SOLUTIONS OF THE BRAID EQUATION WITH SET-TYPE SQUARE

JORGE A. GUCCIONE, JUAN J. GUCCIONE, AND CHRISTIAN VALQUI

Abstract. For a family of height one orders \((X, \leq)\) and each non-degenerate solution \(r_0: X \times X \to X \times X\) of the set-theoretic braid equation on \(X\) satisfying suitable conditions, we obtain all the non-degenerate solutions of the braid equation on the incidence coalgebra of \((X, \leq)\) that extend \(r_0\).

Contents

1 Preliminaries ........................................ 2
2 Factorization of solutions .............................. 4
3 Conditions for solutions with set-type square ....... 7
4 A family of examples ................................... 11

Introduction

Let \(V\) be a vector space over a field \(K\) and let \(r: V \otimes_K V \to V \otimes_K V\) be a bijective linear operator. We say that \(r\) satisfies the braid equation if

\[ r_{12} \circ r_{23} \circ r_{12} = r_{23} \circ r_{12} \circ r_{23}, \]

where \(r_{ij}\) denotes \(r\) acting on the \(i\)-th and \(j\)-th coordinates. Since the eighties many solutions of the braid equation have been found, many of them being deformations of the flip. It is interesting to obtain solutions that are not of this type, and in [6], Drinfeld proposed to study the most simple of them, namely, the set-theoretic ones, i.e. pairs \((X, r_0)\), where \(X\) is a set and

\[ r_0: X \times X \to X \times X \]

is an invertible map satisfying the braid equation. Each one of these solutions yields in an evident way a linear solution on the vector space with basis \(X\). From a structural point of view this approach was considered first by Etingof, Schedler and Soloviev [7] and Gateva-Ivanova and Van den Bergh [8] for involutive solutions, and then by Lu, Yan and Zhu [11] and Soloviev [13] for non-degenerate not necessarily involutive solutions. In the last two decades the theory has developed rapidly, and now it is known that it has connections with bijective 1-cocycles, Bierbach groups and groups of I-type, involutive Yang-Baxter groups, Garside structures, biracks, cyclic sets, braces, Hopf algebras, matched pairs, left symmetric algebras, etcetera (see, for instance [1], [4], [3], [2], [5], [9], [12], [14]).

2010 Mathematics Subject Classification. 16T25.

Key words and phrases. Orders, Braid equation, Non-degenerate solution.

Jorge A. Guccione and Juan J. Guccione were supported by UBACyT 20020150100153BA (UBA) and PIP 11220110100800CO (CONICET).

Christian Valqui was supported by PUCP-DGI- ID 453 - CAP 2017-1-0035.
Suppose now that \((X, \leq)\) is a locally finite poset and consider its incidence coalgebra \(D\). We identify each \(a \in X\) with the pair \((a, a)\) in \(D\). In [10] the following problem was posed:

Let \(r_0: X \times X \longrightarrow X \times X\) be a non-degenerated solution of the set theoretical braid equation. Find necessary and sufficient conditions in order that \(r_0\) is the restriction of a non-degenerate coalgebra automorphism \(r\) of \(D \otimes D\), which is a solution of the braid equation, and then find all such extensions.

Let \(r: D \otimes D \longrightarrow D \otimes D\) be a linear map. For \(a \leq b\) and \(c \leq d\) write

\[
r((a, b) \otimes (c, d)) = \sum_{e \leq f \leq g \leq h} \lambda_{a|b|c|d}^{e|f|g|h} (e, f) \otimes (g, h),
\]

with \(\lambda_{a|b|c|d}^{e|f|g|h} \in K\). Assume that \(r\) is a non-degenerate coalgebra automorphism that induces a non-degenerate solution \(r_0: X \times X \longrightarrow X \times X\) of the braid equation. In [10] Proposition 4.3] we give the equations that the coefficients \(\lambda_{a|b|c|d}^{e|f|g|h}\)'s must satisfy in order that \(r\) is a solution of the braid equation. In Corollary 4.3] we prove that in fact it suffices to solve a relatively small subset of these equations, corresponding to lower extremal inclusions (see Definition 2.3). For instance, when \(X = \{x, y\}\) with \(x < y\), then by [10] Corollary 2.5], necessarily \(r_0\) is the flip and the number of equations we must solve according to [10] Proposition 4.3] is 125. From these 8 are trivially true and 36 are solved by a general result in [10]. Our result shows that it suffices to solve 7 of the remaining 81 equations.

Although the general problem seems to be difficult even with this reduction, the above mentioned result allows us in Section 4 to make significant progress towards the solution of the following problem:

Given \(r_0\) as above, find all the non-degenerate coalgebra automorphisms \(r: D \otimes D \longrightarrow D \otimes D\) fulfilling the following conditions: it is a solution of the braid equation, it induces \(r_0\) on \(X \times X\) and has set-type square up to height 1 (see Definition 3.1), and then determine which ones of these maps have set-type square (see Definition 3.4).

In this section we consider a height 1 poset \((X, \leq)\) with cardinal \(u + v\), having \(u\) minimal elements \(a_0, \ldots, a_{u-1}\) and \(v\) maximal elements \(b_0, \ldots, b_{v-1}\) such that \(a_i < b_j\) for all \(i, j\). We assume that \(u\) and \(v\) are coprime, and we consider a non-degenerate bijective set-theoretical solution \(r_0\) of the braid equation on \(X\). Moreover, we also assume that there exist poset automorphisms \(\phi_r\) and \(\phi_l\) of \(X\) such that

\[
r_0(x, y) = (\phi_l(y), \phi_r(x))
\]

and \(\phi_r \circ \phi_l\) induces an \(uv\)-cycle on the set of all the pairs \((a_i, b_j)\). Our main result is Theorem 4.1 in which we determine all non-degenerate coalgebra automorphisms of \(D \otimes D\) with set-type square up to height 1, that are solutions of the braid equation and induce \(r_0\) on \(X \times X\). This gives various infinite families of solutions of the braid equation. Finally, in Proposition 4.6 we determine which ones of these solutions have set-type square.

1 Preliminaries

A partially ordered set or poset is a pair \((X, \leq)\) consisting of a set \(X\) endowed with a binary relation \(\leq\), called an order, that is reflexive, antisymmetric and transitive. A connected component of \(X\) is an equivalence class of the equivalence relation generated by the relation \(x \sim y\) if \(x\) and \(y\) are comparable. The height of a finite chain \(a_0 < \cdots < a_n\) is \(n\). The height \(b(X)\) of a finite poset \(X\) is the height of its
largest chain. Let \( a, b \in X \). The closed interval \([a, b]\) is the set of all the elements \( c \) of \( X \) such that \( a \leq c \leq b \). We say that \( b \) covers \( a \), and we write \( a \prec b \) (or \( b \succ a \)), if \([a, b] = \{a, b\}\). A poset \( X \) is locally finite if \([a, b]\) is finite for all \( a, b \in X \).

In the sequel \((X, \leq)\) is a locally finite poset and \( Y := \{(a, b) \in X \times X : a \leq b\} \). It is well known that \( D := KY \) is a counitary coalgebra, called the incidence coalgebra of \( X \), via

\[
\Delta(a, b) := \sum_{c \in [a, b]} (a, c) \otimes (c, b) \quad \text{and} \quad \epsilon(a, b) = \delta_{ab}.
\]

Consider \( KX \) endowed with the coalgebra structure determined by requiring that each \( x \in X \) is a group like element. The \( K \)-linear map \( \iota : KX \to D \) defined by \( \iota(x) := (x, x) \) is an injective coalgebra morphism, whose image is the subcoalgebra of \( D \) spanned by its group like elements.

Recall from [7] that a map \( r_0 : X \times X \to X \times X \) is called non-degenerate if the maps \( \lambda(-) \) and \( (-)^b \) from \( X \) to \( X \), defined by \((a, b) := r_0(a, b) \) are bijective for all \( a, b \in X \).

Let \( r \) be a coalgebra automorphism of \( D \otimes D \) and let

\[
\sigma := (D \otimes \epsilon) \circ r \quad \text{and} \quad \tau := (\epsilon \otimes D) \circ r.
\]

We say that \( r \) is non-degenerate if the maps

\[
(D \otimes \sigma) \circ (D \otimes D) \quad \text{and} \quad (\tau \otimes D) \circ (D \otimes D)
\]

are isomorphisms (see [10] Subsection 1.1).

Let \( r : D \otimes D \to D \otimes D \) be a linear map and let

\[
\left(\lambda^c|f|g|h \right)_{(a, b), (c, d), (e, f), (g, h) \in Y}
\]

be as in equality (1.1). In [10] Section 2 and Theorem 3.4 we prove that \( r \) is a non-degenerate coalgebra automorphism if and only if it induces by restriction a non-degenerate bijection \( r_0 : X \times X \to X \times X \) and

1) for \( a \leq b \) and \( c \leq d \),

\[
\sum_{c \leq e} \lambda^c|e|f|g|h |_{a|b|c|d} = \delta_{ab} \delta_{cd};
\]

2) the maps \( \lambda(-) \) and \( (-)^b \) are automorphisms of posets;

3) if \( a \) and \( b \) belong to the same component of \( X \), then \( \lambda(-) = b(-) \) and \((-)^a = (-)^b \);

4) if \( a \leq b \), \( c \leq d \), \( e \leq f \), \( g \leq h \) and \( \lambda^e|f|g|h |_{a|b|c|d} \neq 0 \), then \( a^e \leq g \leq h \leq b^f \) and \( a^c \leq e \leq f \leq a^d \);

5) if \( a \leq b \), \( c \leq d \), \( e \leq f \), \( g \leq h \), \( a^e \leq g \leq h \leq b^f \) and \( a^c \leq e \leq f \leq a^d \), then

\[
\lambda^e|f|g|h |_{a|b|c|d} = \lambda^e|f|g|h |_{a^e|b^e|c^e|d^e}
\]

for each \( g, z \in X \) such that \( e \leq g \leq f \) and \( g \leq z \leq h \).

