Random Cayley Graphs II:
Cutoff and Geometry for Abelian Groups

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Abstract
Consider the random Cayley graph of a finite group $G$ with respect to $k$ generators chosen uniformly at random, with $1 \ll \log k \ll \log |G|$. A conjecture of Aldous and Diaconis asserts, for $k \gg \log |G|$, that the random walk on this graph exhibits cutoff at a time which is a function only of $|G|$ and $k$. The conjecture is verified for all Abelian groups.

We extend the conjecture to all $1 \ll \log k \ll \log |G|$, and prove it for a large class of Abelian groups: the cutoff time is the time at which the entropy of simple random walk on $\mathbb{Z}^k$ is $\log |G|$. We also show that the graph distance from the identity for all but $o(|G|)$ of the elements of $G$ lies in $[M - o(M), M + o(M)]$, where $M$ is the minimal radius of a ball in $\mathbb{Z}^k$ of cardinality $|G|$. In the spirit of the Aldous–Diaconis conjecture, this $M$ depends only on $|G|$ and $k$.

Keywords: cutoff, mixing times, random walk, random Cayley graphs, concentration of measure, entropy, diameter, typical distance

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1 Introduction and Statement of Results

Consider a finite group $G$. Let $Z$ be a multiset of $G$, called the generators. We consider the (nearest-neighbour) random walk both on the directed and on the undirected Cayley multigraph generated by the multiset $Z$, denoted $G^+(Z)$ and $G(Z)$, respectively: this is the undirected, respectively directed, multigraph whose vertex set is $G$ and whose edge multiset is

$$ [(g, g \cdot z) \mid g \in G, z \in Z], \quad \text{respectively} \quad [(g, g \cdot z) \mid g \in G, z \in Z]; $$

if the walk is at $g \in G$, then a step involves choosing $z \in Z$ uniformly at random and moving to $g \cdot z$, respectively to one of $g \cdot z$ or $g \cdot z^{-1}$ each with probability $\frac{1}{2}$. We focus on the case where $Z = \{Z_1, ..., Z_k\}$ with $Z_1, ..., Z_k \sim \text{iid Unif}(G)$, and denote $G_k := G(Z)$ and $G_k^+ := G^+(Z)$.

We say that a statistic is “independent of the algebraic structure of the group” when, up to subleading order terms, it is fully determined by $n := |G|$ and $k$, the number of generators used; eg, such a statistic would be the same for both $G := \mathbb{Z}_n$ and $G := \mathbb{Z}_{\sqrt{n}} \oplus \mathbb{Z}_{\sqrt{n}}$.

In this article, our focus is directed towards cutoff and typical distance statistics when the underlying group $G = \oplus_{j=1}^d \mathbb{Z}_{m_j}$ is a general Abelian group; in a companion article [19], we consider the same statistics for Heisenberg matrix groups.

Write $n := |G|$. Our results are asymptotic as $n \to \infty$; we require $1 \ll \log k \ll \log n$, and emphasise that we do not assume that either of $d$ or $\min_j m_j$ is bounded as $n \to \infty$. We say that an event holds with high probability, abbreviated whp, if its probability tends to 1 as $n \to \infty$. See §1.3 for standard usage of asymptotic notation; in particular, “$\ll$” means “up to a $1 + o(1)$ multiplicative factor” and “$\lesssim$, $\simeq$ and $\gtrsim$” mean, respectively, “$\ll$, $\sim$ and $\gg$”, up to constants”.

1.1 Summarised Statements of Results

We now state our results, in summarised form. More refined statements are given later.

Cutoff Phenomenon

A sequence of Markov chain is said to exhibit cutoff when, in a short time-interval, known as the cutoff window, the total variation (TV) distance of the distribution of the chain from its invariant distribution drops from close to 1 to close to 0. For the random walk on the Cayley graph with generators $Z$, write $d_Z(t)$ for this TV distance at time $t$; this is a random variable (in $Z$). We use standard notation and definitions for mixing, cutoff and total variation distance; see, eg, [25].

For the random walk on a random Cayley graph, Aldous and Diaconis [1] conjectured that, for $k \gg \log n$, there is cutoff at a time independent of the algebraic structure of the group; see §1.2 for the history of this conjecture, and progress made towards verifying it or rebuking it.

For $n \in \mathbb{N}$, let $t_0(k, n)$ denote the time at which the entropy of rate-1/k simple random walk (abbreviated SRW) on $Z$ becomes $\log n/k$, and $t_0^+(k, n)$ the corresponding time for a rate-1/k Poisson process (abbreviated PP). We call $t_0$ an entropic time. It will be the mixing time (with $n = |G|$); in accordance with the Aldous–Diaconis conjecture, it depends only on $k$ and $n$.

A more refined statement than the one below is given in Theorem 3.1, with sharper conditions on $k$, $n$ and $d$; see also Hypotheses A.

Theorem A (Cutoff). Let $G := \oplus_{j=1}^d \mathbb{Z}_{m_j}$. Assume that $1 \ll \log k \ll \log n$. Further assume that $\min_j m_j > n^{1/k}(\log k)^2$ and $d \ll \frac{1}{\log n} \min\{k, \log n/\log k\}$. Then, whp over $Z$, the random walk on $G_k$, respectively $G_k^+$, exhibits cutoff at time $t_0(k, n)$, respectively $t_0^+(k, n)$. Moreover, when $k \lesssim \log n$, the cutoff times satisfy $t_0(k, n) \simeq t_0^+(k, n) \simeq kn^{2/k}$, and the window is order $\sqrt{kn^{2/k}}$.

We provide a very precise and detailed description of the cutoff, finding the correct time and even the profile inside the cutoff window. Perhaps surprisingly, the outline of our proof is identical for the un- and directed cases, and for all regimes of $k$.

Our entropic method is applicable not only to Abelian groups, but also to some non-Abelian groups, for which much less is known (see §1.2). We study these in a companion article, [19, Theorem A]; there, the Aldous–Diaconis conjecture is not satisfied.
Typical Distance

Our second result concerns typical distance in the random Cayley graph, for $k \lessapprox \log n$: for an undirected Cayley graph, when there are $k$ generators, for $R \geq 0$ and $\beta \in (0, 1)$, write

$$B_k(R) := \{ x \in G_k \mid \text{dist}(0, x) \leq R \} \quad \text{and} \quad D_k(\beta) := \min \{ R \geq 0 \mid |B_k(R)| \geq \beta|\Gamma| \}. $$

Informally, we show the mass (in terms of number of vertices) concentrates at a thin ‘slice’, or ‘shell’, consisting of vertices at a distance $M(1 + o(1))$ from the origin, where $M \propto kn^{1/k}$.

For directed graphs, add a +-superscript, giving $B_k^+(\cdot)$ and $D_k^+(\cdot)$. When our results apply for both the un- and directed cases, we indicate this by putting the +-superscript in brackets, eg $D^+(\cdot)$, as is standard; write $Z_k^{(1)}$ similarly.

Investigating this typical distance when $k$ diverges with $n$ was suggested to us by Benjamini [7].

Previous work had concentrated on fixed $k$, ie independent of $n$; see §1.2.

A more refined statement than the one below is given in Theorem 4.1 in §4; see also Hypotheses B and C. There we extend the concept of graph distance from the $L_1$ sense to a general $L_p$ sense, and prove an analogous result; see §4.1.

Theorem B (Typical Distance). Let $G := \oplus Z_{m_j}$. Assume that $1 \ll k \lessapprox \log n$, $\min_j m_j \gg kn^{1/k}$ and $\lim sup \min_j (1, 1/k, \log n) / d < 1$. Then there exists a constant $\alpha^+ \in \mathbb{R}$ so that, if $M^+(\cdot) := \alpha^+ kn^{1/k}$, then, for all constants $\beta \in (0, 1)$, we have

$$|D^+(\beta) - M^+(\beta)|/M^+(\beta) = o(1) \quad \text{whp over } Z. $$

Furthermore, if $k \lessapprox \log n$, then $\alpha = 1/(2e)$ and $\alpha^+ = 1/e$. Alternatively, $M^+(\cdot)$ can be given by the minimal radius of a ball in $Z_k^{(1)}$ with volume at least $n$.

Interesting is the way we prove this theorem. It is quite common in mixing proofs to use geometric properties of the graph, such as expansion or distance properties. We, in essence, do the opposite: we adapt the mixing proof to prove this geometric result. (We give a proof-outline in §4.3.) This is in the same spirit as [28]; see §1.2.

Combining Theorem B with the quantitative version of Theorem A, namely Proposition 2.2 and Theorem 3.1 to come, shows that $t_{\text{mix}}(G_k) \approx (\text{diam } G_k)^2/k$ whp.

One can consider non-Abelian groups; see our companion article, [19, Theorem B]. Again, while the typical distance statistic is inline with the Aldous–Diaconis conjecture when the underlying group is Abelian, this turns out not to be the case when the group of non-Abelian.

1.2 Historic Overview

In this section, we give a fairly comprehensive account of previous work on random Cayley graphs, for cutoff and then for typical distance, and compare our results with existing ones. The cutoff phenomenon in particular for this model has received a great deal of attention over the years.

Cutoff Phenomenon

Aldous and Diaconis [1] conjectures that there should exist a time $t_*(k, n)$ so that, for any group $G$ with $|G| = n$, when there are $k$ generators, the random walk exhibits cutoff at time $t_*(k, n)$.

There has since been much work on this conjecture and related problems, almost all of which deals with the regime $k \gg \log n$. Our work is the only one to consider general groups and $k \lessapprox \log n$.

An upper bound, valid for any group, has been proved in the regime $k \gg \log n$: it proves that the mixing time is upper bounded by $\tilde{O}(\log n)$, where $\rho$ is defined by $k = (\log n)^\rho$, and $k \gg \log n$; see Dou and Hildebrand [17, Theorem 1] and Roichman [34, Theorems 1 and 2]. Combined with a basic diameter lower bound of $\log_k n$, this establishes the conjecture when $\rho \to \infty$, as then $\rho \to 1$, with $t_*(k, n) := \log_k n$; see [16, Theorem 4.1]. A matching lower bound, valid only for Abelian groups, again in the regime $k \gg \log n$, was established by Hildebrand [21, Theorem 3]; see also Hildebrand [22, Theorem 5]. (In §3.3, we give a lower bound, valid for any Abelian group.
and $1 \ll \log k \ll \log n$; further, it is concise.) This established the Aldous–Diaconis conjecture for all Abelian groups when $k \gg \log n$, with $\tau_t(k,n) := \frac{1}{\log k} \log n$, where $k = (\log n)^\rho$.

Counter to this, for certain (non-Abelian) Heisenberg matrix groups, we shown in [19, Theorem A] that the general upper bound does not even capture the correct order for mixing.

As Theorem A above suggests, the case $\rho < 1$ is qualitatively different, and the regime $k \asymp \log n$ can be viewed as the order at which a phase transition occurs. Hildebrand [21, Theorem 4] showed that when $\rho < 1$ the mixing time of $\mathbb{Z}_n$ is super-polylogarithmic in $n$. In fact, for $k \lesssim \log n$ the order of the mixing time should depend on the algebraic structure of the group to some extent, even for the class of Abelian groups: for $\mathbb{Z}_d^2$ clearly $d = \log_2 n$ generators are necessary in order to generate the group. We prove, though, that there is cutoff at a time independent of the group, depending only on $k$ and $n$, for a wide class of groups—the constraints are that $k$ must be large enough in terms of $d$ and small enough in terms of $m_1, \ldots, m_d$; if $d \asymp 1$ and $\log(\min_j m_j)/\log n \asymp 1$, then we only require $k \gg 1$. The above observation shows that a result of this type is really the best that one can hope to prove.

Wilson [40, Theorem 1] established the cutoff phenomenon for $\mathbb{Z}_d^2$, for $k$ random generators conditioned to generate the group. He conjectured, in [40, Conjecture 7], that, up to smaller order terms, the mixing time on $G_k$ is maximised (among groups of a given size) by $G := \mathbb{Z}_d^2$. Pak [32] proved various results for groups when $k$ is close to $\log_2 n$. We use methods inspired by [32] to prove [40, Conjecture 7], in a companion article [20]; we defer discussion of [32, 40] to [20].

In terms of cutoff, the case $k \ll \log n$ remained open for a long time with no progress. By the above discussion the best one can hope for in this regime is to establish universality of cutoff for a large class of groups. A breakthrough in 2017 by Hough [23, Theorem 1.7]: he considered the cyclic group $\mathbb{Z}_p$ with $p$ prime and the regime $1 \ll k \lesssim \log p / \log \log p$; using an elegant argument based on representation theory, he proved cutoff at time $t_0 := kp^2/\pi e$ with window $O(t_0/\sqrt{k})$.

The methods of both Hough [23] and Wilson [40] are specialised to their respective situations. Our cutoff result significantly improves Hough’s result by essentially eliminating the restrictions on $p$ and by applying to a wide class Abelian groups $G = \mathbb{Z}_p \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_d}$, subject only to mild conditions on $d$ and $\min_j m_j$; we also remove the restriction $k \lesssim \log |G|/\log \log |G|$. Our proof is significantly simplified if we assume that $m_j$ is prime for each $j$; this is elaborated on in §5.2. In a companion article [20, Theorem B], we extend Wilson’s result from $\mathbb{Z}_2^d$ to $\mathbb{Z}_p^d$, for a prime $p$; here both $d$ and $p$ are allowed to diverge with $n$.

An advantage of our approach is in being unified over all regimes of $k$, i.e those satisfying $1 \ll \log k \ll \log n$, and for the undirected case simultaneously. In particular, other than for the special case of $\mathbb{Z}_2^d$ considered by Wilson [40], to the best of our knowledge, we prove the first cutoff results for the regime $k \asymp \log n$, which is of particular interest, which stems from the celebrated work of Alon and Roichman [3], who showed that, for an arbitrary group, the Cayley graph is an expander whp when $k \gg \log n$.

A further advantage of our approach is in demonstrating that the convergence within the cutoff window exhibits a Gaussian profile; none of the aforementioned results were so refined.

When $k = n^\alpha$ for a constant $\alpha \in (0,1)$, the mixing time is order 1, and hence there is no cutoff in continuous-time. In discrete-time there is cutoff around $1/\alpha$ when $1/\alpha \notin \mathbb{Z}$; see [20, §5.1].

Regarding generating the group, choose $Z_1, Z_2, \ldots \sim \text{Unif}(G)$ and write

$$\varphi_k(G) := \mathbb{P}\left(\{Z_1, \ldots, Z_k\} \text{ generates } G\right).$$

Pak [31] studied the probability $\varphi_k(G)$, and showed that $\varphi_k(G) \geq 1 - \varepsilon$ when $k > \log_2 |G| + 2 + \log_2(1/\varepsilon)$, for $\varepsilon \in (0,1)$; he also showed that $\varphi_m(G) > \frac{1}{2}$ and $\varphi_{m+1}(G) \geq \frac{1}{2}$ where $m := \lfloor \log_2 |G| \rfloor$.

A related quantity was studied by Pomerance [33]: define

$$k(Z) := \min\{k \in \mathbb{N} \mid \{Z_1, \ldots, Z_k\} \text{ generates } G\}.$$ 

If $G$ is Abelian, then there is a minimal $d$ so that we can write $G = \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2}, \ldots$, for some $\{m_j\}$. (Denote this $d(G)$; we necessarily have $k(Z) \geq d(G)$.) He showed that $\mathbb{E}(k(Z)) - d(G) \leq \sigma$, where $\sigma \approx 2.11846$. (An explicit summation form of $\sigma$ is given in terms of the $\zeta$-function.)

For non-Abelian groups, it was conjectured by Dixon [15] that $k(Z) = 2$ whp (as $|G| \rightarrow \infty$), for any finite simple group $G$; this was verified by Liebeck and Shalev [26]. There is a large amount of literature on the number of elements required to generate a group.
We now put our results into a broader context. A common theme in the study of mixing times is that ‘generic’ instances often exhibit the cutoff phenomenon. In this set-up, a family of transition matrices chosen from a certain family of distributions is shown to, whp, give rise to a sequence of Markov chains which exhibits cutoff. A few notable examples include random birth and death chains [14, 37], the simple or non-backtracking random walk on various models of sparse random graphs, including random regular graphs [29], random graphs with given degrees [5, 6, 8, 9], the giant component of the Erdős–Rényi random graph [8] (where the authors consider mixing from a ‘typical’ starting point) and a large family of sparse Markov chains [9], as well as random walks on a certain generalisation of Ramanujan graphs [10] and random lifts [10, 12].

A recurring idea in the aforementioned works establishing the cutoff phenomenon for certain families of random instances is that the cutoff time can be described in terms of entropy: one can look at some auxiliary random process which up to the cutoff time can be coupled with, or otherwise related to, the original Markov chain—often in the above examples this is the random walk on the corresponding Benjamini–Schramm local limit; this technique has been used recently in [8, 12]. The cutoff time is then shown to be (up to smaller order terms) the time at which the entropy of the auxiliary process equals the entropy of the invariant distribution of the original Markov chain. This is the case in the present work also: in the undirected case, we use the simple random walk on $2^k$ with jump rate 1 as our auxiliary random process; in the directed case, we use a $k$-dimensional rate-1 Poisson process; more details are given in §2.1.

In all previous examples, the Benjamini–Schramm limit had been a tree, eg a Poisson Galton–Watson tree in [8] and a deterministic period tree in [12]. Ours is the first example where the graphs are not close in the local topology to a tree.

**Typical Distance**

As well as determining cutoff for these random Cayley graph, we study a geometric property of a diameter flavour; recall the concept of typical distance from Theorem B. Previous work (detailed below) had concentrated on the case where the number of generators $k$ is a fixed number, ie one which does not increase as the size $n$ of the group increases. In contrast, our results are in the situation where $k \to \infty$ as $n \to \infty$; this line of enquiry was suggested to use by Benjamini [7].

Amir and Gurel-Gurevich [4] studied the diameter of the random Cayley graph of cyclic groups of prime order. They prove (for fixed $k$) that the diameter is order $n^{1/k}$; see [4, Theorems 1 and 2]. They conjecture that the diameter divided by $n^{1/k}$ converges in distribution to some non-trivial random variable as $n \to \infty$; see [4, Conjecture 3].

Shapira and Zuck [36] verify this conjecture on the diameter; they also consider the girth. Further, they consider a concept similar to our $L_p$ typical distance (defined in §4.1); see their “(II)”, [36, Page 2]. They find the limiting distribution for all three of these statistics (for fixed $k$).

Marklof and Strömbergsson [30] consider, as a consequence of a quite general framework, the diameter of the random Cayley graph of $\mathbb{Z}_n$, without any assumption that $n$ is prime.

Lubetzky and Peres [28] derive an analogous typical distance result for $n$-vertex, $d$-regular Ramanujan graphs: whp all by $o(n)$ of the vertices lie at a distance $\log_{d−1} n + \mathcal{O}(\log \log n)$; they establish this by proving cutoff for the non-backtracking random walk at time $\log_{d−1} n$.

Related work on the diameter of random Cayley graphs, including concentration of certain measures, can be found in [27, 35].

The Aldous–Diaconis conjecture for mixing can be transferred naturally to typical distance: the mass should concentrate at a distance $M$, where $M$ can be written as a function only of $k$ and $n$; ie there is concentration of mass at a distance independent of the algebraic structure of the group.

In a companion article [19, Theorem B] we consider typical distance analogously to this paper, but there the underlying group is a non-Abelian Heisenberg matrix group. The $M$ for these groups cannot be written as a function only of $k$ and $n$, while for the Abelian groups in Theorem B above this is not the case.
1.3 Additional Remarks

Asymptotic Results

Our results are asymptotic as the size of the group diverges. More formally, we consider a sequence \((G_N)_{N \in \mathbb{N}}\) of groups, which satisfies \(n_N := |G_N| \to \infty\) as \(N \to \infty\); also indexed by \(N\) is \(k_N\). (We require \(1 \ll \log k \ll \log n\), which translates to \(\lim_N k_N = \infty\) and \(\lim_N \log k_N / \log n_N = 0\).)

Instead of writing statements like \(G_N = \oplus_{j=1}^{m_N} \mathbb{Z}_{m_j}\), we drop the \(N\) and just write \(G = \oplus_{j}^{m_j}\); however, the parameters \(d, m_1, ..., m_d\) need not be constant, but may diverge as \(n = |G|\)

Similarly, we sometimes drop the \(t\)-dependence from the notation, eg writing \(S\) instead of \(S(t)\).

Invariant Distribution

For statistics regarding the Cayley graph generated by \(Z\), we add \(Z\) to the notation, eg writing \(d_Z(\cdot)\) and \(t_{\text{mix}}^Z(\cdot)\); when \(Z\) is chosen randomly, these statistics are thus random.

We also denote by \(S = (S(t))_{t \geq 0}\) the random walk and by \(\pi_G\) the uniform distribution on \(G\), which is invariant for \(S\); it is the unique invariant distribution when the graph is connected.

Additional Notation

We define the dimension of an Abelian group \(G\), which we denote \(d(G)\), to be the minimal number of generators required to generate the group. (This concept does not have an agreed upon name in the literature; the term dimension may be used elsewhere with another meaning.) Write

\[ m_* := \max\{\min_j m_j | \oplus_j^{m_j} \mathbb{Z}_{m_j} \text{ is isomorphic to } G\}; \]

we refer to this as the minimal side-length of \(G\) (there is no standard terminology for this either). The dimension and minimal side-length are can be obtained by the same decomposition: such a decomposition \(\oplus_j^{m_j} \mathbb{Z}_{m_j}\) is minimal in the sense that it has \(d = d(G)\) and \(\min_j m_j = m_*\); it is not difficult to see that this happens if and only if \(\gcd(m_1, ..., m_d) > 1\). In the proofs, for concreteness we fix such a minimal decomposition \(G = \oplus_j^{m_j} \mathbb{Z}_{m_j}\) with \(d = d(G)\) and \(\min_j m_j = m_*\).

If \(f, g\) are functions, write \(f \asymp g\) if \(f(N)/g(N) \to 1\) as \(N \to \infty\); also write \(f \ll g\), or \(g \gg f\), if \(f(N)/g(N) \to 0\) as \(N \to \infty\). Write \(f \lesssim g\), or \(g \gtrsim f\), if there exists a constant \(C\) so that \(f(N) \leq Cg(N)\) for all \(N\); also write \(f \asymp g\) if \(g \lesssim f \lesssim g\). Also write \(f = O(g)\) if \(f \lesssim g\), and \(f = o(g)\) if \(f \ll g\). Throughout the paper, unless otherwise explicitly mentioned all limits will be as the size of the group diverges; so if a term is \(o(1)\), then it tends to 0 as the group gets larger.

Write \(N(\mu, \sigma^2)\) for normal with mean \(\mu\) and variance \(\sigma^2\).

Throughout the paper, any 2-norm is with respect to the uniform distribution: \(\|\|_2 := \|\|_{2, \pi}\).

Simple Graphs

The Cayley graph is simple if and only if no generator is picked twice, ie \(Z_i \neq Z_j\) for all \(i \neq j\), and, in the undirected case, additionally no generator is the inverse of another, ie \(Z_i \neq Z_j^{-1}\) for all \(i\) and \(j\). Since, by assumption, \(k/\sqrt{n} \to 0\) as \(n \to \infty\), the probability of this event tends to 1 as \(n \to \infty\). Hence our “whp over \(Z\)” results all also hold when the generators are chosen uniformly at random from \(G\) but conditional on giving rise a simple Cayley graph.

Supplementary Material

In Appendix A, the entropic times from Theorem A are found. There are a few key ideas; we sketch these, non-rigorously, in §2.3. The sketch gives all the ideas, using the correct approximations, but does not justify these approximations. In the directed case, where the Poisson distribution is studied, the times are found rigorously in [19, §2.3].

In Appendix B, some primarily technical proofs from §3 are deferred, as well as some large deviation results for the simple random walk on \(Z\) and the Poisson distribution.

In Appendix C, some primarily technical proofs from §4 are deferred, as well as some estimates on the size of discrete balls in \(\mathbb{Z}^k\).
2 Entropic Method and Times

2.1 Description of Entropic Methodology

We use an ‘entropic method’, as mentioned in §1.2; cf [8, 9, 10, 12]. The method is fairly general; we now explain the specific application in a little more depth.

For mixing, we define an auxiliary random process \((W(t))_{t \geq 0}\), recording how many times each generator has been used. More precisely, for \(t \geq 0\), for each generator \(i = 1, \ldots, k\), write \(W_i(t)\) for the number of times that it has been picked by time \(t\). By independence, \(W(\cdot)\) forms a rate-1 PP on \(\mathbb{Z}_+^k\); each \(W_i(\cdot)\), for \(I = 1, \ldots, k\), is an independent rate-1/k PP on \(\mathbb{Z}_+\). For the undirected case, recall that we either apply a generator or its inverse; when we apply the inverse of generator \(i\), increment \(W_i \rightarrow W_i - 1\) (rather than \(W_i \rightarrow W_i + 1\)). In this case, \(W(\cdot)\) is a SRW on \(\mathbb{Z}^k\) (rather than a PP on \(\mathbb{Z}_+^k\)).

Since the underlying group is Abelian, the order in which the generators are applied is irrelevant for \(n\log\lambda\). We do not apply an entropic method here, per se, but the underlying principles of the proof are extremely similar.

2.2 Definition of Entropic Times and Concentration

In this section, we define the notion of entropic times. For \(t \geq 0\), write \(\mu_t\), respectively \(\nu_t\), for the law of \(W(t)\), respectively \(W_1(t)\); so \(\mu_t = \nu_t^\otimes k\). Also, for each \(i = 1, \ldots, k\), define

\[ Q_i(t) := -\log \nu_t(W_i(t)), \quad \text{and set} \quad Q(t) := -\log \mu_t(W(t)) = \sum_{i=1}^k Q_i(t). \]

Definition 2.1. For \(k, n \in \mathbb{N}\) and all \(\alpha \in \mathbb{R}\), define \(t_\alpha := t_\alpha(k, n)\) so that

\[ \mathbb{E}(Q_1(t_\alpha)) = (\log n + \alpha \sqrt{vk})/k \quad \text{where} \quad v := \text{Var}(Q_1(t_0)). \]

We call \(t_\alpha\) the entropic time and the \(t_\alpha\) cutoff times.

Direct calculation with the Poisson distribution and SRW on \(\mathbb{Z}\) gives the following relations. These calculations are sketched below in §2.3; the rigorous arguments are deferred to Appendix A.

