Decoupled and unidirectional asymptotic models for the propagation of internal waves

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Abstract

We study the relevance of various scalar equations, such as inviscid Burgers’, Korteweg-de Vries (KdV), extended KdV, and higher order equations, as asymptotic models for the propagation of internal waves in a two-fluid system. These scalar evolution equations may be justified in two ways. The first method consists in approximating the flow by two uncoupled, counter-propagating waves, each one satisfying such an equation. One also recovers these equations when focusing on a given direction of propagation, and seeking unidirectional approximate solutions. This second justification is more restrictive as for the admissible initial data, but yields greater accuracy. Additionally, we present several new coupled asymptotic models: a Green-Naghdi type model, its simplified version in the so-called Camassa-Holm regime, and a weakly decoupled model. All of the models are rigorously justified in the sense of consistency.

1 Introduction

In this paper, we study asymptotic models for the propagation of internal waves in a two-fluid system, which consists in two layers of immiscible, homogeneous, ideal, incompressible and irrotational fluids under the only influence of gravity. We assume that there is no topography (the bottom is flat) and that the surface is confined by a flat rigid lid, although the case of a free surface could be handled with our method. Finally, as we will focus on scalar models, we restrict ourselves to the case of horizontal dimension one. The precise equations of such a system are presented in Appendix A, and we refer to the survey article [31] and references therein for a good overview of the relevance and significance of such system in oceanography.

The mathematical theory for the governing equations of our system is extremely challenging (see [43] in particular), which has led to extensive works concerning simplified, asymptotic models, aiming at capturing the main characteristics of the flow with much simpler equations, provided the size of given parameters (typically measuring the amplitude of the deformation at the interface, and/or shallowness of the two layers of fluid) is small. Among other works, we would like to highlight [8], where many asymptotic models are presented and justified, in a wide range of such regimes. In each case, the resulting model consists in two relatively simple evolution equations coupling the shear velocity and the deformation at the interface.

However, in this work, we are mainly interested in decoupled scalar models, which can be used to describe the unidirectional propagation of gravity waves. The derivation and study of such models have a very rich and ancient history, starting with the work of Boussinesq [11] and Korteweg-de Vries [41] which introduced the famous Korteweg-de Vries equation

\[ \partial_t u + c \partial_x u + \alpha u \partial_x u + \nu \partial_x^3 u = 0, \]

in the context of propagation of surface gravity waves, above one layer of homogeneous fluid (which we refer as “water-wave problem”). However, let us note that the complete rigorous justification of

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such model is recent: [37, 56, 7]. Such a justification is to be understood in the following sense: in the long wave regime\(^1\)
\[ \epsilon \sim \mu \ll 1, \]
the flow can be accurately approximated as a decomposition into two counter-propagating waves, each component satisfying a KdV equation (or more generally, with the same order of accuracy, a Benjamin-Bona-Mahony equation [5, 54]).

\[(BBM) \]
\[ \partial_t u + c \partial_x u + \alpha u \partial_x u + \nu_x \partial_x^2 u - \nu_t \partial_x^2 \partial_t u = 0, \]
The situation is similar in the case of two layers of inmiscible fluids, with a rigid lid (as well as in the case of two layers with a free surface, except the flow is then decomposed into four propagating waves), and correspondent results are presented in [23].

Of course the coefficients \((c, \alpha, \nu, \text{etc.})\) depend on the situation, and a striking difference between the water-wave case and the case of internal waves is that in the latter case, there exists a critical ratio for the two layers of fluid (depending on the ratio of the mass densities) for which the nonlinearity coefficient \(\alpha\) vanishes. In that case, heuristic arguments support the inclusion of the next order (cubic) nonlinearity, which yields the modified KdV equation (see [52] and references therein):

\[(mKdV) \]
\[ \partial_t u + c \partial_x u + \alpha_2 u^2 \partial_x u + \nu_x \partial_x^3 u - \nu_t \partial_x^2 \partial_t u = 0. \]

For the cubic nonlinearity to be of the same order of magnitude as the dispersive terms, this urges to consider the so-called (refering to [19] for this non-standard denomination) Camassa-Holm regime:

\[ \epsilon^2 \sim \mu \ll 1. \]

Note that this regime is interesting by itself as it allows larger amplitude waves than the long wave regime, and in particular yields models developing finite time breaking wave singularities [18, 19]. Thus we ask: Can we extend the results of the long wave regime to the Camassa-Holm regime?

As we shall see, the answer is not straightforward, as several evolution equations are in competition, with an accuracy depending on the situation, and in particular the criticality of the depth ratio, and localization in space of the initial data. In addition to the equations already presented above, we consider the extended KdV (or Gardner) equation

\[(eKdV) \]
\[ \partial_t u + c \partial_x u + \alpha_1 u \partial_x u + \alpha_2 u^2 \partial_x u + \nu_x \partial_x^3 u - \nu_t \partial_x^2 \partial_t u = 0, \]
as presented in [36, 21, 46], studied in [29, 53, 55] (among other works), and tested against experiments in [40, 32, 45]. More generally, we work with the higher order model presented in the water-wave setting in [19] (which we therefore call Constantin-Lannes)

\[(CL) \]
\[ \partial_t u + c \partial_x u + \alpha_1 u \partial_x u + \alpha_2 u^2 \partial_x u + \alpha_3 u^3 \partial_x^3 u + \nu_t \partial_x^2 u - \nu_t \partial_x^2 \partial_t u \]
\[ + \partial_x (\kappa_1 \partial_x^2 u + \kappa_2 (\partial_x u)^2) = 0, \]
which is related, though slightly different from the Camassa-Holm equation obtained for example in [13, 35] or higher order models (see various such models in [61, 25, 26, 30, 51, 24, 20], concerning the water wave or internal wave problem). However, each of these works is limited to the restrictive assumption that only one direction of propagation is non-zero.

We propose to study in detail the justification of the decomposition of the flow (although the unidirectional case is also treated) as presented above, in the case of internal waves. Our results are expressed in a very general setting: assuming only
\[ \mu \ll 1 \text{ and } \epsilon \ll 1, \]
and expressing the accuracy of the different models as a function of these parameters. Although this complicates the expression of our results, and renders the proofs fairly technical, such choice allows to cover general regimes, including the long-wave as well as the Camassa-Holm regimes, described above. We recover in particular the relevance of the KdV approximation in the long wave regime. In the Camassa-Holm regime, the situation will depend on the criticality of the depth ratio, the localization in space of the initial data, as well as the time-scale which is considered.

\(^1\)In the following, \(\mu\) is the shallowness parameter, and \(\epsilon\) is the nonlinearity parameter, as defined precisely (in the bi-fluidic case, see [7] for the water-wave case) in Appendix A.
Outline of the paper, and overview of the results. For the sake of readability, the precise presentation and justification of the governing (full Euler) equations of our system is postponed to Appendix A. Using the shallowness assumption ($\mu \ll 1$), we then introduce the so-called Green-Naghdi model (1.1), written out below. This system is related to the one introduced by Choi and Camassa in [16]. We present its first rigorous justification, in the sense of consistency: roughly speaking, any sufficiently smooth and uniformly bounded solution of the full Euler system satisfies the (A.19) model, up to a residual of size $O(\mu^2)$. Our decoupled models will be constructed from, and compared with the Green-Naghdi model.

In section 2.1, we present a formal argument which allows to derive the decoupled models at stake, that we define precisely in Definition 2.3 and Definition 2.4.

The main results of the paper are presented in Section 2.2. We first show in Proposition 2.5 that the equation (CL), and therefore any simplified equation, is well-posed and uniformly controlled in Sobolev norms over long times, for sufficiently smooth initial data. We also prove that the localization in space of the initial data (that we express by the fact that the function lives in weighted Sobolev norms) propagates over long times.

We express our main result in Proposition 2.7 and Corollary 2.10, stating that for any sufficiently smooth initial data, an approximate solution of the Green-Naghdi equation (in the sense of consistency) consists into two decoupled waves satisfying (CL) (or a lower order equation), plus an explicit, small coupling term. The precision in the sense of consistency and estimates on the size of the coupling term depend on

1. the evolution equation considered (CL,eKdV,KdV, etc.);
2. the size of the parameters $\epsilon, \mu$ as well as $\delta^2 - \gamma$ (critical ratio);
3. the localization in space of the initial data;
4. the time-scale considered.

As announced above, our result is as extensive as possible, and a discussion on several important cases (long-wave regime, Camassa-Holm regime with critical or non-critical ratio) follows in Section 2.3, with several numerical simulations to support our conclusions.

As intermediary steps for the justification of decoupled asymptotic models, we introduce two coupled asymptotic models, which are interesting by themselves. The first one, (2.3a)-(2.3b) is a simplified version of the Green-Naghdi equation, with same precision provided that we are in the Camassa-Holm regime (precisely defined in Remark 2.1), as stated in Proposition 2.2. We also introduce weakly coupled models, which add first order corrections of the fully decoupled models, in the spirit of [62]. These corrections are explicitly obtained using the solutions of the decoupled models; see Remark 2.8.

In Section 3, we turn to the case of a unidirectional flow. More precisely, and in the spirit of [35, 19], we show in Proposition 3.1 that one can construct very precise unidirectional approximate solutions: the interface deformation satisfies a Constantin-Lannes equation (CL), and the shear velocity is given as a function of the interface deformation. As we follow a specific direction of propagation, the coupling terms which were the main contributors for the error in the previous section vanish, and the accuracy of the approximation is therefore considerably improved. We then numerically investigate if the quite restricting condition on the initial data arises naturally, that is if after some time, the flow generated by any initial perturbation will eventually decompose into two almost purely unidirectional waves.

Finally, Appendix B contains the proof of Proposition 2.5, and Appendix C the proof of Proposition 2.7.

The Green-Naghdi coupled model. Let us display here the Green-Naghdi model (A.19), as derived in Appendix A. The system has two unknowns: $\zeta$ representing the deformation at the
interface, and \( \bar{v} \) the shear layer-mean velocity, as defined precisely in (A.18).

\[
\begin{align*}
\partial_t \zeta + \partial_x \left( \frac{h_1 h_2}{h_1 + \gamma h_2} \bar{v} \right) &= 0, \\
\partial_t (\bar{v} + \mu \bar{Q}[h_1, h_2][\bar{v}]) + (\gamma + \delta) \partial_x \zeta + \frac{\epsilon}{2} \partial_x \left( \frac{h_1^2 - \gamma h_2^2}{(h_1 + \gamma h_2)^2} |\bar{v}|^2 \right) &= \mu \partial_x (\bar{R}[h_1, h_2][\bar{v}]),
\end{align*}
\]

(1.1)

where \( \gamma \) and \( \delta \) are, respectively, the mass density and depth ratio (see Appendix A); \( h_1 = 1 - \epsilon \zeta \) and \( h_2 = \frac{1}{\eta} + \epsilon \zeta \) are the depth of, respectively, the upper and the lower layer, and where we define the following operators:

\[
\bar{Q}[h_1, h_2] V \equiv -\frac{1}{3h_1 h_2} \left( h_1 \partial_x \left( h_2^3 \partial_x \left( \frac{h_1 V}{h_1 + \gamma h_2} \right) \right) + \gamma h_2 \partial_x \left( h_1^3 \partial_x \left( \frac{h_2 V}{h_1 + \gamma h_2} \right) \right) \right),
\]

\[
\bar{R}[h_1, h_2] V \equiv \frac{1}{2} \left( h_2 \partial_x \left( \frac{h_1 V}{h_1 + \gamma h_2} \right)^2 - \gamma \left( h_1 \partial_x \left( \frac{h_2 V}{h_1 + \gamma h_2} \right) \right)^2 \right)
+ \frac{1}{3} \frac{V}{h_1 + \gamma h_2} \left( h_1 \partial_x \left( h_2^3 \partial_x \left( \frac{h_1 V}{h_1 + \gamma h_2} \right) \right) - \gamma h_2 \partial_x \left( h_1^3 \partial_x \left( \frac{h_2 V}{h_1 + \gamma h_2} \right) \right) \right).
\]

This model is justified the consistency of the solutions of the full Euler system towards the Green-Naghdi models, as expressed precisely in Proposition A.5.

As it has been said above, our models will be justified with consistency results, with respect to the Green-Naghdi model (1.1). Let us now define precisely what we denote by consistency in the core of this paper.

**Definition 1.1 (Consistency).** Let \((\zeta^p, v^p)_{p \in P}\) be a family of pair of functions, uniformly bounded in \(L^\infty([0,T);H^{s+r})\) \((s \geq 0\) to be determined), depending on parameters

\[
\begin{align*}
p \in P \equiv & \left\{ (\epsilon, \mu, \delta, \gamma), \ 0 \leq \mu \leq \mu_{\text{max}}, \ 0 \leq \epsilon_{\text{max}}, \ \delta \in (\delta_{\text{min}}, \delta_{\text{max}}), \ 0 < \gamma < 1 \right\}.
\end{align*}
\]

We say that \((\zeta^p, v^p)\) is consistent with Green-Naghdi system (1.1) at precision \(O(\epsilon^p)\), of order \(s\) and on \([0,T]\), if \((\zeta^p, v^p)\) satisfies, for \(\epsilon^p\) sufficiently small,

\[
\begin{align*}
\partial_t \zeta^p + \partial_x \left( \frac{h_1 h_2}{h_1 + \gamma h_2} v^p \right) &= \epsilon^p r_1, \\
\partial_t (v^p + \mu \bar{Q}[h_1, h_2] v^p) + (\gamma + \delta) \partial_x \zeta^p + \frac{\epsilon}{2} \partial_x \left( \frac{h_1^2 - \gamma h_2^2}{(h_1 + \gamma h_2)^2} |v^p|^2 \right) &= \mu \partial_x (\bar{R}[h_1, h_2] v^p) = \epsilon^p r_2,
\end{align*}
\]

where \( h_1 = 1 - \epsilon \zeta^p \) and \( h_2 = \frac{1}{\eta} + \epsilon \zeta^p \), and with

\[
\| (r_1, r_2) \|_{L^\infty([0,T), H^{s+r})^2} \leq C^{\left( \mu_{\text{max}}, \epsilon_{\text{max}}, 1/\delta_{\text{min}}, \delta_{\text{max}}, \| \zeta^p \|_{L^\infty([0,T), H^{s+r})}, \| v^p \|_{L^\infty([0,T), H^{s+r})} \right)}.
\]

Here, and in the following, we denote by \( C(\lambda_1, \lambda_2, \ldots) \) any positive constant, depending on the parameters \( \lambda_1, \lambda_2, \ldots \), and whose dependence on \( \lambda_j \) is assumed to be nondecreasing. Moreover, for \( 0 < T \leq \infty \) and \( f(t,x) \) a function defined on \([0,T] \times \mathbb{R} \), we write \( f \in L^\infty([0,T);H^s) \) if \( f \) is uniformly (with respect to \( t \in [0,T] \)) bounded in \( H^s = H^s(\mathbb{R}) \) the \( L^2 \)-based Sobolev space. Its norm is denoted \( \| \cdot \|_{L^\infty([0,T);H^s)} \), as the Sobolev norms are denoted with simple bars: \( | \cdot |_{H^s} \).

## 2 Decomposition of the flow

In this section, our aim is to obtain approximate solutions of (1.1), through a decomposition of the flow into two independent waves, each one satisfying a scalar evolution equation. Our aim is dual. First, we want to investigate which scalar equation each of these waves should to satisfy, in order to be as accurate as possible. Then, we want to estimate the size of the error we commit by neglecting
The coupling between the two components. Section 3 is focused on the case where coupling terms vanish, as only one of the components is non-zero.

The strategy presented here has been used by [7], where the authors present a rigorous justification of the KdV equation for the (one fluid) water wave problem in the long wave regime, and in [23] in the bi-fluidic case. Our work extends the latter results to more general regimes and scalar equations.

We first give a fairly simple formal approach in section 2.1, which allows to heuristically construct the decoupled equations at stake, and consequently the approximate solutions. These approximate solutions are precisely defined in Definitions 2.3 and 2.4. In section 2.2, we give the rigorous justification of such approximations; see Proposition 2.7 and Corollary 2.10. Section 2.3 contains a discussion on our result, considering various different regimes and decoupled models, and supported with numerical simulations.

### 2.1 Formal approach

The main idea of the decomposition is that at first order (that is, setting \( \epsilon = \mu = 0 \)), our system of equations (1.1) is simply a linear wave equation

\[
(2.1) \quad \partial_t U + A_0 \partial_x U = \mathcal{O}(\epsilon, \mu), \quad \text{with} \quad A_0 = \begin{pmatrix} 0 \\ \frac{1}{\gamma + \delta} \end{pmatrix}
\]

and \( U = (\zeta, \bar{v})^T \). It is straightforward to check that \( A_0 \) has two distinct eigenvalues, therefore, we can find a basis of \( \mathbb{R}^2 \) such that (2.1) reduces to two decoupled equations. More precisely, there exists

\[
P = \begin{pmatrix} \gamma + \delta & 1 \\ -(\gamma + \delta) & 1 \end{pmatrix}, \quad P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & \frac{1}{\gamma + \delta} \\ 1 & \frac{1}{\gamma + \delta} \end{pmatrix} \text{ such that } P^{-1} A_0 P = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}.
\]

Now, let us define \((u_l, u_r) = P^{-1} U = \frac{1}{2}(\zeta + \frac{\epsilon}{\gamma + \delta}, \zeta - \frac{\epsilon}{\gamma + \delta})\). Multiplying (2.1) by \( P^{-1} \) (on the left), and keeping only first order terms, yields

\[
(2.2) \quad \partial_t \begin{pmatrix} u_l \\ u_r \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x \begin{pmatrix} u_l \\ u_r \end{pmatrix} = \mathcal{O}(\epsilon, \mu).
\]

As a conclusion, the solution \( U = (\zeta, \bar{v})^T \) may be decomposed in the following way:

\[
U \approx \begin{pmatrix} v_+(x-t) + v_-(x+t) \\ (\gamma + \delta)(v_+(x-t) + v_-(x+t)) \end{pmatrix}^T.
\]

Now, let us take into account the higher order terms in (1.1). One will make use of the following straightforward expansions:

\[
\begin{align*}
\frac{h_1 h_2}{h_1 + \gamma h_2} &= \frac{1}{\gamma + \delta} + \epsilon \frac{\delta^2 - \gamma}{(\gamma + \delta)^2} \zeta - \epsilon^2 \frac{\gamma \delta(\delta + 1)^2}{(\gamma + \delta)^3} \zeta^2 - \epsilon^3 \frac{\gamma \delta^2(\delta + 1)^2(1 - \gamma)}{(\gamma + \delta)^4} \zeta^3 + \mathcal{O}(\epsilon^4), \\
\frac{h_1^2 - \gamma h_2^2}{(h_1 + \gamma h_2)^2} &= \frac{\delta^2 - \gamma}{(\gamma + \delta)^2} - 2\epsilon \frac{\gamma \delta(\delta + 1)^2}{(\gamma + \delta)^3} \zeta - 3\epsilon^2 \frac{\gamma \delta^2(\delta + 1)^2(1 - \gamma)}{(\gamma + \delta)^4} \zeta^2 + \mathcal{O}(\epsilon^3), \\
\mathcal{L}[h_1, h_2] \bar{v} &= \frac{1 + \gamma \delta}{3\delta(\gamma + \delta)} \partial_x^2 \bar{v} - \epsilon \frac{\gamma + \delta}{3} ((\beta - \alpha) \bar{v} \partial_x^2 \zeta + (\alpha + 2\beta) \partial_x (\zeta \partial_x \bar{v}) - \beta \zeta \partial_x^2 \bar{v}) + \mathcal{O}(\epsilon^2), \\
\mathcal{L}[h_1, h_2] \bar{v} &\equiv \alpha \left( \frac{1}{2} (\partial_x \bar{v})^2 + \frac{1}{3} \bar{v} \partial_x^2 \bar{v} \right) + \mathcal{O}(\epsilon)
\end{align*}
\]

with \( \alpha = \frac{1 - \gamma}{(\gamma + \delta)^2} \) and \( \beta = \frac{(1 + \gamma \delta)(\delta^2 - \gamma)}{\delta(\gamma + \delta)^3} \).

Using the decomposition as above: \((u_l, u_r) = P^{-1}(\zeta, \bar{v})^T = \frac{1}{2}(\zeta + \frac{\epsilon}{\gamma + \delta}, \zeta - \frac{\epsilon}{\gamma + \delta})\), and withdrawing every term of size \( \mathcal{O}(\mu \epsilon^2, \epsilon^4) \), one can check that the Green-Naghdi system (1.1) becomes the
Additionally, we assume that there exists $h > 0$.

Proposition 2.2 is straightforward once one obtains a rigorous statement of the expansions $f_1^{\epsilon, \mu}$ and $f_2^{\epsilon, \mu}$ are defined below:

$$f_1^{\epsilon, \mu} = \epsilon \frac{\alpha_1}{2} \partial_x ((u_t + \frac{1}{3}u_r)(u_t - u_r)) + \epsilon \frac{\alpha_2}{3} \partial_x ((u_t - u_r)u_t(u_t + u_r))$$

$$f_2^{\epsilon, \mu} = -\epsilon \frac{\alpha_1}{2} \partial_x ((\frac{1}{3}u_t + u_r)(u_t - u_r)) - \epsilon \frac{\alpha_2}{3} \partial_x ((u_r - u_t)u_r(u_t + u_r))$$

where we used the notation

$$\alpha_1 = \frac{3}{2} \frac{\delta_2-\gamma}{\gamma+\delta}, \quad \alpha_2 = -3 \frac{\delta_2(\delta_2+1)^2}{(\gamma+\delta)^2}, \quad \alpha_3 = -\frac{5}{2} \frac{\delta_2(\delta_2+1)^2(1-\gamma)}{(\gamma+\delta)^2}, \quad \nu = \frac{1}{2} \frac{1+\gamma\delta}{\delta(\gamma+\delta)}.$$

Remark 2.1. One could use higher order expansions with respect to the parameter $\epsilon$, which would lead to decoupled models with formally higher accuracy. However, let us note that our results (see discussion in section 2.3) show that the main error of such decoupled models comes from neglecting coupling terms which arise at low order, so that including higher order terms in the evolution equation is unlikely to produce substantial improvement.

Recall that the Green-Naghdi system is consistent with the full Euler system at precision $O(\mu^2)$; Proposition A.5. Therefore, choosing to keep only terms of order $O(\mu^2, \epsilon^4)$ in (2.3a)–(2.3b) is especially appropriate for parameters in the so-called Camassa-Holm regime:

$$\mathcal{P}_{CH} \equiv \{ (\epsilon, \mu, \delta, \gamma), 0 \leq \mu \leq \mu_{max}, 0 \leq \epsilon \leq M, \delta \in [\delta_{min}, \delta_{max}], 0 < \gamma < 1 \},$$

which is natural to consider, as discussed in the introduction.

Proposition 2.2 (Consistency of (2.3a)–(2.3b)). Let $(u_t^0, u_r^0)^T$ be strong solutions of (2.3a)–(2.3b), depending on sets of parameters $(\epsilon, \mu, \delta, \gamma) = \mathbf{p} \in \mathcal{P}$, as defined in (1.2). We assume that for $s \geq s_0 > 1/2$, $u_t^s, u_r^s \in W^1([0, T); H^{s+1}(\mathbb{R}))$, uniformly in $\mathbf{p}$, where $W^1([0, T); H^{s+1}(\mathbb{R}))$ denotes the space of the functions $f(t, x)$ such that

$$\|f\|_{W^1([0, T); H^{s+1}(\mathbb{R}))} \equiv \|f\|_{L^\infty([0, T); H^{s+1}(\mathbb{R}))} + \|\partial_tf\|_{L^\infty([0, T); H^s(\mathbb{R}))} < \infty.$$  

Additionally, we assume that there exists $h > 0$ such that

$$h_1^R = 1 - \epsilon (u_t^R + u_r^R) \geq h > 0, \quad h_2^R = \frac{1}{\delta} + \epsilon (u_t^R + u_r^R) \geq h > 0.$$ 

Then $(\zeta^R, \bar{v}^R)^T \equiv (u_t^R + u_r^R, (\gamma + \delta)(u_t^R - u_r^R))$ is consistent with Green-Naghdi equations (1.1) of order $s$ on $[0, T)$, at precision $O(\mu^2 + \epsilon^4)$ (in the sense of Definition 1.1).

