Symmetric Monopoles

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Abstract We discuss $SU(2)$ Bogomolny monopoles of arbitrary charge $k$ invariant under various symmetry groups. The analysis is largely in terms of the spectral curves, the rational maps, and the Nahm equations associated with monopoles. We consider monopoles invariant under inversion in a plane, monopoles with cyclic symmetry, and monopoles having the symmetry of a regular solid. We introduce the notion of a strongly centred monopole and show that the space of such monopoles is a geodesic submanifold of the monopole moduli space.

By solving Nahm’s equations we prove the existence of a tetrahedrally symmetric monopole of charge 3 and an octahedrally symmetric monopole of charge 4, and determine their spectral curves. Using the geodesic approximation to analyse the scattering of monopoles with cyclic symmetry, we discover a novel type of non-planar $k$-monopole scattering process.

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1 Introduction

In recent years, there has been considerable interest in monopoles, which are particle-like solitons in a Yang-Mills-Higgs theory in three spatial dimensions. In this paper, we shall consider $SU(2)$ Bogomolny monopoles, which are the finite energy solutions of the Bogomolny equations (1), [5]. Solutions are labelled by their magnetic charge, a non-negative integer $k$, and are physically interpreted as static, non-linear superpositions of $k$ unit charge magnetic monopoles. There is a $4k$-dimensional manifold of solutions up to gauge equivalence, known as the $k$-monopole moduli space $M_k$, and on this there is a naturally defined Riemannian metric, which is hyperkähler [1].

For monopoles moving at modest speeds compared with the speed of light, it is a good approximation to model $k$-monopole dynamics by the geodesic motion on the moduli space $M_k$. This was conjectured some time ago [20], and the consequences explored in some detail [1, 10, 28, 3, 26]. Very recently, the validity of the geodesic approximation has been proved analytically by Stuart [24].

Most studies of Bogomolny monopoles have been concerned either with the general structure of the $k$-monopole moduli space $M_k$ and its metric, or with a detailed study of the geodesics on it for small values of $k$. Little work has been done on $k$-monopole dynamics for $k > 2$. In this paper, we investigate classes of $k$-monopole solutions which are invariant under various symmetry groups and derive results on their scattering. We consider monopoles invariant under inversion in a fixed plane, monopoles invariant under a cyclic group of rotations about a fixed axis, and monopoles invariant under the symmetry groups of the regular solids, that is, the tetrahedral, octahedral and icosahedral groups. The existence of $k$-monopoles with cyclic symmetry was previously shown in [22]. Each submanifold of the moduli space $M_k$ consisting of all $k$-monopoles invariant under a fixed symmetry group is a totally geodesic submanifold. We therefore obtain various examples of monopole scattering with symmetry, by finding geodesics on such submanifolds.

Among our most interesting results are proofs of existence of a tetrahedrally symmetric charge 3 monopole and an octahedrally symmetric charge 4 monopole. We give explicit formulae for the spectral curves and the solutions of Nahm’s equations corresponding to these monopoles. Our approach rather strongly indicates that there should be an icosahedrally symmetric monopole of charge 6, but a detailed study of Nahm’s equations shows, surprisingly, that such an object does not exist.

Much of the motivation for the present study of symmetric monopoles came from results concerning Skyrmions. Skyrmions are $SU(2)$-valued scalar fields in $\mathbb{R}^3$ which minimize Skyrme’s energy functional. They have an integer topological charge $B$, physically identified with baryon number. Numerical work by Braaten et al. [6] has established that the Skyrmions of charges one to four have, respectively, spherical symmetry, toroidal symmetry, tetrahedral symmetry and octahedral symmetry. Sim-
ilar results have subsequently been obtained using instanton-generated Skyrmions \[2, 18\]. Since a unit charge monopole is spherically symmetric, and the maximal symmetry of a charge 2 monopole is that of a torus, we were led to seek monopoles of charge 3 and charge 4 with tetrahedral and octahedral symmetry, respectively. The Skyrmions of charge 5 and charge 6 are also known, and have rather low symmetry, so the absence of an icosahedrally symmetric charge 6 monopole is not so surprising. We should remark that the relationship between Skyrmions and monopoles is not systematically understood. A \( B = 1 \) Skyrmion has six degrees of freedom, whereas a unit charge monopole has four. The moduli space of charge \( k \) monopoles has dimension \( 4k \). There is a less well-defined moduli space of Skyrme fields of baryon number \( B \), of dimension \( 6B \), and a well-defined space of instanton-generated Skyrme fields of dimension \( 8B - 1 \). It would be interesting if the charge \( B \) monopole moduli space could be identified as a submanifold of either of these latter spaces. This is certainly possible for \( B = 2 \) \[2\].

In Section 2 we review monopoles and their moduli spaces. Further details of this material can be found in the book \[1\] and in the references contained therein. In Section 3 we review the spectral curves and rational maps associated with monopoles. In Section 4 we show how the rational map changes when a monopole is inverted in the plane with respect to which the rational map is defined, and we investigate the monopoles which are invariant under this inversion. In Section 5 we consider the holomorphic geometry associated with the centre of a monopole. We define the total phase of a monopole, and introduce the notion of a strongly centred monopole – one whose centre is at the origin and whose total phase is 1. The manifold of strongly centred monopoles is totally geodesic in \( M_k \); in fact up to a \( k \)-fold covering, it splits off isometrically.

Spectral curves of \( k \)-monopoles are curves in \( TP_1 \), the tangent bundle to the complex projective line, satisfying a number of constraints. In Section 6 we consider the action of symmetry groups on general curves in \( TP_1 \), and present various classes of curves with cyclic or dihedral symmetry, and with the symmetries of regular solids. These are candidates for the spectral curves of symmetric monopoles. In Section 7 we show that the simplest curves with the symmetries of the regular solids are related to elliptic curves.

In Sections 8 to 11 we review the Nahm equations associated with monopoles and consider the existence of symmetric monopoles and the corresponding solutions of Nahm’s equations. These equations are in general very difficult to solve explicitly, involving theta functions of curves of high genus \[8\]. Rather conveniently, the symmetry conditions imposed here reduce the solutions to ones written in terms of elliptic functions. We prove thereby the existence of a tetrahedrally symmetric monopole of charge 3 (Theorem 1), and an octahedrally symmetric monopole of charge 4 (Theorem 2), and we determine their spectral curves. We also prove the non-existence of
an icosahedrally symmetric monopole of charge 6 (Theorem 3).

In Section 12, we investigate \( k \)-monopoles symmetric under the cyclic group \( C_k \).
By considering the \( C_k \)-invariant rational maps, we show that the strongly centred monopoles with \( C_k \) symmetry are parametrized by a number of geodesic surfaces of revolution in the moduli space \( M_k \). Geodesic motions on these surfaces model either a purely planar \( k \)-monopole scattering process, or a novel type of \( k \)-monopole scattering, in which \( k \) unit charge monopoles collide simultaneously in a plane and an \( l \)-monopole and a \((k − l)\)-monopole emerge back-to-back along the line through the \( k \)-monopole centre, perpendicular to the plane. The outgoing monopole clusters both become axisymmetric about this line as their separation increases to infinity. When \( k = 3 \) and \( l = 1 \) or \( l = 2 \), this geodesic motion passes instantaneously through the tetrahedrally symmetric 3-monopole (oppositely oriented in the two cases), and when \( k = 4 \) and \( l = 2 \), through the octahedrally symmetric 4-monopole.

Finally a warning is necessary for the reader who wishes to delve into the literature on this subject. There are a number of places in the theory of monopoles where one has to make choices and establish conventions. Most of these are to do with the orientation of \( \mathbb{R}^3 \) and the induced complex structure on the twistor space \( TP_1 \) of all oriented lines in \( \mathbb{R}^3 \). Different authors have made different conventions, and minor inconsistencies can appear to result if the literature is only read in a cursory manner.

2 Monopoles

To define a monopole we start with a pair \((A, \phi)\) consisting of a connection 1-form \( A \) on \( \mathbb{R}^3 \) with values in \( \mathfrak{su}(2) \), the Lie algebra of \( SU(2) \), and a function \( \phi \) (the Higgs field) from \( \mathbb{R}^3 \) into \( \mathfrak{su}(2) \). The Yang-Mills-Higgs energy on this pair is

\[
\mathcal{E}(A, \phi) = \int_{\mathbb{R}^3} (|F_A|^2 + |\nabla_A \phi|^2) d^3x
\]

where \( F_A = dA + A \wedge A \) is the curvature of \( A \), \( \nabla_A \phi = d\phi + [A, \phi] \) is the covariant derivative of the Higgs field, and we use the usual norms on 1-forms and 2-forms and the standard inner product on \( \mathfrak{su}(2) \). The energy is minimized by the solutions of the Bogomolny equations \[ \star F_A = \nabla_A \phi \] (1)

where \( \star \) is the Hodge star on forms on \( \mathbb{R}^3 \). These equations, and the energy, are invariant under gauge transformations, where the gauge group \( G \) of all maps \( g \) from \( \mathbb{R}^3 \) to \( SU(2) \) acts by

\[
(A, \phi) \mapsto (gAg^{-1} - dgg^{-1}, g\phi g^{-1}).
\]
Finiteness of the energy, and the Bogomolny equations, imply certain boundary conditions at infinity in $\mathbb{R}^3$ on the pair $(A, \phi)$ which are spelt out in detail in [1]. In particular, $|\phi| \to c$ for some constant $c$ which cannot change with time. Following [1], we fix $c = 1$.

A monopole, then, is a gauge equivalence class of solutions to the Bogomolny equations subject to these boundary conditions. In some suitable gauge there is a well-defined Higgs field at infinity

$$\phi^\infty: S^2_\infty \to S^2 \subset \mathfrak{su}(2)$$

going from the two sphere of all oriented lines through the origin in $\mathbb{R}^3$ to the unit two-sphere in $\mathfrak{su}(2)$. The degree of $\phi^\infty$ is a positive integer $k$ called the magnetic charge of the monopole.

Before discussing the moduli space of all solutions of the Bogomolny equations we need to be a little more precise and talk about framed monopoles. We say a pair $(A, \phi)$ is framed if

$$\lim_{x_3 \to \infty} \phi(0, 0, x_3) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$ 

The gauge transformations fixing such pairs are those $g$ with $\lim_{x \to \infty} g(0, 0, x)$ diagonal. Notice that every monopole can be gauge transformed until it is framed. So the space of monopoles modulo gauge transformations is the same as the space of framed monopoles modulo those gauge transformations that fix them. We define a framed gauge transformation to be one such that $\lim_{x \to \infty} g(0, 0, x) = 1$. The quotient of the set of all framed monopoles of charge $k$ by the group of framed gauge transformations is a manifold called the moduli space of (framed) monopoles of charge $k$ and denoted $M_k$. The group of constant diagonal gauge transformations (a copy of $U(1)$) acts on $M_k$ and the quotient is called the reduced moduli space $N_k$. This action is not quite free, because the element $-1$ acts trivially, but the group $U(1)/\{\pm 1\}$ acts freely on $M_k$.

The dimension of $M_k$ is $4k$, which can be understood as follows. In the case that $k = 1$ there is a spherically symmetric monopole called the Bogomolny-Prasad-Sommerfield (BPS) monopole, or unit charge monopole. Its Higgs field has a single zero at the origin, and its energy density is peaked there so it is reasonable to think of the origin as the centre or location of the monopole. The Bogomolny equations are translation invariant so this monopole can be translated about $\mathbb{R}^3$ and also rotated by the circle of constant diagonal gauge transformations. This in fact generates all of $M_1$ which is therefore diffeomorphic to $S^1 \times \mathbb{R}^2$. The coordinates on $M_1$ specify the location of the monopole and what can be thought of as an internal phase. More generally there is an asymptotic region of the moduli space consisting of approximate superpositions of $k$ unit charge monopoles located at $k$ widely separated points and with $k$ arbitrary phases.
Although it is not possible to assign precisely to a charge \( k \) monopole \( k \) points or locations in \( \mathbb{R}^3 \) it is possible to assign to the monopole a centre which can be thought of as the average of the locations of the \( k \) particles making up the monopole. The important property of this centre is that if we act on the monopole by an isometry of \( \mathbb{R}^3 \) the centre moves by the same isometry. It is also possible to assign to a \( k \)-monopole a total phase; this is essentially the product of the phases of the \( k \) unit charge monopoles. If we act on the monopole by a constant diagonal gauge transformation corresponding to an element \( \mu \) of \( U(1) \) then the total phase changes by \( \mu^{2k} \).

The natural metric on the moduli space \( M_k \) is obtained as follows. There is a flat \( L_2 \) metric on the space of fields \( (A, \phi) \), and this descends to a curved metric on the space of gauge equivalence classes of fields. The latter metric, restricted to the \( k \)-monopole solutions of the Bogomolny equations is the metric on \( M_k \). Since a large part of the moduli space \( M_k \) describes \( k \) well-separated unit charge monopoles, many geodesics on \( M_k \) correspond to the scattering of \( k \) unit charge monopoles, and we shall discuss below some particularly symmetric cases of such scattering.

3 Spectral curves and rational maps

It is not easy to study charge \( k \) monopoles directly in terms of their fields \( (A, \phi) \). However, there are various ways of transforming monopoles to other types of mathematical objects. There is a twistor theory for monopoles and the result of applying this shows that monopoles are equivalent to a certain class of holomorphic bundles on the so-called mini-twistor space \( TP_1 \). The boundary conditions of the monopole imply that the holomorphic bundle is determined by an algebraic curve, called the spectral curve. Monopoles that differ only by a constant diagonal gauge transformation have the same spectral curve, see \([11, 12]\).

