A Faster FPTAS for Knapsack Problem With Cardinality Constraint

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Abstract

We study the $K$-item knapsack problem (i.e., 1.5-dimensional knapsack problem), which is a generalization of the famous 0-1 knapsack problem (i.e., 1-dimensional knapsack problem) in which an upper bound $K$ is imposed on the number of items selected. This problem is of fundamental importance and is known to have a broad range of applications in various fields such as computer science and operation research. It is well known that, there is no fully polynomial time approximation scheme (FPTAS) for the $d$-dimensional knapsack problem when $d \geq 2$, unless $P = NP$. While the $K$-item knapsack problem is known to admit an FPTAS, the complexity of all existing FPTASs have a high dependency on the cardinality bound $K$ and approximation error $\varepsilon$, which could result in inefficiencies especially when $K$ and $\varepsilon^{-1}$ increase. The current best results are due to [Mastrolilli and Hutter, 2006], in which two schemes are presented exhibiting a space-time tradeoff—one scheme with time complexity $O(n + K\varepsilon^2/\varepsilon^2)$ and space complexity $O(n + \varepsilon^3/\varepsilon)$, while another scheme requires a run-time of $O(n + (K\varepsilon^2 + \varepsilon^3)/\varepsilon^2)$ but only needs $O(n + \varepsilon^2/\varepsilon)$ space, where $z = \min\{K, 1/\varepsilon\}$.

In this paper we close the space-time tradeoff exhibited in [Mastrolilli and Hutter, 2006] by designing a new FPTAS with a running time of $\tilde{O}(n + z^2/\varepsilon^2)$, while simultaneously reaching the $O(n + z^2/\varepsilon)$ space complexity.\footnote{$\tilde{O}$ notation hides terms poly-logarithmic in $n$ and $1/\varepsilon$.} Our scheme provides $\tilde{O}(K)$ and $O(z)$ improvements on the long-established state-of-the-art algorithms in time and space complexity respectively, and is the first scheme that achieves a running time that is asymptotically independent of the cardinality bound $K$ under fixed $\varepsilon$. Another salient feature of our algorithm is that it is the first FPTAS, which achieves better time and space complexity bounds than the very first standard FPTAS over all parameter regimes.
1 Introduction

The famous 0-1 Knapsack Problem (0-1 KP), also known as the Binary Knapsack Problem (BKP), is a classical combinatorial optimization problem that has been studied for more than a century, which often arises when there are resources to be allocated within a budget. The 0-1 knapsack problem can be also viewed as the most fundamental non-trivial Integer Linear Programming (ILP) problem, and can be formally formulated as follows:

\[
\begin{align*}
\text{max} & \quad \sum_{i \in E} p_i x_i, \\
\text{s.t.} & \quad \sum_{i \in E} w_i x_i \leq W \quad \text{and} \quad x_i \in \{0, 1\}.
\end{align*}
\]

Throughout this paper, set \(E = [n]\) denotes the ground set\(^2\) which includes all possible items. The value and size of each item \(i\) is called profit \((p_i)\) and weight \((w_i)\) respectively. Our goal is to make a binary choice for each item \(i\) to maximize the overall profit subject to a budget constraint \(W\). Beyond this basic binary knapsack model, there are a number of interesting practical extensions and variations [Kellerer et al., 2003].

In this paper, we study the \(K\)-item Knapsack Problem (K KP), a well known generalization of the famous 0-1 KP that can be formulated as (1)-(2) with the additional constraint \(\sum_{i \in E} x_i \leq K\), which means that the number of items in any feasible solutions is upper bounded by \(K\). The KKP can be cast as a special case of the two-dimensional knapsack problem, which is a knapsack problem with two different packing constraints. In this context the KKP problem can also be interpreted as a 1.5-dimensional knapsack problem (1.5-KP) [Kellerer et al., 2003, p. 269]. Another closely related problem is the Exact \(K\)-item Knapsack Problem (E-KKP), for which our results still hold and detailed discussions are included in Appendix A.1.

The KKP (and E-KKP) represents many practical applications in various fields ranging from assortment planning [Désir et al., 2016] to multiprocessor task scheduling [Caprara et al., 2000], and crowdsourcing [Wu et al., 2015]. As an interesting illustrative example, consider the worker selection problem in crowdsourcing systems, which provides a platform to engage the intelligence of a large crowd of users [Gong and Shroff, 2018]. Karger et al. [2013], Abraham et al. [2013], Wu et al. [2015] considers the problem of maximizing opinion diversity in constructing a wise crowd, which is finally reduced to E-KKP. On the other hand, KKP also appears as a key subproblem in the solutions of even more complex problems [Ahuja et al., 2004], Epstein and Levin [2012], Martello et al., 1999, Jansen and Porkolab 2006, Aardal et al. 2015]. For example, in the bin packing problem [Epstein and Levin, 2012], to be able to apply the ellipsoid algorithm to solve the linear program within a factor of \(1 + \varepsilon\), the (approximation) algorithm to the KKP is utilized to construct a polynomial time (approximate) separation oracle. In many such practical and theoretical applications, the subroutine utilized to solve KKP frequently appears to be the main complexity bottleneck, e.g., the single-sink capacitated K-facility location problem in [Aardal et al., 2015] and the resource constrained scheduling problem in [Jansen and Porkolab, 2006]. The aforementioned observations and facts motivate our study of designing a faster algorithm for this classic problem.

Additional Related Work. There has been a long history and substantial interests on improving upon the computational efficiency of knapsack problems [Ibarra and Kim, 1975, Lawler, 1977, Jansen

\(^2\)Throughout this paper, we use \([n]\) to denote the set \(\{1, 2, \ldots, n\}\) for any positive integer \(n\).
For the 0-1 KP, Ibarra and Kim [1975] proposed the first FPTAS with a time complexity of $O(n \log n + \min \{n, \log(1/\varepsilon)/\varepsilon^2\})$ and space complexity of $O(n + 1/\varepsilon^3)$. The running time was later improved to $O(n \log(1/\varepsilon) + 1/\varepsilon^3) - k$ by Lawler [1977] via a different scaling method, while the space complexity achieved in Lawler [1977] remains the same as Ibarra and Kim [1975]. Magazine and Oguz [1981] further proposed a new FPTAS that reduces the space requirement to $O(n/\varepsilon)$ and the running time of 0-1 KP was improved to $O(n \min \{\log n, \log(1/\varepsilon)\} + 1/\varepsilon^2 \log(1/\varepsilon) \cdot \min \{n, 1/\varepsilon \log(1/\varepsilon)\})$ in Kellerer et al. [2003].

As for the UKP, the earliest FPTAS was due to Ibarra and Kim [1975], which is an extension of their algorithm for 0-1 KP. The scheme achieves a time complexity of $O(n + 1/\varepsilon^4 \log(1/\varepsilon))$ and space complexity of $O(n + 1/\varepsilon^3)$. A more efficient FPTAS was designed by Lawler [1977], which runs in $O(n + 1/\varepsilon^3)$ and requires $O(n + 1/\varepsilon^2)$ space. Recently an $O(1/\varepsilon)$ improvement on both time and space complexity was made by Jansen and Kraft [2018], in which a new FPTAS was presented with running time of $O(n + 1/\varepsilon^2 \log^3(1/\varepsilon))$ and $O(n + 1/\varepsilon \log^2(1/\varepsilon))$ space bound.

Complexity of Knapsack Problems. An FPTAS is highly desirable for NP-hard problems. Unfortunately, it has been shown that there exists no FPTAS for $d$-dimensional knapsack problem for $d \geq 2$, unless $P=NP$ [Magazine and Chern 1984, Gens and Levner 1980].

1.1 Motivations and our contributions

Known results of KKP: In this paper we will mainly focus on FPTAS for KKP (and E-KKP). The first FPTAS for KKP was proposed in Caprara et al. [2000], by utilizing standard dynamic programming and profit scaling techniques, which runs in $O(nK^2/\varepsilon^2)$ time and requires $O(n + K^3/\varepsilon)$ space. This algorithm was later improved by Mastrolilli and Hutter [2003, 2006], in which a novel idea named Hybrid Rounding is proposed. Based on this technique, Mastrolilli and Hutter [2006] present two alternative FPTASs named Scheme A and Scheme B, which significantly accelerates the dynamic programming procedure while exhibiting a space-time tradeoff. More specifically, Scheme A achieves a time complexity of $O(n + K z^2/\varepsilon^2)$ and space complexity of $O(n + z^3/\varepsilon)$, while Scheme B needs $O(n + z^2/\varepsilon)$ space but requires a run-time complexity of $O(n + K z^2 + z^4/\varepsilon^2)$. We remark that Krishnan [2006] also investigated this problem, under an additional assumption that item profits follow an underlying distribution. This assumption enables the design of a fast algorithm via rounding the item profits adaptively according to the profit distribution.

However, one disadvantages of the aforementioned state-of-the-art results in Mastrolilli and Hutter [2006] is that, both schemes slow down significantly when either the cardinality upper bound or the desired accuracy increases. Furthermore, the current fastest FPTAS (Scheme A) sacrifices its space complexity in order to improve run-time performance. This may not be desirable as the space requirement is often a more serious bottleneck for practical applications than running time [Kellerer et al. 2003, p. 168]. Despite the recent widespread applications of the KKP problem [Ahuja et al. 2004, Désir et al. 2016, Epstein and Levin 2012, Jansen and Porkolab 2006, Nobibon et al. 2011, Soldo et al. 2012, Wu et al. 2015], the state-of-the-art complexity results established in Mastrolilli and Hutter [2006] have not been improved since then. This lack of progress brings us to our first key question.
Q1: Is it possible to design a more efficient FPTAS with lower time and/or space complexity to enhance practicality?

Moreover, although the two schemes in [Mastrolilli and Hutter, 2006] achieve substantial improvements compared with [Caprara et al., 2000], it is worth noting that there exists a hard parameter regime $H = \{(n, K, \varepsilon) | K = \Theta(n), \varepsilon^{-1} = \Omega(n)\}$ in the space of the problem parameters (instance size, cardinality bound, error), in which all existing FPTASs in the literature fail to surpass both the time and space complexity barriers guaranteed by the standard scheme in [Caprara et al., 2000]. For example, the run-time of Scheme B is higher than that of [Caprara et al., 2000]. Hence from a theoretical point of view, it is natural to ask:

Q2: Can we design a new FPTAS that has lower time complexity or space complexity than the standard FPTAS [Caprara et al., 2000] over all parameter regimes?

Last but not least, there are broad classes of problems (e.g., dynamic programming) for which the algorithm can be accelerated with the help of more memory. For instance, the improved time complexity of many of the known FPTASs is paid by a considerable increase in the space requirement, as noted in [Kellerer et al., 2003, p. 59]. This brings us to our third question.

Q3: Does the space-time tradeoffs exhibited in the barrier [Mastrolilli and Hutter, 2006] characterize a fundamental bound of the problem’s complexity?

| Reference                  | Time Complexity $^1$ | Space Complexity $^1$ |
|----------------------------|----------------------|-----------------------|
| Caprara et al. [2000]      | $O(\frac{nK^2}{\varepsilon})$ | $O(n + \frac{K^3}{\varepsilon})$ |
| Mastrolilli and Hutter 2003 | $O(n + \frac{K^2}{\varepsilon} + \frac{1}{\varepsilon})$ | $O(n + \frac{1}{\varepsilon})$ |
| Mastrolilli and Hutter 2006 (Scheme A) | $O(n + \frac{K^2}{\varepsilon})$ | $O(n + \frac{z^2}{\varepsilon})$ |
| Mastrolilli and Hutter 2006 (Scheme B) | $O(n + \frac{K^2}{\varepsilon} + \frac{z^2}{\varepsilon})$ | $O(n + \frac{z^2}{\varepsilon})$ |
| This Paper $^2$            | $O(n + \frac{z^2}{\varepsilon})$ | $O(n + \frac{z^2}{\varepsilon})$ |

$^1 z = \min\{K, \varepsilon^{-1}\}$

$^2$ Our time complexity can be refined to $\tilde{O}(n + z^4 + \frac{z^2}{\varepsilon} \cdot \min\{n, \varepsilon^{-1}\})$, as shown in Theorem 19.

