Robustly inverse shadowing diffeomorphisms

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Abstract

Let $M$ be a closed smooth Riemannian manifold $M$, and let $f : M \to M$ be a diffeomorphism. Herein, we demonstrate that (i) if $f$ has the $C^1$ robustly inverse shadowing property on the chain recurrent set $\text{CR}(f)$, then $\text{CR}(f)$ is hyperbolic and (ii) if $f$ has the $C^1$ robustly inverse shadowing property on a nontrivial transitive set $\Lambda \subset M$, then $\Lambda$ is hyperbolic for $f$. Especially, the item (ii) is a proof of the conjecture of Lee and Lee [11].

1 Introduction

The inverse shadowing property is a dual notion of the shadowing property that was introduced by Corless and Pilyugin [3]. However, the notions are not the same in general. Kloeden and Ombach [9] proved that if an expansive diffeomorphism $f$ has the shadowing property, then it has the inverse shadowing property with respect to the continuous method $T_h$ (see the definition in section 2). Regarding Lewowicz’s results [16], the Pseudo-Anosov map $f$ of a compact surface $S$ contains the inverse shadowing property with respect to the class of the continuous method $T_h$; however, it is expansive and not topologically stable. Therefore, it does not have the shadowing property. To study the hyperbolic structure (Anosov, structurally stable, Axiom A, $\Omega$-stable, hyperbolic, etc.), the shadowing theories are highly useful concepts. In fact, the concepts are close to the hyperbolic structure. Robinson [23] and Sakai [26] proved that a diffeomorphism $f$ of a compact smooth manifold $M$ belongs to the $C^1$ interior of the set of all diffeomorphisms having the shadowing property if and only if it has the hyperbolic structure. Pilyugin [22] proved that a diffeomorphism $f$ of a compact smooth manifold $M$ belongs to the $C^1$ interior of the set of diffeomorphisms having the inverse shadowing property with respect to the continuous method $T_c$ (see the definition in section 2) if and only if it has the hyperbolic structure.

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It is shown that if a diffeomorphism $f$ of a compact smooth manifold $M$ is topologically stable, then it has the inverse shadowing property with respect to the class of the continuous method $T_d$ (see the definition in section 2). That is, the inverse shadowing property with respect to the class of the continuous method $T_d$ implies that it is topologically stable. Bowen [2] proved that if a diffeomorphism $f$ of a compact smooth manifold $M$ is hyperbolic, then it has the shadowing property. Lee [10] proved that if a diffeomorphism $f$ of a compact smooth manifold $M$ is hyperbolic, then it has the inverse shadowing property with respect to the class of the continuous method $T_d$. Therefore, we know that if a diffeomorphism $f$ has the hyperbolic structure, then it has the shadowing and inverse shadowing properties with respect to the class of the continuous method $T_d$.

However, regarding the local dynamical systems with the $C^1$ robust property (see definition 3.1), the results of two concepts are different. Lee [14] proved that if a diffeomorphism $f$ has the $C^1$ robustly shadowing property on the transitive set $\Lambda$, then $\Lambda$ is a hyperbolic for $f$. Lee and Lee [11] proved that if a diffeomorphism $f$ has the $C^1$ robustly inverse shadowing property with respect to the class of the continuous method $T_d$ on the transitive set $\Lambda$, then $\Lambda$ admits a dominated splitting for $f$. However, it is still unclear if a diffeomorphism $f$ has the inverse shadowing property with respect to the class of the continuous method $T_d$ on the transitive set $\Lambda$, thus causing $\Lambda$ to be hyperbolic. Therefore, we will prove the problem herein, which is the primary theorem.

The paper is organized as follows. In section 2, we introduce the shadowing and inverse shadowing properties. In section 3, we introduce the basic notions and primary theorems. In section 4, we prove Theorem A. Finally, in section 5, we prove Theorem B.

### 2 Shadowing and Inverse shadowing properties

Let $M$ be a compact smooth Riemannian manifold without boundary, and let $\text{Diff}(M)$ be the space of $C^1$ diffeomorphisms of $M$ with the $C^1$ topology. Let $\Lambda \subset M$ be a closed $f$-invariant set. For any $\delta > 0$, a sequence of points $\{x_i\}_{i \in \mathbb{Z}} \subset \Lambda$ is regarded as the $\delta$ pseudo orbit of $f$ if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. We say that a diffeomorphism $f$ has the shadowing property on $\Lambda$ if for any $\epsilon > 0$, we can find $\delta > 0$ such that for any $\delta$ pseudo orbit $\{x_i\}_{i \in \mathbb{Z}} \subset \Lambda$, a point $y \in M$ exists such that $d(f^i(y), x_i) < \epsilon$ for all $i \in \mathbb{Z}$. If $\Lambda = M$, then we say that a diffeomorphism $f$ has the shadowing property. It is known that a diffeomorphism $f$ has the shadowing property if and only if $f^n$ has the shadowing property for all $n \in \mathbb{Z} \setminus \{0\}$; further, it $f$ has the shadowing property, then $f$ has the shadowing property on $\Lambda$.

Let $M^\mathbb{Z}$ be the space of all two-sided sequences $\{x_i : i \in \mathbb{Z}\} \subset M$ endowed with the product topology. For any $\delta > 0$, we define

$$
\Gamma_f(\delta) = \{\{x_i : i \in \mathbb{Z}\} : \{x_i : i \in \mathbb{Z}\} \text{ is a } \delta \text{ pseudo orbit of } f\}.
$$

A mapping $\xi : M \to \Gamma_f(\delta) \subset M^\mathbb{Z}$ is regarded as $\delta$-method for $f$ if $\xi(x)_0 = x$, and $\xi(x)$ is a $\delta$ pseudo orbit of $f$ through $x$, where $\xi(x)_0$ means that the 0th component of $\xi(x)$. Herein,
we set $\xi(x) = \{\xi(x)_i : i \in \mathbb{Z}\}$. We say that $\xi$ is a continuous $\delta$-method for $f$ if the map $\xi$ is continuous. We denote by $\mathcal{T}_0(f, \delta)$ the set of all $\delta$ methods, and by $\mathcal{T}_c(f, \delta)$ the set of all continuous $\delta$ methods. For a homeomorphism $g : M \to M$ with $d_0(f, g) < \delta$, $g$ induces a continuous $\delta$ method $\xi(g)$ for $f$ such that

$$\xi(g(x)) = \{g^n(x) : n \in \mathbb{Z}\},$$

where $d_0$ is the $C^0$ metric. For a homeomorphism $g : M \to M$ with $d_0(f, g) < \delta$, we denote by $\mathcal{T}_h(f, \delta)$ the set of all continuous $\delta$ methods $\xi(g)$ for $f$.

