A TROPICAL CALCULATION OF THE WELSCHINGER INVARIANTS OF REAL TORIC DEL PEZZO SURFACES

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Abstract. The Welschinger invariants of real rational algebraic surfaces are natural analogues of the genus zero Gromov-Witten invariants. We establish a tropical formula to calculate the Welschinger invariants of real toric Del Pezzo surfaces for any conjugation-invariant configuration of points. The formula expresses the Welschinger invariants via the total multiplicity of certain tropical curves (non-Archimedean amoebas) passing through generic configurations of points, and then via the total multiplicity of some lattice paths in the convex lattice polygon associated with a given surface. We also present the results of computation of Welschinger invariants, obtained jointly with I. Itenberg and V. Kharlamov.

Introduction

One of the important questions of real enumerative geometry is: for a real algebraic surface Σ, how many real rational curves in an ample linear system |D| pass through a conjugation-invariant configuration of $-K_\Sigma D - 1$ distinct generic points in Σ? Similarly to the complex case, where the answer is given by Gromov-Witten invariants, the recently discovered Welschinger invariants [16, 17] appear to be an ultimate tool to handle the question over the reals. In particular, they led to non-trivial positive lower bounds for the numbers in question, provided that the configuration consisted of only real points [3].

So far no closed or recursive formula is found for the Welschinger invariants, and tropical enumerative geometry [8, 9, 13] provides the only known approach to compute them. In the present paper we express the Welschinger invariants for configurations of real and imaginary conjugate points on real toric Del Pezzo surfaces via the number of certain subdivisions of the corresponding convex lattice polygons.

Welschinger invariants. Let Σ be $\mathbb{P}^2$, or the hyperboloid $(\mathbb{P}^1)^2$, or the plane $\mathbb{P}^2_k$ blown up at k generic real points, equipped with the standard real structure. Let D be a real ample divisor on Σ, and let the non-negative integers $r', r''$ satisfy

$$r' + 2r'' = -K_\Sigma D - 1.$$  

Denote by $\Omega_{r''}(\Sigma, D)$ the set of configurations of $-K_\Sigma D - 1$ distinct points of Σ such that $r'$ of them are real and the rest form $r''$ pairs of imaginary conjugate points. The Welschinger number $W_{r''}(\Sigma, D)$ is the sum of weights of all the real rational curves in |D|, passing through a generic configuration $p \in \Omega_{r''}(\Sigma, D)$, where the weight of a real rational curve C is 1 if it has an even number of real solitary nodes, and is $-1$ otherwise. The surfaces Σ as above are the only toric surfaces, whose complex structure determines a symplectic structure which is generic in its deformation class, and thus, by Welschinger’s theorem [16, 17], $W_{r''}(\Sigma, D)$ does not depend on the choice of a generic element $p \in \Omega_{r''}(\Sigma, D)$ (a simple proof of the latter independence in the algebraic setting is found in [4]). The importance of the Welschinger invariant comes from an immediate inequality

$$|W_{r''}(\Sigma, D)| \leq R_{\Sigma, D}(p) \leq N_{\Sigma, D},$$

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1The generality means here that all the complex rational curves through the configuration are nodal irreducible, and their number is equal to the corresponding Gromov-Witten invariant.
where \( R_{\Sigma,D}(\mathcal{P}) \) is the number of real rational curves in \(|D|\) passing through a generic configuration \( \mathcal{P} \in \Omega_{r''}(\Sigma, D) \), and \( N_{\Sigma,D} \) is the number of complex rational curves in \(|D|\), passing through generic \(-K_{\Sigma}D - 1\) points in \( \Sigma \). We should notice that even the existence of a positive lower bound for \( R_{\Sigma,D}(\mathcal{P}) \) was not reached by any other method, beginning with the case of plane quartics.

**A tropical calculation of the Welschinger invariant.** Our approach to calculating the Welschinger invariant is quite similar to that in \([3]\), and it heavily relies on the enumerative tropical algebraic geometry developed in \([8, 9, 13]\). More precisely, we replace the complex field \( \mathbb{C} \) by the field \( \mathbb{K} = \bigcup_{m \geq 1} \mathbb{C}\{t^{1/m}\} \) of the complex locally convergent Puiseux series equipped with the standard complex conjugation and with a non-Archimedean valuation

\[
\text{Val}: \mathbb{K}^* \to \mathbb{R}, \quad \text{Val} \left( \sum_k a_k t^k \right) = -\min \{ k : a_k \neq 0 \}.
\]

A rational curve over \( \mathbb{K}_R \) passing through a generic configuration \( \mathcal{P} \in \Omega_{r''}(\Sigma_R, D) \) is represented as an equisingular family of real rational curves in \( \Sigma \) over the punctured disc, and its limit at the disc center determines a tropical curve in \( \mathbb{R}^2 \) (called a real rational tropical curve), which passes through the configuration \( \text{Val}(\mathcal{P}) \subset \mathbb{R}^2 \), coordinate-wise Val-projection of \( \mathcal{P} \).

Our first main result is Theorem 4.2 (section 4.2), which reduces the count of real rational tropical curves, passing through generic configurations of points in \( \mathbb{Q}^2 \), and, for any such real rational tropical curve \( A \), determines the contribution to the Welschinger invariant of all the real algebraic curves, projecting to \( A \). The proof is based on the techniques of \([13]\).

In the case of configurations, consisting of only real points, this result has been obtained in \([9, 13]\). We notice that an extension of the tropical formula to the case of configurations of real and imaginary conjugate points, requires some further development of the techniques of tropical enumerative geometry, which we present in this paper. The principal difficulty in the latter case is that the configuration \( \text{Val}(\mathcal{P}) \) contains fewer points than \( \mathcal{P} \), since a pair of conjugate points projects to the same point in \( \mathbb{R}^2 \).

The second main result is Theorem 1.2 (section 1.2), which reduces the count of real rational tropical curves, passing through a generic configuration in \( \mathbb{Q}^2 \), to the count of the total weight of certain lattice paths in the convex lattice polygon corresponding to the divisor \( D \). The weight of a lattice path is the sum of Welschinger numbers of certain subdivisions of the given polygon into convex lattice subpolygons, which can be produced from the lattice path in a finite combinatorial algorithm (section 1.2). Here we follow the Mikhalkin’s idea to place the configuration \( \text{Val}(\mathcal{P}) \subset \mathbb{R}^2 \) on a straight line.

**Applications.** The tropical formula turns the computation of Welschinger invariants into a purely combinatorial problem on geometry of lattice paths and lattice subdivisions of convex lattice polygons. We present here some results, obtained in this way jointly with I. Itenberg and V. Kharlamov (see \([5]\) for a detailed presentation and proofs). These results concern the positivity of Welschinger invariants, their monotone behavior with respect to the number of imaginary points in configurations, and the asymptotics with respect to the growing degree of the divisor \( D \). We formulate a few natural conjectures.

A. The positivity and monotonicity of the Welschinger invariant. The positivity of \( W_0(\Sigma, D) \) for all real toric Del Pezzo surfaces and ample divisors \( D \) was shown in \([3]\). For \( r'' \neq 0 \), the invariants \( W_{r''}(\Sigma, D) \) can vanish (see, for example, computation of \( W_{r''}(\mathbb{P}^2, 3L) \), \( L \) being a line in the plane, in section 1.3).

**Theorem 0.1.** (\([3]\)) (1) The invariants \( W_{r''}(\mathbb{P}^2, D) \) are positive for all ample divisors \( D \) and all \( r'' \geq 0 \), satisfying (\([3]\)).

(2) The following inequalities hold:

(i) \( W_0(\Sigma, D) > W_1(\Sigma, D) > W_2(\Sigma, D) > W_3(\Sigma, D), \) for \( \Sigma = \mathbb{P}^2, (\mathbb{P}^1)^2, \mathbb{P}_1^1, \) or \( \mathbb{P}_2^1 \), with \( D \) an ample divisor such that \( D(K_{\Sigma} + K_{\Sigma}) \geq 0 \);

(ii) \( W_3(\mathbb{P}^2, D) > 0, \) and \( W_2(\Sigma, D) > 0, \) for \( \Sigma = \mathbb{P}_k^1, k = 1,2,3, \) and an ample divisor \( D \).

**Conjecture 0.2.** (\([3]\)) For a real unnodal Del Pezzo surface \( \Sigma \) and any ample divisor \( D \) on \( \Sigma \), the Welschinger invariants \( W_{r''}(\Sigma, D) \) are positive as \( 0 \leq r'' < [(-K_{\Sigma}D - 1)/2] \), and are non-negative for \( r'' = [(-K_{\Sigma}D - 1)/2] \). Furthermore, they satisfy the monotonicity relation

\[
W_{r''}(\Sigma, D) \geq W_{r''+1}(\Sigma, D), \quad 0 \leq r'' < \left[ \frac{-K_{\Sigma}D - 1}{2} \right].
\]
We notice that the monotonicity and non-negativity of Welschinger invariants are closely related, since, by [10], Theorem 2.2, the first difference of the function \( r'' \mapsto W_{r''}(\Sigma, D) \) is twice the Welschinger invariant for the surface \( \Sigma \) blown up at one real point.

**B. The asymptotics of the Welschinger invariants.**

**Theorem 0.3.** (3) The Welschinger invariants of the plane satisfy the relation

\[
\lim_{n \to \infty} \frac{\log W_{r''}(\mathbb{P}^2, nL)}{n \log n} = \lim_{n \to \infty} \frac{\log N_{\mathbb{P}^2.nL}}{n \log n} = 3, \quad 0 \leq r'' \leq 3.
\]

For \( \Sigma = (\mathbb{P}^1)^2, \) or \( \mathbb{P}^2_k, 1 \leq k \leq 3, \) it holds that

\[
\lim_{d \to \infty} \frac{\log W_{r''}(\Sigma, nD)}{n \log n} = \lim_{n \to \infty} \frac{\log N_{\Sigma.nD}}{n \log n} = -K_{\Sigma}D, \quad 0 \leq r'' \leq 2.
\]

This means that the number of real rational curves passing through any generic conjugation-invariant configuration of points in \( \Sigma, \) where \( r'' \) is bounded as in the assertion, is asymptotically equal to the number of all complex rational curves in the logarithmic scale.

We propose a natural extension of Theorem 0.3 to all Del Pezzo surfaces and other values of \( r'' \):

**Conjecture 0.4.** (3) Let \( D \) be an ample divisor on a real unnodal Del Pezzo surface \( \Sigma. \) Then, for any fixed \( r'' \geq 0, \)

\[
\lim_{n \to \infty} \frac{\log W_{r''}(\Sigma, nD)}{n \log n} = \lim_{n \to \infty} \frac{\log N_{\Sigma.nD}}{n \log n} = -K_{\Sigma}D.
\]

The following statement implies Conjecture 0.4 for the hyperboloid:

**Theorem 0.5.** (3) Let \( D \) be an ample divisor on \( (\mathbb{P}^1)^2 \) of bi-degree \( (a_1,a_2), \) \( 0 < a_1 \leq a_2, \) and let a sequence of positive integers \( r''(n) < (a_1 + a_2)n \) satisfy \( \lim_{n \to \infty} r(n)/n = r''_0. \) Then

\[
\lim_{n \to \infty} \inf \frac{\log W_{r''(n)}((\mathbb{P}^1)^2, nD)}{n \log n} \geq -K_{(\mathbb{P}^1)^2}D - r''_0 = 2a_1 + 2a_2 - r''_0.
\]

**Organization of the material.** The paper is structured as follows: in section 1 we introduce tropical curves, in section 2 we describe tropical limits of rational curves defined over the field \( \mathbb{R} \), in section 3 we prove the tropical formula for Welschinger invariants, in section 4 we obtain an explicit combinatorial description for Welschinger invariants via lattice paths and subdivisions of convex lattice polygons.

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1. **Non-Archimedean amoebas and tropical curves**

1.1. **Basic definitions and notation.** In sections 1 and 3 we assume that \( \Delta \) is a non-degenerate lattice polygon in \( \mathbb{R}^2, \Sigma = \text{Tor}(\Delta) \) is a toric surface associated with \( \Delta, \) and \( |D| \) is the tautological linear system, generated by the monomials \( x^iy^j, (i,j) \in \Delta \cap \mathbb{Z}^2. \)

In particular, the surfaces \( \mathbb{P}^2, (\mathbb{P}^1)^2, \mathbb{P}^2_k, k = 1,2,3, \) are naturally associated with the polygons shown in Figure 1:

- the triangle \( \text{Conv}\{(0,0),(0,d),(d,0)\}, \) if \( \Sigma = \mathbb{P}^2, \) deg \( D = d, \)
- the rectangle \( \text{Conv}\{(0,0),(d_2,0),(d_2,d_1),(0,d_1)\}, \) if \( \Sigma = (\mathbb{P}^1)^2, \) deg \( D = (d_1,d_2), \)
- the trapeze \( \text{Conv}\{(0,0),(d-d_1,0),(d-d_1,d_1),(0,d)\}, \) if \( \Sigma = \mathbb{P}^2_1, D \sim dL-d_1E_1, \)
- the pentagon \( \text{Conv}\{(0,0),(d-d_1,0),(d-d_1,d_1),(d_2,d-d_2),(0,d-d_2)\}, \) if \( \Sigma = \mathbb{P}^2_2, D \sim dL-d_1E_1-d_2E_2, \)
- the hexagon \( \text{Conv}\{(d_3,0),(d-d_1,0),(d-d_1,d_1),(d_2,d-d_2),(0,d-d_2),(0,d_3)\}, \) if \( \Sigma = \mathbb{P}^2_3, D \sim dL-d_1E_1-d_2E_2-d_3E_3. \)
Figure 1. Polygons associated with \( \mathbb{P}^2, (\mathbb{P}^1)^2, \mathbb{P}^2_1, \mathbb{P}^2_2, \) and \( \mathbb{P}^2_3 \)

Observe that

\begin{equation}
(1.1) \quad r' + 2r'' = -K_C D - 1 = |\partial \Delta \cap \mathbb{Z}^2| - 1.
\end{equation}

The amoeba \( A_C \) of a curve \( C \in |D|_K \) is defined as the closure of the set \( \text{Val}(C \cap (\mathbb{K}^*)^2) \subset \mathbb{R}^2 \). By Kapranov’s theorem (see [2, 6, 7]) the amoeba \( A_C \) of a curve \( C \in |D|_K \), given by the equation

\begin{equation}
(1.2) \quad f(x, y) := \sum_{(i,j) \in \Delta} a_{ij} x^i y^j = 0
\end{equation}

with \( \Delta \) as the Newton polygon of \( f \), is the corner locus of the convex piece-wise linear function

\begin{equation}
(1.3) \quad N_f(x, y) = \max_{(i,j) \in \Delta \cap \mathbb{Z}^2} (x_i + y_j + \text{Val}(a_{ij})), \quad x, y \in \mathbb{R}.
\end{equation}

In particular, \( A_C \) is a planar graph with all vertices of valency \( \geq 3 \), consisting of closed segments and rays.

