Berry phase and quantum criticality in Yang–Baxter systems

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Spin interaction Hamiltonians are obtained from the unitary Yang–Baxter \( \hat{R} \)-matrix. Based on which, we study Berry phase and quantum criticality in the Yang–Baxter systems.

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I. INTRODUCTION

It is well known that the Yang–Baxter equation (YBE) plays a fundamental role in the theory of (1 + 1)- or 2-dimensional integrable quantum systems, including lattice statistical models and nonlinear field theory. The YBE was originated in solving the \( \delta \)-function interaction model by Yang \cite{Yang} and the statistical models by Baxter \cite{Baxter} and introduced to solve many quantum integrable models by Faddeev and Leningrad Scholars \cite{Faddeev}. Through the RIT relation \cite{Drinfeld} the new algebraic structures (quaternion groups) were established by V. Drinfeld \cite{Drinfeld}. The usual YBE takes the form

\[
\hat{R}_{12}(x)\hat{R}_{23}(xy)\hat{R}_{12}(y) = \hat{R}_{23}(y)\hat{R}_{12}(xy)\hat{R}_{23}(x),
\]

which is valid for three types of \( \hat{R} \)-matrices, i.e., the rational, the trigonometric and the elliptic solutions of YBE. The spectral parameter \( x \) plays an important role that is related the one-dimensional momentum (or the rapidity) in some typical models. An alternatively equivalent form of the YBE reads

\[
\hat{R}_{12}(u)\hat{R}_{23}(u+v)\hat{R}_{12}(v) = \hat{R}_{23}(v)\hat{R}_{12}(u+v)\hat{R}_{23}(u),
\]

if one denotes \( x = e^u \) and \( y = e^v \) (or \( x = e^{-u} \) and \( y = e^{-v} \)). The asymptotic behavior of \( \hat{R}_{i,i+1}(x) \) is \( x \)-independent:

\[
\lim_{x \to \infty} \hat{R}_{i,i+1}(x) = B_i.
\]

The braiding operators \( B_i \)'s satisfy the following braid relations

\[
B_i B_{i+1} B_i = B_{i+1} B_i B_{i+1}, \quad 1 \leq i \leq n - 1, \\
B_i B_j = B_j B_i, \quad |i - j| \geq 2,
\]

where the notation \( B_i \equiv B_{i,i+1} \) is used, \( B_{i,i+1} \) implies \( 1_1 \otimes 1_2 \otimes 1_3 \cdots \otimes B_{i,i+1} \otimes \cdots \otimes 1_n \), and \( 1_j \) represents the unit matrix of the \( j \)-th particle. The usual permutation operator

\[
P_{i,i+1} = \frac{1}{2}(1 + \hat{\sigma}_i \cdot \hat{\sigma}_{i+1}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

for the particles \( i \) and \( i+1 \) is a solution of Eq. \( \text{(4)} \) with the constraint \( P_{i,i+1}^2 = 1 \), where \( \hat{\sigma}_i \) is the vector of the Pauli matrices. The permutation operator \( P_{i,i+1} \) exchanges the spin state \( |k_i \otimes |l\rangle \) to \( |l\rangle \otimes |k_i \rangle \). For a statistical model all the elements of \( \hat{R}(u) \)-matrix should be positive because they are related to the Boltzmann weights. The relationship between \( \hat{R}(u) \) and \( B \) was set up by Jimbo \cite{Jimbo}, Jones \cite{Jones} and others \cite{Kaufman}. We call the process of obtaining the matrix \( \hat{R}(u) \) from a given braiding matrix \( B \) as the “Yang-Baxterization”, which depends on the number of the distinct eigenvalues of matrix \( B \). As was pointed out by Kauffman et al. \cite{Kaufman1, Kaufman2} that the braiding matrix \( B^{\frac{1}{2}} \) (here the superscript “\( \frac{1}{2} \)” means that the spin values of two particles are both \( \frac{1}{2} \)) transforms the “natural basis” \( \{ |1\rangle, |1\rangle \}, \{ |1\rangle, |1\rangle \}, \{ |1\rangle, |1\rangle \}, \{ |1\rangle, |1\rangle \} \) to the Bell states \( \left( \frac{1}{\sqrt{2}} \{ |1\rangle \pm |1\rangle \}, \frac{1}{\sqrt{2}} \{ |1\rangle \pm |1\rangle \} \right) \). It is emphasized that the elements of \( B^{\frac{1}{2}} \) are no longer positive here. However, a braiding matrix \( B \) is nothing to do with the dynamics. To do so, we should Yang-Baxterize the matrix \( B \) to be the \( \hat{R}(x) \)-matrix and look for its resultant consequence in physics, such as Berry phase, quantum criticality and so on.

