ON THE RAMSEY-TURÁN DENSITY OF TRIANGLES

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Abstract. One of the oldest results in modern graph theory, due to Mantel, asserts that every triangle-free graph on \( n \) vertices has at most \( \lfloor n^2/4 \rfloor \) edges. About half a century later Andrásfai studied dense triangle-free graphs and proved that the largest triangle-free graphs on \( n \) vertices without independent sets of size \( \alpha n \), where \( 2/5 \leq \alpha < 1/2 \), are blow-ups of the pentagon. More than 50 further years have elapsed since Andrásfai’s work. In this article we make the next step towards understanding the structure of dense triangle-free graphs without large independent sets.

Notably, we determine the maximum size of triangle-free graphs \( G \) on \( n \) vertices with \( \alpha p \leq n/8 \) and state a conjecture on the structure of the densest triangle-free graphs \( G \) with \( \alpha p > n/3 \). We remark that the case \( \alpha p \geq n/3 \) behaves differently, but due to the work of Brandt this situation is fairly well understood.

§1. INTRODUCTION

1.1. Ramsey-Turán theory. Mantel [15] proved in 1907 that balanced complete bipartite graphs maximise the number of edges among all triangle-free graphs on a given set of vertices. Later this result was generalized to \( K_\ell \)-free graphs by Turán [18] and, asymptotically, to \( H \)-free graphs by Erdős and Stone [10], and by Erdős and Simonovits [7]. All of these works had a decisive impact on the development of extremal graph theory.

Here we deal with one particular line of research systematically initiated by Vera T. Sós and known as Ramsey-Turán theory (for a survey on this fascinating area we refer to Simonovits and Sós [16]). For \( \ell \geq 3 \) and \( n \geq s \geq 0 \) the Ramsey-Turán number \( \text{ex}_\ell(n, s) \) is the maximum number of edges in a \( K_\ell \)-free graph on \( n \) vertices which contains no independent set consisting of more than \( s \) vertices, i.e.,

\[
\text{ex}_\ell(n, s) = \max_{G=(V,E)} \{|E| : K_\ell \not\subseteq G, |V| = n, \text{ and } \alpha(G) \leq s\}.
\]

One usually considers the case that \( n \) is large and thus, instead of \( \text{ex}_\ell(n, s) \), one rather studies the Ramsey-Turán density function \( f_\ell : (0, 1] \to \mathbb{R} \) defined by

\[
f_\ell(\alpha) = \lim_{n \to \infty} \frac{\text{ex}_\ell(n, \alpha n)}{\binom{n}{2}}.
\]

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For the existence of this limit we refer to [6]. Notice that Turán’s theorem implies
\[
\text{ex}_\ell(n, s) = (1 - o(1)) \frac{\ell - 2}{\ell - 1} \binom{n}{2} \quad \text{for } s \geq \left\lfloor \frac{n}{\ell - 1} \right\rfloor,
\]
whence \( f_\ell(\alpha) = \frac{\ell - 2}{\ell - 1} \) for all \( \alpha \geq \frac{1}{\ell - 1} \). The central problem to determine the limit
\[
\varrho(K_\ell) = \lim_{\alpha \to 0} f_\ell(\alpha)
\]
was solved for odd \( \ell \) in [9] and proved to be very difficult for even \( \ell \). Following the ingenious contributions by Szemerédi [17] and by Bollobás and Erdős [2], this problem was finally solved by Erdős, Hajnal, Sós, and Szemerédi [6].

Recently Lüders and Reiher [14] determined the value of \( f_\ell(\alpha) \) whenever \( \alpha \) is sufficiently small depending on \( \ell \), so the extremal behaviour of \( K_\ell \)-free graphs with small (linear) independence number is now well understood. Despite this fact, we firmly believe that the problem to determine \( f_\ell(\alpha) \) for all \( \alpha \in (0, \frac{1}{\ell - 1}) \) remains interesting.

1.2. Results on triangle-free graphs. In this article we restrict our attention to the innocent looking case \( \ell = 3 \), which seems surprisingly intricate to us. In other words, we concentrate on triangle-free graphs and, eliminating some indices, we study the behaviour of the function \( \text{ex}(n, s) = \text{ex}_3(n, s) \) and its ‘scaled’ version \( f(\alpha) = f_3(\alpha) \). We also write
\[
\mathcal{E}(n, s) = \{ G = (V, E) : K_3 \not\subseteq G, \ |V| = n, \ \alpha(G) \leq s, \text{ and } |E| = \text{ex}(n, s) \}
\]
for the corresponding families of extremal graphs.

In an early contribution from 1962 Andrásfai [1] studied the question of how many edges a triangle-free graph on \( n \) vertices, whose independence number is at most \( \alpha n \) for some given constant \( \alpha \), can have. As a special case of (1.1), for \( \alpha \geq \frac{1}{2} \) Mantel’s Theorem yields \( \left\lfloor \frac{n^2}{4} \right\rfloor \) as an answer and for \( \alpha \leq \frac{1}{3} \) the “trivial” upper bound \( \frac{1}{2} \alpha n^2 \) is essentially optimal (see the discussion at the end of this subsection), so the question is most interesting when \( \alpha \in \left( \frac{1}{3}, \frac{1}{2} \right) \). Andrásfai settled this problem for all \( \alpha \in \left[ \frac{2}{5}, \frac{1}{2} \right] \) and conjectured for \( \alpha \in \left( \frac{1}{3}, \frac{2}{3} \right) \) that the answer is describable in terms of appropriate blow-ups of certain graphs nowadays bearing his name (see Conjecture 1.4 below).

**Theorem 1.1** (Andrásfai). For every nonnegative integer \( n \) and every integer \( s \in \left[ \frac{2}{5} n, \frac{1}{2} n \right] \) we have
\[
\text{ex}(n, s) = n^2 - 4ns + 5s^2.
\]
In particular, \( f(\alpha) = 2 - 8\alpha + 10\alpha^2 \) holds for every \( \alpha \in \left[ \frac{2}{5}, \frac{1}{2} \right] \).

Andrásfai [1] also determined the extremal families \( \mathcal{E}(n, s) \) for \( s \in \left[ \frac{2}{5} n, \frac{1}{2} n \right] \) and it turned out that all extremal graphs in these families are blow-ups of the pentagon. We display a sample case in Figure 1.1 and defer a more detailed discussion to Subsection 1.3. Our main result is a similar quadratic formula applicable to every \( \alpha \in \left[ \frac{3}{8}, \frac{2}{5} \right] \).
Theorem 1.2. If $n \geq 0$ and $s \in \left[\frac{3}{8}n, \frac{3}{5}n\right]$, then
\[
\text{ex}(n, s) = 3n^2 - 15ns + 20s^2.
\]
Consequently, we have
\[
f(\alpha) = 6 - 30\alpha + 40\alpha^2 \quad \text{for every} \quad \alpha \in \left[\frac{3}{8}, \frac{2}{5}\right].
\]

It follows from our proof that all extremal graphs for this result, i.e., all graphs in a class of the form $\mathcal{E}(n, s)$ with $s \in \left[\frac{3}{8}n, \frac{3}{5}n\right]$ are blow-ups of the well-known Wagner graph here denoted by $\Gamma_3$, which is a triangle-free cubic graph on 8 vertices. A special case is shown in Figure 1.2.