Our contributions: To this end, we design a more efficient FPTAS through the lens of numerical analysis in this paper, which answers the aforementioned three questions. Table 1.1 summarizes the comparison between our scheme and results in existing literature. The detailed technical contributions are summarized as follows.

- As summarized in Table 1.1, for over thirteen years the best FPTASs are due to [Mastrolilli and Hutter, 2006], their time complexity scales linearly in the total number of items $n$ but has a pretty strong dependence on the cardinality bound $K$, as the coefficient of $K$ is at least in the order of $O(z^2) = O(\min\{K^2, \varepsilon^{-2}\})$. We break this longstanding barrier and answer question Q1 in the affirmative. In particular, we present a new FPTAS with $\tilde{O}(n + z^2/\varepsilon^2)$ running time and $O(n + z^2/\varepsilon)$ space requirement, which offers $\tilde{O}(K)$ and $O(z)$ improvements in time and space complexity respectively, and is the first to achieve time complexity that is asymptotically independent of $K$ (for a given $\varepsilon$).
Based on Equation (29) in Theorem 19, we show that the time complexity of our algorithm can be refined to $\tilde{O}(n + z^4 + \frac{z^2}{\varepsilon} \cdot \min\{n, \varepsilon^{-1}\})$. From this refined bound, it can be seen that even in the hard regime $\mathcal{H}$, our algorithm has the same time complexity (up to log factors) as the standard FPTAS [Caprara et al., 2000], while improving its space complexity by a factor of $n$, which implies that our algorithm is the first FPTAS which outperforms the standard FPTAS [Caprara et al., 2000] over all parameter regimes, thus answering question Q2 in the affirmative. Moreover, our algorithm performs efficiently in both the worlds of space and time complexity, which answers question Q3 and suggests that the space-time tradeoff in [Mastrolilli and Hutter, 2006] is not necessary.

Our new scheme also helps to improve the state-of-the-art complexity results of several classic problems belonging to other fields, owing to the widespread applications of KKP. In this paper we take the resource constrained scheduling problem [Jansen and Porkolab, 2006] as an illustrative example. More specifically, we show that for the scheduling problem, we can indeed provide a time complexity improvement that is roughly on the same order as that of KKP.

**Organization of the paper:** The rest of the paper is structured as follows. In Section 2 we give an overview of our scheme and introduce the item preprocessing procedure. In Section 3 we introduce the algorithm for items with large profits from a numerical point of view. Section 4 is dedicated to the method designed for items with small profits, and in Section 5 we present our main algorithm. We discuss and apply our new FPTAS to problems in a classic resource constrained scheduling problem in Section 6 and a constrained submodular minimization problem in Appendix D. We conclude our work in Section 7.

2 Technique Overview and Item Preprocessing

In this section, we present an overview of our approach and the preprocessing part of our theoretical developments.

2.1 Technique Overview

Mastrolilli and Hutter [2006] proposed a novel hybrid rounding technique, which transforms the input instance into one with simpler structure while approximately guaranteeing the objective value. Different from this hybrid rounding technique developed in [Mastrolilli and Hutter, 2006], we show that it is possible to achieve our desired complexity results solely via the geometric rounding. We divide items into two classes according to their profits and present distinct methods for each class of items. To solve the subproblem for items with low profit, we carefully design a continuous relaxation function which well approximates the optimal objective value of the subproblem. We show that this can be computed in sublinear time for every new input parameters, owing to our design of continuous relaxation, which allows us to exploit the redundancy among various input. As for items with large profit, we derive an efficient approximation scheme via the lens of numerical analysis. More specifically, the problem can be exactly solved by a computationally intractable algebraic computation process. To make it tractable, we discretize the value space to transform the continuous process to a polynomial time approximate scheme, whose time complexity is guaranteed and bounded by the convergence rate of the discretization procedure. By further exploiting the structure of
the item profits after rounding, a fast discrete algorithm executing the algebraic computation is presented. Finally, an approximate solution can be obtained by appropriately putting these modules together.

2.2 Item Preprocessing

Item Partition and Geometric Rounding: We first partition the items in $E$ into different classes according to their profits.

**Definition 1.** (Item Partition) Let $L$ and $S$ to denote the set of large and small items, respectively. An item $e \in E$ is called a small item if its profit is no more than $\epsilon \cdot OPT$, otherwise it is called a large item, i.e., $S = \{ e \in E| \epsilon \cdot OPT / K \leq p_e \leq \epsilon \cdot OPT \}$ and $L = \{ e \in E| p_e \in \Xi \}$, where $\Xi = [\epsilon \cdot OPT, OPT]$. We further divide $L$ and $S$ into different classes, $\{L^i\}_{i \in [r_L]}$ and $\{S^i\}_{i \in [r_S]}$, where $L^i = \{ e \in L| p_e \in (\epsilon (1 + \epsilon)^i \cdot OPT, \epsilon (1 + \epsilon)^i OPT) \} (i \in [r_L])$ and $S^i = \{ e \in S| p_e \in (\epsilon (1 + \epsilon)^{-i} OPT, \epsilon (1 + \epsilon)^{-i+1} OPT) \} (i \in [r_S])$. Let $r$ denote the number of non-empty classes in $E$, as shown in Appendix A.3, we have $r = O(\min\{r_L + r_S, n\}) = O(\min\{\log(K/\epsilon) / \epsilon, n\}) = \tilde{O}(\min\{1/\epsilon, n\})$. (3)

**Definition 2.** (Geometric Rounding) Without loss of generality, we can assume that elements in the same class have the same profit value. More specifically, we let $p_e = p^i_e = \epsilon (1 + \epsilon)^i OPT$, ($\forall e \in L_i$) and $p_e = p^i_e = \epsilon (1 + \epsilon)^{-i} OPT$ ($\forall e \in S_i$).

The simplification in Definition 2 does not hurt the solution since it will incur a loss of $O(\epsilon \cdot OPT)$ in the objective value. Exploiting the simple structure of item profits after item partition and profit rounding, we are able to derive a more fine-grained bound on the size of $|O^* \cap L|$ and $S$.

**Proposition 3.** There are no more than $|O^* \cap L| \leq z$ large items in the optimal solution set $O^*$. And without loss of generality, we can assume that the number of small items $|S| = O(\min\{K \cdot \log(K/\epsilon) / \epsilon, n\}) = \tilde{O}(\min\{K/\epsilon, n\})$.

**Proof.** See Appendix B.

3 Algorithm for Large Items From a Numerical Point of View

To approximately solve the $K$-item knapsack problem on ground set $E$, the first step of our approach is to divide this problem into two smaller $K$KP problems defined on the large item set $L$ and small item set $S$, respectively. The subproblem on $L$ is the same as the original problem, except that the ground set is substituted by $L$ and the cardinality upper bound $k$ must be no less than $z$, which is based on the Proposition 3.

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3We discuss the method of obtaining $OPT$ in Appendix A.2.

4Without loss of generality, we can assume that $1/\epsilon$ is an integer, otherwise we can replace it with the nearest integer, the results in this paper still hold.
3.1 An Abstract Algorithm Based on Convolution

3.1.1 Introduction to \( \varphi_E \) and \( \phi_L \)

In this subsection we first define the simple profit function \( \varphi(\cdot, \cdot) : 2^{L} \times \mathbb{R}^+ \times [z] \to \mathbb{R}^+ \).

Definition 4. (Profit Function) For any given set \( T \subseteq E \), real number \( \omega \), and integer \( k \), \( \varphi_T(p, k) \) is given by \( \varphi_T(\omega, k) = \max \{ \sum_{e \in T'} p_e | \sum_{e \in T'} w_e \leq \omega | |T'| \leq k, T' \subseteq T \} \), which denotes the optimal objective value of the \( K \)-item knapsack problem that is defined on set \( T \), while the budget and cardinality are \( \omega, k \) respectively.

It is clear that \( \varphi_L(\omega, k) \) is equal to the optimal objective value of the subproblem considered in this section. Our objective is to approximately compute the array \( A_L = \{ \varphi_L(\omega, k) \} \omega \in X, k \in [z] \), in which the value of \( X \) will be specified in Section 3.3. This array plays an important role in our final item combination procedure, as we will show later in Section 5.

To compute the profit function efficiently, we introduce the following inverse weight function \( \phi(\cdot, \cdot) : 2^{L} \times \Xi \times [z] \to \mathbb{R}^+ \), which is one of the key ingredients in computing the profit function.

Definition 5. (Inverse Weight Function) For any given set \( T \subseteq E \), real number \( p \) and integer \( k \), \( \phi_T(p, k) \) is given by \( \phi_T(p, k) = \min \{ \sum_{e \in T} w_e | \sum_{e \in T} p_e \geq p, |T'| \leq k, T' \subseteq T \} \), which characterizes the minimum possible total weights under which there exists a subset of \( T \) with total profit being no less than \( p \) and cardinality no more than \( k \).

An immediate consequence of Definitions 4 and 5 is that we can easily obtain the value of \( \varphi_L(\omega, k) \) based on \( \phi \), via equation \( \varphi_L(\omega, k) = \sup \{ p \in \mathbb{R}^+ | \phi_L(p, k) \leq \omega \} \). Therefore it suffices to derive the inverse weight function \( \phi_L(\cdot, \cdot) \) to compute \( A_L \), for which we design Algorithm 1 in Section 3.1.2.

Remark. If we obtain the profit function by directly breaking the original problem into subproblems with smaller weights and cardinality upper bounds, by combining them, we can get a solution to the original problem. However, the combination step will suffer from an expensive computational cost, because we may need to enumerate all the possibilities of budget and cardinality splitting. However, considering the weight function instead will allow us to exploit the special structure of the rounded item profits to accelerate the combination process, which will be discussed in Section 3.3.

3.1.2 Algorithm for computing \( \phi_L \)

If we partition the large item set \( L \) into \( \ell \) disjoint subsets as \( L = \bigcup_{i=1}^{\ell} L^{(i)} \), then \( \phi_L \) can be computed by performing convolution operations sequentially as shown in Algorithm 1. Here the convolution operation \( \otimes \) is defined as follows.

**Algorithm 1:** Computing \( \phi_L(\cdot, \cdot) \)

1. **Input:** Partition scheme \( L = \bigcup_{i=1}^{\ell} L^{(i)} \), Convolution operator \( \otimes \);
2. **Output:** \( \phi_L(\cdot, \cdot) \)
3. **for** \( i = 1 \) **to** \( \ell \) **do**
   4. \( \phi_{\bigcup_{j=1}^{i-1} L^{(j)}} = \phi_{\bigcup_{j=1}^{i-1} L^{(j)}} \otimes \phi_{L^{(i)}}(\cdot, \cdot) \);
5. **Return** \( \phi_L(\cdot, \cdot) \)
At the current stage, it is worth pointing out that in the convolution operation between weight Lemma 7. 

φ for fixed ω,k.

