Conformal relations and Hamiltonian formulation of fourth–order gravity

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Abstract

The conformal equivalence of fourth–order gravity following from a non–linear Lagrangian $L(R)$ to theories of other types is widely known, here we report on a new conformal equivalence of these theories to theories of the same type but with different Lagrangian.

For a quantization of fourth-order theories one needs a Hamiltonian formulation of them. One of the possibilities to do so goes back to Ostrogradski in 1850. Here we present another possibility: A Hamiltonian $H$ different from Ostrogradski’s one is discussed for the Lagrangian $L = L(q, \dot{q}, \ddot{q})$, where $\partial^2 L / \partial (\ddot{q})^2 \neq 0$. We add a suitable divergence to $L$ and insert $a = q$ and $b = \dot{q}$. Contrary to other approaches no constraint is needed because $\ddot{a} = \ddot{b}$ is one of the canonical equations. Another canonical equation becomes equivalent to the fourth–order Euler–Lagrange equation of $L$.

Finally, we discuss the stability properties of cosmological models within fourth–order gravity.

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1 Introduction

The purpose of this review is to present some of the points which have been discussed in recent years in connection with cosmological models following from fourth–order theories of gravity.

Usually, one applies some version of a conformal equivalence theorem between these theories and Einstein’s theory with additional scalar fields. But there exists also another type of conformal equivalence: In [1], we have shown that for non–linear \( L = L(R) \), \( G = dL/dR \neq 0 \) the Lagrangians \( L \) and \( \hat{L}(\hat{R}) \) with \( \hat{L} = 2R/G^3 - 3L/G^4 \), \( \hat{g}_{ij} = G^2 g_{ij} \) and \( \hat{R} = 3R/G^2 - 4L/G^3 \) give conformally equivalent fourth–order field equations being dual to each other. The proof represents a new application of the fact that the operator \( \Box - \frac{R}{6} \) is conformally invariant.

The Ostrogradski approach [2] to find a Hamiltonian formulation for a higher–order theory is the most famous (see [2-9]) but not the only method. In sect. 2 we present an alternative Hamiltonian formalism for fourth–order theories. It systematizes what has been sporadically done in the literature for special examples.

Sect. 3 deals with fourth–order gravity following from a non–linear Lagrangian \( L(R) \). The instability of these theories from the point of view of the Cauchy problem is discussed.

Sect. 2 applies to general higher–order theories, sect. 3 to gravity, and both are combined to fourth–order cosmology in sect. 4. We discuss known exact solutions under the stability criteria mentioned before.

In the final sect. 5 we list some open problems for future research.

The rest of this introduction shortly reviews papers on higher–order theories. Eliezer and Woodard [2] and Jaen, Llosa, Molina and Vives [3] represent standard papers for the generalization of the Ostrogradski approach to non–local systems and to systems with constraints (see also [4-6]) applying Dirac’s approach.

Let the Lagrangian \( L \) be a function of the vector \( q_\alpha \) and its first \( n \) temporal derivatives \( \dot{q}_\alpha, \ddot{q}_\alpha, \ldots, \dot{q}_\alpha^{(n)} \). The Hessian is

\[
H_{\alpha\beta} = \frac{\partial^2 L}{\partial q_\alpha^{(n)} \partial q_\beta^{(n)}} \quad (1.1)
\]

and the non–vanishing of its determinant defines the regularity of \( L \). In the following we do not write the subscript \( \alpha \); one can think of \( q \) as being a point particle in a (one– or higher–dimensional) space. In the Ostrogradski approach, \( Q = \dot{q} \) is taken as additional position variable. This leads to an ambivalence of the procedure, because it is not trivial to see at which places
\( \dot{q} \) has to be replaced with \( Q \), cf. [7]. We prevent this ambivalence in our alternative Hamiltonian, cf. sct. 2, by putting \( Q = \ddot{q} \).

Ref. [8] discusses higher–order field theories. The problem is the lack of an energy bound, typically two kinds of oscillators with different signs of energy exist. Usually, one restricts the space of initial conditions to prevent negative energy solutions. The authors of ref. [8] redefine the energy analogous to the Timoshenko model, so one gets a positive mechanical energy inspite of an indefinite Ostrogradski Hamiltonian, they write: “An appealing aspect of this approach is the absence of any constraint.” So it has this property in common with our approach sct. 2, but it is otherwise a different one.

