Shallow Characters and Supercuspidal Representations

Stella Sue Gastineau
stellasuegastineau@gmail.com
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Abstract

In 2014, Reeder and Yu constructed epipelagic representations of a reductive $p$-adic group $G$ from stable functions on shallowest Moy-Prasad quotients. In this paper, we extend these methods when $G$ is split. In particular, we classify all complex-valued characters vanishing on a slightly deeper Moy-Prasad subgroup and show that, while sufficient, a naive extension of Reeder-Yu’s stability condition is not necessary for constructing supercuspidal representations.

1 Introduction

1.1 Notation

Let $k$ be a non-archimedean local field with value group $\mathbb{Z}$ and ring of integers $\mathfrak{o}$ with prime ideal $\mathfrak{p}$ and residue field $\mathfrak{f} := \mathfrak{o}/\mathfrak{p}$ of finite cardinality $q$ and characteristic $p$. Let $K$ be a maximal unramified extension of $k$, with algebraically closed residue field $\mathfrak{f}$. Let $G$ be an absolutely simple, simply connected algebraic group defined and splitting over $k$. We fix the following subgroups of $G$ for consideration:

- $T$ a maximal torus, defined and splitting over $k$.
- $B$ a Borel subgroup of $G$, containing $T$ and defined over $k$.
- $U$ the unipotent radical of $B$, defined over $k$.

We will also use unbolded letters $G, B, T, U$ to denote the $k$-rational points of $G, B, T, U$ respectively. We will be assuming the basic structure of such groups, which can be found in [3, 5].

1.2 Motivation

The group $G$ acts on its Bruhat-Tits building $B = B(G, k)$ and for each point $\lambda \in B$, the stabilizer $P := G_\lambda$ has a filtration by open **Moy-Prasad subgroups:**

$$P > P_{r_1} > P_{r_2} > \cdots$$

indexed by an increasing, discrete sequence $r(\lambda) = (r_1, r_2, \ldots)$ of positive real numbers. The first Moy-Prasad subgroup $P_{r_1}$ is called the **pro-unipotent radical** of $P$, and will be denoted by $P_+$. In their papers, Gross-Reeder [2] and Reeder-Yu [4] study complex characters of

$$\chi : P_+ \to \mathbb{C}^\times$$

that are trivial on the Moy-Prasad subgroup $P_{r_2}$. In this paper we will go a little bit deeper down the Moy-Prasad filtration and classify all **shallow characters**, those being characters that are trivial on Moy-Prasad subgroup $P_1 \subseteq P_{r_2}$.
In §2.2, we show that a shallow character on $P_{+}$ can be recovered from its restrictions to its affine root subgroups and extended to a group homomorphism. In particular, in Theorem 4 we show that in order to define a shallow character, it is both necessary and sufficient that the extension be trivial on commutators

$$[U_{\beta}, U_{\alpha}] \subseteq \prod_{i,j>0} U_{i\alpha+j\beta}$$

where $\alpha$ and $\beta$ are affine roots whose gradients are not linearly dependant.

Following a classification of shallow characters, we ask for which shallow characters $\chi : P_{+} \to \mathbb{C}^{\times}$ is the compactly-induced representation

$$\text{ind}_{P_{+}}^{G}(\chi) = \left\{ \phi : G \to \mathbb{C} \mid \begin{array}{c} \phi(hx) = \chi(h) \cdot \phi(x) \\ \phi \text{ compactly supported} \end{array} \right\}$$

a supercuspidal representation of $G$. In their papers, Gross-Reeder and Reeder-Yu give a classification of supercuspidal representations of $G$ via stable orbits in a related graded Lie algebra. In Proposition 8 of §3.2, we look at a naive generalization of [4, Proposition 2.4] and show that it is sufficient but not necessary for determining which shallow characters induce up to supercuspidal representations of $G$.

2 Shallow Characters

Throughout this paper we will fix an alcove of the apartment $\mathcal{A} \subseteq \mathcal{B}$ corresponding to $T$, and we will let

$$\Delta = \{\alpha_0, \alpha_1, \ldots, \alpha_\ell\}$$

denote the corresponding set of simple affine roots. We will also fix a point $\lambda$ contained in the closure of this alcove. We will denote by $F_{J} \subseteq \mathcal{A}$ the facet containing $\lambda$ given by the non-vanishing of the simple affine roots $\Delta_{J} \subseteq \Delta$, where

$$J \subset \{0, 1, \ldots, \ell\}.$$

We will also let $P = G_{\lambda}$ denote the stabilizer of $\lambda$ in $G$.

2.1 Shallow affine roots

Given an affine root $\alpha : \mathcal{A} \to \mathbb{R}$, we say that its depth (at $\lambda$) is the real number $\alpha(\lambda)$. Then we say that $\alpha$ is shallow (at $\lambda$) if its depth is strictly between 0 and 1. We also say that $\alpha$ is decomposable (as a shallow affine root) if there exists another shallow affine root $\beta$ such that $\alpha - \beta$ is a shallow affine root. Otherwise, we say that $\alpha$ is indecomposable (as a shallow affine root).

Note that the depth of a shallow affine root precisely depends on $\lambda$; whereas, the set of decomposable and indecomposable shallow affine roots depend only on the the facet $F_{J}$ and not on the point $\lambda$ itself. In fact, setting

$$n_{J}(n_{0}\alpha_{0} + n_{1}\alpha_{1} + \cdots + n_{\ell}\alpha_{\ell}) := \sum_{j \in J} n_{j}$$

for $n_{j} \in \mathbb{Z}$, we can characterize the indecomposable shallow affine roots as follows:
Lemma 1. A shallow affine root $\alpha$ is indecomposable if and only if $n_J(\alpha) = 1$.

Proof. Let $\alpha$ be a shallow root. First note that if $n_J(\alpha) = 1$, then $\alpha$ must be indecomposable as a shallow affine root: Indeed, if $\beta, \alpha - \beta$ is an affine root, then exactly one of $\alpha - \beta$ and $\beta$ is shallow since

$$n_J(\alpha - \beta) = n_J(\alpha) - n_J(\beta).$$

Therefore, for the remainder of the proof we suppose that $n_J(\alpha) \geq 2$.

