UNIVERSAL COVERING CALABI-YAU MANIFOLDS OF THE HILBERT SCHEMES OF \( n \) POINTS OF ENRIQUES SURFACES

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Abstract. The purpose of this paper is to investigate the Hilbert scheme of \( n \) points of an Enriques surface from the following three points of view: (i) the relationship between the small deformation of the Hilbert scheme of \( n \) points of an Enriques surface and that of its universal cover (Theorem 1.1), (ii) the natural automorphisms of the Hilbert scheme of \( n \) points of an Enriques surface (Theorem 1.4), and (iii) the number of distinct Hilbert schemes of \( n \) points of Enriques surfaces, which has the same universal covering space (Theorem 1.7).

Key words. Calabi-Yau manifold, Enriques surface, Hilbert scheme.

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1. Introduction. Throughout this paper, we work over \( \mathbb{C} \), and \( n \) is an integer such that \( n \geq 2 \). A K3 surface \( K \) is a compact complex surface with \( \omega_K \simeq \mathcal{O}_K \) and \( H^1(K, \mathcal{O}_K) = 0 \). An Enriques surface \( E \) is a compact complex surface with \( H^1(E, \mathcal{O}_E) = 0, H^2(E, \mathcal{O}_E) = 0 \), and \( \omega_{E}^{\otimes 2} \simeq \mathcal{O}_E \). A Calabi-Yau manifold \( X \) is an \( n \)-dimensional compact kähler manifold such that it is simply connected, there is no holomorphic \( k \)-form on \( X \) for \( 0 < k < n \), and there is a nowhere vanishing holomorphic \( n \)-form on \( X \). By Oguiso and Schröer [11, Theorem 3.1], the Hilbert scheme of \( n \) points of an Enriques surface \( E^{[n]} \) has a Calabi-Yau manifold \( X \) as the universal covering space of degree \( 2 \). Recall that when \( n = 1 \), \( E^{[1]} \) is an Enriques surface \( E \), and \( X \) is a K3 surface.

In this paper, we study the Hilbert scheme of \( n \) points of an Enriques surface \( E^{[n]} \) from the relationship between \( E^{[n]} \) and its universal covering space \( X \) (Theorem 1.1 and 1.7) and the natural automorphisms of \( E^{[n]} \) (Theorem 1.4).

Section 2 is a preliminary section. We prepare and recall some basic facts on the Hilbert scheme of \( n \) points of a surface and show that for the universal covering space \( X = E^{[n]} \), there is a quotient singular variety \( Z \) such that \( X \) is a resolution of \( Z \) (Theorem 2.7).

In Section 3, we investigate the relationship between the small deformation of \( E^{[n]} \) and that of \( X \). When \( n = 1 \), \( E^{[1]} \) is an Enriques surface \( E \), and \( X \) is a K3 surface. An Enriques surface has a 10-dimensional deformation space and a K3 surface has a 20-dimensional deformation space. Thus the small deformation of \( X \) is much bigger than that of \( E \). For \( n \geq 2 \), by using the result of Göttsche and Soergel [7, Theorem 2] and the properties of the covering space \( X \to E^{[n]} \), we compute the dimension of the deformation space of \( X \). Consequently, we obtain Theorem 1.1 which is different from the case of \( n = 1 \):

**Theorem 1.1.** For \( n \geq 2 \), let \( E \) be an Enriques surface, \( E^{[n]} \) the Hilbert scheme of \( n \) points of \( E \), and \( X \) the universal covering space of \( E^{[n]} \). Then every small deformation of \( X \) is induced by that of \( E^{[n]} \).

**Remark 1.2.** By Fantechi [4, Theorems 0.1 and 0.3], every small deformation of \( E^{[n]} \) is induced by that of \( E \). Thus for \( n \geq 2 \), every small deformation of \( X \) is induced by that of \( E^{[n]} \).
induced by that of $E$.

In Section 4, we study the natural automorphisms of $E^{[n]}$.

**Definition 1.3.** For $n \geq 2$ and $S$ a smooth compact surface, any automorphism $f \in \text{Aut}(S)$ induces an automorphism $f^{[n]} \in \text{Aut}(S^{[n]})$. An automorphism $g \in \text{Aut}(S^{[n]})$ is called natural if there is an automorphism $f \in \text{Aut}(S)$ such that $g = f^{[n]}$.

When $S$ is a $K3$ surface, the natural automorphisms of $S^{[n]}$ were studied by Boissière and Sarti [3]. They showed that an automorphism of $S^{[n]}$ is natural if and only if it preserves the exceptional divisor of the Hilbert-Chow morphism [3, Theorem 1]. We obtain Theorem 1.4 which is similar to [3, Theorem 1]:

**Theorem 1.4.** For $n \geq 2$, let $E$ be an Enriques surface, and $D$ the exceptional divisor of the Hilbert-Chow morphism $\pi_E : E^{[n]} \to E^{(n)}$. An automorphism $f$ of $E^{[n]}$ is natural if and only if $f(D) = D$.

In Section 5, we compute the number of distinct Enriques surface type quotients of $X$ for a fixed $X$.

**Definition 1.5.** For $n \geq 1$, let $E$ be an Enriques surface, $E^{[n]}$ the Hilbert scheme of $n$ points of $E$, and $X$ the universal covering space of $E^{[n]}$. A variety $Y$ is called an Enriques surface type quotient of $X$ if there is an Enriques surface $E'$ and a free involution $\tau$ of $X$ such that $Y \simeq E'^{[n]}$ and $E'^{[n]} \simeq X/\langle \tau \rangle$. Here we call two Enriques surface type quotients of $X$ distinct if they are not isomorphic to each other.

Recall that when $n = 1$, $E^{[1]}$ is an Enriques surface $E$ and $X$ is a $K3$ surface. In [12, Theorem 0.1], Ohashi showed the following theorem:

**Theorem 1.6.** For any nonnegative integer $l$, there exists a $K3$ surface with exactly $2^{2l+10}$ distinct Enriques quotients. In particular, there does not exist a universal bound for the number of distinct Enriques quotients of a $K3$ surface.

We obtain Theorem 1.7 which is different from Theorem 1.6 in the sense of the Enriques surface type quotient:

**Theorem 1.7.** For $n \geq 2$, let $E$ be an Enriques surface, $E^{[n]}$ the Hilbert scheme of $n$ points of $E$, and $X$ the universal covering space of $E^{[n]}$. Then the number of distinct Enriques surface type quotients of $X$ is one.

**Remark 1.8.** When $n = 2$, we do not count the number of distinct Enriques surface type quotients of $X$. We compute the Hodge numbers of the universal covering space $X$ of $E^{[2]}$ (Appendix A).

In Proposition 5.2, we show that for $n \geq 3$, the covering involution of $\pi : X \to E^{[n]}$ acts on $H^2(X, \mathbb{C})$ as the identity. In Proposition 5.5, by using Theorem 2.7 and 1.4, we show that for $n \geq 2$, if an automorphism $\varphi$ of $X$ acts on $H^2(X, \mathbb{C})$ as the identity, then $\varphi$ is a lift of a natural automorphism of $E^{[n]}$. In Proposition 5.9, by using Proposition 5.5 and checking the action to $H^1(X, \Omega_X^{2n-1})$, we classify involutions of $X$ which act on $H^2(X, \mathbb{C})$ as the identity. We prove Theorem 1.7 using those results.

In addition, let $Y$ be a smooth compact Kähler surface. For a line bundle $L$ on $Y$, by using the natural map $\text{Pic}(Y) \to \text{Pic}(Y^{[n]}))$, $L \mapsto L_n$, we put

$$h^{p,q}(Y^{[n]}, L_n) := \dim_{\mathbb{C}} H^q(Y^{[n]}, \Omega_{Y^{[n]}}^p \otimes L_n),$$
\[ h^{p,q}(Y, L) := \dim \mathbb{C}H^q(Y, \Omega_Y^p \otimes L), \]

\[ A := \sum_{n,p,q=0}^{\infty} h^{p,q}(Y[n], L_n) x^p y^q t^n, \]

\[ B := \prod_{k=1}^{\infty} \prod_{p,q=0}^{2} \left( \frac{1}{1 - (-1)^{p+q} x^{p+k-1} y^{q+k-1} t^k} \right)^{(-1)^{p+q} h^{p,q}(Y, L)}. \]

In [2, Conjecture 1], S. Boissière conjectured that

\[ A = B. \]

In the proof of Theorem 1.1, we obtain the counterexample to this conjecture for \( Y \) an Enriques surface and \( L = \Omega_Y^2 \). See Appendix B for details.

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2. Preliminaries. Let \( S \) be a nonsingular surface, \( S[n] \) the Hilbert scheme of \( n \) points of \( S \), \( \pi_S : S[n] \to S^{(n)} \) the Hilbert-Chow morphism, and \( p_S : S^n \to S^{(n)} \) the natural projection. We denote the exceptional divisor of \( \pi_S \) by \( D \). By Fogarty [5, Theorem 2.4], \( S[n] \) is smooth of \( \dim \mathbb{C}S[n] = 2n \).

Let \( \Delta^n \) be the set of \( n \)-uples \((x_1, \ldots, x_n) \in S^n\) with at least two \( x_i \)'s equal, \( S^n_s \) the set of \( n \)-uples \((x_1, \ldots, x_n) \in S^n\) with at most two \( x_i \)'s equal. We put

\[ S_s^{(n)} := p_S(S^n_s), \]
\[ \Delta^{(n)} := p_S(\Delta^n), \]
\[ S_s^{[n]} := \pi_S^{-1}(S_s^{(n)}), \]
\[ \Delta_s^{(n)} := \Delta^n \cap S_s^{(n)}, \]
\[ \Delta_s^{(n)} := p_S(\Delta_s^{(n)}), \] and

\[ F := S[n] \setminus S_s^{[n]} . \]

When \( n = 2 \), \( S_s^2 = S^2 \), \( F = \emptyset \) and \( \text{Blow}_{\Delta\Delta^2} S^n_s/S_n \simeq S^{[2]} \). For \( n \geq 3 \), we have \( \text{Blow}_{\Delta_2} S_s^n/S_n \simeq S_s^n \), and \( F \) is an analytic closed subset and its codimension is 2 in \( S[n] \) by Beauville [1, page 767-768]. Here \( S_n \) is the symmetric group of degree \( n \) which acts naturally on \( S^n \) by permuting of the factors.

Let \( E \) be an Enriques surface, \( E^{(n)} \) the Hilbert scheme of \( n \) points of \( E \), and \( \pi : X \to E^{(n)} \) the universal covering space. Let \( \mu : K \to E \) be the universal covering space of \( E \) where \( K \) is a K3 surface, and \( \Lambda \) the pullback of \( \Delta^{(n)} \) by the morphism:

\[ \mu^{(n)} : K^{(n)} \ni [(x_1, \ldots, x_n)] \mapsto [(\mu(x_1), \ldots, \mu(x_n))] \in E^{(n)}. \]
Then we get a $2^n$-sheeted unramified covering space:

$$\mu^{(n)}|_{K^{(n)} \setminus \Lambda} : K^{(n)} \setminus \Lambda \to E^{(n)} \setminus \Delta^{(n)}.$$ 

Furthermore, let $\Gamma$ be the pullback of $\Lambda$ by the natural projection $p_K : K^n \to K^{(n)}$. Since $\Gamma$ is an algebraic closed set with codimension 2, then

$$\mu^{(n)} \circ p_K : K^n \setminus \Gamma \to E^{(n)} \setminus \Delta^{(n)}$$

is the $2^n n!$-sheeted universal covering space. Since $E^{[n]} D = E^{(n)} \Delta^{(n)}$ where $D = \pi^{-1}(\Delta^{(n)})$, we regard the universal covering space

$$\mu^{(n)} \circ p_K : K^n \setminus \Gamma \to E^{[n]} D.$$

as the universal covering space of $E^{[n]} D$. 

