PERIODIC ORBITS ON A 120-ISOSCELES TRIANGLE, 60-RHOMBUS, 60-90-120-KITE, AND 30-RIGHT TRIANGLE

BENJAMIN R. BAER, FAHEEM GILANI, ZHIGANG HAN, AND RONALD UMBLE

Abstract. A periodic orbit on a frictionless billiard table is a piecewise linear path of a billiard ball that begins and ends at the same point with the same angle of incidence. The period of a primitive periodic orbit is the number of times the ball strikes a side of the table as it traverses its trajectory exactly once. In this paper we find and classify the periodic orbits on a billiard table in the shape of a 120-isosceles triangle, a 60-rhombus, a 60-90-120-kite, and a 30-right triangle. In each case, we use the edge tessellation (also known as tiling) of the plane generated by the figure to unfold a periodic orbit into a straight line segment and to derive a formula for its period in terms of the initial angle and initial position.

1. Introduction

In their 2006 Math Monthly article “Periodic Orbits for Billiards on an Equilateral Triangle” [2], A. Baxter and R. Umble classified and counted equivalence classes of periodic orbits of a billiard ball in motion on a frictionless equilateral triangular table. During an REU at Rutgers University in the summer of 2011, A. Baxter and students E. McCarthy and J. Eskreis-Winkler solved the analogous problem on billiard tables in the shape of a rectangle and an isosceles right triangle [1]. In this paper we consider the problem on a 120-isosceles triangle, a 60-rhombus, a 60-90-120-kite, and a 30-right triangle.

In general, it is known that rational polygons, i.e., polygons whose angles are rational multiples of $\pi$, admit periodic trajectories (see [3] for a proof that periodic trajectories are dense in rational polygons). In this paper, we find and classify the periodic orbits on the aforementioned polygons and briefly discuss partial progress on the counting problem. Our analysis on a 120-isosceles triangle appears in Section 2; the principles applied in Section 2 carry over to Section 3 in which we analyze the other three cases.

The trajectory of a free particle inside a polygon $G$ with boundary $\partial G$ is the piecewise linear path $\gamma : \mathbb{R} \to G$ of that particle subject to elastic collision at the boundary, i.e., the particle following the trajectory moves in a straight line and reflects at $\partial G$ so that the angle of incidence at $\partial G$ equals the angle of reflection. Note that in this paper we do not consider trajectories that encounter a vertex of $G$ (so called singular trajectories). We are interested in understanding periodic trajectories, i.e., trajectories $\gamma$ such that for some $T < \infty$, we have $\gamma(t + T) = \gamma(t)$ for all $t$. 

Date: October 29, 2019.

2010 Mathematics Subject Classification. Primary 37E15; Secondary 97G50.

Key words and phrases. Billiards, edge tessellating polygon, periodic orbit.
Let $\gamma$ be a periodic trajectory and $S$ be the smallest positive real number such that $\gamma(t + S) = \gamma(t)$ for all $t$. If $R$ is a multiple of $S$, the restriction of $\gamma$ to $(0, R]$ is a periodic orbit; it is primitive if $R = S$. In this paper, we wish to study the geometric properties of periodic trajectories, so we only consider the trace of $\gamma$, which is independent of parameterization. Consequently, we define the period of a primitive periodic orbit $\gamma$ as the number of times $\gamma$ strikes the boundary of $G$.

An edge tessellation of the plane is a tessellation or tiling generated by reflecting some polygon in its edges. In [5], M. Kirby and R. Umble proved that edge tessellations are generated by eight types of polygons, namely, an equilateral triangle, an isosceles right triangle, a 30-right triangle, a rectangle, a 60-rhombus, a 60-90-120-kite, and a regular hexagon. Periodic orbits have now been found and classified on all of these except the regular hexagon. Note that the billiards system on an edge tessellating polygon is integrable if and only if the polygon is non-obtuse [4].

Let $T$ be the edge tessellation generated by $G$, where $G$ is a 120-isosceles triangle, a 60-rhombus, a 60-90-120-kite, or a 30-right triangle (Figures 4-7 in Sections 2 and 3 display the edge tessellations generated by these four polygons). The edges of $T$, called inclines, form several infinite families of parallels.

A restricted trajectory is the restriction of some trajectory $\gamma$ to an interval $(0, R]$. We will also use the symbol $\tau$ to denote a restricted trajectory when the context is clear. The unfolding $\upsilon$ of a restricted trajectory $\gamma$ is the line segment in $T$ produced by successively reflecting the trace of $\gamma$ and its images in the inclines of $T$. Let $\sigma$ be this sequence of reflections. The unfolding $\upsilon$ in $T$ allows us to leverage the geometry of $T$ to characterize periodicity and calculate the length and period of primitive orbits.

2. Periodic orbits on a 120-isosceles triangle

Consider a 120-isosceles $\triangle ABC$ positioned and labeled so that $\overrightarrow{AC}$ is horizontal with $A$ to the left of $C$, the apex $B$ is positioned above $\overrightarrow{AC}$, and $\angle B$ is obtuse. Let $\gamma : (0, T] \rightarrow \triangle ABC$ be a restricted trajectory with initial point $\lim_{t \searrow 0} \gamma(t)$ and terminal point $\gamma(T)$.

The inclines of the edge tessellation $T$ generated by the initial triangle $\triangle ABC$ form six infinite families of parallels with angles of inclination 0, 30, 60, 90, 120, and 150 degrees. Write the terminal triangle $\triangle A'B'C' = \sigma(\triangle ABC)$. Let $\upsilon$ be the unfolding of $\gamma$ with initial point $P = \lim_{t \searrow 0} \upsilon(t)$ and terminal point $Q = \upsilon(T) = \sigma(\gamma(T))$.

2.1. Preliminary exploration of periodic orbits. We will find necessary and sufficient conditions on the unfolding $\upsilon$ that ensure that $\gamma$ is a periodic orbit. These conditions will be particularly amenable to later analysis. First, let $\upsilon$ be the unfolding of a periodic orbit with initial point $P$ on $\overrightarrow{AC}$, terminal point $Q$, and initial angle $\Theta = \angle QPC$. Observe that if $\Theta \neq 90$, the reflection of $\overrightarrow{PQ}$ in the vertical line through $P$ is a different unfolding of $\gamma$. Thus we may restrict our considerations to initial angles in the range $0 < \Theta \leq 90$. Note that $\upsilon = \overrightarrow{PQ}$ crosses the interior of $\triangle ABC$ by construction, and the edge reflections of $\overrightarrow{AC}$ generate the horizontal, 60°, and 120° inclines of $T$. We claim that $Q$ is on a horizontal incline.