We suspect that some of the tools we have developed for proving Theorem 1.2 will be relevant for a complete determination of the function $f$, even though some new ideas will certainly be required. Before stating our version of Andrásfai’s conjecture on $f$ we briefly recall another known result on this function.

Notice that every triangle-free graph $G$ satisfies
\[
\Delta(G) \leq \alpha(G),
\]
for the neighbourhood of every vertex is an independent set. Therefore
\[ \text{ex}(n, s) \leq \frac{1}{2} ns \]
holds for all \( n \geq s \geq 0 \) and \( f(\alpha) \leq \alpha \) for \( \alpha > 0 \) follows. We call these estimates the trivial bounds on \( \text{ex}(n, s) \) and \( f(\alpha) \), respectively.

In the regime \( s < \frac{1}{3} n \) Brandt [3] provided several constructions of \( s \)-regular graphs on \( n \) vertices whose independence number is equal to \( s \), and his work implies \( f(\alpha) = \alpha \) for all \( \alpha \in (0, \frac{1}{3}] \). In view of this result and the above theorems it remains to study the behaviour of \( f(\alpha) \) for \( \alpha \in \left(\frac{1}{3}, \frac{3}{5}\right) \). The next subsection offers a conjecture for this range.

1.3. A conjecture on triangle-free graphs. Let us introduce one more piece of terminology. By a blow-up of a graph \( G \) we mean any graph \( \hat{G} \) obtained from \( G \) by replacing each of its vertices \( v_i \) by an independent set \( V_i \) (that can be empty) and joining two subsets \( V_i \) and \( V_j \) of vertices of \( \hat{G} \) by all \( |V_i||V_j| \) possible edges whenever the pair \( \{v_i, v_j\} \) is an edge of \( G \). For instance, Figure 1.1 shows a blow-up of the pentagon, where three consecutive vertices are replaced by independent \((n - 2s)\)-sets (drawn red) while the remaining two vertices are enlarged to \((3s - n)\)-sets.

Let us recall that Andrásfai graphs are Cayley graphs \((\mathbb{Z}/(3k - 1)\mathbb{Z}, S)\) with \( k \geq 1 \) and \( S \subseteq \mathbb{Z}/(3k - 1)\mathbb{Z} \) being a sum-free subset of size \( k \). For definiteness, we denote the graph obtained for \( S = \{k, \ldots, 2k - 1\} \) by \( \Gamma_k \). So explicitly two vertices \( i \) and \( j \) of \( \Gamma_k \) are adjacent if and only if \( i - j \in S \). Following \( \Gamma_1 = K_2 \) the first few Andrásfai graphs are depicted in Figure 1.3.

![Figure 1.3. Andrásfai graphs \( \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \) and \( \Gamma_6 \).](image)

Notice that \( \Gamma_k \) is a triangle-free, \( k \)-regular graph on \( 3k - 1 \) vertices whose independence number is exactly \( k \). Therefore, balanced blow-ups of \( \Gamma_k \) show that the trivial bound on \( f(\alpha) \) is optimal if \( \alpha \) is of the form \( \frac{k}{3k - 1} \), i.e., that we have
\[ f\left(\frac{k}{3k - 1}\right) = \frac{k}{3k - 1} \quad \text{for all } k \geq 1 . \] (1.2)

Like Andrásfai we believe that whenever \( n \geq 0 \) and \( s \in \left(\frac{1}{3} n, \frac{1}{2} n\right] \) there exists a graph \( G \in \mathcal{E}(n, s) \) which is a blow-up of an appropriate Andrásfai graph. This leads to \( f(\alpha) \) being
piecewise quadratic on \((\frac{1}{3}, \frac{1}{2})\) with critical values at \(\alpha_k = \frac{k}{3k-1}\) for \(k \geq 2\). By optimizing over all blow-ups of Andrásfai graphs we were led to the following function.

**Definition 1.3.** For integers \(n \geq s \geq 0\) we set

\[
g(n, s) = \begin{cases} 
\frac{1}{2}ns, & \text{if } s \leq \frac{1}{3}n \\
g_k(n, s), & \text{if } \frac{k}{3k-1}n \leq s < \frac{k-1}{3k-4}n \text{ for some } k \geq 2 \\
\left\lfloor \frac{n^2}{4} \right\rfloor, & \text{if } \frac{s}{2} \leq s \leq n,
\end{cases}
\]

where

\[
g_k(n, s) = \frac{1}{2}k(k-1)n^2 - k(3k-4)ns + \frac{1}{2}(3k-4)(3k-1)s^2. \tag{1.3}
\]

**Conjecture 1.4.** For all integers \(n \geq s \geq 0\) we have \(\text{ex}(n, s) \leq g(n, s)\). In other words, every triangle-free \(n\)-vertex graph \(G\) with \(\alpha(G) \leq s\) has at most \(g(n, s)\) edges.

Admittedly, one needs some time to get used to the functions \(g_k(n, s)\) introduced in (1.3) but we believe that the motivation in terms of optimal blow-ups of Andrásfai graphs renders the conjecture sufficiently natural (see also [13, Lemma 3.3]). Observe that the functions \(g_2(n, s)\) and \(g_3(n, s)\) are precisely the quadratic forms appearing in the Theorems 1.1 and 1.2. Therefore, Conjecture 1.4 is only open for \(s \in (\frac{1}{3}n, \frac{3}{8}n)\). Let us briefly indicate one construction showing that, if true, Conjecture 1.4 is optimal for the most interesting range of \(\frac{s}{n}\).

**Fact 1.5.** If \(s \in (\frac{1}{3}n, \frac{1}{2}n)\), then \(\text{ex}(n, s) \geq g(n, s)\).

**Proof.** Let \(k \geq 2\) be the unique integer with \(s \in \left[\frac{k}{3k-1}n, \frac{k-1}{3k-4}n\right)\). Take a blow-up \(G\) of \(\Gamma_k\) obtained by replacing the vertices 1, \(k\), and \(2k\) by sets of size \((k-1)n - (3k-4)s\) and the remaining vertices by sets of size \(3s - n\). Clearly, \(G\) is triangle-free. One can check that \(G\) has \(n\) vertices, independence number \(s\), and \(g_k(n, s)\) edges. \(\Box\)

Let us observe that Conjecture 1.4 yields a precise prediction on the Ramsey-Turán density function \(f\), namely the following.

