A central limit theorem in the $\beta$-model for undirected random graphs with a diverging number of vertices

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Summary

The $\beta$-model provides a convenient tool for analyzing network data and Chatterjee, Diaconis and Sly (2011) recently established the consistency of the maximum likelihood estimate (MLE) in the $\beta$-model when the number of vertices goes to infinity. In this note, by effectively approximating the inverse of the Fisher information matrix, we further obtain its asymptotic normality under mild conditions. Simulation studies and a data example are also provided to illustrate the theoretical results.

Some key words: Asymptotics; $\beta$-model; Central limit theorem; Diverging number of vertices; Undirected random graph;

1. Introduction

For an undirected random graph on $t$ vertices, the $\beta$-model (Chatterjee, Diaconis and Sly, 2011) assumes that for each $1 \leq i \neq j \leq t$, there exists an edge between $i$ and $j$ with probability

$$p_{i,j} = \frac{e^{\beta_i + \beta_j}}{1 + e^{\beta_i + \beta_j}},$$

independently of all other edges where $\beta_i$ is the influence parameter of the vertex $i$. This model of random graphs is closely related to the Bradley-Terry model for rankings (Bradley and Terry, 1952) and is actively used for analyzing network data (Newman et al., 2001; Jackson, 2008; Robins et al., 2007). For many real world networks, the number of vertices $t$ is large and hence it is necessary to consider the asymptotics with a diverging number of vertices. In the Bradley-Terry model for paired comparisons, Simons and Yao (1999) proved that the MLE is consistent and asymptotically normal when the number of parameters goes to infinity. This contrasts with the well-known Neyman-Scott problem under which the MLE fails even to attain consistency when the number of parameters goes to infinity. More recently, Chatterjee, Diaconis and Sly (2011) proved that the MLE of the $\beta$-model is consistent when the number of vertices $t$ goes to infinity. In this note, by effectively approximating the inverse of the Fisher information matrix, we further establish its asymptotic normality under mild conditions.
The rest of the paper is organized as follows. In Section 2, we present a central limit theorem in the $\beta$-model. Simulation studies and a real example are given in Section 3. All the proofs are relegated to the Appendix.

2. MAIN RESULTS

Suppose that $G$ is an undirected graph on $t$ vertices generated from the $\beta$-model where $\beta = (\beta_1, \ldots, \beta_t)^T \in \mathbb{R}^t$ is unknown. Let $d_1, \ldots, d_t$ be the degrees of the vertices of $G$, and the likelihood is then

$$e^{\sum \beta_i d_i \prod_{i<j} (1 + e^{\beta_i + \beta_j})}.$$  

The maximum likelihood estimate $\hat{\beta}$ of $\beta$ can be obtained by solving the equations

$$d_i = \sum_{j \neq i} \frac{e^{\hat{\beta}_i + \hat{\beta}_j}}{1 + e^{\hat{\beta}_i + \hat{\beta}_j}}, \ i = 1, \ldots, t. \quad (1)$$

For the generalized $\beta$-model, Rinaldo, Petrović and Fienberg (2011) obtained the necessary and sufficient conditions for the existence of the maximum likelihood estimate (MLE) $\hat{\beta} = (\hat{\beta}_1, \ldots, \hat{\beta}_t)$. When $t \to \infty$, Chatterjee, Diaconis and Sly (2011) established the following theorem:

**THEOREM 1.** Define $L_t = \max_{1 \leq i \leq t} |\beta_i|$. 

1. If $L_t = o(\log t)$, then $\hat{\beta}$ uniquely exists with probability tending to one. 
2. If $L_t = o(\log \log t)$, then

$$\max_{1 \leq i \leq n} |\hat{\beta}_i - \beta_i| \leq O_p(e^{c_1 e^{c_2 t} + c_3 L_t} \sqrt{\log t}) = o_p(1),$$

where $c_1, c_2$ and $c_3$ are positive constants. Hence $\hat{\beta}$ is uniformly consistent.

Denote the covariance matrix of $d = (d_1, \ldots, d_t)$ by $V_t = (v_{i,j})_{t \times t}$, where

$$v_{i,j} = \frac{e^{\beta_i + \beta_j}}{(1 + e^{\beta_i + \beta_j})^2}, \ i, j = 1, \ldots, t; i \neq j \text{ and } v_{i,i} = \sum_{j \neq i} v_{i,j}, \ i = 1, \ldots, t.$$  

Note that $V_t$ is also the Fisher information matrix for $\beta$. To establish the asymptotic normality of $\hat{\beta}$, we first obtain an accurate approximation of $V_t^{-1}$. Let $S_t = (s_{i,j})_{t \times t}$, where

$$s_{i,j} = \frac{\delta_{i,j} - \frac{1}{v_{..}}}{v_{i,j}} \quad (2)$$

$\delta_{i,j}$ is the Kronecker delta function and $v_{..} = \sum_{i,j=1}^t v_{i,j}$. In the following proposition whose proof is given Appendix 1, we obtain an upper bound on the error of using $S_t$ to approximate $V_t^{-1}$.