If $Q$ is on some 60° incline, the terminal angle at $Q$ is $60 - \Theta$, $90 + 120 - \Theta$, $60 - \Theta$, or $240 - \Theta$, which equals the initial angle $\Theta$ at $P$. Since $0 < \Theta \leq 90$, we have $\Theta = 30$ (see Figure [1]). However, an initial angle of $\Theta = 30$ produces a period 8
orbit, which terminates on a horizontal incline (see Figure 2). Thus, Q is not on a 60° incline.

Figure 1. Equal initial and terminal angles of an unfolding.

Figure 2. An unfolding with initial angle 30° terminates on a horizontal.

Suppose Q is on some 120° incline of T. Then the terminal angle at Q is 120 − Θ, Θ + 60, Θ − 120, or 300 − Θ, which equals the initial angle Θ at P. Since 0 < Θ ≤ 90, we have Θ = 60. But an initial angle of Θ = 60 produces an orbit of either period 4 or period 10, both of which terminate on a horizontal (see Figure 3a and Figure 3b). Thus Q is not on a 120° incline. Therefore Q is on a horizontal incline as claimed.

The primitive periodic orbits displayed in Figure 8 have initial angle 60° and respective periods 4 and 10. This motivates our next definition. A periodic trajectory with initial angle Θ is monoperiodic if every primitive periodic orbit with initial angle Θ has the same period p; otherwise, it is biperiodic. Indeed, the period 8 orbit in Figure 2 is monoperiodic; the period 4 and period 10 orbits in Figure 3a and Figure 3b are biperiodic. Note that in the latter two cases, the period is 4 if and only if the apex B of △ABC is in the interior of the angle formed by the components of the orbit that meet at P.

Since Q is on a horizontal incline of T, the base A'C' is horizontal. Furthermore, since PQ does not cross the interior of △A'B'C', its apex B' lies above the base. Thus the composition σ is a translation or a glide reflection. However, if σ is a glide reflection, it reverses orientation and the initial angle Θ = 90°. But a periodic orbit with initial angle 90° coincides with the monoperiodic period 8 orbit with
initial angle $30^\circ$, in which case $\sigma$ is a composition of eight reflections and preserves orientation. Therefore $\sigma$ is the translation by vector $PQ$, which we denote by $\tau_{PQ}$. Consequently, the period $p$ of $\gamma$ is even.

Next, we show that every periodic orbit has an unfolding with initial angle $\Theta$ in the range $60^\circ \leq \Theta \leq 90$. We have already restricted $\Theta$ to the range $0 < \Theta < 90$.

Suppose $0 < \Theta \leq 30$. Since $Q$ is on a horizontal incline of $T$, $\overrightarrow{PQ}$ cuts a $120^\circ$ incline with angle of incidence $\Phi = 60 + \Theta$. By assumption and the symmetry of $T$, there is an unfolding $\overrightarrow{P'Q'}$ of $\gamma$ with initial angle $\Phi$ in the range $60 < \Phi < 90$. For example, the period 8 orbit with initial angle $\Theta = 30$ displayed in Figure 2 is represented by the unfolding $\overrightarrow{PQ}$. However, $\gamma$ can also be represented by the unfolding $\overrightarrow{P'Q'}$ whose initial angle $\Phi = 90$, where $P'$ is the midpoint of $\overrightarrow{PC}$.

Now suppose that $30 < \Theta < 60$. Let $\overrightarrow{PQ}$ be the reflection of $\overrightarrow{PQ}$ in $\overrightarrow{AC}$. Then $\overrightarrow{P'Q'}$ cuts a $60^\circ$ incline at a point $P'$ with angle of incidence $\Psi = 120 - \Theta$. Thus $\gamma$ can be represented by an unfolding $\overrightarrow{P'R}$ with initial angle $\Psi$ in the range $60 < \Psi < 90$. Therefore every periodic orbit has an unfolding with initial angle in the range $60 \leq \Theta \leq 90$ as claimed.

We summarize the remarks above as a proposition:

**Proposition 1.** Let $\gamma$ be a periodic orbit on a 120-isosceles $\triangle ABC$ positioned and labeled so that $\overrightarrow{AC}$ is horizontal, $B$ is the apex, and $\angle B$ is obtuse. Let $T$ be the edge tessellation generated by $\triangle ABC$. Position the initial point $P$ on $\overrightarrow{AC}$, and let $\upsilon$ be an unfolding of $\gamma$ in $T$. Then

1. the terminal triangle $\triangle A'B'C' = \tau_{PQ} (\triangle ABC)$,
2. the period of $\gamma$ is even, and
3. the periodic unfolding $\upsilon$ can be uniquely chosen with initial angle $\Theta \in [60, 90]$.

In view of Proposition 1 parts (1) and (3), we restrict our considerations to trajectories $\gamma$ with initial angle in the range $(60, 90]$ and unfoldings $\upsilon$ with terminal point on a horizontal incline of $T$.  

![Figure 3](image-url)

(a) An unfolding with initial angle $60^\circ$ and period 10.

(b) An unfolding with initial angle $60^\circ$ and period 4.
Since $\tau_{PQ}(B) = B'$, we can further restrict our considerations to trajectories $\gamma$ for which the unfolding $v$ satisfies $PQ = BB'$, then determine whether or not the corresponding composition of reflections $\sigma$ is a translation; when it is, Proposition 1 part (1) holds. To this end, we say that the initial point $P$ and the terminal point $Q$ are linked if $PQ = BB'$ and that the initial triangle $\Delta ABC$ and the terminal triangle $\Delta A'B'C'$ are consistently oriented if $\sigma$ is orientation preserving. These fundamental ideas characterize periodicity and will provide the basis for later analysis.

**Proposition 2.** Let $v = PQ$ be the unfolding of a restricted trajectory $\gamma$ with initial and terminal points positioned on horizontals, and let the initial triangle be as in Proposition 1. Then $\gamma$ is a periodic orbit if and only if $P$ and $Q$ are linked and $\Delta A'B'C'$ and $\Delta ABC$ are consistently oriented.

**Proof.** If $\gamma$ is a periodic orbit, then $P$ and $Q$ are linked and the initial and terminal triangles are consistently oriented by definition.

