Classical string propagation in gravitational fields

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The motion of a string in curved spacetime is discussed in detail. The basic formalism for string motion in Minkowski spacetime and in curved spacetime is presented. The description applies to cosmic strings as well as to fundamental strings. Major ansätze for solving the string equations of motion are reviewed. In particular, the “null string ansatz”, which is relevant to strings in strong gravitational fields, is emphasised. The formalism is applied to the motion of strings in 5-dimensional Kaluza-Klein black hole backgrounds with electric and magnetic charge. Such background spacetimes have been of interest lately, particularly from the point of view of fundamental string theory. It is shown that interesting results, relating to the extra dimension as seen by the string, are obtained even at the classical level.

I. INTRODUCTION

Perhaps the most important goal of theoretical physics in recent times has been to develop a unified theory of all the forces of nature. The unification of the electric and magnetic forces was achieved for the first time in Maxwell’s theory of electromagnetism. Weak interactions were successfully included in the unification scheme with the formulation of the electroweak theory. In the 1970’s, the so-called Grand Unified Theories (GUTs), which sought to unify the three forces - electromagnetic, weak and strong - were mooted. The search is on for a ‘theory of everything’ to successfully unify all the forces of nature (for recent reviews, see Refs. [1,2]) - specifically, to unify the three forces with the fourth, i.e., gravity.

The ubiquitous nature of gravity led Einstein to postulate the General Theory of Relativity [3]. He argued that gravitation must be an intrinsic feature of space and time. The Einstein equations relate the spacetime metric to the stress-energy tensor of the matter distribution. Gravitation, therefore, has a geometric origin in General Relativity. However, there are two kinds of problems with Einstein’s theory of gravitation. Firstly, when quantised as a local field theory, it is not renormalisable, unlike standard electroweak theory. This limits its predictive power. Secondly, the standard cosmological model based on Einstein’s theory is plagued by a number of problems, viz. the flatness, horizon and initial singularity problems [4]. These considerations lay open the possibility that Einstein’s theory needs to be modified. There have been many attempts to find a consistent alternative to Einstein’s gravity at high energies, which reverts to general relativity at low energy. The most attractive alternatives seem to be those in which the modification of gravity is linked with its unification with other forces.

There are two approaches to the unification of all forces—strong, weak, electromagnetic and gravitational. In one, all the forces including gravitation are treated at par, ignoring the geometric nature of gravity. This approach accommodates the ideas of supergravity and superstrings. Of these, by far the most successful idea is that of superstring theory [5–7]. The other approach to unification is the Kaluza-Klein idea that quantities like electric charge arise out of the geometry of spacetime. (For a review, see [8].) In fact, the original Kaluza-Klein theory is almost as old as the theory of relativity. While the two approaches are different in perspective, there is one common feature in that they are both formulated in a spacetime of dimensions higher than four. The extra dimensions are postulated to be small in size and – in most scenarios – compact in the topological sense. Gravitation in the large, in both cases, is described by Einstein’s theory, while at short distances we expect departures as the effect of the extra dimensions becomes apparent.

The natural extension of a relativistic point particle is a one-dimensional object, viz. a relativistic string. Such strings have acquired great importance for two distinct reasons. Firstly, one-dimensional topological defects arise generically in a wide class of GUT models [9]. Such defects are expected to be of macroscopic size (hence the name cosmic strings) and produced copiously in the early universe. Secondly, the most promising candidate for unification of all forces appears to be superstring theory. Here the strings are microscopic in size, of the order of the Planck length. If strings, rather than particles, are the fundamental entities, the resulting theory is free from many of the short-distance problems associated with quantum field theory [10].

There are several reasons why a study of classical string propagation in external gravitational fields is of interest. Firstly, such a study is natural from the point of view of classical dynamics. String theory in its simplest description is the theory of a one-dimensional body, its propagation and its interactions with other extended bodies [11]. As a string moves, it traces out a two-dimensional surface (worldsheet) in spacetime, which is the two-dimensional
analogue of a worldline. Its motion in an external gravitational field thus generalises the basic problem of classical dynamics - viz. the motion of a particle in a gravitational field - and hence forms an interesting area of study in its own right.

The other motivations for pursuing this study arise from high-energy physics. One of them has to do with cosmic strings, which, as mentioned above, arise naturally in many GUT models. Since they are expected to be produced copiously in the early universe, it is important to be able to describe their dynamics in cosmological backgrounds as well as in the vicinity of massive objects such as black holes.

Fundamental string theory provides the deepest motivation for the study of string motion in curved spacetimes. String theories possess a much richer set of symmetries than theories of relativistic point particles. Of course, in order to describe microphysics, string theory has to be quantised. Classically, string theory is defined in any number of dimensions, but quantum considerations require it to be formulated in higher dimensions - 26 for bosonic strings and 10 for superstrings. To make a connection with physical four-dimensional physics, the extra dimensions have to be compactified. The mechanism of this compactification is an important issue in string theory, and needs to be studied using a variety of approaches.

The stringy effects are expected to show up in extremely high gravitational fields as in the early universe and near black holes. Thus string quantisation in curved spacetime is expected to be central to (for instance) Planck scale cosmology. An understanding of classical string propagation in curved spacetime must form the basis of any such quantisation programme.

While the energy scales at which string theory is formulated are of the order of the Planck scale, it is not unreasonable to expect that there would be some residual low energy effects. The study of string dynamics in curved spacetime can also be undertaken with the aim of looking for such effects. Thus what appears as a study of classical gravitational dynamics becomes a tool for probing physics at the highest energy scales.

An extensive literature deals with the study of classical string dynamics in curved spacetime. Solutions have been obtained for the string equations of motion in four-dimensional curved spacetime, in particular cosmological and black hole spacetimes. String propagation near a black hole is of particularly great interest because of the interplay between the extended probe and the nontrivial background geometry. The formalism developed in the literature is applicable to both cosmic strings and fundamental strings. However, as described earlier, in the latter case the theory is formulated in higher dimensions. The extra dimensions are expected to contribute nontrivially in the strong gravity regime, e.g., in the vicinity of a black hole. It would be interesting to study how the compact extra dimensions unfold as a string falls into a black hole. The hope is that a string can be used as a probe to understand how ordinary spacetime arises dynamically from an underlying higher dimensional theory. However, the four-dimensional solutions described in the literature can throw no light on the mechanism of compactification. Strictly speaking, for this we should solve the equations of motion in $D$-dimensional ($D = 26$ for bosonic strings and $D = 10$ for superstrings) spacetime, which includes the compact manifold. As can be expected, this problem is technically quite forbidding.

A possible way out is provided by studying a five-dimensional Kaluza-Klein background instead of the full $D$-dimensional spacetime. These five-dimensional spacetimes are the minimal departure from ordinary four-dimensional spacetime. Here the more general class of black holes includes Schwarzschild and Reissner-Nordstrom solutions as well as Pollard-Gross-Perry-Sorkin monopole solutions. The extra fifth dimension is compact and winds around a circle of radius much smaller than the usual four spacetime dimensions. The question is how the extra dimension appears from the point of view of a string falling into the black hole. An attempt in this direction was made by solving the string equations of motion in Kaluza-Klein black hole backgrounds. It was shown that, even at the classical level, the extra compact dimension contributes nontrivially to string propagation.

By now the literature on string propagation in curved spacetime is very extensive and includes a collected volume of papers. In the present article, which necessarily reflects the bias of the authors, we can only review some of the main developments. A number of ansatze and expansion schemes are touched upon here but not elaborated in detail; the interested reader can refer to the papers cited. We emphasise the null string expansion scheme since it is the approximation method which is most relevant to the study of strings in strong gravitational fields. We have chosen to describe in detail the application of this formalism to the propagation of strings near Kaluza-Klein black holes.

