On vector-valued inequalities for Sidon sets and sets of interpolation

N.J. Kalton
University of Missouri-Columbia

Abstract. Let $E$ be a Sidon subset of the integers and suppose $X$ is a Banach space. Then Pisier has shown that $E$-spectral polynomials with values in $X$ behave like Rademacher sums with respect to $L_p$-norms. We consider the situation when $X$ is a quasi-Banach space. For general quasi-Banach spaces we show that a similar result holds if and only if $E$ is a set of interpolation ($I_0$-set). However for certain special classes of quasi-Banach spaces we are able to prove such a result for larger sets. Thus if $X$ is restricted to be “natural” then the result holds for all Sidon sets. We also consider spaces with plurisubharmonic norms and introduce the class of analytic Sidon sets.

1. Introduction.

Suppose $G$ is a compact abelian group. We denote by $\mu_G$ normalized Haar measure on $G$ and by $\Gamma$ the dual group of $G$. We recall that a subset $E$ of $\Gamma$ is called a Sidon set if there is a constant $M$ such that for every finitely nonzero map $a : E \to \mathbb{C}$ we have

$$\sum_{\gamma \in E} |a(\gamma)| \leq M \max_{g \in G} \sum_{\gamma \in E} |a(\gamma)\gamma(g)|.$$ 

We define $\Delta$ to be the Cantor group i.e. $\Delta = \{\pm 1\}^\mathbb{N}$. If $t \in \Delta$ we denote by $\epsilon_n(t)$ the $n$-th co-ordinate of $t$. The sequence $(\epsilon_n)$ is an example of a Sidon set. Of course the sequence $(\epsilon_n)$ is a model for the Rademacher functions on $[0,1]$. Similarly we denote the co-ordinate maps on $T^\mathbb{N}$ by $\eta_n$.

Suppose now that $G$ is a compact abelian group. If $X$ is a Banach space, or more generally a quasi-Banach space with a continuous quasinorm and $\phi : G \to X$ is a Borel map we define $\|\phi\|_p$ for $0 < p \leq \infty$ to be the $L_p$-norm of $\phi$ i.e. $\|\phi\|_p = (\int_G \|\phi(g)\|^pd\mu_G(g))^{1/p}$ if $0 < p < \infty$ and $\|\phi\|_\infty = \text{ess sup}_{g \in G} \|\phi(g)\|$.

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It is a theorem of Pisier [12] that if $E$ is a Sidon set then there is a constant $M$ so that for every subset $\{\gamma_1, \ldots, \gamma_n\}$ of $E$, every $x_1, \ldots, x_n$ chosen from a Banach space $X$ and every $1 \leq p \leq \infty$ we have

\[ M^{-1} \left\| \sum_{k=1}^{n} x_k \epsilon_k \right\|_p \leq \left\| \sum_{k=1}^{n} x_k \gamma_k \right\|_p \leq M \left\| \sum_{k=1}^{n} x_k \epsilon_k \right\|_p. \]

Thus a Sidon set behaves like the Rademacher sequence for Banach space valued functions. The result can be similarly stated for $(\eta_n)$ in place of $(\epsilon_n)$. Recently Asmar and Montgomery-Smith [1] have taken Pisier’s ideas further by establishing distributional inequalities in the same spirit.

It is natural to ask whether Pisier’s inequalities can be extended to arbitrary quasi-Banach spaces. This question was suggested to the author by Asmar and Montgomery-Smith. For convenience we suppose that every quasi-Banach space is $r$-normed for some $r < 1$ i.e. the quasinorm satisfies $\|x + y\|^r \leq \|x\|^r + \|y\|^r$ for all $x, y$; an $r$-norm is necessarily continuous. We can then ask, for fixed $0 < p \leq \infty$ for which sets $E$ inequality (*) holds, if we restrict $X$ to belong to some class of quasi-Banach spaces, for some constant $M = M(E, X)$.

It turns out Pisier’s results do not in general extend to the non-locally convex case. In fact we show that if we fix $r < 1$ and ask that a set $E$ satisfies (*) for some fixed $p$ and every $r$-normable quasi-Banach space $X$ then this condition precisely characterizes sets of interpolation as studied in [2], [3], [4], [5], [8], [9], [13] and [14]. We recall that $E$ is called a set of interpolation (set of type $(I_0)$) if it has the property that every $f \in \ell_\infty(E)$ (the collection of all bounded complex functions on $E$) can be extended to a continuous function on the Bohr compactification $b\Gamma$ of $\Gamma$.