An amoeba \( A \) with Newton polygon \( \Delta \) is called reducible, if \( A \) is the union of two amoebas \( A', A'' \neq A \) with Newton polygons \( \Delta', \Delta'' \) such that \( \Delta = \Delta' + \Delta'' \) (Minkowski sum).

Take the convex hull \( \tilde{\Delta} \) of the set \( \{(i,j, -\text{Val}(a_{ij})) \in \mathbb{R}^3 : (i,j) \in \Delta \cap \mathbb{Z}^2\} \) and define the function

\begin{equation}
(1.4) \quad \nu_f : \Delta \to \mathbb{R}, \quad \nu_f(x, y) = \min \{\gamma : (x, y, \gamma) \in \tilde{\Delta}\}.
\end{equation}

This is a convex piece-wise linear function, whose linearity domains form a subdivision \( S_C \) of \( \Delta \) into convex lattice polygons \( \Delta_1, ..., \Delta_N \). The function \( \nu_f \) is Legendre dual to \( N_f \) (see, for instance, [2]), and thus, the subdivision \( S_C \) is combinatorially dual to the pair \( (\mathbb{R}^2, A_C) \).

We define the tropical curve, corresponding to the algebraic curve \( C \), as a balanced graph, supported at \( A_C \) (cf. [2, 11]), i.e., this is the non-Archimedean amoeba \( A_C \), whose edges are assigned weights equal to the lattice lengths \( \sum p \) of the dual edges of \( S_C \). The subdivision \( S_C \) can be uniquely restored from the tropical curve \( A_C \) (we denote a tropical curve and the supporting amoeba by the same symbol, no confusion will arise).

1.2. **Rank of a tropical curve.** Let \( A \) be a tropical curve with Newton polygon \( \Delta \), given by a tropical polynomial \( (1.3) \). On the right-hand side of formula \( (1.3) \) we remove unnecessary linear functions, take the terms \( \text{Val}(a_{ij}) \) in the remaining linear functions as variables, and factorizing by common shift, obtain the space \( \mathbb{R}^{V(S)-1} \), where \( V(S) \) is the set of the vertices of the dual subdivision \( S \). The set \( \text{Iso}(A_C) \) of tropical curves with Newton polygon \( \Delta \), which are combinatorially isotopic to the given curve (or, equivalently, are dual to the same subdivision \( S \) of \( \Delta \)), is defined in \( \mathbb{R}^{V(S)-1} \) by certain linear relations and inequalities, and thus forms a convex polyhedron. Its dimension is called the rank of the tropical curve \( A \). Clearly, the rank of a tropical curve \( \text{rk}(A_C) \) is bounded from below by the virtual rank

\[ \text{rk}_{vir}(A) = \left| \pi_0(\mathbb{R}^2 \backslash A) \right| - 1 - \sum_{p \in V(A)} (V_p - 3), \]

Clearly, \( S_C \) does not depend on the choice of a polynomial \( f \) defining the curve \( C \).

\[ 3 \]We define the lattice length \( |\sigma| \) of a segment \( \sigma \) with integral endpoints as \( |\sigma \cap \mathbb{Z}^2| - 1. \]
where $V(A)$ is the set of vertices of $A$, and $V_p$ is the valency of the vertex $p$. In terms of $S$,
\[
\text{rk}_{eir}(A) = \text{rk}_{eir}(S) = |V(S)| - 1 - \sum_{k=1}^{N} (|V(\Delta_k)| - 3),
\]
where $V(S)$, $V(\Delta_k)$ are the sets of vertices of $S$ and $\Delta_k$, respectively.

**Definition 1.1.** Let $x_1, ..., x_m \in \mathbb{Q}^2$ be distinct points, $A$ a tropical curve with Newton polygon $\Delta$. The set of tropical curves $A' \in \text{Iso}(A)$ such that $x_1, ..., x_k$ are vertices of $A'$, and $x_{k+1}, ..., x_m \in A'$ imposes a number of linear conditions on the variables $\text{Val}(a_{ij})$, introduced above. We say, that the pair of configurations $(\{x_1, ..., x_k\}, \{x_{k+1}, ..., x_m\})$ is in $A$-generic position if the aforementioned set of curves $A'$ either is empty, or has codimension $\geq 2k + (m - k) = m + k$ in $\text{Iso}(A)$. We say that a configuration $x_1, ..., x_m$ is $A$-generic, if it is generic for any division into a pair of configurations and any tropical curve with Newton polygon $\Delta$.

**Lemma 1.2.** The set of $\Delta$-generic configurations of $m$ points in $\mathbb{Q}^2$ is dense in the set of all $m$-tuples in $\mathbb{Q}^2$. In particular, if a tropical curve $A$ with Newton polygon $\Delta$ passes through a $\Delta$-generic configuration $x_1, ..., x_k \in \mathbb{Q}^2$ so that $x_1, ..., x_k$ are vertices of $A$, then
\[
m + k \leq \text{rk}(A).
\]

**Proof.** The requirements, imposed by the pair of configurations $(\{x_1, ..., x_k\}, \{x_{k+1}, ..., x_m\})$ on $A' \in \text{Iso}(A)$, can be written as linear conditions on the variables $\text{Val}(a_{ij})$ in $\mathbb{R}^{|V(S)| - 1}$, two for a vertex $x_i$ of $A'$, and one for $x_i \in A'$. The failure of generality in this case means just a linear relation to the coordinates of $x_1, ..., x_m$. Since there are only finitely many combinatorial types of tropical curves with Newton polygon $\Delta$, we obtain that the set of $\Delta$-generic $m$-tuples is the complement of finitely many hyperplanes in $(\mathbb{Q}^2)^m$.

A maximal straight line interval, contained in $A_C$, is called an extended edge of $A_C$. The edges of $A_C$, forming an extended edge, are dual to a sequence of parallel edges of $S_C$ ordered so that each two successive edges of this sequence belong to one polygon $A_i$. We say that an extended edge of $A_C$ is dual to each of the edges of $S_C$ in the corresponding sequence.

Having a tropical curve $A_C$ with Newton polygon $\Delta$ and a $\Delta$-generic configuration of point on it, we call the set of vertices of $A_C$, coinciding with points of the configuration, and the set of extended edges of $A_C$, containing points of the configuration in their interior, a basic set of extended edges and vertices of $A_C$.

## 2. TROPICALIZATION OF REAL RATIONAL ALGEBRAIC CURVES

We work over the field $K$ and its real subfield $K_R = \bigcup_{m \geq 1} \mathbb{R}\{t^{1/m}\}$.

### 2.1. Tropical limit.

Fix some $\Delta$-generic collections of points $x' = \{x'_1, ..., x'_{r'}\}$ and $x'' = \{x''_1, ..., x''_{r''}\}$ in $\mathbb{Q}^2$. Let $p'_1, ..., p'_{r'} \in (K_R)^3$ be generic points, satisfying $\text{Val}(p'_i) = x'_i, i = 1, ..., r'$, and let $p''_j, i = 1, ..., r''$, $j = 1, 2$, be generic points in $(K^*)^2 \backslash (K_R)^2$, satisfying $\text{Conj}(p''_1) = p''_2$, $\text{Val}(p''_j) = x''_j, i = 1, ..., r'', j = 1, 2$. That is
\[
x'_i = (-\alpha'_i, -\beta'_i), i = 1, ..., r', \quad x''_j = (-\alpha''_j, -\beta''_j), i = 1, ..., r'',
\]
and
\[
p'_i = (t^{\xi'_i}(\xi'_i + o(1)), t^{\eta'_i}(\eta'_i + o(1))), \quad \xi'_i, \eta'_i \in \mathbb{R}^*, \quad i = 1, ..., r',
\]
\[
p''_j = (t^{\xi''_j}(\xi''_j + o(1)), t^{\eta''_j}(\eta''_j + o(1))), \quad \xi''_j, \eta''_j \in \mathbb{C} \backslash \mathbb{R}, \quad i = 1, ..., r''.
\]

Let a rational curve $C \in |D|_K$, given by a polynomial
\[
f(x, y) := \sum_{(i, j) \in \Delta} a_{ij}(t)x^iy^j = 0,
\]
be defined over $K_R$, and pass through $p'_i, i = 1, ..., r'$, and $p''_j, k = 1, ..., r'', j = 1, 2$.

Changing the parameter $t \mapsto t^m$, we make all the exponents of $t$ in $a_{ij}(t), (i, j) \in \Delta$, integral and the function $\nu_f$ integral-valued at integral points. Introduce the polyhedron
\[
\tilde{\Delta} = \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : (\alpha, \beta) \in \Delta, \gamma \geq \nu_f(\alpha, \beta)\}.
\]
It defines a toric variety $Y = \text{Tor}(\Delta)$, which is naturally fibred over $\mathbb{C}$ so that the fibres $Y_t$, $t \neq 0$, are isomorphic to $\text{Tor}(\Delta)$, and $Y_0$ is the union of toric surfaces $\text{Tor}(\Delta_i)$, $i = 1, \ldots, N$, with $\Delta_1, \ldots, \Delta_N$ being the faces of the graph of $\nu_f$. By the choice of $\nu_f$, $\text{Tor}(\Delta) \simeq \text{Tor}(\Delta_i)$, and we shall write that $Y_0 = \bigcup_i \text{Tor}(\Delta_i)$. Then (2.2) defines an analytic surface $\{ f(x, y) = 0 \}$ in a neighborhood of $Y_0$, which, by (1.3), Lemma 2.3, fibers into equsingular rational curves $C(\ell) \subset Y_\ell \simeq \text{Tor}(\Delta)$, and whose closure intersects $Y_0$ along the curve $C(0)$ that can be identified with $\bigcup_i C_i \subset \bigcup_i \text{Tor}(\Delta_i)$. Passing if necessary to a finite cyclic covering ramified along $Y_0$, we can make $\text{Tor}(\Delta)$ non-singular everywhere but may be at finitely many points, corresponding to the vertices of $\Delta$, and, in addition, make the surfaces $\text{Tor}(\Delta) \setminus \text{Sing}(\text{Tor}(\Delta))$, $k = 1, \ldots, N$, smooth and intersecting transversally in $\text{Tor}(\Delta) \setminus \text{Sing}(\text{Tor}(\Delta))$. As in [13], we define the tropical limit (tropicalization) of the curve $C$ to be the pair, consisting of the tropical curve $A_C$, and a collection of real curves $C_k = C(0) \cap \text{Tor}(\Delta_k)$, $k = 1, \ldots, N$, which are defined by

$$f_k(x, y) := \sum_{(i, j) \in \Delta_k} a_{ij}^0 x^i y^j = 0, \quad k = 1, \ldots, N,$$

respectively, where $a_{ij}(t) = (a_{ij}^0 + O(t))t^{\nu_{ij}(i, j)}$, $(i, j) \in \Delta$.

### 2.2. Tropical limits of real rational curves.

A tropical curve $A_C$ is called *nodal* if the polygons of the dual subdivision $S_C$ are triangles and parallelograms. Recall that, by [13], Lemma 2.2, $\text{rk}(A_C) = \text{rk}_{\text{vir}}(A_C)$ if $A_C$ is nodal. Notice also that the weights of the edges of a nodal tropical curve $A_C$ are constant along its extended edges, and thus, one can speak of weights of extended edges of nodal tropical curves.

An irreducible nodal tropical curve $A_C$ is called real rational of type $(r', r'', s'')$ with $r', r''$, satisfying (0.1), and $0 \leq s'' \leq r''$, if

- $\text{rk}(A_C) = \text{rk}_{\text{vir}}(A_C) = |\partial \Delta \cap \mathbb{Z}^2| - 1 - r'' + s''$,
- the weights of the semi-infinite edges of $A_C$ are 1 or 2, i.e., the edges of $S_C$ lying on $\partial \Delta$ are of length 1 or 2.

**Proposition 2.1.** Under the assumptions of section 2, $A_C$ is a real rational tropical curve of type $(r', r'', s'')$ for some $0 \leq s'' \leq r''$, passing through $x_1', \ldots, x_{r'}'$ and $x_{r'+1}'$, $\ldots, x_{r''}'$, in such a way that precisely $s'$ of the points $x_1', \ldots, x_{r'}'$ are trivalent vertices of $A_C$, whereas $x_{r'+1}'$, $\ldots, x_{r''}'$, and the remaining $r'' - s''$ points among $x_1''$, $\ldots, x_{r''}''$ (which we denote $w_1, \ldots, w_{r''-s''}$) are not vertices of $A_C$, and, moreover, $w_1, \ldots, w_{r''-s''}$ lie on edges of $A_C$ of even weight. Furthermore, for $k = 1, \ldots, N$,

- if $\Delta_k$ is a parallelogram, then $f_k$ is the product of a monomial and few irreducible binomials,
- if $\Delta_k$ is a triangle, then $C_k$ is either a real rational curve crossing $\text{Tor}(\partial \Delta_k) := \bigcup_{\sigma \subset \partial \Delta_k} \text{Tor}(\sigma)$ at 3, 4, or 5 points, at which it is unibranch, or $C_k$ is the union of two imaginary conjugate rational curves such that any component of $C_k$ crosses $\text{Tor}(\partial \Delta_k)$ at precisely three points and is unibranch there.

**Proof.** Our argumentation is similar to that in [13], section 3.3, used to establish that the non-Archimedean amoebas of toric surfaces passing through a respective number of generic points, are nodal.

