Lyapunov functions via Whitney’s size functions

Alfonso Artigue

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Abstract

In this paper we present a technique for constructing Lyapunov functions based on Whitney’s size functions. Applications to asymptotically stable equilibrium points, isolated sets, expansive homeomorphisms and continuum-wise expansive homeomorphisms are given.

1 Introduction

In Dynamical Systems and Differential Equations it is important to determine the stability of trajectories and a well known technique for this purpose is to find a Lyapunov function. In order to fix ideas consider a continuous flow $\phi : \mathbb{R} \times X \rightarrow X$ on a compact metric space $(X, \text{dist})$ with a singular (or equilibrium) point $p \in X$, i.e., $\phi_t(p) = p$ for all $t \in \mathbb{R}$. A Lyapunov function for $p$ is a continuous non-negative function that vanishes only at $p$ and strictly decreases along the orbits close to $p$. Recall that $p$ is stable if for all $\varepsilon > 0$ there is $\delta > 0$ such that if $\text{dist}(x, p) < \delta$ then $\text{dist}(\phi_t(x), p) < \varepsilon$ for all $t \geq 0$. We say that $p$ is asymptotically stable if it is stable and there is $\delta_0 > 0$ such that if $\text{dist}(x, p) < \delta_0$ then $\phi_t(x) \to p$ as $t \to +\infty$. The existence of a Lyapunov function for an equilibrium point implies the asymptotic stability of the equilibrium point.

A remarkable result, first proved by Massera in [6], is the converse: every asymptotically stable singular point admits a Lyapunov function. Later, other authors obtained Lyapunov functions with different methods, see for example [1, 2]. In [3] a generalization is proved in the context of arbitrary metric spaces. The purpose of the present paper is to develop a different technique that allows us to construct Lyapunov functions for different dynamical systems as: isolated sets, expansive homeomorphisms and continuum-wise expansive homeomorphisms. Our techniques are based on the size function $\mu$ introduced by Whitney in [8].

In order to motivate our work let us show how to construct a Lyapunov function for an asymptotically stable singular point. Denote by $\mathcal{K}(X)$ the set of non-empty compact subsets of $X$. In the set $\mathcal{K}(X)$ we consider the Hausdorff distance $\text{dist}_H$ making $(\mathcal{K}(X), \text{dist}_H)$ a metric space. Recall that

$$\text{dist}_H(A, B) = \inf\{\varepsilon > 0 : A \subset B_\varepsilon(B) \text{ and } B \subset B_\varepsilon(A)\},$$

where $B_\varepsilon(C) = \bigcup_{x \in C} B_\varepsilon(x)$ and $B_\varepsilon(x)$ is the usual ball of radius $\varepsilon$ centered at $x$. See [7] for more on the Hausdorff metric. A size function is a continuous map $\mu : \mathcal{K}(X) \rightarrow \mathbb{R}$ satisfying:

1. $\mu(A) \geq 0$ with equality if and only if $A$ has only one point,
2. if $A \subset B$ and $A \neq B$ then $\mu(A) < \mu(B)$.

In [8] it is proved that size functions exists for every compact metric space.

**Theorem 1.1.** If $\phi$ is a continuous flow on $X$ with an asymptotically stable singular point $p$ then there are an open set $U$ containing $p$ and a continuous function $V: U \to \mathbb{R}$ satisfying:

1. $V(x) \geq 0$ for all $x \in U$ with equality if and only if $x = p$ and
2. if $t > 0$ and $\{\phi_s(x) : s \in [0, t]\} \subset U$ then $V(\phi_t(x)) < V(x)$.

**Proof.** By the conditions on $p$ there are $\delta_0, \delta > 0$ such that if $\text{dist}(x, p) < \delta$ then $\phi_t(x) \in B_{\delta_0}(p)$ for all $t \geq 0$ and $\phi_t(x) \to p$ as $t \to \infty$. Define $U = B_\delta(p)$ and $V: U \to \mathbb{R}$ as

$$V(x) = \mu(\{\phi_t(x) : t \geq 0\} \cup \{p\})$$

where $\mu$ is a size function. Since $\phi_t(x) \to p$ we have that

$$O(x) = \{\phi_t(x) : t \geq 0\} \cup \{p\}$$

is a compact set for all $x \in U$. Notice that if $t > 0$ then $O(\phi_t(x)) \subset O(x)$ and the inclusion is proper. Therefore, $V(\phi_t(x)) < V(x)$ because $\mu$ is a size function. Also notice that $V(p) = 0$ and $V(x) > 0$ if $x \neq p$. In order to prove the continuity of $V$, we will prove the continuity of $O: U \to K(X)$, the map defined by (1). Since $\mu$ is continuous we will conclude the continuity of $V$.

