Loop Variables and Gauge Invariant Interactions - II *

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Abstract

We continue the discussion of our previous paper on writing down gauge invariant interacting equations for a bosonic string using the loop variable approach. In the earlier paper the equations were written down in one higher dimension where the fields are massless. In this paper we describe a procedure for dimensional reduction that gives interacting equations for fields with the same spectrum as in bosonic string theory. We also argue that the on-shell scattering amplitudes implied by these equations for the physical modes are the same as for the bosonic string. We check this explicitly for some of the simpler equations. The gauge transformation of space-time fields induced by gauge transformations of the loop variables are discussed in some detail. The unintegrated (i.e. before the Koba-Nielsen integration), regularized version of the equations, are gauge invariant off-shell (i.e. off the free mass shell).

*This is a detailed description of an approach, outlined in a talk at the Puri Workshop in 1996, to use loop variables to describe string interactions.
1 Introduction

In an earlier paper [1] (hereafter ‘I’) we had described a method of writing down gauge invariant interacting equations of motion for the modes of the bosonic string using the loop variable approach [2, 3, 4, 5]. As in the free case the equations were written down in one higher dimension where all the modes are massless. Interactions were introduced by the simple prescription of thickening the string to a “band”.

In the free case the dimensional reduction can easily be done, leading to equations for massive modes [2]. The masses are essentially put in by assigning some fixed value to the momentum “$p_5$” in the internal direction and choosing $(p_5)^2 = m^2$ where $m$ is the required mass. While the procedure is adhoc and does not admit any simple geometric interpretation, it nevertheless has the advantage that it provides a simple method of writing down gauge invariant equations for the massive modes of the string.

When we consider interactions, this dimensional reduction is not so straightforward and it remained to be demonstrated that some generalization of this method would work. This is done in this paper. We describe a dimensional reduction procedure that gives the same free spectrum and also show that the scattering amplitudes that can be deduced from the non-linear terms in the equations of motion are consistent with the scattering amplitudes of string theory for on-shell physical states. The dimensional reduction procedure is also consistent with gauge invariance. Furthermore, as in the higher dimensional situation, gauge invariance does not require that the fields be on-shell. The gauge transformations and gauge invariance are easy to describe when the Koba-Nielsen integrals have not been performed, and there is a regulator on the world sheet. We thus have a gauge invariant set of equations that are valid off the (free) mass shell, i.e where the fields are not forced to satisfy $p^2 = m^2$. This is essentially what string field theory gives [3, 4, 5]. We should hasten to add that unlike string field theory we do not have an action that gives the equation of motion. It remains an open problem to find such an action. After the Koba-Nielsen integrals are done and the continuum limit on the world sheet taken, we get a “low energy” effective equation of motion. The issue of gauge invariance of these equations is subtle. A preliminary calculation was done in I, but we do not pursue this in this paper. This issue is analogous to the question in string field theory where the higher order non-linearity of the both the equations and gauge transformations can emerge only after integrating out degrees of freedom.

Apart from the fact that it is always a good idea to have different ways of
solving a given problem, we believe that a utility of the present approach is that the gauge transformations have a simple form when expressed in terms of loop variables. In fact, gauge invariance is manifest in these variables and the problem is to show that a consistent set of gauge transformations can be defined for the space time fields.

The gauge transformations in the free case look like local (along the string) scale transformations [2]. This continues to be so even in the interacting case and also after dimensional reduction. However this is not at all manifest when we work with space time fields. It suggests that working directly with loop variables rather than with space time fields might provide some understanding of the underlying principles of string theory.

This paper is organized as follows. In Sec 2 we describe the dimensional reduction procedure. We also show that it does not affect arguments for gauge invariance given in I. In Sec 3 we give an explicit calculation of some three and four point functions in the case of the tachyon and massless vector.

In Sec 4 we conclude with a summary and future directions.

2 Dimensional Reduction

2.1 Reproducing String Amplitudes

Let us first state our requirements as a motivation for the dimensional reduction prescription given below. We emphasize that our attitude here is that ultimately what we want are space-time gauge invariant (or coordinate invariant - for closed strings) equations for the modes of the string with the caveat that the on shell S-matrix elements implied by these equations should be the same as that given by conventional string theory. Any set of rules that achieves this is alright as long as they are self consistent. In particular we do not worry about world sheet reparametrizations or BRST invariance or any such elegant geometrical property. If they exist that would be a bonus, but we do not demand any such interpretation at this stage.

Thus, the dimensional reduction procedure thus should not violate the gauge invariance that is built into the loop variable approach. At the free level we simply required that \((p_5)^2 = m^2\) where \(p_5\) is the momentum in the internal direction and \(m\) is the mass of the field, which is also related to the naive dimension ‘P’ (more precisely, \(m^2 = \frac{P-1}{\alpha'}\)) of the vertex operator. We rewrite this as \((p_5)^2 g^{55} = m^2\) to emphasize that there is a metric that could play a role. When interactions are taken into account one expects that \(p_5\) of the different interacting fields will add up as required by momentum.
conservation. It would seem that all values of \( p_5 \) have to be allowed. But we do not want this. Because only the string oscillators in the higher dimension, and not the zero mode, are expected to contribute space-time degrees of freedom in the usual covariant string field constructions [7]. In order to retain this feature we will set \( p_5 = 1 \) for all fields. We will further require that when there are interactions

\[
g^{55} = \frac{P - 1}{N^2} \tag{2.1.1}
\]

where \( N \) is the number of fields at the interaction. Thus \( N = 1 \) for the free case and \( N = 2 \) for the quadratic term in the equation of motion, etc.

The net effect of this is that \((p_5 + q_5 + k_5 \ldots)^2 g^{55} = P - 1\) is true for every term in the equation of motion. This is a very peculiar looking ansatz without any obvious geometrical interpretation. Nevertheless it serves the purpose of providing us the massive equation starting from the massless equation in one higher dimension.

As will be seen below, this is crucial in deriving the equation of motion. We will also set \( X^5 = 0 \) at the end in order that there be no momentum conservation in the internal direction.

In the correlation functions we have terms of the type \((z - w)^{p.q} \ldots\). On expanding the dot product we get \( p_\mu q_\mu + p_5 q_5 \). Again we do not want the \( p_5 q_5 \) terms if we want to recover the Veneziano amplitudes. One way to achieve this is to set, in the definition of the space time fields,

\[
<k_n(\sigma_i) k_m(\sigma_j) \ldots> = \ldots S_n S_m (z_i - z_j)^{-\eta_5 p_5} \ldots \tag{2.1.2}
\]

This will ensure that all unwanted factors of \((z - w)^{p_5 q_5}\) are canceled. We will come back to this later.

These set of rules will be applied below in deriving the equations of motion.

Let us see why these rules reproduce the Veneziano amplitudes. The argument is very similar to that made in [8]. For an \( N \)-point amplitude there are \( N-3 \) Koba-Nielsen variables that need to be integrated. Three of them can be fixed. In our case the \( N \)-point amplitude gives rise to a term in the equation of motion with \( N-1 \) fields and therefore \( N-1 \) Koba-Nielsen variables.

\footnote{There are other ways of achieving these ends, that might make it look a little more geometrical - for instance one can modify the range of the \( \sigma \) integration in \( \int d\sigma p_5 X^5(\sigma) \) relative to the \( \sigma \) integration for the other directions. For all directions (other than 5) we let the range of \( \sigma \) integration be 0 to \( N \). For the 5 direction we assume it is 0 to 1 always. This brings in a relative factor of \( N^2 \) that we need.}
One of these is trivial because of translational invariance. Thus we have \(N-2\) integrations. The last (i.e. \((N-2)\)th) integration will actually just produce (when the particles are on-shell) a term of the form
\[
\frac{1}{(p_1 + p_2 + p_3 + \ldots + p_{N-1})^2 - m^2}.
\]
(We have used \((p_{1,5} + p_{2,5} + \ldots + p_{N-1,5})^2 g^{55} = P - 1 = m^2\). This is just the propagator for the external \(N\)th field, whose equation of motion we are writing down, and whose mass is \(m^2\). When we vary w.r.t the Liouville mode we bring down a factor of \((p_1 + p_2 + \ldots + p_{N-1})^2 - m^2\) which precisely cancels this propagator. The remaining \(N-3\) integrals then produce the Veneziano amplitude. As explained in [9] if we regularize the integral we end up subtracting the intermediate poles in the amplitude. This is also exactly what needs to be done when constructing an effective action.

Note that varying the Liouville mode in terms involving its derivative, also contributes terms to the equation of motion, but these are essentially gauge covariantizations of the previous term. These terms are not there for on shell physical states. Thus from this we conclude that for the on-shell physical states this calculation give you the correct result. The rest of the terms are fixed by gauge invariance.

