Towards Understanding Learning in Neural Networks with Linear Teachers

Roei Sarussi 1 Alon Brutzkus 1 Amir Globerson 1

Abstract

Can a neural network minimizing cross-entropy learn linearly separable data? Despite progress in the theory of deep learning, this question remains unsolved. Here we prove that SGD globally optimizes this learning problem for a two-layer network with Leaky ReLU activations. The learned network can in principle be very complex. However, empirical evidence suggests that it often turns out to be approximately linear. We provide theoretical support for this phenomenon by proving that if network weights converge to two weight clusters, this will imply an approximately linear decision boundary. Finally, we show a condition on the optimization that leads to weight clustering. We provide empirical results that validate our theoretical analysis.

1. Introduction

Neural networks have achieved remarkable performance in many machine learning tasks [Krizhevsky et al., 2012; Silver et al., 2016; Devlin et al., 2019]. Although their success has already transformed technology, a theoretical understanding of how this performance is achieved is not complete. Here we focus on one of the simplest learning settings that is still not understood. We consider linearly separable data (i.e., generated by a “linear teacher”) that is being learned by a two layer neural net with leaky ReLU activations and minimization of cross entropy loss using gradient descent or its variants. Two key questions immediately come up in this context:

• The Optimization Question: Will the optimization succeed in finding a classifier with zero training error, and arbitrarily low training loss?

• The Inductive Bias Question: With a large number of hidden units, the network can find many solutions that will separate the data. Which of these will be found by gradient descent?

Our work addresses these questions as follows.

The Optimization Question: We prove that stochastic gradient descent (SGD) will converge to arbitrary low training loss. Concretely, we show that for any $\epsilon > 0$, SGD will converge to $\epsilon$ cross-entropy loss in $O\left(\frac{1}{\epsilon^2}\right)$ iterations. We consider SGD which performs multiple passes over the data and we devise a novel variant of the perceptron proof to analyze this setting. Our analysis bounds the number of epochs that have high loss examples, and uses this to show convergence to a low loss solution. Importantly, our result holds for any network size and scale of initialization. Therefore, our analysis goes beyond the Neural Tangent Kernel (NTK) analyses which require large network sizes and relatively large initialization scales.

The Inductive Bias Question: We empirically observe that when a small initialization scale is used, the learned network converges to a decision boundary that is very close to linear. See Figure 1b for a 2D example. We also observe that all neurons cluster nicely into two sets of vectors (i.e., they form two groups of well-aligned neurons) as in Figure 1d. To support these empirical findings, we provide the following theoretical results:

(1) We prove that an approximate clustering of the neurons implies that the decision boundary of the network is approximately linear. This is a result of a nice property of leaky ReLU networks which we prove in Section 5.

(2) We provide a novel sufficient condition on the optimization path of gradient flow which implies convergence to clustered solutions. With the result above, it implies convergence to a linear decision boundary. The condition states that from a certain iteration on, all neurons with the same output sign “agree” on the classification of the data. We observe that this condition holds empirically for several synthetic and real datasets. Finally, we use the latter result to prove that under certain assumptions, the learned network is a solution to an SVM problem with a specific kernel.

Our results above make significant headway in understanding why optimization is tractable with linear teachers, and why convergence is to approximately linear boundaries. We

1See Section 5.1 and Section 6.3 for more empirical examples.
also provide empirical evaluation that confirms that weight clustering indeed explains why approximate linear decision boundaries are learned.

2. Related Work

Since training neural networks is NP-Hard for worst-case datasets (Blum & Rivest, 1992), recent works have analyzed neural networks under certain data assumptions to better understand their performance in practice. One common assumption is to analyze neural networks when the data is linearly separable. Even in this case, the theoretical analysis of optimization and generalization is far from resolved. In a work closely related to ours, Brutzkus et al. (2018) consider this setting and show that SGD converges to zero loss for linearly separable data (which was later extended to ReLU activations using noisy SGD in Wang et al., 2019). The key difference from our work is that they use the hinge loss instead of the cross entropy loss. The cross entropy loss creates unique challenges for proving convergence as we show in Section 4. Thus, their results cannot be directly applied for the cross entropy loss and we use novel techniques to guarantee convergence of SGD in this case. The second key difference from their work is that we present novel insights on the inductive bias of SGD using results of Lyu & Li (2020) and Ji & Telgarsky (2020) which hold for the cross entropy loss and not for the hinge loss. Finally, our result which shows that a network with clustered neurons has an approximate linear decision boundary is new and holds irrespective of the loss used.

Recently, Phuong & Lampert (2021) analyzed a subclass of linear teachers where data is “orthogonally separable”. In this case they show that training a ReLU network with the cross entropy loss results in a solution where weights are aligned. In terms of our results, this can be viewed as a case of convergence to a particular PAR (see Section 6). Several other works assume that the data is linearly separable but also that the networks are linear (Ji & Telgarsky, 2019a; Moroshko et al., 2020; Gunasekar et al., 2018). We study the more challenging and realistic setting of two layer nonlinear networks with Leaky ReLU activations. Several works (Lyu & Li, 2020; Ji & Telgarsky, 2020; Nacson et al., 2019) studied the inductive bias of two-layer homogeneous networks and showed connections between gradient methods and margin maximization. Their results hold under the assumption that gradient methods achieve a certain loss value. However, we provide a convergence proof for SGD that shows that it can obtain arbitrary low loss values. Furthermore, we use the results of Lyu & Li (2020) and Ji & Telgarsky (2020) to obtain a more fine grained analysis of the inductive bias of gradient flow for linear teachers. Other works considered the inductive bias of infinite two-layer networks (Chizat & Bach, 2020; Weinberger et al., 2019; Mei et al., 2018). Our results hold for networks of any size. An inductive bias towards clustered solutions has been observed in Brutzkus & Globerson (2019) and proved for a simple setup with nonlinear data.

Fully connected networks were also analyzed via the NTK approximation (Du et al., 2019; 2018; Arora et al., 2019; Ji & Telgarsky, 2019b; Cao & Gu, 2019; Jacot et al., 2018; Fiat et al., 2019; Allen-Zhu et al., 2019; Li & Liang, 2018; Daniely et al., 2016). However other works (Yehudai & Shamir, 2019; Daniely & Malach, 2020) have highlighted limitations of the NTK framework, suggesting that it does not accurately model neural networks as they are used in practice. Our convergence analysis in Section 4 holds for any initialization scale and network size and therefore goes beyond the NTK analysis.

Recently, Li et al. (2020) analyzed two-layer networks beyond NTK in the case of Gaussian inputs and squared loss. We assume linearly separable inputs and the cross entropy loss. Allen-Zhu & Li (2019) analyze a three layer ResNet and provide generalization guarantees for sufficiently wide networks in a regression setting. Woodworth et al. (2020) study the inductive bias of gradient methods for a simplified nonlinear model.

3. Preliminaries

Notations: We use \(|| \cdot |||_2\) to denote the \(L^2\) norm on vectors and Frobenius norm on matrices. For a vector \(v\) we denote \(\hat{v} = \frac{v}{||v||_2}\).

Data Generating Distribution: Define \(X = \{x \in \mathbb{R}^d : ||x|| \leq R_x\} \) and \(Y = \{\pm 1\}\). We consider a distribution of linearly separable points. Formally, let \(D(x, y)\) be a distribution over \(X \times Y\) such that there exists \(w^* \in \mathbb{R}^d\) for which \(\mathbb{P}_{(x,y) \sim D}[yw^* \cdot x \geq 1] = 1\). Let \(S := \{(x_1, y_1), \ldots, (x_n, y_n)\} \subset X \times Y\) be a training set sampled IID from \(D(x, y)\). Let \(S_+ \subset S\) denote the training points with positive labels and \(S_-\) with negative labels. We denote \(x \in S_+\) if there exists \(y \in \{\pm 1\}\) such that \((x, y) \in S\).

Network Architecture: We consider a two-layer neural network with \(2k > 0\) hidden units, where the second layer is fixed and the first layer is learned. Formally, we denote the parameters of the network that are learned by \(W \in \mathbb{R}^{2k \times d}\) and for the second layer we define the fixed vector \(v \in \mathbb{R}^{2k}\), where \(v = (v_1, v_2, \ldots, v_k, -v_1, \ldots, -v_k)\) and \(v > 0\). The network output is given by the function \(N_W : \mathbb{R}^d \to \mathbb{R}\) defined as \(N_W(x) = v \cdot \sigma(Wx)\), where \(\sigma(x) = \max\{x, \alpha\}\) is the Leaky-ReLU activation function applied element-wise, parameterized by \(0 < \alpha < 1\) and \(\cdot\) denotes dot product.

\(^2\)We do not introduce a bias term, but all our results extend to using bias (see Supplementary for a formal justification).
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(a) Decision boundary (large initialization)
(b) Decision boundary (small initialization)
(c) Learned Neurons (large initialization)
(d) Learned Neurons (small initialization)

Figure 1: Results for training a Leaky-ReLU network on linearly separable data, with different initialization scales. Figures (a)+(b) show the resulting decision boundary. It can be seen that large initialization leads to a non-linear boundary, whereas small initialization leads to a linear boundary. Figures (c)+(d) show the learned weight vectors (normalized to unit norm). It can be seen that small initialization leads to two tight clusters of neurons whereas large initialization does not lead to clustering. The network has 100 neurons, initialized from a Gaussian with standard deviation 0.001 for small initialization and 3.0 for large initialization.

It is easy to see that such a network is as expressive as a standard two-layer network where the second layer vector is not fixed [Brutzkus et al., 2018]. Furthermore, the assumption that the second layer is fixed is common in previous works (e.g., see Du et al., 2018; Brutzkus et al., 2018; Ji & Telgarsky, 2019b). We denote row \( i \) of \( W \) by \( w^{(i)} \) and row \( k+i \) by \( u^{(i)} \) for \( 1 \leq i \leq k \). We say that \( w^{(i)} \) are the \( w \) neurons and \( u^{(i)} \) are the \( u \) neurons. Then the network is given by:

\[
N_W(x) = v \sum_{i=1}^{k} \sigma \left( w^{(i)} \cdot x \right) - v \sum_{i=1}^{k} \sigma \left( u^{(i)} \cdot x \right)
\]

We focus on two different gradient-based methods in different parts of the paper. First, we consider the case where \( L_{\Theta}(W) \) is minimized using SGD in epochs with a batch size of one and a learning rate \( \eta \). Data points are sampled without replacement at each epoch. Denote by \( W_t \) the parameters after \( t \) updates.

Our main optimization result, described in Section 4 is shown for SGD. When studying convergence to clustered solutions, we consider gradient flow, because there we can use recent strong results from Lyu & Li (2020) and Ji & Telgarsky (2020). Recall that gradient flow is the infinitesimal step limit of gradient descent where \( W_t \) changes continuously in time and satisfies the differential inclusion

\[
\frac{d}{dt} W_t \in -\partial^o L_{\Theta}(W_t)
\]

Here \( \partial^o L_{\Theta}(W_t) \) stands for Clarke’s sub-differential which is a generalization of the differential for non-differentiable functions.

The importance of gradient flow is that it can be shown to maximize margin in the following sense. Define the network margin for a single data point \((x_i, y_i)\) by

\[
q_i(W) := y_i N_W(x_i)
\]

and the normalized network margin as:

\[
\gamma(W) := \frac{1}{\|W\|} \min_{(x, y) \in S} y_i N_W(x)
\]
where $\|W\|$ is the Frobenius norm of $W$.

The smoothed margin is defined as:

$$\tilde{\gamma}(W) := \frac{1}{\|W\|} \log \left( \frac{1}{\exp(nL_2(W))} - 1 \right)$$ (5)

From Lyu & Li (2020) and Ji & Telgarsky (2020) it follows that gradient flow converges to KKT points of the network margin maximization problem (see Supplementary for details). Here we will use this result in Section 6 to characterize the linear decision boundaries of learned networks.

4. Risk Convergence

We next prove that for any $\varepsilon > 0$, SGD converges to $\varepsilon$ empirical-loss (see Eq. (2)) within $O\left(\frac{\eta}{\varepsilon^2} \right)$ updates.

Let $\tilde{W}_t = \left( w_1^{(1)}, \ldots, w_k^{(k)}, u_1^{(1)}, \ldots, u_k^{(k)} \right) \in \mathbb{R}^{2kd}$ be the vectorized version of $W_t$. We assume that the network is initialized such that the norms of all rows of $W_0$ are upper bounded by some constant $R_0 > 0$. Namely for all $1 \leq i \leq k$ it holds that $\|w_0^{(i)}\|, \|u_0^{(i)}\| \leq R_0$.

Define $M(n, \varepsilon) = Cn^2$, where $C$ is a constant that depends polynomially on $R_\varepsilon, \ga, R_0, k, \eta, v$ and $\|w^*\|$. See the supplementary for the exact definition of $M(n, \varepsilon)$.

The following theorem states that SGD will converge to $\varepsilon$ loss within $M(n, \varepsilon)$ updates.

**Theorem 4.1.** For any $\varepsilon > 0$, there exists an iteration $t \leq M(n, \varepsilon)$ such that $L_\varepsilon(W_t) < \varepsilon$.