In this paper, we mainly focus on the trigonometric Yang-Baxterization. The rational Yang-Baxterization will be also discussed, but the elliptic solutions of YBE will be ignored since it is more complicated. If \( B \) has only two distinct non-zero eigenvalues \( \lambda_1 \) and \( \lambda_2 \), one then simply has the trigonometric Yang-Baxterization approach \cite{Jones, Drinfeld, Jimbo} as

\[
\hat{R}(x) = \rho(x) (\lambda_1 x B + \lambda_2^{-1} x^{-1} B^{-1}),
\]

where \( \rho(x) \) is a normalization factor (one may choose an appropriate \( \rho(x) \) so that \( \hat{R}(x) \) becomes a unitary matrix). Generally a solution of \( \hat{R}(x) \) depends on two parameters: the first is \( \theta \) (or \( x \), which is a function of \( \theta \)); the second is \( \phi \) contained in the braiding matrix \( B \) (the free
where the Hamiltonian reads

$$|\Phi(\theta, \phi(t))\rangle = \tilde{R}(\theta, \phi(t))|\Phi(0)\rangle,$$

where $|\Phi(0)\rangle$ is the initial state independent of $t$, and $\theta$ is time-independent. The normalization condition of the quantum states $\langle \Phi(\theta, \phi(t))|\Phi(\theta, \phi(t))\rangle = \langle \Phi(0)|\Phi(0)\rangle = 1$ requires the unitary condition $\tilde{R}^\dagger(\theta, \phi(t)) = \tilde{R}^{-1}(\theta, \phi(t))$. It follows from Eq. (7) that

$$i\hbar \frac{\partial \Phi(\theta, \phi(t))}{\partial t} = i\hbar \left[ \frac{\partial}{\partial \theta} \tilde{R}(\theta, \phi(t)) \right] \tilde{R}(\theta, \phi(t))|\Phi(0)\rangle = H(t)|\Phi(\theta, \phi(t))\rangle,$$

where the Hamiltonian reads

$$H(t) = i\hbar \frac{\partial \tilde{R}(\theta, \phi(t))}{\partial t} \tilde{R}^\dagger(\theta, \phi(t)).$$

Thus, through the Yang-Baxterization approach $B(\phi) \rightarrow \tilde{R}(\theta, \phi)$, Eq. (9) defines the Hamiltonian for the Yang-Baxter systems.

The purpose of this paper is to investigate some physical consequences such as Berry phases (BP) \[12\] \[13\] in Yang-Baxter systems, quantum criticality (QC) phenomenon \[14\] \[15\] \[16\] \[17\] \[18\] is also discussed. The paper is organized as follows. In Sec. II we study the Berry phase for a kind of Yang–Baxter Hamiltonian related to the extra-special two-group. In Sec. III, we study Berry phase for a Yang–Baxter Hamiltonian related to the well-known six-vertex model and the Temperley-Lieb (TL) algebra. Conclusion and discussion are made in the last section.

II. BP AND QC FOR HAMILTONIAN $H_1(\theta, \phi(t))$

Let us consider the following type braiding matrix for two spin-1/2 particles \[8\] \[10\]

$$B^{\pm \pm} = \frac{1}{\sqrt{2}} (I + M^{\pm \pm}),$$

where $I$ is the $4 \times 4$ unit matrix,

$$M^{\pm \pm} = \begin{pmatrix} \epsilon & e^{i\phi} \\ -e^{-i\phi} & -\epsilon \end{pmatrix},$$

$\epsilon = \pm 1$, and $\phi = \phi(t)$ represents the arbitrary flux. The braiding matrix $B^{\pm \pm}$ has special significance in quantum information and quantum computation, because it can be identified to the universal quantum gate (i.e., the CNOT gate) \[9\] \[10\]. In additional, the braiding matrix $B^{\pm \pm}$ may produce the maximally entangled states (or the Bell states) from the separable ones $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle |\downarrow\uparrow\rangle$, and $|\downarrow\downarrow\rangle$ \[9\] \[10\] \[11\] \[12\].