**PROPOSITION 1.**

$$||V_t^{-1} - S_t|| = O\left(\frac{e^{6L_t}}{(t - 1)^2}\right), \quad (3)$$

where $||A|| = \max_{i,j} |a_{i,j}|$ for a matrix $A = (a_{i,j})$. 

Since
\[
e^{2L_t} \leq e^{\beta_i + \beta_j} \left(1 + e^{2L_t} \right)^2 \leq \frac{1}{4}, \quad i \neq j,
\]
we have
\[
\frac{(t-1)e^{2L_t}}{1 + e^{2L_t}} \leq v_{i,t} \leq \frac{t-1}{4}, \quad i = 1, 2, \ldots, t.
\]
If \( e^{L_t} = o(\sqrt{t-1}) \),
\[
\max_{i=1,\ldots,t} \frac{v_{i,i}}{v_{..}} \leq \frac{(1 + e^{2L_t})^2}{4te^{2L_t}} = o(1).
\]
Noticing that \( d_i = \sum_{j \neq i} d_{i,j} \) is a sum of \( t-1 \) independent binomial random variables, it is easy to get the following proposition by the central limit theorem for the bounded case (page 289, Loève’s, 1977).

**Proposition 2.** If \( e^{L_t} = o(\sqrt{t-1}) \), then for any fixed \( r \geq 1 \), as \( t \to \infty \), the first \( r \) elements of \( S_t(d - E(d)) \) are jointly asymptotically normal with mean zero and covariance structure given by \((G_t)_{r \times r})\), where \( G_t = \text{diag}(v_{1,1}, v_{2,2}, \cdots, v_{t,t}) \).

We now establish a central limit theorem for the MLE in the \( \beta \) model, whose proof is given in Appendix 2.

**Theorem 2.** If \( L_t = o(\log(\log t)) \), then for any fixed \( r \geq 1 \), as \( t \to \infty \), the first \( r \) elements of \( \hat{\beta} - \beta \) are jointly asymptotically normal with mean 0 and covariance structure given by \((G_t)_{r \times r})\).

3. **Numerical examples**

We first conduct simulation studies to illustrate our theoretical results. By Theorem 2, we construct approximate 95% confidence intervals for \( \beta_i \) and \( \beta_i - \beta_j \). We report the coverage probabilities for certain \( \beta_i - \beta_j \) and the average coverage probabilities (ACP) for \( \beta_i, i = 1, \ldots, t \) as well as the probabilities that the MLE does not exist. Let \( \beta_i = 2iL_t/t - L_t, i = 1, \ldots, t \) and \( L_t \) is chosen to be 0, \( \log(\log t) \), \( (\log t)^{1/2} \), and \( \log t \) respectively. From Table 1 we see that when \( L_t = 0 \) or \( \log(\log t) \), the coverage probabilities are very close to the nominal level, indicating the adequacy of the constructed confidence intervals. When \( L_t = (\log t)^{1/2} \) or \( \log t \), the MLE does not exist with nonzero probabilities and the coverage probabilities deviate much from the nominal level. This demonstrates that the condition on \( L_t \) in Theorem 2 is critical in ensuring the existence of the MLE and its asymptotic normality.

[Table 1 about here]

Next, we analyze the food web dataset in Blitzstein and Diaconis (2009), which contains 33 organisms in the Chesapeake Bay and each organism is represented by a vertex in the graph. As in Blitzstein and Diaconis (2009), we study the simple graph after omitting the self-loop at vertex 19.

[Table 2 about here]
The degree sequence $d$ of the graph is summarized as below

$$d = (7, 8, 5, 1, 1, 2, 8, 10, 4, 2, 4, 5, 3, 6, 7, 3, 2, 7, 6, 1, 2, 9, 6, 1, 3, 4, 6, 3, 3, 2, 4, 4).$$

The influence parameters along with their standard errors are reported in Table 2. The largest four degrees are 8, 8, 10, 9 for vertices 2, 7, 8, 22, which also have the largest four influence parameters $-0.083, -0.083, 0.275, 0.102$ from Table 2. On the other hand, the four vertices with the smallest influence parameter $-2.602$ all have degree 1. This indicates that the larger influence parameter the vertex has, the more it is linked with the other vertices as described by the $\beta$-model.