Conversely, if $\Delta A'B'C'$ and $\Delta ABC$ are consistently oriented then $\sigma$ is orientation preserving and is therefore a translation that maps $P$ to $\sigma(P)$. If $P$ and $Q$ are linked, $\sigma(P) = Q$ so that the the initial and terminal points of $\gamma$ are in the same position and the initial and terminal angles are equal [8]. Thus $\gamma$ is a periodic orbit. $\square$

Note that if the initial point of a restricted trajectory $\gamma$ is perturbed along the base of the triangle, the resulting sequence of reflections does not change so long as $\gamma$ does not encounter a vertex. Also note that if $P$ and $Q$ are linked, then they remain linked when translated horizontally. Consequently, if $\gamma$ is a primitive periodic orbit then, by Proposition 2, there is an open neighborhood around the initial point of $\gamma$ in which every trajectory with the same initial angle as $\gamma$ will have the same same period as $\gamma$. Consequently, there are no Fagnano orbits [6] in the 120-isosceles triangle.

### 2.2. Counting the number of inclines cut by an unfolding.

Let $AC$ denote the length of $AC$ and let $u = \frac{1}{2}AC$. Impose a right-hand rectangular coordinate system on $T$ with horizontal axis $\overrightarrow{AC}$, origin $O$ at the midpoint of $\overrightarrow{AC}$, horizontal unit of length $u$, and vertical unit of length $\sqrt{3}u$. Then adjacent horizontal inclines lie one (vertical) unit apart. If $v$ is an unfolding of a periodic orbit with initial point $P$ and terminal point $Q$, Proposition 1 part (1) implies $PQ = (x, y)$ for some $x, y \in \mathbb{N}$ so that $\Theta = \arctan \left(\frac{x}{\sqrt{3}}\right)$. Furthermore, $\Theta \in (60, 90]$ implies $x \leq y$, and we may assume that $\gcd(x, y) = 1$.

Let $P = (a, 0)$ and choose the parametrization $v(t) := (t + a, \frac{\sqrt{3}}{2}t)$, where $t \in (0, T]$ and $a \in (-1, 1)$. Then $P = \lim_{t \to 0} v(t)$ and $Q = v(T)$. Between any two consecutive horizontal inclines of $T$, the unfolding $v$ cuts four non-vertical inclines and possibly a fifth vertical incline. Since points on vertical inclines have integer first-coordinates, it is important to understand the behavior of $v(t)$ when its first-coordinates $t + a$ are integers. A closer inspection of the coordinate system imposed on the tessellation $T$ reveals a fundamental vertical segment of length 2 that occurs repeatedly whenever the horizontal coordinate is an integer. Determining the behavior of the unfolding on these segments is essential for determining the number of times a fifth vertical incline is cut and motivates the following definition.
Let $\mathcal{F} = [0, 2)$. Define a function $f : \mathbb{Z} \times \mathbb{R} \to \mathcal{F}$ by

$$f(\alpha, \beta) := \begin{cases} \beta \mod 2, & \text{if } 2 | \alpha \\ (\beta + 1) \mod 2, & \text{otherwise.} \end{cases}$$

The codomain $\mathcal{F}$ is called the fence, the set $\mathcal{B} = \left(\frac{1}{2}, \frac{3}{2}\right)$ is called the barrier, and its relative complement $\mathcal{B}^c = \mathcal{F} \setminus \mathcal{B}$ is called the gate. Consider the points at which $\upsilon$ cuts a vertical incline and form the set $J_{a,T} = \mathbb{Z} \cap (a, a + T + 1]$ of their first coordinates. For each $i \in J_{a,T}$, let $t_i = i - a$ and define the set of contact points of $\upsilon$ to be

$$\mathcal{C}_{a,T} := \left\{ f\left(i, \frac{y}{x}t_i\right) : i \in J_{a,T} \right\}.$$

Note that $(i, \frac{y}{x}t_i) = \upsilon(t_i)$ when $t_i \leq T$. Now consider a contact point $c \in \mathcal{C}_{a,T}$. The multiplicity of $c$ is the integer

$$m_{a,T}(c) := \# \left\{ t_i : f\left(i, \frac{y}{x}t_i\right) = c \right\},$$

which is the number of times $\upsilon$ cuts a vertical incline at the same relative point $c$. For an illustration of these definitions, see Figure 4.

![Figure 4](image-url)

(a) Geometric motivation for the fence.  
(b) The fence for (A) with contact points. The first contact point is labeled 1.

Without loss of generality, we may assume that the unfolding $\upsilon$ of a periodic orbit $\gamma$ has domain $(0, 1]$ so that its $k$-fold iterate $\upsilon^k$ has domain $(0, k]$.

**Lemma 3.** Let $\gamma$ be a restricted trajectory with linked initial and terminal points $P$ and $Q$, initial angle $\Theta \in (60, 90)$, unfolding $\upsilon$, and contact points $C$. For any natural number $k$, let $\gamma^k$ be the $k$-fold iterate of the trajectory $\gamma$. Then
(1) its set of contact points is \( C \),

(2) its contact points have constant multiplicity, which is \( k \) times the multiplicity of the contact points of \( v \),

(3) the unfolding terminates at the terminal point of the position vector \( kPQ \), and

(4) its contact points are equally spaced along the fence.

Proof. First, consider the \( k \)-fold iterate \( \gamma^k \). Since each of the \( k \) iterates is identical to \( \gamma \) and has contact points \( C \), the unfolding of \( \gamma^k \) terminates at the terminal point of \( kPQ \) in standard position and each contact point of \( \gamma^k \) has constant multiplicity.

In the basis \( B = \{ \frac{1}{2}, \frac{\sqrt{3}}{2} \} \), a point \( (x, y) \) in \( B \)-coordinates projects to the point \( (0, y) \) along a line parallel to a 60° incline and its image on the fence is \( y \mod 2 \). Given a restricted trajectory \( \gamma \) with unfolding \( \gamma^k \) satisfying the hypothesis, reposition \( \gamma^k \) with \( P \) and \( Q \) on vertical inclines. Then \( \gamma^k \) cuts the vertical inclines at points \( P = (x_1, y_1), (x_2, y_2), \ldots, Q = (x_n, y_n) \) in \( B \)-coordinates, where \( y_i < y_{i+1} \) for all \( i \), and \( C = \{ y_i \mod 2 \}_{1 \leq i \leq n} \). If \( n = 2 \), there is nothing to prove. Otherwise, let \( \Delta y = y_{i+1} - y_i \) and let \( m \) be the smallest positive integer such that \( \Delta y + 2Z = m\Delta y + 2Z \). Then \( \Delta y \in \mathbb{Q} \) since \((m-1)\Delta y \in 2Z \). Consider the cyclic subgroup \( G \subset \mathbb{Q}/2Z \) generated by \( \gamma \Delta y + 2Z \). Then \( G = \{ i\Delta y \Delta y + 2Z : 1 \leq i \leq m \} \), \( y_i - y_1 \in i\Delta y + 2Z \) for all \( i \), and \( C = (G + y_1) \cap [0, 2) \). Since the distance between adjacent cosets in \( G \) is constant, and each translated coset in \( G + y_1 \) contains exactly one contact point, the distance between adjacent contact points is constant as well. \( \square \)

An illustration of the contact points along the fence appears in Figure 4b. The contact points are equally spaced when the initial and terminal points are linked. Lemma 3 demonstrates the fundamental nature of contact points. Understanding primitive periodic orbits reduces to understanding contact points and their multiplicity for any linked trajectory.

