Module Intersection and Uniform Formula for Iterative Reduction of One-loop Integrals

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Abstract: In this paper, we develop an iterative sector-level reduction strategy for Feynman integrals, which bases on module intersection in the Baikov representation and auxiliary vector for tensor structure. Using this strategy we have studied the reduction of general one-loop integrals, i.e., integrals having arbitrary tensor structures and arbitrary power for propagators. Inspired by these studies, a uniform and compact formula that iteratively reduces all one-loop integrals has been written down, where messy polynomials in integration-by-parts (IBP) relations have organized themselves to Gram determinants.
1 Introduction

Particle physics is highly developed and comes to the era of precise measurement, which calls for high precision theoretical predictions. One of the most important parts of theoretical predictions is perturbation computations in quantum field theory. For such computations, reducing general Feynman integrals into master integrals plays an important role [1]. It not only greatly reduces the number of integrals to be calculated, but also produces the differential equations [2–5] for solving the master integrals. Based on differential equations, many analytical or numerical methods and packages for solving master integrals have been developed [6–17], which can be widely applied to multi-loop and multi-scale processes.

The reduction problem can be roughly divided into the reduction of loop momenta in the numerator (called ”tensor reduction” or TR) and the reduction of propagators in the denominator with general powers (called ”denominator reduction” or DR). These two kinds of reductions (i.e., TR and DR) are tightly connected\(^1\). The well-established method to do reduction is the integration-by-parts (IBP) method introduced in [1]. When combining with the Laporta algorithm [19], several widely used packages have been developed such as [20–23]. IBP relations give many linear relations between various scalar integrals and solving them gives relations between a given integral and the master integrals (i.e., the reduction). When it comes to multi-loop and multi-scale integrals required by higher precision theoretical predictions, the size of linear equations generated by IBP relations

\(^1\)For example, in [18] one has used the tensor reduction to solve reduction with general denominators for one-loop integrals.
increases dramatically. Solving them becomes more and more difficult. Motivated by this, many developments of reduction methods have been explored for higher loops \cite{29-50}.

Most of the above methods solve a hugely redundant linear system with many equations using programs. On one side, it makes the computation automatically. On the other side, it works like in a black box. Although we have obtained the wanted results, we lost a full understanding of these equations. Knowing these details not only helps overcome difficulties faced by these methods but also gives a better understanding of the mathematical structures of general Feynman integrals.

For example, there are some iterative structures in reduction. Let us consider the bubble topology with propagators:

\[
z_1 = l^2 - m_1^2, \quad z_2 = (l - p)^2 - m_2^2.
\]  

(1.1)

There are three master integrals defined by

\[
f \rightarrow \{ I_{0,0}, I_{0,1}, I_{1,1} \}, \quad I_{a_1, a_2} = \int \frac{d^dl}{i(2\pi)^d/2} \frac{1}{z_1^{a_1} z_2^{a_2}}.
\]  

(1.2)

One kind of differential equations of master integrals is given by

\[
\partial_{m_1} \tilde{f} = A_{m_1} \tilde{f}.
\]  

(1.3)

It is easy to see when taking \( \partial_{m_1} \) on \( \tilde{f} \) iteratively we have

\[
\partial_{m_1}^n \tilde{f} = \{(n - 1)! I_{n+1,0}, 0, (n - 1)! I_{n+1,1} \} = A_{m_1}^{(n)} \tilde{f}
\]  

(1.4)

where

\[
A_{m_1}^{(n)} = \partial_{m_1} A_{m_1}^{(n-1)} + A_{m_1}^{(n-1)} A_{m_1}.
\]  

(1.5)

The iterative structure of \( A_{m_1}^{(n)} \) in (1.5) gives a very simple way to find the reduction of \( I_{n+1,0} \) and \( I_{n+1,1} \) (DR type) using (1.4). This is an iterative reduction relation at topology-level. While the iterative relation at the topology-level is powerful, it is also not easy to get. As in this example, the \( A_{m_1} \) is usually calculated by the traditional IBP method, which is difficult to get for more complicated cases.

Another well-known phenomenon of reduction is that the reduction can be organized into different sectors. Usually, the reduction in the top sector can be done easily using some tricks, for example, the maximum cut for the one-loop box in \cite{26}. However, such a trick lost the information of sub-sectors. Thus if there is a way such that the reduction to its top sector is not much harder than the maximal cut, but the complete information of sub-sectors has been kept, we can carry out the whole reduction top-down (or triangulated)

\[2\] For one-loop integrals, two very efficient methods are the unitarity cut method \cite{24-27} for reduction at the integral level and OPP method \cite{28} for reduction at the integrand level.

\[3\] When we say a reduction is at the topology-level, we mean that the integral is written as the combination of master integrals of the same topology and master integrals of the sub-topologies.
by treating each sub-sector as another "top sector". This motivates us to consider the iterative structure at the sector-level but not the topology-level.

In this paper, we will explore more information hidden inside the IBP relations. The method we will introduce mainly bases on syzygy and module intersection [29, 40–49, 51] in Baikov representation [52]. We will give a quick review in Sec.2 for Baikov representation and module intersection. The main point of our method comparing to the module intersection in [47] is that we will not generate all IBP relations with basis obtained by the module intersection and solve these redundant linear equations. On the contrary, we pick some elements in the module intersection. These elements generate iterative reduction relations for corresponding sector. Using just a few relations, one can iteratively reduce any integral in this sector to master integrals of this sector, and keep the complete information of sub-sectors. In such a strategy of iteration at the sector-level, iterative reduction relations are easier to obtain and they avoid the huge redundancy in linear system of traditional IBP methods. To demonstrate our method, we will present one TR example and one DR example in Sec.4 and discuss the similarity of these two examples. Inspired by examples in the previous section, we construct an uniform formula in Sec.5. It solves both TR and DR for general one-loop integrals. In Sec.6, we give a brief discussion for degenerated situations, i.e., kinematics and masses take some specific values, such as null momenta or on-shell momenta. We will find that naively applying the uniform formula laid out in Sec.5 sometimes does not lead to the simplest iterative relation and more careful treatment is needed. However the method introduced in Sec.3 still works well by giving the simplest iterative relation. Finally, we give a brief discussion in the Sec.7.