More specifically, we start with functions δ

Discretizing the Profit Space: To implement the convolution in polynomial time, we discretize the interval Ξ with the points \{x_i\}_{i \in [m]} as \(X = \{x_i : \varepsilon OPT = x_1 < x_2 < \ldots < x_m = OPT\} \subseteq \Xi\). We denote the discretization parameter of X by discretization parameter \(\delta_X = \max_{1 \leq i \leq m-1} \{x_{i+1} - x_i\}\). To tackle the computational challenge induced by the continuity of profit \(p\), we execute the convolution operation over the discrete functions, i.e.,

\[(\phi_{S_1} \otimes \phi_{S_2})^X(p, k) = \min \left\{ \phi_{S_1}(p_1, k_1) + \phi_{S_2}(p_2, k_2) \mid k_1 + k_2 \leq k, p_1 + p_2 \geq p \ (p_1, p_2 \in X) \right\}\]

More specifically, we start with functions \(\phi_{\mathcal{L}(i)}^X\), and compute \(\phi_{\bigcup_{j=1}^{l} \mathcal{L}(j)} \) iteratively until \(\phi_{\mathcal{L}(i)}^X\) is obtained. Here \(\phi_{\mathcal{L}(i)}^X : X \times [z] \to \mathbb{R}^+\) represents the restriction of base function \(\phi_{\mathcal{L}(i)}\) to set \(X\), i.e., \(\phi_{\mathcal{L}(i)}^X\) equals to \(\phi_{\mathcal{L}(i)}\), but is only defined on \(X \times [z]\). For any \(I \subseteq [l]\), we let \(\phi_{\bigcup_{i \in I} \mathcal{L}(i)}(\cdot, \cdot) = (\otimes_{i \in I} \phi_{\mathcal{L}(i)}^X)(\cdot, \cdot)\), and the discrete profit function \(\varphi_{S_1}^X(\cdot, \cdot)\) can be recovered by its relation with the inverse weight function \(\varphi_{S_1}^X(\omega, k) = \max \{p \in X : \phi_{S_1}(p, k) \leq \omega\}, \forall S \subseteq E\).

Convergence Behaviour of \(\varphi^X(\cdot, \cdot)\): We now first show point-wise convergence of \(\{\varphi_{\mathcal{L}(i)}^X(\cdot, \cdot)\}_X\) towards \(\varphi(\cdot, \cdot)\) when \(\delta_X\) goes to zero.

Lemma 7. For any finite index set \(I\) and \(\omega, k\), we have \(\lim_{\delta_X \to 0} \varphi_{\bigcup_{i \in I} \mathcal{L}(i)}^X(\omega, k) = \varphi_{\bigcup_{i \in I} \mathcal{L}(i)}(\omega, k)\) for fixed \(\omega, k\).
Proof. It suffices to prove the case when \(|I| = 2\), because for the case when \(|I| > 2\), convergence can be proven by induction, using the result we have for \(|I| = 2\). For the case when \(|I| = 2\), it is easy to check that \(\varphi_{\cup_{i\in I} L(i)}(\omega, k) - 2\delta_X \leq \varphi^X_{\cup_{i\in I} L(i)}(\omega, k) \leq \varphi_{\cup_{i\in I} L(i)}(\omega, k)\), thus the proof is complete. 

Note that the straightforward intuition that if discretization is small then convergence occurs, may not always hold. Indeed we can verify that that the weight function \(\phi^X\) does not converges to \(\phi\) through following example.

**Example 8.** (\(\phi^X\) does not converge to \(\phi\)) Considering sets \(S_i = \{e^{(i)}_1, e^{(i)}_2\} \) (\(i = 1, 2\)), where the item profits and weights are given by \((p^{(i)}_{e_1}, w^{(i)}_{e_1}) = (OPT/8, \omega/2) \) (\(i = 1, 2\)), \((p^{(i)}_{e_2}, w^{(i)}_{e_2}) = (OPT/3, \omega/4)\) (\(i = 1, 2\)), \((p^{(i)}_{e_2}, w^{(i)}_{e_2}) = (OPT/6, \omega/4)\). According to Definition \([2]\), we know that \(\phi_{S_1\cup S_2}(OPT/2, 3) = w^{(1)}_{e_2} + w^{(2)}_{e_2} = \omega/2\). Let the discretization set \(X_d = \Xi \cap \{i \cdot \frac{OPT}{2^d} | i \in [2^d]\}\), then it follows that the spacing \(\delta_{X_d} \leq \frac{OPT}{2^d}\) and \(\delta_{X_d} \to 0\) as \(d \to \infty\). However, we can verify that \(\phi^X_{S_1\cup S_2}(OPT/2, 3) = \omega/2 + w^{(1)}_{e_2} + w^{(2)}_{e_2} = \omega = \phi_{S_1\cup S_2}(OPT/2, 3)\), based on the fact that \(p^{(i)}_{e_2} \notin X_d (i = 1, 2)\). Hence \(\phi^X_{S_1\cup S_2}\) does not converge to \(\phi_{S_1\cup S_2}\).

Up to this point, we have proven the theoretical convergence of \(\varphi^X(\cdot, \cdot)\), which ensures that near-optimality of the solution is obtained after discretization, as long as \(X\) is dense enough in \(\Xi\). However, what matters greatly is the order of the accuracy, which refers to how rapidly the error decreases in the limit as the discretization parameter tends to zero. The convergence speed is formally defined as follows.

**Definition 9.** (Michelle, 2002) (Convergence Speed for discretization methods) Let \(n\) be the number of grid points in the discretization process, the discretization method is said to converge with order \(p\) if for the relevant sequence \(\{x_n\}_{n \geq 0}\), there exists \(L\) such that \(|x_n - L| = O(n^{-p})\) holds.

This speed is directly related to the complexity of our algorithm, and from the following lemma we can conclude that the method of discretizing \(X\) by a uniform grid set converges with order 1, as \(\delta_X = O(1/|X|)\) for uniform grid set \(X\).

**Lemma 10.** Let \(\phi^X_L\) be the weight function returned by Algorithm \([2]\), then for any given budget \(\omega \leq W,\) cardinality upper bound \(k \leq z,\) and discretization set \(X\), we have \(|\varphi^X_L(\omega, k) - \varphi_L(\omega, k)| \leq C \delta_X\), where the coefficient \(C = z + 1\). As a consequence, \(|X|\) must be of order \(\Omega(z/\varepsilon)\) to ensure an error of order \(O(\varepsilon OPT)\).

*Proof.* An important observation in the proof is the following: there are \(O(z)\) number of “effective” convolutions as the number of large items is always no more than \(z\), which enables us to relate the convergence rate with the number of large items instead of the number of classes in \(\{L_i\}_{i \in [\Xi]}\). In the following we formalize our intuition and present a rigorous proof.

Let \(O^*_{\omega, k}\) denote the optimal solution to subproblem for large items and \(L^*_{\omega, k} = L^{(i)} \cap O^*_{\omega, k}\). For any set \(S \subseteq E\), we use \(p(S)\) and \(w(S)\) to denote the total profits and weights in \(S\) respectively. For notational convenience, we let \(x^*_{\omega, k} = \{x \in X | x \leq p(L^*_{\omega, k})\}\), and \(x^*_{\omega, k} = \{x \in X | x \leq \sum_{i=1}^\ell x^*_{\omega, k}\}\). Observe that for set \(L^{(i)}\), we have

\[
\phi^X_{L^{(i)}}(x^*_{\omega, k}, L^{*}_{\omega, k}) = \phi_{L^{(i)}}(x^*_{\omega, k}, L^{*}_{\omega, k}) \leq \phi_{L^{(i)}}(p(L^*_{\omega, k}), L^{*}_{\omega, k}) = w(L^*_{\omega, k}),
\]
where the equality is due to the fact that \( \phi_{L(i)}^X \) is the restriction of \( \phi_{L(i)} \) to \( X \), and the inequality follows from the fact that \( \phi_{L(i)}(\cdot, \cdot) \) is monotone non-decreasing with respect to profit \( p \). We remark that the inverse weight function is a subadditive function with respect to the profit parameter. Combining this fact with (4), we obtain
\[
(\otimes_{i=1}^{\ell} \phi_{L(i)}^X)(x_{\omega,k}^*, x_{\omega,k}^* \rho, k) \leq \sum_{i=1}^{\ell} \phi_{L(i)}^X (x_{\omega,k}^*, |L_{\omega,k}^*|) \leq w(L_{\omega,k}^*) \leq \omega, \tag{5}
\]
which further implies that \( \varphi_{L}^X(\omega, k) \geq x_{\omega,k}^* \geq \sum_{i=1}^{\ell} x_{\omega,k}^* - \Delta_i \). Hence, we are able to lower bound the error incurred by discretization as,
\[
\varphi_{L}^X(\omega, k) - \varphi_{L}(\omega, k) \geq \sum_{i=1}^{\ell} x_{\omega,k}^* - \sum_{i=1}^{\ell} p(L_{\omega,k}^*) - \delta X \geq -\delta X \left( 1 + \sum_{i=1}^{\ell} \delta_i \neq 0 \right), \tag{6}
\]
where \( \Delta_i = L_{\omega,k}^* - x_{\omega,k}^* \), and the second inequality holds because \( \Delta_i \leq \delta_X \). To bound the RHS of (6), the key observation here is, \( \Delta_i \neq 0 \) only if \( L_{\omega,k}^* \) is non-empty, i.e.,
\[
\sum_{i=1}^{\ell} \delta_i \neq 0 \leq \sum_{i=1}^{\ell} \delta_i \neq 0 \leq |L \cap O_{\omega,k}^*| \leq z. \tag{7}
\]
Hence the error brought by discretization is lower bounded by \( -(z + 1)\delta_X \). On the other hand, it is clear that \( \phi_{L}^X(\omega, k) \leq \phi_{L}(\omega, k) \), which follows by applying induction on \( |L| \). Therefore the absolute value of the error is no more than \( (z + 1)\delta_X \). Combining this with the fact that \( \delta_X \geq \frac{OPT}{2} \), the proof is complete.

### 3.3 Fast Convolution Algorithm

In this subsection we settle the problem of designing a fast convolution algorithm, which is the last remaining issue that has a critical impact on the efficiency of Algorithm \[1\]. To this end, we establish an inherent connection between convolution results under different inputs \( p \) and \( k \), which is formally described in Lemma \[12\]. Owing to this crucial observation, we are able to remove a large amount of redundant calculations when facing new input parameters, to obtain a fast convolution algorithm. To start with, we first sort items in each \( L_i^* \) in non-increasing order of weights which takes \( O(z \log z) \) time. We define the optimum index function as follows.