Since $\pi : X \setminus \pi^{-1}(D) \to E^{[n]} \setminus D$ is a covering space, and $\mu^{(n)} \circ p_K : K^n \setminus \Gamma \to E^{[n]} \setminus D$ is the universal covering space, there is a morphism

$$\omega : K^n \setminus \Gamma \to X \setminus \pi^{-1}(D)$$

such that $\omega : K^n \setminus \Gamma \to X \setminus \pi^{-1}(D)$ is the universal covering space and $\mu^{(n)} \circ p_K = \pi \circ \omega$:

$$K^n \setminus \Gamma \xrightarrow{\omega} X \setminus \pi^{-1}(D) \xrightarrow{\mu^{(n)} \circ p_K} E^{[n]} \setminus D.$$ 

We denote the covering transformation group of $\pi \circ \omega$ by

$$G := \{ g \in Aut(K^n \setminus \Gamma_K) : \pi \circ \omega \circ g = \pi \circ \omega \}.$$ 

Since $\deg(\mu^{(n)} \circ p_K) = 2^n n!$, the order of $G$ is $2^n n!$. Let $\sigma$ be the covering involution of $\mu : K \to E$. For

$$1 \leq k \leq n, \ 1 \leq i_1 < \cdots < i_k \leq n,$$

we define automorphisms $\sigma_{i_1 \ldots i_k}$ of $K^n$ in the following way: for $x = (x_i)_{i=1}^{n} \in K^n$,

the j-th component of $\sigma_{i_1 \ldots i_k}(x) = \begin{cases} \sigma(x_j) & j \in \{i_1, \ldots, i_k\} \\
 x_j & j \notin \{i_1, \ldots, i_k\}, \end{cases}$

then $S_n \subset G$, and $\{\sigma_{i_1 \ldots i_k}\}_{1 \leq k \leq n, \ 1 \leq i_1 < \cdots < i_k \leq n} \subset G$. Let $H$ be the subgroup of $G$ generated by $S_n$ and $\{\sigma_{ij}\}_{1 \leq i < j \leq n}$.

**PROPOSITION 2.1.** $G$ is generated by $S_n$ and $\{\sigma_{i_1 \ldots i_k}\}_{1 \leq k \leq n, \ 1 \leq i_1 < \cdots < i_k \leq n}$.

**Proof.** We assume that

$$s \circ t = s' \circ t'$$
for some \( s, s' \in S_n \) and \( t, t' \in \{ \sigma_{i_1 \ldots i_k} \}_{1 \leq k \leq n, 1 \leq i_1 < \ldots < i_k \leq n} \). If \( s \neq s' \), then \( s'^{-1} \circ s \neq \text{Id}_{K^n} \). We take an element \( \tilde{x} = (\tilde{x}_i)_{i=1}^n \in K^n \) with \( \tilde{x}_i \neq \tilde{x}_j \) for \( 1 \leq i < j \leq n \) and \( \sigma(\tilde{x}_i) \neq \tilde{x}_i \) for \( 1 \leq i \leq j \leq n \). Since \( s \circ t = s' \circ t' \), we have \( s'^{-1} \circ s(x) = t' \circ t^{-1}(x) \). Thus for some \( i \) where \( 1 \leq i \leq n \),

\[
\sigma(\tilde{x}_i) \in \{ \tilde{x}_j \}_{j=1}^n.
\]

This contradicts the definition of \( \tilde{x} \). Therefore we get \( s = s' \) and \( t = t' \). Since \( |S_n| = n! \), \( \{ \sigma_{i_1 \ldots i_k} \}_{1 \leq k \leq n, 1 \leq i_1 < \ldots < i_k \leq n} = 2^n \), and \( |G| = 2^n n! \), \( G \) is generated by \( S_n \) and \( \{ \sigma_{i_1 \ldots i_k} \}_{1 \leq k \leq n, 1 \leq i_1 < \ldots < i_k \leq n} \).

**Proposition 2.2.** \(|H| = 2^{n-1} n!\).

**Proof.** For \( s \in S_n \) and \( \sigma_{j_1 \ldots j_l} \in \{ \sigma_{i_1 \ldots i_k} \}_{1 \leq k \leq n, 1 \leq i_1 < \ldots < i_k \leq n} \), there are positive numbers \( u_1, \ldots, u_k \) such that

\[
\{ u_1, \ldots, u_k \} = \{ s^{-1}(j_1), \ldots, s^{-1}(j_l) \}, \quad \text{and} \quad u_1 \cdots < u_k.
\]

Then we get \( \sigma_{j_1 \ldots j_l} \circ s = s \circ \sigma_{u_1 \ldots u_k} \). For arbitrary \( j, (i, j) \circ i \circ (i, j) = \sigma_j \). Since \( H \) is generated by \( S_n \) and \( \{ \sigma_{ij} \}_{1 \leq i < j \leq n} \), from Proposition 2.1 we obtain \(|G/H| = 2^n = 2^n n!\).

Recall that \( \mu : K \to E \) is the universal covering and \( \sigma \) is the covering involution of \( \mu \). We put

\[
K_{\mu}^n := (\mu^n)^{-1}(E^n),
\]

where \( \mu^n : K^n \ni (x_i)_{i=1}^n \to (\mu(x_i))_{i=1}^n \in E^n \),

\[
T_{ij} := \{ (x_i)_{i=1}^n \in K_{\mu}^n : \sigma(x_i) = x_j \},
\]

\[
U_{ij} := \{ (x_i)_{i=1}^n \in K_{\mu}^n : x_i = x_j \},
\]

\[
T := \bigcup_{1 \leq i \leq j \leq n} T_{i,j}, \quad \text{and} \quad U := \bigcup_{1 \leq i < j \leq n} U_{ij}.
\]

When \( n = 2 \), \( K_{\mu}^2 = K^2 \), \( U = \Delta^2 \), and \( T = \{(x, y) \in K^2 : \sigma(x) = y \} \). By the definition of \( K_{\mu}^n \), \( H \) acts on \( K_{\mu}^n \). For an element \( \tilde{x} := (\tilde{x}_i)_{i=1}^n \in U \cap T \), some \( i, j, k, l \) with \( k \neq l \) such that \( \sigma(\tilde{x}_i) = \tilde{x}_j \) and \( \tilde{x}_k = \tilde{x}_l \). Since \( \sigma \) does not have fixed points. Thus \( \tilde{x}_i \neq \tilde{x}_j \). Therefore \( \mu^n(\tilde{x}) \notin E^n \). This is a contradiction. We obtain \( T \cap U = \emptyset \).

**Lemma 2.3.** For \( t \in H \) and \( 1 \leq i < j \leq n \), if \( t \in H \) has a fixed point on \( U_{ij} \), then \( t = (i, j) \) or \( t = \text{id}_{K^n} \).

**Proof.** Let \( t \in H \) be an element of \( H \) where there is an element \( \tilde{x} = (\tilde{x}_i)_{i=1}^n \in U_{ij} \) such that \( t(\tilde{x}) = \tilde{x} \). By Proposition 2.1, for \( t \in H \), there are \( \sigma_{i_1 \ldots i_k} \in \{ \sigma_{i_1 \ldots i_k} \}_{1 \leq k \leq n, 1 \leq i_1 < \ldots < i_k \leq n} \) and \( (j_1, \ldots, j_l) \in S_n \) such that

\[
t = (j_1, \ldots, j_l) \circ \sigma_{i_1 \ldots i_k}.
\]
From the definition of $U_{ij}$, for $(x_l)_{l=1}^{n} \in U_{ij}$,
\[
\{x_1, \ldots, x_n\} \cap \{\sigma(x_1), \ldots, \sigma(x_n)\} = \emptyset.
\]
Suppose $\sigma_{i_1 \ldots i_k} \neq \text{id}_{K^n}$. Since $t(\bar{x}) = \bar{x}$, we have
\[
\{\bar{x}_1, \ldots, \bar{x}_n\} \cap \{\sigma(\bar{x}_1), \ldots, \sigma(\bar{x}_n)\} \neq \emptyset.
\]
This is a contradiction. Thus we have $t = (j_1, \ldots, j_l)$. Similarly from the definition of $U_{ij}$, for $(x_l)_{l=1}^{n} \in U_{ij}$, if $x_s = x_t$ $(1 \leq s < t \leq n)$, then $s = i$ and $t = j$. Thus we have $t = (i, j)$ or $t = \text{id}_{K^n}$. \(\Box\)

**Lemma 2.4.** For $t \in H$ and $1 \leq i < j \leq n$, if $t \in H$ has a fixed point on $T_{ij}$, then $t = \sigma_{i,j} \circ (i, j)$ or $t = \text{id}_{K^n}$.

**Proof.** Let $t \in H$ be an element of $H$ where there is an element $\tilde{x} = (\tilde{x}_i)_{i=1}^n \in T_{ij}$ such that $t(\tilde{x}) = \tilde{x}$. By Proposition 2.1, for $t \in H$, there are $\sigma_{i_1 \ldots i_k} \in \{\sigma_{i_1 \ldots i_k}\} \leq k \leq n$, $1 \leq i_1 < \ldots < i_k \leq n$ and $(j_1, \ldots, j_l) \in S_n$ such that
\[
t = (j_1 \ldots j_l) \circ \sigma_{i_1 \ldots i_k}.
\]
Since $(j, j + 1) \circ \sigma_{i,j} \circ (j, j + 1) : U_{ij} \to T_{ij}$ is an isomorphism, and by Lemma 2.3, we have
\[
(j, j + 1) \circ \sigma_{i,j} \circ (j, j + 1) \circ t \circ (j, j + 1) \circ \sigma_{i,j} \circ (j, j + 1) = (i, j) \text{ or } \text{id}_{K^n}.
\]

If $(j, j + 1) \circ \sigma_{i,j} \circ (j, j + 1) \circ t \circ (j, j + 1) \circ \sigma_{i,j} \circ (j, j + 1) = \text{id}_{K^n}$, then $t = \text{id}_{K^n}$. If $(j, j + 1) \circ \sigma_{i,j} \circ (j, j + 1) \circ t \circ (j, j + 1) \circ \sigma_{i,j} \circ (j, j + 1) = (i, j)$, then
\[
t = (j, j + 1) \circ \sigma_{i,j} \circ (j, j + 1) \circ (i, j) \circ (j, j + 1) \circ \sigma_{i,j} \circ (j, j + 1)
= (j, j + 1) \circ \sigma_{i,j} \circ (i, j + 1) \circ \sigma_{i,j} \circ (j, j + 1)
= (j, j + 1) \circ \sigma_{i,j+1} \circ (i, j + 1) \circ (j, j + 1)
= \sigma_{i,j} \circ (i, j).
\]
Thus we have $t = \sigma_{i,j} \circ (i, j)$. \(\Box\)

From Lemma 2.3 and Lemma 2.4, the universal covering map $\mu$ induces a local isomorphism
\[
\mu_*^{[n]} : \text{Blow}_{\cup U} K_{s\mu}^n / H \to \text{Blow}_{E} E_s^n / S_n = E_*^{[n]}.
\]
Here Blow$_A B$ is the blow up of $B$ along $A \subset B$.