**Conjecture 1.6.** The function \(f : (0, 1] \rightarrow \mathbb{R}\) is given by

\[
f(\alpha) = \begin{cases} 
\alpha, & \text{if } \alpha \leq \frac{1}{3} \\
f_k(\alpha), & \text{if } \frac{k}{3k-1} \leq \alpha < \frac{k-1}{3k-4} \text{ for some } k \geq 2 \\
\frac{1}{2}, & \text{if } \frac{1}{2} \leq \alpha \leq 1,
\end{cases}
\]

where

\[
f_k(\alpha) = k(k-1) - 2k(3k-4)\alpha + (3k-4)(3k-1)\alpha^2. \tag{1.4}
\]

Notice that at the critical values \(\alpha_k = \frac{k}{3k-1}\) this function agrees with (1.2). The remainder of this introduction discusses further evidence in support of Conjecture 1.4.
1.4. **Minimum degree.** There appears to be a mysterious analogy between the Ramsey-Turán problem for triangles and the more thoroughly studied problem to describe the structure of triangle-free graphs of large minimum degree. For instance, there is a similar transition from chaos to structure occurring at $\frac{1}{3}n$. As reported in [8] Hajnal constructed triangle-free graphs $G$ with $\delta(G) \geq \left(\frac{1}{3} - o(1)\right)|V(G)|$ of arbitrarily large chromatic number. On the other hand, Łuczak [12] proved that for every $\varepsilon > 0$ all triangle-free graphs $G$ with $\delta(G) \geq \left(\frac{1}{3} + \varepsilon\right)|V(G)|$ admit a homomorphism into a triangle-free graph whose order can be bounded in terms of $\varepsilon$. The ultimate variant of Łuczak’s result is due to Brandt and Thomassé [5], who proved that, actually, such graphs either admit a homomorphism into some Andrásfai graph or into some so-called Vega graph (see [4]).

We will not introduce Vega graphs properly here and only remark that they can be obtained from Andrásfai graph by adding 8 vertices forming a cube as well as several new edges, and possibly deleting at most two special vertices afterwards. For $k \geq 10$ with $k \equiv 1, 2, 3 \pmod{9}$ there are blow-ups of appropriate Vega graphs with $3k - 1$ vertices and independence number $k$ that furnish additional extremal cases for (1.2).

Now several questions present themselves. Are Vega graphs so special that their only appearances in extremal families $\mathcal{E}(n, s)$ are the aforementioned regular ones? Or, perhaps, on the contrary, because of their more sophisticated structure they admit blow-ups falsifying Conjecture 1.4? Both these speculations seem to be false. There are non-trivial blow-ups of Vega graphs which are in $\mathcal{E}(n, s)$. Nonetheless, it can be proved that, in a sense made precise in [13], no ‘natural’ blow-up of an Andrásfai graph or Vega graph can be a counterexample to Conjecture 1.4.

These facts have an interesting consequence. Together with the structure theorem of Brandt and Thomassé they allow us to prove Conjecture 1.4 for $\alpha \in \left[\frac{k}{3k-1}, \frac{k}{3k-1} + \varepsilon_k\right]$, where $k \geq 2$ and $\varepsilon_k$ is sufficiently small. Further details and a conjectural explicit description of the extremal graph families $\mathcal{E}(n, s)$ will be presented in our forthcoming article [13].

This article is organized as follows. In the next section we sketch some tools and observations which we shall use later and which, hopefully, could be useful in the quest of proving Conjecture 1.4 in full generality. The last section is devoted to the proof of Theorem 1.2.

### §2. Preliminaries

The goal of this section is to gather several results that we believe to be relevant in general to the problem of determining $\text{ex}(n, s)$ for $s > n/3$. These preliminaries fall naturally into three groups. We start in Subsection 2.1 with some facts concerning matchings and independent sets in arbitrary, not necessarily triangle-free, graphs. Subsection 2.2 proceeds...
with a discussion of symmetrisation operations – a device we shall use for “simplifying” extremal graphs. Finally in Subsection 2.3 we use this technique for investigating the structure of graphs that are extremal for the problem to determine ex(n, s).

Throughout the article we follow standard graph theoretical notation. Thus, for instance, \( \deg_G(v) \) stands for the degree of a vertex \( v \) of a graph \( G \), and by \( N_G(S) \) we mean the neighbourhood of the set of vertices \( S \). Moreover, we omit subscripts unless they are necessary to avoid confusion. Given two disjoint sets \( A \) and \( B \) we define

\[
K(A, B) = \{ \{a, b\} : a \in A \text{ and } b \in B \}.
\]

This is the edge set of the complete bipartite graph with vertex partition \( A \cup B \). For two disjoint subsets \( A \) and \( B \) of vertices of \( G \) we say that \( A \) is matchable into \( B \) if the bipartite graph induced in \( G \) by the sets \( A \) and \( B \) contains a matching saturating \( A \).

### 2.1. Matchings and independent sets.

Clearly if \( A \) and \( Y \) are two disjoint independent sets in a graph \( G \) with \( |Y| \leq |A| \), then there is an injective map from \( Y \) to \( A \). The following simple consequence of Hall’s theorem, due to Andrásfai (see [1, Lemma 2.3]), ensures that in case \( |A| = \alpha(G) \) one such injection is exemplified by a matching. For the reader’s convenience we include a short proof.

**Fact 2.1.** Let \( A \) and \( Y \) be two disjoint independent sets in a graph \( G \). If \( |A| = \alpha(G) \), then \( Y \) is matchable into \( A \).

**Proof.** In the light of Hall’s theorem [11] it suffices to prove that for an arbitrary \( D \subseteq Y \) and its neighbourhood \( A \cap N(D) \) in \( A \) we have \( |D| \leq |A \cap N(D)| \). Since \( D \cup (A \setminus N(D)) \) is independent in \( G \), we have indeed

\[
|D| + |A| - |A \cap N(D)| = |D \cup (A \setminus N(D))| \leq \alpha(G) = |A|.
\]

In general, deleting edges from a graph may increase its independence number. For our purposes it will be important to know that the following type of edge deletions leave the independence number invariant.

**Lemma 2.2.** Given a graph \( G \), suppose

- that \( A \subseteq V(G) \) is an independent set of size \( \alpha(G) \),
- and that \( M \) is a matching in \( G \) from \( V(G) \setminus A \) to \( A \), the size of which is as large as possible.

If \( G' \) denotes the graph obtained from \( G \) by isolating the vertices in \( A \setminus V(M) \), i.e., by deleting all edges incident with them, then \( \alpha(G') = \alpha(G) \).
Proof. Since $G'$ is a subgraph of $G$, we have $\alpha(G') \geq \alpha(G)$. For the converse direction we consider any set $U \subseteq V(G)$ which is independent in $G'$. Since $U \setminus A$ is independent in $G$, Fact 2.1 tells us that in $G$ there exists a matching $N$ from $U \setminus A$ to $A$ covering all vertices of $U \setminus A$. We contend that

$$M' = \{ e \in M : A \cap U \cap e \neq \emptyset \} \cup N$$

is a matching in $G$. Otherwise, there had to exist two edges, $e \in M$ with $A \cap U \cap e \neq \emptyset$ and $f \in N$, sharing a vertex $x$. By $x \in f \in N$ we have either $x \in U \setminus A$ or $x \in A$. In the former case the vertex of $e$ distinct from $x$ needs to be in $A \cap U$. In particular, both ends of $e$ are in $U$, contrary to $e \in M \subseteq E(G')$ and $U$ being independent in $G'$.