Another standard procedure [3,8,9] for dealing with higher–order Lagrangians is to consider them as a sequence in a parameter \( \epsilon \), so one can break the Euler–Lagrange–equation into a sequence of second order ones. In [9] this is called “reduction of higher–order Lagrangians by a formal power series in an ordering parameter.” [9] deals also with the Lie–Königs theorem: a local Hamiltonian is always possible, and they consider some global questions.

Let us show the famous counter–example [10]: it is an example of a second order system not following from a Lagrangian.

\[
\ddot{x} + \dot{y} = 0 \quad \dot{y} + y + \epsilon \dot{x} = 0
\]

It follows from the Lagrangian

\[
L = \frac{1}{2} [\dot{y}^2 - y^2 + \epsilon (x\dot{y} - \dot{x}y - \dot{x}^2)]
\]

for \( \epsilon \neq 0 \) and has no Lagrangian otherwise. We mention this example to show that the following recipe need not to work always. Recipe for higher–order theories: “Write down the Euler–Lagrange equations, break them into a sequence of second order ones by introducing further coordinates. Find Lagrangians for these second order equations.”

A powerful method for dealing with a classical Lagrangian

\[
L = \frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j - V(q)
\]

is given in [11]. The Euler–Lagrange equation to Lagrangian (1.2) reads

\[
\ddot{q}^i + \Gamma^i_{jk} \dot{q}^j \dot{q}^k = -g^{ik} V_k
\]

and is fulfilled for geodesics in the Jacobi–metric

\[
\hat{g}_{ij} = (E - V) g_{ij}
\]
Remark: For constant potentials $V$ this is trivial, for non-constant potentials the constant $E$ must be correctly chosen to get the result, for $E = V$ it breaks down, of course.

Stelle [12] cites Ostrogradski [2] but uses other methods to extract different spin modes for fourth–order gravity. In [13], a regular reduction of fourth–order gravity similar to the method with an ordering parameter mentioned above has been proposed as follows: In the Newtonian limit one has

$$\Delta \Phi + \beta \Delta \Delta \Phi = 4\pi G \rho,$$

then one restricts to solutions which can be expanded into powers of the coupling parameter $\beta$. Argument: If $\beta$ is a parameter, this is well justified, if it is a universal constant, then this restriction is less satisfying. Comment: This restriction excludes the usual Yukawa–like potential $\frac{1}{r} \exp(-r/\sqrt{\beta})$, so one may doubt whether this method gives the right solutions. Let us further mention ref. [14] for non–local gravitational Lagrangians like $L = R \Box^{-1} R + \Lambda$ in two dimensions and refs. [15-18] for the linearized $R^2$-theory.

To facilitate the reading of sect. 2, we pick up the example eq. (5) of [5]:

$$\tilde{L} = [\ddot{q}^2 + 4\dot{q}q^2 + 4\dot{q}^4]e^{3q}$$  \hspace{1cm} (1.3)

The equation of motion is [5, eq. (6)]

$$2q^{(4)} + 12\dot{q}q^{(3)} + 9(q)q^2 + 18\dot{q}^2\ddot{q} = 0$$  \hspace{1cm} (1.4)

A good check of the validity of the formalism is the following: For a constant $c > 0$ and $\dot{q} > 0$, each solution of

$$\ddot{q} = -2\dot{q}^2 + c\sqrt{\dot{q}}$$

is also a solution of eq. (1.4).

By adding a divergence to eq. (1.3) one gets $L = (\ddot{q})^2 e^{3q}$. The alternative formalism requires to use $q^1 = q$ and $q^2 = \dot{q}$ as new coordinates. So we get

$$L = (q^2)^2 \exp(3q^1)$$  \hspace{1cm} (1.5)

Eq. (1.5) represents the ultralocal Lagrangian mentioned in [5]. It is correctly stated in [5], that the alternative formalism does not work for this version eq. (1.5) of the system. This clarifies that the addition of a divergence to a higher–order Lagrangian sometimes influences the applicability of the alternative Hamiltonian formalism. So one should add a “suitable” total derivative to the Lagrangian. “Suitable” means, that the space of solutions
is the same at both sides, and that the relation between the various coordinates is ensured without imposing any constraints. It turns out, that the Lagrangian $\hat{L}$ differing from $\tilde{L}$, eq. (1.3) by a divergence only

$$\hat{L} = -[\dddot{q}^2 + 6\dddot{q}\dddot{q}^2 + 2\dddot{q}\dddot{q}^{(3)}]e^{3q}$$