**Lemma 2.5.** For every $x \in E_*^{[n]}$, $|\mu_*^{[n]}(x)| = 2$.

**Proof.** For $(x_l)_{l=1}^{n} \in \Delta^n_0 \subset E^n$ with $x_1 = x_2$, there are $n$ elements $y_1, \ldots, y_n$ of $K$ such that $y_1 = y_2$ and $\mu(y_i) = x_i$ for $1 \leq i \leq n$. Then
\[
(\mu^n)^{-1}((x_l)_{l=1}^{n}) = \{y_1, \sigma(y_1)\} \times \ldots \times \{y_n, \sigma(y_n)\}.
\]
Since $H$ is generated by $S_n$ and $\{\sigma_{i,j}\}_{1 \leq i < j \leq n}$, for $(z_l)_{l=1}^{n} \in (\mu^n)^{-1}((x_l)_{l=1}^{n})$, if the number of $i$ with $z_i = y_i$ is even, then
\[
(z_l)_{l=1}^{n} = \{\sigma(y_1), \sigma(y_2), y_3, \ldots, y_n\} \text{ on } K_{s\mu}^n / H,
\]
if the number of $i$ with $z_i = y_i$ is odd, then
\[(z_i)_{i=1}^n = \{ \sigma(y_1), y_2, y_3, \ldots, y_n \} \text{ on } K^*_n/H.\]

Furthermore since $\sigma_i \not\in H$ for $1 \leq i \leq n$,
\[\{ \sigma(y_1), \sigma(y_2), y_3, \ldots, y_n \} \neq \{ \sigma(y_1), y_2, y_3, \ldots, y_n \}, \text{ on } K^*_n/H.\]

Thus for every $x \in E^*_n$, we get $| [\mu^*_n]^{-1}(x) | = 2$. ☐

**Proposition 2.6.** $\mu^*_n : \text{Blow}_{T \cup U} K^*_n/H \to \text{Blow}_{\Delta^2} E^*_n/S_n$ is the universal covering space, and $X \setminus \pi^{-1}(F) \simeq \text{Blow}_{T \cup U} K^*_n/H$. When $n = 2$, we have $X \simeq \text{Blow}_{T \cup U} K^2/H$.

**Proof.** Since $\mu^*_n$ is a local isomorphism, from Lemma 2.5 we get that $\mu^*_n$ is a covering map. Furthermore $\pi : X \setminus \pi^{-1}(F) \to E^*_n$ is the universal covering space of degree 2, $\mu^*_n : \text{Blow}_{T \cup U} K^*_n/H \to \text{Blow}_{\Delta^2} E^*_n/S_n$ is the universal covering space. By the uniqueness of the universal covering space, we have $X \setminus \pi^{-1}(F) \simeq \text{Blow}_{T \cup U} K^*_n/H$. When $n = 2$, since $E^*_2 = E^2$, $K^*_2 = K^2$ and $\text{Blow}_{\Delta^2} E^2/S_2 \simeq E^2$, we have $X \simeq \text{Blow}_{T \cup U} K^2/H$. ☐

**Theorem 2.7.** For $n \geq 2$, let $E$ be an Enriques surface, $E^{[n]}$ the Hilbert scheme of $n$ points of $E$, and $\pi : X \to E^{[n]}$ the universal covering space of $E^{[n]}$. Then there is a birational morphism $\varphi_X : X \to K^n/H$ such that $\varphi_X^{-1}(\Gamma/H) = \pi^{-1}(D)$.

**Proof.** When $n = 2$, this is proved by Proposition 2.6. From here we assume that $n \geq 3$. From Proposition 2.6, we have $X \setminus \pi^{-1}(F) \simeq \text{Blow}_{T \cup U} K^*_n/H$. Since the codimension of $F$ is 2, there is a meromorphic $f$ of $X$ to $K^n/H$ which satisfies the following commutative diagram:

\[
\begin{array}{ccc}
E^{[n]} \setminus F & \xrightarrow{\pi_E} & E^{[n]} \\
\uparrow \pi & & \downarrow \quad p_H \quad & \\
X \setminus \pi^{-1}(F) & \xrightarrow{f} & K^n/H
\end{array}
\]

where $\pi_E : E^{[n]} \to E^{[n]}$ is the Hilbert-Chow morphism, and $p_H : K^n/H \to E^{(n)}$ is the natural projection. For an ample line bundle $L$ on $E^{(n)}$, since the natural projection $\pi^*_H : K^n/H \to E^{(n)}$ is finite, $p_H^* L$ is ample. From the above diagram, we have $\pi^*(\pi^*_E L) |_{X \setminus \pi^{-1}(F)} = f^*(p^*_H L)$. Since $\pi^{-1}(F)$ is an analytic closed subset of codimension 2 in $X$ and $p_H^* L$ is ample, there is a holomorphism $\varphi_X \mid_{X \setminus \pi^{-1}(F)}$ such that $\varphi_X \mid_{X \setminus \pi^{-1}(F)} = f \mid_{X \setminus \pi^{-1}(F)}$. Since $f : X \setminus \pi^{-1}(D) \cong (K^n \setminus \Gamma)/H$, this is a birational morphism. ☐

**3. Proof of Theorem 1.1.** Let $E$ be an Enriques surface, $E^{[n]}$ the Hilbert scheme of $n$ points of $E$, and $\pi : X \to E^{[n]}$ the universal covering space of $E^{[n]}$.

**Proposition 3.1.** For $n \geq 2$, we have $\dim \mathbb{C} H^1(E^{[n]}, \Omega^{2n-1}_{E^{[n]}}) = 0$.

**Proof.** For a smooth projective manifold $S$, we put
\[h^{p,q}(S) := \dim \mathbb{C} H^q(S, \Omega^p_S)\]
and
\[h(S, x, y) := \sum_{p,q} h^{p,q}(S) x^p y^q.\]
By [7, Theorem 2] and [6, page 204], we have the equation (1):

\[
\sum_{n=0}^{\infty} \sum_{p,q} h^{p,q}(E[n]) x^p y^q t^n = \prod_{k=1}^{\infty} \prod_{p,q=0}^{2} \left( \frac{1}{1 - (-1)^{p+q} x^p k^{-1} y^q k^{-1} t^k} \right) (-1)^{p+q} h^{p,q}(E).
\]

Since an Enriques surface $E$ has Hodge numbers $h^{0,0}(E) = h^{2,2}(E) = 1$, $h^{1,0}(E) = h^{0,1}(E) = 0$, $h^{2,0}(E) = h^{0,2}(E) = 0$, and $h^{1,1}(E) = 10$, the equation (1) is

\[
\sum_{n=0}^{\infty} \sum_{p,q} h^{p,q}(E[n]) x^p y^q t^n = \prod_{k=1}^{\infty} \left( \frac{1}{1 - x^k y^k t^k} \right) \left( \frac{1}{1 - x^{k-1} y^{k-1} t^k} \right) \left( \frac{1}{1 - x^{k+1} y^{k+1} t^k} \right).
\]

It follows that

\[
h^{p,q}(E[n]) = 0 \text{ for all } p, q \text{ with } p \neq q.
\]

Thus we have $\dim_{\mathbb{C}} H^1(E[n], \Omega^{2n-1}_{E[n]}) = 0$ for $n \geq 2$. \qed

**Theorem 3.2.** For $n \geq 2$, let $E$ be an Enriques surface, $E[n]$ the Hilbert scheme of $n$ points of $E$, and $X$ the universal covering space of $E[n]$. Then every small deformation of $X$ is induced by that of $E[n]$.

**Proof.** In [4, Proposition 4.2 and Theorems 0.3], Fantechi showed that for a smooth projective surface with $H^0(S, T_S) = 0$ or $H^1(S, O_S) = 0$, and $H^1(S, O_S(-K_S)) = 0$, where $K_S$ is the canonical divisor of $S$, then we get

\[
\dim_{\mathbb{C}} H^1(S, T_S) = \dim_{\mathbb{C}} H^1(S[n], T_{S[n]}).
\]

Since an Enriques surface $E$ satisfies $H^0(E, T_E) = 0$ or $H^1(E, O_E) = 0$, and $H^1(E, O_E(-K_E)) = 0$, we have $\dim_{\mathbb{C}} H^1(E[n], T_{E[n]})) = 10$. Since $K_{E[n]}$ is not trivial and $2K_{E[n]}$ is trivial, we have

\[
T_{E[n]} \simeq \Omega^{2n-1}_{E[n]} \otimes K_{E[n]}.
\]

Therefore we have $\dim_{\mathbb{C}} H^1(E[n], \Omega^{2n-1}_{E[n]} \otimes K_{E[n]})) = 10$. Since $K_X$ is trivial, then we have $T_X \simeq \Omega^{2n-1}_X$. Since $\pi : X \to E[n]$ is the covering map, we have

\[
H^k(X, \Omega^{2n-1}_X) \simeq H^k(E[n], \pi_* \Omega^{2n-1}_X).
\]

Since $X \simeq \text{Spec} O_{E[n]} \oplus O_{E[n]}(K_{E[n]})$ ([11, Theorem 3.1]), we have

\[
H^k(E[n], \pi_* \Omega^{2n-1}_X) \simeq H^k(E[n], \Omega^{2n-1}_{E[n]} \oplus (\Omega^{2n-1}_{E[n]} \otimes K_{E[n]}))).
\]

Thus

\[
H^k(X, \Omega^{2n-1}_X) \simeq H^k(E[n], \Omega^{2n-1}_{E[n]} \oplus (\Omega^{2n-1}_{E[n]} \otimes K_{E[n]}))) \simeq H^k(E[n], \Omega^{2n-1}_{E[n]} \oplus H^k(E[n], \Omega^{2n-1}_{E[n]} \otimes K_{E[n]})))
\]

Combining this with Proposition 3.1, we obtain

\[
\dim_{\mathbb{C}} H^1(X, \Omega^{2n-1}_X) = \dim_{\mathbb{C}} H^1(E[n], \Omega^{2n-1}_{E[n]} \otimes K_{E[n]})) = 10.
\]
Let \( p : \mathcal{Y} \to U \) be the universal family of \( E[n] \) and \( f : \mathcal{X} \to \mathcal{Y} \) be the universal covering space. Then \( q : \mathcal{X} \to U \) is a flat family of \( X \) where \( q := p \circ f \). Then we have a commutative diagram:

\[
\begin{array}{ccc}
T_{U,0} & \xrightarrow{\rho^*} & H^1(\mathcal{Y}_0, T_{\mathcal{Y}_0}) \\
\downarrow \rho_q & & \downarrow \pi \\
H^1(\mathcal{X}_0, T_{\mathcal{X}_0}) & \xrightarrow{\pi^*} & H^1(X, T_X).
\end{array}
\]

Since \( H^1(E[n], T_{E[n]}) \simeq H^1(X, T_X) \) by \( \pi^* \), the vertical arrow \( \tau \) is an isomorphism and

\[
\dim_{\mathbb{C}} H^1(\mathcal{X}_u, T_{\mathcal{X}_u}) = \dim_{\mathbb{C}} H^1(\mathcal{X}_u, \Omega_{\mathcal{X}_u}^{2n-1})
\]

is a constant for some neighborhood of \( 0 \in U \), it follows that \( q : \mathcal{X} \to U \) is the complete family of \( \mathcal{X}_0 = X \), therefore \( q : \mathcal{X} \to U \) is the versal family of \( \mathcal{X}_0 = X \). Thus every small deformation of \( X \) is induced by that of \( E[n] \).