Note that \(P\) was set equal to the naive dimension of the vertex operator. We have been using \(k_n Y_n\) as our vertex operator. But \(Y_n\) does not have a well defined dimension. It contains all \(\tilde{Y}_n\) in it. Furthermore, even \(\tilde{Y}\) doesn’t have an unambiguous dimension because of the \(z\) dependence. Thus \(\tilde{Y}_n(z) = \tilde{Y}_n(w) + (z - w)(n)\tilde{Y}_{n+1} + \ldots\) Both these ambiguities have to be removed. The first will be removed by using \(K_m\) defined in I by
\[
\sum_n k_n Y_n = \sum_m K_m \tilde{Y}_m
\]  
This gives: \((\alpha_0 = 1)\)
\[
K_n = k_n + \alpha_1 k_{n-1} + \alpha_2 k_{n-2} + \ldots + \alpha_n k_0
\]
They have the same gauge transformation law as \(k_n\).

The second ambiguity will be removed by translating all vertex operators to a canonical location on the world sheet that we call \(z\). Thus
\[
\sum_n K_n \tilde{Y}_n(w) = \sum_m K_m(z - w)\tilde{Y}_m(z)
\]
which gives:
\[
K_q(z) = \sum_{n=0}^{q=0} K_n(0) D_n^q(-1)^{q-n} z^{q-n}
\]  
See Appendix A for the basic definitions.
where

\[ D_n^q = q^{-1} C_{n-1}, \quad n, q \geq 1 \]
\[ = \frac{1}{q}, \quad n = 0 \]
\[ = 1, \quad n = q = 0 \] (2.1.7)

Now \( P\tilde{Y}_n(z) = n\tilde{Y}_n(z) \) is an unambiguous equation.

The gauge transformation law for \( K_n(0) \) is the same as for \( k_n \):

\[ K_n \rightarrow K_n + \lambda_p K_{n-p} = K_n + \lambda_p \frac{d}{dx_p} K_n \] (2.1.8)

Of course the second form of the gauge transformation rule cannot be written for \( k_n \) because they are not functions of \( \alpha_n \). The gauge transformation of \( K_n(z) \) however, can only be written in this form:

\[ K_n(z) \rightarrow K_n(z) + \lambda_p \frac{d}{dx_p} K_n(z) \] (2.1.9)

In view of the above it is worth examining afresh the proof of gauge invariance given in I.

Our starting point was the loop variable \( e^A \) given by:

\[ e^{i \int d\sigma k_0(\sigma) Y(\sigma) + i \sum_{n>0} k_n(\sigma) \frac{\partial Y(\sigma)}{\partial x_n(\sigma)}} \]

\[ e^{i \int d\sigma_1 d\sigma_2 \{ k_0(\sigma_1) k_0(\sigma_2) [\tilde{G} + \tilde{\Sigma}](\sigma_1, \sigma_2) + (\sum_{n>0} k_n(\sigma_1) k_0(\sigma_2) \frac{\partial^2 [\tilde{G} + \tilde{\Sigma}](\sigma_1, \sigma_2)}{\partial x_1(\sigma_1) \partial x_1(\sigma_2)} \}} \]

\[ e^{i \int d\sigma_1 d\sigma_2 \{ \sum_{n,m>0} k_n(\sigma_1) k_m(\sigma_2) \frac{\partial^2 [\tilde{G} + \tilde{\Sigma}](\sigma_1, \sigma_2)}{\partial x_1(\sigma_1) \partial x_1(\sigma_2)} \}} \] (2.1.10)

The only change we make is that \( e^B \equiv e^{i \sum_n k_n Y_n} \) is rewritten as \( e^B \equiv e^{i \int d\sigma_1 d\sigma_2 \{ \sum_{n,m>0} k_n(\sigma_1) k_m(\sigma_2) \frac{\partial^2 [\tilde{G} + \tilde{\Sigma}](\sigma_1, \sigma_2)}{\partial x_1(\sigma_1) \partial x_1(\sigma_2)} \}} \)

The change in B is \( \lambda_p \frac{d}{dx_p} B \) by eqn (2.1.9), which is just what it was earlier. Also the operator \( \frac{P-1}{N-1} \) commutes with \( \frac{d}{dx_n} \), because \( P \) acts only on \( \tilde{Y}_n \) which has no \( x_n \) dependence.

Thus the crucial point in the proof, which was that

\[ \delta A = \int d\sigma_1 \lambda_1(\sigma) \left[ \frac{\partial}{\partial x_1(\sigma_1)} + \frac{\partial}{\partial x_1(\sigma_2)} \right] A \] (2.1.11)

is not affected.

It was also shown in I, that while gauge invariance at the level of loop variables is almost manifest, what is non-trivial is that one can consistently
define gauge transformation rules for space-time fields, i.e., the consistency of the map from loop variables and their transformations to fields and their transformation is a non-trivial issue. In I we mapped each loop variable expression of the form \( k_n(\sigma_1)k_m(\sigma_2)G(\sigma_1,\sigma_2) \) to a space time field or fields \( S_nS_mG(\sigma_1,\sigma_2) \) \[ \text{[3]} \]. This whole procedure has to be reexamined for the following reason: When we substitute \( g^{55} = \frac{P - 1}{N^2} \), the numerical value depends on which equation we are considering. Thus the same term, say, \( S_{55}g^{55} \) occurring in an equation for the tachyon will have a value of \( \frac{1}{N^2} \) whereas in an equation for a massive particle could be \( \frac{1}{N^2} \). The question is are we going to get mutually inconsistent transformation rules for the field \( S_{55} \)? We describe below a way out of these problems.

### 2.2 Consistent Definition of Gauge Transformation

#### 2.2.1 Before Dimensional Reduction

Let us begin with a general discussion of the consistency issue. Let us consider a generic term

\[
k_n(\sigma_i)Y_n(z_i)k_m(\sigma_j)Y_m(z_j)....
\]  

(2.2.12)

We have suppressed the Lorentz indices. Under a gauge transformation it transforms to

\[
\int d\sigma \lambda_p(\sigma)[k_{n-p}(\sigma_i)Y_n(\sigma_i,z_i)k_m(\sigma_j)Y_m(\sigma_j,z_j) + ...]
\]  

(2.2.13)

The reason for retaining the \( Y_n \) in the above is that when one maps the above gauge transformation to space time fields one has to be careful about factors of the form \( \tilde{G}(z_i - z_j) \) that arise out of the contraction of the \( Y'_n \)s. The resulting \( z_i - z_j \) dependence will, after doing the Koba Nielsen integrals (over \( z_i, z_j \)) introduce some non trivial momentum dependence in the gauge transformation laws. These are the factors \( F(p,q) \) in eqn.(6.0.9) and (6.0.14) of I.

We will first go over to the \( K_n \) variables of \([2.1.4]\). One can then transfer the \( z_i \) dependence into the \( K_n \) and use \( \tilde{Y}(z) \) with some canonical \( z \) as in \([2.1.6]\). The point \( z \) is arbitrary. We will remove this arbitrariness by defining it to be the location of \( \lambda(\sigma) \), i.e. \( z(\sigma) = z \)

\[ \text{[3]} \] We also kept the cutoff dependent second term which would normally be discarded during the normal ordering process
Thus we replace the above with:

\[ K_n(\sigma_i, z_i - z) \tilde{Y}_n(z) K_m(\sigma_j, z_j - z) \tilde{Y}_m(z) \rightarrow K_n(\sigma_i, z_i - z) \tilde{Y}_n(z) K_m(\sigma_j, z_j - z) \tilde{Y}_m(z) + \int d\sigma \lambda_p(\sigma) \frac{d}{dx} (K_n(\sigma_i, z_i - z) K_m(\sigma_j, z_j - z) \tilde{Y}_n(z) \tilde{Y}_m(z)) \] (2.2.14)

When we contract the \( \tilde{Y} \)'s we will have to do some regularization. We also have to introduce vertex operators \( e^{i \int k_0 Y} \) for the momentum dependence. We will worry about each of these issues in turn. But first let us see how gauge transformation laws for space time fields can be consistently defined from the above. The first step is to substitute (2.1.4) and (2.1.6) in the above as the space time fields are defined in terms of \( k_n \) and not \( K_n \).