So we are left with the case $x \in A$. Now we have in fact $A \cap U \cap e = \{ x \}$ and both ends of $f$ are in $U$. Moreover, $x \in A \cap U \cap e \subseteq A \cap V(M)$ shows that $f$ does not get deleted when we pass from $G$ to $G'$. This contradiction to $U$ being independent in $G'$ proves that $M'$ is indeed a matching in $G$.

Therefore the maximality of $M$ yields

$$|N| + |V(M) \cap A \cap U| = |M'| \leq |M| = |V(M) \cap A \cap U| + |V(M) \cap (A \setminus U)|,$$

i.e.,

$$|U \setminus A| = |N| \leq |V(M) \cap (A \setminus U)|.$$

Thus

$$|U \setminus A| \leq |A \setminus U|,$$

and, consequently, $|U| \leq |A| = \alpha(G)$, as desired. \hfill \Box

To unleash the full power of the foregoing lemma it is useful to know that certain subsets of $V(G)$ can be forced to be subsets of $V(M)$. For such purposes we shall employ the following observation.

Fact 2.3. Let $H$ be a bipartite graph with vertex classes $R$ and $S$ in which the largest matching has size $m$. If some set $R' \subseteq R$ is matchable into $S$, then $H$ contains a matching of size $m$ saturating all vertices of $R'$.

Proof. Let $\mathcal{M}$ be the set of all matchings in $H$ having size $m$. Choose first a matching $N$ from $R'$ into $S$ saturating all vertices in $R'$, and then a matching $M \in \mathcal{M}$ for which $|M \cap N|$ is maximal. We will show that $M$ covers $R'$. Otherwise there was a vertex $x \in R'$ not covered by $M$. Let $z$ be the unique vertex with $xz \in N$. By the maximality of $m$, the $m + 1$ edges in $M \cup \{xz\}$ cannot form a matching in $H$ and, consequently, we have $z \in V(M)$. So there is a vertex $u$ such that $uz \in M$. But now $M' = (M \setminus \{uz\}) \cup \{xz\}$ is a matching in $\mathcal{M}$ satisfying $|M' \cap N| > |M \cap N|$, contradicting our choice of $M$. \hfill \Box
2.2. Symmetrisation. One of the standard proofs of Turán’s theorem [18], due to Zykov [19], shows that every $K\ell$-free graph $G$ can be transformed into an $(\ell - 1)$-partite graph on the same vertex set by a sequence of symmetrisation operations, in such a way that throughout the whole process the number of edges never decreases. Indeed, this implies that $G$ does have at most as many edges as the corresponding Turán graph. The *symmetrisation* step employed by Zykov consists in taking two non-adjacent vertices $u$ and $v$, deleting all edges incident with $u$, and then adding all edges from $u$ to the neighbours of $v$.

Our proof of Theorem 1.2 utilises a modest generalisation of this idea. Given a graph $G$ and two disjoint sets $A, B \subseteq V(G)$, we say that a graph $G'$ on the same vertex set as $G$ arises from $G$ by the *generalised Zykov symmetrisation* $\text{Sym}(A, B)$ if it is obtained by deleting all edges incident with $B$ and afterwards adding all edges from $A$ to $B$. Explicitly, this means

$$V(G') = V(G) \quad \text{and} \quad E(G') = (E(G) \setminus \{e \in E(G) : e \cap B \neq \emptyset\}) \cup K(A, B).$$

We will express this state of affairs by writing $G' = \text{Sym}(G \mid A, B)$. In the special case where $B = \{v\}$ is a singleton, we will often abbreviate $\{v\}$ to $v$, thus speaking, e.g. of the operation $\text{Sym}(A, v)$. For later use we record the following obvious properties of these operations.

**Fact 2.4.** Given a graph $G$ and two disjoint sets $A, B \subseteq V(G)$, let $G' = \text{Sym}(G \mid A, B)$.

(i) If $|A| \geq \deg_G(b)$ holds for all $b \in B$, then $e(G') \geq e(G)$. If equality occurs, then all vertices in $B$ have degree $|A|$ in $G$ and $B$ is an independent set in $G$.

(ii) If $A$ is independent and $G$ is triangle-free, then so is $G'$.

**Proof.** Part (i) follows from the estimate

$$e(G') - e(G) \geq \sum_{b \in B}(|A| - \deg_G(b)) \geq 0,$$

where the first “$\geq$” sign takes into account that edges both of whose ends are in $B$ are subtracted twice in the sum over $b \in B$. The statement addressing the equality case should now be clear.

To prove part (ii) we assume for the sake of contradiction that $xyz$ was a triangle in $G'$. Owing to $G - B = G' - B$ we may suppose further that $x \in B$. Now $y$ and $z$ are neighbours of $x$ in $G'$ and, hence, both of them are in $A$. But $A$ is still independent in $G'$ and thus $yz$ cannot be an edge of $G'$.

Our next result describes a case where symmetrisation preserves the independence number.
Lemma 2.5. Let $A$ and $B$ be two disjoint independent sets in a graph $G$ such that $|A| = |B| = \alpha(G)$. If $M$ is a matching from $V(G) \setminus (A \cup B)$ to $B$, whose size is as large as possible, $B' \subseteq B \setminus V(M)$, and $G' = \text{Sym}(G \mid A, B')$, then $\alpha(G') = \alpha(G)$.

Proof. Since $A$ is independent in $G'$, we have $\alpha(G') \geq \alpha(G)$. Now suppose conversely that $U \subseteq V(G)$ is independent in $G'$. We are to prove $|U| \leq \alpha(G)$.

If $A$ and $U$ are disjoint it follows from Lemma 2.2 (applied to $G - A$, $B$, and $M$ here in place of $G$, $A$, and $M$ there) that $|U| \leq \alpha(G - A) = \alpha(G)$, meaning that we are done.

It remains to consider the case $U \cap A \neq \emptyset$. Now $K(A, B') \subseteq E(G')$ implies that $U$ is disjoint to $B'$. Therefore $U$ is independent in $G$ and so $|U| \leq \alpha(G)$. \hfill \Box

2.3. Some general results. Recall that for $n \geq s \geq 0$ we are interested in the quantity

$$
\text{ex}(n, s) = \max\{|E| : G = (V, E) \text{ is a triangle-free graph with } |V| = n \text{ and } \alpha(G) \leq s\}
$$

and that

$$
\mathcal{E}(n, s) = \{G = (V, E) : K_3 \notin G, \ |V| = n, \ \alpha(G) \leq s, \ \text{and} \ |E| = \text{ex}(n, s)\}
$$

denotes the corresponding family of extremal graphs.

We begin by observing that the estimate $\text{ex}(n, s) \leq g_k(n, s)$ holds whenever $s$ is outside the range required by Conjecture 1.4.