However, in spite of this result, there are specific classes of quasi-Banach spaces for which (*) holds for a larger class of sets $E$. If we restrict $X$ to be a natural quasi-Banach space then (*) holds for all Sidon sets $E$. Here a quasi-Banach space is called natural if it is linearly isomorphic to a closed linear subspace of a (complex) quasi-Banach lattice $Y$ which is $q$-convex for some $q > 0$, i.e. such that for a suitable constant $C$ we have

\[ \left\| \left( \sum_{k=1}^{n} |y_k|^q \right)^{1/q} \right\| \leq C \left( \sum_{k=1}^{n} \|y_k\|^q \right)^{1/q} \]

for every $y_1, \ldots, y_n \in Y$. Natural quasi-Banach spaces form a fairly broad class including almost all function spaces which arise in analysis. The reader is referred to [6] for a discussion of examples. Note that, of course, the spaces $L_q$ for $q < 1$ are natural so that, in particular, (*) holds for all $p$ and all Sidon sets $E$ for every $0 < p \leq \infty$. The case $p = q$
here would be a direct consequence of Fubini's theorem, but the other cases, including $p = \infty$ are less obvious.

A quasi-Banach lattice $X$ is natural if and only if it is $A$-convex, i.e. it has an equivalent plurisubharmonic quasi-norm. Here a quasinorm is plurisubharmonic if it satisfies

$$\|x\| \leq \int_0^{2\pi} \|x + e^{i\theta} y\| \frac{d\theta}{2\pi}$$

for every $x, y \in X$. There are examples of A-convex spaces which are not natural, namely the Schatten ideals $S_p$ for $p < 1$ \[7\]. Of course, it follows that $S_p$ cannot be embedded in any quasi-Banach lattice which is A-convex when $0 < p < 1$. Thus we may ask for what sets $E$ (*) holds for every A-convex space. Here, we are unable to give a precise characterization of the sets $E$ such that (*) holds. In fact we define $E$ to be an analytic Sidon set if (*) holds, for $p = \infty$ (or, equivalently for any other $0 < p < \infty$), for every A-convex quasi-Banach space $X$. We show that any finite union of Hadamard sequences in $N \subset Z$ is an analytic Sidon set. In particular a set such as $\{3^n\} \cup \{3^n + n\}$ is an analytic Sidon set but not a set of interpolation. However we have no example of a Sidon set which is not an analytic Sidon set.

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2. The results.

Suppose $G$ is a compact abelian group and $\Gamma$ is its dual group. Let $E$ be a subset of $\Gamma$. Suppose $X$ is a quasi-Banach space and that $0 < p \leq \infty$; then we will say that $E$ has property $C_p(X)$ if there is a constant $M$ such that for any finite subset $\{\gamma_1, \ldots, \gamma_n\}$ of $E$ and any $x_1, \ldots, x_n$ of $X$ we have (*) i.e.

$$M^{-1} \| \sum_{k=1}^n x_k \epsilon_k \|_p \leq \| \sum_{k=1}^n x_k \gamma_k \|_p \leq M \| \sum_{k=1}^n x_k \epsilon_k \|_p.$$  

(Note that in contrast to Pisier’s result (*), we here assume $p$ fixed.) We start by observing that $E$ is a Sidon set if and only if $E$ has property $C_\infty(C)$. It follows from the results of Pisier \[12\] a Sidon set has property $C_p(X)$ for every Banach space $X$ and for every $0 < p < \infty$. See also Asmar and Montgomery-Smith \[1\] and Pelczynski \[11\].

Note that for any $t \in \Delta$ we have that $\| \sum \epsilon_k(t) x_k \epsilon_k \|_p = \| \sum x_k \epsilon_k \|_p$. Now any real sequence $(a_1, \ldots, a_n)$ with $\max |a_k| \leq 1$ can be written in the form $a_k = \sum_{j=1}^\infty 2^{-j} \epsilon_k(t_j)$ and it follows quickly by taking real and imaginary parts that there is a constant $C = C(r, p)$ so that for any complex $a_1, \ldots, a_n$ and any $r$-normed space $X$ we have

$$\| \sum_{k=1}^n a_k x_k \epsilon_k \|_p \leq C \| a \|_\infty \| \sum_{k=1}^n x_k \epsilon_k \|_p.$$  

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From this it follows quickly that $\| \sum_{k=1}^{n} x_k \eta_k \|_p$ is equivalent to $\| \sum_{k=1}^{n} x_k \epsilon_k \|_p$. In particular we can replace $\epsilon_k$ by $\eta_k$ in the definition of property $C_p(X)$.

We note that if $E$ has property $C_p(X)$ then it is immediate that $E$ has property $C_p(\ell_p(X))$ and further that $E$ has property $C_p(Y)$ for any quasi-Banach space finitely representable in $X$ (or, of course, in $\ell_p(X)$).