Proposition 2.2. Assume that \(1 \ll \log k \ll \log n\). Write \(\kappa := k/\log n\). For all \(\alpha \in \mathbb{R}\) and \(\lambda > 0\), the following relations hold, for some functions \(f\) and \(g\): we have \(t_\alpha \approx t_0\);

for \(k \ll \log n\), we have \(t_0 \approx k \cdot n^{2/\kappa}/(2\pi e)\) and \(t_\alpha - t_0 \approx \sqrt{2} \cdot \alpha t_0/\sqrt{k}\); \hspace{1cm} (2.1a)

for \(k \approx \lambda \log n\), we have \(t_0 \approx k \cdot f(\lambda)\) and \(t_\alpha - t_0 \approx g(\lambda) \cdot \alpha t_0/\sqrt{k}\); \hspace{1cm} (2.1b)

for \(k \gg \log n\), we have \(t_0 \approx k \cdot 1/(\kappa \log k)\) and \(t_\alpha - t_0 \approx \sqrt{\kappa \log k \cdot \alpha t_0/\sqrt{k}}\). \hspace{1cm} (2.1c)

Moreover, \(f, g : (0, \infty) \to (0, \infty)\) are continuous functions, whose value differs between the un- and directed cases. In particular, for all \(\alpha \in \mathbb{R}\), in all cases, we have \(t_\alpha \approx t_0\).

By a standard argument considering appropriate subsequences, to cover the general case \(k = \log n\), it suffices to assume that \(k/\log n\) actually converges, say to \(\lambda \in (0, \infty)\).

Since the \(W_i\), and hence the \(Q_i\), are iid, \(Q\) is a sum of \(k\) iid random variables. It will also turn out that \(\text{Var}(Q(t)) \approx \text{Var}(Q(t_0)) \gg 1\) when \(t \approx t_0\); see Appendix A.2. Proposition 2.2 above shows that \(t_\alpha \approx t_0\) for all \(\alpha \in \mathbb{R}\). The following CLT, proved in Appendix A.1, will be of great importance.

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Proposition 2.3. For $k$ with $1 \ll \log k \ll \log n$, for each $\alpha \in \mathbb{R}$, we have
\[
\mathbb{P}(Q(t_\alpha) \leq \log n \pm \omega) \to \Psi(\alpha) \quad \text{for} \quad \omega := \mathbb{V}\text{ar}(Q(t_0))^{1/4} = (vk)^{1/4}.
\]
(There is no specific reason for choosing this $\omega$. We just need some $\omega$ with $1 \ll \omega \ll (vk)^{1/2}$.)

2.3 Finding the Entropic Times

In this section, we sketch details towards a proof of Proposition 2.2. The full, rigorous details can be found in Appendix A.3, where all of the approximations below are justified. We separate the sketch into three regimes.

Recall that $t_0$ is the time $t$ at which the entropy of $W_1(t)$, which is a rate-$1/k$ process, is $\log n/k$. We need to find the variance $\mathbb{V}\text{ar}(Q_1(s_0k))$, as this is used in the definition of $t_\alpha$, given in Definition 2.1.

In the sketch below, we replace $\mathbb{V}\text{ar}(Q_1(t_0))$ by an approximation.

Write $s_\alpha := t_\alpha/k$. For $s \geq 0$, denote $X_s := W_1(sk)$ for $s \geq 0$ and the entropy of $X_s$ as $H(s)$.

Sketch when $k \ll \log n$. In this regime, the target entropy $\log n/k \gg 1$, and so $s_0 \gg 1$. For $s \gg 1$, we find that $X_s$ has approximately the normal $N(\mathbb{E}(X_s), s)$ distribution. Translating the random variable has no affect on its entropy, and so we approximate the entropy of $X_s$, denoted $H(s)$, by the entropy of a $N(0, s)$ random variable, denoted $\hat{H}(s)$. Direct calculation with the normal distribution shows that
\[
\hat{H}(s) = \frac{1}{2} \log(2\pi es) \quad \text{and hence} \quad \hat{H}'(s) = 1/(2s).
\]

Define $\tilde{s}_\alpha$ as the entropic times for the approximation:
\[
\hat{H}(\tilde{s}_0) = (\log n + \alpha \sqrt{vk})/k \quad \text{where} \quad v := \mathbb{V}\text{ar}(\tilde{Q}_1(\tilde{s}_0k)),
\]
where $\tilde{Q}_1(sk)$ is the analogue of $Q_1(sk)$, except with $W_1(sk)$ replaced by $N(0, s)$. Hence $\tilde{s}_0 = n^{2/k}/(2\pi e)$. Direct calculation with the normal distribution, one finds
\[
\mathbb{V}\text{ar}(\tilde{Q}_1(sk)) = \frac{1}{2}.
\]

As mentioned above, for this sketch, to ease the calculation of $t_\alpha$ in Definition 2.1, we replace $\mathbb{V}\text{ar}(Q_1(t_0))$ by its approximation $\frac{1}{2}$, and assume the above normal distribution approximation.

In order to find the window, assuming for the moment that $\alpha > 0$, we write
\[
s_\alpha - s_0 = \int_{0}^{s_\alpha} \frac{da}{\mathbb{V}\text{ar}} da.
\]
Again, we replace $s_\alpha$ with $\tilde{s}_\alpha$. By definition, $\tilde{s}_\alpha$ satisfies
\[
\hat{H}(\tilde{s}_\alpha) = \log n/k + \alpha/\sqrt{2k}, \quad \text{and hence} \quad \frac{d}{da}\hat{H}'(\tilde{s}_\alpha) = 1/\sqrt{2k}.
\]

Using the expressions for $d\tilde{s}_\alpha/da$ and $\hat{H}'(s)$ above, we find that
\[
\tilde{s}_\alpha - \tilde{s}_0 = (2k)^{-1/2} \int_{0}^{s_\alpha} \frac{2\tilde{s}_\alpha da}{\mathbb{V}\text{ar}} \approx (2k)^{-1/2} \int_{0}^{s_\alpha} \frac{2\tilde{s}_0 da}{\mathbb{V}\text{ar}} = \alpha \tilde{s}_0 \sqrt{2/2k},
\]

since $\tilde{s}_\alpha$ only varies by subleading order terms over $a \in [0, \alpha]$. The argument is analogous for $\alpha < 0$.

We have now shown the desired result for $\tilde{s}_\alpha$, i.e when approximating $W_1(sk)$ by $N(\mathbb{E}(X_s), s)$. It will turn out that this approximation is sufficiently good for the results to pass over to the original case, i.e to apply to $s_0$ and $t_0 = s_0k$. This is made rigorous with a local CLT.

Sketch when $k \gg \log n$. In this regime, the target entropy $\log n/k \gg 1$, and so $s_0 \gg 1$. Hence all the random variables in question are order 1 random variables, in the sense that they do not tend to 0 or $\infty$ as $n \to \infty$. When $k \approx \lambda \log n$, all the desired expressions are continuous functions of $\lambda$. The fact that $t_\alpha - t_0 \asymp t_0/\sqrt{k}$ comes from the fact that the CLT for a sum of $k$ iid random variables, whose mean and variance are constant, has mean order $k$ and exhibits Gaussian fluctuations of order $\sqrt{k}$; the linear dependence on $\alpha$ is obtained in the same way as in the regime $k \ll \log n$. □
\section*{Sketch when \( k \gg \log n \).} In this regime, the target entropy \( \log n/k \ll 1 \), and so \( s_0 \ll 1 \). Thus we approximate \(|X_s|\) by Bernoulli with success probability \( p_s \), where \( p_s \) is the probability that any steps have been made: \( p_s = \mathbb{P}(\text{Poisson}(s) \geq 1) = 1 - e^{-s} \approx s \). In the undirected case, there is also variation due to the sign; an easy calculation shows, however, that this is a subleading order effect on the entropy. Using a tilde to denote the \( \{0,1\} \)- or \( \{0,\pm 1\} \)-valued distribution by which we approximate \( X_s \), in a similar fashion to before, direct calculation shows that

\[ \tilde{H}(s) \approx s \log(1/s) \quad \text{and} \quad \tilde{H}'(s) \approx \log(1/s). \]

Again using the same technique as when \( k \ll \log n \), one finds that the variance is approximately \((\log \kappa)/\kappa\) where \( \kappa := k/\log n \gg 1 \), and from this derives the desired results. \hfill \( \square \)

\section{Total Variation Mixing}

This section focuses on mixing time for SRW on the random Cayley graph of a general Abelian group, which we write in the decomposition \( G = \bigoplus_i^d \mathbb{Z}_{m_i} \), for some integers \( d \) and \( m_1, \ldots, m_d \).

\subsection*{3.1 Precise Statement and Remarks}

In this section, we state the more refined version of Theorem A. There are some simple conditions that the Abelian groups under consideration must satisfy for our proof to be valid.

\begin{hyp}[Abelian Group] \label{hyp:abelian_group} \text{The Abelian group } G, \text{ integer } k \text{ and real } \eta \in (0,1) \text{ jointly satisfy Hypotheses A if } G \text{ admits a decomposition } \bigoplus_{j=1}^d \mathbb{Z}_{m_j}, \text{ so that } \prod_j m_j = n, \min_j m_j > n^{1/k}(\log k)^2 \text{ and either of the following conditions hold:}

\begin{itemize}
  \item \( k \leq \frac{1}{2} \eta \log n \) and \( d(1/k + 2 \log \log k/\log n) \leq 1 - \eta; \)
  \item \( k \geq \frac{1}{4} \log n^{\log \log n} \) and \( d \leq \frac{1}{30} \log n/\log k. \)
\end{itemize}
\end{hyp}

\begin{remark} We can allow \( \eta \) above to tend to 0 (sufficiently slowly) as \( n \to \infty \); see Remark \ref{rem:eta}. In particular, when \( d \ll \log n/\log \log n \), we may take \( k = d + o(d) \) for some \( o(d) \) term. \triangle \end{remark}

Recall that we consider sequences. For ease of notation, when \( k = k_N \) and \( G = G_N \), we write

\[ d_Z^N(t) := \left| \mathbb{P}(S(t) \in \cdot | Z) - \pi_G \right|_{TV} \quad \text{with} \quad Z = \{Z_1, \ldots, Z_{k_N}\} \quad \text{and} \quad \pi_G \sim \text{Unif}(G_N). \]

Also, for all \( \alpha \in \mathbb{R} \), write \( t_\alpha := t_\alpha(k_N,|G_N|) \), suppressing the \( N \)-dependence.

In summary, we prove that whp we have cutoff at the entropic time \( t_0 \) with window given by \( t_\alpha \) and with Gaussian shape; see \eqref{eq:eta0}, \ref{eq:eta}. We now state our full result in generality. Here and from now on, we write \( \Psi \) for the tail distribution function of \( N(0,1) \), ie

\[ \Psi(\alpha) := \mathbb{P}(N(0,1) \geq \alpha) = (2\pi)^{-1/2} \int_0^\infty e^{-y^2/2}dy \quad \text{for} \quad \alpha \in \mathbb{R}. \]

\begin{thm}[Cutoff] \label{thm:cutoff} \text{Let } \{k_N\}_{N \in \mathbb{N}} \text{ be a sequence of integers, } \eta \in (0,1) \text{ and } \{G_N\}_{N \in \mathbb{N}} \text{ be a sequence of finite, Abelian groups. Suppose } G_N, k_N \text{ and } \eta \text{ jointly satisfy Hypotheses A for each } N; \text{ also require } k_N \to \infty \text{ and } \log k_N/\log |G_N| \to 0 \text{ as } N \to \infty. \text{ Then the constant } \eta \text{ can be chosen sufficiently small (independently of } N \text{) so that the following statements hold.}

\begin{enumerate}
  \item For both the un- and the directed cases, whp, the random walk on \( G_k \) exhibits cutoff at \( t_0 \) with window and Gaussian shape given by \( (t_\alpha)_{\alpha \in \mathbb{R}} \); that is, for all \( \alpha \in \mathbb{R} \), we have \( t_\alpha \approx t_0 \) and

\[ d_Z^N(t_\alpha) \to^d \Psi(\alpha) \quad \text{as } N \to \infty. \tag{3.1} \]

Moreover, the implicit lower bound holds in more generality and deterministically: for all \( \alpha \in \mathbb{R} \), all Abelian groups \( G \) (which need not satisfy Hypotheses A) and all multisets \( Z \) of size \( k \), we have

\[ d_Z(t_\alpha) \geq \Psi(\alpha) - o(1). \tag{3.2} \]
\end{enumerate}
\end{thm}
Remark 3.2. We can write the cutoff statement, emphasising the $n$-dependence, in the form
\[
(I^2_{\text{mix}}(\varepsilon; n) - t_0(n))/w(n) \xrightarrow{p} \Psi^{-1}(\varepsilon) \quad \text{for} \quad \varepsilon \in (0, 1),
\]
where $t_0$ is the mixing time and $w$ is the window, given by (2.1). Eg, when $k \ll \log n$, we have
\[
I_{\text{mix}}^2(\varepsilon; n) - \frac{k n^{2/k}}{(2\pi \varepsilon)} \to \frac{\Psi^{-1}(\varepsilon)}{\sqrt{2\pi \varepsilon}} \quad \text{for} \quad \varepsilon \in (0, 1).
\]

We prove Theorem 3.1 by showing, separately, a matching upper and lower bound on the limit (in distribution) of $d_2^2(t_0)$; we show the lower bound in §3.3 and the upper bound in §3.4.

Throughout this section (§3), we shall always be assuming the conditions of this theorem. Only in Proposition 3.9 will the conditions on $d$ be required, and they will be restated there.

We now make some remarks regarding our main theorem, Theorem 3.1.

Remarks 3.3. (i) The CLT (2.2) will give the dominating term in the TV distance (3.1):
\begin{itemize}
  \item on the event \(\{Q(t_0) \leq \log n - \omega\}\), we lower bound the TV distance by \(1 - o(1)\);
  \item on the event \(\{Q(t_0) \geq \log n + \omega\}\), we upper bound the expected TV distance by \(o(1)\).
\end{itemize}

Combined with Proposition 2.3, we deduce that the $d(z(t_0)) \to \Psi(\alpha)$ in distribution.

(ii) The conditions in Hypotheses $A$ include $\min_j m_j > n^{1/k}(\log k)^2$. For $k \approx \log n$, this can be relaxed to only require that $\min_j m_j$ diverges with $n$. We sketch the details in §5.1. $\triangle$

3.2 Outline of Proof

We now give a high-level description of our approach, introducing notations and concepts along the way. No results or calculations from this section will be used in the remainder of the document; rather, this section merely introduces ideas. Recall the definitions from the previous section.

In all cases we show that cutoff occurs around the entropic time. As $Q(t)$ is a sum of iid random variables, we expected it to be concentrated around its mean. Loosely speaking, we show that the shape of the cutoff, ie the profile of the convergence to equilibrium, is determined by the fluctuations of $Q(t)$ around its mean, which in turn, by the CLT (Proposition 2.3), are determined by $\text{Var}(Q(t))$, for $t$ ‘close’ to $t_0$; note that $\text{Var}(Q(t)) = k \text{Var}(Q_1(t))$ since the $Q_i$ are iid.

Throughout this section (§3.2), we write “$\equiv$” to mean “equivalent modulo $n$”, while “$=\ldots$” is equality, either in $Z$ or in the group $G$. We now outline the proof in more detail.

We start by discussing the lower bound. We show in §3.3 that, for any $\omega$ with $1 \ll \omega \ll \log n$ and all $t$ and all $Z = [Z_1, ..., Z_k]$, we have
\[
d(z(t)) \geq P(Q(t) \leq \log n - \omega) - e^{-\omega}.
\]
Observe that the probability on the right-hand side is deterministic, independent of $Z$. Thus we are naturally interested in the fluctuations of $Q(t)$ for $t$ close to $t_0$. Using the CLT application above (Proposition 2.3), with $\omega := \text{Var}(Q(t_0))^{1/4}$, we deduce the lower bound (3.2).

We now turn to discussing the upper bound. As opposed to the lower bound, here we exploit the uniform randomness of $Z$. For clarity of presentation, we concentrate here on $G = \mathbb{Z}_n$.

Let $W'(t)$ be an independent copy of $W(t)$, and let $V(t) := W(t) - W'(t)$. Observe that, in both the un- and directed case, the law of $V(t)$ is that of the SRW in $\mathbb{Z}_k$ with jump rate $2/k$ in each coordinate, evaluated at time $t$. For now, we suppress the $t$ from the notation. It is standard that the TV distance $\|\zeta - \pi_G\|_{TV}$ can be upper bounded by half the $L_2$ distance:
\[
2 \|\zeta - \pi_G\|_{TV} \leq \|\zeta - \pi_G\|_2 = \sqrt{n \sum_{x \in G} (\zeta(x) - \frac{1}{n})^2},
\]
recalling that $\pi_G(x) = 1/n$ for all $x \in G$. A standard, elementary calculation shows that
\[
\|P(S(t) \in \cdot \mid Z) - \pi_G\|_2 = \sqrt{n P(V(t) \cdot Z \equiv 0 \mid Z)} - 1.
\]
Unfortunately, writing $X = (X(s))_{s \geq 0}$ for a rate-1 SRW on $\mathbb{Z}$, a simple calculation shows that

$$\mathbb{P}(V(t_0) \cdot Z \equiv 0 \mid Z) \geq \mathbb{P}(V(t_0) = (0, \ldots, 0) \in \mathbb{Z}^k) = \mathbb{P}(X(2t_0/k) = 0)^k \gg 1/n.$$  

(This calculation differs among the regimes of $k$.) Moreover, the $L_2$-mixing time can then be shown to be larger than the TV-mixing time by at least a constant factor; this is insufficiently precise for showing cutoff in TV. (We drop the $t$-dependence from the notation from now on.)

This motivates the following type of modified $L_2$ calculation. First let $\mathcal{W} \subseteq \mathbb{Z}^k$, and write

$$\text{typ} := \{W, W' \in \mathcal{W}\}, \quad \tilde{\mathbb{P}}(\cdot) := \mathbb{P}(\cdot \mid \text{typ}) \quad \text{and} \quad \tilde{\mathbb{E}}(\cdot) := \mathbb{E}(\cdot \mid \text{typ});$$

note that here we are (implicitly) averaging over $Z$. (The set $\mathcal{W} \subseteq \mathbb{Z}^k$ will be chosen later.)  We now perform the same type of $L_2$ calculation, but for $\tilde{\mathbb{P}}$ rather than $\mathbb{P}$:

$$\|\mathbb{P}(S \in \cdot \mid Z) - \pi_G\|_{\text{TV}} \leq \|\mathbb{P}(S \in \cdot \mid Z, W \in \mathcal{W}) - \pi_G\|_{\text{TV}} + \mathbb{P}(W \notin \mathcal{W});$$

$$2\mathbb{E}(\|\mathbb{P}(S \in \cdot \mid Z, W \in \mathcal{W}) - \pi_G\|_{\text{TV}}) \leq \mathbb{E}(\sqrt{n} \tilde{\mathbb{P}}(V \cdot Z \equiv 0 \mid Z) - 1) \leq \sqrt{n} \tilde{\mathbb{P}}(V \cdot Z \equiv 0) - 1,$$

using Jensen in the final inequality. We think of $W$ as a set of ‘typical values’ for $W$. To have $w \in \mathcal{W}$, we impose local and global typicality requirements. The global ones say that

$$-\log \mu(w) \geq \log n + \omega \quad \text{for all} \quad w \in \mathcal{W},$$

where $\omega := (\nu k)^{1/4}$ as above; the local ones will come later. For a precise statement of the typicality requirements, see Definitions 3.4 and 3.6. These have the property that $\tilde{\mathbb{P}}(W \notin \mathcal{W}) = \Psi(n) + o(1) \approx 1$ when $t = t_\alpha$; see Lemma 3.7. This has the advantage that now

$$\tilde{\mathbb{P}}(V = (0, \ldots, 0)) \asymp \mathbb{P}(W = W' \mid W' \in \mathcal{W}) \leq n^{-1}e^{-\omega},$$

since $-\log x \geq \log n + \omega$ if and only if $x \leq n^{-1}e^{-\omega}$.

By taking expectation over $Z$ and doing a modified $L_2$ calculation, we transformed the quenched estimation of the mixing time into an annealed calculation concerning the probability that a random word involving random generators is equal to the identity. This is a key step.

Of course, there are other scenarios in which we may have $V \cdot Z \equiv 0$. To deal with these, we observe that, conditional on $\{V_i\}_{i=1}^k$ and $V \neq 0$, we have $V \cdot Z \sim gU$ where $g := \gcd(V_1, \ldots, V_k, n)$ and $U \sim \text{Unif[1, \ldots, n/g]}$; see Lemma 3.11. We then deduce that

$$\tilde{\mathbb{P}}(V \cdot Z \equiv 0 \mid V \neq 0) = \tilde{\mathbb{E}}(g/n \mid V \neq 0).$$

We then need to bound the expectation of the gcd; see Lemma 3.12. If $n$ is prime, then the calculation of this gcd is straightforward: it can only not equal 1 if each $V_1, \ldots, V_k$ is a multiple of $n$ (including the 0-multiple); the only reasonable way for this to happen is to have $V_1 = \cdots = V_k = 0$, as $t_0/k$ will be much smaller than $n^2$.

Bounding $\mathbb{E}(g \mid V \neq 0)$ requires significantly more effort when $n$ is not prime. We noted in the introduction that there is a significant simplification in the the proof when the ‘side lengths’, ie the $m_j$ in the decomposition $G = \bigoplus d \mathbb{Z}_{m_j}$, are prime. The outline is the proof with or without primality; it is only in bounding this term where there is a significant difference. We elaborate now. For $v \in \mathbb{Z}^k$, write

$$\mathcal{I}(v) := \{i \in [k] \mid v_i \equiv 0 \pmod{n}\} \quad \text{and} \quad \mathcal{I} := \mathcal{I}(V(t_\alpha)).$$

Alongside the global ones, we also impose local typicality requirements, namely that each coordinate of $W$ has absolute value at most $r_* := \frac{1}{2}n^{1/k}(\log k)^2$, and hence each coordinate of $V$ has absolute value at most $2r_*$; for now, we implicitly assume this. This implies that

$$\mathcal{I} = \{i \in [k] \mid V_i \neq 0\} \quad \text{given local typicality.}$$

Take $t := t_\alpha$, and decompose $D_\alpha := n \tilde{\mathbb{P}}(V \cdot Z \equiv 0) - 1$ according to the size of $\mathcal{I}$:

$$D_\alpha + 1 = n \sum_{1 \leq |I| < L} \tilde{\mathbb{P}}(V \cdot Z \equiv 0 \mid \mathcal{I} = I)\tilde{\mathbb{P}}(\mathcal{I} = I)$$

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\[ + n \sum_{|I| \geq L} \hat{\mathbb{P}}(V \cdot Z \equiv 0 \mid I = I) \hat{\mathbb{P}}(I = I) + n \mathbb{P}(I = \emptyset). \]

The \( I = \emptyset \) case has just been considered above; call that the ‘empty’ case.

Recall that, in both the un- and directed cases, each coordinate \( V_i(t) \) has the distribution of a SRW on \( \mathbb{Z} \) run at rate \( 2/k \) evaluated at time \( t \). We then use the fact that the distribution of the SRW is unimodal to write \( |V_1| \) conditioned to be non-zero as a mixture of uniforms, say \( \text{Unif}(\{1, \ldots, N\}) \), where \( Y_1 \) has some distribution on \( \mathbb{N} \). Then, for any \( \gamma \in \mathbb{N} \), we have

\[ \hat{\mathbb{P}}(\gamma \text{ divides } V_1 \mid Y_1, V_1 \neq 0) = [Y_1/\gamma]/Y_1 \leq 1/\gamma. \]

Using the independence of coordinates, for any \( I \subseteq \{1, \ldots, k\} \), we obtain

\[ \hat{\mathbb{P}}(g = \gamma \mid I = I) \leq \hat{\mathbb{P}}(\gamma \text{ divides } V_i \text{ for all } i \in I \mid I = I) \leq \gamma^{-|I|}. \]

Hence we see that for large \( |I| \) even \( \gamma = 2 \) gives a very small contribution. So, for large \( |I| \), we have \( \mathbb{E}(g \mid I = I) = 1 + o(1) \). Summing over all \( I \) with \( |I| \geq L \), with \( L \to \infty \), we obtain \( \mathbb{E}(g \mid |I| \geq L) = 1 + o(1) \); see Corollary 3.13, where, in the general case of \( G = \oplus \mathbb{F}_q \mathbb{Z}_{m_i} \), the requirement is that \( L - d \to \infty \). (This is not completely precise, since we are implicitly conditioning on typicality; the precise calculation is carried out in \( \S 3.4 \).

So we have now considered \( I = \emptyset \) and \( I \) with \( |I| \geq L \), which we think of as ‘large \( I \)’. We now consider the probability that \( I = I \), for a relatively small set \( I \) (with \( 1 \leq |I| \leq L \)); we call this the ‘small \( I \)’ case. In this case, all the coordinates outside \( I \), which is a large number of coordinates, are \( 0 \) (mod \( n \)). This will cause the probability to be small; see Lemma 3.15. Using independence, we obtain

\[ \mathbb{P}(I = I \mid W) \leq \mathbb{P}(W' \equiv W' \mid W')/\prod_{i \in I} \mathbb{P}(W_i' = W_i \mid W_i). \]

Local typicality turns the “\( \equiv \)” into “\( = \)”. It will also allow us to lower bound the probabilities in the product: given typical \( W_i \), we shall have \( \mathbb{P}(W_i' = W_i \mid W_i) \geq p_* \), where \( p_* := n^{-1/k} \). Using independence, the trivial bound on the gcd provided by local typicality:

\[ \hat{\mathbb{P}}(V \cdot Z \equiv 0 \mid I = I) = \mathbb{E}(g/n \mid V \neq 0) \leq 2r_*/n = n^{-1+1/k}(\log k)^2. \]

The gain in doing so is insignificant in what comes.) If \( L \) is sufficiently small in terms of \( k \), then we find that

\[ \sum_{1 \leq |I| \leq L} \mathbb{P}(V \cdot Z \equiv 0, I = I) = o(1/n); \]

see (3.17) and (B.1).

Combining the ‘large’, ‘small’ and ‘empty’ cases, we deduce that \( D_\alpha = o(1) \) for any \( \alpha \in \mathbb{R} \).