According to the notions above, we define a strong continuous method that is induced by diffeomorphisms. For any $\delta > 0$ and a diffeomorphism $g : M \to M$ with $d_1(f, g) < \delta$, $g$ induces a continuous $\delta$ method $\xi(g)$ for $f$ such that

$$\xi(g(x)) = \{g^n(x) : n \in \mathbb{Z}\},$$

where $d_1$ is the $C^1$ metric. For a diffeomorphism $g : M \to M$ with $d_1(f, g) < \delta$, we denote by $\mathcal{T}_d(f, \delta)$ the set of all continuous $\delta$ methods $\xi(g)$ for $f$. We set

$$\mathcal{T}_a(f) = \bigcup_{\delta > 0} \mathcal{T}_a(f, \delta),$$

where $a = 0, c, h, d$. It is clear that

$$\mathcal{T}_a(f) \subset \mathcal{T}_h(f) \subset \mathcal{T}_c(f) \subset \mathcal{T}_0(f).$$

We say that a diffeomorphism $f$ has the $\mathcal{T}_a$-inverse shadowing property if for any $\epsilon > 0$, $\delta > 0$ such that for any $\delta$ method $\xi \in \mathcal{T}_a(f, \delta)$ and any point $x \in M$, a point $y \in M$ exists such that

$$d(f^n(x), \xi(y)_n) < \epsilon,$$

for all $n \in \mathbb{Z}$, where $a = 0, c, h, d$.

We say that a diffeomorphism $f$ has the inverse shadowing property with respect to the class of the methods $\mathcal{T}_a$ if it has the $\mathcal{T}_a$ inverse shadowing property, where $a = 0, c, h, d$.

Let $S^1 = \{(x, y) : (x - 1/2)^2 + y^2 = 1/4\}$ be a circle and let $r \in [0, 1)$. Subsequently, we define the homeomorphism $f$ on $S^1$ by (i) $f(r) = r$ if $r = 0$ or $r = 1/2$, (ii) $f(r) < r$ if $r \in (0, 1/2)$ and (iii) $f(r) > r$ if $f \in (1/2, 1)$. Let $L = \{(x, 0) : x \in [0, 1]\}$ and define a homeomorphism $g$ on $L$ by $g((x, 0)) = (x^2, 0)$. Let $X = S^1 \cup L$. Therefore, we define a homeomorphism $h : X \to X$ such that

$$h(x) = \begin{cases} f(x), & \text{if } x \in S, \\ g(x), & \text{if } x \in L. \end{cases}$$

Diamond, Lee, and Han [4] demonstrated that a homeomorphism $f : S^1 \to S^1$ has the inverse shadowing property with respect to the class of the continuous method $\mathcal{T}_h$. However, it does not have the shadowing property.
Lee and Park [13] proved that for a unit circle $S$, a diffeomorphism $f : S \to S$ has the shadowing property if and only if $f$ has the inverse shadowing property with respect to the class of the continuous method $\mathcal{T}_h$. Sakai [25] proved that a diffeomorphism $f$ of a compact smooth manifold $M$ belongs to the $C^1$ interior of the set of diffeomorphisms having the inverse shadowing property with respect to the class of the continuous method $\mathcal{T}_h$. It was also proved in [10] that a diffeomorphism $f$ of a compact smooth manifold $M$ belongs to the $C^1$ interior of the set of diffeomorphisms having the inverse shadowing property with respect to the class of the continuous method $\mathcal{T}_d$. We denote by $\mathcal{ISP}_a$ the set of all diffeomorphisms having the inverse shadowing property with respect to the class of the continuous method $\mathcal{T}_h$. This means that a diffeomorphism $f$ of a compact smooth manifold $M$ belongs to the $C^1$ interior of the set of all diffeomorphisms having the inverse shadowing property with respect to the class of the continuous methods $\mathcal{T}_a(a = 0, c, h, d)$. According to the results of Pilyugin [22], Sakai [25], and Lee [10],

$$int\mathcal{ISP}_c = int\mathcal{ISP}_h = int\mathcal{ISP}_d.$$ 

By definition, we know that $\mathcal{ISP}_c \subset \mathcal{ISP}_h \subset \mathcal{ISP}_d$. However, $\mathcal{ISP}_c \neq \mathcal{ISP}_h \neq \mathcal{ISP}_d$, in general. It is noteworthy that $f$ has the inverse shadowing property with respect to the class of the continuous method $\mathcal{T}_d$ if and only if $f^n$ has the inverse shadowing property with respect to the class of the continuous method $\mathcal{T}_d$, for all $n \in \mathbb{Z} \setminus \{0\}$ (see [10]). It is clear that if $f$ has the inverse shadowing property with respect to the class of the continuous method $\mathcal{T}_d$, then $f$ has the inverse shadowing property on $\Lambda \subset M$ with respect to the class of the continuous method $\mathcal{T}_d$.

In this study, we consider the inverse shadowing property with respect to the class of the continuous method $\mathcal{T}_d$. Therefore, we use the following expression: a diffeomorphism $f$ has the inverse shadowing property. This means that a diffeomorphism $f$ has the inverse shadowing property with respect to the class of the continuous method $\mathcal{T}_d$.

### 3 Basic notions and Theorems

In this section, we introduce some notions and primary theorems. Let $M$ be as before, and let $f \in \text{Diff}(M)$. For any $x \in M$, $\text{Orb}(x) = \{f^n(x) : n \in \mathbb{Z}\}$ denotes the orbit of $x$. A point $p \in M$ is called periodic if $\pi(p) > 0$ such that $f^{\pi(p)}(p) = p$, where $\pi(p)$ is the period of $p$. We denote by $P(f)$ the set of all periodic points of $f$. A point $x \in M$ is called nonwandering if in a neighborhood $U$ of $x$, $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. We denote by $\Omega(f)$ the set of all nonwandering points of $f$. It is known that $P(f) \subset \Omega(f)$.

For a given $x, y \in M$, we write $x \rightsquigarrow y$ if for any $\delta > 0$, a $\delta$-pseudo orbit $\{x_i\}_{i=0}^n(n > 1)$ of $f$ exists such that $x_0 = x$ and $x_n = y$. We write $x \rightsquigarrow y$ if $x \rightsquigarrow y$ and $y \rightsquigarrow x$. The set $\{x \in M : x \rightsquigarrow x\}$ is called the chain recurrent set of $f$ and is denoted by $\mathcal{CR}(f)$. It is known that $\Omega(f) \subset \mathcal{CR}(f)$, and $\mathcal{CR}(f)$ is a closed $f$-invariant set.

A closed $f$-invariant set $\Lambda \subset M$ is called hyperbolic for $f$ if the tangent bundle $T\Lambda M$ exhibits a $Df$-invariant splitting $E^s \oplus E^u$ and constants $C > 0$ and $0 < \lambda < 1$ exist such
that
\[ \|D_x f^n|_{E_x^u}\| \leq C\lambda^n \quad \text{and} \quad \|D_x f^{-n}|_{E_x^s}\| \leq C\lambda^n \]
for all \( x \in \Lambda \) and \( n \geq 0 \).

We say that \( f \) satisfies Axiom A if the nonwandering set \( \Omega(f) \) is hyperbolic and it is the closure of \( P(f) \).

According to Smale [27], if \( f \) satisfies Axiom A, then the nonwandering set \( \Omega(f) = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_m \), where \( \Lambda_i \) are compact, disjoint, invariant sets, and each \( \Lambda_i \) contains dense periodic orbits. The sets \( \Lambda_1, \ldots, \Lambda_m \) are called the basic sets. For a basic set \( \Lambda_i \), we define the following:

\[ W^s(\Lambda_i) = \{ x \in M : \lim_{n \to \infty} d(f^n(x), \Lambda_i) = 0 \} \]
\[ W^u(\Lambda_i) = \{ x \in M : \lim_{n \to -\infty} d(f^n(x), \Lambda_i) = 0 \} \]

For the basic sets \( \Lambda_i (1 \leq i \leq n) \), we define \( \Lambda_i > \Lambda_j \) if

\[ (W^s(\Lambda_i) \setminus \Lambda_i) \cap W^u(\Lambda_j) \neq \emptyset. \]

We say that \( f \) satisfies the no-cycle condition if \( \Lambda_{i_0} > \Lambda_{i_1} > \cdots > \Lambda_{i_j} > \Lambda_{i_0} \) cannot occur among the basic sets.