*Step 1.* Let $x_1', \ldots, x_{r'}'$, and $x_1'', \ldots, x_{r''}'$, be vertices of $A_C$ for some $0 \leq s' \leq r'$, $0 \leq s'' \leq r''$, and $x_1'$, $i > s'$, $x_{r'}'$, $j > s''$, not be vertices. By (1.3),

$$r' + r'' + s' + s'' \leq \text{rk}(A_C).$$

By (1.3), Lemma 2.2, we have

$$\text{rk}(A_C) = \text{rk}_{\text{vir}}(A_C) + d(A_C) = |V(S_C)| - 1 - \sum_{k=1}^N (|V(\Delta_k)| - 3) + d(A_C),$$

where

$$2d(A_C) \leq \begin{cases} 0, & \text{if } A_C \text{ is nodal,} \\ \sum_{m \geq 2}(2m - 3)N_{2m} - N^2_{2m} + \sum_{m \geq 2}(2m - 2)N_{2m+1} - 1, & \text{otherwise}, \end{cases}$$
$N_l$ being the number of $l$-gons in $S_C$, and $N_{2m}'$ being the number of $2m$-gons in $S_C$ whose opposite edges are parallel. Substituting the two latter relations into (2.5) and using the Euler formula for the subdivision $S_C$, one obtains

$$2(r' + r'' + s' + s'') \leq |V(S_C) \cap \partial \Delta| + \begin{cases} N_3, & \text{if } A_C \text{ is nodal}, \\ N - \sum_{m \geq 2} N_{2m}' - 1, & \text{otherwise}. \end{cases}$$

Denote by $\tilde{\chi}$ the Euler characteristic of the normalization. Let $C_{ij}, 1 \leq j \leq n_i$, be all the components of $C_i, 1 \leq i \leq N$, repeating each component $C_{ij}$ with its multiplicity in $C_i$, and let $s_{ij}$ be the number of local branches of $C_{ij}$ centered along $\text{Tor}(\partial \Delta_i)$. Since the rational curves $C^{(t)}, t \neq 0$, degenerate into $C^{(0)}$ in a flat family, all the components $C_{ij}$ of $C^{(0)}$ are rational in view of the inequality for geometric genera $g(C^{(t)}) \geq \sum s_{ij} g(C_{ij})$ (see [13], Proposition 2.4, or [10][4]). Denote by $C_0$ the set of components $C_{ij}$, defined by irreducible binomials, and by $C_{nb}$ the set of remaining components $C_{ij}$. Let $U$ be the union of regular neighborhoods in the three-fold $Y$ of the intersection points of $C^{(0)}$ with $\bigcup \text{Tor}(\partial \Delta_k)$. Then

$$2 = \tilde{\chi}(C^{(t)}) \leq \sum_{k=1}^{N} \sum_{C_{ij} \in C_{0}} (2 - s_{kj}) + \tilde{\chi}(C^{(0)} \cap U)$$

$$\leq \sum_{k=1}^{N} \sum_{C_{ij} \in C_{nb}} (2 - s_{kj}) + s(\partial \Delta) + s(U),$$

the latter inequality following from [13], Lemma 3.2 and Remark 3.4, where $s(\partial \Delta)$ stands for the number of local branches (counting multiplicities) of the curves $C_1, ..., C_N$ centered at $\text{Tor}(\sigma)$ with $\sigma$ running over all the edges of $\Delta$.

**Step 2.** For an estimation of the right-hand side of (2.7) we shall construct some graph $G$.

First, we compactify $\mathbb{R}^2$ into $S^2$ by adding an infinite point $v_\infty$. The tropical curve $A_C$ is then compactified by closing the semi-infinite edges with the point $v_\infty$.

For any $k = 1, ..., N$, the reduced curve $C_k^{\text{red}}$ is split into irreducible real components and pairs of imaginary conjugate irreducible components. All such real components or pairs of imaginary conjugate components for all $k = 1, ..., N$, except for real irreducible components from the set $C_0$, and the point $v_\infty$ are taken as the vertices of the graph $\tilde{G}$. If $\Delta_k \cap \Delta_l = \sigma$ is a common edge, and components $C_k'$ or $C_k''$ and $C_l'$ or $C_l''$ of $C_l$ contain a common pair of imaginary conjugate points in $\text{Tor}(\sigma)$, then we join $C_k'$ and $C_l'$ by an edge. If $\sigma \subset \partial \Delta$ is an edge of $\Delta_k$, and a component $C_k'$ of $C_k$ contains a pair of imaginary conjugate points in $\text{Tor}(\sigma)$, then we join $C_k'$ and $v_\infty$ by an edge.

The constructed graph $\tilde{G}$ is then transformed as follows. Let $C_k'$ be a part of $C_k$, consisting of two imaginary conjugate components from the set $C_0$. The curve $C_k'$ intersects with $\text{Tor}(\sigma_1)$ and $\text{Tor}(\sigma_2)$, where $\sigma_1, \sigma_2$ is a pair of parallel edges of $\Delta_k$. If $\Delta_k \cap \Delta_p = \sigma_1, \Delta_k \cap \Delta_q = \sigma_2$, then we remove the vertex $C_k'$ and all ending at it edges, and instead join by an edge any two curves $C_p'$ and $C_q'$ which have been joined with $C_k'$. If $\Delta_p \cap \Delta_q = \sigma_1$ and $\Delta_2 \subset \partial \Delta$, then we get rid of the vertex $C_k'$ of the graph and all adjacent edges, instead connecting $v_\infty$ with any curve $C_p'$, which has been joined with $C_k'$. In this manner, we get rid of one-by-one all the vertices of $\tilde{G}$, corresponding to the curves $C_k'$ which consist of two imaginary conjugate components from the set $C_0$.

We observe that the graph $\tilde{G}$ naturally projects to $A_C \cup \{v_\infty\}$, when sending any vertex $C_k'$ of $\tilde{G}$ to the vertex $v$ of $A_C$ dual to $\Delta_k$, and sending edges of $\tilde{G}$ to respective segments of the compactified extended edges of $A_C \cup \{v_\infty\}$.

The required graph $G$ will be a subgraph of $\tilde{G}$.

First, we define a subgraph $G_0$ of $\tilde{G}$ as follows. Each point $x_j''$, $j > s''$, lies on an edge of $A_C$. Let $\sigma$ be the dual edge of $S_C$. By [13], formula (3.7.17), $f_k''(\xi_j'', \eta_j'') = f_k''(\xi_j', \eta_j') = 0$, where $f_k''$ is the truncation of the polynomial $f_k(x, y)$, defined by (2.4), on the edge $\sigma$, and $\xi_j'', \eta_j''$ are taken from formula (2.2). That is, $\text{Tor}(\sigma)$ contains a pair of distinct imaginary conjugate points, $z$ and $\text{Conj}(z)$, which belong to curves $C_k$. 

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4The hypotheses of Proposition 2.4 [1] require that all members of the family are reduced curves, which can be achieved in our situation by the normalization of the whole family $C$. 7
such that $\Delta_k \supset \sigma$. Hence the edge of $A_C$, containing $x''_j$, is covered by the projection of some edges of $\tilde{G}$.

We choose one such covering edge of $\tilde{G}$, denote it by $\varepsilon_j$, and build the graph $G_0$ from the edges $\varepsilon_j$, $j > s''$, with their endpoints as vertices of $G_0$.

Notice that due to the $\Delta$-generic position of the points $x''_j$, $j > s''$, the graph $G_0 \setminus \{v_\infty\}$ is the union of trees, the valency of its vertices that differ from $v_\infty$ is at most two, and the intersection of the projections of any two of its edges is finite.

A vertex $v \neq v_\infty$ of $G_0$ of valency two corresponds to some curve $C_k$, $1 \leq k \leq N$. If $C_k$ is the union of two imaginary conjugate rational curves such that each of them crosses $\text{Tor}(\partial \Delta_k)$ at precisely three points, and is unibranch at these intersection points, then we call the vertex $v$ of $G_0$ an extendable vertex. Notice that the components of $C_k$ are defined by polynomials with the same Newton triangle, and are determined by $x''_j$, $j > s''$, up to a finite choice in view of [13], Lemma 3.5. Furthermore, due to the generality of the coefficients $\xi''_i, \eta''_i$ in (2.2), the intersection $C_k \cap \text{Tor}(\sigma)$ is either empty or a pair of imaginary conjugate points for any $\sigma \subset \partial \Delta_k$. Exactly one of these pairs lies on an edge of $\Delta_k$, whose dual $A_C$-edge is not covered by $G_0$. However, the latter $A_C$-edge is covered by some edges of $\tilde{G}$ with an endpoint $C_k$. We choose one such edge of $\tilde{G}$ and append it to $G_0$.

Performing this procedure for all extendable vertices of $G_0$, we obtain a graph $G_1$. Next we determine the extendable vertices of $G_1$, using the same definition, and append, if necessary, new edges to $G_1$, obtaining $G_2$. Repeating this procedure, we finally end up with some subgraph $G$ of $\tilde{G}$.

**Step 3.** An important observation is that no new edge in $G_1 \setminus G_0$ can join two extendable vertices of $G_0$. Indeed, the position of two non-adjacent extendable vertices of $G_0$ is uniquely determined by the position of the corresponding four points among $x''_1, \ldots, x''_s$, and thus, due to the generality of the latter points, the straight line through the given extendable vertices is not orthogonal to any of the segments joining integral points in $\Delta$. By a similar reason, no edge from $G_{i+1}\setminus G_i$ can join two extendable vertices of $G_i$, $i \geq 1$. In particular, this means that the number of edges in $G \setminus G_0$ is equal to the total number of extendable vertices in $G_0, G_1, \ldots$.

Furthermore, assume that $C_k'$ is a $q$-valent vertex of $G$, $q \geq 3$, which is not extendable for any $G_i$, $i \geq 0$. Then $C_k'$ cannot be the union of two imaginary conjugate rational curves, such that each of them crosses $\text{Tor}(\partial \Delta_k)$ at precisely $q$ points and is unibranch there. Indeed, if the situation were as described, the position of any $q - 1$ of the intersection points of a component of $C_k'$ with $\text{Tor}(\Delta_k)$ would determine this component and the fourth intersection point uniquely up to a finite choice. Thus, the relation on the coordinates of the intersection points of $C_k'$ with $\text{Tor}(\partial \Delta_k)$ would imply a relation on the coordinates of the points $p''_j', j > s''$, contrary to their general choice. In particular, this implies that the contribution of such a vertex to the sum on the right-hand side of (2.1) is $\leq 2 - 2q$, and the equality corresponds to the case of $C_k'$ being the union of two imaginary conjugate components such that a component crosses $\text{Tor}(\partial \Delta_k)$ at $q + 1$ points and is unibranch there.

We also observe that the points $x''_1, \ldots, x''_s$ and $x''_1, \ldots, x''_s$, which are vertices of $A_C$ by assumption, are not projections of the vertices of $G$. Indeed, the points $x''_1, \ldots, x''_s$ and $x''_1, \ldots, x''_s$, cannot be projections of the vertices of $G_0$ in view of a general position of these points with respect to $x''_j, j > s''$, which in turn determine the projections of the edges of $G_0$. Next, the projections of the edges of $G_1$, and thus, the projections of the extendable vertices of $G_1$ are determined by $G_0$ up to a finite choice, where the number of choices is bounded from above by the total number of possible lattice edges in $\Delta$. By induction we get that the projections of the edges and vertices of $G$ are determined by $x''_j, j > s''$, up to a finite choice, and hence the general position of the points $x''_1, \ldots, x''_s$ and $x''_1, \ldots, x''_s$ does not allow them to be the projections of the vertices of $G$.

**Step 4.** Denote by $N_q(G)$, $q \geq 1$, the number of $q$-valent vertices of $G$ different from $v_\infty$. Denote by $N_3(G)$ the number of trivalent vertices of $G$ which are extendable for some $G_i$, $i \geq 0$. At last, denote by $V_{\infty}$ the valency of the vertex $v_\infty$ of $G$. Then the contribution of a vertex of $G$ to the right-hand side of (2.1) is

- $\leq -q - 1$, if it is a vertex of valency $q = 1, 2$,
- $\leq 2 - 2q$, if it is a vertex of valency $q \geq 3$, which is not extendable for any $G_i$, $i \geq 0$,
- $-2$, if it is a trivalent vertex, which is extendable for some $G_j$, $j \geq 0$,
and thus, the total contribution of the vertices of $G$ is
\[
\leq - \sum_{q \geq 1} (q + 1)N_q(G) - \sum_{q \geq 4} (2q - 2)N_q(G) + 2N_{q}^r(G).
\]
Since the total number of edges of $G$ is $r'' - s'' + N_{q}^r(G)$, we can rewrite the latter inequality as
\[
\leq - \sum_{q \geq 1} N_q(G) - \sum_{q \geq 4} (2q - 2)N_q(G) - 2r'' + 3s'' + V_{\infty}.
\]

Denote by $\tilde{N}$ the number of polygons in $S_C$, which are dual to the vertices of $A_C$ not covered by the vertices of $G$, and which are not even-gons, whose pairs of opposite sides are parallel. Clearly, a curve $C_k$ corresponding to such a polygon $\Delta_k$, contains a component from the set $C_{nb}$, and thus, contributes $\leq -1$ to the sum on the right-hand side of (2.7). That is, inequality (2.7) implies
\[
2 \leq - \sum_{q \geq 1} N_q(G) - \sum_{q \geq 4} (2q - 2)N_q(G) - 2r'' + 3s'' + V_{\infty} - \tilde{N} + s(\partial \Delta)
\]
\[
(2.8) \quad \Rightarrow \quad 2 \leq -N - \sum_{q \geq 4} (2q - 2)N_q(G) + \sum_{m \geq 2} N_{g}^m - 2r'' + 2s'' + V_{\infty} + s(\partial \Delta),
\]
with an equality only if
- the vertices of $G$ injectively project to the set of vertices of $A_C$, which are dual to polygons different from even-gons, whose all pairs of opposite sides are parallel;
- if $\Delta_k$ is a odd-gon, whose dual vertex of $A_C$ is not covered by the vertices of $G$, then $C_k$ contains precisely one (counting multiplicities) component from $C_{nb}$, and, moreover, this component crosses Tor($\partial \Delta_k$) at precisely three points and is unibranch there.

Combining (2.6) and (2.8), we derive
\[
2r' + 4r'' + 2s' + \sum_{q \geq 4} (2q - 2)N_q(G) \leq -2 + |V(S_C) \cap \partial \Delta| + s(\partial \Delta) + V_{\infty} + \begin{cases} 0, & \text{if } A_C \text{ is nodal}, \\ -1, & \text{otherwise} \end{cases}
\]
\[
\leq \sum_{q \geq 4} (2q - 2)N_q(G) 2s' + |\partial \Delta \cap \mathbb{Z}^2 \setminus V(S_C)| + (|\partial \Delta \cap \mathbb{Z}^2| - s(\partial \Delta))
\]
\[
(2.9) \quad \leq V_{\infty} + \begin{cases} 0, & \text{if } A_C \text{ is nodal}, \\ -1, & \text{otherwise}. \end{cases}
\]

The immediate relation
\[
(2.10) \quad V_{\infty} \leq |\partial \Delta \cap \mathbb{Z}^2 \setminus V(S_C)|
\]

excludes the case of a non-nodal tropical curve $A_C$, and, in the case of a nodal tropical $A_C$, implies the equality in (2.9) and in (2.10) with the restriction $q \leq 3$ on the valency of vertices of $G$.