\section*{Appendix 1}

\textbf{Proof of Proposition 1} Define $m = \min_{1 \leq i < j \leq n} v_{i,j}$ and $M = \max_{1 \leq i < j \leq n} v_{i,j}$. By (4), we have $m \geq e^{2L}/(1 + e^{2L})^2$ and $M \leq 1/4$. Denote the $t \times t$ identity matrix by $I_t$. Write $F_t = V_t^{-1} - S_t$, $R_t = (r_{i,j}) = I_t - V_tS_t$ and $W_t = (w_{i,j}) = S_tR_t$. We have the recursion

$$F_t = (V_t^{-1} - S_t)(I_t - V_tS_t) + S_t(I_t - V_tS_t) = F_tR_t + W_t,$$

and it follows that, for any $i$,

$$f_{i,j} = \sum_{k=1}^{t} f_{i,k}[(\delta_{k,j} - 1)\frac{v_{k,j}}{v_{i,j}} + \frac{2v_{k,k}}{v_{i,j}}] + w_{i,j}, \quad j = 1, \ldots, n.$$

Fixing $i$, let $f_{i,\alpha} = \max_{1 \leq k \leq t} f_{i,k}$ and $f_{i,\beta} = \min_{1 \leq k \leq t} f_{i,k}$. Since $2 \sum_{k=1}^{t} f_{i,k}v_{k,k} = 1$, we have $f_{i,\beta} \leq 1/(2v_{i,j})$ and $f_{i,\alpha} \geq 0$. By calculation, it can be shown that

$$\max(|w_{i,j}|, |w_{i,j} - w_{i,k}|) \leq \frac{M}{m^{2(t-1)^2}} \quad \text{for all } i, j, k,$$

and

$$f_{i,\alpha} - f_{i,\beta} = \sum_{k=1}^{t} (f_{i,k} - f_{i,\beta})[(1 - \delta_{k,\beta})\frac{v_{k,\beta}}{v_{i,\beta}} - (1 - \delta_{k,\alpha})\frac{v_{k,\alpha}}{v_{i,\alpha}}] + w_{i,\alpha} - w_{i,\beta}. \quad \text{(A2)}$$

Define $a = \frac{M}{m^{2(t-1)^2}}$. $\Omega = \{k : (1 - \delta_{k,\beta})v_{k,\beta}/v_{i,\beta} \geq (1 - \delta_{k,\alpha})v_{k,\alpha}/v_{i,\alpha}\}$ and $|\Omega| = \lambda$. It follows that

$$\sum_{k \in \Omega} (f_{i,k} - f_{i,\beta})[(1 - \delta_{k,\beta})\frac{v_{k,\beta}}{v_{i,\beta}} - (1 - \delta_{k,\alpha})\frac{v_{k,\alpha}}{v_{i,\alpha}}]$$

$$\leq (f_{i,\alpha} - f_{i,\beta})[\sum_{k \in \Omega} v_{k,\beta}/v_{i,\beta} - \sum_{k \in \Omega} (1 - \delta_{k,\alpha})v_{k,\alpha}/v_{i,\alpha}]$$

$$\leq (f_{i,\alpha} - f_{i,\beta})f(\lambda), \quad \text{(A3)}$$

where $f(\lambda) = \frac{\lambda M}{m^{2(t-1)^2}m} - \frac{(\lambda - 1)M}{m^{2(t-1)^2}m}$. Note that $f(\lambda)$ takes its maximum at $\lambda = t/2$ when $\lambda \in [1, t-1]$ and $f(t/2) = \frac{tM - (t-2)m}{tM + (t-2)m}$. By (A1), (A2), and (A3),

$$f_{i,\alpha} - f_{i,\beta} \leq \frac{tM - (t-2)m}{tM + (t-2)m} \times (f_{i,\alpha} - f_{i,\beta}) + a.$$

Hence,

$$f_{i,\alpha} - f_{i,\beta} \leq \frac{M(tM + (t-2)m)}{2(t-2)m^3(t-1)^2}. \quad \Box$$
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\[ f_{i, \alpha} \leq \frac{M(tM + (t-2)m)}{2(t-2)m^3(t-1)^2} + \frac{1}{2v} \leq \frac{M(tM + (t-2)m)}{2(t-2)m^3(t-1)^2} + \frac{1}{2m(t-1)^2} \]

\[ = O\left(\frac{e^{6L_i}}{(t-1)^2}\right). \]

**APPENDIX 2**

We first prove two lemmas.