Furthermore, the matrix $M^{\pm \pm}$ satisfies the algebraic relation of the extra-special two-group \[20\] \[21\] \[22\] \[23\]. More explicitly, the matrices $M_1^{\pm \pm}$ and $M_2^{\pm \pm}$ satisfy the following algebraic relations:

$$M_i^2 = -1, \quad M_i M_j = M_j M_i, \quad |i - j| \geq 2, \quad M_{i+1} M_i = -M_i M_{i+1}, \quad 1 \leq i, j \leq n - 1. \quad (12)$$

It is easy to verified that the braiding matrix $B^{\pm \pm}$ has two distinct eigenvalues with $\lambda_1 = (1 + i)/\sqrt{2}$, $\lambda_2 = (1 - i)/\sqrt{2}$ and $\lambda_1 \lambda_2 = 1$, then the trigonometric Yang-Baxterization approach is applicable. Based on which one obtains

$$\tilde{R}(x) = [2(x^2 + x^{-2})]^{-1/2}(x + x^{-1})I^{\pm \pm} + (x - x^{-1})M^{\pm \pm},$$

$$[\tilde{R}(x)]^{-1} = [2(x^2 + x^{-2})]^{-1/2}(x + x^{-1})I^{\pm \pm} - (x - x^{-1})M^{\pm \pm}. \quad (13)$$

The unitary condition $[\tilde{R}(x)]^{-1} = \tilde{R}(x^{-1})$ leads to $\phi(t)$ = real. In addition, the Yang–Baxter matrix $\tilde{R}(x)$ may produce the non-maximally entangled states when it acts on the separable ones $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle |\downarrow\uparrow\rangle$, and $|\downarrow\downarrow\rangle$ \[9\] \[10\] \[11\].

Equation (11) can be rewritten as

$$M^{\pm \pm} = e^{i\phi(t)} S^+_i S^+_j - e^{-i\phi} S^-_i S^-_j + \epsilon (S^+_i S^-_j - S^-_i S^+_j), \quad (14)$$

where

$$S^+_i = S^+_i + i S^z_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$S^-_i = S^+_i - i S^z_i = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (15)$$

are the raising and lowering operators of spin-1/2 angular momentum for the $i$-th particle, respectively. We then have from Eq. (9) that

$$H_1(x, \phi(t)) = -\hbar \dot{\phi} [2(x^2 + x^{-2})]^{-1}(x - x^{-1}) \times (x - x^{-1})(S^+_1 S^+_2 + S^-_1 S^-_2) + (x + x^{-1})(e^{i\phi} S^+_1 S^+_2 + e^{-i\phi} S^-_1 S^-_2)]. \quad (16)$$
By using
\[ x = [-\cos 2\theta]^{-1/2}(\cos \theta + \sin \theta), \]
\[ x^{-1} = [-\cos 2\theta]^{-1/2}(\sin \theta - \cos \theta), \]
Equation (17) can be recast to
\[ H_1(\theta, \phi(t)) = -\hbar \dot{\phi} \cos \theta [\cos \theta (S_1^x + S_2^x) + \sin \theta (e^{i\phi} S_1^+ S_2^+ + e^{-i\phi} S_1^- S_2^-)], \tag{18} \]
or in the matrix-form it reads
\[ H_1(\theta, \phi) = -\hbar \dot{\phi} \cos \theta \begin{pmatrix} \cos \theta & 0 & e^{i\phi} \sin \theta \\ 0 & 0 & 0 \\ 0 & 0 & -e^{-i\phi} \sin \theta \end{pmatrix}. \tag{19} \]

The physical consequence of Berry phase for the above Yang–Baxter Hamiltonian system, i.e., \( H_1(\theta, \phi(t)) \), has been discussed in [19]. Namely, from the definition of Berry phase
\[
\gamma(c) = i \int_0^T dt \langle \hat{r}(\vec{R}) \frac{\partial}{\partial \theta} |n(\vec{R})\rangle = i \int_0^T dt A(t),
\]
where \( \hat{r} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \) and \( |n(\vec{R})\rangle = |\Phi_\pm(\theta, \phi)\rangle \), one then obtains the Berry phases for the Yang–Baxter system as
\[
\gamma_\pm^1 = (\pm \int_0^{2\pi} d\phi) \sin^2 \frac{\theta}{2} = \pm \pi(1 - \cos \theta) = \pm \Omega \frac{\theta}{2}, \tag{24} \]
where \( \Omega = 2\pi(1 - \cos \theta) \) is the familiar solid angle enclosed by the loop on the Bloch sphere.