(1.6)
does the job. Of course, the variations of $L$, $\tilde{L}$, and $\hat{L}$ with respect to $q$ all give the same equation of motion (1.4). But only in version (1.6) the alternative formalism (insertion of the equation $Q = \dddot{q}$ and then apply the usual formulas of classical mechanics - to avoid ambiguities with the square–sign we have replaced $q^1$ by $q$ and $q^2$ by $Q$) leads correctly to the Hamiltonian [5, eq. (7)]:

$$H = -\frac{1}{2}(pP - 3P^2Q)e^{-3q} + Q^2e^{3q}$$

(1.7)
It essentially differs from the Ostrogradski approach because terms only linear in the momenta do not appear. The integrability condition $Q = \dddot{q}$ and the equation of motion (1.4) both follow from the canonical equations of eq. (1.7); no constraint is necessary to get this.

Wagoner [19] showed conformal relations between different types of second order theories (Einstein, Brans-Dicke), whereas refs. [20-23] discussed such relations between fourth–order and second order theories. Refs. [19-23] also discuss which of the conformally equivalent metrics can be considered physical.

2 The alternative Hamiltonian formalism

Let us consider the Lagrangian

$$L = L(q, \dot{q}, \dddot{q})$$

(2.1)
for a point particle $q(t)$, a dot denoting $\frac{d}{dt}$ and

$$q^{(n)} = \frac{d^n q}{dt^n}$$

The corresponding Euler–Lagrange equation reads

$$0 = \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \dddot{q}}$$

(2.2)
We suppose this Lagrangian to be non-degenerated, i.e., $L$ is non-linear in $\ddot{q}$. The highest-order term of eq. (2.2) is

$$q^{(4)} \frac{\partial^2 L}{\partial (\dot{q})^2}$$

therefore, non-degeneracy (= regularity, cf. eq. (1.1)) is equivalent to require that eq. (2.2) is of fourth order, i.e.

$$\frac{\partial^2 L}{\partial (\dot{q})^2} \neq 0$$

(If $q$ is a vector consisting of $m$ real components then this condition is to be written as Hessian determinant.)

If we add the divergence $\frac{d}{dt} G(q, \dot{q})$ to $L$, we do not alter the Euler–Lagrange equation (2.2). Furthermore, the expression $\frac{d}{dt} G$ is linear in $\ddot{q}$, and so its addition to $L$ does not influence the condition of non-degeneracy. The addition of such a divergence can therefore simply absorbed by a suitable redefinition of $L$.

In the next two subsections we add a special and a more general divergence to get a Hamiltonian formulation different from Ostrogradski’s one. In Kasper [4] a similar consideration has been made at the Lagrangian’s level. Subsection 2.1 represents only a special case of subsection 2.2, but we write it down, because it has the advantage that the formulas can be given explicitly, and so the formalism becomes more transparent.

## 2.1 A special divergence

The addition of the following divergence is no more done by a redefinition of $L$

$$L_{\text{div}} = \frac{d}{dt}[f(q)\dot{q}\ddot{q}] , \quad f(q) \neq 0$$

and we consider $\hat{L} = L + L_{\text{div}}$. The Euler–Lagrange equation is again eq. (2.2). Using

$$f'(q) \equiv \frac{df}{dq}$$

we get

$$\hat{L} = L + f'(q)q\ddot{q} + f(q)[(\dot{q})^2 + \dot{q}q^{(3)}]$$

which contains third derivatives of $q$.

We introduce new coordinates

$$a = q , \quad b = \ddot{q}$$

(2.5)
(In the Ostrogradski approach, the second coordinate is $\dot{q}$, instead.) It is obvious that there is exactly this one compatibility condition:

$$\ddot{a} = b \quad (2.6)$$

Let us insert eq. (2.5) into eq. (2.4). This insertion becomes unique by the additional requirement that $\hat{L}$ does not depend on second and higher derivatives of $a$ and $b$, i.e.,

$$\hat{L} = \hat{L}(a, \dot{a}, b, \dot{b})$$

giving

$$\hat{L} = L(a, \dot{a}, b) + f'(a)\dot{a}^2 b + f(a)[b^2 + \dot{a}\dot{b}] \quad (2.7)$$

(In the Ostrogradski approach, there remains an ambivalence which of the $\dot{q}$ in the original Lagrangian is to be interpreted as second coordinate and which as time derivative of the first one.)

