).$$

The computation for $S^3E$ is much more complicated. With the help of Maple we get the following
If $x_1, x_2, x_3$ are the Chern roots of $E$, and if $L$ is a line bundle with first Chern class $h$, we can also compute

$$c(E \otimes L^*)^{-1} = \prod_i (1 + x_i - h)^{-1} = (1 - h)^{-3} \prod_i (1 + \frac{x_i}{1-h})^{-1} = \sum_{k \geq 0} (-1)^k s_k (E)(1 - h)^{-k-3},$$

where $s_k$ denotes the $k$-th Segre class of $E$.

Now we specialize to $E = \pi^* T^*$ (for simplicity we omit the symbol $\pi^*$ in the sequel) and $L = O(1)$, and we deduce that

$$c_7(M) = \sum_{j,k \geq 0} (-1)^k \binom{j + k + 2}{j} c_{7-j-k}(S^3 T^*) s_k(T^*) h^j.$$  

We use the basis of $A(G)$ given by the Schubert classes $\sigma_{ijk}$, $3 \geq i \geq j \geq k \geq 0$. Among these are the Chern classes $c_i(T^*) = \sigma_i$ and the Segre classes $s_k(T^*) = \sigma_{1k}$ of $T^*$. In the Schubert basis the total Chern class of $S^2 T^*$ is

$$c(S^2 T^*) = 1 + 4\sigma_1 + 5\sigma_2 + 5\sigma_{11} + 20\sigma_3 + 15\sigma_{21} + 30\sigma_{31} + 10\sigma_{22} + 6\sigma_{211} + 20\sigma_{32} + 12\sigma_{311} + 4\sigma_{221} + 8\sigma_{321}.$$  

Denote by $c_7(M)_{(j)}$ the coefficient of $h^j$ in that sum, for $0 \leq j \leq 7$. Using the relation $hc_5(Q) = -c_6(S^2 T^*) = -8\sigma_{321}$, we get

$$[F(Y)] = c_5(Q) c_7(M)_{(0)} - 8\sigma_{321} \sum_{j=1}^{7} c_7(M)_{(j)} h^{j-1}.$$  

This almost expresses $[F(Y)]$ in normal form, except for the term with $j = 7$ in the sum. Since $c_7(M)_{(7)} = 36$, the normal form is

$$[F(Y)] = c_5(Q) c_7(M)_{(0)} - 8\sigma_{321} \sum_{j=1}^{6} c_7(M)_{(j)} h^{j-1} + 288\sigma_{321} \sum_{i=0}^{5} c_{6-i}(S^2 T^*) h^i.$$  

The computations that remain are to be done in $A(G)$, which is zero in degree greater than nine. Since $[F(Y)]$ has degree 12, it must be a combination of the four classes $\sigma_{333} h^3$, $\sigma_{332} h^4$, $\sigma_{331} h^5$ and $\sigma_{322} h^5$.  

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First we compute $c_7(M)_{(0)} = c_7(S^3T^*) - c_6(S^3T^*)\sigma_1 + c_5(S^3T^*)\sigma_{11} - c_4(S^3T^*)\sigma_{111}$. The formulas for the Chern classes of a third symmetric power give

\[
\begin{align*}
    c_7(S^3T^*) &= 8820\sigma_{331} + 6342\sigma_{322}, \\
    c_6(S^3T^*) &= 3675\sigma_{333} + 7140\sigma_{321} + 1302\sigma_{221}, \\
    c_5(S^3T^*) &= 2870\sigma_{332} + 2436\sigma_{321} + 1442\sigma_{211}, \\
    c_4(S^3T^*) &= 1155\sigma_{331} + 560\sigma_{22} + 588\sigma_{211},
\end{align*}
\]

hence $c_7(M)_{(0)} = 1757\sigma_{331} + 1190\sigma_{332}$. The other relevant classes for $M$ are

\[
\begin{align*}
    c_7(M)_{(6)} &= 196\sigma_1 \\
    c_7(M)_{(5)} &= 595\sigma_2 + 406\sigma_{11} \\
    c_7(M)_{(4)} &= 1375\sigma_3 + 1500\sigma_{21} + 404\sigma_{111}
\end{align*}
\]

The final result of the computation is:

**Proposition 9** The fundamental class of the surface $F(Y) \subset H$ is

\[
[F(Y)] = 15840\sigma_{333}h^3 + 8100\sigma_{332}h^4 + (1341\sigma_{331} + 774\sigma_{322})h^5.
\]

In particular, we have the following intersection numbers in the Chow ring of $F(Y)$:

\[
h^2 = 0, \quad h\sigma_1 = -360, \quad \sigma_2 = 1341, \quad \sigma_1^2 = 2115.
\]

Now we can compute the Chern numbers of $F(Y)$. From the exact sequences

\[
0 \longrightarrow TF(Y) \longrightarrow T\mathcal{H}_F \longrightarrow \mathcal{Q}_F \oplus M_{[F} \longrightarrow 0, \\
0 \longrightarrow T^*\mathcal{H} \longrightarrow T\mathcal{H} \longrightarrow \pi^*TG \longrightarrow 0
\]

and the observation that the vertical tangent bundle of the fibration $\pi$ is nothing else than $\mathcal{Q}(1)$, we deduce that

\[
\begin{align*}
    c_1(F(Y)) &= 2h - 3\sigma_1, \\
    c_2(F(Y)) &= -3h^2 - 9h\sigma_1 + 13\sigma_{11} - \sigma_2.
\end{align*}
\]

**Corollary 10** The Chern numbers and the arithmetical genus of the Fano surface $F(Y)$ are

\[
c_1(F(Y))^2 = 23355, \quad c_2(F(Y)) = 11961, \quad p_a(F(Y)) = 2942.
\]