The Hamiltonian (15) is obtained through the Schrödinger evolution of the Bell state with \( \phi = \omega t \), which does have a nice physical interpretation. Since \( S_1 \) and \( S_2 \) are two-dimensional representation operators of \( SU(2) \) for particles 1 and 2 respectively, we then have \( (S_i^\pm)^2 = 0, \) (i = 1, 2). It is convenient to introduce the following fermionic operators:
\[ S_i^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad S_i^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad i = 1 \tag{25} \]

Then by means of \([S_i^3, S_j^\pm] = \pm S_j^\pm \delta_{ij} \) and \([S_i^+, S_j^-] = 2\delta_{ij} S_i^3 \), we have
\[
\{ \hat{f}_i, \hat{f}_j \} = \hat{f}_i \hat{f}_j + \hat{f}_j \hat{f}_i = 1, \]
\[
\{ \hat{f}_i, \hat{f}_j \} = [\hat{f}_i, \hat{f}_j] = 0, \quad \text{(for } i \neq j), \tag{26} \]
i.e., \( \hat{f}_i \)’s satisfy the fermionic anticommutator for the same \( i \)-th lattice and the bosonic commutator for different sites of the lattices, and \( \hat{n}_i = \hat{f}_i^\dagger \hat{f}_i \) is the number operator that can be 0 and 1. It is easy to check that the following three operators
\[
S^+ = \prod_{i=1}^{2} \hat{f}_i, \quad S^- = \prod_{i=1}^{2} \hat{f}_i, \]
\[
S^3 = \frac{1}{2}(S_1^3 + S_2^3) = \frac{1}{2}(\hat{n}_1 + \hat{n}_2 - 1), \tag{27} \]
form an \( SU(2) \) group satisfying \([S^3, S^\pm] = \pm S^\pm \), and \([S^+, S^-] = 2S^3 \). By the way, its Casimir operator is \( \frac{1}{2}(S^+ S^- + S^- S^+) + (S^3)^2 = \frac{1}{2}[(\hat{n}_1 + \hat{n}_2 - 1) - \hat{n}_1 \hat{n}_2] = \frac{1}{4}(1 - (\hat{n}_1 + \hat{n}_2 - 1))^2 \), which equals to \( \frac{1}{4}(\Omega^2 + 1) \) for \( \hat{n}_1 = \hat{n}_2 \), and 0(0+1) for \( \hat{n}_1 \neq \hat{n}_2 \), respectively.

In terms of Eqs. (24-27) the Hamiltonian (15) can be recast to the form
\[
H_1(\theta, \phi(t)) = -\hbar \omega \cos \theta [\cos \theta (\hat{n}_1 + \hat{n}_2 - 1) + \sin \theta (e^{i\phi} S^+ + e^{-i\phi} S^-)], \tag{28} \]
or
\[
H_1(\theta, \phi(t)) = -\hbar \omega \epsilon(\theta) H_0(\theta, \phi(t)) \tag{29} \]
where
\[
H_0(\theta, \phi(t)) = 2\epsilon(\theta) S^3 + (\Delta(t) + S^+) S^- \tag{30} \]
\[
\epsilon(\theta) = \cos \theta, \quad \Delta(t) = \sin \theta e^{i\phi(t)} \tag{31} \]
The standard procedure of diagonalizing \( H_0(\theta, \phi(t)) \) is
\[
W^\dagger H_0 W = 2\epsilon S_3, \quad \mathcal{E} = \sqrt{(\epsilon(\theta))^2 + |\Delta(t)|^2} \tag{32} \]
and the eigenstate is
\[
|\xi(\theta)\rangle = W|\text{vacuum}\rangle = \exp(\xi S_+ - \xi^* S_-)|\text{vacuum}\rangle, \quad S_-|\text{vacuum}\rangle = 0, \tag{33} \]
with
\[
\xi = re^{i\phi(t)}, \quad \cot(2r) = -\frac{\epsilon(\theta)}{|\Delta(t)|}. \tag{34} \]
Substituting Eq. (31) into Eq. (32) and Eq. (34) we obtain
\[
\mathcal{E} = 1, \quad r = -\frac{\theta}{2}. \tag{35} \]
In other words, we have

\[ W^\dagger HW|\xi(\theta)\rangle = -\hbar \omega \cdot 2 \cos \theta S_3 |\xi(\theta)\rangle = -\hbar \omega \cos \theta (\hat{n}_1 + \hat{n}_2 - 1) |\xi(\theta)\rangle. \]  

(35)