The leading term in (powers of \( \alpha_n \) and \( z_i - z_j \)) is \( k_n(\sigma_i) k_m(\sigma_j) \) which leads to \( S_{n,m} \) (and \( S_n S_m \)). All other space time fields have smaller quantum numbers \( n, m \). We get something that looks like

\[ \delta S_{n,m} = \Lambda_p S_{n-p,m} + \alpha_1 \Lambda_p S_{n-p-1,m} + \ldots \] (2.2.15)

where we have ignored the powers of \( z_i - z_j \). Thus if we assume that we have already defined a gauge transformation for all the lower quantum number space-time fields, then (2.2.14) defines uniquely a gauge transformation law for \( S_{n,m} \). Of course the powers of Koba-Nielsen variables \( z_i - z \) will translate to appropriate dependences on momenta when the integrals are done. For the moment let us leave the dependences on \( z_i \) as they are. Let us turn to the issue of contractions. We can contract the \( \tilde{Y} \)'s after point splitting or some other regularization. A simple regularization is to use \( \ln((w-z)^2 + \epsilon^2) \) as the Green function. Thus in (2.2.14) we will replace \( \tilde{Y}_n(z) \tilde{Y}_m(z) \) by \( \tilde{Y}_n(z) \tilde{Y}_m(z') \) where \( z' = z + \epsilon \) or use the regularized Green function. In either case contraction will produce \( \approx \epsilon^{-n-m} \). We will get an equation for the traces, \( S_{n,m}^\mu, S_{n,m} \), etc., which will have an overall power of \( (\epsilon)^{-n-m} \) multiplying the whole equation, (instead of \( \tilde{Y}_n^\mu \tilde{Y}_m^\nu \)).

Suppose we had gone back one step and kept the \( z_i \) dependence in the \( \tilde{Y} \)'s. Then, instead of (2.2.14) we would have:

\[ K_n(0) \tilde{Y}_n(z_i) K_m(0) \tilde{Y}_m(z_j) \rightarrow K_n(0) \tilde{Y}_n(z_i) K_m(0) \tilde{Y}_m(z_j) + \lambda_p \frac{d}{dx} (K_n(0) K_m(0) \tilde{Y}_n(z_i) \tilde{Y}_m(z_j)) \] (2.2.16)

Contracting the \( \tilde{Y}_n \)'s now give derivatives of \( G(z_i - z_j) \). If we now expand \( G(z_i - z_j) \) in powers of \( z - z_i \) and \( z - z_j \) (where we have regularized as
before) and keep track of powers of $\epsilon$ we will recover the contracted version of (2.2.14). We are guaranteed this because of the following simple fact:

$$G(z_i - z_j) = \langle \hat{Y}_n(z_i) \hat{Y}_m(z_j) \rangle = \langle \hat{Y}_n(z + z_i - z) \hat{Y}_m(z + z_j - z) \rangle = \langle \hat{Y}_n(z) \hat{Y}_m(z) \rangle + (z_i - z) \langle n \hat{Y}_{n+1}(z) \hat{Y}_m(z) \rangle + \ldots$$

This is exactly the equation that fixes the $z$ dependence of $K_n$ and gives

$$G(z_i - z_j) \approx \frac{(n + m - 1)!}{(n - 1)!(m - 1)!} (\epsilon)^{-n-m} + (z_i - z) n \frac{(n + m)!}{n!(m - 1)!} (\epsilon)^{-n-m-1} + \ldots \quad (2.2.17)$$

More generally we can define $\epsilon_{n,m} = \langle \hat{Y}_n(0) \hat{Y}_m(0) \rangle$ and write the Taylor expansion in terms of them. For the regularized Green function introduced above, viz $\ln((z - w)^2 + \epsilon^2)$ we can see that $\epsilon_{n,m} = \epsilon_{m,n}$. If we insert this in (2.2.16) we reproduce the contracted version of (2.2.14).

Two comments are in order before we proceed. As stated in I, the fields $S_{n,m}$ must have an $x_n$-dependence because the RHS of the gauge transformation (2.2.15) has $x_n$ dependence. This can be done as in I by making the string field, $\Psi$, $x_n$-dependent. Second, the grouping of terms used in defining gauge transformation for space time fields is different from that used in I. The one used here is manifestly consistent so we will use this from now on. The method used in I is only correct to lowest order as it compares terms involving $V(z_i)V'(z_j)$ with $\hat{V}(z_i)V'(z_i)$ (for some vertex operators $V, V'$), which can only be true to lowest order. There is also one very important point: Here we are dealing with (regularized) operator products. But in an actual equation of motion the locations of the operators have to be integrated over. Furthermore we may also be interested in taking $\epsilon \to 0$. Verifying the gauge invariance of the final equation starting from the gauge transformations defined here may be quite a complicated process. In this paper we will not attempt this.

Now we turn to the second issue of introducing the vertex operators $e^{ik_0Y}$. This is very important when we discuss gauge transformations because when the $\sigma$ in $\lambda_\mu(\sigma)$ is not equal to any of the $\sigma'$ in $k_0(\sigma')$ then we have an extra vertex operator $e^{ik_0(\sigma)X(z(\sigma))}$ associated with $\lambda(\sigma)$. (In the contracted version this introduces extra factors of $(z(\sigma) - z(\sigma'))^{k_0(\sigma),k_0(\sigma')}$.) Also when we Taylor expand $e^{ik_0(\sigma)X(z(\sigma))}$ we introduce an infinite number of new vertex operators. This latter fact in particular implies that we cannot define the gauge transformation of an object such as (2.2.12). We need to include all the vertex operators on both sides i.e. in (2.2.12) and in (2.2.13).
for consistency. Thus we have to define gauge transformation for the entire interacting loop variable \( e^i \sum_{n \geq 0} \int d\sigma k_n(\sigma)Y_n(\sigma) \). Thus we define

\[
\delta[e^i \sum_{n \geq 0} \int d\sigma' k_n(\sigma')\tilde{Y}_n(\sigma')] = \int d\sigma d\sigma' \lambda_p(\sigma)K_{n-p}(\sigma_1)\tilde{Y}_n(\sigma_1)e^i \sum_{n \geq 0} \int d\sigma' k_n(\sigma')\tilde{Y}_n(\sigma')
\]

(2.2.18)

If we allow contractions we just have to keep track of \( \epsilon_{n,m} \) on both sides. This does not produce anything different. We will use the uncontracted version to define gauge transformations for simplicity.

We will assume that \( z(\sigma) = z = 0 \) for convenience. We also have the following relations:

\[
\sum_q K_q(-w)\tilde{Y}_q(0) = \sum_n K_n(0)\tilde{Y}_n(w)
\]

which defines \( K_q(-w) \) to be given by

\[
K_q(-w) = \sum_{n=0}^q K_n(0)D^q_n(w)^{q-n}
\]

(2.2.19)

where the \( D^q_n \) have been defined earlier. Thus, for instance

\[
K_1(-w) = K_1 + k_0 w
\]

\[
K_2(-w) = K_2 + K_1 w + K_0 \frac{w^2}{2}
\]

\[
K_3(-w) = K_3 + 2K_2 w + K_1 w^2 + K_0 \frac{w^3}{3}
\]

Thus in (2.2.18) we can specify more clearly the dependence on \( z(\sigma) \) by writing it as

\[
\delta[e^i \int d\sigma \sum_n K_n(\sigma, -z(\sigma))\tilde{Y}_m(0)] = \int d\sigma_1 \sum_n \int d\sigma \lambda_p(\sigma) \frac{d}{d\sigma} [K_n(\sigma_1, -z(\sigma_1))\tilde{Y}_n(0)] e^i \int d\sigma' \sum_m K_m(\sigma', -z(\sigma'))\tilde{Y}_m(0)
\]

(2.2.20)

**Vector - A_1^\mu**

For the vector we compare coefficients of \( \tilde{Y}_1(0) \) on both sides. This gives

\[
\delta[\int d\sigma_1 K_1(\sigma_1, -z(\sigma_1))\tilde{Y}_1(0)e^i \int k_0(\sigma')\tilde{Y}_0(0)]
\]
Thus
\[
\delta \left[ \int d\sigma \lambda_1(\sigma) \int d\sigma_1 k_0(\sigma_1) \tilde{Y}_1(0) e^{i \int k_0(\sigma') \tilde{Y}_0(0)} \right] = \int d\sigma \lambda_1(\sigma) \int d\sigma_1 k_0(\sigma_1) \tilde{Y}_1(0) e^{i \int k_0(\sigma') \tilde{Y}_0(0)}
\]

Taking the expectation value on both sides gives
\[
\delta \left[ A_1^\mu(p) \tilde{Y}_1^\mu(0) e^{ip\tilde{Y}_0(0)} \right] = \Lambda(p) p^\mu \tilde{Y}_1^\mu(0) e^{ip\tilde{Y}_0(0)} \quad (2.2.21)
\]