2 Baikov representation and module intersection

In this section, we will review the Baikov representation of integrals [52]. In this frame, it is easier to implement the module intersection as will be discussed shortly. The Baikov representation transforms integrals in the standard form obtained from Feynman rules by changing integral variables from $\prod_i d^d l_i$ to $\prod_j d z_j$, where each $z_i$ represents a propagator (or related Lorentz invariant scalar product involving loop momenta). For one-loop integrals, which are the focus of our current paper, we denote propagators and integrals as

$$
I_{\{a_i\} \equiv I_{a_1, a_2, \cdots, a_n} \equiv \int \frac{d^d l}{i(\pi)^d/2} \prod_{i=1}^n \frac{1}{z_i^{a_i}},
$$

(2.1)

where $E$ is the number of independent external momenta and $n = E + 1$. The Baikov representation of integrals is

$$
I_{a_1, a_2, \cdots, a_{E+1}} = \int C_n(d) K^{-(d-n)/2} G(z)^{(d-n-1)/2} \prod_{i=1}^n \frac{dz_i}{z_i^{a_i}}
$$

(2.2)

4When we say the reduction is at the sector-level, we mean that the integral is written as the combination of master integrals of the same topology and integrals (not need to be master integrals) of the sub-topologies.
where the constant-coefficient \(C_n(d)\) and the Gram determinant \(K\) of external momenta do not involve \(z_i\), so they do not affect our later discussions and can be ignored. The \(\mathcal{G}(z)\) is another Gram determinant depending on both loop momentum and external momenta, i.e.,

\[
\mathcal{G}(z) = G(l, p_1, \cdots, p_E) \tag{2.3}
\]

with \(G\) defined as

\[
G(q_1, \ldots, q_n) \equiv \det(q_i \cdot q_j) \equiv \det \begin{pmatrix}
q_1 \cdot q_1 & q_1 \cdot q_2 & \cdots & q_1 \cdot q_n \\
q_2 \cdot q_1 & q_2 \cdot q_2 & \cdots & q_2 \cdot q_n \\
\vdots & \vdots & \ddots & \vdots \\
n_1 \cdot q_1 & \cdots & \cdots & q_n \cdot q_n
\end{pmatrix} \tag{2.4}
\]

The well-established IBP relations can also be easily implemented in the Baikov representation. However, the differentiation on \(G\) will change its power, which is equivalent to shifting the space-time dimension \(d\) to different values.

To avoid such a situation, the syzygy module is introduced [40]. Let us consider the IBP relation

\[
C \int \sum_{i=1}^{n} \left[ \partial_{z_i} \left( P_i \prod_{i=1}^{n} \frac{1}{z_i^{w_i}} G(z)^{(d-n-1)/2} \right) \right] \prod_{i=1}^{n} dz_i \tag{2.5}
\]

with \(P_i\)s being polynomials of \(z_i\), one can see that if these \(P_i\)’s are properly chosen, i.e., they satisfy

\[
\sum_{i=1}^{n} (P_i \partial_{z_i} G) + P_0 G = 0, \tag{2.6}
\]

the power of \(G\) will not be shifted. The relation (2.6) is a syzygy equation for the set of \((n + 1)\) polynomials

\[
\langle \partial_{z_1} G, \cdots, \partial_{z_n} G, G \rangle \tag{2.7}
\]

All solutions of (2.6) give the syzygy module of the set (2.7). Putting every solution back to (2.5) we get an IBP relation with a given \(\{a_i\}\) set, which does not involve dimension shift.

For later convenience we define the following notations:

\[
\langle P \rangle = \langle P_1, P_2, \cdots, P_n, P_0 \rangle
\]

\[
D_{\langle P \rangle} \equiv \{ D_{P_1}, \cdots, D_{P_n}, D_{P_0} \} \equiv \left\{ \partial_{z_1} (P_1 \cdot), \cdots, \partial_{z_n} (P_n \cdot), \frac{d-n-1}{2} P_0 \cdot \right\}
\]

\[
D_{\langle P \rangle} \cdot Q \equiv -\sum_{i=1}^{n} [\partial_{z_i} (P_i \cdot Q)] + \frac{d-n-1}{2} P_0 \cdot Q. \tag{2.8}
\]
Then the IBP relation (2.5) can be written in a more compact form

\[
C \int \left\{ D(P) \cdot \frac{1}{\prod_{i=1}^{n} z_i^{a_i}} \right\} G(z)^{(d-n-1)/2} \prod_{i=1}^{n} dz_i. \tag{2.9}
\]

The syzygy module is a linear space with a basis of generators\(^5\)

\[\{e_1, \cdots, e_n\}\] \tag{2.10}

and the general solution of (2.6) can be written as \(\langle P \rangle = \sum_{i=1}^{n} f_i e_i\) with \(f_i\)s being arbitrary polynomials of \(z_i\).

Another well-known phenomenon in IBP relation is the changing of power of propagators in (2.9). The power can be increased or decreased. Among them, only \(\partial_z z_i^{-a_i}\) increases the power. To avoid the increase, we can do a similar thing by requiring \(\langle P \rangle\) in (2.8) to be the module generated by the following basis

\[d_1 = \{z_1, 0, \cdots, 0, 0\} \]
\[d_2 = \{0, z_2, \cdots, 0, 0\} \]
\[\cdots \]
\[d_n = \{0, 0, \cdots, z_n, 0\} \]
\[d_{n+1} = \{0, 0, \cdots, 0, 1\}. \tag{2.11}\]

Up to now, we have two modules: one is given by (2.6) and avoids the dimension shift of space-time, while another is given by (2.11) and avoids the increase of power of propagators. If we want to avoid both things, we just take the intersection of the above two modules, i.e., \(\{h_i\} \equiv \{e_i\} \cap \{d_i\}\). Notice that the syzygy module (2.6) and the module intersection \(h_i\) can be solved by computational algebraic geometry [42], and in this work, we use the package Singular [53] to do this. In the examples given in this paper, it takes only seconds or even less to finish the computation. The syzygy of Gram determinant can also easily be obtained by Laplace expansion of the determinant [45].

### 3 The method

In this paper, we will re-investigate the reduction problem for general one-loop integrals, i.e., with arbitrary tensor structure and arbitrary power of propagators. As one can see, these two different reductions, i.e., the tensor reduction (TR) and denominator reduction (DR) can be treated uniformly by module intersection method [40, 48].