First write

$$\alpha = \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_m},$$

so that

$$\beta_j = \alpha_{i_1} + \cdots + \alpha_{i_j}$$
$$\alpha - \beta_j = \alpha_{i_{j+1}} + \cdots + \alpha_{i_m}$$

are an affine roots for all $j = 1, 2, \ldots, m$. Such a decomposition is possible, for example, by Lemma 3.6.2 in [1]. Since $n_J(\alpha) \geq 2$, we know that there exists a $j = 1, 2, \ldots, m$ such that both $\beta_j$ and $\alpha - \beta_j$ are shallow. For instance, we can choose $j$ to be minimal such that $\alpha_{i_j}$ is a shallow affine root in $\Delta_J$. Thus, by setting $\beta = \beta_j$, we have given a decomposition

$$\alpha = \beta + (\alpha - \beta)$$
as shallow affine root whenever $n_J(\alpha) \geq 2$. □

Lemma 2. Suppose that $\alpha, \beta$ are shallow affine roots such that there are positive integers $i, j > 0$ such that $i\alpha + j\beta$ are shallow affine roots. Then $\alpha + \beta$ is a shallow affine root.

Proof. Suppose that $i\alpha + j\beta$ is a shallow affine root for positive integers $i, j > 0$. If both $i, j = 1$, then $\alpha + \beta$ is a shallow affine root and we are done. Therefore, without loss of generality, we will assume that $i > 0$. Note that in this case, we then have the following chain of inequalities:

$$0 < \alpha(\lambda) < \alpha(\lambda) + \beta(\lambda) < i\alpha(\lambda) + j\beta(\lambda) < 1. \quad (2.1)$$

Thus, if we can show that $\alpha + \beta$ is an affine root, then it must be shallow.

First, we note that $\alpha + \beta$ cannot be a constant function. Since $G$ is split, the minimal relation of the affine root group is of the form

$$1 = m_0\alpha_0 + m_1\alpha_1 + \cdots + m_\ell\alpha_\ell.$$

In particular, if $\alpha + \beta$ was a positive constant function, then it must take value at least 1. But this contradicts the inequalities in (2.1).

Let $a, b$ be the respective gradients of $\alpha, \beta$. The subroot system of $R$ generated by $a$ and $b$ must have rank at most 2. In fact, its rank must be exactly 2, since $\alpha + \beta$ is not a constant function. We know that this rank 2 subsystem is not of type $A_2$, since we are assuming that $ia + jb$ is a root for $i > 1$. Therefore, we only need to consider the case where $a$ and $b$ generate a root system of type $C_2$ or $G_2$. In both cases, one can check directly that if $ia + jb$ is a root for positive integers $i, j$ then $a + b$ is a root as well. □
2.2 Shallow characters

A **shallow character** of the pro-unipotent radical $P_+ \subseteq P$ is any group homomorphism

$$\chi : P_+ \to \mathbb{C}^\times$$

that is trivial on the the Moy-Prasad subgroup

$$P_1 = \langle T_0, U_\alpha \mid \alpha(\lambda) \geq 1 \rangle,$$

where $T_0 = T(1 + p)$ is the maximal compact subgroup of $T$ and $U_\alpha$ is the affine root subgroup of $G$ corresponding to the affine root $\alpha$. Since $P_1$ is a normal subgroup of $P_+$, any shallow character of $P_+$ must factor through the quotient $P_+/P_1$, a finite group generated by subgroups

$$U_\alpha P_1/P_1 \cong U_\alpha/U_{\alpha+1} \cong \mathfrak{f}$$

with $\alpha$ being shallow affine roots. Indeed, given any coset $gP_1$ in $P_+/P_1$, there is a unique decomposition

$$gP_1 = \left( \prod_\alpha u_\alpha(x_\alpha)P_1 \right), \quad (2.2)$$

where the product is relative to some fixed order over all shallow affine roots $\alpha$ \([5, \S 3.1.1]\). Therefore, any shallow character $\chi$ can be recovered from its restriction to the shallow affine root groups via the formula:

$$\chi(gP_1) := \prod_\alpha \chi_\alpha(\bar{x}_\alpha),$$

where $\chi_\alpha : \mathfrak{f} \to \mathbb{C}^\times$ is the additive character defined by setting

$$\chi_\alpha(\bar{x}) := \chi(u_\alpha(x)P_1)$$

for any lift $x \in \mathfrak{o}$ of $\bar{x} \in \mathfrak{f}$.

**Lemma 3.** Let $\chi : P_+/P_1 \to \mathbb{C}^\times$ be a shallow character of $P_+$ given by additive characters as above. Then for any shallow affine roots $\alpha, \beta$ we have the following identities:

$$1 = \prod_{i,j} \chi_{i\alpha+j\beta}(C_{\alpha\beta ij} \bar{x}^i \bar{y}^j),$$

where the product is over all $i, j > 0$ such that $i\alpha + j\beta$ is a shallow affine root and the constants $C_{\alpha\beta ij}$ are given as in the Chevalley Commutator Formula \([1, \text{Theorem 5.2.2}]\).

**Proof.** Let $\alpha, \beta$ be two shallow affine roots such that $i\alpha + j\beta$ is a shallow affine root for some positive integers $i, j > 0$. Then by Lemma\[2\] we know that $\alpha + \beta$ is a shallow affine root. Therefore, we can apply the Chevalley commutator formula \([1, \text{Theorem 5.2.2}]\), which says that

$$[u_\beta(y), u_\alpha(x)]P_1 = \prod_{i,j} u_{i\alpha+j\beta}(C_{\alpha\beta ij} x^i y^j)P_1$$

for all $x, y \in \mathfrak{o}$. Here the product is in increasing order over all $i, j > 0$ such that $i\alpha + j\beta$ is an affine root. But if any $i\alpha + j\beta$ is not shallow, then $U_{i\alpha+j\beta} \subseteq P_1$. Therefore, we can assume that the product is only over $i, j > 0$ such that $i\alpha + j\beta$ is a shallow affine root.
Now let $\chi : P_+/P_1 \to \mathbb{C}^\times$ be any shallow character of $P_+$. Since $\chi$ is a group homomorphism, we know that

$$
\chi([u_\beta(y), u_\alpha(x)]P_1) = \prod_{i,j} \chi(u_{i\alpha + j\beta}(C_{\alpha\beta ij}x^i y^j)P_1)
= \prod_{i,j} \chi_{i\alpha + j\beta}(C_{\alpha\beta ij}x^i y^j)
$$

where the product is over all $i, j > 0$ such that $i\alpha + j\beta$ is a shallow affine root. Finally, since $\chi$ maps into an abelian group $\mathbb{C}^\times$, we know that

$$
\chi([u_\beta(y), u_\alpha(x)]P_1) = 1,
$$

finishing our proof.