Fact 2.6. Let integers $n \geq s \geq 0$ and $k \geq 2$ be given. If $s \notin \left(\frac{k}{3k-1}n, \frac{k-1}{3k-4}n\right)$, then

$$
\text{ex}(n, s) \leq g_k(n, s)
$$

and equality can only hold if $s \in \left\{\frac{k}{3k-1}n, \frac{k-1}{3k-4}n\right\}$.

Proof. We check that under our assumption on $s$ the trivial upper bound $\text{ex}(n, s) \leq \frac{1}{2}ns$ is at least as good as $g_k(n, s)$. Since for $s \notin \left(\frac{k}{3k-1}n, \frac{k-1}{3k-4}n\right)$,

$$
ns \leq ns + (kn - (3k - 1)s)((k - 1)n - (3k - 4)s)
$$

$$
= k(k - 1)n^2 - 2k(3k - 4)ns + (3k - 4)(3k - 1)s^2 = 2g_k(n, s)
$$

this is indeed the case and the statement about the equality case also easily follows. \hfill \Box

In combination with Fact 1.5 this leads to the following alternative way of defining $g(n, s)$ in case $\frac{s}{n} \in \left(\frac{1}{3}, \frac{1}{2}\right)$.

Corollary 2.7. If $\frac{1}{3}n < s < \frac{1}{2}n$, then

$$
g(n, s) = \min\{g_k(n, s) : k \geq 2\}.
$$

Our first structural result on graphs in $\mathcal{E}(n, s)$ asserts that they contain two disjoint independent sets of size $s$ (provided there is enough space for them).
Lemma 2.8. If \( n \) and \( s \) are two integers with \( n \geq 2s \geq 0 \), then every graph \( G \in \mathcal{E}(n, s) \) contains two disjoint independent sets of size \( s \).

Proof. Let \((X, Y)\) be a pair of disjoint independent sets in \( G \) such that \(|X| = \alpha(G)\) and subject to this \(|Y|\) is as large as possible. Clearly \(|Y| \leq |X| \leq s\) and we are to prove that equality holds throughout. This could fail in two different ways.

Case 1. \(|Y| < |X|\)

Owing to \( \alpha(G - X) = |Y| \leq |X| - 1 \leq s - 1 < n - s \leq v(G - X) \) there is an edge \( ab \) of \( G \) with \( a, b \notin X \). If both of \( a \) and \( b \) have degree \( \alpha(G) \), then their neighbourhoods are two disjoint independent sets of size \( \alpha(G) \), thus contradicting our choice of the pair \((X, Y)\).

It follows that we may assume, without loss of generality, that \( \deg(a) < \alpha(G) \). By Fact 2.4 the graph \( G' = \text{Sym}(G \mid X, a) \) is triangle-free and has more edges than \( G \). So the extremality of \( G \) entails that \( G' \) contains an independent set of size \( \alpha(G) + 1 \). Owing to \( G - a = G' - a \) any such set needs to be of the form \( Z \cup \{a\} \), where \( Z \) is an independent set in \( G \) of size \( \alpha(G) \). Due to the construction of \( G' \) the sets \( X \) and \( Z \) need to be disjoint and thus the pair \((X, Z)\) contradicts our choice of \((X, Y)\).

Case 2. \(|X| = |Y| = \alpha(G) < s\)

Now \(|V(G) \setminus (X \cup Y)| \geq n - 2(s - 1) \geq 2\) and thus there are two distinct vertices in this set, say \( a \) and \( b \). Let \( G' \) be the result of applying first \( \text{Sym}(Y \cup \{b\}; a) \) and then \( \text{Sym}(X \cup \{a\}; b) \) to \( G \) (see Figure 2.1).

![Figure 2.1. Possibly new edges of \( G' \) are drawn green.](image)

Observe that the sets \( Y \cup \{b\} \) and \( X \cup \{a\} \) are independent in \( G' \), whence \( G' \) is triangle-free. Since \( ab \) is an edge of \( G' \), we have \( \alpha(G') \leq \alpha(G) + 1 \leq s \). Notice that for a vertex \( v \) of \( G \) we have \( \deg_{G'}(v) \leq \alpha(G) = |X| = |Y| \) and so

\[
e(G') \geq e(G) - \deg_G(a) - \deg_G(b) + |X| + |Y| + 1 > e(G),
\]

which clearly contradicts the fact that \( G \in \mathcal{E}(n, s) \).

We conclude this section with a result that provides additional information on the structure of some graphs in \( \mathcal{E}(n, s) \).
Lemma 2.9. Given two integers \( n \geq 0 \) and \( s \in \left[ \frac{1}{3}n, \frac{1}{2}n \right] \), there exists a graph \( G \in \mathcal{E}(n, s) \) containing two disjoint independent sets \( A \) and \( B \) of size \( |A| = |B| = s \) having subsets \( A' \subseteq B \) and \( B' \subseteq A \) with \( |A'| = |B'| = 3s - n \) and

\[
K(A', A) \cup K(B', B) \subseteq E(G).
\]

Proof. Let \( G' \in \mathcal{E}(n, s) \) and let \( A, B \subseteq V(G') \) be any two disjoint independent sets of size \( s \), the existence of which is guaranteed by Lemma 2.8. Set \( X = V(G') \setminus (A \cup B) \) and denote by \( M_A \) and \( M_B \) maximum matchings in \( G' \) from \( X \) to \( A \) and from \( X \) to \( B \), respectively.

Since \( |X| = n - 2s \) we have \( |M_A|, |M_B| \leq n - 2s \), wherefore the sets \( A_* = B \setminus V(M_B) \) and \( B_* = A \setminus V(M_A) \) satisfy \( |A_*|, |B_*| \geq s - (n - 2s) = 3s - n \). Take arbitrary subsets \( A' \subseteq A_* \), \( B' \subseteq B_* \) of size \( 3s - n \), and let \( G \) denote the graph arising from \( G' \) by applying first \( \text{Sym}(A, A') \) and then \( \text{Sym}(B, B') \). Two successive applications of Fact 2.4 and Lemma 2.5 tell us that \( G \) is triangle-free, \( \alpha(G) = s \), and \( e(G) \geq e(G') \). Thus, we arrive at a graph \( G \in \mathcal{E}(n, s) \) for which (2.1) holds. \( \square \)

§3. The proof of Theorem 1.2

3.1. Andrásfai’s result. Let us recall that \( g_2(n, s) = n^2 - 4ns + 5s^2 \). The main result of this subsection, Proposition 3.2 below, asserts that this expression is an upper bound on the number of edges of triangle-free \( n \)-vertex graphs satisfying a less restrictive condition than \( \alpha(G) \leq s \). Therefore, this result provides a technical strengthening of Andrásfai’s theorem quoted in the introduction. Our reason for dealing with such a statement here is that we refer to it in the proof that \( \text{ex}(n, s) \leq g_3(n, s) \). We start with the following observation.