The holomorphic bundle of a \( k \)-monopole is defined as follows \([11]\). Let \( \gamma \) be an oriented line in \( \mathbb{R}^3 \) and let \( \nabla_\gamma \) denote covariant differentiation using the connection \( A \) along \( \gamma \). One considers the ordinary differential equation

\[
(\nabla_\gamma - i\phi)v = 0
\]

where \( v : \gamma \to \mathbb{C}^2 \). The vector space \( E_\gamma \) of all solutions to equation (2) is two-dimensional and the union of all these spaces forms a rank two smooth complex vector bundle \( E \) over the space of all oriented lines in \( \mathbb{R}^3 \). It can be shown that this space of all oriented lines is a complex manifold, in fact isomorphic to \( TP_1 \). One may define on \( E \) a holomorphic structure if the monopole satisfies the Bogomolny equations. The bundle \( E \) has two holomorphic sub-bundles \( E_1^\pm \) whose fibres \( (E_1^\pm)_\gamma \) at \( \gamma \) are defined to be the spaces of solutions that decay as \( \pm\infty \) is approached along
the line $\gamma$. The set of $\gamma$ where $(E_1^+)_{\gamma} = (E_1^-)_{\gamma}$, so there is a solution decaying at both ends, forms a curve $S$ in $TP_1$ called the spectral curve of the monopole. It is possible to show that a decaying solution decays exponentially so the spectral curve is also the set of all lines along which there is an $L_2$ solution. Intuitively one should think of the spectral lines as being the lines going through the locations of the monopoles. In the case of charge 1, the spectral lines are precisely those going through the centre of the monopole.

If we describe a typical point in $P_1$ by homogeneous coordinates $[\zeta_0, \zeta_1]$ then we can cover $P_1$, in the usual way, by two open sets $U_0$ and $U_1$ where $\zeta_0$ and $\zeta_1$ are non-zero, respectively. On the set $U_0$ we introduce the coordinate $\zeta = \zeta_1/\zeta_0$. Let us also denote by $U_0$ and $U_1$ the pre-images of these sets under the projection map from $TP_1$ to $P_1$. Then a tangent vector $\eta \partial/\partial \zeta$ at $\zeta$ in $U_0$ can be given coordinates $(\eta, \zeta)$.

These coordinates allow us to describe an important holomorphic line bundle $L$ on $TP_1$ which has transition function $\exp(\eta/\zeta)$ on the overlap of $U_0$ and $U_1$. Similarly for any complex number $\lambda$ we define the bundle $L^\lambda$ by the transition function $\exp(\lambda \eta/\zeta)$. Finally, if $n$ is any integer we define the line bundle $L^0(n)$ to be the tensor product of $L^\lambda$ with the $n$-th power of the pull-back under projection $TP_1 \to P_1$ of the dual of the tautological bundle on $P_1$. This has transition function $\zeta^{-n} \exp(\lambda \eta/\zeta)$. The line bundle $L^0$ is clearly trivial so we denote it by $O$, and $L^0(n)$ is denoted by $O(n)$.

Another way of introducing the twistor theory for monopoles is to note, as in [19], that the Bogomolny equations on $\mathbb{R}^3$ are equivalent to the self-dual Yang-Mills equations on $\mathbb{R}^4$, invariant under translation in the fourth direction, so monopoles are equivalent to $S^1$-invariant instantons on $S^1 \times \mathbb{R}^3$. The twistor space $Z$ for $S^1 \times \mathbb{R}^3$ is the quotient of $P_3 \setminus P_1$ by a free $\mathbb{Z}$-action, and is a bundle of groups $\mathbb{C}^* \times \mathbb{C}$ over $P_1$. By the original Atiyah-Ward construction, an instanton corresponds to a holomorphic bundle on the twistor space $Z$, and if it is $S^1$-invariant, it descends to $Z/\mathbb{C}^* = TP_1$. The space $Z$ itself is the total space of the principal bundle for the line bundle $L$ defined above by transition functions.

To avoid the potential ambiguity in what we mean by ‘transition function’ let us be more explicit. The line bundle $L^\lambda(n)$ has non-vanishing holomorphic sections $\chi_0$ and $\chi_1$ over $U_0$ and $U_1$ respectively and for points in $U_0 \cap U_1$ these satisfy

$$\chi_0 = \zeta^{-n} \exp(\frac{\lambda \eta}{\zeta}) \chi_1. \quad (3)$$

If we consider an arbitrary holomorphic section $f$ of this line bundle its restriction to $U_0$ and $U_1$ can be written as $f = f_0 \chi_0$ and $f = f_1 \chi_1$ respectively where $f_0$ and $f_1$ are holomorphic functions on $U_0$ and $U_1$. As a consequence of equation (3) these functions must satisfy

$$f_0 = \zeta^n \exp(-\frac{\lambda \eta}{\zeta}) f_1 \quad (4)$$

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at points in the intersection $U_0 \cap U_1$.

With these definitions we can present the results that we need. The sub-bundles $E_1^\pm$ satisfy $E_1^\pm \simeq L_1^\pm(-k)$ and the quotients satisfy $E/E_1^\pm \simeq L_1^\pm(k)$. For a framed monopole there are explicit isomorphisms so we shall write $= \text{ instead of } \simeq$. The curve $S$ is defined by the vanishing of the map $E_1^+ \to E/E_1^-$ and hence by a section of $(E_1^+)^* \otimes E/E_1^- = \mathcal{O}(2k)$. In terms of the coordinates $(\eta, \zeta)$, $S$ is defined by an equation of the form

$$P(\eta, \zeta) \equiv \eta^k + \eta^{k-1}a_1(\zeta) + \ldots + \eta a_{k-1}(\zeta) + a_k(\zeta) = 0,$$

where, for $1 \leq r \leq k$, $a_r(\zeta)$ is a polynomial in $\zeta$ of degree at most $2r$.

The space $TP_1$ has a real structure $\tau$, namely, the anti-holomorphic involution defined by reversing the orientation of the lines in $\mathbb{R}^3$. In coordinates it takes the form $\tau(\eta, \zeta) = (-\bar{\eta}/\bar{\zeta}^2, -1/\bar{\zeta})$. The curve $S$ is fixed by this involution, so we say that it is real. The reality of $S$ implies that for $1 \leq r \leq k$,

$$a_r(\zeta) = (-1)^r\zeta^{2r}a_r(-1/\bar{\zeta}).$$

If $k = 1$ the spectral curve has the form

$$\eta = (x_1 + ix_2) - 2x_3\zeta - (x_1 - ix_2)\zeta^2$$

where $x = (x_1, x_2, x_3)$ is any point in $\mathbb{R}^3$, [11, eq. (3.2)]. Such a curve is called a real section as it defines a section of the bundle $TP_1 \to P_1$, and is real in the sense given above. In terms of the geometry of $\mathbb{R}^3$ this curve is the set of all oriented lines through the point $x$, so it is the spectral curve of a BPS monopole located at $x$. We refer to this curve as the “star” at $x$.

In [11, 12] one can find listed all the properties that a curve in $TP_1$ has to satisfy to be a spectral curve. We are interested in one of these here. From the definition of the spectral curve we see that over the spectral curve the line bundles $E_1^+$ and $E_1^-$ coincide as sub-bundles of $E$; in particular they must be isomorphic. This is equivalent to saying that the line bundle $E_1^+ \otimes (E_1^-)^* = L^2$ is trivial over the curve or that it admits a non-vanishing holomorphic section $s$. The real structure $\tau$ can be lifted to an anti-holomorphic, conjugate linear map between the line bundles $L^2$ and $\mathcal{O}$ and hence the section $s$ can be conjugated to define a new (holomorphic) section $\tau(s) = \tau \circ s \circ \tau$ of $L^{-2} \otimes L^2 = \mathcal{O}$ and because $S$ is compact and connected this is a constant. Because of the framing this constant will be $1$. Notice that given only $S$ and the fact that $L^2$ is trivial over $S$, if we can choose a section $s$ such that $\tau(s)s = 1$ then it is unique up to multiplication by a scalar of modulus one. This circle ambiguity in the choice of $s$ corresponds to
the framing of the monopole. In fact, let $\mu$ be a complex number of modulus one corresponding to a constant diagonal gauge transformation with diagonal entries $\mu$ and $\mu^{-1}$. Then it is possible to follow through the proof in ([11], pp. 593-4) and show that if we phase rotate a framed monopole by $\mu$, the isomorphism $E^+_1 \rightarrow L\langle-k\rangle$ is multiplied by $\mu$ and the isomorphism $E^-_1 \rightarrow L^\ast\langle-k\rangle$ is multiplied by $\mu^{-1}$. The section $s$ of $E^+_1 \otimes (E^-_1)^\ast = L^2$ is therefore multiplied by $\mu^2$. Notice that this is consistent with the fact that the group $U(1)/\{\pm 1\}$ acts freely on the moduli space $M_k$ of framed monopoles.

The rational map of a monopole was originally described by Donaldson in terms of solutions to Nahm’s equations [7]. Hurtubise then showed how it relates to scattering in $\mathbb{R}^3$ and to the spectral curve of the monopole [14]. It will be convenient for our purposes to use the description in terms of spectral curves.

The rational map of a charge $k$ monopole is from $\mathbb{C}$ to $\mathbb{C} \cup \infty$, and is simply a polynomial $p$ of degree less than $k$ divided by a monic (leading coefficient = 1) polynomial $q$ of degree $k$ which has no factor in common with $p$,

$$R(z) = \frac{p(z)}{q(z)}.$$  

We shall denote by $R_k$ the space of all these based rational maps. Donaldson has proved that any such rational map arises from some unique charge $k$ monopole [7], so $R_k$ is diffeomorphic to $M_k$. The disadvantage of characterising a monopole by its rational map is that the definition of the map requires choosing a line and an orthogonal plane in $\mathbb{R}^3$, and this breaks the symmetries of the problem. Whereas the Bogomolny equations are invariant under all the isometries of $\mathbb{R}^3$, the transformation to a rational map commutes only with those isometries that preserve the direction of the line.

To define the rational map we fix the fibre $F$ of $TP_1 \rightarrow P_1$ where $\zeta = 0$ and identify it with $\mathbb{C}$. The fibre consists of all lines in the $x_3$-direction. This corresponds to picking an orthogonal splitting of $\mathbb{R}^3$ as $\mathbb{C} \times \mathbb{R}$. Each point $z$ in $\mathbb{C}$ is identified with a point in $F$ by setting $z = \eta$, and hence with an oriented line, the line $\{(x_1, x_2, x_3) \mid x_3 \in \mathbb{R}\}$ with $z = x_1 + ix_2$. The intersection of $F$ with $S$ defines $k$ points counted with multiplicity and $q(z)$ is defined to be the unique monic polynomial of degree $k$ which has these $k$ points as its roots. Thus $q(z) = P(z, 0)$, where $P$ is given by eq. (4). Recall from (4) that a holomorphic section $s$ of the bundle $L^2$ is determined locally by functions $s_0$ and $s_1$, on $U_0 \cap S$ and $U_1 \cap S$ respectively, such that

$$s_0(\eta, \zeta) = \exp\left(\frac{-2\eta}{\zeta}\right)s_1(\eta, \zeta).$$  

Let $p(z)$ be the unique polynomial of degree $k - 1$ such that $p(z) = s_0(z, 0) \mod q(z)$. The rational map of the monopole is then $R(z) = p(z)/q(z)$. If the roots of $q(z)$ are
distinct complex numbers $\beta_1, \ldots, \beta_k$ then the polynomial $p(z)$ is determined by its values $p(\beta_i) = s_0(\beta_i, 0)$ for all $i = 1, \ldots, k$.

Let us make a brief remark about the construction of the rational map as scattering data in $R^3$. More details are given in [14] and [1]. The points where $S$ intersects $F$, the zeros of $q$, correspond to the lines in the $x_3$-direction admitting a solution of eq. (2) decaying at both ends. Assume these lines are distinct and label them by the corresponding complex numbers $\beta_i$. Pick for each line a solution $v(\beta_i, x_3)$ decaying at both ends. In the regions where $x_3$ is large positive and large negative there are choices of asymptotically flat gauge such that

$$\lim_{x_3 \to \infty} (x_3)^{-k/2} e^{x_3} v(\beta_i, x_3) = v^+_i \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\lim_{x_3 \to -\infty} (x_3)^{-k/2} e^{-x_3} v(\beta_i, x_3) = v^-_i \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$ 

The rational map with our conventions is determined by

$$p(\beta_i) = \frac{v^+_i}{v^-_i}.$$ 

This agrees with the results stated in Chapter 16 of ref. [1], although Hurtubise’s conventions give $p(\beta_i) = v^-_i/v^+_i$.

We deduce from these formulae the action of certain isometries on the rational maps of monopoles. Let $\lambda \in U(1)$ and $w \in C$ define a rotation and translation, respectively, in the plane $C$. Let $t \in R$ define a translation perpendicular to the plane and let $\mu \in U(1)$ define a constant diagonal gauge transformation. A rational map $R(z)$ then transforms under the composition of all these transformations to

$$\tilde{R}(z) = \mu^2 \exp(2it) \lambda^{-2k} R(\lambda^{-1}(z - w)).$$

Note that this is slightly different to the action described in [1], eq. (2.11), because of different conventions.

### 4 Inverting monopoles

Consider the inversion map $I: R^3 \to R^3$ defined by $I(x_1, x_2, x_3) = (x_1, x_2, -x_3)$. This inverts $R^3$ in the $(x_1, x_2)$ plane. The inversion map reverses orientation, and so induces an anti-holomorphic map on the twistor space $TP_1$ which we shall denote by the same symbol and which in the standard coordinates on $TP_1$ is

$$I(\eta, \zeta) = \left( \frac{-\bar{\eta}}{\zeta^2}, \frac{1}{\zeta} \right).$$
To see this note that the real section defined by the point \( I(x_1, x_2, x_3) \) has equation
\[
\eta = (x_1 + ix_2) + 2x_3\zeta - (x_1 - ix_2)\zeta^2.
\]
So a point \( I(\eta, \zeta) \) is on this curve if and only if
\[
-\bar{\eta}^2 = (x_1 + ix_2) + 2x_3\frac{1}{\zeta} - (x_1 - ix_2)\frac{1}{\zeta^2}.
\]
Conjugating this equation and clearing the denominators we recover
\[
\eta = (x_1 + ix_2) - 2x_3\zeta - (x_1 - ix_2)\zeta^2,
\]
the equation of the real section defined by the point \((x_1, x_2, x_3)\). This confirms the formula for \( I \). Notice that \( I \) is very similar to the real structure \( \tau \); in fact \( I \circ \tau(\eta, \zeta) = (\eta, -\zeta) \).

