A NOTE ON LOCALIZATIONS OF MAPPING SPACES

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Abstract. We show that if $A$ is a simply connected, finite, pointed CW-complex then the mapping spaces $\text{Map}_*(A, X)$ are preserved by the localization functors only if $A$ has the rational homotopy type of a wedge of spheres $\bigvee_j S^k$.

1. Introduction

The motivation for this brief note comes from the following well known property of localization functors [2, Thm 3.A.2]. Given a map of pointed spaces $f$ consider the localization functor $L_f: \text{Spaces}_* \to \text{Spaces}_*$. For any $X \in \text{Spaces}_*$ we have a weak equivalence

$$L_f \Omega X \simeq \Omega L\Sigma f X$$

This shows that localizations preserve loop spaces.

It is natural to ask if this preservation property can be extended. This leads to the following

Definition 1.1. We say that a finite, connected, pointed CW-complex $A$ is $L$-good if for any pointed map $f$ and any $X \in \text{Spaces}_*$ we have

$$L_f \text{Map}_*(A, X) \simeq \text{Map}_*(A, Y)$$

for some $Y \in \text{Spaces}_*$.

The weak equivalence (1) shows that $S^1$ is $L$-good. We would like to know what other spaces have this property. This is in fact one of the questions posed by Dror Farjoun in [2, 9.F]. Since $\Omega^k X \cong \Omega(\Omega^{k-1}X)$, applying iteratively the weak equivalence (1) we get that $S^k$ is $L$-good for all $k \geq 1$. Also, since $\text{Map}_*(\bigvee_l S^k, X) \cong \prod_l \text{Map}_*(S^k, X)$, and since localization functors preserve finite products up to a weak equivalence, we obtain that the class of $L$-good spaces contains all spaces $\bigvee_l S^k$ for $k > 0$, $l \geq 0$. Our goal here is to show that, rationally, every $L$-good space will resemble $\bigvee_l S^k$.

Theorem 1.2. Let $A$ be a finite, connected, pointed CW-complex such that for some $p > q > 0$ we have $H^p(A, \mathbb{Q}) \neq 0 \neq H^q(A, \mathbb{Q})$. Then $A$ is not an $L$-good space.

Equivalently, for an $L$-good space $A$ we have $H^i(A, \mathbb{Q}) \neq 0$ for at most one $i > 0$. As a consequence we obtain

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Corollary 1.3. If $A$ is a simply connected $L$-good space then $A$ has the rational homotopy type of $\bigvee_l S^k$ for some $k > 0$, $l \geq 0$.

We note here that the formula (1) follows from the existence of the loop space machines (see e.g. [1], [5], [6]) which describe the structure of spaces $\Omega X$ in terms of maps of finite products $(\Omega X)^m \to (\Omega X)^n$. An analogous description of mapping spaces $\text{Map}_*(A, X)$ for some $A$ would similarly imply that $A$ is an $L$-good space. Theorem 1.2 shows then that finite product "mapping space" machines do not exist for any finite CW-complex $A$ whose rational cohomology is non-trivial in more than one dimension.

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2. Proof of Theorem 1.2

Let $A$ be a CW-complex as in the statement of Theorem 1.2. Since $A$ is finite we can choose $p$ so that $H^i(A, \mathbb{Q}) = 0$ for all $i > p$. For $n > p$ we have a weak equivalence

$$\text{Map}_*(A, K(\mathbb{Q}, n)) \simeq \prod_{i=n-p}^n K(H^{n-i}(A, \mathbb{Q}), i)$$

Consider the constant map $f : S^k \to *$. In this case the localization $L_f$ is the nullification functor $P_{S^k}$. We have

$$P_{S^{n-p+1}} \text{Map}_*(A, K(\mathbb{Q}, n)) \simeq K(H^p(A, \mathbb{Q}), n - p)$$

If follows that if $A$ was an $L$-good space then for every $N > 0$ we would be able to find a space $Y$ such that

$$\text{Map}_*(A, Y) \simeq K(H^p(A, \mathbb{Q}), N)$$

(2)

We will show that this is impossible arguing by contradiction. Assume first that $A$ is simply connected, $0 \neq V = H^p(A, \mathbb{Q})$, and that for some fixed $N > p + 1$ we have a space $Y$ satisfying (2).

