The topology of U-duality (sub-)groups

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Abstract

We discuss the topology of the symmetry groups appearing in compactified (super-)gravity, and discuss two applications. First, we demonstrate that for 3 dimensional sigma models on a symmetric space $G/H$ with $G$ non-compact and $H$ the maximal compact subgroup of $G$, the possibility of oxidation to a higher dimensional theory can immediately be deduced from the topology of $H$. Second, by comparing the actual symmetry groups appearing in maximal supergravities with the subgroups of $SL(32, \mathbb{R})$ and $Spin(32)$, we argue that these groups cannot serve as a local symmetry group for M-theory in a formulation of de Wit-Nicolai type.

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1 Introduction

Since the construction of supergravity theories, and the discovery of the Cremmer-Julia groups of compactified 11 dimensional supergravity [1, 2], it has been clear that Lie groups and algebra’s play an important role in this field. However, most of the attention is confined to the subject of Lie algebra’s. In this paper we will study the topology of some of the (sub-)groups present in (super-)gravity, and hope to convince the reader that examination of these global properties leads to relevant information about the theory.

In section 2 we give some technical background, and quote a useful theorem. Then we apply the methods to two different, but related topics: The theory of oxidation, and a recent proposal for a “generalized holonomy”-group as a symmetry of 11 dimensional supergravity and M-theory.

Dimensional reduction of a theory leads to a lower dimensional theory. Oxidation is the inverse to this process, the reconstruction of a higher dimensional theory from the lower dimensional one. One of the most interesting cases to consider is a 3 dimensional sigma model on a coset $G/H$, coupled to gravity [3, 4, 5, 6]. In [7], it was demonstrated how the group $G$ encodes all the higher dimensional theories that lead to a $G/H$ coset upon dimensional reduction to 3 dimensions, including the possibility of inequivalent dual theories (see [8] for an alternative approach, covering truncations of maximal supergravity).

In section 3 of this paper, we look at the maximal compact subgroup $H$, and argue that the structure of this group is crucial to the inclusion of fermionic representations. As a consequence, the topology of $H$ inside $G$ immediately gives a criterion for oxidizability (even without further study of the group $G$).

A second topic will be the study of proposals for symmetry groups of M-theory [9, 10] (see [11, 12] for subsequent work). An important concept in the study of backgrounds for M-theory is the holonomy (of the spin bundle) on the (geometric) background; it determines the amount of preserved supersymmetries, and many properties of the low-dimensional theory. However, a background may not be “purely” geometrical, but instead include fluxes. In compactification, the fields giving rise to these fluxes merge together with geometric degrees of freedom and are mixed by (subgroups of) the Cremmer-Julia groups. The existence of formulations of 11 dimensional supergravity with the compact subgroup of a Cremmer-Julia group appearing as a local symmetry\(^2\) [13], gives rise to the “generalized holonomy”-conjecture: There exists a suitable extension of the structure group on the spin bundle, such that (in a suitable formulation) a generic background can be described as having “generalized holonomy”, that is holonomy in this extended structure group [9].

The immediate question is whether 11 dimensional supergravity allows a formulation,

\(^2\)Strictly speaking it is the universal covering of the compact subgroup of the Cremmer-Julia group, as we will explain in sections 3 and 4.
such that all possible local symmetries from space-, time- or null-like reduction can be realized within some group, and what this structure group should be; because it represents a \textit{local} symmetry it seems unavoidable that it must be a symmetry of some formulation of the non-perturbative extension of 11 dimensional supergravity, M-theory. In \[10\] it is proposed that the structure group may be a simple finite dimensional group; based on the requirements that such a group must meet, and an analysis of symmetries generated by the 11 dimensional Clifford algebra, $SL(32, \mathbb{R})$ is put forward as a candidate.

In section 4 we examine the topology of the symmetry groups appearing in maximal supergravities, and find that the group $SL(32, \mathbb{R})$ does not contain them all. Similarly, the group $Spin(32)$ proposed in \[9\] also appears problematic. We conclude that these groups cannot be straightforwardly promoted to symmetry groups of M-theory.

2 The topology of (sub-)groups

Although often neglected in the physics literature, the topology of (sub-)groups is relevant to a variety of effects. Elements of the discussion here were crucial in the discovery of a new class of vacua for Yang-Mills theory on a 3-torus \[14, 15, 16\]. An application to string and M-theories leads to new string- and M-theory vacua \[17\]. The first part of our discussion follows \[15\], similar results can be found in \[16\].

The groups we will be discussing are compact groups. A central theorem in Lie-group theory says that every compact simple Lie-group $H$ has a universal covering $\tilde{H}$, which is simply connected. The fundamental group of $H$ is given by a subgroup $\pi_1(H) = Z$ of the center of $\tilde{H}$, and the group $H$ is isomorphic to $\tilde{H}/Z$.

Therefore, to deduce the fundamental group of $H$, it suffices to compare the center of $H$ with the center of $\tilde{H}$. In practice, we are not dealing with abstract groups, but with representations of $H$. These are always representations of $\tilde{H}$, and it will be necessary to check whether the elements of the center of $\tilde{H}$ are realized non-trivially on these representations. The different topologies that can be realized are enumerated by the possible different subgroups of $Z$.