Proof. Proposition 2.2 is straightforward once one obtains a rigorous statement of the expansions $h_1^R h_2^R$, $(h_1^R)^2 - \gamma (h_2^R)^2$, $\mathcal{Q}(h_1^R, h_2^R)^p$, $\mathcal{R}(h_1^R, h_2^R)^p$, as stated above, the residual being estimated in $W^1([0, T); H^{s+1})$ norm. These expansions are easily checked, provided the assumptions of the proposition $(u_t^s, u_r^s) \in W^1([0, T); H^{s+1}(\mathbb{R}))$, $h_1^R, h_2^R \geq h > 0$ are satisfied, and using uniformly continuous (for $s \geq s_0 > 1/2$) Sobolev embedding $H^s \hookrightarrow L^\infty$. \qed
The uncoupled approximation consists simply in neglecting all coupling terms in (2.3a)–(2.3b), that is replacing \( f_\tau^\mu(u_\tau,u_\tau) \) by \( f_\tau^\mu(u_\tau,0) \), and \( f_r^\mu(u_\tau,u_\tau) \) by \( f_r^\mu(0,u_\tau) \). This yields

\[
\begin{aligned}
\partial_t u_\tau + \partial_x u_\tau + \epsilon \alpha_1 u_\tau \partial_x u_\tau + \epsilon^2 \alpha_2 u_\tau^2 \partial_x u_\tau + \epsilon^3 \alpha_3 u_\tau^3 \partial_x u_\tau - \mu \nu \partial_x^2 \partial_t u_\tau \\
- \mu \epsilon \partial_t ((\kappa_1 + 1/2\kappa_2)(\partial_x u_\tau)^2) - \mu \kappa_3 \partial_x (\frac{1}{3} u_\tau^2 \partial_x u_\tau + \frac{1}{2} (\partial_x u_\tau)^2) = 0,
\end{aligned}
\]

(2.6a)

\[
\begin{aligned}
\partial_t u_r - \partial_x u_r - \epsilon \alpha_1 u_r \partial_x u_r - \epsilon^2 \alpha_2 u_r^2 \partial_x u_r - \epsilon^3 \alpha_3 u_r^3 \partial_x u_r - \mu \nu \partial_x^2 \partial_t u_r \\
- \mu \epsilon \partial_t ((\kappa_1 + 1/2\kappa_2)(\partial_x u_r)^2) - \mu \kappa_3 \partial_x (\frac{1}{3} u_r^2 \partial_x u_r + \frac{1}{2} (\partial_x u_r)^2) = 0,
\end{aligned}
\]

(2.6b)

and (2.6a)–(2.6b) are the decoupled equations we consider; see Definition 2.3, below.

Let us now reckon that one can deduce from (2.6a)–(2.6b) a large family of formally equivalent models, with different parameters, following the techniques used for example in \([6, 7, 19, 23]\), and that we discuss below.

- **The BBM trick** (from Benjamin-Bona-Mahony \([5]\)). Keeping only the first order terms in equations (2.6a)–(2.6b), one has the simple transport equations

\[
\begin{aligned}
\partial_t u_\tau + \partial_x u_\tau = O(\mu, \epsilon), \quad \text{and} \quad \partial_t u_r - \partial_x u_r = O(\mu, \epsilon).
\end{aligned}
\]

It follows that one can replace time derivatives in higher order terms (of order \( O(\mu \epsilon) \)) by spatial derivatives (up to a sign), and both equations have formally the same order of accuracy. Throughout this paper, we consider only equations with spatial derivatives in \( O(\mu, \epsilon) \) terms (if they exist), to simplify the rigorous approach of Section 2.2. In particular, (2.6a)–(2.6b) becomes

\[
\begin{aligned}
\partial_t u_\tau + \partial_x u_\tau + \epsilon \alpha_1 u_\tau \partial_x u_\tau + \epsilon^2 \alpha_2 u_\tau^2 \partial_x u_\tau + \epsilon^3 \alpha_3 u_\tau^3 \partial_x u_\tau - \mu \nu \partial_x^2 \partial_t u_\tau \\
+ \mu \epsilon \partial_t ((\kappa_1 + 1/3\kappa_2)u_\tau^2 \partial_x u_\tau + (\kappa_1 + 1/2\kappa_2 + 1/2\kappa_3)(\partial_x u_\tau)^2) = 0,
\end{aligned}
\]

(2.7a)

\[
\begin{aligned}
\partial_t u_r - \partial_x u_r - \epsilon \alpha_1 u_r \partial_x u_r - \epsilon^2 \alpha_2 u_r^2 \partial_x u_r - \epsilon^3 \alpha_3 u_r^3 \partial_x u_r - \mu \nu \partial_x^2 \partial_t u_r \\
- \mu \epsilon \partial_t ((\kappa_1 + 1/3\kappa_3)u_r^2 \partial_x u_r + (\kappa_1 + 1/2\kappa_2 + 1/2\kappa_3)(\partial_x u_r)^2) = 0.
\end{aligned}
\]

(2.7b)

Following a similar idea, we make use of the low order identity obtained from (2.7a)–(2.7b):

\[
\begin{aligned}
\partial_k u_\tau + \partial_x u_\tau + \epsilon \alpha_1 u_\tau \partial_x u_\tau = O(\mu, \epsilon^2),
\end{aligned}
\]

so that one has, for any \( \theta \in \mathbb{R} \),

\[
\partial_t u_\tau = \theta \partial_x u_\tau + (\theta - 1)(\partial_x u_\tau + \epsilon \alpha_1 u_\tau \partial_x u_\tau) + O(\mu, \epsilon^2).
\]

Plugging back into (2.7a) and withdrawing \( O(\mu^2, \epsilon^2) \) terms yields

\[
\begin{aligned}
\partial_t u_\tau + \partial_x u_\tau + \epsilon \alpha_1 u_\tau \partial_x u_\tau + \epsilon^2 \alpha_2 u_\tau^2 \partial_x u_\tau + \epsilon^3 \alpha_3 u_\tau^3 \partial_x u_\tau - \mu \nu \partial_x^2 \partial_t u_\tau \\
+ \mu \epsilon \partial_t ((\kappa_1 + 1/3\kappa_2)u_\tau^2 \partial_x u_\tau + \kappa_3 (\partial_x u_\tau)^2) = 0,
\end{aligned}
\]

(2.8a)

\[
\begin{aligned}
\partial_t u_r - \partial_x u_r - \epsilon \alpha_1 u_r \partial_x u_r - \epsilon^2 \alpha_2 u_r^2 \partial_x u_r - \epsilon^3 \alpha_3 u_r^3 \partial_x u_r - \mu \nu \partial_x^2 \partial_t u_r \\
- \mu \epsilon \partial_t ((\kappa_1 + 1/3\kappa_3)u_r^2 \partial_x u_r + \kappa_3 (\partial_x u_r)^2) = 0,
\end{aligned}
\]

(2.8b)

where we have defined, after parameters (2.4),

\[
\begin{aligned}
\nu_\theta^0 \equiv \theta \nu, \quad \nu_\tau^0 \equiv (1 - \theta) \nu, \quad \kappa_1^0 \equiv \kappa_1 + \frac{\kappa_3}{3} + (1 - \theta) \alpha_1 \nu, \quad \kappa_3^0 \equiv \kappa_1 + \frac{\kappa_2}{2} + (1 - \theta) \alpha_1 \nu.
\end{aligned}
\]

- **Near identity changes of variables.** We used system (1.1) as our reference system, and therefore the unknowns we consider are \( (\zeta, \tau) \), where \( \bar{v} = \bar{u}_2 - 2 \gamma \bar{u}_1 \) is the shear layer-mean velocity.
Decoupled and unidirectional asymptotic models for the propagation of internal waves

However, other natural variables may also be used, such as the shear velocity at the interface (leading to system (A.17))

\[ v_0 = \partial_x \left( (\phi_2 - \gamma \phi_1) \right)_{z = \zeta_0} - \partial_x \left( \phi_2(x, r_2(x, 0)) - \gamma \phi_1(x, r_1(x, 0)) \right), \]

(where we use the change of coordinate flattening the fluid domains: \( r_i(x, \tilde{z}) = \tilde{z} h_i(x) + \varepsilon \zeta(x) \); see [22]), or more generally, using the horizontal derivative of the potential at specific heights \((z_1, z_2) \in [0, 1) \times (-1, 0)\):

\[ \nu^{z_1, z_2} = \partial_x \left( \phi_2(x, r_2(x, z_2)) - \gamma \phi_1(x, r_2(x, z_1)) \right). \]

Using the expansion of the velocity potentials and layer-mean velocities, as obtained in [22], as well as the identity \( h_1 u_1 + h_2 u_2 = 0 \) (obtained through the rigid lid assumption), yields the following approximation:

\[ \nu^{z_1, z_2} = \bar{v} + \mu \lambda^{z_1, z_2} \partial_x^2 \bar{v} + \mu \mathcal{O}(\varepsilon^2, \mu^2), \]

with \( \lambda^{z_1, z_2} = \frac{(3z_2^2 + 6z_2 + 2) + \gamma \delta(3z_1^2 - 6z_1 + 2)}{6 \delta (\gamma + \delta)} \), and \( \mathcal{T}^{z_1, z_2} \) a bi-linear differential operator whose precise formula do not play a significant role in our work as we shall discuss below.

Following this idea, we consider

\[ u_i^\lambda \equiv u_i + \mu \lambda \partial_x^2 u_i, \]
\[ u_r^\lambda \equiv u_r - \mu \lambda \partial_x^2 u_r. \]

If \((u_i, u_r)\) satisfies (2.8a)-(2.8b), then \((u_i^\lambda, u_r^\lambda)\) satisfies the following equations, up to terms of order \( \mathcal{O}(\mu^2, \mu^2) \):

\[ \begin{align*}
\partial_t u_i^\lambda + \partial_z u_i^\lambda + \epsilon \alpha_1 u_i^\lambda \partial_z u_i^\lambda + \epsilon^2 \alpha_2 u_i^\lambda \partial_z^2 u_i^\lambda + \epsilon^3 \alpha_3 u_i^\lambda \partial_z^3 u_i^\lambda - \mu \nu^0 \lambda \partial_x^2 u_i^\lambda \\
+ \mu \nu^0 \lambda \partial_x^2 u_i^\lambda + \mu \mathcal{O}(\kappa_2, \kappa_3 u_i^\lambda) = 0,
\end{align*} \]

and

\[ \begin{align*}
\partial_t u_r^\lambda - \partial_z u_r^\lambda - \epsilon \alpha_1 u_r^\lambda \partial_z u_r^\lambda - \epsilon^2 \alpha_2 u_r^\lambda \partial_z^2 u_r^\lambda - \epsilon^3 \alpha_3 u_r^\lambda \partial_z^3 u_r^\lambda - \mu \nu^0 \lambda \partial_x^2 u_r^\lambda \\
- \mu \nu^0 \lambda \partial_x^2 u_r^\lambda - \mu \mathcal{O}(\kappa_2, \kappa_3 u_r^\lambda) = 0,
\end{align*} \]

where we have defined, after parameters (2.9),

\[ \begin{align*}
u_t^0 & \equiv \nu_t^0 + \lambda, & \nu_x^0 & \equiv \nu_x^0 - \lambda, & \kappa_1^0 & \equiv \kappa_1^0 + \alpha_1 \lambda, \end{align*} \]

Note that in order to fit as much as possible with variables \((\zeta, \nu^{z_1, z_2})\), one could have used more complex change of variables, such as

\[ u_i^\lambda \equiv u_i + \mu \lambda \partial_x^2 u_i + \mu \mathcal{O}(\lambda_2 \partial_x^2 \nu^0 + \lambda_3 \partial_x(u^2)), \]
\[ u_r^\lambda \equiv u_r - \mu \lambda \partial_x^2 u_r - \mu \mathcal{O}(\lambda_2 \partial_x \nu^0 + \lambda_3 \partial_x(u^2)), \]

with \( \lambda = \frac{\lambda^{z_1, z_2}}{2(\gamma + \delta)} \), and \( \lambda_2, \lambda_3 \) obtained through \( \mathcal{T}^{z_1, z_2} \). It turns out \( u_i^\lambda \) and \( u_r^\lambda \) would then satisfy the same equation as above: the new parameters do not depend on \( \lambda_2 \) and \( \lambda_3 \), as their contribution is of order \( \mathcal{O}(\mu^2, \mu^4) \), after using BBM trick to suppress higher order derivatives with respect to time. We thus do not consider such changes of variable.

Ultimately, one obtains the following family of approximations:
Definition 2.3 (Constantin-Lannes approximation). Let $\zeta^0, v^0$ be given scalar functions, and set parameters $(\epsilon, \mu, \gamma, \delta) \in \mathcal{P}$ (see (1.2)), $(\lambda, \theta) \in \mathbb{R}^2$. The Constantin-Lannes approximation (CL) is

$$ U_{\text{CL}} \equiv \left( v_+(t, x - t) + v_-(t, x + t), (\gamma + \delta)(v_+(t, x - t) - v_-(t, x + t)) \right), $$

where $v_\pm |_{t=0} = \frac{1}{2}(\zeta^0 \pm \frac{v^0}{\sqrt{\epsilon}})$ and $v_\pm = (1 \pm \mu \lambda \partial^2_x)^{-1}v^\lambda_\pm$ with $v^\lambda_\pm$ satisfying

$$ (2.12) \quad \partial_t v^\lambda_\pm + \epsilon \alpha_1 v^\lambda_\pm \partial_x v^\lambda_\pm \pm \epsilon^2 \alpha_2 (v^\lambda_\pm)^2 \partial_x v^\lambda_\pm \pm \epsilon^3 \alpha_3 (v^\lambda_\pm)^3 \partial_x v^\lambda_\pm $$

$$ \pm \mu \nu^{\lambda,\lambda}_x \partial^2_x v^\lambda_\pm \pm \mu \nu^{\lambda,\lambda}_t \partial t v^\lambda_\pm \pm \mu \nu^{\lambda,\lambda}_\theta \partial \theta v^\lambda_\pm \pm \mu \nu^{\lambda,\lambda}_\nu \partial \nu v^\lambda_\pm \pm \nu^{\lambda,\lambda}_\nu \partial \nu v^\lambda_\pm \pm \nu^{\lambda,\lambda}_\nu \partial \nu v^\lambda_\pm = 0, $$

with parameters defined as

$$ \alpha_1 = \frac{3 \delta^2 - \gamma}{2 \gamma + \delta}, \quad \alpha_2 = -3 \gamma (\delta + 1)^2 (\gamma + \delta)^2, \quad \alpha_3 = -5 \delta^2 (\delta + 1)^2 \gamma (1 - \gamma), $$

$$ \nu^{\lambda,\lambda}_t \equiv \theta \frac{1 + \gamma \delta}{6 \delta (\delta + \gamma)} + \lambda, \quad \nu^{\lambda,\lambda}_x \equiv \frac{1 - \theta}{6 \delta (\delta + \gamma)} - \lambda, $$

$$ \kappa^{\lambda,\lambda}_1 \equiv \frac{(1 + \gamma \delta)(\delta^2 - \gamma)}{3 \delta (\gamma + \delta)^2} \left( 1 + \frac{1 - \theta}{4} \right) - \frac{(1 - \gamma)^2}{12 (\gamma + \delta)}, $$

$$ \kappa^{\lambda,\lambda}_2 \equiv \frac{(1 + \gamma \delta)(\delta^2 - \gamma)}{3 \delta (\gamma + \delta)^2} \left( 1 + \frac{1 - \theta}{4} \right) - \frac{(1 - \gamma)^2}{12 (\gamma + \delta)}. $$

In this paper, we also consider models with (formally) lower order accuracy, as defined below.

Definition 2.4 (lower order decoupled approximations). Let $\zeta^0, v^0$ be given scalar functions, and set parameters $(\epsilon, \mu, \gamma, \delta) \in \mathcal{P}$, $(\lambda, \theta) \in \mathbb{R}^2$. A decoupled approximate solution of the system (1.1) is

$$ U \equiv \left( v_+(t, x - t) + v_-(t, x + t), (\gamma + \delta)(v_+(t, x - t) - v_-(t, x + t)) \right), $$

where $v_\pm |_{t=0} = \frac{1}{2}(\zeta^0 \pm \frac{v^0}{\sqrt{\epsilon}})$ and $v_\pm = (1 \pm \mu \lambda \partial^2_x)^{-1}v^\lambda_\pm$ with $v^\lambda_\pm$ satisfying a scalar evolution equation. In the following, we consider

- the inviscid Burgers’ equation:

$$ (2.13) \quad \partial_t v^\lambda_\pm \pm \epsilon \alpha_1 v^\lambda_\pm \partial_x v^\lambda_\pm = 0; $$

- the Korteweg-de Vries (or more precisely Benjamin-Bona-Mahony) equation:

$$ (2.14) \quad \partial_t v^\lambda_\pm \pm \epsilon \alpha_1 v^\lambda_\pm \partial_x v^\lambda_\pm \pm \mu \nu^{\lambda,\lambda}_x \partial^2_x v^\lambda_\pm \pm \mu \nu^{\lambda,\lambda}_t \partial t v^\lambda_\pm \pm \mu \nu^{\lambda,\lambda}_\theta \partial \theta v^\lambda_\pm \pm \nu^{\lambda,\lambda}_\nu \partial \nu v^\lambda_\pm \pm \nu^{\lambda,\lambda}_\nu \partial \nu v^\lambda_\pm = 0; $$

- the extended Korteweg-de Vries equation:

$$ (2.15) \quad \partial_t v^\lambda_\pm \pm \epsilon \alpha_1 v^\lambda_\pm \partial_x v^\lambda_\pm \pm \epsilon^2 \alpha_2 (v^\lambda_\pm)^2 \partial_x v^\lambda_\pm \pm \mu \nu^{\lambda,\lambda}_x \partial^2_x v^\lambda_\pm \pm \mu \nu^{\lambda,\lambda}_t \partial t v^\lambda_\pm \pm \mu \nu^{\lambda,\lambda}_\theta \partial \theta v^\lambda_\pm \pm \nu^{\lambda,\lambda}_\nu \partial \nu v^\lambda_\pm \pm \nu^{\lambda,\lambda}_\nu \partial \nu v^\lambda_\pm = 0; $$

where the parameters satisfy the identities of Definition 2.3.

The existence, uniqueness and rigorous justification of such family of approximate solutions of system (1.1), is investigated in the following section.

2.2 Rigorous statements

In this section, we rigorously justify the decoupled approximations defined in Definition 2.3 and 2.4, in the sense of consistency. We first prove that the evolution equations we consider are well-posed for large times, as they satisfy energy estimates in Sobolev norms. Persistence property of estimates in weighted Sobolev norms are also investigated, as spatial decay at infinity (i.e. localization in space) plays an important role in the accuracy of the approximations, as we shall see.

These estimates are valid for long times, and we demand the approximation to be valid for such times. In order to deal with this, we introduce (following the decomposition introduced
in the previous section) an explicit first order corrector, which roughly reproduces the coupling effects between the two counterpropagating waves. Including this correction yields an approximate solution, which is precise (in the sense of consistency) at high order. The key ingredient is then to obtain precise estimates on this corrector, and especially control its the secular growth (which should be sublinear in time).

Let us first study the well-posedness of the Constantin-Lannes equations (2.12), that we reproduce below:

\[
(1 - \mu \beta \partial_x^2)\partial_t u + \epsilon \alpha_1 u \partial_x^2 u + \epsilon^2 \alpha_2 u^2 \partial_x^2 u + \epsilon^3 \alpha_3 u^3 \partial_x u + \mu \nu \partial_x^2 u + \mu \nu \partial_x (k u \partial_x^2 u + \nu (\partial_x u)^2) = 0.
\]

Our result, concerning the existence and uniqueness of strong solutions of (2.16), requires that \(\mu \beta > 0\), as it relies heavily on \textit{a priori} estimates of the solution in the following scaled Sobolev norm:

\[
|u|^2_{H^{s+1}} = |u|_{H^s}^2 + \mu \beta |u|_{H^{s+1}}^2, \quad \text{for some } s \geq 0.
\]

We will also use weighted Sobolev norm, defined through the following norm:

\[
|u|_{X^s_{n,\mu}} = \sum_{j=0}^n |x^j u|_{H^{s+2(n-j)}}.
\]

**Proposition 2.5** (Well-posedness). Let \(u^0 \in H^{s+1}\), with \(s \geq s_0 > 3/2\). Let the parameters be such that \(\beta, \mu, \epsilon > 0\), and define \(M > 0\) such that

\[
\beta + \frac{1}{\beta} + \mu + \epsilon + |\alpha_1| + |\alpha_2| + |\alpha_3| + |\nu| + |\kappa| + |t| \leq M.
\]

Then there exists \(T = C\left(\frac{1}{s_0 - 3/2}, M, |u^0|_{H^{s+1}}\right)\) and a unique \(u \in C^0([0, T/\epsilon); H^{s+1}) \cap C^1([0, T/\epsilon); H^s)\) such that \(u\) satisfies (2.16) and initial condition \(u|_{t=0} = u^0\).

Moreover, \(u\) satisfies the energy estimate for \(0 \leq t \leq T/\epsilon\):

\[
\begin{align*}
\|\partial_t u\|_{L^\infty([0, T/\epsilon); H^s)} + \|u\|_{L^\infty([0, T/\epsilon); H^{s+1})} & \leq C\left(\frac{1}{s_0 - 3/2}, M, |u^0|_{H^{s+1}}\right) \\
\end{align*}
\]

Assume additionally that for fixed \(n, k \in \mathbb{N}\), the function \(x^n u^0 \in H^{s+k}\), with \(0 \leq j \leq n\) and \(s = k + 1 + 2(n - j)\). Then there exists \(T = C\left(\frac{1}{s_0 - 3/2}, M, n, k, \sum_{j=0}^n |x^n u^0|_{H^{s+k+2(n-j)}}\right)\) such that for \(0 \leq t \leq T \times \min(1/\epsilon, 1/\mu)\), one has

\[
\begin{align*}
\|x^n \partial_k \partial_t u\|_{L^\infty([0, T/\epsilon); H^s)} + \|x^n \partial_k u\|_{L^\infty([0, T/\epsilon); H^{s+1})} & \leq C\left(\frac{1}{s_0 - 3/2}, M, n, k, \sum_{j=0}^n |x^n u^0|_{H^{s+k+2(n-j)}}\right).
\end{align*}
\]

In particular, one has, for \(0 \leq t \leq T \times \min(1/\epsilon, 1/\mu)\),

\[
\begin{align*}
\|\partial_t u\|_{L^\infty([0, T); X^s_{n,\mu})} + \|u\|_{L^\infty([0, T); X^{s+1}_{n,\mu})} & \leq C\left(\frac{1}{s_0 - 3/2}, M, n, |u^0|_{X^{s+1}_{n,\mu}}\right).
\end{align*}
\]

**Remark 2.6.** Note that the above result allows the parameters (except \(\mu \beta\)) to vanish, so that we ensure the well-posedness and control in Sobolev norms of all our decoupled models defined in Definition 2.3 and 2.4.

Of course, such results are already well-known for inviscid Burgers’, KdV and eKdV equations (see for example [10, 38, 39, 27, 17] and references therein), and actually do not require \(\mu \beta > 0\). The case of Constantin-Lannes equation is given in Proposition 4 in [19].

The persistence of the solution in weighted Sobolev norms for the Camassa-Lannes equation is new, as far as we know. Similar results in the case of (eventually extended) Korteweg-de Vries equations are obtained in slightly different setting (for the most part using weighted \(L^2\) spaces intersected with non-weighted Sobolev spaces \(H^s, s > 0\)) in [33, 9, 56, 60, 14, 49, 50, 12].
Proposition 2.5 is proved in Appendix B. We now state our main result, whose proof is postponed to Appendix C.

