Generalized Yang-Mills Theory under Rotor Mechanism

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Abstract

This paper follows the previous work on generalized abelian gauge field theory of higher-order derivatives under rotor model and extends the study to the most generalized non-abelian case. We find that the rotor mechanism from the abelian case applies nicely to the non-abelian case under the Lorentz gauge condition. Under the rotor mechanism, the gauge field transforms as $T^a_\mu \rightarrow \Box^n T^a_\mu$. When the order of field derivative is $n = 0$, this restores back to the original Yang-Mills action. Our work gives an extensive generalization of the Yang-Mills theory with higher-order field derivatives. We also compute the equation of motion and Noether’s current of the generalized non-abelian gauge field theory. Finally, we study the dynamic instability issue of the theory by the Ostrogradsky construction and the analysis of the 00-component of the energy-momentum tensor.

Keywords: Yang-Mills theory; non-abelian gauge field; rotor mechanism; dynamic instability

1 Introduction

The study of high-order derivatives system has a long history dated back to 1950s, with the classic development of Pais-Uhlenbeck oscillator [1] and Podolsky electrodynamics [2, 3, 4, 5]. Quantum field theories with high-order derivatives are of appealing interest because they have the possibilities to eliminate ultraviolet divergences in the calculation of scattering amplitudes [2, 3, 4, 5, 6, 7, 8, 9]. Vast amount of studies in higher order derivative field theories including both scalar fields and gauge fields have been performed [1, 10, 11, 12]. However, there are difficulties in establishing the formalism as these theories are often non-renormalizable and dynamically unstable with unbounded Hamiltonian [13, 14, 15, 16, 17]. Yet, higher derivative field theories give insight to the study of quantum gravity and modified theories of gravity and hence it is worth to establish new formalisms on high-order theories [18, 19, 20, 21, 22].

In our previous paper [23], we have established the formalism of generalized abelian gauge field theory under rotor model with higher-order derivatives. We have shown the following theorem:

$$S = -\frac{1}{4} \int d^D x G_{\mu\nu} G^\mu_\nu = \frac{1}{4^n} \int d^D x (\Box^n T^\mu) R^\mu_\nu (\Box^n T^\nu) = -\frac{1}{4^{n+1}} \int d^D x \Box^n G_{\mu\nu} \Box^n G^{\mu\nu},$$

(1)

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where \( \hat{R}_{\mu
u} \) is the projection tensor \( \hat{R}_{\mu
u} = \frac{1}{2}(\Box \eta_{\mu\nu} - \partial_\mu \partial_\nu) \) and \( G_{n\mu\nu} = \partial_\mu T_{n\nu} - \partial_\nu T_{n\mu} \) is the field strength of the \( n \)-th order rotor model. We have the \( n \)-th order gauge field strength identified as \[23]\]

\[
G_{n\mu\nu} \equiv \frac{1}{2n} \Box^n G_{\mu\nu}.
\]

The projection is regarded as the second-order rotation \[23\]. Therefore, the \( n \)-th order rotation of gauge field \( T_{n\mu\nu} \) is attained by successive second-order rotations given by \[23\]

\[
T_{n\mu\nu} = \hat{R}_{\mu\nu_{n-1}} \hat{R}_{\mu_{n-2} \nu} \cdots \hat{R}_{\mu_2 \nu_1} \hat{R}_{\mu_1 \nu_0} T_{\mu_0} = \frac{1}{2n-1} P_{\mu_{n-1} \nu} P_{\mu_{n-2}} \cdots P_{\mu_2} P_{\mu_1} P_{\mu_0} T_{\mu_0},
\]

for which

\[
P_{\mu_j} = \Box \delta_{\mu_j}^{-1}
\]

is called the propagator. This is known as the rotor transformation which generates high-order derivative gauge fields. In the generalized theory, the action changes by the transformation of gauge field \( T_{n\mu} \rightarrow \Box^n T_{\mu} \). The working dimension \( D \) for renormalizability is \( 4n + 4 \) for unity gauge field dimension \[23\]. When \( n = 0 \) this reduces back to the conventional Maxwell action

\[
S = -\frac{1}{4} \int d^4x G_{\mu\nu} G^{\mu\nu} = \int d^4x T_{\mu} \hat{R}_{\mu\nu} T_{\nu}.
\]

In this article, we aim at constructing a generalized non-abelian gauge field theory (Yang-Mills theory) under the rotor mechanism. There is no guarantee that the old mechanism applies due to the existence of extra terms in the action for the non-abelian case. For the non-abelian case, we have the Yang-Mills action in general \( D \) dimensional spacetime as \[24\],

\[
S_{YM} = -\frac{1}{2} \int d^Dx Tr G_{\mu\nu} G^{\mu\nu} = -\frac{1}{4} \int d^Dx G_{\mu\nu} G^{\mu\nu a} a
\]

where

\[
G_{\mu\nu} = \partial_\mu T_\nu - \partial_\nu T_\mu - ig[T_\mu, T_\nu],
\]

for which \( G_{\mu\nu} = F_{\mu\nu}^a t^a \) and \( T_\mu = T_\mu^a t^a \) are matrices with \( t^a \) the generators of \( SU(N) \) Lie group. We have to sum over repeated generator indices \( a \). Using the Lie algebra \([t^a, t^b] = if^{abc} t^c\), this gives the gauge field strength as,

\[
G_{\mu\nu}^a = \partial_\mu T_{\nu}^a - \partial_\nu T_{\mu}^a + gf^{abc} T_{\mu}^b T_{\nu}^c.
\]

Using integration by parts on the kinetic term, equation \[10\] gives

\[
S_{YM} = \int d^Dx \left(T_{\mu a} \hat{R}_{\mu\nu} T_{\nu a} - \frac{g}{2} f^{abc}(\partial_\mu T_{\nu a} - \partial_\nu T_{\mu a}) T_{\mu}^b T_{\nu}^c + \frac{g^2}{4} f^{abc} f^{def} T_{\mu}^b T_{\nu}^c T_{\rho}^d T_{\sigma}^e \right),
\]

where the first term is Maxwell-like, and we have the second term and the third as the extra self-coupling terms compared to the abelian case. Let’s define the Maxwell-like action as

\[
S_{Maxwell} = \int d^Dx (T_{\mu a} \hat{R}_{\mu\nu} T_{\nu a}),
\]
in such a way that we can use all the results we have in our previous work\textsuperscript{23}. We also define the Maxwell-like gauge field strength as \( \tilde{G}_{\mu\nu} = \partial_\mu T_\nu - \partial_\nu T_\mu \), so that
\[
G_{\mu\nu} = \tilde{G}_{\mu\nu} - ig[T_\mu, T_\nu],
\]
and
\[
G_{\mu\nu}^a \to \tilde{G}_{\mu\nu}^a + g f^{abc} T_\mu^b T_\nu^c.
\]
Under the rotor mechanism, the non-abelian gauge field transforms as
\[
T_{\mu\nu}^a \to T_{n\mu\nu}^a = (\hat{R}_{\mu_1\mu_2\mu_3} \hat{R}_{\nu_1\nu_2\nu_3} \ldots \hat{R}_{\mu_{n-2}\mu_{n-1}} \hat{R}_{\nu_{n-2}\nu_{n-1}} \hat{R}_{\mu_{n-1}\nu_{n-1}}} T_{\mu\nu}^a.
\]

2 The generalized non-abelian gauge field theorem under rotor model

In this paper, in analogy to the abelian case of our previous work in \textsuperscript{23}, we aim at proving the following theorem for the non-abelian case,
\[
S_{\text{YM}}^{(n)} = -\frac{1}{4} \int d^D x \frac{g}{n} G_{n\mu\nu} G_{n}^{\mu\nu} - \frac{1}{2} \cdot \frac{g}{n} f^{abc} (\partial_\mu \square T_\nu^a - \partial_\nu \square T_\mu^a) \square T_\mu^b \square T_\nu^c
\]
\[
\quad - \frac{g^2}{4 \cdot 16^n} f^{abc} f^{ade} \square T_\mu^a \square T_\nu^c \square T_\mu^b \square T_\nu^c
\]
\]
(14)
We will see that indeed the rotor mechanism in the abelian case applies to the non-abelian case as well without modification. The only necessary condition is the Lorentz gauge condition \( \partial_\mu T_\mu^a = 0 \). In such way the rotor mechanism functions the gauge field to transform as
\[
T_\mu^a \to \square^n T_\mu^a.
\]
(15)
And the gauge field strength transforms as
\[
G_{\mu\nu} \to \partial_\mu \square^n T_\nu - \partial_\nu \square^n T_\mu - ig[\square^n T_\mu, \square^n T_\nu] = \square^n \tilde{G}_{\mu\nu} - ig(\square^n T_\mu T_\nu - \square^n T_\nu T_\mu)\text{.}
\]
(16)
or explicitly
\[
G_{\mu\nu}^a \to \partial_\mu \square^n T_\nu^a - \partial_\nu \square^n T_\mu^a + g f^{abc} \square^n T_\mu^b \square^n T_\nu^c = \square^n \tilde{G}_{\mu\nu}^a + g f^{abc} \square^n T_\mu^b \square^n T_\nu^c.
\]
(17)
When \( n = 0 \), then general form of equation (14) will reduce back to (6). We will first prove the theorem for \( n = 1 \), followed by \( n = 2 \) case and finally the general \( n \) case as we do in \textsuperscript{23}.