By [10] Remark 2.1, we know that \( \lambda^e|f|g|h |_{x|y|z} = 1 \) for all \( x, y \in X \). We will use freely this fact.

2 Factorization of solutions

Let \( r : D \otimes D \to D \otimes D \) be a non-degenerate coalgebra automorphism that induces a non-degenerate solution \( r_0 : X \times X \to X \times X \) of the braid equation.

Let \((a, b), (c, d), (e, f), (g, h), (i, j), (k, l) \in Y\) and let

\[
T := [a, b] \times [c, d] \times [e, f] \quad \text{and} \quad S := [g, h] \times [i, j] \times [k, l].
\]
We consider $X \times X \times X$ endowed with the product order. Note that $S$ and $T$ are the closed intervals $[(g, i, j), (h, k, l)]$ and $[(a, c, e), (b, d, f)]$ in $X \times X \times X$. Clearly $S \subseteq T$ if and only if

$$[g, h] \subseteq [a, b], \quad [i, j] \subseteq [c, d] \quad \text{and} \quad [k, l] \subseteq [e, f].$$

(2.1)

Note also that

$$b(T) = b([a, b]) + b([c, d]) + b([e, f]).$$

For $S \subseteq T$ as above we define

$$\text{LBE}(S, T) := \sum_{x \in [a, b]} \sum_{w \in [c, d]} \sum_{y \in [e, f]} \sum_{z \in [f, e]} \lambda^x_{w|u|^z|x|z} \lambda^y_{z|v|^y|z} \lambda^y_{z|v|^y|z} \lambda^z_{a|w|a|^z|x}$$

and

$$\text{RBE}(S, T) := \sum_{x \in [a, b]} \sum_{w \in [c, d]} \sum_{y \in [e, f]} \sum_{z \in [f, e]} \lambda^x_{w|u|^z|x} \lambda^y_{z|v|^y|z} \lambda^z_{a|w|a|^z|x} \lambda^z_{a|w|a|^z|x}.$$

In [10] Proposition 4.3 the following result is proved:

**Proposition 2.1.** The map $r$ is a solution of the braid equation if and only if

$$\text{LBE}(S, T) = \text{RBE}(S, T) \quad \text{for all} \quad S \subseteq T.$$  (2.2)

For each $S := [(g, i, j), (h, k, l)]$, we set

$$\psi(S) := (\delta^k_i, \delta^l_j) \otimes (\delta^h_k, \delta^j_i) \otimes (\delta^i_j, \delta^k_l).$$

Let $\Sigma := (a, b) \otimes (c, d) \otimes (e, f)$. A direct computation shows that

$$(r \otimes D) \circ (D \otimes r) \circ (r \otimes D)(\Sigma) = \sum_{S \subseteq T} \text{LBE}(S, T)\psi(S)$$

and

$$(D \otimes r) \circ (r \otimes D) \circ (D \otimes r)(\Sigma) = \sum_{S \subseteq T} \text{RBE}(S, T)\psi(S)$$

(see the proof of [10] Proposition 4.3). Since $r \otimes D$ and $D \otimes r$ are coalgebra morphisms, this implies that

$$\delta_{ab}\delta_{cd}\delta_{ef} = (\epsilon \otimes \epsilon \otimes \epsilon)(\Sigma) = \sum_{S \subseteq T} \text{LBE}(S, T),$$

(2.3)

and similarly

$$\delta_{ab}\delta_{cd}\delta_{ef} = \sum_{S \subseteq T} \text{RBE}(S, T).$$

(2.4)

Assume $S \subseteq T$ and let $(p, q, s) \in S$. We define the splitting of the inclusion $S \subseteq T$ at $(p, q, s)$ as the pair $(S_1 \subseteq T_1, S_2 \subseteq T_2)$, where

$$S_1 := [(g, i, j), (p, q, s)], \quad T_1 := [(a, c, e), (p, q, s)],$$

$$S_2 := [(p, q, s), (h, k, l)], \quad T_2 := [(p, q, s), (b, d, f)].$$

**Theorem 2.2.** The following equalities hold:

$$\text{LBE}(S, T) = \text{LBE}(S_1, T_1) \text{LBE}(S_2, T_2)$$

and

$$\text{RBE}(S, T) = \text{RBE}(S_1, T_1) \text{RBE}(S_2, T_2).$$
Proof. Since by definition we have

$$\text{LBE}(S, T) = \sum_{x \in \{a, g\}} \sum_{w \in \{c, d\}} \sum_{y \in \{h, b\}} \sum_{z \in \{e, f\}} \lambda_{u|b|c|d}^{au} \lambda_{a|c|d}^{au} \lambda_{a|c|d}^{au} \lambda_{a|c|d}^{au},$$

$$\text{LBE}(S_1, T_1) = \sum_{x \in \{a, g\}} \sum_{w \in \{c, d\}} \sum_{y \in \{h, b\}} \sum_{z \in \{e, f\}} \lambda_{u|b|c|d}^{au} \lambda_{a|c|d}^{au} \lambda_{a|c|d}^{au} \lambda_{a|c|d}^{au},$$

and

$$\text{LBE}(S_2, T_2) = \sum_{y \in \{b, h\}} \sum_{z \in \{d, f\}} \sum_{x \in \{i, f\}} \lambda_{p|q|b|d}^{pq} \lambda_{p|q|b|d}^{pq} \lambda_{p|q|b|d}^{pq} \lambda_{p|q|b|d}^{pq},$$

in order to prove the first equality, it suffices to note that, by [10, Proposition 2.10], [10, Corollary 2.5] and [10, Remark 4.2], the equalities

$$\lambda_{u|b|c|d}^{au} \lambda_{a|c|d}^{au} \lambda_{a|c|d}^{au} \lambda_{a|c|d}^{au} = \lambda_{x|y|z}^{x|y|z} \lambda_{y|z}^{y|z} \lambda_{z|y}^{z|y} \lambda_{y|z}^{y|z} = \lambda_{x|y|z}^{x|y|z} \lambda_{y|z}^{y|z} \lambda_{z|y}^{z|y} \lambda_{y|z}^{y|z},$$

and

$$\lambda_{u|b|c|d}^{au} \lambda_{a|c|d}^{au} \lambda_{a|c|d}^{au} \lambda_{a|c|d}^{au} = \lambda_{x|y|z}^{x|y|z} \lambda_{y|z}^{y|z} \lambda_{z|y}^{z|y} \lambda_{y|z}^{y|z}$$

hold. We leave the proof of the second equality to the reader. □

**Definition 2.3.** We say that an inclusion of intervals $[\alpha, \beta] \subseteq [\gamma, \delta]$ with $\gamma < \delta$ is lower extremal if $\alpha = \beta = \gamma$ and that is upper extremal if $\alpha = \beta = \delta$.

**Remark 2.4.** Note that $S \subseteq T$ is extremal if either the three inclusions in (2.1) are lower extremal or the three inclusions are upper extremal.

**Corollary 2.5.** The map $r$ is a solution of the braid equation if and only if identity (2.2) hold for all $S \subseteq T$ lower extremal with $h(T) \geq 1$ or $S = T$ and $h(T) = 1$.

**Proof.** By Proposition 2.1 we must show that $\text{LBE}(S, T) = \text{RBE}(S, T)$ for all $S \subseteq T$. When $h(T) = 0$ this is true since $r_0: X \times X \rightarrow X \times X$ is a solution of the braid equation, while for $n = 1$ it is true by hypothesis. Assume now that identity (2.2) hold for $S \subseteq T$ with $h(T) \leq n$ for some $n \geq 1$ and set $T = [(a, c, e), (b, d, f)]$ with $h(T) = n + 1$. Let $\mathcal{S} := [(a, c, e), (a, c, e)]$ and $\mathcal{S} := [(b, d, f), (b, d, f)]$. By the hypothesis, we know that (2.2) is satisfied for $S = \mathcal{S}$. Moreover, for $S \subseteq T$ with $S \not\in \mathcal{S}$ there exists a splitting $(S_1 \subseteq T, S_2 \subseteq T_2)$ of $S \subseteq T$ with $h(T_1), h(T_2) < h(T)$. Hence, in this case the result follows by induction on $h(T)$, using Theorem 2.2.

Finally we have

$$\text{LBE}(\mathcal{S}, T) = \sum_{S \subseteq T \subseteq \mathcal{S}} \text{LBE}(S, T) - \sum_{S \subseteq T \subseteq \mathcal{S}} \text{LBE}(S, T),$$

$$= \sum_{S \subseteq T \subseteq \mathcal{S}} \text{RBE}(S, T) - \sum_{S \subseteq T \subseteq \mathcal{S}} \text{RBE}(S, T),$$

$$= \text{RBE}(\mathcal{S}, T),$$

where in the second equality we have used equalities (2.3) and (2.4). □
2.1 Braid equation for lower extremal inclusions in height one orders.