**Proposition 2.7** (Consistency). Let $\zeta^0, \epsilon^0 \in H^{s+6}$, with $s \geq s_0 > 3/2$. For $(\epsilon, \mu, \delta, \gamma) = p \in P$, as defined in (1.2), we denote $U^p_{CL}$ the unique solution of the CL approximation, as defined in Definition 2.3. For some given $M^*_{s+6} > 0$, sufficiently big, we assume that there exists $T^* > 0$ and a family $(U^p_{CL})_{p \in P}$ such that

$$T^* = \max \left( T \geq 0 \text{ such that } \left\| U^p_{CL} \right\|_{L^\infty([0,T];H^{s+6})} + \left\| \partial_t U^p_{CL} \right\|_{L^\infty([0,T];H^{s+5})} \leq M^*_{s+6} \right).$$

Then there exists $U^c = U^c[U^p_{CL}]$ such that $U \equiv U^p_{CL} + U^c$ is consistent with Green-Naghdi equations (1.1) of order $s$ on $[0,t]$ for $t < T^*$, at precision $O(\varepsilon^*_{CL})$ with

$$\varepsilon^*_{CL} = C \max(\epsilon^2(\delta^2 - \gamma)^2, \epsilon^4, \mu^2)(1 + \sqrt{t}),$$

with $C = C(\frac{1}{s_0 - 3/2}, M^*_{s+6}, \frac{1}{\delta_{\min}}, \delta_{\max}, \epsilon_{\max}, \mu_{\max}, |\lambda|, |\theta|)$, and the corrector term $U^c$ is estimated as

$$\left\| U^c \right\|_{L^\infty([0,T^*];H^{s+5})} + \left\| \partial_t U^c \right\|_{L^\infty([0,T^*];H^{s+4})} \leq C \max(\epsilon(\delta^2 - \gamma), \epsilon^2, \mu) \min(t, \sqrt{t}).$$

Additionally, if there exists $\alpha > 1/2$, $M^2_{s+6}, T^2 > 0$ such that

$$\sum_{k=0}^{6} \left\| (1 + x^2)^{\alpha} \partial_x^k U^p_{CL} \right\|_{L^\infty([0,T];H^{s+6-k})} + \sum_{k=0}^{5} \left\| (1 + x^2)^{\alpha} \partial_x^k \partial_t U^p_{CL} \right\|_{L^\infty([0,T];H^{s+7-k})} \leq M^2_{s+6},$$

then $U \equiv U^p_{CL} + U^c$ is consistent with Green-Naghdi equations (1.1) of order $s$ on $[0,t]$ for $t < T^2$, at precision $O(\varepsilon^1_{CL})$ with

$$\varepsilon^1_{CL} = C \max(\epsilon^2(\delta^2 - \gamma)^2, \epsilon^4, \mu^2),$$

with $C = C(\frac{1}{s_0 - 3/2}, M^2_{s+6}, \frac{1}{\delta_{\min}}, \delta_{\max}, \epsilon_{\max}, \mu_{\max}, |\lambda|, |\theta|)$ and $U^c$ is uniformly estimated as

$$\left\| U^c \right\|_{L^\infty([0,T^*];H^{s+5})} + \left\| \partial_t U^c \right\|_{L^\infty([0,T^*];H^{s+4})} \leq C \max(\epsilon(\delta^2 - \gamma), \epsilon^2, \mu) \min(t, 1).$$

**Remark 2.8.** The function $U^c = U^c[U^p_{CL}]$ is given explicitly in the proof of the Proposition; see Appendix C, and more precisely Definition C.1.

**Remark 2.9.** In the estimates presented in Proposition 2.7 and Corollary 2.10 below, the terms growing in $O(\sqrt{T})$ come from coupling effects between the two propagating waves, that are neglected in our decoupled models; this is why the accuracy is significantly better if the initial data is sufficiently spatially localized, as the two counterpropagating waves will be located far away from each other after long times.

The function $U^c$, which depends only on the decoupled waves $v_\pm$, is a first order corrector which allows to take into account the leading order coupling effects, and therefore reach the desired accuracy.

Uniformly bounded terms are the contribution of unidirectional errors, generated by the different manipulations on the equation (e.g. BBM trick), and eventual neglected terms in lower order approximations in Corollary 2.10. The magnitude of each contribution depends on the situation (small amplitude of deformation, shallow water, critical ratio, etc.) and is discussed for different scenario in subsection 2.3, below.

Equivalent results may be obtained for lower order approximations, when following the steps of the strategy presented in Appendix B. More precisely, this modifies mostly the last term of (C.5) in Lemma C.3, adding the contribution of terms that are neglected in the chosen evolution equation. The additional error is therefore uniformly bounded on times $[0, T^*]$ and $[0, T^2]$. The Corollary below details the accuracy of such approximations, and its proof is omitted.
Corollary 2.10. Assume that the hypotheses of Proposition 2.7 hold. Denote $U_{e\text{KdV}}^p$, $U_{KdV}^p$ and $U_{iB}^p$, respectively, the solutions of the eKdV, KdV and iB approximations, as defined in Definition 2.4. In each case, we assume that the decoupled approximation is uniformly estimated in $[0,T^*)$, as in Proposition 2.7. Then

i. there exists $U^c = U^c[U_{e\text{KdV}}^p]$ such that $U \equiv U_{e\text{KdV}}^p + U^c$ is consistent with Green-Naghdi equations (1.1) of order $s$ on $[0,t]$ for $t < T^*$, at precision $O(\varepsilon_{e\text{KdV}}^s)$, with

$$\varepsilon_{e\text{KdV}}^s = C \times \left( \max(\varepsilon^2(\delta^2 - \gamma)^2, \varepsilon^4, \mu^2) (1 + \sqrt{t}) + \max(\varepsilon^3, \mu) \right);$$

ii. there exists $U^c = U^c[U_{KdV}^p]$ such that $U \equiv U_{KdV}^p + U^c$ is consistent with Green-Naghdi equations (1.1) of order $s$ on $[0,t]$ for $t < T^*$, at precision $O(\varepsilon_{KdV}^s)$ with

$$\varepsilon_{KdV}^s = C \times \left( \max(\varepsilon^2(\delta^2 - \gamma)^2, \varepsilon^4, \mu^2) (1 + \sqrt{t}) + \varepsilon^2 \right);$$

iii. there exists $U^c = U^c[U_{iB}^p]$ such that $U \equiv U_{iB}^p + U^c$ is consistent with Green-Naghdi equations (1.1) of order $s$ on $[0,t]$ for $t < T^*$, at precision $O(\varepsilon_{iB}^s)$ with

$$\varepsilon_{iB}^s = C \times \left( \max(\varepsilon^2(\delta^2 - \gamma)^2, \varepsilon^4, \mu^2) (1 + \sqrt{t}) + \max(\varepsilon^3, \mu) \right);$$

where $C = C(\frac{1}{s_0 - 3/2}, M_{s+\theta}^2, \frac{1}{\sigma_{\min}}, \delta_{\max}, \varepsilon_{\max}, \mu_{\max}, \lambda, |\theta|).$ Each time, the corrector term $U^c$ is estimated as follows:

$$\|U^c\|_{L^\infty([0,T^*];H^s)} + \|\partial_t U^c\|_{L^\infty([0,T^*];H^{s-1})} \leq C \max(\varepsilon(\delta^2 - \gamma)^2, \varepsilon^2, \mu)(t, \sqrt{t}).$$

Moreover, if the uncoupled approximation is sufficiently localized in space, then all the estimates are improved as in the second part of the Proposition 2.7 (that is replacing $\sqrt{t}$ by 1).

Note that Proposition 2.5 ensures that Proposition 2.7 and Corollary 2.10 are not empty, but on the contrary are valid for long times, provided that $\nu^{\theta,\lambda}_i > 0$ and the initial data sufficiently smooth. More precisely, the following result is a straightforward consequence of Proposition 2.5.

Corollary 2.11 (Existence and magnitude of $T^*$, $T^0$). Let $(\xi^0, \nu^0) = U^0 \in H^{s+7}, s \geq s_0 > 3/2$, and let $p \in T$, with additional restriction $\nu^{\theta,\lambda}_i > \nu_0 > 0$.

Then there exists $C_1, C_2 = C(\frac{1}{s_0 - 3/2}, \mu_{\max}, \varepsilon_{\max}, 1/\delta_{\min}, \delta_{\max}, \lambda, |\theta|, 1/\nu_0, |U^0|_{H^{s+7}},)$, independent of $p$, such that for any $M^i_{s+\theta} \geq C_1$, the decoupled approximate solutions defined in Proposition 2.7 and Corollary 2.10 are uniquely defined and satisfy the uniform bound of the Proposition, with

$$T^* \geq C_2/\epsilon.$$

If $U^0 \in X^{s+7}_2$, then there exists $C_1, C_2 = C(\frac{1}{s_0 - 3/2}, \mu_{\max}, \varepsilon_{\max}, \delta_{\min}^{\theta,\lambda}, \delta_{\max}, \lambda, |\theta|, \nu_0^{-1}, |U^0|_{X^{s+7}_2}),$ such that for any $M^i_{s+\theta} \geq C_1$, one has

$$T^0 \geq C_2/\max(\epsilon, \mu).$$

One question, which is essential in the discussion of section 2.3, is whether these above estimates are optimal. Let us discuss some elements of answer. First, it is well-known that inviscid Burgers’ equation $\partial_t u + u \partial_x u$ will generate a shock in finite time, for any non-trivial, decreasing at infinity initial data. A simple scaling arguments shows that (2.13) will therefore generates shock in finite time $T \approx 1/\epsilon$. Adding a linear dispersive term as in the KdV equation regularizes the solution, and in particular induces global well-posedness for sufficiently smooth initial data. Moreover, there exists an infinite number of conserved quantities [47, 48], which allow to control uniformly in time the Sobolev norm of the solution $\|u\|_{H^s}, s \in N$. This result is true for the whole class of extended Korteweg-de Vries equations (2.15), if $\nu^{\theta,\lambda}_i = 0$.

On the contrary, it is known that Camassa-Holm family of equations, related to (2.12) can develop singularities in finite time in the form of wave breaking [18]. In [19], it is shown that wave
breaking occurs for the solution of (2.12) for a specific set of parameters, in time $T \approx 1/\epsilon$, provided that the initial data is sufficiently big in $L^\infty$ norm. However, as the justification of our model assumes that the initial deformation is bounded in Sobolev norm, uniformly with respect to the parameters, assumptions of [19, Proposition 6] cannot be justified. We also let the reader refer to [12], and references therein, for an in-depth study of the conditions (in term of decay at infinity of the initial data) for finite time blowup of the solution of Camassa-Holm equation.

2.3 Discussion

Our result takes its full meaning in the light of a conjectured stability result on Green-Naghdi equations (1.1), or any consistent model, that would have the following formulation:

**Hypothesis 2.12** (Stability). For $(\epsilon, \mu, \delta, \gamma) = p \in \mathcal{P}$, as defined in (1.2), let $U^p_{GN}$ be a solution of the Green-Naghdi system (1.1) such that $(U^p_{GN})$ is uniformly bounded on $H^s$, $s$ sufficiently big, over time interval $[0, T]$. Then let $(U^p)$ satisfy $U^p |_{t=0} = U^p_{GN} |_{t=0}$, be consistent with Green-Naghdi equations (1.1) on $[0, T]$, of order $s$ at precision $O(\epsilon)$ (see definition 1.1). Then the difference between the two family of functions is estimated as

$$\|U^p - U^p_{GN}\|_{L^\infty([0,\epsilon],H^s)} \leq C \epsilon t,$$

with $C = C(\sup_p \|U^p_{GN}\|_{L^\infty([0,T],H^s)}) \sup_p \|U^p\|_{L^\infty([0,T],H^{s+\gamma})} \frac{1}{\min \epsilon, \epsilon_{\max}, \mu_{\max}}$.

Such a result has been obtained for a well-chosen Green-Naghdi system in the case of the water-wave problem in [44, 2, 34], and is under current investigation by the author and collaborators.

If the stability Hypothesis 2.12 holds, one can deduces from the consistency result of Proposition 2.2 hold over time interval $[0, T]$, and the solution $U \equiv U_{CL} + U^c$ is small. The estimates concerning the difference between the solution of (1.1) and the fully decoupled solution $U_{CL}$ simply follows from the estimates on $U_c$, in Proposition 2.7. This strategy has been used in the water-wave case in [7], and in the case of internal waves (when restricting to the long wave regime) in [23].

Throughout this section, we assume that Hypothesis 2.12 holds, and study the convergence results between solutions of the Green-Naghdi model and the different approximate solutions which proceed.

Let us first state the convergence results concerning coupled models.

**Corollary 2.13** (Convergence between coupled models). For $(\epsilon, \mu, \delta, \gamma) = p \in \mathcal{P}$, as defined in (1.2), let $U^p_{GN}$ be a solution of Green-Naghdi equations (1.1) such that the family $(U^p_{GN})$ is uniformly bounded on $H^s$, $s$ sufficiently big, over time interval $[0, T_{GN}]$. Let $U^p \equiv (u^p_t + u^p_r, (\gamma + \delta)(u^p_t - u^p_r))$ where $u^p_t, u^p_r$ are the solutions of coupled equations (2.3a)–(2.3b) such that $U^p |_{t=0} = U^p_{GN} |_{t=0}$, and assume that the hypotheses of Proposition 2.2 hold over time interval $[0, T_c]$. Then if Hypothesis 2.12 is valid, one has for any $t \leq \min(T_{GN}, T_c)$:

$$\|U^p_{GN} - U^p\|_{L^\infty([0,t],H^s)} \leq C (\epsilon^4 + \mu \epsilon^2) t,$$

with $C = C(\sup_p \|U^p_{GN}\|_{L^\infty([0,T_{GN}],H^s)}) \sup_p \|U^p\|_{L^\infty([0,T_{c}],H^{s+\gamma})} \frac{1}{\min \epsilon, \epsilon_{\max}, \mu_{\max}}$.

**Proof.** This is a direct application of the consistency Proposition 2.2, together with the stability Hypothesis 2.12.

**Remark 2.14.** Note that the consistency result of Proposition A.5 allows to obtain a similar estimate on the difference between the solution of the Green-Naghdi system (1.1) and the solution of the full Euler system (A.4) with same initial data, if the latter exists (see the discussion in Remark A.6). This estimate is of size $O(\epsilon^2 t)$, so that triangular inequalities allow to extend the estimates of Corollary 2.15 and 2.16, below, replacing $U_S$ by the solution of the full Euler system.

Let us now turn to the weakly coupled model, defined in Proposition 2.7.
Decoupled and unidirectional asymptotic models for the propagation of internal waves

Corollary 2.15 (Convergence of weakly coupled model). For \((\epsilon, \mu, \delta, \gamma) = p \in \mathcal{P}\), as defined in (1.2), let \(U_{GN}^p\) be a solution of Green-Naghdi equations (1.1) such that the family \(U_{GN}^p\) is uniformly bounded on \(H^s\), sufficiently big, over time interval \([0, T_{GN}]\). Assume that hypotheses of Proposition 2.7 hold, and denote \(U^p = U_{GN}^p + U^p \) the approximate solution, satisfying \(U_{GN}^p(t) = U^p(t)\). Then if Hypothesis 2.12 is valid, for any \(t \leq \min(T_{GN}, T_{s+6})\), one has

\[
\|U^p - U_{GN}^p\|_{H^s} \leq C \max(\epsilon^2(\delta^2 - \gamma)^2, \epsilon^4, \mu^2) t (1 + \sqrt{t}),
\]

with \(C = C(\sup_p \|U_{GN}^p\|_{H^s}, M_{s+6}, \frac{1}{\min}, \delta_{\max}, |\lambda|, |\theta|)\).

If moreover, the initial data is sufficiently localized in space, then one has for \(t \leq \min(T_{GN}, T_{s+6})\),

\[
\|U^p - U_{GN}^p\|_{H^s} \leq C \max(\epsilon^2(\delta^2 - \gamma)^2, \epsilon^4, \mu^2) t,
\]

with \(C = C(\sup_p \|U_{GN}^p\|_{H^s}, M_{s+6}, \frac{1}{\min}, \delta_{\max}, |\lambda|, |\theta|)\).

We therefore see that the weakly coupled model may achieve the same accuracy as the fully coupled model, in the critical case, and if the initial data is sufficiently localized in space.

Let us now turn to the fully decoupled models. The following result is a straightforward application of the above Corollary, together with the estimate of Proposition 2.7. Estimates concerning lower order decoupled models are obtained in the same way, using Proposition 2.10.

Corollary 2.16 (Convergence of uncoupled models). For \((\epsilon, \mu, \delta, \gamma) = p \in \mathcal{P}\), as defined in (1.2), let \(U_{GN}^p\) be a solution of Green-Naghdi equations (1.1) such that the family \(U_{GN}^p\) is uniformly bounded on \(H^s\), sufficiently big, over time interval \([0, T_{GN}]\). Denote respectively \(U_{CL}^p, U_{KDV}^p, U_{KADV}^p, U_{IB}^p\) the decoupled approximations defined in Definitions 2.3 and 2.4. Assume that the hypotheses of Proposition 2.7 hold, and Hypothesis 2.12 is valid. Define \(\epsilon_0 = \max(\epsilon(\delta^2 - \gamma), \epsilon^2, \mu)\). Then for any \(t \leq \min(T_{GN}, T_{s+6})\), one has

\[
\|U_{CL}^p - U_{GN}^p\|_{H^s} \leq C \epsilon_0 \min(t, t^{1/2})(1 + \epsilon_0 t),
\]

\[
\|U_{KDV}^p - U_{GN}^p\|_{H^s} \leq C \epsilon_0 \min(t, t^{1/2})(1 + \epsilon_0 t) + C \max(\epsilon^3, \mu \epsilon) t,
\]

\[
\|U_{KADV}^p - U_{GN}^p\|_{H^s} \leq C \epsilon_0 \min(t, t^{1/2})(1 + \epsilon_0 t) + C \epsilon^2 t,
\]

\[
\|U_{IB}^p - U_{GN}^p\|_{H^s} \leq C \epsilon_0 \min(t, t^{1/2})(1 + \epsilon_0 t) + C \max(\epsilon^2, \mu) t,
\]

with \(C = C(\sup_p \|U_{GN}^p\|_{H^s}, M_{s+6}, \frac{1}{\min}, \delta_{\max}, |\lambda|, |\theta|)\).

If the initial data is sufficiently localized in space, then one has for \(t \leq \min(T, T^2)\),

\[
\|U_{CL} - U_{S}\|_{H^s} \leq C \epsilon_0 \min(t, 1)(1 + \epsilon_0 t),
\]

\[
\|U_{KDV} - U_{S}\|_{H^s} \leq C \epsilon_0 \min(t, 1)(1 + \epsilon_0 t) + C \max(\epsilon^3, \mu \epsilon) t,
\]

\[
\|U_{KADV} - U_{S}\|_{H^s} \leq C \epsilon_0 \min(t, 1)(1 + \epsilon_0 t) + C \epsilon^2 t,
\]

\[
\|U_{IB} - U_{S}\|_{H^s} \leq C \epsilon_0 \min(t, 1)(1 + \epsilon_0 t) + C \max(\epsilon^2, \mu) t,
\]

with \(C = C(\sup_p \|U_{GN}^p\|_{H^s}, M_{s+6}, \frac{1}{\min}, \delta_{\max}, |\lambda|, |\theta|)\).

Corollary 2.16 exhibits different sources of error with different time scales.

- The first (and often main) source of error comes from the coupling between the two counterpropagating waves which are neglected by our decoupled models, but recovered at first order by the weakly coupled model (see Corollary 2.15). Following [62], we name the contribution of this term the counterpropagation error.
  - This error grows linearly in time, for times of order \(O(1)\), as the two waves are located at the same position; and coupling effects are strong.
  - However, one is able to control sublinearly the secular growth of such coupling terms, which is the key ingredient of our result, and yields a contribution of size \(O(\epsilon \sqrt{t})\) in general, \(O(\epsilon)\) if the initial data is sufficiently localized.
At very long time, one sees the effect of the precision of the consistency result, through the stability hypothesis. Such a contribution is unavoidable, as it appears after many manipulations of the equations, such as the use of BBM trick, or near-identity change of variables. We call these contributions residual errors. The error generated in that way affects both the solution of the scalar evolution equation and the coupling term, which creates two contributions.

— The so-called unidirectional error is linear in time as long as the solution of the decoupled model is uniformly bounded, and varies greatly with the choice of the model.

— The residual error coming from the coupling term may be superlinear (if the initial data is not localized in space), as the coupling term grows in time.

All of these contributions are summarized in Figure 1. The total error of a decoupled models is the sum of the green and the red curves. The green curve represents the error generated by neglecting the coupling between two waves, and therefore is affected by three of the contributions listed above, at different time scales. The red curve is the unidirectional contribution, and varies with the choice of the model.

However, let us precise that the full pattern is not always visible, as in many cases, one source of error will be negligible in front of the others. In particular, keep in mind that we obtained existence and uniform control in weighted Sobolev spaces of the solutions of decoupled models only over times of order $O(1/\max(\epsilon, \mu))$ (Proposition 2.5), thus the time where residual contributions appear is always out of the scope of our rigorous results.

In order to understand the relevance of a specific evolution equation, one has to look whether the unidirectional contribution, which depends on the evolution equation itself, is smaller or greater than the different sources of error due to coupling. For example, unidirectional error is always negligible in front of coupling terms in the Constantin-Lannes model, whereas it is expected to be preponderant in the inviscid Burgers’ model.

Let us look precisely at several interesting scenarios, in the following subsections. Each discussion is supported by numerical simulations. Each time, we compute the coupled Green-Naghdi model and various evolution equations —among iB, KdV, eKdV, CL—, for different values of $\epsilon$ and corresponding $\mu$ ($\mu = \epsilon$ or $\mu = \sqrt{\epsilon}$, depending on the situation), and over times $O(\epsilon^{-3/2})$. As pointed out above, such time scale is out of the scope of our rigorous result. However, interesting behavior appears after times $O(1/\epsilon)$, as we shall see. Each of the figures below contains three panels. On the left-hand side, we represent the difference between the Green-Naghdi model and decoupled models, with respect to time and for $\epsilon = 0.1, 0.05, 0.035$. Values at times $1/\epsilon$ and $1/\epsilon^{3/2}$ are marked. In the two right-hand-side panels, we plot the difference in a log-log scale for several values of $\epsilon$ (the markers reveal the positions which have been simulated), at given times $1/\epsilon$ and $1/\epsilon^{3/2}$. The pink triangles express the convergence rate.

The initial data is fixed such that the left-going wave is initially two-third the right going wave, to avoid symmetry cancellations: $v_-|_{t=0} = 2/3 v_+|_{t=0}$. For localized initial data, we choose initial data $v_+|_{t=0} = \exp(-(x/2)^2)$ and for non-localized initial data, we choose $v_+|_{t=0} = (1 + 10x^2)^{-1/3}$. We set $\delta^2 = \gamma = 0.64$ in the critical ratio setting, and $\delta = 0.5, \gamma = 0.9$ in the non-critical ratio setting. Finally, we set $\theta = 1/2$ and $\lambda = 0$. 

![Figure 1: Sketch of the error](image-url)
The major difficulty of the numerical simulations computed throughout this paper is the fact that as ε is small (down to 0.035 in our simulations), the time domain $t \in [0, 1/\epsilon^{3/2}]$, and therefore the space domain (as the two waves are moving at velocity $c_+ \approx \pm 1$) of computation becomes very large. The decoupled models can be solved very efficiently by using a frame of reference moving with the decoupled wave, but solving numerically the Green-Naghdi scheme has to be carried out on the full time/space domain. Thus we need a scheme which allows a great accuracy, with a relatively small computation cost. With this in mind, we turn to multi-step, explicit and spectral methods. The space discretization, and in particular the discrete differentiation matrices use trigonometric polynomial on an equispaced grid, as described in [59] (precisely (1.5)). This yields an exponential accuracy with the size of the grid $\Delta x$, if the signal is smooth (note that the major drawback is that the discrete differentiation matrices are not sparse). It turns out setting $\Delta x = 0.2$ is sufficient for the numerical errors to be undetectable in our computations. After this space discretization, one has to solve a system of ordinary differential equations in time, and we use the Matlab solver ode113, which is based on the explicit, multistep, Adams-Bashforth-Moulton method [57], with a stringent tolerance of $10^{-8}$.

In the following discussion, we denote by $C_0$ a constant, which do not depend on all the other parameters involved, and we write $A = O(B)$ if $A \leq C_0B$, and $A \approx B$ if $A = O(B)$ and $B = O(A)$.

### 2.3.1 The long wave regime.

Let us assume that we are in the long wave regime: $\epsilon = O(\mu)$, and therefore $\varepsilon_0 \approx \mu$ in Corollary 2.16.

It follows that the decoupled KdV approximation has the same order of accuracy as the higher order models, whatever the initial data or critical ratio is. Indeed, one has the same estimates for the decoupled approximations $U = U_{KdV}$ or $U = U_{eKdV}$ or $U = U_{CL}$:

$$\|U - U_{GN}\|_{L^\infty([0,t];H^1)} \leq C_0 \mu \min(t, t^{1/2})(1 + \mu t),$$

for any $t \in [0, T_{s+6}^\ast)$, and if the initial data is localized,

$$\|U - U_{GN}\|_{L^\infty([0,t];H^s)} \leq C_0 \mu \min(t, 1)(1 + \mu t),$$

for any $t \in [0, T_{s+6}^\ast)$. Let us note that one recovers the results of [23], for the KdV approximation.

---

**Figure 2:** Long wave regime, critical ratio, localized initial data

More precisely, a unidirectional error of size $O(\mu^2 t)$ is not seen, as it is smaller than the coupling terms, presented above, so that the additional error produced by neglecting higher order terms in the Constantin-Lannes approximation 2.12, and considering only the KdV equation, does not change the accuracy of the model. However, let us note that for localized initial data, unidirectional error of size $O(\mu^2 t)$ is the limiting case. This is why one sees in Figure 2(a) a noticeable difference between
the KdV approximation and higher order approximations (eKdV, CL). The rate of convergence in figure 2(b), however, is identical: the error is of size $O(\mu)$ at time $T = O(1/\mu)$ and $O(\mu^{1/2})$ at time $T = O(1/\mu^{3/2})$. We remark that the last panel of Figure 2 shows a slight discrepancy with respect to the predicted convergence rate in $O(\mu^{1/2})$. This may be due to the fact that higher order sources of error (typically of size $O(\mu^3)$) are detectable for larger values of $\mu$ ($\mu = 0.1$), and eventually becomes negligible for smaller values ($\mu = 0.035$), therefore artificially improving the convergence rate. Let us note also that the criticality of the depth ratio do not play a role in this analysis, and that simulations in the case of non-critical ratio gives similar outcome.