2.1 The \( n = 1 \) case

For the first-ordered rotation, the first-ordered rotated field is
\[
L_\mu^a = \hat{R}_{\mu\nu} T_\nu^a.
\]
(18)
The new gauge field strength is
\[
H_{\mu\nu}^a = \partial_\mu L_\nu^a - \partial_\nu L_\mu^a + g f^{abc} L_\mu^b L_\nu^c.
\]
(19)
The new action becomes
\[ S_{YM}^{(1)} = \int d^4x \left( L^{\mu a} \hat{R}_{\mu a} L^{\nu a} - \frac{g}{2} f^{abc}(\partial^\mu L^{\nu a} - \partial^\nu L^{\mu a}) L_b^c L^\nu_\nu - \frac{g^2}{4} f^{abc} f^{ade} L_b^c L^\mu_\nu L^{\mu d} L^{\nu e} \right) , \] (20)

For the first Maxwell-like term, we can apply the result we obtained in our previous work [23] (equation 27), in which we have
\[ \int d^4x L^{\mu a} \hat{R}_{\mu a} L^{\nu a} = \frac{1}{4} \int d^4x \Box T^{\mu a} \hat{R}_{\mu a} \Box T^{\nu a} . \] (21)

Then we have
\[ S_{YM}^{(1)} = \int d^4x \left( \frac{1}{4} \Box T^{\mu a} \hat{R}_{\mu a} \Box T^{\nu a} - \frac{g}{2} f^{abc}(\partial^\mu \hat{R}^{\nu a} - \partial^\nu \hat{R}^{\mu a}) \hat{R}_{\mu a} T^{ab} \hat{R}_{\nu b} T^{bc} \right) \]
\[ = \int d^4x \left( \frac{1}{4} \Box T^{\mu a} \hat{R}_{\mu a} \Box T^{\nu a} - \frac{g}{2} f^{abc}(\partial^\mu \hat{R}^{\nu a} T_\delta^a - \partial^\nu \hat{R}^{\mu a} T_\delta^a) \hat{R}_{\mu a} T^{ab} \hat{R}_{\nu b} T^{bc} \right) \]
\[ \quad - \frac{g^2}{4} f^{abc} f^{ade} \hat{R}_{\mu a} T^{ab} \hat{R}_{\nu b} T^{bc} R^{\nu e} T_\gamma^d R^{\gamma d} T_\delta^e \] .
(22)

Now we investigate the second term and the third term of the last line. For the second term, we have
\[ - \frac{g}{2} f^{abc}(\partial^\mu \hat{R}^{\nu a} T_\delta^a - \partial^\nu \hat{R}^{\mu a} T_\delta^a) \hat{R}_{\mu a} T^{ab} \hat{R}_{\nu b} T^{bc} \]
\[ = - \frac{1}{16} g f^{abc} \left( \partial^\mu \Box T^{\nu a} - \partial^\nu \Box T^{\mu a} \right) \hat{R}_{\mu a} T^{ab} \hat{R}_{\nu b} T^{bc} \]
\[ \times \left( \partial^\mu \square T_\nu^a - \partial^\nu \square T_\mu^a \right) (\Box \eta_{\mu a} - \partial_{\nu} \partial_{\mu} T^{\nu a} \hat{R}_{\mu a} T^{ab} \hat{R}_{\nu b} T^{bc} ) \]
\[ = - \frac{1}{16} g f^{abc} \left( \partial^\mu \Box T^{\nu a} - \partial^\nu \Box T^{\mu a} \right) \hat{R}_{\mu a} T^{ab} \hat{R}_{\nu b} T^{bc} \]
\[ \times \left( \partial^\mu \square T_\nu^a - \partial^\nu \square T_\mu^a \right) (\Box \eta_{\mu a} - \partial_{\nu} \partial_{\mu} T^{\nu a} \hat{R}_{\mu a} T^{ab} \hat{R}_{\nu b} T^{bc} ) . \] (23)

Now we impose the Lorentz gauge condition, then the last three terms vanish. Therefore we have
\[ - \frac{g}{2} f^{abc}(\partial^\mu \hat{R}^{\nu a} T_\delta^a - \partial^\nu \hat{R}^{\mu a} T_\delta^a) \hat{R}_{\mu a} T^{ab} \hat{R}_{\nu b} T^{bc} \]
\[ = - \frac{1}{16} g f^{abc}(\partial^\mu \Box T^{\nu a} - \partial^\nu \Box T^{\mu a}) \Box T_\mu^a \Box T_\nu^a . \] (24)

For the third term, we would also impose the Lorentz gauge condition,
\[ - \frac{g^2}{4} f^{abc} f^{ade} \hat{R}_{\mu a} T^{ab} \hat{R}_{\nu b} T^{bc} R^{\nu c} T_\gamma^d R^{\gamma d} T_\delta^e \]
\[ = - \frac{g^2}{64} f^{abc} f^{ade} (\Box \eta_{\mu a} T^{ab} - \partial_{\nu} \partial_{\mu} T^{ab} ) (\Box \eta_{\nu b} T^{bc} - \partial_{\gamma} \partial_{\nu} T^{bc} ) \]
\[ \times (\Box \eta_{\gamma} T_\delta^e - \partial_{\delta} \partial_{\gamma} T_\delta^e ) \] (25)
\[ = - \frac{g^2}{64} f^{abc} f^{ade} T_\mu^b T_\nu^c T^{ab} T^{bc} \Box T^{\mu d} \Box T^{\nu c} . \]
Therefore, for the \(n = 1\) case, we obtain
\[
S_{YM}^{(1)} = \int d^Dx \left( \frac{1}{4} T^{\mu a} \hat{R}_{\mu\nu} \Box T^{\nu a} - \frac{g}{16} f^{abc}(\Box^\mu T^{\nu a} - \partial^\mu \Box T^{\nu a}) \Box T^b \Box T^c - \frac{g^2}{64} f^{abc} f^{ade} \Box T^b \Box T^c \Box T^{\mu d} \Box T^{\nu e} \right) \tag{26}
\]
Thus the proof is completed.