Let us prove the continuity of $O$ at $x \in U$. Take $\varepsilon > 0$. By the asymptotic stability of $p$ there are $\rho, T > 0$ such that if $y \in B_\rho(x)$ then $\phi_t(y) \in B_{\varepsilon/2}(p)$ for all $t \geq T$. By the continuity of the flow, there is $r > 0$ such that if $y \in B_r(x)$ then $\text{dist}(\phi_t(x), \phi_t(y)) < \varepsilon$ for all $t \in [0, T]$. Now it is easy to see that if $y \in B_{\min(|r, x|)}(x)$ then $\text{dist}_H(O(x), O(y)) < \varepsilon$, proving the continuity of $O$ at $x$ and consequently the continuity of $V$.

Let us recall that size functions can be easily defined. A variation of the construction given in [8], adapted for compact metric spaces, is the following.

Let $q_1, q_2, q_3, \ldots$ be a sequence dense in $X$. Define $\mu_i: K(X) \to \mathbb{R}$ as

$$\mu_i(A) = \max_{x \in A} \text{dist}(q_i, x) - \min_{x \in A} \text{dist}(q_i, x).$$

The following formula defines a size function $\mu: K(X) \to \mathbb{R}$

$$\mu(A) = \sum_{i=1}^{\infty} \frac{\mu_i(A)}{2^i},$$

as proved in [8]. In Section 2 we extend Theorem 1.1 by constructing a Lyapunov function for an isolated invariant sets.

For the study of expansive homeomorphisms (see Definition 3.1) Lewowicz introduced in [5] Lyapunov functions. He proved that expansiveness is equivalent with the existence of such function. In Section 3 we give a different proof of this result by constructing a Lyapunov function defined for compact subsets of the space. In [4] Kato introduced another form of expansiveness called continuum-wise expansiveness (see Definition 3.2). With our techniques we prove that continuum-wise expansiveness is equivalent with the existence of a Lyapunov function on continua subsets of the space.
2 Lyapunov Functions for Isolated Sets

In this section we consider continuous flows on compact metric spaces. The purpose is to construct a Lyapunov function for an isolated set of the flow using a size function. First we consider the case of an isolated set consisting of a point.

2.1 Isolated Singularities

Let \( \phi \) be a continuous flow on a compact metric space \((X, \text{dist})\). A point \( p \in X \) is singular for \( \phi \) if \( \phi_t(p) = p \) for all \( t \in \mathbb{R} \). A singular point \( p \in X \) is isolated if there is an open isolating neighborhood \( U \) of \( p \) such that if \( \phi_k(x) \subset U \) then \( x = p \).

**Definition 2.1.** An open set \( U \) is an adapted neighborhood of an isolated singular point \( p \in U \) if for every orbit segment \( l \subset \text{clo}(U) \) with extreme points in \( U \) it holds that \( l \subset U \).

Given a set \( A \subset X \) and \( x \in A \) denote by \( \text{comp}_x(A) \) the connected component of \( A \) that contains the point \( x \).

**Proposition 2.1.** Every isolated singular point has an adapted neighborhood.

**Proof.** Let \( r > 0 \) be such that \( \text{clo}(B_r(p)) \) is contained in an isolating neighborhood of \( p \). For \( \rho \in (0, r) \) define the set

\[
U_\rho = \{ x \in B_r(p) : \text{comp}_x(\phi_\rho(x) \cap B_r(p)) \cap B_\rho(p) \neq \emptyset \}.
\]