We then collect all the equality conditions mentioned above into the following restrictions:

(R1) relations (2.5) and (2.7) are equalities with $s' = 0$ and $s(\partial \Delta) = |\partial \Delta \cap \mathbb{Z}^2|$;
(R2) the edges of $S_{C}$ lying in $\partial \Delta$ have length 1 or 2;
(R3) the semi-infinite extended edges of $A$, which are dual to the edges of $S$ of length 1 lying on $\partial \Delta$, belong to the graph $G$;
(R4) any curve $C_k$, $1 \leq k \leq N$, crosses the divisors Tor($\sigma$), $\sigma \subset \partial \Delta_k \cap \partial \Delta$ only at its non-singular points, and all these intersections are transversal;
(R5) all the components of the curves $C_k$ with Newton parallelograms $\Delta_k$ belong to $C_{k}$;
(R6) the curves $C_k$, $1 \leq k \leq N$, such that $\Delta_k$ is a triangle dual to a vertex of $A_C$ outside $G$, are rational, intersecting Tor($\partial \Delta_k$) at precisely three points being unibranch there;
(R7) if $\Delta_k$, $1 \leq k \leq N$, is dual to a univalent vertex of $G$, then $\Delta_k$ is a triangle, $C_k$ is rational and crosses Tor($\partial \Delta_k$) at precisely four points being unibranch there;
(R8) if $\Delta_k$, $1 \leq k \leq N$, is dual to a bivalent vertex of $G$, then $\Delta_k$ is a triangle, $C_k$ is rational and crosses Tor($\partial \Delta_k$) at precisely five points being unibranch there;
(R9) if $\Delta_k$, $1 \leq k \leq N$, is dual to a trivalent vertex of $G$, extendable for some $G_i$, then $\Delta_k$ is a triangle, $C_k$ is the union of two imaginary conjugate rational curves, each of them crossing $\text{Tor}(\partial \Delta_k)$ at precisely three points being unibranched there.

**Step 5.** Now we show that $A_C$ is irreducible, and that any trivalent vertex of $G$ is extendable for some $G_i$, $i \geq 0$.

First, the equality in (2.7) means that, in the deformation $C^{(t)}$, $t \geq 0$, no intersection point of any two distinct components of $C_i$, $1 \leq i \leq N$, is smoothed out. Assuming that $A_C$ is reducible, and using the description of the curves $C_k$, obtained in Step 4, we immediately conclude that then the curve $C^{(t)}$, $t \neq 0$, becomes reducible.

Second, the existence of a trivalent vertex of $G$, which was not extendable for all $G_i$, $i \geq 0$, would mean a non-trivial relation to the position of the points $x''_j$, $j > s''$, contrary to their $\Delta$-generality.

**Step 6.** To complete the proof of Proposition 2.1, it remains to verify that the points $x''_1, \ldots, x''_{s''}$ cannot be four-valent vertices of $A_C$.

Assume on the contrary that, say, $x''_1$ is a four-valent vertex of $A_C$, which is dual to a parallelogram $\Delta_k$. The points $p''_1$ and $p''_2$ pick out two conjugate binomial components of the curve $C_k$, which cross two toric divisors of $\text{Tor}(\Delta_k)$, corresponding to a pair of parallel sides of $\Delta_k$. Let these sides be dual to an extended edge $\sigma''$ of $A_C$; denote by $\sigma''$ the other extended edge of $A_C$, passing through $x''_1$. We claim that $A_C$ can be deformed inside $\text{Iso}(A_C)$ so that the deformed tropical curves will pass through the configuration $\mathcal{F} \cup \mathcal{F}'$, the points $x''_1, \ldots, x''_{s''}$ will remain their vertices, and the edge $\sigma''$ will move out of $x''_1$. Indeed, by (2.5), reducing the requirement that $\sigma''$ passes through $x''_1$, we obtain at least one-dimensional family of curves in $\text{Iso}(A_C)$, whereas, keeping that requirement we obtain only the curve $A_C$. By Lemma 3.2(1), presented below in section 3, any of the above deformed tropical curves equipped with the same limit curves $C_1, \ldots, C_N$ gives rise to a real rational curve in $|D|_K$, passing through $p''_i$, $i = 1, \ldots, r''$, and $p''_k$, $k = 1, \ldots, r''$, $j = 1, 2$. Thus, we obtain infinitely many such curves in $|D|_K$, contradicting to the generality of $p''_i$, $i = 1, \ldots, r''$, and $p''_k$, $k = 1, \ldots, r''$, $j = 1, 2$. \hfill $\square$

**Remark 2.2.** We point out that, given a real tropical curve as in the assertion of Proposition 2.1, and a collection of curves $C_1, \ldots, C_N$, as described in the proof, the curves $C^{(t)}$, $t \neq 0$, which may appear from these data, are irreducible provided that the graph $G$ differs from $A_C$. The latter holds if, for example, $r'' + s'' > 0$.

Given a real rational tropical curve $A$ of type $(r', r'', s'')$ with Newton polygon $\Delta$, passing through the points $x'_1, \ldots, x'_r$, and $x''_1, \ldots, x''_{s''}$, as described in Proposition 2.1, to decide what are the curves $C_1, \ldots, C_N$, we have to know the graph $G$, which appeared in the proof of Proposition 2.1.

We can reconstruct $G$ in the following way. Take all the extended edges of $A$ passing through those points of $x''_1, \ldots, x''_{s''}$, which are not vertices of $A$. Recall that all these edges are of even weight. Their union together with endpoints (including $v_\infty$ if necessary) constitutes the graph $G_0$. Let $V^2(G_0)$ be the set of bivalent vertices of $G_0$, different from $v_\infty$. Take any subset $V_0 \subset V^2(G_0)$ and, for each $v \in V_0$, add to $G_0$ the extended edge of $A$ with endpoint $v$, which is not in $G_0$. The new graph is denoted by $G_1$. Then we take any subset $V_1 \subset V^2(G_1) \setminus V^2(G_0)$, and append the extended edges to $G_1$ in a similar manner. In finitely many steps we end up with some graph $G$, whose vertices are some trivalent vertices of $A$ and, perhaps, $v_\infty$, whose edges are some extendable edges of $A$. Notice that all the edges of $G$ necessarily have even weight. By the restrictions obtained in Step 4 of the proof of Proposition 2.1, the set of trivalent vertices of $G$ must be $\bigcup_{i \geq 0} V_i$.

**2.3. Real rational algebraic curves in the tropical limits of rational curves over $\mathbb{K}_\mathbb{R}$.**

**Lemma 2.3.** In the notation of section 2.2 let $\Delta_k$ be a triangle in $S_C$, dual to a vertex of $A_C$, which differs from $x''_1, \ldots, x''_{s''}$, and is not a vertex of $G$. Then $C_k$ is a real rational nodal curve, crossing $\text{Tor}(\partial \Delta_k)$ at precisely three points, where it is non-singular. The number of real solitary nodes of $C_k$ has the same parity as $|\text{Int}(\Delta_k) \cap \mathbb{Z}^2|$. Furthermore, if the lengths of edges of $\Delta_k$ are odd, and we are given the coefficients of the polynomial $f_k$ at the vertices of $\Delta_k$, then $f_k$ and thereby $C_k$ are determined uniquely.

**Proof.** The statements immediately follow from [13], Lemma 3.5 and Proposition 6.1. \hfill $\square$
Lemma 2.4. In the notation of section 2.2 let $\Delta_k$ be a triangle in $S_C$, dual to a vertex $x_j''$, $j \leq s''$, of $A_C$. Then $C_k$ is a real rational nodal curve, crossing Tor$(\partial \Delta_k)$ at precisely three points, where it is non-singular. The number of real solitary nodes of $C_k$ has the same parity as $|\text{Int}(\Delta_k) \cap \mathbb{Z}^2|$. Furthermore, if the lengths of edges of $\Delta_k$ are odd, then there are precisely $|\Delta_k|$ real curves $C_k$ as above, where $|\Delta_k|$ is the double Euclidean area of $\Delta_k$.

Proof. The fact that $C_k$ is non-singular along Tor$(\partial \Delta_k)$, is nodal, and the number of solitary real nodes is $|\text{Int}(\Delta_k) \cap \mathbb{Z}^2| \mod 2$, follows from [13], Lemma 3.5 and Proposition 6.1. To establish the last statement of the lemma, we perform an affine transformation of the lattice $\mathbb{Z}^2$, which takes $\Delta_k$ into the triangle with vertices $(0,m_1)$, $(m_2,0)$, $(m_3,0)$ with odd $m_1$, $m_2$ and even $m_3$ such that $0 \leq m_2 < m_3$, $0 < m_1 < m_3$. Then $C_k$ has parameterization $x(\rho) = a\rho^{m_1}$, $y(\rho) = b\rho^{m_2}(\rho^2 - 1)^{m_3-m_2}$ in suitable affine coordinates $x, y$, where $a, b \in \mathbb{R} \setminus \{0\}$. The condition that $\Delta_k$ is dual to $x_j''$ means that there is $\tau \in \mathbb{C}$ such that $x(\tau) = \xi_j''$, $y(\tau) = \eta_j''$, where $\xi_j'', \eta_j''$ are defined by (2.2). Due to the generality of $\xi_j'', \eta_j''$, the two latter equations have $m_1(m_3 - m_2) = |\Delta_k|$ distinct solutions $(a,b,\tau) \in \mathbb{R}^2 \times \mathbb{C}$.

Lemma 2.5. In the notation of section 2.2 let $\Delta_k$ be a triangle in $S_C$, dual to a univalent vertex of the graph $G$. Then $C_k$ is a real rational nodal curve, crossing Tor$(\partial \Delta_k)$ at precisely four points, two real and two imaginary conjugate, where it is non-singular. The number of real solitary nodes of $C_k$ is equal to $|\text{Int}(\Delta_k) \cap \mathbb{Z}^2| \mod 2$. Assume that $\Delta_k$ has two edges of odd length and one edge of even length, and we are given the intersection points of $C_k$ with Tor$(\partial \Delta_k)$. Then there are $|\Delta_k|/|\sigma|$ such curves $C_k$, where $\sigma \subset \Delta_k$ is the edge of even length.

Proof. A suitable affine automorphism of the lattice $\mathbb{Z}^2$ takes $\Delta_k$ into the triangle with vertices $(0,m_1)$, $(2m_2,0)$, $(2m_3,0)$, where $0 \leq m_2 < m_3$, $m_1 < 2m_3$. We choose a parameterization $x(\rho), y(\rho)$ of $C_k$ such that the value $\rho = \infty$ corresponds to the intersection point of $C_k$ with Tor($[[0,m_1],(2m_3,0)]$), and the values $\rho = \pm \sqrt{-1}$ correspond to the intersection points with Tor($[[2m_2,0],(2m_3,0)]$). Then

$$x(\rho) = a(\rho - c)^{m_1}, \quad y(\rho) = b(\rho - c)^{2m_2}(\rho^2 - 1)^{m_3-m_2},$$

where $a, b, c \in \mathbb{R}^+$. This immediately proves that $C_k$ is non-singular along Tor$(\partial \Delta_k)$, as well as the fact that $C_k$ is nodal and has no real non-solitary nodes (cf. [13], Proposition 6.1). Assume now that $m_1$ is odd. The conditions of the fixed intersection points with Tor$(\partial \Delta_k)$ reduce to the system of equations

$$\frac{a}{b(c^2 + 1)^{m_3-m_2}} = \text{const} \in \mathbb{R}^+,$$  

which has $m_1 = |\Delta_k|/|\sigma|$ distinct solutions $(a,b,c) \in \mathbb{R}^3$.

Lemma 2.6. In the notation of section 2.2 let $\Delta_k$ be a triangle in $S_C$, dual to a bivalent vertex of the graph $G$. Then $C_k$ is a real rational nodal curve, crossing Tor$(\partial \Delta_k)$ at precisely five points, one real and two pairs of imaginary conjugate points, where it is non-singular.

Proof. As in the proof of Lemma 2.5 we can assume that $\Delta_k = \text{Conv} \{0,2m_1),(2m_2,0),(2m_3,0)\}$ with integers $0 \leq m_2 < m_3$, $m_1 < 2m_3$. We choose a parameterization $x(\rho), y(\rho)$ of $C_k$ so that the value $\rho = \infty$ corresponds to the intersection point of $C_k$ with Tor($[[0,m_1],(2m_3,0)]$), the values $\rho = \pm \sqrt{-1}$ correspond to the intersection points with Tor($[[2m_2,0],(2m_3,0)]$), and the values $\rho = \kappa, \pi$ for some $\kappa \in \mathbb{C} \setminus \{\pm \sqrt{-1}\}$ correspond to the intersection points with Tor($[[0,2m_1],(2m_2,0)]$). Then

$$x(\rho) = a(\rho - \kappa)^{m_1}(\rho - \pi)^{m_1}, \quad y(\rho) = b(\rho - \kappa)^{m_2}(\rho - \pi)^{m_2}(\rho^2 - 1)^{m_3-m_2},$$

with some $a, b \in \mathbb{R}^+$. It is immediate that $C_k$ is non-singular at $C_k \cap \text{Tor}(\partial \Delta_k)$. Along the proof of Proposition 2.1 the two pairs of imaginary intersection points of $C_k$ with Tor$(\partial \Delta_k)$ are generic, which reads as a system of equations on $a, b, \kappa, \lambda$:

$$\frac{b}{a}(\kappa - \pi)^{m_2-m_1}(\kappa^2 + 1)^{m_3-m_2} = \lambda, \quad a(\sqrt{-1} - \kappa)^{m_1}(\sqrt{-1} - \pi)^{m_1} = \mu$$

with some generic $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$. This results in a finitely many generic values for $\kappa$. In particular, we obtain that $x'(\rho), y'(\rho)$ do not vanish simultaneously in $(\mathbb{C}^*)^2 \subset \text{Tor}(\Delta_k)$, or, equivalently, $C_k$ is an immersed line.
\( \mathbb{C}P^1 \) into \( \text{Tor}(\Delta_k) \). The condition that three local branches of \( C_k \) are centered at some point of \( (\mathbb{C}^*)^2 \) reduces to a system of four equations

\[
\begin{align*}
(\rho_1 - \kappa)^m_1(\rho_1 - \overline{\kappa})^m_1 &= (\rho_2 - \kappa)^m_1(\rho_2 - \overline{\kappa})^m_1 = (\rho_3 - \kappa)^m_1(\rho_3 - \overline{\kappa})^m_1, \\
(\rho_1 - \kappa)^m_2(\rho_1 - \overline{\kappa})^m_2(\rho_2 + 1)^{m_3 - m_2} &= (\rho_2 - \kappa)^m_2(\rho_2 - \overline{\kappa})^m_2(\rho_3 + 1)^{m_3 - m_2} = (\rho_3 - \kappa)^m_2(\rho_3 - \overline{\kappa})^m_2
\end{align*}
\]

which leads to an algebraic relation on \( \kappa \) and \( \overline{\kappa} \), contradicting their generality.