We note that the convergence analysis holds for any $\eta > 0$. This is in line with other analyses of learning linearly separable data, which show that convergence holds for any $\eta > 0$ (Brutzkus et al. 2018). We next briefly sketch the proof of Theorem 4.1. The full proof is deferred to the supplementary.

Our proof is based on the proof for the hinge loss in Brutzkus et al. (2018) with several novel ideas that enable us to show convergence for the cross entropy loss.

For the hinge loss proof, Brutzkus et al. (2018) consider the vector $\tilde{W}^* = (\tilde{w}^1, \ldots, \tilde{w}^k, \tilde{u}^1, \ldots, \tilde{u}^k) \in \mathbb{R}^{2kd}$ and define $F(W_t) = \tilde{W}^t, \tilde{W}^* \in \mathbb{R}^{2kd}$ and $G(W_t) = \|\tilde{W}^t\|$. Using an online perceptron proof and the fact that $\frac{\|F(W_t)\|}{G(W_t)} \leq 1$, they obtain a bound on the number of points with non-zero loss that SGD samples, which provides the convergence guarantee. This proof is unique to the hinge loss setting,

where points can have exactly zero loss. However, in the case of the cross entropy loss, every update has a non-zero loss. Therefore, the online proof for the hinge loss cannot be applied in this case. To overcome this we (1) use an "epoch-based" analysis that is tailored to the SGD variant we use here, that samples data without replacement in each epoch. (2) bound the number of epochs where there exists a point with loss at least $\varepsilon$. By applying these key ideas with further technical analyses that are unique to the cross entropy loss, we prove Theorem 4.1.

5. Weight Clustering and Linear Separation

As shown in Figure 1, learning with SGD can result in a linear decision boundary, despite the existence of zero-loss solutions that are highly non-linear. In what follows, we provide theoretical and empirical insights into why an approximately linear boundary is learned.

We next show a nice property of Leaky-ReLU networks that can explain why they converge to linear decision boundaries. Assume that a learned network in Eq. (1) is such that all of its $w$ neurons form a ball of “small” radius (i.e., they are well clustered) and likewise all the $u$ neurons (see Figure 1 and Figure 2 for simulations that show such a case). Then, as we show in Theorem 5.1 this implies that the resulting decision boundary will be approximately linear. Later, we give further empirical and theoretical support that learned networks indeed have this clustering structure, and together with Theorem 5.1 this explains the approximate linearity.

Consider the network in Eq. (1). Denote $\overline{w} = \frac{1}{k} \sum_{i=1}^k w^{(i)}$ and $\overline{u} = \frac{1}{k} \sum_{i=1}^k u^{(i)}$. Also, let $r$ denote the maximum radius of the positive and negative weights around their averages. Namely:

$$\|w^{(i)} - \overline{w}\|_2 \leq r, \quad i = 1, \ldots, k$$

$$\|u^{(i)} - \overline{u}\|_2 \leq r, \quad i = 1, \ldots, k$$

The following result says that the decision boundary will be linear except for a region whose size is determined by $r$.

**Theorem 5.1.** Consider the linear classifier $f(x) = \text{sign}(\langle \overline{w} - \overline{u}, x \rangle)$. Then $\text{sign}(\langle N(W) x \rangle) = f(x)$ for all $x$ such that $|\langle \overline{w} - \overline{u}, x \rangle| \geq 2r\|x\|$. The theorem has a simple intuitive implication. The smaller $r$ is, the closer the classifier is to linear. In particular when $r = 0$ the classifier is exactly linear.

An alternative interpretation of the theorem comes from rewriting the condition as:

$$\frac{|\langle \overline{w} - \overline{u}, x \rangle|}{\|x\|} \leq \frac{r}{\|\overline{w} - \overline{u}\|}$$ (6)

Namely that linearity holds whenever the absolute value of
the cosine of the angle between $x$ and $\overline{w} - \overline{u}$ is greater than $\frac{r}{\|\overline{w} - \overline{u}\|}$.

The proof is in the supplementary and is somewhat technical, but a brief outline is as follows. First we show that $\forall x \in \mathbb{R}^d$ such that $|\langle \overline{w} - \overline{u} \rangle \cdot x \rangle \geq r \|x\|$ or it holds that $\forall 1 \leq j \leq k \ w^{(j)} \cdot x < 0$ and similarly for the $u$ neurons. Using this, we show that $\forall x \in \mathbb{R}^d$ such that $|\langle \overline{w} - \overline{u} \rangle \cdot x | \geq r \|x\|$. We show this by dividing the input space to four regions based on the classification of the $w$ and $u$ neurons and using properties of Leaky ReLU. Then via an involved analysis, we proceed to prove that $\langle N_W(x) \rangle = \text{sign} \left( \langle (\overline{w} - \overline{u}) \cdot x \rangle \right)$. We show this by dividing the input space to four regions based on the classification of the $w$ and $u$ neurons and using properties of Leaky ReLU. Then via an involved analysis, we proceed to prove that $\langle N_W(x) \rangle = \text{sign} \left( \langle (\overline{w} - \overline{u}) \cdot x \rangle \right)$.

We note that the proof strongly relies on two assumptions. The first is that the activation function is Leaky ReLU. The result is not true for ReLU networks (see supplementary for an example). The second is that the clusters correspond to the $w$ and $u$ sets of neurons.

5.1. Experiments

Theorem 5.1 states that if neurons cluster, the resulting decision boundary will be approximately linear. But do neurons actually cluster in practice, and what is the resulting $r$? In Figure 2, we show the value of $r$ during training. We use this $r$ to calculate the linear regime in Theorem 5.1 and the fraction of train and test points that fall outside this regime. It can be seen that for small initialization, this fraction converges to zero, implying that the learned classifiers are effectively linear over the data. Additional experiments in the supplementary provide support for the neurons being tightly clustered and $r$ being very small.

Theorem 5.1 shows that a well clustered network leads to a linear decision boundary. However, it does not imply that the network output itself is a linear function of the input. Figure 3 provides a nice illustration of this fact.

6. On Conditions for Convergence to Clustered Solutions

Figure 1 suggests that gradient methods converge to a network with a linear decision boundary when trained on linearly separable data. Understanding when this occurs is important, because a model with a linear decision boundary has good generalization guarantees.

In the previous section we saw that clustering of neurons to two directions implies that the network has an approximate linear decision boundary. Therefore, this reduces the problem of proving that the network has a linear decision boundary to proving that the network neurons are well clustered. It remains to show under which conditions gradient methods converge to clustered solutions.

Providing an end-to-end analysis which shows that gradient methods converge to clustered solutions is a major challenge. In this section we provide initial results for tackling this problem. In Section 6.1 we derive a novel condition on the optimization trajectory which implies that the network converges to a clustered solution and therefore to a linear decision boundary. In Section 6.2 we study a special case where a more fine-grained characterization of the linear decision boundary can be derived using a convex optimization program. Finally, we empirically validate our findings in Section 6.3.

To obtain the results in this section, we apply recent results of Lyu & Li (2020) and Ji & Telgarsky (2020) and therefore make the same assumptions presented in these papers. Specifically, we assume that we run gradient flow (GF) as defined in Section A. We further assume that we are in the late phase of training:

**Assumption 6.1.** There exists $t_0$ such that $L_{\theta}(W_{t_0}) < \frac{1}{n}$.

We note that by the results in Section 4, SGD can attain the loss value in Assumption 6.1. However, in this section we need this assumption because we consider gradient flow and not SGD.

6.1. A Sufficient Condition

We first observe that using Theorem 5.1, we can conclude that when the neurons are perfectly clustered around two directions (i.e., $r = 0$), the decision boundary is linear. We formally define this below.

**Definition 6.1.** A network $N_W(x)$ is perfectly clustered if for all $1 \leq i, j \leq k$ it holds that: $w^{(i)} = w^{(j)}$ and $u^{(i)} = u^{(j)}$.

By applying Theorem 5.1 with $r = 0$, we have:

**Corollary 6.1.** If a network $N_W$ is perfectly clustered, then its decision boundary is linear for all $x \in \mathbb{R}^d$.

For completeness we provide a proof in the supplementary (this result is easier to prove directly than Theorem 5.1).

The key question that remains is under which conditions is the learned network perfectly clustered? To address this, we define a novel condition on the optimization trajectory that implies clustering. We define the Neural Agreement Regime (NAR) of weights of a network as follows. Informally, a network is in the NAR regime if all the $w$ neurons “agree” on the classification of the training data and likewise for the $u$ neurons. Classification in both
Assume that Assumption 6.1 holds and consider the NAR regime $\mathcal{N}$ with parameters $(\beta, c^w, c^u)$. Assume that there exists a time $T_{NAR} \geq t_0$ such that for all $t \geq T_{NAR}$ it holds that $\hat{W} \in \mathcal{N}$. Then, gradient flow converges to a solution in $\mathcal{N}$ and at convergence the network with normalized parameters $N_{\hat{W}}(x)$ is perfectly clustered.

Theorem 6.1 says that if training is such that the trajectory enters an NAR and never leaves it, then the network will become perfectly clustered. The proof uses results from Lyu & Li (2020) and Ji & Telgarsky (2020) that together guarantee convergence of gradient flow to a KKT point of a minimum norm optimization problem. The theorem then follows from a simple observation that in an NAR, the KKT conditions imply that the network is perfectly clustered. The proof is in the supplementary.

Using Corollary 6.1 we immediately obtain the following.

**Corollary 6.2.** Under the assumptions in Theorem 6.1 $\text{GF}$ converges to a network with a linear decision boundary.

Therefore, we see that if a network is at an NAR from some time $T_{NAR}$, then it will converge to a solution with a linear decision boundary. The question that remains is whether networks indeed converge to an NAR and remain there.

### 6.2. The Perfect Agreement Regime

To better understand convergence to NARs, in this section we study a specific NAR for which we provide a more fine-grained analysis. We identify conditions on the training data and optimization trajectory that imply that gradient flow converges to an NAR which we call the Perfect Agreement Regime (PAR). Using Theorem 6.1 and results from Lyu & Li (2020), Ji & Telgarsky (2020), we provide a complete characterization of the weights that gradient flow converges to in this case. Admittedly, the conditions on the data and optimization trajectory are fairly strong. Nonetheless, we show that our theoretical results accurately predict the dynamics that we observe in experiments. Indeed, in Section 6.3 we show empirically that for certain linearly separable datasets, gradient flow converges to a solution in the PAR which is in agreement with our results.

In the PAR, each neuron classifies the data perfectly. Namely, all $w$ neurons classify like the ground truth $w^*$, and all $u$ neurons classify like $-w^*$. Formally, let $y = \ldots$.
Theorem 6.2. Assume that:

\[ V \]

Thus, sufficient condition for convergence to PAR.

\[ S \]

are with margin \( \beta \) the negative points as a positive one, where all classifications converge to a PAR. The conditions require a lower bound on the network smoothed margin (Eq. (4)), as well as a separability condition on the data. To define the separability condition we consider the following:

\[
V^+_\beta (S) := \{ \mathbf{v} \in \mathbb{R}^d | \forall \mathbf{x} \in S_+ \quad \hat{\mathbf{v}} \cdot \mathbf{x} \geq \beta, \\
\exists \mathbf{x} \in S_- \quad \text{s.t.} \quad \hat{\mathbf{v}} \cdot \mathbf{x} \geq \beta \}
\]

Namely, \( V^+_\beta (S) \) is the set of vectors that classifies the positive points correctly and incorrectly classifies at least one of the negative points as a positive one, where all classifications are with margin \( \beta \). Similarly we define:

\[
V^-_\beta (S) := \{ \mathbf{v} \in \mathbb{R}^d | \forall \mathbf{x} \in S_- \quad \hat{\mathbf{v}} \cdot \mathbf{x} \geq \beta, \\
\exists \mathbf{x} \in S_+ \quad \text{s.t.} \quad \hat{\mathbf{v}} \cdot \mathbf{x} \geq \beta \}
\]

Thus, \( V^-_\beta (S) \) is the same as \( V^+_\beta (S) \) but with the roles of \( S_+ \) and \( S_- \) reversed. With these definitions we can provide a sufficient condition for convergence to PAR.

**Theorem 6.2.** Assume that:

1. Assumption [6.1] holds.

2. There exists an NAR \( \mathcal{N} \) and \( T_{NAR} \geq t_0 \) such that for all \( t \geq T_{NAR} \) it holds that \( \mathbf{W}_t \in \mathcal{N} \).

3. There exists \( T_{Margin} \geq T_{NAR} \) such that \( \tilde{\gamma}_{T_{Margin}} > \sqrt{K_\alpha} \max_{\mathbf{x} \in \mathcal{S}} || \mathbf{x} || \)

4. The training data \( \mathcal{S} \) satisfies \( V^+_\beta (\mathcal{S}) = V^-_\beta (\mathcal{S}) = \emptyset \).

Then \( \mathcal{N} \) is a PAR(\( \beta \)) for all \( t > T_{Margin} \), and there exists \( \delta_w, \delta_u > 0 \) such that gradient flow converges to a network whose normalized version is perfectly clustered with neuron directions \( \hat{\mathbf{w}}, \hat{\mathbf{u}} \), where \( (\delta_w \hat{\mathbf{w}}, \delta_u \hat{\mathbf{u}}) \) is the solution to the following convex optimization problem:

\[
\arg \min_{\mathbf{w} \in \mathbb{R}^d, \mathbf{u} \in \mathbb{R}^d} || \mathbf{w} ||^2 + || \mathbf{u} ||^2 \\
\forall \mathbf{x}_+ \in S_+ : \mathbf{w} \cdot \mathbf{x}_+ - \alpha \mathbf{u} \cdot \mathbf{x}_+ \geq 1 \\
\forall \mathbf{x}_- \in S_- : \mathbf{u} \cdot \mathbf{x}_- \geq 1
\]

We first comment on the assumptions. The first two assumptions are the same assumptions on the optimization trajectory as in Theorem [6.1]. Assumption 3 is another assumption on the trajectory that says that sufficiently large smoothed margin is achieved at some stage of the optimization. We note that the lower bound on the smoothed margin can be made small by considering a small \( \alpha \).