**Theorem 4.** Suppose that for each shallow affine root $\alpha$, we are given an additive character $\chi_\alpha : \mathfrak{f} \to \mathbb{C}^\times$. Suppose further that for each pair of shallow affine roots $\alpha, \beta$ we have the following relation:

$$
1 = \prod_{i,j} \chi_{i\alpha + j\beta}(C_{\alpha\beta ij}x^i y^j),
$$

(2.3)

where the product is over all $i, j > 0$ such that $i\alpha + j\beta$ is a shallow affine root. Then there exists a unique shallow character $\chi : P_+/P_1 \to \mathbb{C}^\times$ such that

$$
\chi(u_\alpha(x)P_1) = \chi_\alpha(\bar{x})
$$

for all $x \in \mathfrak{o}$ and shallow affine root $\alpha$. Moreover, any shallow affine root is of this form.

**Proof.** For the proof of this theorem, we will fix an enumeration of the shallow affine roots $\alpha_1, \ldots, \alpha_n$ so that $i < j$ whenever $\alpha_i(\lambda) < \alpha_j(\lambda)$. Then we construct the well-defined function $\chi : P_+/P_1 \to \mathbb{C}^\times$ by setting

$$
\chi\left(\prod_{i=1}^n u_{\alpha_i}(x_i)P_1\right) := \prod_{i=1}^n \chi_{\alpha_i}(\bar{x}_i)
$$

(2.5)

for all $x_1, \ldots, x_n \in \mathfrak{o}$. Indeed, this function is well-defined since each coset in $P_+/P_1$ has a unique decomposition of the form $\prod_{i=1}^n u_{\alpha_i}(x_i)P_1$ with respect to this shallow affine root ordering. What follows is a proof that $\chi$ defines a group homomorphism, and thus, is the unique shallow character satisfying (2.4). Since $P_+/P_1$ is generated by the subgroups $U_{\alpha}P_1/P_1$ for shallow affine roots, it will be sufficient to show that

$$
\chi(gu_\alpha(x)P_1) = \chi(gP_1) \cdot \chi_\alpha(\bar{x})
$$

(2.6)

for all cosets $gP_1$ in $P_+/P_1$ and all shallow affine roots $\alpha$.

Let $\alpha = \alpha_j$ be a shallow affine root. We now show that (2.6) holds via descending induction on $j$. For the base case, we let $j = n$ so that

$$
\chi\left(\prod_{i=1}^n u_{\alpha_i}(x_i)P_1\right) u_{\alpha_n}(x)P_1 = \chi\left(\prod_{i=1}^{n-1} u_{\alpha_i}(x_i)P_1\right) u_{\alpha_n}(x_n + x)P_1
= \prod_{i=1}^{n-1} \chi_{\alpha_i}(\bar{x}_i) \cdot \chi_{\alpha_n}(\bar{x}_n + \bar{x})
= \prod_{i=1}^n \chi_{\alpha_i}(\bar{x}_i) \cdot \chi_{\alpha_n}(\bar{x})
$$

(2.7)
for all $x_1, \ldots, x_n, x \in \mathfrak{o}$. For the induction step, assume that
\[
\chi(gu_{\alpha_i}(x)P_1) = \chi(gP_1) \cdot \chi_{\alpha_i}(\bar{x})
\]
for all cosets $gP_1$ in $P_+/P_1$ and every shallow affine root $\alpha_i$ with $i > j$. In this case, we look at products of the form
\[
\prod_{i=1}^n u_{\alpha_i}(x_i)P_1\]
\[
\cdot u_{\alpha_j}(x_j + x)P_1\cdot \prod_{i=j+1}^n u_{\alpha_i}(x_i)[u_{\alpha_j}(x), u_{\alpha_i}(x)]P_1
\]
If $\alpha_i + \alpha_j$ is a constant, then
\[
[u_{\alpha_j}(x), u_{\alpha_i}(x_i)]P_1 = P_1.
\]
Otherwise, we can use the Chevalley commutator formula to say that
\[
[u_{\alpha_j}(x), u_{\alpha_i}(x_i)]P_1 = \prod_{k,l} u_{k\alpha_j + l\alpha_i} (C_{\alpha_j \alpha_i} \bar{x}^{k \alpha_j} x_i^l)P_1
\]
where the product is in increasing order over over all $k, l > 0$ such that $k\alpha_j + l\alpha_i$ is a shallow affine root. Note that each such $k\alpha_j + l\alpha_i$ must occur later than $\alpha_i$ in the enumeration of shallow affine roots since $k\alpha_j(\lambda) + l\alpha_i(\lambda) > \alpha_j(\lambda)$. By repeatedly applying the induction hypothesis and using relation (2.3), we have that
\[
\chi(g[u_{\alpha_j}(x), u_{\alpha_i}(x_i)]P_1) = \chi(gP_1) \left( \prod_{k,l} \chi_{k\alpha_j + l\alpha_i} (C_{\alpha_j \alpha_i} \bar{x}^{k \alpha_j} x_i^l) \right) = \chi(gP_1) \quad (2.7)
\]
for all cosets $gP_1$ in $P_+/P_1$. Thus, repeatedly applying the induction hypothesis and (2.7), we have
\[
\chi \left( \prod_{i=1}^n u_{\alpha_i}(x_i)P_1 \right) u_{\alpha_j}(x_j)P_1
\]
\[
= \chi \left( \prod_{i=1}^{j-1} u_{\alpha_i}(x_i)P_1 \right) u_{\alpha_j}(x_j + x)P_1 \left( \prod_{i=j+1}^n u_{\alpha_i}(x_i)[u_{\alpha_j}(x), u_{\alpha_i}(x)]P_1 \right)
\]
\[
= \chi \left( \prod_{i=1}^{j-1} u_{\alpha_i}(x_i)P_1 \right) u_{\alpha_j}(x_j + x)P_1 \left( \prod_{i=j+1}^{n-1} u_{\alpha_i}(x_i)[u_{\alpha_j}(x), u_{\alpha_i}(x)]P_1 \right) u_{\alpha_n}(x_n)P_1
\]
\[
= \chi \left( \prod_{i=1}^{j-1} u_{\alpha_i}(x_i)P_1 \right) u_{\alpha_j}(x_j + x)P_1 \left( \prod_{i=j+1}^{n-1} u_{\alpha_i}(x_i)[u_{\alpha_j}(x), u_{\alpha_i}(x)]P_1 \right) \cdot \chi_{\alpha_n}(\bar{x}_n)
\]
\[
\vdots
\]
\[
= \chi \left( \prod_{i=1}^{j-1} u_{\alpha_i}(x_i)P_1 \right) u_{\alpha_j}(x_j + x)P_1 \left( \prod_{i=j+1}^{n-1} \chi_{\alpha_i}(\bar{x}_i) \right).
\]
Finally, using the definition of $\chi$ given in (2.5), we arrive at