Recall the modified \( L_2 \) calculation above and that \( \mathbb{P}(W(t_n) \not\in W(t_n)) = \Psi(\alpha) + o(1) \). From these the upper bound in (3.1) follows.

We now briefly outline what changes when we allow general \( d \), with \( G = \oplus \mathbb{F}_q \mathbb{Z}_{m_i} \), instead of just \( d = 1 \) and \( G = \mathbb{Z}_m \). The main difference is in the calculation with the gcd, but first we define

\[ \mathcal{I}(n) := \{ i \in [k] \mid v_i \neq 0 \text{ mod } m_j \forall j = 1, \ldots, d \} \quad \text{and} \quad \mathcal{I}(V(t_n)). \]

We require \( \min_i m_j > n^{1/k}(\log k)^2 \), and so the local typicality conditions again imply that

\[ \mathcal{I} = \{ i \in [k] \mid V_i \neq 0 \} \quad \text{given local typicality.} \]

Then define \( g_j := \gcd(V_1, \ldots, V_k, m_j) \) and \( g := \gcd(V_1, \ldots, V_k, n) \), so \( g_j \leq g \). For any \( I \neq \emptyset \), we get

\[ \hat{\mathbb{P}}(V \cdot Z \equiv 0 \mid I = I) = \hat{\mathbb{E}}(\prod_{j} g_j/m_j \mid I = I) \leq \hat{\mathbb{E}}(g^d/n \mid I = I). \]

The analysis of \( \hat{\mathbb{E}}(g^d \mid I = I) \) continues as before, but now we divide into the case where \( |I| - d \) is large or small, controlling the latter under the assumption that \( k \) is sufficiently large in terms of \( d \) in some precise, quantitative manner. The main place where the conditions on \( d \) in terms of \( k \) arise is in the analysis of the \( I \) with \( d - |I| \geq 1 \).

This concludes the outline; we now move onto the formal proofs.
3.3 Lower Bound

In this section we prove the lower bound (3.2), which holds for every choice of $Z$.

**Proof of Lower Bound** (3.2). For this proof, we assume that $Z$ is given, and suppress it.

We convert the CLT (2.2) from a statement about $Q$ into one about $W$. Let $\alpha \in \mathbb{R}$ and write

$$E_{\alpha} := \{ \mu(W(t_{\alpha})) \geq n^{-1} e^{\omega} \} = \{ Q(t_{\alpha}) \leq \log n - \omega \};$$

recall that $\omega \gg 1$. From the CLT (2.2), we have $\mathbb{P}(E_{\alpha}) \to \Psi(\alpha)$. Consider the set

$$A_{\alpha} := \{ x \in G \mid \exists w \in \mathbb{Z}^{d} \text{ st } \mu_{\alpha}(w) \geq n^{-1} e^{\omega} \text{ and } x = w \cdot Z \}.$$

Since we use $W$ to generate $S$, we have $\mathbb{P}(S(t_{\alpha}) \in A_{\alpha} \mid E_{\alpha}) = 1$. Every element $x \in A_{\alpha}$ can be realised as $x = w_{x} \cdot Z$ for some $w_{x} \in \mathbb{Z}^{k}$ with $\mu_{\alpha}(w_{x}) \geq n^{-1} e^{\omega}$. Hence, for all $x \in A_{\alpha}$, we have

$$\mathbb{P}(S(t_{\alpha}) = x) \geq \mathbb{P}(W(t_{\alpha}) = w_{x}) = \mu_{\alpha}(w_{x}) \geq n^{-1} e^{\omega}.$$

From this we deduce that

$$1 \geq \sum_{x \in A_{\alpha}} \mathbb{P}(S(t_{\alpha}) = x) \geq |A_{\alpha}| \cdot n^{-1} e^{\omega}, \quad \text{and hence } |A_{\alpha}| / n \leq e^{-\omega} = o(1).$$

Finally we deduce the lower bound (3.2) from the definition of TV distance:

$$\left\| \mathbb{P}(S(t_{\alpha}) \in \cdot \mid Z) - \pi_{G} \right\|_{TV} \geq \mathbb{P}(S(t_{\alpha}) \in A_{\alpha}) - \pi_{G}(A_{\alpha}) \geq \mathbb{P}(E_{\alpha}) - \frac{1}{n}|A_{\alpha}| \geq \Psi(\alpha) - o(1).$$

**Remark.** Using a variant of this argument, in [19, §3.2] we prove a lower bound for nilpotent non-Abelian groups: where $t_{0}(k, |G|)$ was the lower bound above (for Abelian groups), we establish a lower bound of $t_{0}(k, |G|)$ for any group. (If a group is Abelian, then $G = G/\{G, G\}$.) In many cases, this is a significant improvement over previous best-known bound of $\log_{k-1}|G|$. △

3.4 Upper Bound

We define a set $W_{\alpha}$ in which the auxiliary walk $W$ will ‘typically’ lie in at time $t_{\alpha}$, in the sense that $\mathbb{P}(W(t_{\alpha}) \notin W_{\alpha}) \to \Psi(\alpha)$ as $n \to \infty$. Given that $W(t_{\alpha}) \in W(t_{\alpha})$, we show that the TV distance has expectation $o(1)$. Using the upper bound

$$\left\| \mathbb{P}(S(t_{\alpha}) \in \cdot \mid Z) - \pi_{G} \right\|_{TV} \leq \left\| \mathbb{P}(S(t_{\alpha}) \in \cdot \mid Z, W(t_{\alpha}) \in W_{\alpha}) - \pi_{G} \right\|_{TV} + \mathbb{P}(W(t_{\alpha}) \notin W_{\alpha}),$$

this shows an upper bound of $\Psi(\alpha)$ in the limit in probability. By considering all $\alpha \in \mathbb{R}$, we are able to find the shape of the cutoff. If we only desire the order of the window, then we need only consider the limit $\alpha \to \infty$; in this case, $\mathbb{P}(W(t_{\alpha}) \notin W_{\alpha}) \approx \Psi(\alpha) \approx 0$, which explains the use of the word ‘typically’ in describing $W_{\alpha}$. As in the outline, in order to control the TV distance (given typicality), we actually upper bound it first by the $L_{2}$ distance.

Next we define two parameters $r$ and $p$ which will be used in our definition of typicality.

**Definition 3.4.** For all $\alpha \in \mathbb{R}$, define $r_{\alpha}(k, n)$ and $p_{\alpha}(k, n)$ as follows:

$$r_{\alpha}(k, n) := \min \{ r \in \mathbb{Z}_{+} \mid \mathbb{P}\{ |W_{1}(t_{\alpha}) - E(W_{1}(t_{\alpha}))| > r \} \leq 1/k^{3/2} \};$$

$$p_{\alpha}(k, n) := \min \{ \mathbb{P}(W_{1}(t_{\alpha}) - E(W_{1}(t_{\alpha})) = j) \mid |j| \leq r_{\alpha}(k, n) \}.$$

Also define $r_{\ast}(k, n) := \frac{1}{2} n^{1/k}(\log k)^{2}$ and $p_{\ast}(k, n) := n^{-1/k}k^{-2}$.

**Proposition 3.5.** For all $\alpha \in \mathbb{R}$, we have

$$r_{\alpha}(k, n) \leq r_{\ast}(k, n) \quad \text{and} \quad p_{\alpha}(k, n) \geq p_{\ast}(k, n). \tag{3.3}$$

This proposition follows from standard large deviation theory. The exponent $2$ in $(\log k)^{2}$ is not optimal, but is chosen for convenience of proof and to enable us to deal with all regimes of $k$ simultaneously. We give the details in Appendix B.2.

The typicality conditions will be a combination of ‘local’ (coordinate-wise) and ‘global’ ones.
**Definition 3.6.** For all $\alpha \in \mathbb{R}$, define the local and global typicality conditions, respectively:

\[
\mathcal{W}_{\alpha, \ell} := \{ w \in \mathbb{Z}^k \mid |w_i - \mathbb{E}(W_1(t_\alpha))| \leq r_\alpha(k, n) \forall i = 1, \ldots, k \};
\]

\[
\mathcal{W}_{\alpha, g} := \{ w \in \mathbb{Z}^k \mid P(W(t_\alpha) = w) \leq n^{-1}e^{-\omega} \}.
\]

Define $\mathcal{W}_\alpha := \mathcal{W}_{\alpha, \ell} \cap \mathcal{W}_{\alpha, g}$, and say that $w \in \mathbb{Z}^d$ is $(\alpha)$-typical if $w \in \mathcal{W}_\alpha$.

**Lemma 3.7.** For each $\alpha \in \mathbb{R}$, we have

\[
P(W(t_\alpha) \notin \mathcal{W}_\alpha) \to \Psi(\alpha).
\]

**Proof.** By our application of the CLT (2.2), the probability that the global conditions hold converges to $1 - \Psi(\alpha)$. By the union bound, the probability that local typicality fails to hold is at most $k^{-1/2} = o(1)$. The claim follows.

Throughout this section, we fix $\alpha \in \mathbb{R}$ and set $t := t_\alpha$, and suppress this from the notation at various points. First, we condition that $W$ is typical:

\[
\|P(S \in \cdot \mid Z) - \pi_G\|_{TV} \leq \|P(S \in \cdot \mid Z, W \in \mathcal{W}) - \pi_G\|_{TV} + P(\mathcal{W}^c).
\]  

(3.4)

The second term is determined in Lemma 3.7. For the first term, we use a modified $L_2$ calculation, as referenced in the outline (3.2). To do this, let $W'$ be an independent copy of $W$; then $S' := W' \cdot Z$ is an independent copy of $S$. Also let $V := W - W'$. Write

\[
D_\alpha := nP(V(t_\alpha) \cdot Z = 0 \mid \text{typ}_\alpha) - 1 \quad \text{where} \quad \text{typ}_\alpha := \{ W(t_\alpha), W'(t_\alpha) \in \mathcal{W}_\alpha \}.
\]

We also sometimes drop the subscript $\alpha$ from $D_\alpha$ and $\text{typ}_\alpha$. We also decompose the typicality requirements into the local and global parts, as defined in Definition 3.6:

\[
\text{typ}_\ell := \{ W, W' \in \mathcal{W}_\ell \} \quad \text{and} \quad \text{typ}_g := \{ W, W' \in \mathcal{W}_g \}; \quad \text{then} \quad \text{typ} = \text{typ}_\ell \cap \text{typ}_g.
\]

**Lemma 3.8.** For all $\alpha \in \mathbb{R}$, we have

\[
\mathbb{E}\left( \|P(S(t_\alpha) \in \cdot \mid Z, W(t_\alpha) \in \mathcal{W}_\alpha) - \pi_G\|_{TV} \right) \leq \frac{1}{2}\sqrt{D_\alpha}.
\]  

(3.5)

**Proof.** This is an easy application of Cauchy-Schwarz to bound the TV by the $L_2$ distance.

The equalities for $S$ and $Z$ are all in our group $G$; eg $V \cdot Z = 0$ means ‘equal to the identity of $G$’. Recall that we are considering groups of the form $G := \mathbb{Z}_m$. In the same way as when embedding a torus in $\mathbb{Z}^d$, we consider elements of $G$ as elements of the torus $\mathbb{Z}_m^d$, and embed it in $\mathbb{Z}^d$. Under this embedding, we identify each $x \in \mathbb{Z}^d$ with the unique $y \in G$ so that $x_j \equiv y_j \mod m_j$ for all $j$, and write $x \equiv y$. Accordingly, we often treat $V \cdot Z$ as an element of $\mathbb{Z}^d$.

We prove the following upper bounds on $D_\alpha$. The proposition has two parts, according to $k$.

**Proposition 3.9a.** Write $\mathcal{L}_n := d(1/k + 2 \log \log k / \log n)$. Suppose that $0 < \eta \leq 1 - \lim sup_n \mathcal{L}_n$, and that $k \leq \frac{1}{2}\eta \log n$, with $\eta$ sufficiently small. Then, for all $\alpha \in \mathbb{R}$, we have $D_\alpha = o(1)$.

**Proposition 3.9b.** Suppose that $k \geq \frac{1}{2}\eta \log n / \log \log \log n$ and that $d \leq \frac{1}{2\eta} \log k / \log k$. Then, for all $\alpha \in \mathbb{R}$, we have $D_\alpha = o(1)$.

Once we prove these propositions, we have enough to prove the main theorem, Theorem 3.1.

**Proof of Theorem 3.1 Given Proposition 3.9.** The “either/or” conditions in Hypotheses A correspond precisely to the conditions in Propositions 3.9a and 3.9b, respectively. The probability of $\text{typ}_\alpha$ is given by Lemma 3.7. Combined with Lemma 3.8 and (3.4), we deduce the upper bound implicit in (3.1). (The lower bound (3.2) was proved earlier.)
We now state a simple lemma; its proof is deferred to the Appendix.

We now split the sum into 'large' and 'small' subranges. Let

\[ I(v) := \{ i \in [k] \mid v_i \neq 0 \mod m_j \text{ for all } j = 1, \ldots, d \}. \]  

(3.6)

We shall always be considering \( V \) conditioned on typicality. Note that local typicality says that \(|V_i| \leq 2r < m_j\) for all \( i \) and \( j \). Thus, conditioned on local typicality, ie \( \text{typ} \), we have

\[ \mathcal{I}(V) = \{ i \in [k] \mid V_i \neq 0 \}. \]

Also, write \( \mathcal{I} := \mathcal{I}(V) \) for ease of notation. Thus we may write

\[ D + 1 = n \sum_{|I| \leq |I|} \mathbb{P}(V \cdot Z \equiv 0, \mathcal{I} = I \mid \text{typ}) \]

We now split the sum into 'large' \( \mathcal{I} \), 'small' \( \mathcal{I} \) and 'empty' \( \mathcal{I} \). In the sums below, we always have \( I \subseteq [k] \). Let \( L \) be a number greater than 1, allowed to depend on \( n \). We then have

\[ D + 1 \leq n \sum_{|I| < L} \mathbb{P}(V \cdot Z \equiv 0 \mid I = I, \text{typ}) \mathbb{P}(I = I \mid \text{typ}) + n \mathbb{P}(I = \emptyset \mid \text{typ}), \]  

(3.7)

noting that if \( I = \emptyset \) then \( V = 0 \) (as a vector), and hence \( V \cdot Z = 0 \).

We first bound the third term on the right-hand side of (3.7), ie consider \( I = \emptyset \).

**Lemma 3.10.** We have

\[ n \mathbb{P}(I = \emptyset \mid \text{typ}) \leq e^{-\omega}/\mathbb{P}(\text{typ}). \]  

(3.8)

**Proof.** By direct calculation, we have

\[ \mathbb{P}(I = \emptyset, \text{typ}) = \mathbb{P}(V = 0, \text{typ}) = \mathbb{P}(W = W', W \in \mathcal{W}) = \sum_{w \in \mathcal{W}} \mathbb{P}(W = w) \mathbb{P}(W' = w) = \sum_{w \in \mathcal{W}} \mathbb{P}(W = w)^2, \]

since \( W \) and \( W' \) are iid copies. Recall global typicality: \( \mathbb{P}(W = w) \leq n^{-1}e^{-\omega} \) for all \( w \in \mathcal{W} \). Thus

\[ n \mathbb{P}(I = \emptyset \mid \text{typ}) \leq n \sum_{w \in \mathcal{W}} \mathbb{P}(W = w)^2/\mathbb{P}(\text{typ}) \leq e^{-\omega}/\mathbb{P}(\text{typ}). \]

We now turn our attention to \( I \neq \emptyset \), where we must also analyse \( \mathbb{P}(V \cdot Z \equiv 0 \mid I = I, \text{typ}) \). For \( r_1, \ldots, r_\ell \in \mathbb{Z} \setminus \{0\} \), we use the convention \( \gcd(r_1, \ldots, r_\ell, 0) := \gcd(|r_1|, \ldots, |r_\ell|) \). Define

\[ g_j := \gcd(V_1, \ldots, V_k, m_j) \quad \text{for } j = 1, \ldots, d, \]

and also define \( g := \gcd(V_1, \ldots, V_k, n) \).

We now state a simple lemma; its proof is deferred to the Appendix B.1.

**Lemma 3.11.** Conditional on \( V = v \in \mathbb{Z}^k \) with \( \mathcal{I}(v) \neq \emptyset \), we have

\[ v \cdot Z \sim \text{Unif}(\prod_1^d g_j Z_{m_j/g_j}) \sim \text{Unif}(\prod_1^d g_j 2g_j, \ldots, m_j). \]

Note that \( g_j \leq g \) since \( m_j \) divides \( n \), for all \( j = 1, \ldots, d \). From the lemma we now deduce that

\[ n \mathbb{P}(V \cdot Z \equiv 0 \mid I = I, \text{typ}) = n \mathbb{E}(\prod_1^d g_j / m_j \mid I = I, \text{typ}) \leq \mathbb{E}(g^d \mid I = I, \text{typ}), \]  

(3.9)

as \( \prod_1^d m_j = n \) and since, by local typicality, we have \( |V_i| < m_j \) for all \( i \) and \( j \) and observing that the conditioning affects \( V \), but not \( Z \). We now bound the expectation of this gcd.

**Lemma 3.12.** There exists a constant \( C \) so that, for any \( I \subseteq [k] \) with \( \mathcal{I} = I \cap \text{typ} = \emptyset \), we have

\[ \mathbb{E}(g^d \mid I = I, \text{typ}) \leq \begin{cases} C(2r_*)^{d-|I|+2}/\mathbb{P}(\text{typ} | I = I, \text{typ} | I = I) & \text{when } |I| \leq d + 1, \\ 1 + 3 \cdot 2^{d-|I|}/\mathbb{P}(\text{typ} | I = I, \text{typ} | I = I) & \text{when } |I| \geq d + 2. \end{cases} \]  

(3.10a)

(3.10b)

Furthermore, recalling the definition of \( r_* \) from Definition 3.4, we also have

\[ \mathbb{E}(g^d \mid I = I, \text{typ}) \leq (2r_*)^d = n^{d/k} (\log k)^{2d}. \]  

(3.11)

These bounds are used with (3.9) which says that \( n \mathbb{P}(V \cdot Z \equiv 0 \mid I = I, \text{typ}) \leq \mathbb{E}(g^d \mid I = I, \text{typ}). \)
An easy corollary of this says that the contribution to (3.7) by ‘large $I$’ is $1 + o(1)$.

**Corollary 3.13.** For any $L$ with $L \geq d + 2$, we have

$$n \sum_{|I| \geq L} \mathbb{P}(V \cdot Z \equiv 0, I = I \mid \text{typ}) \leq 1 + 3 \cdot 2^{d-L}/\mathbb{P}(\text{typ}).$$

**Proof.** This proof is a direct calculation. By (3.10b), using Bayes’ rule, specifically the fact that $\mathbb{P}(B \mid C)/\mathbb{P}(C \mid B) = \mathbb{P}(B)/\mathbb{P}(C)$ for non-null events $B$ and $C$, for $L \geq d + 2$ we deduce that

$$n \sum_{|I| \geq L} \mathbb{P}(V \cdot Z \equiv 0, I = I \mid \text{typ}) = n \sum_{|I| \geq L} \mathbb{P}(V \cdot Z \equiv 0 \mid I = I, \text{typ})\mathbb{P}(I = I \mid \text{typ})$$

$$\leq \sum_{|I| \geq L} \mathbb{P}(I = I \mid \text{typ}) + 3 \cdot 2^{d-|I|}\mathbb{P}(I \geq L)/\mathbb{P}(\text{typ}) \leq 1 + 3 \cdot 2^{d-L}/\mathbb{P}(\text{typ}).$$

In order to prove Lemma 3.12, we use the following divisibility property of the coordinates of $V$, which we recall are independent. Below, we write $\alpha \beta$ if $\alpha$ divides $\beta$.

**Lemma 3.14.** For all non-empty $I \subseteq [k]$ and all $\gamma \in \mathbb{N}$, we have

$$\mathbb{P}(\gamma \mid V_i \forall i \in I \mid I = I, \text{typ}_I) \leq (1/\gamma)^{|I|}.$$

We defer the proof of this statement until the end of the section.

**Proof of Lemma 3.12.** The definition of $r_r$ from Equation (3.11) along with (3.9) immediately imply the final claim (3.11). Write $\tilde{\mathbb{P}}$ and $\tilde{\mathbb{E}}$ to denote probability and expectation, respectively, conditioned on $I = I$ and $\text{typ}_I$ (ie local typicality). As for (3.9), we obtain

$$n \mathbb{P}(V \cdot Z \equiv 0 \mid I = I, \text{typ}) \leq 1 + \mathbb{E}(g^d - 1) = 1 + \tilde{\mathbb{E}}(g^d - 1)/\tilde{\mathbb{P}}(\text{typ}_I).$$

Hence, to prove (3.10a, 3.10b), we need to bound $\tilde{\mathbb{E}}(g^d)$. To do this, note that

$$\tilde{\mathbb{E}}(g^d) = \sum_{\gamma=1}^{2^r} \gamma^d \tilde{\mathbb{P}}(g = \gamma) \leq \sum_{\gamma=1}^{2^r} \gamma^d \tilde{\mathbb{P}}(\gamma \mid V_i \forall i \in I).$$

Applying Lemma 3.14, we obtain

$$\tilde{\mathbb{E}}(g^d) \leq \sum_{\gamma=1}^{2^r} \gamma^{d-|I|}.$$

To bound this sum, we now consider separate cases, according to the value of $d - |I|$. In particular, we can summarise all these cases in the following way:

$$\tilde{\mathbb{E}}(g^d) \leq \begin{cases} 1 + 3 \cdot 2^{d-|I|} & \text{when } |I| - d \geq 2, \\ C(2r_{\alpha})^{d-|I| + 2} & \text{when } |I| - d \leq 1, \end{cases}$$

where $C$ is the implicit constant in the previous equation. We thus deduce (3.10a, 3.10b).

We now consider the probability of a given realisation of $I$. Recall that $t := t_0$ still.

**Lemma 3.15.** We have

$$\mathbb{P}(I = I, \text{typ}) \leq n^{-1} e^{-\omega}/p^{|I|} = e^{-\omega} n^{-1+|I|}/k^{2|I|}.$$  

Further, if $k \leq \lambda_0 \log n$ with $\lambda_0$ sufficiently small, we have

$$\mathbb{P}(I = I, \text{typ}_I) \leq 2^{-|I|} n^{-1+|I|}/k.$$  

**Proof.** Requiring $I = I$ places restrictions on the coordinates in $I^c$, but not on the coordinates of $I$ other than that they are non-zero; we ignore the latter to get an upper bound (see below).

We prove (3.14) first. If $k \ll \log n$, then $t_0 \approx k n^{2/k}/(2\pi)$, by Proposition 2.2. By the local CLT (see Theorem A.1) letting $X = (X_s)_{s \geq 0}$ be a rate-1 SRW on $\mathbb{Z}$, we have

$$\mathbb{P}(V_1(t) = 0) = \mathbb{P}(X_{2t/k} = 0) = (2\pi \cdot 2t/k)^{-1/2} (1 + o(1)) \leq 2n^{-1/k},$$

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Hence, using the independence of the coordinates of $W$.

Then, using the independence of $\eta$,

\[ W_t(w) := \{ w' \in Z^k_{(+)}, | I(w - w') = I \} \]

An immediate consequence of the definitions of $r$ and $p$, in Definition 3.4, is that,

for all $\alpha \in \mathbb{R}$, if \[ |w_1 - \mathbb{E}(W_1(t_\alpha))| \leq r_\alpha(k,n) \] then \[ P(W_1(t_\alpha) = w_1) \geq p_\alpha(k,n) \]

Hence, for $w \in W$, we then obtain

\[ P(W' \in W_t(w)) \leq P(W' = w)/p_\alpha^{[I]} \leq n^{-1}e^{-\omega}/p_\alpha^{[I]} \]

From this and the sum above, (3.13) follows by summing over all $w \in W$:

\[ P(I = I, \text{typ}) \leq P(I = I, W \in W) \leq n^{-1}e^{-\omega}p_\alpha^{-[I]} \sum_{w \in W} P(W = w) \leq n^{-1}e^{-\omega}p_\alpha^{-[I]}; \]

finally we substitute the definition $p_\alpha = n^{-1/2}k^{-2}$ from Definition 3.4.

We have now done all the hard work in proving Propositions 3.9a and 3.9b, from which we deduced Theorem 3.1. It remains to go through the details of how to combine the previous results; there are no more interesting ideas to prove the propositions, but the details are quite technical.

**Proof of Proposition 3.9a.** Recall that here $0 < \eta \leq 1 - \limsup_n \mathcal{L}_n$ and $k \leq \frac{1}{2}\eta \log n$. We also assume that $\eta$ is sufficiently small so that Lemma 3.15 may be applied with $\lambda_0 := \frac{1}{2}\eta$.

Consider first $I \subseteq [k]$ with $1 \leq |I| \leq d + 1$. We have

\[ nP(V \cdot Z \equiv 0, I = I, \text{typ}) = nP(V \cdot Z \equiv 0 | I = I, \text{typ}) P(I = I, \text{typ}) \]

\[ \leq C(2r_\tau)^{d-|I|+2} P(I = I, \text{typ}_\tau, \text{typ}_g) / P(\text{typ}_g | I = I, \text{typ}) \]

\[ = C(2r_\tau)^{d-|I|+2} \]

\[ \leq C2^{k-|I|}(\log k)^2(2d-|I|+2)n^{-1+(d+2)/k} \]

\[ \leq 2k n^{-1+c+o(1)} \leq 2k n^{-\eta+o(1)} \] (3.15)

Consider now $I \subseteq [k]$ with $d + 2 \leq |I| \leq (1 - \frac{1}{2}\eta)k$. Similarly to above, we have

\[ nP(V \cdot Z \equiv 0, I = I, \text{typ}) = nP(V \cdot Z \equiv 0 | I = I, \text{typ}) P(I = I, \text{typ}) \]

\[ \leq 2k^{-|I|+1}n^{-1+|I|/k} \leq 2n^{-\eta+o(1)} \] (3.16)

We now sum over all $I$ with $1 \leq |I| \leq (1 - \frac{1}{2}\eta)k$, using (3.15, 3.16):

\[ n \sum_{1 \leq |I| \leq (1 - \frac{1}{2}\eta)k} P(V \cdot Z \equiv 0, I = I, \text{typ}) \leq 2 \cdot 2^k \cdot 2k n^{-\eta+o(1)} \leq 2 n^{-\eta/4} = o(1), \] (3.17)
since \( \eta \) is a constant and \( k \leq \frac{1}{2}\eta \log n \) (so \( 2^k \leq n^{2/3} \)), noting that there are at most \( 2^k \) such \( I \).