Section II reviews string propagation in Minkowskian and curved spacetime and describes various ansatze used in solving string equations of motion. In Section III, null strings are defined and the null string expansion presented in detail. Section IV is devoted to the study of string propagation in electrically and magnetically charged Kaluza-Klein black hole backgrounds. The concluding Section V contains the summary and some remarks. In Appendix A, black hole solutions to five-dimensional Kaluza-Klein theory are reviewed.

II. STRING PROPAGATION IN MINKOWSKIAN AND CURVED SPACETIME

A point particle sweeps out a line as it propagates in spacetime and the action is proportional to the length of the worldline. A string, being an extended one-dimensional object, sweeps out a sheet, the worldsheet.
The simplest choice for the action in this case is, that it is proportional to the area of the worldsheet. This action, known as the Nambu-Goto action, is given by

$$S_{NG} = -T_0 \int dA,$$  \hspace{1cm} (1)

where $T_0$ is the string tension.

The worldline traced out by a point particle is characterised by a single variable $\tau$; a corresponding description of a worldsheet requires two variables, $\sigma$ and $\tau$. In order to express the Nambu-Goto action in terms of the spacetime coordinates $X^\mu(\tau, \sigma)$, we need to define an induced metric given by

$$g_{ab} = G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu$$  \hspace{1cm} (2)

where the indices $a, b, \ldots$ run over values $(\tau, \sigma)$. The metric $G_{\mu\nu}$ represents the background spacetime, which is taken to have arbitrary dimension $D$.

In terms of the induced metric, the Nambu-Goto action becomes

$$S_{NG} = -T_0 \int \sqrt{-det g_{ab}} d\sigma d\tau$$  \hspace{1cm} (3)

where $\dot{X}^\mu = \partial X^\mu / \partial \tau$ and $X'^\mu = \partial X^\mu / \partial \sigma$, and the dot product is taken with respect to the metric $G_{\mu\nu}$.

In two dimensions, any metric can be transformed into the so-called conformal gauge which is given by

$$g_{ab} = \exp[\phi(\sigma, \tau)] \eta_{ab}.$$  

The equations of motion from the string action take a simple form in this gauge and are given by

$$\partial^\pm X^\mu - \partial^\sigma X^\mu + \Gamma^\mu_{\nu\rho} [\partial_\sigma X^\nu \partial_\pm X^\rho - \partial_\pm X^\nu \partial_\sigma X^\rho] = 0,$$  \hspace{1cm} (4)

where $\Gamma^\mu_{\nu\rho}$ are Christoffel symbols for the background metric. Because of the high degree of symmetry the system is constrained. The constraint equations, in the conformal gauge, are

$$\partial_\tau X^\mu \partial_\sigma X^\nu G_{\mu\nu} = 0$$  \hspace{1cm} (5)

$$[\partial_\tau X^\mu \partial_\sigma X^\nu + \partial_\sigma X^\mu \partial_\tau X^\nu] G_{\mu\nu} = 0.$$  \hspace{1cm} (6)

We can impose different boundary conditions depending on the kind of strings. For a closed string, the worldsheet is a tube. If $\sigma$ runs from 0 to $\bar{\sigma} = 2\pi$, the boundary condition is periodic, i.e.,

$$X^\mu(\sigma + \bar{\sigma}) = X^\mu(\sigma).$$  \hspace{1cm} (7)

For open strings, the worldsheet is a strip and $\bar{\sigma} = \pi$. Here we have two different types of boundary conditions. One is the Neumann condition $\partial_{\bar{\sigma}} X^\mu = 0$ which implies that there is no net momentum flow off the ends of the string. The other boundary condition, called the Dirichlet condition, $\partial_{\bar{\sigma}} X^\mu = 0$ implies that the end points of the string are fixed.

A. Strings in Minkowski spacetime

In flat spacetimes Eqs. (4) become linear and one can solve them explicitly along with the constraints. If one chooses ‘light-cone’ variables, $x_\pm \equiv \sigma \pm \tau$, on the worldsheet, the equations of motion can be recast as

$$\partial_- X^\mu + \Gamma^\mu_{\nu\rho} [\partial_+ X^\nu \partial_- X^\rho] = 0.$$  \hspace{1cm} (8)

and

$$T_{\pm\pm} \equiv G_{\mu\nu}(X) \partial_\pm X^\mu(\sigma, \tau) \partial_\pm X^\nu(\sigma, \tau)$$  \hspace{1cm} (9)

The equations are still invariant under the conformal reparametrisation

$$\sigma + \tau \rightarrow \sigma' + \tau' = f(\sigma + \tau),$$  \hspace{1cm} (10)

$$\sigma - \tau \rightarrow \sigma' - \tau' = g(\sigma - \tau)$$

with $f$ and $g$ being arbitrary functions of $x$.

For Minkowski spacetime, the equations of motion are linear and take the form

$$\partial_- X^\mu(\sigma, \tau) = 0,$$  \hspace{1cm} (11)

with the quadratic constraint given by

$$[\partial_\pm X^0(\sigma, \tau)]^2 - \sum_{j=1}^{D-1} [\partial_\pm X^j(\sigma, \tau)]^2 = 0.$$  \hspace{1cm} (12)

The solutions of Eqs. (11) are written for a closed string as

$$X^\mu(\sigma, \tau) = q^\mu + 2p^\mu \alpha^\tau,$$  \hspace{1cm} (13)

where $q^\mu$ and $p^\mu$ are the centre of mass position and momentum. The left and right oscillator modes of the string is independent and are described respectively by $\alpha_+^\mu$ and $\alpha_-^\mu$. Since the string coordinates are real, $\alpha_+^\mu = \alpha_-^{\mu*}$ and $\tilde{\alpha}_+^\mu = \tilde{\alpha}_-^{\mu*}$. The picture is not so simple in curved spacetime; this is because the right and left movers interact with themselves and with each other.

The Minkowskian solutions can also be written in the form

$$X^\mu(\sigma, \tau) = l^\mu(\sigma + \tau) + r^\mu(\sigma - \tau),$$  \hspace{1cm} (14)

with $l^\mu$ and $r^\mu$ being arbitrary functions. Making an appropriate conformal transformation one can express one of the string coordinates as a constant times $\tau$.

A convenient gauge choice is the light cone gauge

$$U \equiv X^0 - X^1 = 2p^U \alpha^\tau.$$  \hspace{1cm} (15)

The constraints look like
This implies that \( X^0 + X^1 \) is not an independent quantity since it can be expressed in terms of the transverse coordinates \( X^2, ..., X^{D-1} \). This gauge choice \([13]\) implies that there are no string oscillations along the \( U \) direction. The string moves at a constant speed while oscillating around its centre of mass; the number of normal modes for this oscillation is infinite. The physical modes are the transverse modes in the \( X^2, ..., X^{D-1} \) directions.

The spacetime stress energy-momentum tensor can be obtained by varying the action with respect to the metric \( G_{\mu\nu} \) at the spacetime point \( X \). We have

\[
\sqrt{-G} T^{\mu\nu}(X) = \frac{1}{2\pi\alpha'} \int d\sigma d\tau \left( \dot{X}^\mu \dot{X}^\nu - X^\mu X^\nu \right) \times \delta^{(D)}(X - X(\sigma, \tau)).
\]

Another physically relevant quantity is the invariant string size which is defined by using the induced metric on the worldsheet

\[
ds^2 = G_{\mu\nu} dX^\mu dX^\nu. \tag{18}
\]

Substituting \( dX^\mu = \partial_+ X^\mu d\tau^+ + \partial_- X^\mu d\tau^- \) and using the constraint equation, we have

\[
ds^2 = 2G_{\mu\nu}(X) \partial_+ X^\mu \partial_- X^\nu (d\tau^2 - d\sigma^2) \tag{19}
\]

Therefore, the string size \( l \) can be defined in the following manner

\[
l = \int dl \equiv \int \sqrt{G_{\mu\nu} \dot{X}^\mu \dot{X}^\nu d\sigma}. \tag{20}
\]

One can obtain the invariant string size by substituting the general solution in \( \partial_+ X^\mu \partial_- X^\mu \). The invariant string size is always bounded in Minkowskian spacetime. The picture is entirely different if one considers strings in curved spacetime. The next subsection reviews some aspects of string propagation in curved spacetime.