$$\chi \left( \prod_{i=1}^{n} u_{\alpha_i} (x_i) P_1 \right) u_{\alpha_j} (x) P_1 = \chi \left( \prod_{i=1}^{j-1} u_{\alpha_i} (x_i) P_1 \right) u_{\alpha_j} (x_j + x) P_1 \left[ \prod_{i=j+1}^{n-1} \chi_{\alpha_i} (\bar{x}_i) \right]$$

$$= \left[ \prod_{i=1}^{j-1} \chi_{\alpha_i} (\bar{x}_i) \right] \chi_{\alpha_j} (\bar{x}_j + \bar{x}) \left[ \prod_{i=j+1}^{n-1} \chi_{\alpha_i} (\bar{x}_i) \right]$$

$$= \left[ \prod_{i=1}^{n} \chi_{\alpha_i} (\bar{x}_i) \right] \chi_{\alpha_j} (\bar{x})$$

for all $x_1, \ldots, x_n, x \in \mathfrak{o}$ as desired.

This finishes our proof that there is a unique shallow character of $P_+$ satisfying (2.4). To see that every shallow character of $P_+$ is of this form, we note Lemma 3 says that its restrictions to shallow affine root groups must satisfy (2.3).

**Corollary 5.** Suppose that for each shallow affine root $\alpha$, we are given an additive character

$$\chi_\alpha : \mathfrak{f} \to \mathbb{C}^\times.$$

Suppose further that $\chi_\alpha$ is trivial whenever $\alpha$ is decomposable as a shallow affine root. Then there exists a unique shallow character $\chi : P_+/P_1 \to \mathbb{C}^\times$ such that

$$\chi (u_{\alpha} (x) P_1) = \chi_\alpha (\bar{x})$$

for all $x \in \mathfrak{o}$ and shallow affine roots $\alpha$.

**Proof.** By the previous theorem, we only need to show that given any shallow affine roots $\alpha, \beta$ we have the following relations:

$$1 = \prod_{i,j} \chi_{i\alpha + j\beta} (C_{\alpha ij} \bar{x}_i \bar{y}_j),$$

(2.8)

where the product is in increasing order over all $i, j > 0$ such that $i\alpha + j\beta$ is a shallow affine root. But this is true because each $i\alpha + j\beta$ is a decomposable shallow affine root, and thus each $\chi_{i\alpha + j\beta}$ is trivial. Thus (2.8) naturally holds.

### 2.3 The space of shallow characters

Let $\hat{V}$ be the set of all shallow characters of $P_+$. Then $\hat{V}$ has a natural abelian group structure given by

$$(\chi_1 + \chi_2)(g) = \chi_1 (g) \cdot \chi_2 (g).$$

Moreover, the group $\hat{V}$ can be endowed with the structure of a $\mathfrak{f}$-vector space as shown below: The finite group $P_+/P_1$ is generated by subgroups of the form

$$U_{\alpha} P_1 / P_1 \cong U_{\alpha}/U_{\alpha+1} \cong \mathfrak{f}$$

for shallow affine roots $\alpha$. Once a pinning of $G$ has been chosen, there is a natural action of $\mathfrak{f}$ on each of these subgroups by setting

$$\tilde{z} \cdot u_{\alpha} (x) P_1 := u_{\alpha} (zx) P_1$$
for all $x, z \in \mathfrak{o}$ and shallow affine roots $\alpha$. This action can be extended to the full group $P_+/P_1$ via distribution by setting

$$\tilde{z} \cdot (u_\alpha(x)u_\beta(y)P_1) = u_\alpha(zx)u_\beta(zy)P_1$$

for all $x, y, z \in \mathfrak{o}$ and shallow affine roots $\alpha, \beta$. This in turn endows the abelianization $V := \frac{P_+/P_1}{[P_+/P_1, P_+/P_1]}$ with the structure of a $\mathfrak{f}$-vector space spanned by vectors $v_\alpha$, the image of $u_\alpha(1)P_1$ under the quotient $P_+/P_1 \to V$. Finally, this action endows $\tilde{V}$ with the structure of a $\mathfrak{f}$-vector space with $\mathfrak{f}$-action given via

$$[\tilde{z} \cdot \chi](gP_1) := \chi(\tilde{z}^{-1} \cdot gP_1).$$

Thus, we have shown that $\tilde{V}$ is a $\mathfrak{f}$-vector space.

### 2.3.1 Epipelagic characters

Recall that for real number $0 < r < 1$, we say that a shallow affine root $\alpha$ has depth $r$ provided that $\alpha(\lambda) = r$. We now say that a shallow character $\chi \in \tilde{V}$ has depth $r$ provided that the following hold:

- $\chi_\alpha$ is non-trivial for some shallow affine root $\alpha$ of depth $r$.
- $\chi_\alpha$ is trivial for all shallow affine roots $\alpha$ of depth greater than $r$.

