ASYMPTOTIC CHOW POLYSTABILITY IN KÄHLER GEOMETRY

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Abstract. It is conjectured that the existence of constant scalar curvature Kähler metrics will be equivalent to K-stability, or K-polystability depending on terminology (Yau-Tian-Donaldson conjecture). There is another GIT stability condition, called the asymptotic Chow polystability. This condition implies the existence of balanced metrics for polarized manifolds \((M, L^k)\) for all large \(k\). It is expected that the balanced metrics converge to a constant scalar curvature metric as \(k\) tends to infinity under further suitable stability conditions. In this survey article I will report on recent results saying that the asymptotic Chow polystability does not hold for certain constant scalar curvature Kähler manifolds. We also compare a paper of Ono with that of Della Vedova and Zuddas.

1. Introduction

A pair \((M, L)\) of a compact complex manifold \(M\) and a positive line bundle \(L\) over \(M\) is called a polarized manifold. Here a positive line bundle means a holomorphic line bundle \(L\) such that its first Chern class \(c_1(L)\) is represented, as a de Rham class, by a positive closed \((1, 1)\)-form. Therefore we can find a closed 2-form \(\omega\) of the form

\[
\omega = \frac{i}{2\pi} \sum_{i,j=1}^{m} g_{ij} dz^i \wedge d\bar{z}^j
\]

with \(g = (g_{ij})\) being pointwise a positive definite Hermitian matrix, and \(z^1, \cdots, z^m\) local holomorphic coordinates. Then \(g\) defines a Hermitian metric of \(M\), and \(\omega\) is regarded as its fundamental 2-form. Since \(\omega\) is closed, \(g\) becomes a Kähler metric.

Hence, for a polarized manifold \((M, L)\), \(c_1(L)\) is regarded as a Kähler class. We seek a constant scalar curvature Kähler (cscK) metric with its Kähler form in \(c_1(L)\).

There are known obstructions related to holomorphic vector fields. One is reductiveness of the Lie algebra \(\mathfrak{h}(M)\) of all holomorphic vector fields on \(M\) ([18], [19]), and the other is certain Lie algebra character \(f : \mathfrak{h}(M) \to \mathbb{C}\) ([11], [5]). Besides them, there are obstructions related to GIT stability. A well-known conjecture due to Yau, Tian, and Donaldson says the existence of constant scalar curvature metrics in \(c_1(L)\) will be equivalent to K-(poly)stability ([3]). K-stability is defined using the so-called DF-invariant as a numerical invariant for the Hilbert-Mumford criterion, see Definition 5.2. At the moment of this writing, it has been proved that the existence implies K-stability ([6], [10], [38], [22]), but it is still open whether K-stability implies the existence. Therefore at least K-stability is an obstruction to

Date: May 24, 2011.

2000 Mathematics Subject Classification. Primary 53C55, Secondary 53C21, 55N91.

Key words and phrases. Kähler-Einstein manifold, Chow stability, toric Fano manifold.
the existence. But there is another stability condition which is an obstruction to the existence of cscK metrics when the automorphism group \( \text{Aut}(M, L) \) is discrete. Here \( \text{Aut}(M, L) \) is the subgroup of the automorphism group \( \text{Aut}(L) \) of \( L \) consisting of all automorphisms of \( L \) commuting with the \( \mathbb{C}^* \)-action on the fibers. Notice that such automorphisms descend to automorphisms of \( M \). Therefore \( \text{Aut}(M, L) \) is naturally identified with a subgroup of the automorphism group \( \text{Aut}(M) \) of \( M \). From now on we regard \( \text{Aut}(M, L) \) as a subgroup of \( \text{Aut}(M) \) in this way, and also the Lie algebra \( h_0 \) of \( \text{Aut}(M, L) \) as a Lie subalgebra of the Lie algebra \( h(M) \) of \( \text{Aut}(M) \). The following result due to Donaldson shows in fact asymptotic Chow stability is an obstruction to the existence of cscK metrics.

**Theorem 1.1** (Donaldson [8]). Let \( (M, L) \) be a polarized manifold with \( \text{Aut}(M, L) \) discrete. Suppose there exists a cscK metric in \( c_1(L) \). Then \( (M, L) \) is asymptotically Chow stable.

Note that if \( (M, L^k) \) is Chow stable then there exists a “balanced metric” for \( L^k \). Donaldson further proved in the same paper [8] that as \( k \to \infty \), the balanced metrics converge to the cscK metric (assuming the existence of a cscK metric). Because of this result, we may have an expectation of a possibility to use the convergence of the balanced metrics as a one step in the proof of the implication of stability implying existence.

But the claim of this talk is that Donaldson’s theorem does not hold if \( \text{Aut}(M, L) \) is not discrete. In fact we explain the following result.

**Theorem 1.2** (Ono-Sano-Yotsutani [31]). There is a toric Fano 7-manifold (suggested by Nill and Paffenholtz in [25]) which is Kähler-Einstein but not asymptotically Chow-semistable (polystable).

This result relies on our earlier works [13] and [14]. The following result of Della Vedova and Zuddas, which is also related to our work [13], claims that there are two dimensional examples.

**Theorem 1.3** (Della Vedova-Zuddas [7]). There are constant scalar curvature Kähler surfaces which admit an asymptotically Chow unstable polarization.

The following result of Odaka uses a formula of DF-invariant for blow-ups along the flag ideals due to Wang [41] and Odaka [26].

**Theorem 1.4** (Odaka [27]). There are examples of K-stable polarized orbifolds which are asymptotically Chow unstable. In fact, these examples are Kähler-Einstein orbifolds with finite automorphisms. Hence Donaldson’s theorem does not hold for orbifolds.

Note that there is an argument without using balanced metrics to show that cscK metrics minimize the K-energy when the automorphism group is not discrete, see Li [17].

