EXTINCTION OF MULTIPLE SHOCKS
IN THE MODULAR BURGERS EQUATION

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Abstract. A traveling viscous shock was previously studied in the Burgers equation with the modular advection term. It was shown that small, smooth, and exponentially decaying in space perturbations to the viscous shock decay in time. The present work addresses multiple shocks of the same model. We first prove that no traveling viscous shocks with multiple interfaces exist. We then suggest with the help of a priori energy estimates and numerical simulations that the evolution of viscous shocks with multiple interfaces leads to the finite-time extinction of compact regions between two consequent interfaces. We specify a precise scaling law of the finite-time extinction supported by the interface equations and by numerical simulations.

1. Introduction

The present work addresses the modular Burgers equation

$$\frac{\partial u}{\partial t} = \frac{\partial |u|}{\partial x} + \frac{\partial^2 u}{\partial x^2},$$

(1.1)

which is different from the classical Burgers equation by the modular advection term. This model was previously used to describe inelastic dynamics of particles with piecewise interaction potentials [4, 13]. Generalizations of this model with additional terms were also discussed in [10, 11, 12].

Some preliminary results were obtained for the modular Burgers equation (1.1) both analytically and numerically. Traveling wave solutions were constructed in [8, 9] from differential equations by matching solutions of linear equations with a suitable condition at the interface where the modular nonlinearity jumps. Collisions of compactly supported pulses and dynamics near the single viscous shock were considered in [4] by using qualitative approximations. Numerical approximations of time-dependent solutions to the modular Burgers equation (1.1) were constructed in [7] with the Fourier sine series.

Asymptotic stability of a single viscous shock was proven in [5] by considering dynamics of small perturbations in exponentially weighted spaces. It was shown that the evolution of such perturbations is well defined on both sides of the interface and the perturbations decay in time. A finite-difference numerical method which incorporates the nonlinear dynamics of the interface to the linear dynamics of the advection–diffusion equation was implemented in [5] to illustrate the asymptotic stability of a single viscous shock.

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The purpose of this work is to study multiple viscous shocks of the modular Burgers equation (1.1). We prove by using ODE methods that there exist no traveling solutions with multiple interfaces where the modular nonlinearity jumps. Energy estimates show that the compact areas between interfaces disappear in the time evolution of the model but do not suggest whether it happens in a finite or infinite time. We elaborate the finite-difference numerical method to discover that the two consequent interfaces coalesce in a finite time. Postprocessing data analysis suggests a precise scaling law of the finite-time extinction which agrees with the nonlinear equation for the interface position. These results open a road for future work to prove the finite-time extinction and the scaling law analytically.

We note that although the finite-difference method is rather elementary, it allows us to capture the main feature of the dynamics of the modular Burgers equation (1.1), where the linear equations between interfaces are coupled together by the nonlinear interface equations. It is unclear how else the numerical modelling of the time evolution can be performed due to the singular contribution of the modular nonlinearity unless it is replaced by a smooth approximation.

Before closing the introduction, we mention some contemporary work on other related problems. A diffusion equation with piecewise defined nonlinearity, namely, the KPP model with the cutoff reaction rate, was studied in [14, 15], where matched asymptotic expansions in the dynamically moving coordinate frame have been used both for the existence and the asymptotic stability of traveling viscous shocks. Metastable N-waves of the classical Burgers equation were studied in [2, 6] by using dynamical system methods.

The ultimate goal of these studies is to understand dynamics of the logarithmic Burgers equations [3], which commonly arises in modeling of granular chains in viscous systems. The logarithmic nonlinearity is more singular than the modular nonlinearity, hence it presents further challenges in the analysis of the travelling viscous shocks. Compared to the logarithmic Burgers equation, the logarithmic diffusion equation has been analyzed in many details [1].

The paper is organized as follows. Section 2 presents the existence theory for traveling viscous shocks with multiple interfaces. Section 3 contains a priori energy estimates for the modular Burgers equation. Section 4 gives details of the finite-difference method. Section 5 presents outcomes of numerical simulations and the precise scaling law of the finite-time extinction of multiple shocks. Section 6 concludes the paper with discussion of open problems.