As pointed out in several papers [18, 50, 54–57], arbitrary tensor structure can be compactly organized using an auxiliary vector \(R\). Thus for TR of one-loop \(n\)-point integrals, we enlarge the set of propagators given in (2.1) by adding one new propagator, i.e.,

\[z_1 = l^2 - m_1^2, \quad z_2 = (l + p_1)^2 - m_2^2, \quad z_3 = (l + p_1 + p_2)^2 - m_3^2, \cdots \]

---

\(^5\)For one-loop integrals, it has been proved that the number of generators is exactly the number of propagators.
\[ z_n = (l + p_1 + \cdots + p_{n-1})^2 - m_{n_1}^2, \quad z_{n+1} = l \cdot R. \] (3.1)

with the power of \( z_{n+1} \) to be non-positive integer. Now the Baikov representation becomes\(^6\)

\[
I_{a_1, a_2, \ldots, a_n, a_{n+1}} = C \int \mathcal{G}(z)^{(d-n-2)/2} \frac{dz}{\prod_{i=1}^{n+1} z_{a_i}}
\]

\[
\mathcal{G}(z) = G(l, p_1, \cdots, p_{n-1}, R)
\] (3.2)

Let us denote the basis of syzygy module corresponding to

\[
\langle \partial_{z_1} \mathcal{G}, \ldots, \partial_{z_{n+1}} \mathcal{G}, \mathcal{G} \rangle
\] (3.3)

as \( \{ e_i \} \), while the basis of another module is \( \{ d_i \} \) with

\[
d_1 = \{ z_1, 0, \cdots, 0, 0, 0 \}
\]

\[
\ldots
\]

\[
d_n = \{ 0, 0, \cdots, z_n, 0, 0 \}
\]

\[
d_{n+1} = \{ 0, 0, \cdots, 0, 1, 0 \}
\]

\[
d_{n+2} = \{ 0, 0, \cdots, 0, 0, 1 \}
\] (3.4)

After obtained the module intersection \( \{ h_i \} \equiv \{ e_i \} \cap \{ d_i \} \), we use elements in \( \{ h_i \} \) to generate differential operators as in (2.8) and produce corresponding IBP relations like this\(^7\):

\[
I_{a, -r_{\text{max}}} = \sum_{j=1}^{m} c_j I_{a, -r_{\text{max}}+j} + \text{l.p.p.t.},
\] (3.5)

with \( r_{\text{max}} > 0 \) and \( a_i > 0 \) for \( i < n \), where the l.p.p.t. denote terms with lower power of propagators. More explicitly, we say \( I_{b, -r_b} \) is a l.p.p.t. corresponding to \( I_{a, -r_a} \), if it satisfies \( b_i \leq a_i \) for all \( i \leq n \), and \( \sum_i^n b_i < \sum_i^n a_i \). Notice that when propagators’ power is

\[
\{ a, -r \} = \{ 1, \cdots, 1, -r \},
\] (3.6)

the l.p.p.t. are all terms of sub-sectors.

To have a nice tensor reduction relation, there are some requirements for the (3.5). Firstly, the sign of power \( a_{n+1} \) of \( z_{n+1} \) indicates it is a numerator or a denominator. Since we want to discuss the tensor reduction, \( a_{n+1} \) should be a non-positive integer and relation (3.5) should not include any term with \( a_{n+1} \) positive. Thus for any \( r_{\text{max}} > 0 \), we should require

\[
c_j = 0 \quad \text{when} \quad j > r_{\text{max}},
\] (3.7)

Secondly, no \( c_j \) becomes infinity for any \( r_{\text{max}} > 0 \) for (3.5) to be well defined.

\(^6\)By (2.2), the \( C \) of (3.2) will depend on \( R \) also, but it will not influence IBP relations derived later.

\(^7\)It is possible that the IBP relation cannot be written to the form (3.5). For such a situation, we just throw away this IBP relation.
For the reduction of propagators with arbitrary powers (i.e., the DR), the idea is similar. Without loss of generality, let us consider how to reduce the general power $a_n$ of the $n$-th propagator to one. With propagators given by
\[
\begin{align*}
z_1 = l^2 - m_1^2, \\
z_2 = (l + p_1)^2 - m_2^2, \\
z_3 = (l + p_1 + p_2)^2 - m_3^2, \\
\vdots \\
z_n = (l + p_1 + \cdots + p_{n-1})^2 - m_n^2.
\end{align*}
\] (3.8)
the Baikov representation is given by
\[
I_{a_1, a_2, \ldots, a_n} = C \int G(z)^{(d-n-1)/2} \frac{dz}{\prod_{i=1}^{n} z_i^{a_i}}
\]
\[
G(z) = G(l, p_1, \cdots, p_{n-1}).
\] (3.9)
Noting that for $G$, when writing using momentum variables, it is the same as the one given in (3.2). However, when writing using the $z$ variables, the linear form in (3.1) and the quadratic form in (3.8) do make some differences. Again first we find the syzygy module for relation (2.6) to avoid the shift of the space-time dimension. However, the second module will be a little different from the one given in (2.11). More explicitly, generators become
\[
\begin{align*}
d_1 &= \{z_1, 0, \cdots, 0, 0, 0\} \\
\vdots \\
d_{n-1} &= \{0, 0, \cdots, z_{n-1}, 0, 0\} \\
d_n &= \{0, 0, \cdots, 0, 1, 0\} \\
d_{n+1} &= \{0, 0, \cdots, 0, 0, 1\},
\end{align*}
\] (3.10)
where the $d_n$ is different. The reason is that now we do not ask to avoid the increase of the power of the $n$-th propagator. Finding the module intersection $\{h_i\} \equiv \{e_i\} \cap \{d_i\}$ we will get IBP relations of the form
\[
I_{a_n, n, \text{max}} = \sum_{j=1}^{m} c_j I_{a_n, n, \text{max} - j} + l.p.p.t.,
\] (3.11)
where no $c_j$ becomes infinity for any $a_{n, \text{max}} > 1$. This equation is similar to (3.5), but with the following difference: in (3.5) it is the smallest power $-r_{\text{max}}$ at the left-hand side while in (3.11) it is the maximum power $a_{n, \text{max}}$ at the left-hand side. Another difference is that in (3.11) we do not need to require $c_j = 0$ when $j > a_{n, \text{max}}$ since now it becomes the tensor of the sub-sector.

Before ending this section, let us emphasize that the reason we are able to treat the TR and DR uniformly using module intersection is following two key points. First, we have introduced the auxiliary vector $R$ to represent all tensor structures\(^8\). Secondly, we are not write down all relations coming from module intersection, but select minimum ones, which have nice property, i.e., giving iteration at the sector-level. The meaning of this point will be clear by examples in the Sec.4.