Finally we consider \( I \subseteq [k] \) with \((1 - \frac{1}{\eta})k \leq |I| \leq k \). Setting \( L := (1 - \frac{1}{\eta})k \), (3.12) says that

\[
\sum_{|I| \leq k} \Pr(V \cdot Z \equiv 0, I = I | \text{typ}) \leq 1 + 3 \cdot 2^{-\eta k/2} / \Pr(\text{typ}).
\]  

Plugging (3.8, 3.17, 3.18) into (3.7), we obtain

\[
D = \sum_{|I| \leq k} \Pr(V \cdot Z \equiv 0, I = I | \text{typ}) - 1 = o(1) / \Pr(\text{typ}) = o(1).
\]

**Remark 3.16.** This proof works also when \( \eta \to 0 \); inspection reveals that \( \eta \geq 4d \log \log k / \log n \) and \( \eta \gg 1/k \) is sufficient. In particular, if \( d \ll \log n / \log \log n \) then we may take \( k = d + o(d) \). \( \triangle \)

The proof of Proposition 3.9b is very similar; we give the skeleton of the argument below, with the full details deferred to the Appendix B.1.

**Proof of Proposition 3.9b (Skeleton).** Recall that here \( k \geq \frac{1}{3} \log n / \log \log n \) (and \( \log k \ll \log n \)) and \( d \leq \frac{1}{20} \log n / \log k \). Set \( L \) such that \( \log n / \log k \). Then \( d \leq \frac{1}{2}L \) and \( L - d \gg 1 \).

For \( I \) with \( |I| \leq L \), replace the use of (3.10a, 3.14) by (3.11, 3.13), to obtain

\[
\sum_{|I| \leq L} \Pr(V \cdot Z \equiv 0, I = I | \text{typ}) \leq n^{-1/5} = o(1).
\]

For \( I \) with \( L \leq |I| \leq k \), using (3.12) gives an analogue of (B.1), replacing \( \eta k / 2 \) by \( L - d \gg 1 \).

The proof is completed by combining these, as for Proposition 3.9a. \( \square \)

It remains to give the deferred proof of Lemma 3.14. (Lemma 3.11 is proved in the appendix.)

**Proof of Lemma 3.14.** Let \( X = (X_s)_{s \geq 0} \) be a rate-1 SRW on \( \mathbb{Z} \). To calculate the expectation, we use that \( V = W - W' \) has the distribution of a SRW run at twice the speed; in particular, \( V_i(t) \sim X_{2i/k} \), and that coordinates of \( V \) are independent. (This holds for both the undirected cases.) Clearly the distribution of \( X \) is symmetric about 0. Also, Claim 3.17, proved in the appendix, asserts that the map \( m \mapsto \Pr(|X_0| = m) : N \to [0,1] \) is decreasing for any \( s \geq 0 \). Further, the local typicality conditions \( \text{typ}_k \) confines \( V_i \) to a symmetric interval.

It is easy to see that any monotone decreasing distribution on \( \mathbb{N} \) can be written as a mixture of \( \text{Unif}\{1, ..., Y\} \) distributions, for different \( Y \in \mathbb{N} \). The above says that \( |V_i| \) conditional on \( i \in I \) and \( \text{typ}_k \) is such a monotone distribution, hence we can write \( |V_i| \sim \text{Unif}\{1, ..., Y\} \), where \( Y \) has some distribution. Hence we have

\[
\Pr(\gamma | V_i \mid i \in I, \text{typ}_k, Y) = |Y/\gamma|/Y \leq 1/\gamma.
\]

The lemma follows from the independence of the coordinates of \( V \). \( \square \)

The following claim, referenced above, is proved in the Appendix B.1

**Claim 3.17.** The map \( m \mapsto \Pr(|X_0| = m) : N \to [0,1] \) is decreasing.

## 4 Typical Distance

This section focuses on distances from a fixed point in the random Cayley graph of a general Abelian group, which we write in the decomposition \( G = \oplus^d \mathbb{Z}_{m_i} \), for some integers \( d \) and \( m_1, ..., m_d \).

### 4.1 Definition of \( L_p \) Typical Distance

In the introduction, we stated the typical distance results, Theorem B, for the standard \( (L_1) \) graph distance, and commented that we actually consider a more general \( L_p \) type graph distance. We define this carefully now.

Consider first the undirected case. Consider graph distances and balls in the graph \( G_k \). As well as the usual graph distance, we consider a type of \( L_p \) graph distance, for \( p \in [1, \infty] \): for \( x \in \mathbb{Z}^k \), let \( \|x\|_p \) denote its (usual) \( p \)-norm; for \( v \in G \), let \( P^k_v \) be the set of all paths from 0 to \( v \) in \( G_k \); for a
path $\rho$, let $x(\rho) \in \mathbb{Z}^k$ be such that for each generator $i \in \{1, \ldots, k\}$ the number of times generator $i$ (or its inverse) is used is $x_i(\rho)$; for $v \in G_k$, define
\[
\text{dist}_{k,p}(0, v) := \inf \{\|x(\rho)\|_p \mid \rho \in P_v^k\}.
\]

Note that $\text{dist}_{k,1}$ is just the usual graph distance (regardless of $k$). By transitivity, when the group is Abelian a geodesic will, for each generator index $i$, not use both $Z_i$ and $Z_i^{-1}$ (unless $Z_i = Z_i^{-1}$).

For $p \in [1, \infty]$ and $R > 0$, write $B_{k,p}(R)$ for the $L_p$ ball of radius $R$ in (the group) $G_k$, i.e $B_{k,p}(R) := \{v \in G \mid \text{dist}_{k,p}(0, x) \leq R\}$. Then, for $\beta \in (0, 1)$ and $p \in [1, \infty]$, write
\[
\mathcal{D}_{k,p}(\beta) := \min \{R \geq 0 \mid |B_{k,p}(R)| \geq \beta n\}.
\]

It will also be convenient to write $B_{k,p}(R)$ for the $L_p$ ball of radius $R$ in (the lattice) $\mathbb{Z}^k$, i.e $B_{k,p}(R) := \{x \in \mathbb{Z}^k \mid \|x\|_p \leq R\}$, where $\|x\|_p$ is the $p$-norm of the real-valued vector $x$.

Make analogous definitions for the directed case, writing $G_{k}^+$, $k$, $\text{dist}_{k,p}^+$, $B_{k,p}^+$, and $\mathcal{D}_{k,p}^+$; for the directed case, the lattice balls are subsets not only of $\mathbb{Z}^k$, but in fact of $\mathbb{Z}^k_\leq$. When our results apply for cases, we indicate this by putting the + in brackets, e.g $G_{k}^{(+)}$, $\mathcal{D}_{k,p}^{(+)}$ or $\mathbb{Z}_{k}^{(+)}$.

Throughout this section, we consider any $p \in [1, \infty]$, but the reader is strongly encouraged to, on first read, concentrate on the case $p = 1$, for which we give the most detailed analysis and results. The other cases are conceptually similar, but require additional calculations. Similarly, the specific Abelian group $\mathbb{Z}_n$, rather than a general $\oplus \mathbb{Z}_m$, can be borne in mind.

### 4.2 Precise Statement

In this section, we state the more refined version of Theorem B. Like previously, we require the Abelian groups under consideration to satisfy certain hypotheses. Recall that, for an Abelian group, we define the dimension and minimal side-length, respectively, as follows:
\[
d(G) := \min \{d \in \mathbb{N} \mid \oplus \mathbb{Z}_m \text{ is a decomposition of } G\};
\]
\[
m_* := \max \{\min_{j=1, \ldots, d} m_j \mid \oplus \mathbb{Z}_m \text{ is a decomposition of } G\}.
\]

Our main constraints will be $\lim sup d/k < 1$ and $k^{1/p} n^{1/k} / m_* \ll 1$.

**Hypotheses B.** The integer sequences $(k_N, n_N)_{N \in \mathbb{N}}$ and real $p \in [1, \infty]$ jointly satisfy Hypotheses B if $\lim_{N} n_N = \infty = \lim_{N} k_N$ and one of the following conditions hold:
- $p = 1$ and $\lim_{N} k_N / \log n_N$ exists in $[0, \infty)$;
- $p = \infty$ and $\lim_{N} k_N / \log n_N = 0$;
- $p \in (1, \infty)$ and $k_N \leq \log n_N / \log \log n_N$ for all $N$.

**Hypotheses C.** The integer sequences $(k_N, n_N, d_N, m_N)_{N \in \mathbb{N}}$ and real $p \in [1, \infty]$ jointly satisfy Hypotheses C if $\lim_{N} k_N^{1/p} n^{1/k} / m_N^{1/p} = 0$ and one of the following conditions holds:
- $\lim_{N} k_N / \log n_N = 0$ and $\lim sup_{N} d_N / k_N < 1$;
- $\lim_{N} k_N / \log n_N \in (0, \infty)$, $p = 1$ and $d_N \leq \frac{1}{\log n_N / \log \log n_N}$ for all $N \in \mathbb{N}$.

Finally we set up a little more notation. Let $p \in [1, \infty)$. Make the following definitions:
\[
C_p := 2 \Gamma(1/p + 1) (pe)^{1/p}, \quad C_\infty := \lim_{p \to \infty} C_p = 2 \quad \text{and} \quad C_p^+ := 2C_p
\]
\[
M_{k,p}^{(\pm)} := k^{1/p} n^{1/k} / C_p^{(\pm)}, \quad M_{k,\infty} := n^{1/k}
\]

**Theorem 4.1 (Typical Distance).** Let $(k_N, n_N, m_N)_{N \in \mathbb{N}}$ be integer sequences and let $p \in [1, \infty]$; suppose these jointly satisfy Hypotheses B and C. Let $(G_N)_{N \in \mathbb{N}}$ be a sequence of finite, Abelian groups such that $|G_N| = n_N$ and $G_N$ has dimension $d_N$ and minimal side-length $m_N$ for each $N \in \mathbb{N}$. Write $n := n_N$ and $k := k_N$.

Suppose first that $k \ll \log n$. Then there exists a term $\xi_{p,k}$, satisfying $\xi_{p,k} \ll 1$, so that for all $\beta \in (0, 1)$, we have
\[
|\mathcal{D}_{k,p}^{(\pm)}(\beta) - M_{k,p}^{(\pm)}| / M_{k,p}^{(\pm)} \leq \xi_{p,k} \quad \text{whp over } Z.
\]
Moreover, we may take \(\xi_{p,k} := K_p \max\{ (\log k)^2, k/n^{1/(2k)} \}/k\) for a sufficiently large constant \(K_p\).

Now suppose that \(\lim k/\log n \in (0, \infty)\) and \(p = 1\). Then there exists a term \(\xi_{1,k}\), satisfying \(\xi_{1,k} \ll 1\), and a constant \(\alpha^{(+)} \in (0, \infty)\) so that, writing \(\mathcal{M}_{k,1}^{(+)} = \alpha^{(+)} k\), for all \(\beta \in (0, 1)\), we have

\[
|\mathcal{D}_{k,1}^{(+)}(\beta) - \mathcal{M}_{k,1}^{(+)}|/\mathcal{M}_{k,1}^{(+)} \leq \xi_{1,k} \quad \text{whp over } Z.
\]

Moreover, the implicit lower bound holds in more generality and deterministically: assume that \((k, n, p)\) jointly satisfy Hypotheses \(B\); all \(\beta \in (0, 1)\), all Abelian groups \(G\) (which need not satisfy Hypotheses \(C\)) of size \(n\) and all multisets \(Z\) of size \(k\), we have

\[
\mathcal{D}_{k,p}^{(+)}(\beta) \geq \mathcal{M}_{k,p}^{(+)} \cdot (1 - \xi_{p,k}),
\]

using the same definition of \(\mathcal{M}_{k,p}^{(+)}\), in the appropriate regimes, as above.

**Remark.** We initially prove this theorem for undirected Cayley graphs. In §4.6, we explain how to adapt the proof from the undirected case to the directed case. Doing this, rather than making every statement apply for both the un- and directed cases, significantly increases the readability. \(\triangle\)

### 4.3 Outline of Proof

As remarked after the summarised statement (in §1.1), when considering the mixing of SRW on a graph, geometric properties of the graph are often derived and used. In a reversal of this, we use knowledge about the mixing properties of the SRW to derive a geometric result; the style of proof is similar enough that we even quote lemmas from the mixing section.

The main difference between the proofs is the following: as described in §2.1, \(W(t)\) is a SRW on \(\mathbb{Z}^k\); we replace this \(W(t)\) by \(A(t)\) which is uniformly distributed on a ball of radius \(R\), and then choose \(R\) appropriately. We interpret \(A_i(t)\) as the number of times generator \(i\) has been chosen minus the number of times its inverse has been, exactly as for \(W_i(t)\).

We choose \(M\) so that this ball has size \(ne^r\), for some \(\omega \gg 1\) diverging reasonably slowly. If \(R := M(1 - \xi)\), then we use a counting argument to show that the ball cannot cover more than a proportion \(o(1)\) of the vertices of the graph; hence this gives a deterministic lower bound, valid for all \(Z\). If \(R := M(1 + \xi)\), then we show that not only does the ball cover (almost) all the graph, but the random variable \(A \cdot Z\) is well-mixed whp, in the sense that it is very close to the uniform distribution. From this we deduce that, for a proportion \(1 - o(1)\) of the vertices, there is a non-zero probability that \(A \cdot Z\) is at that vertex, and hence a path to it must exist; furthermore, by choice of \(A\), the path must have length at most \(R = M(1 + \xi)\). To prove this, we even use an analogous \(L_2\) calculation to that used for the mixing, namely Propositions 3.9a and 3.9b.

Note the trivial inequality \(|B_{k,p}(R)| \leq |B_{k,p}^1(R)|\) (valid since the group is Abelian). It will follow from our analysis that up to the radius \(M_{k,p}\) at which the latter has volume close to \(n\), the two balls grow at roughly the same rate. More precisely, for all \(\xi > 0\), we shall have

\[
|B_{k,p}((1 - \xi)M_{k,p})|/n = o(1) \quad \text{and} \quad |B_{k,p}((1 + \xi)M_{k,p})|/n = 1 - o(1) \quad \text{whp over } Z,
\]

subject to some technical conditions; this says that almost all the vertices lie at a distance \(M_{k,p}(1 \pm o(1))\) from the group element \(0\). This will also holds for the directed case.

We give \(M_{k,p}\) explicitly when \(k \ll \log n\), up to subleading order, capturing (4.3).

In the lemmas below, used to prove this theorem, instead of writing one lemma with multiple parts, we split into separate lemmas according to \(p\) and \(k\), eg \(p \in (1, \infty)\) or \(k \asymp \log n\); these parts are indexed with letters, eg Lemmas 4.2a, 4.2b and 4.2c.

### 4.4 Size of Ball Estimates and Lower Bound

We wish to determine the size of the \(L_p\) balls in \(\mathbb{R}^k\). This is done by Lemmas 4.2 and 4.4; the statements are given below, with proofs are deferred to Appendix C.2.

For \(p \in [1, \infty)\), write \(V_{k,p}(R)\) for the (Lebesgue) volume of the \(L_p\) ball of radius \(R\) in \(\mathbb{R}^k\), ie

\[
V_{k,p}(R) := \text{vol}\{x \in \mathbb{R}^k \mid \|x\|_p \leq R\};
\]
also write $V_{k,p} := V_{k,p}(1)$ and note that $V_{k,p}(R) = R^k V_{k,p}$. It is known (see [39]) that

$$V_{k,p} = 2^k \Gamma(1/p + 1)^k / \Gamma(\ell/p + 1). \quad (4.4)$$

We can use this, along with Lemma 4.2b below, to well-approximate $|B_{k,p}(R)|$ when $p \notin \{1, \infty\}$; for $p = 1$ we directly bound $|B_{k,1}(\cdot)|$, while for $p = \infty$ we have an exact expression.

**Lemma 4.2a.** For $p = 1$ and all $R \geq 0$, we have

$$2^k \binom{|R|}{1} 1(R \geq k) \leq |B_{k,1}(R)| \leq 2^k \binom{|R| + k}{k}. \quad (4.5a)$$

**Lemma 4.2b.** For $p \in (1, \infty)$ and all $R \geq k^{1+1/p}$, we have

$$|B_{k,p}(R)| = V_{k,p}(R) \left(1 + O(k^{1+1/p}/R)\right). \quad (4.5b)$$

**Lemma 4.2c.** For $p = \infty$ and all $R \geq 0$, we have

$$|B_{k,\infty}(R)| = (2|R| + 1)^k. \quad (4.5c)$$

We use this lemma to find an $M$ so that $|B_{k,p}(M)| \approx n$.

**Definition 4.3.** Set $\omega := \max\{(\log k)^2, k/n^{1/(2k)}\}$, and choose $M_{k,p}$ to be the minimal integer satisfying $|B_{k,p}(M_{k,p})| \geq ne^\omega$. Note that $\omega$ satisfies $1 \ll \omega \ll k$ if $k \ll \log n$.

Recall that $M_{k,p} = k^{1/p}n^{1/k}/C_p$, and that $C_p = 2 \Gamma(1/p + 1)(pe)^{1/p}$. The next lemma shows that the difference between $M$ and $M$ is only by subleading order terms, and can be absorbed into the error terms in (4.1). Also, let $K$ be a constant, assumed to be as large as required, and let $\xi := 1 - e^{-K\omega/k}$ when $k \ll \log n$. (As such, we can always replace $1 \pm \xi$ by $e^{\pm \xi}$.)

**Lemma 4.4a.** For $k \ll \log n$ and $p = 1$, we have

$$M_{k,1} \leq \left\lceil M_{k,1}(1 + \xi) \right\rceil \quad \text{and} \quad |B_{k,1}(M_{k,1}(1 - \xi))| \ll n. \quad (4.6a)$$

**Lemma 4.4b.** For $k \leq \log n/ \log \log n$ and all $p \in [1, \infty)$, we have

$$M_{k,p} \leq \left\lceil M_{k,1}(1 + \xi) \right\rceil \quad \text{and} \quad |B_{k,p}(M_{k,p}(1 - \xi))| \ll n. \quad (4.6b)$$

**Lemma 4.4c.** For $p = \infty$, we have

$$M_{k,\infty} \leq \left\lceil \frac{1}{2} n^{1/k} e^{\omega/k} - \frac{1}{2} \right\rceil \quad \text{and} \quad |B_{k,\infty}(M_{k,\infty}(1 - \xi))| \ll n. \quad (4.6c)$$

Moreover, if $k \ll \log n$ then $M_{k,\infty} \approx M_{k,\infty}$.

**Lemma 4.4d.** For all $\lambda > 0$, for $k \approx \lambda \log n$, there exists a function $\omega \gg 1$ and a constant $\alpha$ so that, for all $\epsilon \in (0, 1)$, the minimal integer $M_1$ satisfying $|B_{k,1}(M_1)| \geq ne^\omega$ satisfies

$$M_{k,1} \approx \alpha k \approx \alpha \lambda \log n \quad \text{and} \quad |B_{k,1}(\alpha k(1 - \epsilon))| \ll n. \quad (4.6d)$$

In fact, the result holds for any $1 \ll \omega \ll k$.

From this lemma, it is straightforward to deduce the lower bound in Theorem 4.1.

**Proof of Lower Bound, Equation (4.2).** Observe that $|B_{k,p}(M)| \leq |B_{k,p}(M)|$. By Lemma 4.4, the right-hand side is $o(n)$ when $M = M_{k,p}(1 - \xi)$ when $k \ll \log n$. The same holds when $k \approx \lambda \log n$, with $\lambda \in (0, \infty)$, and $p = 1$. We deduce that $D_{k,p}(\beta) \geq M$ for all $Z$. 

\[ \square \]
4.5 Mixing-Type Results and Upper Bound

This section is highly related to the proof of the upper bound on the TV mixing for general Abelian groups, §3.4. Analogously to the situation there, specifically Propositions 3.9a and 3.9b, the main ingredient in the proof of Theorem 4.1 bounds the $L_2$ distance of a certain function of $Z$ from the uniform distribution on $G$. In Proposition 3.9, we considered $S(t) = W(t) \cdot Z$, where $W(t)$ was the SRW on $Z^k$ evaluated at time $t$ (in the undirected case, which we consider here); here we replace $W$ with $A \sim \text{Unif}(B_{k,p}(M_{k,p}))$, ie uniform on the ball of radius $M_{k,p}$, where $M_{k,p}$ is defined in Definition 4.3—it is the minimal integer satisfying $|B_{k,p}(M_{k,p})| \geq ne^\omega$.

**Proposition 4.5a.** Let $p \in [1, \infty]$. Suppose that $k \ll \log n$. If $p \in (1, \infty)$, then further restrict to $k \leq \log n / \log \log n$. Suppose also that $\limsup_n d/k < 1$. Then $\mathbb{E}_Z(\|\mathbb{P}(A \cdot Z = \cdot | Z) - \pi_G\|_2^2) = o(1)$.

**Proposition 4.5b.** Let $p = 1$. Suppose that $k \sim \lambda \log n$ for some $\lambda \in \mathbb{R}$ and $d \leq \frac{1}{2} \log n / \log \log n$. Then $\mathbb{E}_Z(\|\mathbb{P}(A \cdot Z = \cdot | Z) - \pi_G\|_2^2) = o(1)$.

Once we prove these propositions, we have all we need to prove Theorem 4.1.

**Proof of Theorem 4.1 Given Lemma 4.4 and Proposition 4.5.** If $\|\mathbb{P}(A \cdot Z = \cdot | Z) - \pi_G\|_2 \leq \varepsilon$, then the support $S$ of $A \cdot Z$ satisfies $\pi_G(S^c) \leq \varepsilon$. Combining this with Lemma 4.4 and Proposition 4.5, we deduce the upper bounds implicit in (4.1). (The lower bound (4.2) was proved in §4.4.) □

**Remark.** Proposition 4.5a actually holds even if $\eta := 1 - d/k \downarrow 0$, provided it does so sufficiently slowly and $k/\log n$ is sufficiently small. It turns out that $k/\log n \ll \eta$ and $\eta \gg 1/\sqrt{k}$ is sufficient; this allows $k$ very close to both $d$ and $\log n$.

Let $A, A' \sim \text{Unif}(B_{k,p}(M))$, and let $B := A - A'$. Then, like in Lemma 3.8, we have

$$\mathbb{E}_Z(\|\mathbb{P}(A \cdot Z = \cdot | Z) - \pi_G\|_2^2) = n\mathbb{P}(B \cdot Z = 0) - 1.$$  

We then look to upper bound $\mathbb{P}(B \cdot Z = 0)$, using similar techniques to those from §3.

First, it is immediate to see that

$$\mathbb{P}(A = A') = |B_{k,p}(M)|^{-1} \leq n^{-1}e^{-\omega}.$$  

Similarly to before, the side-lengths $\{m_j\}_{j=1}^d$ satisfy $\min_j m_j > 2M$. Then write

$$I := \{i \in [k] \mid B_i \neq 0 \text{ mod } m_j \forall j = 1, \ldots, d\} = \{i \in [k] \mid A_i \neq A'_i\}.$$  

Also, for each $j = 1, \ldots, d$, write

$$g_j := \text{gcd}(B_1, \ldots, B_k, m_j) \quad \text{and} \quad g := \text{gcd}(B_1, \ldots, B_k, n).$$  

We consider $\mathbb{P}(I = I)$ and $\mathbb{E}(g^d | I = I)$. (There will not be any need for ‘typicality’ here.)

**Lemma 4.6a.** For all $k$ and all $p$, we have

$$\mathbb{P}(I = 0) \leq n^{-1}e^{-\omega}. \quad (4.7a)$$

**Lemma 4.6b.** Suppose that $k \ll \log n$ and $p \in [1, \infty)$. If $p \in (1, \infty)$, then restrict further to $k \leq \log n / \log \log n$. Then, for all $I \subseteq [k]$, we have

$$\mathbb{P}(I = I) \leq e^{k(1/(ep) + \xi_p)}n^{-1+|I|/k} \quad (4.7b)$$

where $\xi_p := K_p\omega/k \ll 1$, for some constant $K_p$.

**Lemma 4.6c.** For $p = \infty$, for all $I \subseteq [k]$, we have

$$\mathbb{P}(I = I) \leq e^{-\omega(1-|I|/k)}n^{-1+|I|/k}. \quad (4.7c)$$
Lemma 4.6d. For $p = 1$, for all $I \subseteq [k]$ with $|I| \ll k$, if $k \gg \lambda \log n$ for some $\lambda \in \mathbb{R}$, then we have
\[
\mathbb{P}(I = I) \leq n^{-1+o(1)}. \tag{4.7d}
\]

The following lemma is based on a gcd calculation, and is very similar to Lemma 3.12; its proof is deferred to the Appendix C.1.

Lemma 4.7. There exists a constant $C$ so that, for any $I \subseteq [k]$, we have
\[
n \mathbb{P}(B \cdot Z = 0 \mid I = I) \leq \begin{cases} C(2M)^{d-|I|+2} & \text{when } |I| \leq d + 1, \\ 1 + 3 \cdot 2^{d-|I|} & \text{when } |I| \geq d + 2, \end{cases} \tag{4.8a}
\]

We first prove the results on $\mathbb{P}(I = I)$. For a set $I \subseteq [k]$ and $A \in \mathbb{Z}^{k}$, write $A_I = (A_i)_{i \in I}$ and $A \setminus I = A_{I'}$. Recall that if $C \subseteq C'$ and $U \sim \text{Unif}(C')$, then $(U \mid U \in C) \sim \text{Unif}(C)$. Hence we have
\[
\mathbb{P}(A_I = A'_I) = \frac{\mathbb{P}(A = A')}{\mathbb{P}(A_I = A'_I \mid A \setminus I = A_{I'})} = \frac{|B_{k,p}(M)|^{-1}}{\mathbb{E}(|B_{I',p}(M - ||A_{I'}||_1)|^{-1})} \leq \frac{|B_{k,p}(M)|}{|B_{k,p}(M)|}. \tag{4.9}
\]

Write $\ell := |I|$. Recall that, by choice of $M$, we have $|B_{k,p}(M)| \geq ne^{\omega}$, and so
\[
\mathbb{P}(A_I = A'_I) \leq n^{-1}e^{-\omega}|B_{k,p}(M)|.
\]

**Proof of Lemma 4.6a.** Recall the choice of $M_{k,p}$, from Definition 4.3. Then (4.7a) follows:
\[
\mathbb{P}(I = 0) = \mathbb{P}(A = A') = |B_{k,p}(M_{k,p})|^{-1} \leq n^{-1}e^{-\omega}.
\]

**Proof of Lemma 4.6b.** Consider first $p = 1$. From Lemma 4.4a, recall that $M_1 \leq (2e)^{-1}kn^{1/k}e^{\xi}$ with $\xi \sim \omega/k$. Using Lemma 4.2a, for $\ell \leq k$, we have
\[
|B_{\ell,1}(M_1)| \leq 2e^{\ell(M_1 + \xi)} \leq (2e(M_1/\ell + 1))^\ell \leq e^{\ell(k/\ell)^{+}n^{\ell/k} \leq e^{k(1+\ell/\ell)}n^{\ell/k},
\]

using the fact that $(N/\ell)^x \leq (eN/\ell)^x$, that $\ell \mapsto (k/\ell)^{+}$ is maximised by $\ell = k/e$ and that $1 + x \leq e^x$. The proof is completed by noting that $\{I = I\} \subseteq \{A_I = A'_I\}$, and applying (4.9).