### B. Strings in curved spacetime

The equations of motion of a string are nonlinear and can be brought to a tractable form under some approximations or string coordinate expansion scheme. A number of papers have investigated string propagation in curved spacetime and this study is an active area of research. The string equations of motion have been solved exactly for a restricted class of metrics, the gravitational wave backgrounds \([10,13]\), conical spacetimes \([21,22]\), black holes and cosmological spacetimes.

#### 1. de Sitter spacetime

Among the cosmological scenarios, de Sitter spacetime holds a special place as far as string propagation is concerned. On the one hand, de Sitter spacetime is relevant for inflationary solutions to problems of standard cosmology \([4]\). On the other hand there are new and interesting results in string propagation. It has been shown that the string equations of motion are exactly integrable in \( D \)-dimensional de Sitter spacetime \([23,24]\). The metric is given by

\[
ds^2 = -dt^2 + e^{2Ht} \sum_{i=1}^{D-1} dX_i^2 \tag{21}
\]

The string equations of motion in \( D \)-dimensional de Sitter spacetime correspond to noncompact \( O(D,1) \)-symmetric \( \sigma \)-model. The equations and the constraints are equivalent to a generalised sinh-Gordon equation \([23]\). They reduce to the usual sinh-Gordon equation in \( D = 3 \) dimensions.

#### 2. Gravitational shock wave background

Gravitational shock wave backgrounds are described by the Aichelburg-Sexl metric \([28]\); they represent the gravitational field of a neutral spinless ultrarelativistic particle. This metric is relevant to particle scattering at Planck energy. The Aichelburg-Sexl metric in \( D \) dimensions is

\[
ds^2 = dV dU - dX^2 + f_D(\rho) \delta(U)dU^2 \tag{22}
\]

where \( U \) and \( V \) are null coordinates,

\[
U \equiv X^0 - X^{D-1}, \quad V \equiv X^0 + X^{D-1},
\]

with \( X^i, \quad i = 1, 2, ..., D - 2 \) being the transverse spatial coordinates. Here \( \rho = \sqrt{\sum_{j=1}^{D-2} X_j^2} \) and the function \( f_D(\rho) \) obeys the equation

\[
\nabla^2 f_D(\rho) = 16\pi G \tilde{\rho} \delta^{(D-2)}(X^i) \tag{23}
\]

where \( \tilde{\rho} \) is the momentum in the \( X^{D-1} \) direction. Classical and quantum string scattering has been extensively studied in this background in Refs. \([16,19]\). In Ref. \([20]\), the authors study propagation of a charged closed string through a shock wave in Aichelburg-Sexl spacetime.

#### 3. Conical spacetime

Conical spacetime \([21,22]\) is geometry around a straight cosmic string. In this case, again, there is no need to make a perturbation expansion as the string equations can be solved exactly. The geometry describes a straight
cosmic string of zero thickness. The spacetime being locally flat but has a nontrivial topology globally. The conical spacetime is described by the metric (in $D$ dimensions)

$$ds^2 = -dX^0^2 + dR^2 + R^2 d\phi^2 + dZ^2$$

where the cylindrical coordinates are

$$R = \sqrt{X^2 + Y^2} \quad \text{and} \quad \phi = \tan^{-1} \left( \frac{Y}{X} \right),$$

with range

$$0 \leq \phi \leq 2\pi \alpha, \quad \alpha = 1 - 4G\mu,$$

with $dZ^2$ being the flat ($D - 3$)-dimensional space with $Z^i, 3 \leq i \leq D - 1$ being the Cartesian coordinates. The spatial points $(R, \phi, Z)$ and $(R, \phi + 2\pi \alpha, Z)$ are identified and the spacetime is locally flat for $R \neq 0$ with a conelike singularity at $R = 0$ with azimuthal deficit angle

$$\delta\phi = 2\pi(1 - \alpha) = 8\pi G\mu.$$

Here $G\mu$ is the dimensionless cosmic string parameter, $G$ is the gravitational constant and $\mu$ is the tension of the cosmic string. In this spacetime, bosonic and fermionic strings propagate freely, with the condition of periodicity under rotations by $2\pi \alpha$.

C. String ansatze

In most spacetimes, quite general families of exact solutions can be found by making an appropriate ansatz, which exploits the underlying symmetry of the background.

In axially symmetric backgrounds, one such valid approximation is the circular string ansatz [26] which is given as

$$t = t(\tau), \quad r = r(\tau), \quad \phi = \sigma, \quad \theta = \pi/2$$

This describes a circular string in the equatorial plane with $r(\tau)$ being the only physical mode. The ansatz is suitable if the background is axially symmetric. This ansatz decouples the equations and we have ordinary differential equations for $t$ and $r$. The circular string ansatz has been used to analyse string propagation in cosmological backgrounds, especially in the context of a de-sitter universe. In this background, several new features arise. Apart from the equations being exactly integrable, as mentioned above, the solutions show multistring behaviour [24]. These multistrings are different strings characterised by the same worldsheet. Exact circular string solutions were found (in de Sitter spacetime) which describe two different strings (in fact, it was later shown to be infinitely many different and independent strings); one of them being stable, i.e., string size becomes constant as time grows and the other showing unstable behaviour i.e., string proper size blows up (for a detailed review see [27]).

As the name suggests, the stationary string ansatz [28] is suitable to describe strings in stationary spacetimes. The string coordinates are expressed in terms of worldsheet coordinates in the following manner

$$t = x^0 = \tau, \quad x^i = x^i(\sigma)$$

The string coordinates are obtained in terms of the worldsheet parameter $\sigma$. The circular and stationary strings are related to each other by a transformation in which $\tau$ and $\sigma$ interchange along with an interchange of azimuthal angle $\phi$ with time $t$.

Two other ways to make the equations separable is to use the ring ansatz given by

$$X^0 = X^0(\sigma), \quad X^1 = f(\tau) \cos \sigma, \quad X^2 = f(\tau) \sin \sigma, \quad X^i = \text{const}, \quad i \geq 3$$

or the planetoid string ansatz [29,30] given by

$$t = t_0 + \alpha \tau, \quad \phi = \phi_0 + \beta \tau, \quad r = r(\sigma)$$

String solutions have also been studied in dyonic (electrically and magnetically charged) black hole backgrounds [31] in considerable detail.

D. Perturbation expansion of string coordinates

To solve string equations of motion in more general backgrounds, one has to resort to a perturbative analysis.

1. $\tau$-expansion

The $\tau$ expansion method [1] provides “exact local” solutions for any background. If one is interested in the string behaviour near a given point of the curved spacetime, then one chooses a conformal gauge such that $\tau = 0$ at that point. Once this gauge choice is done, the string equations of motion and the constraints are solved in powers of $\tau$.