\(^8\)We want to emphasize that introducing $R$ is different from introducing irreducible scalar products in usual IBP method. For later, if there are $m$ ISP’s we need to introduce $m$ factors, but for $R$, we need to just introduce one for each loop momentum.
4 Pedagogical examples

Having discussed the method in the previous section, we will present two examples to demonstrate our method.

4.1 tensor reduction of bubbles

In this subsection, we consider the tensor reduction of one-loop bubble integrals. The propagators are

\[ z_1 = l^2 - m_1^2, \quad z_2 = (l + p_1)^2 - m_2^2, \quad z_3 = l \cdot R. \]  

and the Gram determinant in Baikov representation is

\[ G = \det \begin{pmatrix} m_1^2 + z_1 & \frac{1}{2} (-m_1^2 + m_2^2 - p_1^2 - z_1 + z_2) & z_3 \\ \frac{1}{2} (-m_1^2 + m_2^2 + p_1^2 + z_1 - z_2) & p_1^2 & R \cdot p_1 \\ z_3 & R \cdot p_1 & R^2 \end{pmatrix}. \]  

Using the expression in (4.2) we can find

\[ \partial z_1 G = \frac{1}{2} R^2 \left( -m_1^2 + m_2^2 + p_1^2 + z_1 - z_2 \right) - z_3 R \cdot p_1 - (R \cdot p_1)^2 \\
\partial z_2 G = \frac{1}{2} R^2 \left( m_1^2 - m_2^2 + p_1^2 + z_1 - z_2 \right) + z_3 R \cdot p_1 \\
\partial z_3 G = -R \cdot p_1 \left( m_1^2 - m_2^2 + p_1^2 + z_1 - z_2 \right) - 2 p_1^2 z_3 \]  

and the solutions of equation

\[ \sum_1^3 (P_i \partial z_i G) + P_0 G = 0 \]  

can be solved by the syzygy module with three generators

\[ \{e_i\} = \begin{pmatrix} 2 z_3 \\ m_1^2 + m_2^2 - p_1^2 + z_1 + z_2 \\ -2 (m_1^2 - p_1^2 + z_2) \\ m_1^2 - 3 m_2^2 - p_1^2 + z_1 - 3 z_2 - 2 R \cdot p_1 - z_3 \end{pmatrix}, \]  

\[ \{h_i\} = \begin{pmatrix} 2 (R \cdot p_1 + z_3) \\ 2 \left( m_1^2 - z_2 \right) \\ R \cdot p_1 + z_3 \end{pmatrix}. \]  

Meanwhile, the module \( \{d_i\} \) is generated by

\[ \text{DM}[z_1, z_2, 1, 1], \]  

where the DM denotes the diagonal matrix. The module intersection of them is given by \( \{h_i\} \) with 10 basis as polynomials of variables \( \{z_1, z_2, z_3, m_1^2, m_2^2, p_1^2, R \cdot p_1, R^2\} \) when using Singular [53]. Among them, the one with the lowest total power of \( z_i \) is given by

\[ h_{1,1} = 2 z_1 \left( R \cdot p_1 \left( m_1^2 - m_2^2 + p_1^2 + z_1 - z_2 \right) + 2 p_1^2 z_3 \right) \]

\[ \text{The reason not using } \{z_1, z_2, z_3\} \text{ is explained in [47].} \]
\[ h_{1,2} = 2z_2 \left( R \cdot p_1 (m_1^2 - m_2^2 + p_1^2 + z_1 - z_2) + 2p_1^2 z_3 \right) \]
\[ h_{1,3} = R^2 \left( -p_1^2 (2m_2^2 + 2m_2^2 + z_1 + z_2) + (m_1^2 - m_2^2) (m_1^2 - m_2^2 + z_1 - z_2) + (p_1^2)^2 \right) \]
\[ + 2 \left( (2m_1^2 + z_1) (R \cdot p_1)^2 + z_3 R \cdot p_1 (2m_1^2 - 2m_2^2 + 2p_1^2 + z_1 - z_2) + 2p_1^2 z_3^2 \right) \]
\[ h_{1,4} = -4 \left( R \cdot p_1 (m_1^2 - m_2^2 + p_1^2 + z_1 - z_2) + 2p_1^2 z_3 \right). \] (4.7)

Using \((4.7)\), one can check that the IBP relation generated by \(D(h_1)\) acting on \(I_{a_1,a_2,-r-1}\) can be rewritten as

\[ I_{a_1,a_2,-r} = \frac{1}{4p_1^2 (a_1 + a_2 - r - d + 1)} \times \]
\[ \left[ -2 (m_1^2 - m_2^2 + p_1^2) R \cdot p_1 (a_1 + a_2 - d - 2r + 2) I_{a_1,a_2,-(r-1)} \right. \]
\[ + (r - 1) \left( 4m_1^2 (R \cdot p_1)^2 - 2m_1^2 R^2 (m_2^2 + p_1^2) + R^2 (m_2^2 - p_1^2)^2 + m_1^4 R^2 \right) I_{a_1,a_2,-(r-2)} \]
\[ + l.p.p.t. \right], \] (4.8)

where

\[ l.p.p.t. = -2R \cdot p_1 (a_1 + a_2 - r - d + 1) I_{a_1-1,a_2,-(r-1)} \]
\[ + 2R \cdot p_1 (a_1 + a_2 - r - d + 1) I_{a_1,a_2-1,-(r-1)} \]
\[ - (r - 1) R^2 (m_1^2 - m_2^2 + p_1^2) I_{a_1,a_2-1,-(r-2)} \]
\[ + (r - 1) \left( R^2 (m_1^2 - m_2^2 - p_1^2) + 2 (R \cdot p_1)^2 \right) I_{a_1-1,a_2,-(r-2)}. \] (4.9)

Result \((4.8)\) is the recursive relation we are looking for. Notice that when \(r = 1\), the coefficient of \(I_{a_1,a_2,-(r-2)} = I_{a_1,a_2,1}\) is zero by the \(r - 1\) factor, which satisfies the condition \((3.5)\). Since \(r\) is the rank of tensor in the numerator, the relation \((4.8)\) tells us that the integrals of tensor rank \(r\) can be written as the sum of integrals of tensor rank \((r - 1)\) and \((r - 2)\) with proper rational coefficients. This kind of relations has been firstly observed in \([50]\) for the case \(a_1 = a_2 = 1\) and then has been proved in \([57]\). Here we give a simple derivation using module intersection with generalization to arbitrary power of propagators, as well as given the analytic l.p.p.t part missed in \([50, 57]\). It is obvious that applying this kind of second-order iterative relation, one can immediately reduce any tensor integrals of this sector to scalar integrals of the same sector and tensor integrals of sub-sectors.