At the level of the lattices associated to the Lie algebra $h$ that generates $H$, this has a nice realization \[15\]. Let $Q^\vee(H)$ be the coroot-lattice of $h$, and let $P^\vee(H)$ be the coweight lattice of $h$

$$P^\vee(H) = \{ \zeta \in h | \exp(2\pi i \zeta) = 1 \}$$

(1)

Then the fundamental group of $H$ is given by

$$\pi_1(H) = P^\vee(H)/Q^\vee(H)$$

(2)

The groups we are interested in are subgroups of larger groups. We quote a useful theorem
from [15], which is devoted to subgroups of compact groups (similar results can be found in [16]): It summarizes many computations that can also be done case by case [14].

Let $r$ be the rank of the algebra $h$, and $\alpha_i$ ($1 \leq i \leq r$) be a set of simple roots for $h$. Define the fundamental coweights by $\langle \alpha_i, \omega_j \rangle = \delta_{ij}$ (where $\langle , \rangle$ is the bilinear Killing form on $h$). The root integers $a_i$ (resp. coroot integers $a^\vee_i$) are given by expanding the highest root $\theta$ (resp. its coroot $\theta^\vee = 2\theta/\langle \theta, \theta \rangle = \theta$) in the simple roots (coroots):

$$\theta = \sum_{j=1}^{r} a_j \alpha_j = \sum_{j=1}^{r} a^\vee_j \frac{2\alpha_j}{\langle \alpha_j, \alpha_j \rangle}$$  \hspace{1cm} (3)

Furthermore, one introduces $a_0 = a_0^\vee = 1$.

Then one has the following

**Theorem 1** ([15] theorem 1b/c) Let $H$ be a connected simply connected (almost) simple Lie group. Consider the group element

$$\sigma_{\vec{s}} = \exp\{2\pi i \sum_{j=1}^{r} s_j \omega_j \}$$  \hspace{1cm} (4)

where

$$\vec{s} = (s_0, s_1, \ldots, s_r), \quad s_j \in \mathbb{R}, s_j \geq 0, \quad \sum_{j=0}^{r} a_j s_j = 1. \hspace{1cm} (5)$$

The centralizer of $\sigma_{\vec{s}}$ (in $H$) is a connected compact Lie group which is a product of $U(1)^{n-1}$, where $n$ is the number of non-zero $s_j$, and a connected semi-simple group whose Dynkin diagram is obtained from the extended Dynkin diagram of $H$ by removing nodes $i$ for which $s_i \neq 0$. Moreover, the fundamental group of the centralizer of $\sigma_{\vec{s}}$ is isomorphic to a direct product of $\mathbb{Z}^{n-1}$ and a cyclic group of order $a_{\vec{s}}$ where

$$a_{\vec{s}} = \gcd\{a^\vee_j | s_j \neq 0\} \hspace{1cm} (6)$$

**Proof** See [15] and references therein. Note the different ranges appearing in the sums in equations (4) and (5).

We have depicted the extended Dynkin diagrams, root and coroot integers in figure 1. Extended nodes are shaded, omitting them leads to the standard Dynkin diagram. In case root and coroot integers differ, we have denoted them as $a_j/a^\vee_j$.

Theorem 1 allows an easy computation of the topology of the compact subgroup $H$ of a non-compact group $G$ that is generated by a Cartan involution that is inner. Note that if exactly one of the $s_j \neq 0 = \frac{1}{2}$, then $\sigma_{\vec{s}}$ squares to 1 (acting in the adjoint representation), and can be used as a Cartan involution. This generates a non-compact real form $G$, and its maximal compact subgroup $H$ is the centralizer of $\sigma_{\vec{s}}$, which is easily found by using theorem 1. There are 3 ways to arrange this, and meet the other constraints.
First, suppose the node \( j \) corresponds to a long root, with \( a_j = 2 \). This situation occurs in all extended \( B, D, E, F, G \) diagrams (see figure 1). Necessarily, \( a_j^\vee = 2 \) and theorem 1 tells us that the fundamental group of \( H \) is \( \pi_1(H) = \mathbb{Z}_2 \).

Second, suppose that the node \( j \) represents a short root with \( a_j = 2 \). This situation occurs in the \( B, C \) and \( F \) diagrams. Now \( a_j^\vee = 1 \), and the fundamental group of the maximal compact subgroup is given by \( \pi_1(H) = 0 \).

Third, suppose that the node \( j \) represents a long root with \( a_j = 1 \). Then we solve (5) by setting \( s_0 = s_j = \frac{1}{2} \). This can be done for all groups except \( E_8, F_4 \) and \( G_2 \). As \( a_0^\vee = a_j^\vee = 1 \), theorem 1 tells us that the maximal compact subgroup has \( \pi_1(H) = \mathbb{Z} \).

These results cover all cases where the Cartan involution is an inner automorphism (see table 1, this can also easily be proven by invoking Theorem 1a from [15]). In the remaining cases, the Cartan involution is an outer automorphism. We know no general theorem applicable to this situation, and will compute the fundamental group case by case instead. We do not have to compute the fundamental group for all (infinitely many) representations \( H \), but can restrict to a few well chosen irreducible representations (irreps), as we will know explain.