In terms of fields
\[
\delta \left[ \int d\sigma_1 K_2(\sigma_1, -z(\sigma_1)) \tilde{Y}_2(0) e^{i \int k_0(\sigma') \tilde{Y}_0(0)} \right] = \delta \left[ \int d\sigma_1 (K_2(\sigma_1) + K_1(\sigma_1) z(\sigma_1) + k_0(\sigma_1) z(\sigma_1)^2) \tilde{Y}_2(0) e^{i \int k_0(\sigma') \tilde{Y}_0(0)} \right]
\]

In terms of fields
\[
\delta \left[ S_2^\mu(p) e^{ip\tilde{Y}_0(0)} \tilde{Y}_2(0) + \int d\sigma_1 A_1^\mu(p) z(\sigma_1) \tilde{Y}_2(0) e^{ip\tilde{Y}_0(0)} \right] = \Lambda_1(p) A_1^\mu(q) \tilde{Y}_2(0) e^{i(p+q)\tilde{Y}_0(0)} + \int d\sigma p^\mu \Lambda_1(p) z(\sigma) \tilde{Y}_2(0) e^{ip\tilde{Y}_0(0)} \quad (2.2.22)
\]

The last term in the RHS of \((2.2.22)\) is clearly the variation of the second term in the LHS. Thus we are left with
\[
\delta S_2^\mu(k) e^{ik\tilde{Y}_0(0)} \tilde{Y}_2(0) = \Lambda_1^\mu(k) \tilde{Y}_2(0) e^{ik\tilde{Y}_0(0)} + \Lambda_1(p) A_1^\mu(q) \tilde{Y}_2(0) e^{i(p+q)\tilde{Y}_0(0)}
\]

with the understanding that \(p + q = k\).

Note that there is no \(z\)-dependence in the transformation law. It is the same as the naive transformation law introduced in [3].

11
Let us compare the coefficients of $\tilde{Y}_1(0)\tilde{Y}_1(0)$.

\[ \delta \left[ \frac{1}{2!} \int d\sigma_1 \int d\sigma_2 K_{1}\mu_1(\sigma_1, z(\sigma_1))K_{1}\nu_1(\sigma_2, z(\sigma_2))\tilde{Y}_{1\mu}(0)\tilde{Y}_{1\nu}(0)e^{i\int k_0(\sigma')\tilde{Y}_0(0)} \right] \]

\[ = \frac{1}{2!} \int d\sigma_1 \int d\sigma_2 \int d\sigma_1(\sigma)[k_{0\mu_1}^\nu(\sigma_1)(K_{1\nu}^\mu(\sigma_2) + k_{0\nu_1}^\mu(\sigma_2)z(\sigma_2)) + k_{0\nu}(\sigma_2)(K_{1\mu}^\mu(\sigma_1) + k_{0\mu}(\sigma_1)z(\sigma_1))]e^{i\int k_0(\sigma')\tilde{Y}_0(0)} \]

On taking expectation values to bring in space-time fields, the LHS gives

\[ \delta \left[ \frac{1}{2!} \int d\sigma_1 \int d\sigma_2 \int d\sigma_1(\sigma)\left[ \tilde{S}_{\mu\nu}^1(p)A_{1\mu_1}^\nu(p)e^{i(p+q)\tilde{Y}_0(0)} + \tilde{A}_{1\mu_1}^\nu(p)e^{i(p+q)\tilde{Y}_0(0)} \right] \right] \]

\[ = \frac{1}{2} \delta \left[ \tilde{S}_{1,1}(k)e^{ik\tilde{Y}} + \tilde{A}_{1\mu}(p)A_{1\nu}(q)e^{i(p+q)\tilde{Y}} + \int d\sigma z(\sigma_1)p^{(\nu}A_{1\mu_1}^{\mu)}(p)e^{ip\tilde{Y}} \right] \]

RHS

\[ \frac{1}{2} \int d\sigma_1 \int d\sigma_2 \int d\sigma \left[ \frac{D(\sigma - \sigma_2)}{a} \frac{D(\sigma - \sigma_1)}{a} \tilde{A}_{1,1}(p)p^{\mu} + \right. \]

\[ \left. \frac{D(\sigma - \sigma_2)}{a} \tilde{A}_{1}(p)p^{\mu} + \frac{D(\sigma - \sigma_2)}{a} \tilde{A}_{1}(p)q^{\mu}A_{1\nu}(q) + 2 \int d\sigma_2 \tilde{A}_1(p)p^{\mu}p^{\nu}z(\sigma_2) \right] \]

Notice that the $z$-dependent terms are the same on the LHS and RHS we get

\[ \delta S_{1,1}^{\mu\nu} = \Lambda_{1,1}(\mu)p^{\nu} + \Lambda(p)q^{(\mu}A_{1\nu}^{\nu)}(q)s \tag{2.2.23} \]

Once again there are no $z$-dependent terms in the transformation law - it is the same as the one defined in [3].

**S**$_{2,1}^{\mu\nu}$

One can go through the same analysis for $\tilde{Y}_2(0)\tilde{Y}_1(0)$. One finds terms that are $z$-independent, linearly dependent and quadratically dependent on
Using (2.2.22), (2.2.23) we find that the \( z \)-dependent terms cancel and the net result is,

\[
\delta S_{2,1}^{\mu\nu} = \Lambda_{1,1,1}^{\mu\nu} + \Lambda_1^\mu A_1^\nu + \Lambda_1^\nu S_2^{\mu\nu} + \Lambda_1^\nu (p) p^\nu + \Lambda_1(p) S_2^{\mu\nu}(q) q^\nu \quad (2.2.24)
\]

It is very interesting that the \( z \)-dependence cancels out in each case. This is in contrast to the gauge transformation defined in I. The point being that the vertex operators being compared there were different. Thus for instance \( Y_2(z) = Y_2(0) + azY_3(0) + bz^2Y_4(0) + \ldots \) (for some \( a, b, \ldots \)). This implies that when we write down equations of motion in one basis, it is a \( z \)-dependent linear combination of the equations in the other basis.

In the basis used in this paper the \( z \)-dependence cancels out of the gauge transformation laws. They are the naive gauge transformation laws first introduced in [3]. To see the invariance of the equations of I, one would have to combine all the equations for the vertex operators \( Y_1(z), Y_2(z), \ldots \) and extract from them the coefficients of, say, \( \tilde{Y}_n(0) \). One should also keep track of factors of \( \epsilon_{n,m} \) - thus we might consider the coefficient of say, \( \tilde{Y}_n(0) \epsilon_{p,q} \). This has to be invariant under the gauge transformations defined in this paper. It is actually quite easy to see why this must be true. We know that the full equation defined by \( \frac{\delta}{\delta \sigma} e^{\int d\sigma K_n(\sigma') Y_n(\sigma')} = 0 \) is invariant. Thus if we split up this equation into various pieces labelled by \( \tilde{Y}_n(0) \epsilon_{p,q} \) they must individually vanish. However as mentioned earlier the situation after the Koba-Nielsen variables are integrated is more subtle and needs further study. As mentioned in the introduction, this is analogous to the situation in string field theory, where gauge invariance of the string field cannot be directly used to describe gauge invariance of the “low energy” equations of motion. One has to solve for all the fields that have been integrated out.

We show next that the dimensional reduction procedure given before is consistent with this definition.

### 2.2.2 After Dimensional Reduction

The prescription for dimensional reduction is to set \( g_{55}^5 = \frac{P-1}{N^2} \), and also to set internal momenta \( k_{05} \) for all fields to 1. One can also introduce a “vielbein” \( e^{5V} = \sqrt{\frac{P-1}{N^2}} \) so that \( e^{5V} e_{5V}^5 = g_{55}^5 \). This will prove necessary in what follows.

We can separate (2.2.14) into different sets of equations, depending on the number of \( \tilde{Y}_5 \)'s. By Lorentz covariance these equations are independently satisfied. In equations with more than one \( \tilde{Y}_5 \) one can contract them
to get a factor of $g^{55}$ for each pair of $\tilde{Y}^5$. Clearly these are overall multiplicative factors in any equation. Actually $P-1$ is an overall multiplicative factor and will not affect the gauge transformation law. However $N$ can depend on the particular term being considered. Thus for instance in the ‘55’ equation if the LHS has $S_{55}$ then $N = 1$ and if the RHS has $S_5 S_5$ then $N = 2$. Thus the ‘55’ eqn is not obtainable from the ‘$\mu \nu$’ equation by Lorentz covariance. This by itself is not a problem of course, since we don’t expect Lorentz invariance in the $5-\mu$ directions. However when we consider the vertex operator $e^{ik_0 Y^5_0}$ and consider contractions involving $X^5$ we encounter a problem. Consider an equation that has the following form

$$\delta[k_{n\mu}(\sigma_1)k_{m5}(\sigma_2)]Y^\mu_nY^5_m e^{i k_0 Y_0} = \lambda_{p}(\sigma)k_{n-p\mu}(\sigma_1)k_{m5}(\sigma_2)Y^\mu_n Y^5_m e^{i k_0 Y_0} + ...$$

(2.2.25)

Note that there is no momentum conservation in the ‘5’ direction. Also from now on to be consistent we use upper Lorentz indices on the $\tilde{Y}$ and lower indices on $k$.