The minimal depth $\alpha(\lambda) = r$ for shallow affine roots $\alpha$ is $r = r_1$, the index of the pro-unipotent radical $P_+ = P_{r_1}$ in the Moy-Prasad filtration. The affine roots at this depth are said to be epipelagic, and since any epipelagic affine root is necessarily indecomposable as a shallow affine root, Corollary\ref{cor:epipelagic} implies that the set of all shallow characters of depth $r_1$ form a non-trivial subspace of $\tilde{V}$, denoted

$$\tilde{V}_+ := \tilde{V}_{r_1},$$

whose dimension is equal to the non-zero number of epipelagic affine roots. More generally, for all real numbers $0 < r < 1$, we let

$$\tilde{V}_r := \{ \chi \in V \mid \chi \text{ is trivial on } P_s \text{ for all } s > r \}$$

be the subspace of all shallow characters of depth at most $r$.

### 3 Supercuspidal Representations

Recall that a smooth representation of $G$ is a group homomorphism

$$\pi : G \to \GL(V),$$

where $V$ is a complex vector space, such that for every $v \in V$ there is a compact open subgroup $H \subseteq G$ such that $\pi(g)v = v$ for every $g \in H$. We say that a smooth representation $\pi$ is supercuspidal is every matrix coefficient of $G$ is compactly supported modulo the center $Z(G)$. We will now investigate which shallow characters of $P_+$ give rise to supercuspidal representations of $G$ via compact induction.
3.1 Compact Induction

In this section we will recall some basic facts about compact induction: Let $\chi : P_+/P_1 \to \mathbb{C}^\times$ be a shallow character of $P_+$, and consider the **compactly-induced representation** of $G$

$$\pi(\chi) := \text{ind}^G_{P_+}(\chi) = \left\{ \phi : G \to \mathbb{C} \mid \begin{array}{c} \phi(hg) = \chi(h) \cdot \phi(g) \\ \phi \text{ compactly supported} \end{array} \right\},$$

with $G$-action given by right translations:

$$[n \cdot \phi](g) := \phi(gn)$$

for all $n, g \in G$. Given any $n \in G$, we set $n^P_+ := nP_+n^{-1}$ and let $n\chi$ be the conjugate character of $n^P_+$ given by setting

$$n\chi(g) := \chi(n^{-1}gn)$$

for all $g \in n^P_+$. We then define the **intertwining set** to be

$$\mathcal{I}(G, P_+, \chi) := \{ n \in G \mid n\chi \simeq \chi \text{ on } n^P_+ \cap P_+ \}.$$

Then we have the following basic result:

**Lemma 6.** Let $\chi : P_+/P_1 \to \mathbb{C}^\times$ be a shallow character of $P_+$. Then the following are equivalent:

a. $\mathcal{I}(G, P_+, \chi) = P_\chi$.

b. $\pi(\chi)$ is irreducible.

c. $\pi(\chi)$ is supercuspidal.

Recall that the parahoric subgroup $P$ normalizes Moy-Prasad subgroups $P_+, P_1$, and so the conjugate character $n\chi$ is then a shallow character of $P_+$ for any $n \in P$. We therefore consider the stabilizer of $\chi$ in $P$:

$$P_\chi := \{ n \in N \mid n\chi = \chi \} \subseteq \mathcal{I}(G, P_+, \chi).$$

The finite quotient $P_\chi/P_+$ has order equal to the dimension of the semisimple **intertwining algebra**

$$\mathcal{A}_\chi := \text{End}_{P_\chi}(\text{ind}^G_{P_+}(\chi)).$$

There is a bijection $\rho \mapsto \chi_\rho$ between equivalence classes of irreducible $\mathcal{A}_\chi$-modules and the irreducible $P_\chi$ representations appearing in the isotypic decomposition

$$\text{ind}^G_{P_+}(\chi) = \bigoplus_{\rho} \dim(\rho) \cdot \chi_\rho.$$

Then we have the following result, whose proof can be found in [4, §2.1]:

**Lemma 7.** Let $\chi : P_+/P_1 \to \mathbb{C}^\times$ be a shallow character of $P_$. If $\mathcal{I}(G, P_+, \chi) = P_\chi$, then we have the following isotypic decomposition:

$$\pi(\chi) = \bigoplus_{\rho} \dim(\rho) \cdot \text{ind}^G_{P_+}(\chi_\rho),$$

where the direct sum is over all simple $\mathcal{A}_\chi$ modules $\rho$. Moreover, each compactly induced representation

$$\pi(\chi, \rho) := \text{ind}^G_{P_+}(\chi_\rho)$$

are inequivalent irreducible supercuspidal representations of $G.$
3.2 Supercuspidal representations coming from shallow characters

Let \( \mu \) be any point in the apartment \( \mathcal{A} \). For all positive real numbers \( s > 0 \), let

\[
V_{\mu, s} := \text{span}\{v_\alpha \in V \mid 0 < \alpha(\lambda) < 1 \text{ and } \alpha(\mu) \geq s\}
\]

be the \( f \)-span of the vectors \( v_\alpha \) for shallow affine roots \( \alpha \) such that \( \alpha(\mu) \geq s \). Then we have the following sufficient condition for constructing supercuspidal representations:

**Proposition 8.** Let \( \chi \in \hat{V}_r \) be any depth \( r \) shallow character such that the following holds:

(\( * \)) If \( n \in N_G(T) \) and \( \chi \) identically vanishes on \( V_{n\lambda, s} \) for all \( s > r \), then \( n\lambda = \lambda \).

Then \( \mathcal{I}(G, P_+, \chi) = P_\chi \).

**Proof.** Let \( \chi \in \hat{V}_r \) be a depth \( r \) shallow character of \( P_+ \) satisfying (\( * \)). Since \( P \) contains an Iwahori subgroup, the affine Bruhat decomposition [3] implies that in order to show that \( \mathcal{I}(G, P_+, \chi) = P_\chi \), it will be sufficient to consider \( n \in N_G(T) \) and show that if

\[
n\chi = \chi \text{ on } nP_+ \cap P_+
\]

then \( n \in P \).

