Domain Wall/Cosmology correspondence in \((AdS/dS)_6 \times S^4\) geometries

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**ABSTRACT**

We investigate the ten dimensional origin of six dimensional \(F_4\) variant supergravity with supersymmetric de Sitter background. We address first the issue of spontaneous compactification, showing that it consists of a warped compactification on a four sphere of a variant massive type IIA supergravity. Moreover we illustrate how the known D4-D8 brane solution, whose near horizon geometry yields \(AdS_6 \otimes S^4\), is accordingly modified to a system including Euclidean branes. Finally, we discuss the relation between this latter solution and the D4-D8 brane system, showing how it represents a generalisation of the DW/Cosmology correspondence.
1 Introduction

Among the supergravity theories with supersymmetric AdS vacua, \( D = 6 \) \( N = 2 \), supergravity based on the exceptional supergroup \( F_4 \) [1] is somehow peculiar. Indeed, \( F_4 \) appears to be the only supergroup admitting two real sections whose bosonic generators span respectively the algebra \( \text{SO}(2, 5) \otimes \text{SU}(2) \) and \( \text{SO}(1, 6) \otimes \text{SU}(2) \). This reflects into the existence of two version of \( F_4 \) supergravity: the standard one \( \text{F}_4(1, 5) \), with supersymmetric AdS\(_6\) background [1, 4, 5], and a variant version, \( \text{F}^*_4(1, 5) \), with supersymmetric dS\(_6\) background [6]. \( \text{F}^*_4(1, 5) \) is a "variant" theory in the sense discussed by Hull [7]. Variant type II supergravities were introduced [7] considering T–duality transformations involving timelike circles; consequently lower dimensional variant supergravities naturally arise e.g. from compactifications on non–Euclidean tori. Hence, variant supergravities can occur in non–Lorentzian signatures. Quite generally they also have ghosts, and in lower dimensions they may have non–compact \( R \)–symmetry groups.

\( \text{F}^*_4(1, 5) \) supergravity has Lorentzian signature, nevertheless is a variant theory since its vector fields are ghosts. Remarkably the \( R \)–symmetry group is compact, since it is SU(2) for both real sections. This fact turns out to be quite relevant in the understanding of its ten dimensional origin.

It is in fact well known [8] that \( \text{F}_4(1, 5) \) supergravity can be obtained from a consistent Kaluza–Klein compactification of massive IIA\(_m\)(1,9) [9] on a four–sphere. More precisely not on the whole sphere, rather on an hemisphere \( \tilde{S}^4 \) viewed as a foliation of three–spheres \( S^3 \), whose rigid deformations parametrise SU(2). This observation strongly suggests that \( \text{F}^*_4(1, 5) \) must come from a similar compactification where at least the foliating \( S^3 \) is a genuine compact three-sphere.

In Section 1 we will see that this is actually the case. Modifying the ansatz in [8] we show that \( \text{F}^*_4(1, 5) \) can be obtained from a compactification of IIA\(_m^*\)(5,5) on a timelike \( \tilde{S}^4 \), the signature (5,5) is rather peculiar: it is in fact the only signature, together with (1,9), for which we can impose (pseudo–)Majorana and Weyl conditions at the same time. Moreover, apart from the space–time signature, the action of IIA\(_m\)(5,5) coincides with the action of IIA\(_m^*\)(1,9). The same happens for IIA\(_m^*\)(5,5) and IIA\(_m\)(1,9), since the action of both theories exhibits reversed sign for the kinetic terms of the RR fields [10] and the scalar potential.

Matter coupled \( \text{F}_4(1, 5) \) supergravity [4, 5] admits as well an AdS\(_6\) supersymmetric background, which has an holographic description in terms of a five dimensional superconformal field theory [11]. This result can be interpreted as the correspondence between the near horizon geometry and world–volume theory of a system of D4 – D8 branes [11] [12]. It is therefore natural to ask oneself whether a similar holographic description can be found as well for the dS\(_6\) vacuum of matter coupled \( \text{F}^*_4(1, 5) \) supergravity [6]. In particular, given that \( \text{F}^*_4(1, 5) \) supergravity can be obtained by compactification of IIA\(_m^*\)(5,5), we expect the relevant brane system to be a solution of the latter [7].