Next we analyze exhaustively the meaning of equalities \(\text{2.2}\) when the order has height one, the sum of the lengths of the intervals \([a, b]\), \([c, d]\) and \([e, f]\) is greater than 1 and the inclusions are lower extremal:

1) When \(g = h = a < b, i = j = c < d\) and \(e = f = k = l\), then \(\text{2.2}\) reduces to
   \[
   \sum_{y \in [a, b]} \sum_{z \in [c, d]} \lambda^e \lambda^c \frac{v(z)}{v(b)} |a^e| |a^c| |b^e| |b^c| = \sum_{y \in [a, b]} \lambda^e \lambda^c \frac{v(z)}{v(b)} |a^e| |a^c| |b^e| |b^c| .
   \] \(\text{(2.5)}\)

2) When \(g = h = a < b, i = j = c < d\) and \(k = l = e < f\), then \(\text{2.2}\) reduces to
   \[
   \sum_{y \in [a, b]} \sum_{v \in [e, f]} \lambda^e \lambda^c \frac{v(z)}{v(b)} |a^e| |a^c| |b^e| |b^c| = \sum_{y \in [a, b]} \lambda^e \lambda^c \frac{v(z)}{v(b)} |a^e| |a^c| |b^e| |b^c| .
   \] \(\text{(2.6)}\)

3) When \(g = h = a = b, i = j = c < d\) and \(l = e < f\), then \(\text{2.2}\) reduces to
   \[
   \sum_{z \in [c, d]} \sum_{v \in [e, f]} \lambda^e \lambda^c \frac{v(z)}{v(b)} |a^e| |a^c| |b^e| |b^c| = \sum_{z \in [c, d]} \lambda^e \lambda^c \frac{v(z)}{v(b)} |a^e| |a^c| |b^e| |b^c| .
   \] \(\text{(2.7)}\)

4) When \(g = h = a < b, i = j = c < d\) and \(k = l = e < f\), then \(\text{2.2}\) reduces to
   \[
   \sum_{y \in [a, b]} \sum_{z \in [c, d]} \lambda^e \lambda^c \frac{v(z)}{v(b)} |a^e| |a^c| |b^e| |b^c| = \sum_{y \in [a, b]} \lambda^e \lambda^c \frac{v(z)}{v(b)} |a^e| |a^c| |b^e| |b^c| .
   \] \(\text{(2.8)}\)

**Theorem 2.6.** Assume that \((X, \leq)\) has height one and that equality \(\text{2.2}\) hold for all \(S \subseteq T\) with \(h(T) = 1\), and \(S \subseteq T\) lower extremal or \(S = T\). Then \(r\) is a solution of the braid equation if and only if

- for all \(a < b, c < d\) and \(e \in X\), the equality \(\text{2.5}\) is satisfied,
- for all \(a < b, c \in X\) and \(e < f\), the equality \(\text{2.4}\) is satisfied,
- for all \(a \in X, c < d\) and \(e < f\), the equality \(\text{2.7}\) is satisfied,
- for all \(a < b, c < d\) and \(e < f\), the equality \(\text{2.8}\) is satisfied.
Proof. By Corollary 2.5.

3 Conditions for solutions with set-type square

Let $r: D \otimes D \to D \otimes D$ be a non-degenerate coalgebra automorphism and let $r_0: X \times X \to X \times X$ be the map induced by $r$. In the sequel we assume that $r_0$ is a non-degenerate solution of the set-theoretical braid equation and we set $r^2 := r \circ r$ and $Y \otimes Y := \{a \otimes b : a, b \in Y\}$.

**Definition 3.1.** We say that $r$ has set-type square if $r^2(Y \otimes_k Y) \subseteq Y \otimes_k Y$.

**Remark 3.2.** By [10, Remark 3.1], the map $r$ has set-type square if and only if

$$r^2((a, b) \otimes (c, d)) = (a_c a^c, a_d a^d),$$

(3.1)

for all $(a, b), (c, d) \in Y$. Consequently, if $r$ has set-type square, then $r$ permutes the elements of $Y \otimes_k Y$.

**Remark 3.3.** When $a = b$ and $c = d$, then equality (3.1) holds since $r_0$ is a solution of the set-theoretical braid equation.

**Definition 3.4.** We say that $r$ has set-type square up to height 1 if equality (3.1) holds for all the $(a, b), (c, d) \in Y$ with $h([a, b]) + h([c, d]) = 1$.

Let $(a, b), (c, d), (e, f), (g, h) \in Y$ and let

$$T := [a, b] \times [c, d] \quad \text{and} \quad S := [e, f] \times [g, h].$$

(3.2)

We consider $X \times X$ endowed with the product order. Note that $S$ and $T$ are the closed intervals $[(e, g), (f, h)]$ and $[(a, c), (b, d)]$ in $X \times X$. Clearly $S \subseteq T$ if and only if $[e, f] \subseteq [a, b]$ and $[g, h] \subseteq [c, d]$.

Note also that $h(T) = h([a, b]) + h([c, d])$. For $S \subseteq T$ as above we define

$$LSQ(S, T) := \sum_{x \in \{a, b\}, w \in \{c, d\}} \sum_{y \in \{f, g\}, z \in \{e, h\}} \lambda_{w, u, v, z}^{x, y, z} \lambda_{w, u, v, z}^{x, y, z} \lambda_{w, u, v, z}^{x, y, z} \lambda_{w, u, v, z}^{x, y, z} (a_c e, a_d f) \otimes (a_h x, a_h x),$$

and

$$RSQ(S, T) := \delta_{a, b} \delta_{c, d} \delta_{e, f} \delta_{y, z}. \delta_{x, w}.$$

Let $\mathfrak{V} := (a, b) \otimes (c, d)$. Applying twice [10, Corollary 2.9], we obtain

$$r^2(\mathfrak{V}) = \sum_{\{a, b\} \subseteq [e, f], \{c, d\} \subseteq [g, h], \{a, b\} \subseteq [w, z], \{c, d\} \subseteq [x, y]} \lambda_{w, u, v, z}^{x, y, z} \lambda_{w, u, v, z}^{x, y, z} \lambda_{w, u, v, z}^{x, y, z} \lambda_{w, u, v, z}^{x, y, z} (a_c e, a_d f) \otimes (a_h x, a_h x).$$

Consequently, since by [10, Corollary 2.5]

$$(a_c e, a_d f) \otimes (a_h x, a_h x) = (a_c e, a_d f) \otimes (a_h x, a_h x),$$

we have

$$r^2(\mathfrak{V}) = \sum_{S \subseteq T} \sum_{h(S) = 0} LSQ(S, T) \phi(S),$$

(3.3)

where for $S := [e, f] \times [g, h]$ we set $\phi(S) := (a_c e, a_d f) \otimes (a_h x, a_h x)$. Since $r^2$ is a coalgebra morphism, this implies that

$$\delta_{a, b} \delta_{c, d} = (\epsilon \otimes \epsilon)(\mathfrak{V}) = \sum_{S \subseteq T} \sum_{h(S) = 0} LSQ(S, T).$$
Let $S \subseteq T$ and let $(p, q) \in S$. We define the splitting of the inclusion $S \subseteq T$ at $(p, q)$ as the pair $(S_1 \subseteq T_1, S_2 \subseteq T_2)$, where

$$
S_1 := [(e, g), (p, q)], \quad T_1 := [(a, c), (p, q)],
$$
$$
S_2 := [(p, q), (f, h)], \quad T_2 := [(p, q), (b, d)].
$$

Note that

$$
\text{RSQ}(S, T) = \text{RSQ}(S_1, T_1) \text{RSQ}(S_2, T_2) \quad \text{(3.4)}
$$

for each splitting of $S \subseteq T$.

**Proposition 3.5.** Equality (3.4) is true for $(a, b) \otimes (c, d)$ if and only if

$$
\text{LSQ}(S, T) = \text{RSQ}(S, T) \quad \text{for all } S \subseteq T,
$$

where $T := [a, b] \times [c, d]$.

**Proof.** Since the map

$$(e, f) \otimes (g, h) \mapsto (e, f, e) \otimes (g, f, g, h)$$

is injective, the result follows comparing coefficients in equalities (3.1) and (3.3). ◻

**Proposition 3.6.** Let $S \subseteq T$, $(p, q) \in S$ and let $((S_1, T_1), (S_2, T_2))$ be the splitting of $S \subseteq T$ at $(p, q)$. The following equality hold:

$$
\text{LSQ}(S, T) = \text{LSQ}(S_1, T_1) \text{LSQ}(S_2, T_2).
$$

**Proof.** Since, by definition

$$
\text{LSQ}(S, T) = \sum_{x \in [a, c]} \sum_{y \in [c, g]} \lambda_{u[p|b|c|d]}^u \lambda_{u[p|b|c|d]}^{u|z|y},
$$

$$
\text{LSQ}(S_1, T_1) = \sum_{x \in [a, c]} \sum_{y \in [c, g]} \lambda_{u[p|b|c|d]}^u \lambda_{u[p|b|c|d]}^{u|z|y},
$$

and

$$
\text{LSQ}(S_2, T_2) = \sum_{x \in [a, c]} \sum_{y \in [c, g]} \lambda_{u[p|b|c|d]}^u \lambda_{u[p|b|c|d]}^{u|z|y},
$$

in order to prove the first equality, it suffices to note that, by [10] Proposition 2.10 and [10] Corollary 2.5, the equalities

$$
\lambda_{u[p|b|c|d]}^{u|z|y} = \lambda_{u[p|b|c|d]}^{u|z|y} \lambda_{u[p|b|c|d]}^{u|z|y},
$$

and

$$
\lambda_{u[p|b|c|d]}^{u|z|y} \lambda_{u[p|b|c|d]}^{u|z|y} = \lambda_{u[p|b|c|d]}^{u|z|y} \lambda_{u[p|b|c|d]}^{u|z|y},
$$

hold. ◻

As in Definition 2.23 we say that $[\alpha, \beta] \subseteq [\gamma, \delta]$ with $\gamma < \delta$ is lower extremal if $\alpha = \beta = \gamma$ and that it is upper extremal if $\alpha = \beta = \delta$.