**Remark.** We can compare with the invariants of the Fano surface $F_2(Z)$ of planes in a cubic fivefold $Z \subset \mathbb{P}^6$. As a subvariety of $G(3,7)$, $F_2(Z)$ is defined as the zero locus of the global section of the rank ten vector bundle $S^3T^*$, defined by an equation of $Z$. Therefore

\[
[F_2(Z)] = c_{10}(S^3T^*) = 1134\sigma_{442} + 1701\sigma_{433},
\]

as we can easily deduce from the expression given above for the Chern classes of a third symmetric power. On $F_2(Z)$ we thus have $\sigma_2 = 1134$ and $\sigma_1^2 = 2835$. Since $c(TF) = 1 - 3\sigma_1 + 13\sigma_{11} - \sigma_2$, we conclude that

\[
\begin{align*}
    c_1(F_2(Z))^2 &= 25515, \quad c_2(F_2(Z)) = 13041, \quad p_a(F_2(Z)) = 3212.
\end{align*}
\]
In particular we have the relation $p_a(F(Y)) = p_a(F_2(Z)) + 1 - g(\Gamma(Y))$, as expected.

We can also deduce the number $\nu$ of conics in $Y$ passing through a general point. For this consider the universal conic $C \to \mathcal{H}$. This is a subscheme of the universal supporting plane $\mathcal{P} = \mathbb{P}(\pi^*T) \to \mathcal{H}$. Let $\gamma = \pi \circ \rho$. On $\mathcal{P}$ we have a tautological line bundle $\mathcal{O}_\mathcal{P}(-1) \subset \gamma^*T$, and a tautological section $\tau$ of $\rho^*\mathcal{O}_\mathcal{H}(1) \otimes \mathcal{O}_\mathcal{P}(2)$ defined by the composition

$$\rho^*\mathcal{O}_\mathcal{H}(-1) \hookrightarrow \gamma^*S^2T^* \longrightarrow \mathcal{O}_\mathcal{P}(2);$$

the zero locus of $\tau$ is precisely $C$.

Let $V \subset \mathbb{C}^6$ be a general three-dimensional subspace. The zero locus $\mathcal{P}_V$ of the induced morphism

$$\mathcal{O}_\mathcal{P}(-1) \subset \gamma^*T \longrightarrow (\mathbb{C}^6/V) \otimes \mathcal{O}_\mathcal{P}$$

is the set of points in $\mathcal{P}$ mapped to $\mathbb{P}V$ by the natural morphism $\mathcal{P} \longrightarrow \mathbb{P}^5$. In particular the class of $\mathcal{P}_V$ is $H^3$, if $H$ denotes the first Chern class of $\mathcal{O}_\mathcal{P}(1)$, and the class of its intersection $\mathcal{C}_V$ with $\mathcal{C}$ is $(2H + h)H^3$.

Now the intersection of the cycle $\mathcal{C}_V$ with $\rho^{-1}(F(Y))$ is the number of conics in $Y$ meeting $\mathbb{P}V$. Since $Y$ has degree 6 this number is equal to $6\nu$, and therefore

$$6\nu = (2H + h)H^3 \rho^*[F(Y)] = \rho_*(2H^4 + hH^3)[F(Y)].$$

But $\rho_*H^3 = -\sigma_1$ and $\rho_*H^4 = \sigma_{11}$, so using Proposition 11 we obtain:

**Proposition 11** There are $\nu = 318$ conics in $Y$ passing through a given general point.

Along the same line of ideas, one can consider in $F(Y)$ the curve $\Delta$ of degenerate conics, and the Steiner map $s : \Delta \to \mathbb{P}^5$ mapping each such conic to its vertex. The same arguments as in [CMW] yield

$$\deg s(\Delta) = (2h + 2\sigma_1)(3h + 2\sigma_1) = 4860.$$

### 3.2 The Abel-Jacobi isomorphism for $F(Y)$

Since Clemens and Griffiths proved that the Abel-Jacobi mapping $Alb F(X) \to J(X)$ is an isomorphism for $X$ a general cubic threefold and $F(X)$ is Fano variety of lines, similar statements have been obtain for several other Fano manifolds. We fill in a gap in the literature by proving the following statement.

**Theorem 12** Let $Y$ be a general prime Fano threefold of genus 4. Then the Abel-Jacobi mapping

$$Alb F(Y) \longrightarrow J(Y)$$

is an isomorphism; and so by duality $J(Y) = J(Y)^* \simeq Alb F(Y)^* = \text{Pic}^0 F(Y)$.

We adopt the strategy developed by Clemens and Griffiths, and used in [Led] for the quartic threefold, in [CV] for the sextic double solid and in [Col] for the cubic fivefold. The claim is that the theorem will follow from the existence of a Lefschetz pencil $(Y_t)_{t \in \mathbb{P}^1}$ such that

1. $Y_t = Q \cap C_t$ is contained in a fixed smooth quadric $Q$,
2. \( Y_t \) and \( F(Y_t) \) are smooth for general \( t \),

3. if \( Y_t \) is smooth but not \( F(Y_t) \), then the later only has isolated singularities,

4. if \( Y_t \) is singular, then the singular locus of \( F(Y_t) \) is a smooth curve, along which \( F(Y_t) \) has two smooth branches intersecting transversely.

Consider a generic quadric hypersurface \( Q \subset \mathbb{P}^5 \), a generic pencil of cubics \( C_t \subset \mathbb{P}^5 \), \( t \in \mathbb{P}^1 \), and then the pencil of complete intersections \( Y_t = Q \cap C_t \). Condition (2) clearly holds, and (3) follows from a simple dimension count. We focus on condition (4). The pencil \( (Y_t) \) meets the discriminant hypersurface \( \Delta \) parameterizing singular complete intersections of type \( (2, 3) \) at a finite number of points. These points must belong to the dense open subset of \( \Delta \) parameterizing complete intersections with a single node. So we are reduced to proving that for a general nodal complete intersection \( Y \), the Fano surface of conics \( F(Y) \) has the type of singularities allowed by condition (4).