We have suppressed the $\sigma$ arguments and written $k_{05} = k_0^5$ on the RHS. If we use the naive extension of

$$<k_{n\mu}>, <k_{n\mu}k_{m\nu}> = S_{nm\mu\nu}$$

(2.2.27)

to $\mu = 5$ we

$$<k_{n\mu}k_{m5}> = S_{n,m5}$$

(2.2.28)

etc and also $k_{05} = 1$.

This gives

$$\delta S_{n,m5} \frac{g^{55}}{p-1} k_{05} = \Lambda_{p+m}(p) S_{n-p\mu}(q) \frac{g^{55}}{\frac{p-1}{2}} (p_0 + q_0) + ...$$

(2.2.29)

We have used on the LHS $k_{05}g^{55} = P-1$ and on the RHS $(p_0 + q_0)g^{55} = \frac{P-1}{2}$ (because $g^{55} = \frac{P-1}{2}$).

Whereas earlier without any contractions,

$$\delta[k_{n\mu}(\sigma_1)k_{m5}(\sigma_2)] = \lambda_{p}(\sigma)k_{n-p\mu}(\sigma_1)k_{m5}(\sigma_2) + ...$$

(2.2.30)
gave
\[ \delta S_{nm\mu} = \Lambda_{m+pV} S_{n-p\mu} + \ldots \] (2.2.31)
which is not consistent with \((2.2.29)\).

Thus we have a contradiction between the two equations. The source of the contradiction is the factor of \(N\) in the definition of \(g^{55}\). While we are free at the level of loop variables to multiply by \(g^{55}\) when we convert to space time fields we cannot expect meaningful results if we use the naive expression to define space time fields when \(\mu = 5\). In other words \(g^{55}\) is not just a number since it keeps track of how many different fields there are.

A resolution of this problem is obtained by using the vielbein to convert everything to a flat ‘V’ index. This means a different definition of space-time fields that have an index in the 5-direction. Define
\[ \langle k_m e^{5V} \rangle = S_m^V \]
\[ \langle k_m e^{5V} \rangle = S_{h,m}^V \] (2.2.32)
Write \((2.2.25)\)
\[
\delta[k_{\mu}(\sigma_1)k_m V (\sigma_2)] Y_{k\mu} Y_{k m} e^{i k_0 V} = \lambda_p (\sigma) k_{n-p\mu} (\sigma_1) k_{m V} (\sigma_2) Y_{k\mu} Y_{k m} e^{i (p_0 + q_0 V)} + \ldots
\] (2.2.33)
On contraction of the \(Y\)’s gives:
\[
\delta[k_{\mu} k_m V] Y_{k\mu} e^{-m} = \lambda_p k_{n-p\mu} k_{m V} (p_0 + q_0 V) Y_{k\mu} e^{-m} + \ldots
\] (2.2.34)
where \(k_0 V = k_{05} e^{5V}\). Converting to space time fields
\[ \delta S_{nm\mu V} = \Lambda_{m+pV} S_{n-p\mu} + \ldots \] (2.2.35)
On the LHS \(k_0 V\) has the value of \(\sqrt{P - 1}\). On the RHS \(e^{5V} = \frac{\sqrt{P - 1}}{2}\), so \(p_0 V + q_0 V = \sqrt{P - 1}\). Thus there is no contradiction.
In equations involving two or more \(Y_n^5\) we do the same thing, namely go to flat indices. So the equations take the form
\[ \delta S_{nmVV Y_{m} Y_{n} V} = \Lambda_{n+pV} S_{m-pV} Y_{m} Y_{n} V \] (2.2.36)
Contraction of the \(Y_V\)’s does not introduce any factors of \(g^{55}\) and thus there is no problem. The upshot of this discussion is that by going to flat coordinates the ‘N’ dependence is completely absent and there is thus no source of inconsistency from such contractions.
We have to now turn to the extra requirement (2.1.2). This requirement is effectively saying that we should include the factor $e^{i \int d\sigma k_0, V \tilde{Y}^V_0(z(\sigma))}$ in the definition of say $S_{11}$ as:

$$< k_{1\mu}(\sigma_1)k_{1\nu}(\sigma_2)e^{i \int d\sigma k_0, V \tilde{Y}^V_0(z(\sigma))} > \tilde{Y}^\mu_1(0) \tilde{Y}^\nu_1(0) = \left[ S_{11\mu\nu} \frac{D(\sigma_1 - \sigma_2)}{a} + S_{1\mu}S_{1\nu} \right] \tilde{Y}^\mu_1(0) \tilde{Y}^\nu_1(0)$$  \hspace{1cm} (2.2.37)

Note that the location of the exponential factor (which is \textit{not} normal ordered) is at $z(\sigma)$ rather than 0. One can immediately see that when $\mu$ is replaced by $V$ the resulting contractions will induce inconsistencies of the type mentioned above, to avoid which we went to great lengths. Thus the only solution is to have no possibility of contraction between $\tilde{Y}^V_0$'s. Thus we set

$$< \tilde{Y}^V_0(z) \tilde{Y}^V_0(w) > \equiv G^{VV}(z, w) = 0$$  \hspace{1cm} (2.2.38)

Since gauge transformations do not mix Lorentz indices, we are perfectly free to choose this. Thus our dimensionally reduced loop variable will take the form:

$$\exp\left\{ \int d\sigma_1 \int d\sigma_2 \sum_{n,m \geq 0} (k_n(\sigma_1),k_m(\sigma_2)) \frac{\partial^2 [\tilde{G} + \tilde{\Sigma}(\sigma_1, \sigma_2)]}{\partial x_n(\sigma_1) \partial x_m(\sigma_2)} + k_{n,V}(\sigma_1)k_{m,V}(\sigma_2) \frac{\partial^2 \tilde{\Sigma}(\sigma_1, \sigma_2)}{\partial x_n(\sigma_1) \partial x_m(\sigma_2)} \right\} \exp\left\{ i \int d\sigma \sum_{n \geq 0} k_n Y_n(z(\sigma)) \right\}$$  \hspace{1cm} (2.2.39)

In this case we do not need to modify our definition of space-time fields as in (2.2.37), nor do we need (2.1.2). We will use (2.2.39) as our loop variable.

We thus conclude that the dimensional reduction procedure does not introduce any inconsistency in the definition of gauge transformation of space-time fields. Thus there is a consistent map from loop variables to space time fields. Since the equations were manifestly gauge invariant in terms of loop variables, we conclude that they are gauge invariant even when written in terms of space-time fields.

3 Examples
3.1 Tachyon

We begin with a discussion of the tachyon in order to illustrate the dimensional reduction procedure.

3.1.1 Preliminaries

Let us fix some conventions first:

\[
< X^\mu(z_1)X^\nu(z_2) > = -2g^{\mu\nu} \ln (z_1 - z_2) \tag{3.1.40}
\]

with \( g_{00} = -1 \); \( g_{ii} = +1 \). The mass shell condition in this convention is \( p^2 = -m^2 \) and the tachyon has \( m^2 > 0 \). Using these conventions

\[
< e^{ik.X(z_1)} e^{-ik.X(z_2)} > = (z_1 - z_2)^{-2k^2}
\]

From this it is clear that \( e^{ik.X} \) has mass dimension \( k^2 \), and furthermore that \( \int dz e^{ik.X(z)} \) is a marginal operator when \( k^2 = 1 \). This fixes \( m^2 = -1 \) for the (open bosonic string) tachyon.

Another way to see this is to note that

\[
e^{ik.X(z)} = e^{-\frac{1}{2} k^\mu k^\nu <X^\mu(z)X^\nu(z)>} ; e^{ik.X(z)} := \epsilon^{k^2} ; e^{ik.X(z)} : \tag{3.1.41}
\]

where we have normal ordered the vertex operator and introduced a lattice spacing \( \epsilon \) as the ultraviolet cutoff. Scaling \( \epsilon \to \lambda \epsilon \) scales the ultraviolet momentum cutoff by \( \frac{1}{\lambda} \) or equivalently in units of the cutoff increases the momentum of the operator by \( \lambda \). This shows that the mass dimension is \( k^2 \).