In Section 2, we propose a system of D(p,q)–branes of IIA\(_m^*\)(5,5) whose near horizon geometry is actually dS\(_6\) \( \otimes_w \tilde{S}^4 \), with timelike \( \tilde{S}^4 \), in the same way AdS\(_6\) \( \otimes_w \tilde{S}^4 \) with spacelike

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\(^1\)See for instance [2, 3].

\(^2\)We specify for every theory the space–time signature \((t, s)\) in which is formulated.

\(^3\)IIA\(_m^*\)(5,5) is the massive version of IIA\(^*\)(5,5) introduced in [10].
\( \hat{S}^4 \) arises as near horizon geometry of the D4–D8 system of IIA\(_m(1,9)\) \[^{12}\]. Moreover the localised D4–brane of IIA\(_m(1,9)\) corresponds to an Euclidean E5–brane of IIA\(_m^*(5,5)\), suggesting that if an holographic description is actually possible, this should correspond to the fixed point of an Euclidean Super Yang–Mills theory \[^{7}\]; hence the natural candidate would be an Euclidean, eventually variant, version of \(^{11}\).

In Section 3 the system of D(p,q)–branes of IIA\(_m^*(5,5)\) is discussed in the context of the Domain–Wall/Cosmology correspondence \[^{13}\]. In fact, in both F\(_4^*(1,5)\) and IIA\(_m^*(5,5)\) spinors enjoy pseudo–reality conditions. This in turn means that the aforementioned brane system admits a set of of pseudo–real Killing spinors. Similarly the pseudo–Killing spinors of cosmological solutions in the DW/Cosmology correspondence enjoy pseudo–reality conditions. There is in fact an example \[^{14}\] of correspondence between a DW solution of IIA\(_m(1,9)\) and a cosmological solution of IIA\(_m^*(5,5)\) with pseudo–real Killing spinors, suggesting that pseudo–supersymmetry can be quite generally realised as supersymmetry of a variant theory . Similarly , the construction of F\(_4^*(1,5)\) \[^{6}\] is based on the choice of pseudo–reality condition on the spinors, contrary to F\(_4(1,5)\), where reality is imposed \[^{6}\]. This kind of relation between supersymmetric AdS and dS vacua was also recently discussed in the context of DW/Cosmology correspondence in \[^{15}\].

Here we generalise the correspondence between pseudo–supersymmetric solutions of ordinary supergravity theories and supersymmetric solutions of variant supergravity theories, providing as a specific example the D(p,q)–branes system of IIA\(_m^*(5,5)\) under discussion.

2 Spontaneous compactification

In this section we show, in the same spirit of \[^{16}\], how to modify the compactification ansatz of \[^{8}\] in order to obtain a spontaneous compactification of IIA\(_m^*(5,5)\) to F\(_4^*(1,5)\).

In \[^{17}\] it was shown how to map a standard supergravity into a variant supergravity analysing the properties of the spinors, which depend on the signature of the space–time and if reality or pseudo–reality conditions are imposed. The map between F\(_4(1,5)\) and F\(_4^*(1,5)\) was derived in \[^{6}\], while the map between IIA(5,5) and IIA\(_m^*(5,5)\) was derived in \[^{17}\]; the generalisation to the massive case can be easily done with the same techniques of \[^{17}\].

Let us stress from the very beginning, that for convenience we are going to use the same convention for the metric as in \[^{8,12}\], that is

\[
\eta(t,s) = (_{t\times} \ldots \; -_{s\times} \ldots) \quad (2.1)
\]

while in \[^{6,17}\] we used the opposite one.

Let us start by presenting the bosonic Lagrangians of IIA\(_m(1,9)\) in the conventions of \[^{8}\]

\[
\mathcal{L}_{1,9} = \hat{R} - \frac{1}{2} \hat{e} \hat{d} \hat{\phi} \wedge d\hat{\phi} - \frac{1}{2} \hat{e}^{2\hat{\phi}} \hat{d} \hat{F}_2 \wedge d\hat{F}_2 - \frac{1}{2} \hat{e}^{-\hat{\phi}} \hat{d} \hat{H}_3 \wedge d\hat{H}_3 - \frac{1}{2} \hat{e}^{4\hat{\phi}} \hat{d} \hat{F}_4 \wedge d\hat{F}_4 - \frac{1}{2} \hat{d} \hat{A}_3 \wedge d\hat{A}_3 \wedge \hat{B}_2 - \frac{1}{6} \hat{m} \hat{d} \hat{A}_3 \wedge (\hat{B}_2)^2 - \frac{1}{40} m^2 (\hat{B}_2)^5 - \frac{1}{2} m^2 \hat{e}^{5\hat{\phi}} \hat{d} \hat{1} \quad (2.2)
\]

\(^4\text{Reality and pseudo–reality correspond to "convention I" and "convention II" respectively in \[^{17}\].}\)
Note that there is some freedom in the choice of the signs in (2.4), the only relevant thing is the reality of the coefficient in the redefinition. We made this choice in order to have homogeneous scaling of (2.3) and for later consistency with the six dimensional map.