For a fixed quasi-Banach space $X$ and a fixed subset $E$ of $\Gamma$ we let $P_E(X)$ denote the space of $X$-valued $E$-polynomials i.e. functions $\phi : G \to X$ of the form $\phi = \sum_{\gamma \in E} x(\gamma) \gamma$ where $x(\gamma)$ is only finitely nonzero. If $f \in \ell_\infty(E)$ we define $T_f : P_E(X) \to P_E(X)$ by

$$T_f(\sum x(\gamma) \gamma) = \sum f(\gamma) x(\gamma) \gamma.$$ 

We then define $\| f \|_{M_p(E,X)}$ to the operator norm of $T_f$ on $P_E(X)$ for the $L_p$-norm (and to be $\infty$ is this operator is unbounded).

**Lemma 1.** In order that $E$ has property $C_p(X)$ it is necessary and sufficient that there exists a constant $C$ such that

$$\| f \|_{M_p(E,X)} \leq C \| f \|_{\infty}$$

for all $f \in \ell_\infty(E)$.

**Proof:** If $E$ has property $C_p(X)$ then it also satisfies (*) for $(\eta_n)$ in place of $(\epsilon_n)$ for a suitable constant $M$. Thus if $f \in \ell_\infty(E)$ and $\phi \in P_E(X)$ then

$$\| T_f \phi \|_p \leq M^2 \| f \|_{\infty} \| \phi \|_p.$$ 

For the converse direction, we consider the case $p < \infty$. Suppose $\{ \gamma_1, \ldots, \gamma_n \}$ is a finite subset of $E$. Then for any $x_1, \ldots, x_n$

$$C^{-p} \int_{\mathcal{T}_N} \| \sum_{k=1}^{n} x_k \eta_k \|_p \mu_{\mathcal{T}_N} = C^{-p} \int_{\mathcal{T}_N} \int_{G} \| \sum_{k=1}^{n} x_k \eta_k(s) \gamma_k(t) \|_p \mu_{\mathcal{T}_N}(s) \mu_G(t)$$

$$\leq \int_{G} \| \sum_{k=1}^{n} x_k \gamma_k \|_p \mu_G$$

$$\leq C^p \int_{\mathcal{T}_N} \int_{G} \| \sum_{k=1}^{n} x_k \eta_k(s) \gamma_k(t) \|_p \mu_{\mathcal{T}_N}(s) \mu_G(t)$$

$$\leq C^p \int_{\mathcal{T}_N} \| \sum_{k=1}^{n} x_k \eta_k \|_p \mu_{\mathcal{T}_N}.$$ 

This estimate together with a similar estimate in the opposite direction gives the conclusion. The case $p = \infty$ is similar. ■
If $E$ is a subset of $\Gamma$, $N \in \mathbb{N}$ and $\delta > 0$ we let $\text{AP}(E, N, \delta)$ be the set of $f \in \ell_\infty(E)$ such that there exist $g_1, \ldots, g_N \in G$ (not necessarily distinct) and $\alpha_1, \ldots, \alpha_N \in \mathbb{C}$ with $\max_{1 \leq j \leq N} |\alpha_j| \leq 1$ and

$$|f(\gamma) - \sum_{j=1}^{N} \alpha_j \gamma(g_j)| \leq \delta$$

for $\gamma \in E$.

The following theorem improves slightly on results of Kahane [5] and Mélá [8]. Perhaps also, our approach is slightly more direct. We write $B_{\ell_\infty(E)} = \{ f \in \ell_\infty(E) : \|f\|_\infty \leq 1 \}$.

**Theorem 2.** Let $G$ be a compact abelian group and let $\Gamma$ be its dual group. Suppose $E$ is a subset of $\Gamma$. Then the following conditions on $E$ are equivalent:

1. $E$ is a set of interpolation.
2. There exists an integer $N$ so that $B_{\ell_\infty(E)} \subset \text{AP}(E, N, 1/2)$.
3. There exists $M$ and $0 < \delta < 1$ so that if $f \in B_{\ell_\infty(E)}$ then there exist complex numbers $(c_j)_{j=1}^\infty$ with $|c_j| \leq M \delta^j$ and $(g_j)_{j=1}^\infty$ in $G$ with

$$f(\gamma) = \sum_{j=1}^\infty c_j \gamma(g_j)$$

for $\gamma \in E$.