Let us set \( \epsilon = e^{2\sigma} \). If we change \( k^2 \) to \( k^2 + k_5^2 \), then \( (3.1.41) \) becomes

\[
e^{ik.X(z)} = e^{(k^2+k_5^2)\sigma} ; e^{ik.X(z)} : \tag{3.1.42}
\]

Imposing \( \delta \sigma e^{ik.X(z)} = 0 \) gives the equation

\[
k^2 + k_5^2 = 0
\]

Thus we let \( k_5^2 = -1 \) we get the tachyon mass shell condition. In the general case we need \( k_5^2 = (P-1) \) where \( P \) is the engineering dimension of the vertex operator.

\[4\text{The regularized propagator is } -g^{\mu\nu} \ln((z_1 - z_2)^2 + \epsilon^2).\]
3.1.2 Loop Variables

Let us work out the tachyon equation using the loop variable approach. We start with

\[ e^{\int d\sigma_1 \int_0^1 d\sigma_2 k_0(\sigma_1) k_0(\sigma_2) [\tilde{G}(\sigma_1, \sigma_2) + \tilde{\Sigma}(\sigma_1, \sigma_2)]} \]

\[ e^{\int d\sigma_1 \int_0^1 d\sigma_2 k_{0V}(\sigma_1) k_{0V}(\sigma_2) [\tilde{G}^{V}(\sigma_1, \sigma_2)]} \cdot e^{\int d\sigma_0 \sigma(\sigma) + I(\sigma)} . \quad (3.1.43) \]

As explained in [2] the tachyon vertex operator can be represented by \( I(\sigma) \).

**Free Equation**

For the free equation we have \( 0 < \sigma_1 < 1 \) and \( \sigma_1 = \sigma_2 \). and we bring down only one power of \( I \). Thus we can set \( k_0(\sigma_1) = k_0(\sigma_2) = k_0 \). Using \( \tilde{G}(\sigma_1, \sigma_1) \approx \epsilon \), we get

\[ \int \frac{dz}{\epsilon} I(\sigma) e^{(k_0, k_0 + k_0 V_k^V)} \Sigma_0 e^{ik_0 Y_0} \quad (3.1.44) \]

where we have set all the \( x_n \) to zero. Using \( < I(\sigma) > = \Phi(k_0) \) we get on applying \( \frac{d}{d\Sigma} \) the equation

\[ \int dz e^{k_0, k_0 - 1} (k_0, k_0 + k_0 V_k^V) \Phi(k_0) = 0 \quad (3.1.45) \]

Further since \( k_0 V_k^0 k_0^V = k_0 5 k_0 5 V = P - 1 = -1 \) we finally get

\[ (k_0^2 - 1) \Phi(k_0) = 0 \quad (3.1.46) \]

**Cubic Interaction**

We let

\[ 0 < \sigma_1 < 1 : k_0(\sigma_1) = p : z(\sigma_1) = z \]

\[ 1 < \sigma_1 < 2 : k_0(\sigma_1) = q : z(\sigma_1) = w \]

\[ f_0^1 d\sigma_1 f_0^1 d\sigma_2 k_0(\sigma_1) k_0(\sigma_2) [\tilde{G}(\sigma_1, \sigma_2) + \tilde{\Sigma}(\sigma_1, \sigma_2)] \]

becomes

\[ \{ p,p [\tilde{G}(\sigma_1, \sigma_1) + \tilde{\Sigma}(\sigma_1, \sigma_1)] + p V q V \tilde{\Sigma}(\sigma_1, \sigma_1) + q q V [\tilde{G}(\sigma_2, \sigma_2) + \tilde{\Sigma}(\sigma_2, \sigma_2)] + q V q V \tilde{\Sigma}(\sigma_2, \sigma_2) + 2 p q [\tilde{G}(\sigma_1, \sigma_2) + \tilde{\Sigma}(\sigma_1, \sigma_2)] + 2 p V q V \tilde{\Sigma}(\sigma_1, \sigma_2) \} \]

\[ \quad (3.1.43) \]

becomes

\[ \frac{I(\sigma_1) I(\sigma_2)}{2!} e^{\{ p,p [\tilde{G}(\sigma_1, \sigma_1) + \tilde{\Sigma}(\sigma_1, \sigma_1)] + p V p V \tilde{\Sigma}(\sigma_1, \sigma_1) + q q [\tilde{G}(\sigma_2, \sigma_2) + \tilde{\Sigma}(\sigma_2, \sigma_2)] + q V q V \tilde{\Sigma}(\sigma_2, \sigma_2) \}} \]
\[ e^{[(2p\cdot q + 2p\cdot q')\Sigma(\sigma_1, \sigma_2)]} (z - w)^{2p\cdot q} e^{i(p\tilde{Y}_0(\sigma_1) + q\tilde{Y}_0(\sigma_2))} \]  
\[ (z - w)^2 p\cdot q e^{i(p + q)\tilde{Y}_0(\sigma_1) + q\tilde{Y}_0(\sigma_2))} \]

(3.1.47)

We have set \( x_n = 0 \) and hence written \( G \) and \( \tilde{Y} \). \( G(z, z) = \ln \epsilon \). \( \Sigma \) has to be Taylor expanded, but for the present we can just use the lowest order approximation and replace them all by \( \sigma(z) \). \( < I(\sigma_1)I(\sigma_2) > = \Phi(p)\Phi(q) \).

Also \( g_{55} = \frac{-1}{9} \). Using all this we get to lowest order,

\[ e^{[(p^2 - 1/4) + (q^2 - 1/4) + (2p\cdot q - 2/4)]}\sigma \int \frac{dz}{\epsilon} \int \frac{dw}{\epsilon} (z - w)^{2p\cdot q} e^{i(p + q)\tilde{Y}_0(\sigma_1) + q\tilde{Y}_0(\sigma_2))} \]

(3.1.48)

The integral over \( w \) is from \( z + \epsilon \) to \( \infty \). At infinity we assume the contribution is zero. \( \int \frac{dz}{\epsilon} \int \frac{dw}{\epsilon} (z - w)^{2p\cdot q} e^{i(p + q)\tilde{Y}_0(\sigma_1) + q\tilde{Y}_0(\sigma_2))} \approx \Phi^2(X) \)

(3.1.49)

The conclusion is that we get for the cubic interaction between three on shell tachyons a momentum independent constant as expected from bosonic string theory.

**Quartic Interaction**

Analogous to the previous calculation we introduce

\[ 0 < \sigma < 1 : k_0(\sigma) = p : z(\sigma) = z_1 \]

\[ 1 < \sigma < 2 : k_0(\sigma) = q : z(\sigma) = z_2 \]

\[ 2 < \sigma < 3 : k_0(\sigma) = k : z(\sigma) = z_3 \]

Furthermore we have,

\[ < I(\sigma_1)I(\sigma_2)I(\sigma_3) > \approx \Phi(p)\Phi(q)\Phi(k) \]

Also as before \( G(\sigma, \sigma) \approx \epsilon \) and \( \tilde{\Sigma} \approx \sigma \). \( g_{55} = \frac{-1}{9} \). This gives

\[ e^{[(p^2 - 1/9) + (q^2 - 1/9) + (k^2 - 1/9) + (2p\cdot q - 2/9) + (2p\cdot k - 2/9) + (2q\cdot k - 2/9)]}\sigma \int \frac{dz_1}{\epsilon} \int \frac{dz_2}{\epsilon} \int \frac{dz_3}{\epsilon} (z_1 - z_2)^{2p\cdot q} (z_2 - z_3)^{2q\cdot k} (z_3 - z_1)^{2p\cdot k} \]

(3.1.49)

\[ ^5 \text{Actually one should put an infrared cutoff. In the proper time formalism} \] this is automatic. Here it has to be done by hand. If we are going off shell this is important. In this paper we will ignore this issue by assuming that we are always close to being on shell.
\[ e^{(p+q+k)^2} \Phi(p) \Phi(q) \Phi(k) e^{i(p+q+k)} X \]

The integrals can be evaluated in the usual way. First we use translation invariance to set \( z_3 = 0 \). Then define \( z'_2 = \frac{z_2}{z_1} \) to get

\[ \int \frac{dz_1}{\epsilon} (\frac{z_1}{\epsilon})^{2p+k+2q,q+2} \int \frac{dz'_2}{\epsilon} (1 - z'_2)^{2p,q(z'_2)^{2q,k}} \]

The integral over \( z_1 \) gives \( \frac{1}{2} p.q + 2 q.k + 2 k.p + 2 \). The action of \( \frac{d}{\sigma} \) gives \((p + q + k)^2 - 1\). These two factors cancel on shell. The resultant integral over \( z'_2 \) is nothing but the usual Veneziano amplitude. As shown in [9] regularizing the integrals by point splitting subtracts the poles corresponding to on shell intermediate states. This gives the effective action.