This involves the following steps, which we only sketch since the arguments are close from those of \([\text{Let, CV, Col, PB}]\).

1. A conic \( q \subset Y \) not passing through the vertex \( v \) of \( Y \), defines a smooth point of \( F(Y) \). To check this we need to characterize, for a conic \( q \subset Y_{\text{reg}} \) (\( Y \) having arbitrary singularities), the fact that it defines a smooth point in \( F(Y) \). This goes as follows. Choose coordinates in \( \mathbb{P}^5 \) such that \( q \) be defined by the equations \( x_3 = x_4 = x_5 = 0 \) and \( \bar{q}(x_0, x_1, x_2) = 0 \). If \( Y = Q \cap C \), write the equations of \( Q \) and \( C \) as

\[
Q(x) = \alpha \bar{q}(x_0, x_1, x_2) + x_3l_3 + x_4l_4 + x_5l_5,
C(x) = l\bar{q}(x_0, x_1, x_2) + x_3q_3 + x_4q_4 + x_5q_5.
\]

Suppose that \( \alpha \neq 0 \), which means that the supporting plane of \( q \) is not contained in \( Q \). We say in that case that \( q \) is a non isotropic conic. Then there is a unique cubic hypersurface \( \alpha C - lQ \) containing \( Y \) and this supporting plane. Replacing \( C \) by this cubic we may suppose that \( l = 0 \). Then we consider the exact sequence of normal bundles \( 0 \rightarrow N_{q/X} \rightarrow N_{q/\mathbb{P}^5} \rightarrow N_{X/\mathbb{P}^5|q} \rightarrow 0 \). We have \( N_{q/\mathbb{P}^5} = \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(2) \) and \( N_{X/\mathbb{P}^5} = \mathcal{O}(2) \oplus \mathcal{O}(3) \). Moreover, \( F(Y) \) is smooth at \( q \) if and only if \( h^0(N_{q/X}) = 2 \), which is equivalent to the surjectivity of the map \( H^0(N_{q/\mathbb{P}^5}) \rightarrow H^0(N_{X/\mathbb{P}^5|q}) \). This map is easy to identify in terms of \( \alpha, l_3, l_4, l_5, q_3, q_4, q_5 \), and we come to the following conclusion: \( q \) defines a smooth point of \( F(Y) \) if and only if \( q_3, q_4, q_5 \) define linearly independent sections of \( \mathcal{O}(2)_{|q} \).

Now suppose that \( \alpha \neq 0 \), that is \( q \) is an isotropic conic. We checked that the map \( H^0(N_{q/\mathbb{P}^5}) \rightarrow H^0(N_{X/\mathbb{P}^5|q}) \) is always surjective under the hypothesis that \( q \subset Y_{\text{reg}} \).

2. For \( Y \) nodal but general, \( F(Y) \) is smooth at any point defined by some conic \( q \subset Y_{\text{reg}} \). Indeed we can call a conic in some (arbitrary) \( Y \) special, either if it is non isotropic but does not verify the condition above, or if it is isotropic and meets the singular locus of \( Y \). Then the space of complete intersections \( Y \) containing a special conic has at most two irreducible components, none of which coincides with the discriminant hypersurface.

3. The set of conics in \( Y \) passing through the vertex \( v \), is the union of a complete curve \( D \) parameterizing conics which are all smooth at \( v \), and of 66 conics \( q(l, l') = l + l' \), where \( l \)
and \(l'\) are lines in \(Y\) passing through \(v\). Moreover the conics \(q(l,l')\) define smooth points of \(F(Y)\). Indeed \(Y\) contains 12 lines passing through \(v\); hence the 66 reducible conics. The fact that for \(Y\) general (among nodal complete intersections), a conic of type \(q(l,l')\) defines a smooth point of \(F(Y)\) is a boring computation. This computation also shows that the (linear) condition for a conic to pass through \(v\) is transverse to the tangent space to \(F(Y)\) at \(q(l,l')\), which is thus isolated among the conics in \(Y\) passing through \(v\).

4. Let \(\mathbb{P}^+ \to \mathbb{P}^5\) be the blow-up of \(v\), and \(Y^+ \to Y\) the strict transform of \(Y\). Let \(F(Y^+) \xrightarrow{\pi} F(Y)\) be the Fano surface of conics in \(Y^+\). Then \(\pi^{-1}(D) = D_+ \cup D_-\) is the union of two curves, and the restriction of \(\pi\) to \(D_\pm\) is a bijection with \(D\). More precisely, \(F(Y^+)\) is the component of the Hilbert scheme of \(Y^+\) containing the strict transforms of the conics in \(Y\) not passing through \(v\). If \(\tilde{q} \in \pi^{-1}(q)\), where \(q\) is a conic passing through \(v\) but smooth at that point, then \(\tilde{q}\) must be the union of the strict transform \(\tilde{q}\) of \(q\), with a line \(\ell\) in the exceptional divisor. Moreover \(\ell\) must meet \(\tilde{q}\) at the point defined by the tangent line to \(q\) at \(v\). But \(S = E \cap Y^+\) is a smooth quadric surface, hence through that point pass exactly two lines \(\ell_+\) and \(\ell_-\), one from each of the two rulings of \(S\). This is why \(\pi^{-1}(q) = \{\tilde{q} + \ell_+, \tilde{q} + \ell_-\}\), and \(\pi^{-1}(D) = D_+ \cup D_-\).

5. The Fano surface \(F(Y^+)\) of conics in \(Y^+\) is smooth along \(D_\pm\), and the differential of the birational morphism \(F(Y^+) \to F(Y)\) maps the tangent planes to \(F(Y^+)\) along \(D_\pm\) to planes in the Zariski tangent space to \(F(Y)\), meeting exactly along the tangent space to \(D\).