4. Proof of Theorem 1.4. For \( n \geq 2 \), let \( E \) be an Enriques surface, \( E[n] \) the Hilbert scheme of \( n \) points of \( E \), \( \pi : X \to E[n] \) the universal covering space of \( E[n] \), and \( D \) the exceptional divisor of the Hilbert-Chow morphism \( \pi_E : E[n] \to E(n) \). First we show that for an automorphism \( f \) of \( E[n] \), \( f(D) = D \) if and only if \( f^*(\mathcal{O}_{E[n]}(D)) = \mathcal{O}_{E[n]}(D) \) in \( H^2(E[n], \mathbb{C}) \). Next, we show Theorem 1.4.

Proposition 4.1. For any positive integer \( l \in \mathbb{N} \) we have

\[
\dim_{\mathbb{C}} H^0(E[n], \mathcal{O}_{E[n]}(lD)) = 1.
\]

Proof. Since \( D \) is effective, we obtain \( \dim_{\mathbb{C}} H^0(E[n], \mathcal{O}_{E[n]}(lD)) \geq 1 \). Since \( E[n] \setminus D \simeq E(n) \setminus \Delta(n) \), and \( \mathcal{O}_{E[n]}(lD) \simeq \mathcal{O}_{E[n]} \) on \( E[n] \setminus D \), we have

\[
(\pi_E)_*(\mathcal{O}_{E[n]}(lD)) \simeq \mathcal{O}_{E(n)} \text{ on } E(n) \setminus \Delta(n),
\]

where \( \pi_E : E[n] \to E(n) \) is the Hilbert-Chow morphism. Since the codimension of \( \Delta(n) \) is 2, and \( E(n) \) is normal, we have \( \Gamma(E(n) \setminus \Delta(n), \mathcal{O}_{E(n)}) = \Gamma(E(n), \mathcal{O}_{E(n)}) \). Since \( \mathcal{O}_{E(n)}(lD) \) is a local free sheaf, the restriction map:

\[
\Gamma(E[n], \mathcal{O}_{E[n]}(lD))) \to \Gamma(E[n] \setminus D, \mathcal{O}_{E[n]}(lD)))
\]

is injective. Thus we obtain \( \dim_{\mathbb{C}} H^0(E[n], \mathcal{O}_{E[n]}(lD)) = 1 \).

Remark 4.2. Since \( H^1(E[n], \mathcal{O}_{E[n]}) = 0 \), the map \( \text{Pic}(E[n]) \to H^2(E[n], \mathbb{C}) \) is injective. By Proposition 4.1, and \( D \) is effective, we have that for an automorphism \( \varphi \in \text{Aut}(E[n]) \), the condition \( \varphi^*(\mathcal{O}_{E[n]}(D)) = \mathcal{O}_{E[n]}(D) \) in \( H^2(E[n], \mathbb{C}) \) is equivalent to the condition \( \varphi(D) = D \).

Recall that \( \sigma \) is the covering involution of \( \mu : K \to E, \pi \circ \omega : K^n \setminus \Gamma \to E[n] \setminus D \) is the universal covering space, and \( G := \{ g \in \text{Aut}(K^n \setminus \Gamma) : \pi \circ \omega \circ g = \pi \circ \omega \} \) is the covering transformation group of \( \pi \circ \omega \).

Proposition 4.3. Let \( f \) be an automorphism of \( E[n] \setminus D \), and \( g_1, \ldots, g_n \) automorphisms of \( K \) such that \( (\pi \circ \omega) \circ (g_1 \times \cdots \times g_n) = f \circ (\pi \circ \omega) \), where \( (g_1 \times \cdots \times g_n) \) is the automorphism of \( K^n \). Then we have \( g_i = g_1 \) or \( g_i = g_1 \circ \sigma \) for each \( 1 \leq i \leq n \). Moreover \( g_1 \circ \sigma = \sigma \circ g_1 \).
Proof. We show the first assertion by contradiction. Without loss of generality, we may assume that $g_2 \neq g_1$ and $g_2 \neq g_1 \circ \sigma$. Let $h_1$ and $h_2$ be two morphisms of $K$ where $g_i \circ h_i = \operatorname{id}_K$ and $h_i \circ g_i = \operatorname{id}_K$ for $i = 1, 2$. We define two morphisms $H_{1,2}$ and $H_{1,2,\sigma}$ from $K$ to $K^2$ by

$$H_{1,2} : K \ni x \mapsto (h_1(x), h_2(x)) \in K^2$$

$$H_{1,2,\sigma} : K \ni x \mapsto (h_1(x), \sigma \circ h_2(x)) \in K^2.$$ Let $S_\sigma := \{(x, y) : y = \sigma(x)\}$ be the subset of $K^2$. Since $h_1 \neq h_2$ and $h_1 \neq \sigma \circ h_2$, $H_{1,2}(\Delta^2) \cup H_{1,2,\sigma}(S_\sigma)$ do not coincide with $K$. Thus there is $x' \in K$ such that $H_{1,2}(x') \notin \Delta^2$ and $H_{1,2,\sigma}(x') \notin S_\sigma$. For $x' \in K$, we put $x_i := h_i(x') \in K$ for $i = 1, 2$. Then there are some elements $x_3, \ldots, x_n \in K$ such that $(x_1, \ldots, x_n) \in K^n \setminus \Gamma$. We have $g((x_1, \ldots, x_n)) \notin K^n \setminus \Gamma$ by the assumption of $x_1$ and $x_2$. It is contradiction, because $g$ is an automorphism of $K^n \setminus \Gamma$. Thus we have $g_i = g_1$ or $g_i = g_1 \circ \sigma$ for $1 \leq i \leq n$.

We show the second assertion. Since the covering transformation group of $\pi \circ \omega$ is $G$, the liftings of $f$ are given by

$$\{g \circ u : u \in G\} = \{u \circ g : u \in G\}.$$ Thus for $\sigma_1 \circ g$, there is an element $\sigma_{i_1 \ldots i_k} \circ s$ of $G$ where $s \in S_n$ and $t \in \{\sigma_{i_1 \ldots i_k}\}_{1 \leq k \leq n, 1 \leq i_1 < \cdots < i_k \leq n}$ such that $\sigma_1 \circ g = g \circ \sigma_{i_1 \ldots i_k} \circ s$. If we think about the first component of $\sigma_1 \circ g$, we have $s = \operatorname{id}$ and $t = \sigma_1$. Therefore $g \circ \sigma_1 \circ g^{-1} = \sigma_1$, we have $\sigma \circ g_1 = g_1 \circ \sigma$. \(\blacksquare\)

**Theorem 4.4.** For $n \geq 2$, let $E$ be an Enriques surface, $D$ the exceptional divisor of the Hilbert-Chow morphism $\pi_E : E^{[n]} \rightarrow E^{(n)}$. An automorphism $f$ of $E^{[n]}$ is natural if and only if $f(D) = D$, i.e. $f^*(\mathcal{O}_{E^{[n]}}(D)) = \mathcal{O}_{E^{[n]}}(D)$ in $H^2(E^{[n]}, \mathbb{C})$.

**Proof.** Let $f$ be an automorphism of $E^{[n]}$ with $f(D) = D$. Then $f$ induces an automorphism of $E^{[n]} \setminus D$. Since the uniqueness of the universal covering space, there is an automorphism $g$ of $K^n \setminus \Gamma$ such that $\pi \circ \omega \circ g = f \circ \pi \circ \omega$:

$$
\begin{array}{ccc}
E^{[n]} \setminus D & \xrightarrow{f} & E^{[n]} \setminus D \\
\pi \circ \omega \downarrow & & \pi \circ \omega \\
K^n \setminus \Gamma & \xrightarrow{g} & K^n \setminus \Gamma.
\end{array}
$$

Since $\Gamma$ is an analytic set of codimension 2, and $K^n$ is projective, $g$ can be extended to a birational automorphism of $K^n$. By Oguiso [10, Theorem 4.1], $g$ is an automorphism of $K^n$, and there are some automorphisms $g_1, \ldots, g_n \in \operatorname{Aut}(K)$ and $s \in S_n$ such that $g = s \circ g_1 \times \cdots \times g_n$. Since $S_n \subset G$, we can assume that $g = g_1 \times \cdots \times g_n$. By Proposition 4.3, we have $g_i = g_1$ or $g_i \circ \sigma$ for $1 \leq i \leq n$ and $g_1 \circ \sigma = \sigma \circ g_1$. We denote $g_1^{[n]}$ the induced automorphism of $E^{[n]}$ given by $g_1$. Then $g_1^{[n]}|_{E^{[n]} \setminus D} = f|_{E^{[n]} \setminus D}$. Thus $g_1^{[n]} = f$, i.e. $f$ is natural. The other implication is obvious. \(\blacksquare\)

**5. Proof of Theorem 1.7.** Let $E$ be an Enriques surface, $E^{[n]}$ the Hilbert scheme of $n$ points of $E$, and $\pi : X \rightarrow E^{[n]}$ the universal covering space of $E^{[n]}$.

In Proposition 5.2, we shall show that for $n \geq 3$, the covering involution of $\pi : X \rightarrow E^{[n]}$ acts on $H^2(X, \mathbb{C})$ as the identity. In Proposition 5.5, by using Theorem
2.7 and 1.4, we shall show that for \( n \geq 2 \), if an automorphism \( \varphi \) of \( X \) acts on \( H^2(X, \mathbb{C}) \) as the identity, then \( \varphi \) is a lift of a natural automorphism of \( E[\nu] \). In Proposition 5.9, by using Proposition 5.5 and checking the action to \( H^1(X, \Omega_2^{2n-1}) \cong H^{2n-1,1}(X) \), we classify involutions of \( X \) which act on \( H^2(X, \mathbb{C}) \) as the identity. We prove Theorem 1.7 using those results.