Assumption 4 refers to the training set. Informally, it corresponds to requiring that the two classes are approximately symmetric with respect to the origin. The next lemma shows that a certain symmetric training set satisfies Assumption 4:

**Lemma 6.1.** Assume that for any \( \mathbf{x} \in \mathcal{S} \) it holds that \( -\mathbf{x} \in \mathcal{S} \). Then, for any \( \beta > 0 \), \( V^+_\beta (\mathcal{S}) = V^-_\beta (\mathcal{S}) = \emptyset \).

The proof is given in the supplementary. This example suggests that we should observe PAR in symmetric distributions, which produce approximately symmetric training sets. Indeed, we empirically show in Section 6.3 that gradient flow converges to a solution in PAR for a distribution with
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(a) Learned Decision Boundary and PAR Solution

(b) Learned Neurons and PAR Solutions

Figure 4: Illustration of the Perfect Agreement Regime (PAR): A network with 100 neurons is trained on linearly separable data sampled from two Gaussians, and points inside a linear margin are excluded. Figure (a) shows that the network learns a linear decision boundary. Furthermore, the green arrow shows the decision boundary predicted by the PAR result (optimization problem Eq. (7)), and it agrees with the learned boundary. Figure (b) shows the learned neurons (yellow lines for \( u \) neurons and grey lines for \( w \) neurons), as well as the theoretical PAR solutions. It can be seen that neurons indeed converge to the PAR solution.

two symmetric Gaussians. We note that this example shows that Assumption 4 is independent of the maximum margin attainable on the training set. Indeed, by scaling the points, we can obtain any margin and still satisfy the assumption.

The theorem not only implies that convergence will be to a PAR, but it provides the solution that GF will converge to. The optimization problem in Eq. (7) is an SVM optimization problem with the kernel: 

\[
K(x, x') = \sum_{y \in \{-1, 1\}} \sigma'(yw^* \cdot x) \sigma'(yw^* \cdot x') x \cdot x'.
\]

The corresponding feature map is: 

\[
\phi(x) = [\sigma'(w^* \cdot x), -\sigma'(-w^* \cdot x)] \in \mathbb{R}^{2d}.
\]

We prove Theorem 6.2 in the supplementary, and provide a sketch next. First, we use Theorem 6.1 to show that gradient flow converges to an NAR and the neurons are clustered. Then we show that under Assumption 3 and using the monotonicity of the smoothed margin (Eq. (5)), by Lyu & Li (2020), all \( w \) neurons classify the positive points correctly and all \( u \) neurons classify the negative points correctly for all \( t > T_{\text{Margin}} \). Then, using Assumption 4 we show that the solution is in PAR. Finally, we use results of Lyu & Li (2020) to show that the network directions solve the convex optimization problem in the theorem.

6.3. Experiments

In Theorem 6.2 we show that when learning enters the PAR regime the solution will be given by Eq. (7). We performed experiments in several settings that show the above behavior is observed in practice when classes are sampled from Gaussians. Figure 4a shows the decision boundary (Figure 4a) and learned weights (Figure 4b), for learning from points sampled from two classes corresponding to Gaussians. The figure also shows the PAR predictions for the decision boundary and learned weights, and these show excellent agreement with the empirical results. We have also verified that in this case convergence is indeed to a PAR solution. We performed such experiments also for higher dimensional settings, and the results are in the supplementary. Finally, note that we do not expect learning to always converge to a PAR. In the supplementary we show an example where this does not happen.

7. Conclusions

Optimization and generalization are closely coupled in deep-learning. Yet both are little understood even for simple models. Here we consider perhaps the simplest “teacher” model where the ground truth is linear. We prove that cross-entropy can be globally minimized by SGD, despite the non-convexity of the loss, and for any initialization scale. We are not aware of any such result for non-linear networks (for example NTK optimization results require large initialization scale, and sufficiently wide networks (Ji & Telgarsky, 2019b)). Our novel proof technique analyzes SGD in an off-line setting and uses the notion of loss-violation per epoch, which we believe could be useful elsewhere.

In our setting, small initialization scale leads empirically to approximately linear decision boundaries. We prove that such boundaries are obtained when neurons with same output-weight sign are clustered. Empirically we show that such clustering indeed occurs. Moreover, we provide sufficient conditions for converging to such clustered solutions.

Several open questions remain. The first is reducing the assumptions when proving convergence to a clustered solution. Another interesting direction is extending our results to simple non-linear teachers.
8. Acknowledgements

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A. Gradient Flow Definitions

We next formally define gradient flow. A function \( f : \mathbb{X} \to \mathbb{R} \) is locally Lipschitz if for every \( x \in \mathbb{X} \) there exists a neighborhood \( U \) of \( x \) such that the restriction of \( f \) on \( U \) is Lipschitz continuous. For a locally Lipschitz function \( f : \mathbb{X} \to \mathbb{R} \), the Clarke subdifferential at \( x \in \mathbb{X} \) is the convex set:

\[
\text{\partial}^o f(x) := \operatorname{conv}\left\{ \lim_{k \to \infty} \nabla f(x_k) : x_k \to x, f \text{ is differentiable at } x_k \right\}
\]  

(8)

As in (Lyu & Li, 2020) and (Ji & Telgarsky, 2020), a curve \( z \) from an interval \( I \) to a real space \( \mathbb{R}^m \) is called an arc if it is absolutely continuous on any compact subinterval of \( I \). For an arc \( z \) we use \( z'(t) \) (or \( \frac{dz}{dt}(t) \)) to denote the derivative at \( t \) if it exists. We say that a locally Lipschitz function \( f : \mathbb{R}^d \to \mathbb{R} \) admits a chain rule if for any arc \( z : [0; +\infty) \to \mathbb{R}^d \), \( \forall h \in \partial^o f(z(t)) : (f \circ z)'(t) = \langle h, z'(t) \rangle \) holds for a.e. \( t \geq 0 \). It holds that an arc is a.e. differentiable, and the composition of an arc and a locally Lipschitz function is still an arc.

Given the definitions above, we define gradient flow \( \mathbf{W} : [0, \infty) \to \mathbb{R}^k \) to be an arc that satisfies the following differential inclusion for a.e. \( t \geq 0 \):

\[
\frac{dW}{dt} \in -\partial^o L_S(W_t)
\]

(9)

B. Proof of Theorem 4.1

Throughout this proof we will sometimes use the notation \( \langle x, y \rangle \) as the dot product between two vectors \( x \) and \( y \) for readability purposes.

Let \( \mathbf{W} = (w^* \cdots w^*, -w^* \cdots -w^*) \in \mathbb{R}^{2kd} \).

Define the following two functions:

\[
F(W_t) = \langle \mathbf{W}_t, \mathbf{W}^* \rangle = \sum_{i=1}^k \langle w^{(i)}_t, w^* \rangle - \sum_{i=1}^k \langle u^{(i)}_t, w^* \rangle
\]

and

\[
G(W_t) = ||\mathbf{W}_t|| = \sqrt{\sum_{i=1}^k ||w^{(i)}_t||^2 + \sum_{i=1}^k ||u^{(i)}_t||^2}
\]

Then, from Cauchy-Schwartz inequality we have:

\[
\frac{|F(W_t)|}{G(W_t)||\mathbf{W}^*||} \leq 1
\]  

(10)

Recall we define: \( N_W(x) = v \sum_{j=1}^k \sigma(w^{(j)} \cdot x) - v \sum_{j=1}^k \sigma(u^{(j)} \cdot x) \).

We consider minimizing the objective function:

\[
L_E(W) = \frac{1}{n} \sum_{i=1}^n \log \left( 1 + e^{-y_i N_W(x_i)} \right)
\]
We start by showing Eq. (11).

We first outline the proof structure. Let’s assume we run SGD for $p$ where \( W \) is sampled in the epoch, it holds that:

\[
\ell(y_i, W_{t-1}(x_i)) \leq \varepsilon_0
\]

Next, using the Lipschitzness of \( \ell(x) \) we will show that the loss on points cannot change too much during an epoch. Specifically, we will use this to show that at the end of epoch \( t_e \), which we denote by time \( T^* \), it holds for all \( (x_i, y_i) \in S \):

\[
\ell(y_i, W_{T^*}(x_i)) \leq (1 + 2v^2 R^2 \eta kn) \varepsilon_0
\]

now by choosing \( \varepsilon_0 = \frac{\varepsilon}{1 + 2v^2 R^2 \eta kn} \) we will get that \( \forall 1 \leq i \leq n \ell(y_i, W_{T^*}(x_i)) \leq \varepsilon \) which shows that \( L_S(W_{T^*}) \leq \varepsilon \) as required.

We start by showing Eq. (11).

For the gradient of each neuron we have:

\[
\frac{\partial L((x_i, y_i))(W)}{\partial w^{(j)}} = \frac{e^{-y_i, W(x_i)}}{1 + e^{-y_i, W(x_i)}} \cdot \frac{-y_i \partial N_W(x_i)}{\partial w^{(j)}} = -y_i e^{-y_i, W(x_i)} \cdot v x_i \sigma'(w^{(j)} \cdot x_i) = -vy_i x_i |\ell'(y_i, N_W(x_i))| \sigma'(w^{(j)} \cdot x_i)
\]

and similarly:

\[
\frac{\partial L((x_i, y_i))(W)}{\partial u^{(j)}} = vy_i x_i |\ell'(y_i, N_W(x_i))| \sigma'(u^{(j)} \cdot x_i)
\]

where \( \ell'(x) = -\frac{e^{-x}}{1+e^{-x}} = -\frac{1}{1+e^{-x}} \) and \( \ell(x) = \log(1+e^{-x}) \).

Optimizing by SGD yields the following update rule:

\[
W_t = W_{t-1} - \eta \frac{\partial}{\partial W} L((x_i, y_i))(W_{t-1})
\]

where \( W_i = (w^{(1)}_t, ..., w^{(k)}_t, u^{(1)}_t, ..., u^{(k)}_t) \).

For every neuron we get the following updates:

1. \( w^{(j)}_t = w^{(j)}_{t-1} + \eta y_i x_i |\ell'(y_i, N_{W_{t-1}}(x_i))| p^{(j)}_{t-1} \)
2. \( u^{(j)}_t = u^{(j)}_{t-1} - \eta y_i x_i |\ell'(y_i, N_{W_{t-1}}(x_i))| q^{(j)}_{t-1} \)

where \( p^{(j)}_t := \sigma'(u^{(j)}_t \cdot x_{t+1}) ; q^{(j)}_t := \sigma'(u^{(j)}_t \cdot x_{t+1}) \).
Next we will show recursive upper bounds for \( G(W_t) \) and \( F(W_t) \).

\[
G(W_t)^2 = \sum_{j=1}^{k} ||w_t^{(j)}||^2 + \sum_{j=1}^{k} ||u_t^{(j)}||^2 \\
\leq \sum_{j=1}^{k} ||w_{t-1}^{(j)}||^2 + \sum_{j=1}^{k} ||u_{t-1}^{(j)}||^2 \\
+ 2\eta y_t|\ell'(y_t N_{W_{t-1}}(x_t))| \left( \sum_{j=1}^{k} \langle w_{t-1}^{(j)}, x_t \rangle p_{t-1}^{(j)} v - \sum_{j=1}^{k} \langle u_{t-1}^{(j)}, x_t \rangle q_{t-1}^{(j)} v \right) \\
+ 2k\eta^2 v^2 ||x_t||^2 |\ell'(y_t N_{W_{t-1}}(x_t))|^2 = G(W_{t-1})^2 + 2\eta|\ell'(y_t N_{W_{t-1}}(x_t))|y_t N_{W_{t-1}}(x_t) + 2k\eta^2 v^2 ||x_t||^2 |\ell'(y_t N_{W_{t-1}}(x_t))|^2
\]

On the other hand,

\[
F(W_t) = \sum_{j=1}^{k} \langle w_t^{(j)}, w^* \rangle - \sum_{j=1}^{k} \langle u_t^{(j)}, w^* \rangle = \sum_{j=1}^{k} \langle w_{t-1}^{(j)}, w^* \rangle - \sum_{j=1}^{k} \langle u_{t-1}^{(j)}, w^* \rangle + \eta|\ell'(y_t N_{W_{t-1}}(x_t))| \sum_{j=1}^{k} \langle y_t x_t, w^* \rangle p_{t-1}^{(j)} v + \eta|\ell'(y_t N_{W_{t-1}}(x_t))| \sum_{j=1}^{k} \langle y_t x_t, w^* \rangle q_{t-1}^{(j)} v \\
\geq \sum_{j=1}^{k} \langle w_{t-1}^{(j)}, w^* \rangle - \sum_{j=1}^{k} \langle u_{t-1}^{(j)}, w^* \rangle + 2k\eta v|\ell'(y_t N_{W_{t-1}}(x_t))|
\]

Where we used the inequalities \( \langle y_t x_t, w^* \rangle \geq 1 \) and \( q_{t}^{(j)}, p_{t}^{(j)} \geq \alpha \).