**Definition 11.** (Optimum index function) \( \psi : X \times [K] \rightarrow [K] \) is defined as,
\[
\psi(p, k) = \arg\min \left\{ \theta \in [k] \mid \phi_{L_a}^X (\max \{ x \in X : x \leq \theta \cdot p_a^* \}, \theta) + \phi_S^X (\max \{ x \in X : x \leq p - \theta \cdot p_a^* \}, k - \theta) \right\} \tag{8}
\]

Here (8) benefits from our partition in which all items in the same set \( L_i^* \) have equal profit value. Specifically, when we derive the result of \( (\phi_{L_a}^X \otimes \phi_S^X)(p, k) \), there is indeed only one decision variable \( \theta \), i.e., the number of elements selected from \( L_i^* \), that should be figured out. Hence, we denote the optimal value of \( \theta \) by the index function \( \psi \). Therefore, our primary objective is reduced to figure out all the indices \( \{ \psi(p, k) \}_{p \in X, k \in [z]} \), for which we give a graphic illustration in Figure \[1(a)\]. It can be regarded as finding *column minimums* in the cube, here column minimum refers to the optimal indices defined in Definition \[11\].
Consider the problem in parallel slices: We divide the cube into parallel slices. Consider slice
\[ H = \{ (p,k) | p = p_0 + \zeta \lambda_0, k = k_0 + \zeta \} \cap (\Xi \times [0,z]), \] as shown in Figure 1(b), where \( (p_0,k_0) \) denotes the boundary point of slice \( H \), hence \( p_0k_0 = 0 \), and \( \zeta \) represents the drift of point \( (p,k) \) from boundary. It can be seen that the angle between slice \( H \) and the frontal plane is equal to \( \arctan \lambda_0^{-1} \), and there are \( O(|X|) = O(z/\varepsilon) \) such parallel slices in the cube. On the other hand, plugging (9) into (8), the index function can be simplified to
\[ \chi_H(\zeta) = \arg\min_{\theta \in [z]} \phi_{\xi}^X(\lambda_0 \theta, \theta) + \phi_S^X(p_0 + \lambda_0[\zeta - \theta], k_0 + [\zeta - \theta]). \]

**Figure 1:** Graphic Illustrations of the Convolution Operation

**Bounded “gradient” of \( \chi_H \):** Without loss of generality we could assume that there exists an integer \( \tau_\alpha \in \mathbb{Z}^+ \) such that \( p_0^1 = \tau_\alpha \cdot \frac{\varepsilon \text{OPT}}{\varepsilon} \), otherwise we can always modify \( p_0^1 \) by an \( O(\varepsilon \text{OPT}) \) additive factor to meet this criteria while inducing a \( O(\varepsilon \text{OPT}) \) loss in the objective function. Consequently we have \( \lambda_0 = \tau_\alpha \varepsilon \text{OPT} \). We consider the case when \( \Xi \) is discretized by the uniform grid set \( X = \{ i : \frac{\varepsilon \text{OPT}}{\varepsilon} | i \in [z/\varepsilon] \} \). Then the following key observation about the distribution of column minima in slice \( H \) holds.

**Lemma 12.** Consider two columns in \( H \) that are indexed by \( \zeta_1 \) and \( \zeta_2 \). We have \( \frac{\chi_H(\zeta_2) - \chi_H(\zeta_1)}{\zeta_2 - \zeta_1} \leq 1. \)

**Proof.** We finish the proof by contradiction. Let \( \Delta = [\chi_H(\zeta_2) - \chi_H(\zeta_1)] - [\zeta_2 - \zeta_1] \) denote the difference between the numerator and denominator in Lemma 12 and suppose that \( \Delta > 0 \). Without loss of generality we can assume that \( \zeta_2 \geq \zeta_1 \). We first observe points \( (p_0 + \lambda \zeta_1, k_0 + \zeta_1, \chi_H(\zeta_1)) \) and \( (p_0 + \lambda_0 \zeta_1, k_0 + \zeta_1, \chi_H(\zeta_1) + \Delta) \) in column indexed by \( \zeta_1 \), since \( \chi_H(\zeta_1) \) is the index of column minimum, it follows that
\[
\phi_{\xi}^X(\lambda_0 \chi_H(\zeta_1), \chi_H(\zeta_1)) + \phi_S^X(p_0 + \lambda_0[\zeta_1 - \chi_H(\zeta_1)], k_0 + [\zeta_1 - \chi_H(\zeta_1)])
\leq \phi_{\xi}^X(\lambda_0[\chi_H(\zeta_1) + \Delta], \chi_H(\zeta_1) + \Delta) + \phi_S^X(p_0 + \lambda_0[\zeta_1 - (\chi_H(\zeta_1) + \Delta)], k_0 + [\zeta_1 - (\chi_H(\zeta_1) + \Delta)]).
\]
We remark that which suggests that the second term in (13) is identical to that in (10). In a similar way we have the following proposition.

It can be computed by enumerating the interval $\chi$ columns in the slice searching space in one column, given that we have figured out the optimum indices at some other rate of the index function. Taking advantage of this lemma, we are able to reduce the size of the Divide-and-Conquer on slice $H$: In Lemma 12, we establish an upper bound on the growth rate of the index function. Taking advantage of this lemma, we are able to reduce the size of the searching space in one column, given that we have figured out the optimum indices at some other columns in the slice $H$. More specifically, consider columns indexed by $\zeta_1 \leq \zeta_2 \leq \zeta_3$, as shown in Figure 1(e), the information of $\chi_H(\zeta_1)$ and $\chi_H(\zeta_3)$ indeed provide two cutting planes to help us locate $\chi_H(\zeta_2)$ in a smaller interval $[\chi_H(\zeta_3) + 2 - \zeta_2, \chi_H(\zeta_1) - 1]$. Inspired by this observation, for any slice in the form of (9), we design a divide-and-conquer procedure to compute the optimum indices efficiently, as shown in Algorithm 2. A column is called even (odd) column if and only if its corresponding $\zeta$ value in (9) is even (odd). In Algorithm 2, we start with a recursive call to determine the optimum indices of all the even-indexed columns. Then for each odd column $\chi_H(2i)$, it can be computed by enumerating the interval $[\chi_H(2i + 1) - 1, \chi_H(2i - 1) + 1]$. Consequently, we have the following proposition.

**Algorithm 2: SliceIndex($H$)**

1. **Input:** $H$;
2. **Output:** $\chi_H(\cdot)$
3. $H \leftarrow$ The even fibers in $H$;
4. Compute the optimum indices of fibers in $H'$ via SliceIndex($H'$);
5. **for** each odd fiber $2i$ in $H$ **do**
6. 
   - Enumerate $[\chi_H(2i + 1) - 1, \chi_H(2i - 1) + 1]$ to find the minimum index in the $2i$-th fiber of $H$.
Proposition 13. It takes \(O(z \log z) = \tilde{O}(z)\) time to compute \(\chi_H(\cdot)\) via Algorithm 2.

Proof. We denote the running time of computing the index function when there are \(c\) columns in a slice by \(T_H(c)\), and let \(c_H\) denote the number of columns in slice \(H\). Then

- Line 2 requires \((c_H/2)\) time. Without loss of generality we can assume that \(c_H\) is even, otherwise it can be verified that the corresponding total time complexity is within the same order.

- Each iteration in line 4 takes

\[
O\left(\left|\chi_H(2i - 1) + 1 \right| - \left|\chi_H(2i + 1) - 1 \right| + 1\right) = O\left(\chi_H(2i - 1) - \chi_H(2i + 1) + 3\right)
\]

(17)

time. We remark that the RHS of (17) is non-negative according to Lemma 12. Taken together, the running time of computing column minimum in odd columns can be upper bounded as,

\[
\sum_{i=1}^{O(c_H/2)} O\left(\chi_H(2i - 1) - \chi_H(2i + 1) + 3\right) = O(c_H + z) = O(z),
\]

(18)

which holds because both \(c_H\) and \(\chi_H(\zeta)\) are no more than \(z\).

To summarize, the total running time satisfies the recurrence relation \(T_H(c) = T_H(c/2) + O(z)\). Solving this equation we have \(T_H(c_H) = O(z \log z)\), the proof is complete.

Fast Convolution Operation: We are ready to introduce our convolution algorithm, using the concepts and algorithms developed in the previous subsections. The details of the convolution operation are specified in Algorithm 3. The complexity of Algorithm 1 when the convolution operation is specified as Algorithm 3 is presented as follows.

Algorithm 3: Convolution Algorithm

1. Input: \(\phi^X_{L_a}(\cdot, \cdot), \phi^S_{L_a}(\cdot, \cdot)\);
2. Output: \((\phi^X_{L_a} \otimes \phi^S_{L_a})(\cdot, \cdot)\);
3. for each slice \(H\) in the form of \([\widetilde{H}]\) do
   4. Compute \(\chi_H(\cdot)\) using SliceIndex\((H)\);
5. for \(p \in X, k \in [K]\) do
   6. \((\phi^X_{L_a} \otimes \phi^S_{L_a})(p, k) = \phi^X_{L_a}(\max\{x \in X : x \leq \psi(p, k) \cdot p^1_a\}, \psi(p, k))\)
   7. \(
      \phi^S_{L_a}(\max\{x \in X : x \leq \psi(p, k) \cdot p^1_a\}, k - \psi(p, k))
   \) \((p \in X)\)
6. Return \((\phi^X_{L_a} \otimes \phi^S_{L_a})(\cdot, \cdot)\).

Lemma 14. Algorithm 4 runs in \(O(n + (z^2 \log z / \varepsilon) \cdot \min\{\log(1 / \varepsilon) / \varepsilon, n\}) = O(n) + \tilde{O}(n \min\{z^2 / \varepsilon^2, n z^2 / \varepsilon\})\) time and requires \(O(n)\) space.

Proof. Since there are \(O(|X|)\) slices in the searching space, based on Proposition 13 it can be seen that a single convolution operation takes \(O(|X| \cdot z \log z) = \tilde{O}(z^2 / \varepsilon)\) time. Additionally, in Algorithm 1 we need to
Appendix C.1, which relies on our analysis in the following two subsections. The proof is complete.

Generally speaking, for given $p \in X$ and $k \in \mathbb{Z}^+$, it requires $O(z \cdot |X|)$ arithmetic operations to compute $(\phi_{S_1} \otimes \phi_{S_2})(p, k)$ if we enumerate all possible pairs of $(p_1, k_1)$, which further results in a total complexity of $O(z^2|X|^2)$ for operator $\otimes$. Compared with our Algorithm 3, this is unnecessarily inefficient, since it restarts all the arithmetic operations when the input parameters varies.

4 Continuous Relaxation for Small Items

In Section 3 we have shown how to approximately select the most profitable large items under any given budget and cardinality constraints. One important task left is to solve the subproblem with only small items involved. In this section we show how to approximately solve this subproblem efficiently. Similar to Definition 5, the profit function of small items, $\varphi_S\left(\cdot, \cdot \right) : \mathbb{R}^+ \times [K] \rightarrow \mathbb{R}^+$, is given by $\varphi_S(\omega, k) = \max \{\sum_{e \in S} p_e x_e | \sum_{e \in S} x_e \leq k; \sum_{e \in S} w_e x_e \leq \omega; x_e \in \{0, 1\} \}$. The main spirit of our approach for small items is similar to that of Section 3, i.e., find a new function $\tilde{\varphi}_S$, which is a good approximation of $\varphi_S$ and is economical in computations. To this end, our main result in this subsection is formally stated in the following lemma. We leave the proof of this lemma in Appendix C.1, which relies on our analysis in the following two subsections.

Lemma 15. There exists a relaxation $\tilde{\varphi}_S\left(\cdot, \cdot \right) : \mathbb{R}^+ \times [K] \rightarrow \mathbb{R}^+$ that satisfies $|\tilde{\varphi}_S - \varphi_S| = O(\varepsilon OPT)$, and the corresponding array $A_S = \{\tilde{\varphi}_S(\omega, k) | \omega \in W, k \in K \}$ can be computed within $\tilde{O}(n + z^4 + \min\{\frac{z^2}{\varepsilon^2}, \frac{nz}{\varepsilon}\})$ time when $|W| = O(\varepsilon^{-1})$ and $|K| = O(z)$, while requiring $O(z/\varepsilon)$ space.