The smoothness assertion follows from a computation with normal bundles, using the fact that \(Y^+\) is a complete intersection in \(\mathbb{P}^+ \subset \mathbb{P}^5 \times \mathbb{P}^4\), of type \((1,1),(1,2)\). Moreover the conic \(\tilde{q} + \ell_\pm\) is also a complete intersection, of type \((1,0),(0,1),(0,1),(1,1)\). If \(p = \tilde{q} \cap \ell_\pm\), we have a commutative diagram

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & H^0(N) & H^0(N(\ell_\pm)) & H^0(N(\tilde{q})) & N_p & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & H^0(E) & H^0(E(\ell_\pm)) & H^0(E(\tilde{q})) & E_p & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & H^0(F) & H^0(F(\ell_\pm)) & H^0(F(\tilde{q})) & F_p & 0 \\
\end{array}
\]

where \(N\) denotes the normal bundle of \(\tilde{q}\) in \(Y^+\) (recall that \(Y^+\) is smooth and \(\tilde{q}\) is a locally complete intersection), \(E\) denotes the normal bundle of \(\tilde{q}\) in \(\mathbb{P}^+\), and \(F\) the normal bundle of \(Y^+\) in \(\mathbb{P}^+\), restricted to \(\tilde{q}\). Moreover \(N(\ell_\pm)\) denotes the restriction of \(N\) to \(\ell_\pm\), and so on.

Denote by \(\phi(\ell_\pm)\) the map \(H^0(E(\ell_\pm)) \to H^0(F(\ell_\pm))\). The smoothness of the tangent cone to \(C\) at the vertex \(v\) is enough to imply that \(\phi(\ell_\pm)\) is surjective, hence \(h^0(N(\ell_\pm)) = 2\). Moreover there is no non trivial section vanishing at \(p\), and the evaluation map \(H^0(N(\ell_\pm)) \to N_p\) is an isomorphism. Therefore \(H^0(N) \simeq H^0(N(\tilde{q}))\).

Now denote by \(\phi(\tilde{q})\) the map \(H^0(E(\tilde{q})) \to H^0(F(\tilde{q}))\). The fact that \(q\) is non special implies that \(\phi(\tilde{q})\) is surjective. So \(h^0(N) = h^0(N(\tilde{q})) = 2\) and \(F(Y^+)\) is smooth at \(\tilde{q}_\pm\).

All this also holds for the finite number of isotropic conics passing through the vertex.
Indeed one can check that if it did not hold for such an isotropic conic, then the conic would have to meet another singular point of \( Y \), which is impossible.

Finally, we check that the images of the differentials of \( \pi \) at \( \bar{q} + \ell_{\pm} \) meet along a line, which must be the Zariski tangent space to \( D \). So \( D \) is smooth, and isomorphic with \( D_{\pm} \).

This concludes the proof. \( \square \)

4 Application: Two integrable systems

4.1 Cubic fivefolds containing a given fourfold

By a result of Z. Ran, deformations of smooth Lagrangian subvarieties are unobstructed [Ran]. In particular the component \( \mathcal{H} \) of the Hilbert scheme of \( F(X) \) containing \( F_2(Z) \), for \( Z \) a general cubic fivefold containing \( X \) as hyperplane section, is generically smooth. Consider the universal family \( F \to \mathcal{H} \). Donagi and Markman proved that in such a situation, the relative Picard bundle \( \text{Pic}^0 F \to \mathcal{H} \), is an algebraic completely integrable Hamiltonian system (ACIHS), at least over the open subset \( \mathcal{H}_0 \subset \mathcal{H} \) parameterizing smooth deformations ([DM1], Theorem 8.1). This means that the total space of the fibration admits a symplectic structure such that the fibers are Lagrangian subvarieties. Note that it follows in particular that the dimension of \( \mathcal{H} \) equals \( h^{1,0}(F_2(Z)) = 21 \).

The family \( \mathcal{A} \) of cubic fivefolds \( Z \) containing \( X = Z \cap H \) as a hyperplane section is a parameterized by the linear system

\[ |I_X(3)| = \langle c_X, x_0|\mathcal{O}(2)\rangle, \]

where \( c_X \) is an equation of \( X \) in \( H \) and \( x_0 \) is an equation of \( H \). This linear system has dimension 28 and admits a natural action of the seven-dimensional group \( G \), consisting of automorphisms of \( \mathbb{P}^6 \) whose restriction to \( H \) is trivial. Moreover the subvariety \( j_Z(F_2(Z)) \subset F(X) \) does not change when \( Z \) moves in a \( G \)-orbit. If we denote by \( |I_X(3)|^0 \) the open subset of \( |I_X(3)| \) parameterizing smooth cubic fivefolds \( Z \) such that \( F_2(Z) \) is a smooth surface and \( i_Z \) embeds \( F_2(Z) \) in \( F(X) \), then the induced map

\[ |I_X(3)|^0 \to \mathcal{H} \]

is constant on the \( G \)-orbits, and we may therefore consider \( \mathcal{H} \) as a substitute for the moduli space of isomorphism classes of cubic fivefolds containing \( X \).

Remark. Note that the tangent space to the corresponding deformation is \( H^1(Z, T_Z \otimes I_X) = H^1(Z, T_Z(-1)) \), which can be computed from the exact sequence

\[ 0 \to T_Z(-1) \to T_{\mathbb{P}^6}(-1)|_Z \to \mathcal{O}_Z(2) \to 0. \]

The Euler exact sequence restricted to \( Z \) gives \( H^0(Z, T_{\mathbb{P}^6}(-1)|_Z) = H^0(\mathbb{P}^6, \mathcal{O}(1))^* := V \) and \( H^1(Z, T_{\mathbb{P}^6}(-1)|_Z) = 0 \). Since \( H^0(Z, \mathcal{O}_Z(2)) = H^0(\mathbb{P}^6, \mathcal{O}(2)) = S^2 V^* \), we get that

\[ H^1(Z, T_Z \otimes I_X) = \text{Coker}(V \xrightarrow{c_Z} S^2 V^*) \]

is the cokernel of the map \( c_Z \) defined as the differential of an equation of \( Z \). This map is always injective for \( Z \) smooth, hence \( h^1(Z, T_Z \otimes I_X) = 21 \). In particular we can easily obtain locally
complete families for our deformation problem, parameterized by linear subsystems of \(|I_X(3)|\) transverse to \(\mathbb{P}c_X(V)\).