Lemma 2.7. In the notation of section 2.4 let \( \Delta_k \) be a triangle in \( S_C \), dual to a trivalent vertex of the graph \( G \). Then \( C_k \) is the union of two imaginary conjugate nodal rational curves, which cross each other transversally at their non-singular points. Furthermore, each of these rational curves crosses \( \text{Tor}(\partial \Delta_k) \) at precisely three imaginary points, where it is non-singular. The number of real solitary nodes of \( C_k \) is equal to \( |\text{Int}(\Delta_k) \cap \mathbb{Z}^2| + 1 \mod 2 \). Given the intersection points of \( C_k \) with \( \text{Tor}(\sigma_1), \text{Tor}(\sigma_2) \), \( \sigma_1, \sigma_2 \) being two edges of \( \Delta_k \), the number of curves like \( C_k \) is equal to \( 2|\text{Int}(\Delta_k)/(|\sigma_1| \cdot |\sigma_2|)|^{-1} \).

Proof. From the proof of Proposition 2.1 we know that \( C_k \) must be the union of two conjugate rational curves, each of them crossing \( \text{Tor}(\partial \Delta_k) \) at three points, where they are unibranch. Again along the proof of Proposition 2.1 the two pairs of conjugate intersection points of \( C_k \) with \( \text{Tor}(\sigma_1) \cup \text{Tor}(\sigma_2) \) are generic, and thus, the components of \( C_k \) are nodal and non-singular along \( \text{Tor}(\partial \Delta_k) \), they intersect transversally and only at their non-singular points. In particular, \( C_k \) has no non-solitary real nodes, and hence the number of its solitary real nodes has parity of \( |\text{Int}(\Delta_k) \cap \mathbb{Z}^2| + 1 \). For a component of \( C_k \), we have two choices of a pair of intersection points with \( \text{Tor}(\sigma_1) \cup \text{Tor}(\sigma_2) \) up to conjugation. Furthermore, fixing these intersection points, we have by [13], Lemma 3.5, \(|\text{Int}(\Delta_k)/|\sigma_1| \cdot |\sigma_2|)|^{-1}\) choices for a given component, and the proclaimed number of choices for \( C_k \) follows.

2.4. Refinement of the tropical limit. Using the structure of the curves \( C_1, ..., C_N \) which appear in the tropical limit of the curve \( C \subset \text{Tor}_G(\Delta) \), we refine the tropical limit along the procedures described in [13], sections 3.5 and 3.6. Namely, the local branches of the curves \( C_1, ..., C_N \), which are tangent to the divisors \( \text{Tor}(\partial \Delta_1), ..., \text{Tor}(\partial \Delta_N) \), are naturally combined into disjoint pairs \( (B, B') \) such that either

- \( B \) and \( B' \) are centered at the same point \( z \in \text{Tor}(\sigma) \), where \( \sigma \) is a common edge of two triangles, or
- \( B \) is centered at a point \( z \in \text{Tor}(\sigma) \); \( B' \) is centered at a point \( z' \in \text{Tor}(\sigma') \) with \( \sigma, \sigma' \) being parallel edges of two triangles, dual to the same extended edge of \( A_C \), and such that \( z \) and \( z' \) are joined by a sequence of components of \( (\mathbb{C}^*)^0 \) from the set \( C_b \).

Along the procedures of [13], sections 3.5 and 3.6, for any pair \( (B, B') \), the polynomial \( f(x, y) \), defining the curve \( C \), determines a complex polynomial \( f_{B, B'}(x, y) \) such that (i) it has Newton triangle \( \Delta_{B, B'} = \text{Conv}\{0, 2\}, \{0, 0\}, \{m, 1\}\) where \( m \) is the intersection number of \( B \) with \( \text{Tor}(\sigma) \), and the coefficient of \( x^{m-1}y \) is zero, (ii) the polynomial \( f_{B, B'} \) defines a rational nodal curve \( C_{B, B'} \) in \( \text{Tor}(\Delta_{B, B'}) \). Furthermore, the polynomials \( f_1, ..., f_N \), defining the curves \( C_1, ..., C_N \) and given by (2.4), determine the coefficients of \( f_{B, B'} \) at the vertices of \( \Delta_{B, B'} \).

Among the results of [13], Lemma 3.9 and Proposition 6.1, we shall use the following one:

Lemma 2.8. In the above notation, assume that \( m \geq 2 \) and \( a, b, c \in \mathbb{C}^* \). Then there are precisely \( m \) polynomials

\[
\varphi(x, y) = ay^2 + y \left( bx^m + \sum_{i=0}^{m-2} b_i x^i \right) + c,
\]

which define nodal rational curves in \( \text{Tor}(\Delta_{B, B'}) \). If \( a, b, c \) are real and \( m \) is even, then there are precisely two real polynomials (2.11), which define rational nodal curves in \( \text{Tor}(\Delta_{B, B'}) \); furthermore, one of these curves has \( m - 1 \) real solitary nodes, and the other has no such nodes. If \( a, b, c \) are real and \( m \) is odd, then there is only one real polynomial (2.11), which defines a rational nodal curve, and this curve has \( m - 1 \) real solitary nodes.
2.5. The Welschinger number of a real rational tropical curve. Let \( A \) be a real rational tropical curve of type \((r', r'', s'')\) with Newton polygon \( \Delta \), passing through \( \mathcal{F} = \{x'_1, ..., x'_r\} \) and \( \mathcal{F}' = \{x''_1, ..., x''_{r''}\} \) in such a way that \( x'_1, ..., x'_{r'} \) are trivalent vertices of \( A \), and the points \( x''_1, ..., x''_{r''} \) and \( x''_{r''+1}, ..., x''_{r'''} \) lie in the interior parts of some edges of \( A \). We intend to define the Welschinger number \( w(A, \mathcal{F}, \mathcal{F}') \), which later will be equated with the contribution to the Welschinger number of all the real rational curves in the linear system \(|D|\) on the surface \( \Sigma = \text{Tor}_K(\Delta) \), which pass through the points \( p'_i, i = 1, ..., r' \), and \( p''_j, p''_{j'}, j = 1, ..., r'' \), and project onto the tropical curve \( A \).

Among the possible graphs \( G \), inscribed in the given tropical curve \( A \) along the rules presented at the end of section 2.2, there is the maximal graph \( G_{\text{max}} \). It is uniquely determined by the disposition of the sets \( \mathcal{F} \) and \( \mathcal{F}' \) on \( A \), and is obtained by letting \( V_0 = V^2(G_0), V_1 = V^2(G_1) \), and so on, in the notation of section 2.2.

If there is an even weight edge of \( A \) lying outside \( G_{\text{max}} \), we put \( w(A, \mathcal{F}, \mathcal{F}') = 0 \).

Assume that the edges of \( A \) outside \( G_{\text{max}} \) have odd weights. Let \( S \) be the dual subdivision of \( \Delta \). Then the triangles of \( S \) dual to the vertices \( x'_1, ..., x'_{r'} \) of \( A \) have all edges of odd length, the triangles dual to the univalent vertices of \( G_{\text{max}} \) have two odd length edges and one edge of even length, the triangles dual to the trivalent vertices of \( G_{\text{max}} \) are even, that is, all edges are of even length. Then we put

\[
w(A, \mathcal{F}, \mathcal{F}') = (-1)^{a+b-2} \prod \{\Delta'\},
\]

where \( a \) is the number of integral points, which lie strictly inside the triangles of \( S \), \( b \) is the number of the even triangles, \( c \) is the number of of the triangles of even lattice area, and \( \Delta' \) runs over all the triangles of even lattice area, or dual to \( x'_1, ..., x'_{r'} \).

3. The correspondence theorem

3.1. Calculation of the Welschinger number via tropical curves. Let \( \Sigma \) be a Del Pezzo toric surface, associated with a lattice polygon \( \Delta \) as defined in section 2.2 and \( D \subset \Sigma \) be the corresponding real ample divisor. Let the non-negative integers \( r', r'' \) satisfy (1.1).

**Theorem 3.1.** In the above notation and assumptions, let a collection of distinct points \( x'_1, ..., x'_{r'}, x''_1, ..., x''_{r''} \) in \( Q^2 \) be \( \Delta \)-generic, and the distinct points \( p'_i \in (K^2)^2, i = 1, ..., r' \), and \( p''_j, p''_{j'} \in (K^2)^2 \), \( j = 1, ..., r'' \), be generic among those, satisfying

\[
\text{Val}(p'_i) = x'_i, \text{ Val}(p''_j) = x''_j, \text{ Conj}(p''_{j'}) = p''_{j'}, \quad j = 1, ..., r''.
\]

Then one has

\[
W_{r''}(\Sigma, D) = \sum_{s''=0}^{r''} \sum_{A(s'')} w(A(s''), \mathcal{F}, \mathcal{F}'),
\]

where \( \mathcal{F} = \{x'_1, ..., x'_{r'}\}, \mathcal{F}' = \{x''_1, ..., x''_{r''}\}, \) and \( A(s'') \) ranges over all real rational tropical curves of type \((r', r'', s'')\) with Newton polygon \( \Delta \), which pass through \( \mathcal{F} \cup \mathcal{F}' \) in such a way that precisely \( s'' \) points of \( \mathcal{F}' \) are trivalent vertices of \( A(s'') \), and the other points of \( \mathcal{F}' \) are interior points of edges of \( A(s'') \), having even weight.

**Proof.** Step 1. A \( \Delta \)-general position of the points \( \mathcal{F} \cup \mathcal{F}' \) and the general choice of the respective points \( p'_i, i = 1, ..., r' \), and \( p''_{j}, p''_{j'}, j = 1, ..., r'' \), ensure all the generality assumptions used in the proof of Proposition 2.1. Hence, in particular, the real rational curves in the linear system \(|D|\) on the surface \( \text{Tor}_K(\Delta) \), passing through \( p'_i, i = 1, ..., r' \), and \( p''_j, p''_{j'}, j = 1, ..., r'' \), are projected by \( \text{Val} : (K^2)^2 \rightarrow K^2 \) to the real rational tropical curves as described in the assertion of Theorem 3.1.

Step 2. Let \( A \) be a real rational tropical curve of type \((r', r'', s'')\), satisfying the conditions of Theorem 3.1. S the dual subdivision of \( \Delta \) into polygons \( \Delta_1, ..., \Delta_N \), and \( G \) a graph constructed as described at the end of section 2.2. Without loss of generality assume that \( x''_1, ..., x''_{r''} \) are vertices of \( A \). A collection of real polynomials \( f_1(x, y), ..., f_N(x, y) \) with Newton polygons \( \Delta_1, ..., \Delta_N \), respectively, is called an admissible tropicalization if

- for any common edge \( \sigma = \Delta_k \cap \Delta_l \), the truncations \( f''_k \) and \( f''_l \) coincide;
- if an edge \( \sigma \subset \Delta_k \) is dual to an edge of \( A \), passing through \( x'_i, 1 \leq i \leq r' \) (or through \( x''_i, i > s'' \)), then \( f''_k(\xi'_i, \eta'_i) = 0 \) (resp., \( f''_k(\xi''_i, \eta''_i) = f''_k(\xi''_i, \eta''_i) = 0 \));
If a triangle $\Delta_k$ is dual to $x''_i$, $1 \leq i \leq s''$, then $f_k(\xi''_i, \eta''_i) = f_k(\overline{\xi''_i}, \overline{\eta''_i}) = 0$;
- if $\Delta_k$ is a parallelogram, then $f_k$ is split into the product of a monomial and few binomials;
- if a triangle $\Delta_k$ is dual to a vertex of $A$ outside the graph $G$, then $f_k$ defines a real rational curve in $\text{Tor}(\partial \Delta_k)$, which crosses $\text{Tor}(\partial \Delta_k)$ at precisely three points and is non-singular there;
- if a triangle $\Delta_k$ is dual to a univalent (or bivalent) vertex of $G$, then $f_k$ defines a real rational curve in $\text{Tor}(\partial \Delta_k)$, which crosses $\text{Tor}(\partial \Delta_k)$ at two real and a pair of imaginary conjugate points (resp., at one real and two pairs of imaginary conjugate points) and is non-singular there;
- if a triangle $\Delta_k$ is dual to a trivalent vertex of $G$, then the curve $f_k = 0$ in $\text{Tor}(\partial \Delta_k)$ is split into a pair of imaginary conjugate rational curves, each of them crossing $\text{Tor}(\partial \Delta_k)$ at precisely three imaginary points being non-singular there.

Next we define an admissible refined tropicalization associated with $A$ and $G$. First, we consider the set of the local branches of curves $C_k := \{ f_k = 0 \} \subset \text{Tor}(\Delta_k)$, $k = 1, ..., N$, which are tangent to $\bigcup_k \text{Tor}(\partial \Delta_k)$, and, following the instructions of section 3.5, distribute these branches into disjoint pairs. With each pair $(B, B')$ we associate a triangle $\Delta_{B, B'} = \text{Conv}\{ (0,0), (0,2), (m,1) \}$, where $m$ is the intersection number of $B$ (or $B'$) with $\bigcup_k \text{Tor}(\partial \Delta_k)$. If $B$ and $B'$ are branches of curves $C_k$ and $C_l$, respectively, centered at the same point on $\text{Tor}(\sigma)$, $\sigma = \Delta_k \cap \Delta_l$, we perform the refinement procedure described in [13], section 3.5. The corresponding transformations of the polygons $\Delta_k, \Delta_l$ and the polynomials $f_k, f_l$, namely, the coordinate change induced by $M_{\sigma} \in \text{Aff}(\mathbb{Z}^2)$ and the shift $x \mapsto x + \xi$, are illustrated in Figure 2 (see [13], section 3.5, for all details). Notice that the (transformed) polynomials $f_k, f_l$ determine the coefficients of $y^2$, $yx^m$, and 1.