2. **What is (asymptotic) Chow stability ?**

Let \( V_k := H^0(M, O(L^k))^* \) be the vector space of all holomorphic sections of \( L^k \), \( M_k \subset \mathbb{P}(V_k) \) the image of Kodaira embedding by \( L^k \), and \( d_k \) the degree of \( M_k \) in \( \mathbb{P}(V_k) \).
Denote by $m$ the dimension of $M$: $m = \dim C M$. An element of $P(V_k^* \times \cdots \times P(V_k^*))$ $(m + 1$ times) defines $m + 1$ hyperplanes $H_1, \cdots , H_{m+1}$ in $P(V_k)$. Then the set
\[ \{(H_1, \cdots , H_{m+1}) \in P(V_k^* \times \cdots \times P(V_k^*))| H_1 \cap \cdots \cap H_{m+1} \cap M_k \neq \emptyset \} \]
becomes a divisor in $P(V_k^* \times \cdots \times P(V_k^*)$, and this divisor is defined by a polynomial $\bar{M}_k \in (\text{Sym}^d_k(V_k))^\otimes (m+1)$, called the Chow form. Consider the $SL(V_k)$-action on $(\text{Sym}^d_k(V_k))^\otimes (m+1)$. Stabilizer of $M_k$ under $SL(V_k)$-action is $\text{Aut}(M, L)$. In Theorem 1.1 by Donaldson, “Aut$(M, L)$ is discrete” means “the stabilizer is finite”.

**Definition 2.1.** Let $(M, L)$ be a polarized manifold.

1. $M$ is said to be Chow polystable w.r.t. $L^k$ if the orbit of $\bar{M}_k$ in $(\text{Sym}^d_k(V_k))^\otimes (m+1)$ under the action of $SL(V_k)$ is closed.
2. $M$ is Chow stable w.r.t $L^k$ if $M$ is polystable and the stabilizer at $\bar{M}_k$ of the action of $SL(V_k)$ is finite.
3. $M$ is Chow semistable w.r.t. $L^k$ if the closure of the orbit of $\bar{M}_k$ in $(\text{Sym}^d_k(V_k))^\otimes (m+1)$ under the action of $SL(V_k)$ does not contain $0 \in (\text{Sym}^d_k(V_k))^\otimes (m+1)$.
4. $M$ is asymptotically Chow polystable (resp. stable or semistable) w.r.t. $L$ if there exists a $k_0 > 0$ such that $M$ is Chow polystable (resp. stable or semistable) w.r.t. $L^k$ for all $k \geq k_0$.

In the case when $\text{Aut}(M, L)$ is not discrete Mabuchi tried to extend Theorem 1.1 by Donaldson. He first showed that in this case there is an obstruction to asymptotic Chow semistability:

**Theorem 2.2** (Mabuchi [20]). Let $(M, L)$ be a polarized manifold. If $\text{Aut}(M, L)$ is not discrete then there is an obstruction to asymptotic Chow semistability.

This obstruction is expressed in the paper [13] as a series of integral invariants, which are explained later in the next section. Mabuchi then proved the following result.

**Theorem 2.3** (Mabuchi [21]). Let $(M, L)$ be a polarized manifold, and suppose $\text{Aut}(M, L)$ is not discrete. If there exists a constant scalar curvature Kähler metric in $c_1(L)$ and if the obstruction in Theorem 2.2 vanishes then $(M, L)$ is asymptotically Chow polystable.

3. **Obstructions to asymptotic Chow semistability**

The Lie algebra $\mathfrak{h}_0$ of $\text{Aut}(M, L)$ is expressed in various ways. Recall that $\mathfrak{h}(M)$ is the Lie algebra of all holomorphic vector fields on $M$, which is the Lie algebra of $\text{Aut}(M)$. First of all it can be expressed as
\[ \mathfrak{h}_0 = \{ X \in \mathfrak{h}(M) \mid \text{zero}(X) \neq \emptyset \}. \]
Secondly it can be expressed also as
\[ \mathfrak{h}_0 = \{ X \in \mathfrak{h}(M) \mid \exists u \in C^\infty(M) \otimes \mathbb{C} \text{ s.t. } X = \text{grad}^u \in g^7 \frac{\partial u}{\partial z^j} \frac{\partial}{\partial z^j} \}. \]
Or we may say that $\text{Aut}(M, L)$ is the linear algebraic part of $\text{Aut}(M)$. Mabuchi’s obstruction to asymptotic Chow semistability can be re-stated in terms of integral invariants $\mathcal{F}_{\text{Tu}}$’s, which are explained below, as follows.
Theorem 3.1 \([13]\). Let \((M, L)\) be a polarized manifold with \(\dim_{\mathbb C} M = m\).

(a) The vanishing of Mabuchi’s obstruction is equivalent to the vanishing of Lie algebra characters \(\mathcal{F}_{\text{Td}^i} : \mathfrak{h}_0 \to \mathbb{C}\), for \(i = 1, \cdots, m\).

(b) \(\mathcal{F}_{\text{Td}^1}\) is an obstruction to the existence of a constant scalar curvature Kähler metric in \(c_1(L)\), which is sometimes called the classical Futaki invariant.

The Lie algebra characters \(\mathcal{F}_{\text{Td}^i}\) are defined as follows. For \(X \in \mathfrak{h}_0\) we have

\[ i(X)\omega = -\bar{\partial}u_X. \]

Assume the normalization

\[ \int_M u_X \omega^m = 0. \tag{3.1} \]

Choose a type \((1, 0)\)-connection \(\nabla\) in \(T'M\). Put

\[ L(X) = \nabla_X - L_X \in \Gamma(\text{End}(T'M)) \]

and let

\[ \Theta \in \Gamma(\Omega^{1,1}(M) \otimes \text{End}(T'M)) \]

be the \((1,1)\)-part of the curvature form of \(\nabla\).