To summarize we have:

\[
G(W_t)^2 \leq G(W_{t-1})^2 + 2\eta|\ell'(y_t N_{W_{t-1}}(x_t))|y_t N_{W_{t-1}}(x_t) + 2k\eta^2 v^2 R^2_x |\ell'(y_t N_{W_{t-1}}(x_t))|^2 \tag{13}
\]

\[
F(W_t) \geq F(W_{t-1}) + 2k\eta v|\ell'(y_t N_{W_{t-1}}(x_t))| \tag{14}
\]

For an upper bound on \( G(W_t) \) we use the following inequalities (which hold for the cross entropy loss):

\[
\forall x \in \mathbb{R} \quad \frac{x}{1+e^x} \leq 1 \Rightarrow |\ell'(y_t N_{W_{t-1}}(x_t))|y_t N_{W_{t-1}}(x_t) = \frac{y_t N_{W_{t-1}}(x_t)}{1+e^{\ell'(y_t N_{W_{t-1}}(x_t))}} \leq 1 \text{ and } |\ell'(y_t N_{W_{t-1}}(x_t))| \leq 1. \text{ Together we have for any } t:
\]

\[
G(W_t)^2 \leq G(W_{t-1})^2 + 2\eta + 2k\eta^2 v^2 R^2_x
\]

Using this recursively up until \( T = nN_e \) we get:

\[
G(W_T)^2 \leq G(W_0)^2 + T(2k\eta^2 v^2 R^2_x + 2\eta) \tag{15}
\]

Now, for \( F(W_t) \), let \( \varepsilon_0 > 0 \), under our assumption, in any epoch \( i_e \) until \( N_e (1 \leq i_e \leq N_e) \) there exists at least one point in the epoch \((y_{i_e}, x_{i_e}) \in S \) s.t. \( \ell(y_{i_e}, N_{W_{i_e}}(x_{i_e})) > \varepsilon_0 \).

Now, since in our case \( \ell(x) = log(1 + e^{-x}) \) and \( \ell'(x) = -\frac{1}{1+e^{-x}} \), we see that the condition \( \ell(x) > \varepsilon_0 \) implies that:

\[
|\ell'(x)| > 1 - e^{-\varepsilon_0} \tag{16}
\]
In any other case $|\ell'(y_{t_e}N_{W_{t_e-1}}(x_{t_e}))| \geq 0$, so if we assume at least one point violation per epoch (i.e. $\ell(y_{t_{e-1}}N_{W_{t_{e-1}}}(x_{t_{e-1}})) \geq \varepsilon_0$ for some point $(y_{t_{e-1}}, x_{t_{e-1}})$ in the epoch) we would get that at the end of epoch $N_e$:

$$F(W_T) \geq F(W_{T-n}) + 2k\eta \alpha (1 - e^{-\varepsilon_0})$$

This implies that (recursively using Eq. (17)):

$$F(W_T) \geq F(W_0) + 2k\eta \alpha N_e (1 - e^{-\varepsilon_0})$$

where $N_e$ is the number of epochs and $n$ the number of training points, $T = nN_e$.

Now, using the Cauchy-Schwartz, Eq. (15) and Eq. (18) we have:

$$-G(W_0)||\overrightarrow{w}^*|| + 2k\eta \alpha N_e (1 - e^{-\varepsilon_0}) \leq F(W_0) + 2k\eta \alpha N_e (1 - e^{-\varepsilon_0})$$

Using $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$ the above implies:

$$-G(W_0)||\overrightarrow{w}^*|| + 2k\eta \alpha N_e (1 - e^{-\varepsilon_0}) \leq ||\overrightarrow{w}^*||G(W_T) \leq ||\overrightarrow{w}^*||\sqrt{G(W_0)^2 + T(2k\eta^2 \alpha^2 R_x^2 + 2\eta)}$$

Now using $||w^{(i)}_0||/||u^{(i)}_0|| \leq R_0$ we get $G(W_0) \leq \sqrt{2k}R_0$.

Noting that $||\overrightarrow{w}^*|| = \sqrt{2k}||w^*||$ and that $N_e = T/n$, we get:

$$\left(\frac{2k\eta \alpha (1 - e^{-\varepsilon_0})}{n}\right)T \leq \sqrt{4k^2 \eta^2 \alpha^2 R_x^2 + 4k\eta}||w^*||\sqrt{T + 4kR_0}||w^*||$$

Therefore, we have an inequality of the form:

$$aT \leq b\sqrt{T} + c$$

where $a = \frac{2k\eta \alpha (1 - e^{-\varepsilon_0})}{n}$, $b = \sqrt{4k^2 \eta^2 \alpha^2 R_x^2 + 4k\eta}||w^*||$ and $c = 4kR_0||w^*||$.

By inspecting the roots of the parabola $P(X) = x^2 - \frac{b}{a}x - \frac{c}{a}$ we conclude that:

$$T \leq \left(\frac{b}{a}\right)^2 + \sqrt{\left(\frac{b}{a}\right)^2 + \frac{c}{a}} = \left(\frac{4k^2 \eta^2 \alpha^2 R_x^2 + 4k\eta}{4k^2 \eta^2 \alpha^2 (1 - e^{-\varepsilon_0})^2}\right) + \sqrt{\left(\frac{4k^2 \eta^2 \alpha^2 R_x^2 + 4k\eta}{2k\eta \alpha (1 - e^{-\varepsilon_0})}\right) + \frac{4kR_0||w^*||n}{2k\eta \alpha (1 - e^{-\varepsilon_0})}}$$

By the inequality $1 - e^{-x} > \frac{x}{1+x}$ for $x > 0$ (which is equivalent to $\frac{1}{1-e^{-x}} < \frac{x+1}{x}$, with $x = \varepsilon_0 > 0$ we get $\frac{1}{1-e^{-\varepsilon_0}} < \frac{\varepsilon_0+1}{\varepsilon_0} = 1 + \frac{1}{\varepsilon_0}$. Therefore for $\beta > 0$ (all arguments are positive):

$$\frac{1}{(1 - e^{-\varepsilon_0})^\beta} < \left(1 + \frac{1}{\varepsilon_0}\right)^\beta$$

By using the above inequality we can reach a polynomial bound on $T$:

$$T \leq \left(\frac{R_x^2}{\alpha^2} + \frac{1}{k\eta \alpha^2}\right)||w^*||^2n^2 \left(1 + \frac{1}{\varepsilon_0}\right)^2 + \frac{\sqrt{R_0(8k^2 \eta^2 \alpha^2 R_x^2 + 8k\eta)}||w^*||^{1.5}n^{1.5}(1 + \frac{1}{\varepsilon_0})^{1.5}}{2k(\eta \alpha)^{1.5}(1 - e^{-\varepsilon_0})} + \frac{2R_0||w^*||n(1 + \frac{1}{\varepsilon_0})}{\eta \alpha}$$

(19)
We have shown that there is at most a finite amount of epochs $N_e = \frac{T}{n}$ such that there exists at least one point in each of them with a loss greater than $\varepsilon_0$. Therefore, there exists an epoch $1 \leq i_e \leq N_e + 1$ such that each point sampled in the epoch has a loss smaller than $\varepsilon_0$. Formally, for any $(i_e-1)n+1 \leq t \leq i_en$, $\ell(y_t N_{W_{i_e-1}}(x_i)) \leq \varepsilon_0$. Recall that SGD samples without replacement and therefore, each point is sampled at some $t$ in the epoch $i_e$.

Next, we will show that there exists a time $t$ such that $L_\delta(W_t) < \varepsilon$ by bounding the change in the loss values during the epoch. We'll start by noticing that our loss function $\ell(x)$ is locally Lipschitz with coefficient 1, that is because $\forall x, |\ell'(x)| = \frac{1}{1+\|x\|} \leq 1$. With this in mind for any point $(y_t, x_i) \in \mathcal{S}$ if we can bound $|y_t N_{W_{i_e}}(x_i) - y_t N_{W_i}(x_i)|$ we would also bound $|\ell(y_t N_{W_{i_e}}(x_i)) - \ell(y_t N_{W_i}(x_i))|$. 

For any iteration $(i_e-1)n+1 \leq t \leq i_en$ and $1 \leq s \leq n$ we have:

$$
|y_t N_{W_{i_e}}(x_i) - y_t N_{W_i}(x_i)| = |N_{W_{i_e}}(x_i) - N_{W_i}(x_i)|
$$

$$
= v \sum_{j=1}^{k} \left( \sigma(w_{t+s}^{(j)} \cdot x_i) - \sigma(w_t^{(j)} \cdot x_i) \right) + v \sum_{j=1}^{k} \left( \sigma(u_{t+s}^{(j)} \cdot x_i) - \sigma(u_t^{(j)} \cdot x_i) \right)
$$

$$
\leq v \sum_{j=1}^{k} \left| \sigma(w_{t+s}^{(j)} \cdot x_i) - \sigma(w_t^{(j)} \cdot x_i) \right| + v \sum_{j=1}^{k} \left| \sigma(u_{t+s}^{(j)} \cdot x_i) - \sigma(u_t^{(j)} \cdot x_i) \right|
$$

$$
\leq v \sum_{j=1}^{k} \left( \|w_{t+s}^{(j)} - w_t^{(j)}\| \cdot |x_i| + v \sum_{j=1}^{k} \left| u_{t+s}^{(j)} - u_t^{(j)} \right| \cdot |x_i| \right)
$$

$$
\leq v R_x \sum_{j=1}^{k} \|\sum_{h=1}^{s} \eta v y_{t+h} x_{t+h} \|_{\ell_t^{(j)}(y_{t+h} N_{W_{i_e-1}}(x_{t+h}))} p_{t+h-1}^{(j)} \|
$$

$$
+ v R_x \sum_{j=1}^{k} \|\sum_{h=1}^{s} \eta v y_{t+h} x_{t+h} \|_{\ell_t^{(j)}(y_{t+h} N_{W_{i_e-1}}(x_{t+h}))} q_{t+h-1}^{(j)} \|
$$

$$
\leq v R_x \sum_{j=1}^{k} \sum_{h=1}^{s} \eta v |\ell_t^{(j)}(y_{t+h} N_{W_{i_e-1}}(x_{t+h}))| \cdot |x_{t+h}| + v R_x \sum_{j=1}^{k} \sum_{h=1}^{s} \eta v |\ell_t^{(j)}(y_{t+h} N_{W_{i_e-1}}(x_{t+h}))| \cdot |x_{t+h}|
$$

$$
\leq 2v^2 R_x^2 \eta k \sum_{h=1}^{s} |\ell_t^{(j)}(y_{t+h} N_{W_{i_e-1}}(x_{t+h}))| \leq 2v^2 R_x^2 \eta k s (1 - e^{-\varepsilon_0}) \leq 2v^2 R_x^2 \eta k s (1 - e^{-\varepsilon_0}) \leq 2v^2 R_x^2 \eta k s \varepsilon_0
$$

Where in Eq. (20) we used the Lipschitzness of $\sigma(\cdot): \forall x_1, x_2 \in \mathbb{R} |\sigma(x_1) - \sigma(x_2)| \leq |x_1 - x_2|$, in Eq. (21) we used the Cauchy-Schwartz inequality, in Eq. (22) we used the bound for $\ell_t^{(j)}(y_{t+h} N_{W_{i_e-1}}(x_{t+h}))$ recursively and finally in Eq. (23) we used that if $\ell(x) \leq \varepsilon_0$ then $|\ell'(x)| \leq 1 - e^{-\varepsilon_0}$ (follows from a similar derivation to Eq. (16)) and that $1 - e^{-\varepsilon_0} \leq \varepsilon_0$.

Now we can use the bound we just derived and the Lipschitzness of $\ell$ and reach

$$
|\ell(y_t N_{W_{i_e}}(x_i)) - \ell(y_t N_{W_i}(x_i))| \leq 2v^2 R_x^2 \eta k s \varepsilon_0
$$

for any time $(i_e-1)n+1 \leq t \leq i_en$ and $1 \leq s \leq n$. We know that for all $1 \leq i \leq n$, there exists $(i_e-1)n+1 \leq t_e^* \leq i_en$ such that $\ell(y_t N_{W_{i_e}}(x_i)) \leq \varepsilon_0$. Therefore, by Eq. (24), for time $T^* = i_en + 1$ and any $(y_t, x_i) \in \mathcal{S}$ we have:

$$
\ell(y_t N_{W_{i_e}}(x_i)) \leq \ell(y_t N_{W_{i_e}}(x_i)) + 2v^2 R_x^2 \eta k s \varepsilon_0 \leq \varepsilon_0 + 2v^2 R_x^2 \eta k s \varepsilon_0
$$

If $\forall 1 \leq i \leq n \ell(y_t N_{W}(x_i)) \leq \varepsilon$ we would get our bound $L_\delta(W_t) \leq \varepsilon$.