2. Traveling waves

Here we prove that the only nonzero bounded traveling wave solution of the modular Burgers equation (1.1) is the single viscous shock constructed in [8, 9] and [5]. Traveling waves \( u(t, x) = U_c(x - ct) \) are obtained from the second-order differential equation

\[
U''_c(x) + (|U_c|)'(x) + cU'_c(x) = 0, \quad x \in \mathbb{R}.
\] (2.1)
If $\pm U_c(x) > 0$ for every $x \in \mathbb{R}$, the differential equation (2.1) is linear and no bounded solutions exist. Therefore, if the bounded solutions exist, then there exists at least one $x_0 \in \mathbb{R}$ for which $U_c(x_0) = 0$.

By the order of the differential equation (2.1), we consider piecewise $C^2$ functions satisfying the following jump condition at every single viscous shock constructed in [8, 9] and [5] and reproduced here. 

$$[U''_c]_\pm(x_0) := U''_c(x_0 + 0) - U''_c(x_0 - 0) = -2|U'_c(x_0)|. \tag{2.2}$$

The jump condition (2.2) follows from the jump of the modular nonlinearity in (2.1) both for $U'_c(x_0) > 0$ (when $U_c(x) > 0$ for $x > x_0$) and $U'_c(x_0) < 0$ (when $U_c(x) < 0$ for $x > x_0$). If $U'_c(x_0) = 0$, then uniqueness of the zero solution both for $x > x_0$ and $x < x_0$ implies that $U_c(x) = 0$ for every $x \in \mathbb{R}$. Hence it suffices to consider a simple zero of $U_c$ at $x = x_0$.

Since $U_c \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, the first integration of the second-order equation (2.1) yields

$$U'_c(x) + |U_c(x)| + cU_c(x) = d, \quad x \in \mathbb{R}. \tag{2.3}$$

where the constant $d$ is identical to the left and to the right of the point $x_0$ with $d = U'_c(x_0)$.

Let $U_\pm := \lim_{x \to \pm \infty} U_c(x)$ for $U_c \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and assume first that $U_+U_- > 0$. Since $d$ does not change and $\lim_{x \to \pm \infty} U'_c(x) = 0$, it follows from (2.3) that $U_+ = U_-$. Assuming that there exist $x_- < x_+$ such that $U_c(x)$ has no zeros in $(-\infty, x_-) \cup (x_+, \infty)$ and attains $U_c(x_-) = U_c(x_+) = 0$ leads immediately to a contradiction. Indeed, we have $d = (c + \text{sgn}(U_\pm))U_\pm$ and

$$U'_c(x) = (c + \text{sgn}(U_\pm))(U_\pm - U_c(x)), \quad x \in (-\infty, x_-) \cup (x_+, \infty),$$

so that the sign of $U'_c$ on $(-\infty, x_-) \cup (x_+, \infty)$ coincides with the sign of $d$. Therefore, it has the same monotonicity on $(-\infty, x_-)$ and $(x_+, \infty)$ and it cannot approach to $U_-$ and $U_+$ from the same side both as $x \to -\infty$ and $x \to +\infty$.

The case $U_+U_- = 0$ is also impossible since $d = 0$ and $U'_c(x) = -(c + \text{sgn}(U_c))U_c$ has only exponentially growing or decaying solutions which do not admit points $x_0 \in \mathbb{R}$ for which $U_c(x_0) = 0$.