The map for going to \( IIA^*_0(5,5) \) is given by

\[
\hat{A}_1(1) \rightarrow i\hat{A}_1(1); \quad \hat{B}_2(2) \rightarrow -\hat{B}_2(2); \quad \hat{A}_3(3) \rightarrow -i\hat{A}_3(3); \quad m \rightarrow -im
\]  

(2.4)

together with a signature redefinition \( \eta_{(1,0)} \rightarrow \eta_{(5,5)} \).

Note that there is some freedom in the choice of the signs in (2.4), the only relevant thing is the reality of the coefficient in the redefinition. We made this choice in order to have homogeneous scaling of (2.3) and for later consistency with the six dimensional map.

The corresponding Lagrangian is easily obtained

\[
\mathcal{L}^*_{5,5} = \hat{R} - \frac{1}{2} *d\hat{\phi} \wedge d\hat{\phi} + \frac{1}{2} e^{-\frac{1}{2} \phi} *d\hat{f}_2 \wedge d\hat{f}_2 - \frac{1}{2} e^{-\hat{\phi}} *d\hat{H}_3 \wedge d\hat{H}_3 + \frac{1}{2} e^{\frac{1}{2} \phi} *d\hat{f}_4 \wedge d\hat{f}_4 \\
- \frac{1}{2} (\hat{A}(3) \wedge d\hat{A}(3) \wedge \hat{B}(2) + \frac{1}{6} md\hat{A}(3) \wedge (\hat{B}(2))^2 - \frac{1}{40} m^2 (\hat{B}(2))^5 + \frac{1}{2} m^2 e^{-\frac{1}{2} \phi} *1
\]

(2.5)

with the same definition (2.3) for the field strengths.

In the notations of [8], the bosonic Lagrangian of pure \( F_4(1,5) \) supergravity [1] is

\[
\mathcal{L}_{1,5} = R - \frac{1}{2} *d\phi \wedge d\phi - \frac{1}{2} e^{-\frac{1}{2} \phi} *dF_2 \wedge dF_2 + *dF_2 \wedge (2) - \frac{1}{2} e^{-\sqrt{2}\phi} *dH_3 \wedge dH_3 \\
- \frac{1}{2} B_2(2) \wedge (\frac{1}{2} da_1(1) \wedge da_1(1) + \frac{1}{3} gb_2(2) \wedge da_1(1) + \frac{2}{27} g^2 b_2(2) \wedge b_2(2) + \frac{1}{2} f_2(2) \wedge f_2(2)) \\
- g^2 (\frac{2}{e} e^{-\sqrt{2}\phi} - \frac{8}{3} e^{-\sqrt{2}\phi} - 2e^{-\sqrt{2}\phi}) *1
\]

(2.6)

where the field strengths are defined in terms of the potentials according to

\[
F_2(2) = da_1(1) + \frac{2}{3} gb_2(2), \quad \ H_3(3) = db_2(2) \\
F_2(2) = da_1(1) + \frac{1}{2} g \epsilon_{ijk} A_{1}(1) \wedge A_{1}(1); \quad i = 1, 2, 3
\]

(2.7)

The map for going to \( F^*_4(1,5) \) is given by

\[
(A_1(1), A^i_1(1)) \rightarrow i(A_1(1), A^i_1(1)); \quad (B_2(2) \rightarrow -B_2(2); \quad g \rightarrow -ig
\]

(2.8)

and the corresponding Lagrangian

\[
\mathcal{L}^*_{1,5} = R - \frac{1}{2} *d\phi \wedge d\phi + \frac{1}{2} e^{-\frac{1}{2} \phi} *dF_2 \wedge dF_2 + *dF_2 \wedge (2) - \frac{1}{2} e^{-\sqrt{2}\phi} *dH_3 \wedge dH_3 \\
- \frac{1}{2} B_2(2) \wedge (\frac{1}{2} da_1(1) \wedge da_1(1) + \frac{1}{3} gb_2(2) \wedge da_1(1) + \frac{2}{27} g^2 b_2(2) \wedge b_2(2) + \frac{1}{2} f_2(2) \wedge f_2(2)) \\
+ g^2 (\frac{2}{e} e^{-\sqrt{2}\phi} - \frac{8}{3} e^{-\sqrt{2}\phi} - 2e^{-\sqrt{2}\phi}) *1
\]

(2.9)

with the same definitions (2.7) for the field strengths.
The compactification ansatz of \[8\] substituted into the ten dimensional equations of motion of (2.2) implies the six dimensional equations of motion of (2.6).