Let \( n \in N_G(T) \) be such that (3.1) holds, and fix a real number \( s > r \). It is certainly true that

\[
n\chi = \chi \text{ on } nP_s \cap P_+
\]

for the Moy-Prasad subgroup \( P_s \subseteq P \). Let \( \alpha \) be any shallow root such that \( \alpha(n\lambda) \geq s \). Since it has depth \( r \), \( \chi \) must then be trivial on \( U_{n-1}\alpha \subseteq P_s \). Therefore, \( \chi_\alpha \) must be the trivial additive character, since [3.2] requires that

\[
\chi_\alpha(\bar{x}) = \chi(u_\alpha(x)) = n\chi(u_\alpha(x)) = \chi(u_{n-1}\alpha(\pm x)) = 1
\]

for all \( x \in \mathfrak{o} \). But this holds for all \( s > r \) and all shallow affine roots \( \alpha \) such that \( \alpha(n\lambda) \geq s \), and thus \( \chi \) vanishes identically on \( V_{n\lambda, s} \) for all \( s > r \). Consequently, (\( * \)) implies that \( n\lambda = \lambda \) so that \( n \in P \).

**Remark.** In the remainder of this subsection we study condition (\( * \)) of Proposition 8 in further detail. In particular, we first show in \( \S 3.2.1 \) how (\( * \)) is a necessary condition for constructing simple supercuspidal representations of \( G \). Then in \( \S 3.2.2 \) we show how, when leaving the epipelagic case, condition (\( * \)) is no longer necessary for constructing supercuspidal representations of \( G \).

3.2.1 Simple supercuspidal representations

In this subsubsection only, we will make the additional assumption that \( \lambda \) is the barycenter of the fundamental open alcove in \( \mathcal{A} \) bonded by \( \Delta \). If

\[
1 = m_0\alpha_0 + m_1\alpha_1 + \cdots + m_\ell\alpha_\ell
\]

(3.3)

is the minimal integral relation on simple affine roots with \( m_i > 0 \), then \( \lambda \) is the unique point such that for all simple \( \alpha_i \in \Delta \),

\[
\alpha_i(\lambda) = 1/h,
\]

where \( h := m_0 + m_1 + \cdots + m_\ell \) is the Coxeter number of \( R \). In this case, the parahoric subgroup \( P = G_\lambda \) is an Iwahori subgroup of \( G \).
Lemma 9. Let \( \lambda \) be the barycenter of the fundamental open alcove in \( A \). Then for any \( n \in N_G(T) \) such that \( n\lambda \neq \lambda \), there must exist a simple affine root \( \alpha_i \in \Delta \) such that \( \alpha_i(n\lambda) > 1/h \).

Proof. Let \( n \in N_G(T) \) be such that \( n\lambda \neq \lambda \). The difference \( \mu = \lambda - n\lambda \) belongs to the translation group

\[ E := \mathbb{R} \otimes \mathbb{Z} \text{Hom}(k, T), \]

so that we can write \( \mu = sc \) for some real number \( s > 0 \) and non-trivial cocharacter \( c \in \text{Hom}(k, T) \). For all simple affine roots \( \alpha_i \in \Delta \), we have

\[ \alpha_i(n\lambda) = \alpha_i(\lambda + sc) = \alpha_i(\lambda) + s\langle a_i, c \rangle, \]

where \( a_i \) is the gradient of \( \alpha_i \). Since \( \Delta \) forms a base of the affine root system, the gradients \( a_0, a_1, \ldots, a_\ell \) form a spanning set of the \( \ell \)-dimensional vector space

\[ E^* := \mathbb{R} \otimes \mathbb{Z} \text{Hom}(T, k), \]

which is dual to \( E \) under the natural pairing \( \langle \cdot, \cdot \rangle \). Therefore, there must be some \( \alpha_i \) such that \( \langle a_i, c \rangle \neq 0 \). Without loss of generality, we can assume that \( \langle a_i, c \rangle > 0 \) so that \( \alpha_i(n\lambda) > 1/h \); otherwise, if \( \langle a_j, c \rangle \leq 0 \) for all \( \alpha_j \in \Delta \), then (3.3) implies that

\[ 0 = m_0\langle a_0, c \rangle + m_1\langle a_1, c \rangle + \cdots + m_\ell\langle a_\ell, c \rangle < 0, \]

a contradiction.

Lemma 10. Let \( \lambda \) be the barycenter of the fundamental open alcove in \( A \). Then given any non-empty, proper subset

\[ I \subset \{0,1,\ldots,\ell\}, \]

there must exist an element \( n \in N_G(T) \) such that \( \alpha_i(n\lambda) < 0 \) for all \( i \in I \).

Proof. Consider the affine Weyl group

\[ W := N_G(T)/T_0 \]

and the subgroup \( W_I \) of \( W \) generated by simple reflections along the simple affine roots \( \alpha_i \) for \( i \in I \). Note that \( W_I \) is a non-empty, finite Coxeter group, since \( I \) is a non-empty, proper subset of \( \{0,1,\ldots,\ell\} \). Let \( w := w_I \) be the long element in \( W_I \); that is, \( w \) is the unique element on \( W_I \) such that \( w\alpha_i \) is a negative affine root for all \( i \in I \). Such an element has order 2, so that

\[ w^{-1}\alpha = w\alpha \]

for all affine roots \( \alpha \). Moreover, since an affine root is negative if and only if it takes negative values on the open fundamental alcove, we have

\[ \alpha_i(w\lambda) = (w^{-1}\alpha_i)(\lambda) = (w\alpha_i)(\lambda) < 0 \]

for all \( i \in I \). Thus, letting \( n \in N_G(T) \) be any lift of \( w \), we are done.

Proposition 11. Let \( \lambda \) be the barycenter of the fundamental open alcove in \( A \). Then given any epipelagic character \( \chi \in \hat{V}_{1/h} \), the following are equivalent:

a. \( \chi_{\alpha_i} \) is non-trivial for all \( \alpha_i \in \Delta \).
b. If \( n \in N_G(T) \) and \( \chi \) vanishes identically on \( V_{n\lambda,s} \) for all \( s > 1/h \), then \( n\lambda = \lambda \).

**Proof.** (\( a \Rightarrow b \)): Suppose that \( \chi_{\alpha_i} \) is non-trivial for all \( \alpha_i \in \Delta \), and let \( n \in N_G(T) \). By Lemma 9, there exists some \( \alpha_i \) such that \( \alpha_i(n\lambda) > 1/h \). Since \( \chi_{\alpha_i} \) is non-trivial, there must exist some \( s > 1/h \) such that \( \chi \) does not vanish identically on \( \mathfrak{f}v_{\alpha_i} \subseteq V_{n\lambda,s} \).