If branches $B, B'$, forming a pair, are centered at points $z \in \text{Tor}(\sigma), z' \in \text{Tor}(\sigma')$, where $\sigma \neq \sigma'$, then we perform the refinement procedure, described in [13], section 3.6, and illustrated in Figure 3. Namely, Figures 3a,b,c indicate the transforms induced by some $M \in \text{Aff}(\mathbb{Z}^2)$ and by a shift $x \mapsto x + \xi$. The convex piece-wise linear function $\nu : \Delta \to \mathbb{R}$, whose graph projects onto the subdivision $S$, induces a convex piece-wise linear function $\nu'$ on the polygons, shown in Figure 3c except for the trapezoid $\theta$. The generality of the configuration $\overline{\sigma'} \cup \overline{\sigma''}$ yields, in particular, the generality of the function $\nu$ in the following sense: no four parallel edges of the graph of $\nu$ lie on two parallel planes. Thus, $\nu$ can uniquely be extended to $\theta$ as a convex piece-wise linear function, which defines a subdivision of $\theta$ into parallelograms and a translate of $\Delta_{B, B'}$ (see, for example, Figure 3d)). Furthermore, those polynomials $f_1, ..., f_N$, whose Newton polygons contain $\sigma, \sigma'$ and all the parallel to them edges of $S$ (see Figure 3a)), after transformations determine the coefficients at the integral points on the part of the boundary of $\theta$ common with other polygons. This uniquely determines the polynomials with Newton parallelograms inside $\theta$, which are split into products of a monomial and binomials. Finally, we determine the (non-zero) coefficients corresponding to the vertices of the triangle inside $\theta$, and which we respectively assign to the vertices of $\Delta_{B, B'}$. 
Denote by $\mathcal{B}$ the set of all pairs $(B, B')$ of local branches as above. To obtain an admissible refined tropicalization, we extend an admissible tropicalization $f_1, \ldots, f_N$ by adding (complex) polynomials $f_{B,B'}(x,y)$ with Newton polygons $\Delta_{B,B'}$, $(B, B') \in \mathcal{B}$, such that

- any $f_{B,B'}$ is given by formula (2.11) with the coefficients $a, b, c \in \mathbb{C}^*$, prescribed as explained above;
- the curve $\{f_{B,B'} = 0\} \subset \text{Tor}(\Delta_{B,B'})$ is rational nodal;
- if $(B, B')$ is real then $f_{B,B'}$ is real;
- if $(B, B')$ is imaginary, and $(B, B')$ is the conjugate pair, then $f_{B,B'} = f_{B,B'}$.

The curves $C_k = \{f_k = 0\} \subset \text{Tor}(\Delta_k)$, $k = 1, \ldots, N$, and $C_{B,B'} = \{f_{B,B'} = 0\} \subset \text{Tor}(\Delta_{B,B'})$, $(B, B') \in \mathcal{B}$, are called admissible tropicalization curves.

Step 3. Let us be given a tropical curve $A$ and an admissible refined tropicalization $f_i$, $i = 1, \ldots, N$, $f_{B,B'}$, $(B, B') \in \mathcal{B}$, as described in Step 2. We state

Lemma 3.2. (1) Under the above assumptions, there exists a polynomial $f \in \mathbb{K}[x,y]$ with Newton polygon $\Delta$ such that $\{f(x,y) = 0\}$ is a rational nodal curve $C$ in $\text{Tor}(\Delta)$, and the refined tropical limit of the polynomial $f(x,y)$ consists of the tropical curve $A$ and the polynomials $f_i$, $i = 1, \ldots, N$, $f_{B,B'}$, $(B, B') \in \mathcal{B}$. All these curves $C$ have the same number of real solitary nodes, which is equal to the total number of such nodes over all the curves $C_i = \{f_i = 0\} \subset \text{Tor}(\Delta_i)$, $i = 1, \ldots, N$, and $C_{B,B'} = \{f_{B,B'} = 0\} \subset \text{Tor}(\Delta_{B,B'})$, $(B, B') \in \mathcal{B}$.

(2) The number of the above curves $C$, passing through $p'_i$, $i = 1, \ldots, r'$, and $p''_1, p''_2, i = 1, \ldots, r''$, is finite, and depends only on $A$ and on the polynomials $f_i$, $i = 1, \ldots, N$. Furthermore, if all the edges of $A$ outside the graph $G_{\text{max}}$ have odd weight, then the number of such curves $C$ is equal to $2^{s'' - r''} \prod_{j > s''} |\sigma_j|$, where $\sigma_j$ is the edge of $S$, dual to the edge of $A$ passing through $x''_j$, $j > s''$. 

Figure 3. Refinement of the tropicalization, II
Step 4. By Proposition 2.4, the real rational curves in the linear system $|D|$ on $\text{Tor}(\Delta)$, passing through $p_i^t, i = 1, ..., r'$, and $p_{i1}^{t''}, i = 1, ..., r''$, project onto real rational tropical curves as described in the assertion of Theorem 3.1, and their refined tropical limits determine (up to a real constant factor) an admissible refined tropicalization, associated with $A$ and a respective graph $G$.

We claim that, if there is an edge of $A$ of even weight, which is not included in $G$, then the total contribution to $W_{r''}(\Sigma, D)$ of the real rational curves in $\text{Tor}(\Delta)$, having $A$ and $G$ in their tropical limits, is zero.

Indeed, by restriction (R3) obtained in Step 4 of the proof of Proposition 2.4, semi-infinite edges of $A$ of even weight belong to $G$. Let $f_k, k = 1, ..., N$, $f_{B, B'}$, $(B, B') \in B$, be an admissible tropicalization associated with $A$ and $G$. An edge of $A$ of even weight outside $G$ must have two endpoints, and hence there is a pair $(b_0, b_0') \in B$ of real local branches, having an even intersection multiplicity with $\bigcup_k \text{Tor}(\partial \Delta_k)$. By Lemma 2.8, the polynomial $f_{b_0, b_0'}$ can be replaced by another polynomial $f_{b_0, b_0'}$ keeping the admissibility property. By Lemma 3.2, these admissible refined tropical limits produce an equal number of real rational curves in $|D|$, but they have different parity of the number of real solitary nodes (see again Lemma 2.8).

Thus, the only pairs $(A, G)$, which contribute to $W_{r''}(\Sigma, D)$ is $(A, G_{\text{max}})$ such that all edges of $A$ outside $G_{\text{max}}$ have odd weight.

Step 5. In the previous notation and assumptions, let $A$ have no edge of even weight outside $G_{\text{max}}$. By Lemmas 2.3, 2.7, and 2.8, the real rational curves in $|D|$ arising along Lemma 3.2 from admissible refined tropicalizations associated with $A$ and $G_{\text{max}}$, have the same number of real solitary nodes mod 2, and the parity is given by the sign of $w(A, \bar{\mathbf{x}}, \bar{\mathbf{x}}')$ as defined in section 2.5. Thus, to complete the proof of Theorem 3.1, we have to count the number of admissible refined tropicalizations associated with $A$ and $G_{\text{max}}$ and considered up to multiplication by a non-zero real constant.

First, by Lemma 2.4, we have $\prod |\Delta'|$ choices for admissible curves with Newton triangles, dual to $x''_i, ..., x''_r$, where $\Delta'$ runs over all these triangles.

Second, the above choice and the coordinates of the points $p_i^t, i = 1, ..., r'$, and the points $p_{i1}^{t''}, i > s''$, determine the intersections of the possible admissible tropical curves $C_1, ..., C_N$ with $\bigcup \text{Tor}(\sigma')$, where $\sigma'$ runs over all edges of the subdivision $S$, dual to the extended edges of $A$, passing through $x'_i, i = 1, ..., r'$, and $x''_i, i = 1, ..., r''$. Let $E_0$ be the union of all these extended edges of $A$.

The graph $E_0$ is a union of trees, whose vertices differing from $x'_1, ..., x''_r$, have valency at most two. Notice that the position of the remaining edges of $A$ is prescribed by $E_0$ (provided that the combinatorial type of the pair $(A, \bar{\mathbf{x}}')$ is fixed). This yields that the graph $E_0$ has a bivalent vertex $v_0$, and there is an extended edge $\varepsilon_1$ in $A \setminus E_0$ starting at $v_0$. Notice that $\varepsilon_1$ cannot join two vertices of $E_0$ of valency two (this can be verified precisely as a similar statement on the graphs $G_i, i \geq 0$ in Step 3 of the proof of Proposition 2.4. Put $E_1 = E_0 \cup \varepsilon_1$.

For an admissible tropicalization curve with Newton triangle $\Delta'$, dual to $v_0$, we know the intersections with $\text{Tor}(\sigma') \cup \text{Tor}(\sigma'')$ with $\sigma', \sigma''$, being any two edges of the Newton triangle. Then, by Lemmas 2.3, 2.6, and 2.7, the choice of such an admissible tropicalization curve can be made (i) in a unique way, if all the edges of $\Delta'$ have odd length, (ii) in $|\Delta'|/|\sigma'|$ ways, if $\sigma'$ has an even length and $\sigma''$ has an odd length, (iii) in $2|\Delta'|/(|\sigma'| \cdot |\sigma''|)$ ways, if both $\sigma'$ and $\sigma''$ have even length.

Next, by a similar reason, there is a bivalent vertex $v_1$ of $E_1$, and the extended edge $\varepsilon_2$ of $A \setminus E_1$, starting at $v_1$, which does not end up at another bivalent vertex of $E_1$. We then reconstruct a compatible admissible tropicalization curve with Newton triangle dual to $v_1$. Proceeding in the same manner, we complete the reconstruction of all the admissible tropicalization curves $C_1, ..., C_N$. It follows immediately that the reconstruction of $C_1, ..., C_N$ can be done in $2^e \prod |\Delta'| \cdot (\prod \omega(\varepsilon') \cdot \prod \omega(\varepsilon''))^{-1}$ ways, where $e$ is the number of even triangles in $S$, $\Delta'$ runs over all triangles in $S$, containing an even length edge, $\varepsilon'$ runs over all the extended edges of $A$ of even weight with two endpoints, $\varepsilon''$ runs over all the extended edges of $A$, passing through $x''_j$, $j > s''$, and at last, $\omega(*)$ denotes the weight.

By Lemma 2.3 for any collection of admissible tropicalization curves $C_1, ..., C_N$, we can find $\prod \omega(\varepsilon')/2$ compatible collections of admissible refining tropicalization curves $C_{B, B'}, (B, B') \in B$, where $\varepsilon'$ runs over all the extended edges of $A$ of even weight.

Combining these computations with the result of Lemma 3.2 we complete the proof. \qed
Step 1. The subdivision $S$, the convex piece-wise linear function $\nu : \Delta \to \mathbb{R}$, and the collection of admissible tropicalization curves $C_i, i = 1, \ldots, N, C_{B, B'}$, $(B, B') \in \mathcal{B}$, satisfy the conditions of the patchworking Theorem 5 from [13], section 5.3. Indeed, the assumptions of [13], section 5.1, are easily verified. An orientation $\Gamma$ of the tropical curve $A$, required in the assertion of Theorem 5 [13], can be chosen so that all edges of $A$ form angles $\alpha \in (-\pi/2, \pi/2]$ with the horizontal coordinate axis. Then all the triads $(\Delta_k, \Delta_k^-(\Gamma), C_k), k = 1, \ldots, N$, and all the deformation patterns, given by the curves $C_{B, B'}$, $(B, B') \in \mathcal{B}$, are transversal (in the sense of [13], section 5.2) by [13], Lemmas 5.5 and 5.6. Hence [13], Theorem 5, provides the existence of a real nodal rational curve $C$ in the linear system $|D|$ on $\operatorname{Tor}_K(\Delta)$, whose refined tropical limit consists of $A$ and $C_i, i = 1, \ldots, N, C_{B, B'}$, $(B, B') \in \mathcal{B}$. Moreover, the number of real solitary nodes of any of such curves $C$ is equal to the total number of real solitary nodes of the refined tropicalization $C_i, i = 1, \ldots, N, C_{B, B'}$, $(B, B') \in \mathcal{B}$.

To further impose the conditions $p_1', p_2', p_3'' \in C, i = 1, \ldots, r', j = 1, \ldots, r''$, we need a description of the complete family of curves $C \subset \operatorname{Tor}_K(\Delta)$ as above. Such a description is not given in the assertion of Theorem 5 [13], but we shall extract it from the proof of that theorem, for which we refer to [12]. The proof goes as follows. The curves $C$ are represented by polynomials

$$f(x, y) = \sum_{(i, j) \in \Delta} (a_{ij} + c_{ij})x^i y^j \in \mathbb{K}_R[x, y],$$

where

$$f_k(x, y) = \sum_{(i, j) \in \Delta_k} a_{ij}x^i y^j, \quad k = 1, \ldots, N, \quad c_{ij} = c_{ij}(t) \in \mathbb{K}_R, \quad \mathbb{V}(c_{ij}) < 0, \quad (i, j) \in \Delta.$$

The rationality requirement is expressed as a system of equations on the unknowns $c_{ij}(t)$, whose linearization at $t = 0$ appears to be independent, and thus, the implicit function theorem applies, resulting in the existence of a family of the aforementioned curves $C \subset \operatorname{Tor}_K(\Delta)$, smoothly depending on some parameters. Thus, we shall assume that $c_{00} = 0$, where $v_0$ is some vertex of $\Delta$, since the polynomials in $\mathbb{K}[x, y]$ are considered up to a constant factor, and then we have only to indicate, which of $c_{ij}, (i, j) \in \Delta \setminus \{v_0\}$, can be chosen as independent parameters.

Step 2. We start by reducing the set of variables $c_{ij}, (i, j) \in \Delta \setminus \{v_0\}$ up to the following set.

Denote by $V'(S)$ the set of (integral) middle points $v(\sigma)$ of all the edges $\sigma$ of $S$, dual to the extended edges of $A$ belonging to the graph $G_{\text{max}}$. We shall show that independent variables can be chosen among

$$(3.1) \quad c_{ij}, \quad (i, j) \in V(S) \cup V'(S) \setminus \{v_0\},$$

where $V(S)$ stands for the set of vertices of $S$.