By a simple rewriting of (2), the fundamental group of \( H \) is also given by

\[
\pi_1(H) = P(H)/Q(H)
\]

where \( P \) is the lattice generated by the weights of the representations present in the theory, and \( Q \) is the root lattice. Consider the full weight lattice of \( \tilde{H} \), which is the dual lattice.
to the coroot lattice, and includes representations that may not be present in our theory of interest. The root lattice is a sublattice of the full weight lattice, and quotienting the full weight lattice by the root lattice introduces a grading on the weight lattice. Because in an irrep, the various weights differ by a root, all weights of one irrep belong to a single equivalence class, called a congruence class. The lattice $P$ is a sublattice of the full weight lattice, and only a subset of the possible congruence classes may be present. It then suffices to check the decomposition into subgroups for one representative of each congruence class present. Better still, it is sufficient to restrict to those congruence classes that generate all the other classes. This reduces the problem to a finite computation. It gives the full answer, because the non-trivial realization of the elements of the center of the $\tilde{H}$, which gives the fundamental group, depends only on the congruence class.

3 Oxidation and the maximal compact subgroup

In this section we apply the math of section 2 to the theory of oxidation. Consider a 3 dimensional sigma model on a coset $G/H$, coupled to gravity. Is it possible to interpret this as the effective theory resulting from toroidal compactification of a higher dimensional theory? The answer is often yes, as demonstrated by the explicit dimensional reduction of higher dimensional theories [3, 4, 5, 6]. In [7] we gave a constructive approach, in which one can derive the existence of the higher dimensional theory, and the details (such as field content, dynamical and constraint equations) from an analysis of the properties of $G$. Here instead, we focus on the maximal compact subgroup $H$, and show that its topology immediately tells us whether the 3 dimensional coset model on $G/H$ can be oxidized.

To motivate our discussion, we consider the possibility of adding fermions to the theory (the papers [3, 4, 5, 6, 7, 8] focus almost exclusively on bosons). Consider the reduction of General Relativity from $d$ to 3 dimensions [18]. This gives rise to a sigma model on $SL(d-2)/SO(d-2)$. The $SO(d-2)$ group appearing here can be thought of as the remnant of the helicity group for massless fields [3]. The actual maximal compact subgroup in $SL(n, \mathbb{R})$ is $SO(n)$ (see appendix [A]), and $\pi_1(SO(n)) = \mathbb{Z}_2$ (for $n > 2$), which is of course crucially related to fermions. If we want to add massless fermions to the theory, these necessarily transform in representations of the helicity group $Spin(d-2)$ that are not representations of $SO(d-2)$. In the special case $d = 4$ we have $SO(2)$, and should normalize the unit of charge. It is customary to normalize these in such a way that the bosons in the theory have integral charge (spin). Then the fermions turn out to have half-integral spins, so also here we are dealing with a two-fold cover of the group relevant to the bosons.

In [7] it was argued that the index 1 embeddings of $SL(d-2, \mathbb{R}) \subset G$ give all the relevant information to reconstruct the higher dimensional theories. Via the embedding of $H \subset G$, and the embedding of $SO(d-2) \subset SL(d-2, \mathbb{R}) \subset G$, the representations of the helicity
group are encoded in $H$.

But $SO(d-2)$ is also relevant to the existence of fermions. The fact that these transform in the double cover of the group remains true after dimensional reduction. But then it is crucial that $H$ must have a topology that is compatible with that of $SO(d-2)$. Roughly, the $2\pi$ rotation that leaves bosons invariant and multiplies fermions with a minus sign, must be realized in the local symmetry groups $H$, in the compactified theory. More precise, for a theory that has its origin in $d$ dimensions with $d > 4$, the topology of $H$ has to be such that it can accommodate the $\mathbb{Z}_2 = \pi_1(SO(d-2))$, otherwise one can violate spin-statistics via compactification. The same is true in $d = 4$; the $SO(2)$ helicity group must allow a double cover, which is necessarily also an $SO(2)$. Hence if the sigma model on $G/H$ can be oxidized to $d$ dimensions, we expect

$$\pi_1(H) \supset \mathbb{Z} \text{ (for } d = 4), \quad \pi_1(H) \supset \mathbb{Z}_2 \text{ (for } d > 4).$$  

(8)

This gives a necessary criterion. We now study the possible symmetric spaces $G/H$ with simple non-compact $G$, and the topology of the maximal compact subgroup $H$. The results are given in table 1. The list of symmetric spaces is taken from [19]. We have listed whether the non-compact form is generated by an inner or outer (Cartan) involution. The topologies of compact subgroups were computed with theorem 1 of section 2 (for inner Cartan involutions), and in appendix A (for outer Cartan involutions). The maximal oxidation dimension for various theories can be found in [7], and references therein. Multiple entries indicate branches with different end-points.

Interestingly, restricting to simple Lie groups, the necessary criterion (8) turns out to be a sufficient criterion; the subset-symbols can be changed to equal signs! Hence we have:

**Theorem 2:** Consider a sigma model in 3 dimensions on a symmetric space $G/H$, with $G$ a simple non-compact group and $H$ its maximal compact subgroup, coupled to gravity. This sigma model can be oxidized to a higher dimensional model if and only if the group $H$, as embedded in $G$, is not simply connected. Moreover, the maximal oxidation dimension $d$ is given by:

$$d = 3 \quad \text{if } \quad \pi_1(H) = 0$$

$$d = 4 \quad \text{if } \quad \pi_1(H) = \mathbb{Z}$$

$$d > 4 \quad \text{if } \quad \pi_1(H) = \mathbb{Z}_2$$

(9)

**Proof:** By inspection of table 1.

This can be immediately extended to non-simple groups. When the symmetric space is of the form $\prod_i G_i/H_i$, the compact subgroup has fundamental group $\prod_i \pi_1(H_i)$, and one can choose to oxidize from any simple factor, to which the above criterion can be applied.