One question that may arise is the following: can the methods in Section 3 still work for the small item set $S$, i.e., can we apply Algorithm 1 over $S$ and use the output discrete function as an approximation of $\varphi_S$? The answer is that it can be verified that $O(n) + \tilde{O}(K^2/\varepsilon^2)$ time is required, which is significantly high especially when $K$ is large, and fails to provide the desired complexity result. This is because there could be many more small items than large items, which will result in a larger searching space.

To construct the new function $\tilde{\varphi}_S$, we turn to the continuous relaxation of the subproblem, as the continuous problem is much easier to deal with. More importantly, the boundness of small item profits will ensure that the gap between the optimal values of the two problems is small. In the remaining of this section we will show the correctness of Lemma 15 step by step. We first present the details of $\tilde{\varphi}_S$, then analyze its approximation error and computational complexity.

4.1 Continuous Relaxation Design and Error Analysis

Designing $\tilde{\varphi}_S$. In our algorithm, we let $\tilde{\varphi}_S(\omega, k) = \tilde{\varphi}_S^{(1)}(\omega, k) \cdot 1_{\{K \leq \varepsilon^{-1}\}} + \tilde{\varphi}_S^{(2)}(\omega, k) \cdot 1_{\{K > \varepsilon^{-1}\}}$, in which the two building block functions $\tilde{\varphi}_S^{(i)} (i = 1, 2)$ are specified in the following definition.
Definition 16. (Definition of $\tilde{\varphi}^{(1)}_S$, $\tilde{\varphi}^{(2)}_S$) Functions $\tilde{\varphi}^{(i)}_S(\cdot, \cdot) : \mathbb{R}^+ \times [K] \to \mathbb{R}^+$ ($i = 1, 2$) are constructed as $\tilde{\varphi}^{(1)}_S(\omega, k) = \max \{ \sum_{e \in S} p_e x_e | \sum_{e \in S} x_e \leq k, \sum_{e \in S} w_e x_e \leq \omega, x_e \in [0, 1] \}$. and $\tilde{\varphi}^{(2)}_S(\omega, k) = \max_{0 \leq t \leq k} \{ \tilde{\varphi}^{(2)}_S(\omega, k-t) + \tilde{\varphi}^{(2)}_S(0) \}$. Here set $\tilde{S}_\omega = \{ e \in S | w_e \leq \varepsilon \omega / K \}$ represents the set of elements in $S$ with weight less than a threshold $\varepsilon \omega / K$, and $\tilde{S}_\omega = (S \setminus \tilde{S}_\omega) \setminus \{ e \in S | w_e > \omega \}$. Function $\tilde{\varphi}^{(2)}_S(t) = \max \{ \sum_{e \in T} p_e | T \subseteq \tilde{S}_\omega, |T| \leq t \}$ denotes the total profits of the top $t$ elements in $\tilde{S}_\omega$. In addition, $\tilde{\varphi}^{(2)}_{S^*}(\omega, t)$ is given by

$$
\tilde{\varphi}^{(2)}_{S^*}(\omega, t) = \max \left\{ \sum_{e \in S^*} p_e x_e \middle| \sum_{e \in S^*} x_e \leq t, \sum_{e \in S^*} w'_e x_e \leq (1-\varepsilon) \omega, x_e \in [0, 1] \right\}
$$

where $w'_e = \frac{\omega(1+\varepsilon)^{\floor{\frac{\log(1+\varepsilon)\omega}{K\varepsilon}}} \cdot \lfloor \cdot \rfloor}$ refers to the ceiling function.

Note that the first function $\tilde{\varphi}^{(1)}_S$ is the most natural linear programming relaxation of $\tilde{\varphi}_S$, in which all the integer variables are relaxed to real numbers in $[0, 1]$. In the second function $\tilde{\varphi}^{(2)}_S$, we only relax variables corresponding to elements in $\tilde{S}_\omega$, while the element weights are rounded to an integer power of $(1 + \varepsilon)$, and the budget is given by $(1 - \varepsilon) \omega$ instead of $\omega$.

Error of Approximation. We show that $\tilde{\varphi}_S$ provides a good approximation of $\varphi_S$ in the following lemma.

Lemma 17. The differences between functions $\tilde{\varphi}_S$ and $\varphi_S$ is bounded as $|\tilde{\varphi}_S(\omega, k) - \varphi_S(\omega, k)| \leq 4\varepsilon OPT$.

Proof. It suffices to show the following bounds on the differences between functions $\varphi_S$, $\tilde{\varphi}^{(1)}_S$ and $\tilde{\varphi}^{(2)}_S$:

$$
|\tilde{\varphi}^{(1)}_S(\omega, k) - \varphi_S(\omega, k)| \leq 2\varepsilon OPT, \tag{20}
$$

$$
|\tilde{\varphi}^{(2)}_S(\omega, k) - \varphi_S(\omega, k)| \leq 4\varepsilon OPT, \tag{21}
$$

Assuming inequalities above, we can complete the proof thus:

$$
|\tilde{\varphi}_S(\omega, k) - \varphi_S(\omega, k)| = \left| \left( \tilde{\varphi}^{(1)}_S(\omega, k) \cdot 1_{\{K \leq \varepsilon - 1\}} + \tilde{\varphi}^{(2)}_S(\omega, k) \cdot 1_{\{K > \varepsilon - 1\}} \right) - \varphi_S(\omega, k) \right|
$$

$$
\leq \max \left\{ \left| \tilde{\varphi}^{(1)}_S(\omega, k) - \varphi_S(\omega, k) \right|, \left| \tilde{\varphi}^{(2)}_S(\omega, k) - \varphi_S(\omega, k) \right| \right\} \leq 4\varepsilon OPT.
$$

Now we proceed to prove the bounds (20) and (21).

(I) Proof of bound (20): We first make the observation that $\varphi_S(\omega, k) \leq \tilde{\varphi}^{(1)}_S(\omega, k)$, since the feasible region in $\varphi_S$ is a subset of that in $\tilde{\varphi}^{(1)}_S$. As it has been shown in Caprara et al. 2000, there are at most two fractional components in $x^*$, the optimal solution to the LP relaxation problem in $\tilde{\varphi}_S$. Hence, the objective value will suffer a loss of at most $2\varepsilon OPT$, if we set all the fractional entries in $x^*$ to be 0, i.e.,

$$
\sum_{i=1}^{n} p_i x'_i \geq \tilde{\varphi}_S(\omega, k) - 2\varepsilon OPT
$$

holds for the integer vector $x^\prime'$. On the other hand, notice that $x^\prime''$ is also a feasible solution to the subproblem $\varphi_S(\omega, k)$, it follows that $\sum_{i=1}^{n} p_i x'_i \leq \varphi_S(\omega, k)$. Relating $\varphi_S(\omega, k)$ and $\tilde{\varphi}^{(1)}_S(\omega, k)$ to the total profits of $x^\prime$, (20) follows.
(II) Proof of bound (21): To show the correctness of (21), observe that each one of the following two operations appearing in the definition of $\tilde{\varphi}_{\mathcal{S}_k}^{(2)}$, will incur a loss of at most $(1 - \varepsilon)$, compared with $\varphi_{\mathcal{S}_k}^{(1)}(\omega, k)$:

- Increasing the weight $w_e (e \in \mathcal{S}_\omega)$ to $w'_e \in [w_e, (1 + \varepsilon)w_e]$;
- Scaling the budget $\omega$ by a factor of $(1 - \varepsilon)$.

Therefore $\tilde{\varphi}_{\mathcal{S}_k}^{(2)}$ can be lower bounded using $\varphi_{\mathcal{S}_k}$:

$$
\tilde{\varphi}_{\mathcal{S}_k}^{(2)}(\omega, k) \geq (1 - \varepsilon)^2 \varphi_{\mathcal{S}_k}^{(1)}(\omega, k) \geq \varphi_{\mathcal{S}_k}(\omega, k) - 4\varepsilon \text{OPT}. 
$$

(22)

In (a), we use the fact that $(1 - \varepsilon)^2 \geq 1 - 2\varepsilon$, together with inequality $\varphi_{\mathcal{S}_k}^{(1)}(\omega, t) \geq \varphi_{\mathcal{S}_k}(\omega, t) - 2\varepsilon \text{OPT}$, whose proof goes along the same lines as the proof of (20).

Let $\mathcal{S}_{\omega, k}^{*}\in \mathcal{S}_{\omega, k}$ be the optimal solution set to $\varphi_{\mathcal{S}}(\omega, k)$. Observe that the profit function $\varphi_{\mathcal{S}}$ can be expressed as

$$
\varphi_{\mathcal{S}}(\omega, k) = \varphi(\omega - w(\tilde{\mathcal{S}}_{\omega, k}^{*}), k - |\tilde{\mathcal{S}}_{\omega, k}^{*}|) + p(\tilde{\mathcal{S}}_{\omega, k}^{*}), 
$$

where $\tilde{\mathcal{S}}_{\omega, k}^{*} = \mathcal{S}_{\omega, k}^{*}\in \mathcal{S}_{\omega, k}$ with cost no more than $\epsilon \omega / K$. As a consequence, the difference between $\tilde{\varphi}_{\mathcal{S}}^{(2)}$ and $\varphi_{\mathcal{S}}$ can be lower bounded as,

$$
\tilde{\varphi}_{\mathcal{S}}^{(2)}(\omega, k) - \varphi_{\mathcal{S}}(\omega, k) 
\geq [\tilde{\varphi}_{\mathcal{S}}^{(2)}(\omega - w(\mathcal{S}_{\omega, k}^{*}), k - |\mathcal{S}_{\omega, k}^{*}|) - \varphi(\omega - w(\mathcal{S}_{\omega, k}^{*}), k - |\mathcal{S}_{\omega, k}^{*}|)] + [\tilde{\varphi}_{\mathcal{S}}^{(2)}(|\mathcal{S}_{\omega, k}^{*}|) - p(\tilde{\mathcal{S}}_{\omega, k}^{*})] 
\geq - 4\varepsilon \text{OPT}.
$$

(23)

The above, step (a) is based on (23) and definition of $\tilde{\varphi}^{(2)}$; (b) follows from (22) and the fact that $\tilde{\varphi}_{\mathcal{S}_k}^{(2)}(|\mathcal{S}_{\omega, k}^{*}|) \geq p(\tilde{\mathcal{S}}_{\omega, k}^{*})$.

Finally we conclude that $\varphi_{\mathcal{S}}^{(2)}(\omega, k) \leq \varphi_{\mathcal{S}}(\omega, k) + 2\varepsilon \text{OPT}$. We denote the optimal index in $\tilde{\varphi}_{\mathcal{S}_k}^{(2)}(\omega, k)$ by $t_{\omega, k}^{*}$, and consider that the set $\mathcal{S}^{**}$ consists of the following two types of items:

- Top $t_{\omega, k}^{*}$ elements in $\tilde{\mathcal{S}}_{\omega}$;
- Elements corresponding to the integer entries in the optimal solution to $\tilde{\varphi}_{\mathcal{S}_k}^{(2)}(\omega, k - t_{\omega, k}^{*})$.

Observe that $\mathcal{S}^{**}$ is a feasible set to $\varphi_{\mathcal{S}}(\omega, k)$, hence $p(\mathcal{S}^{**}) \leq \varphi_{\mathcal{S}}(\omega, k)$. Combining with the fact that $p(\mathcal{S}^{**}) \geq \varphi_{\mathcal{S}}^{(2)}(\omega, k) - 2\varepsilon \text{OPT}$, the proof is complete.