2. Centre of mass expansion

One way which is intuitively appealing is to go about solving the equations by expanding perturbatively around the centre of mass motion of the string. The worldsheet parameter $\tau$ is the proper time of the centre of mass trajectory. In this approach the string oscillations around the centre of mass are taken as perturbations [32,33]. The appropriate expansion parameter for string coordinates is $\alpha'/R$ where $\alpha'$ is the inverse
string tension and \( R \) is the radius of curvature of the background geometry. This expansion is valid as long as the background metric \( G_{\mu\nu} \) does not change appreciably over distances of the order of the length of the string. If \( \alpha' \) approaches zero, the string collapses to a point and the expansion is no longer valid.

To study string propagation in the strong gravity regime, one has to go to the opposite limit, the limit in which the string tension vanishes \([34]\). In this regime, the length of the string becomes infinite compared to a typical fixed length. In other words, information takes an infinite time to reach from one end of the string to the other. This expansion scheme would therefore be appropriate to study string propagation in the strong gravitational fields. In this paper we will elaborate the procedure employed to solve the string equations of motion in strong gravitational field regimes.

III. NULL STRING EXPANSION

Clearly, the limit \( T_0 \to 0 \) cannot be reached using the Nambu-Goto action. We have to reformulate the lagrangian, in the manner illustrated below. The momentum \( \Pi_\mu \) conjugate to coordinate \( X_\mu \) is given by

\[
\Pi_\mu = \frac{\delta S}{\delta \dot{X}_\mu} = \frac{T_0}{\sqrt{-det g}} \left[ (\dot{X}^\alpha X^{\beta\gamma} G_{\alpha\beta}) \dot{X}^\nu G_{\mu\nu} - (\dot{X}^\alpha X^{\beta\gamma} G_{\alpha\beta}) X^\nu G_{\mu\nu} \right],
\]

where dot represents derivatives with respect to \( \tau \) and prime denotes \( \sigma \) derivatives.

The constraints are given by

\[
\begin{align*}
\psi_1 &\equiv \Pi^\mu X^\nu G_{\mu\nu} = 0, \\
\psi_2 &\equiv \Pi^\mu \Pi^\nu G_{\mu\nu} + T_0^2 X^\mu X^\nu G_{\mu\nu} = 0.
\end{align*}
\]

As in the case of a massless particle, the hamiltonian density is written in terms of constraints as

\[
H = \lambda \psi_2 + \rho \psi_1.
\]

The lagrangian density can then be re-written as

\[
L = \Pi^\mu \dot{X}^\nu G_{\mu\nu} - \lambda \psi_2 - \rho \psi_1
\]

Using \( X^\mu = H/\Pi_\mu \) the reformulated lagrangian is

\[
L = \frac{1}{4\lambda} \left[ \left( \dot{X}^\mu \dot{X}^\nu G_{\mu\nu} \right) - 4\lambda^2 T_0^2 \left( \dot{X}^\mu \dot{X}^\nu G_{\mu\nu} \right) \right]
\]

Here \( c \equiv 2\lambda T_0 \) is the ‘worldsheet velocity of light’, i.e. the velocity of wave propagation along the length of the string.

The classical equations of motion are derived from the reformulated lagrangian, and are given by

\[
\partial_\nu \dot{X}^\mu - c^2 \partial_\sigma \dot{X}^\mu + \Gamma^\mu_{\nu\rho} \left[ \partial_\sigma X^\nu \partial_\tau X^\rho - c^2 \partial_\sigma X^\nu \partial_\sigma X^\rho \right] = 0,
\]

where \( \Gamma^\mu_{\nu\rho} \) are Christoffel symbols for the background metric. The constraint equations are

\[
\partial_\nu X^\mu \partial_\rho X^\nu G_{\mu\nu} = 0 \quad \text{(35)}
\]

\[
[\partial_\nu X^\mu \partial_\rho X^\nu + c^2 \partial_\sigma X^\nu \partial_\sigma X^\rho] G_{\mu\nu} = 0. \quad \text{(36)}
\]

The limit of vanishing string tension, \( T_0 \to 0 \), corresponds to \( c \to 0 \). In this limit, the length of the string becomes infinite compared to a typical fixed length. In other words, it takes an infinite time for information to reach from one point of the string to the other. Every point on the string moves independently along a null geodesic. Such tensionless strings, appropriately called null strings, have been known for over two decades \([33]\).

Null string propagation in Schwarzschild black hole background and near Kerr black holes have been studied.

De Vega and Nicolaidis \([34]\) suggested the use of the worldsheet velocity of light as an expansion parameter. The scheme involves systematic expansion in powers of \( c \). The limit of small worldsheet velocity of light corresponds to that of small string tension. If \( c \ll 1 \), the coordinate expansion is suitable to describe strings in a strong gravitational background (see \([36]-[37]\)). This point perhaps needs a little explanation. The parameter \( c \) is an artificial one as explained in detail in Ref. \([34]\). The appropriate physical dimensionless parameter is \( R_c \sqrt{T_0} \), where \( R_c \) is a typical radius of curvature for the system \([36]\). The formalism uses a trick by introducing \( c \), which is a dimensionless quantity proportional to \( T_0 \), in which an expansion is carried out.

The “null string expansion” is an expansion around the null string configuration. The derivatives w.r.t. \( \tau \) and \( \sigma \) decouple. For \( c \ll 1 \), the zeroth order dominates, in which only \( \tau \) (the proper time of the string) derivatives are present. At the first order and higher orders, the string deviates from its null behaviour. Therefore, this scenario gives a dynamical picture. In the opposite case \( c \gg 1 \), the classical equations of motion give us a stationary picture as the \( \sigma \) derivatives dominate. The case \( c = 1 \) corresponds to the centre of mass expansion of the string. The scheme of using \( c \) as an expansion parameter, therefore, is more general than the other expansion schemes.

We restrict ourselves to the case where \( c \) is small. Using this expansion scheme, the string coordinates are expressed as

\[
X^\mu(\sigma, \tau) = X^\mu_0(\sigma, \tau) + c^2 X^\mu_1(\sigma, \tau) + c^4 X^\mu_2(\sigma, \tau) + \cdots
\]

The zeroth order \( X^\mu_0(\sigma, \tau) \) satisfies the following set of equations:

\[
\begin{align*}
\ddot{X}^\mu_0 + \Gamma^\mu_{\nu\rho} \dot{X}^\nu_0 \dot{X}^\rho_0 &= 0, \\
\dot{X}^\mu_0 \dot{X}^\nu_0 G_{\mu\nu} &= 0, \\
X^\mu_0 \dot{X}^\nu_0 G_{\mu\nu} &= 0.
\end{align*}
\]
The question one could ask is whether the expansion in powers of parameter $c$ preserves the invariance of the action under general coordinate transformations $(\sigma \tau) \rightarrow (\sigma' \tau')$. The answer could be given in analogy with linear approximation to Einstein's gravity. The post-Newtonian expansion in Einstein's theory of gravitation hinges on the split $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. This is not invariant under general coordinate transformations, yet it cannot be said to be without physical content. The case of the null-string expansion is very similar. The Lagrangian given above is fully reparametrisation invariant, although the truncation of the expansion at any finite power of $c$ is not.

The first order fluctuations can be obtained by retaining terms of order $c^2$ and the equations are given by [34]

$$\ddot{X}^\rho + 2\Gamma^\rho_{\kappa\lambda} \dot{X}^\kappa \dot{X}^\lambda + \Gamma^\rho_{\kappa\lambda} \dot{X}^\kappa \ddot{X}^\lambda = 0,$$

with the constraints being

$$2(\dot{X}_0^\mu \dot{X}_0^\nu + \dot{X}_0^\mu \dot{X}_1^\nu) G_{\mu\nu} + G_{\mu\nu,\alpha} \dot{X}_0^\mu \dot{X}_0^\nu \dot{X}_1^\alpha = 0,$$

and

$$\dot{X}_1^\mu \dot{X}_1^\nu + \dot{X}_0^\mu \ddot{X}_1^\nu = 0.$$  

As mentioned earlier, a physically interesting quantity is the invariant or proper string size $l$, which is given by [1]

$$dl^2 = X^\mu X^\nu G_{\mu\nu}(X)d\sigma^2.$$  

The differential string size has the form of an effective mass for the geodesic motion [32]. At the zeroth order, the proper string length is indeterminate. At the first order and higher orders, string length varies as a string propagates in curved spacetime.