The complex groups are generated from two copies of the same algebra, with the outer automorphism that exchanges the two copies as Cartan involution. This gives a symmetric space $H^C/H$, where $H$ is the simply connected compact form of the group. Hence criterion (8) suggests they cannot be oxidized, which is indeed true [7].
| $g$       | $h$          | Cartan | $\pi_1(H)$ | $d$          |
|-----------|--------------|--------|------------|--------------|
| $sl(2)$   | $u(1)$       | in     | $\mathbb{Z}$ | 4            |
| $sl(n > 2)$ | $so(n)$     | out    | $\mathbb{Z}_2$ | $n + 2$      |
| $su^*(2n)$ | $sp(n)$     | out    | 0          | 3            |
| $su(p, q)$ | $su(p) \oplus su(q) \oplus u(1)$ | in | $\mathbb{Z}$ | 4            |
| $so(1, 2) \cong sl(2)$ | $u(1)$ | in | $\mathbb{Z}$ | 4            |
| $so(1, n), n > 2$ | $so(n)$ | in/out | 0 | 3            |
| $so(2, n), n > 2$ | $u(1) \oplus so(n)$ | in | $\mathbb{Z}$ | 4            |
| $so(p, q), p, q > 2$ | $so(p) \oplus so(q)$ | in/out | $\mathbb{Z}_2$ | max($p + 2, q + 2$), 6 (iff. $p$ or $q \geq 4$) |
| $so^*(2n)$ | $u(1) \oplus su(n)$ | in | $\mathbb{Z}$ | 4            |
| $sp(n, \mathbb{R})$ | $u(1) \oplus su(n)$ | in | $\mathbb{Z}$ | 4            |
| $sp(p, q)$ | $sp(p) \oplus sp(q)$ | in | 0 | 3            |
| $e_6(6)$   | $sp(4)$      | out    | $\mathbb{Z}_2$ | 8            |
| $e_6(2)$   | $su(6) \oplus su(2)$ | in | $\mathbb{Z}_2$ | 6            |
| $e_6(-14)$ | $so(10) \oplus u(1)$ | in | $\mathbb{Z}$ | 4            |
| $e_6(-26)$ | $f_4$        | out    | 0          | 3            |
| $e_7(7)$   | $su(8)$      | in     | $\mathbb{Z}_2$ | 10,8         |
| $e_7(-5)$  | $so(12) \oplus su(2)$ | in | $\mathbb{Z}_2$ | 6            |
| $e_7(-25)$ | $e_6 \oplus u(1)$ | in | $\mathbb{Z}$ | 4            |
| $e_8(8)$   | $so(16)$     | in     | $\mathbb{Z}_2$ | 11,10        |
| $e_8(-24)$ | $e_7 \oplus su(2)$ | in | $\mathbb{Z}_2$ | 6            |
| $f_4(4)$   | $sp(3) \oplus su(2)$ | in | $\mathbb{Z}_2$ | 6            |
| $f_4(-20)$ | $so(9)$      | in     | 0          | 3            |
| $g_2(2)$   | $su(2) \oplus su(2)$ | in | $\mathbb{Z}_2$ | 5            |

Table 1: Non-compact algebra’s $g$, maximal compact subalgebra’s $h$, Cartan involution, the fundamental group of $H$, and the maximal oxidation dimension $d$. 
We conclude with a remark relevant to oxidation of theories with fermions. In \cite{7} we have explained how the decomposition of the non-compact group $G$ into subgroups leads to information on the bosonic content of the higher dimensional theories. As the fermions do not arise in representations of $G$, they were not considered. When one instead turns to the compact groups $H$ and $\tilde{H}$, it remains true that $\text{Spin}(d - 2)$ should play a role as helicity group, and therefore the decompositions of $H$ into $\text{Spin}(d - 2)$ are relevant.

A well known example is the embedding $\text{Spin}(16) \rightarrow \text{Spin}(9)$ that is relevant for retrieving the 11 dimensional supergravity from maximal 3 dimensional supergravity. The bosonic degrees of freedom of the 3-d theory are organized in the $128_s$ of $\text{Spin}(16)$, while the fermions are in the other spin irrep $128_c$ \cite{20}. Decomposing to $\text{Spin}(9)$ this gives

$$128_s \rightarrow 44 \oplus 84 \quad 128_c \rightarrow 128$$

(10)

Actually, all irreps of $\text{Spin}(16)/\mathbb{Z}_2 \subset E_{8(8)}$ (which are bosons, like the $128_s$) decompose to irreps of $SO(9)$. Similar decompositions can be made for $\text{Spin}(16) \rightarrow \text{Spin}(d - 2) \times \tilde{H}_d$ for maximal supergravity, but also for the other oxidizable theories in table \ref{table:1}. This seems to hint at a generalization of the concept of “spin structure” with other groups than orthogonal ones.

### 4 Holonomy and symmetries of M-theory

Before investigating the proposal of \cite{10}, let us reexamine its logic.