If we invert the monopole defined by the spectral curve \( S \) and section \( s \) we obtain a new curve \( I(S) \) and a new section \( I(s) \). The definition of \( I(S) \) is straightforward; it is just the image of \( S \) under the map \( I \). We shall consider \( I(s) \) in a moment. Because \( \tau(S) = S \) it follows that \((\eta, \zeta) \in I(S) \) precisely when \((\eta, -\zeta) \in S \). In particular, the intersection of \( I(S) \) and the fibre \( F \) over \( \zeta = 0 \) is just the intersection of \( S \) and \( F \). So if we denote by \( I(p) \) and \( I(q) \) the numerator and denominator of the rational map for the inverted monopole, we see that \( I(q) = q \).

Now consider the section \( s \). Notice that both \( \tau \) and \( I \) interchange the two coordinate patches \( U_0 \) and \( U_1 \). The section \( \tau(s) \) is defined locally by
\[
\tau(s)_0(\eta, \zeta) = \bar{s}_1(\tau(\eta, \zeta)) \quad , \quad \tau(s)_1(\eta, \zeta) = \bar{s}_0(\tau(\eta, \zeta))
\]
and the section \( I(s) \) by
\[
I(s)_0(\eta, \zeta) = \bar{s}_1(I(\eta, \zeta)) \quad , \quad I(s)_1(\eta, \zeta) = \bar{s}_0(I(\eta, \zeta)).
\]
Hence \( I(p) \) is defined by
\[
I(p)(z) = I(s)_0(z, 0) \mod q(z) = \bar{s}_1 \circ \tau(z, 0) \mod q(z)
\]
using the fact that \( \tau(\eta, 0) = I(\eta, 0) \). From the relation \( \tau(s) s = 1 \) and (8) it follows that \((\bar{s}_1 \circ \tau)s_0 = 1\) and hence
\[
(I(p)p)(z) = (\bar{s}_1 \circ \tau(z, 0))s_0(z, 0) \mod q(z) = 1 \mod q(z).
\]
Eq. (10), and the requirement that the degree of \( I(p) \) is less than \( k \), determine \( I(p) \) uniquely. If the roots of \( q \) are the distinct complex numbers \( \beta_1, \ldots, \beta_k \), a useful alternative way of obtaining \( I(p) \) is to notice that it is the unique polynomial of degree less than \( k \) such that \( I(p)(\beta_i)p(\beta_i) = 1 \) for all \( i = 1, \ldots, k \).

We summarize these results as:
Proposition 1 Let a monopole have spectral curve \( P(\eta, \zeta) = 0 \) and rational map \( p/q \). The inverted monopole has spectral curve \( P(\eta, -\zeta) = 0 \) and rational map \( I(p)/q \), where \( I(p)p = 1 \mod q \).

It is interesting to consider the subset of monopoles that are invariant under inversion. Their spectral curves are given by polynomials \( P(\eta, \zeta) \) which are even in \( \zeta \). Their rational maps satisfy \( p^2 = 1 \mod q \), so that \( I(p) = p \). This fixed-point set is described by the following:

Proposition 2 The moduli space \( IM_k \) of \( k \)-monopoles invariant under inversion is a totally geodesic submanifold of \( M_k \) of dimension \( 2k \). It has \( (k+1) \) connected components \( IM^m_k \) for \( 0 \leq m \leq k \). The component \( IM^m_k \) is diffeomorphic to the set of coprime pairs \( (r, s) \) of monic polynomials of degree \( m \) and \( (k-m) \) respectively.

Proof: It is a standard fact from differential geometry that the fixed point set of a finite group of isometries of a Riemannian manifold is a totally geodesic submanifold. For any rational function \( R = p/q \) defining a \( k \)-monopole, consider

\[
f(R) = \sum_i p(\beta_i)
\]

This is a symmetric polynomial in the roots of \( q \) and hence is polynomial in the coefficients of \( p \) and \( q \), and so is continuous on \( M_k \). On \( IM_k \), \( p^2 = 1 \mod q \), hence \( p(\beta_i) = \pm 1 \). Thus, restricted to \( IM_k \), \( f \) takes the values \( k, k-2, \ldots, -k \), and we define \( IM^m_k = f^{-1}(k-2m) \).

If \( p^2 - 1 = 0 \mod q \) then \( q \) divides \((p-1)(p+1)\). Since these factors are coprime, any irreducible factor of \( q \) divides one or the other. Hence we have monic polynomials \( r, s \) of degrees \( m, k-m \) respectively, such that \( q = rs \), and \( p+1 = 2ar, p-1 = 2bs \) for polynomials \( a, b \). Hence \( p = ar + bs \) and \( ar - bs = 1 \).

Conversely, given two coprime monic polynomials \( r, s \), the division algorithm implies that there exist polynomials \( a, b \) such that \( ar - bs = 1 \), and moreover \( a \) can be chosen uniquely to have degree less than that of \( s \). Now define \( p = ar + bs \). This has degree less than \( k \) and

\[
p^2 = (ar + bs)^2 = 1 + 4abq
\]

where \( q = rs \) is monic of degree \( k \). Clearly, from (11), \( p \) and \( q \) are coprime and so define a rational map \( R \).

Now the space of pairs \( (r, s) \) of coprime polynomials is the complement of a hypersurface in \( \mathbb{C}^m \times \mathbb{C}^{k-m} \) and so is a connected \( 2k \)-dimensional manifold. Moreover, when \( r, s \) both have distinct roots, the roots of \( q \) are \( \beta_1, \ldots, \beta_m \) and \( \beta_{m+1}, \ldots, \beta_k \) (the roots of \( r \) and \( s \) respectively). Hence on this manifold \( f(R) = k - 2m \), and so \( IM^m_k \) is connected, and as described in the proposition.
Note that $IM_k^m$ and $IM_k^{k-m}$ are isomorphic; one is obtained from the other by multiplying $p$ by $-1$. The simplest of the components is $IM_k^0$. Here $p(z) \equiv 1$, so the rational maps are of the form

$$R(z) = \frac{1}{q(z)}.$$  

The space $IM_k^0$ is naturally diffeomorphic to the moduli space of $k$ flux vortices in the critically coupled abelian Higgs model, since $k$-vortex solutions are also parametrised by a single monic polynomial of degree $k$ [25]. However, the metrics in the monopole and vortex cases will be different.

We are not sure what kind of monopole configurations lie in the various spaces $IM_k^m$, but we conjecture that for $m = 0$ (or $m = k$), the energy density is always confined to a finite neighbourhood of the plane $x_3 = 0$, whereas for $0 < m < k$ it is possible for there to be monopole clusters arbitrarily far from the plane $x_3 = 0$, arranged symmetrically with respect to inversion in this plane. The examples discussed in Section 12 are consistent with this conjecture. If the roots of $q$ are distinct and well-separated, then the configurations always consist of a set of unit charge monopoles with their centres in the $x_3 = 0$ plane.

Our inversion formula is inconsistent with Proposition 3.12 of [1]. There it was suggested that if

$$R(z) = \frac{p(z)}{q(z)} = \sum_i \frac{\alpha_i}{z - \beta_i}$$

is the rational map of a charge $k$ monopole, which consists of $k$ well-separated unit charge monopoles, then (using our conventions) the individual monopoles are approximately located at the points $(\beta_i, (1/2) \log |\alpha_i|)$ and have phases $\alpha_i|\alpha_i|^{-1}$. Consideration of our inversion formula suggests however that the individual monopoles are located at the points $(\beta_i, (1/2) \log |p(\beta_i)|)$ and have phases $p(\beta_i)|p(\beta_i)|^{-1}$. This has recently been proved by Bielawski in [4]. Interestingly the spaces $IM_k^m$ play a distinguished role in Bielawski’s work.

Finally notice that it follows from equations (8) and (9) and the fact that $\tau(\eta, 0) = I(\eta, 0)$ that using $\tau(s)$ to construct the rational map is the same as using $I(s)$, and hence the $p(\beta_i)$ occurring in the rational map defined using $\tau(s)$ would be the reciprocal of the $p(\beta_i)$ we use, and would give the rational map as defined by Hurtubise.

5 Centred monopoles and rational maps

We remarked earlier that although the positions and internal phases of the $k$ ‘particles’ in a charge $k$ monopole are only asymptotically well-defined, every monopole has a well-defined centre and total phase. This arises naturally in the twistor picture. If $S$ is the spectral curve of a monopole then it intersects every fibre of $TP_1 \to P_1$ in
$k$ points counted with multiplicity. If we add these points together we obtain a new curve which is given by an equation $\eta + a_1(\zeta) = 0$. This curve is a real section and hence $a_1$ is of the form

$$a_1(\zeta) = -k((c_1 + ic_2) - 2c_3\zeta - (c_1 - ic_2)\zeta^2).$$

The point $c = (c_1, c_2, c_3)$ is the centre of the monopole. To define the total phase requires a little more work.

Recall that the twistor space $Z$ for $S^1 \times \mathbb{R}^3$ is the principal bundle for the line bundle $L$ over $TP_1$. Its quotient $\bar{Z}$ by $\pm 1 \in \mathbb{C}^\ast$ is the principal bundle for $L^2$. The spectral curve $S \subset TP_1$ has a trivialization $s$ of $L^2$ and hence lifts to $\bar{Z}$. Thus, over each point in $P_1$, the spectral curve defines $k$ points counted with multiplicity in the fibre. This fibre is the group $\mathbb{C}^\ast \times \mathbb{C}$ and we take the product of the points. This is a section $s^k$ of $L^2$ over the curve $\eta + a_1(\zeta) = 0$.

The bundle $L^2$ over any real section is trivial and we fix as a choice of trivialisation $f$ over $\eta - k((c_1 + ic_2) - 2c_3\zeta - (c_1 - ic_2)\zeta^2) = 0$

$$f_0(\eta, \zeta) = \exp 2k(c_3 + (c_1 - ic_2)/\zeta)$$

$$f_1(\eta, \zeta) = \exp 2k(-c_3 + (c_1 + ic_2)/\zeta).$$

It is easy to check that this non-vanishing section $f$ satisfies $\tau(f)f = 1$. Moreover because $\tau(s)s = 1$ we must have $\tau(s^k)s^k = 1$. If we divide $s^k$ by $f$ we obtain a holomorphic function which must be constant. In fact because $\tau(s^k)s^k = 1$ and $\tau(f)f = 1$ this constant is a complex number of modulus 1. We define $s^k/f$ to be the total phase of the monopole. Notice that if we act on the monopole by a constant diagonal gauge transformation $\mu$ then $s$ is replaced by $\mu^2s$ and the total phase is multiplied by $\mu^{2k}$.

Let us now see how to construct the centre and total phase of a monopole from its rational map. Notice first that if we restrict the equation of the spectral curve to the fibre $\zeta = 0$ we obtain an equation of the form

$$\eta^k - k(c_1 + ic_2)\eta^{k-1} + \ldots = 0$$

and hence $c_1 + ic_2$ is the average of the points of intersection of the spectral curve with $\zeta = 0$ or the average of the zeros of $q$.

Comparing the construction of the rational map of a monopole we see that

$$s^k_0(k(c_1 + ic_2), 0) = \prod_i p(\beta_i) = \Delta(p, q)$$

the resultant of $p$ and $q$. It follows that

$$s^k_f = \Delta(p, q) \exp(-2kc_3).$$

So:
Proposition 3 If $R(z) = p(z)/q(z)$ is the rational map of a monopole with $q_0$ the average of the roots of $q$ and $\Delta(p, q)$ the resultant of $p$ and $q$, then the centre of the monopole is
\[(q_0, (1/2k) \log |\Delta(p, q)|)\]
and the total phase is
\[\Delta(p, q) |\Delta(p, q)|^{-1}.\]

It follows that a monopole is centred if and only if the zeroes of $q$ sum to zero and $|\Delta(p, q)| = 1$. It will be useful to use a stronger notion of centring than this. We call a monopole strongly centred if it is centred and the total phase is 1. From what we have just proven a monopole is strongly centred if and only if its rational map satisfies
\[q_0 = 0 \quad \text{and} \quad \Delta(p, q) = 1.\] (12)

The resultant condition $\Delta(p, q) = 1$ was used in [1], p.30 to identify the universal covering of the moduli space of centred monopoles, but for a fixed complex structure in the hyperkähler family. Our description of strong centring gives an invariant approach, valid for all complex structures. Considered in the context of the twistor space of a hyperkähler metric, it identifies the factor $X$ in the isometric splitting [[1], p.34] $\tilde{M}_k = X \times S^1 \times \mathbb{R}^3$ of a $k$-fold covering of $M_k$ with the space of strongly centred monopoles. It follows that the space of strongly centred monopoles is a geodesic submanifold of $M_k$.

Remark: Note that the twistor space $\tilde{Z}$ is the twistor space for the (trivial) hyperkähler metric on the moduli space of 1-monopoles. A $k$-monopole’s centre and total phase then associates a 1-monopole with a $\mathbb{Z}_k$-ambiguity of phases to a $k$-monopole.

6 Symmetric curves in $TP_1$

In eq. (5) we presented the general form of curves in $TP_1$ that occur as spectral curves of charge $k$ monopoles. The coefficients $a_r(\zeta)$ must satisfy the reality condition (3), and the curve is centred at the origin in $\mathbb{R}^3$ if $a_1(\zeta) = 0$. Here we shall discuss the form of these curves when they are required to be invariant under certain groups of rotations about the origin.