Thus, nonzero bounded solutions may only exist for $U_+U_- < 0$. We claim that no compact interval $[x_-, x_+]$ with $-\infty < x_- < x_+ < \infty$ may exist. Indeed, since the sign of $U_c$ is definite on $(x_-, x_+)$, we have

$$U'_c(x) = d - (c + \text{sgn}(U_c))U_c(x), \quad x \in (x_-, x_+),$$

which shows that the turning point $x_* \in (x_-, x_+)$, where $U'_c(x_*) = 0$, exists only if $c + \text{sgn}(U_c) \neq 0$ and corresponds to the equilibrium point $U_* = \frac{d}{c + \text{sgn}(U_c)}$. However, solutions of the linear first-order differential equation cannot reach the equilibrium point in finite $x$ so that either $x_- = -\infty$ or $x_+ = \infty$.

We conclude that nonzero bounded solutions may only exist for $U_+U_- < 0$ and with a unique $x_0 \in \mathbb{R}$ such that $U_c(x)$ has different signs for $x < x_0$ and $x > x_0$. This is the single viscous shock constructed in [8, 9] and [5] and reproduced here.

Let us first take $U_- < 0 < U_+$. Since $d$ does not change across $x = x_0$, we have

$$d = (c + 1)U_+ = (c - 1)U_- \Rightarrow c = -\frac{U_+ + U_-}{U_+ - U_-}. \tag{2.4}$$
so that the speed of the viscous shock \( c \in (-1, 1) \) is uniquely determined in terms of \( U_\pm \) and \( U_\perp \). Solving (2.3) for \( x > x_0 \) and \( x < x_0 \) with \( U_c(x_0) = 0 \) yields the unique expression

\[
U_c(x) = \begin{cases} 
U_+(1 - e^{(c+1)(x_0-x)}), & x > x_0, \\
U_-(1 - e^{(1-c)(x-x_0)}), & x < x_0,
\end{cases}
\]

(2.5)

where the boundary conditions \( U_\pm = \lim \limits_{x \to \pm \infty} U_c(x) \) are satisfied and \( U_c \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \).

The jump condition (2.2) yields

\[
U_-(1-c)^2 - U_+(1+c)^2 = -2U_+(c+1),
\]

which is trivially satisfied in view of (2.4). This construction yields a viscous shock with the profile \( U_c \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) and a unique \( c \in (-1, 1) \).

Let us now rule out the case \( U_+ < 0 < U_- \). Since \( d \) does not change across \( x = x_0 \), we have

\[
d = (c-1)U_+ = (c+1)U_- \quad \Rightarrow \quad c = -\frac{U_+ + U_-}{U_- - U_+},
\]

so that \( c \in (-1, 1) \). Solving (2.3) for \( x > x_0 \) and \( x < x_0 \) yields with \( U_c(x_0) = 0 \) the unique expression

\[
U_c(x) = \begin{cases} 
U_+(1 - e^{(c-1)(x_0-x)}), & x > x_0, \\
U_-(1 - e^{-(c+1)(x-x_0)}), & x < x_0,
\end{cases}
\]

which diverges as \( x \to \pm \infty \) since \( c \in (-1, 1) \).

We conclude that the viscous shock exists only for \( U_- < 0 < U_+ \) with the unique speed \( c \in (-1, 1) \) given by (2.4) and the unique profile \( U_c \) given by (2.5). This viscous shock is the only traveling wave solution of the modular Burgers equation (1.1) with nonzero profile \( U_c \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \).

3. A PRIORI ENERGY ESTIMATES

Here we use energy estimates to show that a compact region between two consequent interfaces shrinks and eventually disappears in the time evolution of the modular Burgers equation (1.1).

We assume that the modular Burgers equation (1.1) admits a solution \( u \in C^1((0, T) \times \mathbb{R}) \) with piecewisely continuous \( u_{xx} \) for some \( T > 0 \) satisfying \( u(t, \xi_1(t)) = u(t, \xi_2(t)) = 0 \) for some \( -\infty < \xi_1(t) < \xi_2(t) < \infty \) and \( t \in (0, T) \). Without loss of generality, we can consider \( u(t, x) > 0 \) for \( \xi_1(t) < x < \xi_2(t) \). Combining assumptions together yields the following linear evolutionary boundary-value problem:

\[
\begin{cases}
  u_t = u_x + u_{xx}, & \xi_1(t) < x < \xi_2(t), \quad 0 < t < T, \\
  u(t, \xi_1(t)) = 0, & 0 < t < T, \\
  u(t, \xi_2(t)) = 0, & 0 < t < T.
\end{cases}
\]