Definition 3.2. For \(\phi \in I^p(GL(m, \mathbb{C}))\), we define

\[ \mathcal{F}_{\phi}(X) = (m - p + 1) \int_M \phi(\Theta) \wedge u_X \omega^{m-p} \]

\[ + \int_M \phi(L(X) + \Theta) \wedge \omega^{m-p+1}. \]

Notice that \(\mathcal{F}_{\phi}(X)\) is linear in \(X\). One can show that \(\mathcal{F}_{\phi}\) is independent of choices of \(\omega\) and \(\nabla\), from which it follows that \(\mathcal{F}_{\phi}\) is invariant under the adjoint action of \(\text{Aut}(M)\). In particular \(\mathcal{F}_{\phi}\) is a Lie algebra character.

Outline of the proof of Theorem 3.1. To show (a), suppose we have a \(\mathbb{C}^*\)-action on \(M\). Asymptotic Chow semistability implies that there is a lift of the \(\mathbb{C}^*\)-action to \(L\) such that it induces \(\text{SL}(H^0(L^k))\)-action for all \(k\). So, the weight \(w_k\) of the action on \(H^0(L^k)\) is zero for all \(k\). But \(w_k\) can be expressed using the equivariant index formula. The coefficient of \(k^j\) is \(\mathcal{F}_{\text{Td}^i}(X)\) where \(X\) is the infinitesimal generator of the \(\mathbb{C}^*\)-action.

To show (b), recall that the first Todd class \(\text{Td}^1\) is equal to \(\frac{1}{2}c_1\). Thus it corresponds to one half of the trace. Hence the second term of \(\mathcal{F}_{\text{Td}^1}(X)\) in Definition 3.2 is one half of the integral of the divergence of \(X\), which of course vanishes by the divergence theorem. Hence we have

\[ \mathcal{F}_{\text{Td}^1}(X) = \frac{m}{2} \int_M u_X c_1 \wedge \omega^{m-1} \]

where \(c_1\) denotes the first Chern form, or the Ricci form. Since \(mc_1 \wedge \omega^{m-1} = S \omega^m\) where \(S\) is the scalar curvature, the last integral becomes zero if \(S\) is constant because of the normalization \(3.1\). This completes the outline of the proof of Theorem 3.1. See \([13]\) or \([14]\) for the detail of the proof. \(\square\)
Now we have natural questions:

Question (a) In Theorem 2.3, can’t we omit the assumption of the vanishing of the obstruction? That is to say, if there exists a constant scalar curvature Kähler metric in $c_1(L)$ then doesn’t the obstruction necessarily vanish?

Question (b) In Theorem 3.1, if $F_{Td^1} = 0$ then $F_{Td^2} = \cdots = F_{Td^m} = 0$?

In [14] we studied the characters $F_{Td^i}$’s in terms of Hilbert series for toric Fano manifolds. We showed that the linear span of $F_{Td^1}, \ldots, F_{Td^m}$ coincides with the linear span of the characters obtained as derivatives of the Hilbert series. Note that the derivatives of the Hilbert series are computed by inputing toric data into a computer. We saw that, up to dimension three among toric Fano manifolds, there are no counterexamples to Question (b). But later a seven dimensional example of Nill and Paffenholz [25] appeared, and Ono, Sano and Yotsutani [31] checked that this seven dimensional example shows that the answers to Questions (a) and (b) are No. Now we turn to the Hilbert series.

4. Hilbert series.

Let $M$ be a toric Fano manifold of dim $M = m$. We take $L = K_M^{-1}$. Then $L$ is a very ample line bundle. Since $M$ is toric, the real $m$-dimensional torus $T^m$ acts on $M$ effectively. Since we have a natural $S^1$-action on $K_M^{-1}$, the real $(m+1)$-dimensional torus $T^{m+1}$ acts on $K_M^{-1}$ effectively so that $K_M^{-1}$ is also toric.

For $g \in T^{m+1}$, we put

$$L(g) := \sum_{k=0}^{\infty} \text{Tr}(g|H^0(M, K_M^{-k})).$$

Because of Kodaira vanishing theorem we may regard $L(g)$ as a formal sum of the Lefschetz numbers. We may analytically continue $L(g)$ to the algebraic torus $T^{m+1}_C$, and write it as $L(x)$ for an element $x \in T^{m+1}_C$.

Let $\{v_j \in \mathbb{Z}^m\}_j$ be the generators of the fan of $M$. Then the moment polytope of $M$ can be expressed as

$$P^* := \{w \in \mathbb{R}^m|v_j \cdot w \geq -1, \forall j\}.$$ 

Let

$$C^* \subset \mathbb{R}^{m+1}(= \text{Lie}(T^{m+1}))^*$$

be the cone over $P^*$. The integral points in $C^*$ corresponds bijectively to the set of all bases of $H^0(M, K_M^{-k})$ for all $k$.

For $x \in T^{m+1}_C$ and $a = (w,k) \in \mathbb{Z}^{m+1} \cap C^*$, we put

$$x^a = x_1^{a_1} \cdots x_{m+1}^{a_{m+1}}.$$

**Definition 4.1.** The Hilbert series $C(x, C^*)$ is defined by

$$C(x, C^*) := \sum_{a \in C^* \cap \mathbb{Z}^{m+1}} x^a.$$ 

The following fact is nontrivial, but is well-known in combinatorics.

**Fact 4.2.** $C(x, C^*)$ is a rational function of $x$.

It is easy to show the following lemma.

**Lemma 4.3.** $C(x, C^*) = L(x)$. 
For \( b \in \mathbb{R}^{m+1} \cong g = \text{Lie}(T^{m+1}) \), put
\[
e^{-tb} := (e^{-b_1t}, \ldots, e^{-b_{m+1}t}).
\]
Then we have
\[
\mathcal{C}(e^{-tb}, C^*) = \sum_{a \in \tilde{C}^* \cap \mathbb{Z}^{m+1}} e^{-ta \cdot b}.
\]
This is a rational function in \( t \) by Fact (4.2). Let \( P \) be the dual polytope of \( P^* \), and put \( C_R := \{(b_1, \ldots, b_m, m+1) | (b_1, \ldots, b_m) \in (m+1)P \} \subset g \).