### 2.2 The \(n = 2\) case

For the second-ordered rotation, the second-ordered rotation field is
\[
J^{\mu a} = R^{\mu a} \hat{R}_{\mu\nu} T^{\nu a} . \tag{27}
\]
Using the result in [23] (equation 30),
\[
J^{\mu a} = \frac{1}{4}(\Box^2 \delta^\mu_a - \Box \partial^\mu \partial_a)T^{\mu a} . \tag{28}
\]
Also we have
\[
J^a_\sigma = \eta_{\sigma \rho} \hat{R}^{\rho a} \hat{R}_{\mu\nu} T^{\nu a} = \frac{1}{4}(\Box^2 \eta_{\alpha \sigma} - \Box \partial_\sigma \partial_\alpha)T^{\alpha a} . \tag{29}
\]
The new gauge field strength is
\[
K^a_{\mu\nu} = \partial_\mu J^a_\nu - \partial_\nu J^a_\mu + g f^{abc} J^b_\mu J^c_\nu . \tag{30}
\]
The new action becomes
\[
S_{YM}^{(2)} = \int d^Dx \left( J^{\mu a} \hat{R}_{\mu\nu} J^{\nu a} - \frac{g}{2} f^{abc}(\partial^\mu J^{\nu a} - \partial^\nu L^{\mu a})J^b_\mu J^c_\nu - \frac{g^2}{4} f^{abc} f^{ade} J^b_\mu J^c_\nu J^d_\sigma J^e_\tau \right) , \tag{31}
\]
For the first Maxwell-like term, we can apply the result we obtained in our previous paper [23] (equation (43)), in which we have,
\[
\int d^Dx J^{\mu a} \hat{R}_{\mu\nu} J^{\nu a} = \frac{1}{16} \int d^Dx \Box^2 T^{\mu a} \hat{R}_{\mu\nu} \Box^2 T^{\nu a} . \tag{32}
\]
Explicitly, this is
\[
S_{YM}^{(2)} = \int d^Dx \left( \frac{1}{16} \Box^2 T^{\mu a} \hat{R}_{\mu\nu} \Box^2 T^{\nu a} - \frac{g}{2} f^{abc}(\partial^\mu \hat{R}^{\nu a} \hat{R}_{\rho a} T^{\rho a} - \partial^\nu \hat{R}^{\mu a} \hat{R}_{\rho a} T^{\rho a}) \right.
\]
\[
\times \eta_{\mu \rho} \eta_{\nu \sigma} \hat{R}^{\rho a} \hat{R}_{\sigma a} T^{\sigma b} \hat{R}^{\kappa c} \hat{R}_{\tau b} T^{\tau d} \hat{R}^{\lambda e} \hat{R}_{\lambda e} \hat{R}^{\nu \tau} \hat{R}_{\tau \lambda} T^{\lambda e} \right) , \tag{33}
\]
Then we have
\[
S_{YM}^{(2)} = \int d^Dx \left( \frac{1}{16} \Box^2 T^{\mu a} \hat{R}_{\mu\nu} \Box^2 T^{\nu a} - \frac{1}{128} g f^{abc} \left( \partial^\mu (\Box^2 \delta^\nu_a - \Box \partial^\nu \partial_a)T^{\nu a} - \partial^\nu (\Box^2 \delta^\mu_a - \Box \partial^\mu \partial_a)T^{\mu a} \right) \right.
\]
\[
\times (\Box^2 \eta_{\mu a} T^{\nu b} - \Box \partial^\mu \partial_a T^{\nu b})(\Box^2 \eta_{\nu \beta} T^{\beta b} - \Box \partial_\nu \partial_\beta T^{\beta b}) \right.
\]
\[
- \frac{g^2}{1024}(\Box^2 \eta_{\mu a} T^{\nu b} - \Box \partial^\mu \partial_a T^{\nu b})(\Box^2 \eta_{\nu \beta} T^{\beta b} - \Box \partial_\nu \partial_\beta T^{\beta b})(\Box^2 \delta^\nu_T T^{\gamma d} - \Box \partial^\nu \partial_a T^{\gamma d}) \right.
\]
\[
\times (\Box^2 \delta^\lambda_T T^{\lambda e} - \Box \partial^\lambda \partial_a T^{\lambda e}) \right) . \tag{34}
\]
Applying Lorentz gauge condition, we obtain

\[
S^{(2)}_{YM} = \int d^Dx \left( \frac{1}{16} \Box^2 T^{\mu\nu} \hat{R}_{\mu\nu} \Box^2 T^{\rho\sigma} - \frac{g}{128} f^{abc} \left( \partial \Box^2 \delta_\alpha^\gamma \partial T^{\alpha\beta} \right) T^{\rho\sigma} \eta_{\mu\rho} \eta_{\nu\beta} T^{\delta\epsilon} \right)
- \frac{g^2}{1024} \eta_{\mu\rho} \eta_{\nu\beta} \partial T^{\delta\epsilon} \partial T^{\gamma\delta} \partial T^{\lambda\epsilon}
\]

\[
= \int d^Dx \left( \frac{1}{16} \Box^2 T^{\mu\nu} \hat{R}_{\mu\nu} \Box^2 T^{\rho\sigma} - \frac{g}{128} f^{abc} \left( \partial \Box^2 T^{\mu\nu} - \partial T^{\lambda\mu} T^{\sigma\lambda} \right) \partial T^{\mu\nu} \right)
- \frac{g^2}{1024} \eta_{\mu\rho} \eta_{\nu\beta} \partial T^{\delta\epsilon} \partial T^{\gamma\delta} \partial T^{\lambda\epsilon} \right).
\]

which completes the proof for \( n = 2 \) case.

### 3 The general \( n \) case

To prove the general \( n \) case, we use the results in our previous paper \[23\] for \( n \)-th order rotation and apply to the non-abelian case. Equation (3) and equation give (4)

\[
T^{\alpha}_{\nu \mu} = (\hat{R}_{\nu \mu \rho \sigma} \hat{R}^{\rho \sigma}_{\nu \mu - 2} \ldots \hat{R}_{\nu \mu \rho} \hat{R}^{\rho}_{\nu \mu}) T^{\mu \nu} = \frac{1}{2n-1} \left( \Box^{n-1} \delta^{\mu \nu}_{\mu \nu} \right) \hat{R}_{\nu \mu \nu} T^{\nu \mu}.
\]

The new gauge field strength is

\[
G^{a}_{\mu \nu} = \partial \mu T^{a}_{\nu} - \partial \nu T^{a}_{\mu} + g f^{abc} T^{b}_{\mu \nu} T^{c}_{\nu \mu}.
\]

The new action becomes

\[
S^{(2)}_{YM} = \int d^Dx \left( T^{\mu \nu} \hat{T}_{\mu \nu} T^{\nu \sigma} - \frac{g}{2} f^{abc} \left( \partial \mu T^{\nu a} - \partial T^{\lambda \mu} T^{b}_{\mu \nu} \right) T^{b}_{\mu \nu} T^{c}_{\nu \mu} - \frac{g^2}{2} f^{abc} f^{ade} T^{a}_{\mu \nu} T^{b}_{\nu \rho} T^{c}_{\rho \mu} \right),
\]

For the first Maxwell-like term, we can apply the result we obtained in our previous paper \[23\] (equation (75)), in which we have,

\[
\int d^Dx T^{\mu \nu} \hat{R}_{\mu \nu} T^{\nu \rho} = \frac{1}{4n} \int d^Dx \Box^n T^{\mu \nu} \hat{R}_{\mu \nu} \Box^n T^{\nu \rho}.
\]

Therefore the action is

\[
S^{(2)}_{YM} = \int d^Dx \left( \frac{1}{4n} \Box^n T^{\mu \nu} \hat{R}_{\mu \nu} \Box^n T^{\rho \sigma} - \frac{1}{2n+1} g f^{abc} \left( \partial \Box^n \delta^{\nu \mu}_{\nu \mu} \hat{R}_{\mu \nu \rho} T^{\mu \nu} \right)
\]

\[
- \partial \partial^n \eta^{\mu \nu} \hat{R}_{\mu \nu \rho} T^{\mu \nu} \right)(\Box^{n-1} \delta^{\mu \nu}_{\mu \nu} \hat{R}_{\nu \mu \rho} T^{\mu \nu})(\Box^{n-1} \delta^{\nu \mu}_{\nu \mu} \hat{R}_{\mu \nu \rho} T^{\mu \nu})(\Box^{n-1} \delta^{\mu \nu}_{\mu \nu} \hat{R}_{\nu \mu \rho} T^{\mu \nu})
\]

\[
= \frac{g^2}{2n+1} f^{abc} f^{ade} \left( \Box^{n-1} \delta^{\mu \nu}_{\mu \nu} \hat{R}_{\nu \mu \rho} T^{\mu \nu} \right)(\Box^{n-1} \delta^{\nu \mu}_{\nu \mu} \hat{R}_{\mu \nu \rho} T^{\mu \nu})(\Box^{n-1} \delta^{\mu \nu}_{\mu \nu} \hat{R}_{\nu \mu \rho} T^{\mu \nu})
\]