Let us recall that in $TP_1$, the $P_1$ of lines through the origin are parametrized by $\zeta$ with $\eta = 0$. The line in the direction of the Cartesian unit vector $(x_1, x_2, x_3)$ has $\zeta = (x_1 + ix_2)/(1 + x_3)$. It will be important to consider the homogeneous coordinates $[\zeta_0, \zeta_1]$ on $P_1$, as well as the inhomogeneous coordinate $\zeta = \zeta_1/\zeta_0$. An $SU(2)$ Möbius
transformation on the homogeneous coordinates, $[\zeta_0, \zeta_1] \rightarrow [\zeta'_0, \zeta'_1]$, of the form

\[
\begin{align*}
\zeta'_0 &= -(b + ia)\zeta_1 + (d - ic)\zeta_0 \\
\zeta'_1 &= (d + ic)\zeta_1 + (b - ia)\zeta_0
\end{align*}
\]

(13)

where $a^2 + b^2 + c^2 + d^2 = 1$, corresponds to an $SO(3)$ rotation in $\mathbb{R}^3$. The rotation is by an angle $\theta$ about the unit vector $(x_1, x_2, x_3)$, where $x_1 \sin \frac{\theta}{2} = a$, $x_2 \sin \frac{\theta}{2} = b$, $x_3 \sin \frac{\theta}{2} = c$, $\cos \frac{\theta}{2} = d$. The inhomogeneous coordinate $\zeta$ transforms to

\[
\zeta' = \frac{(d + ic)\zeta + (b - ia)}{-(b + ia)\zeta + (d - ic)}.
\]

(14)

Since $\eta$ is the coordinate in the tangent space to $P_1$ at $\zeta$, it follows that if $\zeta$ transforms to $\zeta'$ as in (14) then $\eta$ transforms to $\eta'$ via the derivative of (14), that is

\[
\eta' = \frac{\eta}{(-(b + ia)\zeta + (d - ic))^2}.
\]

(15)

A curve $P(\eta, \zeta) = 0$ in $TP_1$ is invariant under the Möbius transformation if $P(\eta', \zeta') = 0$ is the same curve. If the curve is the spectral curve of a monopole, then the monopole is invariant under the associated rotation.

The simplest group of symmetries is the cyclic group of rotations about the $x_3$-axis, $C_n$. The generator is the Möbius transformation

\[
\zeta' = e^{2\pi i/n}\zeta, \quad \eta' = e^{2\pi i/n}\eta.
\]

A curve $P(\eta, \zeta) = 0$ is invariant if all terms of $P$ have the same degree, mod $n$. A curve of the form (15) is $C_n$-invariant if all terms have degree $k$, mod $n$. In particular, it is $C_k$-invariant if all terms have degree zero, mod $k$.

For there to be axial symmetry about the $x_3$-axis, with symmetry group $C_\infty$, the curve must be invariant under $\zeta \rightarrow e^{i\theta}\zeta$, $\eta \rightarrow e^{i\theta}\eta$, for all $\theta$. This requires that all terms in $P(\eta, \zeta)$ have degree $k$. There is a unique axially symmetric, strongly centred monopole for each charge $k$. It is shown in [11] that its spectral curve is

\[
\eta \prod_{l=1}^{m} (\eta^2 + l^2 \pi^2 \zeta^2) = 0 \quad \text{for} \quad k = 2m + 1
\]

\[
\prod_{l=0}^{m} \left(\eta^2 + (l + \frac{1}{2})^2 \pi^2 \zeta^2\right) = 0 \quad \text{for} \quad k = 2m + 2.
\]

Notice that these curves are not determined by symmetry alone, and that the coefficients of $P$ are transcendental numbers. The only curve of the form (15) which has full $SO(3)$ symmetry is $\eta^k = 0$. This is the spectral curve of a unit charge monopole at the origin when $k = 1$, but for $k > 1$ it is not the spectral curve of a monopole.
The groups $C_n$ and $C_\infty$ are extended to the dihedral groups $D_n$ and $D_\infty$ by adding a rotation by $\pi$ about the $x_1$-axis. This rotation corresponds to the transformation on $TP_1$

$$\zeta' = \frac{1}{\zeta}, \quad \eta' = -\frac{\eta}{\zeta^2}.$$ 

Under this transformation, and for any constant $\nu$,

$$(\eta^2 + \nu\zeta^2)' = \frac{1}{\zeta^4}(\eta^2 + \nu\zeta^2),$$

so each of the axially symmetric monopoles has symmetry group $D_\infty$.

Recall from Section 4 that $P(\eta, \zeta) = 0$ is reflection symmetric under $x_3 \to -x_3$ if $P$ is even in $\zeta$. By a similar argument to that in Section 4, the reflection $x_2 \to -x_2$ corresponds to $\zeta \to \frac{1}{\zeta}, \eta \to \eta$, so a curve $P(\eta, \zeta) = 0$ is invariant under this reflection if all coefficients in $P(\eta, \zeta)$ are real. The axially symmetric monopoles therefore have these reflection symmetries too.

As an example of finite cyclic or dihedral symmetry, let us consider centred $k = 3$ curves with either $C_3$ or $D_3$ symmetry. Before imposing the symmetry, the curves are of the form

$$\eta^3 + \eta(\alpha_4 \zeta^4 + \alpha_3 \zeta^3 + \alpha_2 \zeta^2 + \alpha_1 \zeta + \alpha_0) + (\beta_6 \zeta^6 + \beta_5 \zeta^5 + \beta_4 \zeta^4 + \beta_3 \zeta^3 + \beta_2 \zeta^2 + \beta_1 \zeta + \beta_0) = 0 \quad (16)$$

subject to the reality conditions

$$\alpha_4 = \overline{\alpha_0}, \quad \alpha_3 = -\overline{\alpha_1}, \quad \alpha_2 = \overline{\alpha_2},$$

$$\beta_6 = -\overline{\beta_0}, \quad \beta_5 = \overline{\beta_1}, \quad \beta_4 = -\overline{\beta_2}, \quad \beta_3 = \overline{\beta_3}.$$ 

$C_3$ symmetry implies that (16) reduces to

$$\eta^3 + \alpha \eta \zeta^2 + \beta \zeta^6 + \gamma \zeta^3 - \bar{\beta} = 0 \quad (17)$$

where $\alpha$ and $\gamma$ are real. By a rotation about the $x_3$-axis, we can orient the curve so that $\beta$ is real, too, and then there is reflection symmetry under $x_2 \to -x_2$. There is $D_3$ symmetry if $\gamma = 0$; then the curve reduces to

$$\eta^3 + \alpha \eta \zeta^2 + \beta(\zeta^6 - 1) = 0 \quad (18)$$

with $\alpha$ and $\beta$ real.

The axisymmetric charge 3 monopole has a spectral curve of type (18) with $\alpha = \pi^2$ and $\beta = 0$. Also, three well-separated unit charge monopoles at the vertices of an
equilateral triangle can have $D_3$ symmetry. The spectral curve is asymptotic to the product of three stars at
\[
(x_1, x_2, x_3) = \{(a, 0, 0), \ (a \cos \frac{2\pi}{3}, a \sin \frac{2\pi}{3}, 0), \ (a \cos \frac{4\pi}{3}, a \sin \frac{4\pi}{3}, 0)\},
\]
that is,
\[
(\eta - a(1 - \zeta^2))(\eta - a\omega(1 - \omega^2))(\eta - a\omega^2(1 - \omega^2\zeta^2)) = 0,
\]
where $\omega = e^{2\pi i/3}$. Multiplied out, this is a curve of the form (18) with $\alpha = 3a^2$ and $\beta = a^3$, or equivalently $\alpha^3 = 27\beta^2$. We shall find out more about the spectral curves of charge 3 monopoles with symmetry $C_3$ or $D_3$ when we consider the rational maps associated with the monopoles (see Section 12).

Symmetry under $C_4$ is rather a weak constraint on curves with $k = 4$. On the other hand $D_4$ symmetry implies that a $k = 4$ curve is of the form
\[
\eta^4 + \alpha\eta^2\zeta^2 + \beta\zeta^8 + \gamma\zeta^4 + \beta = 0
\]
with $\alpha$, $\beta$ and $\gamma$ real. The axisymmetric charge 4 monopole has this form of spectral curve, with $\alpha = (5/2)\pi^2$, $\beta = 0$ and $\gamma = (9/16)\pi^4$. Four well-separated unit charge monopoles at the vertices of the square $\{(\pm a, 0, 0), (0, \pm a, 0)\}$ can have $D_4$ symmetry. The spectral curve is asymptotic to a product of stars, and is of the form (19), with $\alpha = 4a^2$, $\beta = -a^4$ and $\gamma = 2a^4$. After a $\pi/4$ rotation, the monopoles are at $(\pm a/\sqrt{2}, \pm a/\sqrt{2}, 0)$, and $\alpha = 4a^2$, $\beta = a^4$ and $\gamma = 2a^4$.

There is another interesting asymptotic 4-monopole configuration, with a spectral curve of type (19). Consider two well-separated axisymmetric charge 2 monopoles, centred at $(0, 0, b)$ and $(0, 0, -b)$, and with the $x_3$-axis the axis of symmetry. The spectral curve is asymptotic to a product of curves associated with the charge 2 monopoles. The spectral curve of a centred axisymmetric charge 2 monopole is $\eta^2 + \frac{1}{4}\pi^2\zeta^2 = 0$. This factorizes as $(\eta + \frac{1}{2}i\pi\zeta)(\eta - \frac{1}{2}i\pi\zeta) = 0$, which is a product of stars at the complex conjugate points $(0, 0, \pm i\pi/4)$. Translation by $b$ gives the curve
\[
\eta^2 + 4b\eta\zeta + (4b^2 + \frac{1}{4}\pi^2)\zeta^2 = 0
\]
which is the product of stars at $(0, 0, b \pm i\pi/4)$. Similarly, translation by $-b$ gives
\[
\eta^2 - 4b\eta\zeta + (4b^2 + \frac{1}{4}\pi^2)\zeta^2 = 0
\]
and the product of these is the curve
\[
\eta^4 + (\frac{1}{2}\pi^2 - 8b^2)\eta^2\zeta^2 + (4b^2 + \frac{1}{4}\pi^2)^2\zeta^4 = 0.
\]
Since all terms have degree 4 this curve is axisymmetric; however, the actual spectral curve of the charge 4 monopole has symmetry $D_4$, as we shall see in Section 12, becoming axisymmetric only in the limit of infinite separation.

Let us now investigate the curves in $TP_1$ with the symmetries of a regular solid. Some of these are special cases of the curves we have already discussed. There are three rotational symmetry groups to consider, those of a tetrahedron, an octahedron and an icosahedron. The direct way to construct a symmetric curve is to find Möbius transformations which generate the symmetry group, and calculate the conditions for the curve to be invariant under all of them. For example, a curve of type (19), with $D_4$ symmetry, has octahedral symmetry if it is invariant under the transformation

$$
\zeta' = \frac{i\zeta + 1}{\zeta + i}, \quad \eta' = \frac{-2}{(\zeta + i)^2} \eta,
$$

which corresponds to a $\pi/2$ rotation about the $x_1$-axis, and this requires that the curve reduces to

$$
\eta^4 + \beta(\zeta^8 + 14\zeta^4 + 1) = 0.
$$

A more powerful and less laborious approach is to use the theory of invariant bilinear forms and polynomials on $P_1$, as expounded in Klein’s famous book [15].

Consider a homogeneous bilinear form $Q_r(\zeta_0, \zeta_1)$ of degree $r$, and its associated inhomogeneous polynomial $q_r(\zeta)$ defined by

$$
Q_r(\zeta_0, \zeta_1) = \zeta_0^rq_r(\zeta).
$$

Generally $q_r$ has degree $r$, but it may have lower degree. Suppose $Q_r(\zeta_0, \zeta_1)$ is invariant under a Möbius transformation of the form (13). Then $q_r(\zeta)$ transforms in a simple way under the corresponding transformation (14), namely

$$
q'_r(\zeta) = \frac{q_r(\zeta)}{(-(b + ia)\zeta + (d - ic))^r}.
$$

(20)

On the other hand, $\eta$ transforms as in (15). Consider a centred curve in $TP_1$,

$$
P(\eta, \zeta) \equiv \eta^k + \eta^{k-2}q_4(\zeta) + \eta^{k-3}q_6(\zeta) + \ldots + q_{2k}(\zeta) = 0.
$$

If, under a Möbius transformation, each polynomial $q_r(\zeta)$ transforms as in (20), and $\eta$ as in (15), then each term in the polynomial $P(\eta, \zeta)$ is multiplied by the same factor $(-(b + ia)\zeta + (d - ic))^{-2k}$, so the curve is invariant. It follows that curves invariant under the rotational symmetry group of a regular solid can be constructed from the inhomogeneous polynomials $q_r$ derived from the bilinear forms $Q_r$ invariant under the group.
Let $G$ denote the tetrahedral, octahedral or icosahedral group. Klein has described the ring of bilinear forms, $\text{Inv}_G$, which change only by a constant factor under each transformation of $G$ — for each form these factors define an abelian character of $G$. Let $\text{Inv}_G^\star$ be the subring of strictly invariant forms. A form $Q$ is in $\text{Inv}_G$ if the roots of the associated polynomial $q$ are invariant under $G$, that is, if they are the union of a set of $G$-orbits on $P_1$.

Generic $G$-orbits on $P_1$ consist of $|G|$ points, i.e. 12, 24 and 60 points respectively for the three groups. The associated forms of degree $|G|$ are always strictly invariant under $G$, and they span a vector space of forms, of dimension two. For each group $G$, there are also three forms of degree less than $|G|$ associated with special orbits of $G$, and these generate the ring $\text{Inv}_G$. Let $V$, $E$ and $F$ be the set of vertices, mid-points of edges, and centres of faces of the centred regular solid (tetrahedron, octahedron or icosahedron) invariant under $G$. Centrally project these points onto the unit sphere, identified with $P_1$, denoting the resulting sets of points again by $V$, $E$ and $F$.

V is a $G$-orbit, so there is a form $Q_V$ in $\text{Inv}_G$ and an associated polynomial $q_V$, such that $Q_V$ has degree $|V|$ and $Q_V = 0$ at all points of $V$. Similarly, there are forms and polynomials $Q_E$, $Q_F$ and $q_E$, $q_F$. Table 1 gives the polynomials $q_V$, $q_E$ and $q_F$ for the three groups $G$, and a star indicates that the associated form ($Q_V$, $Q_E$ or $Q_F$) is strictly $G$-invariant. [A choice of orientation has been made for the solids: the tetrahedron has its vertices at $(1/\sqrt{3})(\pm1, \pm1, \pm1)$, with either two or no signs negative; the octahedron has its vertices on the Cartesian axes; the icosahedron has two vertices on the $x_3$-axis and is invariant under the dihedral group $D_5$.]