(3.1)

The linear problem (3.1) is not closed as we need to find the evolution of \( \xi_{1,2}(t) \) from the boundary conditions at \( x = \xi_{1,2}(t) \) and the evolutionary boundary-value problems satisfied by \( u(t, x) \) for \( x < \xi_1(t) \) and \( x > \xi_2(t) \). At each interface \( x = \xi_{1,2}(t) \), additional two boundary conditions are needed. In the class of piecewise \( C^2 \) functions, these two
conditions are given by the continuity of \( u_x(t, x) \) across \( x = \xi_{1,2} \) and by the jump conditions for \( u_{xx}(t, x) \):

\[
[u_x]^+(t, \xi_{1,2}(t)) = 0, \quad [u_{xx}]^+(t, \xi_{1,2}(t)) = -2\{u_x(t, \xi_{1,2}(t))\}, \quad 0 < t < T. \tag{3.2}
\]

A priori energy estimates are derived from the linear boundary-value problem (3.1) by ignoring the global information from other boundary conditions (3.2). Consequently, the time evolution of \( \xi_{1,2}(t) \) is not relevant for energy estimates.

Integrating (3.1) on \([\xi_1(t), \xi_2(t)]\) yields

\[
\frac{d}{dt} \int_{\xi_1(t)}^{\xi_2(t)} u(t, x) \, dx = u_x(t, \xi_2(t)) - u_x(t, \xi_1(t)) \leq 0, \tag{3.3}
\]

where we have used the conditions \( u_x(t, \xi_2(t)) \leq 0 \) and \( u_x(t, \xi_1(t)) \geq 0 \), which follow from the assumption that \( u(t, x) > 0 \) for \( \xi_1(t) < x < \xi_2(t) \). Hence, the positive mass \( \int_{\xi_1(t)}^{\xi_2(t)} u(t, x) \, dx \) is monotonically decreasing as a function of \( t \) as long as the slopes at the end points of the compact region are nonzero.

Integrating (3.1) multiplied by \( u \) on \([\xi_1(t), \xi_2(t)]\) yields

\[
\frac{d}{dt} \int_{\xi_1(t)}^{\xi_2(t)} u^2(t, x) \, dx = -2 \int_{\xi_1(t)}^{\xi_2(t)} u_x^2(t, x) \, dx \leq 0. \tag{3.4}
\]

Hence, the positive energy \( \int_{\xi_1(t)}^{\xi_2(t)} u^2(t, x) \, dx \) is monotonically decreasing as a function of \( t \) as long as \( \xi_1(t) < \xi_2(t) \).

Energy estimates involving spatial derivatives of \( u(t, x) \) can not be derived for the linear boundary-value problem (3.1) because of the lack of information on the derivatives of \( u(t, x) \) in \( x = \xi_{1,2}(t) \).

The two estimates (3.3) and (3.4) suggest that no new compact regions may be formed dynamically in time since the mass and energy of the compact region with positive \( u(t, x) \) can not increase from zero to positive values. However, the argument does not clarify if the mass and energy extinguish in finite or infinite times or if the two interface \( \xi_{1,2}(t) \) coalesce when the mass and energy vanish. We will answer these questions with numerical experiments.