Let \( T \) be a positive integer. Observe that the initial and terminal points are linked if and only if \( f \left( T, \frac{x}{T} (T - a) \right) = f \left( 0, -\frac{x}{a} \right) \), which is equivalent to the condition

\[
2 \mid \begin{cases} \frac{x}{T}, & \text{if } 2 \mid T \\ \frac{x}{T} + 1, & \text{if } 2 \not\mid T. \end{cases}
\]

Thus whenever \( T \) is an even multiple of \( x \), the initial and terminal points of a restricted trajectory are linked and its contact points have constant multiplicity by Lemma 3.

We can now compute the number of inclines cut by \( v \).

**Proposition 5.** Choose \( T \) so that all contact points in \( \mathcal{C}_{a,T} \) have the same multiplicity \( m_{a,T} \), and let \( b_{a,T} = |B \cap \mathcal{C}_{a,T}| \) be the number of contact points on the barrier. Then, the number of inclines of \( T \) cut by the unfolding \( v(t) \), \( 0 < t \leq T \), is

\[
N_{a,T} = \frac{4y}{x} T + m_{a,T} \cdot b_{a,T}.
\]

Proof. Recall that, between any two consecutive horizontal inclines of \( T \), the unfolding \( v \) cuts four non-vertical inclines and possibly a fifth vertical incline at a point corresponding to a contact point on the barrier. Note that there are a total of \( m_{a,T} \cdot b_{a,T} \) contact points on the barrier, counting multiplicities. Thus \( v \) crosses
the barrier. Proposition 8. The number $N = N_{a,2x}$ of inclines of $T$ cut by $v(t)$, $0 < t \leq 2x$, is given by the following table:

| $N \equiv 1, 3 \mod 4$ | $N \equiv 2 \mod 4$ | $N \equiv 0 \mod 4$ |
|------------------------|------------------------|------------------------|
| 8$y + \frac{3}{4}x$ | 8$y + \frac{5}{4}x$ | 8$y + \frac{7}{4}x$ |
| $x \equiv 0 \mod 3$ and $x \equiv y \mod 2$ | $x \equiv 0 \mod 3$ and $x \equiv y \mod 2$ | $x \equiv 1 \mod 3$ and $x \equiv y \mod 2$ |
| $\frac{5}{4}y + 2x$ | $\frac{5}{4}y + 2x$ | $\frac{5}{4}y + 2x$ |
| $x \equiv 1 \mod 3$ and $x \equiv y \mod 2$ | $x \equiv 2 \mod 3$ and $x \equiv y \mod 2$ | $x \equiv 2 \mod 3$ and $x \equiv y \mod 2$ |
Remark. The columns of the table are thus arranged for easier reference when proving Proposition 7.

Proof. Let \( a = -\frac{2}{5} \epsilon \), where \( 0 \leq \epsilon \leq \frac{1}{3} \) for convenience. Since the trajectory cuts the vertical axis at the point \( v(t_0) = (0, -\frac{2}{5}a) \), the image of \( v(t_0) \) on the fence is \( f(v(t_0)) = \epsilon \). Let \( s \) be the distance between consecutive contact points of \( C_{a,2x} \), and recall that the barrier \( B = \{ \frac{1}{5}, \frac{5}{3} \} \). Thus, \( b_{-\frac{2}{5} \epsilon, 2x} \) equals

\[
\left\{ w \in \mathbb{Z} : \frac{1}{3} < sw + \epsilon \leq \frac{5}{3} \right\} = \# \left( \mathbb{Z} \cap \left( 0, \frac{5 - \epsilon}{s} \right) \right) - \# \left( \mathbb{Z} \cap \left( 0, \frac{1 - \epsilon}{s} \right) \right)
\]

\[
= \left\lfloor \frac{5 - \epsilon}{s} \right\rfloor - \left\lfloor \frac{1 - \epsilon}{s} \right\rfloor,
\]

where \( \lfloor \cdot \rfloor \) denotes the floor function. By Lemma 7 when \( x \neq y \mod 2, s = \frac{1}{3} \) and \( m_{-\frac{2}{5} \epsilon, 2x} = 1 \); when \( x \equiv y \mod 2, s = \frac{2}{5} \) and \( m_{-\frac{2}{5} \epsilon, 2x} = 2 \). Therefore, the number of vertical inclines cut by \( v(t) \), \( 0 < t \leq 2x \), is

\[
m_{-\frac{2}{5} \epsilon, 2x} \cdot b_{-\frac{2}{5} \epsilon, 2x} = \begin{cases} \left\lfloor \frac{5 - \epsilon}{s} \right\rfloor - \left\lfloor \frac{1 - \epsilon}{s} \right\rfloor & x \neq y \mod 2 \\ 2 \left\lfloor \frac{5 - \epsilon}{s} \right\rfloor - 2 \left\lfloor \frac{1 - \epsilon}{s} \right\rfloor & x \equiv y \mod 2 \end{cases}
\]

In view of Lemma 7, all possible values of \( b_{-\frac{2}{5} \epsilon, 2x} \) are given by one value of \( \epsilon \) when \( 3 \mid x \) and two values of \( \epsilon \) when \( 3 \nmid x \). It turns out that \( \epsilon = 0 \) is a choice for which formula (2.9) is readily computable when \( 3 \mid x \); similarly, \( \epsilon \in \{0, \frac{1}{3}\} \) is the desired choice (and leads to distinct counts) when \( 3 \nmid x \). Therefore, the table for \( N_{a,2x} \) follows from repeated application of Proposition 5. The only non-routine part of the computation follows from the fact that \( x \equiv y \mod 2 \) and \( \gcd(x,y) = 1 \) imply \( x \) and \( y \) are odd. The first column (where \( N_{a,2x} \equiv 1, 3 \mod 4 \) of the table used the value \( \epsilon = 0 \), the second column used \( \epsilon = \frac{1}{3} \), and the third column used \( \epsilon = 0 \). \( \square \)

2.3. Determining the period of a primitive periodic orbit. We now turn our attention to primitive periodic orbits. In this setting, \( T \) is the smallest positive integer such that \( v(t) \) is the unfolding of a primitive periodic orbit.