2.3.2 The Camassa-Holm regime, with non-critical ratio.

We are now in the case where $|\delta^2 - \gamma| \geq \alpha_0 > 0$, and $\epsilon \approx \sqrt{\mu}$, thus $\varepsilon_0 \approx \epsilon \approx \sqrt{\mu}$ in Corollary 2.16. In that case, the contribution of the coupling error are always greater than the one of the unidirectional error, and one has the same estimates as for the decoupled approximations $U = U_{iB}$, $U = U_{KdV}$, $U = U_{eKdV}$ or $U = U_{CL}$:

$$
\|U - U_{GN}\|_{L^\infty([0,t];H^s)} \leq C_0 \sqrt{\mu} \min(t, t^{1/2})(1 + \sqrt{\mu}) t),
$$

for any $t \in [0, T^\star_\delta + \delta)$, and if the initial data is localized,

$$
\|U - U_{GN}\|_{L^\infty([0,t];H^s)} \leq C_0 \sqrt{\mu} \min(t, 1)(1 + \sqrt{\mu}) t),
$$

for any $t \in [0, T^\star_\delta + \delta)$. As a consequence, the inviscid Burgers’ approximation is as precise as any higher order decoupled model.

![Figure 3: Camassa-Holm regime, non-critical ratio, localized initial data](image)

Here, the unidirectional error of the inviscid Burgers’ approximation is of the same order of magnitude as the residual coupling error, if the initial data is sufficiently localized. Therefore, in figure 3, one can observe a noticeable difference between the iB approximation and higher order models for long times, although the convergence rate is similar at time $T = 1/\epsilon = 1/\sqrt{\mu}$. For longer times, the decoupled approximate solutions do not seem to converge, which may indicate that the exact solution $U_{GN}$ cannot be controlled over times $O(\epsilon^{-3/2})$, in the setting we use.

2.3.3 The Camassa-Holm regime, critical case.

Now let us assume that $\delta^2 - \gamma = O(\sqrt{\mu})$, and $\epsilon \approx \sqrt{\mu}$, so that $\varepsilon_0 \approx \mu$ in Corollary 2.16. We recall that $T^\star$ and $T^\sharp$ are known to exist and to be of size $T = O(1/\max(\epsilon, \mu)) = O(\mu^{-1/2})$, as a result of Proposition 2.5. Over such time, the unidirectional error in the eKdV model is smaller than the coupling errors: one has for both $U = U_{eKdV}$ and $U = U_{CL}$,

$$
\|U - U_{GN}\|_{L^\infty([0,t];H^s)} \leq C_0 \mu \min(t, t^{1/2}),
$$
for any $t \in [0,T^*_s+6)$ if $T^*_s = \mathcal{O}(\mu^{-1/2})$. If moreover, the initial data is localized, then
\[
\|U - U_{GN}\|_{L^\infty([0,t],H^s)} \leq C_0 \mu \min(t,1),
\]
for any $t \in [0,T^*_s+6)$ if $T^*_s = \mathcal{O}(\mu^{-1/2})$. The accuracy of both eKdV and CL approximations are $p = \mathcal{O}(\mu^{3/4})$ at time $T = \mathcal{O}(\mu^{-1/2}) = \mathcal{O}(\epsilon^{-1})$, or $p = \mathcal{O}(\mu)$ if the initial data is sufficiently localized. The accuracy of the KdV and iB approximation is $\mathcal{O}(\mu^{1/2})$ at that time.

However, if one looks at longer times, then the picture is different. In our simulations, we looked at times up to $T = \mathcal{O}(\epsilon^{-3/2}) = \mathcal{O}(\mu^{-3/4})$. At that time, the localization in space of the initial data plays an important role. If the initial data is non-localized in space, then contribution of the coupling terms are predicted to be greater than the unidirectional error of the eKdV approximation: the secular coupling error is dominant up to $T = \mathcal{O}(\mu^{-1})$ ($\mu \sqrt{t} \geq \mu^{3/2} t$), after what the residual coupling term is dominant ($\mu^{3/2} t^{3/2} \geq \mu^2 t^2$ for $t \geq \mu^{-1}$). Thus the eKdV approximation is predicted to be as precise as the CL approximation. On the contrary, if the initial data is localized, then the unidirectional error is dominant: $\mu^{3/2} t^{3/2} \geq \mu(1 + \mu t)$ for any $t \geq \mu^{-1/2}$. This leads, at time $T = \mathcal{O}(\mu^{-3/4})$, to an error of size $\mathcal{O}(\mu)$ for the CL approximation, and of size $\mathcal{O}(\mu^{3/4})$ for the eKdV approximation. This phenomenon is clearly seen when comparing Figure 4 (localized initial data) and Figure 5 (non-localized initial data).

![Figure 4: Camassa-Holm regime, critical ratio, localized initial data](image)

![Figure 5: Camassa-Holm regime, critical ratio, non-localized initial data](image)
3 Unidirectional propagation

In the previous section, we proved that approximate solutions of the internal wave problem consists in two decoupled, counterpropagating waves, each of them evolving according to a given scalar evolution equation (Definitions 2.3 and 2.4). However, it has been seen that the accuracy of such approximation may be low, especially in the Camassa-Holm regime $\epsilon \sim \sqrt{\mu}$, if the ratio is non-critical ($\delta^2 \neq \gamma$) and/or the initial perturbation is not spatially localized (see section 2.3.2), as significant coupling effects develop between the two counterpropagating waves.

In this section, we show that if one chooses carefully the initial perturbation (deformation of the interface as well as shear layer-mean velocity) in order to root out one of the two waves, then the accuracy of the uncoupled model (that we call unidirectional approximation in that case) can be considerably improved. This study follows the strategy developed for the water-wave problem in [19, 35].

The result is precisely stated below, followed by its proof and a brief discussion.

**Proposition 3.1.** Set $\lambda, \theta \in \mathbb{R}$, and $\zeta^0 \in H^{s+5}$ with $s \geq s_0 > 3/2$. For $(\epsilon, \mu, \delta, \gamma) = p \in P$, as defined in (1.2), denote $(\zeta^p)_{p \in P}$ the unique solution of the equation

\[
\partial_t \zeta + \partial_x \zeta + e \alpha_1 \partial_x^2 \zeta + e^2 \alpha_2 \partial_x^2 \partial_x \zeta + e^3 \alpha_3 \partial_x^3 \zeta + \mu \nu_2^\xi_\lambda \partial_x^2 \zeta - \mu \nu_1^\xi_\lambda \partial_x^2 \partial_x \zeta + \mu \epsilon \partial_x \left( k_1^\xi_\lambda \zeta \partial_x^2 \zeta + e_2^\xi_\lambda (\partial_x \zeta)^2 \right) = 0,
\]

with

\[
\alpha_1 = \frac{3 \delta^2 - \gamma}{2 \gamma + \delta}, \quad \alpha_2 = \frac{21 (\delta^2 - \gamma)^2}{8 (\gamma + \delta)^2} - \frac{3 \delta^3 + \gamma}{\gamma + \delta},
\]

\[
\nu_2^\xi_\lambda = (1 - \theta - \lambda) \frac{4(\gamma + \delta)(\delta^2 - \gamma)(1 + \delta)}{\delta(\gamma + \delta)}, \quad \nu_1^\xi_\lambda = (\theta + \lambda) \frac{1 + \gamma \delta}{\delta(\gamma + \delta)},
\]

\[
k_1^\xi_\lambda = \frac{(14 - 6(\theta + \lambda))(\delta^2 - \gamma)(1 + \gamma \delta)}{24(\gamma + \delta)^2}, \quad k_2^\xi_\lambda = \frac{(17 - 12\theta)(\delta^2 - \gamma)(1 + \gamma \delta)}{48(\gamma + \delta)^2}.
\]

For given $M_{s+5}, h > 0$, assume that there exists $T_{s+5} > 0$ such that

\[
T_{s+5} = \max \{ T \geq 0 \text{ such that } \|\zeta^p\|_{L^\infty([0,T],H^{s+5})} \leq M_{s+5} \},
\]

and for any $(t,x) \in [0,T_{s+5}) \times \mathbb{R}$,

\[
h_1(t,x) = 1 - \epsilon \zeta^p(t,x) > h > 0, \quad h_2(t,x) = \frac{1}{\delta} + \epsilon \zeta^p(t,x) > h > 0.
\]

Then define $\nu^p$ as $\nu^p = \frac{h_1^p + h_2^p}{h_1^p h_2^p} \nu(\zeta^p)$, with

\[
\nu(\zeta) = \zeta + e \frac{\alpha_1}{2} \zeta^2 + e^2 \frac{\alpha_2}{3} \zeta^3 + e^3 \frac{\alpha_3}{4} \zeta^4 + \mu \epsilon \zeta^2 \zeta + \mu \epsilon \zeta \partial_x^2 \zeta + \mu \epsilon (k_1 \zeta \partial_x^2 \zeta + k_2 \partial_x \zeta)^2,
\]

where parameters $\alpha_1, \alpha_2, \alpha_3$ are as above, and

\[
\nu = \frac{1 + \gamma \delta}{\delta(\gamma + \delta)}, \quad k_1 = \frac{14(\delta^2 - \gamma)(1 + \gamma \delta)}{24(\gamma + \delta)^2}, \quad k_2 = \frac{17(\delta^2 - \gamma)(1 + \gamma \delta)}{48(\gamma + \delta)^2}.
\]

Then $(\zeta^p, \nu^p)$ is consistent with Green-Naghdi equations (1.1), of order $s$ and on $[0,T_{s+5})$, with precision $O(\epsilon)$, with

\[
e = C(M_{s+5}, \frac{1}{s_0 - 3/2}, h^{-1}, \frac{1}{\delta_{\min}}, \delta_{\max}, \epsilon_{\max}, \mu_{\max}, |\lambda|, |\theta|) \times \max(e^4, \mu^2).
\]

**Remark 3.2.** When $\gamma \to 0$ and $\delta \to 1$, one recovers the one-fluid model presented in [19, section 2.2], with $q = \frac{1}{\nu}$ and $\lambda = 0$, using notations therein.
Remark 3.3. In Proposition 3.1, the approximation consists in a scalar evolution equation satisfied by the deformation at the interface, and we reconstruct the shear layer-mean velocity from the deformation (and in particular, this determines the initial shear velocity, from the initial deformation). Following [19], a similar strategy could consist in looking for an evolution equation for the shear layer-mean velocity, and reconstruct the deformation at the interface. We decide not to present the outcome of such strategy, as the result is very similar, and calculations somewhat heavier in that case.

Remark 3.4. As discussed in [19, Proposition 5], specific values of parameters in (3.1) yield equations with different properties, especially concerning the behavior near the maximal time of definition (if it is finite). Indeed, the proof of [19, Proposition 5] can easily be adapted to more general coefficients, and one obtains:

- If \( \nu_1^{\theta,\lambda} > 0, \nu_2^{\theta,\lambda} = 2\kappa_1^{\theta,\lambda} > 0 \) and \( \alpha_3 > 0 \), then singularities can develop in finite time only in the form of surging wave breaking. In other words, if the maximal time of existence of \( \zeta \) is finite, \( T < \infty \), then
  \[
  \sup_{t \in [0,T), x \in \mathbb{R}} |\zeta(t, x)| < \infty \quad \text{and} \quad \sup_{x \in \mathbb{R}} \{|\partial_x \zeta(t, x)|\} \uparrow \infty \quad \text{as} \quad t \uparrow T.
  \]

- If \( \nu_1^{\theta,\lambda} > 0, \nu_2^{\theta,\lambda} = 2\kappa_1^{\theta,\lambda} < 0 \) and \( \alpha_3 < 0 \), then singularities can develop in finite time only in the form of plunging wave breaking. In other words, if the maximal time of existence of \( \zeta \) is finite, \( T < \infty \), then
  \[
  \sup_{t \in [0,T), x \in \mathbb{R}} |\zeta(t, x)| < \infty \quad \text{and} \quad \inf_{x \in \mathbb{R}} \{|\partial_x \zeta(t, x)|\} \downarrow -\infty \quad \text{as} \quad t \uparrow T.
  \]

Identity \( \kappa_1^{\theta,\lambda} = 2\kappa_2^{\theta,\lambda} \) holds in the line \( \theta - \lambda = 1/2 \), and in that case, \( \nu_2^{\theta,\lambda} > 0 \) if and only if \( \theta > 1/4 \). Restricting to \( \theta \leq 1 \) as natural values for the use of BBM trick, one can easily check that if \( \gamma = 0 \), then singularities may occur only as surging wave breaking, as it is the case in the one-layer situation. On the contrary, if \( \gamma \sim 1 \), as is the case in the stratified ocean (small variation of densities) then singularities will occur as surging wave breaking if \( \delta > 1 \) (thicker upper layer), and plunging wave breaking will occur for \( \delta < 1 \) (thicker lower layer).

Proof of Proposition 3.1. In order to simplify the calculations, we use the Green-Naghdi system (1.1) expressed using the variables \( (\zeta, \underline{u}) \) where we define \( \underline{v} = \frac{h_1 h_2}{\kappa_1 + \gamma h_2} \bar{v} \). The system reads

\[
(3.3) \quad \partial_t \zeta + \partial_x \underline{v} = 0,
\]

\[
\left\{ \begin{array}{l}
\partial_t \left( \frac{h_1 + \gamma h_2}{h_1 h_2} \underline{v} + \mu Q[h_1, h_2] \underline{v} \right) + (\gamma + \delta) \partial_x \zeta + \frac{\epsilon}{2} \partial_x \left( \frac{h_1^2 - \gamma h_2^2}{(h_1 h_2)^2} |\underline{v}|^2 \right) = \mu \epsilon \partial_x (R[h_1, h_2] \underline{v}),
\end{array} \right.
\]

with the following operators:

\[
Q[h_1, h_2] V = -\frac{1}{3 h_1 h_2} \left( h_1 \partial_x \left( h_2^3 \partial_x \left( \frac{V}{h_2} \right) \right) + \gamma h_2 \partial_x \left( h_1^3 \partial_x \left( \frac{V}{h_1} \right) \right) \right),
\]

\[
R[h_1, h_2] V = \frac{1}{2} \left( h_2 \partial_x \left( \frac{V}{h_2} \right)^2 - \gamma \left( h_1 \partial_x \left( \frac{V}{h_1} \right)^2 \right) \right) + \frac{1}{3} \left( \frac{h_1}{h_2} \partial_x \left( h_2^3 \partial_x \left( \frac{V}{h_2} \right) \right) - \gamma \frac{h_2}{h_1} \partial_x \left( h_1^3 \partial_x \left( \frac{V}{h_1} \right) \right) \right).
\]

Using this system simplifies considerably the analysis. Indeed, it is clear that if \( \zeta \) is to satisfy the following scalar evolution equation,

\[
(3.4) \quad \partial_t \zeta + \partial_x \zeta + \alpha_1 \zeta \partial_x \zeta + \epsilon^2 \alpha_2 \zeta^2 \partial_x \zeta + \epsilon^3 \alpha_3 \zeta^3 \partial_x \zeta
\]

\[
+ \mu \epsilon \partial_x^3 \zeta + \mu \epsilon \partial_x \left( \kappa_1 \zeta \partial_x^2 \zeta + \kappa_2 (\partial_x \zeta)^2 \right) = 0,
\]

then it must satisfy the system (3.3).
then $\psi$ shall satisfy (using the first equation of (3.3), and the fact that the system is at rest at infinity: $\zeta, \psi \to 0$ when $x \to \pm \infty$)
\begin{equation}
\psi = \zeta + \epsilon \frac{\alpha_1}{2} \zeta^2 + \frac{\epsilon^2 \alpha_2}{3} \zeta^3 + \frac{\epsilon^3 \alpha_3}{4} + \mu \nu_x \partial_x^2 \zeta + \mu \epsilon \left(\kappa_1 \zeta \partial_x^2 \zeta + \kappa_2 (\partial_x \zeta)^2\right).
\end{equation}

Now we will show that one can choose coefficients $\alpha_1, \alpha_2, \epsilon$, such that the second equation of (3.3) is satisfied up to a small remainder.

Indeed, when plugging (3.4) and (3.5), and expanding in terms of $\epsilon$ and $\mu$, one obtains
\begin{align*}
\epsilon (3(\delta^2 - \gamma) - 2(\delta + \gamma)\alpha_1) \zeta \partial_x \zeta \\
- \epsilon^2 (6(\delta^2 - \gamma) - 5(\delta^2 - \gamma)\alpha_1 + (\alpha_1^2 + 2\alpha_2)(\gamma + \delta)) \zeta^2 \partial_x \zeta \\
+ \epsilon^3 (10(\delta^2 - \gamma) - 9(\delta^2 + \gamma)\alpha_1 + (2\alpha_1^2 + 14/3 \alpha_2)(\delta^2 - \gamma) - 2(\alpha_3 + \alpha_1 \alpha_2)(\delta + \gamma)) \zeta^3 \partial_x \zeta \\
&+ \mu \left(1 + \frac{\gamma \delta}{3\delta} - 2\nu_x (\delta + \gamma)\right) \partial_x^2 \zeta \\
&+ \mu \epsilon (4(\delta^2 - \gamma)\nu_x - 2(\alpha_1 \nu + \kappa_1)(\delta + \gamma) - \frac{1 - \gamma}{3} + \frac{2(1 + \gamma \delta)}{3\delta} \alpha_1) \zeta \partial_x^2 \zeta \\
&+ \mu \epsilon (2(\delta^2 - \gamma)\nu_x - (3\alpha_1 \nu_x + 2\kappa_1 + 4\kappa_2)(\delta + \gamma) - \frac{2(1 - \gamma)}{3} + \frac{2(1 + \gamma \delta)}{3\delta} \alpha_1) \partial_x \zeta \partial_x^2 \zeta.
\end{align*}
\begin{equation}
= R
\end{equation}
where the remainder $R$ can be estimated, provided $h_1, h_2 \geq h > 0$, as
\[ |R|_{H^{\nu}} \leq C(\|\zeta\|_{H^{\nu+2}}, \frac{1}{s_0 - 3/2}, \frac{1}{h^{1/2}}, \frac{1}{\delta_{\min}}, \delta_{\max}, \epsilon_{\max}, \mu_{\max}) \times \max(\epsilon^4, \mu^2). \]

The left hand side of (3.6) vanishes when we set
\begin{align*}
\alpha_1 &= \frac{3 \delta^2 - \gamma}{2 \gamma + \delta}, \quad \alpha_2 = \frac{5(\delta^2 - \gamma)\alpha_1 - 6(\delta^2 + \gamma) - \alpha_1^2}{2 \gamma + \delta}, \\
\alpha_3 &= \frac{10(\delta^2 - \gamma) - 9(\delta^2 + \gamma)\alpha_1 + (14/3 \alpha_2 + 2\alpha_1)(\delta^2 - \gamma) - \alpha_1 \alpha_2}{2 \gamma + \delta}, \\
\nu &= \frac{1 + \gamma \delta}{6 \delta (\gamma + \delta)}, \\
\kappa_1 &= \frac{4(\delta^2 - \gamma)\nu_x - \frac{1 - \gamma}{3} + \alpha_1 \frac{2(1 + \gamma \delta)}{3 \delta}}{2 \gamma + \delta} - \alpha_1 \nu, \\
\kappa_2 &= \frac{(\delta^2 - \gamma)\nu_x - \frac{1 - \gamma}{3} + \alpha_1 \frac{1 + \gamma \delta}{\delta}}{2 \gamma + \delta} - \frac{3 \alpha_1 \nu}{4} - \frac{\kappa_1}{2}.
\end{align*}
Such a choice corresponds to parameters of Proposition 3.1, with $\theta = \lambda = 0$.

The cases $\theta \neq 0$ and $\lambda \neq 0$ are obtained as in section 2.1, using BBM trick and near-identity change of variables. We detail the calculations below, using for simplicity the notation $O_\delta(\epsilon)$ for any term bounded by $\epsilon C(\|\zeta\|_{H^{\nu+1}})$.

**BBM trick.** We make use of the first order approximation in (3.4):
\[ \partial_t \zeta + \partial_x \zeta + \epsilon \alpha_1 \zeta \partial_x \zeta = O_\delta(\max(\mu, \epsilon^2)), \]
so that one has, for any $\theta \in \mathbb{R}$,
\[ \partial_x \zeta = (1 - \theta) \partial_t \zeta - \theta (\partial_t \zeta + \epsilon \alpha_1 \zeta \partial_x \zeta) + O_\delta(\max(\mu, \epsilon^2)). \]
Plugging this identity into (3.4) yields
\begin{equation}
\partial_t \zeta + \partial_x \zeta + \epsilon \alpha_1 \zeta \partial_x \zeta + c^2 \alpha_2 \zeta^2 \partial_x \zeta + \epsilon^3 \alpha_3 \zeta^3 \partial_x \zeta + \mu (1 - \theta) \nu \partial_x^2 \zeta - \mu \theta \nu \partial_x^2 \partial_x \zeta
\end{equation}
\begin{equation}
+ \epsilon \mu \alpha_2 \left( (\kappa_1 - \theta \alpha_1) \frac{\zeta \partial_x^2 \zeta + (\kappa_2 - \theta \alpha_1) (\partial_x \zeta)^2}{\kappa_1} \right) = O_\delta(\max(\mu^2, \epsilon^2)).
\end{equation}
Conversely, $\zeta_\theta$, a solution of (3.7) (with zero on the right hand side), satisfies (3.4) with a remainder bounded by $O_\delta(\max(\mu^2, \epsilon^2))$. One can easily check that, defining $\nu_\theta$ as a function of $\zeta_\theta$
Decoupled and unidirectional asymptotic models for the propagation of internal waves through (3.5), \((\zeta_\theta, v_\theta)\) satisfies (3.6) up to a remainder \(\mathcal{R}_\theta = O_5(\max(\mu^2, \mu^3))\). Proposition 3.1 is now proved for \(\theta \in \mathbb{R}\) and \(\lambda = 0\).

Near identity change of variable. Let us consider

\[
\zeta_{\theta,\lambda} \equiv \zeta_\theta - \nu \lambda \partial_\zeta^2 \zeta_\theta,
\]

where we recall \(\nu = \frac{1 + \gamma^2}{6a(1 + \gamma)}\), and with \(\zeta_\theta\) satisfying (3.7) (with zero on the right hand side). Then \(\zeta_{\theta,\lambda}\) satisfies

\[
\partial_t \zeta + \partial_x \zeta + \epsilon^2 \alpha_2 \partial_x \theta + \epsilon^3 \alpha_3 \partial_x \zeta + \mu((1 - \theta)\nu - \lambda \nu) \partial_\zeta^2 \zeta - \mu(\theta \nu + \lambda \nu) \partial_\zeta^2 \partial_\zeta^2 \zeta
\]

\[
+ \mu \partial_\zeta \left( (\kappa_1 - \theta \alpha_1 - \lambda \nu \alpha_1) \zeta \partial_\zeta^2 \zeta + (\kappa_2 - \theta \nu \alpha_1)(\partial_\zeta^2 \zeta)^2 \right) = O_5(\max(\mu^2, \mu^3)),
\]

(3.8)

Again, it is now straightforward though technical to check that, denoting \(\zeta_{\theta,\lambda}\) the solution of (3.8) and defining \(\mathcal{R}_{\theta,\lambda}\) as a function of \(\zeta_{\theta,\lambda}\) through (3.5), then \((\zeta_{\theta,\lambda}, v_{\theta,\lambda})\) satisfies (3.6) up to a remainder \(\mathcal{R}_{\theta,\lambda} = O_5(\max(\mu^2, \mu^3))\). Proposition 3.1 is now proved for any \(\theta, \lambda \in \mathbb{R}\).

\(\square\)

Discussion. We note that the accuracy of the unidirectional approximate solution, described in Proposition 3.1 is tremendously improved, when compared with the results of section 2.2. Indeed, using the stability Hypothesis 2.12, one would obtain the following result.

Corollary 3.5 (Convergence of unidirectional approximation). For \((\epsilon, \mu, \delta, \gamma) = p \in \mathcal{P}\), as defined in (1.2), let \(U_{p,GN}^N\) be a solution of Green-Naghdi equations (1.1) such that the family \((U_{p,GN}^N)\) is uniformly bounded on \(H^s\), a sufficiently big, over time interval \([0, T_{GN}]\), and with initial data satisfying (3.2). Assume that hypotheses of Proposition 3.1 hold, and denote \(U_{p,CL}^N\) the unidirectional approximation defined therein. Then if Hypothesis 2.12 is true, one has for any \(t \leq \min(T_{GN}, T_{s+5})\),

\[
\|U_{CL} - U_S\|_{L^\infty([0, t]; H^s)} \leq C \max(\epsilon^4, \mu^2) t,
\]

with \(C = C\left(\sup_p \|U_{p,GN}^N\|_{L^\infty([0, T]; H^s)}, M_{s+5}, h^{-1}, \delta_{\min}, \delta_{\max}, \epsilon_{\max}, |\lambda|, |\theta|\right)\).