### 4. Finite-difference method

Here we consider the simplest problem with the multiple shock wave. Since the modular Burgers equation (1.1) preserve odd functions in the time evolution, we restrict solutions to the class of odd functions \( u(t, -x) = -u(t, x) \) closed on \((0, \infty)\) subject to Dirichlet condition at \( x = 0 \) and the normalized boundary condition \( u(t, x) \to 1 \) as \( x \to +\infty \). We will assume that there exists a single interface at \( x = \xi(t) \in (0, \infty) \) and will simulate its dynamics. Due to the oddness condition, the multiple shock wave consists of three interfaces at \( x = \pm \xi(t) \) and \( x = 0 \).
The mathematical formulation of the evolutionary boundary-value problem is given by
\[
\begin{aligned}
\begin{cases}
    u_t = -u_x + u_{xx}, & u(t, x) < 0, \quad 0 < x < \xi(t), \\
    u_t = u_x + u_{xx}, & u(t, x) > 0, \quad \xi(t) < x < \infty, \\
    u(t, 0) = 0, & u(t, \xi(t)) = 0, \quad \lim_{x \to +\infty} u(t, x) = 1,
\end{cases}
\end{aligned}
\tag{4.1}
\]
where \(u(t, \cdot)\) is a piecewise \(C^2\) functions satisfying the interface conditions
\[
u_{xx}(t, +0) = u_x(t, 0), \quad [u_{xx}]^+(t, \xi(t)) = -2u_x(t, \xi(t)). \tag{4.2}
\]
The first interface condition \((4.2)\) is consistent with the Dirichlet condition \(u(t, 0) = 0\) and the evolution system in \((4.1)\). The second interface condition \((4.2)\) in combination with the Dirichlet condition \(u(t, \xi(t)) = 0\) and the evolution system in \((4.1)\) can be rewritten as the differential equation on \(\xi(t)\):
\[
\xi'(t) = -1 - \frac{u_{xx}(t, \xi(t) + 0)}{u_x(t, \xi(t))} = +1 - \frac{u_{xx}(t, \xi(t) - 0)}{u_x(t, \xi(t))}, \tag{4.3}
\]
provided that \(u_x(t, \xi(t)) > 0\).

To implement the finite-difference method, we assume that \(\xi(t) > 0\) and use \(y := x/\xi(t)\). This transformation scales the domain of the boundary-value problem \((4.1)\) to the time-independent regions \((0, 1)\) and \((1, \infty)\). Abusing notations, we rewrite the evolutionary boundary-value problem for \(u = u(t, y)\) in the form:
\[
\begin{aligned}
\begin{cases}
    u_t = \xi^{-1}(\xi'y - 1)u_y + \xi^{-2}u_{yy}, & u(t, y) < 0, \quad 0 < y < 1, \\
    u_t = \xi^{-1}(\xi'y + 1)u_y + \xi^{-2}u_{yy}, & u(t, y) > 0, \quad 1 < y < \infty, \\
    u(t, 0) = 0, & u(t, 1) = 0, \quad \lim_{y \to +\infty} u(t, y) = 1,
\end{cases}
\end{aligned}
\tag{4.4}
\]
whereas the interface equation \((4.3)\) is rewritten in the form
\[
\xi'(t) = -1 - \frac{u_{yy}(t, 1 + 0)}{\xi(t)u_y(t, 1)} = +1 - \frac{u_{yy}(t, 1 - 0)}{\xi(t)u_y(t, 1)}. \tag{4.5}
\]

By using an equally spaced grid with the step size \(h\) on \([0, 1]\) and \([1, L]\) for sufficiently large \(L\), we replace the first and second spatial derivatives in \((4.4)\) by the central differences. We can do this for every interior point of the grid since there are no evolution equations at \(y = 0\) and \(y = 1\) due to Dirichlet conditions. Neumann condition \(u_{yy}(t, L) = 0\) is used at \(y = L\). It remains to derive discretization of the interface condition \((4.5)\).