### 3.2 Vector

For the massless vector \( g^{55} = 0 \) and so it is fairly obvious that the pole structure comes out right given the arguments given in the first section. The only point of the calculation below is to illustrate the use of the \( K_n \bar{Y}_n \) variables as compared to \( k_n Y_n \) used in I.

We start with the loop variable

\[ e^\int d\sigma_1 \int d\sigma_2 [k_0(\sigma_1,\sigma_2) \bar{G}(\sigma_1,\sigma_2) + \bar{\Sigma}(\sigma_1,\sigma_2)] \]

\[ e^\int d\sigma_1 \int d\sigma_2 [k_1(\sigma_1,\sigma_2) \frac{\partial}{\partial x_1} \bar{G}(\sigma_1,\sigma_2) + \bar{\Sigma}(\sigma_1,\sigma_2)] + \sigma_1 \leftrightarrow \sigma_2 \]

\[ e^i \int d\sigma [k_0(\sigma) \bar{Y}_0(z) + K_1(\sigma, z - z(\sigma)) \bar{Y}_1(z)] \]

where \( K_1(\sigma, z - z(\sigma)) = k_1(\sigma) + k_0(\sigma) \alpha_1 + k_0(\sigma)(z - z(\sigma)) \).

#### 3.2.1 Free Theory

For the free theory there is only one point.

\[ 0 < \sigma < 1 : k_0(\sigma) = k_0, k_1(\sigma) = k_1; z(\sigma) = z. \]

\( G(\sigma, \sigma) \approx ln \epsilon \). In evaluating the derivative of the Green function one has to be more careful. Thus before we set \( \sigma_1 = \sigma_2 = \sigma \) we must either do point splitting or use the regularized Green function \( ln((z - w)^2 + \epsilon^2) \).

Thus let \( \sigma_A \) and \( \sigma_B \) be the split points with \( z(\sigma_{A,B}) = z_{A,B} \). Of course \( k_0(\sigma_A) = k_0(\sigma_B) = k_0 \) and the same for \( k_1 \).

Then we have two possibilities
\[ \sigma_1 = \sigma_A, \text{ and } \sigma_2 = \sigma_B \]

or

\[ \sigma_1 = \sigma_B, \text{ and } \sigma_2 = \sigma_A \]

We refer to \( x_n(\sigma_1) \) by \( x_n \) and \( x_n(\sigma_2) \) as \( y_n \).

Thus we have

\[
k_1(\sigma_1)k_0(\sigma_2) \frac{d}{dx_1} G(\sigma_1, \sigma_2, x, y) = k_1(\sigma_A)k_0(\sigma_B) \frac{d}{dx_1} G(\sigma_A, \sigma_B, x, y) = k_1k_0 \frac{1}{z_A - z_B}
\]

When we consider the second possibility we get

\[
k_1(\sigma_1)k_0(\sigma_2) \frac{d}{dx_1} G(\sigma_1, \sigma_2, x, y) = k_1(\sigma_B)k_0(\sigma_A) \frac{d}{dx_1} G(\sigma_B, \sigma_A, x, y) = k_1k_0 \frac{1}{z_B - z_A}
\]

Adding the two gives zero. This is because of the antisymmetric property of the derivative of the Green function.

If we use \( \ln((z - w)^2 + \epsilon^2) \) it is easy to see that the derivative vanishes when \( z \to w \). Thus using either method we get the same result.

The derivative of \( \tilde{\Sigma} \) does not have any subtlety and we simply get

\[
k_1k_0 \frac{\partial}{\partial x_1} \Sigma(\sigma, \sigma, x, y) = k_1k_0 \frac{1}{2} \frac{\partial}{\partial x_1} \Sigma(\sigma, \sigma, x, x).
\]

We have used the fact that \( \Sigma \) is symmetric in its arguments. Thus to lowest order in \( x_n \) we simply get

\[
k_1k_0 \frac{\partial}{\partial x_1} \Sigma
\]

Thus \((3.2.52)\) becomes for the term proportional to \( \tilde{Y}_1 \) (For this term \( g_{55} = 0 \)),

\[
\epsilon^{k_0} e^{k_0} \Sigma k_1k_0 \frac{\partial}{\partial x_1} \Sigma e^{i(k_0\tilde{Y}_0 + K_1\tilde{Y}_1)} + \epsilon^{k_0} e^{k_0} \Sigma iK_1 \tilde{Y}_1 e^{i(k_0\tilde{Y}_0)}
\]

(3.2.53)

When we vary w.r.t. \( \Sigma \) we get (using \( \frac{\partial}{\partial x_1} K_1 = k_0 \))

\[
[-ik_0(k_1k_0) + k_0^2iK_1] \tilde{Y}_1 e^{i(k_0\tilde{Y}_0)} = 0
\]

(3.2.54)

Using \( k_0k_0g_{55} = 0 \) and \( < k_1^\mu > = A^\mu \), and setting \( x_n = 0 \), we recover Maxwell’s equation

\[
\partial_\mu F^{\mu\nu} = 0
\]
### 3.2.2 Cubic term

The first correction comes in at cubic order. The calculation is done in the Appendix as an illustration. The terms that have a tachyon pole is specific to the bosonic string. The other terms correspond to superstrings (and also to Yang-Mills). One can compare these terms with known results as given for instance in \([10]\) and see that they agree.

Thus we see in these examples that on shell S-matrix elements are correctly reproduced by this theory. General arguments were given in an earlier section. While this does not constitute a proof, hopefully that can also be done with some more work.

In this paper, we have not attempted to calculate higher order terms using the Taylor expansion of \(I\). These higher order contributions (coming from derivatives of \(\Sigma\)) to the equations of motion are additions that are dictated by gauge invariance. The point that needs to be stressed is that any term that comes from derivatives of \(\Sigma\) are “longitudinal” or “gauge” pieces that can be set to zero if we are only interested in S-matrix elements of physical degrees of freedom. Thus they do not affect the arguments presented in this section, which were to show that the S-matrix elements are reproduced correctly. Nevertheless these terms are necessary when one wants manifest gauge invariance.

### 4 Conclusions

We have discussed how one can use loop variables to derive gauge invariant equations of motion for all the modes of the bosonic string. The first step was taken in I. There the theory was written in one higher dimension where the modes are all massless. In this paper we have extended the results of I by explaining how, by a particular type of dimensional reduction of the massless theory, we get the right spectrum and on shell scattering amplitudes of bosonic string theory. The theory is gauge invariant even off shell. Thus in principle we have an off shell gauge invariant formulation. We have not attempted a proof of this result to all orders and for all modes. However we have given some explicit examples and also arguments on why the method should work in general.

What the Koba-Nielsen integration does to gauge transformation and equation, in the continuum limit has not been discussed in this paper. This is an important question that deserves further study. Another issue that needs further work is the evaluation of higher order corrections arising from the
Taylor expansion of \( \tilde{\Sigma} \). We also need to work out the dimensionally reduced versions of the equations of motion for massive particles (“\( S_2 \)” and “\( S_{11} \)” of I) where one has to worry about the degrees of freedom corresponding to \( k_5^5 \), (they were called \( q_n \) in [2]). It would also be interesting to get some geometrical interpretation for (2.1.1) and (2.2.38).

We would like to recollect some of the intriguing features that have emerged from this study. First, the theory is written more elegantly as a massless theory in one higher dimension. Second, the interactions are obtained simply by extending the string to a band. Written in terms of bands it appears just like a free theory. Both the above features are reminiscent of M-theory [11]. The third and probably most intriguing feature is the form of the gauge transformation. It looks like a local rescaling of the generalized momenta. This is a space-time scale transformation. The global version of this is of course just the usual renormalization group (in space-time). The local (i.e. local along the string) version of this seems to be a (part of the) gauge group for the string. This was of course one of the original motivations for this approach [2].