Another interesting property of relation \((4.8)\) is that in \((4.8)\) the \(a_1, a_2\) of bubbles are invariant, while for l.p.p.t. in \((4.9)\), \((a_1 + a_2)\) has changed only by minus one. This observation will be explained in Sec. 5.

Before ending this subsection, let us emphasize that to deal with tensor reduction of bubble topology, we have required \(a_3 = -r < 0\). If we consider the case \(a_3 > 0\), we will get the triangle topology, and the relation becomes IBP relation for triangles, but with the third propagator is not the standard quadratic one. In next subsection, we will consider triangle topology with the standard Feynman propagators, thus it will be useful to compare results in these two subsections.
4.2 DR reduction of triangle

In this subsection, we discuss the reduction of triangles with arbitrary powers for propagators. The propagators are

\[ z_1 = l^2 - m_1^2, \quad z_2 = (l + p_1)^2 - m_2^2, \quad z_3 = (l + p_1 + p_2)^2 - m_3^2. \]  

and the corresponding Gram determinant \( \mathcal{G} \) in Baikov representation is

\[
\det \begin{pmatrix} m_1^2 + z_1 & \cdots & \cdots \\ \frac{1}{2} (-m_1^2 + m_2^2 - p_1^2 - z_1 + z_2) & p_1^2 & \cdots \\ \frac{1}{2} (-m_1^2 + m_3^2 - p_2^2 - 2p_1 \cdot p_2 - z_2 + z_3) & p_1 \cdot p_2 & p_2^2 \end{pmatrix}
\]

(4.11)

where the \( \cdots \) denote the terms, which can be obtained by symmetry. With

\[
\partial_{z_1} \mathcal{G} = \frac{1}{2} p_2^2 \left( -m_1^2 + m_2^2 + p_1^2 - z_1 + z_2 \right) + p_1 \cdot p_2 \left( m_2^2 - m_3^2 + p_2^2 + z_2 - z_3 \right),
\]

\[
\partial_{z_2} \mathcal{G} = \frac{1}{2} \left( p_1 \cdot p_2 \left( m_1^2 - 2m_3^2 + m_2^2 - p_1^2 - p_2^2 + z_1 - 2z_2 + z_3 \right) + m_1^2 p_1^2 + m_1^2 p_2^2 \\
- m_2^2 (p_1^2 + p_2^2) + p_2^2 z_1 - p_1^2 z_2 - p_2^2 z_2 + p_2^2 z_3 - 2 (p_1 \cdot p_2)^2 \right),
\]

\[
\partial_{z_3} \mathcal{G} = \frac{1}{2} \left( p_1^2 (m_2^2 - m_3^2 + p_2^2 + z_2 - z_3) + p_1 \cdot p_2 \left( -m_1^2 + m_2^2 + p_1^2 - z_1 + z_2 \right) \right). 
\]

(4.12)

one can solve

\[
\sum_1^3 \left( P_i \partial_{z_i} \mathcal{G} \right) + P_0 \mathcal{G} = 0,
\]

(4.13)

with the basis \( \{ e_i \} \) of the syzygy module is

\[
\begin{pmatrix}
 m_1^2 + m_2^2 - s + z_1 + z_3 & m_2^2 + m_3^2 - p_2^2 + z_2 + z_3 & 2 \left( m_3^2 + z_3 \right) \\
 m_1^2 + m_2^2 - p_1^2 + z_1 + z_2 & 2 \left( m_2^2 + z_2 \right) & m_2^2 + m_3^2 - p_2^2 + z_2 + z_3 \\
 2 \left( m_1^2 + z_1 \right) & m_2^2 + m_3^2 - p_1^2 + z_1 + z_2 & m_1^2 + m_3^2 - s + z_1 + z_3 \\
\end{pmatrix}
\]

(4.14)

where \( s = (p_1 + p_2)^2 \). Another module is generated by (see (3.10))

\[
\{ d_i \} = \text{DM}[z_1, z_2, 1, 1],
\]

(4.15)

From them, we can compute \( \{ h_i \} \) as the module intersection of them. Among them, the one with the lowest total power of \( z_i \) is given by

\[
h_{1,1} = 2z_1 \left( p_1^2 \left( m_2^2 - m_3^2 + p_2^2 + z_2 - z_3 \right) + p_1 \cdot p_2 \left( -m_1^2 + m_2^2 + p_1^2 - z_1 + z_2 \right) \right),
\]

\[
h_{1,2} = 2z_2 \left( p_1^2 \left( m_2^2 - m_3^2 + p_2^2 + z_2 - z_3 \right) + p_1 \cdot p_2 \left( -m_1^2 + m_2^2 + p_1^2 - z_1 + z_2 \right) \right),
\]

\[
h_{1,3} = -2 \left( m_1^2 p_2 z_1 - m_2^2 p_2 z_2 - m_3^2 p_1^2 z_2 + 2 (p_1 \cdot p_2)^2 (2m_2^2 + z_2) + 2m_3^2 p_1^2 z_3 \\
+ m_2^2 \left( -2m_3^2 p_1^2 - 2m_1^2 p_2^2 + p_1^2 z_2 - 2p_1^2 z_3 - p_2^2 z_1 + p_2^2 z_2 \right) \\
p_1 \cdot p_2 \left( -m_2^2 (2m_1^2 - p_1^2 - 2z_2 + z_3) - m_2^2 (2m_2^2 - 2m_3^2 + 2p_2^2 + z_2 - 2z_3) \\
- 2m_2^2 p_1^2 + m_3^2 z_1 - m_2^2 z_2 + m_4^2 - p_2^2 z_1 + p_1^2 z_2 + p_2^2 z_2 - 2p_1^2 z_3 + 2p_1^2 p_2^2 + z_1 z_3 - z_2 z_3 \right) \right)
\]
\[ + m_1^2 p_1^2 - 2 m_1^2 p_1^2 p_2^2 + m_3^2 p_1^2 - 2 m_3^2 p_1^2 p_2^2 + m_4^4 (p_1^2 + p_2^2) + p_1^2 z_3^2 - p_1^2 p_2^2 z_1 - 2 p_1^2 p_2^2 z_3 - p_1^2 z_2 - p_1^2 p_2^2 + p_1^2 p_2^2) ,
\]
\[ h_{1,4} = 4 p_1 \cdot p_2 (m_1^2 - m_2^2 - p_1^2 + z_1 - z_2) - 4 p_1^2 (m_2^2 - m_3^2 + p_2^2 + z_2 - z_3) . \] (4.16)