If we apply the map (2.8) to the compactification ansatz of \[8\] we easily realise that the equations of motion of (2.9) imply the six dimensional equations of motion of (2.6). Some more details are given in Appendix A.

Consider now the Einstein equation and the metric ansatz of \[8\]
\[
ds_{10}^2 = h(\zeta, \phi)) [ds_6^2 + 2g^{-2}(f_1(\zeta, \phi)d\zeta^2 + f_2(\zeta, \phi)cos^2\zeta h^1h^1)]
\]
(2.10)
where \(h^i = \sigma^i - gA^i(1)\), with \(\sigma^i\) left invariant 1-forms on the 3-sphere \(S^3\) and \(ds_6^2\) an Einstein metric. For \(A^i(1) = \phi = 0\) we have that \(f_1 = 1, = f_2 = \frac{1}{4}\). The ten dimensional geometry becomes
\[
ds_{10}^2 = h(\zeta, 0)[ds_6^2 + 2g^{-2}(d\zeta^2 + \frac{1}{4}cos^2\zeta \sigma^i \sigma^i)]
\]
(2.11)
where \(ds_6^2 = d\zeta^2 + cos^2\zeta \sigma^i \sigma^i\) is the metric of the four-sphere \(\tilde{S}^4\), with Ricci tensor \(R_{\alpha\beta} = \frac{4}{3}g^2g_{\alpha\beta}\) and \(ds_6^2\) is an \(AdS_4\) metric with Ricci tensor \(R_{\alpha\beta} = -\frac{10}{9}g^2g_{\alpha\beta}\).

Applying the map (2.8) to the metric (2.10), the functions \(h, f_1, f_2\) remain untouched, while \(g^{-2} \rightarrow -g^{-2}\). This turns (2.10) into
\[
ds_{10}^* = h(\zeta, \phi)[ds_6^2 - 2g^{-2}(f_1(\zeta, \phi)d\zeta^2 + f_2(\zeta, \phi)cos^2\zeta h^1h^1)]
\]
(2.12)
The change of sign in the metric ansatz means that four out of the nine spacelike directions have become timelike. This implies the change of signature \(\eta(1,9) \rightarrow \eta(5,5)\) in the ten dimensional theory and ensures that the Einstein equation is satisfied since the stress–energy tensor of (2.5) is clearly obtained applying the map (2.4) to the stress–energy tensor of (2.2).

For \(A^i(1) = \phi = 0\) the metric (2.12) becomes
\[
ds_{10}^* = h(\zeta, 0)[ds_6^2 - 2g^{-2}(d\zeta^2 + \frac{1}{4}cos^2\zeta \sigma^i \sigma^i)]
\]
(2.13)
In this case \(ds_6^2\) is a \(dS_6\) metric with Ricci tensors \(R_{\alpha\beta} = -\frac{4}{3}g^2g_{\alpha\beta}\), while \(ds_6^2\) is still the metric of a four–sphere \(\tilde{S}^4\) with Ricci tensor \(R_{ab} = \frac{10}{9}g^2g_{ab}\). The change of sign of the Ricci tensor of \(\tilde{S}^4\) is due to the fact that now the ten dimensional metric has signature \((5,5)\) which splits \((5,5) \rightarrow (1,5) + (4,0)\); the Lorentzian part \((1,5)\) pertains to \(dS_6\), while \(\tilde{S}^4\) is completely timelike, hence the Ricci tensor has the opposite sign.

As discussed before, the fact that the internal manifold has to be a foliation of three–spheres, is related to the six dimensional \(R\)–symmetry group, which is SU(2) for both \(F_4(1,5)\) and \(F_4^*(1,5)\).