An intrinsic meaning of \( C_R \) can be explained as follows. The unit circle bundle associated with \( K_M \) is considered as a Sasaki manifold with the regular Reeb vector field. But the Reeb vector field can be deformed in \( g \). The subset \( C_R \) consists of those which are critical points for the volume functional when we take the variation of the Reeb vector field to be constant multiple of the Reeb vector field itself (see [23]). In other words, \( C_R \) is a natural deformation space of the Reeb vector fields of the toric Sasaki manifold.

Put \( b = (0, \ldots, 0, m+1) \).

**Theorem 4.4 ([14]).** The coefficients of the Laurent series of the rational function \( \sum_{s=0}^{|t|} \mathcal{C}(e^{-t(b+se)}, C^*) \) in \( t \) span the linear space spanned by \( \mathcal{F}_{Td^1}, \ldots, \mathcal{F}_{Td^m} \).

This theorem is a generalization of a result of Martelli, Sparks and Yau [23], which says the classical Futaki invariant is obtained as a derivative of the Hilbert series. Our computations show that the question is closely related to a question raised by Batyrev and Selivanova: Is a toric Fano manifold with vanishing \( f = F_{Td^1} \) for the anticanonical class necessarily symmetric? Recall that a toric Fano manifold \( M \) is said to be symmetric if the trivial character is the only fixed point of the action of the Weyl group on the space of all algebraic characters of the maximal torus in \( \text{Aut}(M) \). The question of Batyrev and Selivanova is natural because it is proved by Batyrev and Selivanova [4] that if a toric Fano manifold is symmetric then there exists a Kähler-Einstein metric. Later Wang and Zhu [42] proved that a toric Fano manifold admits a Kähler-Einstein metric if and only if \( f = F_{Td^1} \) vanishes.

Nill and Paffenholz [25] gave a counterexample to the question of Batyrev-Selyvanova. Namely they gave an example of a non-symmetric seven dimensional toric Kähler-Einstein Fano manifold on which we have \( F_{Td^1} = 0 \). Ono, Sano and Yotsutani showed that, in this example, other \( F_{Td^i} \)'s are non-zero and all proportional.

### 5. Higher integral invariants and higher CM lines

The invariant \( \mathcal{F}_{Td^i} \) is considered as the Mumford weight of the CM line \( \lambda_{CM} \) on the Hilbert scheme \( \mathcal{H} \) of subschemes of \( \mathbb{P}^N \) with Hilbert polynomial \( \chi \) as shown by Paul and Tian [32], [33]. Recently Della Vedova and Zuddas showed that the same is true for higher \( \mathcal{F}_{Td^i} \)'s. This section is based on their paper [7].

Let \( (M, L) \) be an \( m \)-dimensional polarized variety or scheme. For a one parameter subgroup \( \rho : C^* \to \text{Aut}(M, L) \) with a lifting to an action \( \tilde{\rho} : C^* \to \text{Aut}(L) \) on \( L \) we denote by \( w(M, L) \) the weight of the induced action on the determinant line \( \bigotimes_{i=0}^m (\det H^i(M, L))^{-1} \), and by \( \chi(M, L) \) the Euler-Poincare characteristic \( \sum_{i=0}^m (-1)^i \dim H^i(M, L) \). Of course if we replace \( L \) by its sufficiently high
power we may assume $H^i(M, L) = 0$ for $i > 0$. It is known by the general theory that we have polynomial expansions

(5.1) $\chi(M, L^k) = a_0(M, L)k^m + a_1(M, L)k^{m-1} + \cdots + a_m(M, L)$,

(5.2) $w(M, L^k) = b_0(M, L)k^{m+1} + b_1(M, L)k^m + \cdots + b_{m+1}(M, L)$.

We define the Chow weight $Chow(M, L^k)$ of $(M, L^k)$ by

$$Chow(M, L^k) = \frac{w(M, L^k)}{k\chi(M, L^k)} - \frac{b_0(M, L)}{a_0(M, L)}.$$  

One easily gets

$$Chow(M, L^k) = \frac{b_{m+1}(M, L)}{k\chi(M, L^k)} + \frac{a_0(M, L)}{k\chi(M, L^k)} \sum_{\ell=1}^{m} \frac{a_0(M, L)b_{\ell}(M, L) - b_0(M, L)a_{\ell}(M, L)}{a_0(M, L)^2} k^{m+1-\ell}.$$  

The first term $b_{m+1}$ is known to vanish in the smooth case, see [12]. We then define $F_\ell(M, L)$ by

$$F_\ell(M, L) = \frac{a_0(M, L)b_{\ell}(M, L) - b_0(M, L)a_{\ell}(M, L)}{a_0(M, L)^2}.$$  

If $M$ is smooth, $\chi(M, L)$ is expressed using Todd classes and $c_1(L)$ by Riemann-Roch theorem and $w(M, L)$ is expressed using Todd classes, $c_1(L)$, connections in the tangent bundle of $M$ and $L$ with the infinitesimal action of $X$. The connection term in $L$ makes its appearance as the Hamiltonian function $u_X$ in Definition 3.2 of $F_\ell(X)$. Hence the terms $a_i(M, L)$ and $b_j(M, L)$ are written in terms those classes and connections. Della Vedova and Zuddas show that $F_\ell(M, L)$ is independent of the choice of a lifting $\hat{\rho} : \mathbb{C}^* \to Aut(L)$ of $\rho$ and that

(5.3) $F_\ell(M, L) = \frac{1}{vol(M, L)} F_{T\rho^t}(X)$

when $M$ is smooth and $X$ is the infinitesimal generator of the action $\rho : \mathbb{C}^* \to Aut(M, L)$. We give here the case when $\ell = 1$. Refer to [7] for general $\ell$.

**Lemma 5.1 ([9]).** If $M$ is a nonsingular projective variety then

$$F_1(M, L) = \frac{1}{vol(M, L)} F_{T\rho^t}(X)$$

where $X$ is the infinitesimal generator of the $\mathbb{C}^*$-action.