**Lemma 1.** Let \( F_t = V_t^{-1} - S_t \) and \( U_t = \text{Cov}\{F_t(d - E(d))\}. \) Then

\[ ||U_t|| \leq ||V_t^{-1} - S_t|| + \frac{(1 + e^{2L_t})^4}{4e^{4L_t}(t-1)^2}. \]

**Proof.** Note that

\[ U_t = F_t V_t F_t^T = (V_t^{-1} - S_t) - S_t(I_t - V_t S_t), \]

and

\[ (S_t(I_t - V_t S_t))_{i,j} = \frac{(\delta_{i,j} - 1)v_{i,j}}{v_{i,i}v_{j,j}} + \frac{1}{v}. \]

By (A1),

\[ |(S_t(I_t - V_t S_t))_{i,j}| \leq \max\{\frac{(1 + e^{2L_t})^4}{4e^{4L_t}(t-1)^2}, \frac{(1 + e^{2L_t})^2}{t(t-1)e^{2L_t}}\} \leq \frac{(1 + e^{2L_t})^4}{4e^{4L_t}(t-1)^2}, \]

Thus,

\[ ||U_t|| \leq ||V_t^{-1} - S_t|| + ||S_t(I_t - V_t S_t)|| \]

\[ \leq ||V_t^{-1} - S_t|| + \frac{(1 + e^{2L_t})^4}{4e^{4L_t}(t-1)^2}. \]

**Lemma 2.** Assume that Theorem 1 (2) holds. Then, for any \( i, \)

\[ \hat{\beta}_i - \beta_i = (V_t^{-1}(d - E(d)))_i + o_p\left(\frac{1}{t^{1/2}}\right). \]

**Proof.** By Theorem 1 (2), we know that

\[ \lambda_t = \max_{1 \leq i \leq n} |\hat{\beta}_i - \beta_i| = O_p(e^{c_1 e^{2L_t} + c_3 L_t} \frac{\log t}{t}). \]

Let \( \tilde{\gamma}_{i,j} = \hat{\beta}_i + \hat{\beta}_j - \beta_i - \beta_j. \) By Taylor expansion, for any \( i \neq j, \)

\[ \frac{e^\hat{\beta}_i + \hat{\beta}_j}{1 + e^\hat{\beta}_i + \hat{\beta}_j} - \frac{e^{\beta_i + \beta_j}}{1 + e^{\beta_i + \beta_j}} = \frac{e^{\beta_i + \beta_j}}{(1 + e^{\beta_i + \beta_j})^2} \tilde{\gamma}_{i,j} + h_{i,j}, \]

where

\[ h_{i,j} = \frac{e^{\beta_i + \beta_j + \theta_{i,j}\tilde{\gamma}_{i,j}}(1 - e^{\beta_i + \beta_j + \theta_{i,j}\tilde{\gamma}_{i,j}})}{2(1 + e^{\beta_i + \beta_j + \theta_{i,j}\tilde{\gamma}_{i,j}})^3} \tilde{\gamma}_{i,j}^2, \]

and \( 0 \leq \theta_{i,j} \leq 1. \) Rewrite (3) as

\[ d - E(d) = V_t(\hat{\beta} - \beta) + h, \]
where \( \mathbf{h} = (h_1, \cdots, h_t)^T \) and \( h_i = \sum_{j \neq i} h_{i,j} \). Equivalently,
\[
\hat{\beta} - \beta = V_t^{-1} (\mathbf{d} - E(\mathbf{d})) + V_t^{-1} \mathbf{h}.
\]  
(\ref{prop6})

Since \(|e^x(1 - e^x)/(1 + e^x)^2| \leq 1\), we have
\[
|h_{i,j}| \leq |\hat{\gamma}_{i,j}^2|/2 \leq 2\lambda_t^2,
\]
and
\[
|h_i| \leq \sum_{j \neq i} |h_{i,j}| \leq 2(t - 1)\lambda_t^2.
\]

Note that
\[
(S_t \mathbf{h})_i = \frac{h_i}{v_{i,i}} - \frac{1}{v_i} \sum_{j=1}^t h_{j,j}, \quad \text{and} \quad (V_t^{-1} \mathbf{h})_i = (S_t \mathbf{h})_i + (F_t \mathbf{h})_i.
\]