**Proof:** (1) $\Rightarrow$ (2). It follows from the Stone-Weierstrass Theorem that

$$\mathbf{T}^E \subset \bigcup_{m=1}^\infty \text{AP}(E, m, 1/5).$$

Let $\mu = \mu_{\mathbf{T}^E}$. Since each $\text{AP}(E, m, 1/5) \cap \mathbf{T}^E$ is closed it is clear that there exists $m$ so that 

$$\mu(\text{AP}(E, m, 1/5) \cap \mathbf{T}^E) > 1/2.$$ 

Thus if $f \in \mathbf{T}^E$ we can find $f_1, f_2 \in \text{AP}(E, m, 1/5) \cap \mathbf{T}^E$ so that $f = f_1 f_2$. Hence $f \in \text{AP}(E, m^2, 1/2)$. This clearly implies (2) with $N = 2m^2$.

(2) $\Rightarrow$ (3). We let $\delta = 2^{-1/N}$ and $M = 2$. Then given $f \in B_{\ell_\infty(E)}$ we can find $(c_j)_{j=1}^N$ and $(g_j)_{j=1}^N$ with $|c_j| \leq 1 \leq M \delta^j$ and

$$|f(\gamma) - \sum_{j=1}^{N} c_j \gamma(g_j)| \leq 1/2$$

for $\gamma \in E$. Let $f_1(\gamma) = 2(f(\gamma) - \sum_{j=1}^{N} c_j \gamma(g_j))$ and iterate the argument.

(3) $\Rightarrow$ (1): Obvious.
\textbf{Theorem 3.} Suppose $G$ is a compact abelian group, $E$ is a subset of the dual group $\Gamma$ and that $0 < r < 1$, $0 < p \leq \infty$. In order that $E$ satisfies $C_p(X)$ for every $r$-normable quasi-Banach space $X$ it is necessary and sufficient that $E$ be a set of interpolation.

\textbf{Proof:} First suppose that $E$ is a set of interpolation so that it verifies (3) of Theorem 2. Suppose $X$ is an $r$-normed quasi-Banach space. Suppose $f \in B_{\ell_\infty(E)}$. Then there exist $(c_j)_{j=1}^\infty$ and $(g_j)_{j=1}^\infty$ so that $|c_j| \leq M\delta^j$ and $f(\gamma) = \sum c_j \gamma(g_j)$ for $\gamma \in E$. Now if $\phi \in \mathcal{P}_E(X)$ it follows that

$$T_f \phi(h) = \sum_{j=1}^\infty c_j \phi(g_j h)$$

and so

$$\|T_f \phi\|_p \leq M \left( \sum_{j=1}^\infty \delta^{js} \right)^{1/s} \|\phi\|_p$$

where $s = \min(p, r)$. Thus $\|f\|_{\mathcal{M}_{p(E,X)}} \leq C$ where $C = C(p, r, E)$ and so by Lemma 1 $E$ has property $C_p(X)$.

Now conversely suppose that $0 < r < 1$, $0 < p \leq \infty$ and that $E$ has property $C_p(X)$ for every $r$-normed space $X$. It follows from consideration of $\ell_\infty$-products that there exists a constant $C$ so that for every $r$-normed space $X$ we have $\|f\|_{\mathcal{M}_{p(E,X)}} \leq C\|f\|_\infty$ for $f \in \ell_\infty(E)$.

Suppose $F$ is a finite subset of $E$. We define an $r$-norm $\|\|_A$ on $\ell_\infty(F)$ by setting $\|f\|_A$ to be the infimum of $(\sum |c_j|^r)^{1/r}$ over all $(c_j)_{j=1}^\infty$ and $(g_j)_{j=1}^\infty$ such that

$$f(\gamma) = \sum_{j=1}^\infty c_j \gamma(g_j)$$

for $\gamma \in F$. Notice that $\|f_1 f_2\|_A \leq \|f_1\|_A \|f_2\|_A$ for all $f_1, f_2 \in A = \ell_\infty(F)$.

For $\gamma \in F$ let $e_\gamma$ be defined by $e_\gamma(\gamma) = 1$ if $\gamma = \chi$ and 0 otherwise. Then for $f \in A$, with $\|f\|_\infty \leq 1$,

$$\left( \int_G \| \sum_{\gamma \in F} f(\gamma) e_\gamma \|_A^p d\mu_G \right)^{1/p} \leq C \left( \int_G \| e_\gamma \|_A^p d\mu_G \right)^{1/p}.$$ 