Using (4.16) the action of \( D(h_1) \) on \( I_{a_1,a_2,a_3-1} \) gives the wanted IBP relation
\[ I_{a_1,a_2,a_3} = \frac{1}{Q_1} \times \left[ - p_1^2 (a_1 + a_2 + a_3 - d - 1) I_{a_1,a_2,a_3-2} 
\right.
\]
\[ (2a_3 + a_1 + a_2 - d - 2) (p_1^2 (m_2^2 - m_3^2 + p_2^2) + p_1 \cdot p_2 (-m_1^2 + m_2^2 + p_1^2)) I_{a_1,a_2,a_3-1} 
\]
\[ + l.p.p.t. \] (4.17)

where
\[ l.p.p.t. = (a_3 - 1) \left( p_1^2 (-m_1^2 + m_2^2 + p_1^2) + p_1 \cdot p_2 (m_2^2 - m_3^2 + p_2^2) \right) I_{a_1-1,a_2,a_3} 
\]
\[ - (a_3 - 1) (p_1 \cdot p_2 (-m_1^2 + 2m_2^2 - m_3^2 + p_1^2 + p_2^2) - m_3 p_1^2 - m_1 p_2^2 + m_2^2 (p_1^2 + p_2^2) 
\]
\[ + 2 (p_1 \cdot p_2)^2) I_{a_1-1,a_2-1,a_3} 
\]
\[ - p_1 \cdot p_2 (a_1 + a_2 + a_3 - d - 1) I_{a_1-1,a_2,a_3-1} 
\]
\[ + (p_1^2 + p_1 \cdot p_2) (a_1 + a_2 + a_3 - d - 1) I_{a_1,a_2-1,a_3-1} 
\]
\[ Q_1 = (a_3 - 1) \left( m_1^2 p_1^2 - 2 m_1^2 p_1^2 p_2^2 + 4 m_2^2 (p_1 \cdot p_2)^2 + m_3^2 p_1^2 - 2 m_3^2 p_1^2 p_2^2 + m_4^2 (p_1^2 + p_2^2) 
\right.
\]
\[ - 2 p_1 \cdot p_2 (m_1^2 - m_2^2 - p_1^2) (m_2^2 - m_3^2 + p_2^2) - 2 m_2^2 (m_3^2 p_1^2 + m_3^2 p_2^2) + p_1^2 p_2^2 + p_1^2 p_2^2 \] . (4.18)

Notice that when \( a_3 = 2 \), the coefficient of \( I_{a_1,a_2,a_3-2} = I_{a_1,a_2,0} \) is a term of sub-sector (here is the bubble topology). Also for \( a_3 = 1 \) we don’t need to apply this relation for DR since it is already the final goal we want to achieve. The relation (4.17) is also a second-order iterative relation relating \( I_{a_1,a_2,a_3} \) to \( I_{a_1,a_2,a_3-1} \), \( I_{a_1,a_2,a_3-2} \) and l.p.p.t. Similar to (4.8), the \( a_1, a_2 \) of triangles in (4.17) are invariant while for l.p.p.t. in (4.18), \( (a_1 + a_2) \) has changed only by minus one.

Thus applying the relation (4.17) iteratively one can immediately reduce any higher power of \( z_3 \) to one. Similar iterative relations for reducing the power of other propagators can be obtained. Combining them, we can reduce integrals with arbitrary high power of propagators in this sector.

5 Uniform formula for general one-loop reduction

The method laid out in section 3 and related computations done in section 4 look fresh, but maybe not so surprising. However, as we will show in this section, for one-loop integrals, we can write down explicit recursive relations for both TR and DR uniformly for any general \( n \)-point one-loop integrals. In other words, we have solved IBP relations analytically.

The key of our method is to select particular elements in module intersection \( \{ h_i \} \).

Let’s make an observation for two examples in section 4, i.e., both results (4.7) and (4.16), we find that
\[ \{ h_{1,1}, h_{1,2}, h_{1,4} \} = C \times \{ z_1 \partial_{z_3} G, z_2 \partial_{z_3} G, -2 \partial_{z_3} G \} \] (5.1)
and then by (4.13), we have
\begin{equation}
\mathbf{h}_{1,3} = C \times (2\mathcal{G} - z_1\partial_{z_1}\mathcal{G} - z_2\partial_{z_2}\mathcal{G}).
\end{equation}

This pattern indicates that for DR of the \(N\)-th propagator of the \(N\)-point one-loop integrals or the TR of \((N-1)\)-point one-loop integrals (where the \((R \cdot \ell)\) has been considered as the \(N\)-th propagator), the wanted element in the intersection module is given by
\begin{equation}
P_{ui} = z_i\partial_{z_i}\mathcal{G} \quad \text{for} \quad 1 \leq i \leq N-1,
\end{equation}
\begin{equation}
P_{uN} = 2\mathcal{G} - \sum_{i=1}^{N-1} z_i\partial_{z_i}\mathcal{G}, \quad P_{u0} = -2\partial_{z_N}\mathcal{G}.
\end{equation}

It is easy to check that (5.3) satisfies (2.6) and belongs to the module
\begin{equation}
\{d_i\} = DM[z_1, z_2, \cdots, z_{N-1}, 1, 1],
\end{equation}
then the differential operator \(D(P_u)\) gives the generic iterative relation for both TR and DR of general one-loop integrals.