But since under the Lorentz gauge, we immediately have

\[
T^{\mu \nu} = \frac{1}{2n-1} \left( \Box^{n-1} \delta^{\mu \nu}_{\mu \nu} \right) \hat{R}_{\mu \nu} \Box^n T^{\mu \nu} = \frac{1}{2n} \Box^n T^{\mu \nu}.
\]
Therefore we have
\[ S_{\text{YM}}^{(n)} = -\frac{1}{4} \int d^D x G_{n \mu \nu}^a G_{n \mu \nu}^a = \int d^D x \left( \frac{1}{4^n} \Box^n T^{\mu a} \bar{R}_{\mu \nu} \Box^n T^{\nu a} - \frac{1}{2} 8^n g f^{abc} (\partial^\rho \Box^n T^{\rho a} - \partial^\rho \Box^n T^{\rho a}) \Box^n T^{\mu b} \Box^n T^{\nu c} - \frac{g^2}{4} f^{abc} f^{def} \Box^n T^{\mu b} \Box^n T^{\nu d} \Box^n T^{\rho e} \right), \]

which completes the proof of the general \( n \) case. Therefore, under the rotor mechanism, the non-abelian gauge field transforms as \( T^{\mu a} \rightarrow \Box^n T^{\mu a} \) as we can compare to the original action in equation (42). When \( n = 0 \), this would give us back the original Yang-Mills action. The renormalization dimension that requires to keep unity dimension of \( T^{\mu a} \) field would be same as the abelian case, which is \( D = 4n + 4 \) [23].

4 Equation of motion and Noether’s current of the generalized abelian gauge field theory

The equation of motion of the original Yang-Mill’s theory is given by [24, 25],
\[ \partial_\mu F^{\mu \nu a} + g f^{abc} T^b_\mu G^{\mu \nu c} = 0. \]

(43)

The covariant derivative is defined by
\[ D_\mu = \partial_\mu - ig T^c_\mu T^a_c. \]

(44)

In tensor form,
\[ D^a_\mu = \partial_\mu \delta^{ab} - ig T^c_\mu (T^a_c)^{ab}. \]

(45)

Using the adjoint representation \((t^k)^{ij} = -if^{kij}\), we would have
\[ D^a_\mu = \partial_\mu \delta^{ab} + g f^{abc} T^c_\mu. \]

(46)

Therefore, the equation of motion in (43) can be neatly written by the covariant derivative acting on the non-abelian gauge field strength,
\[ D_\mu G^{\mu \nu a} = 0. \]

(47)

Expanding (43) or (47) explicitly gives
\[ 2 \hat{R}^{\mu \nu a} T^{\mu a} + g f^{abc}(2T^a_\mu \partial_\mu T^{\nu c} + (\partial_\mu T^{\nu b})T^{\nu c} + T^b_\mu \partial_\nu T^{\mu k} + g f^{cde} T^b_\mu T^{\mu d} T^{\nu e}) = 0. \]

(48)

Under Lorentz gauge, the equation of motion simplifies to
\[ 2 \Box T^{\mu a} + g f^{abc}(2T^a_\mu \partial_\mu T^{\nu c} + T^b_\mu \partial_\nu T^{\mu k} + g f^{cde} T^b_\mu T^{\mu d} T^{\nu e}) = 0. \]

(49)

When we compare it to the equation of motion of the abelian case \( 2 \hat{R}^{\mu \nu} T_\mu = 0 \), we see that the non-abelian case contains extra non-linear terms.

To find the equation of motion for our generalized rotor model for non-abelian case, we need to vary the action as follow,
\[ \delta S_{\text{YM}}^{(n)} = -\frac{1}{2} \int d^D x G^{\mu \nu a} \delta G_{n \mu \nu}^a. \]

(50)
The remaining work is to compute $\delta G^a_{\mu \nu}$. Using the transformation of field under the Lorentz gauge as in (12),

$$T_{n \mu} = \frac{1}{2^n} \Box^n T_{\mu},$$

(51)

first we have,

$$G^a_{n \mu \nu} = \frac{1}{2^n} \partial_{\mu} \Box^n T_{n \nu} - \frac{1}{2^n} \partial_{n} \Box^n T_{\mu} + \frac{1}{4^n} g f^{abc} \Box^n T_{n b} \Box^n T_{n c}.$$  

(52)

Then we have the variation as

$$\delta G^a_{n \mu \nu} = \frac{1}{2^n} \partial_{\mu} \delta \Box^n T_{n \nu} - \frac{1}{2^n} \partial_{n} \delta \Box^n T_{\mu} + \frac{1}{4^n} g f^{abc} (\Box^n T_{n b} \delta \Box^n T_{n c} + \delta \Box^n T_{n b} \Box^n T_{n c})$$

$$= \frac{1}{2^n} \partial_{\mu} \delta \Box^n T_{n \nu} - \frac{1}{2^n} \partial_{n} \delta \Box^n T_{\mu} + \frac{1}{4^n} g f^{abc} \partial_{\nu} \Box^n T_{n b} \delta \Box^n T_{n c} + \frac{1}{4^n} g f^{abc} \delta \Box^n T_{n b} \Box^n T_{n c}$$

$$= \frac{1}{2^n} \partial_{\mu} \delta \Box^n T_{n \nu} - \frac{1}{2^n} \partial_{n} \delta \Box^n T_{\mu} + \frac{1}{4^n} g f^{abc} \partial_{\nu} \Box^n T_{n b} \delta \Box^n T_{n c} - \frac{1}{4^n} g f^{abc} \delta \Box^n T_{n b} \Box^n T_{n c}$$

$$= \left( \frac{1}{2^n} \partial_{\mu} \delta \Box^n T_{n \nu} + \frac{1}{4^n} g f^{abc} \partial_{\nu} \Box^n T_{n b} \delta \Box^n T_{n c} \right) - \left( \frac{1}{2^n} \partial_{n} \delta \Box^n T_{\mu} + \frac{1}{4^n} g f^{abc} \partial_{\nu} \Box^n T_{n b} \delta \Box^n T_{n c} \right)$$

$$:= \frac{1}{2^n} (D_{n \mu} \delta \Box^n T_{n \nu} - D_{n \nu} \delta \Box^n T_{\mu}),$$

(53)

where we define the covariant derivative of the rotor model as

$$D_{n \mu} = \partial_{\mu} - \frac{i}{2^n} g \Box^n T_{n \mu} t_{c},$$

(54)

or in tensor form

$$D^{ab}_{n \mu} = \partial_{\mu} \delta^{ab} - \frac{i}{2^n} g \Box^n T_{n \mu} (t^c)^{ab},$$

(55)

and using the adjoint representation $\left(t^c\right)^{ab} = -i f^{cab}$, then we have

$$D_{n \mu}^{ab} = \partial_{\mu} \delta^{ab} - \frac{1}{2^n} g f^{cab} \Box^n T_{n c} = \partial_{\mu} \delta^{ab} - \frac{1}{2^n} g f^{abc} \Box^n T_{n c}. $$

(56)

Therefore the covariant derivative in adjoint representation is

$$D^{ab}_{n \mu} = \partial_{\mu} \delta^{ab} + \frac{1}{2^n} g f^{abc} \Box^n T_{n \mu}.$$  

(57)

Therefore we have

$$\frac{1}{2^n} D_{n \mu}^{ab} \delta \Box^n T_{n \nu} = \frac{1}{2^n} D_{n \mu} \delta \Box^n T_{n \nu} = \frac{1}{2^n} \left( \partial_{\mu} \delta^{ab} + \frac{1}{2^n} g f^{cab} \Box^n T_{n c} \right) \delta \Box^n T_{n \nu}$$

$$= \frac{1}{2^n} \partial_{\mu} \delta \Box^n T_{n \nu} + \frac{1}{4^n} g f^{abc} \partial_{\nu} \Box^n T_{n b} \delta \Box^n T_{n c}$$

(58)

as expected. Similarly,

$$\frac{1}{2^n} D_{n \nu}^{ab} \delta \Box^n T_{n \mu} = \frac{1}{2^n} \partial_{\nu} \delta \Box^n T_{n \mu} + \frac{1}{4^n} g f^{abc} \partial_{\mu} \Box^n T_{n b} \delta \Box^n T_{n c}$$