By the continuity of the flow we have that \( U_\rho \) is an open set for all \( \rho \in (0, r) \). Let us prove that if \( \rho \) is sufficiently small then \( U_\rho \) is an adapted neighborhood. By contradiction, suppose that there are \( \rho_n \to 0 \), \( a_n, b_n \in U_{\rho_n}, t_n \geq 0 \) such that \( b_n = \phi_{t_n}(a_n) \) and \( t_n = [0, t_n]|(a_n) \subset \text{clo}(U_{\rho_n}) \) but \( t_n \) is not contained in \( U_{\rho_n} \). Then there is \( s_n \in (0, t_n) \) such that \( \phi_{s_n}(a_n) \subset \partial B_r(p) \). Also, there must be \( u_n < 0 \) and \( v_n > 0 \) such that \( \phi_{u_n}(a_n), \phi_{v_n}(b_n) \in B_{\rho_n}(p) \). But a limit point of \( \phi_{s_n}(a_n) \) contradicts that \( \text{clo}(B_r(p)) \) is contained in an isolating neighborhood of \( p \).

Fix an isolated point \( p \) with an adapted neighborhood \( U \). Consider the sets

\[
W^+(U) = \{ x \in U : \lim_{t \to +\infty} \phi_t(x) = p \text{ and } \phi_{\text{clo}(U)}(x) \subset U \},
\]

\[
W^-(U) = \{ x \in U : \lim_{t \to -\infty} \phi_t(x) = p \text{ and } \phi_{\text{clo}(U)}(x) \subset U \}.
\]

For \( x \in U \) define the orbit segments

\[
O^+_x(U) = \text{comp}_x(U \cap \phi_{[0, +\infty]}(x)),
\]

\[
O^-_x(U) = \text{comp}_x(U \cap \phi_{(-\infty, 0)}(x)).
\]

Define \( C = X \setminus U \) and let \( V^+_p, V^-_p : U \to K(X) \) be defined as

\[
\begin{cases}
V^+_p(x) = \text{clo}(O^+_x(U) \cup W^+_p(U)) \cup C, \\
V^-_p(x) = \text{clo}(O^-_x(U) \cup W^-_p(U)) \cup C.
\end{cases}
\]
Definition 2.2. A Lyapunov function for an isolated point $p$ is a continuous map $V: U \rightarrow \mathbb{R}$ defined in a neighborhood of $p$ such that if $t > 0$ and $\phi_{[0,t]}(x) \subset U \setminus \{p\}$ then $V(x) > V(\phi_t(x))$.

Theorem 2.2. If $p$ is an isolated point and $U$ is an adapted neighborhood of $p$ then the maps $V^+_p$ and $V^-_p$ are continuous in $U$. If in addition, $\mu$ is a size function on $K(X)$ then $V: \hat{U} \rightarrow \mathbb{R}$ defined as

$$V(x) = \mu(V^+_p(x)) - \mu(V^-_p(x))$$

is a Lyapunov function for $p$.

Proof. Let us prove the continuity of $V^+_p$ by contradiction. Assume that $x_n \rightarrow x \in U$ and $V^+_p(x_n) \rightarrow K$ with the Hausdorff distance but $K \neq V^+_p(x)$. By definitions we have that

$$\text{clos}(W^o_U(p)) \cup C \subset K \cap V^+_p(x).$$

Recall that $C$ was defined as the complement of $U$ in $X$. Take a point $y \in K \setminus V^+_p(x) \cup V^+_p(x) \setminus K$. By the inclusion (2) we know that $y \notin \text{clos}(W^o_U(p)) \cup C$.

We divide the proof in two cases.

Case 1. Suppose first that $y \in K \setminus V^+_p(x)$. Since $y \in K$ there is a sequence $t_n \geq 0$ such that $\phi_{t_n}(x_n) \rightarrow y$ and $\phi_{[0,t_n]}(x_n) \subset U$. If $t_n \rightarrow \infty$ then $x \in W^o_U(p)$. Consequently, $y \in W^o_U(p)$, which is a contradiction. Therefore $t_n$ is bounded. Without loss of generality assume that $t_n \rightarrow t \geq 0$ and then $\phi_t(x) = y$. Thus $\phi_{[0,t]}(x) \subset \text{clos}(U)$. Since $y \notin C$ we have that $y \in U$. Now, since $U$ is an adapted neighborhood we conclude that $\phi_{[0,t]}(x) \subset U$ and then $y \in O^+(x) \subset V^+_p(x)$. This contradiction finishes this case.