3 Brane solutions and near–horizon geometry

In this section we will discuss how the D4–D8 system of type II \(A_m(1,9)\) \[12\] which gives as near horizon geometry a warped product \(AdS_6 \otimes \tilde{S}^4\) is mapped into a system of branes.
in IIA\textsuperscript{\textregistered}_m(5,5) giving as near horizon geometry a warped product \(dS_6 \otimes \tilde{S}^4\) with timelike four–sphere.

Let us start with the metric of the D4-D8 brane system \cite{12}

\[
d_s_{10}^2 = (H_4 H_8)^{-\frac{1}{2}} ( -dt^2 + d\tilde{\omega}^2 ) + H_4^2 H_8^{-\frac{1}{2}} d\tilde{x}^2 + (H_4 H_8)^{\frac{1}{2}} dz^2
\]

where we have defined

\[
dt = dt \cdot dt, \quad d\tilde{\omega} = \sum_{i=1}^{4} d\omega_i d\omega_i, \quad \tilde{x}^2 = \sum_{i=1}^{4} dx_i dx_i, \quad dz^2 = dz \cdot dz.
\]

where the indexes (including the dots \(\cdot\) which we introduced for sake of clarity) are raised and lowered with the ten dimensional metric \(\eta_{(10)}\) \cite{21}, i.e. \(dt = -dt\cdot, \quad dx = dx^i\), etc. The harmonic function \(H_8\) depends on the sole coordinate \(z\) which is transverse to the D8–brane, while \(H_4\) depends on \((z, \tilde{x})\) which are transverse to the D4–brane. For localised D4–branes, they have to satisfy \cite{13,12}

\[
\partial_z \partial_{\tilde{x}} H_8(z) = 0, \quad \partial_z \partial_{\tilde{x}} H_4(z, \tilde{x}) + H_8(x) \sum_{i=1}^{4} \partial_x \partial_{\tilde{x}} H_4(z, \tilde{x}) = 0
\]

Performing a change of coordinates \(\tilde{z} = \frac{\tilde{r}}{\sqrt{2}} \tilde{z}\), the metric can be put into the form

\[
d_{s_{10}}^2 = (\hat{H}_8)^{-\frac{1}{2}} \left[ \hat{H}_4^{-\frac{1}{2}} ( -dt^2 + d\tilde{\omega}^2 ) + \hat{H}_4^\frac{1}{2} (d\tilde{x}^2 + d\tilde{z}^2) \right]
\]

where at the near horizon

\[
\hat{H}_8(\tilde{z}) = Q_8 \left( \frac{9}{4} \tilde{z}^2 \right)^\frac{3}{2}; \quad \hat{H}_4(\tilde{z}, \tilde{x}) = \frac{Q_4}{\sqrt{2 \tilde{r}^2 + \tilde{z}^2}}
\]

Again, we define \(x^2 = \sum_{i=1}^{4} x_i x_i\), and \(z^2 = z \cdot z\).

Performing first the change of coordinates \(x = r \sin \alpha, \quad \tilde{z} = r \cos \alpha, \quad r \geq 0, \quad 0 \leq \alpha \leq \frac{\pi}{2}\) and afterwards defining \(r^2 = U^3\), together with a rescaling \(\omega_i \rightarrow \frac{2}{3} Q_4^\frac{3}{2} \omega_i\), we can put \((3.1)\) into a convenient near horizon form \cite{12}

\[
d_{s_{10}}^2 = \frac{9}{4} Q_4^\frac{1}{2} \left( \frac{3}{2} Q_8^\frac{1}{2} \sin \alpha \right)^{-\frac{3}{2}} \left[ U^2 dy_\parallel^2 + \frac{dU^2}{U^2} + d\Omega_3^2 \right]
\]

where \(dy_\parallel^2 = -dt^2 + d\tilde{\omega}^2\) and \(d\Omega_3^2 = d\alpha^2 + \cos^2 \alpha d\Omega_3^2\), which coincides with \cite{21}.