To couple the solutions on \([0, 1]\) and \([1, L]\), we use the central difference approximation of the first and second spatial derivatives at \(y = 1\) in \((4.3)\). This can only be done if additional grid points are added to the left and to the right of the interface point \(y = 1\). In other words, we augment \(\{u_k\}_{k=0}^{k=N}\) for \(y_k = hk\) with \(v_{N+1}\) for \(y_{N+1} = 1 + h\) and \(\{u_k\}_{k=M}^{k=M}\) for \(y_k = hk\) with \(v_{N-1}\) for \(y_{N-1} = 1 - h\), where \(h = \frac{1}{N} = \frac{L}{M}\), \(u_0 = u_N = 0\) due to the Dirichlet conditions at \(y = 0\) and \(y = 1\). For the Neumann condition at \(y = L\), we use an additional grid point at \(y_{M+1} = L + h\) with \(u_{M+1} = u_{M-1}\).
The continuity of \( u_y(t, y) \) and the jump of \( u_{yy}(t, y) \) across \( y = 1 \) are expressed in the central difference approximation by the linear equation

\[
\frac{v_{N+1} - u_{N-1}}{2h} - \frac{v_{N+1} + u_{N-1}}{h^2} = -2\xi \frac{v_{N+1} - u_{N-1}}{2h}.
\]

These linear equations admit the unique solution for the additional variables \( v_{N+1} \) and \( v_{N-1} \) given by

\[
v_{N+1} = \frac{2u_{N+1} - h\xi u_{N-1}}{2 - h\xi}, \quad v_{N-1} = \frac{2u_{N-1} - h\xi u_{N+1}}{2 - h\xi},
\]

where \( h\xi(t) < 2 \) is assumed by smallness of \( h \). Substituting these solutions to the central difference approximation of the interface equation (4.5) yields the approximation

\[
\xi'(t) = -\frac{(2 - h\xi)(u_{N+1} + u_{N-1})}{h\xi(u_{N+1} - u_{N-1})}.
\]

The time evolution of the linear system (4.4) was approximated by the implicit Crank–Nicolson method based on the trapezoidal rule of numerical integration. The Crank–Nicolson method is unconditionally stable for the linear advection–diffusion equations. However, the stability of iterations was affected by the approximation (4.6) since \( \xi(t) \) and \( \xi'(t) \) were used in the evolutionary system (4.4) explicitly based on the predictor–corrector pair (with \( \xi(t) \) obtained from \( \xi'(t) \) by using Heun’s method).

It remains to provide the initial data \( u_0(x) := u(0, x) \) which would be consistent with the interface conditions (4.5). Without loss of generality, we assume \( \xi(0) = 1 \) so that \( y = x \) at \( t = 0 \). For faster convergence of \( u_0(x) \rightarrow 1 \) as \( x \rightarrow +\infty \), we consider the Gaussian function on \((1, \infty)\) concatenated with a quartic polynomial on \((0, 1)\):

\[
u_0(x) = \begin{cases} 
  x(1 - x)(ax^2 + bx + c), & 0 < x < 1, \\
  1 - e^{-\alpha(x^2 - 1)}, & 1 < x < \infty,
\end{cases}
\]

where the boundary conditions \( u_0(0) = u_0(1) = 0 \) and \( \lim_{x \to +\infty} u_0(x) = 1 \) have been satisfied. Parameters \( a, b, \) and \( c \) are found uniquely in terms of \( \alpha \) by using the interface conditions (4.2). The condition \( u_0''(0) = u_0'(0) \) yields \( 2b = 3c \). The condition \( u_0'(1 + 0) = u_0'(1 - 0) \) yields \( a + b + c = -2\alpha \). Finally, the condition \( u_0''(1 + 0) - u_0''(1 - 0) = -2u_0'(1) \) yields \( 2a + b = 2\alpha^2 - \alpha \). Solving all three conditions together, we obtain

\[
a = \frac{\alpha(10\alpha + 1)}{7}, \quad b = -\frac{3\alpha(2\alpha + 3)}{7}, \quad c = -\frac{2\alpha(2\alpha + 3)}{7},
\]

which complete the construction of the initial condition for an arbitrary \( \alpha > 0 \). Since \( \xi'(0) = 2(\alpha - 1) \), the interface expands initially if \( \alpha > 1 \) and contracts initially if \( \alpha < 1 \). Outcomes of numerical simulations of the central difference method with the initial data (4.7) are summarized in the next section.
5. Scaling law of the finite-time extinction