Let \( \Lambda \subset M \) be a closed \( f \)-invariant set. We say that \( \Lambda \) is locally maximal if a neighborhood \( U \) of \( \Lambda \) exists such that \( \Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U) \).

**Definition 3.1** Let \( f \in \text{Diff}(M) \). We say that \( f \) has the \( C^1 \) robustly \( P \) property on \( \Lambda \) if a \( C^1 \) neighborhood \( U(f) \) of \( f \) and a neighborhood \( U \) of \( \Lambda \) exist such that (i) \( \Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U) \), and (ii) for any \( g \in U(f) \), \( g \) has the \( P \) property on \( \Lambda_g \) where \( \Lambda_g \) is the continuation of \( \Lambda \).

In the definition, if \( P \) is the shadowing, then it was defined by Lee, Moriyasu, and Sakai [12]. If \( P \) is the inverse shadowing, then it was defined by Lee and Lee [11]. Herein, we use the second case where \( P \) is the inverse shadowing.

It is known that if a closed set \( \Lambda \) is hyperbolic for \( f \), then \( f \) has the inverse shadowing property on \( \Lambda \). By the stability of hyperbolic invariant sets for \( f \) ([24 Theorem 7.4]), if a closed \( f \)-invariant set \( \Lambda \) is hyperbolic for \( f \), then a \( C^1 \) neighborhood \( U(f) \) and a neighborhood \( U \) of \( \Lambda \) exist such that \( \Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U) \); further, for any \( g \in U(f) \), \( \Lambda_g = \bigcap_{n \in \mathbb{Z}} g^n(U) \) is hyperbolic. Therefore, \( g \) has the inverse shadowing property on \( \Lambda_g \). Hence, we have the following.

**Theorem A** Let \( f \in \text{Diff}(M) \), and let \( \mathcal{CR}(f) \) be the chain recurrent set of \( f \). If \( f \) has the \( C^1 \) robustly inverse shadowing property on \( \mathcal{CR}(f) \), then \( \mathcal{CR}(f) \) is hyperbolic.

A closed \( f \)-invariant set \( \Lambda \) is called transitive for \( f \) if a point \( x \in \Lambda \) exists such that \( \omega(x) = \Lambda \), where \( \omega(x) \) is the omega limit set of \( x \). In this study, we consider that a transitive set \( \Lambda \) is nontrivial as it is not one orbit. We say that a compact invariant set \( \Lambda \)
admits a dominated splitting for \( f \) if the tangent bundle \( T_\Lambda M \) exhibits a continuous \( Df \) invariant splitting \( E \oplus F \) and \( C > 0, 0 < \lambda < 1 \) such that for all \( x \in \Lambda \) and \( n \geq 0 \), we have
\[
\|Df^n|_{E(x)}\| \cdot \|Df^{-n}|_{F(f^n(x))}\| \leq C\lambda^n.
\]
As mentioned in the previous section, if a diffeomorphism \( f \) has the inverse shadowing property on a transitive set \( \Lambda \), then it admits a dominated splitting for \( f \) (see [11]). According to the results, we prove the following.

**Theorem B** Let \( f \in \text{Diff}(M) \) and let \( \Lambda \) be a transitive set of \( f \). If \( f \) has the \( C^1 \) robustly inverse shadowing property on \( \Lambda \), then \( \Lambda \) is hyperbolic for \( f \).

## 4 Proof of Theorem A

In this section, we prove the hyperbolicity of the chain recurrent set \( \mathcal{CR}(f) \) with the \( C^1 \) robustly inverse shadowing property. To prove, we use a \( C^1 \) perturbation lemma, called Franks’ lemma. The following is Franks’ lemma (see [5]):

**Lemma 4.1** Let \( \mathcal{U}(f) \) be any given \( C^1 \) neighborhood of \( f \). Therefore, \( \epsilon > 0 \) and a \( C^1 \) neighborhood \( \mathcal{U}_0(f) \subset \mathcal{U}(f) \) of \( f \) exists such that for a given \( g \in \mathcal{U}_0(f) \), a finite set \( \{x_1, x_2, \ldots, x_N\} \), a neighborhood \( U \) of \( \{x_1, x_2, \ldots, x_N\} \), and linear maps \( L_i : T_{x_i}M \to T_{g(x_i)}M \) satisfying \( \|L_i - D_{x_i}g\| \leq \epsilon \) for all \( 1 \leq i \leq N \), there exists \( \tilde{g} \in \mathcal{U}(f) \) such that \( \tilde{g}(x) = g(x) \) if \( x \in \{x_1, x_2, \ldots, x_N\} \cup (M \setminus U) \) and \( D_{x_i}\tilde{g} = L_i \) for all \( 1 \leq i \leq N \).

Using lemma [4.1] and the \( C^1 \) robustly inverse shadowing property, an important lemma exists as follows. From the lemma, we can demonstrate that if a diffeomorphism \( f \) has the \( C^1 \) robustly inverse shadowing property on \( \mathcal{CR}(f) \), then \( \mathcal{CR}(f) \) is hyperbolic.

**Lemma 4.2** Let \( \Lambda \subset M \) be a closed \( f \)-invariant set. If \( f \) has the \( C^1 \) robustly inverse shadowing property on \( \Lambda \), then for any \( g \) \( C^1 \) close to \( f \), every \( p \in \Lambda_g \cap P(g) \) is hyperbolic, where \( P(g) \) is the set of periodic points for \( g \).

**Proof.** Let \( \mathcal{U}(f) \) be a \( C^1 \) neighborhood of \( f \) and \( U \) a locally maximal neighborhood of \( \Lambda \). Suppose that \( g \in \mathcal{U}(f) \) exists such that \( g \) contains a nonhyperbolic periodic point \( p \in \Lambda_g = \bigcap_{n \in \mathbb{Z}} g^n(U) \). Because \( p \in \Lambda_g \cap P(g) \) is not hyperbolic, an eigenvalue \( \lambda \) of \( D_pg^{\pi(p)} \) exists such that \( |\lambda| = 1 \), where \( \pi(p) \) is the period of \( p \). For simplicity, we may assume that \( g^{\pi(p)}(p) = g(p) = p \). Because \( p \in \Lambda_g \cap P(g) \) is not hyperbolic, an eigenvalue \( \lambda \) of \( D_pg \) exists such that \( |\lambda| = 1 \). Therefore, \( T_pM = E^c_p \oplus E^s_p \oplus E^n_p \) is the \( D_pg \)-invariant splitting of \( T_pM \), where \( E^c_p \) corresponds to eigenvalues \( |\lambda| = 1 \) of \( D_pg \), \( E^s_p \) corresponds to eigenvalues \( |\lambda| < 1 \) of \( D_pg \), and \( E^n_p \) corresponds to eigenvalues \( |\lambda| > 1 \) of \( D_pg \). According to lemma [4.1], \( g_0 \) \( C^1 \) close to \( g \) exists such that \( g_0(p) = g(p) = p \) and \( p \) is not hyperbolic for \( g_0 \). Therefore, we have only one eigenvalue \( \lambda \) of \( D_pg_0 \) such that \( |\lambda| = 1 \) and \( T_pM = E^c_p \oplus E^s_p \oplus E^n_p \). If \( \lambda \in \mathbb{R} \),
We define a diffeomorphism $h$. Given $\alpha > \epsilon > 0$ such that $g$ has the inverse shadowing property on $\Lambda$, we know there are corresponding neighborhoods. Subsequently, we define $p$ as a point in the corresponding neighborhoods. We identify $p$ with a nonzero vector $u \in \mathbb{R}^n$. Using lemma 4.1 again, we obtain $\alpha > 0$ with $B(p, \alpha) \subset U$ and $g_1 C^1$ close to $g_0 (g_1 \in \mathcal{U}(f))$, satisfying