Upon compactification of 11 dimensional supergravity, there exist formulations of the resulting effective theories exhibiting a symmetry $\text{Spin}(1, d - 1) \times \tilde{H}_d$, where the first factor is the Lorentz-group, and $\tilde{H}_d$ is (the double cover of) the maximal compact subgroup of a Cremmer-Julia group \cite{2, 18}. The existence of these “hidden symmetries” prompts the question whether they are a consequence of compactification, or are already present in some form in the higher dimensional theory. An answer to this question was given in \cite{13}, where formulations of 11 dimensional supergravity with a local $\text{Spin}(1, d - 1) \times \tilde{H}_d$ invariance were constructed.

Now because $\tilde{H}_d$ is a local gauge symmetry in such formulations, it is hard to conceive of a way of breaking it. In compactification it is “broken” by the presence of boundary conditions. More accurately, this means that there is non-trivial holonomy in the group $\tilde{H}_d$, such that it is no longer a manifest symmetry of the lower dimensional theory. The group $\tilde{H}_d$ includes the Lorentz group $\text{Spin}(11 - d)$ of the compactified dimensions, but as the Cremmer-Julia analysis shows, it is almost always bigger. This allows the possibility of a background with holonomy in (a subgroup of) $\text{Spin}(1, d - 1) \times \tilde{H}_d$ \cite{9}.

It is now proposed to generalize this concept to a new formulation of 11 dimensional supergravity, with a symmetry group $\tilde{H}_0$ which contains all possible $\tilde{H}_d$ \cite{11}. Clearly, $\tilde{H}_0$,
Table 2: Above the double line: Cremmer-Julia groups $G_d$; their compact subgroups $H_d$. Below the double line: Candidate “generalized holonomy”-groups in lower dimensions

should include all groups $Spin(1, d - 1) \times \tilde{H}_d$ from space-like compactifications, as well as analogous groups for time-like, and null-compactifications. In [10] it is claimed that the finite-dimensional group $SL(32, \mathbb{R})$ meets all the required properties.

We will here re-examine this claim. The works [9, 10, 11] study the symmetry at the level of the supercovariant derivative, acting on spinors. These spinors form the supersymmetry parameters, and their $\tilde{H}_d$ representation carries over to their gauge fields, the gravitini. In this study of the global properties of the symmetry groups, we also have to look at their realization on the other fields, organized in (other) $\tilde{H}_d$ representations.

In table 2 we have listed the Cremmer-Julia groups $G_d$ and their maximal compact subgroup $H_d$ that appear in space-like reduction. We will not discuss the groups for time-like reductions\(^3\) [21], and null reductions [9]. We have taken special care to list the subgroups with their proper topologies. Below the double line, we have listed the candidate “generalized holonomy”-groups, without making claims on their topology: Until further notice, we remain open-minded on what these are subgroups of, or what they are acting on, and as such, it is not yet clear what their topology should be.

We should emphasize that the groups $H_d$ listed are subgroups of the Cremmer-Julia groups, and as such only refer to the bosonic sectors of the theory. The fermions in the theory are not in representations of $H_d$; instead they form representations of a double cover of $H_d$, $\tilde{H}_d$. This sharpens the well-known fact that the fermions do not form

\(^3\)The table for these groups would look similar to table 2 with the groups $H_d$ replaced by the appropriate (quotients of covering-)groups of [21], but would be incomplete without a discussion of the possibility of disconnected components of the group, which is outside the scope of this paper.
representations of $G_d$; actually, there exist no representations of $G_d$ that include the representations of the fermions in their $H_d$ decompositions. Hence, no additional auxiliary fields can ever cure this, and the symmetry groups $G_d$ can represent at most symmetries of the bosonic sector of the theory, never of the full theory.

Therefore, the situation is much like in standard General Relativity where fermions do not form representations of $SO(1, d-1)$, but only of its simply connected cover $Spin(1, d-1)$. When fermions are included we have to use a formalism with vielbeins $\in GL(d, \mathbb{R})/SO(1, d-1)$, local frames, and gamma matrices. Similar concepts should be invoked in the proposed formulation of 11 dimensional supergravity/M-theory.

We discuss some theories from table 2 in somewhat more detail, to give the reader an impression what these statements on topology actually mean.

- In 6 dimensions, the scalars are in the $(5, 5)$ of $Sp(2) \times Sp(2)$, there are vectors in the $(4, 4)$, while the antisymmetric tensors (self-dual and anti self-dual) transform as $(5, 1) \oplus (1, 5)$. None of these representations realizes the diagonal $\mathbb{Z}_2$ in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ center of $Sp(2) \times Sp(2)$, so they are all irreps of $(Sp(2) \times Sp(2))/\mathbb{Z}_2$. The gravitini can be split in parts with opposite chirality; they transform in $(4, 1) \oplus (1, 4)$. The remaining fermions transform as $(5, 4) \oplus (4, 5)$. The fermionic irreps realize all center elements, hence the total symmetry of the theory is $Sp(2) \times Sp(2)$.

- In 5 dimensions [22] the scalars transform in the $42$, and the vectors in the $27$. Neither of the 2 realizes the $\mathbb{Z}_2$ center of $Sp(4)$, hence they are irreps of $Sp(4)/\mathbb{Z}_2$. The gravitini are in the $8$, while the other fermions are in the $48$. These realize the full center, and the full symmetry is $Sp(4)$.