It was proved by Collino that the Abel-Jacobi morphism

\[ \text{Alb} F_2(Z) \longrightarrow J(Z) \]

to the intermediate jacobian, is an isomorphism for a general cubic fivefold \(Z\) \text{[Co]}]. In particular \(\text{Alb} F_2(Z)\), like \(J(Z)\), is self-dual, hence naturally isomorphic with the Picard variety \(\text{Pic}^0 F_2(Z)\).

Denote by \(\mathcal{H}_s \subset \mathcal{H}\) the open subset parameterizing the Lagrangian surfaces \(i_Z(F_2(Z)) \subset F(X)\), for \(Z\) smooth with \(F_2(Z)\) smooth. Denote by \(\mathcal{H}_a \subset \mathcal{H}_s\) the open subset over which the Abel-Jacobi theorem does hold. We conclude:

**Theorem 13** The relative intermediate jacobian \(J(\mathcal{H}_a)\) over the family \(\mathcal{H}_a\) of smooth cubic fivefolds containing \(X\), is an algebraic completely integrable Hamiltonian system.

In particular, the cubic condition of Donagi and Markman must hold (see \text{[DM1]}, section 7.2). That is, the map

\[ T_{J(Z)} \mathcal{A} = \text{Sym}^2 H^3(Z, \Omega^2_Z) \longrightarrow T_{[Z]} \mathcal{H}_a \simeq H^1(Z, TZ \otimes I_X), \]

where \(\mathcal{A}\) denotes the moduli space of polarized abelian varieties, must be completely symmetric. To be precise, note that \(H^3(Z, \Omega^2_Z)\) can easily be computed with the help of the normal exact sequence for \(Z \subset \mathbb{P}^6\) and the Euler sequence restricted to \(Z\). We obtain

\[ H^3(Z, \Omega^2_Z) = \text{Ker}(S^2V \xrightarrow{c^2} V^*) = H^1(Z, TZ \otimes I_X)^*. \]

Let \(H := H^3(Z, \Omega^2_Z)\). The cubic condition is that the map \(S^2H \rightarrow H^*\) introduced above, which is just the differential of the induced map from \(\mathcal{H}_a\) to the moduli space \(\mathcal{A}\), comes from a cubic form \(\theta_Z \in S^3 H^*\). This form can be described as the tautological cubic

\[ \theta_Z : S^3H \hookrightarrow S^3(S^2V) \longrightarrow S^2(S^3V) \xrightarrow{c^2} \mathbb{C}. \]

### 4.2 Fano threefolds of genus four contained in a given cubic fourfold

We want to extend our ACIHS to an open subset \(\mathcal{H}_a \subset \mathcal{H}\), containing \(\mathcal{H}_s\), parameterizing Lagrangian surfaces \(i_Z(F_2(Z)) \subset F(X)\), where \(Z\) is allowed to be a general cubic fivefold with a node. In this case we know that \(F_2(Z)\) gets singular along a curve, and that its normalization is the Fano surface of conics \(F(Y)\) of a general prime Fano threefold of genus 4.

We begin with a few observations. Let \(Z\) be a general nodal cubic fivefold containing \(X\).

1. The singular surface \(F_2(Z)\) is a flat deformation of the smooth \(F_2(Z')\), with \(Z'\) smooth. This follows e.g. from Kollar’s criterion (6.1.3) in \text{[Ko]}, which applies since \(F_2(Z)\) is a locally complete intersection, hence has property \(S_2\).

2. The Hilbert scheme \(\mathcal{H}\) remains smooth at \(i_Z(F_2(Z))\). This is because \(F_2(Z)\) is a locally complete intersection and \(i_Z\) is a closed embedding. So \(i_Z(F_2(Z))\) is again a locally complete intersection in \(F(X)\), again Lagrangian, so Ran’s result on the unobstructedness of deformations remains valid (see \text{[Kaw]}, and also \text{[PM]}).
Proposition 14 The ACIHS $\text{Pic}^0 F_2(Z) \to \mathcal{H}_s$ extends to $\mathcal{H}_n$.

Proof. Over $\mathcal{H}_s$ we have a natural identification between $H^{0,1}(F_2(Z))$ (the underlying vector space of $\text{Pic}^0 F_2(Z)$) and the dual to the tangent space of the base, that is $H^0(N_{F_2(Z)/F(X)})$, which is isomorphic to $H^{1,0}(F_2(Z))$ since $i_2 F_2(Z)$ is Lagrangian. We need to prove that this identification extends to $\mathcal{H}_n$. Then the claim follows since the cubic condition of Donagi and Markman is closed, and also the property of the symplectic form of being closed.

First we describe, for $Z$ a general nodal cubic, the relevant data in terms of the associated Fano threefold $Y$ of genus 4. Recall that the map $\nu : F(Y) \to F_2(Z)$ restricts to a birational isomorphism between $F(Y) - B_+(Y) \to F_2(Z) - \Gamma(Y)$, and maps the curves $B_+(Y)$ and $B_-(Y)$ isomorphically to $\Gamma(Y)$. Moreover the two branches of $F_2(Z)$ along $\Gamma(Y)$ intersect transversally at every point, and this remains true for $i_2 F_2(Z)$ since $i_2 : F_2(Z) \to F(X)$ is a closed embedding. We denote for simplicity $\Gamma = i_2(\Gamma(Y))$ and $F = i_2 F_2(Z)$.