**Corollary 3.7.** The following assertions hold:

1. The map $r$ has set-type square up to height 1 if and only if the equality $\text{LSQ}(S, T) = \text{RSQ}(S, T)$ is true for all $S \subseteq T$ such that $h(T) = 1$, and $S = T$ or $S \subseteq T$ is lower extremal.

2. The map $r$ has set-type square if and only if it has set-type square up to height 1 and $\text{LSQ}(S, T) = \text{RSQ}(S, T)$ for all $S \subseteq T$ lower extremal, such that $h(T) \geq 2$. 

Proof. Mimic the proof of Corollary 2.5 using equality (3.4) and Proposition 3.6. □

For the rest of the section we will assume that $(X, \leq)$ is connected.

**Remark 3.8.** By [10] Corollary 2.5 there exist order automorphisms $\phi_l$ and $\phi_r$ of $X$ such that

$$b^a = \phi_l(a) \quad \text{and} \quad a^b = \phi_r(a) \quad \text{for all } a, b \in X.$$ 

Moreover, since $r_0 : X \times X \to X \times X$ is a solution of the set-theoretic braid equation, $\phi_l$ and $\phi_r$ commute. Consequently, $r$ has set-type square if and only if

$$r^2((a, b) \otimes (c, d)) = (\varphi(a), \varphi(b)) \otimes (\varphi(c), \varphi(d)) \quad \text{for all } (a, b), (c, d) \in Y,$$

where $\varphi := \phi_l \circ \phi_r$.

**Notation 3.9.** For all $s, a, b, c, d \in X$ with $a \prec b$, $c \prec d$ and $i \in \mathbb{Z}$, we will write

$$s^{(i)} := \phi^i(s), \quad \alpha_r(s)(a, b) := \lambda^{(i)}_{a|b|s|a(1)|b(1)}, \quad \beta_r(s)(a, b) := \lambda^{(i)}_{a|b|s|a(1)|b(1)}, \quad \alpha_l(s)(a, b) := \lambda^{(i)}_{a|b|s|a(1)|b(1)} = 1,$$

(3.6) 

Moreover, since $\alpha_r(s)(a, b) = 1$, equality (3.5) reduces to equality (3.12). On the other hand, when

$$r^2((a, b) \otimes (c, d)) = (\varphi(a), \varphi(b)) \otimes (\varphi(c), \varphi(d)) \quad \text{for all } (a, b), (c, d) \in Y,$$

where $\varphi := \phi_l \circ \phi_r$.

**Proposition 3.10.** The map $r$ has set-type square up to height 1 if and only if for all $a \prec b$ and $c \in X$

$$\alpha_r(c)(a, b)\alpha_r(1)(c)(a(1), b(1)) = 1, \quad (3.10)$$

$$\alpha_r(c)(a, b)\beta_r(1)(c)(a(1), b(1)) + \beta_r(c)(a, b) = 0, \quad (3.11)$$

$$\alpha_l(c)(a, b)\alpha_l(1)(c)(a(1), b(1)) = 1, \quad (3.12)$$

and

$$\alpha_l(c)(a, b)\beta_r(1)(c)(a(1), b(1)) + \beta_l(c)(a, b) = 0. \quad (3.13)$$

**Proof.** When $a \prec b$, $e = a$, $f = b$ and $c = d = g = h$, equality (3.5) becomes

$$\lambda^{(i)}_{a|b|c|d|a(1)|b(1)} = 1,$$

which coincides with equality (3.10). Similarly, when $a = b = e = f$, $c \prec d$, $c = d$ and $g = h$, equality (3.5) reduces to equality (3.12). On the other hand, when $c = d = g = h$, equality (3.5) gives

$$\lambda^{(i)}_{a|b|c|d|a(1)|b(1)} = 0,$$

which coincides with equality (3.11). A similar computation shows that when $a = b = e = f$ and $g = h = c \prec d$, equality (3.5) reduces to equality (3.13). By Corollary 3.11 this finishes the proof. □

**Corollary 3.11.** If $r$ has set-type square up to height 1, then

$$\alpha_l(1)(c)(1)(a(1), b(1)) = \alpha_l(c)(a, b) \quad \text{and} \quad \beta_l(1)(c)(1)(a(1), b(1)) = \beta_l(c)(a, b), \quad \text{for } h \in \{r, l\}, \ a \prec b \in X \text{ and } c \in X.$$

**Proof.** We only consider the case $h = l$ since the case $h = r$ is similar. By equalities (3.10) and (3.12),

$$\alpha_l(1)(c)(1)(a(1), b(1)) = \frac{1}{\alpha_r(c(1))(1)(a(1), b(1))} = \alpha_l(c)(a, b),$$

which proves the first equality, and by equalities (3.11), (3.12) and (3.13),

$$\beta_l(1)(c)(1)(a(1), b(1)) = \alpha_l(c)(a, b)\alpha_r(c(1))(1)(a(1), b(1))\beta_l(1)(c)(1)(a(1), b(1))$$

SOLUTIONS OF THE BRAID EQUATION WITH SET-TYPE SQUARE 9
Assume now that all the hypothesis in the statement is equivalent to the fact that $LSQ(S,T)$ is satisfied if and only if for all $a \prec b$ in $X$ there exists a constant $C_m(a,b) \in K^r$ such that

$$\frac{\alpha_l(s^{(1)})(a^{(1)}, b^{(1)})}{\alpha_l(s)(a, b)} = \frac{\alpha_r(t^{(1)})(a^{(1)}, b^{(1)})}{\alpha_r(t)(a, b)} = C_m(a, b),$$

for all $s, t \in X$.

**Proof.** Mimic the proof of [10, Proposition 4.5(1)].

**Remark 3.13.** By items 1)–4) of [10, Proposition 4.5] and Proposition 3.12 if $r$ is a solution of the braid equation, then necessarily

$$\alpha_r(s)(a, b) = \alpha_r(t^{(1)})(a^{(1)}, b^{(1)}), \quad \alpha_l(s)(a, b) = \alpha_l(t^{(1)})(a^{(1)}, b^{(1)}), \quad \beta_r(s)(a, b) = \beta_r(t^{(1)})(a^{(1)}, b^{(1)}), \quad \beta_l(s)(a, b) = \beta_l(t^{(1)})(a^{(1)}, b^{(1)}),$$

and for all $a \prec b$ there exist constants $C_r(a, b)$, $C_l(a, b)$ and $C_m(a, b)$ such that

$$C_r(a, b) = \frac{\alpha_r(s^{(1)})(a^{(1)}, b^{(1)})}{\alpha_r(s)(a, b)} \quad \text{for all } s,$$

$$C_l(a, b) = \frac{\alpha_l(t^{(1)})(a^{(1)}, b^{(1)})}{\alpha_l(t)(a, b)} \quad \text{for all } s,$$

and

$$C_m(a, b) = \frac{\alpha_l(s^{(1)})(a^{(1)}, b^{(1)})}{\alpha_l(s)(a, b)} = \frac{\alpha_r(t^{(1)})(a^{(1)}, b^{(1)})}{\alpha_r(t)(a, b)}$$

for all $s$ and $t$.

Assume now that $r$ has set-type square up to height 1. Then, by Corollary 3.11,

$$C_r(a, b)C_m(a, b) = \frac{\alpha_r(s^{(1)})(a^{(1)}, b^{(1)})}{\alpha_r(s)(a, b)} \frac{\alpha_l(s)(a, b)}{\alpha_r(t)(a, b)} \frac{\alpha_r(t^{(1)})(a^{(1)}, b^{(1)})}{\alpha_l(s)(a, b)} = 1$$

and, similarly, $C_l(a, b)C_m(a, b) = 1$. So, $C_r(a, b) = C_m(a, b)^{-1} = C_l(a, b)$.

**Proposition 3.14.** Assume that $(X, \leq)$ has height 1. If $r$ has set-type square up to height 1, then $r$ has set-type square if and only if

$$\frac{\alpha_l(a)(a, b, c, d)}{\alpha_l^{(1)}(a^{(1)}, b^{(1)}, c^{(1)}, d^{(1)})} \sum_{a^{(1)}, b^{(1)}, c^{(1)}, d^{(1)}} \alpha_l(a)(a^{(1)}, b^{(1)}, c^{(1)}, d^{(1)}) = - \sum_{a^{(1)}, b^{(1)}, c^{(1)}, d^{(1)}} \frac{\beta_l(b)(c, d)}{\alpha_l^{(1)}(b^{(1)})} \frac{\beta_l(a)(c, d)}{\alpha_l^{(1)}(a^{(1)}, b^{(1)}, c^{(1)}, d^{(1)})}$$

for all $a, b, c, d \in X$ with $a \prec b$ and $c \prec d$.