But for any $g \in G \| \sum \gamma(g) e_\gamma \|_A \leq 1$. Define $H$ to be subset of $h \in G$ such that $\| \sum_{\gamma \in F} f(\gamma) \gamma(h) e_\gamma \|_A \leq 3^{1/p} C$. Then $\mu_G(H) \geq 2/3$. Thus there exist $h_1, h_2 \in H$ such that $h_1 h_2 = 1$ (the identity in $G$). Hence by the algebra property of the norm

$$\|f\|_A \leq 3^{2/p} C^2$$

and so if we fix an integer $C_0 > 3^{2/p} C^2$ we can find $c_j$ and $g_j$ so that $\sum |c_j|^r \leq C_0^r$ and

$$f(\gamma) = \sum c_j \gamma(g_j)$$
for $\gamma \in F$. We can suppose $|c_j|$ is monotone decreasing and hence that $|c_j| \leq C_0 j^{-1/r}$. Choose $N_0$ so that $C_0 \sum_{j=N_0+1}^{\infty} j^{-1/r} \leq 1/2$. Thus

$$|f(\gamma) - \sum_{j=1}^{N_0} c_j \gamma(g_j)| \leq 1/2$$

for $\gamma \in F$. Since each $|c_j| \leq C_0$ this implies that $B_{\ell_\infty(F)} \subset AP(F, N, 1/2)$ where $N = C_0 N_0$.

As this holds for every finite set $F$ it follows by an easy compactness argument that $B_{\ell_\infty(E)} \subset AP(E, N, 1/2)$ and so by Theorem 2 $E$ is a set of interpolation.

**Theorem 4.** Let $X$ be a natural quasi-Banach space and suppose $0 < p \leq \infty$. Then any Sidon set has property $C_p(X)$.

**Proof:** Suppose $E$ is a Sidon set. Then there is a constant $C_0$ so that if $f \in \ell_\infty(E)$ then there exists $\nu \in C(G)^*$ such that $\hat{\mu}(\gamma) = f(\gamma)$ for $\gamma \in E$ and $|\mu| \leq C_0 \|f\|_\infty$. We will show the existence of a constant $C$ such that $\|f\|_{\mathcal{M}_p(E,X)} \leq C \|f\|_\infty$. If no such constant exists then we may find a sequence $E_n$ of finite subsets of $E$ such that $\lim C_n = \infty$ where $C_n$ is the least constant such that $\|f\|_{\mathcal{M}_p(E_n,X)} \leq C_n \|f\|_\infty$ for all $f \in \ell_\infty(E_n)$.

Now the spaces $\mathcal{M}_p(E_n, X)$ are each isometric to a subspace of $\ell_\infty(L_p(G,X))$ and hence so is $Y = c_0(\mathcal{M}_p(E_n, X))$. In particular $Y$ is natural. Notice that $Y$ has a finite-dimensional Schauder decomposition. We will calculate the Banach envelope $Y_c$ of $Y$. Clearly $Y_c = c_0(Y_n)$ where $Y_n$ is the finite-dimensional space $\mathcal{M}_p(E_n, X)$ equipped with its the envelope norm $\|f\|_c$.

Suppose $f \in \ell_\infty(E_n)$. Then clearly $\|f\|_\infty \leq \|f\|_{\mathcal{M}_p(E_n,X)}$ and so $\|f\|_\infty \leq \|f\|_c$. Conversely if $f \in \ell_\infty(E_n)$ there exists $\nu \in C(G)^*$ with $|\nu| \leq C_0 \|f\|_\infty$ and such that $f \gamma d\nu = f(\gamma)$ for $\gamma \in E_n$. In particular $C_0^{-1} \|f\|_\infty f$ is in the absolutely closed closed convex hull of the set of functions $\{\hat{g} : g \in G\}$ where $\hat{g}(\gamma) = \gamma(g)$ for $\gamma \in E_n$. Since $\|\hat{g}\|_{\mathcal{M}_p(E,X)} = 1$ for all $g \in G$ we see that $\|f\|_\infty \leq \|f\|_c \leq C_0 \|f\|_\infty$. This implies that $Y_c$ is isomorphic to $c_0$. Since $Y$ has a finite-dimensional Schauder decomposition and is natural we can apply Theorem 3.4 of [6] to deduce that $Y = Y_c$ is already locally convex. Thus there is a constant $C'_0$ independent of $n$ so that $\|f\|_{\mathcal{M}_p(E_n,X)} \leq C'_0 \|f\|_\infty$ whenever $f \in \ell_\infty(E_n)$. This contradicts the choice of $E_n$ and proves the theorem.

We now consider the case of A-convex quasi-Banach spaces. For this notion we will introduce the concept of an analytic Sidon set. We say a subset $E$ of $\Gamma$ is an **analytic Sidon set** if $E$ satisfies $C_\infty(X)$ for every A-convex quasi-Banach space $X$.

**Proposition 5.** Suppose $0 < p < \infty$. Then $E$ is an analytic Sidon set if and only if $E$ satisfies $C_p(X)$ for every A-convex quasi-Banach space $X$.