Lemma 3.1. Let \( n \) and \( s \) be positive integers with \( s \in \left[ \frac{1}{3}n, \frac{1}{2}n \right] \) and suppose that \( G \) is a triangle-free graph on \( n \) vertices containing two disjoint independent sets \( A \) and \( A' \) with \( |A| = s \) and \( |A'| \geq 3s - n \). If

(i) \( \deg(a) \leq s \) for all \( a \in A \),

and

(ii) \( N(a') = A \) for all \( a' \in A' \),

then \( e(G) \leq n^2 - 4ns + 5s^2 \).

Proof. By passing to a subset if necessary we may assume \( |A'| = 3s - n \). The graph \( G - (A \cup A') \) has \( n - s - (3s - n) = 2(n - 2s) \) vertices, so by Mantel’s theorem it has at most \( (n - 2s)^2 \) edges. By (ii) all edges of \( G - A \) are actually edges of \( G - (A \cup A') \) and thus we have

\[
e(G) \leq \sum_{a \in A} \deg(a) + (n - 2s)^2.
\]
Owing to (i) this leads to \( e(G) \leq s^2 + (n - 2s)^2 = n^2 - 4ns + 5s^2 \), as desired.

We remark that the tools we have developed so far lead to a short proof of Andrásfai’s main result in [1].

**Proof of Theorem 1.1.** Due to Lemma 2.9 there exists a graph \( G \in \mathcal{E}(n, s) \) containing two disjoint independent sets \( A \) and \( A' \) with

\[
|A| = s, \quad |A'| = 3s - n, \quad \text{and} \quad K(A', A) \subseteq E(G). \tag{3.1}
\]

Recall that the absence of triangles in \( G \) implies \( \Delta(G) \leq \alpha(G) \leq s \). Thus \( G \) and the sets \( A, A' \) satisfy the hypothesis of Lemma 3.1 and, consequently,

\[
ex(n, s) = e(G) \leq n^2 - 4ns + 5s^2.
\]

On the other hand, Fact 1.5 shows \( \ex(n, s) \geq n^2 - 4ns + 5s^2 \). (See also the blow-up of the pentagon presented in Figure 1.1.)

The main result of this subsection is a generalisation of Theorem 1.1 that allows vertices whose degree is larger than \( s \). On the other hand, if one applies the statement that follows to a triangle free graph \( G \) with \( \alpha(G) = s \leq \frac{1}{2}n \), then \( Z = \emptyset \) and an arbitrary choice of \( Q \) leads to the estimate \( e(G) \leq n^2 - 4ns + 5s^2 \).

**Proposition 3.2.** For \( s \in \left[ \frac{1}{3}n, \frac{1}{2}n \right] \), let \( G \) be a triangle-free graph on \( n \) vertices containing an independent set \( A \) of size \( s \), for which \( \alpha(G - A) \leq s \). Let

\[
Z = \{v \in V(G) \setminus A : \deg(v) > s\}
\]

and let \( Q \subseteq V(G) \setminus (A \cup Z) \) be any set with \( |Q| \geq 3s - n \). If every independent set \( Z' \subseteq Z \) is matchable into \( V(G) \setminus (A \cup Z' \cup Q) \), then

\[
e(G) \leq n^2 - 4ns + 5s^2.
\]

**Proof.** Let \( n \) and \( s \) be fixed and assume for the sake of contradiction, that there exists a counterexample, i.e., a triple \( (G, A, Q) \) satisfying all the assumptions of Proposition 3.2, but for which \( G \) has more than \( n^2 - 4ns + 5s^2 \) edges. Among all such counterexamples we choose one for which the size of the set

\[
Q' = \{q \in Q : N(q) = A\}
\]

is as large as possible. Observe that Lemma 3.1 applied to \( Q' \) here in place of \( A' \) there yields \( |Q'| < 3s - n \), whence

\[
Q' \neq Q. \tag{3.2}
\]

Next we use the maximality of \( Q' \) for showing that the assumption \( \alpha(G - A) \leq s \) holds with equality.
Claim 3.3. There is an independent set \( B \subseteq V(G) \setminus A \) of size \( s \).

Proof. Assume \( \alpha(G - A) \leq s - 1 \). Owing to (3.2) we may pick a vertex \( q_0 \in Q \setminus Q' \) and apply \( \text{Sym}(A, q_*^0) \) to \( G \), thus getting a graph \( G^* \). By Fact 2.4 and \( q_*^0 \notin Z \) we know that \( G^* \) is triangle-free and has at least as many edges as \( G \). Clearly, \( \alpha(G^* - A) \leq \alpha(G - A) + 1 \leq s \) and the assumption of Proposition 3.2 holds for \( G^* \) and the sets \( A, Q \), and

\[
Z^* = \{ v^* \in V(G^*) \setminus A : \deg_{G^*}(v^*) > s \} \subseteq Z.
\]

Since \( |Q' \cup \{ q_*^0 \}| > |Q'| \), the maximality of \( |Q'| \) implies that \( (G^*, A, Q) \) cannot be a counterexample to our result, which yields

\[
e(G) \leq e(G^*) \leq n^2 - 4ns + 5s^2,
\]

contrary to the choice of \( (G, A, Q) \). Thereby Claim 3.3 is proved. \( \square \)

Working with the set \( B \) obtained in the previous claim we define \( X = V(G) \setminus (A \cup B) \) and \( Z' = Z \cap B \). Due to the independence of \( B \) and our hypothesis \( Z' \) is matchable into \( X \). So Fact 2.3 applied to \( Z' \), \( B \), and \( X \) here in place of \( R' \), \( R \), and \( S \) there yields a maximum matching \( M \) between \( B \) and \( X \) which covers \( Z' \).

Now we set \( B' = B \setminus V(M) \) and look at the graph \( G' = \text{Sym}(G|A, B') \). Note that

\[
|B'| \geq s - |X| = s - (n - 2s) = 3s - n.
\]

Fact 2.4 reveals that \( G' \) is triangle-free and satisfies \( e(G') \geq e(G) \). Lemma 2.2 applied to \( G - A, B \), and \( M \) reveals \( \alpha(G' - A) = \alpha(G - A) = s \) and, consequently, every \( a \in A \) has at most \( s \) neighbours in \( G' \). Thus \( G', A, \) and \( B' \) here in place of \( G, A, \) and \( A' \) there satisfy the assumptions of Lemma 3.1, meaning that

\[
e(G) \leq e(G') \leq n^2 - 4ns + 5s^2.
\]

This contradiction to \( (G, A, Q) \) being a counterexample establishes Proposition 3.2. \( \square \)

3.2. Blow-ups of Wagner graphs. The present subsection completes the proof of Theorem 1.2, asserting that every graph \( G \in \mathcal{E}(n, s) \) satisfies \( e(G) \leq 3n^2 - 15ns + 20s^2 \). Recall that by Lemma 2.8 any such graph \( G \) contains two disjoint independent sets \( A \) and \( B \) of size \( s \). Our next result shows that if there exists a further independent set of size \( s \) having appropriate intersections with \( A \) and \( B \), then we can reach our goal.