Along the proof of Theorem 5 as presented in [12], the rationality condition for $C$ reduces to equations for $c_{ij}, (i, j) \in \Delta \setminus \{v_0\}$, which, up to terms containing $t$ to a positive power, coincide with the equations of the transversality of all the triples $(\Delta_k, \Delta_k^-(\Gamma), C_k), k = 1, \ldots, N$, which are written for the coefficients of the polynomials

$$\tilde{f}_k = \sum_{(i, j) \in \Delta_k} (a_{ij} + c_{ij})x^i y^j, \quad k = 1, \ldots, N, \quad c_{v_0} = 0.$$

The transversality condition for a parallelogram $\Delta_k$ (recall that $\Delta_k^-(\Gamma)$ consists of two non-parallel edges) implies by [13], Lemma 5.1,

$$(3.2) \quad c_{ij} = L_{ij} \left( \{c_{\alpha \beta} : (\alpha, \beta) \in \Delta_k^-(\Gamma) \setminus \{v_0\} \} \right) + \text{h.o.t.} + O(t), \quad (i, j) \in \Delta_k \setminus \Delta_k^-(\Gamma),$$

with some real linear functions $L_{ij}, (i, j) \in \Delta_k \setminus \Delta_k^-(\Gamma)$, as well as the following relation: for any pair $\sigma, \sigma'$ of parallel edges of $\Delta_k$, it holds that

$$(3.3) \quad \frac{a_{ij} + c_{ij}}{a_{i'j'} + c_{i'j'}} = \frac{a_{i_1j_1} + c_{i_1j_1}}{a_{i'_1j'_1} + c_{i'_1j'_1}} + O(t), \quad (i, j, (i_1, j_1) \in \sigma, (i', j'), (i'_1, j'_1) \in \sigma',$$

where $(i, j)$ and $(i_1, j_1)$ run over the integral points of $\sigma$, and $(i', j'), (i'_1, j'_1) \in \sigma'$ are obtained from $(i, j), (i_1, j_1)$, respectively, by a shift taking $\sigma$ to $\sigma'$.\]
By [13], Definition 5.2, the transversality of a triple \((\Delta_k, \Delta_k^c(\Gamma), C_k)\) with a triangle \(\Delta_k\) means that the condition to pass through the nodes of the curve \(C_k\), and the relations

\[
(3.4) \quad \begin{aligned}
(C \cdot \text{Tor}(\sigma))_z \geq (C_k \cdot \text{Tor}(\sigma))_z, & \quad z \in C_k \cap \text{Tor}(\sigma), & \sigma \subset \Delta_k^c(\Gamma), \\
(C \cdot \text{Tor}(\sigma))_z \geq (C_k \cdot \text{Tor}(\sigma))_z - 1, & \quad z \in C_k \cap \text{Tor}(\sigma), & \sigma \not\subset \Delta_k^c(\Gamma),
\end{aligned}
\]

are independent for curves \(C\) in the linear system \(|C_k|\) on \(\text{Tor}(\Delta_k)\). From the description of the curves \(C_k\) and the Riemann-Roch theorem (in the form of Lemma 5.5 [13]), it can easily be verified that conditions (3.4) result in

\[
(3.5) \quad c_{ij} = L_{ij}(\{c_{\alpha\beta} : (\alpha, \beta) \in \Delta_k^c(\Gamma) \cup ((V(S) \cup V'(S)) \cap \partial \Delta_k) \setminus \{v_0\}\})
\]

Combining (3.2) and (3.5), we obtain that the variables \(c_{ij}, (i, j) \in \Delta \setminus (V(S) \cup V'(S))\), can be expressed via variables (3.1).

**Step 3.** We finally select an independent set of variables among (3.1).

In each parallelogram \(\Delta_k\) there is a unique vertex, which does not belong to \(\Delta_k^c(\Gamma)\). We remove all such vertices from \(V(S)\) and denote the complement set by \(V_1(S)\). Without loss of generality, assume that \(v_0 \in V_1(S)\).

The extended edge of \(A\), passing through a point \(x'_j, j > s''\), is dual to one or a few parallel edges of \(S\); we pick up one of these edges of \(S\), denote it by \(\sigma_j\), and introduce the set \(V''(S) = \{v(\sigma_j) : s'' < j \leq r''\} \subset V'(S)\).

We claim that \(c_{ij}, (i, j) \in V_1(S) \cup V''(S) \setminus \{v_0\}\), is a complete set of independent variables, parameterizing the family of real rational curves \(C\) as in the assertion of Lemma 3.2.

Indeed, first, we notice that \(\Gamma\) defines a partial order on the set of parallelograms of \(S\), which we complete up to a linear order, and then, following the chosen order, we can restore all the variables \(c_{ij}, (i, j) \in V(S) \setminus \{v_0\}\), by means of relations (3.3), restricted to the vertices of the parallelograms.

Second, we notice that if an extended edge \(\varepsilon\) of \(A\) is dual to several (parallel) edges of \(S\), say \(\sigma^{(1)}, ..., \sigma^{(k)}\), then we can find \(c_{ij}\) with \((i, j) = v(\sigma^{(i)})\), \(1 \leq i < k\), using relations (3.3) and the knowledge of \(c_{ij}, (\alpha, \beta) \in \{v(\sigma^{(k)})\} \cup V(S) \setminus \{v_0\}\).

Third, having the variables \(c_{ij}, (\alpha, \beta) \in V(S) \cup V''(S) \setminus \{v_0\}\), and using relations (3.5), we can find all the variables \(c_{ij}, (i, j) \in V(S) \cup V''(S)\). Namely, we follow the algorithm of construction of the graph \(G_{\alpha}\), starting with the graph \(G_0\) (see Steps 2 and 3 of the proof of Proposition 2.1 and the beginning of section 2.3). That is, we begin with the knowledge of \(c_{ij}, (i, j) = v(\sigma)\) for all edges \(\sigma\) of \(S\), dual to the edges of the graph \(G_0\). We successively append extended edges of \(A\) to \(G_0\) so that each time we have a triangle \(\Delta_k\) and the variables \(c_{ij}\), where \((i, j)\) ranges over the vertices of \(\Delta_k\), and over \(v(\sigma'), v(\sigma'')\), \(\sigma', \sigma''\) being any two edges of \(\Delta_k\). Then relations (3.5) allow us to find \(c_{ij}\) with \((i, j) = v(\sigma'')\), \(\sigma''\) being the third edge of \(\Delta_k\).

Thus, the variables \(c_{ij}, (i, j) \in V_1(S) \cup V''(S) \setminus \{v_0\}\), determine the variables \(c_{ij}, (i, j) \in V(S) \cup V''(S) \setminus \{v_0\}\), and hence, as shown before, all the variables \(c_{ij}, (i, j) \in \Delta_k \setminus \{v_0\}\). At last, we observe that \#((\bar{V}_1(S) \cup V''(S)) \setminus \{v_0\}) = \text{rk}(A) + r'' - s'' = -K_2D - 1\) is the dimension of the variety of rational curves in the linear system \(|D|\) on the surface \(\Sigma\), which confirms the independence of the variables \(c_{ij}, (i, j) \in V_1(S) \cup V''(S) \setminus \{v_0\}\).

A particular consequence of the result obtained in this step is that the variables \(c_{ij}, (i, j) \in V''(S)\) are independent of the variables \(c_{ij}, (i, j) \in V(S) \setminus \{v_0\}\), and that equations (3.3), restricted to the vertices of all the parallelograms in \(S\), impose \(N_4\) independent conditions on the latter set of variable.

**Step 4.** The requirement to pass through the points \(p_i', i = 1, ..., r',\) and \(p_{i1}, p_{i2}', i = 1, ..., r''\), applied to the family of the real rational curves \(C\) with the given refined tropical limit, can be formalized as follows.

Let a vertex \(x_i'', 1 \leq i \leq s''\), of \(A\) be dual to a triangle \(\Delta_k\). Then the equations \(f(p_{i1}') = f(p_i') = 0\), \(f_k(\xi_i'''. \eta''') = f'_k(\xi'_{i''}. \eta'_{i''}) = 0\) with \(\xi_i''', \eta''''\) introduced in (2.2). By Riemann-Roch (see Lemma 2.4) the two latter equations together with (3.5),

\[\footnote{Recall that \(N_4\) is the number of parallelograms in \(S\).} \]

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restricted to $\Delta_k$, impose $|\Delta_k \cap \mathbb{Z}^2| - 1$ independent conditions on the variables $c_{\alpha \beta}$, $(\alpha, \beta) \in \Delta_k \setminus \{v_0\}$, and thus, one can conclude with

$$
\begin{align*}
\frac{a_{\alpha_1 \beta_1} + c_{\alpha_1 \beta_1}}{a_{\alpha_2 \beta_2} + c_{\alpha_2 \beta_2}} &= O(t), \\
\frac{a_{\alpha_2 \beta_2} + c_{\alpha_2 \beta_2}}{a_{\alpha_3 \beta_3} + c_{\alpha_3 \beta_3}} &= O(t),
\end{align*}
$$

where $(\alpha_1, \beta_1, (\alpha_2, \beta_2), (\alpha_3, \beta_3)$ are the vertices of $\Delta_k$.

Let a point $x_i^\prime$, $1 \leq i \leq r'$, belong to an extended edge $\varepsilon$ of $A$, whose endpoints are dual to triangles $\Delta_k, \Delta_l$ (the fact that $\varepsilon$ has two endpoints is established in Step 4 of the Proof of Proposition 2.1). Then the condition $f(p_i^\prime) = 0$ can be expressed by an equation of type (5.4.26) in [13], section 5.4, which, in our notation, is interpreted as

$$
\begin{align*}
c_{\alpha_1 \beta_1} + \varphi_1(\xi_i^\prime, \eta_i^\prime) c_{\alpha_2 \beta_2} &= \varphi_2(\xi_i^\prime, \eta_i^\prime) \left( \varphi_3(\xi_i^\prime, \eta_i^\prime, a_{\alpha_2 \beta_2}, a')^{1/m} \right) t^{p(1 + \Phi) + \Psi},
\end{align*}
$$

where $m$ is the length of the edge $\sigma$ of $\Delta_k$, which is dual to $\varepsilon$, the points $(\alpha_1, \beta_1)$ and $(\alpha_2, \beta_2)$ are the vertices of $\sigma$, the non-zero real numbers $\xi_i^\prime, \eta_i^\prime$ are defined by (2.1), $\varphi_1, \varphi_2, \varphi_3$ are monomials of the corresponding variables, $a'$ is a linear combination of certain coefficients of $f_k$ (or $f_l$) such that $\tilde{a}$ does not vanish$^6$, $p$ is a positive integer, and, at last, $\Phi$ and $\Psi$ are certain real analytic functions of the variables $a_{ij}, c_{ij}, t$, whose terms either contain $t$ to a positive power, or are at least quadratic in $c_{ij}$. We observe that equation (3.7) is split into $m$ equations such that, for an odd $m$, only one of them is real, and, for an even $m$, not one nor two of them are real.

Assume now that a point $x_i^\prime$, $s'' < i \leq r''$, belongs to an extended edge $\varepsilon$ of $A$ with endpoints dual to triangles $\Delta_k, \Delta_l$ (the case of a semi-infinite edge $\varepsilon$ can be treated in the same way). Again by formula (5.4.26) of [13], section 5.4, the conditions $f(p_i^\prime) = f(p_i^\prime) = 0$ reduce to a system of equations, which can be written as

$$
\begin{align*}
c_{\alpha_1 \beta_1} + \varphi_1(\xi_i^\prime, \eta_i^\prime) c_{\alpha_2 \beta_2} &= \varphi_2(\xi_i^\prime, \eta_i^\prime) \left( \varphi_3(\xi_i^\prime, \eta_i^\prime, a_{\alpha_2 \beta_2}, a')^{1/m} \right) t^{p(1 + \Phi) + \Psi},
\end{align*}
$$

where $2m$ is the length of the edge $\sigma$ of $\Delta_k$, which is dual to $\varepsilon$, the point $(\alpha_1, \beta_1)$ is a vertex of $\sigma$, different from $v_0$, and $(\alpha_2, \beta_2) = v(\sigma)$ is the middle point of $\sigma$, the non-zero numbers $\xi_i^\prime, \eta_i^\prime$ are defined by (2.2), $\varphi_1, \varphi_2, \varphi_3$ are monomials of the corresponding variables with real coefficients, $a'$ is a linear combination of certain coefficients of $f_k$ (or $f_l$) such that $\tilde{a} \neq 0$, $p$ is a positive integer, and, at last, $\Phi$ and $\Psi$ are certain real analytic functions of the variables $a_{ij}, c_{ij}, t$, whose terms either contain $t$ to a positive power, or are at least quadratic in $c_{ij}$. We observe that system (3.8) is split into $m$ systems invariant with respect to the complex conjugation.

Take equations (3.8), restricted to the vertices of all the parallelograms in $S$, and equations (3.9) for all $i = 1, ..., s''$, and then append one real equation (3.7) for each $i = 1, ..., r'$, and one ConJ-invariant system (3.8) for each $i = s'' + 1, ..., r''$. Observe that the linearizations of the $|V(S)| - 1 + |V''(S)|$ chosen equations impose independent conditions on the variables $c_{ij}, (i, j) \in V(S) \cup V''(S) \setminus \{v_0\}$. Indeed, using these linearizations and following the algorithm, which restores the tropical curve $A$ from the graph $E_0$ (see Step 5 in the proof of Theorem 3.1), we can determine all the aforementioned variables.

4. Counting the Welschinger invariant via lattice paths in the Newton polygon

4.1. Reconstruction of tropical curve passing through generic points on a straight line. Following the ideas of [8, 9], we intend to choose a specific configuration of the points $x_i^\prime, ..., x_i^{r'}$, and $x_i^\prime, ..., x_i^{r''},$ for which all the subdivisions $S$ of $\Delta$, dual to the real rational tropical curves mentioned in Theorem 3.1 can be obtained in a simple combinatorial algorithm. In this section we describe a reconstruction of a real rational tropical curve.

---

$^6$The coefficient $\tilde{a}$ appears as $a_{01}''$ in formula (5.4.26) of [13].
We introduce an orthogonal system of coordinates $\lambda = \alpha x + \beta y$, $\mu = \beta x - \alpha y$ in $\mathbb{R}^2$ with generic $\alpha, \beta \in \mathbb{Q}$. The coordinate line $\Lambda := \{\mu = 0\}$ is then not orthogonal to any of the segments joining integral points in $\Delta$.