(\( \neg a \Rightarrow \neg b \)): Suppose that there exists some simple affine root \( \alpha_i \in \Delta \) such that \( \chi_{\alpha_i} \) is trivial. Setting

\[
I := \{ i \mid \chi_{\alpha_i} \text{ is non-trivial} \} \subseteq \{0, 1, \ldots, \ell\}
\]

and applying Lemma 10, we see that there must exist some \( n \in N_G(T) \) such that \( \alpha_i(n\lambda) < 0 \) whenever \( \chi_{\alpha_i} \) is non-trivial. In this case, for all \( s > 1/h \), the vector space \( V_{n\lambda,s} \) is contained within the span of subspaces \( \mathfrak{f}v_\alpha \) for shallow affine roots \( \alpha \) such that \( \chi_\alpha \) is trivial. Thus, \( \chi \) identically vanishes on \( V_{n\lambda,s} \) while \( n\lambda \neq \lambda \).

**Corollary 12.** Let \( \lambda \) be the barycenter of the fundamental open alcove in \( \mathcal{A} \), and let \( \chi \in \bar{\mathcal{V}}_{1/h} \) be any epipelagic character such that \( \chi_{\alpha_i} \) is non-trivial for all \( \alpha_i \in \Delta \). Then \( \mathcal{J}(G, P_+, \chi) = P_\chi \).

**Remark.** In the case given by the above corollary, the supercuspidal representations \( \pi(\chi, \rho) \) obtained from compact induction are called simple supercuspidal representations, and they were first studied by Gross-Reeder in [2]. This is a special class of epipelagic representations which were later studied by Reeder-Yu in [4].

### 3.2.2 A supercuspidal representation of \( \text{Sp}_4(\mathbb{Q}_2) \)

Let \( G = \text{Sp}_4(k) \) be the simply connected Chevalley group consisting of matrices in \( \text{SL}_2(k) \) which are fixed under the endomorphism

\[
X \mapsto Q^{-1}(X^\dagger)^{-1}Q,
\]

where \([x_{ij}]^\dagger = [x_{ji}]\) denotes transposition and \( Q \) is the skew-symmetric matrix

\[
Q = \begin{bmatrix}
0 & 1 \\
-1 & 0 \\
-1 & 1 \\
1 & -1
\end{bmatrix}.
\]

Alternatively, \( G \) is seen as the group of isometries with respect to the Hermitian form given by \( Q \). We fix the diagonal maximal torus

\[
T = \left\{ t = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} \middle| t_1, t_2, t_3, t_4 \in \mathbb{Q}_2^\times \text{ with } t_1t_4 = 1 \text{ and } t_2t_3 = 1 \right\}
\]

The root system \( R = R(G, T) \) of \( G \) relative to \( T \) has type \( C_2 \) with base given by short root \( a_1(t) = t_1/t_2 \) and long root \( a_2(t) = t_2/t_3 \). For convenience, we will denote by \( a_0(t) = t_4/t_1 \) the lowest long root in \( R \) relative to this chosen base. A base \( \Delta \) of the affine root system of \( G \) relative to \( T \) can be given by the following three affine functionals:

\[
\begin{align*}
\alpha_0 &= a_0 + 1 \\
\alpha_1 &= a_1 + 0 \\
\alpha_2 &= a_2 + 0
\end{align*}
\]
It should be noted that these simple affine roots satisfy the minimal relation
\[ \alpha_0 + 2\alpha_1 + \alpha_2 = 1. \]

By fixing a pinning of \( G \) via the following root group morphisms:

\[
\begin{align*}
u_{\alpha_1}(x) &= \begin{bmatrix} 1 & x \\ 1 & 1 -x \\ 1 & \end{bmatrix}, \\
u_{\alpha_0+a_1+a_2}(x) &= \begin{bmatrix} 1 & x \\ 1 & 1 \\ 1 & \end{bmatrix}, \\
u_{\alpha_2}(x) &= \begin{bmatrix} 1 & x \\ 1 & 1 \\ 1 & \end{bmatrix}, \\
u_{2\alpha_2+a_1}(x) &= \begin{bmatrix} 1 & x \\ 1 & 1 \\ 1 & \end{bmatrix}.
\end{align*}
\]

for \( x \in k \), we are able to directly compute the structure constants in the Chevalley commutator formulas:

\[
\begin{align*}
[u_{\alpha_1}(y), u_{\alpha_2}(x)] &= u_{\alpha_1+a_2}(+xy)u_{2\alpha_1+a_2}(-xy^2) \\
[u_{\alpha_1}(y), u_{\alpha_0}(x)] &= u_{\alpha_0+a_1}(-xy)u_{\alpha_0+2\alpha_1}(-xy^2) \\
[u_{\alpha_1}(y), u_{\alpha_1+a_2}(x)] &= u_{2\alpha_1+a_2}(+2xy) \\
[u_{\alpha_1}(y), u_{\alpha_0+a_1}(x)] &= u_{\alpha_0+2\alpha_1}(-2xy) \\
[u_{\alpha_2}(y), u_{\alpha_0+a_1}(x)] &= u_{\alpha_0+a_1+a_2}(-xy)u_{\alpha_0+1}(-x^2y) \\
[u_{\alpha_0}(y), u_{\alpha_1+a_2}(x)] &= u_{\alpha_0+1+a_2}(-xy)u_{\alpha_2+1}(-x^2y) \\
[u_{\alpha_1+a_2}(y), u_{\alpha_0+2\alpha_1}(x)] &= u_{\alpha_1+1}(-xy)u_{\alpha_1+a_2+1}(+xy^2) \\
[u_{\alpha_0+a_1}(y), u_{2\alpha_1+a_2}(x)] &= u_{\alpha_0+1}(-xy)u_{\alpha_0+2\alpha_1+1}(+xy^2) \\
[u_{\alpha_1+a_2}(y), u_{\alpha_0+a_1+a_2}(x)] &= u_{\alpha_2+1}(-2xy) \\
[u_{\alpha_0+a_1}(y), u_{\alpha_0+a_1+a_2}(x)] &= u_{\alpha_0+1}(+2xy) \\
[u_{2\alpha_1+a_2}(y), u_{\alpha_0+a_1+a_2}(x)] &= u_{\alpha_1+a_2+1}(-xy)u_{\alpha_2+2}(+x^2y) \\
[u_{\alpha_0+2\alpha_1}(y), u_{\alpha_0+a_1+a_2}(x)] &= u_{\alpha_0+a_1+1}(+xy)u_{\alpha_0+2}(+x^2y)
\end{align*}
\]

for any \( x, y \in \mathfrak{o} \).