**Proof.** Let us denote by $m$ the complex dimension of $M$. Expand $h^0(L^k)$ and $w(k)$ as

$$h^0(L^k) = a_0k^m + a_1k^{m-1} + \cdots,$$

$$w(k) = b_0k^{m+1} + b_1k^m + \cdots.$$

Then by the Riemann-Roch and the equivariant Riemann-Roch formulae

$$a_0 = \frac{1}{m!} \int_M c_1(L)^m = vol(M),$$

$$a_1 = \frac{1}{2(m-1)!} \int_M \rho \wedge c_1(L)^{m-1} = \frac{1}{2m!} \int_M i_\omega^m,$$
Considering the determinant we see from (15) that there are Q embedding. Then we have (5.4) rank \( f \) such that \( \text{det} f \) (5.5) det \( f \) Mumford criterion for Chow stability. see next section for the detail. The idea is the similar to the following Hilbert- line bundle over \( U \), \( L \) invariant of \( (M, L) \). Let \( (M, L) \) be a polarized scheme. We call \( \text{Chow}(M, L) \) Chow-line \( U \rightarrow \text{H} \) be the universal flat family over the Hilbert scheme \( \text{H} \) of subschemes of \( \mathbb{P}^N \) with Hilbert polynomial \( \chi \), and \( \iota : \mathcal{U} \rightarrow \mathcal{H} \times \mathbb{C}
abla \) be the relatively ample line bundle over \( \mathcal{U} \). For \( k \) sufficiently large we have rank \( f_\ast(L^k) = \dim H^0(U_x, \mathcal{L}_x^k) \) and \( \text{det} f_\ast(L^k) = \text{det} H^0(U_x, \mathcal{L}_x^k) \) for all \( x \in \mathcal{H} \). Hence we have

\[
(5.4) \quad \text{rank} f_\ast(L^k) = a_0 k^n + a_1 k^{n-1} + \cdots + a_n.
\]

Considering the determinant we see from (15) that there are Q-line bundles \( \mu_0, \ldots, \mu_{m+1} \) such that

\[
(5.5) \quad \text{det} f_\ast(L^k) = \mu_0^{k_{m+1}} \otimes \mu_1^{k_{m}} \otimes \cdots \otimes \mu_{m+1}.
\]

By definition Chow-line is the Q-line bundle \( \lambda_{\text{Chow}}(\mathcal{H}, L) \) over \( \mathcal{H} \)

\[
(5.6) \quad \lambda_{\text{Chow}}(\mathcal{H}, L) = \text{det} f_\ast(L) \otimes \text{rank} f_\ast(L) \otimes \mu_0^{-\frac{1}{a_0}}.
\]

It is easy to see that \( \text{Chow}(M, L) \) is the Mumford weight of the Chow-line \( \lambda_{\text{Chow}}(\mathcal{H}, L) \).
By (5.7) and (5.5) one can show

\[
(5.7) \quad \lambda_{\text{Chow}}(\mathcal{H}, L) = \mu_{m+1}^{\frac{1}{a_0}} \otimes \left( \bigotimes_{\ell=1}^{m} \left( \mu_{\ell}^{\frac{1}{a_0}} \otimes \mu_0^{-\frac{a_0}{\ell\ell(\ell)}} \right) \right).
\]

We define the \( \ell \)-th CM-line \( \lambda_{\text{CM}, \ell}(\mathcal{H}, L) \) on the Hilbert scheme \( \mathcal{H} \) by

\[
\lambda_{\text{CM}, \ell}(\mathcal{H}, L) = \mu_{m+1}^{\frac{1}{a_0}} \otimes \mu_0^{-\frac{a_0}{\ell\ell(\ell)}}.
\]

It is also easy to see that \( \lambda_{\text{CM}, \ell}(\mathcal{H}, L) \) is the weight of the \( \ell \)-th CM-line \( \lambda_{\text{CM}, \ell}(\mathcal{H}, L) \). Della Vedova and Zuddas then compute \( \text{Chow}(M, L) \) and \( \lambda_{\text{CM}, \ell}(\mathcal{H}, L) \) for projective bundles over curves and for polarized manifolds blown-up at finite points.
Let $\Sigma$ be a genus $g$ smooth curve and $E$ a rank $n \geq 2$ vector bundle over $\Sigma$. Let $M = \mathbb{P}(E)$ be the projective bundle associated to $E$ and denote by $\pi : M \to \Sigma$ the projection. A line bundle $L$ on $M$ is the form $L = \mathcal{O}_{\mathbb{P}(E)}(r) \otimes \pi^*B$ where $B$ is a line bundle over $\Sigma$. We assume that $L$ is ample. We also assume that $E$ is decomposed as $E = E_1 \oplus \cdots \oplus E_s$ into indecomposable components $E_i$, and that we are given a $\mathbb{C}^*$ action on $E$ written in terms of this decomposition

$$t \cdot (e_1, \cdots, e_s) = (t^{\lambda_1}e_1, \cdots, t^{\lambda_s}e_s).$$

In this situation $\text{Chow}(M, L^k)$ is given by

\begin{equation}
\text{Chow}(M, L^k) = \frac{\frac{(m-1+kr)}{m+1}}{\chi(\Sigma, \det(E \otimes B^{-\frac{1}{r}})) \chi(\Sigma, S^k(E^* \otimes B^\frac{1}{r}))} \cdot \sum_{j=1}^{s} \lambda_j \text{rank}(E_j)(\mu(E_j) - \mu(E))
\end{equation}

where $\mu(F) = \deg(F)/\text{rank}(F)$ is the slope of the bundle $F$. On the other hand $F_\ell(M, L)$ is computed for some positive rational number depending only on $m$ as

\begin{equation}
F_\ell(M, L^k) = -C_\ell \frac{\chi(\Sigma, \det(E \otimes B^{-\frac{1}{r}}))}{\mu(E \otimes B^{-\frac{1}{r}})^2} \sum_{j=1}^{s} \lambda_j \text{rank}(E_j)(\mu(E_j) - \mu(E)).
\end{equation}

By (5.8) and (5.9) we see that $F_\ell(M, L^k)$ are proportional for all $\ell$, that they vanish if and only if $\mu(E_j) = \mu(E)$ for all $j = 1, \cdots, s$, and that $\text{Chow}(M, L^k) = 0$ if and only if $F_\ell(M, L^k) = 0$ for some (and hence any) $\ell$.