All the icosahedral forms are strictly invariant because the icosahedral group $A_5$ is simple, and has no non-trivial abelian characters. The tetrahedral forms $Q_V$ and $Q_F$ are not strictly invariant, but acquire factors of $e^{\pm 2\pi i/3}$ under a $2\pi/3$ rotation about a 3-fold symmetry axis; so $Q_V, Q_F$ is strictly invariant. In fact, the polynomial associated with $Q_V, Q_F$ is $\zeta^8 + 14\zeta^4 + 1$, which has octahedral symmetry. Similarly, the octahedral forms $Q_V$ and $Q_E$ acquire factors of $-1$ under a rotation by $\pi/2$ around a 4-fold symmetry axis, and $Q_V, Q_E$ is strictly invariant.

There are remarkable identities satisfied by the forms $Q_V, Q_E$ and $Q_F$ (which remain true if the forms $Q$ are replaced by the associated polynomials $q$), namely

\begin{align*}
Q_V^3 - Q_F^3 - 12\sqrt{3}i \; Q_E^2 &= 0 \quad \text{for the tetrahedral group} \\
108 \; Q_V^4 - Q_F^3 + Q_E^2 &= 0 \quad \text{for the octahedral group} \\
1728 \; Q_V^5 - Q_F^5 - Q_E^2 &= 0 \quad \text{for the icosahedral group}.
\end{align*}

These identities occur, because each term is a strictly invariant form of degree $|G|$, lying in the two-dimensional vector space of forms associated with the generic $G$-orbits.
Polynomials associated with the special orbits V, E and F of the rotational symmetry groups of the regular solids. A star (⋆) denotes that the homogeneous bilinear form \( Q \) related to the polynomial \( q \) is strictly invariant.

**Table 1**

We can now write down some examples of invariant curves in \( TP_1 \), also satisfying the reality conditions (6). Recall that invariant curves in \( TP_1 \) must be constructed from polynomials derived from strictly invariant forms. The simplest curves with tetrahedral symmetry are

\[
\eta^3 + ia\zeta(\zeta^4 - 1) = 0
\]

where \( a \) is real. After a rotation, these become

\[
\eta^3 + a(\zeta^6 + 5\sqrt{2}\zeta^3 - 1) = 0,
\]

which are of the form (17), exhibiting manifest \( C_3 \) symmetry about the \( x_3 \)-axis. The simplest curves in \( TP_1 \) with octahedral symmetry are

\[
\eta^4 + a(\zeta^8 + 14\zeta^4 + 1) = 0
\]

with \( a \) real. More generally, the \( k = 4 \) curves with tetrahedral symmetry are of the form

\[
\eta^4 + ibn\zeta(\zeta^4 - 1) + a(\zeta^8 + 14\zeta^4 + 1) = 0
\]
with $a$ and $b$ real. Finally, the simplest curves with icosahedral symmetry are
\[ \eta^6 + a\zeta(\zeta^{10} + 11\zeta^5 - 1) = 0 \]
with $a$ real.

We shall discuss in the next Sections the possibility that some of these curves are spectral curves of monopoles.

### 7 Symmetric curves and elliptic curves

Let $G \subset SO(3)$ be the symmetry group of a regular solid and $\tilde{G} \subset SU(2)$ the corresponding binary group. Let $V$ be the 2-dimensional defining representation of $SU(2)$. The $n$-th symmetric power $S^nV$ may be considered as the action of $SU(2)$ on homogeneous polynomials of degree $n$ in $\zeta_0, \zeta_1$. Alternatively, the representation is on the space of holomorphic sections of the line bundle $\mathcal{O}(n)$ on $P_1$, which is described as the polynomials of degree $\leq n$ in the affine parameter $\zeta = \zeta_1/\zeta_0$.

Let $n$ be the smallest degree of a homogeneous polynomial invariant under $\tilde{G}$. For the tetrahedral, octahedral and icosahedral group the value of $n$ is $2k$ where $k$ is respectively 3, 4, 6 and in each case there is a unique (up to a multiple) invariant polynomial as given in Table 1. Note that in all cases $|G| = 2k(k-1)$. In the previous section, we defined curves in $TP_1$ by

\begin{align*}
\eta^3 + ia\zeta(\zeta^4 - 1) &= 0 \quad (21) \\
\eta^4 + a(\zeta^8 + 14\zeta^4 + 1) &= 0 \quad (22) \\
\eta^6 + a\zeta(\zeta^{10} + 11\zeta^5 - 1) &= 0 \quad (23)
\end{align*}

where $a$ is real. Each such curve $S$ is invariant by the appropriate group $G$ and satisfies the reality conditions for a spectral curve. We shall prove that the first two are indeed spectral curves for non-singular monopoles of charges 3 and 4, for suitable values of $a$. On the other hand, by the same methods we shall see that there is no charge 6 monopole which has icosahedral symmetry.

The method consists of finding explicitly the solution to Nahm’s equations with these spectral curves, thereby solving the non-linear part of the monopole problem. To find the monopole configuration itself involves solving an associated linear differential equation \[1\].

The solution to Nahm’s equations will be expressed in terms of elliptic functions, and the constant $a$ in terms of the periods of an elliptic curve. The advance evidence for this fact lies in the following result:

**Proposition 4** The curve $S$ in (21), (22), (23) is smooth and of genus $(k-1)^2$. Its quotient by $G$ is an elliptic curve.
If we think of the solution to Nahm’s equations given in [12] as a linearization of the equations on the Jacobian of the spectral curve, then the above proposition implies that a $G$–invariant solution is linearized on the Jacobian of the elliptic curve and hence expressible in terms of elliptic functions. We shall achieve this directly, however, without making use of the general method of [12].

Proof of Proposition: Smoothness is immediate since the polynomials by inspection have distinct roots. The genus formula is standard [12, 1].

Now consider the action of $G$ on $S$. Let $m$ be a fixed point of an element of $G$. Consider its image in $P_1$. The group $G$ acts on the spectral curve through the natural action on the tangent bundle $TP_1$, but for isometries the action of an isotropy group on the tangent space of the point is faithful. Thus the zero vector is the only one fixed. But this is given by $\eta = 0$ and from the equation of the curve these points are the $2k$ zeros of the polynomial.

It follows that $G$ acts freely except at the $2k$ points $\eta = 0$, and since $|G| = 2k(k-1)$, the stabilizer of each is of order $(k-1)$. These are the stabilizers of vertices of the regular solids, and hence cyclic. Thus by the Riemann-Hurwitz formula, the genus $g$ of the quotient satisfies

$$2 - 2(k-1)^2 = 2k(k-1)(2 - 2g) - 2k(k-2)$$

and hence $g = 1$.

8 Special solutions to Nahm’s equations

Recall that a centred charge $k$ monopole may be obtained from a solution of Nahm’s equations [12].

$$\frac{dT_1}{ds} = [T_2, T_3]$$
$$\frac{dT_2}{ds} = [T_3, T_1]$$
$$\frac{dT_3}{ds} = [T_1, T_2]$$

where $T_i(s)$ is a function with values in the Lie algebra $su(k)$. It must satisfy moreover a reality condition $T_i(2 - s) = T_i(s)^T$ with respect to an orthonormal basis compatible with the unitary structure. (In the explicit formulae which follow, we have not always used such a basis, preferring one which is simpler for calculations. We rely on the reality of the spectral curve and the description of [12] to assure the existence of such
a basis.) The solution to Nahm’s equations must be regular for \( s \in (0, 2) \) and have simple poles at \( s = 0 \) and \( s = 2 \). At \( s = 0 \), we have an expansion

\[
T_i = R_i/s + \ldots .
\]

Nahm’s equations themselves imply that, at any simple pole, the residues satisfy

\[
R_1 = -[R_2, R_3] \\
R_2 = -[R_3, R_1] \\
R_3 = -[R_1, R_2]
\]

and so define a representation of the Lie algebra \( \mathfrak{so}(3) \). In order for a solution to give a monopole, the representation at \( s = 0 \) and \( s = 2 \) must (see [12] or [1] Chapter 16) be the unique irreducible representation \( S^{k-1}V \) of dimension \( k \). In fact, as shown in [12], this space is canonically isomorphic to \( H^0(P_1, \mathcal{O}(k - 1)) \) under the projection from the spectral curve \( S \). With any solution to Nahm’s equations, the coefficients of the polynomial

\[
P(\eta, \zeta) = \det(\eta + i(T_1 + iT_2) - 2iT_3\zeta - i(T_1 - iT_2)\zeta^2)
\]  

are independent of \( s \), and indeed for a monopole \( P(\eta, \zeta) = 0 \) defines \( S \).

We may regard the triple of Nahm matrices as a function with values in

\[
\mathbb{R}^3 \otimes \mathfrak{su}(k).
\]

The action of the rotation group \( SO(3) \) on a monopole then appears as the tensor product of its natural action on \( \mathbb{R}^3 \) and \( \mathfrak{su}(k) \). In terms of the irreducibles \( S^{m}V \), this is the representation

\[
S^2V \otimes \text{End}_0(S^{k-1}V)
\]

where \( \text{End}_0 \) denotes trace zero endomorphisms.

In the above situation where the monopole is \( G \)-invariant, the Nahm matrices lie in a subspace of \( S^2V \otimes \text{End}_0(S^{k-1}V) \) which is fixed by \( G \).

**Proposition 5** The fixed point set of \( G \) in \( S^2V \otimes \text{End}_0(S^{k-1}V) \) is a 2-dimensional vector space.

**Proof:** First take the Clebsch-Gordan decomposition of \( \text{End}(S^{k-1}V) \cong S^{k-1}V \otimes S^{k-1}V \) into irreducibles of \( SO(3) \). We obtain

\[
\text{End}_0(S^{k-1}V) \cong S^{2k-2}V \oplus S^{2k-4}V \oplus \ldots \oplus S^2V.
\]
The single $S^2V$ factor gives a 1-dimensional subspace of $S^2V \otimes S^{k-1}V \otimes S^{k-1}V$ fixed by $SO(3)$. This is simply the homomorphism of Lie algebras

$$\mathfrak{so}(3) \cong S^2V \rightarrow \text{End}_0(S^{k-1}V)$$

defined by the representation $S^{k-1}V$. Now the Clebsch-Gordan decomposition also gives

$$S^{2m}V \otimes S^2V = S^{2m+2}V \oplus S^{2m}V \oplus S^{2m-2}V.$$  

But $2k$ is, by choice, the smallest positive integer $n$ such that $S^nV$ has a $G$-invariant vector. Thus $S^{2m}V \otimes S^2V$ has no invariants for $1 < m < k - 1$, and for $m = k - 1$ there is a unique one lying in $S^{2k}V$. This gives another 1-dimensional fixed subspace, and therefore a 2-dimensional space altogether.

We use this fact next to give a substantial simplification of Nahm’s equations in the $G$-invariant case. From Proposition 5, any $G$-invariant element $T = (T_1, T_2, T_3)$ of $\mathbf{R}^3 \otimes \mathfrak{su}(k)$ can be written as

$$T_i = x \rho_i + y S_i$$

where $\rho : \mathbf{R}^3 \rightarrow \mathfrak{su}(k)$ is the representation of $\mathfrak{so}(3)$ on $\mathbf{C}^k$ and $(S_1, S_2, S_3)$ is the $G$-invariant vector in $S^{2k}V \subset \mathbf{R}^3 \otimes \mathfrak{su}(k)$. In particular the Nahm matrices for a $G$-invariant monopole can be expressed as

$$T_i(s) = x(s) \rho_i + y(s) S_i \quad \text{for} \quad i = 1, 2, 3. \quad (26)$$

Given one invariant element $T$ of $\mathbf{R}^3 \otimes \mathfrak{su}(k)$, we can use the cross product on $\mathbf{R}^3$ and the Lie bracket on $\text{End}_0(\mathbf{C}^k)$ to define another, $T \times T$. Since this again lies in the two-dimensional fixed subspace generated by $\rho$ and $S$, there must be constants $\alpha, \beta, \gamma, \delta$ such that:

$$[S_1, \rho_2] + [\rho_1, S_2] = \alpha \rho_3 + \beta S_3$$
$$[S_1, S_2] = \gamma \rho_3 + \delta S_3$$

and corresponding expressions obtained by cyclic permutation. The analogous term for $\rho_i$ is determined by the fact that it gives a representation of $\mathfrak{so}(3)$:

$$[\rho_1, \rho_2] = 2 \rho_3 \quad \text{etc.}$$

using the standard basis of the Lie algebra. From this, Nahm’s equations (24) become

$$\frac{dx}{ds} = 2x^2 + \alpha xy + \gamma y^2 \quad (27)$$
$$\frac{dy}{ds} = \beta xy + \delta y^2. \quad (28)$$
Equations in this general form can always be reduced to quadratures, but in our case we shall calculate the precise values of the constants $\alpha, \beta, \gamma$ and $\delta$ and find $x(s)$ and $y(s)$ exactly.