Obtaining the Final Solution Set. Recall that our ultimate objective is to retrieve an solution set that has near optimal objective function value. To this end, for the subproblem of small items, we can solve the continuous problem and return the corresponding integer components $S = \{e | x^*_e = 1\}$ as an approximate solution, where $x^*$ denotes the optimal fractional solution.
4.2 Computing $\tilde\varphi_S$ Efficiently

In this subsection, we consider how to efficiently compute the function $\tilde\varphi_S(\cdot, \cdot)$. More specifically, our objective is to compute set $\{\tilde\varphi_S(\omega, k) | \omega \in \mathcal{W}, k \in \mathcal{K}\}$, for given $\mathcal{K} \subseteq \mathbb{Z}^{|\mathcal{K}|}$ and $\mathcal{W} \subseteq \mathbb{R}^{|\mathcal{W}|}$.

Computing Relaxation $\tilde\varphi_S^{(1)}(\cdot, \cdot)$. One straightforward approach is to utilize the linear time algorithm [Caprara et al., 2000, Megiddo and Tamir, 1993, Megiddo, 1984] to solve $\tilde\varphi_S^{(1)}$ under distinct parameters in $\mathcal{W}, \mathcal{K}$ separately, which will result in a total complexity of $O(\min\{K/\varepsilon, n\})$. Note that under this approach, the complexity has a high dependence on the parameter $K$.

Computing Relaxation $\tilde\varphi_S^{(2)}(\cdot, \cdot)$. In the rest of the subsection, let $f_t(\omega, k)$ as $f_t(\omega, k) = \tilde\varphi_S^{(2)}(\omega, k - t) + \tilde\varphi_S^{(2)}(t)$, then $\tilde\varphi_S^{(2)}(\omega, k) = \max_{0 \leq t \leq k} f_t(\omega, k)$ according to Definition 16.

Applying Binary Search to Compute $\tilde\varphi_S^{(2)}$. We first claim the following key observation with regard to $\{f_t(\omega, k)\}_{0 \leq t \leq k}$. Basically this concavity property enables us to compute $\tilde\varphi_S^{(2)}$ using $O(\log k)$ calls to the subroutine of computing $f_t(\omega, k)$.

Proposition 18. (Concavity of $f_t$) The sequence $\{f_t(\omega, k)\}_{t \in [k]}$ is a concave sequence with respect to $t$. As a result, $\tilde\varphi_S^{(2)}(\omega, k)$ can be computed in $O(T_f \log k) = O(T_f)$ time, where $T_f$ represents the worst case time complexity for computing $f_t(\omega, k)$ under fixed values of $t, \omega, k$.

Proof. The correctness of Proposition 18 mainly follows from the fact that sequence concavity is preserved under summation.

We first show that $\{\tilde\varphi_S^{(2)}(t)\}_{t \in [k]}$ is a concave sequence. Observe that the first order difference $\Delta \tilde\varphi_S^{(2)}(t) = \tilde\varphi_S^{(2)}(t) - \tilde\varphi_S^{(2)}(t - 1)$ is equal to the $t$-th largest profit in $\tilde{S}_\omega$. Thus, the first order sequence $\{\tilde\varphi_S^{(2)}(t)\}_{t \in [k]}$ is a non-increasing, i.e., $\Delta \tilde\varphi_S^{(2)}(t) \geq \Delta \tilde\varphi_S^{(2)}(t + 1)$ and the concavity of $\{\tilde\varphi_S^{(2)}(t)\}_{t \in [k]}$ follows. For sequence $\{\tilde\varphi_S^{(2)}(\omega, k - t)\}_{t \in [k]}$, let $x^*_\omega,t$ denote the optimal fractional solution to (19). Observe that $(x^*_\omega,k-t+1 + x^*_\omega,k-t-1)/2$ is a feasible solution to (19) under cardinality bound $k - t$, thus $\tilde\varphi_S^{(2)}(\omega, k - t) \geq [\tilde\varphi_S^{(2)}(\omega, k - t + 1) + \tilde\varphi_S^{(2)}(\omega, k - t - 1)]/2$. Based on which, $\{f_t(\omega, k)\}_{t \in [k]}$ is also concave.

For the complexity result, notice that sequence concavity implies monotonicity of the first order difference sequence $\{\Delta f_t(\omega, k)\}_t$. Hence $t^* = \operatorname{argmax}_{t \in [k]} f_t(\omega, k)$ can be derived via a binary search procedure while using the sign of $\Delta f_t(\omega, k)$ as indication information. This makes $O(\log k)$ queries to sequence $\{f_t(\omega, k)\}_{t \in [k]}$.

Computing $f_t(\omega, k)$. At the current stage, the problem of computing $\tilde\varphi_S^{(2)}(\omega, k)$ has been shown to have the same time complexity (up to a factor of $O(\log k)$) as computing $f_t(\omega, k)$, which is further determined by the following two subroutines—calculating $\tilde\varphi_S^{(2)}(\omega, t)$ and $\tilde\varphi_S^{(2)}(t)$. To figure out the first function under multiple input parameters, we dualize the budget constraint as $L(\mu, \omega, t) = \max_{x_e \in [0,1]} \sum_{e \in \tilde{S}_\omega} (p_e - \mu w_e) x_e + \mu \omega \sum_{e \in \tilde{S}_\omega} x_e \leq t$, and we can always apply binary search on set $B' = \{(1 + \varepsilon)^b \cdot \frac{(1 + \varepsilon)^c - 1}{(1 + \varepsilon) - 1} | b, | c, | d \leq \log(K/\varepsilon)/\varepsilon, \text{ and } b, c, d \in \mathbb{Z}\}$ to figure out the
Theorem 19. The total profits of items in set $(2)$ follows from the definition of \( \{ \varphi^{(2)}_{S_t}(t) \}_{t \in |S|} \) can be computed together to reduce running time, under the same budget \( \omega \). We present the details and complexity analysis in Appendix C.2.

5 Putting The Pieces Together—Combining Small and Large Items

The main idea of this section is to utilize our two algorithms established in Section 3 and 4 as two basic building blocks, and to approximately enumerate all the possible profit allocations among \( L \) and \( S \), which is formally described in Algorithm 4 and its performance guarantee is given by the Theorem 19.

**Algorithm 4: Main Algorithm**

1. **Input:** Function \( \phi_S^X(\cdot, \cdot), \varphi_S(\cdot, \cdot) \);
2. **Output:** Near optimal solution \( T_o \);
3. \((k^*, x^*) \leftarrow \text{argmax}_{k \in [\xi], x \in X'} \left\{ x + \varphi_S(W - \phi_L^X(x, k), K - k) \right\}; \)
4. \( T_S^S \leftarrow \text{The solution set in } S \text{ corresponding to } \varphi_S(W - \phi_L(x^*, k^*), K - k^*); \)
5. \( T_o^L \leftarrow \text{The solution set in } L \text{ corresponding to } \phi_L(x^*, k^*); \)
6. **Return** \( S_o \leftarrow T_S^S \cup T_o^L. \)

We remark that the set \( X' \) in Algorithm 4 is not equal to \( X \) but a subset of \( X \), and is given by \( X' = \{ i \in OPT | i \in [1/\varepsilon] \}. \)

**Theorem 19.** The total profits of items in set \( S_o \) given in Algorithm 4 is no less than \( p(S_o) \geq (1 - O(\varepsilon))OPT, \) while requires \( \tilde{O}(n + z^4 + \frac{z^2}{\varepsilon} \cdot \min\{n, \varepsilon^{-1}\}) \) time, which is within the order of \( \tilde{O}(n + \frac{z^2}{\varepsilon}) \).

**Proof.** Without loss of generality we can assume that

\[
\phi_L^X(OPT - \delta_{X'}, |O^* \cap L|) > w(O^* \cap L). \tag{24}
\]

Otherwise, the optimal solution in \( S \) already achieves a near optimal approximation. In the following we let \( x^{(s)} \) be the best approximation of \( \sum_{e \in O^* \cap L} p_e \) in \( X' \), i.e., \( x^{(s)} \in X' \) and

\[
\phi_L^X(x^{(s)}, k^{(s)}) \leq w(O^* \cap L) \leq \phi_L^X(x^{(s)} + \delta_{X'}, k^{(s)}), \tag{25}
\]

where \( k^{(s)} = |O^* \cap L| \). Notice that \( \phi_L^X \) is non-decreasing with respect to profit, we can conclude that such an \( x^{(s)} \) exists. In addition, \( x^{(s)} + \delta_{X'} \in X' \). On the other hand, we have

\[
x^{(s)} + \delta_{X'} \overset{(a)}{\geq} \varphi_L^X \left( \phi_L^X(x^{(s)} + \delta_{X'}, k^{(s)}) \right) \overset{(b)}{\geq} \varphi_L^X \left( w(O^* \cap L), k^{(s)} \right) \overset{(c)}{\geq} \varphi_L \left( w(O^* \cap L), k^{(s)} \right) - \varepsilon OPT, \tag{26}
\]

where (a) follows from the definition of \( \phi_L \) and \( \varphi_L \); (b) is based on the monotonicity of \( \varphi_L^X(\cdot, |L \cap O^*|) \) and RHS of (25); In (c) we utilize the point-wise convergence property of \( \varphi^X \) claimed in Lemma 10.
To summarize, the total profits of the selected set \( S_0 \) can be lower bounded as,

\[
p(S_0) = p(S^c_0) + p(S^S_0)
\geq (a) \left[ \tilde{\varphi}_S \left( W - \phi^X_k(x^{(s)}, k^{(s)}), K - k^{(s)} \right) - 4\varepsilon OPT \right] + x^{(s)}
\geq (b) \left[ \tilde{\varphi}_S \left( w(S \cap O^i), K - k^{(s)} \right) - 4\varepsilon OPT \right] + \left[ \varphi^C \left( w(O^i \cap L), k^{(s)} \right) - \delta_{X^i} - \varepsilon OPT \right]
\geq \varphi_S \left( w(O^i \cap L), K - k^{(s)} \right) + \varphi^C \left( w(O^i \cap L), k^{(s)} \right) - 6\varepsilon OPT
= p(O^i) - 6\varepsilon OPT = (1 - 6\varepsilon)OPT,
\]

where (a) comes from Lemma 17 and the fact that \( (k^{(s)}, x^{(s)}) \) is a candidate pair in the 3-th line of Algorithm 4. In (b), the first term follows from inequality (26), the second term is due to LHS of (25) and the monotonicity of \( \tilde{\varphi}_S \).

**Complexity Results.** The time complexity result directly follows from Lemmas 14 and 15:

\[
O(n) + \tilde{O} \left( \min \left\{ \frac{z^2}{\varepsilon^2}, \frac{n z^2}{\varepsilon} \right\} \right) + \tilde{O} \left( n + \min \left\{ \frac{z^2}{\varepsilon^2}, \frac{n z}{\varepsilon} \right\} + z^4 \right)
= \tilde{O} \left( n + z^4 + \frac{z^2}{\varepsilon} \cdot \min \left\{ n, \varepsilon^{-1} \right\} \right),
\]

which is within the order of \( \tilde{O}(n + z^2/\varepsilon^2) \). For the space requirement, we need to store the information about \( \phi^X_k \) to implement the new convolution operation in the current stage, which requires \( O(|X|z) = O(z^2/\varepsilon) \) space. Combining with Lemmas 14 and 15, it can be seen that \( O(n + z^2/\varepsilon) \) space is sufficient.

\[ \square \]

6 Application of Our New Scheme

In this section we present an application of our new scheme. We revisit the classic resource constrained scheduling problem \cite{JansenPorkolab2006}, in which the objective is to design a preemptive scheduling policy that minimizes the maximum completion time, while satisfying a resource constraint. More specifically, for a given set of tasks \( T = \{T_1, T_2, \ldots, T_n\} \) and \( m \) identical machines, it requires \( p_j \) \((j \in T)\) units of time and \( r_j \) \((j \in T)\) units of resources for processing task \( j \), while there are only \( c \) units of resources available at each time slot. The problem is to design a scheduling algorithm to minimize \( C_{\text{max}} \), the maximum completion time. As in the literature, the problem is denoted by \( P|\text{res1}, \ldots, \text{pmttn}|C_{\text{max}} \).