The momenta are defined as in classical mechanics by

$$p_a = \frac{\partial \hat{L}}{\partial \dot{a}}, \quad p_b = \frac{\partial \hat{L}}{\partial \dot{b}} \quad (2.8)$$

(In the Ostrogradski approach, an additional term is necessary.) Inserting eq. (2.7) into eqs. (2.8) we get

$$p_a = \frac{\partial L}{\partial \dot{a}} + 2f'(a)\dot{a}b + f(a)\dot{b} \quad (2.9)$$

and

$$p_b = f(a)\dot{a} \quad (2.10)$$

Because of $f(a) \neq 0$, cf. eq. (2.3), we can invert eq. (2.10) to

$$\dot{a} = \frac{p_b}{f(a)} \quad (2.11)$$

Inserting eq. (2.11) into eq. (2.9) and dividing by $f(a)$ we get

$$\dot{b} = \frac{1}{f(a)}[p_a - \frac{\partial L}{\partial \dot{a}} - 2f'(a)b\frac{p_b}{f(a)}] \quad (2.12)$$

It is instructive to make a more general consideration: The question, whether eqs. (2.9, 10) can be inverted to $\dot{a}$, $\dot{b}$, can be answered by calculating the Jacobian (by the way: Carl Gustav Jacob Jacobi was born in Potsdam 10th December 1804.)

$$J = \frac{\partial (p_a, p_b)}{\partial (\dot{a}, \dot{b})} = \frac{\partial p_a \partial p_b}{\partial \dot{a} \partial \dot{b}} - \frac{\partial p_a \partial p_b}{\partial \dot{b} \partial \dot{a}} \quad (2.13)$$
We insert eqs. (2.9, 10) into eq. (2.13) and get
\[ J = - [f(a)]^2 \] (2.14)

Because of \( f \neq 0 \) one has also \( J \neq 0 \) and the inversion is possible. This more general consideration gave the additional information that the Jacobian is always negative; this is one of the several possibilities to give the phrase “fourth–order theories are unstable” a concrete meaning.

We define the Hamiltonian \( H \) as usual by
\[ H = \dot{a} p_a + \dot{b} p_b - \hat{L} \]
i.e., with eq. (2.7) we get
\[ H = \dot{a} p_a + \dot{b} p_b - L - f'(a) \dot{a}^2 b - f(a) [b^2 + \dot{a} \dot{b}] \] (2.15)

Here we insert \( \dot{a} \) according to eq. (2.11) and get the Hamiltonian \( H = H(a, p_a, b, p_b) \). The factor of \( \dot{b} \) in \( H \) automatically vanishes, so we do not need eq. (2.12). The canonical equations read
\[ \frac{\partial H}{\partial p_a} = \dot{a} \] (2.16)

further
\[ \frac{\partial H}{\partial p_b} = \dot{b} \] (2.17)

and
\[ \frac{\partial H}{\partial a} = - \dot{p}_a \] (2.18)

and
\[ \frac{\partial H}{\partial b} = - \dot{p}_b \] (2.19)

The whole procedure is intended to give the following results: The Hamiltonian \( H \) shall be considered to be a usual Hamiltonian for two interacting point particles \( a(t) \) and \( b(t) \). One of the canonical equations shall be equivalent to the compatibility condition eq. (2.6) and another one shall be equivalent to the original Euler–Lagrange equation (2.2), whereas the two remaining canonical equations are used to eliminate the momenta \( p_a \) and \( p_b \) from the system. The next step is to find those Lagrangians \( L \) which make this procedure work. From eqs. (2.15) and (2.11) we get
\[ H = \frac{p_a p_b}{f(a)} - L(a, \frac{p_b}{f(a)}, b) - \frac{p_b^2 f'(a) b}{f(a)^2} - f(a) b^2 \] (2.20)
In this form, eq. (2.16) coincides with eq. (2.11) and (2.17) with (2.12). So we may use eqs. (2.9, 10) in the following, because they are equivalent to eqs. (2.11, 12).