Such a result is supported by numerical simulations. In Figures 6 and 7, we compute the decoupled Constantin-Lannes approximation of Definition 2.3, as well as the unidirectional approximation described in Proposition 3.1, and compare them with the the solution of the Green-Naghdi system (1.1), making sure that initial data satisfies (3.2), and in the Camassa-Holm regime \(\epsilon^2 = \mu\) (see Section 2.3 for a description of our numerical method). The display of the results is similar to the ones in section 2.3, except we add a plot of the deformation at final time in the bottom panel. We also choose to limit the time domain of our numerical simulations to \(1/\epsilon\) (instead of \(1/\epsilon^{3/2}\) previously) as more interest is given to short time scales, where decoupled and unidirectional models are significantly different, rather than very long time scales, where nothing relevant happens.

In Figure 6, we choose a set up which is favorable to the CL approximation of Definition 2.3; critical ratio \(\delta^2 = \gamma\) and localized initial data. We see that the unidirectional model is still a much better approximation. In particular, the CL decoupled model predicts a small wave moving on the left (of size \(O(\epsilon^2)\), as \(\alpha_1 = 0\) in (3.2)), which is not predicted in the unidirectional model, and almost nonexistent in the solution of the Green-Naghdi system, as we can see in Figure 6(c). This short-time \(O(\epsilon^2)\) error of the CL decomposition is preserved over times of order \(T = O(1/\epsilon)\). As for the unidirectional model, the produced error is clearly, and as expected, of size \(O(\epsilon t)\). In the Camassa-Holm regime, such accuracy is of the same order of magnitude as the full Euler system itself. This means that if the initial data satisfies (3.2), then the very simple unidirectional model approximate solution described in Proposition 3.1 is as precise an approximation of the solution of the full Euler system (A.4) as the solution of the coupled Green-Naghdi system (1.1) (provided that the stability Hypothesis 2.12 holds); see Remark 2.14.

In Figure 7, the ratio is non critical \(\delta \neq \gamma\). The accuracy of the CL approximate solution of Definition 2.3 is worse than in the critical case, as the short-time error is of size \(O(\epsilon)\). Once again, the same error estimates holds over times of order \(T = O(1/\epsilon)\). The accuracy of the unidirectional
Figure 6: Unidirectional, localized initial data, critical ratio, Camassa-Holm regime

Figure 7: Unidirectional, localized initial data, non-critical ratio, Camassa-Holm regime
Decoupled and unidirectional asymptotic models for the propagation of internal waves

model is not affected, and is still of size \( O(\epsilon^4) \): the criticality of the depth-ratio do not play a role in the accuracy of the unidirectional approximation.

Let us now turn to the following question: is it true that after a certain time, any perturbation will decompose into two waves, each one satisfying (approximatively) an equation of the form (3.2)? Our answer is numerical. We use the numerical simulations of section 2.3.2 (non-critical ratio, localized initial data), and test the right wave of the numerical solution of the Green-Naghdi system against (3.2). As we can see in Figure 8(a) (where the log of the error is plotted to ease the viewing), a very good agreement appears after a given time \( T_0 \), which is independent of \( \epsilon \) (but rather depends on the thickness, or wavelength of the initial data). The accuracy of this agreement is in our simulation of size \( O(\epsilon^4) \), and valid for long times; see Figure 8(b).

![Figure 8: Validity of (3.2) for generic initial data](image)

**Figure 8:** Validity of (3.2) for generic initial data

### A Derivation of the Green-Naghdi system

This section is dedicated to the construction and justification of the Green-Naghdi model (1.1), which is the groundwork of our study. We first recall the so-called full Euler system, governing the behavior of two layers of immiscible, homogeneous, ideal, incompressible fluids only under the influence of gravity. The derivation of such equation is not new, and we refer to [8] for more details.

This system can be written as two evolution equations using Zakharov’s canonical variables [63], namely \( \zeta \), the deformation at the interface, and the trace of a velocity potential at the interface. Such a formulation relies on so-called Dirichlet-to-Neumann operators, solving Laplace’s equation on the two domains of fluid, with suitable Neumann or Dirichlet boundary conditions.

We non-dimensionalize the system in order to put forward the relevant dimensionless parameters of the system, and in particular \( \epsilon \), the nonlinearity parameter, and \( \mu \), the shallowness parameter.

The key ingredient in the construction of the Green-Naghdi model is to obtain an expansion of the Dirichlet-to-Neumann operators, with respect to the shallowness parameter, \( \mu \). Our result is disclosed in Proposition A.2, below.

When replacing the Dirichlet-to-Neumann operators by the truncated expansion, one obtains the Green-Naghdi models: (A.14), (A.17) and (A.19) (each version handles different velocity variables as unknown. The latter is the system we base our study on, and considers the shear layer-mean velocity, obtained after integrating the velocity potential across the vertical layer in each fluid \(^2\)).

Note that one cannot write the full Euler system in the simple form of two evolution equations using layer-mean velocity variables, as the pressure cannot be eliminated from the equation. The nice formulation of the Green-Naghdi system with the shear layer-mean velocity relies on the assumption of shallow water, \( \mu \ll 1 \), which allows to approximate Zakharov’s canonical variables in terms of layer-mean velocity variables. See also [15] and [16] for the formal construction of Green-Naghdi models using layer-mean velocity variables from the beginning.

\(^2\)
All of these asymptotic models are justified by a consistency result, stating that any solution of the full Euler system will satisfy the Green–Naghdi systems up to a small remainder (of size $O(\mu^2)$).

**A.1 The Full Euler system**

![Figure 9: Sketch of the domain](image)

The system we study consists in two layers of immiscible, homogeneous, ideal, incompressible fluids only under the influence of gravity (see Figure 9). We restrict ourselves to the two-dimensional case, i.e. to horizontal dimension $d = 1$.

We assume that the interface is given as the graph of a function $\zeta(t,x)$ which expresses the deviation from its rest position $\{(x,z), z = 0\}$ at the spatial coordinate $x$ and at time $t$. The bottom and surface are assumed to be flat. Therefore, at each time $t \geq 0$, the domains of the upper and lower fluid (denoted, respectively, $\Omega^1_t$ and $\Omega^2_t$), are given by

$$
\Omega^1_t = \{(x,z) \in \mathbb{R}^d \times \mathbb{R}, \quad \zeta(t,x) \leq z \leq d_1 \},
$$

$$
\Omega^2_t = \{(x,z) \in \mathbb{R}^d \times \mathbb{R}, \quad -d_2 \leq z \leq \zeta(t,x) \}. 
$$

We assume that the two domains are strictly connected, that is

$$
d_1 + \zeta(t,x) \geq h > 0, \quad d_2 + \zeta(t,x) \geq h > 0.
$$

We denote by $(\rho_1, \mathbf{v}_1)$ and $(\rho_2, \mathbf{v}_2)$ the mass density and velocity fields of, respectively, the upper and the lower fluid. The two fluids are assumed to be homogeneous and incompressible, so that the mass densities $\rho_1$, $\rho_2$ are constant, and the velocity fields $\mathbf{v}_1$, $\mathbf{v}_2$ are divergence free. As we assume the flows to be irrotational, one can express the velocity field as gradients of a potential: $\mathbf{v}_i = \nabla \phi_i$ ($i = 1, 2$), and the velocity potentials satisfy Laplace’s equation

$$
\partial^2_x \phi_i + \partial^2_z \phi_i = 0.
$$

The fluids being ideal, they satisfy the Euler equations; the momentum equations can be integrated, which yields the Bernoulli equation in terms of potentials:

$$
\partial_t \phi_i + \frac{1}{2} |\nabla_{x,z} \phi_i|^2 = -\frac{P}{\rho_i} - gz \quad \text{in} \quad \Omega^1_t \quad (i = 1, 2),
$$

where $P$ denotes the pressure inside the fluids.

From the assumption that no fluid particle crosses the surface, the bottom or the interface, one deduces kinematic boundary conditions, and the set of equations is closed by the continuity of the pressure at the interface, assuming that there is no surface tension.\(^3\)

\(^3\)The surface tension effects should be included for our system to be well-posed. However, the surface tension is very small in practice, and does not play any role in our asymptotic analysis. See [43] for in depth study of this phenomenon.
Altogether, the governing equations of our problem are the following:

\[
\begin{aligned}
\partial_x^2 \phi_i + \partial_z^2 \phi_i &= 0 \quad &\text{in } \Omega_i^t, \ i = 1, 2, \\
\partial_t \phi_i + \frac{1}{2} |\nabla_{x,z} \phi_i|^2 &= -\frac{P}{\rho_i} - gz \quad &\text{in } \Omega_i^t, \ i = 1, 2, \\
\partial_z \phi_1 &= 0 \quad &\text{on } \Gamma_t, \\
\partial_z \phi_2 &= 0 \quad &\text{on } \Gamma_t, \\
\partial_t \zeta &= \sqrt{1 + |\partial_z \zeta|^2} \partial_n \phi_1 = \sqrt{1 + |\partial_z \zeta|^2} \partial_n \phi_2 \quad &\text{on } \Gamma = \{(x, z) : z = \zeta(t, x)\}, \\
\partial_z \phi_2 &= 0 \quad &\text{on } \Gamma_b = \{(x, z) : z = -d_2\}, \\
P \text{ continuous}
\end{aligned}
\]

(A.1)

where \( n \) is the unit upward normal vector at the interface.

**Rewriting the system as evolution equations.** The key point is to remark that the system (A.1) can be reduced into two evolution equations coupling Zakharov’s canonical variables [63], namely the deformation of the free interface from its rest position, \( \zeta \), and the trace of the upper potential at the interface, \( \psi \), defined as follows:

\[
\psi \equiv \phi_1(t, x, \zeta(t, x)).
\]

Indeed, \( \phi_1 \) and \( \phi_2 \) are uniquely deduced from \( (\zeta, \psi) \) as the unique solutions of the following Laplace’s problems:

\[
\begin{aligned}
&\left\{ \begin{array}{l}
(\partial_x^2 + \partial_z^2) \phi_1 = 0 &\quad \text{in } \Omega_1, \\
\phi_1 = \psi &\quad \text{on } \Gamma, \\
\partial_z \phi_1 = 0 &\quad \text{on } \Gamma_1,
\end{array} \right. &\quad &\left\{ \begin{array}{l}
(\partial_x^2 + \partial_z^2) \phi_2 = 0 &\quad \text{in } \Omega_2, \\
\partial_n \phi_2 = \partial_n \psi &\quad \text{on } \Gamma_t, \\
\partial_z \phi_2 = 0 &\quad \text{on } \Gamma_b.
\end{array} \right.
\end{aligned}
\]

(A.2)

More precisely, we define the so-called Dirichlet-Neumann operators.

**Definition A.1** (Dirichlet-Neumann operators). Let \( \zeta \in W^{1,\infty}(\mathbb{R}) \), and \( \partial_z \psi \in H^{1/2}(\mathbb{R}) \). Then we define

\[
\begin{aligned}
G[\zeta] \psi &= \sqrt{1 + |\partial_z \zeta|^2} \partial_n \phi_1 |_{z=\zeta} = (\partial_t \phi_1) |_{z=\zeta} - (\partial_x \zeta)(\partial_x \phi_1) |_{z=\zeta}, \\
H[\zeta] \psi &= \partial_x (\phi_2 |_{z=\zeta}) = \partial_x (\phi_2(t, x, \zeta(t, x))),
\end{aligned}
\]

where \( \phi_1 \) and \( \phi_2 \) are uniquely defined (up to a constant for \( \phi_2 \)) as the solutions in \( H^2(\mathbb{R}) \) of (A.2).

The well-posedness of Laplace’s problem (A.2), and therefore the Dirichlet-Neumann operators, follow from classical arguments detailed, for example, in [22, Proposition 2.1].

One can then rewrite the conservation of momentum equations in (A.1) at the interface, thanks to chain rule:

\[
\partial_t (\phi_1 |_{z=\zeta}) + g \zeta + \frac{1}{2} |\partial_x (\phi_2 |_{z=\zeta})|^2 + \frac{G[\zeta] \psi + (\partial_x \zeta)(\partial_x (\phi_2 |_{z=\zeta}))^2}{2(1 + |\partial_z \zeta|^2)} = \frac{P |_{z=\zeta}}{\rho_i}.
\]

Using the continuity of the pressure at the interface, one deduces from the identities above

\[
\partial_t (\rho_2 H[\zeta] \psi - \rho_1 \partial_z \psi) + g(\rho_2 - \rho_1) \partial_z \zeta + \frac{1}{2} \partial_x \left( \rho_2 H[\zeta] \psi^2 - \gamma |\partial_x \psi|^2 \right) = \partial_x N(\zeta, \psi),
\]

where \( N \) is defined as

\[
N(\zeta, \psi) \equiv \frac{\rho_1 (G[\zeta] \psi + (\partial_x \zeta)(\partial_x \psi))^2 - \rho_2 (G[\zeta] \psi + (\partial_x \zeta)H[\zeta] \psi)^2}{2(1 + |\partial_z \zeta|^2)}.
\]

The kinematic boundary condition at the interface is obvious, and the system (A.1) is therefore rewritten as

\[
\begin{aligned}
\partial_t (\rho_2 H[\zeta] \psi - \rho_1 \partial_x \psi) + g(\rho_2 - \rho_1) \partial_x \zeta + \frac{1}{2} \partial_x \left( \rho_2 H[\zeta] \psi^2 - \rho_1 |\partial_x \psi|^2 \right) &= \partial_x N(\zeta, \psi), \\
\partial_t \zeta &= G[\zeta] \psi.
\end{aligned}
\]

(A.3)

which is exactly system (9) in [8].
Nondimensionalization of the system. Thanks to an appropriate scaling, the two-layer full Euler system (A.3) can be written in dimensionless form. The study of the linearized system (see [43], for example), which can be solved explicitly, leads to a well-adapted rescaling.

Let $a$ be the maximum amplitude of the deformation of the interface. We denote by $\lambda$ a characteristic horizontal length, say the wavelength of the interface. Then the typical velocity of small propagating internal waves (or wave celerity) is given by

$$c_0 = \sqrt{g \left( \frac{\rho_2 - \rho_1}{\rho_2} \frac{d_1 d_2}{d_2 + \rho_1} \right)}.$$ 

Consequently, we introduce the dimensionless variables

$$\tilde{z} \equiv \frac{z}{d_1}, \quad \tilde{x} \equiv \frac{x}{\lambda}, \quad \tilde{t} \equiv \frac{c_0 t}{\lambda},$$

the dimensionless unknowns

$$\tilde{\zeta}(\tilde{x}) \equiv \frac{\zeta(x)}{a}, \quad \tilde{\psi}(\tilde{x}) \equiv \frac{d_1}{a \lambda c_0} \psi(x),$$

and the four independent dimensionless parameters

$$\gamma = \frac{\rho_1}{\rho_2}, \quad \epsilon \equiv \frac{a}{d_1}, \quad \mu \equiv \frac{d_1^2}{\lambda^2}, \quad \delta \equiv \frac{d_1}{d_2}.$$ 

With this rescaling, the system (A.3) becomes (we withdraw the tildes for the sake of readability)

$$\begin{cases} 
\partial_t \zeta - \frac{1}{\mu} G^{\mu,\epsilon} \psi = 0, \\
\partial_t \left(H^{\mu,\epsilon} \psi - \gamma \partial_x \zeta\right) + (\gamma + \delta) \partial_x \zeta + \frac{\epsilon}{2} \partial_x \left(|H^{\mu,\epsilon}\psi|^2 - \gamma |\partial_x \psi|^2\right) = \mu \epsilon \partial_x \mathcal{N}^{\mu,\epsilon},
\end{cases}$$

with

$$\mathcal{N}^{\mu,\epsilon} \equiv \frac{\left(\frac{1}{n} G^{\mu,\epsilon} \psi + \epsilon (\partial_x \zeta) H^{\mu,\epsilon} \psi\right)^2 - \gamma \left(\frac{1}{n} G^{\mu,\epsilon} \psi + \epsilon (\partial_x \zeta) (\partial_x \psi)\right)^2}{2(1 + \mu |\partial_x \zeta|^2)},$$

and the dimensionless Dirichlet-to-Neumann operators defined by

$$G^{\mu,\epsilon} \psi \equiv \sqrt{1 + |\partial_x \zeta|^2} (\partial_\nu \phi_1) \left|_{z = \epsilon \zeta} \right. = -\mu \epsilon (\partial_x \zeta) (\partial_x \phi_1) \left|_{z = \epsilon \zeta} \right. + (\partial_x \phi_1) \left|_{z = \epsilon \zeta} \right.,$$

$$H^{\mu,\epsilon} \psi \equiv \partial_x (\phi_1 \left|_{z = \epsilon \zeta} \right. ) = (\partial_x \phi_1) \left|_{z = \epsilon \zeta} \right. + \epsilon (\partial_x \zeta) (\partial_x \phi_1) \left|_{z = \epsilon \zeta} \right.,$$

where $\phi_1$ and $\phi_2$ are the solutions of the rescaled Laplace problems

$$\begin{cases} 
(\mu \partial_x^2 + \partial_z^2) \phi_1 = 0 \quad &\text{in } \Omega_1 \equiv \{(x, z) \in \mathbb{R}^2, \epsilon \zeta(x) < z < 1\}, \\
\partial_z \phi_1 = 0 \quad &\text{on } \Gamma_1 \equiv \{z = 1\}, \\
\phi_2 = \psi \quad &\text{on } \Gamma \equiv \{z = \epsilon \zeta\},
\end{cases}$$

$$\begin{cases} 
(\mu \partial_x^2 + \partial_z^2) \phi_2 = 0 \quad &\text{in } \Omega_2 \equiv \{(x, z) \in \mathbb{R}^2, -\frac{1}{\epsilon} < z < \epsilon \zeta\}, \\
\partial_z \phi_2 = \partial_z \phi_1 \quad &\text{on } \Gamma_2, \\
\partial_x \phi_2 = 0 \quad &\text{on } \Gamma_b \equiv \{z = -\frac{1}{\epsilon}\}.
\end{cases}$$

Again, the Dirichlet-Neumann operators are well defined, provided that one has $\zeta \in W^{1,\infty}(\mathbb{R})$, $\partial_x \psi \in H^{1/2}(\mathbb{R})$, and the following condition holds: there exists $\epsilon > 0$ such that

$$h_1 \equiv 1 - \epsilon \zeta \geq \epsilon \zeta \geq \epsilon \zeta \geq h > 0.$$ 

---

\(^5\)We choose $d_1$ as the reference vertical length. By doing so, we implicitly assume that the two layers of fluid have comparable depth, and therefore that the depth ratio $\delta$ do not approach zero or infinity. Note also that $c_0 \to 0$ as $\rho_2 \to \rho_1$. 

---
A.2 Asymptotic models

Our aim is now to obtain asymptotic models for the system (A.4), using smallness of dimensionless parameters. In the following, we will consider the case of shallow water, namely

$$\mu \ll 1.$$  

The key ingredient is to obtain an expansion of the Dirichlet-Neumann operators, in terms of $\mu$. Replacing the operators by the first order terms of these expansions allow to obtain the desired asymptotic models, which is justified in the sense of consistency (as precisely explained below).

As a second step, we will rewrite the equations in terms of the shear layer-mean velocities. One benefit of such a choice is that it yields to a much better behavior concerning the linear well-posedness, as we discuss at the end of this section.

Our method has been used by Alvarez-Samaniego and Lannes [1] in the case of the water-wave problem (one layer of fluid, with free surface), and lead the authors to a complete rigorous justification of the so-called Green-Naghdi equations [28]. In the case of two layers with a free surface, a shallow water model (first order) and Boussinesq-type models (in the long wave regime) have been derived and justified in the sense of consistency in [8], the analysis below is therefore an extension of their work. Similar models as our Green-Naghdi system have been formally obtained in [16], as well as in [51] (with the additional assumption of $\gamma \approx 1$) and in [20]. Let us also mention the work concerning the case of two layers of fluids with an interface and a free surface: Green-Naghdi-type models have been derived in [3, 4], and justified in the sense of consistency in [22]. One could formally recover our models from [22, (44) and (60)] by forcing the surface to be flat ($\alpha \equiv 0$ using notation therein).

Expansion of the Dirichlet-Neumann operators.

The main ingredients of the following Proposition are given in [8], and we extend their result one order further.

Proposition A.2 (Expansion of the Dirichlet-Neumann operators). Let $s_0 > 1/2$ and $s \geq s_0 + 1/2$. Let $\psi$ be such that $\partial_x \psi \in H^{s+11/2}(\mathbb{R})$, and $\zeta \in H^{s+9/2}(\mathbb{R})$. Let $h_1 = 1 - \epsilon \zeta$ and $h_2 = 1/\delta + \epsilon \zeta$ such that (A.7) is satisfied. Then

$$\frac{1}{\mu} G^{\mu,\psi} - \partial_x(h_1 \partial_x \psi)|_{H^s} \leq \mu C_0,$$

$$\frac{1}{\mu} G^{\mu,\psi} - \partial_x(h_1 \partial_x \psi) - \frac{\mu}{3} \partial_x^2(h_1^3 \partial_x^2 \psi)|_{H^s} \leq \mu^2 C_1,$$

$$\left| H^{\mu,\psi} + \frac{h_1}{h_2} \partial_x \psi - \frac{\mu}{3h_2} \partial_x \left(h_1^3 \partial_x \left(\frac{h_1}{h_2} \partial_x \psi\right) - h_1^4 \partial_x^2 \psi\right)\right|_{H^s} \leq \mu^2 C_1,$$

with $C_j = C(\frac{1}{\beta}, \epsilon_{\text{max}}, \mu_{\text{max}}, \frac{1}{\delta_{\text{max}}}, \delta_{\text{max}}, |k|_{H^{s+7/2+2}}, |\partial_x \psi|_{H^{s+7/2+2}})$. The estimates are uniform with respect to the parameters $\epsilon \in (0, \epsilon_{\text{max}}), \mu \in (0, \mu_{\text{max}}), \delta \in (\delta_{\text{min}}, \delta_{\text{max}})$ and $\gamma \in (0, 1)$.

Proof. As remarked in [8], the operator $G^{\mu,\psi}$ can be deduced from similar operator in the (one layer) water wave case with flat bottom:

$$G^{\mu,\psi} = -\mathcal{G}[-\epsilon \zeta] \psi,$$

where $\mathcal{G}$ is defined in [1, section 3], and estimates (A.8),(A.9) follows from Proposition 3.8 therein.

Estimate (A.10) is given in [8, section 2.2.2], and we obtain (A.11) using the same method, expanding one order further. Let us detail the strategy.

The first step consists in rewriting the scaled Laplace problem (A.6) into a variable-coefficient, boundary-value problem on the flat strip $S := \mathbb{R} \times (-1,0)$ using the diffeomorphism

$$S \rightarrow \Omega_2,$$

$$\sigma : (x,z) \mapsto \sigma(x,z) \equiv (x, (1/\delta + \epsilon \zeta)z + \epsilon \zeta).$$
Now, one can check (see [8] or [22, Proposition 2.1]) that $\phi_2$ solves (A.6) if and only if $\phi_2 \equiv \phi_2 \circ \sigma$ satisfies

$$
(A.12) \quad \begin{cases} 
\nabla_{x,z} \cdot Q^\mu [\epsilon \zeta] \nabla_{x,z} \phi_2 = 0 & \text{in } S \\
\partial_n \phi_2 \big|_{z=0} = G^\mu \psi \\
\partial_n \phi_2 \big|_{z=-1} = 0,
\end{cases}
$$

with

$$
Q^\mu [\epsilon \zeta] \equiv \begin{pmatrix} \mu \partial_z \sigma & -\mu \partial_x \sigma \\
1 & 0 \end{pmatrix} + \mu \begin{pmatrix} 0 & 0 \\
1/\delta + \epsilon \zeta & 0 \end{pmatrix} + \mu \begin{pmatrix} 1/\delta + \epsilon \zeta & -(z+1)\epsilon \partial_x \zeta \\
-(z+1)\epsilon \partial_x \zeta & 1/\delta + \epsilon \zeta \end{pmatrix},
$$

and where $\partial_n \phi_2$ stands for the upward conormal derivative associated to the elliptic operator involved:

$$
\partial_n \phi_2 \big|_{z=0} = \begin{pmatrix} 0 \\
1 \end{pmatrix} \cdot Q \nabla_{x,z} \phi_2 \big|_{z=0}.
$$

The asymptotic expansion of the Dirichlet-Neumann operator $H^\mu \epsilon$ is deduced from the identity

$$
H^\mu \epsilon \psi = \partial_x (\phi_2 \big|_{z=0}).
$$

The second step consists in computing the formal expansion of $\phi_2$ the solution of (A.12), as

$$
\phi_2 = \phi^{(0)} + \mu \phi^{(1)} + \mu^2 \phi^{(2)} + \mu^3 \phi_r,
$$

One can solving (A.12) at each order, using the obvious expansion of $Q^\mu$, as well as the known expansion of the operator $G^\mu \epsilon \psi$. This yields explicit formulas for $\phi^{(i)}$, and the estimate follows from adequate control of the residual $\phi_r$.