It is nothing but an oscillator Hamiltonian formed by two fermions with the frequency \( \omega \cos \theta \). When \( \theta = 0 \) Eq. (30) reduces to the standard oscillator for \( \Delta(t) = 0 \). However, when \( \theta \neq 0 \), \( \Delta(t) \) plays a role of the “energy gap” and the wave function takes the form of spin coherent state \([24][25]\). We know that the eigenfunction of oscillator is the Hermitean polynomial whereas the wave function of Eq. (30) with \( \Delta(t) \neq 0 \) is the spin coherent state shown by

\[ |\xi\rangle = \frac{1}{\sqrt{1 + |\tau|^2}} \exp(\tau S_+)|\text{vacuum}\rangle = \frac{1}{\sqrt{1 + |\tau|^2}} \left\{ |0, 0\rangle + \tau |1, 1\rangle \right\}, \]

(36)

where

\[ \tau = -e^{i\phi} \tan \frac{\theta}{2}, \]

\[ |1, 1\rangle = |n_1 = 1, n_2 = 1\rangle, \]

\[ |0, 0\rangle = |n_1 = 0, n_2 = 0\rangle. \]

(37)

We thus conclude that \( \theta = 0 \) is a point of quantum criticality. It is not caused by temperature, but by the degree of entanglement related to the parameter \( \theta \).

The degree of entanglement (or the concurrence \([26]\)) for an arbitrary two-qubit state \( |\psi\rangle = a |\uparrow\uparrow\rangle + b |\uparrow\downarrow\rangle + c |\downarrow\uparrow\rangle + d |\downarrow\downarrow\rangle \) is \( C = 2|ad - bc| \). The Berry phases in Eq. (24) can be expressed in terms of the concurrence of the states \( |\Phi_{\pm}(\theta, \phi)\rangle \) as

\[ \gamma_{\pm}^1 = \mp \pi(1 - \sqrt{1 - C^2}), \]

(38)

where \( C = |\sin \theta| \) is the concurrence of \( |\Phi_{\pm}(\theta, \phi)\rangle \). Interestingly, one may observe that when \( \theta = 0 \) or \( C = 0 \), the quantum criticality occurs in the Hamiltonian system \( H_1(\theta, \phi(t)) \) and at the same time Berry phases vanish correspondingly.

### III. BP AND QC FOR HAMILTONIAN \( H_2(\theta, \phi(t)) \)

In this section, we come to study the Berry phase and also the quantum criticality for a kind of Yang–Baxter Hamiltonian related to the well-known six-vertex model \([4]\) and the Temperley-Lieb algebra.

For the well-known six-vertex model, the braiding matrix reads

\[ B = S^{\frac{\pm}{2}} = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 0 & -\eta & 0 \\ 0 & -\eta & q & -1 \\ 0 & 0 & 0 & q \end{bmatrix}, \]

(39)

where

\[
U^{\frac{\pm}{2}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q & \eta & 0 \\ 0 & \eta^{-1} & q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

(40)

The matrix \( U^{\frac{\pm}{2}} \) satisfies the Temperley-Lieb algebra, i.e., \( U_i U_{i+1} U_i = U_i \), \( U_i^2 = d U_i \) (for the above matrix \( U^{\frac{\pm}{2}}, d = q + q^{-1} \)). The above braiding matrix \( B \) has two distinct non-zero eigenvalues with \( \lambda_1 = q, \lambda_2 = -q^{-1} \) and \( \lambda_1 \lambda_2 = -1 \), so we can perform the trigonometric Yang-Baxterization approach. It gives

\[
\tilde{R}(x) = [q^2 + q^{-2} - (x^2 + x^{-2})]^{-1/2}[(qx - q^{-1}x^{-1})I
- (x - x^{-1})U^{\frac{\pm}{2}}],
\]

(41)

\[
[\tilde{R}(x)]^{-1} = [q^2 + q^{-2} - (x^2 + x^{-2})]^{-1/2}[(qx^{-1} - q^{-1}x)I + (x - x^{-1})U^{\frac{\pm}{2}}].
\]

(42)

It is easy to check that \([\tilde{R}(x)]^{-1} = [\tilde{R}(x)]^{-1} = \tilde{R}(-x)\) for \( x = e^{i\theta}, \eta = e^{i\varphi(t)}, \) and \( \theta, \varphi(t), \eta \) in real.