**Proof.** By Corollary 3.7(2) and the definition of $RSQ(S, T)$, we must show that the hypothesis in the statement is equivalent to the fact that $LSQ(S, T) = 0$ for all $T := [a, b] \times [c, d]$ with $a \prec b$ and $c \prec d$ and $S \subseteq T$ lower extremal. By the very definition of $LSQ(S, T)$, we have

$$LSQ(S, T) = \sum_{a^{(1)}, b^{(1)}, c^{(1)}, d^{(1)}} \frac{\alpha_l(a)(a^{(1)}, b^{(1)}, c^{(1)}, d^{(1)})}{\alpha_l^{(1)}(a^{(1)}, b^{(1)}, c^{(1)}, d^{(1)})}$$

which proves the second equality. □
where the last equality follows from the definitions of $\Gamma$, $\beta_l$ and $\beta_r$. Since, by (3.19) and the fact that the maps $(-r)$ and $(-r)^0$ are automorphisms of posets,

\[
\lambda^{(1)c}_{a|b|c}\lambda^{(1)a|b|c} = \lambda^{(1)c}_{a|b|c}, \quad \lambda^{(1)a|b|d} = \lambda^{(1)a|b|d}, \\
\lambda^{(1)a|b|c}\lambda^{(1)\beta_l(b)(c,d)} = \alpha_r(c)(a,b)\beta_l(b)(c,d), \\
\lambda^{(1)c}_{a|b|c}\lambda^{(1)\beta_l(b)(c,d)} = \alpha_r(c)(a,b)\beta_l(b)(c,d),
\]

we obtain

\[
\text{LSQ}(S,T) = \Gamma_{a|b|c} + \alpha_r(c)(a,b)\beta_l(b)(c,d)\beta_l(1)(c)(a^{(1)},b^{(1)}) \\
+ \alpha_r(c)(a,b)\beta_l(c,d)\beta_l(1)(c)(a^{(1)},b^{(1)}) \\
+ \alpha_r(c)(a,b)\alpha_l(b)(c,d)\Gamma_{a|b|d}, \\
= \Gamma_{a|b|c} + \frac{\beta_l(b)(c,d)\beta_l(1)(c)(a^{(1)},b^{(1)})}{\alpha_l(1)c(a^{(1)},b^{(1)})} \\
+ \frac{\alpha_l(1)c(a^{(1)},b^{(1)})}{\alpha_l(1)c(a^{(1)},b^{(1)})} \Gamma_{a|b|d}, \\
= \Gamma_{a|b|c} + \frac{\beta_l(b)(c,d)\beta_l(1)(c)(a^{(1)},b^{(1)})}{\alpha_l(1)c(a^{(1)},b^{(1)})} \\
+ \frac{\alpha_l(1)c(a^{(1)},b^{(1)})}{\alpha_l(1)c(a^{(1)},b^{(1)})} \Gamma_{a|b|d},
\]

where the second equality holds by Proposition 3.10 and the third one, by Corollary 3.11. Hence $\text{LSQ}(S,T) = 0$ if and only if equality (3.19) is true.

**Corollary 3.15.** If $(X,\leq)$ has height 1, then $r$ has set-type square if and only if equalities (3.10)–(3.14) are fulfilled.

**Proof.** This follows immediately from Propositions 3.5, 3.10 and 3.14. \qed

4 A family of examples

In this section we assume that $u,v \in \mathbb{N}$ are coprime and for each $l \in \mathbb{N}$ we set $\mathbb{N}_l := \{0,\ldots, l-1\}$. Define on the set

\[
X = \{a_0, \ldots, a_{u-1}, b_0, \ldots, b_{v-1}\}
\]

the partial order $a_i < b_j$ for all $i, j$. For the rest of the paper we assume that each element of $K^x$ has $uv$ distinct $uv$-th roots, and we fix a primitive $uv$-th root of unity $w$.

Let $\phi_r, \phi_l : X \to X$ be two commuting poset automorphisms and let

\[
r_0 : X \times X \to X \times X
\]
be the map defined by \( r_0(x, y) := (\phi_l(y), \phi_r(x)) \). It is well-known that \( r_0 \) is a non-degenerate solution of the braid equation. Moreover, by definition, \( \varepsilon_l(-) = \phi_l \) and \( \varepsilon_r(-) = \phi_r \) for all \( x \).

Since \( \phi_l \) and \( \phi_r \) are poset automorphisms there exist permutations \( \sigma_a, \tau_a \) of \( \mathbb{N}_u \), and \( \sigma_v, \tau_v \) of \( \mathbb{N}_v \) such that

\[
\phi_l(a_i) = a_{\sigma_a(i)}, \quad \phi_l(b_j) = b_{\sigma_v(j)}, \quad \phi_r(a_i) = a_{\tau_a(i)} \quad \text{and} \quad \phi_r(b_j) = b_{\tau_v(j)}.
\]

(4.1)

Note that \( \sigma_a \circ \tau_a = \tau_a \circ \sigma_a \) and \( \sigma_v \circ \tau_v = \tau_v \circ \sigma_v \) because \( \phi_l \) and \( \phi_r \) commute. From now on we assume that \( \varsigma_a := \sigma_a \circ \tau_a \) is an \( u \)-cycle and \( \varsigma_v := \sigma_v \circ \tau_v \) is a \( v \)-cycle.

Since \( \gcd(u, v) = 1 \) from this it follows that the map \((i, j) \mapsto (\varsigma_a(i), \varsigma_v(j))\) is an \( uv \)-cycle.

Let \( r : D \otimes D \rightarrow D \otimes D \) be a non-degenerate coalgebra automorphism that has set-type square up to height 1 and induces \( r_0 \) by restriction.

**Proposition 4.1.** If \( r \) satisfies the braid equation, then

\[
\alpha_l(a_i)(a_k, b_i) = \alpha_l(a_k)(a_i, b_i), \quad \alpha_r(a_i)(a_k, b_i) = \alpha_r(a_k)(a_i, b_i),
\]

\[
\alpha_l(b_j)(a_k, b_i) = \alpha_l(a_k)(b_j, b_i), \quad \alpha_r(b_j)(a_k, b_i) = \alpha_r(a_k)(b_j, b_i),
\]

\[
\beta_l(a_i)(a_k, b_i) = \beta_l(a_k)(a_i, b_i), \quad \beta_r(a_i)(a_k, b_i) = \beta_r(a_k)(a_i, b_i),
\]

\[
\beta_l(b_j)(a_k, b_i) = \beta_l(a_k)(b_j, b_i), \quad \beta_r(b_j)(a_k, b_i) = \beta_r(a_k)(b_j, b_i),
\]

(4.2)

for all \( i, j, k \) and \( l \).

**Proof.** Since \( \varphi = \phi_l \circ \phi_r \) acts as an \( u \)-cycle on \( \{a_0, \ldots, a_{u-1}\} \) and as an \( v \)-cycle on \( \{b_0, \ldots, b_{v-1}\} \) and \( \gcd(u, v) = 1 \), this follows by Corollary 3.11 and equalities (3.14) and (3.15).

\( \square \)

In the sequel we assume that equalities (4.2) are fulfilled. For the sake of simplicity, from now on we write

\[
\alpha_{ta} := \alpha_l(a_0)(a_0, b_0), \quad \alpha_{ra} := \alpha_r(a_0)(a_0, b_0),
\]

\[
\alpha_{tb} := \alpha_l(b_0)(a_0, b_0), \quad \alpha_{rb} := \alpha_r(b_0)(a_0, b_0),
\]

\[
\beta_{ta} := \beta_l(a_0)(a_0, b_0), \quad \beta_{ra} := \beta_r(a_0)(a_0, b_0),
\]

\[
\beta_{tb} := \beta_l(b_0)(a_0, b_0), \quad \beta_{rb} := \beta_r(b_0)(a_0, b_0).
\]

By equalities (4.2) and Proposition 3.10

\[
\alpha_{ta} = \alpha_{ta}^{-1}, \quad \alpha_{tb} = \alpha_{tb}^{-1}, \quad \beta_{ta} = -\beta_{tb} \alpha_{ta}^{-1} \quad \text{and} \quad \beta_{rb} = -\beta_{rb} \alpha_{tb}^{-1}.
\]

(4.3)

Combining this with equality (1.2) we obtain that

\[
\alpha_{ta}^2 = \alpha_{tb}^2.
\]

(4.4)

**Remark 4.2.** Let \( C_r(a_i, b_j), C_l(a_i, b_j) \) and \( C_m(a_i, b_j) \) be as in Remark 3.3. By equalities (1.2), we know that \( C_r(a_i, b_j) = C_l(a_i, b_j) = C_m(a_i, b_j) = 1 \) for all \( i \) and \( j \).

**Notation 4.3.** Set \( \alpha := \alpha_{ta}, \beta_a := \beta_{ta}, \beta_b := \beta_{tb} \) and \( \Gamma_{ij|kl} := \Gamma_{a_i|b_j|a_k|b_l} \).

**Remark 4.4.** By equalities (4.3) and equality (4.4) there exists \( \varepsilon \in \{\pm 1\} \) such that the following equalities hold:

\[
\alpha_{tb} = \varepsilon \alpha, \quad \alpha_{ra} = \frac{1}{\alpha}, \quad \alpha_{rb} = \frac{\varepsilon}{\alpha}, \quad \beta_{ra} = -\frac{\beta_a}{\alpha} \quad \text{and} \quad \beta_{rb} = -\frac{\varepsilon \beta_b}{\alpha}.
\]

(4.5)

**Proposition 4.5.** Let \( \varepsilon, \alpha, \beta_a \) and \( \beta_b \) be as above. Equality (2.2) holds for all \( S \subseteq T \) such that \( b(T) \leq 1 \) if and only if

\[
\beta_b(a - 1) = \beta_a(\varepsilon \alpha - 1).
\]

(4.6)
Proof. By \cite{10} Proposition 4.5, Proposition 3.12 and the discussion in \cite{10} Subsection 4.1 we know that equality (2.2) is satisfied for all \(S \subseteq T\) such that \(h(T) \leq 1\) if and only if the conditions in \cite{10} Proposition 4.5, Proposition 4.12 and item 5) of \cite{10} Subsection 4.1 are fulfilled.