At first order, one has

$$
\begin{cases} 
\partial_x \left( \frac{1}{\partial_z + \epsilon \zeta} \partial_z \phi^{(0)} \right) = 0 & \text{in } S \\
\frac{1}{\partial_z + \epsilon \zeta} \partial_z \phi^{(0)} \big|_{z=0} = 0 \\
\frac{1}{\partial_z + \epsilon \zeta} \partial_z \phi^{(0)} \big|_{z=-1} = 0,
\end{cases}
$$

so that

$$
\phi^{(0)}(x,z) = \phi^{(0)}(x),
$$

independent of $z$. At next order, one has (denoting $h_2(x) = 1/\delta + \epsilon \zeta(x)$)

$$
\begin{cases} 
\frac{1}{h_2} \partial_z^2 \phi^{(1)} = -\nabla_{x,z} \cdot \left( -\frac{h_2}{(z+1)\epsilon \partial_x \zeta} \frac{-(z+1)\epsilon \partial_x \zeta}{(z+1)\epsilon \partial_x \zeta} \right) \nabla_{x,z} \phi^{(0)} = -h_2 \partial_z^2 \phi^{(0)} & \text{in } S \\
\frac{1}{h_2} \partial_z \phi^{(1)} \big|_{z=0} = (\epsilon \partial_x \zeta) (\partial_x \phi^{(0)}) + \partial_x (h_1 \partial_x \psi) \\
\frac{1}{h_2} \partial_z \phi^{(1)} \big|_{z=-1} = 0,
\end{cases}
$$

where we used that $\phi^{(0)}$ is independent of $z$. The above system is a second order ordinary differential equation, which is solvable under the condition

$$
\partial_x (h_1 \partial_x \psi) = -\partial_x (h_2 \partial_x \phi^{(0)}),
$$

and whose solution is then

$$
\phi^{(1)}(x,z) = -\frac{1}{2} (z+1)^2 h_2^2 \partial_x^2 \phi^{(0)} + \phi^{(1)}(x),
$$

with $\phi^{(1)}(x)$ being a function independent of $z$, to be determined later. Note that since the horizontal dimension is one,\(^6\) and using the fact that the fluids are at rest at infinity, integrating the compatibility conditions yields

$$
\partial_x \phi^{(0)} = -\frac{h_1}{h_2} \partial_x \psi.
$$

\(^6\)In the 2d case, one should introduce the non-local operator of orthogonal projection onto the gradient vector fields, as in [8], to ensure that the right hand side is a gradient.
Let us turn to the next order. One has
\[ \begin{cases} 
\frac{1}{h_2^2} \partial_x^2 \phi^{(2)} = -h_2 \partial_x^2 \phi^{(1)} + (z + 1)^2 F(x) & \text{in } S \\
\frac{1}{h_2} \partial_x \phi^{(1)} |_{z=0} = G(x) + (\partial_z h_2)(\partial_x \phi^{(1)}) + \frac{1}{3} \partial_x^2 (h_1^3 \partial_x^2 \psi) & \frac{1}{h_2} \partial_x \phi^{(0)} |_{z=-1} = 0,
\end{cases} \]
with
\[ F(x) = \frac{h_2^3}{2} \partial_x^2 \phi^{(0)} \quad \text{and} \quad G(x) = -\frac{1}{2} \partial_x (h_2^3 \partial_x \phi^{(0)}). \]
Solving the first identity with boundary condition \( \partial_z \phi^{(1)} |_{z=-1} = 0 \) yields
\[ \phi^{(2)}(x, z) = -\frac{(z + 1)^2}{2} h_2^2 \partial_x^2 \phi^{(1)} F(x) + \frac{(z + 1)^4}{12} h_2 F(x) + \phi_0^{(2)}(x), \]
with \( \phi_0^{(2)}(x) \) independent of \( z \) (and which can be set to zero for simplicity), and solving the boundary condition at \( z = 0 \) yields the compatibility condition:
\[ -h_2 \partial_x^2 \phi^{(1)} + \frac{1}{3} F(x) = G(x) + (\partial_z h_2)(\partial_x \phi^{(1)}) + \frac{1}{3} \partial_x^2 (h_1^3 \partial_x^2 \psi), \]
or, equivalently,
\[ \partial_x (h_2 \partial_x \phi^{(1)}) = \frac{1}{6} \partial_x (h_2^3 \partial_x^3 \phi^{(0)}) - \frac{1}{3} \partial_x^2 (h_1^3 \partial_x^2 \psi). \]
Finally, integrating this identity and using the expression of \( \partial_x \phi^{(0)} \) obtained above, one deduces
\[ h_2 \partial_x \phi^{(1)} = -\frac{1}{3} \partial_x (h_1^3 \partial_x^2 \psi) - \frac{1}{6} h_2^3 \partial_x^2 (h_1 \partial_x \psi). \]

The final step is as follows. Let us define
\[ \phi_{2, \text{app}} \equiv \phi^{(0)} + \mu \phi^{(1)}, \]
where \( \phi^{(0)} \) and \( \phi^{(1)} \) have been obtained by the above calculations. Note that
\[ H_{\text{app}} = \partial_x (\phi_{2, \text{app}} |_{z=0}) = \partial_x \phi^{(0)} + \mu \partial_x \left( -\frac{1}{2} h_2 \partial_x^2 \phi^{(0)} + \phi_0^{(1)}(x) \right) = -\frac{h_1}{h_2} \partial_x \psi + \frac{\mu}{3 h_2} \partial_x \left( -h_1^3 \partial_x^2 \psi + h_2^3 \partial_x (h_1 \partial_x \psi) \right), \]
which is exactly the expansion in (A.11). Therefore, the result follows from an adequate estimate on
\[ u \equiv \partial_x (\phi_{2, \text{app}} |_{z=0}) - \partial_x (\phi_{2, \text{app}} |_{z=0}). \]

Let us first note that one has straightforwardly
\[ |\phi^{(2)}|_{H^s} \leq C \left( \frac{1}{h}, \epsilon_{\text{max}}, \mu_{\text{max}}, |\psi|_{H^{s+9/2}}, |\partial_x \psi|_{H^{s+11/2}} \right). \]
Then, using previous calculations, \( v \equiv \phi_2 - \phi_{2, \text{app}} - \mu^2 \phi^{(2)} \) satisfies the system
\[ \begin{cases} 
\nabla \cdot Q^n [\epsilon \nabla \cdot \nabla v = \mu^2 \nabla \cdot \mathbf{h} & \text{in } S \\
\partial_n \phi_2 |_{z=0} = V + \mu^2 \left( \frac{0}{1} \right) \cdot \mathbf{h} |_{z=0} & \partial_n \phi_2 |_{z=-1} = \mu^2 \left( \frac{0}{1} \right) \cdot \mathbf{h} |_{z=-1},
\end{cases} \]
with \( \mathbf{h} = Q^n \nabla \cdot \phi^{(2)}, \) and \( V = G^{n+1} + \mu \partial_x (h_1 \partial_x \psi) + \mu^2 \frac{1}{2} \partial_x^2 (h_1^3 \partial_x^2 \psi) \), so that (A.9) yields
\[ |V|_{H^s} \leq \mu^3 C \left( \frac{1}{h}, \epsilon_{\text{max}}, \mu_{\text{max}}, |\psi|_{H^{s+9/2}}, |\partial_x \psi|_{H^{s+11/2}} \right). \]
One can now apply Proposition 3 of [8], after straightforward adjustments, and deduce
\[ |\partial_x v |_{z=0} |_{H^s} \leq \mu^2 C \left( \frac{1}{h}, \epsilon_{\text{max}}, \mu_{\text{max}}, |\psi|_{H^{s+9/2}}, |\partial_x \psi|_{H^{s+11/2}} \right). \]
The Proposition is proved. \( \square \)
Let us now plug the expansions of Proposition A.2 into the full Euler system (A.4), and withdraw $O(\mu^2)$ terms. One obtains

\[ \begin{aligned}
\partial_t \zeta - \partial_x (h_1 \partial_x \psi) - \frac{1}{3} \partial_x^2 (h_1^3 \partial_x^2 \psi) &= 0,
\end{aligned} \]

(A.14)

\[ \partial_t \left( - \frac{h_1 + \gamma h_2}{h_2} \partial_x \psi + \mu \frac{1}{h_2} \partial_x (\mathcal{P} \partial_x \psi) \right) + (\gamma + \delta) \partial_x \zeta 
+ \frac{\epsilon}{2} \partial_x \left( | - \frac{h_1}{h_2} \partial_x \psi + \mu \frac{1}{h_2} \partial_x (\mathcal{P} \partial_x \psi) |^2 - \gamma |\partial_x \zeta|^2 \right)
= \mu \epsilon \partial_x (\mathcal{N}[h_1, h_2] \partial_x \psi), \]

where we denote $\mathcal{P} \partial_x \psi = \mathcal{P}[h_1, h_2] \partial_x \psi$, and with the following operators

\[ \mathcal{P}[h_1, h_2] V = \frac{1}{3} \left( h_2^3 \partial_x \left( \frac{h_1}{h_2} V - h_1^3 \partial_x V \right) \right), \]

\[ \mathcal{N}[h_1, h_2] V = \frac{1}{2} \left( \partial_x (h_1 V) - \epsilon (\partial_x \zeta) \frac{h_1}{h_2} V \right)^2 - \gamma (h_1 \partial_x^2 \psi)^2. \]

Our model is justified by the following consistency result.

**Proposition A.3.** Let $U \equiv (\zeta, \psi)$ be a solution of the full Euler system (A.4) such that (A.7) is satisfied, and $\zeta \in W^1((0, T); H^{s+1/2})$, $\partial_x \psi \in W^1((0, T); H^{s+13/2})$ with $s \geq s_0 + 1/2, s_0 > 1/2$, and where $W^1((0, T); H^{s+1}(\mathbb{R})$ denotes the space of the functions $f(t, x)$ such that

\[ \|f\|_{W^1((0, T); H^{s+1}(\mathbb{R})}) \equiv \|f\|_{L^\infty((0, T); H^{s+1}(\mathbb{R}))} + \|\partial_t f\|_{L^\infty((0, T); H^{s}(\mathbb{R}))} < \infty. \]

Then $U$ satisfies (A.14), up to a remainder $R$, bounded by

\[ \|R\|_{L^\infty((0, T); H^s)} \leq \mu^2 C, \]

with $C = C(s_0 - 1/2, \frac{1}{3}, \epsilon_{\max}, \mu_{\max}, \frac{1}{11}, \delta_{\max}, \|\zeta\|_{W^1((0, T); H^{s+1/2})}, \|\partial_x \psi\|_{W^1((0, T); H^{s+13/2})})$, uniformly with respect to the parameters $\epsilon \in (0, \epsilon_{\max}), \mu \in (0, \mu_{\max}), \delta \in (\delta_{\min}, \delta_{\max})$ and $\gamma \in (0, 1)$.

**Proof.** When plugging $U$ into (A.14), and after some straightforward computation, one can clearly estimate the remainder using the expansions of the Dirichlet-Neumann operators in Proposition A.2. The only non-trivial term comes from $\partial_t \left( H^{\mu, \epsilon} \psi - \gamma \partial_x \psi \right)$, which requires the corresponding expansion of $\partial_t H^{\mu, \epsilon} \psi$. This can be obtained as in Proposition A.2, using the elliptic problems satisfied by the time derivative of the potentials. The expansion follows in the same way, provided that $\zeta \in W^1((0, T); H^{s+1/2}), \partial_x \psi \in W^1((0, T); H^{s+13/2})$. See the proof of [22, Proposition 2.12] for more details.

In [8], the authors use the shear velocity as for the velocity variable:

\[ v \equiv \partial_x \left( (\phi_2 - \gamma \phi_1)|_{z = \epsilon \zeta} \right) = H^{\mu, \epsilon} \psi - \gamma \partial_x \psi. \]

The expansion of $H^{\mu, \epsilon}$ in (A.11) allows to obtain approximately $\partial_x \psi$ as a function of $v$:

\[ \partial_x \psi = - \frac{h_2}{h_1 + \gamma h_2} v + \mu \frac{1}{h_1 + \gamma h_2} \partial_x \left( \mathcal{P}[h_1, h_2] \left( - \frac{h_2}{h_1 + \gamma h_2} v \right) \right) + O(\mu^2). \]

(A.16)

Here and in the following, we use the notation $O(\cdot)$ for estimates as in Proposition A.2.

Plugging (A.15) into (A.14), and again withdrawing $O(\mu^2)$ terms, yields

\[ \begin{aligned}
\partial_t \zeta + \partial_x \left( \frac{h_1 h_2}{h_1 + \gamma h_2} v \right) - \mu \partial_x \left( \mathcal{Q}[h_1, h_2] v \right) &= 0,
\end{aligned} \]

(A.17)

\[ \partial_t v + (\gamma + \delta) \partial_x \zeta + \frac{\epsilon}{2} \partial_x \left( \frac{h_1^2 - \gamma h_2^2}{(h_1 + \gamma h_2)^2} |v|^2 \right) = \mu \epsilon \partial_x (\mathcal{R}[h_1, h_2] v), \]
with the following operators:

\[ Q[h_1, h_2]V = \frac{-1}{3(h_1 + \gamma h_2)} \left( h_1 \partial_x \left( h_2^3 \partial_x \left( \frac{h_1}{h_1 + \gamma h_2} V \right) \right) + \gamma h_2 \partial_x \left( h_1^3 \partial_x \left( \frac{h_2}{h_1 + \gamma h_2} V \right) \right) \right), \]

\[ R[h_1, h_2]V = \frac{1}{2} \left( h_2 \partial_x \left( \frac{h_1}{h_1 + \gamma h_2} V \right)^2 - \gamma \left( h_1 \partial_x \left( \frac{h_2}{h_1 + \gamma h_2} V \right) \right)^2 \right) \]

\[ + \gamma \frac{h_1 + h_2}{3(h_1 + \gamma h_2)^2} V \partial_x \left( h_1 \frac{V}{h_1 + \gamma h_2} \right) - h_1^3 \partial_x \left( \frac{h_2}{h_1 + \gamma h_2} V \right) \].

System (A.17) is justified as an asymptotic model for the full Euler system (A.4), by a consistency result, with the same precision as (A.14).

**Proposition A.4.** Let \( U \equiv (\zeta, \psi) \) be a solution of the full Euler system (A.4) such that (A.7) is satisfied, and \( \zeta \in W^1([0,T]; H^{s+1/2}) \), \( \partial_x \psi \in W^1([0,T]; H^{s+1/2}) \) with \( s \geq s_0 + 1/2 \), \( s_0 > 1/2 \). Define \( v \) as in (A.15). Then \((\zeta, v)\) satisfies (A.17), up to a remainder \( R \), bounded by

\[ \|R\|_{L^\infty([0,T]; H)} \leq \mu^2 C, \]

with \( C = C \left( \frac{1}{s_0 - 1/2}, \frac{1}{s}, \epsilon_{\text{max}}, \mu_{\text{max}}, \frac{1}{s_{\text{min}}}, \delta_{\text{max}}, \|\zeta\|_{W^1([0,T]; H^{s+1/2})}, \|\partial_x \psi\|_{W^1([0,T]; H^{s+1/2})} \right) \), uniformly with respect to the parameters \( \epsilon \in (0, \epsilon_{\text{max}}), \mu \in (0, \mu_{\text{max}}), \delta \in (\delta_{\text{min}}, \delta_{\text{max}}) \) and \( \gamma \in (0,1) \).

The proof of Proposition A.4 is identical to the proof of Proposition A.3, once one obtains a rigorous statement of (A.16) in \( W^1([0,T]; H^{s+1}) \) norm, thus we omit it.

**Using layer-mean velocities.**

As we have seen, several different velocity variables are natural when expressing the Green-Naghdi equations. In the following, we choose to use, as in [15, 16] for example, the *shear layer-mean velocity*, defined by

\[ \bar{v} \equiv \bar{u}_2 - \gamma \bar{u}_1, \]

where \( \bar{u}_1, \bar{u}_2 \) are the layer-mean velocities integrated across the vertical layer in each fluid:

\[ \bar{u}_1(t,x) = \frac{1}{h_1(t,x)} \int_{\zeta(t,x)-1}^{\zeta(t,x)} \partial_z \phi_1(t,x,z) \, dz, \quad \text{and} \quad \bar{u}_2(t,x) = \frac{1}{h_2(t,x)} \int_{\zeta(t,x)-\frac{1}{2}}^{\zeta(t,x)} \partial_z \phi_2(t,x,z) \, dz. \]

We see two main benefits for such a choice. First, the equation describing the evolution of the deformation of the interface is an exact equation, and not an \( \mathcal{O}(\mu^2) \) approximation. What is more, the system obtained using layer-mean velocities have a nicer behavior as for the linear well-posedness. These two facts shall be discussed in more details below.

When integrating Laplace’s equation in (A.1) against a test function \( \tilde{\varphi}(x,z) = \varphi(x) \) on the lower domain \( \Omega_2 \), and using boundary kinematic equations, one has

\[ 0 = \int_{\Omega_2} \varphi \left( \mu \partial_z^2 + \partial_x^2 \right) \phi_2 \, dx \, dz = -\int_{\Omega_2} \nabla_{x,z}^\mu \tilde{\varphi} \cdot \nabla_{x,z}^\mu \phi_2 \, dx \, dz + \int_{\Gamma} \varphi \partial_n \phi_2 - \int_{\Gamma_x} \varphi \partial_z \phi_2 \]

\[ = -\int_{\mathbb{R}} dx \sqrt{\mu} \partial_z \varphi \int_{-1/\delta}^{\zeta} \sqrt{\mu} \partial_z \phi_2(x,z) \, dz - \int_{\mathbb{R}} dx G^{\mu,c} \psi, \]

where \( \nabla_{x,z} = (\sqrt{\mu} \partial_z, \partial_x)^T \), so that one deduces that for any \( x \in \mathbb{R} \),

\[ -\partial_z \left( h_2 \bar{u}_2 \right) = \frac{1}{\mu} G^{\mu,c} \psi. \]

In the same way, one has

\[ 0 = \int_{\Omega_1} \varphi \left( \mu \partial_z^2 + \partial_x^2 \right) \phi_1 \, dx \, dz = -\int_{\Omega_1} \nabla_{x,z}^\mu \tilde{\varphi} \cdot \nabla_{x,z}^\mu \phi_1 \, dx \, dz - \int_{\Gamma} \varphi \partial_n \phi_1 + \int_{\Gamma_x} \varphi \partial_z \phi_1 \]

\[ = -\int_{\mathbb{R}} dx \partial_x \sqrt{\mu} \varphi \int_{\zeta}^{1} \sqrt{\mu} \partial_z \phi_1(x,z) \, dz + \int_{\mathbb{R}} dx G^{\mu,c} \psi, \]
so that
\[ \partial_x (h_1 u_1) = \frac{1}{\mu} G^{\mu, \epsilon} \psi. \]

One deduces from the two above identities, imposing zero boundary conditions at infinity (the fluid is at rest at infinity),
\[ h_2 \bar{u}_2 = -h_1 \bar{u}_1, \quad \text{and} \quad \bar{v} = \frac{h_1 + \gamma h_2}{h_1} \bar{u}_2 = -\frac{h_1 + \gamma h_2}{h_2} \bar{u}_1. \]

It follows that the first equation in (A.4) becomes
\[ \partial_t \zeta = \frac{1}{\mu} G^{\mu, \epsilon} \psi = -\partial_x \left( \frac{h_1 h_2}{h_1 + \gamma h_2} \bar{v} \right). \]

Let us retell that this identity is exact, as opposed to the \( O(\mu^2) \) approximations in previous asymptotic models.

One also deduces an expansion of \( \bar{v} \) in terms of \( v \), using Proposition A.2 (or, equivalently, identifying the above identity with the first line of (A.17)). It follows
\[ v = \bar{v} + \frac{h_1 + \gamma h_2}{h_1 h_2} Q[h_1, h_2] \bar{v} + O(\mu^2). \]

System (A.17) therefore becomes
\[
\begin{cases}
\partial_t \zeta + \partial_x \left( \frac{h_1 h_2}{h_1 + \gamma h_2} \bar{v} \right) = 0, \\
\partial_t \left( \bar{v} + \frac{\mu}{h_1 h_2} \bar{Q}[h_1, h_2] \bar{v} \right) + (\gamma + \delta) \partial_x \zeta + \frac{\epsilon}{2} \partial_x \left( \frac{h_1^2 - \gamma h_2^2}{(h_1 + \gamma h_2)^2} |\bar{v}|^2 \right) = \mu \epsilon \partial_x (\bar{R}[h_1, h_2] \bar{v}),
\end{cases}
\]
with the following operators:
\[
\begin{align*}
\bar{Q}[h_1, h_2] V &\equiv -\frac{1}{3 h_1 h_2} \left( h_1 \partial_x \left( h_2^3 \partial_x \left( \frac{h_1 V}{h_1 + \gamma h_2} \right) \right) + \gamma h_2 \partial_x \left( h_1^3 \partial_x \left( \frac{h_2 V}{h_1 + \gamma h_2} \right) \right) \right), \\
\bar{R}[h_1, h_2] V &\equiv \frac{1}{2} \left( \left( h_2 \partial_x \left( \frac{h_1 V}{h_1 + \gamma h_2} \right) \right)^2 - \gamma \left( h_1 \partial_x \left( \frac{h_2 V}{h_1 + \gamma h_2} \right) \right)^2 \right) \\
&\quad + \frac{1}{3} \frac{V}{h_1 + \gamma h_2} \left( h_2^3 \partial_x \left( \frac{h_1 V}{h_1 + \gamma h_2} \right) \right) - \frac{h_2}{h_1} \partial_x \left( h_1^3 \partial_x \left( \frac{h_2 V}{h_1 + \gamma h_2} \right) \right).
\end{align*}
\]

Again, one obtains the consistency of the full Euler system towards our Green-Naghdi model, after several technical but straightforward computations.

**Proposition A.5.** Let \( U \equiv (\zeta, \psi) \) be a solution of the full Euler system (A.4) such that (A.7) is satisfied, and \( \zeta \in W^1((0, T); H^{s+1/2}) \), \( \partial_x \psi \in W^1((0, T); H^{s+13/2}) \) with \( s \geq s_0 + 1/2 \), \( s_0 > 1/2 \). Define \( \bar{v} \) as in (A.18) or, equivalently, by
\[ \frac{1}{\mu} G^{\mu, \epsilon} \psi = -\partial_x \left( \frac{h_1 h_2}{h_1 + \gamma h_2} \bar{v} \right). \]

Then \( (\zeta, \bar{v}) \) satisfies (A.19), up to a remainder \( R \), bounded by
\[ \| R \|_{L^\infty([0, T); H^s)} \leq \mu^2 C, \]
with \( C = C(s_0^{-1/2}, \frac{1}{\pi}, \epsilon_{\max}, \mu_{\max}, \frac{1}{\delta_{\max}}, \delta_{\max}, \| \zeta \|_{W^1([0, T); H^{s+1/2})}, \| \partial_x \psi \|_{W^1([0, T); H^{s+13/2})}), \) uniformly with respect to the parameters \( \epsilon \in (0, \epsilon_{\max}), \mu \in (0, \mu_{\max}), \delta \in (\delta_{\min}, \delta_{\max}) \) and \( \gamma \in (0, 1) \).
Remark A.6. Proposition A.5 is one of the steps leading to a stronger justification of system (A.19) (or any consistent model with the same precision), in the sense of convergence, as it has been achieved in the water wave case in [1]. As described in [43, Section 6.3], the procedure would follow from

- (Consistency) This is Proposition A.5.
- (Convergence) One proves that the consistent solutions of the full Euler system remain close to the exact solutions of (A.19) (and in particular, one shall prove that the asymptotic model is well-posed). Such a result is assumed in Hypothesis 2.12, and is currently under investigation by the author and collaborators.
- (Existence) One proves that smooth, uniformly bounded family of solutions to the full Euler system exists. This is ensured by Theorem 5.8 in [43], provided that a small surface tension is added, and that an additional stability criterion is satisfied.

See Theorem 6.1 in [43] for the application of such procedure on the so-called “shallow-water/shallow-water” asymptotic model (which corresponds to (A.19), when withdrawing $O(\mu)$ terms).