The motion of null strings in curved backgrounds has been studied in cosmological and black hole backgrounds [34,36,37]. Applying the formalism to FRW geometry, it was shown in [37] that the string expands or contracts at the same rate as the whole universe. It was observed that the total energy of the string grows linearly while the momentum grows quadratically with cosmic time; the energy comes from the contracting geometry. De Vega, Giannakis and Sánchez [36] made a study of null string propagation in the de Sitter background. It was shown that the emergence of conformal anomaly is due to the dimensionfull string tension. Lousto and Sánchez [37] have made an extensive study of string propagation in conformally flat FRW spacetime and in black hole spacetimes. We study null string propagation in the Kaluza-Klein black hole backgrounds. Appendix A gives a brief review of black hole solutions to five dimensional Kaluza-Klein gravity.

IV. STRING PROPAGATION IN KALUZA-KLEIN BLACK HOLE BACKGROUND

The Kaluza-Klein black hole, in general, has both magnetic and electric charges along with a scalar charge (see Appendix A). Out of these three charges, two are independent as clearly shown in Eq. (15). We seek to solve the equations of motion for the string coordinates in the exterior of the black hole. For simplicity, we consider the magnetically and electrically charged cases separately. The main results obtained in this section have been reported earlier [13]; however, it contains technical details which have not been presented before.

A. Magnetically charged black hole

The zeroth order equations of motion for the string coordinates are obtained by substituting the above metric in Eqs. (13). For an electrically neutral ($Q = 0$) background the equations of motion are

$$\frac{\partial^2 t}{\partial \tau^2} + 2 \left( \frac{f'}{f} \right) \frac{\partial t}{\partial \tau} \frac{\partial r}{\partial \tau} = 0,$$

$$\frac{\partial^2 r}{\partial \tau^2} + \frac{f^3}{2AB^2(B'f - 2f'B)} \left( \frac{\partial t}{\partial \tau} \right)^2 + \frac{(A'f - 2f'A)}{2Af} \left( \frac{\partial r}{\partial \tau} \right)^2 - \frac{f^2}{2A} \left( \frac{\partial \phi}{\partial \tau} \right)^2 = 0,$$

$$\frac{\partial^2 \phi}{\partial \tau^2} + \frac{A'}{A} \left( \frac{\partial r}{\partial \tau} \right) \left( \frac{\partial \phi}{\partial \tau} \right) = 0,$$

$$\frac{\partial^2 x_5}{\partial \tau^2} + \left( - \frac{A'}{A} + \frac{B'}{B} \right) \left( \frac{\partial r}{\partial \tau} \right) \left( \frac{\partial x_5}{\partial \tau} \right) = 0,$$

and the constraint equation is

$$\frac{f^2}{B} \left( \frac{\partial t}{\partial \tau} \right)^2 - A \left( \frac{\partial r}{\partial \tau} \right)^2 - A \left( \frac{\partial \phi}{\partial \tau} \right)^2 - B \left( \frac{\partial x_5}{\partial \tau} \right)^2 = 0.$$  

Here we have taken the string to be propagating in the equatorial plane, i.e. $\theta = \pi/2$. Hence the equation of motion for the coordinate $\theta$ vanishes.

The functions $A$, $B$ and $f$, for the magnetically charged black hole case, are given by

$$A = (r + \Sigma_1)(r - 3\Sigma_1),$$

$$B = (r + \Sigma_1)^2,$$

$$f^2 = (r + \Sigma_1)(r - 2M - \Sigma_1).$$

where $\Sigma_1 = \Sigma/\sqrt{3}$.

The first integrals of motion are
\[ \frac{\partial t}{\partial \tau} = \frac{c_1 B}{f^2}, \quad \frac{\partial \phi}{\partial \tau} = \frac{c_2}{A^2}, \quad \frac{\partial x_5}{\partial \tau} = \frac{c_3}{B} A \] (46)
\[
\left( \frac{\partial r}{\partial \tau} \right)^2 = \frac{B}{A} c_1^2 - \frac{f^2}{A^2} c_2^2 - \frac{f^2}{B} c_3^2.
\]

where \( c_1, c_2 \) and \( c_3 \) are functions of \( \sigma \). Since, at the zeroth order, only derivatives with respect to \( \tau \) are present, we can treat \( c_1, c_2 \) and \( c_3 \) as constants. The constants \( c_1 \) and \( c_2 \) correspond respectively to the energy \( E(\sigma) \) and the angular momentum \( L(\sigma) \). The first three equations are obtained by direct integration of the \( t, \phi \) and \( x_5 \) equations. The constraint equation is then used to obtain the equations for \( \partial r/\partial \tau \).

Since \( A, B \) and \( f^2 \) are all functions of \( r \), it is convenient to change all the derivatives with respect to \( \tau \) to those with respect to \( r \),
\[
\frac{\partial t}{\partial \tau} = \frac{dt}{dr} \frac{\partial r}{\partial \tau}, \quad \frac{\partial x_5}{\partial \tau} = \frac{dx_5}{dr} \frac{\partial r}{\partial \tau}, \quad \frac{\partial \phi}{\partial \tau} = \frac{d\phi}{dr} \frac{\partial r}{\partial \tau}. \] (47)

Here we have assumed that the trajectory can be written in the parametric form,
\[
t = t(r) \quad x_5 = x_5(r), \quad \phi = \phi(r) \] (48)

From Eqs. (46) we have,
\[
\frac{\partial r}{\partial \tau} = \pm \sqrt{\frac{B}{A} c_1^2 - \frac{f^2}{A^2} c_2^2 - \frac{f^2}{B} c_3^2}.
\]

We choose the negative sign as we consider an in-falling string.

This change of variables enables us to reduce the equations to quadratures:
\[
\tau = - \int \frac{dr}{\sqrt{\frac{B}{A} c_1^2 - \frac{f^2}{A^2} c_2^2 - \frac{f^2}{B} c_3^2}}, \] (49)
\[
x_5 = - \int \frac{c_3 A dr}{B \sqrt{\frac{B}{A} c_1^2 - \frac{f^2}{A^2} c_2^2 - \frac{f^2}{B} c_3^2}},
\]
\[
t = - \int \frac{c_1 B dr}{f^2 \sqrt{\frac{B}{A} c_1^2 - \frac{f^2}{A^2} c_2^2 - \frac{f^2}{B} c_3^2}}.
\]

up to constants of integration which depend on \( \sigma \). Here we have taken \( c_2 = 0 \), i.e. the string is falling in 'head-on'. The quadratures can be solved numerically to obtain \( t, r, \) and \( x_5 \) as functions of \( \tau \). It is clear from Eq. (46) that, for \( P^2 \) to be positive, we have
\[
\Sigma < 0 \quad \text{or} \quad \Sigma > \sqrt{3} M. \] (50)

The integrals (49) have been evaluated numerically and inverted to obtain the coordinates as functions of \( \tau \). We confine ourselves to the region \( r > M \) and we assume \( r \gg \Sigma \), i.e., from the integrals we drop terms of \( O(\Sigma^2/r^2) \).