Case 2. Now assume that $y \in V^+_p(x) \setminus K$. In this case we have that $y = \phi_s(x)$ for some $s \geq 0$ and $\phi_{[0,s]}(x) \subset U$. Then $\phi_s(x_n) \rightarrow y$ and $y \in K$. This contradiction proves that $V^+_p$ is continuous in $U$.

The continuity of $V^-_p$ is proved in a similar way. Let us show that $V$ is a Lyapunov function for $p$. The continuity of $V$ in $U$ follows by the continuity of $V^+_p$, $V^-_p$ and the size function $\mu$.

Now take $x \notin U \setminus \{p\}$. We will show that $V$ decreases along the orbit segment of $x$ contained in $U$. Notice that for all $t > 0$, $O^+_U(\phi_t(x)) \subset O^+_U(x)$ if $\phi_{[0,t]}(x) \subset U$. Therefore $V^+_p(\phi_t(x)) \leq V^+_p(O^+_U(x))$. The equality can only hold if $x \in W^o_U(p)$. But in this case we have that $x \notin W^o_U(p)$ because $W^o_U(p) \cap W^o_U(p) = \{p\}$. Then $V^-_p(\phi_t(x)) > V^-_p(x)$. Therefore, $V(\phi_t(x)) < V(x)$ and $V$ is a Lyapunov function for $p$. □

2.2 Isolated Sets

Let $\phi: \mathbb{R} \times X \rightarrow X$ be a continuous flow on a compact metric space $X$. Consider a $\phi$-invariant set $\Lambda \subset X$, i.e., $\phi_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$. We say that $\Lambda$ is an isolated set with isolating neighborhood $U$ if $\phi_x(x) \subset U$ implies $x \in \Lambda$.

Definition 2.3. A Lyapunov function for an isolated set $\Lambda$ is a continuous function $V: U \rightarrow \mathbb{R}$ defined on an open set $U$ containing $\Lambda$ such that:

1. $V(x) = 0$ if and only if $x \in \Lambda$,
2. if $\phi_{[0,t]}(x) \subset U \setminus \Lambda$ then $V(x) > V(\phi_t(x))$.  

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Let us show how the construction of a Lyapunov function for an isolated set can be reduced to the case of an isolated singular point.

**Theorem 2.3.** Every isolated set admits a Lyapunov function.

**Proof.** Consider the set $Y = (X \setminus \Lambda) \cup \{\Lambda\}$. On $Y$ define the distance $d$ as

$$d(x, y) = \min\{\text{dist}(x, y), \text{dist}(x, \Lambda) + \text{dist}(y, \Lambda)\}.$$  

It is easy to see that $(Y, d)$ is a compact metric space. Also, the flow $\phi$ induces naturally a flow $\phi'$ on $Y$ with $\Lambda$ as an isolated singular point. Consider from Theorem 2.2 a Lyapunov function for $\Lambda$ as an isolated singular point of $\phi'$. This function naturally defines a Lyapunov function for $\Lambda$ as an isolated set of $\phi$.

### 3 Applications to homeomorphisms

Let $f: X \to X$ be a homeomorphism of a compact metric space $(X, \text{dist})$. An $f$ invariant set $\Lambda$ is isolated if there is an open neighborhood $U$ of $\Lambda$ such that $f^n(x) \in U$ for all $n \in \mathbb{Z}$ implies that $x \in \Lambda$.

**Theorem 3.1.** Every isolated set $\Lambda$ for a homeomorphism $f$ admits a Lyapunov function, that is, a continuous map $V: U \subset X \to \mathbb{R}$ defined on a neighborhood of $\Lambda$ such that:

1. $V(x) = 0$ if and only if $x \in \Lambda$,
2. $V(x) > V(f(x))$ if $x, f(x) \in U \setminus \Lambda$.

**Proof.** Consider $\phi: \mathbb{R} \times X \to X$ the suspension of $f$. Consider $i: X \to X_f$ a homeomorphism onto its image such that $i(X)$ is a global cross section of $\phi$. It is easy to see that $\Lambda$ is an isolated set for $f$ if and only $\Lambda_f = \phi_\mathbb{R}(i(\Lambda))$ is an isolated set for $\phi$. Now consider a Lyapunov function $V'$ for $\Lambda_f$. A Lyapunov function for $f$ can be defined by $V(x) = V'(i(x))$.