To obtain a faster FPTAS for this problem, we basically follow the approach proposed in \cite{JansenPorkolab2006}, which is mainly based on the linear programming formulation \cite{JansenPorkolab2006}, Eq (1.1)]. Though the LP has exponentially many variables in general, by exploiting its underlying structures, an approximate solution could be computed using binary search and solving the special max-min resource sharing problem \cite{JansenPorkolab2006}, Eq (1.2)] at each stage. Interestingly, for the case when there is only one resource constraint, the subproblem that we
need to solve turns out to be the $K$-item knapsack problem studied in this paper. In addition, we have following the proposition based on \cite{Jansen and Porkolab, 2006}.

**Proposition 20.** \cite{Jansen and Porkolab, 2006} A FPTAS for $K$-item knapsack problem with time complexity $T(n,m,1/\varepsilon)$ implies a FPTAS for problem $P|res_1,\ldots,pmtn|C_{\text{max}}$ with time complexity $O((T(n,m,1/\varepsilon) + n \log \log (n/\varepsilon)) n \log (1/\varepsilon)(1/\varepsilon^2 + \log n))$.

Consequently, the following theorem holds.

**Theorem 21.** For problem $P|res_1,\ldots,pmtn|C_{\text{max}}$, there exists a FPTAS with time complexity

$$O((z^2/\varepsilon^2 \log z \log(1/\varepsilon) + n \log \log (n/\varepsilon)) n \log (1/\varepsilon)(1/\varepsilon^2 + \log n)),$$

where $z = \min\{m,1/\varepsilon\}$. This improves the $O(n^2 \log(1/\varepsilon) \max(m^2/\varepsilon, \log \log(n/\varepsilon)) \log(1/\varepsilon)(1/\varepsilon^2 + \log n))$ time complexity result established in \cite{Jansen and Porkolab, 2006}.

**Proof.** This conclusion directly follows from equation (20) and Theorem 19.

**Remark 22.** Note that the complexity term in Proposition 20 is proportional to $T(n,m,1/\varepsilon) + n \log \log (n/\varepsilon)$. In most parameter regimes, it is dominated by $T(n,m,1/\varepsilon)$, the complexity of $KKP$. Roughly speaking, the complexity reduction achieved in $P|res_1,\ldots,pmtn|C_{\text{max}}$ is always in the same order as the improvement we obtain in $E-KKP$ ($KKP$). In addition, we claim that the complexity term in Theorem 21 can be further refined based on (29), from which we know that the (time or space) complexity results of the FPTAS can be improved over parameter regimes via our FPTAS.

## 7 Conclusion

In this paper we proposed a new FPTAS for the $K$-item Knapsack Problem (and Exactly $K$-item Knapsack Problem) that exhibits $\tilde{O}(K)$ and $O(z)$ improvements in time and space complexity respectively, compared with the state-of-the-art \cite{Mastrolilli and Hutter, 2006}. More importantly, our result suggests that for a fixed value of $\varepsilon$, an $(1-\varepsilon)$-approximation solution of $KKP$ can be computed in time asymptotically independent of cardinality bound $K$. Our scheme is also the first FPTAS that achieves better time and space complexity (up to logarithmic factors) than the standard dynamic programming scheme in \cite{Caprara et al., 2000} over all parameter regimes.

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A Supplementary Preliminaries

A.1 Exact $K$-item Knapsack Problem

The Exact $K$-item Knapsack Problem (E-KKP) is another variant of the knapsack problem which has a deep connection with KKP, and can be formally formulated via replacing the cardinality upper bound constraint by an equality constraint $\sum_{i \in E} x_i = K$. It has been shown in Caprara et al. [2000] that E-KKP and KKP can be converted into each other, i.e., any instance of one problem can be solved by using the algorithm of the other problem. We claim that our results presented in this paper work for E-KKP as well, which is straightforward to verify.

A.2 Knowledge of the value of $OPT$

Notice that a 1/2-approximate solution could be obtained in $O(n)$ time by properly rounding the real-valued solution of its linear programming relaxation to a feasible solution set [Caprara et al., 2000]. Hence, in the following analysis, for clarity of presentation, without loss of generality we can assume that we know the value of $OPT$. Indeed it can be verified that, if we replace $OPT$ by $2OPT'$ in the analysis, where $OPT'$ denotes the objective value of the 1/2-approximate solution, all of the analyses in this paper will still hold.

A.3 Upper bound on the number of non-empty classes

The correctness of bound (3) is straightforward. Based on the definition of $r_L, r_S$ in Definition 1, it can be seen that $(1 + \varepsilon)\max\{r_L, r_S\} \leq K/\varepsilon$ holds, combining this with the fact that there are at most $n$ non-empty classes, we conclude that (3) is true.

B Proof of Proposition 3

Proof. The first result is due to the simple fact that in each class $S^*_i$, we can retain the $K$ most profitable items, i.e., items with the smallest weights, and eliminate the other ones. Hence we have $|S| \leq \min\{Kr, n\}$. On the other hand, notice that $|O^* \cap L| \leq OPT/\min_{e \in O^* \cap L} p_e \leq \varepsilon^{-1}$, together with the fact that $|O^* \cap L| \leq |O^*| \leq K$, we know that Proposition 3 follows.

C Supplementary Materials of Section 4

C.1 Proof of Lemma 15

Proof. The complexity results in Lemma 15 can be achieved by letting $\tilde{\varphi}_S = \tilde{\varphi}_S^{(1)}$ when $K \leq \varepsilon^{-1}$ and $\tilde{\varphi}_S = \tilde{\varphi}_S^{(2)}$ otherwise. Therefore, the total running time is bounded by

$$O\left(\frac{z}{\varepsilon} \cdot \min\left\{\frac{K}{\varepsilon}, n\right\}\cdot 1_{\{K \leq \varepsilon^{-1}\}}\right) + \tilde{O}\left(\min\left\{\frac{K}{\varepsilon^2}, \frac{n}{\varepsilon}\right\}\right) + O\left(\frac{z}{\varepsilon}\right) \cdot 1_{\{K > \varepsilon^{-1}\}}$$

$$= \tilde{O}\left(\min\left\{\frac{z^2}{\varepsilon^2}, \frac{nz}{\varepsilon}\right\} + z^4 + \min\left\{Kz^2, nz\right\}\right).$$

The space required is in the order of $O(|\mathcal{K}||\mathcal{W}|) = O(z/\varepsilon)$. 

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C.2 Computing $f_t(\omega, k)$

- **Computing $\varphi_{S_\omega}^{(2)}(\omega, t)$**. We begin our method by dualizing the budget constraint through any non-negative Lagrangian multiplier $\mu$. It holds that $\varphi_{S_\omega}^{(2)}(\omega, t) = \min_{\mu \geq 0} L(\mu, \omega, t)$, where

$$L(\mu, \omega, t) = \max_{x_e \in [0, 1]} \left\{ \sum_{e \in S_\omega} p_e x_e + \mu \left( \omega - \sum_{e \in S_\omega} w_e x_e \right) \right\},$$

$$= \max_{x_e \in [0, 1]} \left\{ \mu \omega + \sum_{e \in S_\omega} p'_e(\mu) \cdot x_e \right\},$$

where $p'_e(\mu) = p_e - \mu w_e$. We remark that there are $O(\log^2(K/\varepsilon)/\varepsilon^2)$ types of elements in $S_\omega$. Here, two elements are of the same type iff both their weights and profits are identical to each other. This is because the profits and weights are rounded into an integer power of $(1 + \varepsilon)$ and there are $O(\log(K/\varepsilon)/\varepsilon)$ types of weights of profits and weights. For a fixed value of $\mu$, $\omega$ and $k$, function $L(\mu, \omega, k)$ can be computed by first sorting elements in $S_\omega$ in non-increasing order of $p'_e(\mu)$, and then selecting the top $k$ elements with non-negative value of $p'_e(\mu)$. This can be done within $\tilde{O}(\log^2(K/\varepsilon)/\varepsilon^2)$ time.

Note that the Lagrange function $L(\mu, \omega, k)$ is convex with respect to $\mu$, as it is the point-wise supremum of a family of linear functions in $\mu$. Moreover, as long as the order of the elements remain unchanged, $L(\mu, \omega, k)$ is a linear function with respect to $\mu$, with slope equal to $\omega - \sum_{e \in S} x_e$. Hence $L(\mu, \omega, k)$ is a piecewise linear function of $\mu$. As a sequence, the optimal multiplier $\mu^*$ must belong to set $B = \{\mu \mid \text{there exists } e^{(1)}, e^{(2)} \in S_\omega \text{ such that } p_{e^{(1)}}(\mu) = p_{e^{(2)}}(\mu)\}$. In particular,

$$B = \left\{ \frac{p_{e^{(1)}} - p_{e^{(2)}}}{w_{e^{(1)}} - w_{e^{(2)}}} \mid e^{(1)}, e^{(2)} \in S_\omega \right\} \subseteq \left\{ \frac{OPT}{\omega} \cdot b \mid b \in B' \right\},$$

where

$$B' = \left\{ (1 + \varepsilon)^b \cdot \frac{(1 + \varepsilon)^c - 1}{(1 + \varepsilon)^d - 1} \mid |b|, |c|, |d| \leq \log(K/\varepsilon)/\varepsilon, \text{ and } b, c, d \in \mathbb{Z} \right\}. \quad (32)$$

This follows from the facts that $p_{e^{(i)}} = \frac{\varepsilon\text{OPT}}{\kappa}(1 + \varepsilon)^{b_i}$ and $w_{e^{(i)}} = \frac{\varepsilon\omega}{\kappa}(1 + \varepsilon)^{c_i}$ ($i = 1, 2$) for some integers $b_i, c_i \in [\log(K/\varepsilon)/\varepsilon]$. Therefore $|B| = |B'| = O(\log^2(K/\varepsilon)/\varepsilon^2) = \tilde{O}(\varepsilon^{-3})$. Utilizing the convexity of $L(\mu, \omega, k)$, $\varphi_{S_\omega}^{(2)}(\omega, t)$ can be computed in $O(|B| \cdot \log^2(K/\varepsilon)/\varepsilon^2) = \tilde{O}(1/\varepsilon^2)$ time, by figuring out $\mu^*$ via binary search over set $B'$. It is worth pointing out that $B'$ must be computed and sorted in advance, which takes $O(|B'| \log |B'|) = \tilde{O}(\varepsilon^{-1})$ time.