To write down explicit relation, let us compute \(P_u\) in (5.3). For one-loop integrals, Gram determinant \(\mathcal{G}\) is always quadratic polynomial of \(z_i\)s
\begin{equation}
\mathcal{G}(z) = \sum_{i,j \leq i} C_{2}^{(ij)} z_i z_j + \sum_{i} C_{1}^{(i)} z_i + C_0
\end{equation}
where
\begin{equation}
C_{2}^{(ii)} = \partial_{z_i}^2 \mathcal{G}/2, \quad C_{1}^{(i)} = (\partial_{z_i} \mathcal{G})|_{z=0}, \quad C_0 = \mathcal{G}|_{z=0},
\end{equation}
\begin{equation}
C_{2}^{(ij)} = C_{2}^{(ji)} = \partial_{z_i}\partial_{z_j}\mathcal{G} \quad \text{for} \quad j \neq i.
\end{equation}
This leads to
\begin{equation}
\partial_{z_i}\mathcal{G} = 2C_{2}^{(ii)} z_i + \sum_{j \neq i} C_{2}^{(ij)} z_j + C_{1}^{(i)},
\end{equation}
\begin{equation}
\sum_{i=1}^{N} z_i\partial_{z_i}\mathcal{G} = 2 \sum_{i,j \leq i} C_{2}^{(ij)} z_i z_j + \sum_{i} C_{1}^{(i)} z_i,
\end{equation}
\begin{equation}
P_{uN} = z_N\partial_{z_N}\mathcal{G} + \sum_{i} C_{1}^{(i)} z_i + 2C_0.
\end{equation}
The rewriting of \(P_{uN}\) in (5.7) tells us that \(P_{uN}\) depends on \(z_i, i = 1, \ldots, N-1\) only linearly.

Combining \(P_{ui}\) in (5.3) and \(\partial_{z_i}\mathcal{G}\) in (5.7), one can see that the power of \(z_i\)s will lead all \(I_{a_1'} \cdots a_{N-1}' a_N'\) to satisfy \(0 \leq \sum_{i=1}^{N-1} a_i - \sum_{i=1}^{N-1} a_i' \leq 1\). To show that, let us do the following explicit computations.

For \(i \leq N-1\), carrying out
\begin{equation}
- DP_{ui} \cdot \frac{1}{\prod_{i=1}^{N} z_i^{a_i}} = -\partial_{z_i} \left( \frac{z_i \partial_{z_N} \mathcal{G}}{\prod_{i=1}^{N} z_i^{a_i}} \right),
\end{equation}
we find
\[(a_i - 1) \left( 2C_2^{(NN)}I_{\cdots,a_{N-1}} + C_1^{(N)}I_{\cdots,a_{N}} \right)
+(a_i - 2)C_2^{(N)}I_{\cdots,a_{i-1},\cdots,a_{N}} + (a_i - 1) \sum_{j=1, j \neq i}^{N-1} C_2^{(Nj)}I_{\cdots,a_{j-1},\cdots,a_{N}}. \tag{5.9}\]

For \(i = N\), action of \(D_{PN}\) gives
\[2 \left( (a_N - 2)C_2^{(NN)}I_{\cdots,a_{N-1}} + (a_N - 1)C_1^{(N)}I_{\cdots,a_{N}} + a_N C_0 I_{\cdots,a_{N+1}} \right)
+ \left( (a_N - 1) \sum_{j \neq N} C_2^{(Nj)}I_{\cdots,a_{j-1},\cdots,a_{N}} + a_N \sum_{j \neq N} C_1^{(j)}I_{\cdots,a_{j-1},\cdots,a_{N-1}} \right). \tag{5.10}\]

Finally action of \(D_{P_0}\) gives
\[-(d - N - 1) \left( C_1^{(N)}I_{\cdots,a_{N}} + 2C_2^{(NN)}I_{\cdots,a_{N-1}} + \sum_{j=1}^{N-1} C_2^{(Nj)}I_{\cdots,a_{j-1},\cdots,a_{N}} \right). \tag{5.11}\]

Combining all together, we have
\[2a_N C_0 I_{\cdots,a_{N+1}} + \left( \sum_{i=1}^{N-1} a_i + 2a_N - d \right) C_1^{(N)}I_{\cdots,a_{N}}
+ \left( \sum_{i=1}^{N} a_i - d \right) 2C_2^{(NN)}I_{\cdots,a_{N-1}} + l.p.p.t. = 0 \tag{5.12}\]

where
\[l.p.p.t. = \left( \sum_{i=1}^{N} a_i - d \right) \sum_{j=1}^{N-1} C_2^{(Nj)}I_{\cdots,a_{j-1},\cdots,a_{N}} + a_N \sum_{j=1}^{N-1} C_1^{(j)}I_{\cdots,a_{j-1},\cdots,a_{N-1}}. \tag{5.13}\]

Expressions (5.12) and (5.13) are our main results for this paper.

When \(a_N < 0\), it corresponds to the numerator and we should use (5.12) to express \(I_{\cdots,a_{N-1}}\) by others for the TR, while when \(a_N > 0\) it corresponds to the denominator with a higher power and we should use (5.12) to express \(I_{\cdots,a_{N+1}}\) by others for the DR. When \(a_N = 0\), the (5.12) will give the relation between \(I_{\cdots,-1}\) and \(I_{\cdots,0}\), which is just the reduction of tensor with rank one.

6 Example of degenerated case

In section 5, we have assumed that the kinematics and masses are general. But when kinematics and masses take some special values, the Gram determinant may be zero and we meet the degenerated situations. For these cases, directly applying (5.12) and (5.13) still work, but why it works should be carefully explained. We will show that there are three different possibilities. The first one is that while it works, it may not give the simplest
iterative relation. The second one is that while it does not work for some situations, there is at least one situation it works. The third one is that it does not work at all, but for this case, there are no master integrals, so it does not matter. No matter which situation one meets, directly applying the method of module intersection presented in the Sec.3 gives an alternative way to do reduction than using (5.12) and (5.13).

Now we give an example to elaborate on the above claim. Let us consider the reduction of triangles

\[ z_1 = l^2, \quad z_2 = (l + p_1)^2, \quad z_3 = (l + p_1 + p_2)^2, \]
\[ z_4 = l \cdot R, \quad p_1^2 = p_2^2 = 0. \]

(6.1)

with specific kinematics. As shown later, by IBP relations one can see that all integrals in this sector can be reduced to sub-sectors.

For TR of (6.1), \( G = G(l, p_1, p_2, R) \). Directly applying the formula (5.12) and (5.13), we will get

\[ I_{a_1,a_2,a_3,-r-1} = \frac{R \cdot p_1 \left( r \cdot R \cdot p_1 I_{a_1,a_2,a_3,1-r} - \left( \sum_i^3 a_i - d - 2r \right) I_{a_1,a_2,a_3,,-r} \right)}{\sum_i^3 a_i - d - r} + \text{l.p.p.t.} , \]

(6.2)

which is a second-order iterative relation. Using (6.2), one can still reduce all \( I_{1,1,1,-r} \) to \( I_{1,1,1,0} \). However, \( I_{1,1,1,0} \) is not a master integral in this example. To reduce \( I_{1,1,1,0} \) we should use (5.12) and (5.13) for DR, which will be explained shortly.