(a) $g_1(p) = g_0(p) = p$,
(b) $g_1(x) = \exp_p \circ D_p g_0 \circ \exp_p^{-1}(x)$ if $x \in B(p, \alpha)$, and
(c) $g_1(x) = g_0(x)$, if $x \in B(p, 4\alpha)$.

We use a nonzero vector $u \in \mathbb{E}_p^c \subset T_p M$ such that $\|u\| = \alpha/4$. Subsequently, $g_1(\exp_p(u)) = \exp_p(D_p g(\exp_p^{-1}(\exp_p(u)))) = \exp_p(u)$.

We set $J_p = \exp_p\{t \cdot u : -\frac{\alpha}{4} \leq t \leq \frac{\alpha}{4}\}$.

For the small arc $J_p$, the following properties hold:
(a) $J_p \subset B(p, \alpha) \cap \exp_p(\mathbb{E}_p^c(\alpha))$ with the center at $p$,
(b) $J_p \subset \Lambda_{\alpha}$, and
(c) $g_1\mid_{J_p} : J_p \to J_p$ is the identity map,

where $\mathbb{E}_p^c(\alpha)$ is the $\alpha$-ball in $\mathbb{E}_p^c$ centered at the origin $O_p$.

We denote $\mathbb{E}_p^c = \{u \in T_p M : u_1 \neq 0, u_2 = \cdots = u_n = 0\}$ in the coordinates of the corresponding neighborhoods. We identify $p$ with $O_p$ and $T_p M$ with $\mathbb{R}^n$ in the coordinates of the corresponding neighborhoods. Subsequently, we know $p = (0, \cdots, 0)$ and $\mathbb{E}_p^c = \{x \in \mathbb{R}^n : x_1 \neq 0, x_2 = \cdots = x_n = 0\}$. Because $f$ has the $C^1$ robustly inverse shadowing property on $\Lambda$, $g_1$ has the inverse shadowing property on $\Lambda_{\alpha} = \bigcap_{n \in \mathbb{Z}} g_0^1(U)$. We use $0 < \epsilon < \alpha/16$ and let $0 < \delta < \epsilon$ be the number of inverse shadowing properties for $g_1$. Given $\alpha > 0$, we define the map $g_1\mid_{B(p, \alpha)} : B(p, \alpha) \to B(p, \alpha)$ by $g_1(x) = (x_1, Cx')$, where $C$ is the hyperbolic part of $D_p g_1$ and $x' = (x_2, x_3, \ldots, x_n)$. We define a diffeomorphism $h : M \to M$ having the following property,

$h(x) = (x_1 + \frac{\delta}{4}, Cx')$ and $h^{-1}(x) = (x_1 - \frac{\delta}{4}, C^{-1}x')$,

for all $x = (x_1, x_2, x_3, \ldots, x_n) = (x_1, x') \in B(p, \alpha)$. Therefore, we can obtain a class of the continuous $\delta$ method $\varphi_h \in \mathcal{T}_d(g_1)$ that is induced by $h$ such that $\varphi_h(x)_n = \{h^n(x) : n \in \mathbb{Z}\}$,
for any $x \in M$. Because $\mathcal{J}_p \subset \Lambda_{g_1}$ and $g_1$ has the inverse shadowing property on $\Lambda_{g_1}$, $g_1$ must have the inverse shadowing property on $\mathcal{J}_p$.

We prove that if $g_1|_{\mathcal{J}_p} : \mathcal{J}_p \to \mathcal{J}_p$ is the identity map, then $g_1$ does not have the inverse shadowing property on $\mathcal{J}_p$.

If the pseudo point $y \in \mathcal{J}_p$, then because $g_1|_{\mathcal{J}_p} : \mathcal{J}_p \to \mathcal{J}_p$ is the identity map, we can easily demonstrate that $g_1$ does not have the inverse shadowing property on $\mathcal{J}_p$. Indeed, we choose $x_0 = (2\epsilon, 0, \ldots, 0) \in \mathcal{J}_p$ such that $d(x_0, p) = 2\epsilon$. Because $g_1$ has the inverse shadowing property on $\mathcal{J}_p$, we can use a pseudo point $y \in \mathcal{J}_p$ such that $y = p = (0, 0, \ldots, 0)$. Subsequently, we know that for $n \geq 0$,

$$d(g_1^n(x_0), \varphi_h(y)_n) = d(x_0, h^n(y)) = d(2\epsilon, \delta_4) > \epsilon.$$

Because $g_1$ has the inverse shadowing property on $\mathcal{J}_p$, this is a contradiction. If a pseudo point $y = (y_1 + \delta/4, 0, \ldots, 0) \in \mathcal{J}_p$ with $d(x_0, y) < \epsilon$, then $d(2\epsilon, y_1 + \delta/4) < \epsilon$. By our construction map $h : M \to M$, $j > 0$ exists such that $y_1 + (\delta/4)j > 3\epsilon$. Thus, $j > 0$ exists such that

$$d(g_1^j(x_0), \varphi_h(y)_j) = d(x_0, h^j(y)) = d(2\epsilon, y_1 + \delta_4j) > \epsilon.$$ 

According to the facts, $g_1$ does not have the inverse shadowing property on $\mathcal{J}_p$. Therefore, for the chosen point $x_0 \in \mathcal{J}_p$, if a pseudo point $y \in \mathcal{J}_p$, then $g$ does not have the inverse shadowing property on $\mathcal{J}_p$. Hence, the pseudo point $y \in M$ has to remain in $B(x_0, \epsilon) \setminus \mathcal{J}_p$. Consequently, for any pseudo point $y \in B(x_0, \epsilon) \setminus \mathcal{J}_p$, because $g_1$ has the inverse shadowing property on $\mathcal{J}_p$, the following inequalities hold:

$$d(g_1^n(x_0), \varphi_h(y)_n) = d(g_1^n(x_0), h^n(y)) < \epsilon$$

for all $n \in \mathbb{Z}$. Subsequently, by our defined map $h : M \to M$, for $z = (z_1, z_2, \ldots, z_n) \in M$, we know that for $n \geq 0$,

$$h^n(z) = (z_1 + \frac{\delta}{4}n, C^nz'),$$

and

$$h^{-n}(z) = (z_1 - \frac{\delta}{4}n, C^{-n}z'),$$

where $z' = (z_2, \ldots, z_n)$. Therefore, we find that $k > 0$ such that $z_1 + (\delta/4)k > 3\epsilon$. Thus, $k > 0$ exists such that

$$d(g_1^k(x_0), h^k(z)) = d(x_0, h^k(z))$$

$$= d((2\epsilon, 0, \ldots, 0), (z_1 + \frac{\delta}{4}k, C^kz'))$$

$$\geq d(2\epsilon, z_1 + \frac{\delta}{4}k) > \epsilon.$$ 

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For the point \( y \in B(x_0, \epsilon) \) with \( d(x_0, y) < \epsilon \), by \( g_1 \) has the inverse shadowing property on \( J_p \), the following inequality \( d(g_1^n(x_0), \varphi_h(y)k) < \epsilon \) holds, for all \( n \in \mathbb{Z} \). However, by the arguments above, \( k > 0 \) such that \( y_1 + (\delta/4)k > 3\epsilon \). Thus,

\[
d(g_1^n(x_0), \varphi_h(y)k) = d(x_0, h^k(y)) = d(2\epsilon, y_1 + \frac{\delta}{4}k) > \epsilon.
\]