Therefore, if we set $\varepsilon_0 = \frac{\varepsilon}{1+2v^2 R_x^2 \eta k s}$ in Eq. (25) we’ll reach our result.
Setting this $\varepsilon_0$ at Eq. (19) leads to:

$$
T \leq \left( \frac{R_0^2}{\alpha^2} + \frac{1}{k\eta v^2\alpha^2} \right) \|w^*\|^2 n^2 \left( 1 + \frac{1 + 2v^2 R_0^2 n \eta k n}{\varepsilon} \right)^2 \\
+ \sqrt{R_0(8k^2 \eta^2 v^2 R_0^2 + 8k \eta)} \|w^*\|^{1.5} \left( 1 + \frac{1 + 2v^2 R_0^2 n \eta k n}{\varepsilon} \right)^{1.5} 2R_0 \|w^*\| n \left( 1 + \frac{1 + 2v^2 R_0^2 n \eta k n}{\eta v \alpha} \right)^{1.5} 2k(\eta v \alpha)^{1.5} \\
\tag{26}
$$

We denote the right hand side of Eq. (26) plus $n$ by $M(n, \epsilon)$.\(^6\) Note that $M(n, \epsilon) = O(n^4 \varepsilon^2)$ and therefore for simplicity we can alternatively denote $M(n, \epsilon)$ to be a less tight bound of the form $Cn^4 \varepsilon^2$ where $C$ is a constant that depends polynomially on $R_\alpha, R_0, k, \frac{1}{\alpha}, \max\left\{n, \frac{1}{\eta} \right\}, \max\left\{v, \frac{1}{\eta} \right\}$ and $\|w^*\|$. Overall, we proved that after $O(n^4 \varepsilon^2)$ steps, SGD will converge to a solution with $L_\phi(W_t) < \varepsilon$ empirical loss for some $t \leq M(n, \epsilon)$.

\(^6\)We need to add $n$ to Eq. (25) because we may consider the epoch immediately after $T$. 

---

Towards Understanding Learning in Neural Networks with Linear Teachers
C. Proof of Theorem [5.1]

Before we start proving the main theorem we will prove some useful lemmas and corollaries.

We first show the following.

**Corollary C.1.** if \( |(w - u) \cdot x| \geq 2r ||x|| \) then \( \frac{r}{|w \cdot x|} \geq r ||x|| \vee |(w \cdot x) \geq r ||x|| \).

**Proof.** Assume in contradiction that \( |w \cdot x| < r ||x|| \wedge |w \cdot x| < r ||x|| \), then by the triangle inequality and the Cauchy-Schwartz inequality we’ll get:
\[
|w \cdot x| < |w \cdot x| + |w \cdot x| < r ||x|| + r ||x|| = 2r ||x|| \quad \text{in contradiction to the assumption } |(w - u) \cdot x| \geq 2r ||x||. \]

Next, we prove the following lemma, which will be used throughout the proof of the main theorem. The lemma ties the dot products with the center of the cluster to the dot products with the individual neurons:

**Lemma C.1.** If \( \forall 1 \leq j \leq k : \ w^{(j)} \in \text{Ball}(w, r) \land u^{(j)} \in \text{Ball}(u, r) \) then: \( \forall x \in \mathbb{R}^d \ s.t. \ |w \cdot x| \geq r ||x|| \) : \( \forall 1 \leq j \leq k \ w^{(j)} \cdot x > 0 \lor \forall 1 \leq j \leq k \ w^{(j)} \cdot x < 0 \) and similarly for \( u \) type neurons \( \forall x \in \mathbb{R}^d \ s.t. \ |u \cdot x| \geq r ||x|| \) : \( \forall 1 \leq j \leq k \ u^{(j)} \cdot x > 0 \lor \forall 1 \leq j \leq k \ u^{(j)} \cdot x < 0 \).

**Proof.** Let’s assume that \( w \cdot x \geq r ||x|| \), therefore \( \forall 1 \leq j \leq k : \ w^{(j)} \cdot x = (w^{(j)} - w) \cdot x + w \cdot x \geq -||w^{(j)} - w|| \cdot ||x|| + r ||x|| > r ||x|| \) where we had used Cauchy-Schwartz inequality and that \( ||w^{(j)} - w|| < r \).

If \( w \cdot x \leq r ||x||, \forall 1 \leq j \leq k : \ w^{(j)} \cdot x = (w^{(j)} - w) \cdot x + w \cdot x < ||w^{(j)} - w|| \cdot ||x|| - r ||x|| < r ||x|| - r ||x|| = 0 \) the same derivation would work for \( u \).

We are now ready to move forward with proving the main lemma.

By Corollary [C.1] we see that \( \{ x \in \mathbb{R}^d \mid |(w - u) \cdot x| \geq 2r ||x|| \} \subseteq \{ x \in \mathbb{R}^d \mid |w \cdot x| \geq r ||x|| \vee |u \cdot x| \geq r ||x|| \} \) so if we prove that:
\[
\forall x \in \mathbb{R}^d \in \{ x \in \mathbb{R}^d \mid |(w - u) \cdot x| \geq 2r ||x|| \} \cap \{ x \in \mathbb{R}^d \mid |w \cdot x| \geq r ||x|| \vee |u \cdot x| \geq r ||x|| \} \quad : \quad \text{sign}(N_W(x)) = \text{sign}((w - u) \cdot x)
\]

We’ll start by showing first our lemma holds \( \forall x \in \mathbb{R}^d \ s.t. |w \cdot x| \geq r ||x|| \land |u \cdot x| \geq r ||x|| \) and then deal with the points in which only one of the above conditions holds.

**Proposition C.1.** \( \forall x \in \mathbb{R}^d \ s.t. |w \cdot x| \geq r ||x|| \land |u \cdot x| \geq r ||x|| \) : \( \text{sign}(N_W(x)) = \text{sign}((w - u) \cdot x) \)

**Proof.** Under our clusterization assumption \( \forall 1 \leq j \leq k : \ w^{(j)} \in \text{Ball}(w, r) \land u^{(j)} \in \text{Ball}(u, r) \) so we can use Lemma (C.1) and we are left with proving that \( \forall x \in \mathbb{R}^d \) such that for the \( w \) neurons \( \{ [1 \leq j \leq k \ w^{(j)} \cdot x > 0] \lor [1 \leq j \leq k \ w^{(j)} \cdot x < 0] \} \) and for the \( u \) neurons \( \{ [1 \leq j \leq k \ u^{(j)} \cdot x > 0] \lor [1 \leq j \leq k \ u^{(j)} \cdot x < 0] \} \) we get \( \text{sign}(N_W(x)) = \text{sign}((w - u) \cdot x) \).

We can represent \( \{ x \in \mathbb{R}^d \mid |w \cdot x| \geq r ||x|| \lor |u \cdot x| \geq r ||x|| \} \) as a union of \( C^+_+ , C^- , C^+_\ominus , C^+_+ \) where:
\[
\begin{align*}
C^+_+ &= \{ x \in \mathbb{R}^d \mid \forall 1 \leq j \leq k \ w^{(j)} \cdot x > 0 \land \forall 1 \leq j \leq k \ u^{(j)} \cdot x > 0 \}
\end{align*}
\]
\[
\begin{align*}
C^- &= \{ x \in \mathbb{R}^d \mid \forall 1 \leq j \leq k \ w^{(j)} \cdot x < 0 \land \forall 1 \leq j \leq k \ u^{(j)} \cdot x < 0 \}
\end{align*}
\]
\[
\begin{align*}
C^\ominus &= \{ x \in \mathbb{R}^d \mid \forall 1 \leq j \leq k \ w^{(j)} \cdot x > 0 \land \forall 1 \leq j \leq k \ u^{(j)} \cdot x < 0 \}
\end{align*}
\]
\[
\begin{align*}
C^+_\ominus &= \{ x \in \mathbb{R}^d \mid \forall 1 \leq j \leq k \ w^{(j)} \cdot x < 0 \land \forall 1 \leq j \leq k \ u^{(j)} \cdot x > 0 \}
\end{align*}
\]

Now we will show that \( \text{sign}(N_W(x)) = \text{sign}((w - u) \cdot x) \) in each region, from which the claim follows.

1. If \( x \in C^+_+ \) then \( N_W(x) = v \left( \sum_{j=1}^{k} \sigma(w^{(j)} \cdot x) - \sigma(u^{(j)} \cdot x) \right) = v \left( \sum_{j=1}^{k} w^{(j)} - u^{(j)} \right) \cdot x \) and therefore \( \text{sign}(N_W(x)) = \text{sign}((w - u) \cdot x) \).
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2. If \( x \in C^- \) then \( N_W(x) = v \left( \sum_{j=1}^{k} \sigma(w^{(j)} \cdot x) - \sigma(u^{(j)} \cdot x) \right) = \alpha v \left( \sum_{j=1}^{k} w^{(j)} - u^{(j)} \right) \cdot x \) and therefore \( \text{sign} \,(N_W(x)) = \text{sign} \,(\langle w - u \rangle \cdot x) \).

3. If \( x \in C^+ \) then both \( N_W(x) = v \left( \sum_{j=1}^{k} \sigma(w^{(j)} \cdot x) - \sigma(u^{(j)} \cdot x) \right) = v \left( \sum_{j=1}^{k} w^{(j)} \cdot x - \alpha u^{(j)} \cdot x \right) > 0 \) and \( \langle w - u \rangle \cdot x > 0 \). Therefore, \( \text{sign} \,(N_W(x)) = \text{sign} \,(\langle w - u \rangle \cdot x) \).

4. If \( x \in C^\pm \) then both \( N_W(x) = v \left( \sum_{j=1}^{k} \sigma(w^{(j)} \cdot x) - \sigma(u^{(j)} \cdot x) \right) = v \left( \sum_{j=1}^{k} \alpha w^{(j)} \cdot x - u^{(j)} \cdot x \right) < 0 \) and \( \langle w - u \rangle \cdot x < 0 \). Therefore, \( \text{sign} \,(N_W(x)) = \text{sign} \,(\langle w - u \rangle \cdot x) \).

\( \square \)

We are left with proving \( \text{sign} \,(N_W(x)) = \text{sign} \,(\langle w - u \rangle \cdot x) \) holds when exactly one condition holds, i.e., either \( \langle w \cdot x \rangle \geq r \) or \( \langle w \cdot x \rangle \geq r \).

**Proposition C.2.**

\[ \forall x \in \{ x \in \mathbb{R}^d \mid \langle w \cdot x \rangle < r|x| \wedge \langle w \cdot x \rangle \geq r|x| \wedge (\langle w - u \rangle \cdot x) \geq 2r|x| \} : \text{sign} \,(N_W(x)) = \text{sign} \,(\langle w - u \rangle \cdot x) \]

and similarly our decision boundary is linear for points in which our condition only holds for \( w \):

\[ \forall x \in \{ x \in \mathbb{R}^d \mid \langle w \cdot x \rangle < r|x| \wedge \langle w \cdot x \rangle \geq r|x| \wedge (\langle w - u \rangle \cdot x) \geq 2r|x| \} : \text{sign} \,(N_W(x)) = \text{sign} \,(\langle w - u \rangle \cdot x) \]

**Proof.** We start with the domain \( \{ x \in \mathbb{R}^d \mid \langle w \cdot x \rangle < r|x| \wedge \langle w \cdot x \rangle \geq r|x| \wedge (\langle w - u \rangle \cdot x) \geq 2r|x| \} \)

i.e. our condition only holds for \( w \).

There are two cases, and we’ll prove the result for each of them:

**If \( \langle w \cdot x \rangle \geq r \):**

In this case \( N_W(x) = v \left( \sum_{j=1}^{k} \sigma(w^{(j)} \cdot x) - \sigma(u^{(j)} \cdot x) \right) = v \left( \sum_{j=1}^{k} \sigma(w^{(j)} \cdot x) - k\langle u \rangle \cdot x \right) \).

Next, for any \( x \) in the domain, we’ll denote \( J^+(x) := \{ j | w^{(j)} \cdot x > 0 \} \) and \( k^+(x) := |J^+(x)| \) similarly \( J^-(x) := \{ j | w^{(j)} \cdot x < 0 \} \) and \( k^-(x) := |J^-(x)| \). Using these definitions, our network has the following form:

\[ N_W(x) = v \left( \sum_{j \in J^+(x)} w^{(j)} \cdot x + \alpha \sum_{j \in J^-(x)} w^{(j)} \cdot x - k\langle u \rangle \cdot x \right) = v \left( k\langle w \rangle \cdot x - k\langle u \rangle \cdot x + (\alpha - 1) \sum_{j \in J^-(x)} w^{(j)} \cdot x \right) \]

Next, we bound \( \forall j \mid \langle w^{(j)} \rangle \cdot x \mid = \langle (w^{(j)} - \langle w \rangle + \langle u \rangle) \cdot x \rangle \leq \|w^{(j)} - \langle w \rangle\| \cdot \|x\| + \|\langle u \rangle \cdot x \| < 2\|x\| \) where we used \( \|w^{(j)} - \langle w \rangle\| < r \) and \( \|w \cdot x\| < r \cdot \|x\| \).