- In 4 dimensions [2], the scalars are in the $70$ of $SU(8)$ (the antisymmetric 4-tensor), the vectors in the $28$ (the antisymmetric 2 tensor); each of these is an irrep of $SU(8)/\mathbb{Z}_2$. The gravitini are in the $8$, the other fermions in the $56$, which are not irreps of $SU(8)/\mathbb{Z}_2$, but of $SU(8)$.

- In 3 dimensions [20], the compact subgroup of $E_{8(8)}$ is $Spin(16)/\mathbb{Z}_2$; the $\mathbb{Z}_2$ in the quotient is not the one that leads to $SO(16)$, but one of the other $\mathbb{Z}_2$ generators\(^4\) in the $\mathbb{Z}_2 \times \mathbb{Z}_2$ center of $Spin(16)$. The scalars are in the $128_s$, which is the only spin irrep of $Spin(16)/\mathbb{Z}_2$; the gravitini are in the $16$, and the fermions are in the $128_c$ (the other spin irrep), which are not irreps of $Spin(16)/\mathbb{Z}_2$. The full symmetry is therefore $Spin(16)$.

The reason to highlight these examples is that they are of direct relevance to the formulations of 11 dimensional supergravity proposed in [13], that form an important ingredient in the “generalized holonomy”-proposal. All these examples feature bosons transforming\(^4\)The $Spin(32)/\mathbb{Z}_2$ group appearing in heterotic string theory is defined in the same way.

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in representations of an $H_d$ which is not simply connected. The fermions transform in irreps of the simply connected cover $\tilde{H}_d$ of $H_d$, making the actual symmetry group a simply connected one.

The next element in the analysis is restriction to higher dimensional groups. Clearly, each $H_d (\tilde{H}_d)$ must contain all higher dimensional entries $H_{d+n} (\tilde{H}_{d+n})$, leading to a chain

$$Spin(16) \supset SU(8) \supset Sp(4) \supset Sp(2) \times Sp(2) \supset \ldots$$

(11)

As the reader can verify, in this chain we have mentioned the correct topologies, e.g. the subgroup of $Spin(16)$ with an $su(8)$ algebra is really $SU(8)$, and not some non-simply connected version. There exists also a chain where all groups in (11) are quotiented by a $\mathbb{Z}_2$, that describes the embeddings of the bosons only.

We expect that this line of argument can be extended to the lower dimensional part of the chain, the hypothetical “generalized holonomy” groups; even though we do not know their explicit realization, they have to contain the higher dimensional groups (the tildes can be omitted for the bosons)

$$\tilde{H}_d \supset \tilde{H}_{d+1} \supset \tilde{H}_{d+2} \supset \ldots$$

(12)

Hence we turn to $\tilde{H}_0 = SL(32, \mathbb{R})$. Looking one entry higher in table 2, we should require that it contains a group with algebra $so(32)$. Now $SL(32, \mathbb{R})$ has as maximal compact subgroup $SO(32)$, so there is only one possible embedding. But, this compact group is really $SO(32)$, and not $Spin(32)$ (see appendix A); there is no decomposition of any $SL(32, \mathbb{R})$ irrep that will ever give a spin irrep of $Spin(32)$.

It is not necessary to spend much time on the other decompositions; $SO(32)$ decomposes in $SO(16) \times SO(16)$, and this group has no $Spin(16)$ subgroups, only $SO(16)$ can be a subgroup. Irrespective of the way in which the hypothetical $SL(32, \mathbb{R})$ symmetry is realized, there does not exist an $SL(32, \mathbb{R})$ irrep that has the representation of the scalars and the fermions in 3 dimensional supergravity in its decomposition (since these are in irreps of $Spin(16)$, but not of $SO(16)$). Hence we conclude that $SL(32, \mathbb{R})$ cannot be the final answer for a symmetry group of M-theory.

The above argument relies on the embedding

$$SL(32, \mathbb{R}) \supset SO(32) \supset (SO(16) \times SO(16)) \supset SO(16)$$

(13)

as the motivation for the original introduction of $SL(32, \mathbb{R})$ did [10]. Although this would be less attractive, one might hope that another embedding chain $SL(32, \mathbb{R}) \supset \ldots \supset Spin(16)$ with other intermediate groups exists, but this is not so. This can be argued by noticing that $SL(32, \mathbb{R})$ has no index 1 orthogonal subgroups; every orthogonal subgroup has to be index 2 or higher. But every index 2 orthogonal subgroup is an $SO(n)$ and never a $Spin(n)$ group. Yet higher index subgroups can be $Spin$-groups, for example,
$SO(16) \subset SL(16)$ has a $Spin(9)$ subgroup for which the $16$ dimensional vector irrep of $SO(16)$ decomposes into the spin irrep of $Spin(9)$ of equal dimension. We however require $128$-dimensional irreps, and hence would need at least $SO(128)$ for this type of construction to work. It therefore seems impossible that $Spin(16)$ is a subgroup of $SL(32, \mathbb{R})$.

Another possibility that could save (part of) the proposal is, that the realization of $SL(32, \mathbb{R})$ on the theory is such that the method of restriction to higher dimensional $H_d$ is invalid, but this also seems an unlikely, and highly unattractive option.