**Lemma 5.1.** Let \( X \) be a smooth complex manifold, \( Z \subset X \) a closed submanifold whose codimension is 2, \( \tau : X_{\mathbb{Z}} \rightarrow X \) the blow up of \( X \) along \( Z \), \( E = \tau^{-1}(Z) \) the exceptional divisor, and \( h \) the first Chern class of the line bundle \( O_{X_{\mathbb{Z}}}(E) \). Then \( \tau^* : H^2(X, \mathbb{C}) \rightarrow H^2(X_{\mathbb{Z}}, \mathbb{C}) \) is injective, and

\[
H^2(X_{\mathbb{Z}}, \mathbb{C}) \simeq H^2(X, \mathbb{C}) \oplus Ch.
\]

**Proof.** Let \( U := X \setminus Z \) be an open set of \( X \). Then \( U \) is isomorphic to an open set \( U' = X_{\mathbb{Z}} \setminus E \) of \( X_{\mathbb{Z}} \). As \( \tau \) gives a morphism between the pair \((X_{\mathbb{Z}}, U')\) and the pair \((X, U)\), we have a morphism \( \tau^* \) between the long exact sequence of cohomology relative to these pairs:

\[
\begin{array}{cccc}
H^k(X, U, \mathbb{C}) & \rightarrow & H^k(X, \mathbb{C}) & \rightarrow & H^k(U, \mathbb{C}) & \rightarrow & H^{k+1}(X, U, \mathbb{C}) \\
\downarrow \tau_{X, U} & & \downarrow \tau_X & & \downarrow \tau_U & & \downarrow \tau_{X, U} \\
H^k(X_{\mathbb{Z}}, U', \mathbb{C}) & \rightarrow & H^k(X_{\mathbb{Z}}, \mathbb{C}) & \rightarrow & H^k(U', \mathbb{C}) & \rightarrow & H^{k+1}(X_{\mathbb{Z}}, U', \mathbb{C}).
\end{array}
\]

By Thom isomorphism, the tubular neighborhood Theorem, and Excision theorem, we have

\[
H^q(Z, \mathbb{C}) \simeq H^{q+4}(X, U, \mathbb{C}), \quad \text{and}
\]

\[
H^q(E, \mathbb{C}) \simeq H^{q+2}(X_{\mathbb{Z}}, U', \mathbb{C}).
\]

In particular, we have

\[
H^l(X, U, \mathbb{C}) = 0 \text{ for } l = 0, 1, 2, 3, \quad \text{and}
\]

\[
H^l(X_{\mathbb{Z}}, U', \mathbb{C}) = 0 \text{ for } l = 0, 1.
\]

Thus we have

\[
\begin{array}{cccc}
0 & \rightarrow & H^1(X, \mathbb{C}) & \rightarrow & H^1(U, \mathbb{C}) & \rightarrow & 0 \\
\downarrow \tau_{X, U} & & \downarrow \tau_X & & \downarrow \tau_U & & \downarrow \tau_{X, U} \\
0 & \rightarrow & H^1(X_{\mathbb{Z}}, \mathbb{C}) & \rightarrow & H^1(U', \mathbb{C}) & \rightarrow & H^0(E, \mathbb{C}),
\end{array}
\]

and

\[
\begin{array}{cccc}
0 & \rightarrow & H^2(X, \mathbb{C}) & \rightarrow & H^2(U, \mathbb{C}) & \rightarrow & 0 \\
\downarrow \tau_{X, U} & & \downarrow \tau_X & & \downarrow \tau_U & & \downarrow \tau_{X, U} \\
H^0(E, \mathbb{C}) & \rightarrow & H^2(X_{\mathbb{Z}}, \mathbb{C}) & \rightarrow & H^2(U', \mathbb{C}) & \rightarrow & H^3(X_{\mathbb{Z}}, U', \mathbb{C}).
\end{array}
\]
Since $\tau|_{U'} : U' \xrightarrow{\sim} U$, we have isomorphisms $\tau^*_v : H^k(U, \mathbb{C}) \simeq H^k(U', \mathbb{C})$. Thus we have
\[
\dim_C H^2(X_Z, \mathbb{C}) = \dim_C H^2(X, \mathbb{C}) + 1, \quad \text{and}
\]
\[
\tau^* : H^2(X, \mathbb{C}) \to H^2(X_Z, \mathbb{C}) \text{ is injective},
\]
and therefore we obtain
\[
H^2(X_Z, \mathbb{C}) \simeq H^2(X, \mathbb{C}) \oplus \mathbb{C}.
\]

PROPOSITION 5.2. Suppose $n \geq 3$. For the covering involution $\rho$ of the universal covering space $\pi : X \to E^{[n]}$, the induced map $\rho^* : H^2(X, \mathbb{C}) \to H^2(X, \mathbb{C})$ is the identity.

Proof. Since the codimension of $\pi^{-1}(F)$ is 2, we get
\[
H^2(X, \mathbb{C}) \cong H^2(X \setminus \pi^{-1}(F), \mathbb{C}).
\]
By Proposition 2.6, $X \setminus \pi^{-1}(F) \simeq \text{Blow}_{T \cup U} K^n_{s^*} / H$.

Let $\tau : \text{Blow}_{T \cup U} K^n_{s^*} \to K^n_{s^*}$ be the blow up of $K^n_{s^*}$ along $T \cup U$,
\[
h_{ij} \text{ the first Chern class of the line bundle } \mathcal{O}_{\text{Blow}_{T \cup U} K^n_{s^*}}(\tau^{-1}(U_{ij})),
\]
and
\[
k_{ij} \text{ the first Chern class of the line bundle } \mathcal{O}_{\text{Blow}_{T \cup U} K^n_{s^*}}(\tau^{-1}(T_{ij})).
\]
By Lemma 5.1, we have
\[
H^2(\text{Blow}_{T \cup U} K^n_{s^*}, \mathbb{C}) \cong H^2(K^n, \mathbb{C}) \oplus \left( \bigoplus_{1 \leq i < j \leq n} \mathbb{C} h_{ij} \right) \oplus \left( \bigoplus_{1 \leq i < j \leq n} \mathbb{C} k_{ij} \right).
\]
Since $n \geq 3$, there is an isomorphism
\[
(j, j+1) \circ \sigma_{ij} \circ (j, j+1) : U_{ij} \xrightarrow{\sim} T_{ij}.
\]
Thus we have $\dim_C H^2(\text{Blow}_{T \cup U} K^n_{s^*} / H, \mathbb{C}) = 11$, i.e. $\dim_C H^2(X, \mathbb{C}) = 11$. Since $H^2(E^{[n]}, \mathbb{C}) = H^2(X, \mathbb{C}) \rho^*$, $\rho^*$ is the identity.

PROPOSITION 5.3. For any positive integer $l \in \mathbb{N}$ we have
\[
\dim_C H^0(X, \mathcal{O}_X(l \pi^*(D))) = 1.
\]

Proof. From Propositin 4.1 we obtain $\dim_C H^0(X, \mathcal{O}_X(l \pi^*(D))) \geq 1$. Like the proof of Proposition 4.1, we have $\dim_C H^0(X, \mathcal{O}_X(l \pi^*(D))) = 1$. from Theorem 2.7.

REMARK 5.4. Since $H^1(X, \mathcal{O}_X) = 0$, the map $\text{Pic}(X) \to H^2(X, \mathbb{C})$ is injective. By Proposition 5.3, and $\pi^{-1}(D)$ is effective, for an automorphism $\varphi \in \text{Aut}(X)$, the condition $\varphi^*(\mathcal{O}_X(\pi^*D)) = \mathcal{O}_X(\pi^*D)$ in $H^2(X, \mathbb{C})$ is equivalent to the condition $\varphi(\pi^{-1}(D)) = \pi^{-1}(D)$.
Recall that \( \omega : K^n \setminus \Gamma \to X \setminus \pi^{-1}(D) \) is the universal covering space.

**Proposition 5.5.** For \( n \geq 2 \), let \( E \) be an Enriques surface, \( E^{[n]} \) the Hilbert scheme of \( n \) points of \( E \), \( \pi : X \to E^{[n]} \) the universal covering space of \( E^{[n]} \), and \( D \) the exceptional divisor of the Hilbert-Chow morphism \( \pi_E : E^{[n]} \to E^{(n)} \).

For an automorphism \( \varphi \) of \( X \) with \( f^*(\Omega^*(\pi^*D)) = \Omega^*(\pi^*D) \) in \( H^2(X, \mathbb{C}) \), there is an automorphism \( \phi \) of \( E \) such that \( \varphi \) is a lift of \( \phi^{[n]} \) where \( \phi^{[n]} \) is the natural automorphism of \( E^{[n]} \) induced by \( \phi \). Furthermore, if the order of \( \varphi \) is 2, then the order of \( \phi \) is at most 2.

**Proof.** Let \( \varphi \) be an automorphism of \( X \) with \( \varphi^*(\Omega^*(\pi^*D)) = \Omega^*(\pi^*D) \) in \( H^2(X, \mathbb{C}) \). By Remark 5.4, \( \varphi|_{X \setminus \pi^{-1}(D)} \) is automorphism of \( X \setminus \pi^{-1}(D) \). By the uniqueness of the universal covering space, there is an automorphism \( g \) of \( K^n \setminus \Gamma \) such that \( \varphi \circ \omega = \omega \circ g \):

\[
\begin{array}{c}
X \setminus \pi^{-1}(D) 
\xrightarrow{\varphi} X \setminus \pi^{-1}(D) \\
\omega \uparrow & \uparrow \omega \\
K^n \setminus \Gamma 
\xrightarrow{g} K^n \setminus \Gamma.
\end{array}
\]

Like the proof of Proposition 4.3, we can assume that there are some automorphisms \( g_i \) of \( K \) such that \( g = g_1 \times \cdots \times g_n \), for each \( 1 \leq i \leq n \), \( g_i = g_1 \) or \( g_i = g_1 \circ \sigma \), and \( g_1 \circ \sigma = \sigma \circ g_1 \). Since \( g_1 \circ \sigma = \sigma \circ g_1 \), \( g_1 \) induces an automorphism \( \phi \) of \( E \). Let \( \phi^{[n]} \) be the natural automorphism of \( E^{[n]} \) induced by \( \phi \). Then we have \( \pi \circ \omega \circ g = \phi^{[n]} \circ \pi \circ \omega \):

\[
\begin{array}{c}
E^{[n]} \setminus D 
\xrightarrow{\phi^{[n]}} E^{[n]} \setminus D \\
\pi \circ \omega \uparrow & \uparrow \pi \circ \omega \\
K^n \setminus \Gamma 
\xrightarrow{g} K^n \setminus \Gamma.
\end{array}
\]

Since \( \varphi \circ \omega = \omega \circ g \), we have \( \phi^{[n]} \circ \pi = \pi \circ \varphi \):

\[
\begin{array}{c}
E^{[n]} \setminus D 
\xrightarrow{\phi^{[n]}} E^{[n]} \setminus D \\
\pi \uparrow & \uparrow \pi \\
X \setminus \pi^{-1}(D) 
\xrightarrow{\varphi} X \setminus \pi^{-1}(D).
\end{array}
\]

We assume that the order of \( \varphi \) is 2. Since \( \omega = \varphi^2 \circ \omega = \omega \circ g^2 \), we get \( g^2 \in H \). Now \( g = g_1 \times \cdots \times g_n \), for each \( 1 \leq i \leq n \), \( g_i = g_1 \) or \( g_i = g_1 \circ \sigma \), and \( g_1 \circ \sigma = \sigma \circ g_1 \). Thus we have \( g_1^2 = \text{id}_K \) or \( \sigma \). By [9, Lemma 1.2], we have \( g_1^2 = \text{id}_K \). Therefore the order of \( \phi \) is at most 2.