To evaluate $\alpha, \beta, \gamma$ and $\delta$ we must find the $k \times k$ matrices $\rho_i$ and $S_i$. If $e_1, e_2, e_3$ is the standard basis of the Lie algebra $\mathfrak{so}(3)$, with $[e_1, e_2] = 2e_3$ etc. then $\rho_i = \rho(e_i)$ where

$$\rho : \mathfrak{so}(3) \to \text{End}_0(\mathbb{C}^k)$$

is the irreducible $k$-dimensional representation. If we regard this as the action on homogeneous polynomials

$$a_0\zeta_0^{k-1} + a_1\zeta_0^{k-2}\zeta_1 + \ldots + a_{k-2}\zeta_0^{k-2} + a_{k-1}\zeta_1^{k-1}$$

then setting

$$X = \frac{1}{2}(e_1 - ie_2), \quad Y = -\frac{1}{2}(e_1 + ie_2), \quad H = -ie_3$$

we have the Lie brackets

$$[X, Y] = H, \quad [H, X] = 2X, \quad [H, Y] = -2Y \tag{29}$$

and the representation is defined on polynomials by the operators

$$X = \zeta_1 \frac{\partial}{\partial \zeta_0}, \quad Y = \zeta_0 \frac{\partial}{\partial \zeta_1}, \quad H = -\zeta_0 \frac{\partial}{\partial \zeta_0} + \zeta_1 \frac{\partial}{\partial \zeta_1}.$$ 

To determine the $S_i$, we have to reinterpret each of the polynomials in $\zeta$ in \(21, (22), (23)\), using the inclusions

$$S^{2k}V \subset S^2V \otimes S^{2k-2}V \subset S^2V \otimes \text{End}_0(S^{k-1}V).$$

The first inclusion is simply polarization (or differentiation) of a homogeneous polynomial $Q_{2k}(\zeta_0, \zeta_1)$ of degree $2k$

$$Q_{2k} \mapsto \zeta_0^2 \otimes \frac{\partial^2 Q_{2k}}{\partial \zeta_0^2} + 2\zeta_0\zeta_1 \otimes \frac{\partial^2 Q_{2k}}{\partial \zeta_0 \partial \zeta_1} + \zeta_1^2 \otimes \frac{\partial^2 Q_{2k}}{\partial \zeta_1^2}.$$ 

The second inclusion comes from $S^{2k-2}V \subset \text{End}_0(S^{k-1}V)$. A useful way to view this is to see the image $\rho(\mathfrak{so}(3))$ as the principal 3-dimensional subalgebra of $\mathfrak{sl}(k, \mathbb{C})$ (see \(10\)).

Any complex simple Lie algebra $\mathfrak{g}$ of rank $r$ breaks up into $r$ irreducible representations under the action of its principal 3-dimensional subalgebra $\langle X, Y, H \rangle$. Moreover, the nilpotent element $X$ is a regular nilpotent in $\mathfrak{g}$, and belongs to an $r$-dimensional abelian nilpotent subalgebra. The weight spaces of this algebra under
the action of $H$ are the highest weight spaces of the irreducible representations into which $\mathfrak{g}$ breaks up.

This is all for a general Lie algebra. In our case, for $\mathfrak{sl}(k, \mathbb{C})$, the decomposition is into representation spaces

$$S^{2k-2}V \oplus S^{2k-4}V \oplus \ldots \oplus S^2V$$

so that the subspace $S^{2k-2}V$ is the representation which contains the vector of highest weight among all elements in the Lie algebra. The nilpotent element $X$ lies in the 3-dimensional Lie algebra $S^2V$, and, being regular, has rank $k - 1$. It acts cyclically on $\mathbb{C}^k$ and so its centralizer is spanned by the powers of $X$. The element of highest weight which commutes with it is thus $X^{k-1}$, of rank 1. This is the highest weight vector of $S^{2k-2}V$ and so applying $Y$ to this element we generate the whole subspace $S^{2k-2}V \subset \text{End}_0(S^{k-1}V)$. Thus, given a homogeneous polynomial $Q(\zeta_0, \zeta_1) \in S^{2k-2}V$, we define a $k \times k$ matrix $S(Q)$ by finding the polynomial $\tilde{Q}$ such that

$$Q(\zeta_0, \zeta_1) = \tilde{Q}(\zeta_0 \frac{\partial}{\partial \zeta_1})\zeta_1^{2k-2}$$

and then setting

$$S(Q) = \tilde{Q} \text{ad}Y X^{k-1}.$$ 

In this way we evaluate the matrices $S_1, S_2, S_3$ for the three cases in (21), (22), (23).

9 Tetrahedral symmetry

In the tetrahedral case, $k = 3$ and the irreducible representation $\rho$ on $\mathbb{C}^3$ is the adjoint representation. From (29), relative to the basis $X, Y, H$, the action of $X$ is given by the matrix

$$
\begin{pmatrix}
0 & -2 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
$$

and then a highest weight vector is a multiple of

$$X^2 = 
\begin{pmatrix}
0 & 0 & -2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
$$

The matrix $Y$ in this representation is

$$
Y = 
\begin{pmatrix}
0 & 0 & 0 \\
-1 & 0 & 0 \\
0 & 2 & 0
\end{pmatrix}
$$
and so, using the above procedure, we can evaluate \( \rho_i \) and \( S_i \). For the representation \( \rho \) we use a basis such that

\[
\rho_1 = \begin{pmatrix} 0 & 2 & 0 \\ -1 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & 2i & 0 \\ i & 0 & 2i \\ 0 & i & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 2i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2i \end{pmatrix}.
\]

To find \( S_i \) we first take the polynomial

\[ Q_6(\zeta_0, \zeta_1) = \zeta_0 \zeta_1 (\zeta_1^4 - \zeta_0^4) \]

and polarize it to obtain

\[ -20\xi_0^2 \otimes \zeta_0^3 \zeta_1 + 10\xi_0 \xi_1 \otimes (\zeta_1^4 - \zeta_0^4) + 20\xi_1^2 \otimes \zeta_0 \zeta_1^3. \]

Relating the basis \( \xi_0, \xi_0 \xi_1, \xi_1^2 \) to the standard orthonormal basis of \( S^2V \cong so(3) \), we have \( (S_1, S_2, S_3) \) given as a multiple of

\[ (2i(\zeta_0 \xi_1^3 - \zeta_0^3 \xi_1), 2(\zeta_0 \xi_1^3 + \zeta_0^3 \xi_1), (\zeta_1^4 - \zeta_0^4)) \]

where we now have to interpret the quartic polynomials as \( 3 \times 3 \) matrices. Following the procedure described above, we obtain

\[
S_1 = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}, \quad S_2 = i \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} \\ 0 & \frac{1}{4} & 0 \end{pmatrix}, \quad S_3 = i \begin{pmatrix} 0 & 0 & -1 \\ \frac{1}{4} & 0 & 0 \end{pmatrix}.
\]

**Remark**: The above form is a direct consequence of choosing a convenient basis for calculations, but the reality conditions are more evident if we change basis. We then find

\[
\rho_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};
\]

\[
S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}.
\]

With these explicit matrices, we can calculate:

\[ [S_1, S_2] = \frac{1}{8} \rho_3, \quad [S_2, S_3] = \frac{1}{8} \rho_1, \quad [S_3, S_1] = \frac{1}{8} \rho_2 \]

from which we deduce that \( \gamma = \frac{1}{8} \) and \( \delta = 0 \). We also find

\[ [S_1, \rho_2] + [\rho_1, S_2] = -4S_3 \quad \text{etc.} \]
and thereby obtain $\alpha = 0$ and $\beta = -4$. Hence, in the tetrahedral case, Nahm’s equations are reduced via $\eqref{eq:27}, \eqref{eq:28}$ to

$$\frac{dx}{ds} = 2x^2 + \frac{1}{8}y^2$$
$$\frac{dy}{ds} = -4xy.$$  \hspace{1cm} (30)

When $T_i = x\rho_i + yS_i$ in the present situation, a straightforward calculation gives the polynomial $\eqref{eq:25}$ as

$$\eta^3 - \frac{1}{2}(48x^2 + y^2)y\zeta(\zeta^4 - 1),$$

so $(48x^2 + y^2)y$ is a constant of integration for $\eqref{eq:30}, \eqref{eq:31}$. The requirement that the Nahm matrices are antihermitian means that $x$ is real, and $y = iv$ with $v$ real. The constant of integration is $(48x^2 - v^2)v = c$, where $c$ is real, and the polynomial $\eqref{eq:32}$ is the same as in $\eqref{eq:21}$, if we identify $a = -\frac{1}{2}c$.

Using this constant of integration and substituting in $\eqref{eq:31}$ gives

$$\sqrt{\frac{3}{\sqrt{v^4 + cv}}} = -ds.$$  

Putting $v = c^{1/3}\wp(u)^{-1}$, where the Weierstrass elliptic function $\wp(u)$ satisfies the equation $\wp'(u)^2 = 4\wp^3(u) + 4$, we find

$$s = 2\sqrt{3}c^{-1/3}u + K$$

for some constant $K$. Thus from these substitutions and $\eqref{eq:31}$ we obtain the general solution to the equations

$$x = \frac{c^{1/3}}{8\sqrt{3}} \frac{\wp'(u)}{\wp(u)}, \quad y = i\frac{c^{1/3}}{\wp(u)}.$$

Now, the period lattice of $\wp(u)$ is (equilaterally) triangular, with triangle edges along the imaginary axis, and $\wp$ has double poles at the vertices of the triangles. (Near $u = 0$, $\wp(u) = u^{-2} + \ldots$) $\wp' = 0$ at the mid-points of the edges, and $\wp = 0$ at the centres of the triangles. Let $2\omega_1$ denote the real period. On the interval $[0, 2\omega_1]$ there are zeros of $\wp(u)$ at $u = \frac{2}{3}\omega_1$ and $u = \frac{4}{3}\omega_1$. Between $\frac{2}{3}\omega_1$ and $\frac{4}{3}\omega_1$, $\wp(u)$ is negative, reaching its minimum value $\wp(\omega_1) = -1$.

To fit the boundary conditions of a monopole, we require $T_i(s)$ to be regular for $0 < s < 2$ and to have poles at $s = 0$ and $s = 2$ whose residues define the irreducible three-dimensional representation of $\mathfrak{so}(3)$. We can satisfy these conditions if we choose $K$ so that $u = \frac{2}{3}\omega_1$ when $s = 0$ and $u = \frac{4}{3}\omega_1$ when $s = 2$, that is

$$s = \frac{1}{\omega_1}(3u - 2\omega_1).$$
with $2\omega_1 = \sqrt{3}e^{1/3}$. Both $x$ and $y$ have simple poles at $s = 0$ and $s = 2$, and one can verify that the residues of the Nahm matrices there define an irreducible representation of $\mathfrak{so}(3)$. Moreover, $x$ and $y$ have no singularities for $0 < s < 2$. Since $y(2-s) = y(s)$ and $x(2-s) = -x(s)$, it follows that $T_i(2-s) = T_i(s)^T$.

It is straightforward to see that another solution with these boundary conditions is obtained by replacing $2\omega_1$ by $-2\omega_1$; the result is a reflection of the monopole about the origin.

Yet another solution of Nahm’s equations is obtained by choosing $K = 0$, with $u = \frac{2}{3}\omega_1$ at $s = 2$. This gives an equivalent monopole, with the same spectral curve. Although the symmetry $s \rightarrow 2 - s$ is no longer manifest, it can be obtained after a unitary transformation of the matrices.

The elliptic curve featured in the solution is a very special one, and the period $2\omega_1$ may be evaluated explicitly. We have (see e.g. [27])

$$2\omega_1 = \int_{-1}^{\infty} \frac{dt}{\sqrt{t^3 + 1}} = \frac{1}{2\sqrt{\pi}} \Gamma\left(\frac{1}{6}\right) \Gamma\left(\frac{1}{3}\right)$$

and consequently

**Theorem 1** The curve $\eta^3 + ia\zeta(\zeta^4 - 1)$ is the spectral curve of a charge 3 monopole with tetrahedral symmetry if $a = \pm\Gamma\left(\frac{1}{6}\right)^3 \Gamma\left(\frac{1}{3}\right)^3 / 48\sqrt{3\pi}^{3/2}$.

### 10 Octahedral symmetry

In the case of the octahedral group, $k = 4$, and we consider the irreducible representation of $SU(2)$ on homogeneous cubic polynomials. In a suitable basis, we can express the representation by

$$\rho_1 = \begin{pmatrix} 0 & 3 & 0 & 0 \\ -1 & 0 & 4 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \rho_2 = i \begin{pmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \rho_3 = i \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$ 

The matrices $S_i$ are found by polarizing the polynomial $\zeta_1^8 + 14\zeta_1^4\zeta_0^4 + \zeta_0^8$, and representing the resulting three sextic polynomials as $4 \times 4$ matrices. We find

$$S_1 = \begin{pmatrix} 0 & -6 & 0 & -60 \\ 2 & 0 & 12 & 0 \\ 0 & -3 & 0 & -6 \\ 5/3 & 0 & 2 & 0 \end{pmatrix}, \quad S_2 = i \begin{pmatrix} 0 & -6 & 0 & 60 \\ -2 & 0 & 12 & 0 \\ 0 & 3 & 0 & -6 \\ 5/3 & 0 & -2 & 0 \end{pmatrix}.$$
and

\[ S_3 = 4i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]

and from these expressions we obtain

\[ [S_1, S_2] = -48\rho_3 - 8S_3 \quad \text{and} \quad [S_1, \rho_2] + [\rho_1, S_2] = -6S_3 \quad \text{etc.} \]

from which it follows that

\[ \alpha = 0, \quad \beta = -6, \quad \gamma = -48, \quad \delta = -8. \]

Using (27) and (28), Nahm’s equations then reduce to

\[
\frac{dx}{ds} = 2x^2 - 48y^2 \quad (33) \\
\frac{dy}{ds} = -6xy - 8y^2. \quad (34)
\]

In this case,

\[
\det(\eta + i(T_1 + iT_2) - 2iT_3\zeta - i(T_1 - iT_2)\zeta^2) = \eta^4 - 960y(x + 3y)(x - 2y)^2(\zeta^8 + 14\zeta^4 + 1) \quad (35)
\]

so that we have an integral of the equations

\[ y(x + 3y)(x - 2y)^2 = c. \quad (36) \]

To solve the equations, put \( x = ty \), then (34) and (36) give

\[ y^4(t + 3)(t - 2)^2 = c \quad \text{and} \quad \frac{d}{ds} \left(\frac{1}{y}\right) = 2(3t + 4) \]

and hence

\[ \frac{d}{ds}((t + 3)^{1/4}(t - 2)^{1/2}) = 2c^{1/4}(3t + 4). \]

Now making the substitution \( t = 5w^2 - 3 \), we obtain

\[
\frac{dw}{\sqrt{w(w^2 - 1)}} = 4(5c)^{1/4}ds.
\]

This is an elliptic integral and is solved by setting \( w = \varphi(u) \) with \( \varphi'(u)^2 = 4\varphi(u)^3 - 4\varphi(u) \) to get

\[ u = 2(5c)^{1/4}s + K. \quad (37) \]
The general solution to the equation is then
\[ x = \frac{2c^{1/4}(5\varphi^2(u) - 3)}{5^{3/4}\varphi'(u)}, \quad y = \frac{2c^{1/4}}{5^{3/4}\varphi'(u)}. \]

At \( u = 0 \), \( y \) vanishes and so if \( K = 0 \), \( x = -(5c)^{1/4}/u + \ldots = -1/2s + \ldots \) from (37). Thus \( T_i = -\rho_i/2s + \ldots \) and the boundary condition at \( s = 0 \) is satisfied since again \( \rho \) is an irreducible representation. Now, however, \( x \) and \( y \) acquire simple poles at each half-period, where \( \varphi'(u) = 0 \). Since \( \varphi'(u)^2 = 4(\varphi(u)^3 - \varphi(u)) \) this occurs where \( \varphi(u) = 0, -1, +1 \). Using \( \varphi''(u) = 6\varphi(u)^2 - 2 \), and (37), we find the residue of \( x \) to be \( \frac{(5\varphi^2 - 3)}{10(3\varphi^2 - 1)} \) and that of \( y \) to be \( \frac{1}{10(3\varphi^2 - 1)} \). Thus, if \( \varphi(u) = 0 \), the residue \( R_3 \) of the Nahm matrix \( T_3 = x\rho_3 + yS_3 \) is
\[
\frac{i}{2} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & -3 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]
which identifies the representation at this pole as \( S^3V \), the irreducible one. On the other hand, if \( \varphi^2(u) = 1 \), the residue is
\[
\frac{i}{2} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]
which are the weights of \( V \oplus V \), which is reducible. It follows that only when \( \varphi(u) = 0 \) does the solution to Nahm’s equations have a pole with the correct residue.