Let \(w\) be a fixed primitive \(uv\)-root of unity and let \(\ell_j, \varphi_j, \gamma_r\) and \(\gamma_l\) be as in items 3) and 4) of \cite{10} Proposition 4.5. By Remark 4.2 we can take \(\gamma_l = \gamma_r = 1\) and \(\ell_j = \varphi_j = 1\).

Assume that equality (2.2) hold for all \(S \subseteq T\) such that \(h(T) \leq 1\). By item 4) of \cite{10} Proposition 4.5 we know that, for all \(i,\)

\[
\left( \alpha_{ia} - w^i, \sum_{j=0}^{w-1} w^{ij} \beta_{ja} \right) \sim \left( \alpha_{ib} - w^i, \sum_{j=0}^{w-1} w^{ij} \beta_{jb} \right),
\]

(4.7)

Taking \(i = 0\) and using equalities (4.5), we obtain

\[
(\alpha - 1, w\beta_a) \sim (\varepsilon \alpha - 1, w\beta_b),
\]

which is equivalent to equality (4.6).

Conversely, assume that (4.6) holds. By equalities (4.2), in order to check that equality (2.2) is satisfied for all \(S \subseteq T\) such that \(h(T) \leq 1\) it suffices to verify item 5) of \cite{10} Subsection 4.1 and that, for each \(0 \leq i < uv,\)

\[
\left( \alpha_{ra} - w^i, \sum_{j=0}^{v-1} w^{ij} \beta_{ra} \right) \sim \left( \alpha_{rb} - w^i, \sum_{j=0}^{v-1} w^{ij} \beta_{rb} \right),
\]

(4.8)

and

\[
\left( \alpha_{ia} - w^i, \sum_{j=0}^{v-1} w^{ij} \beta_{ia} \right) \sim \left( \alpha_{ib} - w^i, \sum_{j=0}^{v-1} w^{ij} \beta_{ib} \right),
\]

(4.9)

When \(i \neq 0\), the second coordinate of all the vectors in (4.8) and (4.9) vanishes, so conditions (4.5) and (4.6) hold. When \(i = 0\), the computation above shows that (4.9) is equivalent to (4.6), and similarly (4.8) is equivalent to (4.6). Finally, item 5) of \cite{10} Subsection 4.1 holds if and only if

\[
\beta_{i\ell} + \alpha_{i\beta} \beta_{j\ell} = \beta_{j\tau} + \alpha_{j\tau} \beta_{i\ell} \quad \text{for all} \ i, j \in \{a, b\}.
\]

But these equalities follow from a straightforward computation using (4.3). \(\square\)

Remark 4.6. Since \(\varepsilon = \pm 1\) and \(\alpha \neq 0\), equality (4.6) yields the following possible cases for the values of \(\alpha, \beta_a\) and \(\beta_b:\)

1) If \(\varepsilon = 1\), then \(\alpha = 1\) or \(\beta_a = \beta_b.\)

2) If \(\varepsilon = -1\) and char \(K \neq 2\), then either \(\alpha = 1\) and \(\beta_a = 0\), or \(\alpha \neq 1\) and \(\beta_b = \beta_a \frac{1+\alpha}{1-\alpha}.\)

In the expression \(\Gamma_{ijk}\) the indices \(i\) and \(k\) belong to \(\{0, \ldots, v - 1\}\) and the indices \(j\) and \(l\) belong to \(\{0, \ldots, u - 1\}.\) So, the first ones can be affected by the maps \(\sigma_a\) or \(\tau_a\), while the second one, by the maps \(\sigma_b\) or \(\tau_b.\) Since there is not danger of confusion, we will write \(\Gamma_{\tau_a(i)\tau_a(j)\tau_a(k)\tau_a(l)}\) etcetera.

Proposition 4.7. Let \(\alpha, \beta_a\) and \(\beta_b\) be as in Notation 3.4. Recall that by Remark 4.4 there exists \(\varepsilon \in \{\pm 1\}\) such that equalities (4.5) are satisfied. Then equalities (4.2) - (4.8) hold if and only if for all \(i, k, m \in \{0, \ldots, uv - 1\}\) and \(j, l, n \in \{0, \ldots, v - 1\}\) the following equalities are fulfilled:

\[
\alpha^2 \Gamma_{ijk\ell} - \alpha \Gamma_{\tau_a(i)\tau_a(j)\tau_a(k)\tau_a(l)} = \beta_a(1 - \varepsilon)\beta_a(1 + \alpha\beta_b),
\]

(4.10)

\[
\alpha^2 \Gamma_{ijk\ell} - \alpha \Gamma_{\tau_a(i)\tau_a(j)\tau_a(k)\tau_a(l)} = \beta_b(1 - \alpha)\beta_a\beta_b,
\]

(4.11)

\[
\alpha^3 \Gamma_{ijk\ell} - \alpha \Gamma_{\tau_a(i)\tau_a(j)\tau_a(k)\tau_a(l)} = \beta_a(\alpha\beta_b(1 - \varepsilon) + \beta_a(1 - \alpha^2)),
\]

(4.12)
\[ a^3 \Gamma_{\sigma | \varepsilon \sigma_m | \varepsilon_n} - a \Gamma_{\tau | \tau | \varepsilon_n} = \beta_0 \alpha (\varepsilon \beta_n + \alpha \beta_n)(1 - \varepsilon^2), \] (4.13)

\[ a^3 \Gamma_{\sigma_k | \varepsilon | \varepsilon_n} - a \Gamma_{\tau | \ell | \tau} = \beta_a (\varepsilon \beta_n + \alpha \beta_n)(1 - \alpha^2), \] (4.14)

\[ a^3 \Gamma_{\sigma_k | \varepsilon | \varepsilon_n} - a \Gamma_{\tau | \ell | \tau} = \beta_b (\varepsilon \beta_n + \alpha \beta_n)(1 - \alpha), \] (4.15)

\[ \beta_a \beta_b (\varepsilon (1 + \alpha) - \beta_a (1 + \varepsilon \alpha)) = a^2 \beta_a (1 + \alpha) \Gamma_{i | j | k | l} + \beta_b (1 + \varepsilon \alpha) \Gamma_{\tau | \tau | \varepsilon_n} \] (4.16)

Proof. By (1.2), we have

\[ \frac{\varepsilon \beta_a \beta_b}{\alpha} + \frac{\beta_a \beta_b}{\alpha} + \frac{\beta_a \beta_b}{\alpha} = \frac{\beta_a \beta_b}{\alpha} + \frac{\beta_a \beta_b}{\alpha} + \frac{\beta_a \beta_b}{\alpha} + \frac{1}{\alpha^2} \Gamma_{\tau | \tau | \varepsilon_n | \varepsilon_n}. \] (4.17)

Multiplying by \( \alpha^3 \) and reordering we see that equality (2.5) with these values of \( a, b, c, d \) and \( e \) is fulfilled if and only if

\[ a^3 \Gamma_{i | j | k | l} - a \Gamma_{\tau | \tau | \varepsilon_n | \varepsilon_n} = \alpha \beta_a \beta_b - \varepsilon \alpha^2 \beta_a \beta_b - \varepsilon \alpha \beta_a^2 + \beta_a^2 \]

\[ = \beta_a (1 + \varepsilon \alpha) \beta_b + \alpha \beta_b, \]

as desired. Similar arguments prove that equality (2.5) with \( a \) as in (1.2), \( b, c, d \) and \( e \) is fulfilled if and only if equality (1.11) holds, etcetera.

For the rest of the section we assume that equality (2.2) is satisfied for all \( S \subseteq T \), such that \( b(T) \leq 1 \), or equivalently, that (4.16) holds.

Let \( \alpha, \beta_a \) and \( \beta_b \) be as in Notation (2.2) and let \( \varepsilon \in \{ \pm 1 \} \) be as in Remark (2.3).

In order to abbreviate the expressions we write \( \Gamma \) instead of \( \Gamma_{\varepsilon \varepsilon_n \varepsilon_n \varepsilon_n} \).

Lemma 4.8. The right hand sides of equalities (1.10)–(1.15) are equal to

\[ \beta_a \beta_b (1 - \varepsilon \alpha^2). \]

Proof. Straightforward using (1.11).

Lemma 4.9. Equalities (1.10)–(1.15) are fulfilled if and only if \( \Gamma_{i | j | k | l} = \Gamma \) for all \( i, j, k, l \) and \( (\alpha^2 - 1) \Gamma = \frac{\beta_a \beta_b}{\alpha} (1 - \varepsilon \alpha^2) \).