**Definition 5.6.** Let \( S \) be a smooth surface. An automorphism \( \varphi \) of \( S \) is numerically trivial if the induced automorphism \( \varphi^* \) of the cohomology ring over \( \mathbb{Q} \), \( H^*(S, \mathbb{Q}) \) is the identity.

We suppose that an Enriques surface \( E \) has numerically trivial involutions. By [9, Proposition 1.1], there is just one numerically trivial involution of \( E \), denoted \( \nu \). Now \( \nu \) is a lifting of \( \nu \), one acts on \( H^0(K, \Omega^2_K) \) as the identity, and another acts on \( H^0(K, \Omega^2_K) \) as \( -\text{id}_{H^0(K, \Omega^2_K)} \), we denote by \( \nu^+ \) and \( \nu^- \), respectively. Then they satisfies \( \nu^+ = \nu^- \circ \sigma \).
Let \( \nu^{[n]} \) be the automorphism of \( E^{[n]} \) which is induced by \( \nu \). For \( \nu^{[n]} \), there are just two automorphisms of \( X \) which are liftings of \( \nu^{[n]} \), denoted \( \varsigma \) and \( \varsigma' \), respectively:

\[
\begin{array}{c}
E^{[n]} \xrightarrow{\nu^{[n]}} E^{[n]} \\
\pi \Downarrow \quad \Downarrow \pi \\
X \xrightarrow{\varsigma} X \quad X \xrightarrow{\varsigma'} X.
\end{array}
\]

Then they satisfies \( \varsigma = \varsigma' \circ \sigma \). As the proof of Proposition 5.5, each order of \( \varsigma \) and \( \varsigma' \) is 2. From here, we classify involutions acting on \( H^2(X, \mathbb{C}) \) as the identity by checking the action to \( H^{2n-1,1}(X, \mathbb{C}) \).

**Lemma 5.7.** \( \dim_{\mathbb{C}} H^{2n-1,1}(K^n/H, \mathbb{C}) = 10 \).

**Proof.** Let \( \sigma \) be the covering involution of \( \mu : K \to E \). Put

\[
H_{\pm}^{p,q}(K, \mathbb{C}) := \{ \alpha \in H^{p,q}(K, \mathbb{C}) : \sigma^*(\alpha) = \pm \alpha \}
\]

and

\[
h_{\pm}^{p,q}(K) := \dim_{\mathbb{C}} H_{\pm}^{p,q}(K, \mathbb{C}).
\]

Since \( K \) is a K3 surface, we have

\[
h^{0,0}(K) = 1, \ h^{1,0}(K) = 0, \ h^{2,0}(K) = 1, \ h^{1,1}(K) = 20,
\]

\[
h_+^{0,0}(K) = 1, \ h_+^{1,0}(K) = 0, \ h_+^{2,0}(K) = 0 \ h_+^{1,1}(K) = 10,
\]

\[
h_-^{0,0}(K) = 0, \ h_-^{1,0}(K) = 0, \ h_-^{2,0}(K) = 1, \ and \ h_-^{2,0}(K) = 10.
\]

Let

\[
\Lambda := \{(s_1, \ldots, s_n, t_1, \ldots, t_n) \in \mathbb{Z}_{\geq 0}^n : \sum_{i=1}^{n} s_i = 2n - 1, \ \sum_{j=1}^{n} t_j = 1\}.
\]

From the Künneth Theorem, we have

\[
H^{2n-1,1}(K^n, \mathbb{C}) \cong \bigoplus_{(s_1, \ldots, s_n, t_1, \ldots, t_n) \in \Lambda} \left( \bigotimes_{i=1}^{n} H^{s_i, t_i}(K, \mathbb{C}) \right).
\]

We take a base \( \alpha \) of \( H^{2,0}(K, \mathbb{C}) \) and a base \( \{ \beta_i \}_{i=1}^{20} \) of \( H^{1,1}(K, \mathbb{C}) \) such that \( \{ \beta_i \}_{i=1}^{10} \) is a base of \( H_+^{1,1}(K, \mathbb{C}) \) and \( \{ \beta_i \}_{i=11}^{20} \) is a base of \( H_-^{1,1}(K, \mathbb{C}) \). Let

\[
\tilde{\beta}_i := \bigotimes_{j=1}^{n} \epsilon_j
\]

where \( \epsilon_j = \alpha \) for \( j \neq i \) and \( \epsilon_j = \beta_i \) for \( j = i \), and

\[
\gamma_i := \bigoplus_{j=1}^{n} \tilde{\beta}_j.
\]

Then \( \{ \gamma_i \}_{i=1}^{20} \) is a base of \( H^{2n-1,1}(K^n, \mathbb{C}) \). Since \( \sigma^* \alpha = -\alpha, \ \sigma^* \beta_i = -\beta_i \) for \( 1 \leq i \leq 10 \), and \( \sigma^* \beta_i = \beta_i \) for \( 11 \leq i \leq 20 \), we obtain

\[
\sigma_{ij}^* \gamma_i = \gamma_i \text{ for } 1 \leq i \leq 10, \text{ and}
\]

\[
\sigma_{ij}^* \gamma_i = 0 \text{ for } 11 \leq i \leq 20.
\]
σ^1_{ij} γ_i = -γ_i for 11 ≤ i ≤ 20.

Since H^{2n-1,1}(K^n/H, C) ∼ H^{2n-1,1}(K^n, C)^H and H = ⟨S_n, {σ_{ij}}⟩_{1≤i<j≤n}, we obtain

\[ H^{2n-1,1}(K^n/H, C) = \bigoplus_{i=1}^{10} \mathbb{C}γ_i. \]

Thus we get dimC H^{2n-1,1}(K^n/H, C) = 10. □

**Remark 5.8.** By Theorem 2.7, there is a resolution φ_X : X → K^n/H. Then φ^*_X : H^{p,q}(K^n/H, C) → H^{p,q}(X, C) is an injective (see [13]). By Lemma 5.7, φ^*_X : H^{2n-1,1}(K^n/H, C) → H^{2n-1,1}(X, C) is an isomorphism.

Recall that π ◦ ω : K^n \ Γ → E[n] \ D is the universal covering space.

**Proposition 5.9.** We suppose that E has a numerically trivial involution, denoted υ. Let v^n be the natural automorphism of E[n] which is induced by υ. Since the degree of π : X → E[n] is 2, there are just two involutions ζ and ζ' of X which are lifts of v^n. Then ζ and ζ' do not act on H^{2n-1,1}(X, C) as −id_{H^{2n-1,1}(X, C)}.

**Proof.** Since v^n(D) = D, v^n|_{E[n] \ D} is an automorphism of E[n] \ D. By the uniqueness of the universal covering space, there is an automorphism g of K^n \ Γ such that v^n ◦ π ◦ ω = π ◦ ω ◦ g:

\[ \begin{array}{ccc}
E[n] \ D & x_n \rightarrow & E[n] \ D \\
\pi ◦ ω & \uparrow & \pi ◦ ω \\
K^n \ Γ & \xrightarrow{g} & K^n \ Γ.
\end{array} \]

By Proposition 4.3, there are some automorphisms g_i of K such that g = g_1 × ··· × g_n for each 1 ≤ i ≤ n, g_i = g_1 or g_i = g_1 ◦ σ, and g_1 ◦ σ = σ ◦ g_1. By Theorem 2.7, we get K^n \ Γ/H ∼ X \ π^{-1}(D). Put

\[ v_{+,even} := u_1 × ··· × u_n \]

where

\[ u_i = v_+ or u_i = v_- and the number of i with u_i = v_+ is even. \]

v_{+,even} is an automorphism of K^n and induces an automorphism v_{+,even} of K^n \ Γ/H. We define automorphisms v_{+,odd}, v_{-,even}, and v_{-,odd} of K^n \ Γ/H in the same way. Since σ_{ij} ∈ H for 1 ≤ i < j ≤ n, and v_+ = v_- ◦ σ, if n is odd,

\[ v_{+,odd} = v_{-,even}, \ v_{-,even} = v_{-,odd}, \ and \ v_{+,odd} ≠ v_{+,even}, \]

and if n is even,

\[ v_{+,odd} = v_{-,odd}, \ v_{-,even} = v_{-,even}, \ and \ v_{+,odd} ≠ v_{+,even}. \]

Since v^n ◦ π_E = π_E ◦ v^n and K^n \ Γ/H ∼ X \ π^{-1}(D), we have v^n ◦ π = π ◦ v_{+,odd} and v^n ◦ π = π ◦ v_{+,even} where π_E : E[n] → E(n) is the Hilbert-Chow morphism, and v^n is the automorphism of E(n) induced by υ. Since the degree of π is 2, we have
\[\{\varsigma, \varsigma'\} = \{\nu_{+, odd}, \nu_{+, even}\}.\] By \cite[page 386-389]{H}, there is an element \(\alpha_{\pm} \in H^{1,1}_1(K, \mathbb{C})\) such that \(\nu_{+}^*(\alpha_{\pm}) = \pm \alpha_{\pm}\). We fix a basis \(\alpha\) of \(H^{0,2}(K, \mathbb{C})\), and let

\[\tilde{\alpha}_{\pm} := \bigotimes_{j=1}^n \epsilon_j\]

where \(\epsilon_j = \alpha\) for \(j \neq i\) and \(\epsilon_j = \alpha_{\pm}\) for \(j = i\), and

\[\tilde{\alpha}_{\pm} := \bigoplus_{j=1}^n \tilde{\alpha}_{\pm}\]

Since \(K^n \setminus \Gamma/H \simeq X \setminus \pi^{-1}(D)\), and by the definition of \(\nu_{+, odd}\) and \(\nu_{+, even}\), we have

\[\nu_{+, odd}^*(\varphi_X^*(\alpha_{\pm})) = \varphi_X^*(\tilde{\alpha}_{\pm})\] and \(\nu_{+, even}^*(\varphi_X^*(\tilde{\alpha}_{\pm})) = \varphi_X^*(\tilde{\alpha}_{\pm}).\]

Thus \(\varsigma\) and \(\varsigma'\) do not act on \(H^{2n-1,1}(X, \mathbb{C})\) as \(-\text{id}_{H^{2n-1,1}(X, \mathbb{C})}\). \(\square\)

**Definition 5.10.** For \(n \geq 1\), let \(E\) be an Enriques surface, \(E^{[n]}\) the Hilbert scheme of \(n\) points of \(E\), and \(X\) the universal covering space of \(E^{[n]}\). A variety \(Y\) is called an Enriques surface type quotient of \(X\) if there is an Enriques surface \(E'\) and a free involution \(\tau\) of \(X\) such that \(Y \simeq E'^{[n]}\) and \(E'^{[n]} \simeq X/\langle \tau \rangle\). Here we call two Enriques surface type quotients of \(X\) distinct if they are not isomorphic to each other.

**Theorem 5.11.** For \(n \geq 3\), let \(E\) be an Enriques surface, \(E^{[n]}\) the Hilbert scheme of \(n\) points of \(E\), and \(X\) the universal covering space of \(E^{[n]}\). Then the number of distinct Enriques surface type quotients of \(X\) is one.