(59)
as expected. Now the variation of action becomes

\[ \delta S^{(n)}_{YM} = -\frac{1}{2} \int d^D x G^\mu_\nu a_n \delta G^a_{n \mu 
u} \]

\[ = -\frac{1}{2} \int d^D x G^\mu_\nu (D_n \mu \delta \square^n T^a_\nu - D_n \nu \delta \square^n T^a_\mu) \]

\[ = \int d^D x G^\mu_\nu D_n \mu \delta \square^n T^a_\nu \]

\[ = - \int d^D x (D_n \mu G^\mu_\nu a_n) \delta \square^n T^a_\nu , \]

where from the third line to the forth line we have performed integration by parts. The action is extremeized when

\[ \frac{\delta S^{(n)}_{YM}}{\delta \square^n T^a_\nu} = 0 . \] (61)

Thus we obtain the equation of motion for the generalized non-abelian gauge field theory as follow

\[ D_n \mu G^\mu_\nu a_n = 0 , \] (62)

or using equation (57), acting on \( G^b_\mu \nu \)

\[ \partial_\mu G^a_\nu a_n + \frac{1}{2n} g f^{abc} \square^n T^b_\mu \square^n T^c_\nu = 0 . \] (63)

Explicitly expanding equation (63), the original equation of motion becomes (48) under the rotor mechanism, which is as follow,

\[ 2 \hat{R}^{\mu \nu} \square^n T^a_\mu + g f^{abc} \frac{1}{2n} (2 \square^n T^a_\mu \partial_\mu \square^n T^{bc} + (\partial_\mu \square^n T^{ab}) \square^n T^{bc} + \square^n T^{ab} \partial_\nu \square^n T^{\mu c}) + \frac{1}{8n} g f^{cde} \square^n T^b_\mu \square^n T^{cd} \square^n T^{\nu e} = 0 . \] (64)

Under the Lorentz gauge condition, the equation of motion simplifies to

\[ 2 \square^{n+1} T^a_\nu + g f^{abc} \frac{1}{2n} (2 \square^n T^a_\mu \partial_\mu \square^n T^{bc} + \square^n T^b_\mu \partial_\nu \square^n T^{\mu c} + \frac{1}{8n} g f^{cde} \square^n T^b_\mu \square^n T^{cd} \square^n T^{\nu e} ) = 0 . \] (65)

The general covariant derivative of the rotor model in (54) in matrix form satisfies the Jacobi identity,

\[ [D_n \rho, [D_n \mu, D_n \nu]] + [D_n \mu, [D_n \nu, D_n \rho]] + [D_n \nu, [D_n \rho, D_n \mu]] = 0 . \] (66)

And since the commutator of covariant derivative is promoted to

\[ [D_n \mu, D_n \nu] = -ig F^a_{n \mu \nu} , \] (67)

therefore we obtain the Bianchi identity of high-order covariant derivative as

\[ D_n \mu F^a_{n \mu \nu} + D_n \nu F^a_{n \nu \rho} + D_n \rho F^a_{n \rho \mu} = 0 . \] (68)

Finally, we will compute the Noether’s current and the associated Noether’s charge of our high-order non-abelian gauge field theory under rotor model. First we know
that by the transformation of field due to the rotor mechanism $T^a_\mu \to \Box^n T^a_\mu$, the Euler-Lagrangian equation becomes,

$$\frac{\partial \mathcal{L}_{YM}^{(n)}}{\partial \Box^n T^a_\nu} = \partial_\mu \frac{\partial \mathcal{L}_{YM}^{(n)}}{\partial (\partial_\mu \Box^n T^a_\nu)}.$$ \hspace{1cm} (69)

Using the Euler-Lagrangian equation, it can be shown that by the same derivation as in [23] (equation 103), that the Noether current is identified by

$$J^\mu = \frac{\partial \mathcal{L}_{YM}^{(n)}}{\partial (\partial_\mu \Box^n T^a_\nu)} \partial \Box^n T^a_\nu.$$ \hspace{1cm} (70)

The Lagrangian can be broken down into three separate parts: the Maxwell part, the mixing part and the self-coupling part.

$$\mathcal{L}_{YM}^{(n)} = \mathcal{L}_{\text{Maxwell}}^{(n)} + \mathcal{L}_{\text{mixing}}^{(n)} + \mathcal{L}_{\text{self-coupling}}^{(n)}.$$ \hspace{1cm} (71)

The Maxwell part is in $2n$-order derivatives,

$$\mathcal{L}_{\text{Maxwell}}^{(n)} = \frac{1}{4^n} \Box^n T^{\mu a} \tilde{F}_{\mu \nu} \Box^n T^a_\nu = -\frac{1}{4^{n+1}} \Box^n \tilde{G}^{\mu a} \Box^n \tilde{G}^{a \nu}.$$ \hspace{1cm} (72)

The mixing part is in $n$-order derivatives,

$$\mathcal{L}_{\text{mixing}}^{(n)} = -\frac{1}{2^{3n+1}} g f^{abc} (\partial^\mu \Box^n T^{\mu a} - \partial^\nu \Box^n T^{\mu a}) \Box^n T^b_\mu \Box^n T^c_\nu.$$ \hspace{1cm} (73)

The self-coupling part is in 0-order derivatives,

$$\mathcal{L}_{\text{self-coupling}}^{(n)} = -\frac{g^2}{2^{3n+2}} f^{abc} \Box^n T^b_\mu \Box^n T^c_\nu \Box^n T^{\mu a}.$$ \hspace{1cm} (74)

Hence the Noether current is

$$J^\mu = \left( \frac{\partial \mathcal{L}_{\text{Maxwell}}^{(n)}}{\partial (\partial_\mu \Box^n T^a_\nu)} + \frac{\partial \mathcal{L}_{\text{mixing}}^{(n)}}{\partial (\partial_\mu \Box^n T^a_\nu)} + \frac{\partial \mathcal{L}_{\text{self-coupling}}^{(n)}}{\partial (\partial_\mu \Box^n T^a_\nu)} \right) \Box^n T^a_\nu.$$ \hspace{1cm} (75)

The first term is the standard Maxwell case, which is evaluated as

$$\frac{\partial \mathcal{L}_{\text{Maxwell}}^{(n)}}{\partial (\partial_\alpha \Box^n T^a_\beta)} = -\frac{1}{4^n} \Box^n \tilde{G}^{\alpha \beta k}.$$ \hspace{1cm} (76)

The second term is evaluated to be

$$\frac{\partial \mathcal{L}_{\text{mixing}}^{(n)}}{\partial (\partial_\alpha \Box^n T^a_\beta)} = \frac{\partial}{\partial (\partial_\alpha \Box^n T^a_\beta)} \left( -\frac{1}{2^{3n+1}} g f^{abc} \left( \frac{\partial^\mu \Box^n T^{\mu a}}{\partial (\partial_\alpha \Box^n T^a_\beta)} - \frac{\partial^\nu \Box^n T^{\mu a}}{\partial (\partial_\alpha \Box^n T^a_\beta)} \right) \Box^n T^b_\mu \Box^n T^c_\nu \right) + 0$$

$$= \frac{1}{2^{3n+1}} g f^{abc} \left( \delta^{\nu a} \delta^{\mu} k - \delta^{\mu a} \delta^{\nu k} \right) \Box^n T^b_\mu \Box^n T^c_\nu$$

$$= \frac{1}{2^{3n+1}} g f^{abc} \left( \Box^n T^{\alpha b} \Box^n T^{\beta c} - \Box^n T^{\alpha c} \Box^n T^{\beta b} \right)$$

$$= \frac{1}{2^{3n+1}} g f^{abc} \Box^n T^{\alpha b} \Box^n T^{\beta c}.$$ \hspace{1cm} (77)
The third term is zero as there are no derivatives of the rotor field,

$$\frac{\partial L^{(n)}}{\partial (\partial^\mu \square n T^\alpha_\mu)} = 0.$$  \tag{78}

Therefore, the Noether’s current is

$$J^\alpha = \left(-\frac{1}{4^n} \square n G^{\alpha\beta k} - \frac{1}{2 \cdot 8^n} g f^{kbc}(\square n T^{\alpha b} \square n T^{\beta c} - \square n T^{\alpha c} \square n T^{\beta b})\right) \delta \square n T^k. \tag{79}$$