One may symmetrize the matrix \( \tilde{R}(x) \) given by Eq. (41) (i.e., to make the matrix elements \([\tilde{R}(x)]_{1/2, -1/2}^{1/2, -1/2}\) through the following unitary transformation

\[
\tilde{R}_i \ i+1(V(x)) = V(x) \tilde{R}_i \ i+1(V(x)^i),
\]

(43)

where \( V(x) = V_{i}(x) \otimes [V_{i+1}(x)]^{-1} \) and

\[
V_{i}(x) = \begin{pmatrix} 0 & x^{-\frac{\pm}{2}} \\ \eta x^{-\frac{\pm}{2}} & 0 \end{pmatrix}.
\]

(44)

The resultant \( \tilde{R}_i \ i+1(V(x)) \) is still a solution of YBE. Let only the parameter \( \eta = e^{i\varphi(t)} \) be time-dependent, it yields from Eq. (9) and Eq. (13) that

\[
H_2(x, \varphi(t)) = \hbar \dot{\varphi} \left[ q^2 + q^{-2} - (x^2 + x^{-2}) \right]^{-1} \left[ (x - x^{-1})(S_1^3 - S_2^3) + (q - q^{-1})(e^{i\varphi} S_1^3 S_2^3 - e^{-i\varphi} S_1^3 S_2^{-3}) \right].
\]

(45)

Putting \( x = e^{i\vartheta}, \vartheta = \pi/2 - \theta \) and \( \varphi(t) = \phi(t) - \pi/2 = \omega t \), we have

\[
H_2(\theta, \phi(t)) = -4\hbar \omega \left[ q^2 + q^{-2} + 2 \cos 2\vartheta \right]^{-1} \cos \theta \times \left[ \cos \theta (S_1^3 - S_2^3) + \frac{1}{2}(q - q^{-1}) (e^{i\varphi} S_1^3 S_2^3 - e^{-i\varphi} S_1^3 S_2^{-3}) \right],
\]

(46)

whose two nonzero eigenvalues are

\[
E_{\pm}^2 = -4\hbar \dot{\varphi} \left[ q^2 + q^{-2} + 2 \cos 2\theta \right]^{-1} \cos \theta \lambda_{\pm} = -\frac{4\hbar \dot{\varphi} \cos \theta}{\lambda_{\pm}}.
\]

(47)
with
\[ \lambda_\pm = \pm \sqrt{\cos^2 \theta + (q - q^{-1})^2/4}. \]

Under the adiabatic approximation the corresponding eigenstates are
\[ |\Phi_+(\theta, \phi)\rangle = \frac{1}{\sqrt{2\lambda_+}} [(\lambda_+ - \cos \theta)^{-1/2} (a_{q^{-1}} - b) \uparrow \downarrow] + i(\lambda_+ - \cos \theta)^{1/2} e^{i\phi} \uparrow \uparrow], \]
\[ |\Phi_- (\theta, \phi)\rangle = \frac{1}{\sqrt{2\lambda_+}} [i(\lambda_+ - \cos \theta)^{-1/2} (q^{-1} - a) \uparrow \downarrow] - (\lambda_+ - \cos \theta)^{1/2} e^{i\phi} \downarrow \uparrow]. \]

The corresponding Berry phases for the Yang–Baxter system are
\[ \gamma^2_\pm = \pm \pi \left(1 - \frac{1}{\lambda_+ \cos \theta}\right), \]
\[ = \pm \pi \left[1 - \frac{\cos \theta}{(\cos^2 \theta + (q - q^{-1})^2/4)^{1/2}}\right]. \]

The above Berry phases have been “q-deformed”, when \( \lambda_+ = 1 \), or \( q = \sqrt{1 + \sin^2 \theta} \pm \sin \theta \). Eq. (50) reduces to Eq. (24). Remarkably the Berry phases in Eq. (50) can still be expressed in terms of the concurrence of the states \(|\Phi_\pm(\theta, \phi)\rangle\) in Eq. (19) as \( \gamma^2_\pm = \mp \pi (1 - \sqrt{1 - C^2}) \), where \( C = (q - q^{-1})/(2\lambda_+) \).