**B. Electrically charged black hole**

For the electrically charged \((P=0)\) black hole, the equations of motion in the zeroth order take the form
\[
AB[Af^2 + 12Q^2(r - \Sigma_1)^2] \frac{\partial^2 r}{\partial \tau^2} = \]
\[
- [12ABQ^2(r - \Sigma_1)^2 + B'A^2 f^2 - 4B'AQ^2(r - \Sigma_1)^2 - 2f'A^2 B f - 8ABQ^2(r - \Sigma_1)] \frac{\partial t}{\partial \tau} \frac{\partial r}{\partial \tau},
\]
\[
+ 4QAB[B'(r - \Sigma_1) - B] \frac{\partial r}{\partial \tau} \frac{\partial x_5}{\partial \tau} = 0,
\]
\[
2A^3 B^2 f \frac{\partial^2 r}{\partial \tau^2} + f^3[4A'B^2 Q^2(r - \Sigma_1)^2 - B'A^2 f^2 + 4B' AQ^2(r - \Sigma_1)^2 + 2f'A^2 B f - 8ABQ^2(r - \Sigma_1)] \left( \frac{\partial t}{\partial \tau} \right)^2
\]
\[
- 8ABQ^2(r - \Sigma_1) \left( \frac{\partial t}{\partial \tau} \right)^2
\]
\[
- 8f^3 B^2 Q[A'(r - \Sigma_1) - A] \frac{\partial t}{\partial \tau} \frac{\partial x_5}{\partial \tau} + A^2 B^2 [A' f - 2f'A] \left( \frac{\partial r}{\partial \tau} \right)^2
\]
\[
- A^2 B^2 f^3 A' \left( \frac{\partial \phi}{\partial \tau} \right)^2 + f^3 B^2 [A'B - B'A] \left( \frac{\partial x_5}{\partial \tau} \right)^2 = 0,
\]
\[
AB^2 [Af^2 + 12Q^2(r - \Sigma_1)^2] \frac{\partial^2 x_5}{\partial \tau^2} =
\]
\[
- 4QA[A'B^2 f^2(r - \Sigma_1) - B'A^2 f(r - \Sigma_1) + 4B'Q^2(r - \Sigma_1)^3 + 2f'A^2 B f - 8ABQ^2(r - \Sigma_1)] \frac{\partial t}{\partial \tau} \frac{\partial r}{\partial \tau}
\]
\[
+ B[A'B^2 f^2 + 12A'BQ^2(r - \Sigma_1)^2 - B'A^2 f^2 + 4B' AQ^2(r - \Sigma_1)^2 - 16ABQ^2(r - \Sigma_1)] \frac{\partial r}{\partial \tau} \frac{\partial x_5}{\partial \tau} = 0,
\]

and the constraint equation is
\[
\left\{ f^2 \frac{\partial^2}{\partial \tau^2} - \frac{4Q^2}{AB} (r - \Sigma_1)^2 \right\} \left( \frac{\partial t}{\partial \tau} \right)^2
\]
\[
- \frac{8Q}{A} (r - \Sigma_1) \frac{\partial t}{\partial \tau} \frac{\partial x_5}{\partial \tau} - \frac{A}{f^2} \left( \frac{\partial \phi}{\partial \tau} \right)^2
\]
\[
- A \left( \frac{\partial \phi}{\partial \tau} \right)^2 - \frac{B}{A} \left( \frac{\partial x_5}{\partial \tau} \right)^2 = 0,
\]

where \( \Sigma_1 = \frac{m^2}{\gamma} \).

The structure of the equations of motion, in this case, is such that they are not reducible to quadratures and we have to solve the differential equations numerically.
Again, we consider an in-falling string in the region where $r >> \Sigma$ and $\theta = \pi/2$.

The leading-order analysis of Eqs. (51) and Eq. (52), which is required for numerical solution, is rather involved. The detailed calculations are reported in [15]. The set of Eqs. (51) and Eq. (52) have to be solved numerically to obtain the coordinates as functions of $\tau$. Again we have a two-parameter family of solutions.

C. Kaluza-Klein radius

As mentioned earlier, the interest lies in seeing the effect of the extra dimension on string propagation. The behaviour of the extra dimension is different in the two cases [13]. However, the picture is easier to interpret if we study the Kaluza-Klein radius, which is related to its asymptotic value $R_0$ as

$$R(r) = R_0 \left( \frac{B}{A} \right)^{1/2}. \quad (53)$$

The radius $R(r)$ has an implicit dependence on $\tau$ through $R(\tau) = R(r(\tau))$, and is hence a dynamical quantity.

The effect of the magnetic field is to shrink the extra dimension (as already indicated in [38]), i.e., as the string approaches the black hole, the value of the Kaluza-Klein radius which it sees becomes smaller than its asymptotic value. The presence of electric charge tends to expand the extra dimension. The opposite behaviour in the two cases was illustrated in Ref. [13] and it was shown that even at the classical level, there is a nontrivial contribution of the extra dimension on string propagation.

Fig. 1 and Fig. 2 clearly show that the behaviour of the Kaluza-Klein radius is opposite in the electrically and magnetically charged cases. It is clear from the above that, even at the classical level, the extra dimensions make a nontrivial contribution to string propagation. In other words, it is possible for the string probe to observe the unfolding/shrinking of the extra dimension.

FIG. 1. Kaluza-Klein radius as viewed by a string falling into a magnetic black hole.

FIG. 2. Kaluza-Klein radius as viewed by a string falling into an electrically charged black hole.
D. Analytical Results for Strings near Magnetic Black Holes

The quadratures (49) can be solved to obtain analytical counterparts of the solutions presented in the previous Section for the magnetically charged black hole. The integrals can be reduced to combinations of elliptical integrals, depending on the relative values of the constants $\Sigma_1$ and $M$ (see [13]). As an illustrative example we consider the case when $c_1^2 = c_3^2$. We can then write the integrals as

$$\tau = \frac{1}{c_1} \int dr \sqrt{(r - 3 \Sigma_1)(r + \Sigma_1)} \quad (54)$$

$$x_5 = \int dr \frac{(r - 3 \Sigma_1)^{3/2}}{\sqrt{(r + \Sigma_1)(r - \alpha)}} \quad (55)$$

$$t = \int dr \frac{(r + \Sigma_1)^{3/2}}{(r - 2M - \Sigma_1)\sqrt{(r - \alpha)}}$$

where $\alpha = \frac{\Sigma_1(S_1 + 3M)}{(3\Sigma_1 + 3M)}$.

In general, these quadratures can be reduced to combinations of elliptical integrals depending on the relative values of the constants (see for example [15]). However, the solutions reduce to elementary functions in the region where $r$ is very large compared to the scalar charge. We take for instance the case when $c_1 = c_3 = 1$. Up to the first order in $\Sigma_1/r$, the solutions are

$$\tau = -\frac{2}{3\sqrt{2}(M + \Sigma_1)} \left\{ \frac{r + \frac{M\Sigma_1}{M + 3\Sigma_1}}{3/2} \right\} \quad (56)$$

$$x_5 = -\frac{\sqrt{2}r}{\sqrt{2}(M + \Sigma_1)} \left[ \left( \frac{r}{3} + \frac{3\Sigma_1}{3M} \right) (r - \Sigma_1)^{1/2} \right]$$

$$t = -\frac{2}{\sqrt{2}(M + 3\Sigma_1)^2} \left( \frac{r - \alpha}{3/2} + 2M + \Sigma_1 \right)$$

$$+ \frac{2(2M + \Sigma_1)^2}{\sqrt{2}(M + 3\Sigma_1)^2} \frac{r - \alpha}{\sqrt{\Sigma_1 - \alpha}}$$

where $\Sigma_1 = \Sigma/\sqrt{3}$ and $\alpha = \frac{3MS_1}{3\Sigma_1 + M}$. These solutions are valid in the region outside the horizon but not asymptotically far from the black hole. The negative sign comes because we consider an in-falling string. These solutions match with the numerical solutions presented in the last section, for the corresponding values of $c_1$ and $c_3$ [4].