Since $F$ is a locally complete intersection in $F(X)$, its normal sheaf $N_F = \mathcal{I}_F/\mathcal{T}_F^2, \mathcal{O}_F$ is a locally free $\mathcal{O}_F$-module of rank two. Since the two branches of $F$ along $\Gamma$ are Lagrangian, their tangent spaces generate in $TF(X)$ the orthogonal $TT^\perp$ to $TT$ with respect to the symplectic form. The quotient of $TT^\perp$ by $TT$ is the direct sum of the normal bundles $N_+$ and $N_-$ of $B_+(Y)$ and $B_-(Y)$ in $F(Y)$. In particular the symplectic form restricts to a non degenerate pairing $N_+ \otimes N_- \to \mathcal{O}_T$.

Consider the tangent map $TF(Y) \to \nu^* TF(X) \simeq \nu^* \Omega^1_F(X)$. Dualizing and pushing forward, we get a map $TF(X) \otimes \mathcal{O}_F \to \nu_* \nu^* TF(X) \to \nu_* \Omega^1_F(X)$, whose image we denote by $N_0 \subset \nu_* \Omega^1_F(Y)$. Clearly $N_0$ and $\nu_* \Omega^1_F(Y)$ are isomorphic outside $\Gamma$. At a point $p \in \Gamma$, an element of the fiber of $\nu_* \Omega^1_F(Y)$ consists in pairs of differential forms on the two branches of $F_2(Z)$ at that point. Such a pair is in the image of $TF(X) \otimes \mathcal{O}_F \simeq \Omega^1_F(X) \otimes \mathcal{O}_F$ if and only if the restrictions of the two forms to $T_p \Gamma$ coincide. So the quotient of $\nu_* \Omega^1_F(Y)$ by $N_0$ is naturally identified with $\Omega^1_T$.

On the other hand, there is a natural map from $N_0$ to $N$, which is again an isomorphism outside $\Gamma$. At a point $p \in \Gamma$, we can choose local coordinates $x, y, z, t$ on $F(X)$ such that $F$ is given by the equations $xy = z = 0$, and $\Gamma$ by $x = y = z = 0$. An element in the fiber of $N_0$ at $p$ can be seen as a linear form on $\langle dx, dy, dz \rangle$, and it is mapped in $N$ to the corresponding linear form on $\langle d(xy) = xdy + ydx, dz \rangle$. This maps $N_0$ injectively in $N$, with image the subsheaf of $N$ defined locally by the condition that the image of $xy$ vanishes along $\Gamma$. Globally, this means that $N_0$ is the kernel of the natural map $N \to N_+ \otimes N_-$. And we have noticed that this line bundle on $\Gamma$ is trivialized by the symplectic form. Note that in terms of deformations, $N_0$ is the subsheaf of $N$ that preserves the singularity $xy = 0$ at first order.

We get the following diagram:

$$
\begin{array}{ccccccc}
0 & \rightarrow & N_0 & \rightarrow & \nu_* \Omega^1_F(Y) & \rightarrow & \Omega^1_T & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
N & \rightarrow & \mathcal{O}_T & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & & & & & \\
\end{array}
$$

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Taking global sections, we get that $h^0(N) \leq 1 + h^0(N_0) \leq 1 + h^0(\Omega^1_{F(Y)})$. But $h^0(N) = 21$ and $h^{1,0}(F(Y)) = 20$, so that these inequalities are equalities, and we obtain the exact sequence

$$0 \to H^0(N_0) \simeq H^0(\Omega^1_{F(Y)}) \to H^0(N) \to H^0(\mathcal{O}_T) \to 0.$$ 

This must be interpreted as follows: $H^0(N)$ is the tangent space to the Hilbert scheme of $F(X)$ at the point defined by $F_2(Z)$ (recall that the Hilbert scheme is smooth at that point); $H^0(N_0) \simeq H^0(\Omega^1_{F(Y)})$ is the tangent space to the hypersurface parameterizing the surfaces $i_ZF_2(Z')$ for $Z'$ a nodal cubic, or equivalently the images of the Fano surfaces $F(Y)$ for $Y$ a Fano threefold of genus 4 contained in $X$. And $H^0(\Omega^1_{F(Y)})$ is naturally isomorphic with the first order deformations of $Y$ (see ?).

On the other hand, we have an exact sequence $0 \to \mathcal{O}_F \to \nu^*\mathcal{O}_{F(Y)} \to \mathcal{O}_T \to 0$, (where the map to $\mathcal{O}_T$ is given by the difference of the restrictions to $B_+(Y)$ and $B_-(Y)$). Hence the exact sequence

$$0 \to H^0(\mathcal{O}_T) \to H^1(\mathcal{O}_F) \to H^1(\mathcal{O}_{F(Y)}) \to 0.$$ 

Now we consider the relative Picard fibration $Pic^0(F) \to \mathcal{H}_n$. The symplectic form defined over $\mathcal{H}_s$ has a natural extension to $\mathcal{H}_n$ which can be defined as follows (see [DXT], section 8.4). If $L$ is an invertible sheaf over $F_2(Z)$, considered as a torsion sheaf on $F(X)$ then the tangent space to $Pic^0(F)$ at $L$ is naturally identified with $Ext^1_{\mathcal{O}_{F(X)}}(L, L)$. And there is a natural skew-symmetric form on this space, defined by the composition

$$\wedge^2 Ext^1_{\mathcal{O}_{F(X)}}(L, L) \to Ext^2_{\mathcal{O}_{F(X)}}(L, L) \xrightarrow{c_1(\mathcal{O}(1))} Ext^4_{\mathcal{O}_{F(X)}}(L, L) \simeq Hom(L, L)^* = \mathbb{C},$$