The slope stability of $E$ is related to the existence of cscK metric as in the following theorem.

**Theorem 5.3** ([1]). A projective bundle $\mathbb{P}(E)$ over a smooth curve of genus $g \geq 2$ admits a Kähler metric of constant scalar curvature in some (and hence any) Kähler class if and only if $E$ is slope polystable.

We will not reproduce the formulas of $\text{Chow}(M, L)$ and $F_\ell(M, L)$ for polarized manifolds obtained by blowing-up at finite points, but the consequences of the formulas are summarized as follows. By a result of LeBrun and Simanca [16] the cone $\mathcal{E}$ of extremal Kähler classes is open in the Kähler cone, and the locus where the Futaki invariant $F_1$ vanishes is the set $\mathcal{C}$ of all cscK classes. By the results of Arezzo and Pacard [2, 3] there is a non-empty open set of cscK classes under mild conditions. Under such conditions we may be able to show that the locus $\mathcal{Z}$ where $F_2 = \cdots = F_m = 0$ is a Zariski closed subset in $\mathcal{C}$. Then a rational point in $\mathcal{C} \setminus \mathcal{Z}$ will be a cscK but asymptotically unstable polarization. This idea works for the blow-up of $\mathbb{CP}^2$ at four points with all but one aligned. See [7] for the detail.

6. Toric case

In this section we compare H. Ono’s paper [29] with the work of Della Vedova and Zuddas [7]. Let $\Delta \subset \mathbb{R}^m$ be an $m$-dimensional integral Delzant polytope. Namely,

(i) $\Delta$ has integral vertices $w_1, \cdots, w_d$,

(ii) $m$ edges of $\Delta$ emanate from each vertex $w_i$, and

(iii) primitive vectors along those edges generate the lattice $\mathbb{Z}^m \subset \mathbb{R}^m$. 

To a Delzant polytope there correspond a nonsingular toric variety and an ample line bundle $L$. The Ehrhart polynomial of $\Delta$

$$E_P(k) = \text{Vol}(\Delta) k^n + \sum_{j=0}^{m-1} E_{P,j} k^j$$

has the property that

$$E_P(i) = \sharp(i P \cap \mathbb{Z}^m).$$

It is also known that there exists an $\mathbb{R}^m$-valued polynomial

$$s_\Delta(k) = k^{n+1} \int_\Delta x \, dv + \sum_{j=1}^{m} k^j s_{\Delta,j}$$

such that

$$s_\Delta(i) = \sum_{a \in i \Delta \cap \mathbb{Z}^m} a.$$

Then Ono [29] proves that if, for each $i$, $(M_\Delta, L_\Delta^i)$ is (not necessarily asymptotically) Chow semistable, we have

$$s_\Delta(i) = \frac{E_\Delta(i)}{\text{Vol}(i \Delta)} \int_{i \Delta} x \, dv.$$

Hence if $(M_\Delta, L_\Delta)$ is asymptotically Chow semistable, we have the equality

$$\text{Vol}(\Delta) s_\Delta(k) - k E_\Delta(k) \int_{\Delta} x \, dv = \sum_{j=0}^{m} k^j \left( \text{Vol}(\Delta) s_{\Delta,j} - E_{\Delta,j-1} \int_{\Delta} x \, dv \right) = 0$$

as a polynomial in $k$. But the Ehrhart polynomial is equal to the Hilbert polynomial $\chi(M_\Delta, L_\Delta^k)$. Moreover, $s_\Delta(k)$ can be regarded as a character of the torus and gives the weight $w(M_\Delta, L_\Delta^k)$ on $L_\Delta^k$ when restricted to a one parameter subgroup. Therefore, as a character,

$$\text{Vol}(k \Delta) s_\Delta(k) - k E_\Delta(k) \int_{\Delta} x \, dv$$

is equal to

$$\text{Vol}(\Delta) w(M_\Delta, L_\Delta^k) - k \chi(M_\Delta, L_\Delta^k) \int_{M_\Delta} u_X \omega^m$$

when restricted to the one parameter group generated by an infinitesimal generator $X$. Put

$$F_{\Delta,j} := \text{Vol}(\Delta) s_{\Delta,j} - E_{\Delta,j-1} \int_{\Delta} dv \in \mathbb{R}^m.$$

By (6.5), $F_{\Delta,j}$ vanishes if $(M_\Delta, L_\Delta^k)$ is Chow semistable. But (6.3) shows

$$\text{Lin}_C \{ F_{\Delta,j}, \ j=1, \cdots, m \} = \text{Lin}_C \{ F_{T_\Delta(p)} |_{\mathbb{C}^m}, \ p=1, \cdots, m \}$$

where $\text{Lin}_C$ stands for the linear hull in $\mathbb{C}^m$. This gives a proof to Conjecture 1.6 in [29].

In [30], Ono further gives a necessary and sufficient condition for Chow semistability condition for $(M_\Delta, L_\Delta^k)$ in terms of toric data. Shelukhin [37] also expresses $F_1(M, -K_M)$ for a toric Fano manifold $M$ in terms of toric data of $M$. 
7. K stability

The notion of K-stability was first introduced by Tian in [40] for Fano manifolds and proved that if a Fano manifold carries a Kähler-Einstein metric then $M$ is weakly K-stable. Tian’s K-stability considers the degenerations of $M$ to normal varieties and uses a generalized version of the invariant $\mathcal{F}_1$ which were defined for normal varieties. Donaldson re-defined in [9] the invariant $\mathcal{F}_1$ for general polarized varieties (or even projective schemes) as introduced in the previous section, and also re-defined the notion of K-stability for a polarized manifold $(M, L)$.

For a polarized variety $(M, L)$, a test configuration of degree $r$ consists of the following.