Linear dispersion relations. The linearized system from (A.17) is

$$\begin{cases}
\partial_t \zeta + \frac{1}{\gamma + \delta} \partial_x v + \mu \frac{1 + \gamma \delta}{3(\gamma + \delta)} \partial_x^2 v = 0, \\
\partial_t v + (\gamma + \delta) \partial_x \zeta = 0,
\end{cases}$$

Let us look for solutions of the form $\zeta = \zeta^0 e^{i(kx - \omega t)}$, $v = v^0 e^{i(kx - \omega t)}$. This leads to the algebraic system

$$\begin{cases}
-i\omega \zeta^0 + \frac{i k}{\gamma + \delta} v^0 - \mu ik^2 \frac{1 + \gamma \delta}{3(\gamma + \delta)} v^0 = 0, \\
-i\omega v^0 + ik(\gamma + \delta) \zeta^0 = 0,
\end{cases}$$

which yields the dispersion relation (with $\omega v^0 = k(\gamma + \delta) \zeta^0$)

$$\omega^2 = k^2 - \mu k^4 \frac{1 + \gamma \delta}{3(\gamma + \delta)}.$$

This equation does not have any real solution $\omega(k)$ if $\mu k^2 \frac{1 + \gamma \delta}{3(\gamma + \delta)} > 1$, thus the system (A.17) is linearly ill-posed.

The linearized system from (A.19) is

$$\begin{cases}
\partial_t \zeta + \frac{1}{\gamma + \delta} \partial_x \bar{v} = 0, \\
\partial_t \left( \bar{v} - \mu \frac{1 + \gamma \delta}{3(\gamma + \delta)} \partial_x^2 \bar{v} \right) + (\gamma + \delta) \partial_x \zeta = 0,
\end{cases}$$

Same calculations as above yield the algebraic system

$$\begin{cases}
-i\omega \bar{v}^0 + \frac{i k}{\gamma + \delta} \zeta^0 = 0, \\
-i\omega \left( \bar{v}^0 + \mu k^2 \frac{1 + \gamma \delta}{3(\gamma + \delta)} \zeta^0 \right) + ik(\gamma + \delta) \zeta^0 = 0,
\end{cases}$$

so that $\omega \zeta^0 = \frac{k}{\gamma + \delta} \zeta^0$ and the dispersion relation is

$$\omega^2 \left( 1 + \mu k^2 \frac{1 + \gamma \delta}{3(\gamma + \delta)} \right) = k^2.$$

This equations has always solutions: $\omega_\pm(k) = \pm \frac{k}{\sqrt{1 + \mu k^2 \frac{1 + \gamma \delta}{3(\gamma + \delta)}}}$. Thus the system using layer-mean velocity variables (A.19) is linearly well-posed.
B Proof of Proposition 2.5

Here, we study the well-posedness as well as the persistence of spatial decay in Sobolev norms, of the following equation:

\[(1 - \mu \beta \partial^2_x)\partial_t u + \epsilon \alpha_1 u \partial_x u + \epsilon^2 \alpha_2 u^2 \partial_x u + \epsilon^3 \alpha_3 u^3 \partial_x u + \mu \nu \partial^2_x u + \mu \epsilon \partial_x (\kappa u \partial_x^2 u + \iota (\partial_x u)^2) = 0.\]

More precisely, we prove the assertions of Proposition 2.5.

The existence and uniqueness of \( u \in C^0([0,T/\epsilon]; H^{s+1}_\mu) \cap C^1([0,T/\epsilon]; H^s_{\mu}) \) such that \( u \) satisfies (B.1) and initial condition \( u|_{t=0} = u^0 \in H^s \) \( s \geq s_0 > 3/2 \) has been obtained in [19, Proposition 4] (where the authors used slightly different parameters).

The proof is based on an iterative scheme, which relies heavily on the following energy estimate:

\[ \frac{1}{1 + \beta - 1} | \partial_t u |_{H^{s+1}_{\mu}} \leq \left( |u(t, \cdot)|^2_{H^s, \mu} + \beta |u(t, \cdot)|^2_{H^{s+1}, \mu} \right)^{1/2} \leq C M, |u_0|_{H^{s+1}_{\mu}}, \]

which is proved to be valid for \( t \leq T_e \equiv C \left( \frac{1}{s_0 - 3/2}, M, |u_0|_{H^{s+1}_{\mu}} / \epsilon \right) \). One proves in the same way

\[ \frac{1}{1 + \beta - 1} | \partial_t u |_{H^s_{\mu}} \leq \tilde{E}^{s-1}(\partial_t u)(t) \leq C \left( \frac{1}{s_0 - 3/2}, M, E^s(u) \right) \leq C \left( \frac{1}{s_0 - 3/2}, M, |u_0|_{H^{s+1}_{\mu}} \right), \]

so that the estimate (2.17) follows.

In order to obtain the estimates in the weighted Sobolev norms, we introduce the function \( v_n = x^n u \), where \( u \) satisfies (B.1). We will prove that the following estimate holds for any \( n, k \in \mathbb{N} \), and for \( 0 \leq t \leq T_{e, \mu} \equiv C \left( s_0 - 3/2, \sum_{j=0}^{n} \frac{|x^{n-j} u^0|_{H^{s+2+j}_{\mu}}}{\mu} \right) \times \min(1/\epsilon, 1/\mu) \) by induction on \( n \in \mathbb{N} \).

(B.2) \[ \| x^n \partial_k u \|_{L^\infty([0,T]; H^s_{\mu})} + \| x^n \partial_k u \|_{L^\infty([0,T]; H^{s+1}_{\mu})} \leq C \left( M, n, k, \sum_{j=0}^{n} |x^{n-j} u^0|_{H^{s+2+j}_{\mu}} \right). \]

Proposition 2.5 follows straightforwardly.

The case \( n = 1 \). One can easily check that \( v_1 \equiv x u \) satisfies the equation

\[ (1 - \mu \beta \partial^2_x)\partial_t v_1 + \epsilon \alpha_1 v_1 \partial_x v_1 + \epsilon^2 \alpha_2 v_1 u \partial_x u + \epsilon^3 \alpha_3 v_1 u^2 \partial_x u + \mu \nu \partial^2_x v_1 + \mu \epsilon \partial_x (\kappa v_1 \partial_x^2 u + \iota (\partial_x u)(\partial_x v_1)) = R[u], \]

with

\[ R[u] \equiv -2 \mu \beta \partial_x \partial_t u + 3 \mu \nu \partial_x^2 u + \mu \epsilon \left( (\kappa + \iota) u \partial_x^2 u + 2 \iota (\partial_x u)^2 \right). \]

When taking the \((L^2, -)\)inner product of (B.3) with \( \Lambda^s v_1 \), one obtains (using that the operator \( \Lambda^s \) is symmetric for the \((L^2, -)\)inner product, and \( \partial_x \) is anti-symmetric)

\[ \frac{1}{2} \frac{d}{dt} \left( E^s(v_1) \right)^2 + \Lambda^s (\epsilon \alpha_1 v_1 \partial_x u + \epsilon^2 \alpha_2 v_1 u \partial_x u + \epsilon^3 \alpha_3 v_1 u^2 \partial_x u) , \Lambda^s v_1 \]

\[ - \mu \epsilon \left( (\kappa v_1 \partial_x^2 u + \iota (\partial_x v_1)(\partial_x u)) , \Lambda^s \partial_x v_1 \right) = (\Lambda^s R[u], \Lambda^s v_1). \]

Now, we use that \( u \) is uniformly bounded through (2.17) for \( t \leq T_e \).

- Using Cauchy-Schwarz inequality, and Moser estimates: \( |f|_{H^s} \leq C \left( \frac{1}{s_0 - 1/2} \right) |f|_{H^s} \) for \( s \geq s_0 > 1/2 \), one has

\[ \left| (\Lambda^s (\epsilon \alpha_1 v_1 \partial_x u + \epsilon^2 \alpha_2 v_1 u \partial_x u + \epsilon^3 \alpha_3 v_1 u^2 \partial_x u)) , \Lambda^s v_1 \right| \]

\[ \leq \left| v_1 \right|_{H^s} C \left( \frac{1}{s_0 - 1/2} \right) \left| u \right|_{H^{s+1}_{\mu}} \leq C \left( \frac{1}{s_0 - 3/2}, M, |u_0|_{H^{s+1}_{\mu}} \right) (E^s(v_1))^2. \]
In the same way, and using the definition $(E^s(\cdot))^2 = |\cdot|_{H^s}^2 + \beta \mu |\cdot|_{H^{s+1}}^2 \geq \frac{1}{c(M)} |\cdot|_{H^{s+1}}^2$,
$$
\left|\left(\Lambda^s (\kappa v_1 \partial_x^2 u + \epsilon (\partial_x v_1) (\partial_x u)) , \Lambda^s \partial_x v_1 \right)\right| \leq C |v_1|_{H^{s+1}} \left(|v_1|_{H^s} |u|_{H^{s+2}} + |v_1|_{H^{s+1}} |u|_{H^{s+1}}\right) \\
\leq \frac{1}{\mu} C \left(\frac{1}{s_0 - 3/2}, M, |u^0|_{H^{s+2}}\right)(E^s(v_1))^2.
$$

Finally, one can check
$$
\left|\left(\Lambda^s R[u] , \Lambda^s v_1 \right)\right| \leq \mu |v_1|_{H^s} C \left(\frac{1}{s_0 - 1/2}, |u|_{H^{s+2}} + |\partial_x u|_{H^{s+1}}\right) \leq \mu C \left(\frac{1}{s_0 - 3/2}, M, |u^0|_{H^{s+3}}\right) E^s(v_1).
$$

Altogether, one obtains the following differential inequality, valid for all $t \leq T$: 
$$
\frac{1}{2} \frac{d}{dt} (E^s(v_1))^2 \leq \epsilon C \left(\frac{1}{s_0 - 3/2}, M, |u^0|_{H^{s+2}}\right)(E^s(v_1))^2 + \mu C \left(\frac{1}{s_0 - 3/2}, M, |u^0|_{H^{s+3}}\right) E^s(v_1).
$$

Gronwall-Bihari's inequality (see [58] for example) allows to deduce:
$$
E^s(v_1)(t) \leq E^s(v_1)|_{t=0} + \frac{\mu}{\epsilon} C \left(\frac{1}{s_0 - 3/2}, M, |u^0|_{H^{s+3}}\right) \left(\exp \left(\frac{1}{c(M)} \left|u^0\right|_{H^{s+3}} et\right) - 1\right).
$$

When restricting $t$ to $t \leq T_{c, \mu} \equiv C \left(\frac{1}{s_0 - 3/2}, M, |u^0|_{H^{s+3}}\right) \times \min(1/\epsilon, 1/\mu)$, it follows that for any $s \geq s_0 > 3/2,$
$$
|xu|_{H^{s+1}} \leq |xu|_{H^{s+1}} + C \left(\frac{1}{s_0 - 3/2}, M, |u^0|_{H^{s+3}}\right).
$$

In order to control the time derivative of $v_1$, we take the $(L^2)$-inner product of (B.3) with $\Lambda^{2s-2}(\partial_x v_1)$. Estimating each term as above, one obtains
$$
\left(\Lambda^{s-1}(\partial_t v_1)\right)^2 \leq \epsilon |v_1|_{H^{s-1}} \left|\partial_t v_1\right|_{H^s} C \left(\frac{1}{s_0 - 1/2}, |u|_{H^{s+2}}\right) + \mu \nu \left|\Lambda^{s-1} \partial_x^2 v \right|_{H^{s+1}} \left|\Lambda^{s-1} \partial_x \partial_t v_1\right| + \mu \nu C \left|\partial_t v_1\right|_{H^s} \left(\left|v_1\right|_{H^{s-1}} \left|u\right|_{H^{s+2}} + \left|v_1\right|_{H^{s+1}} \left|u\right|_{H^{s+2}}\right) + \left|\Lambda^{s-1} R[u], \Lambda^{s-1} \partial_x v_1\right| \\
\leq \left(E^{s-1}(\partial_t v_1)\right) \left(C \left(\frac{1}{s_0 - 3/2}, M, |u^0|_{H^{s+2}}\right) E^s(v_1) + \mu C \left(\frac{1}{s_0 - 3/2}, M, |u^0|_{H^{s+2}}\right)\right).
$$

Estimate (B.2) thus follows for $n = 1$ and $k = 0$.

We know turn to $x \partial_x^k u$, for $k \in \mathbb{N}$, $k \geq 1$. Note that
$$
x \partial_x^k u = \partial_x^k (x u) - k \partial_x^{k-1} u,
$$

so that
$$
|x \partial_x^k u|_{H^{s+1}} \leq |xu|_{H^{s+k+1}} + k |u|_{H^{s+k}}
$$

One deduces, for $0 \leq t \leq T(M) \min(1/\epsilon, 1/\mu),$
$$
|x \partial_t u|_{H^{s+1}} \leq |xu|_{H^{s+k+1}} + C_1 \left(\frac{1}{s_0 - 3/2}, M, |u^0|_{H^{s+k+1}}\right)
$$

In the same way, one has $x \partial_x^k \partial_t u = \partial_x^k (x \partial_t u) - k \partial_x^{k-1} \partial_t u$, so that estimate (B.2) hold for $n = 1$ and any $k \in \mathbb{N}$.

The case $n \geq 2$. Let us assume that (B.2) holds for any $k \in \mathbb{N}$ and $n \leq m - 1$ (with $m \geq 2$). One can easily check that $v_m$ satisfies the equation

(B.4) 
$$
(1 - \mu \beta \partial_x^2) \partial_t v_m + \epsilon \alpha_1 v_m \partial_x u + \epsilon^2 \alpha_2 v_m u \partial_x u + \epsilon^3 \alpha_3 v_m u^2 \partial_x u + \mu \nu \partial_x^2 v_m + \mu \nu \partial_x (\kappa v_m \partial_x^2 u + \epsilon (\partial_x v_m) (\partial_x u)) = R_m[u],
$$
with
\[ R_m[u] \equiv -2 \mu \beta m x^{m-1} \partial_x \partial_t u - 2 \mu \beta m (m-1) x^{m-2} \partial_t u + 3 \mu \nu m x^{m-1} \partial_x^2 u + 3 \mu \nu m (m-1) x^{m-2} \partial_x u + \mu m (m-1) (m-2) x^{m-3} u + \mu m x^{m-1} ((\kappa + \iota) u \partial_x^2 u + 2 \iota (\partial_x u)^2) + \mu m (m-1) x^{m-2} \iota u \partial_x u. \]

Taking the \((L^2-)\)inner product of (B.4) with \(\Lambda^{2s}(v_m)\), one obtains as above
\[
\frac{1}{2} \frac{d}{dt} (E^s(v_m))^2 \leq C (\frac{1}{s_0 - 3/2}, M, t) |u^0|_{H^{s_0+2}} (E^s(v_m))^2 + \left| \left( \Lambda^s R_m[u], \Lambda^s v_m \right) \right|.
\]

Now, one can easily check
\[
\left| \left( \Lambda^s R_m[u], \Lambda^s v_m \right) \right| \leq \mu C (\frac{1}{s_0 - 1/2}, M, t) |v_m|_{H^2} C \left( \left| x^{m-1} \partial_x \partial_t u \right|_{H^2}, \left| x^{m-2} \partial_t u \right|_{H^2}, \left| x^{m-2} \partial_x u \right|_{H^2}, \left| x^{m-3} u \right|_{H^2} \right) \leq \mu E^s(v_m) C \left( \frac{1}{s_0 - 3/2}, M, m, \sum_{j=0}^{m-1} \left| x^{m-1-j} u^0 \right|_{H^{s_0+2j+1}} \right).
\]

As above, Gronwall-Bihari’s inequality allows to deduce
\[
E^s(v_m)(t) \leq E^s(v_m) \big|_{t=0} + C \left( \frac{1}{s_0 - 3/2}, M, m, \sum_{j=0}^{m-1} \left| x^{m-1-j} u^0 \right|_{H^{s_0+2j+1}} \right),
\]
for \(0 \leq t \leq T_{\epsilon, \mu} \equiv C \left( \frac{1}{s_0 - 3/2}, M, m, \sum_{j=0}^{m-1} \left| x^{m-1-j} u^0 \right|_{H^{s_0+2j+1}} \right) \times \min(1/\epsilon, 1/\mu) \) and any \(s \geq s_0 > 3/2\). Therefore,
\[
(B.5) \quad \left| x^m u \right|_{H^{s_0+1}} \leq C \left( \frac{1}{s_0 - 3/2}, M, m, \sum_{j=0}^{m} \left| x^{m-j} u^0 \right|_{H^{s_0+2j+1}} \right) \quad \text{for} \quad 0 \leq t \leq T_{\epsilon, \mu}.
\]

The similar estimate on the time derivative of \(v_m\), is obtained as above by taking the \((L^2-)\)inner product of (B.4) with \(\Lambda^{2s-2}(\partial_t v_m)\). Estimate (B.2) follows for any \(n = m \) and \(k = 0\).

We know turn to \(x^m \partial_x^k u\). Note that for any \(k \in \mathbb{N}, k \geq 1\),
\[
x^m \partial_x^k u = \partial_x^k (x^m u) - \sum_{j=0}^{k-1} C \binom{k}{j} (\partial_x^{k-j} x^m)(\partial_x^j u)
\]
so that
\[
\left| x^m \partial_x^k u \right|_{H^{s_0+1}} \leq \left| x^m u \right|_{H^{s_0+k+1}} + C(k, m) \sum_{j=0}^{k-1} \left| x^{m-k+j} \partial_x^j u \right|_{H^{s_0+1}}.
\]

Using (B.5), one deduces by induction on \(k\) that
\[
\left| x^m \partial_x^k u \right|_{H^{s_0+1}} \leq C \left( \frac{1}{s_0 - 3/2}, M, m, \sum_{j=0}^{m} \left| x^{m-j} u^0 \right|_{H^{s_0+k+2j+1}} \right) + \sum_{j=0}^{k-1} C \left( \frac{1}{s_0 - 3/2}, M, m, k \right) \sum_{j=0}^{m-k+j} \left| x^{m-k+j-i} u^0 \right|_{H^{s_0+j+2i+1}}
\]
\[
\leq C \left( \frac{1}{s_0 - 3/2}, M, m, k, \sum_{j=0}^{m} \left| x^{m-j} u^0 \right|_{H^{s_0+k+2j+1}} \right) \quad \text{for} \quad 0 \leq t \leq T_{\epsilon, \mu}.
\]

In the same way, one estimates \(x^m \partial_x^k \partial_t u\) by induction, using Leibniz rule of differentiation.

Estimate (B.2) therefore holds for \(n = m\) and any \(k \in \mathbb{N}\).

By induction, we proved that (B.2) holds for any \(k \in \mathbb{N}\) and \(n \in \mathbb{N}\). Estimate (2.18) follows as a direct consequence (using the case \(k = 0\)), and Proposition 2.5 is proved.
C Proof of Proposition 2.7

Strategy of the proof. Our strategy is the following. Inspired by the calculations of Section 2.1, we seek an approximate solution of (1.1) under the form

\[ U_{\text{app}} \equiv \left( v_+(t,x-t) + v_-(t,x+t), (\gamma + \delta)(v_+(t,x-t) - v_-(t,x+t)) \right) + \sigma U_c(v_\pm), \]

where \( v_\pm \) satisfies the Constantin-Lannes equation (2.12), and \( U_c \) contains the leading order coupling terms. The parameter \( \sigma \) is assumed to be small, and we want to justify our approximate solutions over times of order \( O(1/\sigma) \).

Precisely, our aim is to prove that

i. the coupling term \( \sigma U_c \) can be controlled, and grows sublinearly in time;

ii. the approximate function \( U_{\text{app}} \) solves the coupled equation (1.1), up to a small remainder.

As we shall see, controlling the secular growth of \( U_c \) requires to consider separately the short time scale, where the coupling effects may be strong, and long time scale, where the coupling between two localized waves moving in opposite directions can be controlled. Thus we introduce the long time scale \( \tau = t/\sigma \), and will seek an approximate solution of (1.1) as

\[
U_{\text{app}}(t,x) = U_{\text{app}}(\tau, t, x), \quad \text{with}
\]

\[
U_{\text{app}}(\tau, t, x) = \left( v_+(\tau, t, x) + v_-(\tau, t, x), (\gamma + \delta)(v_+(\tau, t, x) - v_-(\tau, t, x)) \right) + \sigma U_c(v_\pm)(\tau, t, x),
\]

with obvious misuses of notation.

Plugging the Ansatz into the coupled Green-Naghdi equation (1.1) yields at first order

\[
\partial_\tau v_+ + \partial_x v_+ = 0 \quad \text{and} \quad \partial_\tau v_- - \partial_x v_- = 0,
\]

so that \( v_\pm(\tau, t, x) = \tilde{v}_\pm(\tau, x, \mp \tau) \).

At next order, and following the calculations of Section 2.1, one obtains the decoupled equations

\[
(\gamma + \delta)(v_+(\tau, t, x) - v_-(\tau, t, x)) = \sigma \left( \frac{\kappa_1 v_\pm_+}{\alpha_1} \frac{\kappa_2 v_\pm_-}{\alpha_2} \right) \equiv \tilde{v}_\pm(\tau, x, \mp \tau),
\]

where parameters satisfy identities of Definition 2.3. In order to deal with the coupling terms, we introduce the following first order correction.

Definition C.1. We denote \( U_c \equiv (u_\pm^+ + u_\pm^-, (\gamma + \delta)(u_\pm^+ - u_\pm^-)) \) where \( u_\pm^\pm(v_\pm)(\tau, t, x) \) satisfies initial condition \( u_\pm^\pm|_{\tau=0} \equiv 0 \), and equation

\[
(\gamma + \delta)(u_\pm^+ + f^\prime(\tilde{v}_+, \tilde{v}_-)) - f^\prime(\tilde{v}_+, 0) = 0 \quad \text{and} \quad \sigma((\partial_\tau - \partial_x)u_\pm^+ + f^\prime(\tilde{v}_+, \tilde{v}_-)) - f^\prime(0, \tilde{v}_-) = 0,
\]

where \( f^\prime \) and \( f^\prime \) are defined as in (2.3a)–(2.3b).

Remark C.2. The above discussion does not take into account the use of near identity change of variable as described in Section 2.1. In that case, we set the initial data as follows:

\[
v_\pm(\tau, t, x) = \tilde{v}_\pm(\tau, x, \mp \tau) = v_\pm^\lambda(t,x \mp \tau),
\]

where \( v_\pm^\lambda \) is the solution of (2.12) as defined in Definition 2.3.

The proof is now as follows. First, we state that the approximate solution, \( U_{\text{app}}(\tau, t, x) \), constructed as above, does satisfy (1.1), up to a small remainder (actually depending on the size of \( v_\pm \), \( u_\pm^\pm \), and their derivatives). The fact that \( v_\pm \) is uniformly bounded follows from the assumptions of the proposition. The core of the proof consists in estimating the growth over long times (that is, in the variable \( \tau \), of \( u_\pm^\pm \) satisfying (C.3)). Each of these steps are described in details in the following.
Construction and accuracy of the approximate solution $U_{\text{app}}$. The following Lemma states carefully the definition of our approximate solution $U_{\text{app}}$, and its precision in the sense of consistency.

**Lemma C.3.** Let $\zeta^0, v^0 \in H^{s+6}$, with $s \geq s_0 > 3/2$, and $(\epsilon, \mu, \delta, \gamma) = \mu \in \mathcal{P}$, as defined in (1.2). Let $v_{\pm}^1(\tau, t, x)$ be defined by (C.1), (C.2) and with initial data (C.4). Then define $v_{\pm}(\tau, t, x)$ as

$$v_{\pm}(\tau, t, x) = (1 \pm \mu \lambda \partial^2_x)^{-1} v_{\pm}^1(\tau, t, x),$$

and set

$$U_{\text{app}}(\tau, t, x) = \left( v_{+}(\tau, t, x) + v_{-}(\tau, t, x), (\gamma + \delta)(v_{+}(\tau, t, x) - v_{-}(\tau, t, x)) \right) + \sigma v_{\pm}(\tau, t, x),$$

where $U^c$ is defined in Definition C.1. Then for $\epsilon$ small enough, $U_{\text{app}}(\sigma t, t, x)$ satisfies the coupled equations (1.1), up to a remainder, $R$, bounded by

$$|R|_{H^{s+1}} \leq F \left( \epsilon^4 |v_{\pm}|_{H^{s+1}} + \sigma |u_{\pm}^c|_{H^{s+1}} + \mu \epsilon^2 (|v_{\pm}|_{H^{s+1}} \epsilon + |\partial_\tau v_{\pm}|_{H^{s+1}} + \sigma |u_{\pm}^c|_{H^{s+1}}) \right) + \sigma \epsilon \gamma h^{s+1} + \sigma \epsilon h^{s+1} + \max(\epsilon^4, \mu \epsilon^2) C |(v_{\pm} + |H^{s+1}) + \partial_\tau v_{\pm}|_{H^{s+1}}|,$$

with a function $F$ satisfying $F(X) \leq C(\frac{1}{s_0-3/2}, M^{s+6}, \delta_{\max}, \epsilon_{\max}, \mu_{\max}, |\lambda|, |\theta|) X$.