We performed iterations on the domain $[0, 10]$ discretized with the step size $h = 0.02$. The time step was selected at $\tau = 0.0001$ to provide better accuracy in the evolution of the interface $\xi(t)$ within the finite-difference approximation (4.6). Nevertheless, the accuracy was decreasing when $\xi(t)$ and $u_y(t, 1)$ were getting smaller and iterations eventually broke up and stopped before $\xi(t)$ could reach 0. It was partly related to the fact that the Neumann condition at the end point $L = 10$ has been preserving the initial value $u_0(L) \approx 1$ for a while, after which the value of $u(t, L)$ started to decrease during the extinction stage.

Figure 5.1 shows outcomes of numerical simulations with the initial data (4.7) for $\alpha = 1.5$ (top) and $\alpha = 0.5$ (bottom). The left panels show the profile $u(t, y)$ for $y > 0$ and two values of time: $t = 0$ (dashed line) and $t = T$ (solid line), where $T = 0.25$ for $\alpha = 1.5$ and $T = 0.15$ for $\alpha = 0.5$. The right panels show the numerically computed evolution of the interface $\xi(t)$ versus $t$. The numerical pictures suggest extinction in a finite time, for which $\xi(t) \to 0$ as $t \to t_0$ with $t_0 < \infty$. The evolution of $\xi(t)$ is non-monotone for $\alpha = 1.5$ and monotone for $\alpha = 0.5$.

Figure 5.1. Evolution of the boundary-value problem (4.4) from the initial data (4.7) for $\alpha = 1.5$ (top) and $\alpha = 0.5$ (bottom).
Performing computations for longer times with these and other values of $\alpha$ shows that in all cases there exists $t_0 \in (0, \infty)$ such that

$$\xi(t) \to 0, \quad u_x(t, \xi(t)) \to 0, \quad u_{xx}(t, \xi(t) \pm 0) \to 0,$$

as $t \to t_0$,

where the spatial derivatives were computed in the original variable $x$ by using the chain rule and the numerical approximations:

$$u_x(t, \xi(t)) = \frac{u_{N+1} - u_{N-1}}{h\xi(t)(2 - h\xi(t))}, \quad u_{xx}(t, \xi(t) - 0) = \frac{2u_{N+1} + u_{N-1}(1 - h\xi(t))}{h^2\xi^2(t)(2 - h\xi(t))}. \quad (5.1)$$

We claim based on the postprocessing data analysis that the following scaling law of extinction holds as $t \to t_0$:

$$\xi(t) \sim \sqrt{t_0 - t}, \quad u_x(t, \xi(t)) \sim (t_0 - t), \quad u_{xx}(t, \xi(t) - 0) \sim \sqrt{t_0 - t}. \quad (5.2)$$

This scaling law is in agreement with the interface equation (4.3), which suggests that $\xi'(t)$ diverges as $t \to t_0$:

$$\xi'(t) \sim -\frac{1}{\sqrt{t_0 - t}}. \quad (5.2)$$

For postprocessing data analysis, we use the linear regression in the log-log variable, e.g.

$$\log(\xi(t)) \quad \text{versus} \quad c_1 \log(t_0 - t) + c_2, \quad (5.3)$$

where the coefficient $c_1$ determines the power of the scaling law (5.2). The only obstacle with this method is that the value of $t_0$ is unknown and cannot be approximated well because the iterations break down at time $t_1$ when $\xi(t_1) \approx 0.3$.

![Figure 5.2](attachment:figure52.png)

**Figure 5.2.** Power of the linear regression (left) and the approximation error (right) versus $t_0$ for (5.3) from the initial data (4.7) with $\alpha = 0.1$.