Since $A$ is simply connected we have $\text{Map}_*(A, Y) \simeq \text{Map}_*(A, \tilde{Y})$ where $\tilde{Y}$ is the universal cover of $Y$. Therefore we can assume that $Y$ is simply connected.

Next, let $Y_{(0)}$ denote the rationalization of $Y$. By [4] Thm.3.11, p.77 $\text{Map}_*(A, Y_{(0)}) \simeq \text{Map}_*(A, Y)_{(0)}$, and since $\text{Map}_*(A, Y) \simeq K(V, N)$ is a rational space thus $\text{Map}_*(A, Y_{(0)}) \simeq \text{Map}_*(A, Y)$. As a consequence we can assume that $Y$ is a simply connected rational space.

By [3] Corollary p. 229) we have

$$\Omega Y \simeq \prod_{n \geq 1} K(V_n, n)$$
where $V_n$ is a $\mathbb{Q}$-vector space and $\prod$ denotes the weak product of pointed spaces: $\prod_{n \geq 1} K(V_n, n) = \text{colim}_{M \geq 1} \left( \prod_{n=1}^{M} K(V_n, n) \right)$. We obtain

(3) \hspace{1cm} K(V, N - 1) \simeq \text{Map}_*(A, \Omega Y) \simeq \text{Map}_*(A, \prod_{n \geq 1} K(V_n, n))

We claim that there exists $n_0 \geq N - 1$ such that $V_{n_0} \neq 0$. Indeed, if $V_n = 0$ for all $n \geq N - 1$ then $\prod_{n \geq 1} K(V_n, n) = \prod_{n=1}^{N-2} K(V_n, n)$ so

$$\text{Map}_*(A, \prod_{n \geq 1} K(V_n, n)) = \prod_{n=1}^{N-2} \text{Map}_*(A, K(V_n, n))$$

This would give

$$\pi_i(\text{Map}_*(A, \prod_{n \geq 1} K(V_n, n))) \cong \bigoplus_{n=1}^{N-2} \tilde{H}^{n-i}(A, V_n)$$

In particular we would have $\pi_i(\text{Map}_*(A, \prod_{n \geq 1} K(V_n, n))) = 0$ for $i \geq N - 1$ which contradicts [3].

Since $n_0 \geq N - 1 > p, q$ we have

$$\pi_{n_0 - p}(\text{Map}_*(A, K(V_{n_0}, n_0))) \cong H^p(A, V_{n_0}) \neq 0$$

and

$$\pi_{n_0 - q}(\text{Map}_*(A, K(V_{n_0}, n_0))) \cong H^q(A, V_{n_0}) \neq 0$$

where the inequalities on the right hold by our assumption that $H^p(A, \mathbb{Q}) \neq 0$, $H^q(A, \mathbb{Q}) \neq 0$. Also, the space $\text{Map}_*(A, K(V_{n_0}, n_0))$ is a retract of $\text{Map}_*(A, \prod_{n \geq 1} K(V_n, n))$ so this last space must have non-trivial homotopy groups in at least two dimensions $n_0 - p$ and $n_0 - q$. This however contradicts the formula [3]. The contradiction shows that $\text{Map}_*(A, Y) \neq K(V, N)$ for any space $Y$, and so $A$ is not an $L$-good space.

Assume now that $A$ is not simply connected. If $A$ was an $L$-good space then again we would be able to find a space $Y$ such that $\text{Map}_*(A, Y) \simeq K(V, N)$, where $V = H^p(A, \mathbb{Q})$, $N > p + 2$. This would give

$$\text{Map}_*(\Sigma A, Y) \simeq \Omega \text{Map}_*(A, Y) \simeq K(V, N - 1)$$

Since $\Sigma A$ is a simply connected space this is however impossible by the argument above. It follows that $\text{Map}_*(A, Y) \neq K(V, N)$ for any $Y \in \text{Spaces}_*$, and so $A$ is not an $L$-good space.

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