The associated Noether’s charge is given by

$$Q = \int d^{D-1}x J^0 = \int d^{D-1}x \left(-\frac{1}{4^n} \square n G^{0\beta k} - \frac{1}{2 \cdot 8^n} g f^{kbc}(\square n T^{0 b} \square n T^{\beta c} - \square n T^{0 c} \square n T^{\beta b})\right) \delta \square n T^k. \tag{80}$$

## 5 The issue of dynamic instability of generalized Yang-Mills theory under rotor mechanism

It is well known that quantum field theory with high-order derivative suffers from dynamic instability, in which the canonical Hamiltonian (energy) is unbounded below \[14, 15, 16, 17\]. The idea of dynamic instability aroused from high-order derivative systems was first established by Ostrogradsky, which is famously known as the Ostrogradsky theorem \[13\]. In this section, we will first give the Ostrogradsky construction of generalized Yang-Mills theory under rotor mechanism, then study the 00-component of the energy-momentum tensor which is regarded as the energy density of the system. Generally, if the 00-component of the energy-momentum tensor $T_{0 \mu}$ is greater or equal than 0, i.e. $T_{0 0} \geq 0$, the system is bounded and still considered as stable \[26, 27, 28, 29\], even though the canonical energy is unbounded below. In other words, the energy is given by

$$E = \int d^{D-1}x T^0$$ \tag{81}

is greater or equal than zero, which is bounded.

Now we will first give an review of Ostrogradsky construction, then apply it to our case of generalized Yang-Mills theory. Consider a Lagrangian with higher-order time derivatives up to $n$, $L(x, \dot{x}, \ddot{x}, \cdots, x^{(n)})$, the Euler-Lagrange equation (equation of motion) reads \[1, 12, 14, 30\],

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{x}} + \cdots + (-1)^n \frac{d^n}{dt^n} \frac{\partial L}{\partial x^{(n)}} = \sum_{i=0}^n \left(-\frac{d}{dt}\right)^i \frac{\partial L}{\partial x^{(i)}} = 0. \tag{82}$$

The canonical variables are defined as follow \[1, 12, 14, 30\],

$$X_i = x^{(i-1)}, \quad P_i = \sum_{j=i}^n \left(-\frac{d}{dt}\right)^{j-i} \frac{\partial L}{\partial x^{(j)}}. \tag{83}$$

The canonical Hamiltonian (energy) is then given by \[1, 12, 14, 30\],

$$H = \sum_{i=1}^n P_i x^{(i)} - L = P_1 X_2 + P_2 X_3 + \cdots + P_{n-1} X_n + P_n \dot{X}_n - L(X_1, X_2, \cdots, X_n, \dot{X}_n). \tag{84}$$
The Ostrogradsky Hamiltonian in equation (84), which is known as the canonical energy, is linear in canonical momentum variable $P_i$, implying the energy system can be lowered without any bound. Hence, the inclusion of higher order derivatives make the system unstable.

The Ostrogradsky construction for our generalized Yang-Mills theory under rotor mechanism is as follow. Consider the Lagrangian density as a function of all orders of rotored fields $L(A_\nu, \Box A_\nu, \Box^2 A_\nu, \cdots, \Box^n A_\nu)$, for example,

$$L(A_a^{\nu}, \Box A_a^{\nu}, \Box^2 A_a^{\nu}, \cdots, \Box^n A_a^{\nu}) = -\frac{1}{4} \sum_{k=1}^{n} a_k G_k^{a, \mu\nu} G_k^{\mu\nu a},$$

(85)

where $a_k$ is some coefficient. The Euler-Lagrangian equation is

$$\frac{\partial L}{\partial A_\nu} + \Box \frac{\partial L}{\partial \Box A_\nu} + \Box^2 \frac{\partial L}{\partial \Box^2 A_\nu} + \cdots + \Box^n \frac{\partial L}{\partial \Box^n A_\nu} = \sum_{k=0}^{n} \Box^k \left( \frac{\partial L}{\partial \Box^k A_\nu} \right) = 0.$$  

(86)

Note that there are no negative terms because the $\Box^k$ operator is in even order of derivative. The canonical variables are,

$$X_{i\nu} = \Box^{-1} A_\nu, \quad P_i^{\nu} = \sum_{j=i}^{n} \Box^{-i-j} \frac{\partial L}{\partial \Box^{i+j} A_\nu}.$$  

(87)

The Ostrogradsky Hamiltonian, or the canonical energy is

$$\mathcal{H} = \sum_{i=1}^{n} P_i^{\nu} \Box^i A_\nu - L = \sum_{i=1}^{n} \sum_{j=i}^{n} \left( \Box^{-i-j} \frac{\partial L}{\partial \Box^{i+j} A_\nu} \right) \Box^i A_\nu - L.$$  

(88)

Since the Hamiltonian is linear in $P_i^{\nu}$, it is unbounded below. Hence, the generalized Yang-Mills theory under rotor model is unstable in terms of the canonical energy.

Next, we would evaluate the conserved energy-momentum tensor (also known as the stress-energy tensor). Consider the generic Yang-Mills action under rotor mechanism in curved spacetime,

$$S^{(n)}_{\text{YM}} = -\frac{1}{4} \int d^D x \sqrt{-g} G_{n, \mu\nu} G^{n, \mu\nu a},$$

(89)

where $g = \det g_{\mu\nu}$ is the determinant of the metric. The energy momentum tensor is given by

$$T_{\mu\nu}^{(n)} = -\frac{2}{\sqrt{-g}} \frac{\delta S^{(n)}_{\text{YM}}}{\delta g_{\mu\nu}}.$$  

(90)

This gives the energy-momentum tensor of generalized Yang-Mills theory under the rotor model as

$$T_{\mu\nu}^{(n)} = G_{n, \mu\nu} G^{n, \mu\nu a} - \frac{1}{4} g_{\mu\nu} G_{n, \alpha\beta} G^{n, \alpha\beta a}.$$  

(91)

Alternatively, this can be obtained by using the modified Noether’s theorem in reference [31]. Here as we are interested in the case of flat spacetime, we take $g_{\mu\nu} = \eta_{\mu\nu}$, therefore we have

$$T_{\mu\nu} = G_{n, \mu\nu} G^{n, \mu\nu a} - \frac{1}{4} \eta_{\mu\nu} G_{n, \alpha\beta} G^{n, \alpha\beta a}.$$  

(92)

In terms of rank(1,1) tensor, we have

$$T^{\mu}_{\nu} = G_{n, \mu\nu} G^{n, \mu\nu a} - \frac{1}{4} \eta_{\mu\nu} G_{n, \alpha\beta} G^{n, \alpha\beta a}.$$  

(93)
The total action is given by the sum of the action in (89) for all orders, i.e.

\[ S_{YM_{tot}} = \sum_{k=0}^{n} a_k S^{(k)}_{YM} = -\frac{1}{4} \int d^D x \sum_{k=0}^{n} a_k G^a_{k \mu \nu} G^{\mu \nu a}_k. \] (94)

The total energy-momentum tensor is

\[ (T^\mu_\nu)_{tot} = \sum_{k=0}^{n} \beta_k T^\mu_\nu_k, \] (95)

where \( \beta_k \) is the linear coefficient of the corresponding energy-momentum tensor. As we are interested in the 00-component, which is the energy density, therefore the total energy density of the system is

\[ \mathcal{E}_{tot} = (T^0_0)_{tot} = \sum_{k=0}^{n} \beta_k T^0_0_k. \] (96)

The explicit evaluation of \( T^0_0 \) for the non-abelian case is rather complicated. It is more convenient to study its abelian counterpart first such that we get a brief idea, and return to the non-abelian case afterwards.