**Proof.** Let \(\rho\) be the covering involution of \(\pi : X \to E^{[n]}\) for \(n \geq 3\). Since for \(n \geq 3\) \(\dim_{\mathbb{C}}H^{2}(E^{[n]}, \mathbb{C}) = \dim_{\mathbb{C}}H^{2}(X, \mathbb{C}) = 11\), \(\dim_{\mathbb{C}}H^{2n-1,1}(E^{[n]}, \mathbb{C}) = 0\), and \(\dim_{\mathbb{C}}H^{2n-1,1}(X, \mathbb{C}) = 10\), we obtain that \(\rho^*\) acts on \(H^{2}(X, \mathbb{C})\) as the identity, and \(H^{2n-1,1}(X, \mathbb{C})\) as \(-\text{id}_{H^{2n-1,1}(X, \mathbb{C})}\).

Let \(\varphi\) be an involution of \(X\), which acts on \(H^{2}(X, \mathbb{C})\) as the identity and on \(H^{2n-1,1}(X, \mathbb{C})\) as \(-\text{id}_{H^{2n-1,1}(X, \mathbb{C})}\). By Proposition 5.5, for \(\varphi\), there is an automorphism \(\phi\) of \(E\) such that \(\varphi\) is a lift of \(\phi^{[n]}\) where \(\phi^{[n]}\) is the natural automorphism of \(E^{[n]}\) induced by \(\phi\). Furthermore since the order of \(\phi\) is at most 2, the order of \(\varphi\) is 2. Since \(\phi^{[n]} \circ \pi = \pi \circ \varphi\), \(\phi^{[n]}\) acts on \(H^{2}(E^{[n]}, \mathbb{C})\) as the identity. Thus \(\phi^*\) acts on \(H^{2}(E, \mathbb{C})\) as the identity. If \(E\) does not have numerically trivial automorphisms, then \(\phi = \text{id}_E\). Thus \(\varphi = \rho\).

We assume that \(\phi\) does not act on \(H^{2n-1,1}(X, \mathbb{C})\) as \(-\text{id}_{H^{2n-1,1}(X, \mathbb{C})}\). This is a contradiction. Thus \(\phi = \text{id}_E\), and we get \(\varphi = \rho\). This proves the theorem. \(\square\)

**Theorem 5.12.** For \(n \geq 2\), let \(\pi : X \to E^{[n]}\) be the universal covering space. For any automorphism \(\varphi\) of \(X\), if \(\varphi^*\) is acts on \(H^{*}(X, \mathbb{C}) := \bigoplus_{i=0}^{2n} H^{i}(X, \mathbb{C})\) as the identity, then \(\varphi = \text{id}_X\).

**Proof.** By Proposition 5.5, for \(\varphi\), there is an automorphism \(\phi\) of \(E\) such that \(\varphi\) is a lift of \(\phi^{[n]}\) where \(\phi^{[n]}\) is the natural automorphism of \(E^{[n]}\) induced by \(\phi\). Since \(\varphi^*\) acts on \(H^{2}(X, \mathbb{C})\) as the identity, \(\phi^*\) acts on \(H^{2}(E, \mathbb{C})\) as the identity. From \cite[page 386-389]{H} the order of \(\phi\) is at most 4.
If the order of $\phi$ is 2, by Proposition 5.9 $\varphi$ does not act on $H^{2n-1,1}(X, \mathbb{C})$ as the identity. This is a contradiction.

If the order of $\phi$ is 4, there is an element $\alpha' \in H^{1,1}_-(K, \mathbb{C})$ such that $g_1^*(\alpha'_\pm) = \pm \sqrt{-1}\alpha'$ from [9, page 390-391]. Like the proof of Proposition 5.9, $\varphi$ does not act on $H^{2n-1,1}(X, \mathbb{C})$ as the identity. This is a contradiction. Thus we have $\phi = \text{id}_E$ and $\varphi \in \{\text{id}_X, \rho\}$. Since $\rho$ does not act on $H^{2n-1,1}(X, \mathbb{C})$ as the identity, we have $\varphi = \text{id}_X$. □

**Corollary 5.13.** For $n \geq 2$, let $\pi : X \to E^{[n]}$ be the universal covering space. For any two automorphisms $f$ and $g$ of $X$, if $f^* = g^*$ on $H^*(X, \mathbb{C})$, then $f = g$.

**Theorem 5.14.** For $n \geq 3$, let $E$ be an Enriques surfaces, $E^{[n]}$ the Hilbert scheme of $n$ points of $E$, $\pi : X \to E^{[n]}$ the universal covering space. Then there is an exact sequence:

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \text{Aut}(X) \to \text{Aut}(E^{[n]}) \to 0.$$

**Proof.** Let $f$ be an automorphism $f$ of $X$. We put $g = f^{-1} \circ \rho \circ f$. Since for $n \geq 3$ $\rho^*$ acts on $H^2(X, \mathbb{C})$ as the identity and on $H^{2n-1,1}(X)$ as $-\text{id}_{H^{2n-1,1}(X)}$, we get that $g^* = \rho^*$ as automorphisms of $H^2(X, \mathbb{C}) \oplus H^{2n-1,1}(X)$. Like the proof of Theorem 5.12, we have $g = \rho$, i.e. $f \circ \rho = \rho \circ f$. Thus $f$ induces an automorphism of $E^{[n]}$, and we have an exact sequence:

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \text{Aut}(X) \to \text{Aut}(E^{[n]}) \to 0.$$

□

**6. Appendix A.** We compute the Hodge number of the universal covering space $X$ of $E^{[2]}$. Let $\sigma$ be the covering involution of $\mu : K \to E$, and $\tau : \text{Blow}_{\Delta \cup T} K^2 \to K^2$ the natural map, where $T = \{(x, y) \in K^2 : y = \sigma(x)\}$ and $\Delta = \{(x, x) \in K^2\}$. By Proposition 2.6, we have

$$X \simeq \text{Blow}_{\Delta \cup T} K^2 / H.$$  

We put

$$D_\Delta := \tau^{-1}(\Delta) \quad \text{and} \quad D_T := \tau^{-1}(T).$$

For two inclusions

$$j_{D_\Delta} : D_\Delta \hookrightarrow \text{Blow}_{\Delta \cup T} K^2, \quad \text{and}$$

$$j_{D_T} : D_T \hookrightarrow \text{Blow}_{\Delta \cup T} K^2,$$

let $j_{*D_\Delta}$ be the Gysin morphism

$$j_{*D_\Delta} : H^p(D_\Delta, \mathbb{C}) \to H^{p+2}(\text{Blow}_{\Delta \cup T} K^2, \mathbb{C}),$$

and

$$j_{*D_T} : H^p(D_T, \mathbb{C}) \to H^{p+2}(\text{Blow}_{\Delta \cup T} K^2, \mathbb{C}),$$

the Gysin morphism
From \(K\sigma \in H\beta \in H^{k-2}(\Delta, \mathbb{C}) \oplus H^{k-2}(T, \mathbb{C}) \oplus H^{k}(\text{Blow}_{\Delta UT}K^{2}, \mathbb{C}).\) From [14, Theorem 7.31], we have isomorphisms of Hodge structures by \(\psi:\)

\[
H^{k}(K^{2}, \mathbb{C}) \oplus H^{k-2}(\Delta, \mathbb{C}) \oplus H^{k-2}(T, \mathbb{C}) \simeq H^{k}(\text{Blow}_{\Delta UT}K^{2}, \mathbb{C}).
\]

Furthermore, for automorphism \(f\) of \(K\), let \(\bar{f}\) (resp. \(\bar{f}_{\sigma}\)) be the automorphism of \(\text{Blow}_{\Delta UT}K^{2}\) which is induced by \(f \times f\) (resp. \(f \times (f \circ \sigma)\)). \(f_{\Delta}\) is the automorphism of \(\Delta\) which is induced by \(f \times f\), \(f_{T}\) the automorphism of \(T\) which is induced by \(f \times f\), and \(\bar{f}\) the isomorphism from \(T\) to \(\Delta\) which is induced by \(f \times (f \circ \sigma)\). For \(\alpha \in H^{*}(K^{2}, \mathbb{C}), \beta \in H^{*}(\Delta, \mathbb{C}), \) and \(\gamma \in H^{*}(T, \mathbb{C})\), we obtain

\[
\bar{f}^{*}(\tau^{*}\alpha) = \tau^{*}((f \times f)^{*}\alpha),
\]

\[
\bar{f}^{*}(j_{*}D_{\Delta} \circ \tau|_{D_{\Delta}} \beta) = j_{*}D_{\Delta} \circ \tau|_{D_{\Delta}}(f_{\Delta}^{*}\beta),
\]

\[
\bar{f}^{*}(j_{*}D_{T} \circ \tau|_{D_{T}} \gamma) = j_{*}D_{T} \circ \tau|_{D_{T}}(f_{T}^{*}\gamma),
\]

\[
\bar{f}_{\sigma}^{*}(\tau^{*}\alpha) = \tau^{*}((f \times (f \circ \sigma)^{*}\alpha),
\]

\[
\bar{f}_{\sigma}^{*}(j_{*}D_{\Delta} \circ \tau|_{D_{\Delta}} \beta) = j_{*}D_{T} \circ \tau|_{D_{T}}(f_{\sigma}^{*}\beta),
\]
in \(H^{*}(\text{Blow}_{\Delta UT}K^{2}, \mathbb{C}).\)

**Theorem 6.1.** For the universal covering space \(\pi: X \rightarrow E^{[2]}\), we have \(h^{0,0}(X) = 1, h^{1,0}(X) = 0, h^{2,0}(X) = 0, h^{1,1}(X) = 12, h^{3,0}(X) = 0, h^{2,1}(X) = 0, h^{4,0}(X) = 1, h^{3,1}(X) = 10, \) and \(h^{2,2}(X) = 131.\)

**Proof.** Since \(X \simeq \text{Blow}_{\Delta UT}K^{2}/H\), we have

\[
h^{p,q}(X) = \dim_{\mathbb{C}}\{\alpha \in H^{p,q}(\text{Blow}_{\Delta UT}K^{2}, \mathbb{C}) : h^{*}\alpha = \alpha \text{ for } h \in H\}.
\]

Let \(\sigma\) be the covering involution of \(\mu: K \rightarrow E\). We put

\[
H^{p,q}_{\pm}(K, \mathbb{C}) := \{\alpha \in H^{p,q}(K, \mathbb{C}) : \sigma^{*}(\alpha) = \pm \alpha\}
\]

and

\[
h^{p,q}_{\pm}(K) := \dim_{\mathbb{C}}H^{p,q}_{\pm}(K, \mathbb{C}).
\]

From \(E = K/\langle \sigma \rangle\), we have

\[
H^{p,q}(E, \mathbb{C}) \simeq H^{p,q}_{+}(K, \mathbb{C}).
\]