We now want to map it into a solution of IIA\textsuperscript{\textregistered}_m(5,5). In order to do that, we first transform it into a D–branes solution solution of IIA\textsuperscript{\textregistered}_m(5,5). This is quite immediate, since the action of IIA\textsuperscript{\textregistered}_m(5,5) coincides with the standard action IIA\textsuperscript{\textregistered}_m(1,9), a part from the space–time signature. Since the timelike directions belong to the world–volume of D–branes, we have just to change the signature, that is \(d\tilde{\omega}^2 \to -d\tilde{\omega}^2\), which doesn’t affect the equations of motion, the latter depending explicitly just on the transverse coordinates. Therefore the D(1,4)–D(1,8) brane system is mapped into a D(5,0)–D(5,4) system

\[
d_{s_{10}}^2 = (\tilde{H}_8)^{-\frac{1}{2}} \left[ \tilde{H}_4^{-\frac{1}{2}} ( -dt^2 + d\tilde{\omega}^2 ) + \tilde{H}_4^\frac{1}{2} (d\tilde{x}^2 + d\tilde{z}^2) \right]
\]

\footnote{In order to avoid confusions, in Lorentzian signatures we indicate a Dp–brane as a D(1,p)–brane.}
We indicate these exotic objects as Dirichlet \((p,q)\)–branes as in \([19]\) since they are extended objects of dimension \(p+q\), with Neumann boundary conditions in the worldvolume directions and Dirichlet boundary conditions in the transverse directions.

In order to obtain a solution of IIA ∗ \(m\) \((5,5)\), it is enough to know that IIA ∗ \(m\) \((5,5)\) can also be obtained from IIA \(m\) \((5,5)\) by reversing the signature \([10]\). Therefore one immediately obtains a system of D\((0,5)\)–D\((4,5)\) branes

\[
ds_{10}^2 = (\tilde{H}_8)^{-\frac{1}{2}} \left[ H_4^{-\frac{7}{2}} (dt^2 + d\tilde{x}^2) - H_4^{\frac{1}{2}} (d\tilde{x}^2 + dz^2) \right] \tag{3.8}
\]

where the intersections condition are the same of \((3.3)\), since on the D4–D8 system we have to implement both \(\sum_{i=1}^{4} \partial_{x_i} \partial_{x_i} \rightarrow -\sum_{i=1}^{4} \partial_{x_i} \partial_{x_i}\) and \(\partial_{\tilde{z}} \partial_{\tilde{z}} \rightarrow -\partial_{\tilde{z}} \partial_{\tilde{z}}\). Therefore \(\tilde{H}_4\) and \(\tilde{H}_8\) can be read from \((3.5)\).

The near horizon geometry is therefore given by

\[
ds_{10}^2 = \frac{9}{4} \Omega_4^2 \left( \frac{3}{2} \Omega_8^2 \sin\alpha \right)^{-\frac{1}{2}} \left[ U^2 dy^2_{\|} - \frac{dU^2}{U^2} - d\Omega_4^2 \right] \tag{3.9}
\]

where \(dy^2_{\|} = dt^2 + d\tilde{x}^2\) and \(d\Omega_4^2 = d\alpha^2 + \cos^2\alpha d\Omega_3^2\), which coincides with \((2.13)\). Note that the five dimensional world–volume of the localised D\((0,5)\)–brane is Euclidean.

Therefore if the F\(_1\)\((1,5)\) supergravity can be obtained as the near horizon geometry of a configuration of D\((0,5)\)–D\((4,5)\) branes, there should be a correspondence between its \(dS_6\) background and the fixed point of an Euclidean Super Yang–Mills theory leaving on the D\((0,5)\)–brane.

### 4 Supersymmetry and pseudo–supersymmetry

Let us briefly comment on the D\((0,5)\) and D\((4,5)\) branes in the context of Domain–Wall/ Cosmology correspondence.

Stable DW solutions of a system of gravity coupled to scalars, with a potential \(V(\phi)\), are generally "fake" supersymmetric \([20]\). Which means that it is possible to introduce a real "superpotential \(W\) such that \(V = 2[(W')^2 - W^2]\), where \(W' = \frac{\delta V}{\delta \phi}\) and that the DW solution implies the existence of a Killing spinor obeying

\[
D_{\mu} \varepsilon + W_{\gamma \mu} \varepsilon = 0 \tag{4.1}
\]

that is a supersymmetry–like condition.

The mapping between DW and cosmology \([13]\) implies that a very similar property holds ad well for cosmological solutions with potential \(-V(\phi)\). This is implemented introducing a pure imaginary "superpotential \(\tilde{W} = i\tilde{W}\) such that \(V = -2[(W')^2 - \tilde{W}^2]\) and the cosmological solution implies the existence of a Killing spinor obeying

\[
D_{\mu} \varepsilon + i \tilde{W}_{\gamma \mu} \varepsilon = 0 \tag{4.2}
\]

This latter condition is called pseudo–supersymmetry.