Now, we use eqs. (2.19), cancel $p_b$ by use of eq. (2.10) and get

$$0 = \frac{\partial L}{\partial b} + 2bf(a) - \ddot{a}f(a) \tag{2.21}$$

In order that the compatibility relation eq. (2.6) follows automatically from eq. (2.21), one has to ensure that $f(a) \neq 0$ (which is already assumed) and that

$$0 = \frac{\partial L}{\partial b} + bf(a)$$

identically takes place. The condition of non–degeneracy,

$$\frac{\partial^2 L}{\partial b^2} \neq 0$$

is then also automatically fulfilled. One has the following possible Lagrangian

$$L = -\frac{1}{2}f(a)b^2 + K(a, \dot{a}) \tag{2.22}$$

where $K$ is an arbitrary function, but, for simplicity, we put $K = 0$.

The last of the four canonical equations to be used is eq. (2.18) reading now with eqs. (2.9, 10, 20)

$$0 = f\dot{b} + 2f'\dot{a}\dot{b} + \frac{3}{2}f''b^2 + f''\dot{a}^2b \tag{2.23}$$

If we insert here eq. (2.5) we get exactly the same as the Euler–Lagrange equation (2.2) following from the Lagrangian

$$L = -\frac{1}{2}f(q)(\dot{q})^2 \tag{2.24}$$

Result: For every Lagrangian of type (2.1) which can be brought into type (2.24) with $f \neq 0$ the addition of the divergence (2.3) makes it possible to apply the new coordinates (2.5). Then the system becomes equivalent to a classical Hamiltonian of two particles, and the relation (2.6) between them follows without imposing an additional constraint.
2.2 A general divergence

In this subsection we try to generalize the result of the previous subsection by avoiding to prescribe the special structure (2.3) of the divergence to be added. We substitute eq. (2.3) by

\[ L_{\text{div}} = \frac{d}{dt} h(q, \dot{q}, \ddot{q}) \] (2.25)

Keeping eqs. (2.5) we get instead of eq. (2.7) now

\[ \hat{L} = L(a, \dot{a}, b) + h_1 \dot{a} + h_2 b + h_3 \dot{b} \] (2.26)

where \( h_n \) denotes the partial derivative of \( h \) with respect to its \( n \)th argument.

Using eqs. (2.8), (2.10) is now replaced with

\[ p_b = h_3(a, \dot{a}, b) \] (2.27)

Eq. (2.13) is kept, and (2.14) is replaced with

\[ J = -(h_{23})^2 \] (2.28)

We have to require that \( h_{23} \neq 0 \), and then the equation \( p_b = h_3 \) is locally invertible as \( \dot{a} = F(p_b, a, b) \). From this definition one immediately gets the identity \( F_1 h_{23} = 1 \). Two further identities to be used later are not so trivial to guess. To derive them, let us for a moment fix \( p_b \) and then calculate the increase of \( h_3 \) with increasing \( a \) and \( b \) resp. The assumed constancy of \( h_3 \) yields the equations

\[ h_{13} + F_2 h_{23} = 0 \] (2.29)

and

\[ h_{33} + F_3 h_{23} = 0 \] (2.30)

resp. to be used for deducing the generalization of eq. (2.21). One gets the result: For \( h_{23} \neq 0 \) (which is already presumed), the compatibility relation (2.6) follows automatically from the canonical equation (2.19) if and only if

\[ 0 = L_3 + h_2 \] (2.31)

is identically fulfilled. One can see: The condition of non–degeneracy of the Lagrangian (2.1) namely

\[ L_{33} \neq 0 \]

is equivalent to the condition \( h_{23} \neq 0 \). For any given non–degenerate Lagrangian we can find the appropriate divergence by solving eq. (2.31) as follows

\[ h(q, \dot{q}, \ddot{q}) = -\int_0^\ddot{q} L_3(q, x, \dot{q}) dx \] (2.32)
All other things are fully analogous:

\[ H = [p_a - h_1(a, F, b)]F - h_2(a, F, b)b - L(a, F, b) \] (2.33)

where \( F = F(p_b, a, b) \). Eq. (2.19) with (2.30) gives the compatibility condition (2.6). Eq. (2.18) with (2.29) is equivalent to the Euler–Lagrang e equation (2.2).