(a) A flat family of schemes $\pi : \mathcal{M} \to \mathbb{C}$;

(b) $\mathbb{C}^*$-action on $\mathcal{M}$ such that $\pi : \mathcal{M} \to \mathbb{C}$ is $\mathbb{C}^*$-equivariant with respect to the usual $\mathbb{C}^*$-action on $\mathbb{C}$:

(c) $\mathbb{C}^*$-equivariant relatively ample line bundle $\mathcal{L} \to \mathcal{M}$ such that for $t \neq 0$ one has $M_t = \pi^{-1}(t) \cong M$ and $(\mathcal{M}_t, \mathcal{L}|_{\mathcal{M}_t}) \cong (M, L^r)$.

$\mathbb{C}^*$-action on $(\mathcal{M}, \mathcal{L})$ induces a $\mathbb{C}^*$-action on the central fiber $L_0 \to M_0 = \pi^{-1}(0)$. Moreover if $(M, L)$ admits a $\mathbb{C}^*$-action, then one obtains a test configuration by taking the direct product $L^r \times \mathbb{C} \to M \times \mathbb{C}$. This is called a product configuration. A product configuration endowed with the trivial $\mathbb{C}^*$ action is called the trivial configuration.

**Definition 7.1.** Let $(M, L)$ be a polarized variety, and $(\mathcal{M}, \mathcal{L})$ a test configuration of $(M, L)$. We define DF-invariant $DF(\mathcal{M}, \mathcal{L})$ to be the DF-invariant $F_1(M_0, L_0)$ of the central fiber $(M_0, L_0)$.

**Definition 7.2.** A polarized variety $(M, L)$ is said to be K-polystable (resp. stable) if the DF-invariant $DF(\mathcal{M}, \mathcal{L})$ is negative or equal to zero for all test configurations $(\mathcal{M}, \mathcal{L})$, and the equality occurs only if the test configuration is product (resp. trivial).

**Conjecture ([9]):** Let $(M, L)$ be a nonsingular polarized variety. Then a Kähler metric of constant scalar curvature will exist in the Kähler class $c_1(L)$ if and only if $(M, L)$ is K-polystable.

Let us recall the following general terminology. Let $V$ be a vector space over $\mathbb{C}$ and $\rho$ a one parameter subgroup of $SL(V)$. Let $[v] \in \mathbb{P}(V)$ and $\lambda \in \mathbb{C}^*$. Suppose $[\rho(\lambda)v] \to [v_0] \in \mathbb{P}(V)$ as $\lambda \to 0$. Then we have an endomorphism $\rho(\lambda) : V_0 \to V_0$. The weight of this endomorphism is called Mumford weight of $(v, \rho)$ and is denoted by $\mu(v, \rho)$. We say that $[v] \in \mathbb{P}(V)$ is semistable (resp. stable) with respect to $\rho$ iff $\mu(v, \rho) \leq 0$ (resp. $\mu(v, \rho) < 0$). We also say that $[v] \in \mathbb{P}(V)$ is polystable iff $\mu(v, \rho) < 0$ or $\rho(\mathbb{C}^*)$ is contained in $Stab(v)$. The Hilbert-Mumford criterion says that $[v] \in \mathbb{P}(V)$ is semistable (resp. polystable) with respect to a subgroup $G$ of $SL(V)$ iff $[v] \in \mathbb{P}(V)$ is semistable (resp. polystable) with respect to arbitrary one parameter subgroup of $G$.

Let us define Hilbert stability of a polarized variety $(M, L)$. Suppose $L^r$ is a very ample line bundle with $h^i(L^r) = 0$ for $i > 0$. Then $\chi(r) := h^0(L^r)$ can be computed by Riemann-Roch theorem. If we fix an isomorphism $H^0(L^r) \cong \mathbb{C}^{\chi(r)}$ this gives an embedding $\Phi_{L^r} : M \to \mathbb{P}^{\chi(r)-1}$. A different choice of the isomorphism gives a transformation by an element of $SL(\chi(r))$. When $k$ is sufficiently large we have an
exact sequence

\[ 0 \to I_k \to S^k H^0_M(L^r) \to H^0_M(L^{kr}) \to 0, \]

where \( I_k \) denotes the set of all polynomials of degree \( k \) vanishing along the image of \( M \). The \( k \)-th Hilbert point of \((M, L^r)\) is the point in the Grassmannian

\[ x_{k,r} \in G = G(S^k \mathbb{C} \chi(r); \chi(rk)) \]
determined by the identification \( H^0_M(L^r) \cong \mathbb{C} \chi(r) \).

We say that \((M, L)\) is Hilbert (semi)stable with respect to \( r \) iff the image of \( x_{r,k} \in G \) of the Plücker embedding \( G \to \mathbb{P}^{(s\chi(r)+k-1)} \) is (semi)stable for all large \( k \).

**Fact 7.3** (c.f. [24], Proposition 2.1). Let \( L \) be a very ample line bundle with \( h^i(L) = 0 \) for \( i > 0 \), and \( \rho \) a one parameter subgroup of \( SL(h^0(L)) \). Let \( \tilde{w} \) be the Mumford weight of the Hilbert point \( x_k \in G(S^k \mathbb{C} \chi^h(L^r); \chi(k)) \) with respect to \( \rho \), and \( e \) be the Mumford weight of the Chow point of \((M, L)\) with respect to \( \rho \). Then we have

\[ \tilde{w}(k) = Cek^{m+1} + O(k^m) \]

with positive constant \( C \).

This says if \( e < 0 \) then \( \tilde{w}(k) < 0 \) for large \( k \), namely Chow stability implies Hilbert stability. If \( \tilde{w}(k) \leq 0 \) for all \( k \), then \( e \leq 0 \), namely Hilbert semistable implies Chow semistable.