In the course of the argument that establishes the next proposition, we make two critical observations: (1) when the initial and terminal points of a restricted trajectory \( \gamma \) are linked, doubling the length produces a periodic orbit, and (2) when \( \gamma \) is biperiodic, horizontally translating \( \gamma \) produces a periodic orbit.

Proposition 10. Let \( x, y \in \mathbb{N} \) with \( x \leq y \) and \( \gcd(x, y) = 1 \), and let \( a \in (-1,1) \). Then a primitive periodic orbit \( \gamma \) with initial angle \( \Theta = \arctan(\frac{2}{5}\sqrt{3}) \) and initial point \( P = (a,0) \) has an unfolding \( v(t) = (t + a, \frac{2}{5}t) \), \( 0 < t \leq T \), where \( T = x, 2x \) or \( 4x \). In particular, if \( p_a(x, y) \) is the period of \( \gamma \), then

1. If \( x \neq y \mod 2 \) and \( N_{a,2x} \) is even, then \( T = 2x \) and \( p(x, y) = N_{a,2x} \).
2. If \( x \neq y \mod 2 \) and \( N_{a,2x} \) is odd, then \( T = 4x \) and \( p(x, y) = 2N_{a,2x} \).
3. If \( x \equiv y \mod 2 \) and \( N_{a,2x} \equiv 0 \mod 4 \), then \( T = x \) and \( p(x, y) = \frac{1}{2}N_{a,2x} \).
4. If \( x \equiv y \mod 2 \) and \( N_{a,2x} \equiv 2 \mod 4 \), then \( T = 2x \) and \( p(x, y) = N_{a,2x} \).

Proof. Note that if \( v(t) = (t + a, \frac{2}{5}t), 0 < t \leq T \), is an unfolding of a primitive periodic orbit if and only if among all \( T \) for which the initial and terminal points are linked, \( T \) is the minimum such that \( v \) cuts an even number of inclines of \( T \).
Let $S$ be the smallest positive integer such that the initial and terminal points of the unfolding $(t + a, \frac{2}{y} t)$, $0 < t \leq S$ are linked. Recall from Lemma 10 that $S = 2x$ when $x \not\equiv y \mod 2$, and $S = x$ when $x \equiv y \mod 2$.

**Case 1.** If $x \not\equiv y \mod 2$, then $S = 2x$. If $N_{a,2x}$ is even, then $v(t)$, $0 < t \leq 2x$, is an unfolding of a primitive periodic orbit. Thus $T = 2x$ and $p(x, y) = N_{a,2x}$. If $N_{a,2x}$ is odd, then $v(t)$, $0 < t \leq 4x$, is an unfolding of a primitive periodic orbit. Thus $T = 4x$ and $p(x, y) = 2N_{a,2x}$. 

**Case 2.** If $x \equiv y \mod 2$, then $S = x$. Note that since the multiplicity of each contact point on the barrier is 2, the number of inclines of $T$ cut by $v(t)$, $0 < t \leq x$, is $\frac{1}{2} N_{a,2x}$. If $\frac{1}{2} N_{a,2x}$ is even (i.e., $N_{a,2x} \equiv 0 \mod 4$), then $v(t)$, $0 < t \leq x$, is an unfolding of a primitive periodic orbit. Thus $T = x$ and $p(x, y) = \frac{1}{2} N_{a,2x}$. If $\frac{1}{2} N_{a,2x}$ is odd (i.e., $N_{a,2x} \equiv 2 \mod 4$), then $v(t)$, $0 < t \leq 2x$, is an unfolding of a primitive periodic orbit. Thus $T = 2x$ and $p(x, y) = N_{a,2x}$. \hfill \Box

We can now determine the period of every primitive periodic orbit.

**Theorem 11.** Let $x, y \in \mathbb{N}$ with $x \leq y$ and $\gcd(x, y) = 1$. Then the period $p_a(x, y)$ of a primitive periodic orbit with initial angle $\Theta = \arctan(\frac{2}{\sqrt{3}})$ and initial point $P = (a, 0)$ is

$$p_a(x, y) = \begin{cases} 4y + \frac{2x}{3} & \text{if } x \equiv 0 \mod 3 \text{ and } x \equiv y \mod 2 \\ 8y + \frac{4x}{3} & \text{if } x \equiv 0 \mod 3 \text{ and } x \not\equiv y \mod 2 \\ 4y + \frac{2x+2}{3} \text{ or } 8y + \frac{4x+2}{3} & \text{if } x \equiv 1 \mod 3 \text{ and } x \equiv y \mod 2 \\ 16y + \frac{8x}{3} \text{ or } 8y + \frac{4x+2}{3} & \text{if } x \equiv 1 \mod 3 \text{ and } x \not\equiv y \mod 2 \\ 4y + \frac{2x+2}{3} \text{ or } 8y + \frac{4x-2}{3} & \text{if } x \equiv 2 \mod 3 \text{ and } x \equiv y \mod 2 \\ 16y + \frac{8x+2}{3} \text{ or } 8y + \frac{4x-2}{3} & \text{if } x \equiv 2 \mod 3 \text{ and } x \not\equiv y \mod 2. \end{cases}$$

Furthermore, if the orbit is biperiodic, the period $p_a(x, y)$ is the first of the two periods when $a \in \left(\frac{-2x}{3y}, \frac{2x}{3y}, \frac{3x}{y}, \frac{x}{3y} \right) + \frac{x}{2} \mathbb{Z}$ and is the second of the two periods when $a \in \left(\frac{-x}{3y}, \frac{x}{3y}, \frac{x}{3y} \right) + \frac{x}{2} \mathbb{Z}$, where $s$ is the spacing between consecutive contact points given by Lemma 10 and $d = s \left(\frac{1}{3} \left\lfloor \frac{2x}{3} \right\rfloor + 1 \right) - \frac{2}{3}$.

**Proof.** The formula for $p_a(x, y)$ follows immediately from Proposition 8 and Proposition 10. We now find the period as a function of initial position in the case when the orbit is biperiodic.