**Proof.** This result a direct application of the definitions above, and calculations presented in Section 2.1. More precisely, one obtains

$$R = R_0 + \sigma^2 \partial_\tau u_{\pm}^c + (f(v_{+} + \sigma u_{\pm}^c, v_{-} + \sigma u_{\pm}^c) - f(v_{+}, v_{-})) + R_\vartheta + R_\lambda,$$

where

- $R_0$ is the contribution due to the expansion of $\frac{h_{1}h_{2}}{h_{1} + \gamma h_{2}}$, $\frac{h_{1}^2 - \gamma h_{2}^2}{h_{1} + \gamma h_{2}}$, $\mathcal{Q}[h_{1}, h_{2}]\bar{v}$ and $\mathcal{Q}[h_{1}, h_{2}]\bar{v}$

- $f$ is a linear combination of $f_{1}^{c, \mu}$ and $f_{2}^{c, \mu}$ as defined in (2.3a)–(2.3b);

- $R_\vartheta$ and $R_\lambda$ are respectively the components due to the use of the BBM trick and near-identity change of variable.

Let us detail the first terms leading to $R_0$. We want to control the contribution of the expansion of $\frac{h_{1}h_{2}}{h_{1} + \gamma h_{2}}$, that is, more precisely, estimate

$$\left| R_0^{(1)} \right|_{H^{s+1}} = \left| \phi_{\lambda} \phi_{\gamma} \phi_{\mu} \frac{h_{1}h_{2}}{h_{1} + \gamma h_{2}} \frac{1}{\gamma + \delta} \frac{\delta^2 - \gamma}{(\gamma + \delta)^2} \zeta + \epsilon \gamma \delta^2(\delta + 1)^2(1 - \gamma) \frac{(1 + \gamma)}{(\gamma + \delta)^2} \zeta^3 \right|_{H^{s+1}},$$

where we denote $\zeta = \zeta_{\text{app}} = (v_{+} + \sigma u_{\pm}^c)(\sigma t, t, x)$, and $h_{1} = 1 - \epsilon \zeta_{\text{app}}, h_{2} = \frac{1}{\epsilon} + \epsilon \zeta_{\text{app}}$.

Note that, as we shall prove that $\zeta_{\text{app}}$ is bounded in $L^\infty([0, T^*]; H^{s+1})$, there exists $\epsilon_0 > 0$ such that for any $0 \leq \epsilon < \epsilon_0$, one has

$$|\zeta|_{L^\infty} < \min(1, 1/\delta) \quad \text{and} \quad \left| \frac{h_{1} + \gamma h_{2}}{h_{1} + \gamma h_{2}} \right|_{H^{s+1}} \leq \frac{1}{2(1 + \gamma/\delta)}.$$

Now, one can easily check that, in that case, $R^{(1)}_0 = \epsilon^4 P^{(1)}(\zeta)$, where $P^{(1)}(X)$ is a polynomial of degree 4, and estimate $\left| R_0^{(1)} \right|_{H^{s+1}} \leq \epsilon^4 F \left| v_{\pm} \right|_{H^{s+1}} + \sigma |u_{\pm}^c|_{H^{s+1}}$ follows.

As $\frac{h_{1}h_{2}}{h_{1} + \gamma h_{2}}$ gets multiplied by $\bar{v}$ and differentiated once, the first term in (C.5) follows.

The contributions due to the expansion of $\frac{h_{1}^2 - \gamma h_{2}^2}{(h_{1} + \gamma h_{2})^2}$ is estimated in the same way. The contribution due to the expansion of $\mathcal{Q}[h_{1}, h_{2}]\bar{v}$ requires more derivatives, but may be estimated as above.
by $\mu^2 F \left( |v_{\pm}|_{H^{s+3}} + \sigma |u_{\pm}|_{H^{s+3}} \right)$. The contribution due to the expansion of $\mathcal{C}[h_1, h_2]$ involves one time derivative, and is controlled by $\mu^2 F \left( |\partial_t v_{\pm}|_{H^{s+3}} + \sigma |u_{\pm}|_{H^{s+3}} \right)$. This yields the first line of (C.5).

Then, one can check that

$$f^\pm_{\mu}(v_{\pm} + \sigma u_{\pm}, v_{\pm} + \sigma u_{\pm} - f^\pm_{\mu}(v_{\pm}, v_{\pm}) = \partial_x P(\sigma u_{\pm}^c, v_{\pm}) + \mu \nu \partial_x^2 \partial_t u_{\pm}^c$$

$$+ \mu \partial_x \left( Q_1(\sigma u_{\pm}^c, \partial_x^2 u_{\pm}^c) + Q_2(\sigma \partial_x u_{\pm}^c, \partial_x v_{\pm}) + Q_3(\sigma \partial_x^2 u_{\pm}^c, v_{\pm}) + Q_4(\sigma u_{\pm}^c, \partial_x^2 u_{\pm}^c) + Q_5(\sigma \partial_x u_{\pm}^c, \partial_x v_{\pm}) \right)$$

$$+ \mu \partial_t \left( Q_6(\sigma u_{\pm}^c, \partial_t u_{\pm}^c) + Q_7(\sigma \partial_x u_{\pm}^c, \partial_x v_{\pm}) + Q_8(\sigma \partial_x^2 u_{\pm}^c, v_{\pm}) + Q_9(\sigma u_{\pm}^c, \partial_t u_{\pm}^c) + Q_{10}(\sigma \partial_x u_{\pm}^c, \partial_x v_{\pm}) \right),$$

where $P$ is a bivariate polynomial of degree 4, and whose leading order terms are

$$P(\sigma u_{\pm}^c, v_{\pm}) = \alpha_1 \epsilon(\sigma u_{\pm}^c)^2 + \alpha_2 \epsilon^2 \sigma u_{\pm}^c v_{\pm}^2 + \ldots,$$

and $Q_i$ are bilinear forms. Each of these terms are bounded as in (C.5).

Now, $R_\theta$ and $R_\Lambda$ depend uniquely on decoupled terms $v_{\pm}$, and one has similarly

$$|R_\theta|_{H^s} + |R_\Lambda|_{H^s} \leq C_0 \max(\epsilon^4, \mu^2, \mu^2)(|v_{\pm}|_{H^{s+5}} + |\partial_t v_{\pm}|_{H^{s+4}}),$$

with $C_0 = C \left( \frac{1}{\tau_0 - \tau^2}, \frac{1}{\delta_{\min}}, \epsilon_{\max}, \mu_{\max}, |\lambda|, |\theta| \right)$. The proposition follows.

Our aim is now to estimate each of the terms in (C.5).

**Estimates of the decoupled approximation $v_{\pm}$.** The bound on $v_{\pm}$ is a direct consequence of the assumptions of the Proposition, using the identity

$$v_{\pm}(\sigma t, t, x) = \tilde{v}_{\pm}(\sigma t, x \mp t) = v_{\pm}(t, x \mp t),$$

where $v_{\pm}(t, x \mp t)$ is defined through Definition 2.3, and therefore is uniformly controlled as assumed in the Proposition. It follows

$$\left\| \tilde{v}_{\pm} \right\|_{L^\infty((0, \sigma T^*) \times (0, T^*); H^{s+6})} + \sigma \left\| \partial_t \tilde{v}_{\pm} \right\|_{L^\infty((0, \sigma T^*) \times (0, T^*); H^{s+7})} \leq M_{s+6}.$$ 

However, let us note that one can gain extra smallness on $\sigma \partial_t \tilde{v}_{\pm}$ (trading with a loss of derivatives) from the fact that $\tilde{v}_{\pm}$ satisfies (C.2). Indeed, one has for any $k \in \mathbb{N}$,

$$\sigma \left\| \partial_t \tilde{v}_{\pm} \right\|_{H^{s+k}} \leq \max(\epsilon \alpha_1, \epsilon^2, \mu) P(\left\| \tilde{v}_{\pm} \right\|_{H^{s+k+3}} + \mu \nu \sigma \left\| \partial_t^2 \tilde{v}_{\pm} \right\|_{H^{s+k}},$$

where $P(X)$ is a polynomial, so that

$$\sigma \left\| \partial_t \tilde{v}_{\pm} \right\|_{L^\infty((0, \sigma T^*) \times (0, T^*); H^{s+k})} \leq \max(\epsilon \alpha_1, \epsilon^2, \mu) C(M_{s+6}).$$

As for the case of localized initial data, the assumption of the Proposition yields

$$\left( \sum_{k=0}^{5} \left\| (1 + x^2)^{\alpha} \partial_x^k v_{\pm} \right\|_{L^\infty((0, \sigma T) \times [0, T]; H^s)} ^2 + \sum_{k=0}^{5} \sigma \left\| (1 + x^2)^{\alpha} \partial_t v_{\pm} \right\|_{L^\infty((0, \sigma T) \times [0, T]; H^s)} ^2 \right) \leq M_{s+6}^2,$$

and one deduces as above, for $k = 0, 1, 2, 3$

$$\sigma \left\| (1 + x^2)^{\alpha} \partial_x^k \partial_t \tilde{v}_{\pm} \right\|_{H^{s}} \leq \max(\epsilon \alpha_1, \epsilon^2, \mu) \left( \sum_{k=0}^{3} \left\| (1 + x^2)^{\alpha} \partial_x^{k+1} \tilde{v}_{\pm} \right\|_{H^{s}} + \mu \sigma \left\| (1 + x^2)^{\alpha} \partial_x^{k+2} \partial_t \tilde{v}_{\pm} \right\|_{H^{s}} \right) \leq \max(\epsilon \alpha_1, \epsilon^2, \mu) C(M_{s+6}).$$

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7 Here and in the following, we do not explicitly keep track of the dependence with respect to all the parameters; as $(\epsilon, \mu, \delta, \gamma) = \Xi \in \mathcal{P}$, as defined in (1.2), and parameters satisfy identities of Definition 2.3, one should replace $C(M_{s+6})$ by $C \left( \frac{1}{\tau_0 - \tau^2}, M_{s+6}, \frac{1}{\delta_{\min}}, \epsilon_{\max}, \mu_{\max}, |\lambda|, |\theta| \right)$. 

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Control of the secular growth of the coupling terms \(u_\gamma^2\). Let us now study the term
\[ U^c \equiv (u^c_0 + u^c) \left( (\gamma + \delta)(u^c_0 - u^c) \right), \]
which contains all the coupling effects between the different components. Our aim is to control the secular growth of this term. This will be achieved through the following two Lemmas.

**Lemma C.4.** Let \( s \geq 0 \), and \( f^0 \in H^{s+1}(\mathbb{R}) \). Then there exists a unique global strong solution, \( u(t, x) \in C^0(\mathbb{R}; H^{s+1}) \cap C^1(\mathbb{R}; H^s) \), of
\[
\begin{cases}
(\partial_t + c_1 \partial_x)u = \partial_x f \\
u \bigg|_{t=0} = 0
\end{cases}
\]
with \( (\partial_t + c_2 \partial_x)u = 0 \) \( f \bigg|_{t=0} = f^0 \)

where \( c_1 \neq c_2 \).

Moreover, one has the following estimates for any \( t \in \mathbb{R} \):
\[
\|u(t, \cdot)\|_{H^{s+1}(\mathbb{R})} \leq \frac{2}{|c_1 - c_2|} \|f^0\|_{H^{s+1}(\mathbb{R})}, \quad \|u(t, \cdot)\|_{H^s(\mathbb{R})} \leq |t| \|f^0\|_{H^{s+1}(\mathbb{R})}.
\]

**Lemma C.5.** Let \( s \geq s_0 > 1/2 \), and \( v^0_1, v^0_2 \in H^{s+1}(\mathbb{R}) \). Then there exists a unique global strong solution, \( u \in C^0(\mathbb{R}; H^{s+1}) \cap C^1(\mathbb{R}; H^s) \), of
\[
\begin{cases}
(\partial_t + c_1 \partial_x)u = g(v_1, v_2) \\
u \bigg|_{t=0} = 0
\end{cases}
\]
with \( c_1 \neq c_2 \), and \( g \) is a bilinear mapping defined on \( \mathbb{R}^2 \) and with values in \( \mathbb{R} \).

Moreover, one has the following estimates:

i. \[
\|u\|_{L^\infty([0,t], H^s(\mathbb{R}))} \leq C_{s_0} t \|v^0_1\|_{H^s(\mathbb{R})} \|v^0_2\|_{H^s(\mathbb{R})}, \quad \|\partial_t u\|_{L^\infty([0,t], H^s(\mathbb{R}))} \leq C_{s_0} t \|v^0_1\|_{H^{s+1}(\mathbb{R})} \|v^0_2\|_{H^{s+1}(\mathbb{R})}.
\]

ii. Using \( c_1 \neq c_2 \), one has the sublinear growth
\[
\|u(t, \cdot)\|_{H^s(\mathbb{R})} = C_{s_0} \frac{\sqrt{t}}{|c_1 - c_2|} \|v^0_1\|_{H^s} \|v^0_2\|_{H^s}, \quad \|\partial_t u(t, \cdot)\|_{H^s} = C_{s_0} \frac{1 + \sqrt{t}}{\sqrt{|c_1 - c_2|}} \|v^0_1\|_{H^{s+1}} \|v^0_2\|_{H^{s+1}},
\]

with \( C_{s_0} = C(\frac{1}{s_0 - 1/2}, \frac{1}{c_1 - c_2}, \frac{1}{\alpha - 1/2}) \).

iii. If moreover, there exists \( \alpha > 1/2 \) such that \( v^0_1(1 + x^2)^\alpha \), and \( v^0_2(1 + x^2)^\alpha \in H^s(\mathbb{R}) \), then one has the (uniform in time) estimate
\[
\|u\|_{L^\infty(\mathbb{R})} \leq C_{s_0} \|v^0_1(1 + x^2)^\alpha\|_{H^s(\mathbb{R})} \|v^0_2(1 + x^2)^\alpha\|_{H^s(\mathbb{R})},
\]

with \( C_{s_0} = C(\frac{1}{s_0 - 1/2}, \frac{1}{c_1 - c_2}, \frac{1}{\alpha - 1/2}) \).

**Proof.** Lemma C.4 is straightforward, since one has an explicit expression for the solution:
\[
\begin{align*}
u(t, x) &= \frac{1}{c_2 - c_1} \left( f(t, x + (c_2 - c_1)t) - f(t, x) \right) = \frac{1}{c_2 - c_1} \left( f^0(x - c_1 t) - f^0(x - c_2 t) \right) \end{align*}
\]

It follows \( \|u\|_{H^{s+1}} \leq \frac{2}{|c_2 - c_1|} \|f^0\|_{H^{s+1}} \), and
\[
\|u\|_{H^s} = \frac{1}{|c_2 - c_1|} \int_{c_2 t}^{c_1 t} \partial_x f^0(x - y) \ dy \|_{H^s} \leq \frac{1}{|c_2 - c_1|} \int_{c_2 t}^{c_1 t} \|\partial_x f^0(x - y)\|_{H^s} \ dy \leq |t| \|\partial_x f^0\|_{H^{s+1}}.
\]

As for Lemma C.5, the well-posedness as well as estimate i. are standard; the remaining estimates, controlling the secular growth, are proved in \([42]\), Propositions 3.2 and 3.5.
Lemma C.6. Let $v_{±}$ defined as in Lemma C.3, for $(τ,t,x) ∈ [0,σT^∗] × [0,T^∗] × R$. Then there exists a unique $U_ε(τ,t,x) ∈ L^N((0,T) × R; H^s)$ such that $U_ε ≡ \{u_{±}^ε + w_ε^c, (γ + δ)(u_{±}^ε - u_ε^c)\}$, with $u_{±}^ε$ satisfying (C.3) and $U_ε|_{t=0} ≡ 0$. Moreover, one has the estimate for and $t ∈ [0,T^∗]$

\begin{align}
\sigma\|u_{±}^ε\|_{L^∞([0,σT^∗] × [0,T^∗]; H^s)} & \leq C(M_{s+3})\max(ε_1, ε^2, \mu)\min(t, \sqrt{t}), \\
\sigma^2\|∂_τ u_{±}^ε\|_{L^∞([0,T] × [0,σT^∗]; H^s)} & \leq C(M_{s+6})\max(ε_1, ε^2, \mu)^2\min(t, \sqrt{t}).
\end{align}

Moreover, if $(1 + x^2)ζ_0^0, (1 + x^2)ζ_0^0 ∈ H^{s+3}$, $s ≥ s_0 > 1/2$, and $ζ_0, ζ_0^0 ∈ H^{s+3+2m}$, then one has the uniform estimate

\begin{align}
\sigma\|u_{±}^ε\|_{L^∞([0,σT^∗]; H^s)} & \leq C(M_{s+3})\max(ε_1, ε^2, \mu)\min(t, 1), \\
\sigma^2\|∂_τ u_{±}^ε\|_{L^∞([0,T]; H^s)} & \leq C(M_{s+6})\max(ε_1, ε^2, \mu)^2\min(t, 1).
\end{align}

Proof. The function $u_{±}^ε$ are defined as the solutions of a transport equation, with a known and controlled forcing term, thus the existence and uniqueness of $u_{±}^ε$ and $U_ε$ is straightforward.

As for the estimates, we focus below on $u_{±}^ε$. Estimates for $u_{±}^c$ follow in the same way.

By definition, $u_{±}^ε$ satisfies

$$\sigma(∂_t + ∂_x)u_{±}^ε = f^i(\bar{v}_+, 0) - f^j(\bar{v}_+, \bar{v}_-),$$

and the right-hand-side can be decomposed in the following way:

$$f^i(v_+, 0) - f^j(v_+, v_-) = \sum_{j=1}^{3} a_j \epsilon_j^i ∂_x(v_-^{j+1}) + \sum_{j=1}^{3} b_j ∂_x^2 (v_-^{j+1}) + \mu b \partial_\delta^j v_- + \mu \partial_\delta^j (v_-^{j+1})\partial_\delta^j v_- + c_2 (\partial_\delta v_-)^2 + \mu \sum_{k=0}^{3} d_k (\partial_\delta^k v_-) (∂_\delta^{3-k} v_-)$$

where the parameters $a_j, b_j, c_j$ and $d_k$ depend on $γ, δ, λ, \theta$, and where we used $∂_\delta v_- = ∂_\delta v_-$. Note that the first order terms $ε_1$ and $ε_1(1)$ are factored by $α_1 = \frac{3}{2} \frac{\delta^2 - γ}{\sqrt{γ}}$.

The contribution from $f_j, j = 1, \ldots, 5$ will be estimated thanks to Lemma C.4, as we will use Lemma C.5 for the contribution of $g_{i,j}$ and $g_k$.

For all $0 ≤ j ≤ 4$, one has $∂_t f_j - ∂_x f_j = 0, and

$$|f_j|_{H^{s+1}} ≤ C|v_-|_{L^∞} |v_-|_{H^{s+1}} ≤ C(∥\bar{v}_-(τ, \cdot)∥_{H^{s+3}}), for j = 1, 2, 3;$$

$$|f_4|_{H^{s+1}} ≤ C|\partial_\delta^2 v_-|_{H^{s+1}} ≤ C(∥\bar{v}_-(τ, \cdot)∥_{H^{s+3}});$$

$$|f_5|_{H^{s+1}} ≤ C|v_-|_{L^∞} |\partial_\delta^2 v_-|_{H^{s+1}} + (∂_\delta v_-)^2|_{H^{s+1}} ≤ C(∥\bar{v}_-(τ, \cdot)∥_{H^{s+3}}).$$

It is then straightforward to check that $g_{i,j}$ and $g_k$ are the sum of bilinear mappings, applied to functions $u_{±}^ε$ satisfying $∂_t u_{±}^ε = ∂_x u_{±}^ε = 0$, with $u_{±}^ε$ bounded as below:

$$|v_i^ε|_{H^{s+1}} ≤ C(v_i^ε |_{H^{s+1}} ≤ C(∥\bar{v}_-(τ, \cdot)∥_{H^{s+1}}), for i = 1, 2, 3.$$  

$$|∂_t v_i^ε|_{H^{s+1}} ≤ C(∥v_i^ε |_{H^{s+1}} ≤ C(∥\bar{v}_-(τ, \cdot)∥_{H^{s+1}}), for j = 1, 2, 3.$$  

$$|∂_\delta^j v_i^ε|_{H^{s+1}} ≤ μC(v_i^ε |_{H^{s+1}} ≤ C(∥\bar{v}_-(τ, \cdot)∥_{H^{s+1}}), for k = 0, 1, 2, 3.$$  

It follows form Lemmata C.4 and C.5 that $u_{±}^ε$ satisfies

$$\sigma|u_{±}^ε(τ, t, \cdot)∥_{H^{s+1}} ≤ \min(t, \sqrt{t})\max(α_1, ε_1, ε_2, μ, ϵ_ε)C(∥\bar{v}_-(τ, \cdot)∥_{H^{s+3}}),$$

and the last term is uniformly estimated through (C.6), so that (C.10) follows.
As for the case of localized initial data, we make use of the fact that for $k = 0, 1, 2, 3$, one has
$$
| (1 + x^2)^a \partial_x^k \tilde{v} \pm (\tau, t, \cdot) |_{H^s} = | (1 + x^2)^a \partial_x^k \tilde{v} \pm (\tau, 0, \cdot) |_{H^s} = | (1 + x^2)^a \partial_x^k \tilde{v} \pm (\tau/\sigma, \cdot) |_{H^s} \leq M_{s+3}^2,
$$
so that (C.12) follows in the same way for $\tau \leq \sigma T^p$.

The estimates on $\partial_x v_\pm (\tau, x)$ are obtained similarly, as
$$
(\partial_t + \partial_x) \partial_x u^c_+ \equiv \sum_{j=1}^3 c_j \partial_x \partial_t f_j + \sum_{j=1}^3 c_j \sum_{i=1}^3 \partial_{x_t} g_{i,j} + \mu \partial_x \partial_x g_{i,j} + \mu \epsilon \sum_{k=0}^3 \partial_x g_{k}.
$$
One can check that each term of the right hand side satisfies the hypotheses of Lemmata C.4 or C.5, and estimate (C.7) allows to obtain (C.11).

The case of weighted Sobolev spaces (C.13) follows as above, using estimate (C.9). The Lemma is proved.

**Completion of the proof.** The consistency result stated in our Proposition is now a straightforward consequence of Lemma C.3, together with estimates (C.6)–(C.8), and Lemma C.6.

One can check that the remainder in can be estimated as
$$
| R |_{H^s} \leq C(M_{s+6}^*) \left( \max(\alpha_1^4 \epsilon^2, \epsilon^4, \mu^2) \min(t, \sqrt{t}) + \max(\epsilon^4, \mu^2) \right)
$$
Note that we use
$$
\partial_t (v_\pm(\sigma t, t, x)) = \sigma \partial_x \tilde{v} \pm + \partial_t v_\pm, \quad \text{and} \quad \sigma | \partial_x \tilde{v} \pm |_{H^s} \leq \max(\epsilon\alpha_1, \epsilon^2, \mu \epsilon) | v_\pm |_{H^{s+2}},
$$
as above, as well as a uniform estimate
$$
\sigma \left\| u^c_\pm \right\|_{L^\infty([0,T^*]; H^{s+1})} \leq M.
$$
The latter estimate can be enforced using Lemma C.6, by restricting the time interval with
$$
(T^*)^{1/2} \leq \frac{M}{\max(\epsilon\alpha_1, \epsilon^2, \mu)},
$$
although such condition is not necessary, as Proposition 2.7 is empty for $t \geq M(\max(\epsilon\alpha_1, \epsilon^2, \mu))^{-2}$, the accuracy of the uncoupled solutions being of order $O(1)$.

More precisely, we proved the following Lemma.

**Lemma C.7.** For $\zeta^0, v^0 \in H^{s+6}$, with $s \geq s_0 > 3/2$, and $(\epsilon, \mu, \delta, \gamma) = p \in A$, as defined in 1.2, let $U_{app}$ be defined as in Lemma C.3. Then $U_{app}(\sigma t, t, x)$ satisfies the coupled equations (1.1), up to a remainder, $R$, estimated for $t \in [0, T^*)$ by
$$
\left\| R \right\|_{L^\infty([0,t]; H^s)} \leq C(M_{s+6}^*) \left( \max(\alpha_1^2 \epsilon^2, \epsilon^4, \mu^2) \min(t, \sqrt{t}) + \max(\epsilon^4, \mu^2) \right),
$$
and on $[0, T^*)$ by
$$
\left\| R \right\|_{L^\infty([0,t]; H^s)} \leq C(M_{s+6}^*) \left( \max(\alpha_1^2 \epsilon^2, \epsilon^4, \mu^2) \min(t, 1) + \max(\epsilon^4, \mu^2) \right).
$$
The Proposition is a consequence of the above Lemma, together with the estimates of Lemma C.6 (see also Remark C.2).
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