Proof. Note that by Lemma 4.8, equalities (1.10), (1.12) and (1.14) yield

\[ a^2 \Gamma_{i | j | k | l} - \Gamma_{\tau | \tau | \varepsilon_n | \varepsilon_n} = \frac{\beta_a \beta_b}{\alpha} (1 - \varepsilon \alpha^2), \] (4.20)

\[ a^2 \Gamma_{\sigma_1 | \varepsilon | \varepsilon_n} - \Gamma_{\tau | \tau | \varepsilon_n | \varepsilon_n} = \frac{\beta_a \beta_b}{\alpha} (1 - \varepsilon \alpha^2), \] (4.21)

and

\[ a^2 \Gamma_{\sigma_1 | \varepsilon | \varepsilon_n | \varepsilon_n} - \Gamma_{i | j | k | l} = \frac{\beta_a \beta_b}{\alpha} (1 - \varepsilon \alpha^2). \] (4.22)
Recall that the map \((i,j) \mapsto (\omega_i(i), \omega_j(j))\) is an \(uv\)-cycle. From equality \(4.20\) we obtain that
\[
\alpha^2 \Gamma_{\sigma_i|\sigma_j|\sigma_i \sigma_j} - \Gamma_{\sigma_i|\sigma_i|\sigma_j} = \frac{\beta_a \beta_b}{\alpha} (1 - \varepsilon \alpha^2).
\]
Combining this with \(4.22\) we deduce that \(\Gamma_{\sigma_i|\sigma_i|\sigma_j} = \Gamma_{\sigma_i|\sigma_j|\sigma_i}\) for all \(i\) and \(j\), which implies that
\[
\Gamma_{\sigma_i|\sigma_i|\sigma_j} = \Gamma \quad \text{for all } i \text{ and } j.
\]
On the other hand, from equality \(4.21\) we obtain that
\[
(\alpha^2 - 1) \Gamma = \frac{\beta_a \beta_b}{\alpha} (1 - \varepsilon \alpha^2).
\]
Combining this with \(4.20\), we deduce that \(\Gamma_{\sigma_i|\sigma_i|\sigma_j} = \Gamma \quad \text{for all } i, j, k \text{ and } l\).

From this and \(4.23\) it follows that \(\Gamma_{\sigma_i|\sigma_j|\sigma_i} = \Gamma \quad \text{for all } i, j, k \text{ and } l\). Combining this with equality \(4.20\) we obtain that \((\alpha^2 - 1) \Gamma = \frac{\beta_a \beta_b}{\alpha} (1 - \varepsilon \alpha^2)\), as desired.

\[\left.\Gamma_{\sigma_i|\sigma_j|\sigma_i} = \Gamma \quad \text{for all } i, j, k \text{ and } l\right.\]

\[\left.(\alpha^2 - 1) \Gamma = \frac{\beta_a \beta_b}{\alpha} (1 - \varepsilon \alpha^2)\right.\]

\[\beta_a \beta_b (\varepsilon (1 + \alpha) - \beta_a (1 + \varepsilon \alpha)) = \Gamma (\beta_a (1 + \alpha)(1 - \varepsilon \alpha + \alpha^2) - \beta_k (1 + \varepsilon \alpha)(1 - \alpha + \alpha^2)).\]

\[\left.\left(\alpha^2 - 1\right) \Gamma = \frac{\beta_a \beta_b}{\alpha} (1 - \varepsilon \alpha^2)\right.\]

\[\beta_a \beta_b (\varepsilon (1 + \alpha) - \beta_a (1 + \varepsilon \alpha)) = \Gamma (\beta_a (1 + \alpha)(1 - \varepsilon \alpha + \alpha^2) - \beta_k (1 + \varepsilon \alpha)(1 - \alpha + \alpha^2)).\]

Recall that \(X\) is the set \(\{a_0, \ldots, a_{u-1}, b_0, \ldots, b_{v-1}\}\), where \(u, v \in \mathbb{N}\) are coprime, endowed with the height one order given by \(a_i < b_j\) for all \(i, j\). Recall also that \(\phi_r, \phi_t: X \to X\) are two commuting poset automorphisms and
\[
r_0: X \times X \to X \times X
\]
is the map defined by \(r_0(x, y) := (\phi_r(y), \phi_t(x))\). Let \(\sigma_a, \sigma_b, \tau_0 \text{ and } \tau_0\) be as at the beginning of the section. Recall finally that \((i, j) \mapsto (\omega_i(i), \omega_j(j))\) is an \(uv\)-cycle. Let
\[
r: D \otimes D \to D \otimes D
\]
be a non-degenerate coalgebra automorphism that has set-type square up to height 1 and induces \(r_0\) by restriction. By Theorem 2.6 and Propositions 4.5, 4.7 and 4.10 we know that \(r\) satisfies the braid equation if and only if conditions \(4.6\), \(4.20\), \(4.25\) and \(4.26\) are fulfilled. In Remark 4.6 we described the solutions of equation \(4.6\). In the sequel we are going to construct the families of solutions of the braid equation corresponding to each of the cases of that remark. By condition \(4.20\) we assume that \(\Gamma_{\sigma_i|\sigma_j|\sigma_i} = \Gamma \quad \text{for all } i, j, k \text{ and } l\).

\[\left.\Gamma_{\sigma_i|\sigma_j|\sigma_i} = \Gamma \quad \text{for all } i, j, k \text{ and } l\right.\]

\[\left.(\alpha^2 - 1) \Gamma = \frac{\beta_a \beta_b}{\alpha} (1 - \varepsilon \alpha^2)\right.\]

\[\beta_a \beta_b (\varepsilon (1 + \alpha) - \beta_a (1 + \varepsilon \alpha)) = \Gamma (\beta_a (1 + \alpha)(1 - \varepsilon \alpha + \alpha^2) - \beta_k (1 + \varepsilon \alpha)(1 - \alpha + \alpha^2)).\]

\[\left.\left(\alpha^2 - 1\right) \Gamma = \frac{\beta_a \beta_b}{\alpha} (1 - \varepsilon \alpha^2)\right.\]

\[\beta_a \beta_b (\varepsilon (1 + \alpha) - \beta_a (1 + \varepsilon \alpha)) = \Gamma (\beta_a (1 + \alpha)(1 - \varepsilon \alpha + \alpha^2) - \beta_k (1 + \varepsilon \alpha)(1 - \alpha + \alpha^2)).\]
1) If $\alpha = -1$, then $r$ satisfies the braid equation.

2) If $\alpha \neq -1$, then $r$ satisfies the braid equation if and only if $\Gamma = -\frac{\beta^2}{\alpha(1-\alpha)}$.

**Proof.** Because under the conditions of item 1), equations (4.25) and (4.26) are always satisfied, while under the conditions of item 2), equations (4.25) and (4.26) are satisfied if and only if $\Gamma = -\frac{\beta^2}{\alpha(1-\alpha)}$.

**Proposition 4.13.** Assume that $\varepsilon = -1$ and $\text{char } K \neq 2$.

1) If $\alpha = 1$ and $\beta_b = 0$, then $r$ satisfies the braid equation.

2) If $\alpha = -1$ and $\beta_b = 0$, then $r$ satisfies the braid equation.

3) If $\alpha^2 \neq 1$ and $\beta_b = \beta_a \frac{1+\alpha}{1-\alpha}$, then $r$ satisfies the braid equation if and only if $\Gamma = -\frac{\beta^2}{\alpha(1-\alpha)}$.

**Proof.** Clearly, under the conditions of item 1), equations (4.25) and (4.26) are always satisfied. Assume we are under the hypotheses of item 3). Then Equality (4.26) holds if and only if

$$\Gamma = -\frac{\beta_a \beta_b \frac{1+\alpha}{1-\alpha} - \beta^2}{\alpha(1-\alpha)} = -\frac{\beta^2(1+\alpha^2)}{\alpha(1-\alpha)^2}.$$ 

Replacing $\Gamma$ by this, $\beta_b$ by $\beta_b \frac{1+\alpha}{1-\alpha}$ and $\varepsilon$ by $-1$ in both sides of (4.26), we obtain

$$-2\beta^2(1+\alpha^2) \frac{1+\alpha}{1-\alpha} = -2\beta^2(1+\alpha^2) \frac{1+\alpha}{1-\alpha}.$$ 

Hence $r$ satisfies the braid equation if and only if $\Gamma = -\frac{\beta^2(1+\alpha^2)}{\alpha(1-\alpha)}$. 

For each $(a, b), (c, d), (e, f)$ and $(g, h)$ in $Y$, let $\lambda^{(1)}_{a|b|c|d}$ be as in formula (4.11).