If we calculate the module intersection follow the method in Sec.3, one can find another element \( h_i \) different from the one given in (5.3), which will give us the following iterative reduction relation

\[ I_{a_1,a_2,a_3,-r} = \frac{rR \cdot p_1 I_{a_1,a_2,a_3,1-r}}{2a_2 + 2a_3 - d - r} + \text{l.p.p.t.} . \]

(6.3)

Obviously, as a first-order iterative relation, (6.3) is simpler than (6.2), which supports the claim of the first possibility.

For DR\(^{11}\), the Gram determinant in Baikov representation is

\[ G = G(l, p_1, p_2) = \frac{1}{2} p_1 \cdot p_2 \left( (z_1 - z_2)(z_2 - z_3) - 2p_1 \cdot p_2 z_2 \right) . \]

(6.4)

To determine the number of master integrals, we can consider the maximum cut of this sector in Baikov representation, i.e., setting \( z_i = 0 \) in \( G \). If \( G|_{z=0} \) is a nonzero constant, the number of master integrals is just one by counting critical points \([33, 58, 59]\). But if \( G|_{z=0} = 0 \), there is no master integral in this sector. It is easy to see that \( G|_{z=0} = 0 \) in (6.4), thus \( I_{1,1,1} \) can be reduced to sub-sectors.

One can check that if we regard \( z_1 \) or \( z_3 \) as the \( z_N \) and apply (5.12), the \( C_0, C_1^{(N)}, C_2^{(NN)} \) are all zero and only l.p.p.t. is left. In other words, for these cases, (5.12) does not produce

\(^{10}\)R\(^2\) does not appear in (6.2) because \( G(l, p_1, p_2)|_{z=0} = 0 \) when using (5.12) and (5.13).

\(^{11}\)Now there is no \( z_4 \) in (6.1).
the wanted relation for reduction purpose. However, if we regard $z_2$ as the $z_N$, the iterative 
relation reduce to first-order due to $C_0 = 0$, $C_1^{(2)} \neq 0$ and $C_2^{(22)} \neq 0$. Using it, we can reduce 
$I_{a_1,a_2,a_3,0}$ (including the $I_{1,1,1,0}$ discussed in previous paragraph) to sub-sectors $I_{a_1,0,a_3,0}$ and 
others. The $a_1$ and $a_3$ are left to be reduced by DR of (5.12) in the sub-sector. So, as 
pointed out for the second possibility, although using (5.12) for the DR does not work for $z_1$ and $z_3$, there is $z_2$ it works.

Nevertheless, even if we regard $z_3$ as the $z_N$, we can do similar module computation 
proposed in Sec.3. The syzygy module is generated by 

$$\{e_i\} = \begin{pmatrix} 
z_2 & z_2 & z_3 & -1 
2p_1 \cdot p_2 - z_3 & -z_3 & z_2 - 2z_3 & 1 
-2p_1 \cdot p_2 + z_1 - z_2 + z_3 & z_3 & z_3 & -1 
4p_1 \cdot p_2 + z_2 - 2z_3 & z_1 - 2z_3 -2p_1 \cdot p_2 + z_1 - 2z_3 & 1 \end{pmatrix}, \quad (6.5)$$

while the element in the intersection module is taken to be 

$$\langle P \rangle = \langle z_1, z_2, z_2, 1 \rangle. \quad (6.6)$$

Using (6.6), the iterative relation is 

$$I_{a_1,a_2,a_3,0} = -\frac{2a_3I_{a_1,a_2-1,a_3+1,0}}{2a_1 + 2a_2 - d} \quad (6.7)$$

where the $(a_1 + a_2)$ has been reduced by one at the right-hand side, which is just a l.p.p.t.. Although this relation raises the power of $z_3$ in denominator, it will lower $a_2$ to 0 finally (i.e., reduce to sub-sector), so it shows that method of module intersection still works.

From this example, it is easy to see that only for more degenerated case, i.e., all $C_i^{(ii)}$, 
$C_1^{(i)}$ and $C_0$ in $G$ equal zero, (5.12) can not reduce integrals in this sector to sub-sectors. But for this situation, all sub-sectors have no master integrals by counting the critical 
points. To be more explicitly, the number of master integrals in the corresponding sector is equal to the number of solutions to the equations 

$$0 = \partial z_i \log \left( G^{(d-n-1)/2} \right) \bigg|_{z'=0}, \quad \text{for all } z_i \notin \left\{ z' \right\}, \quad (6.8)$$

Now the $G$ takes the form 

$$G = \sum_{i,j \neq i} c_{ij} z_i z_j \quad (6.9)$$

and the equations (6.8) take the form 

$$\frac{C_i}{z_i - C_i} = 0, \quad (6.10)$$

which obviously has no solution. It means that integrals in this topology are all scaleless 
integrals, and this topology has no master integral. People can test this conclusion by 
taking $p_1 \cdot p_2$ to zero in (6.4) and immediately find this topology to be scaleless. This is 
the third possibility we have mentioned.
7 Summary and Outlook

In this paper, a natural extension of syzygy and module intersection has been shown that it can uniformly reduce one-loop integrals with arbitrary tensor (by using auxiliary vector) and propagators with high powers. With a nice observation, powerful iterative relation can be written down even without doing explicit computations using computational algebraic geometry and module intersection. Furthermore, in this formula, the polynomials that look messy in the traditional reduction methods are arranged themselves to Gram determinants. Such a property will not only speed up the analytic reduction of Feynman integrals, but also help to investigate mathematical structures of Feynman integrals and amplitudes in the future.

It is obvious that the motivation of our study is the reduction for high loops. If the method is good, it must work well at the one-loop level. Results in this paper have demonstrate this point. To generalize this method to high loops, although steps of computational algebraic geometry can be easily applied, some nontrivial problems will arise. There are irreducible scalar products of loop momenta and more than one master integrals for a given sector in most multi-loop integrals. This suggests that for multi-loop integrals, elements of module intersection needed may also be more than one. More importantly, could we obtain the general reduction formulas like (5.12) for multi-loops? Are there also some hidden informations or structures in the messy polynomials in the IBP relation? These problems are definitely interesting for future exploration.

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