Now, if \( \langle w - u \rangle \cdot x \geq 2r\|x\| > 0 \) we get that \( N_W(x) = v \left( k\langle w \rangle \cdot x - (1 - \alpha) \sum_{j \in J^-(x)} w^{(j)} \cdot x \right) > v \left( 2r\|x\| |k - 2r||x||k^+(x)(1 - \alpha) \right) > 0 \) since \( (1 - \alpha) < 1 \) and \( k^+(x) \leq k \) and therefore \( \text{sign} \,(N_W(x)) = \text{sign} \,(\langle w - u \rangle \cdot x) = 1 \) for this case.

If \( \langle w - u \rangle \cdot x \leq -2r\|x\| < 0 \) we get that \( N_W(x) = v \left( k\langle w - u \rangle \cdot x - (1 - \alpha) \sum_{j \in J^-(x)} w^{(j)} \cdot x \right) < v \left( -2r\|x\| |k + 2r||x||k^-(x)(1 - \alpha) \right) < 0 \) since \( (1 - \alpha) < 1 \) and \( k^-(x) \leq k \). Therefore, we get that \( \text{sign} \,(N_W(x)) = \text{sign} \,(\langle w - u \rangle \cdot x) = -1 \) in this case.
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At any rate, we have shown that \( \forall x \in \{ x \in \mathbb{R}^d \mid k \cdot r \cdot ||x|| \geq 2r||x|| \} \), \( \text{sign}(N_W(x)) = \text{sign} ((\overline{w} - \overline{u}) \cdot x) \).

If \( \overline{w} \cdot x \leq -r||x|| \):

First, we notice that \((\overline{w} - \overline{u}) \cdot x > -r||x|| + r||x|| = 0 \) so \( \text{sign} ((\overline{w} - \overline{u}) \cdot x) = 1 \) again we use Lemma. (C.1) and from our assumption \( \overline{w} \cdot x \leq -r||x|| \) we have \( \forall 1 \leq j \leq k \cdot u^{(j)} \cdot x < 0 \) and we can see that our network takes the form:

\[
N_W(x) = v \left( \sum_{j=1}^{k} \sigma(w^{(j)} \cdot x) - \sigma(u^{(j)} \cdot x) \right) = v \left( \sum_{j=1}^{k} \sigma(w^{(j)} \cdot x) - \alpha \cdot k \overline{u} \cdot x \right) \geq v \left( \sum_{j=1}^{k} \sigma(w^{(j)} \cdot x) + \alpha kr||x|| \right). 
\]

Next, we prove the following lemma:

**Lemma C.2.** If \( ||w \cdot x|| < r||x|| \) then \( \alpha \cdot k \cdot r||x|| > - \sum_{j=1}^{k} \sigma(w^{(j)} \cdot x) \).

**Proof.** Let’s assume by contradiction that \( - \sum_{j=1}^{k} \sigma(w^{(j)} \cdot x) > \alpha \cdot k \cdot r||x|| \). We notice that regardless of the sign of the dot product \( \forall j : -\sigma(w^{(j)} \cdot x) \leq -\alpha w^{(j)} \cdot x \) so we have \(-\alpha \sum_{j=1}^{k} w^{(j)} \cdot x \geq - \sum_{j=1}^{k} \sigma(w^{(j)} \cdot x) \geq \alpha \cdot k \cdot r||x|| \), which leads to \(-\alpha k \overline{w} \cdot x \geq \alpha \cdot k \cdot r||x|| \) (where we used the definition of \( \overline{w} \)) finally we reach \( \overline{w} \cdot x \leq -r||x|| \). This contradicts \( ||w \cdot x|| < r||x|| \).

Therefore, we have \(- \sum_{j=1}^{k} \sigma(w^{(j)} \cdot x) < \alpha \cdot k \cdot r||x|| \) and \( \text{sign}(N_W(x)) = \text{sign} ((\overline{w} - \overline{u}) \cdot x) = 1 \) as desired.

To conclude we proved that \( \forall x \in \{ x \in \mathbb{R}^d \mid k \cdot r \cdot ||x|| \geq 2r||x|| \}, \text{sign}(N_W(x)) = \text{sign} ((\overline{w} - \overline{u}) \cdot x) \).

Next we look at \( \forall x \in \{ x \in \mathbb{R}^d \mid k \cdot r \cdot ||x|| \leq 2r||x|| \} \) and through a similar derivation of two cases we will prove that \( \text{sign}(N_W(x)) = \text{sign} ((\overline{w} - \overline{u}) \cdot x) \).

If \( \overline{w} \cdot x \geq r||x|| \):

Through a similar derivation for the case of \( \overline{w} \cdot x \geq r||x|| \), our network has the following form:

\[
N_W(x) = v \left( \sum_{j=1}^{k} \sigma(w^{(j)} \cdot x) - \sigma(u^{(j)} \cdot x) \right) = v \left( k \overline{w} \cdot x - \sum_{j=1}^{k} \sigma(u^{(j)} \cdot x) \right)
\]

\[
= v(k \overline{w} \cdot x - \sum_{j \in J^u_+(x)} u^{(j+)} \cdot x + \sum_{j \in J^u_-(x)} \alpha u^{(j-)} \cdot x)
\]

\[
= v(k \overline{w} \cdot x - \sum_{j \in J^u_+(x)} u^{(j+)} \cdot x + \sum_{j \in J^u_-(x)} u^{(j-)} \cdot x + (1 - \alpha) \sum_{j \in J^u_-(x)} u^{(j-)} \cdot x)
\]

\[
= v(k \overline{w} \cdot x - k \overline{u} \cdot x + (1 - \alpha) \sum_{j \in J^u_-(x)} u^{(j-)} \cdot x)
\]

where \( J^u_+(x) := \{ j \mid u^{(j)} \cdot x > 0 \} \), \( J^u_-(x) := \{ j \mid u^{(j)} \cdot x < 0 \} \) and \( k^u(x) = |J^u_+(x)| \), \( k^u(x) = |J^u_-(x)| \).
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If \((\bar{w} - \bar{u}) \cdot x \geq 2r||x|| > 0\) then \(N_W(x) = v \left( k(\bar{w} - \bar{u}) \cdot x + (1 - \alpha) \sum_{j \in J^+} u^{(j)} \cdot x \right) \geq v (2kr||x|| - 2r||x||/(1 - \alpha)k_{u}^+ (x)) > 0\) (because \((1 - \alpha) < 1\) and \(k_{u}^+ (x) \leq k\) and \(\text{sign}(N_W(x)) = \text{sign}((\bar{w} - \bar{u}) \cdot x) = 1\) (where we used the fact that \(\forall j : |u^{(j)} \cdot x| < 2r||x||\) which follows from \(\bar{u} \cdot x < r||x||\) and \(||u^{(j)} - \bar{u}|| < r\).

If \((\bar{w} - \bar{u}) \cdot x \leq -2r||x|| < 0\) we get that \(N_W(x) \leq v (-2r||x||k + 2r||x||/(1 - \alpha)k_{u}^+ (x)) < 0\) and \(\text{sign}(N_W(x)) = \text{sign}((\bar{w} - \bar{u}) \cdot x) = 1\).

To summarize, we showed that \(\forall x \in \{ x \in \mathbb{R}^d | \bar{w} \cdot x < r||x|| \land \bar{u} \cdot x \geq r||x|| \land |(\bar{w} - \bar{u}) \cdot x| \geq 2r||x||\}, \text{sign}(N_W(x)) = \text{sign}((\bar{w} - \bar{u}) \cdot x).

\(\bar{w} \cdot x \leq -r||x||:\)

We again use Lemma \([\text{C.1}]\) which yields from \(\bar{w} \cdot x \leq -r||x||\) that \(\forall 1 \leq j \leq k \ u^{(j)} \cdot x < 0\) and we can see that our network takes the form:

\[N_W(x) = v \left( \sum_{j=1}^{k} \sigma(u^{(j)} \cdot x) - \sigma(u^{(j)} \cdot x) \right) = \alpha k v \bar{u} \cdot x - v \left( \sum_{j=1}^{k} \sigma(u^{(j)} \cdot x) \right) \leq v \left( -\alpha k r||x|| - \sum_{j=1}^{k} \sigma(u^{(j)} \cdot x) \right)\]

If \(- \sum_{j=1}^{k} \sigma(u^{(j)} \cdot x) < \alpha k r||x||\) we have \(\text{sign}(N_W(x)) = \text{sign}((\bar{w} - \bar{u}) \cdot x) = -1\) as desired.

The same contradiction proof from \(\bar{w} \cdot x \leq -r||x||\) segment above (Lemma. \([\text{C.2}]\)) would show

\[- \sum_{j=1}^{k} \sigma(u^{(j)} \cdot x) < \alpha k r||x||\] (just exchange \(w\) and \(u\)) and we’ll get \(\text{sign}(N_W(x)) = \text{sign}((\bar{w} - \bar{u}) \cdot x) = -1\).

Finally, we proved that

\(\forall x \in \{ x \in \mathbb{R}^d | \bar{w} \cdot x < r||x|| \land |\bar{u} \cdot x| \geq r||x|| \land |(\bar{w} - \bar{u}) \cdot x| \geq 2r||x||\}, \text{sign}(N_W(x)) = \text{sign}((\bar{w} - \bar{u}) \cdot x)\)

and that

\(\forall x \in \{ x \in \mathbb{R}^d | \bar{u} \cdot x < r||x|| \land |\bar{w} \cdot x| \geq r||x|| \land |(\bar{w} - \bar{u}) \cdot x| \geq 2r||x||\}, \text{sign}(N_W(x)) = \text{sign}((\bar{w} - \bar{u}) \cdot x)\)

as required. 

We can now combine Corollary \([\text{C.1}]\), Proposition \([\text{C.1}]\) and Proposition \([\text{C.2}]\) and prove Theorem. \([\text{C.1}]\):

We have \(\forall x \in \mathbb{R}^d \) s.t. \(|(\bar{w} - \bar{u}) \cdot x| \geq 2r||x||\) then \(|\bar{w} \cdot x| \geq r||x|| \lor |\bar{w} \cdot x| \geq r||x||\). If \(x\) is such that \(|\bar{w} \cdot x| \geq r||x|| \lor |\bar{w} \cdot x| \geq r||x||\) we can use Proposition \([\text{C.1}]\) and get \(\text{sign}(N_W(x)) = \text{sign}((\bar{w} - \bar{u}) \cdot x)\).

If only one condition holds i.e. \(x \in \{ x \in \mathbb{R}^d | |\bar{w} \cdot x| < r||x|| \lor |\bar{w} \cdot x| \geq r||x|| \lor |(\bar{w} - \bar{u}) \cdot x| \geq 2r||x||\}\) or \(x \in \{ x \in \mathbb{R}^d | |\bar{w} \cdot x| < r||x|| \land |\bar{w} \cdot x| \geq r||x|| \land |(\bar{w} - \bar{u}) \cdot x| \geq 2r||x||\}\) then we can use Proposition \([\text{C.2}]\) and get 

\(\text{sign}(N_W(x)) = \text{sign}((\bar{w} - \bar{u}) \cdot x)\).

Therefore, overall for \(|(\bar{w} - \bar{u}) \cdot x| \geq 2r||x||\) we get \(\text{sign}(N_W(x)) = \text{sign}((\bar{w} - \bar{u}) \cdot x)\) as required.

\textbf{C.1. Proof of Corollary 6.1}

Since the network is perfectly clustered, the corollary follows by Proposition \([\text{C.1}]\) with \(r = 0\).
D. Additional Experiments - Linear Decision Boundary

In this section we provide additional empirical evaluations of the decision boundary that SGD converges to in our setting.

D.1. Leaky ReLU vs ReLU decision boundary

Theorem. (5.1) addresses the case of Leaky ReLU activation. Here we show that the result is indeed not true for ReLU networks. We compare two perfectly clustered networks (i.e., each with two neurons) one with a Leaky ReLU activation and the other with a ReLU activation. Figure 5 shows a decision boundary for a two neuron network, in the case of Leaky ReLU (Figure 5a) and ReLU (Figure 5b). It can be seen that the leaky ReLU indeed provides a linear decision boundary, as predicted by Theorem 5.1, whereas the ReLU case is non-linear (we explicitly show the regime where the network output is zero. This can be orange or blue, depending on whether zero is given label positive or negative. In any case the resulting boundary is non-linear).

![Leaky ReLU network - Linear Decision Boundary](image1)

![ReLU network - Non Linear Decision Boundary](image2)

Figure 5: The prediction landscape for two neuron networks with Leaky ReLU and ReLU activations. Orange for positive prediction, blue for a negative prediction and grey for zero prediction. The \( w \) neuron is \((1, 0) \in \mathbb{R}^2 \) and the \( u \) neuron is \((0, 1) \in \mathbb{R}^2 \).

D.2. MNIST - Linear Regime

In Figure 2 in the main text we saw how for MNIST digit pairs (0,1) and (3,5) the network enters the linear regime at some point in the training process. In Figure 6 we see the robustness of this behavior across the MNIST data-set by showing the above holds for more pairs of digits.

![Convergence to a classifier that is linear on the data, for MNIST pairs](image3)

Figure 6: Convergence to a classifier that is linear on the data, for MNIST pairs. Each line corresponds to an average over 5 initializations.
D.3. Clustering of Neurons - Empirical Evidence

In Section 5 in the main text and Figure 6 above, we saw that learning converges to a linear decision boundary on the train and test points. Theorem (5.1) suggests that this will happen if neurons are well clustered (in the $w$ and $u$ groups). Here we show that indeed clustering occurs.