Given $(x, y)$, $\Theta$, and $P$ as in the hypotheses, a downward perturbation by $h$ of $C_{a,T}$ shifts the initial point $(a, 0)$ horizontally to the right by $\frac{h}{2} y h$, so that $C_{a,T} - h = C_{a+\frac{h}{2} y, T}$. Assume $(x, y)$ corresponds to a biperiodic periodic orbit. We will determine three increasing values $a_0$, $a_1$, and $a_2$ such that

1. $\frac{1}{3} \in C_{a_0,T} = C_{a_2, T}$ and $\frac{2}{3} \in C_{a_1,T}$, and
2. there is no $a \in (a_0, a_2)$ such that $\frac{1}{3} \in C_{a,T}$.

It follows from the symmetry of the contact points under translation by the spacing $s$ that the periods $p_a(x, y)$ and $p_b(x, y)$ are both constants but $p_a(x, y) \neq p_b(x, y)$ (whenever they are defined) for all $a \in (a_0, a_1) + \frac{s}{2} \mathbb{Z}$ and for all $b \in (a_1, a_2) + \frac{s}{2} \mathbb{Z}$.

We now determine the values of $a_0$, $a_1$, and $a_2$ that satisfy conditions (1) and (2) above. First, from the proof of Proposition 8, we know that $\frac{1}{3} \in C_{a_0,T}$ when $a_0 = -\frac{2}{3y}$. Second, since the spacing of consecutive contact points is $s$, we have
\[ a_2 = a_0 + \frac{s}{3} \text{.} \] Third, to determine \( a_1 \), we must calculate the distance \( d \) between the upper limit of the barrier \( \frac{3}{2} \) and the closest contact point in \( C_{a_0,T} \) above \( \frac{3}{2} \). With \( d \) so defined, a downward perturbation by \( d \) of the contact points satisfies \( C_{a_0,T} - d = C_{a_1,T} \), so that \( a_1 = a_0 + \frac{d}{5} \). Since the contact points \( C_{a_0,T} \) can be parameterized by \( \frac{1}{2} + s \mathbb{Z} \pmod{2} \), the distance \( d = \left( \frac{1}{2} + s \left( \frac{4}{3} \right) + 1 \right) - \frac{3}{5} = s \left( \frac{4}{3} \cdot 1 \right) - \frac{3}{5} \).

Finally, we determine the period at any point. Since the barrier \( B = \left( \frac{4}{3}, \frac{4}{3} \right) \) is open on the left, a downward perturbation of the contact points \( C_{a_0,T} \) corresponds to the same periods as calculated for \( \epsilon = \frac{1}{3} \) in the proof of Proposition 8. Therefore the period \( p_a(x, y) \) is the second of the two periods in the theorem statement when \( a = (a_0, a_1) + \frac{s}{5} \mathbb{Z} \), while \( p_a(x, y) \) is the first of the two periods in the theorem statement when \( a = (a_1, a_2) + \frac{s}{5} \mathbb{Z} \). Substituting for \( a_0 \), \( a_1 \), and \( a_2 \) completes the proof.

Note that when \( \Theta = 60 \), the formula in Theorem 11 is consistent with Proposition 1 part (4); when \( \Theta = 90 \), the formula is also consistent with Proposition 1 part (5) by setting \( x = 0 \) and \( y = 1 \).

**Corollary 12.** A biperiodic primitive periodic orbit with initial angle \( \Theta = \arctan \left( \frac{2}{3} \right) \) has one of two possible periods \( p_1 < p_2 \), where \( p_2 = 2p_1 + 2 \) or \( p_2 = 2p_1 - 2 \).

### 2.4. The number of periodic orbits of a given period.

In this subsection we consider the following counting problem: Given a natural number \( 2m \), count the number of periodic orbits of period \( 2m \) up to horizontal translations of their unfoldings.

Let \( \gamma \) be a periodic orbit on the 120 isosceles triangle. Then \( \gamma \) is a \( k \) fold iterate of some primitive periodic orbit \( \beta \). If we represent \( \beta \) by \( a \) and \( (x, y) \) as prescribed by Theorem 11, then the number of edge hitting events of \( \gamma \) are \( kp_a(x, y) \). Thus we can find primitive orbits \( \beta \) for which there exists a \( k \in \mathbb{N} \) such that \( kp_a(x, y) = 2m \) for some \( a \). This yields the following lemma:

**Lemma 13.** The number of classes of periodic orbits with a given period \( 2m \) is

\[
(2.14) \quad \sum_{k=1}^{\infty} \# \{(x, y) \in \text{dom}(p_a) \mid p_a(x, y) = 2m/k \text{ for some } a\},
\]

where \( p_a \) is the period function in Theorem 11.

Note that since periodic orbits have even period, we can restrict the sum in equation (2.14) to those \( k \) for which \( 2m/k \) is even. The six congruence conditions in the formula for \( p_a \) partition the domain of \( p_a \) into six parts. We can evaluate the cardinality of the set in the summation in equation (2.14) by decomposing the set over these parts and evaluating the cardinality over each part. When \( 3 \nmid x \), we can further decompose the corresponding parts into two disjoint sets, corresponding to the two periods given by the \( p_a \). Consequently, the set in equation (2.14) can be decomposed into ten disjoint sets.

We now focus on one of these ten parts. Restrict the domain of \( p_a \) to \( (x, y) \) for which \( 3 \mid x \) and \( x = y \pmod{2} \). Then given a divisor \( k \) of \( 2m \) such that \( 2m/k \) is even, we must find the number of elements \( (x, y) \) for which \( 4y + \frac{2a}{k} = \frac{4m}{k} \). But, in order to meet the domain restrictions, we must further have \( y > x \), \( \text{gcd}(x, y) = 1 \), \( 3 \mid x \), and \( x = y \pmod{2} \). Let \( x = 3a \), where \( a \in \mathbb{N} \). Then \( y = 3a + 2b \) for some \( b \in \mathbb{N} \).
The equation \( 4y + \frac{2\pi}{r} = \frac{2\pi}{r} \) becomes \( 4(3a + 2b) + 2a = \frac{2\pi}{r} \), which simplifies to \( 7a + 4b = \frac{2\pi}{r} \). Thus we must determine the cardinality of the set

\[
\left\{ (a, b) \in \mathbb{N} \times \mathbb{N} \mid \gcd(3a, 2b) = 1, 7a + 4b = \frac{m}{k} \right\},
\]

which seems to be intractable.