By calculation, we have
\[
|S_t \mathbf{h}_i| \leq \frac{8\lambda_t^2(1 + e^{2Lt})^2}{e^{2Lt}} = O(e^{2c_1 e^{2Lt} + (2c_3 + 2)Lt} \times \frac{\log t}{t}),
\]
and, by Proposition 1,
\[
|(F_t \mathbf{h})_i| \leq |F_t| \times (t \max_i |h_i|) \leq O(e^{6Lt} \times \lambda_t^2) \leq O(e^{2c_1 e^{2Lt} + (2c_3 + 6)Lt} \times \frac{\log t}{t}).
\]

If \( L_t = o(\log(\log t)) \), then \(|V_t^{-1} \mathbf{h})_i| \leq |(S_t \mathbf{h})_i| + |(F_t \mathbf{h})_i| = o(1/t^{1/2})\). This completes the proof. \( \Box \)

**Proof of Theorem 2** By (\ref{prop6}),
\[
(\hat{\beta} - \beta)_i = (S_t(\mathbf{d} - E(\mathbf{d})))_i + (F_t \mathbf{d})_i + (V_t^{-1} \mathbf{h})_i.
\]

By Lemmas 1 and 2, if \( L_t = o(\log(\log t)) \), then
\[
(\hat{\beta} - \beta)_i = (S_t(\mathbf{d} - E(\mathbf{d})))_i + o(1/t^{1/2}).
\]

Theorem 2 follows directly from Proposition 2. \( \Box \)

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Table 1. Coverage probabilities and probabilities that the MLE does not exist (in parentheses).

| $t$ | $(i,j)/ACP$ | $L_t = 0$ | $L_t = \log(\log t)$ | $L_t = (\log t)^{1/2}$ | $L_t = \log t$ |
|-----|--------------|-----------|------------------|------------------|-----------------|
| (50) | (1,50) | 0.955(0) | 0.955(0) | 0.98(0.081) | 0(1) |
| (25,26) | 0.951(0) | 0.952(0) | 0.955(0.081) | 0(1) |
| (49,50) | 0.951(0) | 0.956(0) | 0.992(0.081) | 0(1) |
| ACP | 0.950(0) | 0.953(0) | 0.96(0.081) | 0(1) |
| (100) | (1,100) | 0.964(0) | 0.949(0) | 0.978(0.004) | 0(1) |
| (50,51) | 0.952(0) | 0.943(0) | 0.96(0.004) | 0(1) |
| (99,100) | 0.953(0) | 0.955(0) | 0.981(0.004) | 0(1) |
| ACP | 0.952(0) | 0.951(0) | 0.957(0.004) | 0(1) |
| (200) | (1,200) | 0.948(0) | 0.964(0) | 0.984(0) | 0(1) |
| (100,101) | 0.954(0) | 0.948(0) | 0.945(0) | 0(1) |
| (199,200) | 0.951(0) | 0.947(0) | 0.966(0) | 0(1) |
| ACP | 0.951(0) | 0.951(0) | 0.953(0) | 0(1) |

Table 2. The food web dataset: the estimated influence parameters and their standard errors (in parentheses).

| Vertex | $\beta_1$ | Vertex | $\beta_2$ | Vertex | $\beta_3$ | Vertex | $\beta_4$ |
|--------|-----------|--------|-----------|--------|-----------|--------|-----------|
| 1      | -0.285(2.233) | 2      | -0.083(2.332) | 3      | -0.754(1.981) | 4      | -2.602(0.977) |
| 5      | -2.602(0.977) | 6      | -1.853(1.349) | 7      | -0.083(2.332) | 8      | 0.275(2.486) |
| 9      | -1.041(1.816) | 10     | -1.853(1.349) | 11     | -1.041(1.816) | 12     | -0.754(1.981) |
| 13     | -1.389(1.612) | 14     | -0.506(2.118) | 15     | -0.285(2.233) | 16     | -1.389(1.612) |
| 17     | -1.853(1.349) | 18     | -0.285(2.233) | 19     | -0.506(2.118) | 20     | -2.602(0.977) |
| 21     | -1.853(1.349) | 22     | 0.102(2.415) | 23     | -0.506(2.118) | 24     | -2.602(0.977) |
| 25     | -1.389(1.612) | 26     | -1.041(1.816) | 27     | -0.506(2.118) | 28     | -1.389(1.612) |
| 29     | -1.389(1.612) | 30     | -1.389(1.612) | 31     | -1.853(1.349) | 32     | -1.041(1.816) |
| 33     | -1.041(1.816) |