Lemma 3.4. Given \( n \geq s \geq 0 \) let \( G \) be a triangle-free graph on \( n \) vertices with \( \alpha(G) = s \). If \( G \) contains three independent sets \( A, B, \) and \( C \) of size \( s \) such that \( A \cap B, A \cap C = \emptyset \) and \( |B \cap C| \leq n - 2s \), then

\[
e(G) \leq 3n^2 - 15ns + 20s^2.
\]
Proof. Recall that by the case $k = 3$ of Fact 2.6 we may assume $\frac{3}{8}n < s < \frac{2}{3}n$. Our argument is reminiscent of the proof of Lemma 2.9. Fix three independent sets $A$, $B$, and $C$ in $G$ such that $A$ is disjoint to $B$, $C$ and $|B \cap C| \leq n - 2s$.

**Claim 3.5.** We may assume that there are sets $B' \subseteq A$, $C' \subseteq A \setminus B'$, and $A' \subseteq B \cap C$ of size $|A'| = |B'| = |C'| = 3s - n$ such that

$$K(A', A) \cup K(B', B) \cup K(C', C) \subseteq E(G).$$

Proof. Take a maximum matching $M_B$ from $V(G \setminus (A \cup B))$ into $A$ and consider the graph $G_1 = \text{Sym}(G | B, A \setminus V(M_B))$. By Fact 2.4 this graph is triangle-free and satisfies $e(G_1) \geq e(G)$. Lemma 2.5 yields $\alpha(G_1) = s$ and one checks easily that $A$, $B$, and $C$ are still independent in $G_1$. Moreover,

$$|A \setminus V(M_B)| \geq |A| - |V(G \setminus (A \cup B)| = s - (n - 2s) = 3s - n > 0$$

so there is a set $B' \subseteq A \setminus V(M_B)$ with $|B'| = 3s - n$ and clearly we have $K(B', B) \subseteq E(G_1)$.

Next we observe that the assumption $|B \cap C| \leq n - 2s$ entails $|B \setminus C| \geq 3s - n = |B'|$ and therefore in $G_1$ the set $B'$ is matchable into $B \setminus C$, which is a subset of $V(G \setminus (A \cup C))$. So by Fact 2.3 there is a maximum matching $M_C$ from $V(G \setminus (A \cup C))$ to $A$ which covers all vertices in $B'$. As in the previous paragraph one proves that the graph $G_2 = \text{Sym}(G_1 | C, A \setminus V(M_C))$ is triangle-free, has independence number $s$ and at least as many edges as $G_1$. Also, as before one finds a set $C' \subseteq A \setminus V(M_C)$ with $|C'| = 3s - n$ and observes $K(C', C) \subseteq E(G_2)$. The sets $A$, $B$, and $C$ are still independent in $G_2$ and our reason for insisting on $B' \subseteq V(M_C)$ was that it ensures $C' \subseteq A \setminus B'$. Finally, we let $M_A$ be a maximum matching in $G_2$ from $V(G \setminus (A \cup B))$ to $B$ and put $G_3 = \text{Sym}(G_2 | A, B \setminus V(M_A))$. Standard arguments show that $G_3$ is triangle-free and satisfies $\alpha(G_3) = s$ as well as $e(G_3) \geq e(G_2)$. Furthermore, there is a set $A' \subseteq B \setminus V(M_A)$ with $|A'| = 3s - n$ such that $K(A', A) \subseteq E(G_3)$. Moreover, since $C \cup A'$ is an independent set, we have $A' \subseteq C$. Altogether, the graph $G_3$ has all desired properties and owing to $e(G_3) \geq e(G)$ we may continue the proof with $G_3$ instead of $G$. This proves Claim 3.5. \[ \square \]

Now we set $n^* = 4n - 9s$, $s^* = n - 2s$, $G^* = G - (A' \cup B' \cup C')$, and $A^* = B \setminus A'$, $Q^* = C \setminus B$.

**Claim 3.6.** The numbers $n^*$ and $s^*$ as well as the triple $(G^*, A^*, Q^*)$ satisfy the hypothesis of Proposition 3.2.

Proof. A quick calculation based on $s \in [\frac{1}{3}n, \frac{2}{5}n]$ shows that $n^* > 0$ and $s^* \in [\frac{1}{3}n^*, \frac{1}{2}n^*]$, i.e., that the size of $s^*$ is in the appropriate range. Owing to

$$|V(G^*)| = n - (|A'| + |B'| + |C'|) = n - 3s = 4n - 9s = n^*$$
and $|A^*| = |B| - |A'| = s - (3s - n) = n - 2s = s^*$ the sets $V(G^*)$ and $A^*$ have the correct size. Next, we would like to show $\alpha(G^* - A^*) \leq s^*$. If $J \subseteq V(G^*) \setminus A^*$ is independent in $G^*$, then $J \cup B'$ is independent in $G$, and thus we have $|J| \leq s - |B'| = s^*$, as desired.

Regarding the set $Z^* = \{v \in V(G^*) \setminus A^* : \deg_{G^*}(v) > s^*\}$ we contend

$$Z^* \subseteq V(G) \setminus (A \cup B \cup C). \quad (3.3)$$

Indeed, if $v \in V(G^*) \cap A$, then $\deg_{G^*}(v) = \deg_G(v) - |A'| \leq s - (3s - n) = s^*$ and the same reasoning applies to $B$ and $C$ in place of $A$ as well.

As a consequence of (3.3) we have $Q^* \subseteq V(G^*) \setminus (A^* \cup Z^*)$, as required. The assumption $|B \cap C| \leq n - 2s$ implies $|Q^*| = s - |B \cap C| \geq 3s - n = 3s^* - n^*$, so $Q^*$ is sufficiently large for our purposes.

Finally, if $Z' \subseteq Z^*$ is independent in $G^*$, then $Z'$ is also independent in $G$ and (3.3) shows that $Z'$ has to be disjoint to $A$. Thus, Fact 2.1 tells us that in $G$ there is a matching $M$ from $Z'$ to $A$. By Claim 3.5 and (3.3) such a matching can only use the part $A \setminus (B' \cup C')$ of $A$, meaning that $Z'$ is indeed matchable into $V(G^*) \setminus (A^* \cup Z' \cup Q^*)$. Thereby Claim 3.6 is proved.

Now the foregoing claim and Proposition 3.2 result in

$$e(G^*) \leq n^* s^* - 4n^* s^* + 5s^* = 5n^2 - 24ns + 29s^2,$$

so

$$e(G) = |B||B'| + |C||C'| + |A \setminus (B' \cup C')||A'| + e(G^*)$$

$$\leq 2s(3s - n) + (2n - 5s)(3s - n) + (5n^2 - 24ns + 29s^2)$$

$$= 3n^2 - 15ns + 20s^2. \quad \square$$

Perhaps somewhat surprisingly, the previous result allows us to study a similar configuration, where $C$ is no longer disjoint to one of $A$ and $B$.