Let us summarize this section: For the Lagrangian \( L = L(q, \dot{q}, \ddot{q}) \) where \( \partial^2 L/\partial(\ddot{q})^2 \neq 0 \) we define \( \hat{L} = L + L_{\text{div}} \) where

\[ L_{\text{div}} = -\frac{d}{dt} \int \frac{\partial L}{\partial \ddot{q}}(q, x, \ddot{q})dx \] (2.34)

We insert \( a = q \) and \( b = \ddot{q} \), define the momenta \( p_a = \frac{\partial \hat{L}}{\partial \dot{a}} \) and \( p_b = \frac{\partial \hat{L}}{\partial \dot{b}} \) and get the Hamiltonian \( H = \dot{a}p_a + \dot{b}p_b - \hat{L} \). One of its canonical equations is \( \ddot{a} = b \) and another one is equivalent to the fourth–order Euler–Lagrange equation following from \( L \). By these properties, \( L_{\text{div}} \) is uniquely determined up to the integration constant. Contrary to other approaches, no constraint is needed.

### 3 Instability of \( R^2 \)-theories

This section deals with the classical instability of fourth–order theories following from a non–linear Lagrangian \( L(R) \).

Teyssandier and Tourrenc [24], cf. also [25], solved the Cauchy–problem for this theory, let us shortly repeat the main ingredients.

The Cauchy problem is well–posed (a property which is usually required to take place for a physically sensible theory) in each interval of \( R \)-values where both \( dL/dR \) and \( d^2 L/dR^2 \) are different from zero. The constraint equations are similar as in General Relativity: the four 0-\( i \)-component equations. What is different are the necessary initial data to make the dynamics unique. More exactly: Besides the data of General Relativity one has to prescribe the values of \( R \) and \( \frac{dR}{dt} \) at the initial hypersurface. This coincides with the general experience: Initial data have to be prescribed till the highest–but–one temporal derivative appearing in the field equation (here: fourth–order field equation, \( \frac{dR}{dt} \) contains third–order temporal derivatives of the metric). Under this point of view, classical stability of the field equation means that a small change of the Cauchy data implies also a small change of the solution.

Now we are prepared to classify the stability claims found in refs. [26-34]. To simplify we specialize to the Lagrangian \( L = R - \epsilon R^2 \) with the non–tachyonic sign \( \epsilon > 0 \) and restrict to the range \( \frac{dL}{dR} > 0 \), i.e. \( R < \frac{1}{2\epsilon} \).
On the one hand, refs. [26,27,28] find a classical instability of the Minkowski space–time for this case of fourth–order gravity. Mazzitelli and Rodrigues [29] cite ref. [30] with the sentence “The Minkowski solution in general relativity has been proven to be stable.” which refers to the positive energy theorem of general relativity.

On the other hand, refs. [31-34] find out that the Minkowski space–time is not more unstable in this type of fourth–order gravity than in General Relativity itself. What looks like a contradiction from the first glance is only a notational ambivalence as can be seen now: The main argument in refs. [26-29] is that an arbitrarily large value $\frac{dR}{dt}$ is compatible with small values of $H^2$ and $R^2$. In refs. [31-34] however, following the Cauchy–data argument [24,25], ($\frac{dR}{dt}$ being part of the Cauchy data which are presumed to be small) stability of the Minkowski space–time is obtained in the version: If the Cauchy data are small (meaning: close to the Cauchy data of the Minkowski space–time) then the fourth–order field equation bounds the solution to remain close to the Minkowski space–time.

The argument of ref. [31] is a little bit different: There the conformal transformation to Einstein’s theory with a scalar field $\Phi$ [20] is applied; it is observed that in the $F(R)$-theory there are never ghosts which implies stability. Now, $\Phi$ and $\frac{d\Phi}{dt}$ belong to the Cauchy data which is equivalent to the data $R, \frac{dR}{dt}$ in the conformal picture thus supporting the Cauchy data argument given at the beginning of this section.

4 Cosmology

Several papers [35-43] apply the conformal transformation theorem [20] to cosmology; so for interpreting the cosmological singularity [35], for dealing with anisotropic models [36,37], with transformation to Brans–Dicke extended inflation [38]. Other papers apply this theorem as a mathematical device to transform exact solutions of one of the theories to solutions of the other theory.