E. String Equations in Extremal Black Hole backgrounds

In the last sub-section, we discussed analytical solutions of the equations of motion for a string propagating in a magnetically charged black hole background. The solutions reduce to elementary functions in a suitable large distance approximation, i.e., the scalar charge is very small compared to the distance $r$. In fact, the black hole backgrounds in this case can be thought of as being small deviations from the Schwarzschild black holes. The numerical results of the previous Section were also obtained in this limit.

However, the integrals of motion (49) can be solved analytically without resorting to the limit $r \gg \Sigma$, if $P = 2M$ and $Q = 0$, i.e., for an extremal magnetically charged black hole. The constraint Eq. (A5) implies that $\Sigma_1 = -M$ or $\Sigma_1 = 2M$. The former case is that of the much-studied Pollard-Gross-Perry-Sorkin monopole [40 42]. In that case, the metric reduces to the form reported in Ref. [10]. The solutions are

$$\tau = t = \frac{1}{\sqrt{c_1^2 - c_3^2}} (r - \beta M)^{1/2} (r + 3M)^{1/2} \quad (57)$$

$$+ \frac{(3 + \beta)M}{\sqrt{c_1^2 - c_3^2}} \ln \left[ (r - \beta M)^{1/2} + (r + 3M)^{1/2} \right]$$

$$x_5 = \frac{1}{\sqrt{c_1^2 - c_3^2}} (r - \beta M)^{1/2} (r + 3M)^{1/2}$$

$$+ \frac{16M}{\sqrt{\beta - 1} \sqrt{c_1^2 - c_3^2}} \left[ \frac{2r - \beta M}{\sqrt{\beta - 1} \sqrt{r + 3M}} \right]$$

where $\beta = \frac{c_1^2 + 3c_3^2}{c_1^2 - c_3^2}$. We choose $c_1 = 1$; the condition of reality of the solutions then forces $c_3 < 1$ and consequently $\beta > 1$. Here the time coordinate $t$ is the same as the proper time $\tau$ of the string, as in this case $f^2 / B = 1$.

In the case of PGPS extremal black hole, the string has a decelerated fall into the black hole. This is not surprising, as the ‘repulsive’ or ‘anti-gravity’ effect of extremal black holes has been commented on in the literature (see, for example, [43] and [44]). The effect of the gauge field is opposite to that of gravity.

In addition to the above case, there is another extremal black hole solution (which has not been mentioned hitherto in the literature) corresponding to $\Sigma_1 = 2M$. The integrals can be solved in terms of elliptical functions, the solutions being

$$\tau = \frac{1}{3} \sqrt{\frac{2}{7M}} \sqrt{(r - 6M)(r + 2M) \left( r - \frac{10M}{7} \right)} \quad (58)$$

$$x_5 = \frac{1}{3} \sqrt{\frac{2}{7M}} \left\{ \left[ E\left(g(r), \frac{3}{7}\right) - 4F\left(g(r), \frac{3}{7}\right) \right] \right\}$$

$$- \frac{32M}{21\sqrt{7}} \left[ \left\{ 22 E\left(g(r), \frac{3}{7}\right) - 4F\left(g(r), \frac{3}{7}\right) \right\} \right]$$

$$- \frac{32M}{21\sqrt{7}} \left[ \left\{ 22 E\left(g(r), \frac{3}{7}\right) - 4F\left(g(r), \frac{3}{7}\right) \right\} \right]$$
Using the constraint Eq. (40), the first order correction to the invariant string size which can be calculated once the first order equations are solved. We will not go into details of this calculations as they have been reported in [12].

V. SUMMARY

This article reviews some aspects of classical string propagation in curved spacetime. This is an important field of research, with the long-term goal to understand string quantisation in curved spacetime. Although these procedures have their limitations, it may still be reasonable to say that this is right now the best available framework to study the physics of gravitation in the context of string theory.

We have described the general features of string propagation in Minkowskian and curved spacetime. A number of exact solutions and approximation schemes have been mentioned without any attempt at being comprehensive. We have focused on the null string expansion because it has the potential to be applied to string motion in strong gravitational fields. We believe that this is likely to be more useful in the near future than a wide variety of exact solutions which cannot be generalised. The zeroth order solutions in this scheme can be obtained using the methods and intuition of point particle dynamics with stringy corrections in the first order. For reasons discussed above, it is unlikely that one will need to go beyond the first order.

We have studied, in detail, the propagation of a null string in five-dimensional, electrically and magnetically charged, Kaluza-Klein black hole backgrounds. Here, we have tried to explore the behaviour of the extra fifth dimension as the string approaches the black hole horizon. It is shown that, even at the classical level, the string probe is affected by the extra dimension.

Here we have considered only the classical picture. In principle, however, one expects quantum effects to be dominant in the strong gravity regime. Nevertheless, one can hope that the classical picture will give an intuitive idea of the mechanism of compactification. The work has a natural extension in making a similar study in Kaluza-Klein cosmological backgrounds.

Another interesting investigation would be to study string propagation in the scenario suggested by Randall and Sundrum [43,44]. These authors proposed a five-dimensional scenario, in which the background metric is a slice of Anti De Sitter spacetime. The four-dimensional metric is multiplied by a rapidly changing function of the extra dimension. A simple solution to the gauge hierarchy problem is provided as exponentially big ratios of energy scales can be generated because of the exponential compactification factor. The work reported in this paper can be extended to null p-branes. An extensive study in this regard has been reported in Refs. [52,54]. Another related work is reported in [57], where the canonical analysis of strings propagating in arbitrary backgrounds is presented. Recent literature reveals interesting aspects of string propagation in gravitational wave backgrounds. String propagation in these backgrounds would reveal interesting features and merits further study.
APPENDIX A: KALUZA-KLEIN BLACK HOLES

Stationary Kaluza-Klein solutions with spherical symmetry were studied systematically by Chodos and Detweiler and Dobaish and Maison. These black holes are characterised by the mass, the electric charge and the scalar charge. It was shown by Gross and Perry and in independent works by Pollard and by Sorkin that five-dimensional magnetic monopoles (electrically neutral) exist as solutions to five-dimensional Kaluza-Klein theories. The solutions in Ref. [47] were generalised by Gibbons and Wiltshire to those with four parameters. It was shown that, in general, Kaluza-Klein black holes possess both electric and magnetic charge, with the solutions mentioned above as special cases (see also [49]).

We consider the metric background as given in [38]

\[ ds^2 = -e^{2k \varphi / \sqrt{A}} (dx_5 + 2k A_\alpha dx^\alpha)^2 \]

where \( k^2 = 4 \pi G; x_5 \) is the extra dimension and should be identified modulo \( 2 \pi R_0 \), where \( R_0 \) is the radius of the circle about which the coordinate \( x_5 \) winds.

The demand that the black hole solutions be regular in four dimensions (changing to units where \( G = 1 \) [50]) implies

\[ e^{A \varphi / \sqrt{A}} = \frac{B}{A}, \]

\[ A_\alpha dx^\alpha = \frac{Q}{B} (r - \sqrt{3} \Sigma) dt + P \cos \theta d\phi, \]

and

\[ g_{\alpha \beta} dx^\alpha dx^\beta = \frac{f^2}{\sqrt{AB}} dt^2 - \frac{\sqrt{AB}}{f^2} dr^2 \]

\[ - \sqrt{AB} \left( dt^2 + \sin^2 \theta d\phi^2 \right), \]

where \( A, B \) and \( f^2 \) depend on \( r \) and are given by

\[ A = r - \frac{\Sigma}{\sqrt{3}}, \quad B = r + \frac{\Sigma}{\sqrt{3}}, \]

\[ f^2 = (r - M)^2 - (M^2 + \Sigma^2 - P^2 - Q^2). \]

If \( P = Q = 0 \), we regain the usual Schwarzschild black holes.