- **Calculating $\varphi_{S_\omega}^{(2)}(t)$**. Let threshold set $W = \{\omega_1 \leq \omega_2 \leq \cdots \leq \omega_{|W|}\}$. We partition and store small item set $S$ as $S = \cup_{i=1}^{|W|-1} S^{(i)}$, where $S^{(i)} = \{e \in S \mid w_e \in [\omega_i, \omega_{i+1}]\}$ and

$$S^{(i)} = \left\{ e \in S \mid w_e \in (\omega_i, \omega_{i+1}] \right\} \quad (2 \leq i \leq |W| - 1), \quad (34)$$

which takes $O(|S| \cdot \log |W|) = \tilde{O}(|S|)$ time and $O(|S|)$ space. Without loss of generality, we can assume that items in sets $S = \{e_1, \ldots, e_{|S|}\}$ and $S^{(i)} = \{e_{(i)} \mid i \in [S]|\}$ are in
non-increasing order of profit values, as sorting takes $\tilde{O}(|S|)$ time, which is a low order term. We also store the partial summation sequence $\{A_{i,j}\}$, where

$$A_{i,j} = \sum_{t=1}^{j} p_{e_{s_{i}(t)}} (1 \leq i \leq |W| - 1, 1 \leq j \leq |S^{(i)}|)$$

represents the total profits of the first $j$ items in $S^{(i)}$. To compute functions $\varphi_{S_{\omega_{i}}}^{(2)}$, we deal with functions in non-decreasing value of $\omega$. More specifically, we first obtain the index of the $t$-th largest item in $S^{(i)} = \bigcup_{j=1}^{j} S^{(j)}$, denoted by $\eta_{i}(t)$. Then $\varphi_{S_{\omega_{i}}}^{(2)}(t)$ can be expressed as

$$\varphi_{S_{\omega_{i}}}^{(2)}(t) = \sum_{j=1}^{i} A_{j,\tau_{j}},$$

where $\tau_{j} = \max\{\sigma_{k}^{(j)} | \sigma_{k}^{(j)} \leq \tau_{i}(t)\}$ represents the largest index of items in $S^{j}$ which does not exceed $\eta_{j}(t)$, and it can be found in $O(\log |S^{j}|)$ time. Utilizing (36), $\varphi_{S_{\omega_{i}}}^{(2)}(t)$ can be computed in $O(|W|)$ time, under a given value of $\omega$ and $t$. The total complexity is

$$O(|W| \cdot \log K) \cdot O(|W|) + \sum_{j=1}^{|W| - 1} O(|S^{(j)}| \cdot \log |S^{(j)}|) = \tilde{O}(|W|^{2}) = \tilde{O}(n + \varepsilon^{-2}).$$

To summarize, our second type of relaxation $\{\varphi_{S}^{(2)}(\omega, k)\}_{\omega \in \mathcal{W}, k \in \mathcal{K}}$ can be computed in

$$\tilde{O}(\varepsilon^{-3}) + O(|\mathcal{K}| |W| \cdot \varepsilon^{-2}) + \tilde{O}(n + \varepsilon^{-2})$$

$$= \tilde{O}(n) + O(z/\varepsilon^{3})$$

(38)

time, and requires $O(n + |\mathcal{K}| \cdot |W|) = O(n + \frac{z}{\varepsilon})$ space.

D Application in Constrained submodular minimization

In this subsection, we present an application that lies in the field of submodular optimization, to show the power of $K$-item knapsack problem. We first give the formal definition of a submodular function.

**Definition 23.** (Submodular function) A set function $f(\cdot) : 2^{E} \rightarrow \mathbb{R}^{+}$ is submodular if for all subsets $S, T \subseteq E$, the inequality $f(S) + f(T) \geq f(S \cup T) + f(S \cap T)$ holds. And $f(\cdot)$ is monotone non-decreasing if $f(S) \geq f(T)$ holds for $\forall T \subseteq S$.

We aim to minimize function $F(\cdot) : 2^{E} \rightarrow \mathbb{R}^{+}$ subject to a cardinality constraint,

$$\min_{S \subseteq E} F(S) = f(S) + g(S),$$

$s.t. \ |S| = K$,

(39)

(40)

where $f(\cdot) : 2^{E} \rightarrow \mathbb{R}^{+}$ is a monotone non-decreasing submodular function, $g(\cdot) : 2^{E} \rightarrow \mathbb{R}^{+}$ is a monotone non-increasing and non-negative modular function.
Motivation: In practice this problem is motivated by the file selection problem in network caching. As the Internet traffic is dominated by popular contents (e.g., YouTube, Netflix videos) and the price of storage gets cheaper, recent Internet architectures such as Content-Centric Networking (CCN) suggest storing popular contents in network caches or routers, which could significantly reduce network congestion [Jacobson et al., 2007]. The main problem is how to choose files to store in a network cache so as to maximize the hit ratio (i.e., the probability that a requested file is stored in the cache). Returning to problem (39), the submodular function $f(S)$ corresponds to the required storage size to store the chosen file set $S$ and $g(S)$ represents the cache miss probability for the chosen files. In this problem, $f(\cdot)$ is a submodular function since we assume that caches compress the files to maximize the remaining storage and the compression efficiency increases as more files are compressed together [Nam et al., 2017], and $g(\cdot)$ is a modular function since the cache miss probability is the sum of the hit probabilities of the uncached files. We note that several techniques such as deduplication (used in Dropbox) [Meyer and Bolosky, 2012] and SyncCoding [Nam et al., 2017] enable caches to compress and decompress files instantly with low complexity. In this context, problem (39) aims to store $K$ files in order to minimize the sum of the storage space and the cache miss probability.

D.0.1 An optimal algorithm via ellipsoid relaxation and E-KKP

In this section, we present a near-optimal algorithm in which the solution to E-KKP plays an important role. One of the most important ingredients in the algorithm is the ellipsoid approximation [Goemans et al., 2009] of a submodular function.

Ellipsoid relaxation [Goemans et al., 2009]: For any monotone submodular function $f(\cdot)$, we can construct a function $f^2(\cdot): 2^E \rightarrow \mathbb{R}^+$ which approximates $f$ by a factor of $\alpha(n) = O(\sqrt{n \log n})$ via a polynomial number of queries to $f(\cdot)$, i.e., $f^2(S) \leq f(S) \leq \alpha(n) \cdot f^2(S)$. Moreover, $f^2$ has a particular simple form [Goemans et al., 2009], $f^2(S) = \sqrt{\sum_{e \in S} c_e}$, where constant $c_e > 0$. Utilizing this relaxation, we can reduce the problem to E-KKP as follows.

Reduction to E-KKP: We consider the following problem with arbitrarily given $B$ and $K$,

$$\min f^2(S),$$

$$\text{s.t. } g(S) \leq B,$$ (41)

$$|S| = K,$$ (42)

where constraint (42) is soft. It can be reduced to the E-kKP since $f^2(S)(\cdot)$ is the square root of a modular function and $g(\cdot)$ can represented as a constant minus a monotone increasing modular function, i.e., $g(\cdot) = C - g'(\cdot)$. Let $L^2 = \min_{|S| = K} f^2(S), U^2 = \max_{|S| = K} f^2(S)$. More specifically, we can solve the problem

$$\max g'(S),$$

$$\text{s.t. } f(S) \leq B',$$ (44)

$$|S| = K,$$ (45)

where $B'$ is the ratio between the compressed size and the sum of original file sizes.

[5] The compression efficiency is the ratio between the compressed size and the sum of original file sizes [Nam et al., 2017].

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for each $B' \in \{L^3, L^2(1 + \varepsilon), \ldots, U^2\}$ using the FPTAS for E-KKP, then find the minimum value of $B'$, under which the maximum value of $g'(S)$ is larger than $(1 - \varepsilon)(C - B)$.

**Performance Analysis:** We consider the iteration in which the parameter $B = \omega^*$ satisfies $g(O^*) \in [(1 - \varepsilon)\omega^*, \omega^*]$. Observe that under this given value of $B$, we can conclude that the returning solution $S_{\omega^*}$ of the FPTAS satisfies

$$f^2(S_{\omega^*}) = \sqrt{\sum_{e \in S_{\omega^*}} c_e} \leq \sqrt{(1 + \varepsilon) \sum_{e \in O^*} c_e} \leq (1 + \varepsilon)f^2(O^*),$$

which follows from the fact that the optimal solution $O^*$ is a candidate solution of problem (41). Moreover,

$$g(S_{\omega^*}) \leq \omega^* \leq \frac{g(S_{O^*})}{1 - \varepsilon}. \quad (47)$$

Consequently, we know that

$$\frac{F(O^*)}{F(S^*)} \geq \frac{F(O^*)}{F(S_{\omega^*})} = \frac{f(O^*) + g(O^*)}{f(S_{\omega^*}) + g(S_{\omega^*})} \geq (1 - \varepsilon)\frac{f(S_{\omega^*})/\alpha(n) + g(S_{\omega^*})}{f(S_{\omega^*}) + g(S_{\omega^*})} \geq (1 - \varepsilon) \left[1 - \left(1 - \frac{1}{\alpha(n)}\right) \frac{1}{1 + \eta}\right]$$

where $(a)$ is due to inequality (47) and the fact that

$$f(O^*) \geq f^2(O^*) \geq \frac{f^2(S_{\omega^*})}{(1 + \varepsilon)} \geq \frac{f(S_{\omega^*})}{(1 + \varepsilon)\alpha(n)}$$

And $\eta = \min_{S, S \subseteq E} \{g(S)/f(S)\}$ denotes the minimum possible value of the ratio between $f$ and $g$. Notice that $g(S_{\omega^*}) \geq \eta f(S_{\omega^*})$, hence $(b)$ follows. Intuitively the difficulty of the submodular minimization is related to $\eta$. For example, when $\eta$ increases, the problem becomes easier since the proportion of the modular function increases.

**Lower Bound on the Performance Guarantee:** Here we show the following lower bound on the performance guarantee of any polynomial time algorithm for problem (39).

**Proposition 24.** Given a submodular function $f$ with curvature $\kappa$ and modular function $g$, no polynomial time algorithm can achieve performance guarantee better than $1 - \left(1 - \sqrt{\frac{\log n}{n}}\right) \frac{1}{1 + \eta}$.

**Proof.** The proof of the lower bound is tailored from [Goemans et al. 2009] [Svitkina and Fleischer 2013]. The main idea is to construct a set of submodular functions with the property that, if we query the function on a polynomial number of sets, there always exists two functions which are indistinguishable under the query sequence, while their optimal solution differs greatly.

Consider the following two functions,

$$F_1(S) = r_U(S) + \eta \alpha, \quad F_R(S) = r_R(S) + \eta \alpha,$$

where the cardinality $K = \alpha$, $\alpha = \Theta(\sqrt{n}\theta)$, $\beta = \omega(\log^2 n)$ and $r_U(S) = \min\{|S|, \alpha\}$ denotes the rank function of a uniform matroid; $r_R(S) = \min\{|S|, \beta + |S \cap \hat{R}|, \alpha\}$ represents the rank function

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of matroid \((E, I)\) where \(I = \{T \subseteq E : |T| \leq \alpha, |T \cap R| \leq \beta\}\). Note that \(f_1(S) \neq f_R(S)\) is equivalent to \(r_U(S) \neq r_R(S)\), which implies that \(|S \cap R| > \beta\). Using Chernoff bound we can show that \(\mathbb{P}(|S \cap R| > \beta) \leq n^{-\omega(1)}\). Hence if we make a polynomial number of queries, say \(n^p\), with probability at least \(1 - n^{-\omega(1)} \cdot n^p > 0\), the algorithm fails to distinguish \(f_1(\cdot)\) and \(f_R(\cdot)\), which implies that the approximation guarantee of any polynomial time algorithm is bounded by

\[
\frac{\min_{S:|S|=K} f_R(S)}{\min_{S:|S|=K} f_1(S)} = \frac{\beta + \eta \alpha}{(1 + \eta) \alpha} = 1 - \left(1 - \sqrt{\frac{\log n}{n}}\right) \frac{1}{1 + \eta}.
\]

The proof is complete. \(\square\)