**Theorem 4.14.** Let $r$ be as above of Proposition 4.11. If $r$ is a solution of the braid equation, then the possibly nonzero coefficients $\lambda^{(1)}_{a|b|c|d}$ depend on the parameters $\varepsilon, \alpha, \beta_a, \beta_b$ and $\Gamma$ via the following formulas, in which we use the notation (5.6):

$$\lambda^{(1)}_{a|b|c|d} = 1,$$

$$\lambda^{(1)}_{a|b|c|d} = \beta_a,$$

$$\lambda^{(1)}_{a|b|c|d} = \frac{\beta_a}{\alpha},$$

$$\lambda^{(1)}_{a|b|c|d} = -\beta_a,$$

$$\lambda^{(1)}_{a|b|c|d} = \frac{\beta_a}{\alpha},$$

$$\lambda^{(1)}_{a|b|c|d} = \alpha,$$

$$\lambda^{(1)}_{a|b|c|d} = \frac{1}{\alpha},$$

$$\lambda^{(1)}_{a|b|c|d} = \Gamma,$$

$$\lambda^{(1)}_{a|b|c|d} = -\varepsilon \beta_b,$$

$$\lambda^{(1)}_{a|b|c|d} = -\varepsilon \beta_b,$$
Proposition 4.11 is a solution of the braid equation that induces
\[ \lambda_{a_i|b_j|a_k|b_l}^{(1)b_i(1)b_j|a(1)|b(1)} = -\varepsilon\beta_a, \]
\[ \lambda_{a_i|b_j|a_k|b_l}^{(1)a(1)|b(1)|b_j} = \varepsilon\beta_a, \]
\[ \lambda_{a_i|b_j|a_k|b_l}^{(1)|b(1)|b_j} = \varepsilon. \]

Proof. The first equality holds because \( r \) induces \( r_0 \). The other formulas can be obtained using (4.2) and the discussion at the beginning of Section 4. Proposition 4.1 and equalities (4.5). We prove for example that
\[ \lambda_{a_i|b_j|a_k|b_l}^{(1)b_i(1)b_j|a(1)|b(1)} = -\varepsilon\beta_a, \] (4.27)
By (4.2) and the discussion at the beginning of Section 4 we know that
\[ \lambda_{a_i|b_j|a_k|b_l}^{(1)b_i(1)b_j|a(1)|b(1)} = \lambda_{a_i|a_j|a_k|b_l}^{(1)b_i(1)b_j|a(1)|b(1)} \lambda_{a_i|b_j|b_l}^{(1)b_i(1)b_j|a(1)|b(1)} = -\beta_a\alpha rb. \]
We finish the computation of the equality (4.27), noting that by Proposition 4.1 and equalities (4.5) we have \( \beta_a = \beta_a \) and \( \alpha rb = \alpha. \)

Collecting the results in this section we arrive at the following complete description:

Theorem 4.15. The non-degenerate coalgebra automorphism \( r \) introduced above Proposition 4.11 is a solution of the braid equation that induces \( r_0 \) on \( X \times X \) and has set-type square up to height 1 if and only if the parameters \( \varepsilon, \alpha, \beta_a, \beta_b \) and \( \Gamma \) belong to one of the families given in the following table:

| #  | Fixed values in each family | Dependent values in each family | Parameters | char \( K \) |
|----|----------------------------|-------------------------------|------------|------------|
| 1. | \( \varepsilon = 1, \alpha = 1 \) | \( \beta_b = \beta_a \) | \( \beta_a, \Gamma \in K \) | arbitrary |
| 2. | \( \varepsilon = 1, \alpha = 1 \) | \( \Gamma = -\beta_a\beta_b \) | \( \beta_a, \beta_b \in K \) | \( \text{char } K \neq 2 \) |
|    |                             | \( \beta_b \neq \beta_a \) |            |            |
| 3. | \( \varepsilon = 1, \alpha = 1 \) | \( \beta_a, \beta_b, \Gamma \in K \) | \( \beta_b \neq \beta_a \) | \( \text{char } K = 2 \) |
| 4. | \( \varepsilon = 1, \alpha = -1 \) | \( \beta_b = \beta_a \) | \( \beta_a, \Gamma \in K \) | arbitrary |
| 5. | \( \varepsilon = 1 \) | \( \beta_b = \beta_a \) | \( \beta_a \in K, \alpha \in K^\times \) | arbitrary |
|    |                             | \( \Gamma = -\frac{\beta^2}{\alpha} \) | \( \alpha^2 \neq 1 \) |            |
| 6. | \( \varepsilon = -1, \alpha = 1, \beta_b = 0 \) | \( \beta_b, \Gamma \in K \) | arbitrary |
| 7. | \( \varepsilon = -1, \alpha = -1, \beta_b = 0 \) | \( \beta_b, \Gamma \in K \) | arbitrary |
| 8. | \( \varepsilon = -1 \) | \( \beta_b = \beta_a \frac{1+\alpha}{1-\alpha} \) | \( \beta_a, \alpha \in K^\times \) | arbitrary |
|    |                             | \( \Gamma = -\frac{\beta^2(1+\alpha^2)}{\alpha(1-\alpha)^2} \) | \( \alpha^2 \neq 1 \) |            |

Proposition 4.16. The solutions \( r \) of Theorem 4.15 have set-type square if and only if either \( \varepsilon = -1 \) and \( 2\Gamma = 0 \) or \( \varepsilon = 1 \) and \( 2\Gamma = -\frac{2\beta_a\beta_b}{\alpha} \).
Proof. Let \( r \) be as Theorem 4.1.5. By Proposition 4.11 and equality (4.24), all the equations (3.19) reduce to
\[
2\Gamma = -\frac{\beta_a \beta_d}{\alpha} - \frac{\beta_b \beta_d}{\varepsilon \alpha}
\]
which is clearly equivalent to \( \varepsilon = -1 \) and \( 2\Gamma = 0 \) or \( \varepsilon = 1 \) and \( 2\Gamma = -\frac{2\beta_a \beta_d}{\alpha} \). By Proposition 3.14, this finishes the proof. \( \square \)

Examples 4.17. The hypothesis of Theorems 4.14 and 4.15 are satisfied for instance in the following cases

- \( u = 4, v = 1 \), \( \phi_r(a_i) = a_{i+1} \) and \( \phi_l(a_i) = a_{i+2} \), where the sums are taken modulo 4.
- \( u = 4, v = 1 \), \( \phi_r(a_i) = a_{i+1} \) and \( \phi_l(b_i) = b_{i+1} \), where the sums are taken modulo 4.
- Any example can be constructed in the following way: take an \( u \)-cycle of \( \mathbb{N}_u \) and a \( v \)-cycle of \( \mathbb{N}_v \), and then take \( \sigma_a \) and \( \tau_a \) any powers of \( \varsigma_a \) such that \( \sigma_a \circ \tau_a = \varsigma_a \), and take \( \sigma_b \) and \( \tau_b \) any powers of \( \varsigma_b \) such that \( \sigma_b \circ \tau_b = \varsigma_b \).

In fact, if \( \varsigma_a = \sigma_a \circ \tau_a = \tau_a \circ \sigma_a \) is an \( u \)-cycle, then \( \sigma_a \) and \( \tau_a \) are in the centralizer of \( \varsigma_a \) in \( S_u \) which is \( \langle \varsigma_a \rangle \). The same argument proves that both \( \sigma_b \) and \( \tau_b \) are powers of \( \varsigma_b \). For instance, one can take \( \sigma_a = \varsigma_a, \sigma_b = \varsigma_b, \tau_a = \text{Id} \) and \( \tau_b = \text{Id} \).

References

[1] Iván Angiono, César Galindo, and Leandro Vendramín, Hopf braces and Yang-Baxter operators (2016), to appear in Proc. Amer. Math. Soc., available at arXiv:1604.02098
[2] Ferran Cedó, Eric Jespers, and Ángel del Río, Involutive Yang-Baxter groups, Trans. Amer. Math. Soc. 362 (2010), no. 5, 2541–2558, DOI 10.1090/S0002-9947-09-04927-7. MR2584610
[3] Ferran Cedó, Eric Jespers, and Jan Okniński, Braces and the Yang-Baxter equation, Comm. Math. Phys. 327 (2014), no. 1, 101–116, DOI 10.1007/s00220-014-1935-y. MR3177933
[4] Patrick Dehornoy, A combinatorial approach to the set-theoretic solutions of the Yang-Baxter equation, Adv. Math. 224 (2010), no. 6, 2472–2484, DOI 10.1016/j.aim.2010.02.001. MR2652212
[5] V. G. Drinfel’d, On some unsolved problems in quantum group theory, Quantum groups (Leningrad, 1990), Lecture Notes in Math., vol. 1510, Springer, Berlin, 1992, pp. 1–8. MR1183474
[6] Pavel Etingof, Travis Schedler, and Alexandre Soloviev, Set-theoretical solutions of the Yang-Baxter equation, RC-calculus, and Garside germs, Adv. Math. 282 (2015), 93–127, DOI 10.1016/j.aim.2015.05.008. MR3374524
[7] Tatiana Gateva-Ivanova and Michel Van den Bergh, Semigroups of I-type, J. Algebra 206 (1998), no. 1, 97–112, DOI 10.1006/jabr.1997.7399. MR1637256
[8] Tatiana Gateva-Ivanova, A combinatorial approach to the set-theoretic solutions of the Yang-Baxter equation, J. Math. Phys. 45 (2004), no. 10, 3828–3858, DOI 10.1063/1.1788848. MR2095675
[9] Jorge Alberto Guccione, Juan José Guccione, and Christian Valqui, Solutions of the braid equation and orders, Algebr Represent Theor (2018).
[10] Jiang-Hua Lu, Min Yan, and Yong-Chang Zhu, On the set-theoretical Yang-Baxter equation, Duke Math. J. 104 (2000), no. 1, 1–18, DOI 10.1215/S0012-7094-00-10411-5. MR1769723
[11] Wolfgang Rump, A decomposition theorem for square-free unitary solutions of the quantum Yang-Baxter equation, Adv. Math. 193 (2005), no. 1, 40–55, DOI 10.1016/j.aim.2004.03.019. MR2132760
[12] Alexander Soloviev, Non-unitary set-theoretical solutions to the quantum Yang-Baxter equation, Math. Res. Lett. 7 (2000), no. 5-6, 577–596, DOI 10.4310/MRL.2000.v7.n5.a4. MR1809984
[13] Mitsurio Takeuchi, Survey on matched pairs of groups—an elementary approach to the ESS-LYZ theory, Noncommutative geometry and quantum groups (Warsaw, 2001), Banach Center Publ., vol. 61, Polish Acad. Sci. Inst. Math., Warsaw, 2003, pp. 305–331. MR2024436