**Proof:** Suppose first $E$ is an analytic Sidon set, and that $X$ is an A-convex quasi-Banach space (for which we assume the quasinorm is plurisubharmonic). Then $L_p(G, X)$ also has
a plurisubharmonic quasinorm and so $E$ satisfies (1) for $X$ replaced by $L_p(G, X)$ and $p$ replaced by $\infty$ with constant $M$. Now suppose $x_1, \ldots, x_n \in X$ and $\gamma_1, \ldots, \gamma_n \in E$. Define $y_1, \ldots, y_n \in L_p(G, X)$ by $y_k(g) = \gamma_k(g)x_k$. Then

$$\max_{g \in G} \| \sum_{k=1}^n y_k \gamma_k(g) \|_{L_p(G, X)} = \| \sum_{k=1}^n x_k \gamma_k \|_p$$

and a similar statement holds for the characters $e_k$ on the Cantor group. It follows quickly that $E$ satisfies (1) for $p$ and $X$ with constant $M$.

For the converse direction suppose $E$ satisfies $C_p(X)$ for every A-convex space $X$. Suppose $X$ has a plurisubharmonic quasinorm. We show that $M_\infty(E, X) = \ell_\infty(E)$. In fact $M_\infty(F, X)$ can be isometrically embedded in $\ell_\infty(X)$ for every finite subset $F$ of $E$. Thus (1) holds for $X$ replaced by $M_\infty(F, X)$ for some constant $M$, independent of $F$. Denoting by $e_\gamma$ the canonical basis vectors in $\ell_\infty(E)$ we see that if $F = \{\gamma_1, \ldots, \gamma_n\} \subset E$ then

$$\left( \int_{\Delta} \| \sum_{k=1}^n \epsilon_k(t) e_{\gamma_k} \|_{M_\infty(F, X)}^p d\mu(t) \right)^{1/p} \leq M \max_{g \in G} \| \sum_{k=1}^n \gamma_k(g) e_{\gamma_k} \|_{M_\infty(F, X)} = M.$$

Thus the set $K$ of $t \in \Delta$ such that $\| \sum_{k=1}^n \epsilon_k(t) e_{\gamma_k} \| \leq 3^{1/p} M$ has measure at least $2/3$. Arguing that $K.K = \Delta$ we obtain that

$$\| \sum_{k=1}^n \epsilon_k(t) e_{\gamma_k} \|_{M_\infty(F, X)} \leq 3^{2/p} M^2$$

for every $t \in \Delta$. It follows quite simply that there is a constant $C$ so that for every real valued $f \in \ell_\infty(F)$ we have $\| f \|_{M_\infty(E, X)} \leq C \| f \|_\infty$. In fact this is proved by writing each such $f$ with $\| f \|_\infty = 1$ in the form $f(\gamma_k) = \sum_{j=1}^\infty 2^{-j} \epsilon_j(t_j)$ for a suitable sequence $t_j \in \Delta$. A similar estimate for complex $f$ follows by estimating real and imaginary parts. Finally we conclude that since these estimates are independent of $F$ that $\ell_\infty(E) = M_\infty(E, X)$.

Of course any set of interpolation is an analytic Sidon set and any analytic Sidon set is a Sidon set. The next theorem will show that not every analytic Sidon set is a set of interpolation. If we take $G = T$ and $\Gamma = Z$, we recall that a Hadamard gap sequence is a sequence $(\lambda_k)_{k=1}^\infty$ of positive integers such that for some $q > 1$ we have $\lambda_{k+1}/\lambda_k \geq q$ for $k \geq 1$. It is shown in [10] and [14] that a Hadamard gap sequence is a set of interpolation. However the union of two such sequences may fail to be a set of interpolation; for example $(3^n)_{n=1}^\infty \cup (3^n + n)_{n=1}^\infty$ is not a set of interpolation, since the closures of $(3^n)$ and $(3^n + n)$ in $b\mathbb{Z}$ are not disjoint.
Theorem 6. Let $G = T$ so that $\Gamma = \mathbb{Z}$. Suppose $E \subset \mathbb{N}$ is a finite union of Hadamard gap sequences. Then $E$ is an analytic Sidon set.

Proof: Suppose $E = (\lambda_k)_{k=1}^\infty$ where $(\lambda_k)$ is increasing. We start with the observation that $E$ is the union of $m$ Hadamard sequences if and only there exists $q > 1$ so that $\lambda_{m+k} \geq q^m \lambda_k$ for every $k \geq 1$.

We will prove the theorem by induction on $m$. Note first that if $m = 1$ then $E$ is a Hadamard sequence and hence [14] a set of interpolation. Thus by Theorem 2 above, $E$ is an analytic Sidon set.