where we have used Serre duality and the fact that the canonical sheaf of $F(X)$ is trivial. Now the local to global spectral sequence for $Ext^*$ easily yields an exact sequence

$$0 \to H^1(\mathcal{O}_F) \to Ext^1_{\mathcal{O}_{F(X)}}(L, L) \to H^0(N) \to 0,$$

which is nothing else than the tangent sequence of the Picard fibration. Since the fibration is Lagrangian over $\mathcal{H}_s$, by continuity $H^1(\mathcal{O}_F)$ is isotropic. So the skew-symmetric form on $Ext^1_{\mathcal{O}_{F(X)}}(L, L)$ is non degenerate if and only if the induced pairing

$$H^1(\mathcal{O}_F) \otimes H^0(N) \to H^1(N) \xrightarrow{c_1(\mathcal{O}(1))} H^2(N \otimes \Omega^1_{F(X)}|_F) \to H^2(det N) = H^2(\omega_F) = \mathbb{C}$$

is non degenerate. To state it in a more convenient form, we need to prove that the map

$$\mathbb{C} \xrightarrow{c_1(\mathcal{O}(1))} H^1(\Omega^1_{F(X)}|_F) \to H^1(N) \to Hom(H^1(\mathcal{O}_F), H^0(N)^*)$$

maps $c_1(\mathcal{O}(1))$ to an isomorphism. Over $\mathcal{H}_s$, $N$ is identified with $\Omega^1_F$ and this follows from the Hard Lefschetz theorem. Over $\partial \mathcal{H}_n := \mathcal{H}_n - \mathcal{H}_s$ we use our exact sequences

$$0 \to H^0(\mathcal{O}_T) \to H^1(\mathcal{O}_F) \to H^1(\mathcal{O}_{F(Y)}) \to 0$$
$$0 \to H^1(\Omega^1_T) \to H^0(N)^* \to H^0(\Omega^1_{F(Y)})^* \to 0$$
The restricted map $H^0(\mathcal{O}_\Gamma) \to H^0(N)^* \to H^0(\Omega_{\Gamma}^1)^*$ is zero. This can be seen as follows: there is a commutative diagram 

\[
\begin{array}{cccc}
H^0(N) \otimes H^1(\mathcal{O}_F) & \longrightarrow & H^1(N) & \longrightarrow & c_1(\mathcal{O}(1)) \longrightarrow H^2(\omega_F) \\
\uparrow & & \uparrow & & \uparrow \\
H^0(N_0) \otimes H^0(\mathcal{O}_\Gamma) & \longrightarrow & H^1(N_0 \otimes \mathcal{O}_\Gamma) & \longrightarrow & c_1(\mathcal{O}(1)) \longrightarrow H^1(\omega_F \otimes \mathcal{O}_\Gamma)
\end{array}
\]

where the right arrow of the bottom line involves the sheaf morphisms 

$N_0 \otimes \Omega_{\Gamma}^1 \to N_0 \wedge N \to \det N = \omega_F$.

But the image of $\Omega_{\Gamma}^1$ in $N$ is precisely $N_0$, and $\wedge^2 N_0 \otimes \mathcal{O}_\Gamma = 0$.

Now, the induced map

$H^1(\mathcal{O}_{\Gamma}(Y)) \longrightarrow H^0(\Omega_{\Gamma}^1)^* \simeq H^2(\Omega_{\Gamma}^1)$

is defined by the cup product with $c_1(\nu^* \mathcal{O}(1))$. Since $\nu$ is finite $\nu^* \mathcal{O}(1)$ is still ample, so Hard Lefschetz applies and this map is an isomorphism. On the other hand the other induced map $H^0(\mathcal{O}_\Gamma) \longrightarrow H^1(\Omega_{\Gamma}^1)$ is given by $c_1(\mathcal{O}(1)|_\Gamma)$, and again it is an isomorphism. This concludes the proof. \qed

**Proposition 15** Let $Z$ be a general nodal cubic and $Y$ the associated prime Fano threefold of genus 4. There is an exact sequence

$0 \longrightarrow \mathbb{C}^* \longrightarrow Pic^0 F_2(Z) \xrightarrow{\nu^*} Pic^0 F(Y) \simeq J(Y) \longrightarrow 0$.

**Proof.** The fact that $Pic^0 F(Y) \simeq J(Y)$ is Theorem 12. Now consider the exact sequence $0 \to \mathcal{O}_{\Gamma}^* \to \nu_* \mathcal{O}_{\Gamma}(Y) \to \mathcal{O}_{\Gamma}(Y) \to 0$, and the associated long exact sequence

$0 \longrightarrow H^0(\mathcal{O}_{\Gamma}(Y)) = \mathbb{C}^* \longrightarrow Pic F_2(Z) \xrightarrow{\nu^*} Pic F(Y)$.

The pull-back of a Weil divisor from $F_2(Z)$ to $F(Y)$ has the same intersection number with the curves $B_+(Y)$ and $B_-(Y)$, and this property characterizes the image of $\nu^*$. In particular it contains $Pic^0 F(Y)$. Finally, $H^0(\mathcal{O}_{\Gamma}^*)$ being connected obviously maps to $Pic^0 F_2(Z)$. Hence the claim. \qed

Recall that the Fano threefold $Y$ has been defined as the intersection in $H = \mathbb{P}^5$ of the fixed cubic fourfold $X$, with the base $Q$ of the tangent cone to $Z$ at its unique node $p_Z$. Since $G$ fixes $H$, the map $Z \mapsto Q$ is constant along the $G$-orbits.

**Proposition 16** There is a birational isomorphism

$|\mathcal{O}_{\mathbb{P}^5}(2)| \longrightarrow \partial \mathcal{H}_n = \mathcal{H}_n - \mathcal{H}_s$.