Suppose that \( \lambda \) is contained within the closure of the alcove bounded by the vanishing hyperplanes of the simple affine roots in \( \Delta \). The set of positive affine roots which take value at most 1 at \( \lambda \) is therefore
\[
\{\alpha_0, \alpha_1, \alpha_2, \alpha_0 + \alpha_1, \alpha_1 + \alpha_2, \alpha_0 + 2\alpha_1, \alpha_0 + \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\},
\]
and those which take non-zero value at $\lambda$ form the shallow affine roots. Thus, in order to define a shallow character

$$\chi : P_+/P_1 \rightarrow \mathbb{C}^\times,$$

one only needs to verify that the restrictions to the shallow affine root groups satisfy the following relations coming from the commutators in (3.4):

$$\begin{align*}
1 = \chi_{\alpha_1 + \alpha_2}(xy) & \cdot \chi_{2\alpha_1 + \alpha_2}(xy^2) & \text{if } \alpha_1, \alpha_2 \text{ are shallow} \\
1 = \chi_{\alpha_0 + \alpha_1}(xy) & \cdot \chi_{\alpha_0 + 2\alpha_1}(xy^2) & \text{if } \alpha_0, \alpha_1 \text{ are shallow} \\
1 = \chi_{\alpha_0 + \alpha_1 + \alpha_2}(xy) & & \text{if } \alpha_2, \alpha_0 + \alpha_1 \text{ are shallow} \\
1 = \chi_{\alpha_0 + \alpha_1 + \alpha_2}(xy) & & \text{if } \alpha_0, \alpha_1 + \alpha_2 \text{ are shallow}
\end{align*}$$

(3.5)

for all $x, y \in \mathfrak{f}$.

**Example 13.** Suppose that the residue field of $k$ has order $q = 2$, and let $\lambda$ be the barycenter of the open alcove. Then consider the shallow character

$$\chi : P_+/P_1 \rightarrow \mathbb{C}^\times$$

given by additive characters

| $\alpha$       | $\chi_\alpha(1)$ |
|---------------|-----------------|
| $\alpha_0$    | $-1$            |
| $\alpha_1$    | $+1$            |
| $\alpha_2$    | $+1$            |
| $\alpha_0 + \alpha_1$ | $-1$       |
| $\alpha_1 + \alpha_2$ | $-1$       |
| $\alpha_0 + 2\alpha_1$ | $-1$   |
| $\alpha_0 + \alpha_1 + \alpha_2$ | $+1$    |
| $2\alpha_1 + \alpha_2$ | $-1$    |

Note that $\chi$ has depth $3/4$, but if

$$n_1 = \begin{bmatrix}
-1 & 1 & 1 \\
1 & 1 & -1 \\
1 & -1 & 1
\end{bmatrix} \in N_G(T)$$

is a lift of the simple reflection about the vanishing hyperplane of $\alpha_1$, then for any $s > 3/4$

$$V_{n_1, s} \subseteq f\nu_{\alpha_0 + \alpha_1 + \alpha_2},$$

over which $\chi$ vanishes identically; thus $\chi$ does not satisfy condition (*) in Proposition [S]. Despite this, we see that $\chi$ compactly induces to give a supercuspidal representation of $\text{Sp}_4(k)$. To see this, we first make the following observations:

- If $\alpha$ is a short affine root, then $n\alpha$ is also short for all $n \in N_G(T)$.
- The only positive, short affine roots $\alpha$ for which $\chi_\alpha(1) = -1$ are $\alpha_0 + \alpha_1$ and $\alpha_1 + \alpha_2$.
- For any $n \in N_G(T)$, either $n(\alpha_0 + \alpha_1)$ or $n(\alpha_1 + \alpha_2)$ is a positive affine root.
Consequently, for any \( n \in N_G(T) \),
\[
{n_\chi}_1 = \chi \text{ on } nP_+ \cap P
\]
only if \( n \) either fixes both \( \alpha_0 + \alpha_1 \) and \( \alpha_1 + \alpha_2 \) or swaps them. If \( n \) fixes both short affine roots, then either
\[
\begin{align*}
\{ n(\alpha_0) &= \alpha_0 - 2m \\
n(2\alpha_1 + \alpha_2) &= 2\alpha_1 + \alpha_2 + 2m \}
\end{align*}
\]
\[
\begin{align*}
\{ n(\alpha_0) &= 2\alpha_0 + \alpha_1 - 2m \\
n(2\alpha_1 + \alpha_2) &= \alpha_2 + 2m \}
\end{align*}
\]
holds for some \( m \in \mathbb{Z} \); if \( n \) swaps the short affine roots, then either
\[
\begin{align*}
\{ n(\alpha_0) &= 2\alpha_1 + \alpha_2 - 2m + 1 \\
n(\alpha_0 + 2\alpha_1) &= \alpha_2 + 2m + 1 \}
\end{align*}
\]
\[
\begin{align*}
\{ n(\alpha_0) &= \alpha_2 - 2m + 1 \\
n(\alpha_0 + 2\alpha_1) &= 2\alpha_1 + \alpha_2 + 2m + 1 \}
\end{align*}
\]
holds for some \( m \in \mathbb{Z} \). In all cases, if \( n \) does not act trivially on the affine roots, there exists some long shallow affine root \( \alpha \) such that \( n\alpha \) is also a positive affine root with
\[
-1 = \chi_{\alpha}(1) \neq \chi_{n\alpha}(1) = 1.
\]
Thus, given any \( n \in N_G(T) \), there exists some positive affine root \( \alpha \) such that \( \chi_{\alpha}(1) \neq \chi_{n\alpha}(1) \). Finally, the affine Bruhat decomposition
\[
G = P N_G(T) P
\]
implies that \( \mathcal{I}(G, P_+, \chi) = P_+ = P_+ \), where the last equality holds since \( q = 2 \). Hence, we have constructed a supercuspidal representation \( \pi(\chi) \) of \( \text{Sp}_4(k) \) coming from a shallow character of \( I \) not satisfying condition \((\ast)\) in Proposition 8.

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