**Lemma 3.7.** Suppose $n \geq s \geq 0$ and let $G$ be a triangle-free graph on $n$ vertices with $\alpha(G) = s$ containing two disjoint independent sets $A$ and $B$ of size $s$, which in turn have subsets $A' \subseteq B$, $B' \subseteq A$ with $|A'| = |B'| = 3s - n$ and

$$K(A', A) \cup K(B', B) \subseteq E(G).$$

If $G$ contains a further independent set $C$ of size $s$ intersecting both $A$ and $B$, then

$$e(G) \leq 3n^2 - 15ns + 20s^2.$$
Proof. Regarding \( n \) and \( s \) as being fixed we consider a counterexample with \(|Q|\) maximal, where
\[
Q = \{ q \in V : N(q) = C \}.
\]

Let us start with a few basic observations. First of all, we have \( C \cap A' = \emptyset \), because \( K(A', A) \subseteq E(G) \) and \( A \) intersects \( C \). Similarly, one checks \( C \cap B' = \emptyset \). Furthermore, since \( \Delta(G) \leq \alpha(G) = s \), the sets \( (C \setminus A) \cup A' \) and \( (C \setminus B) \cup B' \) are independent. Hence they consist of at most \( s = |C| \) vertices each, which yields
\[
|A \cap C| \geq |A'| = 3s - n \quad \text{and} \quad |B \cap C| \geq |B'| = 3s - n. \tag{3.4}
\]
Finally, using the definition of \( Q \), the independence of \( A, B, \) and \( C \), and the fact that \( C \) meets \( A \) and \( B \), one obtains
\[
Q \cap (A \cup B \cup C) = \emptyset. \tag{3.5}
\]
The maximality of \( Q \) together with Lemma 3.4 leads to the following statement.

Claim 3.8. We have \( V(G) = A \cup B \cup C \cup Q \).

Proof. Assume contrariwise, that there exists a vertex \( q \in V(G) \setminus (A \cup B \cup C \cup Q) \) and set \( G' = \text{Sym}(G \mid C, q) \). The sets \( A, B, A', B' \), and \( C \) still satisfy the hypothesis of Lemma 3.7 in \( G' \). Moreover, Fact 2.4 tells us that \( G' \) is a triangle-free graph with \( e(G') \geq e(G) \). Due to the maximality of \(|Q|\) all this is only possible if \( \alpha(G') > s \). This means that there exists an independent set \( D' = \{ q \} \cup D \subseteq V(G') \) in \( G' \) with \(|D'| > s \). As usual, \( D \) needs to be an independent set in \( G \) with \(|D| = s \) and \( C \cap D = \emptyset \).

The case \( k = 3 \) of Fact 2.6 allows us to assume \( s > 3n/8 \), which leads to
\[
|A'| + |B'| + |C| + |D| = 8s - 2n > n. \]
Thus \( D \cap (A' \cup B') \neq \emptyset \) and without loss of generality we may suppose \( D \cap A' \neq \emptyset \), which in turn implies \( D \cap A = \emptyset \). Moreover, we have
\[
|B \cap D| \leq |B \setminus C| = |B| - |B \cap C| \overset{(3.4)}{\leq} n - 2s.
\]

So altogether the graph \( G \) and the sets of vertices \( A, B, \) and \( D \) satisfy the assumption of Lemma 3.4 with \( D \) here in place of \( C \) there. But now \( e(G) \leq 3n^2 - 15ns + 20s^2 \) contradicts \( G \) being a counterexample and, hence, establishes Claim 3.8. \( \Box \)

Continuing the proof of Lemma 3.7 we observe that, owing to the definition of \( Q \), the set \((A \setminus C) \cup Q\) is independent, and so \(|Q| \leq |A \cap C|\). In combination with Claim 3.8 this yields
\[
n \overset{(3.5)}{=} |A \cup B \cup C| + |Q| \leq 3s - |A \cap C| - |B \cap C| + |A \cap C| = 3s - |B \cap C| \overset{(3.4)}{\leq} n,
\]
and equality holds throughout. In particular, we have $|Q| = |A \cap C|$ and consequently $E = (A \setminus C) \cup Q$ is an independent set consisting of $s$ vertices. To complete the proof it suffices to show that the assumptions of Lemma 3.4 are satisfied for $B$, $A$, and $E$ here in place of $A$, $B$, and $C$ there. The condition $B \cap E = \emptyset$ is clear and in the light of (3.4) we obtain

$$|A \cap E| = |A \setminus C| = |A| - |A \cap C| \leq s - (3s - n) = n - 2s.$$  \hfill \Box

We proceed to the main result of this article, which we reformulate as follows.

**Theorem 3.9.** If $n \geq s \geq 0$, then

$$\text{ex}(n, s) \leq g_3(n, s) = 3n^2 - 15ns + 20s^2.$$  \hfill (3.6)

Moreover, for $s \in \left[\frac{3}{8}n, \frac{2}{5}n\right]$ equality holds.

**Proof.** The statement on equality follows from the first part in view of Fact 1.5 (see also Figure 1.2), so it remains to establish (3.6).

Arguing indirectly we take a counterexample $(n, s)$ with $n$ minimum. The case $k = 3$ of Fact 2.6 tells us

$$\frac{n}{3} < \frac{3n}{8} < s < \frac{2n}{5} < \frac{n}{2}. \quad (3.7)$$

By Lemma 2.9 there exists a graph $G \in \mathcal{E}(n, s)$ which contains two disjoint independent sets $A$ and $B$ of size $s$ such that there are sets $A' \subseteq B$, $B' \subseteq A$ with $|A'| = |B'| = 3s - n > 0$ and

$$K(A', A) \cup K(B', B) \subseteq E(G).$$

Pick two arbitrary vertices $a \in A'$, $b \in B'$ and set $G' = G - \{a, b\}$. We shall consider two cases depending on the independence number of $G'$.

**Case 1.** $\alpha(G') = s$.

Take an independent set $C \subseteq V(G')$ of size $s$. Since the set $C \cup \{a\}$ has size $s + 1$ and thus fails to be independent in $G$, we have $C \cap A \neq \emptyset$. A similar argument shows $C \cap B \neq \emptyset$ and, hence, $C$ is as demanded by Lemma 3.7. Consequently we have $e(G) \leq 3n^2 - 15ns + 20s^2$, which contradicts $(n, s)$ being counterexample.

**Case 2.** $\alpha(G') \leq s - 1$.

Observe that the minimality of $n$ leads to

$$e(G') \leq 3(n - 2)^2 - 15(n - 2)(s - 1) + 20(s - 1)^2$$

$$= (3n^2 - 15ns + 20s^2) + (3n - 10s + 2).$$

Since (3.7) yields $3n + 1 \leq 8s$, this implies

$$e(G') \leq (3n^2 - 15ns + 20s^2) + (1 - 2s)$$
and consequently we have
\[ e(G) = e(G') + (2s - 1) \leq 3n^2 - 15ns + 20s^2, \]
which again contradicts the assumption that \((n, s)\) is a counterexample. □

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