We now turn to the proposed generalized holonomy group $\tilde{H}_1 = Spin(32)$. It leads to similar problems as $\tilde{H}_0 = SL(32, \mathbb{R})$. The $so(16) \oplus so(16)$-subalgebra of $so(32)$ generates $(Spin(16) \times Spin(16))/\mathbb{Z}_2$ which is not simply connected. If we are willing to accept $(Spin(16) \times Spin(16))/\mathbb{Z}_2$ as the 2 dimensional symmetry group, it turns out to be incompatible with the symmetry in 3 dimensions. If the gravitini are in the $(16, 1) \oplus (1, 16)$ of $(Spin(16) \times Spin(16))/\mathbb{Z}_2$, as suggested in [9], then the 3-d symmetry-group with $so(16)$ algebra must be embedded diagonally in $(Spin(16) \times Spin(16))/\mathbb{Z}_2$ (because both chiralities of the 2-d gravitini must give rise to the 16 of the 3-d gravitini). But the diagonal group in $(Spin(16) \times Spin(16))/\mathbb{Z}_2$ is $SO(16)$, as a computation demonstrates. A set of representatives generating all congruence classes of $(Spin(16) \times Spin(16))/\mathbb{Z}_2$ is $(16, 1)$, $(128_s, 128_s)$ and $(128_s, 128_c)$. These decompose as:

\begin{align*}
(16, 1) & \rightarrow 16 \\
(128_s, 128_s) & \rightarrow 1 \oplus 120 \oplus 1820 \oplus 8008 \oplus 6435 \\
(128_s, 128_c) & \rightarrow 16 \oplus 560 \oplus 4368 \oplus 11440
\end{align*}

which are all $SO(16)$ irreps. Hence also assuming $Spin(32)$ as a symmetry-group appears to be inconsistent with the established higher dimensional symmetry groups, at least with the embeddings suggested in [9].

We cannot find an objection against $\tilde{H}_2 = Spin(16) \times Spin(16)$ this way, but it seems appropriate to conclude here with the observation that the caveats mentioned by Duff and Liu [9] are very real, and seem fatal in some cases.

5 Concluding remarks

In this article we have tried to draw attention to the often neglected global properties of Lie-groups and their subgroups, as they appear in supergravity and related theories. To demonstrate that this is not merely a mathematical enterprise, but actually can give decisive tools, we have illustrated our methods by two applications.

First we have discussed how the inclusion of fermions in theories that give rise to cosets $G/H$ in 3 dimensions leads to restrictions on the topology of the compact subgroups $H$. 

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Therefore, the first fundamental group \( \pi_1(H) \) of \( H \) indicates whether the theory can be interpreted as a reduction of a higher dimensional theory.

In a second application, we have studied the actual topology of the compact subgroups \( H_d \) of the Cremmer-Julia groups \( G_d \) of maximal supergravity. The semi-simple groups appearing here are two-fold connected (for \( d < 8 \)). Additional fermions transform in the double cover \( \tilde{H}_d \) of \( H_d \), and hence do not form representations of \( H_d \), and cannot, even with the inclusion of additional auxiliary fields, be promoted to representations of the Cremmer-Julia groups \( G_d \). In particular, this identifies the symmetry group of 3 dimensional supergravity as \( Spin(16) \) (and not \( SO(16) \) or \( Spin(16)/\mathbb{Z}_2 \)).

We then showed that the recently proposed \( SL(32, \mathbb{R}) \) \[10\] does not contain \( Spin(16) \); it does contain \( SO(16) \) but this is not the proper symmetry group of 3 dimensional supergravity. This demonstrates that there are symmetries of supergravity not contained in \( SL(32, \mathbb{R}) \). The group \( \tilde{H}_1 = Spin(32) \) does have \( Spin(16) \) subgroups, but their embedding seems incompatible with the representations of the gravitini; an embedding compatible with the gravitini representations gives \( SO(16) \) and not \( Spin(16) \). Hence \( SL(32, \mathbb{R}) \) and \( Spin(32) \) seem to be ruled out as candidate symmetries of 11 dimensional supergravity, and M-theory, at least as local symmetries in formulations of the type presented in \[13\].

This does not mean of course that the idea of “generalized holonomy” is wrong, but indicates that the concept is much more involved. As 11 dimensional Lorentz-spinors are real and have 32 components, \( GL(32, \mathbb{R}) \) or \( SL(32, \mathbb{R}) \) and their subgroups may seem natural guesses for symmetry groups of M-theory. Indeed, \( SL(32, \mathbb{R}) \) acts as an automorphism on the supersymmetry algebra but this is not sufficient to elevate it to a symmetry of the theory\(^5\); one also has to require that the symmetry is properly represented on (all fields in) the theory. Even then, this is still not sufficient to elevate it to a local symmetry.

It is interesting to compare with the situation for supergravity in 3 dimensions, which played an important part in our story. There, the spinor parameters have a symmetry group \( SO(16) \), they are real and come in 16 dimensional irreps. That does not imply however that \( SO(16) \) is a symmetry of the theory; the bosons and fermions of 3-d supergravity transform in irreps of \( Spin(16) \), that are not irreps of \( SO(16) \). Another counterexample to the assertion “automorphism of symmetry algebra implies symmetry of the theory” is given by a theory with a symmetry group \( G \) allowing an outer automorphism \( T \). Such a theory need not be invariant under the outer automorphism, because nothing guarantees that the irreps of \( G \) fill out multiplets of \( G \times T \).