Because \( g_1 \) has the inverse shadowing property on \( J_p \), this is a contradiction. Thus, if \( g_1|_{J_p} : J_p \to J_p \) is the identity map, then \( g_1 \) does not have the inverse shadowing property on \( J_p \).

**Case 2.** Consider \( \lambda \in \mathbb{C} \). To avoid complexity, we assume that \( g^{\pi(p)}(p) = g(p) = p \). According to lemma \ref{lem:4.1}, \( \alpha > 0 \) exists with \( B(p, \alpha) \subset U \) and \( g_1 \) close to \( g \) exhibiting the following properties:

(a) \( g_1(x) = \exp_p \circ D_p g \circ \exp_p^{-1}(x) \), if \( x \in B(p, \alpha) \),

(b) \( g_1(x) = g(x) \), if \( x \notin B(p, 4\alpha) \), and

(c) \( g_1(p) = g(p) = p \).

By modifying the map \( D_p g_1 \), \( l > 0 \)Exists such that \( D_p g_1(v) = v \) for any \( v \in E^c_p(\alpha) \cap \exp_p^{-1}(B(p, \alpha)) \). Thus, a small arc \( C_p \subset \exp_p(E^c_p(\alpha)) \cap B(p, \alpha) \) can be obtained such that \( g_1|_{C_p} = C_p \) and \( g_1|_{C_p} : C_p \to C_p \) is the identity map. Because \( g_1 \) has the inverse shadowing property, it is evident that \( g_1 \) has the inverse shadowing property for \( i \in \mathbb{Z} \setminus \{0\} \). Let \( g_1 = g_2 \). Therefore, \( g_2|_{C_p} : C_p \to C_p \) is the identity map. Thus, as in the proof of case 1, a contradiction will be shown.

We say that a diffeomorphism \( f \) is a star if a \( C^1 \) neighborhood \( U(f) \) of \( f \) exists such that for any \( g \in U(f) \), every periodic point in \( P(g) \) is hyperbolic. We denote by \( F(M) \) the set of all star diffeomorphisms. Aoki \cite{Aoki} and Hayashi \cite{Hayashi} proved that if a diffeomorphism \( f \) is a star, then \( f \) satisfies Axiom A and no-cycle condition. It is well known that if \( f \) satisfies Axiom A, then \( P(f) = \Omega(f) = CR(f) \) (see \cite{Yu} and the chain recurrent set \( CR(f) \) is upper semi-continuous, that is, for any neighborhood \( U \) of \( CR(f) \), \( \delta > 0 \) such that \( d_{C^0}(f, g) < \delta \) \( g \in \Diff(M) \)), then \( CR(g) \subset U \), where \( d_{C^0} \) is the \( C^0 \)-metric on \( \Diff(M) \) (see \cite{Yu} Corollary 3 (a)]).

**Proof of Theorem A.** The arguments above are sufficient to demonstrate that \( f \) is a star. Let \( U(f) \) be a \( C^1 \) neighborhood of \( f \) and a neighborhood \( U \) of \( CR(f) \). Because the chain recurrent set \( CR(f) \) is upper semi-continuous, we know that \( CR(g) \subset U \); therefore, \( P(g) \subset CR(g) \subset \Lambda_g = \bigcap_{n \in \mathbb{Z}} g^n(U) \). Because \( f \) has the \( C^1 \) robustly inverse shadowing property on \( CR(f) \), according to lemma \ref{lem:4.2}, every \( p \in \Lambda_g \cap P(g) = P(g) \) is hyperbolic for any \( g \in U(f) \). Therefore, \( f \) is a star, that is, \( f \) satisfies Axiom A and the no-cycle condition. Thus, the chain recurrent set \( CR(f) \) is hyperbolic.
5 Proof of Theorem B

In this section, we introduce a local star condition. Using the condition, we demonstrate that if a diffeomorphism \( f \) has the \( C^1 \) robustly inverse shadowing property on a transitive set \( \Lambda \), then \( f \) is a star on \( \Lambda \). Therefore, the transitive set \( \Lambda \) is hyperbolic for \( f \). Let \( \Lambda \subset M \) be a closed \( f \)-invariant set. We say that a diffeomorphism \( f \) is a star on \( \Lambda \) if a \( C^1 \) neighborhood \( \mathcal{U}(f) \) of \( f \) and a neighborhood \( U \) of \( \Lambda \) exist such that for any \( g \in \mathcal{U}(f) \), every \( p \in \Lambda_g \cap P(g) \) is hyperbolic, where \( \Lambda_g = \Lambda_g(U) = \bigcap_{n \in \mathbb{Z}} g^n(U) \) is the continuation of \( \Lambda \). It is clear that if \( \Lambda = M \), then \( f \) is a star. We denote by \( \mathcal{F}(\Lambda) \) the set of all diffeomorphisms that are stars on \( \Lambda \).

**Lemma 5.1** Let \( \Lambda \) be a closed invariant set of \( f \). If \( f \) has the \( C^1 \) robustly inverse shadowing property on \( \Lambda \), then \( f \in \mathcal{F}(\Lambda) \).

**Proof.** Suppose that \( f \) has the \( C^1 \) robustly inverse shadowing property on \( \Lambda \). By the definition of \( \mathcal{F}(\Lambda) \), a \( C^1 \) neighborhood \( \mathcal{U}(f) \) of \( f \) and a neighborhood \( U \) of \( \Lambda \) exist such that for any \( g \in \mathcal{U}(f) \), every \( p \in \Lambda_g \cap P(g) \) is hyperbolic. Subsequently, the proof is the same as that of lemma 4.2. \( \square \)

From now, we prove that if a diffeomorphism \( f \) is a star on a transitive set \( \Lambda \), that is, \( f \in \mathcal{F}(\Lambda) \), then \( \Lambda \) is hyperbolic for \( f \).