3. Application to Related Polygons

In this section we apply the methods in Section 2 to derive formulas analogous to the one in Theorem 11 that give the periods of primitive periodic orbits on a 60-rhombus, a 60-90-120-kite, and a 30-right triangle. For \( i = 1, 2, 3 \), let \( T_i \) be the edge tessellation generated by a 60-rhombus, a 60-90-120-kite, and a 30-right triangle, respectively. In all three cases, arguments analogous to the one in Section 2 establish the fact that such periodic orbits can be represented with initial angle \( \Theta \in [60, 90] \). Again, we express \( \Theta \in [60, 90] \) in the form \( \Theta = \arctan(\frac{x}{y} \sqrt{3}) \) with \( x \leq y \) and \( \gcd(x, y) = 1 \). Note that when \( \Theta = 90 \), the period of the primitive periodic orbit can be determined by inspection and also fits the general formula as we shall see.

3.1. The 60-rhombus. Note that the tessellation \( T_1 \) can be obtained from the tessellation \( T \) by removing all of the horizontal, 60°, and 120° inclines (see Figure 5). We impose the same coordinate system on \( T_1 \) as we did on \( T \). Note that the barrier and gate on \( T_1 \) completely coincide with the barrier and gate on \( T \) so all of the definitions and methods from the previous sections apply to this case as well. Although there are no horizontal inclines on \( T_1 \), we can position the initial point of an unfolding \( \nu \) of a trajectory on a horizontal incline of \( T \). The 60-rhombus exhibits both line and rotational symmetry but a quick check shows that the initial and terminal rhombuses corresponding to an unfolding with linked initial and terminal points can differ by a reflection (i.e., an orientation reversal) but not by a rotation.

![Figure 5. The tessellation \( T_1 \) generated by a 60-rhombus. The dotted lines are horizontal inclines of \( T \) but not inclines of \( T_1 \).](image)

Note that between any two consecutive horizontal inclines of \( T \) (which are not inclines of \( T_1 \)), an unfolding \( \nu \) on \( T_1 \) cuts two non-vertical inclines and possibly a
third vertical incline at a point corresponding to a contact point on the barrier. Thus we have the following formula analogous to the one in Proposition 5:

\[ N_{a,T} = \frac{2y}{x} T + m_{a,T} \cdot b_{a,T}, \]

where \( b_{a,T} \) is the number of contact points on the barrier and \( m_{a,T} \) is the multiplicity of the contact points. The following table is analogous to the one in Proposition 5:

| \( N \equiv 1, 3 \mod 4 \) | \( N \equiv 2 \mod 4 \) | \( N \equiv 0 \mod 4 \) |
|-----------------------------|-----------------------------|-----------------------------|
| \( 4y + \frac{x}{2} \)       | \( 4y + \frac{x}{2} \)       | \( x \equiv 0 \mod 3 \) and \( x \equiv y \mod 2 \) |
| \( 4y + \frac{x}{2} \)       | \( 4y + \frac{x}{2} \)       | \( x \equiv 0 \mod 3 \) and \( x \not\equiv y \mod 2 \) |
| \( 4y + \frac{x}{2} \)       | \( 4y + \frac{x}{2} \)       | \( x \equiv 1 \mod 3 \) and \( x \equiv y \mod 2 \) |
| \( 4y + \frac{x}{2} \)       | \( 4y + \frac{x}{2} \)       | \( x \equiv 1 \mod 3 \) and \( x \not\equiv y \mod 2 \) |
| \( 4y + \frac{x}{2} \)       | \( 4y + \frac{x}{2} \)       | \( x \equiv 2 \mod 3 \) and \( x \equiv y \mod 2 \) |
| \( 4y + \frac{x}{2} \)       | \( 4y + \frac{x}{2} \)       | \( x \equiv 2 \mod 3 \) and \( x \not\equiv y \mod 2 \) |

Just as in the 120-isosceles triangle case, the initial and terminal triangles are consistently oriented if and only if \( N_{a,T} \) is even, so Proposition 8 applies, and we have the following formula analogous to the one in Theorem 11.

\[ p_a(x, y) = \begin{cases} 
2y + \frac{2x}{3} & \text{if } x \equiv 0 \mod 3 \text{ and } x \equiv y \mod 2 \\
4y + \frac{4x}{3} & \text{if } x \equiv 0 \mod 3 \text{ and } x \not\equiv y \mod 2 \\
2y + \frac{2x+2}{3} & \text{if } x \equiv 1 \mod 3 \text{ and } x \equiv y \mod 2 \\
4y + \frac{4x+2}{3} \text{ or } 8y + \frac{8x+2}{3} & \text{if } x \equiv 1 \mod 3 \text{ and } x \not\equiv y \mod 2 \\
2y + \frac{2x+2}{3} \text{ or } 4y + \frac{4x+2}{3} & \text{if } x \equiv 2 \mod 3 \text{ and } x \equiv y \mod 2 \\
4y + \frac{4x-2}{3} \text{ or } 8y + \frac{8x+2}{3} & \text{if } x \equiv 2 \mod 3 \text{ and } x \not\equiv y \mod 2.
\end{cases} \]

Note that when \( \Theta = 90 \), a direct inspection shows that the period \( p = 4 \), which is consistent with the above formula for \( x = 0 \) and \( y = 1 \).

3.2. The 60-90-120-kite. As in the case of the 60-rhombus, we can also relate the tessellation \( T_2 \) to the tessellation \( T \). Consider the 60-90-120 kite positioned as \( \square AOBX \), where \( A \) and \( B \) coincide with the two vertices of the 120-isosceles triangle \( \triangle ABC \) and \( O \) is the midpoint of \( AC \). Impose the same coordinate system on \( T_2 \) as we did on \( T \). Note that while the fence in this case remains the same, the barrier and gate from the previous section are interchanged, i.e., the barrier \( B = (-\frac{1}{3}, \frac{1}{3}) \) and the gate is \( (\frac{1}{3}, \frac{5}{3}) \). We again position the initial point of an unfolding \( v \) of a trajectory on a horizontal incline, and we define contact points and their multiplicity exactly the same way we did in the previous section. Results concerning the multiplicity of the contact points and monoperiodicity of \( b_{a,2x} \) are the same as in the previous section.