The black hole solutions are characterised by the mass \( M \) of the black hole, the electric charge \( Q \), the magnetic charge \( P \) and the scalar charge \( \Sigma \). Out of the charges, only two are independent [38,50]. The constant parameters are constrained by the relation

\[ \frac{2}{3} \Sigma = \frac{Q^2}{\Sigma + \sqrt{3} M} + \frac{P^2}{\Sigma - \sqrt{3} M}, \]

where the scalar charge is defined by

\[ k \varphi \to \frac{\varphi}{\sqrt{A}} + O \left( \frac{1}{\sqrt{A}} \right) \text{ as } r \to \infty. \]

Eq. \( \text{A5} \) is invariant under the duality transformation

\[ Q \to P, \quad P \to Q, \quad \Sigma \to -\Sigma. \]

which relates "electric-like" and "magnetic-like" black holes. It is worth noting that physically distinct black holes are related by the duality.

The black hole solutions listed by Gibbons and Wiltshire include the much studied Pollard-Gross-Perry-Sorkin (PGPS) extremal magnetic black hole. Recently, it was shown that the PGPS monopole arises as a solution of a suitable dimensionally reduced string theory [51]. This provides an additional motivation to study such black hole backgrounds in the context of string theory. One way is by finding out stringy corrections to the five-dimensional black hole backgrounds [50]. A complementary approach is to study string propagation in Kaluza-Klein black hole backgrounds, as described here.

[1] K. R. Dienes, Phys. Rep. 287, 447 (1997).
[2] L. O’Raifeartaigh and N. Straumann, Rev. of Mod. Phys. 72, 1 (2000).
[3] S. Weinberg, Gravitation and Cosmology (John Wiley & Sons, Inc., 1972).
[4] E. W. Kolb and M. Turner, The Early Universe (Addison-Wesley Publishing Company, 1990).
[5] M. B. Green, J. H. Schwarz and E. Witten, Superstring Theory (Cambridge Univ. Press, 1987).
[6] J. Polchinski, Superstring Theory (Cambridge Univ. Press, 1998).
[7] E. Kiritsis, hep-th/9709062.
[8] J. M. Overduin and P. S. Wesson, Phys. Rep. 283 (1997) 303.
[9] A. Vilenkin and E. P. S. Shellard, Cosmic strings and other topological defects (Cambridge University Press, 1994).
[10] Strings in curved spacetime ed. H. J. De Vega and N. Sánchez (World Scientific, 1996).
[11] H. J. De Vega and N. Sánchez, Lectures on String Theory in Curved Spacetimes, in: Proc. Third Paris Cosmology Colloquium (Paris, June 1995), ed. H. J. De Vega and N. Sánchez, (World Scientific, 1996).
[12] For a different approach, leading to possible observable consequences at particle accelerators, L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 3370 (1999) and L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 4690 (1999).
[13] H. K. Jassal, A. Mukherjee, and R. P. Saxena, Phys. Lett. B 462, 55 (1999).
[14] H. K. Jassal and A. Mukherjee, Phys. Rev. D 62, 067501 (2000).
[15] H. K. Jassal and A. Mukherjee, hep-th/0008166.
[16] D. Amati and C. Klimčík, Phys. Lett. B 210, 92 (1988).
[17] M. E. V. Costa and H. J. De Vega, Nucl. Phys. B 211, 223 (1991).
[18] H. J. De Vega and N. Sanchez, Phys. Lett. B 244, 215 (1990).
[19] M. E. V. Costa and H. J. De Vega, Nucl. Phys. B 211, 235 (1991).
[20] K. Maeda, T. Torii, M. Narita, S. Yahikozawa, Nucl. Phys. B 598, 115 (2001).
[21] H. J. De Vega and N. Sanchez, Phys. Rev. D 42, 2754 (1990).
[22] H. J. De Vega and N. Sanchez, Nucl. Phys. B 374, 405 (1992).
[23] H. J. De Vega and N. Sanchez, Phys. Rev. D 47, 3394 (1993).
[24] F. Combes, H. J. De Vega, V. Mikhailov and N. Sanchez, Phys. Rev. D 50, 2754 (1994).
[25] P. C. Aichelburg and R. U. Sexl, Gen. Rel. Grav. 2, 303 (1971).
[26] A. L. Larsen and N. Sanchez, Phys. Rev. D 51, 6929 (1995).
[27] H. J. De Vega, A. L. Larsen and N. Sanchez, String theory in curved spacetime (World Scientific, 1996).
[28] P. Frolov, V. D. Skarzinsky, A. I. Zelnikov and O. Heinrich, Phys. Lett. B 224, 225 (1989).
[29] H. J. De Vega and I. L. Egusquiza, Phys. Rev. D 54, 7513 (1996).
[30] S. Kar and S. Mahapatra, Class. Quantum Grav. 15, 421 (1998).
[31] S. Mahapatra, Phys. Rev. D 55, 6403 (1997).
[32] H. J. De Vega and N. Sanchez, Phys. Lett. B 197, 320 (1987).
[33] H. J. De Vega and N. Sanchez, Phys. Lett. B 309, 552, 557 (1988).
[34] H. J. De Vega and A. Nicolaidis, Phys. Lett. B 295, 241 (1992).
[35] A. Schild, Phys. Rev. D 16, 1722 (1977).
[36] H. J. De Vega, I. Giannakis and A. Nicolaidis, Mod. Phys. Lett. A 10, 2479 (1995).
[37] C. O. Lousto and N. Sanchez, Phys. Rev. D 54, 6399 (1996).
[38] G. W. Gibbons and D. L. Wiltshire, Ann. Phys. 167, 201 (1986); 176, 393 (E) (1987).
[39] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series and Products (Academic Press, 1994).
[40] D. J. Gross and M. J. Perry, Nucl. Phys. B 226, 29 (1983).
[41] D. Pollard, J. Phys. A 16, 565 (1983).
[42] R. D. Sorkin, Phys. Rev. Lett. 51, 87 (1983).
[43] G. ’t Hooft, Int. J. Mod. Phys. A 11, 4623 (1996).
[44] G. W. Gibbons and P. K. Townsend, Phys. Lett. B 356, 472 (1995).
[45] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 3370 (1999).
[46] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 4690 (1999).
[47] A. Chodos and S. Detweiler, Gen. Rel. and Gravitation, 14, 879 (1982).
[48] P. Dobaisch and D. Maison, Gen. Rel. and Gravitation, 14, 879 (1982).
[49] M. Cvetič and D. Youm, Phys. Rev. Lett. 75, 4165 (1995).
[50] N. Itzhaki, Nucl. Phys. B 508, 700 (1997).
[51] S. Roy, Phys. Rev. D 60, 082003 (1999).
[52] P. Bozhilov, hep-th/0103154.
[53] P. Bozhilov, hep-th/0011032.
[54] P. Bozhilov, Phys. Rev. D 62, 105001 (2000).
[55] P. Bozhilov and B. Dimitrov, Phys. Lett., B 472, 54 (2000).
[56] P. Bozhilov, Phys. Rev. D 60, 125011 (1999).
[57] M. Montesinos and J. D. Vergara, hep-th/0105026.