**Definition 3.1.** A homeomorphism $f: X \to X$ of a compact metric space is expansive if there is $\alpha > 0$ (an expansive constant) such that if $x \neq y$ then there is $n \in \mathbb{Z}$ such that $\text{dist}(f^n(x), f^n(y)) > \alpha$.

Recall that $\mathcal{K}(X)$ denotes the compact metric space of compact subsets of $X$ with the Hausdorff metric. Denote by $\mathcal{F}_1 = \{A \in \mathcal{K}(X) : |A| = 1\}$ where $|A|$ denotes the cardinality of $A$. Given a homeomorphism $f: X \to X$ define the homeomorphism $f': \mathcal{K}(X) \to \mathcal{K}(X)$ as $f'(A) = \{f(x) : x \in A\}$. Notice that $\mathcal{F}_1$ is invariant under $f'$.

**Corollary 3.2.** For a homeomorphism $f: X \to X$ the following statements are equivalent:

1. $f$ is an expansive homeomorphism,
2. $\mathcal{F}_1$ is an isolated set for $f'$,
3. there is a continuous function $V: U \subset \mathcal{K}(X) \to \mathbb{R}$ defined on a neighborhood of $\mathcal{F}_1$ such that $V(A) = 0$ if and only if $A \in \mathcal{F}_1$ and $V(A) > V(f'(A))$ if $A, f'(A) \in U \setminus \mathcal{F}_1$. 

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Proof. \((1 \rightarrow 2)\). Let \(\delta\) be an expansive constant and define

\[ U = \{ A \in \mathcal{K}(X) : \text{diam}(A) < \delta \}. \]

It is easy to see that \(U\) is an isolating neighborhood of \(\mathcal{F}_1\).

\((2 \rightarrow 3)\). It follows by Theorem 3.1.

\((3 \rightarrow 1)\). Take \(\delta > 0\) such that if \(\text{dist}(x, y) \leq \delta\) then \(\{x, y\} \in U\). Let us prove that \(\delta\) is an expansive constant for \(f\). Assume by contradiction that \(\text{dist}(f^n(x), f^n(y)) \leq \delta\) for all \(n \in \mathbb{Z}\) and \(x \neq y\). Define \(A = \{x, y\}\). We have that \(V(f^n(A))\) is a decreasing sequence. Without loss of generality assume that \(V(A) < 0\). Suppose that \(f^n(A)\) accumulates in \(B\). Now it is easy to see that \(B \in U \setminus \mathcal{F}_1\) and also \(V(B) = V(f'(B))\). This contradiction proves the theorem.

Recall that a continuum is a compact connected set. Denote by \(C(X) = \{C \in \mathcal{K}(X) : C \text{ is connected}\}\) the space of continua of \(X\).

Definition 3.2. A homeomorphism \(f : X \to X\) is continuum-wise expansive if there is \(\delta > 0\) such that if \(C \in C(X)\) and \(\text{diam}(f^n(C)) \leq \delta\) for all \(n \in \mathbb{Z}\) then \(C \in \mathcal{F}_1\).

A Lyapunov function for a continuum-wise expansive homeomorphism is a continuous function \(V : U \subset C(X) \to \mathbb{R}\) defined on an open set \(U \subset C(X)\) containing \(\mathcal{F}_1\) and \(C, f(C) \in U\).

Corollary 3.3. For a homeomorphism \(f : X \to X\) the following statements are equivalent:

1. \(f\) is a continuum-wise expansive homeomorphism,
2. \(\mathcal{F}_1\) is an isolated set for \(f' : C(X) \to C(X)\),
3. there is a continuous function \(V : U \subset C(X) \to \mathbb{R}\) defined on an open set \(U \subset C(X)\) containing \(\mathcal{F}_1\) such that \(V(\{x\}) = 0\) for all \(x \in X\) and \(V(f(C)) < V(C)\) if \(C \notin \mathcal{F}_1\) and \(C, f(C) \in U\).

Proof. The proof is similar to the proof of Corollary 3.2.

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