Note that between any two consecutive horizontals, an unfolding \( v \) on \( T_2 \) cuts three non-vertical inclines and possibly a fourth vertical incline at a point corresponding to a contact point on the barrier. Thus we have the following formula analogous to the one in Proposition 5:

\[ N_{a,T} = \frac{3y}{x} T + m_{a,T} \cdot b_{a,T}, \]

where \( b_{a,T} \) is the number of contact points on the barrier and \( m_{a,T} \) is the multiplicity of the contact points. A similar computation of \( b_{a,2x} \) leads to the following table analogous to the one in Proposition 5.
Figure 6. The tessellation $\mathcal{T}_2$ generated by a 60-90-120-kite.

| $N \equiv 1, 3 \mod 4$ | $N \equiv 2 \mod 4$ | $N \equiv 0 \mod 4$ |
|-------------------------|-------------------------|-------------------------|
| $6y + \frac{y}{3}$     | $x \equiv 0 \mod 3 \text{ and } x \equiv y \mod 2$ |
| $6y + \frac{2x}{3}$    | $x \equiv 0 \mod 3 \text{ and } x \not\equiv y \mod 2$ |
| $6y + \frac{x+2}{3}$   | $x \equiv 1 \mod 3 \text{ and } x \equiv y \mod 2$ |
| $6y + \frac{2x+2}{3}$  | $x \equiv 1 \mod 3 \text{ and } x \not\equiv y \mod 2$ |
| $6y + \frac{2x+2}{3}$  | $x \equiv 2 \mod 3 \text{ and } x \equiv y \mod 2$ |
| $6y + \frac{2x+2}{3}$  | $x \equiv 2 \mod 3 \text{ and } x \not\equiv y \mod 2$ |
| $6y + \frac{2x+2}{3}$  | $x \equiv 2 \mod 3 \text{ and } x \not\equiv y \mod 2$ |

Just as in the case of the 120-isosceles triangle and the 60-rhombus, the initial and terminal triangles are consistently oriented if and only if $N_{a,T}$ is even, so Proposition 10 applies again, and we have the following formula analogous to the one in Theorem 11:

$$p_a(x, y) = \begin{cases} 
3y + \frac{x}{3} & \text{if } x \equiv 0 \mod 3 \text{ and } x \equiv y \mod 2 \\
6y + \frac{2x}{3} & \text{if } x \equiv 0 \mod 3 \text{ and } x \not\equiv y \mod 2 \\
6y + \frac{2x+2}{3} \text{ or } 6y + \frac{x+2}{3} & \text{if } x \equiv 1 \mod 3 \text{ and } x \equiv y \mod 2 \\
6y + \frac{2x+2}{3} \text{ or } 12y + \frac{x+2}{3} & \text{if } x \equiv 1 \mod 3 \text{ and } x \not\equiv y \mod 2 \\
3y + \frac{x+2}{3} \text{ or } 6y + \frac{2x+2}{3} & \text{if } x \equiv 2 \mod 3 \text{ and } x \equiv y \mod 2 \\
6y + \frac{2x+2}{3} \text{ or } 12y + \frac{x+2}{3} & \text{if } x \equiv 2 \mod 3 \text{ and } x \not\equiv y \mod 2 
\end{cases}$$

Note that when $\Theta = 90$, a direct inspection shows that the period $p = 6$, which is consistent with the above formula for $x = 0$ and $y = 1$.

3.3. The 30-right triangle. The proof strategy for this shape will be distinct from and simpler than the other shapes since there is no reflectional symmetry. We again relate the tessellation $\mathcal{T}_3$ to the tessellation $\mathcal{T}$ by positioning the 30-right triangle as $\triangle AOB$, where $A$ and $B$ coincide with the two vertices of the 120-isosceles triangle $\triangle ABC$ and $O$ is the midpoint of $\overline{AC}$. Impose the same coordinate system on $\mathcal{T}_3$ as we did on $\mathcal{T}$. Note that in this case, the notions of fence, barrier, and gate are irrelevant, and, between any two consecutive horizontals, an unfolding $v$ on $\mathcal{T}_3$ cuts five non-vertical inclines and possibly a sixth vertical incline. Thus, the number of inclines of $\mathcal{T}_3$ cut by an unfolding $v(t) = (t + a, \frac{b}{c}t), 0 < t < x$, is $N(x, y) = x + 5y$.

Since the 30-right triangle has no line symmetry, the initial and terminal triangles corresponding to an unfolding are consistently oriented if and only if the initial and
terminal points are linked. Thus, to determine the period of a primitive periodic orbit, we only need to determine when the initial and terminal points are linked for the first time. Inspection shows that if \( x \) and \( y \) have the same parity, the initial point links with some point at \( t = x \); otherwise, they first link for some point at \( t = 2x \). Thus, we have

\[
p_a(x, y) = \begin{cases} 
  x + 5y & \text{if } x \equiv y \mod 2 \\
  2x + 10y & \text{if } x \not\equiv y \mod 2.
\end{cases}
\]

Note that when \( \Theta = 90 \), a direct inspection shows that the period \( p = 10 \), which is consistent with the above formula for \( x = 0 \) and \( y = 1 \).

3.4. **Regular Hexagon**. While the techniques in the previous section can be used to determine the number of inclines cut by an unfolding of a trajectory on a regular hexagon, this information is insufficient to determine where an unfolding of a primitive periodic orbit terminates. In addition to requiring that the initial and terminal points be linked and that the initial and terminal hexagon be consistently oriented, one must also consider the rotational symmetries of the terminal hexagon. We leave this interesting case for future investigation, which, if resolved, will complete the classification of periodic orbits on all edge tessellating polygons.

**Acknowledgment**

We wish to thank David Brown for his participation in a seminar during the 2011-2012 academic year in which this problem was first considered and Joshua Pavoncello for writing a computer program \[7\] that visualizes the trajectories on an edge tessellating polygon and collects related experimental data.

**References**

[1] Baxter, A., Eskreis-Winkler, J., McCarthy, E. (2011). Periodic Billiard Paths in Edge-Tessellating Polygons. *Unpublished Manuscript*.

[2] Baxter, A., Umble, R. (2008). Periodic Orbits for Billiards on an Equilateral Triangle. *American Mathematical Monthly*, 115 (6) 479-491.

[3] Boshernitzan, M., Galperin, G., Krüger, T., Troubetzkoy, S. (1998). Periodic Billiard Orbits are Dense in Rational Polygons. *American Mathematical Monthly*, 115 (9) 3523-3535.

[4] Kenyon, R., Smillie, J. (2000). Billiards on Rational-Angled Triangles. *Commentarii Mathematici Helvetici*. 75 (1) 65-108.

[5] Kirby, M., Umble, R. (2011). Edge Tessellations and Stamp Folding Puzzles. *Mathematics Magazine*. 84 (4) 283-289.
Masur, H., Tabachnikov, S. (2002). Rational Billiards and Flat Structures. *Handbook of dynamical systems* (Vol. 1, pp. 1015-1089). Elsevier Science.

Pavonecello, J. (2013). Orbit Mapper. Retrieved from: https://github.com/jrpavonecello/OrbitMapper

Umble, R.N, Han, Z. (2015). *Transformational Plane Geometry*. Chapman and Hall/CRC.