Suppose now that $E$ is the union of $m$ Hadamard sequences and that the theorem is proved for all unions of $l$ Hadamard sequences where $l < m$. We assume that $E = (\lambda_k)$ and that there exists $q > 1$ such that $\lambda_{k+m} \geq q^m \lambda_k$ for $k \geq 1$. We first decompose $E$ into at most $m$ Hadamard sequences. To do this let us define $E_1 = \{\lambda_1\} \cup \{\lambda_k : k \geq 2, \lambda_k \geq q\lambda_{k-1}\}$. We will write $E_1 = (\tau_k)_{k \geq 1}$ where $\tau_k$ is increasing. Of course $E_1$ is a Hadamard sequence.

For each $k$ let $D_k = E \cap [\tau_k, \tau_{k+1})$. It is easy to see that $|D_k| \leq m$ for every $k$. Further if $n_k \in D_k$ then $n_{k+1} \geq \tau_{k+1} \geq qn_k$ so that $(n_k)$ is a Hadamard sequence. In particular $E_2 = E \setminus E_1$ is the union of at most $m - 1$ Hadamard sequences and so $E_2$ is an analytic Sidon set by the inductive hypothesis.

Now suppose $w \in T$. We define $f_w \in l_\infty(E)$ by $f_w(n) = w^{n-\tau_k}$ for $n \in D_k$. We will show that $f_w$ is uniformly continuous for the Bohr topology on $\mathbb{Z}$; equivalently we show that $f_w$ extends to a continuous function on the closure $\tilde{E}$ of $E$ in the Bohr compactification $b\mathbb{Z}$ of $\mathbb{Z}$. Indeed, if this is not the case there exists $\xi \in \tilde{E}$ and ultrafilters $\mathcal{U}_0$ and $\mathcal{U}_1$ on $E$ both converging to $\xi$ so that $\lim_{n \in \mathcal{U}_0} f_w(n) = \zeta_0$ and $\lim_{n \in \mathcal{U}_1} f_w(n) = \zeta_1$ where $\zeta_1 \neq \zeta_0$. We will let $\delta = \frac{1}{3} |\zeta_1 - \zeta_0|$.

We can partition $E$ into $m$ sets $A_1, \ldots, A_m$ so that $|A_j \cap D_k| \leq 1$ for each $k$. Clearly $\mathcal{U}_0$ and $\mathcal{U}_1$ each contain exactly one of these sets. Let us suppose $A_{j_0} \in \mathcal{U}_0$ and $A_{j_1} \in \mathcal{U}_1$.

Next define two ultrafilters $\mathcal{V}_0$ and $\mathcal{V}_1$ on $\mathbb{N}$. $\mathcal{V}_0 = \{V : \cup_{k \in V} D_k \in \mathcal{U}_0\}$ and $\mathcal{V}_1 = \{V : \cup_{k \in V} D_k \in \mathcal{U}_1\}$. We argue that $\mathcal{V}_0$ and $\mathcal{V}_1$ coincide. If not we can pick $V \in \mathcal{V}_0 \setminus \mathcal{V}_1$. Consider the set $A = (A_{j_0} \cap (\cup_{k \in V} D_k)) \cup (A_{j_1} \cap (\cup_{k \notin V} D_k))$. Then $A$ is a Hadamard sequence and hence a set of interpolation. Thus for the Bohr topology the sets $A_{j_0} \cap (\cup_{k \in V} D_k)$ and $A_{j_1} \cap (\cup_{k \notin V} D_k)$ have disjoint closures. This is contradiction since of course $\xi$ must be in the closure of each. Thus $\mathcal{V}_0 = \mathcal{V}_1$.

Since both $\mathcal{U}_0$ and $\mathcal{U}_1$ converge to the same limit for the Bohr topology we can find sets $H_0 \in \mathcal{U}_0$ and $H_1 \in \mathcal{U}_1$ so that if $n_0 \in H_0$, $n_1 \in H_1$ then $|w^{n_1} - w^{n_0}| < \delta$ and further $|f_w(n_0) - \zeta_0| < \delta$ and $|f_w(n_1) - \zeta_1| < \delta$.