Similarly, the Hamiltonian \( H_2(\theta, \phi(t)) \) can be rewritten in terms of \( SU(2) \) generators \( J^+ = S^+_1 S^+_2 = \hat{f}_2 \hat{f}_3 \), \( J^- = S^-_1 S^-_2 = \hat{f}_1 \hat{f}_3 \), \( J^z = (S^+_1 - S^-_1) = (\hat{n}_1 - \hat{n}_2)/2 \) as
\[ H_2(\theta, \phi(t)) = -4\hbar \omega \left(\frac{\cos \theta}{q^2 + q^{-2}} + 2 \cos 2\theta \right) H'_0(\theta, \phi(t)), \]
where
\[ H'_0(\theta, \phi(t)) = 2\varepsilon(\theta) J^+ + \Delta(t) J^- + \Delta^*(t) J^-, \]
\[ \varepsilon(\theta) = \cos \theta, \quad \Delta(t) = e^{i\phi(t)} (q - q^{-1})/2. \]

When \( q - q^{-1} = 0 \), or \( q = \pm 1 \), the Hamiltonian \( H'_0(\theta, \phi(t)) \) contracts to \( H'_0(\theta, \phi(t)) = \varepsilon(\theta)(\hat{n}_1 - \hat{n}_2) \), thus the quantum criticality occurs. Correspondingly, one may easily see that the Berry phases in Eq. (50) vanish.

**IV. CONCLUSION AND DISCUSSION**

In summary, we have obtained some spin-1/2 interaction Hamiltonians from the unitary Yang–Baxter \( \hat{R} \)-matrix. Based on which, Berry phases and quantum criticality in the Yang–Baxter systems have been studied.

Let us make three discussions to end this paper.

(i) In Sec. II and Sec. III, we have focused on the trigonometric Yang-Baxterization approach \( B(\phi) \rightarrow \hat{R}(\theta, \phi) \), based on which the Yang–Baxter Hamiltonians \( H_1(\theta, \phi(t)) \) and \( H_2(\theta, \phi(t)) \) have been established. Now let us come to discuss another approach called the rational Yang-Baxterization \( \hat{R}(\theta, \phi(t)) \). Actually, the first \( \hat{R}(u) \)-matrix discovered in Ref. [1] is a good and simple example for the rational Yang-Baxterization approach. The \( \hat{R}(u) \)-matrix reads
\[ \hat{R}_{i,i+1}(u) = \frac{-c + i u P_{i,i+1}}{c + i u}, \]
where \( c \) is a parameter appeared in the one-dimensional \( \delta \)-function interaction Hamiltonian: \( H = -\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + 2\varepsilon \sum_{i<j} \delta(x_i - x_j) \), and \( u = k_1 - k_{i+1} \) is the relative momentum between the \( i \)-th particle and the \( (i + 1) \)-th particle. If \( c = 0 \), the Hamiltonian represents \( N \) free particles without any interaction, and correspondingly \( \hat{R}_{i,i+1}(u) \)-matrix reduces to \( P_{i,i+1} \). Generally, for a given braiding matrix \( B_{i,i+1} \), we may perform the following transformation
\[ \hat{R}_{i,i+1}(u) = \rho(u) \frac{\delta + u B_{i,i+1}}{\alpha + \beta u}, \]
if the \( \hat{R}_{i,i+1} \) matrices obey the YBE, then we call Eq. (54) as the rational Yang-Baxterization approach.

The Yang–Baxter Hamiltonians are induced from the \( \hat{R}(\theta, \phi) \)-matrix via Eq. (9). It is natural to ask whether the same Hamiltonian, e.g., \( H_1(\theta, \phi(t)) \), can be induced from different matrices of \( \hat{R}(\theta, \phi) \). The answer is yes. Here we would like to provide such an example.

Considering the following braiding matrix
\[ B = S^{\frac{1}{2}} = I + f U^\frac{1}{2} \]
where
\[ U^\frac{1}{2} = \begin{pmatrix} \epsilon & 0 & 0 & e^{i\varphi} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-i\varphi} \\ e^{i\varphi} & 0 & 0 & \epsilon \end{pmatrix} \]
satisfies the TL algebra, and \( d = 2c, \beta = -d/2 = -\epsilon, \epsilon = \pm 1, f = (-d \pm \sqrt{d^2 - 4})/2 = \beta. \) After performing the rational Yang–Baxterization, from Eqs. (54) and (55) one obtains the \( \hat{R}(u) \) satisfying the YBE as
\[ \hat{R}(u) = I + G(u) U, \quad G(u) = \frac{u}{\alpha + \beta u}. \]

Furthermore, the unitary condition \( [\hat{R}(u)]^\dagger = [\hat{R}(u)]^{-1} = \hat{R}(-u) \) leads to \( G(-u) = G(u)^*, \) or \( (\alpha^{-1} u)^* = -\alpha^{-1} u. \) We choose
\[ \alpha u^{-1} = i \tan \theta, \]
then it is easy to have from Eq. (9) and Eq. (57) that
\[ H(u, \phi) = i \hbar G(u) G(-u) \frac{\partial U}{\partial t} [G(u)^{-1} I + U]. \]
Substituting Eq. (57) into Eq. (59) one obtains
\[ H_3(\theta, \varphi(t)) = -i\hbar \varphi \cos \theta \{ \cos \theta (S^z_1 + S^z_2) \}
- i \sin \theta \{ e^{i\varphi} S^+_1 S^+_2 - e^{-i\varphi} S^-_1 S^-_2 \} \}. \] (60)