We pick $r' + r''$ distinct points $y_1, \ldots, y_{r'} + y_{r''}$ in $\Lambda \cap \mathbb{Q}^2$ one by one so that $0 < \lambda(y_i)$ and $\lambda(y_i) \ll \lambda(y_{i+1})$ for all $i \geq 1$.

(4.1) $0 < \lambda(y_i)$ and $\lambda(y_i) \ll \lambda(y_{i+1})$ for all $i \geq 1$, then move them slightly in their neighborhoods in $\mathbb{Q}^2$, obtaining a $\Delta$-generic configuration $\mathfrak{y}$. At last, we declare $r'$ points in $\mathfrak{y}$ as $x'_1, \ldots, x'_{r'}$ and the remaining $r''$ points as $x''_1, \ldots, x''_{r''}$.

Remark 4.1. In fact, one could leave the set $\mathfrak{y}$ on the line $\Lambda$, and satisfy the generality conditions required in Theorem 3.1. In the construction of a real rational tropical curve through $x' \cup x''$, presented below, we suppose that the fixed points lie on the line $\Lambda$, since a small variation of the configuration does not affect the construction.

Let $A$ be a real rational tropical curve of type $(r', r'', s'')$ with Newton polygon $\Delta$, which passes through $x' \cup x''$ so that some $s''$ points among $x''_1, \ldots, x''_{r''}$ are its vertices. As established in Step 3 of the proof of Proposition 2.1, the other points in $x' \cup x''$ are not vertices of $A$.

The curve $A$ can be restored in finitely many ways, and we shall enumerate them.

Consider the germs of edges of $A$ passing through $x' \cup x''$. We have a finite choice of these germs which should be orthogonal to some segments with integral endpoints in $\Delta$. Some $s''$ of the points $x'_1, \ldots, x'_{r'}$, should be trivalent vertices of $A$. In view of the general position of the line $\Lambda$, the germs do not lie on $\Lambda$. Furthermore, we orient the germs or their halves to start at the points $x'_1, \ldots, x'_{r'}, x''_1, \ldots, x''_{r''}$ as shown, for example, on Figure 4(a) (cf. [9], Lemma 4.17 and Figure 11).

Given a combinatorial type of $A$ and a combinatorial type of the pair $(A, x' \cup x'')$, the configuration $x' \cup x''$ determines $A$ uniquely. In particular, some extended edges of $A$ arising from the given edge germs at $x' \cup x''$ must intersect. For such an intersection point we have two possibilities: (i) it is either a four-valent vertex of $A$ (see Figure 4(b)), or (ii) it is a trivalent vertex of $A$ (see Figure 4(c)). In the latter case, another extended edge of $A$ starts at the intersection point, which we orient as shown in Figure 4(c). For this edge germ we again have a finite choice, determined by the orthogonality to a segment with integral endpoints in $\Delta$ and by the equilibrium relation (see, for instance, [11], formula (10) in section 3), which, in particular, says that the directing vector of the new edge germ is a linear combination of the directing vectors of the two edges, coming to the given vertex of $A$, with positive coefficients.
We proceed in the same manner with the new collection of edge germs of $A$, and reconstruct the whole tropical curve $A$. In fact, this procedure coincides with the reconstruction of $A$ in the sequence of graphs $E_0, E_1, \ldots$ as appears in Step 5 of the proof of Theorem 3.1.

We make a few observations:

(O1) The directing vectors of the extended edges of $A$, which do not intersect with $\mathbb{F}' \cup \mathbb{F}''$, and have an endpoint in the half plane $\mu > 0$ (or $\mu < 0$), have a positive (resp., negative) $\mu$-component, and hence $A \cap \Lambda = \mathbb{F}' \cup \mathbb{F}''$. This means that the components of $\Lambda \setminus (\mathbb{F}' \cup \mathbb{F}'')$ lie entirely in the components of $\mathbb{R}^2 \setminus A$, and are dual to some $r' + r'' + 1$ integral points in $\Delta$ ordered by the linear function $\lambda$ in the same way as the components of $\Lambda \setminus (\mathbb{F}' \cup \mathbb{F}'')$. Furthermore, the semi-infinite components of $\Lambda \setminus (\mathbb{F}' \cup \mathbb{F}'')$ correspond to the (integral) points of $\Delta$, where $\lambda_{\Delta}$ takes its maximal and minimal values.

(O2) If $x''_i$ is a trivalent vertex of $A$, and $\omega_1, \omega_2, \omega_3$ are the integral points in $\Delta$, which correspond to the components of $\mathbb{R}^2 \setminus A$ adjacent to $x''_i$, so that the first two components intersect $\Lambda$, then $\lambda(\omega_1) > \lambda(\omega_2) > \lambda(\omega_3)$.

(O3) Let $\mathcal{R}$ be the rays in the half-plane $\mu > 0$, starting at $\mathbb{F}' \cup \mathbb{F}''$ and at the trivalent vertices of $A$, and generated by the oriented edge germs of $A$. We introduce a partial order in $\mathcal{R}$ as follows. The rays, starting at $\mathbb{F}' \cup \mathbb{F}''$, one orders by successive intersections with a line close and parallel to $\Lambda$ and oriented by $\lambda$. If a ray $l_0 \in \mathcal{R}$ starts at a trivalent vertex, where rays $l', l'' \in \mathcal{R}$ merge, and $l' > l''$, then we assume $l_0 > l$ for all $l \in \mathcal{R}$ such that $l' > l$, and $l > l_0$ for all $l \in \mathcal{R}$ such that $l > l'$. Then relation (4.1) yields that, if some rays $l_1, l_2 \in \mathcal{R}$ cross a ray $l_3 \in \mathcal{R}$, then $l_1$ and $l_2$ are ordered, say, $l_1 > l_2$, and, respectively, $\mu(l_1 \cap l_3) > \mu(l_2 \cap l_3)$. A similar claim holds for rays in the half-plane $\mu < 0$.

4.2. Reconstruction of subdivisions dual to real rational tropical curves passing through generic points on a straight line. Observations O1, O2, O3 allow us to convert the preceding construction of $A$ into the dual language of subdivisions of $\Delta$. The dual construction is performed as follows:

Step 1. The linear function $\lambda$ defines a linear order of the set $\Delta \cap \mathbb{Z}^2$. We choose $r' + r'' + 1$ distinct successive points $v_i \in \Delta \cap \mathbb{Z}^2$, $i = 0, \ldots, r' + r''$, such that $\lambda(v_0) = \min \lambda(\Delta)$ and $\lambda(v_{r' + r''}) = \max \lambda(\Delta)$. These points are dual to the components of $\mathbb{R}^2 \setminus A$ containing the components of $\Lambda \setminus (\mathbb{F}' \cup \mathbb{F}'')$. We then connect the points $v_0, \ldots, v_{r' + r''}$ by segments $\sigma_i = [v_{i-1}, v_i]$, $i = 1, \ldots, r' + r''$, obtaining a broken line, called a lattice path (see Figure 4.1(a)).
Step 2. The segments $\sigma_1, \ldots, \sigma_{r'+r''}$ naturally correspond to the points $y_1, \ldots, y_{r'+r''} \in \Lambda$, respectively. We take disjoint sets $V, W \subset \{1, 2, \ldots, r'+r''\}$ with $|V| = s'$, $|W| = r'$, and declare

- the points $y_i, i \in W$, to be (somehow ordered) points $x'_1, \ldots, x'_r$,
- the points $y_i, i \notin W$, to be (somehow ordered) points $x''_1, \ldots, x''_{r''}$,
- the points $y_i, i \in V$, to be trivalent vertices of $A$, respectively dual to the lattice triangles in $\Delta$.

That is, the segments $\sigma_j, j \notin V$, are dual to some edges of $A$, whose germs pass through the corresponding points $y_j$, and are orthogonal to $\sigma_j, j \notin V$, respectively (see Figure 4(a)). In turn, if $j \in V$, then we pick a point $\hat{v}_j \in \Delta \cap \mathbb{Z}^2 \setminus \sigma_j$ such that $\lambda(v_j-1) < \lambda(\hat{v}_j) < \lambda(v_j)$, and obtain a triangle $T_j = \text{Conv}(v_j-1, \hat{v}_j, v_j)$ dual to $y_j$ (see Figure 4(b), where $T_j$ is designated by a shadow). The sides of $T_j$ are dual to the edges of $A$, whose germs start at $y_j$, in the normal directions oriented outside of $T_j$.

In the absence of suitable points $\hat{v}_j$ for $j \in V$, or in the case of an odd length edge $\sigma_i$ for $i \notin V \cup W$, we stop the construction and say that the lattice path $\bigcup \sigma_i$ and the sets $V, W$ are inconsistent.

Step 3. Put $S_0 = \bigcup_1 \sigma_i \cup \bigcup_j T_j$. This is a partial subdivision of $\Delta$, and we extend it in the following inductive process. Let $S_k, k \geq 0$, be a contractible union of some lattice segments, triangles and parallelograms, such that $S_k \supset S_0, S_k \neq \Delta$, and, for each connected component $\Delta'$ of $\Delta \setminus S_k$, the intersection $\delta = S_k \cap \partial \Delta'$ is a lattice path in $\Delta$ along which the function $\lambda$ is strongly monotone.

If $\text{Conv}(\delta) \cap \Delta' = \emptyset$ we stop the construction and say that the sequence $S_0, ..., S_k$ is inconsistent.

If $\text{Conv}(\delta) \cap \Delta' \neq \emptyset$, there a pair of successive segments $\sigma'$ and $\sigma''$ in $\delta$ such that $\text{Conv}(\sigma' \cup \sigma'') \cap \Delta' \neq \emptyset$, and $\lambda(\sigma' \cap \sigma'')$ is minimal among all such pairs. The pair $\sigma', \sigma''$ corresponds to a pair of germs of edges of $A$, generating rays which intersect each other, and, moreover, along the partial order of rays as defined in (O3) above, this pair of intersecting rays is minimal among all pairs of intersecting rays which can be constructed in the current stage.

Then we define $S_{k+1}$ by adding to $S_k$ either the triangle $\text{Conv}(\sigma' \cup \sigma'')$, or the parallelogram, built on $\sigma', \sigma''$. For $A$ this means adding a new tri- or four-valent vertex as shown in Figure 4(b,c).

If the polygon, added to $S_k$, is not contained in the closure of $\Delta'$, we stop the construction and call the sequence $S_0, ..., S_{k+1}$ inconsistent.

Step 4. Consider a sequence $S_0, ..., S_k$ which ends up with $S_k = \Delta$. In parallel, the dual construction gives us a graph $A \subset \mathbb{R}^2$. We assign corresponding weights to the edges of $A$ and obtain a weighted rational graph, which by construction satisfies the equilibrium condition at each vertex (see [11], formula (10) in section 3), and thus, by [11], Theorem 3.6, is a tropical curve with Newton polygon $\Delta$.

We call the obtained subdivision consistent if the tropical curve $A$ is real rational of type $(r', r'', s')$. For all the consistent subdivisions of $\Delta$, obtained in the above algorithm, starting with the initial data $v_0, ..., v_{r'+r''}, V, W$, we sum up the Welschinger numbers and denote the result by $W_{\Delta, r', r''}(v_0, ..., v_{r'+r''} | V, W)$. Moreover, we can get rid of the dependence on $V$, noticing that the non-zero Welschinger numbers come only from the sets $V$, defined as follows: $j \in V$ if $j \notin W$ and $|\sigma_j|$ is odd.

Since all the real rational tropical curves passing through $\mathbb{P} \cup \mathbb{P}'$ are obtained in the above construction, we conclude

**Theorem 4.2.** Given a linear function $\lambda(x, y) = \alpha x + \beta y$ with generic $\alpha, \beta \in \mathbb{Q}$, and a set $W \subset \{1, ..., r'+r''\}$ with $|W| = r'$, one has

$$W_{r', r''}(\Sigma, D) = \sum W_{\Delta, r', r''}(v_0, ..., v_{r'+r''} | W),$$

where the sum ranges over all sequences of points $v_0, ..., v_{r'+r''} \in \Delta \cap \mathbb{Z}^2$ such that

$$\min \lambda(\Delta) = \lambda(v_0) < ... < \lambda(v_{r'+r''}) = \max \lambda(\Delta).$$

**4.3. Example:** real rational plane cubics. Let the non-negative integers $r'$ and $r''$ satisfy $r' + 2r'' = 8$. Consider the pencil of real plane cubics passing through $r'$ generic real points and through $r''$ generic pairs of imaginary conjugate points. Clearly, it has $r' + 1$ real base points. Denote by $n_+$ and $n_-$ the numbers of real cubics in the pencil, which have a non-solitary real node or a solitary real node, respectively. Blowing up the $r' + 1$ real base points of the pencil and integrating with respect to the Euler characteristic over the blown-up real projective plane, we obtain (cf. [16] [17])

$$W_{r', r''}(\mathbb{P}^2, 3L) = n_+ - n_- = r' = 8 - 2r''.$$
Figure 6. Tropical count of $W_{r''}(\mathbb{P}^2, 3L)$
Take a linear function
\[ \lambda(x, y) = x - \beta y \]
with \( 0 < \beta \ll 1 \), and put \( \mathcal{W} = \{1, \ldots, r'\} \). Figure 3 presents all consistent subdivisions of the Newton triangle \( \Delta = \text{Conv}\{((0, 0), (0, 3), (3, 0))\} \), constructed along the procedure of section 4.2 and their Welschinger numbers, confirming formula (4.2).

Notice that \( W_4(P^2, 3) = 0 \). In fact, there is a configuration of four pairs of imaginary conjugate points in \( P^2 \), which defines a pencil of non-singular real cubics. We present the following example:
\[ F := \alpha(x^3 + x^2y + x^2z) + \beta(x_0 x_2 + x_1 y + x_0 z) = 0. \]
By a routine computation, the system of equations for singular points
\[
\begin{align*}
F_{x_0} &= 3\alpha x_0^2 + \alpha x_1^2 + 2\beta x_0 x_2 = 0, \\
F_{x_1} &= 3\alpha x_1^2 + 2\beta x_1 x_2 = 0, \\
F_{x_2} &= 2\alpha x_0 x_2 + \beta x_0^2 + \beta x_1^2 + 3\beta x_2 = 0
\end{align*}
\]
can be reduced to a relation for \( \alpha \) and \( \beta \), which has no real solutions except for \( \beta = 0 \), which leaves the only real singular curve
\[ x^3 + x^2y + x^2z = 0. \]
The latter curve has two imaginary nodes. Hence, by a small variation of the pencil, we get rid of the real singular curves.

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