Now consider $p \in (1, \infty)$. Justified by Lemma 4.2b and Lemma 4.4b, which shows that $M_{k,p} \gg k^{1+1/p}$ for all $p$, we replace this discrete ball by the continuous ball, and lose only a factor $1 + o(1)$; for readability, we do not carry this factor in subsequent formulæ.

Using Stirling’s formula and the upper for $M_{k,p}$ from Lemma 4.4b gives
\[
V_{k,p}(M_{k,p}) \leq V_{k,p} \cdot ((1 + \xi)k^{1/p}n^{1/k} / C_p)^{\ell} \leq p^{1/2}eK_{\omega}(k/\ell)^{\ell/p}n^{\ell/k}.
\]

From this, similarly to in Lemma 4.6a, using (4.9), we deduce that
\[
\mathbb{P}(A_I = A'_I) \leq p^{1/2}eK_{\omega}(k/\ell)^{\ell/p}n^{1+\ell/k} \leq e^{(1+\ell/\ell)}n^{1+\ell/k},
\]

where $\xi = K_{\omega}k/\ell \ll 1$, using again the fact that $(N/\ell)^x \leq (eN/\ell)^x$ and that $\ell \mapsto (k/\ell)^{+}$ is maximised by $\ell = k/e$. The proof is completed by noting that $\{I = I\} \subseteq \{A_I = A'_I\}$.

**Proof of Lemma 4.6c.** The coordinates of $A$ satisfy $A_i \sim \text{Unif}([0, \pm 1, \ldots, \pm M_\infty])$, for $i = 1, \ldots, k$. Write $\ell := |I|$. Hence, by (4.9) and (4.7a), we have
\[
\mathbb{P}(A_I = A'_I) \leq |B_{k,\infty}(M_\infty)| / |B_{k,\infty}(M_\infty)| = (2M_\infty + 1)^{\ell-k}.
\]

By (4.6c), we have $2M_\infty + 1 \geq n^{1/k}e^{\omega/k}$. Hence
\[
\mathbb{P}(I = I) \leq \mathbb{P}(A_I = A'_I) \leq e^{\omega(1+\ell/k)n^{1+\ell/k}}.
\]
Proof of Lemma 4.6d. As above, by (4.9) and (4.7a), it suffices to upper bound $|B_{c,1}(M_1)|$ where $\ell = |I|$, and assume $\ell \leq c|I|$ for a small $c$. By (4.5a) and (4.6d), we have $M = M_1 \leq 2\alpha k$, and so

$$|B_{c,1}(M)| \leq 2^{\ell}(M+\ell) \leq (2c(2\alpha k/\ell + 1))^\ell \leq (8\alpha k/\ell)^\ell,$$

with the last inequality requiring $2\alpha k/\ell \geq 1$, which holds for $c$ sufficiently small, as $\ell \leq ck$. Now, for $c$ sufficiently small (in terms of $\alpha$), the map $\ell \mapsto (8\alpha k/\ell)^\ell$ is increasing on $[1, ck]$. Hence

$$|B_{c,1}(M)| \leq (8\alpha k/\ell)^\ell \leq (9\alpha k/\ell)^{2\ell\log n}.$$

By taking $c$ sufficiently small (in terms of $\lambda$ and $\alpha$), we can upper bound this by an arbitrarily small power of $n$. The result now follows from (4.9) and (4.7a).

We now prove the results on the gcd, i.e. Lemma 4.7. The way to prove this is analogous to the proof of Lemma 3.12. For this reason, the proof is deferred to the appendix.

We have now done all the hard work in proving Propositions 4.5a and 4.5b, from which we deduced Theorem 4.1. It remains to go through the details of how to combine the previous results; there are no more interesting ideas to prove the propositions, but the details are quite technical.

Proof of Proposition 4.5a (when $p < \infty$). Recall that here $k \ll \log n$ and $\limsup d/k < 1$.

Similarly to the mixing proof, we use an $L_2$ calculation:

$$E_Z(\|P(A \cdot Z = \cdot | Z) - \pi_G\|_2^2) = n \sum_{I} P(B \cdot Z = 0, I = I) - 1. \tag{4.10}$$

If $\eta := 1 - \limsup d/k > 0$ then, setting $L := d + \frac{1}{\eta}(k - d)$, we have

$$L/k = \frac{1}{\eta} + (1 - \frac{1}{\eta})d/k, \quad \text{and so} \limsup L/k \leq \frac{1}{\eta} + (1 - \frac{1}{\eta})(1 - \eta) \leq \frac{2}{\eta};$$

also, $L - d \gg 1$. (We use this in (4.13) below.) Also, recall from Lemma 4.4 that we can write

$$M = (1 + \epsilon_p)k^{1/p}n^{1/k}/C_p, \quad \text{where} \ C_p = 2 \Gamma(1/p + 1)(pe)^{1/p} \text{ and } \epsilon_p = \mathcal{O}(\omega/k) = o(1).$$

It can be shown that $C_p \geq 2$ for all $p \in [1, \infty]$, and so

$$2M \leq e^{\epsilon_p}k^{1/p}n^{1/k}. \tag{4.11}$$

Recall that when we consider $p = 1$, we only require $k \ll \log n$; when we consider $p \in (1, \infty)$, we ask further that $k \ll \log n/\log \log n$. Note that if $I = \emptyset$ then $B = 0$, and so $B \cdot Z = 0$. Hence

$$n \mathbb{P}(B \cdot Z = 0 | I = \emptyset) = n \mathbb{P}(I = \emptyset) \leq e^{-\omega},$$

by the choice of the radius $M_{k,p}$.

Consider $I \subseteq [k]$ with $1 \leq \ell = |I| \leq d + 1$. There are at most $2^k$ such sets $I$. Recall $\xi_p$ given in Lemma 4.6b, and that $\xi_p = \mathcal{O}(\omega/k) = o(1)$. Applying (4.7b, 4.8a, 4.11), we obtain

$$n \mathbb{P}(B \cdot Z = 0, I = I) \leq C e^{k\epsilon_p}e^{\xi_p}k^{d-\ell+1}n^{d-\ell}/\mathcal{O}(\omega/k) = 2^{-k}o(1). \tag{4.12}$$

Consider now $I \subseteq [k]$ with $d + 2 \leq \ell = |I| \leq L = d + \frac{1}{\eta}(k - d)$. Applying (4.7b, 4.8b), we obtain

$$n \mathbb{P}(B \cdot Z = 0, I = I) \leq (1 + 3 \cdot 2^{d-\ell}) \cdot e^{k(1/\epsilon_p) + \xi_p}n^{-1+k/\ell} = 2^{-k}o(1). \tag{4.13}$$

We now sum over all $I$ with $1 \leq |I| \leq L$, using (4.12, 4.13):

$$n \sum_{1 \leq |I| \leq L} \mathbb{P}(B \cdot Z = 0, I = I) = o(1). \tag{4.14}$$

By the arguments in Corollary 3.13, with (4.8b) playing the role of (3.10b), we have

$$n \sum_{L \leq |I| \leq k} \mathbb{P}(B \cdot Z = 0, I = I) \leq 1 + 3 \cdot 2^{d-L} = 1 + o(1), \tag{4.15}$$

using the condition $L - d \gg 1$. This last result actually holds for all $p \in [1, \infty]$ and all $k \ll \log n$. The proof is completed by combining (4.14, 4.15) with (4.10).
We discuss some statistics in the regime where \( 3.1 \) can be relaxed: when \( 1 \leq \ell = |I| \leq d + 1 \). There are at most \( 2^k \) such sets \( I \). Applying \( 4.6a, 4.7c, 4.8a \), we obtain
\[
\begin{align*}
 n \mathbb{P}(B \cdot Z = 0, \mathcal{I} = I) & \leq C n^{(d - \ell + 2)/k} e^{\omega(d + 2 - \ell)/k} (1 + e^{-\omega/k} n^{1/k})^{d - 2 - \ell} e^{-\omega(1 - \ell/k)n^{-1 + \ell/k}} \\
 & \leq n^{-1 + \ell/k + o(1)} = 2^{-k} n^{-\eta + o(1)}. \tag{4.16}
\end{align*}
\]
For \( I \subseteq [k] \) with \( d + 2 \leq \ell = |I| \leq L \), applying \( 4.7c, 4.8b \) we have
\[
\begin{align*}
 n \mathbb{P}(B \cdot Z = 0, \mathcal{I} = I) & \leq 2 n^{-1 + \ell/k} \leq 2^{-k} n^{-1 + L/k + o(1)}, \tag{4.17}
\end{align*}
\]
since \( k \ll \log n \). We now sum over the \( I \) with \( 1 \leq |I| \leq L = (1 - \eta)k \), using \( 4.16, 4.17 \):
\[
\begin{align*}
 n \sum_{1 \leq |I| \leq L} \mathbb{P}(B \cdot Z = 0, \mathcal{I} = I) & \leq 2^k \cdot 2^{k} n^{-1 + L/k + o(1)} \leq n^{-\eta + o(1)} = o(1). \tag{4.18}
\end{align*}
\]

The proof is completed by combining \( 4.15, 4.18 \) with \( 4.10 \).

**Proof of Proposition 4.5b.** \( \) Recall that here \( k \asymp \log n \) and \( d \leq \frac{1}{2} \log n / \log \log n \).

Consider \( I \subseteq [k] \) with \( 1 \leq \ell = |I| \leq d + 1 \). There are at most \( (d + 1)^k \leq k^{d + 2} \) such sets \( I \). Since \( \log k = \log \log n + \log \lambda + o(1) \), we have \( k^{d + 2} \leq n^{2/3} \). Applying \( 4.6d, 4.7d, 4.8a \), we obtain
\[
\begin{align*}
 n \mathbb{P}(B \cdot Z = 0, \mathcal{I} = I) & \leq C(3\alpha \lambda \log n)^{d + 2 - \ell} n^{-1 + o(1)} \leq k^{-d - 2} n^{-1/4}. \tag{4.19}
\end{align*}
\]
We now sum over all \( I \) with \( 1 \leq |I| \leq d + 1 \):
\[
\begin{align*}
 n \sum_{1 \leq |I| \leq d + 1} n \mathbb{P}(B \cdot Z = 0, \mathcal{I} = I) & \leq n^{-1/4} = o(1). \tag{4.19}
\end{align*}
\]
Consider \( I \subseteq [k] \) with \( d + 2 \leq |I| \leq L \), for some \( d \ll L \ll k \). Similarly to above, there are at most \( L(k+1) \leq k^{L+1} \) such sets \( I \). Applying \( 4.6d, 4.7d, 4.8b \), we obtain
\[
\begin{align*}
 n \mathbb{P}(B \cdot Z = 0, \mathcal{I} = I) & \leq n^{-1 + o(1)} \leq k^{-L} n^{-1/2}. \tag{4.20}
\end{align*}
\]
We now sum over all \( I \) with \( d + 2 \leq |I| \leq L \):
\[
\begin{align*}
 n \sum_{d + 2 \leq |I| \leq L} n \mathbb{P}(B \cdot Z = 0, \mathcal{I} = I) & \leq n^{-1/2}. \tag{4.20}
\end{align*}
\]

The proof is completed by combining \( 4.15, 4.19, 4.20 \) and using \( 4.10 \).

### 4.6 Adapting Proof to Directed Cayley Graphs

Where the random variable \( A \) was uniform on a certain undirected lattice ball, it is now uniform on a directed ball (of a different radius). Other than this, the only adaptation that needs to be made is in determining the sizes of the discrete lattice balls: now instead of being a subset of \( \mathbb{Z}^k \), for some \( k \), they are restricted to the first quadrant, i.e. to \( \mathbb{Z}^k_+ \). Assuming that their radius is large enough, this simply reduces their size by a factor (roughly) \( 2^k \).

Since all the sizes in question scale like \( R^k \) when the ball-radius is \( R \), when \( k \ll \log n \) (and so \( R \gg 1 \), the desired radius for the directed ball is twice that of the undirected ball. When \( k \asymp \log n \) (and we consider the \( L_1 \) ball), the directed ball has size \( (R+k)^k \), so we are still interested in \( R \asymp k \asymp \log n \), just the constant is different for directed compared with directed.

### 5 Concluding Remarks and Open Questions

\( \S 5.1 \) We elaborate on Remarks 3.3(iii), where we claimed that the condition \( \min_j m_j > n^{1/k}(\log k)^2 \) in Theorem 3.1 can be relaxed: when \( k \asymp \log n \), we require \( \min_j m_j > r_n \), where \( (r_n) \) is a sequence diverging arbitrarily slowly; when \( k \gg \log n \), we completely remove it.

\( \S 5.2 \) We discuss briefly simplifications when assume the side-lengths \( m_j \) are prime.

\( \S 5.3 \) We discuss some statistics in the regime where \( k \) is a fixed constant.

\( \S 5.4 \) To conclude, we discuss some questions which remain open and gives some conjectures. Throughout this section, we only sketch details.
5.1 Relaxed Minimal Side-Length Condition

First consider \( k \approx \lambda \log n \), for some \( \lambda \in (0, \infty) \). Let \((r_n)\) be a sequence of integers with \( r_n \to \infty \), but arbitrarily slowly. Take \( t \geq t_0 \), and so \( t/k \geq 1 \), and write

\[
\mathcal{J}_0 := \{i \in [k] \mid |W_i(t)| \leq r_n\}; \quad \text{then} \quad |[k] \setminus \mathcal{J}_0| = o(k) \text{ whp}.
\]

Define \( \mathcal{J}_0' \) similarly for \( W' \), and set \( \mathcal{J} := \mathcal{J}_0 \cap \mathcal{J}_0' \). Recall from Lemma 3.8 that we wish to bound the probability \( W(t) = W'(t) \), under certain typicality assumptions. Now only consider coordinates \( i \in \mathcal{J} \): define \( \overline{W}(t) \) and \( \overline{W}'(t) \) by setting \( \overline{W}_i(t) = W_i(t) \) and \( \overline{W}'_i(t) = W'_i(t) \) for \( i \in \mathcal{J} \), but \( \overline{W}_i(t) = 0 = \overline{W}'_i(t) \) for \( i \notin \mathcal{J} \); we then see that \( \{W(t) = W'(t)\} \subseteq \{\overline{W}(t) = \overline{W}'(t)\} \). This alleviates the local typicality requirement, in essence replacing it. Define \( \overline{W} := \overline{W} - \overline{W}' \).

This decreases the entropy: previously each (independent) coordinate had entropy \( \log n/k \) at the entropic time, and there were \( k \) of them; now we have removed some. We define the new entropic time to be so that \( \overline{W}(t) \) has entropy \( \log n \). Since \( W \) and \( W' \) are independent, as are the coordinates of each, the law of \( \overline{W}(t) \) is that of \( W(t) \), but with entries with absolute value greater than \( r_n \) removed, and then further entries removed independently with probability \( \mathbb{P}(|W_i(t)| > r_n) \).

Once we have done this, we proceed as before, but now only considering \( i \in \mathcal{J} \). In the definition (3.6) of \( \mathcal{I}(\nu) \), we restrict to only coordinates \( i \in \mathcal{J} \). We sketch the changes which need to be made.

In Lemma 3.12, we used local typicality to say that \( |V_i(t)| \leq 2r_n \) for all \( i \); this allowed us to bound the gcd in question by \( 2r_n \). Here, instead, for \( i \in \mathcal{J} \) we have, by definition, \( |V_i(t)| \leq 2r_n \); so local typicality is no longer needed here. The gcd analysis follows exactly as before.

The calculation of \( \mathbb{P}(\mathcal{I} = I \mid \text{typ}) \), from Lemma 3.15, specifically (3.13), also needs to change. In essence, instead of trying to match \( W \) to a specific vector \( w \) for all coordinates \( i \in [k] \), we need only match for \( i \in \mathcal{J} \). Rectifying this involves dividing the final expression by \( p_{k-|\mathcal{J}|} = p_{\text{typ}(k)} \). Since \( p_*=n^{-1/k}k^{-2} \), we multiply the right-hand side of (3.13) by \( n^{\text{typ}(k)/k} = n^{\text{typ}(1)} \).

In the calculation of the modified \( L_2 \) distance, Proposition 3.9, whenever we use \( \mathbb{P}(\mathcal{I} = I \mid \text{typ}) \) we upper bound by a term including the factor \( n^{-\text{typ}(1)} \). Hence this additional \( n^{\text{typ}(1)} \) gets absorbed into this factor. Thus the proof that the TV distance from the uniform distribution is small shortly after the new entropic time now follows through similarly.

It remains to determine the new entropic time. Using similar arguments to those used in §2.3 to determine the original entropic time, which we denote \( t_{\text{ent}} \), it is not difficult to see that the new entropic time, which we denote \( t_{\text{ent}}' \), satisfies \( t_{\text{ent}}' \approx f(\lambda)k \), where \( f \) given by (2.1b); in words, the entropic time has not changed, up to subleading order terms. (Cf Proposition A.8.) This proves cutoff at the entropic time, under these milder conditions, however we no longer have bounds on the window (or shape).

For the regime \( k \gg \log n \), we can completely remove the minimal side-length condition. Recall, from §1.2, that cutoff had already been established for all Abelian groups in this regime. However, if \( k \gg l^2 \), eg if \( k \gtrsim (\log n)^2 \), then replacing \( r_n \) with 1, the new \( \mathcal{J}_0 \) satisfies \( \mathcal{J}_0 = [k] \) whp; hence our method also determines the window and shape of the cutoff.

Finally, consider \( k \ll \log n \). In this regime, the argument does not work: from §2.3 (cf Proposition A.5), the entropy is growing logarithmically at the entropic time, and so replacing \( k \) by \( k(1-o(1)) \) in (2.1a) changes \( t_0/k \) by more than a \( 1 + o(1) \) factor. This is insufficiently precise.

5.2 Prime Side-Lengths

In the §3.2, the outline of the TV mixing proof where we considered the group \( \mathbb{Z}_n \), we remarked that the proof is easier when one assumes that \( n \) is prime; in general, one can consider \( \bigoplus_{j=1}^d \mathbb{Z}_{m_j} \) and assume that the integers \( m_1, ..., m_d \) are prime.

There are only two other papers considering \( k \lesssim \log n \): Hough [23] considered the group \( \mathbb{Z}_p \) with \( p \) prime; Wilson [40] considered the group \( \mathbb{Z}_p^d \), but comments that his method should apply to \( \mathbb{Z}_p^d \) for fixed \( p \). An inspection of Wilson’s proof suggests that, in its current form, it requires \( p \) to be prime—cf Lemma 3.11 where if \( m_j \) is prime then the gcds are all 1 and the uniform distribution is uniform over the whole group, not just a subgroup. Relatedly, Hough requires \( p \) prime so that \( \mathbb{Z}_p^d \) is a multiplicative group, to apply representation and lattice ideal theory.
Not only would this primality assumption significantly simplify our proof, but also give rise to weaker restrictions on \( d \). We now elaborate.

In Lemma 3.12 we calculate the expectation of \( \gcd(V_1, \ldots, V_k) \), where \( V \) is a rate-\( 2/k \) SRW on \( \mathbb{Z}^k \). Further, we assume that \( |V_i| < m_j \) for all \( i \) and \( j \). Bounding this gcd took some considerable analysis, and moreover added restriction to \( d \); if \( m_j \) is prime, then the gcd (under the above assumptions) is always equal to 1. This significantly simplifies the analysis, and furthermore removes some conditions on \( d \). We do not go into further details here.

In a companion article [20, Theorem B], we analyse, in great detail, mixing statistics when the underlying group is \( \mathbb{Z}_p^d \) for a prime \( p \), which may be small; we place only light restrictions on \( d \), allowing even \( k - d = 1 \) when \( p \gg 1 \). We refer the reader there for further details and examples.

### 5.3 Mixing, Relaxation and Diameter Bounds for Constant \( k \)

Throughout the paper we have always been assuming that \( k \to \infty \) as \( n \to \infty \). We can consider also the \( k \) fixed (independent of \( n \)). There will be no cutoff in this regime. We sketch some details here; a more refined presentation can be found in our companion article [19, §5.1].

Via a concept called moderate growth, Diaconis and Saloff-Coste [13] give a condition so that

\[
t_{\text{mix}}/k \lesssim \Delta^2 \lesssim t_{\text{rel}} \lesssim t_{\text{mix}},
\]

with implicit constants depending on the moderate growth parameters. Breuillard and Tointon [11] give a simple criterion for moderate growth to hold: \( n \leq \beta \Delta^n \) for some \( \alpha, \beta > 0 \), where \( n := |G| \) and \( \Delta \) is the diameter of the graph. Since graph-balls have cardinality bounded by that of lattice-balls (for Abelian groups), it is straightforward to see that \( \Delta \gtrsim kn^{1/k} \) when \( k \gg 1 \); we can hence take \( \alpha := k \approx 1 \) and some \( \beta \) sufficiently large. From this, it follows that for constant \( k \) the product condition for cutoff \( t_{\text{rel}} \ll t_{\text{mix}} \) fails, and so there is no cutoff; see, eg, [25, Proposition 18.4].

### 5.4 Open Questions and Conjectures

In this section, we briefly discuss some questions which we have left open. For some of these questions, we conjecture an answer; for others, it is not clear a priori what the answer should be.

Below PP stands for Poisson process and SRW for simple random walk.

**1: Cutoff for Arbitrary General Abelian Groups**

We have studied cutoff for Abelian groups \( G \) of the form \( \bigoplus_{d} \mathbb{Z}_{m_j} \) with conditions on the dimension \( d \) and the minimal side-length \( m = \min m_j \). These conditions can be relaxed; see Hypotheses A. Note that the lower bound (given in §3.3) is valid for all Abelian groups and all \( k \). In our companion article [20, Theorem B] we consider groups \( \mathbb{Z}_p^d \); we can allow \( p \) constant and also \( k \) very close to the dimension \( d \), and the mixing time is a different entropic time to \( t_0(k, n) \).

**Conjecture 1.** Let \( (G_N)_{N \in \mathbb{N}} \) be a sequence of Abelian groups and \( (k_N)_{N \in \mathbb{N}} \) a sequence of integers with \( \log k_N / \log |G_N| \to 0 \) and \( k_N - d(G_N) \to \infty \) as \( N \to \infty \). Then, whp, the random walk on the associated sequence of random Cayley graphs exhibits cutoff.

This says that all Abelian groups exhibit cutoff provided \( k - d(G) \gg 1 \) (which ensures that the group is generated whp), but does not require the cutoff to be at a specific (eg entropic) time.

We now discuss the conditions of Hypotheses A are used in the method of proof. The main purpose of the minimal side-length condition is so that the auxiliary walk \( W \) on \( \mathbb{Z}^k \) is indistinguishable from one on \( \mathbb{Z}_{m_j} \), for each \( k \). Recalling the definition

\[
\mathcal{I} = \{ i \in [k] \mid V_i \neq 0 \mod m_j \text{ for all } j = 1, ..., d \},
\]

this side-length condition, when we conditioned on local typicality, meant we could write

\[
\mathcal{I} = \{ i \in [k] \mid V_i \neq 0 \};
\]
this latter event is much easier to get our hands on. If we worked with the former (if the minimal side-length condition does not hold), then the analysis of Lemma 3.15 (and Lemma 3.10) would need to change. To illustrate this, consider just Lemma 3.10 for the moment and assume that $m_1 = 2$, and ignore typicality. The probability that a SRW, evaluated at any time, is even is at least $\frac{1}{2}$. This means that the size $|I^c|$ $\leq \text{Geom}(\frac{1}{2})$; thus $\Pr(I = \emptyset) \geq 2^{-k}$. For relatively small $k$, eg $k = \log \log n$, this is much larger than $n^{-1}$, which is the bound we desire.

The purpose of the bound on the dimension $d$ is less subtle. In the bounds of Lemma 3.15, we have the desired $e^{-\omega n^{-1}}$ term, but also a factor $n^{d/|I|} / k^{2d/|I|}$. We only need this when $|I| \leq d + 1$, and so the conditions on $d$ simply ensure that $n^{d/k} k^{2d}$ is not too large.

2: Diameter for Abelian Groups for Diverging $k$

We have shown concentration of typical distance, but never considered the diameter. It is trivial that the typical distance is a lower bound on the diameter, and that twice the typical distance is an upper bound. Can more be determined? Recall that $d(G)$ is the minimal size of a generating set.

**Conjecture 2.** For an Abelian group $G$ and $Z_1, \ldots, Z_k \sim \text{iid Unif}(G)$, write $\Delta_Z$ for the diameter of the Cayley graph with generators $Z$. Assume that $k$ diverges, sufficiently rapidly in terms of $d(G)$. Does the law of $\Delta_Z$ concentrate? If so, does it concentrate around some value $\Delta_k$, $|G|$, which depends only on $k$ and $|G|$?

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[12] G. Conchon-Kerjan (2019). Cutoff for Random Lifts of Weighted Graphs. Available at arXiv:1908.02898
A Deferred Proofs from §2: CLT and Entropic Time Calculations

This part of the appendix (§A) is devoted to properties of the entropic time $t_0$ and cutoff window $t_n - t_0$; this is done through analysis of a CLT for $Q$, Proposition 2.3, and variance of $Q_1$ at the entropic time, $\text{Var}(Q_1(t_0))$. Accordingly, here we mainly derive properties of the SRW on $\mathbb{Z}$ evaluated at $t/k$ or of $\text{Po}(t/k)$, for $t$ around the entropic time.