After redefining the parameter \( \varphi(t) = \phi(t) - \pi/2 \), one may find that the Hamiltonian \( H_3(\theta, \varphi(t)) \) is identical to the Hamiltonian \( H_1(\theta, \phi(t)) \) as shown in Eq. (15).

(ii) The same matrix \( U^{\uparrow \downarrow \downarrow \uparrow} \) may yield inequivalent Yang–Baxter Hamiltonians. For instance, Let
\[ U^{\uparrow \downarrow \downarrow \uparrow} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon & e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} & \epsilon & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \] (61)

which is a special form of Eq. (40) by taking \( q = \epsilon = \pm 1 \). Based on the same matrix \( U^{\uparrow \downarrow \downarrow \uparrow} \), one can have two kinds of inequivalent braiding matrices, one is \( B_1 = q(I - q^{-1} U^{\uparrow \downarrow \downarrow \downarrow}) \) in Eq. (59), the other is \( B_2 = I + f U^{\uparrow \downarrow \downarrow \uparrow} \) in Eq. (55). After making the trigonometric Yang-Baxterization for the former one \( B_1 \), it yields the Yang–Baxter Hamiltonian \( H_2(\theta, \phi(t)) \) as in Eq. (40); similarly, after making the rational Yang-Baxterization for the latter one \( B_2 \), it yields the following Yang–Baxter Hamiltonian:
\[ H_4(\theta, \phi(t)) = -\hbar \varphi \cos \theta \{ \cos \theta (S^z_1 - S^z_2) \}
+ \sin \theta \{ e^{i\varphi} S^+_1 S^-_2 + e^{-i\varphi} S^-_1 S^+_2 \} \}. \] (62)

The Hamiltonian \( H_4(\theta, \phi(t)) \) is inequivalent to Hamiltonian \( H_3(\theta, \phi(t)) \), and it has the same eigenvalues and Berry phases as those of the Hamiltonian \( H_1(\theta, \phi(t)) \).

(iii) For the spins at \( N \)-lattices one may define the following \( SU(2) \) generators as
\[ S^+ = H_{N} \prod_{i=1}^{N} S^+_i = \prod_{i=1}^{N} f^+_i = \begin{pmatrix} 0 & 1 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}, \]
\[ S^- = \prod_{i=1}^{N} S^-_i = \prod_{i=1}^{N} f_i = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \]
\[ S^3 = [S^+, S^-]/2 = \left( \prod_{i=1}^{N} \hat{n}_i - \prod_{i=1}^{N} (1 - \hat{n}_i) \right)/2 \]
\[ = \frac{1}{2} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 \end{pmatrix}. \] (63)

Similarly, the \( SU(2) \) Casimir operator is \( \frac{1}{2}(S^+ S^- + S^- S^+) + (S^3)^2 = \frac{1}{2}(\prod_{i=1}^{N} \hat{n}_i + \prod_{i=1}^{N} (1 - \hat{n}_i)) \), which equals to \( \frac{1}{2}(N + 1) \) if all \( \hat{n}_i \)'s are equal, and otherwise \( 0(0 + 1) \).

When \( S^+ \) acts on the vacuum state, it produces the state with all spins up \( | \uparrow \uparrow \cdots \uparrow \rangle \), similarly, \( S^- \) produces the state with all spins down \( | \downarrow \downarrow \cdots \downarrow \rangle \). The states \( | \uparrow \uparrow \cdots \uparrow \rangle \) and \( | \downarrow \downarrow \cdots \downarrow \rangle \) are the chiral spin states, or the “chiral photons”. All the similar discussion on quantum criticality and Berry phase can be extended to multipartite spin-1/2 systems. Eventually, people have currently found that braiding operators have some important physical applications in non-Abelian quantum Hall systems, topological quantum field theory and topological quantum computation \[25, 29, 30, 31\], how to apply the Yang–Baxter \( \hat{R}(\theta, \phi) \)-matrix (that is a generalization of the braiding operator) to these fields is an interesting and significant topic. We shall investigate this subject subsequently.

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