Since \(K\) is a K3 surface, we have

\[
h^{0,0}(K) = 1, h^{1,0}(K) = 0, h^{2,0}(K) = 1, \text{ and } h^{1,1}(K) = 20, \text{ and}
\]

\[
h^{0,0}_{+}(K) = 1, h^{1,0}_{+}(K) = 0, h^{2,0}_{+}(K) = 0, \text{ and } h^{1,1}_{+}(K) = 10, \text{ and}
\]

\[
h^{0,0}_{-}(K) = 0, h^{1,0}_{-}(K) = 0, h^{2,0}_{-}(K) = 1, \text{ and } h^{2,0}_{-}(K) = 10.
\]
Recall that $H$ is generated by $S_2$ and $\sigma_{1,2}$. Since $\sigma \times \sigma(\Delta) = \Delta$ and $\sigma \times \sigma(T) = T$, from $E = K/\langle \sigma \rangle$ we have $\Delta/H \simeq E$ and $T/H \simeq E$. Thus we have

$$h^{0,0}(\Delta/H) = 1, h^{1,0}(\Delta/H) = 0, h^{2,0}(\Delta/H) = 0, h^{1,1}(\Delta/H) = 10,$$

$$h^{0,0}(T/H) = 1, h^{1,0}(T/H) = 0, h^{2,0}(T/H) = 0, \text{ and } h^{1,1}(T/H) = 10.$$

From the Künneth Theorem, we have

$$H^{p,q}(K^2, \mathbb{C}) \simeq \bigoplus_{s+u=p,t+v=q} H^{s,t}(K, \mathbb{C}) \otimes H^{u,v}(K, \mathbb{C}),$$

and

$$H^{p,q}(K^2/H, \mathbb{C}) \simeq \{ \alpha \in H^{p,q}(K^2, \mathbb{C}) : s^\ast(\alpha) = \alpha \text{ for } s \in S_2 \text{ and } \sigma_{1,2}^\ast(\alpha) = \alpha \}.$$ 

Thus we obtain

$$h^{0,0}(K^2/H) = 1, h^{1,0}(K^2/H) = 0, h^{2,0}(K^2/H) = 0, h^{1,1}(K^2/H) = 10,$$

$$h^{3,0}(K^2/H) = 0, h^{2,1}(K^2/H) = 0, h^{4,0}(K^2/H) = 1,$$

$$h^{3,1}(K^2/H) = 10, \text{ and } h^{2,2}(K^2/H) = 111.$$

We fix a basis $\beta$ of $H^{2,0}(K, \mathbb{C})$ and a basis $\{\gamma_i\}_{i=1}^{10}$ of $H^{1,1}(K, \mathbb{C})$, then we have

$$H^{3,1}(K^2/H, \mathbb{C}) \simeq \bigoplus_{i=1}^{10} \mathbb{C}(\beta \otimes \gamma_i + \gamma_i \otimes \beta).$$

By the above equation, we have

$$h^{0,0}(\text{Blow}_{\Delta \cup T} K^2/H) = 1, h^{1,0}(\text{Blow}_{\Delta \cup T} K^2/H) = 0,$$

$$h^{2,0}(\text{Blow}_{\Delta \cup T} K^2/H) = 0, h^{1,1}(\text{Blow}_{\Delta \cup T} K^2/H) = 12,$$

$$h^{3,0}(\text{Blow}_{\Delta \cup T} K^2/H) = 0, h^{2,1}(\text{Blow}_{\Delta \cup T} K^2/H) = 0,$$

$$h^{4,0}(\text{Blow}_{\Delta \cup T} K^2/H) = 1, h^{3,1}(\text{Blow}_{\Delta \cup T} K^2/H) = 10, \text{ and }$$

$$h^{2,2}(\text{Blow}_{\Delta \cup T} K^2/H) = 131.$$ 

Thus we obtain $h^{0,0}(X) = 1, h^{1,0}(X) = 0, h^{2,0}(X) = 0, h^{1,1}(X) = 12, h^{3,0}(X) = 0, h^{2,1}(X) = 0, h^{4,0}(X) = 1, h^{3,1}(X) = 10, \text{ and } h^{2,2}(X) = 131.$
7. Appendix B. Now we show that the conjecture in [2, Conjecture 1] is not established for $Y$ an Enriques surface and $L = \Omega^2_Y$.

Let $Y$ be a smooth compact Kähler surface. Recall that $Y^{[n]}$ is the Hilbert scheme of $n$ points of $Y$, $\pi_Y : Y^{[n]} \to Y^{(n)}$ the Hilbert-Chow morphism, and $p_Y : Y^n \to Y^{(n)}$ the natural projection. For a line bundle $L$ on $Y$, there is a unique line bundle $L$ on $Y^{(n)}$ such that $p^*_Y L = \bigotimes_{i=1}^n p^*_i L$. By using pull back we have the natural map

$$\text{Pic}(Y) \to \text{Pic}(Y^{[n]}), \quad L \mapsto L_n := \pi^*_Y L,$$

we put

$$h^{p,q}(Y^{[n]}, L_n) := \dim_{\mathbb{C}} H^q(Y^{[n]}, \Omega^p_{Y^{[n]}} \otimes L_n),$$

$$h^{p,q}(Y, L) := \dim_{\mathbb{C}} H^q(Y, \Omega^p_Y \otimes L),$$

$$A := \sum_{n,p,q=0}^\infty h^{p,q}(Y^{[n]}, L_n) x^p y^q t^n, \quad \text{and}$$

$$B := \prod_{k=1}^2 \prod_{p,q=0}^2 \left( \frac{1}{1 - (-1)^{p+q} x^p y^q k^k} \right) (-1)^{p+q} h^{p,q}(Y, L).$$

Then in [2, Conjecture 1] S. Boissière conjectured that

$$A = B.$$

For $Y$ an Enriques surface and $L = \Omega^2_Y$, as in the proof on Theorem 3.2 and the Serre duality, we have

$$h^{2n-1,1}(Y^{[n]}, (\Omega^2_Y)_n) = \dim_{\mathbb{C}} H^1(Y^{[n]}, \Omega^{2n-1}_{Y^{[n]}} \otimes \Omega^{2n}_{Y^{[n]}})$$

$$= \dim_{\mathbb{C}} H^1(Y^{[n]}, T_{Y^{[n]}})$$

$$= 10.$$

for $n \geq 2$. It follows that the coefficient of $x^3 y t^2$ of $A$ is 10.

We show that the coefficient of $x^3 y t^2$ of $B$ is not 10.

$$h^{0,0}(Y, \Omega^2_Y) = \dim_{\mathbb{C}} H^0(Y, \mathcal{O}_Y \otimes \Omega^2_Y) = \dim_{\mathbb{C}} H^0(Y, \Omega^2_Y) = 0.$$

$$h^{0,1}(Y, \Omega^2_Y) = \dim_{\mathbb{C}} H^1(Y, \mathcal{O}_Y \otimes \Omega^2_Y) = \dim_{\mathbb{C}} H^1(Y, \Omega^2_Y) = 0.$$

$$h^{0,2}(Y, \Omega^2_Y) = \dim_{\mathbb{C}} H^2(Y, \mathcal{O}_Y \otimes \Omega^2_Y) = \dim_{\mathbb{C}} H^2(Y, \Omega^2_Y) = 1.$$

By Serre duality, we get

$$\Omega_Y \otimes \Omega^2_Y \simeq T_Y.$$

Since $Y$ is an Enriques surface, we have

$$h^{1,0}(Y, \Omega^2_Y) = \dim_{\mathbb{C}} H^0(Y, \Omega_Y \otimes \Omega^2_Y) = \dim_{\mathbb{C}} H^0(Y, T_Y) = 0.$$
Thus we obtain

\[ h^{1,1}(Y, \Omega_Y^2) = \dim \mathbb{C} \text{H}^1(Y, \Omega_Y \otimes \Omega_Y^2) = \dim \mathbb{C} \text{H}^1(Y, T_Y) = 10. \]

\[ h^{1,2}(Y, \Omega_Y^2) = \dim \mathbb{C} \text{H}^2(Y, \Omega_Y \otimes \Omega_Y^2) = \dim \mathbb{C} \text{H}^2(Y, T_Y) = 0. \]

Since \( Y \) is an Enriques surface, we obtain

\[ \Omega_Y^2 \otimes \Omega_Y^2 \simeq \mathcal{O}_Y. \]

\[ h^{2,0}(Y, \Omega_Y^2) = \dim \mathbb{C} \text{H}^0(Y, \Omega_Y^2 \otimes \Omega_Y^2) = \dim \mathbb{C} \text{H}^0(Y, \mathcal{O}_Y) = 1. \]

\[ h^{2,1}(Y, \Omega_Y^2) = \dim \mathbb{C} \text{H}^1(Y, \Omega_Y^2 \otimes \Omega_Y^2) = \dim \mathbb{C} \text{H}^1(Y, \mathcal{O}_Y) = 0. \]

\[ h^{2,2}(Y, \Omega_Y^2) = \dim \mathbb{C} \text{H}^2(Y, \Omega_Y^2 \otimes \Omega_Y^2) = \dim \mathbb{C} \text{H}^2(Y, \mathcal{O}_Y) = 0. \]

Thus we obtain

\[
B = \prod_{k=1}^{\infty} \prod_{p,q=0}^{2} \left( \frac{1}{1 - (-1)^{p+q} x^{p+k-1} y^{q+k-1} t^k} \right) (-1)^{p+q} h^{p,q}(E, \Omega_E^2) \\
= \prod_{k=1}^{\infty} \left( \frac{1}{1 - x^{k-1} y^{k+1} t^k} \right) \left( \frac{1}{1 - x^k y^k t^k} \right)^{10} \left( \frac{1}{1 - x^{k+1} y^{k-1} t^k} \right) \\
= \prod_{k=1}^{\infty} \left( \sum_{a=0}^{\infty} (x^{k-1} y^{k+1} t^k)^a \right) \left( \sum_{b=0}^{\infty} (x^k y^k t^k)^b \right)^{10} \left( \sum_{c=0}^{\infty} (x^{k+1} y^{k-1} t^k)^c \right).
\]

Thus we have

\[
B \equiv \prod_{k=1}^{2} (1 + x^{k-1} y^{k+1} t^k + x^{2k-2} y^{2k+2} t^{2k}) \times (1 + x^k y^k t^k + x^{2k} y^{2k} t^{2k})^{10} \times \\
(1 + x^{k+1} y^{k-1} t^k + x^{2k+2} y^{2k-2} t^{2k}) \pmod{3^3} \\
\equiv \left( 1 + y^2 t + y^4 t^2 \right) \times (1 + x y^3 t^2) \\
\times \left( 1 + 10(xyt + x^2 y^2 t^2) + 45(xyt + x^2 y^2 t^2)^2 \right) \times (1 + x^2 y^2 t^2) \\
\times \left( 1 + x^2 t + x^4 t^2 \right) \pmod{3^3} \\
\equiv \left( 1 + y^2 t + (xy^3 + y^4) t^2 \right) \times \left( 1 + 10 xyt + 56 x^2 y^2 t^2 \right) \times \\
\left( 1 + x^2 t + (x^3 y + x^4) t^2 \right) \pmod{3^3} \\
\equiv \left( 1 + (10 xyt + y^2) t + (56 x^2 y^2 + 11 xy^3 + y^4) t^2 \right) \times \\
\left( 1 + x^2 t + (x^3 y + x^4) t^2 \right) \pmod{3^3} \\
\equiv 1 + (x^2 + 10 xyt + y^2) t + (x^4 + 11 x^3 y + 56 x^2 y^2 + 11 xy^3 + y^4) t^2 \pmod{3^3}.
\]

Therefore the coefficient of \( x^3 y t^2 \) of \( B \) is 11. The conjecture in [2, Conjecture 1] is not established for \( Y \) an Enriques surface and \( L = \Omega_Y^2 \).
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