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A Appendix: Basic Definitions

The basic loop variable is
\[ e^{i \int c \alpha(t)k(t)\partial_z X(z+t)dt + ik_0 X} = e^{i \sum_n k_n Y_n} \]
where \( \alpha(t) \) is an einbein and \( k(t) \) is a distributed momentum. They have mode expansions:
\[ \alpha(t) = \sum_{n \geq 0} \alpha_n t^{-n} \]
\[ k(t) = \sum_{n \geq 0} k_n t^{-n} \]

\( Y_n \) is defined by
\[ Y_n = \frac{\partial Y}{\partial x_n} \]
where
\[ Y = \sum_n \alpha_n \frac{\partial^n X}{(n-1)!} \equiv \sum_n \alpha_n \tilde{Y}_n \]
and \( x_n \) are defined by
\[ \sum_{n \geq 0} \alpha_n t^{-n} = e^{\sum_{n \geq 0} x_n t^{-n}}. \]
The \( \alpha_n \) satisfy
\[ \frac{\partial \alpha_n}{\partial x_m} = \alpha_{n-m}. \]

Using the above definitions one can easily show that
\[ \frac{\partial^2}{\partial x_n \partial x_m} Y = \frac{\partial}{\partial x_{n+m}} Y. \]

The definitions of \( \tilde{\Sigma} \) and \( \tilde{G} \) are as follows. Define, (using the notation \( z_i = z(\sigma_i) \))
\[ D_{z_1} = D_{z(\sigma_1)} = 1 + \alpha_1(\sigma_1) \frac{\partial}{\partial z(\sigma_1)} + \alpha_2 \frac{\partial^2}{\partial z^2(\sigma_1)} + ... \]  \( \) (A.1)
so that
\[ Y(z(\sigma)) = Dz(\sigma)X(z(\sigma)) \] (A.2)
then,
\[ \tilde{G}(z_1, z_2) = D_{z_1} D_{z_2} G(z_1, z_2) \] (A.3)
\[ \tilde{\Sigma}(\sigma_1, \sigma_2) = D_{z_1} D_{z_2} \rho(\sigma_1, \sigma_2) \] (A.4)
where
\[ \rho(\sigma_1, \sigma_2) = \frac{\mu(z(\sigma_1)) - \mu(z(\sigma_2))}{z(\sigma_1) - z(\sigma_2)} \] (A.5)
is the generalization of the usual Liouville mode \( \rho(\sigma) \) which is equal to \( \frac{d\mu}{dz} \).

The \( \tilde{\Sigma} \) dependence in the loop variable is obtained by the following step:
\[ e^{ik_0 \sigma x_n} D_{z_1} X e^{ip_m \sigma x_m} D_{z_2} X \] (A.6)
defines the action of the Virasoro generators on the two sets of vertex operators.
\[ e^{ik_0 \sigma x_n} D_{z_1} D_{z_2} \int du \mu(u) [z_1 + u - z_2 + u] \] (A.7)
\[ = e^{ik_0 \sigma x_n} \tilde{\Sigma} \] (A.8)
This expression is only valid to lowest order in \( \mu \) which is all we need here.\(^6\)

B Appendix: Cubic Term in Vector Particle Equation

We start with loop variable (3.2.52),
\[ e \int d\sigma_1 \int d\sigma_2 k_0(\sigma_1), k_0(\sigma_2) [\tilde{G}(\sigma_1, \sigma_2) + \tilde{\Sigma}(\sigma_1, \sigma_2)] \]
\[ = e^{ik_0 \sigma x_n} \tilde{\Sigma}(\sigma_1, \sigma_2) + \tilde{\Sigma}(\sigma_1, \sigma_2) + \tilde{\Sigma}(\sigma_1, \sigma_2) \]
\[ e^{i k_1(\sigma_1).k_0(\sigma_2) \sigma x_1(\sigma_1)} [\tilde{G}(\sigma_1, \sigma_2) + \tilde{\Sigma}(\sigma_1, \sigma_2) + \sigma_1 + \sigma_2] \]
\[ e^{i k_0(\sigma) \tilde{Y}_0(z) + K_1(\sigma, z - z(\sigma)) \tilde{Y}_1(z)} \] (B.1)
and keep terms involving three \( k_1 \)'s. This will produce terms of the form \( A.pA.qA \). We can compare these with standard results from say [10]. If they match, then by gauge invariance, other terms are bound to agree as well.
\[ e \int d\sigma_1 \int d\sigma_2 k_0(\sigma_5), k_0(\sigma_6) [\tilde{G}(\sigma_5, \sigma_6) + \tilde{\Sigma}(\sigma_5, \sigma_6)] \]
\(^6\)The exact expression is given in [12]
\[
\frac{1}{2!} \int d\sigma_1 \int d\sigma_2 \{ [k_1(\sigma_1), k_0(\sigma_2)] \frac{\partial}{\partial x_1(\sigma_1)} \bar{G}(\sigma_1, \sigma_2) \} + \sigma_1 \leftrightarrow \sigma_2 \]
\[
\int d\sigma_3 \int d\sigma_4 \{ [k_1(\sigma_3), k_0(\sigma_4)] \frac{\partial}{\partial x_1(\sigma_3)} \bar{G}(\sigma_3, \sigma_4) \} + \sigma_1 \leftrightarrow \sigma_2 \]
\[
i K_1(\sigma, -z) \tilde{\mathcal{Y}}^\mu_1(0) e^{i k_0 \tilde{Y}_0(0)} \quad (B.2)
\]

Let us call the three distinct locations \(\sigma_I, \sigma_{II}, \sigma_{III}\).

Let
\[
k_0(\sigma_I) = p; \quad z(\sigma_I) = z_1
\]
\[
k_0(\sigma_{II}) = q; \quad z(\sigma_{II}) = z_2
\]
\[
k_0(\sigma_{III}) = k; \quad z(\sigma_{III}) = z_3
\]

The following is one possible assignment:
\[
\sigma_1 = \sigma_I; \quad \sigma_2 = \sigma_{II} \text{ or } \sigma_{III}
\]
\[
\sigma_3 = \sigma_{II}; \quad \sigma_4 = \sigma_I \text{ or } \sigma_{III}
\]
\[
\sigma = \sigma_{III}
\]

\(\sigma_5\) and \(\sigma_6\) can equal any of them.

When \(\sigma_5 = \sigma_6\) we need to point split. This gives us
\[
(\varepsilon)^{p^2 + q^2 + k^2} (z_1 - z_2)^2 p.q (z_2 - z_3)^2 q.k (z_3 - z_1)^2 k.p
\]
\[
\frac{1}{2!} \left\{ \frac{p_1.q}{z_1 - z_2} + \frac{p_1.k}{z_1 - z_3} \left\{ \frac{q_1.p}{z_2 - z_1} + \frac{q_1.k}{z_2 - z_3} \right\} K_1^\mu \tilde{\mathcal{Y}}^\mu_1(0) e^{i (p + q + k) \tilde{Y}_0(0)} \right\}
\]

The other assignments of \(\sigma\)'s give similar terms which include all permutations of \(p, q, k\).

We are left with integrals over \(z_1\) and \(z_2\). (we set \(z_3 = 0\)). After changing variables to \(z'_2 = \frac{z_2}{z_1}\) and integrating over \(z'_2\), we get the following terms (and permutations to symmetrize in \(p, q, k\)):

i) \[
- \int dz'_2 (1 - z'_2)^2 p.q - (z'_2)^2 q.k p_1.q q_1.p = -B(2p.q - 1, 2q.k + 1)p_1.q q_1.p \quad (B.3)
\]

ii) \[
\int dz'_2 (1 - z'_2)^2 p.q - 1 (z'_2)^2 q.k - 1 p_1.q q_1.k = -B(2p.q, 2q.k)p_1.q q_1.k \quad (B.4)
\]

iii) \[
- \int dz'_2 (1 - z'_2)^2 p.q - 1 (z'_2)^2 q.k p_1.k q_1.p = -B(2p.q, 2q.k + 1)p_1.k q_1.p \quad (B.5)
\]
iv) $\int dz_2 (1 - z_2')^{2q} (z_2')^{2q,k-1} p_1 . k q_1 . k = B(2p.q + 1, 2q.k)p_1 . k q_1 . k$  (B.6)

Using the expansion

$$B(x, z) = \left( \frac{1}{x} + \frac{1}{z} \right) (1 - \zeta(2) x z) + ...$$

$$= \frac{1}{x} + \frac{1}{z} - \zeta(2)(z + x) + ...$$

we see that i) corresponds to a tachyon pole. So we will not compare it with the results of [10]. The remaining three and their permutations can be compared. We also use the on-shellness conditions $p^2 = q^2 = k^2 = 0$ as well as transversality, $p_1 . p = q_1 . q = k_1 . k = 0$. One can explicitly check that the leading pole terms corresponding to vector exchange as well as the contact terms (proportional to $\zeta(2)$ in the above equations) agree with the corresponding expressions in [10].