We repeatedly use a local CLT for Poisson and simple random walk distributions. We state it here precisely; the particular version is given in [24, Theorem 2.5.6].

**Theorem A.1** (Local CLT). Let $s \geq 0$. Let $X_s$ be a random variable with one of the following distributions: $\text{Po}(s) - s$; the location of a rate-1 continuous-time, one-dimensional, symmetric, simple random walk started from 0 and run for time $s$. If $|x| \leq \frac{s^7}{12}$, then

$$P(X_s = x) = \frac{1}{\sqrt{2\pi s}} \exp \left( -\frac{x^2}{2s} \right) \exp \left( O \left( \frac{1}{\sqrt{s}} + \frac{|x|^3}{s^2} \right) \right).$$

In particular, if $|x| \leq s^{7/12}$, then

$$P(X_s = x) = \frac{1}{\sqrt{2\pi s}} \exp \left( -\frac{x^2}{2s} \right) \exp(O(s^{-1/4})). \quad (A.1)$$

### A.1 Justification of CLT Application

We first justify our CLT application in (2.2). The distribution of $Q_1(t_0)$ depends on $k$ (and $n$), and so we cannot apply the standard CLT. Instead, we apply a CLT for 'triangular arrays'; specifically, we now state a special case of the Lindeberg–Feller theorem.

**Theorem A.2.** For each $k \in \mathbb{N}$, let $\{Y_{i,k}\}_{i=1}^k$ be an iid sequence of centralised, normalised random variables, and suppose that $\mathbb{E}(Y_{1,k}^4) \ll k$. Then

$$\sum_{i=1}^k Y_{i,k}/\sqrt{k} \rightarrow^d N(0,1) \quad \text{as } k \rightarrow \infty,$$

where $N(0,1)$ is a standard normal.

This version can be deduced easily using the Markov (or Chebyshev) inequality from, for example, the version given in [18, Theorem 3.4.5]. Using this, we can deduce our CLT for $Q$.

**Proof of Proposition 2.3.** For our application, for each $\alpha \in \mathbb{R}$, we take

$$Y_{i,k} := \frac{Q_i(t_0) - \mathbb{E}(Q_i(t_0))}{\sqrt{\mathbb{Var}(Q_i(t_0))}}. \quad (A.2)$$

Observe that $\mathbb{E}(Y_{i,k}) = 0$ and $\mathbb{Var}(Y_{i,k}) = \mathbb{E}(Y_{1,k}^2) = 1$. Assuming that $\mathbb{E}(Y_{1,k}^4) \ll k$, we deduce the following result: for any sequence $(\zeta_n)_{n \geq 1}$ which converges to $\zeta$, we deduce that

$$P(Q(t) - \mathbb{E}(Q(t)) \geq \zeta_n \sqrt{\mathbb{Var}(Q(t))}) \rightarrow \Psi(\zeta). \quad (A.3)$$

(We are also using Slutsky’s theorem to allow $\zeta_n$ to depend on $n$, and, of course, the fact that $k \rightarrow \infty$ as $n \rightarrow \infty$.) We also further rely on the following claim:

if $t \approx t_0$, then $\mathbb{Var}(Q_1(t)) \approx \mathbb{Var}(Q_1(t_0))$; also $\mathbb{Var}(Q(t_0)) \gg 1.$ \quad (A.4)

We prove these two statements in this context (independently of the application of the CLT) in Corollary A.4 in §A.2. Now recall Equation (2.1), which says that $t_\alpha \approx t_0$ for all $\alpha \in \mathbb{R}$. Taking

$$\zeta_n := -\alpha \sqrt{\mathbb{Var}(Q(t_0))/\mathbb{Var}(Q(t_\alpha))} \pm \omega / \sqrt{\mathbb{Var}(Q(t_\alpha))} \quad \text{with } \omega := \mathbb{Var}(Q(t_0))^{1/4} \gg 1, \quad (A.5)$$

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applying (A.3, A.4) along with the above recollections we obtain the desired result:

\[
\mathbb{P}(Q(t_\alpha) \leq \log n \pm \omega) \to \Psi(\alpha). \tag{2.2}
\]

It remains to verify that \( \mathbb{E}(Y_{i,k}^2) \ll k \). Roughly, \(|W_1(t)| \) is ‘well-approximated’ by the following:

\[
\begin{aligned}
&|N(\mathbb{E}(W_1(t)), t/k)| \quad \text{when } t/k \gg 1, \quad \text{ie } k \ll \log n; \\
&\text{Bernoulli}(t/k) \quad \text{when } t/k \ll 1, \quad \text{ie } k \gg \log n.
\end{aligned}
\]

In the interim regime \( k \asymp \log n \), we have that \( W_1 \) behaves like an ‘order 1’ random variable, in the sense that its mean and variance are order 1 in \( n \) (ie do not converge to 0 or diverge to \( \infty \)). It will actually turn out that the normal approximation is sufficient in the \( k \asymp \log n \) regime also. Below, we abbreviate \( Q(t_\alpha) \) by \( Q_1, W_1(t_\alpha) \) by \( W_1 \) and \( t_\alpha \) by \( t \).

Write \( s := t/k \). We shall consider separately the cases \( s \geq 1 \) and \( s \ll 1 \). When \( s \geq 1 \), we have \( t \gg k \gg 1 \); when considering \( s \ll 1 \), however, we shall only consider \( t \ll 1 \ll k \). We shall be interested in \( t := t_\alpha \approx t_0 \), and Proposition 2.2 says that \( t_0 \gg 1 \) in all regimes; hence we need only consider \( t \gg 1 \).

Consider first \( s = t/k \) with \( s \geq 1 \). In this regime, we approximate \( W_1(t) \) by \( N(\mathbb{E}(W_1(s)), s) \) distribution, where \( s = t/k \). Let \( Z \sim N(\mathbb{E}(W_1), s) \), and write \( f \) for its density function:

\[
f(x) := (2\pi s)^{-1/2} \exp\left(-\frac{1}{2s}(x - \mathbb{E}(W_1))^2\right) \quad \text{for } x \in \mathbb{R}. \tag{A.6}
\]

Let \( R_1 \) be a real valued random variable defined so that

\[
R_1 = -\log f(x) \quad \text{when } W_1 = x. \tag{A.7}
\]

Also write \( G := W_1 + U \), where \( U \sim \text{Unif}\left(-\frac{1}{2}, \frac{1}{2}\right) \) is independent of \( W_1 \); then \( G \) has density function

\[
g(x) := P(W_1 = [x]) \quad \text{for } x \in \mathbb{R}, \tag{A.8}
\]

where \([x] \in \mathbb{Z}\) is \( x \in \mathbb{R} \) rounded to the nearest integer (rounding up when \( x \in \mathbb{Z} + \frac{1}{2} \)). Using convexity of the 4-norm, we have

\[
(a - b)^4 \leq 2^8((a - a')^4 + (a' - b')^4 + (b' - b)^4) \quad \text{for all } a, a', b, b' \in \mathbb{R}.
\]

Applying this inequality with \( a = Q_1, a' = R_1, b = \mathbb{E}(Q_1) \) and \( b' = \mathbb{E}(R_1) \), we obtain

\[
2^{-8}\mathbb{E}((Q_1 - \mathbb{E}(Q_1))^4) \leq \mathbb{E}((Q_1 - R_1)^4) + \mathbb{E}((R_1 - \mathbb{E}(R_1))^4) + \mathbb{E}(R_1 - Q_1)^4 \\
\leq \mathbb{E}((R_1 - \mathbb{E}(R_1))^4) + 2\mathbb{E}((Q_1 - R_1)^4), \tag{A.9}
\]

with the second inequality following from Jensen (or Cauchy-Schwarz twice). We study these terms separately. Approximately, the local CLT will say that the second term is small; up to an error term which we control with the local CLT, the first term we can calculate directly using properties of the normal distribution.

We consider first the first term of (A.9). In terms of an integral, it is given by

\[
\mathbb{E}((R_1 - \mathbb{E}(R_1))^4) = \int_{\mathbb{R}} g(x)(-\log f(x) - \mathbb{E}(R_1))^4 \, dx.
\]

The local CLT suggests that we can approximately replace the \( g(x) \) factor by \( f(x) \), at least for a large range of \( x \). So let us first study

\[
\int_{\mathbb{R}} f(x)(-\log f(x) - \mathbb{E}(R_1))^4 \, dx = \int_{\mathbb{R}} f(x + \mathbb{E}(W_1))( -\log f(x + \mathbb{E}(W_1)) - \mathbb{E}(R_1))^4 \, dx.
\]

This expression depends only on properties of the normal distribution, and direct calculation, via expanding the fourth power and using moments of \( \mathbb{N}(0, 1) \), finds that this equals \( \frac{4}{s^7} \).

Now, by the local CLT (A.1), we have

\[
\int_{-s^{7/12}}^{s^{7/12}} g(x)(-\log f(x) - \mathbb{E}(R_1))^4 \, dx = \left(1 + \mathcal{O}(s^{-1/4})\right) \int_{-s^{7/12}}^{s^{7/12}} f(x)(-\log f(x) - \mathbb{E}(R_1))^4 \, dx
\]

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\begin{align}
\int_{R([-s^{7/12}, s^{7/12}])} f(x)(-\log f(x) - \mathbb{E}(R_1))^4 \, dx = o(s^{-10}). \tag{A.10}
\end{align}

(In fact, it is easy to see that it is \(O(\exp(-cs^{7/6}))\) for some sufficiently small constant \(c\). Hence
\begin{align}
\mathbb{E}((R_1 - \mathbb{E}(R_1))^4) &= \frac{4}{3} (1 + O(s^{-1/4})) = \frac{4}{3} (1 + o(1)) \tag{A.11}
\end{align}

We now turn to the second term of (A.9). In terms of an integral, it is given by
\begin{align}
\mathbb{E}((Q_1 - R_1)^4) &= \int_R g(x) \log (f(x)/g(x))^4 \, dx.
\end{align}

Again by the local CLT (A.1), we have
\begin{align}
\int_{-s^{7/12}}^{s^{7/12}} g(x) \log (f(x)/g(x))^4 \, dx = O(s^{-1/4}) \int_{-s^{7/12}}^{s^{7/12}} g(x) \, dx \leq O(s^{-1/4}),
\end{align}

and a similar application of the tail bounds in Propositions B.2 and B.3 shows that
\begin{align}
\int_{\mathbb{R}([-s^{7/12}, s^{7/12}])} g(x) \log (f(x)/g(x))^4 \, dx = o(s^{-10}) = O(s^{-1/4}). \tag{A.12}
\end{align}

Hence, combining (A.11, A.12) into (A.9), we obtain
\begin{align}
\mathbb{E}((Q_1 - \mathbb{E}(Q_1))^4) &= \frac{4}{3} \cdot 2^8 + o(1) \leq 1000
\end{align}

We must now consider \(\mathbb{V}ar(Q_1)\). We do this in Corollary A.4 in §A.2. Recall also that \(t_0 \geq k\) when \(k \lesssim \log n\); this follows from the continuity of the function \(f\) in (2.1). Using then the continuity of the function \(g\) in (A.15), we see that if there exists a constant \(c\) so that \(s = t/k \geq c\), then there exists a constant \(C\) (depending on \(c\)) so that \(\mathbb{E}(Y_{1,k}^4) \leq C\); in particular we certainly have \(\mathbb{E}(Y_{1,k}^4) \ll k\). The completes the proof for the regime \(k \lesssim \log n\).

Consider now \(s = t/k\) with \(s \ll 1\) but \(t \gg 1\). In this regime, we shall approximate the number of steps taken by Bernoulli\((t/k)\). Indeed, we have
\begin{align}
\mathbb{E}(W_1 = 0) = 1 - s + O(s) \quad \text{and} \quad \mathbb{E}(|W_1| = 1) = s + O(s^2).
\end{align}

We also use the fact that, for both the undirected and directed cases, for \(x \geq 0\) we have
\begin{align}
\mathbb{P}(W_1 = x) &\geq \mathbb{P}(\text{Poi}(s) = x) \cdot 2^{-x} = 2^{-x} e^{-s} s^x/x! \geq (s^2/x^2)^x; \tag{A.13}
\end{align}

from this one deduces that \(-\log \mathbb{P}(W_1 = x) \leq x \log(x/s^2) = x(x + 2 \log(1/s))\). We use this to show that the terms with \(|x| \geq 2\) contribute subleading order to the expectation
\begin{align}
\mathbb{E}(Q_1) = \sum_x \mathbb{P}(W_1 = x) \log 1/\mathbb{P}(W_1 = x) = s \log(1/s) + O(s).
\end{align}

Similarly, we can use (A.13) to ignore the terms with \(|x| \geq 2\) in
\begin{align}
\mathbb{E}(|Q_1 - \mathbb{E}(Q_1)|^r) &= \sum_x \mathbb{P}(W_1 = x) |-\log \mathbb{P}(W_1 = x) - s \log(1/s) + O(s)|^r \\
&= s \log(1/s)^r (1 + O(s)), \tag{A.14}
\end{align}

for any fixed \(r \in \mathbb{N}\) with \(r \geq 2\), say \(r \in \{2, 3, 4\}\).

In particular, this says that \(\mathbb{V}ar(Q_1) \approx s \log(1/s)^2\), and so
\begin{align}
\mathbb{E}(Y_{1,k}^4) \approx (s \log(1/s)^4)/(s \log(1/s)^2)^2 = 1/s = k/t \ll k,
\end{align}

with the final relation holding since \(s \ll 1\) we do have \(t \gg 1\).

We now have all that we need to get on and calculate the entropic time \(t_0\) in the three regimes of \(k\). However, in order to find the cutoff times \(t_\alpha\), we need to know what the variance of the terms in the sum \(Q(t)\), ie \(\mathbb{V}ar(Q(t))\), is for \(t \approx t_0\). 

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A.2 Variance of \( Q_1(t) \)

Recall that, for all \( t \geq 0 \), we have

\[
Q(t) = -\log \mu(t) = -\sum_{i=1}^{k} \log \nu_i(W_i(t)) = \sum_{i=1}^{k} Q_i(t),
\]

and that the \( Q_i(t) \)-s are iid (for fixed \( t \)). We now determine what its variance is at the entropic time \( t_0 \), and how the variance changes around this time. Note that \( \text{Var}(Q(t)) = k \text{Var}(Q_1(t)) \).

**Proposition A.3.** In both the undirected and the directed case,

\[
\text{Var}(Q_1(sk)) \simeq \begin{cases} 
1/2 & \text{as } s \to \infty, \\
 s \log(1/s)^2 & \text{as } s \to 0; 
\end{cases} \tag{A.15a}
\]

furthermore, the map \( s \mapsto \text{Var}(Q_1(sk)) : [0, \infty) \to \mathbb{R}_+ \) is continuous.

From this, it is easy to calculate the variance at the entropic time \( t_0 \). Note that knowledge of the variance is not required to calculate \( t_0 \). Recall (2.1):

\[
t_0 \approx \begin{cases} 
 kn^{2/k}/(2\pi e) & \text{when } k \ll \log n, \\
 f(\lambda) k & \text{when } k \approx \lambda \log n, \\
 \log n \log(k/\log n) k & \text{when } k \gg \log n. 
\end{cases} \tag{2.1a}
\]

\[
(\text{Knowledge of the variance is required in calculating } t_0 \text{ with } \alpha \neq 0, \text{ but not with } \alpha = 0.)
\]

**Corollary A.4.** For all regimes of \( k \), in both the undirected and directed case,

\[
\text{if } t \approx t_0, \text{ then } \text{Var}(Q_1(t)) \approx \text{Var}(Q_1(t_0)) \gg 1/k. \tag{A.16}
\]

Moreover, for all \( \lambda > 0 \), we have

\[
\text{Var}(Q_1(t_0)) \simeq \begin{cases} 
1/2 & \text{when } k \ll \log n, \\
 v(\lambda) & \text{when } k \approx \lambda \log n, \\
 \log n \log(k/\log n)/k & \text{when } k \gg \log n,
\end{cases}
\]

where \( v : (0, \infty) \to (0, \infty) \) is a continuous function whose value differs between the undirected and directed cases.

**Proof of Corollary A.4.** The first claim is immediate from the limiting expressions (A.15) for the variance and the continuity. The claim for \( k \ll \log n \) is immediate from (2.1a) and Proposition A.3.

For \( k \gg \log n \), there is a small amount of work to do. Set \( s_0 := t_0/k \), and so

\[
s_0 = \frac{t_0}{k} \approx \frac{\log n/k}{\log(k/\log n)} = \frac{1}{\kappa \log \kappa} \quad \text{where } \kappa := \frac{k}{\log n} \gg 1.
\]

We then also have

\[
\log(1/s_0) = -\log \log \kappa - \log \kappa \approx -\log \kappa,
\]

and hence

\[
s_0 \log(1/s_0)^2 \approx (\log \kappa)^2 / (\kappa \log \kappa) = \log \kappa / \kappa = \log n \log(k/\log n)/k.
\]

Note that while this has \( \text{Var}(Q_1(t_0)) \ll 1 \), it does have \( \text{Var}(Q(t_0)) = k \text{Var}(Q_1(t_0)) \gg 1 \).

Finally consider \( k \approx \lambda \log n \). By (2.1b), we have \( s = t_0/k \to f(\lambda) \), and each coordinate runs at rate \( 1/k \). By the continuity of \( Q_1(\cdot) \), and hence given \( C > 0 \) there exists an \( M \) so that

\[
1/M \leq \text{Var}(Q_1(sk)) \leq M \quad \text{for all } s \text{ with } 1/C \leq s \leq C.
\]

Hence, by continuity, \( \text{Var}(Q_1(t_0)) \to v \) for some constant \( v \in (0, \infty) \) depending only on \( \lambda \). Note that this \( v \) is not (necessarily) the same in the directed and undirected cases. \( \square \)
Proof of Proposition A.3. Consider first \( s \to \infty \), i.e. (A.15a). This proof is similar to the \( s \gtrsim 1 \) case, in justifying the CLT application. In particular, if
\[
g(x) := \mathbb{P}(W_1(sk) = [x]) \quad \text{and} \quad f(x) := (2\pi s)^{-1/2} \exp\left(-\frac{1}{2s}(x - \mathbb{E}(W_1(sk)))^2\right),
\]
then the local CLT (A.1) says, for \( s \gtrsim 1 \), that
\[
g(x) = f(x) \left(1 + \mathcal{O}(s^{-1/4})\right) \quad \text{for} \quad x \in \mathbb{R} \quad \text{with} \quad |x - \mathbb{E}(W_1(sk))| \leq s^{7/12}.
\]
Under the assumption that \( W_1(sk) \) is actually distributed as \( N(0,s) \), direct calculation as in the previous section shows that the variance is then \( \frac{1}{2} \). Considering the same approximations as before, namely splitting the integration range into \(|x - \mathbb{E}(W_1)| \leq s^{7/12}\) and \(|x - \mathbb{E}(W_1)| > s^{7/12}\), and using the local CLT to argue that \( \log(g(x)/f(x)) = \mathcal{O}(s^{-1/4}) \) for \( x \) in the first range, we obtain
\[
\mathbb{Var}(Q_1(sk)) = \frac{1}{2} + \mathcal{O}(s^{-1/4}\log s) \approx \frac{1}{2} \quad \text{when} \quad s \gg 1.
\]
Consider next \( s \to 0 \), i.e. (A.15a). In the CLT justification in the case \( s \gtrsim 1 \), we showed that
\[
\mathbb{E}\left((Q_1(sk) - \mathbb{E}(Q_1(sk)))^r\right) = s \log(1/s)^r + \mathcal{O}(s^2 \log(1/s)^r),
\]
(A.14)
and in particular deduced that \( \mathbb{Var}(Q_1(sk)) \approx s \log(1/s)^2 \). This applies for \( s \ll 1 \) also.

The continuity claim of the map \( s \mapsto \mathbb{Var}(Q_1(sk)) \) is immediate. \( \square \)

A.3 Calculating the Entropic and Cutoff Times

In this section we calculate the entropic time \( t_0 \), and the cutoff times \( t_\alpha \). Write
\[
h(t) := \mathbb{E}(Q_1(t));
\]
note that \( h(t) \) is the entropy of \( W_1(t) \), which is either a rate-1/k SRW on \( \mathbb{Z} \) or a rate-1/k Poisson process (abbreviated \( PP \)) on \( \mathbb{Z} \). We prove Proposition 2.2, which the reader should recall.

We first consider the regime \( t \gg k \), which corresponds to \( k \ll \log n \).

Proposition A.5. Write \( s := t/k \). For \( s \gtrsim 1 \), the entropy \( h \) of a rate-1/k SRW or \( PP \) on \( \mathbb{Z} \) satisfies
\[
h(t) = \frac{1}{2} \log(2\pi es) + \mathcal{O}(s^{-1/4}). \tag{A.17}
\]

Proof. We consider both the directed and undirected cases together. Write \( s := t/k \). Define \( f, R_1 \) and \( g \) as in (A.6), (A.7) and (A.8), respectively. By (A.12), we have
\[
|\mathbb{E}(Q_1) - \mathbb{E}(R_1)| \leq \mathbb{E}((Q_1 - R_1)^4)^{1/4} = o(s^{-5/2}) = \mathcal{O}(s^{-1/4}) \quad \text{when} \quad s \gtrsim 1.
\]
A similar calculation as used for (A.11) gives
\[
\mathbb{E}(R_1) = (1 + \mathcal{O}(s^{-1/4})) \cdot \log(2\pi es).
\]
Hence we obtain our desired expression, \( (A.17) \). \( \square \)

We now calculate the derivative of this entropy.

Proposition A.6. Write \( s := t/k \). For \( s \gg 1 \), the entropy \( h \) of a rate-1/k SRW or \( PP \) on \( \mathbb{Z} \) satisfies
\[
h'(t) = (2t)^{-1}(1 + \mathcal{O}(s^{-10})).
\]

Proof. Write \( s := t/k \). Define \( f, R_1 \) and \( g \) as in (A.6), (A.7) and (A.8), respectively. We have
\[
h(t) = h(sk) = -\sum_{x \in \mathbb{Z}} \mathbb{P}(X_s = x) \log \mathbb{P}(X_s = x).
\]
Differentiating this with respect to \( t \) we obtain
\[
k h'(t) = \frac{d}{dt} h(sk) = -\sum_{x \in \mathbb{Z}} \frac{d}{dt} \mathbb{P}(X_s = x) (\log \mathbb{P}(X_s = x) + 1).
\]
Consider first the undirected case. Let $X$ be a rate-$1$ continuous-time SRW on $\mathbb{Z}$. Then $W_1(t) \sim X_s$. Using the Kolmogorov backward equations, we obtain
\[
\frac{d}{ds} \mathbb{P}(X_s = x) = \frac{1}{2} \mathbb{P}(X_s = x + 1) + \frac{1}{2} \mathbb{P}(X_s = x - 1) - \mathbb{P}(X_s = x).
\]
Now write $p_s(x) := \mathbb{P}(X_s = x)$ and $g_s(x) := p_s([x])$. Since $\sum_{x \in \mathbb{Z}} p_s(x) = 1$, we obtain
\[
k \frac{d}{ds} = \sum_{x \in \mathbb{Z}} \left( p_s(x) - \frac{1}{2}(p_s(x + 1) + p_s(x - 1)) \right) \log p_s(x)
= \int_{\mathbb{R}} \left( g_s(x) - \frac{1}{2}(g_s(x + 1) + g_s(x - 1)) \right) \log g_s(x) \, dx
= \int_{\mathbb{R}} \left( g_s(x) - \frac{1}{2}(g_s(x + 1) + g_s(x - 1)) \right) \log f_s(x) \, dx \quad \text{(A.18a)}
+ \int_{\mathbb{R}} \left( g_s(x) - \frac{1}{2}(g_s(x + 1) + g_s(x - 1)) \right) \log (g_s(x)/f_s(x)) \, dx, \quad \text{(A.18b)}
\]
where $f_s(x) := (2\pi s)^{-1/2} \exp(-x^2/(2s))$. The same arguments as used for (A.10) show that the integral in (A.18b) is $o(s^{-10})$. Now consider the integral in (A.18a). Using a simple shift, we have
\[
\int_{\mathbb{R}} g_s(x + 1) \log f_s(x) \, dx = \int_{\mathbb{R}} g_s(x) \log f_s(x) \, dx - \int_{\mathbb{R}} g_s(x) \log (f_s(x) - f_s(x)) \, dx,
\]
and we consider $\int_{\mathbb{R}} g_s(x - 1) \log f_s(x) \, dx$ similarly; hence we have
\[
\int_{\mathbb{R}} g_s(x) \left( \log f_s(x) - \frac{1}{2}(g_s(x + 1) + g_s(x - 1)) \right) \log f_s(x) \, dx
= \frac{1}{2} \int_{\mathbb{R}} g_s(x) \left( \log (f_s(x) - f_s(x)) + \log (f_s(x) + f_s(x)) \right) \, dx
= \frac{1}{2} \int_{\mathbb{R}} g_s(x) \log (f_s(x - 1) f_s(x + 1) / f_s(x)^2) \, dx.
\]
Since $f_s(x) = (2\pi s)^{-1/2} \exp(-x^2/(2s))$, this log is precisely $1/s$ (independent of $x$). Since it is a distribution, $g_s$ integrates to $1$, so the integral equals $1/(2s)$. Combining the bounds for (A.18) and dividing through by $k$ proves the undirected case.