Now if \( u \) is real, \( \varphi'(u)^2 \geq 0 \), so \( \varphi^3(u) \geq \varphi(u) \). Since \( \varphi(u) \to +\infty \) as \( u \) approaches 0 or \( 2\omega_1 \), when \( 2\omega_1 \) is the real period, the turning point \( u = \omega_1 \) is where \( \varphi(u) = 1 \). Similarly if \( u \) is imaginary, and the imaginary period is \( 2\omega_3 \), we must have \( \varphi(\omega_3) = -1 \). Thus the required pole is at \( u = \omega_1 + \omega_3 = \omega_2 \). From (37) this is possible if \( c < 0 \) since we can then take the argument of \( c^{1/4} \) to be \( \pi/4 \), and as \( s \) takes real values, \( u \) lies on the line from 0 to \( \omega_2 \). The boundary condition at \( s = 2 \) can then be satisfied if
\[
\omega_2 = 4(5c)^{1/4}
\]
which determines the constant \( c \). To put this in a more concrete form, note that the substitution
\[
w = \frac{z - i}{1 - iz}
\]
transforms the differential
\[ du = \frac{\psi' du}{\sqrt{4(\psi^3 - \psi)}} = \frac{dw}{2\sqrt{w^3 - w}} \]
into
\[ (1 + i) \frac{dz}{2\sqrt{z^4 - 1}} \]
and so
\[ \omega_2 = (1 + i) \int_0^1 \frac{dt}{\sqrt{1 - t^4}}. \]
From [27], we also have the formula
\[ \int_0^1 \frac{dt}{\sqrt{1 - t^4}} = \frac{1}{\sqrt{8\pi}} \Gamma\left(\frac{1}{4}\right)^2. \]
Using this, together with (38) and (35), we obtain

**Theorem 2** The curve \( \eta^4 + a(\zeta^8 + 14\zeta^4 + 1) \) is the spectral curve of a charge 4 monopole with octahedral symmetry if \( a = 3\Gamma\left(\frac{1}{4}\right)^8 / 64\pi^2 \).

### 11 Icosahedral symmetry

For the icosahedral group, \( k = 6 \), and the representation is defined by

\[
\rho_1 = \begin{pmatrix}
0 & 5 & 0 & 0 & 0 & 0 \\
-1 & 0 & 8 & 0 & 0 & 0 \\
0 & -1 & 0 & 9 & 0 & 0 \\
0 & 0 & -1 & 0 & 8 & 0 \\
0 & 0 & 0 & -1 & 0 & 5 \\
0 & 0 & 0 & 0 & -1 & 0 \\
\end{pmatrix},
\]

\[
\rho_2 = i \begin{pmatrix}
0 & 5 & 0 & 0 & 0 & 0 \\
1 & 0 & 8 & 0 & 0 & 0 \\
0 & 1 & 0 & 9 & 0 & 0 \\
0 & 0 & 1 & 0 & 8 & 0 \\
0 & 0 & 0 & 1 & 0 & 5 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix},
\]

\[
\rho_3 = i \begin{pmatrix}
5 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -3 & 0 \\
0 & 0 & 0 & 0 & 0 & -5 \\
\end{pmatrix}.
\]
Using Maple, we obtain the matrices $S_i$ by polarizing the polynomial $\zeta_0^5 + 11\zeta_0^6 - \zeta_0^7$.

$$S_1 = \begin{pmatrix}
0 & -240 & 0 & 0 & -40320 & 0 \\
48 & 0 & 960 & 0 & 0 & 40320 \\
0 & -120 & 0 & -1440 & 0 & 0 \\
0 & 0 & 160 & 0 & 960 & 0 \\
14 & 0 & 0 & -120 & 0 & -240 \\
0 & -14 & 0 & 0 & 48 & 0
\end{pmatrix},$$

$$S_2 = \begin{pmatrix}
0 & -240 & 0 & 0 & 40320 & 0 \\
-48 & 0 & 960 & 0 & 0 & -40320 \\
0 & 120 & 0 & -1440 & 0 & 0 \\
0 & 0 & -160 & 0 & 960 & 0 \\
14 & 0 & 0 & 120 & 0 & -240 \\
0 & -14 & 0 & 0 & -48 & 0
\end{pmatrix},$$

$$S_3 = \begin{pmatrix}
48 & 0 & 0 & 0 & 0 & 40320 \\
0 & -240 & 0 & 0 & 0 & 0 \\
0 & 0 & 480 & 0 & 0 & 0 \\
0 & 0 & 0 & -480 & 0 & 0 \\
0 & 0 & 0 & 0 & 240 & 0 \\
14/5 & 0 & 0 & 0 & 0 & -48
\end{pmatrix}. $$

From these expressions, we obtain the commutation relations

$$[S_1, S_2] = -230400\rho_3 + 480S_3 \quad \text{etc.}$$

and

$$[S_1, \rho_2] + [\rho_1, S_2] = -10S_3 \quad \text{etc.}.$$

From (27) and (28), and putting $z = 480y$, Nahm’s equations reduce to

$$\frac{dx}{ds} = 2x^2 - z^2 \quad (39)$$

$$\frac{dz}{ds} = -10xz + z^2 \quad (40)$$

and the polynomial (25) gives the constant of integration

$$c = 336z(z - 3x)^2(z + 4x)^3. \quad (41)$$

To solve the equations, put

$$w^3 = \frac{4}{i}(1 - \frac{3x}{z}).$$
then (41) becomes
\[ z^6w^6(1 - w^3)^3 = \frac{9c}{7^6} = \kappa^6 \]
and so
\[ z = \frac{\kappa}{w\sqrt{1 - w^3}}. \] (42)

Substituting in (40), we obtain
\[ \frac{d}{ds} \left( \frac{w\sqrt{1 - w^3}}{\kappa} \right) = \frac{7}{3} \left( 1 - \frac{5w^3}{2} \right) \]
but, expanding the derivative,
\[ \frac{d}{ds} \left( w\sqrt{1 - w^3} \right) = \frac{dw}{ds} \left( 1 - \frac{5w^3}{2} \right) \]
and hence
\[ \frac{dw}{\sqrt{1 - w^3}} = \frac{7\kappa}{3} ds. \]

Making the substitution \( w = \phi(u) \) where the Weierstrass elliptic function \( \phi(u) \) satisfies the equation \( \phi'(u)^2 = 4\phi^3(u) - 4 \) we integrate this by
\[ u = i\frac{7\kappa}{6}s + K. \] (43)

Using these elliptic functions, we can compute \( z \) from (42):
\[ z = -\frac{2\kappa i}{\phi(u)\phi'(u)} \]
and from the definition of \( w \),
\[ x = i\kappa \left( \frac{\phi'(u)}{6\phi(u)} + \frac{\phi(u)^2}{2\phi'(u)} \right). \]

At \( u = 0, \phi(u) = 1/u^2 + \ldots \), so \( z \) vanishes and \( x \) has a simple pole. In fact, putting \( K = 0 \) in (43),
\[ x = -\frac{7i\kappa}{12u} + \ldots = -\frac{1}{2s} + \ldots \]
and so \( T_i = -p_i/2s \ldots \) Since \( \rho \) is irreducible, this is the required behaviour at \( s = 0 \).

The other poles occur at \( u = u_0 \), where \( u_0 \) is one of the two values where \( \phi(u_0) = 0 \),
and at the half-periods \( u = \omega_i \). Near \( u = u_0 \), we have
\[ \phi(u) = \phi'(u_0)(u - u_0) + \ldots \]
and since \( \varphi'(u)^2 = 4\varphi(u)^3 - 4 \), we obtain

\[
x = \frac{i\kappa}{6(u-u_0)} + \ldots = \frac{1}{7(s-s_0)} + \ldots
\]

and

\[
z = \frac{i\kappa}{2(u-u_0)} + \ldots = \frac{3}{7(s-s_0)} + \ldots
\]

using (13). Using this, we calculate the residues \( R_i \) of the matrices \( T_i = x\rho_i + yS_i \) and find the eigenvalues of \( iR_3 \) to be \(-1, -1, 0, 0, 1, 1\), thus identifying the representation as \( S^2V \oplus S^2V \). At a half-period \( \omega_i \),

\[
\varphi(u) = \varphi(\omega_i) + \varphi''(\omega_i)(u - \omega_i)^2/2 + \ldots
\]

but since \( \varphi'(u)^2 = 4\varphi(u)^3 - 4 \), \( \varphi''(\omega_i) = 6\varphi(\omega_i)^2 \) and this provides \( x(s) \) with a simple pole of residue \( 1/14 \) and \( z \) a simple pole of residue \( -2/7 \) at the corresponding value of \( s \). The eigenvalues of \( iR_3 \) are then \(-1/2, -1/2, -1/2, 1/2, 1/2, 1/2, 1/2\), giving the representation \( V \oplus V \oplus V \).

It follows that only the poles at the period points give irreducible representations. But any line joining two periods passes through a half-period. We conclude that there is no solution to Nahm’s equations of this form which is smooth in the interval \((0, 2)\), has poles at the end-points, and whose residues give an irreducible representation. Consequently

**Theorem 3** There does not exist a monopole of charge 6 with icosahedral symmetry.

## 12 Cyclically symmetric scattering of monopoles

In this Section, we shall investigate the rational maps of monopoles with cyclic symmetry, and shall discover some novel types of geodesic monopole scattering. Recall that there is a \( 1-1 \) correspondence between the maps and monopoles. Also, cyclic or axial symmetry about the \( x_3 \)-axis, if present, is manifest.

The rational map of a charge \( k \) monopole takes the form

\[
R(z) = \frac{p(z)}{q(z)},
\]

with \( q \) monic of degree \( k \) and \( p \) of degree less than \( k \). Let \( \omega = e^{2\pi i/k} \). Consider the cyclic group of rotations about the \( x_3 \)-axis, \( C_k \), generated by the transformation \( z \to \omega z \). The monopole with rational map \( R(z) \) is \( C_k \) symmetric if \( R(\omega z) \) differs
from $R(z)$ only by a constant phase. We get a class of charge $k$ monopoles with $C_k$ symmetry for each irreducible character of $C_k$. Let us denote the $l$th such class of monopoles by $M^l_k (0 \leq l < k)$. These are the monopoles whose rational maps are of the form

$$R(z) = \frac{\mu z^l}{z^k - \nu}$$

where $\mu$ and $\nu$ are complex parameters. For these monopoles, $R(\omega z) = \omega^l R(z)$. $M^l_k$ is a 4-dimensional totally geodesic submanifold of the moduli space $M_k$, since it arises by imposing a symmetry on the monopoles. Its metric is also Kähler, because the set of rational maps $M^l_k$ is a complex submanifold of the set of all rational maps.

Since the strongly centred monopoles are geodesic in the moduli space, we shall now restrict attention to rational maps of strongly centred, $C_k$-symmetric monopoles. There is no essential loss of generality in doing this. For a monopole with a rational map of type $M^l_k$, the criterion for it to be strongly centred (12), reduces to

$$\mu^k \prod_{i=1}^{k} (\beta_i)^l = 1$$

(44)

where $\{\beta_i : i = 1, \ldots, k\}$ are the $k$ roots of $z^k - \nu = 0$. Eq. (44) is equivalent to $\mu^k \nu^l = \pm 1$, with the lower sign if both $k$ is even and $l$ odd, and the upper sign otherwise. The magnitude of $\mu$ is $|\mu| = |\nu|^{-l/k}$, and there are $k$ choices for the phase of $\mu$. The rational maps we obtain are parametrised by several surfaces of revolution. For given $k$ and $l$ there may be one or more surfaces. For $l = 0$, for example, there are $k$ distinct surfaces, each with $\nu$ a good coordinate; $\mu$ is a distinct, and constant, $k$th root of unity on each surface. If $l \neq 0$, and $k$ and $l$ have highest common factor $h$, there are $h$ distinct surfaces. As $\arg \nu$ increases by $2\pi$, $\arg \mu$ decreases by $2\pi l/k$, so $\arg \nu$ must increase by $2\pi k/h$ for $\mu$ to return to its initial value. $\nu$ is therefore a good coordinate on each surface, but the range of $\arg \nu$ is $2\pi k/h$.