The group \( SL(32, \mathbb{R}) \) was also suggested in \[23\] as a possible symmetry group of M-theory;

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\(^5\)The proposal that one may enlarge the tangent space group to a generalized structure group if the Killing spinor equation has hidden symmetries \[9\] is essentially a rephrasing of this, and the same criticism applies: Hidden symmetries of the Killing spinor equations are a necessary, but far from sufficient ingredient for enlarging the structure group. Note that the supercovariant derivative used in \[9, 10, 11\] is not \( SL(32, \mathbb{R}) \) covariant.
and identified \[12\] with the \( \text{SL}(32, \mathbb{R}) \) of \[10\]. This claim is due to the identification of parts of the symmetry of \( \text{SL}(32, \mathbb{R}) \). Since it is believed that M-theory has large symmetry algebras, there seems to be no problem in promoting these symmetries to be substructures of some bigger group/algebra.

That the “natural” symmetry groups \( \text{SL}(32, \mathbb{R}) \) and \( \text{GL}(32, \mathbb{R}) \) do not contain all known symmetries of 11 dimensional supergravity is an indication that we might have to transcend beyond Lorentz-covariant formalisms, and turn to bigger finite, or infinite groups (with the immediate question how to realize them on a 32 component spinor). This brings the infinite-dimensional groups \( E_{10} \) and \( E_{11} \) (and their subgroups) back into the picture \[4 \] \[12 \] \[24 \]. Note however, that in the light of the previous discussion it seems unavoidable that also these groups can at most appear in the numerator of some quotient, and that fermions in the theory should appear in representations of a covering of the quotient group in the denominator. This in turn seems to imply that not \( E_{11} \), but at most the “simply connected covering” of its “maximal compact subgroup” can give a symmetry of M-theory. However, in absence of a more detailed understanding of these groups, it is not even clear what the precise meaning of the phrases in parentheses (which were written by analogy to finite-dimensional groups) is.

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A Compact subgroups and outer automorphisms

Theorem 1 and its applications mentioned in section \[2\] cover all the cases where the non-compact form of \( G \) is generated by an inner automorphism. The other case, where the non-compact form is generated by an outer automorphism will be treated in this appendix. Only the groups corresponding to the Dynkin diagrams of \( A_n \), \( D_n \), \( E_6 \) allow outer automorphisms. We discuss these case by case.

A.1 \( A_n \)

\( sl(n, \mathbb{R}), (n > 2) \): The maximal compact subgroup is \( SO(n) \). The embedding proceeds via the fundamental \( n \) irrep of \( SL(n, \mathbb{R}) \), that descends to the \( n \) vector irrep of \( SO(n) \). All irreps of \( SL(n, \mathbb{R}) \) can be found by tensoring the fundamental \( n \) with copies of itself and (anti-)symmetrizing. The irreps of \( SO(n) \) as embedded in \( SL(n, \mathbb{R}) \) are found in
the same way. This procedure never gives any of the spin irreps of Spin(n), and hence \( \pi_1(SO(n)) = \mathbb{Z}_2 \).

\( su^\ast(2n) \): This has maximal compact subgroup \( Sp(n) \). The embedding proceeds via the fundamental \( 2n \) irrep of \( SU^\ast(2n) \) that descends to the fundamental of \( Sp(n) \). Again all other irreps can be found by tensoring. The fundamental of \( Sp(n) \) realizes the full center, hence we are dealing with \( Sp(n) \) and not \( Sp(n)/\mathbb{Z}_2 \).

**A.2 \( D_n \)**

\( so(1, n) \): This real form is generated by an outer automorphism for \( n \) odd. Three representatives generating all congruence classes are the vector irrep \( (n + 1) \) and the spin irreps \( (2^{n-1})_{s,c} \). These decompose to the compact subgroup \( Spin(n) \) as

\[
(n + 1) \to n \oplus 1 \quad (2^{n-1})_{s,c} \to 2^{n-1}
\]

The presence of the spin irrep of \( Spin(n) \) implies that the subgroup is simply connected.

\( so(p, q) \): This real form is generated by an outer automorphism if \( p \) and \( q \) are both odd. Taking again the vector and spin irreps of as representatives, decompositions into the \( Spin(p) \times Spin(q) \) subgroup are

\[
(p + q) \to (p, 1) \oplus (1, q) \quad (2^{n-1})_{s,c} \to (2^{n-1}, 2^{n-1})
\]

These irreps do not realize the diagonal \( \mathbb{Z}_2 \) in the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) center of \( Spin(p) \times Spin(q) \), hence the proper subgroup is \( (Spin(p) \times Spin(q))/\mathbb{Z}_2 \).

**A.3 \( E_6 \)**

\( E_6(6) \): The maximal compact subgroup is \( Sp(4) \). A possible representative is the 27 dimensional irrep of \( E_6(6) \) from which all other irreps can be generated by tensoring. But this irrep descends to the 27, which is an irrep of \( Sp(4)/\mathbb{Z}_2 \). Hence, the compact subgroup is a two-fold connected group.

\( E_6(-26) \): The compact subgroup is \( F_4 \). The universal cover of \( F_4 \) has a trivial center, and hence all possible forms of the compact group \( F_4 \) are simply connected.

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