If \( p \) is a hyperbolic periodic point, then a \( C^1 \) neighborhood \( \mathcal{U}(f) \) and a neighborhood \( U \) of \( p \) exist such that for any \( g \in \mathcal{U}(f) \), a hyperbolic periodic point \( p_g \in P(g) \) exists, where \( p_g = \bigcap_{n \in \mathbb{Z}} g^n(U) \) is called the continuation of \( p \). Mañé [20, Lemma II.3] and Lee and Park [15, Lemma 2.3] proved the following:

**Proposition 5.2** Let \( \Lambda \) be a transitive set of \( f \). Suppose that \( f \in \mathcal{F}(\Lambda) \). Therefore, a \( C^1 \) neighborhood \( \mathcal{U}(f) \) of \( f \), constants \( C > 0, 0 < \lambda < 1 \), and \( m \in \mathbb{Z}^+ \) exist such that

(a) for each \( g \in \mathcal{U}(f) \), if \( p \) is a periodic point of \( g \) in \( \Lambda_g \) with period \( \pi(p,g) \geq m \). Therefore,

\[
\prod_{i=0}^{k-1} \| Dg^m|_{E^s(g^{im}(p))} \| < C \lambda^k \quad \text{and} \quad \prod_{i=0}^{k-1} \| Dg^{-m}|_{E^u(g^{-im}(p))} \| < C \lambda^k,
\]

where \( k = \lceil \pi(p,g)/m \rceil \).

(b) \( \Lambda \) admits a dominated splitting \( T\Lambda M = E \oplus F \) with \( \dim E = \text{index}(p) \).

A compact invariant set \( \Lambda \) of \( f \) is called an \( i \)-fundamental limit set of \( f \) if sequences \( g_n \to f \) exist as \( n \to \infty \) and periodic orbits \( P_n \) of \( g_n \) with index \( i \) exist such that \( \Lambda \) is the Hausdorff limit of \( P_n \). It is noteworthy that the fundamental \( i \)-limit \( \Lambda \) of \( f \) is \( f \)-invariant [17].
Lemma 5.3 Let $\Lambda$ be a transitive set and $f \in \mathcal{F}(\Lambda)$. Therefore, a $C^1$ neighborhood $\mathcal{V}(f)$ of $f$ and a neighborhood $\mathcal{V}$ of $\Lambda$ exist such that for any integer $i$, if $g \in \mathcal{U}(f)$ exists such that $g$ exhibits a hyperbolic periodic point $q \in \mathcal{U}$ of index $i$, then $f$ also exhibits a hyperbolic periodic point of index $i$ in $\Lambda$ and $\Lambda$ is an $i$-fundamental limit set, where $\mathcal{U}(f)$ and $\mathcal{U}$ are as the definition of $f \in \mathcal{F}(\Lambda)$.

Proof. Set $\mathcal{V} \subset \mathcal{V} \subset \mathcal{U}$ with an open neighborhood of $\Lambda$. Let $\mathcal{U}(f)$ be a neighborhood of $f$ with the following properties: (a) for any $g \in \mathcal{V}(f) \subset \mathcal{U}(f)$, a continuous path $\{F_t : 0 \leq t \leq 1\} \subset \text{Diff}(\mathcal{M})$ connecting $f$ and $g$ exists such that any $F_t$ contains no nonhyperbolic periodic orbits in the neighborhood $\mathcal{V}$ of $\Lambda$. (b) for any $g \in \mathcal{V}(f)$, $\bigcap_{i \in \mathbb{Z}} g^i(\mathcal{V}) = \Lambda_g(\mathcal{U}) = \Lambda_g$. We assume that $g \in \mathcal{U}(f)$ exists such that $g$ contains a hyperbolic periodic point $q \in \mathcal{U}$ of index $i$. Subsequently, we consider a continuous path $\{F_t : 0 \leq t \leq 1\} \subset \text{Diff}(\mathcal{M})$ connecting $f$ and $g$ such that any $F_t$ contains no nonhyperbolic periodic orbit in the neighborhood $\mathcal{V}$ of $\Lambda$. If $f$ contains no hyperbolic periodic orbits of index $i$ in $\Lambda$, then a time $t_0$ exists such that the hyperbolic periodic orbits of index $i$ is vanished. Without loss of generality, let $t_0$ be the first time. Therefore, we know that $F_{t_0}$ contains a nonhyperbolic periodic orbit in $\mathcal{U}$; this contradicts with the path choice. Hence, $f$ also contains a hyperbolic periodic point of index $i$ in $\Lambda$.

Let $P \subset \Lambda$ be a hyperbolic periodic orbit of $f$ with index $i$. By the standard arguments of the connecting lemma (for instance, see Lemma 2.2 of [4]), we can apply an arbitrarily small perturbation $g$ of $f$ such that a homoclinic orbit $\text{Orb}(x)$ exists with respect to $P$ in $\mathcal{U}$, such that the closure of $\text{Orb}(x)$ is arbitrarily close to the set $\Lambda$ (in Hausdorff metrics). Applying another arbitrarily small perturbation if necessary, we can assume that $x$ is a transversal homoclinic point of $P$. Subsequently, by the shadowing lemma of hyperbolic set $\text{Orb}(x) \cup P$, we can obtain hyperbolic periodic orbits of $g$ with index $i$ of $\Lambda$ that are arbitrarily close to $\text{Orb}(x) \cup P$, and hence close to $\Lambda$. This ends the proof of the second part of lemma 5.3.

For any $f \in \text{Diff}(\mathcal{M})$ and $x \in \mathcal{M}$, we denote

$$D^s(f, x) = D^s(x, f) = \{v \in T_x \mathcal{M} : \|Df^n(v)\| \to 0 \text{ as } n \to +\infty\},$$

$$D^u(f, x) = D^u(x, f) = \{v \in T_x \mathcal{M} : \|Df^n(v)\| \to 0 \text{ as } n \to -\infty\}.$$

In [15], a characterization of hyperbolicity is detailed as follows:

Proposition 5.4 A compact invariant set $\Lambda \subset \mathcal{M}$ of $f$ is hyperbolic if and only if $T_x \mathcal{M} = D^s(x) \oplus D^u(x)$ for any $x \in \Lambda$.

A point $x \in \mathcal{M}$ without the property $T_x \mathcal{M} = D^s(x) \oplus D^u(x)$ is called a resisting point. A compact invariant set $K$ is called a minimally nonhyperbolic set if $K$ is nonhyperbolic and every compact invariant proper subset of $K$ is hyperbolic. In [10], minimally nonhyperbolic sets are divided into two types. If a resisting point $a$ exists in a minimally nonhyperbolic set $K$ such that $\omega(a)$ and $\alpha(a)$ are all proper subsets of $K$, then $K$ is called the simple type. Otherwise, the nonhyperbolic set is called the nonsimple type.
5.1 Non-existence of heterodimensional cycle

In this Section, we prove the following proposition: no heterodimensional cycle exists near $\Lambda$ for the system close to $f$.

**Proposition 5.5** Let $\Lambda$ be a transitive set and $f \in \mathcal{F}(\Lambda)$. Therefore, a $C^1$ neighborhood $\mathcal{U}(f)$ of $f$ and a neighborhood $U$ of $\Lambda$ exist such that for any $g \in \mathcal{U}(f)$, $g$ has no a heterodimensional cycles in $U$.

**Proof.** To derive a contradiction, we may assume that hyperbolic periodic points $p, q$ exist with different indices and $x \in W^s(p) \cap W^u(q), y \in W^u(p) \cap W^s(q)$ such that $\text{Orb}(p) \cup \text{Orb}(q) \cup \text{Orb}(x) \cup \text{Orb}(y) \subset U$. We denote by $K = \text{Orb}(p) \cup \text{Orb}(q) \cup \text{Orb}(x) \cup \text{Orb}(y)$ and $k$ the index of $p$ and $l$ the index of $q$. Without loss of generality, we can assume that $p, q$ are fixed points of $f$ and $k < l$.