Now let \( \tilde{w}(r, k) \) be the Mumford weight of \( x_{r,k} \). We wish to express this in terms of \( w(r) \) which was the weight for \( H^0(L^r) \) of the one parameter group \( \rho \) in \( SL(h^0(L)) \). As \( \rho \) lies in \( SL(h^0(L)) \) we have to renormalize the one parameter group so that it lies in \( SL(h^0(L^r)) \). After this renormalization we find by putting \( s = rk \)

\[ \tilde{w}(r, k) = -w(s) + \frac{w(r)}{r\chi(r)} s\chi(s) \]

\[ = s\chi(s) \left( \frac{w(r)}{r\chi(r)} - \frac{w(s)}{s\chi(s)} \right) \]

\[ = s\chi(s) \left( F_1(r^{-1} - s^{-1}) + O(r^{-2} - s^{-2}) \right). \]

**Theorem 7.4** ([34], [35]). If we put \( \tilde{w}(r, k) = \frac{1}{r\chi(r)} \sum_{i,j=0}^{m+1} a_{i,j} r^i k^j \) then

(1) \( a_{m+1,m+1} = 0 \);

(2) The Chow weight \( e_r := Chow(M, L^r) \) of \((M, L^r)\) is given by

\[ e_r = \frac{C r^m}{\chi(r)} \sum_{i=0}^{m} a_{i,m+1} r^i \]

with a positive constant \( C \);

(3) \( a_{m,m+1} \) and \( F_1(M, L) \) have the same sign.

This result says that if \( e_r \leq 0 \) for all large \( r \) then \( F_1(M, L) \leq 0 \), namely that asymptotic Chow semistability implies K-semistability.

Now we turn to the computation of \( F_1(M, L) \). The following result of Wang gives a way of computing \( F_1(M, L) \). Note that the sign convention for the DF-invariant is opposite in [41], [26] and [27].
**Theorem 7.5** (Wang [41]). For any test configuration \((\mathcal{M}, \mathcal{L})\) of a polarized variety \((M, L)\) we consider its natural compactification \((\overline{\mathcal{M}}, \overline{\mathcal{L}})\). Then \(F_1(\mathcal{M}, \mathcal{L})\) is computed by

\[
DF(\mathcal{M}, \mathcal{L}) = \frac{-1}{2(m!)((m+1)!)}(-m(L^{m-1}.K_{\mathcal{M}})(\overline{\mathcal{L}}^{m+1} + (m+1)(L^m)(\overline{\mathcal{L}}^m.K_{\overline{\mathcal{M}}}))
\]

where \(K_{\overline{\mathcal{M}}/\mathcal{P}^1} = K_{\overline{\mathcal{M}}} - f^*K_{\mathcal{P}^1}\) with the projection \(f: \overline{\mathcal{M}} \to \mathbb{P}^1\). The notation \((L^m)\) means the intersection number \(L \cdots L\) \((m\text{ times})\) in \(M\), and so on.

With different technicalities Odaka extends and applies this result to the semi test configuration \(\mathcal{B} := Bl_\mathcal{J}(M \times \mathbb{C})\) obtained by blowing up the flag ideal \(\mathcal{J} \subset \mathcal{O}_{X \times \mathbb{C}}\) of the form

\[
\mathcal{J} = I_0 + I_1t + I_2t^2 + \cdots + I_{N-1}t^{N-1} + (t^N)
\]

where \(I_0 \subset I_1 \subset \cdots \subset I_{N-1} \subset \mathcal{O}_M\) is a sequence of coherent ideals of \(M\). Denote this blow-up by \(\Pi: \mathcal{B} \to M \times \mathbb{C}\) and by \(E\) the exceptional divisor, i.e., \(\mathcal{O}(-E) = \Pi^{-1}\mathcal{J}\). We also put \(\mathcal{L} := p_1^*L\) where \(p_1\) is the projection of \(M \times \mathbb{C}\) or \(M \times \mathbb{P}^1\) to the \(i\)-th factor. We assume that the restriction of \(\mathcal{L}(-E)\) to \(\mathcal{B}\) is relatively semiample, and hence we have a semi test configuration \((\mathcal{B}, \mathcal{L}(-E))\|\mathcal{B}\). \(\mathcal{B}\) is compactified to \(\overline{\mathcal{B}} := Bl_\mathcal{J}(X \times \mathbb{P}^1)\). Then the DF-invariant \(DF(\mathcal{B}, \mathcal{L}(-E))\) is computed as follows.

**Theorem 7.6** (Odaka [26]).

\[
DF(\mathcal{B}, \mathcal{L}(-E)) = \frac{-1}{2(m!)((m+1)!)}(-m(L^{m-1}.K_{\mathcal{M}})(\mathcal{L}(-E))^{m+1} + (m+1)(L^m)(\mathcal{L}(-E)^m.p_1^*K_{\mathcal{M}}) + (m+1)(L^m)(\mathcal{L}(-E)^m.K_{\mathcal{B}/M \times \mathbb{C}}))
\]

where the intersection numbers are taken on \(M\) or \(\overline{\mathcal{B}}\).

The next theorem shows that this computation is sufficient to check \(K\)-(semi)stability.

**Theorem 7.7** (Odaka [26]). The negativity (resp. nonpositivity) of all the DF-invariance of the semi test configurations of the above blow-up type \((\mathcal{B}, \mathcal{L}(-E))\) with \(\mathcal{B}\) Gorenstein in codimension 1 is equivalent to \(K\)-stability (resp. \(K\)-semistability) of \((M, L)\).

In [27], Odaka proves Theorem 1.4 using Theorem 7.6 and 7.7. He also proves in [27]

- A semi-log-canonical canonically polarized variety \((X, \mathcal{O}_X(mK_X))\) with \(m \in \mathbb{Z}_{>0}\) is \(K\)-stable.
- A log-terminal polarized variety \((X, L)\) with numerically trivial canonical divisor \(K_X\) is \(K\)-stable.

These results are expected to be true because of Calabi-Yau theorem [43]. In [28], Odaka and Sano give an algebro-geometric proof of the fact that if the alpha invariant of a Fano manifold \(M\), which is equal to the log canonical threshold, is bigger than \(m/(m+1)\) then \((M, -K_M)\) is \(K\)-stable. This is of course another proof of a consequence of a theorem of Tian [39].

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