We consider two different measures of clustering. The first is the ratio $r = \|w - u\|$, and the second is the maximum angle between the neurons of the same type (i.e., the maximal angle between vectors in the same cluster). Figure 7 shows these two measures as a function of the training epochs. They can indeed be seen to converge to zero, which by Theorem (5.1) implies convergence to a linear decision boundary.

Figure 7: Evaluation of clustering measures during training. We consider two different clustering measures in (a) and (b) (see text). It can be seen that both measures converge to zero.

E. Assumptions for Gradient Flow Analysis

In the paper we use results from (Lyu & Li, 2020) and (Ji & Telgarsky, 2020). Here we show that the assumptions required by these theorems are satisfied in our setup.

The assumptions in (Lyu & Li, 2020) and (Ji & Telgarsky, 2020) are:

(A1) . (Regularity). For any fixed $x$, $\Phi(\cdot; x)$ is locally Lipschitz and admits a chain rule;

(A2) . (Homogeneity). There exists $L > 0$ such that $\forall \alpha > 0 : \Phi(\alpha W; x) = \alpha L \Phi(W; x)$;

(B3) . The loss function $\ell(q)$ can be expressed as $\ell(q) = e^{-f(q)}$ such that

(B3.1) $f : \mathbb{R} \to \mathbb{R}$ is $C^1$-smooth.

(B3.2) $f'(q) > 0$ for all $q \in \mathbb{R}$.

(B3.3) There exists $b_f \geq 0$ such that $f'(q)q$ is non-decreasing for $q \in (b_f, +\infty)$, and $f'(q)q \to +\infty$ as $q \to +\infty$.

(B3.4) Let $g : [f(b_f), +\infty) \to [b_f, +\infty)$ be the inverse function of $f$ on the domain $[b_f, +\infty)$. There exists $b_g \geq \max\{2f(b_f), f(2b_f)\}$, $K \geq 1$ such that $g'(x) \leq Kg'(\theta x)$ and $f'(y) \leq Kf'((\theta y)$ for all $x \in (b_g, +\infty)$, $y \in (g(b_g), +\infty)$ and $\theta \in [1/2, 1)$

(B4) . (Separability). There exists a time $t_0$ such that $\mathcal{L}(W) < e^{-f(b_f)} = \ell(b_f)$

We next show that these are satisfied in our setup.
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Proof. (A1). (Regularity) first we show that $\Phi(\cdot; x)$ is locally Lipschitz, with slight abuse of notations, let $W_1 = \tilde{W}_1, W_2 = \tilde{W}_2 \in \mathbb{R}^{2kd}$ so in our case:

$$
\Phi(W_1; x) - \Phi(W_2; x) = v \cdot \sigma(W_1 \cdot x) - v \cdot \sigma(W_2 \cdot x)
$$

$$
= v \left[ \sum_{j=1}^{k} \sigma \left( u_1^{(j)} \cdot x \right) - \sigma \left( u_2^{(j)} \cdot x \right) - \left( \sigma \left( u_1^{(j)} \cdot x \right) - \sigma \left( u_2^{(j)} \cdot x \right) \right) \right]
$$

and therefore

$$
||\Phi(W_1; x) - \Phi(W_2; x)||
$$

$$
= \left| v \left[ \sum_{j=1}^{k} \sigma \left( u_1^{(j)} \cdot x \right) - \sigma \left( u_2^{(j)} \cdot x \right) - \left( \sigma \left( u_1^{(j)} \cdot x \right) - \sigma \left( u_2^{(j)} \cdot x \right) \right) \right] \right|
$$

$$
\leq v \left[ \sum_{j=1}^{k} ||\sigma \left( u_1^{(j)} \cdot x \right) - \sigma \left( u_2^{(j)} \cdot x \right)|| + ||\sigma \left( u_1^{(j)} \cdot x \right) - \sigma \left( u_2^{(j)} \cdot x \right)|| \right]
$$

$$
\leq 2v||x|| \left[ \sum_{j=1}^{k} ||u_1^{(j)} - u_2^{(j)}|| + ||u_1^{(j)} - u_2^{(j)}|| \right] = 2v \cdot ||x|| \cdot ||\tilde{W}_1 - \tilde{W}_2||
$$

And we showed $\Phi(\cdot; x)$ is globally Lipschitz (and therfor locally Lischitz). Next for the chain rule, as shown in Davis et al. [2018] (corollary for deep learning therein), any function definable in an o-minimal structure admits a chain rule. Our network is definable because algebraic, composition, inverse, maximum and minimum operations over definable functions are also definable. Leaky ReLUs are definable as maximum operations over two linear functions (linear functions are definable).and because Leaky ReLUs are definable our network is also definable.

(A2). (Homogeneity). It is easy to see from the definition that in our case, the trainable parameters are only the first layer weights and the network $\Phi(\cdot; x)$ is $L = 1$ homogeneous.

(B3). As seen in Lyu & Li [2020] (Remark A.2. therein) the logistic loss $\ell(q) = \log(1 + e^{-q})$ satisfies (B3) with $f(q) = -\log \left( \log(1 + e^{-q}) \right), g(q) = -\log \left( e^{e^{-q}} - 1 \right), b_f = 0.$

(B4). (Separability). This is Assumption 6.1 in the main text. As we mentioned in the main text, this assumption is satisfied with SGD by Theorem. 4.1.

$$
\square
$$

F. Proof of Theorem 6.1

In this proof we will show that the normalized parameters $\tilde{W}_t := \frac{W_t}{||W_t||}$ under gradient flow optimization, converges to a solution in $\mathcal{N}$ and that the network $N_{\tilde{W}}$ at convergence is perfectly clustered. Under our assumption $\forall t \geq T_{NAR} \tilde{W}_t \in \mathcal{N}$. From the definition of the NAR it’s easy to see that the NAR is a closed domain. Therefore any limit point of $\tilde{W}_t$ is also in the NAR. From Ji & Telgarsky (2020) (Theorem 3.1. therein) we have that the normalized parameters flow converges when using gradient flow. To conclude so far, we had shown that $\tilde{W}_t$ converges to a point inside the NAR $\mathcal{N}$.

We are left with showing that the limit point of $\lim_{t \to \infty} \tilde{W}_t := \tilde{W}$, has a perfectly clustered form.

Lyu & Li (2020) (Theorem A.8. therein) shows that every limit point of $\tilde{W}_t$ is along the direction of a KKT point of the following optimization problem $(P)$:

$$
\min \frac{1}{2}||W||^2
$$

s.t. $q_i(W) \geq 1 \quad \forall i \in [n]$
where \( q_i(W) = y_i N_W(x_i) \) is the network margin on the sample point \((y_i, x_i)\). We are left with showing that at convergence the neurons align in two directions. We will use a characterization of the KKT points of \( (P) \) and show that they are perfectly clustered. Since every limit point of the normalized parameters flow is along the direction of a KKT point of \( (P) \) that would mean \( \hat{W}_* \) has a perfectly clustered form.

A feasible point \( W \) of \( (P) \) is a KKT point if there exist \( \lambda_1, \ldots, \lambda_n \geq 0 \) such that:

1. \( W - \sum_{i=1}^{n} \lambda_i h_i = 0 \) for some \( h_1, \ldots, h_n \) satisfying \( h_i \in \partial q_i(W) \)
2. \( \forall i \in [n] : \lambda_i (q_i(W) - 1) = 0 \)

From Lyu & Li [2020] (Theorem A.8, therein) we know \( \exists \beta \) s.t. \( \beta \hat{W}_* \) is a KKT point of \( (P) \). Since our limit point is in an NAR we don’t need to worry about the non differential points of the network because \( \forall 1 \leq j \leq k, i \in [n] : w_*^{(j)} \cdot x_i \neq 0 \wedge u_*^{(j)} \cdot x_i \neq 0 \) (where \( w_*^{(j)} \) and \( u_*^{(j)} \) stands for the \( w \) and \( u \) type neurons of \( W_* \), respectively). Therefore the Clarke subdifferential coincides with the gradient in our domain, and we can derive it using calculus rules.

By looking at the gradient of the margin for any point \((y_i, x_i)\):

\[
\frac{\partial q_i(W)}{\partial u^{(j)}} = y_i v x_i \sigma'(u^{(j)} \cdot x_i) = y_i v x_i \sigma'(w^{(j)} \cdot x_i)
\]

\[
\frac{\partial q_i(W)}{\partial \bar{u}^{(j)}} = -y_i v x_i \sigma'(u^{(j)} \cdot x_i) = -y_i v x_i \sigma'(w^{(j)} \cdot x_i)
\]

Now using the above gradients implies that:

\[
\frac{\partial q_i(W)}{\partial u^{(j)}} = y_i v x_i \left( \sigma'(u^{(1)} \cdot x_i), \ldots, \sigma'(u^{(k)} \cdot x_i), -\sigma'(u^{(1)} \cdot x_i), \ldots, -\sigma'(u^{(k)} \cdot x_i) \right)
\]

By the definition of the NAR \( N \) with parameters \( (\beta, c_i^w, c_i^u) \) the dot product of a point \( x_i \) with all neurons of the same type is of the same sign, i.e.:

\[
\forall i \in [n], \forall 1 \leq l, p \leq k : \sigma'(u^{(l)} \cdot x_i) = \sigma'(u^{(p)} \cdot x_i) = c_i^w
\]

and

\[
\forall i \in [n], \forall 1 \leq l, p \leq k : \sigma'(u^{(l)} \cdot x_i) = \sigma'(u^{(p)} \cdot x_i) = c_i^u
\]

It follows that for \( W \in N, \frac{\partial q_i(W)}{\partial u^{(j)}} = y_i v \cdot x_i (c_i^w, \ldots, c_i^w, -c_i^u, \ldots, -c_i^u) \).

Therefore, by the definition of a KKT point we have:

\[
\hat{W}_* = \frac{1}{\beta} \left( \sum_{i=1}^{n} \lambda_i y_i v x_i c_i^w, \ldots, \sum_{i=1}^{n} \lambda_i y_i v x_i c_i^w, -\sum_{i=1}^{n} \lambda_i y_i v x_i c_i^u, \ldots, -\sum_{i=1}^{n} \lambda_i y_i v x_i c_i^u \right) \in \mathbb{R}^{2kd}
\]

We can see that the first \( k \) entries are equal, as well as the next \( k \) entries (equal to each other and not to the first \( k \) entries).

Therefore the normalized parameters flow \( \hat{W}_t \) converges to a perfectly clustered solution.

F.1. Proof Of Corollary 6.2.

By Theorem 6.1, we know the normalized parameters \( \hat{W}_t \) are perfectly clustered at convergence so by Corollary 6.1 we get that the decision boundary of \( N_W(x) \) is linear at convergence. From the homogeneity of the network we have \( N_W(x) = ||W|| N_{W_*}(x) \) for any \( W \in \mathbb{R}^{2kd} \) and because the norm is a non negative scalar we get \( \text{sign}(N_W(x)) = \text{sign}(N_{W_*}(x)) \), i.e. \( N_W \) and \( N_{W_*} \) are the same classifiers. Therefore, this implies that the decision boundary of \( N_W \) is linear at convergence.

\(^8\) It is not hard to see that given that the solution is in an NAR, then this optimization problem is convex.

\(^9\) We use \( \text{sign}(\infty) = 1 \) and \( \text{sign}(-\infty) = -1 \), since the norm \( ||W|| \) diverges.
We divide the proof of Theorem (6.2) into two parts. First, we show that the NAR is a PAR, and then we show that if a network enters and remains in the PAR the network weights at convergence are proportional to the solutions of the SVM problem we defined in the main text.