A point $x \in M$ is $C^1$ preperiodic if for any $C^1$ neighborhood $\mathcal{U}(f)$ of $f$ and any neighborhood $U$ of $x$, $g \in \mathcal{U}(f)$ and $y \in U$ exist such that $y$ is a periodic of $g$. We denote by $P_r(f)$ the set of $C^1$ preperiodic points of $f$. A point $x \in M$ is called an $i$-preperiodic of $f$ ($0 \leq i \leq \dim M$) if for any $C^1$ neighborhood $\mathcal{U}(f)$ of $f$ and any neighborhood $U$ of $x$, $g \in \mathcal{U}(f)$ and $y \in U$ exist such that $y$ is a hyperbolic periodic point of $g$ of index $i$ (see [29]).

**Lemma 5.6** $K$ is contained in the $k, l$-fundamental limits of $f$. Precisely, $g_n \to f$ exists with hyperbolic periodic orbits $p_n$ of index $k$, such that $K$ is the Hausdorff limit $p_n$. Similarly, $g'_n \to f$ exists with hyperbolic periodic orbits $q_n$ of index $l$, such that $K$ is the Hausdorff limit $q_n$.

**Proof.** Because $x \in W^s(p) \cap W^u(q), y \in W^u(p) \cap W^s(q)$, for any neighborhoods $U_x$ of $x$, $U_y$ of $y$, and $U_q$ of $q$, one can obtain a point $z$ with integers $i_1 < i_2 < i_3$ such that $f^{i_1}(z) \in U_y, f^{i_2}(z) \in U_q$ and $f^{i_3}(z) \in U_x$ by Palis’ $\lambda$-lemma. By small perturbations, we can create jumps near $x$ and $y$ such that $z$ is a transversal homoclinic point of $p$ for a diffeomorphism $g$ close to $f$. Because the intersection is transversal, we know that the set $\text{Org}(z, g) \cup \text{Orb}(p, g)$ is a hyperbolic set. By the shadowing lemma, a hyperbolic periodic orbit $p'$ of $g$ with the same index of $p$ exists such that it is arbitrarily close to the set $\text{Org}(z, g) \cup \text{Orb}(p, g)$. By choosing sufficiently small $U_x, U_q$, and $U_y$, we can cause the set $\text{Org}(z, g) \cup \text{Orb}(p, g)$ to be arbitrarily close to $K$. This proves that $K$ is the $k$-preperiodic set of $f$. Similarly, we can prove that $K$ is the $l$-preperiodic set of $f$. This ends the proof of lemma 5.6.

Let us consider a sequence of periodic pseudo orbits.

**Lemma 5.7** Set any small $\delta > 0$ and $x_p \in \text{Orb}^+(x), y_p \in \text{Orb}^-(y), x_q \in \text{Orb}^-(x)$ and $y_q \in \text{Orb}^+(y)$ with $x_p, y_p \in B(\delta, p), x_q, y_q \in B(\delta, q)$. Subsequently, for any $\epsilon > 0$, $L > 0$ such that for any $n \geq L$, $p_n, q_n$ exist with the following properties
Lemma 5.8 Set any small \( \delta > 0; \epsilon > 0 \) and \( N > L \) exist such that if \( n \geq N, m \geq N \), then \( g \in C^1 \) close to \( f \) exists such that \( \mathcal{P}(m, n) \) is a periodic orbit of \( g \).

Proof. Let any small \( \delta > 0 \) be fixed and \( N > L \). Because \( \mathcal{P}(m, n) \) is a periodic \( \epsilon \)-pseudo orbit of \( f \), for some \( 0 < \epsilon \leq \delta \), we can create four small perturbations in a neighborhood of \( \{x_p, x_q, y_p, y_q\} \). Subsequently, the pseudo orbit \( \mathcal{P}(m, n) \) can be a periodic orbit for the perturbation. \( \square \)

Lemma 5.9 If \( \delta > 0 \) is sufficiently small, then for a fixed \( n \), the index of \( \mathcal{P}(m, n) \) (with respect to \( g \)) will equal to the index of \( q \) as \( m \) becomes sufficiently large.

Proof. From lemma 5.6, we know that the set \( K \) contains a dominated splitting \( T_K M = E \oplus F \) with \( \dim E = l \). Because \( g \) can be chosen arbitrarily close to \( f \) and \( \mathcal{P}(m, n) \) arbitrarily close to \( K \), the dominated splitting can continue for the periodic orbit \( \mathcal{P}(m, n) \) with respect to \( g \). Without loss of generality, we still use \( E \oplus F \) to denote the dominated splitting. Because \( x_q \) is close to \( q \), we know that \( Dg_{E(x_q)} \) is close to \( Df_{E(x_q)} \). By the contraction of \( Df_{E(x_q)} \), after an easy calculation, we find that \( E_{\mathcal{P}(m, n)} \) is contracting with respect to \( g \) if \( m \) is sufficiently large. Similarly, \( F_{\mathcal{P}(m, n)} \) is expanding if \( m \) is sufficiently large. This proves that the periodic orbit \( \mathcal{P}(m, n) \) of \( g \) contains an index equal to \( l \). This ends the proof of the lemma. \( \square \)

Now, we can complete the proof of proposition 5.5. We set \( m_0, n_0 \) to be sufficiently large. By lemma 5.9, we know that \( m > m_0 \) exists such that the index of \( \mathcal{P}(m, n_0) \) is
equal to \( l \). Subsequently, we set \( m \) and increase \( n \). In this process, the index of \( \mathcal{P}(m, n) \) decreases as \( n \) increases. If \( m_0, n_0 \) is chosen sufficiently large, we can find that \( n > n_0 \) such that \( \mathcal{P}(m_0, n) \) contains the index \( k + 1 \) and \( \mathcal{P}(m_0, n + 1) \) contains index \( k \). By an easy calculation, we know that if \( m_0, n_0 \) is sufficiently large, then \( \mathcal{P}(m, n) \) must contain an eigenvalue \( \lambda \) such that \( |\lambda|^{\frac{1}{\pi \mathcal{P}(m_0, n_0)}} \) is close to 1. This is a contradiction because the set \( \Lambda \) satisfies the local star condition. \( \square \)

5.2 Hyperbolicity of local star transitive sets

In this section, we will prove that if \( f \) satisfies the local star condition, i.e., the transitive \( \Lambda \), then it is hyperbolic. Assume that \( \Lambda \) is not a hyperbolic set. By Zorn’s lemma, we know that a minimally nonhyperbolic set \( K \subset \Lambda \) exists.

**Proposition 5.10** \( K \) cannot be a nonsimple-type minimally nonhyperbolic set.

**Proof.** Assume that \( K \) is a nonsimple-type minimally nonhyperbolic set. Without loss of generality, we assume that a resisting point \( a \) exists such that \( K = \omega(a) \). Let \( k = \min\{i : \text{there is a } i\text{-fundamental limit set contained in } K\} \). From proposition 5.2 and lemma 5.6, we know that a dominated splitting \( T_K M = E \oplus F \) exists with \( \dim E = k \). Therefore, by ergodic closing lemma [20], we know that \( E \) is contracting.

Now, let

\[
G = \{x \in K : \limsup_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(\|Df^m|_{F(f^m(x))}\|) \geq -\log \lambda\}
\]

where \( m, \lambda \) are the constants in proposition 5.2. It is obvious that \( G \) is a nonempty compact invariant subset of \( K \).