For the abelian case, the gauge symmetry is U(1) and there is only one generator. For further simplicity we work for the \( n = 0 \) case first, which is the original Maxwell theory. The Maxwell stress-energy tensor in (93) reduces to

\[ T^\mu_\nu = G^{\mu \lambda} G_{\nu \lambda} - \frac{1}{4} \alpha^{\mu \nu} \alpha_{\alpha \beta} G^{\alpha \beta}. \] (97)

By explicitly expanding the gauge field strength tensor of the first term, and carry out integration by parts of the last term, we have

\[ T^\mu_\nu = (\partial^\mu T^\lambda - \partial^\lambda T^\mu)(\partial_\nu T_\lambda - \partial_\lambda T_\nu) + \delta^\mu_\nu T^\alpha \hat{R}_{\alpha \beta} T^\beta \]

\[ = \partial^\mu T^\lambda \partial_\nu T_\lambda - \partial^\lambda T^\mu \partial_\nu T_\lambda - \partial^\lambda T^\mu \partial_\nu T_\nu + \partial^\lambda T^\mu \partial_\nu T_\nu + \delta^\mu_\nu T^\alpha \hat{R}_{\alpha \beta} T^\beta. \] (98)

The 00-component is

\[ T^0_0 = \partial^0 T^\lambda \partial_\lambda T_0 - \partial^\lambda T^0 \partial_\lambda T_0 - \partial^\lambda T^0 \partial_0 T_\lambda + \partial^\lambda T^\lambda \partial_\lambda T_0 + T^\alpha \hat{R}_{\alpha \beta} T^\beta \]

\[ = \partial^0 T^\lambda \partial_\lambda T_0 - \partial^0 T^\lambda \partial_\lambda T_0 - \partial_\lambda T_0 \partial^0 T^\lambda + \partial^\lambda T^\lambda \partial_\lambda T_0 + T^\alpha \hat{R}_{\alpha \beta} T^\beta = \partial^0 T^\lambda \partial_\lambda T_0 - 2 \partial^0 T^\lambda \partial_\lambda T_0 + \partial^\lambda T^\lambda \partial_\lambda T_0 + T^\alpha \hat{R}_{\alpha \beta} T^\beta. \] (99)

As we use the +−−− convention of the metric, we have \( \partial_0 = \partial^0 = \frac{\partial}{\partial t} \), and \( A^0 = A^0 = V \) is the potential, we have

\[ T^0_0 = \frac{\partial T^\lambda}{\partial t} \frac{\partial T_\lambda}{\partial t} - 2 \frac{\partial T^\lambda}{\partial t} \partial_\lambda V + \partial^\lambda V \partial_\lambda V + T^\alpha \hat{R}_{\alpha \beta} T^\beta \]

\[ \equiv \tilde{T}^\lambda \tilde{T}_\lambda - 2 \tilde{T}^\lambda \partial_\lambda V + \partial^\lambda V \partial_\lambda V + T^\alpha \hat{R}_{\alpha \beta} T^\beta \]

\[ = (\tilde{T}^\lambda - \partial^\lambda V)^2 + T^\alpha \hat{R}_{\alpha \beta} T^\beta. \] (100)

\(^1\text{When we calculate the actual energy-momentum tensor (not density), we have } T^\mu_\nu = \int d^{D-1} x T^\mu_\nu, \text{ so the presence of the integral allows us to do integration by parts.}\)
Hence, the energy density of the rotor model is bounded. When the second term is quadratic in nature, we have

$$T^\mu_\nu = G_n^{\nu\lambda} G_{n\nu\lambda} - \frac{1}{4} \delta^{\mu}_{\nu} G_a^{\nu \beta} G_{n\alpha \beta} = \frac{1}{4n} \left( \square^n G^{\mu \lambda} \square^n G_{\nu \lambda} - \frac{1}{4} \delta^{\mu}_{\nu} \square^n G^{\nu \beta} \square^n G_{\alpha \beta} \right).$$  \hspace{1cm} (101)

The 00-component is

$$T^0_0 = \frac{1}{4n} \left( \frac{\partial \square^n T^{\lambda}_{\nu}}{\partial t} - 2 \frac{\partial \square^n T^{\lambda}}{\partial t} \partial \square^n V + \partial \lambda \square^n V \partial \lambda \square^n V + \square^n T^\rho \hat{R}_{\alpha \beta} \square^n T^\beta \right)$$

$$= \frac{1}{4n} \left[ \left( \frac{\partial \square^n T^{\lambda}}{\partial t} - \partial \lambda \square^n V \right)^2 + \square^n T^\alpha \hat{R}_{\alpha \beta} \square^n T^\beta \right].$$  \hspace{1cm} (102)

Since both terms are quadratic in nature, therefore we must have $T^0_0 \geq 0$ for all $n$. Hence, the energy density of the rotor model is bounded. When $n \to \infty$, we have $T^0_0 = 0$, which is the lower limit. Now we have the total energy density as

$$E_{tot} = (T^0_0)_{tot} = \sum_{k=0}^{n} \beta_k \frac{1}{4k} \left[ \left( \frac{\partial \square^n T^{\lambda}}{\partial t} - \partial \lambda \square^n V \right)^2 + \square^n T^\alpha \hat{R}_{\alpha \beta} \square^n T^\beta \right]$$  \hspace{1cm} (103)

If $\beta_k \geq 0$ for all $k$, it is guaranteed that $E_{tot} \geq 0$.

Finally, we proceed to work for the general non-abelian case. Starting from equation (93), and using the result in (12), we expand explicitly to give

$$T^\mu_\nu = (\tilde{G}_n^{\mu \alpha} + \frac{1}{4n} g f^{abc} \square^n T^{\mu b} \square^n T^{\lambda c}) (\tilde{G}_n^{a \lambda} + \frac{1}{4n} f^{ade} \square^n T^a \square^n T^\lambda)$$

$$+ \delta^{\mu}_{\nu} \left( \frac{1}{4n} \square^n T^{\rho a} \hat{R}_{\alpha \beta} \square^n T^\beta a - \frac{1}{2 \cdot 4n} g f^{abc} \tilde{G}_n^{\alpha \beta} \square^n T^{\rho a} \square^n T^\beta c \right.$$  

$$\left. - \frac{g^2}{4 \cdot 16n} f^{abc} f^{ade} \square^n T^{\rho a} \square^n T^{\beta b} \square^n T^{\beta c} \right)$$

$$= \tilde{G}_n^{\mu \lambda} \tilde{G}_n^{\nu \lambda} + \frac{1}{4n} g f^{abc} \tilde{G}_n^{\mu \lambda} \square^n T^{\rho a} \square^n T^{\beta c} + \frac{1}{4n} g f^{abc} \tilde{G}_n^{\mu \lambda} \square^n T^{\rho b} \square^n T^\lambda c$$

$$+ \frac{1}{16n} g^2 f^{abc} f^{ade} \square^n T^{\rho a} \square^n T^{\beta b} \square^n T^{\beta c} \square^n T^\rho$$

$$+ \delta^{\mu}_{\nu} \left( \frac{1}{4n} \square^n T^{\rho a} \hat{R}_{\mu \nu} \square^n T^\rho a - \frac{1}{2 \cdot 4n} g f^{abc} \tilde{G}_n^{\mu \lambda} \square^n T^{\rho b} \square^n T^\lambda c \right.$$  

$$\left. - \frac{g^2}{4 \cdot 16n} f^{abc} f^{ade} \square^n T^{\rho b} \square^n T^{\rho c} \square^n T^{\rho d} \square^n T^{\rho e} \right).$$  \hspace{1cm} (104)

Next we evaluate the 00-component,

$$T^0_0 = \frac{1}{4n} \left[ \left( \frac{\partial \square^n T^{\lambda a}}{\partial t} - \partial \lambda \square^n V^a \right)^2 + \square^n T^{\mu a} \hat{R}_{\alpha \beta} \square^n T^\beta a \right]$$

$$+ \frac{2}{4n} g f^{abc} \tilde{G}_n^{\mu 0 \lambda} \square^n V^{b} \square^n T^{\lambda c} + \frac{1}{16n} g^2 f^{abc} f^{ade} \square^n V^{b} \square^n V^{d} \square^n T^{\lambda c} \square^n T^\lambda$$

$$- \frac{1}{2 \cdot 4n} g f^{abc} \tilde{G}_n^{\mu \alpha} \square^n T^{\rho a} \square^n T^\beta c - \frac{g^2}{4 \cdot 16n} f^{abc} f^{ade} \square^n T^{\rho a} \square^n T^{\rho c} \square^n T^{\rho d} \square^n T^{\rho e}$$  \hspace{1cm} (105)
Then finally we obtain,

\[ T^0_{n0} = \frac{1}{4^n} \left[ \left( \frac{\partial n^n T^{\lambda a}}{\partial t} - \partial^\lambda n^n V^a \right)^2 + \partial^n T^{\alpha a} \hat{R}_{\alpha \beta} \partial^n T^{\beta a} \right] \]

\[ + \frac{2}{4^n} g f^{abc} \left( \hat{G}^a_{n0 \lambda} \partial^n V^b \partial^n T^{\lambda c} - \frac{1}{4} \hat{G}^{\mu \nu}_{n} \partial^n T^{b}_{\mu} \partial^n T^{c}_{\nu} \right) \]

\[ + \frac{g^2}{16^n} f^{abcdef} \left( \partial^n V^b \partial^n V^d \partial^n T^{\lambda c} \partial^n T^e_{\lambda} - \frac{1}{4} \partial^n T^{b}_{\mu} \partial^n T^{c}_{\nu} \partial^n T^{d}_{\mu} \partial^n T^{e}_{\nu} \right) \]  