The natural question which arises is if it is possible to embed such "fake" (pseudo–) supersymmetric solutions into "true" supergravity solutions.
It is clear that it is not possible to embed both solutions in the same supergravity theory. This is because in supergravity one has to impose reality conditions on the spinors, hence it is immediate to understand that just one between (4.1) and (4.2) can be compatible with a given reality condition.

As pointed out in [14] one could provide an embedding for each solution if there exist two theories in the same signature in which the spinors obey different reality conditions, in particular one compatible with (4.1) and the other with (4.2). They also provide an example, that is the D8–brane solution of IIA\(_m(1, 9)\) corresponds to a cosmological solution of IIA\(_n(1, 9)\).

In the present paper we found another example of this kind of correspondence. Consider in fact the Killing spinor equation for a IIA\(_m(1, 9)\) D–brane; we can schematically write it as

\[
D_\mu \varepsilon + \alpha_q \varepsilon e^{\frac{2\pi i}{2} \phi} G^{(q)} \gamma_\mu (\gamma_{11}) \frac{\varepsilon}{2} = 0; \quad q = 0, 2, 4; \quad \alpha_q \in \mathbb{R}
\]  

(4.3)

where \(G^0 = m\) and \(\gamma_{11} = \gamma_0 \cdots \gamma_{10}\).

Equation (4.3) has to be consistent with the Majorana reality condition on \(\varepsilon\), which for IIA\(_m(1, 9)\) is given by

\[
\psi^\dagger G^{-1} \psi = \psi^T C_-
\]

(4.4)

where \(C_\) is the charge conjugation matrix and \(G_I = \gamma_0\). Consistency can be checked taking into account that for a space–time signature \((t, s)\) we have

\[
(\gamma_{a_1} \cdots \gamma_{a_n})^T = (-1)^n C_1^{-1} \gamma_{a_n} \cdots \gamma_{a_1} C_-
\]

(4.5)

\[
(\gamma_{a_1} \cdots \gamma_{a_n})^\dagger = (-1)^n G_I^{-1} \gamma_{a_n} \cdots \gamma_{a_1} G_I
\]

(4.6)

and that for type IIA D–branes \(q\) is even, therefore \(n\) is odd (4.3).

Note that (4.4) can be generalised to arbitrary space–time signature by defining \(G_I\) as the product of all timelike gamma matrices. Therefore, as discussed in [17], equation (4.3) remains valid for IIA\(_m(5, 5)\).

On the other hand, theories like IIA\(^*\)\(_m(1, 9)\), IIA\(^*\)\(_m(5, 5)\), F\(_4^*\)(1, 5) are characterised by a different reality condition of spinors [10, 6, 17]. It is usually referred as "pseudo–Majorana condition" and can be imposed according to

\[
\psi^\dagger G_I^{-1} = \psi^T C_-
\]

(4.7)

where \(G_I\) is the product of all the spacelike gamma matrices. In this case the hermitian conjugate of a product of \(n\) gamma matrices becomes

\[
(\gamma_{a_1} \cdots \gamma_{a_n})^\dagger = (-1)^{n(t-1)} G_I^{-1} \gamma_{a_n} \cdots \gamma_{a_1} G_I
\]

(4.8)

For signatures (1, 9) and (5, 5), \(t – 1\) is even, while \(s\) is odd. Which means that (4.6) and (4.8) differ for a sign, therefore for IIA\(^*\)\(_m(1, 9)\) and IIA\(^*\)\(_m(5, 5)\), the Killing spinor equation for a D–brane is given by

\[
D_\mu \varepsilon \pm i \alpha_q \varepsilon e^{\frac{2\pi i}{2} \phi} G^{(q)} \gamma_\mu (\gamma_{11}) \frac{\varepsilon}{2} = 0; \quad q = 0, 2, 4; \quad \alpha_q \in \mathbb{R}
\]

(4.9)

\[\text{See [21, 22] for examples of the embedding of "fake" supersymmetric DW into the DW solution in five dimensional supergravities.}\]

\[\text{Remember that here we are using the opposite convention for the space–time signature with respect to [6, 17], therefore (4.6), (4.8) differ from the ones presented in [17].}\]
where the $\pm$ sign depends on how the RR fields are redefined \(^{(2,4)}\) and has no consequences on the reality condition. This is the generalisation of the DW/Cosmology map $iW = \bar{W}$, which we retrieve for the case $q = 0$, corresponding to the example in \(^{(14)}\).