For given $k$, and each $l$ in the range $0 \leq l < k$, let us choose one of the surfaces just described, say, the one containing the rational map in $M^l_k$ with $\nu = 1$ and $\mu = e^{\pi i/k}$ (if $k$ is even and $l$ odd) or $\mu = 1$ (otherwise). Denote this surface by $\Sigma^l_k$. If there is another surface, for a particular value of $l$, then it is isomorphic to $\Sigma^l_k$, as $\mu$ differs on it simply by a constant phase. Let us now consider the geodesics on $\Sigma^l_k$, and the associated $C_k$-symmetric monopole scattering. The simplest geodesic is when $\nu$ moves along the real axis – the monopole then has no angular momentum.

On $\Sigma^0_k$ the rational maps are of the form

$$R(z) = \frac{1}{z^k - \nu},$$

where $\nu$ is an arbitrary complex number. $\Sigma^0_k$ is therefore a submanifold of the space of inversion symmetric monopoles $IM^0_k$. For $\nu = 0$, the rational map is that of a
strongly centred axisymmetric charge \( k \) monopole. If \( |\nu| \) is large, there are \( k \) well-separated unit charge monopoles at the vertices of an \( k \)-gon in \( \mathbb{R}^3 \), with \( x_1 + ix_2 \) a \( k \)th root of \( \nu \), and \( x_3 = 0 \). The geodesic where \( \nu \) moves along the entire real axis corresponds to a simultaneous scattering of \( k \) unit charge monopoles in the \( (x_1, x_2) \) plane, where the incoming and outgoing trajectories are related by a \( \pi/k \) rotation. The configuration is instantaneously axially symmetric when \( \nu = 0 \). This kind of symmetric planar scattering of \( k \) solitons has been observed in a number of models, and can be understood in a rather general way [17, 8].

On \( \Sigma^l_k \), with \( l \neq 0 \), \( \nu \) is a non-zero complex number. \( \nu = 0 \) is forbidden, as the numerator and denominator of \( R(z) \) would have a common factor \( z^l \). A simple geodesic is with \( \nu \) moving along the positive real axis, say towards \( \nu = 0 \). The rational map is

\[
R(z) = \frac{l}{\nu^{l/k}} \frac{z^l}{z^k - \nu}
\]  

(45)

where \( \nu = e^{\pi i/k} \) (if \( k \) is even and \( l \) is odd) or \( \nu = 1 \) (otherwise). Then the initial motion is again \( k \) unit charge monopoles at the vertices of a contracting \( k \)-gon in the \( (x_1, x_2) \) plane. As \( \nu \to 0 \), the map approaches

\[
R(z) = \frac{l}{\nu^{l/k}} \frac{1}{z^k - \nu}
\]

which is the map of a charge \( (k - l) \) axisymmetric monopole, centred at the point \( (0, 0, (-l/2k) \log \nu) \). This is a positive distance along the \( x_3 \)-axis as \( \nu \) is small. Following an argument in [1], pp.25-6, we deduce that the charge \( k \) monopole has split up, with one cluster the charge \( k - l \) monopole just described, and a further cluster (or clusters) near the \( x_3 \)-axis, but not so far up. In fact, there is just one other cluster, which is an axisymmetric charge \( l \) monopole at a negative distance along the \( x_3 \)-axis. This is seen by inverting the original monopole in the \( (x_1, x_2) \) plane. The procedure described in Section 4 shows that the rational map (45) transforms under inversion to

\[
R(z) = \frac{l}{\nu^{(k-l)/k}} \frac{z^{k-l}}{z^k - \nu}
\]

where \( l \tilde{l} = 1 \), because

\[
l \tilde{l} (z^l / \nu^{l/k}) (z^{k-l} / \nu^{(k-l)/k}) = z^k / \nu = 1 \text{ mod } z^k - \nu.
\]

The inverted monopole therefore has an axisymmetric charge \( l \) monopole cluster at \( (0, 0, -((k-l)/2k) \log \nu) \), as \( \nu \to 0 \), while the original monopole has this axisymmetric charge \( l \) cluster at \( (0, 0, ((k-l)/2k) \log \nu) \).
In the geodesic motion, $k$ unit charge monopoles come in, but the outgoing configura- tion is of two approximately axisymmetric monopole clusters, of charges $k - l$ and $l$, at distances $ld$ and $-(k - l)d$ along the $x_3$-axis, with $d$ increasing uniformly. This geodesic motion can, of course, also be reversed. The centre of mass of these clusters remains at the origin.

If $k$ is even and $l = k/2$ then the rational maps, and the geodesic monopole motion we have described, have an additional inversion symmetry. $R(z) = z^{k/2}/(\nu^{1/2}(z^k - \nu))$ lies in the space of inversion symmetric maps $IM_k^{k/2}$, and the factor $\iota$ makes no essential difference. Consequently, the outgoing clusters have the same charges and equal speeds. Since $\nu$ was assumed to be real, there is reflection symmetry under $x_2 \rightarrow -x_2$. Together with the inversion symmetry, $x_3 \rightarrow -x_3$, we obtain an additional rotational symmetry, by $\pi$ about the $x_1$-axis. Hence, monopoles with rational maps of the form (43) have $D_k$ symmetry if $k$ is even and $l = k/2$. There is also $D_k$ symmetry if $l = 0$, for any $k$.

The surfaces $\Sigma^0_k$ and $\Sigma^l_k$ are the “rounded cone” and “trumpet” described in [1]. These surfaces are not isomorphic, but the geodesics with $\nu$ real (on $\Sigma^0_k$) and $\nu$ real and positive (on $\Sigma^l_k$) are isomorphic. Along the first, two unit charge monopoles scatter through $\pi/2$ in the $(x_1, x_2)$ plane, and along the second they scatter through $\pi/2$ in the $(x_1, x_3)$ plane. There are no analogous isomorphisms in the higher charge cases.

The general geodesics on the surfaces $\Sigma^0_k$ and $\Sigma^l_k$ ($l \neq 0$) are presumably analogous to those on the cone $\Sigma^0_2$ or trumpet $\Sigma^l_2$. On $\Sigma^0_k$, they correspond to $k$ unit charge monopoles scattering in the $(x_1, x_2)$ plane with net orbital angular momentum. On $\Sigma^l_k$ ($l \neq 0$), $k$ unit charge monopoles again come in with net orbital angular momentum. If this is small, the geodesic passes through the trumpet-like surface and two monopole clusters with magnetic charges $l$ and $k - l$ emerge back-to-back on the $x_3$-axis. They also have opposite electric charges, which accounts, physically, for angular momentum conservation. If the initial angular momentum is large, then the geodesic does not pass through the trumpet, but is reflected, and there are $k$ outgoing unit charge monopoles in the $(x_1, x_2)$ plane.

What can we learn about the spectral curves of centred $C_k$-symmetric monopoles from this discussion of rational maps? First, recall that monopoles whose rational maps differ only by a phase have the same spectral curves. We need therefore only consider the chosen surfaces of rational maps, $\Sigma^l_k$, and their associated monopoles. Let us also restrict attention to monopoles which are oriented to be reflection symmetric under $x_2 \rightarrow -x_2$, which requires $\nu$ to be real, and choose a fixed phase for $\mu$ as $\nu$ varies in magnitude. This restricts us to $2k - 1$ disjoint curves in the surfaces $\Sigma^l_k$, ($\nu$ real in $\Sigma^0_k$, $\nu$ positive and $\nu$ negative in $\Sigma^l_k$ ($l \neq 0$)), and these curves are geodesics. It follows that among the centred curves in $TP_1$ of the form (5) with $C_k$ symmetry and
oriented, there are $2k-1$ disjoint loci of spectral curves. (We refer to a connected, one-dimensional submanifold of spectral curves as a locus in the space of curves in $TP_1$.) All these spectral curves will have real coefficients because of the reflection symmetry. We have been unable to determine, in general, for which parameter values a curve is a spectral curve, but we can make some qualitative assertions, based on knowledge of the asymptotic monopole configurations, the axisymmetric configurations, and the monopoles with the symmetries of the regular solids. We restrict our remarks to the cases $k=3$ and $k=4$.

For $k=3$, and $l=0, 1$ or 2, there are five loci of spectral curves of the form (17), with $\beta$ real. When $l=0$ there is $D_3$-symmetry, so $\gamma=0$. The locus is asymptotic at both ends to $\alpha^3 = 27\beta^2$, with $\beta$ large and positive at one end, and $\beta$ large and negative at the other. The axisymmetric monopole, half-way along the locus, has $\beta=0$ and $\alpha=\pi^2$. Presumably, $\alpha$ is positive along the whole locus. The four remaining loci, for $l=1$ and $l=2$, are isomorphic. This is because $\nu \to -\nu$ corresponds to a reflection $x_1 \to -x_1$, and because the $l=2$ monopoles are obtained from the $l=1$ monopoles by inversion ($x_3 \to -x_3$). Under the first symmetry $\beta \to -\beta$, and under the second $\gamma \to -\gamma$. Each of the four loci is asymptotic at one end to $\alpha^3 = 27\beta^2, \gamma = 0$, with $\beta$ either positive or negative, and at the other to $\alpha = \frac{1}{4}\pi^2 - 3b^2, \beta = 0, \gamma = 2b(b^2 + \frac{1}{4}\pi^2)$, with $b$ either positive or negative. These latter parameters result from taking the product of the spectral curve of a unit charge monopole at $(0,0,b)$ with the spectral curve of an axisymmetric charge 2 monopole at $(0,0,-b/2)$, that is

$$P(\eta, \zeta) = (\eta + 2b\zeta)(\eta^2 - 2b\eta\zeta + (b^2 + \frac{1}{4}\pi^2)\zeta^2)$$

$$= \eta^3 + \left(\frac{1}{4}\pi^2 - 3b^2\right)\eta\zeta^2 + 2b(b^2 + \frac{1}{4}\pi^2)\zeta^3 = 0.$$  

Since the tetrahedrally symmetric charge 3 monopole has $C_3$ symmetry about various axes, there must be a point on each of these four loci corresponding to such a monopole (in four distinct orientations). The calculations of Section 9 show that the four loci pass through the four points (one on each locus) $(\alpha, \beta, \gamma) = (0, \pm a, \pm 5\sqrt{2}a)$, where $a = \Gamma(\frac{1}{6})^3\Gamma(\frac{1}{3})^3 / 48\sqrt{3}\pi^{3/2}$. The picture of the monopole scattering, corresponding to geodesic motion along one of these loci, is as follows. Three unit charge monopoles come in at the vertices of a (horizontal) equilateral triangle, moving towards its centre. They then coalesce instantaneously into a tetrahedron, with the base triangle oriented the same way as the initial triangle, but somewhat below the initial plane. Finally the tetrahedron breaks up with the top vertex moving up and becoming a unit charge monopole, and the base triangle descending and becoming a toroidal charge two monopole.

In the case $k=4$, we have seven loci of spectral curves with $C_4$ symmetry. Only three of these are essentially different. The four corresponding to the rational maps with $l=1$ and $l=3$, and $\nu$ positive or negative, are isomorphic. The $l=1$ and $l=3$ maps, and hence the corresponding monopoles and spectral curves, are related
by inversion, and the sign of \( \nu \) can be reversed by a \( \pi/4 \) rotation. The spectral curves along these four loci have no higher symmetry than \( C_4 \) symmetry.

There are two isomorphic loci of spectral curves corresponding to the \( l = 2 \) maps. Here there is inversion symmetry, and the spectral curves are therefore \( D_4 \) symmetric and of the form \( \text{[!]}. \) Reversing the sign of \( \nu \) again corresponds to a \( \pi/4 \) rotation, and \( \beta \) changes sign. Along these two loci, we will find the spectral curves corresponding to the octahedrally symmetric 4-monopole (in two orientations). The locus with \( \nu \) negative interpolates between the asymptotic parameter values \( \alpha = 4a^2, \beta = a^4, \gamma = 2a^4 \) with \( a \) large (corresponding to four stars at \((1/\sqrt{2})(\pm a, \pm a, 0)\)) and the asymptotic values \( \alpha = \frac{1}{2} \pi^2 - 8b^2, \beta = 0, \gamma = (4b^2 + \frac{1}{4} \pi^2)^2 \) with \( b \) large (corresponding to two axisymmetric charge 2 monopole clusters on the \( x_3 \)-axis). Along the locus, \( \alpha \) changes sign, and when \( \alpha = 0 \) the locus passes through the spectral curve of the monopole with octahedral symmetry, so \( \gamma = 14\beta - 21\Gamma(\frac{1}{4})^8/32\pi^2 \).

Finally, there is a single locus corresponding to the \( l = 0 \) maps. This interpolates between the asymptotic parameter values \( \alpha = 4a^2, \beta = -a^4, \gamma = 2a^4 \) and \( \alpha = 4a^2, \beta = a^4, \gamma = 2a^4 \), with \( a \) large, and passes through the values \( \alpha = 5\pi^2/2, \beta = 0, \gamma = 9\pi^4/16 \), corresponding to the axisymmetric charge 4 monopole. Presumably, \( \alpha \) and \( \gamma \) are positive along the entire locus.

In summary, our main result is that in the geodesic scattering of monopoles of charge \( k \), with \( C_k \) symmetry and angular momentum zero, there are two kinds of motion. First, there is the well-known possibility of \( k \) monopoles, of unit charge, scattering in the \((x_1, x_2)\) plane through an angle \( \pi/k \). Second, there is the novel possibility of \( k \) unit charge monopoles coming in as before, but emerging as charge \( l \) and charge \( k-l \) axisymmetric monopoles moving back-to-back along the \( x_3 \)-axis. \( l \) can have any integer value in the range \( 0 < l < k \). In the special case of 3-monopole scattering, with \( l = 1 \) or \( l = 2 \), the field configuration passes through the tetrahedrally symmetric 3-monopole. In 4-monopole scattering, with \( l = 2 \), the configuration passes through the octahedrally symmetric 4-monopole.

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