Chimento [44] found an exact solution for fourth–order gravity in a spatially flat Friedmann model. He also found out that in the tachyonic–free case the asymptotic matter-dominated Friedmann solution is stable, and no fine–tuning of initial conditions is necessary to get the final (oscillating) Friedmann stage; particle production of non–conformal fields may backreact to damp the oscillations. [45] generalizes [44]: here the Dirac equation is considered, the result is that there appear also spinor field oscillations.

Let us present the exact solution of [44]. For the spatially flat Friedmann model with Hubble parameter $H = \dot{a}/a$ he solves the fourth–order field
equation with vacuum polarization term. The zero–zero component equation reads

\[ 2H \ddot{H} - \dot{H}^2 + 6H^2 \dot{H} + \frac{9}{4}H^4 + H^2 = 0 \]  

(4.1)

The \( H^4 \)-term stems from the vacuum polarization and the \( H^2 \)-term from the Einstein tensor. The remaining ingredients of eq. (4.1) come from the term \( R^2 \) in the Lagrangian. (Here we only present the tachyonic–free case with \( \Lambda = 0 \) and \( \frac{9}{4} \) in front of \( H^4 \).) The factor in front of \( H^4 \) should not influence the weak–field behaviour because for \( H \approx 0 \) this factor only changes the effective gravitational constant.

From eq. (4.1) the discussion of section 3 becomes obvious: (4.1) represents a third–order equation for the cosmic scale factor \( a \); it is a constraint and not a dynamical equation. (It is only due to the high symmetry, that accidentally the validity of the constraint implies the validity of the dynamical equation.) Supposed, eq. (4.1) would be the true dynamical equation for a theory, then the instability argument of [26-28] could apply.

The ansatz for solving eq. (4.1)

\[ H = \frac{2\dot{s}}{3s} \]

leads to a non–linear third–order equation for \( s \)

\[ 2\dot{s}s^{(3)} - \ddot{s}^2 + s^2 = 0 \]  

(4.2)

Derivative with respect to \( t \) yields the equation \( s^{(4)} + \ddot{s} = 0 \) being linear in \( s \) and having the solution

\[ s = c_1 + c_2 t + c_3 \sin(t + c_4) \]

Inserting this solution into the original equation gives the restriction \( |c_2| = |c_3| \). Let us discuss this solution: \( c_2 = 0 \) leads to the uninteresting flat space–time. So, now let \( c_2 \neq 0 \). Adding \( \pi \) to \( c_4 \) can be absorbed by a change of the sign of \( c_3 \). Therefore, \( c_2 = c_3 \) without loss of generality. Multiplication of \( s \) by a constant factor does not change the geometry, so let \( c_2 = 1 \). A suitable time–translation leads to \( c_1 = 0 \). Finally, the cosmic scale factor is calculated as \( a = s^{2/3} \) leading to

\[ a = [t + \sin(t + c_4)]^{2/3} \sim t^{2/3} [1 + \frac{2}{3t} \sin(t + c_4)] \]  

(4.3)

The r.h.s. of eq. (4.3) gives the late–time behaviour deduced in [46]. The factor \( 1/t \) in front of the “sin”–term shows that the oscillations due to the
higher–order terms are damped. The total energy “sitting” in these oscillations, however, remains constant in time (because of the volume–expansion), cf. [27], and can be converted into classical matter by particle creation.

Let us mention some further cosmological solutions with higher–order gravity: [47] discusses the \( L(R) \)-stability with a conformally coupled scalar field. Ref. [48] (partial results of it can be found in [49]) deals with fourth–order cosmological models of Bianchi–type I and power–law metrics, i.e.

\[
ds^2 = dt^2 - \sum_{i=1}^{3} t^{2p_i} (dx^i)^2
\]

with real parameters \( p_i \). The suitable notation

\[
a_k = \sum_{i=1}^{3} p_i^k
\]

gives the following: \( a_1 = a_2 = 1 \) is the usual Kasner solution for Einstein’s theory. \( a_1^2 + a_2 = 2a_1 \) is the condition to be fulfilled for a solution in \( L = R^2 \). Refs. [50, 51] also discuss \( R^2 \)-models. Ref. [33] considers inflationary cosmology with a Lagrangian

\[
L = R + \lambda R_{\mu\nu}R^{\mu\nu}/R.
\]