In conclusion, the observation in \(^{(14)}\) that pseudo–supersymmetry in IIA$_m$\,(1,9) cosmologies corresponds to supersymmetry in IIA$_m^*(1,9)$ is actually more general. In particular, it is possible to check if given a fake supersymmetric solution, supported by a field strength $G^{(q)}$, which can be embedded into a real supergravity theory, the corresponding fake pseudo–supersymmetric solution can be embedded into a variant theory. It is in fact sufficient to check using \(^{(4,5)}\), \(^{(4,6)}\) and \(^{(4,8)}\) if the fake pseudo Killing spinor equation can be obtained from the supersymmetry variation of a variant theory, in the same spirit of \(^{(17)}\). Note that it is not excluded that the corresponding variant theory has a different space–time signature. This seems to be the case e.g. for M–branes in eleven dimensions and for NS–branes in ten dimensions, since a change of reality in the three–form $C^{(3)} \rightarrow iC^{(3)}$ and in NSNS three–form $H^{(3)} \rightarrow iH^{(3)}$ respectively, are always associated with a change in the space–time signature \(^{(10, 17)}\).

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### Appendix A

For completeness we write the compactification ansatz of IIA$^*_m$\,(5,5). This can be obtained applying the maps \(^{(2,4)}\), \(^{(2,8)}\) to the ansatz in \(^{(8)}\).

$$ds^2_{10} = (\sin \zeta)^{\frac{1}{2}} X \left[ \Delta^{\frac{3}{2}} d\phi^2 - 2g^{-2}\Delta^{\frac{3}{2}} X^2 d\zeta^2 - \frac{1}{2}g^{-2}\Delta^{\frac{3}{2}} X^{-1}\cos^2 \zeta \sum_{i=1}^{3} (\sigma^i - g A_i^{(1)})^2 \right],$$

$$\hat{F}_{(4)} = \frac{\sqrt{2}}{6} g^{-3} s^{1/3} c^3 \Delta^{-2} U d\zeta \wedge \epsilon_{(3)} - \sqrt{2} g^{-3} s^{4/3} c^4 \Delta^{-2} X^{-3} dX \wedge \epsilon_{(3)}$$

$$- \sqrt{2} g^{-1} s^{1/3} c X^4 \ast H_{(3)} \wedge d\zeta + \frac{1}{\sqrt{2}} s^{4/3} \Delta^{2} \ast F_{(2)}$$

$$+ \frac{1}{\sqrt{2}} g^{-2} s^{1/3} c F_{(2)}^i \ast h^i \wedge d\zeta - \frac{1}{4\sqrt{2}} g^{-2} s^{4/3} c^2 \Delta^{-1} X^{-3} F_{(2)}^i \ast h^j \wedge h^k \epsilon_{ijk}, \quad (A.1)$$

$$\hat{H}_{(3)} = s^{2/3} \Delta^{1/2} \ast H_{(3)} + g^{-1} s^{-1/3} c F_{(2)} \ast d\zeta,$$

$$\hat{F}_{(2)} = \frac{1}{\sqrt{2}} s^{2/3} F_{(2)}, \quad e^{\phi} = s^{-5/6} \Delta^{-1/4} X^{-5/4},$$

where $X$ is related to the four dimensional dilaton $\phi$ by $X = e^{-\frac{1}{2}\sqrt{2} \sqrt{g} \phi}$, and

$$\Delta \equiv X \cos^2 \zeta + X^{-3} \sin^2 \zeta,$$

$$U \equiv X^{-6} s^2 - 3X^2 c^2 + 4X^{-2} c^2 - 6X^{-2}. \quad (A.2)$$
Furthermore, the functions $h, f_1, f_2$ introduced for brevity in (2.10) are given by

$$h(\zeta, \phi) = (\sin \zeta)^{1/12} X^{1/18} \Delta^{1/8}; \quad f_1(\zeta, \phi) = X^2; \quad f_2(\zeta, \phi) = \frac{1}{4} \Delta^{-1} X^{-1} \quad (A.3)$$

while $\epsilon(3) \equiv h_1 \wedge h_2 \wedge h_3$, and $s = \sin \zeta$ and $c = \cos \zeta$. The gauge coupling constant $g$ is related to the mass parameter $m$ by $m = \frac{\sqrt{2}}{3} g$.

Note, that the compactification ansatz differs from the one in [8] just for a few signs, which take into account the corresponding modifications of the equations of motion.

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