**Claim.** \( G = K \).

**Proof of Claim.** Assume \( G \) is a proper subset of \( K \). Subsequently, we know that \( G \) is hyperbolic because \( K \) is a minimally nonhyperbolic set. It is easy to verify that \( E \oplus F \) restricted on \( G \) is only the hyperbolic splitting over \( G \).

Because \( K = \omega(a) \), we know that \( a \notin G \). One can apply a small neighborhood \( W \) of \( G \) such that \( a \notin W \) and the locally maximal invariant set in \( W \) is hyperbolic. Because \( a \notin W \) and \( G \subset \omega(a) \), we can obtain a point \( b \in K \) such that \( b \in W \setminus f(W) \) and \( \text{Orb} b^+(b) \subset W \). We know that \( b \notin G \). Therefore, we can obtain

\[
\limsup_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(\|Df^m|_{F(f^m(b))}\|) < -\log \lambda.
\]

Let \( \{n_i\} \) be a sequence of positive integers such that \( f^{n_i m}(b) \to c \in \omega(b) \) as \( i \to \infty \). Subsequently, we can apply \( 1 > \lambda' > \lambda \) and \( n_s > n_t \) with \( s, t \) arbitrarily large such that

\[
\frac{1}{n_s - n_t} \sum_{i=n_s}^{n_t-1} \log(\|Df^m|_{F(f^m(b))}\|) < -\log \lambda'.
\]
Subsequently, by the shadowing property of the hyperbolic sets, we can obtain a hyperbolic periodic point \( p \) with an arbitrarily large period that shadows the orbit segment
\[
\{ f^{n_1 m}(b), f^{(n_1+1)m}(b), \ldots, f^{(n_t-1)m}(b), f^{n_t m}(b) \}
\]
such that
\[
\frac{1}{\pi(p)} \sum_{i=0}^{\pi(p)-1} \log(\|Df^m|_{E^u(f^m(p))}\|) < -\log \lambda.
\]
This contradicts with proposition 5.2. This ends the proof of claim.

Further, \( K \) is shown as a hyperbolic set by the following conclusion proven in [19]. This contradicts that \( K \) is a nonhyperbolic set. This ends the proof of proposition 5.10.

**Theorem 5.11** [19] Let \( K \) be a compact invariant set of \( f \) and assume that \( f \) is a local star in the neighborhood \( U \) of \( K \). If a dominated splitting \( T_K M = E \oplus F \) exists with the following two properties:

(a) \( E \) is contracting, and

(b) constants \( m \in \mathbb{N} \) and \( \lambda \in (0,1) \), and a dense subset \( G \subset \Lambda \) exist such that for any \( x \in G \),
\[
\limsup_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(\|Df^m|_{F(f^m(x))}\|) \geq -\log \lambda,
\]
then \( F \) is expanding and \( K \) is hyperbolic.

**Proposition 5.12** If \( K \) is a simple-type minimally nonhyperbolic set of \( f \), then \( g \in C^1 \) close to \( f \) exists such that \( g \) has a heterodimensional cycle.

**Proof.** Let \( a \) be a resisting point such that \( \omega(a) \) and \( \alpha(a) \) are both the proper subsets of \( K \). From the definition of a minimally nonhyperbolic set, we know that \( K = \omega(a) \cup Orb(a) \cup \alpha(a) \) and both \( \omega(a) \) and \( \alpha(a) \) are hyperbolic sets.

**Claim.** The index of \( \omega(a) \) and \( \alpha(a) \) are different.

**Proof of Claim.** Assume that the index of \( \omega(a) \) and \( \alpha(a) \) are same. We denote by \( i \) the index of \( \omega(a) \). Subsequently, by the shadowing lemma of the hyperbolic sets, we know that \( \Lambda \) contains hyperbolic periodic points with index \( i \). From lemma 5.3 we know that \( \Lambda \) is an \( i \)-fundamental limit. From proposition 5.2 we know that \( \Lambda \) contains a dominated splitting \( T_{\Lambda} M = E \oplus F \) with \( \text{dim} E = i \). One can easily verify that \( E(x) = D^s(x) \) and \( F(x) = D^u(x) \). This contradicts with \( x \) being a resisting points. This ends the proof of claim.
We denote by $i$ the index of $\omega(a)$ and $j$ the index of $\alpha(a)$. Let $W_1$ be a small neighborhood of $\omega(a)$ such that the maximal invariant set in $W_1$ is hyperbolic and any two periodic orbits in $W_1$ are homoclinically related. Let $W_2$ be a small neighborhood of $\alpha(a)$ such that the maximal invariant set in $W_2$ is hyperbolic and any two periodic orbits in $W_2$ are homoclinically related. We can small $W_1, W_2$ such that $W_1 \cap W_2 = \emptyset$ and $\Lambda \setminus (W_1 \cup W_2) \neq \emptyset$. Let $P$ be a hyperbolic periodic orbit in $W_1$, and $Q$ be a hyperbolic periodic orbit in $W_2$. By the standard argument of connecting lemma, we can perform a perturbation $g$ such that $g = f$ in $W_1 \cup W_2 \cup \text{Orb}(a)$ and $W^u(P, g) \cap W^s(Q, g) \neq \emptyset$. It is noteworthy that $g = f$ in $W_1 \cup W_2 \cup \text{Orb}(a)$, we also have $\omega(a, g) = \omega(a, g)$ and $\alpha(a, g) = \alpha(a, f)$.

**Lemma 5.13** $a \in \overline{W^s(P, g)} \cap \overline{W^u(Q, g)}$.

**Proof.** For an arbitrarily small $\delta > 0$, we can apply $b \in \omega(a)$ and $n \in \mathbb{Z}$ such that $d(f^n(a), b) < \delta$; subsequently, we can construct a $\delta$-pseudo orbit as
\[\{\cdots, f^{-2}(b), f^{-1}(b), f^n(a), f^{n+1}(a), \cdots\}.\]
By the shadowing property of the hyperbolic set, we can find $y \in W_1$ such that the orbit of $y$ traces the pseudo orbit. If $\delta$ is sufficiently small, we can obtain $a \in W^s(y)$ by the expansivity of the hyperbolic set.

Because $\alpha(y) = \alpha(c)$ and $\omega(y) = \omega(a)$, we know that $\alpha(y) \cap \alpha(y) \neq \emptyset$. By the shadowing property, we can obtain the periodic points $q_n$ with orbits in $W_1$ such that $q_n \to y$ as $n \to \infty$. It is obvious that $a$ is close to $\bigcup_n W^s(q_n)$. Because $\{q_n\}$ are pairwise homoclinically related, we know that $\bigcup_n W^s(\text{Orb}(q_n)) = \overline{W^s(\text{Orb}(q_n))}$ for any $n$. Further, we know that $a \in \overline{W^s(P)}$. Similarly, we have $a \in \overline{W^u(Q)}$. This ends the proof of the lemma. \qed

Subsequently, we can perform a perturbation in a tube of $a$ such that $W^s(P) \cap W^u(Q) \neq \emptyset$ and maintain the existing $W^u(P) \cap W^s(Q) \neq \emptyset$. Subsequently, we can obtain a heterodimensional cycle. From proposition 5.12, we know that $\Lambda$ does not admit the simple-type nonhyperbolic set. Hence, $\Lambda$ should be a hyperbolic set. \qed

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