Ref. [52] deals with anisotropic Bianchi–type IX solutions for \( L = R^2 \). They look for chaotic behaviour analogous to the mixmaster model in Einstein’s theory. Ref. [53] gives exact solutions for \( L = R^2 \) and a closed Friedmann model, ref. [54] discusses the bounce in closed Friedmann models for \( L = R - \epsilon R^2 \). Supplementing the discussion of [54, eq.(1)] let us mention: In the non–tachyonic case, there exist periodically oscillating models with an always positive scale factor \( a \). Ref. [55] looks for chaos in isotropic models, e.g. by conformally coupled massive scalar fields in the closed universe. The papers [56,57] consider the stability of power–law inflation for \( L = R^m \) within the set of spatially flat Friedmann models. Refs. [58] give overviews on higher–order cosmology, especially chaotic inflation as an attractor solution in initial–condition space. [59] deals with quantum gravitational effects in the de Sitter space–time, and [60] gives a classification of inflationary Einstein–scalar–field–models via catastrophe theory. Ref. [61] considers Chern–Simon terms in Bianchi cosmologies and the cosmic no-hair conjecture.

5 Summary

The scope of this paper was to review recent work connected with higher–order theories, especially fourth–order gravity theories and their application
to cosmology. We presented a step necessary to deduce the Wheeler–de Witt equation for a cosmological minisuperspace model in fourth–order gravity and discussed the several stability claims and conformal transformation theorems.

The method (first used in [51] for $L = R^2$ and a spatially flat Friedmann model) to handle with eqs. (1.3 - 1.6) was systematically generalized in sct. 2 (cf. also [62]) to give a Hamiltonian formulation of a general fourth–order theory. The possibility of deducing this method makes it clear that the method of ref. [51] is not restricted to highly symmetric models. The alternative Hamiltonian formulation has some advantages in comparison with Ostrogradski’s one: No constraint is needed, the Hamiltonian is typically a quadratic function in the momenta. (Ostrogradski’s approach leads always to a Hamiltonian linear in the momenta which gives artificial factors $i$ in the Schrödinger equation.) The calculation of the momenta from the Lagrangian follows the usual equations (2.8) whereas the Ostrogradski approach needs some additional terms. Our approach is less ambiguous, cf. eq. (2.7).

The fact that the Jacobian eq. (2.28) is always negative excludes the possibility to get a positive definite Jacobi metric in eq. (1.2). This is one of the many possibilities to say what is meant by the phrase “fourth–order theories are always unstable”. The Jacobi metric plays the role of the conformally transformed superspace–metric used in quantum cosmology. And here the circle can be closed: In Einstein’s theory (both for Lorentzian and Euclidean signature of the underlying manifold) the superspace–metric has Lorentzian signature and cannot be positive definite. So we get once more the result: Fourth–order gravity contains instabilities, but for the non–tachyonic case of $L = R - \epsilon R^2$ only those which it has in common with General Relativity.

Let us finish by mentioning some open questions worth being attacked in the future:

For other types of fourth– and higher–order theories the singularity behaviour is not yet understood.

To decide the quantum instability of the Minkowski or de Sitter spacetimes in fourth–order gravity one must solve the corresponding Wheeler–de Witt equations. In [29] they are deduced for the spatially flat Friedmann model and the Lagrangian $L = R - \epsilon R^2$.

In [50] it is mentioned that a classical theory with higher derivatives has instabilities: “At the quantum level, the difference is even more dramatic. Noncommuting variables in the lower–derivative theory, such as position and velocities, become commuting in the higher–derivative theory.” Remark of U. Kasper to this sentence: “The uncertainty relation is primarily between
positions and momenta. If the momentum is independent of the velocity then commuting position and velocity need not bother.”

Refs. [62-64] discuss the conformal transformations between fourth–order gravity to Einstein’s theory with a scalar field. Reuter [62] proposed to use the notion “Bicknell–theorem” to state that conformal equivalence. Dick [64] represents a valuable update of the discussion which of the two conformally equivalent metrics shall be considered physical.

Because of the frequent appearance of singular points of the corresponding differential equations it is especially useful to have a growing set of exact solutions (see also [65]) of the non-linear gravity models.

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