**Proof.** Let $\Delta_X \subset |I_X(3)|$ denote the discriminant hypersurface, parameterizing the singular cubics containing $X$. For $X$ general $\Delta_X$ is irreducible. Let $\Delta_X^0$ denote the open subset parameterizing cubics with a single node. The map $\Delta_X^0 \to |\mathcal{O}_{\mathbb{P}^5}(2)|$ sending the nodal cubic $Z$ to the
trace on $H$ of its tangent cone at the node, is constant on the $G$-orbits. More precisely, it is easy to check that its fibers are precisely the $G$-orbits. This implies that there exists a $G$-stable open subset $\Delta_X^0 \subset \Delta_X^n$ such that the restriction map $\Delta_X^0 \to |O_{\mathbb{P}^5}(2)|$ is a good quotient of the $G$-action. In particular, there is an induced rational map $|O_{\mathbb{P}^5}(2)| \to \partial H_n$, which is clearly dominant. So we just need to check that this map is bijective, that is, a general Fano threefold $Y = X \cap Q$ is uniquely determined by the singular Lagrangian surface $i_Z(F_2(Z)) \subset F(X)$ associated to the nodal cubic $Z$ whose equation is $P + x_0 Q = 0$ for some equation $x_0$ of $H \subset \mathbb{P}^6$. By construction, the singular locus of $i_Z(F_2(Z))$ is nothing else than the curve $\Gamma(Y) \subset F(X) \subset G(2,6)$ of lines in $Y$. So we just need to check that $\Gamma(Y)$ defines uniquely the quadric $Q$ such that $Y = X \cap Q$. That is, we must prove that

$$H^0(G(2,6), I_{\Gamma(Y)} \otimes S^2T^*)_Y = \mathbb{C}.$$ 

For $Y$ generic, $I_{\Gamma(Y)}$ can be resolved by the Koszul complex of the section of the vector bundle $E = S^2T^* \oplus S^3T^*$ that defines $\Gamma(Y)$. The claim then follows from the identities $H^0(G(2,6), E^* \otimes S^2T^*) = \mathbb{C}$ and $H^k(G(2,6), \Lambda^{k+1}E^* \otimes S^2T^*) = 0$ for $k > 0$. Both facts are easy consequences of Bott’s theorem on the Grassmannian. \hfill $\square$

Now we are exactly in the situation considered in [DP]: an ACIHS $\mathcal{F} \to B$ is defined over some smooth base $B$, and there is a hypersurface $\Delta \subset B$ over which the fibers degenerate to extensions

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \mathcal{F} \longrightarrow J(\mathcal{F}) \longrightarrow 0,$$

where $J(\mathcal{F})$ is a family of abelian varieties over $\Delta$. If $h$ is a local equation of $\Delta$, suppose that the corresponding Hamiltonian vector field, which is a vertical vector field for the fibration, is tangent to the $\mathbb{C}^*$ direction. Then symplectic reduction applies, and there is an induced ACIHS $J(\mathcal{F}) \to \Delta$.

We apply this result to our setting: $\mathcal{F} \to B$ is our relative Picard fibration $Pic^0(F_2(Z)) \to \mathcal{H}_n$. Over the hypersurface $\Delta = \partial H_n$ defined by nodal cubics, the fibers $Pic^0(F_2(Z))$ are $\mathbb{C}^*$-extensions of the intermediate jacobians $J(Y)$ of the associated Fano threefolds $Y$ of genus 4. Moreover we have seen that the $\mathbb{C}^*$ factor is orthogonal, with respect to the extended symplectic form, to the tangent hyperplane to $\Delta$. So symplectic reduction applies and we conclude:

**Theorem 17** Let $X \subset \mathbb{P}^5$ be a general cubic fourfold. Consider inside the 20-dimensional linear system $|O_{\mathbb{P}^5}(2)|$, the open subset $|O_{\mathbb{P}^5}(2)|_X$ of smooth quadrics $Q$ transverse to $X$. Denote by

$$J(Y_X) \longrightarrow |O_{\mathbb{P}^5}(2)|_X$$

the family of intermediate jacobians of the genus 4 prime Fano threefolds $Y = X \cap Q$, where $Q \in |O_{\mathbb{P}^5}(2)|_X$. Then $J(Y_X)$ is an algebraic completely integrable Hamiltonian system.

Here again the cubic form can easily be identified. Let $W = H^0(O_{\mathbb{P}^5}(1))^*$. From the normal sequence of $Y$ we can easily deduce an exact sequence

$$H^0(Y, T^{\mathbb{P}^5}(-1)|_Y) \longrightarrow W \overset{c_Q,c_X}{\longrightarrow} W^* \oplus S^2W^*/Q \longrightarrow H^1(Y, TY(-1)) \simeq H^1(Y, \Omega^2_Y) \longrightarrow 0.$$

Since $Q$ smooth, $c_Q$ is an isomorphism and we get an isomorphism $S^2W^*/Q \simeq H^1(Y, \Omega^2_Y)$. The dual space $H^1(Y, \Omega^2_Y)$ can therefore be identified with the with the space $A_Q \subset S^2W$.
corresponding to quadrics in the dual projective space which are apolar to \( Q \). The cubic form \( \theta_Q \) on \( A_Q \) is then simply given by the composition

\[
\theta_Q : S^3 A_Q \hookrightarrow S^3(S^2W) \longrightarrow S^2(S^3W) \overset{c_2}{\longrightarrow} \mathbb{C}.
\]

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Atanas Iliev
Institute of Mathematics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str., bl. 8
1113 Sofia, Bulgaria

E-mail: ailiev@math.bas.bg

Laurent Manivel
Institut Fourier, Laboratoire de Mathématiques, UMR 5582 (UJF-CNRS), BP 74
38402 St Martin d’Hères Cedex, France

E-mail: Laurent.Manivel@ujf-grenoble.fr