(106)

The first line of the final equation (106) is Maxwellian-like, which is greater or equal than zero. However, the non-linear terms provided by the structural constant are not necessarily positive, unless the following constraint is satisfied,

\[ \hat{G}^a_{n0 \lambda} \partial^n V^b \partial^n T^{\lambda c} - \frac{1}{4} \hat{G}^{\mu \nu}_{n} \partial^n T^{b}_{\mu} \partial^n T^{c}_{\nu} \geq 0 \]  

(107)

and

\[ \partial^n V^b \partial^n V^d \partial^n T^{\lambda c} \partial^n T^e_{\lambda} - \frac{1}{4} \partial^n T^{b}_{\mu} \partial^n T^{c}_{\nu} \partial^n T^{d}_{\mu} \partial^n T^{e}_{\nu} \geq 0 \]  

(108)

The only certain thing is when \( n \to \infty, T^0_{n0} = 0 \) is bounded. And for the total energy density, we demand all \( \beta_k \geq 0 \). Therefore, while the abelian case is bounded, there is no guarantee for the non-abelian case is also bounded, unless the constraints by (107) and (108) are satisfied. This marks the difference between the abelian case and the non-abelian case.

6 Conclusion

In this paper, we have established the generalized non-abelian gauge field theorem under the rotor mechanism. Under the Lorentz gauge condition, the rotor transformation of gauge field for the general non-abelian case is same as the abelian case. The gauge field transforms as \( T^a_{\mu} \to \partial^n T^a_{\mu} \) under the rotor mechanism. When \( n = 0 \), this restores back to the original Yang-Mills theory. We also compute the equation of motion and Noether’s current for our theory. Finally, we study the dynamic stability issue of both the abelian case and the non-abelian (Yang-Mills) case. Although the canonical energy is unbounded below, the 00-component of the energy-momentum tensor is still positive for the abelian case and thus can still considered as stable. However, the non-abelian case is much more complicated and the system is considered stable only if a certain criteria in the non-linear terms is satisfied. In both case when the rotor order \( n \) is large enough and tends to infinity, the 00-component of the energy-momentum tensor is bounded to zero. In the future, this theory can help to develop the generalized field theory of higher order derivatives for the standard model of particles.

Declaration

I declare that there are no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
References

[1] A. Pais and G.E. Uhlenbeck. On field theories with non-localized action, *Phys. Rev.* **79**, 145-165. 1980.

[2] B. Podolsky, A Generalized Electrodynamics Part I—Non-Quantum. *Phys. Rev.* 62, 68. 1942.

[3] B. Podolsky and C. Kikuchi. A Generalized Electrodynamics Part II-Quantum. *Phys. Rev.* **65**, 228. 1944.

[4] B. Podolsky and C. Kikuchi. Auxiliary Conditions and Electrostatic Interaction in Generalized Quantum Electrodynamics. *Phys. Rev.* **67**, 184. 1945.

[5] B. Podolsky and P. Schwed. Review of a Generalized Electrodynamics. *Rev. Mod. Phys.* 20, 40. 1948.

[6] D. J. Montgomery. Relativistic Interaction of Electrons on Podolsky’s Generalized Electrodynamics. *Phys. Rev.* **69**, 117. 1946.

[7] T. D. Lee and G. C. Wick *Nucl.Phys.B* **9**, 209. 1969.

[8] T. D. Lee and G. C. Wick. *Phys.Rev.D* **2**, 1033. 1970.

[9] B. Grinstein, D. O’Connell, M. B. Wise. The Lee-Wick Standard Model. *Phys.Rev.D* **77**:025012. 2008.

[10] G.W. Gibbons, C.N. Pope and Sergey Solodukhin. Higher Derivative Scalar Quantum Field Theory in Curved Spacetime. *Phys.Rev.D* **100**, 2019.

[11] D.S. Kaparulin, S.L. Lyakhovich, O.D. Nosyrev, Extended Chern-Simons model for a vector multiplet, Symmetry 2021, 13(6) 1004.

[12] D.S. Kaparulin. A stable higher-derivative theory with the Yang-Mills gauge symmetry. arXiv:2011.12928 [hep-th]

[13] M. Ostrogradsky. Mem. Ac. St. Petersbourg VI 4 (1850) 385.

[14] R. P. Woodard. The Theorem of Ostrogradsky. arXiv:1506.02210 [hep-th].

[15] V.V. Nesterenko. On the instability of classical dynamics in theories with higher derivatives, *Phys.Rev.D* **75**, 2007.

[16] N.G. Stephen. On the Ostrogradski instability for higher-order derivative theories and pseudo-mechanical energy. *J. Sound. Vib* **310**(3): 729-739iįñ. 2008.

[17] H. Motohashi and T. Suyama. Third order equations of motion and the Ostrogradsky instability, *Phys.Rev.D* **91**, 2015.

[18] K. S. Stelle. Renormalization of higher-derivative quantum gravity. *Phys.Rev.D* **16**, 953. 1977.

[19] E. S. Fradkin and A.A. Tseytlin. Renormalizable asymptotically free quantum theory of gravity. *Nucl. Phys.B* 201. 1982. 469.
[20] S. Nojiri and S. D. Odintso. Introduction to Modified Gravity and Gravitational Alternative for Dark Energy. *Int. J. Geom. Meth. Mod. Phys.* **4**, 2007.

[21] T. P. Sotiriou. $f(R)$ Theories of Gravity. *Rev. Mod. Phys.* **82**, 451-497, 2010.

[22] S. Nojiri and S. D. Odintsov. *Phys. Rept.* **505**, 2011. 59.

[23] B. T. T. Wong. Generalized abelian gauge field theory under rotor model. *Mod. Phys. Lett. A.* Vol. 36, No. 27, 2150194. 2021.

[24] Yang, C. N.; Mills, R. Conservation of Isotopic Spin and Isotopic Gauge Invariance. *Physical Review.* **96** (1): 191–195, 1954

[25] M. E. Peskin and D. V. Schroeder. An introduction to quantum field theory. *ABP.* 1995.

[26] J. Dai. Stability in the higher derivative Abelian gauge field theory. *Nuclear Physics B.* Vol 961. 2020.

[27] D. S. Kaparulin, S. L. Lyakhovich, A. A. Sharapov. Classical and quantum stability of higher-derivative dynamics. *Eur. Phys. J. C* **74**, 3072. 2014.

[28] D. S. Kaparulin, I. Yu. Karataeva, S. L. Lyakhovich. Higher derivative extensions of 3d Chern-Simons models: conservation laws and stability. *Eur. Phys. J. C* **75**, 552. 2015.

[29] V. A. Abakumova, D. S. Kaparulin, S. L. Lyakhovich. Multi-Hamiltonian formulations and stability of higher-derivative extensions of 3d Chern-Simons. *The EPJ C* **78**, 115. 2018.

[30] F. J. de Urries and J. Julve. Ostrogradski Formalism for Higher-Derivative Scalar Field Theories. *J. Phys. A* **31**, 6949-6964. 1998.

[31] M. Montesinos and E. Flores. Symmetric energy-momentum tensor in Maxwell, Yang-Mills, and Proca theories obtained using only Noether’s theorem. *Rev. Mex. Fis.* **52**, 29-36. 2006.