Now consider the directed case. Let $X_s \sim \text{Po}(s)$, which has $\mathbb{P}(X_s = x) = e^{-s} x^s / s!$. Then $W_1(t) \sim X_s$. Direct differentiation shows that
\[
\frac{d}{ds} \mathbb{P}(X_s = x) = \mathbb{P}(X_s = x - 1) - \mathbb{P}(X_s = x) = e^{-s} s^{x-1} (x-s)/x! \quad \text{for } x \in \mathbb{N},
\]
and $\frac{d}{ds} \mathbb{P}(X_s = 0) = \mathbb{P}(X_s = 0) = e^{-s}$. (These are the backward equations for the Markov chain which starts at $0$ and jumps to the right at rate $1$.) Hence, as above, we have
\[
k \frac{d}{ds} = \sum_{x \in \mathbb{N}} (p_s(x) - p_s(x - 1)) \log p_s(x)
= \int_{1/2}^{\infty} (g_s(x) - g_s(x - 1)) \log f_s(x) \, dx \quad \text{(A.19a)}
+ \int_{1/2}^{\infty} (g_s(x) - g_s(x - 1)) \log (g_s(x)/f_s(x)) \, dx. \quad \text{(A.19b)}
\]
As for (A.18b) above, the same arguments as used for (A.10) show that the integral in (A.19b) is $o(s^{-10})$. Note also that $se^{-s} = o(s^{-10})$. Now consider the integral in (A.19a). Using a simple shift as before, we have
\[
\int_{1/2}^{\infty} (g_s(x) - g_s(x - 1)) \log f_s(x) \, dx = -\int_{1/2}^{\infty} g_s(x) \log (f_s(x) - f_s(x)) \, dx
= \int_{1/2}^{\infty} g_s(x) ((x-s)/x + 1/(2s)) \, dx = 1/(2s),
\]
recalling that here $f_s(x) = (2\pi s)^{-1/2} \exp(-(x-s)^2/(2s))$, $\mathbb{E}(X_s) = s$ and $g_s$ integrates to $1$. As above, combining the bounds for (A.19) and dividing through by $k$ proves the directed case. \qed

We wish to find the times $t_\alpha$ defined so that
\[
h(t_\alpha) = (\log n + \alpha \sqrt{k})/k \quad \text{where } \ v := \text{Var}(Q_1(t_0));
\]
in this case, we have $v \approx \frac{1}{4}$, recalling (A.15b) in the previous section.

**Proposition A.7.** For $k \ll \log n$, we have
\[
t_0 \approx kn^{2/k}/(2\pi e), \quad \text{(2.1a)}
\]
and, for each $\alpha \in \mathbb{R}$, we have $t_\alpha \approx t_0$, and furthermore
\[
t_\alpha/t_0 - 1 \approx \alpha \sqrt{2/k}. \quad \text{(2.1a)}
\]
Proof. We consider the directed and undirected cases simultaneously. Directly manipulating (A.17), we see that if $h(t_0) = \log n/k$ then $$t_0 = kn^{2/k}/(2\pi e) \cdot (1 + O((t_0/k)^{-1/4})) \approx kn^{2/k}/(2\pi e),$$ with the final relation holding since $k \ll \log n$.

We now turn to finding $t_\alpha$. Fix $\alpha \in \mathbb{R}$. Note that $t \mapsto h(t)$ is increasing and $\alpha \sqrt{v/k} = o(1)$, and so from the form of $h(t)$ we see that, for all $\varepsilon > 0$, we have $$(1 - \varepsilon)t_0 \leq t_\alpha \leq (1 + \varepsilon)t_0$$ for $n$ sufficiently large (depending on $\alpha$); hence $t_\alpha \approx t_0$ for all $\alpha \in \mathbb{R}$.

By definition of $t_\alpha$, we have $$h(t_\alpha) - h(t_0) = \alpha \sqrt{v/k},$$ and hence $$\frac{d}{\alpha} h(t_\alpha) = \sqrt{v/k}.$$ Hence we have $$t_\alpha - t_0 = \int_0^\alpha \frac{dh(t_\alpha)}{\alpha} = \int_0^{v/k} \frac{1}{h'(t_\alpha)}\,d\alpha.$$ But, by Proposition A.6, we may write $h'(t) = (2t)^{-1}(1 + o(1))$ with $o(1)$ term uniform over $t \in [1/2t_0, 2t_0]$, which is an interval containing the cutoff window. Hence, recalling that $v \approx \frac{1}{2}$ in this regime, (2.1a) follows:

$$t_\alpha - t_0 = 2\alpha \sqrt{v/k} t_0 \left(1 + o(1)\right) = \alpha \sqrt{2/k} t_0 \left(1 + o(1)\right).$$

We next consider the regime $k \approx \lambda \log n$ with $\lambda \in (0, \infty)$. For $s \geq 0$, write $H(s) := E(Q_1(sk))$, the entropy of a rate-1 SRW or Poisson process in the undirected or directed case, respectively.

**Proposition A.8.** There exists a decreasing, continuous bijection $f : (0, \infty) \rightarrow (0, \infty)$, whose value differs between the undirected and directed cases, so that, for all $\lambda > 0$, for $k \approx \lambda \log n$, we have $$t_0 \approx f(\lambda)k \ \text{where} \ \ f(\lambda) := H^{-1}(1/\lambda),$$ and, for each $\alpha \in \mathbb{R}$, we have $t_\alpha \approx t_0$, and furthermore $$t_\alpha / t_0 - 1 \approx o(g(\lambda)/\sqrt{k}) \ \text{where} \ \ g(\lambda) := \sqrt{\text{Var}(Q_1(f(\lambda)k))/(f(\lambda)H'(f(\lambda)))}.$$ **Proof.** Since $\log n/k \approx 1/\lambda$, we must choose $t := t_0$ so that $h(t/k) \approx 1/\lambda$. From this, and the fact that each coordinate runs at rate $1/k$, we deduce that $t_0/k$ must also converge as $n \rightarrow \infty$, and so $t_0/\log n$ converges as $n \rightarrow \infty$, with limit depending only on $\lambda$, say $f(\lambda)$. This theory holds for both the directed and undirected cases, but the limit is not (necessarily) the same in each case.

Moreover, the increasing and continuity properties of the entropy say that $f(\lambda) = H^{-1}(1/\lambda)$ and that $f$ is a decreasing bijection from $(0, \infty)$ to $(0, \infty)$.

We wish to find times $t_\alpha$ defined so that $$H(t_\alpha/k) = h(t_\alpha) = (\log n + \alpha \sqrt{v/k})/k \ \text{where} \ \ v := \text{Var}(Q_1(t_0));$$ in this case, we have $v \approx v_\star$ for some constant $v_\star$, whose value differs between the undirected and directed cases, recalling Corollary A.4 in the previous section.

We now turn to finding $t_\alpha$. Fix $\alpha \in \mathbb{R}$. Note that $h$ is increasing and $\alpha \sqrt{v/k} = o(1)$, and so from the continuity of $s \mapsto H(s)$ and the fact that the function $H$ is independent of $n$, we see that, for all $\varepsilon > 0$, we have $$(1 - \varepsilon)t_0 \leq t_\alpha \leq (1 + \varepsilon)t_0$$ for $n$ sufficiently large (depending on $\alpha$); hence $t_\alpha \approx t_0$ for all $\alpha \in \mathbb{R}$.

Similarly to in the previous derivative proof, noting that $h'(t) = k^{-1}H'(t/k)$, we have $$t_\alpha - t_0 = \sqrt{v/k} \int_0^\alpha \frac{dh(t_\alpha)}{\alpha} = \sqrt{v/k} \int_0^\alpha 1/H'(t_\alpha/k)\,da.$$ Continuity now of $H'(\cdot)$ along with the fact that $t_\alpha \approx t_0 \approx f(\lambda)k$ then says that $$t_\alpha - t_0 \approx \alpha \sqrt{v_\star k}/H'(t_\alpha/k) \approx \alpha \sqrt{v_\star k}/H'(f(\lambda))$$ $$\approx \alpha t_0 \sqrt{\text{Var}(Q_1(f(\lambda)k))/f(\lambda)H'(f(\lambda))}.$$
Noting that $v = \text{Var}(Q_1(t_0))$ and $t_0 \approx f(\lambda)k$ and using continuity proves (2.1b).

Finally, it remains to show that the expression in (2.1b) is indeed $o(1)$. This again follows straightforwardly since $g$ is a function independent of $n$. Indeed, the expressions for $Q_1$ and $h$ are defined by running a single coordinate at rate $1/k$, so $H(s) = h(sk)$ and $Q_1(sk)$ are independent of $n$ if $s$ is independent of $n$.

Finally we consider the regime $k \gg \log n$. We have to handle the directed and undirected cases slightly differently here. The entropic time $t_0$ and cutoff times $t_s$ will be the same (up to smaller order terms), but the technical details of the proofs will differ ever so slightly.

**Proposition A.9.** Write $s := t/k$. For $s \ll 1$, the entropy $h$ of a rate-1/k SRW or PP on $\mathbb{Z}$ satisfies

$$h(t) = s \log(1/s) + O(s). \quad (A.20)$$

**Proof.** This follows immediately from (A.14) given in the justification of the CLT when $s \ll 1$. \qed

**Proposition A.10.** Write $s := t/k$. For $s \ll 1$, the entropy $h$ of a rate-1/k SRW or PP on $\mathbb{Z}$ satisfies

$$h'(t) = k^{-1} \left( \log(1/s) + O(1) \right). \quad (A.21)$$

**Proof.** We proceed as in the previous derivative proof, i.e, the proof of Proposition A.6.

Consider first the undirected case. Let $X$ be a rate-1 SRW on $\mathbb{Z}$. Then $W_1(t) \sim X_s$ where $s = t/k$. Using the backward equations as in the proof of Proposition A.6, we have

$$k h'(t) = \sum_{x \in \mathbb{Z}} \left( p_s(x) - \frac{1}{2} (p_s(x+1) + p_s(x-1)) \right) \log p_s(x).$$

Recall that we have

$$\mathbb{P}(X_s = 0) = 1 - s + O(s^2) \quad \text{and} \quad \mathbb{P}(X_s = x) = \frac{1}{2} s + O(s^2) \text{ for } x \in \{\pm 1\},$$

and hence $\mathbb{P}(X_s = x) = O(s^2)$ for $x \notin \{0, \pm 1\}$. Also, as previously, in the above sum we may ignore the $x$ with $x \notin \{0, \pm 1\}$ to give an error $O(s \log(1/s))$. (Note that it is not $O(s^2 \log(1/s))$, since the $x$-th term of the sum contains $p_s(x+1)$ and $p_s(x-1)$. Direct calculation then gives

$$k h'(t) = \log(1/s) + \log 2 + O(s) = \log(1/s) + O(1).$$

This proves the undirected case.

We now consider the directed case. Let $X_s \sim \text{Po}(s)$, which has $\mathbb{P}(X_s = x) = e^{-s}x^x/x!$. Then $W_1(t) \sim X_s$. Direct differentiation shows that

$$\frac{d}{ds} \mathbb{P}(X_s = x) = \mathbb{P}(X_s = x - 1) - \mathbb{P}(X_s = x) = e^{-s}s^{x-1}(x-s)/x! \quad \text{for } x \in \mathbb{N},$$

and $\frac{d}{ds} \mathbb{P}(X_s = 0) = -\mathbb{P}(X_s = 0) = -e^{-s}$, as in the previous derivative proof. As there, we have

$$k h'(t) = - \sum_{x \in \mathbb{Z}^+} \frac{d}{ds} \mathbb{P}(X_s = x) (\log \mathbb{P}(X_s = x) + 1).$$

As previously, we may ignore the terms with $x \notin \{0, \pm 1\}$, giving an error $O(s \log(1/s))$. Plugging in the derivative, we obtain

$$k h'(t) = -e^{-s} \log(e^{-s}) - e^{-s}(1-s) \log(se^{-s}) + O(s \log(1/s))$$

$$= s(1-s + O(s^2)) - (1-s)(1-s + O(s^2))(\log s - s) + O(s \log(1/s))$$

$$= \log(1/s) + O(s \log(1/s)) = \log(1/s) + O(1).$$

This proves the directed case. \quad \Box

We wish to find the times $t_s$ defined so that

$$h(t_s) = (\log n + \alpha \sqrt{vk})/k \quad \text{where} \quad v := \text{Var}(Q_1(t_0));$$

in this case, we have $v \approx (\log n/k) \log(k/\log n)$, recalling (A.15a) in the previous section.
Proposition A.11. For $k \gg \log n$, we have
\[ t_0 \approx \log n / \log(k / \log n), \tag{2.1c} \]
and, for each $\alpha \in \mathbb{R}$, we have $t_\alpha \approx t_0$, and furthermore
\[ t_\alpha / t_0 - 1 \approx \alpha \sqrt{\log(k / \log n) / \log n} = o(1). \tag{2.1c} \]

Proof. By (A.20), we desire $s := t/k$ with $t := t_0$ to satisfy
\[ s \log(1/s) \approx \log n/k. \]
Taking logs of this, we obtain
\[ \log(1/s) \approx \log(k / \log n). \]
Hence we obtain
\[ t_0 = sk \approx \log n / \log(k / \log n). \]
We now turn to finding $t_\alpha$. Fix $\alpha \in \mathbb{R}$. From the form (A.20) of $h(t)$, observe that
\[ h(t_0(1 \pm \varepsilon)) = (1 \pm \varepsilon)h(t_0) + O(s_0) = (1 \pm \varepsilon)h(t_0) \cdot (1 + o(1)), \]
where $s_0 := t_0/k$, and $s_0 \ll 1$ so $h(t_0) \approx s_0 \log(1/s_0) \gg s_0$. Note also that
\[ \sqrt{ek} \approx \sqrt{s_0 \log(k / \log n)} \ll \log n, \]
since $k = n^{o(1)}$, and so $h(t_\alpha) = h(t_0) \cdot (1 + o(1))$. Hence, for all $\varepsilon > 0$, we have $t_0(1 - \varepsilon) \leq t_\alpha \leq t_0(1 + \varepsilon)$ for $n$ sufficiently large (depending on $\alpha$); hence $t_\alpha \approx t_0$ for all $\alpha \in \mathbb{R}$.

As in the previous derivative proofs, we have
\[ t_\alpha - t_0 = \sqrt{v/k} \int_0^1 \int_0^a 1/h'(t_a) \, da. \]
But, by Proposition A.10, we may write $h'(t) = k^{-1} \log(1/s)(1 + o(1))$ with $o(1)$ term uniform over $t \in [1/t_0, 2t_0]$, which is an interval containing the cutoff window. Hence, recalling the expression for $v$, (2.1c) follows:
\[ |t_\alpha - t_0| = |\alpha| \sqrt{v/k} \log(k/t_0) (1 + o(1)) \]
\[ = |\alpha| \sqrt{\log n / \log(k / \log n) / \log(k / \log n)} (1 + o(1)) \]
\[ = |\alpha| \sqrt{\log n / \log(k / \log n)} (1 + o(1)) \]
\[ = |\alpha| \sqrt{\log(k / \log n) / \log(n t_0 (1 + o(1))}. \]

Note that $k = n^{o(1)}$, and so $\log(k / \log n) / \log n \ll 1$. (Recall that $k \gg \log n$.) So we do indeed have $|t_\alpha - t_0| = o(t_0)$. Finally, note that $\text{sgn}(t_\alpha - t_0) = \text{sgn}(\alpha). \quad \square$

Remark. In the directed case, we can actually find an explicit closed-form solution for the entropy. The method there does not, though, find the exact shape, i.e. determine $(t_\alpha - t_0)/t_0$; there we do not need this refined information. \triangle
B  Deferred Proofs from §3: Lemmas and Bounds on \( r_\alpha \) and \( p_\alpha \)

B.1  Mixing-Related Proofs

The following claim, Claim B.1, was referred to in the proof of Lemma 3.15.

Claim B.1. Let \( X = (X_s)_{s \geq 0} \) be a rate-1 SRW on \( \mathbb{Z} \). There exists a constant \( \lambda_0 \) sufficiently small so that if \( k \leq \lambda_0 \log n \) then, writing \( s := t_0/k \), we have \( P(X_{2s} = 0) \leq 2n^{-1/k} \).

Proof. It is straightforward to check, using Stirling’s approximation, that if \( \tilde{X} = (\tilde{X}_N)_{N \geq 0} \) is a discrete-time SRW then \( P(\tilde{X}_N = 0) \leq 1/\sqrt{\pi N} \). If \( N = (N_s)_{s \geq 0} \) is a rate-1 PP, then we have \( (X_s)_{s \geq 0} = (\tilde{X}_{N_s})_{s \geq 0} \). Using Poisson concentration (see, e.g., [38, Lemma 2.6]), we obtain

\[
P(X_s = 0) \leq P(\tilde{X}_{N_s} = 0, N_s \geq \frac{5}{6}s) + P(N_s \leq \frac{5}{6}s) \leq (\pi \cdot \frac{5}{6}s)^{-1/2} + \exp(-\frac{1}{4}s) \leq (\frac{5}{6}s)^{-1/2},
\]

with the final inequality requiring \( s \) sufficiently large.

As sketched in §2.3, and made rigorous in Appendix A, if \( H(s) \) is the entropy of \( X_s \), then

\[
H(s) = \frac{1}{2} \log(2\pi e s) + O(s^{-1/4}).
\]

Hence we have \( H^{-1}(\mu) \geq (2\pi e^{1+\varepsilon})^{-1} e^{2\mu} \) for any \( \varepsilon > 0 \), provided \( s \) is sufficiently large.

By definition of the entropic times, writing \( k \approx \lambda \log n \), we have

\[
s := t_0/k = H^{-1}(\log n/k) \approx H^{-1}(1/\lambda) \geq (2\pi e^{1+\varepsilon})^{-1} e^{2\lambda};
\]

We now combine these two results: note that \( n^{1/k} = \exp(\log n/k) \approx e^{1/\lambda} \);

\[
P(X_{2s} = 0) \leq \left( \frac{5}{6} \cdot 2\pi \cdot e^{2/\lambda + \varepsilon}/(2\pi e) \right)^{-1/2} = \sqrt{5e/4} \cdot e^{1/\lambda} \leq 2n^{-1/k}. \]

Proof of Proposition 3.9b. Set \( L := \frac{1}{16} \log n/\log k \); this satisfies \( d \leq \frac{1}{2} L \). Also, \( \log k \ll \log n \), so \( L \gg 1 \) and hence also \( L - d \gg 1 \).

Consider first \( I \subseteq [k] \) with \( 1 \leq |I| \leq L \). We have

\[
n \sum_{1 \leq |I| \leq L} P(V \cdot Z \equiv 0, I = I, \text{typ}) \leq n \sum_{|I| \leq L} P(V \cdot Z \equiv 0, I = I, \text{typ}) \leq n \sum_{|I| \leq L} P(V \cdot Z \equiv 0, I = I, \text{typ}) \leq LkL \cdot n^{1-1+(d+L)/k} \cdot 2L \cdot d \log k / \log k,
\]

We now sum over the \( I \) with \( 1 \leq |I| \leq L \):

\[
n \sum_{1 \leq |I| \leq L} P(V \cdot Z \equiv 0, I = I, \text{typ}) \leq LkL \cdot n^{1-1+(d+L)/k} \cdot 2L \cdot d \log k / \log k,
\]

since \( \left( \begin{array}{c} n \\ 1 \end{array} \right) \leq k^d \leq k^L \) for \( \ell \leq L \). We now use the fact that \( d + |I| \leq \frac{1}{2} L \leq \frac{1}{16} \log n/\log k \) and \( k \geq \frac{1}{4} \log n/\log \log n \) to deduce that \( (d + |I|)/k \leq \frac{2}{5} \). Also, since \( d \leq \frac{1}{2} L \), we have \( k^{3|I|+d} \leq e^{4L \log k} = e^{4 \log n/10} = n^{2/5} \).

by definition of \( L \). Hence

\[
n \sum_{1 \leq |I| \leq L} P(V \cdot Z \equiv 0, I = I, \text{typ}) \leq n^{-1+2/5+2/5} = n^{-1/5}. \tag{B.1}
\]

Finally we consider \( I \subseteq [k] \) with \( L \leq |I| \leq k \). (3.12) says that

\[
n \sum_{L \leq |I| \leq k} P(V \cdot Z \equiv 0, I = I, \text{typ}) \leq 1 + 3 \cdot 2^{d-L}/P(\text{typ}). \tag{B.2}
\]

Plugging (3.8, B.1, B.2) into (3.7), recalling that \( L - d \gg 1 \), we obtain

\[
D = n \sum_{I} P(V \cdot Z \equiv 0, I = I, \text{typ}) - 1 = o(1)/P(\text{typ}) = o(1). \]

\[ \square \]
Proof of Lemma 3.11. For this proof, we actually need only assume that for all \( j \in [d] \) there exists an \( i \in [k] \) so that \( v_i \not\equiv 0 \mod m_j \); in particular, this is implied by the condition \( \mathcal{I}(v) \neq \emptyset \).

First recall that \( Z_i \sim^{iid} \text{Unif}(G) \) where we think of \( G \) as \( \prod_{j=1}^d \mathbb{Z}_{m_j} \). Hence, for each \( i = 1, \ldots, k \), we may write \( Z_i = (\zeta_{i,1}, \ldots, \zeta_{i,d}) \) with \( \zeta_{i,j} \sim \text{Unif}(\mathbb{Z}_{m_j}) \) with all the \( \zeta_{i,j} \) independent. Then
\[
(v \cdot Z)_j = \sum_{i=1}^k v_i \zeta_{i,j},
\]
where \( (v \cdot Z)_j \) is the \( j \)-th component of \( v \cdot Z = \mathbb{Z}^d \), and in particular \( (v \cdot Z)_j \) are independent.

Assuming the \( d = 1 \) case, the above then shows that \( (v \cdot Z)_j \sim \text{Unif}(g_1 \mathbb{Z}_{m_j} / g_j) \) for each \( j \). Hence it suffices to prove the \( d = 1 \) case.

We now prove the \( d = 1 \) case. Since any \( i \in [k] \) with \( v_i \equiv 0 \mod m \) does not contribute to the sum, by passing to a subsequence, wlog we may assume that \( v_i \not\equiv 0 \mod m \) for all \( i \in [k] \).

We use induction on \( |\mathcal{I}| \). Let \( U \sim \text{Unif}\{1, \ldots, n\} \) and set \( R := mU \) where \( m \in \{1, \ldots, n\} \). Define
\[
g := \gcd(m, n) \text{ and } r := m / g \text{ so that } R = mU = g \cdot (rU).
\]
We then have \( \gcd(r, n) = 1 \), and so \( rU \sim \text{Unif}\{1, \ldots, n\} \): indeed, for any \( x \in \{1, \ldots, n\} \), we have
\[
\mathbb{P}(rU = x) = \mathbb{P}(U = x r^{-1}) = \frac{1}{n} \quad \text{where} \quad r^{-1} \text{ is the inverse of } r \text{ mod } n.
\]
Thus we have \( R = g \cdot (rU) \sim \text{Unif}\{g, 2g, \ldots, n\} \), since \( g \mid n \). This proves the base case \( |\mathcal{I}| = 1 \).

Now consider independent \( X, Y \sim \text{Unif}\{1, \ldots, n\} \) and set \( R := aX + bY \). Pulling out a constant as above, we may assume that \( a, b \perp \iota \). Write \( c := \gcd(a, b, n) \). Then there exist \( r, s \in \{1, \ldots, n\} \) with
\[
ar + bs \equiv c \mod n, \quad \text{and hence} \quad a(mr) + b(ms) \equiv cm \mod m \text{ for any } m \in \{1, \ldots, n\}.
\]
Thus \( \{c, 2c, \ldots, n\} \subseteq \text{supp}(R) \). By writing \( R := c(ac^{-1}X + bc^{-1}Y) \), with \( c^{-1} \) the inverse mod \( n \), we see that in fact \( \text{supp}(R) = \{c, 2c, \ldots, n\} \). It remains to show that \( R \) is uniform on its support.

Pulling out the factor \( c \), it is enough to consider \( \gcd(a, b, n) = 1 \). For \( m \in \{0, 1, \ldots, n-1\} \), set
\[
\Omega_m := \{(x, y) \in [n]^2 \mid ax + by \equiv m \mod n\}.
\]
We show that \( |\Omega_m| \) is the same for all \( m \), and hence deduce that \( R \) is uniform on \( \{c, 2c, \ldots, n\} \).

Indeed, for every \( m \) there exists a pair \( (x_m, y_m) \in [n]^2 \) so that \( ax_m + by_m = m \). If also \( (x, y) \in \Omega_m \), then letting \( x' = x - x_m \) and \( y' = y - y_m \), we see that \( (x', y') \in \Omega_0 \). This proves the case \( |\mathcal{I}| = 2 \).

Now suppose that \( X_1, \ldots, X_L \sim^{iid} \text{Unif}\{1, \ldots, n\} \) and \( a_1, \ldots, a_L \in \{1, \ldots, n-1\} \). By the hypothesis,
\[
\sum_{l=1}^{L-1} a_l X_l \sim \text{Unif}\{1, \ldots, n\} \quad \text{and} \quad c_0 := \gcd(a_1, \ldots, a_{L-1}, n).
\]
Now, \( X_L \) is independent of this sum, and so the previous case applies to say that
\[
\sum_{l=1}^{L} a_l X_l \sim \iota U \quad \text{where} \quad U \sim \text{Unif}\{1, \ldots, n\} \quad \text{and} \quad c := \gcd(c_0, a_L, n) = \gcd(a_1, \ldots, a_L, n).
\]
This completes the induction, and hence proves the claim.

Proof of Claim 3.17. First note that the SRW on \( \mathbb{Z} \) is vertex-transitive and reversible. Fix \( s \geq 0 \).

Write \( P_t(x, y) := \mathbb{P}(X_t = y \mid X_0 = x) \). It is then well-known—see, for example, [2, Lemma 3.20, (3.60)]—that, for any \( z \), we have
\[
\max_{x, y} P_t(x, y) = \max_x P_t(x, x) = P_t(z, z).
\]
Since \( X \) starts from 0, for each \( m \in \mathbb{N}_0 \) this says that \( \mathbb{P}(X_t = 0) \geq \mathbb{P}(X_t = m) \).

For \( m \in \mathbb{N}_0 \), let \( \tau := \inf\{t \geq 0 \mid X_t = m\} \), and let \( f \) be its density function. Then
\[
P_t(0, m) = \int_0^t f(s) P_{t-s}(0, 0) \, ds \geq \int_0^t f(s) P_{t-s}(0, 1) \, ds = P_t(0, m + 1).
\]
Since \( X \) has the same law as \(-X\), the proof is completed.
B.2 Proving Bounds on \( r \) and \( p \): Proposition 3.5

The aim of this section is to prove the bounds on \( r \) and \( p \) from Proposition 3.5. The reader is encouraged to review the definitions of \( r \) and \( p \) from Definition 3.4; for convenience, Proposition 3.5 is restated here.