To deal with this numerical issue, we introduce a grid of values of $t_0$ past the terminal time $T$ and use the linear regression (5.3) with $t_0$-dependent values of $c_1$ and the approximation error. The outcomes of these computations for $\alpha = 0.1$ are shown on Figure 5.2, where the left panel shows the coefficient $c_1$ versus $t_0$ and the right panel shows the
corresponding approximation error versus \( t_0 \). The minimum error of the size \( 10^{-9} \) is attained at \( t_0 = 0.1738 \) and this value of \( t_0 \) corresponds to \( c_1 = 0.4917 \), which is close to the claimed value \( \frac{1}{2} \) in (5.2).

Using similar ideas for \( u_x(t, \xi(t)) \) and \( u_{xx}(t, \xi(t) - 0) \), we have found that the minimal approximation errors of the size \( 10^{-9} \) and \( 10^{-6} \) correspond to \( t_0 = 0.1750 \) and \( t_0 = 0.1675 \) respectively. The corresponding coefficients for the power are \( c_1 = 1.0125 \) and \( c_1 = 0.4503 \), which are close to the claimed values \( 1 \) and \( \frac{1}{2} \) in (5.2). It is not surprising that the approximation error for the second derivative \( u_{xx}(t, \xi(t) - 0) \) is significantly larger than that for the first derivative \( u_x(t, \xi(t)) \) since we use the central difference approximations. Consequently, the coefficient \( c_1 = 0.4503 \) deviates from \( \frac{1}{2} \) more significantly than the coefficient \( c_1 = 1.0125 \) deviates from 1.

The accuracy is lower for larger values of \( \alpha \) in the initial data (4.7). For instance, computations at \( \alpha = 0.5 \) shows that the linear regression (5.3) gives the best approximation result at \( t_0 = 0.2008 \) with the error at the level of \( 10^{-6} \). The coefficient \( c_1 = 0.4510 \) corresponds to the power \( \frac{1}{2} \) worse than in the case of \( \alpha = 0.1 \). Similar discrepancy was found for \( u_x(t, \xi(t)) \) with the corresponding approximation of \( c_1 = 1.0609 \) instead of 1. It was surprising, however, that the accuracy of computations for \( u_{xx}(t, \xi(t) - 0) \) was comparable between the cases \( \alpha = 0.1 \) and \( \alpha = 0.5 \). The minimal error was found in the latter case at the level of \( 10^{-6} \) and the coefficient \( c_1 = 0.4589 \) instead of \( \frac{1}{2} \) deviates comparably from the former case.

![Figure 5.3](image)

**Figure 5.3.** Mass (left) and energy (right) versus \( t \) for the time evolution from the initial data (4.7) with \( \alpha = 0.1 \).

We have also computed the numerical approximations for the mass and energy integrals for the compact area on \([0, \xi(t)]\). After the change of variables, these quantities are given by

\[
M(t) := \xi(t) \int_0^1 u(t, y) dy, \quad E(t) := \xi(t) \int_0^1 u^2(t, y) dy.
\]  

(5.4)

Figure 5.3 shows evolution of the mass and energy integrals versus \( t \) for the initial data (1.7) with \( \alpha = 0.1 \). The numerically detected best power fits suggest that

\[
|M(t)| \sim (t_0 - t)^2, \quad E(t) \sim \sqrt{(t_0 - t)^2},
\]  

(5.5)
which are also in agreement with the balance equations \((3.3)\) and \((3.4)\) under the scaling laws \((5.2)\).

6. Conclusion

To summarize, we have shown analytically and numerically that viscous shocks of the modular Burgers equations with multiple interfaces cannot propagate with a common speed and their dynamics lead to the finite-time extinction of compact regions by means of coalescence of two consequent interfaces. We have specified the very precise scaling laws for the finite-time extinction based on numerical simulations with the central difference method adopted to deal with the nonlinear interface equations.

We end this paper with the list of open questions. Further work is needed to illustrate universality of the scaling laws and to prove them analytically in some specific settings. One can consider extensions of these results to the modular Burgers equation with additional terms and to the logarithmic Burgers equation. Dynamics of solitary waves in the modular Korteweg–de Vries equation and other Hamiltonian systems with modular nonlinearity have not been investigated so far and can be an attractive subject.

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