G.1. The NAR is a PAR

In this subsection we will prove the NAR is in fact a PAR under the conditions of the theorem. In the first step we show that for all \( w^{(i)} \)'s, \( \left( \frac{w^{(i)}}{\|w^{(i)}\|} \right) \cdot x_+ \geq \beta \) for all positive \( x_+ \in S_+ \) and times \( t \geq T_{\text{Margin}} \). Assume by contradiction that the latter does not hold. Thus, by assumption 2 the network is in a NAR(\( \beta \)) and there exists a positive \( x_+ \in S_+ \) such that \( \left( \frac{w^{(i)}}{\|w^{(i)}\|} \right) \cdot x_+ \leq -\beta \) for all \( w^{(i)} \). Denote by \( \tau_t \{ x \} \) the margin of the network at time \( t \geq T_{\text{Margin}} \) on the point \( x \). We notice that \( \tau_t \leq \tau_{t \{ x \}} \) by definition. Then:

\[
\tilde{\gamma}_t \leq \gamma_t \leq \gamma_{t \{ x \} +} = \frac{1 + N_W(x_+)}{||W||} = \frac{v \left( \sum_{i=1}^{k} \sigma \left( w^{(i)}_t \cdot x_+ \right) - \sum_{i=1}^{k} \sigma \left( u^{(i)}_t \cdot x_+ \right) \right)}{\sqrt{\sum_{i=1}^{k} ||w^{(i)}||^2 + ||u^{(i)}||^2}}
\]

(27)

\[
\leq \frac{v \left( \sum_{i=1}^{k} \sigma \left( w^{(i)}_t \cdot x_+ \right) - \sum_{i=1}^{k} \sigma \left( u^{(i)}_t \cdot x_+ \right) \right)}{\sqrt{\sum_{i=1}^{k} ||u^{(i)}||^2}} \leq \frac{v \left( - \sum_{i=1}^{k} \sigma \left( u^{(i)}_t \cdot x_+ \right) \right)}{\sqrt{\sum_{i=1}^{k} ||u^{(i)}||^2}} \leq \frac{v \alpha \left( \sum_{i=1}^{k} ||u^{(i)}|| \right)}{\sqrt{\sum_{i=1}^{k} ||u^{(i)}||^2}}
\]

(28)

\[
v \cdot \alpha \cdot \sqrt{\left( \sum_{i=1}^{k} ||u^{(i)}|| \right)} \cdot \text{max}_{i \in [n]} ||x_i|| \leq \frac{v \alpha \cdot \sqrt{k} \cdot \text{max}_{i \in [n]} ||x_i||}{\sqrt{\sum_{i=1}^{k} ||u^{(i)}||^2}}
\]

(29)

where the first inequality follows by Lyu & Li (2020) (Theorem A.7. therein). In Eq. 28 we noticed that \( - \sum_{i=1}^{k} \sigma \left( u^{(i)}_t \cdot x_+ \right) \) is largest when \( \forall 1 \leq i \leq k \quad u^{(i)}_t \cdot x_+ < 0 \) and therefore \( \sigma \left( u^{(i)}_t \cdot x_+ \right) = \alpha u^{(i)}_t \cdot x_+ \). Therefore, by the inequality \( \forall v \in \mathbb{R}^k \quad ||v||_1 \leq \sqrt{k} \cdot ||v||_2 \), we have:

\[
\tilde{\gamma}_t \leq \frac{v \cdot \alpha \cdot \sqrt{k} \cdot \text{max}_{i \in [n]} ||x_i||}{\sqrt{\sum_{i=1}^{k} ||u^{(i)}||^2}} = \sqrt{k} \cdot \alpha \cdot v \cdot \text{max}_{i \in [n]} ||x_i||
\]

(29)

Now under assumption 3 there exists a time \( T_{Margin} \geq T_{\text{NAR}} \) such that \( \gamma_{T_{\text{Margin}}} > \sqrt{k} \cdot \alpha \cdot v \cdot \text{max}_{i \in [n]} ||x_i|| \). By Lyu & Li (2020) (Theorem A.7. therein) the smoothed margin \( \tau_t \) is a non-decreasing function and we will get that \( \forall t \geq T_{Margin} \) :
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Consider the inequality \( \tilde{\gamma}_t > \sqrt{k} \alpha v \cdot \max_{i \in [n]} ||x_i|| \) which is a contradiction to Eq. (29). Hence, \( \forall 1 \leq i \leq k \land x \in S_+ : \left( \frac{w_i^{(t)}}{||u_i^{(t)}||} \right) \cdot x \geq \beta \).

In a similar fashion, assume there is some \( x_- \in S_- \) such that for \( \left( \frac{w_i^{(t)}}{||u_i^{(t)}||} \right) \cdot x_- \geq \beta \) doesn’t hold. Then by assumption 1 the network is in a NAR, \( \left( \frac{w_i^{(t)}}{||u_i^{(t)}||} \right) \cdot x_- \leq -\beta \) and by symmetry again we get:

\[
\tilde{\gamma}_t \leq \tilde{\gamma}_t \leq \tilde{\gamma}_{t, (x_-)} = -1 \cdot N_W(x_-) = -\frac{1}{||W||^2} - v \left( \sum_{i=1}^k \sigma (w_i^{(t)} \cdot x_-) - \sum_{i=1}^k \sigma (w_i^{(t)} \cdot x_-) \right)
\]

\[
\cdots \leq \sqrt{k} \alpha v \cdot \max_{i \in [n]} ||x_i||
\]

By Lyu et al. (2020) (Theorem A.7. therein) we reach a contradiction to the network margin assumption again, so \( \forall x \in S_- : \left( \frac{w_i^{(t)}}{||u_i^{(t)}||} \right) \cdot x \geq \beta \).

To conclude, we have proven so far for all \( t > T_{Margin} : \)

1. \( \forall 1 \leq i \leq k : \forall x \in S_+ : \left( \frac{w_i^{(t)}}{||u_i^{(t)}||} \right) \cdot x \geq \beta \).
2. \( \forall 1 \leq i \leq k : \forall x \in S_- : \left( \frac{w_i^{(t)}}{||u_i^{(t)}||} \right) \cdot x \geq \beta \).

Now, by assumption 4, \( \forall x \in S_- : \left( \frac{w_i^{(t)}}{||u_i^{(t)}||} \right) \cdot x \not\geq \beta \) and similarly \( \forall x \in S_+ : \left( \frac{w_i^{(t)}}{||u_i^{(t)}||} \right) \cdot x \not\geq \beta \). This follows since otherwise \( V^{T_{Margin}}(S) \) and \( V^{T_{Margin}}(S) \) would not be empty in contradiction to assumption 4.

Next, under the network being in an NAR assumption we have for all \( t > T_{Margin} : \)

1. \( \forall x \in S_- : \left( \frac{w_i^{(t)}}{||u_i^{(t)}||} \right) \cdot x \leq -\beta \)
2. \( \forall x \in S_+ : \left( \frac{w_i^{(t)}}{||u_i^{(t)}||} \right) \cdot x \leq -\beta \)

Thus, for all \( t > T_{Margin} \), the network is in \( \text{PAR}(\beta) \).

G.2. PAR alignment direction

Now we will find where the parameters converge to when the network is in the \( \text{PAR}(\beta) \). By Theorem. [6.1], the normalized gradient flow converges to a perfectly clustered solution, i.e., \( \lim_{t \to \infty} W_t := W_* \) is of a perfectly clustered form. Formally that means \( \exists \beta \) and \( \exists \delta \) such that the normalized parameters \( W \) are of the form \( W_* = (\beta \hat{w}, \ldots, \beta \hat{w}, \delta \hat{u}, \ldots, \delta \hat{u}) \in \mathbb{R}^{2kd} \) and WLOG we can assume \( ||\hat{w}|| = ||\hat{u}|| = 1 \).

Because the solution is in the \( \text{PAR}(\beta) \), the network margins are given as follows for positive points:

\[
\forall x_i \in S_+ : q_i(W) = y_i N_W(x_i) = y_i ||W|| N_W(x_i) = v ||W|| \left( \sum_{i=1}^k \sigma (\beta \hat{w} \cdot x_i) - \sigma (\delta \hat{u} \cdot x_i) \right)
\]

\[
= v ||W|| (k \beta \hat{w} \cdot x_i - \alpha k \beta \hat{u} \cdot x_i)
\]
We obtained a reformulation of (P) as an SVM problem with variables $V$.

Assume which is a concatenated version of the original data $\tilde{\varphi}$ where we used the fact we know the normalized solution would have a perfectly clustered form. We denote $\forall \in x_i \in S_-, q_i(W) = y_iN_W(x_i) = y_i||W||N_W(x_i) = v||W|| \left( \sum_{i=1}^{k} \sigma(\delta \tilde{u} \cdot x_i) - \sigma(\beta \tilde{w} \cdot x_i) \right)

= v||W|| (k\delta \tilde{u} \cdot x_i - \alpha k\beta \tilde{w} \cdot x_i)$

where we used the fact we know the normalized solution would have a perfectly clustered form. We denote $\tilde{\beta} := ||W|| \cdot \beta$ and similarly $\tilde{\delta} := ||W|| \cdot \delta$.

Using the above notations, the max margin problem in [Lyu & Li (2020)] (Theorem A.8. therein) takes the form:

$$\arg min_{\beta, \delta} k\tilde{\beta}^2 + k\tilde{\delta}^2 = \arg min_{\beta, \delta} v^2 k^2 \beta^2 + v^2 k^2 \delta^2$$

$$\forall x_+ \in S_+ : v k\tilde{\beta} \tilde{w} \cdot x_+ - \alpha v k\delta \tilde{u} \cdot x_+ \geq 1$$

$$\forall x_- \in S_- : v k\delta \tilde{u} \cdot x_- - \alpha v k\beta \tilde{w} \cdot x_- \geq 1$$

Now we can denote $w := v k\beta \tilde{w}$ and $u := v k\delta \tilde{u}$ and reach the desired formulation:

$$\arg min_{w, u} ||w||^2 + ||u||^2$$

$$\forall x_+ \in N_+ : w \cdot x_+ - \alpha u \cdot x_+ \geq 1$$

$$\forall x_- \in N_- : u \cdot x_- - \alpha w \cdot x_- \geq 1$$

We obtained a reformulation of (P) as an SVM problem with variables $(w, u) \in \mathbb{R}^{2d}$ and with a transformed dataset which is a concatenated version of the original data $\phi(x) = \left[ \sigma'(w \cdot x) x, -\sigma'(-w^* \cdot x) x \right] \in \mathbb{R}^{2d}$, where for $x_+ \in N_+$, $\phi(x_+) = (x_+, -\alpha x_+) \in \mathbb{R}^{2d}$ and for $x_- \in N_-$, $\phi(x_-) = (-\alpha x_-, x_-) \in \mathbb{R}^{2d}$.

**H. Proof of Lemma 6.1**

Assume $\mathbb{V}^+_\beta(S) \neq \emptyset$, i.e. $\exists v \in S$, s.t. $\forall x \in S_+ : v \cdot x \geq \beta$ and $\exists x_+ \in S_- \ s.t. \ v \cdot x_+ \geq \beta$. This means that $\hat{v} \cdot x_+ \leq -\beta$, because the data is linearly separable $-x_+ \in S$ has to be a positive point and by the definition of $\mathbb{V}^+_{\beta}(S)$ that would mean $\hat{v} \cdot x_+ \geq \beta$ in contradiction.

By symmetry, if we assume $\mathbb{V}^-_{\beta}(S) \neq \emptyset$ by taking the positive point which $\hat{v} \in \mathbb{V}^-_{\beta}(S)$ mistakenly classifies as a negative one, we'll reach a contradiction again.

Therefore if $\forall x \in S, -x \in S$ we have $\mathbb{V}^+_\beta(S) = \emptyset$ and $\mathbb{V}^-_{\beta}(S) = \emptyset$ and Assumption 4 in Theorem. 6.2, holds in this case.

![Figure 8: The ratio of neurons from each type in the PAR throughout the training process. We sample 400 data points from two antipodal separable Gaussians (one for each label) in $\mathbb{R}^{30}$. Our network is of 100 neurons (50 of each type) optimized on the data using SGD with batch size 1 with learning rate $\eta = 10^{-5}$.](image)
I. Entrance to PAR - High Dimensional Gaussians

We will show that the entrance to the PAR indeed happens empirically for two separable Gaussians. We measure the percentage of neurons which are in the PAR of both types. A $w$ type neuron is considered in the PAR if it classifies like the ground truth $w^*$. A $u$ type neuron is considered in the PAR if it classifies like $-w^*$.

The percentage of neurons in the PAR throughout the training process is given in Figure 8. We can see that the network enters the PAR.

J. Entrance to NAR which is not a PAR

In this section we show that learning can enter an NAR which is not a PAR. We sample two antipodal Gaussians and add one outlier positive point. Then for each neuron type ($w$ or $u$) we measure the maximum amount of data points classification disagreements between neurons of the same type denoted $\max(n_{diff})$ and the percentage of neurons which are in the PAR.

In Figure 9a we can see that the network yields 100% prediction accuracy. In Figure 9b we can see the directions of the neurons ($w$ type in black and $u$ type in yellow). In Figure 9c we can see that the maximal number of points which neurons of the same type classified differently goes to zero, therefore all neurons of the same type agree on the classification of the data points. In Figure 9d we can see that the ratio of $w$ type neurons which perfectly classifies the data does not increase to 1 so the network does not enter the PAR.
K. Extension - First Layer Bias Term

In order to extend our results to include a bias term in the first layer, we would just need to reformulate our data points $S$ to $S'$ by

$$(x, y) \in S \subseteq \mathbb{R}^d \times \mathbb{Y} \mapsto ((x, 1), y) \in S' \subseteq \mathbb{R}^{d+1} \times \mathbb{Y}$$

and extend our neurons to include a bias term:

$$\forall 1 \leq i \leq k \quad w^{(i)}_t \in \mathbb{R}^d \mapsto (w^{(i)}_t, b^{(i)}_w) \in \mathbb{R}^{d+1}, \quad u^{(i)}_t \in \mathbb{R}^d \mapsto (u^{(i)}_t, b^{(i)}_u) \in \mathbb{R}^{d+1}$$

This is equivalent to reformulating the first weights matrix $W \in \mathbb{R}^{2k \times d} \mapsto W' \in \mathbb{R}^{2k \times (d+1)}$.

This reformulation is equivalent to adding a bias term for every neuron in the first layer, and all of the following results would still hold under the above reformulation.

The proofs of Theorem. (4.1) and Theorem. (5.1) follow exactly if we exchange $W$ with $W'$ while for the proofs of Theorem. (6.1) and Theorem. (6.2) we use results from [Lyu & Li, 2020] and [Ji & Telgarsky, 2020] that require the model to be homogeneous. Note that if we add a bias in the first layer, the model remains homogeneous and the proofs of Theorem. (6.1) and Theorem. (6.2) still hold for those cases as well.