**Proposition 3.5.** For all \( \alpha \in \mathbb{R} \), we have
\[
r_\alpha(k, n) \leq r_\ast(k, n) := \frac{1}{2}n^{1/k}(\log k)^2 \quad \text{and} \quad p_\alpha(k, n) \geq p_\ast(k, n) := n^{-1/k}k^{-2}.
\]

The following propositions provide asymptotic estimates for tails of the Poisson distribution and for continuous-time SRW on \( \mathbb{Z} \), as well as for the ratio between the ‘tail’ and ‘point’ probabilities. We note that in the regime \( r \in [\sqrt{s}, s^{2/3}] \) stronger assertions can be made via the local CLT (A.1).

Below, for \( a, b \in \mathbb{R} \), we write \( a \vee b := \max\{a, b\} \) and \( a \wedge b := \min\{a, b\} \).

**Proposition B.2 (Poisson Bounds).** For \( s \in (0, \infty) \), let \( X_s \sim \text{Po}(s) \). Then, uniformly in \( s \in (0, \infty) \) and in \( r \geq \sqrt{s} \) and \( s + r \in \mathbb{Z} \), we have the following relations:
\[
- \log \mathbb{P}(X_s \geq s + r) \asymp r((r/s) \wedge 1) \log((r/s) \vee e); \quad (B.3a)
\]
\[
\frac{\mathbb{P}(X_s \geq s + r)}{\mathbb{P}(X_s = s + r)} \asymp (s/r) \vee 1. \quad (B.4a)
\]
Moreover, uniformly in \( s \in (0, \infty) \) and in \( r \in [\sqrt{s}, s] \) with \( s - r \in \mathbb{Z} \) we have the following relations:
\[
- \log \mathbb{P}(X_s \leq s - r) \asymp r((r/s) \wedge 1) \log((r/s) \vee e); \quad (B.3b)
\]
\[
\frac{\mathbb{P}(X_s \leq s - r)}{\mathbb{P}(X_s = s - r)} \asymp (s/r) \vee 1. \quad (B.4b)
\]

**Proposition B.3 (SRW Bounds).** Let \( X = (X_s)_{s \geq 0} \) be a rate-1 SRW on \( \mathbb{Z} \) started at 0. Then, uniformly in \( s \in (0, \infty) \) and in \( r \geq \sqrt{s} \) and \( s + r \in \mathbb{Z} \), we have the following relations:
\[
- \log \mathbb{P}(X_s \geq r) \asymp r((r/s) \wedge 1) \log((r/s) \vee e); \quad (B.5)
\]
\[
\mathbb{P}(X_s \geq r)/\mathbb{P}(X_s = r) \asymp (s/r) \vee 1. \quad (B.6)
\]

From these, we can deduce the proof of Proposition 3.5.

**Proof of Proposition 3.5.** Here we take \( s := t_\alpha/k \approx t_0/k \). Writing \( \kappa := k/\log n \), we have
\[
t_0/k \leq n^{2/k} \quad \text{when} \quad k \lesssim \log n, \quad (B.7a)
\]
\[
t_0/k \approx 1/\kappa \log \kappa \ll 1 \quad \text{when} \quad k \gg \log n. \quad (B.7b)
\]

Consider the SRW, which corresponds to the undirected case. Equations (B.3–B.6) are all “\( f \gtrsim g \)”-type statements; let \( c > 0 \) be a universal constant such that \( c \) is a lower and \( C := 1/c \) an upper bound for these relations.

For \( r \), it is enough to find an \( \tilde{r} \) so that
\[
- \log \mathbb{P}(X_s \geq \tilde{r}) \geq 2 \log k.
\]

For \( p \), since we only consider \( j \) with \( |j| \leq r \), and \( r \) is defined as a minimum, we have \( \mathbb{P}(X_s \geq |j|) \geq k^{-3/2} \) for all such \( j \). We split into two regimes, namely \( s \geq 2C \log k \) and \( s < 2C \log k \).

First suppose that \( s \geq 2C \log k \). Set \( \tilde{r} := \sqrt{2Cs \log k} \). Then \( \tilde{r} \leq s \), and so, by (B.5), we have
\[
- \log \mathbb{P}(X_s \geq \tilde{r}) \geq c\tilde{r}((\tilde{r}/s) \wedge 1) \log((\tilde{r}/s) \vee e) = c\tilde{r}^2/s \geq 2 \log k.
\]

For \( p_j \), since \( \tilde{r} \leq s \), by (B.6), we have
\[
\mathbb{P}(X_s = j) \gtrsim (s/r)\mathbb{P}(X_s \geq j) \gtrsim (\log k)^{1/2}n^{-1/k} \cdot k^{-3/2} \gg n^{-1/k}k^{-2}.
\]

Suppose now that \( s < 2C \log k \). Set \( \tilde{r} := 2C \log k \). Then \( \tilde{r} \geq s \), and so, by (B.5), we have
\[
- \log \mathbb{P}(X_s \geq \tilde{r}) \geq c\tilde{r}((\tilde{r}/s) \wedge 1) \log((\tilde{r}/s) \vee e) \geq c\tilde{r}^2 = 2 \log k.
\]
For $p_*$, since $\tilde{r} \geq s$, by (B.6), we have
\[ \mathbb{P}(X_s = j) \geq \mathbb{P}(X_s = j) \geq k^{-3/2} \gg k^{-2} \geq n^{-1/4}k^{-2}. \]

Observe that, in either regime, we have $\tilde{r} \leq r_*$, with $r_*$ defined in Definition 3.4. This completes the proof of (3.3) in the undirected case.

The proof of the directed case, i.e., using Poisson instead of SRW, is in essence the same, due to the similarity of Propositions B.2 and B.3, albeit slightly messier to write down, since one must take care that $s + r \geq 0$.

**Proof of Proposition B.2 (Poisson).** For $s \leq 10$, all that is needed is the observation that
\[ \mathbb{P}(X_s = r) \asymp \mathbb{P}(X_s = r) = \frac{e^{r/(s/r)} p^{r}}{\sqrt{2\pi} s/(r)}, \]

We now consider the case $s \geq 1$. First we state that, for all $r \geq 0$, we have
\[ \max\{\mathbb{P}(X_s \geq s + r), \mathbb{P}(X_s \leq s - r)\} \leq \exp(-\frac{1}{2}s^2/(s + r)/3); \]

this follows from Bernstein’s inequality, by taking an appropriate limit.

A direct calculation involving Stirling’s approximation shows, uniformly in $s$ and in $r$ with $r \geq \frac{1}{2}s$ and $s + r \in \mathbb{Z}$, respectively $\frac{1}{2}s \leq r \leq s$, the following relations:
\[ \mathbb{P}(X_s \geq s + r) \asymp \mathbb{P}(X_s = s + r) = \frac{e^{r/(s/(s + r))}}{\sqrt{2\pi} s/(s + r)}, \]
\[ \mathbb{P}(X_s \leq s - r) \asymp \mathbb{P}(X_s = s - r) = \frac{e^{-r/(s/(s - r))}}{\sqrt{2\pi} s/(s - r)}; \]

from these, one can verify (B.4a, B.4b) for such $r$.

We can obtain lower bounds on $\mathbb{P}(X_s \geq s + r)$ and $\mathbb{P}(X_s \leq s - r)$ for $r \leq \frac{1}{2}s$, from which, together with (B.8), we can verify (B.4a, B.4b) for such $r$:
\[ \mathbb{P}(X_s = s + r) \sqrt{2\pi} s/(s + r) \asymp e^{r/(s/(s + r))} \geq \exp\left(-\frac{r^2}{2s(s + r)} - O\left(\frac{r^3}{(s + r)^2}\right)\right), \]
\[ \mathbb{P}(X_s = s - r) \sqrt{2\pi} s/(s - r) \asymp e^{-r/(s/(s - r))} \geq \exp\left(-\frac{r^2}{2s(s - r)} - O\left(\frac{r^3}{(s - r)^2}\right)\right); \]

these are found using Stirling’s approximation, and both hold uniformly for $r \leq \frac{1}{2}s$.

We now prove (B.3a); the proof of (B.3b) is similar and is omitted. We consider $s \geq 10$, having already considered $s \leq 10$ initially. Observe that $r \mapsto \mathbb{P}(X_s = s + r)$ is decreasing on $r \geq 0$ with $s \pm r \in \mathbb{Z}$. Using the formula for $\mathbb{P}(\text{Po}(\lambda) = k)$, we have
\[ \frac{\mathbb{P}(X_s = s + r)}{\mathbb{P}(X_s = s + r + 1)} = \frac{s + r + 1}{s}. \]

If $r \geq \frac{1}{2}s$, then this ratio is at least $11/9$, when $s \geq 10$, from which one can readily see that (B.3a) holds. Now suppose that $r \in [\sqrt{s}, \frac{1}{2}s]$. To conclude the proof, we show that there exist universal constants $c_1, c_2 \in (0, 1)$ so that, for such $r$, we have
\[ c_1 \mathbb{P}(X_s = s + r) \leq \mathbb{P}(X_s = s + r + \lfloor s/(2r) \rfloor) \leq c_2 \mathbb{P}(X_s = s + r). \]  

This, together with the decreasing statement above, can easily be seen to imply (B.3a). We now prove (B.9). If $\sqrt{s} \leq r \leq \frac{1}{2}s$, then
\[ \frac{\mathbb{P}(X_s = s + r)}{\mathbb{P}(X_s = s + r + j)} = \prod_{i=1}^{j} \frac{s + r + i}{s} = \prod_{i=1}^{j} (1 + (r + i)/s) \leq \exp(\sum_{i=1}^{j}(r + i)/s) = \exp(\frac{1}{2}j(r + 1) + 1)/s). \]

If in addition $j \leq \frac{1}{2}s/r$, then the last estimate is tight up to a constant factor. Indeed, in this case we have $\exp(\frac{1}{2}j(r + 2r + 1)/s) \leq e^j$. Conversely, using the fact that $1 + \theta \geq \exp(\theta - 2\theta^2)$ for $\theta \in [0, \frac{1}{2}]$, we find some universal constant $c_0 > 1$ so that $\exp(\frac{1}{2}j(r + 2r + 1)/s) \geq c_0$. \qed
Proof of Proposition B.3 (SRW). Fix an \( s \in (0, \infty) \); wlog assume \( r \geq 0 \). Recall that \( X \) has the same law as \( Y_N := \sum_1^N \xi_i \), where \((\xi_i)_{i \in \mathbb{N}}\) is an iid sequence of random variables with \( \mathbb{P}(\xi_1 = +1) = \frac{1}{2} = \mathbb{P}(\xi_1 = -1) \) and \( N \sim \text{Po}(s) \), independent of \((\xi_i)_{i \in \mathbb{N}}\). Then, setting \( Y_k := \sum_1^k \xi_i \) for \( k \in \mathbb{Z}_+ \), we have that \((Y_k)_{k \in \mathbb{Z}_+}\) is a discrete-time SRW on \( \mathbb{Z} \) started at the origin.

We first prove (B.5). Observe that \( \mathbb{E}(e^{\lambda Y_k}) = \frac{1}{2} e^\lambda + \frac{1}{2} e^{-\lambda} \leq e^{\lambda^2/2} \), and so \( \mathbb{E}(e^{\lambda Y_k}) \leq e^{\lambda^2 k/2} \), and hence \( \mathbb{P}(Y_k \geq r) \leq \exp\left(-r^2/(2k)\right) \), by taking \( \lambda := r/k \). Further, an elementary calculation involving Stirling’s approximation shows, uniformly over \( r \) with \( \sqrt{k} \log k \leq r \leq k \), that

\[
- \log \mathbb{P}(Y_k \geq r) \leq - \log \mathbb{P}(Y_k \in \{r, r+1\}) \approx r^2/k;
\]

for \( \sqrt{k} \leq r \leq \sqrt{k} \log k \) one can use the local CLT (A.1) to verify that

\[
- \log \mathbb{P}(Y_k \geq r) \approx r^2/k.
\]

For \( r \leq \sqrt{2} s \), we average over \( N \) and use the above bounds on \( Y_k \). In particular, we have

\[
\mathbb{E}(e^{\lambda X_s}) = \sum_{r=0}^{\infty} \mathbb{P}(N = k) e^{\lambda^2 k/2} = \mathbb{E}(e^{\lambda^2 N/2}) = \exp(s(e^{\lambda^2 /2} - 1)) \leq \exp(s(\lambda^2/2 + (\lambda^2/2)^2)),
\]

with the final inequality holding when \( \lambda^2 \leq 2 \), applying the inequality \( e^\theta - 1 \leq \theta + \theta^2 \) valid for \( \theta \in [-1, 1] \). We now set \( \lambda := r/s \) and use Chernoff to deduce that

\[
\mathbb{P}(X_s \geq r) \leq \exp\left(-\frac{1}{2}(r^2/s)(1 - \frac{1}{2}(r/s))\right) \leq \exp\left(-\frac{1}{4}(r^2/s)\right).
\]

For \( r \geq \sqrt{2} s \), we use the inequalities

\[
\mathbb{P}(X_s \geq r) \leq \mathbb{P}(\text{Po}(s) \geq r) \quad \text{and} \quad \mathbb{P}(X_s \geq r) \geq \mathbb{P}(N = 2r) \mathbb{P}(Y_{2r} \geq r).
\]

This case is completed by applying (B.3, B.4), ie Proposition B.2.

We now prove (B.6). For \( r \leq \frac{1}{2} s \), this follows from the fact that \( r \mapsto \mathbb{P}(X_s = r) \) is decreasing (in \( r \)) and that

\[
\sup_{s, r \text{ s.t. } r \geq s/2} \frac{\mathbb{P}(X_s = r + 2)\mathbb{P}(X_s = r)}{\mathbb{P}(Y_{2k} \geq 2r)} \leq \mathbb{P}(Y_{2k+1} \geq 2r + 1) \frac{\mathbb{P}(Y_{2k+1} = 2r + 1)}{\mathbb{P}(Y_{2k+1} = 2r + 1)},
\]

from this, the original claim follows by averaging over \( N \). Using Stirling’s approximation, it is not hard to verify for \( r \in [\sqrt{k}, \frac{1}{2} k] \) that there exist universal constants \( c_1, c_2 \in (0, 1) \) such that the following hold:

\[
c_1 \mathbb{P}(Y_{2k} = 2r) \leq \mathbb{P}(Y_{2k} = 2(r + \lfloor k/r \rfloor)) \leq c_2 \mathbb{P}(Y_{2k} = 2r);
\]

\[
c_1 \mathbb{P}(Y_{2k+1} = 2r + 1) \leq \mathbb{P}(Y_{2k+1} = 2(r + \lfloor k/r \rfloor + 1)) \leq c_2 \mathbb{P}(Y_{2k+1} = 2r + 1).
\]

This, together with the fact that both \( r \mapsto \mathbb{P}(Y_{2k} = 2r) \) and \( r \mapsto \mathbb{P}(Y_{2k+1} = 2r + 1) \) are decreasing on \([0, k]\), is easily seen to imply (B.10). \( \square \)

C Deferred Proofs from §4: Lemmas and Size of Balls Estimates

C.1 Mixing-Related Proofs

Proof of Lemma 4.7. Write \( M := M_{k,p} \). Since \( B = A - A' \), it is a symmetric and unimodal distribution, with support \( 2B_{k,p}(M) \subseteq Z^k \). Note, however, that the coordinates are not independent (except when \( p = \infty \)). But we can still apply Lemma 3.11 to say that

\[
n \mathbb{P}(B \cdot Z = 0 \mid Z = I) = n \mathbb{E}(\prod_{j=1}^d (g_j/m_j) \mid Z = I) \leq n \mathbb{E}(g^d \mid Z = I).
\]
Trivially $B_i \leq 2M$ for all $i$, and so this expectation is at most $(2M)^d$. Note that

$$\mathbb{E}(g^d \mid \mathcal{I} = \mathcal{I}) \leq \sum_{i=1}^{2M} \gamma^d \mathbb{P}(\gamma \mid B_i \forall i \in I \mid \mathcal{I} = \mathcal{I})$$

Note first that $\mathcal{I} = \mathcal{I}$ implies that $B_i = 0$, ie $A_i = A_i'$, for all $i \notin I$. We condition on the values of the coordinates which are outside $I$:

$$A_I \sim \text{Unif}(B_{\mathcal{I}} \mid p)((M^p - \|A_I\|_p^p)^{1/p}) \quad \text{given } A_I,$$

an analogous statement holds for $A_I'$. So now, given that $A_{-I} = A_I'$, and the value $A_{-I}$, we see that $A_I$ and $A_I'$ are independent and uniformly distributed on the $L_p$ ball of dimension $|I|$ with the above radius, and hence $B_I$ is still a symmetric, unimodal distribution. Now let $U$ and $U'$ be copies of $A_I$ and $A_I'$ given the above conditioning, and write $V := U - U'$. Write

$$D_i := D_i(\gamma) := \{ \gamma \mid V_i \} = \{ \gamma \mid (U_i - U_i') \}.$$ 

Then, for all $p \in [1, \infty]$, we have

$$\mathbb{P}(\gamma \mid B_i \forall i \in I \mid \mathcal{I} = \mathcal{I}, \|A_{-I}\|_p) = \mathbb{P}(D_i \forall i \in I).$$

Assume that $I \neq \emptyset$. By exchangeability, it suffices to consider the case $I = \{1, \ldots, \ell\}$. Define $M := (M^p - \|A_{\{1,\ldots,\ell\}\|_p^p)^{1/p}$, and $M'$ analogously. We then have

$$\mathbb{P}(D_i \forall i \in I) = \mathbb{P}(D_i) \mathbb{P}(D_{i+1} \mid D_i) \cdots \mathbb{P}(D_1 \mid D_2, \ldots, D_{\ell}) = \prod_{i=1}^{\ell} \mathbb{P}(D_i \mid D_{i+1}, \ldots, D_\ell).$$

Let $i \in [\ell - 1]$. Let $(u_{i+1}, \ldots, u_{\ell})$ and $(u'_{i+1}, \ldots, u'_{\ell})$ be two vectors in the support of $(U_{i+1}, \ldots, U_{\ell})$. Conditional on $(U_{i+1}, \ldots, U_{\ell}) = (u_{i+1}, \ldots, u_{\ell})$ and $(U'_{i+1}, \ldots, U'_{\ell}) = (u'_{i+1}, \ldots, u'_{\ell})$, we have $(U_1, \ldots, U_{i}) \sim \text{Unif}(B^p_\ell(R))$ and $(U''_{i+1}, \ldots, U''_{\ell}) \sim \text{Unif}(B^p_\ell(R'))$, for some $R, R' \in \mathbb{R}$, and the law of $U - U'$ is symmetric and unimodal on $\mathbb{Z} \setminus \{0\}$. It follows, as in Lemma 3.14, that $\mathbb{P}(D_i \mid D_{i+1}, \ldots, D_\ell) \leq 1/\gamma$. By the same reasoning, $\mathbb{P}(D_1) \leq 1/\gamma$. Hence

$$\mathbb{P}(D_i \forall i \in I) = \mathbb{P}(\gamma \mid B_i \forall i \in I \mid \mathcal{I} = \mathcal{I}) \leq \gamma^{-|I|},$$

from which we deduce that

$$\mathbb{E}(g^d \mid \mathcal{I} = \mathcal{I}) \leq \sum_{|I|=1}^{2M} \gamma^{|I|}.$$ 

As in the proof of Lemma 3.12 (ie the gcd for mixing), we deduce that

$$n \mathbb{P}(B \cdot Z = 0 \mid \mathcal{I} = \mathcal{I}) \leq \begin{cases} 1 + 3 \cdot 2^{d-|I|} & \text{when } d - |I| \leq -2, \\ C(2M)^{|I|+2} & \text{when } d - |I| \geq -1, \end{cases}$$

for some constant $C$. We thus deduce (4.8a, 4.8b). \hfill \square

### C.2 Size of Balls Estimates

**Proof of Lemma 4.2a.** Assume $R \in \mathbb{N}$. Observe that

$$|B_\ell^1(R)| = \left| \{ a \in \mathbb{Z}^k \mid \sum_{i=1}^k |a_i| \leq R \} \right|.$$ 

Moreover, it is a standard combinatorial identity that

$$\left| \{ a \in \mathbb{Z}^k_+ \mid \sum_{i=1}^k a_i \leq R \} \right| = \binom{R+k}{k}.$$ 

The upper and lower bounds will follow easily from this viewpoint, setting $\alpha_i := |a_i|$.

For the upper bound, note that $\alpha_i = |\pm a_i|$, and so given the value of $\alpha_i$, there are two choices for $a_i$ if $\alpha_i > 0$, otherwise there is only one (since $0 = -0$). Hence

$$\left| \{ a \in \mathbb{Z}^k \mid \sum_{i=1}^k |a_i| \leq R \} \right| \leq 2^k \left| \{ a \in \mathbb{Z}^k_+ \mid \sum_{i=1}^k a_i \leq R \} \right| = 2^k \binom{R+k}{k}.$$ 

For the lower bound, we get the factor of $2^k$ by only considering $a \in \mathbb{Z}^k$ with $|a_i| > 0$ for all $i$, and then setting $\beta_i := \alpha_i - 1$. Concretely, for $R \geq k$, we have

$$\left| \{ a \in \mathbb{Z}^k \mid \sum_{i=1}^k |a_i| \leq R \} \right| \geq \left| \{ a \in \mathbb{Z}^k \mid \sum_{i=1}^k |a_i| \leq R, a_i \neq 0 \forall i = 1, \ldots, k \} \right|$$

$$= 2^k \left| \{ a \in \mathbb{Z}^k \mid \sum_{i=1}^k a_i \leq R, \alpha_i > 0 \forall i = 1, \ldots, k \} \right|$$

$$= 2^k \left| \{ \beta \in \mathbb{Z}^k \mid \sum_{i=1}^k \beta_i \leq R - k, \beta_i \geq 0 \forall i = 1, \ldots, k \} \right| = 2^k \binom{R}{k}. \hfill \square$$
Proof of Lemma 4.2b. For any $R$, writing $\text{diam}_p$ for the $L_p$ diameter (in $\mathbb{R}^k$), we have

$$B_p^k(R - \text{diam}_p[-\frac{1}{2}, \frac{1}{2}]^k) \subseteq B_p^k(R) \subseteq B_p^k(R + \text{diam}_p[-\frac{1}{2}, \frac{1}{2}]^k).$$

Note that $\text{diam}_p[-\frac{1}{2}, \frac{1}{2}]^k = k^{1/p}$. Hence, for $R$ with $R \geq k^{1+1/p}$, we have

$$|B_p^k(R)| = (1 + O(k^{1/p}/R))^k = 1 + O(k^{1+1/p}/R).$$

Cf [23, Lemma 2.5], where the case $p = 2$ is considered; there, convolutions are employed.

Proof of Lemma 4.2c. In the $L_\infty$ norm, the coordinates are independent. The claim follows.

Proof of Lemma 4.4a. Upper Bound. Write $M := [e^{\xi}kn^{1/k}/(2e)]$. Note that $k \ll \log n$, and so $n^{1/k} \gg 1$, and so $M \gg k$. Then, by (4.5a) and Stirling’s formula, we have

$$|B_p^k(M)| \geq 2^k\left(\frac{M}{k}\right)^k \geq 2^k(M-k)^k/k! \gtrsim k^{-1/2}(1-k/M)^k(2eM/k)^k \geq k^{-1/2}\exp\left(-k(2k/M + \xi)\right) \cdot n.$$

Take $\xi := 2\omega/k$: then $k/M \approx n^{-1/k} \ll n^{-1/(2k)} \ll \xi$ and $e^{-\xi/k} \gg k^{1/2}$. Hence $|B_p^k(M)| \geq ne^\omega$.

Lower Bound. Set $M := kn^{1/k}e^{-K\omega/k}/(2e)$. Using $(\frac{N}{k}) \leq (eN/k)^k$ and (4.5a), we have

$$|B_p^k(M)| \leq (2e(M/k + 1))^k \leq ne^{-K\omega}\exp(6k/n^{1/k}) \ll n,$$

using $1 + x \leq e^x$ with $x = k/M$, $(\frac{N}{k}) \leq (eN/r)^r$ and $\omega \geq k/n^{1/(2k)} \gg k/n^{1/k}$ as $k \ll \log n$.

Proof of Lemma 4.4b. Upper Bound. From the formula (4.4), we see that

$$M_{p,k} := n^{1/k}e^{2\omega/k}/V_{k,p}^1 = \frac{1}{2}n^{1/k}e^{2\omega/k}\Gamma(k/p + 1)^{1/k}/\Gamma(1/p + 1)$$

satisfies $V_p^k(M_{p,k}) = ne^{2\omega}$. Using Stirling’s formula, and the fact that $k \gg 1$, we then deduce that

$$M_{p,k} \leq n^{1/k}k^{1/p}e^\xi/C_p.$$
Proof of Lemma 4.4d. We first prove that there exists a strictly increasing function $c : (0, \infty) \to (0, \infty)$ so that, for all $a > 0$, omitting here and below all ceiling signs, we have

$$|B_1^k(ak)| = \exp(k(c(a) + o(1))).$$

By considering the number $i$ of coordinates which equal 0, we have $|B_1^k(ak)| = \sum_{i=0}^{k} A_i$, where

$$A_i := A_i(k, a) := \binom{k}{i} 2^{k-i} \binom{k-i+a}{ak}.$$

Choose $i_* := i_*(k, a)$ that maximises $A_i$. Then $A_{i_*} \leq |B_1^k(ak)| \leq (k+1)A_{i_*}$. Observe that

$$\frac{A_{i+1}}{A_i} = \frac{(k-i)^2}{2(i+1)(k+1-a-i)},$$

and hence one can determine $i_*$ as a function of $k$ and $a$, conclude that $i_*(a, k)/k$ converges as $k \to \infty$ and thus determine $c(a)$ (in terms of the last limit). We omit the details. Knowing this limit allows us to plug this into the definition of $A_i$ and use Stirling’s approximation to get

$$A_{i_*} = \exp(k(c(a) + o(1))),$$

for some strictly increasing function $c : (0, \infty) \to (0, \infty)$. Since $k + 1 = o(k)$, the claim follows.

**Upper Bound.** Since $k \approx \lambda \log n$, we have $M_1/k \to c^{-1}(1/\lambda)$ as $n \to \infty$; set $\alpha := c^{-1}(1/\lambda)$.

**Lower Bound.** It follows from the exponential increase in the size of the $L_1$ ball that $|B_1^k((1-\varepsilon)ak)| = o(n)$ for all $\varepsilon > 0$, where $M_1 \approx ak$ and $\alpha = c^{-1}(1/\lambda)$. □