Let $V_0 = \{k \in \mathbb{N} : D_k \cap H_0 \neq \emptyset\}$ and $V_1 = \{k \in \mathbb{N} : D_k \cap H_1 \neq \emptyset\}$. Then $V_0 \in \mathcal{V}_0$ and $V_1 \in \mathcal{V}_1$. Thus $V = V_0 \cap V_1 \in \mathcal{V}_0 = \mathcal{V}_1$. If $k \in V$ there exists $n_0 \in D_k \cap H_0$ and $n_1 \in D_k \cap H_1$.
\( n_1 \in D_k \cap H_1 \). Then

\[
3\delta = |\zeta_1 - \zeta_0| \\
< |f_w(n_1) - f_w(n_0)| + 2\delta \\
= |w^{n_1} - w^{n_0}| + 2\delta \\
< 3\delta.
\]

This contradiction shows that each \( f_w \) is uniformly continuous for the Bohr topology.

Now suppose that \( X \) is an \( r \)-normed \( A \)-convex quasi-Banach space where the quasi-norm is plurisubharmonic. Since both \( E_1 \) and \( E_2 \) are analytic Sidon sets we can introduce a constant \( C \) so that if \( f \in \ell_\infty(E_j) \) where \( j = 1, 2 \) then \( \|f\|_{M_{\infty}(E_j, X)} \leq C\|f\|_\infty \). Pick a constant \( 0 < \delta < 1 \) so that \( 3.4^{1/r}\delta < C \).

Let \( K_l = \{ w \in T : f_w \in AP(E, l, \delta) \} \). It is easy to see that each \( K_l \) is closed and since each \( f_w \) is uniformly continuous by the Bohr topology it follows from the Stone-Weierstrass theorem that \( \bigcup K_l = T \). If we pick \( l_0 \) so that \( \mu_T(K_{l_0}) > 1/2 \) then \( K_{l_0} K_{l_0} = T \) and hence since the map \( w \rightarrow f_w \) is multiplicative \( f_w \in AP(E, l_0^2, 3\delta) \) for every \( w \in T \).

Let \( F \) be an arbitrary finite subset of \( E \). Then there is a least constant \( \beta \) so that \( \|f\|_{M_{\infty}(F, X)} \leq \beta\|f\|_\infty \). The proof is completed by establishing a uniform bound on \( \beta \).

For \( w \in T \) we can find \( c_j \) with \( |c_j| \leq 1 \) and \( \zeta_j \in T \) for \( 1 \leq j \leq l_0^2 \) such that

\[
|f_w(n) - \sum_{j=1}^{l_0^2} c_j \zeta_j^n| \leq 3\delta
\]

for \( n \in E \). If \( \tilde{\zeta}_j \) is define by \( \tilde{\zeta}_j(n) = \zeta_j^n \) then of course \( \|\tilde{\zeta}_j\|_{M_{\infty}(E, X)} = 1 \). Restricting to \( F \) we see that

\[
\|f_w\|_{M_{\infty}(F, X)} \leq l_0^2 + \beta^r (3\delta)^r.
\]

Define \( F : C \rightarrow M_{\infty}(F, X) \) by \( F(z)(n) = z^{n-rk} \) if \( n \in D_k \). Note that \( F \) is a polynomial. As in Theorem 5, \( M_{\infty}(F, X) \) has a plurisubharmonic norm. Hence

\[
\|F(0)\|_r \leq \max_{|w|=1} \|F(w)\|_r \leq l_0^2 + (3\delta)^r \beta^r.
\]

Thus, if \( \chi_A \) is the characteristic function of \( A \),

\[
\|\chi_{E_1 \cap F}\|_{M_{\infty}(F, X)} \leq l_0^2 + (3\delta)^r \beta^r.
\]

It follows that

\[
\|\chi_{E_2 \cap F}\|_{M_{\infty}(F, X)} \leq l_0^2 + (3\delta)^r \beta^r + 1.
\]
Now suppose \( f \in \ell_\infty(F) \) and \( \|f\|_\infty \leq 1 \). Then
\[
\|f\chi_{E_j \cap F}\|_{\mathcal{M}_\infty(F,X)} \leq \|f\chi_{E_j \cap F}\|_{\mathcal{M}_\infty(E_j \cap F,X)}\|\chi_{E_j \cap F}\|_{\mathcal{M}_\infty(F,X)}
\]
for \( j = 1, 2 \). Thus
\[
\|f\|_{\mathcal{M}_\infty(F,X)} \leq C^r(1 + 2l_0^2 + 2(3\delta)^r\beta^r).
\]
By maximizing over all \( f \) this implies
\[
\beta^r \leq C^r(1 + 2l_0^2 + 2(3\delta)^r\beta^r)
\]
which gives an estimate
\[
\beta^r \leq 2C^r(1 + 2l_0^2)
\]
in view of the original choice of \( \delta \). This estimate, which is independent of \( F \), implies that \( E \) is an analytic Sidon set.\( \blacksquare \)

Remark: We know of no example of a Sidon set which is not an analytic Sidon set.

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